Calabi quasimorphism and quantum homology

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Dedicated to the memory of Robert Brooks

Abstract

We prove that the group of area-preserving diffeomorphisms of the 2-sphere admits a non-trivial homogeneous quasimorphism to the real numbers with the following property. Its value on any diffeomorphism supported in a sufficiently small open subset of the sphere equals to the Calabi invariant of the diffeomorphism. This result extends to more general symplectic manifolds: If the symplectic manifold is monotone and its quantum homology algebra is semi-simple we construct a similar quasimorphism on the universal cover of the group of Hamiltonian diffeomorphisms.

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1 Introduction and results

A quasimorphism on a group $G$ is a function $r : G \to \mathbb{R}$ which satisfies the homomorphism equation up to a bounded error: there exists $R > 0$ such that

$$|r(fg) - r(f) - r(g)| \leq R$$

for all $f, g \in G$ (see [4] for preliminaries on quasimorphisms). A quasimorphism $r_h$ is called homogeneous if $r_h(g^m) = mr_h(g)$ for all $g \in G$ and $m \in \mathbb{Z}$. Homogeneous quasimorphisms are invariant under conjugations in $G$. Every quasimorphism $r$ gives rise to a homogeneous one

$$r_h(g) = \lim_{m \to +\infty} \frac{r(g^m)}{m},$$

called the homogenization of $r$. Starting from the classical work of R.Brooks [9], who explicitly constructed a non-trivial quasimorphism on a free group, quasimorphisms are playing an important role in the study of groups. In particular, they appear in the bounded cohomology theory and in the geometry of the commutator norm (see e.g. [4] and Section 1.2.1 below).

In the present paper we focus on the cases when $G$ is either the group of Hamiltonian diffeomorphisms of a symplectic manifold or its universal cover. For a class of manifolds, which for instance includes complex projective spaces, we give an explicit construction of a non-trivial quasimorphism on $G$. The construction involves some tools from "hard" symplectic topology, in particular, Floer and quantum homology. Interestingly enough, our quasimorphism is closely related to the classical Calabi invariant.

1.1 Extending the Calabi homomorphism

Let $G = Ham(M, \omega)$ be the group of Hamiltonian diffeomorphisms of a closed connected symplectic manifold $M^{2n}$ (see e.g. [25], [35] for preliminaries on $G$). The group $G$ has a natural class of subgroups $G_U$ associated to non-empty open subsets $U \subset M$, $U \neq M$. The subgroup $G_U$ consists of all elements $f \in G$ generated by a time-dependent Hamiltonian

$$F_t : M \to \mathbb{R}, \ t \in [0; 1] \text{ with support}(F_t) \subset U \text{ for all } t.$$  \hspace{1cm} (1)

Consider the map $Cal_U : G_U \to \mathbb{R}$ given by

$$f \mapsto \int_0^1 dt \int_M F_t \omega^n.$$  \hspace{1cm} (2)
When the symplectic form $\omega$ is exact on $U$, this map is well defined, meaning that it does not depend on the specific choice of the Hamiltonian $F$ generating $f$. In fact, $Cal_U$ is a homomorphism called the Calabi homomorphism $[3]$, $[10]$. Note that $G_U \subset G_V$ for $U \subset V$, and in this case $Cal_U = Cal_V$ on $G_U$.

In what follows we deal with the class $\mathcal{D}$ of all non-empty open subsets $U$ which can be displaced by a Hamiltonian diffeomorphism:

$$hU \cap \text{Closure } (U) = \emptyset \text{ for some } h \in G.$$  

(3)

Put

$$\mathcal{D}_{\text{ex}} = \{U \in \mathcal{D} \mid \omega \text{ is exact on } U\}.$$ 

(4)

A celebrated result due to A. Banyaga $[3]$ states that the group $G$ is simple and therefore does not admit a non-trivial homomorphism to $\mathbb{R}$. In this paper we observe the following phenomenon: for certain symplectic manifolds the family of homomorphisms $\{Cal_U : G_U \to \mathbb{R}\}_{U \in \mathcal{D}_{\text{ex}}}$ extends to a quasimorphism from $G$ to $\mathbb{R}$.

Given a Hamiltonian $H : M \times S^1 \to \mathbb{R}$ denote by $\psi_H \in G$ the Hamiltonian symplectomorphism generated by $H$, i.e. time-1 map of the Hamiltonian flow of $H$.

**Definition 1.1.** Suppose that a function $r : G \to \mathbb{R}$ satisfies the following condition: if a sequence $\{H_i\}$ of smooth (possibly time-dependent) Hamiltonians $H_i : M \times S^1 \to \mathbb{R}$ converges $C^0$-uniformly to a smooth function $H : M \times S^1 \to \mathbb{R}$ then

$$r(\psi_{H_i}) \to r(\psi_H) \text{ as } i \to \infty.$$ 

(4)

In such a case the function $r$ will be called continuous.

**Definition 1.2.** A quasimorphism on $G$ coinciding with the Calabi homomorphism $\{Cal_U : G_U \to \mathbb{R}\}$ on any $U \in \mathcal{D}_{\text{ex}}$ will be called a Calabi quasimorphism.

**Theorem 1.3.** Let $(M, \omega)$ be one of the following symplectic manifolds:

- the 2-sphere $S^2$ with an area form $\omega$;
- $S^2 \times S^2$ with the split symplectic form $\omega \oplus \omega$;
- the complex projective space $\mathbb{C}P^n$ endowed with the Fubini-Study form.
Then there exists a continuous homogeneous Calabi quasimorphism \( \mu : G \to \mathbb{R} \).

For all the cases except \( \mathbb{C}P^n, n \geq 3 \), such a quasimorphism \( \mu \) will be constructed in Section 3 below. For the case \( \mathbb{C}P^n, n \geq 3 \), see Section 4.3.

**Remark 1.4.**
1) We do not know whether such a quasimorphism \( \mu \) is unique.
2) In the case of \( S^2 \) one can show that all continuous homogeneous Calabi quasimorphisms on \( G \) coincide on elements of the group generated by autonomous Hamiltonians: given such a quasimorphism \( \mu \) and an autonomous Hamiltonian \( H : S^2 \to \mathbb{R} \), one can explicitly compute \( \mu(\psi_H) \) in terms of combinatorics of the level sets of \( H \) – see Section 5.
3) The specific homogeneous quasimorphism \( \mu : G \to \mathbb{R} \) constructed in the proof of Theorem 1.3 is not only continuous but also Lipschitz with respect to the famous Hofer metric on \( G \) – see Section 3.6.

In fact, the "natural environment" in which one can look for a Calabi quasimorphism is the universal cover \( \tilde{G} \) of \( G \) rather than \( G \) itself. Namely, for a non-empty open subset \( U \subset M, U \neq M \), consider a subgroup \( \tilde{G}_U \subset \tilde{G} \) defined as follows. An element of \( \tilde{G} \) lies in \( \tilde{G}_U \) if and only if it can be represented by a Hamiltonian flow \( \{f_t\}_{t \in [0;1]} \), with \( f_0 = 1 \), generated by a Hamiltonian \( F_t \) satisfying condition (1). Formula (2) gives rise to a well defined homomorphism

\[ \tilde{\mathcal{C}}al : \tilde{G}_U \to \mathbb{R}. \]

We wish to extend the family of Calabi homomorphisms

\[ \{\tilde{\mathcal{C}}al_U\}_{U \in \mathcal{D}} \]

to a quasimorphism on \( \tilde{G} \). If such an extension is possible the resulting quasimorphism on \( \tilde{G} \) is also called a *Calabi quasimorphism*. The definition of a continuous function on \( \tilde{G} \) virtually repeats Definition 1.1.

The mere existence of a continuous homogeneous Calabi quasimorphism on \( \tilde{G} \) can be shown for a larger class of symplectic manifolds than the previous theorem (see below). For the manifolds listed in Theorem 1.3 such a quasimorphism on \( \tilde{G} \) actually descends to \( G \): for the cases other than \( \mathbb{C}P^n \) \( (n \geq 3) \) this is due to finiteness of the fundamental group \( \pi_1(G) \); in the case
$\mathbb{C}P^n$ ($n \geq 3$) when $\pi_1(G)$ is unknown, the proof relies on a delicate argument due to P. Seidel and based on his work [43].

Now we are going to formulate a result concerning the existence of a homogeneous Calabi quasimorphism on $\tilde{G}$. It will hold for so called spherically monotone symplectic manifolds. Recall that a closed connected symplectic manifold $(M, \omega)$ is called spherically monotone if there exists a real constant $\kappa > 0$ such that

$$(c_1(M), A) = \kappa \cdot ([\omega], A) \text{ for all } A \in \pi_2(M).$$

Here $c_1(M)$ stands for the first Chern class of the symplectic bundle $TM \to M$ equipped with an $\omega$-compatible almost complex structure $J$ on $M$, where $\omega$-compatibility means that the form $\omega(\cdot, J\cdot)$ is a Riemannian metric on $M$ (such an almost complex structure is homotopically unique [17]).

A crucial character of our story is the even-dimensional quantum homology algebra $QH_{ev}(M)$ (see [22], [24], [38], [39], [46]) over the field $k = \mathbb{C}[\![s]\!]$. Elements of $k$ are formal Laurent series $\sum_{j \in \mathbb{Z}} z_j s^j$ where $z_j \in \mathbb{C}$, $s$ is a formal variable and all $z_j$ vanish for large enough positive $j$. The even-dimensional quantum homology is a commutative Frobenius algebra with unity whose vector space structure is given by $H_{ev}(M) \otimes_k C$. The product on $QH_{ev}(M)$ is a certain deformation of the homological intersection product

$$\cap : H_{ev}(M) \otimes H_{ev}(M) \to H_{ev}(M).$$

Set $P = [\text{point}] \in H_0(M)$. The Frobenius structure on $QH_{ev}(M)$ is given by a non-degenerate bilinear $k$-valued form $\Delta$ which associates to a pair of quantum homology classes $a, b \in QH_{ev}(M)$ the coefficient at $P$ in their quantum product $a \ast b \in H_{ev}(M) \otimes_k C$. We refer to 2.3 below for brief preliminaries on quantum homology and to [24] for a detailed exposition.

Recall that a commutative algebra $Q$ over a field $k$ is called semi-simple if it splits into a direct sum of fields as follows: $Q = Q_1 \oplus \ldots \oplus Q_d$, where

- each $Q_i \subset Q$ is a finite-dimensional linear subspace over $k$;
- each $Q_i$ is a field with respect to the induced ring structure;
- The multiplication in $Q$ respects the splitting:

$$(a_1, \ldots, a_d) \cdot (b_1, \ldots, b_d) = (a_1 b_1, \ldots, a_d b_d).$$

\(^1\)By $H_*(M)$ we always denote the singular homology groups of $M$ with complex coefficients, and $H_{ev}(M)$ stands for its even part.
The semi-simplicity of a Frobenius algebra $Q$ over a field $k$ can be checked using a criterion due to L.Abrams [1] which says that a Frobenius algebra is semi-simple if and only if its Euler class is invertible. Recall that the Euler class $E$ of a Frobenius algebra $Q$ is defined as

$$E = \sum_i e_i e_i^\#,$$

where $\{e_i\}$ is a basis of $Q$ over $k$ and $\{e_i^\#\}$ is the dual basis with respect to the non-degenerate bilinear form on $Q$ defining the Frobenius structure. The Euler class does not depend on the choice of the basis $\{e_i\}$.

**Theorem 1.5.** Let $(M, \omega)$ be a closed connected spherically monotone symplectic manifold. Suppose that the quantum homology algebra $QH_{ev}(M)$ is semi-simple. Then there exists a continuous homogeneous Calabi quasimorphism $\tilde{\mu} : \tilde{G} \to \mathbb{R}$.

**Remark 1.6.**

1) We do not know whether such a quasimorphism $\tilde{\mu}$ is unique.

2) The quasimorphism $\tilde{\mu}$ can be calculated on the subgroup $\pi_1(G) \subset \tilde{G}$ in terms of the Seidel action of $\pi_1(G)$ on the quantum homology of $M$ (see Section 4).

3) The Hofer metric on $G$ can be lifted to a (bi-invariant) pseudo-metric on $\tilde{G}$. The quasimorphism $\tilde{\mu}$ is Lipschitz with respect to this pseudo-metric (see Section 3.6).

Examples of symplectic manifolds $M$ with semi-simple quantum homology algebra $QH_{ev}(M)$ include, in particular, $S^2$, $S^2 \times S^2$, $\mathbb{C}P^n$, $\mathbb{C}P^2$ blown up at one point and complex Grassmannians with the usual monotone symplectic structures. To get the semi-simplicity of $QH_{ev}(M)$ in these cases one can use the known explicit descriptions of the multiplicative structure of $QH_*(M)$ to check that the Euler class is invertible so that the Abrams criterion can be applied. For more details on the first four examples see Section 2.3. In the case of a complex Grassmannian the structure of the quantum homology algebra is described in [3], [14], [17]. In such a case the Euler class is an integral multiple of $P = [\text{point}]$ (see [1], [3]) which is invertible according to a computation based on [7] and due to A.Postnikov (see [13], also see [36]).
It is known that when the class of the symplectic form [ω] vanishes on π₂(\(M\)) then the product structure on \(QH_{ev}(M) = H_{ev}(M) \otimes k\) is given by the ordinary intersection product \(\cap\), and so is never semi-simple. Thus our result does not apply to those spherically monotone symplectic manifolds \((M,\omega)\) where [ω] vanishes identically on π₂(\(M\)).

