The transverse Kähler-Ricci flow on Vaisman manifolds

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Abstract We define the transverse Kähler-Ricci equation on the canonical foliation of a closed Vaisman manifold, and investigate the existence and the convergence of the solution. Using the transverse Kähler-Ricci flow, in a particular case, we deform the Vaisman metric into another Vaisman metric with a transverse Kähler-Einstein structure. In the final part of the article we discuss an example.

Keywords: Vaisman manifolds, transverse Kähler-Ricci flow, transverse Kähler-Einstein metric

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1 Introduction

Let us consider a Hermitian manifold \((M, J, g)\) and \(\omega(\cdot, \cdot) := g(\cdot, J\cdot)\) the corresponding fundamental 2-form. We assume throughout the paper that the manifold \(M\) is closed, connected, with \(\dim \mathbb{R} M \geq 2\).

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If there is a nonzero 1-form $\theta$ such that $d\theta = \theta \wedge \omega$, with $\theta$ parallel with respect to the Levi-Civita connection of the metric $g$, then the manifold is said to be a Vaisman manifold. The 1-form $\theta$ is called the Lee form.

Being parallel, the Lee form is also closed, and Vaisman manifolds are locally conformally Kähler (LCK). We refer to [Dr-Or, Or-Ve3] for definition and properties of LCK manifolds. Note that most part of known LCK manifolds are in fact Vaisman. Actually, it is known that all compact homogeneous LCK manifolds are Vaisman, according to a result proved in [Ga-Mo-Or].

Many significant results on the geometry of Vaisman manifolds have been obtained over time (see, e.g., the recent works [Al-Ha-Ka, Ca-Ni-Ma-Yu2, Ga-Mo-Or, In-Le, Ma-Mo-Pi, Or-Ve2, Pi] and the references therein). A remarkable feature of the Vaisman manifolds are the canonical foliated structure [Ch-Pi, Ts]. The leaves of the foliations are two-dimensional, local Euclidean, while the transverse part of the two-fundamental form, denoted in this paper by $\omega^T_0$ (see Section 3), is in fact Kähler [Va]. Note that it is a commune feature that Vaisman manifolds share with Sasakian manifolds. We would also like to point out that the relation between Vaisman and Sasakian manifolds via the mapping torus construction was explored in [Ba-Ma-Op]. It was obtained that a compact Vaisman manifold is finitely covered by the product of a compact Sasakian manifold and a circle. This nice result gives topological information about compact Vaisman manifolds. We also note that a Hard Lefschetz Theorem for the de Rham cohomology and for the basic cohomology with respect to the Lee vector field on compact Vaisman manifolds was proved in [Ca-Ni-Ma-Yu1]. The proof of this theorem uses results regarding the basic cohomologies of compact oriented Riemannian manifolds with respect to some Riemannian foliations.

In the Kähler geometry an issue of interest is represented by the Kähler-Ricci flow. The importance was revealed in [Ca], where the author shows that it can be used as an alternative for proving the Calabi-Yau theorem [Ya], at least in the case when the first Chern class is null or negative. In particular, in this setting a complex version of the Hamilton’s Ricci flow was used to deform a Kähler metric into an Einstein-Kähler one. We indicate as reference [To, We, So-We] for fundamental facts on Kähler-Ricci flow and also for further developments.

The motivation for our work is represented by the results from [Sm-Wa-Zh]. In the setting of compact Sasaki manifolds, the authors define a transverse Kähler-Ricci flow (called Sasaki-Ricci flow) for the transverse Kähler structure. Using this flow, under assumption that the basic Chern class $c_1^b(M) = k[\omega^T_0]$ (see Section 2 for the definition of basic Chern class), with $k = 0$ or $k = -1$, they obtain Sasaki metrics which are transversally Kähler-Einstein (a notion which is equivalent to $\eta$-Einstein metrics (see [Bo-Ga-Ma] for definition and properties)). Note that in [Wa-Zh], for a Sasaki manifold of dimension 3 with $c_1^b(M) > 0$ it was proven that a Sasaki-Ricci soliton can be obtained.

It is thus natural to ask if on a Vaisman manifold (which is also endowed with a transverse Kähler structure, but has a foliated structure of dimension 2, more complicated than a Sasakian Riemannian flow) a transverse Kähler-Ricci flow can be defined. Furthermore, one can ask if a counterpart of a $\eta$-Einstein manifold can be obtained.
Note that in [El Ka], using the Molino structure theorem for Riemannian foliations [Mo], the author devised a Calabi-Yau theory for the transverse geometry of a foliation using the continuity method. Also, a transverse Ricci flow was defined in [Lo-Mi-Ru] without the assumption of a transverse Kähler structure. As remarked in [Sm-Wa-Zh, Section 3], to check if these methods preserve other structures (such as Sasakian or Vaisman) is not possible by direct computation. Also, we should emphasize the fact that, in general, the class of Vaisman structures is not stable under small deformations [Be].

As in [Sm-Wa-Zh], we solve this problem by defining a Kähler-Ricci flow only on the transverse part of the metric. The flow deforms it into a transverse Kähler-Einstein metric. Then, for the particular case when $c^1(M) = -[\omega^T_0]$, we associate to the flow a compatible deformation of the Vaisman structure. As a consequence we obtain a new Vaisman structure on the manifold with a Kähler-Einstein transverse metric.

We also hope that this approach will be useful in the context of an increased interest in the study of Chern-Ricci type flows on non-Kähler manifolds (see e.g. [To-We1, Gi-Sm]).

The paper is organized as follows. In Section 2 we outline the basic properties of Vaisman manifolds and the possibility to deform such manifolds using a differential basic function. In Section 3 we introduce the transverse Kähler-Ricci flow and prove the short time existence. In Section 4 we prove the long time convergence of the solution of the rescaled transverse Kähler-Ricci flow on Vaisman manifolds. In this part we chose to use local estimates which are similar to the setting represented by Kähler manifolds. The basic references for this type of results are [To, We, So-We]. As many results are similar, we do not include the extended proofs and just outline the differences to our setting of transverse Kähler structures.

In the final part of the paper an example of Vaisman manifold with a transverse Kähler-Einstein metric is constructed.

## 2 Preliminaries

### 2.1 Vaisman manifolds

In the following we present the definition and the main properties of Vaisman manifolds. For a more detailed exposure we indicate [Dr-Or, Or-Ve1].

Assume that $(M, J)$ is a complex manifold which is connected and closed, of real dimension $2n + 2$, with $n \geq 1$. Let $g$ be a Hermitian metric on $M$. We define the fundamental 2-form $\omega(X, Y) := g(X, JY)$, with $X, Y \in \Gamma(TM)$. By abuse, we sometimes call “metric” the fundamental form $\omega$. We also denote by $\nabla^\omega$ the Levi-Civita connection of $\omega$.

**Definition 2.1:** The manifold $(M, J, g)$ is called Vaisman if there exists a 1-form $\theta$, parallel with respect to $\nabla^\omega$, which satisfies the condition $d\omega = \theta \wedge \omega$.

**Remark 2.2:** The 1-form $\theta$ is closed, as it is parallel, and is called the Lee form.
We can also normalize the Lee form such that \( \|\theta\| = 1 \). We denote by \( \theta^c := \theta \circ J \) the anti-Lee form. Let \( U = \theta^2 \) and \( V = \theta^c \) denote the \( g \)-dual vector fields, thus

\[
\theta(U) = 1, \quad \theta^c(V) = 1, \quad \theta(V) = 0, \quad \theta^c(U) = 0.
\]

The fundamental 2-form is determined by the Lee form and the complex structure. We have

\[
\omega = d\theta^c - \theta \wedge \theta^c,
\]

(2.1)

where \( d \) is the de Rham differential operator. This above relation is due to I. Vaisman [Va].

**Remark 2.3:** The link between Sasaki and Vaisman structures is establish by the fact that the universal Riemannian cover of a Vaisman manifolds is a metric cone over a Sasaki manifold [Or-Ve]. Thus, locally, on a Vaisman manifold we have a local Sasaki structure transverse to the flow determined by \( U \).

The fact that \( \theta \) is parallel implies [Va]:

\[
\nabla^g_U U = \nabla^g_V U = \nabla^g_U V = \nabla^g_V V = 0.
\]

(2.2)

### 2.2 The foliated structure on a Vaisman manifold

In particular, \( U \) and \( V \) generate a foliation \( \mathcal{F} \) with leafwise dimension 2 (called the canonical foliation [Ch-Pi, Ts]). From (2.2), we see that the leaves are minimal submanifolds.

As this foliated structure plays an important role in our further considerations, we describe next the main properties.

A metric defined on a foliated manifold is said to be bundle-like if locally it can be identified with a Riemannian submersion [Re]. In the case of a Vaisman manifolds, the metric falls in this category with respect to the canonical foliation [Va].

