In this paper, we obtain a reverse version of the integral Hardy inequality on metric measure space with two negative exponents. For applications we show the reverse Hardy–Littlewood–Sobolev and the Stein–Weiss inequalities with two negative exponents on homogeneous Lie groups and with arbitrary quasi-norm, the result of which appears to be new in the Euclidean space. This work further complements the ranges of $p$ and $q$ (namely, $q \leq p < 0$) considered in the work of Ruzhansky & Verma (Ruzhansky & Verma 2019 Proc. R. Soc. A 475, 20180310 (doi:10.1098/rspa.2018.0310); Ruzhansky & Verma. 2021 Proc. R. Soc. A 477, 20210136 (doi:10.1098/rspa.2021.0136)), which treated the cases $1 < p \leq q < \infty$ and $p > q$, respectively.

1. Introduction

In the famous work [1], G. H. Hardy showed the following (direct) integral inequality:

$$
\int_a^\infty \frac{1}{x^p} \left( \int_a^\infty f(t) \, dt \right)^p \, dx \leq \left( \frac{p}{p-1} \right)^p \int_a^\infty f^p(x) \, dx, \quad (1.1)
$$

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where \( f \geq 0 \), \( p > 1 \) and \( a > 0 \). The subject of the Hardy inequalities has been extensively investigated and we refer to the book [2].

We refer to direct inequalities [2–9] and to the reverse inequalities [10–14].

The main goal of this paper is to extend the reverse Hardy inequalities to general metric measure space with two negative exponents. More specifically, we consider metric spaces \( X \) with a Borel measure \( dx \) allowing for the following polar decomposition at \( a \in X \): we assume that there is a locally integrable function \( \lambda \in L^1_{\text{loc}}(X) \) such that for all \( f \in L^1(X) \) we have

\[
\int_X f(x) \, dx = \int_0^\infty \int_{\Sigma_r} f(r, \omega) \lambda(r, \omega) \, d\omega_r \, dr,
\]

for some set \( \Sigma_r = \{ x \in X : d(x, a) = r \} \subset X \) with a measure on it denoted by \( d\omega_r \), and \( (r, \omega) \to a \) as \( r \to 0 \).

The condition (1.2) is rather general (see [15]) since we allow the function \( \lambda \) to depend on the whole variable \( x = (r, \omega) \). Since \( X \) does not necessarily have a differentiable structure, the function \( \lambda(r, \omega) \) cannot be in general obtained as the Jacobian of the polar change of coordinates. However, if such a differentiable structure exists on \( X \), the condition (1.2) can be obtained as the standard polar decomposition formula. In particular, let us give several examples of \( X \) for which the condition (1.2) is satisfied with different expressions for \( \lambda(r, \omega) \):

(I) Euclidean space \( \mathbb{R}^n \): \( \lambda(r, \omega) = r^{n-1} \).

(II) Homogeneous groups: \( \lambda(r, \omega) = r^{Q-1} \), where \( Q \) is the homogeneous dimension of the group. Such groups have been consistently developed by Folland & Stein [16] (see also an up-to-date exposition in [17,18]).

(III) Hyperbolic spaces \( \mathbb{H}^n \): \( \lambda(r, \omega) = (\sinh r)^{-n-1} \).

(IV) Cartan–Hadamard manifolds: let \( K_M \) be the sectional curvature on \( (M, g) \). A Riemannian manifold \( (M, g) \) is called a Cartan–Hadamard manifold if it is complete, simply connected and has non-positive sectional curvature, i.e. the sectional curvature \( K_M \leq 0 \) along each plane section at each point of \( M \). Let us fix a point \( a \in M \) and denote by \( \rho(x) = d(x, a) \) the geodesic distance from \( x \) to \( a \) on \( M \). The exponential map \( \exp_a : T_a M \to M \) is a diffeomorphism (see e.g. Helgason [19]). Let \( f(\rho, \omega) \) be the density function on \( M \) (see e.g. [20]). Then we have the following polar decomposition:

\[
\int_M f(x) \, dx = \int_0^\infty \int_{\mathbb{S}^{n-1}} f(\exp_a(\rho\omega))|f(\rho, \omega)|\rho^{n-1} \, d\rho \, d\omega,
\]

so that we have (1.2) with \( \lambda(\rho, \omega) = f(\rho, \omega)\rho^{n-1} \).

In [15,21], the (direct) integral Hardy inequality on metric measure spaces was established with applications to homogeneous Lie groups, hyperbolic spaces, Cartan–Hadamard manifolds with negative curvature and on general Lie groups with Riemannian distance for \( 1 \leq p < q < +\infty \) and \( p > q \), respectively. Also, in [22], the authors showed the integral Hardy inequality for \( p \in (0, 1) \) and \( q < 0 \) on metric measure space. In this paper, we continue the investigation of the integral Hardy inequality on a metric measure space, i.e. we show the reverse integral Hardy inequality with negative exponents.

In [23], Hardy and Littlewood considered the one-dimensional fractional integral operator on \((0, \infty)\) given by

\[
T_{\lambda} u(x) = \int_0^x \frac{u(y)}{|x-y|^\lambda} \, dy, \quad 0 < \lambda < 1,
\]

(1.3)

where they also showed the following \( L^q - L^p \) boundedness of this operator \( T_{\lambda} \):

**Theorem 1.1.** Let \( 1 < p < q < \infty \) and \( u \in L^p(0, \infty) \) with \( 1/q = (1/p) + \lambda - 1 \). Then

\[
||T_{\lambda} u||_{L^q(0, \infty)} \leq C||u||_{L^p(0, \infty)},
\]

(1.4)

where \( C \) is a positive constant independent of \( u \).
The multi-dimensional analogue of (1.3) can be represented by the formula:

\[ I_{k}u(x) = \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^k} \, dy, \quad 0 < \lambda < N. \quad (1.5) \]

In [24], Sobolev generalized theorem 1.1 for the multi-dimensional case in the following form:

**Theorem 1.2.** Let \( 1 < p < q < \infty \), \( u \in L^p(\mathbb{R}^n) \) with \( 1/q = (1/p) + (\lambda/N) - 1 \). Then

\[ ||I_{k}u||_{L^{q}(\mathbb{R}^n)} \leq C||u||_{L^{p}(\mathbb{R}^n)}, \quad (1.6) \]

where \( C \) is a positive constant independent of \( u \).

In [25], Stein and Weiss obtained the following radially weighted Hardy–Littlewood–Sobolev inequality, which is known as the Stein–Weiss inequality.

