Classification of initial state granularity via 2d Fourier expansion

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Abstract.
A new method to quantify fluctuations in the initial state of heavy ion collisions is presented. The initial state energy distribution is decomposed with a set of orthogonal basis functions which include both angular and radial variation. The resulting two dimensional Fourier coefficients provide additional information about the nature of the initial state fluctuations compared to a purely angular decomposition. We apply this method to ensembles of initial states generated by both Glauber and Color Glass Condensate Monte-Carlo codes. In addition initial state configurations with varying amounts of fluctuations generated by a dynamic transport approach are analyzed to test the sensitivity of the procedure. The results allow for a full characterization of the initial state structures that is useful to discriminate the different initial state models currently in use.

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Ultra-Relativistic nearly-ideal fluid dynamics has proven to be a very successful tool for modeling the bulk dynamics of the hot dense matter formed during a heavy ion collision \cite{1,2,3,4,5,6}. The major uncertainty in determining transport properties of the QGP, such as the ratio of shear viscosity to entropy, lies in the specification of the initial conditions of the collision. The initial conditions have been mainly assumed to be smooth distributions that are parametrized implementations of certain physical assumptions (e.g., Glauber/CGC). Within the last 2 years the importance of including fluctuations in these distributions has been recognized, leading to a whole new set of experimental observations of higher flow coefficients and their correlations \cite{7,8,9,10}. On the theoretical side there has been a lot of effort to refine the previously schematic models with fluctuation inducing corrections and to employ adynamical descriptions of the early non-equilibrium evolution \cite{11,12,13,14}.

Hydrodynamical simulations can take these fluctuations into account by generating an ensemble of runs each with a unique initial condition, so-called event by event simulations. This is in contrast to event averaged simulations where an ensemble of fluctuating initial conditions is generated, and then a single initial condition corresponding to this set’s ensemble average is subject to evolution. Event by event modeling has proven to be essential for correctly describing all the details of the bulk behavior of heavy ion collisions \cite{15,16,17,18,19,20,21,22,23,24,25}.

The two main models for the generation of hydrodynamic initial conditions are the Glauber \cite{26,27,28,12} and color glass condensate (CGC) models \cite{29,30,31,32,33,34}. The Glauber model samples a Woods-Saxon nuclear density distribution for each nucleus.

Color glass condensate models are \textit{ab initio} calculations motivated by the idea of gluon saturation of parton distribution functions at small momentum scales $x$. In CGC models the gluon distribution for each nucleon is computed and the nuclear collision is modeled as interactions between these coherent color fields. Each of these models generates spatial fluctuations whose details depend on the assumptions made in the specific implementation. Glauber fluctuations come from Monte-Carlo (MC) sampling the nuclear density distribution. CGC fluctuations arise similarly with additional contributions from the self interaction of the color fields.

We present a method for generating a 2d decomposition of fluctuations in the initial state energy density of a heavy-ion collision. We apply this framework to the ensembles of initial states generated by UrQMD \cite{35,36,37}, by an MC Glauber code of Qin \cite{12}, and by the MC-KLN code of Drescher and Nara \cite{29,30,38} which is based on CGC ideas. The events compared are generated for Au+Au collisions at $\sqrt{s} = 200$ AGeV at two impact parameters $b = 2, 7$ fm. We introduce summary statistics for the Fourier expansion and show how the radial information exposes clear differences between the fluctuations generated by Glauber and CGC based codes. We note that the cumulant expansion of Teaney and Yan \cite{10} provides an alternative 2d decomposition of the initial state density. The cumulant basis used therein has many appealing properties and the familiar 1d moments can be readily obtained from it. The norms which arise from the expansion we propose below have similar invariance properties and as we shall show very naturally reveal the underlying roughness of the event.

Since colliding nuclei are strongly Lorentz contracted along the beam axis, we neglect the longitudinal dimension. A priori we expect fluctuations in both radial and azimuthal directions. Experiments make measurements in a 2d transverse momentum space, and further analysis of the final state may reveal correlations with quantities
derived from a fully 2d decomposition of the initial state. Recent work [39] has shown that events can be constructed which have identical angular Fourier moments $\epsilon_2, \epsilon_3$ but dramatically different energy distributions leading to different final state flow coefficients. The structure of such events is not sensitive to a purely azimuthal decomposition.

