Proposal for precise spectroscopy of high-frequency oscillating fields with a single qubit sensor

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Precise spectroscopy of oscillating fields plays significant roles in many fields. Here, we propose an experimentally feasible scheme to measure the frequency of a fast-oscillating field using a single-qubit sensor. By invoking a stable classical clock, the signal phase correlations between successive measurements enable us to extract the target frequency with extremely high precision. In addition, we integrate dynamical decoupling technique into the framework to suppress the influence of slow environmental noise. Our framework is feasible with a variety of atomic and single solid-state-spin systems within the state-of-the-art experimental capabilities as a versatile tool for quantum spectroscopy.

Introduction.— Sensors working in quantum regime with new capabilities, large bandwidths, extremely high spatial and spectral resolutions attract increasingly interest in the field of precision metrology [1–17]. Frequency spectroscopy with single-qubit probes (e.g., nitrogen-vacancy centers in diamond) as one major branch in this field has achieved a significant breakthrough since the quantum heterodyne (Qdyne) technique was developed [18–20]. It enables a quantum sensor for precise spectroscopy to go beyond its coherence time by nonlinearly mixing a target signal field with a stable local oscillator [19].

In particular, precise frequency determination of fast-oscillating fields can have plenty of significant applications, e.g., high-resolution microwave field spectrum analysis [21–23], detection of electron spin motions in solids [24–26] and nuclear magnetic resonance in the high magnetic field regime [27, 28]. Conventional quantum lock-in detection, which allows a probe to cumulatively sense an oscillating signal, usually requires implementing a sequence of periodic spin-flipping π-pulses, the repetition rate of which should be resonant with the target field [29–38]. However, the accessible time duration to implement a sharp π pulse in experiments is not infinitesimal but a finite interval e.g. a few tens nanoseconds in solid-state-spin systems, due to the power limitations of the control fields [28, 39] and the deleterious high-power heating effects, e.g. in biological environments [40], which makes quantum heterodyne detection of high-frequency oscillating fields a big challenge.

In this work, we propose an experimentally feasible scheme for measuring the frequency of a fast-oscillating field using a single qubit sensor. In each single measurement, we effectively obtain a much slower radio-frequency (RF) signal field in a rotating frame, which is then spectrally separated from its noise environment and measured using a dynamical decoupling sequence. Extremely sharp π-pulses to be synchronized with the fast-oscillating field are thus not required and the sensor’s working region can be extended to the high-frequency range under ambient conditions. We further extract information on the target frequency by harnessing phase correlations between successive single measurements based on the Qdyne technique. The proposal is feasible with a solid-state-spin system formed by negatively charged nitrogen-vacancy (NV) center in diamond as well as other atomic qubit systems, and thus serves as a versatile tool for high-resolution quantum spectroscopy.

Effective RF signal in the transverse plane.— We consider a generic two-level system acting as a quantum sensor described by the Hamiltonian $H_p = (\omega_0/2)\sigma_z$, with the Pauli operator $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$. The sensor is exposed to a fast-oscillating field, see Fig. 1 (a), with the interaction Hamiltonian assumed as $H_s = \vec{b} \cdot \vec{\sigma} \cos(\omega t + \varphi_s)$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, where $\omega$ is the target frequency to be estimated and $\vec{b} = (b_x, b_y, b_z)$ is a vector of signal cou-
pling strength. Conventional quantum lock-in detection usually measures the longitudinal field component of the target field, i.e., \( z \)-component, which is parallel to the quantization axis of the probe [1, 11, 19], which however becomes invalid in the case of a fast-oscillating field. It can be seen from Fig.1 (b) that experimentally accessible finite-width \( \pi \) pulses fail to flip the probe spin fast enough to be resonant with the target field. As a result, actions on the probe carrying information about the target frequency can not coherently adds up.

