On uniqueness of maximal coupling for diffusion processes with a reflection

Kazumasa Kuwada *

Abstract

A maximal coupling of two diffusion processes makes two diffusion particles meet as early as possible. We study the uniqueness of maximal couplings under a sort of ‘reflection structure’ which ensures the existence of such couplings. In this framework, the uniqueness in the class of Markovian couplings holds for the Brownian motion on a Riemannian manifold whereas it fails in more singular cases. We also prove that a Kendall-Cranston coupling is maximal under the reflection structure.

Key words: Diffusion process, maximal coupling, mirror coupling, Kendall-Cranston coupling

1 Introduction

The concept of coupling is very useful in various problems in probability. Given probability measures $\mu_1$ on $(\Omega_1, \mathcal{F}_1)$ and $\mu_2$ on $(\Omega_2, \mathcal{F}_2)$, we say $\mu$ a coupling of $\mu_1$ and $\mu_2$, or $\mu \in C(\mu_1, \mu_2)$, when $\mu$ is a probability measure on $(\Omega_1, \mathcal{F}_1) \times (\Omega_2, \mathcal{F}_2)$ so that its marginal distributions coincide with $\mu_1$ and $\mu_2$ respectively. That is, $\mu(A_1 \times \Omega_2) = \mu_1(A_1)$ for $A_1 \in \mathcal{F}_1$ and $\mu(\Omega_1 \times A_2) = \mu_2(A_2)$ for $A_2 \in \mathcal{F}_2$.

We consider couplings of a diffusion process $(\{Z(t)\}_{t \geq 0}, \{P_x\}_{x \in M})$ on a topological space $X$. A coupling $P \in C(P_x, P_y)$ determines a stochastic process $(Z_1, Z_2)$ on $X \times X$ so that each individual component moves as the diffusion process starting at $x$ and $y$ respectively. A characteristic of couplings on which we concentrate our attention is the coupling time $T(Z_1, Z_2)$, the time when $Z_1$ and $Z_2$ coalesce (defined in (2.1)). In many applications, we would like to make the coupling probability $P[T > t]$ small by constructing a suitable coupling $P$. In these ways, one can obtain various estimates for heat kernel, harmonic functions(or harmonic maps), eigenvalues etc. by means of the geometry of $X$. These results indicate that the existence of a good coupling reflects the nature of $Z$ or $X$.

Our interest in this paper is the problem of the uniqueness. More precisely, we would like to know what properties of $Z$ or $X$ are related to the uniqueness of couplings which

*Department of Mathematics, Faculty of Science, Ochanomizu University, Tokyo 112-8610, Japan
† Tel:+81-3-5978-5300, e-mail: kkuwada@math.ocha.ac.jp
minimize the coupling probability. At this moment, however, the existence of such a good
coupling is not obvious at all in general. Thus we confine ourselves in a special situation
where the existence is ensured.

In the preceding work by E. P. Hsu and K.-Th. Sturm [6], they discussed the uniqueness
of maximal coupling when \( X = \mathbb{R}^d \) and \( Z \) is the Brownian motion on it. Motivated by
the coupling inequality, they defined a maximal coupling as it minimizes the coupling
probability. In their framework, there is a natural maximal coupling \( P_M \in \mathcal{C}(P_{x_1}, P_{x_2}) \)
called “mirror coupling” defined by using the reflection with respect to the hyperplane
which maps \( x_1 \) to \( x_2 \). They showed that the mirror coupling is the unique maximal
coupling in the class of Markovian couplings \( \mathcal{C}_0(P_{x_1}, P_{x_2}) \) (see Definition 2.4). They also
showed by examples that the uniqueness no longer holds when we are allowed to take
non-Markovian couplings. Their argument to derive the uniqueness uses the explicit form
of the transition density of the Brownian motion. In this sense, their argument depends
on the nature of the Euclidean Brownian motion.

In order to investigate how such a uniqueness depends on the nature of \( Z \) or \( X \), we
discuss the same uniqueness problem in a similar, but more general, situation. That is, we
assume a sort of ‘reflection structure’ like a reflection in Euclidean spaces for given initial
points \( x_1, x_2 \in X \). Then we can naturally define a mirror coupling \( P_M \in \mathcal{C}(P_{x_1}, P_{x_2}) \) as
a maximal Markovian coupling. In this situation, we consider the uniqueness of maximal
couplings in \( \mathcal{C}_0(P_{x_1}, P_{x_2}) \). As a result, the Brownian motion on a Riemannian manifold
enjoys the uniqueness. But, as we will see, the uniqueness no longer holds if we consider
more singular cases. These observations show that the uniqueness is related to the nature
of \( Z \) or \( X \) even when the mirror coupling exists.

The organization of this paper is as follows. In the next section, we introduce our
framework including the notion of ‘reflection structure’, maximal coupling and Markovian
coupling. Our main theorem gives a sufficient condition to the uniqueness of maximal
couplings in \( \mathcal{C}_0(P_{x_1}, P_{x_2}) \) (Theorem 2.6). We will prove Theorem 2.6 in section 3 following
the idea of [6]. Section 4 is devoted to some examples. On one hand, we will show that
the uniqueness holds under the assumption on the short time asymptotic behavior of \( Z \)
and the geometry of \( X \) (Theorem 4.1). A typical example satisfying these conditions is the
Brownian motion on a complete Riemannian manifold (Corollary 4.3). This framework
includes the Euclidean Brownian motion as discussed in [6]. There we exhibit complete
Riemannian manifolds which have the reflection structure with respect to specified initial
points. On the other hand, we also show two easy examples where the uniqueness of
maximal Markovian coupling fails (see Example 4.10 and Example 4.11). At the end of
this section, we consider the case for the Brownian motion on 2-dimensional Sierpinski
gasket. We show that the uniqueness holds while this case is not included in the framework
of Theorem 4.1. In section 5 we show that the Kendall-Cranston coupling coincides with
our mirror coupling under the existence of the reflection structure. The Kendall-Cranston
coupling is originally introduced by Kendall [8] and Cranston [3] for the Brownian motion
on an arbitrary complete Riemannian manifold. Their coupling is useful to estimate
analytic quantities by means of the geometric quantity such as Ricci curvature. But,
in general, there is no reason why the Kendall-Cranston coupling should be maximal.
The construction of their coupling is based on a sort of reflection of infinitesimal motion
by means of the Riemannian geometry. Thus, our result is quite natural. It should
be remarked that our result implies that the Kendall-Cranston coupling is the unique maximal coupling if there is a reflection structure.

## 2 Coupling of diffusions and its properties

Throughout this paper, we assume \( X \) to be an arcwise-connected Hausdorff topological space with the second countability axiom. For a coupled diffusion process \((Z_1(t), Z_2(t))\), the coupling time \(T(Z_1, Z_2)\) is defined by

\[
T(Z_1, Z_2) := \inf \{t > 0 ; Z_1(s) = Z_2(s) \text{ for all } s \geq t \}.
\]

We set

\[
\varphi_t(x, y) := \frac{1}{2} \| \mathbb{P}_x \circ Z(t)^{-1} - \mathbb{P}_y \circ Z(t)^{-1} \|_{\var{\text{var}}}.
\]

Here \( \| \cdot \|_{\var{\text{var}}} \) stands for the total variation norm. By using this function, the coupling inequality is written as follows: for every \( x, y \in X \) and \( \mathbb{P} \in \mathcal{C}(\mathbb{P}_x, \mathbb{P}_y) \),

\[
\mathbb{P}[T(Z_1, Z_2) > t] \geq \varphi_t(x, y).
\]  (2.3)

For the proof of (2.3), it suffices to remark that

\[
\mathbb{P}[T(Z_1, Z_2) > t] \geq \mathbb{P}[Z_1(t) \neq Z_2(t)]
\]

\[
\geq \mathbb{P}[Z_1(t) \in A, Z_2(t) \notin A]
\]

\[
\geq \mathbb{P}[Z_1(t) \in A] - \mathbb{P}[Z_2(t) \in A]
\]

holds for arbitrary \( A \in \mathcal{B}(X) \).

### Definition 2.1 (cf. [5, 6])

For \( t > 0 \), we say \( \mathbb{P} \in \mathcal{C}(\mathbb{P}_x, \mathbb{P}_y) \) maximal at \( t \) when the equality holds in (2.3). We say \( \mathbb{P} \in \mathcal{C}(\mathbb{P}_x, \mathbb{P}_y) \) maximal when the equality holds in (2.3) for each \( t > 0 \).

Let us fix \( x_1, x_2 \in X \). The reflection structure with respect to \( x_1 \) and \( x_2 \) stated in section \[1\] means the following two properties assigned on \( X \) and \( Z \):

(A1) There is a continuous map \( R : X \to X \) with \( R \circ R = \text{id} \) so that \( \mathbb{P}_{x_1} \circ R^{-1} = \mathbb{P}_{x_2} \),

(A2) The set of fixed points \( H := \{ x \in X ; R(x) = x \} \) separates \( X \) into two disjoint open sets \( X_1 \) and \( X_2 \) (i.e., \( X \setminus H = X_1 \cup X_2 \)) with \( R(X_1) = X_2 \).

As an easy but significant consequence of (A2), every continuous path in \( X \) joining \( x \in X_1 \) and \( y \in X_2 \) must intersect \( H \). In general, it highly depends on the choice of \( x_1, x_2 \in X \) whether (A1) and (A2) hold or not (see Example \[4, 8\]). But, we can easily verify that (A1) and (A2) are satisfied for the Euclidean Brownian motion for any \( x_1, x_2 \in X \). In that case, \( R \) is an reflection with respect to a hyperplane \( H \). Under (A1) and (A2), we can construct a mirror coupling of \( \mathbb{P}_{x_1} \) and \( \mathbb{P}_{x_2} \). Let \( \tau := \inf \{ t > 0 ; Z_1(t) \in H \} \) be a hitting time to \( H \). We define the mirror coupling \( \mathbb{P}_M \) as the law of \((Z_1, Z_2)\) where \( Z_1 \) is a copy of \((Z, \mathbb{P}_{x_1})\) and

\[
Z_2(t) = \begin{cases} 
RZ_1(t) & \text{if } t < \tau, \\
Z_1(t) & \text{if } t \geq \tau.
\end{cases}
\]  (2.4)

By definition, \( \mathbb{P}_M \in \mathcal{C}(\mathbb{P}_{x_1}, \mathbb{P}_{x_2}) \) and \( \tau = T(Z_1, Z_2) \) under \( \mathbb{P}_M \).
Proposition 2.2 \( \mathbb{P}_M \) is maximal.

