Mean-Field Reflected BSDEs: the general Lipschitz case

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Abstract
In [9], a mean-field type reflected backward stochastic differential equation was formulated, motivated by applications in pricing life insurance contracts with surrender options. The uniqueness and existence result was established through a fixed point method when the driver is independent of the second unknown z. The existence result alone was generalized using a penalization method under some additional assumptions. In this note, we develop a new fixed point method to establish existence as well as uniqueness of the solution of the mean-field reflected BSDEs, removing these additional regularity assumptions.

Key words: mean-field, reflected BSDEs, fixed point method

MSC-classification: 60H10, 60H30

1 Introduction
Let (Ω, F, P) be a given complete probability space under which B is a d-dimensional standard Brownian motion. Suppose (F_t)_{0≤t≤T} is the natural filtration generated by B augmented by the P-null sets and P the corresponding sigma algebra of progressive sets of Ω × [0, T]. This paper is devoted to the study of the following mean-field type reflected backward stochastic differential equations (BSDEs):

$$\begin{aligned}
&Y_t = \xi + \int_t^T f(s, Y_s, P_{Y_s}, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t, \quad 0 \leq t \leq T, \\
&Y_t \geq h(t, Y_t, P_{Y_t}), \quad \forall t \in [0, T] \quad \text{and} \quad \int_0^T (Y_t - h(t, Y_t, P_{Y_t}))dK_t = 0,
\end{aligned}$$

(1)

where P_{Y_t} is the marginal probability distribution of the process Y at time t, the terminal condition ξ is a scalar-valued F_T-measurable random variable, the driver f : Ω × [0, T] × R × P_1(R) × R^d → R and the constraint h : Ω × [0, T] × P_1(R) × R → R are progressively measurable maps with respect to F × B(R) × B(P_1(R)) × B(R^d) and P × B(P_1(R)) × B(R) respectively.

It is well known that El Karoui et al. [10] introduced the following reflected BSDE

$$\begin{aligned}
&Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t, \quad 0 \leq t \leq T, \\
&Y_t \geq L_t, \quad \forall t \in [0, T] \quad \text{and} \quad \int_0^T (Y_t - L_t)dK_t = 0,
\end{aligned}$$

(2)

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in which the obstacle \( L \) is a given stochastic process. Great progress has since then been made in this field, as it has rich connections with obstacle problems of partial differential equations and American option pricing (see [11]). In particular, the term \( Y \) can be seen as a solution of an optimal stopping problem

\[
Y_t = \underset{\text{stopping time} \geq t}{\text{ess sup}} E_t \left[ \eta 1_{\{\tau = T\}} + L_\tau 1_{\{\tau < T\}} + \int_t^\tau f(s, Y_s, Z_s) ds \right], \forall t \leq T.
\]

(3)

For more details on this topic, we refer the reader to [1, 8, 12–14] and the references therein.

Recently, in order to study partial hedging of financial derivatives, various mean-field type reflected BSDEs were introduced, in which the driver \( f \) and the obstacle \( h \) may depend on the law of the term \( Y \). For example, Briand, Elie and Hu [6] considered BSDEs with mean reflection to study the super-hedging problem under running risk management constraint. We refer the reader to [2–4, 7, 15] and the references therein for some other important contributions.

In particular, motivated by applications in pricing life insurance contracts with surrender options, Djehiche, Elie and Hamadène formulated in [9] mean-field reflected BSDEs of the form (1). When the driver \( f \) is independent of the second unknown \( z \), they used a fixed point method to prove the existence and uniqueness result for mean-field reflected BSDEs similar to (1) via the Snell envelope representation (3):

\[
\Gamma(\Upsilon)_t = \underset{\text{stopping time} \geq t}{\text{ess sup}} E_t \left[ \xi 1_{\{\tau = T\}} + h(\tau, \Upsilon_\tau, (P_{\Upsilon_s})_{s=\tau}) 1_{\{\tau < T\}} + \int_t^\tau f(s, \Upsilon_s, P_{\Upsilon_s}) ds \right], \forall t \leq T.
\]

(4)

