AN INTERPRETATION OF TEMAM’S STABILIZATION TERM IN THE QUASI-INCOMPRESSIBLE NAVIER-STOKES SYSTEM

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Abstract. Using a characterization of inertial forces proposed by Paolo Podio-Guidugli we provide an interpretation of the stabilization term

\[ f_{st} = -\frac{1}{2}(\nabla \cdot v)v \]

in Roger Temam’s quasi-incompressible approximation

\[
\begin{align*}
\frac{\partial v}{\partial t} + (v \cdot \nabla) v + \nabla p - \mu \Delta v &= f + f_{st}, \\
\varepsilon \frac{\partial p}{\partial t} + \nabla \cdot v &= 0
\end{align*}
\]

of the incompressible Navier-Stokes system, showing that this term is a manifestation of inertia.

1. THE COMPRESSIBLE NAVIER-STOKES SYSTEM

The Navier-Stokes (N-S) equation, whose most popular version is

(\text{NS}) \quad \frac{\partial v}{\partial t} + \rho (v \cdot \nabla) v + \nabla p - \mu \Delta v - \mu \nabla (\nabla \cdot v) = f,

constitutes a basic model in fluid dynamics. In this expression, \(\rho(x,t)\) and \(v(x,t)\) are, respectively, the mass density (kg/m\(^3\)) and the velocity (m/s) fields; \(p(x,t)\) is the pressure field (Pa), and \(f(x,t)\) is the non-inertial body force field (N/m\(^3\)). The convective term \((v \cdot \nabla)v\) on the left-hand side can be written, using the conventional notation of continuum mechanics, as

\[ (v \cdot \nabla)v = \sum_{i=1}^{3} v_i \frac{\partial v}{\partial x_i} = \nabla v[v]. \]

The terms \(\Delta v\) and \(\nabla (\nabla \cdot v)\) are, respectively, the Laplacian of the velocity field and the gradient of its divergence. These terms carry dissipative effects proportional to the dynamic viscosity \(\mu\) (Pa/s).

In its many variants, the N-S equation has been systematically and successfully used as a model in engineering applications with relevant impact on our everyday’s life, even though its mathematics is still not completely understood: as of today, it is not known whether its (weak) solutions are unique and smooth for general initial data at all places and at all times, so much so that a one-million-dollar prize has been offered by the Clay Mathematics Institute for a proof or a counterexample.

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The N-S equation \((\text{NS})\) is a consequence of the pointwise equilibrium equation
\[
f_i + f + \nabla \cdot \sigma = 0,
\]
which dictates that the inertial body force \(f_i\), the (prescribed) non-inertial body force \(f\), and the system of internal forces be in equilibrium, the system of internal forces being equipollent to a body force whose density is the divergence of the Cauchy stress \(\sigma\). In particular, \((\text{NS})\) follows from the constitutive prescriptions
\[
(1) \quad f_i = -\rho \left( \frac{\partial v}{\partial t} + (v \cdot \nabla)v \right),
\]
and
\[
\sigma = -pI + \mu(\nabla v + \nabla v^T)
\]
for the inertial force and for the Cauchy stress, respectively.

The density and velocity fields are not independent, being related by the mass-balance equation:
\[
(\text{MBE}) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0.
\]

The system \((\text{NS}) - (\text{MBE})\) is closed by the state equation
\[
(\text{SE}) \quad p = \hat{p}(\rho),
\]
which makes explicit the dependence of pressure on density through the constitutive mapping \(\hat{p}\). We refer to the system \((\text{NS}), (\text{MBE}), (\text{SE})\) as the compressible Navier–Stokes system.

2. The incompressible Navier-Stokes system and its approximation

If we replace the state equation \((\text{SE})\) with the constraint
\[
(2) \quad \rho(x, t) = \rho_*
\]
then pointwise equilibrium and mass balance yield the incompressible Navier-Stokes system:
\[
(3) \quad \begin{cases} 
\rho_* \frac{\partial v}{\partial t} + \rho_* (v \cdot \nabla)v + \nabla p - \mu \Delta v = f, \\
\nabla \cdot v = 0,
\end{cases}
\]
the pressure \(p\) being now a reactive field.