1.2 Applications and discussion

1.2.1 The commutator norm

Let \(G\) be a group and \([G,G]\) be its commutator subgroup. Every element \(h \in [G,G]\) can be written as a product of simple commutators \(fgf^{-1}g^{-1}, f,g \in G\). The commutator norm \(||h||\) is by definition the minimal number of simple commutators needed in order to represent \(h\). It is known (see e.g. [3], [4]) that for a homogeneous quasimorphism \(\mu : G \to \mathbb{R}\) one has \(^{2}\)

\[ ||h|| \geq \text{const}(\mu) \cdot \mu(h), \quad h \in [G,G]. \]

In particular, existence of a homogeneous quasimorphism which does not vanish on \([G,G]\) implies that the diameter of the group \([G,G]\) with respect to the commutator norm is infinite. Recall that \(G\) is called perfect if \(G = [G,G]\).

A.Banyaga [2] proved that the group \(G = \text{Ham}(M,\omega)\) and its universal cover \(\tilde{G}\) are perfect for every closed symplectic manifold \((M,\omega)\). As an immediate consequence of our results we get the following

**Corollary 1.7.** Let \((M,\omega)\) be a closed connected spherically monotone symplectic manifold, \(G = \text{Ham}(M,\omega)\). Suppose that the quantum homology algebra \(QH_{ev}(M)\) is semi-simple. Let \(U \in \mathcal{D}\) be a displaceable open subset. Then

\[ ||\tilde{f}|| \geq \text{const} \cdot |\text{Cal}(\tilde{f})| \]

for every \(\tilde{f} \in \tilde{G}_U\). If in addition the fundamental group \(\pi_1(G)\) is finite then

\[ ||f|| \geq \text{const} \cdot |\text{Cal}(f)| \]

for every \(f \in G_U\) provided \(U \in \mathcal{D}_{ex}\).

The second part of the corollary follows from Proposition 3.4 below which says that if \(\pi_1(G)\) is finite then the Calabi quasimorphism \(\tilde{\mu}\) on \(\tilde{G}\) descends

\(^{2}\)Here and below \(\text{const}\) stands for a positive constant.
to a homogeneous Calabi quasimorphism $\mu$ on $G$. Corollary 1.7 generalizes a result obtained in a recent work [13] which served as the starting point for the present research.

1.2.2 Quantitative fragmentation lemma

Let $\{U_1, ..., U_m\}$ be an open covering of a closed connected symplectic manifold $(M, \omega)$. Banyaga’s fragmentation lemma states that any element $f \in G$ can be written as a product of diffeomorphisms $g_i$ as follows. Each $g_i$ lies in $G_{U_j}$ for some $j \in \{1; ..., m\}$ and moreover it is contained in the kernel of the Calabi homomorphism. Denote by $l(f)$ the minimal number of $g_i$’s needed in order to represent $f$.

Suppose now that $(M, \omega)$ is one of the manifolds listed in Theorem 1.3, and all the sets $U_j$ lie in $D_{ex}$. The following is an immediate consequence of Theorem 1.3.

**Corollary 1.8.**

$$l(f) \geq \text{const} \cdot |\text{Cal}(f)|$$

for every $f \in G_{U}$ provided $U \in D_{ex}$.

1.2.3 Asymptotic growth of one-parametric subgroups

We recall a few known definitions. Denote by $\mathcal{F}$ the space of all smooth Hamiltonian functions $F : M \times S^1 \to \mathbb{R}$ which satisfy the following normalization condition: $\int_M F_t \omega^n = 0$ for all $t \in S^1$, where $F_t = F(\cdot, t)$. Introduce the $C^0$-norm on $\mathcal{F}$ by

$$\|F\|_{C^0} = \max_{M} F - \min_{M} F. \quad (5)$$

A distance between the identity $1$ and an element $f$ of the group $G$ is defined [18] as

$$\rho(1, f) = \inf_{F} \int_{S^1} \|F_t\|_{C^0} \, dt,$$

where the infimum is taken over all time-dependent Hamiltonians $F \in \mathcal{F}$ generating $f$. The distance function $\rho$ gives rise to a bi-invariant non-degenerate metric on $G$ [18], [20], [33], called the *Hofer metric*.

A *time-independent* Hamiltonian $F \in \mathcal{F}$ generates a one-parametric subgroup $\{\psi^F_t\}$ of $G$ so that $\psi^F_t = \psi^1_t$. The *asymptotic growth* of the subgroup
\{\psi'_F\} is defined as

\[ \zeta(F) = \lim_{t \to +\infty} \frac{\rho(1, \psi'_F)}{t\|F\|_{C^0}}. \]

Such a limit always exists and belongs to [0, 1].

**Corollary 1.9.** Let \( M = S^2 \). Then for a generic \( F \)

\[ \zeta(F) > 0. \]

The proof can be found in Section 5.5. It relies on the estimate

\[ \zeta(F) \geq \frac{|\mu(\psi_F)|}{\|F\|_{C^0}}, \tag{6} \]

which holds for the specific quasimorphism \( \mu \) constructed in the proof of Theorem 1.3 (see Section 5.5), and on the explicit computation of the value of \( \mu : G \to \mathbb{R} \) on \( \psi'^m_F \), \( m = 1, 2, \ldots \), in the case \( M = S^2 \) made in Section 5.4.

**Remark 1.10.**

1) In fact, the inequality \( \zeta(F) > 0 \) for a generic Hamiltonian \( F \) is valid on all closed symplectic surfaces. The case of the 2-torus is settled in [35], Section 8.4. The argument given in [35] actually works for any closed symplectic surface of positive genus.

2) For symplectic manifolds listed in Theorem 1.3 inequality (6) immediately produces examples of 1-parametric subgroups of \( G \) with positive \( \zeta(F) \) – for example, take an autonomous Hamiltonian \( F \) supported in a sufficiently small ball and such that the Calabi invariant of \( \psi_F \) is non-zero. This shows, in particular, that for these manifolds the group \( G \) has infinite diameter with respect to the Hofer metric.

### 1.2.4 Other quasimorphisms?

J.Barge and E.Ghys [3] constructed a quasimorphism of a different nature on the group of compactly supported symplectomorphisms of a standard symplectic ball. The Barge-Ghys quasimorphism is closely related to the Maslov class in symplectic geometry. This construction was later generalized in [13] to closed symplectic manifolds \((M, \omega)\) with \( c_1(M) = 0 \) (e.g. tori
and K3-surfaces). As a result one gets a homogeneous quasimorphism on \( \tilde{\text{Symp}}_0 (M, \omega) \) which does not vanish on \( \tilde{\text{Ham}} (M, \omega) \) \cite{13}. Here \( \tilde{\text{Symp}}_0 (M, \omega) \) is the universal cover of the identity component of the group of symplectomorphisms of \((M, \omega)\).

Interestingly enough, this class of manifolds is disjoint from the one considered in the present paper. We believe however that the class of manifolds admitting a Calabi quasimorphism can be enlarged, namely the spherical monotonicity condition can be removed. Such a generalization should go along the same lines though the technicalities will become more complicated. On the other hand, semi-simplicity of the quantum homology algebra seems to be a crucial assumption.

No other quasimorphism of \( G \) and \( \tilde{G} \) is known at the moment. The simplest symplectic manifolds for which no information on quasimorphisms and the commutator norm is available at all are closed oriented surfaces of genus \( \geq 2 \). Any progress in this direction would be very interesting.

1.2.5 Continuum of Calabi quasimorphisms on an open surface

The notion of Calabi quasimorphism can be extended in a straightforward way to open symplectic manifolds. It turns out that even for very simple manifolds Calabi quasimorphisms can form an infinite-dimensional affine space. For an open connected symplectic manifold \( M \) denote by \( G_M \) the group of all Hamiltonian diffeomorphisms of \( M \) generated by Hamiltonians with compact support in \( M \times S^1 \). It is known that formula (2) gives rise to the well defined Calabi homomorphism \( \text{Cal}_M : G_M \to \mathbb{R} \). Up to a multiple, this is the only homomorphism \( G_M \to \mathbb{R} \).

**Theorem 1.11.** Suppose that either \( M \subset \mathbb{R}^2 \) is an open disk of finite area, or \( M \subset T^* S^1 \) is an open annulus of finite area. There exists a family \( \mu_\epsilon, \epsilon \in \mathbb{R} \), of continuous homogeneous Calabi quasimorphisms on \( G_M \) with the following properties:

- Given a finite subset \( I \subset \mathbb{R} \), the quasimorphisms \( \mu_\epsilon, \epsilon \in I \), are linearly independent over \( \mathbb{R} \). In particular, the 2-nd bounded cohomology of \( G_M \) is an infinite-dimensional space over \( \mathbb{R} \);

- Moreover, if \( M \) is an annulus, the quasimorphisms can be chosen so that every \( \mu_\epsilon \) coincides with the Calabi homomorphism \( \text{Cal}_M \) on the
subgroup $G_U$, where $U$ is the interior of any (not necessarily displaceable!) embedded closed disk in $M$.

The proof is given in Section 5.6 below. It seems likely that analogous results hold true for any other open surface of genus 0 with finite area. It would be also interesting to find a generalization to higher dimensions, for instance to the cases when $M$ is either the standard symplectic open ball, or the open unit coball bundle of the flat $n$-dimensional torus. Let us mention also that J.-M. Gambaudo [15] suggested a different approach which could lead to an infinite sequence of non-trivial quasimorphisms on $G_M$ in the case when $M$ is a 2-dimensional disk.

2 Symplectic preliminaries

2.1 Starting notations

Let $(M^{2n}, \omega)$ be a closed spherically monotone symplectic manifold. Consider

\[ \tilde{\pi}_2(M) = \pi_2(M) \sim, \]

where by definition $A \sim B$ iff $([\omega], A) = ([\omega], B)$. Clearly, both $[\omega]$ and $c_1(M)$ descend to homomorphisms of $\tilde{\pi}_2(M)$. In view of the comment at the very end of Section 1.1, we will always assume that $[\omega]$ does not vanish on $\tilde{\pi}_2(M)$. In particular, the group $\tilde{\pi}_2(M)$ is the infinite cyclic group, and it has a generator $S$ so that $\Omega := ([\omega], S) > 0$. Set $N := (c_1(M), S) > 0$. As above $\tilde{G}$ stands for the universal cover of $\text{Ham}(M, \omega)$.

2.2 Gromov-Witten invariants

The Gromov-Witten invariant $GW_j$, $j \in \mathbb{N}$, is a 3-linear (over $\mathbb{C}$) $\mathbb{C}$-valued form on $H_*(M)$. It is defined along the following lines (see [24], [38], [39] for the precise definition). First of all, $GW_j(A,B,C) = 0$ unless

\[ \deg A + \deg B + \deg C = 4n - 2Nj. \]

If the equality above holds, we assume without loss of generality that the homology classes $A, B$ and $C$ are represented by smooth submanifolds $\hat{A}, \hat{B}$ and $\hat{C}$ respectively. Take an $\omega$-compatible almost complex structure $J$ on $M$. Consider the following elliptic problem:
Find all $J$-holomorphic maps $\mathbb{CP}^1 \to (M, J)$ which represent the class $jS \in \bar{\pi}_2(M)$ and which send the points $0, 1, \infty \in \bar{\mathbb{C}} \cup \{\infty\} = \mathbb{CP}^1$ to $\hat{A}, \hat{B}$ and $\hat{C}$ respectively.

When the almost complex structure $J$ and the submanifolds $\hat{A}, \hat{B}, \hat{C}$ are chosen in a generic way, the set of solutions of this problem is finite. The number $GW_j(A, B, C)$ is defined as the number of the solutions counted with an appropriate sign. It is useful to have in mind that if $J$ is a genuine complex structure and $\hat{A}, \hat{B}, \hat{C}$ are generic complex submanifolds, the sign in question is positive.

### 2.3 Quantum homology algebra

As a vector space over $\mathbb{C}$ the quantum homology $QH_*(M)$ is isomorphic to $H_*(M) \otimes \mathbb{C} k$, where $k$ stands for the field $\mathbb{C}[[s]]$ which appeared in Section 1.1. The quantum multiplication $a \ast b$, $a, b \in QH_*(M)$, is defined as follows. For $A, B \in H_*(M)$ and $j \in \mathbb{N}$ define $(A \ast B)_j \in H_*(M)$ as the unique class which satisfies

$$(A \ast B)_j \circ C = GW_j(A, B, C)$$

for all $C \in H_*(M)$. Here $\circ$ stands for the ordinary intersection index in homology. Now for any $A, B \in H_*(M)$ set

$$A \ast B = A \cap B + \sum_{j \in \mathbb{N}} (A \ast B)_j s^{-j} \in QH_*(M).$$

By $k$-linearity extend the quantum product to the whole $QH_*(M)$. As a result one gets a correctly defined skew-commutative associative product operation on $QH_*(M)$ which is a deformation of the classical $\cap$-product in singular homology $[22], [24], [38], [39], [46]$.

The field $k$ has a ring grading defined by the condition that the grade of $s$ equals $2N$. Such a grading on $k$ together with the usual grading on $H_*(M)$ define a grading on $QH_*(M) = H_*(M) \otimes k$. If $a, b \in QH_*(M)$ have graded degrees $\deg(a), \deg(b)$ then $\deg(a \ast b) = \deg(a) + \deg(b) - 2n$.

The fundamental class $[M]$ is the unity with respect to the quantum multiplication. If $A \in H_*(M)$, $\varsigma \in k$, we will denote the elements $A \otimes 1, [M] \otimes \varsigma$ of $QH_*(M) = H_*(M) \otimes k$ respectively by $A$ and $\varsigma$. The even part $QH^e_*(M) : = H^e_*(M) \otimes k$ is a commutative subalgebra of $QH_*(M)$. 

