LEFSCHETZ PROPERTIES FOR CW-COMPLEX NAGATA IDEALIZATIONS

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Abstract. We introduce a novel construction which allows us to identify the elements of the skeletons of a CW-complex $P(m)$ and the monomials in $m$ variables. From this, we infer that there is a bijection between finite CW-subcomplexes of $P(m)$, which are quotients of finite simplicial complexes, and some bigraded standard Artinian Gorenstein algebras, generalizing previous constructions in [3, 4] and [8].

We apply this to a generalization of Nagata idealization for level algebras. These algebras are standard graded Artinian algebras whose Macaulay generator is given explicitly as a bigraded polynomial of bidegree $(1, d)$. We consider the algebra associated to polynomials of the same type of bidegree $(d_1, d_2)$.

Introduction

Let $X = V(f) \subset \mathbb{P}_K^N$ be a hypersurface, where the underlying field $K$ has characteristic 0; the Hessian determinant of $f$ (for short Hessian of $f$ or, by abusing language, Hessian of $X$) is the determinant of the Hessian matrix of $f$.

Hypersurface with vanishing Hessian were studied for the first time in 1851 by O. Hesse; he wrote two papers ([12, 13]) according to which these hypersurfaces should be necessarily cones. But in 1876 Gordan and Noether ([9]) proved that Hesse’s claim is true for $N \leq 3$, and it is false for $N \geq 4$. They and Franchetta classified all the counterexamples to Hesse’s claim in $\mathbb{P}^4$ (see [9, 4, 5, 7]). In 1900, Perazzo classified cubic hypersurfaces with vanishing Hessian for $N \leq 6$ ([15]). This work was studied and generalized in [6, and the problem is still open in the spaces of greater dimension.

Hessians of higher degree were introduced in [14] and used to control the so called Strong Lefschetz Properties (for short, SLP). The Lefschetz properties have attracted a great attention in the last years. The basic papers of the algebraic theory of Lefschetz properties were the original ones of Stanley [16, 17, 18] and the book of Watanabe and others [10].

An algebraic tool that we find many times in these papers is the Nagata Idealization: it is a tool to convert any module $M$ over a (commutative) ring (with unit) $R$ in an ideal of another ring $R \ltimes M$. The starting point is the isomorphism between the idealization of an ideal $I = (g_0, \ldots, g_n)$ of $K[u_1, \ldots, u_m]$ and its level algebra see [10] Definition 2.72. In this way, the new ring is a Standard Graded Artinian Gorenstein Algebra (SGAG algebra, for short). An explicit formula for the Macaulay generator $f$ is:

$$f = \sum x_i g_i \in K[x_0, \ldots, x_n, u_1, \ldots, u_m]_{(1,d)}.$$  

A generalization of this construction can be to consider polynomials of the form:

$$f = \sum x_i^d g_i \in K[x_0, \ldots, x_n, u_1, \ldots, u_m]_{(d,d+1)};$$

these are called Nagata polynomials of degree $d$. The Lefschetz properties for the relevant associated algebras $A$, the geometry of Nagata hypersurfaces of degree $d$, the interaction between the combinatorics of $f$ and the structure of $A$ were studied in [11], where the $g_i$’s are square free monomials, using a simplicial complex associated to $f$. 

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Date: May 5, 2020.

2010 Mathematics Subject Classification. Primary 13A30, 05E40; Secondary 57Q05, 13D40, 13A02, 13E10.

Key words and phrases. Lefschetz properties, Artinian Gorenstein Algebra, Nagata idealization, CW-complex.

P.D.P. & G.I. are members of INdAM - GNSAGA and P.D.P is supported by PRIN2017 “Advances in Moduli Theory and Birational Classification”.

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In this paper we use the CW-complexes, a natural generalization of the simplicial complex to study Nagata polynomials of bidegree \((d_1, d_2)\). We study the Hilbert vector and we give a complete description of the ideal \(I\) for every case, also if the \(q_i\)'s are not square free monomials.

The geometry of the Nagata hypersurface is very similar to the geometry of the hypersurfaces with vanishing Hessian.

More precisely, we introduce a novel Construction 3.14 which allows us to identify each (monic) monomial of degree \(d\) in \(m\) variables with an element of the \((d - 1)\)-skeleton of an interesting CW-complex that we have called \(P(m)\). This CW-complex is constructed by generalizing the construction introduced in [3] which associates to a (monic) square-free monomial in \(m\) variables of degree \(d\) a unique \((d - 1)\)-cell of the simplex of dimension \(m - 1\), and vice versa. In a few words, we consider an \(h\)-power \(u^h\) as a product of \(h\) linear forms: \(\hat{u}_1 \cdots \hat{u}_h\); this corresponds to a \((h - 1)\)-simplex, and we identify all the \(\delta\)-faces, with \(\delta < h - 1\), of this simplex to just one \(\delta\)-face, recursively, starting from \(\delta = 0\) to \(\delta = h - 2\): for \(\delta = 0\) we identify all the points to one, then if \(\delta = 1\) we obtain a bouquet of \(h\)-circles, and we identify all these circles, and so on. Generalizing this construction to a general monic monomial and attaching the corresponding CW-complexes along the common skeletons, we obtain \(P(m)\).

The paper is organized as follows: in Section 1 we recall some generalities about graded Artinian Gorenstein Algebras and Lefschetz Properties, with their connections with the vanishings of higher order Hessians. In Section 2 we remind what the Nagata idealization is, what we intend for higher Nagata idealization and we show its connection with the Lefschetz Properties for bihomogeneous polynomials. Section 3 is the core of this article, and contains its original part. After recalling some generalities about bigraded algebras and the topological definitions that we need, we give the construction of the CW-complex \(P(m)\); then, we apply it to the so called Nagata polynomials (see Definition 2.9) in Theorems 3.20 and 3.22 which give Theorem 3.20 a precise description of the Artinian Gorenstein Algebra associated to a Nagata polynomial and Theorem 3.22 the generators of the annihilator of the polynomial. We show that from these theorems follows a generalization of the principal results of [1], see Corollaries 3.21 and 3.23.

We think that the study of the Nagata hypersurfaces can be—among other things—an useful tool for the classification of the hypersurfaces with vanishing Hessian in \(\mathbb{P}^n\).

**Notations.** In this the paper we fix the following notations and assumptions:

- we will work on \(\mathbb{K}\), a field of characteristic 0.
- \(R := \mathbb{K}[x_0, \ldots, x_n]\) will always be the ring of polynomials in \(n + 1\) variables \(x_0, \ldots, x_n\).
- \(Q := \mathbb{K}[X_0, \ldots, X_n]\) will be the ring of differential operators of \(R\), where \(X_i = \frac{\partial}{\partial x_i}\).
- The subscript of a graded \(\mathbb{K}\)-algebra will indicate, as it is custom, the part of that degree; so, for instance, \(R_d\) is the \(\mathbb{K}\)-vector space of the homogeneous polynomials of degree \(d\), and \(Q_\delta\) the \(\mathbb{K}\)-vector space of the homogeneous differential operators of order \(\delta\).

### 1. Graded Artinian Gorenstein Algebras and Lefschetz Properties

#### 1.1. Graded Artinian Gorenstein Algebras are Poincaré Algebras.

**Definition 1.1.** Let \(I\) be a homogeneous ideal of \(R\) such that \(A = R/I = \bigoplus_{i=0}^d A_i\) is a graded Artinian \(\mathbb{K}\)-algebra, where \(A_d \neq 0\). \(d\) is the socle degree of \(A\). \(A\) is said standard if \(I_1 = 0\) and therefore \(A\) is generated in degree 1. Setting \(h_i = \dim \mathbb{K} A_i\), the vector \(\text{Hilb}(A) = (1, h_1, \ldots, h_d)\) is called Hilbert vector of \(A\). Since \(I_1 = 0\), then \(h_1 = n + 1\) is called codimension of \(A\).

We also recall the following definitions.

**Definition 1.2.** A graded Artinian \(\mathbb{K}\)-algebra \(A = \bigoplus_{i=0}^d A_i\) is a Poincaré algebra if \(\cdot : A_i \times A_{d-i} \to A_d\) is a perfect pairing for \(i \in \{0, \ldots, d\}\).

**Definition 1.3.** A graded Artinian \(\mathbb{K}\)-algebra \(A\) is Gorenstein if (and only if) \(\dim \mathbb{K} A_d = 1\) and it is a Poincaré algebra.
Remark 1.4. The Hilbert vector of a Poincaré algebra $A$ is symmetric with respect to $h$,
that is $\text{Hilb}(A) = (1, h_1, h_2, \ldots, h_2, h_1, 1)$.

