1. INTRODUCTION

The kaon B parameter, $B_K$, is an essential bridge between $K_0 - \bar{K}_0$ mixing experiments and the CKM matrix. Using domain wall fermions (DWF), quenched calculations of $B_K$ have been done \[1,2,3\], where operator mixing and $O(a)$ errors are small because of the good chiral properties of DWF. Including dynamical fermions is an obvious way to improve these calculations.

Using an improved HMC algorithm for DWF \[4,5\], we have generated three ensembles of $b = 3 \times 32$ volumes with $m_{\text{dyn}} = 0.02, 0.03, 0.04$. Our fermion action is the standard DWF action with $L_s = 12$ and $M_5 = 1.8$ and our gauge action is DBW2 with $\beta = 0.80$. Extrapolating the rho mass in $m_{\text{dyn}}$ to the chiral limit, we find a lattice spacing of $a^{-1} = 1.81(6)$ and a residual mass $m_{\text{res}} = 0.00136(5)$. We measure on lattices separated by 50 trajectories and have 32, 84, 57 lattices for $m_{\text{dyn}} = 0.02, 0.03$ and 0.04, respectively.\[^3\] For more details see \[5\] and for results with $\Delta S = 1$ matrix elements see \[6\].

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\[^3\]This includes some data generated after the conference.

2. CALCULATION OF $B_K$

$B_K$ is defined by

$$B_K = \frac{\langle K^0 | O_{LL} | K^0 \rangle}{\frac{2}{3} \langle K^0 | A_{\mu} | 0 \rangle \langle 0 | A_{\mu} | K^0 \rangle},$$

(1)

where $A_{\mu} = \bar{s}\gamma_\mu \gamma_5 d$ and $O_{LL}$ is the $\Delta S = 2$ four-quark operator $(\bar{s}d)_{V-A} (\bar{s}d)_{V-A}$. For a general value of the mass of the pseudoscalars entering $B_{PS}$, we will use the symbol $B_K$ for the case where the pseudoscalars have the kaon mass.

We first evaluate this by the conventional method, where the ratio of a three-point Green’s function and two pseudoscalar–axial-vector correlators is taken:

$$R(t) = \frac{\langle J^0_\mu(t_0)O_{LL}(t)J^0_\mu(t_1) \rangle}{\frac{2}{3} \langle J^0_\mu(t_0)A_{LL}^\mu(t)A_{LL}^\mu(t) \rangle \langle A_{LL}^\mu(t)J^0_\mu(t_1) \rangle}.$$  

(2)

(no sum on $\alpha$). We use Coulomb gauge fixed wall sources at $t_0 = 4$ and $t_1 = 28$ for valence masses, $m_{\text{val}}$, equal to 0.01, 0.02, 0.03, 0.04 and 0.05. Propagators are found with both periodic and anti-periodic temporal boundary conditions and averaged to double the temporal length.

In Figure \[4\] we show $R(t)$. We choose to fit data from $t = 14$ to 17, same fitting range as \[4\], to determine $B_{PS}^\text{lat}$. The error bars in this figure, and throughout this report, assume that our lattices are decorrelated. The fits shown in Figure \[4\] give values of $B_{PS}^\text{lat} = 0.557(10), 0.611(6)$ and 0.645(7) for $m_{\text{dyn}} = m_{\text{val}} = 0.02, 0.03$ and 0.04 respectively. Variations in the plateaus are

$B_K$ from Two-flavor Dynamical Domain Wall Fermions

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We report preliminary results from an ongoing calculation of $B_K$ for $N_f = 2$ dynamical QCD with domain wall fermions. Simulations have been done with three dynamical quark masses on $16^3 \times 32$ volumes with $L_s = 12$, where the lattice spacing is $a^{-1} = 1.81(6)$ GeV. Using measurements on $\sim 70$ lattices for each dynamical mass and extrapolating $m_{\text{dyn}} = m_{\text{val}}$ to the kaon point, we find $B_K^{\text{lat}}(\mu = 2\text{GeV}) = 0.503(20)$. 

