A Polynomial-time Algorithm for Detecting the Possibility of Braess Paradox in Directed Graphs

Pietro Cenciarelli        Daniele Gorla
Ivano Salvo
Sapienza University of Rome, Department of Computer Science
cencia@di.uniroma1.it, gorla@di.uniroma1.it, salvo@di.uniroma1.it

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Abstract

A directed multigraph is said vulnerable if it can generate Braess paradox in Traffic Networks. In this paper, we give a graph-theoretic characterisation of vulnerable directed multigraphs; analogous results appeared in the literature only for undirected multigraphs and for a specific family of directed multigraphs. The proof of our characterisation also provides an algorithm that checks if a multigraph is vulnerable in $O(|V| \cdot |E|^2)$; this is the first polynomial time algorithm that checks vulnerability for general directed multigraphs. The resulting algorithm also contributes to another well known problem, i.e. the directed subgraph homeomorphism problem without node mapping, by providing another pattern graph for which a polynomial time algorithm exists.

1 Introduction

Traffic Networks [1, 2] provide a model for studying selfish routing: non-cooperative agents travel from a source node $s$ to a destination node $t$. Since the cost (or latency) experienced by an agent while traveling along a path depends on network congestion (and hence on routes chosen by other agents), traffic in a network stabilizes to the equilibrium of a non-cooperative game, where all agents experience the same latency. This phenomenon has been defined by Wardrop [24] in the context of transport analysis.

For example, consider Fig. 1(1), that depicts the so called Wheatstone network. In the model, edges are labeled by a function (here, the constants 1 and 0, and the identity function $x$) that specifies the latency of each edge in terms of the flow it experiences. So, for example, the edge of latency 0 is “ideal”, in the sense that, independently on how much traffic travels along it, the passage from $b$ to $c$ is instantaneous. By contrast, the delay on passing from $a$ to $b$ and from $c$ to $d$ is linear in the amount of flow traveling along such edges. Thus, every small autonomous flow particle $\epsilon$ will try to use the edge $b \rightarrow c$, because
in this way it will experience a delay of $2\epsilon$ (instead of $1 + \epsilon$, if it had avoided that edge). Consequently, a Wardrop flow of value 1 assigns all the flow to the path $a b c d$ in the picture, and the overall latency is 2.

In traffic networks, a well known and counterintuitive phenomenon is the Braess paradox [1, 3], that originates when latency at Wardrop equilibrium decreases because of removing edges. The Wheatstone network is a minimal example of Braess paradox: Fig. 1(2) shows its optimal subnet. There, a Wardrop flow of value 1 assigns $1/2$ to both paths in the network ($a b d$ and $a c d$), thus obtaining a latency of $3/2$.

Braess paradox has been studied for decades. The results that are most strongly related to ours start with [20], where it is shown that, given a multigraph, a latency function on its edges and the total amount of flow, it is NP-hard to prove whether the resulting net suffers from the Braess paradox or not. An intriguing question raised in [20] (Open Question 1 in Sect. 6.1) is to study Braess paradox from a graph-theoretical perspective, i.e. by considering only a graph and studying whether it admits instances that generate the paradox. This property of a graph has been called vulnerability.

A characterisation of vulnerable undirected multigraphs is presented in [17], where it is proved that an undirected graph is vulnerable if and only if it is not series-parallel [19]. This characterisation only holds for graphs where every node and every edge lie on at least one simple $st$-path. Later, in [4], the same characterisation is proved for directed multigraphs that satisfy the very same condition, called irredundancy therein. However, while checking redundancy and calculating the maximal irredundant subgraph can be efficiently done in undirected graphs [5], the same does not hold for directed graphs. Indeed, in [5] the authors hint at the difficulty of this problem and note that their solution is tractable only for planar digraphs. They also conjecture that the problem of recognizing vulnerable digraphs is untractable, in general.

**Contribution** In this paper, we disprove this conjecture. First of all, we prove NP-hardness of checking and removing redundancy in directed multigraphs, thus formalizing the informal claim in [5]. Then, we provide a graph-theoretic characterisation of vulnerable directed nets, by proving that a directed multigraph is vulnerable if and only if it contains the graph $\mathcal{W}$ underlying the Wheatstone network.
network and depicted in Fig. 1(3). Our constructive proof provides an algorithm to check vulnerability, with an execution time that is $O(|V| \cdot |E|^2)$. This is the first polynomial time algorithm we are aware of for checking vulnerability of general directed multigraphs and fixes the issue left open in [5].

Finally, our characterisation and the resulting polynomial time algorithm contribute to another well known problem: the directed subgraph homeomorphism problem without node mapping [8]. This problem consists in fixing a pattern graph ($\mathcal{W}$ in our case) and try to find an homeomorphic copy of it within an input graph. In [8], it is proved that the general problem is NP-hard unless the pattern graph has all nodes with indegree at most 1 and outdegree at most 2, or indegree at most 2 and outdegree at most 1 (as for $\mathcal{W}$); pattern graphs of this kind, for which a polynomial time algorithm exists, are known in the literature [10]. However, as far as we know, no polynomial time algorithm for all this class of pattern graphs has been devised so far. Thus, our work also contributes to this research line, by providing another pattern graph for which a polynomial time algorithm exists.

**Related work** As we already said, the most strongly related papers are [4, 5, 17].

In [17], vulnerable undirected multigraphs are characterised as the non-series-parallel ones that, in turn, are those containing $\mathcal{W}$ as homeomorphic subgraph. Moreover, this work also characterises the undirected networks where all Wardrop equilibria are weakly Pareto efficient; these turn out to be the nets with linearly independent routes (i.e., those nets where every st-path has at least one edge that belongs only to that path). Linear independence of routes is also shown to be the characterisation of those undirected nets that are efficient under heterogeneous players (i.e., nets where different players can have different latencies on the same edges). Furthermore, linear independence is characterised by not having as homeomorphic subgraph any of three elementary nets (one of which is $\mathcal{W}$).

In [4, 5], directed networks are considered, both in their single commodity version (i.e., with just one pair of source and target) and in the multicommodity one (i.e., with many such pairs). Assuming irredudancy of the net (i.e., that every node and edge lies on at least one st-path, for some st pair), they prove that, for single commodity nets, vulnerability coincides with not being series-parallel (like in [17]); the result is then properly generalised to multicommodity nets. As we show in this paper, checking irredudancy of a net and calculating the maximal irredudant subnet are computationally difficult problems. Indeed, in [5] an algorithm for checking vulnerability is provided only for undirected and for planar directed nets.

An orthogonal bunch of works [14, 20, 21] has been devoted to the complexity of estimating the Braess ratio (i.e., the maximum ratio between the Wardrop latency of $G$ and of any its subnet) and the price of anarchy (i.e., the worst-case ratio between the values of any Wardrop flow and of the optimal one). These works show that both measures have very strong inaproximability, both for the single and for the multicommodity directed scenarios.
Other works have been carried out to graph-theoretically characterise networks with similar kinds of games [7, 11, 12, 16, 18]. These works differ from ours in the kind of efficiency one aims at or in the model of the game. For example, in [16] the aim is to have all players with the same latency in all equilibria, whereas in [7] the aim is to characterise nets whose equilibria all minimize the maximum latency of every path, in a framework where every player can only choose one path and send along it just one information unit.

Finally, an orthogonal paper is [9], where Braess paradox is generalized to all congestion games; structures that do not suffer of Braess paradox are then characterised in terms of matroids. Their elegant result, differently from ours, it is not directly related to graph-theoretic concepts, nor it provides any algorithmic procedure.

**Organization of the paper** We start in Section 2 by giving the basic notions on traffic networks and Braess paradox; this leads to the definition of vulnerability. Then, in Section 3 we show that the only existing characterisation of vulnerability for directed multigraphs [4, 5] cannot yield a polynomial algorithm for general multi-digraphs, unless P = NP. In Section 4 we provide our characterisation. In Section 5 we show how this characterisation yields a polynomial time algorithm and then show how this can be used for the directed subgraph homeomorphism problem. In Section 6 we conclude the paper.

## 2 Traffic Networks, Braess Paradox and Vulnerability

We start by providing the necessary background, by essentially following the presentation in [20]. A directed multigraph (or multi-digraph) $G = (V, E)$ consists of a set $V$ of vertices (or nodes) and a set $E$ of edges. Every edge relates a pair of vertices; if $e$ relates $(u, v)$, we say that $e$ is an output of $u$ and an input of $v$. We denote with $in(x)$ and $out(x)$ the edges entering into $x$ and exiting from $x$, respectively. When the name of an edge is not relevant but only its extremes are, we denote an edge that relates $(u, v)$ as $u \rightarrow v$.

A path is a sequence $u_1e_1u_2 \ldots u_{n-1}e_{n-1}u_n$ (for $n \geq 1$) of nodes and edges such that $e_i$ relates $(u_i, u_{i+1})$, for all $i < n$; $u_1$ and $u_n$ are called extremes and $u_2, \ldots, u_{n-1}$ are called internal nodes. When only the extremes of the path $p$ are relevant, we shall write $u_1 \xrightarrow{p} u_n$ (or simply $u_1 \sim u_n$ if we ignore the name given to the path). If each $u_i$ appears just once in $p$, we say that $p$ is simple (or acyclic). We fix a source node, written $s$, and a target node, written $t$. The resulting triple $(G, s, t)$ will be called net; we shall sometimes simply write it $G$, i.e., without specifying the source and target (that we leave understood). An st-path is a path from $s$ to $t$. The set of st-paths in $G$ is denoted by $P(G)$; the set of simple st-paths is denoted by $SP(G)$. We say that a net is st-connected if all vertices of $G$ belong to an st-path.

