Wave propagation and its stability for a class of discrete diffusion systems

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Abstract. This paper is devoted to investigating the wave propagation and its stability for a class of two-component discrete diffusive systems. We first establish the existence of positive monotone monostable traveling wave fronts. Then, applying the techniques of weighted energy method and the comparison principle, we show that all solutions of the Cauchy problem for the discrete diffusive systems converge exponentially to the traveling wave fronts when the initial perturbations around the wave fronts lie in a suitable weighted Sobolev space. Our main results can be extended to more general discrete diffusive systems. We also apply them to the discrete epidemic model with the Holling-II-type and Richer-type effects.

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1. Introduction

In this paper, we study the wave propagation and its stability for a class of discrete diffusive systems. Such discrete systems arise in many applications, e.g., the pulse propagation through myelinated nerves [1], the motion of domain walls in semiconductor superlattices [3], the sliding of charge density waves [11], and so on. Among these models, one can see the spatial discrete effects play important roles. However, due to the special and poorly understood phenomena occurring in these systems, the mathematical study of spatially discrete models is more difficult than that spatially continuous models. Of particular phenomena is the pinning or propagation failure of wave fronts in spatially discrete equations. In past years, there was some significant progress on these subjects. We only illustrate some related works in the sequel.

Keener [19] studied the propagation failure of wave fronts in coupled FitzHugh–Nagumo systems of discrete excitable cells

\[
\begin{align*}
\frac{dv_{1,j}(t)}{dt} &= d[v_{1,j+1}(t) - 2v_{1,j}(t) + v_{1,j-1}(t)] + h(v_{1,j}(t), v_{2,j}(t)), \\
\frac{dv_{2,j}(t)}{dt} &= g(v_{1,j}(t), v_{2,j}(t)),
\end{align*}
\]

where the subscript \(j\) indicates the \(j\)th cell in a string of cells, \(v_{1,j}\) represents the membrane potential of the cell and \(v_{2,j}\) comprises additional variables (such as gating variables, chemical concentrations, etc.) necessary to the model. The constant \(d\) means the coupling coefficient. Especially, if (1.1) is in the absent of the recovery, that is, \(g \equiv 0\) and \(v_{2,j}\) is the constant independent of \(j\), then (1.1) can reduce to a simple but typical spatially discrete equation

\[
\frac{dv_j(t)}{dt} = d[v_{j+1}(t) - 2v_{j}(t) + v_{j-1}(t)] + f(v_{j}(t)).
\]

If the nonlinearity \(f(\cdot)\) is a bistable function (e.g., \(f(x) = x(x-a)(1-x)\)), Bell and Cosner [2] obtained the threshold properties, that is, conditions forcing non-convergence to zero of solutions as time approaches
phenomena of traveling wave fronts for more general monostable discrete equations, see, e.g., [6–8,10]. Following the work of [37], there have been extensive studies on the propagation of traveling wave fronts for the following two-component discrete diffusion system:

\[
\begin{align*}
\partial_t v_1(x,t) &= d_1 \mathcal{D}[v_1](x,t) + h(v_1(x,t), v_2(x,t)), \\
\partial_t v_2(x,t) &= d_2 \mathcal{D}[v_2](x,t) + g(v_1(x,t), v_2(x,t)),
\end{align*}
\]

where \( t > 0, x \in \mathbb{R}, d_i \geq 0, h(u,v), g(u,v) \in C^2(\mathbb{R}^2, \mathbb{R}) \) and

\[ \mathcal{D}[v_i](x,t) := v_i(x+1,t) - 2v_i(x,t) + v_i(x-1,t), \quad i = 1, 2. \]

System (1.3) can be considered as the continuum version of the lattice differential system

\[
\begin{align*}
v'_{1,j}(t) &= d_1 \mathcal{D}_j[v_1](t) + h(v_{1,j}(t), v_{2,j}(t)), \\
v'_{2,j}(t) &= d_2 \mathcal{D}_j[v_2](t) + g(v_{1,j}(t), v_{2,j}(t)),
\end{align*}
\]

where \( t > 0, j \in \mathbb{Z} \) and (1.4) is a spatial discrete version of the following reaction–diffusion system

\[
\begin{align*}
\partial_t v_1(x,t) &= d_1 \partial_{xx} v_1(x,t) + h(v_1(x,t), v_2(x,t)), \\
\partial_t v_2(x,t) &= d_2 \partial_{xx} v_2(x,t) + g(v_1(x,t), v_2(x,t)),
\end{align*}
\]

where \( t > 0, x \in \mathbb{R} \). Systems (1.5) with special kinds of nonlinearities arises from many biological, chemical models, and so on (see [4,20,28]). For example, the system

\[
\begin{align*}
\partial_t v_1(x,t) &= d_1 \partial_{xx} v_1(x,t) - \alpha v_1(x,t) + h_1(v_2(x,t)), \\
\partial_t v_2(x,t) &= d_2 \partial_{xx} v_2(x,t) - \beta v_2(x,t) + g_1(v_1(x,t)),
\end{align*}
\]

with \( \alpha, \beta > 0 \) describes the spread of an epidemic by oral-fecal transmission. Here, \( -\alpha v_1 \) means the natural death rate of the bacterial population; \( -\beta v_2 \) represents the natural diminishing rate of the infective population due to the finite mean duration of the infectious population. The nonlinearity \( h_1(v_2) \in C^2(\mathbb{R}, \mathbb{R}) \) is the contribution of the infective humans to the growth rate of the bacterial, while \( g_1(v_1) \in C^2(\mathbb{R}, \mathbb{R}) \) is the infection rate of the human population. System (1.6), Hsu and Yang [17] investigated the existence, uniqueness and asymptotic behavior of traveling waves for (1.6). More recently, using the monotone iteration scheme via an explicit construction of a pair of upper and lower solutions, the properties for the two-component spatially discrete competitive system

\[
\begin{align*}
v'_{1,j}(t) &= \mathcal{D}_j[v_1](t) + v_{1,j}(t)(1 - v_{1,j}(t) - b_2 v_{2,j}(t)), \\
v'_{2,j}(t) &= d \mathcal{D}_j[v_2](t) + rv_{2,j}(t)(1 - v_{2,j}(t) - b_1 v_{1,j}(t)),
\end{align*}
\]

for \( d > 0 \). In addition, if we replace the terms \( \mathcal{D}_j[v_1](t) \) and \( \mathcal{D}_j[v_2](t) \) of (1.7) by \( \mathcal{D}[v_1](t) \) and \( \mathcal{D}[v_2](t) \), respectively, one can see that the profile equations of the new system are the same with those of (1.7) (cf. Sect. 2). Therefore, the new system also admits traveling wave fronts.

It is known that traveling wave solutions of biological models always correspond to the distribution of species and dynamics of phenomena. Therefore, it is significant to see whether the traveling wave
solutions are stable or not. Motivated by [13,17,18], we will investigate the existence and stability of traveling wave fronts of system (1.3).

Recently, Hsu et al. [16] considered the existence of traveling wave solutions for the following lattice differential system:

\[ U_{i,j}'(t) = d_i D_j [U_i](t) + f_i(U_{1,j}, \ldots, U_{n,j}), \quad (1.8) \]

for \( 1 \leq i \leq n, \ j \in \mathbb{Z} \) and \( t \geq 0 \), where \( d_i > 0, \ U_{i,j} \in C^2(\mathbb{R}, \mathbb{R}) \) and \( f_i(\cdot) \in C(\mathbb{R}, \mathbb{R}) \). Suppose the nonlinearities \( f_i(\cdot) \) satisfy the following assumptions:

(A1) System (1.8) has two homogeneous equilibria \( 0 := (0, \ldots, 0) \) and \( \mathbf{E} := (e_1, \ldots, e_n) \) with each \( e_i > 0 \), i.e., \( f_i(0) = f_i(\mathbf{E}) = 0 \), for \( 1 \leq i \leq n \).

(A2) Assume that \( \partial f_i(u)/\partial u_k \geq 0 \) for all \( u \in [0, \mathbf{E}] \) with \( i \neq k, \ i, k = 1, \ldots, n \). Here, the closed rectangle \([0, \mathbf{E}]\) denotes the set \( \{u \in \mathbb{R}^n : 0 \leq u \leq \mathbf{E}\} \).

