Abstract. The Smoluchowski’s aggregation equation has applications in the field of bio-pharmaceuticals [1], financial sector [2], aerosol science [3] and many others. Several analytical, numerical and semi-analytical approaches have been devised to calculate the solutions of this equation. Semi-analytical methods are commonly employed since they do not require discretization of the space variable. The article deals with the introduction of a novel semi-analytical technique called the optimized decomposition method (ODM) (see [4]) to compute solutions of this relevant integro-partial differential equation. The series solution computed using ODM is shown to converge to the exact solution. The theoretical results are validated using numerical examples for scientifically relevant aggregation kernels for which the exact solutions are available. Additionally, the ODM approximated results are compared with the solutions obtained using the Adomian decomposition method (ADM) in [5]. The novel method is shown to be superior to ADM for the examples considered and thus establishes as an improved and efficient method for solving the Smoluchowski’s equation.

Keywords. Integro-Partial Differential Equations, Smoluchowski’s Equation, Optimized Decomposition Method, Adomian Decomposition Method, Semi-Analytical Approximations, Convergence Analysis

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1. Introduction

The population balance equations (PBEs) have several applications in modelling real world phenomena including milling process [6], protein filament division [7], fluidized bed wet granulation [8], fibrin clot formation [9] and aggregation of magnetic nanoflowers in biofluids [10], among many others (see [11] for detailed survey on applications). The PBE involves expressions that explain relevant processes such as aggregation, breakage, growth and nucleation. In this paper, we focus on the continuous version of the Smoluchowski’s equation modelling aggregation phenomenon [12]. The equation was first introduced by Smoluchowski [13] in a discrete form to study the Brownian motion of particles. Over the years, both the discrete and continuous forms of the Smoluchowski’s equation have been widely studied by many researchers (see [14–19]) and the references therein. The aggregation equation is written as

\[ \frac{\partial u(\tau, x)}{\partial \tau} = \frac{1}{2} \int_0^x a(x-y, y)u(\tau, x-y)u(\tau, y)dy - u(\tau, x) \int_0^\infty a(x, y)u(\tau, y)dy, \] (1.1)

for \( x \in \mathbb{R}^+ := [0, \infty] \), \( \tau \in [0, \infty] \) and with initial datum

\[ u(0, x) = u_0(x) \geq 0. \] (1.2)

The equation represents the rate of change in the concentration of particles of size \( x \) when the particles of different sizes undergo aggregation. The first term explains the formation of an \( x \) sized cluster due to the aggregation of clusters of sizes \( (x-y) \) and \( y \), whereas the second expression denotes the removal of the particles of size \( x \). Here, the function \( a(x, y) \geq 0 \) is known as the aggregation kernel and this symmetric quantity describes the rate at which particles of sizes \( x \) and \( y \) coagulate to form a particle of size \( x + y \).
Owing to the wide range of applications of the aforementioned equation, it becomes important to explore the techniques for solving the problem. Various numerical and semi-analytical methods have been employed to solve the aggregation model. The numerical methods include the finite volume scheme (FVS) \[18\,20\], the sectional methods \[21\,22\], fast Fourier transform method \[23\] and quadrature method of moments (QMOM) (see \[24\,25\]) and the references therein. There are several interesting numerical techniques that can be explored for the implementation on the aggregation equation such as the spectral element method (SEM) (see \[26\,30\]), haar wavelet schemes \[31\], collocation methods \[32\], Daftardar-Gejji and Jafari numerical technique \[33\] and compact decompositions \[34\]. As far as the semi-analytical methods are concerned, various decomposition schemes have been used to calculate series approximated solutions for the aggregation equation such as Laplace decomposition method \[35\] and tensor decomposition \[36\]. The homotopy perturbation method (HPM) was used in \[37\] to obtain the series solution for the aggregation equation for 

\[
a(x, y) = 1, \quad a(x, y) = xy
\]

with \(u_0(x) = e^{-x}, \frac{e^{-x}}{x}\) respectively. Hammouch and Toufika \[38\] established that the Laplace-variational iteration method (LVIM) is more efficient compared to HPM for the problem (1.1). The other homotopy methods are also among the popular ones to analyze the series solutions of aggregation equation, namely, the homotopy analysis method (HAM), optimal homotopy asymptotic method (OHAM) and the homotopy analysis transform method (HATM) (see \[39\,40\]). Singh et. al. \[5\] implemented ADM for 

\[
a(x, y) = 1, \quad a(x, y) = (x + y), \quad a(x, y) = xy, \quad a(x, y) = x^2/3 + y^2/3
\]

with initial value \(u_0(x) = e^{-x}\).

Hasseine et al. \[41\] established the comparison between the semi-analytical solutions obtained using VIM, HPM and ADM for 

\[
a(x, y) = xy \quad \text{with} \quad u(0, x) = e^{-x}.
\]

In 2019, Kaur et al. \[42\] found the series solution for equation (1.1) using HPM considering four aggregation kernels namely 

\[
a(x, y) = 1, \quad a(x, y) = (x + y), \quad a(x, y) = xy, \quad a(x, y) = x^2/3 + y^2/3
\]

with initial value \(u_0(x) = e^{-x}\).

