ALMOST ALL WORDS ARE SEEN
IN CRITICAL SITE PERCOLATION
ON THE TRIANGULAR LATTICE

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Abstract We consider critical site percolation on the triangular lattice, that is, we choose $X(v) = 0$ or 1 with probability 1/2 each, independently for all vertices $v$ of the triangular lattice. We say that a word $(\xi_1, \xi_2, \ldots) \in \{0, 1\}^\mathbb{N}$ is seen in the percolation configuration if there exists a selfavoiding path $(v_1, v_2, \ldots)$ on the triangular lattice with $X(v_i) = \xi_i, i \geq 1$. We prove that with probability 1 ‘almost all’ words, as well as all periodic words, except the two words $(1, 1, 1, \ldots)$ and $(0, 0, 0, \ldots)$, are seen. ‘Almost all’ words here means almost all with respect to the measure $\mu_\beta$ under which the $\xi_i$ are i.i.d. with $\mu_\beta\{\xi_i = 0\} = 1 - \mu_\beta\{\xi_i = 1\} = \beta$ (for an arbitrary $0 < \beta < 1$).
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1. Introduction.

We take up a problem of Benjamini and Kesten (1995) for site percolation on a graph $G$ with vertex set $V$. It is assumed throughout that all vertices are independently occupied or vacant, each with probability 1/2. The corresponding measure on occupancy configurations is denoted by $P$; $E$ denotes expectation with respect to $P$. Set $X(v) = 1(0)$ if $v$ is occupied (vacant, respectively). We say that a word $\xi = (\xi_1, \xi_2, \ldots)$, with $\xi_i \in \{0, 1\}$, is seen along a path $\pi = (v_0, v_1, \ldots)$ on $G$ if $\pi$ is selfavoiding and $X(v_i) = \xi_i, i \geq 1$ (note that the state of $v_0$ does not figure in this definition). Benjamini and Kesten (1995) studied which words $\xi$ are seen along some path on $G$. The classical percolation question is whether the word $(1, 1, \ldots)$ with all $\xi_i = 1$ is seen on $G$. The question whether so-called AB-percolation occurs (see Mai and Halley (1980) and Wierman and Appel (1987)) becomes in the present terminology the question whether the word $(1, 0, 1, 0, \ldots)$ is seen. Let

$$\rho(\xi) = P\{\xi \text{ is seen along some path}\},$$

and let

$$\Xi = \{0, 1\}^\mathbb{N}$$

be the space of all words. On this space we put the measure

$$\mu = \prod_{i=1}^{\infty} \mu_i,$$

where for each $i, \mu_i(\{0\}) = 1 - \mu(\{1\}) = \beta$ for some $0 < \beta < 1$. Benjamini and Kesten (1995) only considered the case $\beta = 1/2$, but W. Thurston asked us what the influence of $\beta$ is. It turns out that everything in this paper works for $0 < \beta < 1$. On 'nice' graphs, $\rho(\xi)$ can only take the values 0 or 1. Moreover, $\xi \mapsto \rho(\xi)$ is a tail function, that is, it depends only on $\{\xi_i\}_{i \geq n}$ for any $n$. It therefore follows from Kolmogorov’s zero-one law that

$$\mu\{\xi: \rho(\xi) = 1\} = 0 \text{ or } 1.$$  

In particular this is the case when $G$ is the triangular lattice $T$ in the plane (see Proposition 3 in Benjamini and Kesten (1995)). If $\mu\{\xi: \rho(\xi) = 1\} = 1$, we say that 'the random word percolates'. Benjamini and Kesten (1995) raised the question whether for $G = T$ this is the case or not. This special case is of interest because site percolation on $T$ with each vertex occupied with probability 1/2 is critical (see Kesten (1982), Grimmett (1989)), and it is known (see Harris (1960), Fisher (1961), Kesten (1982)) that one does not see the word $(1, 1, \ldots)$ (i.e., $\rho(1, 1, \ldots) = 0$ or ordinary percolation does not occur), but one does see the word $(1, 0, 1, 0, \ldots)$ (i.e., $\rho(1, 0, 1, 0, \ldots) = 1$ or AB-percolation does occur)(see Wierman and Appel (1987)). So now one wants to know whether almost all $[\mu]$
words are seen or almost no words are seen; these are the only possibilities by (1.3).
Here we prove that almost all words are seen on $T$ and also generalize considerably
the result of Wierman and Appel (1987) that AB-percolation occurs (albeit only
when the occupation probability of each vertex equals 1/2, while AB-percolation
is known to occur for the occupation probability in a small interval around 1/2).
To formulate our result we define the lengths of the runs,
$$r_m = r_m(\xi),$$ of a word
$\xi$ by the requirement that
$$\sum_{1}^{2k} r_m < i \leq \sum_{1}^{2k+1} r_m \quad \text{and} \quad \sum_{1}^{2k+1} r_m < i \leq \sum_{1}^{2k+2} r_m, k \geq 0,$$
and $r_1 \geq 0, r_m > 0$ for $m > 1$. We shall prove
the following result.

**Theorem.** On $T$, $\rho(\xi) = 1$ for almost all $[\mu]$ words $\xi$. Also $\rho(\xi) = 1$ for all words $\xi$ with $r_m(\xi)$ bounded in $m$.

It becomes an interesting question to find more explicitly for which words $\xi$, $\rho(\xi) = 1$ and for which words $\rho(\xi) = 0$ on $T$. By the theorem $\rho(\xi) = 1$ for all words whose letters $\xi_i$ are eventually periodic with the exception of words which have $\xi_i = 1$ or $\xi_i = 0$ from some point on. The latter words are excluded by boundedness of the $r_m$, and in fact, as we already stated, $\rho((1,1,\ldots)) = \rho((0,0,\ldots)) = 0$. There certainly are uncountably many $\xi$’s for which $\rho(\xi) = 0$. Indeed, the almost sure existence of infinitely many occupied and vacant circuits surrounding 0 on $T$ quickly shows that $\rho(\xi) = 0$ if $\xi$ has long runs of ones and zeroes. That is, if $r_m(\xi)$ grows fast enough with $m$, then $\rho(\xi) = 0$.

The idea of the proof of the Theorem is rather combinatorial and strongly
uses special properties of the triangular lattice. We do not know whether one sees
almost all words in bond percolation on $\mathbb{Z}^2$ at the critical point (or more accurately
in the equivalent site percolation on the covering graph of $\mathbb{Z}^2$). We almost explicitly
find occupancy configurations inside of which we can see many words. The key idea
is that of a double path. This is a pair of selfavoiding paths $(\pi', \pi'')$ such that $\pi'(\pi'')$
is occupied (vacant) and such that their initial points $u'$ and $u''$ are neighbors, as
well as their final points $v'$ and $v''$. The ‘region between the paths’ should also
have a certain minimality property (see (5.4)). One shows fairly simply, by means
of Proposition 2.2 of Kesten (1982), that for any word $\xi = (\xi_1, \xi_2, \ldots)$ there is
a path $(w_0, w_1, \ldots, w_\nu)$ in the region between $\pi'$ and $\pi''$ along which one sees an
initial piece of $\xi$. This path starts at $w_0 = u'$ and ends at $w_\nu$ which is either $v'$ or $v''$. Thus one sees at least a piece of $\xi$ of length min($||v' - u'||, ||v'' - u'||$). The same
is true when $u'$ is replaced by $u''$. Unfortunately, with probability 1 there do not
exists infinite double paths in critical percolation on $T$; in fact, with probability 1
there are not even infinite occupied or vacant paths. There exist infinitely many
occupied and vacant circuits surrounding the origin and these form obstructions
to double paths. In order to see a word $\xi$ along a path which crosses an occupied
circuit at a vertex $v$, the word must have a 1 at the position corresponding to
the vertex $v$. On the other hand, Proposition 2.2 of Kesten (1982) easily shows
that there exist at least some double paths between successive occupied or vacant
circuits $C'$ and $C''$ (see Section 5 after Corollary 1). Practically all the work is to show that there exist many suitable double paths between $C'$ and $C''$ when these circuits are ‘far apart’. These double paths need to have special configurations near their endpoint. These special configurations will allow one to connect in many ways a double path between $C'$ and $C''$ and another double path between $C''$ and the next circuit $C'''$ in such a way that one sees a random word $\xi$ with high probability along one of the many pairs of connected double paths. There is also considerable technical work needed to show that in general successive occupied or vacant circuits are far apart (see Lemmas 5-7).

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2. Notation and some preliminaries about occupied and vacant circuits. We write $\mathcal{T}$ for the triangular lattice. For the present purposes we shall assume that $\mathcal{T}$ is imbedded in the plane such that its vertex set is $\mathbb{Z}^2$, and its edges are all edges between pairs $(i_1,j_1)$ and $(i_2,j_2)$ with either

$$|i_1 - i_2| + |j_1 - j_2| = 1$$

or

$$i_2 = i_1 + 1, j_2 = j_1 - 1.$$  
(These are the usual edges on $\mathbb{Z}^2$ plus the NW diagonals; see Figure 1).

![Figure 1. A piece of the triangular lattice $\mathcal{T}$.](image)

Most terminology about graphs and paths on graphs follows Kesten (1982) and for seeing words or percolation of words, the terminology is taken from Benjamini and Kesten (1995). We point out that in this paper a path will always mean a selfavoiding path even if this is not explicitly indicated. $\|x\|$ will denote the Euclidean norm of $x$.

Let $S(m)$ be the square

$$S(m) = [-m,m]^2,$$  
(2.1)

and let $A_m$ be the annulus

$$A_m = S(2^m) \setminus S(2^{m-1}).$$  
(2.2)

For a subset $G$ of $\mathbb{R}^2$ we denote its topological boundary by $\Delta G$. We shall say that a circuit $J$ on $\mathcal{T}$, or a Jordan curve $J$ in $\mathbb{R}^2$ surrounds $G$, if $J \cap G = \emptyset$ and if there is no continuous path in $\mathbb{R}^2 \setminus J$ from $G$ to $\infty$ (in other words, if $J$ separates $G$ from $\infty$).

For any circuit $C$ we denote by $\overset{\circ}{C}$ and $C^{\text{ext}}$ the region in the interior and exterior of $\mathbb{R}^2 \setminus C$, respectively. $\overline{C}$ will stand for $C \cup \overset{\circ}{C}$. 
We also need a partial order of circuits. If $C', C''$ are circuits on $T$ we say that $C''$ is larger than $C'$ (or $C'$ is smaller than $C''$) if $\hat{C}' \subset \hat{C}''$, or equivalently, if $C' \subset \hat{C}''$. In accordance with this ordering, we say that $C''$ is larger than $C'$ (or $C'$ is smaller than $C''$) if $\circ C' \subset \circ C''$, or equivalent, if $C' \subset \circ C''$. In accordance with this ordering, we say that $C'$ is the largest, or outermost occupied circuit in $C''$ if $C'$ is occupied, lies inside $\hat{C}''$ (in particular, $C'$ and $C''$ should be disjoint), and there is no other occupied circuit $C''' \subset \hat{C}''$ such that $C'''$ is larger than, but not equal to $C'$. If there is any occupied circuit in $\hat{C}''$, then there is a largest such circuit, by the argument of Lemma 1 in Kesten (1980). Similarly if ‘occupied’ is replaced by ‘vacant’.

Remark 1. We shall repeatedly need the following fact. Let $C' \subset \hat{C}''$ for two given circuits $C', C''$ such that $C'$ is the largest occupied circuit in $C''$. Then $C'$ must be occupied, and every vertex $v \in C'$ must have a neighbor $w \in \hat{C}'' \setminus C'$ which belongs to $C''$ or is connected by a vacant path to a neighbor of $C''$. (It is not important for this what the state of the vertices on $C''$ is.) Even though the statement is similar to Proposition 2.2 in Kesten (1982), it does not seem to follow from this Proposition. Instead we shall use Corollary 2.2 of Kesten (1982).

Assume that $C'$ is occupied, but there is no occupied circuit in $\circ C'' \setminus \circ C'$ which is larger than but not equal to $C'$. Change all vertices in $\hat{C}'$ (if any) to vacant. This can be done because it influences neither our hypothesis nor conclusion. We also change all vertices in $C'' \cup (C'')^{\text{ext}}$ to vacant. Then there will be no occupied circuit in all of $T$ larger than but not equal to $C'$. Now let $v \in C'$. If we also change $v$ to vacant, then there does not exist an occupied circuit in $T$ which surrounds $v$ and therefore, by Corollary 2.2 of Kesten (1982), the vacant cluster of $v$ is unbounded (in our changed configuration). But this means that in the changed configuration there is a vacant selfavoiding path on $T$ from $v$ to $\infty$. Such a path has to stay in $(C')^{\text{ext}}$ except for its initial vertex $v$, because it cannot exit from $C''$ at any point but $v$ (recall that $C' \setminus \{v\}$ is occupied). This is equivalent to the existence of a neighbor $w$ of $v$ which belongs to $C''$ or has a vacant connection to a neighbor of $C''$ in the original configuration, as claimed.

A similar argument applies to show that if $C'$ is an occupied circuit in $\hat{C}''$ then there is no vacant circuit in $\hat{C}'' \setminus \hat{C}'$ if and only if there is an occupied path from some vertex $w \in C'$ to a vertex $u$ adjacent to $C''$. For the ‘only if’ direction in this situation, make all vertices in $\hat{C}' \cup C'' \cup (C'')^{\text{ext}}$ occupied. Then, if there is no vacant circuit in $\hat{C}'' \setminus \hat{C}'$, the occupied cluster of $C'$ has to be infinite.

We next define $C_1^{(N)}, C_2^{(N)}, \ldots$ to be the successive disjoint occupied or vacant circuits in $S(2^N)$, working from the outside in. More precisely, $C_1^{(N)}$ is the outermost occupied or vacant circuit, as the case may be, in $S(2^N)$. $C_{k+1}^{(N)}$ is the
outermost occupied or vacant circuit inside $C_k^{(N)}$ and disjoint from $C_k^{(N)}$. We continue as long as there are such circuits surrounding the origin, $0$. It is more elegant to let $N \to \infty$, to obtain a system of circuits which does not depend on $N$, and we shall do this now. Note that in contrast to the $C_i^{(N)}$, this new system of circuits will be numbered from the inside out, that is we will have $C_i \subset \mathring{C}_{i+1}$. We use the notation $\kappa(C) = 0$ and $\kappa(C) = 1$ for the events \{C is vacant\} and \{C is occupied\}, respectively.

**Lemma 1.** With probability 1 there exists a sequence of random disjoint circuits $C_k, k \geq 1$, surrounding 0 and so that each of these circuits is occupied or vacant,

$$C_k \subset \mathring{C}_{k+1}, \quad (2.3)$$

and

$$C_k \text{ is the largest circuit in } \mathring{C}_{k+1} \text{ which is entirely occupied or vacant.} \quad (2.4)$$

Moreover, for a fixed circuit $C$ surrounding 0 and $\varepsilon \in \{0, 1\}$, the event

$$\bigcup_k \{C_k = C, \kappa(C) = \varepsilon\} = \{C = \text{some } C_k, \kappa(C) = \varepsilon\} \quad (2.5)$$

depends only on the vertices in $C^\text{ext} \cup C$.

**Proof.** For some fixed $M$, let $C_1^{(M)}$ be the outermost occupied or vacant circuit in $S(2^M)$. For the sake of argument, let $C_1^{(M)}$ be occupied. Next let $D^{(M)}$ be the largest vacant circuit in $\mathring{C}_1^{(M)}$. We claim that then for all $N \geq M$ the outermost vacant circuit in $S(2^M)$ from the sequence $C_k^{(N)}, k \geq 1$, is $D^{(M)}$; in particular, $D^{(M)}$ will be one of the $C_k^{(N)}$. Indeed, this outermost vacant circuit in $S(2^M)$ among the $C_k^{(N)}$ cannot intersect the occupied $C_1^{(M)}$ and hence must lie entirely inside or outside it. It cannot lie outside $C_1^{(M)}$, because there is neither an occupied nor a vacant circuit outside $C_1^{(M)}$ in $S(2^M)$. Thus the outermost vacant circuit in $S(2^M)$ from the sequence $C_k^{(N)}$ must lie inside $C_1^{(M)}$ and equal $D^{(M)}$. This proves our claim.

Now suppose $D^{(M)} = C_k^{(N)}$ and let the last circuit which exists in the sequence $C^{(N)}$ be $C_{\ell(N)}^{(N)}$. Then we can find the successive circuits $C_i^{(N)}, k(N) \leq i \leq \ell(N)$, as the sequence of successive occupied or vacant circuits in $D^{(M)}$. Thus the circuits in the sequence $C_k^{(N)}$ inside $D^{(M)}$ are the same for all $N \geq M$. Since, with probability 1, $D^{(M)}$ lies outside any fixed square for all large $M$, we can take for the circuits $C_k$ inside a fixed square $S$ simply the circuits from the sequence $C_k^{(M)}$ for any $M$ for
which \( \mathcal{D}(M) \) lies outside \( S \). If we index these circuits so that \( C_k \subset \mathcal{C}_{k+1} \), then (2.3) automatically holds and (2.4) follows because \( C_k \) and \( C_{k+1} \) are successive circuits in the sequence \( C_k(M) \) for large \( M \). Finally, the event in (2.5) equals

\[
\lim_{M \to \infty} \bigcup_k \{ C_k = C, \kappa(C) = \varepsilon \},
\]

and for fixed \( M \) the event here depends only on the vertices in \( C_{\text{ext}} \cup C \). Indeed, the event \( \{ C_k = C, \kappa(C) = \varepsilon \} \) is itself a disjoint union of

\[
\{ C_k = C, C_i^{(M)} = C, \kappa(C) = \varepsilon, \kappa(C_i) = \varepsilon_i \}
\]

over \( C_i, \varepsilon_i, 1 \leq i \leq k - 1 \), with \( C_1 \subset S(2^M), C \subset C_{k-1}, C_{k+1} \subset C_i, 1 \leq i < k \). Moreover, one can show by induction on \( k \) that the event in (2.7) depends only on vertices in \( C_{\text{ext}} \cup C \). E.g., the event \( \{ C_1^{(M)} = C_1, \kappa(C_1) = 1 \} \) occurs if and only if \( C_1 \) is occupied and there does not exist an occupied circuit \( C' \) in \( S(2^M) \) which is larger than, but not equal to \( C_1 \). Thus the last statement of the lemma also holds.

From now on \( \{ C_k \} \) will be the ‘growing’ sequence of circuits constructed in Lemma 1 and no further reference to the \( C_k(N) \) will be made. In the next lemma we formulate some more distributional properties for the vertices between the successive circuits \( C_k \).

**Lemma 2.** The events \( \{ \kappa(C_k) = \varepsilon_k \}, k \geq 1 \), are independent with

\[
P\{ \kappa(C_k) = \varepsilon_k \} = \frac{1}{2}
\]

for all choices of \( \varepsilon_k \in \{0, 1\} \).

Moreover, if \( \{ C_i : 0 \leq i \leq n \} \) are fixed circuits surrounding the origin such that \( C_i \subset \mathcal{C}_{i+1} \), then conditionally on the event

\[
\{ C_i = C_i, \kappa(C_i) = \varepsilon_i, 1 \leq i \leq n \},
\]

the collections of vertices

\[
\{ v : v \in C_i^{\text{ext}} \cap \mathcal{C}_i \}, 1 \leq i \leq n \text{ (with } C_0^{\text{ext}} = T) \text{,}
\]

are independent, and the distribution of the occupancy of \( \{ v : v \in C_i^{\text{ext}} \cap \mathcal{C}_i \} \), given (2.9), is simply the conditional distribution of \( \{ v : v \in C_i^{\text{ext}} \cap \mathcal{C}_i \} \) given that there is no occupied or vacant circuit in \( C_i^{\text{ext}} \cap \mathcal{C}_i \).

**Proof.** We only prove the independence of the events \( \{ \kappa(C_k) = \varepsilon_k \} \). We do this by interchanging occupied and vacant vertices in a certain region. Condition on
\{C_{k+i} = C_{k+i}, \kappa(C_{k+i}) = \varepsilon_{k+i}, 0 \leq i \leq r\} \text{ for any } r < \infty. \text{ This event depends only on the vertices in } C^\text{ext}_k \cup C_k. \text{ Thus conditionally on this event, the vertices in } C^\text{ext}_k \text{ are still independently occupied or vacant with probability } 1/2 \text{ each. In particular, there is still symmetry between ‘occupied’ and ‘vacant’ in } C^\text{ext}_k. \text{ Consequently, the events } \{C_{k-1} \text{ is occupied}\} \text{ and } \{C_{k-1} \text{ is vacant}\} \text{ still have the same conditional probability, which must then be equal to } 1/2. \text{ This proves the stated independence and (2.8).}

The rest of the lemma follows from Remark 1. We leave the details to the reader.

\section{Spacings between circuits.}
This section discusses how close together the circuits \(C_k\) are. Throughout, \(c_i\) will denote a strictly positive finite constant. We begin with a few very simple lemmas of this kind.

\textbf{Lemma 3.} For \(0 \leq \ell < k\), let \(\sigma(\ell, k)\) be the maximal number of disjoint occupied circuits surrounding \(0\) in \(S(2^k) \setminus S(2^{\ell})\). Then there are constants \(0 < c_i < \infty\) such that for \(\ell < k\),

\[
P\{\sigma(\ell, k) \leq c_1(k - \ell)\} \leq 2e^{-c_2(k - \ell)} \tag{3.1}
\]

and

\[
P\{\sigma(\ell, k) \geq x(k - \ell)\} \leq c_3e^{-c_4x}, \quad x \geq 0. \tag{3.2}
\]

\textit{Proof.} By the Russo-Seymour-Welsh lemma, the probability that there exists an occupied circuit in \(A_m\) is bounded below by a strictly positive constant, \(c_5\) say. Since \(S(2^k) \setminus S(2^{\ell})\) is the disjoint union of the annuli \(A_i, \ell < i \leq k\), it follows that \(\sigma(\ell, k)\) is stochastically larger than a binomial variable with \((k - \ell)\) trials and success probability \(c_5\). Thus (3.1) with \(c_1 = (1/2)c_5\) follows from standard exponential bounds for the binomial distribution.

On the other hand, the argument in Chayes, Chayes and Durrett (1986) shows that \(E\sigma(\ell, k) \leq c_6(k - \ell)\). Thus by Markov’s inequality

\[
P\{\sigma(\ell, k) \geq 2c_6(k - \ell)\} \leq \frac{1}{2}.
\]

By the BK inequality (cf. Grimmett (1989), Sect. 2.3) it then follows that

\[
P\{\sigma(\ell, k) \geq 2mc_6(k - \ell)\} \leq \left(\frac{1}{2}\right)^m, \quad m \geq 0.
\]

The inequality (3.2) easily follows from this.

Define the \textit{diameter} of a set of vertices \(D\) by

\[
diam(D) = \sup\{\|x - y\| : x, y \in D\}.
\]
We shall use without proof that
\[ \text{diam} \left( C_k \right) \leq \text{diam} \left( C_{k+1} \right); \]
this follows from \( C_k \subset \circ C_{k+1} \).

**Lemma 4.** There exist constants \( c_i \) such that with probability 1
\[ c_7 \leq \lim \inf_{k \to \infty} \frac{\log(\text{diam}(C_k))}{k} \leq \lim \sup_{k \to \infty} \frac{\log(\text{diam}(C_k))}{k} \leq c_8. \] (3.3)

Also with probability 1 there exists a \( k_0 = k_0(\omega) < \infty \) such that for all \( k \geq k_0 \) the following properties hold:
\[ \min_{x \in C_k} \|x\| \geq \frac{c_9}{k^{c_{10}}} \max_{x \in C_k} \|x\|, \] (3.4)
and even
\[ \min_{x \in C_{k-1}} \|x\| \geq \frac{c_9}{k^{c_{10}}} \max_{x \in C_k} \|x\|. \] (3.5)

**Proof.** For the upper bound in (3.3), note that if there exists a vacant circuit in \( A_m \) and an occupied circuit in \( A_{m+1} \), both surrounding 0, then at least one of the vacant circuits among the \( C_k \) has to lie in \( A_m \cup A_{m+1} \), by the argument used in the beginning of Lemma 1. Since
\[ P\{\text{there exists a vacant circuit in } A_m \text{ and an occupied circuit in } A_{m+1} \text{ } \]
both surrounding 0\}
\[ \geq c_5^2, \]
it follows from the independence of the configurations in different \( A_m \) and the strong law of large numbers that with probability 1 there are, for \( \varepsilon > 0 \) and \( n \) large, at least \((c_5^2 - \varepsilon)n/2\) vacant circuits in \( S(2^n) \) among the \( C_k \). This implies the last inequality in (3.3).

