GENERALIZED CONVEXITY OF THE INVERSE HYPERBOLIC COSINE FUNCTION

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Received 08 May, 2018

Abstract. The generalized convexity of the inverse hyperbolic cosine function related to the hyperbolic metric is investigated in this paper.

2010 Mathematics Subject Classification: 33B10; 26D07

Keywords: Hölder mean, convexity, concavity, inverse hyperbolic cosine function

1. INTRODUCTION

The hyperbolic functions and their inverses play an important role in the study of the hyperbolic geometry and quasiconformal mappings [1, 4, 5, 8, 9, 11, 12]. For example, the explicit formulas for the hyperbolic metric in the unit disk $B^2$ and the upper half plane $H^2$ are given in terms of the inverse hyperbolic sine and cosine functions, respectively, as follows [5, p.35, p.40]:

$$
\rho_{B^2}(x, y) = 2\text{arsh} \frac{|x - y|}{\sqrt{(1 - |x|^2)(1 - |y|^2)}},
$$

$$
\rho_{H^2}(x, y) = \text{arch} \left( 1 + \frac{|x - y|^2}{2\text{Im}x \text{Im}y} \right).
$$

In recent papers [9,11], the authors investigated the properties of hyperbolic Lambert quadrilaterals in the unit disk by studying the inverse hyperbolic tangent and sine functions.

The study of the convexity/concavity with respect to Hölder means, or simply $H_{p,q}$-convexity/concavity, of special functions has attracted attentions of many researchers, see [2, 6, 7, 9–11, 13–17]. In particular, the convexity/concavity of the inverse hyperbolic tangent and sine functions has been studied in [9] and [11], respectively. For the definition of the above so-called generalized convexity/concavity, the reader is referred to Section 2.

In this paper, we continue the work of [9, 11] to study the generalized convexity for the inverse hyperbolic cosine function. Our main result is stated in the following theorem.
Theorem 1. For \( p, q \in \mathbb{R} \), the inverse hyperbolic cosine function \( \text{arch} \) is strictly \( H_{p,q} \)-convex on \((1, +\infty)\) if and only if \((p, q) \in D_1\), while arch is strictly \( H_{p,q} \)-concave on \((1, +\infty)\) if and only if \((p, q) \in D_2 \cup D_3\), where
\[
D_1 = \{(p, q) | -\infty < p \leq 0, 2 \leq q < +\infty\},
\]
\[
D_2 = \{(p, q) | 0 \leq p \leq \frac{2}{3}, -\infty < q \leq C(p)\},
\]
\[
D_3 = \{(p, q) | \frac{2}{3} < p < +\infty, -\infty < q \leq 2\},
\]
and \( C(p) \) is the same as in Lemma 3(4) with \( C(0) = 1 \) and \( C\left(\frac{2}{3}\right) = 2 \). In particular, for all \( x, y \in (1, +\infty) \), there hold
\[
\text{arch}\sqrt{xy} \leq \sqrt{\frac{\text{arch}^2 x + \text{arch}^2 y}{2}} \leq \text{arch}\left(\sqrt[3]{\frac{\sqrt{x^2} + \sqrt{y^2}}{2}}\right), \tag{1.1}
\]
with equalities if and only if \( x = y \).

2. Preliminaries

For \( r, s \in (0, +\infty) \), the Hölder mean of order \( p \) is defined by
\[
H_p(r, s) = \left(\frac{r^p + s^p}{2}\right)^{\frac{1}{p}} \quad \text{for} \ p \neq 0, \quad H_0(r, s) = \sqrt{rs}.
\]
For \( p = 1 \), we get the arithmetic mean \( A = H_1 \); for \( p = 0 \), the geometric mean \( G = H_0 \); and for \( p = -1 \), the harmonic mean \( H = H_{-1} \). It is well known that \( H_p(r, s) \) is continuous and increasing with respect to \( p \).

A function \( f : I \to J \) is called \( H_{p,q} \)-convex (concave) if it satisfies
\[
f\left(H_p(r, s)\right) \leq (\geq)H_q\left(f(r), f(s)\right)
\]
for all \( r, s \in I \), and strictly \( H_{p,q} \)-convex (concave) if the inequality is strict except for \( r = s \).