(A3) Each \( f_i(\cdot) \) is Lipschitz continuous on \([0, \mathbf{E}]\), and there exists a continuous function \( \mathbf{r} = (r_1, \ldots, r_n) : [0,1] \to [0, \mathbf{E}] \) with \( \mathbf{r}(0) = \mathbf{0}, \mathbf{r}(1) = \mathbf{E} \) such that each \( r_n \) is increasing and \( f_i(\mathbf{r}(\varepsilon)) > 0 \), for \( 1 \leq i \leq N \) and \( \varepsilon \in (0,1) \).

Then, the authors [16] applied the truncated method to derive the following existence result of traveling wave solutions for system (1.8).

**Theorem 1.1.** Assume that (A1)–(A3) hold. Suppose system (1.8) has no other equilibrium in the closed rectangle \([0, \mathbf{E}]\), then there exists \( c^* > 0 \) such that if \( c > c^* \) then system (1.8) has an increasing traveling wave solution connecting \( \mathbf{0} \) and \( \mathbf{E} \).

Since the profile equation of (1.3) can be considered a special form as that of (1.8), by Theorem 1.1, we can directly obtain the existence of traveling wave fronts of system (1.3). On the other hand, different to the assumption (A3), we can also derive the existence of traveling wave fronts for system (1.3) by using the monotone iteration method (see Theorem 2.1).

The stability of traveling wave fronts for reaction–diffusion equations with monostable nonlinearity has been extensively studied in past years, see [9,18,21,22,24,25,27,31–34] and reference therein. For an example, Guo and Zimmer [14] proved the global stability of traveling wave fronts for a spatially discrete equation by using a combination of the weighted energy method and the Green function technique. However, to the best of our knowledge, the stability of traveling wave solutions for multi-component discrete reaction–diffusion systems is less reported. Recently, by comparison principles, Hsu and Lin [15] established a framework to study the stability of traveling wave solutions of the general system (1.8). Unfortunately, due to different types of diffusion terms, their results cannot be applied to system (1.3). Motivated by these articles [14,15,18,25], we will apply the weighted energy method and comparison principle to prove the stability of traveling wave fronts for the two-component discrete system (1.3). More precisely, we establish the \( L^1_{w_1}, L^1 \) and \( L^2 \)-energy estimates for the perturbation system (see Theorem 2.2 and Sect. 4). Moreover, following the same proof arguments of the main theorem, we can extend the stability result to the continuum version of system (1.8) and more general discrete diffusive system. We also apply our main results to the discrete version of epidemic model (1.6).

### 2. Main results

A solution \((v_1, v_2)\) or \((v_{1,j}, v_{2,j})\) of system (1.3) or (1.4) is called a traveling wave solution if there exists a constant \( c > 0 \) and smooth functions \( \phi_1(\cdot) \) and \( \phi_2(\cdot) \) such that

\[
\begin{align*}
\text{or} & \quad v_1(x, t) = \phi_1(x + ct) \quad \text{and} \quad v_2(x, t) = \phi_2(x + ct) \\
& \quad v_{1,j}(t) = \phi_1(j + ct) \quad \text{and} \quad v_{2,j}(t) = \phi_2(j + ct). \\
\end{align*}

(2.1)
Here, $c$ means the wave speed and $\xi := x + ct$ or $j + ct$ represents the moving coordinate. Substituting the ansätze of (2.1) into (1.3) or (1.4), we can obtain the same profile equations:

$$
\begin{align*}
\begin{cases}
    c\phi'_1(\xi) &= d_1 D[\phi_1](\xi) + h(\phi_1(\xi), \phi_2(\xi)), \\
    c\phi'_2(\xi) &= d_2 D[\phi_2](\xi) + g(\phi_1(\xi), \phi_2(\xi)),
\end{cases}
\end{align*}
$$

(2.2)

where $\mathcal{D}[\phi](\xi) := \phi(\xi + 1) - 2\phi(\xi) + \phi(\xi - 1)$. Moreover, a traveling wave solution $(\phi_1, \phi_2)$ is called a traveling wave front if each $\phi_i$, $i = 1, 2$ is monotone.

To guarantee the existence of traveling wave solutions of (1.3), throughout this article, we assume the nonlinearities $h(\cdot, \cdot)$ and $g(\cdot, \cdot)$ satisfy the following assumptions.

\(\text{(H1)}\) System (1.3) has only two equilibria $0 := (0, 0)$ and $K := (K_1, K_2)$ for some $K_1, K_2 > 0$ in the first quadrant, i.e., $h(0) = g(0) = 0$ and $h(K) = g(K) = 0$.

\(\text{(H2)}\) $h(v) := \partial_v h(v) \geq 0$ and $g(v) := \partial_v g(v) \geq 0$, $\forall v \in I := [0, K]$.

\(\text{(H3)}\) $\alpha_i, \tilde{\alpha}_i < 0$, for $i = 1, 2$, $\alpha_1 \alpha_2 < \beta_1 \beta_2$ and $\tilde{\alpha}_1 \tilde{\alpha}_2 > \beta_1 \beta_2 > 0$, where

\[
\begin{align*}
    \alpha_1 &:= \partial_1 h(0), \quad \alpha_2 := \partial_2 g(0), \quad \beta_1 := \partial_2 h(0), \quad \beta_2 := \partial_1 g(0), \\
    \tilde{\alpha}_1 &:= \partial_1 h(K), \quad \tilde{\alpha}_2 := \partial_2 g(K), \quad \tilde{\beta}_1 := \partial_2 h(K), \quad \tilde{\beta}_2 := \partial_1 g(K).
\end{align*}
\]

Here, we remark that two vectors $(u_1, \ldots, u_n) \leq (v_1, \ldots, v_n)$ in $\mathbb{R}^n$ means $u_i \leq v_i$ for $i = 1, \ldots, n$. An interval of $\mathbb{R}^n$ is defined according to this order. Based on the above assumptions, our first goal is to find solutions of (2.2) satisfying the following conditions:

$$
\lim_{\xi \to -\infty} (\phi_1(\xi), \phi_2(\xi)) = 0 \quad \text{and} \quad \lim_{\xi \to \infty} (\phi_1(\xi), \phi_2(\xi)) = K.
$$

(2.3)

It’s obvious that the (H1) and (H2) imply the assumptions (A1) and (A2), respectively. By the result of Theorem 1.1, we immediately have the following existence result.

**Theorem 2.1.** Assume that $h(\cdot)$ and $g(\cdot)$ satisfy (H1), (H2) and (A3) or (H3). Then, there exists a constant $c^* > 0$ such that for any $c > c^*$, the system (1.3) admits an increasing traveling wave solution satisfying (2.3).

**Remark 2.1.** (1) The assumption (A3) in Theorem 1.1 could be verified for some specific systems, e.g., the Lotka–Volterra system, epidemic model and Nicholson’s Blowflies reaction–diffusion equation (see [16]). However, due to the assumption (A3), one can see from the proof of [16] that (1.3) may have increasing traveling wave solutions satisfying (2.3) when $c < c^*$.

(2) Since the nonlinearities of system (1.3) are not so general as (1.8), to avoid the verification of (A3), we may replace (A3) by the assumption (H3) in Theorem 2.1. Then, following the same ideas of our previous works [17,18], we can also obtain the same assertion of Theorem 2.1. In this situation, the constant $c^*$ is actually the threshold speed for the existence of increasing traveling wave solution satisfying (2.3). More precisely, the assumption (H3) can help us to investigate the characteristic roots of the linearized equations for the profile equations (2.2) at the equilibria $0$ and $K$. According to the local analysis of (2.2) at the equilibria $0$ and $K$ (see Sect. 3) and (H2), $c^*$ is actually the threshold speed such that the linearized equation of (2.2) at $0$ has positive eigenvalues. By the eigenvalues, we can construct a pair of supersolution and subsolution for (2.2), which are the same as those of [17]. Then, employing the monotone iteration scheme, system (1.3) admits traveling wave solutions satisfying (2.3). Since the proof arguments are the same as those of [17,18], we skip the details.

Next, we state the stability result of traveling wave fronts derived in Theorem 2.3. Before that, let us introduce the following notations.

