We prove that a continuous image of a Radon-Nikodým compact of weight less than \(b\) is Radon-Nikodým compact. As a Banach space counterpart, subspaces of Asplund generated Banach spaces of density character less than \(b\) are Asplund generated. In this case, in addition, there exists a subspace of an Asplund generated space which is not Asplund generated which has density character exactly \(b\).

The concept of Radon-Nikodým compact, due to Reynov [12], has its origin in Banach space theory, and it is defined as a topological space which is homeomorphic to a weak* compact subset of the dual of an Asplund space, that is, a dual Banach space with the Radon-Nikodým property (topological spaces will be here assumed to be Hausdorff). In [9], the following characterization of this class is given:

**Theorem 1.** A compact space \(K\) is Radon-Nikodým compact if and only if there is a lower semicontinuous metric \(d\) on \(K\) which fragments \(K\).

Recall that a map \(f : X \times X \rightarrow \mathbb{R}\) on a topological space \(X\) is said to fragment \(X\) if for each (closed) subset \(L\) of \(X\) and each \(\varepsilon > 0\) there is a nonempty relative open subset \(U\) of \(L\) of \(f\)-diameter less than \(\varepsilon\), i.e. \(\sup \{f(x, y) : x, y \in U\} < \varepsilon\). Also, a map \(g : Y \rightarrow \mathbb{R}\) from a topological space to the real line is lower semicontinuous if \(\{y : g(y) \leq r\}\) is closed in \(Y\) for every real number \(r\).

It is an open problem whether a continuous image of a Radon-Nikodým compact is Radon-Nikodým. Arvanitakis [2] has made the following approach to this problem: if \(K\) is a Radon-Nikodým compact and \(\pi : K \rightarrow L\) is a continuous surjection, then we have a lower semicontinuous fragmenting metric \(d\) on \(K\), and if we want to prove that \(L\) is Radon-Nikodým compact, we should find such a metric on \(L\). A natural candidate is:

\[
d_1(x, y) = d(\pi^{-1}(x), \pi^{-1}(y)) = \inf \{d(t, s) : \pi(t) = x, \pi(s) = y\}.\]

The map \(d_1\) is lower semicontinuous and fragments \(L\) and it is a quasi metric, that is, it is symmetric and vanishes only if \(x = y\). But it is not a metric because, in general, it lacks triangle inequality. Consequently, Arvanitakis [2] introduced the

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following concept:

**Definition 2.** A compact space $L$ is said to be *quasi Radon-Nikodým* if there exists a lower semicontinuous quasi metric which fragments $L$.

The class of quasi Radon-Nikodým compacta is closed under continuous images but it is unknown whether it is the same class as that of Radon-Nikodým compacta or even the class of their continuous images. At least other two superclasses of continuous images of Radon-Nikodým compacta appear in the literature. Reznichenko [1, p. 104] defined a compact space $L$ to be *strongly fragmentable* if there is a metric $d$ which fragments $L$ such that each pair of different points of $L$ possess disjoint neighbourhoods at a positive $d$-distance. It has been noted by Namioka [10] that the classes of quasi Radon-Nikodým and strongly fragmentable compacta are equal. The other superclass of continuous images of Radon-Nikodým compacta, called *countably lower fragmentable* compacta, was introduced by Fabian, Heisler and Matoušková [5]. In section 3, we recall its definition and we prove that this class is equal to the other two.

The main result in section 1 is the following:

**Theorem 3.** If $K$ is a quasi Radon-Nikodým compact space of weight less than $b$, then $K$ is Radon-Nikodým compact.

The weight of a topological space is the least cardinality of a base for its topology. We also recall the definition of cardinal $b$. In the set $\mathbb{N}^\mathbb{N}$ we consider the order relation given by $\sigma \leq \tau$ if $\sigma_n \leq \tau_n$ for all $n \in \mathbb{N}$. Cardinal $b$ is the least cardinality of a subset of $\mathbb{N}^\mathbb{N}$ which is not $\sigma$-bounded for this order (a set is $\sigma$-bounded if it is a countable union of bounded subsets). It is consistent that $b > \omega_1$. In fact, Martin’s axiom and the negation of the continuum hypothesis imply that $c = b > \omega_1$, cf. [6, 11D and 14B]. It is also possible that $c > b > \omega_1$, cf. [17, section 5]. On the other hand, cardinal $d$ is the least cardinality of a cofinal subset of $(\mathbb{N}^\mathbb{N}, \leq)$, that is, a set $A$ such that for each $\sigma \in \mathbb{N}^\mathbb{N}$ there is some $\tau \in A$ such that $\sigma \leq \tau$. In a sense, the following proposition puts a rough bound on the size of the class of quasi Radon-Nikodým compacta with respect to Radon-Nikodým compacta.

