Towards Quantum Superpositions of a Mirror: an Exact Open Systems Analysis –
Calculational Details

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We give details of calculations analyzing the proposed mirror superposition experiment of Marshall, Simon, Penrose, and Bouwmeester within different stochastic models for state vector collapse. We give two methods for exactly calculating the fringe visibility in these models, one proceeding directly from the equation of motion for the expectation of the density matrix, and the other proceeding from solving a linear stochastic unravelling of this equation. We also give details of the calculation that identifies the stochasticity parameter implied by the small displacement Taylor expansion of the CSL model density matrix equation. The implications of the two results are briefly discussed. Two pedagogical appendices review mathematical apparatus needed for the calculations.

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I. INTRODUCTION

There is currently much interest in experiments to create quantum superposition states involving large numbers of particles, with the ultimate aim of testing whether quantum superpositions of macroscopic systems can be observed. Recently, Marshall, Simon, Penrose, and Bouwmeester [1], motivated by suggestions of Penrose [2], have proposed a novel interferometric experiment in which a single photon interacts with a miniature mirror mounted on a cantilever in one arm of the interferometer, thus setting up a superposition of states containing of order $10^{14}$ atoms. Since the two superposed states in this experiment have a relative center of mass displacement of order the width of the mirror center of mass wave packet $\sim \sigma \sim 10^{-11}$ cm, the experiment will place new constraints on proposals for modifications to quantum mechanics in which center of mass displacement is the key parameter.

Among the different proposals, collapse models [3, 4, 5, 6] have been extensively studied. The basic idea is to combine the standard Schrödinger evolution and the postulate of wavepacket reduction into one universal dynamical equation, which is assumed to govern all physical processes. Such a dynamics accounts both for the quantum properties of microscopic systems and for the classical properties of macroscopic ones; in particular, it guarantees that measurements made on microscopic systems always have definite outcomes, and with the correct quantum probabilities (Born probability rule).

In a recent Letter [7], we have analyzed the Marshall et al experiment within the framework of the GRW [3], CSL [4] and QMUPL [5] models, and have shown that — within the CLS model, which predicts the largest deviation from standard quantum predictions — one expects the maintenance of coherence to better than 1 part in $10^8$. Our aim in this paper is to give the derivations of formulas presented, without derivation, in our Letter. In Section 2 we give the basic equations for the Marshall et al experiment, first as formulated in their paper, and then as formulated within the collapse models. In Section 3 we solve for the visibility (the physical quantity measured in the experiment) by direct calculation from the density matrix evolution equation, making use of the interaction picture, the Baker-Hausdorff formula, and cyclic permutation under a trace. In Section 4 we give an alternative derivation of the visibility, obtained by solving a linear stochastic unravelling of the density matrix equation, using the Itô stochastic calculus. In Section 5 we compute the stochasticity parameter entering into the visibility formula in terms of the parameters of the CSL model. We briefly summarize our results and their application to the Marshall et al experiment in Section 6. In Appendix A we derive the Baker–Hausdorff formula used in the text, and in Appendix B we review the Itô calculus formulas used in the calculation of Section 4.
II. BASIC FORMALISM

The Hamiltonian for the Marshall et al experiment, with the moving mirror in a cavity in interferometer arm A, is

\[ H = \hbar \omega_c (a_A^\dagger a_A + a_B^\dagger a_B) + \hbar \omega_m b^\dagger b - \hbar G a_A^\dagger a_A (b + b^\dagger). \] (1)

Here \( \omega_c \) is the frequency of the photon, \( a_A^\dagger \) and \( a_B^\dagger \) are the creation operators for the photon in the interferometer arms A and B, respectively, while \( \omega_m \) and \( b^\dagger \) are the frequency and the phonon creation operator associated with the cavity, with \( \sigma = (\hbar/2M\omega_m)^{1/2} \) the width of the mirror wave packet and \( M \) the mass of the mirror.

The semi–silvered beam splitter of the interferometer places the photon in an initial state that is an equal superposition of being in arm A or B,

\[ |\psi_0\rangle = \frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B) |0\rangle_m, \] (2)

and standard quantum mechanics predicts that at time \( t \) the state vector will be

\[ |\psi_t\rangle = e^{-iHt} |\psi_0\rangle = \frac{1}{\sqrt{2}} e^{-i\omega_c t} \left[ |0\rangle_A |1\rangle_B |0\rangle_m \right. \]
\[ + e^{i\kappa^2(\omega_m t - \sin \omega_m t)} |1\rangle_A |0\rangle_B |\alpha_t\rangle_m \right]. \] (3)

Here we have written \( \kappa = G/\omega_m \) and \( |\alpha_t\rangle_m \) denotes a unit normalized mirror coherent state with complex amplitude \( \alpha_t = \kappa(1 - e^{-i\omega_m t}) \). While in state \( |0\rangle_m \) the mirror is fixed in its equilibrium position (the origin of the reference frame), in state \( |\alpha_t\rangle_m \), the mirror oscillates between 0 and \( \ell = 4\kappa \sigma \); in both cases, the shape of the wavefunction (in position) is a Gaussian of width \( \sigma \).

