Central limit theorems for discretized occupation time functionals

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Abstract
The approximation of integral type functionals is studied for discrete observations of a continuous Itô semimartingale. Based on novel approximations in the Fourier domain, central limit theorems are proved for $L^2$-Sobolev functions with fractional smoothness. An explicit $L^2(\mathbb{P})$-lower bound shows that already lower order quadrature rules, such as the trapezoidal rule and the classical Riemann estimator, are rate optimal, but only the trapezoidal rule is efficient, achieving the minimal asymptotic variance.

Keywords: occupation time, semimartingale, integral functionals, Sobolev spaces, lower bound

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1. Introduction

For $T > 0$ let $X = (X_t)_{0 \leq t \leq T}$ be an $\mathbb{R}^d$-valued continuous Itô semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$. Consider the approximation of the occupation time functional

$$\Gamma_t(f) = \int_0^t f(X_r) \, dr, \quad 0 \leq t \leq T,$$

for a function $f$ from discrete observations $X_{t_k}$ at equidistant times $t_k = k\Delta_n$, where $\Delta_n = T/n$ and $k \in \{0, \ldots, n\}$. This discretization problem appears naturally in numerical analysis and statistics for stochastic processes. The mathematical challenge is to determine an optimal approximation method.

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and the rate at which convergence takes place. A canonical choice is the Riemann estimator
\[ \hat{\Gamma}_{t,n}(f) = \Delta_n \sum_{k=1}^{[t/\Delta_n]} f(X_{t_{k-1}}), \]
which has been analysed in several recent papers to obtain weak and strong \( L^p(\mathbb{P}) \)-approximations of \( \Gamma_t(f) \) for non-smooth functions \( f \) with rates of convergence depending on the properties of the process \( X \) and on the regularity of \( f \), cf. [3, 4, 10, 19]. Central limit theorems for occupation and local times have been shown in [13].

For smooth \( f \) more can be said, because then the process \( t \mapsto f(X_t) \) is again a continuous Itô semimartingale. It is well-known (e.g., [12]) that in this case the weak approximation error is of order \( O(\Delta_n) \). A central limit theorem for \( \Delta_n^{-1}(\Gamma_{\Delta_n[t/\Delta_n]}(f) - \hat{\Gamma}_{t,n}(f)) \) was obtained in [16, Chapter 6] with the weak limit depending only on \( \nabla f \). This suggests that the central limit theorem might also hold for less smooth functions, but the proof of [16] relies crucially on Itô’s formula and is therefore restricted to \( f \in C^2(\mathbb{R}^d) \).

The goal of this work is to prove a central limit theorem for the Riemann estimator for general Itô processes in \( \mathbb{R}^d \) and under minimal assumptions on the function \( f \) such that \( t \mapsto f(X_t) \) is not necessarily a semimartingale. Related to the classical work of [11] on occupation densities, the central idea is to express the error \( \Gamma_{\Delta_n[t/\Delta_n]}(f) - \hat{\Gamma}_{t,n}(f) \) in terms of the Fourier transform of \( f \) and the complex exponentials \( e^{i\langle u,X_t \rangle} \) for a frequency \( u \in \mathbb{R}^d \).

The analysis of this requires approximating \( X_t \) depending on \( u \) and is inspired by the one-step Euler approximations of [9], applied here to a discretization problem different from the usual piecewise constant approximation in time of semimartingales. The approximation problem is therefore moved to the frequency domain and regularity of \( f \) is measured in the fractional \( L^2 \)-Sobolev sense. Under regularity assumptions on the coefficients of \( X \) and assuming an additive perturbation by an independent random variable \( \xi \) having bounded Lebesgue density we extend the central limit theorem to \( f \in H^s(\mathbb{R}^d) \), \( s \leq 2 \), at the same rate \( \Delta_n \). If \( X \) has independent increments, then this applies to \( f \in H^1(\mathbb{R}^d) \). The proof ideas for the central limit theorem have also been applied in [2] to obtain generalized Itô formulas for functions \( f \) with fractional Sobolev regularity. We consider here only continuous Itô semimartingales, but extensions to more general processes including jumps seem possible.

One might wonder if the rate of convergence \( \Delta_n \) can be improved using different estimators or quadrature rules. From a probabilistic point of view,
a natural estimator is the conditional expectation $E[\Gamma_t(f)|G_n]$, where $G_n = \sigma(X_{t_k} : k \in \{0, \ldots, n\})$ is the sigma field generated by the data. While there is generally no analytic expression for $E[\Gamma_t(f)|G_n]$, it is a classical result in probabilistic numerics that it is given by the trapezoidal rule

$$\hat{\Theta}_{t,n}(f) = \Delta_n \sum_{k=1}^{[t/\Delta_n]} \frac{f(X_{t_{k-1}}) + f(X_{t_k})}{2}$$

if $f$ is the identity function and $X$ is a Brownian motion [8]. We show that the trapezoidal rule also satisfies a central limit theorem at the rate $\Delta_n$. By proving an $L^2(P)$ lower bound on the estimation error when $X$ is a Brownian motion we show that both $\hat{\Gamma}_{t,n}(f)$ and $\hat{\Theta}_{t,n}(f)$ are rate-optimal and that the latter is also efficient in the sense that its asymptotic variance coincides with the minimal $L^2(P)$ estimation error. Related lower bounds for integral functionals for less smooth functions $f$ and local times have been obtained by [3, 5].

The paper is organized as follows. In Section 2 we review the CLT for $f \in C^2(\mathbb{R}^d)$ and extend it in Section 3 to $f$ with fractional Sobolev regularity. Several special cases are studied to explore or relax the used assumptions. Section 4 presents the lower bound. Proofs of the main results are deferred to Section 5.

Let us introduce some notation. $C$ always denotes a positive absolute constant, which may change from line to line. We write $a \lesssim b$ for $a \leq Cb$ and $Y_n = o_P(a_n)$ if $a_n^{-1}Y_n \xrightarrow{p} 0$ as $n \to \infty$ for a sequence of random variables $(Y_n)_{n \geq 1}$ and real numbers $(a_n)_{n \geq 1}$. If $Z^{(n)}$ and $Z$ are stochastic processes on $[0, T]$, then $Z^{(n)}_{t} \xrightarrow{u} Z_t$ means $\sup_{0 \leq t \leq T} |(Z^{(n)}_t - Z_t)| \xrightarrow{P} 0$. Stable convergence in law is denoted by $Z^{(n)}_{t} \xrightarrow{st} Z_t$ and may refer, depending on the context, to convergence at a fixed time $0 \leq t \leq T$ or to functional convergence on the Skorokhod space $D([0, T], \mathbb{R}^d)$. For details on stable convergence the reader is referred to [17].

2. Central limit theorems for $f \in C^2(\mathbb{R}^d)$

Recall (for example from [16]) that the Itô semimartingale $X$ can be realised as

$$X_t = X_0 + \int_0^t b_r dr + \int_0^t \sigma_r dW_r, \quad 0 \leq t \leq T,$$

where $b_r$ and $\sigma_r$ are functions of $r$. If $f \in C^2(\mathbb{R}^d)$, then the conditional expectation $E[\Gamma_t(f)|G_n]$ can be approximated by the trapezoidal rule

$$\hat{\Theta}_{t,n}(f) = \Delta_n \sum_{k=1}^{[t/\Delta_n]} \frac{f(X_{t_{k-1}}) + f(X_{t_k})}{2}$$

which satisfies a central limit theorem at the rate $\Delta_n$. By proving a lower bound on the estimation error when $X$ is a Brownian motion we show that both $\hat{\Gamma}_{t,n}(f)$ and $\hat{\Theta}_{t,n}(f)$ are rate-optimal and that the latter is also efficient in the sense that its asymptotic variance coincides with the minimal $L^2(P)$ estimation error. Related lower bounds for integral functionals for less smooth functions $f$ and local times have been obtained by [3, 5].

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where \( X_0 \) is \( \mathcal{F}_0 \)-measurable, \((W_t)_{0 \leq t \leq T}\) is a standard \( d \)-dimensional Brownian motion, \( b = (b_t)_{0 \leq t \leq T} \) is a locally bounded \( \mathbb{R}^d \)-valued process and \( \sigma = (\sigma_t)_{0 \leq t \leq T} \) is a càdlàg \( \mathbb{R}^{d \times d} \)-valued process, all adapted to \((\mathcal{F}_t)_{0 \leq t \leq T}\).

For \( f \in C^2(\mathbb{R}^d) \) the process \((f(X_t))_{0 \leq t \leq T}\) is again a continuous Itô semi-martingale. The following result is Theorem 6.1.2 in [16], which itself is based on [14].

**Theorem 1.** For \( f \in C^2(\mathbb{R}^d) \) we have as \( n \to \infty \) the stable convergence

\[
\Delta_n^{-1}(\Gamma_{n|t/\Delta_n]}(f) - \hat{\Gamma}_{t,n}(f)) \xrightarrow{st} \frac{f(X_t) - f(X_0)}{2} + \frac{1}{\sqrt{12}} \int_0^t \langle \nabla f(X_r), \sigma_r d\tilde{W}_r \rangle
\]

as processes on the Skorokhod space \( \mathcal{D}([0,T], \mathbb{R}^d) \), where \( \tilde{W} \) is a \( d \)-dimensional Brownian motion, defined on an independent extension of \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\).

The proof of the CLT is based on the decomposition

\[
\Gamma_{n|t/\Delta_n]}(f) - \hat{\Gamma}_{t,n}(f) = M_{t,n}(f) + D_{t,n}(f) + E_{t,n}(f)
\]

with

\[
M_{t,n}(f) = \sum_{k=1}^{[t/\Delta_n]} \int_{t_{k-1}}^{t_k} (f(X_r) - \mathbb{E}[f(X_r)|\mathcal{F}_{t_{k-1}}])dr,
\]

\[
D_{t,n}(f) = \sum_{k=1}^{[t/\Delta_n]} \int_{t_{k-1}}^{t_k} \mathbb{E} \left[ f(X_r) - f(X_{t_{k-1}}) - \frac{f(X_{t_k}) - f(X_{t_{k-1}})}{2} | \mathcal{F}_{t_{k-1}} \right] dr,
\]

\[
E_{t,n}(f) = \frac{\Delta_n}{2} \sum_{k=1}^{[t/\Delta_n]} \mathbb{E}[f(X_{t_k}) - f(X_{t_{k-1}})|\mathcal{F}_{t_{k-1}}].
\]

The proof proceeds by applying a standard CLT for triangular arrays of martingale differences (cf. Theorem IX.7.28 of [13]) to \((M_{t,n}(f))_{0 \leq t \leq T}\), while the limit process of \((E_{t,n}(f))_{0 \leq t \leq T}\) yields the asymptotic bias in (3). For \( f \in C^2(\mathbb{R}^d), (D_{t,n}(f))_{0 \leq t \leq T}\) is shown to be asymptotically negligible by Itô’s formula.

**Remark 2 (Trapezoidal rule).** The trapezoidal rule from (11) satisfies

\[
\hat{\Theta}_{t,n}(f) = \hat{\Gamma}_{t,n}(f) + \Delta_n \frac{f(X_{[t/\Delta_n]} - f(X_0)}{2}.
\]
Since $f(X_{[t/\Delta_n] \Delta_n})$ converges uniformly to $f(X_t)$, the CLT in Theorem 1 provides us also with a functional CLT for $\hat{\Theta}_{t,n}(f)$:

$$
\Delta_n^{-1}(\Gamma_{n[t/\Delta_n]}(f) - \hat{\Theta}_{t,n}(f)) \overset{st}{\rightarrow} \frac{1}{\sqrt{12}} \int_0^t \langle \nabla f(X_r), \sigma_r d\tilde{W}_r \rangle.
$$

(5)

The trapezoidal rule achieves the same rate as the Riemann estimator, but is asymptotically unbiased, as opposed to (3). Moreover, Theorem 11 below shows for a Brownian motion $X$ that the trapezoidal rule is efficient in the sense that it attains the minimal asymptotic variance among all square-integrable estimators for $\Gamma_t(f)$ from the observations $X_{tk}$. For simplicity, we consider in the following only $\hat{\Gamma}_{t,n}(f)$, but results transfer to $\hat{\Theta}_{t,n}(f)$.

3. Central limit theorems for $f \in H^s(\mathbb{R}^d)$

If $f$ is not smooth, then $(f(X_t))_{0 \leq t \leq T}$ is generally not a semimartingale and it is not clear if the strategy from the last section still applies to prove a CLT. Inspired by the observation that the limit process in (3) requires formally only a (weak) derivative for $f$, we aim now at deriving a CLT for functions in the fractional Sobolev space of regularity $s \geq 0$

$$
H^s(\mathbb{R}^d) = \{ f : \| f \|_{L^2} + \| f \|_{H^s} < \infty \}, \quad \| f \|_{H^s}^2 = \int_{\mathbb{R}^d} |\mathcal{F} f(u)|^2 |u|^{2s} du,
$$

where $\mathcal{F} f$ is the Fourier transform, which for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is defined as $\mathcal{F} f(u) = \int_{\mathbb{R}^d} f(x) e^{i u \cdot x} dx$, $u \in \mathbb{R}^d$. We further say that $f \in H^s_{\text{loc}}(\mathbb{R}^d)$, if $f \cdot \varphi \in H^s(\mathbb{R}^d)$ for all smooth and compactly supported $\varphi \in C^\infty_c(\mathbb{R}^d)$. It is well-known that $C^k(\mathbb{R}^d) \subseteq H^k_{\text{loc}}(\mathbb{R}^d)$ for $k \in \mathbb{N}$ and the Sobolev embedding implies $H^s_{\text{loc}}(\mathbb{R}^d) \subseteq C^k(\mathbb{R}^d)$ if $s > d/2 + k$, cf. Section 2.7 of [21].

