Configuration space approach to nonrelativistic conformal dynamics

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Abstract. Lagrangian point of view on conformally invariant nonrelativistic dynamics is presented.

1. Introduction
We present here the review of the results obtained in our paper [1]. Our aim is to discuss the configuration space point of view of classical dynamical systems invariant under the action of the so-called N-conformal nonrelativistic symmetries. We start with very brief discussion of the orbit method [2] which provides the Hamiltonian description of dynamical systems exhibiting some symmetry. Then we remind the structure of N-conformal Galilean algebras and construct the dynamical systems invariant under the transitive action of these algebras. In the following chapter it is shown that the canonical transformations generated by the action of symmetry algebras can be rewritten as point transformation. We provide also the integrals of motion implied by Noether theory. Last section is devoted to some conclusion.

2. The orbit method
The main idea of orbit method can be described as follows: Given a Lie group $G$ we are looking for all phase manifolds such that:

- $G$ acts as a group of canonical transformations (i.e. it leaves the symplectic form invariant),
- $G$ acts transitively.

The solution of this problem is provided by the orbit method. Namely, one can show that all symplectic manifolds with the symplectic structure invariant under the transitive action of $G$ are coadjoint orbits equipped with the Kirillov form.

The resulting Poisson structure can be described as follows. Let $\{\tilde{X}_i\}$ be a basis in $\tilde{L}$ with the commutation rules

$$[\tilde{X}_i, \tilde{X}_j] = i\epsilon^{ij}_k \tilde{X}_k$$

(1)

Denote by $\{\xi_i\}$ the coordinates in $\tilde{L}$ (dual to $L$) corresponding to the basis $\{\tilde{X}^i\}$. Then the Poisson structure implied by Kirillov symplectic form reads

$$\{\xi_i, \xi_j\} = \epsilon^{k}_{ij} \xi_k.$$  

(2)
With the phase space at our disposal we can define the dynamical system by selecting some function on coadjoint orbit as a Hamiltonian $H$. We assume that $H$ is an element of $L$ (in general one could assume that $H$ belongs to the universal enveloping algebra). We see that, in general, $H$ does not commute with the remaining generators (in fact, it cannot commute with all generators; if it commuted, the dynamics, due to transitivity assumption, would be trivial). As a result, some of the integrals of motion implied by the symmetry group $G$ should be explicitly time-dependent (vide Galilei group).

3. N-conformal Galilean algebras $g_N$

The N-conformal nonrelativistic algebras $g_N$ have been introduced by J. Negro, M.A. del Olmo, A. Rodriguez-Marco [3]. They are defined by the following commutation rules:

$$
[J_j, J_k] = i\epsilon_{jkl} J_l, \quad [D, H] = iH, \quad [D, K] = -iK, \quad [K, H] = 2iD,
$$

$$
[D_s, C^a_j] = iC^a_j, \quad [J^a, C^b_j] = i\epsilon_{abc} C^c_j, \quad [H, C^a_j] = -ijC^a_{j-1},
$$

$$
[D, C^a_j] = i(N - j)C^a_j, \quad [K, C^a_j] = i(N - j)C^a_{j+1},
$$

$a, b, d = 1, 2, 3 \quad j, k, l = 0, 1, ..., N$, all remaining commutators being vanishing.

In what follows we skip $D_s$. The resulting algebra has the following structure $\tilde{g}_N = (su(2) \oplus sl(2, \mathbb{R})) \uplus \{C^a_i\}$; $\{C^a_i\}$ span spin one representation of $su(2)$ and ”spin” $\frac{N}{2}$ representation of $sl(2, \mathbb{R})$.

In some cases the algebras $g_N$ admit central extensions. Namely, this is the case for $N$ odd; then the relevant commutation rule reads

$$
[C^a_j, C^b_k] = i\delta^{ab} \delta^{N,j+k} (-1)^{\frac{k-j+1}{2}} k!j! M
$$

$a, b = 1, 2, 3 \quad j, k = 0, 1, ..., N$ with $M$ being the new central element.

There is one exception: in two-dimensional space the central extension is allowed for all $N$.

In what follows we will be dealing with centrally extended $\tilde{g}_N$ (i.e. $N$ is odd because we assume three space dimensions).