**Theorem 1.3.** Let \( 0 < \lambda < N \), \( 1 < p < \infty \), \( \alpha < N(p-1)/p \), \( \beta < N/q \), \( \alpha + \beta \geq 0 \) and \( 1/q = (1/p) + ((\lambda + \alpha + \beta)/N) - 1 \). If \( 1 < p \leq q < \infty \), then

\[ ||x|^{-\beta}I_{k}u||_{L^{q}(\mathbb{R}^n)} \leq C||I^{\alpha}u||_{L^{p}(\mathbb{R}^n)}, \quad (1.7) \]

where \( C \) is a positive constant independent of \( u \).

To the best of our knowledge, the Hardy–Littlewood–Sobolev inequality on the Heisenberg group was proved by Folland & Stein [26] and the best constants of the Hardy–Littlewood–Sobolev inequality, in the Euclidean space and Heisenberg group were obtained in [27,28], respectively. Also, in [18,29,30], the authors studied the Hardy–Littlewood–Sobolev and the Stein–Weiss inequalities on Heisenberg and homogeneous Lie groups. Note that systematic studies of different functional inequalities on general homogeneous (Lie) groups were initiated by the papers [31–34].

The reverse Stein–Weiss inequality in Euclidean setting has the following form:

**Theorem 1.4 ([35], theorem 1).** For \( n \geq 1 \), \( p \in (0,1) \), \( q < 0 \), \( \lambda > 0 \), \( 0 \leq \alpha < -n/q \) and \( 0 \leq \beta < -n/p' \) satisfying \( \frac{1}{p} + \frac{1}{q} - \frac{\alpha + \beta + \lambda}{n} = 2 \), there is a constant \( C = C(n,\alpha,\beta,\lambda,p,q) > 0 \) such that for any non-negative functions \( f \in L^{\lambda}(\mathbb{R}^n) \) and \( 0 < \int_{\mathbb{R}^n} g^p(y) \, dy < \infty \), we have

\[ \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x|^\alpha |x-y|^\beta f(x)g(y)|y|^\beta \, dy \, dx \right)^{1/q'} \geq C \left( \int_{\mathbb{R}^n} f^\lambda (x) \, dx \right)^{1/q} \left( \int_{\mathbb{R}^n} g^p (y) \, dy \right)^{1/p'}, \quad (1.8) \]

where \( 1/q + 1/q' = 1 \) and \( 1/p + 1/p' = 1 \).

Note, we obtain the reverse Hardy–Littlewood–Sobolev inequality if \( \alpha = \beta = 0 \). Improved Stein–Weiss inequality was obtained in [36] on the Euclidean upper half-space and in [37] on homogeneous Lie groups. For more results about the reverse Hardy–Littlewood–Sobolev inequality in Euclidean space, we refer the reader to Beckner [38], Carrillo et al. [39], Dou & Zhu [40], Ngô & Nguyen [41] and the references therein. Note that the reverse Hardy–Littlewood–Sobolev and Stein–Weiss inequalities were shown in [37] for the case \( p \in (0,1) \) and \( q < 0 \). In this paper, we show the reverse Hardy–Littlewood–Sobolev and Stein–Weiss inequalities with two negative exponents i.e. \( q < p < 0 \), which is also new in the Euclidean space.

2. Main result

Firstly, let us denote by \( B(a,r) \) a ball in \( \mathbb{X} \) with centre \( a \) and radius \( r \), i.e.,

\[ B(a,r) := \{ x \in \mathbb{X} : d(x,a) < r \}, \]

where \( d \) is the metric on \( \mathbb{X} \). Once and for all let us fix some point \( a \in \mathbb{X} \), and denote

\[ |x|_a := d(a,x). \quad (2.1) \]

Let us recall briefly the reverse Hölder inequality.
Theorem 2.1 ([42], theorem 2.12, p. 27). Let \( p < 0, \) so that \( p' = p/(p - 1) > 0. \) If non-negative functions satisfy \( 0 < \int_X f^p(x) \, dx < +\infty \) and \( 0 < \int_X \tilde{g}^{p'}(x) \, dx < +\infty, \) we have

\[
\int_X f(x)g(x) \, dx \geq \left( \int_X f^p(x) \, dx \right)^{1/p} \left( \int_X \tilde{g}^{p'}(x) \, dx \right)^{1/p'}.
\]  

(2.2)

As the main results of this section, we show the reverse integral Hardy inequality as well as its conjugate.

Theorem 2.2. Assume that \( p, q < 0 \) are such that \( q \leq p < 0. \) Let \( \mathbb{X} \) be a metric measure space with a polar decomposition at \( a \in \mathbb{X}. \) Suppose that \( u, v \geq 0 \) are locally integrable functions on \( \mathbb{X}. \) Then the inequality

\[
\left[ \int_X \left( \int_{B(a,|x|_a)} f(y) \, dy \right)^q u(x) \, dx \right]^{1/q} \geq C_1(p, q) \left( \int_X f^p(x)u(x) \, dx \right)^{1/p},
\]  

(2.3)

holds for all non-negative real-valued measurable functions \( f, \) if and only if

\[
0 < D_1 = \inf_{x \neq a} D_1(|x|_a) = \inf_{x \neq a} \left[ \left( \int_{B(a,|x|_a)} u(y) \, dy \right)^{1/q} \left( \int_{B(a,|x|_a)} v^{1-p'}(y) \, dy \right)^{1/p'} \right],
\]  

(2.4)

and \( D_1(|x|_a) \) is non-decreasing. Moreover, the largest constant \( C_1(p, q) \) in (2.3) satisfies

\[
D_1 \geq C_1(p, q) \geq |p|^{1/q}(p')^{1/p'}D_1,
\]  

(2.5)

where \( 1/p + 1/p' = 1. \)

Remark 2.3. In (2.5), by simple calculation, we have that for the case \( q \leq p < 0 \)

\[
|p|^{1/q}(p')^{1/p'} \leq 1.
\]  

(2.6)

Proof. Let us divide a proof of this theorem to two steps.

