Actuator Fault Detection for Discrete-Time Descriptor Systems via a Convex Unknown Input Observer with Unknown Scheduling Variables

Javier Martinez, 1 Braulio Aguiar, 2 Víctor Estrada-Manzo, 1 and Miguel Bernal 2

1Dept. of Mechatronics, Universidad Politécnica de Pachuca, C.P. 43830 Zempoala, Mexico
2Dept. of Electrical and Electronics Engineering, Sonora Institute of Technology, 5 de Febrero 818 Sur, C.P. 85000 Cd Obregon, Mexico

Correspondence should be addressed to Víctor Estrada-Manzo; victor_estrada@upp.edu.mx

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This paper presents actuator fault detection of discrete-time nonlinear descriptor systems by means of nonlinear unknown input observers. The approach is based on the exact factorization of the estimation error in order to overcome the well-known problem of unmeasurable scheduling variables within the observation of convex models, thus avoiding the use of Lipschitz constants, differential mean value theorem, or robust techniques. As a result, the designing conditions are cast in terms of linear matrix inequalities and efficiently solved via commercially available software. Numerical as well as academic setups are provided to illustrate the advantages and performance of the proposal.

1. Introduction

The use of state observers [1] is important for many tasks within control theory. In particular, unknown input (UI) observers whose task is to estimate both the state and unknown inputs (they may be disturbances) have been developed in [2], and they are of particular interest also for fault detection and isolation [3, 4] or fault tolerant control schemes [5, 6]; they can be also applied in other fields [7]. For linear systems, there are several UI observers, e.g., [8–10]; for nonlinear systems, linear methods are mostly chosen because of its simplicity [11]. Nonlinear techniques such as sliding mode [12], adaptive schemes [13], Lipschitz approaches [14], high-gain observers [15], or combinations of them require certain structure of the model or performing nonlinear transformations.

On the other hand, convex models (a convex model is a collection of linear models interconnected by scalar functions (also known as scheduling functions), which are nonlinear and hold the convex sum property in a region [16, 17]. If the convex model is the result of the sector nonlinearity [18], it is an exact representation of the nonlinear system) [19, 20] have been directly combined with the direct Lyapunov method; thus obtaining conditions in terms of linear matrix inequalities (LMIs) [21]. Conditions in the form of LMIs are preferred since they are numerically solvable via convex optimization techniques [22]. Within this context, there exist several state observers for both continuous [23, 24] and discrete-time systems [25–27]. In the case of UI observers, there are some works concerning proportional-integral setups [28, 29] or non-Luenberger forms [30–34].

Nevertheless, within the convex framework, the observer design presents an open problem: if the scheduling variables do not exclusively depend on measurable/available signals, the designing conditions get involve and difficult to cast as LMIs. Recent works have intended to tackle this issue by employing Lipschitz constraints [24, 35], the differential mean value theorem (DMVT) [36, 37], and robust $H_{\infty}$ approaches to mitigate the influence of the unknown scheduling parameters [38]; more recently, in [39, 40], a transformation that enlarges the size of the state is proposed.
In this work, we follow the ideas of [41] for solving this problem: algebraic rearrangements in order to properly factorize the error signal.

A generalized representation for standard state-space models is given by descriptor systems [42]; within this setup, a special case considers that the descriptor matrix is invertible [43]. For this class of descriptor systems, in [27], a convex observer design has been developed, and it considers only available scheduling variables; in [44, 45], an UI observer has been proposed, and it can handle unmeasurable scheduling variables by robust argumntations.

The estimation of the states and parameters makes possible the fault diagnosis, which is divided into fault detection, fault isolation, and fault identification [46, 47]. The fault detection (FD) is used to identify when there is a fault, which is divided in three scenarios: fault detection, fault isolation, and fault identification [46, 47]. Possible the fault diagnosis, which is divided into fault detection, fault isolation, and fault identification [46, 47].

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2.1. An Amenable Descriptor Error Form. In order to obtain an amenable error system for Lyapunov analysis, in [41], a methodology that avoids the use of Lipschitz-like bounds has been presented; it is based on algebraic operations in order to factorize the estimation error at the left side of error dynamic difference equation. Thus, the variation of the Lyapunov function along the trajectories of the error system can be written as $\Delta V (e) = \epsilon_2^T Q (\cdot) \epsilon_2$ and $\Delta V (e) < 0$ if $Q (\cdot) < 0$ and the latter is guaranteed via convex models and linear matrix inequalities. Motivated by these ideas, the following observer structure is adopted:

$$\mathcal{E} (y_k) \tilde{x}_{k+1} = \mathcal{A} (\tilde{x}_k, y_k) \tilde{x}_k + \mathcal{B} (y_k) u_k + L (\tilde{x}_k, y_k) (y_k - \tilde{y}_k), \quad \tilde{y}_k = \mathcal{E} (\tilde{x}_k) \tilde{x}_k, \quad (6)$$

with

$$\tilde{x}_k = \begin{bmatrix} \tilde{x}_k \\ \tilde{d}_k \end{bmatrix},$$

$$\mathcal{A} (\tilde{x}_k, y_k) = \begin{bmatrix} A (\tilde{x}_k) & D (y_k) Q_n B_p \\ 0 & S \end{bmatrix}, \quad (7)$$

$$\mathcal{E} (\tilde{x}_k) = \begin{bmatrix} C (\tilde{x}_k) \end{bmatrix} 0,$$

and the nonlinear observer gain $L (\tilde{x}_k, y_k) \in \mathbb{R}^{mp + pq}$ depending only on all the available signals; it should be designed such that the estimation error

$$e_k = x_k - \tilde{x}_k = \begin{bmatrix} x_k - \tilde{x}_k \\ d_k - \tilde{d}_k \end{bmatrix} \quad (8)$$

satisfies $\lim_{k \to \infty} e_k = 0$. Thus, in [41], it has been proven that, under mild assumptions, it is always possible to write the error dynamics as

$$\mathcal{E} (y_k) e_{k+1} = \mathcal{A} (x_k, \tilde{x}_k) - L (x_k, y_k) \mathcal{E} (x_k, \tilde{x}_k) e_k, \quad (9)$$