**Proposition 4.** Every quasi Radon-Nikodým compact space embeds into a product of Radon-Nikodým compact spaces with at most $d$ factors.

In section 2 we discuss the Banach space counterpart to Theorem 3. A Banach space $V$ is Asplund generated, or $GSG$, if there is some Asplund space $V'$ and a bounded linear operator $T : V' \to V$ such that $T(V')$ is dense in $V$. Our main result for this class is the following:
Theorem 5. Let $V$ be a Banach space of density character less than $b$ and such that the dual unit ball $(B_{V^*}, \omega^*)$ is quasi Radon-Nikodým compact, then $V$ is Asplund generated.

The density character of a Banach space is the least cardinal of a norm-dense subset, and it equals the weight of its dual unit ball in the weak* topology.

Examples constructed by Rosenthal [13] and Argyros [4, section 1.6] show that there exist Banach spaces which are subspaces of Asplund generated spaces but which are not Asplund generated. However, since the dual unit ball of a subspace of an Asplund generated space is a continuous image of a Radon-Nikodým compact [4, Theorem 1.5.6], we have the following corollary to Theorem 5:

Corollary 6. If a Banach space $V$ is a subspace of an Asplund generated space and the density character of $V$ is less than $b$, then $V$ is Asplund generated.

Also, a Banach space is weakly compactly generated (WCG) if it is the closed linear span of a weakly compact subset. The same examples mentioned above show that neither is this property inherited by subspaces. A Banach space $V$ is weakly compactly generated if and only if it is Asplund generated and its dual unit ball $(B_{V^*}, \omega^*)$ is Corson compact [11], [14]. Having Corson dual unit ball is a hereditary property since a continuous image of a Corson compact is Corson compact [7], hence:

Corollary 7. If a Banach space $V$ is a subspace of a weakly compactly generated space and the density character of $V$ is less than $b$, then $V$ is weakly compactly generated.

Corollary 7 can also be obtained from the following theorem, essentially due to Mercourakis [8]:

Theorem 8. If a Banach space $V$ is weakly $K$-analytic and the density character of $V$ is less than $b$, then $V$ is weakly compactly generated.

The class of weakly $K$-analytic spaces is larger than the class of subspaces of weakly compactly generated spaces. We recall its definition in section 2. The result of Mercourakis [8, Theorem 3.13] is that, under Martin’s axiom, weakly $K$-analytic Banach spaces of density character less than $c$ are weakly compactly generated, but his arguments prove in fact the more general Theorem 8. We give a more elementary proof of this theorem, obtaining it as a consequence of a purely topological result: Any $K$-analytic topological space of density character less than $b$ contains a dense $\sigma$-compact subset. We also remark that it is not possible to generalize Theorem 8 for the class of weakly countably determined Banach spaces.

Cardinal $b$ is best possible for Theorem 5, Theorem 8 and their corollaries, as it is shown by slight modifications of the mentioned example of Argyros [4, section 1.6]
and of the example of Talagrand [15] of a weakly $K$-analytic Banach space which is not weakly compactly generated, so that we get examples of density character exactly $b$.

For information about cardinals $b$ and $d$ we refer to [17]. Concerning Banach spaces, our main reference is [4].

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1. **Quasi Radon-Nikodým compacta of low weight**

In this section, we characterize quasi Radon-Nikodým compacta in terms of embeddings into cubes $[0,1]^\Gamma$ and from this, we will derive proofs of Theorem 3 and Proposition 4. Techniques of Arvanitakis [2] will play an important role in this section, as well as the following theorem of Namioka [9]:

**Theorem 9.** Let $K$ be a compact space. The following are equivalent.

1. $K$ is Radon-Nikodým compact.
2. There is an embedding $K \subset [0,1]^\Gamma$ such that $K$ is fragmented by the uniform metric $d(x,y) = \sup_{\gamma \in \Gamma} |x_\gamma - y_\gamma|$.

Let $P \subset \mathbb{N}^\mathbb{N}$ be the set of all strictly increasing sequences of positive integers. Note that this is a cofinal subset of $\mathbb{N}^\mathbb{N}$. For each $\sigma \in P$ we consider the lower semicontinuous non decreasing function $h^\sigma : [0, +\infty] \rightarrow \mathbb{R}$ given by:

- $h^\sigma(0) = 0$,
- $h^\sigma(t) = \frac{1}{\sigma_n}$ whenever $\frac{1}{n+1} < t \leq \frac{1}{n}$,
- $h^\sigma(t) = \frac{1}{\sigma_1}$ whenever $t > 1$.

