ERGODICITY OF BOWEN–MARGULIS MEASURE
FOR THE BENOIST 3-MANIFOLDS

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ABSTRACT. We study the geodesic flow of a class of 3-manifolds introduced by Benoist which have some hyperbolicity but are non-Riemannian, not CAT(0), and with non-$C^1$ geodesic flow. The geometries are nonstrictly convex Hilbert geometries in dimension three which admit compact quotient manifolds by discrete groups of projective transformations. We prove the Patterson–Sullivan density is canonical, with applications to counting, and construct explicitly the Bowen–Margulis measure of maximal entropy. The main result of this work is ergodicity of the Bowen–Margulis measure.

1. INTRODUCTION

In 2004, Yves Benoist released the first results on geodesic flows of compact quotients of properly convex domains in real projective space endowed with the Hilbert metric, proving that strict convexity of the domain is equivalent to an Anosov geodesic flow of the quotient [3]. Not long after, Benoist produced nontrivial examples of nonstrictly convex domains with compact quotients in dimension three, and proved rigid geometric properties for these domains ([4], see Theorem 1.4).

This family of 3-manifolds, whose quotients we call the Benoist 3-manifolds, lack the Anosov property but have have similar topological properties to nonpositively curved manifolds which are rank one. Hence they are promising candidates for studying the geodesic flow. However, the geometry is only Finsler and not Riemannian, meaning angles are not defined, the natural metric is not CAT(0), and the geodesic flow is not $C^1$. In this work, we extend the approach of Knieper for rank one manifolds [18, 19] and study the geodesic flow of properly convex domains in real projective space, known also as Hilbert geometries, without the strictly convex hypothesis for the first time. We prove the following central result:

**Theorem 1.1.** The Bowen–Margulis measure is an ergodic measure of maximal entropy for geodesic flows of the Benoist 3-manifolds.
In seeking the main result, we develop the asymptotic geometry and Patterson–Sullivan measures at infinity. Let \( \delta_\Gamma \) denote the critical exponent of the fundamental group \( \Gamma \) acting on the universal cover. It is known that \( \delta_\Gamma \) is positive in this setting [8].

**Theorem 1.2.** The universal cover of a Benoist 3-manifold admits a Busemann density of dimension \( \delta_\Gamma \) called the Patterson–Sullivan density, and Busemann densities of the same dimension \( \delta > 0 \) are unique up to constant.

Let \( S_\Omega(x, t) \) be the sphere of Hilbert radius \( t \) about \( x \) and let vol be a natural volume on this sphere. As in [26], Theorem 1.2 can be applied to prove:

**Theorem 1.3.** Let \( \Omega \) be a properly convex, indecomposable Hilbert geometry of dimension three which admits a cocompact action by a discrete, torsion-free group \( \Gamma \) of projective transformations. Then for all \( x \in \Omega \), there is a constant \( a(x) > 0 \) such that

\[
\frac{1}{a} \leq \frac{\text{vol} S_\Omega(x, t)}{e^{\delta t}} \leq a.
\]

A corollary of Theorem 1.3 is that the group \( \Gamma \) is divergent.

**Historical remarks.** The Hilbert metric on a properly convex domain in real projective space is named after Hilbert’s proposed solution to his fourth problem; these geometries are examples of affine metric spaces for which lines are always geodesic. Though much work has been done for geodesic flows of strictly convex Hilbert geometries, little is known on the dynamics in the nonstrictly convex case [3, 11, 10, 13, 9]. As alluded to in the introduction, Benoist first proved that for any properly convex domain \( \Omega \) in \( n \)-dimensional real projective space \( \mathbb{RP}^n \) which is divisible, meaning \( \Omega \) admits a discrete, cocompact action by a group \( \Gamma \) of projective transformations, the following are equivalent: (i) \( \Omega \) is strictly convex, (ii) the topological boundary \( \partial \Omega \) is \( C^1 \), (iii) \( \Gamma \) is \( \delta \)-hyperbolic, and (iv) the geodesic flow of \( M = \Omega/\Gamma \) is Anosov [3, Theorem 1.1]. Since the geodesic flow is also topologically transitive (in fact, mixing, [3, Theorem 1.2]), it follows that there is a unique measure of maximal entropy in the strictly convex setting [7, 15].

Benoist then constructed examples of nonstrictly convex, divisible Hilbert geometries in dimension three which have some hyperbolicity but have isometrically embedded flats [4, Proposition 4.2]. These flats appear as properly embedded triangles \( \Delta \) in \( \Omega \), meaning \( \Delta \subset \Omega \) and \( \partial \Delta \subset \partial \Omega \), which are isometric to \( \mathbb{R}^2 \) with the hexagonal norm in the Hilbert metric [14]. Moreover, he showed that any nonstrictly convex, indecomposable, divisible properly convex set in \( \mathbb{RP}^3 \) must have the same structure:

**Theorem 1.4** ([4, Theorem 1.1]). Let \( \Gamma < \text{SL}(4, \mathbb{R}) \) be a discrete torsion-free subgroup which divides an open, properly convex, indecomposable \( \Omega \subset \mathbb{RP}^3 \), and let \( M = \Omega/\Gamma \). Let \( \mathcal{F} \) denote the collection of properly embedded triangles in \( \Omega \). Then

(a) Every subgroup in \( \Gamma \) isomorphic to \( \mathbb{Z}^2 \) stabilizes a unique triangle \( \Delta \in \mathcal{F} \).

(b) If \( \Delta_1, \Delta_2 \in \mathcal{F} \) are distinct, then \( \Delta_1 \cap \Delta_2 = \emptyset \).
(c) For every $\Delta \in \mathcal{T}$, the stabilizer $\text{Stab}_\Gamma(\Delta)$ contains an index-two $\mathbb{Z}^2$ subgroup.

(d) The group $\Gamma$ has only finitely many orbits in $\mathcal{T}$.

(e) The image in $M$ of triangles in $\mathcal{T}$ is a finite collection $\Sigma$ of disjoint tori and Klein bottles, denoted by $T$. If one cuts $M$ along each $T \in \Sigma$, each of the resulting connected components is atoroidal.

(f) Every open line segment in $\partial \Omega$ is included in the boundary of some $\Delta \in \mathcal{T}$.

(g) If $\Omega$ is not strictly convex, then the set of vertices of triangles in $\mathcal{T}$ is dense in $\partial \Omega$.

This structure is essential to make the arguments needed, and we will refer back to parts of this theorem throughout the paper. Since a version of this theorem does not yet exist in higher dimensions, our arguments are valid only in dimension three. We call compact quotient manifolds of nonstrictly convex, properly convex, indecomposable domains the Benoist 3-manifolds.

The existing theory does not apply to studying the geodesic flows of the Benoist 3-manifolds. The geodesic flow of $\Omega/\Gamma$ has the same regularity of $\partial \Omega$, hence by Benoist’s dichotomy, in the nonstrictly convex setting the geodesic flow is not $C^1$. Crampen's Lyapunov exponents cannot be computed [13], and Pesin theory, which requires the flow to be $C^{1+\alpha}$, does not apply. Knieper’s work uses the existence of an inner product and a notion of angle, so his work on Riemannian rank one manifolds cannot be directly applied [18, 19]. The geometry is not CAT(0) because the isometrically embedded flats, which are properly embedded projective triangles, are not CAT(0) so we cannot use results from the thesis of Ricks [14, 23].

Nonetheless, we can adapt the methods of Knieper in rank one following the Patterson–Sullivan approach [22, 26]. The irregularity of the geometry and our techniques to manage this comprise a significant portion of the paper. The Bowen–Margulis measure comes from the Patterson–Sullivan density in a natural way, and ergodicity follows a variation of the Hopf argument [16]. In the setting we study, the stable and unstable sets are not even locally smooth and are not defined for a dense set of directions, but we are still able to adapt this classical proof.

Structure of the paper. We first introduce Hilbert geometries and the central tools in Section 2. In Section 3 we gather lemmas on the asymptotic geometry and the compatible Busemann function, which apply to arbitrary Hilbert geometries. We then construct the Patterson–Sullivan density for the universal cover of a Benoist 3-manifold in Section 4 and prove the Shadow Lemma (Lemma 4.8), and in Section 5 we prove this construction is canonical (Theorem 1.2), with application to growth rates of volumes of spheres and divergence of $\Gamma$ (Theorem 1.3). Lastly, in Section 6, we construct the Bowen–Margulis measure and complete the proof of the main result, Theorem 1.1.

2. Preliminaries

We say a domain $\Omega \subset \mathbb{R}^n$ is properly convex if $\Omega$ can be represented as a bounded convex set in some affine chart, and denote by $\partial \Omega$ the topological
boundary of $\Omega$ in $\mathbb{RP}^n$. Define $H$ to be a supporting hyperplane to a properly convex $\Omega \subset \mathbb{RP}^n$ if $H$ is a codimension 1 projective subspace of $\mathbb{RP}^n$ which intersects $\partial \Omega$ but not $\Omega$. Then a properly convex $\Omega$ is strictly convex if every supporting hyperplane intersects $\partial \Omega$ at a single point.

For any properly convex domain $\Omega$, fix an affine chart in which $\Omega$ is bounded and define the Hilbert $\Omega$-distance between $x, y \in \Omega$ as follows: define $d_{\Omega}(x, x) = 0$ and for $x \neq y$, let $\overline{xy}$ denote the projective line in this affine chart uniquely determined by $x$ and $y$ and take $a, b \in \partial \Omega$ to be the distinct intersection points of $\overline{xy}$ with $\partial \Omega$. Then

$$d_{\Omega}(x, y) = \frac{1}{2} \log \left| \left[ \begin{array}{c} a \\ x \\ y \\ b \end{array} \right] \right|,$$

where $\left[ \begin{array}{c} a \\ x \\ y \\ b \end{array} \right] = \left[ \begin{array}{c} a-y \\ x-b \\ y-b \\ -x \end{array} \right]$. is the Euclidean cross-ratio, a projective invariant. It will be useful to denote by $[xy]$ the segment of $\overline{xy}$ between $x$ and $y$ in $\overline{\Omega} := \Omega \cup \partial \Omega$, and $[xy] = [xy] \setminus \{y\}$. The cross-ratio of four projective lines $L_1, L_2, L_3, L_4$ intersecting at one point is well-defined as $[L_1 : L_2 : L_3 : L_4] = [a_1 : a_2 : a_3 : a_4]$ where $a_i \in L_i$ and $a_1, a_2, a_3, a_4$ are collinear; this can be used to prove that $d_{\Omega}$ is a well-defined metric. The group $\text{Aut}(\Omega) := \{g \in \text{PSL}(n+1, \mathbb{R}) \mid g \Omega = \Omega\}$ is a subgroup of isometries of $(\Omega, d_{\Omega})$. The Hilbert metric comes from a Finsler norm $F_\Omega$ defined on the tangent bundle $T\Omega$ (see [12]). The norm $F_\Omega$ is only Riemannian when $\Omega$ is an ellipsoid and has the same regularity as the boundary of the domain [25, 12]. Projective lines are geodesic and are the only geodesics in the strictly convex case, but in general geodesics are not always lines. The metric space $(\Omega, d_{\Omega})$ is complete and the topology induced by the metric $d_{\Omega}$ coincides with the ambient Euclidean topology on $\Omega$ in this affine chart.