The physically measurable quantity considered by Marshall et al is the maximum interference visibility for the photon \( \nu(t) \), defined as twice the modulus of the off–diagonal element of the reduced density matrix of the photon. The full density matrix for the system is

\[ \rho = \langle \psi_t | \psi_t \rangle \]
\[ = \frac{1}{2} \left[ |0\rangle_A \langle 0| |1\rangle_B \langle 1|0\rangle_m m \langle 0| \right. \]
\[ + |1\rangle_A \langle 1|0\rangle_B \langle 0| |\alpha_t\rangle_m m \langle \alpha_t| \right. \]
\[ + e^{i\kappa^2(\omega_m t - \sin \omega_m t)} |1\rangle_A \langle 0|0\rangle_B \langle 1| |\alpha_t\rangle_m m \langle 0| \]
\[ + e^{-i\kappa^2(\omega_m t - \sin \omega_m t)} |0\rangle_A \langle 1| |1\rangle_B \langle 0| |\alpha_t\rangle_m m \langle \alpha_t| \right]. \] (4)

Thus after tracing over the mirror states, the reduced density matrix has as the coefficient of the off–diagonal term \( |1\rangle_A \langle 0|0\rangle_B \langle 1| \) the factor \( \frac{1}{2} f \), with

\[ f = e^{i\kappa^2(\omega_m t - \sin \omega_m t)} e^{-\kappa^2(1 - \cos \omega_m t)}. \] (5)

which using \( m |0\rangle m \langle \alpha_t| = e^{-\kappa^2 |\alpha_t|^2} \) gives

\[ f = e^{i\kappa^2(\omega_m t - \sin \omega_m t)} e^{-\kappa^2(1 - \cos \omega_m t)}. \] (6)

Thus, under standard quantum mechanical evolution of the state, one has for the time dependence of the visibility

\[ \nu(t) = e^{-\kappa^2(1 - \cos \omega_m t)}. \] (7)

According to the above formula, the visibility starts from its maximal value 1; it then decreases, but after half a period of the mirror’s motion it increases again, reaching the maximal value after one period \( T = 2\pi/\omega_m \). The strategy to test the macroscopic superposition of the mirror then goes as follows. One measures the photon’s visibility after one period \( T \): if it is close to 1, then no collapse of the mirror’s wavefunction has occurred; if on the contrary it is smaller than 1, a spontaneous collapse process is present which reduces the superposition to one of its two terms. Of course, one must keep control of all sources of decoherence, which tend to lower the observed visibility.

We proceed now to reanalyze the experiment using the modified Schrödinger evolution of the QMUPL model of wavefunction collapse [3], this model is particularly useful since, as we shall prove, it allows one to get an exact formula for the visibility when a spontaneous collapse mechanism is present. Moreover, this model corresponds to the leading term in the small–displacement Taylor expansion of both the GRW and the CSL models; such an expansion is particularly suitable to the present case since, according to the parameters of the experiment, the maximum displacement between the two superposed states of the mirror is of order \( 10^{-11} \) cm, which is much smaller than the typical distance of \( 10^{-5} \) cm required for quantum superpositions to be destroyed, in the GRW and CSL models. Under the QMUPL model, the state vector evolves as

\[ d|\psi_t\rangle = \left[ \frac{-i}{\hbar} H dt + \sqrt{\tau} (q - \langle q \rangle t) dW_t \right. \]
\[ \left. - \frac{\eta}{2} (q - \langle q \rangle t)^2 dt \right] |\psi_t\rangle, \] (8)

where \( H \) is given by Eq. (1), and \( \langle q \rangle \equiv \langle \psi_t | q | \psi_t \rangle \) is the quantum mechanical expectation of the position operator \( q = \sigma (b + b^\dagger) \) associated with the center of mass of the mirror. The stochastic dynamics is governed by a standard Wiener processes \( W_t \), defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Using the rules of the Itô calculus (see Appendix B), the density matrix evolution corresponding to Eq. (5) is

\[ d|\rho\rangle = -\frac{i}{\hbar} [H, \rho] dt - \frac{1}{2} \eta [q, [q, \rho]] dt + \sqrt{\eta} [\rho, [\rho, q]] dW_t. \] (9)

Since to observe interference fringes experimentally requires passing to an ensemble of identically prepared photons through the apparatus, the relevant density matrix in the stochastic case is the ensemble expectation \( \rho = E[\hat{\rho}] \), which obeys the ordinary differential equation

\[ \frac{d\rho}{dt} = -\frac{i}{\hbar} [H, \rho] - \frac{1}{2} \eta [q, [q, \rho]] \]
\[ = -\frac{i}{\hbar} [H, \rho] - \frac{1}{2} \eta \sigma^2 [b + b^\dagger, [b + b^\dagger, \rho]]. \] (10)
Defining an off-diagonal density matrix \( \rho_{OD} \) acting in the mirror Hilbert subspace by 
\[ \langle 1 | \rho (0) | 0 \rangle a \left\langle 1 \right| = \frac{1}{2} \rho_{OD} , \]
so that the factor \( f \) introduced above is \( \text{Tr}_m \rho_{OD} \), we can project out from Eq. (10) the evolution equation for \( \rho_{OD} \),
\[ \frac{d \rho_{OD} (t)}{dt} = -i H^A \rho_{OD} (t) + i \rho_{OD} (t) H^B \]
\[ - \frac{1}{2} \eta \sigma^2 [b^\dagger b^\dagger, [b^\dagger, \rho_{OD} (t)]] , \quad \text{Eq. (11)} \]
with \( \hbar H^A \) the effective mirror Hamiltonian acting when the photon passes through interferometer arm A, and with \( \hbar H^B \) the corresponding effective mirror Hamiltonian acting when the photon passes through arm B,
\[ H^A = \omega_m b^\dagger b - G (b + b^\dagger) \quad H^B = \omega_m b^\dagger b. \quad \text{Eq. (12)} \]
We must now solve the dynamics represented by Eqs. (11) and (12), or equivalently by Eq. (10), so as to calculate \( \text{Tr}_m \rho_{OD} \) and obtain the visibility. Additionally, we must calculate the stochastic parameter \( \eta \) entering into Eqs. (8)–(11) in terms of the parameters of the CSL model. These are the issues addressed in the following three sections.

