Exotic branes and mixed-symmetry potentials II: duality rules and exceptional $p$-form gauge fields

José J. Fernández-Melgarejo$^a$, Yuho Sakatani$^b$, Shozo Uehara$^b$

$^a$Departamento de Física, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain

$^b$Department of Physics, Kyoto Prefectural University of Medicine, Kyoto 606-0823, Japan

Abstract

In $U$-duality-manifest formulations, supergravity fields are packaged into covariant objects such as the generalized metric and $p$-form fields $A^I_p$. While a parameterization of the generalized metric in terms of supergravity fields is known for $U$-duality groups $E_n$ with $n \leq 8$, a parameterization of $A^I_p$ has not been fully determined. In this paper, we propose a systematic method to determine the parameterization of $A^I_p$, which necessarily involves mixed-symmetry potentials. We also show how to systematically obtain the $T$- and $S$-duality transformation rules of the mixed-symmetry potentials entering the multiplet. As the simplest non-trivial application, we find the parameterization and the duality rules associated with the dual graviton. Additionally, we show that the 1-form field $A^I_1$ can be regarded as the generalized graviphoton in the exceptional spacetime.
# Contents

1 Introduction 2

2 Parameterization of the 1-form $A^I_1$
   2.1 Supergravity fields ........................................... 3
   2.2 Strategy: Linear map ........................................... 3
   2.3 Detailed procedures .......................................... 5
   2.4 Results ......................................................... 7
   2.5 $T$-duality rule ............................................... 9
   2.6 $S$-duality rule ............................................... 10

3 Another approach based on the generalized metric 14
   3.1 Linear map between generalized metrics ....................... 15
   3.2 Connection between two parameterization ..................... 17
   3.3 1-form $A^I_\mu$ as the generalized graviphoton ............... 18

4 Parameterization of $A^I_\mu$ 21

5 Summary and Discussion 23

A Notation 24

B $E_n$ generators 25
   B.1 M-theory parameterization .................................... 26
   B.2 Type IIB parameterization .................................... 28
1 Introduction

In the previous paper [1], we have conducted a detailed survey of mixed-symmetry potentials in 11D and type II supergravities. By considering their reduction to $d$ dimensions, they yield various $p$-form fields $A^p_i$, which transform covariantly under $E_n$ $U$-duality transformation ($n = 11 - d$). In the $U$-duality-covariant formulation of supergravity known as exceptional field theory (EFT) [2–4] (see [6–13] for earlier fundamental works), and the $U$-duality-manifest approaches to brane actions [15–18], the $p$-form fields $A^p_i$ play an important role in providing $U$-duality-covariant descriptions. However, to make contact with the standard descriptions in supergravity and brane actions, explicit parameterizations of $A^p_i$ are needed. In this paper, we propose a systematic method to determine the parameterization of $A^p_i$ by utilizing the equivalence between M-theory (or type IIA theory) and type IIB theory. In our method, in addition to the parameterization of the $p$-form fields, the duality transformation rules of various potentials can also be obtained. As the first non-trivial example, we obtain the $T$- and $S$-duality rules for the dual graviton, equations (2.50)–(2.53) and (2.65), respectively.

In EFT, the fundamental fields are the generalized metric $M_{IJ}$ and $p$-form fields $A^p_i$, as well as certain auxiliary fields. For $E_n$ EFT with $n \leq 8$, the parameterization of the generalized metric has been determined in [13, 19] by means of the bosonic fields in 11D supergravity. The parameterization in terms of type IIB supergravity has been determined in [20, 21] for $E_n$ EFT with $n \leq 7$. They are nothing more than the two different parameterizations of the same object $M_{IJ}$, and as was concretely realized in [22], we can relate the two parameterizations through some redefinitions of fields. As was shown in [22], by rewriting the M-theory fields in terms of type IIA fields, these field redefinitions are precisely the $T$-duality transformation rule. However, the analysis of [22] is limited to the $E_n$ EFT with $n \leq 7$, where the generalized metric contains only the standard $p$-form potentials. In this paper, we extend their analysis to the case of $E_8$ EFT, and find the $T$-duality and $S$-duality rule for the dual graviton. This gives a non-trivial check of our duality rules for the dual graviton advertised in the first paragraph.

If we look at the explicit parameterization of the 1-form field $A^I_\mu$, its first component $A^I_\mu$ is the graviphoton. In 11D, the graviphoton is defined as $\hat{A}^i_\mu \equiv \hat{g}_{\mu r} \hat{g}^{r i}$, by using the 11D inverse metric $\hat{g}^M \hat{N}$ and the metric $\hat{g}_{\mu \nu}$ in the external spacetime. In this paper, we propose that the 1-form field $A^I_\mu$ can be regarded as a generalized graviphoton in the exceptional spacetime

$$A^I_\mu = m_{\mu \nu} M^{\nu I}, \quad m \equiv (M^{\mu \nu})^{-1}, \quad (1.1)$$

where $M^{IJ} (\hat{I} = \{\mu, I\})$ is the inverse generalized metric in $E_{11}$ EFT. We also find that the parameterizations of the higher $p$-form fields $A^p_\mu (p \geq 2)$ can be easily obtained from that of the 1-form $A^I_\mu$ through a simple antisymmetrization of indices.
2 Parameterization of the 1-form $A^I_I$

In this section, we explain our method to determine the parameterizations of the 1-form $A^I_I$. The index $I$ transforms in a fundamental representation of the $E_n$ algebra with Dynkin label $[1,0,\ldots,0]$, known as the vector representation or the particle multiplet. Our approach relies on the existence of two equivalent descriptions of EFT by deleting different nodes of the $E_n$ Dynkin diagram; M-theory and type IIB theory (see [1] and references therein for details):

As it is explained in the accompanying paper [1], in terms of M-theory, the 1-form field $A^I_I$ is decomposed into SL($n$) tensors as follows:

$$(A^I_i)_\mu = (A^{i}_{\mu}, A^{i}_{\mu; i_1 i_2 \sqrt{2}}, A^{i}_{\mu; i_1 i_2 i_3 \sqrt{5}}, A^{i}_{\mu; i_1 i_2 i_3 i_4 i_5 \sqrt{7}}), \quad (2.2)$$

where $i, j = d, \ldots, 9, z$ are indices of the fundamental representation of SL($n$). On the other hand, in terms of type IIB theory, the 1-form field is decomposed into SL($n-1$) $\times$ SL(2) tensors as follows:

$$(A^I_\alpha)_\mu = (A^\alpha_{\mu}, A^\alpha_{\mu; m_1 m_2 m_3 \sqrt{3}}, A^\alpha_{\mu; m_1 m_2 m_3 m_4 \sqrt{5}}, A^\alpha_{\mu; m_1 m_2 m_3 m_4 m_5 \sqrt{6}}), \quad (2.3)$$

where $\alpha, \beta = 1, 2$ are the SL(2) S-duality indices and $m, n = d, \ldots, 9$ are indices of the fundamental representation of SL($n-1$). In order to stress the difference between the two parameterizations, we have denoted the 1-form in type IIB parameterization by $A^I_\alpha$. Although we know the tensor structures of each component, it is not obvious how to determine the explicit parameterization in terms of the standard supergravity fields, which is the main subject of this paper.

As demonstrated in [22], the two decompositions (2.2) and (2.3) can be related by using the equivalence between M-theory on $T^2$ with coordinates $(x^\alpha) = (x^y, x^z)$ and type IIB theory on $S^1$ with a coordinate $x^y$:

$$\text{M-theory}/T^2 \xrightarrow{\text{compactification on } x^z} \text{Type IIA theory}/S^1 \xrightarrow{T\text{-duality along } x^y/x^z} \text{Type IIB theory}/S^1. \quad (2.4)$$

Here, $x^z$ is a coordinate along the M-theory circle, and the coordinate $x^y$ in M-theory (or type IIA theory) is mapped to the coordinate $x^y$ in type IIB theory under the $T$-duality. By using the map, we can rewrite various quantities in M-theory in terms of type IIB supergravity.

2.1 Supergravity fields

In order to discuss the parameterization, we will briefly explain the supergravity fields considered in this paper. We basically follow the convention of [22].
11D supergravity: In 11D supergravity, we consider the following bosonic fields,

\[ \{ \hat{g}_{\hat{M}\hat{N}}, \hat{A}_3, \hat{A}_6, \hat{A}_{8,1} \} \quad (\hat{M}, \hat{N} = 0, \ldots, 9, z). \]  

The standard potentials \( \hat{A}_3 \) and \( \hat{A}_6 \) couple to M2-brane and M5-brane, respectively, while the dual graviton \( \hat{A}_{8,1} \) couples to the Kaluza–Klein monopole 6\(^1\) (sometimes called MKK) \[23\].

When we consider a compactification to \( d \) dimensions, the 11D metric \( \hat{g}_{\hat{M}\hat{N}} \) is decomposed as

\[ (\hat{g}_{\hat{M}\hat{N}}) = \left( \begin{array}{cc} \hat{g}_{MN} + \hat{A}_k^k \hat{G}_{kl} \hat{A}^l_l & -\hat{A}_k^k \hat{G}_{kL} \\ -\hat{G}_{ik} \hat{A}^l_k & \hat{G}_{ij} \end{array} \right) \quad (\mu, \nu = 0, \ldots, d - 1), \]  

where we have defined the graviphoton as \( \hat{A}^i \equiv -\hat{g}_{\mu k} \hat{G}^{ki} = \hat{g}_{\mu \nu} \hat{g}^{\nu i} \).

Type IIA supergravity: When we consider type IIA supergravity, we use the following standard 11D–10D map:

\[ (g_{MN}) = \left( \begin{array}{cc} \hat{g}_{MN} \hat{g}_{Mz} \hat{g}_{zN} \hat{g}_{zz} \\ \hat{g}_{zN} \hat{g}_{zz} \end{array} \right) = \begin{pmatrix} e^{-\frac{2}{3} \Phi} g_{MN} + e^{\frac{4}{3} \Phi} C_M^N \end{pmatrix} \left( \begin{array}{c} e^{\frac{4}{3} \Phi} C_N \\ e^{\frac{4}{3} \Phi} \end{array} \right), \]  

where \( \hat{A}_3 = \mathcal{C}_3 + \mathcal{B}_2 \wedge dx^5 \), \( \hat{A}_6 = \mathcal{C}_6 + \left( \mathcal{C}_5 - \frac{1}{2!} \mathcal{C}_3 \wedge \mathcal{B}_2 \right) \wedge dx^5 \), and we have added the hat to the subscript, like \( \hat{A}_3 \), to stress that it is a \( p \)-form in 11D. In our convention, the dual graviton \( \hat{A}_{8,1} = \{ \hat{A}_{8,1}, \hat{A}_{8,2} \} \) follows the 11D–10D map,

\[ \hat{A}_{8,1} = \mathcal{A}_{8,1} + \mathcal{A}_{7,1} \wedge dx^5, \quad \hat{A}_{8,2} = \mathcal{A}_{6} + \left( \mathcal{C}_7 - \frac{1}{3!} \mathcal{C}_3 \wedge \mathcal{B}_2 \wedge \mathcal{B}_2 \right) \wedge dx^5, \]  

where \( \hat{A}_{8,2} \) corresponds to the \( \hat{N} \) studied in \[23\]. The metric and the graviphoton are defined as

\[ (g_{MN}) = \begin{pmatrix} g_{\mu \nu} + \omega_{\mu \rho} \omega_{\nu \rho} - \omega_{\mu \nu} \omega_{\rho \sigma} \\ -g_{\mu \rho} \omega_{\nu \rho} + \omega_{\mu \nu} \omega_{\rho \sigma} \end{pmatrix} g_{\rho \sigma}, \quad \omega_{\mu \nu} \equiv g_{\mu \nu} \hat{g}^{\nu \rho} \hat{g}_{\rho \mu}. \]  

Then we find the 11D–10D map for the graviphoton,

\[ \hat{A}_3^\mu = \omega_{\mu \nu} \hat{A}_3^\nu, \quad \hat{A}_6^\mu = - \left( C_\mu + \omega_{\mu \nu} \omega_{\rho \sigma} \right). \]

Type IIB supergravity: In type IIB theory, in addition to the standard Einstein-frame metric \( g_{MN} \), we consider the following SL(2) \( S \)-duality-covariant tensors,

\[ (m_{\alpha \beta}) \equiv e^{\phi} \begin{pmatrix} e^{-2 \varphi} + (C_0)^2 & C_0 \\ C_0 & 1 \end{pmatrix}, \quad (A_2^\alpha) \equiv \begin{pmatrix} B_2 \\ -C_2 \end{pmatrix}, \]  

\[ A_4 \equiv C_4 - \frac{1}{2} C_2 \wedge B_2, \quad (A_6^\alpha) \equiv \begin{pmatrix} C_6 - C_4 \wedge C_2 + \frac{1}{6} B_2 \wedge C_2 \wedge C_2 - (B_6 - C_4 \wedge C_2 + \frac{1}{6} B_2 \wedge C_2 \wedge C_2) \end{pmatrix}. \]

We also consider the dual graviton \( A_{7,1} \), whose behavior under duality transformations is to be determined. Upon compactification to \( d \) dimensions, the graviphoton is introduced as

\[ (g_{MN}) = \begin{pmatrix} g_{\mu \nu} + A_\mu^p g_{pq} A_\nu^q & -A_\mu^p g_{pn} \\ -g_{mp} A_\nu^p & G_{mn} \end{pmatrix}, \quad A_\mu^m \equiv g_{\mu \nu} \hat{g}^{\nu m}. \]
2.2 Strategy: Linear map

Here, let us explain the detailed procedure, how to determine the parameterization of the 1-form in both the M-theory and type IIB languages,

\[
(A^I_\mu) = \begin{pmatrix}
\mathcal{A}^A_{\mu} \\
\mathcal{A}^{A_1}_{\mu} \\
\mathcal{A}^{A_2}_{\mu} \\
\vdots
\end{pmatrix}, \quad \quad (A^m_\mu) = \begin{pmatrix}
\mathcal{A}^{A_1}_{\mu} \\
\mathcal{A}^{A_2}_{\mu} \\
\mathcal{A}^{A_3}_{\mu} \\
\vdots
\end{pmatrix},
\]

(2.14)

where ellipses stand for the rest of components that complete the \(U\)-duality multiplet which, potentially, involve further mixed-symmetry potentials.