The following monotone form of l’Hôpital’s rule is of great use in deriving monotonicity properties and obtaining inequalities. See the extensive bibliography of [3].

Lemma 1 ([1, Theorem 1.25]). For \(-\infty < a < b < \infty\), let functions \( f, g : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\), and be differentiable on \((a, b)\). Let \( g'(x) \neq 0 \) on \((a, b)\). If \( f'(x)/g'(x) \) is increasing (decreasing) on \((a, b)\), then so are
\[
\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.
\]
If \( f'(x)/g'(x) \) is strictly monotone, then the monotonicity in the conclusion is also strict.

We prove the following three lemmas before giving the proof of Theorem 1.
Lemma 2. Let \( r \in (1, +\infty) \).

(1) The function \( f_1(r) = \frac{\text{arch} r}{\sqrt{r^2 - 1}} \) is strictly decreasing with range \((0, 1)\);

(2) The function \( f_2(r) = 2 + \frac{\text{arch} r + r\sqrt{r^2 - 1} - 2r^3\sqrt{r^2 - 1}}{(r^2 - 1)(\text{arch} r + r\sqrt{r^2 - 1})} \) is strictly decreasing with range \((0, \frac{2}{3})\).

Proof. (1) Let \( f_{11}(r) = \text{arch} r \) and \( f_{12}(r) = \sqrt{r^2 - 1} \), then \( f_{11}(1^+) = f_{12}(1^+) = 0 \). By differentiation, we have
\[
\frac{f_{11}'(r)}{f_{12}'(r)} = \frac{1}{r},
\]
which is strictly decreasing. Hence by Lemma 1, the function \( f_1 \) is strictly decreasing with \( f_1(1^+) = 1 \) and \( f_1(+\infty) = 0 \).

(2) Let \( f_{21} = \text{arch} r + r\sqrt{r^2 - 1} - 2r^3\sqrt{r^2 - 1} \) and \( f_{22} = (r^2 - 1)(\text{arch} r + r\sqrt{r^2 - 1}) \), then \( f_{21}(1^+) = f_{22}(1^+) = 0 \). By differentiation, we have
\[
\frac{f_{21}'(r)}{f_{22}'(r)} = \frac{-4}{2 + f_2(r)},
\]
which is strictly decreasing by (1). Hence by Lemma 1, the function \( f_2 \) is strictly decreasing with \( f_2(1^+) = \frac{2}{3} \) and \( f_2(+\infty) = 0 \). \( \square \)

Lemma 3. For \( p \in \mathbb{R} \) and \( r \in (1, +\infty) \), define
\[
h_p(r) = 1 + p\sqrt{r^2 - 1} \cdot \frac{\text{arch} r}{r} + \frac{1}{\sqrt{r^2 - 1}} \cdot \frac{\text{arch} r}{r}.
\]

(1) If \( p \geq \frac{2}{3} \), then \( h_p \) is strictly increasing with range \((2, +\infty)\).

(2) If \( p < 0 \), then \( h_p \) is strictly decreasing with range \((-\infty, 2)\).

(3) If \( p = 0 \), then \( h_p \) is strictly decreasing with range \((1, 2)\).

(4) If \( 0 < p < \frac{2}{3} \), then \( h_p \) is not monotone and the range of \( h_p \) is \([C(p), +\infty)\), where
\[
C(p) = \min_{r \in (1, +\infty)} h_p(r)
\]
with \( 1 < C(p) < 2 \).

Proof. By Lemma 2(1), it is easy to get
\[
h_p(1^+) = 2 \quad \text{and} \quad h_p(+\infty) = \begin{cases} +\infty, & p > 0, \\ 1, & p = 0, \\ -\infty, & p < 0. \end{cases}
\]
By differentiation, we have
\[
h_p'(r) = \frac{1}{r} \left( 1 + \frac{\text{arch} r}{r\sqrt{r^2 - 1}} \right) (p - f_2(r)),
\]
where \( f_2(r) \) is the same as in Lemma 2(2).
By Lemma 2(2), we have (1)–(3).