We seek a two dimensional Fourier expansion on a disk of radius $r_0 > 0$. The orthogonality of Bessel functions of the first kind is the key to this decomposition

$$\int_0^{r_0} J_\alpha(\frac{r}{r_0}\lambda_{\alpha,n}) J_\alpha(\frac{r}{r_0}\lambda_{\alpha,n'}) r dr = \frac{r_0^2}{2} \delta_{nn'}|J_{\alpha+1}(\lambda_{\alpha,n})|^2, \forall n, n' \in \mathbb{Z}, \forall \alpha \in \mathbb{R}, \quad (1)$$

where $\lambda_{\alpha,n}$ is the $n^{th}$ positive zero of $J_\alpha(x)$. For integer $\alpha$, $J_\alpha(-x) = (-1)^\alpha J_\alpha(x)$ and so $|J_{\alpha+1}(\lambda_{\alpha,n})|^2 = |J_{|\alpha|+1}(\lambda_{\alpha,n})|^2$ and $\lambda_{\alpha,n} = \lambda_{-\alpha,n}$. It follows that the functions

$$\phi_{m,n}(r, \theta) := \frac{1}{J_{|m|+1}(\lambda_{m,n})} J_m(\frac{r}{r_0}\lambda_{m,n}) e^{im \theta} \quad (2)$$

form a complete orthonormal set on this disk (under the uniform measure $r dr d\theta/\pi r_0^2$). As such any well behaved (square-integrable and vanishing at the boundary of the disk) function $f$ on this disk admits the convergent expansion

$$f(r, \theta) = \sum_{m,n} A_{m,n} \phi_{m,n}(r, \theta), \quad (3)$$

in terms of these basis functions, with generalized Fourier coefficients $A_{m,n} \in \mathbb{C}$ given by

$$A_{m,n} = \frac{1}{\pi r_0^2} \int f(r, \theta) \phi_{m,n}^*(r, \theta) r dr d\theta. \quad (4)$$

A simple scaling law describes the dependence of coefficients $A_{m,n}$ on $r_0$, which thus is arbitrary so long as the support of $f$ is contained in the ball of radius $r_0$. The parameter $r_0$ sets the maximum radial position of features in the event that will be resolved in the decomposition. Throughout the paper we use $r_0 = 10 \, \text{fm}$, sufficient for all the distributions we consider. We center the coordinate system at the center of mass of the distributions $f$.

The basis functions $\phi_{m,n}$ are solutions to Bessel’s equation on the unit disk [40], eigenfunctions of the Laplacian with Dirichlet bc and eigenvalues $-(\lambda_{m,n}/r_0)^2$. Thus for sufficiently smooth $f$,

$$-\nabla^2 f(r, \theta) = \sum_{m,n} \frac{\lambda_{m,n}^2}{r_0^2} A_{m,n} \phi_{m,n}. \quad (5)$$

The values $r_0/\lambda_{m,n}$ constitute characteristic length scales for the associated Fourier components $A_{m,n}$. Higher orders of $m$ and $n$ are associated with larger $\lambda_{m,n}$ (see Fig.1), corresponding to shorter length scales, and are associated with smaller-scale features (see [41, 42] for a similar perspective in 1d). Other boundary conditions (Neumann, for example) lead to similar countable sets of basis functions.

Higher values of the angular indices $\pm m$ correspond to higher numbers of zero crossings in the angular components of basis functions, or “lumpiness” in rotation, while higher values of the radial indices $n$ are associated with more roughness as one moves closer or farther from the center of mass. The first few basis functions
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Figure 1: The eigenvalues $\lambda_{m,n}^2$ of the negative Dirichlet Laplacian on the unit disk are plotted as a function of $|m|$ (angular) and $n$ (radial). These can be interpreted as an inverse length scale for the coefficients $A_{m,n}$. Asymptotically $|\lambda_{m,n}| \approx \pi(n + |m| - 1/4)$ as $n, m \to \infty$.