We say that $\Omega$ in $\mathbb{RP}^n$ is divisible if there exists a discrete subgroup $\Gamma$ of $\text{Aut}(\Omega)$ acting properly discontinuously and cocompactly on $\Omega$. Also, $\Omega$ is decomposable if the cone over $\Omega$ in $\mathbb{R}^{n+1}$ is decomposable, and indecomposable if $\Omega$ is not decomposable (see [21, Section 3] for more details). Let $M = \Omega/\Gamma$ denote the quotient manifold.

Since geodesics are not unique for the Benoist 3-manifolds, we define the geodesic flow to be flowing along projective lines, as is the case when $\Omega$ is strictly convex. More formally, let $\Omega$ be a divisible properly convex domain with dividing group $\Gamma$ and quotient manifold $M$. Let $\ell_v : \mathbb{R} \to M$ be the projective line parameterized at unit Hilbert speed, uniquely determined by $v \in T^1 M$, the unit tangent bundle to $M$ for the Finsler norm $F_\Omega$. The Finsler unit tangent bundle to $\Omega$ is denoted $T^1 \Omega$. Then the Hilbert geodesic flow of $\Omega$ is $\phi^t : T^1 \Omega \to T^1 \Omega$ defined by $\phi^t(v) = \ell_v(t)$, and this flow descends to the geodesic flow $\phi^t$ on $T^1 M$, the unit tangent bundle to the quotient. Note that this geodesic flow has the same regularity as the boundary of the universal cover $\Omega$ (for more details, see [12, Section 2.4]).

Formally, a geodesic $\gamma$ for the Hilbert metric is a path in $\Omega$ or $M$ such that the length of any segment of $\gamma$ is equal to distance between the endpoints. On occasion we will parameterize $\gamma$ at unit Hilbert speed, and treat $\gamma$ as a mapping from $\mathbb{R}$ to $\Omega$ or $M$ to take advantage of the parameterization.
2.1. **Busemann functions.** For any three points \( x, y, z \in \Omega \), we define the *Busemann function* to be

\[
\beta_z(x, y) = d_\Omega(x, z) - d_\Omega(y, z).
\]

Evidently, for all \( z \in \Omega \), the function \( \beta_z \) is anti-symmetric, meaning \( \beta_z(x, y) = -\beta_z(y, x) \), and satisfies the property of a cocycle, that is \( \beta_z(x, y) + \beta_z(y, w) = \beta_z(x, w) \), for all \( x, y, w \in \Omega \). Also \( |\beta_z(x, y)| \leq d_\Omega(x, y) \) by the triangle inequality. Lastly, since \( \Gamma \) is acting on \( \Omega \) by isometries, \( \beta_{\gamma x}(y, y) = \beta_z(x, y) \) for all \( \gamma \in \Gamma \).

Geometrically, \( \beta_z(x, y) \) describes the signed distance between the Hilbert spheres centered at \( z \) passing through \( x \) and \( y \).

**Definition 2.1.** We wish to extend the Busemann functions to the boundary. Define

\[
\beta^-(\xi, x, y) = \inf_{z_n \to \xi} \liminf_{n \to \infty} \beta_{z_n}(x, y),
\]

\[
\beta^+(\xi, x, y) = \sup_{z_n \to \xi} \limsup_{n \to \infty} \beta_{z_n}(x, y).
\]

These functions exist and are bounded in absolute value by \( d_\Omega(x, y) \) for all \( x, y \in \Omega \) and \( \xi \in \partial \Omega \). It is straightforward to verify for all \( x, y, w \in \Omega \) and \( \xi \in \partial \Omega \) that \( \beta^-(\xi, x, y) = -\beta^+(\xi, x, y) \), and

\[
\beta^-(\xi, x, y) + \beta^-(\xi, y, w) \leq \beta^-(\xi, x, w) \leq \beta^+(\xi, x, w) \leq \beta^+(\xi, x, y) + \beta^+(\xi, y, w).
\]

Since \( \Gamma \) acts by isometries, we also have \( \beta^+(\gamma \xi, y, y) = \beta^+(\xi, x, y) \). Then if indeed \( \beta^+ = \beta^- \), we may define \( \beta^\xi = \beta_x^+ \), and see that the anti-symmetric, cocycle, and \( \Gamma \)-invariance properties of \( \beta_z \) for \( z \in \Omega \) extend to \( \beta^\xi \) for \( \xi \in \partial \Omega \). For such \( \xi \in \partial \Omega \), the *horosphere* through \( x \in \Omega \) based at \( \xi \) is the zero set of \( \beta^\xi(x, \cdot) \), denoted by \( \mathcal{H}^\xi(x) \).

2.2. **Boundary points.** Recall that a supporting hyperplane to \( \Omega \) at a point \( \xi \) in \( \partial \Omega \) is a projective hyperplane \( H \) such that \( H \) contains \( \xi \) and \( H \cap \Omega = \emptyset \). Borrowing language from convex geometry, we introduce the following terms:

**Definition 2.2.** A point \( \xi \) in \( \partial \Omega \) is *smooth* if there is a unique supporting hyperplane to \( \Omega \) at \( \xi \). The point \( \xi \) in \( \partial \Omega \) is *extremal* if \( \xi \) is not contained in any open line segment inside \( \partial \Omega \).

Note that smooth points in \( \partial \Omega \) may not be \( C^1 \) points when \( \partial \Omega \) is treated as a curve in an affine chart. For the examples of interest to this work, there will be a dense set of points in the boundary for which the derivative is not defined. By Benoist, the complement of boundaries of properly embedded triangles in \( \partial \Omega \) is exactly the set of smooth extremal points:

**Proposition 2.3** ([4, Proposition 3.8]). Let \( \Gamma \) be a discrete, torsion-free subgroup of \( \text{PSL}(4, \mathbb{R}) \) which divides an indecomposable, divisible, properly convex domain in \( \mathbb{R}^3 \). Then
a) For every nontrivial line segment $\sigma \subset \partial \Omega$, there exists a properly embedded triangle $\Delta$ such that $\sigma \subset \partial \Delta$.

b) A point $x \in \partial \Omega$ is smooth if and only if $x$ is not the vertex of any properly embedded triangle in $\Omega$.

It follows from Theorem 1.4 that smooth extremal points are dense in $\partial \Omega$ in the setting of interest. We will see that these smooth extremal points carry the hyperbolic behavior of the dynamics, and the Busemann functions will be well-defined for these points.

**Remark 2.4.** We point out a particular feature that is special for the Benoist 3-manifolds, and essential for our study. By Benzecri’s thesis work, [2], there are no angular points in the boundary of the universal cover of a Benoist 3-manifold. Benoist extracts the consequences of this result in Proposition 2.3 and Theorem 1.4 part (b). If $\Omega/\Gamma$ is a Benoist 3-manifold, then every point in the boundary of $\Omega$ is either smooth or extremal; the only exceptions to smoothness are vertices of properly embedded triangles, and the only exceptions to extremality are points in the open edges of properly embedded triangles, and these cannot coincide (distinct properly embedded triangles have disjoint closures).

2.3. **Busemann densities.** We introduce here a nonstandard definition of Busemann densities to address issues with nonsmooth points in the boundary, where the Busemann functions are not well-defined.

**Definition 2.5.** A Busemann density of dimension $\alpha > 0$ for $\Omega$ is a family of finite Borel measures $\{\mu_x\}_{x \in \Omega}$ supported on $\partial \Omega$ which satisfy:

- (quasi-$\Gamma$-invariance) for all $\gamma \in \Gamma$, $\gamma \ast \mu_x = \mu_{\gamma x}$, and
- (transformation rule) for all $x, y \in \Omega$ and $\xi \in \text{supp} \mu_y$, the measures $\mu_x$ and $\mu_y$ are absolutely continuous, and in particular their Radon–Nikodym derivative satisfies
  \[ e^{-\alpha \beta_\gamma(x,y)} \leq \frac{d\mu_x}{d\mu_y}(\xi) \leq e^{-\alpha \beta_\gamma(x,y)}. \]

Note that if the Busemann function was well-defined for every point in $\partial \Omega$, we would recover the standard definition of a Busemann (conformal) density. To prove Theorem 1.2, we will construct a Busemann density for which almost every point is smooth and extremal, and consequently the Busemann functions will be defined almost everywhere and the density will be conformal in the usual sense.

2.4. **Shadow topology.** At times we will take advantage of the ambient Euclidean topology on $\Omega$, represented as a bounded convex domain in an affine chart, and the induced topology on $\partial \Omega$ and $\overline{\Omega} = \Omega \cup \partial \Omega$. We define another topology on $\partial \Omega$ which interacts with the Hilbert geometry inside $\Omega$ as follows.

**Definition 2.6.** Let $B_\Omega(x, r)$ be the open metric $d_\Omega$-ball about $x \in \Omega$ of radius $r$. Then the shadow of radius $r$ from $x$ to $y$ is denoted by $\Theta_r(x, y)$, and is equal to the endpoints of projective rays based at $x$ which pass through the open metric
ball $B_{\Omega}(y, r)$. These shadows generate a possibly basepoint-dependent topology on $\partial \Omega$ called the shadow topology based at $x$. More precisely, the shadow topology based at $x$ is the topology generated by the set

$$\{\Theta_r(x, y) : y \in \Omega, r > 0\}.$$  

It is straightforward to confirm that this topology agrees with the ambient Euclidean topology, and is therefore basepoint independent. If the properly convex domain $\Omega$ was strictly convex with $C^1$ boundary, then a basis for this topology is the set $\{\Theta_r(x, y) : y \in \Omega\}$ for any fixed $r > 0$. We cannot conclude as much in the nonstrictly convex setting, but we will see that this topology is still well-behaved near the smooth extremal points, which will suffice for the development of the Patterson–Sullivan theory.

2.5. **Regular vectors.** For any vector $v \in T^1\Omega$, there is a unique oriented projective line $\ell_v$ determined by $v$, and we let $v^-$ and $v^+$ denote the intersections of $\ell_v$ in $\partial \Omega$ in backward and forward time, respectively.

**Definition 2.7.** A vector $v \in T^1\Omega$ is regular if both $v^-$ and $v^+$ are smooth extremal points. The set of regular vectors in $T^1\Omega$ is denoted by $T^1\Omega_{\text{reg}}$. Regularity is preserved by projective transformations, so a vector in $T^1M$ is regular if any lift in $T^1\Omega$ is regular, and $T^1M_{\text{reg}}$ is the set of all regular vectors in $T^1M$.