A key assumption in this section is to consider instead of $X$ the process $X + \xi$ with an independent random variable $\xi$. $L^2(\mathbb{P})$ bounds on the terms in (4) will be obtained from the following simple lemma.

**Lemma 3.** Let $\xi$ be a random variable, independent of the filtration $\mathcal{F}$ with bounded Lebesgue density. Suppose that $h(x) \equiv h(X, x)$, $x \in \mathbb{R}^d$, is a family of random variables such that $x \mapsto h(x)$ is Borel-measurable and such that $\mathbb{P}$-almost surely $h \in L^2(\mathbb{R}^d)$. Then

$$
\mathbb{E}[h^2(\xi)] \lesssim \int_{\mathbb{R}^d} \mathbb{E}[|h(u)|^2] du = (2\pi)^{-d} \int_{\mathbb{R}^d} \mathbb{E}[|\mathcal{F} h(u)|^2] du.
$$
Proof. By the independence of $X$ and $\xi$ we have

$$E[h^2(\xi)] = E[E[h^2(\xi)|X]] \lesssim E\left[\int_{\mathbb{R}^d} h^2(x)dx\right] = \int_{\mathbb{R}^d} E[|h(u)|^2]du, \quad u \in \mathbb{R}^d.$$ 

The claim follows therefore from the Plancherel theorem. \hfill \Box

Introducing the Fourier transform in this way, combined with the shift property of the Fourier transform

$$Ff(X_t + \cdot)(u) = Ff(u)e^{-i(u,X_t)},$$

allows for separating the function $f$ from the process $X$. This decomposition leads naturally to an analysis depending on the fractional Sobolev regularity of $f$ and on the characteristic function of the marginals of $X$. Since the latter are usually not known explicitly, $X$ is approximated depending on the frequency $u$. For this we make the following assumptions, cf. \cite{15}.

**Assumption $S(\alpha;\beta)$.** Let $0 \leq \alpha, \beta \leq 1$. There exists an increasing sequence of stopping times $(\tau_R)_{R \geq 1}$ with $\tau_R \to \infty$ for $R \to \infty$ such that for all $0 \leq s, t \leq T$ with $t + s \leq T$

$$E\left[\sup_{0 \leq r \leq s} \left|\sigma(t+r)\wedge \tau_R - \sigma(t)\wedge \tau_R\right|^2\right] \leq C_s^{2\alpha}, \quad E\left[\sup_{0 \leq r \leq s} \left|b(t+r)\wedge \tau_R - b(t)\wedge \tau_R\right|^2\right] \leq C_s^{2\beta}.$$

Moreover, $\sup_{0 \leq t \leq T} |(\sigma_t\sigma_t^\top)^{-1}| < \infty$ $\mathbb{P}$-almost surely.

The non-degeneracy of $\sigma_t\sigma_t^\top$ is a technical condition and can probably be relaxed (it is not necessary in Theorem \cite{11}). For Itô semimartingales $b$ and $\sigma$ we have $\alpha = \beta = 1/2$, which also allows for non-predictable jumps. Other important examples are $\sigma$ and $b$ with $\alpha$- and $\beta$-Hölder continuous paths and with integrable Hölder constants, for instance with $\alpha < H$ when $\sigma$ is a fractional Brownian motion of Hurst index $0 < H < 1$.

We can now formulate our first main result.

**Theorem 4.** Let $s \geq 1$ and grant Assumption $S(\alpha;\beta)$ with $\alpha > \max(0, 1 - s/2), \beta > 0$. Let $\xi$ be a random variable, independent of the filtration $\mathcal{F}$ with bounded Lebesgue density. Then we have for $0 \leq t \leq T$ and $f \in H^s(\mathbb{R}^d)$ (or $f \in H^s_{loc}(\mathbb{R}^d)$ and $\xi$ bounded) the stable convergence

$$\Delta_n^{-1}\left(\Gamma_{\Delta_n[t/\Delta_n]} (f(\cdot + \xi)) - \widehat{\Gamma}_{t,n} (f(\cdot + \xi))\right) \overset{st}{\to} f(X_t + \xi) - f(X_0 + \xi) + \frac{1}{\sqrt{12}} \int_0^t \langle \nabla f(X_r + \xi), \sigma_r d\widehat{W}_r\rangle,$$

where $\widehat{W}_t = W_{t-n\Delta_n}$.\hfill \Box
as \( n \to \infty \), where \( \widetilde{W} \) is as in Theorem 7.

We conclude that the Riemann estimator satisfies the CLT in Theorem 1 at the optimal rate \( \Delta_n \) for Sobolev-smooth functions. Compared to Theorem 1, here the stable convergence holds only at a fixed time \( t \), because it is difficult to control the term \( D_{t,n}(f) \) in (4) uniformly in \( t \). It is unclear if this can be achieved for Sobolev functions, in general. Note that the compositions \( \nabla f(X_r + \xi) \) are well-defined random variables for \( f \in H^1(\mathbb{R}^d) \), because by independence \( X_r + \xi \) has a Lebesgue density and so \( \nabla f = \nabla \tilde{f} \) almost surely implies \( \nabla f(X_r + \xi) = \tilde{f}(X_r + \xi) \) \( \mathbb{P} \)-almost surely.

Remark 5 (Regularisation by \( \xi \)). The additive perturbation by \( \xi \) is required in the proofs and is essential to our approach of weakening the regularity conditions on \( f \). This is conceptually related to the averaging by noise phenomenon \[7\] by regularising the underlying discretization problem through convolution smoothing. Indeed, as discussed above \( \mu \) is the Lebesgue density of \( \xi \) and \( X_t \) has marginal density \( p_t \), then \( X_t + \xi \) has density \( p_t \ast \mu \), where \( \ast \) denotes the convolution operator. Alternatively, \( f(X_t + \xi) \) corresponds in average to \( f \ast \mu \). These two different points of views have been explored in \[3\] for the approximation of occupation time functionals and \( f \in H^s(\mathbb{R}^d), 0 \leq s \leq 1 \).

Remark 6 (Regularity of \( X \) and \( f \)). Theorem 4 presents a trade-off between the regularity of \( X \) and \( f \in H^s(\mathbb{R}^d) \). For \( s \leq 2 \) the CLT applies as soon as \( \beta > 0 \) and \( \alpha + s/2 > 1 \). This means, the more regular \( \sigma \) is, the less regular \( f \) can be. If \( \sigma \) is a semimartingale (and thus \( \alpha = 1/2 \)), then we need only \( s > 1 \), while for \( s > 2 \) any \( \alpha \) is admissible.

If the characteristic functions of the \( X_t \) are known explicitly, then regularity conditions in the CLT and its proof simplify. For example, for \( X \) with independent increments (and thus with deterministic \( b \) and \( \sigma \)) independence of \( X_0 = \xi \) and \( X - X_0 \) is trivially true. In this case we only need to require \( b \) and \( \sigma \) to be càdlàg functions and the result applies to any \( f \in H^1(\mathbb{R}^d) \).

**Theorem 7.** Suppose that \( X_0 \) has a bounded Lebesgue density. Assume that \( b, \sigma \) are deterministic càdlàg functions and that \( \sup_{0 \leq t \leq T} |(\sigma_t \sigma_t^\top)^{-1}| < \infty \). Then we have for \( 0 \leq t \leq T \) and \( f \in H^1(\mathbb{R}^d) \) the stable convergence

\[
\Delta_n^{-1} \left( \Gamma_{\Delta_n[t/\Delta_n]}(f) - \widehat{\Gamma}_{t,n}(f) \right) \xrightarrow{st} \frac{f(X_t) - f(X_0)}{2} + \frac{1}{\sqrt{12}} \int_0^t \langle \nabla f(X_r), \sigma_r d\widetilde{W}_r \rangle,
\]

as \( n \to \infty \), where \( \widetilde{W} \) is as in Theorem 7.
Since $X_r$ has a Lebesgue density (e.g., by [20]), we can argue as after Theorem 4 that $\nabla f(X_r)$ is a well-defined random variable for $f \in H^1(\mathbb{R}^d)$ which depends only on the equivalence class of $f$ and not on the chosen representative. In dimension $d = 1$ the condition on $X_0$ can be removed.

**Theorem 8.** If $d = 1$, then Theorem 7 applies to any initial value $X_0$.

Next, for integrable $f$ with integrable Fourier transform we obtain a CLT with functional convergence and without $\xi$. For this define the Fourier Lebesgue spaces of regularity $s \geq 0$

$$FL^s(\mathbb{R}^d) = \{ f : \| f \|_{L^1} + \| f \|_{FL^s} < \infty \}, \quad \| f \|_{FL^s} = \int_{\mathbb{R}^d} |\mathcal{F} f(u)| |u|^s \, du.$$

We further say that $f \in FL^s_{loc}(\mathbb{R}^d)$, if $f \cdot \varphi \in FL^s(\mathbb{R}^d)$ for all $\varphi \in C^\infty_c(\mathbb{R}^d)$.

**Example 9.** If $f \in H^s(\mathbb{R}^d)$ and $\mu \in L^2(\mathbb{R}^d)$, then $f \ast \mu \in FL^s(\mathbb{R}^d)$. Moreover, if $f \in H^s_{loc}(\mathbb{R}^d)$ for $s' > s + d/2$, then $f \in FL^s_{loc}(\mathbb{R}^d)$.

By the Fourier inversion formula it follows $FL^1(\mathbb{R}^d) \subseteq C^1(\mathbb{R}^d)$. In this sense the next theorem generalises Theorem 4 without requiring $f \in C^2(\mathbb{R}^d)$.

**Theorem 10.** Let $s \geq 1$ and grant Assumption $S(\alpha;\beta)$ with $\alpha > \max(0, 1 - s/2), \beta > 0$. Then the functional stable convergence in (3) holds for any $f \in FL^s_{loc}(\mathbb{R}^d)$.

### 4. Optimality for Brownian motion

We study next the optimality in the $L^2(\mathbb{P})$-sense for estimating $\Gamma_T(f)$ at the fixed time $T$ for a given $f \in H^1(\mathbb{R}^d)$.

Considering for $\Gamma_T(f)$ all square-integrable estimators, which are measurable with respect to the sigma field $\mathcal{G}_n = \sigma(X_{t_k} : 0 \leq k \leq n)$, shows that the minimal error is

$$\inf_{\hat{\Gamma}} \| \Gamma_T(f) - \hat{\Gamma} \|_{L^2(\mathbb{P})} = \| \Gamma_T(f) - \hat{\Gamma}^* \|_{L^2(\mathbb{P})},$$

which is attained by the conditional expectation $\hat{\Gamma}^* = \mathbb{E}[\Gamma_T(f)|\mathcal{G}_n]$. The asymptotic estimation error can be computed explicitly when $X$ is a Brownian motion.
Theorem 11. Let $X$ be a Brownian motion and let $f \in H^1(\mathbb{R}^d)$. Suppose that $X_0$ has a bounded Lebesgue density or $d = 1$. Then

$$\lim_{n \to \infty} \left( \Delta^{-1}n\|\Gamma T(f) - \hat{\Gamma}^*\|_{L^2(P)} \right) = \mathbb{E} \left[ \frac{1}{12} \int_0^T |\nabla f(X_t)|^2 \, dt \right]^{1/2}.$$ 

In view of Theorems 7 and 8, this means that both $\hat{\Gamma}_{T,n}(f)$ and the trapezoidal rule estimator $\hat{\Theta}_{T,n}(f)$ are rate optimal for $f \in H^1(\mathbb{R}^d)$ when $X$ is a Brownian motion, while $\hat{\Theta}_{T,n}(f)$ is even efficient. In particular, no other quadrature rule for equidistant observation times can achieve a better rate by exploiting higher smoothness of $f$. The minimal asymptotic $L^2(P)$ error corresponds exactly to the asymptotic variance from (5) with respect to $\hat{\Theta}_{T,n}(f)$.

5. Proofs

In the following we rely on well-known properties of the Fourier transform, cf. [1]. For example, for $a \in \mathbb{R}$

$$h \in L^2(\mathbb{R}^d) : \mathcal{F}h(a + \cdot)(u) = \mathcal{F}h(u)e^{-i\langle u, a \rangle},$$

$$h \in H^1(\mathbb{R}^d) : \mathcal{F}(\partial_j h)(u) = iu_j \mathcal{F}h(u), \quad j = 1, \ldots, d.$$ 

5.1. Proof of Theorem 4

Throughout write $Y = X + \xi$, where $\xi$ is independent of $X$ and has a bounded Lebesgue density.

5.1.1. Localization

By a well-known localization procedure (cf. Lemma 4.4.9 in [1]) and Assumption $S(\alpha; \beta)$, it suffices to prove the CLT under the following stronger Assumption.