Figure 2: Plots of the first few real (left) and imaginary (right) components of $\phi_{m,n}(r, \theta)$. The angular coefficient $m \in [0, 5]$ increases from left to right, the radial coefficient $n \in [1, 3]$ increases from bottom to top.

Figure 3: First: A typical UrQMD ensemble average subtracted event. Second: the reconstructed event generated by (3) after applying the decomposition. Third: The absolute values of the Fourier coefficients $|A_{m,n}|$ for this event.

are plotted in Fig. 2. The lumpy shape of the basis functions suggests that this representation will be very useful for characterizing the hot and cold spot structures in the initial state of a heavy ion reaction.

A typical UrQMD event (after subtracting the ensemble average) is shown in
Fig. 3 alongside its decomposition and the resulting Fourier coefficients $A_{m,n}$. Here we characterize the event in terms of coefficients $m \in [-8, 8]$ and $n \in [1, 8]$, sufficient to capture a detailed image of the original initial state distribution (further terms contribute very little additional information because differences between the original and reconstructed distributions are dominated by numerical noise for coefficients with order higher than $|m|_{\text{max}} = 8, n_{\text{max}} = 8$). Let us now explore different ways to extract useful information from this decomposition.

Using the orthogonality of the basis functions along with the Laplacian we can derive simple expressions for norms of the function $f$ to be expanded in the frequency domain. The $L_2(f)$ norm, a measure of the total mass of $f$, is given by Plancherel’s theorem as:

$$L_2(f) := \langle f, f \rangle^{1/2} = \left[ \sum |A_{m,n}|^2 \right]^{1/2}, \quad (5)$$

where $\langle a, b \rangle = \frac{1}{\pi r_0^2} \int_0^{r_0} a(r, \theta) b^*(r, \theta) r dr d\theta$ is the inner product for functions on the disk. The Sobolev $H_1(f)$ norm gives a measure of roughness, or of how ‘wobbly’ the function is across the disk:

$$H_1(f) := \langle (-\ell^2 \nabla^2 + I) f, f \rangle^{1/2}$$

$$= \left[ \sum \left( \frac{\ell^2 A_{m,n}^2}{r_0^2} + 1 \right) |A_{m,n}|^2 \right]^{1/2}, \quad (6)$$

where $\ell$ is a characteristic length scale introduced to maintain unit consistency (we use $\ell = 1 \text{ fm}$). A variation on the Sobolev norm gives the angular variation $M_1(f)$ which quantifies angular gradients,

$$M_1(f) := \langle \partial_\theta^2 f, f \rangle^{1/2} = \left[ \sum m^2 |A_{m,n}|^2 \right]^{1/2}. \quad (7)$$

We use these below to quantify roughness features of the events considered. Note that although the individual coefficients $\{A_{m,n}\}$ are not invariant under translation or rotation of the coordinate system, the quantities $L_2(f)$ and $H_1(f)$ are invariant (and moreover scale simply with changes in length scale $r_0$), while $M_1(f)$ is invariant to rotations.

To illustrate the usefulness of our proposed method, we apply it to three example models: UrQMD, MC-Glauber and MC-KLN, which may be viewed as representative of the models currently in use.

We consider a set of 100 events generated by each code. UrQMD includes Boltzmann hadronic transport before the hydro begins, while the MC-Glauber code includes simple streaming transport of the nucleons after interaction; both of these will introduce added spatial fluctuation in the energy density. The Glauber code also includes KNO scaling of the multiplicity fluctuations per binary collision. For all models the initial condition was computed in a 200 point grid in the transverse plane at the center of the collision along the beam axis. To explore the centrality dependence of the analysis we consider events at two impact parameters $b = 2, 7 \text{ fm}$. All events are generated for Au+Au collisions at $\sqrt{s_{NN}} = 200 \text{ AGeV}$. To study the fluctuations generated by these events the ensemble averaged event is computed for each model and subtracted from each event in the ensemble before applying the decomposition.