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The algebra $QH_{ev}(M)$ is a Frobenius algebra over $k$. Consider the pairing

$$
\Delta : QH_{ev}(M) \times QH_{ev}(M) \rightarrow k, \\
\Delta\left( \sum A_j s^j, \sum B_l s^l \right) = \sum (A_j \circ B_l) \cdot s^{j+l}.
$$

In fact $\Delta$ associates to a pair of quantum homology classes $a, b \in QH_{ev}(M)$ the coefficient at $P = [\text{point}]$ in their quantum product $a \ast b \in H_{ev}(M) \otimes \mathbb{C}$. The pairing $\Delta$ defines a Frobenius algebra structure, which means that $\Delta$ is non-degenerate and

$$
\Delta(a, b) = \Delta(a \ast b, [M]) \quad \text{for all } a, b \in QH_{ev}(M).
$$

Let $\tau : k \rightarrow \mathbb{C}$ be the map sending $\sum z_j s^j$ to $z_0$. Define a $\mathbb{C}$-valued pairing

$$
\Pi(a, b) = \tau \Delta(a, b) = \tau \Delta(a \ast b, [M])
$$

on $QH_{ev}(M)$. It will play an important role below.

### 2.3.1 Example: $S^2$

Let $M$ be the 2-sphere $S^2$. Set $P = [\text{point}]$. Note that $GW_1(P, P, P) = 1$, and moreover this is the only non-vanishing Gromov-Witten invariant. Hence $P \ast P = s^{-1}$ and $QH_*(M)$ is a field:

$$
QH_*(M) = \frac{k[P]}{P^2 = s^{-1}}.
$$

Thus $QH_*(M) = QH_{ev}(M)$ is semi-simple.

### 2.3.2 Example: $\mathbb{C}P^n$

The previous example can be generalized. Let $M = \mathbb{C}P^n$ be equipped with the standard Fubini-Study symplectic form. This is a spherically monotone symplectic manifold. Let $A \in H_{2n-2}(M)$ be the projective hyperplane class. Then

$$
QH_*(M) = \frac{k[A]}{A^{n+1} = s^{-1}},
$$

(see [38], [39], [46]). One immediately sees that $QH_*(M) = QH_{ev}(M)$ is a field over $k = \mathbb{C}[[s]]$ and therefore it is a semi-simple algebra.
2.3.3 Example: $S^2 \times S^2$

Let $M = S^2 \times S^2$ be equipped with the split symplectic form $\omega \oplus \omega$, where $\omega$ is an area form on $S^2$. This is a spherically monotone symplectic manifold. Let $A, B \in H_2(M)$ be the homology classes of $S^2 \times pt$ and $pt \times S^2$. The elements $P = [pt], A, B, [M]$ form a basis of $QH_*(M)$ over $k$ and the multiplicative relations are completely defined by the identities:

$$A \ast B = P, \quad A^2 = B^2 = s^{-1}.$$

Thus the Euler class is

$$\mathcal{E} = 2P \ast [M] + 2A \ast B = 4P.$$ 

The classes $A, B$ are invertible and so are $P$ and $\mathcal{E}$. Therefore, according to the Abrams criterion, the algebra $QH_*(M) = QH_{ev}(M)$ is semi-simple. Note that $QH_{ev}(M)$ is not a field since it contains divisors of zero: $(A - B) \ast (A + B) = A^2 - B^2 = 0$.

2.3.4 Example: $\mathbb{C}P^2$ blown up at one point

Let $M$ be the complex blow up of $\mathbb{C}P^2$ at one point equipped with a monotone symplectic form (see [23],[34]). Its quantum homology algebra is described in [23] as follows. Let $P = [\text{point}]$. Denote by $A$ the exceptional divisor and set $B = [\mathbb{C}P^1] - A$. Together with the fundamental class $[M]$ the classes $P, A, B$ generate $QH_*(M)$ as a vector space over $k$. The multiplicative relations are as follows (recall that $[M]$ is the unity element in the quantum homology algebra):

$$
\begin{align*}
P \ast P &= (A + B)s^{-3} & A \ast P &= Bs^{-2} \\
P \ast B &= s^{-3} & A \ast A &= -P + As^{-1} + s^{-2} \\
A \ast B &= P - As^{-1} & B \ast B &= As^{-1}.
\end{align*}
$$

The Euler class $\mathcal{E}$ is easily computable:

$$\mathcal{E} = P \ast [M] + A \ast B + B \ast (A + B) + [M] \ast P = 4P - As^{-1}.$$ 

One can check that $\mathcal{E}$ is invertible:

$$\mathcal{E}^{-1} = \frac{1}{283}(-12Ps^4 + 9As^3 + 73Bs^3 + 16s^2).$$

Therefore, according to the Abrams criterion, $QH_*(M) = QH_{ev}(M)$ is a semi-simple algebra.
2.4 The action functional

Let $\Lambda$ be the space of all smooth contractible loops $x : S^1 = \mathbb{R}/\mathbb{Z} \to M$. Consider a covering $\tilde{\Lambda}$ of $\Lambda$ whose elements are equivalence classes of pairs $(x, u)$, where $x \in \Lambda$, $u$ is a disk spanning $x$ in $M$ and the equivalence relation is defined as follows: $(x_1, u_1) \sim (x_2, u_2)$ iff $x_1 = x_2$ and the 2-sphere $u_1 \# (-u_2)$ vanishes in $\tilde{\pi}_2(M)$. The equivalence class of a pair $(x, u)$ will be denoted by $[x, u]$. The group of deck transformations of this covering can be naturally identified with $\tilde{\pi}_2(M)$. The generator $S$ acts by the transformation $s : \tilde{\Lambda} \to \tilde{\Lambda}$ so that

$$s([x, u]) = [x, u\#(-S)]. \tag{8}$$

Recall that by $\mathcal{F}$ we denote the space of all smooth Hamiltonian functions $F : M \times S^1 \to \mathbb{R}$ which satisfy the following normalization condition: $\int_M F(\cdot, t) \omega^n = 0$ for any $t \in S^1$. For $F \in \mathcal{F}$ define the action functional

$$A_F([x, u]) := \int_{S^1} F(x(t), t) dt - \int_u \omega$$

on $\tilde{\Lambda}$. Note that

$$A_F(sy) = A_F(y) + \Omega \tag{9}$$

for all $y \in \tilde{\Lambda}$.

Let $\mathcal{P}_F \subset \Lambda$ be the set of all contractible 1-periodic orbits of the Hamiltonian flow generated by $F$. Its full lift $\tilde{\mathcal{P}}_F$ to $\tilde{\Lambda}$ coincides with the set of critical points of $A_F$. We define the action spectrum $\text{spec}(F)$ as the set of critical values of $A_F$. This is a closed nowhere dense $\Omega \mathbb{Z}$-invariant subset of $\mathbb{R}$ [28], [41].

2.5 Filtered Floer homology

For a generic Hamiltonian $F \in \mathcal{F}$ and $\alpha \in (\mathbb{R} \setminus \text{spec}(F)) \cup \{+\infty\}$ define a complex vector space $C_\alpha(F)$ as the set of all formal sums

$$\sum_{y \in \tilde{\mathcal{P}}_F} z_y y,$$

where $z_y \in \mathbb{C}$, $A_F(y) < \alpha$, which satisfy the following finiteness condition:

$$\# \{ y \mid z_y \neq 0 \text{ and } A_F(y) > \delta \} < \infty$$
for every $\delta \in \mathbb{R}$. Formula (8) defines a structure of the vector space over $k$ on $C_{\infty}(F)$.

Given a loop $\{J_t\}, t \in S^1$, of $\omega$-compatible almost complex structures, define a Riemannian metric on $\Lambda$ by

$$(\xi_1, \xi_2) = \int_0^1 \omega(\xi_1(t), J_t \xi_2(t)) dt,$$

where $\xi_1, \xi_2 \in T\Lambda$. Lift this metric to $\tilde{\Lambda}$ and consider the negative gradient flow of the action functional $\mathcal{A}_F$. For a generic choice of the Hamiltonian $F$ and the loop $\{J_t\}$ the count of isolated gradient trajectories connecting critical points of $\mathcal{A}_F$ gives rise in the standard way [14], [19] to a Morse-type differential

$$d : C_{\infty}(F) \to C_{\infty}(F), \ d^2 = 0. \quad (10)$$

The differential $d$ is $k$-linear. Moreover it preserves $C$-subspaces $C_\alpha(F) \subset C_{\infty}(F)$ for all $\alpha \in \mathbb{R}$.

The complexes $(C_\alpha(F), d)$ have a natural grading according to the Conley-Zehnder index $\text{ind} : \hat{\mathcal{P}}_F :\to \mathbb{Z}$ (see [11]) which satisfies $\text{ind}(sy) = \text{ind}(y) + 2N$ for every $y \in \hat{\mathcal{P}}_F$. Note that different authors use slightly different versions of the Conley-Zehnder index. In order to fix our convention consider the case of a sufficiently $C^2$-small autonomous Morse Hamiltonian $F$. Then the Conley-Zehnder index $\text{ind}(y)$ of an element $y \in \hat{\mathcal{P}}_F$, represented by a pair $(x, u)$ consisting of a critical point $x$ of $F$ and the trivial disk $u$, is equal to the Morse index of $x$. In what follows we are interested in the even part of the homology of these complexes.

Notice that in spite of the involvement of the almost complex structures $\{J_t\}, t \in S^1$, in the definition of the complex $(C_{\infty}(F), d)$, different choices of $\{J_t\}$ lead to complexes whose homologies are related by natural isomorphisms preserving the filtration as long as the Hamiltonian $F$ is fixed [14], [19], [28], [30]. Because of this we will suppress the dependence on $\{J_t\}$, and define

$$V_\alpha(F) = H_{ev}(C_\alpha(F), d) \text{ and } V^\alpha(F) = H_{ev}(C_{\infty}(F)/C_\alpha(F), d).$$

Meanwhile these homology groups have been defined for generic Hamiltonians $F$ only. Using an appropriate continuation procedure one can extend the definition to all $F \in \mathcal{F}$. Namely, let $\alpha \notin \text{spec}(F)$. Pick any generic Hamiltonians $F', F''$ such that the spaces $V_\alpha(F')$ and $V_\alpha(F'')$ are defined. Then if $F', F''$ are sufficiently $C^\infty$-close to $F$ we have that $\alpha \notin \text{spec}(F')$, etc.
\( \alpha \notin \text{spec}(F'') \), and the spaces \( V_{\alpha}(F') \) and \( V_{\alpha}(F'') \) are canonically isomorphic. Thus we can set \( V_{\alpha}(F) \) as \( V_{\alpha}(F') \) for any \( F' \) sufficiently close to \( F \), and this definition is correct. Similarly one can define \( V^\alpha(F) \) for arbitrary \( F \) and \( \alpha \notin \text{spec}(F) \).

Suppose now that two Hamiltonian functions \( F, F' \in \mathcal{F} \) generate the same element \( f \in \tilde{G} \). In this case \( \text{spec}(F) = \text{spec}(F') \). This was proved in \([41]\) (see Lemma 3.3 there) in the case when \( \omega \) vanishes on \( \bar{\pi}_2 \) and the proof readily extends to the general case (see e.g. \([30]\)). The resulting set will be denoted simply by \( \text{spec}(f) \). Moreover the vector spaces \( V_{\alpha}(F) \) and \( V^\alpha(F) \) can be canonically identified, respectively, with \( V_{\alpha}(F') \) and \( V^\alpha(F') \). Therefore we shall denote them, respectively, by \( V_{\alpha}(f) \) and \( V^\alpha(f) \), where \( \alpha \in (\mathbb{R} \setminus \text{spec}(f)) \cup +\infty \) (the dependence on \( \{J_t\} \) is suppressed for the same reasons as above). These spaces are called \textit{filtered Floer homology} of an element \( f \in \tilde{G} \).

### 2.6 Algebraic data and spectral invariants

Filtered Floer homology come with additional algebraic data which we are going to list now.

#### 2.6.1 Identification with quantum homology

All spaces \( V_{\infty}(f) \) are canonically identified (as vector spaces over \( k \)) with \( QH_{ev}(M) \) \([32]\). The different choices of a Hamiltonian \( F \) generating \( f \) and of almost complex structures \( \{J_t\} \) give rise to different Floer complexes whose homologies are related by natural isomorphisms preserving the filtration, thus leading to a well-defined space \( V_{\infty}(f) \). For each such Floer complex its homology can be canonically identified with \( QH_{ev}(M) \) (as in \([32]\)). These identifications agree for different Floer complexes and therefore lead to a well-defined identification of \( V_{\infty}(f) \) with \( QH_{ev}(M) \) which preserves the grading \([32]\). Recall that \( QH_*(M) = H_*(M) \otimes_{\mathbb{C}} k \) carries a grading, while \( V_{\infty}(f) \) is graded by the Conley-Zehnder index \( \text{ind} \).

#### 2.6.2 Natural inclusions

For any \( \beta < \alpha \leq +\infty \) the natural inclusion \( C_{\beta}(F) \to C_{\alpha}(F) \) of Floer complexes leads to a homomorphism

\[ i_{\alpha\beta} : V_{\beta}(f) \to V_{\alpha}(f) \]
between their homology. Moreover \( i_{\alpha \beta} i_{\beta \gamma} = i_{\alpha \gamma} \) for any \( \gamma < \beta < \alpha \). We abbreviate \( i_\alpha \) for \( i_{\alpha \infty} \). In the case we wish to emphasize the dependence of \( i_\alpha \) on the element \( f \) we will write \( i_\alpha \{ f \} \).