where

$$\mathcal{A} (x_k, \tilde{x}_k) e_k = \mathcal{A} (x_k, \tilde{x}_k) y_k - \mathcal{A} (\tilde{x}_k, y_k) \tilde{x}_k \quad \text{and} \quad \mathcal{E} (x_k, \tilde{x}_k) e_k = \mathcal{E} (x_k) \tilde{x}_k - \mathcal{E} (\tilde{x}_k) y_k$$

have bounded entries in $\mathbb{R}^\omega \times \mathbb{R}^\omega$. For instance, consider a polynomial expression $p (x) - p (\tilde{x})$ with $p (x) = x_1 x_2$ and $p (\tilde{x}) = \tilde{x}_1 \tilde{x}_2$; following [41], we have $p (x) - p (\tilde{x}) = 0.5 (x_1 + \tilde{x}_1) e_1 + 0.5 (x_2 + \tilde{x}_2) e_2, e_1 = x_1 - \tilde{x}_1, e_2 = x_2 - \tilde{x}_2$. Now, consider a nonpolynomial expression $\bar{p} (x) - \bar{p} (\tilde{x})$, with $\bar{p} (x) = \sin x_1$ and $\bar{p} (\tilde{x}) = \sin \tilde{x}_1$; a third-order Taylor approximation will give $\bar{p} (x) - \bar{p} (\tilde{x}) = x_1 - (x_1^3/6) - \tilde{x}_1 + (\tilde{x}_1^3/6)$, and then we can apply similar procedure as before.

Now, error system (9) has an amenable form for Lyapunov-based analysis. However, if the aim is to obtain LMI conditions, (9) should be expressed as an exact convex model, and this is the matter of the following section.

2.2. Convex Expressions. The sector nonlinearity [18] is employed to express bounded nonconstant terms $z (\cdot) \in \{ z^0, z^1 \}$ as a convex sums of its bounds, that is, $z (\cdot) = w_0 (z) z^0 + w_1 (z) z^1$, where $z^0$ and $z^1$ are the minimum and maximum of $z (\cdot)$ in a region; the functions $w_0 = (z^1 - z (\cdot))/(z^1 - z^0)$ and $w_1 = 1 - w_0$ hold the convex sum property in the same modeling region, i.e., $w_0 (z) + w_1 (z) = 1$ and $w_0, w_1 \in [0, 1]$.

Note that, in (9), matrices $\mathcal{E} (y_k), \mathcal{A} (x_k, \tilde{x}_k),$ and $\mathcal{E} (x_k, \tilde{x}_k)$ contain nonconstant terms depending on $x_k, \tilde{x}_k,$ and $y_k$; clearly, all the state variables are not fully available; thus, a useful convex rewriting must take this into account. The following steps extend the sector nonlinearity to our case:

Step 1: identify all the nonconstant terms, also known as scheduling variables, in $\mathcal{E} (y_k), \mathcal{A} (x_k, \tilde{x}_k), \mathcal{E} (x_k, \tilde{x}_k)$, and $\mathcal{E} (x_k, \tilde{x}_k)$ depending exclusively on available signals, and capture them in the vector $z (x_k, y_k) \in \mathbb{R}^\nu$ while all the rest of the terms should be grouped in $\zeta (x_k, \tilde{x}_k, y_k) \in \mathbb{R}^\rho$. Each entry is assumed to be bounded in $\Omega_x \times \Omega_{\tilde{x}}$, i.e., $z_i (x_k, y_k) \in \{ z_i^0, z_i^1 \}, i \in \{ 1, 2, \ldots, s \}$, and $\zeta_j (x_k, \tilde{x}_k, y_k) \in \{ \zeta_j^0, \zeta_j^1 \}, j \in \{ 1, 2, \ldots, \sigma \}$.

Step 2: construct, for each $z_i (x_k, y_k), i \in \{ 1, 2, \ldots, s \}$, and $\zeta_j (x_k, \tilde{x}_k, y_k), j \in \{ 1, 2, \ldots, \sigma \}$ a pair of scalar convex functions as follows:

$$w_i^0 (x, y) = \frac{z_i^0 - z_i (x, y)}{z_i^1 - z_i^0}, \quad w_i^1 (x, y) = 1 - w_i^0 (x, y), \quad (10)$$

$$w_i^0 (x, \tilde{x}, y) = \frac{\zeta_i^0 - \zeta_i (x, \tilde{x}, y)}{\zeta_i^1 - \zeta_i^0}, \quad w_i^1 (x, \tilde{x}, y) = 1 - w_i^0 (x, \tilde{x}, y);$$

by construction, these functions hold the convex sum property in $\Omega_x \times \Omega_{\tilde{x}}$, i.e., $w_i^0 (x, y) + w_i^1 (x, \tilde{x}, y) = 1, w_i^0 (x, \tilde{x}, y), w_i^1 (x, \tilde{x}, y) \in [0, 1], w_0^0 (x, \tilde{x}, y) + w_0^1 (x, \tilde{x}, y) = 1, w_0^0 (x, \tilde{x}, y), w_0^1 (x, \tilde{x}, y) \in [0, 1]$.

Step 3: define the scheduling functions as

$$w_j (z) = w_j^1 (z_j), \quad \omega_j (\zeta) = \omega_j^1 (\zeta_j), \quad (11)$$

with $i \in \{ 1, 2, \ldots, r \}, j \in \{ 1, 2, \ldots, \rho \}, \rho > 2^\sigma$; moreover, the sets of indexes $\{ i_1, i_2, \ldots, i_\rho \}$ and $\{ j_1, j_2, \ldots, j_\rho \}$ are $s$-digit and $\sigma$-digit binary representation of $(i - 1)$ and $(j - 1)$, respectively. The scheduling functions also hold the convex sum property in $\Omega_x \times \Omega_{\tilde{x}}$, that is, $\sum_{j=1}^\rho \omega_j (z) = 1, \omega_j (z) \in [0, 1], \sum_{j=1}^\rho \omega_j (\zeta) = 1, \omega_j (\zeta) \in [0, 1]$.