To determine the parameterization, we make the following modest assumptions:

- The M-theory fields \(A_{1;p,q,r,...}\) and the type IIB fields \(A_{1;p,q,r,...}^{a_1,\ldots,a_s}\) are respectively parameterized by the following fields:

  \[
  \text{M-theory:} \quad \{ \hat{A}^i_\mu, \hat{A}_3^i, \hat{A}_5^i, \hat{A}_{5,1}^i, \ldots \},
  \]
  \[
  \text{Type IIB theory:} \quad \{ A^m_n, A^m_2, A^m_4, A^m_6, A^m_{7,1}, \ldots \}.
  \]  
  
(2.15)  
(2.16)

- The top form is normalized with weight one:

  \[
  A_{\mu,p,q,r,...} = \hat{A}_{\mu,p,q,r,...} + \text{(sum of products of potentials)},
  \]
  \[
  A^{a_1,\ldots,a_s}_{\mu,p,q,r,...} = A^{a_1,\ldots,a_s}_{\mu,p,q,r,...} + \text{(sum of products of potentials)}.
  \]

(2.17)

According to these, the first components of the 1-forms should be, respectively,

\[
A^i_\mu = \hat{A}^i_\mu \quad \text{(M-theory)}, \quad A^m_\mu = A^m_\mu \quad \text{(type IIB)}.
\]

(2.18)

In the following, we explain the procedure to determine the components with higher level, which is based on [22]. In order to utilize the map (2.14), we decompose the physical coordinates on the \(n\)-torus in M-theory as \((x^i) = (x^a, x^a)\) \((a, b = 1, \ldots, n - 2)\) and those on the \((n - 1)\)-torus in type IIB theory as \((x^m) = (x^a, x^b)\). Under the decomposition, the 1-form fields (2.14) are decomposed into \(\text{SL}(n - 2) \times \text{SL}(2)\) tensors as follows:

\[
(A^I_\mu) = \begin{pmatrix}
\mathcal{A}^A_{\mu} \\
\mathcal{A}^{A_1}_{\mu} \\
\mathcal{A}^{A_2}_{\mu} \\
\vdots
\end{pmatrix}, \quad \quad (A^m_\mu) = \begin{pmatrix}
\mathcal{A}^{A_1}_{\mu} \\
\mathcal{A}^{A_2}_{\mu} \\
\mathcal{A}^{A_3}_{\mu} \\
\vdots
\end{pmatrix},
\]

\[
(2.19)
\]
where toroidal directions (either compactified or T-dualized) are shown explicitly. In terms of the Dynkin diagram given in (2.1), in M-theory we have first performed the level decomposition associated with the node $\alpha_n$. Secondly, we have done the level decomposition associated with $\alpha_{n-2}$. On the other hand, in type IIB theory the order is reversed. In the end, we obtain the same decomposition. Indeed, the set of $\text{SL}(n-2) \times \text{SL}(2)$ tensors appearing in (2.19) have the same structure. Then, we make the following identifications [22]:

\[
\begin{pmatrix}
A_\mu \\
A_\nu \\
\vdots \\
A_{n+2}
\end{pmatrix}
= 
\begin{pmatrix}
\mathcal{A}_\mu^a \\
\mathcal{A}_\nu^a \\
\vdots \\
\mathcal{A}_{n+2}^a
\end{pmatrix},
\]

or

\[
\begin{pmatrix}
\mathcal{A}_\mu^a \\
\mathcal{A}_\nu^a \\
\vdots \\
\mathcal{A}_{n+2}^a
\end{pmatrix}
= 
\begin{pmatrix}
A_\mu^{\alpha \beta} \\
A_\nu^{\alpha \beta} \\
\vdots \\
A_{n+2}^{\alpha \beta}
\end{pmatrix},
\]

where we have defined

\[
\epsilon \equiv (\epsilon^{\alpha \beta}) \equiv (\epsilon_{\alpha \beta}) \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

We will refer to the set of linear relations established in (2.20) as the linear map. Actually, by using a constant matrix $S^I_J$, it can be rewritten as

\[
\mathcal{A}^I_\mu = S^I_J A^J_\mu, \quad \mathcal{A}^I_\mu = (S^{-1})^I_J A^J_\mu.
\]

We note that this identification has been originally proposed in [21] in the context of $E_{11}$.

Now, for simplicity, we assume the standard T-duality rule for the NS–NS field:

\[
\begin{align*}
g_{AB} &= g_{\text{g}AB} \frac{A-B}{g_{\text{g}YY}}, \\
B_{AB} &= g_{\text{g}AB} \frac{A-B}{g_{\text{g}YY}},
\end{align*}
\]

where $A, B = \{\mu, a\}$ (i.e., nine directions except the T-dual direction $x^y$ or $x^\prime$). From these, we obtain the T-duality rule for the graviphoton

\[
\mathcal{A}_\mu^a A^B = A^B_{\text{g}YY} + \mathcal{A}_\mu^a B^B_{\text{g}YY}.
\]

By using the 11D–10D relation (2.10), the first rule gives

\[
\mathcal{A}_\mu^a = \mathcal{A}_\mu^a M^{-A} = \mathcal{A}_\mu^a A^B = \mathcal{A}_\mu^a,
\]

which is nothing but the first row of (2.20).

\[\footnote{This assumption is not necessary in the approach discussed in Section 3} \]
2.3 Detailed procedures

We will continue this process by considering the index structure. The second components of the 1-form in the M-theory and the type IIB parameterization are generically expanded as

\[ A^{\mu i_1i_2} = \hat{A}^{\mu i_1i_2} + c_1 \hat{A}^k \hat{A}^{ki_1i_2}, \quad A^{\alpha \mu m} = A^{\alpha \mu m} + c_2 A^{\alpha \mu p} A^{\alpha pm}, \]  

(2.26)

where \( c_1 \) and \( c_2 \) are parameters to be determined. From \( A^{\mu \alpha} = B^{\mu \alpha} \) in (2.20), we have

\[ \hat{A}^{\mu \alpha} = B^{\mu \alpha} = A^{\alpha \mu} + c_2 A^{\alpha \mu p} A^{\alpha py}. \]  

(2.27)

On the other hand, the second rule of (2.24) and the 11D–10D relation (2.10) gives

\[ \hat{A}^{\mu y} = A^{\mu y} = B^{\mu y} + A^{\alpha \mu y} A^{\alpha py}. \]  

(2.28)

and by comparing this with the \( \alpha = y \) component of (2.27), we find \( c_2 = 1 \).

Similarly, the map \( A^{\mu \alpha} = B^{\mu \alpha} \) in (2.20) gives

\[ A^{\mu y} = A^{\mu y} + c_1 A^{\mu A y} = A^{\mu y}. \]  

(2.29)

On the other hand, the second line of (2.23) and the 11D–10D relation (2.7) give

\[ \hat{A}^{\alpha a \mu} = B^{\alpha a \mu} = - \frac{g_{\alpha a y} B_{\alpha ay}}{g_{\alpha y y}}, \quad \hat{A}^{\alpha a \mu} = A^{\alpha a \mu} = - \frac{g_{\alpha a y} A_{\alpha ay}}{g_{\alpha y y}}. \]  

(2.30)

By substituting the second relation into the left-hand side of (2.29) and using \( g_{\alpha y y} = - (A_{\mu} g_{\alpha y} + A_{\mu} g_{\alpha y}) \), we obtain

\[ \frac{A_{\mu} g_{\alpha y} + A_{\mu} g_{\alpha y} - c_1 A_{\mu} g_{\alpha y}}{g_{\alpha y y}} = \hat{A}_{\mu y} + c_1 \hat{A}_{\mu a y} = A_{\mu} = A_{\mu y}, \]  

(2.31)

which shows \( c_1 = 1 \). Thus, the parameterizations of \( A^{\mu i_1i_2} \) and \( A^{\alpha \mu m} \) have determined as

\[ A^{\mu i_1i_2} = \hat{A}^{\mu i_1i_2} + \hat{A}^{k} \hat{A}^{ki_1i_2}, \quad A^{\alpha \mu m} = A_{\mu m} + A_{\mu p} A^{\alpha pm}. \]  

(2.32)

In order to determine the parameterization of further components of the 1-forms, the T-duality rules (2.23) are not enough and we need additional T-duality rules. To find the T-duality rules, we assume that

- The T-duality rules have the 9D covariance (in the nine directions \( x^A \) orthogonal to the T-duality direction \( x^y \)).

- The metric appears in the T-duality rule only through the combination \( \frac{g_{\alpha y}}{g_{\alpha y y}} \) and the graviphoton does not appear explicitly.

By using these assumptions, we obtain the set of standard T-duality rules.
For example, from $\alpha = z$ component of (2.27), we find
\[ \hat{A}_\mu^{M-B} = -C_{\mu y} - A_\mu^p C_{py}. \] (2.33)

In terms of the type IIA field, this is equivalent to
\[ C_{\mu y} + A_\mu^a C_a + A_\mu^y C_y A^{A-B} = C_{\mu y} + A_\mu^a C_a, \] (2.34)

and by using the identity $g_{\mu y} = - (A_\mu^a g_{ay} + A_\mu^y g_{yy})$, we obtain
\[ \left( C_{\mu y} - \frac{g_{\mu y}}{g_{yy}} C_y \right) + A_\mu^a \left( C_a - \frac{g_{ay}}{g_{yy}} C_y \right) A^{A-B} = C_{\mu y} + A_\mu^a C_a. \] (2.35)

From the assumption that the $T$-duality rule does not contain the graviphoton explicitly, this implies the standard $T$-duality rule,
\[ C_{Ay} B^A_A = C_{Ay} - \frac{C_{By} g_{Ay}}{g_{yy}}. \] (2.36)

or conversely,
\[ C_{Ay} A^{A-B} = C_{Ay} - C_{0} B_{Ay}, \] (2.37)

where we have employed the standard rule $C_{Ay} A^{A-B} = C_{Ay}$.

Similarly, if we consider the linear map $A_{\mu,\alpha}^{M-B} = A_{\mu,\alpha}^B \epsilon_{\beta\alpha}$ in (2.20), we find
\[ \hat{A}_{\mu,\alpha} + A_{\mu,\alpha}^{B} \hat{A}_{\kappa,\alpha} = (A_{\mu,\alpha}^B + A_{\mu,\alpha}^p A_{\mu,\alpha}^B) \epsilon_{\beta\alpha}. \] (2.38)

In particular, for $\alpha = y$, we obtain a map between the type IIA/IIB fields,
\[ C_{\mu y} + A_\mu^a C_{ay} - (C_{\mu y} + A_\mu^m C_m) B_{ay} A^{A-B} = C_{\mu a} + A_\mu^a C_{ba} + A_\mu^y C_{ya}, \] (2.39)

and this is equivalent to
\[ C_{\mu y} - C_{\mu} B_{ay} + \frac{C_{y} B_{ay} g_{\mu y}}{g_{yy}} + A_\mu^b \left( C_{by} - C_{b y} g_{yy} \right) \]
\[ A^{A-B} = C_{\mu a} - \frac{C_{ya} g_{\mu y}}{g_{yy}} + A_\mu^b \left( C_{ba} - \frac{C_{ya} g_{by}}{g_{yy}} \right). \] (2.40)

Then, we find the $T$-duality rule
\[ C_{AB} A^{A-B} = C_{AB} - 2 \frac{C_{[A]y B[B]y}}{g_{yy}}. \] (2.41)

**Further steps**

We can further proceed by considering a general expansion of the SL(2) singlet $A_{\mu,m_1m_2m_3}$,
\[ A_{\mu,m_1m_2m_3} = A_{\mu,m_1m_2m_3} + c_3 \epsilon_{\alpha\beta} A_{\mu,m_1}^\alpha A_{m_2m_3}^\beta \]
\[ + c_4 A_{\mu}^a A_{m_1m_2m_3} + c_5 \epsilon_{\alpha\beta} A_{\mu}^a A_{m_1}^\alpha A_{m_2m_3}^\beta. \] (2.42)

Similarly, unknown $T$-duality rules can be also expanded by considering possible 9D covariant expressions with parameters. Then, the consistency with the linear map (2.20) determines all of the parameters. In this manner, by using the linear map (2.20), we can find both the parameterization of $A_{\mu}^I$ and $T$-duality rules for the gauge potentials one after another.
2.4 Results

By continuing the above procedure, we have determined the M-theory parameterization as

\[
(A_i^\mu) = \begin{pmatrix}
\frac{A_i^\mu}{\sqrt{2}} \\
\frac{A_{i+1}^\mu}{\sqrt{2}} \\
\frac{A_{i+1}^{\mu+1}}{\sqrt{2}} \\
\vdots
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{\sqrt{2}} (\hat{N}_{\mu,i+1,i+2} + \hat{N}^k_{\mu,k;i+1,i+2}) \\
\frac{1}{\sqrt{2}} (\hat{N}_{\mu,i+1...i_5} + \hat{N}^k_{\mu,k;i+1...i_5}) \\
\frac{1}{\sqrt{2}} (\hat{N}_{\mu,i+1...i_7,i} + \hat{N}^k_{\mu,k;i+1...i_7,i}) \\
\vdots
\end{pmatrix},
\]

(2.43)

Remarkably, the two tensors \(\hat{N}_{\mu,p,q,r,...}\) and \(\hat{N}_{k,p,q,r,...}\), in each row can be regarded as particular components of 11D-covariant tensors:

\[
\hat{N}_{\hat{M}_1;\hat{M}_2;\hat{M}_3} = \hat{A}_{\hat{M}_1;\hat{M}_2;\hat{M}_3},
\]

\[
\hat{N}_{\hat{M}_1;\hat{M}_2...\hat{M}_6} = \hat{A}_{\hat{M}_1...\hat{M}_6} - 5 \hat{A}_{\hat{M}_1;\hat{M}_2;\hat{M}_3} \hat{A}_{\hat{M}_4;\hat{M}_5;\hat{M}_6},
\]

\[
\hat{N}_{\hat{M}_1;\hat{M}_2...\hat{M}_8;\hat{N}} \simeq \hat{A}_{\hat{M}_1...\hat{M}_8;\hat{N}} - 21 (\hat{A}_{\hat{M}_1;\hat{M}_2...\hat{M}_6} \hat{A}_{\hat{M}_7;\hat{M}_8;\hat{N}} - \hat{A}_{\hat{M}_1;\hat{M}_2...\hat{M}_6} \hat{A}_{\hat{M}_7;\hat{M}_8;\hat{N}})
+ 35 \hat{A}_{\hat{M}_1;\hat{M}_2;\hat{M}_3} \hat{A}_{\hat{M}_4;\hat{M}_5;\hat{M}_6} \hat{A}_{\hat{M}_7;\hat{M}_8;\hat{N}},
\]

(2.44)

where the meaning of the equivalence \(\simeq\) is explained below.