A flow for a net $(G, s, t)$ is a function $\varphi : SP(G) \rightarrow \mathbb{R}^+$. The value of a flow is the sum of the values sent over all paths. A flow induces a unique flow $\varphi(e)$
on edges: for any edge \( e \in E \), \( \varphi(e) = \sum_{p \in SP(G): e \in p} \varphi(p) \). A latency function \( l_e : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) assigns to each edge \( e \) a latency that depends on the flow on it; as usual, we only consider continuous and non-decreasing latency functions. The latency of an \( st \)-path \( p \) under a flow \( \varphi \) is the sum of the latencies of all edges in the path under \( \varphi \), i.e., \( l_p(\varphi) = \sum_{e \in p} l_e(\varphi(e)) \). If \( H \) is a subgraph of \( G \), we denote with \( l|_H \) the restriction of the latency function \( l \) on the edges of \( H \).

Given a net \( G \), a real number \( r \in \mathbb{R}^+ \) and a latency function \( l \), we call the triple \((G,r,l)\) an instance. A flow \( \varphi \) is feasible for \((G,r,l)\) if the value of \( \varphi \) is \( r \). Notice that, since we do not have any constraint on edges or vertices, every \( r \) admits at least one feasible flow.

A feasible flow \( \varphi \) for \((G,r,l)\) is at Wardrop equilibrium (or is a Wardrop flow)\(^1\) if, for all pairs \( p,q \) of \( st \)-paths such that \( \varphi(p) > 0 \), we have \( l_p(\varphi) \leq l_q(\varphi) \). In particular, this implies that, if \( \varphi \) is a Wardrop flow, all \( st \)-paths to which \( \varphi \) assigns a positive flow have the same latency. It is known [20] that every instance admits a Wardrop flow and that different Wardrop flows for the same instance have the same latency along all \( st \)-paths with a positive flow. Thus, we denote with \( L(G,r,l) \) the latency of all \( st \)-paths with positive flow at Wardrop equilibrium. In the special case where \( r = 0 \), we let \( L(G,r,l) \) be 0.

Braess paradox [1,3] originates when latency at Wardrop equilibrium decreases because of removing edges (or equivalently, by raising the latency function on edges): an instance \((G,r,l)\) suffers from Braess paradox if there is a subgraph of \( G \) with a lower latency.

In [20] it is shown that, given an instance \((G,r,l)\), it is NP-hard to prove whether it suffers from the Braess paradox or not. An intriguing question raised in [20] is to study the problem from a graph-theoretical perspective, i.e. by only having \( G \) and studying whether \( G \) admits instances that generate the paradox. This leads to the following definition [20].

**Definition 2.1.** A net \( G \) is vulnerable if there exist a value \( r \), a latency function \( l \) and a subgraph \( H \) of \( G \) such that \( L(G,r,l) > L(H,r,l|_H) \).

### 3 On the Complexity of Irredundancy in Directed Nets

All characterisations of vulnerable nets we are aware of are for irredundant nets, that are nets containing only irredundant vertices and edges. As defined in [4] (and derived from [17]), a vertex or an edge is irredundant if it appears in a simple \( st \)-path, and redundant otherwise. Redundant vertices and edges can be safely ignored when studying the Braess phenomenon in a multigraph: they will never be touched by a Wardrop flow, since only acyclic paths have a positive flow. The characterisation given by [4] provides a polynomial time algorithm for checking vulnerability for irredundant nets, where it suffices to check whether the net is series-parallel (this can be done in linear time, by following [23]).

\(^1\) Here we use as definition the characterisation of Wardrop flow put forward by Proposition 2.2 in [20].
a polynomial reduction from a redundant net to an irredundant subnet with the same simple paths existed, then a polynomial time algorithm would exist also for redundant nets. As we now show, this is not possible unless \( P = \mathbb{NP} \). Furthermore, we will show that also verifying if a net is irredundant is an \( \mathbb{NP} \)-complete problem. Therefore, checking if a net satisfies the preconditions prescribed by the algorithm in [5] cannot be done in polynomial time, unless \( P = \mathbb{NP} \).

First, we observe that irredundancy can be checked by only focusing on edges.

**Fact 1.** If every edge of \((G, s, t)\) is irredundant, then every node is irredundant.

*Proof.* Assume that \( v \) is redundant: this means that every \( st \)-path containing \( v \) is not simple. Fix one of these paths \( p \) and consider the edges of \( p \) that are input and output of \( v \): they cannot be irredundant, otherwise also \( v \) would be. Contradiction.

We start with an easy characterisation of irredundant edges. By exploiting it, we then move to the problems of: (1) checking whether an edge is irredundant; (2) deriving from any net its maximal irredundant subnet with the same acyclic paths; and (3) checking whether a net is irredundant. We will prove that all these three problems are \( \mathbb{NP} \)-hard.

**Fact 2.** An edge \( e = u \to v \) is irredundant in the net \((G, s, t)\) if and only if there exist two node-disjoint paths \( s \leadsto u \) and \( v \leadsto t \).

*Proof.* The ‘if’ part is trivial: the \( st \)-path \( s \leadsto u \) \( e \to v \) \( \leadsto t \) is acyclic and contains \( e \). For the ‘only if’ part, by definition, there exists a simple \( st \)-path \( v_0 e_0 v_1 e_1 \ldots v_n \) (for \( v_0 = s \), \( v_n = t \) and \( n \geq 1 \)) such that \( e = e_i \) for some \( i \in \{0, \ldots, n-1\} \). Thus, \( s \leadsto u \) is \( v_0 e_0 v_1 \ldots v_i \) and \( v \leadsto t \) is \( v_{i+1} \ldots v_n \): these paths are node-disjoint because the original path was simple.

**Definition 3.1.** Edge-wise \( st \)-irredundancy (denoted as \( \text{EW-st-IRR} \)) is the problem of deciding whether, given a multi-diagraph \( G = (V, E) \), two distinct nodes \( s, t \in V \) and an edge \( e \in E \), the edge \( e \) is irredundant in the net \((G, s, t)\).

We first observe that \( \text{EW-st-IRR} \) belongs to \( \mathbb{NP} \). By definition, a certificate for the edge \( e \) in \( G \) to be irredundant is a simple path from \( s \) to \( t \) containing \( e \).

**Proposition 3.1.** \( \text{EW-st-IRR} \) is \( \mathbb{NP} \)-hard.

*Proof.* Thanks to Fact 2 we show that \( \text{EW-st-IRR} \) can be easily used to solve \( 2\text{-DPP} \) (the \( 2 \)-disjoint paths problem) [15]:

\( 2\text{-DPP} \): given a directed graph \( G = (V, E) \) and four distinct vertices \( x, y, w, z \), decide whether there exist \( x \leadsto y \) and \( w \leadsto z \) node-disjoint.

The polynomial reduction is the following. If \( G \) contains an edge \( y \to w \), it suffices to check whether such an edge is \( xz \)-irredundant. Otherwise, we consider the graph \( G' = (V, E \cup \{e\}) \), where \( e \) is a new edge relating \( (y, w) \), and check whether \( e \) is \( xz \)-irredundant in \( G' \).
NP-completeness of checking whether an edge is irredundant in a net will now be used to show NP-hardness of the problem of extracting from a given net its maximal equivalent (with respect to vulnerability) irredundant subnet. Indeed, we shall prove that a net and any of its subnets are equivalent with respect to vulnerability if they have the same set of simple paths (see Proposition 4.2 later on).

**Definition 3.2.** The maximal irredundant subnet problem (denoted as MIS) is the problem of calculating, given a net $(G, s, t)$, its maximal irredundant subnet $(G', s, t)$ such that $SP(G) = SP(G')$. We denote $G'$ by $\text{MIS}(G, s, t)$

**Lemma 3.2.** MIS$(G, s, t)$ is unique.

**Proof.** Redundant vertices and edges are univocally determined, once fixed $G$, $s$ and $t$. Moreover, removing redundant edges does not change the set of simple paths in $G$. So, it suffices to delete them from $G$ in order to obtain a subnet $(G', s, t)$ that is irredundant but has the same simple $st$-paths as $G$. Indeed, in passing from $G$ to $G'$, we have only removed edges and vertices that either do not belong to any $st$-path (viz., those vertices that are unreachable from $s$ or that cannot reach $t$, and edges incident to them) or belong only to cyclic $st$-paths. To show uniqueness, notice that: if we remove other edges/vertices, the resulting net will loose some acyclic $st$-path; if we remove not all these edges/vertices, the resulting net will still be redundant. $\blacksquare$

**Proposition 3.3.** MIS is NP-hard.

**Proof.** Thanks to Lemma 3.2, MIS can be easily used to solve EW-$st$-IRR: an edge is $st$-irredundant for $(G, s, t)$ if and only if it appears in MIS$(G, s, t)$. $\blacksquare$

Finally, also the problem of checking whether a given net is irredundant is an NP-complete problem. In principle, this problem could be computationally simpler: there could exist some characterisation that provides a polynomial test to check if a net is redundant or not but without giving any hint on which edges are redundant and which ones are not.

**Definition 3.3.** The $st$-irredundancy problem (denoted as $st$-IRR) is the problem of deciding whether, given a multi-digraph $G = (V, E)$ and two distinct nodes $s, t \in V$, the net $(G, s, t)$ is irredundant.

We first observe that $st$-IRR belongs to NP. A certificate for $G$ to be irredundant are $|E|$ simple $st$-paths such that the $i$-th path contains the $i$-th edge of $E$. We now reduce EW-$st$-IRR to $st$-IRR: given a net $G$ and an edge $u \rightarrow v$, we build a new net $(G^*, s^*, t^*)$ that is redundant if and only if $u \rightarrow v$ is redundant in $(G, s, t)$.

Intuitively, our reduction is based on the following ideas: (1) if $u \rightarrow v$ is irredundant, then there exists a simple path $p$ in $G$ of the form $s \leadsto u \rightarrow v \leadsto t$, and (2) for each vertex $x$, adding new edges of the form $u \rightarrow x$ and $x \rightarrow v$ does not make irredundant the edge $u \rightarrow v$ in $G^*$ if it is redundant in $G$. By using
such additional edges, if \( u \to v \) is irredundant in \( G \), we can make irredundant any edge \( x \to y \) of \( G \) thanks to the path \( s \sim u \to x \to y \to v \sim t \) that is simple provided that \( x \) and \( y \) do not occur in \( p \).