For the proof, we use the following lemma.

Lemma 2.3 Suppose \( \mathbb{P}_x \circ R^{-1} = \mathbb{P}_{Rx} \) for \( x \in X_1 \). Then, for each \( t > 0 \),

\[
\varphi_t(x, Rx) = \mathbb{P}_x[Z(t) \in X_1] - \mathbb{P}_{Rx}[Z(t) \in X_1].
\]

Proof. By (2.2),

\[
\varphi_t(x, Rx) = \sup_{A \in \mathcal{B}(X)} \left( \mathbb{P}_x[Z(t) \in A] - \mathbb{P}_{Rx}[Z(t) \in A] \right). \tag{2.5}
\]

Note that

\[
\mathbb{P}_x[Z(t) \in A] - \mathbb{P}_{Rx}[Z(t) \in A] = \mathbb{P}_x[Z(t) \in A, \tau \leq t] - \mathbb{P}_{Rx}[Z(t) \in A, \tau \leq t] + \mathbb{P}_x[Z(t) \in A, \tau > t] - \mathbb{P}_{Rx}[Z(t) \in A, \tau > t].
\]

First we show

\[
\mathbb{P}_x[Z(t) \in A, \tau \leq t] = \mathbb{P}_{Rx}[Z(t) \in A, \tau \leq t] \tag{2.6}
\]

for each \( A \in \mathcal{B}(X) \). By the strong Markov property,

\[
\mathbb{P}_x[Z(t) \in A, \tau \leq t] = \mathbb{E}_x \left[ \mathbb{P}_{Z(t)}[Z(t - s) \in A] | s = \tau ; \tau \leq t \right].
\]

By assumption, the law of \((Z(\tau), \tau)\) under \( \mathbb{P}_x \) equals that under \( \mathbb{P}_{Rx} \). Thus we have

\[
\mathbb{E}_x \left[ \mathbb{P}_{Z(t)}[Z(t - s) \in A] | s = \tau ; \tau \leq t \right] = \mathbb{E}_{Rx} \left[ \mathbb{P}_{Z(t)}[Z(t - s) \in A] | s = \tau ; \tau \leq t \right] = \mathbb{P}_{Rx}[Z(t) \in A, \tau \leq t].
\]

Next, by (A2), we have

\[
\mathbb{P}_x[Z(t) \in A, \tau > t] - \mathbb{P}_{Rx}[Z(t) \in A, \tau > t]
= \mathbb{P}_x[Z(t) \in X_1 \cap A, \tau > t] - \mathbb{P}_{Rx}[Z(t) \in X_2 \cap A, \tau > t].
\]

These observations imply that the supremum in (2.5) is attained when \( A = X_1 \). \( \square \)

Proof of Proposition 2.2. By (A2),

\[
\mathbb{P}_M[T(Z_1, Z_2) > t] = \mathbb{P}_{x_1}[\tau > t] = \mathbb{P}_{x_1}[Z(t) \in X_1, \tau > t] - \mathbb{P}_{x_2}[Z(t) \in X_1, \tau > t].
\]

By (A1), we can apply (2.6) for \( x = x_1 \). Thus we obtain

\[
\mathbb{P}_{x_1}[Z(t) \in X_1, \tau > t] - \mathbb{P}_{x_2}[Z(t) \in X_1, \tau > t] = \mathbb{P}_{x_1}[Z(t) \in X_1] - \mathbb{P}_{x_2}[Z(t) \in X_1].
\]

Hence Lemma 2.3 yields the conclusion. \( \square \)
Definition 2.4 Let \( Z^* = (Z_1, Z_2) \) be a coupling of diffusion process \( Z \) starting from \( (x_1, x_2) \) under \( \mathbb{P} \). We define a canonical filtration \( \{ \mathcal{F}_t^* \}_{t \geq 0} \) by \( \mathcal{F}_t^* := \sigma\{Z^*(u) ; 0 \leq u \leq t\} \). We say that \( \mathbb{P} \) is Markovian or \( \mathbb{P} \in \mathcal{C}_0(\mathbb{P}_{x_1}, \mathbb{P}_{x_2}) \) if, for each \( s > 0 \), the shifted process \( \{Z^*(t + s)\}_{t \geq 0} \) under \( \mathbb{P} \) conditioned on \( \mathcal{F}_s^* \) is still a coupling of the diffusion process starting from \( Z^*(s) = (Z_1(s), Z_2(s)) \). By using the shift operators \( \{\theta_s\}_{s > 0} \) defined by \( \theta_s(Z^*)(t) = Z^*(s + t), \mathbb{P} \in \mathcal{C}_0(\mathbb{P}_{x_1}, \mathbb{P}_{x_2}) \) means \( \mathbb{P}[\cdot | \mathcal{F}_s] \circ \theta_s^{-1} \in \mathcal{C}(\mathbb{P}_{Z_1(s)}, \mathbb{P}_{Z_2(s)}) \) for each \( s > 0 \).

Obviously, the mirror coupling \( \mathbb{P}_M \) is Markovian. As noted in [6], the condition that \( Z^* \) is a Markovian coupling does not imply that \( Z^* \) is a Markov process in general.

To state our main theorem, we introduce a subclass of \( \mathcal{C}(\mathbb{P}_{x_1}, \mathbb{P}_{x_2}) \).

Definition 2.5 We say \( \mathbb{P} \in \hat{\mathcal{C}}(\mathbb{P}_{x_1}, \mathbb{P}_{x_2}) \) when, for each \( t > 0 \), there is \( \Xi(t) \in \mathcal{B}(X \times X) \) with \( \Xi(t) \subset X_1 \times X_2 \) and \( \mathbb{P}(Z^*(t) \in (\Xi(t)) \cap X_1 \times X_2) = 0 \) so that each \( (x, y) \in \Xi(t) \) satisfies the following: if there is a decreasing sequence \( \{s_n\} \) of positive numbers with \( \lim_{n \to \infty} s_n = 0 \) so that

\[
\begin{align*}
\mathbb{P}_x[Z(s_n) \in A] &\geq \mathbb{P}_y[Z(s_n) \in A], \\
\mathbb{P}_x[Z(s_n) \in A'] &\leq \mathbb{P}_y[Z(s_n) \in A']
\end{align*}
\]

hold for all \( A \subset X_1 \cup H, A' \subset X_2 \cup H \) and all \( n \in \mathbb{N} \), then \( x = R_y \).

We can easily verify \( \mathbb{P}_M \in \hat{\mathcal{C}}(\mathbb{P}_{x_1}, \mathbb{P}_{x_2}) \).

Theorem 2.6 Assume (A1) and (A2) for \( x_1, x_2 \in X \). Let \( \mathbb{P} \in \hat{\mathcal{C}}(\mathbb{P}_{x_1}, \mathbb{P}_{x_2}) \cap \mathcal{C}_0(\mathbb{P}_{x_1}, \mathbb{P}_{x_2}) \). If there is \( t_0 > 0 \) so that \( \mathbb{P} \) is maximal at every \( t \in (0, t_0) \), then the law of \( Z^*(t \wedge t_0) \) under \( \mathbb{P} \) is identical to that under \( \mathbb{P}_M \). In particular, if \( \mathbb{P} \in \hat{\mathcal{C}}(\mathbb{P}_{x_1}, \mathbb{P}_{x_2}) \cap \mathcal{C}_0(\mathbb{P}_{x_1}, \mathbb{P}_{x_2}) \) is maximal, then \( \mathbb{P} = \mathbb{P}_M \). As a result, if

\[
\hat{\mathcal{C}}(\mathbb{P}_{x_1}, \mathbb{P}_{x_2}) \supset \mathcal{C}_0(\mathbb{P}_{x_1}, \mathbb{P}_{x_2}),
\]

then \( \mathbb{P}_M \) is the unique maximal coupling in \( \mathcal{C}_0(\mathbb{P}_{x_1}, \mathbb{P}_{x_2}) \).

Remark 2.7 (i) The conditions (2.7) and (2.8) are equivalent to the fact that a Hahn decomposition of \( \mathbb{P}_x \circ Z(s_n)^{-1} - \mathbb{P}_y \circ Z(s_n)^{-1} \) is given by \( X_1 \) or \( X_1 \cup H \) for each \( n \in \mathbb{N} \).
(ii) We can directly show that the Brownian motion on a Euclidean space satisfy (2.9). Indeed, for every \( x, y \in \mathbb{R}^d, \bar{X} = \{z \in \mathbb{R}^d ; |z - x| \leq |z - y|\} \) gives a Hahn decomposition of \( \mathbb{P}_x \circ Z(s)^{-1} - \mathbb{P}_y \circ Z(s)^{-1} \) for each \( s > 0 \). This is because the transition density depends only on the distance for fixed \( t > 0 \). (iii) In the case of Euclidean Brownian motion, more strong assertion holds: the maximality of \( \mathbb{P} \) only at \( t > 0 \) implies that the law of \( Z^*(\cdot \wedge t) \) under \( \mathbb{P} \) is identical to that under \( \mathbb{P}_M \) (see [6]). But their proof requires some properties derived from the explicit form of the transition density of the Euclidean Brownian motion.
3 Proof of Theorem 2.6

To begin with, we remark that (A1) produces the following auxiliary lemma.

Lemma 3.1 For each $t > 0$, there is $\tilde{X}^{(t)} \in \mathcal{B}(X)$ with $\mathbb{P}_{x_1}[Z(t) \in \tilde{X}^{(t)}] = 1$ so that $\mathbb{P}_x \circ R^{-1} = \mathbb{P}_{R_x}$ for $x \in \tilde{X}^{(t)}$.