Among these methods, HPM, VIM, and ADM seem to attract the most attention of the researchers. However, to the best of our knowledge, it was observed that theoretical convergence of the series solution towards the exact solution is demonstrated only in a few papers \[5\,43\]. In \[5\], the authors proved the theoretical convergence of ADM solution only for the constant kernel \(a(x, y) = 1\). While in \[43\], the convergence of the HPM in general, was proved and validated for a second-order ordinary differential equations. However, as far as the integro-partial differential equations are concerned, ADM seems to be most studied. Recently, an article by Obidat \[4\] pointed out some demerits of ADM such as slow convergence \[44\] and inability to deal with the boundary conditions \[45\]. To overcome these issues, the author \[4\] introduced a new optimized decomposition method (ODM) for solving non-linear ordinary and partial differential equations and showed using numerical examples that ODM is much more efficient than ADM. Further in \[46\], the ODM has been used to evaluate the approximated solutions for the second-order differential equations.

Therefore, it would be interesting to implement the ODM to compute the series solutions for integro-partial differential equations, in particular, aggregation equation and conduct a comparative analysis with ADM \[5\]. So, this article is an attempt to introduce this method for the equation (1.1) and analyze the results theoretically and numerically. The results are validated using several test cases including \(a(x, y) = 1, \quad a(x, y) = (x + y), \quad a(x, y) = xy \quad \text{with} \quad u(0, x) = e^{-x}\). Further, the comparison with the approximated solutions computed using ADM is shown to justify the novelty of the technique.

The article is organized as follows: Section 2 includes the preliminaries for ODM and ADM. The application of the ODM to the aggregation equation and the theoretical results related to the convergence analysis are presented in Section 3. The numerical examples are considered to justify the advantages of using ODM over ADM in Section 4 and some conclusions are included in the last section.
Consider the integro-partial differential equation of the type
\[
\frac{\partial}{\partial \tau} u(\tau, x) = M[u(\tau, x)],
\] (2.1)
with the initial condition as given in (1.2) and where \(M\) is a non-linear function of \(u\). The solution of the above equation is written as
\[
u(\tau, x) = u_0(x) + L^{-1}(M[u(\tau, x)]),
\] (2.2)
where \(L^{-1}\) is the inverse operator of \(L = \frac{\partial}{\partial \tau}\). The main idea of this method revolves around obtaining a linear approximation to the non-linear problem. As in [4], under the assumption that the non-linear function \(F(\frac{\partial}{\partial \tau} u, u) := \frac{\partial}{\partial \tau} u - M[u]\) can be linearized by a first-order Taylor series expansion at \(\tau = 0\), the linear approximation to \(F\) can be obtained as
\[
F \left( \frac{\partial}{\partial \tau} u, u \right) \approx \frac{\partial}{\partial \tau} u - C(x)u,
\]
where
\[
C(x) = \left. \frac{\partial M}{\partial u} \right|_{\tau=0}.
\] (2.3)
The above approximation leads us to a linear operator \(R\) defined as
\[
R[u(\tau, x)] = M[u(\tau, x)] - C(x)u(\tau, x),
\]
which is not easily invertible. Thanks to [4], the solution
\[
u(\tau, x) = \sum_{k=0}^{\infty} u_k(\tau, x)
\] (2.4)
and the coefficients \(u_k(\tau, x)\) are determined by TABLE 1 as

| Table 1. Table of the coefficients for ODM |
|--------------------------------------------|
| \(u_0(\tau, x)\) | \(u_0(x)\) |
| \(u_1(\tau, x)\) | \(L^{-1}(Q_0(\tau, x))\) |
| \(u_2(\tau, x)\) | \(L^{-1}(Q_1(\tau, x) - C(x)u_1(\tau, x))\) |
| \(u_{k+1}(\tau, x)\) | \(L^{-1}(Q_k(\tau, x) - C(x)(u_k(\tau, x) - u_{k-1}(\tau, x)))\), \(k \geq 2\) |

where
\[
Q_k(\tau, x) = \frac{1}{k!} \frac{d^k}{d\theta^k} \left[ M \left( \sum_{i=0}^{k} \theta^i u_i(\tau, x) \right) \right] \bigg|_{\theta=0}
\] (2.5)
and
\[
M \left[ \sum_{k=0}^{\infty} u_k(\tau, x) \right] = \sum_{k=0}^{\infty} Q_k(\tau, x).
\] (2.6)
The following proposition discusses the condition required for the convergence of this method.

**Proposition 2.1.** Let the coefficients of the series solution be determined by TABLE 1 and the series \(\sum_{k=0}^{\infty} u_k(\tau, x)\) is convergent then \(u(\tau, x)\) is the solution of equation (2.1).
Proof. Given that
\[ \sum_{k=0}^{\infty} u_k(\tau, x) = u_0(x) + \mathcal{L}^{-1}\left( Q_0(\tau, x) + [Q_1(\tau, x) - C(x)u_1(\tau, x)] + [Q_2(\tau, x) - C(x)
\]
\[ (u_2(\tau, x) - u_1(\tau, x)) + \ldots + [Q_{n-1}(\tau, x) - C(x)(u_{n-1}(\tau, x) - u_{n-2}(\tau, x))] + \ldots \). \]
Simplifying the above equation followed by using the convergence of the series \( \sum_{k=0}^{\infty} u_k(\tau, x) \) which ensures that \( \lim_{k \to \infty} u_k = 0 \), we get
\[ \sum_{k=0}^{\infty} u_k(\tau, x) = u_0(x) + \mathcal{L}^{-1}\left( \sum_{k=0}^{\infty} Q_k(\tau, x) \right). \]
The equation (2.6) and \( u(\tau, x) = \sum_{k=0}^{\infty} u_k(\tau, x) \) yield
\[ u(\tau, x) = u_0(x) + \mathcal{L}^{-1}\left( M\left[ \sum_{k=0}^{\infty} u_k(\tau, x) \right] \right), \]
and so \( \mathcal{L}[u(\tau, x)] = M[u(\tau, x)] \). Hence, \( u(\tau, x) \) is the solution of \( (2.1) \).

