Exact AdS/CFT spectrum: Konishi dimension at any coupling

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We compute the full dimension of Konishi operator in planar N=4 SYM theory for a wide range of couplings, from weak to strong coupling regime, and predict the subleading terms in its strong coupling asymptotics. For this purpose we solve numerically the integral form of the AdS/CFT Y-system equations for the exact energies of excited states proposed by us and A. Kozak.

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INTRODUCTION

N=4 supersymmetric Yang-Mills theory recently gave us serious hopes for a better understanding of the dynamics of strongly interacting 4-dimensional gauge theories. Due to the AdS/CFT correspondence [1], as well as to the quantum integrability discovered on both sides of this duality in the planar limit [2, 3, 4, 5, 6, 7, 8, 10, 11, 12], we acquire little by little tools for the study of the most important quantities in N=4 SYM, such as the planar spectrum of dimensions $\Delta(\lambda)$ of local operators, where ’t Hooft coupling $\lambda$ is the scale independent parameter in this superconformal 4D theory. Their weak coupling behaviour ($\lambda \rightarrow 0$) was studied by Feynman perturbation theory. The dual string worldsheet $\sigma$-model allows to find the strong coupling asymptotics of various dimensions. Integrability allows us to connect the two regimes. In particular, the asymptotic Bethe ansatz (ABA) of [13] gives us the asymptotic spectrum of single trace operators when the number of elementary fields is very large.

However for short operators, such as Konishi operator $\text{Tr}[D, Z]^2$ [39], the calculation of anomalous dimensions is still an interesting and important challenge.

Recently we proposed the Y-system for the planar AdS/CFT [14], a set of functional equations defining the anomalous dimensions of all operators of planar N=4 SYM theory at any coupling. The integral form of the Y-system for excited states in SL(2) sector, including the one corresponding to Konishi operator, was presented in [15]. The integral equations for the BPS vacuum energy were independently obtained in [16, 17] by the thermodynamical Bethe ansatz (TBA) technique based on the dynamics of bound states [18] in a wide range of the ’t Hooft coupling $\lambda$, compared to the asymptotic Bethe ansatz curve and to the predicted large $\lambda$ asymptotics $\Delta_K(\lambda) \simeq 2\lambda^{1/4} + 2/\lambda^{1/4}$ obtained by fit. Derive the large volume ($L$) [40] asymptotic solution and then, departing from it, we solve the integral form of the Y-system iteratively. As a demonstration of the power of our method, we calculate numerically the dimension of Konishi operator in a wide range of the ’t Hooft coupling covering both the weak and strong coupling regimes. The results appear to be quite satisfactory: we manage to compute the dimension of Konishi operator in the interval of couplings $0 \lesssim \lambda \lesssim 700$ and to confirm, within the precision of our numerics, all the existing data concerning this quantity: the perturbative results [41] up to 4 loops (up to $\lambda^4$ terms) [29, 30, 31, 32] and the large $\lambda$ asymptotics $2\lambda^{1/4}$ predicted by [33]. Fitting our numerical data with $\lambda > 60$ we find (with uncertainty in last digit)

$$\Delta_K = 2\lambda^{1/4} \left( 1.0002 + \frac{0.994}{\lambda^{1/2}} - \frac{1.30}{\lambda} + \frac{3.1}{\lambda^{3/2}} + \ldots \right) \quad \text{(1)}$$
The leading term reproduces indeed the expected large λ behavior within our numerical precision. It was also argued in [34] that the subleading coefficient ought to be integer (the next corrections could be transcendental [35]). Indeed, our numerics seems to indicate that ∆K = 2λ1/4 + 2/λ1/4 + . . . thus predicting the value of this integer! [42] We also obtained predictions for two further subleading corrections (with a lower precision of course). The results are represented in Fig.2.

![Figure 2: Plot of ∆K(λ) − 2λ1/4 from the numerical data compared with the Bethe ansatz prediction and some fits. The fits in this plot are done assuming the asymptotics ∆K(λ) = 2λ1/4 + 2/λ1/4 + . . . and allowing for zero (dashed line), one (dotted line) or two (solid line) further corrections.](image)

**Y-SYSTEM FUNCTIONAL AND INTEGRAL EQUATIONS FOR ADS/CFT**

The Y-system defining the spectrum of all local operators in planar AdS/CFT correspondence has the form [14]

\[
Y_{a,s}^+ Y_{a,s}^- = \left(1 + Y_{a,s+1}\right)\left(1 + Y_{a,s-1}\right)/(1 + Y_{a+1,s})\left(1 + Y_{a-1,s}\right).
\]

where the functions Y_{a,s}(u) correspond only to the nodes marked by · ◦ ⊙ □ on Fig.3 (we will use these notations for Y-functions in what follows). The one particle energy at infinite length

\[
\epsilon^{(a)}(u) = a + \frac{2ig}{x^{[1-a]}} - \frac{2ig}{x^{[a]}}
\]

is defined through the Zhukovski map x(u) + x^{[a]}(u) = \frac{u}{g} and f^{[±]} ≡ f(u ± ia/2), f^± ≡ f(u ± i/2) for any function f(u). A solution of Y-system with a given set of quantum numbers defines the energy of a state (or dimension of an operator in N=4 SYM) through the formula [41] where the Bethe ansatz equations Y_{a_i}(u_j) = -1.