Consider the foliated distribution \( \mathcal{F} \) generated by the vector fields \( U \) and \( V \). If \( Q \) is the transverse orthogonal complement of \( \mathcal{F} \) with respect to \( g \), then we have a canonical splitting of the tangent bundle

\[
TM = Q \oplus T\mathcal{F}.
\]

(2.3)

This also imply a splitting of the metric

\[
g = g^T \oplus g^{T\mathcal{F}},
\]

where \( g^T \) is the transverse metric, \( g^{T\mathcal{F}} \) being the leafwise metric. The later is in fact Euclidean [Va], while the transverse metric, as in the case of a Riemannian submersion, can be locally projected on a local transversal. Here, by a local transversal we mean a local submanifold of dimension \( 2n \), transverse to the leaves.
Other geometric objects that can be projected locally on a local transversal are the basic differential forms.

**Definition 2.4:** The de Rham complex of the basic (projectable) differential forms is defined as

\[ \Omega_b(M) := \{ \alpha \in \Omega(M) \mid \iota_X \alpha = 0, \text{Lie}_X \alpha = 0 \text{ for any } X \in \Gamma(TF) \} . \]

By \( \text{Lie}_X \) we denote the Lie derivative along \( X \), while \( \iota \) stands for interior product. The **basic de Rham derivative** is defined as \( d_b := d|_{\Omega_b(M)} \) (see e.g. [Ton]).

On a Vaisman manifold the complex structure \( J \) is also projectable, as well as the transverse fundamental two form \( \omega \) (which is thus a basic form, according to the above definition). The canonical foliation is endowed with a transverse Kähler structure [Va]. Thus we can consider a local orthonormal frame \( \{ e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}, U, V \} \), such that

\[ J(U) = V, \quad J(V) = -U, \quad J(e_i) = e_{n+i}, \quad J(e_{n+i}) = -e_i. \]

**Definition 2.5:** A convenient connection for the transverse geometry of the foliation is represented by the **Bott connection** \( \nabla^T \) (see [Ton, Chapter 3]) defined by

\[
\begin{align*}
\nabla^T_X Y &= \pi_Q(\nabla^T_X Y) \text{ if } X \in \Gamma(Q), \\
\nabla^T_X Y &= \pi_Q([X,Y]) \text{ if } X \in \Gamma(TF).
\end{align*}
\]

**Remark 2.6:** It is well known that the restriction of the above connection

\[ \nabla^T : \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q) \]

as well as the **transverse Riemann curvature tensor field**

\[ R^T : \Gamma(Q) \times \Gamma(Q) \times \Gamma(Q) \times \Gamma(Q) \to \Omega^1_b(M) \]

are projectable on a local transversal [Ton, Chapter 3].

The transverse Ricci tensor is defined using the transverse Riemann curvature tensor and the above local orthonormal frame.

**Definition 2.7:** For any \( X, Y \in \Gamma(Q) \), the **transverse Ricci tensor field** is defined as

\[ \text{Ric}^T(X,Y) = \sum_{i=1}^{2n} R^T(e_i, X, Y, e_i). \]

The **transverse Ricci operator** \( \text{Ric}^T : \Gamma(Q) \to \Gamma(Q) \) is defined such that

\[ \text{Ric}^T(X,Y) = g^T(\text{Ric}^T(X), Y). \]
Remark 2.8: The transverse Ricci operator defined above can be naturally extended on the complex of basic differential forms, obtaining an operator \( \text{Ric}^T : \Omega_b(M) \to \Omega_b(M) \). In the next subsection we describe more explicitly the above projectable mathematical objects using foliated coordinates.

2.3 Vaisman structure described in a foliated chart

We consider local foliated charts of the form \((U, z^j, \overline{z}^j, x, y)\), \(U\) being homeomorphic to the product \(T \times O\), where for the local transversal \(T\) we can be considered a local chart \((T, z^j, \overline{z}^j)\) on a complex Kähler manifold, while \((O, x, y)\) is a local chart on the Euclidean space \(\mathbb{R}^2\). \(z^j, \overline{z}^j\) are the transverse complex coordinates, \(z^j = x^j + iy^j, \overline{z}^j = x^j - iy^j\) while \(x, y\) are the leafwise real coordinates. More precisely, we choose the coordinates such that

\[
\frac{\partial}{\partial x} = U, \quad \frac{\partial}{\partial y} = V,
\]

where \(U\) is the Lee vector field, \(V\) being the anti-Lee vector field.

Remark 2.9: Considering the complex version of the basic de Rham complex \(\Omega_b(M)\), we see that the differential forms \(dz^j, d\overline{z}^j\) are basic. The dual vector fields \(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^j}\) are basic vector fields. A defining relation for a basic vector field \(X \in \Gamma(Q)\) is \([X, Y] \in \Gamma(Q)\), for any leafwise vector field \(Y \in \Gamma(TF)\) (see e.g. [Mo, Chapter 1]).

As a Vaisman manifold has a transverse Kähler structure, we consider \(h\) a local transverse Kähler potential, and describe the local structure of a Vaisman manifold using \(h\). Note that the local function \(h\) is basic. First we express the complex structure \(J\) using the above local foliated coordinates.

In the local map, the standard complex structure \(J_0\) is defined by the relations

\[
J_0 \left( \frac{\partial}{\partial x^j} \right) = \frac{\partial}{\partial y^j}, \quad J_0 \left( \frac{\partial}{\partial y^j} \right) = -\frac{\partial}{\partial x^j}, \quad J_0 \left( \frac{\partial}{\partial x} \right) = \frac{\partial}{\partial y}, \quad J_0 \left( \frac{\partial}{\partial y} \right) = -\frac{\partial}{\partial x}.
\]

We emphasize in the Remark 2.3, that a Vaisman manifold can be described locally as a Riemannian submersion with the fibers having a local structure of a Sasaki manifold (see also [Or-Ve]). For the local transverse Kähler potential in the case of a Sasaki manifold we refer to [Go-Ko-Nu, Sm-Wa-Zh]. Thus the complex structure \(J\) is expressed as

\[
J = J_0 + \frac{\partial}{\partial x} \otimes d^c(h) + \frac{\partial}{\partial y} \otimes d^c(h) \circ J_0
\]

\[
= J_0 + \frac{\partial}{\partial x} \otimes \left( i \frac{\partial h}{\partial z^j} dz^j - i \frac{\partial h}{\partial \overline{z}^j} d\overline{z}^j \right) + \frac{\partial}{\partial y} \otimes \left( i \frac{\partial h}{\partial z^j} d\overline{z}^j - i \frac{\partial h}{\partial \overline{z}^j} dz^j \right) \circ J_0.
\]

The Lee and anti-Lee forms are

\[
\theta = dx, \quad \theta^c = -\theta \circ J = dx + i \frac{\partial h}{\partial z^j} dz^j - i \frac{\partial h}{\partial \overline{z}^j} d\overline{z}^j.
\]
We introduce the basis of local vector fields \( \{ X_1, \ldots, X_n, \overline{X}_1, \ldots, \overline{X}_n \} \) on \( Q \), where
\[
X_j = \frac{\partial}{\partial z^j} - i \frac{\partial h}{\partial z^j} \frac{\partial}{\partial x}, \quad \overline{X}_j = \frac{\partial}{\partial \overline{z}^j} + i \frac{\partial h}{\partial \overline{z}^j} \frac{\partial}{\partial x}. \tag{2.4}
\]
We can check by a direct computation that
\[
J(X_j) = iX_j, \quad J(\overline{X}_j) = -i\overline{X}_j. \tag{2.5}
\]
The dual of the basis \( \{ X_i, \overline{X}_i, U, \overline{V} \} \) is in fact \( \{ dz^i, d\overline{z}^i, \theta, \theta^c \} \). Notice that the local 1-forms \( dz^i, d\overline{z}^i \) are basic, and thus projectable.

We also denote
\[
g(X_j, \overline{X}_k) = g^T_{jk}.
\]
In local coordinates \( g^T \) is written as below.
\[
g^T = g^T_{jk} dz^j \otimes d\overline{z}^k := g^T_{jk} (dz^j \otimes d\overline{z}^k + d\overline{z}^k \otimes dz^j).
\]
The associated transverse fundamental two-form \( \omega^T \) is written in the following manner.
\[
\omega^T = ig^T_{jk} dz^j \wedge d\overline{z}^k.
\]
We compute now the coefficients of the Bott connection in the local foliated coordinates. First, the following Lie brackets vanish
\[
[X_j, X_k] = [\overline{X}_j, \overline{X}_k] = [U, X_j] = [U, \overline{X}_j] = 0.
\]
For the anti Lee vector filed \( V \) we get a similar result. Using this result, similar to the Kähler case, we see that the only Christoffel coefficients of \( \nabla^T \) that do not vanish are \( \Gamma^k_{jl} \) and \( \Gamma^k_{jl} \) (see also [Sm-Wa-Zh, Section 4]). By a direct computation we get
\[
\Gamma^k_{jl} = (g^T)^{km} \frac{\partial g^T_{mk}}{\partial z^j},
\]
\( (g^T)^{km} \) being the entries of the inverse of the matrix \( g^T_{jk} \). As both Sasaki manifolds and Vaisman manifolds have a transverse Kähler structure, the results and computations are similar to [Sm-Wa-Zh, Section 4] (see also [Sl-Vi-Vil]).

