Fixed versus random triangulations in 2D simplicial Regge calculus

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We study 2D quantum gravity on spherical topologies using the Regge calculus approach with the $dl/l$ measure. Instead of a fixed non-regular triangulation which has been used before, we study for each system size four different random triangulations, which are obtained according to the standard Voronoi-Delaunay procedure. We compare both approaches quantitatively and show that the difference in the expectation value of $R^2$ between the fixed and the random triangulation depends on the lattice size and the surface area $A$. We also try again to measure the string susceptibility exponents through a finite-size scaling Ansatz in the expectation value of an added $R^2$ interaction term in an approach where $A$ is held fixed. The string susceptibility exponent $\gamma_{\text{str}}$ is shown to agree with theoretical predictions for the sphere, whereas the estimate for $\gamma_{\text{str}}$ appears to be too negative.

1. INTRODUCTION

In the past few years Regge calculus has been extensively used in the study of quantum gravity \cite{1}. In its usual form one studies regular simplicial triangulations of manifolds of a given topology, which are mostly hypertori. In this work we will investigate two-dimensional quantum gravity, where analytic calculations have shown that the internal fractal structure of the manifold depends very sensitively on the global topology. One universal quantity is the string susceptibility exponent $\gamma_{\text{str}}$, which is the sub-dominant correction to the large area behavior of the partition function $Z(A) \propto A^{\gamma_{\text{str}}} e^{-\lambda_R A}$, where $\lambda_R$ denotes the renormalized cosmological constant. The exponent $\gamma_{\text{str}}$ depends on the genus $g$ of the surface through the KPZ formula $\gamma_{\text{str}} = 2 - \frac{5}{2}(1-g)$ \cite{2}.

For the torus ($g = 1$) the Regge approach with the $dl/l$ measure gives compatible results. For the sphere ($g = 0$) and topologies of higher gender, however, the situation is still unclear \cite{3,4}. One potential problem is that for the sphere only very small regular triangulations exist, such as the tetrahedron, the octahedron, and the icosahedron. In order to obtain triangulations on larger lattices, one either has to use non-regular triangulations, with a few special vertices which might spoil the finite-size scaling (FSS) behavior, or to resort to random triangulations. In this work we use Monte Carlo (MC) simulations to study in detail random triangulations of a sphere and compare the results with our earlier results obtained by using the triangulated surface of a cube as spherical lattice \cite{5}. In addition we present estimates of $\gamma_{\text{str}}$ and the related exponent $\gamma'_{\text{str}}$ \cite{6} on the random lattices, employing the novel FSS method of Ref. \cite{6}.

2. MODEL

For a measurement of the string susceptibilities in Regge calculus one needs to introduce a curvature square term in the action, and then deduces from its expectation value an estimate of $\gamma_{\text{str}}$ and $\gamma'_{\text{str}}$ through FSS analyses. We therefore considered the partition function

$$Z(A) = \int D\mu(q)e^{-\sum_i (\lambda A_i + aR_i^2)\delta(\sum_i A_i - A)},$$ \hspace{1cm} (1)

where $R_i^2 = \delta_i^2/A_i$ denotes the local squared curvatures. The $A_i$ are barycentric areas and $\delta_i = 2\pi - \sum_{j \geq i} \theta_i(t)$ are the deficit angles, with $\theta_i(t)$ being the dihedral angle at vertex $i$. The dynamical degrees of freedom are the squared link lengths, $q = l^2$, which stand in a linear relation to the components of the metric tensor $g$. We...
used the simple scale invariant “computer measure” $D\mu(q) = \left[ \prod_i \frac{d\nu_i}{\nu_i} \right] F_r(\{\nu_i\})$. The notation is identical to that used in Ref. [9].

The only dynamical term is the $R^2$-interaction, because we held $A$ fixed during the update. The $R^2$ coupling constant $a$ sets a length scale of $\sqrt{a}$, and $\hat{A} := A/a$ can be used to distinguish between the cases of weak $R^2$-gravity ($\hat{A} \gg 1$), where the KPZ scaling is recovered, and strong $R^2$-gravity ($\hat{A} \ll 1$) where it was found [8], that $Z(A) \propto A^{-\gamma_{\text{str}}-3} e^{-S_c/\hat{A}} e^{-\lambda_A A^{k\hat{A}}}$, with the classical action $S_c = 16\pi^2(1-g)^2$, some constant $b$, and $\gamma_{\text{str}} = 2-2(1-g)$.