Assumption $S_{loc}(\alpha; \beta)$. Let $0 \leq \alpha, \beta \leq 1$. Then it holds $P$-a.s.

$$\sup_{0 \leq t \leq T} (|X_t| + |b_t| + |\sigma_t| + |(\sigma_t\sigma^T_t)^{-1}|) \leq C,$$

and for all $0 \leq t, t' \leq T$ with $t + t' \leq T$

$$\mathbb{E} \left[ \sup_{0 \leq r \leq t'} |\sigma_{t+r} - \sigma_t|^2 \right] \leq C(t')^{2\alpha}, \quad \mathbb{E} \left[ \sup_{0 \leq r \leq t'} |b_{t+r} - b_t|^2 \right] \leq C(t')^{2\beta}.$$
When \( f \in H^s_{\text{loc}}(\mathbb{R}^d) \) and \( \xi \) is bounded, this assumption allows us to reduce the argument to \( f \in H^s(\mathbb{R}^d) \). Indeed, let \( \varphi \) be a smooth function with \( \varphi = 1 \) on \( B_{C+C_\xi} = \{ x \in \mathbb{R}^d : |x| \leq C + C_\xi \} \), where \( |\xi| \leq C_\xi \) for a constant \( C_\xi \), and with compact support in \( B_{C+C_\xi+\varepsilon}, \varepsilon > 0 \). If \( f \in H^s_{\text{loc}}(\mathbb{R}^d) \), then \( \tilde{f} = f \varphi \in H^s(\mathbb{R}^d) \) and \( \Gamma_t(f(\cdot+\xi)) = \Gamma_t(\tilde{f}(\cdot+\xi)) \), \( \tilde{\Gamma}_{t,n}(f(\cdot+\xi)) = \Gamma_{t,n}(\tilde{f}(\cdot+\xi)) \).

5.1.2. Approximation results

In this section we prove some useful approximation results for the process \( X \). For \( t, \varepsilon > 0 \) let \( \lfloor t \rfloor_\varepsilon = \lfloor t/\varepsilon \rfloor_\varepsilon, t(\varepsilon) = \max(\lfloor t \rfloor_\varepsilon - \varepsilon, 0) \). \( t(\varepsilon) \) projects \( t \) onto the grid \( \{ 0, \varepsilon, 2\varepsilon, \ldots, \lfloor T/\varepsilon \rfloor_\varepsilon \} \) such that \( t - t(\varepsilon) \leq 2\varepsilon \) and \( t - t(\varepsilon) \geq \varepsilon \land t \). Set

\[
\tilde{X}_t(t') = X_t + b_t(t - t') + \sigma_t(W_t - W_{t'}), \quad 0 \leq t' \leq t.
\]

Lemma 12. Grant Assumption \( S_{\text{loc}}(\alpha;\beta) \). Then we have \( \mathbb{P} \)-a.s. for all \( 0 \leq r, t, t' \leq T, t + r \leq t' \),

\[
(i) \quad \mathbb{E}[\sup_{0 \leq r' \leq r} |X_{t+r'} - X_t|^p |\mathcal{F}_t|^{1/p} \lesssim r^{1/2} \text{ for } p \geq 1,
\]

\[
(ii) \quad \mathbb{E}[\sup_{0 \leq r' \leq r} |X_{t+r'} - \tilde{X}_{t+r'}(t)|^2 |\mathcal{F}_t| \lesssim r^{2\beta+2} + r^{2\alpha+1},
\]

\[
(iii) \quad \mathbb{E}[\sup_{0 \leq r' \leq r} |\tilde{X}_{t'}(t + r') - \tilde{X}_{t'}(t)|^2] \lesssim r + r^{2\beta} + r^{2\alpha}.
\]

Proof. The first two results follow from the conditional Burkholder-Davis-Gundy inequality, applied componentwise, cf. [16, Section 2.1.5], and from Assumption \( S_{\text{loc}}(\alpha;\beta) \). For (iii) write

\[
\tilde{X}_{t'}(t + r') - \tilde{X}_{t'}(t) = X_{t+r'} - X_t + (b_{t+r'} - b_t)(t' - (t + r')) + (\sigma_{t+r'} - \sigma_t)(W_{t'} - W_{t+r'}) - b_{t'} + \sigma_t(W_{t'+r} - W_t),
\]

and conclude by (i) and again Assumption \( S_{\text{loc}}(\alpha;\beta) \).

Lemma 13. Grant Assumption \( S_{\text{loc}}(\alpha;\beta) \). Then the following holds for \( f \in H^1(\mathbb{R}^d) \) and as \( n \to \infty \):

\[
(i) \quad \Delta_n \sum_{k=1}^n \mathbb{E}[|\nabla f(Y_{t_{k-1}})|^2] = O(1),
\]

\[
(ii) \quad \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \mathbb{E}[|\nabla f(Y_{t_{k-1}}), Z_r|^2] \, dr = O(\Delta_n) \text{ for } Z_r = X_r - X_{[r/\Delta_n] \Delta_n} \text{ and } Z_r = X_r - \tilde{X}_r([r/\Delta_n] \Delta_n),
\]

\[
(iii) \quad \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \mathbb{E}[|f(Z_r + Y_{t_{k-1}}) - f(Y_{t_{k-1}}) - \langle \nabla f(Y_{t_{k-1}}), Z_r \rangle|^2] \, dr = o(\Delta_n)
\]

for \( Z_r = X_r - X_{[r/\Delta_n] \Delta_n} \) and \( Z_r = X_r - X_{[r/\Delta_n] \Delta_n} - X_{[r/\Delta_n] \Delta_n} \).
(iv) $\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \mathbb{E}[\gamma_i^t \nabla f(Y_t) - \gamma_i^{t_{k-1}} \nabla f(Y_{t_{k-1}})]^2 dr = o(1)$, where $(\gamma_r)_{0 \leq r \leq T}$ is a càdlàg process such that $\mathbb{P}$-a.s. $\sup_{0 \leq r \leq T} |\gamma_r| \lesssim 1$.

(v) $\sup_{0 \leq t \leq T} \mathbb{E}[|f(Y_{t/\Delta_n}) - f(Y_t)|^2] = o(1)$.

Proof. (i). Write $\nabla f(Y_{t_{k-1}}) = h(\xi)$ with $h(x) = \nabla f(X_{t_{k-1}} + x)$ such that $|\mathcal{F} h(u)^2| = |\mathcal{F} f(u)|^2 |u|^2 e^{-i(u,X_{t_{k-1}})} = |\mathcal{F} f(u)|^2 |u|^2$. The claim follows from Lemma \[3\]

(ii). Let $t_{k-1} \leq r < t_k$ and write $\langle \nabla f(X_{t_{k-1}} + x), Z_r \rangle = h(\xi)$ with $h(x) = \langle \nabla f(X_{t_{k-1}} + x), Z_r \rangle$ such that $\mathcal{F} h(u) = \mathcal{F} f(u) \langle iu, Z_r \rangle e^{-i(u,X_{t_{k-1}})}$. As in (i) we get from Lemma \[3\] that

$$\mathbb{E}[\langle \nabla f(Y_{t_{k-1}}), Z_r \rangle^2] \lesssim \|f\|_H^2 \mathbb{E}[|Z_r|^2].$$

The result follows from $\sup_r \mathbb{E}[|Z_r|^2] \lesssim \Delta_n$ using Lemma \[12\](i,ii).

(iii). Let $t_{k-1} \leq r < t_k$ and take

$$h(x) = f(Z_r + X_{t_{k-1}} + x) - f(X_{t_{k-1}} + x) - \langle \nabla f(X_{t_{k-1}} + x), Z_r \rangle$$

$$= \int_0^1 \langle \nabla f(X_{t_{k-1}} + x + aZ_r) - \nabla f(X_{t_{k-1}} + x), Z_r \rangle da,$$

implying

$$|\mathcal{F} h(u)| = |\mathcal{F} f(u) \int_0^1 \langle iu(e^{-i(aZ_r)} - 1), Z_r \rangle da|.$$

Lemma \[3\] gives

$$\mathbb{E}[(f(Z_r + Y_{t_{k-1}}) - f(Y_{t_{k-1}}) - \langle \nabla f(Y_{t_{k-1}}), Z_r \rangle)^2] = \mathbb{E}[|h(\xi)|^2]$$

$$\lesssim \int_{\mathbb{R}^d} |\mathcal{F} f(u)|^2 |u|^2 \mathbb{E}\left[\int_0^1 |e^{-i(u,aZ_r)} - 1|^2 da|Z_r|^2\right] du$$

$$\lesssim \Delta_n \int_{\mathbb{R}^d} |\mathcal{F} f(u)|^2 |u|^2 \mathbb{E}\left[\int_0^1 |e^{-i(u,aZ_r)} - 1|^4 da\right]^{1/2} du,$$

using in the last line the Cauchy-Schwarz inequality and that $\mathbb{E}[|Z_r|^4] \lesssim \Delta_n^2$ uniformly in $r$ by Lemma \[12\](i). Now, observe that uniformly in $n$ and $r$, $\mathbb{E}\left[\int_0^1 |e^{-i(u,aZ_r)} - 1|^4 da\right] \lesssim 1$ and again that by Lemma \[12\](i) uniformly in $r$ pointwise for $u \in \mathbb{R}^d$ as $n \to \infty$

$$\mathbb{E}\left[\int_0^1 |e^{-i(u,aZ_r)} - 1|^4 da\right] \lesssim |u|^4 \mathbb{E}[|Z_r|^4] \to 0.$$
The claim follows therefore from dominated convergence.

(iv). Let again $t_{k-1} \leq r < t_k$ and take this time $h(x) = \gamma_r^\top \nabla f(X_r + x) - \gamma_{t_{k-1}}^\top \nabla f(X_{t_{k-1}} + x)$ such that

$$
Fh(u) = Ff(u)ig_r(u)\top u, \quad g_r(u) = \gamma_r e^{-i(u,X_r)} - \gamma_{t_{k-1}} e^{-i(u,X_{t_{k-1}})}.
$$

Lemma 3 shows

$$
\mathbb{E}[|\gamma_r^\top \nabla f(Y_r) - \gamma_{t_{k-1}}^\top \nabla f(Y_{t_{k-1}})|^2] = \mathbb{E}[|h^2(\xi)|^2] \lesssim \int_{\mathbb{R}^d} |Ff(u)|^2 |u|^2 \mathbb{E}[|g_r(u)|^2] du.
$$

Here, $|g_r(u)|$ is almost surely bounded. Moreover, for all $u \in \mathbb{R}^d$ and uniformly in $r$ we get $\mathbb{E}[|g_r(u)|^2] \to 0$ by dominated convergence, as both $\gamma$ and $X$ have càdlàg paths. Another application of dominated convergence yields (iv). The proof of (v) is analogous to (iv) and is therefore skipped.

5.1.3. The main decomposition

We apply the decomposition from (4), but with $X$ replaced by $Y$ in the definitions of $M_{t,n}(f)$, $E_{t,n}(f)$ and $D_{t,n}(f)$. Observe first the following two propositions.

**Proposition 14.** Grant Assumption $\mathcal{S}_{loc}(\alpha;\beta)$. Then we have for $f \in H^1(\mathbb{R}^d)$ with $\tilde{W}$ as in Theorem 7 the functional stable convergence

$$
\Delta_{-1} M_{t,n}(f) \xrightarrow{st} \frac{1}{2} \int_0^t \langle \nabla f(Y_r), \sigma_r dW_r \rangle + \frac{1}{\sqrt{12}} \int_0^t \langle \nabla f(Y_r), \sigma_r d\tilde{W}_r \rangle.
$$

**Proposition 15.** Grant Assumption $\mathcal{S}_{loc}(\alpha;\beta)$. If $f \in H^1(\mathbb{R}^d)$, then

$$
\Delta_{-1} E_{t,n}(f) \xrightarrow{ucp} \frac{1}{2} (f(Y_t) - f(Y_0)) - \frac{1}{2} \int_0^t \langle \nabla f(Y_r), \sigma_r dW_r \rangle.
$$

Since the limit process in Proposition 13 is continuous, stable convergence also holds at any fixed $0 \leq t \leq T$, cf. [6]. It follows from Proposition 13 using Slutsky’s lemma that $\Delta_{-1} (M_{t,n}(f) + E_{t,n}(f))$ converges stably to the claimed limit in Theorem 4. The proof of the theorem follows therefore from showing that $\Delta_{-1} D_{t,n}(f)$ vanishes in probability asymptotically as $n \to \infty$, which we will do in Section 5.1.4 below.

