The Harmless Set Problem

Ajinkya Gaikwad* and Soumen Maity**

Indian Institute of Science Education and Research, Pune, India
ajinkya.gaikwad@students.iiserpune.ac.in; soumen@iiserpune.ac.in

Abstract. Given a graph \( G = (V, E) \), a threshold function \( t : V \to N \) and an integer \( k \), we study the Harmless Set problem, where the goal is to find a subset of vertices \( S \subseteq V \) of size at least \( k \) such that every vertex \( v \in V \) has less than \( t(v) \) neighbors in \( S \). We enhance our understanding of the problem from the viewpoint of parameterized complexity. Our focus lies on parameters that measure the structural properties of the input instance. We show that the problem is W[1]-hard parameterized by a wide range of fairly restrictive structural parameters such as the feedback vertex set number, pathwidth, treedepth, and even the size of a minimum vertex deletion set into graphs of pathwidth and treedepth at most three. On dense graphs, we show that the problem is W[1]-hard parameterized by cluster vertex deletion number. We also show that the Harmless Set problem with majority thresholds is W[1]-hard when parameterized by the treewidth of the input graph. We prove that the Harmless Set problem can be solved in polynomial time on graph with bounded cliquewidth. On the positive side, we obtain fixed-parameter algorithms for the problem with respect to neighbourhood diversity, twin cover and vertex integrity of the input graph. We show that the problem parameterized by the solution size is fixed parameter tractable on planar graphs. We thereby resolve two open questions stated in C. Bazgan and M. Chopin (2014) concerning the complexity of Harmless Set parameterized by the treewidth of the input graph and on planar graphs with respect to the solution size.

Keywords: Parameterized Complexity · FPT · W[1]-hard · treewidth · feedback vertex set number

1 Introduction

Social networks are used not only to stay in touch with friends and family, but also to spread and receive information on specific products and services. The

* The first author gratefully acknowledges support from the Ministry of Human Resource Development, Government of India, under Prime Minister’s Research Fellowship Scheme (No. MRF-192002-211).

** The second author’s research was supported in part by the Science and Engineering Research Board (SERB), Govt. of India, under Sanction Order No. MTR/2018/001025.
spread of information through social networks is a well-documented and well-studied topic. Kempe, Kleinberg, and Tardos [18] initiated a model to study the spread of influence through a social network. One of the most well known problems that appear in this context is Target Set Selection introduced by Chen [6] and defined as follows. We are given a graph, modeling a social network, where each node $v$ has a (fixed) threshold $t(v)$, the node will adopt a new product if $t(v)$ of its neighbors adopt it. Our goal is to find a small set $S$ of nodes such that targeting the product to $S$ would lead to adoption of the product by a large number of nodes in the graph. This problem may occur for example in the context of disease propagation, viral marketing or even faults in distributed computing [12, 24]. This problem received considerable attention in a series of papers from classical complexity [12, 5, 7, 25], polynomial time approximability [6, 1], parameterized approximability [3], and parameterized complexity [4, 8, 23].

A natural research direction considering this fact is to look for the complexity of variants or constrained version of this problem. Bazgan and Chopin [2] followed this line of research and introduced the notion of harmless set. Throughout this article, $G = (V, E)$ denotes a finite, simple and undirected graph. We denote by $V(G)$ and $E(G)$ its vertex and edge set respectively. For a vertex $v \in V$, we use $N(v) = \{u : (u, v) \in E(G)\}$ to denote the (open) neighbourhood of $v$ in $G$. The degree $d(v)$ of a vertex $v \in V(G)$ is $|N(v)|$. For a subset $S \subseteq V(G)$, we use $N_S(v) = \{u \in S : (u, v) \in E(G)\}$ to denote the (open) neighbourhood of vertex $v$ in $S$. The degree $d_S(v)$ of a vertex $v \in V(G)$ in $S$ is $|N_S(v)|$. A harmless set consists of a set $S$ of vertices with the property that no propagation occurs if any subset of $S$ gets activated. In other words, a harmless set is defined as a converse notion of a target set. More formally,

**Definition 1.** [2] A set $S \subseteq V$ is a harmless set of $G = (V, E)$, if every vertex $v \in V$ has less than $t(v)$ neighbours in $S$.

Note that in the definition of harmless set, the threshold condition is imposed on every vertex, including those in the solution $S$. As mentioned in [2], another perhaps more natural definition could have been a set $S$ such that every vertex $v \notin S$ has less than $t(v)$ neighbours in $S$. This definition creates two problems. First, it makes HARMLESS SET problem meaningless as the whole set of vertices of the input graph would be a trivial solution. Second, there might be some propagation steps inside $S$ if some vertices are activated in $S$. In this paper, we consider the HARMLESS SET problem under structural parameters. We define the problem as follows:

| HARMLESS SET |
| --- |
| **Input:** A graph $G = (V, E)$, a threshold function $t : V \rightarrow \mathbb{N}$ where $1 \leq t(v) \leq d(v)$ for every $v \in V$, and an integer $k$. |
| **Question:** Is there a harmless set $S \subseteq V$ of size at least $k$? |

The majority threshold is $t(v) = \lceil \frac{d(v)}{2} \rceil$ for all $v \in V$. We now review the concept of a tree decomposition, introduced by Robertson and Seymour in [26]. Treewidth is a measure of how “tree-like” the graph is.
Definition 2. A tree decomposition of a graph \( G = (V, E) \) is a tree \( T \) together with a collection of subsets \( X_t \) (called bags) of \( V \) labeled by the nodes \( t \) of \( T \) such that \( \bigcup_{t \in T} X_t = V \) and (1) and (2) below hold:

1. For every edge \( uv \in E(G) \), there is some \( t \) such that \( \{u, v\} \subseteq X_t \).
2. (Interpolation Property) If \( t \) is a node on the unique path in \( T \) from \( t_1 \) to \( t_2 \), then \( X_{t_1} \cap X_{t_2} \subseteq X_t \).

Definition 3. The width of a tree decomposition is the maximum value of \( |X_t| - 1 \) taken over all the nodes \( t \) of the tree \( T \) of the decomposition. The treewidth \( tw(G) \) of a graph \( G \) is the minimum width among all possible tree decompositions of \( G \).

Example 1. Figure 1 gives an example of a tree decomposition of width 2.

A rooted forest is a disjoint union of rooted trees. Given a rooted forest \( F \), its transitive closure is a graph \( H \) in which \( V(H) \) contains all the nodes of the rooted forest, and \( E(H) \) contain an edge between two vertices only if those two vertices form an ancestor-descendant pair in the forest \( F \).

Definition 4. The treedepth of a graph \( G \) is the minimum height of a rooted forest \( F \) whose transitive closure contains the graph \( G \). It is denoted by \( td(G) \).

Definition 5. A set \( S \subseteq V(G) \) is a vertex cover of \( G = (V, E) \) if each edge in \( E \) has at least one endpoint in \( S \). The size of a smallest vertex cover of \( G \) is the vertex cover number of \( G \).

We recall a natural way of generalizing vertex cover to dense graphs. We relax the definition of vertex cover so that not all edges need to be covered.
Definition 6. An edge is a twin edge if its incident vertices have the same neighborhood (excluding each other).

Definition 7. A set $X \subseteq V(G)$ is a twin-cover of $G$ if every edge in $G$ is either twin or incident to a vertex in $X$. We then say that $G$ has twin-cover number $k$ if $k$ is the minimum possible size of a twin-cover of $G$.

Definition 8. A set $X \subseteq V(G)$ is a cluster vertex deletion set of $G$ if $G \setminus X$ is a union of cliques.

Fig. 2. (a) A minimum size vertex cover (b) a minimum size twin cover of an example graph.

An illustration and comparison is provided in Figure 2.

For the standard concepts in parameterized complexity, see the recent textbook by Cygan et al. [9].

1.1 Our Results:

Our main results are as follows:

– the HARMLESS SET problem with general thresholds is FPT when parameterized by the neighbourhood diversity.
– the HARMLESS SET problem with general thresholds is FPT when parameterized by twin cover of the input graph.
– the HARMLESS SET problem with general thresholds is FPT when parameterized by vertex integrity of the input graph.
– the HARMLESS SET problem with majority thresholds is W[1]-hard when parameterized by the treewidth of the graph.
– the HARMLESS SET problem with general thresholds is W[1]-hard when parameterized by the size of a vertex deletion set into trees of height at most 3, even when restricted to bipartite graphs.
– the HARMLESS SET problem with general thresholds is W[1]-hard when parameterized by cluster vertex deletion set.
– the HARMLESS SET problem with general thresholds can be solved in polynomial time on graph with bounded cliquewidth.
– the HARMLESS SET problem with general thresholds is FPT when parameterized by the solution size when restricted to planar graphs.
Fig. 3. Relationship between vertex cover (vc), neighbourhood diversity (nd), twin cover (tc), modular width (mw), cluster vertex deletion number (cvd), feedback vertex set (fvs), pathwidth (pw), treewidth (tw) and clique width (cw). Note that $A \rightarrow B$ means that there exists a function $f$ such that for all graphs, $f(A(G)) \geq B(G)$. It also gives an overview of the parameterized complexity landscape for the Harmless Set problem with general thresholds. The problem is FPT parameterized by blue colored parameters and W[1]-hard when parameterized by red colored parameters. The problem remains unsettled when parameterized by mw.

1.2 Known Results:

Bazgan and Chopin [2] studied the parameterized complexity of Harmless Set and the approximation of the associated maximization problem. When the parameter is $k$, they proved that the Harmless Set problem is W[2]-complete in general and W[1]-complete if all thresholds are bounded by a constant. When each threshold is equal to the degree of the vertex, they showed that Harmless Set is fixed-parameter tractable for parameter $k$ and the maximization version is APX-complete. They gave a polynomial-time algorithm for graphs of bounded treewidth and a polynomial-time approximation scheme for planar graphs. The parametric dual problem $(n - k)$-Harmless Set asks for the existence of a harmless set of size at least $n - k$. The parameter is $k$ and $n$ denotes the number of vertices in the input graph. They showed that the parametric dual problem $(n - k)$-Harmless Set is fixed-parameter tractable for a large family of threshold functions.

2 FPT algorithm parameterized by neighbourhood diversity

In this section, we present an FPT algorithm for the Harmless Set problem parameterized by neighbourhood diversity. We say that two (distinct) vertices $u$
and \( v \) have the same \textit{neighborhood type} if they share their respective neighborhoods, that is, when \( N(u) \setminus \{v\} = N(v) \setminus \{u\} \). If this is so we say that \( u \) and \( v \) are \textit{twins}. It is possible to distinguish true-twins (those joined by an edge) and false-twins (in which case \( N(u) = N(v) \)).

\textbf{Definition 9.} [20] A graph \( G = (V, E) \) has \textit{neighborhood diversity} at most \( d \), if there exists a partition of \( V \) into at most \( d \) sets (we call these sets \textit{type classes}) such that all the vertices in each set have the same neighborhood type.

