Spectral analysis of deformed random networks

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We study spectral behavior of sparsely connected random networks under the random matrix framework. Sub-networks without any connection among them form a network having perfect community structure. As connections among the sub-networks are introduced, the spacing distribution shows a transition from the Poisson statistics to the Gaussian orthogonal ensemble statistics of random matrix theory. The eigenvalue density distribution shows a transition to the Wigner’s semicircular behavior for a completely deformed network. The range for which spectral rigidity, measured by the Dyson-Mehta $\Delta_3$ statistics, follows the Gaussian orthogonal ensemble statistics depends upon the deformation of the network from the perfect community structure. The spacing distribution is particularly useful to track very slight deformations of the network from a perfect community structure, whereas the density distribution and the $\Delta_3$ statistics remain identical to the undeformed network. On the other hand the $\Delta_3$ statistics is useful for the larger deformation strengths. Finally, we analyze the spectrum of a protein-protein interaction network for Helicobacter, and compare the spectral behavior with those of the model networks.

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I. INTRODUCTION

The network concept has been gaining recognition as a fundamental tool in understanding the dynamical behavior and the response of real systems from different fields such as biology, social systems, technological systems. Examples of biological systems include food-web, nervous system, cellular metabolism, protein-protein interaction network, gene regulatory networks; social systems include scientific collaboration, citation, linguistic networks; and technological systems include internet, power-grid [1]. Many of these networks have been shown to have universal structural properties, such as degree distribution following a power law, small diameter, large clustering coefficient, existence of communities [1, 2, 3, 4].

Different network models have been proposed and investigated in detail to understand systems having an underlying network structure [1, 2, 3, 5]. These models concentrate to capture one or more structural properties of the networks mentioned above [1, 2, 3]. Apart from these direct measurements of structural properties, network spectra are also useful to understand various properties of the underlying system. Eigenvalues of the adjacency matrix of networks form what are called network spectra, and provide information about some basic topological properties of the underlying network [6, 7]. Recently, considerable research has been done in the direction of network spectra [8, 9].

In the following, we mention known results on the spectra of real world and model networks. The spectra of networks have some correspondence with the spectra of random matrices. For instance, the distribution of eigenvalues of a matrix having finite mean number $p$ of nonzero Gaussian distributed random elements per row follows Wigner semicircular law in the limit $p \to N$, where $N$ is the dimension of matrix [10, 11]. For very small $p$, which corresponds to the sparse random matrix, one gets the semicircular law but with peaks at different parts of the spectrum (maximum at the eigenvalue zero) [12]. Recent investigations of the spectral behavior of networks, leading to matrices with entries zero and one, show that the random networks follow Wigner semicircular law as well [13] with degeneracy at the eigenvalue zero. The small-world model networks show a very complex spectral density with many sharp peaks [12], while the spectral density of the scale-free model networks exhibits a triangular distribution [14, 15, 16]. The spectra of real world networks show remarkably different features than that of the model networks [1, 13, 16, 17], and based on this observation a network construction method was proposed which captures a peak at zero property shown by the spectra of many real world networks such as protein-protein interaction networks [17]. Recently, spacing distributions of Erdös-Rényi networks have been studied under random matrix theory (RMT) framework [18]. As connection probability decreases Ref. [18] shows a transition to the Poisson statistics. Additionally, it shows the transition to the Poisson statistics upon the deletion of nodes in the real world networks [19]. Refs. [15, 19] have shown that the spacing distributions of various model networks, namely small-world and scalefree networks, follow the universal behavior of RMT. In contrast to [18], these works [15, 19] have considered only connected networks. Furthermore, spectral rigidity such as the $\Delta_3$ statistics, defined in Eq. 3, provides a qualitative measure of the level of randomness in networks [20]. Recently localization of eigenvectors have also been used to analyze various structural and dynamical properties of real and model networks [21].