Also, if $f : X \times X \rightarrow \mathbb{R}$ is a map and $A, B \subset X$, we will use the notation $f(A,B) = \inf \{ f(x,y) : x \in A, y \in B \}$.

**Theorem 10.** Let $K$ be a compact subset of the cube $[0,1]^\Gamma$. The following are equivalent:

1. $K$ is quasi Radon-Nikodým compact.
2. There is a map $\sigma : \Gamma \rightarrow P$ such that $K$ is fragmented by

$$f(x,y) = \sup_{\gamma \in \Gamma} h^{\sigma(\gamma)}(|x_\gamma - y_\gamma|)$$

which is a lower semicontinuous quasi metric.
PROOF: Observe that $f$ in (2) is expressed as a supremum of lower semicontinuous functions, and therefore, it is lower semicontinuous. Also, $f(x, y) = 0$ if and only if $h^*(\gamma, t) = 0$ for all $\gamma \in \Gamma$ if and only if $|x_\gamma - y_\gamma| = 0$ for all $\gamma \in \Gamma$. Hence, $f$ is indeed a lower semicontinuous quasi metric and it is clear that (2) implies (1). Assume now that $K$ is quasi Radon-Nikodym compact and let $g : K \times K \to [0, 1]$ be a lower semicontinuous quasi metric which fragments $K$.

PROOF: Let $g_0 : [0, 1] \times [0, 1] \to [0, 1]$ be a lower semicontinuous quasi metric on $[0, 1]$. Then, there exists $\tau \in P$ such that $h^*(|t - s|) \leq g_0(t, s)$ for all $t, s \in [0, 1]$.

PROOF: We define $\tau$ recursively. Suppose that we have defined $\tau_1, \ldots, \tau_m$ in such a way that if $|t - s| > \frac{1}{m+1}$, then $h^*(|t - s|) \leq g_0(t, s)$. Let

\[ K_m = \left\{ (t, s) \in [0, 1] \times [0, 1] : |t - s| \geq \frac{1}{m+2} \text{ and } g_0(t, s) \leq \frac{1}{m} \right\} \]

Then, $\{K_m\}_{m=1}^\infty$ is a decreasing sequence of compact subsets of $[0, 1]^2$ with empty intersection. Hence, there is $m_1$ such that $K_m$ is empty for $m \geq m_1$. We define $\tau_{n+1} = \max\{m_1, \tau_n + 1\}$. 

Now, we state a lemma which is just a piece of the proof of [2, Proposition 3.2]. We include its proof for the sake of completeness.

**Lemma 11.** Let $g_0 : [0, 1] \times [0, 1] \to [0, 1]$ be a lower semicontinuous quasi metric on $[0, 1]$. Then, there exists $\tau \in P$ such that $h^*(|t - s|) \leq g_0(t, s)$ for all $t, s \in [0, 1]$.

**Lemma 12.** Let $K, L$ be compact spaces, let $f : K \times K \to \mathbb{R}$ be a symmetric map which fragments $K$ and $p : K \to L$ a continuous surjection. Then $L$ is fragmented by $g(x, y) = f(p^{-1}(x), p^{-1}(y))$ and in particular, $L$ is fragmented by any $g'$ with $g' \leq g$.

**PROOF:** Let $M$ be a closed subset of $L$ and $\varepsilon > 0$. By Zorn’s lemma a set $N \subseteq K$ can be found such that $p : N \to M$ is onto and irreducible (that is, for every $N' \subseteq N$ closed, $p : N' \to M$ is not onto). We find $U \subseteq N$ a relative open subset of $N$ of $f$-diameter less than $\varepsilon$. By irreducibility, $p(U)$ has nonempty relative interior in $M$. This interior is a nonempty relative open subset of $M$ of $g$-diameter less than $\varepsilon$. 

\[ \square \]
In the sequel, we use the following notation: If $A \subset \Gamma$ are sets, $d_A$ states for the pseudometric in $[0,1]^\Gamma$ given by $d_A(x,y) = \sup_{\gamma \in A} |x_\gamma - y_\gamma|$.