Remark 2.2. Note that the iterative scheme for ODM reduces to ADM if \( C(x) = 0 \). Also, let us define the coefficients and the n-term series solutions for ADM as \( v_k(\tau, x) \) and \( \psi_n(\tau, x) \), respectively, where \( \psi_n(\tau, x) \) is given as
\[ \psi_n(\tau, x) = \sum_{k=0}^{n} v_k(\tau, x). \quad (2.7) \]

Remark 2.3. As explained in [4], ODM is an optimized method in the sense that the approximation \( R[u] = \frac{\partial}{\partial \tau} u - C(x)u \) is the best linear approximation to \( F(\frac{\partial}{\partial \tau} u, u) \) near \( \tau = 0 \), i.e., near the initial data \( u(0, x) \).

3. ODM Implementation for Aggregation Equation

In this section, the general expression for the n-term series solution is provided for the aggregation equation (1.1). For this, define the non-linear operator \( M \) as
\[ M[u(\tau, x)] = \frac{1}{2} \int_{0}^{x} a(x - y) u(\tau, x - y) u(\tau, y) dy - u(\tau, x) \int_{0}^{\infty} a(x, y) u(\tau, y) dy, \quad (3.1) \]
then using Leibnitz rule (differentiating wrt \( u(\tau, x) \)), we obtain
\[ C(x) = \frac{1}{2} \int_{0}^{x} a(x - y, y) u(0, y) dy - \int_{0}^{\infty} a(x, y) u(0, y) dy. \quad (3.2) \]
Having (3.1) and (3.2), the linear operator \( R \) becomes
\[ R[u(\tau, x)] = \frac{1}{2} \int_{0}^{x} a(x - y, y) u(\tau, x - y) u(\tau, y) dy - u(\tau, x) \int_{0}^{\infty} a(x, y) u(\tau, y) dy
\]
\[ - \left( \frac{1}{2} \int_{0}^{x} a(x - y, y) u(0, y) dy - \int_{0}^{\infty} a(x, y) u(0, y) dy \right) u(\tau, x). \]
Setting \( k = 0 \) in (2.5) and using (3.1), the term \( Q_0 \) is written as
\[ Q_0(\tau, x) = \left( \frac{1}{2} \int_{0}^{x} a(x - y, y) u_0(\tau, x - y) u_0(\tau, y) dy - u_0(\tau, x) \int_{0}^{\infty} a(x, y) u_0(\tau, y) dy \right), \]
and
\[ u_1(\tau, x) = \mathcal{L}^{-1} \left( \frac{1}{2} \int_0^x a(x - y, y) u_0(\tau, x - y) u_0(\tau, y) dy - u_0(\tau, x) \int_0^\infty a(x, y) u_0(\tau, y) dy \right). \] (3.3)

Further, \( k = 1 \) in (2.5) yields
\[ Q_1(\tau, x) = \frac{1}{2} \int_0^x a(x - y, y) \left( u_0(\tau, x - y) u_1(\tau, y) + u_1(\tau, x - y) u_0(\tau, y) \right) dy \]
\[ - \int_0^\infty a(x, y) \left( u_0(\tau, x) u_1(\tau, y) + u_1(\tau, x) u_0(\tau, y) \right) dy, \]
which using (3.2), gives
\[ u_2(\tau, x) = \mathcal{L}^{-1} \left( \frac{1}{2} \int_0^x a(x - y, y) \left( u_0(\tau, x - y) u_1(\tau, y) + u_1(\tau, x - y) u_0(\tau, y) \right) dy \right. \]
\[ - \int_0^\infty a(x, y) \left( u_0(\tau, x) u_1(\tau, y) + u_1(\tau, x) u_0(\tau, y) \right) dy - u_1(\tau, x) \left( \frac{1}{2} \int_0^x a(x - y, y) u(0, y) dy - \int_0^\infty a(x, y) u(0, y) dy \right) \right). \] (3.4)

Finally, for \( k \geq 2 \) and only when \( i + j = k \), we have
\[ Q_k(\tau, x) = \frac{1}{2} \int_0^x a(x - y, y) \left( \sum_{i=0}^k u_i(\tau, x - y) \sum_{j=0}^k u_j(\tau, y) \right) dy - \int_0^\infty a(x, y) \left( \sum_{i=0}^k u_i(\tau, x) \sum_{j=0}^k u_j(\tau, y) \right) dy, \]
and
\[ u_{k+1}(\tau, x) = \mathcal{L}^{-1} \left( \frac{1}{2} \int_0^x a(x - y, y) \left( \sum_{i=0}^k u_i(\tau, x - y) \sum_{j=0}^k u_j(\tau, y) \right) dy \right. \]
\[ - \int_0^\infty a(x, y) \left( \sum_{i=0}^k u_i(\tau, x) \sum_{j=0}^k u_j(\tau, y) \right) dy - C(x) \left[ u_k(\tau, x) - u_{k-1}(\tau, x) \right] \right). \] (3.5)

Hence, the \( n \)-term series solution for (1.1) becomes
\[ \phi_n(\tau, x) := \sum_{k=0}^n u_k(\tau, x) = u_0(x) + \mathcal{L}^{-1} \left( Q_0(\tau, x) + [Q_1(\tau, x) - C(x) u_1(\tau, x)] + [Q_2(\tau, x) - C(x) (u_2(\tau, x) - u_1(\tau, x))] + \ldots + [Q_{n-2}(\tau, x) - C(x) (u_{n-2}(\tau, x) - u_{n-3}(\tau, x))] + [Q_{n-1}(\tau, x) - C(x) (u_{n-1}(\tau, x) - u_{n-2}(\tau, x))] \right). \]