We generated a series of ensembles of UrQMD events by averaging successively larger samples of independent raw events together before subtracting out the ensemble
average. See [13] for more details on this process and its influence on the ellipticity and triangularity of the UrQMD initial state.

As the number \( N_a \) of events over which we average increases, the scale of the fluctuations will be diminished at approximate rate \( 1/\sqrt{N_a} \). However this rescaling due to averaging events will not substantially change the shape of any particular fluctuation. We have examined events with \( N_a = \{1, 2, 5, 10, 25\} \). While only the \( N_a = 1 \) case represents the true UrQMD output the diminishing scale fluctuation in this sequence of data sets will be used to illustrate different aspects of roughness in our analysis. Any statistic proposed as a measure of event fluctuations should be invariant under this averaging procedure, which affects all frequencies equally, and instead should be sensitive to smoothing procedures that filter higher frequencies.

![Figure 4: (color online) The ensemble averages of the Sobolev norm \( H_1 \) (left) the angular variation \( M_1 \) (center) and the \( L_2 \) norm (right). The labels on the UrQMD glyphs give the number of averagings used to generate the ensemble members. The MC-Glauber and MC-KLN results are plotted with an artificial offset in \( b \) to permit easier comparison.](image)

In Fig. 4 the ensemble averages of each quantity \((H_1, M_1, L_2)\) are plotted for each set of events. Considering first the UrQMD results it is clear that each tends to zero as \( N_a \) increases as expected, confirming that each is quantifying fluctuations. Fluctuations are lower at the larger impact parameter \( b \) but the relative ordering of the models is preserved. For all norms we see good agreement between the \( N_a = 2 \) UrQMD events and the MC-Glauber code.

The \( L_2 \) norm reflects the mean square fluctuation, the \( H_1 \) norm in addition reflects the mean square spatial gradient of the fluctuations, and the \( M_1 \) semi-norm quantifies mean square angular variation. The MC-KLN results show the largest \( H_1 \) values while having \( M_1 \) values comparable to UrQMD \( N_a = 1 \), the MC-KLN \( L_2 \) is somewhat less than UrQMD. The relative ordering of the Glauber and UrQMD events is the same across all norms. The UrQMD and Glauber events are rather similar in the scale and nature of their fluctuations, UrQMD produces slightly larger fluctuations. The MC-KLN model produces fluctuations on a scale comparable to a hypothetical UrQMD \( N_a = 3/2 \). However the large \( H_1 \) and comparable \( M_1 \) indicates that these events exhibit larger radial gradients than the other models. This may be attributed to the very rapid spatial falloff of the gluon density near the edges of the nucleons.

The norms defined and plotted above, however informative, are clearly quite sensitive to the overall scale of the fluctuations. This is shown most clearly in the separation over \( N_a \) for the UrQMD results in Fig. 4. We propose the following quantity (the “roughness ratio”) as an overall scale invariant measure of the roughness in an
event:

\[ R^2 = \frac{H_1^2}{L_2^2} - 1 = \frac{\langle -\ell^2 \nabla^2 f, f \rangle}{\langle f, f \rangle} = \frac{\ell^2 \sum \lambda_{m,n}^2 |A_{m,n}|^2}{r_0^2 \sum |A_{m,n}|^2}, \quad (8) \]

a weighted average of the scale-free squared characteristic inverse lengths \((\lambda_{m,n} \ell/r_0)^2\), weighted by the squared coefficients \(|A_{m,n}|^2\). This is invariant to rescaling of \(f\) or to Euclidean transformation, but smoothing operations that preferentially reduce high-frequency components of \(f\) will reduce \(R^2\) (but never below its minimum of \((\lambda_{0,1} \ell/r_0)^2 \approx 0.05783\)).