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noticeable at the 1-2% level and the assumption of 50 trajectories being sufficient for decorrelation needs further studies and statistics. However, the current data certainly makes a few percent resolution in the lattice values achievable by possibly doubling the statistics.

In Figure 2, we show our results for $B_{\text{PS}}$ for degenerate valence quark masses for each different value of $m_{\text{dyn}}$. The filled points in the figure have $m_{\text{val}} = m_{\text{dyn}}$ and correspond to the fitted lines in Figure 1. (We also have data for non-degenerate valence quarks and with our statistics, only the average quark mass plays a role.) Our simplest result to report for $B_{\text{K}}$ comes from extrapolating the $m_{\text{val}} = m_{\text{dyn}}$ points in Figure 2 to $m_f = \overline{m}_s/2 \equiv 0.018$. (A pseudoscalar with two quarks with $m_f = \overline{m}_s/2$ has the mass of the kaon.) We perform this extrapolation using the partially quenched formula (7)

$$B_{\text{PS}}(m) = b_0 \left( 1 - \frac{6}{(4\pi f)^2} \left( M_{\text{PS}}^2 \log \frac{M_{\text{PS}}^2}{\Lambda_{\chi\text{PT}}^2} \right) \right) + b_1 M_{\text{PS}}^2 ,$$

fitting the three points with $m_{\text{val}} = m_{\text{dyn}}$ to determine $b_0$ and $b_1$. We find $b_0 = 0.267(12)$, $b_1 = 1.256(79)$ and $B_{\text{PS}}^\text{lat} = 0.541(9)$ (statistical error only), with a small $\chi^2$/d.o.f. Note that we are using the known coefficient of the chiral logarithm, $M_{\text{PS}}^2 = 4.06 \times (m_f + m_{\text{res}}), f = 0.0786$ and $\Lambda_{\chi\text{PT}} = 1$ GeV. The statistical error in the lattice spacing gives an additional 2% systematic error in the value of $B_{\text{PS}}^\text{lat}$, due to the uncertainty in the strange quark mass.

While our data makes extrapolation to $m_f = \overline{m}_s/2$ quite robust, the value of $b_0$ is much less solid. A small value in the chiral limit has also been seen in the quenched calculations, but more control over the extrapolation is needed.

We have also fit the dependence of $B_{\text{PS}}$ on $m_{\text{val}}$ to (9) for each fixed $m_{\text{dyn}}$, using the five values for $m_{\text{val}}$. The interpolated values for $B_{\text{K}}^\text{lat}(m_{\text{val}} = \overline{m}_s/2, m_{\text{dyn}})$ are 0.537(11), 0.557(9) and 0.568(10) for $m_{\text{dyn}} = 0.02, 0.03$ and 0.04 respectively. We reduce the fitting range to exclude the heaviest two points and see the results stay same within error.

An alternative method of extracting $B_{\text{K}}$ is to divide the three-point function in the numerator of (1) by a wall-wall pseudoscalar correlator (1). This removes some zero mode effects in the quenched approximation, while giving larger statistical errors. However here we see no difference in the central values for results calculated using these two methods.

3. RENORMALIZATION AND MIXING

To get to $B_{\text{K}}^\text{MS}$ we need to renormalize $B_{\text{PS}}^\text{lat}$ and control mixing with other dimension six, chirally disallowed operators. Using NPR [9] in the quenched theory we have seen that mixing co-
efficients with chirally disallowed operators are small. Preliminary NPR calculations on our ~ 40 dynamical DWF lattices also show that the mixing coefficients are small (less than a few percent).

These NPR results are preliminary, since we have not fully investigated all the mass dependence of the procedure and our statistics are somewhat limited. However, we can also use the \( \Omega \) parameter introduced in [8] to quantify the chiral symmetry breaking of DWF and conclude that the mixing with chirally disallowed operators is of \( \mathcal{O}(m^2_{\text{res}}) \) and hence negligible.

