Critical phenomena in complex networks: from scale-free to random networks

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Received 31 December 2022 / Accepted 17 October 2023
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Abstract. Within the conventional statistical physics framework, we study critical phenomena in configuration network models with hidden variables controlling links between pairs of nodes. We obtain analytical expressions for the average node degree, the expected number of edges in the graph, and the Landau and Helmholtz free energies. We demonstrate that the network's temperature controls the average node degree in the whole network. We also show that phase transition in an asymptotically sparse network leads to fundamental structural changes in the network topology. Below the critical temperature, the graph is completely disconnected; above the critical temperature, the graph becomes connected, and a giant component appears. Increasing temperature changes the degree distribution from power-degree for lower temperatures to a Poisson-like distribution for high temperatures. Our findings suggest that temperature might be an inalienable property of real networks.

1 Introduction

Network science has contributed to diverse fields in the natural and human sciences due to its interdisciplinary nature. The phenomena and processes in networks belonging to nature’s fundamental structures are quite different from those in lattices and fractals. Studying these intriguing effects will lead to a new understanding of a broad class of natural, artificial, and social systems [1–7]. Complex networks typically involve the study of networks with intricate structures, such as social networks, biological networks, or the Internet.

Current research in complex networks focuses on three main classes of models: random graphs, small-world, and scale-free networks [8]. In contrast to regular networks, some properties’ values are fixed in the random graph. Others, such as the number of nodes, edges, and connections between them, are determined randomly [1,4–6,9]. Small-world networks are intermediate between highly clustered regular lattices and random graphs. The coexistence of small path length and clustering may be taken as the main feature of small-world networks [8,10]. In general, small-world networks are characterized by the property that the typical distance between any pairs of nodes is short; it depends logarithmically on the number of nodes. A subclass of the small-world networks, the so-called Watts–Strogatz networks, are additionally characterized by a relatively high clustering level compared to a random graph with the same node-edge density [11]. This subclass is often identified with the whole class of small-world networks. Finally, scale-free networks, having a power-law degree distribution, are inhomogeneous by their nature. Large hubs characterize them, i.e., a few nodes are highly connected to other network nodes, and most nodes have very few connections. Nowadays, scale-free models are of significant interest since many real networks, such as social networks, airline networks, the World Wide Web, computer networks, the Internet, and others, can be treated as scale-free networks [1–6,8,10,12–15].

Given that complex networks are large systems, a deterministic approach falls short in explaining their collective behavior, making a statistical approach a reasonable choice for studying them. The statistical mechanics’ methods allow for creating models and analyzing real-world networks by drawing parallels between network phenomena and physical systems. The theoretical framework and mathematical tools offered by statistical physics are invaluable in comprehending complex networks’ behavior and characteristics. This interdisciplinary method has dramatically advanced our comprehension of complex systems and network science [5,6,12,16–27].
During the last 2 decades, statistical mechanics methods applied to complex networks have become a powerful tool. A few models, including Exponential Random Graphs Models (ERGMs), have been proposed for the study and explanation of the properties of real-world networks [8, 21–24, 28–30]. The recent advances on this road have revealed new and unexpected challenges in the statistical physics of complex networks. One is the concept of network temperature and its function in the formation and dynamics of complex networks [31–36]. In the context of complex networks, the term “temperature” is not a standard or widely recognized parameter. This concept might be used metaphorically or in a specialized context to describe certain dynamic aspects of complex networks. However, it does not have a widely accepted or standardized interpretation as in statistical mechanics (where the temperature is a fundamental parameter). The question arises: is it possible to make the concept of a network’s temperature as meaningful as it is in statistical physics?

This paper explores the relationship between temperature and network formation in ERGMs, focusing on critical phenomena. A careful study of this issue can make clear the meaning of the network’s temperature and provide a bridge between critical phenomena in complex networks and the standard theory of phase transitions. Motivated by the importance of scale-free models for studying real networks, we concentrate on configuration network models consistent with scale-free networks in the limit of low temperatures.

The main result of this study is that the network’s temperature controls the average node degree in the whole network, resulting in fundamental structural changes in the network topology. The contributions of the paper are as follows:

- We develop a detailed analysis of the critical phenomena in this class of network models, providing explicit analytical expressions for thermodynamic potentials.
- We show that in asymptotically sparse networks, with increasing temperature, the degree distribution is changed from power-degree for lower temperatures to a Poisson-like distribution for high temperatures.
- We demonstrate that the network’s temperature is a parameter that controls the topology of a network, particularly the average node degree in the whole network, and governs the transition from unconnected to power-degree (scale-free) and random graphs.
- We show that critical phenomena in asymptotically sparse networks lead to fundamental structural changes in the network topology. The network is completely disconnected below the critical temperature $T_c$. Above $T_c$, the graph becomes connected, and a giant component appears due to the second (structural) phase transition occurring at the critical temperature $T_0$. In the thermodynamic limits, both critical temperatures coincide.
- We provide numerically critical exponent values that describe the behavior of thermodynamic potentials near the critical point.

The paper is organized as follows. Section 2 discusses the statistical properties of complex undirected networks with a fixed number of vertices and the varying number of links described by the grand canonical ensemble. As a particular application, we consider in detail simple (fermionic) graphs with only one edge allowed between any pair of vertices. In Sect. 3, we introduce a model which is studied in this paper. In Sect. 4, we explore critical phenomena in asymptotically sparse networks, characterized by the property that the network becomes empty at zero temperature. We study the network’s topological features, such as degree distribution and formation of a giant component. The transition from an unconnected to a connected network and other essential aspects of network phase transitions are also discussed. In conclusion, we summarize our results and discuss possible generalizations of our approach. In Appendices A–C, technical details are presented concerning the computation of the main results in the text. In the Supplementary Material (SM) at https://doi.org/10.1140/epjb/s10051-023-00612-0, we estimate the critical exponents of the phase transition.

2 Statistical description of complex networks

2.1 General formalism

A network is a set of $N$ nodes (or vertices) connected by $L$ links (or edges). One can describe the network by an adjacency matrix, $a_{ij}$, where each existing or non-existing link between pairs of nodes $(ij)$ is indicated by a 1 or 0 in the $i, j$ entry. Individual nodes possess local properties such as node degree $k_i = \sum_j a_{ij}$, and clustering coefficient $c_i = \sum_{jk} a_{ij} a_{jk} a_{ki}/k_i(k_i - 1)$ [8, 11, 28, 37]. The network as a whole can be described quantitatively by its degree distribution $P(k)$ and connectivity. The connectivity is characterized by the connection probability $p_{ij}$, i.e., the probability that a pair of nodes $(ij)$ is connected.