Step 1. Firstly, let us denote

\[
F(s) := \sum_{\sigma} \lambda(s, \sigma)f^p(s, \sigma)v(s, \sigma) \, d\sigma,
\]  

(2.7)

\[
V(s) := \sum_{\sigma} \lambda(s, \sigma)v^{1-p'}(s, \sigma) \, d\sigma,
\]  

(2.8)

\[
h(t) := \left( \int_0^t \int_{\Sigma} \lambda(s, \sigma)v^{1-p'}(s, \sigma) \, d\sigma \, ds \right)^{1/pp'},
\]  

(2.9)

\[
H_1(t) := \int_0^t \int_{\Sigma} \lambda(s, \sigma)v^{-(p'/p)}(s, \sigma)h^{-p'}(s) \, d\sigma \, ds,
\]  

(2.10)

\[
U_1(s) := \int_{\Sigma} \lambda(s, \sigma)u(s, \sigma) \, d\sigma.
\]  

(2.11)

By using the reverse Hölder inequality (2.2) with the polar decomposition, we compute

\[
\int_{B(a,|x|_a)} f(y) \, dy = \int_{B(a,|x|_a)} \left[ f(y)v^{1/p}(y)h(y) \right] \left[ v^{1/p}(y)h(y) \right]^{-1} \, dy
\]

\[
\geq \left( \int_{B(a,|x|_a)} (f(y)v^{1/p}(y)h(y)) \, dy \right)^{1/p} \left( \int_{B(a,|x|_a)} (v^{1/p}(y)h(y))^{-p'} \, dy \right)^{1/p'}
\]

\[
= \left( \int_0^{\vert x \vert} h^{p'}(s)\lambda(s, \sigma)f^p(s, \sigma)v(s, \sigma) \, d\sigma \, ds \right)^{1/p}
\]  

(2.12)
Multiplying by $u$,

\[ \frac{\lambda(s, \sigma)}{q} \]

By combining (2.13) and (2.12), we get

\[ \int_{0}^{t} h^{p}(s) F(s) \, ds \frac{1}{p} H_{1}^{1/p'}(|x|_{a}). \]

(2.12)

Let us calculate $H_{1}(t)$:

\[
H_{1}(t) = \int_{0}^{t} \lambda(s, \sigma) u^{-q/p}(s, \sigma) h^{-p'}(s) \, ds
\]

\[
= \left( \int_{0}^{r_{s}} h^{p}(s) F(s) \, ds \right) \frac{1}{p} H_{1}^{1/p'}(|x|_{a}).
\]

(2.13)

By combining (2.13) and (2.12), we get

\[
\int_{B(a, |x|_{a})} f(y) \, dy \geq \left( \int_{0}^{r_{s}} h^{p}(s) F(s) \, ds \right) \frac{1}{p} H_{1}^{1/p'}(|x|_{a})
\]

\[
= (p')^{1/p'} \left( \int_{0}^{r_{s}} h^{p}(s) F(s) \, ds \right) \frac{1}{p} h^{q/p'}(|x|_{a}).
\]

(2.14)

Multiplying by $u$, integrating over $\mathbb{X}$ with $q < 0$ and by using (direct) Minkowski’s inequality with $q/p \geq 1$ (see [42], theorem 2.9, p. 26), we compute

\[
\int_{\mathbb{X}} \left( \int_{B(a, |x|_{a})} f(y) \, dy \right) u(x) \, dx = \int_{0}^{\infty} \int_{0}^{\infty} u(z, \omega) \lambda(z, \omega) \left( \int_{0}^{r_{s}} \lambda(s, \sigma) f(s, \sigma) \, ds d\sigma \right)^{q} \\lambda(z, \omega) \, dz d\omega
\]

\[
\leq (p')^{q/p'} \int_{0}^{\infty} U_{1}(z) \left( \int_{0}^{r_{s}} h^{p}(s) F(s) \, ds \right)^{q/p} h^{q/p'}(z) \, dz
\]

\[
= (p')^{q/p'} \int_{0}^{\infty} U_{1}(z) \left( \int_{0}^{r_{s}} h^{p}(s) F(s) \, ds \right)^{q/p} h^{q/p'}(z) \, dz
\]

\[
\leq (p')^{q/p'} \left[ \int_{0}^{\infty} h^{p}(s) F(s) \left( \int_{0}^{\infty} U_{1}(z) h^{q/p'}(z) \, dz \right) \frac{p}{q} \right]^{q/p},
\]

(2.15)
where \( \chi_{[0,r]} \) is the cut-off function. At the same time, one can also estimate

\[
    h^{p/q/(p')} (t) = \left( \left( \int_0^t \int_0^x \lambda(s, \sigma) \rho^{1-p'}(s, \sigma) \, ds \, d\sigma \right)^{q/p'} \right)^{1/p'}
\]

(2.8)

\[
    = \left( \int_0^t V(s) \, ds \right)^{q/p'} \left( \int_0^t U_1(s) \, ds \right)^{1/(1-p')}
\]

(2.15)

where \( D_1(\|t\|) := \left( \int_0^t V(s) \, ds \right)^{1/p'} \left( \int_0^t U_1(s) \, ds \right)^{1/q} \). By using this fact and since \( D_1(\|x\|) \) is non-decreasing, we get

\[
    \int_X \left( \int_{B(a, \|x\|)} f(y) \, dy \right)^q u(x) \, dx \\
    \leq \left( p' \right)^{q/p} \left( \int_0^\infty h^p(s) F(s) \left( \int_s^{\infty} U_1(r) \rho^{1-p'}(r) \, dr \right)^{p/q} \, ds \right)^{q/p}
\]

(2.12)

\[
    \leq \left( p' \right)^{q/p} \left( \int_0^\infty h^p(s) F(s) D_1^{p/q}(p') \left( \int_s^{\infty} U_1(r) \left( \int_0^r U_1(z) \, dz \right)^{(1-p')/(1-p)} \, dr \right)^{p/q} \, ds \right)^{q/p}
\]

(2.16)

\[
    = \left( p' \right)^{q/p} \left( \int_0^\infty h^p(s) F(s) D_1^{p/q}(p') \left( \int_s^{\infty} U_1(r) \left( \int_0^r U_1(z) \, dz \right)^{1/p} \, dr \right)^{p/q} \, ds \right)^{q/p}
\]

\[
    = \left( p' \right)^{q/p} \left( \int_0^\infty h^p(s) F(s) D_1^{p/q}(p') \left( \int_0^s U_1(z) \, dz \right)^{1/p} \, ds \right)^{q/p}
\]

(2.9)

Finally, we obtain

\[
    \left( \int_X \left( \int_{B(a, \|x\|)} f(y) \, dy \right)^q u(x) \, dx \right)^{1/q} \geq |p|^{1/q} (p')^{1/p' D_1} \left( \int_X f^p(x) v(x) \, dx \right)^{1/p'.}
\]

(2.18)

Hence, it follows that (2.3) holds with \( C_1(p, q) \geq |p|^{1/q} (p')^{1/p' D_1} \), proving one of the relations in (2.5).
Step 2.
Now it remains to show that (2.3) yields (2.4). Let us fix \( t > 0 \) and denote the following function:

\[
f(x) := \begin{cases} v^{1-p'}(x), & \text{if } |x|_a \leq t, \\ \alpha f_1(x), & \text{if } |x|_a > t, \end{cases}
\]