If neighbourhood diversity of a graph is bounded by an integer \( d \), then there exists a partition \( \{C_1, C_2, \ldots, C_d\} \) of \( V(G) \) into \( d \) type classes. We would like to point out that it is possible to compute the neighborhood diversity of a graph in linear time using fast modular decomposition algorithms [28]. Notice that each type class could either be a clique or an independent set by definition and two type classes are either joined by a complete bipartite graph or no edge between vertices of the two types is present in \( G \). For algorithmic purpose it is often useful to consider a \textit{type graph} \( H \) of graph \( G \), where each vertex of \( H \) is a type class in \( G \), and two vertices \( C_i \) and \( C_j \) are adjacent iff there is a complete bipartite clique between these type classes in \( G \). The key property of graphs of bounded neighbourhood diversity is that their type graphs have bounded size.

For example, a graph \( G \) with neighbourhood diversity four and its corresponding type graph \( H \) is illustrated in Figure 4.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{A graph \( G \) with neighbourhood diversity 4 and its corresponding type graph \( H \).}
\end{figure}

The following result explains why the vertices with low thresholds are inside the solution.

\textbf{Lemma 1.} Let \( C_i = \{v_1, \ldots, v_{|C_i|}\} \) be a type class in \( G \) such that \( t(v_1) \leq t(v_2) \leq \ldots \leq t(v_{|C_i|}) \). Let \( S \) be a maximum size harmless set in \( G \) and \( x_i = |S_i| = |C_i \cap S| \). Then \( S' = (S \setminus S_i) \cup \{v_1, v_2, \ldots, v_{x_i}\} \) is also a maximum size harmless set in \( G \).
Lemma 2. Let $C_i$ be an independent type class and $x_i \in \{0, 1, \ldots, |C_i|\}$. Let $u_0$ be a vertex in $C_i$ with threshold $t(C_i)$. Then every vertex $u$ in $C_i$ has less than $t(u)$ neighbours in $S$ if and only if $u_0$ has less than $t(C_i)$ neighbours in $S$. 

Proof. Clearly, $|S| = |S'|$. To show $S'$ is a harmless set, it is enough to show that each vertex $v$ in $C_i$ has less than $t(v)$ neighbours in $S'$. Let $v$ be an arbitrary element of $C_i$. If $v \in \{v_1, \ldots, v_{x_i}\}$, we have

$$d_{S'}(v) = \begin{cases} d_S(v) & \text{if } v \in S \\ d_S(v) - 1 & \text{if } v \notin S \end{cases}$$

Therefore, $v$ satisfies the threshold condition $d_{S'}(v) \leq d_S(v) < t(v)$. Suppose $v \in \{v_{x_i+1}, v_{x_i+2}, \ldots, v_{|C_i|}\}$. If $v \notin S$ then $d_{S'}(v) = d_S(v) < t(v)$. If $v \in S$ then, by definition of $S'$, some vertex $v' \in \{v_1, v_2, \ldots, v_{x_i}\}\setminus S$ must have replaced $v$ as $v' \leq t(v)$. We have $d_{S'}(v') = d_{S}(v) + 1$ and also $d_{S'}(v) = d_{S}(v) + 1$. It implies that $d_{S'}(v) = d_{S}(v) + 1 = d_{S}(v') < t(v') \leq t(v)$. Therefore, $S'$ is a harmless set.

In this section, we prove the following theorem:

**Theorem 1.** The **Harmless Set** problem with general thresholds is FPT when parameterized by the neighbourhood diversity.

Given a graph $G = (V, E)$ with neighbourhood diversity $nd(G) \leq d$, we first find a partition of the vertices into at most $d$ type classes $C_1, \ldots, C_d$. Let $C$ be the set of all clique type classes and $I$ be the set of all independent type classes. The case where some $C_i$ are singletons can be considered as cliques or independent sets. For simplicity, we consider singleton type classes as independent sets.

**ILP formulation:** Our goal here is to find a largest harmless set $S$ of $G$. For each $C_i$, we associate a variable $x_i$ that indicates $|S \cap C_i| = x_i$. As the vertices in $C_i$ have the same neighbourhood, the variables $x_i$ determine $S$ uniquely, up to isomorphism. The threshold $t(C_i)$ of a type class $C_i$ is defined to be

$$t(C_i) = \min \{t(v) \mid v \in C_i\}.$$ 

Let $\alpha(C_i)$ be the number of vertices in $C_i$ with threshold value $t(C_i)$. We define $C_1 = \{C_i \in C \mid x_i < \alpha(C_i)\}$ and $C_2 = \{C_i \in C \mid x_i \geq \alpha(C_i)\}$. We next guess if a clique type class $C_i$ belongs to $C_1$ or $C_2$. There are at most $2^d$ guesses as each clique type class $C_i$ has two options: either it is in $C_1$ or in $C_2$. We reduce the problem of finding a maximum harmless set to at most $2^d$ integer linear programming problems with $d$ variables. Since integer linear programming is fixed-parameter tractable when parameterized by the number of variables [21], we conclude that our problem is FPT when parameterized by the neighbourhood diversity $d$. We consider the following cases based on whether $C_i$ is in $I, C_1$ or $C_2$:

**Case 1:** Assume $C_i$ is in $I$.

**Lemma 2.** Let $C_i$ be an independent type class and $x_i \in \{0, 1, \ldots, |C_i|\}$. Let $u_0$ be a vertex in $C_i$ with threshold $t(C_i)$. Then every vertex $u$ in $C_i$ has less than $t(u)$ neighbours in $S$ if and only if $u_0$ has less than $t(C_i)$ neighbours in $S$. 

...
Proof. Suppose each \( u \in C_i \) has less than \( t(u) \) neighbours in \( S \). Then obviously \( u_0 \in C_i \) has less than \( t(u_0) = t(C_i) \) neighbours in \( S \). Conversely, suppose \( u_0 \) has less than \( t(C_i) \) neighbours in \( S \). Let \( u \) be an arbitrary vertex of \( C_i \). As \( u \) and \( u_0 \) are two vertices in the same type class \( C_i \), we have \( d_S(u) = d_S(u_0) \). Moreover, for each \( u \in C_i \), we have \( t(C_i) \leq t(u) \) by definition of \( t(C_i) \). Therefore, \( d_S(u) = d_S(u_0) < t(C_i) \leq t(u) \).

Here \( d_S(u_0) = \sum_{C_j \in N_H(C_i)} x_j \). By Lemma 2 every vertex \( u \in C_i \) has less than \( t(u) \) neighbours in \( S \) if and only if

\[
\sum_{C_j \in N_H(C_i)} x_j < t(C_i).
\]

Case 2: Assume \( C_i \) is in \( C_1 \). That is, \( C_i \) is a clique type class and \( x_i < \alpha(C_i) \). Assuming \( x_i < \alpha(C_i) \) ensures that there exists at least one vertex in \( S^c \cap C_i \) with threshold \( t(C_i) \).

**Lemma 3.** Let \( C_i \in C_1 \) and \( u_0 \) be a vertex in \( S^c \cap C_i \) with threshold \( t(C_i) \). Then every vertex \( u \) in \( C_i \) has less than \( t(u) \) neighbours in \( S \) if and only if \( u_0 \) has less than \( t(C_i) \) neighbours in \( S \).

Proof. Suppose every vertex \( u \) in \( C_i \) has less than \( t(u) \) neighbours in \( S \). Then obviously \( u_0 \) has less than \( t(u_0) = t(C_i) \) neighbours in \( S \). Conversely, suppose \( u_0 \) has less than \( t(C_i) \) neighbours in \( S \). Let \( u \) be an arbitrary vertex of \( C_i \). If \( u \in S \cap C_i \), then Lemma 4 and the condition \( x_i < \alpha(C_i) \) ensure \( u \) has threshold \( t(C_i) \). Note that \( d_S(u) = d_S(u_0) - 1 < t(C_i) - 1 < t(C_i) = t(u) \). If \( u \in S^c \cap C_i \), then we have \( d_S(u) = d_S(u_0) < t(C_i) \leq t(u) \). Therefore, every vertex in \( C_i \) satisfies the threshold condition.

Here \( d_S(u_0) = x_i + \sum_{C_j \in N_H(C_i)} x_j \). By Lemma 3 every vertex \( u \) in \( C_i \) has less than \( t(u) \) neighbours in \( S \) if and only if

\[
x_i + \sum_{C_j \in N_H(C_i)} x_j < t(C_i).
\]

Case 3: Assume that \( C_i \) is in \( C_2 \). That is, \( C_i \) is a clique type class and \( x_i \geq \alpha(C_i) \). By Lemma 4 all the vertices with threshold \( t(C_i) \) are inside \( S \).

**Lemma 4.** Let \( C_i \in C_2 \) and \( u_0 \) be a vertex in \( S \cap C_i \) with threshold \( t(C_i) \). Then every vertex \( u \) in \( C_i \) has less than \( t(u) \) neighbours in \( S \) if and only if \( u_0 \) has less than \( t(C_i) \) neighbours in \( S \).

Proof. Suppose every vertex \( u \) in \( C_i \) has less than \( t(u) \) neighbours in \( S \). Then obviously \( u_0 \) has less than \( t(u_0) = t(C_i) \) neighbours in \( S \). Conversely, suppose \( u_0 \) has less than \( t(C_i) \) neighbours in \( S \). Let \( u \) be an arbitrary vertex of \( C_i \). If \( u \in S \cap C_i \), then \( d_S(u) = d_S(u_0) < t(C_i) \leq t(u) \). Suppose \( u \in S^c \cap C_i \). Note
that such an element \(u\) may not always exist, it is possible that all vertices in \(C_i\) are included in \(S\) (that is, \(x_i = |C_i|\)). Let us assume that such \(u\) exists. Since \(u\) is outside the solution and by Lemma 1, all the vertices with threshold \(t(C_i)\) are inside the solution, we get \(t(u) \geq t(C_i) + 1\). It is easy to note that \(d_S(u) = d_S(u_0) + 1 < t(C_i) + 1 \leq t(u)\).

Here \(d_S(u_0) = (x_i - 1) + \sum_{C_j \in N_H(C_i)} x_j\). By Lemma 2, every vertex \(u\) in \(C_i\) has less than \(t(C_i)\) neighbours in \(S\) if and only if \((x_i - 1) + \sum_{C_j \in N_H(C_i)} x_j < t(C_i)\).

The next lemma follows readily from the three lemmas above and the definition of the sequence \((x_1, x_2, \ldots, x_d)\) and the harmless set.

**Lemma 5.** Let \(G = (V, E)\) be a graph such that \(V\) can be partitioned into at most \(d\) type classes \(C_1, \ldots, C_d\). The sequence \((x_1, x_2, \ldots, x_d)\) represents a harmless set \(S\) of \(G\) if and only if \((x_1, x_2, \ldots, x_d)\) satisfies

1. \(x_i \in \{0, 1, \ldots, |C_i|\}\) for \(i = 1, 2, \ldots, d\)
2. \(\sum_{C_j \in N_H(C_i)} x_j < t(C_i)\) for all \(C_i \in I\).
3. \(x_i + \sum_{C_j \in N_H(C_i)} x_j < t(C_i)\) and \(x_i < \alpha(C_i)\) for all \(C_i \in I_1\)
4. \((x_i - 1) + \sum_{C_j \in N_H(C_i)} x_j < t(C_i)\) and \(\alpha(C_i) \leq x_i \leq |C_i|\) for all \(C_i \in C_2\).