Unlike the conventional approach to the statistical mechanics, where the Gibbs distribution is derived by considering a system in weak interaction with the environment, the statistical description of complex networks is based on the informational Shannon–Gibbs entropy, subject to certain constraints [18, 38, 39]. For a graph $G \in \mathcal{G}$, belonging to ensemble of graphs, $\mathcal{G}$, we denote by $P(G)$ the probability of obtaining this graph. Then, the Shannon–Gibbs entropy reads

$$S = - \sum_{G \in \mathcal{G}} P(G) \ln P(G).$$  (1)
Let us assume that the following constraints are imposed: \( \sum_G P(G) = 1 \) and \( E = \sum_G H(G)P(G) \), where \( H(G) \) is the graph Hamiltonian and \( E \) is the expectation value of the energy of the system. From the principle of the maximum entropy written as
\[
\frac{\partial S}{\partial E} = 0,
\]
we find\( \mathcal{F} = \Omega + \mu L \) and \( \langle E \rangle \), where \( \lambda \) and \( \beta \) are the Lagrange multipliers, we obtain the Gibbs distribution:
\[
P(G) = \frac{1}{Z} e^{-\beta H(G)},
\]
where
\[
Z = \sum_{G \in \mathcal{G}} e^{-\beta H(G)}
\]
is the partition function. Equations (3)–(4) define the exponential random graph model (ERGM) [18,22–26]. The expected value of any function, \( X(G) \), is calculated as follows: \( \langle X \rangle = \sum_{G \in \mathcal{G}} X(G)P(G) \). The computation of the Shannon–Gibbs entropy yields
\[
S = -\sum_{G \in \mathcal{G}} P(G) \ln P(G) = \ln Z + \beta E.
\]
Employing the relation \( \partial S/\partial E = 1/T \), we find that the Lagrange multiplier \( \beta = 1/T \) is the inverse “temperature” of the network. Next, using the thermodynamic relation \( F = E - TS \), where \( F \) is the Helmholtz free energy, we find \( F = -T \ln Z \).

For an undirected network with fixed number of vertices \( N \) and varying number of links, the probability of obtaining a graph \( G \) can be written as [18–20,31]
\[
P(G) = \frac{1}{Z} \exp \left( \beta (\mu L(G) - H(G)) \right),
\]
where \( \beta = 1/T \) is an inverse network temperature, \( \mu \) is the chemical potential, and \( L(G) = \sum_{i<j} a_{ij} \) is number of links in the graph \( G \). The partition function reads
\[
Z = \sum_{G \in \mathcal{G}} \exp \left( \beta (\mu L(G) - H(G)) \right).
\]

To obtain the grand potential, \( \Omega \), which we will refer to as the Landau free energy, we use the relation \( \Omega = -\beta^{-1} \ln Z \). Next, one can recover the Helmholtz free energy \( F \), internal energy \( E \), entropy \( S \), and heat capacity \( C_h \), using the following relations:
\[
F = \Omega + \mu L, \quad E = F + TS, \quad S = -\left. \frac{\partial \Omega}{\partial T} \right|_{\mu}, \quad C_h = T \left. \frac{\partial S}{\partial T} \right|_{\mu}.
\]

Finally, having the Landau free energy, one can find the expected number of links as follows: \( L = -\partial \Omega/\partial \mu \).

### 2.2 Fermionic graphs

A fermionic graph is a graph that can only have a single edge allowed between any two vertices [18]. Thus, in the fermionic model, edges (not nodes) between pairs of nodes are treated as “fermions”. Let us assign \((i,j)\) the “energy” \( \varepsilon_{ij} \) to each edge. Then, the graph Hamiltonian can be written as \( H(G) = \sum_{i<j} \varepsilon_{ij} a_{ij} \), and the partition function can be written as [31]
\[
Z = \sum_{G \in \mathcal{G}} \prod_{i<j} e^{\beta (\mu - \varepsilon_{ij})}.
\]

The computation of the partition function yields
\[
Z = \prod_{i<j} (1 + e^{\beta (\mu - \varepsilon_{ij})}).
\]

With the help of the partition function, one can obtain the connection probability of the existing link between nodes \( i \) and \( j \) as
\[
p_{ij} = -\frac{\partial \ln Z}{\partial (\beta \varepsilon_{ij})}.
\]

Employing Eq. (11), we find
\[
p_{ij} = -\frac{1}{e^{\beta (\mu - \varepsilon_{ij})} + 1}.
\]

Using the relations \( \Omega = -\beta^{-1} \ln Z \) and \( L = -\partial \Omega/\partial \mu \), we obtain
\[
\Omega = -\beta^{-1} \ln \left( \sum_{i<j} (1 + e^{\beta (\mu - \varepsilon_{ij})}) \right),
\]
\[
L = \sum_{i<j} \frac{1}{e^{\beta (\mu - \varepsilon_{ij})} + 1}.
\]

The expected degree of a vertex \( i \) is given by \( \bar{k}_i = \sum_j p_{ij} \). Denoting the average node degree in the whole network with \( \langle k \rangle = (1/N) \sum_i \bar{k}_i \), we get
\[
\langle k \rangle = \frac{2}{N} \sum_{i<j} \frac{1}{e^{\beta (\varepsilon_{ij} - \mu)} + 1},
\]

Comparing this expression with Eq.(15), we obtain the following relation between the expected number of links and average node degree: \( L = N \langle k \rangle / 2 \).

The density of a network is characterized by the connectance, which provides insights into how densely connected a network is relative to its size. A high connectance indicates that a significant portion of possible connections between nodes is realized, suggesting a dense and well-connected network. On the other hand,
a low connectance means a sparser network with fewer connections relative to the potential links. The connectance is defined as the fraction of those edges that are present [6]

\[
\rho = \frac{L}{L_{\text{max}}} = \frac{2L}{N(N-1)}. \tag{17}
\]

The range of density values is \(0 \leq \rho \leq 1\). Using the relation \(L = N\langle k \rangle/2\), one can rewrite the expression for the network’s density (17) in the equivalent form \(\rho = \kappa\), where \(\kappa = \langle k \rangle/(N-1)\) is the average node degree per node. We will say that a network is asymptotically sparse if for large \(N\) the network’s density \(\rho \to 0\) as \(T \to 0\).

Configuration fermionic model

It would be beneficial to discuss in detail the model presented in Ref. [18]. The graph Hamiltonian is defined as follows: \(H(G) = \sum_{i<j}(\epsilon_i + \epsilon_j)a_{ij}\), where \(\epsilon_i\) is an “energy” assigned to each vertex \(i\). It’s important to note that this energy is limited by a maximum value of \(\epsilon_0\), ensuring that \(0 \leq \epsilon_i \leq \epsilon_0\). For \(N \gg 1\) one can replace the sum by an integral, \(\frac{\rho}{N(N-1)} \sum_{i<j} \to \int\).