RMT, initially proposed to explain statistical proper-
ties of nuclear spectra, has also provided successful predictions for the spectral properties of different complex systems such as disordered systems, quantum chaotic systems and large complex atoms among this. It has been followed by numerical and experimental verifications in the last few decades \cite{11, 12}. Quantum graphs, which model the systems of interest in quantum chemistry, solid state physics and transmission of waves, have also been studied under the RMT framework \cite{22}. Recently, RMT has been shown to be useful in understanding the statistical properties of empirical cross-correlation matrices appearing in the study of multivariate time series in several problems: price fluctuations in stock market \cite{23}, Electro encephalogram data \cite{24}, variation of different atmospheric parameters \cite{25}.

In the present work we study spectral behavior of networks having community structure under the framework of RMT. The study of community structure helps to elucidate the organization of networks, and eventually could be related to the functionality of groups of nodes \cite{1, 2, 26}. Regardless of the type of real world networks in terms of the degree and other structural properties \cite{1}, it is possible to distinguish communities in the whole networks \cite{1}. However, the question of definition of the community is problematic, and usually community is assigned to the nodes which are connected densely among themselves, and are only sparsely connected with other nodes outside the community. We therefore model here community structure by sparsely connected Erdős-Rényi random networks. This simple approach considers more densely connected nodes as a definition of community, and does not pay attention to the detailed structure of the connections \cite{2}. Recent literature is largely filled up with methods to detect communities in networks based on structural measures \cite{27, 28}, whereas few works emphasize on the spectral properties such as density distribution and eigenvector analysis as well \cite{29}. The objective of our work is not the detection of communities, rather we show the applicability of spectral methods under the RMT framework to analyze community structures in networks. Instead of paying attention to the nodes forming communities, we look for the signatures of overlapping of communities in the spectra of the corresponding adjacency matrix. We study various spectral behaviors, namely density distribution, nearest neighbor spacing distribution (NNSD) and spectral rigidity for deformed random networks. We find that the NNSD detects even the small mixing of communities in the network, whereas spectral rigidity probed by the $\Delta_3$ statistics is suitable to analyze larger mixing, which is, in general, the case for real world networks. Communities are modeled by random or scale-free sub-networks, and interactions between communities are considered as random. For small interaction strength the NNSD of the network shows the transition from the Poisson to the Gaussian Orthogonal Ensemble (GOE) statistics. For large interactions, the $\Delta_3$ statistics shows systematic increase in the range for which it follows GOE statistics. Finally, as an application, we study the spectral properties of a protein-protein interaction network of Helicobacter under the RMT framework.

![Figure 1](image_url)

**FIG. 1:** (color online) Connection matrices corresponding to $p = 0.01$ and different values of $q$. (a) plots the connection matrix of the two sub-networks which do not have any connection between them. (b) corresponds to $q/p = 0.1$, and (c) depicts the case $q/p = 0.5$, when the connections between the the sub-networks are as large as 50% of the connections inside.

## II. DEFORMED NETWORKS

For an unweighted network, the adjacency matrix is defined in the following way: $A_{ij} = 1$, if $i$ and $j$ nodes are connected and zero otherwise. For undirected networks, this matrix is symmetric and consequently has real eigenvalues. Random matrices corresponding to unweighted random networks have entries 0 and 1, where number of 1’s in a row follows a Gaussian distribution with mean $p$ and variance $p(1-p)$. This type of matrix is very well studied within the RMT framework \cite{11, 12}. We then turn our attention to the following structure: (1) Take $m$ random networks with connection probability $p$; the spectral behavior of the matrix corresponding to each of these sub-networks (blocks) separately follows GOE statistics. The matrix corresponding to the full network would be a $m$ block diagonal matrix. (2) Introduce random connections among these sub-networks with probability $q$. This configuration leads to $m$ block matrix, with blocks having entries one with probability $p$, and off diagonal blocks having entries one with probability $q$. The above networks can be casted in the following form:

$$A = A_0 + A_q$$  \hspace{1cm} (1)