Proof. Take $A_i \in \mathcal{B}(X)$ for $i = 0, \ldots, n$ and $0 < s_1 < s_2 < \cdots < s_n$. Then the Markov property implies

$$\mathbb{P}_{x_1}[Z(t) \in A_0, Z(t + s_1) \in A_1, \ldots, Z(t + s_n) \in A_n] = \mathbb{E}_{x_1}[1_{A_0}(Z(t)) \cdot \mathbb{P}_{Z(t)}[Z(s_1) \in A_1, \ldots, Z(s_n) \in A_n]].$$

By using (A1) twice,

$$\mathbb{P}_{x_1}[Z(t) \in A_0, Z(t + s_1) \in A_1, \ldots, Z(t + s_n) \in A_n]$$

$$= \mathbb{P}_{x_2}[Z(t) \in R^{-1}A_0, Z(t + s_1) \in R^{-1}A_1, \ldots, Z(t + s_n) \in R^{-1}A_n]$$

$$= \mathbb{E}_{x_2}[1_{R^{-1}A_0}(Z(t)) \cdot \mathbb{P}_{RZ(t)}[Z(s_1) \in R^{-1}A_1, \ldots, Z(s_n) \in R^{-1}A_n]]$$

$$= \mathbb{E}_{x_1}[1_{A_0}(Z(t)) \cdot \mathbb{P}_{RZ(t)}[Z(s_1) \in R^{-1}A_1, \ldots, Z(s_n) \in R^{-1}A_n]].$$

Since $A_0$ is arbitrary, there is $\tilde{X}_{s_1, \ldots, s_n; A_1, \ldots, A_n} \in \mathcal{B}(X)$ with $\mathbb{P}_{x_1}[Z(t) \in \tilde{X}_{s_1, \ldots, s_n; A_1, \ldots, A_n}] = 1$ so that

$$\mathbb{P}_x[Z(s_1) \in A_1, \ldots, Z(s_n) \in A_n] = \mathbb{P}_{R_x}[Z(s_1) \in R^{-1}A_1, \ldots, Z(s_n) \in R^{-1}A_n]$$

for $x \in \tilde{X}_{s_1, \ldots, s_n; A_1, \ldots, A_n}$. Since $X$ enjoys the second countability axiom, there is a countable family of open sets $\mathcal{U}$ in $X$ so that $\sigma(\mathcal{U}) = \mathcal{B}(X)$. Thus

$$\tilde{X}^{(t)} = \bigcap_{n \in \mathbb{N}} \bigcap_{s_i \in \mathbb{Q}} \bigcap_{1 \leq i \leq n} \tilde{X}_{s_1, \ldots, s_n; A_1, \ldots, A_n}$$

is what we desired. □

Remark 3.2 In this paper, we used the second countability axiom of $X$ only for the proof of Lemma 3.1. Thus, if $\mathbb{P}_x \circ R^{-1} = \mathbb{P}_{R_x}$ holds for all $x \in X$, then $X$ need not satisfy it.

We write $\mu^t_1 = \mathbb{P}_{x_1} \circ Z(t)^{-1}$ and $\mu^t_2 = \mathbb{P}_{x_2} \circ Z(t)^{-1}$ for simplicity. Let us define $\mu^t_0$ by

$$\mu^t_0(A) = \mu^t_2(A \cap X^t_1) + \mu^t_1(A \cap X^t_1)$$

(3.1)

for each $A \in \mathcal{B}(X)$. By Lemma 2.3 we have $\mu^t_0 \leq \mu^t_1$ and $\mu^t_0 \leq \mu^t_2$.

Definition 3.3 For $t > 0$, the mirror coupling $\mu^t_M \in \mathcal{C}(\mu^t_1, \mu^t_2)$ is the probability measure on $X \times X$ defined by

$$\mu^t_M(dx,dy) = \delta_x(dy)\mu^t_0(dx) + \delta_{R_x}(dy)(\mu^t_1 - \mu^t_0)(dx).$$

(3.2)
Lemma 3.4 Let $s, t > 0$. Then for $x, y \in X$,

$$\inf \left\{ \int_{X \times X} \varphi_s(z_1, z_2) \nu(dz_1dz_2) ; \nu \in \mathcal{C}(\mathbb{P}_x \circ Z(t)^{-1}, \mathbb{P}_y \circ Z(t)^{-1}) \right\} \geq \varphi_{s+t}(x, y). \quad (3.3)$$

In particular, the equality holds when $(x, y) = (x_1, x_2)$. In this case, the infimum is attained at $\mu_M^t$.

**Proof.** Let $u_{s,t}(z) := \mathbb{P}_z[Z(t) \in E]$ for $E \in \mathcal{B}(X)$. Let $\mu^t \in \mathcal{C}(\mathbb{P}_x \circ Z(t)^{-1}, \mathbb{P}_y \circ Z(t)^{-1})$. Then

$$u_{s+t,E}(x) - u_{s+t,E}(y) = \mathbb{E}_x[u_{s,E}(Z(t))] - \mathbb{E}_y[u_{s,E}(Z(t))]$$

$$= \int_{X \times X} \{u_{s,E}(z_1) - u_{s,E}(z_2)\} \, d\mu^t(dz_1dz_2)$$

$$\leq \int_{X \times X} \varphi_s(z_1, z_2) \, d\mu^t(dz_1dz_2).$$

By taking the supremum on $E \in \mathcal{B}(X)$ in the left hand side of the above inequality, we obtain (3.3). We now turn to the latter assertion. We set $x = x_1$ and $y = x_2$. By (3.2), we have

$$\int_{X \times X} \varphi_s(z_1, z_2) \, d\mu_M^t(dz_1dz_2) = \int_X \varphi_s(z, Rz) \mu_1^t(dz) - \int_X \varphi_s(z, Rz) \mu_0^t(dz). \quad (3.4)$$

Set $u_t(z) = u_{t,x_1}(z)$. Let $\tilde{X}^{(t)}$ be as in Lemma 3.1. By Lemma 2.3, we obtain

$$\varphi_s(z, Rz) = u_s(z) - u_s(Rz)$$

for $z \in X_1 \cap (\tilde{X}^{(t)} \cup R\tilde{X}^{(t)})$. Thus

$$\int_X \varphi_s(x, Rx) \mu_0^t(dx) = \int_{X_1} \varphi_s(x, Rx) \mu_1^t(dx) + \int_{X_2} \varphi_s(x, Rx) \mu_2^t(dx). \quad (3.5)$$

Substituting (3.6) to (3.4), we obtain

$$\int_{X \times X} \varphi_s(x, y) \mu_M^t(dxdy) = \int_{X_1} \varphi_s(x, Rx) \mu_1^t(dx) - \int_{X_1} \varphi_s(x, Rx) \mu_2^t(dx)$$

$$= \int_{X_1 \cap \tilde{X}^{(t)}} \varphi_s(x, Rx) \mu_1^t(dx) - \int_{X_1 \cap R\tilde{X}^{(t)}} \varphi_s(x, Rx) \mu_2^t(dx)$$

$$= \int_{X_1} \{u_s(x) - u_s(Rx)\} \mu_1^t(dx) - \int_{X_1} \{u_s(x) - u_s(Rx)\} \mu_2^t(dx)$$

$$= \int_X u_s(x) \mu_1^t(dx) - \int_X u_s(x) \mu_2^t(dx)$$

$$= u_{s+t}(x_1) - u_{s+t}(x_2)$$

$$= \varphi_{s+t}(x_1, x_2).$$

Here the third equality follows from (3.5). \qed
In the following, we show a kind of converse assertion.

**Proposition 3.5** Let $\mathbb{P} \in \mathcal{C}(\mathbb{P}_{x_1}, \mathbb{P}_{x_2})$ and $t > 0$. Suppose that there is a sequence $\{s_n\}_{n \in \mathbb{N}}$ so that

$$E[\varphi_{s_n}(Z_1(t), Z_2(t))] = \varphi_{s_n+t}(x_1, x_2)$$

holds for all $n \in \mathbb{N}$. Then $\mathbb{P} \circ (Z_1(t), Z_2(t))^{-1} = \mu_M'$.

Let $D := \{(x, x) \in X \times X : x \in X\}$ and $t : X \to D$ a canonical injection. For the proof of Proposition 3.5, we show the following lemma.

**Lemma 3.6** Let $\mathbb{P} \in \mathcal{C}(\mathbb{P}_{x_1}, \mathbb{P}_{x_2})$ and $t > 0$. Suppose that there is a sequence $\{s_n\}_{n \in \mathbb{N}}$ so that

$$E[\varphi_{s_n}(Z_1(t), Z_2(t))] = \varphi_{s_n+t}(x_1, x_2)$$

holds for all $n \in \mathbb{N}$. Then $\mathbb{P} \circ (Z_1(t), Z_2(t))^{-1}|_D = \mu_0 \circ t^{-1}$.

**Proof.** Set $\mu := \mathbb{P} \circ (Z_1(t), Z_2(t))^{-1}$. For simplicity, we write $\mu_i' =: \mu_i$ for $i = 0, 1, 2$. By a usual argument, $\mu$ is expressed in the following forms:

$$\mu(dx, dy) = k_1(x, dy)\mu_1(dx) = k_2(y, dx)\mu_2(dy).$$

We define a coupling $\nu \in \mathcal{C}(\mu_1, \mu_2)$ by

$$\nu(dx, dy) = \frac{1}{2}\delta_x(dy)\mu_0(dx) + \frac{1}{2}\int_X k_2(z, dx)\mu_0(dz) - \frac{1}{2}k_1(x, dy)\mu_0(dx) - \frac{1}{2}k_2(y, dx)\mu_0(dy) + \mu(dx, dy)$$

By (3.3) and (3.7), for $s \in \{s_n\}_{n \in \mathbb{N}}$, we have

$$0 \leq \int_{X \times X} \varphi_s(x, y)\nu(dx, dy) - \int_{X \times X} \varphi_s(x, y)\mu(dx, dy)$$

$$= \frac{1}{2}\int_{X \times X \times X} \varphi_s(x, y)k_2(z, dx)k_1(z, dy)\mu_0(dz) - \frac{1}{2}\int_X \varphi_s(x, y)k_1(x, dy)\mu_0(dx) - \frac{1}{2}\int_X \varphi_s(x, y)k_2(y, dx)\mu_0(dy)$$

$$= \frac{1}{2}\int_{X \times X \times X} \{\varphi_s(x, y) - \varphi_s(x, z) - \varphi_s(z, y)\}k_2(z, dx)k_1(z, dy)\mu_0(dz). \quad (3.8)$$

