Nonlinear dynamics of two coupled nano-electromechanical resonators

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Abstract
As a model of coupled nano-electromechanical resonators we study two nonlinear driven oscillators with an arbitrary coupling strength between them. Analytical expressions are derived for the oscillation amplitudes as a function of the driving frequency and for the energy transfer rate between the two oscillators. The nonlinear restoring forces induce the expected nonlinear resonance structures in the amplitude–frequency characteristics with asymmetric resonance peaks. The corresponding multistable behaviour is shown to be an efficient tool to control the energy transfer arising from the sensitive response to small changes in the driving frequency. It is shown that the energy transfer rate between the nano-electromechanical resonators can be controlled by tuning the driving field frequency. In addition, our results imply that the nonlinear response can be exploited to design precise sensors for mass or force detection experiments based on nano-electromechanical resonators.

(Some figures in this article are in colour only in the electronic version)

1. Introduction
Currently experimental efforts are devoted to the fabrication of nanoscale resonators with a precise control of their behaviour. Such nanoscale resonators are ideal prototypical systems for testing fundamental physical concepts, such as entanglement and quantum correlations [1]. By now several types of resonators were successfully considered, like optical two-level atoms in quantum cavities [2], artificial Josephson junction qubits [3], atoms seized in ion traps [4] or nano-optomechanical devices [5]. There is a new trend towards nano-electromechanical resonators. They are widely studied from both the quantum-mechanical [4, 6–10] and a classical point of view [11–18]. These devices are approximately 200 nm in size and consist of three layers of gallium arsenide (GaAs): an $n$-doped layer of width 100 nm is stacked within an insulating layer of 50 nm and a $p$-doped layer of 50 nm [18]. The resonators can be controlled by electric fields via the piezoelectric effect, which fix their mechanical strain [17]. Along with a single resonator, one can consider coupled resonators driven by an additional external field. Coupled resonators show different dynamical regimes dependent on the interplay of their coupling strength and the driving. For the case of moderate coupling between two resonators, this problem was already addressed in a recent paper by Karabalin \textit{et al} [18]. They showed that the linear and weakly nonlinear response of one oscillator can be modified by driving the second oscillator. A complicated frequency-sweep response curve was obtained numerically when both oscillators are driven into the strongly nonlinear regime.

In this paper, we study two nonlinear oscillators allowing for an arbitrary coupling strength between them with a possibility of driving both with the same frequency but different amplitudes. The coupling strength between the oscillators is quantified in terms of the connectivity parameter defined below in section 2. We derive general analytical expressions for the amplitude–frequency characteristics valid for arbitrary (weak as well as strong) connectivity. We analyse the redistribution of energy between the two resonators injected into the system via the external driving fields. We
quantify stable and unstable dynamical regimes, with special focus on the nonlinear response of the system. Our predictions point to possible new applications of nanoscale resonators exploiting their sensitivity in response to external fields and perturbations. In particular, they may be used as sensors for tiny forces or masses [19, 20], which lead to a shift in their resonance frequencies to be identified in the sensitive nonlinear response regime. In addition, we will show that the energy transfer rate between the resonators is controllable via the driving field frequency. The key issue of the proposed control mechanism is the unstable areas of the resonance hysteresis loops that can be utilized for switching between the different energy transport regimes.

Our paper is organized as follows: in the next section, we discuss our fundamental model of two coupled driven oscillators. In section 3, we study the mode frequency shifts and relaxation effects for a non-resonant driving, while in section 4 we address the resonant case with a special focus on the nonlinear shifts of the mode frequencies. In the following sections, we investigate the frequency response function and the key problem of the energy redistribution between the oscillators, before concluding in section 8.

2. Model

We consider a system of two nanomechanical oscillators described by the coordinates \( x_{1,2} \) in the framework of the model outlined in [18]. The Hamiltonian of the system has the form

\[
H = \frac{x_1^2 + x_2^2}{2} + \frac{\omega_1^2 x_1^2}{2} + \frac{\omega_2^2 x_2^2}{2} + \frac{\chi_1 x_1^4}{4} + \frac{\chi_2 x_2^4}{4} + \frac{D(x_1 - x_2)^2}{2}.
\]

(1)

Here, \( \omega_1 \) and \( \omega_2 \) are frequencies of the individual resonators, \( \chi_1 \) and \( \chi_2 \) are nonlinearity parameters, and \( D \) is the coefficient of the coupling between the resonators.

The corresponding dynamical equations can be written in the following form:

\[
\dot{x}_1 + \omega_1^2 x_1 + D(x_1 - x_2) = \varepsilon M,
\]

\[
\dot{x}_2 + \omega_2^2 x_2 + D(x_2 - x_1) = \varepsilon N,
\]

\( \varepsilon M = \varepsilon M(x_1, x_1, t) = -2\gamma_1 x_1 - \chi_1 x_1^3 + F_1 \cos \Omega t, \)

\( \varepsilon N = \varepsilon N(x_2, x_2, t) = -2\gamma_2 x_2 - \chi_2 x_2^3 + F_2 \cos \Omega t, \)

\( \varepsilon \ll 1. \)

(2)

Here in equations (2) we add phenomenological dissipation coefficients \( \gamma_1, \gamma_2 \) and \( F_1, F_2 \) are amplitudes of the external harmonic forces applied to the resonators, \( \Omega \) is the frequency of these forces. As usual, we assume the right-hand sides of equations (3) to be small perturbations, see also [18, 21–24].