III. DIRECT SOLUTION FOR THE VISIBILITY FROM THE DENSITY MATRIX EVOLUTION EQUATION

In this section we give a calculation of the mean visibility directly from the density matrix equation of motion. An essential identity in everything that follows is the Baker-Hausdorff identity, derived in Appendix A,
\[ e^{-i H^A t} e^{i H^B t} = N_t e^{\eta \sigma b^\dagger b}, \quad \text{Eq. (13)} \]
with \( H^A \) and \( H^B \) as given in Eq. (12), and with
\[ N_t = e^{-\kappa^2 (1 - i \omega_m t - e^{-i \omega_m t})} \]
\[ = e^{-\kappa^2 (1 - i \omega_m t) + i \kappa^2 (\omega_m t - \sin \omega_m t)} \]
\[ \alpha_t = \kappa (1 - e^{-i \omega_m t}) \quad \beta_t = -\kappa (1 - e^{-i \omega_m t}) \quad \text{Eq. (14)} \]
Defining the photon off-diagonal part of the density matrix \( \rho_{OD} \) as in Section 2, which obeys the evolution equation of Eq. (11), the visibility is \( \nu = | \text{Tr}_m \rho_{OD} | \); thus what is needed is to calculate \( \text{Tr}_m \rho_{OD} \).
Let us now go to the interaction picture by defining
\[ \rho_{OD}^I (t) = e^{i H^A t} \rho_{OD} (t) e^{-i H^A t} \quad \text{Eq. (15)} \]
so that \( \rho_{OD}^I (0) = \rho_{OD} (0) = | 0 \rangle_m \langle 0 | \). The corresponding differential equation obeyed by \( \rho_{OD}^I \) is
\[ \frac{d \rho_{OD}^I (t)}{dt} = -i H^A \rho_{OD}^I (t) + i \rho_{OD}^I (t) H^B \]
\[ - \frac{1}{2} \eta \sigma^2 [b^\dagger b^\dagger, [b^\dagger, \rho_{OD}^I (t)]] e^{-i H^B t} , \quad \text{Eq. (16)} \]
Multiplying from the left by \( e^{-i H^A t} \) and from the right by \( e^{i H^B t} \), we get the differential equation
\[ e^{-i H^A t} \frac{d \rho_{OD}^I (t)}{dt} e^{i H^B t} = \]
\[ - \frac{1}{2} \eta \sigma^2 e^{i H^A t} [b^\dagger b^\dagger, [b^\dagger, \rho_{OD}^I (t)]] e^{-i H^B t} , \quad \text{Eq. (17)} \]
Now take \( \text{Tr}_m \) of this equation, and use cyclic invariance, to get
\[ \frac{d}{dt} \text{Tr}_m e^{-i H^A u} \rho_{OD}^I (t) e^{i H^B u} = \]
\[ - \frac{1}{2} \eta \sigma^2 \text{Tr}_m \left\{ [b^\dagger b^\dagger, [b^\dagger, e^{-i H^B (t-\omega_m t) H^A (t-\omega_m t)] \rho_{OD}^I (t) \right\} , \quad \text{Eq. (18)} \]
Taking the adjoint of Eq. (18) and setting \( t \to -t \), we get
\[ e^{i H^B t} e^{-i H^A t} = N_t e^{\eta \sigma b^\dagger b} \quad \text{Eq. (19)} \]
from which we easily calculate that the double commutator in Eq. (18) is
\[ \left[ b + b^\dagger, [b + b^\dagger, e^{-i H^B (t-\omega_m t) H^A (t-\omega_m t)] \right] = \]
\[ = (\beta_u - \alpha_u) e^{-i H^B (t-\omega_m t) H^A (t-\omega_m t)} \]
\[ = 4 \kappa^2 (1 - \cos \omega_m (u - t))^2 (1 - \cos \omega_m (u - t))^2 \]
\[ \text{Substituting this into Eq. (18), and using cyclic invariance of the trace and Eq. (18), then gives} \]
\[ \frac{d}{dt} \text{Tr}_m e^{-i H^A u} \rho_{OD}^I (t) e^{i H^B u} = \]
\[ = -2 \eta (\kappa \sigma)^2 (1 - \cos \omega_m (u - t))^2 \]
\[ \text{Tr}_m e^{-i H^A u} \rho_{OD}^I (0) e^{i H^B u} , \quad \text{Eq. (20)} \]
which can be immediately integrated to give
\[ \text{Tr}_m e^{-i H^A u} \rho_{OD}^I (t) e^{i H^B u} = \]
\[ = -2 \eta (\kappa \sigma)^2 J_{u_0} (1 - \cos \omega_m (u - v))^2 \]
\[ \text{Tr}_m e^{-i H^A u} \rho_{OD}^I (0) e^{i H^B u} , \quad \text{Eq. (22)} \]
Setting \( u = t \) in this equation, and using Eq. (19) and cyclic invariance of the trace together with Eq. (18), we get
\[ f = \text{Tr}_m \rho_{OD} (t) \]
\[ = -2 \eta (\kappa \sigma)^2 J_{t_0} (1 - \cos \omega_m (t - v))^2 \]
\[ \text{Tr}_m e^{-i H^A t} \rho_{OD}^I (0) e^{i H^B t} \]
\[ = -2 \eta (\kappa \sigma)^2 J_{t_0} (1 - \cos \omega_m (t - v))^2 \]
\[ \text{Tr}_m e^{-i H^A t} \rho_{OD}^I (0) e^{i H^B t} \]
\[ = e^{-\kappa^2 (1 - i \omega_m t) + i \kappa^2 (\omega_m t - \sin \omega_m t)} . \quad \text{Eq. (23)} \]
Finally, taking the absolute value of Eq. (23), we get for the visibility
\[ \nu (t) = \exp \left[ -\kappa^2 (1 - \cos \omega_m t) \right] \]
\[ \times \exp \left[ - \frac{3 \eta \sigma^2}{16} \left( t - \frac{4 \sin \omega_m t}{3 \omega_m} + \frac{2 \omega_m t}{6 \omega_m} \right) \right] . \quad \text{Eq. (24)} \]
Equations 23 and 24 are the results that we quoted in Ref. 7.