As discussed in [1], for any mixed-symmetry potential, not all of the components couple to supersymmetric branes. For the dual graviton \(A_{\mu;i_1...i_7,i}\), only the components satisfying

\[
i \in \{i_1, \ldots, i_7\},
\]

(2.45)
couples to supersymmetric branes. The components which do not couple to supersymmetric branes correspond to the \(E_{11}\) roots \(\alpha\) satisfying \(\alpha \cdot \alpha < 2\), and they are not connected to the standard \(p\)-form potentials under \(T\)-duality and \(S\)-duality. Since our procedure to determine the parameterization is based on \(T\)-duality and \(S\)-duality, it can only provide the parameterization of the components which couple to supersymmetric branes. In this sense, it is more honest to express the last equation of (2.44) as

\[
\hat{N}_{\hat{M}_1;\hat{M}_2...\hat{M}_7;\hat{x};x} = \hat{A}_{\hat{M}_1...\hat{M}_7;\hat{x};x} - 21 \hat{A}_{\hat{M}_1;\hat{M}_2...\hat{M}_6} \hat{A}_{\hat{x};\hat{M}_7} x + 35 \hat{A}_{\hat{M}_1;\hat{M}_2;\hat{M}_3} \hat{A}_{\hat{M}_4;\hat{M}_5;\hat{M}_6} \hat{A}_{\hat{x};\hat{M}_7} x
\]

\[
= \hat{A}_{\hat{M}_1...\hat{M}_7;\hat{x};x} - 15 \hat{A}_{\hat{M}_1;\hat{M}_2...\hat{M}_5} \hat{A}_{\hat{M}_6;\hat{M}_7} x
\]

\[
- 10 \hat{A}_{\hat{M}_1;\hat{M}_2} \hat{A}_{\hat{M}_3;\hat{M}_4;\hat{M}_5} \hat{A}_{\hat{M}_6;\hat{M}_7} x + 15 \hat{A}_{\hat{M}_1;\hat{M}_2;\hat{M}_3} \hat{A}_{\hat{M}_4;\hat{M}_5;\hat{x}} \hat{A}_{\hat{M}_6;\hat{M}_7} x.
\]

(2.46)

In this paper, equalities that hold under the restriction (2.35) are denoted by \(\simeq\). The parameterizations of mixed-symmetry potentials which do not satisfy the restriction (2.45) are not determined in this paper.
Now we turn to the results in type IIB theory. The parameterization takes the form

\[
(\mathcal{A}_\mu^I) = \left( \begin{array}{c}
\frac{1}{V_3} (N_{\mu;m_1m_2m_3} + A_p^\alpha N_{\mu;m_1m_2m_3}^\alpha) \\
\frac{1}{\sqrt{6}} (N_{\mu;m_1m_2m_3} + A_p^\alpha N_{\mu;m_1m_2m_3}^\alpha) \\
\frac{1}{\sqrt{6}} (N_{\mu;m_1m_2m_3} + A_p^\alpha N_{\mu;m_1m_2m_3}^\alpha)
\end{array} \right),
\]

(2.47)

where

\[
N_{M_1:M_2} = A_{\alpha M_1 M_2},
\]

\[
N_{M_1:M_2 M_3 M_4} = A_{M_1 \ldots M_4} - \frac{3}{2} \epsilon_{\gamma \delta} A_{M_1[M_2} A_{M_3 M_4]}^\gamma A_{M_4]M_1}^\delta
\]

\[
= C_{M_1 \ldots M_4} - 3 C_{M_1[M_2 B_{M_3 M_4]}},
\]

\[
N_{M_1:M_2 M_3 M_4}^\alpha = A_{M_1 \ldots M_4}^\alpha + 5 A_{M_1[M_2 A_{M_3 \ldots M_4]}^\alpha} + 5 \epsilon_{\gamma \delta} A_{M_1[M_2 A_{M_3 M_4]}^\gamma A_{M_4]M_1}^\delta}
\]

\[
= \left( C_{M_1 \ldots M_4} - 10 C_{M_1[M_2 M_3 M_4 B_{M_5 M_6}] + 15 C_{M_1[M_2 B_{M_3 M_4} B_{M_5 M_6}]} - (B_{M_1 \ldots M_4} - 10 C_{M_1[M_2 M_3 M_4 C_{M_5 M_6}]}) \right),
\]

(2.48)

\[
N_{M_1:M_2 \ldots M_7,N} \equiv A_{M_1 \ldots M_7,N} + 6 C_{M_1[M_2 M_3 M_4 B_{M_5 M_6}] N - 6 C_{M_1[M_2 C_{M_3 \ldots M_7} N}
\]

\[
- 60 C_{M_1[M_2 M_3 M_4 C_{M_5 M_6} B_{M_7,N} + 10 C_{M_1[M_2 M_3 M_4 C_{M_5 M_6} B_{M_7,N}]}} + 45 \frac{1}{2} (B_{M_1[M_2 B_{M_3 M_4} C_{M_5 M_6} C_{M_7,N} - C_{M_1[M_2 C_{M_3 M_4} B_{M_5 M_6} B_{M_7,N}]}).
\]

The last component \( N_{1,6,1} \) is relatively long, and the S-duality invariance is not clear. However, this is because of the definition of the dual graviton \( A_{7,1} \). As we will see later [in Eq. (3.17)], a certain redefinition of \( A_{7,1} \) makes the expression of \( N_{1,6,1} \) simpler.

### 2.5 T-duality rule

In addition to the parameterizations, we have obtained the \( T \)-duality rules as follows:

\[
g_{AB} = \frac{A_{AM} B_{MN}}{g_{yy}}, \quad g_{AY} = \frac{A_{AY} B_{MN}}{g_{yy}}, \quad g_{yy} = \frac{A_{AM} B_{MN}}{g_{yy}}, \quad g_{AB} = \frac{1}{g_{yy}},
\]

\[
\alpha \gamma \delta \mu = \alpha \gamma \delta \mu, \quad \alpha \gamma \mu = \alpha \gamma \mu + A_{\mu B_{MN}},
\]

\[
A_{AB} = B_{AB} - \frac{A_{AY} B_{MN}}{g_{yy}}, \quad A_{AY} = - \frac{B_{AY} B_{MN}}{g_{yy}}, \quad g_{yy} = \frac{A_{AM} B_{MN}}{g_{yy}}, \quad e^{2 \phi} = \frac{e^{2 \phi}}{g_{yy}}.
\]

\[
A_{A_1 \ldots A_{n-1}} = C_{A_1 \ldots A_{n-1}} - (n - 1) \frac{C_{A_1 \ldots A_{n-2}} B_{A_{n-1}}}{g_{yy}},
\]

\[
A_{A_1 \ldots A_n} = C_{A_1 \ldots A_n} - n C_{A_1 \ldots A_{n-1}} B_{A_{n-1}} - n (n - 1) \frac{C_{A_1 \ldots A_{n-2}} B_{A_{n-1}}}{g_{yy}}.
\]
For the 6-form potential $B_6$ and the dual graviton $A_{7,1}$, we find

\[
\mathcal{B}_{A_1\ldots A_6} = B_{A_1\ldots A_6} - 5 A_{[A_1\ldots A_4} C_{A_5]y} - 5 A_{[A_1 A_2 A_3]} C_{A_4 A_5} y
\]
\[-\frac{45}{2} C_{[A_1 A_2} B_{A_3 A_4} C_{A_5]y} - \frac{15}{2} C_{[A_1 A_2} C_{A_3 A_4} B_{A_5]y}
\[-\frac{10 A_{[A_1 A_2 A_3]} C_{A_4 A_5} y g_{A_5} y} - \frac{15 C_{[A_1 A_2} B_{A_3 A_4]} y C_{A_5 A_6} y g_{A_5} y} ,
\]

\[
(2.50)
\]

\[
\mathcal{B}_{A_1\ldots A_6} = A_{A_1\ldots A_6 y} - 6 B_{[A_1\ldots A_5] y} B_{A_6]y}
\[+ 30 A_{[A_1 A_2 A_3] y} C_{[A_4 A_5} B_{A_6]y} + 30 A_{[A_1 A_4 C_{A_5] y} B_{A_6]y}
\[- \frac{315}{2} B_{[A_1 A_2 B_{A_3}]} C_{A_4 A_5} C_{A_6]y} + \frac{60 A_{[A_1 A_2 A_3] y} B_{A_4 A_5} y C_{A_5 y} g_{A_6} y} ,
\]

\[
(2.51)
\]

\[
\mathcal{A}_{A_1\ldots A_6 y} = B_{A_1\ldots A_6 y} - 30 B_{[A_1 A_2 A_3 y} C_{A_4 A_5 C_{A_6]y} y - 6 B_{[A_1 A_5 A_2 A_3 y} C_{A_4 A_5 y} g_{A_6} y + \frac{20 A_{[A_1 A_2 A_3]} y C_{A_4 A_5} C_{A_6]y} y g_{A_6} y} ,
\]

\[
(2.52)
\]

\[
\mathcal{A}_{A_1\ldots A_6 y, B} = A_{A_1\ldots A_6 y, B} - 6 C_{[A_1 A_6 y, B} C_{A_5 A_6] B} - 6 B_{[A_1 A_6 y, B} B_{A_6]y}
\[+ 10 C_{[A_1 A_2 B_{A_3} y] C_{A_4 A_5 A_6]} + 20 C_{[A_1 A_2 A_3]} B_{A_4 y} B_{A_5 y} C_{A_6 y}
\[- \frac{45}{2} B_{[A_1 A_2 A_3} C_{A_4 A_5} \left(-B_{A_5 y} y C_{A_6] B} + A_{[A_5 y} y B_{A_6] B} \right)
\[- \frac{A_{[A_1 A_5 A_2 A_3]} y C_{A_6 y} y g_{A_6} y} + \frac{6 C_{[A_1 A_5 A_2 A_3]} y C_{A_6 y} y g_{A_6} y} - \frac{6 C_{[A_1 A_5 A_2 A_3]} y C_{A_6 y} y g_{A_6} y} ,
\]

\[
(2.53)
\]

The $T$-duality rules \(2.51\) and \(2.52\) coincide with the known results \(25\) (see Appendix A therein), for which the following identification of supergravity fields are needed:

\[
\begin{pmatrix}
\mathcal{g}_{\mu \nu} \\
B \\
C^{(1)} \\
C^{(3)} \\
C^{(5)} \\
\mathcal{B} \\
N
\end{pmatrix}_{(1A)} \quad \begin{pmatrix}
g^{MN} \\
B_2 \\
C_1 \\
C_3 \\
C_5 \\
-\mathcal{B}_6 \\
\mathcal{N}_{(1A) \text{ here}} \\
\end{pmatrix}_{(1A) \text{ here}} = \begin{pmatrix}
\mathcal{g}_{\mu \nu} \\
B \\
C^{(0)} \\
C^{(2)} \\
C^{(4)} \\
C^{(6)} \\
\mathcal{B} \\
\mathcal{N}_{(1B) \text{ here}} \\
\end{pmatrix}_{(1B) \text{ here}} = \begin{pmatrix}
g^{MN} \\
B_2 \\
-C_0 \\
-C_2 \\
-A_4 \\
-\left( C_6 - \frac{1}{2} B_2 \land B_2 \land C_2 \right) \\
-\left( B_6 - \frac{1}{2} C_2 \land C_2 \land B_2 \right) \\
\end{pmatrix}_{(1B) \text{ here}} .
\]

\[
(2.54)
\]
On the other hand, (2.53) has been obtained in [20], where $B_2 = 0$ and $C_2 = 0$ are assumed. If we truncate $B_2$ and $C_2$, we have $A_4 = C_4$ and the $T$-duality rule (2.53) reduces to

$$
A_{A_1\cdots A_6} B \frac{A-A_1\cdots A_6 y, B + 10 A_{A_1 A_2 A_3 | B} A_{A_4 A_5 A_6} y}{g_{yy}} - \frac{10 A_{A_1 A_2 y B} A_{A_3 A_4 A_5} y}{g_{yy}} g_{A_6 y}.
$$

(2.55)

More explicitly, according to the restriction (2.43), the direction $x = B$ must be contained in \{$A_1, \ldots, A_6$\} and by choosing $A_6 = x$, we have

$$
A_{A_1\cdots A_5 y x, x} = \frac{A-A_1\cdots A_5 y, x + 5 A_{A_1 A_2 A_3 | x} A_{A_4 A_5} y}{g_{yy}} - \frac{10 A_{A_1 A_2 | x} A_{A_3 A_4 A_5} y}{g_{yy}} g_{x y} \frac{y}{g_{x y}} (2.56)

$$

If we denote $k \equiv \partial_x$ and $h \equiv \partial_y$, and define

$$
N^{(7)}_{M_1\cdots M_6} \equiv A_{M_1\cdots M_7, x}, \quad N^{(7)}_{M_1\cdots M_6} \equiv A_{M_1\cdots M_7, x}, \quad N^{(7)}_{M_1\cdots M_7} \equiv A_{M_1\cdots M_7, y},
$$

(2.57)

the result of [20] [see Eq. (5.13)] is precisely reproduced,

$$
\frac{(t_k N^{(7)}_{A_1\cdots A_5 y})}{A-A_1\cdots A_5} - \frac{5 (t_k A)}{A_1 A_2 A_3 (t_k t_h A) A_4 A_5} - \frac{10 A_{A_1 A_2 A_3 (t_k t_h A) A_4 A_5}}{g_{yy}} g_{x y} \frac{y}{g_{x y}} (2.58)

$$

In the above computation, we have shown only $T$-duality transformations from type IIB to type IIA, but we can easily find the inverse map. The standard rules [2,49] have the same form even for the map from type IIA to type IIB,

$$
g_{AB} B-A = g_{AB} - g_{A y} g_{B y} - g_{A y} g_{B y}, \quad g_{A y} B-A = - g_{A y} B-y, \quad g_{B y} B-A = \frac{1}{g_{y y}}.
$$

(2.59)

Regarding the 6-form potential and the dual graviton, the results are as follows:

$$
B_{A_1\cdots A_6 y} \frac{B-A \frac{A_1 \cdots A_6 y}{y} + 5 \frac{A_{A_1 \cdots A_4 | y} A_{A_5}}{y} + 5 \frac{A_{A_1 A_2 A_3 A_4 A_5 | y}}{y} g_{A_6 y}}{g_{y y}} - 15 \frac{g_{A_1 A_2 | y} A_{A_3 A_4 A_5 | y}}{g_{y y}} - 5 \frac{g_{A_1 \cdots A_4 | y} A_{A_5}}{g_{y y}}
$$