By the triangular inequality for $\|\cdot\|_{\text{var}}$, we have $\varphi_s(x, y) \leq \varphi_s(x, z) + \varphi_s(z, y)$. Thus the left hand side of (3.8) must be 0. Moreover, there is $\Omega_s \subset X \times X \times X$ with $\int_{\Omega_s} k_2(z, dx)k_1(z, dy)\mu_0(dz) = 0$ so that

$$\varphi_s(x, y) = \varphi_s(x, z) + \varphi_s(z, y)$$

holds for each $(x, y, z) \in \Omega_s$. Note that this equality is equivalent to the existence of a Borel subset $E_s(x, y, z) \subset X$ which satisfies

$$\mathbb{P}_x[Z(t) \in A] \leq \mathbb{P}_z[Z(t) \in A] \leq \mathbb{P}_y[Z(t) \in A]$$
for each Borel set $A \subset E^{(x,y,z)}_s$ and

$$\mathbb{P}_y[Z(t) \in A'] \leq \mathbb{P}_x[Z(t) \in A'] \leq \mathbb{P}_x[Z(t) \in A].$$

for each Borel set $A' \subset (E^{(x,y,z)}_s)$. This fact follows from a simple calculation of the total variation norm by using Hahn decompositions. Let $\Omega := \bigcap_{n \in \mathbb{N}} \Omega_n$. We set

$$A_1 = \{(x, y, z) \in X \times X \times X ; x = z\},$$

$$A_2 = \{(x, y, z) \in X \times X \times X ; y = z\}.$$ 

Then we claim

$$\Omega \subset A_1 \cup A_2. \quad (3.9)$$

Let $(z_1, z_2, z_3) \in \Omega$. Suppose $(z_1, z_2, z_3) \notin A_1 \cup A_2$. Take open neighborhoods $V_i$ of $z_i$ ($i = 1, 2, 3$) with $V_i \cap V_3 = \emptyset$ for $i = 1, 2$. We choose $n \in \mathbb{N}$ sufficiently large so that $\mathbb{P}_{z_i}[Z(s_n) \in V_i] \geq 3/4$ for $i = 1, 2, 3$. But, for $E_{s_n} = E_{s_n}^{(z_1, z_2, z_3)}$, we have

$$\frac{3}{4} \leq \mathbb{P}_{z_3}[Z(s_n) \in V_3] = \mathbb{P}_{z_3}[Z(s_n) \in V_3 \cap E_{s_n}] + \mathbb{P}_{z_3}[Z(s_n) \in V_3 \cap E_{s_n}^c]$$

$$\leq \mathbb{P}_{z_2}[Z(s_n) \in V_3 \cap E_{s_n}] + \mathbb{P}_{z_1}[Z(s_n) \in V_3 \cap E_{s_n}^c]$$

$$\leq \frac{1}{2}.$$ 

Of course it is absurd. Now (3.9) yields

$$\mu_0(X) = \int_{\Omega} k_1(z, dy)k_2(z, dx)\mu_0(dz)$$

$$\leq \int_{A_1} k_1(z, dy)k_2(z, dx)\mu_0(dz) + \int_{A_2} k_1(z, dy)k_2(z, dx)\mu_0(dz)$$

$$- \int_{A_1 \cap A_2} k_1(z, dy)k_2(z, dx)\mu_0(dz)$$

$$= \mu_0(X) - \int_X (1 - k_1(z, \{z\})) (1 - k_2(z, \{z\})) \mu_0(dz).$$

This equality asserts that there is $\tilde{\Omega} \in \mathscr{B}(X)$ with $\mu_0(\tilde{\Omega}^c) = 0$ so that $k_1(x, \{x\}) = 1$ or $k_2(x, \{x\}) = 1$ holds for all $x \in \tilde{\Omega}$. Set $\Omega_1 := \{x \in X ; k_1(x, \{x\}) = 1\}$. Let $\iota : X \to X \times X$ be given by $\iota(x) = (x, x)$. For $A \in \mathscr{B}(X)$, (3.1) yields

$$\mu_0(A) = \mu_2(A \cap X_1) + \mu_1(A \cap X_1^c)$$

$$\geq \int_{A \cap X_1} k_2(z, \{z\})\mu_2(dz) + \int_{A \cap X_1^c} k_1(z, \{z\})\mu_1(dz)$$

$$= \mu(\iota(A \cap X_1)) + \mu(\iota(A \cap X_1^c))$$

$$= \mu(\iota(A)).$$
Thus, for \( x, y \),

\[
\tilde{\mu}(\Omega^c) = 0. \quad \text{Thus we have}
\]

\[
\mu(\iota(A)) = \mu(\iota(A \cap \tilde{\Omega})) + \mu(\iota(A \cap \tilde{\Omega}^c \cap \tilde{\Omega}))
\]

\[
= \int_{A \cap \tilde{\Omega}} k_1(z, \{z\})\mu_1(dz) + \int_{A \cap \tilde{\Omega}^c \cap \tilde{\Omega}} k_2(z, \{z\})\mu_2(dz)
\]

\[
= \mu_1(A \cap \tilde{\Omega}) + \mu_2(A \cap \tilde{\Omega}^c \cap \tilde{\Omega})
\]

\[
\geq \mu_0(A).
\]

Thus we obtain \( \mu|_D = \mu_0 \circ \iota^{-1} \).

\[\square\]

**Proof of Proposition 3.5.** We use the same notation as in the proof of Lemma 3.6. We denote \( \hat{\mu} = \mu - \mu_0 \circ \iota^{-1} \). Note that \( \hat{\mu} \) is positive and it is absolutely continuous with respect to \( \mu \) by Lemma 3.6. In order to derive \( \mu = \mu^t_M \), we consider the integration of \( \varphi_s \) by \( \mu^t_M \) for \( s \in \{s_n\}_{n \in \mathbb{N}} \):

\[
\int_{X \times X} \varphi_s(x, y)\mu^t_M(dxdy) = \int_X \varphi_s(x, Rx)\mu_1(dx) - \int_X \varphi_s(x, Rx)\mu_0(dx)
\]

\[
= \int_{X \times X} \varphi_s(x, Rx)\mu(dx) - \int_{X \times X} \varphi_s(x, Ry)\mu_0 \circ \iota^{-1}(dxdy)
\]

\[
= \int_{X \times X} \varphi_s(x, Rx)\hat{\mu}(dxdy).
\]

By virtue of (A1), we also obtain

\[
\int_{X \times X} \varphi_s(x, y)\mu^t_M(dxdy) = \int_{X \times X} \varphi_s(y, Ry)\hat{\mu}(dxdy).
\]

By Lemma 3.4

\[
0 = \int_{X \times X} \varphi_s(x, y)\mu^t_M(dxdy) - \int_{X \times X} \varphi_s(x, y)\mu(dx)
\]

\[
= \frac{1}{2} \int_{X \times X} \{\varphi_s(x, Rx) + \varphi_s(y, Ry) - 2\varphi_s(x, y)\} \hat{\mu}(dxdy). \quad (3.10)
\]

By (A1) and (3.11), \( \hat{\mu}(X_1 \times X) = \hat{\mu}(X \times X_1) = 0 \). This fact together with (3.10) yields

\[
\int_{X_1 \times X^c_1} \{\varphi_s(x, Rx) + \varphi_s(y, Ry) - 2\varphi_s(x, y)\} \hat{\mu}(dxdy) = 0. \quad (3.11)
\]

Let \( \tilde{X}^{(t)} \) be as given in Lemma 3.7. For \( z \in X_1 \cap \tilde{X}^{(t)} \),

\[
\varphi_s(z, Rz) = \mathbb{P}_z[Z(s) \in X_1] - \mathbb{P}_{Rz}[Z(s) \in X_1]
\]

\[
= \mathbb{P}_z[Z(s) \in X_1] - \mathbb{P}_z[Z(s) \in X_2].
\]

Thus, for \( x, y \in (X_1 \times X^c_1) \cap (\tilde{X}^{(t)} \times R\tilde{X}^{(t)}) \),

\[
\varphi_s(x, Rx) + \varphi_s(y, Ry) - 2\varphi_s(x, y)
\]

\[
= \mathbb{P}_x[Z(s) \in A] - \mathbb{P}_y[Z(s) \in A] + \mathbb{P}_y[Z(s) \in X_2] - \mathbb{P}_x[Z(s) \in X_2]
\]

\[
- 2 \sup_{A \in \mathcal{F}(X)} |\mathbb{P}_x[Z(s) \in A] - \mathbb{P}_y[Z(s) \in A]|.
\]

\[
\leq 0. \quad (3.12)
\]
Note that
\[ \mu((\tilde{X}^{(t)} \times R\tilde{X}^{(t)})^c) \leq \mu_1((\tilde{X}^{(t)})^c) + \mu_2((R\tilde{X}^{(t)})^c) = 0 \]

since \( \mu \in \mathcal{C}(\mu_1, \mu_2) \). Hence there is \( \tilde{E}_s \subset (X_1 \times X_1^c) \cap (\tilde{X}^{(t)} \times R\tilde{X}^{(t)}) \) with \( \tilde{\mu}(\tilde{E}_s) = 0 \) so that the equality holds in (3.12) for \( (x, y) \in \tilde{E}_s \). Let \( E = \Xi^{(t)} \cap (\bigcap_{n \in \mathbb{N}} \tilde{E}_n) \). Here \( \Xi^{(t)} \) is given in Definition 2.5 associated with \( P \in \hat{\mathcal{C}}(P_{x_1}, P_{x_2}) \). Then \( \tilde{\mu}(E^c) = 0 \) and
\[ \frac{1}{2} \|P_x \circ Z(s_n)^{-1} - P_y \circ Z(s_n)^{-1}\|_{\text{var}} = P_x[Z(s_n) \in X_1] - P_y[Z(s_n) \in X_1] \]
\[ = P_y[Z(s_n) \in X_2] - P_x[Z(s_n) \in X_2] \]

for all \( (x, y) \in E \) and \( n \in \mathbb{N} \). Hence the property of \( \Xi^{(t)} \) immediately implies \( x = R_y \) for every \( (x, y) \in E \) (cf. Remark 2.7(i)). It yields \( \mu = \mu_M^t \). \( \square \)

