SMOOTH SELECTION FOR INFINITE SETS

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Abstract. Whitney’s extension problem asks the following: Given a compact set $E \subset \mathbb{R}^n$ and a function $f : E \to \mathbb{R}$, how can we tell whether there exists $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on $E$? A 2006 theorem of Charles Fefferman [6] answers this question in its full generality.

In this paper, we establish a version of this theorem adapted for variants of the Whitney extension problem, including nonnegative extensions and the smooth selection problems. Among other things, we generalize the Finiteness Principle for smooth selection by Fefferman-Israel-Luli [9] to the setting of infinite sets.

Our main result is stated in terms of the iterated Glaeser refinement of a bundle formed by taking potential Taylor polynomials at each point of $E$. In particular, we show that such bundles (and any bundles with closed, convex fibers) stabilize after a bounded number of Glaeser refinements, thus strengthening the previous results of Glaeser, Bierstone-Milman-Pawlucki, and Fefferman which only hold for bundles with affine fibers.

1. Introduction

Let $m, n, d$ be positive integers. We write $C^m(\mathbb{R}^n, \mathbb{R}^d)$ to denote the space of all functions $\vec{F} := (F_1, \ldots, F_d) : \mathbb{R}^n \to \mathbb{R}^d$ whose derivatives $\partial^\alpha \vec{F} = (\partial^\alpha F_1, \ldots, \partial^\alpha F_d)$ (for all $|\alpha| \leq m$) are continuous and bounded on $\mathbb{R}^n$. We equip $C^m(\mathbb{R}^n, \mathbb{R}^d)$ with the usual norm

$$\|\vec{F}\|_{C^m(\mathbb{R}^n, \mathbb{R}^d)} := \max_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} |\partial^\alpha F_j(x)|$$

and seminorm

$$\|\vec{F}\|_{\dot{C}^m(\mathbb{R}^n, \mathbb{R}^d)} := \max_{|\alpha| = m} \sup_{x \in \mathbb{R}^n} |\partial^\alpha F_j(x)|.$$
Problem 1.1 (Nonnegative extension). Given integers $m \geq 0, n \geq 1$, an arbitrary subset $E \subset \mathbb{R}^n$, $f : E \rightarrow [0, \infty)$, how can we tell if there exists $F \in C^m(\mathbb{R}^n)$ such that $F \geq 0$ and $F|_E = f$?

Simple examples, such as $E = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ with $f(x) = x$ and $m \geq 1$, demonstrate that the existence of a nonnegative $C^m$ extension does not follow merely from nonnegative $f$ and the existence of a (not necessarily nonnegative) $C^m$ extension. Thus, a solution to Problem 1.1 must address an obstacle absent from the classical Whitney extension problem. Naturally, this obstacle extends to the further generalizations explored in this paper.

Problem 1.1 can be viewed as a special case of the following generalized problem (see [11] for recent progress on the analogous problem for finite sets):

Problem 1.2 (Extension with restricted range). Given extended real numbers $\lambda_1, \lambda_2$ with $-\infty \leq \lambda_1 \leq \lambda_2 \leq \infty$, $E \subset \mathbb{R}^n$, and $f : E \rightarrow [\lambda_1, \lambda_2] \cap \mathbb{R}$, how can we tell if there exists $F \in C^m(\mathbb{R}^n)$ such that $\lambda_1 \leq F \leq \lambda_2$ and $F|_E = f$?

Our solution to Problems 1.1 and 1.2 will result from the solution to an even more generalized problem. In order to present our solution to this problem, we need to introduce some background definitions and recall some facts.

If $\bar{F} \in C^m(\mathbb{R}^n, \mathbb{R}^d)$, we write $J_x \bar{F} = (J_x F_1, \ldots, J_x F_d)$ (the “$m$-th jet”) to denote the $m$-th degree Taylor polynomial of $\bar{F}$ at $x$. Let $\mathcal{P}$ denote the vector space of polynomials in $n$ real variables and of degree at most $M$. Given $x \in \mathbb{R}^n$, we give $\mathcal{P}$ the structure of a ring, denoted $\mathcal{R}_x$, by defining multiplication $\odot_x$ as $R \odot_x R' = J_x (RR')$. We write $\mathcal{R}_x$ for the $\mathcal{R}_x$-module of $m$-jets of functions $\bar{F} \in C^m(\mathbb{R}^n, \mathbb{R}^d)$ at $x \in \mathbb{R}^n$.

As a vector space, $\mathcal{R}_x$ can be identified with $\bar{P} := \mathcal{P} \odot \cdots \odot \mathcal{P}$, the space of $d$-tuples of real $m$-th degree polynomials on $\mathbb{R}^n$. The $\mathcal{R}_x$-multiplication on $\mathcal{R}_x$ is given by $R \odot_x \bar{P} = (R \odot_x P_1, \ldots, R \odot_x P_d)$, where $R \in \mathcal{R}_x$ and $\bar{P} = (P_1, \ldots, P_d)$.

We write $c(m, n, d), C(m, n, d)$, etc., to denote constants that depend on $m, n, d$; they may denote different constants in different appearances.

If $S$ is a finite set, we write $\#(S)$ to denote the number of elements in $S$.

A problem that is closely related to Problems 1.1 and 1.2 is the following

Problem 1.3 (Smooth Selection). Let $E \subset \mathbb{R}^n$ be an arbitrary set. For each $x \in E$, let $\mathcal{K}(x) \subset \mathbb{R}^d$ be closed and convex. How can we tell whether there exists a function $\bar{F} \in C^m(\mathbb{R}^n, \mathbb{R}^d)$ such that $\bar{F}(x) \in \mathcal{K}(x)$ for all $x \in E$?

The pioneer papers [5, 9, 28] studied the analogous version of Problem 1.3 for finite sets $E$. In particular, for finite sets $E \subset \mathbb{R}^n$, the authors in [9] gave an answer to the finite set version of the problem by means of proving a finiteness principle.
See also [16, 25–27, 29–31] for related Lipschitz selection problems and [10, 20–22] for related problems of nonnegative interpolation.

The $C^1$ case of nonnegative extension (Problem 1.1) was addressed in [19], though little other work has directly addressed the infinite set versions of the constrained extension problems addressed here.

In this paper, we modify the approach of [6] to answer Problem 1.3 for infinite sets $E \subset \mathbb{R}^n$.

Before we present our solution to the above problems, we would like to point out another closely related problem for which our result can provide an answer:

**Problem 1.4** (Generalized Brenner-Epstein-Hochster-Kollár Problem). Suppose we are given real-valued functions $\phi_1, \ldots, \phi_d$, an $\mathbb{R}^s$-valued function $\phi$ on $\mathbb{R}^n$, and closed convex subsets $K_1(x), \ldots, K_d(x) \subset \mathbb{R}^s$ for each $x \in E \subset \mathbb{R}^n$. How can we decide whether there exist $f_1, \ldots, f_d \in C^m(\mathbb{R}^n, \mathbb{R}^s)$ such that

\[
\sum_{i=1}^{d} \phi_i f_i \leq \phi \text{ on } E 
\]

and

\[
f_i(x) \in K_i(x), \text{ for } 1 \leq i \leq d, x \in E?
\]

The pioneer papers [2, 14, 15] studied Problem 1.4 where the inequalities in (1.1) are replaced by equations and there are no convex constraints (1.2). Note that the inequalities in (1.1) may be used to restrict the output of the $f_i$ to convex sets depending on $x$, provided said convex sets are the intersection of a bounded number of half-spaces.

In order to present our solutions to Problems 1.2 (a fortiori 1.1), 1.3, and 1.4, we introduce some basic definitions.

Let $E \subset \mathbb{R}^n$ be a compact set. A bundle over $E$ is a family $\mathcal{H} = (H(x))_{x \in E}$ of (possibly empty) subsets $H(x) \subset \mathbb{R}^s$, parameterized by the points $x \in E$. We refer to each $H(x), x \in E$, as a fiber. A $C^m$ section (or “section” for short) of a bundle $\mathcal{H} = (H(x))_{x \in E}$ is a $C^m$ function $\vec{F} : \mathbb{R}^n \to \mathbb{R}^d$ such that $J_x \vec{F} \in H(x)$ for each $x \in E$. The problem of finding a $C^m$ extension satisfying certain conditions reduces to the problem of finding a section of a given bundle (see below).

We say a bundle $(H(x))_{x \in E}$ is convex if $H(x)$ is convex for each $x \in E$. (We note that prior literature has required each fiber of a bundle to be an affine space; we drop the requirement to handle nonlinear problems.)

Fix integers $m \geq 0, n, d \geq 1$. Let $(H(x))_{x \in E}$ be a bundle and let $k^2$ be a positive integer depending only on $m, n, d$. For each $x_0 \in E$, we define the Glaeser refinement of $H(x_0)$, denoted by $\tilde{H}(x_0)$, according to the following rule:
Let \( \vec{P}_0 \in \vec{P} \). We say that \( \vec{P}_0 \in \vec{H}(x_0) \) if and only if given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that, for any \( x_1, \ldots, x_k \in E \cap B(x_0, \delta) \), there exist \( \vec{P}_1, \ldots, \vec{P}_k \in \vec{P} \), with \( \vec{P}_j \in H(x_j) \) for \( j = 0, 1, \ldots, k \) and \( \left| \partial^\alpha (\vec{P}_i - \vec{P}_j)(x_j) \right| \leq \varepsilon |x_i - x_j|^{m-|\alpha|} \)

The Glaeser refinement is used, essentially, to toss polynomials which cannot satisfy the conditions of Taylor’s theorem with the polynomials remaining at nearby fibers.

We say that a bundle \( (H(x))_{x \in E} \) is Glaeser stable if \( H(x) \) is its own Glaeser refinement for each \( x \in E \).

We will often produce repeated Glaeser refinements of the same bundle, in which case \( (H_0(x))_{x \in E} \) will be used to denote the original bundle and \( (H_{k+1}(x))_{x \in E} \) will be used to denote \( (\vec{H}_k(x))_{x \in E} \) for \( k \in \mathbb{N} \).

The Glaeser refinement is a procedure first suggested in 1958 by G. Glaeser [18] for determining whether a function \( f \) defined on an arbitrary subset \( E \subset \mathbb{R}^n \) can be extended to a \( C^1(\mathbb{R}^n) \) function. In their studies of extension problems in \( C^m(\mathbb{R}^n) \) for subanalytic sets, Bierstone, Milman and Pawłucki [1] introduced the notion of iterated paratangent bundles, which is analogous to the Glaeser refinement. Our version of the Glaeser refinement is a vector-valued version of the one used by C. Fefferman for extending a function \( f \) defined on an arbitrary subset \( E \subset \mathbb{R}^n \) to a \( C^m(\mathbb{R}^n) \) function. We also apply our Glaeser refinement to a more general class of bundles than in [6].

Here is an outline for the rest of the introduction. First, we briefly recall Fefferman’s solution [6] to the Whitney’s extension problem for \( C^m \) and then we will point out the major hurdles in adapting the machinery to prove our main theorems (see Theorems 1.5, 1.6 below). We will then state our main results and explain our proof in broad strokes.

Given a function \( f : E \to \mathbb{R}^n \) on an arbitrary subset \( E \subset \mathbb{R}^n \), to determine whether \( f \) can be extended to a function \( F \in C^m(\mathbb{R}^n) \) – This is called the “Whitney Extension Problem for \( C^m \) – C. Fefferman [6] associated an affine subspace \( H(x) \subset \mathcal{P} \) for each \( x \in E \), with the following crucial property:

- If \( F \in C^m(\mathbb{R}^n) \) and \( F = f \) on \( E \), then \( J_x(F) \in H(x) \) for all \( x \in E \).

Evidently, if \( H(x) = \emptyset \) for any \( x \in E \), then \( f \) cannot be extended to a \( C^m \) function \( F \). Fefferman’s solution [6] to the Whitney’s extension problem for \( C^m \) can be summarized as follows:

1. We start with the trivial holding space \( H_0(x) = \{P \in \mathcal{P} : P(x) = f(x)\} \) for all \( x \in E \).

2. We produce a list of holding spaces \( H_0(x) \supset H_1(x) \supset H_2(x) \supset \cdots \) by repeatedly taking the Glaeser refinement (see Definition 1).
(3) After at most \( L = 2 \dim \mathcal{P} + 1 \), we have \( H_L(x) = H_I(x) \) for all \( l \geq L \) and all \( x \in E \). (In other words, \( (H_L(x))_{x \in E} \) is Glaeser stable.)

(4) \( f \) extends to a \( C^m \) function on \( \mathbb{R}^n \) if and only if \( H_L(x) \) is non-empty for all \( x \in E \).

Fefferman’s proof of the solution relies on the affine space structure of the \( H(x) \) (and breaks down for more general sets) in three crucial ways:

1. The proof that iterated Glaeser refinements terminate in a Glaeser stable bundle after a finite number of steps (Step 3) in [6] (adapted from [1]) relies on the affine space structure of \( H_l \) and an induction on the dimensions. It uses the fact that if \( W \subset V \) is a subspace with \( \dim W = \dim V \), then \( W = V \), which is false for the natural notion of dimension for convex sets.

2. The proof that the \( \delta \) in the definition of Glaeser stability may be chosen uniformly in \( x \) and \( P \) (with some adjustment) relies on the finite representation of \( H(x) \) via a linear basis. General convex sets may not be represented in terms of a finite basis.

3. The set \( E \) is partitioned according to the dimension of the \( H(x) \) as affine spaces. If one does this by dimension of a convex set, it must be checked this dimension does not change under natural set operations.

Our circumstances do not guarantee the presence of affine spaces. For instance, considering Problem 1.1 we see that \( x^2 \) and \( 2x^2 \) are potential Taylor polynomials at \( x = 0 \), while \( -x^2 \) is not.

To establish the termination of iterated Glaeser refinements of bundles \( (H(x))_{x \in E} \) whose fibers \( H(x) \) are allowed to be general convex sets, we recognize the following: The Glaeser refinement depends on a local procedure (see Definition 1). More precisely, in order to check if \( \vec{P}_0 \in H(x_0) \), we need to consider only polynomials \( \vec{P}_1 \) that are close to \( \vec{P}_0 \). Furthermore, convex subsets of \( \mathbb{R}^n \) have a notion of dimension which allows us to view them as affine spaces locally. To formalize this idea, we introduce the local dimension in Definition 3.1.

To find uniform \( \delta \) as dictated by (2) above, we pass from the infinite to the finite via compactness of \( E \) and the \( H(x) \), truncating the latter as necessary. An open cover of neighborhoods in \( E \times \mathcal{P} \) must be carefully constructed so that each of these neighborhoods allows for uniform choice of \( \delta \) (for given \( \epsilon \)) and for the necessary quantitative analysis to be conducted. This delicate construction is done in Section 9. Sorting the elements \( x \) of \( E \) by \( \dim H(x) \) is initially easy; any convex set naturally carries a dimension. However, we must show that the truncations described above do not change the dimension and ruin the partition. This is done in Section 7.

To state our main theorems, we need to introduce a few terms.

Let \( E \subset \mathbb{R}^n \) be an arbitrary set. For each \( x \in E \) and \( M \geq 0 \), let \( \Gamma(x,M) \subset \vec{P} \) be a (possibly empty) convex set. We say \( \Gamma = (\Gamma(x,M))_{x \in E, M \geq 0} \) is a shape field if
for all \( x \in E \) and \( 0 \leq M' \leq M < \infty \), we have \( \Gamma(x, M') \subset \Gamma(x, M) \). We say a shape field \( \Gamma = (\Gamma(x, M))_{x \in E, M \geq 0} \) is \textit{closed} if \( \Gamma(x, M) \) is closed for each \( x \in E \) and \( M \geq 0 \).

Any shape field \( (\Gamma(x, M))_{x \in E, M \geq 0} \) on a compact set \( E \) gives rise to a convex bundle \((\Gamma(x))_{x \in E}\) if we set

\[
\Gamma(x) = \bigcup_{M \geq 0} \Gamma(x, M) \text{ for each } x \in E.
\]

We may then define the Glaeser refinement of a shape field \((\Gamma(x, M))_{x \in E, M \geq 0}\) to be \((\tilde{\Gamma}(x, M))_{x \in E, M \geq 0}\), where \( \tilde{\Gamma}(x, M) = \tilde{\Gamma}(x) \cap \Gamma(x, M) \) with \( \tilde{\Gamma}(x) \) being the Glaeser refinement of \( \Gamma(x) \).

We will also define what it means for a shape field to be regular (see Definition 2.1) or \((C, \delta_{\text{max}})\)-convex for \( \delta_{\text{max}} > 0 \) (see Definition 2.2), though in the interest of postponing technicalities this will be presented in Section 2. For now, we simply note that regularity ensures \( \Gamma(x, M) \) “varies continuously” in \( x \) and \( M \), while \((C, \delta_{\text{max}})\) convexity allows us to combine functions via partition of unity as in [9]. Both properties will be satisfied by reasonable examples of shape fields, in particular, the examples arising from Problems 1.1-1.4.

We are now ready to state our main theorems.

**Theorem 1.5 (Qualitative Main Theorem).** Let \( m, n, d \geq 1 \) be integers. There exists \( k^{2} = k^{2}(m, n, d) \) in (1.3) such that the following holds.

Let \( E \subset \mathbb{R}^{n} \) be a closed set. Let \((\Gamma(x))_{x \in E}\) be a bundle arising from a closed regular \((C, 1)\)-convex shape field. Then the iterated Glaeser refinement of \((\Gamma(x))_{x \in E}\) terminates in a Glaeser stable bundle \((\Gamma^{*}(x))_{x \in E}\) in at most \( \dim \mathcal{P} + 1 \) steps and \((\Gamma(x))_{x \in E}\) has a section if and if every fiber of \((\Gamma^{*}(x))_{x \in E}\) is nonempty.

We can further strengthen Theorem 1.5 to have better control on the derivatives. In fact, this step is required for our proof strategy. We need the following definition for this purpose.

Given a shape field \( \Gamma = (\Gamma(x, M))_{x \in E, M \geq 0} \) with nonempty fibers, we define

\[
\|\Gamma\| := \inf \left\{ M \geq 0 : \exists \tilde{\Gamma}_{i} \in \Gamma(x, M) \text{ for } 1 \leq i \leq k^{2} \text{ such that } \left| \partial^{\alpha}(\tilde{\Gamma}_{i} - \tilde{\Gamma}_{j})(x_{i}) \right| \leq M |x_{i} - x_{j}|^{m - |\alpha|}, 1 \leq i, j \leq k^{2}. \right\}
\]

The finiteness of \( \|\Gamma\| \) will be established in Lemma 6.1. (Specifically, we prove this for the scalar-valued case in Section 6 and reduce to this case in Section 4.)

**Theorem 1.6 (Quantitative Main Theorem).** Let \( m, n, d \geq 1 \) be integers. There exists \( k^{2} = k^{2}(m, n, d) \) in (1.3) and (1.4) such that the following holds:
Fix $C_w > 0$. Let $Q_0 \subset \mathbb{R}^n$ be a cube of length 3 and $E \subset Q_0$ be compact. Let $(\Gamma(x, M))_{x \in E, M \geq 0}$ be a closed regular $(C_w, 1)$-convex shape field. Write

$$\Gamma(x) = \bigcup_{M \geq 0} \Gamma(x, M).$$

Then the following hold.

1. The repeated Glaeser refinement of $(\Gamma(x))_{x \in E}$ terminates in a Glaeser stable bundle $(\Gamma^*(x))_{x \in E}$ after at most $2D + 1$ steps, where $D = D(m, n, d) = \dim \vec{P}$.

2. If $\Gamma^*(x)$ is nonempty for all $x \in E$, then there exists $\vec{F} \in C^m(\mathbb{R}^n, \mathbb{R}^d)$ such that

$$\|\vec{F}\|_{C^m(\mathbb{R}^n, \mathbb{R}^d)} \leq C\|\Gamma^*\|$$

and

$$J_x \vec{F} \in \Gamma(x, C\|\Gamma^*\|)$$

for all $x \in E$.

Here the constant $C$ depends only on $m, n, d, C_w$.

In Section 2, we will apply Theorem 1.6 to present our solutions to Problems 1.3, 1.2, and 1.4. In Section 3, we will prove the first part of Theorem 1.6, using the aforementioned concept of local dimension. Section 4 contains a reduction to the case $d = 1$, using the gradient trick as in [15].

In Sections 5 through 10, we prove the second half of Theorem 1.6. We show that for given $\epsilon > 0$, we may find a uniform $\delta$ in the statement that $(\Gamma^*(x))_{x \in E}$ is Glaeser stable and use this to construct a modulus of continuity for which we may apply the $C^{m, \omega}$ shape fields finiteness principle from [9]. The proof that $\delta$ may be taken uniformly will rely on the dimension of $\Gamma(x)$ being the same for all $x \in E$, forcing us to decompose $E$ into smaller sets (strata) for which this property holds. The solutions are then patched together via a partition of unity. The process is formally conducted via induction on the number of strata.

Section 11 is not part of the proof of Theorem 1.6; however, it will provide a drastic decrease in the value of $k^2$ from that given in the proof of the theorem (greater than $(\dim \vec{P} + 1)^{3 \cdot 2^{\dim \vec{P}}}$) to $k^2 = 2^{\dim \vec{P}}$.

This is an overly simplified version of our long story. The details will be presented in the sections below.

This paper is part of a literature on extension and interpolation, going back to the seminal works of H. Whitney [32–34]. We refer the interested readers to [7–10, 12, 13, 17, 21] and references therein for the history and related problems.

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2. Applications

In this section, we explain how to use Theorem 1.5 to answer Problems 1.3, 1.2, and 1.4.

By specializing to $d = 1$, $E = \mathbb{R}^n$, and

$$\mathcal{K}(x) = \begin{cases} \{f(x)\} & \text{for } x \in E_0 \\ [\lambda_1, \lambda_2] \cap \mathbb{R} & \text{for } x \in E \setminus E_0 \end{cases},$$

we see that Problem 1.3 encompasses Problem 1.2.

By specializing to $D = ds$, $E = E_0$ and

$$\mathcal{K}(x) = \left\{(y_1, \cdots, y_d) \in \prod_{i=1}^d K_i(x) : y_i \in \mathbb{R}^s \text{ for } i = 1, \cdots, d, \text{ and } \sum_{i=1}^d \phi_i(x) y_i \leq \phi(x) \right\},$$

we see that Problem 1.3 encompasses Problem 1.4.

Thus, to solve Problems 1.2 and 1.4, it suffices to solve Problem 1.3.

In order to show Theorem 1.5 applies to Problem 1.3, we must provide the definitions missing from the introduction.

**Definition 2.1.** Given a compact set $E \subset \mathbb{R}^n$, we say a shape field $(\Gamma(x, M))_{x \in E, M \geq 0}$, with $\Gamma(x) = \bigcup_{M \geq 0} \Gamma(x, M)$, is regular if the following hold.

1. Given $M \geq 0$, $x \in E$,
   $$\Gamma(x, M) \subset \{\bar{P} \in \mathcal{P} : |\partial^\alpha \bar{P}(x)| \leq M \text{ for } |\alpha| \leq m\}.$$

2. Given $\epsilon > 0$, $M \geq 0$, there exists $\delta > 0$ such that for any $x, x' \in E$ with $|x - x'| \leq \delta$, if $\bar{P} \in \Gamma(x, M)$ and $\bar{P}' \in \Gamma(x')$ with
   $$|\partial^\alpha (\bar{P} - \bar{P})(x)|, |\partial^\alpha (\bar{P} - \bar{P})(x')| \leq \delta |x - x'|^{m-|\alpha|} \text{ for } |\alpha| \leq m,$$
   then $\bar{P}' \in \Gamma(x', M + \epsilon)$.

3. Given $\epsilon > 0$, $M \geq 0$, there exists $\delta > 0$ such that if $\bar{P} \in \Gamma(x, M)$, then for any $\bar{P}' \in \Gamma(x)$ satisfying
   $$|\partial^\alpha (\bar{P} - \bar{P})(x)| < \delta \text{ for } |\alpha| \leq m$$
   implies
   $$\bar{P}' \in \Gamma(x, M + \epsilon).$$
Note that the above conditions say nothing about the content of $\Gamma(x)$, rather the values of $M$ for which a given $\vec{P} \in \Gamma(x)$ lies in $\Gamma(x, M)$.