(2.60)

$$
B_{A_1 \cdots A_6} \frac{B-A \frac{A_1 \cdots A_6 y}{y} - 6 \frac{A_{A_1 \cdots A_5 y} A_{A_6}}{y} - 30 \frac{A_{A_1 \cdots A_4 | y} A_{A_5} A_{A_6}}{y}}{y} + 10 \frac{g_{A_1 A_2 A_3 A_4 A_5 | y}}{g_{y y}} + 30 \frac{g_{A_1 A_2 | y} A_{A_3 A_4 | y} A_{A_5 A_6}}{y}.
$$
\[ A_{A_1 \cdots A_6 y, y} = \mathcal{D}_{A_1 \cdots A_6} - \frac{6 B_{A_1 \cdots A_5} y}{g_{yy}} + \frac{6 B_{A_1 \cdots A_5} A_6 y}{g_{yy}}, \quad (2.61) \]

\[ A_{A_1 \cdots A_6 y, B} \simeq \mathcal{D}_{A_1 \cdots A_6 y, B} - \frac{6 B_{A_1 \cdots A_5} y}{g_{yy}} + \frac{6 B_{A_1 \cdots A_5} A_6 y}{g_{yy}}, \quad (2.62) \]

\[
\begin{align*}
&- \frac{15}{2} C_{[A_1 \cdots A_4 y] C_{A_5 A_6} y} + 20 C_{[A_1 A_2 A_3 C_{A_4 A_5} y] B_{A_6} y} - 20 C_{[A_1 A_2 A_3 C_{A_4} y] B_{A_5 A_6}} - \frac{6 \mathcal{D}_{A_1 \cdots A_5} y, y B_{A_6} y}{g_{yy}} \\
&+ \frac{6 B_{By} B_{[A_1 \cdots A_5] y} A_6 y}{g_{yy}} + \frac{15 C_{[A_1 \cdots A_4 y] C_{A_5} A_6 y} y B_{A_6} y}{g_{yy}} \\
&+ \frac{15}{2} C_{[A_1 \cdots A_4 y] C_{A_5 A_6} y} y B_{A_6} y - \frac{20 C_{[A_1 A_2 A_3 C_{A_4 A_5} y] B_{A_6} y} y B_{A_6} y}{g_{yy}} \\
&- \frac{195}{8} C_{[A_1 A_2 y] C_{A_3 A_4 A_5} y} B_{A_6} y y B_{A_6} y - \frac{45 C_{[A_1 A_2 y] C_{A_3 A_4} y] B_{A_5} y} y B_{A_6} y}{g_{yy}} \\
&- \frac{45 B_{By} C_{[A_1 A_2 y] C_{A_3 A_4} A_5 y} y B_{A_6} y}{g_{yy}} - \frac{45 y C_{[A_1 A_2 y] C_{A_3 A_4} y] B_{A_5} y} y B_{A_6} y}{g_{yy}}. 
\end{align*}
\]

Now, let us comment more on the restriction rule. In the T-duality rule \( (2.53) \), we are assuming that \( B \) is contained in \( \{ A_1, \cdots , A_6 \} \). When the restriction is removed, we expect the right-hand side of the T-duality rule is modified. In general, the components which do not satisfy the restriction is in the same orbit as the \((\alpha \neq \beta)\)-component of the type IIB potential \( \varphi^{\alpha \beta} \), which is electric-magnetic dual to the 0-form potential \( m_{\alpha \beta} \). Therefore, it will be possible that \( \varphi^{\alpha \beta}_{A_1 \cdots A_6 y} \) appears on the right-hand side of \( (2.53) \).

### 2.6 S-duality rule

The standard S-duality transformation rules are reproduced as follows:

\[
g'_{MN} = g_{MN}, \quad A'_\mu = A^m, \quad C'_0 = -\frac{C_0}{(C_0)^2 + e^{-2 \varphi}}, \quad e^{-\varphi'} = \frac{e^{-\varphi}}{(C_0)^2 + e^{-2 \varphi}}, \quad (2.64)
\]

\[
B'_2 = -C_2, \quad C'_2 = B_2, \quad C'_4 = C_4 - B_2 \wedge C_2, \quad A'_4 = A_4, \quad (2.65)
\]

\[
C'_6 = -B_6 + \frac{1}{2} B_2 \wedge C_2 \wedge C_2, \quad B'_6 = C_6 - \frac{1}{2} C_2 \wedge B_2 \wedge B_2, \quad A'_6 = -A_6^2, \quad A'_6 = A_6^1.
\]

From the S-duality invariance of \( A_{\mu m_1 \cdots m_6} \), we also find

\[
A'_{M_1 \cdots M_7, M} \simeq A_{M_1 \cdots M_7, M} + 7 \left( B_{[M_1 \cdots M_6} B_{M_7] M} - B_{[M_1 \cdots M_6} B_{M_7 M]} \right) + 7 \left( C_{[M_1 \cdots M_6} C_{M_7]} M - C_{[M_1 \cdots M_6} C_{M_7 M]} \right) - \frac{105}{2} \left( A_{[M_1 \cdots M_4} B_{M_5 M_6} C_{M_7] M} - A_{[M_1 \cdots M_4} C_{M_5 M_6} B_{M_7 M]} \right) + \frac{945}{4} \left( B_{[M_1 M_2} B_{M_3 M_4} C_{M_5 M_6} C_{M_7] M} - B_{[M_1 M_2} B_{M_3 M_4} C_{M_5 M_6} C_{M_7 M]} \right). \quad (2.65)
\]
3 Another approach based on the generalized metric

In this section, we discuss another derivation of the T-/S-duality transformation rule for the dual graviton, which is based on the generalized metric. We also explain another method to determine the parameterization of the 1-form $A^I$.

In $d$ dimensions, scalar fields are packaged into $U$-duality-covariant object called the generalized vielbein, which are denoted as $\mathcal{M}_{IJ}$ and $M_{IJ}$ in M-theory and type IIB, respectively. The generalized vielbein, $\mathcal{E}_{IJ}$ and $E_{IJ}$ respectively, are defined such that

$$\mathcal{M}_{IJ} \equiv \delta_{KL} \mathcal{E}_K^I \mathcal{E}_L^J, \quad M_{IJ} \equiv \delta_{KL} E^K_1 E^L_J. \tag{3.1}$$

According to [13], the generalized vielbein can be constructed as follows. We first consider the positive-root generators of the $E_n$ algebra, which are summarized as

$$\{E_\alpha\} = \{K^i_j \ (i < j), R^{i_1 i_2 i_3}, R^{i_1 \ldots i_6}, R^{i_1 \ldots i_8, i}, \ldots\}, \tag{3.2}$$

in the M-theory parameterization and as

$$\{E_\alpha\} = \{K^m_n \ (m < n), R_{22}, R^{m_1 m_2}, R^{m_1 \ldots m_4}, R^{m_1 \ldots m_6}, R^{m_1 \ldots m_7, m}, \ldots\}, \tag{3.3}$$

in the type IIB parameterization. We also consider the Cartan generators,

$$\{H_k\} = \{K^d_d - K^{d+1}_{d+1}, \ldots, K^{9}_9 - K^z_z, K^{8}_8 + K^9_9 + K^z_z + \frac{1}{4} D\}, \tag{3.4}$$

in the M-theory parameterization ($D \equiv K^i_i$) and

$$\{H_k\} = \{K^d_d - K^{d+1}_{d+1}, \ldots, K^{7}_7 - K^8_8, K^8_8 + K^9_9 - \frac{1}{4} D - R_{12}, 2 R_{12}, K^s_8 - K^9_9\}, \tag{3.5}$$

in the type IIB parameterization ($D \equiv K^m_m$). Then, we prepare the matrix representations of these generators in the vector representation. In the M-theory parameterization, the matrix representations have been obtained in [13] for $n \leq 7$ and in [19] for $n = 8$. In the type IIB parameterization, they have been determined in [20][21] for $n \leq 7$. The results for $n = 8$ are given in Appendix E. Then, we define the generalized vielbein in the M-theory parameterization as

$$\mathcal{E} \equiv (\mathcal{E}_{IJ}) \equiv \hat{\mathcal{E}} L, \quad \hat{\mathcal{E}} \equiv e^{h_{kj} H_k e^{\sum_{i < j} h_{ij} K^i_j}}$$

$$L \equiv (L_{IJ}) \equiv e^{\frac{1}{2} \hat{A}_{i_1 i_2 i_3} R^{i_1 i_2 i_3} \hat{A}_{i_1 \ldots i_6} R^{i_1 \ldots i_6} \hat{A}_{i_1 \ldots i_8, i} R^{i_1 \ldots i_8, i} \ldots}, \tag{3.6}$$

and the generalized vielbein in the type IIB parameterization as

$$\mathcal{E} \equiv (\mathcal{E}_{IJ}) \equiv \hat{\mathcal{E}} L, \quad \hat{\mathcal{E}} \equiv e^{h_{kj} H_k e^{\sum_{m<n} h_{mn} K^m_n}},$$

$$L \equiv (L_{IJ}) \equiv e^{\frac{1}{2} \hat{A}_{m_1 m_2} R^{m_1 m_2} \hat{A}_{m_1 \ldots m_4} R^{m_1 \ldots m_4} \hat{A}_{m_1 \ldots m_6} R^{m_1 \ldots m_6} \hat{A}_{m_1 \ldots m_7, m} R^{m_1 \ldots m_7, m} \ldots}. \tag{3.7}$$

The objects $\hat{A}$ and $A$ are again the M-theory and type IIB fields respectively, expressed in a new basis. Namely, $\hat{A}$ and $A$ are respectively related to $\tilde{A}$ and $A$ by field redefinitions, as we will show in this section. The ellipses in both parameterizations disappear for $n \leq 8$. 

14
The generalized metrics (3.1) are then expressed as

\[ \mathcal{M}_{IJ} \equiv (E^T E)_{IJ} = (L^T \hat{M} L)_{IJ} \]

where we have defined the untwisted metrics as

\[ \hat{\mathcal{M}}_{IJ} \equiv (\hat{E}^T \hat{E})_{IJ} , \quad \hat{M}_{IJ} \equiv (\hat{E}^T \hat{E})_{IJ} , \]

which are parameterized by the supergravity fields \( h^k \) (vielbein) and \( \delta^k \) (vielbein and \( m_{\alpha \beta} \)). Explicitly, the untwisted metrics take the following form:

\[ \hat{\mathcal{M}} \equiv |\hat{G}|^{-1/2} \begin{pmatrix} \hat{G}_{ij} & 0 & 0 & 0 & \cdots \\ 0 & \hat{G}^{i_1 \rightarrow i_1, j_1 \rightarrow j_1} & 0 & 0 & \cdots \\ 0 & 0 & \hat{G}^{i_1 \rightarrow i_1, j_1 \rightarrow j_2} & 0 & \cdots \\ 0 & 0 & 0 & \hat{G}^{i_1 \rightarrow i_1, j_1 \rightarrow j_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} , \]

\[ \hat{M} \equiv |\hat{G}|^{-1/2} \begin{pmatrix} G_{mn} & 0 & 0 & 0 & 0 \\ 0 & m_{\alpha \beta} G^m_{\alpha \beta} & 0 & 0 & 0 \\ 0 & 0 & G^{m_1, m_2, n_1, n_2, n_3} & 0 & \cdots \\ 0 & 0 & 0 & G^{m_1, m_2, n_1, \cdots} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} , \]

where

\[ \hat{G}^{i_1 \rightarrow i_p, j_1 \rightarrow j_p} \equiv \delta^{i_1 \rightarrow i_p}_{k_1 \rightarrow k_p} \hat{G}^{i_1 k_1} \cdots \hat{G}^{i_p k_p} , \quad |\hat{G}| \equiv \det(\hat{G}_{ij}) , \]

\[ G^{m_1, \cdots, m_p, n_1, \cdots, n_p} \equiv \delta q_1 \rightarrow q_p \hat{G}^{m_1 q_1} \cdots \hat{G}^{m_p q_p} , \quad |G| \equiv \det(G_{mn}) . \]

On the other hand, the twist matrices \( L \) and \( \hat{L} \) contain various gauge potentials, which can be computed by using the matrix representations of the \( E_n \) generators given in Appendix B.

As we have introduced the parameterization of the generalized metrics, let us explain the procedure to obtain the duality rules, which has been proposed in [22] for \( n \leq 7 \).

### 3.1 Linear map between generalized metrics

Here, we explain how to determine the duality transformation rules from the generalized metric. As we have discussed in Section 2 in the M-theory and type IIB parameterizations, we are using different basis, which are related through the linear map (2.22). Accordingly, the generalized metrics in the two parameterizations are related as

\[ \mathcal{M}_{IJ} = (S^{I})^K_L \mathcal{M}_{KL} S^{L} J . \]

The explicit form of \( S^{I} J \) has been obtained in [22] only for \( n \leq 7 \), and in this paper, we extend the result to be applicable to \( n \leq 8 \). Under the linear map, the generalized coordinates are transformed as

\[ x^I = S^{I} J \mathcal{X}^J , \quad \mathcal{X}^I = (S^{-1})^I_J x^J . \]
Since the matrix size of $S^I_J$ is very large $21 \times 21$, we show the linear map as follows:

\[
\begin{pmatrix}
  x^a \\ x^\alpha \\
\end{pmatrix}
\begin{pmatrix}
  y_a \\ y_\alpha \\
\end{pmatrix}
\begin{pmatrix}
  x^a \\ x^\alpha \\
\end{pmatrix}
\begin{pmatrix}
  y_a \\ y_\alpha \\
\end{pmatrix}
\begin{pmatrix}
  x^a \\ x^\alpha \\
\end{pmatrix}
\begin{pmatrix}
  y_a \\ y_\alpha \\
\end{pmatrix}
\begin{pmatrix}
  x^a \\ x^\alpha \\
\end{pmatrix}
\begin{pmatrix}
  y_a \\ y_\alpha \\
\end{pmatrix}

= \quad \Leftrightarrow \quad = \quad (S^{-1})^I_J \begin{pmatrix}
  x^a \\ x^\alpha \\
\end{pmatrix}
\begin{pmatrix}
  y_a \\ y_\alpha \\
\end{pmatrix}
\begin{pmatrix}
  x^a \\ x^\alpha \\
\end{pmatrix}
\begin{pmatrix}
  y_a \\ y_\alpha \\
\end{pmatrix}
\begin{pmatrix}
  x^a \\ x^\alpha \\
\end{pmatrix}
\begin{pmatrix}
  y_a \\ y_\alpha \\
\end{pmatrix}
\begin{pmatrix}
  x^a \\ x^\alpha \\
\end{pmatrix}
\begin{pmatrix}
  y_a \\ y_\alpha \\
\end{pmatrix}
\]

where
\[
\begin{pmatrix}
  Y_{a_1 \cdots a_6} \\
  Y_{a_1 \cdots a_5,a} \\
\end{pmatrix}
= \begin{pmatrix}
  \frac{9\sqrt{7}+1}{28} \delta_{a_1 \cdots a_6} \\
  \frac{3\sqrt{7}-2}{28} \delta_{a_1 \cdots a_5,a} \\
\end{pmatrix},
\]

\[
\begin{pmatrix}
  Y_{a_1 \cdots a_6} \\
  Y_{a_1 \cdots a_5,a} \\
\end{pmatrix}
= \begin{pmatrix}
  \frac{9\sqrt{7}+1}{28} \delta_{a_1 \cdots a_6} \\
  \frac{3\sqrt{7}-2}{28} \delta_{a_1 \cdots a_5,a} \\
\end{pmatrix},
\]

with $\delta^{i_1 \cdots j_n} \equiv n! \delta^{i_1 \cdots j_n}$. The constant matrix $S^I_J$ can be read off from the above map between the coordinates. We can check that the matrix $S^I_J$ satisfies the property

\[
S^I_K (S^T)^K_J = \delta^I_J, \quad (S^T)^I_K S^K_J = \delta^I_J,
\]

under the generalized transpose, which is defined for a matrix $A = (A^I_J)$ as

\[
(A^T)^I_J \equiv \delta^I_K (A^T)^K_L \delta_{L,J} \equiv \delta^I_K A^L_K \delta_{L,J},
\]

namely the standard matrix transpose $^T$ followed by a flip in the position of the indices. This property shows that the flat metric is preserved under the linear map,

\[
\delta_{IJ} = S^K_I S^K_J \delta_{KL}.
\]
Now, the constant matrix $S^I_J$ has been completely determined and the relation (3.13) connects the two parameterizations. By comparing both sides, we can express the M-theory fields in terms of the type IIB fields, and vice versa. In the case $n \leq 7$, the generalized metric does not contain the dual graviton, but it appears in $n \geq 8$ and here we consider the generalized metric in $E_8$ EFT.