Step 4: compute the vertex models $\mathcal{E}_j = \mathcal{E} (y_k) |_{w_j=1}, \mathcal{A}_{ij} = \mathcal{A} (x_k, \tilde{x}_k) |_{w_{ij}=1}, \mathcal{E}_{ij} = \mathcal{E} (x_k, \tilde{x}_k) |_{w_{ij}=1}, i \in \{ 1, 2, \ldots, r \}, j \in \{ 1, 2, \ldots, \rho \}$. 

Thus, an exact convex representation of (9) is
\[
\sum_{i=1}^{C} \mathbf{w}(i) e_{k} e_{k+1} = \sum_{i=1}^{C} \sum_{j=1}^{C} \mathbf{w}(i) \mathbf{w}(j) (\mathbf{A} - L(\mathbf{X}_k, \mathbf{Y}_k)) e_{k} e_{k+1}.
\]
(12)

The nonlinear observer gain is \( L(\mathbf{X}_k, \mathbf{Y}_k) \) and will be defined later on.

2.2.1. Notation. For convex expressions, the following shorthand notation will be employed throughout the manuscript: single convex sums \( \sum_{i=1}^{C} \mathbf{x}_i \mathbf{w}(i) \mathbf{y}_i \) and its inverse \( \sum_{i=1}^{C} \mathbf{x}_i \mathbf{w}(i) \mathbf{y}_i \) denotes \( \mathbf{X}_k \mathbf{Y}_k \), with delayed scheduling functions \( \sum_{i=1}^{C} \mathbf{x}_i \mathbf{w}(i) \mathbf{y}_i \) or depending on nonavailable variables \( \mathbf{X}_k \mathbf{Y}_k \), and so on. Additionally, \( \mathbf{A} > 0 \) means that \( \mathbf{A} \in \mathbb{R}^{n \times n} \) is positive (negative) definite. An asterisk (*) will be used in matrix expressions to denote the transpose of the symmetric element; for in-line expressions, it will denote the transpose of the terms on its left side:
\[
\begin{bmatrix}
  A & B^T \\
  B & C
\end{bmatrix} = \begin{bmatrix}
  A & (*) \\
  B & C
\end{bmatrix} \text{ and } A + B + A^T + B^T + C = A + B + (*) + C.
\]
(13)

Hence, system (12) is shortly written as
\[
\mathbf{G}(\mathbf{w}(i)) e_{k+1} = (\mathbf{A} - L(\mathbf{X}_k, \mathbf{Y}_k)) e_{k} e_{k+1}.
\]
(14)

The following lemmas are useful in order to derive LMI conditions for the design of \( L(\mathbf{X}_k, \mathbf{Y}_k) \). The first one concerns a sum-relaxation scheme based on [59]; the second one allows avoiding the computation of \( \mathbf{G}^{-1}(\mathbf{y}_k) \) while adding slack variables [27].

Lemma 1 (see [59]). Let \( \mathbf{Y}^{m}_{\mathbf{d}} = (\mathbf{Y}^{m}_{\mathbf{d}})^T \), \( (i, l, m) \in \{1, 2, \ldots, r\}^3 \), \( j \in \{1, 2, \ldots, l\} \), be matrices of adequate dimensions. Then, \( \mathbf{Y}^{m}_{\mathbf{d}} + \mathbf{Y}^{m}_{\mathbf{d}} + \mathbf{Y}^{m}_{\mathbf{d}} < 0 \) holds if
\[
\frac{2}{r - 1} \mathbf{Y}^{m}_{\mathbf{d}} + \mathbf{Y}^{m}_{\mathbf{d}} + \mathbf{Y}^{m}_{\mathbf{d}} < 0,
\]
(15)
for all \( (i, l, m) \in \{1, 2, \ldots, r\}^3 \), \( j \in \{1, 2, \ldots, l\} \).

Lemma 2 (see [60]). Let \( \xi \in \mathbb{R}^n \), \( \Theta = \Theta^T \in \mathbb{R}^{m \times n} \), and \( \mathbf{R} \in \mathbb{R}^{m \times n} \), rank \( \mathbf{R} \) \(< n \); then, the following statements are equivalent:
\[ (1) \quad \mathbf{R} = \mathbf{R} \mathbf{R}^T, \quad \mathbf{R} \mathbf{R} \mathbf{R}^T < 0 \]
\[ (2) \quad \exists \mathbf{Z} \in \mathbb{R}^{n \times m} : \mathbf{Z} \mathbf{Z} \mathbf{Z}^T < 0 \]

Now, we are ready to establish LMI conditions for the stabilization of error system (9) at the origin \( e = 0 \) via its exact convex representation (12).

3. LMI-Based Stabilization of the Error System

This section provides LMI conditions to compute \( L(\mathbf{X}_k, \mathbf{Y}_k) \). The developments are based on a convex Lyapunov function candidate (in the context of TS models, it has been introduced as nonquadratic Lyapunov function [61], in the context of fuzzy systems as fuzzy Lyapunov functions [62], in the context of LPV models, as parameter dependent Lyapunov functions [63]). Here, the name convex Lyapunov function is adopted due to its dependence on the convex functions \( \mathbf{w}(i) \).

\[
V(e) = e^T P_{e}(z_k) e_k, \quad P_{e}(z_k) = \sum_{i=1}^{C} \mathbf{w}(i) P_{e}(z_k),
\]
(16)
with \( P_{e} \in \mathbb{R}^{(n+p+q) \times (n+p+q)} \); \( P_i > 0, i \in \{1, 2, \ldots, r\} \); its variation is
\[
\Delta V(e) = e^T P_{e}(z_{k+1}) e_{k+1} - e^T P_{e}(z_k) e_k,
\]
(17)
which can be expressed as follows (without substituting the dynamics of estimation error (9)):
\[
\Delta V(e) = \begin{bmatrix} e_k & e_{k+1} \end{bmatrix}^T \begin{bmatrix} P_{w}(z_k) & 0 \\ 0 & P_{w}(z_{k+1}) \end{bmatrix} \begin{bmatrix} e_k \\ e_{k+1} \end{bmatrix}.
\]
(18)

Note that function (16) is convex and it only depends on available signals. With this in mind, the following result states LMI conditions for the design of the nonlinear gain \( L(\mathbf{X}_k, \mathbf{Y}_k) \).