Employing Eqs. (14) and (15), we obtain

\[
\Omega = -\frac{N(N-1)}{2\beta} \int \int \ln(1 + e^{(\mu-\epsilon-\epsilon')}) \rho(\epsilon) \rho(\epsilon') d\epsilon d\epsilon', \tag{18}
\]

\[
L = \frac{N(N-1)}{2} \int \int p(\epsilon, \epsilon') \rho(\epsilon) d\epsilon \rho(\epsilon') d\epsilon, \tag{19}
\]

where \(\rho(\epsilon)\) denotes the density of states, with the standard normalization \(\int \rho(\epsilon) d\epsilon = 1\), and

\[
p(\epsilon, \epsilon') = \frac{1}{e^{\beta(\epsilon+\epsilon'-\mu)} + 1}. \tag{20}
\]

In the same limit, the expected degree of a vertex with energy \(\epsilon\) is

\[
\bar{k}(\epsilon) = (N-1) \int p(\epsilon, \epsilon') \rho(\epsilon') d\epsilon'. \tag{21}
\]

For the average node degree in the whole network, this yields

\[
\langle k \rangle = \int \bar{k}(\epsilon) \rho(\epsilon) d\epsilon = (N-1) \int \int p(\epsilon, \epsilon') \rho(\epsilon) d\epsilon \rho(\epsilon') d\epsilon'. \tag{22}
\]

3 Description of the model

Motivated by the importance of scale-free networks, we will explore configuration models consistent with scale-free networks. Scale-free networks are characterized by a power-law degree distribution, \(P(k) \propto k^{-\gamma}\). Appendix A shows that the networks with exponential growth (decrease) of the density of states are compatible with the scale-free networks. We will call them as Type A (Type B), respectively. We restrict ourselves by considering the model with the range of the energy limited by the chemical potential: \(0 \leq \epsilon \leq \mu\). A discussion of the more general model is presented in Appendix A.

In this study, we will explore a network model involving the chemical potential defined by \(\mu = T_0 \ln(\nu/\kappa)\), where \(\kappa = \langle k \rangle/(N-1)\) is the average node degree per node, and the constant \(\nu\) is given by (See Appendix B for details.)

\[
\nu = \left(\frac{\gamma - 1}{\gamma - 2}\right)^2, \quad \gamma > 2. \tag{23}
\]

The successful application of the hyperbolic configuration model in describing the Internet [34] motivates one to consider this particular network model. The proposed model can potentially be a valuable addition to the field of network science, and its implications could be significant.

By imposing the standard normalization condition, \(\int_0^\mu \rho(\epsilon) d\epsilon = 1\), we obtain

\[
\rho_{\beta}(\epsilon) = \frac{\alpha \beta e^{\alpha \beta (\epsilon - \mu) / 2}}{2 \sinh(\alpha \beta \mu / 2)}, \quad \text{Type A}, \tag{24}
\]

\[
\rho_{\alpha \beta}(\epsilon) = \frac{\alpha \beta e^{-\alpha \beta (\epsilon - \mu) / 2}}{2 \sinh(\alpha \beta \mu / 2)}, \quad \text{Type B}. \tag{25}
\]

where \(\alpha = \beta_c(\gamma - 1)/\beta\), and \(\beta_c = 1/T_c\) is a constant with dimension of inverse temperature.

Comments. In contrast to Ref. [31], where the temperature of a network is fixed by setting \(T = \gamma - 1\), we allow the temperature to be a free parameter. The parameter \(T_c = \beta_c^{-1}\) depends on the model. For illustrative purposes, throughout the paper, we choose \(T_c = 1\) and \(\gamma = 2.1\) to perform numerical simulations. It is worth noting that the chosen value of \(\gamma\) is typical for many real networks. We also assume that the number of nodes \(N \gg 1\).

We are now able to calculate the expected vertex degree \(\bar{k}(\epsilon)\), the expected number of links \(L\), and the Landau free energy \(\Omega\). After some algebra, we obtain

- Type A

\[
\bar{k}_{\beta}(\epsilon) = \frac{N - 1}{2 \sinh(\alpha \beta \mu / 2)} \left(e^{\alpha \beta \mu / 2} F_1(1, \alpha; 1 + \alpha; -e^{\beta \epsilon}) - e^{-\alpha \beta \mu / 2} F_1(1, \alpha; 1 + \alpha; -e^{\beta \epsilon - \mu})\right), \tag{26}
\]

\[
L_{\beta} = \frac{N(N - 1)}{8 \sinh^2(\alpha \beta \mu / 2)} \left(e^{\alpha \beta \mu} \times 3 F_2(1, \alpha, \alpha; 1 + \alpha, 1 + \alpha; -e^{\beta \mu}) - 2_3 F_2(1, \alpha, \alpha; 1 + \alpha, 1 + \alpha; -e^{\beta \mu}) + e^{-\alpha \beta \mu} 3 F_2(1, \alpha, \alpha; 1 + \alpha, 1 + \alpha; -e^{-\beta \mu})\right), \tag{27}
\]
work employing Eqs. (27) and (30), potential on temperature for Type A and Type B networks. Therefore, our critical phenomena study will focus on asymptotically sparse networks, referred to as Type A graphs.

4 Critical phenomena in asymptotically sparse networks

Our research findings, as detailed in Section 3, indicate that networks of Type B do not display critical phenomena. For derivatives of the generating function:

\begin{align}
\Omega_g &= -\frac{1}{\alpha \beta} L_g - \frac{N(N-1)}{8\beta \sinh^2(\alpha \beta / 2)} (e^{\alpha \beta} \ln (1 + e^{-\beta}) \\
&- 2 \ln 2 + e^{-\alpha \beta} \ln (1 + e^{\beta}) \\
&+ e^{\alpha \beta} \Phi(-e^{\beta}, 1, \alpha) - 2 \Phi(-1, 1, \alpha) \\
&+ e^{-\alpha \beta} \Phi(-e^{-\beta}, 1, \alpha)) \\
\end{align}

where \( k_d(\varepsilon) = \frac{\alpha(N-1)}{2(1+\alpha) \sinh(\alpha \beta / 2)} (e^{\alpha \beta})/2 \\
\times F_1(1, 1 + \alpha; 2 + \alpha; -e^{\beta}(-\varepsilon)) \\
- e^{-\alpha \beta} / 2 F_1(1, 1 + \alpha; 2 + \alpha; -e^{-\beta}(-\varepsilon)),

\begin{align}
\Omega_d &= \frac{1}{\alpha \beta} L_d - \frac{N(N-1)}{8\beta \sinh^2(\alpha \beta / 2)} (e^{-\alpha \beta} \ln (1 + e^{\beta}) \\
&- 2 \ln 2 + e^{-\alpha \beta} \ln (1 + e^{\beta}) \\
&+ e^{-\alpha \beta} \Phi(-e^{-\beta}, 1, -\alpha) - 2 \Phi(-1, 1, -\alpha) \\
&+ e^{\alpha \beta} \Phi(-e^{\beta}, 1, -\alpha)),
\end{align}

4.1 Degree distribution

Critical phenomena in networks with arbitrary degree distribution can be treated successfully using the generating function formalism [28,42]. Following Ref. [28], we define a generating function as

\[ G_0(z) = \sum_k z^k P(k), \]

where \( P(k) \) is the degree distribution (the probability that any given vertex has degree \( k \)).