We summarize the canonical solution of the unperturbed coupled system first. Its dynamics follows the simple equations

\[
\dot{x}_1 + \omega_1^2 x_1 + D(x_1 - x_2) = 0,
\]

\[
\dot{x}_2 + \omega_2^2 x_2 + D(x_2 - x_1) = 0.
\]

(4)

The transition from coupled oscillations to the mode oscillations can be done via the following transformation [24]:

\[
x_1 = q_1 + q_2,
\]

\[
x_2 = -K^{-1} q_1 + K q_2,
\]

where

\[
K = -\frac{1}{\sigma}(1 + \sqrt{1 + \sigma^2}), \quad K^{-1} = \frac{1}{\sigma}(1 - \sqrt{1 + \sigma^2}),
\]

\( KK^{-1} = 1, \)

(6)

\[
\sigma = \frac{2D}{|\omega_1^2 - \omega_2^2|}.
\]

(7)

We call the parameter \( \sigma \) describing the coupling strength between the oscillators connectivity. From now on, we assume that \( \omega_2 > \omega_1 \).

The mode oscillations have the frequencies

\[
v_{1,2}^2 = \dot{\omega}_+^2 + \omega_2^2 \sqrt{1 + \sigma^2},
\]

where

\[
\dot{\omega}_+^2 = \frac{\dot{\omega}_1^2 + \dot{\omega}_2^2}{2},
\]

\[
\dot{\omega}_{1,2}^2 = \omega_{1,2}^2 + D,
\]

\[
\dot{\omega}_1^2 = \frac{\dot{\omega}_1^2 - \dot{\omega}_2^2}{2}.
\]

(8)

\( \dot{\omega}_{1,2} \) are partial frequencies. We would like to stress that the value of the connectivity \( \sigma \) depends not only on the linear coupling term \( D \), but on the proximity of the free oscillation frequencies \( \omega_1 \) and \( \omega_2 \). In the limit of a weak connectivity (\( \sigma \ll 1 \)), the frequencies \( v_{1,2} \) tend to the partial frequencies \( \dot{\omega}_{1,2} \), while in the limit of strong connectivity (\( \sigma \gg 1 \)),

\[
v_1^2 \simeq \dot{\omega}_1^2, \quad v_2^2 \simeq \dot{\omega}_2^2.
\]

(10)

\[
\dot{\omega}_1^2 = \frac{\dot{\omega}_1^2 + \dot{\omega}_2^2}{2}, \quad \dot{\omega}_2^2 = \frac{\dot{\omega}_1^2 - \dot{\omega}_2^2}{2} + 2D.
\]

Obviously, in the limit \( \sigma \gg 1 \), the mode frequency separation attains the maximal possible value which is equal to 2D.

In the case of a finite driving \( F_{1,2} \neq 0 \) but in the linear \( \chi_{1,2} = 0 \) dissipationless regime \( \gamma_{1,2} = 0 \), the particular solutions of the dynamical equations (2) are given by

\[
x_1 = A_1 \cos \Omega t, \quad x_2 = A_2 \cos \Omega t,
\]

\[
A_{1,2} = \frac{F_{1,2}}{d^2} \left( \dot{\omega}_{2,1}^2 - \Omega^2 \right)^2 + F_{2,1} D.
\]

(11)

Here, \( \frac{d^2}{d^2} = \frac{1}{(\Omega^2 - v_1^2)(\Omega^2 - v_2^2)} \)

\[
= \frac{1}{\Omega^2(v_1^2 - v_2^2)} \left( \frac{v_1^2}{v_1^2 - \Omega^2} - \frac{v_2^2}{v_2^2 - \Omega^2} \right)
\]

(12)

and \( A_{1,2} \) are the amplitudes of the induced resonator oscillations. They increase resonantly when the frequency of the external driving tends closer to either of \( v_{1,2} \).

With the above solutions at hand the influence of the damping term as well as of the nonlinear corrections can be
taken into account with the help of a standard substitution in the resonant denominator [24]:
\[
\frac{1}{v_{1,2} - \Omega^2} \rightarrow \frac{1}{2v_{1,2}(v_{1,2} - \Omega)}
\]
\[
\rightarrow \frac{1}{2v_{1,2}(v_{1,2} + \delta_{1,2} + i\Gamma_{1,2} - \Omega)}
\]
\[
\rightarrow \frac{1}{2v_{1,2}\sqrt{(v_{1,2} + \delta_{1,2} - \Omega)^2 + \Gamma_{1,2}^2}}.
\]
where \(\Gamma_{1,2}\) are the mode relaxation rates and \(\delta_{1,2}\) are the nonlinear corrections to the mode frequencies that depend on the oscillation amplitudes \(A_{1,2}\). We would like to point out that the substitutions of equation (13) are correct only if the nonlinearity is not too strong and the decay rate is not too high (\(v_{1,2} \gg \delta_{1,2}; \Gamma_{1,2}\)). Explicit expressions for \(\delta_{1,2}\) and \(\Gamma_{1,2}\), in terms of the system parameters will be given in the next section.

3. Nonlinear shift of the mode frequencies and consequences of the relaxation terms: the non-resonant case

We turn back to the perturbed system of equation (2) making the preliminary assumption that the resonance condition does not hold at the mode frequencies \(v_{1,2} \neq \Omega\). It is clear that in this particular case, the role of the external force is negligible. We concentrate at first on the influence of the relaxation and of the nonlinearity on the oscillation of the coupled resonators.