1.2. Graded Artinian Gorenstein Quotient Algebras of $Q$. For any $d \geq \delta \geq 0$ there exists a natural $\mathbb{K}$-bilinear map $B: R_d \times Q_\delta \rightarrow R_{d-\delta}$ defined by differentiation

$$B(f, \alpha) = \alpha(f)$$

**Definition 1.5.** Let $\mathcal{I}$ be a homogeneous ideal of $Q$. The annihilator of $\mathcal{I}$ in $Q$ is the following homogeneous ideal

$$\text{Ann}(\mathcal{I}) := \{ \alpha \in Q \mid \forall f \in \mathcal{I}, \alpha(f) = 0 \}.$$ 

In particular, if $\mathcal{I}$ is generated by a homogeneous element $f$, we shall write for simplicity $\text{Ann}(\mathcal{I}) = \text{Ann}(f)$.

Let $A = Q/\text{Ann}(f)$, where $f$ is homogeneous, by construction $A$ is a standard graded Artinian $\mathbb{K}$-algebra; moreover $A$ is Gorenstein as the following theorem states.

**Theorem 1.6** ([14], Theorem 2.1 and Remark 2.3). Let $I$ be a homogeneous ideal of $Q$ such that $A = Q/I$ is a standard Artinian graded $\mathbb{K}$-algebra. Then $A$ is Gorenstein if and only if there exist $d \geq 1$ and $f \in R_d$ such that $A \cong Q/\text{Ann}(f)$.

**Remark 1.7.** Using the notation as above, $A$ is called the SGAG $\mathbb{K}$-algebra associated to $f$. The socle degree $d$ of $A$ is the degree of $f$ and the codimension is $n + 1$, since $I_1 = 0$.

1.3. Lefschetz Properties and the Hessian Criterion. Let $A = \bigoplus_{i=0}^{d} A_i$ be a graded Artinian $\mathbb{K}$-algebra.

**Definition 1.8.** If there exists an $L \in A_1$ such that:

a) the multiplication map $\cdot L: A_i \rightarrow A_{i+1}$ is of maximal rank for any $i$, then $A$ has the Weak Lefschetz Property (WLP, for short);

b) the multiplication map $\cdot L^k: A_i \rightarrow A_{i+k}$ is of maximal rank for any $i$ and $k$, then $A$ has the Strong Lefschetz Property (SLP, for short);

**Definition 1.9.** Let $A$ be the SGAG $\mathbb{K}$-algebra associated to an element $f \in R_d$, and let $\mathcal{B}_k = \{ \alpha_j \in A_k \mid j \in \{1, \ldots, \sigma_k\} \}$ be an ordered $\mathbb{K}$-basis of $A_k$. The $k$-th Hessian matrix of $f$ with respect to $\mathcal{B}_k$ is

$$\text{Hess}_f^{\mathcal{B}_k} = (\alpha_i \alpha_j(f))_{i,j=1}^{\sigma_k}.$$ 

The $k$-th Hessian of $f$ with respect to $\mathcal{B}_k$ is

$$\text{hess}_f^{\mathcal{B}_k} = \det \left( \text{Hess}_f^{\mathcal{B}_k} \right).$$

With notations as above, the following theorem holds.

**Theorem 1.10** ([19] Theorem 4). An element $L = a_0 X_0 + \cdots + a_n X_n \in A_1$ is a strong Lefschetz element of $A$ if and only if $\text{hess}_f^{\mathcal{B}_k}(a_0, \ldots, a_n) \neq 0$ for all $k \in \left\{0, \ldots, \left\lfloor \frac{d}{2} \right\rfloor \right\}$. In particular, if for some $k$ one has $\text{hess}_f^{\mathcal{B}_k} = 0$, then $A$ does not have the SLP condition.

2. Higher Order Nagata Idealization

2.1. Nagata Idealizations.

**Definition 2.1.** Let $A$ be a ring and let $M$ be an $A$-module. The Nagata idealization $A \ltimes M$ of $M$ is the ring with support $A \times M$ and operations defined as follow:

$$(r, m) + (s, n) = (r + s, m + n), (r, m) \cdot (s, n) = (rs, sm + rn).$$
We recall that in this way $M \cong \{0\} \times M$ becomes an idempotent ideal of the ring $A \ltimes M$. The following theorem holds.

**Theorem 2.2** ([10], Theorem 2.77). Let $S' := \mathbb{K}[u_1, \ldots, u_m]$ and $S := R \otimes_{\mathbb{K}} S'$ be rings of polynomials, let $T' = \mathbb{K}[U_1, \ldots, U_m]$ and $T := Q \otimes_{\mathbb{K}} T'$ be the associated ring of differential operators, where $X_i = \partial_{x_i}$ and $U_j = \partial_{u_j}$. Let $g_0, \ldots, g_n$ be homogeneous elements of $S'$ of degree $d$, let $I$ be the $T'$-submodule of $S'$ generated by $\{\partial(g_i) \in R \mid \partial \in T, i \in \{0, \ldots, n\}\}$ and let $A' := T'/\text{Ann}(I)$. Let us define $f = x_0 g_0 + \cdots + x_n g_n \in R$, it is a bihomogeneous polynomial of bidegree $(1, d)$, and let $A := T/\text{Ann}(f)$. Considering $I$ as an $A'$-module, one has $A' \ltimes I \cong A$.

### 2.2. Lefschetz Properties for Higher Nagata Idealizations.

**Notations.** Let us fix in what follows the notation as in Theorem 2.2

- $S := R \otimes_{\mathbb{K}} \mathbb{K}[u_1, \ldots, u_m] = \mathbb{K}[x_0, \ldots, x_n, u_1, \ldots, u_m]$ is the bigraded ring of polynomials in $m+n+1$ variables $x_0, \ldots, x_n, u_1, \ldots, u_m$;
- so $S_{(d_1, d_2)}$ is the $\mathbb{K}$-vector space of bihomogeneous polynomials $f$ of bidegree $(d_1, d_2)$; that is, $f$ can be written as $\sum_{i=0}^{p} a_i b_i$, where $a_i \in R_{d_1} = \mathbb{K}[x_0, \ldots, x_n]_{d_1}$ and $b_i \in \mathbb{K}[u_1, \ldots, u_m]_{d_2}$.
- $T := Q \otimes_{\mathbb{K}} \mathbb{K}[U_1, \ldots, U_m] = \mathbb{K}[X_0, \ldots, X_n, U_1, \ldots, U_m]$ is the (bigraded) ring of differential operators of $S$, where $X_i = \partial_{x_i}$ and $U_j = \partial_{u_j}$.

**Definition 2.3.** A bihomogeneous polynomial

$$f = \sum_{i=0}^{n} x_i^{d_1} g_i \in S_{(d_1, d_2)}$$

is called a **Nagata polynomial of degree** $d_1 \geq 1$ if $g_i \in \mathbb{K}[u_1, \ldots, u_m]$, $i = 0, \ldots, n$, are linearly independent monomials of degree $d_2 \geq 2$.

**Remark 2.4.** One needs $n \leq \left(\frac{m + d_2 - 1}{d_2}\right)$ otherwise the $g_i$'s are not linearly independent.

From now on, we assume that $n$ satisfies this condition.

About the SGAG $\mathbb{K}$-algebra $A$, the following propositions hold.

**Proposition 2.5** ([5] Proposition 2.5). Assuming that $n + 1 \geq m \geq 2, d_2 > d_1 \geq 1$ and $s > \left(\frac{m + d_1 - 1}{d_1}\right)$, for any $j \in \{1, \ldots, s\}$, let $f_j \in S_{(d_1, 0)}$, $g_j \in S_{(0, d_2)}$. Then the form $f = f_1 g_1 + \cdots + f_s g_s$, of degree $d_1 + d_2$, satisfies $\text{hess}^{d_1} f = 0$; that is, $A = T/\text{Ann}(f)$ does not have the SLP condition.

**Proposition 2.6** ([11] Proposition 2.7). Assuming that $n + 1 \geq m \geq 2, d_1 \geq d_2$. Then $L = \sum_{i=0}^{n} X_i$ is a Weak Lefschetz Element; that is, $A = T/\text{Ann}(f)$ has the WLP condition.

3. CW-complex Nagata Idealization of Bidegree $(d_1, d_2)$

From now on, $S$ and $T$ are the rings set in the previous subsection.

3.1. Bigraded Artinian Gorenstein Algebras. Let $A = \bigoplus_{i=0}^{d} A_i$ be a SGAG $\mathbb{K}$-algebra, it is bigraded if:

$$A_d = A_{(d_1, d_2)} \cong \mathbb{K}, A_i = \bigoplus_{h=0}^{i} A_{(i, h-i)} \text{ for } i \in \{0, \ldots, d-1\},$$
since $A$ is a Gorenstein ring, and the pair $(d_1, d_2)$ is said the socle bidegree of $A$. In this case, $A$ is said SBAG algebra, for short.

**Remark 3.1.** By Definition 1.3, $A_i \cong A_{d-i}^\vee$ and since the duality commutes with direct sums, one has $A_{(i,j)} \cong A_{(d_1-i,d_2-j)}^\vee$ for any pair $(i, j)$.