### 3.2 Connection between two parameterization

By comparing the two parameterizations (3.13) of the $E_8$ generalized metric, we find the following relation between the M-theory fields and type IIB fields:

\[
(G_{ij}) = \begin{pmatrix} 
\hat{G}_{ab} & \hat{G}_{a\beta} \\
\hat{G}_{ab} & \hat{G}_{a\beta}
\end{pmatrix}
\]

By comparing the two parameterizations (3.13) of the generalized metric in M-B, we can express the M-theory fields in terms of the type IIB fields, and vice versa. In the case $n \leq 7$, the generalized metric does not contain the dual graviton, but it appears in $n \geq 8$ and here we consider the generalized metric in $E_8$ EFT.

### (3.21)

Now, the constant matrix $S^I_J$ has been completely determined and the relation (3.13) connects the two parameterizations. By comparing both sides, we can express the M-theory fields in terms of the type IIB fields, and vice versa. In the case $n \leq 7$, the generalized metric does not contain the dual graviton, but it appears in $n \geq 8$ and here we consider the generalized metric in $E_8$ EFT.
By making identifications

\[
\hat{A}_{M_1 \cdots M_3} = \hat{A}_{M_1 M_2 M_3}, \quad \hat{A}_{M_1 \cdots M_6} = \hat{A}_{M_1 \cdots M_6},
\]

(3.29)

for M-theory fields and

\[
\hat{A}_{\alpha M_1 M_2} = A_{\alpha M_1 M_2}, \quad A_{M_1 \cdots M_4} = A_{M_1 \cdots M_4}, \quad \hat{A}_{\alpha M_1 \cdots M_6} = A_{\alpha M_1 \cdots M_6},
\]

(3.30)

\[
A_{M_1 \cdots M_7, N} = A_{M_1 \cdots M_7, N} + \frac{105}{4} \left[ C_{M_1 \cdots M_4} (B_{M_5 M_6} C_{M_7} - B_{M_5 M_6} B_{M_7 N}) - C_{M_1 \cdots M_4} (B_{M_5 M_6} C_{M_7} - B_{M_5 M_6} B_{M_7 N}) \right]
\]

\[
+ \frac{315}{8} \left[ C_{M_1 M_2} C_{M_3 M_4} B_{M_5 M_6} B_{M_7 N} - C_{M_1 M_2} C_{M_3 M_4} B_{M_5 M_6} B_{M_7 N} \right],
\]

(3.31)

for type IIB fields, and by using the 11D–10D map, these relations are precisely the \( T \)-duality rules obtain in Section 2.5. The \( S \)-duality rule for the new dual graviton is simply

\[
A'_{M_1 \cdots M_7, N} = A_{M_1 \cdots M_7, N},
\]

(3.32)

which is consistent with (2.65) under the identification (3.31).

Note that, in order to obtain the duality rules for the higher mixed-symmetry potentials, we need to consider the \( E_n \) generalized metric with \( n \geq 9 \).

### 3.3 1-form \( \mathcal{A}_I^I \) as the generalized graviphoton

In Section 2 we found that the 1-form gauge field \( \mathcal{A}_I^I \) has a simple structure in terms of the tensors \( \hat{N} \) and \( N \),

\[
\mathcal{A}_I^I = \hat{N}_I^I + \hat{A}_I^k \hat{N}_k^I \quad (\text{M-theory}), \quad \mathcal{A}_I^I = N_I^I + A_p^I N_p^I \quad (\text{type IIB}).
\]

(3.33)

In fact, this combination has a clear origin. The basic idea is as follows.

**Generalized graviphoton in DFT:** In type IIB theory, the graviphoton is given by

\[
\mathcal{A}_I^I = \mathcal{g}^{I\mu} \mathcal{g}\mu\nu \mathcal{g}^{\nu I}.
\]

(3.34)

We can consider a generalization of this graviphoton in double field theory (DFT). In DFT, the inverse of the generalized metric has the form\(^2\)

\[
\mathcal{H}^{IJ} = \begin{pmatrix}
\hat{g}^{MN} & \hat{g}^{MK} \hat{B}_{KN} \\
-\hat{B}_{MK} \hat{g}^{KN} & (\hat{g} - \hat{\tilde{B}} \hat{g}^{-1} \hat{\tilde{B}}_{MN})
\end{pmatrix}, \quad (x^I) \equiv (x^M, \tilde{x}_M).
\]

(3.35)

\(^2\)Note that the \( B \)-field in this paper has the opposite sign to the one usually used in DFT.
We decompose the physical coordinates as \( (x^M) = (x^\mu, x^m) \) and define the generalized coordinates for the compact directions as \( (x^I) \equiv (x^m, \tilde{x}_m) \) \((m = 1, \ldots, n - 1)\). Then, we find

\[
\mathcal{H}^{\mu I} = (\hat{g}^{\mu m}, \hat{g}^{\mu K} \hat{B}_{Km}) = (\hat{g}^{\mu m}, \hat{g}^{\mu \nu} \hat{B}_{\nu m} + \hat{g}^{\mu p} \hat{B}_{p m})
\]

(3.36)

This leads us to define the generalized graviphoton as

\[
A^I_\mu \equiv g_{\mu \nu} \mathcal{H}^{\nu I} = \begin{pmatrix} A^{m}_\mu \\ \hat{B}_{\mu m} + A^{p}_\mu \hat{B}_{p m} \end{pmatrix} \left[ g \equiv (\hat{g}^{\mu \nu})^{-1} \right],
\]

(3.37)

which transforms covariantly under \( O(n - 1, n - 1) \) transformations, and is sometimes used in the double sigma model (see for example [18, 27]). By using

\[
N_N^I \equiv \begin{pmatrix} \delta^m_N \\ \hat{B}_{Nm} \end{pmatrix},
\]

(3.38)

we observe that the generalized graviphoton can be expressed as

\[
A^I_\mu = N^I_\mu \ + A^p_\mu N^I_p,
\]

(3.39)

which has the same structure as (3.33).

**Generalized graviphoton in EFT:** We now consider the case of EFT starting with the generalized metric \( \mathcal{M}_{IJ} \) in \( E_{11} \) EFT. Denoting the inverse matrix of \( \mathcal{M}_{\mu \nu} \) by \( m_{\mu \nu} \), we define the generalized graviphoton as

\[
A^I_\mu \equiv m_{\mu \nu} \mathcal{M}^{\nu I}.
\]

(3.40)

In the following, we show that this \( A^I_\mu \) is precisely the 1-form considered in Section 2. To this end, let us recall that the generalized metric has the structure

\[
\mathcal{M}_{IJ} = (L^T \hat{\mathcal{M}} L)_{IJ}, \quad \mathcal{M}^{IJ} = (L^{-1} \hat{\mathcal{M}}^{-1} L^{-T})^{IJ}.
\]

(3.41)

By using the fact that the matrix \( L \) has a lower-triangular form, we find

\[
\mathcal{M}^{\mu \nu} = \hat{\mathcal{M}}^{\mu \nu} = |\hat{g}|^{-\frac{1}{2}} g^{\mu \nu} \equiv m^{\mu \nu},
\]

(3.42)

\[
\mathcal{M}^{IJ} = (\hat{\mathcal{M}}^{-1} L^{-T})^{\mu J} = \hat{\mathcal{M}}^{\mu N} (L^{-T})^{N J} = |\hat{g}|^{-\frac{1}{2}} g^{\mu \nu} [(L^{-T})^{\nu I} + A^k_\nu (L^{-T})^{k I}],
\]

(3.43)

where we have used

\[
(\hat{\mathcal{M}}^{\mu \nu}) = |\hat{g}|^{-\frac{1}{2}} \begin{pmatrix} \delta^{\mu}_\nu & 0 \\ \hat{A}^{\mu}_\rho & \delta^{\nu}_\rho \end{pmatrix} \begin{pmatrix} g^{\nu \rho} & 0 \\ 0 & \hat{G}^{k l} \end{pmatrix} \begin{pmatrix} \delta^{\rho}_\sigma & \hat{A}^{\rho}_\sigma \\ 0 & \delta^{\sigma}_l \end{pmatrix}.
\]

(3.44)
Then, we obtain

$$A_\mu^I = m_{\mu\nu} A_\nu^I = (L^{-T})_\mu^I + \hat{A}^k_\mu (L^{-T})_k^I. \quad (3.45)$$

In order to show that this is the same as the 1-form considered in Section 2, let us compute the explicit form of $(L^{-T})_\mu^I$ in M-theory/type IIB parameterizations. In the M-theory parameterization, $L$ is defined as $[22]$ and by using the matrix representations of the $E_{11}$ generators given in Appendix $B$ we obtain

$$(L^{-T})_N^I \simeq \begin{pmatrix}
\delta^i_N \\
\hat{A}_{N,1+2} \\
\\
\hat{A}_{N,1+2} - 5 \hat{A}_{N,1+2} \hat{A}_{i,1+2} \\
\hat{A}_{N,1+2} - 7 \hat{A}_{N,i+2} \hat{A}_{i,1+2} + 35 \hat{A}_{N,1+2} \hat{A}_{i,1+2} \hat{A}_{i,1+2} \\
\vdots
\end{pmatrix}, \quad (3.46)$$

where $i \in \{i_1, \ldots, i_7\}$ has been assumed for the fourth row. By using the identification $[3.29]$, $(L^{-T})_N^I$ is precisely the same as $N_N^I$ given in $[2.41]$ and the generalized graviphoton is the same as the 1-form $[2.43]$.

On the other hand, in the type IIB parameterization, $L$ is defined as $[3.7]$ and we obtain

$$(L^{-T})_N^I \simeq \begin{pmatrix}
\delta^m_N \\
A_{N,m}^\alpha \\
\overline{A}_{N,m}^\alpha + 5 A_{N,m}^\alpha A_{N,m}^\alpha \\
\overline{A}_{N,m}^\alpha + 2 A_{N,m}^\alpha A_{N,m}^\alpha \\
\overline{A}_{N,m}^\alpha + 7 A_{N,m}^\alpha A_{N,m}^\alpha \\
\vdots
\end{pmatrix}, \quad (3.47)$$

where $m \in \{m_1, \ldots, m_6\}$ has been assumed for the fifth row. Again by using the identification $[3.31]$ and $m \in \{m_1, \ldots, m_6\}$, $(L^{-T})_N^I$ matches with $N_N^I$ given in $[2.48]$ and the generalized graviphoton in the type IIB parameterization is precisely the 1-form $[2.47]$.

Here, let us comment on the relation to the series of papers $[28-30]$. The standard wave solution in 11D supergravity has the non-vanishing flux associated with the graviphoton $A_\mu^I$. In $[28-30]$, the wave solution was embedded into EFT, which has non-vanishing $A_\mu^I$. Then, by rotating the duality frames, various brane solutions were obtained in EFT. Particularly in $[30]$, the 1-form $A_\mu^I$ was regarded as the graviphoton in the $(4+56)$-dimensional exceptional space. Since all of their brane solutions in EFT couples to the generalized graviphoton $A_\mu^I$, branes were interpreted as a kind of generalized waves in the exceptional space. Although the explicit parameterization of $A_\mu^I$ was not determined there, conceptually, their idea is closely related to the result obtained here.
4 Parameterization of $A_p^{I_p}$

In this section, we study the parameterization of the higher $p$-form fields $A_p^{I_p}$.

2-form $A_2^{I_2}$

The 2-form gauge field $A_2^{I_2}$ transforms in the string multiplet, characterized by the Dynkin label $[0, \ldots, 0, 1, 0]$. It is decomposed as

$$A_2^{I_2} = \left( \begin{array}{c}
\sqrt{\Delta} A_{[\mu \nu], i} \\
\sqrt{\Delta} A_{[\mu \nu], i_{1 \cdots i_4}} \\
\vdots
\end{array} \right), \quad A_{\mu \nu}^{I_2} = \left( \begin{array}{c}
\sqrt{\Delta} A_{[\mu \nu], m_1 m_2} \\
\sqrt{\Delta} A_{[\mu \nu], m_1 \cdots m_4} \\
\vdots
\end{array} \right), \quad \text{(4.1)}$$

and for example, the first component, in each parameterization, can be expanded as

$$\hat{A}_{[\mu \nu], i} = \hat{A}_{\mu \nu i} + m_1 \hat{A}_{[\mu} A_{i | k]} + m_2 \hat{A}_{[\mu} A^k_{i | k]} \hat{A}_{k li} , \quad \text{(4.2)}$$

$$A_{\mu \nu}^{I_2} = A_{\mu \nu}^{I_2} + b_1 A_{[\mu} A_{\nu | p] + b_2 A_p A_{[\mu} A_{| \nu]} A_{p a]} , \quad \text{(4.3)}$$

by introducing parameters $m_1, m_2, b_1$, and $b_2$. We already have the $T$-duality rules, and by following the same procedure as the 1-form, we can determine these parameters.