To correctly handle the case in which \( x \) or \( y \) occur in \( p \), we build a net \((G^*, s^*, t^*)\) that consists of two copies \( G' \) and \( G'' \) of \( G \) obtained by decorating each node in \( G \) with \( ' \) and \( '' \), respectively. Each edge \( x' \to y' \) in \( G' \) (resp. \( x'' \to y'' \) in \( G'' \)) will be irredundant in \( G^* \) whenever \( u \to v \) is irredundant in \( G \) thanks to a path of the form \( s^* \sim s'' \sim u'' \sim x' \to y' \sim v'' \sim t'' \sim t^* \) (resp. \( s^* \sim s' \sim u' \sim x'' \to y'' \sim v' \sim t' \sim t^* \)) obtained by adding in \( G^* \) paths \( u' \sim x'', u'' \sim x', x' \sim v'', \) and \( x'' \sim v' \), for all nodes \( x' \in V', x'' \in V'' \).

To ensure that redundancy of \( u \to v \) in \( G \) implies redundancy of \( u' \to v' \) and \( u'' \to v'' \) in \( G^* \), we will define \( G^* \) in such a way that a simple path entering in \( G' \) via \( s' \) (resp. in \( G'' \) via \( s'' \)) can reach the target \( t^* \) of \( G^* \) only through the node \( t' \) (resp. \( t'' \)), possibly after only one detour in \( G'' \) (resp. \( G' \)). This is guaranteed by additional nodes \( z', z'', r', r'', a', a'' \) and their incident edges (see Fig. 2). A simple path can enter in \( G' \) (resp. \( G'' \)) only through nodes \( z' \) and \( r' \) (resp. \( z'' \) and \( r'' \)): this implies that this path can reach \( t^* \) only through the path \( t' \to a'' \to z'' \to t^* \) (resp. \( t'' \to a' \to z' \to t^* \)). As a consequence, this path can
Theorem 3.4. \textit{st-IRR} is NP-hard.

\textbf{Proof.} Given an instance for \textbf{EW-st-IRR} (i.e., a multi-digraph $G = (V, E)$, two distinct vertices $s, t \in V$ and $u \rightarrow v \in E$), we first observe that edges entering into $s$ or exiting from $t$ are trivially redundant; so, we can assume that $v \neq s$ and $u \neq t$. Now, we build a new net $G''$ such that $(G', s, t, v)$ is irredundant if and only if $u \rightarrow v$ is irredundant in $(G, s, t)$. The construction is as follows:

- Create two isomorphic node-disjoint copies of $G$, call them $G' = (V', E')$ and $G'' = (V'', E'')$, with $V' = \{x' : x \in V\}$, $V'' = \{x'' : x \in V\}$, $E' = \{e' : e \in E\}$ and $E'' = \{e'' : e \in E\}$, where $e'$ and $e''$ relate $(x', y')$ and $(x'', y'')$ respectively, if $e$ relates $(x, y)$.

- Define $G^*$ as $(V^*, E^*)$, where
  \[
  V^* = V' \cup V'' \cup \{s^*, z', z'', a', a'', r', r'', t^*\}
  \]
  \[
  E^* = (E' \setminus (\text{in}(s') \cup \text{out}(t'))) \cup (E'' \setminus (\text{in}(s'') \cup \text{out}(t'')))
  \]
  \[
  \cup \{s^* \rightarrow z', z' \rightarrow r', r' \rightarrow s', t' \rightarrow a'', a'' \rightarrow z'', z'' \rightarrow t^*,
  s^* \rightarrow z'', z'' \rightarrow r'', r'' \rightarrow s'', t'' \rightarrow a', a' \rightarrow z', z' \rightarrow t^*\}
  \]
  \[
  \cup \{u' \rightarrow a', r'' \rightarrow v'\} \cup \bigcup_{x'' \in V''} \{a'' \rightarrow x'', x'' \rightarrow r''\}
  \]
  \[
  \cup \{u'' \rightarrow a'', r' \rightarrow v''\} \cup \bigcup_{x' \in V'} \{a'' \rightarrow x', x' \rightarrow r'\}
  \]

The graphical representation of this construction is given in Fig. 2.

\textbf{Irredundancy:} Let us first assume that $u \rightarrow v$ is irredundant in $(G, s, t)$ and show that every edge in $G^*$ is irredundant. By definition, irredundancy of $u \rightarrow v$ implies the existence of a simple $st$-path in $G$ that contains it; let $s \xrightarrow{p_1} u \xrightarrow{p_2} v \xrightarrow{p_3} t$ be such a path. First, note that no edge belonging to $\text{in}(s') \cup \text{out}(t')$ can appear in this path, since all these edges are redundant; for this reason, we have excluded them in $G^*$.

To show that a generic edge $x' \rightarrow y' \in E' \setminus (\text{in}(s') \cup \text{out}(t'))$ is irredundant, let us consider the path $s^* \rightarrow z' \rightarrow r' \rightarrow s'' \xrightarrow{p_2''} u'' \rightarrow a'' \rightarrow x' \rightarrow y' \rightarrow r' \rightarrow v'' \xrightarrow{p_2'} t'' \rightarrow a' \rightarrow z' \rightarrow t^*$, where $p_1$ and $p_2$ denote $p_1$ and $p_2$ within $G''$. By considering such paths for every $x' \rightarrow y' \in E'$, all edges in $\{s^* \rightarrow z', z'' \rightarrow r'', r'' \rightarrow s'', u'' \rightarrow a'', a'' \rightarrow x', y' \rightarrow r', r' \rightarrow v'', t'' \rightarrow a', a' \rightarrow z', z' \rightarrow t^*\}$ are proved irredundant, for every $x', y' \in V'$.

The remaining edges of $E^*$ are “dual” (i.e. have $'$ and $''$ swapped) of the ones considered so far and can be proved irredundant in $G^*$ by considering the dual version (i.e. with $'$ and $''$ swapped everywhere) of the path above.
**Redundancy:** Let us now assume that \( u \to v \) is redundant in \((G, s, t)\) but, by contradiction, assume that there is a simple \( s^*t^* \)-path \( p \) in \( G^* \) that contains \( u' \to v' \). Let us now yield a contradiction by reasoning on how \( p \) has entered into \( G' \) for the first time:

1. **Through \( r' \):** We first observe that \( r' \) can only be reached through \( z' \): the only edges different from \( z' \to r' \) entering into \( r' \) are those of the form \( x' \to r' \), for \( x' \in V' \); but using these edges would mean that \( p \) has already entered into \( G' \). Second, we observe that the only way for reaching \( t^* \) with a simple path is through the path \( t' \to a'' \to z'' \to t^* \), because \( z' \) has already been touched by \( p \). Finally, since \( G' \) and \( G \) have the same simple paths, \( p \) cannot lie only within \( G' \), otherwise it could be used to show irredundancy of \( u \to v \) in \( G \). So, \( p \) must eventually enter into \( G'' \) (and then come back into \( G' \)). We have two possibilities on how \( p \) has first entered into \( G' \) through \( r' \):

   (a) **through the edge \( r' \to s' \):** in this case, \( p \) cannot reach \( G'' \) through a path of the form \( x' \to r' \to v'' \) otherwise it would be cyclic. Thus, \( p \) has reached \( G'' \) through a path of the form \( u' \to a' \to x'' \), for some \( x'' \in V'' \). However, \( p \) cannot pass through \( u' \to v' \) before leaving \( G' \), otherwise it would be cyclic (since it would touch \( u' \) twice). For the same reason, \( p \) cannot pass through \( u' \to v' \) when it comes back into \( G' \). Contradiction.

   (b) **through the edge \( r' \to v'' \) and by then entering into \( G' \) from \( G'' \):** in this case, we have two possible ways for reaching \( G' \), i.e. through a path of the form \( u' \to a'' \to x' \) or through a path of the form \( x'' \to r'' \to v' \). The first way would force \( p \) to touch \( a'' \) twice (the second one when passing from \( t' \) to \( t^* \)). But also the second way would make \( p \) cyclic because, to include \( u' \to v' \), it should touch \( v' \) twice. Contradiction.

2. **Through \( r'' \):** In this case, \( G' \) cannot be reached through \( r'' \to v' \) otherwise, to include \( u' \to v' \), \( p \) should touch \( v' \) twice. Thus, it must be that, from \( r'' \), \( p \) follows a path of the form \( r'' \to s'' \xleftarrow{y'} u'' \to a'' \to x' \), for some \( x' \in V' \). However, having touched \( a'' \), the only possible way for \( p \) to acyclically reach \( t^* \) is by coming back into \( G'' \), reaching \( t'' \) and then following the path \( t'' \to a' \to z' \to t^* \). Thus, \( p \) cannot come back into \( G'' \) through a path \( u' \to a' \to x'' \); hence, the only possibility is via a path of the form \( y' \to r' \to v'' \), for some \( y' \in V' \). Moreover, to be acyclic, \( p \) has to move from \( v'' \) to \( t'' \) via a path \( p'' \) that only touches vertices that \( p \) has not touched before. However, in this way we would have found an \( s''t'' \)-path \( s'' \xleftarrow{y''} u'' \to v'' \xrightarrow{y''} t'' \) that is acyclic and contains the edge \( u'' \to v'' \). This yields a contradiction because, since \( G'' \) and \( G \) have the same simple paths, such a path could be used to show irredundancy of \( u \to v \) in \( G \).
4 Characterisation of Vulnerable Multi-Digraphs

A characterisation of vulnerable undirected multigraphs is presented in [17]. In particular, in [17] it is proved that an undirected multigraph is vulnerable if and only if it is not series-parallel; moreover, in [17] it is also proved that this holds if and only if the multigraph does not contain a homeomorphic copy of (the undirected version of) \( W \). In [4], it is proved that an irredundant multi-digraph is vulnerable if and only if it is not two-terminal series-parallel (TTSP) [19].