- Let $I$ be an interval, especially $I = \mathbb{R}$, then we denote $L^2(I)$ by the space of the square integrable functions on $I$.
- The space $H^k(I)$ ($k \geq 0$) means the Sobolev space of the $L^2$-functions $f(x)$ defined on $I$ whose derivatives $\frac{d^j}{dx^j} f(i = 1, \ldots, k)$ also belong to $L^2(I)$.
Let’s write $L^2_\omega(I)$ and $W^{k,p}_\omega(I)$ by the weight $L^2$-space and weight Sobolev space with positive weighted function $\omega(x) : \mathbb{R} \to \mathbb{R}$, respectively. For any $f \in L^2_\omega(I)$ or $W^{k,p}_\omega(I)$, its norm is given (resp.) by

$$\|f\|_{L^2_\omega(I)} = \left(\int_I w(x)|f(x)|^2dx\right)^{1/2}, \text{ or } \|f\|_{W^{k,p}_\omega(I)} = \left(\sum_{i=0}^k \int_I \omega(x)|\frac{d^i}{dx^i}f(x)|^pdx\right)^{1/p}.$$ 

Furthermore, we set $H^k_\omega(I) := W^{k,2}_\omega(I)$.

1. In this section, we will investigate the characteristic roots of the linearized equations for the profile $f$. 
2. Let $(\lambda, c)$ be a traveling wave solution of (1.3) satisfying (2.3) with the wave speed $c > c^*$.

Motivated by the work of [18, 25], we will adopt the weighted energy method to establish the stability result. Then, by the comparison principle and Hölder inequality, we can obtain the following stability result.

**(Theorem 2.2)**. Assume that (H1)–(H4) hold and the initial data of (1.3) satisfy

$$0 \leq v_{i0}(x, 0) \leq K_i, \; \forall x \in \mathbb{R} \text{ and } v_{i0}(x, 0) - \phi_i(x) \in C(L^1_\omega(\mathbb{R}) \cap H^1(\mathbb{R}))$$

for $i = 1, 2$. Then, the solution of (1.3) with initial data $(v_{10}(x, 0), v_{20}(x, 0))$ uniquely exists, which satisfies $0 \leq v_i(x, t) \leq K_i, \; \forall (x, t) \in \mathbb{R} \times [0, +\infty)$ and

$$v_i(x, t) - \phi_i(x + ct) \in C([0, +\infty), \; L^1_\omega(\mathbb{R}) \cap H^1(\mathbb{R})), \; i = 1, 2.$$ 

Moreover, there exist positive constants $\mu$ and $C$ such that

$$\sup_{x \in \mathbb{R}} |v_i(x, t) - \phi_i(x + ct)| \leq Ce^{-\mu t}, \; \forall t \geq 0, \; i = 1, 2.$$ 

Moreover, following the same proof arguments of Theorem 2.2, we can generalize the above stability result to the continuum version of system (1.8) (see Sect. 5).

3. **Local analysis for (2.2)**

In this section, we will investigate the characteristic roots of the linearized equations for the profile equations (2.2) at the equilibria $0$ and $K$. From (2.2) and the notations in (H3), one can see the characteristic polynomials of (2.2) at $0$ and $K$ have the form (resp.)

$$P(\lambda, c) := [d_1(e^\lambda + e^{-\lambda} - 2) + \alpha_1 - c\lambda][d_2(e^\lambda + e^{-\lambda} - 2) + \alpha_2 - c\lambda] - \beta_1\beta_2, \text{ and }$$

$$P(\lambda, c) := [d_1(e^\lambda + e^{-\lambda} - 2) + \alpha_1 - c\lambda][d_2(e^\lambda + e^{-\lambda} - 2) + \alpha_2 - c\lambda] - \beta_1\beta_2.$$ 

Then, the threshold speed $c^*$ in Theorem 2.1 can be decided by the following lemma.

**(Lemma 3.1)**. Assume (H1)–(H3) hold.
(1) There exists a positive constant $c^*$ such that if $c > c^*$ then $P(\lambda, c) = 0$ has two positive real roots $\lambda_1(c) < \lambda_2(c) < \lambda_m^+$, i.e., $P(\lambda_1, c) = P(\lambda_2, c) = 0$, and $P(\lambda, c) > 0$ for any $\lambda \in (\lambda_1(c), \lambda_2(c))$. In addition, $\lim_{c \to c^*} \lambda_1(c) = \lim_{c \to c^*} \lambda_2(c) = \lambda^* > 0$, i.e., $P(\lambda^*, c^*) = 0$.

(2) For any $c > 0$, there exists a $\bar{\lambda}(c) > 0$ such that $P(\bar{\lambda}, c) = 0$. Moreover, if $\varepsilon > 0$ and small enough, we have $P(\bar{\lambda} - \varepsilon) < 0$.

**Proof.** Since the proof is similar to [17, Lemma 2.1], we only sketch the proof of part (1) by the following steps.

**Step 1.** Let’s set $f_i(\lambda, c) := d_i(e^{\lambda} + e^{-\lambda} - 2) + \alpha_i - c\lambda$ for $i = 1, 2$. It’s easy to see that there exist $\lambda^-_i(c) < 0 < \lambda^+_i(c), i = 1, 2$ such that $f_i(\lambda^+_i, c) = 0$,

$$f_i(\lambda, c) < 0, \quad \text{for } \lambda \in (\lambda^-_i, \lambda^+_i) \quad \text{and} \quad f_i(\lambda, c) > 0, \quad \text{for } \lambda \in \mathbb{R}\setminus[\lambda^-_i, \lambda^+_i], \quad i = 1, 2. \quad (3.1)$$

**Step 2.** Let’s set $\lambda^+_M := \max\{\lambda^+_1, \lambda^+_2\}$ and $\lambda^-_M := \min\{\lambda^-_1, \lambda^-_2\}$. If $c$ is large enough, we have $P(1/\sqrt{c}, c) > 0$ and $0 < 1/\sqrt{c} < \lambda^-_M < +\infty$.

**Step 3.** Let’s define

$$c^* := \inf\{c_0 > 0 : P(\lambda, c) \text{ has a root } \bar{\lambda} \in (0, \lambda^+_m) \text{ and } P'(\bar{\lambda}, c) > 0 \text{ for } c > c_0\}.$$ If $c > c^*$, then there exists some $\lambda_0 \in (0, \lambda^+_m)$ such that $P(\lambda_0, c) > 0$. Since $P(\pm\infty, c) = +\infty$,

$$P(0, c) = \alpha_1\alpha_2 - \beta_1\beta_2 < 0 \quad \text{and} \quad P(\lambda^+_2, c) = P(\lambda^-_1, c) = -\beta_1\beta_2 < 0.$$ Therefore, if $c$ is large enough, $P(\lambda, c)$ has four roots in the following intervals

$$(-\infty, 0), \quad (0, 1/\sqrt{c}), \quad (1/\sqrt{c}, \lambda^+_M) \quad \text{and} \quad (\lambda^+_M, +\infty).$$

**Step 4.** Since $P(\lambda, c) = f_1(\lambda, c)f_2(\lambda, c) - \beta_1\beta_2$, similar to the proof of [17, Lemma 2.1], $P(\lambda, c)$ has two positive real roots $\lambda_1(c) < \lambda_2(c)$ in $(0, \lambda^+_m)$ which satisfy the assertions. \hfill $\Box$

Here, we mention that the parameter $\gamma$ for the weighted function $\omega(\xi)$ will be chosen by $\gamma = \lambda_1(c) + \varepsilon$ (see Sect. 4), where $\varepsilon > 0$ and small enough. Then, it follows from (1) of Lemma 3.1 that $P(\gamma) > 0$.

Moreover, we recall the following lemma which plays an important role in obtaining the weighted energy estimate for the stability result.

**Lemma 3.2.** (See [17, Lemma 3.1].) Let $A = (a_{ij})$ be a two by two matrix such that $a_{ii} < 0, i = 1, 2$ and $a_{ij} > 0$ for $i \neq j$. Then, the system of the following inequalities

$$a_{11}x_1 + a_{12}x_2 < 0 \quad \text{(} > 0, \text{resp.)} \quad \text{and} \quad a_{21}x_1 + a_{22}x_2 < 0 \quad \text{(} > 0, \text{resp.)}$$

has a solution $(x_1, x_2)$ with $x_i > 0, i = 1, 2$ if and only if $\det A > 0 \quad \text{(} < 0, \text{resp.)}.$

4. Stability of traveling wave fronts

To prove the result of Theorem 2.2, we first give some auxiliary statements about the global solutions of the Cauchy problem for (1.3) and the comparison principle. By the standard energy method and continuity extension method (see, [26]), we have the following result.