**Definition 2.2.** Let $\Gamma = (\Gamma(x, M))_{x \in E, M \geq 0}$ be a shape field. Let $C_w > 0$ and $\delta_{\text{max}} \in (0, 1]$. We say that $\Gamma$ is $(C_w, \delta_{\text{max}})$-convex, if the following condition holds.

Let $0 < \delta \leq \delta_{\text{max}}, x \in E, M \geq 0, \vec{P}_1, \vec{P}_2 \in \vec{P}, Q_1, Q_2 \in \mathcal{P}$. Assume that

\begin{align}
&\vec{P}_1, \vec{P}_2 \in \Gamma(x, M); \\
&|\partial^\alpha (\vec{P}_1 - \vec{P}_2)(x)| \leq M\delta^{|\alpha|} \text{ for } |\alpha| \leq m; \\
&|\partial^\alpha Q_i(x)| \leq \delta^{-|\alpha|} \text{ for } |\alpha| \leq m, i = 1, 2; \text{ and} \\
&Q_1 \circ_x Q_1 + Q_2 \circ_x Q_2 = 1.
\end{align}

Then

\begin{equation}
\sum_{i=1,2} Q_i \circ_x Q_i \circ_x \vec{P}_i \in \Gamma(x, C_w M).
\end{equation}

It is clear that any $(C_w, \delta_{\text{max}})$-convex shape field is also $(C'_w, \delta'_{\text{max}})$-convex for any $C'_w \geq C_w$ and $0 < \delta'_{\text{max}} \leq \delta_{\text{max}}$.

2.1. **Solution to Problem 1.3.** Using a standard argument involving partition of unity, we may assume that $E$ is compact.

Consider the bundle $(\Gamma_K(x))_{x \in E}$ with fiber

\begin{equation}
\Gamma_K(x) = \bigcup_{M \geq 0} \Gamma_K(x, M),
\end{equation}

where

\begin{equation}
\Gamma_K(x, M) = \left\{ \vec{P} \in \vec{P} : \vec{P}(x) \in K(x) \text{ and } \left| \partial^\alpha \vec{P}(x) \right| \leq M \text{ for } |\alpha| \leq m. \right\}.
\end{equation}

A $C^m$-section of the bundle $(\Gamma_K(x))_{x \in E}$ is precisely a $C^m(\mathbb{R}^n, \mathbb{R}^d)$ function $\vec{F}$ such that $\vec{F}(x) \in K(x)$ for each $x \in E$.

To find a $C^m$-section of $(\Gamma_K(x))_{x \in E}$, we use Theorem 1.6. In particular, we will prove the following.

**Lemma 2.1.** $(\Gamma_K(x, M))_{x \in E, M \geq 0}$ is a closed, regular, $(C, 1)$-convex shape field.

**Proof.** We check each condition one at a time.

**Shape field.** It is clear that $(\Gamma_K(x, M))_{x \in E, M \geq 0}$ is a closed shape field.

**Regularity.** Now we show that $(\Gamma_K(x, M))_{x \in E, M \geq 0}$ is regular as in Definition 2.1. In particular, we want to show that $(\Gamma_K(x, M))_{x \in E, M \geq 0}$ satisfies conditions (2.1)–(2.3).

It is clear that $(\Gamma_K(x, M))_{x \in E, M \geq 0}$ satisfies condition (2.1).
We want to show that a small number to be chosen. Let \( C \) where
\[
(2.11) \quad \left| \partial^{\alpha}(\vec{P} - \vec{P}')(x) \right|, \left| \partial^{\alpha}(\vec{P} - \vec{P}')(x) \right| \leq \delta |x - x'|^{m-|\alpha|} \quad \text{for } |\alpha| \leq m
\]

Since \( \vec{P}' \in \Gamma_{\mathcal{K}}(x') \), we have
\[
(2.12) \quad \vec{P}'(x') \in \mathcal{K}(x').
\]

By Taylor’s theorem and the fact that \( \vec{P} \in \Gamma_{\mathcal{K}}(x, M) \) is an \( m \)-jet, we have
\[
(2.13) \quad \left| \partial^{\alpha} \vec{P}(x') \right| \leq \begin{cases} 
M(1 + C |x - x'|^{m-|\alpha|}) & \text{ for } |\alpha| < m \\
M & \text{ for } |\alpha| = m,
\end{cases}
\]

where \( C = C(m, n, d) \).

We see from (2.11) and (2.13) that
\[
(2.14) \quad \left| \partial^{\alpha} \vec{P}(x') \right| \leq \left| \partial^{\alpha}(\vec{P} - \vec{P}')(x') \right| + \left| \partial^{\alpha} \vec{P}(x') \right| \leq \begin{cases} 
M(1 + C \delta^{m-|\alpha|}) + \delta^{m+1-|\alpha|} & \text{ for } |\alpha| < m \\
M + \delta & \text{ for } |\alpha| = m.
\end{cases}
\]

In view of (2.12) and (2.14) while choosing \( \delta \) to be sufficiently small in a manner dependent only on \( M \) and \( C = C(m, n, d) \), we can conclude that \( \vec{P}' \in \Gamma_{\mathcal{K}}(x', M + \epsilon) \).

We now turn to condition (2.3).

Let \( \epsilon > 0 \) and \( M \geq 0 \). Let \( \delta \in (0, \epsilon) \). Let \( x \in E \), \( \vec{P} \in \Gamma_{\mathcal{K}}(x, M) \), \( \vec{P}' \in \Gamma_{\mathcal{K}}(x) = \bigcup_{M \geq 0} \Gamma_{\mathcal{K}}(x, M) \). Assume that \( \left| \partial^{\alpha}(\vec{P} - \vec{P}')(x) \right| \leq \delta < \epsilon \). We immediately see that \( \vec{P}' \in \Gamma_{\mathcal{K}}(x, M + \epsilon) \). Condition (2.3) is satisfied.

Therefore, \( (\Gamma_{\mathcal{K}}(x, M))_{x \in E, M \geq 0} \) is regular.

\((C, 1)\)-convexity. Now we show that \( (\Gamma_{\mathcal{K}}(x, M))_{x \in E, M \geq 0} \) is \((C, 1)\)-convex. Fix the following:

\[
(2.15) \quad \delta \in (0, 1], x \in E, M \geq 0; \\
(2.16) \quad \vec{P}_1, \vec{P}_2 \in \Gamma_{\mathcal{K}}(x, M) \text{ with } \left| \partial^{\alpha}(\vec{P}_1 - \vec{P}_2)(x) \right| \leq M \delta^{m-|\alpha|} \text{ for } |\alpha| \leq m; \\
(2.17) \quad Q_1, Q_2 \in \mathcal{P} \text{ with } \left| \partial^{\alpha}Q_i(x) \right| \leq \delta^{-|\alpha|} \text{ and } Q_1 \circ x Q_1 + Q_2 \circ x Q_2 = 1.
\]

We want to show that
\[
(2.18) \quad \vec{P} := \sum_{i=1,2} Q_i \circ x Q_i \circ x \vec{P}_i \in \Gamma_{\mathcal{K}}(x, CM).
\]
Thanks to (2.17), we have \( \vec{P}(x) \in K(x) \). Thanks to (2.16) and (2.17), we have
\[ |\partial^\alpha \vec{P}(x)| \leq CM. \] Therefore, (2.18) holds.

\[ \square \]

3. Termination Lemma

Let \( D = \dim \vec{P} \). The purpose of this section is to prove the following lemma, which serves as part 1 of Theorem 1.6.

Lemma 3.1 (Termination of Glaser Refinement). If \((H(x))_{x \in E}\) is a convex bundle, then \( H_l(x) = H_{2D+1}(x) \) for all \( l \geq 2D + 1 \).

The origin of Lemma 3.1 goes back to Glaser [18]. Bierstone-Milman and Fefferman also adapted it in their works [1] and [4].

Before beginning the proof of Lemma 3.1, we will show that the convexity hypothesis on the holding spaces extends to the Glaser refinements.

Lemma 3.2. If \((H_0(x))_{x \in E}\) is a convex bundle, then \((H_l(x))_{x \in E}\) is a convex bundle for all \( l \geq 1 \).

Proof. We induct on \( l \geq 0 \). The case \( l = 0 \) is trivial by assumption.

Let \( l \geq 1 \). Suppose \( H_{l-1}(x) \) is convex for each \( x \in E \). We want to show that \( H_l(x) \) is convex for each \( x \in E \).

Fix \( x_0 \in E \). Let \( \vec{P}_0, \vec{Q}_0 \in H_l(x_0) \). Let \( \theta \in [0, 1] \), and set
\[ \vec{R}_0 := (1 - \theta)\vec{P}_0 + \theta \vec{Q}_0. \]
We want to show that \( \vec{R}_0 \in H_l(x_0) \).

Let \( \epsilon > 0 \). Since \( \vec{P}_0, \vec{Q}_0 \in H_l(x_0) \), there exists \( \delta > 0 \) such that, for any \( x_1, \cdots, x_{k^2} \in E \cap B(x_0, \delta) \), there exist \( \vec{P}_1, \cdots, \vec{P}_{k^2}, \vec{Q}_1, \cdots, \vec{Q}_{k^2} \), with
\[ \vec{P}_j, \vec{Q}_j \in H_{l-1}(x_j) \text{ for } j = 1, \cdots, k^2; \]
\[ |\partial^\alpha (\vec{P}_i - \vec{P}_j)(x_j)| \leq \epsilon |x_i - x_j|^{m - |\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j \leq k^2; \]
and
\[ |\partial^\alpha (\vec{Q}_i - \vec{Q}_j)(x_j)| \leq \epsilon |x_i - x_j|^{m - |\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j \leq k^2. \]

We set
\[ \vec{R}_j := (1 - \theta)\vec{P}_j + \theta \vec{Q}_j \text{ for } j = 1, \cdots, k^2. \]
By the induction hypothesis that $H_{l-1}(x)$ is convex for each $x \in E$, (3.1) and (3.4) imply $\tilde{R}_j \in H_{l-1}(x_j)$ for each $j = 1, \ldots, k^z$. Moreover, by (3.4) and the triangle inequality,

\begin{align}
(3.5) & \quad |\partial^\alpha (\tilde{R}_i - \tilde{R}_j)(x_j)| \leq (1 - \theta) |\partial^\alpha (\tilde{P}_i - \tilde{P}_j)(x_j)| + \theta |\partial^\alpha (\tilde{Q}_i - \tilde{Q}_j)(x_j)| \\
(3.6) & \quad \leq \varepsilon |x_i - x_j|^{m-|\alpha|}
\end{align}

for $|\alpha| \leq m$ and $0 \leq i, j \leq k^z$.

Hence, $H_l(x_0)$ is convex. Since $x_0$, $\tilde{P}_0$, and $\tilde{Q}_0$ were arbitrary, the lemma is proved. \hfill \Box

In [1] and [6], the analogous version of Lemma 3.1 for affine holding spaces was proven using an argument which relied on the well-definition of the dimension of vector subspaces. Here, we adapt this argument to the case of non-affine holding spaces so another definition of dimension will be required.

In this section, we will use the notation $B_{\eta}(\tilde{P})$ to denote the open ball of radius $\eta$ in $\tilde{P}$ centered $\tilde{P}$ with respect to the metric

\[ d(\tilde{P}, \tilde{P}') := \max_{|\alpha| \leq m} |\partial^\alpha (\tilde{P} - \tilde{P}') (0)|. \]

$\overline{B}_{\eta}(\tilde{P})$ will denote the analogous closed ball in $\tilde{P}$.

Let $(H(x))_{x \in E}$ be a holding space with Glaeser refinement $(\tilde{H}(x))_{x \in E}$. Observe that whenever $x_0, x_1, x_2, \ldots \in E$, $\tilde{P}_0 \in \tilde{H}(x_0)$ and $x_j \to x_0$ as $n \to \infty$, there exist $\tilde{P}_j \in H(x_j)$ such that $d(\tilde{P}_j, \tilde{P}_0) \to 0$. This property, rather than the definition of Glaeser refinement, will be key in the proof of Lemma 3.1.

**Definition 3.1.** Given $K \subset \tilde{P}$ convex and $\tilde{P} \in \tilde{P}$, we define the local dimension of $K$ at $\tilde{P}$, denoted $\dim_{\tilde{P}} K$, to be the largest integer $k$ such that the following holds:

\[ (3.8) \quad \text{There exists } \eta > 0 \text{ such that } K \cap B_{\eta}(\tilde{P}) \supset W \cap B_{\eta}(\tilde{P}) \text{ for some } k\text{-dimensional affine subspace } W \supset \tilde{P}. \]

We take $\dim_{\tilde{P}} K = -\infty$ in the case where $\tilde{P} \notin K$. Thus, the condition $\dim_{\tilde{P}} K \neq -\infty$ implies $\tilde{P} \in K$.

For an example illustrating the notion of local dimension, consider the half-ball $K = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, y \geq 0 \}$. In this case $\dim_{(0,1/2)} K = 2$ since $B_{1/10}((0, 1/2)) \subset K$ and $\dim_{(0,0)} K = 1$ since any 2-dimensional ball containing $(0,0)$ must intersect the lower half-plane but the line segment from $(-1/2, 0)$ to $(1/2, 0)$ is contained in $K$. Furthermore, $\dim_{(1,0)} K = \dim_{(0,1)} K = 0$ since in either case, any line segment with the given point as its midpoint must intersect $K^c$. 


Remark 3.2.  
(1) If $\vec{P}$ is in the relative interior of $K$, written $\vec{P} \in \text{int}K$, then we may take $K \cap B_\eta(\vec{P}) = W \cap B_\eta(\vec{P})$ in (3.8).

(2) In the case where $K$ is a closed, convex subset of $\vec{P}$, then it has a well-defined dimension as a set, often defined as the smallest dimension of an affine subspace containing the set $K$, which we denote $\dim K$. When $\vec{P} \in \text{int}K$, $\dim_{\vec{P}}K = \dim K$; in particular, for a vector subspace $V \subset \vec{P}$, $\dim_{\vec{P}}V = \dim V$ for any $\vec{P} \subset V$.

(3) Whenever $K \supset L$, we have $\dim_{\vec{P}}K \geq \dim_{\vec{P}}L$.

Lemma 3.1 will follow shortly from the following result:

**Lemma 3.3.** Suppose $(H(x))_{x \in E}$ is a convex bundle. Let $x \in E$, $\vec{P} \in \vec{P}$, and $k \geq 0$ be an integer. If

\[(3.9) \quad \dim_{\vec{P}}H_{2k+1}(x) \geq D - k,\]

then $\dim_{\vec{P}}H_l(x) = \dim_{\vec{P}}H_{2k+1}(x)$ for all $l \geq 2k + 1$.

To see the value of Lemma 3.3, let us use it to prove Lemma 3.1:

**Proof of Lemma 3.1 via Lemma 3.3.** Let $x \in E$ and $\vec{P} \in H_{2D+1}(x)$. Then, $\dim_{\vec{P}}H_{2D+1}(x) \geq 0$ because the zero-dimensional affine space $W = \{\vec{P}\}$ satisfies $B_\eta(\vec{P}) \cap K \supset B_\eta(\vec{P}) \cap W$ for any $\eta > 0$.

Taking $k = D$ in (3.9), $\dim_{\vec{P}}H_l(x) = \dim_{\vec{P}}H_{2k+1}(x) \geq 0$ for all $l \geq 2k + 1$.

The local dimension is a nonnegative integer if and only if $\vec{P} \in H_l(x)$; therefore, $\vec{P} \in H_l(x)$ for all $l \geq 2D + 1 \geq 2k + 1$. \(\square\)

Now, it suffices to prove Lemma 3.3.

**Proof of Lemma 3.3.** We prove this result by induction on $k$. By Lemma 3.2, $H_l(y)$ is convex for all $y \in E$ and $l \geq 0$. This convexity will be used throughout the proof.

**Base case** ($k = 0$): First, suppose $\dim_{\vec{P}}H_1(x) = D$ for some $x \in E$ and $\vec{P} \in H_1(x)$. In such a case $\dim_{\vec{P}}H(x) = D$ as well by Remark 3.2 No. 3. Pick $\eta$ so $H_1(x) \cap B_\eta(\vec{P}) \supset W \cap B_\eta(\vec{P})$ for some $D$-dimensional affine subspace $W$ running through $\vec{P}$. The only $D$-dimensional affine space in $\vec{P}$ is $\vec{P}$ itself, so $B_\eta(\vec{P}) \subset H_1(x)$.

**Claim 3.3.** $B_{\eta/2}(\vec{P}) \subset H(y)$ for all $y$ sufficiently close to $x$.

**Proof of Claim 3.3.** Suppose the contrary. Then, there is a sequence $x_n \rightarrow x$ and $\vec{P}_n \in B_{\eta/2}(\vec{P})$ such that $\vec{P}_n \notin H(x_n)$.

Consider the set $B_\eta(\vec{P}) \cap H(x_n)$. Since $\vec{P}_n \notin H(x_n)$ and $H(x_n)$ is convex, there exists $\vec{Q}_n \in B_\eta(\vec{P})$ such that $B_{\eta/4}(\vec{Q}_n)$ and $H(x_n) \cap B_\eta(\vec{P})$ are disjoint subsets of $B_\eta(\vec{P})$. By passing to a subsequence, we may assume there exists $\vec{Q} \in B_\eta(\vec{P})$ such
that \(d(\vec{Q}_n, \vec{Q}) \to 0\). This contradicts the fact that \(\vec{Q} \in H_1(x)\), thus proving Claim 3.3. \(\square\)

Since \(B_{\eta/2}(\vec{P}) \subset H(y)\) for all \(y\) sufficiently close to \(x\), we see this property also holds for \(H_1(y)\) in place of \(H(y)\). In fact, let \(r > 0\); then for \(y\) in an open subset of \(E\), \(H_{l+1}(y) \cap B_r(\vec{P})\) is determined solely by \(H_l(z) \cap B_r(\vec{P})\) for \(z\) in that open subset by the definition of Glaeser refinement.

Repeating this logic, we see \(B_{\eta/2}(\vec{P}) \subset H_l(y)\) for all \(H_l(y)\) with \(l \geq 2\) whenever \(y \in E\) sufficiently close to \(x\); this includes the case \(y = x\). Thus, the local dimension of \(H_l(x)\) at \(\vec{P}\) remains at \(D\) for all \(l\), establishing the base case.

**Induction Hypothesis:** Assume

\[
\dim_{\vec{P}} H_{2j+1}(x) \geq D - j
\]

implies \(\dim_{\vec{P}} H_l(x) = \dim_{\vec{P}} H_{2j+1}(x)\) for all \(l \geq 2j+1\) whenever \(j \leq k\).

Now suppose \(\dim_{\vec{P}} H_{2k+3}(x) \geq D - k - 1\). If \(\dim_{\vec{P}} H_{2k+3}(x) > D - k - 1\), then the conclusion follows from the induction hypothesis. Thus, we may assume

(3.10) \[
\dim_{\vec{P}} H_{2k+3}(x) = D - k - 1.
\]

Furthermore,

(3.11) \[
\dim_{\vec{P}} H_{2k+1}(x) = \dim_{\vec{P}} H_{2k+2}(x) = \dim_{\vec{P}} H_{2k+3}(x) = D - k - 1,
\]

since if \(\dim_{\vec{P}} H_{2k+1}(x) > D - k - 1\), \(\dim_{\vec{P}} H_l(x)\) would have terminated at some number greater than \(D - k - 1\) starting at some \(l < 2k+1\) by induction hypothesis. We want to show

(3.12) \[
\dim_{\vec{P}} H_l(x) = D - k - 1
\]

whenever \(l \geq 2k + 3\).

**Claim 3.4.** Let \(l = 2k + 1, 2k + 2\). Fix \(x \in E\) and \(\vec{P} \in H_{l+1}(x)\). Pick \(\eta > 0\) so that \(H_{l+1}(x) \cap B_\eta(\vec{P}) \supseteq W \cap B_\eta(\vec{P})\) for some \((D - k - 1)\)-dimensional affine subspace \(W \supseteq P\). For \(y\) sufficiently close to \(x\), either

(3.13) \[
\dim[H_i(y) \cap B_\eta(\vec{P})] > D - k - 1
\]

or

(3.14) \[
H_i(y) \cap B_{\eta/2}(\vec{P}) = V_y \cap B_{\eta/2}(\vec{P})
\]

for some \((D - k - 1)\)-dimensional affine subspace \(V_y\) of \(\vec{P}\). Furthermore, if \(\vec{P}\) is in the relative boundary of \(H_{l+1}(x)\), then in particular, (3.13) holds for \(y\) sufficiently close to \(x\).
Proof of Claim 3.4. Since $l = 2k + 1, 2k + 2$, we have $\dim_\overline{P} H_l(x) = \dim_\overline{P} H_{l+1}(x) = D - k - 1$.

If $\overline{P} \in \text{int} H_{l+1}(x)$, then suppose the contrary. That is, there exists a sequence $x_n \to x$ such that (3.13) and (3.14) fail for $y = x_n$. In particular, since (3.13) does not hold, $H_l(x_n) \cap B_\eta(P)$ is contained in a $(D - k - 1)$-dimensional affine space $V_n$. Here, we use Remark 3.2 No. 1.

By compactness of the Grassmannian of $(D - k - 1)$-planes in $\overline{P}$ with intersection in $\overline{B_\eta(P)}$ and the closure of $\overline{B_\eta(P)}$, we may assume $V_n$ converges to a subspace $V$ by passing to a subsequence.

Since $H_{l+1}(x) \cap B_\eta(\overline{P}) = W \cap B_\eta(\overline{P})$, $V$ must equal $W$. Else, there exists $\overline{P}' \in H_{l+1}(x)$ which is not the limit of $\overline{P}_n \in H_l(x_n)$, contradicting our prior observation on refinements.

Since (3.14) fails, there exists $\overline{P}'_n \in V_n \cap B_{\eta/2}(\overline{P})$ such that $\overline{P}'_n \notin H_l(x_n)$. Furthermore, by the convexity of $H_l(x_n)$, there exists $\overline{Q}_n \in B_\eta(\overline{P}) \cap V_n$ such that $\overline{Q}_n \notin H_l(x_n)$ and $B_{\eta/4}(\overline{Q}_n) \subset B_\eta(\overline{P})$ is disjoint from $H_l(x_n)$.

By passing to another subsequence, we may assume $\overline{Q}_n$ converges to some $\overline{Q} \in V = W$. However, this contradicts the fact that $\overline{Q} \in H_{l+1}(x)$, establishing our claim for $\overline{P} \in \text{int} H_{l+1}(x)$.

If $\overline{P}$ is in the relative boundary of $H_{l+1}(x)$, then $\dim H_{l+1}(x) > D - k - 1$. Taking $\overline{Q} \in \text{int} H_{l+1}(x)$ such that $\dim_\overline{Q} H_{l+1}(x) > D - k - 1$, we may repeat the above argument to deduce, at the very least, that for $y$ sufficiently close to $x$, $\dim[H_l(y) \cap B_\eta(\overline{P})] > D - k - 1$. This completes the proof of Claim 3.4. □

Consider $y$ sufficiently close to $x$. There are now distinct cases based on which of (3.13) or (3.14) holds for $l = 2k + 1$ and $l = 2k + 2$.

First, there is the case that (3.13) holds for $l = 2k + 1$ or $l = 2k + 2$, which means $H_j(y) \cap B_\eta(\overline{P})$ and $H_l(y) \cap B_\eta(\overline{P})$ have equal relative interiors for all $j \geq l$ by induction hypothesis.

Else, (3.14) holds for both $l = 2k + 1, 2k + 2$, meaning $H_{2k+1}(y) \cap B_{\eta/2}(\overline{P}) = V_y \cap B_{\eta/2}(\overline{P})$ for some $(D - k - 1)$-dimensional affine subspace $V_y$ of $\overline{P}$ and $H_{2k+2}(y) \cap B_{\eta/2}(\overline{P}) = V_y' \cap B_{\eta/2}(\overline{P})$ for some $(D - k - 1)$-dimensional affine subspace $V_y'$ of $\overline{P}$. We see that $V_y = V_y'$ since $H_{2k+2}(y) \subset H_{2k+1}(y)$ and $\dim V_y = \dim V_y'$.

