A GEOMETRY WHERE EVERYTHING IS BETTER THAN NICE

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Abstract. We present a geometry in the disk whose metric truth is curiously arithmetic.

1. The better-than-nice metric

Consider the metric
\[ g = \frac{4}{1 - r^2} \left( dx^2 + dy^2 \right) \]
in the unit disk. Here \( r^2 = x^2 + y^2 \) as usual. Everything of interest can be computed explicitly, and with surprising results.

1.1. Hypocycloids in the disk. Consider the curve
\[ c(t) = (1 - a)e^{i\theta(t)} + ae^{-i\phi(t)}, \quad 0 < a < 1, \]
thought of as a point on a circle of radius \( a \) turning at a rate \( \dot{\phi} \) in the clockwise direction as the centre of the circle rotates on a circle of radius \( 1 - a \) rotating at a rate \( \dot{\theta} \) in the counterclockwise direction.

For the small circle of radius \( a \) to roll without slipping on the inside of the circle of radius 1 requires the point \( c(t) \) to have velocity 0 when \( |c(t)| = 1 \), which is the relation
\[ a\dot{\phi} = (1 - a)\dot{\theta}. \]

Because \( g \) is rotationally invariant, without loss of generality we may take
\[ \theta(t) = \frac{at}{2\sqrt{a(1 - a)}}, \quad \phi(t) = \frac{(1 - a)t}{2\sqrt{a(1 - a)}}. \]

Theorem 1.1. The curve \( c(t) \) is a geodesic for the metric \( g \), parameterized by arclength.

Proof. For our \( g \), the equations \( \ddot{u}^i + \Gamma^i_{jk} \dot{u}^j \dot{u}^k = 0 \) determining geodesics \( u(t) = (x(t), y(t)) \) parameterized proportional to arclength are
\[
\begin{align*}
(1 - x^2 - y^2)\ddot{x} + x(x^2 - y^2) + 2y\dot{x}\dot{y} &= 0 \quad \text{and} \\
(1 - x^2 - y^2)\ddot{y} - y(x^2 - y^2) + 2x\dot{x}\dot{y} &= 0,
\end{align*}
\]
which can be expressed in terms of \( z = x + iy \), as the single equation
\[ (1 - \bar{z}z)\ddot{z} + \dddot{z} z^2 = 0. \]
Given the formula for \( c(t) \) it is straightforward to verify that \( c(t) \) satisfies (1) and moreover that \( |\dot{c}| = \sqrt{1 - |c|^2}/2 \), from which it follows that \( ||\dot{c}||_g = 1 \), meaning that the parameterization is by arclength. q.e.d.

**Theorem 1.2.** The closed geodesics (i.e. keep rolling the generating circle of the hypocycloid until it closes up) have length \( 4\pi \sqrt{n} \), and the number of geometrically distinct geodesics of length \( 4\pi \sqrt{n} \) is given by the arithmetic function \( \psi(n) \).

The function \( \psi(n) \) counts the number of different ways that the integer \( n \) may be written as a product \( n = pq \), with \( p \leq q \), \((p,q) = 1\). Values of this function are tabulated in sequence A007875 in the online encyclopedia of integer sequences [1].

**Proof.** Beginning at \( c(0) = 1 \), the geodesic \( c(t) \) first returns to the boundary circle at
\[
c\left(4\pi \sqrt{a(1-a)}\right) = e^{2\pi i(1-a)},
\]
returning again at points of the form \( e^{2\pi im(1-a)} \) \((m \in \mathbb{Z}_+)\). The corresponding succession of cycloidal geodesic arcs winds clockwise around the origin if \( 0 < a < 1/2 \) and counterclockwise if \( 1/2 < a < 1 \); when \( a = 1/2 \), \( c(t) \) traverses back and forth along the \( x \)-axis. Thus \( c(t) \) forms a once-covered closed geodesic precisely when \( 2\pi m(1-a) = q2\pi \) for some relatively prime pair of positive integers \( q < m \), in which case the geodesic has length
\[
4\pi m \sqrt{a(1-a)} = 4\pi \sqrt{(m-q)q}.
\]
To count geometrically distinct closed geodesics we restrict to \( 0 < a \leq 1/2 \), in which case \( p := m - q = ma \leq m(1-a) = q \). Given any relatively prime positive integers \( p \leq q \) the geodesic of length \( 4\pi \sqrt{pq} \) occurs when \( a = p/(p+q) \). q.e.d.