Simplifying the above equation enables us to have
\[ \phi_n(\tau, x) := u_0(x) + \mathcal{L}^{-1} \left( \sum_{k=1}^n Q_{k-1}(\tau, x) - C(x) u_{n-1}(\tau, x) \right) \]
\[ = u_0(x) + \mathcal{L}^{-1} \left( M(\phi_{n-1}(\tau, x)) - C(x) u_{n-1}(\tau, x) \right) \]
\[ = u_0(x) + \mathcal{L}^{-1} \left( M \left( \sum_{k=0}^{n-1} u_k(\tau, x) \right) - C(x) u_{n-1}(\tau, x) \right). \]
3.1. **Convergence Analysis.** Consider the Banach space $B = (C([0, T]) : L^1[0, \infty), || \cdot ||)$ with the norm defined as

$$||u|| = \sup_{s \in [0, \tau_0]} \int_0^\infty |u(s, x)|dx < \infty. \quad (3.6)$$

Using the expression of solution from (2.2) and the definition of function $M$ from (3.1), the equation (1.1) is expressed in the following form

$$u = \mathcal{A}u,$$  

(3.7)

where $\mathcal{A} : B \to B$ is a non-linear operator defined by

$$\mathcal{A}u = u_0(x) + \mathcal{L}^{-1}(M[u(\tau, x)]). \quad (3.8)$$

As presented in [5], to establish the contraction mapping of $\mathcal{A}$, the above equation can be written in the following equivalent form

$$\frac{\partial}{\partial \tau} [u(\tau, x) \exp[H(x, \tau, u)]] = \frac{1}{2} \exp[H(x, \tau, u)] \int_0^x a(x - y, y)u(\tau, x - y)u(\tau, y)dy,$$

where,

$$H(x, \tau, u) = \int_0^\tau \int_0^\infty a(x, y)u(s, y)dyds.$$  

Thus the equivalent operator $\hat{\mathcal{A}}$ is given by

$$\hat{\mathcal{A}}u = u_0(x) \exp[-H(x, \tau, u)] + \frac{1}{2} \int_0^\tau \exp[H(x, s, u) - H(x, \tau, u)] \int_0^x a(x - y, y)u(\tau, x - y, s)u(s, y)dyds. \quad (3.9)$$

Now, taking into account the recursive scheme for (1.1) and using (3.8), the $n$-term series solution becomes

$$\phi_n = \mathcal{A}\phi_{n-1} - \int_0^\tau C(x)u_{n-1}(s, x)ds. \quad (3.10)$$

To establish our main findings in Theorem 3.2 below, the following theorem is required which states an important result regarding the contraction mapping of the operator $\mathcal{A}$. The result plays a significant role in proving that the sequence $\{\phi_n\}$ is a Cauchy sequence which finally proves that the series solution converges towards the exact solution.

**Theorem 3.1.** Let the operator $\mathcal{A}$ be defined in (3.8) such that $a(x, y) = 1$, $\forall x, y \in (0, \infty)$. Then $\mathcal{A}$ is a contraction map, i.e., $||\mathcal{A}u - \mathcal{A}u'|| \leq \delta ||u - u'||$, $\forall u, u' \in B$ if

$$\delta = t_0 \exp(2t_0L)||u_0|| + 2t_0L^2 + 2t_0L < 1,$$

holds, where $L = ||u_0||(T + 1)$ for $T \in (0, \tau)$.  

**Proof.** In the article by Singh et. al. [5] [see Theorem 3.1], the operator $\mathcal{A}$ is proven to be a contraction map by establishing the contraction mapping of its equivalent operator $\hat{\mathcal{A}}$ (defined in (3.9)). □

**Theorem 3.2.** Let the coefficients of the series solution for the aggregation equation (1.1) be determined by the equations (3.3)-(3.5) and $\phi_n$ be the $n$-term series solution defined by (3.10). Then, $\phi_n$ converges to the exact solution $u$ with

$$||u - \phi_n|| \leq \frac{\delta^n}{1 - \delta} ||u_1||, \quad (3.11)$$

if the following conditions hold

(A1) $\delta = t_0 \exp(2t_0L)||u_0|| + 2t_0L^2 + 2t_0L < 1$, where $L = ||u_0||(T + 1)$, $T \in (0, \tau)$ and $||u_1|| < \infty$.
(A2) $\{u_n\}$ is a Cauchy sequence, i.e., for any $n > m$, $||u_n - u_m|| < \varepsilon$, where $\varepsilon = \frac{1}{m^p}$ such that $p > 1$.
(A3) $C(x) \in L^\infty(B, || \cdot ||_\infty)$ where $|| \cdot ||_\infty$ is the essential supremum norm, i.e., $|C(x)| \leq k$ for some $k \in \mathbb{R}^+$.  

