SMOOTH SOLUTION TO HIGHER DIMENSIONAL COMPLEX PLATEAU PROBLEM

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Abstract. Let X be a compact connected strongly pseudoconvex CR manifold of real dimension \(2n - 1\) in \(\mathbb{C}^N\). For \(n \geq 3\), Yau solved the complex Plateau problem of hypersurface type by checking a bunch of Kohn-Rossi cohomology groups in 1981. In this paper, we generalize Yau’s conjecture on some numerical invariant of every isolated surface singularity defined by Yau and the author to any dimension and prove that the conjecture is true for local complete intersection singularities of dimension \(n \geq 3\). As a direct application, we solved complex Plateau problem of hypersurface type for any dimension \(n \geq 3\) by checking only one numerical invariant.

1. Introduction

The famous classical complex Plateau problem asks which odd dimensional real sub-manifolds of \(\mathbb{C}^N\) are boundaries of complex sub-manifolds in \(\mathbb{C}^N\). Harvey and Lawson [Ha-La] proved in their beautiful seminal paper that for any compact connected CR manifold \(X\) of real dimension \(2n - 1\), where \(n \geq 2\), in \(\mathbb{C}^N\), there is a unique complex variety \(V\) in \(\mathbb{C}^N\) such that the boundary of \(V\) is \(X\).

The next question is to determine when \(X\) is a boundary of a complex sub-manifold in \(\mathbb{C}^N\), i.e. when \(V\) is smooth. Suppose \(X\) is a compact connected strongly pseudoconvex CR manifold of real dimension \(2n - 1, n \geq 3\), in the boundary of a bounded strongly pseudoconvex domain \(D\) in \(\mathbb{C}^{n+1}\). In 1981, Yau [Ya] showed that \(X\) is a boundary of the complex sub-manifold \(V \subset D - X\) if and only if Kohn-Rossi cohomology groups \(H^{p,q}_{KR}(X)\) are zeros for \(1 \leq q \leq n - 2\). So Yau’s result solved the classical Plateau problem for hypersurface type as \(n \geq 3\).

For \(n = 2\), i.e. \(X\) is a 3-dimensional CR manifold, the intrinsic smoothness criteria for the complex Plateau problem remains unsolved for over

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a quarter of a century even for the hypersurface type. The main difficulty is that the Kohn-Rossi cohomology groups are infinite dimensional in this case. Let $V$ be a complex variety with $X$ as its boundary, then the singularities of $V$ are surface singularities. In [Lu-Ya], Luk and Yau proved that if $X$ is a compact strongly pseudoconvex Calabi-Yau CR manifold of dimension 3 contained in the boundary of a strongly pseudoconvex bounded domain $D$ in $\mathbb{C}^N$ and the holomorphic De Rham cohomology $H^2_0(X)$ vanishes, then $X$ is a boundary of a complex variety $V$ in $D$ with boundary regularity and $V$ has only isolated singularities which are all Gorenstein surface singularities with vanishing $s$-invariant in the interior. Later, in [Du-Ya], Yau and the author introduced a new CR invariant $g^{(1,1)}(X)$ which is intrinsic in term of $X$. This new invariant gives a necessary and sufficient condition for the variety $V$ bounded by $X$ with the holomorphic De Rham cohomology $H^2_0(X) = 0$ being smooth.

From [Du-Ya], we know that $g^{(1,1)}(X)$ is related to the corresponding invariants of singularities in the variety which $X$ bounds. The main idea for solving the Plateau problem is to show the strict positivity of the invariant $g^{(1,1)}$ of each normal singularity. Yau conjectured that $g^{(1,1)} > 0$ for every isolated normal surface singularity (see [Du-Ga]).

**Conjecture:** For each isolated normal surface singularity, the invariant $g^{(1,1)}$ is strictly positive.

In [Du-Ya], we showed that $g^{(1,1)}(X) > 0$ for every isolated normal singularity with $\mathbb{C}^*$-action. It provides some evidence to make one believe the truth of the conjecture.