In any case, we see that for $y$ sufficiently close to $x$, $H_{2k+1}(y) \cap B_\eta(\overline{P})$ and $H_{2k+2}(y) \cap B_\eta(\overline{P})$ have equal relative interiors. Ideally, one would like to show that these sets are completely equal; however, Lemma 3.4 below will address this disparity.
Lemma 3.4. Let \( (H(x))_{x \in E} \) be a bundle and fix \( y_0 \in E \). Fix \( \delta_0 > 0 \). Let \( \left( \tilde{H}(x) \right)_{x \in E} \) denote the bundle made from taking \( (H(x))_{x \in E} \) and replacing \( H(y) \) by \( \overline{H}(y) \) for all \( y \in B(y_0, \delta_0) \setminus \{y_0\} \). Then \( H_1(y_0) = \tilde{H}_1(y_0) \).

In particular, if two bundles have the same relative interiors of their fibers at every \( y \in B(y_0, \delta_0) \setminus \{y_0\} \) and \( \tilde{P} \) is in both their fibers at \( y_0 \), then \( \tilde{P} \) is either in the fiber at \( y_0 \) of both \( \overline{H} \) or neither.

We recall that given \( r > 0 \), \( y \) in a relatively open subset of \( E \), \( H_{l+1}(y) \cap B_r(\tilde{P}) \) is determined solely by \( H_t(z) \cap B_r(\tilde{P}) \) for other \( z \) in that relatively open subset. Applying Lemma 3.4 with \( y_0 \) ranging over \( y \) sufficiently close to \( x \), we find

\[
\text{int} H_1(y) = \text{int} H_{2k+1}(y) \quad \text{for } l \geq 2k + 1.
\]

Another application of Lemma 3.4 shows that \( H_1(x) \cap B_{\eta/2}(\tilde{P}) \) remains constant in \( l \) for \( l \geq 2k + 1 \) as well, completing the proof of Lemma 3.3.

At this point, it merely remains to prove Lemma 3.4.

Proof of Lemma 3.4. Since \( H(x) \subseteq \hat{H}(x) \) for all \( x \in E \), we have \( H_1(y_0) \subseteq \hat{H}_1(y_0) \).

Suppose \( \tilde{P}_0 \in \hat{H}_1(y_0) \). Let \( \epsilon > 0 \) and pick \( 0 < \delta < \delta_0 \) such that for every \( y_1, \ldots, y_{k^2} \in B(y_0, \delta) \), there exists \( \tilde{P}_j \in \hat{H}(y_j) \) for \( j = 0, 1, \ldots, k^2 \) such that

\[
|\partial^\alpha (\tilde{P}_i - \tilde{P}_j)(y_j)| \leq (\epsilon/2)|y_i - y_j|^{|\alpha|} \quad \text{for } |\alpha| \leq m, 0 \leq i, j \leq k^2.
\]

Now fix a particular choice of \( y_1, \ldots, y_{k^2} \) and make a choice of \( \tilde{P}_j \in \hat{H}(y_j) \) satisfying (3.16). If all the \( \tilde{P}_j \) lie in \( H(y_j) \), then we are done. It is possible to choose \( \tilde{P}_j \not\in H(y_j) \); however, we may always choose \( \tilde{P}_j \in H(y_j) \) such that \( |\partial^\alpha (\tilde{P}_j - \tilde{P}_j')(x)| \leq \eta_j \) for some \( \eta_j > 0 \) and all \( x \in E, 0 \leq |\alpha| \leq m \).

By the triangle inequality, for any \( |\alpha| \leq m, 0 \leq i, j \leq k^2 \),

\[
|\partial^\alpha (\tilde{P}_i' - \tilde{P}_j')(y_j)| \leq |\partial^\alpha (\tilde{P}_i' - \tilde{P}_i)(y_j)| + |\partial^\alpha (\tilde{P}_j - \tilde{P}_j)(y_j)| + |\partial^\alpha (\tilde{P}_i - \tilde{P}_j')(y_j)|
\]

\[
\leq \eta_i + (\epsilon/2)|y_i - y_j|^{|\alpha|} + \eta_j
\]

\[
\leq \epsilon |y_i - y_j|^{|\alpha|}
\]

by taking \( \eta_j \) (\( 0 \leq j \leq k^2 \)) sufficiently small relative to \( \epsilon \) and \( \min_{i \neq j} |y_i - y_j| \).

Since the choices of \( \epsilon \) and \( y_j \) were arbitrary, this shows \( \tilde{P}_0 \in \hat{H}_1(y_0) \).

The second conclusion follows from the fact that if two fibers have the same relative interior, then they have the same closure and by considering the holding space \( (H'(x))_{x \in E} \), where \( H'(x) = H(x) \) for \( x \neq y_0 \) and \( H(y_0) = \{\tilde{P}\} \).
4. Reduction to the scalar-valued case

In this section, we show that the validity of Theorem 1.6 for $C^{m+d}(\mathbb{R}^n, \mathbb{R})$ implies the same for $C^m(\mathbb{R}^n, \mathbb{R}^d)$, thus reducing all future analysis to scalar-valued functions. The main lemma of the section is the following.

**Lemma 4.1.** Let $m, n, d$ be positive integers. Let $k^2$ be a sufficiently large constant depending only on $m, n, d$. If Theorem 1.6 holds for $C^{m+1}(\mathbb{R}^{n+d}, \mathbb{R})$ and shape fields of type $(m+1, n+d, 1)$, then it also holds for $C^m(\mathbb{R}^n, \mathbb{R}^d)$ and shape fields of type $(m, n, d)$.

We use a gradient trick inspired by [15].

To begin with, we put more emphasis on the degree of regularity and dimension: For positive integers $s_1, s_2, s_3$, we write $\mathcal{P}^s_1(\mathbb{R}^{s_2}, \mathbb{R}^{s_3})$ to denote the vector space of $s_3$-tuples of polynomials with degree no greater than $s_1$ on $\mathbb{R}^{s_2}$. We write $\mathcal{O}^s_1$ to denote the ring multiplication of $s_1$-jets at $x \in \mathbb{R}^{s_2}$, and denote this ring by $\mathcal{R}^s_1$. As before, we use the same notation to denote the action of $\mathcal{R}^s_1$ on the module $\mathcal{P}^s_1(\mathbb{R}^{s_2}, \mathbb{R}^{s_3})$.

For the rest of the section, we use $(x, v) = (x_1, \cdots, x_n, v_1, \cdots, v_d)$ to denote a vector in $\mathbb{R}^{n+d}$. We write $\nabla_v$ to denote the operator $(\partial_{v_1}, \cdots, \partial_{v_d})$. Note that if $P$ is an $(m+1)$-jet (of a real-valued function) on $\mathbb{R}^{n+d}$, then $P(\cdot, 0)$ is an $(m+1)$-jet (of a real-valued function) on $\mathbb{R}^n$, and $\nabla_v|_{v=0}P$ is an $m$-jet (of an $\mathbb{R}^d$-valued function) on $\mathbb{R}^n$.

We say a shape field $(\Gamma((x, M))_{x \in E, M \geq 0}$ is of type $(s_1, s_2, s_3)$ if $\Gamma((x, M) \subset \mathcal{P}^s_1(\mathbb{R}^{s_2}, \mathbb{R}^{s_3})$ for all $x \in E$ and $M \geq 0$.

**Lemma 4.2.** Let $E \subset \mathbb{R}^n$ be a compact set. Let $(\Gamma((x, M))_{x \in E, M \geq 0}$ be a regular $(C_w, \delta_{\max})$-convex shape field of type $(m, n, d)$. For each $x \in E$ and $M \geq 0$, define

\[(\hat{\Gamma}((x, M)) \quad \text{where} \quad \hat{\Gamma}((x, M)_{(x,0) \in E \times \{0\}, M \geq 0} \text{ is a regular } (C, \delta_{\max})-\text{convex shape field of type } (m+1, n+d, 1) \text{ on } E \times \{0\} \subset \mathbb{R}^{n+d}.

C^{-1}\|\Gamma\| \leq \|\hat{\Gamma}\| \leq C\|\Gamma\|, \text{ with } \| \cdot \| \text{ as in Definition 1.4}.

As a corollary to part 2 of the above lemma, the Finiteness Lemma (Lemma 6.1) for vector-valued functions immediately follows from the lemma for scalar-valued functions.

**Proof.** We begin with the first statement. It is clear that $(\hat{\Gamma}((x, 0))_{(x,0) \in E \times \{0\}, M \geq 0}$ is a shape field of type $(m+1, n+d, 1)$. 

We now prove that $(\hat{\Gamma}((x, 0), M))_{(x, 0) \in E \times \{0\}, \mathcal{M} \geq 0}$ is regular.

Condition (2.1) is clearly satisfied, thanks to the first constraint in (4.1).

We prove condition (2.2). Let $\epsilon > 0$ and $M \geq 0$. Let $\eta = \eta(\epsilon, M) > 0$ be a small number to be chosen. Let $(x, 0), (x', 0) \in E \times \{0\}$ with $|(x, 0) - (x', 0)| \leq \eta$. Let $\hat{P} \in \hat{\Gamma}((x, 0), M)$ and $\hat{P'} \in \hat{\Gamma}(x', 0) = \bigcup_{M' \geq 0} \hat{\Gamma}((x', 0), \mathcal{M}')$, with

\[
\left| \partial^\alpha (\hat{P} - \hat{P'}) (x, 0) \right|, \left| \partial^\alpha (\hat{P} - \hat{P'}) (x', 0) \right| \leq \delta |x - x'|^{m + 1 - |\alpha|}, \quad \text{for } \alpha \in \mathbb{N}_0^{n+d}, |\alpha| \leq m + 1.
\]

Set $\hat{P} = \nabla_v|_{v=0} \hat{P}$ and $\hat{P'} = \nabla_v|_{v=0} \hat{P'}$. By the definition of $\hat{\Gamma}$, $\hat{P} \in \Gamma(x, M)$ and $\hat{P}' \in \Gamma(x')$. In view of (4.2), we have

\[
\left| \partial^\beta (\hat{P} - \hat{P'}) (x) \right|, \left| \partial^\beta (\hat{P} - \hat{P'}) (x') \right| \leq C \eta |x - x' - m - |\beta|, \quad \text{for } \beta \in \mathbb{N}_0^n, |\beta| \leq m.
\]

Since $(\Gamma(x, M))_{x \in E, \mathcal{M} \geq 0}$ is regular, for sufficiently small $\eta$, (4.3) implies

\[
\hat{P}' \in \Gamma(x, M + \hat{\epsilon}(\eta)), \quad \text{i.e., } \nabla_v|_{v=0} \hat{P} \in \Gamma(x, M + \hat{\epsilon}(\eta)).
\]

Here $\hat{\epsilon}(\cdot)$ is a decreasing function of $\eta$, depending only on $M$.

We choose $\eta$ to be sufficiently small (depending only on $\epsilon$ and $M$), so that (4.2), (4.4), and a similar argument as in Lemma 2.1 altogether imply $\hat{P}' \in \hat{\Gamma}((x', 0), M + \epsilon)$.

Condition (2.2) is satisfied.

For condition (2.3), let $\epsilon > 0$, $M \geq 0$, and let $\eta > 0$ be a sufficiently small number to be determined. Let $(x, 0) \in E \times \{0\}$, $\hat{P} \in \hat{\Gamma}((x, 0), M)$, $\hat{P}' \in \hat{\Gamma}(x, 0)$ with

\[
\left| \partial^\alpha (\hat{P} - \hat{P'}) (x, 0) \right| \leq \eta \text{ for } \alpha \in \mathbb{N}_0^{n+d}, |\alpha| \leq m + 1.
\]

Set $\hat{P} = \nabla_v|_{v=0} \hat{P}$ and $\hat{P'} = \nabla_v|_{v=0} \hat{P'}$. By the definition of $\hat{\Gamma}$, $\hat{P} \in \Gamma(x, M)$ and $\hat{P}' \in \Gamma(x')$. In view of (4.5), we have

\[
\left| \partial^\beta (\hat{P} - \hat{P'}) (x) \right| \leq C \eta \text{ for } \beta \in \mathbb{N}_0^n, |\beta| \leq m.
\]

Since $(\Gamma(x, M))_{x \in E, \mathcal{M} \geq 0}$ is regular, for sufficiently small $\eta$ (depending only on $\epsilon$ and $M$), (4.5), (4.6), and a similar argument as in Lemma 2.1 imply $\hat{P}' \in \hat{\Gamma}((x, 0), M + \epsilon)$.

Condition (2.3) is satisfied.

Thus, we have shown that $(\hat{\Gamma}((x, 0), M))_{(x, 0) \in E \times \{0\}, \mathcal{M} \geq 0}$ is regular.

We now show that $(\hat{\Gamma}((x, 0), M))_{(x, 0) \in E \times \{0\}, \mathcal{M} \geq 0}$ is $(C, \delta_{\max})$-convex for some $C$ depending only on $m, n, d, C_w$. 
Let $\delta \in (0, \delta_{\text{max}}], (x, 0) \in E \times \{0\}, M \geq 0$, $\hat{P}_1, \hat{P}_2, \hat{Q}_1, \hat{Q}_2 \in P^{m+1}(\mathbb{R}^{n+d}, \mathbb{R})$ satisfy
\begin{equation}
\hat{P}_1, \hat{P}_2 \in \hat{\Gamma}((x, 0), M) \tag{4.7}
\end{equation}
\begin{equation}
\left| \partial^\alpha (\hat{P}_1 - \hat{P}_2)(x, 0) \right| \leq M\delta^{m+1-|\alpha|} \text{ for } \alpha \in \mathbb{N}_0^{m+d}, |\alpha| \leq m+1; \tag{4.8}
\end{equation}
\begin{equation}
\left| \partial^\alpha \hat{Q}_i(x, 0) \right| \leq \delta^{-|\alpha|} \text{ for } \alpha \in \mathbb{N}_0^{m+d}, |\alpha| \leq m+1, i = 1, 2; \text{ and } \tag{4.9}
\end{equation}
\begin{equation}
\hat{Q}_1 \circ_{(x, 0)}^{m+1,n+d} \hat{Q}_1 + \hat{Q}_2 \circ_{(x, 0)}^{m+1,n+d} \hat{Q}_2 = 1. \tag{4.10}
\end{equation}

We want to show that
\begin{equation}
\hat{P} = \sum_{i=1,2} \hat{Q}_i \circ_{(x, 0)}^{m+1,n+d} \hat{Q}_i \circ_{(x, 0)}^{m+1,n+d} \hat{P}_i \in \hat{\Gamma}((x, 0), CM), \tag{4.11}
\end{equation}
or equivalently,
\begin{equation}
\left| \partial^\alpha \hat{P}(x, 0) \right| \leq M \text{ for } \alpha \in \mathbb{N}_0^{m+d}, |\alpha| \leq m+1, \tag{4.12}
\end{equation}
\begin{equation}
\hat{P}(\cdot, 0) \equiv 0, \text{ and } \tag{4.13}
\end{equation}
\begin{equation}
\nabla_v |_{v=0} \hat{P} \in \Gamma(x, CM). \tag{4.14}
\end{equation}

Thanks to (4.7), we have
\begin{equation}
\left| \partial^\alpha \hat{P}_i(x, 0) \right| \leq M \text{ for } \alpha \in \mathbb{N}_0^{m+d}, |\alpha| \leq m+1, \tag{4.15}
\end{equation}
\begin{equation}
\hat{P}_i(\cdot, 0) \equiv 0 \text{ for } i = 1, 2, \text{ and } \tag{4.16}
\end{equation}
\begin{equation}
\nabla_v |_{v=0} \hat{P}_i \in \Gamma(x, M) \text{ for } i = 1, 2.. \tag{4.17}
\end{equation}

In view of (4.10) and the definition of $\hat{P}$ in (4.11), we have
\begin{equation}
\partial^\alpha \hat{P}(x, 0) = \sum_{\beta+\gamma=\alpha} C_{\alpha,\beta,\gamma} \cdot \partial^\beta \hat{Q}_1(x, 0) \cdot \partial^\gamma \hat{Q}_1(x, 0) \cdot \partial^{\alpha-\beta-\gamma}(\hat{P}_1 - \hat{P}_2)(x, 0). \tag{4.18}
\end{equation}

Using (4.8) and (4.9) to estimate (4.17), we see that (4.12) follows.

Thanks to (4.15), we have
\begin{equation}
\hat{P}(\cdot, 0) = \sum_{i=1,2} \hat{Q}_i |_{v=0} \circ_{(x, 0)}^{m+1,n+d} \hat{Q}_i |_{v=0} \circ_{(x, 0)}^{m+1,n+d} \hat{P}_i |_{v=0} \equiv 0. \tag{4.19}
\end{equation}

Thus, (4.13) follows.
Let $\pi : P_{(x_0)}^{m+1,n+d} \to P_{(x_0)}^{m,n+d}$ be the natural projection. Thanks to (4.16), we have

$$\nabla_v|_{v=0}\hat{P} = \sum_{i=1,2} \pi \hat{Q}_i|_{v=0} \circ_{(x_0)}^{m,n+d} \pi \hat{P}_i|_{v=0} \circ_{(x_0)}^{m,n+d} \nabla_v|_{v=0}\hat{P}_i$$

(4.19)

$$= \sum_{i=1,2} \pi \hat{Q}_i|_{v=0} \circ_{(x_0)}^{m,n+d} \pi \hat{Q}_i|_{v=0} \circ_{(x_0)}^{m,n+d} \nabla_v|_{v=0}\hat{P}_i$$

We write $\partial^\alpha = \partial^\beta_x \partial^\xi_v$, for $\alpha \in N_0^{n+d}$, $\beta \in N_0^n$, $\xi \in N_0^d$.

If $|\xi| = 1$, i.e., $\partial_v = \partial_{v_s}$ for some $s \in \{1, \ldots, d\}$, then (4.8) implies

$$|\partial^\alpha(\hat{P}_1 - \hat{P}_2)(x,0)| = \left| \partial^\beta_x(\partial_{v_s}|_{v=0}\hat{P}_1 - \partial_{v_s}|_{v=0}\hat{P}_2)(x) \right| \leq M\delta^{m-|\beta|} \text{ for } |\beta| \leq m.$$  

Therefore,

(4.20)  \[ |\partial^\beta_x \left[ \nabla_v|_{v=0}(\hat{P}_1 - \hat{P}_2) \right](x) | \leq CM\delta^{m-|\beta|} \text{ for } |\beta| \leq m. \]

On the other hand, we see from (4.9) that

(4.21)  \[ |\partial^\beta_x \left[ \pi \hat{Q}_i|_{v=0} \right] | = \left| |\partial^\beta_x \hat{Q}(x,0)| \leq \delta^{-|\beta|} \text{ for } |\beta| \leq m. \]

In view of (4.19)-(4.21) and the $(C_u, \delta_{\text{max}})$-convexity of $(\Gamma(x,M))_{x \in E, M \geq 0}$, we can conclude that $\hat{P} \in \hat{\Gamma}((x,0), CM)$, so (4.11) holds. We have shown that $(\hat{\Gamma}((x,0), M))_{(x,0) \in E \times \{0\}, M \geq 0}$ is $(C, \delta_{\text{max}})$-convex.

This concludes the proof of Lemma 4.2(1).

We turn to the second statement.

Let $M = 2\|\hat{\Gamma}\|$. We will show that $\|\Gamma\| \leq CM$.

Let $x_1, \ldots, x_{k^2} \in E$ be given. There exist $\hat{P}_1 \in \hat{\Gamma}((x_1,0), M), \ldots, \hat{P}_{k^2} \in \hat{\Gamma}((x_{k^2},0), M)$ such that

(4.22)  \[ |\partial^\alpha(\hat{P}_i - \hat{P}_j)(x_i,0)| \leq M \| (x_i,0) - (x_j,0) \|^{m+1-|\alpha|} \text{ for } \alpha \in N_0^{n+d}, |\alpha| \leq m + 1. \]

We set

$$\bar{P}_i := \nabla_v|_{v=0}\hat{P}_i \in P^m(\mathbb{R}^n, \mathbb{R}^d) \text{ for } 1 \leq i \leq k^2.$$  

It follows immediately from (4.22) that

$$|\partial^\beta(\bar{P}_i - \bar{P}_j)(x_i)| \leq CM \| x_i - x_j \|^{m-|\beta|} \text{ for } \beta \in N_0^n, |\beta| \leq m.$$  

Therefore, $\|\Gamma\| \leq \|\hat{\Gamma}\|$.

Now we show the reverse direction. Let $M = 2\|\Gamma\|$. We will show that $\|\hat{\Gamma}\| \leq M$. 


Let \( x_1, \ldots, x_{k^2} \in E \) be given. There exist \( \bar{P}_1, \bar{P}_2, \ldots, \bar{P}_{k^2} \in \Gamma(x_{i^2}, M) \) such that
\[
4.23 \quad \left| \partial^\beta(\bar{P}_i - \bar{P}_j)(x_i) \right| \leq M |x_i - x_j|^{m-|\beta|} \quad \text{for } \beta \in \mathbb{N}_0^n, |\beta| \leq m.
\]
We define
\[
\bar{P}_i(\cdot, v) = v \cdot \bar{P}_i(\cdot) \in \mathcal{P}^{m+1}(\mathbb{R}^{n+d}, \mathbb{R}) \quad \text{for } 1 \leq i \leq k^2.
\]
Observe that for \( 1 \leq i \leq k^2 \), \( \xi \in \mathbb{N}_0^d \),
\[
4.24 \quad \partial^|\xi| v|_{v=0} \bar{P}_i = \begin{cases} 0 & \text{if } |\xi| = 0 \\
P_{i,j} \text{ for some } 1 \leq j \leq d & \text{if } |\xi| = 1 \\
0 & \text{if } |\xi| \geq 2
\end{cases}
\]
Therefore, for \( |\xi| \neq 1 \), we have
\[
4.25 \quad \left| \partial^2_x \partial^|\xi| v(\bar{P}_i - \bar{P}_j)(x_i, 0) \right| = 0 \quad \text{for } \beta \in \mathbb{N}_0^n, \xi \in \mathbb{N}_0^d, |\beta| + |\xi| \leq m + 1.
\]
Suppose \( |\xi| = 1 \). Without loss of generality, we may assume \( \partial^\xi v = \partial_{v_s} \) for some \( s \in \{1, \ldots, d\} \). Thanks to (4.23), we have
\[
4.26 \quad \left| \partial^2_x \partial_{v_s} (\bar{P}_i - \bar{P}_j)(x_i, 0) \right| = \left| \partial^2_x (P_{i,s} - P_{j,s})(x_i) \right| \leq M |x_i - x_j|^{m-|\alpha|} = M |x_i - x_j|^{m+1-|\gamma|}.
\]
We see from (4.25) and (4.26) that \( \|\hat{\Gamma}\| \leq M \). This proves the second statement.

With Lemma 4.2 in hand, we are ready to prove Lemma 4.1.

Proof of Lemma 4.1. Let \( Q_0 \subset \mathbb{R}^n \) be a cube of length 3 and \( E \subset Q_0 \) be compact. Let \( E = E \times \{0\} \subset \mathbb{R}^{n+d} \) and \( \bar{Q}_0 = Q_0 \times \{0, 0+3\delta Q_0\} \). Note that \( \bar{Q}_0 \) is a hypercube of sidelength less than 3.

Let \( (\Gamma(x), M))_{x \in E, M \geq 0} \) be a regular \((C_w, 1)\)-convex shape field. Write \( \Gamma(x) = \bigcup_{M \geq 0} \Gamma(x, M) \). Suppose \( \Gamma^*(x) \neq \emptyset \) for all \( x \in E \), where \( (\Gamma^*(x))_{x \in E} \) is the termination of iterated Glaeser refinements of \( (\Gamma(x))_{x \in E} \) guaranteed by Lemma 3.3. We write \( \Gamma^*(x, M) = \Gamma^*(x) \cap \Gamma(x, M) \) for each \( x \in E \) and \( M \geq 0 \).