**Proposition 4.1.** Assume that (H1)–(H3) hold, the initial data $(v_{10}(x, 0), v_{20}(x, 0))$ of system (1.3) satisfy the conditions of (2.4). Then, (1.3) admits a unique solution $(v_1(x, t), v_2(x, t))$ such that $0 \leq v_i(x, t) \leq K_i, \forall (x, t) \in \mathbb{R} \times [0, +\infty)$ and

$$v_i(x, t) - \phi_i(x + ct) \in C([0, +\infty), L^1(\mathbb{R}) \cap H^1(\mathbb{R})), \quad i = 1, 2.$$ Similar to the proofs of [23, Proposition 3] and [29, Lemma 3.2], we easily obtain the following comparison principle.
Proposition 4.2. (Comparison principle) Assume (H1)–(H3)). Let \((v_1^+(x,t), v_2^+(x,t))\) be the solutions of system (1.3) with the initial data \((v_{10}^+(x,0), v_{20}^+(x,0))\), respectively. If \((v_{10}^-(x,0), v_{20}^-(x,0)) \leq (v_{10}^+(x,0), v_{20}^+(x,0))\) for all \(x \in \mathbb{R}\), then it follows that
\[
(v_1^-(x,t), v_2^-(x,t)) \leq (v_1^+(x,t), v_2^+(x,t)), \quad \forall (x,t) \in \mathbb{R} \times \mathbb{R}_+.
\]

Hereinafter, we assume the initial data \((v_{10}(x,0), v_{20}(x,0))\) satisfy (2.4), and set
\[
v_{10}^-(x,0) \triangleq \min\{v_{10}(x,0), \phi_i(x)\}, \quad v_{10}^+(x,0) \triangleq \max\{v_{10}(x,0), \phi_i(x)\}, \quad \text{for } i = 1, 2.
\]

According to Proposition 4.1, we denote \((v_1^+(x,t), v_2^+(x,t))\) by the nonnegative solutions of system (1.3) with the initial data \((v_{10}^+(x,0), v_{20}^+(x,0))\). Then, it follows from Proposition 4.2 that
\[
\begin{align*}
0 \leq v_1^-(x,t) &\leq v_1^+(x,t), \quad 0 \leq v_2^-(x,t) \leq v_2^+(x,t), \\
\forall (x,t) &\in \mathbb{R} \times \mathbb{R}_+.
\end{align*}
\]
(4.1)

Therefore, the stability result of Theorem 2.2 follows provided that \((v_1^+(x,t), v_2^+(x,t))\) converges to \((\phi_1(\xi), \phi_2(\xi))\). For convenience, we denote
\[
V_i^\pm(\xi, t) \triangleq v_i^\pm(\xi - ct, t) - \phi_i(\xi), \quad i = 1, 2,
\]
with the corresponding initial data \(V_{10}^\pm(x,0)\) and \(V_{20}^\pm(x,0)\). Then, our goal is to show that there exist positive constants \(C, \mu\), such that
\[
\sup_{x \in \mathbb{R}} |V_1^\pm(\xi, t)|, \quad \sup_{x \in \mathbb{R}} |V_2^\pm(\xi, t)| \leq Ce^{-\mu t}, \quad \text{for } t \geq 0.
\]
(4.2)

In the sequel, we only prove the assertion of (4.2) for \((V_1^+, \xi, t), V_2^+(\xi, t))\), since the assertion for \((V_1^-(\xi, t), V_2^-(\xi, t))\) can be proved by the same way.

4.1. \(L_{\omega_1}^1\)-energy and \(L^1\)-energy estimates

For convenience, we simplify the notations \((V_1^+(\xi, t), V_2^+(\xi, t))\) by \((V_1(\xi, t), V_2(\xi, t))\), and denote \(X(\xi, t) := (V_1(\xi, t), V_2(\xi, t))^T\) and \(\Phi(\xi) := (\phi_1(\xi), \phi_2(\xi))^T\). By (1.3), (2.2) and elementary computations, \(V_1(\xi, t)\) and \(V_2(\xi, t)\) satisfy the system
\[
\begin{align*}
\partial_t V_1(\xi, t) + c\partial_\xi V_1(\xi, t) - d_1 D[V_1](\xi, t) &= h(\Phi(\xi) + X(\xi, t)) - h(\Phi(\xi)) \\
&= h(\Phi(\xi) + X(\xi, t)) - h(\Phi(\xi)) + h_{11}(\Phi) V_1^2 + h_{22}(\Phi) V_2^2 + 2 h_{12}(\Phi) V_1 V_2, \\
\partial_t V_2(\xi, t) + c\partial_\xi V_2(\xi, t) - d_2 D[V_2](\xi, t) &= g(\Phi(\xi) + X(\xi, t)) - g(\Phi(\xi)) \\
&= g(\Phi(\xi) + X(\xi, t)) - g(\Phi(\xi)) + g_{11}(\Phi) V_1^2 + g_{22}(\Phi) V_2^2 + 2 g_{12}(\Phi) V_1 V_2,
\end{align*}
\]
(4.3)

with initial data \(V_{10}(x,0) = v_{10}^+(x,0) - \phi_1(x)\) and \(V_{20}(x,0) = v_{20}^+(x,0) - \phi_2(x)\), where
\[
\Phi(\xi) \leq \tilde{\Phi}(\xi, t), \quad \tilde{\Psi}(\xi, t) \leq \Phi(\xi) + X(\xi, t).
\]

Obviously, \(V_{10}(x,0), V_{20}(x,0) \in C(L^1_\omega(\mathbb{R}) \cap H^1(\mathbb{R}))\) and Proposition 4.1 implies that the solution \(V_1(\xi, t), V_2(\xi, t) \in C(L^1_\omega(\mathbb{R}) \cap H^1(\mathbb{R}))\) for each \(t \in [0, +\infty)\). Furthermore, in order to establish the energy estimate, technically we need the sufficient regularity for the solution \(V_1(\xi, t)\) and \(V_2(\xi, t)\) of (4.3). To do this, the usual approach is applying the technique of mollification. Let us mollify the initial data as
\[
V_{10}^\varepsilon(x,0) = J_\varepsilon \ast V_{10}(x,0) = \int_\mathbb{R} J_\varepsilon(x-y) V_{10}(y,0) dy \in W^{2,1}_\omega(\mathbb{R}) \cap H^2(\mathbb{R}), \quad i = 1, 2,
\]
where \(J_\varepsilon(\cdot)\) is the usual mollifier. Let \(V_1^\varepsilon(\xi, t)\) and \(V_2^\varepsilon(\xi, t)\) be the solution to (4.3) with the above-mollified initial data. We then have \(V_1^\varepsilon(\xi, t) \in C([0, \infty), W^{2,1}_\omega(\mathbb{R}) \cap H^2(\mathbb{R}))\), \(i = 1, 2\). By taking the limit \(\varepsilon \to 0\),
we can obtain the corresponding energy estimate for original solution \( V_t(\xi, t) \) (cf. [25, Lemma 3.1]). For the sake of simplicity, in the sequel we formally use \( V_t(\xi, t) \) to establish the desired energy estimates.

**Lemma 4.1.** Assume that (H1)–(H4) hold. For any \( c > c^* \) and \( \gamma = \lambda_1(c) + \varepsilon \), where \( \varepsilon > 0 \) is small enough, there exist positive constants \( \mu \) and \( C \) such that

\[
\|V_1(\cdot, t)\|_{L_{t,1}} + \|V_2(\cdot, t)\|_{L_{t,1}} + \int_0^t e^{\mu(s-t)} (\|V_1(\cdot, s)\|_{L_{\omega,1}} + \|V_2(\cdot, s)\|_{L_{\omega,1}}) \, ds < C e^{-\mu t}
\]

for each \( t \geq 0 \), where \( \omega_1(\xi) = e^{-\gamma(\xi - \xi_0)} \).