A homogeneous ideal $I$ of $S$ is a bihomogeneous ideal if:

$$I = \bigoplus_{i,j=0}^{+\infty} I_{(i,j)}, \text{ where } \forall i, j \in \mathbb{N}_{\geq 0}, I_{(i,j)} = I \cap S_{(i,j)}.$$  

Let $f \in S_{(d_1,d_2)}$, then $I = \text{Ann}(f)$ is a bihomogeneous ideal of $T$; using Theorem 1.6, $A = T/I$ is a SBAG $K$-algebra of socle bidegree $(d_1, d_2)$ (and codimension $m + n + 1$).

**Remark 3.2.** Using the above notations, one has:

$$\forall i > d_1, j > d_2, I_{(i,j)} = T_{(i,j)}.$$  

Indeed, for all $\alpha \in T_{(i,j)}$ with $i > d_1, j > d_2$, $\alpha(f) = 0$; as a consequence:

$$\forall k \in \{0, \ldots, d_1 + d_2\}, A_k = \bigoplus_{0 \leq i \leq d_1, 0 \leq j \leq d_2, i+j=k} A_{(i,j)}.$$  

Moreover, the valuation map $\alpha \in T_{(i,j)} \mapsto \alpha(f) \in A_{(d_1-i,d_2-j)}$ provides the following short exact sequence:

$$0 \longrightarrow I_{(i,j)} \longrightarrow T_{(i,j)} \longrightarrow A_{(d_1-i,d_2-j)} \longrightarrow 0.$$  

**Definition 3.3.** A bihomogeneous Nagata polynomial

$$f = \sum_{i=0}^{n} x_i^{d_1} g_i \in S_{(d_1,d_2)}$$

is called a simplicial Nagata polynomial of degree $d_1$ if the monomials $g_i$ are square free.

**Remark 3.4.** One needs $n \leq \binom{m}{d_2}$ otherwise the $g_i$’s are not square free.

**3.2. CW-complexes and bihomogeneous polynomials.**

**3.2.1. Abstract finite simplicial complexes.**

**Definition 3.5.** Let $V = \{u_1, \ldots, u_m\}$ be a finite set. A(n abstract finite) simplicial complex $\Delta$ with vertex set $V$ is a subset of $2^V$ such that

1) $\forall u \in V \Rightarrow \{u\} \in \Delta,$
2) $\forall \sigma \in \Delta, \tau \subsetneq \sigma, \tau \neq \emptyset \Rightarrow \tau \in \Delta.$

The elements $\sigma$ of $\Delta$ are called faces or simplices; a face with $q + 1$ vertices is called $q$-face or face of dimension $q$ and one writes $\dim \sigma = q$; the maximal faces (with respect to the inclusion) are called facets; if all facets have the same dimension $d \geq 1$ then one says that $\Delta$ is of pure dimension $d$. The set $\Delta^k$ of faces of dimension at most $k$ is called $k$-skeleton of $\Delta$. $2^V$ is called simplex (of dimension $m - 1$).

**Remark 3.6.**
a) (cfr. [11 Remark 3.4]) There is a natural bijection, first introduced, as far as we now, in [3], between the square free monomials, of degree d, in the variables \( u_1, \ldots, u_m \) and the \((d-1)\)-faces of the simplex \( 2^V \), with vertex set \( V = \{ u_1, \ldots, u_m \} \). In fact, a square free monomial \( g = u_{i_1} \cdots u_{i_d} \) corresponds to the subset \( \{ u_{i_1}, \ldots, u_{i_d} \} \) of \( 2^V \). Vice versa, to any subset \( F \) of \( V \) with \( d \) elements one associates the free square monomial \( m_F = \prod_{u_i \in F} u_i \) of degree \( d \).

b) Let \( f = \sum_{i=0}^{n} x_i^{d_i} g_i \in S_{(d_1, d_2)} \) be a simplicial Nagata polynomial; by hypothesis there is bijection between the monomials \( g_i \)'s and the indeterminates \( x_i \)'s. From all this, one can associate to \( f \) a simplicial complex \( \Delta_f \) with vertices \( u_1, \ldots, u_m \) where the facet which corresponds to \( g_i \) is identified with \( x_i^{d_i} \).

3.2.2. CW-complexes. For the topological background, we refer to [11].

**Definition 3.7.** Let \( k \in \mathbb{N}_{\geq 1} \). A topological space \( e^k \) homeomorphic to the open (unitary) ball

\[ \mathbb{B}^k := \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_1^2 + \cdots + x_k^2 < 1\} \]

of dimension \( k \) (structured with natural topology induced by \( \mathbb{R}^{k+1} \)) is called \( k \)-cell.

**Definition 3.8.** A CW-complex is a topological space \( X \) constructed in the following way:

1) there exists a fixed and discrete set of points \( X^0 \subset X \), whose elements are called 0-cells;
2) inductively, the \( k \)-skeleton \( X^k \) of \( X \) is constructed from \( X^{k-1} \) by attaching \( k \)-cells \( e^k_\alpha \) (with index set \( A_k \)) via continuous maps \( \varphi^k_\alpha : S^{k-1} \rightarrow X^{k-1} \) (the attaching maps). This means that \( X^k \) is a quotient of \( Y^k = X^{k-1} \bigcup_{\alpha \in A_k} \mathbb{D}^k_\alpha \) under the identification \( x \sim \varphi^k_\alpha(x) \) for \( x \in \partial\mathbb{D}^k_\alpha \); the elements of the \( k \)-skeleton are the (closure of the) attached \( k \)-cells;
3) \( X = \bigcup_{k \in \mathbb{N}_{\geq 0}} X^k \) and a subset \( C \) of \( X \) is closed if and only if \( C \cap X^k \) is closed for any \( k \) (closed weak topology).

**Definition 3.9.** A subset \( Z \) of a CW-complex \( X \) is a CW-subcomplex if it is the union of cells of \( X \), such that the closure of each cells is in \( Z \).

**Definition 3.10.** A CW-complex is finite if it is obtained by a finite number of cells.

We will be interested mainly in finite CW-complexes.

**Remark 3.11.** a) For any \( k \)-cell \( e^k_\alpha \) one considers the characteristic map \( \Phi^k_\alpha : \mathbb{D}^k_\alpha \hookrightarrow Y^k \rightarrow X^k \hookrightarrow X \), which restricted to \( \mathbb{B}^k_\alpha \) induces a homeomorphism onto \( e^k_\alpha \);

b) vice versa, given the characteristic maps \( \Phi^k_\alpha \), a subset \( C \) of \( X \) is closed if and only if \( (\Phi^k_\alpha)^{-1}(C) \) is closed for any indices \( \alpha \) and \( k \);

c) By previous point, \( X \) is a topological quotient space of \( \bigcup_{k \in \mathbb{N}_{\geq 1}, \alpha \in A} \mathbb{D}^k_\alpha \cup X^0 \).

**Example 3.12** (Geometric realization of an abstract simplicial complex). To any simplicial complex \( \Delta \) one can associate a finite CW-complex \( \tilde{\Delta} \).

Of course \( \Delta^0 = \Delta^0 \). Let \( \alpha = \{u, v\} \in \Delta^1 \) and let \( \tilde{\varphi}^1_\alpha : \mathbb{D}^1_\alpha = [-1, 1] \rightarrow \Delta^0 \bigcup_\beta \mathbb{D}^1_\beta \) be the map such that \( \tilde{\varphi}^1_\alpha|_{[-1, 1]} = \text{id} \) and \( \tilde{\varphi}^1_\alpha(-1) = u, \tilde{\varphi}^1_\alpha(1) = v \); passing to the topological quotient space, one has \( \tilde{\Delta}^1 \). In particular, \( \alpha \) is identified with the 1-cell \( e^1_\alpha \) attached to \( \{u, v\} \). Inductively, let \( \tilde{\Delta}^{k-1} \) be the \((k-1)\)-skeleton of \( \tilde{\Delta} \); by induction, a \( k \)-face \( \alpha \) of \( \Delta \) is associated to \( \mathbb{D}^k \) and \( \partial\mathbb{D}^k \) is

\[ 1 \text{ For clarity, the closed (unitary) disk } \mathbb{D}^k \text{ of dimension } k \text{ is } \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_1^2 + \cdots + x_k^2 \leq 1\}; \mathbb{D}^k \setminus \partial\mathbb{D}^k = \mathbb{B}^k \text{, and for } k \geq 1, \partial\mathbb{D}^k = S^{k-1} = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_1^2 + \cdots + x_k^2 = 1\}.\]
homeomorphic to an element $\tilde{\alpha}^{k-1}$ of $\tilde{\Delta}^{k-1}$, which corresponds to $(k-1)$-skeleton of $\alpha$. Let $\varphi^k_\alpha$ be these homeomorphisms: one can attach any $\alpha$ to $\tilde{\Delta}^{k-1}$ via $\varphi^k_\alpha$’s and obtain $\tilde{\Delta}^k$. From all this, $\tilde{\Delta} = \bigcup_{k \in \mathbb{N}_{\geq 0}} \tilde{\Delta}^k$.

\[ \triangle \]

**Remark 3.13.** Of course what is stated in Example 3.12 is just the geometric realization of the abstract simplicial complex $\Delta$ as a simplicial complex (as a topological space).