In the following, we present an ILP formulation for the HARMLESS SET problem parameterized by neighbourhood diversity for a guess:

Maximize \(\sum_{C_i} x_i\)

Subject to

\(x_i \in \{0, 1, \ldots, |C_i|\}\) for \(i = 1, 2, \ldots, d\)

\(\sum_{C_j \in N_H(C_i)} x_j < t(C_i)\) for all \(C_i \in I\),

\(x_i + \sum_{C_j \in N_H(C_i)} x_j < t(C_i)\) and \(x_i < \alpha(C_i)\) for all \(C_i \in I_1\)

\((x_i - 1) + \sum_{C_j \in N_H(C_i)} x_j < t(C_i)\) and \(\alpha(C_i) \leq x_i \leq |C_i|\) for all \(C_i \in C_2\)

**Example 2.** Consider a graph composed of a clique \(C\) of size \(c + 1\) (\(\geq 4\)) plus a vertex \(u\) adjacent to a vertex \(v\) of the clique as shown in Figure 5. We set
unanimity thresholds, so $t(u) = 1$, $t(v) = c + 1$ and $t(x) = c$ for all $x$ in $C \setminus \{v\}$. The type classes are $C_1 = \{u\}$, $C_2 = \{v\}$ and $C_3 = C \setminus \{v\}$. Here $\alpha(C_1) = 1,$

\[
\begin{array}{c}
\text{4} \\
\text{4} \\
\text{4} \\
\text{4} \\
\text{4} \\
\text{4} \\
\end{array}
\]

\[
\begin{array}{c}
v \\
u \\
\text{1} \\
\end{array}
\]

Fig. 5. The graph in Example 2 with $c = 4$.

$\alpha(C_2) = 1$ and $\alpha(C_3) = |C_3| = c$. Now consider the guess $C_1, C_2 \in \mathcal{I}$ and $C_3 \in \mathcal{C}_2$. Then we end up with the following ILP:

$$
\begin{align*}
\text{min} & \quad x_1 + x_2 + x_3 \\
\text{s.t.} & \quad x_2 < 1 \\
& \quad x_1 + x_3 < c + 1 \\
& \quad x_3 - 1 + x_2 < c \text{ and } x_3 = c
\end{align*}
$$

Note that $x_2 < 1$ implies that $x_2 = 0;$ $x_1 + x_3 < c + 1$ and $x_3 = c$ imply $x_1 = 0.$ It is easy to see that $x_1 = x_2 = 0, x_3 = c$ is an optimal solution and represent a valid harmless set for the graph.

**Solving the ILP** Lenstra \[21\] showed that the feasibility version of $p$-ILP is FPT with running time doubly exponential in $p$, where $p$ is the number of variables. Later, Kannan \[17\] proved an algorithm for $p$-ILP running in time $p^{O(p)}$.

In our algorithm, we need the optimization version of $p$-ILP rather than the feasibility version. We state the minimization version of $p$-ILP as presented by Fellows et. al. \[13\].

**$p$-Variable Integer Linear Programming Optimization ($p$-Opt-ILP):**
Let matrices $A \in \mathbb{Z}^{m \times p}$, $b \in \mathbb{Z}^{p \times 1}$ and $c \in \mathbb{Z}^{1 \times p}$ be given. We want to find a vector $x \in \mathbb{Z}^{p \times 1}$ that minimizes the objective function $c \cdot x$ and satisfies the $m$ inequalities, that is, $A \cdot x \geq b.$ The number of variables $p$ is the parameter. Then they showed the following:

**Proposition 1.** \[13\] $p$-Opt-ILP can be solved using $O(p^{2.5p+o(p)} \cdot L \cdot \log(MN))$ arithmetic operations and space polynomial in $L$. Here $L$ is the number of bits in the input, $N$ is the maximum absolute value any variable can take, and $M$ is an upper bound on the absolute value of the minimum taken by the objective function.

In the formulation for Harmless Set problem, we have at most $d$ variables. The value of the objective function is bounded by $n$ and the value of any variable in the integer linear programming is also bounded by $n$. The constraints can be
represented using $O(d^2 \log n)$ bits. Proposition $\text{I}$ implies that we can solve the problem with the guess $\mathcal{P}$ in FPT time. There are at most $2^d$ guesses, and the ILP formula for a guess can be solved in FPT time. Thus Theorem $\text{I}$ holds.

3 FPT algorithm parameterized by twin cover

In this section, we present an FPT algorithm for the HARMLESS SET problem with general thresholds parameterized by twin-cover. That is, we prove the following theorem:

**Theorem 2.** The HARMLESS SET problem with general thresholds is FPT when parameterized by twin cover of the input graph.

*Outline of the algorithm.* Given an $n$-vertex graph $G$ with $\text{tc}(G) \leq k$, we first find a twin cover $X$ of size at most $k$. We next guess $S_X = S \cap X$ where $S$ is a largest harmless set in $G$. There are at most $2^k$ guesses as each member of $X$ has two options: either in $S \cap X$ or $S^c \cap X$. Finally we reduce the problem of finding the rest of $S$ to an integer linear programming (ILP) optimization with at most $2^k$ variables. Since ILP optimization is fixed-parameter tractable when parameterized by the number of variables $\text{I}$, we can conclude that our problem is fixed-parameter tractable when parameterized by the twin cover number.

*Characterizations of a harmless set $S$ with a twin cover $X$.* Let $G = (V, E)$ be a graph and $X \subseteq V$ be a twin cover of $G$. Then $C = G \setminus X$ is a collection of disjoint cliques, that is $C = \{C_1, C_2, \ldots\}$. The threshold $t(C_i)$ of a clique $C_i$ is defined to be

$$t(C_i) = \min\{t(v) \mid v \in C_i\}.$$

Let $\alpha(C_i)$ be the number of vertices in $C_i$ with threshold value $t(C_i)$. It may be observed that from a clique it is always better to include the vertices with lower thresholds in the solution. The reason is this. Suppose $a$ and $b$ are two vertices from the same clique $C$ such that $t(a) < t(b)$. Suppose $a$ is in the solution, whereas $b$ is outside the solution. Then $a$ has one less neighbours in the solution compared to $b$. This one less degree of $a$ in the solution helps the vertex $a$ to satisfy the required threshold condition as $t(a) < t(b)$. We define $C_{> \alpha} = \{C \in C : \alpha(C) > t(C) - |N_{S_X}(C)|\}$ and $C_{\leq \alpha} = \{C \in C : \alpha(C) \leq t(C) - |N_{S_X}(C)|\}$.

**Lemma 6.** Assume that $C$ is in $C_{> \alpha}$. Then every vertex $u$ in $C$ has less than $t(u)$ neighbours in $S$ if and only if $|S \cap C| \leq t(C) - |N_{S_X}(C)| - 1$.

*Proof.* Suppose every vertex $u$ in $C$ has less than $t(u)$ neighbours in $S$, and suppose, for the sake of contradiction, that $|S \cap C| \geq t(C) - |N_{S_X}(C)|$. If $|S \cap C| = t(C) - |N_{S_X}(C)|$ then $|S \cap C| < \alpha(C)$. This implies there exists a vertex $u_0 \in S^c \cap C$ with $t(u_0) = t(C)$. Note that $d_S(u_0) = |S \cap C| + |N_{S_X}(C)| = t(C) - t(u_0)$, a contradiction to the assumption that every vertex in $C$ has less than $t(u)$ neighbours in $S$. Let us assume that $|S \cap C| \geq t(C) - |N_{S_X}(C)| + 1$. Let
Let $u$ be an arbitrary vertex in $C$. Then $d_S(u) \geq |S \cap C| - 1 + |N_{S_X}(C)| \geq t(C) \geq t(u)$, which is again a contradiction. This proves the forward direction. On the other hand, let us assume that $|S \cap C| \leq t(C) - |N_{S_X}(C)| - 1$. It implies that $|S \cap C| < \alpha(C)$ and this ensures that there exists at least one vertex $u_0$ in $S^c \cap C$ with threshold $t(C)$. By Lemma 3 it is enough to check whether $u_0 \in S^c \cap C$ with threshold $t(C)$ satisfies the threshold condition. Clearly, $d_S(u_0) = |S \cap C| + |N_{S_X}(C)| \leq t(C) - |N_{S_X}(C)| - 1 + |N_{S_X}(C)| = t(C) - 1$ satisfies the threshold condition. Therefore, every vertex $u$ in $C$ has less than $t(u)$ neighbours in $S$.

**Lemma 7.** Assume that $C$ is in $\mathcal{C}_<$. Then every vertex $u$ in $C$ has less than $t(u)$ neighbours in $S$ if and only if $|S \cap C| \leq t(C) - |N_{S_X}(C)|$.

**Proof.** Suppose each vertex $u \in C$ has less than $t(u)$ neighbours in $S$. Let $u_0$ be a vertex in $C$ with threshold $t(u_0) = t(C)$. Then $u_0$ also has less than $t(u_0)$ neighbours in $S$, that is, $d_S(u_0) \leq |S \cap C| + |N_{S_X}(C)| < t(u_0) = t(C)$. Therefore, we get $|S \cap C| \leq t(C) - |N_{S_X}(C)|$.

On the other hand, first suppose that $|S \cap C| = t(C) - |N_{S_X}(C)|$. Therefore, we can say that $|S \cap C| \geq \alpha(C)$. It means that all the vertices with least threshold are inside the solution. Let $u$ be an arbitrary vertex in $S \cap C$ with threshold $t(u)$. By Lemma 3 it is enough to check whether $u$ satisfies the threshold condition. It is easy to observe that $d_S(u) = |S \cap C| - 1 + |N_{S_X}(C)| = t(C) - 1 < t(C) = t(u)$, that is, $u$ satisfies the threshold condition. Now suppose $|S \cap C| = t(C) - |N_{S_X}(C)| - \delta$ for some integer $\delta \geq 1$. Take an arbitrary vertex $u \in C$. We have $d_S(u) \leq |S \cap C| + |N_{S_X}(C)| = t(C) - \delta < t(C) \leq t(u)$. This implies that all the vertices in $C$ satisfy the threshold condition.

We partition the family $C$ of cliques into twin classes $C_1, C_2, \ldots, C_t$, where $t \leq 2^k$. Two cliques $C_i$ and $C_j$ are in the same twin class if and only if they have the same neighbours in $X$, that is, $N_X(C_i) = N_X(C_j)$. For each twin class $C_i$, we associate a variable $x_i$ that indicates $|C_i \cap S| = x_i$. The variables $x_i$ determine $S$. The objective is to maximize $\sum_{i=1}^{t} x_i$ under the condition

$$x_i \leq \sum_{C \in C_i \cap C} \left( t(C) - |N_{S_X}(C)| - 1 \right) + \sum_{C \in C_j \cap C} \left( t(C) - |N_{S_X}(C)| \right).$$

The above constraint makes sure that the vertices of twin class $C_i$ satisfy the threshold condition. Next, we add $k$ more constraints to make sure that all the vertices in $X$ satisfy threshold condition. Let $u \in X$ be an arbitrary vertex. We need $d_S(u) < t(u)$. Note that $d(u) = d_X(u) + \sum_{C_i : N(u) \cap C_i \neq \emptyset} |C_i|$. Therefore, for each $u \in X$, we add the constraint $d_S(u) = d_X(u) + \sum_{C_i : N(u) \cap C_i \neq \emptyset} x_i < t(u)$.