Our main result (Theorem 4.10 in Section 4.4) states that any st-connected multi-digraph is vulnerable if and only if it contains a homeomorphic copy of \( W \) (in the sense made precise by Definition 4.2). Therefore, our characterisation extends the result in [17] to multi-digraphs and generalises the result in [4] to possibly redundant graphs, thus answering to Open Question 1 in Sect. 6.1 of [20]. We obtain this result in four steps.

First, in Section 4.1, we characterise vulnerability for acyclic multi-digraphs (Fact 3): this is an immediate corollary of a result in [4] and standard results about two-terminal series-parallel nets. Second, in Section 4.2, we show that two nets with the same set of simple st-paths have exactly the same Wardrop equilibria and, consequently, removing redundant edges does not affect vulnerability. Third, in Section 4.3, we show that, by carefully analysing a simple cycle in a multi-digraph, we can always either find \( W \) as an homeomorphic subgraph (again according to Definition 4.2), or find at least a redundant edge. Finally, in Section 4.4, we apply such cycle analysis to characterise vulnerability of general (i.e., possibly cyclic) graphs \( G \), by either finding a homeomorphic copy of \( W \) in \( G \) or by finding an acyclic graph with the same simple paths as \( G \).

4.1 Characterising Vulnerability for Acyclic Multi-Digraphs

We start by giving the formal definitions of subgraph homeomorphism and st-embedding. In what follows, we always assume that \( G \) is st-connected. This is not restrictive, as we can always find the maximal st-connected subgraph of \( G \) (through a visit from \( s \) and a backwards visit from \( t \)), without affecting its vulnerability. Indeed, as we show in Section 4.2, edges and vertices that do not lie on a simple st-path will never play any role in flows and, consequently, they do not influence vulnerability.

**Definition 4.1** ([13]). A subgraph homeomorphism between \( H \) and \( G \) is a pair of injective mappings \( (\phi, \psi) \) such that \( \phi : V_H \to V_G, \psi : E_H \to SP(G) \), and for every edge \( e = x \to y \in E_H, \psi(e) \) is a path in \( G \) from \( \phi(x) \) to \( \phi(y) \).

A homeomorphism is called node-disjoint if all paths in \( \text{Image}(\psi) \) are pairwise node-disjoint, up-to their end points.

**Definition 4.2.** An st-embedding of the Wheatstone graph \( W \) (see Fig. 1(3)) into a net \( (G,s,t) \) is a node-disjoint subgraph homeomorphism \( (\phi, \psi) \) between
Let \( W \) and \( G \) such that there are (possibly empty) node-disjoint simple paths from \( s \) to \( \phi(a) \) and from \( \phi(d) \) to \( t \) that are pairwise node-disjoint up-to their endpoints with all paths in \( \text{Image}(\psi) \).

By using results in [4] and by applying the standard result from [6], where it is proved that an acyclic directed graph is TTSP if and only if it does not contain a subgraph homeomorphic to \( W \), we can characterise vulnerability for acyclic multi-digraphs, since acyclicity is a special case of irredundancy.

**Fact 3.** Let \((G, s, t)\) be an acyclic st-connected net. \( G \) is vulnerable if and only if there exists an st-embedding of \( W \) into \( G \).

### 4.2 From Cyclic to Acyclic Graphs

Here we show soundness of our approach by showing that removing edges that do not lie on a simple st-path (i.e., redundant edges) does not affect vulnerability of the original graph.

Notationally, we write \( G' \subseteq G \) whenever \( G = (V, E) \) and \( G' = (V, E') \), with \( E' \subseteq E \); notation \( G' \subset G \) has a similar meaning, with \( E' \subset E \).

**Lemma 4.1.** Let \( G' \subseteq G \) and \( \varphi \) be a Wardrop flow for \((G, r, l)\). If \( \text{SP}(G') = \text{SP}(G) \) then \( \varphi \) is a Wardrop flow for \((G', r, l)\).

**Proof.** By definition, \( \varphi \) is a Wardrop flow if and only if, for every \( p, q \in \text{P}(G) \) such that \( \varphi(p) > 0 \), it holds that \( l_p(\varphi) \leq l_q(\varphi) \). By definition, \( \varphi \) assigns positive flow only to acyclic st-paths in \( G \); thus, \( p \in \text{SP}(G) = \text{SP}(G') \). Moreover, since \( G' \subseteq G \), it holds that \( P(G') \subseteq P(G) \). Thus, trivially, for every \( p, q \in \text{P}(G') \) such that \( \varphi(p) > 0 \), it holds that \( l_p(\varphi) \leq l_q(\varphi) \); this means that \( \varphi \) is a Wardrop flow for \((G', r, l)\). \( \square \)

**Proposition 4.2.** Let \( G' \subseteq G \). If \( \text{SP}(G') = \text{SP}(G) \) then \( G \) is vulnerable if and only if \( G' \) is vulnerable.

**Proof.** Vulnerability of \( G' \) trivially entails vulnerability of \( G \). Let us prove the opposite implication. Let \( H \subseteq G \) be such that \( L(H, r, l|_H) < L(G, r, l) \), for some \( r \) and \( l \). It cannot be \( H = G' \), otherwise \( \text{SP}(H) = \text{SP}(G') = \text{SP}(G) \) and, because of Lemma 4.1, we would have \( L(H, r, l|_H) = L(G, r, l) \).

If \( H \subset G' \), let \( \varphi \) be a Wardrop flow for \( G \). By Lemma 4.1 \( \varphi \) is a Wardrop flow for \( G' \) and hence \( L(G, r, l) = L(G', r, l) \); thus, \( L(H, r, l) < L(G', r, l) \), i.e. \( G' \) is vulnerable.

Otherwise, there is an edge \( e \) belonging to \( H \) but not to \( G' \); this means that \( e \) only belongs to cyclic st-paths of \( G \), because by hypothesis \( \text{SP}(G') = \text{SP}(G) \). Then, consider \( H' \), obtained by removing \( e \) from \( H \). Since \( P(H) \subseteq P(G) \), we have that \( e \) only belongs to cyclic st-paths of \( H \); thus, \( \text{SP}(H') = \text{SP}(H) \). Now, let \( \varphi' \) be a Wardrop flow for \( H \). By Lemma 4.1 \( \varphi' \) is a Wardrop flow for \( H' \) and \( L(H, r, l|_H) = L(H', r, l|_{H'}) \). If \( H' \subset G' \), then \( L(H', r, l|_{H'}) < L(G', r, l|_{G'}) \) and \( G' \) is vulnerable. Otherwise, we can find another edge to be removed from \( H' \); but this procedure has to terminate, eventually yielding that \( G' \) is vulnerable, as desired. \( \square \)
4.3 Dealing with Cycles

In this section, we present our analysis of cycles in a graph. We distinguish three kinds of cycles. In all cases, by analysing a cycle \( C \) in a net \((G, s, t)\), we will come up with one of the following outcomes: 1. we find at least one redundant edge in \( C \); or 2. we find an \( st \)-embedding of \( \mathcal{W} \) in \( G \); or 3. we find a strictly smaller (according to Definition 4.4) cycle \( C' \).

As usual, we start by giving some preliminary definitions and results. To this aim, we adopt the following notation: given a path \( p \triangleq \) \( u_1u_2 \ldots u_n \), for \( n \geq 0 \), we let:

\[
\begin{align*}
[u_1 \mapsto u_{n+1}] & \triangleq u_1u_2 \ldots u_nu_{n+1} (=p) \\
(u_1 \mapsto u_{n+1}) & \triangleq e_1u_2e_2 \ldots e_{n}u_{n+1} \\
(u_1 \mapsto u_{n+1}) & \triangleq e_1e_2u_2e_2 \ldots e_{n}u_{n+1}
\end{align*}
\]

**Definition 4.3.** Let \((G, s, t)\) be a net and let \( C \) be a simple cycle in \( G \).

Any simple path of the form \( s \mapsto u \) such that \( u \in C \) and \( C \cap [s \mapsto u] = \emptyset \) is called an entry path (in \( C \)) and \( u \) is said an entry node; notationally, we shall always denote entry nodes with \( \varepsilon \).

Any simple path of the form \( u \mapsto t \) such that \( u \in C \) and \( C \cap [u \mapsto t] = \emptyset \) is called an exit path (from \( C \)) and \( u \) is an exit node; notationally, we shall always denote exit nodes with \( \xi \).

Clearly, any vertex of a cycle \( C \) can be both an entry and an exit for \( C \) (or it can be neither an entry nor an exit for \( C \)). If \( G \) is \( st \)-connected, then every cycle must have at least one entry and one exit node. When the cycle has just one entry node or just one exit node, we can easily find at least one redundant edge (see Lemma 4.3). Otherwise, we distinguish two kinds of cycles, depending on how entry and exit nodes interleave (see Definition 4.5).

**Lemma 4.3.** Let \((G, s, t)\) be an \( st \)-connected net and \( C \) a cycle of \( G \) with at most one entry or one exit node. Then \( C \) contains at least one redundant edge.

**Proof.** Let \( \varepsilon \) be the only entry node of \( C \). The edge of \( C \) entering into \( \varepsilon \) (call it \( x \mapsto \varepsilon \)) is clearly redundant, because \( x \) is reachable from \( s \) only via \( \varepsilon \). Similarly, if \( \xi \) is the only exit node of \( C \), the edge of \( C \) exiting from \( \xi \) (call it \( \xi \mapsto x \)) is redundant, because \( t \) is reachable from \( x \) only via \( \xi \).

To simplify our analysis of cycles, we find it useful to define \( s \)-minimal cycles that have the pleasant property of having at least an entry path that does not touch any exit path.

**Definition 4.4.** Let \((G, s, t)\) be a net, \( C \) be a simple cycle in \( G \) and \( d(u, v) \) be the length of the shortest path (w.r.t. the number of edges) from node \( u \) to node \( v \).

The distance of \( C \) from the source is \( d_s(C) = \min_{u \in C} d(s, u) \). We will denote with \( \varepsilon^* \) an entry node for \( C \) such that \( d_s(C) = d(s, \varepsilon^*) \).