On the other hand, under some additional assumptions, they applied a penalization method to obtain the existence of a solution when the driver \( f \) also depends on the second unknown \( z \). More precisely, they used a global domination condition in the \( z \) component and assumed that

\[
f(s, Y_s, P_{Y_s}, Z_s) = F(s, Y_s, E[Y_s], Z_s), \quad h(t, Y_t, P_{Y_t}) = H(t, Y_t, E[Y_t]),
\]

where \( F \) and \( H \) are non-decreasing with respect to \( E[Y_s] \). Our aim is to establish the existence as well as the uniqueness of the solution to the mean-field reflected BSDE [11] without these additional assumptions. The key point of our method is based on the following representation result for the reflected BSDE (4):

\[
Y_t = \underset{\text{stopping time} \geq t}{\text{ess sup}} y^T_{\tau}, \forall t \leq T,
\]

(5)

where \( y^T_{\tau} \) is the solution to the following standard BSDE:

\[
y^T_{\tau} = \eta 1_{\{\tau = T\}} + L_\tau 1_{\{\tau < T\}} + \int_t^\tau f(s, y^T_s, z^T_s) ds - \int_t^\tau z^T_s dB_s.
\]

Note that equation (5) does not explicitly involve the term \( Z \), which differs from the Snell envelope representation (3). With help of (4), we construct a solution map \( \Gamma \) when the driver \( f \) depends on the second unknown \( z \). By some technical computations, we prove the existence and uniqueness of the solution to mean-field reflected BSDEs of the form (4). In conclusion, we develop a fixed point method to deal with mean-field reflected BSDEs which gives some extension of the result from [9] to the case where the driver depends on the second unknown \( z \).

The paper is organized as follows. In section 2, we state the main result concerning the existence and uniqueness of solutions of mean-field reflected BSDE [11]. Section 3 is devoted to some technical lemmas and to the proof of the main result.
Notation.

For each Euclidian space, we denote by $\langle \cdot, \cdot \rangle$ and $| \cdot |$ its scalar product and the associated norm, respectively. Then, for each $p \geq 1$, we consider the following collections:

- $\mathcal{L}^p$ is the collection of real-valued $\mathcal{F}_T$-measurable random variables $\xi$ satisfying $E[|\xi|^p] < \infty$;
- $\mathcal{H}^{p,d}$ is the collection of $\mathbb{R}^d$-valued $\mathcal{F}$-progressively measurable processes $(z_t)_{0 \leq t \leq T}$ satisfying
  $$E \left[ \left( \int_0^T |z_t|^2 dt \right)^{\frac{p}{2}} \right] < \infty;$$
- $\mathcal{S}^p$ is the collection of real-valued $\mathcal{F}$-adapted continuous processes $(y_t)_{0 \leq t \leq T}$ satisfying
  $$E \left[ \sup_{0 \leq t \leq T} |y_t|^p \right] < \infty;$$
- $\mathcal{A}^p$ is the collection of continuous non-decreasing processes $(K_t)_{0 \leq t \leq T} \in \mathcal{S}^p$ with $K_0 = 0$;
- $\mathcal{P}_p(\mathbb{R})$ is the collection of all probability measures over $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with finite $p$th moment, endowed with the $p$-Wasserstein distance $W_p$;
- $\mathcal{T}_t$ is the collection of $\mathcal{F}$-stopping times $\tau$ such that $\tau \geq t$ $\mathbf{P}$-a.s..

We denote by $\mathcal{H}^{p,d}_{[a,b]}, \mathcal{S}^p_{[a,b]}$ and $\mathcal{A}^p_{[a,b]}$ the corresponding collections for the stochastic processes with time indexes on $[a,b]$.

## 2 The main result

In this section, we study the solvability of the mean-field reflected BSDE (1). In what follows, we make use of the following conditions on the terminal condition $\xi$, the driver $f$ and the constraint $h$.

**H1** There exists a constant $p > 1$ such that the terminal condition $\xi \in \mathcal{L}^p$ with $\xi \geq h(T, \xi, \mathbf{P}_T)$.