![Figure 3: T-shaped “fat hook” (T-hook) uniting two SU(2|2) fat hooks (see [54]). It defines the interactions between Y-functions in the AdS/CFT Y-system equations.](image)

In this paper we restrict ourselves to the integral form of the Y-system for the SL(2|2) excited states obtained in [13]. Furthermore we focus for simplicity on the Konishi operator where we have only two symmetric roots u_1 = -u_2 which we can encode into the “Baxter functions” R^{(±)}(u) = (x(u) − x^±(1)x(u) − x^±(2)) and their complex conjugates B(u) = \bar{R}(u) where x^±_{1,2} = x(u_{1,2} ± i/2) with the physical choice of branches, such that |x(u)| > 1, while for the free variable x(u) we should use the mirror kinematics which corresponds to the branches where Im(x(u)) > 0 [27]. Unless it is explicitly said otherwise, we will be always using this latter choice in what follows. With the mirror choice, x(u) has a semi-infinite cut for u ∈ (-∞, -2g) U (2g, +∞). Notice that above all the cuts, the physical and mirror quantities coincide.

The energy of the Konishi state is computed from

\[
\Delta_K = 2 + 2\epsilon^{(1)}(u_1) + \sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi i} \partial_u \epsilon^{(a)}(1 + Y_{a_i}) ,
\]

where the integral equations defining Y_{a_i} read [15]

log Y_{Θ} = K_{m-1} \ast \log(1 + 1/Y_{Θ_m})/(1 + Y_{Ω_m})

\[ + \mathcal{R}^{(m)} \ast \log(1 + Y_{Θ_m}) + \log \frac{R^{(+)}_{m-1}}{R^{(-)}_{m-2}}\]

log Y_{Θ_m} = M_{nm} \ast \log(1 + Y_{Θ_m} - K_{n-1} \ast \log(1 + Y_{Θ}))

\[ - K_{n-1,m-1} \ast \log(1 + Y_{Ω_m}) + \log \frac{R^{(+)}_{m-1} B^{(+)}_{n-2}}{R^{(-)}_{m-1} B^{(-)}_{n-2}}\]

log Y_{Ω_m} = K_{n-1,m-1} \ast \log(1 + 1/Y_{Θ_m}) + K_{n-1} \ast \log(1 + Y_{Θ})

log Y_{Θ_m} = T_{nm} \ast \log(1 + Y_{Θ_m}) + 2\mathcal{R}^{(n)} \ast \log(1 + Y_{Θ})

\[ + N_{nm} \ast \log(1 + Y_{Ω_m}) + i\Phi_n .\]

We use here the kernels and sources defined in [13] and presented in the appendix for completeness. The integrals in convolutions K * f = \int dv K(u, v) f(v) go along the real axis, but slightly below the poles in the terms involving Y_{Ω_m} (due to the last term in the corresponding integral equation, see [15]). The convolutions @ should
be understood in the sense of a B-cycle (see [13]), e.g. \( \mathcal{R}^{(n_0)} \otimes \log(1 + Y_{\Theta}) \) stands for
\[
\int_{-2g}^{2g} dv \left[ \mathcal{R}^{(n_0)} \log(1 + Y_{\Theta}) - B^{(n_0)} \log(1 + 1/Y_{\Theta}) \right]
\]
where \( 1/Y_{\Theta} \) is the analytical continuation of \( Y_{\Theta} \) across the cut \( u \in (-\infty, -2g) \cup (2g, +\infty) \). Summation over the repeated index \( m \) is assumed with \( m \geq 2 \) for \( \Delta_{\pm m} \) and \( \Theta_{\pm m} \), and \( m \geq 1 \) for \( \Theta_{n_m} \).

A remarkable feature of all these equations, crucial for the success of our numerics and noticed in [12], is the reality of all \( Y \)-functions in the integral equations.