For the lower bound in (3.3) let us count the maximal number of disjoint occupied circuits in \( S(2^k) \). Let \( 1 \leq m_1 < m_2 < \ldots \) be the successive values of \( m \) for which \( A_m \) contains a vacant circuit surrounding 0. Let \( D_i \) denote the smallest vacant circuit surrounding 0 in \( A_{m_i} \). No occupied circuit can cross a \( D_i \). Thus all occupied circuits which intersect \( \cup_{m < m_i} A_m \) must be contained in \( S(2^{m_i}) \). In particular, if \( \nu = \nu(k) \) is the unique index for which \( m_\nu \leq k < m_{\nu+1} \), then
\[ \text{maximal number of disjoint occupied circuits surrounding 0 in } S(2^k) \leq \sum_{i=0}^{\nu} \tilde{\sigma}(m_i, m_{i+1}), \] (3.6)
where
\[ \tilde{\sigma}(m_i, m_{i+1}) = \text{maximal number of disjoint occupied circuits surrounding 0 inside } D_{i+1} \text{ and outside } D_i \]
\((\sigma(m_0, m_1) = \text{maximal number of disjoint occupied circuits inside } D_1.)\) Obviously, \(\nu(k) \leq k\) so that the right hand side of (3.6) is at most
\[ \sum_{i=0}^{k} \tilde{\sigma}(m_i, m_{i+1}). \]
Moreover, conditionally on \(m_i = n, D_i = D\) and all vertices inside \(D\), we have essentially by the same argument as for (3.2) (see also Kesten and Zhang (1997), Lemma 3), that
\[
P\{\tilde{\sigma}(m_i, m_{i+1}) \geq x | m_i = n, D_i = D\} \\
\leq P\{m_{i+1} - m_i > p | m_i = n, D_i = D\} + P\{\tilde{\sigma}(n-1, n+p) \geq x | m_i = n, D_i = D\} \\
\leq \prod_{j=1}^{p} P\{\text{there is no vacant circuit in } A_{n+j}\} + P\{\sigma(n-1, n+p) \geq x\} \\
\leq (1 - c_5)^p + c_3e^{-c_4x/(p+1)} \text{ (by (3.2))} \\
\leq c_{11}e^{-c_{12}\sqrt{x}} \text{ (by taking } p = \lfloor \sqrt{x} \rfloor). \]
Thus, \(\tilde{\sigma}(m_0, m_1), \tilde{\sigma}(m_1, m_2), \ldots\) are stochastically smaller than a sequence of i.i.d. random variables \(\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots\) with
\[
P\{\tilde{\sigma}_i \geq x\} \leq c_{11}e^{-c_{12}\sqrt{x}}. \]
By applying the strong law of large numbers to the \(\tilde{\sigma}_i\) we see that with probability 1
\[
\limsup_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k} \tilde{\sigma}(m_i, m_{i+1}) \leq \limsup_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k} \tilde{\sigma}_i \leq c_{13} < \infty. \]
Consequently, (3.6) yields that
\[
\frac{1}{k}\{\text{maximal number of disjoint occupied circuits surrounding 0 in } S(2^k)\} \leq c_{13} \]
eventually. Adding the same estimate for the vacant circuits we finally see that for large \(k\) there are no more than \(2c_{13}k\) of the \(\{C_i\}\) in \(S(2^k)\). Thus \(\text{diam } (C_{2c_{13}k+1}) \geq 2^k\), which implies the first inequality in (3.3).
Finally we turn to (3.4) and (3.5). Assume that $C_k$ is an occupied circuit which is not contained in $S(2^j)$. For a suitable choice of the constant $c_{14}$ there exists with probability 1, for all large $j$, a vacant circuit $C$ surrounding 0 in some $A_\ell$ with $j - c_{14} \log j < \ell \leq j$. Then, by assumption, $C_k$ contains a point outside $C$. Since the occupied circuit $C_k$ cannot cross the vacant $C$ it must lie entirely outside $C$ and therefore outside $S(2^{j - c_{14} \log j})$. (3.4) is an immediate consequence of this, (3.3) and the fact that $C_k$ is not contained in $S(2^j)$ for any $j$ with $2^j < \frac{1}{2} \max_{x \in C_k} \|x\|$. Essentially the same argument proves (3.5), because if $C_k$ and $C$ are as in the preceding paragraph, then $C_{k-1}$ cannot contain any points of $\overset{\circ}{C}$, because $C$ is already a circuit which surrounds 0 and lies inside $C_k$ and is disjoint from it. ■

Lemma 5. There exist constants $c_{15}, c_{16}$ such that for any vertex $v$ and $n \geq 1$

$$\frac{c_{15}}{n^2} \leq P\{\text{there exists 5 disjoint paths from some neighbors of } v \text{ to } \Delta(v + S(n)), \text{ one of which is vacant and the other four of which are occupied}\} \leq \frac{c_{16}}{n^2}. \quad (3.7)$$

More generally, for $1 \leq k \leq n$

$$P\{\text{there exists 5 disjoint paths from } \Delta(v + S(k)) \text{ to } \Delta(v + S(n)), \text{ one of which is vacant and the other four of which are occupied}\} \leq \frac{k^2}{n^2}. \quad (3.8)$$

Finally,

$$P\{\text{there exist at least 6 disjoint paths from } \Delta(v + S(k)) \text{ to } \Delta(v + S(n)), \text{ at least one of which is vacant and at least four of which are occupied}\} \leq c_{16} \left(\frac{k}{n}\right)^{2+c_2/\log 2}. \quad (3.9)$$

Proof. We begin with the upper bound in (3.7). Consider a vertex $w \in \overset{\circ}{S}(n)$ and assume that the following event occurs:

$$F(w, n) := \{\text{there exist 5 disjoint paths from neighbors of } w, \text{ which consist of two occupied paths with endpoints on the left edge of } S(n), \text{ another occupied path with endpoint on the right edge of } S(n) \text{ and two vacant paths with endpoints on the upper and lower edge of } S(n), \text{ respectively}\}. \quad (3.10)$$
Let $\pi_1, \pi_2 (\pi_3)$ be the three occupied paths with their endpoint on the left edge (respectively, right edge) of $S(n)$, and let $\pi_4, \pi_5$ be the vacant paths with endpoints on the upper and lower edge, respectively (see Figure 2). Let $u_i$ be the endpoint of $\pi_i$ and index the paths $\pi_1, \pi_2$ such that $u_2$ lies above $u_1$ on the left edge. Then the existence of $\pi_1, \pi_3, \pi_4, \pi_5$ shows that $w$ is pivotal for the existence of an occupied left-right crossing of $S(n)$. That is, if $w$ is occupied (vacant) then such a crossing exists (respectively, does not exist). Let us now take $w$ occupied; this can be done because the state of $w$ does not influence the occurrence of $F(w, n)$. Then $\pi_1, w$ and $\pi_3$ together form a left-right crossing, and in fact the existence of the vacant $\pi_4, \pi_5$ shows that any such crossing has to pass through $w$. Let $\gamma = (v_1, v_2, \ldots, v_\lambda)$ be the lowest occupied left-right crossing, with the vertices numbered such that $v_1 (v_\lambda)$ lies in the left (respectively, right) edge of $S(n)$ and such that $v_i \in S(n)$ if $1 < i < \lambda$. In particular, $w$ has to be one of the $v_i$. The existence of $\pi_2, \pi_4$ says that $w$ is the first of the $v_i$ which has a vacant connection to the top edge. This uniquely locates $w$ in $S(n)$. In other words, $F(w,n)$ can occur for at most one $w \in S(n)$ and consequently

$$\sum_{w \in S(n/2)} P\{w \text{ is occupied and } F(w,n)\} = \frac{1}{2} \sum_{w \in S(n/2)} P\{F(w,n)\} \leq 1. \quad (3.11)$$

One next has to prove that for any fixed $0 < \delta < 1$, $P\{F(w,n)\}$ is of the same order for all $w \in S(n(1-\delta))$. More precisely, there exists a constant $0 < c_{17} < \infty$ such that

$$\frac{1}{c_{17}} \leq \frac{P\{F(w',n)\}}{P\{F(w'',n)\}} \leq c_{17} \text{ for all } w', w'' \in S(n(1-\delta)). \quad (3.12)$$
Almost all words are seen 15

(3.12) with $\delta = 1/2$ and (3.11) combined will show that for any $w' \in S(n/2)$,

$$P\{F(w', n)\} \leq \frac{1}{c_{17}} \frac{1}{\text{number of vertices in } S(n/2)} \sum_{w \in S(n/2)} P\{F(w, n)\} \leq \frac{c_{18}}{n^2}.$$  \hspace{1cm} (3.13)

The proof of (3.12) is quite long and involved and we shall not give it here. It is very similar to the proof of Lemma 4 in Kesten (1987), which deals with the probability of four disjoint paths to $\Delta S(n)$, two occupied ones and two vacant ones, with the occupied paths separating the vacant ones. An extra complication arises in the case of five paths, considered here, because the occupied and vacant paths cannot separate each other in this case. One needs to modify the definition of an $(\eta, k)$ fence in Kesten (1987) to deal with this. In the appendix we shall give a brief indication how to do this.

Next consider the event

$$G(w, n) := \{\text{there exist 5 disjoint paths from neighbors of } w,}$$

which consist of two occupied paths with endpoints

on the left edge of $S(n)$, two more occupied paths

with endpoints on the top and right edge of $S(n)$,

respectively, and one vacant path with endpoint

on the bottom edge of $S(n)\}.$$  \hspace{1cm} (3.14)

$G(w, n)$ is quite similar to $F(w, n)$; only the path to the top edge is now assumed occupied instead of vacant. Denote the path to the top edge still by $\pi_4$, though, and leave the meaning of the $\pi_i$ with $i \neq 4$ unchanged. We shall show that in fact $G(w, n)$ and $F(w, n)$ have the same probability, by changing the path to the top edge from occupied to vacant, as we now explain. Assume that $G(w, n)$ occurs. Again the state of $w$ does not influence the occurrence of $F(w, n)$ or $G(w, n)$, so we take it as occupied. Then $w$ again lies on the lowest occupied left-right crossing of $S(n)$. Again denote this lowest crossing by $\gamma = (v_1, \ldots, v_\lambda)$ with $v_1(v_\lambda)$ on the left (right) edge of $S(n)$. $\gamma$ is a crosscut of $\overset{\circ}{S}(n)$ and $\overset{\circ}{S}(n) \setminus \gamma$ consists of two components. (A crosscut of an open connected set $D$ is a simple continuous curve $C$, such that $C$ minus its endpoints lies in $D$ and the endpoints of $C$ lie on the boundary of $D$; see Newman (1951), Section V.11.) We denote the component of $\overset{\circ}{S}(n) \setminus \gamma$ which lies ‘above’ $\gamma$ (that is the one which has the upper edge of $S(n)$ as part of its boundary) by $U = U(\gamma)$ and the one which has the lower edge in its boundary by $L = L(\gamma)$. Similarly, $w$ lies on the occupied left-right crossing $\pi$ which consists of (the reverse of ) $\pi_1, \{w\}$ and $\pi_3$. The corresponding components of $\overset{\circ}{S}(n) \setminus \pi$ are denoted by $U(\pi)$ and $L(\pi)$. Because $\gamma$ is the lowest
left-right crossing, $\pi$ cannot contain any points of $L(\gamma)$, i.e., $\pi \subset \overline{U}(\gamma) := U(\gamma) \cup \gamma$. Equivalently, $U(\pi) \subset U(\gamma)$. Then $\pi_2$, which is disjoint from $\pi$ and has its endpoint on the left edge above that of $\pi_1$, must lie in $U(\pi) \subset U(\gamma)$. In particular the path consisting of the edge from $w$ to the initial point of $\pi_2$, followed by $\pi_2$ itself is an occupied crosscut of $U(\gamma)$ with one endpoint at $w$. For any such crosscut $\delta$ let $V = V(\delta, \gamma)$ be the component of $U(\gamma) \setminus \delta$ which has the left endpoint of $\gamma$, $v_1$, in its boundary, and let $W(\delta, \gamma)$ be the other component of $U(\gamma) \setminus \delta$, with $v_\lambda$ in its boundary. By Proposition 2.3 of Kesten (1982) there then exists an occupied crosscut of $U(\gamma)$, $\delta$ say, with one endpoint at $w$, so that $V(\delta, \gamma)$ is minimal among all such crosscuts.

Finally, define the map $\Phi$ on the occupancy configurations in $S(n)$ which have a lowest occupied crossing $\gamma$ through $w$ and a further crosscut $\delta$ as above. For a given configuration $\omega$, $\Phi(\omega)$ is the configuration obtained from $\omega$ by changing all occupied vertices in $W(\delta, \gamma)$ to vacant and vice versa. Since the lowest crossing $\gamma$ and after that the minimal crosscut $\delta$ are uniquely determined, the map $\Phi$ is well defined. Since it leaves the state of the vertices on $\gamma$ and on $\delta$ unchanged, one easily sees that the lowest occupied crossing in $\Phi(\omega)$ is still $\gamma$, and also the occupied crosscut minimizing $V(\delta, \gamma)$ for $\Phi(\omega)$ is still $\delta$. Thus one can recognize in the configuration $\Phi(\omega)$ the states of which vertices have been flipped by $\Phi$, so that $\Phi$ is one-to-one. Moreover, the event $G(w, n)$ is taken onto the event $F(w, n)$ by $\Phi$, because the occupied path $\pi_4$ in $G(w, n)$ is turned into a vacant one by applying $\Phi$. Indeed, $\pi_4 \setminus \{u_4\}$ must lie in $W(\delta, \gamma)$, because its endpoint $u_4$ lies on the upper edge of $S(n)$ which belongs to $\Delta W(\delta, \gamma)$ and not to $\Delta V(\delta, \gamma)$ whenever the endpoint of $\delta$ lies on the left edge of $S(n)$. Since the probability of the configuration $\Phi(\omega)$ is the same as the probability of $\omega$, it follows that

$$P\{G(w, n)\} = P\{F(w, n)\} \leq \frac{c_{18}}{n^2}, \quad w \in S(n/2) \quad (by \ (3.13)). \quad (3.15)$$

The last inequality seems almost what we want. What must still be shown is that the probability in (3.7) (which is independent of $v$) is at most a constant (independent of $n$) times $P\{G(0, n)\}$. The difference between $G(0, n)$ and the event in (3.7) for $v = 0$ lies in the restrictions on the endpoints of the paths $\pi_i$ imposed in $G(0, n)$. For $G(0, n)$ to occur these endpoints have to lie on certain edges of $S(n)$, while there is no such restriction in (3.7). The proof that removing these restrictions on the endpoints only increases the probability by a bounded factor is basically the same as the proof of (3.12) and is again based on Kesten (1987). We skip this proof, but make some remarks about this in the Appendix. This proves the upper bound in (3.7).

For the lower bound in (3.7) we note that by (3.15) and (3.12)

$$P\{G(0, n)\} = P\{F(0, n)\} \geq \frac{1}{c_{17}n^2} \sum_{w \in S(n(1-\delta))} P\{F(w, n)\}. \quad (3.16)$$
Furthermore, the interpretation of $F(w, n)$ as the event that the first $v_i$ on $\gamma$ with a vacant connection to the top edge of $S(n)$ is $w$, shows that the sum in the right hand side here equals

$$P\{\text{first vertex } v_i \in \gamma \text{ with a vacant connection to the top edge of } S(n) \text{ lies in } S(n(1 - \delta))\}. \tag{3.17}$$

Thus the lower bound in (3.7) for $v = 0$, and hence for any $v$, will follow once we prove that for some small fixed $0 < \delta < 1$, the probability (3.17) is at least $c_{19} > 0$. This follows by combining fairly standard arguments, but since it is not entirely trivial we give most of the proof anyway. First we want to show that for some $\delta > 0$, the probability of the event

$$\{\text{the lowest occupied left-right crossing } \gamma \text{ of } S(n) \text{ exists, lies in } [-n, n] \times [-((1 - 5\delta)n, (1 - 5\delta)n),$$

and does neither return to the vertical line $\{x = -(1 - 5\delta)n\}$ after it first reaches the vertical line $\{x = (1 - 5\delta)n\}$, nor returns to the vertical line $\{x = (1 - 3\delta)n\}$ after it first reaches the vertical line $\{x = (1 - \delta)n\}$ (starting from the left edge of $S(n)$)\} \tag{3.18}

is bounded away from 0 (as $n \to \infty$). To see this we take $\delta < 1/8$ without loss of generality, and assume that the following two paths exist (see Figure 3):

(i) a vacant left-right crossing $\pi'_6$ in $[-n, n] \times [-n(1 - 5\delta), -n(1 - 6\delta)]$;

(ii) a vacant connection $\pi'_7$ from $\pi'_6$ to the bottom edge of $S(n)$.

One easily sees that in this case, the lowest occupied left-right crossing of $S(n)$ (if it exists) must lie above $\pi'_6$ and hence in $[-n, n] \times [-n(1 - 5\delta), -n(1 - 6\delta)]$. The probability that such $\pi'_6, \pi'_7$ exist is at least $c_{20}$ (by the Russo-Seymour-Welsh lemma and the Harris-FKG inequality). If such $\pi'_6, \pi'_7$ exist, then we take $\pi_6$ to be the lowest vacant left-right crossing of $[-n, n] \times [-n(1 - 5\delta), -n(1 - 6\delta)]$. The existence of $\pi'_7$ then guarantees that also $\pi_6$ has a vacant connection to the bottom edge of $S(n)$. Thus we have a probability of at least $c_{21} = c_{21}(\delta) > 0$ that the lowest vacant left-right crossing of $[-n, n] \times [-n(1 - 5\delta), -n(1 - 6\delta)]$ exists and has a vacant connection to the bottom edge of $S(n)$. We shall now condition on this last event and even on the specific value of $\pi_6$. Then we still know nothing about the vertices ‘above $\pi_6$’ and they are independently occupied or vacant with probability 1/2.

Here and in the rest of this lemma, ‘above $\pi_6$’ means in the component of $S(n) \backslash \pi_6$ which has the upper edge of $S(n)$ in its boundary. Define

$$a(\pi_6) = P\{\text{there exists an occupied left-right crossing of } [-n, n] \times [-((1 - 5\delta)n, (1 - 5\delta)n) \text{ above } \pi_6 | \pi_6\}. \tag{3.19}$$
Then it is easy to see that there is some constant $c_{22} > 0$ (independent of $\pi_6$ and of $0 < \delta < 1/8$) such that

$$c_{22} \leq P\{\text{there exists an occupied left-right crossing of } [-n, n] \times [-\delta - 6n, (1 - 6\delta)n] \leq a(\pi_6) \leq 1 - c_{22}. \quad (3.19)$$

We also need the following probabilities:

$$b(\pi_6, \delta) := P\{\text{there exists an occupied left-right crossing of } [-n, (1 - \delta)n] \times [-\delta - 5n, (1 - 5\delta)n] \text{ above } \pi_6|\pi_6 \}.$$  

and

$$c(\pi_6, \delta) := P\{\text{there exists an occupied left-right crossing of } [-\delta - 5\delta n, (1 - 5\delta)n] \times [-\delta - 5\delta n, (1 - 5\delta)n] \text{ or of } (1 - 3\delta)n, (1 - \delta)n] \times [-\delta - 5\delta n, (1 - 5\delta)n] \text{ above } \pi_6|\pi_6 \}.$$  

We claim that uniformly in $\pi_6$,

$$\lim_{\delta \to 0} \left| \frac{b(\pi_6, \delta)}{a(\pi_6)} - 1 \right| = 0, \quad (3.20)$$

and

$$c(\pi_6, \delta) \leq 1 - c_{23}. \quad (3.21)$$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Illustration of $\pi_6 - \pi_{10}$ and $\gamma$. Again solidly drawn (dashed) paths are occupied (vacant). The boundary of $V$ is boldly drawn.}
\end{figure}
Before proving (3.20) and (3.21) we point out that they imply that the probability of the event in (3.18) is bounded away from 0. Indeed, given \( \pi_6 \) and that \( \pi_6 \) has a vacant connection to the bottom edge of \( S(n) \), the conditional probability that there exists a lowest occupied crossing \( \gamma \) of \( S(n) \) in \([-n,n] \times [(1-5\delta)n, (1-5\delta)n] \) is \( a(\pi_6) \). If \( \gamma \) returns to the vertical line \( \{x=-(1-5\delta)n\} \) after first hitting the line \( \{x=(1-5\delta)n\} \) (starting from the left edge of \( S(n) \)), then \( \gamma \) must contain an occupied left-right crossing \( \gamma_1 \) of \([-n,(1-5\delta)n] \times [(1-5\delta)n, (1-5\delta)n] \) above \( \pi_6 \), and in addition two occupied left-right crossings of \([-n, (1-5\delta)n] \times [(1-5\delta)n, (1-5\delta)n] \times [-(1-5\delta)n, (1-5\delta)n] \) which are disjoint from \( \gamma_1 \). Indeed, the piece of \( \gamma \) from the left edge of \( S(n) \) to the first visit to the vertical line \( \{x=(1-5\delta)n\} \) contains such a crossing \( \gamma_1 \), then the return from \( \{x=(1-5\delta)n\} \) to \( \{x=-(1-5\delta)n\} \) contains another occupied crossing of \([-1-5\delta)n, (1-5\delta)n] \times [-(1-5\delta)n, (1-5\delta)n] \), and finally there is a third crossing between the return to \( \{x=-(1-5\delta)n\} \) and the final point on the right edge of \( S(n) \). Similarly, if \( \gamma \) returns to the line \( \{x=(1-3\delta)n\} \) after reaching the line \( \{x=(1-\delta)n\} \), then there are two occupied left-right crossings of \([(1-3\delta)n, (1-\delta)n] \times [(1-5\delta)n, (1-5\delta)n] \), disjoint from \( \gamma_1 \). Hence, by the BK inequality, the conditional probability that \( \gamma \) exists but returns to \( \{x=-(1-5\delta)n\} \) after first visiting \( \{x=(1-5\delta)n\} \) or that \( \gamma \) returns to \( \{x=(1-3\delta)n\} \) after first visiting \( \{x=(1-\delta)n\} \) is for small \( \delta \) at most

\[
\begin{align*}
 b(\pi_6, \delta)c^2(\pi_6, \delta) &\leq (1-c_{23})^2b(\pi_6, \delta) \quad \text{(by (3.21))} \\
 &\leq (1-c_{23})a(\pi_6) \quad \text{(by (3.20) for \( \delta \) small enough).}
\end{align*}
\]

Finally the conditional probability of the event (3.18), given \( \pi_6 \), is at least

\[
 a(\pi_6) - b(\pi_6, \delta)c^2(\pi_6, \delta) \geq c_{23}a(\pi_6) \geq c_{23}c_{22}.
\]

Since \( \pi_6 \) exists with probability at least \( c_{21} \) we find that for small \( \delta \)

\[
P\{(3.18) \text{ occurs} \} \geq c_{21}c_{22}c_{23}.
\]

We still have to prove (3.20) and (3.21). For (3.20) we note first that by definition, \( b(\pi_6, \delta) \geq a(\pi_6) \). On the other hand, it is also clear that

\[
b(\pi_6, \delta) \leq P\{\text{there exists an occupied left-right crossing of} \}[-n, (1-\delta)n] \times [-n, n] \}.
\]

(3.22)

To see that the last probability on the right here is not much larger than \( P\{\text{there exists an occupied left-right crossing of} \ S(n) \} \) when \( \delta \) is small, we can use the argument in the beginning of Lemma 2 of Kesten (1987). If there is a crossing as in the right hand side of (3.22), then the lowest such crossing can with high probability be extended on the right to give a left-right crossing of \( S(n) \). This
is illustrated in Figure 7 of Kesten (1987) and we leave the details to the reader. Then we have for any fixed $\eta > 0$ and $0 < \delta < \delta_0$ for some $\delta_0 = \delta_0(\eta) > 0$, that

$$b(\pi_6, \delta) \leq (1 + \eta) P\{\text{there exists an occupied left-right crossing of } S(n)\}$$

$$\leq (1 + \eta)^2 P\{\text{there exists an occupied left-right crossing of } S(n)\}$$

$$\leq (1 + \eta)^2 a(\pi_6); \quad (3.23)$$

the second inequality here holds by a similar argument as for the first inequality, this time applied to the existence of vacant top-bottom crossings. Thus (3.20) holds.

The inequality (3.21) is completely standard. It follows from the fact that for fixed $0 < \delta_1 < \delta_2$,

$$P\{\text{there exists a vacant top-bottom crossing of } [\delta_1 n, \delta_2 n] \times [-n, n]\}$$

$$\geq c_{24} = c_{24}(\delta_1, \delta_2) > 0$$

(by the Russo-Seymour-Welsh lemma; compare Kesten (1982), Theorem 6.1). Again we leave the details to the reader.

The inequalities (3.20) and (3.21) finally complete the proof of that (3.18) has a probability bounded away from 0. We still have to deduce (3.17) from this lower bound for the probability of (3.18). Fortunately this is fairly easy. Assume that the event (3.18) occurs. We now condition on the lowest occupied left-right crossing $\gamma$, say we condition on $\gamma = (v_1, v_2, \ldots, v_\lambda)$. Again, under this condition the vertices in $S(n)$ above $\gamma$ are still independently occupied or vacant with probability 1/2. Therefore (even conditionally on $\gamma$) there is a probability at least $c_{25} > 0$ that the following paths occur (see Figure 3):

(iii) an occupied connection $\pi_8$ from a vertex $v_k \in \gamma$ to a vertex $u$ in the top edge of $S(n)$ in the vertical strip $[(1 - 5\delta)n, (1 - 4\delta)n) \times [-n, n]$;

(iv) an occupied connection $\pi_9$ from $\pi_8$ to the left edge of $S(n)$; $\pi_9$ lies above $\gamma$ and is disjoint from $\gamma$;

(v) a vacant connection $\pi_{10}$ from a neighbor of a vertex $v_\ell \in \gamma$ to the top edge of $S(n)$ in the vertical strip $[(1 - 4\delta)n, (1 - 3\delta)n]$.

Now $S(n) \setminus \gamma$ consists of the two components with the bottom edge and the top edge of $S(n)$ in their boundary, respectively. Let $U = U(\gamma)$ be the component with the top edge in its boundary. $\pi_8$ is a crosscut of $U$. We consider the component $V$ of $U \setminus \pi_8$ whose boundary consists of the following pieces: the piece of $\gamma$ from $v_1$ on the left edge of $S(n)$ to $v_k$, the piece of the top edge of $S(n)$ from $u$ to the upper left corner of $S(n)$, and finally the piece of the left edge of $S(n)$ from the top left corner back to $v_1$. Now there cannot be a vacant connection from a vertex adjacent to $\gamma$ and located in $V$ to the top edge of $S(n)$, because
any such connection would have to cross \( \pi_8 \cup \pi_9 \) which is impossible. Therefore, any vacant connection from a vertex adjacent to \( \gamma \) to the top edge of \( S(n) \) has to start at a vertex adjacent to one of \( \{v_k, v_{k+1}, \ldots, v_\lambda\} \). However, \( \pi_8 \) lies in the vertical strip \([1-5\delta)n, (1-4\delta)n \times [-n,n] \), so that also \( v_k \) lies in this strip and hence comes at or after \( \gamma \)'s first intersection with the vertical line \( \{x = (1-5\delta)n\} \). But then, on the event \((3.18)\), \( v_k, v_{k+1}, \ldots, v_\lambda \) lie to the right of the vertical line \( \{x = -(1-5\delta)n\} \). Thus the first vertex on \( \gamma \) with a vacant connection to the top edge (if any) lies in \([-1-5\delta)n, n] \times [-(1-5\delta), (1-5\delta)n] \) (for the vertical restriction we used the vertical restriction of \( \gamma \) on \((3.18)) \). The path \( \pi_{10} \) guarantees that there is at least one vacant connection from \( \gamma \) to the top edge. The restrictions on the location of \( \pi_{10} \) imply that \( v_\ell \in [1-4\delta)n, (1-3\delta)n] \). Unfortunately \( v_\ell \) is not necessarily the first point on \( \gamma \) with a vacant connection to the top edge of \( S(n) \). If this first vertex on \( \gamma \) is \( v_f \), then we know from the preceding argument that \( k \leq f \leq \ell \) and that \( v_f \in [-(1-5\delta)n, n] \times [-(1-5\delta), (1-5\delta)n] \), but a priori, \( v_f \) could still lie arbitrarily close to the right edge of \( S(n) \). However, if \( v_f \in [(1-\delta)n, n] \times [-(1-5\delta)n, (1-5\delta)n] \), then \( \gamma \) first reaches the line \( \{x = (1-\delta)n\} \) (at or before \( v_f \)) and then returns to the line \( \{x = (1-3\delta)n\} \) (because \( v_\ell \) lies on or to the left of this line). This cannot occur on \((3.18)\). Therefore

\[
P\{\gamma \text{ and } v_f \text{ exist and } v_f \in [-(1-\delta)n, (1-\delta)n] \times [-(1-5\delta)n, (1-5\delta)n]\} \\
\geq P\{\text{one of (3.18) occurs}\}c_{25}.
\]

This finally proves that \((3.17)\) is bounded below by some \( c_{19} \) as desired.

Now that \((3.7)\) has been established, \((3.8)\) can be deduced by the method of Lemma 6 in Kesten (1987). This method allows one to show that the probability appearing in \((3.7)\) equals at least \( c_{26}P\{G(v,k/2)\} \times \text{the probability in (3.8)} \). Thus if we combine the upper bound of \((3.7)\) with the lower bound

\[
P\{G(v,k/2)\} \geq \frac{c_{27}}{k^2}
\]

(use \((3.16)\) and the fact that \((3.17)\) is bounded below), we obtain \((3.8)\).

Finally, \((3.9)\) follows immediately from \((3.8)\) and Reimer’s inequality (Reimer (1996)), or even by Theorem 4.2 of van den Berg and Fiebig (1987). Indeed, Reimer’s inequality shows that the left hand side of \((3.9)\) is at most equal to the left hand side of \((3.8)\) times

\[
P\{\text{there exists an occupied or vacant path from } \Delta(v+S(k)) \text{ to } \Delta(v+S(n))\}.
\]

But if \( k \leq 2^p < 2k, n/2 < 2^q \leq n \), then

\[
P\{\text{there exists a vacant path from } \Delta(v+S(k)) \text{ to } \Delta(v+S(n))\} \\
\leq P\{\text{there is no occupied circuit in } A_m \text{ for any } A_m \subset S(n) \setminus S(k)\} \\
\leq P\{\sigma(p,q) = 0\} \leq 2e^{-c_2(q-p)} \text{ (see Lemma 3)} \leq 2 \left( \frac{4k}{n} \right)^{c_2/\log 2}.
\]
Adding to this the same estimate for occupied paths we obtain that (3.24) is bounded by $4(4k/n)^{c_2/\log^2}$, so that (3.9) indeed follows from (3.8). ■

**Lemma 6.** For $0 < c < 1$ and $j \geq 1$ define

$$T(r, s) = T(r, s; c, j) = [r2^{c(j+1)}, (r + 3)2^{c(j+1)}] \times [s2^{c(j+1)}, (s + 3)2^{c(j+1)}]$$

(3.25)

and

$$N(j, c) = \text{number of squares } T(r, s) \text{ which intersect } S(2^j + 1) \text{ and which, for some } k,$$

intersect two successive circuits $C_k$ and $C_{k+1}$ with diam $(C_k) \geq 2^j$.

Then, for fixed $c \in (0, 1)$, with probability 1,

$$N(j, c) \leq j^2 \text{ for all large } j.$$  

(3.26)

Furthermore, if

$$N^{(3)}(j, c) = \text{number of squares } T(r, s) \text{ which intersect } S(2^j + 1) \text{ and which, for some } k,$$

intersect three successive circuits $C_k - C_{k+2}$ with diam $(C_k) \geq 2^j - (\log j)^2$, then, with probability 1,

$$N^{(3)}(j, c) = 0 \text{ for all large } j.$$  

(3.27)

Finally, with probability 1,

(number of vertices of $C_k$ which are adjacent to $C_{k+1}$) $\leq k^2$ eventually.  