(2.19)

where \( f_1 \) is any function satisfying \( \int_{B(a,|x|_a)} f_1(y) \, dy < \infty \) and \( \int_{|x|_a \geq t} v(x) f_1^p(x) \, dx < \infty \), and \( \alpha > 0 \). Then we compute

\[
C_1(p, q) \leq \left( \int_X \left( \int_{|y|_a \leq |x|_a} f(y) \, dy \right)^q \, u(x) \, dx \right)^{1/q} \left( \int_X f^p(y) v(y) \, dy \right)^{-1/(p')}
\]

\[
= \left[ \int_X \left( \int_{|y|_a \leq |x|_a} f(y) \, dy \right)^q \, u(x) \, dx \right]^{1/q} \left[ \int_{|y|_a \leq t} v^{1-p'}(y) \, dy + \alpha^p \int_{|y|_a > t} v(y) f_1^p(y) \, dy \right]^{-1/(p')}
\]

\[
\leq \left[ \int_{|x|_a \leq t} \left( \int_{|y|_a \leq |x|_a} v^{1-p'}(y) \, dy \right)^q \, u(x) \, dx \right]^{1/q} \left[ \int_{|y|_a \leq t} v^{1-p'}(y) \, dy + \alpha^p \int_{|y|_a > t} v(y) f_1^p(y) \, dy \right]^{-1/(p')}
\]

\[
\leq \left[ \int_{|x|_a \leq t} u(x) \, dx \right]^{1/q} \left[ \int_{|y|_a \leq t} v^{1-p'}(y) \, dy \right] \left[ \int_{|y|_a \leq t} v^{1-p'}(y) \, dy + \alpha^p \int_{|y|_a > t} v(y) f_1^p(y) \, dy \right]^{-1/(p')}
\]

Summarizing the above facts with \( q \leq p < 0 \) and taking the limit as \( \alpha \to 0 \), we obtain

\[
C_1(p, q) \leq \left( \int_{|y|_a \leq t} v^{1-p'}(y) \, dy \right)^{1/p'} \left( \int_{|x|_a \leq t} u(x) \, dx \right)^{1/q}.
\]

(2.20)

Finally, we get \( C_1(p, q) \leq D_1 \).

Now let us prove the conjugate integral Hardy inequality.

**Theorem 2.4.** Assume that \( p, q < 0 \) such that \( q \leq p < 0 \). Let \( \mathbb{X} \) be a metric measure space with a polar decomposition at \( a \in \mathbb{X} \). Suppose that \( u, v \geq 0 \) are locally integrable functions on \( \mathbb{X} \). Then the inequality

\[
\left( \int_X \left( \int_{X \setminus B(a,|x|_a)} f(y) \, dy \right)^q \, u(x) \, dx \right)^{1/q} \geq C_2(p, q) \left( \int_X f^p(x) v(x) \, dx \right)^{1/p}
\]

(2.21)

holds for all non-negative real-valued measurable functions \( f \), if and only if

\[
0 < D_2 = \inf_{x \neq a} D_2(|x|_a) = \inf_{x \neq a} \left[ \left( \int_{X \setminus B(a,|x|_a)} u(y) \, dy \right)^{1/q} \left( \int_{X \setminus B(a,|x|_a)} v^{1-p'}(y) \, dy \right)^{1/p'} \right],
\]

(2.22)

and \( D_2(|x|_a) \) is non-increasing. Moreover, the largest constant \( C_2(p, q) \) satisfies

\[
D_2 \geq C_2(p, q) \geq |p|^{1/p'}(p')^{1/p} D_2,
\]

(2.23)

where \( 1/p + 1/p' = 1 \).

**Proof.** The main idea of the proof of this theorem is similar to that of theorem 2.2 with the only difference that \( D_2(|x|_a) \) is non-increasing, so we omit the details. ■
3. Consequences on homogeneous groups

In this section, we consider several consequences of the main results for the reverse integral Hardy, Hardy–Littlewood–Sobolev and Stein–Weiss inequalities on homogeneous groups.

Let us recall that a Lie group (on $\mathbb{R}^n$) $G$ with the dilation

$$D_{\lambda}(x) := (\lambda^{v_1}x_1, \ldots, \lambda^{v_n}x_n), \quad v_1, \ldots, v_n > 0, D_{\lambda} : \mathbb{R}^n \to \mathbb{R}^n,$$

which is an automorphism of the group $G$ for each $\lambda > 0$, is called a homogeneous (Lie) group. For simplicity, throughout this paper we use the notation $\lambda x$ for the dilation $D_{\lambda}(x)$. The homogeneous dimension of the homogeneous group $G$ is denoted by $Q := v_1 + \ldots + v_n$. Also, in this paper we denote a homogeneous quasi-norm on $G$ by $|x|$, which is a continuous non-negative function

$$G \ni x \mapsto |x| \in [0, \infty), \quad (3.1)$$

with the following properties

(i) $|x| = |x^{-1}|$ for all $x \in G$,

(ii) $|\lambda x| = \lambda |x|$ for all $x \in G$ and $\lambda > 0$,

(iii) $|x| = 0$ if and only if $x = 0$.

Let us also recall the following well-known fact about quasi-norms.

**Proposition 3.1 (e.g. [17], proposition 3.1.38 and [43], proposition 1.2.4).** If $| \cdot |$ is a homogeneous quasi-norm on $G$, there exists $C > 0$ such that for every $x, y \in G$, we have

$$|xy| \leq C(|x| + |y|). \quad (3.2)$$

The following polarization formula on homogeneous Lie groups will be used in our proofs: there is a (unique) positive Borel measure $\sigma$ on the unit quasi-sphere $S := \{x \in G : |x| = 1\}$, so that for every $f \in L^1(G)$ we have

$$\int_G f(x) \, dx = \int_0^\infty \int_{S} f(ry)r^{Q-1} \, d\sigma \, dr. \quad (3.3)$$

We refer to Folland & Stein [16] for the original appearance of such groups, to Fischer & Ruzhansky [17] and to Ruzhansky & Suragan [43] for a recent comprehensive treatment. Let us define quasi-ball centred at $x$ with radius $r$ in the following form:

$$B(x, r) := \{y \in G : |x^{-1}y| < r\}. \quad (3.4)$$

(a) Reverse integral Hardy inequality

In this subsection, we show the reverse integral Hardy inequality on homogeneous Lie groups.