In the following, we present an ILP formulation for the HARMLESS SET problem for a given $S_X$:
Maximize $\sum_{i=1}^{t} x_i$, $t \leq 2^k$
Subject to
\[
x_i \leq \sum_{C \in C > i \cap C} \left( t(C) - |N_{S_X}(C)| - 1 \right) + \sum_{C \in C \leq i \cap C} \left( t(C) - |N_{S_X}(C)| \right), \text{ for } 1 \leq i \leq 2^k
\]
\[
d_{S_X}(u) + \sum_{C_i : N(u) \cap C_i \neq \emptyset} x_i < t(u) \text{ for all } u \in X
\]

In the ILP formulation for HARMLESS SET problem parameterized by twin cover, we have at most $2^k$ variables. The value of objective function is bounded by $n$ and the value of any variable in the integer linear programming is also bounded by $n$. The constraints can be represented using $O(k^2 \log n)$ bits. Proposition 1 implies that we can solve the problem for given $S_X$ in FPT time. There are at most $2^k$ guesses for $S_X$, and the ILP formula for a guess can be solved in FPT time. Thus Theorem 2 holds.

4 FPT algorithm parameterized by vertex integrity

In this section, we present an FPT algorithm for HARMLESS SET parameterized by vertex integrity.

**Definition 10.** The vertex integrity of a graph $G$, denoted $vi(G)$, is the minimum integer $k$ satisfying that there is $X \subseteq V(G)$ such that $|X| + |V(C)| \leq k$ for each component $C$ of $G - X$. We call such $S$ a $vi(k)$-set of $G$.

This parameter is bounded from above by vertex cover number plus one and from below by treedepth. Gima et al. [15] described an equivalence relation among components. For a vertex set $X$ of $G$, we define an equivalence relation $\sim_{G,X}$ among components of $G - X$ by setting $C_1 \sim_{G,X} C_2$ if and only if there is an isomorphism $g$ from $G[X \cup V(C_1)]$ to $G[X \cup V(C_2)]$ that fixes $X$; that is, $g|_X$ is the identity function. When $C_1 \sim_{G,X} C_2$, we say that $C_1$ and $C_2$ have the same $(G, X)$-type (or just the same type if $G$ and $X$ are clear from the context). See Figure 6. This equivalence relation induces a set of equivalence classes $C_1, C_2, \ldots$. We can choose a representative of each equivalence class.

**Theorem 3.** The HARMLESS SET problem is fixed-parameter tractable when parameterized by the vertex integrity of the input graph.

**Outline of the algorithm.** Let $X$ be a $vi(k)$-set of $G$. Such a set can be found in $O(k^{k+1}n)$ time [11]. We next guess $S_X = S \cap X$ where $S$ is a largest harmless set in $G$. There are at most $2^k$ guesses as each member of $X$ has two options: either in $S \cap X$ or $S^c \cap X$. Finally we reduce the problem of finding the rest of $S$ to an integer linear programming (ILP) optimization with number of variables.
depend only on \( k \).

**Characterizations of a harmless set \( S \) with a \( \text{vi}(k) \)-set \( X \).** Let \( G = (V, E) \) be a graph and \( X \subseteq V \) be a \( \text{vi}(k) \)-set of \( G \). Then \( C = G \setminus X \) is a collection of disjoint components, that is \( \mathcal{C} = \{C_1, C_2, \ldots\} \) such that \( |X| + |C_i| \leq k \) for all \( i \). We know \( \mathcal{C} \) can be partitioned into equivalence classes \( C_1, C_2, \ldots \). Let \( C_l \) be a representative of the equivalence class \( C_l \) and let \( v \in C_l \). Note that \( v \) has neighbours only in \( X \cup V(C_l) \), that is, \( N(v) \subseteq X \cup V(C_l) \). Suppose the intersection of the solution \( S \) with \( X \) is \( S_X = S \cap X \) and the intersection of \( S \) with the component \( C_l \) is \( A = S \cap C_l \subseteq C_l \). Therefore \( v \in C_l \) satisfies the threshold condition if \( d_S(v) = |N_{S_X}(v)| + |N_A(v)| < t(v) \). We say \( A \subseteq C_l \) is a **valid selection** for \( C_l \) if every vertex of \( C_l \) satisfies the threshold condition when the vertices of \( A \cup S_X \) are in the solution. Similarly, we say \( A \subseteq C_l \) is a **valid selection** for \( C \in C_l, C \neq C_l \), if every vertex of \( C \) satisfies the threshold condition when the vertices of \( g(A) \cup S_X \) are in the solution, where \( g \) is an isomorphism from \( G[X \cup V(C_l)] \) to \( G[X \cup V(C)] \) that fixes \( X \). It is important to note that given two connected component \( C_1 \) and \( C_2 \) from the same equivalence class \( C_l \), a subset \( A \subseteq C_l \) might be valid selection for one connected component but may not be valid for the other connected component as the threshold values of vertices in \( C_1 \) and \( C_2 \) can differ.

**Fig. 7.** The components \( C_2 \) and \( C_3 \) of \( G - X \) have the same \((G, X)\)-type. That is, \( C_2 \) and \( C_3 \) are in the same equivalence class. Note that given \( S_X = \{v_3, v_4, v_5\} \), \( A = \{d\} \) is a valid selection for \( C_2 \) but \( g(A) = \{f\} \) is not a valid selection for \( C_3 \); \( B \) is not a valid selection for \( C_1 \); \( C \) is a valid selection for \( C_4 \).
In the following, we present an ILP formulation for the components. Therefore the possible solutions are $(x(A_1), x(A_2), x(A_3)) = (0, 0, 3), (0, 1, 2), (0, 2, 1), (1, 1, 1), (1, 0, 2), (1, 2, 0), (2, 0, 1), (2, 1, 0)$. Note that $(0, 0, 3)$ indicates $A_3$ is valid selection in three components $C_1$, $C_2$ and $C_3$; similarly $(0, 1, 2)$ indicates $A_2$ is a valid selection in one component and $A_3$ is a valid selection in two components.

In the following, we present an ILP formulation for the HARMLESS SET problem for a given $S_X$:
Maximize \[ \sum_{l} \sum_{A \in \mathcal{V}S(l)} |A| \times x(A) \]

Subject to
\[ x(A_i) \leq \rho(A_i) \text{ for all } A_i \in \mathcal{V}S(l) \]
\[ x(A_i) + x(A_j) \leq \rho(A_i) + \rho(A_j) - \rho(A_i A_j) \text{ for all } A_i, A_j \in \mathcal{V}S(l) \]
\[ \sum_{A_i \in \mathcal{V}S(l)} x(A_i) = |C_l| \]
\[ \sum_{l} \sum_{A \in \mathcal{V}S(l)} |N(u) \cap A| \times x(A) < t(u) \text{ for all } u \in \mathcal{X} \]

In the ILP formulation for Harmless Set problem parameterized by vertex integrity, the number of variables are bounded by a computable function \( f(k) \). The value of objective function is bounded by \( n \) and the value of any variable in the integer linear programming is also bounded by \( n \). The constraints can be represented using \( O(k^2 \log n) \) bits. Proposition 1 implies that we can solve the problem for given \( H_X \) in FPT time. There are at most \( 2^k \) guesses for \( H_X \), and the ILP formula for a guess can be solved in FPT time. Thus Theorem 3 holds.

5 \textbf{W[1]-hardness parameterized by treewidth}

In this section we show that the Harmless Set problem with majority thresholds is W[1]-hard when parameterized by the treewidth. To show W[1]-hardness of Harmless Set with majority thresholds, we reduce from the following problem, which is known to be W[1]-hard parameterized by the treewidth of the graph [27]:

| **Minimum Maximum Outdegree** |
|-------------------------------|
| **Input:** An undirected graph \( G \) whose edge weights are given in unary, and a positive integer \( r \). |
| **Question:** Is there an orientation of the edges of \( G \) such that, for each \( v \in V(G) \), the sum of the weights of outgoing edges from \( v \) is at most \( r \)? |

In Minimum Maximum Outdegree problem, every edge weight \( w(u, v) \) of \( G \) is given in unary, that is, every edge weight \( w(u, v) \) is polynomially bounded in \( |V(G)| \). In a weighted undirected graph \( G \), the weighted degree of a vertex \( v \), is defined as the sum of the weights of the edges incident to \( v \) in \( G \). In this section, we prove the following theorem:

**Theorem 4.** The Harmless Set problem with majority thresholds is W[1]-hard when parameterized by the treewidth of the graph.
Proof. Let $G = (V, E, w)$ and a positive integer $r \geq 3$ be an instance $I$ of MINIMUM MAXIMUM OUTDEGREE. We construct an instance $I' = (G', t, k)$ of HARMLESS SET the following way. See Figure 3 for an illustration.

1. For each weighted edge $(u, v) \in E(G)$, we introduce two sets of new vertices $V_{uv} = \{u_1^v, \ldots, u_{w(u,v)}^v\}$ and $V_{vu} = \{v_1^u, \ldots, v_{w(u,v)}^u\}$ into $G'$. Make $u$ and $v$ adjacent to every vertex of $V_{uv}$ and $V_{vu}$, respectively. The vertices of $\bigcup_{(u,v) \in E(G)} V_{uv} \cup V_{vu}$ are called type 1 vertices.

2. For each $1 \leq i \leq w(u,v) - 1$, we introduce $x(u_i^v, v_{i+1}^u)$ into $G'$ and make it adjacent to $u_i^v$ and $v_{i+1}^u$; introduce $x(u_i^v, v_{i+1}^u)$ and make it adjacent to $u_i^v$ and $v_{i+1}^u$. We also add $x(u_i^v, v_{i+1}^u)$ into $G'$ and make it adjacent to $u_i^v$ and $v_{i+1}^u$. We call such vertices, the vertices of type 2.

3. For every vertex $x$ of type 2, we add a triangle (cycle of length 3) and make $x$ adjacent to exactly one vertex of this triangle. For every vertex $x$ of type 1, let $n(x)$ be the number of neighbors of $x$ in $V(G)$ and in the set of type 2 vertices. Note that $2 \leq n(x) \leq 4$. We add $n(x) + 1$ many triangles corresponds to vertex $x$ and make $x$ adjacent to exactly one vertex of each triangle.