$A_0$ is a $m$ blocks diagonal random matrix, where each block represents one community, and the off-diagonal block matrix $A_q$ denotes the interactions among the communities. Each block in $A_q$ is a random matrix, which for large $N$ has mean $q$ and deviation $q(1-q)$. Since the nonzero values of $q$ introduce deformation to the complete block diagonal form, we refer $A$ being a deformed network. This terminology is motivated by the literature on deformed random matrices \cite{30}. Fig. \[\text{II}a\] shows the connection matrices for $m = 2$ and various values of $q$. Fig. \[\text{II}b\] represents the two random sub-networks, each of size $N = 500$, with the connection probability.
inside a sub-network being \( p = 0.01 \) and between the sub-networks being \( q = 0 \). The ratio \( q/p \), which can be considered as the relative strength of \( A_q \) and \( A_0 \), measures the deformation from the block-diagonal form of the matrix, or from the perfect structured network. The value \( q/p = 1 \), which corresponds to equal strength of inter and intra-community connections, yields complete random network. Fig. 1(b) plots the connection matrix for \( q/p = 0.1 \), which implies that inter-community connections are 10\% of the intra-community connections. Fig. 1(c) shows the connection matrix for \( q = 0.005 \); for this value of \( q \), the inter-community strength is 50\% \((q/p = 0.5)\) of the intra-community strength. Note that in numerical simulations we use the value of \( p \) equal to 0.01, which leads to a sparse connected random network \((N_c \sim N)\) with the average degree \(< k > \sim N \times p = 5, N_c \) being the number of connections in the network. Larger value of \( p \) would lead to networks with the larger average degree. Real world networks are sparse \([1]\), and hence we chose such a small value of \( p \).

### III. NUMERICAL SIMULATION RESULTS

We denote the eigenvalues of the network by \( \lambda_i \), \( i = 1, \ldots, m \times N \), where \( N \) is the size of the sub-network, and \( m \) is the number of the sub-networks. Note that the size of each sub-network may be different, but for simplicity we consider here equal size. Fig. 2 plots the spectral density for \( m = 2 \) block matrices having \( qN^2 \) non-zero off diagonal entries, corresponding to the two sub-networks connected with probability \( q \). As discussed earlier \( q \) varies from \( q = 0 \), which corresponds to the two completely disconnected sub-networks (\( A = A_0 \), Fig. 1(a)), to \( q = p \) leading to a single random network. The cases for \( 0 < q << p \) correspond to the configurations when the initial community structure is almost preserved. Increase in the value of \( q \) leads more entries of one in the matrix \( A_q \) (Eq. 1). Finally the \( q = p \) case destroys the community structure completely, and the network can be treated as one single random network. Fig. 2 presents the density distribution of eigenvalues for various values of \( q \). The eigenvalues are scaled with respect to the spectra of the network for \( q/p = 1 \). With this scaling, the density distributions are not semicircular for values of \( q < p \). As the coupling between the two blocks increases \((q > 0)\), the density distribution shows a transition to the semicircular form at \( q = p \):

\[
\rho(\lambda) = \frac{2}{\pi \lambda_0^2} \sqrt{\lambda_0^2 - \lambda^2},
\]

where \( \lambda_0 \) is the radius of the semicircular distribution for \( q = p \) calculated from the spectra of network as \( \lambda_0 = (\lambda_{\text{max}} - \lambda_{\text{min}})/2 \), \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) being the highest and the lowest eigenvalue. Now we turn our attention to the statistics of eigenvalue fluctuations.

#### A. Nearest neighbor spacing distribution

In the following, we study spectral fluctuations of the networks for different values of \( q \). In order to get universal properties of the eigenvalue fluctuations, one has to remove the spurious effects due to variations of the spectral density and to work at the constant spectral density on the average. Thereby, it is customary in RMT to unfold the eigenvalues by a transformation \( \tilde{\lambda}_i = \bar{N}(\lambda_i) \), where \( \bar{N}(\lambda) = \int_{-\infty}^{\lambda} \rho(\lambda') d\lambda' \) is the averaged integrated eigenvalue density \([10]\). Unfolding is a transformation which produces the eigenvalues with a constant average level density. Since we do not have an analytical form for \( \bar{N} \), we numerically unfold the spectrum by polynomial curve fitting.

