Fractional Fokker-Planck equations for subdiffusion and exceptional orthogonal polynomials

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It is pointed out that, for the fractional Fokker-Planck equation for subdiffusion proposed by Metzler, Barkai, and Klafter [Phys. Rev. Lett. 82 (1999) 3563], there are four types of infinitely many exact solutions associated with the newly discovered exceptional orthogonal polynomials. They represent fractionally deformed versions of the Rayleigh process and the Jacobi process.

I. INTRODUCTION

In recent years anomalous diffusions have attracted much interest owing to their ubiquitous appearances in many physical situations. For examples, charge carrier transport in amorphous semiconductors, nuclear magnetic resonance diffusometry in percolative, and porous systems, Rouse or reptation dynamics in polymeric systems, transport on fractal geometries, and many others [1]. Unlike the well-known Brownian motion, anomalous diffusions are characterized by a mean-squared displacement relation \( \langle x^2 \rangle \sim t^\gamma \) which is not linear in time – it is superdiffusive for \( \gamma > 1 \), and subdiffusive for \( \gamma < 1 \).

The Brownian motion is Markovian in nature. This means each new step in the motion depends only on the present state, and is independent of the the previous states. Anomalous diffusion arises, on the contrary, from some memory effect of previous states, or as a result of some fractal structure of the background space, or due to some non-linear interaction inherent in the system, etc. Among various ways to model anomalous diffusions, one interesting and successful way is by fractional differential equations that involve derivatives of fractional order [2-19]. We note here that a type of superdiffusive motion, so-called the Lévy flight, has been modeled by a diffusion equation with the Riesz-Weyl type fractional space-derivative [4, 5], and the subdiffusive diffusion has been described by fractional Fokker-Planck equations (FFPE) [10-14].

It is natural that one would seek exact solutions of these fractional stochastic equations. However, as is often the case, exact solutions are hard to come by. The purpose of this note is to point out that for the FFPE proposed in [10-12], there are four types of infinitely many exact solutions associated with the newly discovered exceptional orthogonal polynomials. This is a direct extension of the results in [21].

II. FRACTIONAL FPE AND SCHRÖDINGER EQUATIONS

The FFPE proposed in [10-12] is

\[
\frac{\partial}{\partial t} P(x, t) = \alpha D_t^{1-\alpha} \left[ -\frac{\partial}{\partial x} D_1(x) + \frac{\partial^2}{\partial x^2} D_2 \right] P(x, t),
\]

for \( 0 < \alpha < 1 \). (2.1)

Here \( P(x, t) \) is the probability density function (PDF), and \( D_1(x) \) and \( D_2 = \) constant are, respectively, the fractional drift and diffusion coefficients. \( \alpha D_t^{1-\alpha} \) is the Riemann-Liouville fractional derivative defined by [21, 22]

\[
\alpha D_t^{1-\alpha} f(x, t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t (t - t')^{\alpha-1} f(x, t') \, dt'
\]

for \( 0 < \alpha < 1 \). The general situation where the drift and diffusion coefficients are space-time dependent is considered in [13, 14].

Without loss of generality, we set \( D_2 = 1 \). Now let

\[
P(x, t) = T(t) \Phi(x),
\]

then we have

\[
\frac{d}{dt} T(t) = -\mathcal{E} \alpha D_t^{1-\alpha} T(t),
\]

\[
\left[ -\frac{\partial}{\partial x} D_1(x) + \frac{\partial^2}{\partial x^2} \right] \Phi(x) = -\mathcal{E} \Phi(x).
\]

Solution of the temporal part [23] is given by \[10]\]

\[
T(t) = E_{\alpha}(-\mathcal{E} t^\alpha),
\]

where \( E_\alpha(z) \) is the Mittag-Leffler function [21, 22]

\[
E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \quad \text{Re} \alpha > 0.
\]

The drift coefficient can be defined by a drift potential \( U(x) \) as \( D_1(x) = -U'(x) \), where the prime denotes derivative with respective to \( x \). Substituting

\[
\Phi(x) \equiv e^{-U(x)/2} \phi(x)
\]

(2.7)
into Eq. (2.1), one finds that $\phi$ satisfies the time-independent Schrödinger equation $H\phi = \mathcal{E}\phi$ with Hamiltonian $H$ [20, 23]

$$H = -\frac{\partial^2}{\partial x^2} + \frac{1}{4} U'(x)^2 - \frac{1}{2} U''(x).$$

and eigenvalue $\mathcal{E}$. $\phi_0 = \exp(-U(x)/2)$ is the zero mode of $H$: $H\phi_0 = 0$.