**EXACT BETHE EQUATIONS**

The \( Y \)-system integral equations for the functions \( Y_{a,s} \) need to be supplemented by the *exact* Bethe equations \( \Phi_{\pm}(u_j) = -1 \) which, as explained in [14], reproduce the asymptotic Bethe equations of Beisert and Staudacher in the large \( L \) limit. To use this equation, we need to analytically continue the last of eq. (5) in the free variable \( u \) from mirror to physical plane and then evaluate it at \( u = u_1 \). After some manipulations with contours we find
\[
\log Y_{\Theta}(u_1) = \tilde{T}_{1m} \log(1 + Y_{\Theta}) + \log Z_{\Delta_m}(u_1) + i\Phi_{\pm}(u_1) + 2(R_{ph,m}^{(n_0)} \otimes K_{m-1} + K_{m-1}^{-1}) \ast_{p,v} \log(Z_{\Delta_m})
\]
\[
+ 2(R_{ph,m}^{(n_0)} \otimes \log(1 + Y_{\Theta}) - 2 \log(u_{-i/2} + 2 - \sum_{j=1}^{2} \log \frac{1}{x_{j}} - \frac{x_{j}}{1})
\]
where \( \ast_{p,v} \) stands for principal value integration, \( \tilde{T}_{1m} \) is the dressing phase in the physical kinematics while \( i\Phi_{\pm} \) is the same as \( i\Phi \) but evaluated in the physical region (see appendix). We used \( 1/Y_{\Theta}(u_1) \rightleftharpoons (u_{-i/2} + 2 - \sum_{j=1}^{2} \log \frac{1}{x_{j}} - \frac{x_{j}}{1}) \) to isolate the poles in \( Y_{\Theta} \) at \( u = u_1 \) and to ensure nicer asymptotics at large \( u \) which is of course useful for the numerics. Finally, in contrast to \( \Phi_{\pm}(u_1) \), the term \( Z_{\Delta_m}(u_1) \) is evaluated in mirror kinematics.

**NUMERICS AND ITS INTERPRETATION**

We solve the integral equations (5) iteratively for the Konishi state. Namely, we find the \( Y \)-functions at a step \( n \) by plugging the \( Y \)-functions of step \( n - 1 \) into the r.h.s. of these equations. As the first step of the iterations we use the large \( L \), ABA solutions of the \( Y \)-system [14]. At each step of iterations we also update the position of the Bethe roots by solving the exact Bethe equation of the previous section, using again the ABA values as the starting point. It is important to note that the r.h.s. of (5) remains, within our precision, purely imaginary in the process of iterations.

We should also truncate the infinite set of \( Y \)-functions to some finite number. We explicitly iterate the first 25 \( Y_{\Omega} \)'s and 25 \( Y_{\Theta} \)'s at each step. Then we also extrapolate them to obtain extra 40 \( Y_{\Omega} \)'s and 40 \( Y_{\Theta} \)'s with higher numbers to replace the infinite sums in (3) on the next step of iteration. Thus in total we sum up 65 \( Y_{\Omega} \) and 65 \( Y_{\Theta} \). For the middle node functions \( Y_{\alpha} \) we can truncate the sums much earlier (typically \( \log Y_{\alpha}/Y_{\alpha} \ll 0 \)): first 5 of them are largely enough for our precision. Finally, the integrals along the real axis are computed along the stretch \((-X,X)\) with \( X \) being a large cut-off. With these approximations, our absolute precision for the energy is around \( \pm 0.001 \).

We solved the integral equations for a wide range of couplings \( 0 \leq \lambda \leq 700 \gg 1 \) stretching from the perturbative region up to this value, which is already a deep strong coupling regime. We found no sign of any singularity and the curve looks perfectly smooth. By this reason we believe that any new singularities, such as those related to the Lüscher \( \mu \)-terms, will not appear. This seems to be the case perturbatively [17] and our numerics seems to indicate that the integral form of \( Y \)-system we are solving describes exactly the full spectrum of planar \( N=4 \) SYM theory in \( SL(2) \) sector. Although we cannot discard a possibility that some new singularities could collide with the integration contours for even larger values of the `t Hooft coupling (such extra singularities could be easily incorporated into our code), our numerical results suggest that this possibility is very unlikely.

We also observed from our numerics that at large \( \lambda \) the phase \( \log Y_{\Theta}(u_1) \) in (5) is not dominated by only the dressing phase \( i\Phi_{\pm}(u_1) \) and thus the coefficient 2 in the asymptotics \( \Delta K \approx 2 \lambda^{1/4} \) cannot be explained by the same arguments as in [11]. Hence the \( L = 2 \) case seems to be very different in this respect from \( L \rightarrow \infty \).

With a precision of 0.001 we can approximate the Konishi dimension in the range we considered by
\[
\left[ \sqrt{g^2 + 1} \right]^{252.934.1+384.748.4+674.139.2+126.173k+29.294l+4} = h \approx g^2/\sqrt{g^2 + 1} \quad \text{and} \quad \lambda = 16\pi^2 g^2. \]

This function is just a shorthand for a table of data, it has little to do with the exact analytical structure of the Konishi dimension.