Using the unfolded eigenvalues, we calculate the NNSD \( P(s) \), where \( s^{(i)} = \tilde{\lambda}_{i+1} - \tilde{\lambda}_i \), for different \( q \) values. Fig. 3 plots the spacing distribution for the two values of \( q \), \( q = 0 \) and \( q = 10^{-4} \). For such small values of \( q \), although the density distributions remain unchanged, the NNSD shows significant changes. Spacing distributions calculated from the network spectra are fitted using Brody formula \([31]\),

\[
P_\beta(s) = As^\beta \exp \left(-\alpha s^{\beta+1}\right),
\]

where \( A \) and \( \alpha \) are determined by the parameter \( \beta \) as follows:

\[
A = (1 + \beta)\alpha \quad \text{and} \quad \alpha = \left[ \Gamma \left( \frac{\beta + 2}{\beta + 1} \right) \right]^{\beta+1}.
\]
Eq. 2 is a semi-empirical formula characterized by the single parameter $\beta$. As $\beta$ goes from zero to one, the Brody formula smoothly changes from Poisson to GOE. As can be seen from Fig. 3 for $q/p \sim 0.001 (q \sim 10^{-5})$, the value of the Brody parameter $\beta \sim 0.2$, which suggests that distribution is very close to the Poisson $[P(s) = \exp(-s)]$ denoted by the dotted curve in the figure. As the value of $q$ increases, $\beta$ also increases, and it is of the order of 1 for the value of $q/p \sim 0.01$ (which corresponds to the value of $q$ as less as $10^{-4}$), and becomes insensitive for a further increase in $q$. For larger values of $q$, we analyze the spectra using the spectral rigidity test of RMT.

**B. Spectral rigidity via $\Delta_3$ statistics**

The spectral rigidity, measured by $\Delta_3$ statistics of RMT, gives information about the long-range correlations among the eigenvalues. The $\Delta_3$ statistics measures the least-square deviation of the spectral staircase function representing the cumulative density $N(\lambda)$ from the best straight line fitting for a finite interval $L$ of the spectrum, i.e.,

$$\Delta_3(L; x) = \frac{1}{L} \min_{c_1, c_2} \int_{x}^{x+L} [N(\lambda) - c_1 \lambda - c_2]^2 \, d\lambda$$

(Eq. 3)

where $c_1$ and $c_2$ are obtained from a least-square fit. Average over several choices of $x$ gives the spectral rigidity $\Delta_3(L)$. For the uncorrelated eigenvalues, $\Delta_3(L) = L/15$, reflecting strong fluctuations around the spectral density $\rho(\lambda)$. For the GOE case, $\Delta_3(L)$ statistics is given by

$$\Delta_3(L) \sim \frac{1}{\pi^2} \ln L.$$  

(Eq. 4)

Fig. 4 plots the $\Delta_3$ statistics for five different values of $q$. Various open symbols are the numerical values of $\Delta_3$ for various $q$ values, and the solid line (merged with the open circles corresponding to $q/p = 1$) is the GOE prediction (Eq. 3). Since for $q = p$ the $\Delta_3$ statistics of network follows the GOE prediction completely, the solid line showing GOE statistics merges with the circles showing numerical values for this $q$. The figure is plotted for an average over 20 realizations of the random set of connections between the networks.
IV. DEFORMED SCALEFREE NETWORKS

