RESEARCH ARTICLE

On Besov regularity and local time of the stochastic heat equation

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doi:10.1007/s10959-021-02813-w

Abstract

Sharp Besov regularities in time and space variables are investigated for \((u(t, x), \ t \in [0, T], \ x \in \mathbb{R})\), the mild solution to the stochastic heat equation driven by space-time white noise. Existence, Hölder continuity, and Besov regularity of local times are established for \(u(t, x)\) viewed either as a process in the space variable or time variable. Hausdorff dimensions of their corresponding level sets are also obtained.

KEYWORDS

Stochastic heat equation; White noise; Besov-Orlicz spaces; Schauder functions; Haar basis; Local times; Hölder continuity; Hausdorff dimension

AMS CLASSIFICATION

60G15; 60G17; 60H05; 60H15

Introduction

Stochastic partial differential equations (SPDEs) model various random phenomena in physics, finance, fluid mechanics, among others, see, e.g., \cite{11, 25, 30}. They have been widely investigated by different approaches in the last three decades. Let us mention the analytic method \cite{31, 33}, the semigroup point of view \cite{23}, and the probabilistic setting using the theory of martingale measures \cite{46}. The particular case of stochastic heat equations has been intensively studied from many perspectives: See, e.g., \cite{2, 4, 22, 39, 40, 47} for regularity investigations and \cite{12, 15, 34} for many other studies. The regularity in Besov spaces of SPDEs has received much attention in the past decade. Among other things, such studies are closely related to the theme of adaptive numerical wavelet methods. For more details on this subject, we refer to \cite{16, 18}.

In this paper, we consider the following linear stochastic heat equation

\[
\begin{align*}
\frac{\partial u}{\partial t}(t, x) &= \frac{1}{2} \Delta u(t, x) + \frac{\partial^2}{\partial t \partial x} W(t, x), \quad t > 0, \ x \in \mathbb{R}, \\
u(0, x) &= 0, \quad x \in \mathbb{R},
\end{align*}
\]

where \(\frac{\partial^2}{\partial t \partial x} W(t, x)\) is a space-time white noise. In Section 2.1, following Walsh’s random field approach \cite{46}, we will briefly give a rigorous formulation of the formal equation

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The purpose of this paper is to investigate the Besov regularity for the process $(u(t, x), t \in [0, T], x \in [a, b])$ as well as its local time, respectively in the space variable $x$ (for fixed $t$) and in the time variable $t$ (for fixed $x$), with $T > 0$ and $a < b$ are arbitrary real values.

Besov spaces, involving a bounded or unbounded interval $I \subseteq \mathbb{R}$, usually noted in the literature by $B_{p,q}^\alpha(I)$ with $0 < \alpha < 1$, $1 \leq p, q \leq +\infty$, are a set of functions of $L^p(I)$ having a smoothness of order $\alpha$. They cover some classical function spaces as special cases. Namely, $B_{p,p}^\alpha(\mathbb{R})$ coincides with the classical Sobolev space $W^{\alpha,p}(\mathbb{R})$ and $B_{\infty,\infty}^\alpha(I)$ is the classical $\alpha$-Hölder space $C^\alpha(I)$. When the Orlicz norm is used in place of the $L^p$-norm, we obtain the well-known Besov-Orlicz spaces.

In this paper, we are essentially concerned by the class of Besov spaces $B_{p,\infty}^\alpha([0,1])$ and also by Besov-Orlicz spaces $B_{\infty,\infty}^\alpha([0,1])$, $\mathcal{N}$ is the Young function $\mathcal{N}(x) = e^{x^2} - 1$. We will note these spaces respectively by $B_p^\alpha$ and $B_{\mathcal{N}}^\alpha$. In this case, for $\alpha p > 1$ and for any $\varepsilon > 0$, we have the following continuous injections:

$$\mathcal{H}^\alpha \hookrightarrow B_p^\alpha \hookrightarrow B_p^\alpha \hookrightarrow \mathcal{H}^{\alpha - 1/p},$$

(2)

where $B_p^{\alpha,0}$ is a separable subspace of $B_p^\alpha$. Furthermore, it is important to recall that the Besov-Orlicz space $B_{\mathcal{N}}^\alpha$ is continuously embedded in $B_p^\alpha$ for any $p \geq 1$. See section [13] for more details and [42] for an introduction to Besov spaces.

Our first main result is to prove that, for any $p \geq 4$ we have:

i) For any fixed space variable $x$,

$$\mathbb{P} \left[ (u(t, x))_{t \in [0,1]} \in B_p^{1/4} \right] = 1 \quad \text{and} \quad \mathbb{P} \left[ (u(t, x))_{t \in [0,1]} \in B_p^{1/4,0} \right] = 0.$$

ii) For any fixed time variable $t$,

$$\mathbb{P} \left[ (u(t, x))_{x \in [0,1]} \in B_p^{1/2} \right] = 1 \quad \text{and} \quad \mathbb{P} \left[ (u(t, x))_{x \in [0,1]} \in B_p^{1/2,0} \right] = 0.$$

In fact, we even get a better result showing that i) and ii) are true respectively in the Besov-Orlicz space $B_{\mathcal{N}}^{1/4}$ and $B_{\mathcal{N}}^{1/2}$. We should point out here that, by a suitable affine change of variables, the interval $[0,1]$ may be replaced in i) (resp. ii)) by any arbitrary compact interval $[0, T]$ (resp. $[a, b]$).

On the other hand, injections (2) show that the obtained regularity results are the best one can get in the scale of Besov spaces. They improve the classical Hölder regularities of $u(t, x)$; namely, the process $u(t, x)$ satisfies a.s. $\alpha$-Hölder condition with $\alpha < \frac{1}{4}$ (resp. $\alpha < \frac{1}{2}$) in the time (resp. space) variable, see e.g. [22], [40] and references therein.

As far as we know, only the spatial Besov regularity of $u(t, x)$ has been partially investigated firstly in Deaconu’s thesis [24], where the author has also claimed the temporal regularity as in i) but she failed to provide a proof for her finding. Our approach is certainly much more technical than that of [24], but it allowed us to obtain sharp results. Our proof is based on the characterization of Besov spaces in terms of sequences spaces (Theorem [14]). We use essentially the same arguments, with some adjustments, like those used by Ciesielski, Kerkyacharian, and Royen in [14], where the authors have investigated Besov regularity for a large class of Gaussian processes.

Recently, a more direct method, which uses the usual modulus-of-continuity definition of the Besov norms, has been employed in [35] to prove the temporal regularity in the
Besov-Orlicz space for solutions to parabolic stochastic differential equations. Many other works investigating different problems have been published using the sequential characterization of Besov topology, we refer e.g., to this non-exhaustive list [38], [14], [9], [7].

Our second aims are to investigate existence, Hölder continuity, and Besov regularity of local times of \( u(t, x) \), viewed as a process respectively in time and space variables. We will use the notion of local nondeterminism (LND), initiated by Berman in [6] and extended later in [19] to the strong local nondeterminism concept (SLND). These two notions are the most important mathematical tools used by several authors to study sample path properties for various Gaussian processes and random fields, see, e.g. [3, 8, 48–52] and references therein.

We must note that, for fixed \( x \), the process \((u(t, x), t \in [0, T])\) is identically distributed as the so-called bifractional Brownian motion (see [41]). So, many sample path properties of the process \((u(t, x), t \in [0, T])\), including the Hölder continuity of its local time, may be deduced from [44]. In this last paper, to prove that the bifractional Brownian motion satisfies the SLND property, the authors have used the Lamperti transformation to connect self-similar and stationary Gaussian processes. In our paper, to verify the LND condition for the processes \((u(t, x), t \in [0, T], x \in \mathbb{R})\) respectively in time and space variables, we use fine estimates on the Green kernel \( G \) and careful calculations. Furthermore, we will improve the Knowledge of the sample paths properties for these two processes by proving regularity results of their local times in the modular Besov spaces \( B^{\omega}_{p, \lambda} \), with respect to modulus \( \omega(t) = t^\alpha (\log 1/t)^{\lambda} \), with \( \alpha = 1/4 \) (resp. \( 1/2 \)) and \( p\lambda > 1 \). In contrast with the Besov regularity obtained for the Brownian local time in [10], our results are not optimal, but they have the merit of being new and sharper than what is known.

Let us give an overview of the results existing in the literature related to the context of our paper. First, it was raised in [51] the relevance of studying the sample paths properties for the solutions to stochastic heat or wave equations through their local times and LND concept. Ouahhabi and Tudor [36] have studied, for fixed space variable, the Hölder regularity of the local time of the solution to the \( d \)-dimensional stochastic linear heat equation driven by a fractional-white noise, with Hurst parameter \( H > 1/2 \). The investigation of the sample paths properties, both in time and space variables, of the solution to the stochastic heat equation driven by a fractional-colored noise is considered in [43] and [52].

While writing this article, we discovered the papers [29, 35, 45] where a direct method was used to handle Besov norms. They gave alternative proofs of Besov regularities of Brownian motion and fractional Brownian motion, without the equivalent wavelet description used in [13] and [14]. By this new approach, we expect that we will establish sharp Besov regularities for the solution and its local time to a \( d \)-dimensional stochastic heat (or wave) equation driven by a fractional-colored noise. We intend to study these questions in a future paper.

This paper is organized as follows: In the first paragraph, we briefly recall some notions on local times, Besov and Besov-Orlicz spaces. The second paragraph is devoted to studying the Besov-Orlicz regularity for the sample paths of \( t \mapsto u(t, x) \) and \( x \mapsto u(t, x) \). In the third paragraph, we investigate the existence and Besov regularity of local times of the process \( u(t, x) \) in time and space variables and conclude by considering Hausdorff dimensions of the level sets \( \{ t \in [0, T]; u(t, x) = u(t_0, x) \} \) and \( \{ x \in [a, b]; u(t, x) = u(t, x_0) \} \).
1. Preliminaries

In this paragraph, we give some basic notions on local times and Besov spaces.

1.1. The local times

We recall here the concept of the local time, in the Fourier analytic sense, as it has been initiated by S. Berman in [3]. Let $I \subset \mathbb{R}$ be a compact interval, and $\theta : s \in I \rightarrow \theta_s \in \mathbb{R}$, a deterministic Borel function. $\mathcal{B}(I)$ be the Borel $\sigma$-algebra on $I$, for any $B \in \mathcal{B}(I)$ we define the occupation measure $\mu_B$ by

$$\mu_B(A) = \lambda\{s \in B, \theta_s \in A\}, \; A \in \mathcal{B}(\mathbb{R}).$$

If $\mu_B$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, we say that $\theta$ has a local time on $B$, and we define its local time $L(\cdot, B)$ as the Radon Nikodym derivative of $\mu_B$ with respect to the Lebesgue measure. If $B = [0, t]$ (resp. $B = I$) we write $L(\xi, t)$ (resp. $L(\xi)$) instead of $L(\cdot, [0, t])$ (resp. $L(\xi, I)$). This definition can be extended to any measurable and bounded (or positive) function $f$ to get the so-called occupation density formula:

$$\int_0^t f(\theta_s)ds = \int_{\mathbb{R}} f(\xi)L(\xi, t)d\xi.$$

The idea of Berman is to relate properties of $L(\cdot, B)$ with the integrability of the Fourier transform of $\theta$. Recall the following essential result:

**Proposition 1.1.** The function $\theta$ has a square integrable local time $L(\xi, B)$ iff

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left| \int_B \exp(iu\theta_s)ds \right|^2 du < \infty.$$

Moreover, we have the following representation of the local time, for almost every $\xi \in \mathbb{R}$,

$$L(\xi, B) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_B \exp(iu(\theta_s - \xi))dsdu. \tag{3}$$

The deterministic function $\theta$ can be chosen to be a sample path of a stochastic process $(X_t, \; t \in [0, T])$. To prove almost sure existence and square integrability of the local time $L(\xi, B)$, it is enough to establish that

$$\mathbb{E} \int_{\mathbb{R}} \int_B \left| \exp(iuX_s)ds \right|^2 du < \infty.$$

When $(X_t, \; t \in [0, T])$ is Gaussian, then we get the well-known Berman’s criterion:
Proposition 1.2 (5). If $(X_t, t \in [0,T])$ is a centred Gaussian process, and satisfies
\[ \int_0^T \int_0^T [\mathbb{E}(X_s - X_t)^2]^{-1/2} \, ds \, dt < \infty. \]
Then for almost all $\omega$, for any $B \in \mathcal{B}([0,T])$, the trajectory $s \mapsto X_s(\omega)$ has a local time $L(\xi,B,\omega)$ which is square integrable with respect to $\xi$.