To study equations (2), we use a modified method of a slowly varying amplitude [21]. Taking into consideration the transformation (5), we reduce the unperturbed system to the mode oscillations and write down the solution in the form
\[
x_1(t) = A_1(t) \sin[v_{1,2}t + \alpha_1(t)] + A_2(t) \sin[v_{2,2}t + \alpha_2(t)],
\]
\[
x_2(t) = -K^{-1}A_1(t) \sin[v_{1,2}t + \alpha_1(t)]
+ K A_2(t) \sin[v_{2,2}t + \alpha_2(t)],
\]
where \(A_{1,2}(t)\), \(\alpha_{1,2}(t)\) are slowly varying amplitudes and phases, respectively. The variables \(\dot{\alpha}_{1,2}(t)\) are the first-order infinitesimal variables and therefore the terms proportional to the second-order derivatives \(\ddot{A}_{1,2}(t)\) can be omitted upon an insertion of the above relations into equation (2). Following the standard procedures, after straightforward but laborious calculations, we find for the slowly varying amplitudes and the phases
\[
\frac{dA_1}{dt} = -\frac{1}{2} \Gamma_1, \quad \frac{dA_2}{dt} = -\frac{1}{2} \Gamma_2,
\]
\[
\frac{d\alpha_1}{dt} = \delta_1, \quad \frac{d\alpha_2}{dt} = \delta_2,
\]
where
\[
\Gamma_{1,2} = \frac{1}{2} \gamma_1 \left[1 \pm \frac{1}{\sqrt{1 + \sigma^2}}\right] + \gamma_2 \left[1 \mp \frac{1}{\sqrt{1 + \sigma^2}}\right],
\]
\[
\delta_{1,2} = \frac{3}{8} \frac{1}{v_{1,2}} \left[\chi_1 A_1^2 \left(1 \pm \frac{1}{\sqrt{1 + \sigma^2}}\right) + \chi_2 A_2^2 \left(1 \mp \frac{1}{\sqrt{1 + \sigma^2}}\right)\right].
\]
are the relaxation rates and the nonlinear shifts of the mode frequencies. An interesting fact is that the shift of the mode frequencies \(\delta_{1,2}\) depends on the square of the amplitudes \(A_{1,2}^2\) as a consequence of the nonlinearity. For details of derivation of equations (15) and (16) see equations (A.1)–(A.3).

In the case of a weak connectivity (\(\sigma \ll 1\)), from equation (17) we deduce
\[
\Gamma_{1,2} \approx \gamma_{1,2}, \quad \delta_{1,2} = \frac{3}{4} \frac{1}{\tilde{\omega}_{1,2}} \chi_{1,2} A_{1,2}^2,
\]
while in case of strong connectivity (\(\sigma \gg 1\))
\[
\Gamma_1 = \Gamma_2 \approx \frac{1}{2}(\gamma_1 + \gamma_2),
\]
\[
\delta_1 \approx \frac{3}{8} \sqrt{\frac{2}{\omega_1^2 + \omega_1^2}} (\chi_1 A_1^2 + \chi_2 A_2^2),
\]
\[
\delta_2 \approx \frac{3}{8} \sqrt{\frac{2}{\omega_2^2 + \omega_2^2} + 4D} (\chi_1 A_1^2 + \chi_2 A_2^2).
\]
Therefore, when \(\sigma \gg 1\) the modes are damped with the equal rates. However, the nonlinear shifts of the mode frequencies are different (\(\delta_1 > \delta_2\)).

4. Frequency response function of two coupled resonators

The amplitudes of the driven oscillations of the two coupled resonators are presented by expressions (11) and (12). Reexpressed in terms of the connectivity \(\sigma\), we can rewrite the amplitudes as
\[
A_{1,2} = \frac{F_{1,2}(\tilde{\omega}_{1,2}^2 - \Omega^2) + F_{2,1} \omega_2^2 \sigma}{2\Omega^2} \times \left[\frac{\omega_2^2}{\omega_1^2 - \Omega^2} \frac{1}{\sqrt{1 + \sigma^2}} \left(1 \mp \frac{1}{\sqrt{1 + \sigma^2}}\right) - \left(\frac{1}{\omega_1^2 - \Omega^2} - \frac{1}{\omega_2^2 - \Omega^2}\right)\right],
\]
(18)

In the limit of a weak connectivity (\(\sigma \ll 1\)) \(A_{1,2} = 2F_{1,2}\sqrt{\omega_{1,2}^2 - \Omega^2}\) whereas in the limit of a strong connectivity (\(\sigma \gg 1\)), we obtain
\[
A_{1,2} = 2\frac{DF_{2,1}}{\Omega^2} \left(\frac{1}{\omega_2^2 - \Omega^2} + \frac{1}{\omega_1^2 - \Omega^2}\right).
\]
(19)

Nano-mechanical resonators for intermediate values of the connectivity (\(\sigma \approx 2\)) were studied numerically and experimentally in [18]. While the general analytical expression (18) is derived for arbitrary values of the connectivity and therefore in the special case of a moderate connectivity the solution recovers the amplitude–frequency characteristics obtained in [18]. Using transformation (13), one can easily modify (18) in order to add corrections describing the damping and nonlinear terms. However, the expressions obtained in this way are rather involved (see appendix B). That is why here we only present the asymptotic
expressions corresponding to the strong connectivity limit ($\sigma \gg 1$):

$$A_{1,2} \approx \frac{DF_{2,1}}{\Omega^2} \left( \frac{1}{\omega_1 + \delta_1 - \Omega^2 + \Gamma_1^2} \right) + \frac{\delta_1}{\omega_1 + \delta_1 - \Omega^2 + \Gamma_1^2},$$

(20)

where $\delta_{1,2}$ and $\Gamma_{1,2}$ are determined in (18).