In order to find the $G_N$-invariant systems we use the orbit method. The construction is highly simplified due to existence of central extension. It appears then that the orbit is uniquely defined by the choice of $sl(2, R)$ and $su(2)$ coadjoint orbits. The variables parameterizing these orbits we will refer to as internal variables. The remaining variables can be chosen in such a way that they form the Darboux coordinates. Let $\vec{c}_j$ be the coordinates on coadjoint orbits related to the generators $\vec{C}$ while $\chi, s$ are the internal coordinates corresponding to $sl(2, R)$ and $su(2)$ respectively.

We define

$$
\vec{c} = \begin{cases} 
(-1)^{j-\frac{N-1}{2}} \bar{p}_j, & j = 0, ..., \frac{N-1}{2} \\
mj!q_{N-j}, & j = \frac{N+1}{2}, ..., N.
\end{cases}
$$
Then \[4], \[5]\

\[
\begin{align*}
    h &= \chi^0 - \chi^1 + \frac{1}{2m} \vec{p}_{N-1} \vec{p}_{N-1} + \sum_{k=1}^{N-1} \vec{q}_k \vec{p}_{k-1}, \\
    d &= \chi^2 + \sum_{k=0}^{N-1} (\frac{N}{2} - k) \vec{q}_k \vec{p}_k, \\
    k &= \chi^0 + \chi^1 + m \left( \frac{N+1}{2} \right)^2 \vec{q}_{N-1} \vec{q}_{N-1} - \sum_{k=0}^{N-3} (N-k)(k+1) \vec{q}_k \vec{p}_{k+1}, \\
    \vec{j} &= \vec{s} + \sum_{k=0}^{N-1} \vec{q}_k \times \vec{p}_k;
\end{align*}
\]

(6)

here \( m \) is the value of central charge \( M \).

The relevant Poisson brackets read off from Kirillov form are

\[
\begin{align*}
    \{ q^a_k, p^b_l \} &= \delta^{ab} \delta_{kl}, \\
    \{ s^a, s^b \} &= \epsilon^{abc} s^c, \quad \vec{s} = \text{const.}
\end{align*}
\]

(7)

(in the last equation raising and lowering indices is performed with the help of \( g_{\mu\nu} = \text{diag}(+-) \))

Summarizing, we arrive at the following structure. There are "external" variables \( \vec{q}_k, \vec{p}_k, k = 0, \ldots, \frac{N-1}{2} \) obeying the canonical equations of motion given by the Hamiltonian

\[
H = \frac{1}{2m} \vec{p}_{N-1} \cdot \vec{p}_{N-1} + \sum_{k=1}^{N-1} \vec{q}_k \cdot \vec{p}_{k-1}
\]

(8)

In addition, there are "internal" variables \( \chi^\mu \) and \( s_a \) related to \( sl(2, \mathbb{R}) \) and \( su(2) \) subalgebras. The dynamics of \( \chi \)'s is governed by the Hamiltonian

\[
H = \chi^0 - \chi^1.
\]

(9)

while the spin variables are constants of motion.

One can easily show [6] that the \( sl(2, \mathbb{R}) \) dynamics is described by the conformal mechanics of Alfaro, Fubini and Furlan [6]. In what follows we select trivial orbit for \( su(2) \) and \( sl(2, \mathbb{R}) \), i.e. there are no internal variables.

We see that (7) is the Ostrogradski Hamiltonian for the system described by higher-derivative Lagrangian:

\[
L = \frac{m}{2} \left( \frac{d \frac{N+1}{2} \vec{q}}{dt} \right)^2.
\]

(10)
This can be easily seen by writing out the canonical equations of motion

\[
\begin{align*}
\dot{q}_k &= q_{k+1}, \quad k = 0, \ldots, \frac{N-3}{2}; \\
\dot{p}_k &= -p_{k-1}, \quad k = 1, \ldots, \frac{N-1}{2}; \\
\dot{p}_0 &= 0,
\end{align*}
\]

which, for the basic variable \( \vec{q} = \vec{q}_0 \), imply \( \vec{q}^{(N+1)} = 0 \).

So, in the particular case of trivial \( SU(2) \) and \( SL(2, \mathbb{R}) \) orbits, \( \hat{g}_N \) is the symmetry algebra of higher derivative free theory [4].