For non-hypersurface type, $n \geq 3$, Yau’s method doesn’t work because his method relies heavily on the Tyurina numbers of hypersurface singularities while they cannot be generalized for arbitrary singularities in an effective way. Moreover, numerical conditions on the boundary $X$ can hardly guarantee the normality of the singularities in the interior which is automatically satisfied for isolated hypersurface singularities though. So it is natural to ask the second best thing that when $X$ is a boundary of a variety $V$ which is smooth after normalization. Recently, Gao, Yau and the author generalized the invariant $g^{(1,1)}$ (resp. $g^{(1,1)}(X)$) to higher dimension as $g^{(\Lambda^1)}$ (resp. $g^{(\Lambda^1)}(X)$) and showed that if $g^{(\Lambda^1)}(X) = 0$, then the interior has at worst finite number of rational singularities ([D-G-Y]). In particular, if $X$ is Calabi–Yau of real dimension 5, then the vanishing of this invariant is equivalent to give the interior regularity up to normalization. The main method in [D-G-Y] relies Reid’s results on the existence of the explicit resolution of 3 dimensional Gorenstein terminal and canonical singularities and
the crucial point is to show the positivity of the numerical invariant $g^{(\Lambda_1)}$ for 3 dimensional isolated Gorenstein singularities. In [D-G-Y], we generalized Yau’s above conjecture to dimension 3 and proved it holds for 3 dimensional isolated Gorenstein singularities.

In this paper we generalize Yau’s conjecture to any dimension and prove that it is true for isolated local complete intersection singularities of dimension $n \geq 3$. More explicitly, we show that $g^{(\Lambda_1)}$ is strictly positive for every isolated local complete intersection singularity of dimension $n \geq 3$.

The crucial point for studying the numerical invariant $g^{(\Lambda_1)}$ is to understand holomorphic 1-forms on the resolution of isolated singularities. So we transfer the original complex Plateau problem to studying holomorphic 1-forms or holomorphic vector fields dually on singular varieties. Holomorphic vector fields on singular varieties are interesting in their own right (cf. [S-S1], [S-S2] or [B-S-S]). Our method is considering the vanishing order of holomorphic 1-forms along some special exceptional component on the resolution manifold of isolated singularities mainly. As a direct application of the strictly positivity of the numerical invariant $g^{(\Lambda_1)}$, we can deal with the complex Plateau problem of hypersurface type for any dimension $n \geq 3$ by checking only one numerical invariant.

**Main Theorem 1** Let $X$ be a strongly pseudoconvex compact CR manifold of real dimension $2n - 1 \geq 5$. Suppose that $X$ is contained in the boundary of a strongly pseudoconvex bounded domain $D$ in $\mathbb{C}^{n+1}$. Then $X$ is a boundary of the complex sub-manifold $V \subset D - X$ with boundary regularity if and only if $g^{(\Lambda_1)}(X) = 0$.

The crucial point for the above theorem is by generalizing Yau’s conjecture to higher dimension and proving it is true for isolated local complete intersection singularities of dimension $n \geq 3$.

**Conjecture:** For each isolated normal singularity of dimension $n$, the invariant $g^{(\Lambda_1)}$ is strictly positive.

**Main Theorem 2** Let $V$ be an $n$-dimensional Stein space with $0$ as its only local complete intersection singular point, then $g^{(\Lambda_1)} \geq 1$.

The proof of theorem relies on Shokurov’s minimal discrepancy conjecture ([Sh]). Up to now, we only know it holds for isolated local complete intersection singularities when the dimension of singularity is great
than 3 (see [E-M], [E-M-Y]). If Shokurov’s minimal discrepancy conjecture were true for isolated Gorenstein terminal singularities, then our Main Theorem 2 is also true for isolated Gorenstein singularities of dimension \( n \geq 3 \) and our Main Theorem 1 holds for Calabi-Yau CR manifold of non-hypersurface type.

In Section 2, we shall recall some basic definitions of a CR manifold. In Section 3, we survey some results of invariant of singularities and CR invariants introduced in [D-G-Y]. Moreover, we show the strictly positivity of the invariant \( g^{(\Lambda^1)} \) for isolated local complete intersection singularities of dimension \( n \geq 3 \). In Section 4, we solve our Main Theorem 1 in this paper.