**Theorem 1.** The origin \( e = 0 \) of error system (9), with an exact convex representation (12), is asymptotically stable if there exist matrices \( P_i \in \mathbb{R}^{(n+p+q) \times (n+p+q)} \), \( N_i \in \mathbb{R}^{(n+p+q) \times n} \), and \( G_i \in \mathbb{R}^{(n+p+q) \times (n+p+q)} \), \( i \in \{1, 2, \ldots, r\} \) such that \( P_i > 0 \) and the LMIs in (15) are satisfied with
\[
\begin{bmatrix} P_i & \ast \\ \ast & -N_i \end{bmatrix} > 0,
\]
(19)
for all \( (i, l, m) \in \{1, 2, \ldots, r\}^3 \), \( j \in \{1, 2, \ldots, l\} \). Then, the observer gain is computed as \( L(\mathbf{X}_k, \mathbf{Y}_k) = G_{w}(z_k) N_{w}(z_k) \). Moreover, any trajectory \( e_k \) starting in the outermost Lyapunov level set \( \{ e : V(e) \leq c \} \subset \Omega_1 \times \Omega_2, c > 0 \) goes to zero as time goes to infinity.

**Proof.** Recall the variation of the Lyapunov function expressed as in (18); thus, error system (9), with exact convex representation (12), is also expressed as
\[
[ \mathbf{A} - L(\mathbf{X}_k, \mathbf{Y}_k) ] e_{k} e_{k+1} = [ e_k e_{k+1} ] = 0.
\]
(20)

By Lemma 2, (18) and (20) can be written together:
\[
\begin{bmatrix} Z_1 & Z_2 \end{bmatrix} \begin{bmatrix} \mathbf{A} - L(\mathbf{X}_k, \mathbf{Y}_k) e_{k} e_{k+1} \end{bmatrix} = \begin{bmatrix} -P_{e}(z_k) & 0 \\ 0 & P_{w}(z_{k+1}) \end{bmatrix} < 0.
\]
(21)

Hence, by choosing \( Z_1 = 0 \) and \( Z_2 = G_{w}(z_k) \), and the definition \( L(\mathbf{X}_k, \mathbf{Y}_k) = G_{w}(z_k) N_{w}(z_k) \), \( G_{w}(z_k) \in \mathbb{R}^{(n+p+q) \times (n+p+q)} \), \( N_{w}(z_k) \in \mathbb{R}^{(n+p+q) \times n} \), and we have
Finally, by means of relaxation Lemma 1, the desired result yields.

Remark 1. The speed convergence of observer (6) can be increased if the following condition is verified \( \Delta V(e) \leq (\alpha^2 - 1)V(e) \), \( 0 < \alpha \leq 1 \) which can be translated into LMIs, that is, solving the LMIs in (15) with

\[
Y^{jm}_{il} := \begin{bmatrix}
-\alpha^2 P_l & * \\
G_l \delta A_{ij} - N_l \delta i & -G_l + G_l ^T + P_m
\end{bmatrix},
\]

for all \( (i,l,m) \in \{1,2,\ldots,r\}^3, j \in \{1,2,\ldots,p\} \).

The following result provides LMI conditions for standard systems and its corresponding observer, i.e., (6) with \( E(y_k) = I_n \), where \( I_n \) is the identity matrix. Hence, the next result follows directly from Theorem 1.

**Corollary 1.** The origin of error system (9), with \( E(y_k) = I_n \), is asymptotically stable if there exist \( P_l \in \mathbb{R}^{(n+p+q+r) \times (n+p+q+r)}, N_l \in \mathbb{R}^{(n+p+q+r) \times (n+q+r)}, \) and \( G_l \in \mathbb{R}^{(n+p+q+r) \times (n+q+r)} \), \( l \in \{1,2,\ldots,r\} \) such that \( P_l > 0 \) and LMIs in (15) hold with

\[
Y^{jm}_{il} := \begin{bmatrix}
-\alpha^2 P_l & * \\
G_l \delta A_{ij} - N_l \delta i & -G_l + G_l ^T + P_m
\end{bmatrix},
\]

for all \( (i,l,m) \in \{1,2,\ldots,r\}^3, j \in \{1,2,\ldots,p\} \).

The nonlinearity observer gain \( L(x_k, y_k) = G_w(x_k)N_w(z_k) \) Moreover, for any trajectory \( e_k \) starting in the outermost Lyapunov level set \( \{e, V(e) \leq c\} \subset \Omega_x \times \Omega_z, c > 0 \) goes to zero as time goes to infinity.

**Proof.** It follows directly by considering \( E(y_k) = I_n \) from the developments in Theorem 1.

Remark 2. The methodology in previous approaches starts by computing a convex model of the given nonlinear one (standard or descriptor), and then, the estimation error is computed; if the problem involves unmeasurable scheduling variables, it is solved by considering the already defined scheduling functions. In contrast with those works, the methodology hereby presented begins by obtaining the nonlinear estimation error dynamics via factorizations; then, it employs the sector nonlinearity approach for writing an equivalent convex representation; thus, a fitter convex model is employed and the observer gain includes all the available signals.

Remark 3. In this work, a convex Lyapunov function such as the one introduced in [61] is employed; naturally delayed scheduling functions can be included similar to [25] or to the generalization [27]. For instance, following [25], the Lyapunov function candidate would be \( V(e) = e^T P_{w(z_k)} e_k \), \( P_{w(z_k)} = \sum_{i=1}^{r} w_m (z_k) P_m \) together with the observer gain \( L(x_k, \tilde{x}_k, y_k) = G_w(z_k) N_w(z_k) \).

Remark 4. The numerical complexity of Theorem 1 can be approximated by \( \log_{10}(n^2 n_l) \) where \( n_l = (n + p + q + r) \) the number of decision variables and \( n_l = 2r^p (n + p + q + o) \) is the number of LMI rows [64]. Moreover, if the number LMI conditions in Theorem 1 is large, one can reduce its complexity by “judiciously eliminating” some convex functions either from the Lyapunov function or the observer gain.