In the following we consider scalefree networks as the sub-networks, and study the spectral behavior for various values of $q$. Again $q$ measures the strength of the off-diagonal block matrix defining the interaction between the sub-networks. Matrix $A_0$ in Eq. 1 corresponding to the scalefree sub-networks, consists of two block diagonal matrices, with entries of one in each block following a power law characteristic of the sub-network. We use Barabási-Albert algorithm [3] to generate the scalefree sub-networks. In scalefree network the probability $P(k)$, that a node has degree $k$, decays as a power law $P(k) \sim k^{-\gamma}$, where $\gamma$ is a constant and for the type of probability law used in the simulations $\gamma = 3$. Other forms for the probability law are also possible which gives different exponent [32]. However, the results reported here are independent of the value of $\gamma$ [33]. Size and average degree of the sub-networks remain the same as for the random sub-networks, i.e. $N = 500$ and $<k> = 5$. The average degree ($<k>$) of a network can be calculated as $<k>= 2\times N_c/N$, where $N_c$ is the number of connections and $N$ is the size of the network. With the increase in the value of $q$, deformation from the network having scalefree community structure also increases. Fig. 5 plots various spectral behavior of deformed networks made of the scalefree sub-networks. Fig. 5(a) plots the density distribution for the various values of $q$. For small values of $q$, the density is very different from that of the deformed random networks (Fig. 4). It has a triangular shape with a peak at zero. This is a well-known shape for sparse scalefree networks [8,13,14]. For $q/p < 0.01$, when the scalefree structure of the sub-networks dominates over the random interaction between them, the eigenvalue density distribution does not show any noticeable change. But the NNSD in Fig. 5(b) suggests a possible structure in the network. As shown in Fig. 5(b), for $q/p = 0.001 (q = 10^{-5})$ the NNSD is close to Poisson statistics with a value of the Brody parameter $\beta \approx 0.21$. As $q$ increases, value of the Brody parameter increases as well, becoming one for $q \sim 10^{-4}$. After this value of $q$, the NNSD does not provide any further insight, and we probe for long-range correlations among eigenvalues. Fig. 5(c) plots the $\Delta_3$ statistics for various values of $q$. It shows similar behavior as for the deformed random networks (see Fig. 4). For $q \sim 0.01$, when the network has distinguishable community structure, the value of $L$ for which $\Delta_3$ follows the GOE statistics is as small as 25. As $q$ is increased, $L$ also increases, becoming $\sim 150$ for $q/p \sim 1$.

Fig. 5(d) shows the density distribution (inset) and the spacing distribution of the protein-protein interaction network of Helicobacter [35]. The largest connected component of the network has dimension $N = 708$ and number of connections $N_c = 2789$. The average degree of this scalefree network is $<k> \sim 4$. The density distribution has triangular form with a peak at zero. This behavior of the density distribution suggests scalefree properties of the network [8,13,14], but does not provide information of randomness or structure in the network. To get further insight, we calculate the NNSD and the spectral rigidity of the network. For this, first we unfold the eigenvalues using the procedure explained earlier. The NNSD of the network follows GOE statistics with the value of $\beta \sim 0.98$, suggesting enough random connections in the network. Further test of long-range correlations among eigenvalues shows that the $\Delta_3$ statistics follows the GOE...
V. CONCLUSIONS AND DISCUSSIONS

The eigenvalue density distribution of networks having two sub-networks tend towards the semi-circular distribution as the random connections between the sub-networks are increased. For very small values of \( q < 10^{-4} \), corresponding to the very small deformation from the community structure, the density distribution does not present any noticeable changes, but the NNSD, which reflects short-range correlations among eigenvalues, show important features. For two random sub-networks, which are almost uncoupled (i.e. \( q \sim 0 \)), the NNSD is very close to the Poisson statistics, and as \( q \) increases, it has a smooth transition to the GOE statistics. Note that this Poisson to GOE transition is found for many different systems, for example spectra of insulator-metal transition, order-chaos transition follow this Poisson-GOE transition \[\text{[11]}.\] Sade et. al \[\text{[37]}\] have studied transition to the GOE statistics as a function of site disorder for the spectra of small-world and scale-free networks. Here, by keeping the network structure fixed, disorder at nodes is increased and depending upon the network average degree transition to GOE statistics is seen. The main difference between \[\text{[37]}\] and the study presented in this paper is the following: we track changes in the spectra with structural changes in the network architecture. As random connections among the sub-networks are increased, first there is transition for the NNSD to the GOE statistics, and this transition occurs for very small value of random connections among networks. This is the crucial and remarkably different result observed here, which suggests that very small random interaction between communities is enough to introduce short-range correlations among them, spreading the randomness in the whole network. Second, further increase in coupling among the sub-networks is reflected by long-range correlations among eigenvalues. For this increase in the value of \( q \), the NNSD does not give additional insight to the deformation of the network, as it remains same with the \( \beta \sim 1 \), so we turned our attention to the \( \Delta_3 \) statistics.