**Lemma 13.** Let $K$ be a compact subset of the cube $[0,1]^\Gamma$ and let $\sigma : \Gamma \rightarrow P$ be a map such that the quasi metric

$$f(x,y) = \sup_{\gamma \in \Gamma} h^{\sigma(\gamma)}(|x_\gamma - y_\gamma|)$$

fragments $K$ and such that $\sigma(\Gamma)$ is a $\sigma$-bounded subset of $\mathbb{N}^\mathbb{N}$. Then, $K$ is Radon-Nikod´ ym compact. In addition, there exist sets $\Gamma_n \subset \Gamma$ such that $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$ and each $d_{\Gamma_n}$ fragments $K$. 

**PROOF:** There is a decomposition $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$ such that each $\sigma(\Gamma_n)$ has a bound $\tau_n$ in $(\mathbb{N}^\mathbb{N}, \leq)$. We choose $\tau_n \in P$. First, we prove that each $d_{\Gamma_n}$ fragments $K$. For every $n \in \mathbb{N}$, $K$ is fragmented by the map

$$f_n(x,y) = \sup_{\gamma \in \Gamma_n} h^{\tau_n}(|x_\gamma - y_\gamma|) \leq f(x,y)$$

and

$$f_n(x,y) = \sup_{\gamma \in \Gamma_n} h^{\sigma(\gamma)}(|x_\gamma - y_\gamma|) \geq \sup_{\gamma \in \Gamma_n} h^{\tau_n}(|x_\gamma - y_\gamma|) = h^{\tau_n} \left( \sup_{\gamma \in \Gamma_n} |x_\gamma - y_\gamma| \right) = h^{\tau_n}(d_{\Gamma_n}(x,y)).$$

Hence, a set of $f_n$-diameter less than $\frac{1}{n}$ in $K$ is a set of $d_{\Gamma_n}$-diameter less than $\frac{1}{n}$ and therefore, since $f_n$ fragments $K$, also $d_{\Gamma_n}$ fragments $K$.

Consider now $p_n : [0,1]^\Gamma \rightarrow [0,1]^\Gamma$ the natural projection and $K_n = p_n(K)$. By Lemma 12, since $K$ is fragmented by $f_n$, $K_n$ is fragmented by

$$g_n(x,y) = \sup_{\gamma \in \Gamma_n} h^{\sigma(\gamma)}(|x_\gamma - y_\gamma|).$$

and hence, $K_n$ is Radon-Nikodým compact. Moreover, since $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$, $K$ embeds into the product $\prod_{n \in \mathbb{N}} K_n$ and the class of Radon-Nikodým compacta is closed under taking countable products and under taking closed subspaces [9], so $K$ is Radon-Nikodým compact.

**PROOF OF THEOREM 3:** If the weight of $K$ is less than $b$, then $K$ can be embedded into a cube $[0,1]^\Gamma$ with $|\Gamma| < b$. Any subset of $\mathbb{N}^\mathbb{N}$ of cardinality less than $b$ is $\sigma$-bounded, so Theorem 3 follows directly from Theorem 10 and Lemma 13. 

**PROOF OF PROPOSITION 4:** Let $K$ be quasi Radon-Nikodým compact, suppose $K$ is embedded into some cube $[0,1]^\Gamma$ and let $\sigma : \Gamma \rightarrow P$ be as in Theorem 10. Let $A \subset P$ be a cofinal subset of $P$ of cardinality $d$. For $\alpha \in A$, let

$$\Gamma_\alpha = \{ \gamma \in \Gamma : \sigma(\gamma) \leq \alpha \},$$

let $p_\alpha : [0,1]^\Gamma \rightarrow [0,1]^{\Gamma_\alpha}$ be the natural projection, and let $K_\alpha = p_\alpha(K)$. Again, since $\Gamma = \bigcup_{\alpha \in A} \Gamma_\alpha$, $K$ embeds into the product $\prod_{\alpha \in A} K_\alpha$. By Lemma 12, $K_\alpha$ is
fragmented by
\[ g_\alpha(x, y) = \sup_{\gamma \in I_\alpha} h^{\sigma(\gamma)}(|x_\gamma - y_\gamma|) \]
The set \( \{\sigma(\gamma) : \gamma \in I_\alpha\} \) is a bounded, and hence \( \sigma \)-bounded, set. Hence, by Lemma 13, \( K_\alpha \) is Radon-Nikodým compact. \( \square \)

We note that from Lemma 13, we obtain something stronger than Theorem 3:

**Theorem 14.** For every quasi Radon-Nikodým compact subset of a cube \([0, 1]^\Gamma\) with \(|\Gamma| < b\) there is a countable decomposition \( \Gamma = \bigcup_{n \in \mathbb{N}} I_n \) such that \( d_{I_n} \) fragments \( K \) for all \( n \in \mathbb{N} \).