The triangle inequality gives us

\[ ||\phi_n - \phi_m|| = ||A\phi_{n-1} - \int_0^\tau C(x)u_{n-1}(s,x)ds - A\phi_{m-1} + \int_0^\tau C(x)u_{m-1}(s,x)ds||. \]

The triangle inequality gives us

\[ ||\phi_n - \phi_m|| \leq ||A\phi_{n-1} - A\phi_{m-1}|| + \left| \int_0^\tau C(x)(u_{n-1}(s,x) - u_{m-1}(s,x))ds \right|. \]

Theorem 3.1 and assumption (A2) yield

\[ ||\phi_n - \phi_m|| \leq \delta||\phi_{n-1} - \phi_{m-1}|| + \varepsilon\tau|C(x)|_\infty. \]

Putting \( n = m + 1 \) in the above expression and further simplifications lead to

\[ ||\phi_{m+1} - \phi_m|| \leq \delta||\phi_m - \phi_{m-1}|| + \varepsilon\tau|C(x)|_\infty \]

\[ \leq \delta\left( \delta||\phi_{m-1} - \phi_{m-2}|| + \varepsilon\tau|C(x)|_\infty \right) + \varepsilon\tau|C(x)|_\infty \]

\[ \vdots \]

\[ \leq \delta^m||\phi_1 - \phi_0|| + \varepsilon\tau k \left( 1 + \delta + \delta^2 + \ldots + \delta^{m-1} \right). \]

Now,

\[ ||\phi_n - \phi_m|| \leq ||\phi_{m+1} - \phi_m|| + ||\phi_{m+2} - \phi_{m+1}|| + \ldots + ||\phi_n - \phi_{n-1}|| \]

\[ \leq \left[ \delta^m||\phi_1 - \phi_0|| + \varepsilon\tau k \left( 1 + \delta + \delta^2 + \ldots + \delta^{m-1} \right) \right] + \left[ \delta^{m+1}||\phi_1 - \phi_0|| + \varepsilon\tau k \left( 1 + \delta + \delta^2 + \ldots + \delta^m \right) \right] + \ldots \]

\[ \leq \delta^m\left( \frac{1 - \delta^{n-m}}{1 - \delta} \right)||\phi_1 - \phi_0|| + \varepsilon\tau k \left[ \frac{(1 - \delta^m)}{1 - \delta} \right] + \left[ \frac{(1 - \delta^{m+1})}{1 - \delta} \right] + \ldots + \left[ \frac{(1 - \delta^{n-1})}{1 - \delta} \right]. \]

For a suitable \( \tau_0 \) and thanks to (A1), the above expression becomes

\[ ||\phi_n - \phi_m|| \leq \frac{\delta^m}{1 - \delta} ||u_1|| + \frac{\varepsilon\tau_0 k}{1 - \delta} (n - m). \quad (3.12) \]

Finally, using (A2) and \( \frac{1}{\delta^m} < \frac{1}{\delta^\infty} \), the above expression converges to zero as \( m \to \infty \). Thus, \( \exists \phi \) such that

\[ \lim_{n \to \infty} \phi_n = \phi \text{ and so } u = \sum_{k=0}^{\infty} u_k = \lim_{n \to \infty} \phi_n = \phi, \]

which is the exact solution of (3.7). Finally, fixing \( m \) and letting \( n \to \infty \) in the equation (3.12), we obtain the theoretical error bound (3.11). □

4. Numerical Examples

This section includes the implementation of ODM for several test cases of aggregation equation (1.1) considering three different kernels, i.e., constant \( a(x,y) = 1 \), sum \( a(x,y) = x + y \) and product \( a(x,y) = xy \) with the initial condition \( u_0(x) = e^{-x} \). The calculations of the approximated solutions and other requisite computations are done with the help of MATHEMATICA. To establish the accuracy of ODM, the series solution is compared with the available exact solution for concentration and moments. Further, the ODM results are also compared with the findings of ADM proposed in [5] and it is shown through graphs and tables of errors that ODM enjoys better estimates than ADM.

Example 4.1. Consider the case of constant aggregation, i.e., \( a(x,y) = 1 \) with \( u_0(x) = e^{-x} \). The exact solution in this case is given in [47] as

\[ u(\tau, x) = \frac{4}{(2 + \tau)^2} e^{-\frac{2x}{2 + \tau}}. \]
Using the equations (3.1), (3.2) and (3.3), one gets
\[ C(x) = \frac{1}{2} (\sinh(x) - \cosh(x)) - \frac{1}{2}, \quad u_0(x) = e^{-x}, \]
\[ u_1(\tau, x) = \frac{\tau e^{-x}(x - 2)}{2!}, \]
\[ u_2(\tau, x) = \frac{e^{-2x\tau^2}}{2^2!}(-2 + x) + \frac{e^{-x\tau^2}}{2^2!}(4 - 5x + x^2), \]
\[ u_3(\tau, x) = \frac{e^{-3x\tau^3}}{2^4!}(-2 + x) + \frac{e^{-2x\tau^3}}{2^3!}(8 - 8x + x^2) + (-1)^33! \frac{e^{-2x\tau^3}}{2^3!3!}(-2 + x) + (-1)^32! \frac{e^{-x\tau^3}}{2^3!3!}(-2 + x) + \frac{\tau^3 e^{-x}}{2^3!3!}(x^3 + 25x - 10x^2 - 29/2), \]
\[ u_4(\tau, x) = \frac{e^{-4x\tau^4}}{2^4!}(-2 + x) + 9e^x(-32(-2 + x) + \tau(23 - 20x + 2x^2)) + 9e^x(-16(4 - 6x + x^2) + \tau(-101 + 138x - 26x^2 + 2x^3)) + e^{3x}(-72(15 - 20x + 4x^2) + \tau(1073 - 2511x + 1494x^2 - 300x^3 + 18x^4))). \]

As we proceed further, the coefficients become more complex but thanks to MATHEMATICA, higher terms can be computed using the equation (3.5).