In what follows we will always identify abstract simplicial complexes with their corresponding simplicial complexes.

\[ \diamond \]

**Construction 3.14.** We recall that, in Remark 3.6, we have seen that to any degree $d$ square-free monomial $u_{i_1} \cdots u_{i_d} \in K[u_1, \ldots, u_m]_d$ one can associate the $(d-1)$-face $\{u_{i_1}, \ldots, u_{i_d}\}$ of the abstract $(m-1)$-dimensional simplex $\Delta(m) := 2\{u_1, \ldots, u_m\}$, and vice versa: if we call

\[
\rho_d := \{ f \in K[u_1, \ldots, u_m]_d \mid f \neq 0 \text{ is a square-free monic monomial} \}
\]

\[
D(m)_d := \Delta(m)^d \setminus \Delta(m)^{d-1},
\]

we have a bijection

\[
\sigma_d: \rho_d \to D(m)_d
\]

\[
u_{i_1}\cdots u_{i_d} \mapsto \{u_{i_1}, \ldots, u_{i_d}\}.
\]

Alternatively, one can associate to $u_{i_1} \cdots u_{i_d}$ the element of the $(d-1)$-skeleton $\{u_{i_1}, \ldots, u_{i_d}\} \in \Delta(m)_{d-1}$, i.e. we have a bijection

\[
\sigma_d: \rho_d \to \Delta(m)_{d-1}
\]

\[
u_{i_1}\cdots u_{i_d} \mapsto \{u_{i_1}, \ldots, u_{i_d}\}.
\]

between the square-free monomials and the $(d-1)$-faces of the (topological) simplex $\widetilde{\Delta(m)}$.

Using CW-complexes, we want to extend this construction to the non-square-free monic monomials. We can proceed as follows. Let $g := u_{i_1}^1 \cdots u_{i_m}^m$ be a generic degree $d := j_1 + \cdots + j_m$ monomial; we consider the following finite set: $W := \{u_{i_1}^1, u_{i_1}^2, \ldots, u_{i_m}^1, u_{i_m}^2\}$, and if $\Delta(d) := 2^W$ is the abstract associated (finite) simplex, we can consider the corresponding (topological) simplex (which is a CW-complex) $\widetilde{\Delta(d)}$.

Then, if $j_k \leq 1$ we do nothing, while if $j_k \geq 2$, we recursively identify, for $\ell$ varying from 0 to $j_k - 2$, the $\ell$-faces of the subsimplex $2\{u_{i_1}^1, \ldots, u_{i_k}^k\} \subset \widetilde{\Delta(d)}$: start with $\ell = 0$, and we identify all the $j_k$ points to one point—call it $u_k$. Then, for $\ell = 1$, we obtain a bouquet of $\binom{j_k + 1}{2}$ circles, and we identify them in just one circle $S^1$ passing through $u_k$, and so on, up to the facets of $2\{u_{i_1}^1, \ldots, u_{i_k}^k\}$, i.e. its $j_k + 1$ $(j_k - 1)$-faces, which, by the construction, have all their boundary in common, and we identify all of them.

Make all these identifications for all $j_1, \ldots, j_m$; in this way, we obtain a finite CW-complex $X = X_g$ of dimension $d - 1$, with 0-skeleton $X^0 = \{u_i \mid j_i \neq 0\} \subset \{u_1, \ldots, u_m\}$, obtained from the $(d-1)$-dimensional simplex $\Delta(d)$, with the above identification.

In this way, we obtain a finite CW-complex $X = X_g$ of dimension $d - 1$, with 0-skeleton $X^0 = \{u_i \mid j_i \neq 0\} \subset \{u_1, \ldots, u_m\}$, obtained from the $(d-1)$-dimensional simplex $\Delta(d)$, with the above identification. We note that each closure of a $(j_k - 1)$-cell $\{u_{i_1}^{j_1}, \ldots, u_{i_k}^{j_k}\}$ becomes, with this identification, a point if $j_k = 1$, a circle $S^1$ if $j_k = 2$, a topological space with fundamental group $\mathbb{Z}_{j_k}$ if $j_k = 2$ (i.e. it is not a topological surface), etc. We will denote these spaces in what follows by $e_k^{j_k - 1}$, i.e. $e_k^{j_k - 1}$ corresponds to $u_{i_k}^{j_k}$, and vice versa.
Proposition 3.15. Every power in \( u_1^{i_1} \cdots u_m^{i_m} \) (up to a permutation of the variables) corresponds to a \( \epsilon_k^{j_k-1} \), and vice versa.

Moreover, we can see \( X_g \) as a \((d-1)\)-dimensional join between these spaces \( \epsilon_k^{j_k-1} \) and the span of the 0-skeleton \( X^0 \) i.e. the simplex \( S_X \subset \Delta(m) \) associated to it; \( S_X \cong \Delta(\ell) \), where \( \ell = \#X^0 \leq m \).

Remark 3.16. This last observation suggests us to consider an alternative construction: we recall that the cellular decomposition of the real projective spaces is obtained attaching a single cell at each passage; indeed, \( \mathbb{P}^n \) is obtained from \( \mathbb{P}^{n-1} \) by attaching one \( n \)-cell with the quotient projection \( \varphi^{n-1}: \mathbb{S}^{n-1} \to \mathbb{P}^{n-1} \) as the attaching map.

Then, to each power \( u_k^{j_k} \) we associate a real projective space of dimension \( j_k - 1 \) \( \mathbb{P}^{j_k-1} \) and immersions \( i_{k-1} : \mathbb{P}^{j_k-1} \to \mathbb{P}^k \); so \( \mathbb{P}^0 = u_k \in \mathbb{P}^{j_k-1} \).

Finally, to \( g = u_1^{i_1} \cdots u_m^{i_m} \) we associate the join between the \( \mathbb{P}^{j_k-1} \) and the \( S_X \) defined above; if we call this join by \( X_g \), we can proceed in an equivalent way, by changing \( \epsilon_k^{j_k-1} \) with \( \mathbb{P}^{j_k-1} \).

\( \diamond \)

It is now clear how to glue two of these finite CW-complexes—say \( X = X_{u_1^{i_1} \cdots u_m^{i_m}} \) and \( Y = Y_{u_1^{k_1} \cdots u_m^{k_m}} \), of degree \( d = j_1 + \cdots + j_m \) and \( d' = k_1 + \cdots + k_m \)—along \( \Delta(m) \): we simply attach \( X \) and \( Y \) via the inclusion maps \( S_X \subset \Delta(m) \) and \( S_Y \subset \Delta(m) \); where \( S_X \) and \( S_Y \) are the simplexes associated to, respectively, \( X \) and \( Y \).

Finally, taking all these finite CW-complexes together, we obtain a CW-complex \( P \) in the following way:

\[
C := \bigcup_{u_1^{i_1} \cdots u_m^{i_m} \in \mathbb{K}[u_1, \ldots, u_m]} X_{u_1^{i_1} \cdots u_m^{i_m}} \quad P(m) := C_{\sim}
\]

where \( \sim \) is the equivalence relation induced by the above gluing. In this way, the have again that

Proposition 3.17. There is bijection between the monomials of degree \( d \) in \( \mathbb{K}[u_1, \ldots, u_m] \) and the elements of the \((d-1)\)-skeleton of \( P(m) \).

In other words, if we define

\[
\rho_d' := \{ f \in \mathbb{K}[u_1, \ldots, u_m] \mid f \neq 0 \text{ is a monic monomial} \}
\]

we have a bijection, using the above notation

\[
\sigma'_d: \rho_d' \to P(m)_{d-1} \quad u_1^{i_1} \cdots u_m^{i_m} \mapsto X_{u_1^{i_1} \cdots u_m^{i_m}}.
\]

Proposition 3.18. \( X_{u_1^{i_1} \cdots u_m^{i_m}} \subset X_{u_1^{j_1} \cdots u_m^{j_m}} \) if and only if \( u_1^{i_1} \cdots u_m^{i_m} \) divides \( u_1^{j_1} \cdots u_m^{j_m} \).

Let \( f = \sum_{i=0}^n x_i^{d_i} g_i \in S_{(d_1, d_2)} \) be a Nagata polynomial; by hypothesis there is bijection between the monomials \( g_i \)’s and the indeterminates \( x_i \)’s. From all this, one can associate to \( f \) a finite \((d_2-1)\)-dimensional, CW-subcomplex of \( P(m) \), \( \Delta_f \) where the \((d_2-1)\)-skeleton is given by the \( X_g \)’s glued together with the above procedure. Each \( X_g \) can be identified with \( x_1^{d_1} \) as before.

The previous construction generalizes the analogous one given in [1].