4. Integrals of motion and canonical transformation

It is a matter of standard procedure to obtain the Noether charges corresponding to the generators of Lie algebra. They read [4], [5]

\[
\begin{align*}
h &= h(t), \\
d &= d(t) - th(t), \\
k &= k(t) - 2td(t) + t^2h(t), \\
\vec{j} &= \vec{j}(t), \\
\vec{c}_j &= \begin{cases} 
(-1)^j \frac{j!}{(j-k)!} \sum_{k=0}^{j} \frac{j!}{(j-k)!} p_k, & 0 \leq j \leq \frac{N-1}{2} \\
(-1)^j \frac{j!}{(j-k)!} \sum_{k=0}^{N-1} \frac{j!}{(j-k)!} p_k + m \sum_{k=0}^{n} (-1)^{j-k} \frac{j!}{(j-k)!} p_k, & \frac{N+1}{2} \leq j \leq N.
\end{cases}
\end{align*}
\]

where \( d(t), h(t), k(t) \) and \( \vec{j}(t) \) are given by eqs. (5) with \( \chi^t \) and \( \vec{s} = 0 \). These charges generate the canonical transformations representing the action of \( N \)-conformal Galilean group on Hamiltonian level. Computing systematically the infinitesimal action of all generators (\( \delta_0 = G, f \)) we find:

- for \( \vec{c}_j \):

\[
\begin{align*}
\delta_0 \vec{q}_n &= \sum_j \vec{x}_j \vec{c}_j, \vec{q}_n = \sum_{k=0}^{N-n} (-1)^{k+n} \frac{N+n}{2} \frac{(k+n)!}{k!} \dot{\vec{x}}_{k+n}, \\
\delta_0 \vec{p}_n &= \sum_j \vec{x}_j \vec{c}_j, \vec{p}_n = m \sum_{k=0}^{n} (-1)^k \frac{(k+N-n)!}{k!} \dot{\vec{x}}_{k+n-N-n},
\end{align*}
\]

- for \( h \):

\[
\begin{align*}
\delta_0 \vec{q}_n &= \{ \tau h, \vec{q}_n \} = -\tau \left( \frac{1}{m} \delta_{\frac{N+n}{2}} \vec{p}_{\frac{N+n}{2}} + (1 - \delta_{\frac{N+n}{2}}) \vec{q}_{n+1} \right), \\
\delta_0 \vec{p}_n &= \{ \tau h, \vec{p}_n \} = \tau (1 - \delta_{n0}) \vec{p}_{n-1},
\end{align*}
\]

- for \( d \):

\[
\begin{align*}
\delta_0 \vec{q}_n &= \{ \lambda d, \vec{q}_n \} = \lambda \left( -\left( \frac{N}{2} - n \right) \vec{q}_n + \frac{t}{m} \vec{p}_{\frac{N-1}{2}+n} + \left( 1 - \delta_{\frac{N-1}{2}+n} \right) \vec{q}_{n+1} \right), \\
\delta_0 \vec{p}_n &= \{ \lambda d, \vec{p}_n \} = \lambda \left( \left( \frac{N}{2} - n \right) \vec{p}_n - t(1 - \delta_{n0}) \vec{p}_{n-1} \right),
\end{align*}
\]

- for \( k \):

\[
\begin{align*}
\delta_0 \vec{q}_n &= \{ \lambda k, \vec{q}_n \} = \lambda \left( -\left( \frac{N}{2} - n \right) \vec{q}_n + \frac{t}{m} \vec{p}_{\frac{N-1}{2}+n} + \left( 1 - \delta_{\frac{N-1}{2}+n} \right) \vec{q}_{n+1} \right), \\
\delta_0 \vec{p}_n &= \{ \lambda k, \vec{p}_n \} = \lambda \left( \left( \frac{N}{2} - n \right) \vec{p}_n - t(1 - \delta_{n0}) \vec{p}_{n-1} \right),
\end{align*}
\]
 transformations rewritten as a canonical ones have a special form: 
the point transformation on Lagrangian level. In most cases the answer is no.