(3.28)

**Proof.** A trivial extension of the argument used for (2.8) shows that,

$$P\{\text{for some } k, T(r, s) \text{ intersects } C_k \text{ and } C_{k+1} \text{ with diam } (C_k) \geq 2^j\}$$

$$= 8P\{\text{for some } k, T(r, s) \text{ intersects } C_k \text{ and } C_{k+1} \text{ and both these circuits are occupied and } C_{k+2} \text{ is vacant and diam } (C_k) \geq 2^j\}.$$  

(3.29)

Now assume that $T(r, s) \text{ intersects } S(2^j + 1)$. If $T(r, s) \text{ intersects the two occupied circuits } C_k, C_{k+1}, \text{ and } C_{k+2}$ is vacant, and diam $(C_k) \geq 2^j$, then there are 5 disjoint paths starting on or adjacent to $\Delta T(r, s)$ and reaching points at distance $2^{j-1} - 6 \cdot 2^{c(j+1)} \geq 2^{j-2}$ from $T(r, s)$. Four of these are occupied and consist of disjoint arcs of $C_k$ and $C_{k+1}$. In addition, by definition, $C_{k+1}$ is the largest circuit in $C_{k+2}$. By Remark 1 and the fact that $C_{k+1}$ is assumed occupied and $C_{k+2}$ is
assumed vacant, there is for any vertex \( w \in T(r, s) \cap C_{k+1} \) a vacant connection from a neighbor \( u \) of \( w \) to \( C_{k+2} \). This vacant connection together with \( C_{k+2} \) contains a vacant path of diameter \( \geq \frac{1}{2} \text{diam } (C_{k+2}) \geq 2^{j-1} \). This is the fifth path starting from a vertex on or adjacent to \( \Delta T(r, s) \); it is disjoint from the other four because it is vacant.

It follows from the preceding argument and (3.8) that for fixed \( r, s \) and large \( j \), the probability of the event in the left hand side of (3.29) is at most \( c_2 2^{-2(1-c)(j+1)} \). The possible number of choices for \( r, s \) is at most \( 10 \cdot 2^{2(1-c)(j+1)} \), because we only consider \( T(r, s) \) which intersect \( S(2^{j+1}) \). Therefore the expected number of pairs \( r, s \) for which the event in the left hand side of (3.29) occurs is at most \( c_2 \). By Markov’s inequality the probability that there are more than \( j^2 \) such pairs \( r, s \) is then at most \( c_2 j^2 / j^2 \). The Borel-Cantelli lemma now proves (3.26).

The proof of (3.27) is similar. If \( C_k, C_{k+1} \) and \( C_{k+2} \) are occupied and intersect \( T(r, s) \) and \( C_{k+3} \) is vacant, then there are seven disjoint paths starting on or adjacent to \( \Delta T(r, s) \) and reaching points at distance \( 2^{j-2} \) from \( \Delta T(r, s) \). Now use (3.9) instead of (3.8).

We shall not prove (3.28). Its proof is essentially the same as that of (3.26), but simpler. One now uses (3.7) directly, instead of (3.8).

The next lemma is a simple deterministic result. For two sets of vertices \( \pi', \pi'' \) we define

\[
d(\pi', \pi'') = \inf \{ \| x - y \| : x \in \pi', y \in \pi'' \}.\]

**Lemma 7.** Let \( C_{k+1} \) and \( C_{k+2} \) be two successive circuits. Assume

\[
2^j \leq \text{diam } (C_{k+1}) < 2^{j+1}.\]

Let \( V_1^{(k+1)}, V_2^{(k+1)}, \ldots, V_M^{(k+1)} \) be \( M \) arbitrary subsets of \( C_{k+1} \) for which

\[
d(V_p^{(k+1)}, V_q^{(k+1)}) := \min \{ \| x - y \| : x \in V_p^{(k+1)}, y \in V_q^{(k+1)} \} \geq 8 \cdot 2^{c'j}, p \neq q. \quad (3.30)\]

If \( N(j, c) \leq j^2 \), then at least \( M - j^2 \) of the sets \( V_m^{(k+1)} \) satisfy

\[
d(V_m^{(k+1)}, C_{k+2}) > 2^{c'j}. \quad (3.31)\]

In particular, if \( M > j^2 \), and \( N(j, c) \leq j^2 \), then there is at least one set \( V_m^{(k+1)} \) which satisfies (3.31)

**Proof.** Assume that for some \( m_0 \) the set \( V_{m_0}^{(k+1)} \) contains a vertex \( x \in C_{k+1} \) for which there exist a \( y \in C_{k+2} \) with \( \| x - y \| \leq 2^{c'j} \). Since \( \text{diam } (C_{k+1}) < 2^{j+1} \), we have \( C_{k+1} \subset S(2^{j+1}) \). Thus \( x \) belongs to some square \([r + 1]2^{c'j}, (r + 2)2^{c'j}] \times [(s + 1)2^{c'j}, (s + 2)2^{c'j}] \) and this square must intersect \( S(2^{j+1}) \). It then follows from \( \| x - y \| \leq 2^{c'j} \) that \( y \in T(r, s; c, j) \) (see (3.25) for \( T(r, s) \)). Thus for each \( V_{m_0}^{(k+1)} \)
which contains an $x$ as just mentioned, there is a $T(r, s)$ which intersects $S(2^{j+1})$ as well as $C_{k+1}, C_{k+2}$, and which contains a point of $V_{m_0}^{r(k+1)}$. Different sets $V_m^{r(k+1)}$ lead to different $T(r, s)$, because two different $V_m^{r(k+1)}$ cannot intersect the same $T(r, s)$ by (3.30).

By definition, $N(j, c) \leq j^2$ means that there are at most $j^2$ squares $T(r, s)$ of the form (3.25) which intersect $S(2^{j+1}), C_{k+1}$ and $C_{k+2}$. By the preceding paragraph this implies that there are at most $N(j, c) \leq j^2$ sets $V_m^{r(k+1)}$ with $d(V_m^{r(k+1)}, C_{k+2}) \leq 2^{cj}$. The remaining $M - N \geq M - j^2$ sets $V_m^{r(k+1)}$ satisfy $d(V_m^{r(k+1)}, C_{k+2}) > 2^{cj}$. ■

4. Almost all circuits are ‘good’.

Let $C$ be a circuit surrounding 0. For certain constants $0 < c_{30}, c_{31}, c_{33} < 1, c_{32} > 0$, still to be determined, we consider vacant connected (on $T$) sets $D$ with some or all of the following properties:

$$D \subset \overset{\circ}{C} \text{ and } D \text{ contains exactly one vertex adjacent to } C;$$

$$\text{diam}(D) \geq (\text{diam}(C))^{c_{30}};$$

the vacant cluster of $D$ in $\overset{\circ}{C}$ contains at least $[\text{diam } (C)]^{c_{31}}$ selfavoiding paths $\theta_m$ which are adjacent to $C$, have length $\theta_m \geq c_{32} \log \log(\text{diam}(C))$ and satisfy $d(\theta_p, \theta_q) \geq (\text{diam } (C))^{c_{33}c_{30}}$ for $p \neq q$;

moreover, there exists a vertex $z \in D$ and for each of the $\theta_m$ a vacant path from $z$ to $\theta_m$ and such that only its endpoint on $\theta_m$ is adjacent to $C$.

We call an occupied circuit $C$ good (or more explicitly $(c_{30} - c_{33})$-good) if it has the following property:

Any vacant connected (on $T$) set $D$ with the properties (4.1) and (4.2) also satisfies (4.3).

We note that ‘$\theta$ adjacent to $C$’ means that for each vertex $u$ of $\theta$, there exists a vertex $v$ in $C$ which is adjacent to $u$. This is not a symmetric relation; $\theta$ adjacent to $C$ does not imply that $C$ is adjacent to $\theta$.

A good vacant circuit $C$ is defined by interchanging ‘occupied’ and ‘vacant’ in this definition.

Here is our principal estimate.
Proposition 1. For any $0 < c_{33} < 1$, and $0 < c_{30} < 1$, but $c_{30}$ sufficiently close to 1, we have

$$P\{\text{there exists an occupied circuit } C_k \text{ with } 2^j \leq \text{diam } (C_k) < 2^{j+1} \text{ and which is not good}\} \leq c_{34} \exp(-c_{35}j).$$

(4.5)

Here $c_{34}, c_{35}$ depend on $c_{30} - c_{33}$ only.

**Proof.** We shall break down the proof into five steps. $c_i$ will denote a strictly positive finite constant throughout this proof. All these constants with $i \geq 37$ will be independent of $c_{30}$, so that $c_{30}$ still can be adjusted at the end of the proof. The proof works for any choice of $c_{32} > 0$. Throughout this proof we tacitly assume that $j$ is large; our estimates may fail for a finite number of $j$’s.

In the first step we show how to replace the random $C_k$ by a fixed circuit $C$ from a specified class. The second step is largely topological and serves to find a certain circuit $J \subset \tilde{C}$ adjacent to $C$. Step (iii) in conjunction with Step (ii) bounds the number of circuits $J$ which have to be considered. Steps (iv) and (v) together estimate the probability that (4.3) fails for $D$ equal to some vacant path with one endpoint on a specific circuit $J$ as constructed in Step (ii).

**Step (i)** Any circuit $C$ which surrounds 0 and has $2^j \leq \text{diam } (C) < 2^{j+1}$ must be contained in $S(2^{j+1})$, but must also contain points outside $S(2^{j-2})$. Now let $C_{\tau}$ be the last circuit in our sequence $C_1, C_2, \ldots$, which is contained in $S(2^{j+1})$. Then the probability in (4.5) is (for any choice of $u \geq 0$) at most

$$\sum_{0 \leq p < u} P\{p < \tau, C_{\tau-p} \text{ is occupied, has diameter } \geq 2^j, \text{ but is not good}\} + P\{u < \tau, C_{\tau-u} \text{ is not contained in } S(2^{j-2})\}.$$  

(4.6)

The second term in (4.6) can be estimated by the argument of Lemma 3. If $C_{\tau-u}$ is not contained in $S(2^{j-2})$, then at least one of the following three events must occur (for any choice of $1 \leq q \leq j - 2$):

- there exists no vacant circuit in $S(2^{j-2}) \setminus S(2^{j-2-q})$,
- there exists no occupied circuit in $S(2^{j-2}) \setminus S(2^{j-2-q})$,

or

$$C_{\tau-u} \text{ lies outside } S(2^{j-2-q}), \text{ so that } C_{\tau}, C_{\tau-1}, \ldots, C_{\tau-u} \subset S(2^{j+1}) \setminus S(2^{j-2-q}).$$  

(4.7)
If (4.7) holds, then \( \sigma(j - 2 - q, j + 1) \geq (u + 1)/2 \) or there are at least \((u + 1)/2\) disjoint vacant circuits in \( S(2^{j+1}) \setminus S(2^{j-2} - q) \). Therefore, with \( c_5 \) as in the proof of Lemma 3,

\[
P\{C_{\tau - u} \not\subset S(2^{j-2})\} \leq 2(1 - c_5)^q + 2P\{\sigma(j - 2 - q, j + 1) \geq (u + 1)/2\} \leq 2(1 - c_5)^q + 2c_3 e^{-c_4(u+1)/(2q+6)} \text{ (by (3.2)).}
\]

By choosing \( q = j - 2 \) and \( u = j^2 \) we find that the second term in (4.6) is at most

\[
(2c_3 + 1)e^{-c_{36}j}.
\]

Each term in the sum in (4.6) will be decomposed into two pieces. We fix

\[
\ell = \lfloor c_{30}j - 2 \log j / \log 2 \rfloor - 4,
\]

and divide \( S(2^{j+1}) \) into the at most

\[
2^{2(j+2-\ell)} \leq c_{37}j^4 2^{2(1-c_{30})j}
\]
squares

\[
S(r, s) := [r2^\ell, (r+1)2^\ell] \times [s2^\ell, (s+1)2^\ell], \quad -2^{j+1-\ell} \leq r, s < 2^{j+1-\ell}.
\]

For each of these squares we take

\[
\tilde{S}(r, s) = [(r - 1)2^\ell, (r + 2)2^\ell] \times [(s - 1)2^\ell, (s + 2)2^\ell];
\]

its topological boundary is denoted by \( \Delta \tilde{S}(r, s) \). Note that by our imbedding of \( \mathcal{T} \), this boundary is a circuit on the triangular lattice. Now if \( C_{\tau - r} = C \) for some fixed circuit \( C \) surrounding the origin and of diameter \( \geq 2^j \), and if \( C \) intersects a given \( S(r, s) \), then \( C \) must also contain points outside \( \tilde{S}(r, s) \) (because its diameter exceeds that of \( \tilde{S}(r, s) \)). Thus in this case, \( C \) must contain a crossing of the annulus \( \tilde{S}(r, s) \setminus S(r, s) \). Here, and in the sequel we define for \( m' < m \), a crossing of the annulus \( S(m) \setminus S(m') \) to be a path in this annulus from its outer boundary \( \Delta S(m) \) to a point adjacent to its inner boundary \( \Delta S(m') \).

We shall first split off the probability of the event that there are ‘too many’ such crossings. Specifically, we shall prove that for some \( c_i \)

\[
P\{\text{there exist more than } c_{38}j^4 2^{2(1-c_{30})j} \text{ disjoint occupied crossings} \}
\]

of some \( \tilde{S}(r, s) \setminus S(r, s), \quad -2^{j+1-\ell} \leq r, s < 2^{j+1-\ell} \}

\[
\leq c_{39} \exp[-c_{40}j^4 2^{2(1-c_{30})j}].
\]

(4.13)
To see (4.13), note that again by the Russo-Seymour-Welsh lemma, for any fixed \( r, s \),
\[
P\{ \text{there exists an occupied crossing of } \tilde{S}(r, s) \setminus S(r, s) \} \leq (1 - c_{41}).
\]

By the BK-inequality we therefore also have, still for fixed \( r, s \),
\[
P\{ \text{there exist } q \text{ disjoint occupied crossings of } \tilde{S}(r, s) \setminus S(r, s) \} \leq (1 - c_{41})^q, \quad q \geq 0.
\]
(4.14)

Thus the number of crossings of each of our annuli is bounded by a geometric variable. Moreover, the number of crossings of two annuli \( \tilde{S}(r', s') \setminus S(r', s') \) and \( \tilde{S}(r'', s'') \setminus S(r'', s'') \) are independent as soon as \( |r'' - r'| \vee |s'' - s'| \geq 3 \). The estimate (4.13) now follows by standard methods for exponential bounds. Combining (4.6) with the estimates (4.8) and (4.13) we find that the probability in (4.5) is at most
\[
(2c_3 + 1)e^{-c_{36}j} + j^2c_{39}e^{-c_{40}j^42^{2(1-c_{30})j}} + \sum_{0 \leq p < j^2} \sum_{C} P\{ p < \tau, C_{\tau-p} \text{ is occupied and equals } C \text{ and } C_{\tau-p} \text{ is not good} \}
\]
\[
= (2c_3 + 1)e^{-c_{36}j} + j^2c_{39}e^{-c_{40}j^42^{2(1-c_{30})j}} + \sum_{0 \leq p < j^2} \sum_{C} P\{ C \text{ is not good} | p < \tau, C_{\tau-p} \text{ is occupied and equals } C \}
\]
\[
\times P\{ p < \tau, C_{\tau-p} \text{ is occupied and equals } C \},
\]
(4.15)

where the sum over \( C \) runs over the circuits contained in \( S(2^{j+1}) \), with diameter \( \geq 2^j \) and with at most \( c_{38}j^42^{2(1-c_{30})j} \) disjoint crossings of all \( \tilde{S}(r, s) \setminus S(r, s) \) together.

**Step (ii)** In this step we make a number of topological preparations. Basically we need a circuit \( J \) on \( T \) which is 'just inside' \( C \) and we need some relation between crossings of squares by \( C \) and \( J \). The statements here are fairly intuitive but the arguments are finicky; one has to make sure that they don’t rely on pictures.

We need some definitions. These follow Kesten (1982). If \( W \) is a set of vertices of \( T \), then its **boundary** is
\[
\partial W := \{ v : v \notin W \text{ but } v \text{ is adjacent to } W \};
\]
the **exterior boundary** is denoted by \( \partial_{\text{ext}} \) and defined as
\[
\partial_{\text{ext}} W = \{ v \in \partial W : \exists \text{ a path } \pi \text{ on } T \text{ from } v \text{ to } \infty \text{ such that the only point of } \pi \text{ in } W \cup \partial W \text{ is } v \}.\]
The following separation property is basic for us:

If $W$ is a nonempty, finite connected set of vertices of $T$,
then there exists a circuit $J$ on $T$ which passes
exactly through all vertices of $\partial_{\text{ext}} W$ and surrounds $W$. \hspace{1cm} (4.16)

Basically this is what the proof of Proposition 2.1 in the Appendix of Kesten (1982)
shows. There are two differences. Firstly, Kesten (1982) works on a graph $M_{\text{pl}}$
and secondly, Kesten (1982) only proves that $J \subset \partial_{\text{ext}} W$, but not the converse.
However, the only property of $M_{\text{pl}}$ which is used in Kesten (1982) is that it is
planar and has triangular faces. Thus the proof there applies to $T$ without changes
and we only need to show that

$$\partial_{\text{ext}} W \subset J. \hspace{1cm} (4.17)$$

Assume that this fails and that there exists a vertex $u \in \partial_{\text{ext}} W \setminus J$. Let $u$ be
adjacent to $w \in W$ and let $\pi$ be a path on $T$ from $u$ to $\infty$ such that $\pi \cap (W \cup \partial W) = u$. Let $\tilde{\pi}$ be the path consisting of the edge from $w$ to $u$ followed by $\pi$. Then $\tilde{\pi}$ is a path from $W$ to $\infty$ which does not intersect $J$ (because $u \notin J$ and $J \subset \partial_{\text{ext}} W \subset \partial W$ and hence $(\pi \setminus \{u\}) \cap J = \emptyset$). But then $J$ does not surround $W$, contrary to what we know about $J$ already. Hence there is no $u \in \partial_{\text{ext}} W \setminus J$
and (4.17) holds. This also proves (4.16).

It is standard in topological considerations of this type to interchange exterior
and interior by letting a finite point play the role of $\infty$. We shall do this here for
the situation where $W = C$, a fixed circuit on $T$. We let a point $z \in \overset{\circ}{C}$
which is not adjacent to $C$ play the role of $\infty$. We define

$$\partial^z_{\text{int}} C = \{ v \in \partial C : \exists \text{ a path } \pi \text{ on } T \text{ from } v \text{ to } z$$
$$\text{such that the only point of } \pi \text{ in } C \cup \partial C \text{ is } v \}. \hspace{1cm} (4.18)$$

It is easy to see that if $z \in \overset{\circ}{C}$, then $\partial^z_{\text{int}} C \subset \overset{\circ}{C}$. The proof in Kesten (1982) of (4.16) above now yields

there exists a circuit $J^z$ on $T$ such that $J^z \subset \overset{\circ}{C} \cap \partial C$
and such that the vertices on $J^z$ are exactly the vertices of $\partial^z_{\text{int}} C$;
moreover any continuous path on $\mathbb{R}^2$ from $C$ to $z$ must intersect $J^z$.

$$\hspace{1cm} (4.19)$$

We now want to show that if $C$ has at most $c_{38}j^42^{2(1-c_{30})j}$ crossings of all
$\tilde{S}(r, s) \setminus S(r, s)$ together, then we only need to consider (in a sense to be made
precise) \(c_3 j^4 2^{(1-c_3)j}\) crossings by the circuit \(J^z\) of all \(\tilde{S}(r, s) \setminus S(r, s)\). For the time being fix \((r, s)\) and abbreviate \(S(r, s)\) and \(\tilde{S}(r, s)\) to \(S\) and \(\tilde{S}\), respectively. We assume that \(C\) has the properties listed at the end of Step (i) and that

\[ z \in \tilde{C} \setminus \tilde{S}, \text{ and } z \text{ is not adjacent to } C. \]  \tag{4.20}

For the time being we further suppress the superscript \(z\) in the notation and write \(\partial_{\text{int}} C\) for \(\partial_{\text{int}}^z C\) and \(J\) for \(J^z\).

Let \(w\) be a vertex of \(C\) which lies in \(S\) or is adjacent to \(S\). Since \(\text{diam } (C) > 2\cdot \text{diam } (\tilde{S})\), \(C\) cannot be contained in \(\tilde{S}\). We can then move counterclockwise and clockwise from \(w\) along \(C\) until we reach \(\Delta \tilde{S}\) for the first time in \(u'\) and \(u''\) say. Write \(E = E(w)\) for the piece of \(C\) from \(u'\) to \(u''\) through \(w\). Then the first piece of \(E\) from \(u'\) to the first vertex of \(E\) adjacent to \(\Delta S\) is a crossing of \(\tilde{S}\setminus S\). Conversely, for any piece of \(C\) which forms a crossing from a point \(u'\) on \(\Delta \tilde{S}\) to a point \(u''\) adjacent to \(\Delta S\), there is only one piece \(E\) of \(C\), as above, which contains the crossing from \(u'\) to \(u''\), to wit \(E(u'')\) (this is found by simply continuing from \(u''\) along \(C\) until we reach \(\Delta \tilde{S}\) again at some point \(u''\)). Thus the number of different pieces \(E(w)\) which can be found is at most equal to the number of crossings by \(C\) of \(\tilde{S}\setminus S\) (even though there may be many more points \(w\) of \(C\) in or adjacent to \(S\)).

We want to establish a similar statement for \(J\). We will be considering vertices \(v\) of \(J\) in \(S\). Let \(v \in J\). We follow \(J\) counterclockwise and clockwise from \(v\) until we first reach vertices in the exterior of \(\tilde{S}\), say the vertices \(x'\) and \(x''\), respectively. Such points must exist, because we cannot have \(J \subset \tilde{S}\) if \(J\) separates \(z\) from \(C\) and hence from \(\infty\) – and hence from \(\infty\) – and \(z \in \tilde{C} \setminus \tilde{S}\) (see (4.20)). Denote the piece of \(J\) from \(x'\) to \(x''\) through \(v\) by \(F = F(v)\). Note that \(v\) is adjacent to some vertex \(w\) of \(C\), because \(J \subset \partial C\). \(F(v)\) is an obvious analogue of \(E(w)\). We are going to show that

the number of different \(F(v)\) with \(v \in J \cap S\) which

can arise is at most equal to the number of different \(E(w)\).

\tag{4.21}

It suffices for this to show that if \(v'\) and \(v''\) are two distinct vertices of \(J\) in \(S\) which are adjacent to \(w'\) and \(w''\) and if \(E(w') = E(w'')\), then also \(F(v') = F(v'')\). To prove this, fix \(v', v'' \in J \cap S\) and let \(w', w'' \in C\) be adjacent to \(v'\) and \(v''\), respectively and assume \(E(w') = E(w'')\). Then, by our construction of the \(E\)'s, there is a piece of \(C\) from \(w'\) to \(w''\) which is entirely contained in the interior of \(\tilde{S}\). Denote this piece of \(C\) by \(B\). Further let \(\pi'\) be a path on \(T\) from \(v' \in J \subset \partial_{\text{int}} C\) to \(z\) such that the only point of \(\pi'\) in \(C \cup \partial C\) is \(v'\). Write \(\tilde{\pi}'\) for the path consisting of the edge \(\{w', v'\}\) followed by \(\pi'\). Choose \(\pi''\) and \(\tilde{\pi}''\) in a similar way for \(v''\) (see Figure 4). Let \(y\) be the first point of \(\pi''\) which is also on \(\pi'\). We can then replace
the part of $\pi''$ from $y$ to $z$ by the piece of $\pi'$ from $y$ to $z$, so that we may assume without loss of generality that $\pi'$ and $\pi''$ coincide from $y$ to $z$ (of course $y = z$ is possible). Finally, let $J_B \subset \partial_{\text{ext}}B$ be the circuit on $T$ which surrounds $B$ as in (4.16) with $W$ replaced by $B$. Now $v' \in \partial_{\text{int}}C$ also belongs to $\partial_{\text{ext}}B$ because $v'$ is adjacent to $B$ and $B \subset C$ (hence $B \cup \partial B \subset C \cup \partial C$); moreover, a path on $T$ from $v'$ to $z$ can be extended to a path on $T$ to $\infty$ without hitting $B \cup \partial B$, because $B \subset$ interior of $\tilde{S}$, hence $B \cup \partial B \subset \tilde{S}$, and $z \notin \tilde{S}$. Thus $v' \in J_B$ and the same holds for $v''$. Hence there are two arcs of $J_B$ from $v'$ to $v''$. We next consider the path $\sigma$ on $T$ from $w'$ to $w''$ which consists of the piece of $\tilde{\pi}'$ from $w'$ to $y$ followed by the reverse of the part of $\tilde{\pi}''$ from $y$ to $w''$. It is immediate that this path intersects $C$ only at its endpoints. This path has no double points, unless $w' = w''$, in which case $w'$ is the unique double point on this path. Moreover, since $z \in \overset{\circ}{C}$, also $y \in \overset{\circ}{C}$ and $\sigma \setminus \{w', w''\} \subset \overset{\circ}{C}$. Thus $\sigma$ is a crosscut of $\overset{\circ}{C}$ and divides it into two components (see Figure 5). Let $G$ be the component of $\overset{\circ}{C} \setminus \sigma$ which has boundary consisting of $\sigma$ and $B$. The other component has boundary consisting of $\sigma$ and $C \setminus B$ (see Newman (1951), Theorem V.11.8). The part $C \setminus B$ of this boundary lies in the exterior of $\Delta G = (\sigma \text{ concatenated with } B)$, since we can connect a point of $C \setminus B$ to $\infty$ without hitting $C \cup \overset{\circ}{C} \supset G \cup \Delta G$.

**Figure 4.** The circuits $C$ (solidly drawn) and $J_B$ (dashed). Also shown are the arc $B$ of $C$ (boldly drawn) and the paths $\tilde{\pi}', \tilde{\pi}''$. 
Figure 5. σ (the dashed path) is a crosscut of $\tilde{C}$. $C$ is the solidly drawn circuit.

We further claim that the interior of one of the arcs of $J_B$ between $v'$ and $v''$ is contained in $G$. Indeed, the only points of $\sigma$ in $\partial C$ are the two points $v'$ and $v''$. Thus also $\sigma \cap J_B = \{v', v''\}$ (recall $J_B \subset \partial B \subset C \cup \partial C$). Moreover, $B$ is disjoint from $J_B$. Thus $J_B \cap (\sigma \cup B) = \{v', v''\}$. It follows that each of the two arcs of $J_B$ from $v'$ to $v''$ (minus their endpoints) lie in the interior or exterior of the Jordan curve made up from $B$ and $\sigma$. If both these arcs would lie in the exterior, then $J_B$ could not separate $B$ from $y$ inside $G$ (except for the endpoints of $\varphi$). $\varphi$ would be disjoint from $J_B$. Moreover, $\varphi$ can be continued by the common piece of $\pi'$ and $\pi''$ from $y$ to $z$; this is still disjoint from $J_B$, because it is disjoint from $(C \cup \partial C) \subset \partial_{ext} B$. Therefore, $J_B$ would not separate $B$ from $z$. But $B \subset$ interior of $\tilde{S}$ and hence

$$J_B \subset \partial B \subset \tilde{S} \quad (4.22)$$

and $z \notin \tilde{S}$ (see (4.20)). Therefore $z$ also lies in the exterior of $J_B$. But then $J_B$ would also not separate $B$ from $\infty$, contrary to the choice of $J_B$. This proves that at least one of the arcs of $J_B$ from $v'$ to $v''$ lies in $G$. We shall denote such an arc by $\tau$. (In fact, $\tau$ is unique, because the other arc of $J_B$ from $v'$ to $v''$ has to lie outside $G$, but we shall not need this.)

We next show that $\tau$ is actually also one of the arcs of $J$ from $v'$ to $v''$. To begin with let $u$ be a vertex on $\tau \setminus \{v', v''\}$ and let $\psi$ be a path on $\mathcal{T}$ from $u$ to $z$ which intersects $B \cup \partial B$ only in $u$; such a path exists, because $u \in J_B = \partial_{ext} B$. We have to consider two cases. First consider the case when $z \in G$. In this case we may assume that

$$\psi \subset G \cup \pi' \cup \pi'' \quad (4.23)$$

Indeed, $\psi$ starts at $u \in G$. If it hits $\partial G$ this must be in a point of $\pi' \cup \pi'' \setminus \{v', v''\}$, since $\psi$ was chosen disjoint from $B$, and because it is a path on $\mathcal{T}$, it does not intersect the edges $\{w', v'\}$ and $\{w'', v''\}$ (whose endpoints lie in $B \cup \partial B$ and not on $\psi$). But then we can replace the piece of $\psi$ between its first intersection with
\(\pi' \cup \pi''\) and \(z\) by a piece of \(\pi' \cup \pi''\), to get (4.23). Note that the change in \(\psi\) will preserve the property

\[
\psi \cap (B \cup \partial B) = \{u\},
\]

(4.24)
because \((\pi' \cup \pi'') \cap (B \cup \partial B) \subset (\pi' \cup \pi'') \cap (C \cup \partial C) = \{v', v''\}\), by choice of \(\pi', \pi''\).