(4) If \( 0 < p < \frac{2}{3} \), since the range of \( f_2 \) is \( (0, \frac{2}{3}) \), there exists one and only one point \( r_p \in (1, +\infty) \) such that \( p = f_2(r_p) \). Then \( h_p \) is strictly decreasing on \((1, r_p)\) and increasing on \((r_p, +\infty)\). Since \( h_p \) is continuous in \( r \), there exists

\[
C(p) = \min_{r \in (1, +\infty)} h_p(r)
\]

and \( 1 < C(p) < 2 \).

**Lemma 4.** Let \( p, q \in \mathbb{R} \), \( r \in (1, +\infty) \), and \( C(p) \) be the same as in Lemma 3(4). Let

\[
g_{p,q}(r) = \frac{\text{arch}^{q-1}r}{r^{p-1}\sqrt{r^2 - 1}}.
\]

(1) If \( p \geq \frac{2}{3} \), then \( g_{p,q} \) is strictly decreasing for each \( q \leq 2 \), and \( g_{p,q} \) is not monotone for any \( q > 2 \).

(2) If \( p < 0 \), then \( g_{p,q} \) is strictly increasing for each \( q \geq 2 \), and \( g_{p,q} \) is not monotone for any \( q < 2 \).

(3) If \( p = 0 \), then \( g_{p,q} \) is strictly increasing for each \( q \geq 2 \), and \( g_{p,q} \) is strictly decreasing for each \( q \leq 1 \), and \( g_{p,q} \) is not monotone for any \( 1 < q < 2 \).

(4) If \( 0 < p < \frac{2}{3} \), then \( g_{p,q} \) is strictly decreasing for each \( q \leq C(p) \), and \( g_{p,q} \) is not monotone for any \( q > C(p) \).

**Proof.** By logarithmic differentiation in \( r \), we have

\[
\frac{g'_{p,q}(r)}{g_{p,q}(r)} = \frac{1}{\sqrt{r^2 - 1} \cdot \text{arch} r} \left(q - h_p(r)\right),
\]

where \( h_p(r) \) is the same as in Lemma 3. Hence the results immediately follow from Lemma 3.

3. PROOF OF MAIN RESULT

We are now in a position to prove Theorem 1.

**Proof of Theorem 1.** Without loss of generality, we may assume that \( 1 < x \leq y < +\infty \). Let \( t = H_p(x, y) \), then \( x \leq t \leq y \) and

\[
\frac{\partial t}{\partial x} = \frac{1}{2} \left( \frac{x}{t} \right)^{p-1}.
\]

The proof is divided into the following four cases.

**Case 1.** \( p \neq 0 \) and \( q \neq 0 \). Define

\[
F(x, y) = \text{arch}^q(H_p(x, y)) - \frac{\text{arch}^q x + \text{arch}^q y}{2}.
\]
By differentiation, we have
\[
\frac{\partial F}{\partial x} = \frac{q}{2} x^{p-1} \left( \frac{\text{arch}^{p-1} t}{t^{p-1} \sqrt{t^2 - 1}} - \frac{\text{arch}^{p-1} x}{x^{p-1} \sqrt{x^2 - 1}} \right) = \frac{q}{2} x^{p-1} \left( g_{p,q}(t) - g_{p,q}(x) \right),
\]
where \( g_{p,q} \) is defined in Lemma 4.

**Case 1.** \( p \geq \frac{2}{3} \) and \( q \leq 2 \).

By Lemma 4(1), the function \( g_{p,q} \) is strictly decreasing on \((1, +\infty)\).

**Case 1.1** If \( q > 0 \), then \( \frac{\partial F}{\partial x} \leq 0 \). Hence \( F(x, y) \) is strictly decreasing and \( F(x, y) \geq F(y, y) = 0 \). Namely,
\[
\text{arch}(H_p(x, y)) \geq \left( \frac{\text{arch}^{q} x + \text{arch}^{q} y}{2} \right)^{\frac{1}{q}} = H_q(\text{arch} x, \text{arch} y),
\]
with equality if and only if \( x = y \).