### 2.6.3 Spectral invariants: finiteness and continuity

Following the works of C. Viterbo [45], Y.-G. Oh [26], [27], [28], [30] and M. Schwarz [41], [42], we give the following definition. Given \( f \in \tilde{G} \) and \( a \in QH_{ev}(M) = V_\infty(f) \), \( a \neq 0 \), set

\[
c(a, f) = \inf \{ \alpha \mid a \in \text{Image} \ i_\alpha \}.
\]

Then \(-\infty < c(a, f) < +\infty\) and for a given \( a \) the function \( c(a, f) \) is continuous with respect to the \( C^\infty \)-topology on \( \tilde{G} \). In the case when \( a \) is a singular homology class this has been proved in [28]. The proof for the general case can be found in [30]. In fact, in the case of a spherically monotone symplectic manifold, when we consider an arbitrary quantum homology class, the only detail that should be added to the proof in [28] is the following one.

A generic element \( f \in \tilde{G} \) can be defined by means of a Hamiltonian flow (generated by a Hamiltonian \( F \)) that has only a finite number of 1-periodic trajectories. Thus, since \( M \) is spherically monotone, there exist constants \( R_1, R_2 > 0 \) such that for any \( y \in \tilde{P}_F \)

\[
R_1 \cdot \text{ind}(y) - R_2 \leq A_F(y) \leq R_1 \cdot \text{ind}(y) + R_2.
\]

(11)

Now write \( a \) as a sum \( a = \sum_m a^{(m)} \) of its homogeneous graded components, with each \( a^{(m)} \) having the grading \( m \). Consider the set \( \mathcal{I} := \{ m \mid a^{(m)} \neq 0 \} \). The definition of the field \( k \) implies that \( m_0 := \max \mathcal{I} < \infty \). We are going to use the grading-preserving identification of \( QH_{ev}(M) \) with \( V_\infty(f) \) (see Section 2.6.1).

First, it follows that

\[
c(a, f) = \max_{m \in \mathcal{I}} c(a^{(m)}, f).
\]

(12)

Further, the classes \( a \) and \( a^{(m)} \) can be viewed as Floer homology classes from \( V_\infty(f) \). Let \( C^{(m)} = \sum_y z_yy, \ z_y \in \mathbb{C}, \ y \in \tilde{P}_F, \) be a Floer chain representing \( a^{(m)} \). Then \( \text{ind}(y) = m \) for any \( y \) entering \( C^{(m)} \). Using [11] we see that for any such \( y \)

\[
R_1 m - R_2 \leq A_F(y) \leq R_1 m + R_2.
\]
Therefore, since the Floer chain $C^{(m)}$ representing $a^{(m)}$ was chosen arbitrarily,

$$ R_1m - R_2 \leq c(a^{(m)}, f) \leq R_1m + R_2 \text{ for every } m \in I. $$

Combining it with (12) we see that

$$ R_1m_0 - R_2 \leq c(a, f) \leq R_1m_0 + R_2, $$

and hence $c(a, f)$ is finite. The rest of the proof in [28] of the finiteness and continuity of spectral invariants can be carried over to our case in a direct fashion. In particular, the continuity follows from the following $C^0$-estimate [28], [41]. For any Hamiltonians $F', F'' \in \mathcal{F}$ and $a \in \mathbb{Q}H_{ev}(M)$ one has

$$ \int_{S^1} - \max_M (F'_t - F''_t) \, dt \leq c(a, \tilde{\psi}_{F'}) - c(a, \tilde{\psi}_{F''}) \leq \int_{S^1} - \min_M (F'_t - F''_t) \, dt. \quad (13) $$

The numbers $c(a, f)$ are called spectral invariants of an element $f \in \tilde{G}$. One can show that they all lie in the action spectrum $\text{spec}(f)$ [28], [30], [41] (this is obviously true for a generic Hamiltonian from the original definition (10) of $d$) and persist under conjugations in $\tilde{G}$ (see [28], [30], [41], cf. [13]). Spectral invariants will play a crucial role below in the construction of the Calabi quasimorphism.

### 2.6.4 Spectral invariants as characteristic exponents

A function $\chi : V \to \mathbb{R} \cup -\infty$ on a vector space $V$ over $\mathbb{C}$ is called a characteristic exponent if

- $\chi(v) \in \mathbb{R}$ for all non-zero $v \in V$ and $\chi(0) = -\infty$;
- $\chi(\delta \cdot v) = \chi(v)$ for every non-zero $\delta \in \mathbb{C}$ and $v \in V$;
- $\chi(v_1 + v_2) \leq \max(\chi(v_1), \chi(v_2))$ for all $v_1, v_2 \in V$.

This notion (which appears in the theory of Lyapunov exponents in Dynamical Systems, see e.g. [12]) is relevant in our study of spectral invariants. It is a straightforward consequence of the definitions that for a given $f \in \tilde{G}$ the function

$$ c(\cdot, f) : \mathbb{Q}H_{ev}(M) \to \mathbb{R}, \ a \mapsto c(a, f) $$

21
is a characteristic exponent on $QH_{ev}(M)$. Starting from this observation, one can apply various known facts about characteristic exponents to the spectral invariants. For instance, for every $f \in \tilde{G}$ and $m \in \mathbb{Z}$ the set

$$\{c(a, f) \mid a \in QH_{2m}(M)\}$$

has at most $\dim CQH_{2m}(M)$ distinct elements. The well known fact which will be used below is as follows. We formulate it in the language of characteristic exponents.

**Proposition 2.1.** Let $\chi : V \to \mathbb{R}$ be a characteristic exponent. Assume that $\chi(v_1) < \chi(v_2)$. Then $\chi(v_1 + v_2) = \chi(v_2)$.

**Proof:** By definition, $\chi(v_1 + v_2) \leq \chi(v_2)$. Assume on the contrary that $\chi(v_1 + v_2) < \chi(v_2)$. Then (using that $\chi(-v_1) = \chi(v_1)$) we have

$$\chi(v_2) = \chi(-v_1 + (v_1 + v_2)) \leq \max(\chi(v_1), \chi(v_1 + v_2)) < \chi(v_2),$$

a contradiction. Hence $\chi(v_1 + v_2) = \chi(v_2)$. □

### 2.6.5 Spectral invariants of the identity

Define a function $\nu : QH_{ev}(M) \to \mathbb{Z}$ as follows. For a non-zero element $a = \sum A_j s^j \in QH_{ev}(M)$ set $\nu(a)$ to be the maximal $j$ such that $A_j \neq 0$. We claim that

$$c(a, 1_{\tilde{G}}) = \Omega \nu(a), \quad (14)$$

where $1_{\tilde{G}}$ stands for the identity in $\tilde{G}$.

Indeed, according to [28] and [11], $c(A, 1_{\tilde{G}}) = 0$ for any singular homology class $A \neq 0$. Suppose now that $a \neq 0$ is an arbitrary quantum homology class. In view of formula (12) above it suffices to prove the claim assuming that $a$ is homogeneous in the sense of the grading. In this case $a$ is given by a finite sum of the form $a = \sum_m A_m s^m$, where $A_m \in H_{ev}(M)$.

Using the $k$-linearity of the identification between $V_{\infty}(f)$ and $QH_{ev}(M)$ and formula (13) one easily gets that

$$c(sb, f) = c(b, f) + \Omega \text{ for all } b \in QH_{ev}(M), \ f \in \tilde{G}.$$ 

Therefore $c(A_m s^m, 1_{\tilde{G}}) = \Omega m + c(A_m, 1_{\tilde{G}}) = \Omega m$ for any $m$ such that $A_m \neq 0$. In view of Proposition 2.1

$$c(a, 1_{\tilde{G}}) = \max_{m : A_m \neq 0} \Omega m = \Omega \nu(a),$$

22
and the claim follows (cf. [30]).

In fact, given an arbitrary $a \in \mathcal{QH}_{ev}(M)$ one can calculate $c(a, f)$ not only for $f = 1_{\tilde{G}}$ but for any $f \in \pi_1(G) \subset \tilde{G}$, i.e. for any lift of $1 \in G$ to $\tilde{G}$ – see Section 4.

### 2.6.6 Pair-of-pants product

There exists a so-called pair-of-pants product [32]

$$V_{\alpha}(f) \times V_{\beta}(g) \to V_{\alpha + \beta}(fg), \quad (v, w) \mapsto v \ast_{PP} w.$$ 

It agrees with the quantum product, namely

$$i_{\alpha + \beta}(v \ast_{PP} w) = i_{\alpha}(v) \ast i_{\beta}(w).$$

This immediately yields the following triangle inequality for spectral invariants:

$$c(a \ast b, fg) \leq c(a, f) + c(b, g), \quad a, b \in \mathcal{QH}_{ev}(M).$$

### 2.6.7 Natural projections

The natural projection of Floer complexes $C_{\infty}(F) \to C_{\infty}(F)/C_{\alpha}(F)$ induces a homomorphism

$$\pi_{\alpha} : \mathcal{QH}_{ev}(M) = V_{\infty}(f) \to V_{\alpha}(f).$$

between their homology. The homological exact sequence yields Kernel $\pi_{\alpha} = \text{Image } i_{\alpha}$. In the case we wish to emphasize the dependence of $\pi_{\alpha}$ on the element $f$ we will write $\pi_{\alpha}\{f\}$.

### 2.6.8 Poincaré duality

Each critical point $y$ of $\mathcal{A}_F$ of Conley-Zehnder index $\text{ind}(y)$ is also a critical point of $\mathcal{A}_{-F}$ of Conley-Zehnder index $2n - \text{ind}(y)$. Moreover there exists a non-degenerate intersection pairing between the Floer complex associated to $F$ and the Floer complex of $-F$ leading to the Poincaré duality in the Floer homology theory similarly to the situation in the classical Morse homology theory. One can show that for every $\alpha \in \mathbb{R}$ the space $V_{\alpha}(f^{-1})$ is canonically isomorphic to $\text{Hom}(V^{-\alpha}(f), \mathbb{C})$. This isomorphism gives rise to a non-degenerate pairing $L : V_{\alpha}(f^{-1}) \times V^{-\alpha}(f) \to \mathbb{C}$ which agrees with the
intersection pairing $\Pi$ on the quantum homology (see equation (7) of Section 2.3):

$$\Pi(i_\alpha\{f\}^{-1}a, b) = L(a, \pi_{-\alpha}\{f\}b)$$

for every $a \in V_\alpha(f^{-1})$ and $b \in QH_{ev}(M)$. These statements can be extracted from [32].

### 2.7 Comparing spectral invariants of $f$ and $f^{-1}$

For an element $b \in QH_{ev}(M) \setminus \{0\}$ denote by $\Upsilon(b)$ the set of all $a \in QH_{ev}(M)$ with $\Pi(a, b) \neq 0$.

**Lemma 2.2.**

$$c(b, f) = -\inf_{a \in \Upsilon(b)} c(a, f^{-1})$$

for all $b \in QH_{ev} \setminus \{0\}$ and $f \in \tilde{G}$.

**Proof:** Set

$$\delta = \inf_{a \in \Upsilon(b)} c(a, f^{-1}).$$

The proof is divided into two steps.

1) Take arbitrary $\epsilon > 0$ and set $\alpha = \epsilon - c(b, f)$. Then $b \not\in$ Image $i_{-\alpha}\{f\}$, so by 2.6.7

$$w := \pi_{-\alpha}\{f\}b \neq 0.$$

Since the pairing $L$ is non-degenerate, there exists $v \in V_\alpha(f^{-1})$ such that $L(v, w) \neq 0$. Put $a_0 = i_\alpha\{f^{-1}\}v$. By 2.6.8 we have $L(v, w) = \Pi(a_0, b) \neq 0$. We see that $c(a_0, f^{-1}) \leq \alpha$, so $\delta \leq \alpha = \epsilon - c(b, f)$. Since this inequality holds for every $\epsilon > 0$ we conclude that $\delta \leq -c(b, f)$.

2) Take arbitrary $\epsilon > 0$ and set $\alpha = -c(b, f) - \epsilon$. Then $b \in$ Image $i_{-\alpha}\{f\}$, so by 2.6.7 $\pi_{-\alpha}\{f\}b = 0$. Assume that there exists $a \in \Upsilon(b)$ such that $c(a, f^{-1}) < \alpha$. Then $a \in$ Image $i_\beta\{f^{-1}\}$ for some $\beta < \alpha$, and hence $a \in$ Image $i_\alpha\{f^{-1}\}$ in view of 2.6.2. Take $v \in V_\alpha(f^{-1})$ so that $a = i_\alpha\{f^{-1}\}v$. By 2.6.8

$$\Pi(a, b) = L(v, \pi_{-\alpha}\{f\}b) = 0,$$

and we get a contradiction with the assumption $\Pi(a, b) \neq 0$. Hence $c(a, f^{-1}) \geq \alpha$ for every $a \in \Upsilon(b)$, so $\delta \geq \alpha = -c(b, f) - \epsilon$. Since this is true for every $\epsilon > 0$ we get that $\delta \geq -c(b, f)$. Combining this with the inequality proved in Step 1 we conclude that $\delta = -c(b, f)$ as required.
3 Constructing the quasimorphism

3.1 Spectral numbers define a quasimorphism $r$ on $\tilde{G}$

Suppose that the algebra $Q = QH_{ev}(M)$ is semi-simple, and let $Q = Q_1 \oplus \ldots \oplus Q_d$ be its decomposition into the direct sum of fields. Denote by $e$ the unity of $Q_1$.