As follows from (20), in the case of a strong connectivity, the force $F_2$ acting on the second oscillator ‘drives’ the first one, and vice versa—$F_1$, acting on the first oscillator, ‘drives’ the second one.

The amplitude–frequency characteristic consists of two tilted peaks with different heights, see figure 1. The first peak corresponds to the frequency $\omega_1$, and is definitely more pronounced than the second peak, corresponding to the frequency $\omega_2$ ($\omega_2 \gg \omega_1$). Furthermore, the first peak is more tilted due to the relation $\delta_1 > \delta_2$. The parts of the plot $CD$ and $IH$, corresponding to unstable oscillations of the system, are dotted. During upward/downward frequency sweeps of $\Omega$, one observes hysteretic behaviour around $\Delta \omega_1$ and $\Delta \omega_2$ along the loops $BCED$ and $G1KH$, respectively [21, 24]. In [18] similar hysteretic loops in amplitude–frequency characteristics of coupled nonlinear oscillators were obtained numerically and were confirmed experimentally for intermediate values of the connectivity $\sigma \approx 2$. In the unstable region, the system is extremely sensitive to the perturbations. This fact can be used for the switching of the oscillation amplitude. After reaching the point $C$, the amplitude of the oscillation decreases sharply to the value $E$. Therefore, a simple and efficient switching protocol can be realized by tuning of the external field frequency only.

Pairwise occurrence of the resonance peaks is connected to the fact that we consider a broad area of modulation of the driving frequency. When the frequency interval for the driving frequency $\Omega$ (see figure 1) is larger than the frequency distance between resonances peaks $\Delta \Omega > |\nu_1 - \nu_2|$ it includes concurrently both resonance peaks on both mode frequencies $\nu_1, \nu_2$. On the narrow frequency interval, $\Delta \Omega < |\nu_1 - \nu_2|$ is not possible to observe concurrently both resonance peaks [18].

We would also like to point out that the domain of the amplitude–frequency characteristics that should be utilized for switching belongs to the unstable area (see the frequency intervals $B–C$ and $G–I$ in figure 1). Therefore, the system can jump to the lower state before reaching the summit of the unstable domain (point $C$). If this occurs the jump of the oscillation amplitude is smaller making the experimental observation of the two different transport regimes difficult. To circumvent this problem the frequency of the driving field should be changed adiabatically. Following our approach here, we seek the criteria of adiabaticity that may be useful for different realizations of the system. A similar problem arises for example when studying nonlinear resonant transport in cold atoms [25]. The method of the slow varying amplitudes implies that the amplitude change rate should be slower than the mode frequencies $\nu_1, \nu_2$. Therefore, the rate of the oscillation amplitude change caused by tuning the frequency of the driving field is limited by the following condition:

$$\frac{dA_{1,2}(\Omega(t))}{dt} \leq \frac{dA_{1,2}(\Omega(t))}{d\Omega(t)} \frac{d\Omega(t)}{dt} < v_{1,2} A_{1,2}(\Omega(t)).$$

The adiabaticity condition can then be simplified taking into account the explicit expressions for the amplitudes given in equation (20). It is not difficult to show that in the vicinity of unstable areas

$$\frac{dA_{1,2}(\Omega(t))}{d\Omega(t)} \approx \frac{2A_{1,2}(\Omega(t))}{\Omega(t)}.$$

Thus, for the adiabaticity criteria, we finally obtain the following estimation:

$$\frac{d\Omega(t)}{dt} < \min(v_{1,2}) \frac{\Omega(t)}{2}.$$

5. Nonlinear shift of the mode frequencies and the influence of the relaxation terms: the resonant case

Let us suppose that the harmonic force $F_1 \cos \Omega t$ is tuned in resonance with one of the modes and that $F_2 = 0$. For this problem, we derive equations for the slowly varying amplitudes in a more straightforward way. Taking into consideration the resonance condition and the transformation (5), we can write down the solution of the equation set (2) in the following form:

$$x_1(t) = A_1(t) \sin \nu_1 t + A_2(t) \cos \nu_2 t + B(t) \sin (\nu_2 t + \psi(t)),$$

$$x_2(t) = -K^{-1} (A_1(t) \sin \nu_1 t + A_2(t) \cos \nu_2 t + B(t) \sin (\nu_2 t + \psi(t))).$$

After application of the standard method outlined in the last section, for equations of slowly varying amplitudes and phases

$$\begin{align*}
A_1(t) &= \frac{DF_{1,2}}{\Omega^2} \left( \frac{1}{\omega_1 + \delta_1 - \Omega^2 + \Gamma_1^2} \right) + \frac{\delta_1}{\omega_1 + \delta_1 - \Omega^2 + \Gamma_1^2}, \\
A_2(t) &= \frac{DF_{1,2}}{\Omega^2} \left( \frac{1}{\omega_2 + \delta_2 - \Omega^2 + \Gamma_2^2} \right) + \frac{\delta_2}{\omega_2 + \delta_2 - \Omega^2 + \Gamma_2^2}, \\
B(t) &= \frac{DF_{1,2}}{\Omega^2} \left( \frac{\nu_1 \nu_2}{\omega_1 - \omega_2} \right) \left( \frac{1}{\omega_1 + \delta_1 - \Omega^2 + \Gamma_1^2} \right) + \frac{\nu_1 \nu_2}{\omega_2 + \delta_2 - \Omega^2 + \Gamma_2^2}.
\end{align*}$$
we derived the system of equations. However, expressions in the general case are very large and therefore an explicit form of these equations is given in equations (C.1)–(C.4). In the limit of a strong connectivity ($\sigma \gg 1$), the expressions simplify to