![Coefficients plot at \( \tau = 1 \)](image1a.png)

![Coefficients plot at \( \tau = 1.5 \)](image1b.png)

**Figure 1.** ODM coefficients plot for Example 4.1

Figure 1(a) and Figure 1(b) graphically depict the coefficients \( u_k(\tau, x) \) of ODM for \( k = 1, 2, \ldots, 8 \) at \( \tau = 1 \) and \( \tau = 1.5 \), respectively. These coefficients plot help in deciding which \( n \) to choose to compute the non-negative approximate solution \( \phi_n(\tau, x) \). For instance, at \( \tau = 1 \), the most negative value contribution is from \( u_1(\tau, x) \) but the positive value contribution from \( u_2(\tau, x) \) and \( u_3(\tau, x) \) can not surpass this negative value. Further, it is easy to see that \( u_4(\tau, x) \) and \( u_5(\tau, x) \) are again negative, but the addition of the positive value of \( u_6(\tau, x) \) gives the non-negative 6-term solution \( \phi_6(\tau, x) \) as the first desired non-negative solution whose value is interestingly close to the exact solution \( u(\tau, x) \). By following Singh et al. [5], Figure 2 shows the plot of the coefficients for the approximate series solutions computed using ADM at \( \tau = 1 \) and \( \tau = 1.5 \). These coefficients are denoted by \( v_k(\tau, x) \) and it can be observed that \( v_k(\tau, x) \) is negative for odd values of \( k \) while positive for even values of \( k \) for both values of \( \tau \). This leads to the negative of \( \psi_k(\tau, x) \) for odd \( k \), which is not the case for the ODM coefficients for any \( k \).

To see these finite term solutions \( \phi_n \) and \( \psi_n \) for various values of \( n \), the comparison with the exact number density is provided at time \( \tau = 1 \) and \( \tau = 1.5 \) for ODM in Figure 3 and ADM in Figure 4. One can also
visualize the decreasing behavior of the concentration \((u, \phi_n, \psi_n)\) as size increases which confirms the aggregation of particles. In addition to this, Figure 3 depicts that as time increases from \(\tau = 1\) to \(\tau = 1.5\), concentration value reduces. It is evident from Figure 5 that the 8-term solution \(\psi_8\) by ADM and 6-term solution \(\phi_6\) by ODM are close to the exact solution \(u(\tau, x)\) at \(\tau = 1\). Further, at \(\tau = 1.5\), \(\psi_{14}\) is nearest to \(u(\tau, x)\) whereas only 8-term solution \(\phi_8\) is required to get the same accuracy with the exact solution. This indicates the advantage of using ODM over ADM. Moving further, the efficacy of the two methods are also compared by calculating the moments of the approximated solutions and comparing them with the moments of the exact solution. The \(r^{th}\) moment of the exact solution is defined as

\[
\mu_{r, Exact}^{Exact}(\tau) = \int_0^\infty x^r u(\tau, x) dx, \tag{4.1}
\]

while for the ODM and ADM approximated solutions, these are given by

\[
\mu_{r,n}^{ODM}(\tau) = \int_0^\infty x^r \phi_n(\tau, x) dx, \quad \mu_{r,n}^{ADM}(\tau) = \int_0^\infty x^r \psi_n(\tau, x) dx. \tag{4.2}
\]
These moments are relevant physical quantities with zeroth moment (obtained by putting $r = 0$ in (4.1)) being the total number of clusters and the first moment ($r = 1$ in (4.1)) gives the total mass (volume) of the system. Putting $r = 2$ gives the second moment which is defined as the energy dissipated by the system [48].

In Figure 6, the zeroth and first moments are computed using 8-term series solutions of ODM and ADM and the results are compared with the exact moments. It is well known that the number of particles in a coagulation system has a decreasing trend and is justified by the moments plotted in Figure 6(a). However, it is visualized here that $\mu_{0,8}^{ADM}(\tau)$ starts to get away from $\mu_{0}^{Exact}(\tau)$ approximately around $\tau = 1$ whereas $\mu_{0,8}^{ODM}(\tau)$ gives nice accuracy. Since, Figure 5 concludes that $\phi_8$ is closest to the exact solution $u$ at $\tau = 1.5$, it is imperative that ODM will give best approximation, but the method is performing well even when time is increased up to $\tau = 2$. The first and second moments using approximated solutions $\phi_8$ and $\psi_8$ are compared with the corresponding exact moments in Figures 6(b) and 7. The increasing nature of the second moment plot as time progresses shows that more energy is dissipated with time. This is due to the formation of
bigger particles due to coagulation process. It needs to be mentioned here that for $i = 1, 2$, $\mu_{i,8}^{ODM}(\tau)$ and $\mu_{i,8}^{ADM}(\tau)$ estimate $\mu_{i}^{Exact}(\tau)$ very well.
Figure 8. Absolute error plots for ODM and ADM series solutions for Example 4.1