1.2. **Eigenfunctions and eigenvalues of the Laplacian.** Set the Laplacian \( \Delta \) to be
\[
\Delta = -g^{ab} \nabla_a \nabla_b
\]
where \( \nabla_a \) is the covariant derivative operator associated to the metric \( g \) via the Levi-Civita connection. Consider the eigenvalue problem
\[
\Delta u = \lambda u
\]
for functions \( u \) with the boundary value \( u(r = 1) = 0 \). This problem has a number of remarkable features.

**Theorem 1.3.** The eigenfunctions and eigenvalues satisfy

1. The eigenvalues \( \lambda_n \) are precisely the positive integers \( n = 1, 2, 3, \ldots \).
2. The eigenfunctions are polynomials.
3. The dimension of the eigenspace for eigenvalue \( n \) is the number of divisors of \( n \). (The number of divisors function is denoted by \( \tau(n) \).)

**Proof.** Since the operator
\[
-\Delta + \lambda = \frac{1 - r^2}{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \lambda
\]
is analytic hypoelliptic, distributional solutions to the eigenvalue equation $\Delta u = \lambda u$
are necessarily real analytic, and representable near $(x, y) = (0, 0)$ by absolutely
convergent Fourier series

$$u(r, \theta) = \sum_{n \in \mathbb{Z}} a_n(r)e^{int}.$$  

Observing that $(-\Delta + \lambda)(a_n(r)e^{int})$ has the form $A_n(r)e^{int}$ for some $A_n$, it follows
that $u$ is an eigenfunction of $\Delta$ only if each summand $a_n(r)e^{int}$ is, so it suffices to
consider just products of the form

$$u(r, \theta) = f(r)e^{int}.$$  

Expressing $-\Delta + \lambda$ in polar coordinates yields

$$(-\Delta + \lambda)(f(r)e^{int}) = \frac{1 - r^2}{4} \left( f''(r) + \frac{1}{r} f'(r) + \left( \frac{4\lambda}{1 - r^2} - \frac{n^2}{r^2} \right) f(r) \right) e^{int}.$$  

Therefore $u(r, \theta) = f(r)e^{int}$ is an eigenfunction of $\Delta$ only if $f$ satisfies

$$r^2(1 - r^2)f'' + r(1 - r^2)f' + \left(4\lambda r^2 - n^2(1 - r^2)\right)f = 0.$$  

Assume for definiteness that $n \geq 0$ and write $f(r) = r^ng(r^2)$, so that $g$ satisfies the
hypergeometric equation

$$r(1 - r)g'' + (c - (a + b + 1)r)g' - abg = 0,$$

where

$$a = \left(n + \sqrt{n^2 + 4\lambda}\right)/2, \quad b = \left(n - \sqrt{n^2 + 4\lambda}\right)/2, \text{ and } c = n + 1.$$  

This has a unique non-singular solution (up to scalar multiplication), the hypergeometric function

$$g(r) = {}_2F_1(a, b; n + 1; r) \quad \text{where} \quad g(1) = \frac{n!}{\Gamma(a)\Gamma(1 + b)},$$

which is zero at $r = 1$ if and only if $b = -m$ for some integer $m \geq 1$. This implies
$\lambda = m(m + n)$ is a positive integer, proving part (1) of the theorem. If $n > 0$ there are two corresponding eigenfunctions $u(r, \theta) = r^ng(r^2)e^{int}$, making a total of
$\tau(\lambda)$ eigenfunctions for each positive integer eigenvalue $\lambda$. That these are linearly
independent (part (3)), and that the eigenfunctions are polynomials (part (2)) can be verified
using an explicit formula for the eigenfunctions, as follows.

Using the formulation $\Delta = -(1 - z\bar{z})\frac{\partial^2}{\partial z\partial\bar{z}}$, where $z = x + iy$, one can check directly
that the Rodrigues-type formula

$$u^{(p, q)}(z) = \frac{(-1)^p}{q(p + q - 1)}(1 - z\bar{z})^\frac{p+q}{2} \frac{\partial^{p+q}}{\partial z^p\partial\bar{z}^q}(1 - z\bar{z})^{p+q-1}$$  

represents eigenfunctions corresponding to $\lambda = pq$, i.e., $\Delta u^{(p, q)} = pq u^{(p, q)}$. q.e.d.
1.3. Two corollaries.

Corollary 1.4. The spectral function is precisely the square of the Riemann zeta function

\[ \sum_n \frac{1}{(\lambda_n)^s} = \sum_n \frac{\tau(n)}{n^s} = (\zeta(s))^2. \]

In a more applied vein, consider a unit radius circular membrane fixed at the boundary (i.e. a drumhead) having constant tensile force per unit length \( S \) and radially varying density \( \rho(r) = 4S/(1 - r^2) \). Small transverse displacements of the membrane \( w(x,y,t) \) are governed by the equation

\[ w_{tt} + \Delta w = 0, \quad (3) \]

so squared eigenfrequencies of the membrane correspond to eigenvalues of \( \Delta \).