When computing numerical solutions of the incompressible N-S system \((3)\), the incompressibility constraint \((3)_2\) poses several issues. One possible approach to alleviate these issues is to relax the incompressibility constraint \((3)_2\). A possible relaxation scheme involves using the compressible N-S system with a reasonably simple choice for the constitutive mapping \(\hat{p}\):
\[
(4) \quad \hat{p}(\rho) = K \left( \frac{\rho}{\rho_*} - 1 \right),
\]
where \(K\), the bulk modulus [Pa], is chosen as large as it takes to nullify, within the required numerical precision, the departure of \(\rho\) from its reference value \(\rho_*\). When \((4)\) is adopted, the inversion of \((\text{SE})\) yields
\[
(5) \quad \rho = \rho_* \left( 1 + \frac{p}{K} \right) =: \hat{\rho}(p/K).
\]
On substituting (5) into (NS) we obtain a system with respect to the unknowns \( v \) and \( p \):

\[
\begin{cases}
\frac{\hat{\varrho}(p/K)}{\partial t} \frac{\partial v}{\partial t} + \hat{\varrho}(p/K)(v \cdot \nabla)v + \nabla p - \mu \Delta v - \mu \nabla(\nabla \cdot v) = f, \\
\frac{\varrho_\ast}{K} \frac{\partial p}{\partial t} + \nabla \cdot (\hat{\varrho}(p)v) = 0.
\end{cases}
\]

In view of (5) one would expect that if the bulk modulus tends to infinity, and if pressure remains uniformly bounded in norm by a constant, then density should tend to the reference value \( \varrho_\ast \):

\[ K \to \infty \quad \text{and} \quad |p| < \text{const.} \quad \Rightarrow \quad \varrho \to \varrho_\ast, \]

so much so the incompressibility constraint would be recovered:

\[ \nabla \cdot v \to 0, \]

according to some notion of weak convergence.

3. **The quasi-incompressible Navier-Stokes system**

In the book [2, Chap. 8], R. Temam proposes the *quasi-incompressible N-S system* [1]

\[
\begin{cases}
\frac{\varrho_\ast}{K} \frac{\partial v}{\partial t} + \varrho_\ast(v \cdot \nabla)v + \nabla p - \mu \Delta v = f - \frac{\varrho_\ast}{2}(\nabla \cdot v)v, \\
\frac{1}{K} \frac{\partial p}{\partial t} + \nabla \cdot v = 0,
\end{cases}
\]

as an alternative to (6) to approximate the solutions of the incompressible N-S system (3). System (7) is obtained from (6) through the following three steps: (i) approximate the actual density \( \varrho = \hat{\varrho}(p) \) with the referential density \( \varrho_\ast \), i.e., perform the formal substitution:

\[ \hat{\varrho}(p/K) \to \varrho_\ast; \]

(ii) dispense of the term \( -\mu \nabla(\nabla \cdot v) \)

(iii) add the *stabilization force*:

\[ f_s = -\frac{\varrho_\ast}{2}(\nabla \cdot v)v \]

to the non-inertial body force \( f \).

One of the advantages of (7) over (6) is that the term that contains the time derivative of the velocity is *linear* with respect to the unknowns \( (v \) and \( p) \), which is a desirable feature from the point of view of both mathematical and numerical analysis. The adoption of (7) is justified by a convergence theorem. Remarkably, this theorem would fail without the stabilization term.

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1. The format adopted in [2] for the quasi-incompressible N-S system — the same format we use in the Abstract of the present paper — is with \( \varrho_\ast = 1 \) and with the bulk modulus written as \( K = 1/\varepsilon \), where \( \varepsilon \) is a small regularization parameter. Our motivation for multiplying the stabilization term by \( \varrho_\ast \) is not only for the sake of dimensional consistency, as will be apparent at the end of the present paper.

2. Perhaps, the omission of Step (ii) would result in a more physically-sound version of the quasi-incompressible N-S system.
4. A MECHANICAL INTERPRETATION OF THE STABILIZATION TERM

We show in this section that, besides analytical convenience, there is a somehow deeper (from the standpoint of mechanics) motivation for the introduction of the stabilization term \( \mathcal{S} \) in (7). For the Reader’s convenience we split our presentation in two parts. In the first part we recapitulate the procedure leading to the standard constitutive prescription (1) for the inertial-force density. In the second part we offer our interpretation of the stabilization term.