4. The weighted degree of a vertex $x \in V$ in $G$ is denoted by $d_w(x; G)$. We partition the vertices of $V(G)$ based on whether $\lceil \frac{d_w(x; G)}{2} \rceil \leq r + 1$ or $\lceil \frac{d_w(x; G)}{2} \rceil > r + 1$. A vertex $x$ in $G$ with $\lceil \frac{d_w(x; G)}{2} \rceil \leq r + 1$ is called a vertex of low-degree-type. For each $x \in V(G)$ of low-degree-type, we add $2 \left[ r + 1 - \lceil \frac{d_w(x; G)}{2} \rceil \right]$ triangles and make $x$ adjacent to exactly one vertex of each triangle. A vertex $x \in V(G)$ with $\lceil \frac{d_w(x; G)}{2} \rceil > r + 1$ is called a vertex of high-degree-type. For each $x \in V(G)$ of high-degree-type, we add a set $V_x^\Delta = \{v_1^\Delta, \ldots, v_{r+1}^\Delta\}$ of $r + 1$ many vertices and make them adjacent to $x$. For each $v \in V_x^\Delta$, we add two triangles and make $v$ adjacent to exactly one vertex of each triangle. For each high-degree-type vertex $x$, we also add a set of $(r + 2)$ many triangles and make $x$ adjacent to exactly one vertex of each triangle.

This completes the construction of graph $G'$.

We set $k = n + W + \sum_{(u,v) \in E(G)} (3w(u,v) - 2) + \sum_{x \in \text{high-degree-type}} (d_w(x; G) - r)$ where $W = \sum_{(u,v) \in E(G)} w(u,v)$. Clearly $I'$ can be computed in polynomial time.

We now show that the treewidth of $G'$ is bounded by a function of the treewidth of $G$. We do so by modifying an optimal tree decomposition $T$ of $G$ as follows:

- For every edge $(u, v)$ of $G$, we take an arbitrary node $t$ in $T$ whose bag $X_t$ contains both $u$ and $v$; attach to this node a chain of nodes $1, 2, \ldots, w(u,v) - 1$ such that the bag of node $i$ is

$$X_i \cup \{u_i^v, v_i^u, u_{i+1}^v, v_{i+1}^u, x(u_i^v, v_{i+1}^u), x(u_{i+1}^v, v_i^u), x(u_i^v, v_{i+1}^u), x(v_i^u, u_{i+1}^v)\}.$$
For every Type 1 vertex $u^v$, take an arbitrary node $t$ in the modified tree decomposition whose bag contains $u^v$; attach to it a chain of at most five nodes $t_1, t_2, \ldots, t_5$ such that the bag $X_{t_i}$ of node $t_i$ contains $u^v$ and the vertices of $i$th triangle corresponds to $u^v$.

- For every Type 2 vertex $x_{(u^v,v^u)}$, take an arbitrary node $t$ in the modified tree decomposition whose bag contains $x_{(u^v,v^u)}$; attach to it another node $t'$ such that the bag $X_{t'}$ of node $t'$ contains $x_{(u^v,v^u)}$ and the vertices of the triangle corresponds to $x_{(u^v,v^u)}$.

- For every edge $(u, v)$ of $G$, we take an arbitrary node $t$ in $T$ whose bag $X_t$ contains $u$. If $u$ is of low-degree-type then attach to it a chain of $2[(r + 1) - \lceil \frac{d_w(x; G)}{2} \rceil]$ nodes such that the bag of node $i$ contains $u$ and the vertices of $i$th triangle corresponds to $u$.

- For every edge $(u, v)$ of $G$, we take an arbitrary node $t$ in $T$ whose bag $X_t$ contains $u$. If $u$ is of high-degree-type then attach to it two chains of node: the first chain of node $1, 2, \ldots, r + 2$ such that the bag $X_i$ of node $i$ contains $u$ and the vertices of $i$ triangle corresponds to $u$; and the second chain of nodes $1, 2, \ldots, d_w(x; G) - r$ such that the bag $X_i$ of node $i$ contains $u, v_i^\Delta$ and the vertices of two triangles corresponds to $v_i^\Delta$.

It is easy to verify that the result is a valid tree decomposition of $G'$ and its width is at most the treewidth of $G$ plus eight.
Now we show that our reduction is correct. That is, we prove that $I$ is a yes instance of MINIMUM MAXIMUM OUTDEGREE if and only if $I'$ is a yes instance of HARMLESS SET. Let $D$ be the directed graph obtained by an orientation of the edges of $G$ such that for each vertex the sum of the weights of outgoing edges is at most $r$. We claim that the set

$$H = V(G) \bigcup_{(u,v) \in E(D)} V_{uv} \bigcup_{x \in \text{high-degree-type}} V_x$$

is harmless set of size at least $k$. Next, we show that all the vertices in $H$ satisfy the threshold condition. It is easy to verify that each $u \in \bigcup_{(u,v) \in E(G)} (V_{uv} \cup V_{vu})$ satisfies the threshold condition as $u$ has $n(u)$ neighbours in $H$ and $n(u) + 1$ neighbours outside $H$, that is, $u$ has less than $\lceil \frac{d(w(u))}{2} \rceil$ neighbours in $H$. Each

$$x \in \bigcup_{(u,v) \in E(G)} \left\{ x(u_i^1,v_i^1), x(u_i^1,v_i^2), x(u_i^2,v_i^2), x(u_i^2,v_i^3), x(v_i^2,v_i^3) \mid 1 \leq i \leq \omega(u,v) - 1 \right\}$$

satisfies the threshold condition as $x$ has only one neighbour in $H$ and two neighbours outside $H$, that is, $x$ has less than $\lceil \frac{d(w(x))}{2} \rceil = \lceil \frac{3}{2} \rceil = 2$ neighbours in $H$. It is also easy to see that the vertices of triangles satisfy the threshold condition. Let $u$ be an arbitrary vertex of low-degree-type. If the weighted degree of $u$ in $G$ is $d_u(u;G)$ then its degree in $G'$ is $d_u(u;G) + 2 \left( r + 1 - \lceil \frac{d_u(u;G)}{2} \rceil \right)$. Observe that the neighbours of $u$ inside $H$ are all of type $1$ which is equal to outdegree of $u$ and we know outdegree of $u$ is bounded by $r$. Thus each low-degree-type vertex has less than $\lceil \frac{d_u(x)}{2} \rceil = r + 1$ neighbours in $H$. Therefore each low-degree-type vertex satisfies the threshold condition. Next, let $x$ be an arbitrary vertex of high-degree-type. If the weighted degree of $x$ in $G$ is $d_u(x;G)$ then its degree in $G'$ is $2d_u(x;G) + 2$. Clearly the neighbours of $x$ inside $H$ are at most $r + (d_u(x;G) - r)$. Therefore the vertices of high-degree-type satisfy the threshold condition. This implies that $I'$ is a yes instance.

Conversely, assume that $G'$ admits a harmless set $H$ of size at least $k$. We make the following observations: (i) let $C$ be the set of all triangles introduced in the reduction algorithm, then $C$ does not intersect with $H$. This is true because any vertex with degree $2$ has threshold equal to $1$. This implies that both the neighbours of that vertex have to be outside the solution as otherwise the vertex will fail to satisfy the threshold condition, (ii) for each $(u,v) \in E(G)$ the set $V_{uv} \cup V_{vu}$ contributes at most half vertices in $H$ as otherwise $x_{(u^i,v^j)}$ for some $1 \leq i \leq w(u,v)$ will fail to satisfy the threshold condition. Note that the total number of vertices in $\bigcup_{(u,v) \in E(G)} V_{uv} \cup V_{vu}$ is $2W$. The above observations imply that the size of harmless set $H$ is at most $|V(G')| - W - 3|C| = n + W + \sum_{(u,v) \in E(G)} (3w(u,v) - 2) + \sum_{x \in \text{high-degree-type}} (d_u(x;G) - r)$, which is equal to $k$. It
implies that either \( V_{uv} \subseteq H \) or \( V_{vu} \subseteq H \) for all \((u, v) \in E(G)\) as otherwise some vertex \( x(u_v^i, v_u^{i+1}) \) will fail to satisfy the threshold condition. Hence the harmless set is of the form

\[
H = V(G) \bigcup_{(u, v) \in E(G)} (V_{uv} \text{ or } V_{vu}) \bigcup_{x \in \text{high-degree-type}} V^x \bigcup_{(u, v) \in E(G)} \left\{ x(u_v^i, v_u^i), x(u_v^{i+1}, v_u^i), x(u_{w(u, v)}^i, v_{w(u, v)}^i) \mid 1 \leq i \leq w(u, v) - 1 \right\}.
\]

Next, we define a directed graph \( D \) by \( V(D) = V(G) \) and

\[
E(D) = \left\{ (u, v) \mid u, v \in V(D) \text{ and } V_{uv} \subseteq H \right\} \bigcup \left\{ (v, u) \mid u, v \in V(D) \text{ and } V_{vu} \subseteq H \right\}.
\]

Let us assume that there exists a vertex \( x \in V(G) \) of low-degree-type such that the outdegree is more than \( r \). We can easily see that \( d_H(x) \geq \left\lceil \frac{d_G(x)}{2} \right\rceil \) which is a contradiction. Let us assume that there exists a vertex \( x \in V(G) \) of high-degree-type such that the outdegree is more than \( r \). We can easily see that \( d_H(x) \geq d_G(x; G) \geq \left\lceil \frac{d_G(x)}{2} \right\rceil \) which is a contradiction. This implies that \( I \) is a yes-instance.

6 Hardness Results

In this section we show that HARMSLESS SET is \( \text{W}[1] \)-hard parameterized by a vertex deletion set to trees of height at most three, that is, a subset \( D \) of the vertices of the graph such that every component in the graph, after removing \( D \), is a tree of height at most three. We show our hardness result for HARMSLESS SET using a reduction from the MULTIDIMENSIONAL RELAXED SUBSET SUM (MRSS) problem.

| MULTIDIMENSIONAL SUBSET SUM (MSS) |
|-----------------------------------|
| **Input:** An integer \( k \), a set \( S = \{s_1, \ldots, s_n\} \) of vectors with \( s_i \in \mathbb{N}^k \) for every \( i \) with \( 1 \leq i \leq n \) and a target vector \( g \in \mathbb{N}^k \). |
| **Parameter:** \( k \) |
| **Question:** Is there a subset \( S' \subseteq S \) such that \( \sum_{s \in S'} s = g ? \) |

We consider a variant of MSS that we require in our proofs. In the MULTIDIMENSIONAL RELAXED SUBSET SUM (MRSS) problem, an additional integer \( k' \) is given (which will be part of the parameter) and we ask whether there is a subset \( S' \subseteq S \) with \( |S'| \leq k' \) such that \( \sum_{s \in S'} s \geq g \). This variant can be formalized as follows:
Multidimensional Relaxed Subset Sum (MRSS)

**Input:** An integer \( k \), a set \( S = \{s_1, \ldots, s_n\} \) of vectors with \( s_i \in \mathbb{N}^k \) for every \( i \) with \( 1 \leq i \leq n \), a target vector \( t \in \mathbb{N}^k \) and an integer \( k' \).

**Parameter:** \( k + k' \)

**Question:** Is there a subset \( S' \subseteq S \) with \( |S'| \leq k' \) such that \( \sum_{s \in S'} s \geq g \)?

It is known that MRSS is \( W[1] \)-hard when parameterized by the combined parameter \( k + k' \), even if all integers in the input are given in unary \([14]\). We now prove the following theorem:

**Theorem 5.** The Harmless Set problem with general thresholds is \( W[1] \)-hard when parameterized by the size of a vertex deletion set into trees of height at most 3, even when restricted to bipartite graphs.