**H2** The process $f(t, 0, 0, 0)$ belongs to $\mathcal{H}^{p,1}$ and there exists a constant $\lambda > 0$ such that for any $t \in [0, T]$, $y_1, y_2 \in \mathbb{R}$, $\nu_1, \nu_2 \in \mathcal{P}_1(\mathbb{R})$ and $z_1, z_2 \in \mathbb{R}^d$

$$|f(t, y_1, \nu_1, z_1) - f(t, y_2, \nu_2, z_2)| \leq \lambda (|y_1 - y_2| + W_1(\nu_1, \nu_2) + |z_1 - z_2|).$$

**H3** The process $h(t, y, \nu)$ belongs to $\mathcal{S}^p$ for any $y \in \mathbb{R}$, $\nu \in \mathcal{P}_1(\mathbb{R})$ and there exist two constants $\gamma_1, \gamma_2 > 0$ such that for any $t \in [0, T]$, $y_1, y_2 \in \mathbb{R}$, $\nu_1, \nu_2 \in \mathcal{P}_1(\mathbb{R})$

$$|h(t, y_1, \nu_1) - h(t, y_2, \nu_2)| \leq \gamma_1 |y_1 - y_2| + \gamma_2 W_1(\nu_1, \nu_2).$$

**Definition 2.1** *By a solution to (1), we mean a triple of progressively measurable processes $(Y, Z, K)$ in the product space $\mathcal{S}^p \times \mathcal{H}^{p,d} \times \mathcal{A}^p$ such that (1) holds.*

We are now ready to state the main result of this paper.

**Theorem 2.2** *Assume that (H1)-(H3) are satisfied. If $\gamma_1$ and $\gamma_2$ satisfy

$$\left( \frac{p+1}{p-1} \right)^{\frac{1}{p-1}} \left( \left( \frac{p}{p-1} \right)^p \gamma_1 + \gamma_2 \right) < 1,$$

then the mean-field reflected BSDE (1) admits a unique solution $(Y, Z, K) \in \mathcal{S}^p \times \mathcal{H}^{p,d} \times \mathcal{A}^p$.*

**Remark 2.3** Note that the enhanced sufficient condition (1) is crucial for Theorem 2.2 as in [8]. Theorem 3.1. In particular, the condition (1) does not depend on the Lipschitz constant of $f$ with respect to the second unknown $z$. 

3
3 The proof

In order to prove Theorem 2.2, we need to state some technical results on the representation of solutions of BSDEs. For each $\mathcal{F}$-stopping time $\tau$ taking values in $[0, T]$ and for every $\mathcal{F}_\tau$-measurable function $\eta \in \mathcal{L}^p$, we first define the following map:

$$E^\eta_{t, \tau} := Y_t, \quad \forall t \in [0, T],$$

where $Y$ is the solution to the following standard BSDE on the random time horizon $[0, \tau]$

$$Y_t = \eta + \int_t^\tau g(s, Z_s)ds - \int_t^\tau Z_s dB_s.$$  \hfill (7)

Suppose that $g$ satisfies Assumption (H2). It follows from [5] or [16] that the BSDE (7) admits a unique solution $(Y, Z) \in \mathcal{S}^p \times \mathcal{H}^{p.d}$. Moreover, it is easy to check that

$$Y_t = Y_{t\wedge \tau}, Z_t = Z_t 1_{[0, \tau]}(t), \quad \forall t \in [0, T].$$

Lemma 3.1 Suppose that $\eta^i \in \mathcal{L}^p$ is $\mathcal{F}_\tau$-measurable and the driver $g^i$ satisfies the Assumption (H2), for $i = 1, 2$. Then for each $t \in [0, T]$ and $\mu \in (1, p]$

$$|E^\eta_{t, \tau}[\eta^i]| \leq \exp \left( \frac{\lambda^2}{2(\mu - 1)}(T - t) \right) E_t \left[ \left( |\eta^i| + \int_t^\tau |g^i(s, 0)|ds \right)^\mu \right]^{\frac{1}{\mu}},$$

$$|E^{\eta^i}_{t, \tau}[\eta^i] - E^{\eta^2}_{t, \tau}[\eta^2]| \leq \exp \left( \frac{\lambda^2}{2(\mu - 1)}(T - t) \right) E_t \left[ \left( |\eta^i - \eta^2| + \int_t^\tau |g^1(s, 1) - g^2(s, 0)|ds \right)^\mu \right]^{\frac{1}{\mu}}.$$

Proof. We only give the proof for the second inequality, since the first one can be proved in a similar way. Let $(Y^1, Z^1)$ be the solution to BSDE (7) corresponding to the terminal condition $\eta^i$, $i = 1, 2$. For each $t \in [0, T]$, denote by

$$\beta_t = \frac{g^1(t, Z^1_t) - g^1(t, Z^2_t)}{|Z^1_t - Z^2_t|^2} (Z^1_t - Z^2_t) 1_{\{|Z^1_t - Z^2_t| \neq 0\}}.$$  