Table 2. Comparison of numerical errors in computing approximate solutions using ADM and ODM for Eqn (1.1) with parameters as given in Example 4.1

| n   | 2   | 3   | 4   | 5   | 6   | 8   |
|-----|-----|-----|-----|-----|-----|-----|
| ADM at $\tau = 0.5$ | 0.023 | 0.006 | 0.002 | $4.5 \times 10^{-4}$ | $1.2 \times 10^{-4}$ | $8.3 \times 10^{-6}$ |
| ODM at $\tau = 0.5$ | 0.054 | 0.014 | 0.007 | 0.002 | $4.6 \times 10^{-4}$ | $3.6 \times 10^{-5}$ |
| ADM at $\tau = 1$ | 0.149 | 0.082 | 0.044 | 0.024 | 0.013 | 0.004 |
| ODM at $\tau = 1$ | 0.178 | 0.086 | 0.036 | 0.015 | 0.008 | 0.003 |
| ADM at $\tau = 1.5$ | 0.428 | 0.353 | 0.287 | 0.231 | 0.184 | 0.115 |
| ODM at $\tau = 1.5$ | 0.328 | 0.230 | 0.010 | 0.075 | 0.060 | 0.027 |
| ADM at $\tau = 2$ | 0.882 | 0.972 | 1.054 | 1.131 | 1.200 | 1.332 |
| ODM at $\tau = 2$ | 0.517 | 0.441 | 0.286 | 0.295 | 0.245 | 0.166 |

The novelty of ODM can also be justified by looking at 3D plots in Figure 8 which provides the absolute error between the exact and the approximated number density computed using 8-term series solutions for both the methods. We also summarize the numerical errors associated with the ADM and ODM at $\tau = 0.5$, $\tau = 1$, $\tau = 1.5$ and $\tau = 2$ in TABLE 2. These errors are computed by dividing the interval $[0, 10]$ into $N$ sub-intervals $[x_{i-1/2}, x_{i+1/2}], i = 1(1)N$. Each interval is represented by the mid-point $x_i = \frac{x_{i-1/2} + x_{i+1/2}}{2}$ and the error is computed using the following rule

$$\text{Error} = \sum_{i=1}^{N} |\xi_n^i - u_i| h_i, \quad (4.3)$$

where $\xi_n^i = \xi_n(\tau, x_i)$ and $u_i = u(\tau, x_i)$ are the series and exact solutions with step size $h_i = x_{i+1/2} - x_{i-1/2}$. All the computations are done by considering $N = 100$ and $h_i = 0.1 \forall i$. The table clearly shows that the error for ADM is less than ODM when $\tau = 0.5$. But when the value of $\tau$ is increased which is a more
realistic scenario, the error for ODM is seen to be lower than the error associated with the ADM. There is a significant difference in the error for ODM and ADM at \( \tau = 2 \) which claims the superiority of the novel method. Furthermore, TABLE 3 presents the order of convergence at \( \tau = 2 \) and it indicates that ADM has slower rate of convergence than ODM. Although, as \( h \) tends to 0, the order of convergence for both the methods approaches 1.

### Table 3. Order of convergence using ADM and ODM at \( \tau = 2 \) for Eqn (1.1) with parameters as given in Example 4.1.

| \( h \)     | ADM         | ODM         |
|------------|-------------|-------------|
| 0.5        | 0.38        | 0.92        |
| 0.25       | 0.63        | 1           |
| \((0.25)/2\)| 0.82        | 1           |
| \((0.25)/4\)| 0.89        | 1           |
| \((0.25)/8\)|            |             |

**Example 4.2.** The computation of ODM series solution for the aggregation parameter \( a(x, y) = (x + y) \) with exponential initial value \( u_0(x) = e^{-x} \) is done and the simulation results are compared with the exact solution defined in [49] as

\[
u(x, t) = \frac{e^{-\tau} \exp(x(e^{-\tau} - 2)) I_1(2x\sqrt{1 - e^{-\tau}})}{x\sqrt{1 - e^{-\tau}}}.
\]

Following equations (3.1), (3.2) and (3.3-3.5) give us

\[
C(x) = \frac{1}{2} x (\sinh(x) - \cosh(x)) - 1, \quad u_0(x) = e^{-x},
\]

\[
u_1(x, t) = \frac{1}{2} \tau e^{-\tau} (x(2x - 2) - 2),
\]

\[
u_2(x, t) = \frac{1}{24} \tau^2 e^{-2x} x (e^x(2x + 3)((x - 6)x + 6) + 3((x - 2)x - 2)),
\]

\[
u_3(x, t) = \frac{1}{576} \tau^3 e^{-3x} (-72e^x(x^2 - 2x - 2)(x + e^x(x + 2) + \tau((x - 2)x - 2))x^2)
+ \frac{1}{576} \tau^3 e^{-3x} (8e^x(x^3 - 9x^2 + 9x)x^2 + e^{2x}(2x(x((x + 2) - 2) + 144) - 84) - 177) - 180).
\]

Figure 9 shows the comparison of the exact concentration with the approximated 10-term series solutions obtained by using ODM and ADM. In addition, it also presents the comparison between the exact and numerically approximated zeroth moment. Since, the aggregation rate is more than the constant rate, it is seen that less number of particles are left at the end of the process. The number obtained after \( \tau = 1 \) is approximately same as in previous case after \( \tau = 2 \). The plots of the first and second moments considering \( \phi_{10} \) and \( \psi_{10} \) are shown in Figure 10 along with the analytical moments. It is observed that ODM predicts all the moments better than ADM. The absolute errors for both the techniques in computing the series solutions using 10-terms are depicted in Figure 11. Again, one can see that ODM enjoys better estimates as compared to ADM.
(a) Comparison of approximated and exact concentration

\[ \phi_{10}, \phi_{10} \]

(b) Zeroth moment

**Figure 9.** Comparison of solutions and zeroth moment for Example 4.2

(a) ODM

\[ \mu_{1,10}^{\text{Exact}}, \mu_{1,10}^{\text{ADM}}, \mu_{1,10}^{\text{ODM}} \]

(b) ADM

\[ \mu_{2,10}^{\text{Exact}}, \mu_{2,10}^{\text{ADM}}, \mu_{2,10}^{\text{ODM}} \]