IV. SOLUTION FOR THE VISIBILITY BY A STOCHASTIC UNRAVELLING METHOD

In this section we give an alternate derivation of Eq. 24, using stochastic methods to solve Eq. 10. We exploit the property that although Eq. 5 for the stochastic evolution of the state vector uniquely implies the evolution of Eq. 11 for the expectation density matrix \( \rho \), this relationship is not one to one: there are in fact an infinite number of different stochastic evolutions (or unravellings) which imply Eq. 11 for the evolution of their expectations 9. In particular, a simple calculation using the Itô calculus shows that the linear stochastic equation

\[
d |\psi_t\rangle = \left[ -\frac{i}{\hbar} H dt + i\sqrt{\eta} g dW_t - \frac{\eta}{2} q^2 dt \right] |\psi_t\rangle \tag{25}
\]

also has Eq. 11 for the evolution equation for \( \rho = E[|\psi_t\rangle\langle\psi_t|] \). This means that, as long as one is interested only in the statistical properties of the system — i.e. expectation values like \( \text{Tr}_m \rho \Omega(t) \) and the visibility — one can choose freely to work either with the stochastic evolution of Eq. 5 or with the stochastic evolution of Eq. 25. Of course, for individual realizations of the stochastic process, the two equations Eq. 5 and Eq. 25 imply radically different dynamics; in particular, Eq. 5 induces the collapse of the wavefunction, while Eq. 25 does not. However, for all physical quantities that depend only on the expectation of the density matrix, the two evolutions give the same answer.

Let us then resort to Eq. 25, since it is linear. According to this equation, the initial state 2 evolves as follows:

\[
|\psi_t\rangle = \frac{1}{\sqrt{2}} e^{-i\omega_0 t} \left[ |0\rangle_A |1\rangle_B |\phi_{t,m}^0\rangle + |0\rangle_A |1\rangle_B |\phi_{t,m}^1\rangle \right],
\]

where the state vectors \( |\phi_{t,m}^0\rangle \) and \( |\phi_{t,m}^1\rangle \) satisfy the following stochastic differential equation for the mirror center of mass:

\[
\frac{d}{dt} \phi_{t,m}^n = \left[ \frac{i}{\hbar} \frac{d^2}{dx^2} + \frac{i M \omega_0^2}{2\hbar} x^2 + \int dx \phi_{t,m}^n \right] dt + \frac{i}{\sqrt{2\eta}} x dW_t - \frac{\eta}{2} x^2 dt \phi_{t,m}^n(x),
\]

with \( n = 0, 1 \), and with the coupling constant \( g = G/\sigma \).

We now have to find the solution for the initial condition \( |\phi_{t,m}^n\rangle_m = |0\rangle_m \).

We take as a trial solution,

\[
\phi_{t}^n(x) = \left( \frac{M \omega_m}{\hbar} \right)^{\frac{n}{2}} \exp \left[ -a_t^n x^2 + b_t^n x + c_t^n \right], \tag{28}
\]

and by substituting it into Eq. 25 and using the rules of the Itô calculus, we get the following set of equations for the parameters \( a_t^n, b_t^n \) and \( c_t^n \):

\[
d a_t^n = -\frac{2i}{M} (a_t^n)^2 dt + \frac{i M \omega^2}{2\hbar} dt \quad a_t^0 = \frac{M \omega_m}{2\hbar},
\]

\[
d b_t^n = \left[ i g - \frac{2i}{M} b_t^n a_t^n \right] dt + i \sqrt{\eta} dW_t \quad b_t^n = 0,
\]

\[
d c_t^n = \frac{i}{2M} (b_t^n)^2 - 2a_t^n dt \quad c_t^n = 0. \tag{29}
\]

The first two equations can be easily integrated and one gets

\[
a_t^n = \frac{M \omega_m}{2\hbar}, \tag{30}
\]

\[
b_t^n = \frac{ng}{\omega_m} \left[ 1 - e^{-i\omega_m t} \right] + i \sqrt{\eta} \int_0^t e^{-i\omega_m (t-s)} dW_s.
\]