**Proof.** Let \((k, k', S, g)\) be an instance of MRSS. From this we construct an instance \((G, t, r)\) of Harmless Set the following way. For each vector \( s \in S \), we introduce a tree \( T^s \) of height three. For \( s \in S \), \( \max(s) \) is the value of the largest coordinate of \( s \) and \( \max(S) \) is maximum of \( \max(s) \) values. The tree \( T^s \) consists of vertices \( V(T^s) = A^s \cup B^s \cup \{e^s\} \) where \( A^s = \{a_1^s, \ldots, a_{\max(S)}^s\} \) and \( B^s = \{b_1^s, \ldots, b_{\max(S)}^s\} \). Make \( e^s \) adjacent to every vertex of \( B^s \). We also make \( a_i^s \) adjacent to \( b_i^s \) for all \( 1 \leq i \leq \max(s) \). Next, we introduce the set \( U = \{u_1, \ldots, u_k\} \) of vertices into \( G \). For each \( 1 \leq i \leq k \) and \( s \in S \), make \( u_i \) adjacent to exactly \( s(i) \) many vertices of \( A^s \) arbitrarily. We introduce three cycles \( C_1, C_2, C_3 \) of length four where \( V(C_1) = \{a_1, a_2, a_3, a_4\} \), \( V(C_2) = \{b_1, b_2, b_3, b_4\} \) and \( V(C_3) = \{c_1, c_2, c_3, c_4\} \). We make \( a_1 \) adjacent to every vertex of \( \bigcup_{s \in S} A^s \), make \( b_1 \) adjacent to every vertex of \( \bigcup_{s \in S} B^s \) and make \( c_1 \) adjacent to every vertex of \( \bigcup_{s \in S} \{e^s\} \). This completes the construction of graph \( G \). Note that \( G \) is a bipartite graph with bipartition

\[
V_1 = U \bigcup_{s \in S} B^s \bigcup \{a_1, a_3, b_2, b_4, c_1, c_3\}
\]

and

\[
V_2 = \bigcup_{s \in S} A^s \bigcup_{s \in S} \{e^s\} \bigcup \{a_2, a_4, b_1, b_3, c_2, c_4\}.
\]

We observe that if we delete the set \( U \cup \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_1\} \) of size \( k + 9 \) from \( G \) then we are left with trees of height at most three. We define the threshold function as follows:

\[
\sum_{s \in S'} s \]

\( \sum_{s \in S'} s \geq g \)

This completes the construction of graph \( G \). Note that \( G \) is a bipartite graph with bipartition

\[
V_1 = U \bigcup_{s \in S} B^s \bigcup \{a_1, a_3, b_2, b_4, c_1, c_3\}
\]

and

\[
V_2 = \bigcup_{s \in S} A^s \bigcup_{s \in S} \{e^s\} \bigcup \{a_2, a_4, b_1, b_3, c_2, c_4\}.
\]

We observe that if we delete the set \( U \cup \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_1\} \) of size \( k + 9 \) from \( G \) then we are left with trees of height at most three. We define the threshold function as follows:
Fig. 9. The graph $G$ in the proof of Theorem 5 constructed from MRSS instance $S = \{(2, 1), (1, 1), (1, 2)\}$, $g = (3, 3), k = 2, k' = 2$. The set $S' = \{(2, 1), (1, 2)\}$ forms a solution of MRSS instance and the set $H = \{u_1, u_2\} \cup B^{a_1} \cup B^{a_3} \cup B^{b_2} \cup B^{b_3} \cup A^{s_2} \cup \{c^{s_1}, c^{s_3}\}$ forms a harmless set in $G$.

We set $r = k + n \times \max(S) + (n - k') \times \max(S) + k'$. Now we show that our reduction is correct. That is, we claim $(k, k', S, g)$ is a yes instance of MRSS if and only if $(G, t, r)$ is a yes instance of HARMLESS SET. Towards showing the forward direction, let $S' \subseteq S$ be such that $|S'| \leq k'$ and $\sum_{s \in S'} s \geq g$. We claim that the set

$$H = U \cup \bigcup_{s \in S} B^s \bigcup_{s \in S \setminus S'} A^s \bigcup_{s \in S'} \{c^s\}$$

is a harmless set of size at least $r$. It is easy to see that $|H| \geq r$. Next, we show that all the vertices in $G$ satisfy the threshold condition. Let $u_1$ be an
arbitrary element in \( U \). We know that \( d(u_i) = \sum_{s \in S} s(i) \) and \( \sum_{s \in S} s \geq g \). Hence, we get \( d_H(u_i) \leq \sum_{s \in S} s(i) - g(i) < t(u_i) = \sum_{s \in S} s(i) - g(i) + 1 \). It is easy to see that every vertex \( u \) of \( \bigcup_{s \in S} A^s \) satisfies the threshold condition. As \( a_1 \notin H \), \( d_H(u) = d(u) - 1 < d(u) = t(u) \). Similarly, every vertex of \( \bigcup_{s \in S} \{ c_s \} \) satisfies the threshold condition as \( c_1 \notin H \). For each \( b^*_i \in \bigcup_{s \in S} B^s \), we have either \( a^*_i \) or \( c^*_i \) inside the solution and \( b_1 \notin H \). Thus \( d_H(b^*_i) = 1 < 2 = t(b^*_i) \). Hence all the vertices in \( \bigcup_{s \in S} B^s \) satisfy the threshold condition. As \( |S'| \leq k' \), we see that \( c_1 \) has at most \( k' \) neighbours inside \( H \), thus \( d_H(c_1) = k' < k' + 1 = t(c_1) \). For the rest of the vertices in \( H \) it is easy to verify that the threshold condition is satisfied.

Towards showing the reverse direction of the claim, let \( H \) be a harmless set of size at least \( r \) in \( G \). It is easy to see that \( H \cap \{ a_i, b_i, c_i \mid 1 \leq i \leq 4 \} = \emptyset \) as otherwise one of the vertex in \( \{ a_i, b_i, c_i \mid 1 \leq i \leq 4 \} \) will fail to satisfy the threshold condition. Clearly \( U \) and \( \bigcup_{s \in S} B^s \) can contribute at most \( k \) and \( n \times \max(S) \) to the solution respectively. We also observe that the set \( \bigcup_{s \in S} A^s \) can contribute at most \( (n - k') \times \max(S) \) to the solution due to the fact that \( t(a_1) = (n - k') \times \max(S) + 1 \). Therefore, the only way to have a harmless set of size at least \( r \) is that the set \( \bigcup_{s \in S} \{ c^s \} \) contribute at least \( k' \) elements to set \( H \).

Since \( t(c_1) = k' + 1 \), the set \( \bigcup_{s \in S} \{ c^s \} \) contributes at most \( k' \) elements to set \( H \). Therefore, the set \( \bigcup_{s \in S} \{ c^s \} \) contributes exactly \( k' \) elements to \( H \). We define

\[
S' = \{ s \in S \mid c^s \in H \}.
\]

Observe that \( \bigcup_{s \in S'} A^s \cap H = \emptyset \) as otherwise one of the vertices in the set \( \bigcup_{s \in S'} B^s \) will not satisfy the threshold condition. From here, we see that \( H = U \cup \bigcup_{s \in S'} A^s \cup \bigcup_{s \in S'} \{ c^s \} \). Since each \( u_i \in U \) satisfies the threshold condition, we have \( d_H(u_i) = \sum_{s \in S, s \notin S'} s(i) = \sum_{s \in S} s(i) - \sum_{s \in S'} s(i) < t(u_i) = \sum_{s \in S} s(i) - g(i) + 1 \). This implies that \( \sum_{s \in S'} s(i) \geq g(i) \) for all \( 1 \leq i \leq k \). Therefore \( (k, k', S, g) \) is a yes-instance.

Clearly trees of height at most three are trivially acyclic. Moreover, it is easy to verify that such trees have pathwidth \([19]\) and treedepth \([22]\) at most three, which implies:

**Theorem 6.** The HARMLESS SET problem with general thresholds is \( W[1] \)-hard when parameterized by any of the following parameters:

- the feedback vertex set number,
– the pathwidth and treedepth of the input graph,
– the size of a minimum set of vertices whose deletion results in components
  of pathwidth/treedepth at most three,
even when restricted to bipartite graphs.

7 W[1]-hardness parameterized by cluster vertex deletion number

The cluster vertex deletion number of a graph is the minimum number of its
vertices whose deletion results in a disjoint union of complete graphs. This gen-
eralizes the vertex cover number, provides an upper bound to the clique-width
and is related to the previously studied notion of the twin cover of the graph
under consideration.

**Theorem 7.** The Harmless Set problem with general thresholds is W[1]-hard
when parameterized by the cluster vertex deletion number of the input graph.

**Proof.** Let \((k, k', S, g)\) be an instance of MRSS. From this we construct an in-
stance \((G, t, r)\) of Harmless Set the following way.

![Graph G](image)

**Fig. 10.** The graph \(G\) in the proof of Theorem 7 constructed from MRSS instance
\(S = \{(2, 1), (1, 1), (1, 2)\}, g = (3, 3), k = 2, k' = 2.\) The set \(S' = \{(2, 1), (1, 2)\}\) forms a
solution of MRSS instance and the set \(H = B^{s_1} \cup B^{s_2} \cup A^{s_2}\) forms a harmless
set in \(G.\)

For each vector \(s \in S\), we introduce a clique \(K^s.\) For \(s \in S, \max(s)\) is
the value of the largest coordinate of \(s\) and \(\text{max}(S)\) is maximum of \(\text{max}(s)\)
values. The clique \(K^s\) consists of vertices \(V(K^s) = A^s \cup B^s\) where \(A^s =
\{a_1^s, \ldots, a_{\text{max}(S)}^s+1\}\) and \(B^s = \{b_1^s, \ldots, b_{\text{max}(S)}^s+1\}\). Next, we introduce the set
\(U = \{u_1, \ldots, u_k\}\) of vertices into \(G.\) For each \(1 \leq i \leq k\) and \(s \in S,\) make \(u_i\)
adjacent to exactly \(s(i)\) many vertices of \(A^s\) arbitrarily. Finally, we add two ver-
tices \(x\) and \(y.\) Next, make \(x\) adjacent to all the vertices in set \(U\) and \(y.\) This
completes the construction of graph \(G.\) Note that deletion of the set \(U\) of size \(k\)
from $G$ results in a disjoint union of complete graphs. We define the threshold function as follows:

$$t(u) = \begin{cases} 
1 & \text{if } u \in \{x, y\} \\
\max(S) + 1 & \text{if } u \in \bigcup_{s \in S} A^s \\
\max(S) + 2 & \text{if } u \in \bigcup_{s \in S} B^s \\
\sum_{s \in S} s(i) - g(i) + 1 & \text{if } u = u_i \text{ for all } 1 \leq i \leq k
\end{cases}$$

We set $r = k' \times \max(S) + (n - k') \times (\max(S) + 1)$. Now we show that our reduction is correct. That is, we claim $(k, k', S, g)$ is a yes instance of MRSS if and only if $(G, t, r)$ is a yes instance of HARMLESS SET. Towards showing the forward direction, let $S' \subseteq S$ be such that $|S'| \leq k'$ and $\sum_{s \in S'} s \geq g$. We claim that

$$H = \bigcup_{s \in S'} B^s \setminus \{b^s\} \bigcup_{s \in S \setminus S'} A^s$$

is a harmless set in $G$ of size at least $r$. Clearly $|H| \geq r$. Next, we show that all vertices of $G$ satisfy the threshold condition. Since $x$ and $y$ have no neighbours in $H$, they satisfy the threshold condition. If $s \in S'$, then every vertex of $A^s \cup B^s$ has at most $\max(S)$ neighbours in $H$. Therefore, they satisfy the threshold condition. If $s \in S \setminus S'$, then every vertex of $A^s$ has exactly $\max(S)$ neighbours in $H$ and every vertex of $B^s$ has exactly $\max(S) + 1$ neighbours in $H$. Therefore, all vertices in cliques $K^s$ satisfy the threshold condition. Let $u_i$ be an arbitrary element in $U$. As $\sum_{s \in S'} s(i) \geq g(i)$, we get $d_H(u_i) = \sum_{s \in S'} s(i) = \sum_{s \in S} s(i) - \sum_{s \in S'} s(i) \leq \sum_{s \in S} s(i) - g(i) < t(u_i)$.