**Theorem 3.2.** Let $G$ be a homogeneous Lie group of homogeneous dimension $Q$ with a quasi-norm $| \cdot |$. Assume that $q \leq p < 0$ and $\alpha, \beta \in \mathbb{R}$. Then the reverse integral Hardy inequality

$$\left[ \int_G \left( \int_{B(0,|x|)} f(y) \, dy \right)^q |x|^{\alpha} \, dx \right]^{1/q} \geq C_1 \left( \int_G f^\beta(x) |x|^\beta \, dx \right)^{1/p}, \quad (3.5)$$

holds for some $C_1 > 0$ and for all non-negative measurable functions $f$, if $\alpha + Q > 0$, $\beta(1-p') + Q > 0$ and $(Q + \alpha)/q + (Q + \beta(1-p'))/p' = 0$, where $1/p + 1/p' = 1$. Moreover, the biggest constant $C_1$ for (3.4) satisfies

$$\left( \frac{|S|}{\alpha + Q} \right)^{1/q} \left( \frac{|S|}{Q + \beta(1-p')} \right)^{1/p'} \geq C_1 \geq |p|^{1/q}(p')^{1/p'} \left( \frac{|S|}{\alpha + Q} \right)^{1/q} \left( \frac{|S|}{Q + \beta(1-p')} \right)^{1/p'}.$$
Proof. Let us show that the condition (2.4) is satisfied with $u(x) = |x|^\alpha$ and $v(x) = |x|^\beta$. We calculate the first integral in (2.4):

$$
\int_{B(0,|x|)} u(y) \, dy = \int_{B(0,|x|)} |y|^\alpha \, dy \overset{(3.2)}{=} r^{|x|} \int_0^1 r^Q \, r^{Q-1} \, dr \, d\sigma = \frac{|\mathcal{S}|}{Q + \alpha} |x|^{Q + \alpha},
$$

(3.6)

where $|\mathcal{S}|$ is the area of the unit quasi-sphere in $G$. Then,

$$
\int_{B(0,|x|)} v^{1-p'}(y) \, dy = \int_{B(0,|x|)} |y|^{\beta(1-p')} \, dy

\overset{(3.2)}{=} \int_0^{|x|} r^{\beta(1-p')} r^{Q-1} \, dr \, d\sigma

= \frac{|\mathcal{S}|}{Q + \beta(1-p')} |x|^{Q + \beta(1-p')}.
$$

Finally, by using the above facts and $(Q + \alpha)/q + (Q + \beta(1-p'))/p' = 0$, we have

$$
D_1(|x|) = \left( \frac{|\mathcal{S}|}{\alpha + Q} \right)^{1/q} \left( \frac{|\mathcal{S}|}{Q + \beta(1-p')} \right)^{1/p'} [ |x|^{(Q+\alpha)/q+(Q+\beta(1-p'))/p'} ]

= \left( \frac{|\mathcal{S}|}{\alpha + Q} \right)^{1/q} \left( \frac{|\mathcal{S}|}{Q + \beta(1-p')} \right)^{1/p'},
$$

which shows that $D_1(|x|)$ is a non-decreasing function. Then

$$
D_1 = \inf_{x \neq 0} D_1(|x|) = \left( \frac{|\mathcal{S}|}{\alpha + Q} \right)^{1/q} \left( \frac{|\mathcal{S}|}{Q + \beta(1-p')} \right)^{1/p'} > 0.
$$

Therefore, by (2.5) we have

$$
D_1 \geq C_1 \geq |p|^{1/q(p')}^{1/p'} D_1,
$$

where $D_1 = (|\mathcal{S}|/(\alpha + Q))^{1/q}(|\mathcal{S}|/(Q + \beta(1-p')))^{1/p'}$ thereby, completing the proof.

Now we obtain the conjugate reverse integral Hardy inequality on homogeneous Lie groups.

**Theorem 3.3.** Let $G$ be a homogeneous Lie group of homogeneous dimension $Q$ with a quasi-norm $| \cdot |$. Assume that $q \leq p < 0$ and $\alpha, \beta \in \mathbb{R}$. Then the reverse conjugate integral Hardy inequality

$$
\left[ \int_G \left( \int_{G \setminus B(0,|x|)} f(y) \, dy \right)^q |x|^\alpha \, dx \right]^{1/q} \geq C_2 \left( \int_G f^p(x) |x|^\beta \, dx \right)^{1/p},
$$

(3.7)

holds for some $C_2 > 0$ and for all non-negative measurable functions $f$, if $\alpha + Q < 0$, $\beta(1-p') + Q < 0$ and $(Q + \alpha)/q + (Q + \beta(1-p'))/p' = 0$. Moreover, the biggest constant $C_2$ for (3.6) satisfies

$$
\left( \frac{|\mathcal{S}|}{|\alpha + Q|} \right)^{1/q} \left( \frac{|\mathcal{S}|}{Q + \beta(1-p')} \right)^{1/p'}

\geq C_2 \geq |p|^{1/q(p')}^{1/p'} \left( \frac{|\mathcal{S}|}{|\alpha + Q|} \right)^{1/q} \left( \frac{|\mathcal{S}|}{Q + \beta(1-p')} \right)^{1/p'}.
$$

Proof. Proof of this theorem is similar to the previous case, where we use theorem 2.4 instead of theorem 2.2.

(b) The reverse Hardy–Littlewood–Sobolev inequality and Stein–Weiss inequality

In this subsection, we obtain the reverse Hardy–Littlewood–Sobolev inequality and Stein–Weiss inequality on Euclidean space and homogeneous Lie groups.
Let us introduce the Riesz operator on homogeneous Lie groups in the following form:

\[ I_{\lambda E}(u)(x) = |x|^{-\lambda} u(x) = \int_{E} |y|^{-\lambda} u(y) \, dy, \quad \lambda < 0, \quad (3.8) \]

where \( \ast \) is the convolution. Hence, by taking \( G = (\mathbb{R}^n, +) \), \( Q = n \) and \( | \cdot | = | \cdot |_E \) (\( | \cdot |_E \) is the Euclidean distance), we get the Riesz operator on Euclidean space:

\[ I_{\lambda E}(u)(x) = |x|^{-\lambda} u(x) = \int_{E} |x - y|^{-\lambda} u(y) \, dy, \quad \lambda < 0. \quad (3.9) \]

Firstly, let us present the Hardy–Littlewood–Sobolev inequality on Euclidean space.