**Proof.** According to (H4) and (4.3), we have

\[
\frac{\partial_t V_1(\xi, t)}{q} + c \partial_{\xi} V_1(\xi, t) - \partial_{\xi} D[V_1](\xi, t) - \nabla h(0, 0) X(\xi, t) \leq 0,
\]

\[
\frac{\partial_t V_2(\xi, t)}{q} + c \partial_{\xi} V_2(\xi, t) - \partial_{\xi} D[V_2](\xi, t) - \nabla g(0, 0) X(\xi, t) \leq 0.
\]

Multiplying (4.5) by \( e^{\mu t} \omega_1(\xi) \) for some \( \mu > 0 \) and integrating it over \( \mathbb{R} \times [0, t] \) with respect to \( \xi \) and \( t \), since \( V_i \in L^1_t(R) \cap H^1(R) \subseteq L^2_{\omega,1}(\mathbb{R}) \cap H^1(\mathbb{R}) \), \( \{e^{\mu t} \omega_1 V_i\}|_{\xi=-\infty}=0 \) \( (i = 1, 2) \), it follows that

\[
0 \geq \int_0^t \int_{-\infty}^\infty \left( \{e^{\mu \xi} \omega_1 V_1(\xi, s)\} + \{ce^{\mu \xi} \omega_1 V_1(\xi, s)\} \right) + e^{\mu \xi} \omega_1 V_1(\xi, s)(-\mu + c \gamma - \alpha_1)
\]

\[
- e_{\mu \xi} d_1 \omega_1 D[V_1](\xi, s) - e_{\mu \xi} \omega_1 V_1(\xi, s)) \right) \, d\xi \, ds
\]

\[
= e^{\mu \xi} \|V_1(\cdot, t)\|_{L_{\omega,1}(R)} - \|V_1(\cdot, 0)\|_{L_{\omega,1}(R)} - \int_0^t \int_{-\infty}^\infty e_{\mu \xi} \omega_1 V_1(\xi, s) \, d\xi \, ds
\]

\[
+ \int_0^t \int_{-\infty}^\infty e_{\mu \xi} \omega_1 V_1(\xi, s)[-\mu - f_1(\gamma, c)] \, d\xi \, ds.
\]

Note that \( f_i(\lambda, c), \ i = 1, 2 \) are defined in Lemma 3.1. Hence, we have

\[
e^{\mu \xi} \|V_1(\cdot, t)\|_{L_{\omega,1}(R)} + \int_0^t \int_{-\infty}^\infty e_{\mu \xi} \omega_1 V_1(\xi, s)[-\mu - f_1(\gamma, c)] \, d\xi \, ds
\]

\[
- \int_0^t \int_{-\infty}^\infty e_{\mu \xi} \omega_1 V_1(\xi, s) \, d\xi \, ds \leq C_1,
\]

for some constant \( C_1 > 0 \). Similarly, from the second equation of (4.5), it yields

\[
e^{\mu \xi} \|V_2(\cdot, t)\|_{L_{\omega,1}(R)} + \int_0^t \int_{-\infty}^\infty e_{\mu \xi} \omega_1 V_2(\xi, s)[-\mu - f_2(\gamma, c)] \, d\xi \, ds
\]

\[
- \int_0^t \int_{-\infty}^\infty e_{\mu \xi} \omega_1 V_2(\xi, s) \, d\xi \, ds \leq C_2,
\]

for some constant \( C_2 > 0 \). Since \( \gamma = \lambda_1 + \varepsilon \), from the proof of Lemma 3.1 we know that \( f_i(\gamma, c) < 0 \) for \( i = 1, 2 \). Then, it follows from Lemma 3.2 that there exist positive constants \( p \) and \( q \) satisfying the inequalities
\[ pf_1(\gamma, c) + q\beta_2 < 0 \quad \text{and} \quad p\beta_1 + qf_2(\gamma, c) < 0. \quad (4.8) \]

Multiplying (4.6) and (4.7) by \( p \) and \( q \) (resp.) and adding them, we can obtain

\[
p\|V_1(\cdot, t)\|_{L^1_\omega(\mathbb{R})} + q\|V_2(\cdot, t)\|_{L^1_\omega(\mathbb{R})} - [pf_1(\gamma, c) + q\beta_2 + p\mu] \int_0^t \int_{-\infty}^\infty e^{-\mu(t-s)}\omega V_1(\xi, s)d\xi ds
\]

\[ - [p\beta_1 + qf_2(\gamma, c) + q\mu] \int_0^t \int_{-\infty}^\infty e^{-\mu(t-s)}\omega V_2(\xi, s)d\xi ds \leq (pC_1 + qC_2)e^{-\mu t}. \]

Then, taking \( \mu = 0 \), it follows that

\[
p\|V_1(\cdot, t)\|_{L^1_\omega(\mathbb{R})} + q\|V_2(\cdot, t)\|_{L^1_\omega(\mathbb{R})} - [pf_1(\gamma, c) + q\beta_2] \int_0^t \|V_1(\cdot, s)\|_{L^1_\omega} ds
\]

\[ - [qf_2(\gamma, c) + p\beta_1] \int_0^t \|V_2(\cdot, s)\|_{L^1_\omega} ds \leq pC_1|\mu=0 + qC_2|\mu=0. \]

By taking \( \mu > 0 \) and small enough, it follows that

\[-pf_1(\gamma, c) - q\beta_2 - p\mu > 0 \quad \text{and} \quad -qf_2(\gamma, c) - p\beta_1 - q\mu > 0.\]

Then, we obtain the key energy estimate (4.4). This completes the proof. \( \square \)

Using the \( L^1_\omega \)-estimate of Lemma 4.1, we further have the following \( L^1 \)-estimate.

**Lemma 4.2.** Assume that (H1)–(H4) hold. Then, for any \( c > c^* \), there exist positive constants \( \mu, \xi_0 \) and \( C \) such that

\[ e^{\mu t}(\|V_1(\cdot, t)\|_{L^1(\mathbb{R})} + \|V_2(\cdot, t)\|_{L^1(\mathbb{R})}) \leq C, \forall t \geq 0. \quad (4.9) \]

**Proof.** Multiplying the equations (4.5) by \( e^{\mu t} \) and integrating it over \( \mathbb{R} \times [0, t] \) with respect to \( \xi \) and \( t \), since \( V_i \in L^1_\omega(\mathbb{R}) \cap H^1(\mathbb{R}), \{e^{\mu t}V_i\}_{\xi=-\infty}^{\xi=+\infty} = 0 \) (i = 1, 2), we can obtain

\[
0 \geq \int_0^t \int_{-\infty}^\infty \left( \{e^{\mu s}V_1(\xi, s)\}_s - \mu e^{\mu s}V_1(\xi, s) + \{ce^{\mu s}V_1(\xi, s)\}_\xi - e^{\mu s}[d_1D[V_1](\xi, s)
\]

\[ + h_1(\Phi(\xi))V_1(\xi, s) + h_2(\Phi(\xi))V_2(\xi, s)] \right) d\xi ds
\]

\[ = e^{\mu t}\|V_1(\cdot, t)\|_{L^1(\mathbb{R})} - \|V_1(\cdot, 0)\|_{L^1(\mathbb{R})} + \int_0^t \int_{-\infty}^\infty e^{\mu s}(F_1(\xi)V_1(\xi, s) + F_2(\xi)V_2(\xi, s))d\xi ds, \quad (4.10)\]

where \( F_1(\xi) := -\mu - h_1(\Phi(\xi)) \) and \( F_2(\xi) := -h_2(\Phi(\xi)). \) Since \( \omega_1(\xi) \geq 1 \) for \( \xi \leq \xi_0 \), by Lemma 4.1, we can obtain

\[ \left| \int_0^t \int_{-\infty}^{\xi_0} e^{\mu s}(F_1(\xi)V_1(\xi, s) + F_2(\xi)V_2(\xi, s))d\xi ds \right|
\]

\[ \leq C_3 \int_0^t e^{\mu s}(\|V_1(\cdot, s)\|_{L^1_{\omega_1}(-\infty, \xi_0)} + \|V_2(\cdot, s)\|_{L^1_{\omega_1}(-\infty, \xi_0)}) ds \leq C_4, \quad (4.11)\]
for some positive constants $C_3$ and $C_4$. Then, it follows from (4.10) and (4.11) that

$$e^{\mu t}\|V_1(\cdot, t)\|_{L^1(\mathbb{R})} + \int_0^t \int_{\xi_0}^\infty e^{\mu s}(F_1(\xi)V_1(\xi, s) + F_2(\xi)V_2(\xi, s)) \, d\xi ds \leq C_5,$$  

(4.12)

for some constant $C_5 > 0$. Similarly, there exists a constant $C_6 > 0$ such that

$$e^{\mu t}\|V_2(\cdot, t)\|_{L^1(\mathbb{R})} + \int_0^t \int_{\xi_0}^\infty e^{\mu s}(G_1(\xi)V_1(\xi, s) + G_2(\xi)V_2(\xi, s)) \, d\xi ds \leq C_6,$$  