Remark 1. In the rest of this paper, we will note the local time of a process by $L(\xi,B)$ instead of $L(\xi,B,\omega)$ unless a confusion exists.

The following theorem will clarify Berman’s principle, providing the link between the regularity of the local time and the irregularity of its underlying stochastic process:

Theorem 1.3 (5). Let $(X_t, t \in [0,T])$ be a centred Gaussian process, such that for some $p \geq 0$,
\[ \int_0^T \int_0^T [\mathbb{E}(X_s - X_t)^2]^{-(p+1)/2} \, ds \, dt < \infty. \] (4)
Then for almost all $\omega$, the trajectory $s \mapsto X_s(\omega)$ has a local time $L(\xi)$, such that the first $\lfloor p/2 \rfloor$ derivatives of $L(\xi)$ exist and are square integrable ($\lfloor v \rfloor = \text{integral part of } v$). Moreover, when $\lfloor p \rfloor$ is even, the sample functions of $X$ are nowhere $\frac{1}{2(p+1)}$-Hölder continuous.

Remark 2. A particular situation is obtained if,
\[ \mathbb{E}(X_t - X_s)^2 \geq c |t - s|^\beta, \]
where $0 < \beta < 2$ and $c$ is a positive constant. We can easily deduce that a.s. the trajectories of the process $X$ are nowhere $\beta$-Hölder continuous.

To derive from Kolmogorov continuity theorem, the joint continuity of the local time and the Hölder continuity in $t$ of $L(\xi,t)$, our starting point is the following identities about the moments of the increments of the local time
\[ \mathbb{E}[L(\xi + k, t + h) - L(\xi, t + h) - L(\xi + k, t) + L(\xi, t)]^n \]
\[ = (2\pi)^{-n} \int_{[t,t+h]^n} \int_{\mathbb{R}^n} \prod_{j=1}^n (e^{-i(\xi+k)u_j} - e^{-i\xi u_j}) \mathbb{E}[\exp(i \sum_{j=1}^n u_j X_{t_j})] \, d\mathbf{u} \, d\mathbf{t}, \]
and
\[ \mathbb{E}[L(\xi, t + h) - L(\xi, t)]^n = (2\pi)^{-n} \int_{[t,t+h]^n} \int_{\mathbb{R}^n} \exp(-i \xi \sum_{j=1}^n u_j) \mathbb{E}[\exp(i \sum_{j=1}^n u_j X_{t_j})] \, d\mathbf{u} \, d\mathbf{t}, \]
where $\mathbf{u} = (u_1, ..., u_n), \mathbf{t} = (t_1, ..., t_n), \xi, \xi + k \in \mathbb{R}$ and $t, t + h \in [0,T]$. To find suitable upper bounds for these increments for a given centered Gaussian process $X$, the expression $\text{Var}[\sum_{j=1}^n v_j (X_{t_j} - X_{t_{j-1}})]$ appearing by an appropriate change of variables in the characteristic function above, is lower bounded by $C_m \sum_{j=1}^n v_j^2 \text{Var}(X_{t_j} - X_{t_{j-1}})$.
when $X$ verifies the LND property (see, [6, Lemma 2.3.]). This last property reads as follows:

$$\lim_{c \to 0} \inf_{0 \leq r \leq s, r < s < t} \frac{\text{Var}(X_t - X_s | X_r, \ r \leq \tau \leq s)}{\text{Var}(X_t - X_s)} > 0.$$  \hspace{1cm} (5)

To analyze the local time of the solution to the stochastic heat equation, the LND property will be verified in the next for $u(t, x)$ viewed either as a process in time variable for any fixed $x \in \mathbb{R}$ or a process in space variable for any fixed $t$.

### 1.2. Modular Besov spaces

In this paragraph, we recall some notions of modular Besov spaces and their characterizations in terms of some sequences spaces. Let $I \subset \mathbb{R}$ be a compact interval, $1 \leq p < \infty$ and $f \in L^p(I; \mathbb{R})$. We can measure the smoothness of $f$ by its modulus of continuity computed in the $L^p$-norm. For this end, let us define for any $t > 0$,

$$\Delta_p(f, I)(t) = \sup_{|s| \leq t} \left\{ \int_{I_s} |f(x + s) - f(x)|^p \, dx \right\}^{\frac{1}{p}},$$

where $I_s = \{ x \in I; \ x + s \in I \}$.

We can find in the literature various ways to define Besov norms. They are generally defined with respect to some moduli, which are non-decreasing and continuous positive functions $\omega$ defined on $[0, +\infty[$, s.t. $\omega(0) = 0$. The most typical example of moduli is:

$$\omega_{\alpha, \lambda}(t) = t^\alpha (\log(1/t))^\lambda, \hspace{1cm} (6)$$

with $0 < \alpha < 1$ and $\lambda \in \mathbb{R}$. We refer to [14] for a more details on this subject.

**Remark 3.** From now on, we will only be concerned with the moduli $\omega_{\alpha, \lambda}$ defined in (6) and write $\omega$ instead of $\omega_{\alpha, \lambda}$ unless a confusion exists.

Let $\omega$ be any modulus defined by (6), we consider the norm

$$\|f\|_{\omega, p} := \|f\|_{L^p(I)} + \sup_{0 < t \leq 1} \frac{\Delta_p(f, I)(t)}{\omega(t)}.$$

The modular Besov space is given by

$$B^\omega_p(I) = \{ f \in L^p(I); \ \|f\|_{\omega, p} < \infty \}.$$

The space $(B^\omega_p(I), \| \cdot \|_{\omega, p})$ is a non separable Banach space. We also define

$$B^{\omega, 0}_p(I) = \{ f \in L^p(I); \ \Delta_p(f, I)(t) = o(\omega(t)) \text{ as } t \to 0^+ \}.$$

$B^{\omega, 0}_p(I)$ is a separable subspace of $B^\omega_p(I)$. For $p = \infty$, the space $B^\omega_\infty(I)$ is defined in the same way by using the usual $L^\infty$-norm. In this case it coincides with $H^\omega(I)$, the
$\omega$-Hölder space defined by
\[ \mathcal{H}^\omega(I) := \left\{ f : I \to \mathbb{R} \mid \text{ess sup}_{x, y \in I, x \neq y} \frac{|f(x) - f(y)|}{\omega(|x - y|)} < \infty \right\}, \quad (7) \]
endowed with the norm $\|f\|_{\omega, \infty} = \text{ess sup}_{x \in I} |f(x)| + \text{ess sup}_{x, y \in I, x \neq y} \frac{|f(x) - f(y)|}{\omega(|x - y|)}$.

As it is mentioned in the introduction, standard changes of variables permit us to restrict ourselves to the unit interval $I = [0, 1]$, so we will omit to specify the interval $I$ in our notations. The following theorem (Theorem 1.4) is a characterization of modular Besov spaces $\mathcal{B}_\omega^p$ in terms of progressive differences of a function in dyadic points. Its proof has been established for general moduli in [14].

**Theorem 1.4.** Let $\omega$ be as in (6), $p > 1$, $1 < p < \alpha < 1$ and $\lambda \geq 0$, then we have

1. $\mathcal{B}_\omega^p$ is linearly isomorphic to a sequences space and we have the following equivalence of norms:

\[ \|f\|_{\omega, p} \sim \max \left\{ |f_0|, |f_1|, \sup_{j \geq 0, 1 \leq k \leq 2^j} \left( \frac{2^{-j(\frac{1}{p} + \frac{1}{\omega})}}{\omega(2^{-j})} \left[ \sum_{k=1}^{2^j} |f_{jk}|^p \right]^{\frac{1}{p}} \right) \right\}, \]

where the coefficients $\{f_0, f_1, f_{jk}, j \geq 0, 1 \leq k \leq 2^j\}$ are given by

\[ f_0 = f(0), \quad f_1 = f(1) - f(0), \]

\[ f_{jk} = 2 \cdot 2^{j/2} \left\{ f\left( \frac{2k - 1}{2^{j+1}} \right) - \frac{1}{2} f\left( \frac{2k}{2^{j+1}} \right) - \frac{1}{2} f\left( \frac{2k - 2}{2^{j+1}} \right) \right\}. \]

2. $f$ is in $\mathcal{B}_\omega^p$ if and only if

\[ \lim_{j \to 0} \frac{2^{-j(\frac{1}{p} + \frac{1}{\omega})}}{\omega(2^{-j})} \left[ \sum_{k=1}^{2^j} |f_{jk}|^p \right]^{\frac{1}{p}} = 0. \]

When $\omega(t) = t^\alpha$, we will use the notations:

\[ \|f\|_{\alpha, p} := \|f\|_{\omega, p}, \quad \mathcal{B}_\alpha^0 := \mathcal{B}_\omega^\alpha \quad \text{and} \quad \mathcal{B}_\alpha^{0, 0} := \mathcal{B}_\omega^{\omega, 0}. \]

**Remark 4.** One can easily show, by Theorem 1.4, the following useful continuous injections

- For any $\varepsilon > 0$, $1 \leq p < \infty$ and $\frac{1}{p} < \alpha < 1$,

\[ \mathcal{H}^{\alpha + \varepsilon} \hookrightarrow \mathcal{B}_\alpha^{0, 0} \hookrightarrow \mathcal{B}_\alpha^0 \hookrightarrow \mathcal{H}^{\alpha - \frac{1}{p}}. \]
Let $1 \leq p < \infty$ and $0 < \alpha < \beta < 1$, we have
\[ B^{\beta}_p \hookrightarrow B^{\alpha}_p. \]

Let $\omega(t) = t^\alpha (\log (1/t))^\lambda$, $0 < \alpha < 1$ and $\lambda \geq 0$, then
\[ B^{\alpha}_p \hookrightarrow H^\gamma, \quad (8) \]
for any $\gamma < \alpha$ and $p$ sufficiently large.

### 1.3. Besov–Orlicz spaces

Let $I \subset \mathbb{R}$, be a compact interval, and $\mathcal{N}$ is the Young function defined by $\mathcal{N}(x) = e^{x^2} - 1$. The Orlicz space $L_{\mathcal{N}}(I)$ is the space of measurable functions $f : I \mapsto \mathbb{R}$, such that
\[ \|f\|_{\mathcal{N}} := \inf_{\lambda > 0} \left\{ \lambda \left[ 1 + \int_I \mathcal{N}(\lambda f(t)) dt \right] \right\} < \infty. \]
It is more suitable to use an equivalent norm to $\| \cdot \|_{\mathcal{N}}$ (see e.g. Ciesielski [13]):
\[ \|f\|_{\mathcal{N}} = \sup_{p \geq 1} \frac{\|f\|_{L^p(I)}}{\sqrt{p}}. \]

Let $\Delta_{\mathcal{N}}(f, I)(t)$ be the modulus of continuity of $f$ in the Orlicz space $L_{\mathcal{N}}(I)$ defined as:
\[ \Delta_{\mathcal{N}}(f, I)(t) = \sup_{p \geq 1} \frac{\Delta_p(f, I)(t)}{\sqrt{p}}. \]
For $0 < \alpha < 1$, we consider the following norm
\[ \|f\|_{\alpha, \mathcal{N}} = \|f\|_{\mathcal{N}} + \sup_{0 < t \leq 1} \frac{\Delta_{\mathcal{N}}(f, I)(t)}{t^\alpha}. \]

The Besov-Orlicz space is defined by
\[ B^{\alpha, \mathcal{N}}_\mathcal{N}(I) := \{ f \in L_{\mathcal{N}}(I); \|f\|_{\alpha, \mathcal{N}} < \infty \}. \]

$B^{\alpha, \mathcal{N}}_\mathcal{N}(I)$ endowed with the norm $\| \cdot \|_{\alpha, \mathcal{N}}$ is a non separable Banach space. We introduce $B^{\alpha, 0, \mathcal{N}}_\mathcal{N}(I) = \{ f \in L_{\mathcal{N}}(I); \Delta_{\mathcal{N}}(f, I)(t) = o(t^\alpha) \text{ as } t \to 0^+ \}$ a separable subspace of $B^{\mathcal{N}}_\mathcal{N}(I)$.