2.6. **Standing assumptions.** In Section 3, we take $\Omega$ to be an arbitrary properly convex set in real projective space in unspecified dimension. In the remaining Sections 4, 5, and 6, we assume $\Omega$ is a nonstrictly convex, properly convex, divisible, indecomposable domain in real projective space of dimension 3, with discrete torsion-free dividing group $\Gamma$, so that $M = \Omega/\Gamma$ is a Benoist 3-manifold.

Throughout the paper, we fix an affine chart in which $\Omega$ is bounded, and work with $\Omega$ in this affine chart.

3. **ASYMPTOTIC GEOMETRY**

In this section, we will prove some straightforward lemmas on the shadow topology and Busemann functions for a Hilbert geometry in any dimension. These results are likely well-understood by experts; the proofs are included for completeness.

3.1. **The shadow topology.** The following lemma confirms that for an extremal point $\xi$ in $\partial \Omega$, shadows of a fixed radius generate the local topology at $\xi$. This fact requires that $\xi$ is both smooth and extremal.

**Lemma 3.1.** Let $y_n \in \Omega$ be a sequence of points converging along a projective line to $\xi \in \partial \Omega$. Then $\xi$ is an extremal point in $\partial \Omega$ if and only if for all $r < 0$, $x \in \Omega$,

$$\bigcap_{n \in \mathbb{N}} \Theta_r(x, y_n) = \{\xi\}.$$
Figure 3.1. For the proof of Lemma 3.2. In the left panel, we take the 2-dimensional intersection of $\Omega$ with the projective plane $P$ determined by $x, \xi,$ and $y.$ In the right panel, we take a sequence 2-dimensional intersections of $\Omega$ with the projective plane $P$ determined by the projective lines $xz$ and $yz,$ and see that $\beta_{\xi}(x, y) = \beta_{\xi}(x, x_n) = \frac{1}{2} \log[x^{-} : x : x_n]$, where $x_n$ is as pictured. In Lemma 3.2 we confirm that if $\xi$ is smooth, then the image on the right converges to the image on the left, and $\beta_{\xi}(x, y) = \frac{1}{2} \log[x^{-} : x : \xi]$ as pictured in the left panel.

**Proof.** Let $\eta$ be a point in $\partial \Omega$ distinct from $\xi$, and let $y = y_0$. For each $n$, let $x_n$ be a closest point to $y_n$ on the projective line from $x$ to $\eta$ in the Hilbert metric. Let $a_n, b_n$ be such that $d_{\Omega}(x_n, y_n) = \frac{1}{2} \log[a_n : x_n : y_n : b_n]$. Since $\xi$ is extremal, $y_n$ converging to $\xi$ implies the same for $b_n$, hence the Hilbert distance between $x_n$ and $y_n$ goes to infinity as $n$ grows. Hence for large $n$, the projective ray $(x\eta)$ does not intersect the ball $B_\Omega(x, y_n)$, and $\eta$ is not in $O_{r}(x, y_n)$.

Conversely, see that if $\xi$ is not extremal, then there is an open line segment contained in the shadow $O_{r}(x, y_n)$ for all $n$.

3.2. **The Busemann function and horospheres.** In this subsection, we verify some regularity properties of the Busemann function.

**Lemma 3.2.** The Busemann function $\beta_{\xi}$ is well-defined on smooth points $\xi$ in $\partial \Omega$, and $\beta_{\xi}(x, y)$ varies continuously over the inputs $x$ and $y$ in $\Omega$.

More specifically, we have the following geometric description of the Busemann function: for any $x, y \in \Omega$, and any smooth boundary point $\xi \in \partial \Omega$, let $H_{\xi}$ be the supporting hyperplane to $\Omega$ at $\xi$, and let $x^{-}, y^{-}$ be the intersection points of the lines $x\xi, y\xi$ respectively with $\partial \Omega$ which are not $\xi$. If $x^{-} \neq y^{-}$, let $q(x, y, \xi)$ be the unique intersection point of the line $\overline{x^{-}y^{-}}$ with the hyperplane $H_{\xi}$ in projective space. Then if $x, y$ and $\xi$ are not collinear,

$$\beta^+_{\xi}(x, y) = \frac{1}{2} \log[\overline{x^{-}q} : \overline{xq} : \overline{yq} : \overline{\xi q}]$$

and otherwise,

$$\beta^-_{\xi}(x, y) = \beta^+(x, y) = \frac{1}{2} \log[x^{-} : x : y : \xi].$$
Proof. Suppose first that \(x, y, \) and \(\xi\) are not collinear, so there is a unique projective plane \(P\) containing \(x, y, \) and \(\xi\). Since \(\xi\) is smooth in \(\partial \Omega\), then \(\xi\) is also smooth in \(\partial \Omega \cap P\), and there is a unique supporting hyperline \(L_\xi\) to \(\partial \Omega \cap P\) at \(\xi\). For each \(n\) sufficiently large, let \(P_n\) be the projective plane containing the three points \(x, y, \) and \(z_n\). As \(n\) goes to infinity, \(P_n\) converges to \(P\) in the Gromov-Hausdorff sense.

In the projective plane \(P_n\), let \(y^+_n\) and \(y^-_n\) be the intersection points of the projective line \(\overline{yz}_n\) with the boundary \(\partial \Omega \cap P_n\), such that \(y^-_n\) is closer to \(y\) than \(z_n\). It follows that the sequence of points \(q_n\) in \(P_n\), which converges to \(P\) in the Gromov-Hausdorff sense, goes to infinity, so \(q_n\) converges to \(P\) in the Gromov-Hausdorff sense.

A quick calculation confirms the cross-ratio has the property that
\[
[a : x_1 : x_2 : b](a : x_2 : x_3 : b) = [a : x_1 : x_3 : b].
\]

Then since the cross-ratio of four lines is well-defined and \(q_n x_n^\xi = q_n y_n^\xi = x_n^\xi y_n^\xi\),
\[
\beta_{z_n}(x, y) = \frac{1}{2} \log[\overline{x_n^\xi y_n^\xi} : \overline{x_n^\xi z_n^\xi} : \overline{x_n^\xi y_n^\xi} : \overline{y_n^\xi z_n^\xi}] = \frac{1}{2} \log[\overline{x_n^\xi y_n^\xi} : \overline{y_n^\xi z_n^\xi} : \overline{x_n^\xi y_n^\xi} : \overline{y_n^\xi z_n^\xi}]
\]

Now, as \(z_n\) converges to \(\xi\), the points \(x_n^-\) and \(y_n^-\) converge to \(x^-\) and \(y^-\) respectively, hence the lines \(\overline{x_n^- y_n^-}\) converge to the line \(\overline{x^- y^-}\). Both \(y_n^+\) and \(x_n^+\) converge to the smooth point \(\xi\) and are contained in the plane \(P_n\) which converges to the plane \(P\), so the lines \(\overline{y_n^+ x_n^+}\) must converge to the unique supporting projective line \(L_\xi\) to \(\xi\) in \(\partial \Omega \cap P\). It follows that the sequence of points \(q_n\) in \(P_n\), which is the unique supporting hyperline of the lines \(\overline{x_n^- y_n^-}\) and \(\overline{x_n^+ y_n^-}\), converges to the intersection point \(q\) of the lines \(\overline{x^+ y^-}\) and \(L_\xi = \overline{\xi q}\). The conclusion follows, in this case where \(x, y, \) and \(\xi\) are not collinear.

If \(x, y, \) and \(\xi\) are collinear, then let \(w\) be any other point in \(\Omega\) which is not collinear. Then a short calculation confirms that \(\beta_{z_n}(x, y) = \beta_{z_n}(x, w) + \beta_{z_n}(y, w)\) converges to \(\beta_{\xi}(x, w) + \beta_{\xi}(y, w) = \frac{1}{2} \log[\overline{x^- : y : \xi}]\) as desired. Finally, it is now clear to see geometrically that the Busemann functions at fixed \(\xi\) vary continuously in the inputs \(x, y\) where \(x, y\) and \(\xi\) are not collinear.

Thus, we have:

**Corollary 3.3.** When \(\xi\) is smooth, the Busemann functions \(\beta_{\xi}\) satisfy the anti-symmetric, cocycle, and \(\Gamma\)-invariance properties, as discussed in Definition 2.1.

**Lemma 3.4.** The Busemann functions restricted to smooth points in \(\partial \Omega\) are continuous.

Proof. Since \(\xi\) is a smooth point in \(\partial \Omega\), the Busemann functions at \(\xi\) are well-defined by Lemma 3.2. Let \(x^-\), \(y^-\) be the other intersection points of \(\overline{\xi x}, \overline{\xi y}\) with \(\partial \Omega\), respectively. Let \(H_\xi\) denote the unique supporting hyperplane to \(\Omega\) at \(\xi\). We proceed under the assumption that \(x^- \neq y^-\), so that \(x, y, \) and \(\xi\) determine a projective plane \(P\), though the arguments easily generalize to the case where
Figure 3.2. For the proof of Lemma 3.4. For clarity and simplicity, the figure only depicts the case where $\Omega$ is two-dimensional and $x, y$ are such that $\beta_\xi(x, y) = 0$. By Lemma 3.2, to show that the Busemann functions $\beta_{\xi_n}(x, y)$ converge to $\beta_\xi(x, y)$ as $\xi_n$ converges to $\xi$, it suffices to show that $q_n$ converges to $q$.

$x, y,$ and $\xi$ are collinear by Lemma 3.2. To complete the setup, let $q$ again be the unique intersection of the line $x^- y^-$ with the hyperplane $H_\xi$ in $P$. See Figure 3.2 for clarity.

Let $\xi_n$ in $\partial \Omega$ be a sequence of smooth boundary points converging to $\xi$ and let $H_{\xi_n}$ be the unique supporting hyperplanes to $\Omega$ at $\xi_n$. Take $x_n^-$ to be the other intersection point of $\overline{\xi x_n}$ with $\partial \Omega$ and $y_n^-$ the other intersection point of $\overline{y \xi_n}$ with $\partial \Omega$. Let $P_n$ be any projective plane containing $x, y$ and $\xi_n$; since $\xi_n$ converges to $\xi$, then $P_n$ also converges to the plane $P$ containing $x, y$ and $\xi$. Lastly, take $q_n$ to be the intersection point in $P_n$ of the line $\overline{x_n y_n}$ with the supporting hyperplane $H_{\xi_n}$.