Instead, we measure transverse field components in the \( x-y \) plane by tuning the energy splitting \( \omega_0 \) of the sensor close to the target frequency \( \omega \). We remark that such a theoretical setting is feasible with a variety of experimental platforms, e.g. the ground-state energy levels of the NV center electron spin are split by a few gigahertz due to zero-field splitting and can be adjusted by an external static magnetic field [6]. Thus, we tune the difference \( \Delta = \omega - \omega_0 \) to be on the order of megahertz and make it satisfy \( k_s = (b_x^2 + b_y^2)^{1/2} \ll |\Delta| \ll \omega + \omega_0 \), which results in an effective RF signal in the interaction picture with respect to \( H_p = (\omega_0/2)\sigma_z \) as follows [41]

\[
H_I \approx \frac{k_s}{2} \left[ \cos(\Delta t + \varphi)\sigma_x + \sin(\Delta t + \varphi)\sigma_y \right],
\]

wherein \( \varphi = \varphi_x + \theta \), \( \tan(\theta) = b_x/b_y \). In order to extract phase information on \( \varphi \) from this RF signal and suppress slow environmental noise, a dynamical decoupling sequence nearly measuring the longitudinal field component of the target field at fixed sequence numbers. (c) Dependence of signal contrast \( C \) on the sequence number \( N_s \). The parameters are chosen as \( k_s/2\pi = 50\text{kHz}, \tau = 0.5\mu\text{s}, \) and \( \Delta/2\pi = 1\text{MHz} + 0.232\text{kHz} \). We model the magnetic noise \( \delta B(t) \) as an O-U process with a correlation time \( \tau_B = 4\text{ms} \) and a standard deviation \( \Delta_B/2\pi = 100\text{kHz} \).

\[
\rho \approx \frac{1}{2} \left[ \cos(\delta t + \varphi)\sigma_x + \sin(\delta t + \varphi)\sigma_y \right],
\]

in superposition state \( |\pm\rangle = (|0\rangle + |1\rangle)/\sqrt{2} \) is found to be

\[
P_n \equiv P(\varphi_n) = \sin^2[\Phi(\varphi_n) - \pi/4],
\]

where \( \varphi_n = \omega t_n + \varphi \) with \( t_n = (n-1)T_L \) denoting the starting time of the \( n \)-th measurement and \( \Phi(\varphi) = k_s T_s \sin(\delta T_s/2) \cos(\delta T_s/2 + \varphi)/\pi \). We note that the phase \( \varphi_n \) can be equivalently rewritten as \( \varphi_n = \delta_L t_n + \varphi \), where \( \delta_L = \omega - 2\pi N_L f_L \) is the reduced signal frequency with \( N_L \) an appropriate integer to guarantee the condition \( |\delta_L/2\pi| < f_L/2 \). The validity of Eq.(3) is testified by our exact numerical simulation governed by the Hamiltonian in Eq.(1), see Fig.2 (a), where we also take into account the influence of slow magnetic noise with and without applying CPMG sequences as a comparison. The noise is described by an additional term.
\[ H_{\text{noise}} = \delta B(t) \sigma_z / 2 \] where the noise \( \delta B(t) \) is phenomenologically modelled by a stochastic Ornstein-Uhlenbeck (O-U) process [46], which is reasonable for several quantum platforms, e.g., NV color centers [47–49] or trapped ion systems [11]. It can be seen that the main effects of noise can be eliminated by dynamical decoupling sequences, resulting in a largely extended coherence time of the sensor, which would play a significant role in the case of weak signal detection. On the other hand, \( P(\varphi) \) at a fixed evolution time, even though with visible fluctuations in the presence of a single noise realization, exhibits an exact \( 2\pi \)-periodicity versus the initial phase \( \varphi \) of the target field, as shown in Fig.2 (b). We remark that this feature is crucial for us to correlate successive single measurements of the whole Qdyne measurement chain by precise timekeeping and extract the information on the target frequency beyond the limitation set by the sensor’s coherence time [18, 19].

In order to find an appropriate number \( N_s \) of CPMG sequences to be applied, we further define the signal contrast as follows

\[
C = \frac{\text{Max}[P(\varphi)] - \text{Min}[P(\varphi)]}{\text{Max}[P(\varphi)] + \text{Min}[P(\varphi)]}
\]

where Max (Min) represents the maximum (minimum) value of \( P(\varphi) \) as a function of the signal phase \( \varphi \). It can be seen from Fig.2 (c) that the contrast \( C \) increases and would nearly saturates to 1 as \( N_s \) grows. Thus, we can set the required sequence number in each single measurement run as the number \( N_s = N_s^* \) at which \( C \) begins to saturate.