We will restrict ourselves to the interval $I = [0, 1]$, so we will omit to precise the interval $I$ in our notations, e.g. we will use $B^{\mathcal{N}}_\mathcal{N}$ to denote the Besov-Orlicz space $B^{\alpha, \mathcal{N}}([0, 1])$. With the same notations as in Theorem 1.4, we have the following isomorphism theorem (see Ciesielski [13] or Ciesielski et al. [14]):

**Theorem 1.5.** We have
(1) $\mathcal{B}_N^{\alpha}$ is linearly isomorphic to a sequences space and we have the following equivalence of norms:

$$||f||_{\alpha,N} \sim \max \left\{ |f_0|, |f_1|, \sup_{p,j} \frac{1}{\sqrt{p}} 2^{-j\left(\frac{1}{2} - \alpha + \frac{1}{p}\right)} \left( \sum_{k=1}^{2^j} |f_{jk}|^p \right)^{\frac{1}{p}} \right\},$$

(2) $f$ belongs to $\mathcal{B}_N^{\alpha,0}$ if and only if

$$\lim_{j \to \infty} \sup_{p \geq 1} \frac{1}{\sqrt{p}} 2^{-j\left(\frac{1}{2} - \alpha + \frac{1}{p}\right)} \left( \sum_{k=1}^{2^j} |f_{jk}|^p \right)^{\frac{1}{p}} = 0.$$

**Remark 5.** For any $p \in [1, \infty)$ and $0 < \alpha < 1$, the following injections are easy to verify

$$\mathcal{B}_N^{\alpha} \hookrightarrow \mathcal{B}_p^\alpha \quad \text{and} \quad \mathcal{B}_N^{\alpha,0} \hookrightarrow \mathcal{B}_p^{\alpha,0}.$$

2. **Besov regularity of solution to stochastic heat equation**

2.1. **Linear stochastic heat equation**

Consider the linear stochastic heat equation defined by (1), where $\frac{\partial^2}{\partial t \partial x} W(t, x)$ is a space-time white noise, i.e. $(W(t, x), t \geq 0, x \in \mathbb{R})$ is a centered Gaussian process with covariance function given by

$$E[W(t, A)W(t', A')] = (t \wedge t') \lambda(A \cap A'), \quad A, A' \in \mathcal{B}_b(\mathbb{R}), \quad t, t' \geq 0,$$

where $\lambda$ is the Lebesgue measure, $\mathcal{B}_b(\mathbb{R})$ is the collection of bounded Borel sets and $W(t, x) = W(t, [0, x])$. It is well known that there exist a unique (mild) solution of this equation given by

$$u(t, x) = \int_0^t \int_{\mathbb{R}} G(t - s, x - y) dW(s, y), \quad (9)$$

where the integral is the Wiener integral with respect to the Gaussian process $W$ and $G$ is the Green kernel of the heat equation given by

$$G(t, x) = \begin{cases} (2\pi t)^{-1/2} \exp(- \frac{|x|^2}{2t}) & \text{if } t > 0, \quad x \in \mathbb{R} \\ 0 & \text{if } t \leq 0, \quad x \in \mathbb{R}. \end{cases} \quad (10)$$

2.2. **Besov regularity of $t \to u(t, x)$**

Our main result in this paragraph, is the following theorem

**Theorem 2.1.** For all $x \in \mathbb{R}$ and $4 < p < \infty$, we have

$$\mathbb{P}(u(., x) \in \mathcal{B}_p^{1/4}) = 1 \quad \text{and} \quad \mathbb{P}(u(., x) \in \mathcal{B}_p^{1/4,0}) = 0,$$
where \( u(., x) \) is the sample path \( t \in [0, 1] \rightarrow u(t, x) \).

To prove this theorem, we are going to adapt the method used in [14]. For any \( x \in \mathbb{R} \), let

\[
u_{jk} = 2 \cdot 2^{j/2} \left\{ u \left( \frac{2k - 1}{2^{j+1}}, x \right) - \frac{1}{2} u \left( \frac{2k}{2^{j+1}}, x \right) - \frac{1}{2} u \left( \frac{2k - 2}{2^{j+1}}, x \right) \right\}.
\]

We can rewrite \( \nu_{jk} \) as

\[
u_{jk} = 2 \cdot 2^{j/2} \left\{ \int_0^{2^k - 1} G \left( \frac{2k - 1}{2^{j+1}} - \tau, x - y \right) G \left( \frac{2k}{2^{j+1}} - \tau, x - y \right) W(d\tau, dy) + \right. \\
\left. + \int_{2^k - 2}^{2^k - 1/2} G \left( \frac{2k - 1}{2^{j+1}} - \tau, x - y \right) G \left( \frac{2k}{2^{j+1}} - \tau, x - y \right) W(d\tau, dy) + \right. \\
\left. + \int_{2^k - 1/2}^{2^k} \left[ - \frac{1}{2} G \left( \frac{2k - 1}{2^{j+1}} - \tau, x - y \right) \right] W(d\tau, dy) \right\}
\]

\( = 2 \cdot 2^{j/2} \{ I_1(j, k) + I_2(j, k) + I_3(j, k) \}. \) (11)

It is important to note that \( I_1(j, k), I_2(j, k) \) and \( I_3(j, k) \) are independent terms. We also put

\[
v_{jk} = \frac{\nu_{jk}}{\sigma_{jk}} \quad \text{with} \quad \sigma_{jk} = \left\{ \mathbb{E}[|\nu_{jk}|^2] \right\}^{1/2}.
\]

We first state some preliminary results.

**Lemma 2.2.** For all \( x \in \mathbb{R}, t, s \in [0, 1] \) and \( 0 \leq r_1 < r_2 \leq t \land s \), we have

\[
\int_{r_1}^{r_2} \int_{\mathbb{R}} G(t - \tau, x - y)G(s - \tau, x - y)dyd\tau = \frac{1}{\sqrt{2\pi}} \left( \sqrt{t + s - 2r_1} - \sqrt{t + s - 2r_2} \right).
\]

**Proof.** The proof is a consequence of successive elementary changes of variables. \( \square \)

The following lemma is a useful tool to get precise estimations in our calculations. For the proof, we refer to [14].

**Lemma 2.3.** Let \((X, Y)\) be a mean zero Gaussian vector such that \( \mathbb{E}(X^2) = \mathbb{E}(Y^2) = 1 \) and \( \rho = |\mathbb{E}XY| \). Then for any measurable functions \( f \) and \( g \) such that \( \mathbb{E}(f(X))^2 < \infty, \mathbb{E}(f(Y))^2 < \infty \) and \( f(X), f(Y) \) are centred, we have

\[
|\mathbb{E}(f(X)g(Y))| \leq \rho \left\{ \mathbb{E}(f(X))^2 \right\}^{1/2} \left\{ \mathbb{E}(g(Y))^2 \right\}^{1/2},
\]

when \( f \) (or \( g \)) is even, we can replace \( \rho \) by \( \rho^2 \) in the previous inequality.
By using the equality (11) and Lemma 2.2, we have for all $j \geq 1$ and $k, k' \in \{1, ..., 2^j\}$ such that $k \neq k'$

$$\mathbb{E}[u_{jk} u_{jk'}] = \frac{2 \cdot 2^{j/2}}{\sqrt{\pi}} (\Delta^4 \Phi_{k,k'}(0) - \Delta^4 \Psi_{k,k'}(0)),$$

where $\Delta^4 \Phi_{k,k'}, \Delta^4 \Psi_{k,k'}$ are the one step progressive differences of order 4 of the functions

$$\Phi_{k,k'}(x) = (2(k + k') - x)^{1/2} \quad \text{and} \quad \Psi_{k,k'}(x) = (2|k - k'| - 2 + x)^{1/2}.$$  

We also have, for all $j \geq 1$ and $k \in \{1, ..., 2^j\}$

$$\mathbb{E}[|u_{jk}|^2] = \frac{2 \cdot 2^{j/2}}{\sqrt{\pi}} (\Delta^4 \Phi_{k,k}(0) + 2 - \frac{1}{\sqrt{2}}).$$

(14)

**Lemma 2.4.** For all $j \geq 1$ and $k, k' \in \{1, ..., 2^j\}$ with $k \neq k'$, there exists a constant $c_{k,k'} \in (0, 1)$ such that

$$|\mathbb{E}[u_{jk} u_{jk'}]| \leq \frac{15}{8 \sqrt{\pi}} \left( 2 \cdot \frac{2^{j/2}}{2^{1/2}} \right).$$

(15)

And there exists a constant $m > 0$ such that, for all $j \geq 1$ and $k \in \{1, ..., 2^j\}$,

$$m 2^{j/2} \leq \mathbb{E}[|u_{jk}|^2] \leq \frac{4 - \sqrt{2}}{\sqrt{\pi}} 2^{j/2}.$$  

(16)

**Proof.** Denote by $\Phi_{k,k'}^{(4)}, \Psi_{k,k'}^{(4)}$, the derivatives of order 4 of respectively $\Phi_{k,k'}$ and $\Psi_{k,k'}$. By successive mean value theorem and (13), there exist two constants $0 < \beta_{k,k'} < 4$ and $0 < \gamma_{k,k'} < 4$ such that

$$\mathbb{E}[u_{jk} u_{jk'}] = \frac{2 \cdot 2^{j/2}}{\sqrt{\pi}} (\Phi_{k,k'}^{(4)}(\beta_{k,k'}) - \Psi_{k,k'}^{(4)}(\gamma_{k,k'}))$$

$$= \frac{15}{8 \sqrt{\pi}} \left( \frac{1}{(2|k - k'| - 2 + \gamma_{k,k'})^{7/2}} - \frac{1}{(2(k + k') - \beta_{k,k'})^{7/2}} \right)$$

$$\leq \frac{15}{8 \sqrt{\pi}} \frac{2^{j/2}}{(2|k - k'| - 2 + \gamma_{k,k'} \wedge 1)^{7/2}}.$$  

Then we get (15) with $c_{k,k'} = \gamma_{k,k'} \wedge 1$.

To show the upper bound in (16), we use the mean value theorem and (14). So,
there exists a constant $0 < \beta_{k,k} < 4$ such that

$$
\mathbb{E}[u_{jk}^2] = \frac{2 \cdot 2^{j/2}}{\sqrt{\pi}} (\Phi_{k,k}^{(4)} (\beta_{k,k}) + 2 - \frac{1}{\sqrt{2}})
$$

(17)

$$
= \frac{2 \cdot 2^{j/2}}{\sqrt{\pi}} (-15 \frac{1}{16} (4k - \beta_{k,k})^{7/2} + 2 - \frac{1}{\sqrt{2}})
$$

(18)

$$
\leq \frac{2}{\sqrt{\pi}} (2 - \frac{1}{\sqrt{2}})^{2j/2}.
$$

On the other hand to prove the lower bound, first we remark that for $k = 1$, we have by (14)

$$
\mathbb{E}[|u_{j1}|^2] = \frac{2}{\sqrt{\pi}} \left( \frac{7\sqrt{2}}{4} - \sqrt{3} + \frac{3}{2} \right) 2^{j/2},
$$

and for $k \in \{2, ..., 2^j\}$, we get by (18)

$$
\frac{2}{\sqrt{\pi}} \left( -\frac{15}{16} \frac{1}{4^{7/2}} + 2 - \frac{1}{\sqrt{2}} \right) 2^{j/2} \leq \mathbb{E}[|u_{jk}|^2].
$$

So the lower bound in (16) is obtained with

$$
m = \frac{2}{\sqrt{\pi}} \left( -\frac{15}{16} \frac{1}{4^{7/2}} + 2 - \frac{1}{\sqrt{2}} \right) \wedge \frac{2}{\sqrt{\pi}} \left( \frac{7\sqrt{2}}{4} - \sqrt{3} + \frac{3}{2} \right).
$$

This finishes the proof of Lemma 2.4.