2. STRONGLY PSEUDOCONVEX CR MANIFOLDS

We will recall the basic definition of strongly pseudoconvex CR manifolds. We recommend [Ta] or the preliminaries in [Du-Ya] for the details.

Definition 2.1. Let \( X \) be a connected orientable manifold of real dimension \( 2n - 1 \). A CR structure on \( X \) is a rank \( n - 1 \) subbundle \( S \) of \( \mathbb{C}T(X) \) (complexified tangent bundle) such that

1. \( S \cap \bar{S} = \{0\} \),
2. If \( L, L' \) are local sections of \( S \), then so is \( [L, L'] \).

Definition 2.2. Let \( L_1, \ldots, L_{n-1} \) be a local frame of the CR structure \( S \) on \( X \) so that \( \bar{L}_1, \ldots, \bar{L}_{n-1} \) is a local frame of \( \bar{S} \). Since \( S \oplus \bar{S} \) has complex codimension one in \( \mathbb{C}T(X) \), we may choose a local section \( N \) of \( \mathbb{C}T(X) \) such that \( L_1, \ldots, L_{n-1}, \bar{L}_1, \ldots, \bar{L}_{n-1}, N \) span \( \mathbb{C}T(X) \). We may assume that \( N \) is purely imaginary. Then the matrix \((c_{ij})\) defined by

\[
[L_i, \bar{L}_j] = \sum_k a_{i,j}^k L_k + \sum_k b_{i,j}^k \bar{L}_k + c_{i,j} N
\]

is Hermitian, and is called the Levi form of \( X \).

Remark 2.3. The number of non-zero eigenvalues and the absolute value of the signature of \((c_{ij})\) at each point are independent of the choice of \( L_1, \ldots, L_{n-1}, N \).

Definition 2.4. \( X \) is said to be strongly pseudoconvex if the Levi form is positive definite at each point of \( X \).

Definition 2.5. Let \( X \) be a CR manifold of real dimension \( 2n - 1 \). \( X \) is said to be Calabi-Yau if there exists a nowhere vanishing holomorphic section in \( \Gamma(\wedge^n \widehat{T}(X)^*) \), where \( \widehat{T}(X) \) is the holomorphic tangent bundle of \( X \).

Remark:
(1) Let $X$ be a CR manifold of real dimension $2n - 1$ in $\mathbb{C}^n$. Then $X$ is a Calabi-Yau CR manifold.

(2) Let $X$ be a strongly pseudoconvex CR manifold of real dimension $2n - 1$ contained in the boundary of bounded strongly pseudoconvex domain in $\mathbb{C}^{n+1}$. Then $X$ is a Calabi-Yau CR manifold.

3. Invariants of singularities and CR invariants

Let $X$ be a compact connected strongly pseudoconvex CR manifold of real dimension $2n - 1$, in the boundary of a bounded strongly pseudo-convex domain $D$ in $\mathbb{C}^N$. By a result of Harvey and Lawson, there is a unique complex variety $V$ in $\mathbb{C}^N$ such that the boundary of $V$ is $X$. Let $\pi : (M, A_1, \cdots, A_k) \to (V, 0_1, \cdots, 0_k)$ be a resolution of all the singularities with $A_i = \pi^{-1}(0_i)$, $1 \leq i \leq k$, as exceptional sets.

In order to solve the classical complex Plateau problem, we need to find some CR-invariant which can be calculated directly from the boundary $X$ and the vanishing of this invariant will give a smooth solution to complex Plateau problem.

For this purpose, we define a new sheaf $\bar{\Omega}_V^1$, a new invariant of surface singularities $g^{(1,1)}$ and a new CR invariant $g^{(1,1)}(X)$ in [Du-Ya]. Recently, we generalized them to higher dimension for dealing with general complex Plateau problem ([D-G-Y]).

**Definition 3.1.** Let $(V, 0)$ be a Stein germ of an $n$-dimensional analytic space with an isolated singularity at $0$. Suppose $\bar{\Omega}_V^1 := \theta_*\Omega^1_{V \setminus V_{\text{sing}}}$ where $\theta : V \setminus V_{\text{sing}} \to V$ is the inclusion map and $V_{\text{sing}}$ is the singular set of $V$. Define a sheaf of germs $\bar{\Omega}_V^{\Lambda p}$ by the sheaf associated with the presheaf

$$U \mapsto \langle \Lambda^p \Gamma(U, \bar{\Omega}_V^1) \rangle,$$

where $U$ is an open set of $V$ and $2 \leq p \leq n$.