(4.13)

where $G_1(\xi) := -g_1(\Phi(\xi))$ and $G_2(\xi) := -\mu - g_2(\Phi(\xi))$. Summing up (4.12) and (4.13), there exists a constant $C > 0$ such that

$$\|V_1(\cdot, t)\|_{L^1(\mathbb{R})} + \|V_2(\cdot, t)\|_{L^1(\mathbb{R})} + \int_0^t \int_{\xi_0}^\infty e^{-\mu(t-s)} [(F_1(\xi) + G_1(\xi))V_1(\xi, s) + (F_2(\xi) + G_2(\xi))V_2(\xi, s)] \, d\xi ds \leq Ce^{-\mu t}.$$  

(4.14)

By the assumption (H4), we have

$$\lim_{\xi \to -\infty} (F_1(\xi) + G_1(\xi)) = -h_1(K) - g_1(K) = -\alpha_1 - \bar{\beta}_2 > 0,$$

$$\lim_{\xi \to -\infty} (F_2(\xi) + G_2(\xi)) = -h_2(K) - g_2(K) = -\bar{\beta}_1 - \alpha_2 > 0.$$

Therefore, choosing $\xi_0$ large enough and $\mu$ small enough, we have $F_1(\xi) + G_1(\xi), F_2(\xi) + G_2(\xi) > 0$ for $\xi \geq \xi_0$. Hence, the assertion of this lemma follows from (4.14).

\[\Box\]

4.2. $L^2$-energy estimate

Based on the $L^1_{\omega_1}$-estimate of Lemma 4.1, we further have the following $L^2$-estimate.

**Lemma 4.3.** Assume that (H1)–(H4) hold. For any $c > c^*$, there exist positive constants $\xi_0$ (for the weight function $w(\xi)$) and $C$ such that

$$\|V_1(\cdot, t)\|_{L^2(\mathbb{R})}^2 + \|V_2(\cdot, t)\|_{L^2(\mathbb{R})}^2 \leq C,$$  

(4.15)

for all $t \geq 0$.

**Proof.** Let’s multiply the equations (4.5) by $V_i(\xi, t)$ and integrating it over $\mathbb{R} \times [0, t]$ with respect to $\xi$ and $t$. Since $V_i \in L^1_{\omega_1}(\mathbb{R}) \cap H^1(\mathbb{R})$, $\{\frac{1}{2} c V_i^2\}_{i=1}^\infty = 0$ $(i = 1, 2)$. Thus, we can obtain that

$$0 \geq \int_0^t \int_{-\infty}^\infty \left( \frac{1}{2} V_1^2(\xi, s) \right)_{s} + \frac{1}{2} c V_1^2(\xi, s) - V_1(\xi, s)[d_1 D[V_1](\xi, s)$$

$$+ h_1(\Phi(\xi))V_1(\xi, s) + h_2(\Phi(\xi))V_2(\xi, s))] \, d\xi ds$$

$$= \frac{1}{2} \left[ \|V_1(\cdot, t)\|_{L^2(\mathbb{R})}^2 - \|V_1(\cdot, 0)\|_{L^2(\mathbb{R})}^2 \right]$$

$$+ \int_0^t \int_{-\infty}^{\infty} \left( - h_1(\Phi(\xi))V_1^2(\xi, s) - h_2(\Phi(\xi))V_1((\xi, s)V_2(\xi, s)) \right) \, d\xi ds$$

$$\geq \frac{1}{2} \left[ \|V_1(\cdot, t)\|_{L^2(\mathbb{R})}^2 - \|V_1(\cdot, 0)\|_{L^2(\mathbb{R})}^2 \right] + \int_0^t \int_{-\infty}^{\infty} (Q_1(\xi)V_1^2(\xi, s) + Q_2(\xi)V_2^2(\xi, s)) \, d\xi ds,$$
where \( Q_1(\xi) := -h_1(\Phi(\xi)) - \frac{1}{2} h_2(\Phi(\xi)) \) and \( Q_2(\xi) := -\frac{1}{2} h_2(\Phi(\xi)) \). Since \( \omega_1(\xi) \geq 1 \) for \( \xi \leq \xi_0 \) and \( 0 \leq V_1(\xi, t) \leq K_1 \), Lemma 4.2 implies that

\[
\int_{-\infty}^{\xi_0} V_1^2(\xi, t) d\xi \leq K_1 \int_{-\infty}^{\xi_0} \omega(\xi) V_1(\xi, t) d\xi \leq K_1 \| V_1(\cdot, t) \|_{L^2_1(\mathbb{R})} \leq K_1 C e^{-\mu t}. \tag{4.16}
\]

Similarly, we can obtain

\[
\int_{-\infty}^{\xi_0} V_2^2(\xi, t) d\xi \leq K_2 C e^{-\mu t}. \tag{4.17}
\]

Then, it follows that

\[
\left| \int_0^t \int_{-\infty}^{\xi_0} e^{\mu s} (Q_1(\xi)) V_1^2(\xi, s) + Q_2(\xi) V_2^2(\xi, s) d\xi ds \right|
\leq C_7 \int_0^t \int_{-\infty}^{\xi_0} V_1^2(\xi, s) + V_2^2(\xi, s) d\xi ds \leq C_8, \tag{4.18}
\]

for some positive constants \( C_7 \) and \( C_8 \). Moreover, we have

\[
\| V_1(\cdot, t) \|_{L^2(\mathbb{R})}^2 + 2 \int_0^t \int_{\xi_0}^{\xi} (Q_1(\xi)) V_1^2(\xi, s) + Q_2(\xi) V_2^2(\xi, s) d\xi ds \leq C_9, \tag{4.19}
\]

for some positive constants \( C_9 \). Similarly, there exists \( C_{10} > 0 \) such that

\[
\| V_2(\cdot, t) \|_{L^2(\mathbb{R})}^2 + 2 \int_0^t \int_{\xi_0}^{\xi} (P_1(\xi)) V_1^2(\xi, s) + P_2(\xi) V_2^2(\xi, s) d\xi ds \leq C_{10}, \tag{4.20}
\]

where \( P_1(\xi) := -\frac{1}{2} g_1(\Phi(\xi)) \) and \( P_2(\xi) := -\frac{1}{2} g_1(\Phi(\xi)) - g_2(\Phi(\xi)) \). Summing up (4.19) and (4.20), it follows

\[
\| V_1(\cdot, t) \|_{L^2(\mathbb{R})}^2 + \| V_2(\cdot, t) \|_{L^2(\mathbb{R})}^2 + 2 \int_0^t \int_{\xi_0}^{\xi} ((P_1(\xi) + Q_1(\xi)) V_1^2(\xi, s) \\
+ (P_2(\xi) + Q_2(\xi)) V_2^2(\xi, s)) d\xi ds \leq C_{11}. \tag{4.21}
\]

for some \( C_{11} > 0 \). By the assumption (H4), we have

\[
\lim_{\xi \to \infty} (Q_1(\xi) + P_1(\xi)) = -h_1(K) - \frac{1}{2} h_2(K) - \frac{1}{2} g_1(K) = -\frac{1}{2} (2\bar{\alpha}_1 + \bar{\beta}_2 + \bar{\beta}_1) > 0,
\]

\[
\lim_{\xi \to \infty} (Q_2(\xi) + P_2(\xi)) = -\frac{1}{2} h_2(K) - \frac{1}{2} g_1(K) + g_2(K) = -\frac{1}{2} (\bar{\beta}_1 + \bar{\beta}_2 + 2\bar{\alpha}_2) > 0.
\]

Therefore, choosing \( \xi_0 \) large enough, we have \( Q_1(\xi) + P_1(\xi), Q_2(\xi) + P_2(\xi) > 0 \) for \( \xi \geq \xi_0 \). Then, the estimate (4.15) follows from (4.21). The proof is complete. \( \square \)

By the same procedure, we can also obtain the \( L^2 \)-estimate for the derivatives \( \partial_\xi V_1(\cdot, t) \) and \( \partial_\xi V_2(\cdot, t) \). Indeed, differentiating the system (4.5) with respect to \( \xi \), we can obtain

\[
\begin{align*}
\partial_\xi V_1(\xi, t) + c \partial_{\xi\xi} V_1(\xi, t) - d_1D[\partial_\xi V_1](\xi, t) - \nabla h(X(\xi, t) + \Phi(\xi)) \partial X_\xi(\xi, t) & \leq 0, \\
\partial_\xi V_2(\xi, t) + c \partial_{\xi\xi} V_2(\xi, t) - d_2D[\partial_\xi V_2](\xi, t) - \nabla g(X(\xi, t) + \Phi(\xi)) \partial X_\xi(\xi, t) & \leq 0.
\end{align*}
\]
In this section, we will generalize the result of Theorem 2.2 to the following discrete system

\[ \partial_t U_i(x, t) = d_i D[U_i](x, t) + f_i(U_1(x, t), \ldots, U_n(x, t)), \quad \text{for } i = 1, \ldots, n. \]  

(5.1)

We assume that the conditions (A1)–(A3) hold for (5.1). Since the profile equations of (5.1) are the same with those of (1.8), we can have the existence result of traveling wave solutions as the statement of Theorem 1.1. According to Sect. 4, we know that the results of Lemmas 3.1–3.2 are significant in proving the estimations of Lemmas 4.1–4.4. Therefore, to obtain the stability of traveling wave solutions of (5.1), we have to generalize the results of Lemma 3.1–3.2.