The Bott connection \( \nabla^T \) defined in the previous part can be naturally extended on basic differential forms, so the differential operators \( \partial_b : \Omega_b(M) \to \Omega_b(M) \) and \( \overline{\partial}_b : \Omega_b(M) \to \Omega_b(M) \) can be defined as
\[
\partial_b = \sum_{i=1}^n dz^i \wedge \nabla^T_{X_i}, \quad \overline{\partial}_b = \sum_{i=1}^n d\overline{z}^i \wedge \nabla^T_{\overline{X}_i}, \tag{2.6}
\]

**Remark 2.10:** In the next section we apply the above derivatives on smooth function which are not necessarily basic. If \( f \) varies along the leaves, then using formulas (2.6), we
can naturally extend the operators $\partial_b, \overline{\partial}_b$ on $f$. Thus $i\partial_b \overline{\partial}_b f$ is a real differential 2-form which vanishes when applied on the leafwise distribution $TF$ but it is not necessarily basic (i.e. invariant along leaves).

We pass to the transverse Ricci operator. Assume that

$$Ric^T(\omega^T) = R^T_{jk} dz^j \wedge d\overline{z}^k.$$  

Then, as in the case of Kähler manifolds (see e.g. [To, Section 2.1]), we obtain

$$R^T_{jk} = - \frac{\partial^2}{\partial z^j \partial \overline{z}^k} \log \det(g^T_{jk}).$$  \hspace{1cm} (2.7)

From (2.3) and (2.7), if $\omega^T$ and $\tilde{\omega}^T$ are two different transverse fundamental forms, then

$$Ric^T(\omega^T) - Ric^T(\tilde{\omega}^T) = i \partial_b \overline{\partial}_b \log \det \tilde{g} \over\det g.$$  \hspace{1cm} (2.8)

Consider now the following class of the basic Dolbeaut cohomology w.r.t. the complex structure of the manifold.

$$H^{1,1}_b(M) := \frac{\{\partial_b - \text{closed real (1,1) basic forms}\}}{\operatorname{Im} \partial_b} = \frac{\{\overline{\partial}_b - \text{closed real (1,1) basic forms}\}}{\operatorname{Im} \partial_b \overline{\partial}_b}.$$  \hspace{1cm} (2.9)

The last equality is based on the basic $\partial \overline{\partial}$-Lemma proved in [El Ka, Proposition 3.5.1].

As the function in (2.8) is globally defined and in a local map has the value

$$\log \det \tilde{g} \over\det g,$$

we see that the basic Chern class $c^1_b(M) \in H^{1,1}_b(M)$ defined as

$$c^1_b(M) := [\text{Ric}^T(\omega^T)]$$

does not depend on the metric (see also [El Ka]).

Similar to [Sm-Wa-Zh], for an smooth function (not necessarily basic) we denote

$$f_{ij} := X_j(f), \quad f_{i\overline{j}} := \overline{X}_j(f),$$

$$f_{i\overline{j}k} := \nabla^g df(X_j, X_k) = \nabla_X df(X_k) - df(\nabla_X X_k).$$  \hspace{1cm} (2.9)

where $\nabla^g df$ is the Hesse tensor field associated to the function $f$, $\nabla^g$ being the Levi-Civita linear connection of the initial metric $g \equiv g_0$.

**Remark 2.11:** If we change the metric, from (2.9) we see that the Hesse tensor field differs by an differential operator of order at most 1.

**Remark 2.12:** Notice that if $f \in \Omega^1_b(M)$, i.e. $f$ is a basic function, then $df$ is a basic 1-form, hence

$$f_{ij} = f_j, \quad f_{i\overline{j}k} = f_{j\overline{k}}.$$
Remark 2.13: In the above considered local foliated map $U$, which is homeomorphic to the product $\mathcal{T} \times \mathcal{O}$, any local basic smooth function on $U$ projects on a smooth function on $\mathcal{T}$. For instance, the local coefficients $g^T_{jk}$ of the transverse metric project on the local coefficients $g^T_{jk}$ of the Kähler metric on $\mathcal{T}$. The same holds for basic 1-forms and 2-forms, as well for (local) basic vector fields. We see in Remark 2.6 that the transverse Bott connection and the associated Riemannian curvature tensor can be projected on the local transversal. Thus locally the transverse Ricci tensor is also projectable on the Ricci tensor on the transversal $\mathcal{T}$.

As a Riemannian foliation are locally described by a Riemannian submersion, the correspondence is in fact one to one, and any geometric object can be lifted on a (local) basic counterpart (see [Ton] for more details).

2.4 Deformation of a Vaisman structure using a differential basic function

In the recent paper [Or-Sl] a method to deform the structure of a Vaisman manifold using a family of basic functions $\varphi(t)$ is described. More precisely, a family of Vaisman structure $(M, J(t), g(t))$ with $J(0) = J$ and $g(0) = g$ for $t$ small enough is constructed, fixing the canonical foliations and deforming only the complex structure and the metric. The deformation of the complex structure is

$$J(t) = J - U \otimes d\varphi(t) - V \otimes (d^c\varphi(t) \circ J).$$

The family of anti-Lee 1-forms is

$$\theta^c(t) = \theta \circ J(t) = \theta^c + d^c\varphi(t),$$

while the metric is deformed as

$$\omega(t) = d\theta^c + d^c d^c\varphi(t) - \theta \wedge \theta^c(t).$$

The transverse metric is deformed in the following way

$$\omega^T(t) = i(g^T_{j\bar{k}} + \varphi(t) g^T_{j\bar{k}})dz^j \wedge d\bar{z}^k.$$

Thus the basic cohomology group $[\omega^T(t)]$ remains fixed. As a consequence, in this paper we use this method to construct a Vaisman metric for which the transverse part is the solution of the transverse Kähler-Ricci flow (for definition see Section 3 below).

3 The transverse Kähler-Ricci flow on a Vaisman manifold

3.1 The transverse Kähler-Ricci and Monge-Ampère equations

Using the foliated structure induced on a Vaisman manifold by the canonical foliation, we present in this subsection the transverse Kähler-Ricci equation and the equivalent
Monge-Ampère equation.

**Definition 3.1:** Let \((M, J, g)\) be a Vaisman manifold and \(\omega^T_0\) the transverse Kähler form with respect to the canonical foliation. The equation

\[
\begin{cases}
\frac{\partial}{\partial t} \omega^T(t) = -\text{Ric}_T(\omega^T(t)), \\
\omega^T(0) = \omega^T_0,
\end{cases}
\]

(3.1)

is called the transverse Kähler-Ricci equation.

From (3.1) we obtain the equation for the cohomology classes.

\[
\begin{cases}
\frac{\partial}{\partial t} [\omega^T(t)] = -c_1^b(M) \\
[\omega^T(0)] = [\omega^T_0]
\end{cases}
\]

(3.2)

with the solution \([\omega^T(t)] = [\omega^T_0] - tc_1^b(M)\). Denote by \(C^b_M\) the transverse Kähler cone:

\[C^b_M := \{\alpha \in H^{1,1}_b(M) \mid \text{there exists a transverse Kähler form in } [\alpha]\}\].

As in the Kähler case (see [To, Section 2.1]), \(C^b_M\) is an open set. We construct the corresponding Monge-Ampère equation. Let

\[T_{\text{max}} := \sup\{t > 0 \mid [\omega^T_0] - tc_1^b(M) \in C^b_M\}\].

For \(s < T_{\text{max}}\) we have \([\omega^T_0] - sc_1^b(M) \in C^b_M\). Let then \(\hat{\omega}^T(s) \in [\omega^T_0] - sc_1^b(M)\) be a Kähler form and define

\[\chi := \frac{1}{s} (\hat{\omega}^T(s) - \omega^T_0)\].

Notice that \(\chi \in C^b_M\). We define now the basic forms

\[\hat{\omega}^T(t) := \omega^T_0 + t\chi, \quad t \in [0, T_{\text{max}})\].

As \([\omega^T(t)] = [\hat{\omega}^T(t)]\) for any \(t\), \(\hat{\omega}^T(t)\) represents the basic cohomology class of \(\omega^T(t)\). Thus there exist a family of smooth basic functions \(\{\varphi(t)\}_t\) such that

\[\omega^T(t) = \hat{\omega}^T(t) + i\partial_b \overline{\partial}_b \varphi(t)\].

(3.3)

Let us consider in the following a transverse volume form \(\Omega\).

**Remark 3.2:** Note that since the foliation \(\mathcal{F}\) is totally geodesic, in particular minimal, \(\mathcal{F}\) is transversally orientable and global transversal volume forms can be defined [Ton].

On a local map \(\mathcal{U}\) with transverse coordinates \(z^1, \overline{z}^1, \ldots, z^n, \overline{z}^n\) there is a local smooth basic function \(f\) such that

\[\Omega = f \cdot i^n dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge dz^n \wedge d\overline{z}^n\].