**Case 1.1.1** If \( q < 0 \), then \( \frac{\partial F}{\partial x} \geq 0 \). Hence \( F(x, y) \) is strictly increasing and \( F(x, y) \leq F(y, y) = 0 \). Namely,
\[
\text{arch}(H_p(x, y)) \geq \left( \frac{\text{arch}^{q} x + \text{arch}^{q} y}{2} \right)^{\frac{1}{q}} = H_q(\text{arch} x, \text{arch} y),
\]
with equality if and only if \( x = y \).

In conclusion, arch is strictly \( H_{p,q} \)-concave on the whole interval \((1, +\infty)\) for \( (p, q) \in \{(p, q) \mid p \geq \frac{2}{3}, 0 < q \leq 2\} \cup \{(p, q) \mid q \geq 2\} \).

**Case 1.2** \( p \geq \frac{2}{3} \) and \( q > 2 \).

By Lemma 4(1), the function \( g_{p,q} \) is not monotone on \((1, +\infty)\). With an argument similar to Case 1.1, it is easy to see that arch is neither \( H_{p,q} \)-concave nor \( H_{p,q} \)-convex on the whole interval \((1, +\infty)\) for \( (p, q) \in \{(p, q) \mid p \geq \frac{2}{3}, q > 2\} \).

**Case 1.3** \( p < 0 \) and \( q \geq 2 \).

By Lemma 4(2), the function \( g_{p,q} \) is strictly increasing on \((1, +\infty)\) and hence \( \frac{\partial F}{\partial x} \geq 0 \). Then \( F(x, y) \) is strictly increasing and \( F(x, y) \leq F(y, y) = 0 \). Namely,
\[
\text{arch}(H_p(x, y)) \leq \left( \frac{\text{arch}^{q} x + \text{arch}^{q} y}{2} \right)^{\frac{1}{q}} = H_q(\text{arch} x, \text{arch} y),
\]
with equality if and only if \( x = y \).

In conclusion, arch is strictly \( H_{p,q} \)-convex on the whole interval \((1, +\infty)\) for \( (p, q) \in \{(p, q) \mid p < 0, q \geq 2\} \).

**Case 1.4** \( p < 0 \) and \( q < 2 \).

By Lemma 4(2), the function \( g_{p,q} \) is not monotone on \((1, +\infty)\). With an argument similar to Case 1.3, it is easy to see that arch is neither \( H_{p,q} \)-concave nor \( H_{p,q} \)-convex on the whole interval \((1, +\infty)\) for \( (p, q) \in \{(p, q) \mid p < 0, q < 2\} \).

**Case 1.5** \( 0 < p < \frac{2}{3} \) and \( q \leq C(p) \).
By Lemma 4(4), the function $g_{p,q}$ is strictly decreasing on $(1, +\infty)$.

**Case 1.5.1** If $q > 0$, then $\frac{\partial F}{\partial x} \leq 0$. Hence $F(x, y)$ is strictly decreasing and $F(x, y) \geq F(y, y) = 0$. Namely,

$$\text{arch}(H_p(x, y)) \geq \left( \frac{\text{arch}^q x + \text{arch}^q y}{2} \right)^{\frac{1}{q}} = H_q(\text{arch} x, \text{arch} y),$$

with equality if and only if $x = y$.

**Case 1.5.2** If $q < 0$, then $\frac{\partial F}{\partial x} \geq 0$. Hence $F(x, y)$ is strictly increasing and $F(x, y) \leq F(y, y) = 0$. Namely,

$$\text{arch}(H_p(x, y)) \geq \left( \frac{\text{arch}^q x + \text{arch}^q y}{2} \right)^{\frac{1}{q}} = H_q(\text{arch} x, \text{arch} y),$$

with equality if and only if $x = y$.

In conclusion, arch is strictly $H_{p,q}$-concave on the whole interval $(1, +\infty)$ for $(p, q) \in \{(p, q) | 0 < p < \frac{2}{3}, 0 < q \leq C(p)\} \cup \{(p, q) | 0 < p < \frac{2}{3}, q < 0\}$.