**Lemma 3.2.** ([D-G-Y]) Let $V$ be an $n$-dimensional Stein space with $0$ as its only singular point in $\mathbb{C}^N$. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with $A$ as exceptional set. Then $\bar{\Omega}_V^{\Lambda p}$ is coherent and there is a short exact sequence

$$0 \to \bar{\Omega}_V^{\Lambda p} \to \bar{\Omega}_V^p \to \mathcal{G}^{(\Lambda p)} \to 0$$

(3.1)

where $\mathcal{G}^{(\Lambda p)}$ is a sheaf supported on the singular point of $V$. Let

$$G^{(\Lambda p)}(M \setminus A) := \Gamma(M \setminus A, \Omega_M^p) / \langle \Lambda^p \Gamma(M \setminus A, \Omega_M^p) \rangle,$$

(3.2)

then $\dim \mathcal{G}_0^{(\Lambda p)} = \dim G^{(\Lambda p)}(M \setminus A)$.

Thus, from Lemma 3.2, we can define a local invariant of a singularity which is independent of resolution.
**Definition 3.3.** Let $V$ be an $n$-dimensional Stein space with $0$ as its only singular point. Let $\pi : (M,A) \to (V,0)$ be a resolution of the singularity with $A$ as exceptional set. Let

$$g^{(A^p)}(0) := \dim \mathcal{G}^{(A^p)}_0 = \dim G^{(A^p)}(M \setminus A).$$
(3.3)

We will omit $0$ in $g^{(A^p)}(0)$ if there is no confusion from the context.

Let $\pi : (M,A_1,\cdots,A_k) \to (V,0_1,\cdots,0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \leq i \leq k$, as exceptional sets. In this case we still let

$$G^{(A^p)}(M \setminus A) := \Gamma(M \setminus A, \Omega^p_M) / \langle \Lambda^p \Gamma(M \setminus A, \Omega^1_M) \rangle,$$
where $A = \cup_i A_i$.

**Definition 3.4.** If $X$ is a compact connected strongly pseudoconvex CR manifold of real dimension $2n-1$, in the boundary of a bounded strongly pseudoconvex domain $D$ in $\mathbb{C}^N$. Suppose $V$ in $\mathbb{C}^N$ such that the boundary of $V$ is $X$. Let $\pi : (M,A_1,\cdots,A_k) \to (V,0_1,\cdots,0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \leq i \leq k$, as exceptional sets. Let

$$G^{(A^p)}(M \setminus A) := \Gamma(M \setminus A, \Omega^p_M) / \langle \Lambda^p \Gamma(M \setminus A, \Omega^1_M) \rangle$$
(3.4)

and

$$G^{(A^p)}(X) := \mathcal{S}^p(X) / \langle \Lambda^p \mathcal{S}^1(X) \rangle,$$
(3.5)

where $\mathcal{S}^q$ are holomorphic cross sections of $\wedge^q(\hat{T}(X)^\ast)$. Then we set

$$g^{(A^p)}(M \setminus A) := \dim G^{(A^p)}(M \setminus A),$$
(3.6)

$$g^{(A^p)}(X) := \dim G^{(A^p)}(X).$$
(3.7)

**Remark 3.5.** Form the definition (cf. Definition 3.7 and 3.8 in [Du-Ya]), $g^{(A^1)} = g^{(1,1)}$ and $g^{(A^2)}(X) = g^{(1,1)}(X)$.

**Lemma 3.6.** ([D-G-Y]) Let $X$ be a compact connected strongly pseudoconvex CR manifold of real dimension $2n-1$ which bounds a bounded strongly pseudoconvex variety $V$ with only isolated singularities $\{0_1, \cdots, 0_k\}$ in $\mathbb{C}^N$. Let $\pi : (M,A_1,\cdots,A_k) \to (V,0_1,\cdots,0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \leq i \leq k$, as exceptional sets. Then $g^{(A^p)}(X) = g^{(A^p)}(M \setminus A)$, where $A = \cup_i A_i$, $1 \leq i \leq k$.