We define the local two-form \(i\partial_b \overline{\partial}_b \log \Omega := i\partial_b \overline{\partial}_b \log f\). The next result can be proved as in the setting of Kähler manifolds [We, Subsection 1.4].

**Claim 3.3:** \(i\partial_b \overline{\partial}_b \log \Omega\) does not depend on the local foliated map and can be globally defined.
We consider also the globally defined two-form $\text{Ric}^T \Omega := -i \partial \bar{\partial} \log f$.

Fix now a transverse volume form $\Omega$ such that $i \partial \bar{\partial} \log \Omega = \chi$.

**Remark 3.4:** A transverse volume form with $i \partial \bar{\partial} \log \Omega = \chi$ can be constructed as follows. If $\omega^T_0$ is the initial transverse Kähler form, consider the transverse volume form $\Omega_0 := (\omega^T_0)^n$. Since both $i \partial \bar{\partial} \log \Omega_0 = i \partial \bar{\partial} \log (\det g^T_{jk})$ and $\chi$ represent $-c_1^{\mathcal{L}} [\mathcal{M}]$, [El Ka, Proposition 3.5.1] implies the existence of a basic smooth function $F$ such that

$$\chi - i \partial \bar{\partial} \log \Omega_0 = i \partial \bar{\partial} F.$$ 

Define the transverse volume form $\Omega := e^F \Omega_0$ (see also [To]).

Adapting the proof from the case of Kähler manifolds (see e.g. [To, Section 3.2]) to our context, we get that the transverse Kähler-Ricci equation (3.1) is equivalent to a Monge-Ampère equation in the following manner.

**Proposition 3.5:** A smooth family of transverse Kähler metrics $\omega^T(t)$ is a solution of equation (3.1) for $t \in [0, \varepsilon)$ if and only if there is a smooth family of smooth basic functions $\varphi(t)$, which satisfies (3.3) and also the associated Monge-Ampère equation

$$\begin{cases} 
\frac{\partial \varphi}{\partial t}(t) = \log \left( \frac{(\hat{\omega}^T(t) + i \partial \bar{\partial} \varphi(t))^n}{\Omega} \right) - F, \\
\varphi(0) = 0
\end{cases}$$

(3.4)

Notice that (3.4) can be rewritten as

$$\begin{cases} 
\frac{\partial \varphi}{\partial t}(t) = \log \left( \frac{(\hat{\omega}^T(t) + i \partial \bar{\partial} \varphi(t))^n}{\Omega_0} \right) - F, \\
\varphi(0) = 0
\end{cases}$$

(3.5)

Let $\omega^T_U$ be the restriction of the transverse Kähler form $\omega^T$ to $\mathcal{U}$, where $(\mathcal{U}, z_j, \bar{z}_j, x, y)$ is a foliated chart (Subsection 2.3). Then

$$\omega^T_U = i(g^T_{jk} + \varphi_{jk}) dz^j \wedge d\bar{z}^k.$$

In foliated coordinates, the Monge-Ampère equation (3.4) becomes

$$\begin{cases} 
\frac{\partial \varphi}{\partial t}(t) = \log \left( \frac{(\det (\hat{\omega}^T(t) + \varphi_{jk}(t)))}{(\det g^T_{jk})} \right) + F, \\
\varphi(0) = 0
\end{cases}$$

(3.6)

where $\hat{g}^T_{jk}(t)$ are the local coefficients of $\hat{\omega}^T(t)$ (as such, they are basic functions).
3.2 Short time existence for transverse Kähler-Ricci flow on a Vaisman manifold

We present now the method used to prove the existence and uniqueness of the solution of equation (3.5) (which is equivalent to (3.4)). The equation is only transversally parabolic. It is important to remark that in order to keep $\omega_t$ invariant along the leaves (thus corresponding to a bundle-like metric) the solution $\varphi(t)$ should be a basic function. As in [Sm-Wa-Zh], we construct an additional equation which is parabolic also in the leafwise directions and which coincides with (3.5) when $\varphi(t)$ is basic. This is in fact a frequent technique employed when investigating the transverse geometry of a Riemannian foliation (see e.g. the extension of the basic Laplace operator in [Ton, Chapter 7]). We obtain the existence and uniqueness of the solution using the theory of parabolic equations (see e.g. [Kr, Chapter 8]). Finally, using maximum principle [Kr, Theorem 8.1.2] we prove that the solution $\varphi(t)$ is actually a basic function.

To the family of Kähler forms $\omega_t$ we associate a family of forms $\tilde{\omega}_t$ which also vanish when applied on the leafwise distribution $T_F$, but are not necessarily basic. We define the forms in the following manner.

**Definition 3.6:** For $\varphi(t) \in C^\infty(M)$, denote by $\tilde{\omega}_t$ the differential 2-forms

$$
\tilde{\omega}_t(X,Y) := \tilde{\omega}_t(t)(X,Y) + i\partial_b\bar{\partial}_b\varphi(t)(\pi_Q(X),\pi_Q(Y)) = \tilde{\omega}_t(t)(X,Y) + \frac{1}{2} (\nabla^\omega d\varphi(t)(J \circ \pi_Q(X),\pi_Q(Y)) - \nabla^\omega d\varphi(t)(\pi_Q(X),J \circ \pi_Q(Y))).
$$

for any vector fields $X,Y \in \Gamma(TM)$.

From Remark 2.10, $\tilde{\omega}_t$ vanishes when applied to leafwise vector fields. Using (2.5) and (2.9), in the above local foliated chart $U$, $\tilde{\omega}_t$ satisfies

$$
\tilde{\omega}_t(X,Y) = \tilde{\omega}_t(Y,X),
$$

$$
\tilde{\omega}_t(X_j,X_k) = \tilde{\omega}_t(X_j,X_k) = 0,
$$

$$
\tilde{\omega}_t(X_j,X_k) = \tilde{\omega}_t(X_j,X_k) = (\tilde{\omega}_t)^j_k + \varphi^j_k.
$$

Thus the local form of $\tilde{\omega}_t$ is

$$
\tilde{\omega}_t = i \left( \partial_{j\overline{k}} + \varphi_{j\overline{k}} \right) dz^j \wedge d\overline{z}^k.
$$

**Remark 3.7:** According to Remark 2.12, if $\varphi(t)$ are basic functions, then $\tilde{\omega}_t \equiv \omega_t$.

We write the new Monge-Ampère equation using the transverse Kähler forms $\tilde{\omega}_t(t)$. We add the corresponding derivates along the leaves, according to the method described above. As the leafwise metric is Euclidean, we obtain the following equation.

$$
\begin{cases}
\frac{\partial \varphi}{\partial t}(t) = \log \frac{(\tilde{\omega}_t(t))^{\bullet \gamma} \wedge \theta \wedge \theta^c(t)}{(\tilde{\omega}_t^{\bullet \gamma} \wedge \theta \wedge \theta^c)} + \frac{1}{2} U(U(\varphi(t))) + \frac{1}{2} V(V(\varphi(t))) + F, \\
\varphi(0) = 0.
\end{cases}
$$
Claim 3.8: If $\varphi$ is a basic function, the equations (3.8) and (3.5) coincide.

Proof. The result follows using Remark 3.7 and the fact that basic functions are constant along leaves.

In the local map $\mathcal{U}$ we obtain the equation

$$
\begin{cases}
\frac{\partial \varphi}{\partial t}(t) = \log \left( \frac{\det (\tilde{g}^T_{jk} + \varphi|_{jk})}{\det \tilde{g}^T_{jk}} \right) + \frac{1}{2} \frac{\partial^2 \varphi}{\partial x^2}(t) + \frac{1}{2} \frac{\partial^2 \varphi}{\partial y^2}(t) + F, \\
\varphi(0) = 0.
\end{cases}
$$

(3.9)

Denote by $\tilde{\omega}(t)$ the family of metrics determined by $\tilde{\omega}^T(t)$ on the transverse complement $Q$ and the initial Euclidean metric on the leafwise distribution $T\mathcal{F}$. The operator in (3.9) is the complex Laplace operator

$$
\tilde{\Delta} = \frac{1}{2} \text{tr}\tilde{\omega}(t) \nabla_{\tilde{\omega}}^2 d.
$$

(3.10)

From Remark 2.11 we see that this differential operator has the same leading symbol as the Laplace operator $\Delta_{\tilde{\omega}(t)} = \frac{1}{2} \text{tr} \tilde{\omega}(t) \nabla_{\tilde{\omega}}^2 d$. Thus $\tilde{\Delta}$ is also an elliptic second order differential operator on $M$. The existence and uniqueness of the solution of equation (3.9) on $\mathcal{U}$ can be obtained using the theory of parabolic equation with variable coefficients (see e.g. [Kr, Chapter 8]).

Remark 3.9: For convenience, in the rest of the paper we write $\Delta = \Delta_{\tilde{\omega}(t)}$.