The second case is when \(z \notin G\). Then \(\psi\) begins in \(G\), but ends at \(z \notin G\) and must therefore hit the (topological) boundary of \(G\) somewhere. As in the first case this intersection point must lie on \(\pi' \cap \pi''\) \(\\setminus\{v', v''\}\). We may replace the piece of \(\psi\) after the intersection with \(\pi' \cap \pi''\) by a piece of \(\pi'\) or of \(\pi''\). Again this will preserve property (4.24). Thus, (4.23) holds in both cases. We now show that this implies

\[
\psi \cap (C \cup \partial C) = \{u\}.
\]

(4.25)
Indeed, we proved before that \(C \setminus B\) is in the exterior of \(\sigma \cup B\), so that no vertex on \(C \setminus B\) can be equal to or adjacent to a vertex in \(G = \) the interior of \(\sigma \cup B\). In particular, (4.23) then tells us that \(\psi\) contains no vertices on or adjacent to \(C \setminus B\).

Together with (4.24) this gives (4.25).

The result (4.25) together with the fact that \(u \in \tau \subseteq J_B\) is adjacent to \(B \subseteq C\) implies that

\[
u \in \partial_{\text{int}} C.
\]

(4.26)
Thus all vertices on \(\tau\) are vertices of \(J\). This almost shows that \(\tau\) is an arc of \(J\). What we still have to rule out is that vertices occur in different order on \(\tau\) than on \(J\). To show that this is not the case we prove that if \(J = (x_0, x_1, \ldots, x_{\nu-1}, x_\nu = x_0)\), then the only neighbors of \(x_i\) on \(J\) are \(x_{i-1}\) and \(x_{i+1}\) (with \(x_{\nu} = x_{\nu-1}, x_{\nu+1} = x_1\)). If this were not the case, there would exist an edge \(e\) between some \(x_i\) and \(x_j\) for which there exist vertices \(y'\) and \(y''\) on each of the two arcs of \(J\) between \(x_i\) and \(x_j\). \(e\) would be a crosscut of \(\overset{\circ}{J}\) or of \(J^{\text{ext}}\). Both of these possibilities lead to a contradiction. Assume first that \(e\) is a crosscut of \(\overset{\circ}{J}\). Then \(\overset{\circ}{J} \setminus e\) consists of two components, one of which contains \(z\) (because of the fact that \(J\) separates \(z\) from \(C\) and \(J \subseteq \overset{\circ}{C}\) implies \(z \in \overset{\circ}{J}\); moreover, \(e\) contains only the vertices \(x_i\) and \(x_j\), so that \(z \notin e\)). Let \(z\) lie in the component bounded by \(e\) and the arc of \(J\) from \(x_i\) to \(x_j\) through \(y'\). Then any path on \(T\) from \(y''\) to \(z\) would have to intersect \(e\) or \(J \subseteq \partial C\) in a point other than \(y''\). A path on \(T\) can intersect \(e\) only in \(x_i\) or \(x_j\). Thus there would not exist a path on \(T\) from \(y''\) to \(z\) which only intersects \(C \cup \partial C\) in \(y''\). This contradicts the fact that \(y'' \in \partial_{\text{int}} C\). If, on the other hand, \(e\) is a crosscut of \(J^{\text{ext}}\), then it divides \(J^{\text{ext}}\) into two components, one of which contains all of \(C\). Let this again be the component bounded by \(e\) and the arc of \(J\) from \(x_i\) to \(x_j\) through \(y'\). Then any path on \(T\) from \(y''\) to \(C\) must intersect \(e\) or \(J\) in a point different from \(y''\), contrary to the fact that \(y''\) is adjacent to \(C\). Thus no neighbors \(x_i, x_j\) as above can exist and the only neighbors of \(x_i\) are \(x_{i-1}, x_{i+1}\), as claimed. But then, if \(\tau\) runs successively through the vertices

\[x_{i_0} = v', x_{i_1}, \ldots, x_{i_\nu} = v''\]
of $J$, we must have $i_{\ell+1} = i_{\ell} + 1 \pmod{\nu}$ or $i_{\ell+1} = i_{\ell} - 1 \pmod{\nu}$ for $0 \leq \ell \leq \rho - 1$, so that $\tau$ is indeed an arc of $J$.

This finally proves (4.21). Indeed, since $\tau \subset J_B$ is adjacent to $B$, and $B \subset \text{interior of } \tilde{S}$, we have $\tau \subset \tilde{S}$. Thus, there is an arc of $J$ from $v'$ to $v''$ which stays in $\tilde{S}$. Hence, when forming $F(v')$ we must pass through $v''$ so that $F(v') = F(v'')$.

Step (iii) The result (4.21) has been proven for a single square $\tilde{S}(r,s)$ and a single $z \notin \tilde{S}(r,s)$, or rather a single $J^z$, only. We already know that there are at most $c_{37}j^{42(1-c_{30})^j}$ different pairs $(r,s)$ (see (4.10)). In this step we show how to control the number of different circuits $J^z$ which can arise.

If $C_{\tau - p} = C$ and $C_{\tau - p}$ is occupied but not good, then there exists a vacant set $D$ which satisfies (4.1) and (4.2), but not (4.3). It is easily seen that there then exists a selfavoiding vacant path, $\zeta = (v_1, \ldots, v_q) \subset D$, such that

\begin{equation}
  v_1 \text{ is the unique vertex of } \zeta \text{ adjacent to } C \text{ and } \zeta \subset \hat{C}; \tag{4.27}
\end{equation}

\begin{equation}
  \text{diam}(\zeta) \geq \frac{1}{2}(\text{diam}(C))^{c_{30}} \tag{4.28}
\end{equation}

and such that (4.3) with $D$ replaced by $\zeta$ fails. The vertex $v_1$ must lie in some $S(r,s)$. Since $\text{diam}(\zeta) > \text{diam}(\tilde{S}(r,s))$, $\zeta$ must contain points outside $\tilde{S}(r,s)$. We shall replace $\zeta$ by its initial piece from $v_1$ to the first exit from $\tilde{S}$. Then (4.27) holds as well as

\begin{equation}
  v_1 \in S(r,s), v_q \notin \tilde{S}(r,s), \text{ but } v_i \in \tilde{S}(r,s) \text{ for } 1 \leq i \leq q - 1. \tag{4.29}
\end{equation}

We further introduce the square

\[ \tilde{S} = \tilde{S}(r,s) = [(r - 1)2^\ell - 1, (r + 2)2^\ell + 1] \times [(s - 1)2^\ell - 1, (s + 2)2^\ell + 1] \]

which just surrounds $\tilde{S}$. Its topological boundary is denoted by $\Delta \tilde{S} = \Delta \tilde{S}(r,s)$. Since $v_q$ is adjacent to $v_{q-1} \in \tilde{S}(r,s) \subset \text{interior of } \tilde{S}(r,s)$, we must have

\begin{equation}
  v_q \in \Delta \tilde{S}. \tag{4.30}
\end{equation}

Again we fix $(r,s)$ so that this holds, but suppress $(r,s)$ in the notation in this step.

We claim that (4.27) implies that

\begin{equation}
  v_1 \in J^{v_q}. \tag{4.31}
\end{equation}

To see this, note that $v_1$ is adjacent to some vertex $w \in C$ and (the reverse of) the path $\zeta$ followed by the edge $\{v_1, w\}$ is a path from $v_q$ to $C$, which starts at
\( v_q \in \tilde{C}. \) It therefore must intersect \( \partial^{v_q} \partial^{v_q} C = J^{v_q} \) (see (4.19)). But the only possible intersection point is \( v_1, \) because that is the only point on \( \zeta \cap \partial C. \) Hence (4.31) holds. Thus if we want to bound the number of possible choices for \( J^z, \) then we only need to consider \( J^z \) with \( z \in \partial \tilde{S} \) for some \( \tilde{S}(r, s) \) and such that \( J^z \) contains a point of \( S(r, s). \)

Now consider two vertices \( z' \) and \( z'' \) on \( \partial \tilde{S} \) with corresponding paths \( \zeta', \zeta'' \) to vertices \( v'_1 \in J^{z'} \cap S, v''_1 \in J^{z''} \cap S, \) respectively. We may assume that these paths satisfy (4.27) and (4.29)-(4.31) with the prime or double prime added at the appropriate places. In particular,

\[
\zeta' \setminus \{v'_1\} \text{ and } \zeta'' \setminus \{v''_1\} \text{ are disjoint from } C \cup \partial C. \tag{4.32}
\]

If there exists a path on \( T \) from \( \zeta' \) to \( \zeta'' \) which is disjoint from \( C \cup \partial C, \) then the definition (4.18) shows that \( \partial^{\zeta', \zeta''} \partial^{\zeta', \zeta''} C = \partial^{\zeta', \zeta''} \partial^{\zeta', \zeta''} C. \) In particular, this is the case if \( C \cup \partial C \) does not intersect a piece of \( \partial \tilde{U} \) which connects \( \zeta' \) to \( \zeta'', \) where

\[
U = U(r, s) := [r2^\ell - 2^{\ell-1}, (r + 1)2^\ell + 2^{\ell-1}] \times [s2^\ell - 2^{\ell-1}, (s + 1)2^\ell + 2^{\ell-1}].
\]

![Figure 6](image)

**Figure 6.** The successive squares from the inside out are \( S, V, U, \tilde{S}. \) \( \delta'' \) is the boldly drawn arc on \( \Delta \tilde{S} \) and \( \delta' \) is just outside \( S. \) Also shown are \( \gamma' \) and \( \gamma''. \) \( R \) is the upper left hand corner of \( \tilde{S} \setminus S \) between \( \gamma' \) and \( \gamma''. \) \( C \) must contain a crossing like one of the dashed ones to separate \( \gamma' \) from \( \gamma''. \)

Note that \( \Delta U \) is the boundary of a square intermediate between \( S \) and \( \tilde{S}, \) and \( \zeta' \) and \( \zeta'' \) both intersect \( \Delta U. \) In fact there exists a piece, \( \gamma' \) say of \( \zeta' \) of the form
(v'ₙ, ..., v'ₘ) which connects a vertex in ∆S to ∆Ŝ and which lies in S \ Ŝ. We will have

\[ v'_n \in \Delta[r2^ℓ - 1, (r + 1)2^ℓ + 1] \times \Delta[2s^ℓ - 1, (s + 1)2^ℓ + 1]. \quad (4.33) \]

Also ζ'' contains a similar piece γ'''. If γ' and γ''' intersect, then ζ' and ζ''' are clearly connected outside C ∪ ∂C (see (4.32) and note that v'_1 \neq v'ₙ because v'_1 \in S and v'_ₙ \in S by (4.33); similarly v''ₙ \neq v''''ₙ). If γ' and γ''' are disjoint, then γ', γ''' partition the interior of

\[ S \setminus \Delta[r2^ℓ - 1, (r + 1)2^ℓ + 1] \times \Delta[2s^ℓ - 1, (s + 1)2^ℓ + 1] \]

into two components (see Newman (1951), exercise V.11.3). Let R be one of these components. Its boundary consists of γ', γ''' and an arc δ' of \( \Delta[r2^ℓ - 1, (r + 1)2^ℓ + 1] \times [s2^ℓ - 1, (s + 1)2^ℓ + 1] \) between v'_1 and v''''ₙ, and an arc δ'' of \( \Delta S \) between v'_1 and v''''ₙ (see Figure 6). As observed above, if C ∪ ∂C does not intersect (R ∪ ∆R) \ ∆U, then J'' = J''''. However, the only way C ∪ ∂C can intersect (R ∪ ∆R) \ ∆U is when C contains a piece in R which connects δ' \subset \Omega[r2^ℓ - 1, (r + 1)2^ℓ + 1] \times [s2^ℓ - 1, (s + 1)2^ℓ + 1] with a point adjacent to ∆U, or C contains a piece in R which connects δ''' \subset \Omega Ŝ with a point adjacent to ∆U (recall that C contains points outside Ŝ, because diam(C) > 2-diam(Ŝ) and C ∪ ∂C cannot intersect the pieces γ', γ''' of ∆R by (4.27)). Thus for any J'''', J'' which are different, C has to have a crossing of V \ S or a crossing of Ŝ \ U in the corresponding region R, where

\[ V = V(r, s) := [r2^ℓ - 2^ℓ - 1 + 1, (r + 1)2^ℓ + 2^ℓ - 1 - 1] \times [s2^ℓ - 2^ℓ - 1 + 1, (s + 1)2^ℓ + 2^ℓ - 1 - 1]. \]

Now choose a maximal collection z₁, ..., zₙ ⊂ ∂Ŝ for which the J''ₙ are distinct. Index the zᵢ so that we meet z₁, z₂, ..., zₙ in this order as we traverse ∂Ŝ clockwise, starting from z₁. Let ζᵢ be a path from zᵢ to some vertex in J''ₙ ∩ S, satisfying (4.27) and (4.29)-(4.31). Then C must have a crossing of the above form between any two successive ζᵢ. Therefore,

\[ \text{the maximal number of distinct } J'' \leq \text{the maximal number of disjoint crossings by C of } V \setminus S \text{ or of } Ŝ \setminus U. \]

(4.34)

We just proved that (for fixed \((r, s)\)) the total number of circuits J'' which can arise as we vary the endpoint v of a path ζ which satisfies (4.27) and (4.29)-(4.31), is at most

\[ \text{[maximal number of disjoint occupied crossings of } V(r, s) \setminus S(r, s) \text{ or of } Ŝ(r, s) \setminus U(r, s)] \]

\[ \times \text{[maximal number of disjoint occupied crossings of } Ŝ(r, s) \setminus S(r, s)]. \]

(4.35)
Finally we take the union over $-2^{j+1-\ell} \leq r, s < 2^{j+1-\ell}$. We find that the total number of choices for $J^z, F(v)$ is the sum of (4.35) over $(r, s)$.

**Step (iv)** We now pick up the estimates from Step (i) again. Essentially the same argument as for (4.14) shows that the random variable in (4.35) has all moments. Therefore, by Chebyshev’s inequality we have

$$P\{\text{sum of (4.35) over } -2^{j+1-\ell} \leq r, s < 2^{j+1-\ell} > c_42j^42^{2(1-c_{30})j}\} \leq c_43j^{-4}2^{-2(1-c_{30})j}. \tag{4.36}$$

Thus after an adjustment of $c_{36}, c_{38}$ the estimate (4.15) remains valid even if we restrict $C$ further so that it contains at most $c_42j^42^{2(1-c_{30})j}$ pairs of crossings, the first one of $V(r, s) \setminus S(r, s)$ or of $\hat{S}(r, s) \setminus U(r, s)$ for a certain $(r, s)$, and the second one of $\tilde{S}(r, s) \setminus S(r, s)$ for the same $(r, s)$. (Note that this makes $c_{36}$ dependent on $c_{30}$.)

The last sum in (4.15) is therefore bounded by

$$j^2 \sup_{p,C} P\{C \text{ is not good} | p < \tau, C_{\tau-p} \text{ is occupied and equals } C\},$$

where the sup is over $0 \leq p < j^2$ and $C$ satisfying the restrictions at the end of Step (i) and the ones just mentioned. For the remainder of this proof $C$ will be a fixed circuit satisfying these conditions. For brevity we define the event

$$E(p, C) = \{p < \tau, C_{\tau-p} \text{ is occupied and equals } C\}.$$

It is important to realize that for a given circuit $C$, the event $E(p, C)$ depends only on the vertices in $C^{\text{ext}} \cup C$. (Note that the condition $\{p < \tau, C_{\tau-p} = C\}$ merely says that $C$ is one of our $C_k$ and there are exactly $p$ of the circuits $C_k$ outside $C$ but inside $S(2^{j+1})$; this only involves vertices on or outside $C$.). Given $E(p, C)$, the further conditions for $C_{\tau-p}$ to be not good, depend only on the vertices in $C$. Therefore, for calculating

$$P\{C \text{ is not good} | E(p, C)\}, \tag{4.37}$$

even with the conditioning on $E(p, C)$, we may assume that all vertices in the interior of $C$ are still independently occupied or vacant with probability 1/2. Now we saw that if $E(p, C)$ occurs, but $C$ is not good, then there exists a pair $(r, s)$, one of the possible arcs $F(v)$, and a vacant path $\zeta = (v_1, \ldots, v_q)$ which satisfy (4.27) and (4.29)-(4.31), but such that (4.3) with $D$ replaced by $\zeta$ fails. If $C$ is fixed, we shall estimate the probability in (4.37) by first picking a pair $(r, s)$ so that $C$ contains a point in or adjacent to $S(r, s)$, and then one of the possible interior
boundaries \( J = J^z \) and finally one of the possible arcs \( F(v) \) of \( J^z \) in \( \hat{S}(r, s) \). Once \((r, s), J \) and \( F = F(v) \) have been chosen we estimate
\[
\sup P \{ \exists \text{ a vacant path } \zeta = (v_1, \ldots, v_q) \text{ which satisfies (4.27), (4.29), (4.30)} \]
and \( v_q \in \hat{o} J, v_1 \in F \), but for which (4.3) fails\}
\[
(4.38)
\]
Here the sup is over all choices of \( C \) as above, over \((r, s)\), and over the possible choices of \( J \) and \( F \) (for the chosen \( C \) and \((r, s)\)). For the class of circuits \( C \) which we allow, the total number of choices for \((r, s), J, F \) is at most \( c_{42} j^6 2^{2(1-c_{30})j} \), so that the last sum in (4.15) is bounded by
\[
c_{42} j^6 2^{2(1-c_{30})j} \times \text{the sup in (4.38)}.
(4.39)
\]
We are going to imitate the proof of Lemma 8.2 in Kesten (1982) to estimate (4.38). For the remainder of this step \( C, (r, s) \) and \( F \) remain fixed. This of course also tells us which part of \( \hat{S} \) belongs to \( \hat{o} C \). By our construction in Step (i), \( F \) is a crosscut of the interior of \( \hat{S} \) which contains some vertex in \( S \). Let \( F = (w_1, w_2, \ldots, w_\nu) \), with \( w_1, w_\nu \in \Delta \hat{S} \), where the vertices are indexed in such a way that we move along \( J \) in the counterclockwise direction as we run successively through \( w_1, w_2, \ldots, w_\nu \). We number the remaining vertices of \( J \), still as we meet them by continuing in the counterclockwise direction along \( J, w_\nu+1, w_\nu+2, \ldots, w_p \). \( w_p \) is adjacent to \( w_1 \).

Let \( I \) be the component of \( \hat{o} J \cap \text{(interior of } \hat{S}) \) which has \( F \) as part of its boundary. There is only one such component. This is relatively easy to see in our situation, because \( F \) is a piecewise linear path, built up from edges of \( T \), and \( F \) is a crosscut of \( \text{(interior of } \hat{S}) \). The topological boundary of \( I \) consists of \( F \) and alternating pieces of \( \Delta \hat{S} \) and \( J \) (see Figure 7). We need a more precise statement, though, which describes the relative order of some of the pieces of \( \Delta I \). First let us prove a general result. Let \( J \) be a Jordan curve and let it be oriented counterclockwise (i. e., with \( \hat{o} \text{ ‘on the left’} \)). Let \( K_1, K_2, \ldots, K_N \) be a number of crosscuts of \( \hat{o} J \) with disjoint interiors, i. e., for which \( \hat{o} \cap K_i \cap K_j = \emptyset \) when \( i \neq j \). Let \( I \) be one of the components of \( \hat{o} J \setminus \cap_{i=1}^N K_i \). We claim that then \( \Delta I \) consists of a number of disjoint arcs \( \alpha_i \) from \( J \) and some of the crosscuts \( K_i \), and we can traverse \( \Delta I \) in one direction so that we traverse all the \( \alpha_i \) in \( \Delta I \) in the same order and in the same direction as they are traversed when \( J \) is traversed in the counterclockwise direction. This claim is easily proven by induction on \( N \). It is trivially true when \( N = 0 \). Now assume it is already proven when \( N = n \) and let \( K_{n+1} \) be a crosscut of \( \hat{o} J \) such that \( K_{n+1} \cap \hat{o} J \) is disjoint from \( \cup_{i=1}^n K_i \). Finally, let \( I_n \) be one of the components of \( \hat{o} J \setminus \cup_{i=1}^n K_i \). If \( K_{n+1} \) is disjoint from \( I_n \), then
this component and its boundary is unchanged by the addition of $K_{n+1}$, so our claim is valid for this component. If $K_{n+1}$ intersects $I_n$, then it is easily seen that $K_{n+1}$ is a crosscut of $I_n$ with its endpoints, $a_1$ and $a_2$ say on two of the arcs $\alpha_i, \alpha_j$ ($i = j$ is possible). Then $K_{n+1}$ divides $I_n$ into two new components. Call them $I'_n$ and $I''_n$. The boundary of each of $I'_n, I''_n$ consists of $K_{n+1}$ plus one of the arcs of $\Delta I_n$ from $a_1, a_2$. Since our claim was known to be true for $\Delta I_n$ it is also true for each of the arcs of $\Delta I_n$ from $a_2$ to $a_1$ followed by $K_{n+1}$, that is for the boundaries of $I'_n$ and $I''_n$. This proves our claim for $N = n + 1$.

We now apply the claim of the last statement to $I$, the component of $\overset{.}{J} \cap (\text{interior of } \overset{.}{S})$ defined above. For the $K_i$ we take all segments of $\Delta \overset{.}{S}$ whose interior lies in $\overset{.}{J}$. $I$ is one of the components of $\overset{.}{J}$ minus these segments. The claim from the preceding paragraph then tells us that $\Delta I$ consists of arcs of $J$ and segments of $\Delta \overset{.}{S}$, and that we can traverse $\Delta I$ in such a way that all the arcs of $J$ in $\Delta I$ are traversed in the same direction and order as when $J$ is traversed counterclockwise. In fact this will necessarily be when we traverse $\Delta I$ counterclockwise, because when traversing the arc $F$ which belongs to $J$ and to $\Delta I$, any vector pointing into $\overset{.}{J}$ also points into $\overset{.}{J}$.

Next, we shall introduce a partial order of the paths $\zeta$ which we are considering. Call a path $\zeta = (v_1, \ldots, v_q)$ permissible, if it satisfies (4.27), (4.29), (4.30) and

$$v_q \in \overset{.}{J}, v_1 \in F \cap S. \tag{4.40}$$

A permissible path minus its endpoints lies in $I$; in fact it is a crosscut of $I$ (note that $\zeta \setminus \{v_1\} \subset \overset{.}{J}$, by virtue of (4.27) and the fact that $v_q \in \overset{.}{J}$; thus, as we move from $v_1 \in F$ along $\zeta$ we enter a component of $\overset{.}{J} \cap (\text{interior of } \overset{.}{S})$ with $F$ in its boundary, and this must be $I$). We shall also need to consider a slightly larger class of paths which includes the permissible paths. This is the collection of paths $\gamma = (y_1, \ldots, y_q)$ with the following three properties:

$$y_1 \in (J \cap \text{interior of } \overset{.}{S}) \text{ and } y_1 \text{ is the only vertex of } \gamma \text{ adjacent to } C, \tag{4.41}$$

$$\gamma \setminus \{y_1, y_q\} \subset \overset{.}{I} \subset \overset{.}{J}, \tag{4.42}$$

$$y_q \in \Delta \overset{.}{S} \cap \overset{.}{J}, \text{ and } y_1 \text{ not adjacent to } y_q. \tag{4.43}$$
Figure 7. Illustration of the regions $I, L(\gamma)$ (hatched) and $R(\gamma)$ (dotted) for a path $\gamma$ as in (4.41)-(4.43). $\gamma$ is the boldly drawn curve and $I$ consists of $R(\gamma) \cup L(\gamma) \cup \gamma$ (minus the endpoints of $\gamma$).

Basically only the requirement (4.40) for permissible paths has been weakened to $y_1 \in J \cap \text{(interior of } \hat{S})$. We shall write $w_t$ for this initial point $y_1$. Such a path $\gamma$ is still a crosscut of $I$ so that $I \setminus \gamma$ consists of two components which we denote as $L(\gamma)$ and $R(\gamma)$. $L(\gamma)$ will be the component of $I \setminus \gamma$ whose boundary contains the endpoint $w_1$ of $F$, and $R(\gamma)$ will be the other component. This implies that the whole arc of $\Delta I$ in the counterclockwise direction starting from $w_1$ till it first reaches a point of $\gamma$, belongs to $\Delta L(\gamma)$, and except for $w_t$ no point of this arc belongs to $\Delta R(\gamma)$. Call this arc $A(\gamma)$. For a permissible path $\gamma$, $A(\gamma)$ is the arc of $F$ from $w_1$ to $w_t = y_1 \in F$. This means that this arc of $F$ belongs to $\Delta L(\gamma)$ and

$$y_q \notin A(\gamma).$$ (4.44)

By the property of $\Delta I$ proven above, we have that all vertices $w_i$ of $\Delta I$ with $i < t$ must be passed before we get to $w_t$ as we move counterclockwise from $w_1$. Thus (4.44) implies

all $w_i$ which belong to $\Delta I$ and which have $i < t$,  
lie in $A(\gamma)$ and belong to $\Delta L(\gamma) \setminus \Delta R(\gamma)$;  
in particular $w_i \in \Delta R(\gamma)$ implies $i \geq t$. (4.45)

(This deduction of (4.45) from (4.44) does not use that $\gamma$ is permissible, but only the properties (4.41)-(4.43).) We shall soon see that all the paths in which we are interested have the properties (4.44) and (4.45).
It may help the reader to form a mental picture by considering the case when $F$ is a left-right crossing of $\hat{S}$ with $w_1$ and $w_t$ on the left and right edge of $\Delta \hat{S}$, respectively. Then according to our numbering of the vertices of $F$, $\hat{J}$ contains the points just above $F$. If $\zeta$ has its last point $v_q$ on the top edge of $\Delta \hat{S}$, then $L(\zeta)$ and $R(\zeta)$ are on the left and right of $\zeta$, respectively. $\zeta^{(1)}$ can be thought of as the ‘leftmost’ vacant permissible path. Of course, the general picture may be much more complicated.

We now define a partial ordering of permissible paths. (Note that we only order permissible paths, not the more general paths $\gamma$ of the preceding paragraph.) If $\zeta'$ and $\zeta''$ are two permissible paths, then we say that $\zeta'$ precedes $\zeta''$ if $L(\zeta') \subset L(\zeta'')$. \hfill (4.46)

It follows from the argument of Lemma 1 in Kesten (1980) or Proposition 2.3 in Kesten (1982) that if $\zeta', \zeta''$ are permissible, then there exists a permissible $\zeta'''$ which precedes both of them. If $\zeta', \zeta''$ are both vacant, then we can take $\zeta'''$ also vacant. In what follows we assume that there exists at least one vacant permissible path. From the preceding remarks it then follows that there also exists a first path (in the above order) among all the vacant permissible paths. This path will be denoted by

$$\zeta^{(1)} = (v^{(1)}_1, \ldots, v^{(1)}_{q^{(1)}}).$$

It is an important fact that for a fixed permissible path $\gamma$ (with initial point on $F$ equal to $w_t$), the event $\{\zeta^{(1)} = \gamma\}$ depends only on the vertices in $L(\gamma) \cup \gamma \cup \{w_1, \ldots, w_t\} \cup (\Delta L(\gamma) \cap \hat{J})$. Indeed this event occurs if and only if $\gamma$ is vacant, but any permissible $\gamma' \neq \gamma$ with $L(\gamma') \subset L(\gamma)$ contains some occupied vertex; any such $\gamma'$ lies in $L(\gamma) \cup \gamma \cup \{w_1, \ldots, w_t\} \cup (\Delta L(\gamma) \cap \hat{J})$. This is so because $\gamma'$ minus its endpoints lies in $L(\gamma)$; moreover, the initial point of $\gamma'$ has to lie in $F \cap \Delta L(\gamma) = \{w_1, \ldots, w_t\}$ (see (4.40), (4.44), (4.45)), and the endpoint $v'_{q'}$ of $\gamma'$ lies in $\hat{J}$, hence not on $J$ itself.