Repeating the procedure, we find the parameterization

$$A_{\mu \nu}^{I_2} = \left( \begin{array}{c}
N_{[\mu | i} + \hat{N}_{\mu | i} \\
\sqrt{\Delta} (N_{[\mu | i_{1 \cdots i_4}} + \hat{N}_{\mu | i_{1 \cdots i_4}}) \\
\vdots
\end{array} \right), \quad \text{(4.4)}$$

$$A_{\mu \nu}^{I_2} = \left( \begin{array}{c}
N_{[\mu | i} + \hat{N}_{\mu | i} \\
\sqrt{\Delta} (N_{[\mu | i_{1 \cdots i_4}} + \hat{N}_{\mu | i_{1 \cdots i_4}}) \\
\vdots
\end{array} \right), \quad \text{(4.5)}$$

Interestingly, the tensors $\hat{N}$ and $N$ are precisely the same as those defined in Section 2.4. The origin of this simple structure can be understood as follows.

For example, let us consider the map (2.38)

$$\hat{A}_{\mu \nu a} + \hat{A}_k A_{k a} = \hat{A}_\mu A_{\beta q} \epsilon_{\beta \alpha} , \quad \text{(4.6)}$$

in which both sides are connected through $T$-duality. However, the $T$-duality rule is 9D covariant, and even if we replace the index $a$ by the 9D index $A = (\mu, a)$, the above relation is still satisfied. Then, choosing $A = \nu$ and antisymmetrizing $\mu$ and $\nu$, we get

$$\hat{A}_{\mu \nu a} + \hat{A}_k A_{k a} = \hat{A}_\mu A_{\beta q} \epsilon_{\beta \alpha} , \quad \text{(4.7)}$$

21
which connects the first row of $A_{\mu}^2$ and the first row of $A_{\mu}^2$. In this manner, simply by replacing an internal index $a$ with an external index $\nu$ and acting the antisymmetrization, we obtain the parameterization of the 2-form from the result of the 1-form.

In the literature, several components of the 1-form and 2-form have been studied for example in [3,31]. By following the notation of [18], their M-theory parameterization are

$$A_{\mu}^m = A_{\mu}^m, \quad A_{\mu \nu} = \frac{\hat{C}_{\mu \rho} - A_{\mu}^k \hat{C}_{\rho \nu}^{mk}}{\sqrt{2}}, \quad B_{\mu \nu} = \frac{\hat{C}_{\mu \nu} - A_{[\mu}^k \hat{C}_{\nu] \alpha}^{mk}}{\sqrt{10}}.$$  (4.8)

while the type IIB parameterizations are

$$A_{\mu}^i = A_{\mu}^i, \quad A_{\mu \alpha =} = \epsilon_{\alpha \beta} (\hat{C}_{\mu \beta}^i - A_{\mu}^k \hat{C}_{\nu \beta}^{ik}), \quad B_{\mu \nu}^\alpha = \frac{C_{\mu \nu}^\alpha - A_{[\mu}^j \hat{C}_{\nu] \alpha}^{jk}}{\sqrt{10}}.$$  (4.9)

By comparing, for example $A_{\mu \nu}$ with $B_{\mu \nu}$, we find that their results also follow the antisymmetrization rule and seem to be consistent with our results up to conventions.

3-form and higher $p$-form

Similar to the case of the 2-form, a parameterization of a general $p$-form can be obtained by acting the antisymmetrization to that of the 1-form. In the case of 3-form, we obtain

$$A_{\mu \nu \rho}^2 = \left( \begin{array}{c} \hat{N}_{[\mu \nu \rho]} + \hat{A}^k_{[\mu} \hat{N}_{k] \nu \rho] \\ \frac{1}{\sqrt{2}} (\hat{N}_{[\mu \nu \rho]1_{1235}} + \hat{A}^k_{[\mu} \hat{N}_{k] \nu \rho]1_{1235}}) \\ \vdots \\ \frac{1}{\sqrt{2}} (\hat{N}_{[\mu \nu \rho]1_{15}} + \hat{A}^k_{[\mu} \hat{N}_{k] \nu \rho]1_{15}}) \end{array} \right),$$  (4.10)

$$A_{\mu \nu \rho}^2 = \left( \begin{array}{c} N_{[\mu \nu \rho]m} + A_{[\mu}^p N_{p] \nu \rho]m} \\ \frac{1}{\sqrt{2}} (N_{[\mu \nu \rho]m_{1235}} + A_{[\mu}^p N_{p] \nu \rho]m_{1235}}) \\ \vdots \\ \frac{1}{\sqrt{2}} (N_{[\mu \nu \rho]m_{15}} + A_{[\mu}^p N_{p] \nu \rho]m_{15}}) \end{array} \right).$$  (4.11)

Compared to the 2-form, the first component in type IIB side $N_{[\mu \nu \rho]}^p$ has disappeared because the number of indices is not enough to account for a 3-form. The 4-form is

$$A_{\mu \nu \rho \sigma}^2 = \left( \begin{array}{c} \frac{1}{\sqrt{2}} (\hat{N}_{[\mu \nu \rho]4_{1234}} + \hat{A}^k_{[\mu} \hat{N}_{k] \nu \rho]4_{1234}}) \\ \vdots \\ \frac{1}{\sqrt{2}} (\hat{N}_{[\mu \nu \rho]4_{15}} + \hat{A}^k_{[\mu} \hat{N}_{k] \nu \rho]4_{15}}) \end{array} \right),$$  (4.12)

$$A_{\mu \nu \rho \sigma}^2 = \left( \begin{array}{c} N_{[\mu \nu \rho]4_{m}} + A_{[\mu}^p N_{p] \nu \rho]4_{m}} \\ \frac{1}{\sqrt{2}} (N_{[\mu \nu \rho]4_{m_{1234}}} + A_{[\mu}^p N_{p] \nu \rho]4_{m_{1234}}}) \\ \vdots \\ \frac{1}{\sqrt{2}} (N_{[\mu \nu \rho]4_{m_{15}}} + A_{[\mu}^p N_{p] \nu \rho]4_{m_{15}}) \end{array} \right).$$  (4.13)

and higher $p$-forms are also obtained in a similar manner.

We note that if there exist certain invariant tensor $f_{I \rho} I^L$, with symmetry $f_{I \rho} I^L = (-1)^{pq} f_{L \rho} I^L$, we can redefine an $r$-form $A_{\mu \nu \cdots \rho}^r$ as

$$A_{\mu \nu \cdots \rho}^r \rightarrow A_{\mu \nu \cdots \rho}^{r'} = f_{I \rho} I^L A_{\mu \nu \cdots \rho}^{I L} \wedge A_{\mu}^I.$$  (4.14)

In such case, the $r$-form field is not unique and we cannot fix the parameterization unambiguously.
5 Summary and Discussion

In this paper, we have proposed a systematic way to determine the parameterization of the $p$-form field $A^I_p$. As a demonstration, we have determined how the dual graviton enters the $p$-form field. We have also determined the duality rules for the dual graviton, which have been partially studied in the literature. Our procedure is based on the (factorized) $T$-duality and $S$-duality transformations, which form a subgroup of the full $U$-duality group. Accordingly, our procedure cannot determine the contribution of the mixed-symmetry potentials which do not couple to any supersymmetric branes. However, we have provided another approach to determine the parameterization of $A^I_p$. We have found that the 1-form field is precisely the generalized graviphoton $A^I_\mu = m_{\mu\nu} \mathcal{M}^{I\nu}$ defined by the $E_{11}$ generalized metric. By following the procedure of [6, 32, 33], we can in principle determine the parameterization of the $E_{11}$ generalized metric level by level. We can then determine the full parameterization of the 1-form field. As we have shown, once the parameterization of the 1-form field is determined, we can easily obtain the parameterization of the $p$-form field by antisymmetrizing the indices.

As future directions, it is interesting to revisit the worldvolume actions of exotic brane. In the case of exotic branes, the Wess–Zumino term contains the mixed-symmetry potentials, but at present, the explicit forms of the brane actions are known for a few examples [26,34–36]. A manifestly $U$-duality-covariant Wess–Zumino term, which employs the $p$-form fields $A^I_p$, has been proposed in [15] and it is important to clarify the connection to the results of [26,34–36] by using the concrete parameterization of $A^I_p$. It is also interesting to develop another $U$-duality-manifest approach to brane actions [17,37] (see also [16,18] for a similar approach).

It is also useful to study the duality transformation rules for more mixed-symmetry potentials beyond the dual graviton. By following the procedure proposed in this paper, it is a straightforward task to determine such duality rules. Recently, $T$-duality manifest formulation for mixed-symmetry potentials has been studied in detail in [38], which aims to be more useful to determine the $T$-duality rules. Nevertheless, in order to consider the $S$-duality rule or the M-theory uplifts, our $U$-duality-based procedure would potentially prove more useful.

Acknowledgments

The work of JJFM is supported by Plan Propio de Investigación of the University of Murcia R-957/2017 and Fundación Séneca (21257/PI/19 and 20949/PI/18). The work of YS is supported by JSPS KAKENHI Grant Numbers 18K13540 and 18H01214.
A Notation

In this appendix, we summarize the notation that has been used along this work to denote various fields corresponding to each theory and each dimension, as well as the different types of indices.

M-theory and type IIA/IIB theory are defined in $D$ dimensions, where $D = 11, 10$ respectively. Upon a dimensional reduction on a torus, we have a $d$-dimensional supergravity theory, with a global symmetry group $E_n$, where $n = D - d$. According to this, all the splittings of the M-theory and type IIB coordinates and the higher/lower-dimensional indices that have been used are shown in Figure 1. The $D$-dimensional coordinates in M-theory and type IIB theory are denoted by $x^M$ and $x^\hat{M}$, respectively.

In addition, indices for the $p$-form multiplet are denoted as $I_p$ in M-theory and as $I_p$ in type IIB theory. In particular, for the 1-form, we denote $I \equiv I_1$ and $l_1 \equiv l$. In type IIB theory, the index of the vector representation of the SL(2) $S$-duality group is represented by $\alpha = 1, 2$.

In Table A.1 we summarize the notation that we have used to represent the fields of various theories. Fields transforming as $U$-duality multiplets are considered. Similarly, standard supergravity fields of M-theory and type II theories, and the lower-dimensional fields that arise after compactification are considered.

![Figure 1: Left: splittings of the M-theory coordinates ($x^M$) and their index notation. A compactification on a circle $S^1$ along the direction $x^z$ and a $T$-duality transformation along the $x^9$ coordinate are considered. Right: splittings of the type IIB coordinates ($x^\hat{M}$) and their notations are shown. A $T$-duality transformation is taken along the coordinate $x^y$.](image)

24
| Field                          | M-theory | Type IIB | Type IIA |
|-------------------------------|----------|----------|----------|
| $U$-duality-covariant $p$-form | $\mathcal{A}_p^I$ | $\mathcal{A}_p^I$ | –        |
| generalized metric           | $\mathcal{M}$   | $\mathcal{M}$   | –        |
| generalized vielbein          | $\mathcal{E}$   | $\mathcal{E}$   | –        |
| twist matrix                  | $L$       | $L$       | –        |
| $D$-dim. metric               | $\hat{g}$   | $g$       | $g$      |
| $D$-dim. fields               | $\hat{A}$   | $A, B, C$  | $\mathcal{A}, \mathcal{B}, \mathcal{C}$ |
| $D$-dim. fields (Section 3)   | $\hat{A}$   | $A$       | –        |
| spacetime metric              | $\hat{g}$   | $g$       | $g$      |
| internal metric               | $\hat{G}$   | $G$       | $G$      |

Table A.1: Summary of the fields that have been used along this work. While the first four lines correspond to $U$-duality multiplets, the rest correspond to standard supergravity fields. In the last two lines we show the $d$-dimensional fields that appear after compactification.

B  $E_n$ generators

In this appendix, we show the explicit matrix representation of the $E_n$ generators in the vector representation. In the M-theory parameterization, our matrices are consistent with [19]. Through the linear map from M-theory parameterization to type IIB parameterization, we find the matrix representations also in the type IIB parameterization, which is new.

Here, we show the results for $E_8$, but the $E_n$ generators with $n \leq 7$ can be easily obtained through a truncation. For example, an $E_8$ generator $R^{i_1 \cdots 8, i}$ disappears in $E_7$ because the index $i$ ranges over seven directions and $i_{1 \cdots 8}$ automatically vanishes. Conversely, our $E_8$ generators can be understood as a truncation of the $E_{11}$ generators. In $E_{11}$, the matrix representation becomes infinite dimensional, but the first several blocks are the same as the $E_8$ generators. Accordingly, although we have computed the matrix $(L^{-\top})$ in (3.46) by using the $E_8$ generators, the first four rows do not change even if we use the matrix representation of the $E_{11}$ generators.\(^3\) In that sense, the results given in this appendix can be understood as a truncation of the $E_{11}$ generators.