**Figure 10.** Moments comparison: ODM, ADM and exact solutions for Example 4.2

| n  | 6   | 7   | 8   | 9   | 10  | 11  |
|----|-----|-----|-----|-----|-----|-----|
| ADM at \( \tau = 0.5 \) | 0.003 | 0.002 | 0.001 | 0.0009 | 0.0006 | 0.0004 |
| ODM at \( \tau = 0.5 \) | 0.008 | 0.0075 | 0.004 | 0.0043 | 0.002 | 0.002 |
| ADM at \( \tau = 1 \) | 0.226 | 0.352 | 0.500 | 0.593 | 0.808 | 1.187 |
| ODM at \( \tau = 1 \) | 0.196 | 0.317 | 0.281 | 0.436 | 0.367 | 0.449 |

**Table 4.** Comparison of numerical errors for ADM and ODM with parameters as given in Example 4.2
Additionally, the numerical errors at $\tau = 0.5$ and $\tau = 1$ are given in TABLE [4] which proves that ODM is a better method to obtain an approximate solution for the Eq. (1.1) having sum aggregation rate. It is worth mentioning that the order of convergence in this case has similar observations as in previous example.

Example 4.3. Now, consider the case of product kernel $a(x,y) = xy$ with initial data $u_0(x) = e^{-x}$. For this case, the analytic number density is provided in [47] as

$$u(\tau, x) = e^{-\tau x - x} \sum_{k=0}^{\infty} \frac{x^{k} e^{3k}}{\Gamma(2k + 2)(k + 1)!}.$$ 

Using the equations (3.1), (3.2) and the recursive scheme from equations (3.3-3.5) yield
Figure 13. Absolute error plots for ODM and ADM series solutions for Example 4.3

\[
C(x) = \frac{1}{2} [(x + 2)(e^{-x} - 1)], \quad u_0(x) = e^{-x},
\]

\[
u_1(\tau, x) = \frac{1}{12} \tau e^{-x} x (x^2 - 12),
\]

\[
u_2(\tau, x) = \frac{1}{720} \tau^2 e^{-2x} x (15(e^x - 1)(x + 2)(x^2 - 12) + e^x x(x^4 - 60x^2 + 360)),
\]

\[
u_3(\tau, x) = \frac{\tau^3 e^{-3x}}{120960} (28(-1 + e^x)x(2 + x)(-15\tau(2 + x)(-12 + x^2) + e^x(-90(-12 + x^2))))
\]

\[
+ \frac{\tau^3 e^{-2x}}{120960} (28(-1 + e^x)x(2 + x)(-360 + x(180 + x(30 - 45x + x^3))))
\]

\[
+ \frac{\tau^3 e^{-x}}{120960} (-1680(264 + x(240 + x(120 + x(38 + 5x)))))
\]

\[
+ \frac{\tau^3 e^{-x}}{120960} (443520 - 86625x + 20160x^2 + x^4(-10080 + x^4(-3360 + x^4(3696 + x^4(56 - 148x + x^3))))).
\]

Figure 12 represents the approximate solutions \(\phi_n\) and \(\psi_n\) when \(n = 3, 4\) and 5. Similar to the previous examples, we notice similar trend for concentration and moments. It is unequivocal that ODM series solution provides a better approximation than ADM. It is mentioned in [5] that ADM needs 8-term solution to get a good agreement with the exact solution. However, here one can notice that ODM needs only 5 terms. Moving further, the absolute errors in computing ODM series solution \(\phi_5\) and ADM solution \(\psi_5\) are presented in Figure 13 which clearly claim the superiority of ODM over ADM. The TABLE 5 depicts the comparison of error (defined in (4.3) with \(N = 1000\) and \(h_i = 0.01\)) values for the ADM and ODM for different values of \(n\). Again, it is noticed that the errors for ODM are lower than the errors of the solution computed using ADM. Note that, in this case as well, both ODM and ADM provide first order convergence but the values due to ODM are slightly closer to 1 as compared to ADM which shows the superiority of ODM. The following Remark 4.1 presents some important observations regarding the advantage of the method.

**Remark 4.1.** It is important to note that in the case of constant and sum kernels, when the value of \(\tau\) is increased, error values using ODM are significantly lower than ADM. However, for the product kernel, where the rate of aggregation is highest among all, the ODM gives better results even at lower values of \(\tau\).
Table 5. Numerical errors in computing approximated solutions using ADM and ODM at $\tau = 0.5$ with parameters as given in Example 4.3.

| n  | ADM    | ODM    |
|----|--------|--------|
| 3  | 0.175  | 0.123  |
| 4  | 0.221  | 0.144  |
| 5  | 0.220  | 0.086  |

5. Concluding Remarks

In this paper, ODM was implemented to compute the series solution for Smoluchowski’s aggregation equation. In addition, the theoretical investigation into the convergence analysis for the method was also carried out. It was observed that for the convergence of series solution to the exact solution, two additional assumptions were required, i.e., $\{u_n\}$ should be a Cauchy sequence, and the parameter $C(x)$ must be essentially bounded. The article included some numerical examples to establish the application and novelty of this method over ADM considering various relevant kernels with the exponential initial distribution. It was demonstrated that for the constant kernel, ODM needed fewer terms as compared to ADM for higher value of time. Also, it predicted the behavior of zeroth moment better than ADM. In the case of sum kernel, ODM proved to be a more reliable method for estimating the number and mass of particles in the system. Finally, we observed that ODM estimated the number density in fewer terms than ADM and both absolute and numerical error values were significantly lower than ADM for product kernel even for small values of time.

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