Having the generating function, one can easily calculate the degree distribution and its moments:

\[ P(k) = \frac{d^k}{dk^k} G_0(z) \bigg|_{z=0}, \]

\[ \langle k^n \rangle = \left( \frac{d}{dz} \right)^n G_0(z) \bigg|_{z=1}. \]

In particular, this yields

\[ \langle k \rangle = G_0'(1), \quad \langle k^2 \rangle = G_0''(1) + G_0'(1). \]

Further, it is convenient to introduce the abbreviation for derivatives of the generating function:

\[ z_n = \left. \frac{d^n}{dz^n} G_0(z) \right|_{z=1}. \]

Then, using Eq. (35), we obtain \( \langle k \rangle = z_1, \langle k^2 \rangle = z_2 + z_1, \) etc. Straightforward computation leads to the following relation: \( z_n = \langle k^n \rangle \), where we denote by \( \langle k^n \rangle \) the \( n \)-th moment of the node degree:

\[ \langle k^n \rangle = \int \hat{k}^n(z) \rho(z) dz. \]

Using Eq. (35), one can calculate the \( n \)-th moment of the degree distribution, \( \langle k^n \rangle \), if we know the corresponding moments for the hidden variables, \( \langle \hat{k}^n \rangle \). In particular, we obtain \( \langle k^2 \rangle = \langle \hat{k}^2 \rangle + \langle k \rangle \).

The computation of the generating function and the degree distribution yields (see Appendix C)

\[ G_0(z) = \int e^{(z-1)k} \rho(z) dz, \]

\[ P(k) = \frac{1}{k!} \int_0^\mu e^{-K(z)} (\hat{k}(z))^k \rho(z) dz. \]
Fig. 1 (a) The chemical potential, $\mu$, as a function of temperature ($T_c = 1$). The upper (blue) line depicts the behavior of the chemical potential for the Type A graph. The lower (red) line presents $\mu$ for the Type B graph. (b) The average node degree per node, $\kappa = \langle k \rangle / N$, as a function of temperature. Upper (red) line: Type B graph. Lower (blue) curve: Type A graph. Inset: Zoom of the main plot for the Type A graph.

Fig. 2 Degree distribution as a function of $k$ and $T$. Number of nodes is $N = 10^7$

In Fig. 2 the degree distribution, $P(k)$, is depicted as a function of $k$ and temperature. As one can see, with increasing temperature, the dependence of $P(k)$ on $k$ is changed from power-degree, for lower temperatures, to a Poisson-like distribution for high temperatures. To prove this guess, below, we will consider two limited cases, $T \approx T_c$ (low temperatures) and $T \gg T_c$ (high temperatures), and derive the approximate formulas for the corresponding degree distribution.

Low temperatures

For low temperatures, $T \approx T_c$, taking into account that $\beta_c \mu \gg 1$, we obtain

$$\rho(\varepsilon) \approx \alpha \beta e^{\alpha \beta (\varepsilon - \mu)},$$

(41)

As one can see, the main contribution in the computation of integrals (114), (40) are given by high energies, $\beta \varepsilon \gg 1$. Using the asymptotic properties of the hypergeometric functions [41,43], we obtain

$$\bar{k}(\varepsilon) \approx N_2 F_1(1, \alpha; 1 + \alpha; -e^{\beta \varepsilon}),$$

(42)

$$\langle k \rangle \approx N_3 F_2(1, \alpha, \alpha; 1 + \alpha, 1 + \alpha; -e^{\beta \mu}).$$

(43)

where $\xi = \alpha / ( \alpha - 1)$. Note that at the critical point $\xi(T_c) = \sqrt{\nu}$. Using these results, we rewrite Eq. (114) as

$$G_0(z) = \alpha x_0^\alpha \int_{x_0}^1 e^{-(1-z)kx/(\xi x_0)x^{-\alpha-1}dx},$$

(46)

where $x = e^{-\beta \varepsilon}$ and $x_0 = e^{-\beta \mu}$. Performing the integration, we obtain

$$G_0(z) = \alpha y^\alpha (\Gamma(-\alpha, y) - \Gamma(-\alpha, y/x_0)),$$

(47)

where $y = (1-z)(k)/\xi$. To verify our results, we derive $G_0(1)$ and $G'_0(1)$. We find that $G_0(1) = 1$ and $G'_0(1) = \langle k \rangle$, as expected.

In order to get the degree distribution, we use expression (46) for the generating function. Taking the derivatives at the point $z = 0$, we find

$$\frac{d^k}{dz^k} G_0(0) = \alpha \left( \langle k \rangle \right)^k x_0^{\alpha-k} \int_{x_0}^1 e^{-(kx/(\xi x_0)x^{-\alpha-1}dx}.$$

(48)
Performing the integration, we obtain
\[
P(k) = \alpha \left( \frac{\langle k \rangle}{\xi} \right)^{\alpha} \frac{\Gamma(k - \alpha, \langle k \rangle/\xi)}{k!} - \alpha \left( \frac{\langle k \rangle}{\xi x_0} \right)^{\alpha} \frac{\Gamma(k - \alpha, \langle k \rangle/\xi x_0)}{k!}.
\]
(49)

where we have used Eq. (34) in computing the degree distribution.

Since near the critical point \(\langle k \rangle/(\xi x_0) \approx N \sqrt{\nu} \gg 1\), one can neglect the last term and write
\[
P(k) = \alpha \left( \frac{\langle k \rangle}{\xi} \right)^{\alpha} \frac{\Gamma(k - \alpha, \langle k \rangle/\xi)}{k!}.
\]
(50)

In the limit of \(k \gg 1\), we obtain
\[
P(k) \sim k^{-\alpha - 1}.
\]
(51)

Thus, near the critical point the degree distribution scales as \(P(k) \sim k^{-d}\), with \(d = \alpha+1 = (T/T_c)(\gamma-1)+1\). At the critical point, we have \(d = \gamma\).