**Case 2.** $0 < p < \frac{2}{3}$ and $q > C(p)$.

By Lemma 4(4), the function $g_{p,q}$ is not monotone on $(1, +\infty)$. With an argument similar to Case 1.5, it is easy to see that arch is neither $H_{p,q}$-concave nor $H_{p,q}$-convex on the whole interval $(1, +\infty)$ for $(p, q) \in \{(p, q) | 0 < p < \frac{2}{3}, q > C(p)\}$.

**Case 2.1** $p \geq \frac{2}{3}$ and $q = 0$.

By Lemma 4(1), the function $g_{p,q}$ is strictly decreasing on $(1, +\infty)$ and hence $\frac{\partial F}{\partial x} \leq 0$. Then $F(x, y)$ is strictly decreasing and $F(x, y) \geq F(y, y) = 1$. Namely,

$$\text{arch}(H_p(x, y)) \geq \sqrt{\text{arch} x \cdot \text{arch} y} = H_0(\text{arch} x, \text{arch} y),$$

with equality if and only if $x = y$.

In conclusion, arch is strictly $H_{p,q}$-concave on the whole interval $(1, +\infty)$ for $(p, q) \in \{(p, q) | p \geq \frac{2}{3}, q = 0\}$.

**Case 2.2** $p < 0$ and $q = 0$.

By Lemma 4(2), the function $g_{p,q}$ is not monotone on $(1, +\infty)$. With an argument similar to Case 2.1, it is easy to see that arch is neither $H_{p,q}$-concave nor $H_{p,q}$-convex on the whole interval $(1, +\infty)$ for $(p, q) \in \{(p, q) | p < 0, q = 0\}$.

**Case 2.3** $0 < p < \frac{2}{3}$ and $q = 0$. 

By logarithmic differentiation, we obtain

$$\frac{1}{F} \frac{\partial F}{\partial x} = x^{p-1}(g_{p,0}(t) - g_{p,0}(x)),$$

where $g_{p,0}$ is defined in Lemma 4.

**Case 2.1** $p \geq \frac{2}{3}$ and $q = 0$.

By Lemma 4(1), the function $g_{p,q}$ is strictly decreasing on $(1, +\infty)$ and hence $\frac{\partial F}{\partial x} \leq 0$. Then $F(x, y)$ is strictly decreasing and $F(x, y) \geq F(y, y) = 1$. Namely,

$$\text{arch}(H_p(x, y)) \geq \sqrt{\text{arch} x \cdot \text{arch} y} = H_0(\text{arch} x, \text{arch} y),$$

with equality if and only if $x = y$.

In conclusion, arch is strictly $H_{p,q}$-concave on the whole interval $(1, +\infty)$ for $(p, q) \in \{(p, q) | p \geq \frac{2}{3}, q = 0\}$.
By Lemma 4(4), the function \( g_{p,q} \) is strictly decreasing on \((1, +\infty)\) and hence \( \frac{\partial F}{\partial x} \leq 0. \) Then \( F(x, y) \) is strictly decreasing and \( F(x, y) \geq F(y, y) = 1. \) Namely,
\[
\text{arch}(H_p(x, y)) \geq \sqrt{\text{arch } x \cdot \text{arch } y} = H_0(\text{arch } x, \text{arch } y),
\]
with equality if and only if \( x = y. \)

In conclusion, \( arch \) is strictly \( H_{p,q} \)-concave on the whole interval \((1, +\infty)\) for \((p, q) \in \{(p, q) | 0 < p < \frac{2}{3}, q = 0\}\).

**Case 3.** \( p = 0 \) and \( q \neq 0. \)

Define
\[
F(x, y) = \text{arch}^q(\sqrt{xy}) = \frac{\text{arch}^q x + \text{arch}^q y}{2}.
\]
By differentiation, we obtain
\[
\frac{\partial F}{\partial x} = \frac{q}{2x}(g_{0,q}(t) - g_{0,q}(x)),
\]
where \( g_{0,q} \) is defined in Lemma 4.