As in Lemma 4.2, we define the following objects.
\[
\hat{\Gamma}((x, 0), M) = \left\{ \bar{P} \in \mathcal{P}^{m+1}(\mathbb{R}^{n+d}, \mathbb{R}) : \left| \partial^\alpha \bar{P}(x, 0) \right| \leq M \text{ for } \alpha \in \mathbb{N}_0^{n+d}, |\alpha| \leq m + 1, \right. \\
\left. \bar{P}(\cdot, 0) \equiv 0, \text{ and } \nabla v|_{v=0} \bar{P} \in \Gamma(x, M) \right\}
\]
was established in Lemma 2.1

Recall that the first part of Theorem 1.6 was established in Lemma 3.1; thus it suffices to prove the second part. By Taylor’s theorem and the definition of Glaeser refinement, we see that if \((\Gamma(x))_{x \in E}\) has a section \(F\) satisfying \(\|F\|_{C^m(\mathbb{R}^{n+d})} \leq M\), then \((\Gamma^*(x))_{x \in E}\) is nonempty and \(\|\hat{\Gamma}^*\| \leq CM\). Thus, it suffices to suppose \((\Gamma^*(x))_{x \in E}\) is nonempty and determine the existence of a section with the appropriate norm bounds.

Thanks to Lemma 4.2 and the assumption that \((\Gamma(x, M))_{x \in E, M \geq 0}\) is a closed regular \((C, 1)\)-shape field, we see that

\[
(\hat{\Gamma}((x, 0), M))_{(x, 0) \in E, M \geq 0} \text{ is a closed regular } (C, 1)\text{-convex shape field.}
\]

We write

\[
(\hat{\Gamma}((x, 0))) = \bigcup_{M \geq 0} \hat{\Gamma}((x, 0), M) \text{ and } \hat{\Gamma}^*((x, 0)) = \bigcup_{M \geq 0} \hat{\Gamma}^*((x, 0), M) \text{ for } (x, 0) \in \hat{E}.
\]

We will show that the hypothesis of Theorem 1.6 is satisfied for \(\hat{E}, m+1, n+d, \hat{\Gamma}, \) and \(\hat{\Gamma}^*\) in place of \(E, m, n, \Gamma, \) and \(\Gamma^*\). In particular, we need to show that

\[
(\hat{\Gamma}^*((x, 0))) \neq \emptyset \text{ for each } (x, 0) \in \hat{E}
\]

(4.31) \(\hat{\Gamma}^*((x, 0)))\) is its own Glaeser refinement, and

(4.32) \(\hat{\Gamma}^*((x, 0)))\) is its own Glaeser refinement, and

(4.33) \(\hat{\Gamma}^*((x, 0))) \subset \hat{\Gamma}^*((x, 0)) \text{ for } x \in E.
\]

Assuming the above, we see that the shape field \((\hat{\Gamma}((x, 0), M))_{(x, 0) \in E, M \geq 0}\) satisfies the hypotheses of Theorem 1.6 for \(C^{m+1}(\mathbb{R}^{n+d}, \mathbb{R})\). While we did not show the iterated Glaeser refinement of \(\hat{\Gamma}\) terminates in \(\hat{\Gamma}^*\), it is clear from definition that it must terminate in a Glaeser stable bundle containing \(\hat{\Gamma}^*\); call this bundle \(\Gamma'\). It follows that \(\Gamma'((x, 0)))\) is nonempty for all \((x, 0) \in \hat{E}\) from the fact that \(\hat{\Gamma}((x, 0))) \subset \Gamma'((x, 0)))\). The appropriate quantitative bounds follow from \(\|\Gamma'\| \leq \|\hat{\Gamma}^*\|\).

Applying Theorem 1.6, there exists \(G \in C^{m+1}(\mathbb{R}^{n+d}, \mathbb{R})\) with

\[
\|G\|_{C^{m+1}(\mathbb{R}^{n+d}, \mathbb{R})} \leq \hat{M} \text{ and } J_+((x, 0), \hat{M}) \text{ for all } (x, 0) \in \hat{E}.
\]

Here, \(J_+((x, 0))\) denotes the\((m+1)\)-jet at \((x, 0), \) and \(\hat{M} = C\|\Gamma'\| \leq C\|\hat{\Gamma}^*\|\) for some controlled constant \(C\). Thanks to the regularity condition (2.1), we can further improve the control of the norm

\[
(4.34) \|G\|_{C^{m+1}(\mathbb{R}^{n+d}, \mathbb{R})} \leq C\hat{M}.
\]
Define
\[ \vec{F}(x) = (\partial_{v_1} G(x, 0), \ldots, \partial_{v_d} G(x, 0)) \].

In view of the definition of \( \hat{\Gamma} \) in (4.28) and (4.30) and (4.34), we have
\[ \|\vec{F}\|_{\dot{C}^m(\mathbb{R}^n, \mathbb{R}^d)} \leq C \hat{M} \text{ and } J_x \vec{F} \in \Gamma^*(x, C\hat{M}). \] (4.35)

In view of (4.35), we see that Lemma 4.1 holds.

It now suffices to establish (4.31), (4.32), and (4.33).

We show (4.31) as follows. Since we assume that \( \Gamma^*(x) \neq \emptyset \) for each \( x \in E \), we can pick \( \vec{P}_x \in \Gamma^*(x) \) for each \( x \in E \). We define
\[ \hat{P}^{(x,0)}(y, v) = \sum_{1 \leq \gamma \leq m+1} \frac{1}{\gamma!} v^\gamma P_\gamma(y) \] for some \( P_\gamma \in P^m(\mathbb{R}^n, \mathbb{R}) \).

We immediately verify that \( \hat{P}^{(x,0)} \in \hat{\Gamma}^*((x,0)) \) for each \( (x,0) \in \hat{E} \). Therefore, \( \hat{\Gamma}^*((x,0)) \neq \emptyset \) for each \( (x,0) \in \hat{E} \), establishing (4.31).

Now, to show (4.32), fix \( (x_0,0) \in \hat{E} \) and \( \hat{P}_0 \in \hat{\Gamma}^*((x_0,0)) \). We will show that \( \hat{P}_0 \) survives the Glaeser refinement procedure. Let \( \epsilon > 0 \).

Since \( \hat{P}_0 \in \hat{\Gamma}^*((x_0,0)) \), we can write
\[ \hat{P}_0(y, v) = \sum_{1 \leq \gamma \leq m+1} \frac{1}{\gamma!} v^\gamma P_\gamma(y) \text{ for some } P_\gamma \in P^m(\mathbb{R}^n, \mathbb{R}). \] (4.36)

We set
\[ \vec{P}_0 = (P_{0,1}, \ldots, P_{0,d}) = \nabla v \big|_{v=0} \hat{P}_0 \]
\[ = \left( \partial_{v_1} \big|_{v=0} \left[ \sum_{1 \leq \gamma \leq m+1} \frac{1}{\gamma!} v^\gamma P_\gamma(x) \right], \ldots, \partial_{v_d} \big|_{v=0} \left[ \sum_{1 \leq \gamma \leq m+1} \frac{1}{\gamma!} v^\gamma P_\gamma(x) \right] \right). \] (4.37)

By construction, we have
\[ \vec{P}_0 \in \Gamma^*(x_0). \] (4.38)

Thanks to (4.36) and (4.37), we can write
\[ \hat{P}_0(y, v) = \sum_{1 \leq j \leq d} v_j P_{0,j}(y) + \sum_{2 \leq \gamma \leq m+1} \frac{1}{\gamma!} v^\gamma P_\gamma(y). \] (4.39)
Since the bundle \((\Gamma^*(x))_{x \in E}\) is Glaeser stable, we know that there exists \(\delta > 0\) such that for all \(x_1, \ldots, x_{k^*} \in E \cap B^n(x_0, \delta)\), there exist
\[
\vec{P}_1 = (P_{1,1}, \ldots, P_{1,d}) \in \Gamma^*(x_1),
\]
(4.40)
\[
\vdots
\]
\[
\vec{P}_{k^*} = (P_{k^*,1}, \ldots, P_{k^*,d}) \in \Gamma^*(x_{k^*}),
\]
with
\[
|\partial^\alpha(\vec{P}_i - \vec{P}_j)(x_i)| \leq \epsilon |x_i - x_j|^{m-|\alpha|} \quad \text{for } |\alpha| \leq m, 0 \leq i, j \leq k^*.
\]
(4.41)

For any \((x_1, 0), \ldots, (x_{k^*}, 0) \in \hat{E} \cap B^{n+d}((x_0, 0), \delta)\), we set
\[
\hat{P}_k(y, v) = \sum_{j=1}^d v_j P_{k,j}(y) + \sum_{2 \leq |\gamma| \leq m+1} \frac{1}{\gamma!} v^\gamma P_{\gamma}(y) \quad \text{for } 1 \leq k \leq k^*.
\]
(4.42)

Here, \(P_{\gamma}\) are as in (4.39), and \(\vec{P}_1, \ldots, \vec{P}_{k^*}\) are as in (4.40).

We claim that
\[
\hat{P}_k \in \hat{\Gamma}((x_k, 0)) \quad \text{for } 1 \leq k \leq k^*.
\]
(4.43)

and
\[
|\partial^\alpha(\hat{P}_i - \hat{P}_j)(x_i, 0)| \leq C\epsilon |x_i - x_j|^{m+1-|\alpha|} \quad \text{for } |\alpha| \leq m+1, 0 \leq i, j \leq k^*.
\]
(4.44)

To verify (4.43), we apply \(\nabla_v\big|_{v=0}\) to (4.42) and see that
\[
\nabla_v\big|_{v=0} \hat{P}_k = (P_{k,1}, \ldots, P_{k,d}) = \vec{P}_k \in \Gamma^*(x_k) \quad \text{for } 1 \leq k \leq k^*.
\]
(4.45)

Therefor, (4.43) follows from (4.28) and (4.45).

To verify (4.44), we write \(\partial^\alpha = \partial_x^\alpha \partial_0^\beta\). Observe that for \(0 \leq k \leq k^*\),
\[
\nabla_v\big|_{v=0} \hat{P}_k = \begin{cases} 
0 & \text{if } |\xi| = 0 \\
\sum_{1 \leq j \leq d} P_{k,j} & \text{if } |\xi| = 1 \\
\sum_{|\xi| \geq 2} P_{\xi} & \text{if } |\xi| \geq 2
\end{cases}
\]
(4.46)

Thanks to (4.46), we see that (4.44) holds trivially for \(|\xi| \neq 1\). Therefore, it suffices to show (4.44) for the case \(|\xi| = 1\). Without loss of generality, we may assume
\[ \partial_s^k = \partial_{v_s} \] for some \( s \in \{1, \ldots, d\} \). Then
\[ \left| \partial^\alpha (\hat{P}_i - \hat{P}_j)(x_i, 0) \right| = \left| \partial_x^\beta \partial_{v_s} (\hat{P}_i - \hat{P}_j)(x_i, 0) \right| \]
\[ = \left| \partial_x^\beta (P_i,s - P_j,s)(x_i) \right| \]
\[ \leq \epsilon |x_i - x_j|^{|\alpha|} \quad \text{for some } s \in \{1, \ldots, d\} \]

We see that (4.44) follows from (4.47). We have shown that the bundle \((\hat{\Gamma}^*((x, 0)))_{(x,0) \in \hat{E}}\) as defined in (4.28) and (4.30) is Glaeser stable, giving us (4.32).

Lastly, we get (4.33) by observing that \(\hat{\Gamma}^*((x, 0)) \subset \hat{\Gamma}((x, 0))\) since \(\Gamma^*((x, 0)) \subset \Gamma((x, 0))\) for all \((x,0) \in \hat{E}\).

\[ \square \]

Remark 4.1. Thanks to Lemma 4.1, to prove Theorem 1.6 for \(C^m(\mathbb{R}^n, \mathbb{R}^d)\), it suffices to prove it for the case \(d = 1\).

For the rest of the paper, we write \(C^m(\mathbb{R}^n)\) instead of \(C^m(\mathbb{R}^n, \mathbb{R}^d)\), and \(F, \mathcal{P}, P\) instead of \(\vec{F}, \vec{\mathcal{P}}, \vec{P}\).

5. Finiteness principle for shape fields

Recall from the previous section that we will be working with real-valued functions for the rest of the paper.

Definition 5.1. A function \(\omega : [0, 1] \to [0, \infty)\) is called a regular modulus of continuity if it satisfies the following conditions:

(\(\omega\)-1) \(\omega(0) = \lim_{t \downarrow 0} \omega(t) = 0\) and \(\omega(1) = 1\);
(\(\omega\)-1) \(\omega(t)\) is increasing on \([0, 1]\); and
(\(\omega\)-1) \(\omega(t)/t\) is decreasing on \((0, 1]\).

Note that in (\(\omega\)-2) and (\(\omega\)-3), we do not demand that \(\omega\) be strictly increasing, or that \(\omega(t)/t\) be strictly decreasing.

Let \(\omega\) be a regular modulus of continuity. We write \(C^m(\mathbb{R}^n)\) to denote the space of all \(C^m\) functions \(F\) on \(\mathbb{R}^n\) for which the norm
\[ \|F\|_{C^m(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} |\partial^\alpha F(x)| + \sup_{\substack{|\alpha| = m \\alpha \in \mathbb{R}^n \\alpha \neq 0 \\alpha \neq 0 \\alpha \neq 0 \\alpha \neq 0}} \frac{|\partial^\alpha F(x) - \partial^\alpha F(x')|}{\omega(|x - x'|)} \]

is finite.
Similarly, we write $\dot{C}^{m,\omega}(\mathbb{R}^n)$ to denote the space of all $C^m$ functions $F$ on $\mathbb{R}^n$ for which the norm

$$
\|F\|_{\dot{C}^{m,\omega}(\mathbb{R}^n)} := \sup_{\|x-x'\| \leq 1} \frac{|\partial^\alpha F(x) - \partial^\alpha F(x')|}{\omega(|x-x'|)}
$$

is finite.

The following lemma was proven in [9].

**Lemma 5.1.** Let $(\Gamma(x, M))_{x \in E, M \geq 0}$ be a $(C_w, \delta_{\max})$-convex shape field. Let $\delta \in (0, \delta_{\max})$, $x \in E$, $M \geq 0$, $A', A'' > 0$, $P_1, \ldots, P_k, Q_1, \ldots, Q_k \in \mathcal{P}$. Assume that

- $P_i \in \Gamma(x, A'M)$ for $i = 1, \ldots, k$;
- $|\partial^\alpha (P_i - P_j)(x)| \leq A'M^{\delta_{\max} - |\alpha|}$ for $|\alpha| \leq m$, $1 \leq i, j \leq k$;
- $|\partial^\alpha Q_i(x)| \leq A''\delta_{\max}^{-|\alpha|}$ for $|\beta| \leq m$ and $1 \leq i \leq k$;
- $\sum_{i=1}^k Q_i \circ_x P_i = 1$.

Then $\sum_{i=1}^k Q_i \circ_x P_i \in \Gamma(x, CM)$, with $C = C(A', A'', C_w, m, n, k)$.

By adapting the proof of the Finiteness Principle for shape fields from [9] (see also [3, 5]), we obtain the following result.

**Theorem 5.2 (\dot{C}^{m,\omega}-Finiteness Principle for Shape Fields).** Let $m, n \in \mathbb{N}$. For sufficiently large $k_{\#SF} = k(m, n)$, the following holds.

Let $E \subset \mathbb{R}^n$ be an arbitrary subset and $\omega$ be a regular modulus of continuity. Let $\Gamma(x, M)_{x \in E, M \geq 0}$ be a $(C_w, \delta_{\max})$-convex shape field. Let $M_0 < \infty$ and $Q_0$ be a cube of sidelength $\delta_{Q_0} \leq \delta_{\max}$. Assume that for each $S \subset E$ with $\#(S) \leq k^{\#SF}$, there exists $(P^x)_{x \in S}$ such that

$$
P^x \in \Gamma(x, M_0) \text{ for } x \in S
$$

and

$$
|\partial^\alpha (P^x - P^y)(x)| \leq M_0\omega(|x-y|)\omega(|x-y|^{m-|\alpha|}) \text{ for } |\alpha| \leq m, x, y \in S, x \neq y.
$$

Then there exists $F \in C^{m,\omega}(Q_0)$, with

$$
J_x F \in \Gamma(x, C^4 M_0) \text{ for } x \in E \cap Q_0
$$

and

$$
\|F\|_{C^{m,\omega}(Q_0)} \leq C^4 M_0.
$$

Here, $C^4 = C^4(m, n, C_w)$. 

6. Other Preliminaries

Lemma 6.1 (Finiteness Lemma). Let $E \subseteq \mathbb{R}^n$ be a compact set. Let $(\Gamma(x, M))_{x \in E, M \geq 0}$ be a regular Glaeser stable shape field. Then there exists a finite constant $A^d$ such that:

Given $x_1, \ldots, x_{k^d} \in E$, there exist $P_i \in \Gamma(x_i, A^d)$ such that

$$|\partial^\alpha (P_i - P_j)(x_j)| \leq A^d |x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 1 \leq i, j \leq k^d.$$  \hspace{1cm} (6.1)

The $A^d$ from Lemma 6.1 will be fixed and referenced throughout the remainder of the paper.

Proof. We slightly modify the proof given in [6].

Suppose towards a contradiction, that for each $\nu = 1, 2, \cdots$ we can find

- $x_1^{(\nu)}, \ldots, x_{k^d}^{(\nu)} \in E$, and
- $A^{(\nu)} > 0$, with $\lim_{\nu \to \infty} A^{(\nu)} = \infty$,

such that for each $\nu$,

(6.2) there do not exist polynomials $P_1, \cdots, P_{k^d} \in \mathcal{P}$ such that

$$P_j \in \Gamma(x_j^{(\nu)}, A^{(\nu)}) \text{ for } j = 1, \ldots, k^d; \text{ and}$$

(6.4) $$|\partial^\alpha (P_i - P_j)(x_j^{(\nu)})| \leq A^{(\nu)} |x_i^{(\nu)} - x_j^{(\nu)}|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j \leq k^d.$$ \hspace{1cm} (6.3)

Since $E$ is compact, by passing to a subsequence, we may assume that

$$x_j^{(\nu)} \to x_j^{(\infty)} \in E \text{ as } \nu \to \infty, \text{ for } j = 1, \cdots, k^d.$$ \hspace{1cm} (6.5)

Let $z_1, \ldots, z_{\mu_{\max}}$ be an enumeration of the distinct elements of $\{x_1^{(\infty)}, \ldots, x_{k^d}^{(\infty)}\}$.

For each $\mu = 1, \cdots, \mu_{\max}$, let $S(\mu) := \{ j : x_j^{(\infty)} = z_\mu, 1 \leq j \leq k^d \}$. Therefore, for sufficiently large $\nu$, we have

$$|x_j^{(\nu)} - z_\mu| \leq \eta_1 \text{ for all } j \in S(\mu),$$ \hspace{1cm} (6.6)

and

$$|x_j^{(\nu)} - x_{j'}^{(\nu)}| \geq \eta_2 \text{ for } j \in S(\mu), j' \in S(\mu') \text{ with } \mu \neq \mu',$$ \hspace{1cm} (6.7)

where $\eta_1$ and $\eta_2 > 0$ are chosen to be independent of $\nu$. For the rest of the proof, we use $A_0, A_1, \text{ etc.}$, to denote constants independent of $\nu$.

We now apply the hypothesis that $(\Gamma(x))_{x \in E}$ is Glaeser-stable.

Fix $\mu \in \{1, \cdots, \mu_{\max}\}$. We set $x_0 = z_\mu, P_0 \in \Gamma(x_0, A_0)$, and $\epsilon = 1$ in the definition of Glaeser refinement. Let $\nu$ be sufficiently large. We set $x_j := x_j^{(\nu)}$ for $j \in S(\mu)$,
\[ x_j := z_{j'} \text{ for } j' \notin S(\mu) \ (1 \leq j \leq k^2). \]  
Since \( \Gamma(x) \) is its own Glaeser refinement, it follows from (6.5) that there exist

\begin{equation}
\tag{6.7}
P_j^{(\nu)} \in \Gamma(x_j^{(\nu)}), \text{ for } j \in S(\mu) \\
\nu \text{ sufficiently large, such that }
\end{equation}

\begin{equation}
\tag{6.8}
\left| \partial^\alpha (P_i^{(\nu)} - P_j^{(\nu)})(x_j^{(\nu)}) \right| \leq \left| x_i^{(\nu)} - x_j^{(\nu)} \right|^{m-|\alpha|} \text{ for } i, j \in S(\mu), |\alpha| \leq m.
\end{equation}

Furthermore, it follows from (6.5), (6.7), (6.8), and (2.2) (upon taking \( \eta_1 \) sufficiently small) that

\begin{equation}
\tag{6.9}
P_j^{(\nu)} \in \Gamma(x_j^{(\nu)}, A_1), \text{ for } j \in S(\mu).
\end{equation}

We repeat the argument above for \( \mu = 1, \cdots, \mu_{\text{max}} \). Thus, for sufficiently large \( \nu \), we can find polynomials \( P_1^{(\nu)}, \cdots, P_{k^2}^{(\nu)} \) that satisfy the following:

\begin{equation}
\tag{6.10}
P_j^{(\nu)} \in \Gamma(x_j^{(\nu)}, A_2) \text{ for } j = 1, \cdots, k^2; \text{ and }
\end{equation}

\begin{equation}
\tag{6.11}
\left| \partial^\alpha (P_i^{(\nu)} - P_j^{(\nu)})(x_j^{(\nu)}) \right| \leq \left| x_i^{(\nu)} - x_j^{(\nu)} \right|^{m-|\alpha|} \text{ for } |\alpha| \leq m, i, j \in S(\mu), \mu = 1, \cdots, \mu_{\text{max}}.
\end{equation}

Observe that by 2.1, (6.10) implies

\begin{equation}
\tag{6.12}
|\partial^\alpha P_j^{(\nu)}(x_j^{(\nu)})| \leq A_2 \text{ for } |\alpha| \leq m.
\end{equation}

Combining (6.6) and (6.12), we see that

\begin{equation}
\tag{6.13}
|\partial^\alpha (P_i^{(\nu)} - P_j^{(\nu)})(x_j^{(\nu)})| \leq A_3 \left| x_i^{(\nu)} - x_j^{(\nu)} \right|^{m-|\alpha|} \text{ for } |\alpha| \leq m, i \in S(\mu), j \in S(\mu'), \mu \neq \mu'.
\end{equation}

Together with (6.11) and (6.13), we see that

\begin{equation}
\tag{6.14}
|\partial^\alpha (P_i^{(\nu)} - P_j^{(\nu)})(x_j^{(\nu)})| \leq A_4 \left| x_i^{(\nu)} - x_j^{(\nu)} \right|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 1 \leq i, j \leq k^2.
\end{equation}

For sufficiently large \( \nu \), we have \( A^{(\nu)} > A_2 \) and \( A^{(\nu)} > A_4 \), with \( A_2 \) and \( A_4 \), respectively, as in and (6.10) (6.14). Then, (6.10), and (6.14) together contradict (*).

The lemma is proved.

\[\square\]

\begin{lemma}
\tag{6.2}
Let \( S \subset \mathbb{R}^n \) with \( 2 \leq \#(S) \leq k^2 \). Then we may partition \( S \) into subsets \( S_1, \cdots, S_{\nu_{\text{max}}} \) with the following properties.

(a) \( \#(S_i) \leq \#(S) \) for each \( \nu = 1, \cdots, \nu_{\text{max}} \).
\end{lemma}
(b) If \( x \in S_\nu \) and \( y \in S_\mu \) with \( \nu \neq \mu \), then \( |x - y| > c(k^2) \cdot \text{diam}(S) \). Here, \( c(k^2) \) depends only on \( k^2 \).

**Lemma 6.3.** Suppose \( W \subset \mathbb{R}^n \) is an \( r \)-dimensional subspace, \( w \in W \), and \( w_0, \ldots, w_r \) form a nondegenerate affine \( r \)-simplex containing \( B(w, \delta) \cap W \) for some \( \delta > 0 \). Then, there exists \( \eta > 0 \) for which the following holds:

Let \( W' \subset \mathbb{R}^n \) be another \( r \)-dimensional subspace. If \( w'_0, \ldots, w'_r \in W' \) satisfy

\[
|w_j - w'_j| < \eta \quad \text{for } j = 0, 1, \ldots, r,
\]

and \( w' \in W' \) satisfies

\[
|w - w'| < \eta,
\]

then \( w' \) is contained in the convex hull of \( w'_0, \ldots, w'_r \).