Then, the pair of processes $(Y^1 - Y^2, Z^1 - Z^2)$ solves the following BSDE:

$$Y_t^1 - Y_t^2 = \eta^1 - \eta^2 + \int_t^\tau \left( \beta_s (Z^1_s - Z^2_s) \right)^\top + g^1(s, Z^1_s) - g^2(s, Z^2_s) 1_{[0, \tau]}(s)ds - \int_t^\tau 1_{[0, \tau]}(s)(Z^1_s - Z^2_s)dB_s.$$  \hfill (8)

Note that $\tilde{B}_t := B_t - \int_0^t \beta_s 1_{[0, \tau]}(s)ds$, defines a Brownian motion under the equivalent probability measure $\tilde{P}$ given by $d\tilde{P} := E(\beta \cdot B)^T_0 d\mathcal{P}$ with

$$E(\beta \cdot B)^T_t = \exp \left( \int_t^\tau \beta_s 1_{[0, \tau]}(s)dB_s - \frac{1}{2} \int_t^\tau |\beta_s|^2 1_{[0, \tau]}(s)ds \right), \quad 0 \leq t \leq T.$$  \hfill (9)

It follows that for every $t \in [0, T]$

$$Y_t^1 - Y_t^2 = E_{\tilde{P}} \left[ \eta^1 - \eta^2 + \int_t^\tau \left( g^1(s, Z^2_s) - g^2(s, Z^2_s) \right) 1_{[0, \tau]}(s)ds \right]$$

$$= E_t \left[ E(\beta \cdot B)^T_t \left( \eta^1 - \eta^2 + \int_t^\tau \left( g^1(s, Z^2_s) - g^2(s, Z^2_s) \right) 1_{[0, \tau]}(s)ds \right) \right].$$
Noting that $|\beta_t| \leq \lambda$ and by a standard computation, we have that for any $q \geq 1$,

$$
E_t\left[|\mathcal{E}(\beta \cdot B)|^q\right] \leq \exp\left(\frac{\lambda^2}{2}(q^2 - q)(T-t)\right).
$$

In view of Hölder’s inequality, we have for any $\mu \in (1, p]$

$$
|Y_t^1 - Y_t^2| \leq \exp\left(\frac{\lambda^2}{2(\mu - 1)}(T-t)\right)E_t\left[\left(|\eta^1 - \eta^2| + \int_t^T |g^1(s, Z_s^2) - g^2(s, Z_s^2)|ds\right)^\mu\right]^{\frac{1}{\mu}}.
$$

The proof is complete. \(\blacksquare\)

Next, we introduce a representation result for the solution of the following reflected BSDE:

$$
\begin{align*}
Y_t &= \eta + \int_t^T g(s, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t, \quad 0 \leq t \leq T, \\
Y_t &\geq L_t, \quad \forall t \in [0, T] \quad \text{and} \quad \int_0^T (Y_t - L_t)dK_t = 0,
\end{align*}
$$

(8)

where $\eta \in \mathcal{L}^p$, $L \in \mathcal{S}^p$ with $L_T \leq \xi$ and the driver $g$ satisfies Assumption (H2). It follows from [10] or [12] that the reflected BSDE (8) admits a unique solution $(Y, Z, K) \in \mathcal{S}^p \times \mathcal{H}^{p,d} \times \mathcal{A}^p$.

**Lemma 3.2** For each $t \in [0, T]$, the solution $Y_t$ to the reflected BSDE (8) satisfies the following property:

$$
Y_t = \text{ess sup}_{\tau \in \mathcal{T}_t} \mathcal{E}^q_{t, \tau}[\eta 1_{\{\tau = T\}} + L_\tau 1_{\{\tau < T\}}].
$$

**Proof.** For any $\tau \in \mathcal{T}_t$, we have

$$
Y_s = Y_\tau + \int_s^\tau g(r, Z_r)dr - \int_s^\tau Z_r dB_r + K_\tau - K_\tau, \quad \forall s \in [t, \tau].
$$

Note that $Y_\tau \geq \eta 1_{\{\tau = T\}} + L_\tau 1_{\{\tau < T\}}$ and $K$ is a non-decreasing process. It follows from a comparison theorem on BSDEs that

$$
Y_t \geq \mathcal{E}^q_{t, \tau}[\eta 1_{\{\tau = T\}} + L_\tau 1_{\{\tau < T\}}], \quad \forall \tau \in \mathcal{T}_t.
$$