\[ \delta_0 \vec{q}_n = \{ ck, \vec{q}_n \} = c \left[ (1 - \delta_{n0}) n(N - n + 1) \vec{q}_{n-1} + 2t \left( \frac{N}{2} - n \right) \vec{q}_n - t^2 \left( \frac{1}{m} \delta_{n, \frac{N-1}{2}} \vec{p}_{\frac{N-1}{2}} + (1 - \delta_{n, \frac{N-1}{2}}) \vec{q}_{n+1} \right) \right], \]

\[ \delta_0 \vec{p}_n = \{ ck, \vec{p}_n \} = c \left[ m \delta_{n, \frac{N-1}{2}} \left( \frac{N + 1}{2} \right)^2 \vec{q}_{\frac{N-1}{2}} - (1 - \delta_{n, \frac{N-1}{2}})(N - n)(n + 1) \vec{p}_{n+1} - 2t \left( \frac{N}{2} - n \right) \vec{p}_n + t^2 (1 - \delta_{n0}) \vec{p}_{n-1} \right]. \]

while \( \vec{j} \) acts in the standard way.

Now, given some canonical transformation one can pose the question whether it comes from
the point transformation on Lagrangian level. In most cases the answer is no, because the point
transformations rewritten as a canonical ones have a special form:

- new coordinates are the functions of old ones and time only
- new momenta are the linear functions of old momenta with coordinate- and time-dependent
coefficients (actually they can be more general affine functions if the Lagrangian transforms
by a total derivative).

However, the above statement is true only provided the time variable remains unchanged. If it
changes the momenta can enter the expression for the variations of coordinate variables provided
they appear only in the form of Poisson brackets of Hamiltonian with coordinates. Then the
terms containing momenta can be removed from transformation formulas at the expense of
admitting the time variation. The resulting modified point transformations coincide "on-shell"
(i.e. provided the equations of motion hold) with the initial canonical ones.

To be more explicit, let us write out the form of infinitesimal transformations both on
Lagrangian and Hamiltonian levels:

- **Lagrangian level**

\[ q'(t') = q(t) + \delta q(t) \quad t'(t) = t + \delta t(t) \]

\[ \delta q(t) = q'(t') - q(t) = q'(t) + q'(t) \delta t - q(t) \]

\[ \simeq q'(t) - q(t) + \delta t \{ q, H \} \]  

- **Hamiltonian level**

\[ q'(t) = q(t) + \delta_0 q(t), \quad \delta_0 q(t) = \delta \lambda \{ G, q(t) \} \]

\[ p'(t) = p(t) + \delta_0 p(t), \quad \delta_0 p(t) = \delta \lambda \{ G, p(t) \} \]

\[ \delta_0 q(t) = \delta q(t) - \delta t \{ q, H \}; \]

Note, that \( q(t) \longrightarrow q'(t') \) is the point transformation. Therefore, the form of \( \delta q(t) \) can be
found on Lagrangian level. Eq.(18) tells us then that the Hamiltonian form of the coordinate
transformation may involve momentum variables provided they enter in specific form.

The careful inspection of formulas defining the canonical action of N-conformal Galilei group
shows that they have the desired property described above and can be therefore rewritten as
the point transformations. Below we quote the final results. Moreover, we write out the global
form of the relevant transformations.

- transformations generated by \( \vec{c}_k \)’s:

\[ t' = t, \]

\[ \vec{q}_n(t') = \vec{q}_n(t) + \sum_{k=0}^{N-n} (-1)^{k+n-\frac{N+1}{2}} \frac{(k + n)!}{k!} t^k \vec{x}_{n+k}. \]
• transformations generated by $h$:

\[
t' = t + \tau, \\
\vec{q}_n'(t') = \vec{q}_n(t).
\]  

(21)

• transformations generated by dilatation $d$:

\[
t' = e^{-\lambda}t, \\
\vec{q}_n'(t') = e^{\lambda(n-\frac{N}{2})}\vec{q}_n(t).
\]  

(22)

• transformations generated by special conformal generator $k$:

\[
t' = \frac{t}{1-ct} \equiv t(c) \\
\vec{q}_n'(t') = \sum_{k=0}^{n} \binom{n}{k} \frac{(N+k-1)!}{(N-1)!} \frac{c^k}{(1-ct)^{N+k}} \vec{q}_{n-k}(t)
\]  

(23)

• finally, the action of rotation subgroup is standard.