**Proof.** Since \( w \in \text{Conv}(w_0, w_1, \ldots, w_r) \), there exist \( \lambda_0, \lambda_1, \ldots, \lambda_r \) such that \( w = \sum_{i=0}^r \lambda_i w_i \). Let \( \xi_1, \ldots, \xi_r \) be an orthonormal basis for \( \text{span}(w_0 - w_1, \ldots, w_0 - w_r) \). Then the coefficients \( \lambda_0, \lambda_1, \ldots, \lambda_r \) satisfy the following constrained linear system.

\[
\sum_{i=0}^r \lambda_i (w_i \cdot \xi_j) = w \cdot \xi_j \quad \text{for } j = 1, \ldots, r; \quad (6.17)
\]

\[
\sum_{i=0}^r \lambda_i = 1; \quad (6.18)
\]

\[
0 \leq \lambda_i \leq 1 \quad \text{for } i = 0, 1, \ldots, r. \quad (6.19)
\]

Note that the system (6.17)–(6.18) is nondegenerate, since \( w_0, w_1, \ldots, w_r \) form a nondegenerate affine \( r \)-simplex.

Moreover, since \( B(w, \delta) \cap W \subset \text{Conv}(w_0, w_1, \ldots, w_r) \), we may replace (6.19) by

\[
c \leq \lambda_i \leq 1 \quad \text{for } i = 0, 1, \ldots, r \quad (6.20)
\]

for some small constant \( c > 0 \).

If \( \eta \) is sufficiently small, (6.15) implies that \( w'_0, \ldots, w'_r \) also form a nondegenerate \( r \)-simplex in \( W' \). We may find \( \lambda'_0, \lambda'_1, \ldots, \lambda'_r \) such that

\[
\sum_{i=0}^r \lambda'_i (w'_i \cdot \xi_j) = w' \cdot \xi_j \quad \text{for } j = 1, \ldots, r; \quad (6.21)
\]

\[
\sum_{i=0}^r \lambda'_i = 1. \quad (6.22)
\]

Note that \( w' \in \text{Conv}(w'_0, \ldots, w'_r) \) if and only if \( \lambda'_0, \lambda'_1, \ldots, \lambda'_r \) satisfy the system (6.21) and (6.22) coupled with the constraint

\[
0 \leq \lambda'_i \leq 1 \quad \text{for } i = 0, 1, \ldots, r. \quad (6.23)
\]
By choosing \( \eta \) to be sufficiently small, (6.16) and (6.15) imply that the coefficients of the linear system (6.17)–(6.18) are sufficiently close to that of (6.21)–(6.22). This forces \( \lambda_i \) to be sufficiently close to \( \lambda_i' \) for \( i = 0, 1, \cdots, r \). Thanks to (6.20), we see that \( \lambda_0', \lambda_1', \cdots, \lambda_r' \) satisfy the constraint (6.23). This proves the lemma. \( \square \)

**Lemma 6.4 (Helly’s Theorem).** Let \( (K_\alpha)_{\alpha \in A} \) be a family of compact, convex sets in \( \mathbb{R}^N \). If any \( N + 1 \) of the \( K_\alpha \) have nonempty intersection, then \( \bigcap_{\alpha \in A} K_\alpha \neq \emptyset \).

For a proof of the above, see [24], for example.

**Lemma 6.5 (Whitney Extension Theorem for Finite Sets).** [32] Let \( S \subset \mathbb{R}^n \) be finite, and for each \( x \in S \), let \( P^x \in \mathcal{P} \). If

\[
(6.24) \quad |\partial^\alpha P^x(x)| \leq M \text{ for } |\alpha| \leq m
\]

and

\[
(6.25) \quad |\partial^\alpha (P^x - P^y)(x)| \leq M|x - y|^{m-|\alpha|} \text{ for } |\alpha| \leq m, x, y \in S, x \neq y,
\]

then there exists \( F \in C^m(\mathbb{R}^n) \) such that \( \|F\|_{C^m(\mathbb{R}^n)} \leq CM \) and \( J_x F = P^x \) for all \( x \in S \). Here, \( C \) depends only on \( m, n \).

7. Convex Sets and Strata

In this section, we write \( A, A', \) etc., to denote quantities that are strictly greater than \( A^\sharp \), with \( A^\sharp \) as in Lemma 6.1.

**Definition 7.1.** For \( x_0 \in E, \ k \in \mathbb{N}_0, \) and \( A > 0 \), we let \( \Gamma(x_0, k, A) \) be the set of \( P_0 \in \Gamma(x) \) such that for any \( x_1, \ldots, x_k \in E \), there exist \( P_1, \cdots, P_k \in \mathcal{P} \), with

- \( P_i \in \Gamma(x, A) \) for \( i = 1, \cdots, k \); and
- \( |\partial^\alpha (P_i - P_j)(x_j)| \leq A|x_i - x_j|^{m-|\alpha|} \) for \( |\alpha| \leq m, 0 \leq i, j \leq k \).

Note that by the Finiteness Lemma, there exists \( A^\sharp < \infty \) such that \( \Gamma(x, k, A^\sharp) \) is nonempty for all \( x \in E \) and \( k \leq k^\sharp \).

Let \( \mathcal{R}_x \) denote the ring of \((m-1)\)-jets of functions at \( x \) and let \( \pi_x : \mathcal{R}_x \to \mathcal{R}_x \) be the natural projection. We identify \( \mathcal{R}_x \) with the space \( \mathcal{P} \) of polynomials with degree no greater than \( (m-1) \). For \( A > 0 \), we define

\[
(7.1) \quad \Gamma(x, k, A) := \pi_x \Gamma(x, k, A). \quad \text{(7.1)}
\]

We note that \( \Gamma(x, k, A) \) is convex, since \( \pi_x \) is linear.

Similarly, let

\[
(7.2) \quad \Gamma(x, A) := \pi_x \Gamma(x, A). \quad \text{(7.2)}
\]
7.1. **Strata.** The following three lemmas will assist in defining the dimension of a Glaeser stable bundle.

**Lemma 7.1.** Let $A \geq 2A^\sharp$, $x_0 \in E$, and $k \in \mathbb{N}$. Then,

1. $\dim \Gamma(x_0) = \dim \Gamma(x_0, A) = \dim \Gamma(x_0, 2A^\sharp)$.
2. $\dim \Gamma(x_0, k, A) = \dim \Gamma(x_0, k, 2A^\sharp)$.

**Proof.** Suppose the contrary to the first conclusion, that there exists $A > 2A^\sharp$ such that $\dim \Gamma(x_0, A) > \dim \Gamma(x_0, 2A^\sharp) = d$ and pick $P \in \Gamma(x_0, A)$ such that $P$ is not contained in the $d$-dimensional hyperplane in $\mathcal{P}$ containing $\Gamma(x_0, 2A^\sharp)$.

By Lemma 6.1, $\Gamma(x_0, A^\sharp)$ is nonempty, so take $P' \in \Gamma(x_0, A^\sharp)$.

Now consider the sequence $P_n = (1/n)P' + (1 - 1/n)P$. By convexity, $P_n \in \Gamma(x_0, A)$ for all $n$. But by supposition, $P_n \notin \Gamma(x_0, 2A^\sharp)$ for all $n$, contradicting (2.3). Thus, $\dim \Gamma(x_0, A) = \dim \Gamma(x_0, 2A^\sharp)$. $\dim \Gamma(x_0) = \dim \Gamma(x_0, 2A^\sharp)$ follows by taking unions.

Now suppose the contrary to the second conclusion. Analogous to the previous situation, there exists $A > 2A^\sharp$ such that $\dim \Gamma(x_0, k, A) > \dim \Gamma(x_0, k, 2A^\sharp) = d'$ and pick $P \in \Gamma(x_0, k, A)$ such that $P$ is not contained in the $d'$-dimensional hyperplane in $\mathcal{P}$ containing $\Gamma(x_0, k, 2A^\sharp)$.

Let $P' \in \Gamma(x_0, k, A^\sharp)$ and again consider the sequence $P_n = (1/n)P' + (1 - 1/n)P$. By our prior reasoning, there exists $N$ such that $n \geq N$ implies $P_n \in \Gamma(x_0, 3/2A^\sharp)$.

Now, let $x_1, ..., x_k \in E$. Choose $Q_j \in \Gamma(x_j, 1)$ such that

\[
|\partial^\alpha(Q_i - Q_j)(x_i)| \leq |x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j \leq k
\]

with $Q_0 = P$. Similarly, choose $Q'_j \in \Gamma(x_j, A)$ such that

\[
|\partial^\alpha(Q'_i - Q'_j)(x_i)| \leq A|x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j \leq k
\]

with $Q'_0 = P'$.

Form sequences $Q_j^{(n)} = (1/n)Q'_j + (1 - 1/n)Q_j$. By (2.3) (and the independence of $\delta$ of $x$ in this condition), we may take $N$ independent of $x_1, ..., x_k$ such that $n \geq N$ implies $Q_j^{(n)} \in \Gamma(x_j, 2A^\sharp)$. Thus, for $n \in N$, $P'_n \in \Gamma(x_0, k, 2A^\sharp)$. This is a contradiction, so $\dim \Gamma(x_0, k, A) = \dim \Gamma(x_0, k, 2A^\sharp)$.

**Lemma 7.2.** Let $A \geq 2A^\sharp$, $x_0 \in E$, and $k \leq k^\sharp$. Then,

\[
\dim \Gamma(x_0, A) = \dim \Gamma(x_0, k, A).
\]

**Proof.** Suppose the contrary. Then by Lemma 7.1, $\dim \Gamma(x_0, k, A') \leq \dim \Gamma(x_0, A) - 1$ for all $A' > A$. In particular, there exists $P \in \Gamma(x_0, A)$ such that for all $A' > 0$ $P \notin \Gamma(x_0, k, A')$.
Define the convex bundle \((\Gamma'(x))_{x \in E}\), where

\[
\Gamma'(x, M) = \begin{cases} 
\Gamma(x, M) & \text{if } M \geq 0, x \neq x_0 \\
\{P\} & \text{if } x = x_0, M \geq A \\
\emptyset & \text{if } x = x_0, M < A.
\end{cases}
\]

Observe that

By Lemma 6.1 and the fact \(k \leq k^\sharp\), there exists \(\tilde{A} > 0\) such that \(\Gamma'(x_0, k, \tilde{A})\) is nonempty (in particular, \(\Gamma'(x_0, k, \tilde{A}) = \{P\}\)), where the \(\Gamma'(x, k, A)\) are defined analogously to the original \(\Gamma(x, k, A)\) but with \(\Gamma'(x, M)\) in place of \(\Gamma(x, M)\).

Since \(\Gamma'(x, A) \subset \Gamma(x, A)\) for all \(x \in E\) and \(A > 0\), \(\Gamma'(x_0, k, \tilde{A}) \subseteq \Gamma(x_0, k, \tilde{A})\) for all \(x \in E, A > 0\), thus \(P \in \Gamma(x_0, k, \tilde{A})\), a contradiction. \(\square\)

Motivated by the prior two lemmas, we make the following definition. Given a graded holding space \((H(x, M))_{x \in E}\), define \(W_H(x)\) as the (unique) \(\dim H(x)\)-dimensional subspace of \(P\) containing \(H(x)\).

Lemma 7.3. Let \(A \geq 2A^\sharp, x \in E,\) and \(k \leq k^\sharp\). Then,

\[
\dim W_{\Gamma}(x) = \dim \Gamma(x) = \dim \Gamma(x, A) = \dim \Gamma(x, k, A) = \dim \Gamma(x, 2A^\sharp) = \dim \Gamma(x, k; 2A^\sharp).
\]

and

\[
\dim \pi_x W_{\Gamma}(x) = \dim \Gamma(x) = \dim \Gamma(x, A) = \dim \Gamma(x, k, A) = \dim \Gamma(x, 2A^\sharp) = \dim \Gamma(x, k; 2A^\sharp).
\]

Proof. Lemmas 7.1 and 7.2 immediately imply (7.7). Since each of the relevant \(\Gamma\)'s is contained in \(\Gamma(x)\), not only are they all the same dimension, but they are contained in the same \(\dim \Gamma(x)\)-dimensional hyperplane. Thus, the dimensions of the projections under \(\pi_x\) are all equal as well, establishing (7.8). \(\square\)

Definition 7.2. Let \(E \subset \mathbb{R}^n\) be a compact set. Let \((H(x))_{x \in E}\) be a convex bundle over \(E\). For each \(x \in E\), we define the signature of \(x\) to be

\[\text{sig}(x) := (\dim H(x), \dim [\ker \pi_x \cap H(x)]).\]

For given integers \(k_1, k_2\), we define

\[E(k_1, k_2) := \{x \in E : \text{sig}(x) = (k_1, k_2)\}.\]

We define the stratum

\[E_1 := E(k_1^*, k_2^*)\]

where \(k_1^*\) is as small as possible, and \(k_2^*\) is as large as possible for this given \(k_1^*\). We call \(E_1\) the lowest stratum.
7.2. Lowest stratum.

**Lemma 7.4.** Let \( E \subset \mathbb{R}^n \) and \((H(x))_{x \in E}\) be a convex, Glaeser stable bundle. Let \( E_1 \) be the lowest stratum as in Definition 7.2. Then \( E_1 \) is compact.

**Proof.** Let \( x \in E \). Suppose \( \dim H(x) = d \). Let \( P_1, \ldots, P_d \) be the vertices of a nondegenerate affine \( d \)-simplex in \( \text{int} H(x) \). Thus, small perturbation of \( \{P_1, \ldots, P_d\} \) remain the vertices of a nondegenerate affine \( d \)-simplex.

By the Glaeser stability of \((H(x))_{x \in E}\), for any \( \tilde{x} \in E \) sufficiently close to \( x \), we may find \( \tilde{P}_1, \ldots, \tilde{P}_d \in H(\tilde{x}) \), such that they are the vertices of a nondegenerate affine \( d \)-simplex. Hence,

\[
\dim(H(\tilde{x})) \geq d \text{ for any } \tilde{x} \text{ sufficiently close to } x.
\]

It follows that

\[
\{x \in E : \dim H(x) < d\}
\]

is a closed set for any \( d \in \mathbb{N}_0 \). In particular, for \( k_1 = \min_{y \in E} \dim H(y) \), the set

\[
E(k_1) := \bigcup_{k_2 \in \mathbb{N}_0} E(k_1, k_2)
\]

is closed.

By hypothesis of Theorem 1.6 again, we see that the map

\[
x \mapsto W_H(x)
\]

is continuous from \( E(k_1) \) to \( \mathcal{G} \), the Grassmannian of \( k_1 \)-planes in \( \mathcal{P} \).

Set \( k_2 = \max_{y \in E(k_1)} \dim(\ker \pi_y \cap W_H(y)) \), noting that \( \dim(\ker \pi_y \cap W_H(y)) = \dim(\ker \pi_y \cap H(y)) \). By Definition 7.2, we see that

\[
E_1 = \{x \in E(k_1, k_2) : \dim(\ker \pi_x \cap H(x)) = k_2\}.
\]

We will show that \( E_1 \) is closed.

Let \( (x_\nu)_{\nu=1}^\infty \) be a sequence in \( E_1 \) converging to \( x \in \mathbb{R}^n \). By (7.9), we know that \( x \in E(k_1) \), and thus, \( W_H(x_\nu) \) converges to \( W_H(x) \) in \( \mathcal{G} \). Passing onto a subsequence, we can further assume that \( \ker \pi_{x_\nu} \cap W_H(x_\nu) \) converges to some (not necessarily unique) \( W \in \text{Conv}(\mathcal{P}) \) with \( \dim W = k_2 \).

Now, \( W \in H(x) \) and \( \pi_x|_W \equiv 0 \). Therefore,

\[
\dim(\ker \pi_x \cap W_H(x)) \geq k_2.
\]

By our choice of \( k_2 \), we must have

\[
\dim(\ker \pi_x \cap W_H(x)) = k_2.
\]

Hence, \( x \in E_1 \) as claimed.

Since \( E \) is compact and \( E_1 \subset E \) is closed, the lemma follows. \( \square \)
Recall from Definition 7.2 that $E_1$ denotes the lowest stratum of $E$. By Lemma 7.4, $E_1$ is compact. Furthermore, for a convex bundle $(H(x))_{x \in E}$, $\dim H(x)$ and $\dim(\ker \pi_x \cap H(x))$ are constant for all $x \in E_1$.

Limiting our attention to the bundle $(\Gamma(x, M))_{x \in E}$ from the hypotheses of Theorem 1.6, we set

$$d := \dim \Gamma(x) \text{ for all } x \in E_1,$$

$$\bar{d} := \dim \Gamma(x) \text{ for all } x \in E_1.$$

### 7.3. Glaeser Stability

Given a shape field $(\Gamma(x, M))_{x \in E}$, $M \geq 0$, $x_0 \in E$, and $A > 0$, define

$$\Gamma^0(x, A) := \bigcup_{M < A} \Gamma(x, M).$$

**Lemma 7.5.** Let $(\Gamma(x, M))_{x \in E}$ be a regular, convex, Glaeser-stable, shape field and fix $A > 0$. The bundle $(\Gamma^0(x, A))_{x \in E}$ is Glaeser stable.

**Proof.** Fix $x_0 \in E$ and $P_0 \in \Gamma^0(x, A)$. Then, there exists $M < A$ such that $P_0 \in \Gamma(x, M)$. By 2.2, there exists $\eta > 0$ such that for any $x' \in E \cap B(x, \eta)$, if $P' \in \Gamma(x')$ with

$$|\partial^\alpha (P - P')(x)|, |\partial^\alpha (P - P')(x')| \leq \eta |x - x'|^{m - |\alpha|} \text{ for all } |\alpha| \leq m,$$

then $P' \in \Gamma(x', \frac{M + A}{2}) \subset \Gamma^0(x', A)$.

Let $0 < \epsilon < \eta$. By the Glaeser stability of $(\Gamma(x))_{x \in E}$, there exists $0 < \delta < \eta$ such that if $x_1, \ldots, x_{k^2} \in E \cap (x, \delta)$, then there exist $P_1, \ldots, P_{k^2} \in \mathcal{P}$ such that

$$P_j \in \Gamma(x_j) \text{ for } j = 1, \ldots, k^2,$$

and

$$|\partial^\alpha (P_i - P_j)(x_i)| < \eta |x_i - x_j|^{m - |\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j \leq k^2.$$

Since (7.13) implies (7.11), we have $P_j \in \Gamma^0(x_j, A)$ for $j = 1, \ldots, k^2$, which is precisely what we wanted to show.

In the sequel, we use $\Gamma(x, A)$ to denote $\Gamma^0(x, A)$, simplifying notation. Note that $\Gamma^0(x)$ may not be a closed subset of $\mathcal{P}$, but since $\Gamma^0(x, A) \subset \Gamma(x, A) \subset \Gamma^0(x, A')$ for $A < A'$, we see

$$\Gamma(x) = \bigcup_{M \geq 0} \Gamma^0(x, M)$$

and $(\Gamma^0(x, M))_{x \in E}$ will share most relevant properties with $(\Gamma(x, M))_{x \in E}$, in particular, $(C, \delta_{\max})$-convexity and regularity.
8. More on Convex Sets

In Section 9, we will establish a uniform choice of $\delta$ in the definition of Glaeser stability which will allow us to choose polynomials from the $\Gamma(x, A)$. The purpose of this section is to show that we may take polynomials from the smaller $\Gamma(x, k, A)$ (where $k$ and $A$ will be chosen appropriately).

Lemma 8.1. Let $x_0 \in E, A > 0$, and $\bar{k} \in \mathbb{N}$. Suppose $P_0 \in \Gamma(x_0, \bar{k}, A)$ and $P_0' \in \Gamma(x_0, A)$ such that $\pi_{x_0}(P_0 - P_0') = 0$. Then, there exists $C$, depending solely on $m, n$ such that $P_0' \in \Gamma(x_0, \bar{k}, CA)$.

Proof. Let $x_1, \ldots, x_\bar{k} \in E$. By the fact that $P_0 \in \Gamma(x_0, \bar{k}, A)$, there exist $P_j \in \Gamma(x, A)$ $(1 \leq j \leq \bar{k})$ such that

$$\sum_{i=0}^{\bar{k}} |\partial^\alpha(P_i - P_j)(x_j)| \leq A|x_i - x_j|^{m - |\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j \leq \bar{k}. \tag{8.1}$$

The conclusion will follow from establishing

$$|\partial^\alpha(P_0' - P_j)(x_j)| \leq A'|x_0 - x_j|^{m - |\alpha|} \text{ for } |\alpha| \leq m, 1 \leq j \leq \bar{k} \tag{8.2}$$

and

$$|\partial^\alpha(P_0' - P_0')(x_j)| \leq A'|x_0 - x_j|^{m - |\alpha|} \text{ for } |\alpha| \leq m, 1 \leq j \leq \bar{k}. \tag{8.3}$$

Since $P_0, P_0' \in \Gamma(x_0, A)$, by 2.1 we have

$$|\partial^\alpha P_0(x_0)|, |\partial^\alpha P_0'(x_0)| \leq A \text{ for } |\alpha| \leq m. \tag{8.4}$$

Therefore, since $\partial^\alpha P_0, \partial^\alpha P_0'$ are constant functions for $|\alpha| = m$, we have

$$|\partial^\alpha(P_0 - P_0')(y)| \leq A_1, y \in \mathbb{R}^n, |\alpha| = m, \tag{8.5}$$

where $A_1$ depends solely on $A, m, n$, and $\text{diam}E$. We are able to ignore the dependence on $\text{diam}E$ in the future, as $E \subset Q_0$, a cube of fixed length.

By (8.1) and (8.5), we see for $1 \leq j \leq \bar{k}$ and $|\alpha| = m$

$$|\partial^\alpha(P_0' - P_j)(x_0)| \leq |\partial^\alpha(P_0' - P_0)(x_0)| + |\partial^\alpha(P_0 - P_j)(x_0)| \leq A_1 + A|x_0 - x_j|^{m - |\alpha|} \tag{8.6}$$

$$= (A_1 + A)|x_0 - x_j|^{m - |\alpha|}, \tag{8.7}$$

establishing (8.2) with $A' = A_1 + A$. (8.3) follows similarly.

It remains to check the cases of $|\alpha| \leq m - 1$. Here, we see that (8.2) follows immediately since $\pi_{x_0}(P_0 - P_0') = 0$ implies $\partial^\alpha P_0(x_0) = \partial^\alpha P_0'(x_0)$ for $|\alpha| \leq m - 1$.

To establish (8.3), we observe that for $1 \leq j \leq \bar{k}$ and $|\alpha| \leq m - 1$, by (8.1) and (8.4)

$$|\partial^\alpha(P_0' - P_j)(x_j)| \leq |\partial^\alpha(P_0' - P_0)(x_j)| + |\partial^\alpha(P_0 - P_j)(x_j)| \leq A_2|x_0 - x_j|^{m - |\alpha|} + A|x_0 - x_j|^{m - |\alpha|}, \tag{8.9}$$

$$\leq A_2|x_0 - x_j|^{m - |\alpha|} + A|x_0 - x_j|^{m - |\alpha|}, \tag{8.10}$$
where \( A_2 \) depends solely on \( A, m, n \). This gives (8.3) with \( A' = A_2 + A \).

\[ \Box \]

**Lemma 8.2.** Let \( A > 0 \) and \( x \in E_1 \). Suppose \( 1 + (D + 1) \cdot \bar{k} \leq \bar{k} \). If \( \overline{P} \in \text{int} \Gamma(x, \bar{k}, A) \), then there exists \( 0 < \delta < 1 \) such that if \( x' \in E_1 \cap B(x, \delta) \), there exists \( \overline{P}' \in \Gamma(x', \bar{k}, A) \) such that

\[ \left| \partial^\alpha (\overline{P} - \overline{P}')(x) \right| , \left| \partial^\alpha (\overline{P} - \overline{P}')(x') \right| \leq A |x - x'|^{m - |\alpha|} \text{ for } |\alpha| \leq m - 1. \]

**Proof.** Choose \( \overline{P}_0, ..., \overline{P}_{\bar{d}} \in \Gamma(x, \bar{k}, A) \) such that \( \text{Conv}(\overline{P}_0, ..., \overline{P}_{\bar{d}}) \) is a \( \bar{d} \)-dimensional simplex containing \( B(\eta, \overline{P}) \cap \pi_\nu W_\Gamma(x) \) for some \( \eta > 0 \). Write \( \overline{P}_j = \pi_x P_j \) and \( \overline{P} = \pi_x P \) for some \( P_j, P \in \Gamma(x, \bar{k}, A), 0 \leq j \leq \bar{d} \).