By Lemma 3.2, to show that $\beta_{\xi_n}(x, y)$ converges to $\beta_\xi(x, y)$, it suffices to show that $q_n$ converges to $q$. The points $q_n$ in the compact complement of $\Omega$ must accumulate on some point $q'$, which must be in the plane $P$ because the planes $P_n$ converge to $P$, and must also be on the line $\overline{x y}$ so it cannot equal $\xi$. The lines $\xi_n q_n$ are disjoint from $\Omega$, hence the same holds for the limiting line $\xi q'$. Then $q'$ lies on the unique supporting hyperline to $\Omega \cap P$ at $\xi$, and must equal the unique intersection of $\overline{x y}$ with this line.

Since horospheres are zero sets of the Busemann function, we have:

**Corollary 3.5.** Horospheres based at smooth boundary points are globally defined, continuous, and vary continuously over smooth points.

4. **Patterson–Sullivan Theory**

In this section, we construct the Patterson–Sullivan density for the universal cover of a Benoist 3-manifold. The density is named for the independent work of Patterson and Sullivan in negative curvature and has since been generalized to many settings, including rank one manifolds [22, 26, 18]. Theorems 1.2 and
follow the study of these measures and their properties. To generalize the results beyond dimension three, we need deeper understanding of the geometry of the flats and hyperbolicity of the group in higher dimensions.

4.1. Poincaré series and the critical exponent. The critical exponent, $\delta_{\Gamma}$, of a group $\Gamma$ acting discretely, properly discontinuously, and by isometries on $(\Omega, d_{\Omega})$ is the critical value of $0 \leq s \in \mathbb{R}$ for the Poincaré series,

$$P(x, y, s) = \sum_{\gamma \in \Gamma} e^{-sd_{\Omega}(x, \gamma y)}.$$

The group $\Gamma$ is of divergent type if $P(x, y, \delta_{\Gamma})$ diverges and convergent type if $P(x, y, \delta_{\Gamma})$ converges. It is straightforward to verify that convergence of $P(x, y, s)$ does not depend on $x$ or $y$ by the triangle inequality and that we can realize

$$\delta_{\Gamma} = \limsup_{t \to \infty} \frac{1}{t} \log N_{\Gamma}(x, t),$$

where $N_{\Gamma}(x, t) := \#\{\gamma \in \Gamma \mid d_{\Omega}(x, \gamma x) \leq t\}$ for some $x \in \Omega$. By previous work we have that $\delta_{\Gamma} > 0$ [8]. When $\Gamma$ is a discrete group acting on $\Omega$ with finite co-volume, $\delta_{\Gamma} \leq \dim(\Omega) - 1$, with equality if and only if $\Omega$ is the ellipsoid; this generalizes a result of Crampon for the strictly convex case [5, 10]. In our setting where the quotient $\Omega/\Gamma$ is compact, the inequality $\delta_{\Gamma} \leq \dim(\Omega) - 1$, without the rigidity statement, follows quickly from a theorem of Tholozan that the volume growth entropy is bounded above by $\dim(\Omega) - 1$ [27]. Although the theorem of Tholozan requires no group action at all, in the cocompact case, the critical exponent and volume growth entropy coincide, hence the result can be applied to produce a bound on the critical exponent.

4.2. Patterson–Sullivan densities. We will now prove that a Busemann density exists for the universal cover of a Benoist 3-manifold. The argument will depend on features of the Benoist 3-manifolds discussed in Remark 2.4.

**Proposition 4.1.** There exists a Busemann density of dimension $\delta_{\Gamma} > 0$ on $\partial \Omega$, called a Patterson–Sullivan density.

**Proof.** The construction follows Patterson and Sullivan [22, 26]. For $s > \delta_{\Gamma}$, choose an observation point $o \in \Omega$ for the measures and for the visual boundary. For each $x \in \Omega$ define a measure on $\overline{\Omega}$ by

$$\mu_{x, s} = \frac{1}{P(o, o, s)} \sum_{\gamma \in \Gamma} e^{-sd_{\Omega}(x, \gamma o)} \delta_{\gamma o},$$

where $\delta_{p}$ is the Dirac mass at $p$. Note that for $s > \delta_{\Gamma}$, $\mu_{x, s}$ is supported on $\Omega$. Also, by definition of the critical exponent, if $s > \delta_{\Gamma}$ then $P(x, y, s)$ is finite for all $x, y \in \Omega$ so $\mu_{x, s}(\overline{\Omega}) = P(x, y, s)/P(o, o, s) < \infty$. By compactness of $\overline{\Omega}$ we may extract a weak limit by choosing a convergent subsequence as $s$ decreases to $\delta_{\Gamma}$ to obtain a finite nontrivial measure,

$$\mu_{x} = \lim_{s_{n} \to \delta_{\Gamma}^{-}} \mu_{x, s_{n}}.$$
If the Poincaré series diverges at $\delta_\Gamma$ (\(\Gamma\) is of divergent type), then the total mass of $\mu_{x,s}$ is pushed to $\partial\Omega$ as $s$ decreases to $\delta_\Gamma$ and $P(o,o,s) \to \infty$. At the limit, supp$\mu_x \subset \partial\Omega$. If the Poincaré series converges at $\delta_\Gamma$ (\(\Gamma\) is of convergent type), then we follow Patterson’s method for Fuchsian groups which generalizes to any manifold group [22]. First, he showed it is possible to construct an increasing function $f: \mathbb{R}^+ \to \mathbb{R}^+$ with subexponential growth: that is, for all $\epsilon < 0$, there exists an $x_0(\epsilon) > 0$ such that for all $x > x_0$, $y > 0$

$$f(x + y) \leq f(x)e^{\epsilon y},$$

and the modified Poincaré series

$$P_f(x, y, s) = \sum_{\gamma \in \Gamma} f(d_{\Omega}(x, \gamma y))e^{-s d_{\Omega}(x, \gamma y)}$$

has the same critical exponent $\delta_\Gamma$ and diverges at $s = \delta_\Gamma$ [22, Lemma 3.1]. Then we denote by $\mu^f_{x,s}$ a weak limit as $s$ decreases to $\delta_\Gamma$ of

$$\mu^f_{x,s} = \frac{1}{P_f(o,o,s)} \sum_{\gamma \in \Gamma} f(d_{\Omega}(x, \gamma o))e^{-s d_{\Omega}(x, \gamma o)} \delta_{\gamma o}.$$

Taking $f \equiv 1$ recovers $\mu_x$, so we will check that these measures satisfy the definition of a Busemann density for the case that $\Gamma$ is convergent.

We remark first that $P_f(o,o,s)$ exhibits the same convergence and divergence behavior as $P(o,o,s)$ for $s \neq \delta_\Gamma$ so $\mu^f_{x,s}$ will be a finite nontrivial measure supported on point masses in $\Omega$ much like $\mu_{x,s}$. Taking a weak-limit then produces a finite nontrivial measure $\mu^f_x$ supported on $\partial\Omega$ by the divergence of $P_f(o,o,s)$ as $s$ decreases to $\delta_\Gamma$. Moreover, for any Borel measurable set $A \subset \Omega$,

$$\mu^{f}_{x,s}(\gamma^{-1} A) = \frac{1}{P_f(o,o,s)} \sum_{\gamma \in \Gamma} f(d_{\Omega}(x, \gamma o))e^{-s d_{\Omega}(x, \gamma o)} \delta_{\gamma o}(\gamma^{-1} A)$$

$$= \frac{1}{P_f(o,o,s)} \sum_{\gamma \in \Gamma} f(d_{\Omega}(x, \gamma o))e^{-s d_{\Omega}(x, \gamma o)} \delta_{\gamma o}(A) = \mu^{f}_{x,s}(A).$$

Then the quasi-$\Gamma$-invariance property from Definition 2.5 holds for any weak limit $\mu^f_x$. Since $\mu^f_{x,s}$ is supported on countably many point masses in $\Omega$ for $s > \delta_\Gamma$, we compute

$$\frac{d\mu^{f}_{x,s}(\gamma o)}{d\mu^{f}_{y,s}(\gamma o)} = \frac{\mu^{f}_{x,s}(\gamma o)}{\mu^{f}_{y,s}(\gamma o)} = \frac{f(d_{\Omega}(x, \gamma o))e^{-s d_{\Omega}(x, \gamma o)}}{f(d_{\Omega}(y, \gamma o))e^{-s d_{\Omega}(y, \gamma o)}} = \frac{f(d_{\Omega}(x, \gamma o))e^{-s d_{\Omega}(x, \gamma o)}}{f(d_{\Omega}(y, \gamma o))e^{-s d_{\Omega}(y, \gamma o)}}.$$

As $s \to \delta_\Gamma$, indeed supp$\mu^{f}_{x,s}$, supp$\mu^{f}_{y,s}$ is pushed to $\partial\Omega$. By the increasing and subexponential properties of $f$, for all $\epsilon > 0$ we have that for all $\gamma o$ such that $d_{\Omega}(\gamma o, y)$ is sufficiently large,

$$f(d_{\Omega}(x, \gamma o)) \leq f(d_{\Omega}(y, \gamma o) + d_{\Omega}(x, y)) \leq f(d_{\Omega}(y, \gamma o)) e^{\epsilon d_{\Omega}(x, y)}. $$
Then for any $\gamma \in \Gamma$ such that $d(x, \gamma o)$ is sufficiently large,
\[
e^{-d_{\Omega}(x, y)} e^{-s\beta_{\gamma o}(x, y)} \leq \frac{d\mu_{\gamma o}^f}{d\mu_{\gamma}^f} \left(\gamma o\right) \leq e^{d_{\Omega}(x, y)} e^{-s\beta_{\gamma o}(x, y)}.
\]

To extend the Radon-Nikodym derivative to the limit, let $D$ be any compact fundamental domain containing the fixed point $o$, and let $\xi \in \partial \Omega$ be arbitrary. If $\xi$ is smooth, by minimality of $\Gamma$ acting on $\partial \Omega$ [4, Proposition 3.10] there exists a sequence of group elements $\gamma_n$ such that $\gamma_n o$ converges to $\xi$. Then apply Lemma 3.2 to the smooth point $\xi$ to conclude $\beta_{\gamma_n o}(x, y)$ converges to the well-defined Busemann function $\beta_\xi(x, y)$ as desired.