**Lemma 2.5.** There exists a constant $M > 0$ such that, for all $j \geq 1$ and $k, k' \in \{1, ..., 2^j\}$, we have

$$
\sum_{k,k'=1}^{2^j} |\mathbb{E}[v_{jk}v_{jk'}]|^2 \leq M 2^j,
$$

(19)

where $v_{jk}$ is given by (12).

**Proof.** Equality (12) gives

$$
\sum_{k,k'=1}^{2^j} |\mathbb{E}[v_{jk}v_{jk'}]|^2 = 2 \sum_{k'=k}^{2^j} |\mathbb{E}[v_{jk}v_{jk'}]|^2 + 2 \sum_{k'=k}^{2^j} |\mathbb{E}[v_{jk}v_{jk'}]|^2 + \sum_{k=1}^{2^j} \{\mathbb{E}[|v_{jk}|^2]\}^2
$$

$$
= 2 \sum_{k'=k}^{2^j} \frac{|\mathbb{E}[u_{jk}u_{jk'}]|^2}{\sigma_{jk}\sigma_{jk'}}^2 + 2 \sum_{k'=k}^{2^j} \frac{|\mathbb{E}[u_{jk}u_{jk'}]|^2}{\sigma_{jk}\sigma_{jk'}}^2 + 2^j
$$

(20)

$$
= 2I_1 + 2I_2 + 2^j.
$$

12
First we are going to estimate $I_1$. Since $k' = k - 1$, we have by (13)\[
\mathbb{E}[u_{jk}u_{jk'}] = \frac{2 \cdot 2^{j/2}}{\sqrt{\pi}} (\Delta^4 \Phi_{k,k'}(0) + 2(1 + 2\sqrt{3} - 3\sqrt{2})), \tag{21}
\]
and
\[
\Delta^4 \Phi_{k,k'}(0) = \Phi^{(4)}_{k,k'}(\beta_{k,k'}) = -\frac{15}{16} \frac{1}{(4k - 2 - \beta_{k,k'})^{7/2}}, \tag{22}
\]
where $\beta_{k,k'} \in (0, 4)$. Combining (21) and (22), we obtain\[
\mathbb{E}[u_{jk}u_{jk'}] = \frac{2 \cdot 2^{j/2}}{\sqrt{\pi}} \left( -\frac{15}{16} \frac{1}{(4k - 2 - \beta_{k,k'})^{7/2}} + 2(1 + 2\sqrt{3} - 3\sqrt{2}) \right). \tag{23}
\]
So, we get\[
\mathbb{E}[u_{jk}u_{jk'}] \leq \frac{4 \cdot 2^{j/2}}{\sqrt{\pi}} (1 + 2\sqrt{3} - 3\sqrt{2}), \tag{24}
\]
and\[
-\mathbb{E}[u_{jk}u_{jk'}] \leq \frac{15 \cdot 2^{j/2}}{8\sqrt{\pi}} \frac{1}{(4k - 2 - \beta_{k,k'})^{7/2}} \leq \frac{15 \cdot 2^{j/2}}{8\sqrt{\pi}} \frac{1}{2^{7/2}}. \tag{25}
\]
Combining (24) and (25), we obtain that\[
|\mathbb{E}[u_{jk}u_{jk'}]| \leq K2^{j/2}, \tag{26}
\]
where $K = \frac{4(1 + 2\sqrt{3} - 3\sqrt{2}) \vee \frac{15}{2^{7/2}} \sqrt{\pi}}{\sqrt{\pi}}$. So by (10) and (26), we have\[
I_1 \leq \left( \frac{K}{m} \right)^2 (2^j - 1).
\]
Now we are going to estimate $I_2$. If we note $C = \left(\frac{15}{m_3} \sqrt{\pi}\right)^2$, we have by (15) and (16),

$$I_2 \leq C \sum_{k' < k\atop k - k' \geq 2} \frac{1}{(2(k - k') - 2 + c_{k,k'})^7}$$

$$= C \sum_{k=3}^{2j} \sum_{k' = 1}^{k-2} \frac{1}{(2(k - k') - 2 + c_{k,k'})^7}$$

$$\leq C \sum_{k=3}^{2j} \sum_{k' = 1}^{k-2} \frac{1}{(2(k - k') - 2)^7}$$

$$\leq C \sum_{k=3}^{2j} \left( \sum_{k' = 1}^{k-2} \int_{2(k - k' - 1)}^{2(k - k' - 3)} \frac{1}{x^7} \, dx \right)$$

$$\leq C \sum_{k=3}^{2j} \left( \int_{1}^{2k-4} \frac{1}{x^7} \, dx \right) \leq \frac{C}{6} (2^j - 2).$$

This finishes the proof of the lemma 2.5. □

**Lemma 2.6.** For all $j \geq 1$ and $k \in \{1, \ldots, 2^j\}$, we have

$$\mathbb{E} \left[ \sum_{k=1}^{2^j} (|v_{jk}| - c_p)^2 \right] \leq (c_{2p} - c_p^2) M 2^j,$$

(27)

where $c_p = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |x|^p e^{-x^2/2} \, dx$.

**Proof.** First we have

$$\mathbb{E} \left[ \sum_{k=1}^{2^j} (|v_{jk}| - c_p)^2 \right] = \sum_{k,k' = 1}^{2^j} \mathbb{E} \left[ (|v_{jk}| - c_p)(|v_{jk'}| - c_p) \right].$$

And by applying Lemma 2.3 with $f(x) = g(x) = |x|^p - c_p$, we get

$$\mathbb{E} \left[ \sum_{k=1}^{2^j} (|v_{jk}| - c_p)^2 \right] \leq (c_{2p} - c_p^2) \sum_{k,k' = 1}^{2^j} \mathbb{E} \left[ |v_{jk}v_{jk'}|^2 \right].$$

Inequality (27) of Lemma 2.5 ends the proof of the lemma 2.6. □

Now, we are ready to finish the proof of Theorem 2.1.

**Proof of Theorem 2.1.** We are going to show that, almost surly

$$2^{-j} \sum_{k=1}^{2^j} |v_{jk}|^p \xrightarrow{j \to \infty} c_p.$$  

(28)
For this end we will prove that for all $\varepsilon > 0$, we have

$$\sum_{j \geq 1} \mathbb{P} \left\{ 2^{-j} \sum_{k=1}^{2^j} |v_{jk}|^p \notin [c_p - \varepsilon, c_p + \varepsilon] \right\} < \infty. \tag{29}$$

Markov’s inequality ensures

$$\mathbb{P} \left\{ 2^{-j} \sum_{k=1}^{2^j} |v_{jk}|^p \notin [c_p - \varepsilon, c_p + \varepsilon] \right\} \leq \frac{1}{\varepsilon^2 2^{2j} \mathbb{E} \left[ \sum_{k=1}^{2^j} |v_{jk}|^p - c_p \right]^2}. \tag{30}$$

Combining the inequality (30) and Lemma 2.6, we get that (29) holds and (28) is then a consequence of Borel-Cantelli Lemma. Finally, our main result, Theorem 2.1, is a simple consequence of Theorem 1.4.

Below we conclude a stronger regularity result than Theorem 2.1.

**Theorem 2.7.** Let $\mathcal{N}(x) = e^{x^2} - 1$. For all $x \in \mathbb{R}$, we have

$$\mathbb{P}(u(., x) \in B_{\mathcal{N}}^{1/4}) = 1 \text{ and } \mathbb{P}(u(., x) \in B_{\mathcal{N}}^{1/4, 0}) = 0,$$

where $u(., x)$ is the sample path $t \in [0, 1] \rightarrow u(t, x)$.

**Proof.** The proof is similar to that one of [14, Theorem II.5]. Indeed, we use Lemma 2.6 and the fact that for positive integer $p$, we have

$$c_2^p = (2p)! / (p! 2^p) \sim e^{-p} 2^{p+1/2} p^p. \quad \text{Therefore, there exists a constant } c \text{ with } c > 1, \text{ such that } c_2^p \leq ce^{-p} (2p)^p. \tag*{□}$$

**Remark 6.** Lemma 2.2 shows clearly that the process $(u(t, x), t \geq 0)$ is, up to a constant, a bifractional Brownian motion with parameters $H = K = \frac{1}{p}$. We believe that, with the same arguments, we can prove that $\mathbb{P}[W_{t}^{H,K} \in B_{p}^{H,K}] = 1$ and $\mathbb{P}[W_{t}^{H,K} \in B_{p}^{H,K,0}] = 0$, where $(W_{t}^{H,K}, t \in [0, 1])$ is a bifractional Brownian motion with parameters $H \in (0, 1)$ and $K \in (0, 1]$. We investigate this result in an upcoming paper.

### 2.3. Besov regularity of $x \rightarrow u(t, x)$

In this paragraph we will study the Besov regularity of $(u(t, x), x \in [0, 1])$ for any fixed $t \in (0, T]$.

**Theorem 2.8.** For all $t \in (0, T]$ and $2 < p < \infty$, we have

$$\mathbb{P}[u(t, .) \in B_p^{1/2}] = 1 \text{ and } \mathbb{P}[u(t, .) \in B_p^{1/2, 0}] = 0,$$

where $u(t, .)$ is the sample path $x \in [0, 1] \rightarrow u(t, x)$.

As before we first state some preliminary lemmas.
Lemma 2.9. For all \( t \in (0, T) \), \( x, y \in [0, 1] \), we have
\[
\int_0^t \int_{\mathbb{R}} G(t - \tau, x - \xi)G(t - \tau, y - \xi) d\xi d\tau = F(x - y),
\]
where
\[
F(u) = \int_0^t \frac{1}{2\sqrt{\pi r}} e^{-\frac{u^2}{4r}} dr.
\]  
(31)

Proof. Straightforward calculations. \( \square \)

Define for any \( t \in (0, T) \),
\[
z_{jk} = 2 \cdot 2^{j/2} \left\{ u\left( t, \frac{2k - 1}{2^{j+1}} \right) - \frac{1}{2} u\left( t, \frac{2k}{2^{j+1}} \right) - \frac{1}{2} u\left( t, \frac{2k - 2}{2^{j+1}} \right) \right\},
\]  
(32)

and set
\[
\omega_{jk} = \frac{z_{jk}}{\theta_{jk}} \quad \text{with} \quad \theta_{jk} = \left\{ \mathbb{E}[|z_{jk}|^2] \right\}^{1/2}.
\]  
(33)

By using the equality (32) and Lemma 2.9 we obtain, for all \( j \geq 1 \) and \( k,k' \in \{1, ..., 2^j\} \),
\[
\mathbb{E}[z_{jk} z_{jk'}] = 2^j \Delta^4 \Lambda_{j,k,k'}(0),
\]  
(34)

where \( \Delta^4 \Lambda_{j,k,k'} \) is the one step progressive difference of order 4 of the function
\[
\Lambda_{j,k,k'}(x) = F\left( \frac{2(k - k') - 2 + x}{2^{j+1}} \right),
\]
where \( F \) is given by (31).