We end this section with the proofs of the two aforementioned propositions. It is worth emphasising that they hold for $f \in H^1(\mathbb{R}^d)$, while the analysis for $D_{t,n}(f)$ requires more smoothness for $f$. The crucial steps in the proof of Proposition 14 are to suitably approximate the summands of $M_{t,n}(f)$ and to conclude by the (stochastic) Fubini theorem in (11).
Proof of Proposition 14. Recall \( \tilde{X}_r(t_{k-1}) \) for \( t_{k-1} \leq r \leq t_k \) from (6). Let

\[
Z_k = \int_{t_{k-1}}^{t_k} (f(Y_r) - f(Y_{t_{k-1}}) - \mathbb{E} [f(Y_r) - f(Y_{t_{k-1}}) | \mathcal{F}_{t_{k-1}}]) \, dr,
\]

\[
\tilde{Z}_k = \int_{t_{k-1}}^{t_k} \langle \nabla f(Y_{t_{k-1}}), \tilde{X}_r(t_{k-1}) - X_{t_{k-1}} - \mathbb{E} [\tilde{X}_r(t_{k-1}) - X_{t_{k-1}} | \mathcal{F}_{t_{k-1}}] \rangle \, dr,
\]

and write \( M_{t,n}(f) = \sum_{k=1}^{[t/\Delta_n]} Z_k, \tilde{M}_{t,n}(f) = \sum_{k=1}^{[t/\Delta_n]} \tilde{Z}_k \) and set \( M_{t,n} = M_{t,n}(f) - \tilde{M}_{t,n}(f) \). \( (M_{t,n}(f))_{k \in \{0,...,n\}} \) is a discrete martingale such that by the Burkholder-Gundy inequality

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} M_{t,n}^2 \right] = \mathbb{E} \left[ \sup_{k \in \{1,...,n\}} M_{k,\Delta_n,n}^2 \right] \leq 2 \sum_{k=1}^{n} \mathbb{E} \left[ (Z_k - \tilde{Z}_k)^2 \right].
\]

In addition, set

\[
\tilde{Z}_k = \int_{t_{k-1}}^{t_k} \langle \nabla f(Y_{t_{k-1}}), X_r - X_{t_{k-1}} - \mathbb{E} [X_r - X_{t_{k-1}} | \mathcal{F}_{t_{k-1}}] \rangle \, dr.
\]

Lemma 13(ii,iii) shows \( \sum_{k=1}^{n} \mathbb{E} [(Z_k - \tilde{Z}_k)^2] = o(\Delta_n^2) \) and \( \sum_{k=1}^{n} \mathbb{E} [(\tilde{Z}_k - \tilde{Z}_k)^2] = o(\Delta_n^3) \), hence we can conclude

\[
\Delta_n^{-1} \sup_{0 \leq t \leq T} |M_{t,n}| \overset{p}{\to} 0.
\]

It is therefore enough to study the limit of \( \Delta_n^{-1} \tilde{M}_{t,n}(f) \). The claim follows from Theorem IX.7.28 of [17], once we have shown for \( 0 \leq t \leq T \) that

\[
\Delta_n^{-2} \sum_{k=1}^{[t/\Delta_n]} \mathbb{E} \left[ \tilde{Z}_k^2 \right] \mathcal{F}_{t_{k-1}} \overset{p}{\to} \frac{1}{3} \int_0^t \langle \sigma_r \nabla f(Y_r) \rangle^2 \, dr,
\]

(7)

\[
\Delta_n^{-2} \sum_{k=1}^{[t/\Delta_n]} \mathbb{E} \left[ \tilde{Z}_k^2 \mathcal{F}_{t_{k-1}} \right] \mathcal{F}_{t_{k-1}} \overset{p}{\to} 0, \text{ for all } \varepsilon > 0,
\]

(8)

\[
\Delta_n^{-1} \sum_{k=1}^{[t/\Delta_n]} \mathbb{E} \left[ \tilde{Z}_k (W_{t_k} - W_{t_{k-1}})^\top \mathcal{F}_{t_{k-1}} \right] \mathcal{F}_{t_{k-1}} \overset{p}{\to} \frac{1}{2} \int_0^t \nabla f(Y_r)^\top \sigma_r \, dr,
\]

(9)

\[
\Delta_n^{-1} \sum_{k=1}^{[t/\Delta_n]} \mathbb{E} \left[ \tilde{Z}_k (N_{t_k} - N_{t_{k-1}}) \right] \mathcal{F}_{t_{k-1}} \mathcal{F}_{t_{k-1}} \overset{p}{\to} 0,
\]

(10)
where (10) has to hold for all bounded (R-valued) martingales N which are orthogonal to all components of W. Note that \( \mathbb{E}[\tilde{Z}_k|\mathcal{F}_k] = 0 \) such that the asymptotic bias \( \Delta_n^{-1} \sum_{k=1}^{[t/\Delta_n]} \mathbb{E}[\tilde{Z}_k|\mathcal{F}_{t_{k-1}}] \) vanishes.

Let us prove (7) through (10). Write \( \tilde{X}_r(t_{k-1}) - X_{t_{k-1}} = X_{t_{k-1}} + b_{t_{k-1}} \int_{t_{k-1}}^r dr' + \sigma_{t_{k-1}} \int_{t_{k-1}}^r dW_{r'} \). The stochastic Fubini theorem thus provides the identity

\[
\tilde{Z}_k = \int_{t_{k-1}}^{t_k} \nabla f(Y_{t_{k-1}}), \sigma_{t_{k-1}} \int_{t_{k-1}}^r dW_{r'} - \sigma_{t_{k-1}} \mathbb{E} \left[ \int_{t_{k-1}}^r dW_{r'} \bigg| \mathcal{F}_{t_{k-1}} \right] ) dr \\
= \langle \nabla f(Y_{t_{k-1}}), \sigma_{t_{k-1}} \int_{t_{k-1}}^{t_k} (t_k - r) dW_r \rangle. \tag{11}
\]

By Itô’s isometry, the boundedness of \( \sigma \) from Assumption \( S_{loc}(\alpha; \beta) \) and Lemma 13, (7) follows from

\[
\Delta_n^{-2} \sum_{k=1}^{[t/\Delta_n]} \mathbb{E} \left[ \tilde{Z}_k \bigg| \mathcal{F}_{t_{k-1}} \right] = \frac{\Delta_n}{3} \sum_{k=1}^{[t/\Delta_n]} \left| \sigma_{t_{k-1}} \nabla f(Y_{t_{k-1}}) \right|^2 + o_p(1) \\
= \frac{1}{3} \int_0^t \left| \sigma_r \nabla f(Y_r) \right|^2 dr + o_p(1),
\]

using Lemma 13 (iv) for the Riemann approximation in the last line. With respect to (8) apply the Cauchy-Schwarz inequality to \( \tilde{Z}_k \) such that using the boundedness of \( \sigma \)

\[
|\tilde{Z}_k| \lesssim h(\xi), \quad h(x) = |\nabla f(x)| \Delta_n^{3/2} |V_k|
\]

and a random variable \( V_k \overset{d}{\sim} N(0, I_d) \), which is independent of \( \mathcal{F}_{t_{k-1}} \). Since also \( \xi \) is independent of \( \mathcal{F}_{t_{k-1}} \), the first inequality of Lemma 3 applied to \( h \) yields for some \( \varepsilon' > 0 \)

\[
\mathbb{E}[\tilde{Z}_k^2 1_{\{|\tilde{Z}_k| > \varepsilon\}} |\mathcal{F}_{t_{k-1}}] \lesssim \mathbb{E}[h^2(X_{t_{k-1}} + \xi) 1_{\{h(X_{t_{k-1}} + \xi) > \varepsilon'\}} |\mathcal{F}_{t_{k-1}}] \\
\lesssim \Delta_n^3 \int_{\mathbb{R}^d} \mathbb{E}[|\nabla f(X_{t_{k-1}} + x)|^2 |V_k|^2 1_{\{h(x) > \varepsilon'\}} |\mathcal{F}_{t_{k-1}}] dx \\
= \Delta_n^3 \int_{\mathbb{R}^d} |\nabla f(x)|^2 \mathbb{E}[|V|^2 1_{\{h(x) > \varepsilon'\}} |\mathcal{F}_{t_{k-1}}] dx.
\]

For fixed \( x \in \mathbb{R}^d \) we have \( h(x) \to 0 \) \( \mathbb{P} \)-a.s., implying \( \mathbb{E}[|V_k|^2 1_{\{h(x) > \varepsilon'\}} |\mathcal{F}_{t_{k-1}}] \to 0 \) by dominated convergence. Hence, (8) is obtained from using dominated convergence.
convergence once more in the last display. (9) follows from Itô’s isometry and Lemma 13(iv):

$$\Delta_n^{-1} \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E} \left[ \tilde{Z}_k (W_{t_k} - W_{t_{k-1}} \right] = \frac{\Delta_n}{2} \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \nabla f(Y_{t_{k-1}})^\top \sigma_{t_{k-1}} + o_P(1)$$

In the same way, (10) follows from $E[\tilde{Z}_k (N_{t_k} - N_{t_{k-1}}) | \mathcal{F}_{t_{k-1}}] = o_P(1)$. □

Proof of Proposition 13: Let $B_k = \langle \nabla f(Y_{t_{k-1}}), X_{t_k} - X_{t_{k-1}} \rangle$, $A_k = f(Y_{t_k}) - f(Y_{t_{k-1}}) - B_k$, and write $E_t(f) = S_1(t) + S_2(t) + S_3(t)$ with

$$S_1(t) = \frac{\Delta_n}{2} \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} (f(Y_{t_k}) - f(Y_{t_{k-1}})) = \frac{\Delta_n}{2} (f(Y_{\lfloor t/\Delta_n \rfloor \Delta_n}) - f(Y_0)),$$

$$S_2(t) = \frac{\Delta_n}{2} \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \left( \mathbb{E} \left[ A_k | \mathcal{F}_{t_{k-1}} \right] - A_k \right),$$

$$S_3(t) = \frac{\Delta_n}{2} \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \left( \mathbb{E} \left[ B_k | \mathcal{F}_{t_{k-1}} \right] - B_k \right).$$

Lemma 13(v) implies $\Delta_n^{-1} S_1(t) \xrightarrow{ucp} \frac{1}{2} (f(Y_t) - f(Y_0))$, while we have by the Burkholder-Gundy inequality

$$\mathbb{E} \left[ \sup_{0 \leq s \leq T} S_2^2(t) \right] = \mathbb{E} \left[ \sup_{k \in \{1, \ldots, n\}} S_2^2(t) \right] \lesssim \Delta_n^2 \sum_{k=1}^{n} \mathbb{E}[A_k^2] = o(\Delta_n^2), \quad (12)$$

such that $\Delta_n^{-1} S_2(t) \xrightarrow{ucp} 0$, concluding with Lemma 13(iii) (taking $Z_r = X_{\lfloor r/\Delta_n \rfloor \Delta_n} - X_{\lfloor r/\Delta_n \rfloor \Delta_n}$). At last, decompose

$$S_3(t) = \frac{\Delta_n}{2} \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \int_{t_{k-1}}^{t_k} \nabla f(Y_{t_{k-1}}), (\mathbb{E}[b_r | \mathcal{F}_{t_{k-1}}] - b_r) \rangle dr$$

$$- \frac{\Delta_n}{2} \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \int_{t_{k-1}}^{t_k} \nabla f(Y_{t_{k-1}}), \sigma_r dW_r \right).$$

Exactly as in (12), but using Lemma 13(i), the first line is of order $o_P(\Delta_n)$ uniformly in $t$, while the second one equals $-\frac{\Delta_n}{2} \int_0^t \langle \nabla f(Y_r), \sigma_r dW_r \rangle + o_P(\Delta_n)$, again uniformly in $t$, by Lemma 13(iv) and Itô’s isometry. □
5.1.4. The term $D_{t,n}(f)$

Since $f$ is not smooth, Itô’s formula cannot be used directly to reduce $D_{t,n}(f)$ to more manageable terms. Instead, write $D_{t,n}(f) = h(\xi)$ with

$$h(x) = \sum_{k=1}^{[t/\Delta_n]} \int_{t_{k-1}}^{t_k} \mathbb{E}[f(X_r + x) - f(X_{t_{k-1}} + x)] - \frac{f(X_{t_k} + x) - f(X_{t_{k-1}} + x)}{2} |\mathcal{F}_{t_{k-1}}| dr$$

such that by linearity of the Fourier transform

$$\mathcal{F}h(u) = \mathcal{F}f(u) \sum_{k=1}^{[t/\Delta_n]} \int_{t_{k-1}}^{t_k} \mathbb{E}[e^{-i\langle u, X_r \rangle} - e^{-i\langle u, X_{t_{k-1}} \rangle}] - \frac{e^{-i\langle u, X_{t_k} \rangle} - e^{-i\langle u, X_{t_{k-1}} \rangle}}{2} |\mathcal{F}_{t_{k-1}}| dr$$

for $u \in \mathbb{R}^d$. For fixed $u$ the function $e^{i\langle u, \cdot \rangle}$ is smooth and so we deduce from Itô’s formula and Fubini’s theorem that

$$\mathcal{F}h(u) = \mathcal{F}f(u) (\mathcal{F}^{(1)}_{t,n}(u) + \mathcal{F}^{(2)}_{t,n}(u))$$

with

$$\mathcal{F}^{(1)}_{t,n}(u) = -\sum_{k=1}^{[t/\Delta_n]} \int_{t_{k-1}}^{t_k} \left( t_k - r - \frac{\Delta_n}{2} \right) \mathbb{E} \left[ ie^{-i\langle u, X_r \rangle} \langle u, b_r \rangle |\mathcal{F}_{t_{k-1}} \right] dr, \quad (13)$$

$$\mathcal{F}^{(2)}_{t,n}(u) = -\frac{1}{2} \sum_{k=1}^{[t/\Delta_n]} \int_{t_{k-1}}^{t_k} \left( t_k - r - \frac{\Delta_n}{2} \right) \mathbb{E} \left[ e^{-i\langle u, X_r \rangle} |\sigma^\top_r u \rangle^2 |\mathcal{F}_{t_{k-1}} \right] dr. \quad (14)$$

Lemma 3 therefore implies

$$\mathbb{E}[|D_{t,n}(f)|^2] \lesssim \int_{\mathbb{R}^d} |\mathcal{F}f(u)|^2 \mathbb{E} \left[ |\mathcal{F}^{(1)}_{t,n}(u) + \mathcal{F}^{(2)}_{t,n}(u)|^2 \right] du. \quad (15)$$

Introduce for $u \in \mathbb{R}^d$ and $i \in \{1, 2\}$ the functions

$$g_n^{(i)}(u) = \Delta_n^{-2} (1 + |u|^2)^{-s} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |F_{t,n}^{(i)}(u)|^2 \right]. \quad (16)$$

This provides us with

$$\mathbb{E}[|D_{t,n}(f)|^2] \lesssim \Delta_n^2 \int_{\mathbb{R}^d} |\mathcal{F}f(u)|^2 (1 + |u|^2)^s (g_n^{(1)}(u) + g_n^{(2)}(u)) du.$$
Proposition 16. Let $s \geq 1$ and grant Assumption $S_{\text{loc}}(\alpha;\beta)$ with $\alpha > \max(0, 1 - s/2)$, $\beta > 0$. Then we have for $f \in H^s(\mathbb{R}^d)$ and $0 \leq t \leq T$

$$\Delta_n^{-1} D_{t,n}(f) \xrightarrow{P} 0. \quad (17)$$

Let us now state and prove the aforementioned two lemmas, as well as two auxiliary lemmas.