In the case without unknown inputs \( (d_k = 0) \) (1), the states of the resulting system can be estimated by the following nonlinear observer:

\[
E(y_k) \hat{x}_{k+1} = A(\hat{x}_k, y_k) \hat{x}_k + B(y_k) u_k + L(x_k, y_k)(y_k - \hat{y}_k), \quad \hat{y}_k = C(\hat{x}_k, y_k) \hat{x}_k,
\]

where \( \hat{x}_k \in \mathbb{R}^n \) is the observer state and \( L(\hat{x}_k, y_k) \in \mathbb{R}^{n \times n} \) is the observer gain to be designed. Following the methodology previously presented, the error dynamics is

\[
E(y_k) e_{k+1} = (\delta A(\hat{x}_k, \hat{x}_k) - L(\hat{x}_k, y_k) \delta C(x_k, \hat{x}_k)) e_k,
\]

where \( \delta A(x_k, \hat{x}_k) e_k = A(x_k) x_k - A(\hat{x}_k, y_k) \hat{x}_k \) and \( \delta C(x_k, \hat{x}_k) e_k = C(x_k) x_k - C(\hat{x}_k, y_k) \hat{x}_k \). Thus, we have the next result.

**Corollary 2.** The origin of error system (26) is asymptotically stable if there exist \( P_l \in \mathbb{R}^{n \times n}, N_l \in \mathbb{R}^{n \times n}, \) and \( G_l \in \mathbb{R}^{n \times n} \), \( l \in \{1,2,\ldots,r\} \) such that \( P_l > 0 \) and LMIs in (15) hold with

\[
Y^{jm}_{il} := \begin{bmatrix}
-\alpha^2 P_l & * \\
G_l \delta A_{ij} - N_l \delta i & -G_l + G_l ^T + P_m
\end{bmatrix},
\]

for all \( (i,l,m) \in \{1,2,\ldots,r\}^3, j \in \{1,2,\ldots,p\} \).

**Proof.** The proof follows a similar path than the one for Theorem 1, with a Lyapunov function candidate \( V(e) = e^T P_{w(z_k)} e_k \) \( P_l \in \mathbb{R}^{n \times n}, \) \( l \in \{1,2,\ldots,r\} \), and the nonlinear error dynamics (26) expressed in a convex form as...
\[
\left[ \bar{A}_w(z_i)w(z_i) - L(\bar{x}_k, y_k)\bar{C}_w(z_i)w(z_i) - E_w(z_i) \right] e_k + e_{k+1} = 0,
\]
and by the use of Finsler’s lemma, with \( L(\bar{x}_k, y_k) = G_{w(z_i)}^{-1}N_w(z_i) \), \( Z_1 = 0 \), and \( Z_2 = G_{w(z_i)} \), we have

\[
Y_w(z_i)w(z_i)w(z_i))w(z_i) = \begin{bmatrix} -P_w(z_i) \\ G_w(z_i)\bar{A}_w(z_i)w(z_i) - N_w(z_i)\bar{C}_w(z_i)w(z_i) - G_w(z_i)E_w(z_i) + (\ast) + P_w(z_k) \end{bmatrix} < 0.
\]

**4. Actuator Fault Detection and Estimation**

In this section, an application of the UI observer developed above is employed in order to estimate and detect actuator faults [46, 47]. Thus, let us now consider nonlinear discrete-time descriptor system (1) with an actuator fault:

\[
E(y_k)x_{k+1} = A(x_k)x_k + B(y_k)u_k + f_a(u_k, y_k) + D(y_k)d_k, \quad y_k = C(x_k)x_k,
\]

where \( f_a(u_k, y_k) \in \mathbb{R}^m \) represents the actuator fault vector; if it can be included inside of the unknown input, then (30) can be expressed as (1); in this case, conditions in Theorem 1 can be used. As customary in fault detection schemes, a residual signal based on the observer estimation of the unknown input is generated [65, 66]:

\[
r_k = \| \bar{d}_k \|.
\]

The residual can be filtered to get a clear signal. The estimation of the fault can be done once the fault is considered as an unknown input [67].

**Remark 5.** It is possible to apply a mix of \( H_{\infty} \) and \( H_\infty \) performances, for instance, \( H_{\infty} \) attenuation in order to make residual (31) robust to uncertainties, noise, and the fault and the \( H_\infty \) index in order to make more sensitive the detection of the fault [68, 69].

**5. Examples**

In this section, two examples are presented. The first one is intended to compare the performance of our proposal in contrast with recent approaches while the second one illustrates the estimation of actuator faults in the train system. The LMI conditions have been implemented in YALMIP [70] with SeDuMi [71], while simulations have been run in Simulink for MATLAB2019b.

**Example 1.** Consider descriptor system (1) with matrices

\[
E(y_k) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad A(x_k) = \begin{bmatrix} \chi_2^2 & -0.5\beta \\ 0.7\sin x_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad D = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, \quad C = \begin{bmatrix} 1^T \\ 0 \end{bmatrix},
\]

with \( \beta > 0 \) is a known parameter. It is assumed that the unknown input has dynamics such as \((z - 1)^2\bar{d}_k = 0\), i.e., \( d_{k+1} = Sd_k \) with \( S = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \). In order to estimate both the state \( \chi_k \) and the unknown input \( d_k \), an extended system of form (4) and its corresponding observer (6) are constructed. Thus, the error dynamics \( e_k = \chi_k - \bar{\chi}_k \) can be computed as

\[
\xi(y_k)e_{k+1} = \begin{bmatrix} x_1\chi_2^2 - 0.5\beta x_2 + 0.2d_1 \\ 0.7x_1 + x_2\sin x_1 + 0.2d_1 \\ d_1 \\ -d_1 + 2d_2 \end{bmatrix} - \begin{bmatrix} \bar{x}_1\chi_2^2 - 0.5\beta \bar{x}_2 + 0.2\bar{d}_1 \\ 0.7\bar{x}_1 + \bar{x}_2\sin \bar{x}_1 + 0.2\bar{d}_1 \\ \bar{d}_2 \\ -\bar{d}_1 + 2\bar{d}_2 \end{bmatrix} - L(\cdot)(y - \bar{y}).
\]
Applying the factorization methodology in [41] and considering that \( y = x_1 \), we have \( x_3^2 - \bar{x}_3^2 = (x_2 + \bar{x}_2)(x_2 - \bar{x}_2) = (x_2 + \bar{x}_2)e_2 \) leading to final error dynamics