By definition of the \( \Gamma(x, \bar{k}, A) \), for any \( x' \in E_1 \), there exists \( P'_j \in \Gamma(x', A) \) \((0 \leq j \leq d)\) such that

\[ |\partial^\alpha (P_j - P'_j)(x')| \leq A |x - x'|^{m - |\alpha|} \text{ for } |\alpha| \leq m. \]

In particular, since \( |\partial^\alpha P_j(x)| \leq A \) for \( |\alpha| \leq m \) by 2.1,

\[ |\partial^\alpha (\overline{P}_j - \overline{P}'_j)(x)| \leq A_1 |x - x'|^{m - |\alpha|} \text{ for } |\alpha| \leq m - 1, \]

where \( \overline{P}_j = \pi_x P'_j \) and \( A_1 \) is a constant depending solely on \( A, m, n \).

Applying Lemma 6.3 to \( \overline{P}_0, ..., \overline{P}_{\bar{d}} \), we see that by taking \( |x - x'| \) sufficiently small, there exists \( 0 < \delta < 1 \) such that if \( x' \in E_1 \cap B(x, \delta) \) and \( \overline{P} \in \pi_\nu W_\Gamma(x') \) satisfies

\[ |\partial^\alpha (\overline{P} - \overline{P})(x)| \leq A_1 |x - x'|^{m - |\alpha|} \text{ for } |\alpha| \leq m - 1, \]

then \( \overline{P} \in \Gamma(x') := \text{Conv}(\overline{P}_0, ..., \overline{P}_{\bar{d}}) \subset \Gamma(x', A) \). \( A_1 \) will be chosen later and depend solely on \( A, m, n \). Here, we implicitly used Lemma 7.3 to show \( \pi_\nu W_\Gamma \) was \( \bar{d} \)-dimensional, allowing us to apply Lemma 6.3.

Fix \( x' \in E_1 \cap B(x, \delta) \). Given a finite set \( S \subset E \), define \( S^+ := \{ x, x' \} \cup S \), and define

\[ \mathcal{K}(S) := \left\{ \overline{P} \in \Gamma(x') : \exists P^y \in \Gamma(y, A) \text{ for all } y \in S^+ \text{ such that } P^x = P, \pi_{x'} P^x = \overline{P}, \right. \]

\[ \left. |\partial^\alpha (P^y - P^x)(z)| \leq A |y - z|^{m - |\alpha|} \text{ for } |\alpha| \leq m, y, z \in S^+ \right\}. \]

Each \( \mathcal{K}(S) \) is compact, convex subset of \( \overline{P} \), which has dimension \( \leq D \). Given \( S \subset S' \), we have \( \mathcal{K}(S') \subset \mathcal{K}(S) \). Given \( S^+ \subset E \) with \( |S^+| \leq \bar{k} + 1 \), we see from \( P \in \Gamma(x, \bar{k}, A) \) there exists \( \tilde{P} \in \mathcal{P} \) such that \( \pi_{x'} \tilde{P} \) satisfies the conditions for membership in \( \mathcal{K}(S^+) \), except it is only known so far that \( \pi_{x'} \tilde{P} \in \overline{P} \), rather than \( \pi_{x'} \tilde{P} \in \Gamma(x') \). However, \( |\partial^\alpha (P - P^{x'})(x')| \leq A |x - x'|^{m - |\alpha|} \) implies (8.14) with \( A_1 \).
chosen accordingly; thus, it is required that \( \pi_x \bar{P} \in \overline{\Gamma}(x') \) thus \( \pi_x \bar{P} \in \mathcal{K}(S^+) \) and \( \mathcal{K}(S^+) \) is nonempty.

Suppose \( S_1, \ldots, S_{D+1} \subset E \) with \( \#(S_i) \leq \tilde{k} \) for \( i = 1, \ldots, D + 1 \). Then \( S := S_1 \cup \cdots \cup S_{D+1} \) satisfies \( \#(S) \leq \tilde{k} + 1 \). Hence, \( \mathcal{K}(S) \neq \emptyset \), and \( \mathcal{K}(S) \subset \mathcal{K}(S_i) \) for each \( i \). Thus, \( \bigcap_{i=1}^{D+1} \mathcal{K}(S_i) \neq \emptyset \). By Helly’s theorem, there exists \( \bar{P} \in \mathcal{K}(S) \) for every \( S \subset E \) with \( \#(S) \leq \tilde{k} \). It follows that \( \bar{P} \in \overline{\Gamma}(x', \tilde{k}, A) \). Taking \( S = \emptyset \) and using the fact that \( \bar{P} \in \mathcal{K}(S) \), we see that the first conclusion follows.

\[ \square \]

**Lemma 8.3.** Suppose \( A > 0 \) and \( 1 + (D + 1)\tilde{k} \leq \overline{k} \). Given \( x \in E_1 \), and \( \overline{Q} \in \text{int}\Gamma(x, \overline{k}, A) \), there exist \( \epsilon_0, \delta_0 > 0 \) such that for any \( x \in E_1 \cap B(x, \delta_0) \) and any \( \overline{Q} \in \overline{\Gamma}(x', A) \), if \( |\partial^\alpha (\overline{Q} - \overline{Q})|(x)| \leq \epsilon_0 \) for \( |\alpha| \leq m - 1 \), then \( \overline{Q} \in \overline{\Gamma}(x', \tilde{k}, A) \).

**Proof.** Choose \( \overline{Q}_0, \ldots, \overline{Q}_d \in \text{int}\Gamma(x, \overline{k}, A) \) which form a \( \tilde{d} \)-dimensional simplex with \( \overline{Q} \) lying in its relative interior. Let \( \eta > 0 \) be as in Lemma 6.3.

Applying Lemma 8.2 to each of the \( \overline{Q}_j \) and taking \( \delta_0 \) to be the minimum of the associated \( \delta \)'s, we see that for any \( x' \in E_1 \cap B(x, \delta_0) \), there exist \( \overline{Q}_0, \ldots, \overline{Q}_d \in \overline{P} \) such that

\[ \overline{Q}_j \in \overline{\Gamma}(x', \overline{k}, A) \] (8.15)

and

\[ |\partial^\alpha (\overline{Q}_j - \overline{Q}_j)(x)|, |\partial^\alpha (\overline{Q}_j - \overline{Q}_j)(x')| \leq A |x - x'|^{m-|\alpha|} \text{ for } |\alpha| \leq m - 1. \] (8.16)

Taking \( \delta_0 > 0 \) small enough, by (8.16), Lemma 6.3 applies. (By Lemma 7.3 and the definition of strata, \( \dim \overline{\Gamma}(x', A) = \dim \overline{\Gamma}(x, A) = \tilde{d} \).) Thus, there exists \( \epsilon_0 > 0 \) such that if \( \overline{Q} \in \overline{\Gamma}(x', A) \), if \( |\partial^\alpha (\overline{Q} - \overline{Q})|(x)| \leq \epsilon_0 \) for \( |\alpha| \leq m - 1 \), then \( \overline{Q} \in \text{Conv}(\overline{Q}_0, \ldots, \overline{Q}_d) \subset \overline{\Gamma}(x', \tilde{k}, A) \). \[ \square \]

**Lemma 8.4.** Suppose \( A, A_1 > 0 \) and \( 1 + (D + 1)\tilde{k} \leq \overline{k} \). Given \( x \in E_1 \), and \( Q \in \text{int}\Gamma(x, \overline{k}, A) \), there exist \( \epsilon, \delta > 0 \) such that for any \( x' \in E_1 \cap B(x, \delta) \) and any \( Q' \in \Gamma(x', A) \), if

\[ |\partial^\alpha (Q - Q')(x)| \leq \epsilon \text{ for } |\alpha| \leq m - 1, \] (8.17)

and

\[ |\partial^\alpha Q'(x)| \leq A \text{ for } |\alpha| = m, \] (8.18)

then \( Q' \in \Gamma(x', \overline{k}, CA) \), with \( C \) depending only on \( m, n \).

**Proof.** Let \( x \in E_1 \). Let \( \epsilon_0, \delta_0 \) be as in Lemma 8.3 with \( A = A_1 \). Let \( \epsilon \) and \( \delta \) be small numbers to be determined, depending only on the parameters appeared above.
Fix \( Q \in \Gamma(x, \tilde{k}, A_1) \), \( x' \in E_1 \cap B(x, \delta) \), and \( Q' \in \Gamma(x', A) \) satisfying (8.17) and (8.18).

Since \( Q \in \Gamma(x, \tilde{k}, A_1) \), we have \( |\partial^a Q(x)| \leq A_1 \) for \( |\alpha| \leq m \). Therefore, (8.17) and (8.18) imply that \( |\partial^a Q'(x)| \leq A_3 \) for \( |\alpha| \leq m \). Taking \( \delta \leq 1 \), we have
\[
|\partial^a Q'(x')| \leq A_4 \quad \text{for} \quad |\alpha| \leq m.
\]

We set \( \bar{Q} := \pi_x Q \) and \( \bar{Q}' := \pi_{x'} Q' \), immediately seeing that by taking \( \delta \leq \delta_0 \), we have
\[
\bar{Q} \in \text{int} \, \bar{\Gamma}(x, \tilde{k}, A_1), \quad x' \in E_1 \cap B(x, \delta_0), \quad \text{and} \quad \bar{Q}' \in \bar{\Gamma}(x', A).
\]

By Taylor’s theorem and (8.19), we have
\[
|\partial^a(\bar{Q} - Q')(x)| \leq A_3 |x - x'|^{m - |\alpha|} \leq A_3 \delta^{m - |\alpha|} \leq A_3 \delta \quad \text{for} \quad |\alpha| \leq m.
\]

Taking \( \epsilon \) and \( \delta \) to be sufficiently small, we see from (8.17) and (8.21) that
\[
|\partial^a(\bar{Q} - Q)(x)| \leq \epsilon_0 \quad \text{for} \quad |\alpha| \leq m - 1.
\]

In view of (8.20) and (8.22), we see that Lemma 8.3 applies. Hence, \( \bar{Q}' = \pi_{x'} \bar{Q} \in \bar{\Gamma}(x', \tilde{k}, A_4) \). This means that there exists some \( \bar{Q} \in \Gamma(x', \tilde{k}, A_4) \) such that \( \bar{Q}' = \pi_{x'} \bar{Q} \).

Now, since \( \pi_{x'}(\bar{Q} - Q') \equiv 0 \), we see from Lemma 8.1 that \( Q' \in \Gamma(x', \tilde{k}, A_5) \). \( \square \)

9. Uniform Delta and Modulus of Continuity

The purpose of this section is to show that, given \( \epsilon > 0 \), we may take the \( \delta \) in the definition of Glaser stable to be uniform in \( x \) and \( P_0 \) (with some change in \( k^2 \)), then use this fact and the results of Section 8 to construct a modulus of continuity that will let us apply Theorem 5.2.

Given \( (x, P), (x', P') \in E \times \mathcal{P} \), we let
\[
\text{dist}((x, P), (x', P')) := |x - x'| + \max_{\alpha} |\partial^a (P - P')(x)| + \max_{\alpha} |\partial^a (P - P')(x')|.
\]

Note that the above turns \( E \times \mathcal{P} \) into a metric space. While the above description of \( \text{dist}(\cdot, \cdot) \) will be most useful, the metric defined by
\[
\rho((x, P), (x', P')) := |x - x'| + \max_{\alpha} |\partial^a (P - P')(0)|.
\]

will be useful for establishing a crucial property of \( \text{dist}(\cdot, \cdot) \).

**Lemma 9.1.** The metrics \( \text{dist}(\cdot, \cdot) \) and \( \rho(\cdot, \cdot) \) are equivalent, that is, given compact \( E \subset \mathbb{R}^n \), there exists \( C > 0 \) such that
\[
C^{-1} \rho((x, P), (x', P')) \leq \text{dist}((x, P), (x', P')) \leq C \rho((x, P), (x', P'))
\]
whenever \( x, x' \in E, P, P' \in \mathcal{P} \).
In particular, a set is compact in the dist metric if and only if it is closed and bounded in the dist metric.

Proof. Write

\[ P - P'(y) = \sum_{\beta} c_{\beta}x^{\beta}, \]

so that

\[ \rho((x, P), (x', P')) = |x - x'| + \max_{\alpha} |c_{\alpha}| \]

and

\[ \partial^{\alpha}(P - P')(y) = \sum_{|\beta| \leq m, \beta \geq \alpha} c_{\beta}!y^{\beta - \alpha}. \]

Then, letting \( M = \sup_{y \in E} |y| < \infty, \)

\[ \text{dist}((x, P), (x', P')) = |x - x'| + \max_{\alpha} |\partial^{\alpha}(P - P')(x)| + \max_{\alpha} |\partial^{\alpha}(P - P')(x')| \]

\[ \leq |x - x'| + 2DM^{m} \max_{\beta \leq m} \max_{|\beta| \leq m} |c_{\beta}| \]

\[ \leq \left[ 2DM^{m} \max_{|\beta| \leq m} \beta! \right] \rho((x, P), (x', P')). \]

The reverse direction may be proven analogously.

The metric \( \rho(\cdot, \cdot) \) is by definition equivalent to a Euclidean metric on \( E \times P \) identified in the natural way with \( E \times \mathbb{R}^{D} \). By the above, so is the metric \( \text{dist}(\cdot, \cdot) \); thus, the Heine-Borel theorem applies, which is the form of our conclusion. \( \square \)

For Lemmas 9.2 through 9.6, fix a Glaeser stable, convex bundle \((H(x))_{x \in E}\) with lowest stratum \( E_{1}\).

**Lemma 9.2.** Suppose \( 1 + (D + 1)\overline{k} \leq k^{2} \). Let \( x \in E, P \in \text{int}H(x), \) and \( \epsilon > 0 \) be given. Then there exists \( 0 < \delta < 1 \) (depending on \( P \)) such that for every \( x' \in E_{1} \cap B(x, \delta), \) there exists \( P' \in H'(x') \) such that

\[ |\partial^{\alpha}(P - P')(x)| \leq \epsilon |x - x'|^{m - |\alpha|} \]

and given \( x_j' \in E \cap B(x, \delta) \) (\( 1 \leq j \leq \overline{k} \)), there exist \( P'_j \in H(x_j) \) such that

\[ |\partial^{\alpha}(P'_i - P'_j)(x_j')| \leq \epsilon |x'_i - x'_j|^{m - |\alpha|}, 0 \leq i, j, \leq \overline{k}. \]
We mimic the proof of Lemma 6.1 in [6], the most significant change being the introduction of simplices so the compactness hypothesis of Helly’s theorem is satisfied.

Proof. Recall that the bundle \((H(x))_{x \in E}\) is assumed to be Glaeser stable.

Suppose \(x' = x\), then we may take \(P' \equiv P\) and (9.7) follows. (9.8) follows from Glaeser stability.

Suppose \(x' \neq x\). Since \(P \in \text{int}H(x)\), choose \(P_0, \ldots, P_t \in H(x)\) which form a \(d\)-dimensional simplex containing \(B(P, \delta) \cap H(x)\). Let \(\eta\) be as in the conclusion of Lemma 6.3, applied to \(P_0, \ldots, P_d\).

Let \(0 < \epsilon' < \min\{\epsilon, \eta\}\) and by the Glaeser stability of \((H(x))_{x \in E}\) pick \(0 < \delta < \eta\) such that for \(j = 0, \ldots, d\), if \(x' \in E \cap B(x', \delta)\) there exists \(P'_j \in H(x')\) such that

\[
|\partial^n (P_j - P_j')(x)|, |\partial^n (P_j - P_j')(x')| \leq \epsilon'|x - x'|^{m-|\alpha|} \leq \eta \quad \text{for } |\alpha| \leq m.
\]

(9.9)

Thus, if \(P' \in H(x')\) such that

\[
|\partial^n (P - P')(x)|, |\partial^n (P - P')(x')| \leq \epsilon|x - x'|^{m-|\alpha|} \leq \eta \quad \text{for } |\alpha| \leq m,
\]

then \(P' \in \Lambda(x') := \text{Conv}(P'_0, \ldots, P'_j)\) (as the definition of lowest stratum implies \(\dim H(x') = \dim H(x)\)).

For any finite set \(S \subset E \cap B(x, \delta)\) with \(S \supset \{x, x'\}\), define

\[
\mathcal{K}(S) := \left\{ P' \in \Lambda(x') : \begin{array}{l}
\text{There exists a map } y \mapsto P^y \text{ from } S \text{ to } \mathcal{P} \text{ such that } \\
P^x = P, P^{x'} = P', P^y \in H(y) \text{ for } y \in S, \text{ and } \\
|\partial^n (P^y - P^z)(z)| \leq \epsilon' |y - z|^{m-|\alpha|} \text{ for } |\alpha| \leq m, y, z \in S. 
\end{array} \right\}
\]

Each \(\mathcal{K}(S)\) is a compact, convex subset of \(\mathcal{P}\), which has dimension \(D\).

Suppose we are given \(S_1, \ldots, S_{D+1} \subset E \cap B(x, \delta)\), each containing \(x\) and \(x'\), with \(#(S_i) \leq \overline{k} + 2\) for each \(i\). Then \(S := \bigcup_{i=1}^{D+1} S_i \subset E \cap B(x, \delta)\), with \(x, x' \in S\), and \(#(S) \leq 2 + (D + 1)\overline{k} \leq 1 + k^2\). Therefore, by the Glaeser stability of \((H(x))_{x \in E}\), there exists \(y \mapsto P^y\), with \(P^x = P\), \(P^x' \in H(y)\) for each \(y \in E\), and

\[
|\partial^n (P^y - P^z)(z)| \leq \epsilon' |y - z|^{m-|\alpha|} \text{ for } |\alpha| \leq m, y, z \in S.
\]

(9.11)

(9.10) forces \(P^{x'} \in \Lambda(x')\).

It is clear that \(P^{x'} \in \mathcal{K}(S_i)\) for each \(i = 1, \ldots, D+1\). Therefore, \(\mathcal{K}(S_1), \ldots, \mathcal{K}(S_{D+1})\) have nonempty intersection. By Helly’s theorem, there exists \(P' \in \mathcal{K}(S') \subset H(x')\), whenever \(S' \subset E \cap B(x, \delta)\), with \(x, x' \in S'\) and \(#(S') \leq \overline{k} + 2\). (9.7) and (9.8) then follow from the definition of \(\mathcal{K}(S')\).

\(\square\)

**Definition 9.1.** Let \((H(x))_{x \in E}\) be a Glaeser stable bundle. Given \(x_0 \in E\), \(P_0 \in H(x_0), \epsilon > 0\) and \(\overline{k} \in \mathbb{N}\), we define \(\Delta(x_0, P_0; \epsilon, \overline{k})\) to be the supremum over all \(\delta > 0\) such that the following holds:

If \(x_1, \ldots, x_k \in E \cap B(x_0, \delta)\), there exist \(P_1, \ldots, P_k \in \mathcal{P}\) such that

\[
P_j \in H(x_j), 1 \leq j \leq \overline{k}
\]

(9.12)
and
\begin{equation}
|\partial^\alpha(P_i - P_j)(x_j)| \leq \epsilon |x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j \leq \overline{k}.
\end{equation}

Note that since the $x_i$ are taken to be in the open ball of radius $\delta$, the above holds with $\delta = \Delta(x_0, P_0; \epsilon, \overline{k})$.

**Lemma 9.3.** Let $x_0 \in E$ and $\overline{k} \leq k^2$. Suppose $P_0^{(0)}, ..., P_0^{(L)} \in H(x)$. Let $P_0 = \sum_{l=0}^{L} \lambda_l P_0^{(l)}$ for $\lambda_l \in [0, 1]$. Then, $\Delta(x, P_0; \overline{k}, \epsilon) \geq \min_{0 \leq l \leq L} \Delta(x, P_0^{(l)}; \overline{k}, \epsilon)$.

**Proof.** Let $\epsilon > 0$. Set $\delta_0 := \min_{0 \leq l \leq L} \Delta(x, P_0^{(l)}; \overline{k}, \epsilon)$.

Let $x_1, ..., x_{\overline{k}} \in E \cap B(x, \delta_0)$. By the definition of Glaeser stability, for each $0 \leq l \leq L$ there exist $P_1^{(l)}, ..., P_{\overline{k}}^{(l)} \in H(x_j)$, and
\begin{equation}
|\partial^\alpha(P_i^{(l)} - P_j^{(l)})(x_j)| \leq \epsilon |x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j \leq \overline{k}, 0 \leq l \leq L.
\end{equation}

For $1 \leq j \leq \overline{k}$ and $0 \leq l \leq L$, we set
\[P_j := \sum_{i=0}^{L} \lambda_l P_j^{(l)}.
\]

Since $H(x_j)$ is convex for each $j$, we have $P_j \in H(x_j)$. Furthermore, thanks to (9.14), we have
\[|\partial^\alpha(P_i - P_j)(x_j)| \leq \sum_{0 \leq l \leq L} |\partial^\alpha(P_i^{(l)} - P_j^{(l)})(x_j)| \leq \epsilon |x_i - x_j|^{m-|\alpha|}
\]
for $|\alpha| \leq m, 0 \leq i, j \leq \overline{k}$.

**Lemma 9.4.** Suppose $1 + (D + 1)\overline{k} \leq k^2$. Let $x \in E_1$ and $P \in \text{int}H(x)$. There exists $\epsilon_0 > 0$ such that when $0 < \epsilon < \epsilon_0$, there exist $\delta, \eta > 0$ such that whenever $x' \in E_1$, $P' \in W_H(x')$ and
\begin{equation}
\text{dist}((x, P), (x, P')) < \eta,
\end{equation}
we have $P' \in H(x')$ and
\begin{equation}
\Delta(x', P'; \overline{k}, \epsilon) \geq \delta.
\end{equation}

**Proof.** Note that if $d = 0$, then $H(x)$ consists of a single polynomial for any $x \in E_1$ and the conclusion follows from Lemma 9.2.

Suppose $d \geq 1$ and let $P_0, ..., P_d \in \text{int}H(x)$ which form a nondegenerate affine simplex with $P$ in its relative interior. Let $\eta_0 > 0$ be as in the conclusion of Lemma 6.3. Noting that $\eta_0$ is dependent solely on $P$, set $\epsilon_0 = \eta_0$.

For each $P_j$, let $\delta_j > 0$ be as in the conclusion of Lemma 9.2 and take $\delta = \frac{1}{2} \min\{\delta_0, ..., \delta_d, 1\}$. Now take $\eta = \min\{\delta, \eta_0\}$.
Suppose Lemma 9.6.