We now wish to estimate first

$$P\{\zeta^{(1)} \text{ exists but does not satisfy (4.3)} | E(p, C)\} \hfill (4.47)$$

(of course we mean here again that (4.3) with $\zeta^{(1)}$ substituted for $D$ fails). It is convenient for this to introduce the following definitions. Let $\gamma = (y_1, \ldots, y_q)$ be a path which has the properties (4.41)-(4.43). Suppose that $y_1 = w_t \in J \cap (\text{interior of } \hat{S})$ and that $G$ is some subset of $\mathbb{R}^2$. Then we say that a vertex $w_m \in J \cap (\text{interior of } \hat{S})$ has a vacant connection in the set $G$ to $\gamma$ if $m = t$ or if there exists a path $\delta = (z_1, \ldots, z_u)$ with

$$\delta \subset G, \hfill (4.48)$$
$z_1 = w_m$, $z_u \in \gamma, z_u \neq w_t$, \hfill (4.49)

$\delta \setminus \{z_1, z_u\} \subset R(\gamma)$, \hfill (4.50)

and

$z_1, \ldots, z_{u-1}$ are vacant \hfill (4.51)

(see Figure 8). Note that we do not require $z_u$ to be vacant here (although it will be vacant in our applications); also we always count $w_t$ (the initial point of $\gamma$) as having a vacant connection to $\gamma$. However, if $w_m \neq w_t$, then (4.51) requires that $w_m = z_1$ be vacant. Note further that $z_1 = w_m \in J \cap (\text{interior of } \hat{S})$ and $\delta \setminus \{z_1, z_u\} \subset I$ by (4.50). Thus each neighborhood of $w_m$ contains points of $\delta$ in $R(\gamma) \subset I$. Of course each neighborhood of $w_m \in J$ also contains points of $J^{\text{ext}} \supset I^{\text{ext}}$. Thus $w_m \in \Delta I$. By (4.50) we then have

$w_m \in \Delta I \cap \Delta R(\gamma)$.

Now assume that $\gamma$ satisfies also the conditions (4.44) and hence (4.45), in addition to (4.41)-(4.43). Then we must have $t \leq m \leq \rho$. If $\delta$ is a vacant connection from $w_m$ to $\gamma$ and $m > t$, define $\gamma^*(\delta)$ as the path which consists of $\delta$, followed by the piece of $\gamma$ from $z_u$ (= the endpoint of $\delta$), to $y_q$ (= the endpoint of $\gamma$). It follows from (4.49), (4.50), (4.42) and (4.43) that $\gamma^*(\delta)$ is again a crosscut of $I$ and that its initial point $z_1 = w_m$ is the only point of $\gamma^*(\delta)$ which is adjacent to $C$. (Note that

$z_2, \ldots, z_{u-1} \in I \subset J$ by (4.50), $z_u \neq w_t$ by (4.49), so that also $z_u \in \gamma \setminus \{y_1\} \subset J$.) Therefore, if $z_1 = w_m$ is not adjacent to $y_q$, then $\gamma^*(\delta)$ again satisfies (4.41)-(4.43) with $y_1$ replaced by $z_1$. We want to show that $\gamma^*(\delta)$ also satisfies (4.44) and (4.45). To this we note that as we move around $\Delta I$ counterclockwise from $w_1$, we first traverse the arc $A(\gamma)$ till we get to $w_t$. As we continue traversing $\Delta I$ we go through an arc of $\Delta I$, $A'$ say, from $w_t$ to $y_q$. This arc must be the part of $\Delta I$ which belongs to $\Delta R(\gamma)$. Finally we will go from $y_q$ back to $w_1$ along another arc $A''$ of $\Delta I$ which belongs to $\Delta L(\gamma)$ again (because it ends in $w_1$). Since $w_m \in \Delta I \cap \Delta R(\gamma)$ we must have $w_m \in A'$ and the counterclockwise arc of $\Delta I$ from $w_t$ to $w_m$ is part of $A'$ and does not contain $y_q$. (Note that $w_m \in J, y_q \in J$, so $w_m \neq y_q$.) Thus $A(\gamma^*(\delta))$, which is the counterclockwise arc of $\Delta I$ from $w_1$ to $w_m$, consists of $A(\gamma)$ plus the counterclockwise subarc of $A'$ from $w_t$ to $w_m$, and does not contain $y_q$. This is precisely the desired analogue of (4.44) for $\gamma^*$, namely

$y_q \notin A(\gamma^*)$.

As we already pointed out, (4.45) for $\gamma^*$ then follows.

We now define for $i + [2c_{32} \log j] \leq \rho$ the event

$Y(w_i, \gamma, h) = \{w_i$ has a vacant connection to $\gamma$ in $w_t + S(2^h),$

and $w_{i+1}, w_{i+2}, \ldots, w_{i+[2c_{32} \log j]}$ are vacant$\}$. \hfill (4.52)
Note that in (4.52) we look for a connection from \( w_i \) to \( \gamma \) in the square \( w_t + S(2^h) \), centered at the initial point \( w_t \) of \( \gamma \). Such a connection can exist only if \( w_i \in w_t + S(2^h) \). We further define for \( D \geq 1 \)

\[
Z(\gamma, h, D) = \text{maximal number of indices } m_i \in [t, \rho - \lceil 2c_{32} \log j \rceil] \\
\text{for which } \|w_{m_i} - w_{m_k}\| > D \text{ for } i \neq k \text{ and for which} \\
w_{m_i} \in \text{(interior of } \hat{S}) \text{ and } Y(w_{m_i}, \gamma, h) \text{ occurs.} 
\tag{4.53}
\]

When \( \gamma \) is vacant and

\[
Z(\gamma, h, D) \geq \exp[c_{31}j]
\]

for

\[
D = 2^{c_{33}c_{30}(j+1)},
\tag{4.54}
\]

then (4.3) with \( D \) replaced by \( \gamma \) and \( z \) by \( y_q \) certainly holds. Indeed, for each of the \( m \)’s for which \( Y(w_m, \gamma, h) \) occurs, \( w_m \) has a vacant connection, \( \delta \) say, to \( \gamma \) and is therefore connected to \( y_q \) by the vacant path \( \gamma^*(\delta) \). The only vertex adjacent to \( C \) on this path is \( w_m \). The path \( (w_m, w_{m+1}, \ldots, w_{m+[2c_{32} \log j]}) \) is therefore one of the possible paths \( \theta \) required in (4.3). The probability in (4.47) can therefore be bounded by

\[
\sum_\gamma P\{Z(\gamma, h, D) < \exp[c_{31}j] | E(p, C), \zeta^{(1)} = \gamma\} P\{\zeta^{(1)} = \gamma | E(p, C)\},
\tag{4.55}
\]

with \( D \) as in (4.54) and the sum over \( \gamma \) running over all permissible paths. We note that for a fixed path \( \gamma \), \( Z(\gamma, h, D) \) depends only on the vertices in \( R(\gamma) \cup \{w_m : t < m \leq \rho\} \subset \overset{o}{J} \cup J \subset \overset{o}{C} \), because any vacant connection has to lie in \( R(\gamma) \) except for its endpoints (by virtue of (4.50)), and no requirements are made on \( w_t = y_1 \) or \( z_u \) in (4.51). Moreover, the initial point \( w_m \) of any such vacant connection has to satisfy \( t \leq m \leq \rho \), as we saw above. On the other hand, as we observed before, \( E(p, C) \) depends only on the vertices in \( C^{\text{ext}} \cup C \), while \( \{\zeta^{(1)} = \gamma\} \) depends only on the vertices in \( L(\gamma) \cup \gamma \cup \{w_1, \ldots, w_t\} \cup (\Delta L(\gamma) \cap \overset{o}{J}) \). Since \( C^{\text{ext}} \cup C \cup L(\gamma) \cup \gamma \cup \{w_1, \ldots, w_t\} \cup (\Delta L(\gamma) \cap \overset{o}{J}) \) and \( R(\gamma) \cup \{w_m : t < m \leq \rho\} \) are disjoint, the conditioning in the first factor in (4.55) has no influence. For similar reasons, the conditioning has no influence on the second factor (recall that \( C \) is fixed). Therefore (4.55) equals

\[
\sum_\gamma P\{Z(\gamma, h, D) < \exp[c_{31}j]\} P\{\zeta^{(1)} = \gamma\} \\
\leq P\{\zeta^{(1)} \text{ exists}\} \sup_\gamma P\{Z(\gamma, h, D) < \exp[c_{31}j]\},
\tag{4.56}
\]
where $\gamma$ still ranges over the permissible paths. In particular, these have initial vertex $y_1 \in S$, and therefore the distance of $y_1$ to $\Delta \hat{S}$ exceeds $2^\ell$. To estimate (4.56) we introduce further

$$\Lambda(\alpha, h, D) = \sup_{\gamma} E \exp[-\alpha Z(\gamma, h, D)],$$

where we now let $\gamma$ range over all paths satisfying (4.41)-(4.45) and

$$\text{distance of } y_1 \text{ to } \Delta \hat{S} > 2^h.$$  \hspace{1cm} (4.58)

We shall choose $\alpha \geq 0$ and $h$ later, but from now on we restrict $h$ to satisfy

$$4 \leq h \leq \ell, \text{ and } 2^h \geq 8D \vee 32c_32 \log j.$$ \hspace{1cm} (4.59)

We then obviously have for any $\alpha \geq 0$ that (4.56), and hence (4.47) is bounded by

$$P\{\zeta^{(1)} \text{ exists}\} \exp[\alpha \exp\{c_{31}j\}] \Lambda(\alpha, h, D).$$ \hspace{1cm} (4.60)

The remainder of this step is devoted to proving that there exists some constant $c_{44} \in (0, 1]$, independent of $\alpha, h, D$ such that under (4.59)

$$\Lambda(\alpha, h, D) \leq (1 - c_{44}) \Lambda(\alpha, h - 4, D) + c_{44} \Lambda^2(\alpha, h - 4, D).$$ \hspace{1cm} (4.61)

We shall prove (4.61) by proving that for fixed $\gamma$, under (4.58), (4.59), $Z(\gamma, h, D)$ is stochastically larger than

$$Z(\gamma, h - 4, D) + \zeta \tilde{Z}(h - 4, D),$$ \hspace{1cm} (4.62)

where $Z(\gamma, h - 4, D), \tilde{Z}(h - 4, D)$ and $\xi$ are independent with

$$P\{\xi = 1\} = 1 - P\{\xi = 0\} \geq c_{44} > 0$$ \hspace{1cm} (4.63)

for a constant $c_{44} \in (0, 1]$ which is independent of $\gamma, h, D$, and where $\tilde{Z}$ has a distribution of the form

$$P\{\tilde{Z}(h - 4, D) \in B\} = \sum_{\gamma} q(\overline{\gamma}) P\{Z(\overline{\gamma}, h - 4, D) \in B\} \quad (B \text{ a Borel set}).$$ \hspace{1cm} (4.64)

Here $q(\overline{\gamma}) \geq 0, \sum_{\overline{\gamma}} q(\overline{\gamma}) = 1$, and $\overline{\gamma}$ runs over all paths of with the properties (4.41)-(4.45) and whose initial point $\overline{y}_1$ satisfies

$$\text{distance of } \overline{y}_1 \text{ to } \Delta \hat{S} > 2^{h - 4}.$$ \hspace{1cm} (4.65)
In other words, the distribution of $\tilde{Z}$ is a mixture of distributions of $Z(\gamma, h - 4, D)$ over $\gamma$ which satisfy (4.41)-(4.45) and (4.65). In particular

$$E \exp[-\alpha \tilde{Z}(h - 4, D)] \leq \Lambda(\alpha, h - 4, D), \quad \alpha \geq 0,$$

and the bound (4.62) implies (under (4.58))

$$E \exp[-\alpha Z(\gamma, h, D)]$$

$$\leq E \exp[-\alpha Z(\gamma, h - 4, D)]\{1 - c_{44} + c_{44}E \exp[-\alpha \tilde{Z}(h - 4, D)]\}$$

$$\leq (1 - c_{44})\Lambda(\alpha, h - 4, D) + c_{44}\Lambda^2(\alpha, h - 4, D).$$

Thus it will indeed suffice for (4.61) to show $Z(\gamma, h, D)$ stochastically larger than (4.62). The proof of this stochastic domination is essentially given in Lemma 8.2 of Kesten (1982) so we will skip some details. Let $\gamma = (y_1, \ldots, y_q)$ be fixed so that (4.41)-(4.45) hold and assume that (4.58) and (4.59) hold. Let $y_1 = w_t$. We now also need a partial order of the paths $\delta = (z_1, \ldots, z_u)$ with $z_1 = w_m \in J \cap$ (interior of $\hat{S}$) and which satisfy (4.48)-(4.50) with

$$G = w_t + S(5 \cdot 2^{h-3}) \setminus S(4 \cdot 2^{h-3}). \quad (4.66)$$

$G$ is an annulus centered at $w_t$. Note that $z_1 \in G$ forces $z_1 \in$ (interior of $\hat{S}$) by (4.58) and (4.59) (recall that $w_t = y_1$). Furthermore, $\|z_1 - y_1\| = \|z_1 - w_t\| > 4 \cdot 2^{h-3} > 2$, so that $z_1$ is not adjacent to $y_1$. Also $z_1$ is not adjacent to $y_q$ because $\|z_1 - y_q\| \geq \|y_1 - y_q\| - \|z_1 - y_1\| > 2^h - 5 \cdot 2^{h-3} \geq 2$ (see (4.58)). Therefore the case that $\delta$ consists of a single edge in $\Delta R(\gamma)$ is excluded. These paths $\delta$ are therefore crosscuts of $R(\gamma)$ and consequently $R(\gamma) \setminus \delta$ consists of two components which we denote by $U(\delta) = U(\delta, \gamma)$ and $V(\delta) = V(\delta, \gamma)$ (these $U, V$ are unrelated to the $U, V$ of Step (iii) which will not be used anymore). $U(\delta)$ is the component of $R(\gamma) \setminus \delta$ whose boundary consists of $\delta$, the piece of $\gamma$ from $z_u$ to $y_1 = w_t$ plus that arc of $\Delta I$ from $w_t$ to $w_m = z_1$ which is contained in $\Delta R(\gamma)$; see Figure 8.
Figure 8. Illustration of $\gamma$ (boldly drawn), $\delta$ (solidly drawn) and the components $U(\delta)$ (the hatched region) and $V(\delta)$ (the dotted region). $F$ is the dashed part of $J$.

The last arc is a subarc of the arc $A'$ introduced in the paragraph following (4.51). As we traverse $\Delta I$, vertices of $J$ are met in the same order as on $J$, so that this boundary arc of $U(\delta)$ contains only vertices $w_i \in J$ with $t \leq i \leq m$ (note that $A'$ does not contain $w_1$, so this subarc of $A'$ cannot be the one which contains $w_i$ with $m \leq i \leq \rho$ and $1 \leq i \leq m$). $V(\delta)$ is the other component of $R(\gamma) \setminus \delta$; its boundary consists of $\delta$ followed by the part of $\gamma$ from $z_u$ to $y_q$, and the part of $A'$ which is not in $\Delta U(\delta)$, i.e., the (reverse of) the subarc from $w_m$ to $y_q$. We say that $\delta'$ precedes $\delta''$ (with respect to $\gamma$) if $U(\delta', \gamma) \subset U(\delta'', \gamma)$. Again, if $\delta', \delta''$ lie in a set $G$, then there exists a path $\delta'' \subset G$ which precedes $\delta'$ and $\delta''$. Also if $\delta'$ and $\delta''$ are vacant, with the possible exception of their endpoint on $\gamma$, then $\delta''$ minus its endpoint on $\gamma$ can also be taken vacant. Assume now that

$$\text{there exists a path } \delta \text{ satisfying (4.48)-(4.51) with } G \text{ given by (4.66).} \quad (4.67)$$

In this case there is a minimal $\delta$ with respect to $\gamma$ (in the above order) with the properties (4.48)-(4.51). Let this minimal path be $\bar{\delta} = (\bar{z}_1, \ldots, \bar{z}_u)$. Let $\bar{z}_u = y_v$. Then consider the path $\bar{\gamma} := \gamma^* (\bar{\delta})$. As we saw before, $\bar{\gamma}$ also has the properties (4.41)-(4.45) with the obvious replacements (e.g., $y_1$ is replaced by the first vertex of $\bar{\gamma}$, that is, by $\bar{z}_1$).

Now write again $w_m$ for $\bar{z}_1$, and consider a vertex $w_i$ for which $Y(w_i, \bar{\gamma}, h - 4)$ occurs. This means that $w_i$ has a vacant connection to $\bar{\gamma}$ in $w_m + S(2^h - 4)$ and that all the vertices $w_1, w_{i+1}, \ldots, w_{i+[2c_{32} \log j]}$ are vacant. Note that this can happen
only for
\[ w_i \in \left( w_m + S(2^{h-4}) \right) \subset [w_t + S(5 \cdot 2^{h-3}) \setminus S(4 \cdot 2^{h-3})] + S(2^{h-4}) \]
(because \( w_m \in G \)) \subset w_t + S(6 \cdot 2^{h-3}) \subset w_t + S(2^h) \subset \text{(interior of } \hat{S}) \],

so that automatically \( w_i \in \text{(interior of } \hat{S}) \). In addition, it must be the case that \( m \leq i \leq \rho \), as in the lines following (4.51), because \( \tilde{\gamma} \) satisfies (4.44), (4.45). For the sake of argument, let \( w_i \) have the vacant connection \( \hat{\delta} \) to \( \tilde{\gamma} \) in \( w_m + S(2^{h-4}) \). Then one can see that \( \hat{\delta} \), possibly followed by a piece of \( \hat{\delta} \), forms a vacant connection from \( w_i \) to \( \gamma \) (see Kesten (1982), equation (8.73) and its proof). Since this vacant connection is made up from \( \hat{\delta} \) and possibly a piece of \( \hat{\delta} \), it lies in
\[
(w_m + S(2^{h-4})) \cup G \subset [w_t + S(5 \cdot 2^{h-3}) \setminus S(4 \cdot 2^{h-3})] + S(2^{h-4})
\]
(because \( w_m \in G \)) \subset w_t + S(2^h).

Thus \( w_i \) has a vacant connection to \( \gamma \) in \( w_t + S(2^h) \). If further \( w_p \) is vacant for all \( i \leq p \leq i + \lceil 2c_{32} \log j \rceil \) then \( Y(w_i, \gamma, h) \) occurs. In other words, \( Y(w_i, \gamma, h) \) occurs if (4.67) holds and \( Y(w_i, \tilde{\gamma}, h - 4) \) occurs.

It is further immediate from the definitions that for any \( w_i \), the occurrence of \( Y(w_i, \gamma, h - 4) \) implies the occurrence of \( Y(w_i, \gamma, h) \). This is true whether (4.67) holds or not.

Now by definition of \( Z \), there exist \( Z(\gamma, h-4, D) \) indices \( m_i \in [t, \rho - \lceil 2c_{32} \log j \rceil] \) with \( w_{m_i} \in \text{(interior of } \hat{S}) \) and which satisfy
\[
\|w_{m_i} - w_{m_k}\| > D \text{ for } i \neq k \tag{4.68}
\]
and
\[
Y(w_{m_i}, \gamma, h - 4) \text{ occurs.} \tag{4.69}
\]
Similarly, if (4.67) holds, then there are \( Z(\tilde{\gamma}, h-4, D) \) indices \( n_i \geq m \) with \( w_{n_i} \in \text{(interior of } \hat{S}) \) which satisfy
\[
\|w_{n_i} - w_{n_k}\| > D \text{ for } i \neq k \tag{4.70}
\]
and
\[
Y(w_{n_i}, \tilde{\gamma}, h - 4) \text{ occurs, and hence } Y(w_{n_i}, \gamma, h) \text{ occurs.} \tag{4.71}
\]
Also \( Y(w_{m_i}, \gamma, h - 4) \) can occur only when \( w_{m_i} - w_t \in S(2^{h-4}) \), while \( Y(w_{n_i}, \tilde{\gamma}, h - 4) \) can occur only when \( w_{n_i} - w_m \in S(2^{h-4}) \). But \( w_m = z_1 \in G \), so that
\[
w_m - w_t \in S(5 \cdot 2^{h-3}) \setminus S(4 \cdot 2^{h-3}),
\]
and in particular $\|w_m - w_t\| > 4 \cdot 2^{h-3}$. Thus for any $m_i, n_k$ which satisfy (4.69) and (4.71), respectively, we must have

$$\|w_{m_i} - w_{n_k}\| \geq \|w_m - w_t\| - \|w_{m_i} - w_t\| - \|w_{n_k} - w_m\|$$

$$> 4 \cdot 2^{h-3} - 2\sqrt{2}2^{h-4} > 2^{h-3} \geq D \text{ (see (4.59)).}$$

(4.72)

Thus each of the vertices $w_{m_i}$ which satisfies (4.69) automatically has distance more than $D$ to each of the vertices $n_k$ which satisfy (4.71). It follows that we can combine the vertices satisfying (4.68), (4.69) with those satisfying (4.70), (4.71) and that under (4.58), (4.59) and (4.67),

$$Z(\gamma, h, D) \geq Z(\gamma, h - 4, D) + Z(\tilde{\gamma}, h - 4, D).$$

If we set

$$\xi = I[\tilde{\gamma} \text{ exists}] = I[(4.67) \text{ occurs}],$$

and

$$\tilde{Z}(h - 4, D) = Z(\tilde{\gamma}, h - 4, D),$$

then this says that the stochastic lower bound (4.62) for $Z(\gamma, h, D)$ is valid when $\xi = 1$. When $\xi = 0$, then (4.62) merely says $Z(\gamma, h, D) \geq Z(\gamma, h - 4, D)$, and as we saw, this always holds.

It remains to verify that $Z(\gamma, h - 4, D), \tilde{Z}(h - 4, D)$ and $\xi$ are independent and that their joint distribution has the properties (4.63)-(4.65). For fixed $\gamma$, $Z(\gamma, h - 4, D)$ depends only on connections inside $(w_t + S(2^{h-4})) \cap (R(\gamma) \cup \Delta R(\gamma))$ and vertices $w_p \in J$ with $t \leq i < p \leq i + \left[2c_{32} \log j\right]$ for some $w_i$ for which $Y(w_i, \gamma, h - 4)$ can occur. Such $w_i$ have to lie in $w_t + S(2^{h-4})$ and $i \geq t$ must hold (see the lines following (4.55)). Thus, $Z(\gamma, h - 4, D)$ depends only on vertices within distance $\sqrt{2} \cdot 2^{h-4} + \left[2c_{32} \log j\right] < 2^{h-2}$ from $w_t$. $\xi$ only depends on the vertices in the annulus $G$ of (4.66), because $\tilde{\delta}$ has to satisfy (4.48). Since the distance from $G$ to $w_t$ is at least $4 \cdot 2^{h-3} > 2^{h-3} + \left[2c_{32} \log j\right]$ for the $h$-values under consideration (see (4.59)), $Z(\gamma, h - 4, D)$ and $\xi$ are indeed independent. For the same reason, the location of the path $\tilde{\delta}$ when it exists, is independent of $Z(\gamma, h - 4, D)$. Moreover,

$$P\{\xi = 1\} \geq P\{\exists \text{ vacant circuit in the annulus } G \text{ of (4.66)}\}$$

(4.73)

for the same reasons as for equation (7.72) in Kesten (1982); see also Lemma 3 in Kesten (1980)). Since the right hand side here is bounded away from 0, (4.63) holds for $c_{44} > 0$ given by the right hand side of (4.73).
Finally we need to consider the conditional distribution of \( \tilde{Z} \). The conditional distribution when \( \xi = 0 \) is irrelevant, because in this case \( \tilde{Z} \) does not figure in (4.62). To finish our proof it suffices to show that

\[
P\{ \tilde{Z}(h-4,D) \in B| \text{vertices within distance } 2^{h-2} \text{ of } w_t, \xi = 1, \tilde{\delta} = \delta \} = P\{ Z(\gamma^*(\delta), h-4, D) \in B \}.
\]

(4.74)

Indeed, (4.74) implies that

\[
P\{ \tilde{Z}(h-4,D) \in B | Z(h-4,\gamma,D), \xi = 1 \} = \sum_\delta P\{ \tilde{\delta} = \delta | \xi = 1 \} P\{ Z(\gamma^*(\delta), h-4, D) \in B \},
\]

so that (4.64) holds with

\[
q(\gamma) = \sum_{\tilde{\delta}: \gamma^*(\tilde{\delta}) = \gamma} P\{ \tilde{\delta} = \delta | \xi = 1 \}.
\]

This distribution also satisfies (4.65), because if (4.58) holds, then the distance of any point in the annulus \( G \) of (4.66) to \( \tilde{\Delta} \) is at least equal to the distance of \( w_t = y_1 \) to \( \tilde{\Delta} \) minus \( 5 \cdot 2^{h-3} > 2^{h-3} \). In particular this holds for the initial point of all possible \( \delta \).

Finally, to prove (4.74), we first check more precisely on which vertices the event \( \{ \xi = 1, \tilde{\delta} = \delta \} \) depends. For fixed \( \gamma \) and \( \tilde{\delta} \), this only depends on the vertices in \( G \cap \left( U(\tilde{\delta}, \gamma) \cup \tilde{\delta} \cup \{ w_t, \ldots, w_{\text{final}} \} \cup \text{interior of the piece of } \gamma \text{ from } w_t = y_1 \text{ to } z_u \right) \), where \( w_{\text{final}} \) is the initial point of \( \tilde{\delta} \) on \( J \) and \( z_u \) its final point on \( \gamma \). Indeed, we only need to check vertices in this set to check the minimality of \( \tilde{\delta} \), as follows from the description of \( \Delta U(\delta) \) in the lines following (4.66). We claim that

\[
U(\tilde{\delta}, \gamma) \cup \text{ (interior of the piece of } \gamma \text{ from } w_t = y_1 \text{ to } z_u \text{) } \subset L(\gamma^*(\tilde{\delta})).
\]

(4.76)

In fact, we claim that for any \( \delta \) which satisfies (4.48)-(4.50), it holds that

\[
L(\gamma) \cup U(\delta, \gamma) \cup \text{ (interior of the piece of } \gamma \text{ from } w_t = y_1 \text{ to } z_u) = L(\gamma^*(\delta)).
\]

(4.77)

(4.76) follows once we have (4.77) by replacing \( \delta \) by \( \tilde{\delta} \). (4.77) is the analogue of equation (8.70) in Kesten (1982). It follows by observing that \( R(\gamma^*(\delta)) \) and \( V(\delta, \gamma) \) have the same boundary and therefore are the same. Hence also

\[
L(\gamma^*(\delta)) = I \setminus (R(\gamma^*(\delta)) \cup \gamma^*(\delta)) = I \setminus (V(\delta, \gamma) \cup \gamma^*(\delta))
\]

\[
= \left( L(\gamma) \cup R(\gamma) \cup (\gamma \setminus \{ y_1, y_q \}) \right) \setminus (V(\delta, \gamma) \cup \gamma^*(\delta))
\]

\[
= L(\gamma) \cup U(\delta, \gamma) \cup \text{ (interior of the piece of } \gamma \text{ from } w_t = y_1 \text{ to } z_u).
\]
Alternatively, we can check directly that the two sides of (4.77) have the same boundary. Thus (4.77) holds, as claimed.

Just before (4.73) we described on which vertices \(Z(\gamma, h - 4, D)\) depends. All these vertices have to lie within distance \(2^{h-2}\) of \(w_t\). For the same reasons, \(Z(\gamma^*(\delta), h - 4, D)\) depends only on the vertices in \(R(\gamma^*(\delta)) \cup \{w_i : \bar{m} < i \leq \rho\}\) which are within distance \(2^{h-2}\) of the initial point \(w_m\) of \(\delta\) (that is, once \(\delta\) is fixed). But

\[
\|w_m - w_t\| > 4 \cdot 2^{h-3},
\]

and no vertex \(w_i \in J\) can lie in \(L(\gamma^*(\delta)) \cup R(\gamma^*(\delta)) \subset I \subset \bar{J}\). Therefore

\[
\left(R(\gamma^*(\delta)) \cup \{w_i : \bar{m} < i \leq \rho\}\right) \cap \{\text{vertices within distance } 2^{h-2} \text{ of } w_m\}
\]

is disjoint from

\[
L(\gamma^*(\delta)) \cup \bar{\delta} \cup \{w_{t+1}, \ldots, w_m\} \cup \{\text{vertices within distance } 2^{h-2} \text{ of } w_t\}.
\]

Thus even under the conditioning in the left hand side of (4.74), the vertices in \(R(\gamma^*(\delta)) \cup \{w_i : \bar{m} < i \leq \rho\}\) at distance \(\leq 2^{h-2}\) from \(w_m\) are still independently occupied or vacant with probability 1/2. Therefore, the conditional distribution of \(Z(\gamma, h - 4, D)\), given the conditions in the left hand side of (4.74) is just the unconditional distribution of \(Z(\gamma^*(\delta), h - 4, D)\). This is precisely what (4.74) states.