**Proof of Theorem 2.6.** Let \( t_0 \in (0, \infty] \) and \( P \in \hat{\mathcal{C}}(P_{x_1}, P_{x_2}) \cap \mathcal{C}_0(P_{x_1}, P_{x_2}) \) maximal at each \( t \in (0, t_0) \). Note that
\[ T(Z_1, Z_2) = \inf\{s > 0 ; Z_1(s) = Z_2(s)\} \quad \text{P-a.s.} \tag{3.13} \]
holds since \( P \) is maximal. Take \( s, t > 0 \) with \( s + t < t_0 \). By the maximality of \( P \) at \( s + t \),
\[ \varphi_{s+t}(x_1, x_2) = P[T(Z_1, Z_2) > s + t] = \mathbb{E}[P[T(Z_1, Z_2) > s + t | \mathcal{F}_t^s]]. \]

Since \( P \in \mathcal{C}_0(P_{x_1}, P_{x_2}) \), (2.23) yields \( P[T(Z_1, Z_2) > s + t | \mathcal{F}_t^s] \geq \varphi_s(Z_1(t), Z_2(t)) \). In addition, by (3.3), \( \mathbb{E}[\varphi_s(Z_1(t), Z_2(t))] \geq \varphi_{s+t}(x_1, x_2) \). Hence we obtain
\[ \mathbb{E}[\varphi_s(Z_1(t), Z_2(t))] = \varphi_{s+t}(x_1, x_2). \]

Letting \( s \to 0 \), Proposition 3.5 yields \( P \circ (Z_1(t), Z_2(t))^{-1} = \mu_M^t \). Since \( t \in (0, t_0) \) is arbitrary, it implies that
\[ P[Z_2(t) = Z_1(t) \lor Z_2(t) = RZ_1(t) \text{ for all } t \in (0, t_0)] = 1. \]

Recall that \( \tau \) is the first hitting time of \( Z_1 \) to \( H \). The above equality implies that \( \tau \) equals the first hitting time of \( Z_2 \) to \( H \) \( \mathbb{P} \)-almost surely. In addition, by (A2), for each \( t \in (0, t_0) \),
\[ \{t \leq \tau\} \subset \{Z_2(t) = RZ_1(t)\} \quad \text{P-a.s.}. \tag{3.14} \]

Thus it suffices to show that
\[ \{\tau < t\} \subset \{Z_2(t) = Z_1(t)\} \quad \text{P-a.s.} \tag{3.15} \]

Note that (3.2) implies \( P[Z_1(t) = Z_2(t)] = \mu_0^t(X) \). By the maximality of \( P \), Lemma 2.3 and (3.1),
\[ P[T(Z_1, Z_2) \leq t] = 1 - \varphi_t(x_1, x_2) = 1 - P_{x_1}[Z(t) \in X_1] + P_{x_2}[Z(t) \in X_1] \]
\[ = P_{x_1}[Z(t) \in X_1^c] + P_{x_2}[Z(t) \in X_1] \]
\[ = \mu_0^t(X). \]
Uniqueness of maximal coupling

It means
\[ P[Z_1(t) = Z_2(t)] = P[T(Z_1, Z_2) \leq t]. \tag{3.16} \]

Since we have (3.14),
\[ \{Z_1(t) = Z_2(t)\} \subset \{\tau < t\} \quad \text{P-a.s.,} \]
\[ \{\tau < t\} \subset \{T(Z_1, Z_2) < t\} \quad \text{P-a.s..} \]

The second inclusion follows from (3.13). Combining them with (3.16), we obtain (3.15) and it completes the proof. \( \square \)

4 Examples and counterexamples

Let us consider several examples of \( X \) and \( Z \) with (A1) and (A2) for given \( x_1, x_2 \in X \).

First we state a sufficient condition for (2.9) to be satisfied. A key ingredient is the Varadhan type short time asymptotic behavior of transition probabilities (4.1). In order to state it in a general form, we introduce some terms concerning the metric geometry.

Let (\( X, d \)) be a metric space. We call a curve \( \gamma : [0, 1] \rightarrow X \) geodesic if, for each \( t \in [0, 1] \), there exists \( \delta > 0 \) so that \( d(\gamma(t), \gamma(s)) = |t - s|d(\gamma(0), \gamma(1)) \) for \( |t - s| < \delta \). Recall that a metric space \( (X, d) \) is geodesic when, for each \( x, y \in X \), there is a rectifiable curve in \( X \) whose length realizes \( d(x, y) \). Note that such a curve always becomes a geodesic by a suitable re-parameterization. We call it a minimal geodesic joining \( \gamma(0) \) and \( \gamma(1) \). A geodesic metric space \( (X, d) \) is said to be non-branching when, for any two geodesics \( \gamma \) and \( \gamma' \) of the same length with \( \gamma(0) = \gamma'(0) \), we have \( \inf \{t > 0; \gamma(t) \neq \gamma'(t)\} = 0 \) or \( \infty \). Here we follow the usual manner \( \inf \emptyset = \infty \).

**Theorem 4.1** Let \( (X, d) \) be a non-branching geodesic metric space and \( (Z, P_x) \) a diffusion process on it. Suppose that (A1) and (A2) hold for given \( x_1, x_2 \in X \). In addition, we assume the following for each \( t > 0 \):

(i) the support of the law of \( Z(t) \) under \( P_{x_1} \) equals \( X \),

(ii) there exist

\begin{itemize}
\item an increasing function \( \rho : (0, \infty) \rightarrow (0, \infty) \) with \( \lim_{s \rightarrow 0} \rho(s) = 0 \),
\item a strictly increasing function \( \Psi : [0, \infty) \rightarrow [0, \infty) \),
\item a sequence \( \{s_n\}_{n \in \mathbb{N}} \) of positive numbers with \( \lim_{n \rightarrow \infty} s_n = 0 \) and
\item \( Y(t) \in \mathcal{B}(X) \) with \( RY(t) = Y(t) \) and \( P_{x_1}[Z(t) \notin Y(t)] = 0 \)
\end{itemize}

so that
\[ -\lim_{n \rightarrow \infty} \rho(s_n) \log P_x[Z(s_n) \in A] = \Psi(d(x, A)) \quad \text{for each} \ A \in \mathcal{B}(X) \tag{4.1} \]

holds for \( x \in Y(t) \).

Then (2.9) holds.
Proof. Take \( z \in \tilde{X}(t) \cap Y(t) \) for \( t > 0 \), where \( \tilde{X}(t) \) is as in Lemma 3.1. Then \( \mathbb{P}_z [Z(s) \in A] = \mathbb{P}_{Rz} [Z(s) \in RA] \) holds for each \( A \in \mathcal{B}(X) \). Thus (4.1) yields \( d(z, A) = d(Rz, RA) \). By taking \( A = B_r(w) \), a ball of radius \( r > 0 \) centered at \( w \in X \), and taking \( r \to 0 \), we obtain \( d(z, w) = d(Rz, Rw) \). Since \( R \) is continuous, the condition (i) implies that \( R \) acts on \( X \) as isometry. Let \( x \in Y(t) \cap X_1 \) and \( y \in Y(t) \cap X_2 \). Suppose (2.7) and (2.8) holds. Take \( z \in X_1 \). Since \( X_1 \) is open, \( B_r(z) \subset X_1 \) for all sufficiently small \( r > 0 \). For such \( r \), (4.1) and (2.4) implies that,
\[
\Psi(d(x, B_r(z))) = - \lim_{n \to \infty} \rho(s_n) \log \mathbb{P}_y [Z(s_n) \in B_r(z)] \\
\leq - \lim_{n \to \infty} \rho(s_n) \log \mathbb{P}_z [Z(s_n) \in B_r(z)] \\
= \Psi(d(y, B_r(z))).
\]
(4.2)
It immediately implies \( d(x, B_r(z)) \leq d(y, B_r(z)) \). By taking \( r \to 0 \), we obtain \( d(x, z) \leq d(y, z) \). In the same way, for \( z \in X_2 \), we obtain \( d(x, z) \geq d(y, z) \). These two estimates yield
\[
d(x, z) = d(y, z) \text{ for } z \in H.
\]
(4.3)
Take a minimal geodesic \( \gamma_0 : [0, 1] \to X \) joining \( x \) and \( y \). By the remark after (A2), there exists \( t_0 \in [0, 1] \) so that \( \gamma_0(t_0) \in H \). By (4.3), we have \( d(x, \gamma_0(t_0)) = d(y, \gamma_0(t_0)) \).
In addition, \( t_0 = 1/2 \) follows. Again by (4.3), \( d(x, \gamma_0(1/2)) = d(x, H) \) holds. Let \( \gamma_1 \) be a curve joining \( x \) and \( Rx \) given by
\[
\gamma_1(t) = \begin{cases} 
\gamma_0(t) & \text{if } t \in [0, 1/2], \\
R(\gamma_0(1-t)) & \text{if } t \in (1/2, 1]. 
\end{cases}
\]
Then \( \gamma_1 \) is a minimal geodesic joining \( x \) and \( Rx \) because \( d(x, \gamma_1(1/2)) = d(Rx, \gamma_1(1/2)) = d(x, H) \). Since \( X \) is non-branching, we obtain \( Rx = y \). Thus, once we set \( \Xi(t) = (Y(t) \cap X_1) \times (Y(t) \cap X_2) \), \( \mathbb{P}[Z^*(t) \in (\Xi(t))^{c} \cap X_1 \times X_2] = 0 \) holds for each \( \mathbb{P} \in C(P_{x_1}, P_{x_2}) \). This means \( \mathcal{E}(P_{x_1}, P_{x_2}) = \mathcal{E}(P_{x_1}, P_{x_2}) \) and therefore (2.9) holds. \( \square \)

Remark 4.2 If our diffusion process \( Z(t) \) has a continuous transition density \( p_t(x, y) \) with respect to a Radon measure \( m \), that is, \( \mathbb{P}_x[Z(t) \in A] = \int_A p_t(x, y) m(dy) \), then (4.1) in Theorem 4.1 is replaced as follows:
\[
- \lim_{n \to \infty} \rho(s_n) \log p_{s_n}(x, y) = \Psi(d(x, y)) \text{ for each } y \in X.
\]
(4.4)
Indeed, (2.7) and (2.8) imply that \( p_{s_n}(x, z) \geq p_{s_n}(y, z) \) for \( z \in X_1 \) and \( p_{s_n}(x, z) \leq p_{s_n}(y, z) \) for \( z \in X_2 \). Thus the same proof works.