A similar result holds also for generalized Cantor cubes (cf. [5, Theorem 3], [2, Theorem 3.6]): If \( K \) is a quasi Radon-Nikodým compact subset of \([0, 1]^\Gamma\), then there is a decomposition \( \Gamma = \bigcup_{n \in \mathbb{N}} I_n \) such that \( d_{I_n} \) fragments \( K \) for all \( n \in \mathbb{N} \). We give now an example which shows that this phenomenon does not happen for general cubes, even if the compact \( K \) has weight less than \( b \) or it is dimensionless:

**Proposition 15.** There exist a set \( \Gamma \) of cardinality \( b \) and a compact subset \( K \) of \([0, 1]^\Gamma\) homeomorphic to the metrizable Cantor cube \([0, 1]^N\) such that for any decomposition \( \Gamma = \bigcup_{n \in \mathbb{N}} I_n \) there exists \( n \in \mathbb{N} \) such that \( d_{I_n} \) does not fragment \( K \).

**Proof:** First, we take \( \Gamma \) a subset of \( \mathbb{N}^3 \) of cardinality \( b \) which is not \( \sigma \)-bounded. We call \( A = \{\gamma_n : \gamma \in I, n \in \mathbb{N}\} \) the set of all terms of elements of \( I \). We define \( K' = \{x \in [0, 1]^\Gamma \times \mathbb{N} : x_{\gamma,n} = x_{\gamma',n'} \ \text{whenever} \ \gamma_n = \gamma'_n\} \).

Observe that \( K' \) is homeomorphic to \([0, 1]^N\): namely, for each \( a \in A \) choose some \( \gamma^a, n^a \in \Gamma \times \mathbb{N} \) such that \( \gamma^a_n = a \); in this case we have a homeomorphism \( K' \rightarrow [0, 1]^A \) given by \( x \mapsto (x_{\gamma^a,n^a})_{a \in A} \).

Now, we consider the embedding \( \phi : [0, 1]^\Gamma \times \mathbb{N} \rightarrow [0, 1]^\Gamma \) given by
\[ \phi(x) = \left( \sum_{n \in \mathbb{N}} \left( \frac{2}{3} \right)^n x_{\gamma,n} \right)_{\gamma \in \Gamma} \]
We claim that the space \( K = \phi(K') \subset \mathbb{N} \) verifies the statement. Let \( \Gamma = \bigcup_{n \in \mathbb{N}} I_n \) any countable decomposition of \( I \). Since \( \Gamma \) is not \( \sigma \)-bounded, there is some \( n \in \mathbb{N} \) such that \( I_n \) is not bounded. For this fixed \( n \), since \( I_n \) is not bounded, there is some \( m \in \mathbb{N} \) such that the set \( S = \{\gamma_m : \gamma \in I_n\} \subset A \) is infinite. We consider \( K_0 = \{x \in K' : x_{\gamma,k} = 0 \ \text{whenever} \ \gamma_k \notin S\} \subset K \).

By the same arguments as for \( K' \), \( K_0 \) is homeomorphic to the Cantor cube \([0, 1]^N\) by a map \( K_0 \rightarrow [0, 1]^S \) given by \( x \mapsto (x_{\gamma,n})_{a \in S} \). Now, we take two different elements \( x, y \in K_0 \). Then, there must exist some \( \gamma \in I_n \) such that \( x_{\gamma,m} \neq y_{\gamma,m} \), and this implies that \( |\phi(x)_\gamma - \phi(y)_\gamma| \geq 3^{-m} \) and therefore \( d_{I_n}(\phi(x), \phi(y)) \geq 3^{-m} \).
This means that any nonempty subset of \( \phi(K_0) \) of \( d_{I_n} \)-diameter less than \( 3^{-m} \) must be a singleton. If \( d_{I_n} \) fragmented \( K \), this would imply that \( \phi(K_0) \) has an isolated point, which contradicts the fact that it is homeomorphic to \([0, 1]^N\). \( \square \)
2. Banach spaces of low density character

In this section we find that cardinal $b$ is the least possible density character of Banach spaces which are counterexamples to several questions. First, we introduce some notation: If $A$ is a subset of a Banach space $V$, we call $d_A$ to the pseudometric $d_A(x^*, y^*) = \sup_{x \in A} |x^*(x) - y^*(x)|$ on $B_{V^*}$. Also, we recall the following definition [4, Definition 1.4.1]:

**Definition 16.** A nonempty bounded subset $M$ of a Banach space $V$ is called an Asplund set if for each countable set $A \subset M$ the pseudometric space $(B_{V^*}, d_A)$ is separable.