Lemma 2.10. For all \( j \geq 1 \) and \( k,k' \in \{1, ..., 2^j\} \) such that \( k \neq k' \), there exists a constant \( \kappa_{j,k,k'} \in (0, 4) \), such that
\[
|\mathbb{E}[z_{jk} z_{jk'}]| \leq \frac{3 \theta}{16 \sqrt{\pi}} \frac{1}{|2(k - k') - 2 + \kappa_{j,k,k'}|^3},
\]  
(35)

where \( \theta = \int_0^\infty e^{-\frac{s^4}{s^{1/2}}} \left[ 1 + \frac{1}{s} + \frac{1}{12 s^2} \right] ds \). And there exist \( m_1, m_2 \) > 0 such that, for all \( j \geq 1 \) and \( k \in \{1, ..., 2^j\} \),
\[
m_1 \leq \mathbb{E}[|z_{jk}|^2] \leq m_2.
\]  
(36)

Proof. First recall that by (34), we have
\[
\mathbb{E}[z_{jk} z_{jk'}] = 2^j \Delta^4 \Lambda_{j,k,k'}(0).
\]  
(37)

Denote by \( \Lambda_{j,k,k'}^{(4)} \) the derivative of order 4 of \( \Lambda_{j,k,k'} \). By the mean value theorem and (37), for all \( j \geq 1 \) and \( k,k' \in \{1, ..., 2^j\} \) such that \( k \neq k' \), there exists a constant
\( \kappa_{j,k,k'} \in (0, 4) \) such that

\[ \mathbb{E}[z_{jk} z_{jk'}] = 2^j \Lambda_{j,k,k'}^{(4)}(\kappa_{j,k,k'}). \tag{38} \]

Taking into account the fact that \( k \neq k' \), elementary calculation of \( \Lambda_{j,k,k'}^{(4)} \) entails

\[ \mathbb{E}[z_{jk} z_{jk'}] = \frac{3}{16\sqrt{\pi}} \frac{1}{|2(k-k') - 2 + \kappa_{j,k,k'}|^3} \times \int_0^{\nu} e^{-\frac{s^2}{5/2}} \left[ 1 - \frac{1}{s} + \frac{1}{12 s^2} \right] ds, \tag{39} \]

where \( \nu = t \left( \frac{2^{(j+1)}}{2(k-k') - 2 + \kappa_{j,k,k'}} \right)^2 \). Then we get (35) by taking \( \theta = \left( \frac{2^{(j+1)}}{2(k-k') - 2 + \kappa_{j,k,k'}} \right)^2 \).

Now we are going to show (36). First, we start by proving the upper bound. From (34) and (38) we get that for all \( j \geq 1 \) and \( k \in \{1, \ldots, 2^j\} \),

\[ \mathbb{E}[|z_{jk}|^2] = 2^j \Delta^4 \Lambda_{j,k,k}(0) = 2^j \int_0^{t} \frac{1}{2\sqrt{\pi r}} \Delta^4 \psi_{j,r}(0) dr, \]

where

\[ \psi_{j,r}(x) = \exp \left( -\frac{1}{4r} \left( \frac{x - 2}{2^{j+1}} \right)^2 \right). \]

By the change of variable \( s = 2^{(j+1)} r \), we have that

\[ \mathbb{E}[|z_{jk}|^2] = \int_0^{2^{(j+1)}} \frac{1}{2\sqrt{\pi s}} \Delta^4 \varphi_{s}(0) ds, \tag{40} \]

where \( \varphi_{s}(x) = \frac{e^{-\frac{(x-2)^2}{2}}}{2} \). Remark that \( \varphi_{s} \) depends neither on \( j \) nor on \( k \). Again by mean value theorem there exists a constant \( \lambda_{s} \in (0, 4) \) depending only on \( s \), such that

\[ \Delta^4 \varphi_{s}(0) = \varphi_{s}^{(4)}(\lambda_{s}) = \frac{3 e^{-\frac{1}{4\lambda_{s} - 2}}}{8 s^2} \left\{ 1 - \frac{(\lambda_{s} - 2)^2}{s} + \frac{1}{12} \frac{(\lambda_{s} - 2)^4}{s^2} \right\}. \tag{41} \]

We use the formula (41) away from the origin together with (40), to get

\[ \mathbb{E}[|z_{jk}|^2] \leq \int_0^{4t} \frac{1}{2\sqrt{\pi s}} \Delta^4 \varphi_{s}(0) ds + \int_0^{\infty} \frac{3 e^{-\frac{1}{4\lambda_{s} - 2}}}{8 s^2} \left\{ 1 - \frac{(\lambda_{s} - 2)^2}{s} + \frac{1}{12} \frac{(\lambda_{s} - 2)^4}{s^2} \right\} ds. \tag{42} \]

Then (42) entails

\[ \mathbb{E}[|z_{jk}|^2] \leq \int_0^{4t} \frac{2}{\sqrt{\pi s}} ds + \int_0^{\infty} \frac{3}{8} \frac{1}{s^2} \left\{ 1 + \frac{4}{3} \frac{1}{s^2} \right\} ds := m_2. \tag{43} \]
For the lower bound of (36), we have by (40)
\[ E|z_{jk}|^2 \]
\[
= \frac{1}{2\sqrt{\pi}} \left\{ \int_0^{t(2^j+1)} \frac{1}{\sqrt{s}} e^{-\frac{1}{2} s} ds - 4 \int_0^{t(2^j+1)} \frac{1}{\sqrt{s}} e^{-\frac{1}{4} s} ds + 3 \int_0^{t(2^j+1)} \frac{1}{\sqrt{s}} ds \right\}
\geq \frac{1}{2\sqrt{\pi}} \int_0^{4t} \frac{1}{\sqrt{s}} \left( e^{-\frac{1}{2} s} - 4e^{-\frac{1}{4} s} + 3 \right) ds := m_1,
\]
where the last inequality holds because the function \( x \in (0, \infty) \rightarrow \int_0^x \frac{1}{\sqrt{s}} e^{-\frac{1}{2} s} ds - 4 \int_0^x \frac{1}{\sqrt{s}} e^{-\frac{1}{4} s} ds + 3 \int_0^x \frac{1}{\sqrt{s}} ds \) is nondecreasing. So we obtain a lower bound \( m_1 \), which finishes the proof of Lemma 2.10.

**Lemma 2.11.** There exists a constant \( K > 0 \) such that, for all \( j \geq 1 \) and \( k, k' \in \{1, \ldots, 2^j\} \), we have
\[
\sum_{k, k'=1}^{2^j} |E_{\omega_{jk} \omega_{jk'}}|^2 \leq K2^j,
\]
where \( \omega_{jk} \) is given by (33).

**Proof.** The proof uses the same calculations as those used in Lemma 2.6.

We utilize similar arguments as in the proof of Lemma 2.6 to obtain

**Lemma 2.12.** For all \( j \geq 1 \) and \( k \in \{1, \ldots, 2^j\} \), we have
\[
E \left[ \sum_{k=1}^{2^j} (|\omega_{jk}|^p - c_p)^2 \right] \leq (c_{2p} - c_p^2)K2^j,
\]
where \( c_p = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} |x|^p e^{-\frac{x^2}{2}} dx \).

**Proof of Theorem 2.8.** We deduce the proof by the same arguments as those used in Theorem 2.1.

As above one can also show the following

**Theorem 2.13.** Let \( N(x) = e^{x^2} - 1 \). For all \( t \in (0, T] \), we have
\[
\mathbb{P}(u(t,.) \in B_{N}^{1/2}) = 1 \quad \text{and} \quad \mathbb{P}(u(t,.) \in B_{N}^{1/2,0}) = 0,
\]
where \( u(t,.) \) is the sample path \( x \in [0,1] \rightarrow u(t,x) \).

**Remark 7.** One can easily deduce a sharp Besov regularity, with respect to space variable, of the following stochastic heat equation with a non-null initial condition:
\[
\frac{\partial v}{\partial t}(t, x) = \frac{1}{2} \Delta v(t, x) + \frac{\partial^2}{\partial t \partial x} W(t, x), \quad t > 0, \ x \in \mathbb{R},
\]
\[
v(0, x) = g(x), \quad x \in \mathbb{R},
\]
where \( g \) is a bounded measurable function. The (mild) solution of (47) is given by,

\[
v(t, x) = \int_{\mathbb{R}} G(t, x - y)g(y) \, dy + \int_0^t \int_{\mathbb{R}} G(t - s, x - y)dW(s, y)
:= J(t, x) + u(t, x).
\]

- Suppose that \( g \in \mathcal{B}^\alpha_{p_0}(\mathbb{R}) \) for some \( 1 < p_0 < \infty \) and \( 0 < \alpha < 1 \). Then one can easily verify that, for any fixed \( t \in (0, T] \), the function \( J(t, \cdot) : x \in [0, 1] \mapsto J(t, x) \) belongs to \( \mathcal{B}^\alpha_{p_0} \). Consequently, standard continuous injections imply that, for any \( 2 < p \leq p_0 \), \( \mathbb{P}\left[ v(t, \cdot) \in \mathcal{B}^{\frac{\alpha}{2}}_{p} \right] = 1 \).
- In addition, if \( g \in \mathcal{B}^\alpha_N(\mathbb{R}) \), then \( \mathbb{P}\left[ v(t, \cdot) \in \mathcal{B}^{\frac{\alpha}{2}}_{N} \right] = 1 \).

3. Existence and regularity of local times

3.1. Local time of the process \( t \to u(t, x) \)

In this section we will consider the process \((u(t, x), t \in [0, T])\), for some fixed \( T > 0 \) and \( x \in \mathbb{R} \).

3.1.1. Existence of the local time

We will need the following estimates, which can be easily shown by standard arguments as change of variables and Parseval’s equality:

**Lemma 3.1.** For any \( x \in \mathbb{R} \) and \( s < t \) with \( s, t \in [0, T] \),

\[
(1) \quad \int_0^s \int_{\mathbb{R}} |G(t - r, x - y) - G(s - r, x - y)|^2 \, dy \, dr \leq C|t - s|^{1/2}
\]

\[
(2) \quad \int_0^s \int_{\mathbb{R}} G^2(t - r, x - y) \, dy \, dr = C_1|t - s|^{1/2}.
\]

**Proposition 3.2.** For any \( x \in \mathbb{R} \) and \( 0 \leq p < 3 \), we have

\[
\int_0^T \int_0^T \mathbb{E}[(u(t, x) - u(s, x))^2]^{(p+1)/2} \, ds \, dt < \infty.
\]

**Proof.** We have for \( s, t \in [0, T] \), such that \( s < t \),

\[
\mathbb{E}(u(t, x) - u(s, x))^2
= \mathbb{E}\left| \int_0^s \int_{\mathbb{R}} (G(t - \tau, x - y) - G(s - \tau, x - y)) \, dW(\tau, y) + \int_s^t \int_{\mathbb{R}} G(t - \tau, x - y) \, dW(\tau, y) \right|^2.
\]

Independence argument together with the point \( (2) \) of Lemma 3.1 give

\[
\mathbb{E}(u(t, x) - u(s, x))^2
= \int_0^s \int_{\mathbb{R}} (G(t - \tau, x - y) - G(s - \tau, x - y))^2 \, dy \, d\tau + \int_s^t \int_{\mathbb{R}} G^2(t - \tau, x - y) \, dy \, d\tau
\geq \int_s^t \int_{\mathbb{R}} G^2(t - \tau, x - y) \, dy \, d\tau \geq C_1 \sqrt{(t - s)},
\]

19
where \( C_1 \) is a positive constant. This ends the proof of Proposition 3.2.

**Theorem 3.3.** Let \( x \in \mathbb{R} \) be fixed, then

1. There exists a square integrable version of the local time of \((u(t, x), t \in [0, T])\).
2. The process \((u(t, x), t \in [0, T])\) satisfies the LND property i.e., formula (5).

**Proof.** The existence of a square integrable version of the local time of \( u(t, x) \) is a consequence of Berman’s theory (cf. Theorem 1.3 and Proposition 3.2). We will denote this version by \((L(\xi, t), t \geq 0, \xi \in \mathbb{R})\).