Thus the FFPE can be exactly solved if the corresponding Schrödinger equation is exactly solvable. One needs only to link the $U(x)$ in the Schrödinger system with the drift coefficient $D_1(x) = -U'(x)$ in the FFPE. Particularly, those related to the exceptional orthogonal polynomials discussed in [20] can be carried over directly, and hence an infinitely number of exactly solvable FFPE are found.

Let $\mathcal{E}_n$ and $\phi_n (n = 0, 1, 2, \ldots)$ be the eigenvalues and eigenfunctions of $H$, then the solution of Eq. (2.1) is [1, 10, 20]

$$P(x, t) = \phi_0(x) \sum_n c_n \phi_n(x) E_\alpha(-\mathcal{E}_n t^\alpha). \quad (2.8)$$

with constant coefficients $c_n$ ($n = 0, 1, \ldots$)

$$c_n = \int_{-\infty}^{\infty} \phi_n(x) \left( \phi_0^{-1}(x) P(x, 0) \right) dx. \quad (2.9)$$

Note that $c_0 = \int_{-\infty}^{\infty} P(x, 0) dx = 1$. The stationary distribution, corresponding to $\mathcal{E}_0 = 0$, is $P_0(x) = \phi_0^2(x) = \exp(-U(x))$ (with $\int P_0(x) dx = 1$), which is obviously independent of $\alpha$.

In this paper we shall be interested in the initial profile given by the $\delta$-function

$$P(x, t) = \delta(x - x_0), \quad (2.10)$$

which corresponds to the situation where the particle performing the stochastic motion is initially located at the point $x_0$. From Eqs. (2.9) and (2.8) we have

$$c_n = \phi_0^{-1}(x_0) \phi_n(x_0), \quad (2.11)$$

and

$$P(x, t) = \phi_0(x) \phi_0^{-1}(x_0) \sum_n \phi_n(x_0) \phi_n(x) E_\alpha(-\mathcal{E}_n t^\alpha). \quad (2.12)$$

The fractional Ornstein-Uhlenbeck process with $U(x) = x^2/2$ have been discussed in [1, 10]. Here we would like to extend the solvable cases to those related to the single-indexed exceptional orthogonal polynomials, which turn out to be fractional generalisation of the Rayleigh process [24] and the Jacobi process [25].

### III. FFPEs WITH EXCEPTIONAL ORTHOGONAL POLYNOMIALS

Discoveries of the so-called exceptional orthogonal polynomials, and the quantal systems related to them, have been among the most interesting developments in mathematical physics in recent years [24-38]. Unlike the classical orthogonal polynomials, these new polynomials have the remarkable properties that they start with degree $\ell = 1, 2, \ldots$ polynomials instead of a constant, and yet they still form complete sets with respect to some positive-definite measure. The first new polynomials are associated with a single index $\ell$, but soon new polynomials with multiple indices were discovered. For a review of these polynomials, please see e.g., Ref. [40-42].

The new polynomials that fit in the construction described in the last section are the single-indexed ones. There are four basic families of the single-indexed exceptional orthogonal polynomials: two associated with the Laguerre types (termed the L1 and L2 types) and the other two with the Jacobi types (termed the J1 and J2 types). Quantal systems associated with the Laguerre-type exceptional orthogonal polynomials are related to the radial oscillator, while the Jacobi-types are related to the Darboux-Pöschl-Teller potential. The corresponding FPEs describe the generalized Rayleigh process, and the generalized Jacobi process, respectively.