The solution also has the property that $\tilde{g}^T_{jk} + \varphi|_{jk}$ are the coefficients of a transverse metric tensor, positively defined on the transverse distribution $Q$. The proof is similar to the case of Kähler manifolds, and we refer to [To, Section 3]. We obtain a solution of class $C^\infty$ of the equation (3.9) on $\mathcal{U} \times [0, \varepsilon_U)$, for $\varepsilon_U > 0$ depending on $\mathcal{U}$. As the manifold $M$ is compact, we use a finite covering we get the existence and uniqueness of the global solution of the equation (3.8) on $M \times [0, \varepsilon)$.

Next we prove that the solution $\varphi(t)$ is a basic function for any $t \in [0, \varepsilon)$. The property is local, so we work on the local foliated map $\mathcal{U}$.

For the computations we use the following results.

Claim 3.10: The following Lie brackets vanish:

$$
\begin{align*}
\left[ \frac{\partial}{\partial x}, X_j \right] &= \left[ \frac{\partial}{\partial x}, \overline{X}_j \right] = \left[ \frac{\partial}{\partial x}, \nabla_{\overline{X}_j} \overline{X}_k \right] = 0, \\
\left[ \frac{\partial}{\partial y}, X_j \right] &= \left[ \frac{\partial}{\partial y}, \overline{X}_j \right] = \left[ \frac{\partial}{\partial y}, \nabla_{\overline{X}_j} \overline{X}_k \right] = 0.
\end{align*}
$$

(3.11)

Proof. In the local chart $\mathcal{U}$ considering the relations (2.4), as $h_j, h_{\overline{j}} \in \Omega_0(M)$, we have

$$
\left[ \frac{\partial}{\partial x}, X_j \right] = \left[ \frac{\partial}{\partial x}, \overline{X}_j \right] = 0,
$$

[13]
The similar result holds for the leafwise vector field $\frac{\partial}{\partial y}$.

Clearly the coefficients $g_{j\overline{k}}$ of the metric $g$ are basic functions, $g_{j\overline{k}} \in \Omega_b(M)$.

We get

$$g \left( X_j, \frac{\partial}{\partial x} \right) = g \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial x} \right) - i \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right) = g_{jx} - \frac{1}{2} h_{jx} = 0,$$

where $g_{jx} = g \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial x} \right)$. As the restriction of the metric to the leaves is Euclidean, we get that all coefficients of the metric are basic functions. Thus all Levi-Civita coefficients are basic functions, and

$$\left[ \frac{\partial}{\partial x}, \nabla_{\overline{X}_j} X_k \right] = 0.$$

For the vector field $\frac{\partial}{\partial y}$ the proof is similar.

**Claim 3.11:** The following relations hold

$$\frac{\partial}{\partial x} (\varphi|_{\overline{E}})(t) = \frac{\partial}{\partial x} (\varphi)_{|\overline{E}}(t), \quad \frac{\partial}{\partial y} (\varphi|_{\overline{E}})(t) = \frac{\partial}{\partial y} (\varphi)_{|\overline{E}}(t). \quad (3.12)$$

**Proof.** The commutation of the derivatives follows directly from (3.11).

**Claim 3.12:** If $\varphi$ is a solution of the equation (3.8) on $M \times [0, \varepsilon)$, then it satisfies also the equation

$$\frac{\partial}{\partial t} (U(\varphi)^2) = 2U(\varphi) U \left( \log \left( \frac{(\tilde{\omega}^T(t))^n \wedge \theta \wedge \theta^c}{(\omega_0^n \wedge \theta \wedge \theta^c)} \right) \right)$$

$$+ \frac{1}{2} U^2(\varphi)^2 + \frac{1}{2} V^2(\varphi)^2 - (U^2(\varphi))^2 - (U(V(\varphi)))^2. \quad (3.13)$$

**Proof.** Step 1: We use equation (3.8) to compute the following.

$$\frac{\partial}{\partial t} (U(\varphi)^2) = 2U(\varphi) \frac{\partial}{\partial t} (U(\varphi)) = 2U(\varphi) U \left( \frac{\partial \varphi}{\partial t} \right)$$

$$= 2U(\varphi) U \left( \log \left( \frac{(\tilde{\omega}^T(t))^n \wedge \theta \wedge \theta^c}{(\omega_0^n \wedge \theta \wedge \theta^c)} \right) \right) + U(\varphi) U^2(\varphi) + U(\varphi) V^2(U(\varphi)) + U(\varphi) U \left( F \right).$$

**Step 2:** For the last three terms we use that $F \in \Omega_b(M)$ and $[U,V] = 0$. The following relations hold.
\[ U(\varphi)U^2(U(\varphi)) = \frac{1}{2}U^2(U(\varphi)^2) - (U^2(\varphi))^2, \]
\[ U(\varphi)V^2(U(\varphi)) = \frac{1}{2}V^2(U(\varphi)^2) - (U(V(\varphi)))^2, \]
\[ U(F) = 0. \]

Thus we obtain equation (3.13).

For convenience we denote in the sequel \( U(\varphi) \) by \( u \). We now express equation (3.13) in the local chart \((U, z^1, \ldots, z^n, z^1, \ldots, z^n, x, y)\).

**Lemma 3.13:** In local coordinates the equation (3.13) becomes
\[
\frac{\partial}{\partial t} u^2 = \left( \frac{\partial}{\partial x} \right)^2 \left( \log \left( \frac{\det(\tilde{g}^T)_{jk} + \varphi_{jk}}{\det g^T_{jk}} \right) \right) + \frac{1}{2} \frac{\partial^2 u^2}{\partial x^2} + \frac{1}{2} \frac{\partial^2 u^2}{\partial y^2} - \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{\partial u}{\partial y} \right)^2.
\] (3.14)

where \((\tilde{g}^T)^{jk}\) are the coefficients of the inverse matrix of \( \tilde{g}^T_{jk} = \hat{g}^T_{jk} + \varphi_{jk} \).

**Proof.** From (3.13) we get on \( U \)
\[
\frac{\partial}{\partial t} u^2 = 2 u \frac{\partial}{\partial x} \left( \log \left( \frac{\det(\tilde{g}^T)_{jk} + \varphi_{jk}}{\det g^T_{jk}} \right) \right) + \frac{1}{2} \frac{\partial^2 u^2}{\partial x^2} + \frac{1}{2} \frac{\partial^2 u^2}{\partial y^2} - \left( \frac{\partial u}{\partial x} \right)^2 - \left( \frac{\partial u}{\partial y} \right)^2.
\]

Thus, using (3.12)
\[
\frac{\partial}{\partial x} \left( \log \left( \frac{\det(\tilde{g}^T)_{jk} + \varphi_{jk}}{\det g^T_{jk}} \right) \right) = \frac{\partial}{\partial x} \left( \log \left( \det(\tilde{g}^T)_{jk} + \varphi_{jk} \right) \right) = \left( \tilde{g}^T \right)^{jk} \frac{\partial}{\partial x} \left( \tilde{g}^T_{jk} + \varphi_{jk} \right) = \left( \tilde{g}^T \right)^{jk} u_{jk}.
\]

Finally \( 2u \cdot u_{jk} = u_{jk}^2 - 2u_{jk}u_{jk} \), and the equation (3.14) is obtained.

In the following denote for convenience the right part of (3.14) by \( Lu \).

**Remark 3.14:** We remark that the terms of order 2 of \( Lu \) are in fact the complex Laplace operator written in (3.10). Thus the equation is of parabolic type.

We prove that if \( \varphi \) is a solution of equation (3.13) on \( M \times [0, \varepsilon) \), then \( u = 0 \). We achieve this using the maximum principle for this type of equation [Kr, Theorem 8.1.2].

**Claim 3.15:** Assume that \( \varphi \) is a solution of equation (3.13) on \( M \times [0, \varepsilon) \). Thus if \( u = 0 \) for \( t = 0 \), then this property holds for any \( t \in [0, \varepsilon) \).
Proof. Step 1: Define an auxiliary function $v = u^2 - \frac{\gamma}{e^t - t}$. Here $\gamma > 0$ is a real number, and the function is defined on the domain $\overline{D}$, where $D = M \times (0, \varepsilon')$, $\varepsilon' \in [0, \varepsilon)$.

Step 2: We prove that the maximum of $v$ is attained on the parabolic boundary $M \times \{0\}$. As $\varphi$ is a smooth function on $\overline{D} = M \times [0, \varepsilon']$, the function $v = u^2 - \frac{\gamma}{e^t - t}$ attains its maximum at a point $p_\gamma \in \overline{D}$. As $v \to -\infty$ when $t \to \varepsilon'$, $p_\gamma$ is not on the upper lid $M \times \{\varepsilon'\}$. Suppose that $p_\gamma \in D$. Let $(\mathcal{U}, z^1, \ldots, z^n, \overline{z}^1, \ldots, \overline{z}^i, x, y)$ be a foliated map around $p_\gamma$. Then, according to Lemma 3.13, the equation has the form (3.14). In the following consider $t$ fixed. At the point $p_\gamma$ of coordinates $(z_1^i, z_2^i, x, y)$, the map $(z^i, \overline{z}^i, x, y) \mapsto v$ has a local maximum. Hence $u^2$ has also a local maximum at $p_\gamma$, while $u$ has a local extremum.