3.3. The Hilbert Function of SBAG Algebras. The first main result of this paper is the following general theorem.

Remark 3.19. In order to state it, we observe that the canonical bases of

\[
S_{(d_1, d_2)} = \mathbb{K}[x_0, \ldots, x_n]_{d_1} \otimes \mathbb{K}[u_1, \ldots, u_m]_{d_2}
\]

and

\[
T_{(d_1, d_2)} = \mathbb{K}[X_0, \ldots, X_n]_{d_1} \otimes \mathbb{K}[U_1, \ldots, U_m]_{d_2}
\]
In the cases

More precisely, a basis for

We divide the proof by computing the dimension of

where \(i_0 + \cdots + i_n = k_0 + \cdots + k_n = d_1, \ j_1 + \cdots + j_m = \ell_1 + \cdots + \ell_m = d_2\) and \(\delta_{k_0,\ldots,k_n,\ell_1,\ldots,\ell_m}\) is the Kronecker delta.

This simple observation allows us to identify—given a Nagata polynomial \(f = \sum_{r=0}^{n} x_r^d g_r \in S_{(d_1,d_2)}\)—also the dual differential operator \(G_r\) of the monomial \(g_r\)—i.e. the monomial \(G_r \in \mathbb{K}[U_1,\ldots,U_m]_{d_2}\) such that \(G_r(g_r) = 1\) and \(G_r(g) = 0\) for any other monomial \(g \in \mathbb{K}[u_1,\ldots,u_m]_{d_2}\)—with the same element of the \((d_2-1)\)-skeleton of \(\Delta_f\) associated to \(g_r\). In other words, we associate to \(g_r = u_1^{i_1} \cdots u_m^{i_m}\) and to \(G_r = U_1^{j_1} \cdots U_m^{j_m}\) the CW-subcomplex of \(\Delta_f \subset P(m), X_{u_1^{i_1} \cdots u_m^{i_m}}\).

**Theorem 3.20.** Let \(f = \sum_{r=0}^{n} x_r^d g_r \in R_{(d_1,d_2)}\), with \(g_r = u_1^{i_1} \cdots u_m^{i_m}\), be a Nagata polynomial of (positive) degree \(d_1\), where \(n \leq \binom{m}{d_2}\), let \(\Delta_f\) be the CW-complex associated to \(f\) and let \(A = Q/\text{Ann}(f)\). Then

\[
A = \bigoplus_{h=0}^{d=d_1+d_2} A_h
\]

where

\[
A_h = A_{(h,0)} \oplus \cdots \oplus A_{(p,q)} \oplus \cdots \oplus A_{(0,h)}, \ p \leq d_1, \ q \leq d_2, \ A_d = A_{(d_1,d_2)}
\]

and moreover, \(\forall j \in \{0,1,\ldots,d_2\}, \ \dim A_{(i,j)} = a_{i,j} = \left\{ \begin{array}{ll}
f_j & i = 0 \\
\sum_{r=0}^{n} f_{j,r} & i \in \{1,\ldots,d_1-1\}, \\
f_{d_2-j} & i = d_1 \\
\end{array} \right.
\]

where:

- \(f_j\) is the number of the elements of the \((j-1)\)-skeleton of the CW-complex \(\Delta_f\) (with the convention that \(f_0 = 1\));
- \(f_{j,r}\) is the number of the elements of the \((j-1)\)-skeleton of the CW complex \(X_{G_r}\) (with the convention that \(f_{0,r} = 1\), so that \(\dim A_{(i,0)} = n+1\)).

More precisely, a basis for \(A_{(i,j)}\), \(\forall j \in \{0,1,\ldots,d_2\}\), is given by

1. if \(i = 0\), \(\{\Omega_1,\ldots,\Omega_f\}\), where any \(\Omega_s := U_1^{s_1} \cdots U_m^{s_m}\), with \(s_1 + \cdots + s_m = j\) is associated to the element \(X_{u_1^{a_1} \cdots u_m^{a_m}}\) of the \((j-1)\)-skeleton of \(\Delta_f\);
2. if \(i = 1,\ldots,d_1-1\), \(\{\Omega_{s_1 \cdots s_m}\}_{s \in \{0,\ldots,n\}}\), where \(\Omega_{s_1 \cdots s_m} := X_{u_1^{s_1} \cdots u_m^{s_m}}\) is associated to the element \(X_{u_1^{a_1} \cdots u_m^{a_m}}\) of the \((j-1)\)-skeleton of \(X_{G_r}\);
3. if \(i = d_1\), \(\{X_{u_1^{a_1}}^{d_1} \Omega_1(f), \ldots, X_{u_1^{a_1}}^{d_1} \Omega_{d_2-j}(f)\}\), where \(\{\Omega_1,\ldots,\Omega_{d_2-j}\}\) is the basis for \(A_{(0,d_2-j)}\) of case 1.

In the cases 1 and 2 the basis are given by monomials, in the case 3, in general, not.

**Proof.** We divide the proof by computing the dimension of \(A_{(i,j)}\) and find a basis for it, as \(i\) varies:

1. if \(i = 0\) first of all, \(A_{(0,0)} \cong \mathbb{K}\).

Then, by definition, if \(j \in \{1,\ldots,d_2\}, A_{(0,j)}\) is generated by the only (canonical images of the) monomials \(\Omega_s \subset Q_j = \mathbb{K}[U_1,\ldots,U_m]_{d_2} \cong Q_{(0,j)}\) that do not annihilate \(f\). This means that, if we write

\[
\Omega_s = U_1^{s_1} \cdots U_m^{s_m} \quad s_1 + \cdots + s_m = j,
\]
there exists an \( r_s \in \{0, \ldots, n\} \) such that \( g_{r_s} = u_1^{s_1} \cdots u_m^{s_m} g'_{r_s} \), where \( g'_{r_s} \in R_{d_2-j} \) is a (nonzero) monomial; this means that \( X_{u_1^{s_1} \cdots u_m^{s_m}} \) is an element of the \((j-1)\)-skeleton of the CW-complex \( \Delta_f \) by Proposition 3.18.

We need now to prove that these monomials are linearly independent over \( \mathbb{K} \): let \( \{\Omega_1, \ldots, \Omega_f\} \) be a system of monomials of \( Q(0,j) \), where any \( \Omega_s = U_1^{s_1} \cdots U_m^{s_m} \) with \( s_1 + \cdots + s_m = j \), is associated to an element of the \((j-1)\)-skeleton of the CW-complex \( \Delta_f \); take a linear combination of them and apply it to \( f \):

\[
0 = \sum_{s=1}^{f_f} c_s \Omega_s(f) = \sum_{s=1}^{f_f} \sum_{r=0}^{n} x_r^{d_1} \Omega_s(g_r) = \sum_{r=0}^{n} x_r^{d_1} \sum_{s=1}^{f_f} c_s \Omega_s(g_r),
\]

by the linear independence of the \( x_r^{d_1} \)'s

\[
(2) \quad \sum_{s=1}^{f_f} c_s \Omega_s(g_r) = 0, \quad \forall r \in \{0, \ldots, n\}.
\]

By hypothesis, for any index \( s \) there exists an \( r_s \in \{0, \ldots, n\} \) such that \( \Omega_s(g_{r_s}) = g'_{r_s} \in R_{d_2-j} \setminus \{0\} \), then for any index \( s \) one has \( c_s = 0 \), since the linear combinations in (2) are formed by linearly independent monomials (\( g_r \) is fixed in each linear combination!). In other words, \( \dim A_{(0,j)} = f_f \).

First of all, we observe that \( X_sX_b(f) = 0 \) if \( a \neq b \). Therefore \( A_{(i,j)} \) is generated by the only (canonical images of) the monomials \( \Omega_s^{i_1} \cdots \Omega_s^{i_m} := X_s^{i_1}U_1^{s_1} \cdots U_m^{s_m} \in Q(i,j) \), with \( s_1 + \cdots + s_m = j \) that do not annihilate \( f \). In particular, a basis for \( A_{(i,0)} \) is given by \( X_0^i, X_n^i \) and we can suppose from now on that \( j > 0 \). Since

\[
\Omega_s^{i_1} \cdots \Omega_s^{i_m}(f) = x_1^{d_1-i} \left(U_1^{s_1} \cdots U_m^{s_m}\right)(g_{s}),
\]

in order to obtain that this is not zero, we must have that \( g_s = u_1^{s_1} \cdots u_m^{s_m} g'_{s} \), where \( g'_{r_s} \in R_{d_2-j} \) is a nonzero monomial. This means \( X_{u_1^{s_1} \cdots u_m^{s_m}} \subset X_{g_s} \) by Proposition 3.18.

As above, we can prove that these monomials are linearly independent over \( \mathbb{K} \): let \( \{\Omega_s^{i_1} \cdots \Omega_s^{i_m}\} \) be a system of monomials of \( Q(i,j) \), where any \( \Omega_s^{i_1} \cdots \Omega_s^{i_m} = X_s^{i_1}U_1^{s_1} \cdots U_m^{s_m} \) is associated to the element \( X_{u_1^{s_1} \cdots u_m^{s_m}} \) of the \((j-1)\) skeleton of \( X_{g_s} \subset \Delta_f \), i.e. \( X_{u_1^{s_1} \cdots u_m^{s_m}} \subset X_{g_s} \subset \Delta_f \) by Proposition 3.18.