State-selective fluorescence detection is an efficient method to readout the state of several quantum systems, such as trapped ions [50] and solid-state spins in diamond [2], which defines a map \( \mathcal{M} \) transforming the population signal \( P_n \) into a random variable \( z_n \) denoting the number of photons collected in each experimental run

\[
z_n = \mathcal{M}(P_n).
\]

As an ideal example, \( z_n = \text{Bn}[P_n] \) represents a Bernoulli random process which takes the value 1 with a probability of \( P_n \) and the value 0 with a probability of \( 1 - P_n \). The Bernoulli process based mapping only involves quantum projection noise. More realistically, we assume that \( z_n = \text{Pois}[\mu_0 + (\mu_1 - \mu_0)\text{Bn}[P_n]] \), or equivalently \( z_n \sim \text{Pois}[\mu_1] + (1 - P_n)\text{Pois}[\mu_0] \), where \( \text{Pois}[\mu] \), \( \mu = 0, 1 \), describes a Poissonian process with the mean value \( \mu \), which is connected with the photon shot noise.

The information about the signal frequency is imparted onto a time trace of \( N \) measurement outcomes \( \{z\}_{n=1}^N \) at sampling times \( \{t\}_{n=1}^N \). We extract the target frequency \( \omega \) by making a discrete Fourier transform \( \hat{z}_k = \sum_{n=1}^{N} z_n e^{i2\pi nk/N} \), \( k = 0, 1, \cdots, N - 1 \), which corresponds to the amplitude at the frequency component \( f = kf_L/N \). The associated power spectrum \( F_k = |\hat{z}_k|^2 \) is found to satisfy

\[
\langle F_k \rangle = \left| \sum_{n=1}^{N} (z_n) e^{i2\pi nk/N} \right|^2 + \sum_{n=1}^{N} \text{Var}[z_n],
\]

with \( \langle z_n \rangle = \mu_0 + (\mu_1 - \mu_0)P_n \). It can be seen that the first term shows a peak at \( \delta L = \delta L/N/(2\pi f_L) \) with a width \( \omega = 1/N \), to which we refer as the signal scaling as \( N^2 \), while the second term is a uniform part for all frequency components which scales as \( N \) [41].

Therefore, when \( N \) is large enough, the average spectral density \( \langle F_k \rangle \) agrees with the spectral density of \( \langle z_n \rangle \), which is in principle allows us to extract \( \delta L \) from \( \langle F_k \rangle \).

**Analysis of sensing performance.**—The power spectrum \( \langle F_k \rangle \) \( N \to 1 \) derived from a Qdyne measurement chain is only a single realization of a stochastic process with the mean value \( \langle F_k \rangle \). In order to verify the possibility of extracting \( \delta L \) from \( \langle F_k \rangle \), we demonstrate that the fluctuation of this stochastic process, defined by \( \langle \Delta F_k \rangle^2 = \langle (F_k - \langle F_k \rangle)^2 \rangle \), is negligible compared to the signal itself, i.e., \( \langle F_k \rangle \) around its peak. Without loss of generality, by assuming a process \( P_n = a + b \cos(\delta t + \varphi) \) with \( a, b \) certain real parameters [41], the signal and its fluctuation are found to be

\[
\langle F_{k_p} \rangle \approx b^2 (\mu_1 - \mu_0)^2 \beta N^2 / 4,
\]

\[
\langle \Delta F_{k_p} \rangle^2 = \eta N^3,
\]