Using Remark 3.14 we also obtain
$$\left( u_{ij} \right)_{p_\gamma} = \left( u_{\overline{z}} \right)_{p_\gamma} = 0, \quad (Lu)_{p_\gamma} \leq 0.$$ for $1 \leq j, k \leq n$.

Now we fix $z^i, \overline{z}^i, x, y$ and consider the map $t \mapsto v$. At $p_\gamma$ we get
$$\left( \frac{\partial v}{\partial t} \right)_{p_\gamma} = \left( \frac{\partial u^2}{\partial t} \right)_{p_\gamma} - \frac{\gamma}{(e^t - t_{p_\gamma})^2} = 0.$$ Thus
$$0 = \left( Lu - \frac{\partial u^2}{\partial t} \right)_{p_\gamma} \leq -\frac{\gamma}{(e^t - t_{p_\gamma})^2} < 0.$$ The contradiction implies that the maximum of $v$ is attained for $t = 0$.

Step 3: Obtain an estimation for the solution. As a consequence $v \leq 0$, and as $\gamma > 0$ is arbitrary, we get that $u = 0$ on $\overline{D}$.

Finally, $\varepsilon' \in [0, \varepsilon)$ can be taken arbitrary and the result follows.

Lemma 3.16: The solution $\varphi \in M \times [0, \varepsilon)$ of the equation (3.8) is a basic function.

Proof. From Claim 3.15 we obtain that $U(\varphi) = 0$. Using a similar argument we obtain $V(\varphi) = 0$. As the leafwise distribution $TF$ is generated by the vector fields $U$ and $V$, we obtain the conclusion.

Proposition 3.17: The transverse Monge-Ampère equation (3.4) has a solution $\varphi$ for a short time $t \in [0, \varepsilon)$, which is a basic function.

Proof. The result is obtained from Lemma 3.16 and Claim 3.8.

Remark 3.18: If the basic Chern class $c_1^b(M)$ and the cohomology class of the initial Kähler form $\omega_0^T$ are connected by the relation
$$c_1^b(M) = k[\omega_0^T], \quad (3.15)$$ where, using a rescaling, $k$ can be taken as $-1, 0$ or $1$, then $[\omega^T(t)]$ is fixed and $\omega^T(t) = \omega_0^T + i\partial \overline{\partial} \varphi(t)$. 

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Finally, we use Proposition 3.5 and the method of deformation of a Vaisman structure with basic functions from Section 2.4 to obtain the following theorem.

**Theorem 3.19:** On a Vaisman manifold $M$ the transverse Kähler-Ricci equation (3.1) has a solution for $t \in [0, \varepsilon)$, for a small $\varepsilon > 0$. Furthermore, if the condition (3.15) is satisfied, the transverse Kähler-Ricci flow can be associated to a family of Vaisman structures defined on $M$.

### 4 Transverse Kähler-Einstein metrics

#### 4.1 Maximal existence time

In the first part of this section we obtain fundamental estimates that allow us to determine the maximal existence time for the transverse Monge-Ampère equation (3.4). In the sequel we assume that the solution exists on a maximum time interval $[0, T_{\text{max}})$, with $T_{\text{max}} < \infty$.

In Remark 2.13 we describe a local one to one correspondence between transverse metric, Bott connection, transverse Ricci operator, on a side, and their projection on a local transversal, on the other side. This correspondence also works for the transverse Laplace operator $\Delta^T := \frac{1}{2} \tr \omega^T(t) \nabla \omega^T(t) d$ which projects on the Laplace operator $\Delta^T := \frac{1}{2} \tr \omega^T(t) \nabla \omega^T(t) d$ on $T$. Thus the local solution $\varphi(t)$ of the Kähler-Ricci equation corresponds to the solution $\varphi^T(t)$ of the local Kähler-Ricci flow on $T$.

We use this correspondence between a foliated map and a local transversal stated in Remark 2.13, to get a local estimate around a fixed point in a foliated chart from a result which holds in a local transversal. Then we apply the min-max principle on the manifold $M$. We exemplify the method to estimate the function $\varphi(t)$.

**Proposition 4.1:** We have the estimate

\[
\sup_{M \times [0, T_{\text{max}}]} |\varphi(t)| \leq C. \tag{4.1}
\]

where $C$ is a constant that does not depend on $t$.

**Proof.** Let $\psi = \varphi - At$. Use (3.4) and get

\[
\frac{\partial}{\partial t} \psi = \log \left( \frac{(\hat{\omega}^T(t) + i \partial_{\bar{b}} \partial_{\bar{b}} \varphi(t))}{\Omega} \right)^n - A = \log \left( \frac{(\hat{\omega}^T(t) + i \partial_{\bar{b}} \partial_{\bar{b}} \psi(t))}{\Omega} \right)^n - A.
\]

Assume that $p_0 = (x_0, t_0)$ is an interior maximum point for $\psi$. Consider a foliated chart $U \simeq T \times O$ around $p_0$.

As $\psi$ is also a basic function, it is projectable on $T$, and its projection $\psi^T$ is a smooth function for which $\pi_T(x_0)$ is a maximum point, $\pi_T : U \to T$ being the canonical projection. Then

\[
i \bar{\partial} \partial \psi^T(\pi_T(x_0)) \leq 0, \quad \frac{\partial}{\partial t} \psi^T(\pi_T(x_0)) \geq 0.
\]
Hence, at \( p_0 \) we obtain
\[
i \partial_b \overline{\partial}_b \psi(p_0) = i \partial \overline{\partial} \psi^T(\pi_T(x_0)) \leq 0 \leq 0, \quad \frac{\partial}{\partial t} \psi(p_0) = \frac{\partial}{\partial t} \psi^T(\pi_T(x_0)) \geq 0.
\]
From this we get
\[
0 \leq \frac{\partial}{\partial t} \psi(p_0) = \log \left( \frac{\hat{\omega}_T + i \partial_b \overline{\partial}_b \psi}{\Omega_{x_0}} \right)_{p_0} - A \leq \log \left( \frac{\hat{\omega}_T}{\Omega_{x_0}} \right)_{p_0} - A,
\]
using the fact that \( \hat{\omega}_T_{p_0} > 0 \) for the transverse directions, as \( \hat{\omega}_T \) is a transverse metric. The last term does not depend on \( t \), and we fix the constant \( A := 1 + \sup_{M \times [0, T_{\max}]} \log \left( \frac{\hat{\omega}_T(t)}{\Omega} \right) \).

The function \( \log \left( \frac{\hat{\omega}_T(t)}{\Omega} \right) \) is smooth and basic, and can be defined on the compact manifold \( M \times [0, T_{\max}] \). We obtain a contradiction. As a consequence, the maximum is attained for \( t_0 = 0 \). Then
\[
\sup_{M \times [0, T_{\max}]} (\varphi(t) - At) \leq \psi(0) = 0,
\]
so
\[
\sup_{M \times [0, T_{\max}]} (\varphi(t)) \leq A \cdot T_{\max} \leq C.
\]
The other estimate \( \inf_{M \times [0, T_{\max}]} (\varphi(t)) \geq C \) is proved in a similar manner.

For the next results, we obtain local estimates in our foliated setting from results which are known to work in the framework of Kähler manifolds. We use the correspondence between a foliated map and a local transversal stated in Remark 2.13. For Kähler manifolds we omit the proofs and only refer to [To, Section 3.4], [We, Chapter 3].

**Proposition 4.2:** There is a constant \( C > 0 \) such that
\[
\sup_{M \times [0, T_{\max}]} |\dot{\varphi}(t)| \leq C, \quad (4.2)
\]
and the constant \( C \) does not depend on the parameter \( t \).

**Remark 4.3:** On a local foliated map, with \( \varphi^T \) defined as above, considering the projection on the local transversal \( T \), we obtain from (4.2) that
\[
\sup_{T} |\dot{\varphi}^T(t)| \leq C, \quad (4.3)
\]

The last estimate we obtain refer to the trace \( \text{tr}_{\omega^T(t)} \omega_0^T \).

**Proposition 4.4:** There is a constant \( C > 0 \), independent of \( t \), such that
\[
\sup_{M} \text{tr}_{\omega^T(t)} \omega_0^T \leq C. \quad (4.4)
\]
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Remark 4.5: On a local foliated map we consider the projection on the local transversal \( T \). From Remark 2.13, we obtain the transverse counterpart of (4.4).

\[
\sup_T \text{tr}_T \omega_T(t) \omega_0^T \leq C. \tag{4.5}
\]

The next result is a direct consequence of local estimates (4.3), (4.5) of the Kähler-Ricci equation on the transversal \( T \).

Claim 4.6: There is a constant \( C_0 > 0 \), independent of \( t \), such that

\[
\frac{1}{C_0} \omega_0^T \leq \omega_T(t) \leq C_0 \omega_0^T. \tag{4.6}
\]

The result is a direct consequence of local estimates (4.3), (4.5) and the Kähler-Ricci equation on the transversal \( T \).