We calculate \( B_K \), the renormalization group invariant (RGI) parameter, from \( B_{K}^{\text{lat}} \) via the multiplicative renormalization

\[
\hat{B}_K = Z^{\text{RGI,RI-MOM}}(\mu) \times (Z_Q^{S=2}(\mu a)/Z_A^{S}(\mu a)) B_{K}^{\text{lat}}(a) \tag{4}
\]

A similar equation is used for \( B_{K}^{\text{MS}} \) except that \( Z^{\text{RGI,RI-MOM}} \rightarrow Z^{\text{MS,RI-MOM}} \). The factor \( Z^{\text{RGI,RI-MOM}} Z_Q^{S=2}(\mu)/Z_A^{S}(\mu) \) is calculated following the techniques in [3] with all quark masses equal to \( m_{\text{dyn}} \). This quantity is extrapolated to \( m_{\text{val}} = -m_{\text{res}} \), then the extrapolation to \( (p a)^2 = 0 \) is done to remove \( \mathcal{O}(a^2) \) errors. The two loop \( \alpha_S(\mu) \) formula with \( N_F = 2, \Lambda_{\text{QCD}} = 300 \) MeV and \( a^{-1} = 1.81 \) GeV is used. The uncertainty due to the choice of \( \Lambda_{\text{QCD}} \) is estimated by changing \( \Lambda_{\text{QCD}} \) by \( \pm 50 \) MeV. We check the results stay same within error by changing the order of the two limits, \((p a)^2 \rightarrow 0 \) and \( m_{\text{dyn}} \rightarrow -m_{\text{res}} \).

Following the procedure above, we find \( Z^{\text{RGI,RI-MOM}} Z_Q^{S=2}/Z_A^{S} = 1.29(4) \) and \( Z^{\text{MS,RI-MOM}} Z_Q^{S=2}/Z_A^{S} = 0.93(2) \) at \( \mu = 2 \) GeV. These numbers agree with the one loop perturbative calculation, which gives the \( \text{MS} \) factor to be 0.92 [3].

4. CONCLUSIONS

Our initial calculation of \( B_K \) with dynamical domain wall fermions has yielded \( B_{K}^{\text{MS}}(2 \) GeV\) = 0.503(20) and \( \hat{B}_K = 0.697(33) \). The errors on these quantities include the statistical error on \( B_{K}^{\text{lat}} \), the error on the \( Z \) factors and a 2% error reflecting the uncertainty in the kaon mass arising from the uncertainty in the lattice scale. These results correspond to an extrapolation of \( m_{\text{dyn}} = m_{\text{val}} \) to \( m_{\text{val}}/2 \).

We summarize results for each dynamical lattice in Table 1. These are obtained by interpolating \( m_{\text{val}} \) to \( m_{\text{val}}/2 \) at fixed \( m_{\text{dyn}} \) and are somewhat smaller than quenched DWF calculations at a similar lattice spacing. We see a trend in Table 1 toward lower values as the dynamical quark mass is reduced. We have likely underestimated our statistical error, due to the long autocorrelation times in dynamical QCD simulations, and we have not yet studied the other methods of chiral extrapolations, the continuum limit or finite volume effects. However, it is clear that these dynamical DWF calculations are possible with current computers and hold promise for the next generation of computers.

| \( m_{\text{dyn}} \) | \( B_{K}^{\text{lat}} \) | \( B_{K}^{\text{MS}}(2 \text{GeV}) \) | \( \hat{B}_K \) |
|---|---|---|---|
| 0.02 | 0.537(11) | 0.499(22) | 0.692(34) |
| 0.03 | 0.557(9) | 0.518(20) | 0.719(32) |
| 0.04 | 0.568(10) | 0.529(21) | 0.733(34) |
| \( \infty \) | [1] | 0.536(6) | |
| \( \infty \) | [2] | 0.564(14) | |

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