**Theorem 3.1.** The function 
\[ r : \tilde{G} \to \mathbb{R}, \quad f \mapsto c_1(e, f) \]

is a quasimorphism.

The proof is given in Section 3.3 below. For the proof we need the following lemma.

3.2 A lemma from non-Archimedean geometry

Let $\nu : Q \to \mathbb{Z}$ be the function introduced in \[2.6.3\].

**Lemma 3.2.** There exists $R > 0$ such that $\nu(b) + \nu(b^{-1}) \leq R$ for every $b \in Q_1 \setminus \{0\}$.

We are grateful to V. Berkovich for explaining to us the proof. The reader is referred to [16] for preliminaries on non-Archimedean geometry.

**Proof:** For $\zeta \in k$ set $|\zeta| = \exp \nu(\zeta)$. Then $||$ is a non-Archimedean absolute value on $k$, and the field $k$ is complete with respect to $|\cdot|$. For $b \in Q_1$ put $||b|| = \exp \nu(b)$. Then $||$ is a norm on $Q_1$, where $Q_1$ is considered as a vector space over $k$. Since the field $Q_1$ is a finite extension of $k$, the absolute value $|\cdot|$ extends to an absolute value $|||\cdot|||$ on $Q_1$. Furthermore, all norms on a finite-dimensional space over $k$ are equivalent. Thus there exists $\delta > 0$ so that 
\[ ||b|| \leq \delta \cdot |||b||| \]
for every $b \in Q_1$. Therefore for $b \neq 0$
\[ ||b|| \cdot ||b^{-1}|| \leq \delta^2 \cdot |||b||| \cdot |||b^{-1}||| \]
where the last equality follows from the definition of the absolute value. Therefore $\nu(b) + \nu(b^{-1}) \leq R$ with $R = 2 \log \delta$. This completes the proof. \[\square\]
3.3 Proof of Theorem 3.1

Note that $e \ast e = e$. By the inequality in Section 2.6.6 we have

$$c(e, fg) = c(e \ast e, fg) \leq c(e, f) + c(e, g).$$  \hspace{1cm} (15)

Similarly,

$$c(e, fg) \geq c(e, f) - c(e, g^{-1}).$$

Applying Lemma 2.2 we get that

$$c(e, fg) \geq c(e, f) + \inf_{b : \Pi(b, e) \neq 0} c(b, g).$$ \hspace{1cm} (16)

Our next goal is to find a lower bound for $c(b, g)$ provided $\Pi(b, e) \neq 0$. Write $b = b_1 + \ldots + b_d$ where $b_i \in Q_i$ for all $i = 1, \ldots, d$. Then formula (7) of Section 2.3 yields

$$\Pi(b, e) = \tau \Delta(b \ast e, [M]) = \tau \Delta(b_1, [M]) \neq 0.$$

This immediately implies that $b_1 \neq 0$ and $\nu(b_1) \geq 0$. Therefore $b_1$ is invertible in $Q_1$ and $\nu(b_1^{-1}) \leq R$, where $R$ is the constant from Lemma 3.2. Applying 2.6.6 we obtain

$$c(b, g) \geq c(b \ast e, g) - c(e, 1_G) = c(b_1 \ast e, g) - c(e, 1_G) \geq c(e, g) - c(b_1^{-1}, 1_G) - c(e, 1_G).$$

Using 2.6.5 we get that

$$c(b_1^{-1}, 1_G) = \Omega \nu(b_1^{-1}) \leq \Omega R,$$

and therefore

$$c(b, g) \geq c(e, g) - \Omega R - c(e, 1_G).$$

Substituting this into (16) we see that

$$c(e, fg) \geq c(e, f) + c(e, g) - \text{const.}$$

Together with (15) this proves that the map $r$ which sends $f$ to $c(e, f)$ is a quasimorphism. \hfill \blacksquare
3.4 Building a Calabi quasimorphism \( \tilde{\mu} \) from \( r \)

Consider now a homogeneous quasimorphism \( \tilde{\mu} : \tilde{G} \to \mathbb{R} \) given by

\[
\tilde{\mu}(\tilde{f}) = -\text{vol}(M) \cdot \lim_{m \to \infty} \frac{r(\tilde{f}^m)}{m},
\]

where \( r(\tilde{f}) = c(e, \tilde{f}) \) and

\[
\text{vol}(M) = \int_M \omega^n.
\]

Let \( \mathcal{D} \) be the class of all displaceable open subsets of \( M \) as in (3).

**Proposition 3.3.** The restriction of \( \tilde{\mu} \) on \( \tilde{G}_U \) coincides with the Calabi homomorphism \( \tilde{\text{Cal}}_U \) for every \( U \in \mathcal{D} \).

**Proof:** We follow closely the work by Yaron Ostrover [31]. Take an open subset \( U \in \mathcal{D} \). By definition there exists a Hamiltonian diffeomorphism \( h \in G \) which displaces \( U \):

\[
h(U) \cap \text{Closure}(U) = \emptyset.
\]

Fix any lift \( \tilde{h} \) of \( h \) to \( \tilde{G} \). Let \( F : M \times \mathbb{R} \to \mathbb{R} \) be a Hamiltonian function which is 1-periodic in time and satisfies \( F(x, t) = 0 \) for all \( t \in \mathbb{R}, x \in M \setminus U \). Write \( f_t \) for the corresponding Hamiltonian flow, and \( \tilde{f}_t \) for its lift to \( \tilde{G} \). Put \( \tilde{f} = \tilde{f}_1 \) and note that the periodicity of \( F \) in \( t \) yields \( \tilde{f}^m = \tilde{f}^m \) for all \( m \in \mathbb{Z} \). Put

\[
F'(x, t) = F(x, t) - \{ \text{vol}(M) \}^{-1} \cdot \int_M F(x, t) \omega^n.
\]

Note that \( F'(x, t) \) generates the same flow \( f_t \) and satisfies the normalization condition \( \int_M F'(x, t) \omega^n = 0 \) which enters the definition of the action functional (see Section 2.4). Consider the family \( \tilde{h} f_t, t \in \mathbb{R} \). Since \( hU \cap U = \emptyset \) and \( f_t(U) = U \) the fixed point set of \( h f_t \) coincides with the fixed point set of \( h \) for every \( t \), and hence lies outside \( U \). Moreover, for every \( t \) and every \( x \in M \setminus U \) we have \( F'(x, t) \equiv u(t) \) where

\[
u(t) = \{ \text{vol}(M) \}^{-1} \cdot \int_M F(x, t) \omega^n.
\]
Using this one can calculate the action spectrum of $\hat{h}f_t$:

$$\text{spec}(\hat{h}f_t) = \text{spec}(\hat{h}) + w(t), \text{ where } w(t) := \int_0^t u(z) dz.$$ 

Consider the function $\psi(t) = r(\hat{h}f_t) = c(e, \hat{h}f_t)$. It is continuous and takes values in $\text{spec}(\hat{h}f_t)$ (see 2.6.3). Since $\text{spec}(\hat{h})$ is a closed nowhere dense subset of $\mathbb{R}$ we conclude that there exists $s_0 \in \text{spec}(h_0)$ such that $\psi(t) = s_0 + w(t)$. Hence $r(\hat{h}f^m) = s_0 + w(m)$. Using that $r$ is a quasimorphism we calculate

$$\hat{\mu}(\hat{f}) = -\text{vol}(M) \cdot \lim_{m \to +\infty} \frac{r(\hat{h}f^m)}{m} = -\text{vol}(M) \cdot \lim_{m \to +\infty} \frac{w(m)}{m} = \int_0^1 dt \int_M F(x, t) \omega^n = \widehat{\text{Cal}}_{\mathcal{U}}(\hat{f}).$$

This completes the proof.

3.5 From $\tilde{G}$ to $G$

**Proposition 3.4.** Suppose that $\pi_1(G)$ is finite. Then the quasimorphism $\tilde{\mu} : \tilde{G} \to \mathbb{R}$ descends to a homogeneous Calabi quasimorphism $\mu : G \to \mathbb{R}$.

**Proof:** The fundamental group $\pi_1(G)$ is the kernel of the natural projection $\tilde{G} \to G$. Note that $\pi_1(G)$ lies in the center of $\tilde{G}$. Then

$$\tilde{\mu}((\phi \hat{f})) = \tilde{\mu}(\phi) + \tilde{\mu}(\hat{f})$$

for every $\phi \in \pi_1(G)$ and $\hat{f} \in \tilde{G}$. This is true since for any $m$ the quantities $m \tilde{\mu}((\phi \hat{f})^m) = \tilde{\mu}((\phi \hat{f})^m)$ and $\tilde{\mu}(\phi^m) + \tilde{\mu}(\hat{f}^m) = m(\tilde{\mu}(\phi) + \tilde{\mu}(\hat{f}))$, which are homogeneous with respect to $m$, differ by a constant which is independent of $m$. Since $\pi_1(G)$ is finite $\tilde{\mu}$ vanishes on $\pi_1(G)$. Hence $\tilde{\mu}(\phi \hat{f}) = \tilde{\mu}(\hat{f})$, so $\tilde{\mu}$ descends to a function $\mu : G \to \mathbb{R}$. Using Proposition 3.3 one readily checks that $\mu$ is the required quasimorphism.

3.6 Hofer metric and the continuity properties of $\tilde{\mu}$ and $\mu$

As before we write $\mathcal{F}$ for the space of all normalized time-dependent Hamiltonians on $M$. This space is equipped with a $C^0$-norm given by formula (5). We denote by $\tilde{\psi}_F$ the element of $\tilde{G}$ generated by a Hamiltonian $F \in \mathcal{F}$.
The Hofer metric on \(G\) (see Section 1.2.3) can be lifted to a bi-invariant pseudo-metric \(\tilde{\rho}\) on \(\tilde{G}\) defined as follows. Let \(\psi \in G\), let \(\tilde{\psi} \in \tilde{G}\) be its lift, let \(1\) be the identity in \(G\) and \(1_{\tilde{G}}\) the identity in \(\tilde{G}\). Then
\[
\tilde{\rho}(1_{\tilde{G}}, \tilde{\psi}) = \inf_F \int_{S^1} \|F_t\|c^0 \, dt,
\]
where the infimum is taken over all \(F \in \mathcal{F}\) such that \(\tilde{\psi} = \tilde{\psi}_F\). In particular,
\[
\rho(1, \psi) = \inf_F \tilde{\rho}(1_{\tilde{G}}, \tilde{\psi}_F), \quad (18)
\]
where the infimum is taken over all Hamiltonians \(F\) generating \(\psi\) (or, equivalently, over all lifts \(\tilde{\psi}_F\) of \(\psi\) to \(\tilde{G}\)).

We will prove now that \(\tilde{\mu}\) is a continuous function on \(\tilde{G}\) and is Lipschitz with respect to \(\tilde{\rho}\).

**Proposition 3.5.** Let \(\tilde{\mu} : \tilde{G} \to \mathbb{R}\) be the Calabi quasimorphism constructed above. Then for any Hamiltonians \(H', H''\)
\[
|\tilde{\mu}(\tilde{\psi}_{H'}) - \tilde{\mu}(\tilde{\psi}_{H''})| \leq |\text{vol}(M)| \cdot \tilde{\rho}(\tilde{\psi}_{H'}, \tilde{\psi}_{H''}) \leq |\text{vol}(M)| \cdot \int_{S^1} \|H'_t - H''_t\|c^0 \, dt, \quad (19)
\]
and therefore \(\tilde{\mu} : \tilde{G} \to \mathbb{R}\) is continuous.

Proposition 3.5 together with (18) immediately lead to the following corollary.

**Corollary 3.6.** If the quasimorphism \(\tilde{\mu} : \tilde{G} \to \mathbb{R}\) constructed above descends to a quasimorphism \(\mu : G \to \mathbb{R}\) then for any Hamiltonians \(H', H''\)
\[
|\mu(\psi_{H'}) - \mu(\psi_{H''})| \leq |\text{vol}(M)| \cdot \rho(\psi_{H'}, \psi_{H''}) \leq |\text{vol}(M)| \cdot \int_{S^1} \|H'_t - H''_t\|c^0 \, dt, \quad (20)
\]
and therefore \(\mu : G \to \mathbb{R}\) is a continuous function.

**Proof of Proposition 3.5:** Suppose that \(H'\) and \(H''\) generate Hamiltonian flows \(\{f_t\}\) and \(\{g_t\}\) which give rise to the elements \(f := \tilde{\psi}_{H'}\) and \(g := \tilde{\psi}_{H''}\). Recall that the Hamiltonian \(H'_t'H''(x, t) := H'(x, t) + H''(g_t^{-1}(x), t)\) generates
the flow \( \{f_t g_t\} \) and the Hamiltonian \( \overline{H'}(x, t) := -H'(f_t(x), t) \) generates the flow \( \{f_t^{-1}\} \). Set \( H := \overline{H''} H \). Thus
\[
\tilde{\psi}_H^{-1} \tilde{\psi}_H'' = \tilde{\psi}_H,
\]
where \( H(x, t) = -H'(f_t(x), t) + H''(f_t(x), t) \). Observe that for each \( t \)
\[
\|H_t\|_{C^0} = \|H'_t - H''_t\|_{C^0}. \tag{21}
\]
Since \( \tilde{\rho} \) is bi-invariant we have
\[
\tilde{\rho}(f, g) = \tilde{\rho}(1_G, f^{-1} g) \leq \int_{S^1} \|H_t\|_{C^0} dt.
\]
Combining this with (21) we get the second inequality in (19).