The eigenfunctions of the corresponding Schrödinger equations are given by

$$\phi_{\ell,n}(x; \lambda) = N_{\ell,n}(\lambda) \phi_\ell(x; \lambda) P_{\ell,n}(\eta(x); \lambda), \quad (3.13)$$

where $\eta(x)$ is a function of $x$, $\lambda$ is a set of parameters of the systems, $N_{\ell,n}(\lambda)$ is a normalization constant, $\phi_\ell(x; \lambda)$ is the asymptotic factor, and $P_{\ell,n}(x; \lambda)$ is the polynomial part of the wave function, which is given by the exceptional orthogonal polynomials. When $\ell = 0$, $P_{0,n}(x; \lambda)$ are simply the ordinary classical orthogonal polynomials of the corresponding quantal systems, with $P_{0,0}(x; \lambda) = \text{constant}$. When $\ell > 0$, $P_{\ell,n}(x; \lambda)$ is a polynomial of degree $\ell + n$. The PDF [21, 22] for these generalized systems are

$$P(x, t) = \phi_\ell^2(x; \lambda) P_{\ell,0}(\eta; \lambda) P_{\ell,0}^{-1}(\eta_0; \lambda) \quad (3.14)$$

$$\times \sum_{n=0}^{\infty} N_{\ell,n}(\lambda)^2 P_{\ell,n}(\eta; \lambda) P_{\ell,n}(\eta_0; \lambda) E_\alpha(-\mathcal{E}_n t^\alpha).$$

Infinite number of exactly solvable FFPE’s can be constructed in terms of these four sets if new polynomial, using the basic data of these four systems summarized in [20].

Here we shall only illustrate the construction with the L1 Laguerre polynomials. The function $U(x)$ that generates the original radial oscillator potential is

$$U_0(x; g) = x^2 - 2g \log x, \quad 0 < x < \infty. \quad (3.15)$$

The exactly solvable Schrödinger equations associated with the deformed radial oscillator potential are defined by ($\ell = 1, 2, \ldots$)

$$U_{\ell}(x; g) = x^2 - 2(g + \ell) \log x - 2 \log \frac{\xi_\ell(\eta; g + 1)}{\xi_\ell(\eta; g)}. \quad (3.16)$$
Here $\lambda = g > 0$ and $\eta(x) \equiv x^2$ and $\xi_\ell(\eta; g)$ is a deforming function, which for $L1$ type is given by
\[
\xi_\ell(\eta; g) = L_\ell^{(g+\ell+\frac{1}{2})}(-\eta). \tag{3.17}
\]
The eigen-energies are $E_{\ell,n}(g) = E_n(g + \ell) = 4n$, which are independent of $g$ and $\ell$. Hence the deformed radial oscillator is iso-spectral to the ordinary radial oscillator.

The eigenfunctions are given by
\[
\phi_{\ell,n}(x; g) = \frac{e^{-\frac{1}{4}x^2 + \eta x^2 + \ell x^2}}{\xi_\ell(x^2; g)}, \tag{3.19}
\]
where the corresponding exceptional Laguerre polynomials $P_{\ell,n}(\eta; g)$ ($\ell = 1, 2, \ldots, n = 0, 1, 2, \ldots$) can be expressed as a bilinear form of the classical associated Laguerre polynomials $L_n^\alpha(\eta)$ and the deforming polynomial $\xi_\ell(\eta; g)$, as given in [33]:
\[
P_{\ell,n}(\eta; g) = \xi_\ell(\eta; g) + 1)P_n(\eta; g + \ell - 1) - \xi_\ell(\eta; g)\partial_\eta P_n(\eta; g + \ell - 1), \tag{3.20}
\]
where $P_n(\eta; g) = L_n^{(g+\frac{1}{2})}(\eta)$. The polynomials $P_{\ell,n}(\eta; g)$ are degree $\ell + n$ polynomials in $\eta$ and start at degree $\ell$: $P_{\ell,0}(\eta; g) = \xi_\ell(\eta; g + 1)$. They are orthogonal with respect to certain weight functions, which are deformations of the weight function for the Laguerre polynomials (for details, see [33]). From the orthogonality relation of $P_{\ell,n}(\eta; g)$, we get the normalization constants [33]
\[
N_{\ell,n}(g) = \left[\frac{2n!(n + g + \ell - \frac{1}{2})}{(n + g + 2\ell - \frac{1}{2})\Gamma(n + g + \ell + \frac{1}{2})}\right]^\frac{1}{2}. \tag{3.21}
\]