\(^3\)To be more precise, in our matrix representations in M-theory, in the fourth row and below that, we have used Schouten-like identities; i.e. terms with antisymmetrized nine indices $(\cdots)_{[i_1 \cdots i_7]}$ has been dropped because they disappear automatically in $n \leq 8$. However, this does not affect the computation of $(L^{-\top})$ in (3.46) because the restriction rule $i \in \{i_1, \ldots, i_7\}$ has been assumed there and terms with the structure $(\cdots)_{[i_1 \cdots i_7]}$ disappear even for $n = 11$. In this sense, (3.46) can be understood as obtained from the $E_{11}$ generalized metric.
B.1 M-theory parameterization

In the M-theory parameterization, the $E_n$ generators are decomposed as

$$\{T_\alpha\} = \{K^i_{j}, R_{i123}, R_{i1-6}, R_{i1-8,i}, R_{i1-23}, R_{i1-6}, R_{i1-8,i}, \ldots\}, \quad (B.1)$$

where the ellipses disappear for $n \leq 8$. In this appendix, we may use a short-hand notation for the multiple indices,

$$(\cdots)_{i_1 \cdots p} \equiv (\cdots)_{i_1 \cdots i_p}. \quad (B.2)$$

We also use a notation,

$$\delta^{ji_1 \cdots j_n}_{i_1 \cdots i_n} \equiv n! \delta^{ji_1 \cdots j_n}_{i_1 \cdots i_n}. \quad (B.3)$$

If we restrict to the case $n \leq 8$, they satisfy the following commutation relations:

$$[K^i_{j}, K^k_{l}] = \frac{\delta^i_j}{2!} K^k_{l} - \frac{\delta^i_l}{2!} K^k_{j}, \quad (B.4)$$

$$[K^i_{j}, R_{k1k2k3}] = \frac{\delta^i_{k2k3}}{2!} R_{ir1r2}, \quad (B.5)$$

$$[K^i_{j}, R_{k1\cdots k6}] = \frac{\delta^i_{k1\cdots k6}}{5!} R_{ir1\cdots r5}, \quad (B.6)$$

$$[K^i_{j}, R_{k1\cdots k8,k}] = \frac{\delta^i_{k1\cdots k8,k}}{7!} R_{ir1\cdots r7,k} + \delta^i_j R_{k1\cdots k8,i}, \quad (B.7)$$

$$[K^i_{j}, R_{k1k2k3}] = -\frac{1}{2!} \delta^i_{k1k2k3} R_{jr1r2}, \quad (B.8)$$

$$[K^i_{j}, R_{k1\cdots k6}] = -\frac{1}{5!} \delta^i_{k1\cdots k6} R_{jr1\cdots r5}, \quad (B.9)$$

$$[K^i_{j}, R_{k1\cdots k8,k}] = -\frac{1}{7!} \delta^i_{k1\cdots k8,k} R_{jr1\cdots r7,k} - \delta^i_j R_{k1\cdots k8,i}, \quad (B.10)$$

$$[R^{i12i3}_{j}, R^{i2j3}_{j}] = -R^{i12i3}_{j1j2j3}, \quad (B.11)$$

$$[R^{i12i3}_{j}, R^{i1j6}_{j6}] = -\frac{1}{3!} \delta^{i1\cdots j6}_{i1\cdots j6} R^{i12i3}_{jr1r2r3}, \quad (B.12)$$

$$[R^{i12i3}_{j}, R^{ij1j2j3}_{j}] = \frac{1}{2!} \delta^{i12i3}_{j1j2j3} \delta^{i1r2}_{j1j2j3} K_{s}^t - \frac{1}{3!} \delta^{i12i3}_{j1j2j3} \delta^{i1j6}_{j1j2j3} K_{s}^t, \quad (B.13)$$

$$[R^{i12i3}_{j}, R^{ij1j6}_{j6}] = \frac{1}{3!} \delta^{i12i3}_{j1\cdots j6} R_{jr1r2r3}, \quad (B.14)$$

$$[R^{i12i3}_{j}, R^{ij1j8}_{j8}] = \frac{1}{2!} \delta^{i12i3}_{j1j8} R_{jr1r2r3}, \quad (B.15)$$

$$[R^{i1\cdots i6}_{j}, R^{ij1j2j3}_{j}] = \frac{1}{3!} \delta^{i1\cdots i6}_{j1j2j3} \delta^{i1r2}_{j1j2j3} R_{jr1r2r3}, \quad (B.16)$$

$$[R^{i1\cdots i6}_{j}, R^{ij1j6}_{j6}] = \frac{1}{2!} \delta^{i1\cdots i6}_{j1\cdots j6} \delta^{i1\cdots j6}_{j1\cdots j6} R_{jr1r2r3}, \quad (B.17)$$

$$[R^{i1\cdots i6}_{j}, R^{ij1j8}_{j8}] = \frac{1}{2!} \delta^{i1\cdots i6}_{j1\cdots j8} R_{jr1r2r3}, \quad (B.18)$$

\footnote{If we consider the $E_{11}$ algebra, for example $[R^{i12i3}_{j}, R_{j1\cdots j8}]$ needs to be modified as $[R^{i12i3}_{j}, R_{j1\cdots j8}] = \frac{1}{2!} \delta^{i12i3}_{j1j2j3} R_{jr1r2r3} - \delta^{i12i3}_{j1j2j3} R_{jr1r2r3}}$.

However, the second term on the right-hand side identically vanishes for $n \leq 8$. In this sense, the commutation relations shown here are valid only in the case $n \leq 8$.}
\[
\begin{align*}
\begin{bmatrix} R_{i_1 \cdots i_8,i} & R_{j_1 j_2 j_3} \end{bmatrix} &= \frac{1}{5!} \delta_{j_1 j_2 j_3 r_1 r_2} \delta^{i_1 \cdots i_8} R_{i_1 \cdots i_8}, \\
\begin{bmatrix} R_{i_1 \cdots i_8,i} & R_{j_1 \cdots j_6} \end{bmatrix} &= \frac{1}{2!} \delta_{j_1 \cdots j_6 r_1 r_2} \delta^{i_1 \cdots i_8} R_{i_1 \cdots i_8}, \\
\begin{bmatrix} R_{i_1 \cdots i_8,i} & R_{j_1 \cdots j_8,j} \end{bmatrix} &= \delta_{j_1 \cdots j_8} K_{i_j}, \\
\begin{bmatrix} R_{i_1 i_2 i_3} & R_{j_1 j_2 j_3} \end{bmatrix} &= R_{i_1 i_2 i_3 j_1 j_2 j_3}, \\
\begin{bmatrix} R_{i_1 i_2 i_3} & R_{j_1 \cdots j_6} \end{bmatrix} &= \frac{1}{5!} \delta_{j_1 \cdots j_6} R_{i_1 i_2 i_3 r_1 r_2} R_{i_1 \cdots i_8}. 
\end{align*}
\]

We note that our convention will be related to that of [19] as follows:

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{Here} & K^i & R_{i,i_3} & R_{i,i_3} & R_{i_3,i} & R_{i_3,i_3} & R_{i_3,i_3} \\
\hline
[19] & K^i & R_{i,i_3} & R_{i,i_3} & 2 R_{i,i_3} & -2 R_{i_3,i} & 2 R_{i_3,i_3} \\
\hline
\end{array}
\]

Now, we show the matrix representations of these generators in the vector representation. In the M-theory parameterization, the vector representation (for \( n \leq 8 \)) is decomposed as

\[
\{ x^i \} = \left\{ x^i, \frac{y_{i_1}}{\sqrt{2!}}, \frac{y_{i_2}}{\sqrt{3!}}, \frac{y_{i_3}}{\sqrt{4!}}, \frac{y_{i_4}}{\sqrt{5!}}, \frac{y_{i_5}}{\sqrt{6!}}, \frac{y_{i_6}}{\sqrt{7!}}, \frac{y_{i_7}}{\sqrt{8!}}, \frac{y_{i_8}}{\sqrt{8!}} \right\},
\]

where \( y_{i_1 \cdots i_7} = 0 \). In this paper, in order to reduce the matrix size, we have combined \( y_{i_1 \cdots i_7} \) and \( y_{i_1 \cdots i_8} \), and our \( y_{i_1 \cdots i_7} \) do not satisfy \( y_{i_1 \cdots i_7} = 0 \). We then find that the following matrices

\[
(T_a)^T \mathbf{j} \text{ satisfy the above } E_8 \text{ algebra:}
\]

\[
K^p q \equiv \text{diag}_{7 \times 7} \begin{pmatrix}
-\delta_p^q & \delta_p^q \\
\delta_{p1}^q & \delta_{p2}^q & \delta_{p3}^q & \delta_{p4}^q & \delta_{p5}^q & \delta_{p6}^q & \delta_{p7}^q & \delta_{p8}^q \\
\delta_{p12}^q & \delta_{p12}^q & \delta_{p12}^q & \delta_{p12}^q & \delta_{p12}^q & \delta_{p12}^q & \delta_{p12}^q & \delta_{p12}^q \\
\delta_{p13}^q & \delta_{p13}^q & \delta_{p13}^q & \delta_{p13}^q & \delta_{p13}^q & \delta_{p13}^q & \delta_{p13}^q & \delta_{p13}^q \\
\delta_{p14}^q & \delta_{p14}^q & \delta_{p14}^q & \delta_{p14}^q & \delta_{p14}^q & \delta_{p14}^q & \delta_{p14}^q & \delta_{p14}^q \\
\delta_{p15}^q & \delta_{p15}^q & \delta_{p15}^q & \delta_{p15}^q & \delta_{p15}^q & \delta_{p15}^q & \delta_{p15}^q & \delta_{p15}^q \\
\delta_{p16}^q & \delta_{p16}^q & \delta_{p16}^q & \delta_{p16}^q & \delta_{p16}^q & \delta_{p16}^q & \delta_{p16}^q & \delta_{p16}^q \\
\delta_{p17}^q & \delta_{p17}^q & \delta_{p17}^q & \delta_{p17}^q & \delta_{p17}^q & \delta_{p17}^q & \delta_{p17}^q & \delta_{p17}^q \\
\delta_{p18}^q & \delta_{p18}^q & \delta_{p18}^q & \delta_{p18}^q & \delta_{p18}^q & \delta_{p18}^q & \delta_{p18}^q & \delta_{p18}^q \\
\delta_{p19}^q & \delta_{p19}^q & \delta_{p19}^q & \delta_{p19}^q & \delta_{p19}^q & \delta_{p19}^q & \delta_{p19}^q & \delta_{p19}^q \\
\end{pmatrix} - \frac{\delta_p^q \delta_p^j}{9 - n},
\]

\[
R_{i_1 i_2 i_3} \equiv \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
(B.26)
\]
We can identify the Cartan generators as
\[
\{ H_k \} = \{ K^1_1 - K^2_2, \ldots, K^7_7 - K^8_8, K^6_6 + K^7_7 + K^8_8 + \frac{1}{2} K^i_i \}, \tag{B.32}
\]
and the positive/negative simple-root generators are
\[
\{ E_k \} = \{ K^1_2, \ldots, K^7_8, R^{678} \}, \quad \{ F_k \} = \{ K^2_1, \ldots, K^8_7, R_{678} \}. \tag{B.33}
\]
They satisfy the relations
\[
[H_k, E_l] = A_{kl} E_l, \quad [H_k, F_l] = -A_{kl} F_l, \quad [E_k, F_l] = \delta_{kl} H_l,
\]
\[\frac{1}{\alpha_n} \text{tr}(H_k H_l) = A_{kl}, \quad \frac{1}{\alpha_n} \text{tr}(E_k F_l) = \delta_{kl}, \tag{B.34}\]

where
\[
(A_{kl}) = \begin{pmatrix}
2 & -1 \\
-1 & 2 & \ddots \\
& \ddots & \ddots & -1 & -1 \\
& & -1 & 2 & -1 & 0 \\
& & & -1 & 2 & 0 \\
& & & & -1 & 0 & 2
\end{pmatrix}, \quad \begin{array}{cccccc}
& 4 & 5 & 6 & 7 & 8 \\
\alpha_n & 3 & 4 & 6 & 12 & 60
\end{array}. \tag{B.35}
\]
The set of the positive/negative root generators can be obtained by taking commutators of
the simple-root generators $E_k/F_k$, and they can be summarized as
\[
\{ E_\alpha \} = \{ K^i_j \ (i < j), R_{123}, R_{1123}, R_{i1\ldots8} \}, \quad \{ F_\alpha \} = \{ K^i_j \ (i > j), R^{i1\ldots6}, R^{i1\ldots8} \}. \tag{B.36}\]

### B.2 Type IIB parameterization

We can transform the $E_n$ generators of the M-theory parameterization into the type IIB parameterization by using the linear map $T(T_a) \equiv (S^T)_K (T_a)^{KL} S^{LJ}$. Namely, we act the following operation to the matrix representations of the generators
\[
\mathcal{T}(T_a)^{IJ} \equiv (S^T)^{K}_{K'} (T_a)^{KL} S^{LJ}. \tag{B.37}
\]
Then, $\mathcal{T}(T_a)^{IJ}$ is the matrix representations in the type IIB parameterization. The explicit form of the constant matrix $S^{IJ}$ has been determined such that the algebra of the type IIB generators is closed. We also change the name of the generators such that the SL(7) × SL(2) symmetry is manifest. Concretely, we convert the non-positive-level generators of the M-theory
parameterization into those of the type IIB parameterization as follows:

\begin{align}
\mathcal{T}(\alpha_{\beta}) = \left[ \mathcal{T}(R_{a_1\cdots a_3}) \right]^T &= \left[ R_{a_1\cdots a_3} \right]^T = R^{a_1 a_2 a_3}. \tag{B.38}
\end{align}

We then obtain the \( E_n \) generators \( n \leq 8 \) in the type IIB parameterization,

\begin{align}
\{ T_\alpha \} = \{ K^m_{\alpha}, R_{a_1\beta}, R_{a_1 \cdots a_2}, R_{a_1 \cdots a_4}, R_{a_1 \cdots a_6}, R_{a_1 \cdots a_7, a}, R_{a_1 \cdots a_8, a}, R_{a_1 \cdots a_6, a}, R_{a_1 \cdots a_6, a} \}. \tag{B.40}
\end{align}

By using the notations,

\begin{align}
\delta^{m_1 \cdots m_n}_{m_1 \cdots m_n} = n! \delta^{m_1 \cdots m_n}_{m_1 \cdots m_n}, \quad (\cdots)_{m_1 \cdots p} = (\cdots)_{m_1 \cdots m_p}, \quad \delta^{\alpha \beta}_{\gamma \delta} = \delta^{(\alpha \beta)}_{(\gamma \delta)}. \tag{B.41}
\end{align}
their matrix representations are found as follows:

\[ K_s \equiv \text{diag} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ R_{s\gamma} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ (R_{s\gamma})^{\alpha\beta}_{12} \equiv \delta^\alpha_1 \delta^\beta_2 + \frac{\delta^{3\alpha2}}{\beta_2(\gamma, \delta) \beta_1}, \quad (B.43) \]

\[ R^\gamma_{12} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ \left[ C_2 \equiv \frac{4 + \sqrt{2}}{14} \right], \quad (B.44) \]
\[
R_{f_{12}}^{\gamma} \equiv (R_{f_{12}}^{\gamma})^T, \tag{B.45}
\]

\[
R^{f_{1-4}} \equiv \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\delta \alpha \gamma \delta \alpha \gamma & 4 \delta \alpha \gamma & 4 \delta \alpha \gamma & 4 \delta \alpha \gamma & 4 \delta \alpha \gamma & 4 \delta \alpha \gamma & 4 \delta \alpha \gamma & 4 \delta \alpha \gamma & 4 \delta \alpha \gamma \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
C_{f_4} \equiv \frac{4 + \sqrt{2}}{7}, \tag{B.46}
\]

\[
R^{f_{1-4}} \equiv \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\delta \alpha \gamma \delta \alpha \gamma & 4 \delta \alpha \gamma & 4 \delta \alpha \gamma & 4 \delta \alpha \gamma & 4 \delta \alpha \gamma & 4 \delta \alpha \gamma & 4 \delta \alpha \gamma & 4 \delta \alpha \gamma & 4 \delta \alpha \gamma \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
\]
\[ R_{1-6}^7 \equiv \left( \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \end{array} \right) \]

\[ c_6 \equiv \frac{2 - 3\sqrt{2}}{14} \quad (B.48) \]

\[ R_{1-6}^7 \equiv \left( R_{1-6}^7 \right)^T, \quad (B.49) \]

\[ R_{1-7,1}^7 \equiv \left( \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \end{array} \right) \]

\[ c_{7,1} \equiv 4 + \sqrt{2}, \quad (B.50) \]
Again, we can identify the Cartan generators as

\[ H_k = \{ K^d_d - K^{d+1}_{d+1}, \ldots, K^7_7 - K^8_8, K^8_8 + K^9_9 - \frac{1}{2} D - R_{12}, 2 R_{12}, K^8_8 - K^9_9 \} , \]  

and the positive/negative simple-root generators are

\[ E_k = \{ K^1_1, \ldots, K^5_6, R^7_1, R_{22}, K^6_7 \}, \quad F_k = \{ K^2_1, \ldots, K^6_5, R^7_7, -R^{11}_1, K^7_6 \} . \]  