Take a linear combination of them and apply it to \( f \):

\[
(3) \quad 0 = \sum_{s \in \{0, \ldots, n\}} \sum_{s_k \leq r_k, k=1, \ldots, m} \sum_{\sum s_k = j} c_s^{i_1, \ldots, i_m} \Omega_s^{i_1} \cdots \Omega_s^{i_m}(f) = \sum_{s=0}^{n} x_r^{d_1-i} \sum_{s_k \leq r_k, k=1, \ldots, m} \sum_{\sum s_k = j} c_s^{i_1, \ldots, i_m} g_s^{i_1, \ldots, i_m}
\]

where \( g_s^{i_1, \ldots, i_m} \in R_{d_2-j} \) is the nonzero monomial such that \( g_s = u_1^{s_1} \cdots u_m^{s_m} g'_s \). From (3) we deduce, as in the preceding case, that

\[
(4) \quad \sum_{s_k \leq r_k, k=1, \ldots, m} \sum_{\sum s_k = j} c_s^{i_1, \ldots, i_m} g_s^{i_1, \ldots, i_m} = 0 \quad s = 0, \ldots, n;
\]

as before, given one choice of \( s_1, \ldots, s_m \) there exists an \( s \in \{0, \ldots, n\} \) such \( g'_s \) is a nonzero monomial, and the (nonzero) \( g_s^{i_1, \ldots, i_m} \)'s in (4) are linearly independent since they are obtained by a fixed \( g_s \).

\( i = d_1 \): by duality, see Remark 3.31 \( A_{(d_1,j)} \cong A_{(0,d_2-j)}^\vee \) so \( \dim A_{(d_1,j)} = f_{d_2-j} \). To find a basis for \( A_{(d_1,j)} \), we consider the exact sequence (1) given by evaluation by \( f \), which in this case reads

\[
(5) \quad 0 \to I_{(0,d_2-j)} \to Q_{(0,d_2-j)} \to A_{(d_1,j)} \to 0,
\]
then a basis for $A_{(d_1,j)}$ is obtained in the following way: if $\{\Omega_1, \ldots, \Omega_{f_{d_2-j}}\}$ is the basis for $A_{(0,d_2-j)} \cong Q_{(0,d_2-j)}/I_{(0,d_2-j)}$ of the case $i = 0$, then a basis for $A_{(d_1,j)}$ is
\[ \left\{ X_{d_1}^{d_1} \Omega_1, \ldots, X_{n_1}^{d_1} \Omega_{f_{d_2-j}}(f) \right\}. \]
\[ \square \]

As a corollary of Theorem 3.20, we see that we can deduce the general case of the simplicial Nagata polynomial, which is a (slightly) improvement of the first part of [1, Theorem 3.5].

**Corollary 3.21.** Let $f = \sum^n_{r=0} x_{r}^{d_1} g_{r} \in R_{(d_1,d_2)}$, with $g_{r} = x_{r_1} \cdots x_{r_{d_2}}$, be a simplicial Nagata polynomial of (positive) degree $d_1$, where $n \leq \left( \frac{m}{d_2} \right)$, let $\Delta_f$ be the simplicial complex associated to $f$ and let $A = Q/\text{Ann}(f)$. Then
\[ A = \bigoplus_{h=0}^{d=d_1+d_2} A_h \]
where $A_h = A_{(h,0)} \oplus \cdots \oplus A_{(p,q)} \oplus \cdots \oplus A_{(0,h)}$, $p \leq d_1$, $q \leq d_2$, $A_d = A_{(d_1,d_2)}$ and moreover, $\forall j \in \{0, 1, \ldots, d_2\}$,
\[ \dim A_{(i,j)} = a_{i,j} = \begin{cases} f_j & i = 0 \\ \sum_{r=0}^n f_{j,r} & i \in \{1, \ldots, d_1-1\}, \\ f_{d_2-j} & i = d_1 \end{cases} \]
where:
- $f_j$ is the number of $(j-1)$-cells of the $\Delta_f$ (with the convention that $f_0 = 1$);
- $f_{j,r}$ is the number of $(j-1)$-subcells of $\Delta_{g_r}$, i.e. the $(d_2-1)$-cell of the $\Delta_f$ associated to $g_r$ (with the convention that $f_0,r = 1$, so that $\dim A_{(i,0)} = n+1$).

More precisely, a basis for $A_{(i,j)}$, $\forall j \in \{0, 1, \ldots, d_2\}$, is given by
1. if $i = 0$, $\{\Omega_1, \ldots, \Omega_{f_{j}}\}$, where any $\Omega_s := U_{s_1} \cdots U_{s_j}$ is associated to the $(j-1)$-subcell $\{u_{s_1}, \ldots, u_{s_j}\}$ of $\Delta_f$;
2. if $i = 1, \ldots, d_1-1$, $\left\{ \Omega_s^{1 \leq s_1 \leq s_j} \right\}$, $s \in \{0, \ldots, n\}$, where $\Omega_s^{1 \leq s_1 \leq s_j} := X_{s_1} U_{s_1} \cdots U_{s_j}$ is associated to the $(j-1)$-subcell $\{u_{s_1}, \ldots, u_{s_j}\}$ of $\Delta_{g_s} (\subset \Delta_f)$;
3. if $i = d_1$, $\left\{ X_{d_1}^{d_1} \Omega_1(f), \ldots, X_{n_1}^{d_1} \Omega_{f_{d_2-j}}(f) \right\}$, where $\{\Omega_1, \ldots, \Omega_{f_{d_2-j}}\}$ is the basis for $A_{(0,d_2-j)}$ of case 1.

In the cases 1 and 2 the basis are given by monomials, in the case 3, in general, not.

**Theorem 3.22.** Let $f = \sum^n_{r=0} x_{r}^{d_1} g_{r} \in S_{(d_1,d_2)}$, with $g_{r} = x_{r_1} \cdots x_{r_{d_2}}$ such that $r_1 + \cdots + r_{d_2} = d_2$, be a Nagata polynomial whose associated CW-complex is $\Delta_f$, as in the preceding theorem.

Then $I := \text{Ann}(f)$ is generated by:
- a) $X_i X_j$ and $X_i^{d_1+1}$, for $i, j \in \{0, \ldots, n\}$, $i < j$;
- b) $\langle U_1, \ldots, U_{d_2+1} \rangle$, i.e. all the (monic) monomials of degree $d_2 + 1$;
- c) the monomials $U_1^{s_1} \cdots U_{d_2}^{s_{d_2}}$ such that $s_1 + \cdots + s_{d_2} = m$, where $X_{u_1^{s_1} \cdots u_{d_2}^{s_{d_2}}}$ is a (minimal) element of the $(j-1)$-skeleton of $P(m)$ not contained in $\Delta_f$ (for $j \in \{1, \ldots, d_2\}$);
- d) the monomials $X_s U_i$, where $u_i$ does not divide $g_r$ (i.e. $\{u_i\}$ is not an element of the 0-skeleton of $X_{g_r}$);
- e) the monomials $X_s U_1^{r_1} \cdots U_{d_2}^{r_{d_2}}$ such that $r_1 + \cdots + r_{d_2} = m$, where $U_1^{r_1} \cdots U_{d_2}^{r_{d_2}}$ is minimal among those that do not divide $g_r$ (i.e. the (minimal) element of the $(j-1)$-skeleton of $P(m)$, $X_{u_1^{r_1} \cdots u_{d_2}^{r_{d_2}}}$, is not contained in $X_{g_r}$), for $j \in \{1, \ldots, d_2\}$;
f) the binomials $X_r^{d_1} U_{1}^{r_1} \cdots U_m^{r_m} - X_s^{d_1} U_{1}^{s_1} \cdots U_m^{s_m}$ with $r_1 + \cdots + r_m = \sigma_1 + \cdots + \sigma_m = j$

such that $g_{r,s} = \gcd(g_r, g_s)$ and $g_r = u_1^{r_1} \cdots u_m^{r_m} g_{r,s}$, $g_s = u_1^{s_1} \cdots u_m^{s_m} g_{r,s}$ (i.e. $X_{gr,s}$ is the element of the $(d_2 - j - 1)$-skeleton of $\Delta_f$ which represents the intersection of $X_{gr}$ and $X_{gs}$: $X_{gr,s} = X_{gr} \cap X_{gs}$).

Proof. Set as before $A := T/I$, where $T = \mathbb{K}[X_0, \ldots, X_n, U_1, \ldots, U_m]$.