If $\xi$ is not smooth, then $\xi$ must be extremal by Benoist’s structure theorems, as discussed in Remark 2.4. Cover the projective ray from $o$ to $\xi$ by orbits $\gamma_n D$ of $D$ under the group $\Gamma$, so $d_{\Omega}(\gamma_n o, o)$ diverges. For each $n$, choose a point $z_n$ on the projective ray from $x$ to $\xi$ that lies in $\gamma_n D$. Then $z_n$ converges to $\xi$ along a projective ray, and by the triangle inequality,
\[
|\beta_{\gamma_n o}(x, y) - \beta_{z_n}(x, y)| \leq 2d_{\Omega}(\gamma_n o, z_n).
\]

Since $\xi$ is extremal, we can in fact choose a sequence of points $z_n$ on the projective ray from $o$ to $\xi$ such that $d_{\Omega}(\gamma_n o, z_n)$ converges to zero, and hence any accumulation of $\beta_{\gamma_n o}(x, y)$ as $n$ goes to $\infty$ is bounded above and below by $\beta_\xi^\pm(x, y)$, as desired. To construct the sequence, let $\xi_n$ be the endpoint in $\partial \Omega$ of the projective ray from $o$ passing through $\gamma_n o$. It suffices to show that $\xi_n$ converges to $\xi$. By contrapositive, suppose $\xi_n$ does not converge to $\xi$. Choose $R$ larger than twice the diameter of the compact fundamental domain $D$. Since $\xi$ is extremal, by Lemma 3.1, the shadows $\sigma_R(o, y)$ around points $y$ on the projective ray from $o$ to $\xi$ form a neighborhood basis for $\xi$. The assumption that $\xi_n$ does not converge to $\xi$ implies that there exists a $T$ and a subsequence $\xi_{n_j}$ such that for all $y$ on the projective ray $(o \xi)$ at least $T$ from $o$, then $\xi_{n_j}$ is not in $\sigma_R(o, y)$, and equivalently, $\gamma_{n_j} o$ is not in the ball $B_{\Omega}(y, R)$. But then, for $j$ large, $\gamma_{n_j} o$ cannot be in the image of the fundamental domain $\gamma_{n_j} D$, and we conclude the argument.

**Remark 4.2.** The Patterson--Sullivan measures are Borel measures on $\partial \Omega$ and have full support by quasi-$\Gamma$-invariance and minimality of the action of $\Gamma$ on $\partial \Omega$ [4, Proposition 3.10].

4.3. **The Shadow Lemma and applications.** In this subsection we prove Sullivan’s Shadow Lemma in the setting of interest [26].

4.3.1. **Geometric lemmas.** Define $\gamma \in \text{Aut}(\Omega)$ to be *hyperbolic* if $\gamma$ has an attracting fixed point and a repelling fixed point in $\partial \Omega$, denoted $\gamma^+$ and $\gamma^-$, which are both smooth and extremal, and $\gamma$ has no other fixed points in $\overline{\Omega}$. This definition diverges from the classical definition that the translation length of $\gamma$ is positive and realized in $\Omega$, which is a consequence but not equivalent. We choose this definition in this setting to separate stabilizers of triangles from group elements that act hyperbolically with north-south dynamics, since both such isometries
have positive translation length realized in \( \Omega \). We will need a proposition from the topological study of the Benoist 3-manifolds, which is straightforward given Theorem 1.4 of [4]:

**Proposition 4.3** ([8]). If \( M = \Omega / \Gamma \) is a Benoist 3-manifold then \( \Gamma \) is the disjoint union of hyperbolic isometries and stabilizers of properly embedded triangles. There are infinitely many conjugacy classes of hyperbolic group elements.

The immediate goal is to prove the following geometric proposition, similar to that in [1], as needed for the Shadow Lemma.

**Proposition 4.4.** Fix \( x \in \Omega \). For any two noncommuting hyperbolic isometries \( g, h \) preserving \( \Omega \) and \( O \) a sufficiently small neighborhood of \( h^+ \), there exists an \( R \) large and \( M \in \mathbb{N} \) such that for all \( r \geq R \) and all \( y \in \Omega \), either \( h^M O \subset \Omega(y, x) \) or \( g^M h^M O \subset \Omega_r(y, x) \).

We first prove two geometric lemmas. Let \( \mathcal{C} A \) be the convex hull of a subset \( A \) in our affine chart for \( \Omega \).

**Lemma 4.5.** If \( h \) is a hyperbolic isometry then for any open sets \( O^+ \subset \partial \Omega \) containing \( h^+ \) and \( O^- \) containing \( h^- \), there exists an \( N \in \mathbb{N} \) such that for all \( n \geq N \),

\[
h^n(\overline{\Omega \setminus \mathcal{C} O^-}) \subset O^+ \quad \text{and} \quad h^{-n}(\overline{\Omega \setminus \mathcal{C} O^+}) \subset O^-.
\]

**Proof.** If \( h \) is a projective transformation preserving \( \Omega \) with only two fixed points in \( \partial \Omega \) (and none inside \( \Omega \) since we assume \( \Gamma \) is torsion-free), then \( h \) is a biproximal matrix, so \( h^+ \) is an attracting eigenline in \( \mathbb{R}^{n+1} \) and \( h^- \) is a repelling eigenline. The result follows since \( h \) preserves \( \partial \Omega \). \( \square \)

**Lemma 4.6.** Suppose \( h, g \) are hyperbolic projective transformations preserving \( \Omega \) such that \( g^+ \neq h^+ \). Then there exist neighborhoods \( V_g, V_h \) of \( g^+, h^+ \) such that \( \overline{\mathcal{C} V_g} \cap \overline{\mathcal{C} V_h} = \emptyset \) and there is no properly embedded triangle which intersects both \( \overline{\mathcal{C} V_g} \) and \( \overline{\mathcal{C} V_h} \).

**Proof.** Since \( g, h \) are hyperbolic, \( g^+, h^+ \) are smooth extremal points. There are disjoint open neighborhoods \( V_g, V_h \) around \( g^+, h^+ \) respectively in \( \partial \Omega \) whose closures are also disjoint, and for which \( g^- \not\in \overline{V_g} \) and \( h^- \not\in \overline{V_h} \). If the lemma was false, by convexity of \( \mathcal{C} g^n V_g, \mathcal{C} h^n V_h \), there would exist a sequence of properly embedded triangles \( \Delta_n \) such that \( \overline{g^n V_g} \cap \partial \Delta_n \neq \emptyset \) and \( \overline{h^n V_h} \cap \partial \Delta_n \neq \emptyset \) for all \( n \). Since the collection of properly embedded triangles is closed in \( \Omega \) [4, Proposition 3.2], the \( \Delta_n \) accumulate on some \( \Delta \) properly embedded in \( \Omega \). Because \( g, h \) are hyperbolic, \( \cap_{n=1}^\infty \overline{g^n V_g} = \{g^+\} \) and \( \cap_{n=1}^\infty \overline{h^n V_h} = \{h^+\} \). Then \( (g^+ h^+) \subset \overline{\Delta} \) which contradicts the smooth extremal property for fixed points of hyperbolic isometries. \( \square \)

**Proof of Proposition 4.4.** Applying Lemma 4.6, there are pairwise disjoint neighborhoods \( V_h^\pm, V_g^\pm \) of \( h^\pm, g^\pm \) respectively such that no properly embedded triangle intersects any pair of convex hulls of these neighborhoods in \( \overline{\Omega} \). In particular, this means for \( V_i, V_j \in \{V_h^\pm, V_g^\pm\} \) with \( V_i \neq V_j \), for any \( x \in \mathcal{C} V_i \) and \( y \in \mathcal{C} V_j \), the
projective line \((x, y)\) is contained in \(\Omega\) and is not contained in any single properly embedded triangle.

By Lemma 4.5, there exists an \(N_1\) such that \(h^{-n}(\overline{\Omega} \setminus \mathcal{C}V^+_h) \subset V^-_h\) for all \(n \geq N_1\). Moreover, there exists an \(N_2\) such that \(g^{-n}(\overline{\Omega} \setminus \mathcal{C}V^+_g) \subset V^-_g\), implying

\[
g^{-n}(\mathcal{C}V^+_g) \subset g^{-n}(\overline{\Omega} \setminus \mathcal{C}V^+_g) \subset V^-_g \subset \overline{\mathcal{C}V^+_h}.
\]

Then for all \(y \in \Omega\) and all \(n \equiv \max\{N_1, N_2\}\), either \(h^{-n}y \in \mathcal{C}V^-_h\) or \(h^{-n}g^{-n}y \in \mathcal{C}V^-_h\).

Moreover, there exists an \(N\) such that the projective ray \((xy, \gamma)\) is contained in \(\Omega\) and is not contained in any single properly embedded triangle by choice of \(V^-_h, V^+_h\) (Lemma 4.5). Then take \(r \equiv \max_{\eta \in \overline{V^+_h}} d_{\Omega}(x, (\gamma y \eta))\) and the leftmost containment is satisfied.

4.3.2. The Shadow Lemma. First, we need a basic lemma:

**Lemma 4.7.** For all \(\xi \in \partial_{\Omega}(x, y)\),

\[
d_{\Omega}(x, y) - 2r \leq \beta^+_\xi(x, y) \leq \beta^+_\xi(x, y) \leq d_{\Omega}(x, y).
\]

**Proof.** The rightmost inequality is immediate from the triangle inequality. For the leftmost inequality, let \(z \in \Omega\) converge to \(\xi\) along the projective line from \(x\) to \(\xi\). Divide the projective line \((xz)\) into two segments by its first intersection \(p\) with the closed ball \(B_\Omega(y, r)\). Then by the triangle inequality, \(d_{\Omega}(x, y) \leq d_{\Omega}(x, p) + r\) and \(d_{\Omega}(z, y) \leq d_{\Omega}(z, p) + r\), so

\[
d_{\Omega}(x, y) - 2r \leq d_{\Omega}(x, p) + d_{\Omega}(p, z) - d_{\Omega}(y, z) = d_{\Omega}(x, z) - d_{\Omega}(y, z) = \beta^+_z(x, y).
\]

The lower bound follows.