Towards showing the reverse direction, let $H$ be a harmless set of size at least $r$ in $G$. It is easy to see that $H \cap (U \cup \{x, y\}) = \emptyset$ as otherwise one of the vertices in $\{x, y\}$ will fail to satisfy the threshold condition. Observe that any clique $K^s$ can contribute at most $\max(S) + 1$ vertices as otherwise vertices in set $A^s$ will fail to satisfy the threshold condition. We prove the following simple claim.

**Claim.** If $|K^s \cap H| = \max(S) + 1$ then $K^s \setminus H = A^s$.

**Proof.** Targeting a contradiction, assume that there exists a vertex $a^s \in A^s$ such that $a^s \not\in H$. As $|K^s \cap H| = \max(S) + 1$, we have $d_H(a^s) = \max(S) + 1 = t(a^s)$, which is a contradiction.

Note that $n$ cliques together contributes at least $r = k' \times \max(S) + (n - k') \times (\max(S) + 1)$ vertices to $H$ and each clique can contributes at most $\max(S) + 1$ vertices. Therefore, by Pigeonhole principle, there are at least $(n - k')$ cliques $K^s$ such that $|H \cap K^s| = \max(S) + 1$. By the above claim, there are at least $(n - k')$ cliques $K^s$ such that $H \cap K^s = A^s$. We define

$$S' = \{s \in S \mid H \cap K^s \neq A^s\}.$$
Clearly for \( s \in S' \), we have \( |K^s \cap H| \leq \max(S) \). It is easy to see that

\[
H' = (H \setminus \bigcup_{s \in S'} A^s) \cup \bigcup_{s \in S'} B^s \setminus \{b^s_i\}
\]

is again a harmless set with \( |H'| \geq |H| \). Every vertex of \( K^s \), \( s \in S' \), satisfies the threshold condition because \( |K^s \cap H'| = \max(S) \). For each \( u_i \in U \), we see that \( d_{H'}(u_i) \leq d_H(u_i) < t(u_i) \). For rest of the vertices, we can easily verify that the threshold conditions are satisfied. This implies that \( H' \) is a harmless set of size at least \( r \). So we consider the harmless set to be of the form

\[
H' = \bigcup_{s \in S \setminus S'} A^s \bigcup_{s \in S'} B^s \setminus \{b^s_1\}
\]

Since each \( u_i \in U \) satisfies the threshold condition, we have

\[
d_{H'}(u_i) = \sum_{s \in S \setminus S'} s(i) = \sum_{s \in S} s(i) - \sum_{s \in S'} s(i) < t(u_i) = \sum_{s \in S} s(i) - g(i) + 1.
\]

This implies that \( \sum_{s \in S'} s(i) \geq g(i) \) for \( 1 \leq i \leq k \). Therefore \( (k, k', S, g) \) is a yes-instance.

\[\square\]

8 Graphs of bounded clique-width

This section presents a polynomial time algorithm for the HARMLESS SET problem for graphs of bounded clique-width. The clique-width of a graph \( G \), denoted by \( \text{cw}(G) \), is the minimum number of labels needed to construct \( G \) using the following four operations:

1. Create a new graph with a single vertex \( v \) with label \( i \) (written \( i(v) \)).
2. Take the disjoint union of two labelled graphs \( G_1 \) and \( G_2 \) (written \( G_1 \cup G_2 \)).
3. Add an edge between every vertex with label \( i \) and every vertex with label \( j \), \( i \neq j \) (written \( n_{i,j} \)).
4. Relabel every vertex with label \( i \) to have label \( j \) (written \( \rho_{i \to j} \)).

We say that a construction of a graph \( G \) with the four operations is a c-expression if it uses at most \( c \) labels. Thus the clique-width of \( G \) is the minimum \( c \) for which \( G \) has a c-expression. A c-expression is a rooted binary tree \( T \) such that

1. each leaf has label \( i \) for some \( i \in \{1, \ldots, c \} \),
2. each non-leaf node with two children has label \( \cup \), and
3. each non-leaf node with only one child has label \( \rho_{i \to j} \) or \( \eta_{i,j} \) (\( i, j \in \{1, \ldots, c \}, i \neq j \)).

Example 4. Consider the graph \( P_n \), which is simply a path on \( n \) vertices. Note that \( \text{cw}(P_1) = 1 \) and \( \text{cw}(P_2) = \text{cw}(P_3) = 2 \). Now consider a path on four vertices \( v_1, v_2, v_3, v_4 \), in that order. Then this path can be constructed using the four operations (using only three labels) as follows:

\[
\eta_{3,2}(3(v_4) \cup \rho_{3 \to 2}(\rho_{2 \to 1}(\eta_{3,2}(3(v_3) \cup \eta_{2,1}(2(v_2) \cup 1(v_1))))))).
\]

This construction can readily be generalized to longer paths for \( n \geq 5 \). It is easy to see that \( \text{cw}(P_n) = 3 \) for all \( n \geq 4 \).
A $c$-expression represents the graph represented by its root. A $c$-expression of a $n$-vertex graph $G$ has $O(n)$ vertices. A $c$-expression of a graph is irredundant if for each edge $\{u, v\}$, there is exactly one node $\eta_{i,j}$ that adds the edge between $u$ and $v$. It is known that a $c$-expression of a graph can be transformed into an irredundant one with $O(n)$ nodes in linear time. Here we use irredundant $c$-expression only.

Computing the clique-width and a corresponding $c$-expression of a graph is NP-hard. For $c \leq 3$, we can compute a $c$-expression of a graph of clique-width at most $c$ in $O(n^2m)$ time, where $n$ and $m$ are the number of vertices and edges, respectively. For fixed $c \geq 4$, it is not known whether one can compute the clique-width and a corresponding $c$-expression of a graph in polynomial time. On the other hand, it is known that for any fixed $c$, one can compute a $(2^{c+1} - 1)$-expression of a graph of clique-width $c$ in $O(n^3)$ time. For more details see [16].

**Theorem 8.** Given an $n$-vertex graph $G$ and an irredundant $c$-expression $T$ of $G$, the Harmless Set problem is solvable in $O(n^{4c})$ time.

For each node $t$ in a $c$-expression $T$, let $G_t$ be the vertex-labeled graph represented by $t$. We denote by $V_t$ the vertex set of $G_t$. For each $i$, we denote the set of $i$-vertices in $G_t$ by $V^i_t$. For each node $t$ in $T$, we construct a table $dp_t(r, s) \in \{\text{true, false}\}$ with indices $r : \{1, \ldots, c\} \to \{0, \ldots, n\}$ and $s : \{1, \ldots, c\} \to \{-n+1, \ldots, n-1\} \cup \{\infty\}$ as follows. We set $dp_t(r, s) = \text{true}$ if and only if there exists a set $S$ in $V_t$ such that for all $i \in \{1, 2, \ldots, c\}$

1. $r(i) = |S \cap V^i_t|$;
2. $s(i) = \min_{v \in V^i_t} \{t(v) - |N_{G_t}(v) \cap S|\}$, otherwise $s(i) = \infty$.

That is, $r(i)$ denotes the number of the $i$-vertices in $S$ and $s(i)$ is the “surplus” at the weakest $i$-vertex in $S$.

Let $\tau$ be the root of the $c$-expression $T$ of $G$. Then $G$ has a harmless set of size $h$ if there exist $r, s$ satisfying

1. $dp_\tau(r, s) = \text{true}$;
2. $\sum_{i=1}^{c} r(i) = h$
3. $\min\{s(i)\} \geq 1$.

In the following, we compute all entries $dp_t(r, s)$ in a bottom-up manner. There are $(n+1)^2 \cdot (2n)^c = O(n^{2c})$ possible tuples $(r, s)$. Thus, to prove Theorem 8 it is enough to prove that each entry $dp_t(r, s)$ can be computed in time $O(n^{2c})$ assuming that the entries for the children of $t$ are already computed.

**Lemma 8.** For a leaf node $t$ with label $i$, $dp_t(r, s)$ can be computed in $O(1)$ time.

**Proof.** Observe that $dp_t(r, s) = \text{true}$ if and only if $r(j) = 0$, $s(j) = \infty$ for all $j \neq i$, and either

1. $r(i) = 0$, $s(i) = \infty$ or
The first case corresponds to $S = \emptyset$, and the second case corresponds to $S = V_t^r$. These conditions can be checked in $O(1)$ time.

**Lemma 9.** For a $∪$ node $t$, $dp_t(r, s)$ can be computed in $O(n^{2c})$ time.

**Proof.** Let $t_1$ and $t_2$ be the children of $t$ in $T$. Then $dp_t(r, s) = \text{true}$ if and only if there exist $r_1, s_1$ and $r_2, s_2$ such that $dp_{t_1}(r_1, s_1) = \text{true}$, $dp_{t_2}(r_2, s_2) = \text{true}$, $r(i) = r_1(i) + r_2(i)$, $s(i) = \min\{s_1(i), s_2(i)\}$ for all $i$. The number of possible pairs for $(r_1, r_2)$ is at most $(n+1)^c$ as $r_2$ is uniquely determined by $r_1$. There are at most $(2n)^c$ possible pairs for $(s_1, s_2)$. In total, there are $O(n^{2c})$ candidates. Each candidate can be checked in $O(1)$ time, thus the lemma holds.

**Lemma 10.** For a $η_{ij}$ node $t$, $dp_t(r, s)$ can be computed in $O(1)$ time.

**Proof.** Let $t'$ be the child of $t$ in $T$. Then, $dp_t(r, s) = \text{true}$ if and only if $dp_{t'}(r', s') = \text{true}$ with the following conditions:

- $s(h) = s'(h)$ hold for all $h \notin \{i, j\}$;
- $s(i) = s'(i) - r(j)$ and $s(j) = s'(j) - r(i)$.