**High temperatures**

Now, let us consider the case of high temperatures, \(T \gg T_c\). Proceeding as above, we obtain
\[
\rho(\varepsilon) = \frac{\alpha \beta e^{\alpha \beta (\varepsilon - \mu/2)}}{2 \sinh(\alpha \beta /2)},
\]
(52)
\[
\bar{k}(\varepsilon) \approx \frac{N}{2} e^{-\beta \varepsilon},
\]
(53)
\[
\langle k \rangle \approx \frac{N e^{-\beta \mu/2} \sinh((\alpha-1)\beta \mu/2)}{2 \sinh(\alpha \beta /2)}.
\]
(54)

When \(\beta \to 0\) \((T \to \infty)\), we have \(\langle k \rangle \to N/2\), as was predicted by the model.

Now, we can repeat the same procedure that we did in the case of low temperatures \((T \simeq T_c)\). Straightforward computation of the generating function yields
\[
G_0(z) = e^{-(1-z)\langle k \rangle}.
\]
(55)

In the calculation of this expression, we have used the asymptotic properties of the incomplete gamma function \(\Gamma(a, z)\) for large \(a\) [41]. Next, using Eqs. (34) and (55), we obtain a Poisson distribution:
\[
P(k) = \frac{e^{-\langle k \rangle} \langle k \rangle^k}{k!}.
\]
(56)

Thus, in the limit of high temperatures, the degree distribution does not depend on \(\gamma\), and the graph becomes a classical random graph described by the Poisson distribution [17].

In Fig. 3, we compare the approximate (red dashed curves) and exact (solid blue curves) expressions for the degree distribution. One can see an excellent agreement between both the approximate and exact results for low and high temperatures.

**4.2 Formation of a giant component**

Most real networks exhibit inhomogeneity in their link distribution leading to the natural clustering of the network into groups or communities. Within the same community, node–node connections are dense, but connections are less dense between groups. A group of nodes forms a component when all of them are connected, directly or indirectly [28, 44]. A “giant component” contains a significant part of the total number of nodes. For instance, if the degree connection \(k = N - 1\), the whole network is a giant component. In particular, this is valid for a Type B graph in the limit of \(T \to 0\). In Sect. 2, we have shown that, for both Type A and Type B graphs, \(k \to N/2\) in the limit of high temperatures. Thus, one can expect that with increasing temperature, the dense network should be fragmented in a finite number of giant components, and in an asymptotically sparse network, giant components should arise. In what follows, we will show that only one giant component appears in our model.

A giant component is formed in the network when the following condition holds [45]:
\[
\langle k^2 \rangle - 2\langle k \rangle > 0.
\]
(57)

Using the relations \(\langle k \rangle = z_1\) and \(\langle k^2 \rangle = z_2 + z_1\), one can recast (57) as
\[
z_2 - z_1 > 0.
\]
(58)

The computation of \(z_n\) employing the generating function (46) yields
\[
z_n \approx \frac{\gamma - 1}{n + 1 - \gamma} \left( \frac{\langle k \rangle}{\sqrt{\nu}} \right)^n \left( e^{(n+1-\gamma)\beta \mu} - 1 \right).
\]
(59)

Now, using (59) in combination with \(\sqrt{\nu} = (\gamma-1)/(\gamma-2)\), we obtain
\[
z_2 - z_1 \sim \begin{cases} 
  \langle k \rangle, & 2 < \gamma < 3 \\
  \frac{\langle k \rangle}{k_0} (\langle k \rangle - k_0), & \gamma > 3
\end{cases}
\]
(60)

where \(k_0 = (\gamma - 3)(\gamma - 1)/(\gamma - 2)^2\) is the threshold. The structural phase transition, related to percolation and formation of the giant component, occurs at the critical temperature \(T_0\) implicitly defined by equation \(z_2 - z_1 = 0\). From Eq. (60) it follows that \(T_0 = T_c\), if \(2 < \gamma < 3\), and \(T_0 > T_c\), when \(\gamma > 3\). We see that for networks with \(2 < \gamma < 3\), the phase transition associated with percolation still exists, though at a vanishing threshold (This result was reported before in Ref. [46]).
The size of the giant component $S$, being defined as the fraction of the graph occupied by the giant component, is given by [28]

$$S = 1 - G_0(u), \quad (61)$$

where $u$ is the smallest non-negative real solution of the equation

$$u = G_1(u) \equiv \frac{G_0'(u)}{\langle k \rangle}. \quad (62)$$

**Low temperatures**

As has been shown above, for low temperatures the generating function can be approximated as

$$G_0(z) = \alpha y^\alpha (\Gamma(-\alpha, y) - \Gamma(-\alpha, y/x_0)), \quad (63)$$

where $y = (1 - z)\langle k \rangle/\xi$ and $x_0 = \langle k \rangle/(N\xi^2)$. In the limit of $N \gg 1$, one can neglect the last term and write

$$G_0(z) = \alpha y^\alpha \Gamma(-\alpha, y). \quad (64)$$

We now examine Eq. (62) in detail. Using formulas for derivatives of the incomplete Gamma function and its series expansion [41],

$$\frac{d^n}{dz^n} (z^{-a} \Gamma(a, z)) = (-1)^n z^{-a-n} \Gamma(a + n, z), \quad (65)$$

$$\Gamma(a, z) = \Gamma(a) - \sum_{k=0}^\infty \frac{(-1)^k z^{a+k}}{k!(a+k)}, \quad (66)$$

we recast Eq.(62) as

$$\epsilon = -\left(\frac{\epsilon(\langle k \rangle)}{\xi}\right)^{\gamma-2} \Gamma(3 - \gamma) + (\gamma - 2) \sum_{k=1}^\infty \frac{(-1)^k (\epsilon(\langle k \rangle))^k}{k!(k + 3 - \gamma)\xi^k}, \quad (67)$$

where $\epsilon = 1 - u$.