The distance of \( C \) from the target is \( d_t(C) = \min_{u \in C} d(u, t) \). We will denote with \( \xi^* \) an exit node for \( C \) such that \( d_t(C) = d(\xi^*, t) \).

We say that \( C \) is \( s \)-minimal if, for every cycle \( C' \) in \( G \), \( d_s(C) \leq d_s(C') \).
Lemma 4.4. Let \((G, s, t)\) be an \(st\)-connected net, \(C\) an \(s\)-minimal cycle of \(G\), and \(\varepsilon^*\) an entry node with minimum distance from \(s\). Then, there exists a path \(v \otimes \xi, \varepsilon^*\) such that, for every cycle \(C'\) that contains \(\varepsilon^*\) and for every \(\xi \sim t\) exit path of \(C'\), it holds that \(p \cap (\xi \sim t) = \emptyset\).

Proof. Let \(s \otimes \varepsilon^*\) be a path of minimum length in \(G\) entering in \(\varepsilon^*\), \(C'\) a cycle that contains \(\varepsilon^*\) and \(\xi \sim t\) an exit path for \(C'\). If \(p\) contains a node \(v\) that appears in \(q\) \((v \notin \{\xi, \varepsilon^*\}\), otherwise \(p/q\) would not be an exit/exit path for \(C'\)), then \(p\) has the form \(s \otimes_1 v \otimes_2 \varepsilon^*\) and \(q\) has the form \(\varepsilon^* \otimes_1 v \otimes_2 t\). Then, the cycle \(C'' = v \otimes_2 \varepsilon^* \otimes_3 \xi \otimes_4 v\) is such that \(d_s(C'') \leq d(s, v) < d(s, \varepsilon^*) = d_s(C)\), thus contradicting \(s\)-minimality of \(C\). \(\square\)

Definition 4.5. Given a simple cycle \(C\), we call it splittable if \(C\) is of the form \(v_1 \ldots v_k v_1\) and there exists an \(h \in \{1, \ldots, k\}\) such that all entry nodes of \(C\) are in \(v_1 \ldots v_h\) and all exit nodes of \(C\) are in \(v_h \ldots v_k\).

For a splittable cycle \(C\), we can define an order relation among its vertices: let \(\varepsilon_1\) be the first entry node in \(v_1 \ldots v_h\) and, for every \(x, y\) occurring as vertices of \(C\) with \(x \notin \varepsilon_1\), let \(x \leq_C y\) if \(C = \varepsilon_1 \sim x \sim y \sim \varepsilon_1\). We will denote with \(\varepsilon\) (resp. \(\xi\)) the last entry (resp. exit) node in \(C\) (w.r.t. \(\leq_C\)). Notice that the definition of splittable cycle allows the last entry coincide with the first exit but does not allow the last exit coincide with the first entry.

Definition 4.6. Given a splittable cycle \(C\), we call:

- \(E\) the entry region of \(C\), that is the set of nodes in \([\varepsilon_1 \otimes \hat{\varepsilon}_1]\);
- \(X\) the exit region of \(C\), that is the set of nodes in \([\xi_1 \otimes \hat{\xi}_1]\), if \(\xi_1 \neq \hat{\varepsilon}\), or in \((\xi_1 \otimes \hat{\xi}_1]\), otherwise;
- \(N\) the neutral region of \(C\), that is the set of nodes in \((\hat{\xi}_1 \otimes \varepsilon_1]\).

We call splitter the edge of \(C\) entering into \(\varepsilon_1\), if \(\xi_1 \neq \hat{\varepsilon}\), or the vertex \(\xi_1(= \hat{\varepsilon})\), otherwise.

Notice that, by the above definition, every \(v \in E\) is such that \(\varepsilon_1 \leq_C v <_C \xi_1\) and every \(v \in X\) is such that \(\hat{\varepsilon} <_C v \leq_C \hat{\xi}_1\). This property is crucial for proving Proposition 4.3 that in turn is a key building block in the proof of Lemma 4.8. To ensure this property, we have been forced to consider the exit region and the splitter in different ways, according to whether \(\xi_1 \neq \hat{\varepsilon}\) or not.

Definition 4.7. A chordal path for a simple cycle \(C\) is a path between two non-adjacent vertices of \(C\) that touches \(C\) only in its extremes.

A hyper-chord for a simple cycle \(C\) from \(x\) to \(y\) \((x, y \in C)\) is a simple path \(x \sim y\) that can be decomposed as a set of simple paths \(x = x_1 \sim x_2 \sim x_3 \sim \ldots \sim x_{k-1} \sim x_k = y\), such that:

- every \(x_i \sim x_{i+1}\) is either a chordal path for \(C\) or is an edge of \(C\);
• at least one \( x_i \sim x_{i+1} \) is a chordal path for \( C \).

A hyper-chord \( x \sim^h y \) is said neutral for a splittable cycle \( C \) if \( C \cap (x \sim^h y) \subseteq \mathbb{N} \).

**Fact 4.** Let \( C \) be a splittable simple cycle. Then,

1. If \( p \) is a path from \( E \) to \( X \) whose only vertices of \( E \cup X \) are its extremes, then either \( p \) touches the splitter or \( p \) is a neutral hyper-chord between \( E \) and \( X \).

2. If \( p \) is a path from \( E \) to \( N \) whose only vertex of \( E \) is its starting vertex, then either \( p \) touches the splitter or starts with a chordal path going from \( E \) to \( X \) or to \( N \).

**Proposition 4.5.** Let \( C \) be a splittable cycle and \( x \sim^h y \) a neutral hyper-chord from \( E \) to \( X \). Then, for every entry or exit path \( p \), it holds that \( p \cap (x \sim^h y) = \emptyset \).

**Proof.** Let us suppose that there exists an internal node \( u \) of \( h \) that belongs to an exit path. If \( u \) belongs to a chordal path \( w \sim z \) with \( w \in E \cup N \) and \( z \in X \cup N \), then the path \( w \sim u \sim t \) would be an exit path, contradicting the fact that \( C \) is a splittable cycle (if \( w \in E \)) or the fact that \( w \in N \) (because \( N \) does contain no exit nor entry nodes). Similarly, if \( u \) belongs to an entry path, then the path \( s \sim u \sim z \) would be an entry path, contradicting the fact that \( C \) is a splittable cycle (if \( z \in X \)) or the fact that \( z \in N \).

Given a simple splittable cycle \( C \), we denote with \( f_{EN} \) the first (w.r.t. \( \leq_C \)) vertex of \( N \) target of a neutral hyper-chord from \( E \); we let \( f_{EN} \) be \( \varepsilon_1 \) if there is no neutral hyper-chord from \( E \) to \( N \). Similarly, we denote with \( \ell_{NX} \) the last (w.r.t. \( \leq_C \)) vertex of \( N \) source of a neutral hyper-chord to \( X \); we let \( \ell_{NX} \) be \( \xi \) if there is no neutral hyper-chord from \( N \) to \( X \). Thus, \( \hat{\xi} <_C f_{EN} \leq_C \varepsilon_1 \) and \( \xi \leq_C \ell_{NX} <_C \varepsilon_1 \).

**Lemma 4.6.** Let \( C \) be a splittable simple cycle. If \( f_{EN} \leq_C \ell_{NX} \), then there exists a neutral hyper-chord for \( C \) from \( E \) to \( X \).

**Proof.** First, observe that there must exist a neutral hyper-chord \( p \) from \( E \) to \( N \), otherwise \( f_{EN} \equiv \varepsilon_1 >_C \ell_{NX} \). Similarly, there must exist a neutral hyper-chord \( q \) from \( N \) to \( X \), otherwise \( \ell_{NX} \equiv \hat{\xi} <_C f_{EN} \).

If \( p \) and \( q \) intersect, call \( a \) the first intersection along \( p \) with \( q \). Then, consider \( p', \) the prefix of \( p \) ending in \( a \), and \( q', \) the suffix of \( q \) starting from \( a \). Then, the path \( p',q' \) is a neutral hyper-chord from \( E \) to \( X \).

Consider now the case in which \( p \) and \( q \) do not intersect; thus, \( f_{EN} <_C \ell_{NX} \). Thus, we can always find a pair of vertices \( x \in p \) and \( y \in q \) such that \( x <_C y \) and there exists no other pair \( x' \in p \) and \( y' \in q \) such that \( x <_C x' <_C y' \leq_C y \). Then, consider \( p', \) the prefix of \( p \) ending in \( x \), and \( q', \) the suffix of \( q \) starting from \( y \). Then, \( p',x \leq_C y,q' \) is a neutral hyper-chord from \( E \) to \( X \).

**Lemma 4.7.** If \( C \) is a splittable simple cycle without neutral hyper-chords from \( E \) to \( X \), then all edges of \( C \) between \( \ell_{NX} \) and \( f_{EN} \) are redundant.
Proof. By Lemma 4.6, $\ell_{NX} < f_{EN}$ and, by contradiction, assume that there exists an irredundant edge $x \to y \in \ell_{NX} \subseteq f_{EN}$. This implies that $\ell_{NX} \leq C x$ and $y \leq C f_{EN}$. Let $\hat{E}$ be $E$, if $\hat{E} \neq \xi_1$, and be $E \cup \{\hat{E}\}$, otherwise. If the edge $x \to y$ is irredundant, there should exist a path $p$ from $\hat{E}$ to $x$; let us call $p'$ the suffix of $p$ starting from the last vertex in $\hat{E}$ touched by $p$. If such a vertex belongs to $E$, then, by Fact 4(2), we only have two possibilities:

(a) $p'$ starts with a chordal path from $E$ to some $x' \in N$. Then, $p'$ has to reach $x$ but this cannot be done by touching vertices of $E \cup X$ (by construction of $p'$ and by hypothesis); hence, $p'$ is a neutral hyper-chord from $E$ to $x$. If $x = \ell_{NX} = \xi \in X$, we would contradict the hypothesis that there is no neutral hyper-chord from $E$ to $X$. Hence $x \in N$ and, by definition, $f_{EN} \leq C x$; since $x < y$, we would contradict $y \leq C f_{EN}$.