\[
\begin{bmatrix}
2 & -(1 + x_1^2)^{-1} & 0 & 0 \\
(1 + x_1^2)^{-1} & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
e_{k+1} \\
e_k(\gamma_1)
\end{bmatrix}
= 
\begin{bmatrix}
0 & x_1(x_2 + \bar{x}_2) - 0.5\beta & 0.2 & 0 \\
0.7 & \sin x_1 & 0.2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 2
\end{bmatrix}
\begin{bmatrix}
e_k \\
-\mathcal{L}(\cdot) \left[ \begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \end{smallmatrix} \right]
\end{bmatrix}
\tag{34}
\]

In order to synthesize the nonlinear observer gain via LMIs, it is necessary to express (34) in a convex form (12); to this end, let us consider the compact sets \( \Omega_1 = \{|x_1| \leq 0.5, |x_2| \leq 0.8\} \) and \( \Omega_2 = \{|\bar{x}_1| \leq 0.8\} \). Thus, the non-constant terms and their bounds are \( z_1 = (1 + x_1^2)^{-1} \in [0.8, 1] \), \( z_2 = \sin x_1 \in [-0.4794, 0.4794] \), \( z_3 = x_1 \in [-0.5, 0.5] \), \( z_4 = \bar{x}_2 \in [-0.8, 0.8] \), and \( \xi_1 = x_1 \in [-0.8, 0.8] \) (nonavailable signals). The scalar convex functions are \( w(z) = w_i^1(z_1)w_i^2(z_2)w_i^3(z_3)w_i^4(z_4), \quad w_i^j(z_i) = (z_i^1 - z_i)/\left( z_i^2 - z_i^0 \right), \quad w_i^j(z_i) = 1 - w_i^j(z_i), \quad i \in \{1, 2, 3, 4\}, \quad \) and \( \omega_i(\xi) = \omega_i^1(\xi_1), \quad \omega_i^j(\xi_1) = (\xi_1^1 - \xi_1)/\left( \xi_1^2 - \xi_1^0 \right), \quad \omega_i^j = 1 - \omega_i \). The vertex matrices are

\[
\begin{pmatrix}
2 & -z_i^0 & 0 & 0 \\
z_i^0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
2 & -0.8 & 0 & 0 \\
0.8 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & z_i^0(\xi_i^0 + \xi_i^4) - 0.5\beta & 0.2 & 0 \\
0.7 & z_i^2 & 0.2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 2
\end{pmatrix}
= 
\begin{pmatrix}
0 & -0.5\beta & 0.2 & 0 \\
0.7 & 0.4794 & 0.2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 2
\end{pmatrix}
\tag{35}
\]

For example, if \( i = 6 \) and \( j = 1 \), we have \( w_6(z) = w_6^1(z_1)w_6^2(z_2)w_6^3(z_3)w_6^4(z_4) \), \( \omega_1(\xi) = \omega_6^1(\xi_1) \), and

\[
\begin{pmatrix}
2 & -z_1^0 & 0 & 0 \\
z_1^0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
2 & 0.8 & 0 & 0 \\
0.8 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Several numerical tests have been performed in order to show the effectiveness of the proposal in contrast with recent works:

(i) LMIs in Theorem 1 have been run as well as the ones in [45], Theorem 2, with a Lyapunov function candidate (16) and decay rate \( \alpha = 0.95 \) (see Remark 1), seeking feasibility for the largest \( \beta > 0 \). It results that our proposal is feasible up to \( \beta = 0.863 \), while the one in [45], Theorem 2, is only up to \( \beta = 0.794 \). Comparisons have been done with the same Lyapunov function (16) and under the same relaxation scheme.

(ii) In terms of the number of decision variables, for this example, we have that numerical complexity for Theorem 1 is 12.86 while for [45] is 10.24.

(iii) A UI observer uses the generalization in [27], and although it is for discrete-time descriptor systems, it cannot be applied as it only considers available scheduling variables.

(iv) If UI observer schemes for standard state-space models are to be applied [33, 39], then the inverse of the matrix \( E(\gamma_i) \) has to be computed. For instance, using conditions in Corollary 1 yields numerical problems because 65568 LMIs have to be solved.
For illustration purposes, the LMI conditions in form (1) have been found feasible for $\beta \approx 0.85$ and a decay rate $\alpha = 0.95$, and some of the computed matrices are:

$$N_1 = 1 \times 10^{-4} \begin{bmatrix} 0.3502 \\ 0.4455 \\ 0.0786 \\ -0.0245 \end{bmatrix},$$

$$G_1 = 1 \times 10^{-4} \begin{bmatrix} 0.4528 & 0.4482 & -0.1117 & -0.0147 \\ -0.4197 & 0.6839 & 0.0704 & -0.0772 \\ 0.0656 & -0.3033 & 0.7189 & -0.5508 \\ -0.0796 & 0.1505 & -0.5751 & 0.4700 \end{bmatrix},$$

$$N_5 = 1 \times 10^{-4} \begin{bmatrix} 0.4936 \\ 0.5295 \\ 0.0875 \\ -0.0429 \end{bmatrix},$$

$$G_5 = 1 \times 10^{-4} \begin{bmatrix} 0.3710 & 0.5399 & -0.2236 & 0.0594 \\ -0.4458 & 0.6316 & 0.0105 & -0.0417 \\ 0.0960 & -0.2168 & 0.6811 & -0.5264 \\ -0.0961 & 0.0893 & -0.5422 & 0.4492 \end{bmatrix},$$

$$N_{16} = 1 \times 10^{-4} \begin{bmatrix} -0.1330 \\ 0.7569 \\ 0.1866 \\ -0.0911 \end{bmatrix},$$

$$G_{16} = 1 \times 10^{-4} \begin{bmatrix} 0.8122 & -0.3260 & 0.0464 & -0.0970 \\ -0.0814 & 0.9592 & -0.2018 & 0.0460 \\ -0.0773 & -0.1013 & 0.6744 & -0.5174 \\ -0.0047 & 0.0364 & -0.5419 & 0.4463 \end{bmatrix}.$$