Lemma 17. Under Assumption $S_{\text{loc}}(\alpha;\beta)$ with $\beta > 0$ the function $g_n^{(1)}$ from (16) with $s \geq 1$ satisfies $g_n^{(1)}(u) \to 0$ as $n \to \infty$ for all $u \in \mathbb{R}^d$ and $\sup_{n \in \mathbb{N}, u \in \mathbb{R}^d} g_n^{(1)}(u) < \infty$.

Proof. Define $\tilde{F}_{t,n}^{(1)}(u)$ exactly as $F_{t,n}^{(1)}(u)$, but with $e^{-i(u,X_t)}\langle u, b_t \rangle$ replaced by $e^{-i(u,X_{\lfloor t/\Delta_n \rfloor} \Delta_n)}\langle u, b_{\lfloor t/\Delta_n \rfloor} \Delta_n \rangle$. From $\int_{t_{k-1}}^{t_k} (t_k - r - \Delta_n/2) dr = 0$ conclude $\tilde{F}_{t,n}^{(1)}(u) = 0$ such that

$$g_n^{(1)}(u) = \Delta_n^{-2} (1 + |u|^2)^{-s} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |F_{t,n}^{(1)}(u) - \tilde{F}_{t,n}^{(1)}(u)|^2 \right]$$

$$\lesssim \mathbb{E} \left[ \sup_{0 \leq t \leq T} |e^{-i(u,X_t)}\langle u/|u|, b_t \rangle - e^{-i(u,X_{\lfloor t/\Delta_n \rfloor} \Delta_n)}\langle u/|u|, b_{\lfloor t/\Delta_n \rfloor} \Delta_n \rangle|^2 \right]$$

using in the last line $s \geq 1$. From Assumption $S_{\text{loc}}(\alpha;\beta)$, the process $b$ is uniformly bounded such that $\sup_{n \in \mathbb{N}, u \in \mathbb{R}^d} g_n^{(1)}(u) < \infty$. On the other hand, as $X$ has càdlàg paths and using the approximation property of $b$ according to $S_{\text{loc}}(\alpha;\beta)$ for $\beta > 0$, conclude also that $g_n^{(1)}(u) \to 0$ as $n \to \infty$ for $u \in \mathbb{R}^d$. \( \Box \)

The next lemma is the key result for general continuous semimartingales and is inspired by the one-step-Euler-approximation of [9] to approximate the characteristic function of the marginals $X_t$.

Lemma 18. Let $s \geq 1$ and grant Assumption $S_{\text{loc}}(\alpha;\beta)$ for $\alpha > \max(0, 1 - s/2)$, $\beta > 0$. Then the function $g_n^{(2)}$ from (16) satisfies $g_n^{(2)}(u) \to 0$ as $n \to \infty$ for all $u \in \mathbb{R}^d$ and $\sup_{n \in \mathbb{N}, u \in \mathbb{R}^d} g_n^{(2)}(u) < \infty$.

Proof. We distinguish the cases $|u| \leq 1$ and $|u| > 1$. For $|u| \leq 1$ the argument is analogous to the proof of Lemma 17. Define $\tilde{F}_{t,n}^{(2)}(u)$ exactly as $F_{t,n}^{(2)}(u)$, but
with \( e^{-i\langle u, X_r \rangle} |\sigma_r^\top u|^2 \) replaced by \( e^{-i\langle u, X_r / \Delta_n \rangle \Delta_n} |\sigma_r^\top \Delta_n u|^2 \). Again \( \tilde{F}_{t,n}^{(2)}(u) = 0 \) and we conclude as above that

\[
g_n^{(2)}(u) = \Delta_n^{-2} (1 + |u|^2)^{-s} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |F_{t,n}^{(2)}(u) - \tilde{F}_{t,n}^{(2)}(u)|^2 \right]
\]

satisfies \( \sup_{n \in \mathbb{N}, u \in \mathbb{R}^d, |u| \leq 1} g_n^{(2)}(u) < \infty \) and \( g_n^{(2)}(u) \to 0 \) as \( n \to \infty \) for \( |u| \leq 1 \).

Let now \( |u| > 1 \). We introduce a new grid depending on the parameters

\[
\varepsilon \equiv \varepsilon(u, \Delta_n) = \nu|u|^{-2}, \quad \nu = \nu(u, \Delta_n) = C_1^{-1} \log(1 + |u|^{2}\eta_n^{1/2}),
\]

where \( C_1 > 0 \) is a constant such that according to Assumption \( S_{\text{loc}}(\alpha, \beta) \)

\[
\inf_r (\frac{1}{2} |\sigma_r^\top u|^2) = \inf_r (\frac{1}{2} \langle \sigma_r \sigma_r^\top u, u \rangle) \geq C_1 |u|^2
\]

and where \( \eta_n \) is a sequence of non-negative real numbers to be determined later. Recall the approximated process from (6) and set for \( 0 \leq r, r', r'' \leq T \)

\[
U_{r, r', r''} = \mathbb{E}[e^{-i\langle u, X_{r'} \rangle}|\sigma_{r''}^\top u|^2|F_{t,n}], \quad U_{r, r'} = U_{r, r', r''}.
\]

With this define

\[
\tilde{F}_{t,n}^{(2)}(u) = \sum_{k=1}^{t/\Delta_n} \int_{t_{k-1}}^{t_k} (t_k - r - \Delta_n^{-2}) U_{r, r', r''} dr.
\]

and obtain the upper bound \( g_n^{(2)}(u) \leq 2g_n^{(2,1)}(u) + 2g_n^{(2,2)}(u) \) with

\[
g_n^{(2,1)}(u) = \Delta_n^{-2} (1 + |u|^2)^{-s} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |F_{t,n}^{(2)}(u) - \tilde{F}_{t,n}^{(2)}(u)|^2 \right],
\]

\[
g_n^{(2,2)}(u) = \Delta_n^{-2} (1 + |u|^2)^{-s} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\tilde{F}_{t,n}^{(2)}(u)|^2 \right].
\]

Upper bounds on these two terms are obtained in Lemmas 19, 20 and 21 below. The null sequence \( \eta_n \) is determined in Lemma 20. In order to conclude when \( |u| > 1 \) note that the conditions \( \alpha > \max(0, 1 - s/2), \beta > 0, s \geq 1 \)
imply for some sufficiently small $\delta > 0$

$$
|u|^{4-2s} \varepsilon^{2\alpha} + |u|^{6-2s} \varepsilon^{2+2\beta} + |u|^{6-2s} \varepsilon^{1+2\alpha} \leq \varepsilon^{\delta} \nu^{2\alpha} + |u|^{4} \varepsilon^{2+2\beta} + |u|^{2} \varepsilon^{1+\delta} \nu^{2\alpha} \\
\lesssim \varepsilon^{\delta} \nu^{2\alpha} + \varepsilon^{\delta} \nu^{1+2\alpha}, \quad (19)
$$

$$
|u|^{4-2s} \varepsilon e^{-C_1|u|^2\varepsilon} + |u|^{2-2s} (1 - e^{-C_1|u|^2\Delta_n}) + |u|^{8-2s} e^{-C_1|u|^2\varepsilon} \eta_n \\
\leq |u|^2 \varepsilon e^{-C_1|u|^2\varepsilon} + (1 - e^{-C_1|u|^2\Delta_n}) + |u|^6 e^{-C_1|u|^2\varepsilon} \eta_n \\
= \nu e^{-C_1\nu} + (1 - e^{-C_1|u|^2\Delta_n}) + \frac{|u|^6 \eta_n}{1 + |u|^6 \eta_n}, \quad (20)
$$

$$
|u|^{4-2s} \varepsilon^2 + |u|^{2-2s} (1 - e^{-C_1|u|^2\varepsilon}) + |u|^{8-2s} e^{-C_1|u|^2\varepsilon} \eta_n \\
\leq |u|^2 \varepsilon^2 + (1 - e^{-C_1|u|^2\varepsilon}) + |u|^6 e^{-C_1|u|^2\varepsilon} \eta_n \\
\leq \varepsilon \nu + (1 - e^{-C_1\nu}) + \frac{|u|^6 \eta_n}{1 + |u|^6 \eta_n}. \quad (21)
$$

We get from (18) that $\varepsilon \leq 1$ and $\nu^p \lesssim \varepsilon$ uniformly in $u, n$ and any $p \in \mathbb{N}$ and that $\varepsilon, \nu \to 0$ for any fixed $u \in \mathbb{R}^d$ as $n \to \infty$. Consequently, the terms in (19), (20) and (21) are uniformly in $u, n$ bounded and converge to zero for any fixed $u \in \mathbb{R}^d$ as $n \to \infty$. Lemmas 19, 20 and 21 then show that $g_n^{(2, 1)}(u)$ and $g_n^{(2, 2)}(u)$ and thus also $g_n^{(2)}(u)$ are uniformly in $u, n$ bounded and converge to zero for any fixed $u \in \mathbb{R}^d$ as $n \to \infty$, which is what we wanted to prove.

**Lemma 19.** In Lemma 18 it holds

$$
g_n^{(2, 1)}(u) \lesssim |u|^{4-2s} \varepsilon^{2\alpha} + |u|^{6-2s} \varepsilon^{2+2\beta} + |u|^{6-2s} \varepsilon^{1+2\alpha}.
$$

**Proof.** Let $k \geq 1$, $t_{k-1} \leq r < t_k$. Since $r - r(\varepsilon) \lesssim \varepsilon$, Assumption $S_{\text{loc}}(\alpha; \beta)$ and Lemma 12(ii) provide the approximation errors

$$
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |\sigma_r - \sigma_r(\varepsilon)|^2 \right] \lesssim \varepsilon^{2\alpha}, \quad \mathbb{E}[|X_r - \tilde{X}_r(r(\varepsilon))|^2] \lesssim \varepsilon^{2\beta+2} + \varepsilon^{2\alpha+1}.
$$

Since $\tilde{X}_r(r) = X_r$ and recalling that $\sigma$ is uniformly bounded by Assumption $S_{\text{loc}}(\alpha; \beta)$ this further gives

$$
\mathbb{E}[|U_{r,r} - U_{r,r}(\varepsilon)|^2] \lesssim \mathbb{E}\left[ |\sigma_r^T u|^2 - |\sigma_r(\varepsilon) u|^2 |^2 \right] + |u|^{4} \mathbb{E}\left[ \left| e^{-i(u,X_r)} - e^{-i(u,\tilde{X}_r(r(\varepsilon)))} \right|^2 \right] \\
\lesssim |u|^{4} \varepsilon^{2\alpha} + |u|^{6} \varepsilon^{2+2\beta} + |u|^{6} \varepsilon^{1+2\alpha}.
$$
The result follows thus from

\[(g_n^{(2,1)}(u))^{1/2} = \Delta_n^{-1} |u|^{-s} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \tilde{F}_{t,n}^{(2)}(u) - \tilde{F}_{t,n}^{(1)}(u) \right|^2 \right]^{1/2} \]

\[\leq \Delta_n^{1/2} |u|^{-s} \sum_{k=1}^{n} \left( \int_{t_{k-1}}^{t_k} \mathbb{E}[|U_{r,r} - U_{r,r(\varepsilon)}|^2] dr \right) \]

\[\lesssim |u|^{2-s} \xi^{1+\beta} + |u|^{2-s} \xi^{1/2+\alpha}.\]

\[\square\]

**Lemma 20.** In the setting of Lemma 18 there exists a sequence \(\eta_n \geq 0\) with \(\eta_n \to 0\) and such that for \(\varepsilon \leq \Delta_n\) we have

\[g_n^{(2,2)}(u) \lesssim |u|^{1-2s} \xi e^{-C_1 |u|^2 \varepsilon} + |u|^{2-2s} (1 - e^{-C_1 |u|^2 \Delta_n}) + |u|^{8-2s} e^{-C_1 |u|^2 \varepsilon} \eta_n.\]

**Proof.** Observe first the following fact by the Burkholder-Gundy inequality:

Assuming this holds, we can apply (22) to

\[\eta \in \{0, \ldots, n\} \text{ such that by (23) and (24)}\]

\[\int_{t_{k-1}}^{t_k} \mathbb{E}[|U_{r,r} - U_{r,r(\varepsilon)}|^2] dr\]

\[\lesssim |u|^{2-s} \xi^{1-\beta} + |u|^{2-s} \xi^{1/2+\alpha} + |u|^{2-s} \xi^{1/2+\alpha} \eta_n.\]

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which proves the claim.