The drift coefficient and PDF of the FFPE generated by the drift potential $U_\ell(x)$ in Eq. 3.10 is
\[
D^{(1)}(x) = -2x + 2\frac{g + \ell}{x} + 4x\left(\frac{\partial_\eta \xi_\ell(\eta; g + 1)}{\xi_\ell(\eta; g + 1)} - \frac{\partial_\eta \xi_\ell(\eta; g)}{\xi_\ell(\eta; g)}\right) \tag{3.22}
\]
and
\[
P(x, t) = \phi_\ell^2(x; g)P_{\ell,0}(x^2; g)P_{\ell,0}^{-1}(x_0^2; g) \times \sum_{n=0}^\infty N_{\ell,n}^2 P_{\ell,n}(x^2; g)P_{\ell,n}(x_0^2; g)E_\alpha(-4nt^\alpha). \tag{3.23}
\]

In the limit $\alpha \to 1$, $\ell \to 0$ for $g = 1/2$, we have $\xi_\ell(x; g+1) \to 0$ and $\xi_\ell(x; g) \to 0$, and Eqs. 3.22 and 3.23 reduce to the drift coefficient and PDF for the ordinary Rayleigh process,
\[
D^{(1)} = -2x + \frac{1}{x}, \tag{3.24}
\]
\[
P(x, t) = 2x e^{-x^2} \sum_{n=0}^\infty L_n(x^2)L_n(x_0^2) \exp(-4nt),
\]
where $L_n(x)$ are the ordinary Laguerre polynomials.

IV. NUMERICAL RESULTS

In [20] we have studied how the Rayleigh process was deformed by $\ell$. Here the effect of different $\alpha$ on the process is considered numerically for the original Rayleigh process ($g = 1/2, \ell = 0$), and the $\ell$-deformed one ($g = 1/2, \ell = 5$). The choice of $\ell = 5$ is due to an observation noted in [20], explained below.

The drift coefficient is related to the negative slope of $U_\ell(x)$ by $D_2(x) = -U'_\ell(x)$. So one can gain some qualitative understanding of how the peak of the probability density moves just from the sign of the slope of $U_\ell(x)$. The peak of $p(x, t)$ tends to move to the right (left) when it is at a position $x$ such that $U'_\ell(x)$ is negative (positive), until the stationary distribution is reached.

Fig. 1 depicts $U_\ell(x)$ for $g = 0.5$ and $\ell = 0$ (the original Rayleigh process), 1 and 5. Note that there can be sign change of the slope of the drift potential in certain region near the left wall. In such region, the peak of the probability density function will move in different directions for different $\ell$. Particularly, at $x_0 = 1.2$, the sign of the slope of $U_{5}(x)$ is different from those of $U_{0}(x)$ and $U_{1}(x)$. Thus one expects that for the initial profile $P(x, 0) = \delta(x-x_0)$ with the peak initially located at $x_0 = 1.2$, the peak will move to the right for $\ell = 5$ system, while for the other two values of $\ell$, the peak will move to the left.

Fig. 2 and 3 show how $\alpha$ affects the Rayleigh process ($g = 1/2, \ell = 0$) and the $\ell$-deformed process ($g = 1/2, \ell = 5$), respectively. As expected, the peak moves to the left for the Rayleigh process, and to the right for the $\ell$-deformed one. Further, it is clear from the graphs that for $0 < \alpha < 1$ the FFPE describes subdiffusive motion – the peaks for $\alpha \neq 1$ move slower to the left that the Rayleigh motion in Fig. 2, while they move faster to the right in Fig. 3. And the smaller the value of $\alpha$, the slower the peak moves. From the last sub-figure it is obvious that the PDF at large times is the stationary distribution which is independent of $\alpha$.