The factor \( f \) previously introduced can be written as:

\[
f = \int_{-\infty}^{+\infty} E \left[ \phi_t^n(x)^* \phi_t^1(x) \right] dx. \tag{31}
\]

We reverse the two operations of computing the statistical average \( E[...] \) and of taking the partial trace; the integration over \( x \) gives

\[
\int_{-\infty}^{+\infty} \phi_t^n(x)^* \phi_t^1(x) dx = \exp \left[ \frac{(b_t^{0*} + b_t^1)^2}{8 a_t} + c_t^{0*} + c_t^1 \right]. \tag{32}
\]

As the final step, we have to take the average of Eq. 32 with respect to the noise. To this end, we compute the stochastic differential of the exponent, obtaining after some algebra

\[
d \left[ \frac{(b_t^{0*} + b_t^1)^2}{8 a_t} + c_t^{0*} + c_t^1 \right] = \frac{i}{2M \omega_m^2} \left[ 1 - e^{-i\omega_m t} \right] dt + \frac{i \sqrt{\eta} g}{M \omega_m} z_t dt,
\]

where \( z_t \) is the stochastic process given by the formula

\[
z_t = \int_0^t \sin \omega_m (t-s) dW_s. \tag{34}
\]

The form of the stochastic integral of Eq. 34 is such that \( z_t \) is a Gaussian stochastic process with zero mean, while the correlation function is

\[
K(t,s) = E[z_t z_s] = \int_{\text{min}(t,s)}^{\text{max}(t,s)} \sin \omega_m (t-u) \sin \omega_m (s-u) du. \tag{35}
\]
Equation (33) shows that, as expected, $f$ is the product of a “deterministic” part $f_D$, which does not depend on the noise $z_t$, and a “stochastic” part $f_S$ which depends on the noise. The deterministic part gives the result of Eq. (33),

$$f_D = \exp \left[ \frac{ihg}{2\omega}\int_0^t \left( 1 - e^{-i\omega m s} \right) ds \right] = e^{i\eta^2(\omega m t - \sin \omega m t)} e^{-\kappa^2(1 - \cos \omega m t)}.$$  \hspace{1cm} (36)

We now have to compute the stochastic part,

$$f_S = E \left[ \exp \left( i\sqrt{\eta}g\int_0^t z_s ds \right) \right].$$  \hspace{1cm} (37)

One easily recognizes, in the above formula, the definition of the characteristic functional $\Phi[k_t]$ of the Gaussian stochastic process $z_t$, with $k_t = \sqrt{\eta}g/M \omega_m$. One then has,

$$f_S = \exp \left[ -\frac{\eta}{2} \left( \frac{g}{M \omega_m} \right)^2 \int_0^t ds_1 \int_0^t ds_2 K(s_1, s_2) \right]$$

$$= \exp \left[ -\frac{3}{16} \eta \ell^2 \left( t - \frac{4}{3\omega_m} \sin \omega m t + \frac{1}{6\omega_m} \sin 2\omega m t \right) \right].$$  \hspace{1cm} (38)

with $\ell = 4\kappa \sigma$ the maximum excursion of the mirror center of mass in its oscillation. (A derivation of Eq. (38) directly from the Itô calculus is given in Appendix B.)

The final result for the visibility $\nu = |f|$ is thus

$$\nu(t) = \exp \left[ -\kappa^2(1 - \cos \omega_m t) \right] \times \exp \left[ -\frac{3}{16} \eta \ell^2 \left( t - \frac{4}{3\omega_m} \sin \omega m t + \frac{1}{6\omega_m} \sin 2\omega m t \right) \right].$$  \hspace{1cm} (39)

as also obtained by the method of Sec. 3.

V. CALCULATION OF THE STOCHASTICITY PARAMETER FROM THE CSL MODEL

In this section we calculate the stochasticity parameter $\eta$ appearing in Eq. (5), in terms of parameters that appear in the CSL model for state vector collapse, which applies to systems of identical particles treated by a field-theoretic approach (for a similar calculation based on properties of the complementary error function, see Ghirardi, Pearle, and Rimini [3, Appendix C]). The relevant CSL equation, taken from Eqs. (8.23) and (8.24) of the review of Bassi and Ghirardi [3], can be written as

$$\frac{\partial}{\partial t} \langle Q' | \rho | Q'' \rangle = -\Gamma(Q', Q'')(\langle Q' | \rho | Q'' \rangle),$$  \hspace{1cm} (40)

where

$$\Gamma(Q', Q'') = \frac{1}{2} \gamma \int d^3 x [F(Q' - x) - F(Q'' - x)]^2,$$  \hspace{1cm} (41)

and where

$$F(z) = \int d^3 y D(y) \left( \frac{\alpha}{2\pi} \right)^2 e^{-\alpha^2/2}.$$  \hspace{1cm} (42)

Letting $d = Q' - Q''$ and using translation invariance and space inversion symmetry, we can rewrite Eq. (41) as

$$\Gamma(Q', Q'') = \frac{1}{2} \gamma \int d^3 x [F(x + d) - F(x)]^2,$$  \hspace{1cm} (43)

so that Taylor expansion gives for the leading small displacement term (with summation on $i, j$ understood)

$$\Gamma(Q', Q'') \simeq \frac{1}{2} \gamma \int d^3 x d_i d_j \frac{\partial^2 F(x)}{\partial x_i \partial x_j} F(x).$$  \hspace{1cm} (44)