Let us next show \((23)\) and \((24)\). Fix \(k \geq 1\) and note that

\[
\mathbb{E}[|Z_k|^2] \lesssim \Delta_n^2 \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} \mathbb{E}[U_{r,r'(\varepsilon)} U_{r',r'(\varepsilon)}^*] dr \, dr'.
\]

Let \(t_{k-1} \leq r' \leq r < t_k\) and \(r_* = \max(r(\varepsilon), r')\). From \((4)\) it follows that \(\langle u, \tilde{X}_r(r(\varepsilon)) - \tilde{X}_{r_*}(r(\varepsilon)) \rangle\) is conditional on \(\mathcal{F}_{r_*}\) \(N((u, b_r(\varepsilon))(r - r_*), |\sigma_r^{\top} u|^2 (r - r_*))\)-distributed. Hence,

\[
\left| \mathbb{E}[e^{-i(u, \tilde{X}_r(\varepsilon)) - \tilde{X}_{r_*}(r(\varepsilon))}] |\mathcal{F}_{r_*}\right| = e^{\frac{1}{2}|\sigma_r^{\top} u|^2 (r - r_*)} \leq e^{-C_1 |u|^2 (r - r_*)}
\]

with \(C_1\) from \((18)\). The tower property of conditional expectation and \(|U_{r,r'}(\varepsilon)| \lesssim |u|^2\) gives

\[
|\mathbb{E}[U_{r,r(\varepsilon)} U_{r',r'(\varepsilon)}^*]| = \left| \mathbb{E}[\mathbb{E}[e^{-i(u, \tilde{X}_r(r(\varepsilon)) - \tilde{X}_{r_*}(r(\varepsilon)))} |\mathcal{F}_{r_*}] e^{\frac{1}{2}|\sigma_r^{\top} u|^2 (r - r_*)}] \right| \lesssim |u|^4 \mathbb{E}[|\mathbb{E}[e^{-i(u, \tilde{X}_r(\varepsilon)) - \tilde{X}_{r_*}(r(\varepsilon)))} |\mathcal{F}_{r_*}^2]|] \lesssim |u|^4 e^{-C_1 |u|^2 (r - r_*)}.
\]

The condition \(\varepsilon \leq \Delta_n\) yields \(\min(\varepsilon, r - r') \leq r - r_* \leq \Delta_n\) such that

\[
\mathbb{E}[|Z_k|^2] \lesssim \Delta_n^2 |u|^4 \left( \Delta_n^2 e^{-C_1 |u|^2 \varepsilon} + \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} e^{-C_1 |u|^2 (r - r')} dr \, dr' \right)
\]

\[
\lesssim \Delta_n^3 |u|^2 \left( |u|^2 e^{-C_1 |u|^2 \varepsilon} + (1 - e^{-C_1 |u|^2 \Delta_n}) \right).
\]

This proves \((23)\). To see why \((24)\) holds, let \(k \geq 2\). Since \(\int_{t_{k-1}}^{t_k} (t_k - r - \Delta_n/2) dr\) vanishes, the same holds for \(U_{t_k,t_{k-2}} = U_{t_k,t_{k-2},t_{k-2}}\) and thus

\[
|\mathbb{E}[Z_k] |\mathcal{F}_{t_{k-2}}^2| | \lesssim \Delta_n \int_{t_{k-1}}^{t_k} |\mathbb{E}[U_{r,r(\varepsilon)} U_{r',r'(\varepsilon)}] - U_{r,r(\varepsilon),t_{k-2}} |\mathcal{F}_{t_{k-2}}| dr
\]

\[
+ \Delta_n \int_{t_{k-1}}^{t_k} |\mathbb{E}[U_{r,r(\varepsilon),t_{k-2}} - U_{r,t_{k-2},t_{k-2}}] |\mathcal{F}_{t_{k-2}}| dr.
\]

Set \(r^* = \max(r(\varepsilon), t_{k-2})\) for \(t_{k-1} \leq r < t_k\) and write

\[
\tilde{X}_r(r(\varepsilon)) = \sigma_r(r)(W_r - W_{r^*}) + b_r(r - r^*) + \tilde{X}_{r^*}(r(\varepsilon)),
\]

\[
\tilde{X}_{t_{k-2}} = \sigma_{t_{k-2}}(W_{t_{k-2}} - W_{r^*}) + \sigma_{t_{k-2}}(W_r - W_{r^*}) + b_{t_{k-2}}(t_k - r^*) + \tilde{X}_{r^*}(t_{k-2}).
\]
Conditioning on $\mathcal{F}_{r^*}$ shows
\[
|\mathbb{E}[U_{r,r} - U_{tk-2}]| \leq |u|^2 |\sigma(r) - \sigma_{tk-2}| |\mathbb{E}[e^{-i(u,\bar{X}(r))}]| \lesssim |u|^2 \Delta_n e^{-C|u|^2 \varepsilon},
\]
using that $2\Delta_n \geq r-r^* \geq \varepsilon$ (because $\varepsilon \leq \Delta_n$). On the other hand, assuming first that $R = |\sigma_{r}^T u|^2 - |\sigma_{tk-2}^T u|^2 \geq 0$, we have
\[
|\mathbb{E}[U_{r,r},t_{k-2}] - U_{tk-2}| \leq |u|^2 |\mathbb{E}[e^{-i(u,\bar{X}(r))}] - e^{-i(u,\bar{X}(t_{k-2}))}| \\
\lesssim |u|^2 e^{-\frac{1}{2}|\sigma_{t_{k-2}}^T u|^2(r-r^*)} e^{-\frac{1}{2}R(r-r^*)} |\mathbb{E}[e^{-i(u,\bar{X}(t_{k-2}))}]| \\
\lesssim |u|^4 e^{-C|u|^2 \varepsilon}(\Delta_n + |\bar{X}(r)| - |\bar{X}(t_{k-2})|),
\]
using in the last line $|u| > 1$, $R \geq 0$ and again $2\Delta_n \geq r-r^* \geq \varepsilon$. The same upper bound is obtained for $R < 0$ by taking $e^{-\frac{1}{2}|\sigma_{t_{k-2}}^T u|^2(r-r^*)}$ out of the absolute value above instead of $e^{-\frac{1}{2}|\sigma_{t_{k-2}}^T u|^2(r-r^*)}$. We thus find
\[
\mathbb{E}\left[\sup_{k \in \{1,\ldots,n\}} |\mathbb{E}[U_{r,r},t_{k-2}] - U_{tk-2}|^2 |\mathcal{F}_{r^*}|^2 \right] \lesssim |u|^8 e^{-C_1|u|^2 \varepsilon} \eta_n, \quad (26)
\]
where $0 \leq \eta_n \to 0$ for $n \to \infty$ due to Lemma 12(iii). In all, this shows (24) and ends the proof. \hfill \square

**Lemma 21.** In the setting of Lemma 18 we have for $\varepsilon > \Delta_n$ that
\[
g_n^{(2)}(u) \lesssim |u|^{4-2s} \varepsilon^2 + |u|^{2-2s} (1 - e^{-C_1|u|^2 \varepsilon}) + |u|^{8-2s} e^{-C_1|u|^2 \varepsilon} \eta_n.
\]

**Proof.** Let $|u| > 1$ and $\varepsilon > \Delta_n$. We first fix some notation. Let
\[
I_j(t) = \{k = 1, \ldots, [t/\Delta_n] : (j-1)\varepsilon < t_k \leq j\varepsilon\}, \quad 1 \leq j \leq [T/\varepsilon],
\]
be the set of those $k \leq [t/\Delta_n]$ such that $t_k \leq t$ lies in the interval $((j-1)\varepsilon, j\varepsilon]$. Let $Z_k$ be as in Lemma 20 and set $A^{(j)}_t = \sum_{k \in I_j(t)} Z_k$ such that $\bar{F}_{tn}^{(2)}(u) = \sum_{j=1}^{[T/\varepsilon]} A^{(j)}_t$. For $t \geq 0$ denote by $j(t)$ the unique $j \in \{1, \ldots, [T/\varepsilon]\}$ with $(j-1)\varepsilon < t \leq j\varepsilon$. If $t \leq (j-1)\varepsilon$, then $I_j(t)$ is empty and $A^{(j)}_t = 0$, while for $t > j\varepsilon$ we have $I_j(t) = I_j(T)$ and $A^{(j)}_t = A^{(j)}_T$. This means $\bar{F}_{tn}^{(2)}(u) =
\[ \sum_{j=1}^{j(t)-1} A_T^{(j)} + A_T^{(j(t))}. \] Using the trivial bound \(|U_{r,v}| \lesssim |u|^2\) for \(r \geq 0\) and the fact that \(I_j(t)\) contains at most \(2\varepsilon \Delta_n^{-1}\) many \(k\), we get \(|A_T^{(j)}| \lesssim \Delta_n |u|^2 \varepsilon\) for all \(1 \leq j \leq \lfloor T/\varepsilon \rfloor\) and therefore

\[
\sup_{0 \leq t \leq T} |\tilde{F}_{t,n}^{(2)}(u)| \lesssim \max_{m \in \{3, \ldots, \lfloor T/\varepsilon \rfloor\}} \left| \sum_{j=3}^m A_T^{(j)} \right| + \Delta_n |u|^2 \varepsilon.
\]

Applying (22) three times (first with \(R_k = A_T^{(k+2)} \in G_k = \mathcal{F}_{(k+2)\varepsilon}, k \in \{1, \ldots, \lfloor T/\varepsilon \rfloor - 2\}\)), then with \(R_k = \mathbb{E}[A_T^{(k+2)} | \mathcal{F}_{(k+1)\varepsilon}] \in \mathcal{G}_k = \mathcal{F}_{(k+1)\varepsilon}\), and finally with \(R_k = \mathbb{E}[A_T^{(k+2)} | \mathcal{F}_{k\varepsilon}] \in \mathcal{G}_k = \mathcal{F}_{k\varepsilon}\) yields

\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |\tilde{F}_{t,n}^{(2)}(u)|^2 \right] \lesssim \varepsilon^{-1} \max_{j \in \{3, \ldots, \lfloor T/\varepsilon \rfloor\}} \mathbb{E}[|A_T^{(j)}|^2] + \varepsilon^{-2} \max_{j \in \{3, \ldots, \lfloor T/\varepsilon \rfloor\}} \mathbb{E}[|\mathbb{E}[A_T^{(j)} | \mathcal{F}_{(j-3)\varepsilon}]|^2] + \Delta_n^2 |u|^4 \varepsilon^2.
\]

We show below for \(j \geq 3\) (cf. (23), (24)) that

\[
\mathbb{E}[|A_T^{(j)}|^2] \lesssim \Delta_n^2 |u|^2 \varepsilon (1 - e^{-C_1 |u|^2 \varepsilon}), \quad (27)
\]

\[
\mathbb{E}[|\mathbb{E}[A_T^{(j)} | \mathcal{F}_{(j-3)\varepsilon}]|^2] \lesssim \Delta_n^2 \varepsilon^2 |u|^8 e^{-C_1 |u|^2 \varepsilon} \eta_n, \quad (28)
\]

with \(\eta_n\) from (24). Plugging these bounds into the last display gives

\[
g_{n,2}^{(2,2)}(u) = \Delta_n^{-2} (1 + |u|^2)^{-s} \mathbb{E}\left[ \sup_{0 \leq t \leq T} |\tilde{F}_{t,n}^{(2)}(u)|^2 \right] \lesssim \Delta_n^{-2} |u|^{-2s} \left( \mathbb{E}[|Z_1|^2] + \Delta_n^{-1} \max_{k \in \{2, \ldots, n\}} \mathbb{E}[|Z_k|^2] + \Delta_n^{-2} \max_{k \in \{2, \ldots, n\}} \mathbb{E}[|\mathbb{E}[Z_k | \mathcal{F}_{k-2}\varepsilon]|^2] \right) \lesssim |u|^{2-2s} (1 - e^{-C_1 |u|^2 \varepsilon}) + |u|^{8-2s} e^{-C_1 |u|^2 \varepsilon} \eta_n + |u|^{4-2s} \varepsilon^2.
\]

Let us now prove (27) and (28). For (27) let \(k, k' \in I_j(T), j \geq 3\) and consider \(t_{k-1} \leq r < t_k, t_{k'-1} \leq r' < t_{k'}, r' \leq r\). Since \(\varepsilon > \Delta_n\), we have \(r(\varepsilon) \leq r'\), implying by (25) with \(r_* = r'\) that \(\mathbb{E}[U_{r(\varepsilon)} U_{r', r(\varepsilon)}] \lesssim |u|^4 e^{-C_1 |u|^2 (r-r')}\). As also \((j-2) \varepsilon \leq t_{k-1}, t_{k'-1},\) this shows

\[
\mathbb{E}[|A_T^{(j)}|^2] \lesssim \Delta_n^2 \sum_{k, k' \in I_j(T)} \int_{t_{k-1}}^{t_k} \int_{t_{k'-1}}^{t_{k'}} |\mathbb{E}[U_{r, r(\varepsilon)} U_{r', r(\varepsilon)}]| \ dr \ dr' \]

\[
\lesssim \Delta_n^2 |u|^4 \int_{(j-2) \varepsilon}^{j \varepsilon} \int_{(j-2) \varepsilon}^{j \varepsilon} e^{-C_1 |u|^2 (r-r')} \ dr \ dr' \lesssim \Delta_n^2 |u|^2 \varepsilon (1 - e^{-C_1 |u|^2 \varepsilon}),
\]

(23)
proving (27). For (28), on the other hand, we have

\[ \mathbb{E}[||E|] \sum_{k \in I_{(T)}} \int_{t_{k-1}}^{t_k} |\mathbb{E}[U_{r,r(e)} - U_{t_{k-3}(j-3)e}]|^2 dr |^2 \lesssim \Delta^4_n (\varepsilon / \Delta_n)^2 \sup_{k \in \{1, \ldots, n\}, \tau_{k-1} \leq r < t_k} \mathbb{E}[||E|] \mathbb{E}[U_{r,r(e)} - U_{t_{k-3}(j-3)e}]|^2 |^2 \lesssim \Delta^2_n \varepsilon^2 |u|^8 e^{-C_1 |u|^2 \varepsilon \eta_n}, \]

using (23) in the last line, which holds here exactly as above if we set \( r^* = \max(r(e), (j-3)e) \) and recall that \( \varepsilon > \Delta_n \).