**Theorem 3.4 (The reverse Hardy–Littlewood–Sobolev inequality on \( \mathbb{R}^n \)).** Assume that \( n \geq 1 \), \( q < p < 0, \lambda < 0 \) such that \( 1/p' + 1/q + \lambda/n = 0 \), where \( 1/p + 1/p' = 1 \) and \( 1/q + 1/q' = 1 \). Then for all non-negative functions \( f \in L^{q'}(\mathbb{R}^n) \) and \( 0 < \int_{\mathbb{R}^n} h^p(x) \, dx < \infty \), we get

\[
\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} f(x) |x - y|^{-\lambda} h(y) \, dy \right) \, dx \geq C \left( \int_{\mathbb{R}^n} f^{q'}(x) \, dx \right)^{1/q} \left( \int_{\mathbb{R}^n} h^p(x) \, dx \right)^{1/p},
\]

where \( C \) is a positive constant independent of \( f \) and \( h \).

**Proof.** By using the reverse Hölder inequality with \( 1/q + 1/q' = 1 \), we calculate

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) |x - y|^{-\lambda} h(y) \, dx \, dy \geq \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |x - y|^{-\lambda} h(y) \, dy \right)^q \, dx \right)^{1/q} \| f \|_{L^{q'}(\mathbb{R}^n)}^{q'}. \]

Thus for (3.9), it is enough to show that

\[
\left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |x - y|^{-\lambda} h(y) \, dy \right)^q \, dx \right)^{1/q} \geq C \left( \int_{\mathbb{R}^n} h^p(x) \, dx \right)^{1/p}.
\]

By direct calculation, we have

\[
\left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |x - y|^{-\lambda} h(y) \, dy \right)^q \, dx \right)^{1/q} \geq \left( \int_{\mathbb{R}^n} \left( \int_{B_{E}(0,|x|_E)} |x - y|^{-\lambda} h(y) \, dy \right)^q \, dx \right)^{1/q},
\]

where \( B_{E}(0,|x|_E) \) is the Euclidean ball centred at \( 0 \) with radius \( |x|_E \). By using \( |y|_E \leq |x|_E \), we get

\[
|x - y|_E \leq |x|_E + |y|_E \leq 2|x|_E. \quad (3.12)
\]

Then for any \( \lambda < 0 \), we have

\[
\left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |x - y|^{-\lambda} h(y) \, dy \right)^q \, dx \right)^{1/q} \geq \left( \int_{\mathbb{R}^n} \left( \int_{B_{E}(0,|x|_E)} |x - y|^{-\lambda} h(y) \, dy \right)^q \, dx \right)^{1/q} \geq 2^{\lambda} \left( \int_{\mathbb{R}^n} |x|^{-\lambda q} \left( \int_{B_{E}(0,|x|_E)} h(y) \, dy \right)^q \, dx \right)^{1/q}, \quad (3.13)
\]

If condition (2.4) in theorem 2.2 with \( u(x) = |x|^{-\lambda q} \) and \( v(x) = 1 \) in (2.3) is satisfied, then we have

\[
\left( \int_{\mathbb{R}^n} |x|^{-\lambda q} \left( \int_{B_{E}(0,|x|_E)} h(y) \, dy \right)^q \, dx \right)^{1/q} \geq C \left( \int_{\mathbb{R}^n} h^p(x) \, dx \right)^{1/p}.
\]

Let us show that the condition (2.4) is satisfied. From the assumption, we have

\[
0 = \frac{1}{p'} + \frac{1}{q} + \frac{\lambda}{n} \geq \frac{1}{q} + \frac{\lambda}{n},
\]

which means \( n + \lambda q > 0 \). By using this fact, we obtain

\[
\int_{B_{E}(0,|x|_E)} u(y) \, dy = \int_{B(0,|x|_E)} |y|^{-\lambda q} \, dy \quad \overset{(3.2)}{=} \int_{0}^{r|x|_E} r^{\lambda q - n - 1} \, dr \, d\sigma
\]
which implies
\begin{equation}
\frac{|\mathcal{S}|}{n + \lambda q} |x|_{E}^{n + \lambda q},
\end{equation}
and
\begin{equation}
\int_{B_{E}(0,|x|_{E})} v^{1-p'}(y) \, dy = \int_{B_{E}(0,|x|_{E})} 1 \, dy = |\mathcal{S}| |x|_{E}^{n}.
\end{equation}

Finally, by using the assumption $1/p' + 1/q + \lambda/n = 0$,
\begin{equation}
D_{1}(|x|_{E}) = \left( \frac{|\mathcal{S}|}{n + \lambda q} \right)^{1/q} (|\mathcal{S}|)^{1/p'} |x|_{E}^{(n/p')+(n+\lambda q)/q} = \left( \frac{|\mathcal{S}|}{n + \lambda q} \right)^{1/q} |\mathcal{S}|^{1/p'},
\end{equation}
which implies $D_{1}(|x|_{E})$ is a non-decreasing function. Thus,
\begin{equation}
D_{1} = \inf_{x \neq 0} D_{1}(|x|_{E}) = \left( \frac{|\mathcal{S}|}{n + \lambda q} \right)^{1/q} |\mathcal{S}|^{1/p'} > 0,
\end{equation}
completing the proof.

\textbf{Remark 3.5.} Inequality (3.10) seems to be new even in the Euclidean space.

Also, let us now present the reverse Hardy–Littlewood–Sobolev inequality on $G$.

\textbf{Theorem 3.6 (The reverse Hardy–Littlewood–Sobolev inequality on $G$).} Let $G$ be a homogeneous Lie group of homogeneous dimension $Q \geq 1$ with arbitrary quasi-norm $| \cdot |$. Assume that $q < p < 0$, $\lambda < 0$ such that $1/p' + 1/q + \lambda/Q = 0$, where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. Then for all non-negative functions $f \in L^{q}(G)$ and $0 < \int_{G} h^{p}(x) \, dx < \infty$, we get
\begin{equation}
\int_{G} \int_{G} f(x)|y|^{-1} |h(y)|^{1/q} \, dy \, dx \geq C \left( \int_{G} f^{q}(x) \, dx \right)^{1/q} \left( \int_{G} h^{p}(x) \, dx \right)^{1/p},
\end{equation}
where $C$ is a positive constant independent of $f$ and $h$.

\textbf{Proof.} The proof of this theorem is similar to theorem 3.4, but here we use proposition 3.1 and the polar decomposition formula (3.3).

Let us now show the reverse Stein–Weiss inequality on $\mathbb{R}^{n}$.