where \( k_p \) denotes the neighboring integers close to \( \delta L = \delta L/N/(2\pi f_L) \), and the coefficients \( \beta, \eta \) depend on the difference value \( k_p - \delta L \) [41]. Based on the above result in Eq. (7), we find that the signal-to-noise ratio near the signal peak scales as \( \mathcal{R} = \langle F_{k_p} \rangle / \langle \Delta F_{k_p} \rangle \sim \mathcal{O}(\sqrt{N}) \gg 1 \), implying that \( F_{k_p} \) obtained from a single stochastic realization agrees well with \( \langle F_{k_p} \rangle \), although accompanied by a relatively small fluctuation, and can thus be exploited to extract \( \delta L \). Such scal-
\[ \delta \text{F}_{\text{peak}}, \text{which is fitted with the formula in Eq.}(8) \text{and yields} \]
\[ \bar{z} \text{emitted photons in a single experimental run generated by a map} \]
\[ \{ \mu \text{relatively small state-dependent fluorescence, e.g.} \]
\[ \text{is governed by the Hamiltonian in Eq.}(1) \text{together with longi-} \]
\[ \text{measurement process, each single measurement run of which} \]
\[ \text{our exact numerical simulations, as shown in Fig.3.} \]
\[ \langle 0 \gamma \text{us}, \text{with a precision} \text{scale} \text{s as} \]
\[ \text{and further the QFI of the whole Qdyne measurement chain as} \]
\[ \mathcal{I}(\omega) = \sum_{n=1}^{N} \mathcal{I}(\omega) \approx 2k^2T^2_{L}N^3/(3\pi^2) \sim k^2T^2_{L}T^{-1}T^3 \]
\[ \text{for} \ N \gg 1. \text{On the other hand, the minimum frequency} \]
\[ \text{change that can be detected above the noise level satisfies} \]
\[ \text{the well-known quantum Cramér-Rao bound [51], i.e.} \delta \omega \geq \frac{1}{\sqrt{\mathcal{I}(\omega)}}. \text{Therefore, the measurement precision scales as} \]
\[ \delta \omega \propto k^{-1}T^{-1/2}T^{-1/2}, \text{which is consistent with the above} \]
\[ \text{analysis in the context of signal-to-noise ratio.} \]

**Conclusions & Outlook.**– We present an experimentally feasible scheme for precise spectroscopy of high-frequency oscillating fields using a single-qubit sensor, which is inaccessible for conventional quantum lock-in detection method. By approximately matching the two-level quantum sensor’s energy splitting with the field frequency, the proposal effectively transforms the transverse field components to the RF range in a rotating frame. Thus, it provides a powerful method for quantum spectroscopy when the signal’s oscillating period is much shorter than experimentally accessible time duration to implement spin-flipping \( \pi \) pulses, which would significantly extend bandwidths of a single-qubit sensor. In combination with the dynamical decoupling technique, our scheme is robust against slow environmental noise acting on the sensor. A further extension to more general control pulse sequences robust to pulse imperfections is possible and may increase the information that can be obtained from the oscillating fields [52]. This result is expected to extend the application of quantum heterodyne detection in the high frequency regime.

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**Note added.**– After the completion of this work, we became aware of the related work [53].

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Supplementary material

1. Derivation of the effective Hamiltonian in the toggling frame

We consider a two-level quantum system acting as a single-qubit sensor, which is exposed to a fast oscillating field and described by a total Hamiltonian as

\[ H_t = \frac{\omega_0}{2} \sigma_z + \vec{b} \cdot \sigma \cos(\omega t + \varphi_x) + \frac{\delta B(t)}{2} \sigma_z, \]

(S.1)

where \( \delta B(t) \) describes slow environmental noise with a correlation time \( \tau_B \) and a standard deviation denoted by \( \Delta_B \). We remark that transverse noise in the \( x-y \) plane is largely suppressed by the energy splitting \( \omega_0 \) of the qubit sensor. By moving to the interaction picture with respect to this control field, we similarly obtain an effective Hamiltonian in the toggling frame as

\[ \mathcal{H}_t \approx \frac{\delta B(t)}{2} \sigma_z + \frac{k_s}{2} \left[ \cos(\Delta t + \varphi)\sigma_x + \sin(\Delta t + \varphi)\sigma_y \right], \]