$$\frac{dA_1}{dr} = \frac{F}{\Omega_1} - \gamma A_1 - \frac{3\chi}{\Omega_1} (A_1^2 + A_2^2 + 2B^2) A_2,$$

$$\frac{dA_2}{dr} = -\gamma A_2 + -\frac{3\chi}{\Omega_1} (A_1^2 + A_2^2 + 2B^2) A_1,$$  \hspace{1cm} (25)

As is evident, a resonant external force $F_1 \cos \Omega t$ ($\Omega = \nu_1$) for $\gamma = \chi = 0$ leads to the simplest form of instability (the secular instability), namely to the linear growth of the oscillation amplitude $A_1 = (F/4\Omega \gamma) t$.

We would like to point out that in the first three equations (the variables $A_1$, $A_2$ and $B$, the right-hand side of the set of equations (25) does not depend on the fourth variable $\psi$. Therefore, the set of equations (25) can be solved self-consistently for the first three variables.

In order to find the stationary values of the slowly varying amplitudes and in order to examine the stability of these values, we utilize the following transformation:

$$A_1 = \rho \cos \theta, \hspace{1cm} A_2 = -\rho \sin \theta.$$ \hspace{1cm} (26)

In the more convenient polar coordinates $\rho$ and $\theta$, we obtain

$$\frac{d\rho}{dr} = -\gamma \rho + \frac{F}{4\Omega} \cos \theta,$$

$$\frac{d\theta}{dr} = -\omega_{NL} - \frac{F}{4\Omega} \rho,$$ \hspace{1cm} (27)

where $\omega_{NL} = \frac{3\chi}{\Omega_1} (\rho^2 + 2B)$. By setting the rhs of equations (27) equal to zero, we obtain equations for the stationary values of amplitudes:

$$B_0 = 0, \hspace{1cm} s \cos \theta_0 = \rho_0, \hspace{1cm} s \sin \theta_0 = -r \rho_0^3,$$ \hspace{1cm} (28)

where $s = \frac{F}{4\Omega} \gamma, \hspace{1cm} r = \frac{3\chi}{2}\Omega_1$. To determine $\rho_0$, we eliminate the variable $\theta_0$ from the set of equations (27) and obtain a cubic equation with respect to $x = \rho_0^2$:

$$x^3 + \frac{x}{r^2} - \frac{s^2}{r^2} = 0.$$ \hspace{1cm} (29)

Equation (29) is a reduced cubic equation. The number of real roots of this equation depends on the sign of the discriminant:

$$D = \frac{1}{3\sqrt{2}} \left( \frac{s^2}{\sqrt{2}r^2} \right)^3 > 0,$$ \hspace{1cm} (30)

which is positive in our case. That is why equation (29) has a real root. Real roots of equation (29) can be identified easily with the help of the well-known Cardano formula [26]. However, as it will become evident below, they are not necessary for a further specification of the expressions for the stationary points needed for the study of the stability conditions.

To address the question concerning the stability of the stationary points more precisely, we linearize the set of equations (25) in the vicinity of the stationary points $A_1^{(0)} = \rho_0 \cos \theta_0, \hspace{1cm} A_2^{(0)} = \rho_0 \sin \theta_0$ and $B_0 = 0$, and obtain

$$\delta \dot{A}_1 = -\gamma (1 + 2r A_2^{(0)} A_2^{(0)}) \delta A_1 - \gamma r (A_2^{(0)} + 3A_1^{(0)}) \delta A_2,$$

$$\delta \dot{A}_2 = \gamma r (A_2^{(0)} + 3 A_1^{(0)}) \delta A_1 - \gamma (1 - 2r A_1^{(0)} A_2^{(0)}) \delta A_2.$$ \hspace{1cm} (31)

Alternatively, by taking into consideration the transformation (26),

$$\delta \dot{A}_1 = R_{11} \delta A_1 + R_{12} \delta A_2,$$

$$\delta \dot{A}_2 = R_{21} \delta A_1 + R_{22} \delta A_2,$$ \hspace{1cm} (32)

where

$$R = \left( \begin{array}{cc}
-\gamma (1 + 2r A_2^{(0)} \sin 2\theta_0) & -\gamma r A_2^{(0)} (3 - \cos 2\theta_0) \\
\gamma r A_2^{(0)} (3 + \cos 2\theta_0) & -\gamma (1 - 2r A_1^{(0)} \sin 2\theta_0) \end{array} \right).$$ \hspace{1cm} (33)

As discussed in [22], the type of the stability is determined by three characteristics of the matrix ($R$):

$$T = R_{11} + R_{22}, \hspace{1cm} d = R_{11} R_{22} - R_{12} R_{21}, \hspace{1cm} T^2 - 4d.$$ \hspace{1cm} (34)

With the help of the matrix ($R$), it is easy to check that in our case the characteristics of the matrix ($R$) are

$$T = -2\gamma < 0, \hspace{1cm} d = \gamma^2 (1 + 8r^2 A_0^2) > 0,$$

$$T^2 - 4d = 32\gamma^2 r^2 A_0^2 > 0$$ \hspace{1cm} (35)

and point towards the condition of a stable focus. Note that conditions (35) do not depend on the field parameter $s = \frac{F}{4\Omega} \gamma$, and therefore hold for arbitrary values of the amplitudes ($A_1, A_2$).