(37)

A first simulation has been performed without unknown inputs, i.e., $d_k = 0$ and the known input $u_k = 0.4 \sin(0.6k)$. Figure 1 shows evolution of the error signals converging asymptotically to the origin. A second simulation considers $u_k = 0$, and the unknown input is defined as

$$d_k = \begin{cases} 0.4k, & \text{if } 3 \leq k < 5, \\ 0.8, & \text{if } 5 \leq k < 12, \\ 0, & \text{otherwise}, \end{cases}$$

and initial conditions $x(0) = [0.4 - 0.6]^T$, $\mathbf{d}(0) = [0 0]^T$, $\mathbf{\dot{x}}(0) = [0 0]^T$, and $\mathbf{\ddot{d}}(0) = [0 0]^T$. Figure 2 shows the state $x_2$ being adequately estimated while in Figure 3, it can be seen that the observer effectively reconstructs the unknown input.

The next example provides an application of the UI observer hereby proposed for the task of actuator fault detection in a train system.

Example 2. This example provides a fault detection scheme via the proposed UI observer for a train system, and see Figure 4 for a diagram. Let us consider first the fault-free case, and thus, a continuous-time model is [72–74]

$$M\ddot{v}(t) = b_T(v(t))u_T(t) - b_B(v(t))u_B(t) - w(v(t)) - g(\theta(p)), \quad \ddot{p}(t) = v(t),$$

(39)

where $v(t)$ is the speed of the train, $p(t)$ is the position of the train, $b_B(v)$ is the maximum coefficient of the braking force, $u_B(t)$ is the relative braking force, $b_T(v)$ is the maximum coefficient of the traction force, $u_T(t)$ is the relative traction force, $g(\theta)$ is the force of declivity or external force, $w(v)$ is the friction force, and $M$ is the mass of the train. In practical cases, the declivity force is considered as $g(\theta(p)) = Mg \sin(\theta(p))$, where $\theta(p)$ is the slope angle on the position $p(t)$ and $g$ is the gravity force [75]. In [76], the friction force $w(v)$ can be estimated via David’s equation:

$$w(v) = w_0 + w_1 v + w_2 v^2,$$

(40)

where $w_0$, $w_1$, and $w_2$ are real coefficients that depend on the train characteristics and rail type; their values are unknown in the practice, but there exist some methods for their estimation [76]. By means of the Euler approximation $\dot{x}(t) = (x_{k+1} - x_k)/T_s$, where $T_s$ is the sampling time. Therefore, a discrete-time train system of (23) is

$$Mv_{k+1} = (b_T(v_k)u_{Tk} - b_B(v_k)u_{Bk} - w(v_k)) - g(\theta(p_k))T_s + Mv_k, \quad p_{k+1} = v_k T_s + p_k.$$  

(41)

The fault studied in this example is the jamming fault, and it may occur only during the braking. This fault depends on the adherence conditions between the wheel and the track, producing that when the brake locks the wheel, the wheel slides on the track [68, 74]; in the real setup, the fault can occur depending on weather conditions. The main issue of a jamming fault is the impact directly on the measurements, producing a wrong estimate of both position and speed of the train. In Figure 5, it...
can be seen that the speed \( v_k \) under the fault is lower, and this is because the wheel is locked during the fault and the sensor is not able to provide the right measure and thus both the speed and position are erroneous; for example, the measured position differs 4.10 meters from the real position, see Figure 6. Therefore, this fault can be considered as an actuator fault or exogenous input, when it occurs, the control is inhibited \([72–74]\).

The actuator fault to be considered is

\[
\begin{aligned}
f_a(u_T, u_B) &= (b_T(v_k)u_T - b_B(v_k)u_B + \xi_k)f_k,
\end{aligned}
\]

Figure 2: State \( x_2 \) and its estimation \( \hat{x}_2 \).

where \( f_k \in [0, 1] \) represents the grade of the fault; if \( f_k = 0 \), then the system is fault-free; if \( f_k = 1 \), a total fault occurs and the control is inhibited. The term \( \xi_k \in \mathbb{R} \) represents a resistive force produced by the fault, and it is unknown \([74]\).

As the previous examples, the unknown input is assumed to comply with \( d_{k+1} = Sd_k \); therefore, we have an augmented system of form (3) with \( \chi = [v_k, p_k, d_k, d_{k+1}]^T \) and matrices

\[
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix}
\]

Figure 1: Error signal \( e_k \) in the absence of unknown inputs.

\[
\begin{bmatrix}
d_k \\
\hat{d}_k
\end{bmatrix}
\]

Figure 3: Unknown input \( d_k \) and its estimation \( \hat{d}_k \).

Figure 4: Fault detection scheme.

Figure 5: Comparison between the time evolution of the speed \( v_k \) with fault and without fault.
and the parameter $b$ is a constant value for $b_B(v)$ and $b_T(v)$, producing a unique control signal $u_k$, where $u_k > 0$ and $u_k < 0$ correspond to $u_T$ and $u_B$, respectively. Let us consider the braking period, i.e., when the train is arriving to the station and there are no big slopes, we have $g(\theta(p_k)) = M g \sin(\theta(p_k))$, with $\theta(p_k) = \beta p_k \in [-1, 1]$, where $\beta$ is a constant.