Now let us prove the first inequality in (19). Indeed, according to (13), for any \( a \in QH_*(M) \)
\[
|c(a, f) - c(a, g)| \leq \int_{S^1} \|H'_t - H''_t\|_{C^0} dt = \int_{S^1} \|H_t\|_{C^0} dt.
\]
This inequality is true for any \( H' \) and \( H'' \) generating, respectively, \( f \) and \( g \), while its left-hand side depends only on the elements \( f, g \in \tilde{G} \) and not on the Hamiltonians that generate them. Thus taking in the right-hand side the infimum over all \( H' \) and \( H'' \) generating, respectively, \( f \) and \( g \), we obtain
\[
|c(a, f) - c(a, g)| \leq \tilde{\rho}(1_G, f^{-1} g)
\]
and hence
\[
|c(a, f^m) - c(a, g^m)| \leq \tilde{\rho}(1_G, f^{-m} g^m) \tag{22}
\]
Now observe that
\[
f^{-m} g^m = \prod_{i=0}^{m-1} g^{-i}(f^{-1} g) g^i.
\]
Thus
\[
\tilde{\rho}(1_G, f^{-m} g^m) \leq \sum_{i=0}^{m-1} \tilde{\rho}(1_G, g^{-i}(f^{-1} g) g^i) \leq m \tilde{\rho}(1_G, f^{-1} g), \tag{23}
\]
where the last inequality holds because $\tilde{\rho}$ is bi-invariant. Combining (22) with (23) we see that

$$\frac{1}{m}|c(a, f^m) - c(a, g^m)| \leq \tilde{\rho}(1_G, f^{-1}g) = \tilde{\rho}(f, g). \quad (24)$$

Now take $a$ to be the unit element $e$ of the field $Q_1$ involved in the definition of the quasimorphism $r = c(e, \cdot) : \tilde{G} \to \mathbb{R}$ (see Section 3.1) and recall that, according to its definition, $\tilde{\mu}(f) = -\text{vol}(M) \lim_{m \to +\infty} c(e, f^m)/m$ (see Section 3.4). Together with (24) this yields

$$|\tilde{\mu}(f) - \tilde{\mu}(g)| \leq |\text{vol}(M)| \cdot \tilde{\rho}(f, g),$$

proving the first inequality in (19). The proposition is proven. \[\square\]

3.7 Proofs of Theorems 1.3 (for $M = S^2, S^2 \times S^2, \mathbb{C}P^2$) and 1.5

Proof of Theorem 1.5: According to Theorem 3.1 and Proposition 3.3, the function $\tilde{\mu} : \tilde{G} \to \mathbb{R}$ constructed above is a homogeneous Calabi quasimorphism. In view of Proposition 3.5 it is continuous (and even Lipschitz with respect to the Hofer pseudo-metric on $\tilde{G}$). This proves Theorem 1.5. \[\square\]

Proof of Theorem 1.3 for $M = S^2, S^2 \times S^2, \mathbb{C}P^2$: Let $(M, \omega)$ be one of the manifolds $S^2, S^2 \times S^2, \mathbb{C}P^2$. Then $(M, \omega)$ is spherically monotone. The quantum homology algebra $QH_{ev}(M)$ is semi-simple (see Section 2.3). Thus Theorem 1.3 gives us a continuous homogeneous Calabi quasimorphism $\tilde{\mu} : \tilde{G} \to \mathbb{R}$. The fundamental group $\pi_1(G)$ is finite for $M = S^2, S^2 \times S^2, \mathbb{C}P^2$ (see [17]). Therefore, according to Proposition 3.4, $\tilde{\mu}$ descends to a homogeneous Calabi quasimorphism on $\mu : G \to \mathbb{R}$. In view of Corollary 3.6, $\mu$ is continuous (and Lipschitz with respect to the Hofer metric on $G$). The theorem is proven. \[\square\]

4 Spectral invariants and Hamiltonian loops

A homogeneous quasimorphism on an abelian group is always a homomorphism (the proof of this simple fact is actually contained in the proof of Proposition 3.4 above). Thus the restriction of the quasimorphism $\tilde{\mu} : \tilde{G} \to \mathbb{R}$
constructed above on the abelian subgroup $\pi_1(G) \subset \tilde{G}$ is a homomorphism. In this section we obtain a formula for this homomorphism in terms of the Seidel action of $\pi_1(G)$ on the quantum homology of $M$ \[43, 21\]. As an application we show that this homomorphism vanishes when $M$ is the projective space $\mathbb{C}P^n$ endowed with the Fubini-Study form and thus complete the proof of Theorem \[1.3\]. The results of this section were communicated to us by Paul Seidel.

4.1 Preliminaries on the Seidel action

4.1.1 Hamiltonian fibrations over $S^2$

There exists a one-to-one correspondence between homotopy classes of loops in $G$ and isomorphism classes of Hamiltonian fibrations over the 2-sphere $S^2$ with the fiber $(M^2, \omega)$, see \[25, 35\]. We denote by $\pi : E_\gamma \rightarrow S^2$ the fibration associated to a loop $\gamma$. An important invariant of such a fibration is its coupling class $W \in H^2(E_\gamma, \mathbb{R})$ which is uniquely defined by the following conditions: the restriction of $W$ to each fiber coincides with the class of the symplectic form, and its top power $W^{n+1}$ vanishes.

Denote by $T^{\text{vert}}E_\gamma$ the vector bundle over $E_\gamma$ formed by all tangent spaces of the fibers of $\pi$ and by $c_1^{\text{vert}}$ the first Chern class of this bundle.

Take a positively oriented complex structure $j$ on $S^2$ and an almost complex structure $\hat{J}$ on $E_\gamma$ whose restriction on each fiber is compatible with the symplectic form on it and such that the projection $\pi$ is a $(\hat{J}, j)$-holomorphic map (see \[43\]).

Two $(j, \hat{J})$-holomorphic sections $v_1, v_2$ of $\pi : E_\gamma \rightarrow S^2$ are said to be equivalent if $W([v_1(S^2)]) = W([v_2(S^2)])$. Since $M$ is assumed to be spherically monotone this condition is equivalent to $c_1^{\text{vert}}([v_1(S^2)]) = c_1^{\text{vert}}([v_2(S^2)])$.

Denote by $\mathcal{S}_\gamma$ the set of all such equivalence classes – it is an affine space modeled on $\bar{\pi}_2(M) \cong \mathbb{Z}$. According to the definition, the maps $v \mapsto W([v(S^2)])$ and $v \mapsto c_1^{\text{vert}}([v(S^2)])$ give rise to some correctly defined functions on $\mathcal{S}_\gamma$. From this moment on, given an equivalence class $\sigma \in \mathcal{S}_\gamma$, we will denote by $W(\sigma)$, $c_1^{\text{vert}}(\sigma)$ the values of those functions on $\sigma$.

Recall that in our notation $S$ is the positive generator of $\bar{\pi}_2(M)$ and $\Omega := (\omega, S) > 0$, $N := (c_1(M), S)$. Thus for a class $\sigma \in \mathcal{S}_\gamma$ a sum $\sigma + mS$, $m \in \mathbb{Z}$, stands for another class in $\mathcal{S}_\gamma$ so that:

$$W(\sigma + mS) = W(\sigma) + m\Omega, \quad c_1^{\text{vert}}(\sigma + mS) = c_1^{\text{vert}}(\sigma) + mN. \quad (25)$$
4.1.2 Gromov-Witten invariants revisited

Given $\sigma \in \mathcal{S}_\gamma$ and homology classes $A, B, C \in H_2(M)$ let $GW_\sigma(A, B, C)$ denote the Gromov-Witten number (cf. Section 2.2), defined as follows. In the fibers of $\pi$ over $0, 1, \infty \in S^2$ pick the cycles $\hat{A}, \hat{B}, \hat{C}$ realizing, respectively, the homology classes $A, B, C$. Consider the sections from the class $\sigma$ whose intersection with the fibers over $0, 1, \infty$ belongs, respectively, to $\hat{A}, \hat{B}, \hat{C}$. If there is a finite number of such sections count them with appropriate signs and set $GW_\sigma(A, B, C)$ equal to the result. Otherwise set $GW_\sigma(A, B, C)$ to be zero. The resulting number does not depend on the choice of cycles $\hat{A}, \hat{B}, \hat{C}$ – for details see [43].

4.1.3 Hamiltonian loops

Consider the space $\Lambda$ of all smooth contractible loops in $M$ (i.e. smooth maps from $S^1$ to $M$). Let $\tilde{\Lambda}$ be the cover of $\Lambda$ introduced in Section 2.4. Its elements are equivalence classes of pairs $(x, u)$, where $x \in \Lambda$, $u$ is an oriented disk spanning $x$ in $M$, and the equivalence relation is defined as follows: $(x_1, u_1) \sim (x_2, u_2)$ iff $x_1 = x_2$ and the 2-sphere $u_1 \# (-u_2)$ vanishes in $\bar{\pi}_2(M)$.

The group $\mathcal{G}$ of all (smooth) identity-based loops in $G$ acts on $\Lambda$: if $\gamma = \{g_t\} \in \mathcal{G}$ then the action $T_\gamma : \Lambda \to \Lambda$ is defined as

$$T_\gamma \{x_t\} = \{g_t(x_t)\}.$$ 

This map can be lifted (not uniquely!) to a map on $\tilde{\Lambda}$. In fact there is a one-to-one correspondence between lifts of $T_\gamma$ and classes of sections $\sigma \in \mathcal{S}_\gamma$. We denote the lift corresponding to $\sigma$ by $\tilde{T}_{\gamma,\sigma}$.

Suppose that Hamiltonian loop $\gamma = \{g_t\} \in \mathcal{G}$ is generated by a normalized Hamiltonian $K : M \times S^1 \to \mathbb{R}$, $K \in \mathcal{F}$, and let $H \in \mathcal{F}$. Consider the action functional $A_H$ on $\tilde{\Lambda}$ (see Section 2.4). The following formula (see [43, 21]) is crucial for our purposes:

$$(\tilde{T}_{\gamma,\sigma}^*)^{-1} A_H - A_{K\#H} = -W(\sigma). \quad (26)$$

In particular, the function in the left hand side is constant on $\tilde{\Lambda}$.

The action of $\tilde{T}_{\gamma,\sigma}$ on $\tilde{\Lambda}$ defines an isomorphism, which we will denote by $\iota$, between the Floer homology of $H$ and the Floer homology of $K\#H$ [43]. According to Seidel’s theorem [43], under the identification of Floer and quantum homology (see Section 2.6.1) this isomorphism between Floer
homology groups corresponds to the multiplication by some class $\Psi$ in the quantum homology of $M$. This class is defined as follows:

$$\Psi = \sum_{m \in \mathbb{Z}} A_{\sigma + mS} s^{-m},$$  

(27)

where $A_{\sigma + mS} \in H_{ev}(M)$ is uniquely determined by the condition

$$A_{\sigma + mS} \circ_M C = GW_{\sigma + mS}([M], [M], C)$$

for any $C \in H_{ev}(M)$.

Note also that in view of (26), the isomorphism $\iota$ shifts the filtration of the Floer homology groups by $W(\sigma)$:

$$\iota : V_\alpha(H) \rightarrow V_{\alpha + W(\sigma)}(K^\sharp H)$$  

(28)

for any $\alpha \in \mathbb{R}$.

### 4.1.4 Extending the field

In what follows it would be convenient to work with the extension $\bar{k}$ of the field $k$, where $\bar{k}$ is formed by semi-infinite sums $\sum_{\alpha \in \mathbb{R}} z_\alpha s^\alpha$, $z_\alpha \in \mathbb{C}$, satisfying the condition that for any $\alpha_0 \in \mathbb{R}$ there is only a finite number of terms with $z_\alpha \neq 0, \alpha \geq \alpha_0$, in the sum. Define

$$QH_{ev}(M) := H_{ev}(M) \otimes_{\mathbb{C}} \bar{k} = QH_{ev}(M) \otimes_k \bar{k}.$$  

Naturally, $QH_{ev}(M) \subset \overline{QH}_{ev}(M)$. The function $\nu$ on $QH_{ev}(M)$ defined in Section 2.6.5 extends to a function on $\overline{QH}_{ev}(M)$ which we will denote by $\nu$:

$$\nu \left( \sum_{\alpha \in \mathbb{R}} A_\alpha s^\alpha \right) = \max \{ \alpha : A_\alpha \neq 0 \} .$$

### 4.1.5 Seidel action

Now we are ready to define the Seidel action which is given by a homomorphism $\Phi$ from the group $\pi_0(\mathcal{G}) = \pi_1(G)$ to the group of invertible elements of $\overline{QH}_{ev}(M)$ (see [13, 21]) It sends the class of a loop $\gamma$ to the element

$$\Phi_\gamma = \sum_{\sigma \in \mathcal{S}_\gamma} A_\sigma s^{-W(\sigma)/\Omega} .$$
4.2 A formula for the restriction of \( \tilde{\mu} \) on \( \pi_1(G) \)

**Proposition 4.1** (cf. [29]). Let \([\gamma] \in \pi_1(G) \subset \tilde{G}\) be represented by a loop \( \gamma \). Then for any \( a \in QH_{\text{ev}}(M) \)

\[
c(a, [\gamma]) = \Omega \tilde{\nu}(a \Phi^{-1}).
\]

**Proof of Proposition 4.1:** As before let \( K \in \mathcal{F} \) be the normalized Hamiltonian generating \( \gamma \). Fix a lift \( T_\gamma \) associated to some section class \( \sigma \).