Corollary 4.3 Let \( X \) be a complete Riemannian manifold and \( Z(t) \) the Brownian motion on \( X \). Assume \( X \) to satisfy (A1) and (A2). Then (2.9) follows.

Proof. In this case, \( Z \) has a continuous transition density \( p_t(x, y) \). Letting \( \rho(s) := 2s \), \( \Psi(v) = v^2 \) and any sequence \( \{s_n\}_{n \in \mathbb{N}} \) with \( \lim_{n \to \infty} s_n = 0 \), (4.4) follows from [15] for every \( x \in X \) (see Remark 4.2). The condition (i) in Theorem 4.1 comes from strict positivity of the transition density. It is well-known that all properties imposed on \( X \) in Theorem 4.1 hold. \( \square \)
We can also apply Theorem 4.1 to Alexandrov spaces. These metric spaces are a generalization of a complete Riemannian manifold with sectional curvature bounded below (see [2] for details).

**Corollary 4.4** Let \((X,d)\) be an Alexandrov space and \(Z(t)\) a diffusion process on \(X\) corresponding to a canonical regular Dirichlet form on \(X\) constructed in [12] (see [13] also). Assume \(X\) to satisfy (A1) and (A2). Then (2.9) follows.

**Proof.** In this case, there is a continuous transition density \(p_t(x,y)\) of \(Z\). Letting \(\rho(s) = 2s, \Psi(v) = v^2\) and any sequence \(\{s_n\}_{n \in \mathbb{N}}\) with \(\lim_{n \to \infty} s_n = 0\), (4.4) follows from Corollary 2 of [16] for every \(x \in X\) (see Remark 4.2). As in the case of Riemannian manifolds, the condition (i) follows from positivity of the transition density. By definition, \(X\) is a geodesic space. The curvature condition on \(X\) easily implies that \(X\) is non-branching. Thus we can apply Theorem 4.1.

**Remark 4.5** Let \(X\) be a Riemannian manifold and \(Z\) the Brownian motion on it. (i) If (A1) and (A2) are satisfied for given initial points, then the argument in the proof of Theorem 4.1 implies that \(R\) is isometry. In this case, \(H\) is a totally geodesic smooth submanifold of \(X\) (see [10] p.61, for example). In particular, \(H\) becomes a complete Riemannian manifold. In addition, \(H\) is of codimension 1. (ii) If (A2) are satisfied with respect to an isometry \(R\) with \(R \circ R = \text{id}\), then (A1) follows for each \(x_1, x_2 \in X\) with \(R x_1 = x_2\).

In what follows, we will see some manifolds satisfying the conditions (A1) and (A2). In all cases, we assume \(Z\) to be the Brownian motion.

**Example 4.6** We consider the case \(X\) is an irreducible Riemannian global symmetric space of constant curvature. We will review that (A1) and (A2) are satisfied for every distinct pair \(x_1, x_2 \in X\) of starting points in these cases. By Remark 4.5 (ii), It suffices to find an isometry \(R\) with \(R \circ R = \text{id}\), \(R x_1 = x_2\) satisfying (A2). The flat case, i.e. \(X = \mathbb{R}^n\), is considered in [6].

In the case of positive curvature, \(X\) is a sphere:

\[
X = S^n = \{ z = (z_0, z_1, \ldots, z_n) \in \mathbb{R}^{n+1}; \ z_0^2 + \cdots + z_n^2 = r \}
\]

with a metric induced from the canonical metric on \(\mathbb{R}^{n+1}\). Take \(x_1, x_2 \in X\) with \(x_1 \neq x_2\). Then we can easily verify that the restriction of the reflection in \(\mathbb{R}^{n+1}\) with respect to a hyperplane fulfills all of our requirements.

In the case of negative curvature, \(X\) is a hyperbolic space:

\[
X = H^n = \{ z = (z_0, z_1, \ldots, z_n) \in \mathbb{R}^{n+1}; \ -z_0^2 + z_1^2 + \cdots + z_n^2 = -r, z_0 > 0 \}
\]

with a metric induced from the Lorentz metric on \(\mathbb{R}^{n+1}\). Take \(x_1, x_2 \in X\) with \(x_1 \neq x_2\). Let \(m\) be the midpoint of \(x_1\) and \(x_2\). By homogeneity, we may assume \(m = (r, 0, \ldots, 0)\).

By arranging the chart appropriately, we may assume \(x_1 = (z_0, z_1, 0, \ldots, 0)\). Then \(x_2 = (z_0, -z_1, 0, \ldots, 0)\). Set \(R : (z_0, z_1, \ldots, z_n) \mapsto (z_0, -z_1, z_2, \ldots, z_n)\). Then \(R\) fulfills all of our requirements.
Uniqueness of maximal coupling

Remark 4.7 The converse of Example 4.6 is true for an irreducible Riemannian global symmetric space \( X \) in the following sense. If there exists an isometry \( R \) satisfying (A1) and (A2) for some pair \( x_1, x_2 \in X \), then \( X \) must be of a constant curvature. It follows from the result in [7] (cf. Remark 4.5(i)).

Example 4.8 Let us consider 2-dimensional torus \( X = \mathbb{T}^2 = (\mathbb{R}/\sim)^2 \), where \( \sim \) identifies \( x \) with \( x + n \) for each \( x \in \mathbb{R} \) and \( n \in \mathbb{N} \). Let \( \pi : \mathbb{R}^2 \to \mathbb{T}^2 \) be the canonical projection. We denote \( \pi(x) \) by \([x]\). Take \( x_1, x_2 \in X \) with \( x_1 \neq x_2 \). By arranging an appropriate chart, we may assume that \( x_1 = [(a, 0)] \) and \( x_2 = [(0, b)] \) for \( 0 \leq b \leq a \leq 1/2 \). Let \( K = \{ z \in \mathbb{T}^2 ; d(z, x_1) = d(z, x_2) \} \). If (A1) and (A2) are satisfied for \( x_1, x_2 \in X \), then \( R \) is isometry and \( H \subset K \) must hold (cf. Remark 4.5). In the following, we will write \( K \) explicitly. First we consider the case \( b \neq 0 \). Take six points \( z_1, z_2, z_3, z_4, z_5, z_6 \in \mathbb{R}^2 \) as follows:

\[
\begin{align*}
  z_1 &= \left( \frac{1}{2a} \left( a^2 + b^2 - b \right), b - \frac{1}{2} \right), \\
  z_2 &= \left( \frac{1}{2a} \left( a^2 - b^2 + b \right), \frac{1}{2} \right), \\
  z_3 &= \left( \frac{1}{2a} \left( a^2 + b^2 - b \right), b + \frac{1}{2} \right), \\
  z_4 &= \left( \frac{1}{2(1-a)} \left( -a^2 - b^2 + b + 1 \right), b - \frac{1}{2} \right), \\
  z_5 &= \left( \frac{1}{2(1-a)} \left( -a^2 + b^2 - b + 1 \right), \frac{1}{2} \right), \\
  z_6 &= \left( \frac{1}{2(1-a)} \left( -a^2 - b^2 + b + 1 \right), b + \frac{1}{2} \right).
\end{align*}
\]

Let \( l_{ij} \) be a line segment in \( \mathbb{R}^2 \) whose endpoints are \( z_i \) and \( z_j \). Then \( K = \pi(l_{12} \cup l_{23} \cup l_{45} \cup l_{56}) \) holds. We can easily verify that \( K \) has singularity at \([z_2]\) or \([z_5]\) (see Fig.1) and \( H \) cannot be contained in \( K \) by Remark 4.5(i). Thus there is no reflection structure. Next we consider the case \( b = 0 \). Then we have \( K = \pi \left( \{(a/2, q) ; q \in [0, 1]\} \cup \{(1+a)/2, q) ; q \in [0, 1]\} \right) \).
We can easily verify that $\mathcal{E}$ is closable in $L^2(X, dx)$ and its closure defines a Dirichlet form on $X$. Thus we can define the corresponding diffusion process $(\{Z(t)\}_{t \geq 0}, \{\mathbb{P}_x\}_{x \in X})$ (see [H]). By using identity maps $\iota_1 : Y_1 \to Y_2$ and $\iota_2 : Y_2 \to Y_1$, we define $R : X \to X$ by
Uniqueness of maximal coupling

Rx = \nu(x) if x \in Y_i. By definition of R and P_{x_i}(i = 1, 2), (A1) and (A2) holds. In this case, H = \{z_1\} = \{z_2\}. Let us define a map \eta : Y_2 \to Y_2 by \eta(x) = 1 - x. We define a Markovian coupling \mathbb{P} \in \mathcal{C}(P_{x_1}, P_{x_2}) as the law of (Z_1, Z_2) where Z_1 is a copy of (Z, P_{x_1}) and

\[
Z_2(t) = \begin{cases} 
\eta \circ R(Z_1(t)) & \text{if } t \leq \tau, \\
Z_1(t) & \text{if } t > \tau.
\end{cases}
\] (4.5)

Then clearly \hat{\mathbb{P}} \neq P_M and

\[
\hat{\mathbb{P}}[T(Z_1, Z_2) > t] = P_{x_1}[\tau > t] = P_M[T(Z_1, Z_2) > t]
\]

for each t > 0. Thus \hat{\mathbb{P}} is also a maximal Markovian coupling. Note that \eta \circ R also satisfies (A1) and (A2) instead of R in this case.

Example 4.11 (Fig.3) Next example is a tree. The space X, given in Fig.3, is a union of nine copies of the unit interval [0, 1] with some identification of these endpoints. X is naturally regarded as a metric space. As in Example 4.10, we can construct a canonical Dirichlet form and the corresponding diffusion process on X. Let x_1 = p_{11} and x_2 = p_{22}. There is an isometry R : X \to X so that R(p_{11}) = p_{22}, R(p_{12}) = p_{21}, R(p_1) = p_2 and R fixes all other endpoints. Then (A1) and (A2) holds. Let \eta be an isometry so that \eta(p_{21}) = p_{22} and \eta fixes all other endpoints. We define a Markovian coupling \hat{\mathbb{P}} \in \mathcal{C}(P_{x_1}, P_{x_2}) as the law of (Z_1, Z_2) where Z_1 is a copy of (Z, P_{x_1}) and

\[
Z_2(t) = \begin{cases} 
R(Z_1(t)) & \text{if } t \leq \tau_{\{p_1\}}, \\
\eta \circ R(Z_1(t)) & \text{if } \tau_{\{p_1\}} \leq t < \tau_{\{p_0\}}, \\
Z_1(t) & \text{if } t \geq \tau_{\{p_0\}},
\end{cases}
\] (4.6)

where \tau_{\{x\}} is the first hitting time to x. Then clearly \hat{\mathbb{P}} \neq P_M and

\[
\hat{\mathbb{P}}[T(Z_1, Z_2) > t] = P_{x_1}[\tau_{\{p_0\}} > t] = P_M[T(Z_1, Z_2) > t]
\]

for each t > 0. Thus \hat{\mathbb{P}} is also a maximal Markovian coupling. Different from Example 4.10, this example essentially has only one reflection structure.