The set of the positive/negative root generators can be summarized as

\[ \{ E_\alpha \} = \{ K^m_n \ (m < n), R_{22}, R^m_{12}, R^{m+1}_{m+4}, R^{m+1}_{m+6}, R^{m+1}_{m+7, m} \} , \]  

\[ \{ F_\alpha \} = \{ K^m_n \ (m > n), R_{11}, R^m_{12}, R^m_{m+1-4}, R^m_{m+1-6}, R^m_{m+1-7, m} \} . \]  

We have checked that the obtained type IIB generators satisfy the following \( E_8 \) algebra:

\[ [K^m_n, K^p_q] = \delta^p_n K^{m}_q - \delta^m_q K^p_n , \]  

\[ [K^m_n, R^p_{12}] = \delta^p_{n} R^m_{12} , \]  

\[ [K^m_n, R^p_{12}] = \frac{1}{3!} \delta^{p}_{n} R^{m}_{123} , \]  

\[ [K^m_n, R^p_{12}] = \frac{1}{3!} \delta^{p}_{n} R^{m}_{123} , \]  

\[ [K^m_n, R^p_{12}] = \frac{1}{6!} \delta^{p}_{n} R^{m}_{123} + \delta^p_n R^{p}_{1, 7, m} , \]  

\[ [K^m_n, R^p_{12}] = -\delta^m_{p} R^m_{12} , \]  

\[ [K^m_n, R^p_{12}] = -\delta^m_{p} R^m_{12} , \]  

\[ [K^m_n, R^p_{12}] = -\delta^m_{p} R^m_{12} , \]  

\[ [K^m_n, R^p_{12}] = -\delta^m_{p} R^m_{12} , \]  

\[ [R_{\alpha \beta}, R_{\gamma \delta}] = \delta^\gamma_{(\alpha \epsilon \beta)\delta} R_{\sigma \delta} + \delta^\sigma_{(\alpha \epsilon \beta)\delta} R_{\gamma \sigma} , \]  

\[ [R_{\alpha \beta}, R^m_{12}] = \delta^m_{(\alpha \epsilon \beta)} R^m_{12} , \]  

\[ (\mathbf{B.51}) \]
\[
\begin{align*}
[R_{\alpha\beta}, R_{\gamma}^{m_{1-6}}] &= \delta^\sigma_{(\alpha\beta)} R_{\sigma}^{m_{1-6}}, \\
[R_{\alpha\beta}, R_{m_{12}}^n] &= -\delta^{\gamma}_{(\alpha\beta)} R_{m_{12}}^\sigma, \\
[R_{\alpha\beta}, R_{m_{1-6}}^\gamma] &= -\delta^{\gamma}_{(\alpha\beta)} R_{m_{1-6}}^\sigma, \\
[R_{m_{12}}, R_{\alpha\beta}^n] &= -\epsilon_{\alpha\beta} R_{m_{12}n}^{12}, \\
[R_{m_{12}}, R_{n_{1-4}}^\alpha] &= R_{m_{12}n_{1-4}}, \\
[R_{m_{12}}, R_{n_{1-4}}^\beta] &= -\frac{1}{3!}\epsilon_{\alpha\beta} \delta^{m_{1-5}}_{n_{1-5}} R_{m_{12}r_{1-4}^5}, \\
[R_{m_{12}}, R_{n_{1-4}}^\gamma] &= \delta^{\beta}_{\alpha} \delta^{\delta}_{\gamma \rho} \delta^{\kappa} \delta^{q} R_{m_{12}r_{1-4}^5} - \frac{1}{4} \delta^{\beta}_{\alpha} \delta^{\delta}_{m_{12}n_{1-4}} \delta^{q} R_{m_{12}r_{1-4}^5}, \\
[R_{m_{12}}, R_{n_{1-4}}^\gamma] &= \frac{1}{3!} \delta^{m_{1-4}}_{n_{1-4}} R_{m_{12}r_{1-4}^5} - \frac{1}{2} \delta^{m_{1-4}}_{n_{1-4}} \delta^{q} R_{m_{12}r_{1-4}^5}, \\
[R_{m_{1-4}}, R_{n_{1-4}}^\alpha] &= \frac{1}{2!} \delta^{m_{1-4}}_{r_{123}} R_{n_{1-6}^a}, \\
[R_{m_{1-4}}, R_{n_{1-4}}^\beta] &= \frac{1}{3!} \delta^{m_{1-4}}_{r_{123}} R_{n_{1-6}^a}, \\
[R_{m_{1-4}}, R_{n_{1-4}}^\gamma] &= \frac{1}{3!} \delta^{m_{1-4}}_{r_{123}} R_{n_{1-6}^a}, \\
[R_{m_{1-6}}, R_{n_{1-7}}^\alpha] &= -\epsilon_{\alpha\beta} \delta^{m_{1-6}}_{n_{1-7}} R_{m_{1-6}}, \\
[R_{m_{1-6}}, R_{n_{1-7}}^\beta] &= \frac{1}{3!} \delta^{m_{1-6}}_{n_{1-7}} R_{n_{1-7}^a}, \\
[R_{m_{1-6}}, R_{n_{1-7}}^\gamma] &= \delta^{m_{1-6}}_{n_{1-7}} R_{n_{1-7}^a}, \\
[R_{m_{1-7}^m}, R_{n_{1-7}}^\alpha] &= \delta^{m_{1-7}}_{n_{1-7}} K_{n_{1-7}}, \\
[R_{m_{1-7}^m}, R_{n_{1-7}}^\beta] &= \epsilon^{\alpha\beta}_{n_{1-7}} R_{m_{12}n_{1-7}}, \\
[R_{m_{1-7}^m}, R_{n_{1-7}}^\gamma] &= -\epsilon_{\alpha\beta} \delta_{n_{1-7}} R_{m_{12}n_{1-7}}, \\
[R_{m_{12}}, R_{n_{1-4}}^\alpha] &= -R_{m_{12}n_{1-4}}, \\
[R_{m_{12}}, R_{n_{1-4}}^\beta] &= \frac{1}{3!} \epsilon^{\alpha\beta}_{n_{1-4}} R_{m_{12}r_{1-4}^5}, \\
[R_{m_{1-4}}, R_{n_{1-4}}^\gamma] &= -\frac{1}{3!} \delta^{r_{123}}_{n_{1-4}} R_{m_{1-4}r_{1-4}^5}. 
\end{align*}
\]
References

[1] J. J. Fernández-Melgarejo, Y. Sakatani and S. Uehara, “Exotic branes and mixed-symmetry potentials I: predictions from $E_{11}$ symmetry,” arXiv:1907.07177 [hep-th].

[2] O. Hohm and H. Samtleben, “Exceptional Form of D=11 Supergravity,” Phys. Rev. Lett. 111, 231601 (2013) [arXiv:1308.1673 [hep-th]].

[3] O. Hohm and H. Samtleben, “Exceptional Field Theory I: $E_{6(6)}$ covariant Form of M-Theory and Type IIB,” Phys. Rev. D 89, no. 6, 066016 (2014) [arXiv:1312.0614 [hep-th]].

[4] O. Hohm and H. Samtleben, “Exceptional field theory. II. $E_{7(7)}$,” Phys. Rev. D 89, 066017 (2014) [arXiv:1312.4542 [hep-th]].

[5] O. Hohm and H. Samtleben, “Exceptional field theory. III. $E_{8(8)}$,” Phys. Rev. D 90, 066002 (2014) [arXiv:1406.3348 [hep-th]].

[6] P. C. West, “$E_{11}$, SL(32) and central charges,” Phys. Lett. B 575, 333 (2003) [hep-th/0307098].

[7] A. Kleinschmidt and P. C. West, “Representations of $G^{+++}$ and the role of space-time,” JHEP 0402, 033 (2004) [hep-th/0312247].

[8] P. C. West, “$E_{11}$ origin of brane charges and U-duality multiplets,” JHEP 0408, 052 (2004) [hep-th/0406150].

[9] F. Riccioni and P. C. West, “$E_{11}$-extended spacetime and gauged supergravities,” JHEP 0802, 039 (2008) [arXiv:0712.1795 [hep-th]].

[10] C. Hillmann, “Generalized $E_{7(7)}$ coset dynamics and D=11 supergravity,” JHEP 0903, 135 (2009) [arXiv:0901.1581 [hep-th]].

[11] D. S. Berman and M. J. Perry, “Generalized Geometry and M theory,” JHEP 1106, 074 (2011) [arXiv:1008.1763 [hep-th]].

[12] D. S. Berman, H. Godazgar, M. Godazgar and M. J. Perry, “The Local symmetries of M-theory and their formulation in generalised geometry,” JHEP 1201, 012 (2012) [arXiv:1110.3930 [hep-th]].

[13] D. S. Berman, H. Godazgar, M. J. Perry and P. West, “Duality Invariant Actions and Generalised Geometry,” JHEP 1202, 108 (2012) [arXiv:1111.0459 [hep-th]].

[14] D. S. Berman, M. Cederwall, A. Kleinschmidt and D. C. Thompson, “The gauge structure of generalised diffeomorphisms,” JHEP 1301, 064 (2013) [arXiv:1208.5884 [hep-th]].
[15] E. A. Bergshoeff and F. Riccioni, “D-Brane Wess-Zumino Terms and U-Duality,” JHEP 1011, 139 (2010) [arXiv:1009.4657 [hep-th]].

[16] A. S. Arvanitakis and C. D. A. Blair, “Unifying Type-II Strings by Exceptional Groups,” Phys. Rev. Lett. 120, no. 21, 211601 (2018) [arXiv:1712.07115 [hep-th]].

[17] Y. Sakatani and S. Uehara, “Exceptional M-brane sigma models and η-symbols,” PTEP 2018, no. 3, 033B05 (2018) [arXiv:1712.10316 [hep-th]].

[18] A. S. Arvanitakis and C. D. A. Blair, “The Exceptional Sigma Model,” JHEP 1804, 064 (2018) [arXiv:1802.00442 [hep-th]].

[19] H. Godazgar, M. Godazgar and M. J. Perry, “E8 duality and dual gravity,” JHEP 1306, 044 (2013) [arXiv:1303.2035 [hep-th]].

[20] A. G. Tumanov and P. West, “Generalised vielbeins and non-linear realisations,” JHEP 1410, 009 (2014) [arXiv:1405.7894 [hep-th]].

[21] K. Lee, S. J. Rey and Y. Sakatani, “Effective action for non-geometric fluxes duality covariant actions,” JHEP 1707, 075 (2017) [arXiv:1612.08738 [hep-th]].

[22] Y. Sakatani and S. Uehara, “Connecting M-theory and type IIB parameterizations in Exceptional Field Theory,” PTEP 2017, no. 4, 043B05 (2017) [arXiv:1701.07819 [hep-th]].

[23] E. Bergshoeff, E. Eyras and Y. Lozano, “The Massive Kaluza-Klein monopole,” Phys. Lett. B 430, 77 (1998) [hep-th/9802199].

[24] P. C. West, “The IIA, IIB and eleven-dimensional theories and their common E11 origin,” Nucl. Phys. B 693, 76 (2004) [hep-th/0402140].

[25] E. Eyras, B. Janssen and Y. Lozano, “Five-branes, KK monopoles and T duality,” Nucl. Phys. B 531, 275 (1998) [hep-th/9806169].

[26] E. Eyras and Y. Lozano, “Exotic branes and nonperturbative seven-branes,” Nucl. Phys. B 573, 735 (2000) [hep-th/9908094].

[27] C. M. Hull, “A Geometry for non-geometric string backgrounds,” JHEP 0510, 065 (2005) [hep-th/0406102].

[28] J. Berkeley, D. S. Berman and F. J. Rudolph, “Strings and Branes are Waves,” JHEP 1406, 006 (2014) [arXiv:1403.7198 [hep-th]].

[29] D. S. Berman and F. J. Rudolph, “Branes are Waves and Monopoles,” JHEP 1505, 015 (2015) [arXiv:1409.6314 [hep-th]].
[30] D. S. Berman and F. J. Rudolph, “Strings, Branes and the Self-dual Solutions of Exceptional Field Theory,” JHEP 1505, 130 (2015) arXiv:1412.2768 [hep-th].

[31] A. Baguet, O. Hohm and H. Samtleben, “$E_{6(6)}$ Exceptional Field Theory: Review and Embedding of Type IIB,” PoS CORFU 2014, 133 (2015) arXiv:1506.01065 [hep-th].

[32] P. C. West, “$E_{11}$ and M theory,” Class. Quant. Grav. 18, 4443 (2001) hep-th/0104081.

[33] P. West, “$E_{11}$, generalised space-time and IIA string theory,” Phys. Lett. B 696, 403 (2011) arXiv:1009.2624 [hep-th].

[34] A. Chatzistavrakidis, F. F. Gautason, G. Moutsopoulos and M. Zagermann, “Effective actions of nongeometric five-branes,” Phys. Rev. D 89, no. 6, 066004 (2014) arXiv:1309.2653 [hep-th].

[35] T. Kimura, S. Sasaki and M. Yata, “World-volume Effective Actions of Exotic Five-branes,” JHEP 1407, 127 (2014) arXiv:1404.5442 [hep-th].

[36] T. Kimura, S. Sasaki and M. Yata, “World-volume Effective Action of Exotic Five-brane in M-theory,” JHEP 1602, 168 (2016) arXiv:1601.05589 [hep-th].

[37] Y. Sakatani and S. Uehara, “Branes in Extended Spacetime: Brane Worldvolume Theory Based on Duality Symmetry,” Phys. Rev. Lett. 117, no. 19, 191601 (2016) arXiv:1607.04265 [hep-th].

[38] E. Bergshoeff, A. Kleinschmidt, E. T. Musaev and F. Riccioni, “The different faces of branes in Double Field Theory,” arXiv:1903.05601 [hep-th].