(b) $p'$ touches the splitter. In this case, $p'$ must reach $x$ by jumping at least one exit node $\xi$, that should be used to reach $t$ after touching $x \to y$. Let $p''$ be the path from $y$ to $\xi$. It cannot touch any vertex in $E$ otherwise, by Fact 4(1), it would touch the splitter (but then $p$ would be cyclic) or would contain a neutral hyper-chord from $E$ to $X$. So, it must be that $p''$ is a neutral hyper-chord from $y$ to $X$. Like before, if $y = f_{EN} = \varepsilon_1 \in E$, we would contradict the hypothesis that there is no neutral hyper-chord from $E$ to $X$. Hence $y \in N$ and, by definition, $y \leq C \ell_{NX}$; since $x < y$, this would contradict $\ell_{NX} \leq C x$.

If $\hat{E} = \xi_1$, it can also be possible that the last vertex of $\hat{E}$ touched by $p$ is the splitter; we reason like in case (b) above.

Now, we are ready to prove our main result about splittable cycles.

Lemma 4.8. Let $G$ be an st-connected multidigraph and $C$ be an s-minimal splittable cycle in $G$ with at least two entry and two exit nodes. Then either:

- $C$ contains at least one redundant edge; or
- $W$ is st-embeddable in $G$; or
- there exists an s-minimal cycle $C'$ such that $d_t(C') < d_t(C)$.

Proof. If there is no neutral hyper-chord from $E$ to $X$, then, by Lemma 4.7, all edges in $C$ between $\ell_{NX}$ and $f_{EN}$ are redundant. Otherwise, if $w \sim z$ is a neutral hyper-chord from $E$ to $X$ in $C$, we first observe that we have the following situation (left) and hence the following homeomorphic copy of $W$ in $G$ (right):

\[
\begin{array}{c}
s \rightarrow s' \quad w \rightarrow z \quad t' \rightarrow t \\
\end{array}
\]

\[
\begin{array}{c}
\varepsilon'' \quad \xi' \\
\end{array}
\]

\[
\begin{array}{c}
\varepsilon' \quad \xi'' \\
\end{array}
\]
Observe that we can always assume that one between $\xi'$ and $\xi''$ is $\xi^*$, the exit node at minimum distance from $t$. Moreover, we choose exit paths $\xi' \leadsto t$ and $\xi'' \leadsto t$ among those of minimum length and that, once they intersect for the first time (in $t'$), they become the same path (at most, $t' = t$ and this path is empty). The same assumption holds for the entry paths $s \leadsto \varepsilon'$ and $s \leadsto \varepsilon''$ (with $s'$ their last intersection), where one between $\varepsilon'$ and $\varepsilon''$ is $\varepsilon^*$. Finally, remember that a neutral hyper-chord from $E$ to $X$ is always disjoint from both entry and exit paths (see Proposition 4.5).

If $s \leadsto \varepsilon'$ is disjoint from $\xi' \leadsto t$ and $\xi'' \leadsto t$ (this happens, e.g., if we can let $\varepsilon^*$ play the role of $\varepsilon'$, i.e., if $\varepsilon^* \leq_C w$ — see Lemma 4.4), what we have just shown is an $st$-embedding of $W$. Otherwise, $\varepsilon'' = \varepsilon^*$ and we distinguish the following scenarios, depending on where the entry path $s' \leadsto \varepsilon'$ touches $\xi' \leadsto t$ and $\xi'' \leadsto t$ (by Lemma 4.3, $s \leadsto s'$ is always disjoint from every exit path; moreover, the intersections can be only in internal nodes, otherwise the paths would not be entry/exit paths). We consider the following cases:

1. $s' \leadsto \varepsilon'$ touches the exit paths only in $(\xi' \leadsto t')$;
2. $s' \leadsto \varepsilon'$ touches the exit paths only in $(\xi'' \leadsto t')$;
3. $s' \leadsto \varepsilon'$ touches both exit paths, but only in $(\xi' \leadsto t')$ and $(\xi'' \leadsto t')$;
4. $s' \leadsto \varepsilon'$ touches the exit paths in $[t' \leadsto t]$.

**Case 1.** Let $\omega$ be the last node of $(\xi' \leadsto t')$ that belongs also to $s' \leadsto \varepsilon'$. By definition of $\omega$, the path $(\omega \leadsto t')$ is disjoint from $s \leadsto \omega$ and $\omega \leadsto \varepsilon'$. This case is depicted in the following diagram (left) and the corresponding $st$-embedding of $W$ is depicted on the right.

![Diagram Case 1](image1)

**Case 2.** As before, by letting $\omega$ be the last node of $(\xi'' \leadsto t')$ that belongs also to $s' \leadsto \varepsilon'$. This case is depicted in the following diagram (left) and the corresponding $st$-embedding of $W$ is depicted on the right.

![Diagram Case 2](image2)
**Case 3**: Let $\alpha$ and $\beta$ two nodes of $s' \sim \varepsilon'$ such that $\alpha$ belongs to $(\xi' \sim t')$ (resp. $(\xi'' \sim t')$) and $\beta$ belongs to $(\xi' \sim t')$ (resp. $(\xi' \sim t')$) with no other nodes of $(\xi' \sim t')$ and $(\xi'' \sim t')$ in $\alpha \sim \beta$. Therefore, we have the $st$-embedding of $W$ below on the right.

![Diagram showing st-embedding](image)

The case where $\alpha \in (\xi'' \sim t')$ and $\beta \in (\xi' \sim t')$ is like the diagrams above, with the positions of $\alpha$ and $\beta$ swapped.

**Case 4**: Since one between $\xi'$ and $\xi''$ is $\xi^*$, let $\omega$ be the last node of $s' \sim \varepsilon'$ that belongs to $\xi^* \sim t$ (it is not important if $\omega$ is on $(\xi' \sim t')$ or in $(\xi'' \sim t')$).

Consider the cycle $C' = \omega \sim \varepsilon' \sim \varepsilon^* \sim \xi^* \sim \omega$ (depicted in black in the figures below): 

![Diagram showing cycle C'](image)

$C'$ is still $s$-minimal, because $\varepsilon^*$ belongs to it. Moreover, since $\omega$ is on a path of minimum length from $\xi^*$ to $t$ and $\omega \sim t$ does not touch $C'$, we have that $d_t(C') \leq d(\omega, t) < d(\xi^*, t) = d_t(C)$. 

**Lemma 4.9.** Let $G$ be an $st$-connected multidigraph and $C$ be an $s$-minimal non-splitting cycle of $G$ with at least two entry and two exit nodes. Then either:

- $W$ is $st$-embeddable in $G$; or
- there exists an $s$-minimal cycle $C'$ such that $d_t(C') < d_t(C)$.

**Proof.** Since $C$ is $s$-minimal, we can fix one entry node to be $\varepsilon^*$ and since $C$ is not splittable we can always find another entry $\varepsilon$ and two exits $\xi', \xi''$ in such a way that $C = \varepsilon^* \sim \xi' \sim \varepsilon \sim \xi'' \sim \xi^*$, with $\xi' \neq \varepsilon$ and one between $\xi', \xi''$ is $\xi^*$ (see picture below (left) – where we assume the same conventions on $s \sim s'$ and $t' \sim t$ as in Lemma 4.8). If $s' \sim \varepsilon$ is disjoint from both exit paths $\xi' \sim t$ and $\xi'' \sim t$, we have the $st$-embedding of $W$ depicted below (right):

![Diagram with st-embedding](image)
Otherwise, we need to distinguish several cases, depending on where $s' \sim \varepsilon$ touches $\xi' \sim t$ and $\xi'' \sim t$. Let $\alpha$ (resp. $\omega$) be the first (resp. last) node along $s' \sim \varepsilon$ that belongs also to an exit path. We will consider all possible positions of $\alpha$ and $\omega$ on exit paths: 1. $\alpha \in (\xi' \sim t')$; 2. $\alpha \in (\xi'' \sim t')$; 3. $\omega \in (\xi' \sim t')$; 4. $\omega \in [t' \sim t]$; 5. $\alpha \in [t' \sim t]$ and $\omega \in (\xi'' \sim t')$.

**Case 1.** In the net depicted below (left), we have the $st$-embedding of $\mathcal{W}$ depicted below (right), independently of other intersections between $\alpha \sim \varepsilon$ and exit paths:

**Case 2.** In the net depicted below (left), we have the $st$-embedding of $\mathcal{W}$ depicted below (right), independently of other intersections between $\alpha \sim \varepsilon$ and exit paths:

**Case 3.** In the net depicted below (left), we have the $st$-embedding of $\mathcal{W}$ depicted below (right), independently of other intersections between $s' \sim \omega$ and exit paths:

**Case 4.** In the net depicted below, we can consider the cycle $C' = \omega \sim \varepsilon \sim \xi'' \sim \varepsilon \sim \xi' \sim t' \sim \omega$ (in black):

19
Since $C'$ contains $\varepsilon^*$, it is still an $s$-minimal cycle; moreover, $d_t(C') \leq d(\omega, t) < \min\{d(\xi', t), d(\xi'', t)\} = d_t(C)$, since $\xi' \sim t$ and $\xi'' \sim t$ are paths of minimum length and $\omega \sim t$ does not touch $C'$.

**Case 5.** If we are not in one of cases 1–4, then the situation is depicted below:

If $\alpha \sim \omega$ does not touch $(\xi' \sim t')$, let $t^*$ be the first node along $t' \sim t$ that appears in $s' \sim \varepsilon$ (of course, $t^*$ can be $\alpha$ or some node between $\alpha$ and $\omega$ along $s' \sim \varepsilon$). We can then consider the cycle $C' = t^* \sim \omega \sim \varepsilon \sim \xi'' \sim \varepsilon^* \sim \xi' \sim t'$ (that has the same shape of the cycle considered in Case 3). $C'$ is $s$-minimal (since it contains $\varepsilon^*$) and $d_t(C') \leq d(t^*, t) < \min\{d(u', t), d(u'', t)\} = d_t(C)$.