Thus, in the stationary resonance regime, when the frequency of the external driving field is in the resonance with one of the modes of the two strongly coupled resonators, the stationary points are characterized by a stable focus. Therefore, we can argue that the dissipation leads to a stabilization of the secular instability regime.

6. Energy redistribution between resonators

In this section, we will address the problem of the energy redistribution between the resonators ($A_1^2/A_2^2$), which are pumped via the external fields. Let us suppose that the harmonic force acts only on the second resonator $F_2 = F, F_1 = 0$. Then, with the help of expression (12) for the ratio of the oscillation amplitudes we obtain the following relation:

$$\left| \frac{A_1}{A_2} \right| = \frac{D}{\omega_1 - \Omega^2}.$$ \hspace{1cm} (36)

We recall that expressions (12) for the amplitudes $A_{1,2}$ with respect to the mode frequencies $\nu_{1,2}$ have the same resonances embedded in the denominators. They naturally compensate each other and therefore do not appear in the ratio $A_1/A_2$. Nevertheless, as we see from equation (36), another resonance $\omega_1 \approx \Omega$ appears in the denominator of the expression $A_1/A_2$.

At first we neglect the influence of the damping and the nonlinearity terms, assuming that the frequency of the harmonic force is tuned with one of the partial frequencies of
the resonators. Then, for the case when $F_2 \equiv F$, $\Omega \simeq \omega_2$ and $F_1 = 0$, we obtain
\[ \left| \frac{A_1}{A_2} \right| = \frac{D}{\omega_1^2 - \omega_2^2} \approx \frac{\sigma}{2}. \tag{37} \]
Hence, the relation between the amplitudes $A_1$ and $A_2$ is linear. From the second resonator a fraction $\frac{\sigma^2}{2}$ of the energy is transferred to the first one.

The damping and the nonlinear corrections can be considered again with the help of the substitution
\[ \frac{1}{\omega_1^2 - \omega_2^2} \rightarrow \frac{1}{2\omega_1 \sqrt{(\beta A_1^2 - \Delta)^2 + y^2}}, \quad \beta = \frac{3}{4} \chi. \tag{38} \]

From now on, we assume that $\gamma_1 \approx \gamma_2 \equiv \gamma$, $\chi_1 = \chi_2 = \chi$ and $\Delta = \omega_2 - \omega_1 > 0$.

The resonant denominator is an important feature of the expression (38). When $A_1$ changes, the resonance condition holds in the expression (38). Performing the substitution (38) in the expression (36) and raising to the square, we obtain
\[ \frac{x}{y} = \frac{f}{(\beta x - \Delta)^2 + y^2}, \tag{39} \]
where $x = A_1^2$, $y = A_2^2$, $f = \frac{\sigma^2}{2\gamma}$. So, instead of studying the dependence $y = Y(x)$, for convenience one can convert the problem to one studying the following implicit function:
\[ F(x, y) = x[(\beta x - \Delta)^2 + y^2] - f y = 0. \tag{40} \]
By setting the derivative $dy/dx$ equal to zero, one obtains an equation for the extrema of the function $y = Y(x)$:
\[ \frac{dy}{dx} = \frac{dF}{dx} - \frac{dF}{dy} \frac{dy}{dx} = f^{-1}(3\beta^2 x^2 - 4\beta x \Delta + \Delta^2 + y^2) = 0. \tag{41} \]

It follows then that the points of extremum
\[ x_{1,2} = \frac{2 \Delta}{3 \beta} \left( 1 \pm \sqrt{1 - \frac{3 \Delta^2 + y^2}{4 \Delta^2}} \right). \tag{42} \]
For simplicity, we consider the limiting case $\gamma \ll \Delta$. In this case, we get two real roots from equation (42):
\[ x_1 = \frac{\Delta}{\beta}, \quad x_2 = \frac{\Delta}{3 \beta}. \tag{43} \]

It is easy to determine the signs of the second derivatives:
\[ \frac{d^2 y}{dx^2} \bigg|_{x=x_1} = \frac{2\beta \Delta}{f} > 0, \quad \frac{d^2 y}{dx^2} \bigg|_{x=x_2} = -\frac{2\beta \Delta}{f} < 0. \tag{44} \]

Therefore, the function $y = Y(x)$ has a maximum at the point $x = x_2$, and a minimum at $x = x_1$. The curve $y = Y(x)$ is characterized by two asymptotes as well. The first one, for small values of $x$ and $y$, is a linear function $y = (\Delta^2/f)x = (4/\sigma^2)x$. The second one, for large values of $x$ and $y$, is a cubic function $y = (x^3/\sigma^2)$.

Using the results obtained in this section, one can plot the curve of the energy redistribution between the resonators, see figure 2.

The anharmonicity of resonators’ oscillations can significantly change the energy redistribution between the resonators. It turns out that the energy pumped into the second resonator via the external energy source $F_2 = F$ is transformed into the first resonator ($F_1 = 0$) in a different way depending on the oscillation amplitude. For small amplitudes of the normal modes, the energy transfer between the resonators is linear, $A_1^2 = \frac{\sigma^2}{2} A_2^2$ and the transfer rate is defined by the values of the connectivity $\sigma$. With increasing oscillation amplitude the linear law is changed and turns nonlinear $A_1^2 = (\frac{1}{4} \times \frac{\sigma^2}{2} A_2^2)^{1/3}$. Therefore, we can conclude that the anharmonicity of the oscillations degrades the energy transfer rate.