On the other hand, we define the stopping time $\tau^* = \inf\{\tau \in [t, T] : Y_\tau = L_\tau\} \wedge T$. Since $Y_\tau \geq L_\tau$ and $\int_t^\tau (Y_\tau - L_\tau)dK_\tau = 0$, we conclude that $K_{\tau^*} = K_t$, which indicates that

$$
Y_s = Y_{\tau^*} + \int_s^{\tau^*} g(r, Z_r)dr - \int_s^{\tau^*} Z_r dB_r, \quad \forall s \in [t, \tau^*].
$$

Note that $Y_{\tau^*} = \xi 1_{\{\tau^* = T\}} + L_{\tau^*} 1_{\{\tau^* < T\}}$ by the definition of $\tau^*$. It follows that

$$
Y_t = \mathcal{E}^q_{t, \tau^*}[\eta 1_{\{\tau^* = T\}} + L_{\tau^*} 1_{\{\tau^* < T\}}],
$$

which completes the proof. \(\blacksquare\)

**Remark 3.3** Unlike the classical Snell envelope method, we establish a representation result for the solution of the reflected BSDE in Lemma 3.2, which enables us to construct a contraction map to solve mean-field reflected BSDEs of the form (1) when the driver $f$ also depends on the second unknown $z$.

We are now ready to prove Theorem 2.2. More precisely, we first state the existence and uniqueness of the solution on a small time interval $[T - h, T]$, in which $h$ is to be determined later. Then, we stitch the local solutions to build the global solution.
Let us start by defining a map \( \Gamma \) on the space \( S^p_{[T-h,T]} \); given for each \( U \in S^p_{[T-h,T]} \) by

\[
\Gamma(U)_t := \text{ess sup}_{\tau \in T_t} \mathcal{E}^U_{t,\tau} \left[ 1_{\{\tau = T\}} + h(\tau, U_\tau, (Pu_\tau)_s = \tau) 1_{\{\tau < T\}} \right], \quad \forall t \in [T-h,T],
\]

where the driver \( f^U \) is given by \( f^U(t, z) := f(t, U_1, Pu_1, z) \). According to Assumption (H3), it is obvious that \((h(s, U_\tau, Pu_\tau))_{s \in [T-h,T]} \in S^p_{[T-h,T]} \). It follows from Lemma \ref{lem:existence_of_solution} that \( \Gamma(U) \) is the \( S^p \)-solution to the reflected BSDE \ref{eq:BSDE} with data \( (\eta, g, L) = (\xi, f^U, h(\cdot, U_\cdot, Pu_\cdot)) \). Thus for any \( h \in (0, T] \), we have

\[
\Gamma \left( S^p_{[T-h,T]} \right) \subset S^p_{[T-h,T]}.
\]

Next, we show the uniqueness and existence of the local solution.

**Lemma 3.4** Assume that (H1)-(H3) hold. If \( \gamma_1 \) and \( \gamma_2 \) satisfy \ref{eq:gamma1_gamma2}, then there exists a constant \( \delta > 0 \) depending only on \( p, \lambda, \gamma_1 \) and \( \gamma_2 \) such that for any \( h \in (0, \delta) \), the mean-field reflected BSDE \ref{eq:BSDE} admits a unique solution \((Y, Z, K) \in S^p_{[T-h,T]} \times \mathcal{H}^d_{[T-h,T]} \times \mathcal{A}^p_{[T-h,T]} \) on the time interval \([T-h, T]\).

**Proof.** The proof will be divided into two steps.

**Step 1 (The contraction).** Let \( U^i \in S^p_{[T-h,T]} \), \( i = 1, 2 \). In view of Lemma \ref{lem:continuity_of_Gamma}, we conclude that for any \( t \in [T-h, T] \), \( \tau \in T_t \) and \( \mu \in (1, p) \),