First, we consider the case with $2 < \gamma < 3$. Keeping only dominant terms as $\epsilon \to 0$, we obtain

$$\epsilon \sim \langle k \rangle^{1/(3-\gamma) - 1} \equiv \langle k \rangle^{(\gamma-2)/(3-\gamma)}. \quad (68)$$

For networks with $\gamma > 3$, a non-vanishing threshold, $k_0$, emerges. This means that now, near the critical point, $\langle k \rangle \ll 1$, and not $\langle k \rangle - k_0 \ll 1$; and therefore $\epsilon \sim (\langle k \rangle - k_0)^{\gamma}$. To find $\sigma$, we rewrite Eq.(67) as follows:

$$\xi \frac{\langle k \rangle - k_0}{k_0} = (\gamma - 3) \sum_{k=1}^\infty \frac{(-1)^k (\epsilon(\langle k \rangle))^k}{(k + 1)(k + 3 - \gamma)\xi^k} \quad (69)$$

Considering only the leading terms as $\epsilon \to 0$, we obtain

$$\epsilon \sim \begin{cases} (\langle k \rangle - k_0)^{1/(\gamma-3)}, & 3 < \gamma < 4 \\ \langle k \rangle - k_0, & \gamma > 4 \end{cases} \quad (70)$$

Returning to the size of the giant component, we find that $S \approx \epsilon(\langle k \rangle)$. In conjunction with (68) and (70), this yields

$$S \sim \begin{cases} (\langle k \rangle - k_0)^{1/(\gamma-3)}, & 2 < \gamma < 3 \\ (\langle k \rangle - k_0)^{1/(\gamma-3)}, & 3 < \gamma < 4 \\ (\langle k \rangle - k_0), & \gamma > 4 \end{cases} \quad (71)$$

It follows that near the critical point, the size of the giant component behaves as $S \sim (\langle z_2 \rangle - \langle z_1 \rangle)^{\delta}$, where the
critical exponent, $\delta$, is given by

$$\delta = \begin{cases} 
1/(3 - \gamma), & 2 < \gamma < 3 \\
1/(\gamma - 3), & 3 < \gamma < 4 \\
1, & \gamma > 4
\end{cases}$$  \hspace{1cm} (72)

To compare our results with those known for the percolation phase transitions, we use the relation

$$z_2 - z_1 \sim q - q_c,$$  \hspace{1cm} (73)

where $q = 1 - p$, $q_c$ is the threshold, and $p$ denotes the fraction of nodes (and their links) removed from the network. Near the critical point, we obtain $S \sim (q-q_c)^\delta$, where $\delta$ is defined by Eq. (72). It is in agreement with the previously reported results [46].

**High temperatures**

In this case, in order to derive the size of the giant component, we employ Eq. (55) for the generating function,

$$G_0(z) = e^{-(1-z)(k)}.$$  \hspace{1cm} (74)

Proceeding as above, we find that $S$ satisfies the functional equation:

$$1 - S = G_0(1 - S) = e^{-(k)S}.$$  \hspace{1cm} (75)

In this limit $S \approx 1$, and the size of the giant component can be estimated as follows:

$$S \approx 1 - e^{-(k)}.$$  \hspace{1cm} (76)

When $T \rightarrow \infty$ we obtain $S = 1 - e^{-N/2}$. Thus, for $T \gg T_c$, almost all nodes of the network belong to the giant component. However, since $S < 1$, the giant component does not fill the entire graph. Moreover, only one giant component can be formed in the network, in agreement with the known results for graphs with purely power-law distributions [28].

Numerical simulations presented in Fig. 4 confirm our analytical findings. Figure 4a shows the giant component’s size as a function of average node degree for a network with $N = 10^4$ nodes. In Fig. 4b the dependence of $S$ on the temperature is depicted for $N = 10^2$ (green) and $N = 10^4$ (blue) nodes. As one can see, the size of the giant component is a rapidly increasing function. The saturation occurs for comparatively low temperatures and values of the average node degree. For $N = 10^4$, $S(T)$ is approximated rather well by the Heaviside step function.

**Formation of the giant component in the thermodynamic limit**

In the thermodynamic limit, we obtain

$$\lim_{N \rightarrow \infty} G_0(z, \mu, N) = 0, \text{ and}$$

$$\lim_{N \rightarrow \infty} G_1(z, \mu, N) = 0, 0 \leq z < 1, T > T_c.$$  \hspace{1cm} (77)

Using these results, we find that the solution of the Eq. (62) is given by $u = 0$, and the giant component size is $S = 1$. For $T < T_c$, we have $S = 0$. Thus, in the thermodynamic limit, the size of the giant component is defined by the Heaviside step function, $S(T) = \Theta(T - T_c)$, where

$$\Theta(x) := \begin{cases} 
1, & x > 0 \\
0, & x \leq 0
\end{cases}.$$  \hspace{1cm} (78)

A short analysis shows that for the infinite network, the critical temperature of the structural phase transition leading to the formation of a giant component is independent on $\gamma$ and coincides with $T_c$. The numerical results presented in Fig. 4b confirm our theoretical predictions.

**4.3 Phase transitions**

Before proceeding, we introduce the quantities:

$$\kappa = \lim_{N \rightarrow \infty} \frac{\langle k \rangle}{N} \text{ and } \omega = \lim_{N \rightarrow \infty} \frac{2\Omega}{N(N-1)}.$$  \hspace{1cm} (79)

where $\kappa$ is the average node degree per site in the limit of $N \rightarrow \infty$ and is called the *specific average node degree*; $\omega$ is the Landau free energy per link and is called the *specific Landau free energy*.

**Remark 1** Since in our model, $\kappa$ does not depend on $N$, we did not introduce a new notation for the average node degree per node in the thermodynamic limit.

Employing Eqs. (26)–(27), we obtain

$$\kappa = \frac{1}{4 \sinh^2(\alpha \beta \mu/2)} \left( e^{\alpha \beta \mu} \frac{3}{2} F_2(1, 1; 1 + \alpha, 1 + \alpha; -e^{\beta \mu}) - 2 F_2(1, 1; 1 + \alpha, 1 + \alpha; -1) + e^{-\alpha \beta \mu} \right) \times 4 \frac{1}{\alpha \beta} \left( e^{\alpha \beta \mu} \ln (1 + e^{-\mu}) - 2 \ln 2 + e^{-\alpha \beta \mu} \ln (1 + e^{\beta \mu}) + e^{\alpha \beta \mu} \Phi(-e^{\beta \mu}, 1, \alpha) - 2 \Phi(1, 1, \alpha) + e^{-\alpha \beta \mu} \Phi(-e^{-\beta \mu}, 1, \alpha) \right).$$  \hspace{1cm} (80)

To define the specific Helmholtz free energy $f$, specific entropy $s$, and specific heat capacity $c$, we use Eqs. (8) and (9),

$$f = \omega + \mu \kappa, \quad s = -\frac{\partial \omega}{\partial T}, \quad c = T \frac{\partial s}{\partial T}.$$  \hspace{1cm} (82)

To describe the phase transition, we introduce the order parameter, $\eta = 2\kappa$, which ranges between zero
Fig. 4 (a) Size of a giant component as a function of average node degree ($N = 10^4$). Inset: Zoom of the main plot. (b) Size of a giant component as a function of temperature. Red-dashed line presents the Heaviside step function. Number of nodes: $N = 10^2$ (green line); $N = 10^4$ (blue line). Inset: Zoom of the main plot.