5. Extension to general discrete diffusive system

In this section, we will generalize the result of Theorem 2.2 to the following discrete system

\[ \partial_t U_i(x, t) = d_i D[U_i](x, t) + f_i(U_1(x, t), \ldots, U_n(x, t)), \quad \text{for } i = 1, \ldots, n. \]  

(5.1)

We assume that the conditions (A1)–(A3) hold for (5.1). Since the profile equations of (5.1) are the same with those of (1.8), we can have the existence result of traveling wave solutions as the statement of Theorem 1.1. According to Sect. 4, we know that the results of Lemmas 3.1–3.2 are significant in proving the estimations of Lemmas 4.1–4.4. Therefore, to obtain the stability of traveling wave solutions of (5.1), we have to generalize the results of Lemma 3.1–3.2.
Recently, Hsu and Yang [5] generalized the statement of Lemma 3.2 to more general cases. Before to cite their results, we first introduce some notations given in [5]. Let \( A = (a_{i,j}) \in M_{n \times n}(\mathbb{R}) \). Given any increasing subsequence \( \{p_i\}_{i=1}^n \) of \( \{1, \ldots, n\} \), let us define \( A(p_1, p_2, \ldots, p_n) \) by the submatrix that lies in the rows and columns of \( A \) which are indexed by \( \{p_1, p_2, \ldots, p_n\} \). In addition, we denote \( A(1 : k, j) = A(1, \ldots, k, j) \) for \( 1 \leq k < j \leq n \).

Using the above notations, Hsu and Yang [5] recently proved the following result.

**Lemma 5.1.** Let \( A = (a_{i,j}) \in M_{n \times n}(\mathbb{R}) \) with \( a_{i,j} \geq 0 \) for \( i \neq j \) and \( a_{i,i} < 0 \) for \( 1 \leq i \leq n \). Then, the inequalities \( Ax < 0 \) has a positive solution \( x \in \mathbb{R}^n \) if and only if the principal minors of \( A \) satisfy

\[
(-1)^{k-1} \det A(1 : k, j) > 0, \quad \text{for } 1 \leq k < j \leq n. \tag{5.2}
\]

Note that Lemma 3.2 is a special case of the above lemma with \( n = 2 \). Now we consider the profile equation for system (5.1), that is

\[
co_1(\xi) = d_iD[\phi_i(\xi)] + f_i(\phi_1(\xi), \ldots, \phi_n(\xi)), \quad i = 1, \ldots, n, \tag{5.3}
\]

where \( \xi = x + ct \) and \( U_i(x, t) = \phi_i(x + ct) \) for \( i = 1, \ldots, n \). Let’s set

\[
\alpha_{i,j} := \partial f_i(0)/\partial u_j, \delta_{i,j} := \alpha_{i,j} \quad \text{if } i \neq j, \quad \text{and} \quad \delta_{i,i} := \delta_{i,j}(\lambda, c) := d_i(e^\lambda + e^{-\lambda} - 2) - c\lambda + \alpha_{i,i},
\]

for \( i, j = 1, \ldots, n \). Then, the characteristic polynomial of (5.3) at 0 has the form

\[
P(\lambda, c) = \det J(\lambda, c) := \det [\delta_{i,j}].
\]

**Lemma 5.2.** There exist \( c_* > 0 \) and \( 0 < \lambda_1(c) < \lambda_2(c) \) such that \( \delta_{i,i} < 0 \) for all \( \lambda \in (\lambda_1(c), \lambda_2(c)) \), \( i = 1, \ldots, n \), provided that \( c > c_* \).

**Proof.** Let’s define \( \alpha_M := \max\{\alpha_{1,1}, \ldots, \alpha_{n,n}\} \), \( d_M := \{d_1, \ldots, d_n\} \) and

\[
c_* := \min_{\lambda > 0} \frac{d_M(e^\lambda + e^{-\lambda} - 2) + \alpha_M}{\lambda}, \quad i = 1, \ldots, n. \tag{5.4}
\]

If \( c > c_* \) there exist \( 0 < \lambda_1(c) < \lambda_2(c) \) such that \( c\lambda > d_M(e^\lambda + e^{-\lambda} - 2) + \alpha_M \), for \( \lambda \in (\lambda_1(c), \lambda_2(c)) \).

Then, for any \( i = 1, \ldots, n \), we have

\[
\delta_{i,i} < d_M(e^\lambda + e^{-\lambda} - 2) - c\lambda + \alpha_M < 0, \quad \text{for } \lambda \in (\lambda_1(c), \lambda_2(c)).
\]

The proof is complete.

As a consequence of Lemmas 5.1 and 5.2, we immediately have the following result.

**Lemma 5.3.** Assume all \( \delta_{i,j} \geq 0 \) with \( i \neq j \), \( c > c_* \) and \( \lambda \in (\lambda_1(c), \lambda_2(c)) \). Then, there exists a vector \( v = (v_1, \ldots, v_n)^T \in \mathbb{R}^n \) with all \( v_i > 0 \) such that \( J(\lambda, c)v < 0 \) if and only if

\[
(-1)^{k-1} \det J(1 : k, j)(\lambda, c) > 0, \quad \text{for } 1 \leq k < j \leq n. \tag{5.5}
\]

Furthermore, we generalize (H4) by the following assumptions

(A4) \( \partial_{u_j} f_i(u) \leq 0, \forall u \in I, i, j, k = 1, \ldots, n \); and

\[
\sum_{i=1}^n \bar{\alpha}_{i,j} < 0 \quad \text{and} \quad 2\bar{\alpha}_{j,j} + \sum_{i \neq j, i=1}^n \bar{\alpha}_{i,k} < 0, \quad j = 1, \ldots, n, \tag{5.6}
\]

where \( \bar{\alpha}_{i,j} := \partial f_i(K)/\partial u_j, i, j = 1, \ldots, n \).
Note that (H4) is a special case of (A4) with \( n = 2 \). The condition (5.5) with \( n = 2 \) is equivalent to Lemma 3.2. According to the above lemmas, we assume (A1)–(A4), (5.5) hold and \( c > \max\{c^*, c_\ast\} \). Then, we can also obtain the estimations of Lemmas 4.1–4.4. More precisely, similar to the previous notations, let’s write \((V_1^+ (\xi, t), \ldots, V_n^+ (\xi, t))\) by \((V_1 (\xi, t), \ldots, V_n (\xi, t))\), and denote \( X (\xi, t) := (V_1 (\xi, t), \ldots, V_n (\xi, t))^T \) and \( \Phi (\xi) := (\phi_1 (\xi), \ldots, \phi_n (\xi))^T \). By elementary computations, (4.3) is generalized to the following system
\[
\partial_t V_i (\xi, t) + c \partial_\xi V_i (\xi, t) - d_i D(V_i)(\xi, t) = f_i (\Phi (\xi) + X (\xi, t)) - f_i (\Phi (\xi)) = \nabla f_i (\Phi) X (\xi, t) + \frac{1}{2} \sum_{j,k=1,2} \frac{\partial^2 f_i (\Phi)}{\partial u_j u_k} V_j V_k,
\]
for \( i = 1, \ldots, n \), where \( \Phi (\xi) \leq \Phi_i (\xi, t) \leq \Phi (\xi) + X (\xi, t) \). Let’s replace the parameters \( \gamma \) and \( (p, q) \) in the proof of Lemma 4.1 by \( \lambda \) and \( v \) of Lemma 5.3, respectively. Then, (4.8) yields to \( J(\lambda, c)v = |\delta_{i,j}|v < 0 \). Then, the proof of Lemma 4.1 also true. In addition, it is easy to see that the proofs of Lemmas 4.2–4.4 also hold under the assumption (A4). Hence, we have the following stability result.