**Case 3.1** \( p = 0 \) and \( q \geq 2. \)

By Lemma 4(3), the function \( g_{p,q} \) is strictly increasing on \((1, +\infty)\) and hence \( \frac{\partial F}{\partial x} \geq 0. \) Then \( F(x, y) \) is strictly increasing and \( F(x, y) \leq F(y, y) = 0. \) Namely,
\[
\text{arch}(H_0(x, y)) \leq \left( \frac{\text{arch}^q x + \text{arch}^q y}{2} \right)^{\frac{1}{q}} = H_q(\text{arch } x, \text{arch } y),
\]
with equality if and only if \( x = y. \)

In conclusion, \( arch \) is strictly \( H_{p,q} \)-convex on the whole interval \((1, +\infty)\) for \((p, q) \in \{(p, q) | p = 0, q \geq 2\}\).

**Case 3.2** \( p = 0 \) and \( q \leq 1. \)

By Lemma 4(3), the function \( g_{p,q} \) is strictly decreasing on \((1, +\infty).\)

**Case 3.2.1** If \( 0 < q \leq 1, \) then \( \frac{\partial F}{\partial x} \leq 0. \) Hence \( F(x, y) \) is strictly decreasing and \( F(x, y) \geq F(y, y) = 0. \) Namely,
\[
\text{arch}(H_0(x, y)) \geq \left( \frac{\text{arch}^q x + \text{arch}^q y}{2} \right)^{\frac{1}{q}} = H_q(\text{arch } x, \text{arch } y),
\]
with equality if and only if \( x = y. \)

**Case 3.2.2** If \( q < 0, \) then \( \frac{\partial F}{\partial x} \geq 0. \) Hence \( F(x, y) \) is strictly increasing and \( F(x, y) \leq F(y, y) = 0. \) Namely,
\[
\text{arch}(H_0(x, y)) \geq \left( \frac{\text{arch}^q x + \text{arch}^q y}{2} \right)^{\frac{1}{q}} = H_q(\text{arch } x, \text{arch } y),
\]
with equality if and only if \( x = y. \)

In conclusion, \( arch \) is strictly \( H_{p,q} \)-concave on the whole interval \((1, +\infty)\) for \((p, q) \in \{(p, q) | p = 0, 0 < q \leq 1\} \cup \{(p, q) | p = 0, q < 0\}\).

**Case 3.3** \( p = 0 \) and \( 1 < q < 2. \)
By Lemma 4(3), the function $g_{p,q}$ is not monotone on $(1, +\infty)$. With an argument similar to Case 3.2, it is easy to see that arch is neither $H_{p,q}$-concave nor $H_{p,q}$-convex on the whole interval $(1, +\infty)$ for $(p, q) \in \{(p, q) | p = 0, 1 < q < 2\}$.

Case 4. $p = 0$ and $q = 0$.

By Case 2.3, for all $x, y \in (1, +\infty)$, we have

$$\text{arch}(H_p(x, y)) \geq H_0(\text{arch} x, \text{arch} y) \quad \text{for} \quad 0 < p < \frac{2}{3},$$

By the continuity of $H_p$ in $p$ and arch in $x$, we have

$$\text{arch}(H_0(x, y)) \geq H_0(\text{arch} x, \text{arch} y),$$

with equality if and only if $x = y$.

In conclusion, arch is strictly $H_{0,0}$-concave on the whole interval $(1, +\infty)$.

By Case 1.1 and Case 3.1, arch is strictly $H_{\frac{2}{3}, 2}$-concave and strictly $H_{0, 2}$-convex on $(1, +\infty)$. Therefore, the inequalities (1.1) hold with equalities if and only if $x = y$. This completes the proof of Theorem 1.

Setting $p = 1 = q$ in Theorem 1, we easily obtain the concavity of arch.

**Corollary 1.** The inverse hyperbolic cosine function arch is strictly concave on $(1, +\infty)$.

**ACKNOWLEDGMENTS**

This research was supported by National Natural Science Foundation of China (NNSFC) under Grant No.11601485 and No.11771400, and Science Foundation of Zhejiang Sci-Tech University (ZSTU) under Grant No.16062023-Y.

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