By (9.17) and Lemma 9.2, there exist \( P'_0, ..., P'_d \in H(x') \) such that

\[
\Delta(x', P'_j, \bar{\kappa}, \epsilon) \geq \delta, 0 \leq j \leq d.
\]

By Lemma 6.3, the definition of lowest stratum, and (9.18), \( P' \) lies in the convex hull of \( P'_0, ..., P'_d \). Thus, by Lemma 9.3,

\[
\Delta(x', P', \bar{\kappa}, \epsilon) \geq \delta.
\]

\( \square \)

Lemma 9.5. Let \( x \in E_1 \) and \( P \in \text{int}H(x) \). If \( \epsilon > 0 \), then there exists \( \tilde{\eta} > 0 \) such that whenever \( x' \in E_1 \cap B(x, \tilde{\eta}) \), there exists \( P' \in H(x') \) such that

\[
dist((x, P), (x', P')) < \epsilon.
\]

Proof. By the definition of Gleaser stability, there exists \( 0 < \tilde{\eta} < \min\{1, \epsilon/2\} \) such that whenever \( x' \in E_1 \cap B(x, \tilde{\eta}) \), there exists \( P' \in H(x') \) such that

\[
|\partial^\alpha (P - P')(x)|, |\partial^\alpha (P - P')(x')| \leq \epsilon/2 |x - x'|^{\alpha} \leq \epsilon/2 \text{ for } |\alpha| \leq m.
\]

Combining (9.23) with the fact \( |x - x'| < \tilde{\eta} < \epsilon/2 \) we obtain the desired conclusion. \( \square \)

Lemma 9.6. Suppose \( 1 + (D + 1)\bar{k} \leq k^2 \). Let \( \epsilon > 0 \) and \( K_0 \) be a compact subset of \( \mathbb{R}^n \times \mathcal{P} \) such that whenever \( (x, P) \in K_0 \), \( P \in \text{int}H(x) \). Then, there exists \( \delta > 0 \) such that whenever \( (x_0, P_0) \in K_0 \) and \( x_1, ..., x_k \in E_1 \cap B(x_0, \delta) \), there exists \( P_1, ..., P_{\bar{k}} \in H(x) \) such that

\[
|\partial^\alpha (P_i - P_j)(x_j)| \leq \epsilon |x_i - x_j|^{\alpha} \text{ for } |\alpha| \leq m, 0 \leq i, j \leq \bar{k}.
\]

Proof. Let \( (x, P) \in K_0 \). Then, since \( P \) lies in the relative interior of \( H(x) \), we may apply Lemma 9.4 to obtain \( \eta_x, \delta_x, (\epsilon_0)_x > 0 \) such that whenever \( (x', P') \) lies in

\[
B_{K_0}((x, P), \eta_x) : \{ (x', P') \in K_0 : \text{dist}((x, P), (x', P')) < \eta_x \},
\]

and \( 0 < \epsilon < (\epsilon_0)_x \) we have

\[
\Delta(x', P'; \bar{k}, \epsilon) \geq \delta_x.
\]
Thus, \( \{B_{K_0}((x, P), \eta_x)\}_{(x, P) \in K} \) forms an open cover of \( K_0 \). By compactness of \( K_0 \), we obtain a finite subcover \( \{B_{K_0}((x_1, P_1), \eta_{x_1}), \ldots, B_{K_0}((x_l, P_l), \eta_{x_l})\} \). Taking \( \delta = \min\{\delta_{x_1}, \ldots, \delta_{x_l}\} > 0 \) and requiring \( 0 < \epsilon < \min\{(\epsilon_0)_{x_1}, \ldots, (\epsilon_0)_{x_l}\} \), we see that

\[
\Delta(x', P'; \delta, \epsilon) \geq \delta
\]

for all \((x', P') \in K_0\), which is equivalent to our desired conclusion.

For \( \epsilon \geq \min\{(\epsilon_0)_{x_1}, \ldots, (\epsilon_0)_{x_l}\} \), apply the conclusion for \( \frac{1}{2} \min\{(\epsilon_0)_{x_1}, \ldots, (\epsilon_0)_{x_l}\} \) and observe

\[
\Delta(x', P'; \delta, \epsilon) \geq \Delta\left(x', P'; \delta, \frac{1}{2} \min\{(\epsilon_0)_{x_1}, \ldots, (\epsilon_0)_{x_l}\}\right) \geq \delta.
\]

\( \square \)

We now consider the case where \((H(x))_{x \in E}\) is the shape field \((\Gamma(x, M))_{x \in E}\). Let \( x \in E_1 \) and \( P \in \text{int} \Gamma(x, k^2, 2A^2) \).

Let \( \epsilon_0 = (\epsilon_0)_{(x, P)} \) be as in Lemma 9.4 applied to the Glaeser stable bundle \((\Gamma(x, 2A^2))_{x \in E}\). (See Lemma 7.5.) Let \( \eta_{(x, P)} \) and \( \delta_{(x, P)} \) be the corresponding \( \eta \) and \( \delta \) from the conclusion of Lemma 9.4 with \( \epsilon \) taken as \( \frac{(\epsilon_0)_{(x, P)}}{2} \).

By Lemma 9.5, there exists \( 0 < \tilde{\eta}_{(x, P)} < \eta_{(x, P)} \) such that whenever \( x' \in E_1 \cap B(x, \tilde{\eta}) \), there exists \( P' \in \Gamma(x', A^2) \) such that

\[
\text{dist}((x, P), (x', P')) < \frac{\eta_{(x, P)}}{4}.
\]

Furthermore, by Lemma 8.4, we may take \( \eta_{(x, P)} \) and \( \tilde{\eta}_{(x, P)} \) small enough that if \( x' \in E_1 \) and \( P \in \Gamma(x') \) satisfies

\[
\text{dist}((x, P), (x', P')) \leq \eta_{(x, P)},
\]

then \( P' \in \Gamma(x', \tilde{k}, CA^2) \), where \( C \) is as in the conclusion of Lemma 8.4 and \( \tilde{k} \) satisfying

\[
1 + (D + 1)\tilde{k} \leq k^2.
\]

Consider the collection of balls \( B(x, \tilde{\eta}_{(x, P)}/4) \subset \mathbb{R}^n \), with \( x \) ranging over \( E_1 \) and given \( x, P \) ranging over \( \text{int} \Gamma(x, k^2, 2A^2) \). This forms an open cover of \( E_1 \). \( E_1 \) is compact, so consider a finite subcover

\[
\{B(x_1, \tilde{\eta}_{(x_1, P_1)}/4), \ldots, B(x_j, \tilde{\eta}_{(x_j, P_j)}/4)\}.
\]

Define

\[
K_j = \{(x, P) \in \overline{B}((x_j, P_j), \eta_{(x_j, P_j)}/2) : x \in E_1 \cap \overline{B}(x, \tilde{\eta}/2), P \in \Gamma(x, 2A^2)\},
\]

where \( \overline{B} \) is used to refer to the closed ball, and

\[
K = \bigcup_{j=1}^J K_j \subset E_1 \times \mathcal{P}.
\]
Note that Lemma 8.4 implies that

\[ (9.35) \quad \text{if } (x, P) \in K, \text{ then } P \in \Gamma(x, \bar{k}, CA^2). \]

with \( C \) depending only on \( m, n \).

**Lemma 9.7.** \( K \) is compact under the topology induced by the \( \text{dist} \) metric defined in (9.1).

**Proof.** It suffices to show \( K_j \) is compact for arbitrary \( 1 \leq j \leq J \). Let \( 1 \leq j_0 \leq J \). By Lemma 9.1, it suffices to show \( K_{j_0} \) is closed and bounded; boundedness follows immediately from 2.1.

To show \( K_{j_0} \) is closed, let \( x \in \mathbb{R}^n, P \in \mathcal{P} \), and \((x_1, P_1), (x_2, P_2), \ldots \in K_{j_0} \) such that

\[ (9.36) \quad \| x_i - x \|, \text{dist}((x_i, P_i), (x, P)) \rightarrow 0 \text{ as } i \rightarrow \infty. \]

It follows that \((x, P) \in \overline{B}(x_{j_0}, P_{j_0}, \eta(x_{j_0}, P_{j_0})/2) \) and \( x \in \overline{B}(x_{j_0}, \tilde{\eta}(x_{j_0}, P_{j_0})/2) \) since the same is true for each \((x_i, P_i)\).

By compactness of \( E_1 \) established by Lemma 7.4, we have \( x \in E_1 \). Our goal is to show \( P \in \Gamma(x, 2A^2) \). By Lemma 9.4 and the definition of \( K \), to show \( P \in \Gamma(x, 2A^2) \), it suffices to show \( P \in W_T(x) \).

By the compactness of the Grassmanian of \( d \)-dimensional hyperplanes of \( \mathcal{P} \) having nontrivial intersection with \( \{ |\partial^\alpha P(y)| \leq 2A^2 \text{ for } |\alpha| \leq m \} \) for some \( y \in E \), we may suppose by passing to a subsequence that \( W_T(x_i) \) converges to some \( d \)-dimensional subspace \( W \subset \mathcal{P} \).

If \( W = W(x) \) we are done, so suppose for the sake of contradiction that \( W \neq W_T(x) \). Then there exists \( \hat{P} \in \Gamma(x, 2A^2) \) such that \( \hat{P} \notin W_T(x) \), hence there cannot exist a sequence \( P_i \in W_T(x_i) \) satisfying \( P_i \rightarrow \hat{P} \). This contradicts the Glaeser-stable property of \( (\Gamma(x, 2A^2))_{x \in E} \) established in Lemma 7.5. Thus, \( W = W_T(x) \). \( \square \)

**Lemma 9.8.** Let \( \bar{k} \) be as in (9.31). There exists \( \gamma > 0 \) such that:

For all \( x \in E_1 \), there exists \( P \in \Gamma(x, \bar{k}, CA^2) \) such that \( (x, P) \in K \) and if \( x' \in E_1, P' \in \Gamma(x', 2A^2) \) such that \( \text{dist}((x, P), (x', P')) < \gamma \), then \( (x', P') \in K \).

**Proof.** Returning to the notation of the construction of \( K \), choose \( 1 \leq j \leq J \) such that \( x \in B(x_j, \tilde{\eta}(x_j, P_j)/4) \). By Lemma 9.5 and the construction of \( K \), there exists \( P \in \Gamma(x, 2A^2) \) such that

\[ (9.37) \quad \text{dist}((x_j, P_j), (x, P)) < \eta(x_j, P_j)/4, \]

thus \((x, P) \in K \) and by (9.35), \( P \in \Gamma(x, \bar{k}, CA^2) \).

Now suppose \( x' \in E_1 \cap B(x_j, \tilde{\eta}(x_j, P_j)/4) \), \( P' \in \Gamma(x') \) such that \( \text{dist}((x, P), (x', P')) < \eta(x_j, P_j)/2 \). Then by the triangle inequality, we have

\[ (9.38) \quad |x_j - x'| < \tilde{\eta}(x_j, P_j) \]
and

\[ \text{dist}((x_j, P_j), (x', P')) < \eta(x_j, P_j)/2. \]  

Thus, by definition, \((x', P') \in K_j\) and the conclusion follows setting \(\gamma = \min_{1 \leq j \leq J} \frac{\eta(x_j, P_j)}{4}. \)

**Lemma 9.9.** There exists \(0 < \delta_0 < 1\) and a regular modulus of continuity \(\omega\) for which the following holds:

Given \(x, x' \in E_1\) with \(|x - x'| \leq \delta_0\) and given \((x, P) \in K\), there exists \(P' \in \Gamma(x', 2A^\#)\) such that

\[ |\partial^\alpha (P' - P)(x)| \leq \omega(|x - x'|) \cdot |x - x'|^{m - |\alpha|} \text{ for } |\alpha| \leq m. \]

**Proof.** We mimic the proof of Lemma 6.4 in [6].

For \(\nu = 0, 1, 2, \ldots\), we set \(\epsilon_\nu := 2^{-\nu}\). By Lemma 9.7, \(K\) is compact and we may apply Lemma 9.6 to successively pick \(\delta_0, \delta_1, \delta_2, \ldots\), such that the following hold.

\((\delta-1)\) \(\delta_0 = 1.\)

\((\delta-2)\) \(0 < \delta_{\nu+1} < \frac{1}{2} \delta_\nu.\)

\((\delta-3)\) If \(\nu \geq 1\), then given \(x, x' \in E_1\) with \(|x - x'| \leq \delta_\nu\), and given \((x, P) \in K\), there exists \(P' \in \Gamma(x', 2A^\#)\) with

\[ |\partial^\alpha (P' - P)(x)| \leq \frac{1}{2} \epsilon_\nu |x' - x'|^{m - |\alpha|} \text{ for } |\alpha| \leq m. \]

We define a regular modulus of continuity \(\omega(t)\) on \([0, 1]\) by setting

\(\omega(0) = 0, \omega(\delta_\nu) = \epsilon_\nu,\) and \(\omega(t)\) linear on each \([\delta_{\nu+1}, \delta_\nu]\) for \(\nu \geq 0.\)

Now suppose \(x, x' \in E_1\).

Suppose \(x = x'\), we may simply take \(P = P'.\)

Suppose \(x \neq x'.\) Furthermore, assume that \(0 < |x - x'| < \delta_1.\) Let \((x, P) \in K.\)

Pick \(\nu \geq 1\) such that \(\delta_{\nu+1} < |x - x'| \leq \delta_\nu.\) By \((\delta-3)\), there exists \(P' \in \Gamma(x', 2A^\#)\) such that

\[ |\partial^\alpha (P' - P)(x)| \leq \frac{1}{2} \epsilon_\nu |x' - x'|^{m - |\alpha|} \text{ for } |\alpha| \leq m. \]

On the other hand, since \(|x' - x| > \delta_{\nu+1}\), we have

\[ \omega(|x' - x|) \geq \omega(\delta_{\nu+1}) = \epsilon_{\nu+1} = \frac{1}{2} \epsilon_\nu. \]

Combining (9.41) and (9.42), we have

\[ |\partial^\alpha (P' - P)(x)| \leq \omega(|x' - x|) |x - x'|^{m - |\alpha|} \text{ for } |\alpha| \leq m. \]

□
Lemma 9.10 (Existence of Modulus of Continuity). Suppose

\[ k^* \geq D + 2, 1 + (D + 1) k \leq k^* \]

and let \( \omega \) be as in Lemma 9.9. Then, given any \( \bar{k} \geq 1 \), there exists a controlled constant \( \hat{C}_{k^*} \) and \( \delta' > 0 \) such that the following holds:

Let \( x_0 \in S \subset E_1 \) with \( \text{diam}(S) \leq \delta' \) and \( |S| \leq \bar{k} \). Then, there exists a map \( x \mapsto P^x \) from \( S \) into \( \mathcal{P} \) such that

\[ P^x \in \Gamma(x, \bar{k}, CA^2), (x, P^x) \in K \text{ for each } x \in S; \text{ and} \]

\[ |\partial^\alpha(P^x - P^y)(y)| \leq \hat{C}_{k^*}\omega(|x - y|) |x - y|^{m-|\alpha|} \text{ for } x, y \in S, |x - y| \leq \delta', |\alpha| \leq m. \]

Proof. In what follows, we first construct the mapping \( x \mapsto P^x \) via Lemma 6.2 and repeated applications of Lemma 9.9 as in [6]. Then, we will show that each step of the construction is valid in the sense that it produces \( P^x \) such that \( (x, P^x) \in K \), justifying the uses of Lemma 9.9 and satisfying 9.44.

The proof is by induction on \( \bar{k} \); the base case \( \bar{k} = 1 \) following from simply taking \( P^{x_0} = P_0 \) as guaranteed by Lemma 9.8.

Now suppose Lemma 9.10 holds for all positive integers less than \( \bar{k} \).

First require that \( \delta_1 < \delta_0 \), where \( \delta_0 \) is as in Lemma 9.9.

By Lemma 6.2, we may partition \( S \) into sets \( S_0, \ldots, S_M \) such that

\[ \#(S_l) \leq \bar{k} - 1 \text{ for each } l, (0 \leq l \leq M), \text{ and} \]

\[ \text{dist}(S_l, S_{l'}) > c_{\bar{k}} \cdot \text{diam}(S) \text{ for } l \neq l'. \]

Without loss of generality, take \( x_0 = x_0 \in S_0 \) and suppose each \( S_l \) is nonempty, fixing \( x_l \in S_l \) \((l \geq 1)\).

Again referring to Lemma 9.8, choose \( \gamma > 0 \) and \( (x_0, P_0) \in K \) such that if \( x' \in E_1, P' \in \Gamma(x', 2A^2) \) such that \( \text{dist}((x_0, P_0), (x', P')) < \gamma \), then \( (x', P') \in K \).

By Lemma 9.9, there exist \( P_{l_1} \in \Gamma(x_{l_1}, 2A^2) \) for \( 1 \leq l_1 \leq M_1 \) such that

\[ |\partial^\alpha(P_0 - P_l(x_0))| < \omega(|x_0 - x_l|) |x_0 - x_l|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 1 \leq l \leq M. \]

Letting \( \delta = \text{diam}(S) \), we have in particular that

\[ |\partial^\alpha(P_0 - P_l(x_0))| < \omega(|x_0 - x_l|) \delta^{m-|\alpha|} \text{ for } |\alpha| \leq m, 1 \leq l \leq M. \]

For each \( 0 \leq l \leq M \), we apply the induction hypothesis to each \( x_l, S_l \) in place of \( x_0, S \). Thus we obtain a map \( x \mapsto P^x \) such that

\[ P^{x_l} = P_l, \]

\[ (x, P^x) \in K, \]
and

\[(9.52) \quad |\partial^\alpha (P^x - P^y)(y)| \leq \hat{C}_{k-1} \omega(|x - y|)|x - y|^{m-|\alpha|} \text{ for } |\alpha| \leq m \]

whenever \(x, y \in S_l\) (same \(l\)).

It remains to show that (9.52) holds in the case where \(x \in S_l, y \in S_{l'} \ (l \neq l')\).

From (9.52) in the case already established, we have

\[(9.53) \quad |\partial^\alpha (P^x - P_l)(x)| \leq \hat{C}_{k-1} \omega(\delta)\delta^{m-|\alpha|} \text{ for } |\alpha| \leq m \]

and

\[(9.54) \quad |\partial^\alpha (P^y - P_{l'})(y)| \leq \hat{C}_{k-1} \omega(\delta)\delta^{m-|\alpha|} \text{ for } |\alpha| \leq m.\]

By (9.53) and the fact \(|x - y| \leq \delta\), we have

\[(9.55) \quad |\partial^\alpha (P^x - P_l)(y)| \leq C''\hat{C}_{k-1} \omega(\delta)\delta^{m-|\alpha|} \text{ for } |\alpha| \leq m \]

and by similar reasoning

\[(9.56) \quad |\partial^\alpha (P_l - P_{l'})(y)| \leq C''\omega(\delta)\delta^{m-|\alpha|} \text{ for } |\alpha| \leq m.\]

Summing (9.54), (9.55), and (9.56) gives

\[(9.57) \quad |\partial^\alpha (P^x - P^y)| \leq C''\hat{C}_{k-1} \omega(|x - y|)|x - y|^{m-|\alpha|} \text{ for } |\alpha| \leq m.\]

Observing that (9.47) implies \(|x - y| \geq c_\delta\delta\) and therefore \(\omega(|x - y|) \geq \omega(c_\delta \delta) \geq c_\delta \omega(\delta)\), we have

\[(9.58) \quad |\partial^\alpha (P^x - P^y)(y)| \leq \hat{C}\hat{C}_{k-1} \omega(|x - y|)|x - y|^{m-|\alpha|} \text{ for } |\alpha| \leq m.\]

In the above, we have implicitly described a recursive method for determining \(P^x\) from repeated applications of Lemma 9.9 giving us \(P^x \in \Gamma(x, 2A^2)\) for \(x \in S\).

Therefore, we must check that at each step we have produced \((x, P^x) \in K\) in order to satisfy the hypotheses of Lemma 9.9.

At each stage, we have the inequality

\[(9.59) \quad |\partial^\alpha (P^x - P^{x_0})(x_0)| < C_{k}\omega(|x - x_0|)|x - x_0|^{m-|\alpha|} \text{ for } |\alpha| \leq m.\]

By Lemma 9.8, we may establish that \((x, P^x) \in K\) by choosing \(\delta'\) small enough such that (9.59) and \(|x - x_0| < \delta'\) implies \(\text{dist}((x, P^x), (x_0, P_0)) < \gamma\). This is allowed since the constant \(C_{k}\) is dependent only on \(k\). Since \((x, P^x) \in K\), \(P^x \in \Gamma(x, \hat{k}, CA^2)\), so our map has the desired properties. \(\square\)
10. Main Proof

In this section, we complete the proof of Theorem 1.6 in the case $d = 1$.

Let $\Gamma'(x, M)$ now refer to the unrefined shape field in the hypothesis of Theorem 1.6; we will use the hypothesis that $\Gamma'(x, M)$ is closed for fixed $x, M$.

Given $x_0 \in E_1, \overline{k} \in \mathbb{N}, A > 0$, define $\Gamma'(x_0, \overline{k}, A) \subset \Gamma'(x_0, A)$ to be the closure of the set of $P_0 \in \Gamma'(x_0, A)$ such that for all $x_1, ..., x_{\overline{k}} \in E$, there exist $P_j \in \Gamma(x_j, A)$ (not $\Gamma'(x, A)$) such that

$$(10.1) \quad |\partial^\alpha (P_i - P_j)(x_j)| \leq A|x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j \leq \overline{k}.$$

By definition, $(\Gamma'(x, \overline{k}, M))_{x \in E_1, M \geq 0}$ is closed, is convex, and satisfies $\Gamma(x, \overline{k}, M) \subset \Gamma'(x, \overline{k}, M)$ for all $M > 1, x \in E_1, \overline{k} \in \mathbb{N}$.

First, we show that a property similar to that described in (10.1) applies to all $P_0 \in \Gamma'(x_0, \overline{k}, M)$ (even after the closure is taken). Then, we will show that the shape fields finiteness principle (Theorem 5.2) applies to the $\Gamma'(x_0, \overline{k}, M)$.

**Lemma 10.1.** Let $X_0 \in E_1, M \geq 0$. If $P_0 \in \Gamma'(x_0, \overline{k}, M)$ and $x_1, ..., x_{\overline{k}}$, then there exist $P_j \in \Gamma(x_j, M) \subset \Gamma(x_j, 2M)$ such that

$$(10.2) \quad |\partial^\alpha (P_i - P_j)(x_j)| \leq 2M|x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j \leq \overline{k}.$$

**Proof.** Fix $x_0 \in E_1$ and $M \geq 0$.

Let $P_0 \in \Gamma'(x_0, \overline{k}, M)$ and $x_1, ..., x_{\overline{k}} \in E_1$. If there do not exist $P_j \in \Gamma(x_j, M)$ such that

$$(10.3) \quad |\partial^\alpha (P_i - P_j)(x_j)| \leq M|x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j \leq \overline{k},$$

then there exist $P^{(1)}, P^{(2)}, ..., P^{(\nu)} \in \Gamma'(x, \overline{k}, M)$ such that $P^{(\nu)} \to P_0$ and $P^{(\nu)}_j \in \Gamma(x_j, M)$ $(\nu = 1, 2, 3, ...)$ such that

$$(10.4) \quad |\partial^\alpha (P^{(1)}_i - P^{(\nu)}_j)(x_j)| \leq M|x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j \leq \overline{k}.$$

By taking $\nu$ sufficiently large and applying the triangle inequality, we find $N \in \mathbb{N}$ such that

$$(10.5) \quad |\partial^\alpha (P_0 - P^{(N)})(x_j)| \leq 2M|x_0 - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 0 \leq j \leq \overline{k}.$$

Combining (10.4) and (10.5) gives the desired conclusion. \hfill \Box

**Lemma 10.2.** $(\Gamma'(x, \overline{k}, M))_{x \in E_1, M \geq 0}$ is $(C, 1)$-Whitney convex, where $C$ depends only on $C_w, m, n$.