Otherwise, $\alpha \sim \omega$ touches $(\xi' \sim t')$ in a set of vertices $B = \{\beta_1, \ldots, \beta_k\}$ and we must have a path from some $\beta \in B$ to some $\gamma$ in $(\xi'' \sim t')$ that has no further intersections with $\xi' \sim t'$ and $\xi'' \sim t'$. If this path does not touch $t' \sim t$ (see picture below, left), we can find an $st$-embedding of $W$ (see picture below, right).

Otherwise, let $\beta^*$ be the first vertex of $B$ along $(\xi' \sim t')$ and $t^*$ the first vertex along $[t' \sim t]$ after $\beta^*$ in $(s' \sim \varepsilon)$:

Then, the path $C' = \beta^* \sim t^* \sim \varepsilon \sim \xi'' \sim \varepsilon^* \sim \xi' \sim \beta^*$ (in black above) is an $s$-minimal cycle (since it contains $\varepsilon^*$) such that $d_t(C') \leq d(t^*, t) < \min\{d(u', t), d(u'', t)\} = d_t(C)$.

### 4.4 Characterising Vulnerability

We are now ready for giving the characterisation of vulnerability for general $st$-multi-digraphs.
Theorem 4.10. Let \((G, s, t)\) be an st-connected net. \(G\) is vulnerable if and only if there exists an st-embedding of \(W\) into \(G\).

Proof. For the ‘if’ part, we know that \(G\) admits a subgraph of the form

\[
s \xrightarrow{u} s' \xrightarrow{v} t' \xrightarrow{v} t
\]

where all paths are node-disjoint, except for the extremal nodes. Let us consider the latency assignment \(l\) that assigns:

- 0 to all edges in \(u \sim v\), in \(s \sim s'\) and in \(t' \sim t\);
- \(x\) to the first edge in \(s' \sim u\) and in \(v \sim t'\) and 0 to all the remaining edges in those paths;
- 1 to the first edge in \(s' \sim v\) and in \(u \sim t'\) and 0 to all the remaining edges in those paths;
- \(\infty\) to all the remaining edges.

In this way, \(G\) behaves like the Wheatstone net; thus, it is vulnerable.

For the ‘only if’ part, if \(G\) is not cyclic, the statement follows by Fact 3. If \(G\) contains cycles, by contradiction, let us suppose that there exists no st-embedding of \(W\) into it. Then, take an s-minimal cycle and, according to its kind, apply one of Lemma 4.3, 4.8 or 4.9. Such results can lead either to deleting redundant edges or to change the cycle. However, since the new cycle is strictly closer to \(t\) than the old one, we must eventually delete some redundant edge(s). By repeating this reasoning, we will eventually transform \(G\) into an acyclic subgraph \(G'\) such that \(SP(G') = SP(G)\). By Proposition 4.2, since \(G\) is vulnerable, \(G'\) is vulnerable too; thus, since \(G'\) is acyclic, there exists an st-embedding of \(W\) into \(G'\) (by Fact 3) that is also an st-embedding of \(W\) into \(G\). Contradiction.

5 Consequences of the Characterisation

5.1 On Polynomially Checking Vulnerability

Stemming from the characterisation of vulnerable nets provided by Theorem 4.10, we define function isVulnerable (its pseudocode is given in Algorithm 1) that detects if \(W\) is st-embeddable in a given net \((G, s, t)\). It essentially implements the procedure hinted to in the ‘only if’ part of the proof of Theorem 4.10.
5.1.1 Algorithm Description

The outer while loop of function isVulnerable (lines 2-20) runs until $G$ is acyclic or an $st$-embedding of $W$ is found (line 18). This loop starts by computing an $s$-minimal cycle $C$ of $G$ (line 3).

The inner repeat loop (lines 4-16) runs until cycle analysis in its body either finds at least a redundant edge to delete or finds an $st$–embedding of $W$. This loop starts by computing the set of entry (line 6) and exit (line 7) nodes of $C$ and then it determines which kind of cycle $C$ is (lines 8-15).

If $C$ has just one entry node (line 9) or just one exit node (line 11), according to Lemma 4.8, a redundant edge is found and it is stored in the set $D$ of edges that will be deleted from $G$.

If $C$ is a splittable cycle, according to Lemma 4.9, function nonSplittableAnalysis (line 13) returns $\langle true, \emptyset, C \rangle$ if it finds a $st$-embedding of $W$ in $G$, $\langle false, D, C \rangle$ if it identifies a set $D \neq \emptyset$ of redundant edges to be deleted from $G$, and $\langle false, \emptyset, C' \rangle$ otherwise, where $C'$ is a cycle closer to $t$ than $C$.

At the beginning of the function (line 1) and after each edge deletion (line 20), we invoke function makeSTConnected in order to ensure the invariant that $G$ is an $st$-connected net.

At the end, if the while loop terminates because $G$ is acyclic, we just verify if $G$ is two-terminal series-parallel or not (line 21).

5.1.2 Complexity Analysis

We start by describing how functions entryNodes, exitNodes, splittableAnalysis and nonSplittableAnalysis can be implemented and by analysing their complexity. Then, we analyse the overall complexity of function isVulnerable.

As it should be clear from the proofs of Lemma 4.8 and 4.9, we only need to solve shortest path problems, reachability problems and to detect intersections between paths. Moreover, since we are dealing with $st$-connected multigraphs, the number of edges $|E|$ is at least $|V| - 1$.

In what follows, to make notation lighter, given $X \subseteq V$, we denote by $in(X)$ (resp, $out(X)$) the set of edges $\{in(x) : x \in X\}$ (resp, $\{out(x) : x \in X\}$).

Fact 5. Functions entryNodes and exitNodes cost $O(|E|)$. Furthermore, they can associate: (1) a minimum length entry/exit path to every entry/exit node; and, (2) the minimum distance from $s$ and from $t$ to every node in such paths.

Proof. Function entryNodes can be implemented via a BFS from $s$ in $(V, E \setminus out(C))$; entry nodes of $C$ are those with finite distance from $s$. Dually, exitNodes computes exit nodes by a backwards BFS from $t$ in $(V, E \setminus in(C))$. By construction, each BFS builds a minimum spanning tree that can be used to find a minimum length path for every visited node; the minimum distance is the height of the node in the tree. This is all done in $O(|E|)$. \qed
Algorithm 1 Checking Vulnerability

Input: A directed multigraph $G = (V, E)$, $s$ is the source and $t$ is the target

function isVulnerable($G, s, t$)
1. $G = makeSTConnected(G, s, t)$
2. while $G$ is cyclic do
3.    $⟨C, ε^∗⟩ = s$-minimalCycle($G, s, t$)
4.    repeat
5.        $C′ = C$; $Vuln = false$
6.        $En = entryNodes(G, C, s)$
7.        $Ex = exitNodes(G, C, t)$
8.        if $En = \{ε\}$ then
9.            $D = \{the \ edge \ of \ C \ that \ enters \ into \ ε\}$
10.       else if $Ex = \{ξ\}$ then
11.          $D = \{the \ edge \ of \ C \ that \ exits \ from \ ξ\}$
12.          if $isSplittable(C, En, Ex, G)$ then
13.              $⟨Vuln, C, D⟩ = splittableAnalysis(C, En, Ex, G)$
14.          else
15.              $⟨Vuln, C⟩ = nonSplittableAnalysis(C, En, Ex, G)$
16.          until $C ≠ C′$
17.         if $Vuln$ then
18.             return true
19.       else
20.           $G = makeSTConnected((V, E \setminus D), s, t)$
21.       return $¬TTSP(G)$

Remark 5.1. We will store the four entry/exit paths needed for the cycle analysis as an array of nodes indexed by the distance of the node from $s$ (for entry paths) or $t$ (for exit paths). These can be efficiently built from the spanning trees produced by the BFSs of Fact 4. In this way, intersections between entry/exit paths within functions splittableAnalysis and nonSplittableAnalysis can be computed in $O(|V|)$. E.g., if we need to find the first (resp. last) intersection along an entry path $p$ with an exit path $q$, it suffices to scan from left to right (resp., from right to left) the array storing $p$: since we are dealing with shortest paths, if the $i$-th node of $p$ has distance $k$ from $t$, it can only occur in the $k$-th position of the array representing $q$. A dual reasoning is needed if $p$ is an exit and $q$ is an entry path.

Lemma 5.1. Function splittableAnalysis costs $O(|E|)$.

Proof. This function implements the case analysis described in the proof of Lemma 4.3. It starts with calculating entry ($E$), exit ($X$), and neutral ($N$) regions of $C$: these are computed just by a scan of $C$ in $O(|V|)$.

The next step is to search for a neutral hyper-chord from $E$ to $X$. This can be done by a backwards BFS in $(V, E \setminus (out(E) \cup in(X)))$ that starts from the set of nodes $X$ and costs $O(|E|)$. This search computes the set $W \subseteq E$ of nodes that are the source of a neutral hyper-chord from $E$ to $X$. 

23
If no such hyper-chord exists \((W = \emptyset)\), we have to compute \(f_{\text{EN}}\), \(f_{\text{NX}}\), and all edges of \(C\) between \(f_{\text{NX}}\) and \(f_{\text{EN}}\) (that are redundant). This can be done by two BFSs (the first starts from all \(E\) and is run in \((V,E \setminus (\text{in}(E) \cup \text{in}(X)))\), the second starts from all \(N\) and is run in \((V,E \setminus (\text{in}(E) \cup \text{out}(X)))\)), followed by a linear scansion of \(C\). The overall cost is again \(O(|E|)\).

Otherwise, we can choose any element of \(W \neq \emptyset\) as \(w\). However, to speed-up convergence of the algorithm, it is convenient to choose \(w\) as the maximum element of \(W\) (w.r.t. \(\leq_C\)). If \(w^* \leq_C w\) (this can be checked in \(O(|V|)\)), we return \((true, \emptyset, C)\). Otherwise, we let \(w''\) be \(w^*\) and consider the first entry node as \(e'\) (this is a simplifying choice: every \(e' \leq_C w\) would work); then, we choose \(\xi'\) and \(\xi''\) so that one of them is \(\xi^*\).