We now use the fact that, acting on the exponential within the integral of Eq. (42), $\frac{\partial^2 F(x)}{\partial x_i \partial x_j}$ is equivalent to $\frac{\partial F(x)}{\partial x_i}$, which can be integrated by parts to act on the density $D$. Since for density distributions with cubic or higher symmetry we expect the coefficient of $d_i d_j$ in Eq. (44) to be proportional to $d_i$, we can extract this coefficient by replacing $d_i d_j$ by $d_i d_j^3/3$, giving

$$\Gamma(Q', Q'') \simeq \frac{1}{2} \gamma C d_i^2,$$  \hspace{1cm} (45)

with the coefficient $C$ given by

$$C = \frac{1}{3} \left( \frac{\alpha}{2\pi} \right)^3 \int d^3 y \int d^3 w \partial_i D(y) \partial_i D(w)$$

$$\times \int d^3 x e^{-(\alpha^2/2)[(x+y)^2 + (x+w)^2]}.$$  \hspace{1cm} (46)

We can now complete the square in the exponent, $$(x+y)^2 + (x+w)^2 = 2[x + \frac{1}{2}(y + w)]^2 + \frac{1}{2}(y - w)^2,$$  \hspace{1cm} (47)

which allows us to do the $x$ integration, giving

$$C = \frac{1}{24} \left( \frac{\alpha}{\pi} \right)^2 \int d^3 y \int d^3 w \partial_i D(y) \partial_i D(w) e^{-(\alpha^2/4)(y - w)^2}.$$  \hspace{1cm} (48)

Let us now assume a cubical volume of uniform density $D_0$ and side $S$, so that we can take

$$D(w) = D_0 \prod_{i=1}^3 \theta(w_i + S/2) \theta(S/2 - w_i).$$  \hspace{1cm} (49)

The three terms summed over $i$ in Eq. (48) give equal contributions, so we have

$$C = \frac{1}{8} D_0^2 \left( \frac{\alpha}{\pi} \right)^{3/2} I_{12} I_3,$$  \hspace{1cm} (50)

with

$$I_{12} = \int_{-S/2}^{S/2} dy_1 dy_2 dw_1 dw_2 e^{-(\alpha/4)(y_1 - w_1)^2}$$

$$e^{-(\alpha/4)(y_2 - w_2)^2},$$  \hspace{1cm} (51)
and with
\[
I_3 = \int_{-\infty}^{\infty} dy_3 \int_{-\infty}^{\infty} dw_3 [\delta(y_3 + S/2) − \delta(S/2 − y_3)] \\
[\delta(w_3 + S/2) − \delta(S/2 − w_3)] e^{−(\alpha/4)(y_3−w_3)^2}(52)
\]
When \(S^2\alpha >> 1\), we can use the fact that the exponentials are sharply peaked to get the approximations
\[
I_{1,2} \simeq S^24\pi/\alpha \quad I_3 \simeq 2 , \quad (53)
\]
giving
\[
C \simeq D_0^2 S^2 \left(\frac{\alpha}{\pi}\right)^{1/2} . \quad (54)
\]
This identifies the parameter \(\eta\) appearing as the coefficient of the \([q, [q, p]]\) term in the density matrix equation of motion \(1\) as
\[
\eta = \gamma C = \gamma S^2 D_0^2 \left(\frac{\alpha}{\pi}\right)^{1/2} , \quad (55)
\]
as used in Eq. (14) of Ref. \(7\).

As a consistency check, let us use Eq. \(55\) to determine the transition regime from quadratic growth of \(\Gamma\) to linear growth. For \(|d|\alpha^{1/2} >> 1\), we know (see Bassi and Ghirardi \(4\), p. 326) that \(\Gamma\) is given by the formula
\[
\Gamma = \gamma n_{\text{out}} D_0 , \quad (56)
\]
where \(n_{\text{out}}\) is the number of nucleons in the displaced cube not lying in the original cube, which is clearly (for a 3 axis displacement) given by \(|d|S^2 D_0\). So equating \((1/2)\gamma S^2 D_0^2 (\alpha/\pi)^{1/2}|d|^2 = \gamma |d|S^2 D_0^2\), we find that the transition from quadratic to linear growth occurs at \(|d| = 2(\pi/\alpha)^{1/2}\), which is of order the width of the Gaussians and so is reasonable.

VI. DISCUSSION

To summarize, we have given details of the calculation of the stochastic reduction in the visibility implied by Eqs. \(8\)–\(10\), leading to the visibility formula of Eqs. \(24\) and \(39\), as well as details of the calculation of the stochasticity parameter \(\eta\) implied by the CSL model, leading to the formula of Eq. \(55\). As already discussed, in the absence of stochastic reduction, the visibility as given by Eq. \(1\) starts at 1 at time \(t = 0\), decreases as \(t\) increases, and then returns to 1 at \(t = 2\pi/\omega_m\), at which point the mirror has completed one period of its oscillation. By contrast, with stochasticity present, we learn from Eqs. \(24\) and \(39\) that at time \(t = 2\pi/\omega_m\) the mirror visibility is damped by a factor \(e^{-\Lambda}\), with
\[
\Lambda = (3/16)\eta\ell^2(2\pi/\omega_m) . \quad (56)
\]
Combining this formula with Eq. \(56\), in the CSL model we get
\[
\Lambda = (3/16)\gamma S^2 D_0^2 \left(\frac{\alpha}{\pi}\right)^{1/2} \ell^2(2\pi/\omega_m) . \quad (57)
\]
As shown in Ref. \(7\), which gives a detailed discussion of the physical context, for the parameter values appropriate to the CSL model and the Marshall et al experiment, Eq. \(57\) gives \(\Lambda \sim 0.2 \times 10^{-8}\), indicating that according to the CSL model, coherence is maintained to an accuracy of better than one part in \(10^8\). Thus the Marshall et al experiment is orders of magnitude away from a capability of testing spontaneous collapse models for state vector reduction.