These examples reveal that maximal Markovian coupling may not be unique if the underlying space is more singular than Riemannian manifolds or Alexandrov spaces. One characteristic property which is common to these examples is the existence of branching geodesics. But, in general, non-branching property of geodesics is not necessary for the uniqueness of maximal Markovian coupling. To see this fact, we consider the Brownian motion on 2-dimensional Sierpinski gasket.

Take three points p_1, p_2, p_3 \in \mathbb{R}^2 with |p_i - p_j| = 1 for all i \neq j. Let us define a contraction map \Psi_i : \mathbb{R}^2 \to \mathbb{R}^2 for i = 1, 2, 3 given by \Psi_i(x) = (x - p_i)/2 + p_i. Obviously, p_i is the unique fixed point of \Psi_i. The Sierpinski gasket is a unique compact set in \mathbb{R}^2 satisfying X = \bigcup_{i=1}^3 \Psi_i(X) (see Fig.4). For detailed properties of the Sierpinski gasket, see [3] for instance. We set V_0 = \{p_1, p_2, p_3\} and V_n = \bigcup_{i=1}^3 \Psi_i(V_{n-1}). The Brownian motion (\{Z(t)\}_{t \geq 0}, \{P_x\}_{x \in X}) on X is given by a suitable scaling limit of a continuous
time random walk on $V_n$ as $n \to \infty$ (see [11,14]). There is a reflection $\hat{R}$ on $\mathbb{R}^2$ so that $\hat{R}(p_1) = p_2$. We denote the fixed points of $\hat{R}$ by $\hat{H}$. The map $\hat{R}$ naturally induces a reflection $R$ on $X$ so that its fixed points $H$ coincides with $X \cap \hat{H}$. Moreover, $X$ and $\{(Z(t))_{t \geq 0}, \{\mathbb{P}_x\}_{x \in X}\}$ fulfills (A1) and (A2) for $x_1 = p_1$ and $x_2 = p_2$. As shown in [9], there is a unique distance $d$ on $X$, called shortest path metric, such that it satisfies

(i) $(X,d)$ becomes a geodesic metric space,

(ii) $d(p_i, p_j) = 1$ for each $i \neq j$,

(iii) $d(z_1, z_2) = 2d(\Psi_i(z_1), \Psi_i(z_2))$ for $z_1, z_2 \in X$ and $i = 1, 2, 3$.

**Theorem 4.12** Let $X$ be the Sierpinski gasket as defined above. Then (2.9) holds.

**Proof.** Let $p_t(x,y)$ be the transition density of the Brownian motion. Then, the main theorem of [11] asserts that, for each $u > 0$ and $x, y \in X$,

$$-\lim_{n \to \infty} \left( \frac{2}{5} \right)^n u^{\frac{1}{d_w-1}} \log p_{(2/5)^n} u(x,y) = d(x,y)^{d_w/(d_w-1)} F \left( \frac{u}{d(x,y)} \right),$$

where $d_w > 2$ is the walk dimension of the Sierpinski gasket and $F$ is an implicitly determined, non-constant, positive continuous function on $(0, \infty)$. For our aim, we need a refined observation on $F$. By the definition of $F$ in [11],

$$F(v) = v^{1/(d_w-1)} \sup_{s > 0} \{ K(s) - vs \},$$

for some positive, concave and real analytic function $K(s)$ on $(0, \infty)$. Thus,

$$d(x,y)^{d_w/(d_w-1)} F \left( \frac{u}{d(x,y)} \right) = u^{1/(d_w-1)} \sup_{s > 0} \{ d(x,y)K(s) - us \}.$$
Since $F$ is continuous on $(0, \infty)$, there is $s_v \in (0, \infty)$ for each $v \in (0, \infty)$ so that $K(s_v) - vs_v = \sup_{s>0} \{K(s) - vs\}$ holds. Indeed, if there exists a sequence $\{s_n\}_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} s_n = \infty$ so that
\[
\lim_{s_n \to \infty} (K(s_n) - vs_n) = \sup_{s>0} \{K(s) - vs\},
\]
then $F(v') = \infty$ for every $v' < v$. These observations imply that, for $0 < a < b$,
\[
\sup_{s>0} \{aK(s) - us\} = aK(s_{a/a}) - us_{a/a} < bK(s_{a/a}) - us_{a/a} \leq \sup_{s>0} \{bK(s) - us\}.
\]

It means that the right hand side in (4.7) is strictly increasing with respect to $d(x, y)$. Thus, the same argument as given in Theorem 4.1 yields that $x \in X_1$ and $y \in X_2$ with (2.7) and (2.8) satisfies
\[
d(x, z) = d(y, z) \text{ for all } z \in H. \tag{4.9}
\]
To complete the proof, we show that (4.9) implies $x = Ry$. It suffices to show that $w = w'$ holds when $w, w' \in X_2$ with $d(w, z) = d(w', z)$ for $z = p_3$ or $z = \Psi_2(p_1)$. In this case, we have
\[
w \in \Psi_2(X) \iff d(w, p_3) \geq 1/2, d(w, \Psi_2(p_1)) \leq 1/2,
w \in \Psi_3(X) \iff d(w, p_3) \leq 1/2, d(w, \Psi_2(p_1)) \geq 1/2.
\]
Thus, $w, w' \in \Psi_2(X)$ or $w, w' \in \Psi_3(X)$. In particular, $w = w' = \Psi_2(p_3)$ if and only if $d(w, p_3) = d(w, \Psi_2(p_1))$. Now we assume $w \in \Psi_2(X) \setminus \Psi_3(X)$. To see the argument below, we easily find that the same argument also works for the case $w, w' \in \Psi_3(X) \setminus \Psi_2(X)$. Since $(X, d)$ is a geodesic space, $d(w, p_3) = d(w, \Psi_2(p_3)) + 1/2$ and therefore $d(w, \Psi_2(p_3)) = d(w', \Psi_2(p_3))$. Then we have
\[
w \in \Psi_2 \circ \Psi_1(X) \iff d(w, \Psi_2(p_1)) \leq 1/4, d(w, \Psi_2(p_3)) \geq 1/4,
w \in \Psi_2 \circ \Psi_2(X) \iff d(w, \Psi_2(p_1)) \geq 1/4, d(w, \Psi_2(p_3)) \geq 1/4,
w \in \Psi_2 \circ \Psi_3(X) \iff d(w, \Psi_2(p_1)) \geq 1/4, d(w, \Psi_2(p_3)) \leq 1/4.
\]
Thus $w, w' \in \Psi_2(\Psi_i(X))$ for some $i \in \{1, 2, 3\}$. In particular, $w = w'$ when $w \in V_2$. Since we have
\[
d(w, \Psi_2(p_1)) = d(w, \Psi_2 \circ \Psi_i(p_1)) + d(\Psi_2 \circ \Psi_i(p_1), \Psi_2(p_1)),
d(w, \Psi_2(p_3)) = d(w, \Psi_2 \circ \Psi_i(p_3)) + d(\Psi_2 \circ \Psi_i(p_3), \Psi_2(p_3))
\]
when $w \in \Psi_2 \circ \Psi_i(X)$, the same argument as above works by replacing $\Psi_2(p_1), \Psi_2(p_3)$ and $\Psi_2(X)$ by $\Psi_2 \circ \Psi_i(p_1), \Psi_2 \circ \Psi_i(p_3)$ and $\Psi_2 \circ \Psi_i(X)$ respectively. When $w \in \bigcup_{n \in \mathbb{N}} V_n$, such a recursive argument ends in a finite step with resulting $w = w'$. When $w \notin \bigcup_{n \in \mathbb{N}} V_n$, we obtain a sequence $\{i_n\}_{n \in \mathbb{N}}$ with $i_n \in \{1, 2, 3\}$ so that $w, w' \in \Psi_{i_1} \circ \Psi_{i_2} \circ \cdots \circ \Psi_{i_n}(X)$ for each $n \in \mathbb{N}$. Since $\bigcap_{n \in \mathbb{N}} \Psi_{i_1} \circ \Psi_{i_2} \circ \cdots \circ \Psi_{i_n}(X)$ is just one point, $w = w'$ follows. \qed
Remark 4.13 As shown in the above proof, (4.7) means that (4.4) holds with \( \rho(s) = s^{1/(d_w-1)} \), \( s_n = u(2/5)^n \) and \( \Psi(v) = v^{d_w/(d_w-1)} F(u/v) \). Thus the Sierpinski gasket satisfies all assumption in Theorem 4.1 except for being non-branching. For example, we consider two minimal geodesics \( \gamma_1 \) and \( \gamma_2 \). \( \gamma_1 \) joins \( p_3 \) and \( p_2 \). \( \gamma_2 \) joins \( p_3 \) and \( \Psi_2(p_1) \) via \( \Psi_2(p_3) \). Then both of \( \gamma_1 \) and \( \gamma_2 \) contains the minimal geodesic joining \( p_3 \) and \( \Psi_2(p_3) \). Thus Theorem 4.12 is not a direct consequence of Theorem 4.1.

5 Kendall-Cranston coupling

Let \( X \) be a \( d \)-dimensional complete Riemannian manifold and \( \{Z(t)\}_{t \geq 0}, \{\mathbb{P}_x\}_{x \in X} \) the Brownian motion on it. In this framework, we construct a Kendall-Cranston coupling following the argument due to von Renesse [17]. As we will see, his argument is based on approximation by coupled geodesic random walks.