If the entry path in \(e'\) does not intersect exit paths, we return \((true, \emptyset, C)\). Otherwise, we check in which case (among \([a, b, c]\) and \([d]\) described in the proof of Lemma 4.8) we are. By Remark 5.1 this can be done in \(O(|V|)\). In this check, also \(\omega\) (cases \([e, f]\) and \([g]\) and \(\alpha/\beta\) (case \([h]\) can be identified. According to the case in which we fall, we return either \((true, \emptyset, C)\) or \((false, \emptyset, C')\), where \(d_i(C') < d_i(C)\), without any additional computational cost.

Summing up, the overall complexity is \(O(|E|)\).

**Lemma 5.2.** Function nonSplittableAnalysis costs \(O(|V|)\).

**Proof.** Function nonSplittableAnalysis implements the case analysis for non-splittable cycles described in the proof of Lemma 4.9. We pay an \(O(|V|)\) for choosing \(\varepsilon, \varepsilon^*, \eta', \eta''\). Then, we need to check in which of the possible situations put forward by the Lemma we fall: this only requires to list all intersections between two minimum paths, that, by Remark 5.1 costs \(O(|V|)\).

**Theorem 5.3.** Checking whether a net \((G,s,t)\) is vulnerable can be solved in time polynomial to the size of \(G\), in particular its complexity is \(O(|V| \cdot |E|^2)\).

**Proof.** Correctness of function isVulnerable in Algorithm 1 stems from the proof of Theorem 4.10.

As for the inner repeat loop, at every iteration \(d_i(C)\) decreases by at least 1 and thus, this loop terminates after at most \(O(|V|)\) iterations. By Fact 5 and Lemma 5.1 and 5.2 the body of this loop costs \(O(|E|)\). Indeed, lines \([5]\) correspond to Lemma 4.8 whereas function isSplittable checks whether all entry nodes come before all exit nodes in \(C\); both these tasks can be done via a scansion of \(C\) and thus cost \(O(|V|)\).

As for the outer while loop, at each iteration the number of edges of \(G\) strictly decreases by at least 1; thus, this loop terminates after at most \(O(|E|)\) iterations. The complexity of the body of this loop is dominated by the cost of the inner repeat loop, that, by the above considerations, is \(O(|V| \cdot |E|)\). Indeed, making an \(st\)-multidigraph \(st\)-connected (function makeSTConnected invoked in line \([1]\) and \([20]\) consists of a DFS from \(s\) and a backwards DFS from \(t\); all nodes not visited in both DFSs can be deleted together with their incident edges. This costs \(O(|E|)\). An \(s\)-minimal cycle \(C\) and the corresponding \(\varepsilon^*\) (function s-minimalCycle in line \([3]\) can be found in \(O(|V| \cdot |E|)\). This covers
also the complexity of the guard of the while loop in line 2. Indeed, to find $C$ and $\varepsilon^*$, consider every successor $v$ of $s$ and see whether $v$ belongs to some cycle (this can be easily done by a DFS in $G$ starting from $v$). If so, $v$ is $\varepsilon^*$ and the cycle found is $C$. Otherwise, iterate the reasoning with the successors of the successors of $s$; and so on until a cycle is found or all vertices have been considered (in this case $G$ is acyclic).

Finally, checking if an acyclic graph is two-terminal series-parallel can be executed in linear time (see [22, 23]).

To sum up, the overall complexity of our algorithm is $O(|V| \cdot |E|^2)$.

5.2 On the Directed Subgraph Homeomorphism Problem

Theorem 4.10 and Algorithm 1 have consequences on another, long-standing problem: the directed subgraph homeomorphism [8]. The main problem is the following: fixed a pattern graph $H$, determine whether $H$ is homeomorphic to a subgraph of any given $G$ with respect to a given mapping from the nodes of $H$ to the nodes of $G$. This problem is shown to be NP-hard, apart from a very simple family of pattern graphs for which a polynomial time algorithm exists. Furthermore, in the conclusions the authors also discuss the problem when the mapping is not given. Our characterisation falls in this latter case.

In [8] it is suggested that the general problem is still NP-hard. However, in the special case when the pattern graph has all nodes with indegree at most 1 and outdegree at most 2, or indegree at most 2 and outdegree at most 1, this is no more necessary, since there exist pattern graphs of this kind for which a polynomial time algorithm exists [10]. However, as far as we know, no polynomial time algorithm for all this class of pattern graphs has been devised so far. Thus, our work also contributes to this research line: $W$ is another pattern graph for which a polynomial time algorithm for the directed subgraph homeomorphism problem without node mapping exists.

Corollary 5.4. Checking whether a multi-digraph $G$ contains a subgraph homeomorphic to $W$ can be solved in time polynomial to the size of $G$.

Proof. Just run Algorithm 1 for every pair of distinct vertices in $G$ (ordinately playing the role of the source and target) and return true if and only if at least one of these calls returns true. This algorithm costs $O(|V|^3 \cdot |E|^2)$.

6 Conclusions

We have proved a graph-theoretic characterisation of vulnerable multi-digraphs that generalizes the analogous one for undirected multigraphs [17]. Our characterisation improves the results in [4, 5] (given only for irredundant directed multigraphs) to general directed multigraphs; indeed, there exist redundant graphs that are not vulnerable, as shown in Fig. 3. Since graphs of this kind can easily appear in real traffic networks, we believe that our characterisation was a necessary step for completing the knowledge of vulnerability.
Also the resulting algorithm was a necessary step: as we proved in this paper, checking irredundancy and calculating the maximal irredundant subnet are NP-hard problems. Interestingly, a crucial part of our approach is the identification (and the consequent deletion) of redundant edges. However, we want to remark that our algorithm neither identify all such edges in a graph nor can be used to decide if a specific edge is redundant. Indeed, it removes all redundant edges only when it cannot find \( W \). Thus, Algorithm 1 can solve (in polynomial time) MIS for non-vulnerable nets only.

Of course, more efficient algorithms for checking vulnerability should be devised to be used in practice, but this was not our aim here.

References

[1] M. Beckmann, C. B. McGuire, and C. B. Winsten. *Studies in the Economics of Transportation*. Yale University Press, 1956.

[2] M. Bell and Y. Iida. *Transportation Network Analysis*. Wiley, 1987.

[3] D. Braess. Über ein paradoxon aus der verkehrsplanung. *Unternehmensforschung*, 12:258–268, 1968.

[4] X. Chen, Z. Diao, and X. Hu. Excluding braess paradox in nonatomic selfish routing. In *Proc. of SAGT15*, volume 9347 of LNCS, pages 219–230. Springer, 2015.

[5] X. Chen, Z. Diao, and X. Hu. Network characterizations for excluding braess’s paradox. *Theory of Computing Systems*, pages 1–34, 2016.

[6] R. J. Duffin. Topology of series-parallel networks. *Journ. of Math Analysis and Applications*, 10:303–318, 1965.
[7] A. Epstein, M. Feldman, and Y. Mansour. Efficient graph topologies in network routing games. *Games and Economic Behavior*, 66(1):115–125, 2009.

[8] S. Fortune, J. Hopcroft, and J. Wyllie. The directed subgraph homeomorphism problem. *Theoretical Computer Science*, 10(2):111 – 121, 1980.

[9] S. Fujishige, M. Goemans, T. Harks, B. Peis, and R. Zenklusen. Matroids are immune to braess paradox. Available at: arXiv:1504.07545, 2015.

[10] M. S. Hecht and J. D. Ullman. Flow graph reducibility. In *Proc. of STOC ’72*, pages 238–250. ACM Press, 1972.

[11] R. Holzman and N. Law-yone. Network structure and strong equilibrium in route selection games. *Mathematical Social Sciences*, 46(2):193–205, 2003.

[12] R. Holzman and D. Monderer. Strong equilibrium in network congestion games: increasing versus decreasing costs. *Int. J. Game Theory*, 44(3):647–666, 2015.

[13] A. S. LaPaugh and R. L. Rivest. The subgraph homeomorphism problem. *Journal of Computer and System Science*, 20(2):133–149, 1980.

[14] H. C. Lin, T. Roughgarden, É. Tardos, and A. Walkover. Stronger bounds on braess’s paradox and the maximum latency of selfish routing. *SIAM J. Discrete Math.*, 25(4):1667–1686, 2011.

[15] M. Middendorf and F. Pfeiffer. On the complexity of the disjoint paths problem. *Combinatorica*, 13(1):97–107, 1993.

[16] I. Milchtaich. Topological conditions for uniqueness of equilibrium in networks. *Math. Oper. Res.*, 30(1):225–244, 2005.

[17] I. Milchtaich. Network topology and the efficiency of equilibrium. *Games and Economic Behavior*, 57:321–346, 2006.

[18] I. Milchtaich. Network topology and equilibrium existence in weighted network congestion games. *Int. J. Game Theory*, 44(3):515–541, 2015.

[19] J. Riordan and C. Shannon. The number of two-terminal series-parallel networks. *Journal of Mathematics and Physics*, 21:83–93, 1942.

[20] T. Roughgarden. On the severity of Braess’s Paradox: Designing networks for selfish users is hard. *J. Comput. Syst. Sci.*, 72(5):922–953, 2006.

[21] T. Roughgarden and É. Tardos. How bad is selfish routing? *Jornal of the ACM*, 49(2):236–259, 2002.

[22] B. Schoenmakers. A new algorithm for the recognition of series parallel graphs. Technical report, CWI - Centrum voor Wiskunde en Informatica, 1995.
[23] J. Valdes, R. Tarjan, and E. Lawler. The recognition of series-parallel digraphs. *SIAM Journal of Computing*, 11:298–313, 1982.

[24] J. Wardrop. Some theoretical aspects of road traffic research. In *Proc. of the Institute of Civil Engineers, Pt. II*, volume 1, pages 325–378, 1952.