Corollary 1.5. Unlike the standard vibrating membrane whose eigenfrequencies are proportional to zeros of \( J_0 \), for each eigenfrequency \( \omega_n = \sqrt{n} \) of (3), all the higher harmonics \( m \omega_n (2 < m \in \mathbb{Z}^+) \) are also eigenfrequencies.

1.4. Acoustic imaging and combinatorics. Supplemented by the sequence of monomials \( u^{(p,0)}(z) := z^p \) for \( p \geq 0 \), the eigenfunctions of \( \Delta \) are the special functions suited to acoustic imaging of layered media. A layered medium consists of a stack of \( n - 1 \) horizontal slabs between two semi-infinite half spaces, where each slab and half space has constant acoustic impedance. Thus impedance as a function of depth is a step function having jumps at the \( n \) interfaces. To image the layers, an impulsive horizontal plane wave is transmitted at time \( t = 0 \) from a reference plane in the upper half space down toward the stack of horizontal slabs, and the resulting echoes are recorded at the reference plane, producing a function \( G(t) \) \( (t > 0) \) (the boundary Green’s function of the medium). Let \( L = (L_1, \ldots, L_n) \), \( u^{(0,q)} \equiv 0 \) if \( q \geq 1 \) and \( k_j = 0 \) if \( j > n \).

Theorem 1.6.

\[ G(t) = \sum_{k \in \mathbb{Z}^n} \left( \prod_{j=1}^{n} u^{(k_j,k_{j+1})}(R_j) \right) \delta(t - \langle L, k \rangle). \]

Here \( L = (L_1, \ldots, L_n) \), \( u^{(0,q)} \equiv 0 \) if \( q \geq 1 \) and \( k_j = 0 \) if \( j > n \).

Proof. Expanding the binomial \( (1 - z\bar{z})^{p+q-1} \) in the formula \( (2) \), and then applying the derivative \( \partial^{p+q}/\partial z^p \partial \bar{z}^q \), yields

\[ u^{(p,q)}(z) = \frac{(-1)^{p+y+1}}{q} (1 - z\bar{z})^{m+y-q+1} z^{m+y-p+1} \sum_{j=0}^{y} (-1)^j \frac{(j + y + m + 1)!}{j!(j + m)!(y - j)!} (z\bar{z})^j, \]
where \( m = |p - q| \) and \( \nu = \min\{p, q\} - 1 \). Switching to polar form \( z = re^{i\theta} \), it follows that
\[
u = \min\{p, q\} - 1.
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\[
u = \min\{p, q\} - 1.
\]
\[

\text{For } \theta = 0, \pi, \text{ the latter coincide with the functions } f^{(p, q)} \text{ occurring in } [2, \text{ Thm. 2.4, 4.3}]. \]

Each term \( \left( \prod_{j=1}^n u^{(k_j, k_j+1)}(R_j) \right) \delta(t - \langle L, k \rangle) \) corresponds to the set of all scattering sequences that have a common arrival time \( t_i = \langle L, k \rangle \), with each individual scattering sequence weighted according to the corresponding succession of reflections and transmissions. A scattering sequence may be represented by a Dyck path as in Figure 1. The tensor products \( \prod_{j=1}^n u^{(k_j, k_j+1)}(R_j) \) thus have a combinatorial interpretation in that they count weighted Dyck paths having \( 2k_j \) edges at height \( j \). See [2].

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{dyck_path.png}
\caption{The Dyck path for a scattering sequence. Time increases to the right. Each node at depth \( x_j \) receives a weight according to the structure of the path at the node; weights \( R_j, -R_j, 1 - R_j, 1 + R_j \) correspond respectively to down-up reflection, up-down reflection, downward transmission, upward transmission. The scattering sequence returns to the reference depth \( x_0 \) at time \( t_i = \langle L, k \rangle \), where \( k = (k_1, \ldots, k_n) \). Its amplitude is the product of the weights.}
\end{figure}

\textbf{References}

[1] The On-Line Encyclopedia of Integer Sequences, published electronically at https://oeis.org, 2016, Sequence A007875.

[2] P. C. Gibson. The combinatorics of scattering in layered media. \textit{SIAM J. Appl. Math.}, 74(4):919–938, 2014.
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