Fig. 5 (a) Specific free energy as a function of temperature. Upper (red) line depicts the Helmholtz free energy. Lower (blue) line presents the Landau free energy. (b) Order parameter as a function of temperature. Inset: Zoom of the main plot.

and one. Using Eq. (80) and relation $\eta = 2\nu e^{-\beta_c \mu}$, we obtain

$$\eta = \frac{1}{2 \sinh^2(\alpha \beta \mu(\eta)/2)} \times \left( e^{\alpha \beta \mu(\eta)} - e^{-\alpha \beta \mu(\eta)} \right) - 2 \eta F_2(1, \alpha, \alpha; 1 + \alpha, 1 + \alpha; -1), \quad (83)$$

where $\mu(\eta) = \beta_c^{-1} \ln(2\nu/\eta)$. The solution of Eq. (83) yields the dependence of the order parameter on temperature. In Fig. 5, the specific Landau and Helmholtz free energies and the order parameter are depicted. Both energies are continuous functions of temperature; however, they get off their analytical properties at the critical point $T_c$. In the range of temperature $0 \leq T \leq T_c$, the graph is completely disconnected $\eta = 0$. Thus, in contrast to conventional phase transitions, for $T < T_c$ we have a disordered (“paramagnetic”) state, and for $T > T_c$ we have an ordered (“ferromagnetic”) state.

We find that near the critical temperature $T_c$ the chemical potential and the specific thermodynamic potentials behave as (see SM, for details)

$$\mu \sim -\lambda \ln \tau, \quad (84)$$

$$\omega \approx -\frac{1}{2\nu \beta_c} \left( A - \tau \ln \left( \frac{\eta}{2\nu} \right) \right) \eta, \quad (85)$$

$$f \approx -\frac{1}{2\nu \beta_c} \left( A + (1 - \tau) \ln \left( \frac{\eta}{2\nu} \right) \right) \eta, \quad (86)$$

$$s \approx -\frac{1}{\nu} \ln \left( \frac{\eta}{2\nu} \right) \eta, \quad (87)$$

$$c \approx -\frac{1}{\nu \beta_c} \left( 1 + \ln \left( \frac{\eta}{2\nu} \right) \right) \frac{d\eta}{dT}, \quad (88)$$

where $\tau = (T - T_c)/T_c$ is the reduced temperature and

$$A = \frac{\gamma^2 - 3\gamma + 3}{(\gamma - 2)^2}. \quad (89)$$
Table 1. Critical exponents for $0 < \tau < 1$

| Thermodynamic functions            | Relation                                      |
|------------------------------------|----------------------------------------------|
| Chemical potential                 | $\mu \propto -\lambda \ln \tau$             |
| Order parameter                    | $\eta \propto \tau^\lambda$                 |
| Landau free energy                 | $\omega \propto -\tau^\lambda$              |
| Helmholtz free energy              | $f \propto -\tau^\lambda \ln \tau$          |
| Entropy                            | $s \propto -\tau^\lambda \ln \tau$          |
| Heat capacity                      | $c \propto -\tau^{\lambda-1} \ln \tau$      |

The constant $\lambda$ is calculated by performing the numerical simulations. The computation yields

$$\lambda = \begin{cases} 
\frac{1}{(\gamma - 2)}, & 2 < \gamma < 3 \\
1, & \gamma \geq 3 
\end{cases}$$  \hspace{1cm} (90)

Finally, employing Eq. (84) and keeping the leading terms in Eqs. (85)–(88), one can obtain the dependence of thermodynamic functions on the reduced temperature. In Table 1, we summarize our results.

Conclusion

We demonstrated diverse critical effects and phenomena occurring in configuration network models with hidden variables, which significantly differ from those in lattices. For instance, the temperature can act contra common sense: increasing the Type A networks’ temperature results in a high connection graph degree; for Type B networks, the opposite is valid. We have shown how to treat random and scale-free networks within the conventional statistical physics approach and elucidated the role of network temperature. The temperature parameter controls the average node degree in the whole network and governs the transition from unconnected to power-degree (scale-free) and random graphs.

We found a finite-temperature phase transition in an asymptotically sparse network, leading to fundamental structural changes in the network topology. Increasing temperature changes the degree distribution from power-degree for lower temperatures to a Poisson-like distribution for high temperatures. The low-temperature phase yields a completely disconnected graph (ordered “paramagnetic” state). Above the critical temperature, the graph becomes connected (“ferromagnetic” state).

We have shown that the second (structural) phase transition occurs at the critical temperature $T_0$. For the finite network, the latter depends on the exponent of the distribution $\gamma$: $T_0 = T_c$, if $2 < \gamma < 3$, and $T_0 > T_c$ when $\gamma > 3$. The structural phase transition is associated with percolation and leads to the formation of the giant component, which contains a significant part of the total number of nodes. A giant component implies the coexistence of many vertices with a low degree and a small number of nodes with a high degree in the network. In other words, all nodes belonging to the giant component are connected directly or indirectly. Near the critical temperature $T_0$, the size of the giant component $S \ll 1$. For high temperatures, almost all network nodes belong to the giant component. However, since we obtain $S = 1 - e^{\gamma / N / 2}$, the giant component does not fill the entire graph, even for the network’s infinite temperature. In the thermodynamic limit, the critical temperature of the structural phase transition coincides with $T_c$.

Our results suggest that a network temperature might be an inalienable property of real networks, placing conditions on degree distribution, the topology of networks, and spreading information across these systems. Our approach provides a unified statistical description of real networks’ properties, from their scale-free and random graph behavior to community structure and topology change. While we restricted ourselves by considering the undirected case only, one can use our approach equally well for directed graphs; the generalization to directed graphs is straightforward. It is worth mentioning that we have considered only unweighted networks in our work. Extending the approach to the weighted network models requires a more profound development of analytical methods.

Acknowledgements. The authors acknowledge the support by the CONACYT.

Author contributions

All the authors contributed equally to the paper.

Data availability statement. No data associated in the manuscript.