By [3, Theorem 2.1], $M$ is an Asplund subset of $V$ if and only if $d_M$ fragments $(B_{V^*}, w^*)$. Also, by [4, Theorem 1.4.4], a Banach space $V$ is Asplund generated if and only if it is the closed linear span of an Asplund subset.

**PROOF OF THEOREM 5:** Let $\Gamma$ be a dense subset of the unit ball $B_V$ of $V$ of cardinality less than $b$. Then, we have a natural embedding $(B_{V^*}, w^*) \subset [-1, 1]^\Gamma$. Since $(B_{V^*}, w^*)$ is quasi Radon-Nikodym compact, we apply Theorem 14 and we have $\Gamma = \bigcup \Gamma_n$ and each $d_{\Gamma_n}$ fragments $(B_{V^*}, w^*)$. This means that for each $n$, $\Gamma_n$ is an Asplund set, and by [4, Lemma 1.4.3], $M = \bigcup_{n \in \mathbb{N}} \Gamma_n$ is an Asplund set too.

Finally, since the closed linear span of $M$ is $V$, by [4, Theorem 1.4.4], $V$ is Asplund generated. □

We recall now the concepts that we need for the proof of Theorem 8. We follow the terminology and notation of [4, sections 3.1, 4.1]. Let $X$ and $Y$ be topological spaces. A map $\phi: X \to 2^Y$ from $X$ to the subsets of $Y$ is said to be an usco if the following conditions hold:

1. $\phi(x)$ is a compact subset of $Y$ for all $x \in X$.
2. $\{x : \phi(x) \subset U\}$ is open in $X$, for every open set $U$ of $Y$.

In this situation, for $A \subset X$ we denote $\phi(A) = \bigcup_{x \in A} \phi(x)$.

A completely regular topological space $X$ is said to be $K$-analytic if there exists an usco $\phi: \mathbb{N}^\mathbb{N} \to 2^X$ such that $\phi(\mathbb{N}^\mathbb{N}) = X$. A Banach space is weakly $K$-analytic if it is a $K$-analytic space in its weak topology.

We note that if a Banach space $V$ contains a weakly $\sigma$-compact subset $M$ which is dense in the weak topology, then $V$ is WCG. This is because if $M = \bigcup_{n=1}^\infty K_n$ being $K_n$ a weakly compact set bounded by $c_n > 0$, then $\{0\} \cup \bigcup_{n=1}^\infty K_n$ is a weakly compact subset of $V$ whose linear span is (weakly) dense in $V$. Hence, Theorem 8 is deduced from the following:

**Proposition 17.** If $X$ is a $K$-analytic topological space which contains a dense subset of cardinality less than $b$, then $X$ contains a dense $\sigma$-compact subset.

**PROOF:** We have an usco $\phi: \mathbb{N}^\mathbb{N} \to 2^X$ with $\phi(\mathbb{N}^\mathbb{N}) = X$ and also a set $\Sigma \subset \mathbb{N}^\mathbb{N}$ such that $|\Sigma| < b$ and $\phi(\Sigma)$ is dense in $X$. Any subset of $\mathbb{N}^\mathbb{N}$ of cardinal less than...
\( b \) is contained in a \( \sigma \)-compact subset of \( \mathbb{N}^\mathbb{N} \) [17, Theorem 9.1]. Uscos send compact sets onto compact sets, so if \( \Sigma' \supset \Sigma \) is \( \sigma \)-compact, then \( \phi(\Sigma') \) is a dense \( \sigma \)-compact subset of \( X \). \( \square \)

We recall that a completely regular topological space \( X \) is \( K \)-countably determined if there exists a subset \( \Sigma \) of \( \mathbb{N}^\mathbb{N} \) and an usco \( \phi : \Sigma \rightarrow 2^X \) such that \( \phi(\Sigma) = X \) and that a Banach space is weakly countably determined if it is \( K \)-countably determined in its weak topology. Talagrand [16] has constructed a Banach space which is weakly countably determined but which is not weakly \( K \)-analytic. A slight modification of this example gives a similar one with density character \( \omega_1 \). This shows that no analogue of Theorem 8 is possible for weakly countably determined Banach spaces. The change in the example consists in substituting the set \( T \) considered in [16, p. 78] by any subset \( T' \subseteq T \) of cardinal \( \omega_1 \) such that \( \{ o(X) : X \in T' \} \) is uncountable and \( A \) by \( A' = \{ A \subseteq T' : A \in A_1 \} \) (the notations are explained in [16]).