4.1. The standard constitutive prescription for the inertial force. First, we assume that the spatial density of kinetic energy be

\[
\kappa = \frac{\vartheta}{2} |v|^2.
\]

Next, we choose any reference configuration where the mass density is \( \varrho^* \). Then, the Jacobian of the deformation map which takes the reference configuration into the current configuration at point \( x \) and time \( t \) is

\[
J(x, t) = \frac{\varrho^*}{\varrho(x, t)}.
\]

Therefore, the kinetic energy per unit referential volume is

\[
\kappa_r = J\kappa = \frac{J\varrho}{2} |v|^2 = \frac{\varrho^*}{2} |v|^2.
\]

Thus, the material time derivative of the referential kinetic energy is:

\[
\dot{\kappa}_r = \varrho^* \dot{v} \cdot v,
\]

where \( \dot{v} \) is the material time derivative of \( v \).

Now, following [1], we require that the rate of change of specific (i.e. per unit referential volume or, equivalently, per unit mass) kinetic energy plus the specific inertial power be null during every possible motion. In the present case, this requirement takes the form:

\[
(f_{i,r} + \varrho^* \dot{v}) \cdot v = 0.
\]

This requirement singles out the inertial force up to a powerless contribution which, when taken to be null, prompts the following constitutive choice for the referential inertial force:

\[
f_{i,r} = -\varrho^* \dot{v},
\]

whence the following constitutive prescription for the inertial force density in the current configuration:

\[
f_i = J^{-1} f_{i,r} = \frac{\varrho}{\varrho^*} f_{i,r} = -\varrho \dot{v}.
\]

The step from (14) to (1) is immediate on recalling that \( \dot{v} = \frac{\partial v}{\partial t} + (v \cdot \nabla) v \).

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3Any reference configuration would do. However, this choice is convenient in view of our foregoing developments.
4.2. **A non standard constitutive prescription for the inertial force.** We show in the foregoing that if instead of (9) we make the following constitutive choice for the spatial density of kinetic energy:

\( \kappa^* = \frac{\rho_s}{2} |v|^2 \),

then the procedure outlined in the previous subsection leads to the following constitutive prescription for the spatial inertial-force density:

\( f^*_i = -\rho_s \dot{v} + f_s \),

where the second term on the right-hand side is the stabilization force (8).

As a start, we recall that the mass-balance equation (MBE) can be written as

\( \dot{\rho} + \rho \nabla \cdot v = 0 \).

Next, on adopting (15), we obtain that the kinetic energy per unit referential volume is

\( \kappa^*_r = J \kappa^* = \frac{J \rho_s}{2} |v|^2 \).

Thus, the rate of change of the referential kinetic-energy density is:

\[ \dot{\kappa}^*_r = \frac{\rho^2_s}{\rho} \dot{v} \cdot v - \frac{\dot{\rho}}{\rho^2} |v|^2 \]

This leads us to the following choice for the referential density of the inertial force:

\( f^*_r = -\frac{\rho_s}{\rho} \left( \frac{\rho_s}{\rho} \dot{v} + \frac{\rho_s}{2} (\nabla \cdot v) v \right) \),

which yields, for the inertial force per unit volume in the current configuration, the expression

\( f^*_i = \frac{\rho}{\rho_s} f^*_r = -\rho_s \dot{v} - \frac{\rho_s}{2} (\nabla \cdot v) v \),

which coincides with (16), in view of (8).

5. **Conclusions**

We have shown that if the approximate expression (15) is adopted for the spatial density of kinetic energy, where \( \rho_s \) is the mass density appearing in the incompressible Navier-Stokes system (3), then the extra stabilization term (8) in the quasi-compressible Navier-Stokes system (7) emerges as a natural consequence of the requirement that the specific power expenditure of the inertial force be equal to minus the rate of change of the specific kinetic energy. In other words, the stabilization term is a manifestation of inertia.

6. **Acknowledgements**

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**References**

[1] P. Podio-Guidugli (1997) *Inertia and Invariance*, Ann. Mat. Pura Appl. 172:103–124.

[2] R. Temam, Navier Stokes equations — Theory and Numerical Analysis, North Holland, 1979.