**Step (v)** In this last step we assemble the pieces proven in Steps (i)-(iv) to complete the proof of Proposition 1. First we must generalize the estimate (4.60) for (4.47), in order to obtain an estimate for (4.38). Recall that \(\zeta^{(1)}\) was the first vacant permissible path (in the ordering introduced in (4.46)). More generally, we now introduce the successive disjoint permissible vacant paths \(\zeta^{(i)}\): \(\zeta^{(i)}\) is the first such path, and \(\zeta^{(i)}\) is the first vacant permissible path which comes after \(\zeta^{(i-1)}\) and is disjoint from \(\zeta^{(i)}\). At the first \(i\) for which there is no such path we say that \(\zeta^{(i)}\) and \(\zeta^{(i')}\) with \(i' > i\) do not exist. Now it is not hard to see that if \(\zeta'\) and \(\zeta''\) are two permissible paths which are either disjoint or have only their endpoint on \(J\) in common, then one must precede the other. In fact, we already stated that Lemma 1 of Kesten (1980) or Proposition 2.3 in Kesten (1982) shows that there exists a path \(\zeta''\) which precedes both \(\zeta'\) and \(\zeta''\). The construction of \(\zeta''\) (or a direct study of the boundaries of \(L(\zeta')\) and \(L(\zeta'')\)) shows that \(\zeta''\) can be taken as one of \(\zeta'\) or \(\zeta''\) when these paths are disjoint or intersect only on \(J\). It follows that if \(\zeta\) is any vacant permissible path, then it must intersect one of the \(\zeta^{(i)}\) in a point not on \(J\), because otherwise \(\zeta\) would be comparable to all of them, and appear as one of the \(\zeta^{(i)}\) itself. But if \(\zeta\) intersects \(\zeta^{(i)}\) in a point not on \(J\), then the vacant cluster of \(\zeta\) in \(\hat{C}\) is the same as the vacant cluster of \(\zeta^{(i)}\) in \(\hat{C}\), and if (4.3) fails for
ζ, then it fails for ζ(i). Consequently, (for fixed C, J, F) the probability in (4.38) is bounded by

\[
\sum_{i \geq 1} P\{\zeta(i) \text{ exists, but (4.3) fails for } \zeta(i)|E(p, C)\}.
\]  

Instead of only (4.47) we should therefore estimate this sum. Now the argument leading to (4.56) remains valid without change when we replace ζ(1) by ζ(i). Indeed, the only property of ζ(1) which is used is that the event \{ζ(1) = γ\} depends only on the vertices in \(L(γ) \cup γ \cup \{w_1, \ldots, w_t\} \cup (ΔL(γ) \cap J)\). However, this is equally true with ζ(i) instead of ζ(1). Therefore (see (4.60)), the sum in (4.78) is at most

\[
\sum_{i \geq 1} P\{\zeta(i) \text{ exists}\} \exp[\alpha \exp\{c_{31}j\}] \Lambda(α, h, D).
\]

But (see (4.14))

\[
P\{\zeta(i) \text{ exists}\} \\
\leq P\{\text{there exist at least } i \text{ disjoint crossings of } S \setminus \bar{S}\} \\
\leq (1 - c_{41})^i.
\]

Combining this with our preceding estimate we find that (4.38) is bounded by

\[
[c_{41}]^{-1} \exp[\alpha \exp\{c_{31}j\}] \Lambda(α, h, D).
\]

Finally this yields the bound

\[
\frac{c_{42}j^6}{c_{41}} 2^{2(1 - c_{30})j} \exp[\alpha \exp\{c_{31}j\}] \Lambda(α, h, D)
\]

for the sum in (4.15) (recall (4.39)).

We shall now use the recurrence relation (4.61) to get an explicit estimate for \(\Lambda(α, h, D)\). To this end we rewrite (4.61) as

\[
\Lambda(α, h, D) \leq f(Λ(α, h - 4, D)),
\]

where

\[
f(s) = (1 - c_{44})s + c_{44}s^2, \quad 0 \leq s \leq 1.
\]

The relation (4.80) is of the form of the recurrence relation for the generating function of a Bienaymé-Galton-Watson branching process in which each individual has one child with probability \(1 - c_{44}\) and two children with probability \(c_{44}\). If
$W_n$ denotes the size of the $n$-th generation in such a process, and we start with $W_0 = 1$ individual, then

$$E_{s}^{W_n} = f^{(n)}(s), \quad |s| \leq 1,$$

where $f^{(n)}$ denotes the $n$-th iterate of $f$. Recall that (4.80) holds if $\alpha \geq 0$ and (4.59) holds. Thus, if

$$\alpha \geq 0, \ 4 \leq h \leq \ell, \ \text{and} \ 2^{h-4n} \geq 8D \vee 32c_{32}\log j,$$  \hspace{1cm} (4.81)

we can iterate (4.80) $n$ times to obtain (for any $A > 0$)

$$\Lambda(\alpha, h, D) \leq f^{(n)}(\Lambda(\alpha, h - 4n, D)) \leq P\{W_n \leq A\} + [\Lambda(\alpha, h - 4n, D)]^A. \hspace{1cm} (4.82)$$

We shall choose our parameters as follows:

$$D = 2^{c_{33}c_{30}j+1}, h = \ell, n = \lfloor (1 - c_{33})c_{30}j/8 \rfloor, A = (1 + c_{45})^n. \hspace{1cm} (4.83)$$

(4.81) will be satisfied for large $j$, provided we fix $c_{33} < 1$ (see (4.9) for $\ell$).

The first term in the right hand side of (4.82) can now be estimated by standard branching process methods. Indeed, it is known (see Athreya and Ney (1972), Corollary 1 in Section I.11) that for our branching process, which has extinction probability 0 and $f'(0) = 1 - c_{44},$

$$f^{(n)}(s) \leq c_{46}(s)[f'(0)]^n = c_{46}(s)[1 - c_{44}]^n, \quad 0 \leq s < 1.$$  

Here $c_{46}(s)$ depends on $s$ only. Therefore, if we fix $s > [1 - c_{44}]^{1/2}$, then we obtain for suitable constants $c_{47} < \infty, c_{48} < 1,$

$$P\{W_{\lfloor n/2 \rfloor} \leq n\} \leq s^{-n}c_{46}(s)[1 - c_{44}]^{\lfloor n/2 \rfloor} \leq c_{47}[c_{48}]^n. \hspace{1cm} (4.84)$$

We improve this further by noting that $W_n$ consists of the offspring of the $W_{\lfloor n/2 \rfloor}$ individuals in the $\lfloor n/2 \rfloor$-th generation. If any one of these individuals has more than $(1 + c_{45})^n$ children in the $n$-th generation, then $W_n > (1 + c_{45})^n.$ Therefore, for any choice of $c_{45} \geq 0,$

$$P\{W_n \leq (1 + c_{45})^n\} \leq P\{W_{\lfloor n/2 \rfloor} \leq n\} + [P\{W_{n-\lfloor n/2 \rfloor} \leq (1 + c_{45})^n\}]^n. \hspace{1cm} (4.85)$$

Since the $W$-process is supercritical and has zero extinction probability, we can choose $c_{45} > 0$ such that

$$P\{W_{n-\lfloor n/2 \rfloor} \leq (1 + c_{45})^n\} \to 0 \quad (n \to \infty). \hspace{1cm} (4.86)$$
Together with (4.84) and (4.85) this gives for large $j$

$$P\{W_n \leq A\} = P\{W_n \leq (1 + c_{45})^n\} \leq 2c_{47}[c_{48}]^n \leq c_{49} \exp[-c_{50}(1 - c_{33})c_{30}j]. \quad (4.87)$$

This is an exponential bound (in $j$) for the first term in the right hand side of (4.82).

To estimate the second term in the right hand side of (4.82) we note that it follows from the definition (4.57) that

$$\Lambda(\alpha, h - 4n, D) \leq 1 - (1 - e^{-\alpha}) \inf_{\gamma} \{Z(\gamma, h - 4n, D) \geq 1\}, \quad (4.88)$$

where $\gamma$ ranges over the paths satisfying (4.41)-(4.45) and
distance of $y_1$ to $\Delta \hat{S} > 2^{h - 4n}. \quad (4.89)$

For such a $\gamma$,

$$P\{Z(\gamma, h - 4n, D) \geq 1\} \geq P\{Y(w_t, \gamma, h - 4n) \text{ occurs}\} \geq P\{w_{t+1}, \ldots, w_{t+\lfloor2c_{32}\log j\rfloor} \text{ are vacant}\} \geq 2^{-2c_{32} \log j - 1} \geq j^{-c_{51}c_{32}}. \quad (4.90)$$

For the second inequality here we note that if the first vertex of $\gamma, y_1 = w_t$ satisfies (4.89), then automatically $t \leq \rho - \lfloor2c_{32}\log j\rfloor$ because $w_{\rho}$ is adjacent to $w_1 \in \Delta \hat{S}$. Thus (4.89) implies $\|w_t - w_{\rho}\| \geq 2^{h - 4n} - 2 > 2[2c_{32}\log j]$. From (4.82) and (4.87)-(4.90), we obtain for $0 \leq \alpha \leq 1/2$,

$$\Lambda(\alpha, h, D) \leq c_{49} \exp[-c_{50}(1 - c_{33})c_{30}j] + \left[1 - \frac{\alpha}{2j^{c_{51}c_{32}}}\right]^A \leq c_{49} \exp[-c_{50}(1 - c_{33})c_{30}j] + \exp[-\frac{(1 + c_{45})^n}\alpha] j^{-c_{51}c_{32}}. \quad (4.91)$$

Finally we take

$$c_{31} = c_{31}(c_{30}, c_{33}) < \frac{(1 - c_{33})c_{30}}{16} \log(1 + c_{45}) \quad (4.92)$$

and $\alpha = \alpha(j, c_{30}, c_{31})$ so that

$$\alpha e^{c_{31}j} = (1 - c_{30})j$$

and $c_{30} \in (0, 1)$ so close to 1 that

$$\frac{1 - c_{30}}{c_{30}} \leq \frac{1}{6} c_{50}(1 - c_{33}).$$

Note that with these choices

$$\frac{(1 + c_{45})^n\alpha}{2j^{c_{51}c_{32}}} = \frac{(1 + c_{45})\lfloor(1 - c_{33})c_{30}j/8\rfloor}{2j^{c_{51}c_{32} - 1}} e^{-c_{31}j} \geq c_{50}(1 - c_{33})c_{30}j$$

for large $j$. Substituting these values of the parameters in the bound (4.79) for the sum in (4.15) finally gives us the exponential bound (4.5).
5. Usefulness and existence of double paths.

We first demonstrate that one can use so-called ‘double-paths’ to write (a piece) of an arbitrary word. We define a double path to be a pair of self-avoiding paths \( \pi', \pi'' \) such that

\[
\pi' \text{ is occupied and } \pi'' \text{ is vacant; (5.1)}
\]

\[
\text{the initial points of } \pi' \text{ and } \pi'', \text{ } u' \text{ and } u'', \text{ are neighbors; (5.2)}
\]

\[
\text{the final points of } \pi' \text{ and } \pi'', \text{ } v' \text{ and } v'', \text{ are neighbors; (5.3)}
\]

and such that the following minimality property holds. For any occupied self-avoiding path \( \hat{\pi}' \) from \( u' \) to \( v' \) and any self-avoiding vacant path \( \hat{\pi}'' \) from \( u'' \) to \( v'' \) denote by \( R(\hat{\pi}', \hat{\pi}'') \) the interior of the Jordan curve formed by concatenating \( \hat{\pi}', \{v', v''\}, (\text{the reverse of}) \hat{\pi}'', \{u'', u'\}. \) (Note that \( \hat{\pi}' \) and \( \hat{\pi}'' \) are automatically disjoint, since one of them is occupied and the other is vacant.) Let \( R(\pi', \pi'') \) be the union of \( R(\pi', \pi'') \) and its boundary (that is the above Jordan curve). Then we further require that

\[
R(\pi', \pi'') \text{ is minimal among all such } R(\pi', \pi''). \quad (5.4)
\]

By means of Proposition 2.2 in Kesten (1982) one can show that this implies the following property:

- every vertex \( u \in \pi' \setminus \{u', v'\} \) has a neighbor \( w \in (R(\pi', \pi'') \cup \pi'') \)
  - such that \( w \) is connected to a vertex of \( \pi'' \) by a vacant path in \( R(\pi', \pi'') \)
  - and such that the edge \( \{u, w\} \subset R(\pi', \pi'') \);
- the same property holds when the single and double prime superscripts are interchanged (and ‘vacant’ is replaced by ‘occupied’). (5.5)

E.g., to obtain the first part interchange occupied and vacant in Proposition 2.2 of Kesten (1982) and take \( A_1 = \{u\}, A_3 = \pi'', A_2 = \{u'', u'\} \) followed by the piece of \( \pi' \) from \( u' \) to \( u \) and \( A_4 = \) piece of \( \pi' \) from \( u \) to \( v' \) followed by \( \{v', v''\} \).

By the minimality property (5.4) there does not exist an occupied path inside \( R(\pi', \pi'') \setminus (\{u\} \cup \pi'') \) from a vertex of \( A_2 \) to a vertex of \( A_4 \). Note that \( J \setminus (A_1 \cup A_3) \) should be \( J \setminus (A_1 \cup A_3) \) on lines 1 and 2 f.b. of p. 30 in Kesten (1982). We point out that in the first part of (5.5) \( w \in \pi'' \) is possible; in this case the vacant path from \( w \) to \( \pi'' \) consists of \( w \) only. A similar comment applies to the second part of (5.5).

We shall repeatedly use the fact that if \( \hat{\pi}', \hat{\pi}'' \) are an occupied and vacant path, respectively, which start at the adjacent points \( u', u'' \) and end at the adjacent points \( v', v'' \), but for which \( R(\hat{\pi}', \hat{\pi}'') \) is not minimal, then there exist an occupied
path $\pi'$ from $u'$ to $v'$ and a vacant path $\pi''$ from $u''$ to $v''$ such that $(\pi', \pi'')$ is a double path with $\overline{R}(\pi', \pi'') \subset \overline{R}(\pi', \pi'')$. To find such paths, first take $\pi'$ as an occupied path from $u'$ to $v'$ in $\overline{R}(\pi', \pi'')$ which minimizes $\overline{R}(\pi', \pi'')$ over all such paths. Such a $\pi'$ exists by the argument for Proposition 2.3 in Kesten (1982) (see also Lemma 1 in Kesten (1980)). After that take $\pi''$ as a vacant path from $u''$ to $v''$ in $\overline{R}(\pi', \pi'')$ which minimizes $\overline{R}(\pi', \pi'')$ over all such paths. One easily checks that (5.4) holds for these $\pi', \pi''$.

Now let $(\pi', \pi'')$ be a double path starting at $(u', u'')$ and let

$$\Theta = \min(||v' - u'||, ||v'' - u''||) - 1.$$  

(5.6)

Finally, let $\xi = (\xi_1, \ldots)$ be any infinite word (here each $\xi_i \in \{0, 1\}$ with 0 (1) corresponding to vacant (respectively, occupied). The next lemma (which is purely deterministic) shows how one can ‘see’ an initial segment of the word $\xi$ inside $\overline{R}(\pi', \pi'')$.

**Lemma 8.** At least the initial segment $(\xi_1, \ldots, \xi_\Theta)$ of $\xi$ can be seen inside $\overline{R}(\pi', \pi'')$ from $u'$ as well as from $u''$. In fact, there exist paths $\sigma' = (\sigma'_0 = u', \sigma'_1, \ldots), \sigma'' = (\sigma''_0 = u'', \sigma''_1, \ldots) \subset \overline{R}(\pi', \pi'')$, which start at $u'$ and $u''$, respectively, and which have as endpoint one of $\{v', v''\}$ such that one sees an initial segment of $\xi$ from $u'(u'')$ along $\sigma'$$^i(\sigma'')$. The length of these paths is at least $\Theta$, so that one can see at least $(\xi_1, \ldots, \xi_\Theta)$ from both $u'$ and $u''$. For a fixed occupancy configuration, and any $i$, the event \{length $(\sigma' \setminus \{u'\}) \leq i$\} depends on $\xi_1, \ldots, \xi_i$ only and not on $\xi_j, j > i$. If length $(\sigma') \geq i$, then $\sigma'_1, \ldots, \sigma'_i$ also depend on $\xi_1, \ldots, \xi_i$ only. The same statement with single and double prime superscripts interchanged is also valid.

**Proof.** We only prove that we can find the path $\sigma'$ from $u'$. This means that we can find a selfavoiding path $\sigma = (\sigma_0 = u', \sigma_1, \ldots, \sigma_\nu) \subset \overline{R}(\pi', \pi'')$ such that $\sigma_1$ is adjacent to $u'$, $\sigma_\nu \in \{v', v''\}$, and such that $\sigma_i$ is occupied (vacant) if $\xi_i = 1$ (respectively $\xi_i = 0$). We prove this in the following recursive way. We find a neighbor $\sigma_1$ of $u' \in \overline{R}(\pi', \pi'')$ such that $\sigma_1$ is occupied (vacant) if $\xi_1 = 1$ ($\xi_1 = 0$) and a new double path $(\pi'_1, \pi''_1)$ with $\sigma_1$ the initial point of one of them and with endpoints $(v', v'')$, and such that

$$\overline{R}(\pi'_1, \pi''_1) \subset \overline{R}(\pi', \pi'') \setminus \{u'\}.$$  

(5.7)

We then repeat this step, that is we find a neighbor $\sigma_2$ of $\sigma_1$ such that $\sigma_2$ is occupied (respectively is vacant) if $\xi_2 = 1$ ($\xi_2 = 0$) and a further double path etc. This construction will continue until we first use a vertex $\sigma_\nu$ from $\{v', v''\}$. It is clear from the recursive nature of the construction that the event $\{\nu \leq i\}$ is independent of $\xi_{i+1}, \ldots$. Similarly, if $\nu \geq i$, then the piece $(u', \sigma_1, \ldots, \sigma_i)$ of $\sigma$ is independent of $\xi_{i+1}, \ldots$. 
Two cases have to be distinguished in our construction, depending on the value of $\xi_1$.

**Case (i) $\xi_1 = 1$.** We now take $\sigma_1$ as the neighbor of $u'$ on $\pi'$. This uniquely determines $\sigma_1$. We take $\hat{\pi}'$ to be the piece of $\pi'$ from $\sigma_1$ to $v'$ (this is just $\pi'$ minus its first edge). We also want a vacant path $\hat{\pi}''$. To choose this, we observe that by (5.5) there must exist a vacant path $\pi_2$ in $R(\pi', \pi'')$ from a neighbor $u_1$ of $\sigma_1$ to $\pi''$. Now form the vacant path $\hat{\pi}''$ from $u_1$ to $v''$ which consists of $\pi_2$ followed by the piece of $\pi''$ from the endpoint of $\pi_2$ to $v''$. Then $(\hat{\pi}', \hat{\pi}'')$ is a pair of paths, occupied and vacant, respectively, from $R(\sigma_1, u_1)$ to $(v', v'')$. By construction $\hat{\pi}'$, $\hat{\pi}''$ and $\{\sigma_1, u_1\}$ are contained in $R(\pi', \pi'')$, so that

$$R(\hat{\pi}', \hat{\pi}'') \subset R(\pi', \pi'')$$

(compare Newman (1951), Theorem 11.1 and its proof). In fact, $u'$ can be connected to $\infty$ by a path whose only point in $R(\pi', \pi'')$ is $u$, and $u \notin \hat{\pi}' \cup \hat{\pi}''$. We therefore even have

$$R(\hat{\pi}', \hat{\pi}'') \subset R(\pi', \pi'') \setminus \{u'\}.$$ 

It is not clear that $(\hat{\pi}', \hat{\pi}'')$ itself has the minimality property corresponding to (5.4), but we take for $(\pi'_1, \pi''_1)$ the occupied and vacant pair of paths from $R(\sigma_1, u_1)$ to $(v', v'')$ which makes $R(\pi'_1, \pi''_1)$ minimal. This will automatically satisfy

$$R(\pi'_1, \pi''_1) \subset R(\hat{\pi}', \hat{\pi}'') \subset R(\pi', \pi'') \setminus \{u'\}.$$ 

Thus (5.7) will be satisfied and we are done with our recursive step in Case (i).

**Case (ii) $\xi_1 = 0$.** This time we take $\sigma_1$ as a vacant neighbor of $u'$ on $\pi''$. However, we do not necessarily take $\sigma_1 = u''$, because it is not clear that $u''$ has a neighbor on $\pi' := \pi' \setminus \{\text{first edge of } \pi'\}$. Instead, we find $\sigma_1$ as follows. Let $u_1$ be the neighbor of $u'$ on $\pi'$ (this was called $\sigma_1$ in Case (i)). Now consider the six neighbors of $u'$ on $T$ (see Figure 9). One of these neighbors is $u_1$, which is occupied. Another one of these neighbors is $u''$, which is vacant. Now $\{u', u_1\}$ is a side of two triangular faces of $T$, one of which belongs to $R(\pi', \pi'')$. Call the third vertex of this triangle $u_2$ (the others are $u'$ and $u_1$). We claim that $u_2$ must be vacant. To see this we must examine two subcases. We assume that $u_2$ is occupied. In each case we shall see that this assumption contradicts the minimality property (5.4).
u does not contain u is vacant, and the path from u (as claimed, then there exists an occupied path π). Proposition 2.2 of Kesten (1982) again. If there does not exist a vacant path R

Indeed, if u vacant). Now to find π removes (the interior of) the triangle ({u, u}). Subcase (a) u2 is a vertex on π. In this case, replace the piece of π from u to u2 by the single edge {u, u2}. This removes at least (the interior of) the triangle (u, u1, u2) from R(π, π′), and yields a strictly smaller R, in contradiction to (5.4).

Subcase (b) u2 is occupied, but is not on π. In this subcase we can replace the edge {u, u1} of π by the two edges {u, u2} and {u2, u1}. This replacement again removes (the interior of) the triangle (u, u1, u2) from R(π, π′′), and leads to a contradiction as before. This proves that u2 must be vacant as claimed.

We next claim that there must exist a vacant path π2 from u2 to π′′ in R(π′, π′′). If u2 ∈ π′′, then this path consists of {u2} only, and the existence of π2 is clear. If u2 /∈ π′′, then u2 must lie in R(π′, π′′), because it is a vertex of a triangle in R(π′, π′′) which does not lie on ΔR(π′, π′′) (u2 /∈ π′ because u2 is vacant). Now to find π2, denote the hexagon whose vertices are the neighbors of u by H and move from u2 towards u′ along the arc of the boundary of H which does not contain u1. We continue till we first hit a vertex, u3 say, of π′ ∪ π′′. Since u′′ ∈ π′′, we must have u3 = u′′ or we must reach u3 before u′′. Let A2 be the arc of H from u2 to u3. Then {u′, u2} and A2, minus the endpoints u′ and u3, lie in R(π′, π′′), because u2 ∈ R(π′, π′′). This shows that u3 ∈ π′ is impossible. Indeed, if u3 ∈ π′, then we can replace the piece of π′ from u′ via u1 to u3 by the single edge from u′ to u3 and so decrease R(π′, π′′). But then u3 ∈ π′′, u3 itself is vacant, and the path from u1 to u3 consisting of {u1, u2} followed by A2 lies in R(π′, π′′). If A2 is vacant, then we can take A2 itself for π2. If not, then apply Proposition 2.2 of Kesten (1982) again. If there does not exist a vacant path π2 as claimed, then there exists an occupied path π3 from a vertex in A2 to a vertex in A4, where

\[ A_4 = \{u', v'\}, \text{ followed by the (reversed) piece of } \pi' \text{ from } v' \text{ to } u_1 \]

followed by the edge \{u_1, u_2\};
this time take \( A_1 = \{ u_2 \} \) and \( A_3 = \text{piece of } \pi'' \) from \( u_3 \) to \( v'' \). But then the path consisting of the edge from \( u' \) to the initial vertex of \( \pi_3 \) on \( \tilde{A}_2 \) followed by \( \pi_3 \) itself is an occupied path in \( \tilde{R}(\pi', \pi'') \), which runs from \( u' \) to a point of \( \pi' \), but which does not start with the edge \( \{ u', u_1 \} \). This again contradicts the minimality property (5.4). Thus \( \pi_3 \) cannot exist, but \( \pi_2 \) does exist.

We now take \( \sigma_1 = u_2, \hat{\pi'} = \text{piece of } \pi' \) from \( u_1 \) to \( v' \), and \( \hat{\pi''} = \pi_2 \) followed by the piece of \( \pi'' \) from the endpoint of \( \pi_2 \) on \( \pi'' \) to \( v'' \). From here on we continue as in Case (i).

We can continue using the procedure of Case (i) or Case (ii) as long as both \( \pi', \pi'' \) each have at least two points. Assume then that we arrive at the situation where one of them has only one point. For the sake of argument let \( \sigma_m \) be occupied (\( \xi_m = 1 \)) and let the remaining path \( \pi'' = \{ v'' \} \) be a one point path only; \( \pi' \) will be an occupied path from \( \sigma_m \) to the neighbor \( v' \) of \( v'' \). If now \( \xi_{m+1} = 0 \), then we simply take as the next point \( \sigma_{m+1} = v'' \) and we stop our procedure. If, however, \( \xi_{m+1} = 1 \), then we are in Case (i). The argument there still applies and \( \sigma_{m+1} \) has to be taken as the neighbor of \( \sigma_m \) on \( \pi' \). Even though \( \pi'' \) has only one vertex, the next vacant path (\( \pi_1'' \) in our notation) may again have more than one vertex, but this causes no problem. At each step \( \tilde{R} \) becomes strictly smaller, so our process must stop at some time. ■

**Corollary 1.** Let \((\pi', \pi'')\) be a double path with initial points \((u', u'')\) and endpoints \((v', v'')\). Let \( y \in \pi' \). Then for all \( \xi = (\xi_1, \xi_2, \ldots) \) there exists a \( \lambda \) and a path \((z_0 = y, z_1, \ldots, z_\lambda) \subset \tilde{R}(\pi', \pi'')\) which starts at \( y \) and ends at \( z_\lambda \in \{v', v''\} \) such that \((\xi_1, \ldots, \xi_\lambda)\) is seen along this path. Moreover \( \lambda \geq \min(\|v' - y\|, \|v'' - y\|) - 1 \). The same holds when \( y \in \pi'' \).