**Lemma 4.8** (Shadow Lemma). Let \(\mu\) be a Busemann density of dimension \(\delta > 0\) on \(\partial \Omega\). Then for every \(x \in \Omega\) and all sufficiently large \(r\), there exists a \(C > 0\) such that for all \(\gamma \in \Gamma\),

\[
\frac{1}{C} e^{-\delta d_{\Omega}(x, \gamma x)} \leq \mu_x(\partial_{\Omega}(x, \gamma x)) \leq Ce^{-\delta d_{\Omega}(x, \gamma x)}.
\]

**Proof.** We follow the elegant proof of Roblin [24]. Since \(\gamma\) is an isometry and by quasi-\(\Gamma\)-invariance,

\[
\mu_x(\partial_{\Omega}(x, \gamma x)) = \mu_x(\gamma \partial_{\Omega}(\gamma^{-1} x, x)) = \mu_{\gamma^{-1} x}(\partial_{\Omega}(\gamma^{-1} x, x)).
\]
By the transformation rule (Definition 2.5),
\[
\int_{\mathcal{O}_r(y^{-1}x,x)} e^{-\beta(x,y)} d\mu_x(\xi) \leq \int_{\mathcal{O}_r(y^{-1}x,x)} e^{-\beta(x,y)} d\mu_x(\xi).
\]
(4.2)

Combining Equations (4.1) and (4.2) with Lemma 4.7,
\[
\int_{\mathcal{O}_r(y^{-1}x,x)} e^{-\delta d_1(y^{-1}x,x)} d\mu_x(\xi) \leq \mu_x(\mathcal{O}_r(x,\gamma x))
\]
\[
\leq \int_{\mathcal{O}_r(y^{-1}x,x)} e^{-\delta d_1(y^{-1}x,x)} d\mu_x(\xi),
\]
(4.3)

so, letting \(\|\mu_x\| := \mu_x(\partial \Omega) < \infty\),
\[
e^{-\delta d_1(y^{-1}x,x)} \mu_x(\mathcal{O}_r(y^{-1}x,x)) \leq \mu_x(\mathcal{O}_r(x,\gamma x)) \leq e^{-\delta d_1(y^{-1}x,x)} e^{2\delta r \|\mu_x\|}.
\]
(4.4)

The rightmost inequality of Equation (4.4) gives us the rightmost inequality of the lemma immediately. By Proposition 4.3 there exist two noncommuting hyperbolic isometries \(g, h\). Then apply Proposition 4.4 to obtain open sets \(O_1 = hM \Omega, O_2 = gM hM \Omega \subset \partial \Omega\) such that for all \(r\) sufficiently large and all \(\gamma \in \Gamma\), either \(O_1 \subset \mathcal{O}_r(y^{-1}x,x)\) or \(O_2 \subset \mathcal{O}_r(y^{-1}x,x)\). The \(\mu_x\) have full support (Remark 2.5) so we may take \(0 < \frac{1}{C} < \min(\mu_x(O_1))\) to complete the proof. \(\square\)

4.3.3. **Boundaries of flats are null sets.** Let \(S_\Delta(x,r) := \{y \in \Delta : d_\Omega(x,y) = r\}\) denote the sphere of Hilbert radius \(r\) about \(x\) restricted to a properly embedded triangle, \(\Delta\). Similarly, \(B_\Delta(x,r)\) is the open ball of Hilbert radius \(r\) about \(x\) restricted to the triangle \(\Delta\). For a properly embedded triangle \(\Delta\) in \(\Omega\), let \(\text{Stab}_\Gamma(\Delta) = \{\gamma \in \Gamma \mid \gamma \Delta = \Delta\}\).

**Lemma 4.9.** Pick a tiling of a properly embedded triangle \(\Delta\) by \(\text{Stab}_\Gamma(\Delta)\) such that \(p\) is in the interior of a fundamental domain in the tiling. Choose \(R\) so that the open \(B_\Delta(p,R)\) covers the compact fundamental domain containing \(p\). If \(N_r\) denotes the minimal number of \(\gamma B_\Delta(p,R)\) which cover \(S_\Delta(p,r)\), where \(\gamma \in \text{Stab}_\Gamma(\Delta)\), then \(N_r\) is quasi-linear in \(r\).

**Proof.** The projective triangle with the Hilbert metric is isometric to \(\mathbb{R}^2\) with a hexagonal norm [14]. By Benoist’s Theorem 1.4(c), \(\text{Stab}_\Gamma(\Delta)\) is isomorphic to \(\mathbb{Z}^2\) up to index 2. Under De la Harpe’s isometry this \(\mathbb{Z}^2\) group acts by translations so the growth of orbits of a fundamental domain under the hexagonal norm is quasi-linear. \(\square\)

**Proposition 4.10.** The boundary of any properly embedded triangle is a null set for any Busemann density of dimension \(\delta > 0\).

**Proof.** Choose a fundamental domain \(T\) for the action of \(\text{Stab}_\Gamma(\Delta)\) on a properly embedded triangle \(\Delta\), a point \(p \in T\) and \(R_0\) as in Lemma 4.9. Let \(x \in \Omega\) be in a fundamental domain \(D\) for the \(\Gamma\)-action on \(\Omega\) such that \(T \subset D\). Choose \(R\) large enough that \(D \subset B_\Omega(x,R)\) and \(B_\Delta(p,R_0) \subset B_\Omega(x,R)\). Then the \(\Gamma \cdot B_\Omega(x,R)\) covers \(\Omega\), and the minimal number of \(\gamma\) such that \(\cup_{i=1}^N \gamma_i B_\Omega(x,R)\) covers \(S_\Delta(p,r)\),
is bounded above by the \( N_r \) in Lemma 4.9. For each \( r \), choose a covering of 
\( S_\Delta(x,r) \) by \( N_r \)-many \( \gamma_i B_\Omega(x,R) \) and assume that \( \gamma_i \in \text{Stab}_R(\Delta) \) for \( i = 1, \ldots, N_r \).

Next, we show for all large enough \( r, \partial \Delta \subset \bigcup_{i=1}^{N_r} \mathcal{O}_{2R}(x,\gamma_i x) \). Let \( r > 2R \). Consider any projective ray \( \eta \) based at \( p \) such that \( \eta^+ \in \partial \Delta \). Let \( \xi \) denote projective ray based at \( x \) such that \( \xi^+ = \eta^+ \). Then parameterizing \( \xi, \eta \) at unit speed, we have that 
\[
d_\Omega(\xi, \eta) \leq d_\Omega(x, p) \leq R \quad \text{for all } t \geq 0.
\]
Since \( p \in \Delta \) and \( \eta^+ \in \partial \Delta \), then \( \eta \cap S_\Delta(p, r) \neq \emptyset \) and there exists a \( \gamma_i \) such that \( \eta \cap B_\Omega(\gamma_i x, R) \neq \emptyset \). Let \( t \geq 0 \) be such that 
\[
d_\Omega(\eta, \gamma_i x) < R.
\]
Then
\[
d_\Omega(\xi, \gamma_i x) \leq d_\Omega(\xi, \eta) + d_\Omega(\eta, \gamma_i x) \leq 2R,
\]
and \( \xi^+ \in \mathcal{O}_{2R}(x,\gamma_i x) \). Lastly, for each \( i = 1, \ldots, N_r \) let \( q_i \in S_\Delta(p, r) \cap B_\Omega(\gamma_i x, R) \neq \emptyset \). Then
\[
r = d_\Omega(p, q_i) \leq d_\Omega(p, x) + d_\Omega(x, \gamma_i x) + d_\Omega(\gamma_i x, q_i) \leq d_\Omega(x, \gamma_i x) + 2R
\]
implying 
\[
- d_\Omega(x, \gamma_i x) \leq 2R - r \quad \text{for all } i = 1, \ldots, N_r.
\]
By Lemma 4.8, 
\[
(4.5) \quad \mu_x(\partial \Delta) \leq \sum_{i=1}^{N_r} \mu_x(\mathcal{O}_{2R}(x, \gamma_i x)) \leq \sum_{i=1}^{N_r} Ce^{-\delta d_\Omega(x,\gamma_i x)} \leq Ce^{\delta 2R - \delta r} N_r.
\]
Given that \( N_r \) is quasi-linear in \( r \) by Lemma 4.9, that \( \delta > 0 \), and that Equation 4.5 holds for all \( r \) sufficiently large, we conclude \( \mu_x(\partial \Delta) = 0 \).

Then the following corollary is immediate after Proposition 2.3:

**Corollary 4.11.** The set of smooth extremal points in \( \partial \Omega \) is full measure for any Busemann density of dimension \( \delta > 0 \).

## 5. Busemann Densities Are Unique

In this section we complete the proof of Theorem 1.2. The arguments in this section follow those of Sullivan and Knieper [26, 18]. We give brief proofs, mainly to point out when we need Corollary 4.11.

**Lemma 5.1** (Local estimates). If \( \mu \) is a Busemann density of dimension \( \delta > 0 \) on \( \partial \Omega \), then for all \( x \) and all sufficiently large \( r \) there exists a constant \( b(r) \) such that for \( y \in \Omega \) with \( d_\Omega(x,y) \) large,
\[
\frac{1}{b(r)} e^{-\delta d_\Omega(x,y)} \leq \mu(x, \mathcal{O}_R(x,y)) \leq b(r) e^{-\delta d_\Omega(x,y)}
\]

**Proof.** Note that if \( y = \gamma x \), then we apply Lemma 4.8 to obtain the result. Else, for some \( \Gamma \)-tiling of \( \Omega \) with compact fundamental domain \( D \), choose \( r \) large enough that for all \( x \in D \), we have \( D \subset B_\Omega(x, \frac{r}{2}) \). Choosing \( D \) such that \( y \in D \), there exists a \( \gamma \in \Gamma \) such that \( \gamma x \in D \subset B_\Omega(y, \frac{r}{2}) \). By the triangle inequality,
\[
\mathcal{O}_{\frac{r}{2}}(x, \gamma x) \subset \mathcal{O}_R(x,y) \subset \mathcal{O}_{\frac{r}{2}}(x, \gamma x).
\]
Applying Lemma 4.8, if \( r \) is sufficiently large then there is a uniform constant \( C \) such that
\[
\frac{1}{C} e^{-\delta d_{3}(x,y)} \leq \mu_{x}(\Theta_{x}(x,y)) \leq C e^{-\delta d_{3}(x,y)}.
\]

Our final observation is that since \( \gamma x \in B_{\Omega}(y, r/2) \),
\[
\frac{1}{C} e^{-\delta d_{3}(x,y)} \leq \mu_{x}(\Theta_{x}(x,y)) \leq C e^{-\delta d_{3}(x,y)}.
\]

It follows that Busemann densities have no atomic part:

**Corollary 5.2.** Busemann densities of dimension \( \delta > 0 \) on \( \partial \Omega \) have no atoms.

**Proof.** It suffices to check for smooth extremal points \( \xi \in \partial \Omega \) by Corollary 4.11. Let \( y_{n} \) be a sequence of points in \( \Omega \) converging to \( \xi \) along a projective line. Then apply the local estimate lemma (Lemma 5.1), for fixed sufficiently large \( R \), to the shadows \( \Theta_{x}(y_{n}, R) \). The conclusion follows Lemma 3.1.

**Corollary 5.3.** Busemann densities of dimension \( \delta > 0 \) are equivalent.

**Proof.** Let \( \{ \mu_{x} \}, \{ \nu_{x} \} \) be Busemann densities of dimension \( \delta \). Let \( \xi \in \partial \Omega \) be a smooth extremal point and take a sequence \( y_{n} \) of points in \( \Omega \) converging to \( \xi \) along a projective line. Then for all sufficiently large \( n \), \( d_{\Omega}(x, y_{n}) \) is large enough to apply Lemma 5.1 to both densities and \( \nu \) and conclude:
\[
\frac{1}{b_{\mu}(r)} \leq \frac{\mu_{x}(\Theta_{x}(x,y_{n}))}{\nu_{x}(\Theta_{x}(x,y_{n}))} \leq b_{\mu}(r).
\]

By Lemma 3.1, since \( y_{n} \) converges to \( \xi \) along a projective line and \( \xi \) is smooth and extremal, the shadows \( \Theta_{x}(x,y_{n}) \) form a nested decreasing sequence with intersection \( \{ \xi \} \). Thus, since smooth extremal points form a set of full measure for any \( \delta \)-dimensional Busemann density by Corollary 4.11, we conclude that \( \mu_{x} \) and \( \nu_{x} \) are equivalent.