We now explain the condition for $s(i)$. Recall that $T$ is irredundant. That is, the graph $G_t$ does not have any edge between the $i$-vertices and the $j$-vertices. In $G_t$, an $i$-vertex has exactly $r(j)$ more neighbours in $S$ and similarly a $j$-vertex has exactly $r(i)$ more neighbours in $S$. Thus we have $s(i) = s'(i) - r(j)$ and $s(j) = s'(j) - r(i)$. The lemma holds as there is only one candidate for each $s'(i)$ and $s'(j)$.

**Lemma 11.** For a $ρ_{i→j}$ node $t$, $dp_t(r, s)$ can be computed in $O(n^2)$ time.

**Proof.** Let $t'$ be the child of $t$ in $T$. Then, $dp_t(r, s) = \text{true}$ if and only if there exist $r', s'$ such that $dp_{t'}(r', s') = \text{true}$, where:

- $r(i) = 0$, $r(j) = r'(i) + r'(j)$, and $r(h) = r'(h)$ if $h \notin \{i, j\}$;
- $s(i) = \infty$, $s(j) = \min\{s'(i), s'(j)\}$, and $s(h) = s'(h)$ if $h \notin \{i, j\}$.

The number of possible pairs for $(r'(i), r'(j))$ is $O(n)$ as $r'(j)$ is uniquely determined by $r'(i)$. There are at most $O(n)$ possible pairs for $(s'(i), s'(j))$. In total, there are $O(n^2)$ candidates. Each candidate can be checked in $O(1)$ time, thus the lemma holds.

## 9 Harmless Set on Planar Graphs

In this section, we propose a fixed parameter tractable algorithm for Harmless Set parameterized by the solution size, even when restricted to planar graphs. Note that the Harmless Set problem parameterized by the solution size is $W[1]$-hard on general graphs even when all thresholds are bounded by a constant.
The Harmless Set Problem 29

Theorem 9. The Harmless Set problem with general thresholds parameterized by solution size is fixed parameter tractable on planar graphs.

Proof. Let \((G, k)\) be an instance of Harmless Set, where \(G\) is a planar graph. First we do some preprocessing on \(G\). If \(v \in V(G)\) has a neighbour with threshold value 1 then clearly \(v\) cannot be part of any harmless set; we color \(v\) red. We color the rest of the vertices green. We observe that if a vertex and all of its neighbours are colored red then its removal does not change the solution. This shows that the following rule is safe.

- Reduction 1: If a vertex \(v\) and its neighbours are colored red then delete \(v\) from \(G\). The new instance is \((G - v, k)\).

Next we claim that after exhaustive application of Reduction 1 if the diameter of the reduced graph \(G\) is greater than or equal to 6\(k\) then we always get a yes instance. Let us assume that the diameter of graph \(G\) is 6\(k\). Then there exists a pair of nonadjacent vertices \(a\) and \(b\) such that \(d(a, b) = 6k\). Let \(P = (v_1, v_2, \ldots, v_{6k+1})\) be a shortest path joining \(a\) and \(b\), where \(v_1 = a\) and \(v_{6k+1} = b\). Now, we construct a harmless set \(S\) of size \(k\) containing only green vertices. Since we cannot apply Reduction 1, every vertex on \(P\) is either green or at least one of its neighbours is green. For every vertex \(v_{6i+1} \in P, 0 \leq i \leq k\), if \(v_{6i+1}\) is colored green we include it in \(S\); otherwise if \(v_{6i+1}\) is colored red, we include one of its green neighbours in \(S\). We claim that \(S\) is a harmless set of size \(k\). Clearly, \(|S| \geq k\). Since we include only green vertices in \(S\), all the vertices with threshold one satisfy the threshold condition. Next, consider a vertex \(u\) with threshold at least two. We show that \(d_S(u) \leq 1\). Assume, for the sake of contradiction, that \(d_S(u) \geq 2\). We can assume that \(s_1 \in N_G[v_{6i+1}]\) and \(s_2 \in N_G[v_{6(i+1)+1}]\) for some \(i\). Note that, we have \(d(v_{6i+1}, v_{6(i+1)+1}) = 5\) by construction. If \(u\) is adjacent to \(s_1\) and \(s_2\) then it implies that \(d(v_{6i+1}, v_{6(i+1)+1}) \leq 4\) and hence \(d(a, b) < 6k\), which is a contradiction. This shows that all the vertices with threshold greater than or equal to two also satisfy the threshold condition. Therefore, we get a harmless set \(S\) of size at least \(k\). Based on the above argument, our second rule is the following.

- Reduction 2: If the diameter of \(G\) is more than or equal to 6\(k\) then we conclude that we are dealing with a yes-instance.

Let \((G, k)\) be an input instance such that Reduction 1 and 2 are not applicable to \((G, k)\). Then \(G\) is a planar graph with diameter at most 6\(k\), it implies that treewidth of \(G\) is at most 18\(k\). Now, we can solve the problem using a standard dynamic programming algorithm technique in time \(2^{O(k \log k)}\). Note that when we are looking for a harmless set of size \(k\), we can assume that \(t_{\max} \leq k + 1\) where \(t_{\max}\) denotes the value of maximum threshold.

10 Conclusion and Future Directions

We have shown that Harmless Set with general thresholds is \(W[1]\)-hard parameterized by the size of a vertex deletion set into trees of height at most 3.
and also the cluster vertex deletion number of the input graph. On the positive side, we have given FPT algorithms when parameterized by any of the following parameters: vertex integrity, neighbourhood diversity and twin cover. To give an upper bound on the complexity, we give an XP-algorithm when parameterized by cliquewidth. For the HARMLESS SET problem with majority thresholds, we have shown W[1]-hardness parameterized by treewidth.

The natural next step is to figure out the parameterized complexity of HARMLESS SET for general thresholds with respect to the parameters such as vertex deletion to disjoint paths, modular width and co-cluster vertex deletion set. For HARMLESS SET with majority thresholds, it will be interesting to see if the parameters such as treedepth, feedback vertex set and cluster vertex deletion set allow FPT algorithms or the problem still remains hard.

References

1. A. Aazami and K. Stilp. Approximation algorithms and hardness for domination with propagation. SIAM Journal on Discrete Mathematics, 23(3):1382–1399, 2009.
2. C. Bazgan and M. Chopin. The complexity of finding harmless individuals in social networks. Discret. Optim., 14(C):170–182, Nov. 2014.
3. C. Bazgan, M. Chopin, A. Nichterlein, and F. Sikora. Parameterized approximability of maximizing the spread of influence in networks. In D.-Z. Du and G. Zhang, editors, Computing and Combinatorics, pages 543–554, Berlin, Heidelberg, 2013. Springer Berlin Heidelberg.
4. O. Ben-Zwi, D. Hermelin, D. Lokshtanov, and I. Newman. Treewidth governs the complexity of target set selection. Discrete Optimization, 8(1):87–96, 2011. Parameterized Complexity of Discrete Optimization.
5. C. C. Centeno, M. C. Dourado, L. D. Penso, D. Rautenbach, and J. L. Szwarcfiter. Irreversible conversion of graphs. Theoretical Computer Science, 412(29):3693–3700, 2011.
6. N. Chen. On the approximability of influence in social networks. SIAM Journal on Discrete Mathematics, 23(3):1400–1415, 2009.
7. C.-Y. Chiang, L.-H. Huang, B.-J. Li, J. Wu, and H.-G. Yeh. Some results on the target set selection problem. Journal of Combinatorial Optimization, 25(4):702–715, 2013.
8. M. Chopin, A. Nichterlein, R. Niedermeier, and M. Weller. Constant thresholds can make target set selection tractable. Theory of Computing Systems, 55(1):61–83, 2014.
9. M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. Parameterized Algorithms. Springer, 2015.
10. R. G. Downey and M. R. Fellows. Parameterized Complexity. Springer, 2012.
11. P. G. Drange, M. Dregi, and P. van ’t Hof. On the computational complexity of vertex integrity and component order connectivity. Algorithmica, 76(4):1181–1202, 2016.
12. P. A. Dreyer and F. S. Roberts. Irreversible k-threshold processes: Graph-theoretical threshold models of the spread of disease and of opinion. Discrete Applied Mathematics, 157(7):1615–1627, 2009.
13. M. R. Fellows, D. Lokshtanov, N. Misra, F. A. Rosamond, and S. Saurabh. Graph layout problems parameterized by vertex cover. In S.-H. Hong, H. Nagamochi,
and T. Fukunaga, editors, *Algorithms and Computation*, pages 294–305, Berlin, Heidelberg, 2008. Springer Berlin Heidelberg.

14. R. Ganian, F. Klute, and S. Ordyniak. On structural parameterizations of the bounded-degree vertex deletion problem. *Algorithmica*, 2020.

15. T. Gima, T. Hanaka, M. Kiyomi, Y. Kobayashi, and Y. Otachi. Exploring the gap between treedepth and vertex cover through vertex integrity. In T. Calamoneri and F. Corò, editors, *Algorithms and Complexity*, pages 271–285, Cham, 2021. Springer International Publishing.

16. M. Kamiński, V. V. Lozin, and M. Milanič. Recent developments on graphs of bounded clique-width. *Discrete Applied Mathematics*, 157(12):2747 – 2761, 2009.

17. R. Kannan. Minkowski’s convex body theorem and integer programming. *Mathematics of Operations Research*, 12(3):415–440, 1987.

18. D. Kempe, J. Kleinberg, and E. Tardos. Maximizing the spread of influence through a social network. In *Proceedings of the Ninth ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, KDD ’03, page 137–146, New York, NY, USA, 2003. Association for Computing Machinery.

19. T. Kloks. *Treedepth, Computations and Approximations*, volume 842 of *Lecture Notes in Computer Science*. Springer, 1994.

20. M. Lampis. Algorithmic meta-theorems for restrictions of treewidth. *Algorithmica*, 64:19–37, 2012.

21. H. W. Lenstra. Integer programming with a fixed number of variables. *Mathematics of Operations Research*, 8(4):538–548, 1983.

22. J. Nesetril and P. O. de Mendez. *Sparsity: Graphs, Structures, and Algorithms*. Springer Publishing Company, Incorporated, 2014.

23. A. Nichterlein, R. Niedermeier, J. Uhlmann, and M. Weller. On tractable cases of target set selection. *Social Network Analysis and Mining*, 3(2):233–256, 2013.

24. D. Peleg. Local majorities, coalitions and monopolies in graphs: a review. *Theoretical Computer Science*, 282(2):231 – 257, 2002.

25. T. Reddy and C. Rangan. Variants of spreading messages. *Journal of Graph Algorithms and Applications*, 15(5):683–699, 2011.

26. N. Robertson and P. Seymour. Graph minors. iii. planar tree-width. *Journal of Combinatorial Theory, Series B*, 36(1):49 – 64, 1984.

27. S. Szeider. Not so easy problems for tree decomposable graphs. *CoRR*, abs/1107.1177, 2011.

28. M. Tedder, D. Corneil, M. Habib, and C. Paul. Simpler linear-time modular decomposition via recursive factorizing permutations. In *Automata, Languages and Programming*, pages 634–645, Berlin, Heidelberg, 2008. Springer Berlin Heidelberg.