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APPENDIX A: BAKER–HAUSDORFF FORMULAS

We derive here the Baker–Hausdorff formula of Eqs. \(18\)–\(19\). Let us define the unitary evolution operator
\[
U = e^{−iH^A t} , \quad (A1)
\]
and the corresponding interaction picture operator
\[
U^I = e^{iH^B t} U = e^{iH^B t} e^{−iH^A t} . \quad (A2)
\]
The operator \(U^I\) obeys the equation of motion
\[
\frac{dU^I}{dt} = e^{iH^B t} i(I(H^B = H^A) e^{−iH^B t} U^I) = e^{iH^B t} iG(b + \lambda^t) e^{−iH^B t} U^I = [A(t) + B(t)] U^I , \quad (A3)
\]
where we have defined
\[
A(t) = iGe^{iH^B t} be^{−iH^B t} = iGe^{−i\omega_m t} b , \quad (A4)
\]
\[
B(t) = iGe^{iH^B t} b_1 e^{−iH^B t} = iGe^{i\omega_m t} b_1 . \quad (A4)
\]
These obey the commutators
\[
[A(s), A(t)] = [B(s), B(t)] = 0, \quad [A(s), B(t)] = −G^2 \exp[−i\omega_m (s − t)] , \quad (A5)
\]
all of which are c-numbers. Integrating Eq. \(A3\) with respect to \(t\), and using \(U^I(0) = 1\), we get
\[
U^I(t) = T \exp \left[ \int_0^t ds (A(s) + B(s)) \right] , \quad (A6)
\]
where \(T\) orders later times to the left.
Consider now the operator $W$ defined as
\[ W = \exp \left[ \int_0^t dsB(s) \right] \exp \left[ \int_0^t dsA(s) \right], \quad (A7) \]
which obeys
\[
\frac{dW}{dt} = \exp \left[ \int_0^t dsB(s) \right] [B(t) + A(t)]
\exp \left[ \int_0^t dsA(s) \right]
= \left\{ B(t) + \exp \left[ \int_0^t dsB(s) \right] A(t) \right\} W.
\]  
(A8)

Now for general $u$ we have
\[
\frac{d}{dt} \exp \left[ \int_0^t dsB(s) \right] A(u) \exp \left[ - \int_0^t dsB(s) \right]
= \exp \left[ \int_0^t dsB(s) \right] [B(t), A(u)] \exp \left[ - \int_0^t dsB(s) \right]
= [B(t), A(u)],
\]  
(A9)

where we have used the fact that the commutator $[B(t), A(u)]$ is a c-number. Integrating on $t$, this gives
\[
\exp \left[ \int_0^t dsB(s) \right] A(u) \exp \left[ - \int_0^t dsB(s) \right]
= A(u) + \int_0^t ds[B(s), A(u)],
\]  
(A10)

and now setting $u = t$ we get
\[
\exp \left[ \int_0^t dsB(s) \right] A(t) \exp \left[ - \int_0^t dsB(s) \right]
= A(t) + \int_0^t ds[B(s), A(t)].
\]  
(A11)

Comparing with Eq. (A8), we have obtained
\[
\frac{dW}{dt} = \left( A(t) + B(t) + \int_0^t ds[B(s), A(t)] \right) W, \quad (A12)
\]
and comparing this with Eqs. (A3) and (A6) for $U^t$, we get
\[
U^t = W \exp \left( - \int_0^t du \int_0^u ds[B(s), A(u)] \right). \quad (A13)
\]

Multiplying Eq. (A2) from the left by $e^{-iH^B_t}$ and from the right by $e^{iH^B_t}$, we then get
\[
e^{-iH^A_t} e^{iH^B_t} = e^{-iH^B_t} U^t e^{iH^B_t} \\
e^{-iH^B_t} \exp \left[ \int_0^t dsB(s) \right] e^{iH^B_t} e^{-iH^B_t} \exp \left[ \int_0^t dsA(s) \right] e^{iH^B_t} \exp \left( - \int_0^t du \int_0^u ds[B(s), A(u)] \right) \\
e\exp \left[ \int_0^t dsB(s) e^{iH^B_t} \right] \exp \left[ -iH^B_t \right] \int_0^t dsA(s)e^{iH^B_t} \exp \left( - \int_0^t du \int_0^u ds[B(s), A(u)] \right) \\
e\exp \left[ \int_0^t ds iGe^{-i\omega_m t} b^\dagger e^{iH^B_t} \right] \exp \left[ -iH^B_t \right] \int_0^t ds iGe^{-i\omega_m s} e^{-iH^B_t} b e^{iH^B_t} \\
\times \exp \left( - \int_0^t du \int_0^u ds G^2 \exp \left( i\omega_m (s-u) \right) \right) \\
= \exp \left[ \int_0^t ds iGe^{-i\omega_m s} e^{-i\omega_m t} b^\dagger \right] \exp \left[ -iH^B_t \right] \int_0^t ds iGe^{-i\omega_m s} e^{i\omega_m t} b \exp \left( - \int_0^t du \int_0^u ds G^2 \right) \\
= \exp \left[ \kappa (1 - e^{-i\omega_m t}) b^\dagger \right] \exp \left[ -\kappa (1 - e^{i\omega_m t}) \right] \exp \left[ -\kappa^2 (1 - i\omega_m t - e^{-i\omega_m t}) \right], \quad (A14)
\]

which is Eq. (14) of Sec. 3.