Claim 4.7: On a compact local transversal \( T \) we obtain the following estimates

(i) If the condition (4.6) is fulfilled, then for \( t \in [0, T_{\text{max}}) \)

\[
\| \omega_T(t) \|_{C^k(T, \omega_0^T)} \leq C_k(T, k). \tag{4.7}
\]

Here the positive constant \( C(T, k) \) does not depend on \( t \).

(ii) As a consequence,

\[
\| \varphi_T(t) \|_{C^k(T, \omega_0^T)} \leq C_k(T, k). \tag{4.8}
\]

For the proof of (4.7), which is a purely local result, we refer to [Sh-We]. The inequality (4.8) is based on local Schauder estimate and uses (4.5) and (4.7). We refer to [To, Section 3.4] for the proof.

Remark 4.8: For the basic function the derivatives along the leafwise directions vanish. Thus on a local foliated map \( U = T \times O \) we get

\[
\| \varphi(t) \|_{C^k(U, \omega_0)} = \| \varphi_T(t) \|_{C^k(T, \omega_0^T)} \leq C(U, \omega_0). \]

Considering a covering argument, the higher order estimate for the solution of the transverse Kähler-Ricci flow is obtained.

Proposition 4.9: For any \( k \geq 0 \) there is a constant \( C_k \) independent on the parameter \( t \), such that

\[
\| \varphi(t) \|_{C^k(M, \omega_0)} \leq C_k, \tag{4.9}
\]

for \( t \in [0, T_{\text{max}}) \).

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We use the above estimates to obtain the maximal time interval for the solution of the transverse Kähler-Ricci equation (3.1).

**Theorem 4.10:** Let $M$ be a Vaisman compact manifold and let $\omega^T(t)$ be the transverse Kähler-Ricci flow with respect to canonical foliation, described by relations (3.1). Then $\omega^T(t)$ is defined on the maximal time interval $[0,T_{\max})$, where $T_{\max} \leq \infty$ is defined as

$$T_{\max} = \sup\{T > 0 \mid [\omega^T_0] - Tc_1^b(M) \in C_M^b\}.$$  

**Sketch of the proof.** As the proof is similar to the case of compact Kähler manifolds (see e.g. [We, Chapter 3] or [To, Section 3.4]), we only emphasize the main steps. We assume that the maximum interval is $[0,S)$, with $S < T_{\max}$ and construct the solution $\varphi$ at $S$ using the Ascoli-Arzela theorem (see e.g. [Ab-Ma-Ra, Theorem 1.5.12]). For any sequence $\varphi(t_j)$ which is bounded in $C^{k+1}(M,\omega_0)$, for any $k \geq 0$ we obtain a subsequence $\varphi(t_{j_k})$ which converges in $C^k(M,\omega_0)$ to a smooth function $\varphi(S)$. Notice that $\varphi(S)$ may depend on the initial sequence. To prove the uniqueness of $\varphi(S)$ we use the estimates (4.1) and (4.2). Finally we construct the solution for $t > S$ changing in Proposition 3.17 the initial condition to $\varphi(S)$, contradicting the maximality of $S$.

Note that with an argument similar to Claim 3.15 the function $\varphi$ can be proved to be basic for $t \in [0,T_{\max})$.

### 4.2 Long time convergence of transverse Kähler-Ricci flow

Throughout this subsection we assume that $c_1^b(M) < 0$, i.e. there is a transverse metric $\alpha^T$ such that $-\alpha^T \in c_1^b(M)$. Thus, from (3.2) we obtain that $[\omega^T(t)] \in C_M^b$, and the transverse Kähler-Ricci flow is defined for any $t \in [0,\infty)$. We investigate in what follows the long time convergence of $\omega^T(t)$.

From the equation (3.2), under the above assumptions $[\omega^T(t)]$ does not converges when $t \to \infty$. Thus, as in the case of Kähler manifolds, we "rescale" the transverse Kähler-Ricci equation. More precisely, we consider the equation

$$\begin{cases}
\frac{\partial}{\partial t} \omega^T(t) = -\text{Ric}(\omega^T(t)) - \omega^T(t),
\omega^T(0) = \omega^T_0.
\end{cases} \tag{4.10}$$

We rescale the flow as follows. Take $\overline{\omega}^T(s) = e^t \cdot \omega^T(t)$, where $s = e^t - 1$. Then

$$\frac{\partial}{\partial t} \omega^T(t) = \frac{\partial}{\partial t} \overline{\omega}^T(s) - \frac{\overline{\omega}^T(s)}{s+1}, \text{ and } \text{Ric}(\omega^T(t)) + \omega^T(t) = \text{Ric}(\overline{\omega}^T(s)) + \frac{\overline{\omega}^T(s)}{s+1}.$$  

From (4.10) we obtain $\frac{\partial}{\partial s} \overline{\omega}^T(s) = -\text{Ric}((\overline{\omega})^T(s))$, so $\omega^T(t)$ verifies the rescaled equation (4.10) if and only if $\overline{\omega}^T(s)$ verifies the transverse Kähler-Ricci equation (3.1).

The equation for the cohomology classes is obtained from (4.10).

$$\begin{cases}
\frac{\partial}{\partial t} [\omega^T(t)] = -c_1^b(M) - [\omega^T(t)],
[\omega^T(0)] = [\omega^T_0].
\end{cases} \tag{4.11}$$
The solution is \( \omega^T(t) = e^{-t} [\omega^T_0] - (1 - e^{-t}) c_1^T(M) \).

As in Section 3, we choose \( \hat{\omega}_\infty \in -c_1^T(M) \). Consider a transverse volume form \( \Omega \) such that \( i \partial_b \overline{\partial}_b \log \Omega = \hat{\omega}_\infty \). We take \( \hat{\omega}_\infty(t) = e^{-t} + (1 - e^{-t}) \hat{\omega}_\infty \) and consider a smooth family of smooth basic functions \( \varphi(t) \) such that \( \omega^T(t) = \hat{\omega}(t) + i \partial_b \overline{\partial}_b \varphi(t) \). Thus for the rescaled equation we have an equivalent Monge-Ampère equation:

\[
\frac{\partial \varphi(t)}{\partial t} = \log \left( \frac{\omega^T(t) + i \partial_b \overline{\partial}_b \varphi(t)}{\Omega} \right) - \varphi(t), \quad (4.12)
\]

The equivalence between (4.10) and (4.12) and the existence of solution for (4.12) for \( t \in [0, \infty) \) can be proved in a similar manner (see also [To, Section 5.1]).

Similar to Section 3, an important step is to show that the solution \( \varphi(t) \) of (4.12) is a basic function. To prove this result for the rescaled Kähler-Ricci equation, we should slightly adjust the proof of Proposition 3.17.

**Proposition 4.11:** The solution of the rescaled transverse Monge-Ampère equation (4.12) is a basic function \( \varphi \) for \( t \in [0, \infty) \).

**Proof.** We use the notations from Subsection 3.2. In local foliated coordinates we obtain for the rescaled equation

\[
\frac{\partial u^2}{\partial t} = \left( g^T \right)^{\overline{\partial}_b \partial_b u^2_{\overline{\partial}_b \partial_b} + \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} - \left( g^T \right)^{\overline{\partial}_b \partial_b} u_{\overline{\partial}_b \partial_b} - 2 \left( \frac{\partial u}{\partial x} \right)^2 - 2 \left( \frac{\partial u}{\partial y} \right)^2 - 2 u^2. \quad (4.13)
\]

Define the function \( v \) as in Subsection 3.2. On \( M \times [0, \varepsilon] \) it attains its maximum at a point \( p_\gamma \), for arbitrary fixed \( \varepsilon \in [0, \infty) \). Assume that \( v_{p_\gamma} \geq 0 \), which means \( u^2_{p_\gamma} \geq \frac{\gamma}{\varepsilon - t_{p_\gamma}} \).

At \( p_\gamma \) we obtain

\[
\left( u_{\overline{\partial}_b \partial_b} \right)_{p_\gamma} = \left( u_{\overline{\partial}_b \partial_b} \right)_{p_\gamma} = \left( \frac{\partial u}{\partial x} \right)_{p_\gamma} = \left( \frac{\partial u}{\partial y} \right)_{p_\gamma} = 0, \quad (Lu)_{p_\gamma} \leq -2 u^2_{p_\gamma},
\]

\[
\left( \frac{\partial u^2}{\partial t} \right)_{p_\gamma} = \left( \frac{\partial u^2}{\partial t} \right)_{p_\gamma} = \frac{\gamma}{(\varepsilon - t_{p_\gamma})^2} = 0.
\]

Hence

\[
0 = (Lu - \frac{\partial u^2}{\partial t})_{p_\gamma} \leq -2 u^2_{p_\gamma} - \frac{\gamma}{(\varepsilon - t_{p_\gamma})^2} \leq -2 \frac{\gamma}{\varepsilon - t_{p_\gamma}} - \frac{\gamma}{(\varepsilon - t_{p_\gamma})^2} < 0.
\]

From the contradiction we infer that \( p_\gamma \in M \times \{0\} \), so \( v < 0 \). As \( \gamma \) and \( \varepsilon \) are arbitrary, for \( \gamma \to 0 \) we get \( u^2 \leq 0 \). Hence \( u = U(\varphi) = 0 \) on \( M \times [0, \infty) \). In a similar manner we show that \( V(\varphi) = 0 \). Thus the function \( \varphi \) is basic. \( \blacksquare \)
Now we need several estimates on the functions $\varphi(t)$ and $\dot{\varphi}(t)$. This allows us to obtain the convergence of the solution when $t \to \infty$. As the proofs are similar to the setting of Kähler manifolds, we only refer to [To, Section 5.1], [We, Chapter 4.1].