By Lemma 3.6 and the proof of Lemma 3.2 we can get the following lemma easily.

**Lemma 3.7.** ([D-G-Y]) Let $X$ be a compact connected strongly pseudoconvex CR manifold of real dimension $2n-1$, which bounds a bounded strongly pseudoconvex variety $V$ with only isolated singularities $\{0_1, \cdots, 0_k\}$ in $\mathbb{C}^N$. Then $g^{(A^p)}(X) = \sum_i g^{(A^p)}(0_i) = \sum_i \dim \mathcal{S}^{(A^p)}(0_i)$. 

The following theorem is the crucial part for solving the classical complex Plateau problem of real dimension 3.

**Theorem 3.8.** ([Du-Ya]) Let \( V \) be a 2-dimensional Stein space with 0 as its only normal singular point with \( \mathbb{C}^* \)-action. Let \( \pi : (M,A) \rightarrow (V,0) \) be a minimal good resolution of the singularity with \( A \) as exceptional set, then \( g^{(\Lambda^1)} \geq 1 \).

**Remark 3.9.** We also show that \( g^{(\Lambda^1)} \) is strictly positive for rational singularities ([Du-Ga]) and minimal elliptic singularities ([Du-Ga2]) and exact 1 for rational double points, triple points and quotient singularities ([Du-Lu-Ya]).

Similarly, the following two theorems are the crucial results for solving the classical complex Plateau problem of real dimension 5.

**Theorem 3.10.** ([D-G-Y]) Let \( V \) be an \( n \)-dimensional Stein space with 0 as its only non-rational singular point, where \( n > 2 \), then \( g^{(\Lambda^1)} \geq 1 \).

**Theorem 3.11.** ([D-G-Y]) Let \( V \) be a 3-dimensional Stein space with 0 as its only normal Gorenstein singular point, then \( g^{(\Lambda^1)} \geq 1 \).

Now we will prove more general result which shows that generalized Yau’s conjecture mentioned in Section 1 is true for isolated local complete intersection singularities of dimension \( n \geq 3 \). The main idea in the proof is to consider the discrepancy along a special prime exceptional divisor.

**Definition 3.12.** Suppose that \( X \) is a normal variety such that its canonical class \( K_X \) is \( \mathbb{Q} \)-Cartier, and let \( f : Y \rightarrow X \) be a resolution of the singularities of \( X \). Then

\[
K_Y = f^*K_X + \sum a_iE_i,
\]

where the sum is over the irreducible exceptional divisors, and the \( a_i \)’s are rational numbers, called the discrepancies.

**Definition 3.13.** The minimal discrepancy of a variety \( X \) at 0, denoted by \( Md_0(X) \) (or \( Md(X) \) for short), is the minimum of all discrepancies of discrete valuations of \( \mathbb{C}(X) \), whose center on \( X \) is 0.

**Remark 3.14.** The minimal discrepancy only exists when \( X \) has log-canonical singularities (see, e.g. [C-K-M]). Whenever \( Md(X) \) exists it is at least \(-1\).

Shokurov conjecture that the minimal discrepancy is bounded above in term of the dimension of a variety.

**Conjecture 3.15.** (Shokurov [Sh]): The minimal discrepancy \( Md_0(X) \) of a variety \( X \) at 0 of dimension \( n \) is at most \( n - 1 \). Moreover, if \( Md_0(X) = n - 1 \), then \( (X,0) \) is nonsingular.
The conjecture was confirmed for surfaces ([Al]) and 3-dimensional singularities after the explicit classification ([Re]) of Gorenstein terminal 3-fold singularities with [E-M-Y] or [Ma]. If \( X \) is a local complete intersection, then the conjecture also holds (see [E-M] and [E-M-Y]).

The following theorem solves generalized Yau’s conjecture for isolated local complete intersection singularities and it is also the crucial point for solving the classical Plateau problem in the next section.