By Theorem 3.20 (1) a basis for $A_{(0,j)}$, $\forall j \in \{1, \ldots, d_2\}$, is given by $\{\Omega_{s}^{1} \cdot \cdots \cdot \Omega_{s}^{n}\}$, where $\Omega_{s} \subseteq U_1^{r_1} \cdots U_m^{r_m}$, with $s_1 + \cdots + s_m = j$ is associated to the element $X_{u_1}^{s_1} \cdots u_m^{s_m}$ of the $(j-1)$-skeleton of $\Delta_f$; therefore, using the identification introduced in Remark 3.19 a basis for $I_{(0,j)}$ is given by the monomials $U_1^{r_1} \cdots U_m^{r_m}$ such that $s_1 + \cdots + s_m = j$, where $X_{u_1}^{s_1} \cdots u_m^{s_m}$ is an element of the $(j-1)$-skeleton of $P(m)$, not contained in $\Delta_f$ (for $j \in \{1, \ldots, d_2\}$).

Then, we observe that $X_{i}X_{j}(f) = 0$ if $i \neq j$ and $X_{d_i+1}^{d_i+1}(f) = 0 = U_{1}^{r_1} \cdots U_m^{r_m} (f)$ with $\sum_{j=1}^{m} i_j = d_2 + 1$, degree reasons.

Set $\beta := (X_0X_1, \ldots, X_{n-1}X_n, X_0^{d_1+1}, \ldots, X_n^{d_1+1}, (U_1, \ldots, U_m)^{d_2+1});$ this is a homogeneous ideal such that $\beta \subseteq I$ and $A \cong \frac{T}{\beta}$.

By Theorem 3.20 (2), if $i = 1, \ldots, d_1 - 1$, a basis for $A_{(i,j)} \forall j \in \{1, \ldots, d_2\}$, is given by $\{\Omega_{s}^{i} \cdot \cdots \cdot \Omega_{s}^{n}\}$, where $\Omega_{s}^{j} := X_{i}^{j} \cdot U_1^{r_1} \cdots U_m^{r_m}$ is associated to the element $X_{u_1}^{s_1} \cdots u_m^{s_m}$ of the $(j-1)$-skeleton of $X_{gs}$.

Therefore, again using the identification introduced in Remark 3.19 a basis for $\left(\frac{I}{\beta}\right)_{(i,j)}$ is given by both

- the monomials $X_{i}^{j} U_1^{r_1} \cdots U_m^{r_m}$ such that $s_1 + \cdots + s_m = j$, with $r \neq s$, where $u_1^{r_1} \cdots u_m^{r_m}$ divides $g_s$ (i.e. $X_{u_1}^{s_1} \cdots u_m^{s_m}$ is an element of the $(j-1)$-skeleton of $X_{gs}$), for $i = 1, \ldots, d_1 - 1$, and

- the monomials $X_{i}^{j} U_1^{r_1} \cdots U_m^{r_m}$ such that $r_1 + \cdots + r_m = j$, where $u_1^{r_1} \cdots u_m^{r_m}$ does not divide $g_s$ (i.e. the element of the $(j-1)$-skeleton of $P(m)$, $X_{u_1}^{s_1} \cdots u_m^{s_m}$ is not contained in $X_{gs}$), for $j \in \{1, \ldots, d_2\}$.

It remains to find the generators of $I$ of bidegree $(d_1, j)$, with $j \in \{1, \ldots, d_2\}$. This is more complicated since the generators of $A_{(d_1,j)}$ are not monomials. Let $\gamma$ be the homogeneous ideal generated by the monomials of the cases (3.3), (3.4), (3.5) and (3.3), i.e. the generators that we have found so far. We have $\beta \subset \gamma \subset I$ and the exact sequence (11) given by evaluation by $f$ becomes

\[ 0 \rightarrow \left(\frac{I}{\gamma}\right)_{(d_1,j)} \rightarrow \left(\frac{T}{\gamma}\right)_{(d_1,j)} \rightarrow A_{(0,d_2-j)} \rightarrow 0, \]

since we identify $A \cong \frac{T}{\gamma}$. Then, if $\rho_1 + \cdots + \rho_m = \sigma_1 + \cdots + \sigma_m = j$, $X_{r}^{d_1} U_1^{r_1} \cdots U_m^{r_m} \in \left(\frac{I}{\gamma}\right)_{(d_1,j)}$ if and only if $X_{r}^{d_1} U_1^{r_1} \cdots U_m^{r_m} = X_{s}^{d_1} U_1^{s_1} \cdots U_m^{s_m} \in A_{(0,d_2-j)}$, which means $U_1^{r_1} \cdots U_m^{r_m} (g_r) = U_1^{s_1} \cdots U_m^{s_m} (g_s)$. Since $A_{(0,d_2-j)}$ is generated by the monomials $\Omega_s := U_1^{s_1} \cdots U_m^{s_m}$, with $s_1 + \cdots + s_m = d_2 - j$ associated to the elements of the $(d_2 - j - 1)$-skeleton of $\Delta_f$, we obtain case (3).

As we have done for Theorem 3.20 we give, as a corollary of Theorem 3.22 the case of the simplicial Nagata polynomial, giving an improving of the second part of [1, Theorem 3.5]; we also correct that statement, since the authors forgot the (trivial) generators $X_i X_j$, $i \neq j$.

Corollary 3.23. Let $f = \sum_{r=0}^{n} x_r^{d_1} g_r \in R_{(d_1, d_2)}$, with $g_r = x_{r_1} \cdots x_{r_{d_2}}$, be a simplicial Nagata polynomial whose associated simplicial complex is $\Delta_f$, as in the preceding theorem.

Then $I := \text{Ann}(f)$ is generated by:
a) $X_iX_j$ and $X_i^{d+1}$, for $i, j, k \in \{0, \ldots, n\}$, $i < j$;

b) $U_i^2, \ldots, U_i^m$;

c) the monomials $U_{s_1}\cdots U_{s_j}$, where $\{u_{s_1}, \ldots, u_{s_j}\}$ is a (minimal) $(j - 1)$-cell of $2^{\{a_1, \ldots, a_m\}}$ not contained in $\Delta_f$ (for $j \in \{1, \ldots, d\}$);

d) the monomials $X_iU_j$, where $u_i$ does not divide $g_j$ (i.e. $\{u_i\} \notin \Delta_{g_j}$);

e) the binomials $X_iU_j', \ldots U_j'^m - X_iU_{s_1}\cdots U_{s_j}$, such that $g_{r,s} \in \text{gcd}(g_r, g_s)$, $g_r = u_{r_1}\cdots u_{r_p}g_{r,s}$, $g_s = u_{s_1}\cdots u_{s_q}g_{r,s}$ (i.e. $g_{r,s}$ represents the $(d - j)\text{-face}$ given by the intersection $\Delta_{g_r} \cap \Delta_{g_s}$).

Proof. We note only that we have to add the squares of case [\[\text{E}3.24\]] although they do not correspond to cells, since the polynomials $g_i$'s are square-free. The rest follows swiftly from Theorem [3.22].

Example 3.24. Let $f = x_0^d u_1 u_2 u_3 + x_1^d u_1 u_2 u_4 + x_2^d u_1 u_3 u_5 + x_3^d u_1 u_3 u_5 + x_4^d u_2 u_3 u_6 + x_5^d u_2 u_4 u_6 + x_6^d u_4 u_5 u_6 + x_7^d u_3 u_5 u_6$ be a bihomogeneous bidegree $(d, 3)$ polynomial with $d \geq 1$; it is a simplicial Nagata polynomial, whose associated simplicial complex is in the following figure:

![Diagram]

We have: $$A = A_0 \oplus A_1 \oplus \cdots \oplus A_{d+3}$$ and we want firstly to compute the Hilbert vector by applying Corollary [3.21] first of all,

$$a_{1,0} = 8 \quad a_{0,1} = 6,$$

and therefore

$$h_0 = h_{d+3} = 1$$

$$h_1 = h_{d+2} = a_{1,0} + a_{0,1} = 8 + 6 = 14.$$  

Then, we analyze the possible cases depending on the degree $d$:

- If $d = 1$, then

  $$a_{1,1} = 8 \cdot 3 = 24$$

  $$a_{0,2} = 12$$

  $$h_2 = a_{1,1} + a_{0,2} = 36$$

  and the Hilbert vector is $(1, 14, 36, 14, 1)$.

- If $d = 2$, then, recalling bigraduate Poincaré duality,

  $$a_{2,0} = a_{0,3} = 8$$

  $$a_{2,1} = a_{0,2} = 12$$

  and therefore

  $$h_2 = a_{2,0} + a_{1,1} + a_{0,2} = 8 + 8 \cdot 3 + 12 = 44,$$

  $$h_3 = 0 + a_{2,1} + a_{1,2} + a_{0,3} = 8 + 8 \cdot 3 + 8 = 44$$

  in accordance with Poincaré duality; so the Hilbert vector is $(1, 14, 44, 44, 14, 1)$ (cfr. [\[\text{E}3.6\]] Example 3.6)].
• If \( d = 3 \), then, again by bigraduate Poincaré duality,

\[
a_{3,0} = a_{0,3} = 8, \quad a_{2,1} = a_{1,2} = 8 \cdot 3 = 24, \quad a_{3,1} = a_{0,2} = 12, \quad a_{2,2} = a_{1,1} = 24, \quad a_{1,3} = a_{2,0} = 8,
\]

therefore

\[
h_2 = a_{2,0} + a_{1,1} + a_{0,2} = 44, \\
h_3 = a_{3,0} + a_{2,1} + a_{1,2} + a_{0,3} = 64 \\
h_4 = 0 + a_{3,1} + a_{2,2} + a_{1,3} = 44
\]

\( h_2 = h_4 \) in accordance with Poincaré duality and the Hilbert vector is \((1, 14, 44, 64, 44, 14, 1)\).