Taking \( H \) to be the zero Hamiltonian generating the identity and applying formulas (26),(27) and (28) we get that for any \( a \in QH_{\text{ev}}(M) \)

\[
c(a, [\gamma]) = c(a \Psi^{-1}, 1_{\tilde{G}}) + W(\sigma) = \Omega \nu(a \Psi^{-1}) + W(\sigma) = \Omega \tilde{\nu}(a \Phi^{-1} s^{W(\sigma)/\Omega}) = \Omega \tilde{\nu}(a \Phi^{-1}).
\]

The proposition is proven. \[\square\]

Now we will derive a formula for the restriction of \( \tilde{\mu} \) on \( \pi_1(G) \). Recall that the algebra \( Q = QH_{\text{ev}}(M) \) is assumed to be semi-simple and thus, as an algebra over \( k \), it decomposes into a direct sum of fields:

\[
Q = Q_1 \oplus \ldots \oplus Q_d
\]

(see Section 3.1). Let \( e \) be the unit element for the field \( Q_1 \) involved in the definition of the quasimorphisms \( r \) and \( \tilde{\mu} \) (see Sections 3.1 and 3.4) so that

\[
\tilde{\mu}(\tilde{f}) = -\text{vol}(M) \cdot \lim_{m \to +\infty} \frac{c(e, \tilde{f}^m)}{m}.
\]

According to Proposition 4.1, for any \([\gamma] \in \pi_1(G)\) one has:

\[
\tilde{\mu}([\gamma]) = -\text{vol}(M) \cdot \Omega \lim_{m \to +\infty} \frac{\tilde{\nu}(e \Phi^{-m})}{m}.
\]

(29)
4.3 Proof of Theorem 1.3 in the case $M = \mathbb{C}P^n$

We consider the manifold $M = \mathbb{C}P^n$ equipped with the Fubini-Study symplectic form $\omega$. We need to show that the Calabi quasimorphism $\tilde{\mu} : \tilde{G} \to \mathbb{R}$ descends to a function on $G$, i.e. $\tilde{\mu}$ vanishes on $\pi_1(G) \subset \tilde{G}$. Indeed, according to Proposition 3.4 and Corollary 3.6, this would give us a continuous homogeneous Calabi quasimorphism on $G$.

Example 2.3.2 tells us that $QH_{ev}(M)$ is a field. Therefore we can assume that, in the notation of Section 4.2, $e = [M]$ and hence, according to (29)

$$\tilde{\mu}([\gamma]) = -\text{vol}(M) \cdot \Omega \lim_{m \to +\infty} \frac{\tilde{\nu}(\Phi_{\gamma}^{-m})}{m},$$

for any $[\gamma] \in \pi_1(G)$ represented by a loop $\gamma$. Thus it suffices to prove the following fact:

$$\lim_{m \to +\infty} \frac{\tilde{\nu}(\Phi_{\gamma}^{-m})}{m} = 0.$$  \hspace{1cm} (30)

The proof of (30) splits into the following two propositions.

Proposition 4.2. $\Phi_{\gamma}$ is a monomial of the type $\delta A^m s^\beta$ for some $0 \neq \delta \in \mathbb{C}, m \in \mathbb{Z}, \beta \in \mathbb{R}$, where $A \in H_{2n-2}(\mathbb{C}P^n)$ is the hyperplane class.

Proposition 4.3. If $\Phi_{\gamma} = \delta s^\alpha$ for some $0 \neq \delta \in \mathbb{C}$ then $\alpha = 0$.

Postponing the proofs of the propositions we first finish the proof of (30). Indeed, recall from Section 4.1.3 that the map $\gamma \mapsto \Phi_{\gamma}$ is a homomorphism from $\pi_1(G)$ to the group of invertible elements of $\bar{QH}_{ev}(M)$. Thus

$$\Phi_{\gamma^i} = \Phi_{\gamma}^i$$

for any $i \in \mathbb{Z}$. Now using the explicit form of $\Phi_{\gamma}$ given by Proposition 4.2 and the equality $A^{-(n+1)} = s[M]$ (see Example 2.3.2), we can write:

$$\Phi_{\gamma^{-(n+1)}} = \Phi_{\gamma}^{-(n+1)} = \delta^{-(n+1)} A^{-(n+1)m} s^{-(n+1)\beta} = \delta^{-(n+1)} [M] s^{-(n+1)\beta+m} = \delta^{-(n+1)} [M],$$

where the last equality holds because of Proposition 4.3. But

$$\tilde{\nu}(\delta^{-(n+1)} [M]) = 0.$$

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Hence $\bar{\nu}(\Phi_{\gamma-(n+1)}) = 0$. Since this is true for every $\gamma$ we get that $\bar{\nu}(\Phi^k) = 0$ provided $(n + 1)$ divides $k$. This immediately proves \((30)\). The proof of Theorem 1.3 in the case $M = \mathbb{C}P^n$ is finished.

**Proof of Proposition 4.2:** One needs to show that there exists at most one class $\sigma \in \mathcal{S}_\gamma$ such that $GW_\sigma([M], [M], C) \neq 0$ for some $C \in H_*(M)$. The proof follows from the dimension count.

Indeed, assume $\sigma_1 = \sigma_2 + mS \in \mathcal{S}_\gamma$, $m \neq 0$, and $GW_{\sigma_1}([M], [M], C_1) \neq 0$, $GW_{\sigma_2}([M], [M], C_2) \neq 0$ for some $C_1, C_2 \in H_*(M)$. Since the Gromov-Witten invariants are non-zero the virtual dimension of the corresponding moduli spaces has to be zero. Using the formula for the virtual dimension we get:

$$\deg(C_i) = 2n + 4 - 2c_{vert}^i(\sigma_i), \ i = 1, 2.$$  

Hence, according to \((23)\),

$$|\deg(C_1) - \deg(C_2)| = 2mN = 2m(n + 1),$$

because the minimal Chern number $N$ of $\mathbb{C}P^n$ is $n + 1$. But on the other hand, $|\deg(C_1) - \deg(C_2)|$ cannot be bigger than $2n$ which leads us to contradiction unless $m = 0$. The proposition is proven.

**Proof of Proposition 4.3:** Suppose $\Phi_\gamma = \delta [M] s^\alpha$, i.e. $A_\sigma = \delta [M]$, $\delta \in \mathbb{C}$. This means that $A_\sigma \circ M P = GW_\sigma([M], [M], P) \neq 0$, where $P = [\text{point}]$. Let $\mathcal{M}$ be the moduli space of $(j, \bar{j})$-holomorphic sections of the fibration $\pi : E_\gamma \to S^2$ belonging to the class $\sigma \in \mathcal{S}_\gamma$. We need the following lemma.

**Lemma 4.4.** $\mathcal{M}$ is a smooth compact manifold.

**Proof of Lemma 4.4:** According to \([43]\), $\mathcal{M}$ is a smooth manifold of dimension $2n$. The Gromov compactness theorem \([17]\) says that the only way the compactness of $\mathcal{M}$ may fail is a so-called bubbling-off, when a sequence of sections from $\mathcal{M}$ converges to a curve in $E_\gamma$ which is a connected union of a pseudo-holomorphic section of $\pi$ representing some $\sigma' \in \mathcal{S}_\gamma$ and a number of pseudo-holomorphic spheres lying in fibers of $\pi$. In such a case the total energy has to be preserved, meaning that $\sigma = \sigma' + mS$, $m \geq 1$. But, just as we already checked in the proof of Proposition 4.2 the virtual dimension of the moduli space of pseudo-holomorphic sections belonging to the class $\sigma'$ equals $2n - 2m(n + 1) < 0$ and therefore such bubbling-off does not happen. Therefore $\mathcal{M}$ is compact. The lemma is proven.
Consider the evaluation map

\[ ev : \mathcal{M} \times S^2 \rightarrow E_\gamma, \ (v,q) \mapsto v(q). \]

Then \( \dim \mathcal{M} = 2n \) and the degree of the map \( ev \) is non-zero because \( GW_\sigma([M],[M],P) \neq 0 \). Let \( \eta \) be the generator of \( H^2(S^2) \) dual to the fundamental class. Represent \( ev^*(W) \in H^2(\mathcal{M}) \oplus H^2(S^2) \) as \( ev^*(W) = \theta + R\eta \) for some \( \theta \in H^2(\mathcal{M}) \), \( R \in \mathbb{R} \). Recall that \( W^{n+1} = 0 \) and that \( \theta^{n+1} = 0 \), \( \eta^2 = 0 \) for dimensional reasons. Therefore

\[ 0 = ev^*(W^{n+1}) = (\theta + R\eta)^{n+1} = (n+1)R\theta^n\eta. \]

On the other hand, the restriction of the coupling class on any fiber of \( \pi \) is the class of the symplectic form on that fiber. Therefore the product of the \( n \)-th power of the coupling class with \( \pi^*\eta \) represents a non-zero multiple of the fundamental class of \( E_\gamma \). The image of this cohomology class under \( ev^* \) is non-zero, because the degree of \( ev \) is non-zero. Note also that \( ev^*\pi^*\eta = \eta \).

Thus

\[ 0 \neq ev^*(W^n\pi^*\eta) = (\theta + R\eta)^n\eta = (\theta^n + nR\theta^{n-1}\eta)\eta = \theta^n\eta. \]

Combining it with \( (n+1)R\theta^n\eta = 0 \) we see that \( R = 0 \). But

\[ R = ev^*(W)([S^2]) = W(\sigma) = 0. \]

Now recall that \( \Phi_\gamma = \delta[M]s^{-W(\sigma)/\Omega} \). Hence \( \Phi_\gamma = \delta[M] \). The proposition is proven. \( \blacksquare \)

## 5 Calabi quasimorphism and combinatorics of level sets of autonomous Hamiltonians on the 2-sphere

### 5.1 Morse functions and abelian subgroups of \( Ham(S^2) \)

Let \( \omega \) be an area form on the 2-sphere \( S^2 \) with total area 1. Fix a Morse function \( F \) on \( S^2 \), and consider the subspace \( \mathcal{H}_F \subset C^\infty(S^2) \) consisting of all functions \( H \) whose Poisson bracket with \( F \) vanishes: \( \{H,F\} = 0 \). For a smooth function \( H \) denote by \( \psi_H \) the time-1-map of the Hamiltonian flow
generated by \( H \). Note that the Poisson bracket of every two functions from \( \mathcal{H}_F \) vanishes, and therefore the subgroup

\[
\Gamma_F = \{ \psi_H \mid H \in \mathcal{H}_F \} \subset \text{Ham}(S^2)
\]

is abelian. Intuitively speaking, \( \Gamma_F \) is a maximal torus in \( \text{Ham}(S^2) \).

Write \( G = \text{Ham}(S^2) \) and let \( \mu \) be a continuous homogeneous Calabi quasimorphism on \( G \). The purpose of this section is to calculate the restriction of any such \( \mu \) on the subgroup \( \Gamma_F \). Note that since \( \Gamma_F \) is abelian the map \( \mu : \Gamma_F \to \mathbb{R} \) is a homomorphism. It turns out that the answer can be given in terms of simple combinatorial data associated to the Morse function \( F \) (see Theorem 5.2 below).

5.2 A measured tree associated to a Morse function

Let \( F : S^2 \to \mathbb{R} \) be a Morse function. Look at connected components of non-empty level sets of \( F \). These components split into three different groups:

I. Points of local maximum/minimum of \( F \);

II. Immersed closed curves whose self-intersections correspond to critical points of index 1 of \( F \);

III. Simple closed curves.

We denote by \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) the sets of all components of types I and II respectively. Put \( \mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \). Let us emphasize that \( \mathcal{V} \) is a finite set.

The set \( S^2 \setminus \bigcup_{P \in \mathcal{V}} P \) is a union of a finite number of pairwise disjoint open cylinders diffeomorphic to \( S^1 \times \mathbb{R} \). Denote their collection by \( \mathcal{E} \). Every cylinder \( C \in \mathcal{E} \) is foliated by simple closed curves of type III. Denote by \( e_C \) the space of leaves of this foliation, which is naturally homeomorphic to \( \mathbb{R} \). Consider the Reeb graph \( T \) associated to the function \( F \) as follows (see [37], cf. [8]). Its vertices \( v_P \) are in one-to-one correspondence with elements \( P \in \mathcal{V} \), and its open edges are \( e_C, C \in \mathcal{E} \). We say that an edge \( e_C \) connects vertices \( v_P \) and \( v_Q \) if \( \partial C \subset P \cup Q \). Note that vertices \( v_P, P \in \mathcal{V}_1 \) are free. This means that \( v_P \) is adjacent to only one edge. If \( P \in \mathcal{V}_2 \) the vertex \( v_P \) lies in the interior of \( T \). Let us emphasize that every point \( x \in T \) corresponds to a subset of \( S^2 \) which is denoted by \( \gamma_x \).

We claim that in fact \( T \) is a tree. Indeed remove a point \( x \in T \) lying on any open edge. Since \( S^2 \setminus \gamma_x \) is disconnected we conclude that \( T \setminus \{ x \} \) is disconnected as well, so \( T \) has no cycles. The claim follows.
Introduce a probability measure $\rho$ on the tree $T$ which is uniquely determined by the following conditions. Given two points $x, y$ lying on the open edge $e_C$, $C \in \mathcal{E}$, we define the measure $\rho([x, y])$ of the segment $[x, y]$ as the area of the subcylinder of $C$ bounded by closed curves $\gamma_x$ and $\gamma_y$. We also require that all the vertices have measure 0.

By definition, a measured tree is a finite tree equipped with a non-atomic Borel probability measure whose restriction on every open edge is homeomorphic to the Lebesgue measure on an open interval. With this language, the construction above associates a measured tree $(T, \rho)$ with any Morse function $F$ on $S^2$.