Let \( D(X) = \{(x, x) \in X \times X \mid x \in X \} \). For each \( (x, y) \in X \times X \setminus D(X) \), we choose a minimal geodesic \( \gamma_{xy} : [0, 1] \to X \) of constant speed with \( \gamma_{xy}(0) = x \) and \( \gamma_{xy}(1) = y \). Let \( H_{xy} \) be the hyperplane in \( T_yX \) of codimension 1 which is perpendicular to \( \gamma_{xy}(1) \) and \( 0 \in H_{xy} \). For each \( v \in T_xX \), take a parallel translation by Levi-Civita connection along \( \gamma_{xy} \) to \( T_yX \) and reflect the resulting vector with respect to \( H_{xy} \). In this way, we obtain a new vector \( w \in T_yX \). We define a map \( m_{xy} : T_xX \to T_yX \) by \( m_{xy}v = w \). Clearly \( m_{xy} \) is isometry. Take a measurable section \( \varphi : X \to \mathcal{O}(X) \) to the orthonormal frame bundle \( \mathcal{O}(X) \). Let us define maps \( \Phi_i : X \times X \to \mathcal{O}(X) \) for \( i = 1, 2 \) satisfying

\[
\begin{align*}
\Phi_1(x, y) &\in \mathcal{O}_x(X) \quad x, y \in X \times X, \\
\Phi_2(x, y) &\in \mathcal{O}_y(X) \quad x, y \in X \times X, \\
\Phi_2(x, y)u &\equiv m_{xy}\Phi_1(x, y)u \quad (x, y) \in X \times X \setminus D(X), u \in \mathbb{R}^d, \\
\Phi_1(x, x) &= \Phi_2(x, x) = \varphi(x) \quad x \in X.
\end{align*}
\]

We can choose \( \gamma_{xy} \) so that \( (x, y) \mapsto \gamma_{xy} \) is measurable as a map from \( X \times X \setminus D(X) \) to \( C^1([0, 1] \to X) \) and \( \gamma_{xy} \) is symmetric, i.e. \( \gamma_{xy}(t) = \gamma_{yx}(1-t) \). Also we can choose \( \Phi_i \) to be measurable for \( i = 1, 2 \). Take a sequence of random variables \( \{\xi_n\}_{n \in \mathbb{N}} \) uniformly distributed on \( d \)-dimensional unit disk. Let us define a coupled geodesic random walk \( Z^\varepsilon(n) = (Z^\varepsilon_1(n), Z^\varepsilon_2(n)) \) on \( X \times X \) with step size \( \varepsilon > 0 \) and starting point \( (x, y) \in X \times X \) inductively by

\[
Z^\varepsilon(0) = (x, y),
\]

\[
Z^\varepsilon(n+1) = \left( \exp_{Z^\varepsilon_1(n)} \left( \varepsilon \sqrt{d+2\Phi_1(Z^\varepsilon(n))\xi_{n+1}} \right), \exp_{Z^\varepsilon_2(n)} \left( \varepsilon \sqrt{d+2\Phi_2(Z^\varepsilon(n))\xi_{n+1}} \right) \right).
\]

Let \( \tau_\lambda(t) \) be the Poisson process with intensity \( \lambda > 0 \) independent of \( \{\xi_n\}_{n \in \mathbb{N}} \). Then the sequence of processes \( \{Z^{k-1/2}(\tau_k(t))\}_{k \in \mathbb{N}} \) is tight in the Skorokhod path space \( D([0, \infty) \to X \times X) \) and \( Z_i^{k-1/2}(\tau_k(t)) \) weakly converges to the Brownian motion on \( X \) as \( k \to \infty \) for \( i = 1, 2 \). Let \( \tilde{Z}(t) = (\tilde{Z}_1(t), \tilde{Z}_2(t)) \) be a (subsequential) limit of \( \{Z^{k-1/2}(\tau_k(t))\}_{k \in \mathbb{N}} \). Let \( \sigma \) be the first hitting time of \( \tilde{Z} \) to \( D(X) \). We set \( Z(t) \) by

\[
Z(t) = \begin{cases}
\tilde{Z}(t) & \text{if } t < \sigma, \\
(\tilde{Z}_1(t), \tilde{Z}_1(t)) & \text{if } t \geq \sigma.
\end{cases}
\]
We call $Z(t)$ a Kendall-Cranston coupling. This is indeed a coupling of two Brownian motions starting at $x$ and $y$ respectively. Our choice of $\xi_n$ is a bit different from that in \cite{[17]}, where $\xi_n$ is uniformly distributed on the unit sphere. But it does not matter since the same argument works.

**Theorem 5.1** Assume (A1) and (A2) for $x_1, x_2 \in X$. Then a Kendall-Cranston coupling of $\mathbb{P}_{x_1}$ and $\mathbb{P}_{x_2}$ is the mirror coupling defined by $R$. In particular, the Kendall-Cranston coupling is unique in the sense that it is independent of the choice of subsequences of approximating geodesic random walks. As the result, the Kendall-Cranston coupling is the unique maximal Markovian coupling of $\mathbb{P}_{x_1}$ and $\mathbb{P}_{x_2}$.

**Proof.** As shown in the proof of Theorem \ref{thm:sym} $R$ is an isometry on $X$. For $x \in X$, set $y = Rx$. We claim

$$dR(u) = m_{xy}u \quad \text{for } u \in T_xX.$$  \hfill (5.1)

In order to complete the proof, it suffices to show \ref{eq:5.1}. Indeed, since the equality $R(\exp_z(w)) = \exp_{Rz}(dR(w))$ holds for $z \in X$ and $w \in T_zX$, \ref{eq:5.1} implies

$$Z^*_2(n) = RZ^*_1(n) \quad \text{for } n \leq T(Z^*_1, Z^*_2).$$  \hfill (5.2)

Note that the coupled geodesic random walks never meet under \ref{eq:5.2}. That is,

$$\mathbb{P}[T(Z^*_1, Z^*_2) = \infty] = 1$$  \hfill (5.3)

for each $\varepsilon > 0$. This fact is shown as follows: by \ref{eq:5.2}, $Z^*_1(T(Z^*_1, Z^*_2)) \in H$ must hold if $T(Z^*_1, Z^*_2) < \infty$. Let $\nu_{z, \varepsilon}$ be the law of $\exp_z(\varepsilon \xi_1)$. Then $\nu_{z, \varepsilon}(H) = 0$ for each $z \in X$ and $\varepsilon > 0$ since $H$ is a submanifold of codimension 1 as mentioned in Remark \ref{rem:5.5}. It easily implies \ref{eq:5.3}. Once we obtain \ref{eq:5.2} and \ref{eq:5.3}, the central limit theorem for geodesic random walks yields that the full sequence of $\{Z^{k-1/2}(\tau_k(t))\}_{k \in \mathbb{N}}$ weakly converges to the image of the Brownian motion by the map $z \mapsto (z, Rz)$ as $k \to \infty$. Thus a Kendall-Cranston coupling is unique and identical to the mirror coupling.

Set $\gamma = \gamma_{xy}$ for simplicity. First we show

$$R(\gamma(t)) = \gamma(1 - t).$$  \hfill (5.4)

By the symmetric choice of $\gamma_{xy}$, we may assume $l := \inf_{z \in H} d(x, z) \leq \inf_{z \in H} d(y, z)$ without loss of generality. By (A2), $l < \infty$ holds. Take $t_0 \in [0, 1]$ so that $d(x, \gamma(t_0)) = l$. Let $\gamma_1 : [0, 2t_0] \to X$ be a curve joining $x$ and $y$ given by

$$\gamma_1(s) = \begin{cases} 
\gamma(s), & s \in [0, t_0], \\
R(\gamma(2t_0 - s)), & s \in (t_0, 2t_0]. 
\end{cases}$$

Then the length of $\gamma_1$ equals $2l$ and the minimality of $\gamma$ implies $2l = d(x, y)$ and $t_0 = 1/2$. Moreover, $\dot{\gamma}_1(1/2) = \dot{\gamma}(1/2)$ must hold and therefore $\gamma_1 = \gamma$. It proves \ref{eq:5.4}. Note that the above discussion implies $\gamma(t) \in X_1$ for $t \in [0, 1/2)$ and $\gamma(t) \in X_2$ for $t \in (1/2, 1]$.

Next we show \ref{eq:5.1} in the case $u = \dot{\gamma}(0)$. It easily follows from \ref{eq:5.4}, that is,

$$dR(u) = dR(\dot{\gamma}(0)) = -\dot{\gamma}(1) = m_{xy}u.$$  \hfill (5.5)
Finally we prove (5.1) for \( u \perp \dot{\gamma}(0) \). Let \( \lVert \cdot \rVert_{s,t} T_{\gamma(s)}X \to T_{\gamma(t)}X \) be the parallel translation along \( \gamma|_{[s,t]} \). It suffices to show that
\[
\lVert 1/2 \circ dR \circ 1/2,0(h) = h \tag{5.6}
\]
of each \( h \in T_{\gamma(1/2)}X \) with \( h \perp \dot{\gamma}(1/2) \). Indeed, once we prove it,
\[
dR(u) = dR(\lVert 1/2,0 \circ 0,1/2(u)) = \lVert 1/2,1 \circ 0,1/2(u) = \lVert 0,1(u) = m_{xy}u.
\]
Now we show (5.6). Take \( \varepsilon > 0 \) so that the exponential map \( \exp_{\gamma(1/2)} : T_{\gamma(1/2)}X \to X \) is diffeomorphic on \( 2\varepsilon \)-ball centered at \( 0 \in T_{\gamma(1/2)}X \). We may assume that \( |h| = \varepsilon \). Let \( h' = \exp_{\gamma(1/2)}(\gamma(1/2 - \varepsilon /d(\gamma(0), \gamma(1)))) \). Note that \( \lVert 1,1/2 \circ dR \circ 1/2,0(h') = -h' \). We consider a curve \( c : [0,1] \to X \) given by \( c(t) = \exp(\cos \pi th' + \sin \pi th) \). Since \( c(0) \in X_1 \) and \( c(1) \in X_2 \), (A2) yields that \( c \) intersects \( H \). By the choice of \( h' \), \( R(c(t)) \neq c(t) \) if \( t \neq 1/2 \). Hence \( c(1/2) \in H \). It implies (5.6) and therefore completes the proof. \( \square \)

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