\textbf{Theorem 3.7 (The reverse Stein–Weiss inequality on $\mathbb{R}^{n}$).} Assume that $n \geq 1$, $q \leq p < 0$, $\lambda < 0$ and $1/p' + 1/q + (\alpha + \beta + \lambda)/n = 0$, where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. Then for all non-negative functions $f \in L^{q}(\mathbb{R}^{n})$ and $0 < \int_{\mathbb{R}^{n}} h^{p}(x) \, dx < \infty$, we have
\begin{equation}
\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} |x|_{E}^{\alpha} f(x) |x - y|_{E}^{\beta} h(y) |y|_{E}^{\beta} \, dx \, dy \geq C \left( \int_{\mathbb{R}^{n}} f^{q}(x) \, dx \right)^{1/q} \left( \int_{\mathbb{R}^{n}} h^{p}(x) \, dx \right)^{1/p},
\end{equation}
if one of the following conditions is satisfied:
\begin{enumerate}
\item[(a)] $\beta > -\frac{n}{p'}$,
\item[(b)] $\alpha > -\frac{n}{q}$.
\end{enumerate}

\textbf{Proof.} Similarly to theorem 3.3, by using the reverse Hölder inequality with $1/q + 1/q' = 1$, we calculate
\begin{align*}
&\int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} |x|_{E}^{\alpha} f(x) |x - y|_{E}^{\beta} h(y) |y|_{E}^{\beta} \, dy \, dx \\
&= \int_{\mathbb{R}^{n}} \left( \int_{\mathbb{R}^{n}} |x|_{E}^{\alpha} |x - y|_{E}^{\beta} h(y) |y|_{E}^{\beta} \, dy \right) f(x) \, dx \\
&\geq \left( \int_{\mathbb{R}^{n}} \left( \int_{\mathbb{R}^{n}} |x|_{E}^{\alpha} |x - y|_{E}^{\beta} h(y) |y|_{E}^{\beta} \, dy \right)^{q} \, dx \right)^{1/q} \|f\|_{L^{q}(\mathbb{R}^{n})}.
\end{align*}
Thus for (3.18), it is enough to show that
\[
\left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |x|^q |x - y|^2 h(y) |y|^\beta \, dy \right)^q \, dx \right)^{1/q} \geq C \left( \int_{\mathbb{R}^n} |h^p(x) \, dx \right)^{1/p},
\]
and by substituting \(z(y) = h(y)|y|^{\beta}\), this is equivalent to
\[
\int_{\mathbb{R}^n} \frac{1}{|x|^q} (|x - y|^2 z(y) \, dy)^q \, dx \leq C \left( \int_{\mathbb{R}^n} |y|^{-\beta p} z^p(x) \, dx \right)^{q/p}.
\]
We have that
\[
\int_{\mathbb{R}^n} \frac{1}{|x|^q} (|x - y|^2 z(y) \, dy)^q \, dx \geq \int_{B(0,|x|)} \frac{1}{|x|^q} (|x - y|^2 z(y) \, dy)^q \, dy,
\]
then
\[
\left( \int_{\mathbb{R}^n} |x|^q |x - y|^2 z(y) \, dy \right)^q \leq \left( \int_{B(0,|x|)} |x|^q |x - y|^2 z(y) \, dy \right)^q.
\]
Therefore, we obtain
\[
\left( \int_{\mathbb{R}^n} |x|^q \left( \int_{\mathbb{R}^n} |x - y|^2 z(y) \, dy \right)^q \, dx \right)^{1/q} \geq \left( \int_{\mathbb{R}^n} |x|^q \left( \int_{B(0,|x|)} |x - y|^2 z(y) \, dy \right)^q \, dx \right)^{1/q} := I_1^n.
\]
Similarly to (3.19), we have
\[
\left( \int_{\mathbb{R}^n} |x|^q \left( \int_{\mathbb{R}^n} |x - y|^2 z(y) \, dy \right)^q \, dx \right)^{1/q} \geq \left( \int_{\mathbb{R}^n} |x|^q \left( \int_{\mathbb{R}^n \setminus B(0,|x|)} |x - y|^2 z(y) \, dy \right)^q \, dx \right)^{1/q} := I_2^n.
\]
From (3.19)–(3.20), we obtain
\[
\left( \int_{\mathbb{R}^n} |x|^q \left( \int_{\mathbb{R}^n} |x - y|^2 z(y) \, dy \right)^q \, dx \right)^{1/q} \geq I_1^n, \quad (3.21)
\]
and
\[
\left( \int_{\mathbb{R}^n} |x|^q \left( \int_{\mathbb{R}^n \setminus B(0,|x|)} |x - y|^2 z(y) \, dy \right)^q \, dx \right)^{1/q} \geq I_2^n. \quad (3.22)
\]