**Proposition 4.12:** The following inequality holds

$$\sup_{M \times [0, \infty)} \varphi(t) \leq C.$$  \hfill (4.14)

where the constant does not depend on $t$.

**Proposition 4.13:** There is a constant $C > 0$ which does not depend on the parameter $t$, such that

$$\sup_{M \times [0, \infty)} \dot{\varphi}(t) \leq C(t + 1)e^{-t},$$  \hfill (4.15)

Next we obtain a lower bound for the function $\varphi(t) + \dot{\varphi}(t)$. We consider a null barrier function (for definition see [To, Section 4.2]). The next result follows now from [To, Section 5.1], using once again Remark 2.13.

**Proposition 4.14:** There is a constant $C > 0$ which does not depend on the parameter $t$, such that

$$\varphi(t) + \dot{\varphi}(t) \geq -C,$$  \hfill (4.16)

**Remark 4.15:** As a direct consequence, on a local foliated map, considering the projection on the local transversal, we obtain from (4.16) that

$$\varphi^T(t) + \dot{\varphi}^T(t) \geq -C,$$  \hfill (4.17)

Finally, we obtain the estimate on the trace $\text{tr}_{\omega^T(t)} \omega_0^T$.

**Proposition 4.16:** There is a constant $C > 0$ such that

$$\sup_M \text{tr}_{\omega_0^T} \omega^T(t) \leq C.$$  \hfill (4.18)

**Remark 4.17:** As in the previous subsection, consider the projection on the local transversal $\mathcal{T}$ in a foliated map $\mathcal{U}$. We obtain the transverse corresponding estimate of (4.18)

$$\sup_{\mathcal{T}} \text{tr}_{\omega_0^T} \omega^T(t) \leq C.$$  \hfill (4.19)

The following results are obtained using a method which is identical to the arguments used in the previous subsection.

**Claim 4.18:** There is a constant $C_0 > 0$, independent of $t$, such that

$$\frac{1}{C_0} \omega_0^T \leq \omega^T(t) \leq C_0 \omega_0^T.$$  \hfill (4.20)
The result is a direct consequence of local estimates (4.17), (4.19) and the Kähler-Ricci equation on the transversal $\mathcal{T}$ and the proof is also similar in the framework of Kähler manifolds, so we refer to [To, Section 5.1] or [We, Section 4.1].

We use (4.20), as in Subsection 4.1, and obtain

**Claim 4.19:** On a local transversal $\mathcal{T}$ we obtain the following higher estimate

$$\|\varphi^T(t)\|_{C^k(\mathcal{T},\omega^T_0)} \leq C_k(\mathcal{T},k).$$

(4.21)

For the basic function $\varphi(t)$ the derivatives along the leaves vanish. Thus, on a local foliated map $U = \mathcal{T} \times \mathcal{O}$ we get

$$\|\varphi(t)\|_{C^k(U,\omega_0)} = \|\varphi^T(t)\|_{C^k(\mathcal{T},\omega^T_0)} \leq C(U,\omega_0).$$

With a covering argument, the following higher order estimate is obtained.

**Proposition 4.20:** For any $k \geq 0$ there is a constant $C_k$ independent on the parameter $t$, such that

$$\|\varphi(t)\|_{C^k(M,\omega_0)} \leq C_k,$$

(4.22)

for $t \in [0, \infty)$.

We state now the main result of this section.

**Theorem 4.21:** Assume that on a Vaisman manifold $M$ the basic Chern class satisfies $c_1^b(M) < 0$. Then the rescaled Kähler-Ricci equation (4.10) has solution for $t \in [0, \infty)$. For $t \to \infty$, $\omega^T(t)$ converges to a Kähler transverse metric $\omega^T_\infty$ which satisfies

$$\text{Ric}(\omega^T_\infty) = -\omega^T_\infty.$$

The limit $\omega^T_\infty$ does not depend on initial metric $\omega^T_0$.

Assume, furthermore, that basic Chern class satisfies $c_1^b(M) = -[\omega^T_0]$. Thus the transverse Kähler-Ricci flow is associated to a families of Vaisman metrics defined on the manifold $M$.

**Sketch of the proof.** The convergence of $\varphi(t)$, when $t \to \infty$ and the uniqueness of the limit is proved identically to the setting of Kähler manifolds. We briefly recall the argument (see [To, Section 3.4] or [We, Section 4.1] for more details).

From (4.15) and (4.16) we obtain that the function $P := \varphi(t) + Ce^{-t}(1 + t)$ is non-increasing, and also bounded from below. Hence $P$ (and thus $\varphi(t)$) converges pointwise to a limit $\varphi_\infty$. Using (4.22), we see that $\varphi(t) \to \varphi_\infty$ as $t \to \infty$ in $C^\infty(M,\omega_0)$ and $\varphi_\infty$ is smooth. We notice also that the derivatives of $\varphi_\infty$ along the leaves vanish, i.e. $\varphi_\infty$ is also a basic function. From (4.6) we get that $\omega_\infty = \hat{\omega}_\infty + i\partial \bar{\partial} \varphi_\infty$ is a transverse Kähler metric.

From the rescaled Monge-Ampère equation (4.12) we obtain that $\dot{\varphi}(t)$ is also convergent. Thus $\dot{\varphi}(t) \to 0$ as $t \to \infty$, so we get

$$0 = \log \frac{(\omega^T_\infty)^n}{\Omega} - \varphi_\infty.$$
Applying the operator $i\partial b \bar{\partial} b$, we obtain the relation $\text{Ric}(\omega_{\infty}^T) = -\omega_{\infty}^T$.

The fact that the limit transverse metric $\omega_{\infty}^T$ does not depend on the initial transverse metric is proved similar to [We, Section 4.1], the only needed ingredient is the remark that the differential operators $i\partial b \bar{\partial} b$ and $i\partial \bar{\partial}$ agree when applied on basic functions.

Finally, for the last assertion we use the deformation of a Vaisman structure described in Section 2.4.

4.3 An example

In the final part of the article we present an example. In accordance with [Or-Ve1, Section 2], we consider a projective manifold $Q$. Assume the existence of a Kähler-Einstein metric $\omega_Q$, with $\text{Ric}_Q(\omega_Q) = -\omega_Q$. Consider $L$ a positive line bundle on $Q$. Denote by $\text{Tot}^\circ(L)$ the total space of all non-zero vectors in $L$. If we denote by $\|v\|$ the length of a vector $v$, then $\tilde{\omega} = dd^c \|v\|^2$ is a Kähler potential on $\text{Tot}^\circ(L)$.

Take $q \in \mathbb{R}^>0$, and choose $\sigma_q : \text{Tot}^\circ(L) \to \text{Tot}^\circ(L)$ to map a vector $v$ in $qv$. Consider also the $\mathbb{Z}$ action on $\text{Tot}^\circ(L)$ determined by $\sigma_q$. $\omega := \tilde{\omega} / \|v\|^2$ is projectable on the quotient manifold $\text{Tot}^\circ(L)/\mathbb{Z}$, which is thus equipped with a natural Vaisman structure.

The leaves of the canonical foliation associated to the Vaisman manifold are in fact the fibers of $L$. We obtain a regular foliation (i.e. the leaves are compact and the projection to the leaf space is a smooth submersion).

Denote the transverse metric on the Vaisman foliation by $\omega_{KE}^T$. Let $\text{Ric}^T$ be the transverse Ricci operator. Locally, $\omega_{KE}^T$ and $\text{Ric}^T$ project on $\omega_Q$ and $\text{Ric}_Q(\omega_Q)$. We obtain

$$\text{Ric}^T(\omega^T) = -\omega^T,$$

so the metric is transversally Kähler-Einstein.

As in Section 2.4, we use a basic function to deform the Vaisman metric to a new Vaisman metric $\omega_0$. Hence, if $\omega_0^T$ is the corresponding transverse metric, we have

$$c_1^T(M) = -[\omega_{KE}^T] = -[\omega_0^T].$$

We can thus apply Theorem 4.21 and obtain a transverse Kähler-Ricci flow. This flow deforms the Vaisman metric $\omega_0$ into a new Vaisman metric which is transversally Kähler-Einstein. Due to the uniqueness of such metric, it should be in fact the initial Vaisman metric (with transverse component $\omega_{KE}^T$).

Remark 4.22: A Sasaki manifold with transverse Einstein metric is called $\eta$-Einstein. (see e.g. [Bo-Ga-Ma]). The above example thus represents the counterpart of this type of metrics in the setting of Vaisman manifolds.

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