\[
\left| \mathcal{E}^U_{t,\tau} 1_{\{\tau = T\}} + h(\tau, U^1_\tau, (Pu^1_\tau)_s = \tau) 1_{\{\tau < T\}} \right| - \mathcal{E}^{U^2}_{t,\tau} 1_{\{\tau = T\}} + h(\tau, U^2_\tau, (Pu^2_\tau)_s = \tau) 1_{\{\tau < T\}} \right|^{\mu} \leq \exp \left( \frac{p\lambda^2 h}{2(\mu - 1)} (T-t) \right) \mathcal{E}^U_{t,\tau} \left[ \left| h(\tau, U^1_\tau, (Pu^1_\tau)_s = \tau) - h(\tau, U^2_\tau, (Pu^2_\tau)_s = \tau) \right| + \int_t^T \left| f^{U^1}(s, Z^1_s) - f^{U^2}(s, Z^2_s) \right| \, ds \right]^{\mu/\gamma}
\]

which implies the following,

\[
|\Gamma(U^1)_t - \Gamma(U^2)_t|^\mu \leq \exp \left( \frac{p\lambda^2 h}{2(\mu - 1)} \right) \mathcal{E}^U_{t,\tau} \left[ \left( \gamma_1 + \lambda h \right) \sup_{s \in [T-h,T]} \left| U^1_{s+} - U^2_{s+} \right| + \left( \gamma_2 + \lambda h \right) \sup_{s \in [T-h,T]} \mathbb{E}[\left| U^1_{s+} - U^2_{s+} \right|] \right]^{\mu/\gamma}.
\]

The convexity inequality \((ax + by)^p \leq (a + b)^{p-1}(ax^p + by^p)\) holds for any non-negative constants \(a, b, x, y\) and \( p \geq 1 \). It follows that

\[
\mathcal{E}^U_{t,\tau} \left[ \left( \gamma_1 + \lambda h \right) \sup_{s \in [T-h,T]} \left| U^1_{s+} - U^2_{s+} \right| + \left( \gamma_2 + \lambda h \right) \sup_{s \in [T-h,T]} \mathbb{E}[\left| U^1_{s+} - U^2_{s+} \right|] \right]^{\mu/\gamma}
\]

\[
\leq (\gamma_1 + \gamma_2 + 2\lambda h)^{\frac{p}{\mu-1}} \left( \gamma_1 + \lambda h \right) \mathcal{E}^U_{t,\tau} \left[ \sup_{s \in [T-h,T]} \left| U^1_{s+} - U^2_{s+} \right|^\mu \right] + (\gamma_2 + \lambda h) \sup_{s \in [T-h,T]} \mathbb{E}[\left| U^1_{s+} - U^2_{s+} \right|^\mu]^{\frac{1}{\mu}}
\]

\[
\leq (\gamma_1 + \gamma_2 + 2\lambda h)^{\mu-1} \left( \gamma_1 + \lambda h \right) \mathcal{E}^U_{t,\tau} \left[ \sup_{s \in [T-h,T]} \left| U^1_{s+} - U^2_{s+} \right|^\mu \right] + (\gamma_2 + \lambda h) \sup_{s \in [T-h,T]} \mathbb{E}[\left| U^1_{s+} - U^2_{s+} \right|^\mu].
\]
Recalling (9) and applying Doob’s maximal inequality, we derive
\[
E \left[ \sup_{t \in [T-h, T]} |\Gamma(U^1)_t - \Gamma(U^2)_t|^p \right] \leq \exp \left( \frac{p\lambda^2 h}{2(\mu - 1)} \right) (\gamma_1 + \gamma_2 + 2\lambda h)^{p-1} \\
\times \left( (\gamma_1 + \lambda h) \left( \frac{p}{p-\mu} \right)^{\frac{p}{p-1}} E \left[ \sup_{s \in [T-h, T]} |U^1_s - U^2_s|^p \right] + (\gamma_2 + \lambda h) \sup_{s \in [T-h, T]} E[|U^1_s - U^2_s|^p] \right).
\]
Consequently, for any \( \mu \in (1, p) \) and \( h \in (0, (\mu - 1)^2] \), we have
\[
E \left[ \sup_{t \in [T-h, T]} |\Gamma(U^1)_t - \Gamma(U^2)_t|^p \right] \leq \Lambda(\mu) E \left[ \sup_{s \in [T-h, T]} |U^1_s - U^2_s|^p \right]^{\frac{1}{p}}
\]
with
\[
\Lambda(\mu) = \exp \left( \frac{\lambda^2(\mu - 1)}{2} \right) (\gamma_1 + \gamma_2 + 2\lambda(\mu - 1)^2)^{\frac{p-1}{p}} \left( (\gamma_1 + \lambda(\mu - 1)^2) \left( \frac{p}{p-\mu} \right)^{\frac{p}{p-1}} + (\gamma_2 + \lambda(\mu - 1)^2) \right)^{\frac{1}{p}}.
\]
Under assumption (6), we can then find a small enough constant \( \mu^* \in (1, p) \) depending only on \( p, \lambda, \gamma_1 \) and \( \gamma_2 \) such that \( \Lambda(\mu^*) < 1 \). Now we define
\[
\delta := (\mu^* - 1)^2.
\]
(10)
It is obvious that \( \Gamma \) is a contraction map on the time interval \([T-h, T] \) for any \( h \in (0, \delta] \).