Thus, with UI observer (4), the error dynamics $e_k = x_k - \hat{x}_k$ is

\[ \mathcal{G} e_{k+1} = \begin{bmatrix} (\omega_1 - \omega_2 y_k) T_s + M T_s g(\theta(p_k)) \theta(p_k) \end{bmatrix} T_s \begin{bmatrix} T_s & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} (-\omega_1 v_k - \omega_2 v_k^2 + d_k) T_s + M v_k - T_s M g \sin(\beta p_k) \end{bmatrix} \begin{bmatrix} T_s v_k + p_k \\ d_{k+1} \\ -d_k + 2d_{k+1} \end{bmatrix} + \begin{bmatrix} (-\omega_1 \tilde{v}_k - \omega_2 \tilde{v}_k^2 + \tilde{d}_k) T_s + M \tilde{v}_k - T_s M g \sin(\beta \tilde{p}_k) \end{bmatrix} \begin{bmatrix} T_s \tilde{v}_k + \tilde{p}_k \\ \tilde{d}_{k+1} \\ -\tilde{d}_k + 2\tilde{d}_{k+1} \end{bmatrix} - \mathcal{L}(\cdot)(y - \hat{y}). \]

(44)

Applying the factorization in [41] and considering that $y_k = v_k$ (the only available signal is the velocity $v_k$), we have $v_k^T - \tilde{v}_k^T = (v_k + \hat{v}_k)e_1 = 2y_k e_1$, $e_1 = v_k - \tilde{v}_k$, as for $\sin(\beta p_k) - \sin(\beta \tilde{p}_k)$, two terms of the series Taylor are taken; then, the error dynamics yields:

\[ \mathcal{G} e_{k+1} = (\mathcal{A}(y_k, x_k, \hat{x}_k) - \mathcal{L}(y_k) \mathcal{G}) e_k, \]

(45) with matrices
\[
\mathcal{F}(y_k, x_k, \tilde{x}_k) = \begin{bmatrix}
-(w_1 + 2w_2 y_k) T_s + M T_s M g \left(-1 - \frac{\beta^2}{6} \left( \bar{p}_k^2 + \bar{p}_k \bar{\bar{p}}_k + \bar{\bar{\bar{p}}}_k^2 \right) \right) T_s & 0 \\
\end{bmatrix}
\]

Therefore, considering that \( v_k \in [0, 30] \text{ (m/s) and } p_k, \bar{p}_k \in [0, 600] \text{ (m)}, \) we have that the nonconstant terms are \( z = v_k \in [0, 30] \) (available), \( \zeta_1 = \bar{p}_k^2 \in [0, 360000], \)
\( \zeta_2 = p_k \in [0, 600], \) \( \zeta_3 = \bar{p}_k \in [0, 600], \) \( \) and \( \zeta_4 = \bar{\bar{p}}_k \in [0, 360000] \) (nonavailable signals). The scheduling functions are defined as \( \omega_j(z) = \omega_j^1 (\zeta_1) \omega_j^2 (\zeta_2) \omega_j^3 (\zeta_3) \omega_j^4 (\zeta_4), \)
\( \omega_0^j (\zeta_j) = (\zeta_j^1 - \zeta_j)/ (\zeta_j^1 - \zeta_j^0), \) \( \omega_1^j = 1 - \omega_0^j (\zeta_j), \) and \( w_j(z) = (z^1 - z)/(z^1 - z^0), \) \( w_2(z) = 1 - w_1(z). \) The vertex matrices are

\[
\overline{\mathcal{F}}_{i,j} = \begin{bmatrix}
-(w_1 + 2w_2 z^i) T_s + M T_s M g \left(-1 - \frac{\beta^2}{6} \left( \bar{\bar{\bar{p}}}_k^2 + \bar{\bar{p}}_k \bar{\bar{p}}_k + \bar{\bar{\bar{p}}}_k^2 \right) \right) T_s & 0 \\
\end{bmatrix}
\]

where \( i = \{1, 2\} \) and \( j = \{1, 2, 3, 4\}. \) The LMI conditions in Theorem 1 have been found feasible with a decay rate \( \alpha = 0.7 \) (see Remark 1), and the parameters are \( M = 408.5 \text{ tons, } \)
\( w_0 = 3.525 \text{ (N/m), } w_1 = 2.98 \times 10^{-3} \text{ (N/m), } \)
\( w_2 = 4.575 \times 10^{-4} \text{ (N/m²), } \) and \( T_s = 0.001 \text{ (s).} \) The computed matrices are
A simulation has been performed for initial conditions $v(0) = 28$, $u(0) = 0$, $d(0) = [0\ 0]^T$, $\tilde{v}(0) = 27.9$, $\tilde{p}(0) = 0.05$, and $\tilde{d}(0) = [0.05\ 0.01]^T$, with a constant braking force $u_k = -0.5$, $\beta = 0.0016$, and the unknown input $d_k = f_d(u_k)$, with $\xi_k = 2M$ and the fault occurrence

$$f_k = \begin{cases} 1, & \text{if } 4 \leq k \leq 4.2, \\ 1, & \text{if } 6 \leq k \leq 6.2, \\ 1, & \text{if } 8 \leq k \leq 8.2, \\ 0, & \text{otherwise}. \end{cases}$$

As the previous example, first simulation has been performed without unknown inputs, i.e., $d_k = 0$. Figure 7 shows the speed being adequately estimated while Figure 8 plots the Lyapunov function, whose signal is always positive and monotonously decreases to zero.

In the case where $d_k = f_d(u_k)$, in order to detect the fault, the unknown input estimation is used as residual $r_k = |\tilde{d}_k|$ with a single threshold of $\epsilon = M/2$. If the residual $r_k$ is bigger than $\epsilon$, then the fault is occurring; otherwise, the fault has finished. Thus, we obtain a delay average of 0.003 seconds to detect when the fault begins and 0.006 to detect when the fault ends. As we can see from Figure 9, our proposal is enough to detect when the fault occurs.
6. Conclusions

It has been presented a methodology for the design of unknown input observers for discrete-time nonlinear descriptor systems. It is based on algebraic rearrangements and allows overcoming the problem of unmeasurable premise variables as to get a compatible error dynamic system with the direct Lyapunov method; thus, sufficient conditions in terms of LMIs have been obtained. The resulting UI observer is proven to be less conservative than those in the literature. Moreover, the proposed scheme has been employed to solve the detection and estimation of actuator faults. The advantages of the proposal have been illustrated via numerical example and applied to the fault detection in train systems.

Data Availability

No data were used to support this study. All the values to reproduce the results are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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