5.3 The median of a measured tree

Let $(T, \rho)$ be a measured tree. A point $x \in T$ is called a median if the measure of each connected component of $T \setminus \{x\}$ does not exceed $\frac{1}{2}$.

Proposition 5.1. Every measured tree has unique median.

Proof:

I. Uniqueness. Assume on the contrary that $x$ and $y$ are two distinct medians. Denote by $Y$ the connected component of $T \setminus \{y\}$ which contains $x$. Consider the collection $X_1, \ldots, X_m$ of all connected components of $T \setminus \{x\}$. Assume without loss of generality that $y \in X_1$. Denote by $\alpha$ the open path connecting $x$ and $y$. Then

$$X_2 \cup \ldots \cup X_m \cup \alpha \subset Y,$$

so

$$\sum_{i=2}^{m} \rho(X_i) + \rho(\alpha) \leq \frac{1}{2}.$$

But $\rho(X_1) \leq 1/2$ as well, and

$$\sum_{i=1}^{m} \rho(X_i) = 1,$$

so

$$\sum_{i=2}^{m} \rho(X_i) \geq \frac{1}{2}.$$
This yields \( \varrho(\alpha) = 0 \), which contradicts to our assumption that \( x \neq y \). Uniqueness follows.

**II. Existence.** For a point \( x \in T \) denote by \( Z_x \) the set of all connected components of \( T \setminus \{x\} \). Put

\[
\phi(x) = \max_{X \in Z_x} \varrho(X).
\]

We claim that \( \phi \) is a lower semicontinuous function on \( T \). Obviously, \( \phi \) is continuous at \( x \) if either \( x \) lies on an open edge of \( T \) or \( x \) is a free vertex. Suppose that \( x \) is an interior vertex and \( Z_x = \{X_1, \ldots, X_m\} \). Then

\[
\phi(x) = \max_{1 \leq i \leq m} \varrho(X_i).
\]

Take sufficiently small \( \epsilon > 0 \) and consider a neighborhood \( U \) of \( x \) in \( T \) consisting of all points \( y \in T \) which belong to the edges adjacent to \( x \) and satisfy \( \varrho([x,y]) < \epsilon \). Assume without loss of generality that \( y \in X_1 \). Then

\[
\phi(y) = \max(\varrho(X_1) - \varrho([x,y]), \sum_{i=2}^{m} \varrho(X_i) + \varrho([x,y]))
\]

for all \( y \in U \). We see that \( \phi(y) \geq \phi(x) - \epsilon \), and the claim on the lower semicontinuity of \( \phi \) follows.

Since \( T \) is compact, the function \( \phi \) attains its minimal value at some point \( x \in T \). Let us check that \( \phi(x) \leq 1/2 \). Indeed, suppose on the contrary that \( \phi(x) = \frac{1}{2} + \delta \) with \( \delta > 0 \). Let \( X \) be the (unique) connected component of \( T \setminus \{x\} \) with \( \varrho(X) = \frac{1}{2} + \delta \). Denote by \( e \) the open edge of \( X \) adjacent to \( x \). Choose any point \( y \in e \) with \( \varrho([x,y]) < \delta \). Note that \( X \setminus [x,y] \) is a connected component of \( T \setminus \{y\} \) whose measure equals

\[
\varrho(X) - \varrho([x,y]).
\]

Since this number is strictly bigger than \( \frac{1}{2} \) we conclude that it is equal to \( \phi(y) \). But then \( \phi(y) < \phi(x) \) which contradicts to the assumption that \( \phi \) attains its minimum at \( x \). Therefore \( \phi(x) \leq 1/2 \), and hence \( x \) is a median. This completes the proof of the proposition. \( \square \)

### 5.4 The calculation

Let \( \mu : Ham(S^2) \to \mathbb{R} \) be any continuous homogeneous Calabi quasimorphism. For a Morse function \( F \) on \( S^2 \) consider the subgroup \( \Gamma_F \) defined

\[
\Gamma_F = \{ \gamma \in \Gamma \mid F \circ \gamma = F \}
\]
in Section 5.2. Recall that $\Gamma_F$ consists of all Hamiltonian diffeomorphisms $\psi_H$, $H \in \mathcal{H}_F$, where the space $\mathcal{H}_F$ consists of all functions whose Poisson bracket with $F$ vanishes. Below we calculate the homomorphism $\mu : \Gamma_F \to \mathbb{R}$ in terms of the measured tree $(T, \varrho)$ associated to $F$.

To state our result we start with the following simple observation. Take any $H \in \mathcal{H}_F$. Since $\{H, F\} = 0$ the function $H$ is constant on each connected component of every level set of $F$. Therefore $H$ descends to a function $\bar{H}$ on the tree $T$. Denote by $x_0$ the median of $(T, \varrho)$.

**Theorem 5.2.** Let $\mu : \text{Ham}(S^2) \to \mathbb{R}$ be any homogeneous continuous Calabi quasimorphism. Then

$$\mu(\psi_H) = \int_{S^2} H \cdot \omega - \bar{H}(x_0)$$

for every function $H \in \mathcal{H}_F$.

**Proof of Theorem 5.2.** Take a sequence of functions $w_i : \mathbb{R} \to \mathbb{R}$, $i \in \mathbb{N}$ such that $w_i(s) \equiv \bar{H}(x_0)$ for $|s - \bar{H}(x_0)| < \frac{1}{i}$ and $w_i$ converges uniformly to $w(s) = s$ as $i \to +\infty$. Take any $H \in \mathcal{H}_F$ and put $H_i = w_i \circ H$. Then the sequence $H_i$ converges uniformly to $H$. Since $\mu$ is continuous, one has $\lim_{i \to +\infty} \mu(\psi_{H_i}) = \mu(\psi_H)$ and hence it suffices to show that

$$\mu(\psi_{H_i}) = \int_{S^2} H_i \cdot \omega - \bar{H}_i(x_0)$$

for all sufficiently large $i$.

Denote by $\gamma_{x_0}$ the level set component of $F$ corresponding to the median $x_0 \in T$. Note that $S^2 \setminus \gamma_{x_0}$ is the disjoint union of a finite number of open disks which we denote by $U_1, ..., U_m$. By definition of the median, the area of each $U_j$ does not exceed $\frac{1}{2}$. Therefore every open subset whose closure lies in $U_j$ is displaceable. Consider the function $K = H_i - \bar{H}_i(x_0)$. Note that $K$ can be decomposed as follows:

$$K = K_1 + ... + K_m,$$

where $\text{supp}(K_j) \subset U_j$, $j = 1, ..., m$.

Since $\mu$ is a Calabi quasimorphism we obtain

$$\mu(\psi_{K_j}) = \int_{S^2} K_j \cdot \omega.$$
Note now that
\[ \psi_{H_i} = \psi_K = \psi_{K_1} \circ \ldots \circ \psi_{K_m}. \]

Therefore
\[ \mu(\psi_{H_i}) = \sum_{j=1}^{m} \mu(\psi_{K_j}) = \sum_{j=1}^{m} \int_{S^2} K_j \cdot \omega = \int_{S^2} H_i \cdot \omega - \bar{H}_i(x_0), \]

because \( \mu \) is homogeneous, all \( \psi_{K_j} \) commute and \( \int_{S^2} \omega = 1 \). The proof is finished. \( \blacksquare \)

5.5 Proof of Corollary 4.9

First assume that \( M \) is any of the manifolds listed in Theorem 1.3 and \( \mu \) is the specific continuous homogeneous Calabi quasimorphism on \( G \) that we constructed in Section 3. Let \( F \) be an autonomous Hamiltonian on \( M \). Corollary 3.6 immediately yields that
\[
\left| \frac{\mu(\psi_F)}{\|F\|_{C^0}} \right| \leq \lim_{m \to +\infty} \frac{\rho(1, \psi^{m}_F)/m}{\|F\|_{C^0}} = \zeta(F). \tag{31}
\]

Now let \( M = S^2 \), \( \int_M \omega = 1 \) and let the rest of the notation be as above. Let \( \mathcal{F}_{\text{aut}} \subset \mathcal{F} \) denote the set of all autonomous Hamiltonians on \( S^2 \) whose integral over \( S^2 \) is zero. Consider the set \( \mathcal{W} \subset \mathcal{F}_{\text{aut}} \) of autonomous Hamiltonians \( F : S^2 \to \mathbb{R} \) which satisfy the following properties:

a) \( F \) is a Morse function;

b) 0 is a regular value of \( F \);

c) no connected component of \( F^{-1}(0) \) divides \( S^2 \) into two parts of area 1/2.

Lemma 5.3. \( \mathcal{W} \) is an open and dense subset in \( \mathcal{F}_{\text{aut}} \) with respect to the \( C^\infty \)-topology.

The proof is easy and left to the reader. Now, according to the lemma, a generic autonomous Hamiltonian \( F \in \mathcal{F}_{\text{aut}} \) belongs to \( \mathcal{W} \). For any such \( F \) the value \( \bar{F}(x_0) \) of \( \bar{F} \) at the median \( x_0 \) is non-zero. (When we talk about \( \bar{F}(x_0) \) we view \( F \) as an element of \( \mathcal{H}_F \).) Therefore, according to Theorem 5.2:
\[ \mu(\psi_F) = -\bar{F}(x_0) \neq 0. \]

Thus \( |\mu(\psi_F)| > 0 \) and hence, in view of (31), \( \zeta(F) > 0 \) for a generic \( F \). This finishes the proof of Corollary 4.9. \( \blacksquare \)
5.6 Proof of Theorem 1.11

1) We think of \( S^2 \) as of the round sphere
\[
\{ x_1^2 + x_2^2 + x_3^2 = 1 \} \subset \mathbb{R}^3
\]
endowed with the symplectic form \( \omega \) which equals the spherical area form divided by \( 4\pi \) (so the total area equals 1). Put \( F(x) = \frac{1}{2} x_3 \). Note that \( F \) is a Morse function on the sphere, which is in fact the moment map of the 1-turn rotation around the vertical axes. Therefore the measured Reeb graph of \( F \) can be identified with the image of \( F \), that is with the segment \([-\frac{1}{2}; \frac{1}{2}]\), endowed with the Lebesgue measure. Its median of course is the point 0.

2) We start with the case when \( M \) is the annulus
\[
\{(p, q) \in T^* S^1 \mid 0 < p < 0.1\}.
\]
Fix \( \epsilon \in (0; 0.1) \). In view of the discussion above there exists a symplectic embedding \( h_\epsilon : M \to S^2 \) which sends each circle \( \{ p = c \} \) to the level set \( \{ F = c - \epsilon \} \).

Write \( G_M \) for the group of Hamiltonian diffeomorphisms of \( M \) generated by Hamiltonians with compact support. The embedding \( h_\epsilon \) induces a monomorphism \( \phi_\epsilon : G_M \to Ham(S^2) \). Let \( \mu \) be a Calabi quasimorphism on \( Ham(S^2) \). Put \( \mu_\epsilon = \mu \circ \phi_\epsilon \). Clearly this is a quasimorphism on \( G_M \). Let \( \bar{U} \subset M \) be a closed disk with smooth boundary, whose interior is denoted by \( U \). Then its image \( h_\epsilon(\bar{U}) \) is displaceable in \( S^2 \), and therefore \( \mu_\epsilon \) coincides with the Calabi homomorphism on \( G_U \).

To show that any finite collection of \( \mu_\epsilon \)'s is linearly independent over \( \mathbb{R} \) we shall proceed as follows. Take any compactly supported Hamiltonian of the form \( H = H(p) \) on \( M \). Denote by \( \psi_H \) the corresponding Hamiltonian diffeomorphism. It follows from Theorem 5.2 that
\[
\mu_\epsilon(\psi_H) = Cal_M(\psi_H) - H(\epsilon).
\]
This immediately yields the linear independence.

Finally, after the obvious change of parameter, we can assume that \( \epsilon \) runs over \( \mathbb{R} \) instead of \( (0; 0.1) \). This completes the proof in the case when \( M \) is the annulus.
3) Let us turn to the case when $M$ is the disk
\[ \{(p, q) \in \mathbb{R}^2 \mid \pi(p^2 + q^2) < 1\}. \]

Fix $\epsilon \in (\frac{1}{2}; 1)$. There exists a conformally symplectic embedding $h_\epsilon : M \to S^2$ which sends each circle $\{\pi(p^2 + q^2) = c\}, \ c \in (0; 1)$ to the level set $\{F = \frac{1}{2} - \epsilon \cdot c\}$. The embedding $h_\epsilon$ induces a monomorphism $\phi_\epsilon : G_M \to \text{Ham}(S^2)$. Let $\mu$ be a Calabi quasimorphism on $\text{Ham}(S^2)$. Then $\mu_\epsilon = \epsilon^{-2} \cdot \mu \circ \phi_\epsilon$ is a Calabi quasimorphism on $G_M$.

To show that any finite collection of $\mu_\epsilon$’s is linearly independent over $\mathbb{R}$ we shall proceed as follows. Take any compactly supported Hamiltonian of the form $H = H(\pi(p^2 + q^2))$ on $M$. Denote by $\psi_H \in \text{Ham}(M)$ the corresponding Hamiltonian diffeomorphism. It follows from Theorem 5.2 that
\[ \mu_\epsilon(\psi_H) = \text{Cal}_M(\psi_H) - \epsilon^{-1} H(\epsilon^{-1}/2). \]

This immediately yields the linear independence.

Finally, after the obvious change of parameter, we can assume that $\epsilon$ runs over $\mathbb{R}$ instead of $(\frac{1}{2}; 1)$. This completes the proof in the case when $M$ is the disk.

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