(S.2)

under the conditions of \( k_s = \sqrt{b_x^2 + b_y^2} \ll |\Delta| \ll \omega + \omega_0 \) and \( b_x \ll \omega \), wherein \( \Delta = \omega - \omega_0 \). More accurately, \( \Delta \) can be defined as \( \Delta = \omega - \omega_d \), where \( \omega_d \) represents the frequency of external control fields, i.e. spin-flipping \( \pi \)-pulses. In this case, we obtain an additional term \( H' = (\delta_d/2)\sigma_z \), \( \delta_d = \omega_0 - \omega_d \) to Eq.(S.2) in the interaction picture with respect to \( H_0 = (\omega_d/2)\sigma_z \). We remark that a small detuning of \( \delta_d \) can be eliminated by dynamical decoupling sequences, which makes the proposal in the main text more feasible and robust in experiments.

CPMG sequence.– In order to extract signal information in Eq.(S.2) and prolong the sensor’s coherence time in a single measurement, we apply a train of CPMG sequences i.e. \((\tau-\pi-\tau-\pi)^N\) along the \( \hat{x} \)-direction, which results in an effective Hamiltonian in the toggling frame [1] as

\[ \mathcal{H}(\varphi) = \frac{k_s}{2} \cos(\Delta t + \varphi)\sigma_x + f(t) \frac{\delta B(t)}{2} \sigma_z + f(t) \frac{k_s}{2} \sin(\Delta t + \varphi)\sigma_y, \]

(S.3)

where \( f(t) \) is a modulation function arising from periodic \( \pi \)-pulses. More precisely, \( f(t) = 1 \) if \( 0 \leq \text{mod}(\omega_0 t, 2\pi) < \pi \) otherwise \( f(t) = -1 \) with \( \omega_s = \pi/\tau \), which can be expanded as \( f(t) = \sum_{n=0}^{\infty} (4/n\pi) \sin(n\omega_0 t) \). Under the conditions of \( \omega_s \approx \Delta \gg \{k_s, \Delta_B\} \) and \( \tau \ll \tau_B \), we obtain (i.e. Eq.2 in the main text)

\[ \mathcal{H}(\varphi) \approx \frac{k_s}{\pi} \cos(\delta t + \varphi)\sigma_y, \]

(S.4)

where \( \delta = \Delta - \omega_s \approx 0 \). If we initialize the sensor in state \( |1\rangle \) and let it evolve under the Hamiltonian in Eq.(S.4) for time \( T_s \), the population in the state \( |+\rangle = (|0\rangle + |1\rangle)/\sqrt{2} \) is found to be

\[ P(\varphi) = \sin^2[\Phi(\varphi) - \pi/4], \]

(S.5)

where \( \Phi(\varphi) = k_s T_s \sin(\delta T_s/2) \cos(\delta T_s/2 + \varphi)/\pi \) [2].

XY-8 sequence.– As an alternative efficient method to decouple environmental noise and measure the effective signal field in Eq.(S.2), we can also use a train of XY-8 sequences, i.e. \((\tau-\pi_x-\tau-\pi_y-\tau-\pi_x-\tau-\pi_y-\tau-\pi_x-\tau-\pi_y-\tau-\pi_x)^N\). By moving to the interaction picture with respect to this control field, we similarly obtain an effective Hamiltonian in the toggling frame as

\[ \tilde{H}(\varphi) = h(t) \frac{\delta B(t)}{2} \sigma_z + f(t) \frac{k_s}{2} \cos(\Delta t + \varphi)\sigma_x + g(t) \frac{k_s}{2} \sin(\Delta t + \varphi)\sigma_y, \]

(S.6)

where \( f(t), g(t) \) and \( h(t) \) are modulation functions associated with the periodic XY-8 sequences. The Fourier expansion coefficients of these functions, corresponding to the harmonics \( \cos(n\omega_0 t) \), \( \tilde{\omega}_n = \pi/(4\tau) \), \( n \in \mathbb{Z} \), can be obtained as follows

\[ f_n = \frac{4}{n\pi} \left[ \sin\left(\frac{3n\pi}{8}\right) - \sin\left(\frac{7n\pi}{8}\right) \right], \]

\[ g_n = \frac{4}{n\pi} \left[ \sin\left(\frac{n\pi}{8}\right) - \sin\left(\frac{5n\pi}{8}\right) \right], \]

\[ h_n = \frac{4}{n\pi} \left[ \sin\left(\frac{7n\pi}{8}\right) - \sin\left(\frac{3n\pi}{8}\right) + \sin\left(\frac{5n\pi}{8}\right) - \sin\left(\frac{n\pi}{8}\right) \right]. \]