**Step 1.** Let us prove (a) for (3.21). By using \(|y|_E \leq |x|_E\), we get
\[
|x - y|_E \leq |x|_E + |y|_E \leq 2|x|_E. \quad (3.23)
\]
Then for any \(\lambda < 0\), we have
\[
2^{\lambda} |x|_E^{\lambda} \leq |x - y|_E. \quad (3.24)
\]
Therefore, we get
\[
I_1 = \int_{\mathbb{R}^n} |x|^q \left( \int_{B(0,|x|)} |x - y|^2 z(y) \, dy \right)^q \, dx \leq 2^{\lambda q} \int_{\mathbb{R}^n} |x|^{(\alpha + \lambda)q} \left( \int_{B(0,|x|)} z(y) \, dy \right)^q \, dx.
\]
If condition (2.4) in theorem 2.2 with \(u(x) = |x|^{\alpha+\lambda} q \) and \(v(y) = |y|^{-\beta p} \) in (2.3) is satisfied, then we have
\[
I_1 \leq C \int_{\mathbb{R}^n} \left( \int_{B(0,|x|)} z(y) \, dy \right)^q |x|^{\alpha+\lambda} q \, dx \leq C \left( \int_{\mathbb{R}^n} |y|^{-\beta p} z^p(y) \, dy \right)^{q/p}.
\]
Let us verify that the condition (2.4) holds. By using the assumption \(\beta > -\frac{n}{p'}\), we obtain
\[
0 = \frac{1}{p'} + \frac{1}{q} + \frac{\alpha + \beta + \lambda}{n} > \frac{1}{q} + \frac{\alpha + \lambda}{n},
\]
that is, \((n + (\alpha + \lambda)q)/nq < 0\), or \(n + (\alpha + \lambda)q > 0\). Then, we get
\[
\left( \int_{B_1(0,|x|_E)} u(y) \, dy \right)^{1/q} = \left( \int_{B_1(0,|x|_E)} |y|^{(\alpha + \lambda)q} \, dy \right)^{1/q}
\]
\[
= \left( \frac{|\mathcal{S}|}{n + (\alpha + \lambda)q} \right)^{1/q} |x|^{(n + (\alpha + \lambda)q)/q}.
\]
Since \(\beta > -\frac{n}{p'}\), we have
\[-\beta p(1 - p') + n = \beta p' + n > 0.\]
Thus \(-\beta p(1 - p') + n > 0\). Then, a direct computation gives
\[
\left( \int_{B_1(0,|x|_E)} v^{1-p'}(y) \, dy \right)^{1/p'} = \left( \int_{B_1(0,|x|_E)} |y|^{-\beta p(1-p')} \, dy \right)^{1/p'}
\]
\[
= \left( \frac{|\mathcal{S}|}{\beta p' + n} \right)^{1/p'} |x|^{(\beta p' + n)/p'}. \tag{3.24}
\]
Therefore by using \(1/p' + 1/q + (\alpha + \beta + \lambda)/n = 0\), we have
\[
\mathcal{D}_1(|x|_E) = \left( \int_{B_1(0,|x|_E)} u(y) \, dy \right)^{1/q} \left( \int_{B_1(0,|x|_E)} v^{1-p'}(y) \, dy \right)^{1/p'}
\]
\[
= \left( \frac{|\mathcal{S}|}{n + (\alpha + \lambda)q} \right)^{1/q} \left( \frac{|\mathcal{S}|}{\beta p' + n} \right)^{1/p'} > 0.
\tag{3.25}
\]
Then by using (2.3), we obtain
\[
I_1^{1/q} \geq C \left( \int_{\mathbb{R}^n} \left| y \right|^{-\beta p} \, dy \right)^{1/p} = C \left( \int_{\mathbb{R}^n} h^p(y) \, dy \right)^{1/p}. \tag{3.26}
\]
**Step 2.** Let us prove (b) for (3.22). From \(|x|_E \leq |y|_E\), we calculate
\[
|x - y|_E \leq |x|_E + |y|_E \leq 2|y|_E,
\]
then
\[
|x - y|_E^2 \geq C |y|_E^2,
\]
where \(C > 0\). Then, if condition (2.4) with \(u(x) = |x|^a_E\) and \(v(y) = |y|^{-(\beta + \lambda)p}\) is satisfied, we have
\[
I_2 \leq C \int_{\mathbb{R}^n} |x|^a_E \left( \int_{\mathbb{R}^n \setminus B_1(0,|x|)} z(y)|y|^{\lambda}_E \, dy \right)^q \, dx \leq C \left( \int_{\mathbb{R}^n} |y|^{-\beta p} \, dy \right)^{q/p}.
\]
Now let us check that the condition (2.4) holds. We have
\[
\left( \int_{\mathbb{R}^n \setminus B_1(0,|x|_E)} u(y) \, dy \right)^{1/q} = \left( \int_{\mathbb{R}^n \setminus B_1(0,|x|_E)} |y|^a_E \, dy \right)^{1/q} = \left( \int_{|x|_E}^{\infty} \int_{|\mathcal{S}|} r^aq \, dr \, d\sigma \right)^{1/q}
\]
\[
= \left( \frac{|\mathcal{S}|}{|n + aq|} \right)^{1/q} |x|^{(n + aq)/q},
\]
where \(n + aq < 0\). From \(\alpha > -n/q\), we have \(0 = 1/p' + 1/q + (\alpha + \beta + \lambda)/n > 1/p' + (\beta + \lambda)/n\), then
\[
(\beta + \lambda)p' + n < 0. \tag{3.27}
\]
By using this fact, we have
\[
\left( \int_{\mathbb{R}^n \setminus B_1(0,|x|_E)} v^{1-p'}(y) \, dy \right)^{1/p'} = \left( \int_{\mathbb{R}^n \setminus B_1(0,|x|)} |y|^{-(\beta + \lambda)(1-p')} \, dy \right)^{1/p'}
\]
Then by using \(1/p' + 1/q + (\alpha + \beta + \lambda)/n = 0\), we get
\[
D_2(|x| E) = \left(\frac{|\mathcal{G}|}{|n + (\beta + \lambda)p'|}\right)^{1/q} \left(\frac{|\mathcal{G}|}{|n + (\beta + \lambda)p'|}\right)^{1/p'}.
\]
which means \(D_2(|x| E)\) is a non-increasing function. Therefore, we have
\[
D_2 = \inf_{x \neq 0} D_2(|x| E) = \left(\frac{|\mathcal{G}|}{|n + \alpha q|}\right)^{1/q} \left(\frac{|\mathcal{G}|}{|n + (\beta + \lambda)p'|}\right)^{1/p'} > 0.
\]
Then, we have
\[
\int_2^{1/q} C\left(\int_{\mathbb{R}^n} |y|_{E}^{-\beta p} f(y) \, dy\right)^{1/p} = C\left(\int_{\mathbb{R}^n} h^p(y) \, dy\right)^{1/p}.
\]

**Remark 3.8.** Inequality (3.18) seems to be new even in the Euclidean space.

Let us now show the reverse Stein–Weiss inequality \(G\).

**Theorem 3.9 (The reverse Stein–Weiss inequality on \(G\)).** Let \(G\) be a homogeneous group of homogeneous dimension \(Q \geq 1\) and let \(| \cdot |\) be an arbitrary homogeneous quasi-norm on \(G\). Assume that \(q \leq p < 0, \lambda < 0\) and \(1/p' + 1/q + (\alpha + \beta + \lambda)/Q = 0\), where \(1/p + 1/p' = 1\) and \(1/q + 1/q' = 1\). Then for all non-negative functions \(f \in L^q(G)\) and \(0 < \int_{G} h^p(x) \, dx < \infty\), we have
\[
\int_{G} \int_{G} |x|_{E}^{-\beta p} f(x) |y|_{E}^{-\alpha p} h(y) \, dx \, dy \geq C\left(\int_{G} f^q(x) \, dx\right)^{1/q} \left(\int_{G} h^p(x) \, dx\right)^{1/p},
\]
if one of the following conditions is satisfied:

(a) \(\beta > -\frac{Q}{p'}\),
(b) \(\alpha > -\frac{Q}{q}\).

**Proof.** The proof is similar to the previous theorem, but here we use proposition 3.1 and the polar decomposition formula (3.3).

Data accessibility. This article does not contain any additional data.

Authors’ contributions. A.K.: writing—original draft; M.R.: supervision, writing—review and editing; D.S.: writing—review and editing.

All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

Conflict of interest declaration. We declare we have no competing interests.

Funding. The first and second authors were supported in parts by the FWO Odysseus 1 grant no. G.0H94.18N: Analysis and Partial Differential Equations, by the Methusalem programme of the Ghent University Special Research Fund (BOF) (grant no. 01M01021) and by the EPSRC (grant no. EP/R003025/2). Also, this research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant no. AP19676031) and partially supported by the collaborative research programme ‘Qualitative analysis for nonlocal and fractional models’ from Nazarbayev University.

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