Let us now prove that \((u(t, x), t \in [0, T])\) satisfies the LND condition. So we have to prove,

\[
\lim_{c \to 0} \inf_{0 \leq t - r \leq c, r < s < t} \frac{\text{Var}(u(t, x) - u(s, x) | u(\tau, x), r \leq \tau \leq s)}{\text{Var}(u(t, x) - u(s, x))} > 0. \tag{49}
\]

First, remark that we have the following inclusion of \( \sigma \)-algebras

\[
\sigma(u(\tau, x), r \leq \tau \leq s) \subset F_s^W,
\]

where we have noted \( F_s^W = \sigma((W(r, y), 0 \leq r \leq s, y \in \mathbb{R}) \lor N) \). We then get

\[
\frac{\text{Var}(u(t, x) - u(s, x) | u(\tau, x), r \leq \tau \leq s)}{\text{Var}(u(t, x) - u(s, x))} \geq \frac{\text{Var}(u(t, x) - u(s, x) | F_s^W)}{\text{Var}(u(t, x) - u(s, x))}. \tag{50}
\]

Now

\[
\text{Var}(u(t, x) - u(s, x) | W(\tau, y), 0 \leq \tau \leq s, y \in \mathbb{R}) = \text{Var} \left( \int_0^s \int_{\mathbb{R}} (G(t - \tau, x - y) - G(s - \tau, x - y)) dW(\tau, y) \right.
\]

\[
+ \int_s^t \int_{\mathbb{R}} G(t - \tau, x - y) dW(\tau, y) \bigg| F_s^W \bigg).
\tag{51}
\]

But since \( \int_0^s \int_{\mathbb{R}} (G(t - \tau, x - y) - G(s - \tau, x - y)) dW(\tau, y) \) is \( F_s^W \)-measurable and \( \int_s^t \int_{\mathbb{R}} G(t - \tau, x - y) dW(\tau, y) \) is independent of \( F_s^W \), we have

\[
\text{Var}(u(t, x) - u(s, x) | W(\tau, y), 0 \leq \tau \leq s, y \in \mathbb{R}) = \int_s^t \int_{\mathbb{R}} G^2(t - \tau, x - y) dyd\tau. \tag{52}
\]

On the other hand

\[
\text{Var}(u(t, x) - u(s, x)) = \int_0^s \int_{\mathbb{R}} (G(t - \tau, x - y) - G(s - \tau, x - y))^2 dyd\tau
\]

\[
+ \int_s^t \int_{\mathbb{R}} G^2(t - \tau, x - y) dyd\tau
\tag{53}
\]

\[
:= A(s, t).
\]
Combining (50), (51), (52) and (53) we obtain

\[
\lim_{c \to 0} \inf_{0 \leq t-r \leq c, r<s<t} \frac{\Var(u(t, x) - u(s, x) | u(\tau, x), \ r \leq \tau \leq s)}{\Var(u(t, x) - u(s, x))} = \frac{\int_s^t \int_\mathbb{R} G^2(t-\tau, x-y)dyd\tau}{A(s, t)}.
\]

Remark that

\[
\lim_{c \to 0} \inf_{0 \leq t-r \leq c, r<s<t} \frac{\int_s^t \int_\mathbb{R} G^2(t-\tau, x-y)dyd\tau}{A(s, t)} > 0 \quad \iff \lim_{c \to 0} \inf_{0 \leq t-r \leq c, r<s<t} \frac{\int_s^t \int_\mathbb{R} G^2(t-\tau, x-y)dyd\tau}{A(s, t)} > 0.
\]

The last property of $G$ is assured by Lemma 3.1, which ends the proof of Theorem 3.3.

**Proposition 3.4.** For all $x, y \in \mathbb{R}$, $t, t+h \in (0, T]$ and for any even positive integer $n$, there exists $C_n > 0$ such that

\[
\mathbb{E}[L(y, t + h) - L(x, t + h) - L(y, t) + L(x, t)]^n \leq C_n |x-y|^\zeta n |h|^{n(3/4-\zeta/4)},
\]

\[
\mathbb{E}[L(x, t + h) - L(x, t)]^n \leq C_n |h|^{3n/4},
\]

where $0 < \zeta < 1$.

**Proof.** We prove just the first inequality; the second one follows the same lines. For simplicity of notations we use $X_t$ to denote the process $(u(t, x), \ t \in [0, T])$. We consider only $h > 0$ such that $t+h \in [0, T]$ the other case follows the same way. Following [28] or [20] we have

\[
\mathbb{E}[L(y, t + h) - L(x, t + h) - L(y, t) + L(x, t)]^n = (2\pi)^{-n} \int_{[t,t+h]^n} \prod_{|j| \neq 0} \left| e^{-iyu_j} - e^{-ixu_j} \right| \mathbb{E}[e^{i \sum_{j=1}^n u_j X_j}] du dt.
\]

The elementary inequality $|1 - e^{i\theta}| \leq 2^{1-\zeta} |\theta|^\zeta$ for any $0 < \zeta < 1$ and $\theta \in \mathbb{R}$, leads to

\[
\mathbb{E}[L(y, t + h) - L(x, t + h) - L(y, t) + L(x, t)]^n \leq 2^{-n\zeta} \pi^{-n} |y-x|^n T(n, \zeta),
\]

where

\[
T(n, \zeta) = \int_{[t,t+h]^n} \prod_{j=1}^n |u_j|^{\zeta} \mathbb{E}[e^{i \sum_{j=1}^n u_j X_j}] du dt.
\]

In order to apply the LND property for the Gaussian process $X_t$, we do two transformations:

- We replace the integration over the domain $[t, t+h]^n$ by the integration over the subset $t < t_1 < t_2 \ldots < t_n < t + h$. 


In the integral over the u’s, we change the variable of integration by the following transformation

\[ u_n = v_n, \quad u_j = v_j - v_{j+1}, \quad j = 1, \ldots, n - 1. \]

We obtain

\[ T(n, \zeta) = n! \int_{t_1 < t_2 < \ldots < t_n < t+h} \prod_{j=1}^{n} |v_j - v_{j+1}|^{\zeta} |v_n|^{\zeta} \mathbb{E}[e^{\sum_{j=1}^{n} \sigma_j (X_{t_j} - X_{t_{j-1}})}] d\varpi dt \]

\[ = n! \int_{t_1 < t_2 < \ldots < t_n < t+h} \prod_{j=1}^{n} |v_j - v_{j+1}|^{\zeta} |v_n|^{\zeta} e^{-\frac{1}{2}\text{var} \sum_{j=1}^{n} \sigma_j (X_{t_j} - X_{t_{j-1}})} d\varpi dt, \quad (55) \]

where \( t_0 = 0 \). Now, since \(|a - b|^{\zeta} \leq |a|^{\zeta} + |b|^{\zeta}\) for all \( 0 < \zeta < 1 \), it follows that

\[ \prod_{j=1}^{n} |v_j - v_{j+1}|^{\zeta} |v_n|^{\zeta} \leq \prod_{j=1}^{n-1} (|v_j|^{\zeta} + |v_{j+1}|^{\zeta}) |v_n|^{\zeta}. \quad (56) \]

Note that the last term in the right is at most equal to a finite sum of terms each of the form \( \prod_{j=1}^{n} |v_j|^{\epsilon_j \zeta} \), where \( \epsilon_j = 0, 1, \) or \( 2 \) and \( \sum_{j=1}^{n} \epsilon_j = n \). Let us write for simplicity \( \sigma^2(j) = \mathbb{E}(X_{t_j} - X_{t_{j-1}})^2 \). Using (56) and the LND property of \( X_t \), i.e. the second point in Theorem 3.3, the term \( T(n, \zeta) \) in (55) is dominated by the sum over all possible choice of \( (\epsilon_1, \ldots, \epsilon_n) \in \{0, 1, 2\}^n \) of the following terms

\[ \int_{t_1 < t_2 < \ldots < t_n < t+h} \prod_{j=1}^{n} |v_j|^{\epsilon_j \zeta} \exp \left( -\frac{C_n}{2} \sum_{j=1}^{n} v_j^2 \sigma^2(j) \right) d\varpi dt, \quad (57) \]

where \( C_n \) is a positive constant and \( h \) is small enough such that \( 0 < h < \delta_n \), (\( \delta_n \) and \( C_n \) are given by the LND property). Now, by the change of variable \( x_j = \sigma(j)v_j \), the term (57) becomes

\[ \int_{t_1 < t_2 < \ldots < t_n < t+h} \prod_{j=1}^{n} \sigma(j)^{-1-\epsilon_j} \prod_{j=1}^{n} |x_j|^{\epsilon_j \zeta} \exp \left( -\frac{C_n}{2} \sum_{j=1}^{n} x_j^2 \right) d\varpi dt. \quad (58) \]

Using the second point in Lemma 3.1 we have \( \sigma^2(j) = \mathbb{E}(X_{t_j} - X_{t_{j-1}})^2 \geq C(t_j - t_{j-1})^{1/2} \), where \( C \) is a positive constant. This implies that the integral in (58) is dominated by

\[ \int_{t_1 < t_2 < \ldots < t_n < t+h} \prod_{j=1}^{n} |t_j - t_{j-1}|^{\zeta} \prod_{j=1}^{n} |x_j|^{\epsilon_j \zeta} \exp \left( -\frac{C_n}{2} \sum_{j=1}^{n} x_j^2 \right) d\varpi dt \]

\[ = C(n, \zeta) \int_{t_1 < t_2 < \ldots < t_n < t+h} \prod_{j=1}^{n} |t_j - t_{j-1}|^{\zeta} d\varpi dt. \quad (59) \]
Now, return to Equation (54). Combining (55), (57) and (59) we obtain

$$E[L(y, t + h) - L(x, t + h) - L(y, t) + L(x, t)]^{n} \leq C(n, \zeta)|y - x|^{n \zeta} \int_{t < t_{1} < t_{2} < \ldots < t_{m} < t + h} \prod_{j=1}^{n}|t_j - t_{j-1}|^{-\frac{1-\zeta}{4}} d\mathbb{P}. \quad (60)$$

Remark that the integral in the right hand side of (60) is finite. Moreover, by using an elementary calculations, we have for any $n \geq 1$, $h > 0$ and $b_{j} < 1$

$$\int_{t < t_{1} < t_{2} < \ldots < t_{m} < t + h} \prod_{j=1}^{n}|t_j - t_{j-1}|^{-b_{j}} d\mathbb{P} = h^{n - \sum_{j=1}^{n} b_{j}} \prod_{j=1}^{n} \frac{\Gamma(1 - b_{j})}{\Gamma(1 + n - \sum_{j=1}^{n} b_{j})}.$$  

Finally, taking $b_{j} = \frac{1+\zeta}{4}$, we get

$$E[L(y, t + h) - L(x, t + h) - L(y, t) + L(x, t)]^{n} \leq C(n, \zeta)|y - x|^{n \zeta} h^{n \left(\frac{1}{2} - \frac{\zeta}{4}\right)}. \quad (61)$$

We can deduce by classical arguments (cf. D. Geman-J. Horowitz [28, Theorem 26.1] or Berman [6, Theorem 8.1]) the following regularity result for the local time of the solution $(u(t, x), t \in [0, T])$

**Theorem 3.5.** For any $x \in \mathbb{R}$, the solution $(u(t, x), t \in [0, T])$ has almost surely a jointly continuous local time $(L(\xi, t), t \in [0, T], \xi \in \mathbb{R})$ which satisfies for all $\alpha < 3/4$,

$$\sup_{\xi} |L(\xi, t + h) - L(\xi, t)| \leq \eta' h^{\alpha}, \quad (62)$$

for any $t, t + h \in [0, T]$ such that $|h| < \eta$, where $\eta$ and $\eta'$ are random variables a.s. positive and finite.

We also obtain by Proposition 3.4 together with a version of Kolmogorov’s continuity theorem in Besov norms (see Boufoussi et al. [9, Lemma 2.1.]) that

**Theorem 3.6.** For all $\lambda > 0$, $p > \frac{1}{\lambda}$ and $\xi \in \mathbb{R}$,

$$\mathbb{P} \left(L(\xi, .) \in \mathcal{B}^{\omega_{\lambda}}_{p}\right) = 1,$$

where $\omega_{\lambda}(t) = t^{3/4}(\log(1/t))^{\lambda}$ and $L(\xi, .)$ is the sample paths $t \to L(\xi, t), t \in [0, 1]$.

**Remark.**

1. Taking $\lambda$ small enough, Theorem 3.6 ensures a more accurate regularity result. Particularly, we deduce by injection (8) that $L(\xi, .)$ satisfies a.s. a $\beta-$Hölder condition for any $\beta < \frac{3}{4}$.