5.2. Proof of Theorem 7

For the proof set \( \xi = X_0 \) and replace \( X \) by \( X - X_0 \). Since \( X \) with deterministic \( b \) and \( \sigma \) has independent increments, \( X \) and \( \xi \) are independent. Under the stated assumptions on \( b \) and \( \sigma \), Assumption \( S_{\text{loc}}(\alpha; \beta) \) holds true with \( \alpha = \beta = 0 \), except for the boundedness of \( X \) on \([0, T]\). Even without this property Lemmas 12 and 13 hold true (the boundedness of \( X \) was not used in the proofs). We can now repeat the arguments in Section 5.1.3 and obtain the claimed result for \( f \in H^1(\mathbb{R}^d) \) by the decomposition (4) and by applying Propositions 14 and 15. We are left with showing \( \Delta_{n}^{-1} D_{t,n}(f) \xrightarrow{\mathbb{P}} 0 \), which follows from the next proposition. As compared to the general semimartingale case the key property for deterministic \( b, \sigma \) is that the characteristic functions of the marginals \( X_t \) can be computed explicitly.

**Proposition 22.** Suppose that \( \xi \) has a bounded Lebesgue density and assume that \( b, \sigma \) are deterministic càdlàg functions and that \( \sup_{0 \leq t \leq T} |(\sigma_t \sigma_t^\top)^{-1}| < \infty \). Then we have for \( f \in H^1(\mathbb{R}^d) \) and \( 0 \leq t \leq T \)

\[ \Delta_{n}^{-1} D_{t,n}(f) \xrightarrow{\mathbb{P}} 0. \]  

**Proof.** As in Section 5.1.4 apply the upper bound (15) and define the \( g_{n}^{(i)}(u) \) this time without the supremum over \( 0 \leq t \leq T \) as

\[ g_{n}^{(i)}(u) = \Delta_{n}^{-2}(1 + |u|^2)^{-1} \mathbb{E} \left[ |F_{t,n}^{(i)}(u)|^2 \right] \]

with \( F_{t,n}^{(i)}(u) \) in (13) and (14). In order to conclude as after (16) by dominated convergence, we need to show \( g_{n}^{(i)}(u) \to 0 \) as \( n \to \infty \) for all \( u \in \mathbb{R}^d \) and
\[ \sup_{n \in \mathbb{N}, u \in \mathbb{R}} g_n^{(i)}(u) < \infty. \] For \( g_n^{(i)}(u) \) this follows from repeating the proof of Lemma \[17\] word for word using instead of the approximation property of \( b \) in Assumption \[S_{loc}(\alpha; \beta)\] that \( b \) is càdlàg.

Next, consider \( g_n^{(2)}(u) \). The assumptions on \( b \) and \( \sigma \) imply that \( \langle u, X_r - X_h \rangle \) is independent of \( \mathcal{F}_h \) for all \( 0 \leq h < r \leq t \) and is \( N(\int_h^r \langle u, b_r \rangle dr', \int_h^r |\sigma_r u|^2 dr') \)-distributed. This means

\[
\mathbb{E} \left[ e^{-i\langle u, X_r - X_h \rangle} \mid \mathcal{F}_h \right] = \mathbb{E} \left[ e^{-i\langle u, X_r - X_h \rangle} \right] = e^{-\frac{1}{2} \int_h^r |\sigma_r u|^2 dr'}.
\] (31)

From this we find that

\[
4 \mathbb{E} \left[ |F_{t,n}^{(i)}(u)|^2 \right] = \mathbb{E} \left[ \left| \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} e^{-i\langle u, X_{tk-1} \rangle} \int_{tk-1}^{tk} \left( tk - r - \frac{\Delta_n}{2} \right) |\sigma_r u|^2 \mathbb{E} \left[ e^{-i\langle u, X_r - X_{tk-1} \rangle} \mid \mathcal{F}_{tk-1} \right] dr \right|^2 \right] = \mathbb{E} \left[ \left| \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} e^{-i\langle u, X_{tk-1} \rangle} \int_{tk-1}^{tk} \left( tk - r - \frac{\Delta_n}{2} \right) |\sigma_r u|^2 e^{-\frac{1}{2} \int_{tk-1}^{r} |\sigma_r u|^2 dr'} dr \right|^2 \right].
\]

Introduce the family of functions \( \kappa(u, r) = |\sigma_r u|^2 e^{-\frac{1}{2} \int_{tk-1}^{r} |\sigma_r u|^2 dr'} \) and with this

\[
\tilde{\kappa}(k, u) = \int_{tk-1}^{tk} \left( tk - r - \frac{\Delta_n}{2} \right) (\kappa(r, u) - \kappa(tk-1, u)) dr.
\]

Since \( \int_{tk-1}^{tk} (tk - r - \Delta_n/2) dr = 0 \), we find that

\[
4 \mathbb{E} \left[ |F_{t,n}^{(i)}(u)|^2 \right] = \mathbb{E} \left[ \left| \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} e^{-i\langle u, X_{tk-1} \rangle} \tilde{\kappa}(k, u) \right|^2 \right] = \mathbb{E} \left[ \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \sum_{k'=1}^{\lfloor t/\Delta_n \rfloor} \tilde{\kappa}(k, u) \tilde{\kappa}(k', u) \mathbb{E} \left[ e^{-i\langle u, X_{tk-1} - X_{tk'-1} \rangle} \right] \right].
\]

The ellipticity of \( \sigma_r \sigma_r^\top \) implies the existence of a constant \( C_1 > 0 \) with \( \inf_r |\sigma_r u|^2 = \inf_r \langle \sigma_r \sigma_r^\top u, u \rangle \geq C_1 |u|^2 \). It thus follows from (31) that

\[
\mathbb{E} \left[ e^{-i\langle u, X_{tk-1} - X_{tk'-1} \rangle} \right] \leq e^{-C_1 |u|^2 |tk-1 - tk'-1|}.
\]
Using this in the last display and upper bounding the integrand in \( \tilde{\kappa}(k, u) \) yields at last

\[
g_n^{(2)}(u) \leq \Delta_n^2 |u|^2 \sum_{k, k' = 1}^n e^{-C_1|u|^2|t_{k-1} - t_{k'-1}|} \sup_{k \in\{1, \ldots\}, t_{k-1} \leq r \leq t_k} |\kappa(r, u) - \kappa(t_{k-1}, u)|^2 / |u|^4.
\]

As \( \sigma \) is càdlàg, observe that \(|u|^{-4} \sup_{k \in\{1, \ldots\}, t_{k-1} \leq r \leq t_k} |\kappa(r, u) - \kappa(t_{k-1}, u)|^2 \to 0 \) for any fixed \( u \in \mathbb{R}^d \) as \( n \to \infty \) and that

\[
\sup_{n \in \mathbb{N}, u \in \mathbb{R}^d} \sup_{k \in\{1, \ldots\}, t_{k-1} \leq r \leq t_k} |\kappa(r, u) - \kappa(t_{k-1}, u)|^2 / |u|^4 < \infty,
\]

while also

\[
\Delta_n^2 |u|^2 \sum_{k, k' = 1}^n e^{-C_1|u|^2|t_{k-1} - t_{k'-1}|} \leq |u|^2 \int_0^T \int_0^T e^{-C_1|u|^2|t - t'|} dt dt' \\
\leq 2|u|^2 \int_0^T \int_0^T e^{-C_1|u|^2(t - t')} dt dt' \lesssim |u|^2 \int_0^T e^{-C_1|u|^2 t} dt.
\]

From this obtain \( g_n^{(i)}(u) \to 0 \) as \( n \to \infty \) for all \( u \in \mathbb{R}^d \) and \( \sup_{n \in \mathbb{N}, u \in \mathbb{R}^d} g_n^{(i)}(u) < \infty \), which is what we still needed to show. \( \square \)

### 5.3. Proof of Theorem 8

Recall that \( X \) has independent increments as stated in the proof of Theorem 7. The estimation error for \( t \in [0, \Delta] \) is treated separately. Write \( \Gamma_t(f) - \hat{\Gamma}_{t,n}(f) = E_{0,n} + E_{1,n} \) with

\[
E_{0,n} = \int_0^{\Delta_n} (f(X_r) - f(0)) dr, \quad E_{1,n} = \sum_{k=2}^{\lfloor t/\Delta_n \rfloor} \int_{t_{k-1}}^{t_k} (f(X_r) - f(X_{t_{k-1}})) dr.
\]

By a Sobolev embedding deduce for \( f \in H^1(\mathbb{R}) \) that \( f \) is \( \gamma = 1/2 \)-Hölder continuous, that is,

\[
\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\gamma}} < \infty. \quad (32)
\]

In particular,

\[
\mathbb{E}[|E_{0,n}|] \lesssim \Delta_n \sup_{0 \leq r \leq \Delta_n} \mathbb{E}[|X_r|^{\gamma}].
\]
implying $E_{0,n} = o_P(\Delta_n)$. For $E_{1,n}$, we use the decomposition (4) (with sums starting at $k = 2$) and aim at applying Propositions 14 and 15 to $M_{t,n}(f)$ and $E_{t,n}(f)$ with $Y = X$ (that is with $\xi = 0$). The respective proofs depend on $\xi$ only through applications of Lemma 13 and a specific argument for (8).

We first check that the statements (i)-(v) of Lemma 13 hold for $f \in H^1(\mathbb{R})$, $Y = X$ and $k \geq 2$. Part (v) of that lemma holds again by the Hölder continuity in (32). The corresponding statements in parts (i)-(iv), by independence of increments, expressions of the form $\sum_{k=2}^n \int_{t_k}^{t_{k-1}} \mathbb{E}[m_r^2(X_{t_k-1})]dr$ need to be bounded for certain random processes $m_r$. Denote the Lebesgue density of $X_{t_k-1}$ by $p_{t_k-1}$. Due to Gaussianity and the non-degeneracy of $\sigma$ we have $p_{t_k-1} \lesssim t_k^{-1/2}$ such that

$$\int_{t_{k-1}}^{t_k} \mathbb{E}[m_r^2(X_{t_k-1})]dr \lesssim t_k^{-1/2} \int_{t_k-1}^{t_k} \int_{\mathbb{R}} \mathbb{E}[m_r^2(x)]dx \, dr.$$  

Using now the second inequality in Lemma 3 the proofs of (i)-(iv) in Lemma 13 provide uniform bounds on $\sup_{0 \leq t \leq T} \int_{\mathbb{R}} m_r^2(x)dx$. Since $\Delta_n \sum_{k=2}^n t_k^{-1/2}$ is summable, (i)-(iv) of Lemma 13 (with $k \geq 2$) remain true. The modification for (8) is analogous and therefore skipped. The conclusions of Propositions 14 and 15 hold true, and we conclude by the following proposition.

**Proposition 23.** Let $d = 1$ and suppose that $b, \sigma$ are deterministic càdlàg functions and that $\inf_{0 \leq t \leq T} \sigma_t^2 > 0$. Then we have for $f \in H^1(\mathbb{R})$, $\xi = 0$ and $0 \leq t \leq T$

$$\Delta_n^{-1} D_{t,n}(f) \xrightarrow{P} 0.$$  

**Proof.** By independence of increments write $D_{t,n}(f) = \sum_{k=2}^n h_k(X_{t_k-1})$ with

$$h_k(x) = \int_{t_{k-1}}^{t_k} \mathbb{E}[f(X_r - X_{t_k-1} + x) - f(x) - \frac{\Delta_n}{2}(f(X_{t_k} - X_{t_k-1} + x) - f(x))]dr.$$  

Decompose $\mathbb{E}[|D_{t,n}(f)|^2] = R_1 + R_2$ with $R_1 = \sum_{k=2}^n \mathbb{E}[h_k(X_{t_k-1})^2]$, $R_2 = \sum_{k \neq j, k,j \geq 2} \mathbb{E}[h_k(X_{t_k-1})h_j(X_{t_j-1})]$. It suffices to show $R_1 = o(\Delta_n^2)$ and $R_2 = o(\Delta_n^2)$.
By Lemma \[13\] (iii) (which can be applied by the arguments at the beginning of this section) and the Cauchy-Schwarz inequality we have \(R_1 = \sum_{k=2}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[\tilde{h}_k^2(X_{t_k-1})] + o(\Delta_n^2)\) with

\[
\tilde{h}_k(x) = \int_{t_{k-1}}^{t_k} \mathbb{E} \left[ f'(x) \left( X_r - X_{t_{k-1}} - \frac{\Delta_n}{2} (X_{t_k} - X_{t_{k-1}}) \right) \right] dr
\]

\[
= f'(x) \mathbb{E} \left[ \int_{t_{k-1}}^{t_k} (t_k - r - \frac{\Delta_n}{2}) dX_r \right] = f'(x) \int_{t_{k-1}}^{t_k} (t_k - r - \frac{\Delta_n}{2}) b_r dr,
\]

concluding by the stochastic Fubini theorem in the last line. Hence,

\[
\sum_{k=2}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[\tilde{h}_k^2(X_{t_k-1})] \lesssim \sum_{k=2}^{\lfloor t/\Delta_n \rfloor} \int_{\mathbb{R}} \tilde{h}_k^2(x) dx \lesssim \Delta_n^3 \|f\|^2_{H^1},
\]

which shows \(R_1 = o(\Delta_n^2)\).