7. Application to mass measurement sensors and the nonlinear shift of the mode frequencies

Nanomechanical resonators can be used as apprehensible sensors in many applications. For a review see [27]. A decisive advantage of the nanomechanical resonators are their resonance frequencies $\omega \approx 1$ GHz and quality factors $Q \approx 10^{3}$–$10^{5}$, which are significantly higher than those of electrical resonant circuits. That is the reason why nanomechanical resonators are sensitive transducers for the detection of molecular systems, in particular for biological molecules [20]. Resonant mass sensor devices operate by measuring the frequency shift which is proportional to the mass of the molecules of the material under investigation [28]. Details of the measurement protocol can be found in [29]. Here we briefly refer to the main facts. Assuming that the added mass $\delta M$ is smaller than the effective resonator mass $M$, one can write a linearized expression $\delta M \approx \frac{\delta M}{M} \delta \omega$. The minimal measurable frequency shift $\delta \omega$ naturally defines the sensitivity of the sensor. Due to thermal fluctuations $\delta \omega > 0$. 

![Figure 2. Energy redistribution curve between the coupled resonators and its asymptotics, plotted using equation (40) for the following values of parameter $\beta = 0.51 \times 10^{10} \text{ Hz/m}$, $\Delta = 0.17 \times 10^{9} \text{ Hz}$, $\gamma = 2.0 \times 10^{10} \text{ Hz}$, $f = 1.64 \times 10^{10} \text{ Hz}^2$.](Image 329x543 to 550x765)
For the single, simple damped harmonic oscillator system, the minimal measurable frequency shift reads [29]

\[
\delta \omega \approx \left[ \frac{k_B T \omega \Delta f}{M \omega^2 A^2} \right]^{1/2},
\]

Here, \( \Delta f \) is the measurement bandwidth, \( M \) is the resonator mass, \( \omega \) is the frequency of the oscillation and \( T \) is the temperature. As follows from the analysis of the preceding sections, at low temperatures the nonlinear effects (that were not considered in [29]) produce a frequency shift larger than the minimal measurable frequency shift associated with the thermal effects (see equation (17)) \( \delta_{1,2} > \delta \omega \). We propose to use the system of coupled nonlinear oscillators to act as an amplifier for the frequency shifts. We are convinced that in this way far better mass measurements are possible for experiments described in [29].

8. Conclusion

We have developed a general analytical treatment of a system of two coupled driven nonlinear nanomechanical resonators, which is valid for an arbitrary coupling strength (connectivity) between them. We derive general analytical expressions for the amplitude–frequency characteristics of the system with a special emphasis on the energy redistribution and the energy transport between the resonators. The obtained results are valid for arbitrary values of the connectivity. In the limit of a weak coupling, one recovers the previously obtained results [18]. In particular, we have shown that the amplitude–frequency characteristic consists of two tilted peaks, the frequency separation between which is equal to twice the value of the resonators coupling constant 2D. If the frequency of the external force \( \Omega \) is swept, the oscillation amplitude shows two hysteresis loops in the vicinity of the mode frequencies. These hysteresis loops contain unstable areas, in which a slight change of the driving frequency is accompanied by an instantaneous and a significant change of the oscillation amplitude. This is an interesting phenomenon, since it can be utilized to switch easily between the energy transport regimes of the resonators. We found that for small oscillation amplitudes the energy transfer between the resonators follows a linear law \( A_1^2 = \frac{2}{7} A_2^2 \) and the transfer rate is entirely defined by the values of the connectivity \( \sigma \). With increasing the oscillation amplitude the energy transfer law turns nonlinear \( A_1^2 = \left( \frac{2}{7} \times \frac{D^2}{2} A_2^2 \right)^{1/2} \) and therefore the transport rate becomes slower. Switching off the energy transfer rate by tuning of the driving field frequency is a simple protocol from an experimental point of view and therefore we expect it to be easily observable.

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Appendix A. Details from section 3

The derivations of equations (15) and (16) are

\[
\frac{dA_1}{dt} = \frac{1}{4\nu_1} \frac{\sigma}{\sqrt{1+\sigma^2}} (-K^{-1}P_1 + Q_1),
\]

\[
\frac{dA_2}{dt} = -\frac{1}{4\nu_2} \frac{\sigma}{\sqrt{1+\sigma^2}} (KP_2 + Q_2),
\]

where

\[
P_1 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} M \cos \xi d\eta d\zeta,
\]

\[
Q_1 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} N \cos \xi d\eta d\zeta,
\]

\[
P_2 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} M \cos \eta d\xi d\zeta,
\]

\[
Q_2 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} N \cos \eta d\xi d\zeta,
\]

\[
P_3 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} M \sin \xi d\eta d\zeta,
\]

\[
Q_3 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} N \sin \xi d\eta d\zeta,
\]

\[
P_4 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} M \sin \eta d\xi d\zeta,
\]

\[
Q_4 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} N \sin \eta d\xi d\zeta,
\]

for \( M \) and \( N \) are determined by equation (3). After inserting \( x_1(t) \) and \( x_2(t) \) (14), from (A.2) we obtain

\[
M(\xi, \eta, \zeta) = -\gamma_1(A_1 + A_2) \cos \xi + A_2 \sin \eta + A_1 \cos \zeta \]

\[
N(\xi, \eta, \zeta) = -\gamma_2(-A_1 K^{-1} \cos \xi + A_2 K^2 \sin \eta - A_2 \Omega \sin \zeta) \]

\[
\xi = v_1 t + \alpha_1; \eta = v_2 t + \alpha_2; \zeta = \Omega t.
\]

Inserting (A.3) into (A.2), after simple integration, from (A.1) one obtains equations (15) and (16).