2. The process $u(., x)$ satisfies (1) of Theorem 1.3 with $0 \leq p < 3$. Then there is a version of the local time $L(\xi, t)$, which is differentiable with respect to the space variable, and a.s. $L^{(1)}(\xi, t) = \partial_{x}L(\xi, t) \in L^{2}(\mathbb{R}, d\xi)$.

It is easy to verify that $L^{(1)}$ satisfies (54) with $T(n, \zeta + 1)$ instead of $T(n, \zeta)$. Following the same arguments as in Proposition 3.4 the finiteness of the integral in (60) (with $\zeta + 1$ in place of $\zeta$) requires that $\zeta < 1/2$. Furthermore, we obtain
that for all \( x, y \in \mathbb{R} \), \( t, t + h \in (0, T] \) and for any positive integer \( n \), there exists \( C_n > 0 \) such that

\[
\mathbb{E}[L^{(1)}(y, t + h) - L^{(1)}(x, t + h) - L^{(1)}(y, t) + L^{(1)}(x, t)]^n \leq C_n |x - y|^{cn} h^{(n/2 - \zeta/4)},
\]

where \( 0 < \zeta < 1/2 \).

Consequently, we have the following regularity result:

**Theorem 3.7.** There is a jointly continuous version of \((L^{(1)}(\xi, t), t \in [0, T], \xi \in \mathbb{R})\) satisfying: For all compact \( U \subset \mathbb{R} \) and for any \( \alpha < 1/2 \)

\[
\sup_{x, y \in U, x \neq y} \frac{|L^{(1)}(x, t) - L^{(1)}(y, t)|}{|x - y|^{\alpha}} < \infty, \text{ a.s.}
\]

### 3.1.2. Hausdorff dimension of level sets

Let \( x \in \mathbb{R} \) be fixed. We define, for any \( \xi \in \mathbb{R} \), the \( \xi \)-level set of \((u(t, x), t \in [0, T])\) by

\[
M^x(\xi) = \{ t \in [0, T] : u(t, x) = \xi \}.
\]

Our goal is to determine the Hausdorff dimension \( \dim_H(M^x(u(t_0, x))) \) of \( M^x(u(t_0, x)) \). We can refer to [27, p. 27] for an introduction to Hausdorff measure and dimension. One of the crucial applications of the joint continuity of the local time is to extend \( L(\xi, .) \) as a finite measure supported on the level set \( M^x(\xi) \) see [1, Theorem 8.6.1].

To find a lower bound of the Hausdorff dimension of the level sets we need first the following Frostman’s Lemma cf. [21, Lemma 6.10.]

**Lemma 3.8.** Let \( E \) be a Borel set of \( \mathbb{R} \). \( \mathcal{H}^s(E) > 0 \) if and only if there exists a finite Borel measure \( \mu \) supported on \( E \) such that \( \mu(E) > 0 \) and a positive constant \( c \) such that

\[
\mu((y - r, y + r)) \leq cr^s,
\]

for all \( y \in \mathbb{R} \) and \( r > 0 \).

**Lemma 3.9.** For all \( x \in \mathbb{R} \), we have almost surely and for almost every \( t_0 \in [0, T] \)

\[
\dim_H(M^x(u(t_0, x))) \geq \frac{3}{4}.
\]

**Proof.** Let \( x \in \mathbb{R} \) be fixed, we have by [5, Lemma 1.1.] that for almost every \( t_0 \)

\[
L(u(t_0, x), T) > 0.
\]

We know that \( L(u(t_0, x), .) \) is a measure supported on \( M^x(u(t_0, x)) \), and [62] entails that \( L(u(t_0, x), .) \) satisfies a.s. a Hölder condition of any order smaller than \( \frac{3}{4} \). So by Lemma [3.8] we have almost surely and for almost every \( t_0 \)

\[
\dim_H(M^x(u(t_0, x))) \geq \frac{3}{4}.
\]
Lemma 3.10. For all $x \in \mathbb{R}$, we have almost surely and for all $t_0 \in [0, T]$

$$\dim_H(M^x(u(t_0, x))) \leq \frac{3}{4}.$$ 

Proof. We know that $u(., x)$ satisfies a.s. a Hölder condition of any order smaller than $\frac{1}{4}$. By Theorem 3.5 its local time is jointly continuous. The result then follows by [1, Theorem 8.7.3].

Combining Lemma 3.9 and Lemma 3.10 we obtain

Corollary 3.11. For all $x \in \mathbb{R}$, we have almost surely and for almost every $t_0$

$$\dim_H(M^x(u(t_0, x))) = \frac{3}{4}.$$ 

3.2. Local time of the process $x \rightarrow u(t, x)$

3.2.1. Existence of the local time

Let $[a, b] \subset \mathbb{R}$, we will prove the existence of the local time of the process $(u(t, x), \; x \in [a, b])$ where $t > 0$ is fixed. First we need the following result

Lemma 3.12. For fixed $t > 0$, and for any $x, y \in [a, b]$, there exists a constant $c_t > 0$ such that

$$c_t|x - y| \leq \mathbb{E}(u(t, x) - u(t, y))^2 \leq \frac{|x - y|}{2\pi}.$$ 

Proof. Let $x, y \in [a, b]$ such that $x > y$, the change of variable $r = t - s$ together with Parseval’s identity give

$$\mathbb{E}(u(t, x) - u(t, y))^2 = \int_0^t \int_{\mathbb{R}} (G(r, x - z) - G(r, y - z))^2 dz dr$$

$$= \frac{1}{2\pi} \int_0^t \int_{\mathbb{R}} |e^{izu} \exp(-ru^2/2) - e^{izu} \exp(-ru^2/2)|^2 dudr$$

$$= \frac{1}{2\pi} \int_0^t \int_{\mathbb{R}} \exp(-ru^2) |e^{i(x-y)u} - 1|^2 dudr.$$ 

Again by the transformations $v = u(x - y)$ and $\tau = \frac{r}{(x-y)^2}$, we get

$$\mathbb{E}(u(t, x) - u(t, y))^2 = \frac{x - y}{2\pi} \int_0^{t(x-y)^2} \int_{\mathbb{R}} \exp(-\tau v^2) |e^{iv} - 1|^2 dv d\tau.$$ 

25
Using Fubini, we obtain

\[
\mathbb{E}(u(t, x) - u(t, y))^2 = \frac{x - y}{2\pi} \int_\mathbb{R} \int_0^t \exp(-\tau v^2) |e^{iv} - 1|^2 d\tau dv
\]

\[
= \frac{x - y}{2\pi} \int_\mathbb{R} (1 - \exp(-\frac{t}{(x - y)^2}v^2)) \frac{|e^{iv} - 1|^2}{v^2} dv
\]

\[
= \frac{x - y}{2\pi} \left\{ \int_\mathbb{R} \frac{|e^{iv} - 1|^2}{v^2} dv - \int_\mathbb{R} \exp(-\frac{t}{(x - y)^2}v^2) \frac{|e^{iv} - 1|^2}{v^2} dv \right\}
\]

\[
= \frac{x - y}{2\pi} \left\{ 1 - \int_\mathbb{R} \exp(-\frac{t}{(x - y)^2}v^2) \frac{|e^{iv} - 1|^2}{v^2} dv \right\},
\]

where in the last line, Parseval’s identity gives

\[
\int_\mathbb{R} \frac{|e^{iv} - 1|^2}{v^2} dv = \int_\mathbb{R} \chi_{[0,1]}(v) dv = 1.
\]

So, on one hand, it is clear that

\[
\mathbb{E}(u(t, x) - u(t, y))^2 \leq \frac{|x - y|}{2\pi}.
\]

On the other hand, to find a lower bound for \(\mathbb{E}(u(t, x) - u(t, y))^2\), we need to get a constant \(0 \leq C_t < 1\) such that

\[
C_t \geq \int_\mathbb{R} \exp(-\lambda v^2) \frac{|e^{iv} - 1|^2}{v^2} dv := A,
\]

where we have used the notation \(\lambda = \frac{t}{(x - y)^2}\).

Now, denote by \(f(v) = \frac{1}{\sqrt{2\pi\lambda}} e^{-v^2/2\lambda}\) and \(g(v) = \chi_{[0,1]}(v)\). It follows by Parseval’s identity,

\[
A = \int_\mathbb{R} \left| \exp\left(\frac{-\lambda v^2}{2}\right) \frac{e^{iv} - 1}{iv} \right|^2 dv = \int_\mathbb{R} |f * g(v)|^2 dv
\]

\[
= \int_\mathbb{R} |f * g(v)|^2 dv.
\]

Then

\[
A = \int_\mathbb{R} \left\{ \int_{[0,1]} \frac{1}{2\pi\lambda} e^{-(v-z_1)^2/2\lambda} e^{-(v-z_2)^2/2\lambda} dz_1 dz_2 \right\} dv.
\]
By Fubini we have

\[
A = \int_{[0,1]^2} \frac{1}{\sqrt{2\pi \lambda}} \left\{ \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi \lambda}} e^{-\frac{(v-z_1)^2}{2\lambda}} e^{-\frac{(v-z_2)^2}{2\lambda}} dv \right\} dz_1 dz_2 \\
= \int_{[0,1]^2} e^{-(z_1-z_2)^2/4\lambda} \left\{ \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi \lambda}} \exp\left(-\frac{1}{\lambda} (v - z_1 z_2)^2 \right) dv \right\} dz_1 dz_2 \\
= \int_{[0,1]^2} \frac{1}{\sqrt{4\pi \lambda}} \exp\left(-\frac{(z_1-z_2)^2}{4\lambda} \right) dz_1 dz_2 \\
= \int_{[0,1]} \left\{ \int_{[0,1]} \frac{1}{\sqrt{2\pi (2\lambda)}} \exp\left(-\frac{(z_1-z_2)^2}{2(2\lambda)} \right) dz_2 \right\} dz_1 \\
= \int_{[0,1]} \mathbb{P}[0 \leq \sqrt{2\lambda N} + z_1 \leq 1] dz_1,
\]

where \( N \) is a standard Normal random variable. Then

\[
A = \mathbb{E} \left[ \int_{[0,1]} \chi_{[-\sqrt{2\lambda N},1-\sqrt{2\lambda N}]}(z_1) dz_1 \right] \\
= \mathbb{E} \left[ (1 - \sqrt{2\lambda N}) \chi_{[0,1]}(\sqrt{2\lambda N}) + (1 + \sqrt{2\lambda N}) \chi_{[-1,0]}(\sqrt{2\lambda N}) \right] \\
= 2\mathbb{E} \left[ (1 - \sqrt{2\lambda N}) \chi_{[0,1]}(\sqrt{2\lambda N}) \right].
\]

The last equality follows by the symmetry of the distribution of \( N \). Now replace \( \lambda \) by its value, since \( x, y \in [a, b] \) we obtain

\[
A = 2\mathbb{E} \left[ (1 - \frac{\sqrt{2t}}{b-a} N) \chi_{[0,\frac{b-a}{\sqrt{2t}}]}(N) \right] \leq 2\mathbb{E} \left[ (1 - \frac{\sqrt{2t}}{b-a} N) \chi_{[0,\frac{b-a}{\sqrt{2t}}]}(N) \right] \\
\leq 2\mathbb{P}[0 \leq N \leq \frac{\sqrt{2t}}{b-a}] < 1.
\]

We then get \( 0 \leq A < 1 \), and this finishes the proof of the lemma.

Consequently, we have

**Proposition 3.13.** For all \( t > 0 \) and \( 0 \leq p < 1 \), we have

\[
\int_a^b \int_a^b \left[ \mathbb{E}(u(t,x) - u(t,y))^2 \right]^{-(p+1)/2} dx dy \leq \infty.
\]

**Proposition 3.14.** For all \( t > 0 \), there exists a square integrable version of the local time of \( (u(t,x), x \in [a,b]) \). We denote this version by \( (L(\xi,y), y \in [a,b], \xi \in \mathbb{R}) \), where \( L(\xi,y) := L(\xi,[a,y]) \).

**Proof.** It is a consequence of Proposition 3.13 together with Theorem 1.3.
3.2.2. Regularity of the local time

In order to study the regularity of the local time, we need to recall the fundamental tool for that, the strong local nondeterminism concept (SLN D). This notion was introduced by Cuzick and DuPreez in [19] (see also [49]), and used by many authors to investigate the law of iterated logarithm, Chung’s law of the iterated logarithm, modulus of continuity for various Gaussian processes.