**Theorem 5.1.** Assume (A1)–(A4), (5.5) hold and \( c > \max\{c^*, c_\ast\} \). System (5.1) admits a traveling wave solution connecting \( 0 \) and \( K \), which is exponential stable in the same sense as that of Theorem 2.2.

### 6. Applications

Let us consider the discrete version of epidemic model (1.6), that is

\[
\begin{align*}
\partial_t v_1 (x, t) &= d_1 D[v_1](x, t) - a_1 v_1 (x, t) + \bar{h}(v_2 (x, t)), \\
\partial_t v_2 (x, t) &= d_2 D[v_2](x, t) - a_2 v_2 (x, t) + \bar{g}(v_1 (x, t)).
\end{align*}
\]

(6.1)

where \( t > 0, x \in \mathbb{R} \). According to [17], we also assume the nonlinearities \( \bar{h} (\cdot) \) and \( \bar{g} (\cdot) \) satisfy the following assumptions:

- **(B1)** \( \bar{h}, \bar{g} \in C^2(\mathbb{R}^+, \mathbb{R}^+) \), \( \bar{h}(0) = \bar{g}(0) = 0 \), \( K_2 = \bar{g}(K_1)/a_2, \bar{h}(\bar{g}(K_1)/a_2) = a_1 K_1 \) and \( \bar{h}(\bar{g}(u)/a_2) > a_1 u \) for \( u \in (0, K_1) \), where \( K_1 \) is a positive constant.
- **(B2)** \( \bar{h}'(0)\bar{g}'(0) > a_1 a_2 \).
- **(B3)** \( \bar{h}''(u) \leq 0, \bar{h}'(u) \geq 0 \) for all \( v \in [0, K_2] \) and \( \bar{g}''(u) \leq 0, \bar{g}'(u) \geq 0 \) for all \( u \in [0, K_1] \).
- **(B4)** \( \min\{a_1, a_2\} > \max\{\bar{g}'(K_1), \bar{h}'(K_2)\} \).

It’s clear that (6.1) has two equilibria (0, 0) and \((K_1, K_2)\). Under assumptions (B1)–(B3), the existence of traveling wave solutions for system (6.1) connecting (0, 0) and \((K_1, K_2)\) was proved by Hsu and Yang [17]. Moreover, we can rewrite (6.1) in the form of (1.3) by setting
\[
\begin{align*}
\bar{h}(v_1 (x, t), v_2 (x, t)) := -a_1 v_1 (x, t) + \bar{h}(v_2 (x, t)), \\
\bar{g}(v_1 (x, t), v_2 (x, t)) := -a_2 v_2 (x, t) + \bar{g}(v_1 (x, t)).
\end{align*}
\]

Obviously, the assumptions (B1)–(B4) imply that the conditions (H1)–(H4) hold. Therefore, we can obtain same assertion of Theorems 2.1 and 2.2 for system (6.1).

Next, we illustrate some examples for \( \bar{h} (x) \) and \( \bar{g} (x) \) which satisfy the assumptions (B1)–(B4).

**Example 6.1.** Assume the Holling-II-type functions
\[
\bar{h}(x) = \frac{\alpha_1 x}{\beta_1 + \gamma_1 x} \quad \text{and} \quad \bar{g}(x) = \frac{\alpha_2 x}{\beta_2 + \gamma_2 x},
\]
(6.2)
where $\alpha_i, \beta_i, \gamma_i, i = 1, 2$ are positive constants. Then, $\bar{h}(0) = \bar{g}(0) = 0$. Furthermore, elementary computations imply that

\[
(K_1, K_2) = \left(\frac{\alpha_1 \alpha_2 - a_1 a_2 \beta_1 \beta_2}{a_1 (a_2 \bar{\beta}_1 + \alpha_2 \gamma_1)}, \frac{\alpha_1 \alpha_2 - a_1 a_2 \beta_1 \beta_2}{a_2 (a_1 \beta_2 \gamma_1 + \alpha_1 \gamma_2)}\right), \quad \bar{h}'(0) = \frac{\alpha_1}{\beta_1}, \quad \bar{g}'(0) = \frac{\alpha_2}{\beta_2},
\]

\[
\bar{h}''(0) = -\frac{2 \alpha_1 \gamma_1}{\beta_1^2}, \quad \bar{g}''(0) = -\frac{2 \alpha_2 \gamma_2}{\beta_2^2}, \quad \bar{g}'(K_1) = \frac{\alpha_2 \beta_2}{(\beta_2 + \gamma_2 K_1)^2}, \quad \bar{h}'(K_2) = \frac{\alpha_1 \beta_1}{(\beta_1 + \gamma_1 K_2)^2}.
\]

Hence, the assumptions (B1)–(B4) hold provided that

\[
\alpha_1 \alpha_2 > a_1 a_2 \beta_1 \beta_2 \quad \text{and} \quad \min\{a_1, a_2\} > \max\left\{\frac{\alpha_2 \beta_2}{(\beta_2 + \gamma_2 K_1)^2}, \frac{\alpha_1 \beta_1}{(\beta_1 + \gamma_1 K_2)^2}\right\}.
\]

Noting that (6.3) hold when $\beta_1$ and $\beta_2$ are small enough. Thus, we have the following result.

**Theorem 6.1.** Let $\bar{h}(x)$ and $\bar{g}(x)$ be the functions given by (6.2), where $\alpha_i, \beta_i, \gamma_i, i = 1, 2$ are positive constants satisfying the conditions of (6.3). Then, the assertions of Theorems 2.1 and 2.2 hold for system (6.1).

**Example 6.2.** Assume

\[
\bar{h}(x) = ax \quad \text{and} \quad \bar{g}(x) = pxe^{-qx^m},
\]

where $p, q$ and $m$ are positive constants. The function $\bar{g}(x)$ is called the Ricker-type function. Of particular, when $m = 1$, $\bar{g}(x)$ is reduced to the Nicholson’s blowflies function. By elementary computations, we have

\[
(K_1, K_2) = \left(\frac{1}{q}, \frac{a_1}{a} K_1\right), \quad \bar{g}'(x) = p(1-qm x^m)e^{-qx^m} \quad \text{and} \quad \bar{g}''(x) = pqm x^{m-1} e^{-qx^m} (qm x^m - 1 - m).
\]

Let us set $u_* := (mq)^{-1/m}$, then (6.6) implies that $\bar{g}(x)$ is non-decreasing on $[0, u_*]$ and non-increasing on $[u_*, \infty]$. Therefore, if $1 < \frac{ap}{a_1 a_2} \leq e^{1/m}$, we have

\[
K_1 \leq u_* \quad \text{and} \quad \bar{g}'(x) \geq 0 \quad \text{for} \quad x \in [0, K_1] \quad \text{and} \quad \bar{h}'(x) = a \geq 0 \quad \text{for} \quad x \in [0, K_2].
\]

Furthermore, for $x \in [0, K_1]$, we know that

\[
\bar{g}'(K_1) = \frac{a_1 a_2}{a}\left(1 - m \ln\left(\frac{ap}{a_1 a_2}\right)\right) \geq 0 \quad \text{and} \quad \bar{g}''(x) \leq 0.
\]

Hence, the assumptions (B1)–(B4) hold provided that

\[
\min\{a_1, a_2\} > \max\left\{a, \frac{a_1 a_2}{a}\left(1 - m \ln\left(\frac{ap}{a_1 a_2}\right)\right)\right\}.
\]

Thus, we can obtain the following result.

**Theorem 6.2.** Let $\bar{h}(x)$ and $\bar{g}(x)$ be the functions given by (6.4), where $p, q, m, i = 1, 2$ are positive constants. Assume $1 < \frac{ap}{a_1 a_2} \leq e^{1/m}$ and the conditions of (6.7) hold. Then, the assertions of Theorems 2.1 and 2.2 hold for system (6.1).
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