**APPENDIX**

We use: \( \mathcal{P}^{(n)}(v) \equiv -\frac{1}{2\pi i} \frac{d}{dv} \log \frac{x_v^{[n]}}{x_v^{[-n]}} \), \( K_n \equiv \frac{2n/\pi}{n^2 + 4\pi^2} \) and
\[
\mathcal{R}^{(nm)}(u, v) = \frac{\partial_u}{2\pi i} \log \frac{x_u^{[n]} - x_v^{[-m]}}{x_u^{[m]} - x_v^{[-n]}} - \frac{1}{2i} \mathcal{P}^{(m)}(v),
\]
\[
\mathcal{B}^{(nm)}(u, v) = \frac{\partial_u}{2\pi i} \log \frac{1/x_u^{[n]} - x_v^{[-m]}}{1/x_u^{[m]} - x_v^{[-n]}} - \frac{1}{2i} \mathcal{P}^{(m)}(v),
\]
\[
M_{nm} = K_{n-1} \mathcal{R}^{(nm)} + K_n^{[n-1-m-1]},
\]
\[
N_{nm} = \mathcal{R}^{(n_0)} \otimes K_{m-1} + K_n^{[n-1-m-1]},
\]
\[ K_{nm} = \mathcal{F}_n^u \circ \mathcal{F}_m^v \circ K_2(u - v), \quad K_{nm}^c = \mathcal{F}_n^u \circ \mathcal{F}_m^v \circ K_1(u - v), \]

where the fusion operation \( \mathcal{F}_n^u \circ A = \sum_{k=-\infty}^{\infty} A(u + ik) \).

Finally, we also use a nice integral representation [15] of the kernel \( T_{nm} \) given by

\[ T_{nm}(u, v) = -K_{nm}(u - v) - \frac{i n}{2} \mathcal{F}(m)(v), \tag{7} \]

\[-2 \sum_{a=1}^{\infty} \int \left[ B_{1m}(10)\left(u, w + i \frac{a}{2}\right) B_{1m}(01)(w - i \frac{a}{2}, v) + c.c. \right] dw, \]

where \( B_{1m}(10) = \mathcal{F}_n^u \circ \mathcal{F}_m^v \circ \mathcal{B}(10) \). For the exact Bethe equations we should use this kernel in the mixed representation,

\[ \tilde{T}_{1m} = -\sum_{a=1}^{\infty} 2B_{1m,\text{mir}}^{(10)}(u, w + i \frac{a}{2}) B_{1m,\text{mir}}^{(01)}(w - i \frac{a}{2}, v) \]
\[-\sum_{a=1}^{\infty} 2B_{1m,\text{mir}}^{(10)}(u, w - i \frac{a}{2} - i 0^+) B_{1m,\text{mir}}^{(01)}(w + i \frac{a}{2}, v) - K_{1m}. \]

Finally the source term \( \Phi_n(u) = \mathcal{F}_n^u \circ \Phi(u) \), with

\[ \Phi(u) = \frac{1}{i} \log \left[ \left( \frac{x^-}{x^+} \right)^{L+M} S^2 \frac{B^{(\text{+})}}{B^{(\text{-})}} \frac{R^{(\text{+})}}{R^{(\text{-})}} \right] \tag{8} \]

where the BES [13] dressing phase \( S(u) = \prod_{j=1}^{\infty} \sigma \left( \frac{u}{x^\pm}, \frac{x^\pm}{u} \right) \) should be taken in the mixed, mirror-physical representation in the integral equations and in the physical-physical representation for \( \Phi_{\text{ph}} \) appearing in the exact Bethe equations. We use the mirror-physical integral representation [13]

\[ \log S = \log \frac{B^{(\text{-})}}{B^{(\text{+})}} + \left( B^{(10)}(u, w + i 0) \ast \mathcal{G} \ast \log \frac{R^{(\text{+})}(u - i 0)}{R^{(\text{-})}(u - i 0)} + c.c. \right) \tag{9} \]

where \( \mathcal{G}(u) \equiv \frac{\partial}{\partial z} \log \frac{\Gamma(1-iu)}{\Gamma(1+iu)} \) while for the physical-physical representation we can use the DHM integral representation [38]. Finally \( K_{\text{ph,ph}}(u, v) \) represents a kernel where we use the physical (mirror) branches for \( u \) and \( v \). For Konishi \( L = 2 \) and the number of derivatives \( M = 2 \).

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Here $Z$ is one of the complex scalars and $D$ is a covariant derivative in a light cone direction.

$L$ is the number of $Z$ fields in an operator in $SL(2)$ sector.

meaningful until the convergency radius $\lambda < \lambda_c = \pi^2$; in this region our numerical data are indistinguishable from the BAE for our accuracy.

Assuming the leading coefficient to be $2\lambda^{1/4}$ one gets 1.001 instead of 0.994 for the subleading term, even closer to 1 - the value predicted here.