Now, we turn to the fact that cardinal \( b \) is best possible in Theorem 5, Theorem 8 and their corollaries. We fix a subset \( S \) of \( \mathbb{N}^\mathbb{N} \) of cardinality \( b \) which is not \( \sigma \)-bounded.

Following the exposition of the example of Argyros in [4, section 1.6] we just substitute the space \( Y = \overline{\prod}_{\sigma} \{ \pi_\sigma : \sigma \in \mathbb{N}^\mathbb{N} \} \) in [4, Theorem 1.6.3] by \( Y' = \overline{\prod}_{\sigma} \{ \pi_\sigma : \sigma \in S \} \) and we obtain a Banach space of density character \( b \) which is a subspace of a WCG space \( C(K) \) but which is not Asplund generated. The same arguments in [4, section 1.6] hold just changing \( \mathbb{N}^\mathbb{N} \) by \( S \) where necessary. Only the proof of [4, Lemma 1.6.1] is not good for this case. It must be substituted by the following:

**Lemma 18.** Let \( \Gamma_n, n \in \mathbb{N} \), be any subsets of \( S \) such that \( \bigcup_{n \in \mathbb{N}} \Gamma_n = S \). Then there exist \( n, m \in \mathbb{N} \) and an infinite set \( A \in \mathcal{A}_m \) such that \( A \subseteq \Gamma_n \).

Here, as in [4, section 1.6], \( \mathcal{A}_m \) is the family of all subsets \( A \subseteq \mathbb{N}^\mathbb{N} \) such that if \( \sigma, \tau \in A \) and \( \sigma \neq \tau \), then \( \sigma_i = \tau_i \) if \( i \leq m \) and \( \sigma_{m+1} \neq \tau_{m+1} \). Also, \( \mathcal{A} = \bigcup_{m=1}^{\infty} \mathcal{A}_m \).

**PROOF OF LEMMA 18:** We consider \( \Gamma_{i,j} = \{ \sigma \in \Gamma_i : \sigma_1 = j \} \), \( i, j \in \mathbb{N} \). Note that \( S = \bigcup_{j} \Gamma_{i,j} \). Since \( S \) is not \( \sigma \)-bounded, there exist \( n, l \) with \( \Gamma_{n,l} \) unbounded. This implies that for some \( m \), the set \( \{ \sigma_m : \sigma \in \Gamma_{n,l} \} \) is infinite. We take \( m \) the least integer with this property \( (m > 1) \). Let \( B \subseteq \Gamma_{n,l} \) be an infinite set such that \( \sigma_m \neq \sigma_m' \) for \( \sigma, \sigma' \in B \), \( \sigma \neq \sigma' \). Since all \( \sigma_k \) with \( \sigma \in B \), \( k < m \), lie in a finite set, an infinite set \( A \subseteq B \) can be chose such that \( A \in \mathcal{A}_{m-1} \). \( \square \)

On the other hand, if we follow the proof in [4, section 4.3] that the Banach space \( C(K) \) of Talagrand is weakly \( K \)-analytic but not WCG, and we change \( K \) in [4, p. 76] by \( K' = \{ X, A : A \in A, A \subseteq S \} \subseteq \{ 0, 1 \}^S \) then \( C(K') \) still verifies this conditions and has density character \( b \). Observe that \( C(K') \) is weakly \( K \)-analytic because \( K' \) is a retract of the original \( K \). The fact that \( C(K') \) is not WCG (not even a subspace of a WCG space) follows from [4, Theorem 4.3.2] and Lemma 18 above by the same arguments as in [4, p. 78].
3. COUNTABLY LOWER FRAGMENTABLE COMPACTA

In this section we prove that the concept of quasi Radon Nikodým compact [2] is equivalent to that of countably lower fragmentable compact [5]. The main result for this class in [5] is that if $K$ is countably lower fragmentable, then so is $(B_{C(K)^*}, w^*)$. We note that, with these two facts at hand, together with the fact that if $C(K)$ is Asplund generated, then $K$ is Radon-Nikodým [4, Theorem 1.5.4], Theorem 3 is deduced from Theorem 5.

We need some notation: if $K$ is a compact space and $A \subset C(K)$ is a bounded set of continuous functions over $K$, we define the pseudometric $d_A$ on $K$ as $d_A(x, y) = \sup_{f \in A} |f(x) - f(y)|$. If $X$ is a topological space, $d : X \times X \rightarrow \mathbb{R}$ is a map, and $\Delta$ is a positive real number, it is said that $d$ $\Delta$-fragments $X$ if for each subset $L$ of $X$ there is a relative open subset $U$ of $L$ of $d$-diameter less than or equal to $\Delta$.