(S.7)
In this case, we require \( \delta = \Delta - \tilde{\omega}_s \approx 0 \). In a similar way we derive Eq. (S.4), the Hamiltonian in Eq. (S.6) can be approximated as

\[
\tilde{H}(\varphi) = \frac{k_s}{4} f_1 \left[ \cos(\delta t + \varphi)\sigma_x - \sin(\delta t + \varphi)\sigma_y \right],
\]

under the conditions of \( \tilde{\omega}_s \gg \{ k_s, \delta, \Delta_B/4 \} \) and \( \tau \ll \tau_B \), where the denominator factor of \( \Delta_B/4 \) results from the fact that \( h_n \) takes nonzero values only at \( n = 4m, m \in \mathbb{Z}^+ \). If the sensor is initialized in the state \( |\uparrow_y\rangle \), the final population in the state \( |0\rangle \) after an evolution time \( t \) is given by

\[
\tilde{P}(\varphi) = \left| \langle 0 | e^{i\frac{\delta}{2} \sigma_z t} e^{-i(\frac{\delta}{2} \sigma_x + \frac{\omega}{2} \sin \varphi \sigma_y) t} | +y \rangle \right|^2 = \frac{1}{2} \left[ 1 + \frac{g}{\Omega} \cos(\varphi) \sin(\Phi) - \frac{2g\tilde{\omega}_s}{\Omega^2} \sin(\varphi) \sin^2(\frac{\Phi}{2}) \right],
\]

where \( g = k_s f_1/2, \Omega = \sqrt{g^2 + \delta^2} \) and \( \Phi = \Omega t \).

2. Fourier analysis of the Qdyne measurement

The state population in Eq. (S.5) for the \( j \)-th measurement (based on CPMG-sequence) takes the form

\[
P_j = c + d \sin(\eta \cos(\omega t_{j-1} + \varphi + \delta T_s/2)),
\]

where \( \eta = k_s T_s \text{sinc}(\delta T_s/2)/\pi \) and \( c, d \) are some real parameters. Provided that \( \eta < 2 \), \( P_j \) can be approximated as

\[
P_j \approx c + 2dJ_1(\eta) \cos(\omega t_{j-1} + \varphi + \delta T_s/2),
\]

based on the formula \( \sin(\eta \cos \theta) = 2 \sum_{n=0}^{\infty} (-1)^n J_{2n+1}(\eta) \cos(2n+1)\theta \). While in the case of XY-8 sequence, we have

\[
P_j \approx [\tilde{c} + \tilde{d} \cos(\varphi_j + \tilde{\theta})],
\]

where \( \tilde{c}, \tilde{d} \) and \( \tilde{\theta} \) are parameters depending on \( g, \delta \) and \( \Phi \). Therefore, without loss of generality, we consider an example of the form \( P_n = a + b \cos(2\pi f t_n + \theta) e^{-\gamma t_n} \) in the following discussion, where \( \gamma \) denotes the intrinsic linewidth of the target field. By assuming a photon emission process as \( z_n \sim P_n \text{Pois}[\mu_1] + (1 - P_n) \text{Pois}[\mu_0] \), we find that

\[
\langle z_n \rangle = a_z + b_z \cos(2\pi f t_n + \theta) e^{-\gamma t_n},
\]

\[
\text{Var}[z_n] = \mu_0^2 + (\mu_1^2 - \mu_0^2) P_n + \langle z_n \rangle (1 - \langle z_n \rangle),
\]

where \( a_z = \mu_0 + a(\mu_1 - \mu_0) \) and \( b_z = b(\mu_1 - \mu_0) \). Moreover, the power spectrum of \( \{ z_n \}_{n=1}^N \) is given by

\[
F_k = \sum_{n,m=1}^{N} z_n z_m e^{i2\pi(n-m)\frac{k}{N}}, \quad k \in (-N/2, N/2].
\]