**APPENDIX B: BASIC ITÔ CALCULUS FORMULAS**

The stochastic differential $dW_t$ behaves heuristically as a random square root of $dt$, as expressed in the Itô
calculus rules
\[ dW_t^2 = dt, \quad dW_t dt = dt^2 = 0. \] (B1)

As a consequence of Eq. (B1), the Leibniz chain rule of the usual calculus is modified to
\[ d(AB) = dA B + A dB + dA dB, \] (B2)

and thus in differentiating a function \( f(A) \), one has
\[ df(A) = f(A + dA) - f(A) = f'(A) dA + \frac{1}{2} f''(A) (dA)^2. \] (B3)

These formulas are used in the calculations leading to Eqs. (29) and (33) of Sec. 4.

The Itô differential \( dW_t \) is statistically independent of the random process up to time \( t \), so we have the definition
\[ E[dW_t C(t)] = 0 \] (B4)

for any stochastic process \( C(t) \) constructed from \( dW_t \) with \( s \leq t \). From Eqs. (B1), (B4), we get useful formulas for expectations of integrals. Consider first
\[ f(t) = E \left[ \int_0^t dW_u A(u) \int_0^t dW_u B(u) \right], \] (B5)

which has the differential
\[ df(t) = E \left[ dW_t A(t) \int_0^t dW_u B(u) \right. \\
+ \left. \left( \int_0^t dW_u A(u) \right) dW_t B(t) + A(t)B(t) dt \right] \\
= E[A(t)B(t)] dt, \] (B6)

which integrates back to give
\[ E \left[ \int_0^t dW_u A(u) \int_0^t dW_u B(u) \right] = \int_0^t du E[A(u)B(u)], \] (B7)

a formula called the Itô isometry. When \( A(u) \) and \( B(u) \) have differing domains of support \( D_A \) and \( D_B \), the integral on the right of Eq. (B7) clearly extends only over the intersection \( D_A \cap D_B \). Applying Eq. (B7) to the definition of \( z_t \) in Eq. (B1) immediately gives the formula for the correlation function \( K(t,s) \) of Eq. (35). Consider next the expectation
\[ f(t) = E \left[ \exp \left( \int_0^t \Phi(u,v) dW_v \right) \right]. \] (B8)

Its differential is, by Eq. (B3),
\[ df = E \left[ \exp \left( \int_0^t \Phi(u,v) dW_v \right) \left( \Phi(u,t) dW_t + \frac{1}{2} \Phi(u,t)^2 dt \right) \right] \\
= \frac{1}{2} f(t) \Phi(u,t)^2 dt, \] (B9)

which integrates back to give
\[ E \left[ \exp \left( \int_0^t \Phi(u,v) dW_v \right) \right] = \exp \left( \frac{1}{2} \int_0^t dv \Phi(u,v)^2 \right). \] (B10)

In particular, setting \( u = t \) we get the useful formula
\[ E \left[ \exp \left( \int_0^t \Phi(t,v) dW_v \right) \right] = \exp \left( \frac{1}{2} \int_0^t dv \Phi(t,v)^2 \right). \] (B11)

As an application of Eq. (B11), consider the expectation
\[ g(t) = E[\exp(C \int_0^t z_s ds)], \] with \( z_t \) given by Eq. (B4).

Since
\[ \int_0^t z_s ds = \int_0^t ds \int_0^s \sin \omega_m(s - v) dW_v \\
= \int_0^t dW_v \int_v^t ds \sin \omega_m(s - v) \\
= \int_0^t \omega_m^{-1} [1 - \cos \omega_m(t - v)] dW_v, \] (B12)

the expectation \( g(t) \) has the form of Eq. (B11), with \( \Phi(t,v) \equiv C \omega_m^{-1} [1 - \cos \omega_m(t - v)] \), and we have
\[ g(t) = \exp \left( \frac{C^2}{2 \omega_m^2} \int_0^t [1 - \cos \omega_m(t - v)]^2 dv \right), \] (B13)

which corresponds to the integral appearing in Eq. (B11). An alternative expression for \( g(t) \) is obtained by using the formula \( \Phi(t,v) = C \int_0^t ds \sin \omega_m(s - v) \), which gives

\[ \int_0^t dv \Phi(t,v)^2 = C^2 \int_0^t dv \int_v^t ds_1 \int_v^t ds_2 \sin \omega_m(s_1 - v) \sin \omega_m(s_2 - v) \\
= C^2 \int_0^t ds_1 \int_0^t ds_2 \int_0^{\min(s_1,s_2)} dv \sin \omega_m(s_1 - v) \sin \omega_m(s_2 - v) \\
= C^2 \int_0^t ds_1 \int_0^t ds_2 K(s_1,s_2), \] (B14)
with $K$ the correlation function defined in Eq. (35). When substituted into Eq. (B11), this corresponds to the integral appearing in Eq. (38).

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