**Step 2 (Uniqueness and existence).** Note that any solution \( Y \) to the mean-field reflected BSDE (11) is a fixed point of the map \( \Gamma \). For any \( h \in (0, \delta] \), \( \Gamma \) has a unique fixed point \( Y \in S_{[T-h, T]}^p \), so that
\[
Y_t = \text{ess sup}_{\tau \in T} E_{\tau, \tau}^{f^p, Y} [1_{\tau = T} + h(\tau, Y, (P_{Y_s})_{s=\tau})1_{\tau < T}], \quad \forall t \in [T-h, T].
\]
On the other hand, the reflected BSDE (8) with data \((\eta, g, L) = (\xi, f^p, h(\cdot, Y, (P_Y)_{\cdot}))\) admits a unique solution
\[
(\tilde{Y}, Z, K) \in S_{[T-h, T]}^p \times \mathcal{H}_{[T-h, T]}^{p, d} \times \mathcal{A}_{[T-h, T]}^p.
\]
It follows from Lemma 3.2 that \( \tilde{Y} = \Gamma(Y) = Y \), which implies that \((Y, Z, K)\) is a solution to the mean-field reflected BSDE (11) on the time interval \([T-h, T] \).

Let us now turn to the proof of uniqueness. Suppose \((Y', Z', K')\) is also a solution to the mean-field reflected BSDE (11) on the time interval \([T-h, T] \). In the spirit of Lemma 3.2, \( Y' \) is the fixed point of the map \( \Gamma \), which indicates that \( Y = Y' \). Applying Itô’s formula to \(|Y - Y'|^2\) yields that \( Z = Z' \) and then \( K = K' \). This completes the proof. \( \blacksquare \)

Now we are in a position to complete the proof of the main result.

**Proof of Theorem 2.2.** The uniqueness of the global solution on \([0, T] \) is inherited from the uniqueness of local solution on each small time interval. It suffices to prove the existence.

By Lemma 3.4 there exists a constant \( \delta > 0 \) depending only on \( p, \lambda, \gamma_1 \) and \( \gamma_2 \), such that the mean-field reflected BSDE (11) admits a unique solution
\[
(Y^1, Z^1, K^1) \in S_{[T-\delta, T]}^p \times \mathcal{H}_{[T-\delta, T]}^{p, d} \times \mathcal{A}_{[T-\delta, T]}^p.
\]
on the time interval $[T - \delta, T]$. Next, taking $T - \delta$ as the terminal time and applying Lemma 3.4 again, the mean-field reflected BSDE (1) admits a unique solution $(Y^2, Z^2, K^2) \in S_{[T-2\delta,T-\delta]}^p \times H_{[T-2\delta,T-\delta]}^{p,d} \times A_{[T-2\delta,T-\delta]}^p$ on the time interval $[T - 2\delta, T - \delta]$. Denote by

$$Y_t = \sum_{i=1}^{2} Y^i_t 1_{[T-i\delta,T-(i-1)\delta]} + Y^1_T, \quad Z_t = \sum_{i=1}^{2} Z^i_t 1_{[T-i\delta,T-(i-1)\delta]} + Z^1_T 1_{[T,T]}, \quad K_t = K^2_t 1_{[T-i\delta,T-(i-1)\delta]} + (K^2_T + K^1_T) 1_{[T-\delta,T]}.$$ 

It is easy to check that $(Y, Z, K) \in S_{[T-2\delta,T]}^p \times H_{[T-2\delta,T]}^{p,d} \times A_{[T-2\delta,T]}^p$ is a solution to the mean-field reflected BSDE (1). Repeating this procedure, we get a global solution $(Y, Z, K) \in S^p \times H^{p,d} \times A^p$. The proof of the theorem is complete.

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