**Definition 3.15.** Let \( \{X_t, \ t \in I\} \) be a gaussian stochastic process with \( 0 < \mathbb{E}(X_t^2) < \infty \) for any \( t \in J \) where \( J \) is a subinterval of \( I \). Let \( \phi \) be a function such that \( \phi(0) = 0 \) and \( \phi(r) > 0 \) for all \( r > 0 \). Then \( X \) is SLND on \( J \) if there exist constants \( K > 0 \) and \( r_0 > 0 \) such that for all \( t \in J \) and all \( 0 < r < \min\{|t|, r_0\} \),

\[
\text{Var}(X_t|X_s : s \in J, r \leq |s-t| \leq r_0) \geq K \phi(r).
\]

**Theorem 3.16.** For all \( t > 0 \), there exists a positive constant \( K = K(t,a,b) \), such that for all \( 0 < r \leq |b-a| \), we have

\[
\text{Var} \left( u(t,y)|u(t, x) : x \in [a,b], r \leq |y - x| \leq |b - a| \right) \geq K r.
\]

**Proof.** It is enough to show that there exists a constant \( K > 0 \) such that,

\[
\mathbb{E} \left( u(t,y) - \sum_{k=1}^{n} a_k u(t, x_k) \right)^2 \geq K r,
\]

for all integers \( n \geq 1 \), \( (a_k)_i \in \mathbb{R} \) and \( (x_k)_i \in [a, b] : r \leq |y - x_k| \leq |b - a|, \forall k \leq n \).

Parseval’s identity implies

\[
\mathbb{E} \left( u(t,y) - \sum_{k=1}^{n} a_k u(t, x_k) \right)^2 = 
\]

\[
= \int_0^t \int_{\mathbb{R}} \left( G(t-s,y-z) - \sum_{k=1}^{n} a_k G(t-s,x_k-z) \right)^2 dzds \]

\[
\]

\[
= \frac{1}{2\pi} \int_0^t \int_{\mathbb{R}} \left| \exp(iyu) - \sum_{k=1}^{n} a_k \exp(ix_k u) \right|^2 \exp(-su^2) du \]

\[
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} \left| \exp(iyu) - \sum_{k=1}^{n} a_k \exp(ix_k u) \right|^2 \frac{1 - \exp(-tu^2)}{u^2} du := Q(r).
\]

So we just need to prove that \( Q(r) \geq Kr \).

Let \( \varphi : \mathbb{R} \to [0,1] \) be a function in \( C^\infty(\mathbb{R}) \) such that \( \varphi(0) = 1 \) and \( \text{supp}(\varphi) \subset [0,1] \). Denote by \( \hat{\varphi} \) the Fourier transform of \( \varphi \). Then \( \hat{\varphi} \in C^\infty(\mathbb{R}) \) and \( \hat{\varphi}(u) \) decays rapidly as \( |u| \to \infty \). Set

\[
\varphi_r(\theta) = r^{-1} \varphi(r^{-1} \theta).
\]
By the inversion theorem we have
\[
\varphi_r(\theta) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iu\theta} \hat{\varphi}(ru) du. \tag{66}
\]

Since \( r \leq |y - x_i| \) and \( \text{supp}(\varphi) \subset [0, 1] \), we have \( \varphi_r(y - x_i) = 0 \) for any \( k = 1, \ldots, n \). This and (66) imply that
\[
B := \int_{\mathbb{R}} \left( \exp(iyu) - \sum_{k=1}^{n} a_k \exp(ix_k u) \right) \exp(-iu\theta) \hat{\varphi}(ru) du
= 2\pi(\varphi_r(0) - \sum_{k=1}^{n} a_k \varphi_r(y - x_k)) = 2\pi r^{-1}. \tag{67}
\]

On the other hand, by (65) and Hölder inequality, we obtain
\[
B^2 \leq \int_{\mathbb{R}} \left| \exp(iyu) - \sum_{k=1}^{n} a_k \exp(ix_k u) \right|^2 \frac{1 - \exp(-tu^2)}{u^2} du
\times \int_{\mathbb{R}} \frac{u^2}{1 - \exp(-tu^2)} |\hat{\varphi}(ru)|^2 du
= \mathbb{E} \left( u(t, y) - \sum_{k=1}^{n} a_k u(t, x_k) \right)^2 \times \int_{\mathbb{R}} \frac{u^2}{1 - \exp(-tu^2)} |\hat{\varphi}(ru)|^2 du
\leq \mathbb{E} \left( u(t, y) - \sum_{k=1}^{n} a_k u(t, x_k) \right)^2 \frac{1}{r^3} \int_{\mathbb{R}} \frac{v^2}{1 - \exp(-tv^2/|b-a|^2)} |\hat{\varphi}(v)|^2 dv,
\]
where last inequality is justified by the change of variable \( v = ru \) and \( 0 < r \leq |b - a| \). So by (67) we get
\[
4\pi^2 \frac{1}{r^2} \leq \mathbb{E} \left( u(t, y) - \sum_{k=1}^{n} a_k u(t, x_k) \right)^2 \frac{1}{r^3} K,
\]
where
\[
K = \int_{\mathbb{R}} \frac{v^2}{1 - \exp(-tv^2/|b-a|^2)} |\hat{\varphi}(v)|^2 dv.
\]

Finally, (68) holds. This finishes the proof of Theorem 3.16. \( \square \)

**Lemma 3.17.** Let \( y, y + h \in [a, b] \). For any even positive integer \( n \), we have
\[
\mathbb{E}[L(\xi, y + h) - L(\xi, y)]^n \leq C_n |h|^{n/2}, \tag{68}
\]
where \( C_n \) is a positive constant.
\textbf{Proof.} For simplicity we will deal with \( h > 0 \) such that \( y + h \in [a,b] \). The other case uses the same calculation. Let \( I = [y, y+h] \), then following \([26],[28]\), we have

\[
\mathbb{E}[L(\xi, I)^n] = (2\pi)^{-n} \int_{I^n} \int_{R^n} e^{-i<u,\xi>} \mathbb{E} \left[ e^{i\sum_{k=1}^{n} u_k u(t,x_k)} \right] d\bar{u}d\bar{x}
\]

\[
= (2\pi)^{-n} \int_{I^n} \int_{R^n} e^{-i<u,\xi>} e^{-\frac{1}{2} \text{Var}(\sum_{k=1}^{n} u_k u(t,x_k))} d\bar{u}d\bar{x},
\]

where \( \bar{\xi} = (\xi, \ldots, \xi) \) and \( \bar{\pi} = (u_1, \ldots, u_n) \), hence

\[
\mathbb{E}[L(\xi, I)^n] \leq (2\pi)^{-n} \int_{I^n} \int_{R^n} e^{-\frac{1}{2} \text{Var}(\sum_{k=1}^{n} u_k u(t,x_k))} d\bar{u}d\bar{x}. \tag{69}
\]

On the other hand, for distinct \( x_1, x_2, \ldots, x_n \), the matrix \( \text{Cov}(u(t,x_1), u(t,x_2), \ldots, u(t,x_n)) \) is invertible. Then the following function is a gaussian density

\[
\frac{[\det \text{Cov}(u(t,x_1), u(t,x_2), \ldots, u(t,x_n))]^{1/2}}{(2\pi)^{n/2}} e^{-\frac{1}{2} \text{Cov}(u(t,x_1), u(t,x_2), \ldots, u(t,x_n))\bar{\pi}}, \tag{70}
\]

where \( \bar{\pi} \) denotes the transpose of \( \pi \). Therefore

\[
\int_{R^n} e^{-\frac{1}{2} \text{Var}(\sum_{k=1}^{n} u_k u(t,x_k))} d\bar{\pi} = \frac{(2\pi)^{n/2}}{[\det \text{Cov}(u(t,x_1), u(t,x_2), \ldots, u(t,x_n))]^{1/2}}. \tag{71}
\]

Combining (69) and (71), we get

\[
\mathbb{E}[L(\xi, I)^n] \leq (2\pi)^{-n/2} \int_{I^n} \frac{1}{[\det \text{Cov}(u(t,x_1), u(t,x_2), \ldots, u(t,x_n))]^{1/2}} d\bar{x}. \tag{72}
\]

It follows from (2.8) in \([6]\) that

\[
\det \text{Cov}(u(t,x_1), u(t,x_2), \ldots, u(t,x_n)) = \text{Var}(u(t,x_1)) \prod_{j=2}^{n} \text{Var}(u(t,x_j)|u(t,x_1), \ldots, u(t,x_{j-1})). \tag{73}
\]

(73) together with (63) imply

\[
\det \text{Cov}(u(t,x_1), u(t,x_2), \ldots, u(t,x_n)) \geq K^n |x_1 - a_1| \prod_{j=2}^{n} \min_{1 \leq i < j} |x_j - x_i|. \tag{74}
\]

By using (74) in (72) we get

\[
\mathbb{E}[L(\xi, I)^n] \leq C_n \int_{I^n} \frac{1}{|x_1 - a_1|^{1/2}} \prod_{j=2}^{n} \frac{1}{\min_{1 \leq i < j} |x_j - x_i|^{1/2}} d\bar{x}
\]

\[
\leq C_n h^{n/2}, \tag{75}
\]

30
where the last inequality is obtained by integrating in the order $dx_n, dx_{n-1}, \cdots, dx_1$ and with the help of some elementary arguments. This finishes the proof of the lemma 3.17.

**Lemma 3.18.** For all $\xi, \xi + k \in \mathbb{R}$, $y, y + h \in [a, b]$ and for all even positive integer $n$, there exists a constant $C_n > 0$ such that

$$E[L(\xi + k, y + h) - L(\xi, y + h) - L(\xi + k, y) + L(\xi, y)]^n \leq C_n |k|^n|h|^n(1/2 - \delta/2),$$

where $0 < \delta < \frac{1}{2}$.

**Proof.** The proof uses the same techniques as those of Proposition 3.4.

We can deduce by classical arguments (cf. Berman [6, Theorem 8.1.] or Geman-J. Horowitz [28, Theorem 26.1]) the following regularity result on the local time of the process $(u(t, x), x \in [a, b])$

**Theorem 3.19.** For any fixed $t > 0$, the process $(u(t, x), x \in [a, b])$ has almost surely, a jointly continuous local time $(L(\xi, y), \xi \in \mathbb{R}, y \in [a, b])$. It satisfies a.s. a $\gamma$-Hölder condition in $y$, uniformly in $\xi$, for every $\gamma < \frac{1}{2}$: there exist random variables $\eta$ and $\eta'$ which are almost surely positive and finite such that

$$\sup_{\xi} |L(\xi, y + h) - L(\xi, y)| \leq \eta' |h|^{\gamma},$$

for all $y, y + h \in [a, b]$ and all $|h| < \eta$.

We also have, by [9, Lemma 2.1.], the following Besov regularity of the local time $L(\xi, y)$ in the space variable $y$

**Theorem 3.20.** For all $\lambda > 0$ and $p > \frac{1}{\lambda}$,

$$\mathbb{P}(L(\xi, \cdot) \in \mathcal{B}^{\omega_\lambda}_{p}) = 1,$$

where $\omega_\lambda(t) = t^{1/2}(\log(1/t)^{\lambda}$ and $L(\xi, \cdot)$ is the sample paths $y \to L(\xi, y), y \in [0, 1]$.

For fixed $t > 0$, let $M_t(\xi) = \{x \in [a, b] : u(t, x) = \xi\}$ be the $\xi$-level set of the process $(u(t, x), x \in [a, b])$. Proceeding in the same way as for $(u(t, x), t \in [0, T])$, we have

**Corollary 3.21.** For all $t > 0$, we have for almost every $x_0$

$$\dim_H(M_t(u(t, x_0))) = \frac{1}{2} \quad a.s.$$ 

**Acknowledgement(s)**

The first author would like to warmly thank Professor M. Dozzi for his fruitful discussions on an earlier version of this article.
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