**Proposition 5.4.** If \( \{ \mu_{x} \} \) is a Busemann density of dimension \( \delta > 0 \) on \( \partial \Omega \), then for all \( x \in \Omega \), the measure \( \mu_{x} \) is ergodic for the \( \Gamma \)-action on \( \partial \Omega \).

**Proof.** Let \( A \subset \partial \Omega \) be a Borel, \( \Gamma \)-invariant set with positive \( \mu_{x} \)-measure for all \( x \), since the measures are equivalent. Define a new density \( \tilde{\mu}_{x}(B) := \mu_{x}(A \cap B) \) for all \( x \in \Omega \). Since \( A \) is \( \Gamma \)-invariant and has positive measure, it suffices to show that \( \tilde{\mu}_{x} \) is a Busemann density also of dimension \( \delta \). Then \( \mu_{x} \) is equivalent to \( \tilde{\mu}_{x} \) by Corollary 5.3, and we conclude that \( \mu_{x}(\partial \Omega \setminus A) = \tilde{\mu}_{x}(\partial \Omega \setminus A) = 0 \), proving ergodicity of \( \mu_{x} \) for \( \Gamma \).

It is clear that \( \tilde{\mu}_{x} \) is nontrivial and finite. Since smooth extremal points are full measure and the transformation rule is well-defined for smooth extremal points, the proof that \( \tilde{\mu}_{x} \) satisfies the transformation rule does not differ significantly from [18, Proposition 4.15], and the proof of quasi-\( \Gamma \)-invariance is unchanged.
THEOREM 5.5. Busemann densities of dimension $\delta > 0$ on $\partial \Omega$ are unique up to a constant.

Proof. Let $[\mu_x], [\nu_x]$ be two Busemann densities of dimension $\delta$. Since $\mu_x$ and $\nu_x$ are equivalent, it suffices to show that the Radon-Nikodym derivative $dv_x/d\mu_x$ is $\Gamma$-invariant on the set of smooth extremal points, which are a set of full measure by Corollary 4.11. Ergodicity of $\mu_x$ then implies that the Radon-Nikodym derivative is constant $\mu_x$-almost everywhere. Since the densities have the same transformation rule almost everywhere, this constant does not depend on $x$. Verifying that the Radon–Nikodym derivative is $\Gamma$-invariant on the set of smooth extremal points is straightforward.

Combining Proposition 4.1 and Theorem 5.5 gives us Theorem 1.2.

5.1. Volume growth and divergence of $\Gamma$. In this section, we see that $\Gamma$ is divergent. With all the tools is place, the proof does not differ from that of Knieper for rank one manifolds, but we include it here for completeness [18, Theorem 5.1].

First we prove Theorem 1.3 on the growth rate of volumes of spheres. Let $\text{vol}_{x,t}$ be a Hilbert volume form on $S_{x}(x, t)$ which is $\Gamma$-equivariant, meaning $\text{vol}_{x,t}^\gamma = \gamma_* \text{vol}_{x,t}$. We abbreviate with vol when the context is clear. For a definition of sphere volume, see the definition of area of a smooth hypersurface in [28, Equation 1.3]. Since the nonsmooth points in the spheres come from nonsmooth points in the boundary of $\Omega$, which form a countable set in this case by Benoist (Theorem 1.4), these nonsmooth points in spheres are measure zero and the definition can be applied. Indeed, Vernicos studies asymptotics of this sphere volume for any Hilbert geometry, not necessarily smooth ones (see [28, Theorem 2.1]).

Proof of Theorem 1.3. Let $[\mu_x]$ denote the Patterson–Sullivan density of dimension $\delta$ of $\Gamma$, existence of which we constructed in Proposition 4.1. By previous work we have that $\delta$ is divergent. Let $\delta = \delta_\Gamma$ throughout the proof. Let $R$ be sufficiently large to apply Sullivan’s Shadow Lemma (Lemma 4.8) and consequently the local estimate in Lemma 5.1. Consider $r \geq 6R$. By compactness of $S_{\Omega}(x, t)$, we can take $\{x_i\}_{i=1}^{N_t}$ to be a maximal $r$-separating set in $S_{\Omega}(x, t)$. In particular, if $i \neq j$, then $d_{\Omega}(x_i, x_j) \geq r$, implying $B_{\Omega}(x_i, r/3) \cap B_{\Omega}(x_j, r/3) = \emptyset$. Maximality implies $S_{\Omega}(x, t) \subset \bigcup_{i=1}^{N_t} B_{\Omega}(x_i, r)$.

By the local estimate (Lemma 5.1), there exists a $b(x)$ such that for all $r \in [2R, 6R]$ and $x_i$, each of which is distance $t$ from $x$, we have

$$\frac{1}{b} e^{-\delta t} \leq \mu_x(\Theta_{r}(x, x_i)) \leq be^{-\delta t}.$$

Then

$$\mu_x(\partial \Omega) \leq \mu_x \left( \bigcup_{i=1}^{N_t} \Theta_{r}(x, x_i) \right) \leq \sum_{i=1}^{N_t} \mu_x(\Theta_{r}(x, x_i)) \leq N_t be^{-\delta t},$$
and
\[ \mu_x(\partial \Omega) \geq \frac{N_t}{b} \cdot e^{-\delta t}. \]

Since \( \mu_x(\partial \Omega) \) is a constant depending on \( x \), there is a number \( b'(x) \) such that
\[ \frac{1}{b'} e^{\delta t} \leq N_t \leq b' e^{\delta t}. \]

By cocompactness of \( \Gamma \) acting on \( \Omega \), and \( \Gamma \)-equivariance of the sphere volumes \( \text{vol}_{x,t} \) on the spheres of radius \( t \) around \( x \), there exists an \( \ell(r) \) such that
\[ \frac{1}{\ell} \leq \text{vol}_{x,t}(B_\Omega(y, r) \cap S_{x,t}(x, t)) \leq \ell. \]

Then we may arrange for an \( \ell' \) such that
\[ \text{vol}_{x,t}(S_{x,t}(x, t)) \leq \frac{N_t}{\ell} \cdot \ell' \leq \ell' b' e^{\delta t} \]
and
\[ \text{vol}_{x,t}(S_{x,t}(x, t)) \geq \frac{N_t}{\ell} \cdot \ell' \geq \frac{1}{\ell' b'} e^{\delta t}, \]
which concludes the proof of the theorem. \( \square \)

**Corollary 5.6.** Let \( \Omega \) be a properly convex, divisible, indecomposable Hilbert geometry of dimension three with dividing group \( \Gamma \). Then \( \Gamma \) is of divergent type.

**Proof.** Let \( D \) be a compact fundamental domain for the \( \Gamma \)-action on \( \Omega \). Then for \( s > \delta \Gamma \), \( P(x, y, s) \) converges so we can apply Fubini’s Theorem to the following integral:
\[ \int_D \sum_{y \in \Gamma} e^{-sd_\Gamma(x, y)} d\text{vol}(y) = \int_{\gamma(D)} \sum_{y \in \Gamma} e^{-sd_\Gamma(x, y)} d\text{vol}(y) = \int_0^\infty e^{-st} \text{vol}(S_{x,t}(x, t)) dt. \]

As \( s \) decreases to \( \delta \Gamma \), the right hand side diverges by Theorem 1.3. \( \square \)

6. The Bowen–Margulis Measure

In this section, we introduce the \( \Gamma \)-invariant Bowen–Margulis measure on \( T^1 \Omega \), denoted \( \mu_{BM} \), following the standard construction [26, 24, 18] and prove Theorem 1.1.
6.1. **Definition and properties.** Let \( \{ \mu_x \} \) be the Patterson–Sullivan density constructed in Proposition 4.1, which is a Busemann density of dimension \( \delta_1 > 0 \). For each \( x \in \Omega \) and Borel set \( A \subset T^1 \Omega \), define

\[
\tilde{\mu}_{BM}^x(A) = \int_{v \in A} \text{length}_\Omega(\ell_v \cap \pi A) e^{\delta_1 (\beta_v(x, \pi v) + \beta_v^-(x, \pi v))} \ d\mu_x(v^-) d\mu_x(v^+)
\]

where \( \pi : T^1 \Omega \to \Omega \) is the footpoint projection and \( \text{length}_\Omega \) is Hilbert length. Since the Busemann function is well-defined almost everywhere (Lemma 3.2 and Corollary 4.11), this definition is valid. Then \( \tilde{\mu}_{BM}^x \) is \( \Gamma \)-invariant by the definition of a \( \delta_1 \)-dimensional Busemann density and the cocycle property of the Busemann function. On \( T^1 M \) the measure is finite, and we may normalize it so \( \mu_{BM}^x(T^1 M) = 1 \).

Recall that \( T^1 \Omega_{\text{reg}} \) is the set of regular vectors, which are vectors \( v \) whose endpoints \( v^-, v^+ \) in \( \partial \Omega \) are both smooth and extremal, and \( T^1 M_{\text{reg}} \) is the projection of these vectors to \( T^1 M \) (Definition 2.7).

The following Lemma is clear given the discussion above.

**Lemma 6.1.** The regular set \( T^1 M_{\text{reg}} \) is a set of full \( \mu_{BM}^x \)-measure.

For the remainder of the paper, we will let \( \mu_{BM} := \mu_{BM}^x \) for some \( x \in \Omega \). Note that since the \( \mu_{BM}^x, \mu_{BM}^y \) are equivalent by construction, we will have that they are in fact equal up to a constant after the proof of ergodicity is complete.

6.2. **Ergodicity.** Let \( d \) be the Finsler metric on \( T^1 M \) discussed in [8, Section 4.1]. Define the \( \varphi^t \)-invariant strong unstable foliations for \( v \in T^1 M_{\text{reg}} \) to be

\[
W^{\text{su}}(v) = \{ w \in T^1 M \mid d(\varphi^{-t} v, \varphi^{-t} w) \to 0 \text{ as } t \to +\infty \}
\]

and similarly for \( W^{\text{ss}}(v) \), the strong stable foliation, which is contracted in forward time. The weak unstable set \( W^{\text{ou}}(v) \) is the disjoint union of \( W^{\text{ss}}(\varphi^t v) \) for all \( t \in \mathbb{R} \), and similarly for the weak stable set \( W^{\text{os}}(v) \). This gives us a flow-invariant foliation of the weak unstable sets by strong unstable leaves, and similarly for the stable foliation.