A Generalized configuration model: $\mu < \varepsilon_0$

In this section, we will discuss a general configuration model with exponential distribution of the density of states, assuming that the chemical potential and energy are limited by a maximum value of $\varepsilon_0$ so that $\mu \leq \varepsilon_0$ and $0 \leq \varepsilon_1 \leq \varepsilon_0$. By imposing the standard normalization condition, $\int_{\varepsilon_0}^{\infty} \rho(\varepsilon) d\varepsilon = 1$, we obtain

$$\rho_\mu(\varepsilon) = \frac{\alpha \beta \epsilon^{\alpha \beta} (\varepsilon - \varepsilon_0 / 2)}{2 \sinh(\alpha \beta \varepsilon_0 / 2)}, \hspace{1cm} \text{Type A}$$  \hspace{1cm} (91)

$$\rho_\mu(\varepsilon) = \frac{\alpha \beta \epsilon^{\alpha \beta} (\varepsilon - \varepsilon_0 / 2)}{2 \sinh(\alpha \beta \varepsilon_0 / 2)}, \hspace{1cm} \text{Type B}$$  \hspace{1cm} (92)

where $\alpha = \beta_0 (\gamma - 1) / \beta$, and $\beta_0 = 1 / T_c$ is a constant with dimension of inverse temperature. The computation of the expected vertex degree $\bar{k}(\varepsilon)$, expected number of links $L$, and the Landau free energy $\Omega$ yields

- Type A

$$\bar{k}(\varepsilon) = \frac{N - 1}{2 \sinh(\alpha \beta \varepsilon_0 / 2)} (e^{\alpha \beta \varepsilon_0 / 2}$$
\[ L_g = \frac{N(N-1)}{8 \sinh^2(\alpha \beta \xi_0/2)} e^{-\alpha \beta \xi_0} \times F_2(1, \alpha, \alpha; 1 \alpha; 1 + \alpha, 1 + \alpha; e^{-\beta(\xi - \xi_0)}) \] 

\[ \Omega_g = -\frac{1}{\alpha \beta} L_g - \frac{N(N-1)}{8 \sinh^2(\alpha \beta \xi_0/2)} \times \ln (1 + e^{\beta(\mu - 2\xi_0)}) \] 

\[ 2 \ln (1 + e^{\beta(\mu - 2\xi_0)}) + e^{-\alpha \beta \xi_0} \ln (1 + e^{\beta \mu}) + e^{\alpha \beta \xi_0} \Phi(-e^{-\beta(2\xi - \xi_0)}, 1, \alpha) \] 

\[ -2 \Phi(-e^{-\beta(\xi_0 - \mu)}, 1, \alpha) + e^{-\alpha \beta \xi_0} \Phi(-e^{-\beta \mu}, 1, \alpha). \] 

\[ \nu = \frac{e^{\beta \mu}}{4 \sinh^2(\alpha \beta \mu/2)} \left( e^{\alpha \beta \mu} 3 F_2(1, \alpha, \alpha; 1 + \alpha, 1 + \alpha; -e^{-\beta \mu}) \right) \] 

We make use of the asymptotic properties of the generalized hypergeometric functions [40, 41, 43] to get

\[ \nu = \left( \frac{\gamma - 1}{\gamma - 2} \right)^2 e^{(\beta_{c} - \beta) \mu} + O(1 - e^{-\beta \mu}). \] 

As seen, the asymptotic series converges when $\gamma > 2$. Still supposing $\gamma > 2$ and, in addition, assuming that $\mu(T) \to \infty$ when $T \to T_c$, we obtain

\[ \nu = \left( \frac{\gamma - 1}{\gamma - 2} \right)^2. \] 

For high temperatures, $T \gg T_c$, similar consideration yields

\[ \nu e^{-\beta \mu} = -\alpha^2 \beta' (\alpha) + O(1 - e^{-\beta \mu}). \] 

Substituting $\alpha = \beta_c (\gamma - 1)/\beta$ and taking the limit of $T \to \infty$, we get

\[ \mu \to \mu_0 = \ln \left( \frac{2(\gamma - 1)^2}{(\gamma - 2)/2} \right) \quad \text{and} \quad \kappa \to \frac{1}{2}. \] 

\section{C Generating function}

Following Ref. [28], we define a generating function as

\[ G_0(z) = \sum_k z^k P(k), \] 

where $P(k)$ is the degree distribution (the probability that any given vertex has degree $k$). Further, all calculations will be confined to the region $0 \leq z \leq 1$.

Having the generating function, one can easily calculate the degree distribution and its moments:

\[ P(k) = \frac{1}{k!} \frac{d^k}{dz^k} G_0(z) \bigg|_{z=0}, \] 

\[ \langle k^n \rangle = \left( \frac{d}{dz} \right)^n G_0(z) \bigg|_{z=1}. \]

In particular, this yields

\[ \langle k \rangle = G'_0(1), \quad \langle k^2 \rangle = G''_0(1) + G'_0(1). \] 

Further, it is convenient to introduce the abbreviation for derivatives of the generating function:

\[ z_n = \frac{d^n}{dz^n} G_0(z) \bigg|_{z=1}. \]

Then, using Eq. (107), we obtain $\langle k \rangle = z_1$, $\langle k^2 \rangle = z_2 + z_1$, etc.
We are now ready to analyze the topological properties of the network. First, we are interested in the degree distribution, \( P(k) \). To proceed, we employ the generating functions approach presented in Appendix C. To compute \( G_0(z) \), we use the generating function formalism for networks with hidden variables developed in Ref. [47]. Following Ref. [47], one can write the degree distribution as

\[
P(k) = \int g(k|\varepsilon)\rho(\varepsilon)d\varepsilon, \tag{110}
\]

where \( g(k|\varepsilon) \) denotes the propagator, with the normalization condition \( \sum_k g(k|\varepsilon) = 1 \). Substituting \( P(k) \) in Eq. (105), we obtain

\[
G_0(z) = \int d\varepsilon \rho(\varepsilon) \sum_k z^k g(k|\varepsilon). \tag{111}
\]

As shown in Ref. [47],

\[
\ln \sum_k z^k g(k|\varepsilon) = N \int d\varepsilon' \rho(\varepsilon') \ln(1-(1-z)p(\varepsilon,\varepsilon')). \tag{112}
\]

Using this result in Eq. (111), we obtain

\[
G_0(z) = \int d\varepsilon \rho(\varepsilon) \exp\left(N \int d\varepsilon' \rho(\varepsilon') \ln(1-(1-z)p(\varepsilon,\varepsilon'))\right). \tag{113}
\]

Employing (113) and the results of Ref. [47], one can write the generating function as

\[
G_0(z) = \int \epsilon^{(\varepsilon-1)\tilde{k}(\varepsilon)} \rho(\varepsilon)d\varepsilon, \tag{114}
\]

where \( \tilde{k}(\varepsilon) = N \int p(\varepsilon,\varepsilon')\rho(\varepsilon')d\varepsilon' \) is the expected degree of the node with the hidden variable \( \varepsilon \).

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