**Theorem 3.16.** Let \( V \) be a \( n \)-dimensional Stein space with \( 0 \) as its only isolated local complete intersection singular point, where \( n \geq 3 \). Then \( g^{\Lambda_1} \geq 1 \).

**Proof.** We only need to show that the result holds for rational singularities from Theorem 3.10.

Let \( \pi : M \to V \) be a resolution such that \( E = \cup E_i \) as the exceptional set of \( \pi \), where each \( E_i \) is the nonsingular irreducible component of dimension \( n-1 \) with normal crossings. It is well known that isolated local complete intersection singularities are Gorenstein and rational Gorenstein singularities are canonical (see [Ko-Mo] Corollary 5.24). Since canonical singularities are log terminal, by Hacon and McKernan’s result (cf. Corollary 1.5 in [H-M]), \( E \) is rational chain connected. So for each \( E_i \), \( H^0(E_i, \Omega^1_{E_i}) = 0 \) (cf. [Ko] Corollary 3.8).

We know that Shokurov’s minimal discrepancy conjecture holds for local complete intersection singularities (cf. [E-M], [E-M-Y]), so there exists \( s \in \Gamma(M, \Omega^s_M) \) such that \( \text{Ord}_F s \leq n-2 \), where \( F \) is some \( E_i \). Take a tubular neighborhood \( N \) of \( F \) such that \( N \subset M \). Consider the exact sequence ([E-V])

\[
0 \to \Omega^1_N(\log F)(-F) \to \Omega^1_N \to \Omega^1_F \to 0. \quad (3.8)
\]

By taking global sections we have

\[
0 \to \Gamma(N, \Omega^1_N(\log F)(-F)) \to \Gamma(N, \Omega^1_N) \to \Gamma(F, \Omega^1_F). \quad (3.9)
\]

Since \( H^0(F, \Omega^1_F) = 0 \),

\[
\Gamma(N, \Omega^1_N(\log F)(-F)) = \Gamma(N, \Omega^1_N) \quad (3.10)
\]

from (3.9).

Suppose \( \eta \in \Gamma(N, \Omega^1_N) \), then \( \eta \in \Gamma(N, \Omega^1_N(\log F)(-F)) \) by (3.10). Chose a point \( P \) in \( F \) which is a smooth point in \( E \). Let \((x_1, x_2, \ldots, x_n)\) be a co-ordinate system center at \( P \) such that \( F \) is given locally by \( x_1 = 0 \). Write \( \eta \) locally around \( P : \eta \overset{=}{=} f_1dx_1 + f_2x_1dx_2 + \cdots + f_ndx_n \), where \( f_1, f_2, \ldots, f_n \) are holomorphic functions and “\( = \)” means local equality around \( P \). So the vanishing order of any elements in \( \Lambda^g\Gamma(M, \Omega^1_M) \) along the irreducible exceptional set \( F \) is at least \( n-1 \) by noticing \( \Gamma(M, \Omega^1_M) \subseteq \Gamma(N, \Omega^1_N) \) under...
natural restriction. Therefore \( s \notin \Lambda^s \Gamma(M, \Omega^1_M) \). Because the singularity is rational,\[ g(\Lambda^1) = \dim \Gamma(M, \Omega^1_M)/\Lambda^n \Gamma(M, \Omega^1_M) \geq 1. \]

Q.E.D.

Remark 3.17. If Shokurov’s minimal discrepancy conjecture were true for isolated Gorenstein terminal singularities, then Theorem 3.16 is also true for isolated Gorenstein singularities of dimension \( n \geq 3 \). Finally, our main Theorem 4.6 holds for Calabi-Yau CR manifold of non-hypersurface type.

4. The classical complex Plateau problem

In 1981, Yau [Ya] solved the classical complex Plateau problem for the case \( n \geq 3 \).

Theorem 4.1. ([Ya]) Let \( X \) be a compact connected strongly pseudoconvex CR manifold of real dimension \( 2n - 1 \), \( n \geq 3 \), in the boundary of a bounded strongly pseudoconvex domain \( D \) in \( \mathbb{C}^{n+1} \). Then \( X \) is the boundary of a complex sub-manifold \( V \subset D - X \) if and only if Kohn–Rossi cohomology groups \( H^{p,q}_{KR}(X) \) are zeros for \( 1 \leq q \leq n - 2 \).