**Proof.** Let $0 < \delta \leq 1, M \geq 0$, and $x_0 \in E_1$. Choose $P^{(0)}_1, P^{(0)}_2 \in \Gamma'(x_0, \overline{k}, M)$ which satisfy

$$(10.6) \quad |\partial^\alpha (P^{(0)}_1 - P^{(0)}_2)(x_0)| \leq M\delta^{m-|\alpha|} \text{ for } |\alpha| \leq m;$$

...
Let \( x_1, \ldots, x_{\overline{k}} \in E \). Since \( P_1^{(0)}, P_2^{(0)} \in \Gamma'(x_0, \overline{k}, A) \), for \( l = 1, 2 \) there exist \( P_l^{(i)} \in \Gamma(x_j, 2M) \) \( (1 \leq j \leq \overline{k}) \) such that
\[
|\partial^\alpha (P_l^{(i)} - P_l^{(j)})| \leq 2M|x_i - x_j|^{m-|\alpha|} \quad \text{for } |\alpha| \leq m, 0 \leq i, j \leq \overline{k}.
\]
Thus by the classical Whitney extension theorem for finite sets (Lemma 6.5), for \( l = 1, 2 \) there exist \( F_l \in C^m(\mathbb{R}^n) \) such that
\[
\|F_l\|_{C^m(\mathbb{R}^n)} \leq CM, l = 1, 2; \quad \text{and}
\]
\[
J_{x_i} F_l = P_l^{(i)}, 0 \leq i \leq \overline{k}, l = 1, 2.
\]
Let \( F = \theta_1 F_1 + \theta_2 F_2 \), so that (10.6), (10.12), and (10.15) imply
\[
\|F\|_{C^m(\mathbb{R}^n)} \leq CM.
\]
Now for \( 1 \leq i \leq \overline{k} \) define
\[
P_i^{(i)} := J_{x_i}(F) = J_{x_i}(\theta_1 F_1 + \theta_2 F_2) = J_{x_i} \theta_1 \odot_{x_i} P_1^{(i)} + J_{x_i} \theta_2 \odot_{x_i} P_2^{(i)}.
\]
If \( x_i \notin B(x_0, \delta) \), then \( J_{x_i} \theta_1 = 0 \), so \( P_i^{(i)} = P_2^{(i)} \in \Gamma(x_i, 2M) \). If \( x_i \in B(x_0, \delta) \), then (10.6) and (10.14) imply
\[
|\partial^\alpha (P_1^{(i)} - P_2^{(i)})(x_i)| \leq CM\delta^{m-|\alpha|} \quad \text{for } |\alpha| \leq m,
\]
so \( P_0^{(i)} \in \Gamma(x_i, CM) \) by the \((C_w, 1)\) convexity of \( (\Gamma(x, M))_{M \geq 0, x \in E} \).

By (10.17), we see
\[
|\partial^\alpha (P_0^{(i)} - P_0^{(j)})(x_j)| \leq CM|x_i - x_j|^{m-|\alpha|} \quad \text{for } |\alpha| \leq m, 0 \leq i, j \leq \overline{k},
\]
establishing (10.9).
\[\square\]
Recall that our proof of Theorem 1.6 is by induction on the number of strata. The following lemma will address the base case.

**Lemma 10.3.** Let $A^\sharp$ be as in Lemma 6.1 and $\bar{k} \geq k^\sharp_{SF}$ satisfy (9.31). Then, there exists $\tilde{F} \in \dot{\mathcal{C}}^m(\mathbb{R}^n, \mathbb{R}^d)$ such that

\begin{equation}
\|\tilde{F}\|_{\dot{\mathcal{C}}^m(\mathbb{R}^n, \mathbb{R}^d)} \leq CA^\sharp
\end{equation}

and

\begin{equation}
J_x(\tilde{F}) \in \Gamma(x, \bar{k}, CA^\sharp) \text{ for all } x \in E_1,
\end{equation}

where $C$ depends solely on $m, n$.

**Proof.** By Lemma 9.10, there exists a modulus of continuity $\omega$ such that the following holds. If $S \subset E_1$ and $|S| \leq \bar{k}$, then there exists a map $x \mapsto P_x$ satisfying

\begin{equation}
P_x \in \Gamma(x, \bar{k}, CA^\sharp) \text{ for each } x \in S; \text{ and}
\end{equation}

\begin{equation}|\partial^\alpha (P_x - P_y)(y)| \leq C \omega(|x-y|)|x-y|^{m-|\alpha|} \text{ for } x, y \in S, |x-y| \leq 1, |\alpha| \leq m.
\end{equation}

In particular, by the classical Whitney extension theorem for finite sets (Lemma 6.5) there exists $F^S$ such that

\begin{equation}
\|F^S\|_{\dot{\mathcal{C}}^m(\mathbb{R}^n, \mathbb{R}^d)} \leq CA^\sharp
\end{equation}

and

\begin{equation}
J_x F^S \in \Gamma(x, \bar{k}, CA^\sharp) \text{ for all } x \in S.
\end{equation}

By Lemmas 10.1 and 10.2 and the fact that $\bar{k}$ has been chosen to be greater than $k^\sharp_{SF}$, by Theorem 5.2, there exists $\bar{F} \in \dot{\mathcal{C}}^m(\mathbb{R}^n, \mathbb{R}^d)$ such that

\begin{equation}
\|\bar{F}\|_{\dot{\mathcal{C}}^m(\mathbb{R}^n, \mathbb{R}^d)} \leq C' A^\sharp
\end{equation}

and

\begin{equation}
J_x(\bar{F}) \in \Gamma'(x, \bar{k}, C' A^\sharp) \text{ for all } x \in E_1.
\end{equation}

The first conclusion (10.21) follows from (10.27).

To obtain (10.22), we observe that if for $x_0 \in E_1$, $J_{x_0} \bar{F} \in \Gamma'(x_0, M)$, then $J_{x_0} \bar{F} \in \Gamma(x_0, M)$ since $\bar{F} \in \dot{\mathcal{C}}^m(\mathbb{R}^n)$ and $(\Gamma(x, M))_{x \in E}$ is obtained through multiple Glaeser refinements of $(\Gamma'(x, M))_{x \in E}$. Thus, (10.22) follows from (10.28) and our careful definition of the $\Gamma'(x, \bar{k}, A)$.

$\square$
Now we move on to the induction step. Suppose we are given a compact set $E$ with a Glaeser stable bundle $(\Gamma(x,M))_{x \in E, M \geq 0}$ which has $\Lambda$ associated strata, and that Theorem 1.6 is known to hold in the case of $\Lambda - 1$ strata with $k^2 = k^2_{\text{old}}$.

Given a cube $Q$, let $rQ$ denote the $r$ times concentric dilation of $Q$. $Q^*$ will be used to denote $1.01Q$.

By Lemma 7.4, $E_1$ is compact; thus $\mathbb{R}^n \setminus E_1$ is open and admits a Whitney decomposition as in [6] and [32]. That is, there exist closed cubes $Q_\nu$ such that

\begin{align}
\delta_\nu := \text{diam}(Q_\nu) &\leq 1. \\
\mathbb{R}^n \setminus E_1 = \bigcup_\nu Q_\nu. \\
3Q_\nu \cap E_1 &\cap E = \emptyset.
\end{align}

If $\delta_\nu < 1$, then there exists $x_0^{(\nu)} \in E_1$ such that $\text{dist}(x_0^{(\nu)}, Q_\nu) < C \delta_\nu$.

Furthermore, there exists a Whitney partition of unity $\{\theta_\nu\}_\nu$ subordinate to $\{Q_\nu\}_\nu$ satisfying

\begin{align}
1 = \sum_\nu \theta_\nu &\text{ on } \mathbb{R}^n \setminus E_1. \\
\text{supp}(\theta_\nu) &\subset Q_\nu^*. \\
|\partial^\alpha \theta_\nu| &\leq C \delta_\nu^{-|\alpha|} \text{ on } \mathbb{R}^n \text{ for } |\alpha| \leq m + 1.
\end{align}

Our current goal is to show that we may apply the induction hypothesis on each of the $E \cap Q_\nu$, but only after subtracting off the function $\tilde{F}$ from Lemma 10.3.

To apply the induction hypothesis, first we must establish Glaeser stability for the desired bundle. To do so, consider the bundle $(H(x))_{x \in E}$ such that

\begin{align}
H(x) = \{J_x \tilde{F}\} &\text{ if } x \in E_1 \\
H(x) = \Gamma(x, C_0 A^2) &\text{ if } x \notin E_1,
\end{align}

where $C = C_0$ is as in the conclusion of Lemma 10.3.

Since $E_1$ is compact and $J_x \tilde{F} \in \Gamma(x, C_0 A^2)$ for $x \in E_1$, we see that the Glaeser stability of $(\Gamma(x, C_0 A^2))_{x \in E}$ implies the Glaeser stability of $(H(x))_{x \in E}$. Furthermore,
\{(x_0, J_{x_0} \tilde{F}) : x_0 \in E_1\} is compact. Thus, by Lemma 9.6, we may choose \(\delta\) to be uniform in \(x_0\) in the following.

For any \(\epsilon > 0\), there exists \(\delta > 0\) for which the following holds: If \(x_0 \in E_1\) and \(x_1, \ldots, x_{\tilde{k}} \in E \cap B(x_0, \delta)\), then there exist

\[(10.33a)\quad P_i \in \Gamma(x_i, C_0 A^\sharp)\]

such that

\[(10.33b)\quad P_0 = J_{x_0} \tilde{F}\]

and

\[(10.33c)\quad |\partial^\alpha (P_i - P_j)(x_j)| \leq \epsilon |x_i - x_j|^{m - |\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j, \leq \tilde{k}.\]

Subtracting off \(J_{x_i} \tilde{F}\) from each of the above polynomials, we obtain the following:

If \(x_0 \in E_1\) and \(x_1, \ldots, x_{\tilde{k}} \in E \cap B(x_0, \delta)\), then there exist

\[(10.34a)\quad P_i \in -J_{x_i} \tilde{F} + \Gamma(x_i, C_0 A^\sharp)\]

such that

\[(10.34b)\quad P_0 = 0\]

and

\[(10.34c)\quad |\partial^\alpha (P_i - P_j)(x_j)| \leq \epsilon |x_i - x_j|^{m - |\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j, \leq \tilde{k}.\]

In particular, by (10.29d), for any \(\epsilon > 0\) there exists \(\delta > 0\) such that if \(x_i \in E \cap Q^*_{\nu}\) and \(\delta_\nu < \delta\), then

\[(10.35)\quad |D^\alpha (P_i)(x_i)| \leq \epsilon \delta_\nu^{m - |\alpha|}.\]

Second, we must determine bounds for the norm of the appropriate holding space. By (10.22), for any \(x_0 \in E_1\) and \(x_1, \ldots, x_{\tilde{k}} \in E\), there exist

\[(10.36a)\quad P_i \in \Gamma(x_i, C_0 A^\sharp)\]

such that

\[(10.36b)\quad P_0 = J_{x_0} \tilde{F}\]

and

\[(10.36c)\quad |\partial^\alpha (P_i - P_j)(x_j)| \leq C_0 A^\sharp |x_i - x_j|^{m - |\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j, \leq \tilde{k}.\]

Again subtracting off \(J_{x_i} \tilde{F}\) from each of the above polynomials, we obtain the following:

If \(x_0 \in E_1\) and \(x_1, \ldots, x_{\tilde{k}} \in E\), there exist

\[(10.37a)\quad P_i \in -J_{x_i} \tilde{F} + \Gamma(x_i, C_0 A^\sharp)\]
such that

\[(10.37b)\quad P_0 = 0\]

and

\[(10.37c)\quad |\partial^\alpha(P_i - P_j)(x_j)| \leq C_0A^j|x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j, \leq \tilde{k}.
\]

In particular, if \(x_i \in Q'\) and \(\delta < 1\), then by (10.29d)

\[(10.38)\quad |\partial^\alpha(P_i)(x_i)| \leq C_0A^j\delta^{m-|\alpha|} \text{ for } |\alpha| \leq m\]

when \(x_1, \ldots, x_{\tilde{k}}\) are taken from \(E \cap Q'\).

Observe that for each scale \(\delta < 1\) there are only finitely many \(Q'\) such that \(\delta = \text{diam}(Q').\) Thus, by (10.34a), (10.34c), (10.35), (10.37a), (10.37c), and (10.38), one may produce a function \(A : (0, 1] \to (0, C_0A^j]\) such that \(\lim_{t \to 0} A(t) = 0\) and:

For any \(Q'\) satisfying \(\delta < 1\) and \(x_1, \ldots, x_{\tilde{k}} \in E \cap Q'\), there exist

\[(10.39a)\quad P_i \in -J_{x_i}\mathbf{F} + \Gamma(x_i, C_0A^j)\]

such that

\[(10.39b)\quad |\partial^\alpha(P_i)(x_i)| \leq A(\delta)\delta^{m-|\alpha|} \text{ for } |\alpha| \leq m\]

and

\[(10.39c)\quad |\partial^\alpha(P_i - P_j)(x_j)| \leq A(\delta)|x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j, \leq \tilde{k}.
\]

We will apply the following rescaled version of the induction hypothesis on each \(E \cap Q'\).

Before its formal statement, we define a shape field \((H(x, M))_{x \in E}\) to be \(\bar{\delta}\text{-regular}\) for \(\delta > 0\) if it satisfies (2.2) and (2.3), but

\[(10.40)\quad H(x, M) \subset \{P \in \mathcal{P} : |\partial^\alpha P(x)| \leq \delta^{m-|\alpha|}M \text{ for } |\alpha| \leq m\}
\]

in place of (2.1). (The usual notion of being regular is equivalent to being 1-regular.)

**Lemma 10.4.** Let \(\tilde{\delta} > 0\), \(Q_0 \subset \mathbb{R}^n\) be a cube of length \(3\tilde{\delta}\) and \(E \subset Q_0\) be compact. Suppose that for each \(x \in E\) we are given a \((C, \tilde{\delta})\) convex, \(\tilde{\delta}\text{-regular}, \) closed shape field \((\Gamma(x, M))_{x \in E, M \geq 0}\) such that

1. The Glaeser refinement of \((\Gamma(x))_{x \in E}\) terminates in a bundle \((\Gamma^*(x))_{x \in E}\) such that \(\Gamma^*(x)\) is nonempty for all \(x \in E\) and

2. There exists \(A > 0\), such that given \(x_1, \ldots, x_{k_{\text{old}}} \in E\), there exists polynomials

\[P_j \in \Gamma^*(x, A)\]

satisfying

\[(10.41)\quad |\partial^\alpha(P_i - P_j)(x_j)| \leq A|x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 1 \leq i, j, \leq k_{\text{old}}^i.
\]

Assume also that \(E\) has fewer than \(\lambda\) strata. Then there exists \(F \in C^m(\mathbb{R}^n)\) such that
Given $x \in E, M \geq 0$, define
\begin{equation}
\tilde{\Gamma}(x, M) := (-J_x \tilde{F} + \Gamma'(x, C_0 A^x M / A(\delta_\nu))) \cap \{ P \in \mathcal{P} : |\partial^\alpha P(x)| \leq \delta^{m-|\alpha|} M \text{ for } |\alpha| \leq m \},
\end{equation}
where again, $(\Gamma'(x, M))_{x \in E}$ refers to the unrefined bundle mentioned in Theorem 1.5.

Then, it is clear from the relevant definitions that $(\tilde{\Gamma}(x, M))_{x \in E} \cap Q^*_\nu \geq 0$ is a $(C_w, \tilde{\delta})$ convex, $\tilde{\delta}$-regular, closed shape field. Furthermore, the Glaeser refinement of $(\tilde{\Gamma}(x))_{x \in E}$ terminates in $(\Gamma^*(x))_{x \in E}$, the Glaeser stable bundle obtained through refinement of $(\Gamma(x))_{x \in E}$. This is because
\begin{equation}
\bigcup_{M \geq 0} \{ P \in \mathcal{P} : |\partial^\alpha P(x)| \leq \delta^{m-|\alpha|} M \text{ for } |\alpha| \leq m \} = \mathcal{P},
\end{equation}
so $\Gamma(x) = \tilde{\Gamma}(x)$ for all $x \in E$. As a further corollary, we see that $\dim \Gamma(x) = \dim \tilde{G}(x)$ and $\dim \pi_x \Gamma(x) = \dim \pi_x \tilde{\Gamma}(x)$ for all $x \in E$, so both bundles have the same number of strata on $E \cap Q^*_\nu$.

Lastly, (10.39) says that given $x_1, \ldots, x_{k^\text{old}} \in E$, there exist polynomials $P_j \in \Gamma^*(x, A(\delta_\nu))$ satisfying
\begin{equation}
|\partial^\alpha (P_i - P_j)(x_j)| \leq A(\delta_\nu)|x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 1 \leq i, j \leq k^\text{old}.
\end{equation}
Thus, the hypotheses of Lemma 10.4 are fully satisfied. We conclude that there exists $F_\nu \in C^m(\mathbb{R}^n)$ such that
\begin{equation}
\|F_\nu\|_{C^m(\mathbb{R}^n)} \leq CA(\delta_\nu)
\end{equation}
and
\begin{equation}
J_x F_\nu \in \tilde{\Gamma}(x, CA(\delta_\nu)), x \in E \cap Q^*_\nu.
\end{equation}

By definition of $\tilde{\Gamma}(x, M)$, (10.46) implies
\begin{equation}
J_x F_\nu \in -J_x \tilde{F} + \Gamma(x, CA^x)
\end{equation}
and
\begin{equation}
|\partial^\alpha (J_x F_\nu)(x)| \leq CA(\delta_\nu)^{m-|\alpha|} \text{ for } |\alpha| \leq m, x \in E \cap Q^*_\nu.
\end{equation}
Combining (10.45) with (10.48) gives
\begin{equation}
|\partial^\alpha (J_x F_\nu)(x)| \leq CA(\delta_\nu)^{m-|\alpha|} \text{ for } |\alpha| \leq m, x \in Q^*_\nu.
\end{equation}

For $\delta > 0$, define
Following [6], one may easily show, using the properties of the Whitney decomposition and associated partition of unity, along with (10.49) that $F^{[\delta]}$ converges in $C^m$ to a function $F^{[0]}$ satisfying

$$
\|F^{[0]}\|_{C^m} \leq CA^2 
$$

and

$$
J_x F^{[0]} = 0, x \in E_1.
$$

For $x \in E \setminus E_1$, $x \in \text{supp}(\theta_\nu)$ for finitely many $\nu$; thus, there exists a $\delta(x) > 0$ such that $F^{[0]} \equiv F^{[\delta(x)]}$ in a neighborhood of $x$. As a result,

$$
J_x F^{[0]} = J_x F^{[\delta(x)]} = \sum_{\text{supp}(\theta_\nu) \ni x} J_x \theta_\nu \otimes J_x F_\nu \in -J_x \tilde{F} + \Gamma(x, CA^2)
$$

by the $(C_w, \tilde{\delta})$-convexity of the shape field, specifically Lemma 5.1.

We now set $F = F^{[0]} + \tilde{F}$ and see that $\tilde{F}$ is the desired section, as

$$
J_x (F(x)) \in \Gamma(x, CA^2) \text{ for all } x \in E
$$

and

$$
\|F\|_{C^m} \leq CA^2.
$$

11. Improvement of $k^\sharp$

**Theorem 11.1.** Theorem 1.6 holds with $k^\sharp = 2^\dim \tilde{P}$ in (1.3) and (1.4).

Given a holding space $H(x)$, we will now let $\tilde{H}(x)$ denote its Glaeser refinement when $k^\sharp$ is taken to be the value in (1.3) initially used to prove Theorem 1.5. Call this value $k_1$. We will use $H'(x)$ to denote the Glaeser refinement when $k^\sharp$ is taken to be $k_0 := 2^\dim \tilde{P}$. Theorem 11.1 is easily seen to be implied by the following proposition.

**Proposition 11.2.** Let $H(x)$ be a holding space. Then $\tilde{H}(x) = H'(x)$.

The containment $\tilde{H}(x) \subset H'(x)$ follows trivially from the definitions, as increasing $k^\sharp$ increases the number of $y_j$ near $x$ for which we must find polynomials in $H(y_j)$. Thus, the essence of the proof will be showing that $\tilde{H}(x) \supset H'(x)$.

A key to proving Proposition 11.2 is the following result proven in [23].
Theorem 11.3. Let $S \subset \mathbb{R}^n$ be a finite set of diameter at most 1. For each $x \in S$, let $\mathcal{G}(x) \subset \overline{P}$ be convex. Suppose that for every subset $S' \subset S$ with $|S| \leq 2\dim \overline{P}$, there exists $F_{S'} \in C^m(\mathbb{R}^n, \mathbb{R}^d)$ such that $\|F_{S'}\|_{C^m(\mathbb{R}^n, \mathbb{R}^d)} \leq 1$ and $J_x F_{S'} \in \mathcal{G}(x)$ for all $x \in S$.

Then, there exists $F \in C^m(\mathbb{R}^n, \mathbb{R}^d)$ such that $\|F\|_{C^m(\mathbb{R}^n, \mathbb{R}^d)} \leq \gamma$ and $J_x F \in \mathcal{G}(x)$ for all $x \in S$.

Here, $\gamma$ depends only on $m, n, d$, and $|S|$.

A scalar-valued version of Theorem 11.3 was originally proven in [28]. However, we state the vector-valued version for full generality, as this is particularly useful for the selection problem.

Proof of Proposition 11.2. Pick $y_0 \in E$ and let $P_0 \in H'(y_0)$. Replacing each $H(y)$ with $H(y) - P_0$, we may assume $P_0$ is the zero polynomial.

Fix $\epsilon > 0$ and let $M \geq 0$ be a constant depending only on $m, n, d$ and to be determined later. Choose $0 < \delta < 1/2$ so that for any $y_1, \ldots, y_k \in B(y_0, \delta) \cap E$ there exist $P_1, \ldots, P_k \in \overline{P}$ such that

$$P_j \in H(y_j)$$

and

$$|\partial^n (P_i - P_j)(y_j)| \leq (\epsilon/M)|y_i - y_j|^{|m - |\alpha||} \quad \text{for } |\alpha| \leq m, 0 \leq i, j \leq k_0.$$  

Let $y_1, \ldots, y_k, z \in B(x, \delta) \cap E$ and write $S = \{y_0, y_1, \ldots, y_k\}$. For $y_j \in S$, define $\mathcal{G}(y_j)$ to be $H(y_j)$ if $j \geq 1$ and $\{P\}$ if $j = 0$. Since $P \in H'(y_0)$, for any $S' \subset S$ satisfying $|S'| = k_0 + 1$ and $y_0 \in S$, there exist polynomials $(P_z)_{z \in S'}$ such that

$$|\partial^n (P_z - P_{z'})(z)| \leq (\epsilon/M)|z - z'|^{m - |\alpha|} \quad \text{for } |\alpha| \leq m, z, z' \in S'.$$

In particular, taking $z' = y_0$ in (11.3), we see that

$$|\partial^n P_z(z)| \leq (\epsilon/M)\delta^{m - |\alpha|} \leq (\epsilon/M).$$

Therefore, by the classical Whitney extension theorem for finite sets (Lemma 6.5), there exists $F_{S'}$ such that

$$J_{y_j} F = P_j \in \mathcal{G}(y_j) \quad \text{for } y_j \in S'$$

and

$$\|F_{S'}\|_{C^m} \leq C(\epsilon/M),$$

where $C$ depends only on $m, n, d$.

By adding $y_0$ if necessary, we see that for any $S' \subset S$ with $|S'| = k_0 + 1$ there exists $F_{S'}$ such that

$$J_{y_j} F_{S'} = P_j \in \mathcal{G}(y_j) \quad \text{for all } y_j \in S'.$$
and
\[(11.8)\quad \|F_S^r\|_{C^m} \leq C(\epsilon/M).\]

Taking $\gamma = \gamma(m, n, d, k_1)$ as in Theorem 11.3, there exists $F$ such that
\[(11.9)\quad J_{y_j}F \in \mathcal{G}(y_j) \subset H(y_j)\]
and
\[(11.10)\quad \|F\|_{C^m(\mathbb{R}^n)} \leq \gamma C(\epsilon/M).\]

Setting $Q_j = J_{y_j}F$ and $Q_0 = P_0$, we see that for all $0 \leq i, j \leq k_1$, there exist $Q_1, ..., Q_{k_1} \in \bar{P}$ with
\[(11.11)\quad Q_j \in H(y_j), 1 \leq j \leq k_1\]
and
\[(11.12)\quad |\partial^{\alpha}(Q_i - Q_j)(y_j)| \leq \gamma C(\epsilon/M)|y_i - y_j|^{m - |\alpha|} \text{ for } |\alpha| \leq m.\]

Picking $M = C\gamma$, we have
\[(11.13)\quad |\partial^{\alpha}(Q_i - Q_j)(y_j)| \leq \epsilon|y_i - y_j|^{m - |\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j \leq k_1.

Since $y_1, ..., y_{k_1} \in B(x, \delta) \cap E$ were arbitrary, $Q_0 = P_0 \in \tilde{H}(y_0)$. Thus, $\tilde{H}(x) \supset H'(x)$.

\[\square\]

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