Appendix B. Details from section 4

By taking into account damping effects, the amplitudes of the forced oscillation of the nonlinear resonators for an arbitrary value of the connectivity \( \sigma \) are

\[
A_{1,2} = \frac{F_{1,2}(\delta^2 - \Omega^2) + F_{2,1} \omega^2 \sigma}{4\Omega^2}
\]

\[
\left[ \frac{\alpha^2}{\omega^2} - \frac{1}{\sqrt{1+\sigma^2}} \frac{1}{\nu_1 \sqrt{(v_1 + \delta_1 - \Omega)^2 + \Gamma_1}} - \frac{1}{\nu_2 \sqrt{(v_2 + \delta_2 - \Omega)^2 + \Gamma_2}} \right]
\]
\[
-M(r) = \frac{1}{v_1 \sqrt{(v_1 + \delta_1 - \Omega)^2 + \Gamma_1}} + \frac{1}{v_2 \sqrt{(v_2 + \delta_2 - \Omega)^2 + \Gamma_2}}.
\]

(B.1)

Appendix C. Details from section 5

The derivations of equation (25) are as follows:

\[
\begin{align*}
\frac{dA_1}{dr} &= \frac{1}{4\nu_1} \frac{\sigma^2}{\sqrt{1 + \sigma^2}} (K^{-1} P_1^{(r)} + Q_1^{(r)}), \\
\frac{dA_2}{dr} &= \frac{1}{4\nu_1} \frac{1}{\sqrt{1 + \sigma^2}} (-K^{-1} P_2^{(r)} + Q_2^{(r)}), \\
\frac{dB}{dr} &= \frac{1}{4\nu_2} \frac{1}{\sqrt{1 + \sigma^2}} (-K^{-1} P_3^{(r)} + Q_3^{(r)}), \\
\frac{d\psi}{B \, dr} &= \frac{1}{4v_2} \frac{1}{\sqrt{1 + \sigma^2}} (K P_4^{(r)} + Q_4^{(r)}),
\end{align*}
\]

where

\[
\begin{align*}
P_1^{(r)} &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} M \cos \eta \, d\eta \, d\eta, \\
Q_1^{(r)} &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} N \cos \eta \, d\eta \, d\eta, \\
P_2^{(r)} &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} M \sin \eta \, d\eta \, d\eta, \\
Q_2^{(r)} &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} N \sin \eta \, d\eta \, d\eta, \\
P_3^{(r)} &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} M \cos \xi \, d\xi \, d\eta, \\
Q_3^{(r)} &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} N \cos \xi \, d\xi \, d\eta, \\
P_4^{(r)} &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} M \sin \xi \, d\xi \, d\eta, \\
Q_4^{(r)} &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} N \sin \xi \, d\xi \, d\eta, \\
\xi &= v_2 t + \psi, \quad \eta = \Omega t,
\end{align*}
\]

\[
M^{(r)} = M^{(r)}(\chi_1, \dot{\chi}_1, t) = -\gamma [\nu_1 (A_1 \cos \eta - \sin \eta) + B \cos \xi] \\
- \chi_1 (A_1 \sin \eta + A_2 \cos \eta + B \cos \xi) F + F \cos \Omega t,
\]

\[
N^{(r)} = N^{(r)}(\chi_2, \dot{\chi}_2, t) = -\gamma [\nu_2 (A_1 \sin \eta - A_2 \cos \eta) + K B \sin \xi].
\]

Upon insertion equation (C3) into (C2) and after an integration, one gets equations for the slowly varying amplitudes $A_1, A_2, B$ and $\psi$. The explicit form of the set of equations is as follows:

\[
\frac{dA_1}{dr} = \frac{1}{4\Omega} \left[ \frac{\sigma^2}{\sqrt{1 + \sigma^2}} \right] F_1 - \frac{1}{2} \frac{1}{\sqrt{1 + \sigma^2}} \times \left[ \gamma_1 (1 + \sqrt{1 + \sigma^2}) A_1 + \gamma_2 (-1 + \sqrt{1 + \sigma^2}) A_1 \\
- \frac{1}{4\Omega} \frac{\sigma^2}{\sqrt{1 + \sigma^2}} \left[ \frac{3}{4} \chi_1 (1 + \sqrt{1 + \sigma^2}) (A_1^2 + A_2^2 + 2B^2) A_2 \\
+ \frac{3}{4} \chi_2 (-1 + \sqrt{1 + \sigma^2}) \right] \right].
\]

\[
\frac{dA_2}{dr} = -\frac{1}{2} \frac{1}{\sqrt{1 + \sigma^2}} \left[ \gamma_1 (1 + \sqrt{1 + \sigma^2}) A_2 \\
+ \frac{1}{4\Omega} \frac{\sigma^2}{\sqrt{1 + \sigma^2}} \left[ \frac{3}{4} \chi_1 (1 + \sqrt{1 + \sigma^2}) (A_1^2 + A_2^2 + 2B^2) A_1 \\
+ \frac{3}{4} \chi_2 (-1 + \sqrt{1 + \sigma^2}) \right] \right].
\]

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