Under the assumption that \( \langle z_n z_m \rangle = \langle z_n \rangle \langle z_m \rangle \) for \( m \neq n \), the average of \( F_k \) is found to be

\[
\langle F_k \rangle = \left| \sum_{n=1}^{N} \left[ a_z + \frac{b_z}{2} (e^{i2\pi n \frac{k}{N} + i\theta - n \frac{\pi}{N}} + e^{-i2\pi n \frac{k}{N} - i\theta - n \frac{\pi}{N}}) \right] e^{i2\pi n \frac{k}{N}} \right|^2 + \sum_{n=1}^{N} \text{Var}(z_n),
\]

with \( \tilde{f} = f N/f_L \) and \( \tilde{\gamma} = \gamma N/f_L \). In order to calculate \( \langle F_k \rangle \), we define the following functions as

\[
f_\alpha(x) = \frac{1}{N} \sum_{n=1}^{N} e^{i2\pi n \frac{x}{N} - n \frac{\pi}{N}},
\]

\[
f_\beta(x) = |f_\alpha(x)|^2 = \frac{1}{N^2} \frac{\cosh(\tilde{\gamma}) - \cos(2\pi x)}{\text{cosh}(\frac{\tilde{\gamma}}{N}) - \cos(2\pi \frac{x}{N})} \approx \frac{2 [\cosh(\tilde{\gamma}) - \cos(2\pi x)]}{\tilde{\gamma}^2 + 4\pi^2 x^2}, \quad x/N \ll 1, \tilde{\gamma}/N \ll 1.
\]

Without loss of generality, we assume that \( \tilde{f} \gg 1, \frac{N}{2} - \tilde{f} \gg 1 \) and \( 0 < k \leq N/2 \). Thus, we can get

\[
\sigma = \sum_n \text{Var}(z_n) \approx \left[ \mu_0^2 + a(\mu_1^2 - \mu_0^2) + a_z(1 - a_z) \right] N - b_z^2 \frac{\sin(\tilde{\gamma})}{2 \sinh(\frac{\tilde{\gamma}}{N})} \sim \mathcal{O}(N),
\]
\[ \langle F_k \rangle \approx \frac{b^2}{4} \beta N^2 \sim O(N^2), \quad (S.19) \]

where \( \beta = f \beta (k_p - \bar{f}) \) and \( k_p \) denotes the neighboring integers of \( \bar{f} \). Similarly the fluctuation of \( F_k \) is given by

\[ (\Delta F_k)^2 \approx 2\sigma \langle F_k \rangle + F^{(3)}(k), \quad (S.20) \]

where

\[ F^{(3)}(k) = \sum_{n,m,q} \left( (z_n^2) - \langle z_n^2 \rangle \right) \langle z_m \rangle \langle z_q \rangle \left( e^{i2\pi(2n-m-q)k/N} + e^{i2\pi(m+q-2n)k/N} \right). \quad (S.21) \]

For \( k = k_p \), we can obtain

\[ F^{(3)}(k_p) \approx \frac{b^4}{8} \Re \left[ f_{\alpha}^2 (\bar{f} - k_p) f_{\alpha} (2k_p - 2\bar{f}) \right] N^3, \quad (S.22) \]

which further leads to that (i.e. Eq.8 in the main text)

\[ (\Delta F_{k_p})^2 \approx \eta N^3, \quad (S.23) \]

with

\[ \eta = \frac{b^2}{2} \beta \left[ \mu_0^2 + a (\mu_1^2 - \mu_0^2) + a_z (1 - a_z) - \frac{b_2^2}{2} \right] + \frac{b^4}{8} \Re \left[ f_{\alpha}^2 (\bar{f} - k_p) f_{\alpha} (2k_p - 2\bar{f}) \right]. \quad (S.24) \]

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