As it is well-known [8] the coordinate space for higher-derivative theories consists of the original coordinate together with the supplementary ones which represent the consecutive time derivatives. It is easy to see that this structure is preserved by the above transformations. Namely, in all cases the following property holds:

\[
\vec{q}_{n+1} = \dot{\vec{q}}_n \quad \text{implies} \quad \vec{q}'_{n+1}(t') = \frac{d\vec{q}_n(t')}{dt'}
\]  

(24)

which allows us to go back from Ostrogradski coordinates to the original one $\vec{q} = \vec{q}_0$ entering the Lagrangian

\[
L = \frac{m}{2} \left( \frac{d^{N+1} \vec{q}}{dt^{N+1}} \right)^2.
\]  

(25)

In terms of this original coordinate $\vec{q}$ and $t$ we find the following transformation rules:

• for $\vec{c}_k$’s:

\[
t' = t, \quad \vec{q}'(t') = \vec{q}(t) + \sum_{k=0}^{N} (-1)^{k-\frac{N+1}{2}} t^k \vec{c}_k;
\]  

(26)

• for $h$:

\[
t' = t + \tau, \quad \vec{q}'(t') = \vec{q}(t);
\]  

(27)

• for $d$:
\( t' = e^{-\lambda t}, \quad \vec{q}'(t') = e^{-\lambda \frac{2}{\lambda}} \vec{q}(t) \); (28)

- for \( k \):

\[
\begin{align*}
 t' &= \frac{t}{1 - ct}, \quad \vec{q}'(t') = \frac{\vec{q}(t)}{(1 - ct)^2}; \\
\end{align*}
\] (29)

The above equations allow us to write out the differential realization of \( n \)-conformal Galilean algebra

\[
\begin{align*}
 \hat{H} &= i \frac{\partial}{\partial t}, \quad \hat{D} = -i \left( \frac{N}{2} \vec{q} \frac{\partial}{\partial \vec{q}} + t \frac{\partial}{\partial t} \right), \\
\hat{K} &= i \left( Nt \vec{q} \frac{\partial}{\partial q} + t^2 \frac{\partial}{\partial t} \right), \quad \hat{C}_k = i(-1)^{k-\frac{N}{2}+1} \frac{\partial}{\partial \vec{q}}. \\
\end{align*}
\] (30)

It is straightforward to check that the above operators obey the relevant algebra without central extension.

This is a general property - the geometrical realization of a given initial symmetry group in coordinate space leads, in general, to centrally extended group acting in phase space (with central charge acting trivially).

To complete the picture we can show that all integrals of motion (11) can be obtained via Noether theorem applied to higher derivative original Lagrangian

\[
L = L(\vec{q}(t), \frac{d\vec{q}(t)}{dt}, \frac{d^2\vec{q}(t)}{dt^2}, ..., \frac{d^{N+1}\vec{q}(t)}{dt^{N+1}}). 
\] (31)

Let us remind the form of Noether identity in the higher-derivative case. The symmetry condition reads

\[
L(\vec{q}'(t'), \frac{d\vec{q}'}{dt'}, \frac{d^{N+1}\vec{q}'}{dt'^{N+1}}) dt' = L(\vec{q}(t), \frac{d\vec{q}}{dt}, \frac{d^{N+1}\vec{q}}{dt^{N+1}}) + \frac{d}{dt}(\delta f), 
\] (32)

where

\[
\vec{q}' = \vec{q} + \epsilon \vec{q}(q, t), \quad t' = t + \epsilon g(t), \quad \text{and} \quad \delta f = \delta f(\vec{q}, \vec{q}', \vec{q}, ..., \vec{q}^{N+1}). 
\] (33)

By a straightforward reasoning, as in the first-order case, one concludes that the corresponding integral of motion is of the form

\[
C = Hg - \sum_{k=0}^{N+1} \tilde{p}_k \chi^{(k)} + \frac{N+1}{2} \sum_{n=2}^{N-1} \sum_{k=1}^{n-1} \sum_{l=0}^{k-l-1} \frac{d^{k-l-1} \delta q^{(n-k)}}{dt^{k-l-1}} \left( - \frac{d^l}{dt^l} \right) \left( \frac{\partial L}{\partial q^{(n)}} \right) + \delta f, 
\] (34)

where

\[
\tilde{p}_n = \sum_{j=0}^{N+1-n} \left( - \frac{d}{dt} \right)^j \left( \frac{\partial L}{\partial q^{n+j+1}} \right), \quad n = 0, 1, ..., \frac{N - 1}{2}; 
\] (35)
are the Ostrogradski momenta, and
\[
H = \sum_{l=0}^{N-1} p_l q_l^{(l+1)} - L,
\]  
(36)
is the Ostrogradski Hamiltonian.