The \( \Delta_3 \) statistics, which measures long-range correlations among the eigenvalues, detects deformation from a network having two coupled sub-networks, to a single random network. More deformation of the network from community structure, leads to a larger range of \( L \) for which \( \Delta_3 \) follows the GOE statistics. Note that, for the case of sub-networks being completely random, the spacing and the \( \Delta_3 \) statistics of each of them follows RMT prediction. Therefore, any deviation from GOE statistics is due to the community structure these two sub-networks form when considered as a single network.

It is interesting to note that our results resemble the behavior of deformed random matrix ensembles (DGOE) introduced to study the effect of isospin symmetry breaking in nuclei \[\text{[30]}\]. The qualitative behavior of the spectral density and the \( \Delta_3 \) statistics of networks presented here is similar to that of deformed matrices studied in \[\text{[38, 39, 40]}\]. The analytical form of the density derived in \[\text{[39]}\] depends on a parameter \( \alpha \) measuring the relative strength of the off-diagonal random matrices to the block diagonal random matrices. In similar lines, for deformed networks, we can compare \( q/p \), relative strength of off-diagonal and diagonal networks, with \( \alpha \). The results presented here suggest that further investigations of complex networks following similar lines as in deformed random matrices \[\text{[39]}\] would be useful to have detailed information of communities in the networks \[\text{[41]}\].

To conclude, we have studied the spectral behavior of networks having community structure, and shown that the NNSD and \( \Delta_3 \) statistics capture features related to the structure in the network. We investigate the spectral properties of a real world network as well, and compare the results with those of the model networks. On the one hand, results presented in this paper advances the studies of the spectral properties of network with the community structure under the universal RMT framework; on the other hand, variations in the correlations among eigenvalues shed light on the coupling among communities. For the simulations, the community structure in network is modeled by the very simple random or scalefree sub-networks, and the interactions among these sub-networks are considered random, whereas real world networks have richer structure \[\text{[8]}\]. However, the results presented here provide a platform to investigate the community structure of networks using a well developed theory of random matrices; the further investigations in this direction would deal with real world networks with richer and more complicated structure under the deformed random ma-
trix framework [36, 41].

VI. ACKNOWLEDGMENTS

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[34] The relation between the preferential attachment rule Π(k) ∝ k + a/m used in the manuscript and the degree distribution exponent γ is, γ = 2 + a/m [33], where a is the initial attractiveness of nodes and m is the number of connections new node makes. For the Barabási-Albert (BA) model [3], a/m = 1 which leads to the degree distribution with γ = 3. In the limit of zero initial connectivity a = 0, all new nodes connect only the first one. This case gives γ = 2, and the network would be star network, with N − 1 nodes having one connection and one node with N − 1 connections. The eigenvalues of this star network are −N−1,0,√N−1, which gives the spectral density with three peaks at these three values. The analysis and the comments about the spectral behavior of scale-free networks presented in the section IV are made for 2 < γ ≤ 3. Most of the real world networks lie between 2 < γ < 3 [3]. For this range the density distribution show typical triangular shape, and the tail of ρ(λ) at large λ is related to the behavior of the degree distribution P(k). In particular, as P(k) ∝ k−γ, ρ(λ) ∝ |λ|1−γ−2. [3]. Nearest neighbor spacing distribution for the individual sub-
network would show GOE statistics of RMT [19]. The spectral behavior of the combined network would show qualitative similar behavior of transition from Poisson to GOE statistics as coupling between the sub-networks is increased, only the range of $\Delta_3(L)$ statistics for which it follows GOE statistics may be different.

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