• In general, let \( d \geq 4 \); by hypothesis

\[
h_{d+1} = h_2 = a_{2,0} + a_{1,1} + a_{0,2} = 44,
\]

and

\[
h_k = a_{k,0} + a_{k-1,1} + a_{k-2,2} + a_{k-3,3} \quad \forall k \in \{3, \ldots, d\},
\]

where

\[
a_{k,0} = 8 \quad a_{k-1,1} = 8 \cdot 3 = 24 \quad a_{k-2,2} = 8 \cdot 3 = 24 \quad a_{k,3} = 8.
\]

Again using the Poincaré duality we have:

\[
h_{d+3-k} = h_k = 64 \quad \forall k \in \left\{3, \ldots, \left\lfloor \frac{d+3}{2} \right\rfloor \right\}
\]

and the Hilbert vector is \((1, 14, 44, 64, \ldots, 64, 44, 14, 1)\).

Now, we want to find the generators of \( \text{Ann}(f) \), by applying Corollary 3.23 also in this case, we study this ideal in dependence of the degree \( d \):

• Let \( d = 1 \), by Corollary 3.23 \( \text{Ann}(f) \) is (minimally) generated by:
  a) \((X_0, \ldots, X_7)^2 = X_0^2, X_0X_1, \ldots \);  
b) \( U_0^2, \ldots, U_6^2 \);  
c) \( U_1U_6, U_2U_5, U_3U_4 \);  
d) \( X_0U_4, X_0U_5, X_0U_6, X_1U_3, X_1U_4, X_1U_5, X_1U_6, X_2U_3, X_2U_4, X_2U_5, X_2U_6, X_3U_4, X_3U_5, X_3U_6, X_4U_3, X_4U_4, X_4U_5, X_4U_6, X_5U_2, X_5U_3, X_5U_4, X_5U_6, X_6U_1, X_6U_2, X_6U_3, X_6U_4, X_6U_5, X_6U_6, X_7U_1, X_7U_2, X_7U_3, X_7U_4, X_7U_5, X_7U_6 \);  
e) \( X_0U_3 - X_1U_4, X_0U_2 - X_2U_3, X_0U_1 - X_3U_2, X_0U_0 - X_4U_1, X_1U_2 - X_5U_0, X_1U_1 - X_6U_1, X_1U_0 - X_7U_0, X_2U_1 - X_3U_2, X_2U_0 - X_4U_1, X_2U_1 - X_5U_2, X_2U_2 - X_6U_3, X_2U_3 - X_7U_4, X_3U_3 - X_4U_2, X_3U_4 - X_5U_3, X_3U_5 - X_6U_4, X_3U_6 - X_7U_5, X_4U_4 - X_5U_5, X_4U_5 - X_6U_6, X_4U_6 - X_7U_7, X_5U_5 - X_6U_6, X_5U_6 - X_7U_7, X_6U_6 - X_7U_8, X_7U_7 - X_8U_9 \);

• Let \( d \geq 2 \), by Corollary 3.23 \( \text{Ann}(f) \) is (minimally) generated by
  a) \((X_0, \ldots, X_7)^{d+1} \) and \( X_hX_k \) where \( h, k \in \{0, \ldots, 7\} \), \( h < k \);  
b) \( U_0^2, \ldots, U_6^2 \);  
c) \( U_1U_6, U_2U_5, U_3U_4 \);  
d) \( X_0U_4, X_0U_5, X_0U_6, X_1U_3, X_1U_4, X_1U_5, X_1U_6, X_2U_3, X_2U_4, X_2U_5, X_2U_6, X_3U_4, X_3U_5, X_3U_6, X_4U_3, X_4U_4, X_4U_5, X_4U_6, X_5U_2, X_5U_3, X_5U_4, X_5U_5, X_5U_6, X_6U_1, X_6U_2, X_6U_3, X_6U_4, X_6U_5, X_6U_6, X_7U_1, X_7U_2, X_7U_3, X_7U_4, X_7U_5, X_7U_6 \);  
e) \( X_0U_3 - X_1U_4, X_0U_2 - X_2U_3, X_0U_1 - X_3U_2, X_0U_0 - X_4U_1, X_1U_2 - X_5U_0, X_1U_1 - X_6U_1, X_1U_0 - X_7U_0, X_2U_1 - X_3U_2, X_2U_0 - X_4U_1, X_2U_1 - X_5U_2, X_2U_2 - X_6U_3, X_2U_3 - X_7U_4, X_3U_3 - X_4U_2, X_3U_4 - X_5U_3, X_3U_5 - X_6U_4, X_3U_6 - X_7U_5, X_4U_4 - X_5U_5, X_4U_5 - X_6U_6, X_4U_6 - X_7U_7, X_5U_5 - X_6U_6, X_5U_6 - X_7U_7, X_6U_6 - X_7U_8, X_7U_7 - X_8U_9 \).

**Example 3.25.** Let

\[
f = x_0^d u_1 u_2 + x_1^d u_2^2 + x_2^d u_2 u_3
\]

be a bihomogeneous bidegree \((d, 2)\) polynomial, with \( d \geq 1 \); it is a Nagata polynomial whose CW-complex is the following:
We have:

\[ A = A_0 \oplus A_1 \oplus \ldots \oplus A_{d+2} \]

and we want to find its Hilbert vector; first of all,

\[ a_{1,0} = 3 \quad a_{0,1} = 3 \]

and therefore

\[ h_0 = h_{d+2} = 1 \quad h_1 = h_{d+1} = a_{1,0} + a_{0,1} = 6. \]

Therefore, if \( d = 1 \), then Hilbert vector is \((1, 6, 6, 1)\).

If \( d = 2 \), we have

\[ a_{1,1} = 2 + 1 + 2 = 5, \]

so

\[ h_2 = \dim A_2 = a_{2,0} + a_{1,1} + a_{0,2} = 3 + 5 + 3 = 11 \]

and the Hilbert vector is \((1, 6, 11, 6, 1)\).

If \( d = 3 \) then, by bigraduate Poincaré duality

\[ a_{3,0} = a_{0,2} = 3 \quad a_{0,3} = 3 \]

so

\[ h_2 = a_{2,0} + a_{1,1} + a_{0,2} = 11 \]
\[ h_3 = a_{3,0} + a_{2,1} + a_{1,2} + a_{0,3} = 3 + 5 + 3 = 11 \]

and the Hilbert vector is \((1, 6, 11, 11, 6, 1)\).

In general, let \( d \geq 4 \); by hypothesis

\[ h_d = h_2 = a_{2,0} + a_{1,1} + a_{0,2} = 11, \]

and

\[ h_k = \dim A_{(k,0)} + \dim A_{(k-1,1)} + \dim A_{(k-2,2)} \quad \forall k \in \{3, \ldots, d\}, \]

so, since

\[ a_{k,0} = 3 \quad a_{k-1,1} = 5 \quad a_{k-2,2} = 3 \]

using Poincaré duality we have:

\[ h_{d+2-k} = h_k = a_{k,0} + a_{k-1,1} + a_{k-2,2} = 11 \quad \forall k \in \left\{3, \ldots, \left\lfloor \frac{d+2}{2} \right\rfloor \right\}, \]

and the Hilbert vector is \((1, 6, 11, \ldots, 11, 6, 1)\).

Let \( d = 1 \), by Theorem 3.22 \( \text{Ann}(f) \) is (minimally) generated by:

- \( \langle X_0, X_1, X_2 \rangle^2, U_2^2, U_3^2, U_1 U_3, U_1^2; \)
- \( X_0 U_1^2, X_0 U_3, X_1 U_2, X_1 U_3, X_2 U_1; \)
- \( X_0 U_2 - X_1 U_1, X_0 U_1 - X_3 U_3. \)

Let \( d \geq 2 \), by Theorem 3.22 \( \text{Ann}(f) \) is (minimally) generated by:

- \( \langle X_0, X_1, X_2 \rangle^{d+1}, X_0 X_1, X_0 X_2, X_1 X_2, U_2^3, U_3^2, U_1 U_3, U_1^3; \)
- \( X_0^2 U_1, X_0^2 U_3, X_1^2 U_2, X_1^2 U_3, X_2^2 U_1; \)
- \( X_0^2 U_2 - X_1^2 U_1, X_0^2 U_1 - X_3^2 U_3. \)


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