**Lemma 6.2.** For all regular \( v, w \) with \( v \neq -w \), we have

\[
W^{\text{ss}}(v) \cap W^{\text{ou}}(w) \neq \emptyset.
\]

**Proof.** If \( v \) is regular then \( W^{\text{ss}}(v) \) is defined by a geometric characterization on the universal cover [3, 8]:

\[
\tilde{W}^{\text{ss}}(v) = \{ w \in T^1 \Omega \mid v^+ = w^+, \pi w \in \mathcal{H}_{v^+}(\pi v) \},
\]

\[
\tilde{W}^{\text{os}}(v) = \{ w \in T^1 \Omega \mid v^+ = w^+ \}.
\]

where \( \mathcal{H}_{v^+}(\pi v) \) is the globally defined horosphere through \( \pi v \) at \( v^+ \) (Corollary 3.5). The result follows the geometric interpretation of the Busemann function in Lemma 3.2. 

\[
\square
\]
6.2.1. The Hopf argument. We first establish or recall basic facts which set up
the ergodicity proof. Let $f: T^1M \to \mathbb{R}$ be integrable. Then the forward and
backward Birkhoff averages of $f$ for $\varphi$ are, respectively,

$$f^+(v) = \lim_{t \to +\infty} \frac{1}{t} \int_0^t f \circ \varphi^s(v) \, ds,$$

$$f^-(v) = \lim_{t \to +\infty} \frac{1}{t} \int_0^t f \circ \varphi^{-s}(v) \, ds.$$

By the Birkhoff ergodic theorem, $f^+$ and $f^-$ exist for $\mu_{BM}$-almost every $v \in T^1M$ (see \cite[Theorem 4.1.2]{17}). The following lemma is straightforward to verify by
compactness of $T^1M$:

**Lemma 6.3.** Forward Birkhoff averages of continuous functions are constant on
strong stable leaves of regular vectors and backward Birkhoff averages as constant
on strong unstable leaves.

Since $\varphi$ is invertible, $f^+ = f^- \mu_{BM}$-almost everywhere (see \cite[Proposition 4.1.3]{17}). We have the following classical lemma, which we do not prove here,
which allows us to verify ergodicity by proving $f^+$ is constant almost everywhere
for all continuous $f$.

**Lemma 6.4.** If $f^+$ is constant $\mu_{BM}$-almost everywhere for all continuous $f$, then
every $\varphi$ -invariant $L^1$-integrable function is constant $\mu_{BM}$-almost everywhere.

We make the arguments locally in the universal cover and conclude ergodicity
by transitivity of the flow on the quotient. We define strong unstable conditional
measures as induced Patterson–Sullivan measure on strong unstable leaves:

$$(6.2) \quad \mu^u_v(A) = \int_{w \in \mathcal{A} \cap W^u(v)} e^{\delta t \beta_{w^+}(x, \pi w)} \, d\mu_x(w^+).$$

We can define the strong stable conditionals $\mu^s_v$ on $W^s(v)$ similarly. Note that
for $w \in W^s(v)$, we have $\pi w \in \mathcal{H}_{\varphi^t}(\pi v)$ and $w^t = v^t$, hence $\beta_{w^+}(x, \pi w)$ is constant
over $w \in W^s(v)$ and the conditional measures will not depend the point
in a leaf of the foliation. We will say the strong unstable foliation is *absolutely
continuous* if the associated strong stable conditionals are absolutely continuous
as measures.

**Lemma 6.5.** The strong unstable foliations are absolutely continuous for all regular
points in $T^1M$.

*Proof.* For each $t$ we have uniform contraction along flow lines:

$$\frac{d\varphi_t^* \mu^s_v}{d\mu^s_{\varphi^t v}} = e^{-\delta t}$$

by the cocycle property of the Busemann function. This gives us absolute continuity
of the strong unstable conditionals along flow lines.

It remains to consider $v, w$ regular vectors on the same strong stable leaf.
To determine absolute continuity of the strong unstable conditionals, we define
a measurable bijection $h: W^u(v) \to W^u(w)$ where, for $u \in W^u(v)$, we
let \( h(u) \) be the unique regular vector such that \( h(u)^- = w^- \), \( h(u)^+ = u^+ \), and \( \beta_w^- (\pi w, \pi h(u)) = 0 \); in other words, \( h(u) \in W^{su}(w) \cap W^{os}(u) \) (Lemma 6.2). Then we compute the density

\[
\rho_{v,w}(u) := \frac{d\mu^{su}_w}{dh_u \mu^{su}_v}(u) = e^{-\delta_v \beta_{u^+} (\pi u, \pi h(u))}
\]

and see that \( 0 < \rho_{v,w}(u) < \infty \) as well. To complete the computation, let \( A \subset W^{su}(w) \). Then

\[
\int_A \rho_{v,w}(u) \, dh_u \mu_v^{su}(u) = \int_{u \in A} e^{-\delta_v \beta_{u^+} (\pi u, \pi h(u))} e^{\delta_v \beta_{u^+} (x, \pi h(u))} \, d\mu(x(u^+))
\]

by the cocycle property of the Busemann function.

**Remark 6.6.** The final remark we make before proving ergodicity is that, locally, the \( \mu_{BM} \)-measure of a Borel set \( A \) agrees with the \( \tilde{\mu}_{BM} \)-measure of a lift of \( A \), so we can exploit the Patterson–Sullivan product structure of \( \mu_{BM} \) on such sufficiently small neighborhoods (see Definition 6.1). We will refer to this feature as the local product structure of \( \mu_{BM} \). Then it is clear that, for such a small Borel measurable set \( N \subset T^1 M \) which we identify with a lift in \( T^1 \Omega \), we have \( \mu_{BM}(N) = 0 \) if and only if \( \mu_v^{su}(N) = 0 \) for \( \mu_{BM} \)-almost every \( v \).

In the arguments below, we abuse notation and treat the measures as conditional measures on a small neighborhood in the universal cover.

**Theorem 6.7.** The Bowen–Margulis measure is ergodic for the geodesic flow.

**Proof.** Let \( f \) be a continuous function and \( \Lambda_q = \{ v \in T^1 M \mid f^+(v) \geq q \} \) for some \( q \in \mathbb{Q} \) such that \( \mu_{BM}(\Lambda_q) > 0 \). Then \( \mu_v^{su}(\Lambda_q) > 0 \) for \( \mu_{BM} \)-almost every \( v \in T^1 M_{reg} \) by the local product structure of the Bowen–Margulis measure and that \( T^1 M_{reg} \) has full \( \mu_{BM} \)-measure (Lemma 6.1). By Lemma 6.5, the unstable conditionals are absolutely continuous for every pair of regular vectors, so \( \mu_v^{su}(\Lambda_q) > 0 \) for every regular vector \( v \). Let \( G \) be the set of full \( \mu_{BM} \)-measure on which \( f^- = f^+ \) (by invertibility of the flow and [17, Proposition 4.1.3]). Then \( G \) is also a set of full \( \mu_v^{su} \)-measure for \( \mu_{BM} \)-almost every \( v \). Then for almost every \( v \), we have \( \mu_v^{su}(\Lambda_q) > 0 \) which implies \( \mu_v^{su}(\Lambda_q \cap G) > 0 \), and so there exists a \( w \in \Lambda_q \cap G \cap W^{su}(v) \). Thus for all \( u \in G \cap W^{su}(v) \), a full \( \mu_v^{su} \)-measure set, we have

\[
f^+(u) = f^-(u) = f^-(w) = f^+(w) \geq q
\]

since \( f^- \) is constant on strong unstable sets by \( q^f \)-invariance of \( f^- \) (Lemma 6.3). Thus, \( u \in \Lambda_q \) and \( \mu^{su}_{v}(\Lambda_q) = 1 \) for \( \mu_{BM} \)-almost every \( v \). This implies \( \mu_{BM}(\Lambda_q) = 1 \) by the local product structure of \( \mu_{BM} \). Since \( \mathbb{Q} \) is dense in \( \mathbb{R} \) we conclude \( f^+ \) is constant on a set of full measure and by Lemma 6.4 the proof is complete. \( \square \)
6.3. A measure of maximal entropy. The measure-theoretical entropy of \( \mu \) with respect to the finite measurable partition \( \mathcal{A} = \{ A_1, \ldots, A_m \} \), also known as the Kolmogorov-Sinai entropy, is

\[
h_{\mu}(\varphi^1, \mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} H_{\mu}(\mathcal{A}^{(n)}_\varphi)
\]

where \( H_{\mu}(\mathcal{B}) = -\sum_{i=1}^{k} \log(\mu(B_i)) \mu(B_i) \) is the entropy of a finite measurable partition \( \mathcal{B} = \{ B_1, \ldots, B_k \} \) and \( \mathcal{A}^{(n)}_\varphi := \bigvee_{i=0}^{n-1} \varphi^{-i} \mathcal{A} \) is the partition consisting of all intersections \( \bigcap_{i=0}^{n-1} \varphi^{-i} A_{j_i} \) over all possible \( \{ j_1, \ldots, j_{n-1} \} \subset \{ 1, \ldots, m \} \). Then the measure-theoretic entropy of the pair \( (\varphi^1, \mu) \) is

\[
h_{\mu}(\varphi^1) = \sup_{\mathcal{A}} h_{\mu}(\varphi^1, \mathcal{A})
\]

and the entropy of \( \mu \) for the geodesic flow \( \varphi^t \) is \( h_{\mu} := h_{\mu}(\varphi^1) \). By work in [8], \( \varphi^t \) is entropy-expansive with expansivity constant \( \epsilon > 0 \). Then by [6, Theorem 3.5], \( h_{\mu} = h_{\mu}(\varphi, \mathcal{A}) \) for \( \text{diam}(\mathcal{A}) < \epsilon \).

**Lemma 6.8.** There exists some \( a > 0 \) such that

\[
\mu_{BM}(\alpha) \leq e^{-\delta \Gamma n} a
\]

for all \( \alpha \in \mathcal{A}^{(n)}_\varphi \).

**Proof.** The proof does not differ from [19, Lemma 2.5]. \( \square \)

**Theorem 6.9.** The Bowen–Margulis measure is a measure of maximal entropy.

**Proof.** The proof is as in [19, Theorem 5.12]. First, using Lemma 6.8 one computes \( H_{BM}(\mathcal{A}^{(n)}_\varphi) \geq \delta \Gamma n - \log a \). In [8, Proposition 7.3], since the quotient is compact and by a technical lemma of Cramp on Hilbert geometries [10, Lemma 8.3], the arguments of Manning extend [20] allowing us to conclude \( \delta \Gamma = h_{top} \). Then by the variational principle,

\[
h_{top} \geq h_{BM} = \lim_{n \to \infty} \frac{1}{n} H_{BM}(\mathcal{A}^{(n)}_\varphi) \geq \lim_{n \to \infty} \frac{1}{n} (\delta \Gamma n - \log a) = \delta \Gamma = h_{top}. \quad \square
\]

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