As global lattice topology we used a randomly triangulated sphere constructed according to the Voronoi-Delaunay procedure, see Fig. 1 for a sample lattice. In this way we can control the influence of non-regular triangulations. For spherical topologies we have the relations $N_0 - 2 = N_2/2$, $N_0 - 2 = N_1/3$, and $2N_1 = 3N_2$, where $N_0$, $N_1$, and $N_2$ denote the number of sites, links and triangles, respectively.

3. FINITE-SIZE SCALING

All previously used methods to extract $\gamma_{\text{str}}$ [3–5] are plagued by inconsistencies, as has been discussed in our earlier work [7]. We suggested a new approach which is in spirit much closer to the continuum analysis of Ref. [5] as it uses $\hat{A}$ as the distinguishing parameter between weak and strong $R^2$ gravity. The dimensionless expectation value $\hat{R}^2 := a(\sum_i R_i^2)$ can be shown to depend only on $N_2$ and the dimensionless parameter $\hat{A}$. Sending $N_2 \to \infty$ one expects $\hat{R}^2(\hat{A}, \infty)$ to be expandable in a power series, whose first three terms read as

$$\hat{R}^2(\hat{A}) = \ldots + b_0 \hat{A} + b_1 + b_2/\hat{A} + \ldots,$$

where $b_0 = -\lambda_R - \lambda a$, $b_1 = \gamma_{\text{str}} - 2$ for $\hat{A} \gg 1$, and $b_0 = -b - \lambda_R - \lambda a$, $b_1 = \gamma'_{\text{str}} - 2$, $b_2 = S_c$ for $\hat{A} \ll 1$.

Expanding $\hat{R}^2(\hat{A}, N_2)$ at constant $\hat{A}$ we obtain

$$\hat{R}^2(\hat{A}, N_2) = N_2 d_0(\hat{A}) + d_1(\hat{A}) + d_2(\hat{A})/N_2 + \ldots (3)$$

The next step is to expand the coefficients $d_i$ as a power series in $\hat{A}$. The coefficient $d_1$ carries all

the necessary information to extract the string susceptibilities. A comparison with (2) yields

$$d_1(\hat{A}) = b_0 \hat{A} + \gamma_{\text{str}} - 2 + O(1/\hat{A})$$

for $\hat{A} \gg 1$ and

$$d_1(\hat{A}) = S_c/\hat{A} + \gamma'_{\text{str}} - 2 + b_0 \hat{A} + O(\hat{A}^2)$$

for $\hat{A} \ll 1$. If we plot $d_1$ versus $\hat{A}$ we expect to see a linear behavior for very large $\hat{A}$, and a divergent behavior for small $\hat{A}$, from which we can extract $\gamma_{\text{str}}$ as well as $\gamma'_{\text{str}}$. Because $\hat{R}^2$ in (4) becomes infinite in the continuum limit $N_2 \to \infty$, it was suggested to add a non-scale invariant part $q_{\text{str}}^\prime$ to the measure and fine-tune $\alpha$ such that it cancels the divergent term $d_0$. However, a trial simulation showed within error bars no change in the relevant coefficient $d_1(\hat{A})$ [7].

3. SIMULATION

For each lattice size we generated four different randomly triangulated spheres constructed according to the Voronoi-Delaunay procedure. Usually, the size of the lattices varied from 218 up to 17 498 lattice sites, corresponding to 648 – 52 488
link degrees of freedom, or $432 - 34,992$ triangles. To update the links we used a standard multi-hit Metropolis algorithm with a hit rate ranging from $1, \ldots, 3$. The area was kept fixed with a value of $\tilde{A}$ in the range of $7 - 1800$. For each run on the four copies we recorded about $20,000 - 50,000$ measurements of the curvature square $R^2 := \sum_i R^2_i$ on every second to fourth MC sweep. The statistical errors for each copy were computed using standard jack-knife errors on the basis of 20 blocks. The integrated autocorrelation time $\tau_{R^2}$ of $R^2$ was usually in the range of $5 - 10$. As final error in the average of the four copies we used the standard root mean square deviation.