To illustrate that \(R\) truly measures the degree of roughness in an event we have successively applied a finite difference smoothing operator (a five point Laplacian stencil) to the event shown in Fig: 3, in effect melting the event. We present the \(H_1\), \(L_2\) and \(R\) measures resulting from this process in Table: 1. All of the norms decrease with the number of smoothing iterations, however as we have shown above the \(H_1\) and \(L_2\) norms do not exhibit an overall scale invariance which is desirable in an explicit measure of roughness. The \(L_2\) norm gives the most natural measure of the overall scale of fluctuations in an event. The roughness ratio \(R\) provides an explicitly scale invariant measure of the gradients, or roughness, in an event.

| \(N_f\) | \(H_1^2\) | \(L_2^2\) | \(R^2\) |
|--------|--------|--------|--------|
| 0.00   | 3855.68| 1754.63| 1.20  |
| 32.00  | 3275.55| 1547.44| 1.12  |
| 128.00 | 2159.49| 1108.36| 0.95  |
| 1024.00| 244.82 | 173.38 | 0.41  |
| 4096.00| 27.87 | 24.80 | 0.12  |

Table 1: The \(H_1^2\), \(L_2^2\) and \(R^2\) measures of a typical UrQMD event as a function of the number \(N_f\) of filtering passes, using a five-point finite difference smoothing filter. The event we smoothed is shown in Fig: 3.

Figure 5: (color online) The distribution of \(R^2\) (left, right) for UrQMD \(N_a = \{1, 5, 25\}\), MC-KLN and MC-Glauber. The left figure shows events at \(b = 2\) fm, the right figure shows \(b = 7\) fm.

In Fig: 3 we show the distribution of the roughness ratio \(R\) for UrQMD, MC-Glauber and KLN events at two centralities. It is invariant to the overall scale of the fluctuations in the event, the curves for each of the \(N_a\) UrQMD classes fall neatly on top of each other. Further we observe that the MC-Glauber and UrQMD events have
very similar distributions, this is reasonable given that the UrQMD events originate from sampling Woods-Saxon profiles as well. The distributions for the KLN events clearly separate from the Glauber curves. Given the scale invariance of this quantity we conclude that the shape and extent of the fluctuations in KLN events is fundamentally different from that in equivalent Glauber events.

We have presented a new method for characterizing the fluctuations in the initial state of heavy ion collisions. The method is simple and general, it can be applied as easily to theoretical models as to the output of event generators. The $R$ measure we have introduced is invariant under changes in coordinates, rotations and the overall scale of the distributions. We have shown the ability of this measure to quantify the broad differences among the physical models we considered. The radial information included provides additional insights into the nature of fluctuations which are not readily attainable by considering quantities derived from angular decompositions alone.

In future work we will examine how this measure passes through the hydrodynamical evolution to the hadronic final-state of the collision. Even if the observables developed here cannot be measured in detectors they provide a useful basis for apples-to-apples comparison of initial state models.

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Appendix A. Angular Decomposition

To connect our analysis with commonly used quantities we quote in Table A1 the values of $\epsilon_2$ and $\epsilon_3$ computed for events without ensemble average subtraction.

|                  | $b$ (fm) | $E[\langle \epsilon_2 \rangle]$ | $E[\langle \epsilon_3 \rangle]$ |
|------------------|----------|---------------------------------|---------------------------------|
| UrQMD, $N_a = 1$ | 2        | 0.096 ± 0.005                   | 0.079 ± 0.004                   |
| MC-KLN           | 2        | 0.084 ± 0.005                   | 0.046 ± 0.003                   |
| MC-GLAUB         | 2        | 0.089 ± 0.005                   | 0.070 ± 0.004                   |
| UrQMD, $N_a = 1$ | 7        | 0.271 ± 0.010                   | 0.117 ± 0.006                   |
| MC-KLN           | 7        | 0.343 ± 0.010                   | 0.111 ± 0.006                   |
| MC-GLAUB         | 7        | 0.231 ± 0.009                   | 0.099 ± 0.006                   |

Table A1: The ensemble average values of $\langle \epsilon_2 \rangle$ and $\langle \epsilon_3 \rangle$ for each of the models at impact parameters $b = 2, 7$ fm.

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