V. SUMMARY

In this note we have discussed the construction of four types of infinitely many exact solutions, associated with the newly discovered exceptional orthogonal polynomials, for the fractional Fokker-Planck equations for sub-diffusion proposed in [10 12]. They represent the fractionally deformed versions of the Rayleigh process and the Jacobi process.
[1] R. Metzler and J. Klafter, Phys. Rep. 339 (2000) 1.
[2] W. Wyss, J. Math. Phys. 27 (1986) 2782.
[3] W. Wyss, J. Math. Phys. 30 (1989) 134.
[4] A. Compte, Phys. Rev. E 53 (1996) 4191.
[5] R. Metzler, J. Klafter, and I. Sokolov, Phys. Rev. E 58 (1998) 1621.
[6] E. Buckwar and Y. Luchko, J. Math Analysis Appl. 227 (1998) 81.
[7] F. Mainardi, Y. Luchko, and G. Pagnini, Frac. Cal. Appl. Anal. 4 (2001) 153.
[8] R. Gorenflo, Y. Luchko, F. Mainardi, J. Comput. Appl. Math. 118 (2000) 175.
[9] J.-S. Duan, A.-P. Guo, and W.-Z. Yun, Abstract and Applied Analysis Vol 2014 (2014) 548126.
[10] R. Metzler, E. Barkai, J. Klafter, Phys. Rev. Lett. 82 (1999) 3563.
[11] R. Metzler, E. Barkai, J. Klafter, Europhys. Lett. 46 (1999) 431.
[12] E. Barkai, R. Metzler, and J. Klafter, Phys. Rev. E 61 (2000) 132.
[13] A. Weron, M. Magdziarz and K. Weron, Phys. Rev. E 77 (2008) 036704
[14] B. I. Henry, T.A.M. Langlands, P. Straka, Phys. Rev. Lett. 105 (2010) 170602.
[15] E. K. Lenzi, R. S. Mendes, Kwok Sau Fa, and L. C. Malacarne, and L.R. da Silva, J. Math. Phys. 44 (2003) 2179.
[16] B.I. Henry and S.L. Wearne, Physica A 276 (2000) 448.
[17] K. Seki, M. Wojcik, and M. Tachiya, J. Chem. Phys. 119 (2003) 2165.
[18] I. M. Sokolov and J. Klafter, Phys. Rev. Lett. 97 (2006) 140602.
[19] B. I. Henry, T. A.M. Langlands, S.L. Wearne, Phys. Rev. E 74 (2006) 031116.
[20] C.-I. Chou and C.-L. Ho, Int. J. Mod. Phys. B 27 (2013) 1350135.
[21] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.

g = 0.5

FIG. 1: Plot of $U_\ell(x)$ versus $x$ for the L1 type FFPE with $g = 1/2$ and $\ell = 0$ (solid curve), 1 (dashed curve) and 5 (dotted curve)
FIG. 2: Plots of $P(x,t)$ at different times for the Rayleigh process with $g = 0.5, \ell = 0, x_0 = 1.2, \alpha = 1$ (solid curve), $0.75$ (dashed curve) and $0.25$ (dotted curve). In Eq. (3.24) 120 terms were used. The solid curve shown in the last sub-figure is the stationary distribution, independent of $\alpha$. 
FIG. 3: Plots of $P(x,t)$ at different times for the $\ell$-deformed Rayleigh process with $g = 0.5$, $\ell = 5$, $x_0 = 1.2$, $\alpha = 1$ (solid curve), $0.75$ (dashed curve) and $0.25$ (dotted curve). In Eq. (3.23) 120 terms were used. The solid curve shown in the last sub-figure is the stationary distribution, independent of $\alpha$. 