**Definition 19.** A compact space $K$ is said to be countably lower fragmentable if there are bounded subsets $\{A_{n,p} : n, p \in \mathbb{N}\}$ of $C(K)$ such that $C(K) = \bigcup_{n \in \mathbb{N}} A_{n,p}$ for every $p \in \mathbb{N}$, and the pseudometric $d_{A_{n,p}} \frac{1}{p}$-fragments $K$.

This is the definition as it appears in [5]. However, variable $p$ is superfluous in it. If the sets $A_{n,1}$ exist, it is sufficient to define $A_{n,p} = \{\frac{1}{p} f : f \in A_{n,1}\}$.

On the other hand, we recall a concept introduced by Namioka [9]: For a topological space $K$, a set $L \subset K \times K$ is said to be an almost neighborhood of the diagonal if it contains the diagonal $\Delta_K = \{(x, x) : x \in K\}$ and satisfies that for every nonempty subset $X$ of $K$ there is a nonempty relative open subset $U$ of $X$ such that $U \times U \subset L$. The use of this was suggested to us by I. Namioka and simplifies our original proof.

**Theorem 20.** For a compact subset $K$ of $[0, 1]^\Gamma$ the following are equivalent:

1. $K$ is quasi Radon-Nikodým compact
2. $K$ is countably lower fragmentable.
3. There are subsets $\Gamma_{n,p}, n, p \in \mathbb{N}$, of $\Gamma$ such that $d_{\Gamma_{n,p}} \frac{1}{p}$-fragments $K$ for every $n, p \in \mathbb{N}$.

**Proof:** Suppose $K$ is quasi Radon-Nikodým compact and let $\phi$ be a lower semicontinuous quasi metric which fragments $K$. Then, we just define $A_{n,p} = \{f \in C(K) : |f(x) - f(y)| < \frac{1}{p}$ whenever $\phi(x, y) \leq \frac{1}{n}\} \cap \{f : \|f\|_{\infty} \leq n\}$

Clearly, $d_{A_{n,p}} \frac{1}{p}$-fragments $K$ because any subset of $K$ of $\phi$-diameter less than $\frac{1}{n}$ has $d_{A_{n,p}}$-diameter less than $\frac{1}{p}$, and we know that $\phi$ fragments $K$. On the other hand, for a fixed $p \in \mathbb{N}$, in order to prove that $C(K) = \bigcup_{n \in \mathbb{N}} A_{n,p}$, observe that, if $f \in C(K)$, then

$$C_n = \{(x, y) \in K \times K : |f(x) - f(y)| \geq \frac{1}{p} \text{ and } \phi(x, y) \leq \frac{1}{n}\}$$
is a decreasing sequence of compact subsets of $K \times K$ with empty intersection so there is some $n > \|f\|_\infty$ such that $C_n$ is empty, and then, $f \in A_{n,p}$.

That (2) implies (3) is evident, just to take $\Gamma_{n,p} = A_{n,p} \cap \Gamma$ whenever $A_{n,p}$, $n, p \in \mathbb{N}$ are the sets in the definition of countably lower fragmentability.

Now, suppose (3). For every $n, p \in \mathbb{N}$, since $d_{A_{n,p}} \frac{1}{p}$-fragments $K$, this means that the set $C_{n,p} = \{(x, y) \in K \times K : d_{\Gamma_{n,p}}(x, y) \leq \frac{1}{p}\}$ is an almost neighborhood of the diagonal which, in addition, is closed. On the other hand, observe that, for each $n, p \in \mathbb{N}$, $(x, y) \in C_{n,p}$ if and only if $|x_\gamma - y_\gamma| \leq \frac{1}{p}$ for all $\gamma \in \Gamma_{n,p}$ so that

$$\bigcap_{n, p \in \mathbb{N}} C_{n,p} = \bigcap_{p \in \mathbb{N}} \left\{ (x, y) : |x_\gamma - y_\gamma| \leq \frac{1}{p} \forall \gamma \in \bigcup_{n \in \mathbb{N}} \Gamma_{n,p} = \Gamma \right\} = \Delta_K$$

Now, $K$ is quasi Radon-Nikodým by virtue of [10, Theorem 1], which states that $K$ is quasi Radon-Nikodým compact if and only if there is a countable family of closed almost neighborhoods of the diagonal whose intersection is the diagonal $\Delta_K$. □

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