**Proof.** There is nothing new to prove if \( y = u' \) or \( y = v' \). So assume \( y \in \pi' \setminus \{u', v'\} \). Let \( \hat{\pi'} \) be the piece of \( \pi' \) from \( y \) to \( v' \). This is an occupied path. We next construct a vacant path from a neighbor \( y'' \) of \( y \) to \( v'' \) inside \( \tilde{R}(\pi', \pi'') \). To this end, recall that there exists a neighbor \( y'' \) of \( y \) such that \( \{y, y''\} \subset \tilde{R}(\pi', \pi'') \) and such that \( y'' \) is connected by a vacant path in \( \tilde{R}(\pi', \pi'') \), \( \pi_1 \) say, to \( v'' \) (see (5.5)). Let the endpoint of \( \pi_1 \) on \( \pi'' \) be \( z'' \). Then take for \( \hat{\pi''} \) the concatenation of \( \pi_1 \) and the piece of \( \pi'' \) from \( z'' \) to \( v'' \). By construction \( \hat{\pi'}, \hat{\pi''} \subset \tilde{R}(\pi', \pi'') \). Hence also \( \tilde{R}(\hat{\pi'}, \hat{\pi''}) \subset \tilde{R}(\pi', \pi'') \). \((\hat{\pi'}, \hat{\pi''})\) is not necessarily a double path, because \( \tilde{R}(\hat{\pi'}, \hat{\pi''}) \) may not be minimal, but we can now find a double path in \( \tilde{R}(\hat{\pi'}, \hat{\pi''}) \) with the starting points \((y, y'')\) and endpoints \((v', v'')\) by the procedure outlined just before Lemma 3. The Corollary now follows by applying Lemma 3 to this new double path. ■

Unfortunately there are with probability 1 no infinite double paths, because we are working with critical percolation on \( T \). We shall now show, though, that we can find suitable double paths between two successive circuits \( C_k \) and \( C_{k+1} \).
Let $C_i$ be fixed as $C_i \subset \mathcal{C}_{i+1}$. We first consider the case that both $C_k$ and $C_{k+1}$ are occupied. Let $v = v^{(k)} \in C_k$. Because $C_k$ is the outermost occupied circuit in $\mathcal{C}_{k+1}$, there exists a vacant path $\mathcal{D} = \mathcal{D}^{(k)}$ from a neighbor $w$ of $v$ to a vertex $u'$ adjacent to $C_{k+1}$ (see Remark 1). Without loss of generality we assume that

$$u' \text{ is the only vertex of } \mathcal{D} \text{ adjacent to } C_{k+1}. \quad (5.8)$$

Since the vacant path $\mathcal{D}$ cannot intersect the occupied circuits $C_k$ and $C_{k+1}$ we may further assume that

$$w \text{ and } \mathcal{D} \subset C_k^{\text{ext}} \cap C_{k+1}. \quad (5.9)$$

In addition there is no vacant circuit between the successive circuits $C_k$ and $C_{k+1}$. Therefore, again by Remark 1, there also exists an occupied path $\rho$ connecting some vertex $a \in C_k$ to some vertex $b$ of $C_{k+1}$. Without loss of generality we may assume that

$$\rho \setminus \{a, b\} \subset C_k^{\text{ext}} \cap C_{k+1}.$$ 

We could construct a double path from $\mathcal{D}, \rho$ and pieces of $C_k, C_{k+1}$. However, we need double paths with some further special properties.

To construct these special double paths we must first fix the constants $c_{30} - c_{33}$ appearing in Proposition 1. We take $c_{32}$ so large that with probability 1

$$\frac{c_{32}}{2} \log \log(\text{diam } (C_k)) > M_k := \left\lceil \frac{2 \log k}{\log(\beta \wedge (1 - \beta))} \right\rceil \quad \text{eventually} \quad (5.10)$$

($\beta$ is the parameter in the measure $\mu$ in (1.2).) This can be done by virtue of (3.3). We shall assume in the sequel that $k$ is so large that (5.10) holds. Next we choose two pairs $c_{30}, c_{33}$ and $\tilde{c}_{30}, \tilde{c}_{33}$ in $(0, 1)$ with corresponding $c_{31}$ and $\tilde{c}_{31}$ (see (4.92)) so that the estimate (4.5) holds for each of these choices of the parameters, and so that

$$\tilde{c}_{33} \tilde{c}_{30} > c_{30}. \quad (5.11)$$

Finally we take for $j$ the unique integer with

$$2^j \leq \text{diam } (C_{k+1}) < 2^{j+1}. \quad (5.12)$$

Now assume that $v = v^{(k)} \in C_k$ satisfies

$$d(v, C_{k+1}) \geq 2[\text{diam}(C_{k+1})]^{\tilde{c}_{30}} \quad (5.13)$$

and that $C_{k+1}$ is $(\tilde{c}_{30} - \tilde{c}_{33})$-good (with $\tilde{c}_{32} = c_{32}$). We shall now use Proposition 1 to construct (in a deterministic way) under assumptions (5.13), (5.10)
and the further assumptions (5.23) and (5.25) below, a double path \((\pi', \pi'') = (\pi'(k, v), \pi''(k, v))\) with the following properties:

the initial points of \(\pi'\) and \(\pi''\) are \(v\) and \(w\), respectively; \hspace{1cm} (5.14)

the final points of \(\pi'\) and \(\pi''\) are some vertices \(y \in C_{k+1}\) and \(u \in \overset{\circ}{C}_{k+1}\), respectively; \hspace{1cm} (5.15)

\(R(\pi', \pi'') \subset C_{k}^{\text{ext}} \cap \overset{\circ}{C}_{k+1}\) and consequently \(R(\pi', \pi'') \subset (C_{k}^{\text{ext}} \cup C_{k}) \cap \overset{\circ}{C}_{k+1}\); \hspace{1cm} (5.16)

there exists a vacant path \(\theta = \theta^{(k+1)} = (z_1, z_2, \ldots, z_{q-1}, u)\) \\(\subset C_{k}^{\text{ext}} \cap \overset{\circ}{C}_{k+1}\) such that \(\theta\) is adjacent to \(C_{k+1}\), \(q = \text{length } (\theta) = M_{k+1}\), and such that \(z_1, \ldots, z_{q-1} \notin R(\pi', \pi'')\); \hspace{1cm} (5.17)

there exists a vacant path \(\hat{\tau} = \hat{\tau}^{(k)}\) from \(w\) to \(\theta\) with only its endpoint on \(\theta\) adjacent to \(C_{k+1}\); \hspace{1cm} (5.18)

if \(v(z_i) \in C_{k+1}\) is a point adjacent to \(z_i, 1 \leq i \leq q\) (with \(z_q = u\) and \(v(z_q) = v(u) = y\), and \(V^{(k+1)} = \{v(z_i) : 1 \leq i \leq q\}\), then

\(d(V^{(k+1)}, C_{k+2}) \geq 2[\text{diam } (C_{k+2})]^{c_{30}}\). \hspace{1cm} (5.19)

To start our construction, consider the vacant path \(\mathcal{D}\) from \(w\) to \(u'\). It satisfies (for large \(k\))

\[\text{diam } (\mathcal{D}) \geq \|w - u'\| \geq \|u' - v\| - \|w - v\| \geq d(v, C_{k+1}) - 4\] (because \(u'\) is adjacent to \(C_{k+1}\))

\[\geq 2[\text{diam } (C_{k+1})]^{c_{30}} - 4 > [\text{diam } C_{k+1}]^{c_{30}}.\] \hspace{1cm} (5.20)

Therefore, the fact that \(C_{k+1}\) is good implies (see (4.4)) that there exist \(\lceil\text{diam } (C_{k+1})\rceil^{c_{31}} \geq 2^{c_{31}j}\) vacant paths \(\theta_m \subset C_{k}^{\text{ext}} \cap \overset{\circ}{C}_{k+1}\) in the vacant cluster of \(\mathcal{D}\) in \(\overset{\circ}{C}_{k+1}\) which are adjacent to \(C_{k+1}\) and satisfy

\[\text{length } (\theta_m) \geq c_{32} \log \log (\text{diam } (C_{k+1})) \geq 2M_{k+1}\] \hspace{1cm} (5.21)
and
\[ d(\theta_p, \theta_q) \geq \lceil \text{diam} (C_{k+1}) \rceil \tilde{c}_{33} \tilde{c}_{30}, \quad p \neq q. \]

We take \( \tilde{c}_{33} \tilde{c}_{30} > \tilde{c} > c_{30} \) so that this implies (for \( k \) large enough)
\[ d(\theta_p, \theta_q) \geq 16\lceil \text{diam} (C_{k+1}) \rceil \tilde{c}, \quad p \neq q. \] (5.22)

For each such \( \theta_m \) and \( z \in \theta_m \) pick a neighbor \( v(z) \) of \( z \) in \( C_{k+1} \) and let \( V_m^{(k+1)} = \{ v(z) : z \in \theta_m \} \). We now also assume that
\[ N(j, \tilde{c}) \leq j^2 \text{ and } N^{(3)}(j, \tilde{c}_{30}) = 0 \text{ for all } j \geq j. \] (5.23)

We know from Lemma 6 that with probability 1 this holds for large \( j \). Also for large \( j, 2\tilde{c}_{31} j > j^2 \), so that by Lemma 7 for at least one \( m \)
\[ d(V_m^{(k+1)}, C_{k+2}) > 2\tilde{c}^2 j. \] (5.24)

Pick such an \( m \) and denote \( \theta_m \) by \( \hat{\theta} \). Let \( \hat{\theta} = (z_1, \ldots, z_p), p \geq 2M_{k+1} \). It is part of (4.3) that there exists a vacant path \( \hat{\tau} \) from some vertex \( z \in D \) to some vertex \( z_q \in \hat{\theta} \) such that \( z_q \) is the only vertex of \( \hat{\tau} \) adjacent to \( C_{k+1} \). We can always extend \( \hat{\tau} \) by the piece of \( D \) from \( z \) to \( w \), and at the other end we may stop at the first vertex of \( \hat{\tau} \) on \( \hat{\theta} \). Thus we may assume without loss of generality that \( z = w \) and
\[ \hat{\tau} \setminus \{ z_q \} \text{ is disjoint from } \hat{\theta} \text{ and contains no vertex adjacent to } C_{k+1}. \]

Also without loss of generality we assume that the vertices of \( \hat{\theta} \) are indexed such that \( q \geq M_{k+1} \).

Now consider the following two paths from \( C_k \) to \( C_{k+1} \) which lie in \( C_k^{\text{ext}} \cap \hat{C}_{k+1} \), except for their initial and final points: the path \( \tau \), which consists of the edge from \( v \) to \( w \), followed by \( \hat{\tau} \), and finally an edge from the endpoint \( z_q \) of \( \hat{\tau} \) to \( y := v(z_q) \in C_{k+1} \) (such a neighbor \( y \) of \( z_q \) exists because \( \hat{\theta} \) is adjacent to \( C_{k+1} \)); and the path \( \rho \), which runs from \( a \in C_k \) to \( b \in C_{k+1} \). These two paths divide the ring \( C_k^{\text{ext}} \cap \hat{C}_{k+1} \) into two components, each of which is bounded by \( \tau, \rho \) and different arcs of \( C_k \) between \( v \) and \( a \) and of \( C_{k+1} \) between \( b \) and \( y \). One of these components contains \( \{ z_1, \ldots, z_{q-1} \} \), because \( \rho \) does not intersect \( \hat{\theta} \) at all (recall that \( \hat{\theta} \) is vacant and \( \rho \) is occupied) and \( \tau \) intersects \( \hat{\theta} \) only in \( z_q \). We denote by \( W \) the component of \( (C_k^{\text{ext}} \cap \hat{C}_{k+1}) \setminus (\tau \cup \rho) \) which does not contain \( z_1, \ldots, z_{q-1} \) (see Figure 10).
Finally we define

\[ \widehat{\pi}' = \text{the path consisting of the arc of } C_k \text{ from } v \text{ to } a \]
\[ \text{in the boundary of } W, \text{ followed by } \rho \text{ and finally the} \]
\[ \text{arc of } C_{k+1} \text{ from } b \text{ to } y \text{ in the boundary of } W, \]
\[ \widehat{\pi}'' = \widehat{\tau}. \]

These paths are occupied and vacant, respectively, and their initial points, \( v, w \) are adjacent, and so are their final points \( y, z_q \). Moreover, the construction of our paths is such that \( \overline{R}(\widehat{\pi}', \widehat{\pi}'') = W \cup \Delta W \) and therefore \( z_1, \ldots, z_{q-1} \) lie outside \( \overline{R}(\widehat{\pi}', \widehat{\pi}'') \). Finally take for \( \pi' \) and \( \pi'' \) an occupied and vacant path from \( v \) to \( y \) and from \( w \) to \( z_q \), respectively, which lie in \( \overline{R}(\widehat{\pi}', \widehat{\pi}'') \) and form a double path. These paths have the properties (5.14)-(5.16) by construction, with the choice \( z_q \) for \( u \).

(5.17) holds if we take \( \theta = (z_{q-M_{k+1}+1}, z_{q-M_{k+1}+2}, \ldots, z_q = u) \) and renumber its vertices. Condition (5.18) was already verified right after (5.24). Lastly, let \( V^{(k+1)} = V_m^{(k+1)} \) for the \( m \) for which \( \widehat{\theta} = \theta_m \). To obtain (5.19) we shall further assume that

\[ \text{diam } (C_{k+i+1}) \leq (k + i)^{c_{10} + 2}\text{diam } (C_{k+i}), \quad i \geq 0. \quad (5.25) \]

We shall show in the next section that this is justified because this holds with probability 1 for all large \( k \). (5.24), (5.12), (5.25) and \( \widehat{c} > c_{30} \) show that for large \( j \)

\[ d(V^{(k+1)}, C_{k+2}) \geq 2[\text{diam } (C_{k+2})]^{c_{30}}, \]

which gives (5.19). This completes the construction of our special double path when \( C_k \) and \( C_{k+1} \) are both occupied (or both vacant) and when (5.13), (5.23) and (5.25) hold.
If $C_{k+2}$ is also occupied, then we can repeat the above construction with $k$ replaced by $k + 1$ and for any $v^{(k+1)} \in V^{(k+1)}$. (5.19) takes the place of (5.13). However, the exponent $\tilde{c}_{30}$ in (5.13) has been replaced by the smaller $c_{30}$, and hence we are only able to construct a double path and sets $\hat{\theta}^{(k+2)}$, $V^{(k+2)}$ which satisfy (5.14)-(5.17) with $k$ replaced by $k + 1$, but with (compare (5.24))

$$d(V^{(k+2)}, C_{k+3}) \geq 2[\text{diam } (C_{k+3})]^c$$

for any $c < c_{33}c_{30}$, instead of (5.19). Thus with each repetition of the construction our estimate seems to deteriorate. We shall now prove that we can actually obtain

$$d(V^{(k+2)}, C_{k+3}) \geq 2[\text{diam } (C_{k+3})]^{\tilde{c}_{30}},$$

(5.26)

so that our estimate does not deteriorate after the first repetition, and we can continue to find double paths from $v^{(k+i)} \in V^{(k+i)} \subset C_{k+i}$ and can satisfy (5.26) with $k + 2$ replaced by $k + i$.

To see (5.26), assume we picked $v^{(k+1)} \in V^{(k+1)}$ and then a neighbor $w^{(k+1)} \in C_{k+1}^{\text{ext}} \cap C_{k+2}$ of $v^{(k+1)}$ so that $w^{(k+1)}$ has a vacant connection to a neighbor of $C_{k+2}$. Then find a tentative $\hat{\theta}^{(k+2)}$ and $\hat{V}^{(k+2)} \subset C_{k+2}$ as in the preceding construction with $k$ replaced by $k + 1$. By (5.18) this comes with a vacant path $\hat{\tau}^{(k+1)}$ from $w^{(k+1)}$ to some vertex in $\hat{\theta}^{(k+2)}$, such that this vertex is the only vertex of $\hat{\tau}^{(k+1)}$ adjacent to $C_{k+2}$. We identify $D^{(k+1)}$ with $\hat{\tau}^{(k+1)}$. Now our construction only guarantees that

$$\text{diam } (D^{(k+1)}) \geq d(v^{(k+1)}, C_{k+2}) - 4 \geq 2[\text{diam } (C_{k+2})]^{\tilde{c}_{30}} - 4.$$ 

However, if actually the better estimate

$$\text{diam } (D^{(k+1)}) \geq 2[\text{diam } (C_{k+2})]^{\tilde{c}_{30}} - M_{k+2} - 4 > [\text{diam } (C_{k+2})]^{\tilde{c}_{30}}$$

holds, then we can simply go through the construction with $k$ replaced by $k + 1$ to find a $V^{(k+2)}$ which satisfies (5.26). (Note that our construction did not use (5.13) itself, but only its consequence (5.20).) We therefore only have to consider the case when

$$d(V^{(k+1)}, C_{k+2}) - 4 \leq \text{diam } (D^{(k+1)}) < 2[\text{diam } (C_{k+2})]^{\tilde{c}_{30}} - M_{k+2} - 4.$$ 

In this case we have for any $v^{(k+2)} \in \hat{V}^{(k+2)}$ some $z \in \hat{\theta}^{(k+2)}$ adjacent to $v^{(k+2)}$ and

$$\|v^{(k+1)} - v^{(k+2)}\| \leq \|v^{(k+1)} - z\| + 2$$

$$\leq \text{diam } (D^{(k+1)}) + \text{length}(\hat{\theta}^{(k+2)}) + 2 < 2[\text{diam } (C_{k+2})]^{\tilde{c}_{30}}.$$
We claim that in this case we must even have
\[ d(v^{(k+2)}, C_{k+3}) \geq 2[diam (C_{k+2})]^{30}. \] (5.27)
Indeed, if this fails, then each of \( C_{k+1}, C_{k+2} \) and \( C_{k+3} \) must have a point in
\[ v^{(k+2)} + S(2[diam (C_{k+2})]^{30}). \]
Therefore, if
\[ 2^j \leq diam (C_{k+2}) < 2^{j+1}, \]
then there would exist some \( r, s \) so that
\[ v^{(k+2)} \in [r2^{30}(j+1), (r + 1)2^{30}(j+1)] \times [s2^{30}(j+1), (s + 1)2^{30}(j+1)]. \]
Therefore, if \( 2^{30}q \geq 4, \)
\[
T([2^{-30}q}(r - 2), [2^{-30}q}(s - 2)]; 2^{30}, j + q) \]
\[ = [2^{-30}q}(r - 2)][2^{30}(j+q+1), (2^{-30}q}(r - 2)] + 3][2^{30}(j+q+1)] \]
\[ \times [2^{-30}q}(s - 2)][2^{30}(j+q+1), (2^{-30}q}(s - 2)] + 3][2^{30}(j+q+1)] \]
\[ \supset [r2^{30}(j+1), (r + 1)2^{30}(j+1)] \times [s2^{30}(j+1), (s + 1)2^{30}(j+1)] + S(2 \cdot 2^{30}(j+1)) \]
would intersect \( S(2^{j+1}) \subset S(2^{j+q+1}), \) as well as \( C_{k+1}, C_{k+2}, C_{k+3}. \) Since \( diam(C_{k+3}) \)
\[ \geq diam (C_{k+2}) \geq 2^j \] and \( diam(C_{k+1}) \geq (k+1)^{-c_{10}-2} diam(C_{k+2}) \geq 2^{-\log j}j^{j+q+1}, \) for
\[ \text{large } j, \] we would have \( N(3)(j+q, c_{30}) \neq 0. \) This is ruled out by (5.23), so that
(5.27) must hold. Since this holds for any \( v^{(k+2)} \in \hat{V}^{(k+2)} \) this implies (5.26) with
\( \theta^{(k+2)}, V^{(k+2)} \) taken equal to \( \hat{\theta}^{(k+2)}, \hat{V}^{(k+2)}. \)

We have described a construction of double paths between two successive
circuits which are occupied. Of course this works equally well (after an interchange
of ‘occupied’ and ‘vacant’) between two successive circuits which are vacant. We
now briefly describe how to find a double path with the properties (5.14) -(5.17)
and (5.19) when \( C_k \) is vacant and \( C_{k+1} \) is occupied. In fact we obtain (5.19) for
any fixed \( c_{30} < 1 \) for large \( k \) even without the assumption (5.13). We merely need
that the simpler (3.28) holds. The treatment of vertices adjacent to \( C_{k+1} \) will be
somewhat different from the preceding construction and this is why (3.28) comes
in.

In the present situation we start again with a vertex \( v = v^{(k)} \in C_k. \) Since
\( C_k \) is the outermost vacant circuit in \( C_{k+1}, \) there is an occupied path \( \tau \) from a
neighbor \( w \) of \( v \) to a vertex \( x \in C_{k+1}. \) The occupied \( \tau \) cannot intersect the vacant
circuit \( C_k \) and we may stop \( \tau \) at its first vertex on \( C_{k+1}. \) We then have
\[ w \text{ and } \tau \setminus \{x\} \subset C_k^\text{ext} \cap \hat{C}_{k+1}. \]
In addition there is no occupied circuit between $C_k$ and $C_{k+1}$. Therefore, again by Remark 1, there also exists a vacant path $\rho'$ from some vertex in $C_k$ to a vertex adjacent to $C_{k+1}$. Let $\rho$ be the piece of $\rho'$ from its last intersection with $C_k$ to its first point adjacent to $C_{k+1}$. Let $\rho$ run from $a \in C_k$ to $z'$ adjacent to $C_{k+1}$. Then

$$\rho \setminus \{z'\}$$
does not contain any vertex adjacent to $C_{k+1}$ and

$$\rho \setminus \{a\} \subset C_k^\text{ext} \cap C_{k+1}.$$

Now define $\hat{D}$ as the path consisting of (the reverse of) $\rho$ plus an arc of $C_k$ starting at $a$ and of diameter $\geq \frac{1}{2} \text{diam } (C_k)$. This path lies in $C_{k+1}$ but it may contain vertices adjacent to $C_{k+1}$ other than $z'$. However, the number of vertices on $\hat{D}$ adjacent to $C_{k+1}$ cannot exceed

$$1 + (\text{the number of vertices of } C_k \text{ adjacent to } C_{k+1}) \leq 1 + k^2,$$

provided (3.28) holds. Thus $\hat{D}$ contains a subpath $\mathcal{D}$ which contains exactly one vertex adjacent to $C_{k+1}$ and satisfies

$$\text{diam } (\mathcal{D}) \geq \frac{1}{2k^2} \text{diam } (\hat{D}) \geq \frac{1}{4k^2} \text{diam } (C_k) \geq 2[\text{diam } (C_{k+1})]^{\hat{c}_30} \text{ (by (5.25)).}$$

Indeed, we can take for $\mathcal{D}$ a piece of $\hat{D}$ between two successive points adjacent to $C_{k+1}$ (and including one of these points) of maximal diameter. As before, we can now apply Proposition 1 to $\mathcal{D}$ and then use Lemma 7 to find in the vacant cluster of $\mathcal{D}$ in $C_{k+1}$ a path $\hat{\theta}^{(k+1)}$ and a set $V^{(k+1)} \in C_{k+1}$ such that

$$\hat{\theta}^{(k+1)}$$
is adjacent to $V^{(k+1)} \subset C_{k+1}$ and such that (5.24) holds. However, $\mathcal{D}$ is a piece of $\hat{D}$, which is part of the vacant cluster of $v$. Therefore $\hat{\theta}$ is also in the vacant cluster of $v$ in $C_{k+1}$. From here on we continue essentially as from (5.24). We merely point out that for $\hat{\pi}''$ we shall take a path from $v$ to a vertex $z_q$ of $\hat{\theta}^{(k+1)}$ in $(C_k^\text{ext} \cup C_k) \cap C_{k+1}$ in the vacant cluster of $v$ (or equivalently of $\mathcal{D}$); we can take this path in $C_k^\text{ext} \cup C_k$ because we can always replace the piece from $v$ to the last intersection of $\hat{\pi}''$ with $C_k$ by an arc of $C_k$. (We are not claiming this time that $z_q$ is the only point of $\hat{\pi}''$ which is adjacent to $C_{k+1}$.) For $\hat{\pi}'$ we shall take a path consisting of the path $\tau$ from $w$ to $x \in C_{k+1}$ followed by a suitable arc of $C_{k+1}$ from $x$ to a neighbor $y$ of $z_q$. 
This arc can be chosen so that $\overline{R}(\tilde{\pi}', \tilde{\pi}'') \subset (C_k^\text{ext} \cup C_k) \cap \overline{C}_{k+1}$. (5.16) will be a consequence of this.

This completes our construction of special double paths. It shows that if (5.13) holds for some $k_0$ and some $v_{k_0}$ and (5.25), (3.28) hold for all $k \geq k_0$ and if (5.23) holds for the corresponding $j$ (see (5.12)), then one can successively choose $\theta^{(k+i)}$, $V^{(k+i)}$ and for $v \in V^{(k+i)}$ a special double path $(\pi'(k+i, v), \pi''(k+i, v))$ so that (5.14)-(5.17) with $k$ replaced by $k+i$ hold.

6. Synthesis.

We now put the various lemmas about circuits and double paths together to show that the probability of seeing a random word on $T$ is bounded away from 0. Our strategy is to first choose all the $C_k$ and to decide whether they are occupied or vacant. In other words, we shall condition on

$$\{C_k = C_k, \kappa(C_k) = \varepsilon_k, k \geq 1\}. \quad (6.1)$$

Here $\{C_k\}_{k \geq 1}$ is a sequence of circuits for which $C_k \subset \overset{\circ}{C}_{k+1}$ and $\varepsilon_k \in \{0, 1\}$. After having chosen $C_k$ and $\kappa(C_k)$ we choose the vertices not on $\cup^\infty_{i=1} C_k$. We shall show that by ignoring a $P$-null set we may assume that the following conditions hold for $k \geq$ some (random) $k_0$, respectively $j \geq$ some (random) $j_0$ and fixed $\overline{c} > \hat{c}_30, \tilde{c} \in (c_{30}, \hat{c}_{33}\hat{c}_{30})$:

$$C_k \text{ is good;} \quad (6.2)$$

$$\text{diam } (C_{k+1}) \leq \text{diam } (C_k)[\log(\text{diam } (C_k))^{c_{10}+1}; \quad (6.3)$$

$$N(j, \tilde{c}) \leq j^2 \text{ and } N(j, \overline{c}) \leq j^2; \quad (6.4)$$

$$N^{(3)}(j, \tilde{c}_{30}) = 0; \quad (6.5)$$

and finally (3.28) and (5.10). For an occupancy configuration which satisfies (6.3)-(6.5), (3.28) and (5.10) we now make a number of choices recursively. At the $k$-th stage we will have singled out a vertex $v^k \in C_k$ and a double path

$$(\pi'(k), \pi''(k)) = (\pi'(k, v^k), \pi''(k, v^k)) \subset (C_k^\text{ext} \cup C_k) \cap \overline{C}_{k+1}$$

such that

$$v^k \in \Delta \overline{R}(\pi'(k, v^k), \pi''(k, v^k)).$$

The endpoints of $\pi'(k)$ and $\pi''(k)$ are on $C_{k+1}$ or adjacent to $C_{k+1}$, depending on whether $C_{k+1}$ is occupied or vacant. For the sake of argument let us describe the situation when $C_{k+1}$ is occupied. Then the endpoint $y^{(k+1)}$ of $\pi'(k)$ will lie on $C_{k+1}$ and the endpoint $u^{(k+1)}$ of $\pi''(k)$ will be in $C_k^\text{ext} \cap \overset{\circ}{C}_{k+1}$, adjacent to $C_{k+1}$. Associated with $(\pi'(k), \pi''(k))$ will be a vacant path $\theta^{(k+1)} = (z_1^{(k+1)}, \ldots, z_{M_{k+1}}^{(k+1)})$
which is adjacent to $C_{k+1}$, ends at $z^{(k+1)} = u^{(k+1)}$ and is such that $\theta^{(k+1)} \setminus \{u^{(k+1)}\}$ lies outside $\overline{R}(\pi'(k), \pi''(k))$. Moreover (5.19) will hold.