When \( n = 2 \), Yau and the author used CR invariant \( g(1,1)(X) \) to give the sufficient and necessary condition for the variety bounded by a Calabi-Yau CR manifold \( X \) being smooth if \( H^2_h(X) = 0 \) ([Du-Ya]).

Theorem 4.2. ([Du-Ya]) Let \( X \) be a strongly pseudoconvex compact Calabi-Yau CR manifold of dimension 3. Suppose that \( X \) is contained in the boundary of a strongly pseudoconvex bounded domain \( D \) in \( \mathbb{C}^N \) with \( H^2_h(X) = 0 \). Then \( X \) is the boundary of a complex sub-manifold (up to normalization) \( V \subset D - X \) with boundary regularity if and only if \( g(1,1)(X) = 0 \).

Corollary 4.3. ([Du-Ya]) Let \( X \) be a strongly pseudoconvex compact CR manifold of dimension 3. Suppose that \( X \) is contained in the boundary of a strongly pseudoconvex bounded domain \( D \) in \( \mathbb{C}^3 \) with \( H^2_h(X) = 0 \). Then \( X \) is the boundary of a complex sub-manifold \( V \subset D - X \) if and only if \( g(1,1)(X) = 0 \).

When \( X \) is a Calabi-Yau CR manifold of dimension 5 (\( n = 3 \)), we give the following necessary and sufficient condition for the variety bounded by \( X \) being smooth in [D-G-Y].

Theorem 4.4. ([D-G-Y]) Let \( X \) be a strongly pseudoconvex compact Calabi-Yau CR manifold of dimension 5. Suppose that \( X \) is contained in the boundary of a strongly pseudoconvex bounded domain \( D \) in \( \mathbb{C}^N \). Then \( X \) is the boundary of a complex sub-manifold (up to normalization) \( V \subset D - X \) with boundary regularity if and only if \( g(\Lambda^4)(X) = 0 \).
The main idea of the proof is based on the following theorem and Reid’s explicit resolutions of Gorenstein terminal and canonical singularities.

**Theorem 4.5.** ([D-G-Y]) Let $X$ be a strongly pseudoconvex compact CR manifold of dimension $2n - 1$, where $n > 2$. Suppose that $X$ is contained in the boundary of a strongly pseudoconvex bounded domain $D$ in $\mathbb{C}^n$. Then $X$ is the boundary of a variety $V \subset D - X$ with boundary regularity and the number of non-rational singularities is not greater than $g^{(n+1)}(X)$. In particular, if $g^{(n+1)}(X) = 0$, then $V$ has at worst finite number of rational singularities.

It is a wonderful idea that Yau related complex Plateau problem of hypersurface type to the Kohn-Rossi cohomology groups for $n \geq 3$ in 1981. To determine if $X$ is a boundary of a complex manifold under some conditions, one only needs to calculate a bunch of Kohn-Rossi cohomology groups. Next theorem shows that we can determine if $X$ is a boundary of a complex manifold by checking only one numerical invariant.

**Theorem 4.6.** Let $X$ be a strongly pseudoconvex compact CR manifold of real dimension $2n - 1 \geq 5$. Suppose that $X$ is contained in the boundary of a strongly pseudoconvex bounded domain $D$ in $\mathbb{C}^n$. Then $X$ is the boundary of a complex sub-manifold $V \subset D - X$ with boundary regularity if and only if $g^{(n+1)}(X) = 0$.

**Proof.** ($\Rightarrow$): Since $V$ is smooth, $g^{(n+1)}(X) = 0$ follows from Lemma 3.7.

($\Leftarrow$): It is well known that $X$ is the boundary of a variety $V$ in $D$ with boundary regularity ([Lu-Ya], [Ha-La2]) and the singularities of $V$ are hypersurface singularities. The the result follows easily from Theorem 3.16 and Lemma 3.7. Q.E.D.

**Remark 4.7.** In this case, $g^{(n+1)}(X) = 0$ is equivalent to vanishing of Kohn-Rossi cohomologies.

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