In our case the generalized momenta and the Hamiltonian \(H\), when written in terms of \(\vec{q}\)'s, read
\[
\vec{p}_n = m \left(-1\right)^{N-1} \frac{N}{2} q^{(N-n)} - \frac{N}{2} \sum_{l=0}^{N-1} \vec{p}_{l} q_{l}^{(l+1)} - L,
\]  
(37)
\[
H = \sum_{n=0}^{N-1} \left(-1\right)^{N-1} \vec{q}_n^{(N-n)} q_{n}^{(n+1)} + \frac{m}{2} \left(q^{(N+1)}\right)^2.
\]  
(38)

Now, one can identify \(\delta f\) for each symmetry transformation and check that the integrals of motion derived previously coincide with those given by (31). In detail, we obtain the following results:

• for the transformation generated by \(\vec{c}_k\)'s
  \[
  g_k = 0, \quad \vec{\chi}_k = (-1)^k \frac{N}{2} \vec{t}_k, \quad k = 0, \ldots, N.
  \]  
  and
  \[
  \delta \vec{f}_k = 0, \quad k = 0, \ldots, \frac{N-1}{2};
  \]  
  \[
  \delta \vec{f}_k = m \sum_{n=0}^{N+1} (-1)^{N-1} \frac{k!}{(k-n)!} q^{(N-n)} q^{(k)}, \quad k = \frac{N}{2}, \ldots, N.
  \]  
  (39)
The corresponding integrals of motion are of the form
\[
\vec{C}_k = m \sum_{n=0}^{N+1} (-1)^{N-1} \frac{k!}{(k-n)!} q^{(N-n)} q^{(k)}, \quad k = 0, \ldots, N.
\]  
(40)

• for time translations obviously \(C = H\),

• for dilatation
  \[
  \delta f = 0, \quad g = -t, \quad \vec{\chi} = -\frac{N}{2} \vec{q}.
  \]  
  (41)
and the integral of motion reads
\[
D = -tH + D(t) = -tH + m \sum_{k=0}^{N-1} (-1)^{N-1} \left(\frac{N}{2} - k\right) q^{(N-k)} q^{(k)}.
\]  
(42)

• for rotation, the conserved angular momentum reads
\[
\vec{J} = m \sum_{k=0}^{N-1} (-1)^{N-1} \vec{q}^{(k)} \times q^{(N-k)}.
\]  
(43)
• for special conformal transformation
\[
\delta f = \frac{m}{2} \left( \frac{N+1}{2} \right)^2 \left( q^{N-1} \right)^2, \quad g = t^2 \quad \text{and} \quad \vec{\chi} = tN\vec{q}
\]
while the integral of motion takes the form
\[
K = t^2 H - 2tD(t) + K(t) = t^2 H - 2tD(t) + \\
\frac{m}{2} \sum_{j=0}^{N-3} (j+1)(N-j)(-1)^{N-j-1} \vec{q}^j \vec{q}^{(N-j+1)} + \\
\frac{m}{2} \left( \frac{N+1}{2} \right)^2 \left( q^{N-1} \right)^2.
\]

As in the case of first order theory, all integrals of motion can be obtained from the ones on the Hamiltonian level by expressing Ostrogradski momenta in terms of \( \vec{q} \) and its time derivatives.

5. Conclusion
In the case of trivial \( SL(2, \mathbb{R}) \) and \( SU(2) \) orbits the action of \( N \)-conformal Galilei group reduces to the one on the standard coordinate space with trivial topology. This is no longer the case if we select nontrivial orbit of \( SL(2, \mathbb{R}) \times SU(2) \) (except the case of positive Casimir of \( sl(2, \mathbb{R}) \) and vanishing one of \( SU(2) \)).

Above we have considered the case when the symmetry algebra \( g_N \) admits central extension (i.e. \( N \)-odd). The case of \( N \)-even (and \( d > 2 \)) is more complicated. The set of coadjoint orbits is more reach and their classification does not seem to be easy. Moreover, in most cases, the global Darboux coordinates do not exist. Therefore, the classification and detailed description of invariant dynamical systems remains in this case open.

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