We now concentrate on the case where $\xi$ is random with the distribution $\mu$ of (1.2). A few words to handle the case of a deterministic $\xi$ with bounded $r_m(\xi)$ will be said at the end. Let

$$\mathcal{F}_n = \sigma\text{-field in } \Xi \text{ generated by } \xi_1, \ldots, \xi_n.$$  

To avoid double subscripts we shall write $\xi(i)$ instead of $\xi_i$ in the remainder of this section. At stage $k$ we will also have found an $\{\mathcal{F}_n\}$-stopping time $\lambda_k$ and we will have chosen $\xi(1), \ldots, \xi(\lambda_k)$ and a path $(w(0), \ldots, w(\lambda_k))$ such that

$$(w(0), \ldots, w(\lambda_k)) \text{ is a selfavoiding path;} \quad (6.6)$$

$w(i) \text{ occupied or vacant according as } \xi_i = 1 \text{ or } \xi_i = 0, \quad 1 \leq i \leq \lambda_k; \quad (6.7)$$

$$w(\lambda_k) = v^{(k)} \in \Delta\overline{R}(\pi'(k), \pi''(k)) \quad (6.8)$$

but

$$w(i) \notin \overline{R}(\pi'(k), \pi''(k)) \cup C_k^{\text{ext}} \text{ for } i < \lambda_k. \quad (6.9)$$

We will have carried out the construction in such a way that

- the construction of $(w(0), w(1), \ldots, w(\lambda_k))$ depends only on $\xi(1), \xi(2), \ldots, \xi(\lambda_k)$, and conditionally on this information $\xi(\lambda_k + 1), \xi(\lambda_k + 2), \ldots$ still have the distribution $\mu$. \hfill (6.10)

The $(k + 1)$-th stage will consist of first choosing the set $V^{k+1} \subset C_{k+1}$ of neighbors to the points in $\theta^{(k+1)}$ as described in (5.19) and then for each $v \in V^{(k+1)}$ construct the special double path $(\pi'(k + 1, v), \pi''(k + 1, v))$ described in the preceding section. We next need to choose for the $(k + 1)$-th stage an $\{\mathcal{F}_n\}$-stopping time $\lambda_{k+1}$ and the path

$$\gamma_{k+1} := (w(\lambda_k + 1), w(\lambda_k + 2), \ldots, w(\lambda_{k+1}))$$

so that (6.6) and (6.7) with $k + 1$ instead of $k$ hold. At the same time we will have to find the new $v^{(k+1)}$, $(\pi'(k + 1, v^{(k+1)}), \pi''(k + 1, v^{(k+1)})) \subset (C_{k+1}^{\text{ext}} \cup C_{k+1}) \cap \overline{C}_{k+2}$ so that also (6.8) and (6.9) with $k + 1$ instead of $k$ hold. Together with this we must also find $\theta^{(k+2)}$ so that (5.17) holds with $k$ replaced by $k + 1$. We shall start this all at some $k_0$ with $\lambda(k_0) = 0$ and some vertex $w(0) \in C_{k_0}$ and with

$$d(w(0), C_{k_0+1}) \geq 2[d(\text{diam } (C_{k_0+1}))]^3. \quad (6.11)$$
We shall make the choice for the starting parameters explicit when we fill in the details. First we note that if we are successful with our construction at every stage, then we see \( \xi = (\xi_1, \xi_2, \ldots) \) along the infinite selfavoiding path \((w(0), w(1), \ldots)\) which is the concatenation of the \( \gamma_k \). The principal estimate will be that the conditional probability of successfully choosing \((\xi(\lambda_k + 1), \ldots, \xi(\lambda_{k+1}))\) and the path \( \gamma_{k+1} \) in the above manner, given the occupancy configuration, \( \lambda_k \) and \( \xi(1), \ldots, \xi(\lambda_k) \), is at least \( 1 - \frac{1}{k^2} \). This will then guarantee that the probability of seeing the random word \((\xi(1), \xi(2), \ldots)\) from some point \( w(0) \) is at least
\[
\prod_{k_0}^{\infty} \left( 1 - \frac{1}{k^2} \right) \geq \frac{1}{2}
\] (6.12)
(if \( k_0 \) is chosen sufficiently large). By the zero-one law (1.3) this will then show that
\[
\mu\{ \xi : \rho(\xi) = 1 \} = 1
\]
and complete the proof of our Theorem.

Let us fill in the remaining details. Most of the work has been done in Sections 2-5 already. We fix \( \hat{c} \in (c_{30}, \hat{c}_{33} \hat{c}_{30}) \) and \( \bar{c} \in (\hat{c}_{30}, 1) \). First we prove that we may assume (6.2)-(6.5), (3.28) and (5.10). Of these, (6.4), (6.5) and (3.28) are explicitly stated in Lemma 6, and (6.2) follows from Proposition 1. As for (6.3), this is almost immediate from Lemma 4. Indeed, for large \( k \),
\[
\text{diam } (C_{k+1}) \leq 2 \max_{x \in C_{k+1}} \|x\| \leq 2 \frac{(k + 1)^{c_{10}}}{c_9} \min_{x \in C_k} \|x\| \text{ (by (3.5))}
\]
\[
\leq 2 \frac{(k + 1)^{c_{10}}}{c_9} \text{ diam } (C_k) \text{ (recall that } C_k \text{ surrounds 0)}
\]
\[
\leq c_{52} [\log (\text{diam } (C_k))]^{c_{10}} \text{ diam } (C_k) \text{ (by (3.3))}.
\]
Finally, (5.10) follows from (3.3) when \( c_{32} \) is sufficiently large. From now on we assume that (6.2)-(6.5), (3.28) and (5.10) hold for \( k \geq k_0, j \geq j_0 \), respectively, where we assume that \( j_0 \) satisfies (5.12) with \( k_0 \) for \( k \). If necessary we raise \( k_0 \) so that
\[
\prod_{k_0}^{\infty} \left( 1 - \frac{1}{k^2} \right) \geq \frac{1}{2} \quad 2^{j_0}/(8 \cdot 2^{j_0} \sqrt{2} + 1) > j_0^2 \text{ and } 2^{j_0} \geq 8 \cdot 2^{j_0}. 
\]
We now begin finding \((w(0), w(1), \ldots)\) and \( \lambda_k \) satisfying (6.6)-(6.10) by an application of Lemma 7 with \( V_m^{(k_0)} = \{v_m\} \) for a collection of vertices \( v_m \in C_{k_0} \) with
\[
\|v_p - v_q\| \geq 8 \cdot 2^{j_0} \text{ for } p \neq q.
\]
Since \( \text{diam} (C_{k_0}) \geq 2^{j_0} \), we can find at least \( 2^{j_0}/(8 \cdot 2^{30j_0} \sqrt{2} + 1) > j_0^2 \) such vertices. Then by virtue of (6.4) and Lemma 7 there exists at least one \( v_m \) with
\[
d(v_m, C_{k_0+1}) \geq 2^{30j_0} \geq 8 \cdot 2^{30j_0}.
\]

We take \( w(0) \in C_{k_0} \) as one of those \( v_m \).

We next use the method described after (5.13) with \( w(0) \) taking the role of \( v^{(k)} \) there, and \( k = k_0 \). We then find a double path \( (\pi'(k_0), \pi''(k_0)) \), a path \( \theta^{(k_0+1)} \) and a set \( V^{(k_0+1)} \subset C_{k_0+1} \) with the properties (5.14)-(5.17) and (5.19) in case \( C_{k_0+1} \) is occupied, and with these properties with ‘occupied’ and ‘vacant’ interchanged in case \( C_{k_0+1} \) is vacant. Now we pick for each \( v \in V^{(k_0+1)} \) a corresponding special double path \( (\pi'(k+1, v), \pi''(k+1, v)) \), again following the recipe at the end of the last section, so that these paths satisfy (5.14)-(5.17) and (5.19) with \( k \) replaced by \( k+1 \). Now define
\[
\mathcal{R}^{(k+1)} = \bigcup_{v \in V^{(k+1)}} \overline{R}(\pi'(k+1, v), \pi''(k+1, v)).
\] (6.13)

At the moment we only know what this means for \( k = k_0 \), but the same definition will apply later for \( k > k_0 \). We also take \( \lambda_{k_0} = 0 \). (We don’t need \( \lambda_i \) for \( i < k_0 \), but if desired one can set those \( \lambda_i = 0 \).) Now we choose \( \xi(1), \xi(2), \ldots \) one at a time and a corresponding path \( (w(0), w(1), \ldots) \) in \( \overline{R}(\pi'(k_0), \pi''(k_0)) \) such that (6.6) and (6.7) hold, following the algorithm of Lemma 8. This lemma tells us that we can continue with this until the \( w \)-path reaches one of the final points of \( \pi'(k_0, w(0)) \) or \( \pi''(k_0, w(0)) \). However, for the present purposes we may want to stop earlier. We define the \( \{F_n\} \)-stopping time
\[
\nu_{k+1} = \min \{ n : w(n) = \text{final point of } \pi'(k, v^{(k)}) \text{ or } \pi''(k, v^{(k)}) \text{ or } w(n) \in \mathcal{R}^{(k+1)} \}.
\] (6.14)

Again we only do this for \( k = k_0 \) with \( v^{(k_0)} = w(0) \) at the moment, but we will use the same definition for general \( k \) later. If
\[
w(\nu_{k_0+1}) \in \mathcal{R}^{(k_0+1)},
\] (6.15)
then we take \( \lambda_{k_0+1} = \nu_{k_0+1} \). By the algorithm of Lemma 8, we still have
\[
w(i) \in \overline{R}(\pi'(k_0, v^{(k_0)}), \pi''(k_0, v^{(k_0)})), \quad i \leq \nu_{k_0+1}.
\] (6.16)
Moreover, (6.15) says that there exists some \( v^{(k_0+1)} \in V^{(k_0+1)} \) so that
\[
w(\nu_{k_0+1}) \in \overline{R}(\pi'(k_0 + 1, v^{(k_0+1)}), \pi''(k_0 + 1, v^{(k_0+1)})).
\]
Also by definition of $\nu$

$$w(i) \notin \overline{R}(\pi'(k_0 + 1, v^{(k_0 + 1)}), \pi''(k_0 + 1, v^{(k_0 + 1)})) \text{ for } i < \nu_{k_0 + 1},$$

so that (6.9) holds for $k = k_0 + 1$ (recall (5.16) for $k = k_0$) and also

$$w(\nu_{k_0 + 1}) \in \Delta \overline{R}(\pi'(k_0 + 1, v^{(k_0 + 1)}), \pi''(k_0 + 1, v^{(k_0 + 1)})). \quad (6.17)$$

In turn, we get from this and (6.16) that

$$w(\nu_{k_0 + 1}) \in \overline{R}(\pi'(k_0, v^{(k_0)}), \pi''(k_0, v^{(k_0)})) \setminus \Delta \overline{R}(\pi'(k_0 + 1, v^{(k_0 + 1)}), \pi''(k_0 + 1, v^{(k_0 + 1)})) \subset C_{k_0 + 1} \text{ (again use (5.16)).}$$

We shall choose future points of $w(\cdot)$ in $C_{k_0 + 1}^\text{ext} \cup C_{k_0 + 1}$; $w(\cdot)$ will not return to $C_{k_0 + 1}$ after time $\nu_{k_0 + 1}$.

The second possibility in (6.14) is that $w(\nu_{k_0 + 1})$ is a final point of $\pi'(k_0, v^{(k_0)})$ or of $\pi''(k_0, v^{(k_0)})$. To discuss this situation more explicitly let us assume $C_{k_0 + 1}$ is occupied. Then the properties (5.14)-(5.17) and (5.19) tell us that the occupied path $\pi'(k_0, v^{(k_0)})$ has its final point at $y = v(z_q)$ in $V^{(k_0 + 1)}$ for some $z_q \in \theta^{(k_0 + 1)}$ (with $q = M_{k_0 + 1}$), and this $y$ is a point of $\pi'(k_0 + 1, v(z_q)) \subset \overline{R}(\pi'(k_0 + 1, v(z_q)), \pi''(k_0 + 1, v(z_q)))$. Thus, if $w(\nu_{k_0 + 1}) = y$, then we still are in the situation (6.15). The only case left to consider is when

$$w(\nu_{k_0 + 1}) = \text{ the final point } z_q \text{ of the vacant path } \pi''(k_0, v^{(k_0)}). \quad (6.18)$$

Since we chose our path so that (6.7) holds for $i \leq \nu_{k_0 + 1}$, this can happen only when $\xi(\nu_{k_0 + 1}) = 0$. We now successively examine $\xi(i)$ for $i > \nu_{k_0 + 1}$ until we come to the smallest $r \geq 1$ with $\xi(\nu_{k_0 + r}) = 1$. We say that we have **success at the $(k_0 + 1)$-th stage** if either (6.15) holds or (6.18) holds and the $r$ there is $\leq M_{k_0 + 1}$. If we are not successful at the $(k_0 + 1)$-th stage, then we stop our construction and we give up on seeing $\xi$ from $w(0)$. If (6.18) holds and $r \leq M_{k_0 + 1}$, then we define $\lambda_{k_0 + 1} = \nu_{k_0 + 1} + r$ and take

$$w(\nu_{k_0 + i}) = z_{q-i}, \quad 1 \leq i < r,$$

$$w(\nu_{k_0 + r}) = v(z_q - r + 1) \in V^{(k_0 + 1)}. \quad (6.19)$$

Let us check that (6.6)-(6.10) hold for $k = k_0 + 1$ with this choice. This is easy when (6.15) applies and will be left to the reader in this case. Assume then that (6.15) does not hold, but (6.18) prevails. Then (6.6) holds because $\theta^{(k_0 + 1)} = (z_1, \ldots, z_q)$, and $\theta^{(k_0 + 1)} \setminus \{z_q\}$ lies outside $\overline{R}(\pi'(k_0, v^{(k_0)}), \pi''(k_0, v^{(k_0)}))$.
so that \((w(0), \ldots, w(\lambda_{k_0+1} - 1))\) is a selfavoiding path. Even \(w(\lambda_{k_0+1})\) has not been visited before, because \(w(\lambda_{k_0+1}) = w(\nu_{k_0+1} + r) \in V'(k_0+1) \subset R'(k_0+1)\) and \(w(i) \notin R'(k_0+1)\) for \(i < \lambda_{k_0+1}\) (recall that (6.15) does not hold and \(\theta(k_0+1) \subset C_{k_0+1}\), while \(R'(k_0+1) \subset C_{k_0+1} \cup C^\text{ext}_{k_0+1}\)). Next, (6.7) holds by construction, since \(C_{k_0+1}\) was occupied and \(\theta(k_0+1)\) was vacant. (6.8) is automatic if we take \(v(k_0+1) = w(\lambda_{k_0+1})\) and \(\pi'(k_0+1) = \pi'(k_0+1, v(k_0+1))\) and similarly for \(\pi''\). In this case also (6.9) holds because we assumed that (6.15) does not occur and \(\theta(k_0+1) \subset C_{k_0+1}\), and therefore is disjoint from

\[ \overline{R}(\pi'(k_0+1, v(k_0+1)), \pi''(k_0+1, v(k_0+1))) \cup C^\text{ext}_{k_0+1} \subset C_{k_0+1} \cup C^\text{ext}_{k_0+1}. \]

(6.10) is clear from the description of the selections which we made.

It is also clear that

\[
\begin{align*}
\mu\{\text{no success at the } (k_0 + 1)\text{-th stage|full occupancy configuration and } \mathcal{F}_{\lambda_{k_0}} \} \\
\leq \mu\{\xi(\nu_{k_0+1} + i) = 0, 1 \leq i \leq M_{k_0+1}\text{full occupancy configuration and } \mathcal{F}_{\lambda_{k_0}} \} \\
\leq [\beta \wedge (1 - \beta)]^{-M_{k_0+1}} \leq \frac{1}{(k + 1)^2} \text{ (see (5.10))}.
\end{align*}
\]

(6.20)

If we were successful at stage \(k_0 + 1\), then we can now go on and repeat the above steps with \(k_0\) replaced by \(k_0 + 1, k_0 + 2, \ldots\) as long as we have success. Only one step deserves further comment. In analogy with (6.17) we will have (6.8) for \(k > k_0\), but there is no reason to believe that \(w(\lambda_k)\) will always be an initial point of \(\pi'(k)\) or \(\pi''(k)\). If this is not the case for a certain \(k\), then for choosing \(\gamma_{k+1}\) we should apply Corollary 1 instead of Lemma 8. See Figure 11 for a typical step.

![Figure 11](image-url)
If we are successful at the \((k_0 + i)\)-stage for all \(i\), then we see \(\xi\) along the path \((w(0), w(1), \ldots)\) and we therefore proved (6.12) and our theorem for almost all \(\xi[\mu]\) follows.

Finally, when \(\xi\) is deterministic with \(r_m(\xi) \leq M\) for some \(M < \infty\) and all \(m\), it is clear that the same method as for the random word will work. We now will be certain of success at the \(k\)-th stage, as soon as \(M_k > M\). Thus \(\rho(\xi) = 1\) for such a \(\xi\).

**7. Appendix to Lemma 5.**

The proof of (3.12) generally follows Kesten (1987), but since a nontrivial change is needed we outline here what this change is. Roughly speaking, we have to show that if \(F(w, n)\) occurs, then we can with a probability bounded away from 0 choose the 5 paths to \(\Delta S(n)\) so that their endpoints are not too close together and so that they can be extended to \(\Delta S(2n)\). This will give

\[
P\{F(w, 2n)\} \geq c_{53}P\{F(w, n)\}, \quad w \in S(n(1 - \delta)).
\]  

(7.1)

Simple monotonicity arguments then give (3.12).

We only consider \(n = 2^k\) for some \(k\). This will be enough, because \(P\{F(w, n)\}\) is decreasing in \(n\). We write \((1 - \delta)2^k\) instead of \(\lfloor(1 - \delta)2^k\rfloor\) to simplify notation. Analogously to Kesten (1987) we introduce the following four strips, whose union is \(S(2^k) \setminus S((1 - \delta)2^k)\):

\[
\begin{align*}
S_R &= [(1 - \delta)2^k, 2^k] \times [-2^k, 2^k]; \\
S_L &= [-2^k, -(1 - \delta)2^k] \times [-2^k, 2^k]; \\
S_T &= [-2^k, 2^k] \times [(1 - \delta)2^k, 2^k]; \\
S_B &= [-2^k, 2^k] \times [-2^k, -(1 - \delta)2^k].
\end{align*}
\]

For any occupied left-right crossing \(r\) of \(S_R\) we denote its endpoint on the right edge of \(\Delta S(2^k)\) by \(a(r) = (a_1(r), a_2(r)) = (2^k, a_2(r))\). For a vacant crossing \(r^*\) of the same rectangle we denote its endpoint on the right edge of \(\Delta S(2^k)\) by \(a^*(r^*)\).

If \(r\) is a left-right crossing of \(S_R\), then \(\hat{S}_R \setminus \{r\}\) has two components, the component below \(r\), denoted by \(\hat{S}_R^-(r)\), and the component above \(r\), denoted by \(\hat{S}_R^+(r)\) (compare Kesten (1987)). We say that a left-right crossing \(r'\) of \(S_R\) lies below another left-right crossing \(r''\) of \(S_R\) if \(S_R^-(r') \subset S_R^-(r'')\). Now define \(r_1\) as the lowest occupied left-right crossing of \(S_R\). When \(r_1\) exists, define \(r_{i+1}\) as the lowest occupied left-right crossing of \(S_R\) in \(\hat{S}_R^+(r_i)\). Note that this requires in particular that \(r_{i+1}\) is disjoint from \(r_i, r_{i-1}, \ldots, r_1\). We continue choosing such \(r_i\) as long as they exist. Let \(r_1, r_2, \ldots, r_\nu\) be all the occupied left-right crossings of \(S_R\) which can be found in this way. We use the same procedure to find a maximal
sequence of disjoint vacant left-right crossings \( r_1^*, \ldots, r_\nu^* \) of \( S_R \), beginning with the lowest vacant left-right crossing \( r_1^* \).

Next we define an \((\eta, k)\)-fence for one of the \( r_i \). We say that an occupied left-right crossing \( r_i \) in our sequence \( r_1, \ldots, r_\nu \) has an \((\eta, k)\)-fence if the following four analogues of (2.26)-(2.28) in Kesten (1987) hold.

\[
\text{if } r_j, j \neq i, \text{ is any other occupied left-right crossing of } S_R \text{ in } \{r_1, \ldots, r_\nu\}, \text{ then } |a_2(r_j) - a_2(r_i)| > 2\sqrt{\eta}2^k; \tag{7.2}
\]

\[
\text{if } r_j^* \text{ is any of the vacant left-right crossings of } S_R \text{ in } \{r_1^*, \ldots, r_\nu^*\}, \text{ then } |a_2^*(r_j^*) - a_2(r_i)| > 2\sqrt{\eta}2^k; \tag{7.3}
\]

there exists an occupied vertical crossing \( s \) of the rectangle
\[
[a_1(r_i), a_1(r_i) + \sqrt{\eta}2^k] \times [a_2(r_i) - \eta 2^k, a_2(r_i) + \eta 2^k]
= [2^k, 2^k + \sqrt{\eta}2^k] \times [a_2(r_i) - \eta 2^k, a_2(r_i) + \eta 2^k]
\]
which is connected to \( r_i \) by an occupied path in \( a(r_i) + S(\sqrt{\eta}2^k); \tag{7.4}\)

\[
\|a(r_i) - (2^k, 2^k)\| > 2\sqrt{\eta}2^k \text{ and } \|a(r_i) - (2^k, -2^k)\| > 2\sqrt{\eta}2^k. \tag{7.5}
\]

The last requirement is just that \( a(r_i) \) should not lie too close to the corners of \( S(2^k) \) and is only needed to keep \( a(r_i) \) away from the endpoints of top-bottom crossings of \( S_T \) and of \( S_B \).

The main step is to prove an analogue of Lemma 2 in Kesten (1987), namely that for all \( \delta > 0, \varepsilon > 0 \) there exists an \( \eta = \eta(\delta, \varepsilon) > 0 \) such that

\[
P\{\text{there exists an occupied } r_i \text{ which does not have an } (\eta, k)\text{-fence}\} \leq \varepsilon. \tag{7.6}
\]

Analogously to the reference we guarantee the existence of an \((\eta, k)\)-fence for all \( r_i \) by the occurrence for each \( r_i, 1 \leq i \leq \nu \), of an event \( E_j(\eta, k, r_i) \) for some \( \eta^{-3/8} \leq 2^j \leq \eta^{-1/2} \), as well as of an \( E_j(1)(\eta, k, r_i) \) and an \( E_j^*(2)(\eta, k, r_i) \), for some \( 2\eta^{-1/2} < 2^{(1)}, 2^{(2)} < \eta^{-5/8} \), and finally the existence of vacant circuits surrounding \( (2^k, 2^k) \) and \( (2^k, -2^k) \) in the annuli

\[
(2^k, 2^k) + [S(\eta^{1/4}2^k) \setminus S(2\sqrt{\eta}2^k)] \text{ and } (2^k, -2^k) + [S(\eta^{1/4}2^k) \setminus S(2\sqrt{\eta}2^k)], \tag{7.7}
\]
respectively. Here,

\[ E_j(\eta, k, r) := \{ \text{there exist top-bottom crossings of the strips} \]
\[ [2^k - \eta 2^{k+j}, 2^k - \eta 2^{k+j-1}] \times [a_2(r) - \eta 2^{k+j}, a_2(r) + \eta 2^{k+j}] \]
\[ \text{and} \ [2^k + \eta 2^{k+j-1}, 2^k + \eta 2^{k+j}] \times [a_2(r) - \eta 2^{k+j}, a_2(r) + \eta 2^{k+j}], \]
\[ \text{and left-right crossings of the strips} \]
\[ [2^k - \eta 2^{k+j}, 2^{k+j} + \eta 2^{k+j-1}] \times [a_2(r) - \eta 2^{k+j}, a_2(r) - \eta 2^{k+j-1}] \]
\[ \text{and} \ [2^k - \eta 2^{k+j}, 2^k + \eta 2^{k+j}] \times [a_2(r) + \eta 2^{k+j-1}, a_2(r) + \eta 2^{k+j}]; \]
\[ \text{all vertices on these crossings which} \]
\[ \text{are outside the closure of} \ S_R^{-}(r) \text{are occupied} \}. \quad (7.8) \]

\( E_j^*(\eta, k, r) \) is defined in the same way as \( E_j(\eta, k, r) \) except that now all vertices outside \( S_R^{-}(r) \) on the relevant crossings have to be vacant instead of occupied. The occurrence of an \( E_j^* \) was not required in Kesten (1987). We add this requirement here because occurrence of an \( E_j^* \) with \( j \) in the prescribed range will prevent another occupied left-right crossing \( t \) of \( S_R \) which lies above \( r \) to have \( |a_2(t) - a_2(r)| \leq 2\sqrt{2^k} \). The necessary \( E_j(\eta, k, r_i) \) and \( E_j^*(\eta, k, r_i) \) for all \( 1 \leq i \leq \nu \) occur with high probability when \( \eta \) is small. Also the vacant circuits surrounding \( (2^k, \pm 2^k) \) in the annuli of (7.7) occur with high probability when \( \eta \) is small. In this way we obtain (7.6) for sufficiently small \( \eta \), entirely as in Kesten (1987).

Finally we must show how to use (7.6) to show that if \( F(w, 2^k) \) occurs, then we can keep the endpoints of the five paths from \( w \) to \( \Delta S(2^k) \) a distance \( 2\sqrt{2^k} \) apart. This rests on the following deterministic facts:

\[ \text{each occupied left-right crossing} \ r \text{ of} \ S_R \text{intersects some} \ r_i; \quad (7.9) \]

if \( s_1, \ldots, s_p \) are disjoint occupied left-right crossings of \( S_R \),
then there exist \( r_{i_1}, \ldots, r_{i_p} \) with \( i_1, \ldots, i_p \) distinct and \( p \) disjoint occupied left-right crossings of \( S_R, t_1, \ldots, t_p \),
such that \( t_q \) has the same initial point as \( s_q \) on the left edge of \( S_R \), and such that \( t_q \) has the same endpoint as \( r_{i_q} \)
on the right edge of \( S_R, 1 \leq q \leq p. \quad (7.10) \]

(7.9) is easy, because for any two disjoint left-right crossings of \( S_R \), one has to lie above the other. Thus any collection of disjoint left-right crossings can be ordered so that if \( r' \) and \( r'' \) belong to this collection and \( r' \) precedes \( r'' \), then \( r' \) lies below \( r'' \). Therefore, if \( r \) is an occupied left-right crossing of \( S_R \) which is disjoint
from all the \( r_i \), then it lies between two successive \( r_i \) and should have been counted as one of the \( r_i \).

(7.10) follows from the fact that there exists a path \( \tau \) which connects the top and bottom edge of \( S_R \) in \( S_R \) such that \( \tau \) intersects each \( r_i \) in exactly one vertex \( v_i \), and except for these vertices \( v_i \), \( \tau \) is vacant. To see that such a \( \tau \) can be found, recall that \( r_1, \ldots, r_\nu \) are all the disjoint \( r_i \). Then by Proposition 2.2 of Kesten (1982) there exists a vacant path in \( S_R \) from some vertex \( w_\nu \) in the top edge of \( S_R \) to a vertex \( u_\nu \) which is adjacent to some \( v_\nu \in r_\nu \). If we have found \( v_j \in r_j \) for \( j \geq i \), then again by Proposition 2.2 of Kesten (1982) there exists a vacant path from some neighbor \( w_i \) of \( v_i \) to a neighbor \( u_{i-1} \) of \( r_{i-1} \). The vacant piece from \( w_i \) to \( u_{i-1} \) lies below \( r_i \) and above \( r_{i-1} \). The concatenation of the pieces from \( w_i \) to \( u_{i-1} \) and the edges \( \{u_{i-1}, v_{i-1}\}, \{v_{i-1}, w_{i-1}\} \) give the desired path \( \tau \).

Once we have \( \tau \) with the properties of the preceding paragraph and disjoint occupied left-right crossings \( s_1, \ldots, s_p \), then we construct the desired left-right crossings \( t_1, \ldots, t_p \) by the following method: \( \tau \) separates the left edge of \( S_R \) from its right edge, so that each \( s_q \) must intersect \( \tau \) in some point. Since \( s_q \) is occupied this intersection must be at one of the \( v_i \). Let the first intersection of \( s_q \) with \( \tau \) be at \( v_{i_q} \). These \( v_{i_q} \) must be distinct for different \( q \), because the \( s_q \) are disjoint. Now take for \( t_q \) the first part of \( s_q \) from its initial point on the left edge of \( S_R \) till \( v_{i_q} \), followed by the part of \( r_{i_q} \) from \( v_{i_q} \) till its endpoint \( a(r_{i_q}) \) on the right edge of \( S_R \). Note that the \( t_q \) for different \( q \) are disjoint, because the piece of \( s_q \) till it reaches \( v_{i_q} \) lies to the 'left of \( \tau \)' and cannot intersect the piece of some \( r_j \) with \( j \neq i_q \) from \( v_j \) to \( a(r_j) \), because this lies to the 'right of \( \tau \').

From here on we can basically follow Kesten (1987). If \( F(w, 2^k) \) occurs, or even just the event in the middle member of (3.7), and all occupied and vacant crossings of the strips \( S_R, S_L, S_T \) and \( S_B \) which are analogues of the \( r_i \), have \((\eta, k)\)-fences, then by (7.9) and (7.10) we can modify the paths from \( w \) to \( \Delta S(2^k) \) so that their last piece coincides with one of the \( r_i \) or \( r_j^* \) or one of the analogues of these for the strips \( S_L, S_T \) or \( S_B \). Now one argues as in Lemma 4 of Kesten (1987) to show that the conditional probability given \( F(w, 2^k) \), or given the event in (3.7), that there are five paths from \( w \) to \( \Delta S(2^k) \) (appropriately occupied or vacant) which end in crossings which have an \((\eta, k)\)-fence, is bounded away from zero (bounded in \( k \), that is). From there one obtains (7.1) and (3.12) as in Corollary 3 of Kesten (1987).

It also follows by the same argument that the probability in (3.7) is at most \( c_{\delta 4} P\{G(0, n)\} \).

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