With respect to \(R_2\) and \(u \in \mathbb{R}\) denote by \(F_{t,n,k}(u), F_{t,n,k}^{(2)}(u)\) the summands in the sums \([13], [14]\) from Section \[5.1.4\] such that \(|\mathcal{F} h_k(u)| = |\mathcal{F} f(u)| |F_{t,n,k}^{(1)}(u) + F_{t,n,k}^{(2)}(u)|\). Let now \(p_{t_{k-1},t_{j-1}}\) denote the joint Lebesgue density of \((X_{t_{k-1}}, X_{t_{j-1}})\). Non-degeneracy of \(\sigma\) and noting that \(X\) is a Gaussian process yields for \(k > j\)

\[
|\mathcal{F} p_{t_{k-1},t_{j-1}}(u,v)| \leq e^{-C_1 |u|^2(t_{k-1}-t_{j-1})} e^{-C_1 |u+v|^2t_{j-1}} \leq e^{-C_1 |u+v|^2t_{j-1}}
\]

for some \(C_1 > 0\), implying uniformly in \(u \in \mathbb{R}\)

\[
\sum_{k \neq j, k,j \geq 2} \int_{\mathbb{R}} |\mathcal{F} p_{t_{k-1},t_{j-1}}(u,v)| dv \lesssim \sum_{k \neq j, k,j \geq 2} \int_{\mathbb{R}} e^{-C_1 |u+v|^2 \min(t_{j-1},t_{k-1})} dv
\]

\[
\lesssim \Delta_n^2 \int_0^T \int_{\mathbb{R}} e^{-C_1 |u+v|^2 r} dr dv \lesssim \Delta_n^{-2} \int_0^T r^{-1/2} dr \lesssim \Delta_n^{-2},
\]

The same result holds uniformly in \(v \in \mathbb{R}\) when integrating with respect to \(u\). The Plancherel theorem and the Cauchy-Schwarz inequality yield

\[
|R_2| = (2\pi)^{-1} \left| \sum_{k \neq j}^{\lfloor t/\Delta_n \rfloor} \mathcal{F} h_k(u) \mathcal{F} h_j(v) \mathcal{F} p_{t_{k-1},t_{j-1}}(u,v) d(u,v) \right|
\]

\[
\lesssim \sum_{k \neq j, k,j \geq 2} \int_{\mathbb{R}} |\mathcal{F} h_k(u)|^2 |\mathcal{F} p_{t_{k-1},t_{j-1}}(u,v)| d(u,v)
\]

\[
\lesssim \int_{\mathbb{R}} |\mathcal{F} f(u)|^2 (1 + |u|^2) g_n(u) du,
\]

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\[ g_n(u) = (1 + |u|^2)^{-1} \sup_{k \in \{1, \ldots, n\}} \Delta_n^{-2} |F_{t,n,k}^{(1)}(u) + F_{t,n,k}^{(2)}(u)|^2. \]

It is easy to check that the upper bounds on the terms \( g_n^{(i)}(u) \) in the proof of Proposition 22 yield for the summands considered here that \( \sup_{n \in \mathbb{N}, u \in \mathbb{R}} g_n(u) < \infty \) and \( g_n(u) \to 0 \) for all \( u \in \mathbb{R} \) as \( n \to \infty \). Consequently, by dominated convergence \( R_2 = o(\Delta_n^2) \).

5.4. Proof of Theorem 10

By arguing as in Section 5.1.1 it is enough to prove the CLT for \( f \in FL^s(\mathbb{R}^d) \) for \( s \geq 1 \) under Assumption \( S_{loc}(\alpha; \beta) \). We use the decomposition (4) and aim at applying Propositions 14 and 15 to \( M_{t,n}^k(f) \) and \( E_{t,n}^k(f) \) with \( Y = X \) (that is with \( \xi = 0 \)). The respective proofs depend on \( f \) only through applications of Lemma 13 and a specific argument for (8). We first check that the statements (i)-(v) of Lemma 13 hold for \( f \in FL^s(\mathbb{R}^d) \) and \( Y = X \).

From the boundedness of \( X \) in Assumption \( S_{loc}(\alpha; \beta) \) and the embedding \( FL^1(\mathbb{R}^d) \subseteq C^1(\mathbb{R}^d) \) we get \( f \in C^1(\mathbb{R}^d) \) and that the processes \( t \mapsto f(X_t) \) and \( t \mapsto |\nabla f(X_t)| \) are uniformly bounded. This and the continuity of paths of \( X \) already imply (i), (iv) and (v) of Lemma 13 while for (ii) we also use \( \langle \nabla f(X_{t_k-1}), Z_r \rangle^2 \lesssim |Z_r|^2 \) and \( \sup_r \mathbb{E}|Z_r|^2 \lesssim \Delta_n \) using Lemma 12(i,ii). At last, (iii) is obtained from the same upper bound on \( \mathbb{E}|Z_r|^2 \) and a Taylor expansion of \( f \). On the other hand, with \( \tilde{Z}_k \) from the proof of Proposition 14 we have for a random variable \( V_k \sim N(0, I_d) \), which is independent of \( F_{t_k-1}^k \), and \( \varepsilon' > 0 \) that

\[ \mathbb{E}[\Delta_n^3 |V_k|^2 1_{\{\Delta_n^{3/2} |V_k| > \varepsilon'\}}] \rightarrow 0, \]

where the conclusion holds by dominated convergence. This proves (8) and the conclusions of Propositions 14 and 15 hold true. We conclude the proof of the CLT by the following proposition.

**Proposition 24.** Let \( s \geq 1 \) and grant Assumption \( S_{loc}(\alpha; \beta) \) with \( \alpha > \max(0, 1 - s/2), \beta > 0 \). Then we have for \( f \in FL^s(\mathbb{R}^d) \)

\[ \Delta_n^{-1} D_{t,n}^k(f) \xrightarrow{ucp} \frac{1}{2} (f(X_t) - f(X_0)) - \frac{1}{2} \int_0^t \langle \nabla f(X_r), \sigma_r dW_r \rangle. \]
Proof. We use the notation from Section 5.1.4. Write $D_{t,n}(f) = h(0)$ with
the function $h$ defined there. By Fourier inversion $h(0) = (2\pi)^{-d} \int_{\mathbb{R}^d} \mathcal{F}h(u) du$
$\mathbb{P}$-almost surely such that by the triangle inequality
\[
\mathbb{E}[\sup_{0 \leq t \leq T} |D_{t,n}(f)|] = \mathbb{E}[\sup_{0 \leq t \leq T} |h(0)|] \lesssim \int_{\mathbb{R}^d} |\mathcal{F}f(u)| \mathbb{E}\left[\sup_{0 \leq t \leq T} |F_{t,n}^{(1)}(u) + F_{t,n}^{(2)}(u)|\right] du
\leq \Delta_n \int_{\mathbb{R}^d} |\mathcal{F}f(u)|(1 + |u|^s)(g_n^{(1)}(u) + g_n^{(2)}(u))^{1/2} du.
\]
The result follows from dominated convergence, using Lemmas 17 and 18.

5.5. The lower bound: Proof of Theorem 11

The sigma field $\mathcal{G}_n$ is generated by $X_0$ and the increments $X_{t_k} - X_{t_{k-1}}$
for $k \in \{1, \ldots, n\}$. Their independence and the Markov property imply
$\mathbb{E}[f(X_t)|\mathcal{G}_n] = \mathbb{E}[f(X_t)|X_{t_{k-1}}, X_{t_k}], t_{k-1} \leq t \leq t_k$. By the same argument
the random variables
\[
Y_k = \int_{t_{k-1}}^{t_k} (f(X_t) - \mathbb{E}[f(X_t)|\mathcal{G}_n]) dt
\]
are uncorrelated, implying
\[
\|\Gamma_T(f) - \mathbb{E} [\Gamma_T(f)|\mathcal{G}_n]\|_{L^2(\mathbb{P})}^2 = \sum_{k=1}^{n} \mathbb{E} [Y_k^2] = \sum_{k=1}^{n} \mathbb{E}\left[\text{Var}_k\left(\int_{t_{k-1}}^{t_k} f(X_t) dt\right)\right],
\]
where $\text{Var}_k(Z) = \mathbb{E}[(Z - \mathbb{E}[Z|X_{t_{k-1}}, X_{t_k}])^2|X_{t_{k-1}}, X_{t_k}]$ is the conditional variance
of a random variable $Z$ with respect to the sigma field generated by $X_{t_{k-1}}$ and $X_{t_k}$. For fixed $k$ write
\[
\text{Var}_k\left(\int_{t_{k-1}}^{t_k} f(X_t) dt\right) = \text{Var}_k\left(\int_{t_{k-1}}^{t_k} (f(X_t) - f(X_{t_{k-1}})) dt\right) = T_k^{(1)} + T_k^{(2)} + T_k^{(3)}
\]
with $T_k^{(1)} = \text{Var}_k\left(\int_{t_{k-1}}^{t_k} \langle \nabla f(X_{t_{k-1}}), X_t - X_{t_{k-1}} \rangle dt\right)$,
$T_k^{(2)} = \text{Var}_k\left(\int_{t_{k-1}}^{t_k} (f(X_t) - f(X_{t_{k-1}}) - \langle \nabla f(X_{t_{k-1}}), X_t - X_{t_{k-1}} \rangle) dt\right)$,
and with the crossterm satisfying $|T_k^{(3)}| \leq 2(T_k^{(1)}T_k^{(2)})^{1/2}$. Conditional on $X_{t_{k-1}}, X_t$, the process $(X_t)_{t_{k-1} \leq t \leq t_k}$ is a Brownian bridge starting from $X_{t_{k-1}}$ and ending at $X_t$. Hence,

$$
\mathbb{E}[X_t - X_{t_{k-1}} | X_{t_{k-1}}, X_t] = \frac{t - t_{k-1}}{\Delta_n}(X_t - X_{t_{k-1}}),
$$

cf. Equation 6.10 of [18]. Write $X_t - X_{t_{k-1}} = \int_{t_{k-1}}^{t_k} 1_{(r \leq t)} dX_r$. Then

$$
\int_{t_{k-1}}^{t_k} (X_t - X_{t_{k-1}} - \mathbb{E}[X_t - X_{t_{k-1}} | X_{t_{k-1}}, X_t]) dt
$$

$$
= \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} (1_{(r \leq t)} - \frac{t - t_{k-1}}{\Delta_n}) dX_r dt = \int_{t_{k-1}}^{t_k} (t_k - r - \frac{1}{2}\Delta_n) dX_r,
$$

using the stochastic Fubini theorem in the last line. From Itô’s isometry and independence of increments obtain

$$
\mathbb{E}[T_k^{(1)}] = \mathbb{E} \left[ |\nabla f(X_{t_{k-1}})|^2 \right] \int_{t_{k-1}}^{t_k} (t_k - r - \frac{1}{2}\Delta_n)^2 dr = \frac{\Delta_n^3}{12} \mathbb{E} \left[ |\nabla f(X_{t_{k-1}})|^2 \right].
$$

Recall from the proofs of Theorems 7 and 8 that the statements of Lemma 13(iii) apply to $X$ with deterministic coefficients $b$ and $\sigma$ when $X_0$ has a bounded Lebesgue density or when $d = 1$. Consequently,

$$
\sum_{k=1}^{n} \mathbb{E}[T_k^{(1)}] = \mathbb{E} \left[ \frac{1}{12} \int_0^T |\nabla f(X_t)|^2 dt \right] + o(\Delta_n^2).
$$

On the other hand, the Cauchy-Schwarz inequality shows

$$
\mathbb{E}\left[ \sum_{k=1}^{n} T_k^{(2)} \right] \leq \sum_{k=1}^{n} \mathbb{E} \left[ \left( \int_{t_{k-1}}^{t_k} (f(X_t) - f(X_{t_{k-1}})) dt \right)^2 \right]
$$

$$
\leq \Delta_n \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \mathbb{E} \left[ (f(X_t) - f(X_{t_{k-1}})) dt \right] dt,
$$

which is of order $o(\Delta_n^2)$ by Lemma 13(iii). Combining the last two displays also shows $\sum_{k=1}^{n} T_k^{(3)} = o_\mathbb{P}(\Delta_n^2)$. The result follows then from

$$
\Delta_n^{-2} \sum_{k=1}^{n} \mathbb{E} \left[ \text{Var}_k \left( \int_{t_{k-1}}^{t_k} f(X_t) dt \right) \right] \rightarrow \mathbb{E} \left[ \frac{1}{12} \int_0^T |\nabla f(X_t)|^2 dt \right].
$$
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