5. RESULTS

We first begin with a comparison of the raw data for $\hat{R}^2/N_0$ obtained on the randomly triangulated sphere and our earlier data produced on the surface of a cube, see Fig. 2. The difference in $\hat{R}^2$ depends, as could be expected, on the lattice size, such that the difference between the two triangulations decreases as $N_0$ increases. This is also true for the difference in $\hat{R}^2$ between different copies of the random triangulations. Noteworthy is, that for the larger system sizes the copies assume almost the same value within their statistical error.

However, the difference in $\hat{R}^2$ also depends on $\tilde{A}$ such that for small $\tilde{A}$ the value of $\hat{R}^2$ is larger on the cube, whereas for large $\tilde{A}$ the value of $\hat{R}^2$ is smaller on the cube than on the randomly triangulated sphere. This effect is not visible among the four copies of the random triangulations. Strictly speaking this shows that $Z$ of eq. (11) and $\hat{R}^2$ of eq. (3) depend also on the way the manifold is triangulated, i.e. the incidence matrix. Especially for large $\tilde{A}$, i.e. the region which determines $\gamma_{str}$, the values for $d_1$ depend on the triangulation, and one can expect similar values only for very large lattices.

We used linear two-parameter fits in eq. (3) to extract from our raw data for $\hat{R}^2(\tilde{A}, N_2)$ the values $d_1(\tilde{A})$. A number of data points on the smaller lattices had to be discarded until the fit reaches a sufficiently high quality. All data points for $d_1$ obtained in this way are shown in Fig. 3.

In the next step we fitted $d_1(\tilde{A})$ for small values of $\tilde{A} < 120$ according to the Ansatz (5), which yields $S_c = 162(4)$ and $\gamma'_{str} = -0.2(4)$ with a total $\chi^2 = 2.4$, see Fig. 4. This result is perfectly compatible with the theoretical prediction $S_c = 16\pi^2 \approx 158$ and $\gamma'_{str} = 0$.

For large $\tilde{A}$ we can employ Ansatz (4). However, there are only three data points with a sufficiently large $\tilde{A}$ available. The linear fit yields $\gamma_{str} = -10(2)$ with $\chi^2 = 0.8$. It is not clear, however, if we are already in the asymptotic regime, which might set in at much larger $\tilde{A}$. A further difficulty is that one is not interested in the slope, but in the intersection of the fit with the $y$–axis, which is far from the location of the points used in the fit. The systematic uncertainty on our estimate of $\gamma_{str}$ is therefore hard to estimate, but it is interesting to note that our value is too negative, which is just opposite to what has been claimed in Ref. [4] by using a different FSS method.

6. CONCLUSIONS

Random triangulations appear to be a good alternative for topologies where no large regular triangulations exist. They show good scaling behavior, and the differences between different copies of the same area decrease as the system size in-
creases. In this way they can provide a “typical” lattice for the evaluation of expectation values with the partition function of eq. (1).

The quantitative difference in $R^2$ between the non-regular triangulation and the random triangulation of the sphere depends on both, $\hat{A}$ and $N_0$. In this way one will obtain on the usually used system sizes different values of $d_1(\hat{A})$. The difference seems to be negligible for small values of $\hat{A}$, so that $\gamma_{\text{str}}'$ can be consistently obtained on both, the cube and the randomly triangulated sphere. However, the difference becomes important for large values of $\hat{A}$, and is thus a potential problem for the determination of $\gamma_{\text{str}}'$.

Our FSS method of fitting at constant values of $\hat{A}$ gives results for $\gamma_{\text{str}}'$ which are compatible with the theoretical prediction. In contrast to Ref. [5], we employ a consistent FSS scheme and also much larger lattices. It would be interesting to test if contrary to [5] also for topologies of higher gender the theoretical expectations for $\gamma_{\text{str}}'$ can be confirmed.

Due to the fewer data points, only a crude estimate for $\gamma_{\text{str}}$ could be obtained which, however, appeared to be too negative compared to the KPZ theory. This is exactly opposite to what has been found in [4] with a different FSS Ansatz. We attribute this discrepancy to their method, which in our opinion [7], bears conceptual problems for large values of $\hat{A}$. It is yet unclear, if our system sizes are already in the asymptotic scaling regime, so that the potential danger of systematic errors is still very large.

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