Convergence of eigenfunction expansions for flexural gravity waves in infinite water depth

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Abstract. In the present paper, point-wise convergence of the eigenfunction expansion to the velocity potential associated with the flexural gravity waves problem in water wave theory is established for infinite water depth case. To take into account the hydroelastic boundary condition at the free surface, a flexible membrane is assumed to float in water waves. In this context, firstly the eigenfunction expansion for the velocity potentials is obtained. Thereafter, an appropriate Green’s function is constructed for the associated boundary value problem. Using suitable properties of the Green’s functions, the vertical components of the eigenfunction expansion is written in terms of the Dirac delta function. Finally, using the property of the Dirac delta function, the convergence of the eigenfunction expansion to the velocity potential is shown.

1. Introduction
A large class of boundary value problems (bvps) arise in the field of ocean engineering in which one or more of the boundary conditions (bcs) are higher-order bcs. Those higher-order bcs are different from the usual bcs like Dirichlet, Neumann, Robin bcs. Although in case of linearized water wave theory and potential flow field, the governing equation is Laplace/Helmholtz types, the bcs may become higher order in nature due to bc on the structure. Hence, the associated bvps will become a non Sturm-Liouville type and the convergence study of the related solutions are utmost important. In general, for a 2D problem with x-axis being horizontal and coincides with the mean free surface and y-axis being positive in the vertically downward direction, the bc at the still water level \( y = 0 \) is of the form \( L \phi_y + M \phi = 0 \) with \( L \) and \( M \) being the differential operator. For a free surface bc, [1] considered \( L = 1 \) and \( M = K \), where \( K = \omega^2/g \) with \( \omega \) and \( g \) being the incident wave frequency and gravitational acceleration. [2] extended the study of [1] in presence of surface tension and took \( L = 1 + c_0 \left( \partial^2 \partial y^2 \right) \) and \( M = K \). However, [2] used the Green’s function technique to solve the associated bvps which is different than the solution technique as used in [1]. The reason behind the same is that it was difficult to form the relevant eigenfunctions which will form an orthogonal basis to the solution space due to the presence of third order bcs at the free surface. Finally, [3] and [4] showed that the vertical eigenfunctions for the the problems as defined in [2] can become orthogonal w.r.t a newly defined inner product. This orthogonality of the eigenfunctions is required to get an algebraic system of equations which forms a diagonally dominant matrix. After the pioneering work of [3] and [4], the obvious question comes that can the vertical eigenfunctions become orthogonal in bvps having more higher order bcs. It was then [5] who presented a new type of orthogonality relation...
Involving the related eigenfunctions and their derivatives for a general class of wave propagation problems through ducts and pipes in which higher-order boundary conditions often arise at the duct and pipe walls. In this work, the boundary conditions on the upper and lower wave guide surfaces were taken of the form $L = \sum_{n=0}^{M} c_n (\partial^{2n}/\partial x^{2n})$ and $M = \sum_{n=0}^{N} d_n (\partial^{2n}/\partial x^{2n})$. A special case of these boundary conditions is considered by [6] in which a fifth order boundary condition is considered by taking $L = c_0 + c_1 (\partial^2/\partial x^2) + c_2 (\partial^4/\partial x^4)$ and $M = d_0$. This particular boundary condition arises due to the presence of a semi-infinite flexible plate floating at the free surface. The study of [5] limited to the problems in which the domain is a strip of finite height and the same generally appears in case of finite water depth. [7] extended the work of [5] for quarter plane problems (arises for infinite water depth case) by deriving a new class of orthogonal mode-coupling relations. It is to be noted that the higher order boundary conditions taken by [7] are of the form $L = \sum_{n=0}^{M} c_n (\partial^{2n}/\partial x^{2n})$ and $M = d_0$. A further extension in quarter plane domain was done by [8] with $L$ being same as in [7], but $M$ was taken as $M = \sum_{n=0}^{N} d_n (\partial^{2n}/\partial x^{2n})$. However, there was a negligible study regarding the formal characteristics of the eigenfunctions associated with the expansion formulae apart from the orthogonal mode-coupling relations. [9] presented a number of significant properties associated with the eigenfunctions arises in the wave propagation problems presented in [5]. In all the aforementioned studies, the physical problems are modeled in two-dimensional Cartesian coordinate system. A similar three dimensional channel flow problem was studied by [10] in which the orthogonal relations are generalized from the previous two-dimensional problems. In the work of [10], the water depth is considered finite. The same was extended by [11] for infinite depth problems. In this work, a generalized higher order boundary condition is considered at the free surface in which $L$ and $M$ have been taken as $L = \sum_{n=0}^{M} c_n (\partial^2/\partial x^2 + \partial^2/\partial z^2)^n$ and $M = \sum_{n=0}^{N} d_n (\partial^2/\partial x^2 + \partial^2/\partial z^2)^n$. Recently, [12] showed the convergence of eigenfunction expansions for the velocity potentials associated with surface gravity waves in two-layer fluid. Further, using the spectral representation of the vertical eigenfunctions, [13] studied the convergence of the Havelock’s expansion formulae in presence of floating flexible plate. It is to be noted that the procedure of the convergence study presents in this manuscript is completely different than those discussed in [12] and [13].

It is well known that the eigenfunction expansion method is widely used to solve various physical and engineering problems. In eigenfunction expansion method, generally, the unknowns are written as an infinite series involving the eigenfunctions. In this context, it’s very important to check whether this infinite series will finally converges to the required solution. In the present paper, a point-wise convergence analysis of the eigenfunction expansion to the required solution is presented. The overall structure of this manuscript is organized as following. In Section 2, the mathematical description i.e., the governing equations and boundary conditions are presented. Section 3 presents the eigenfunction expansion for the velocity potential, the formulation of Green’s function and their characteristics. This section also include the proof of the convergence of the eigenfunction expansion to the velocity potential of the associated bvp. Finally Section 4 contains the Conclusion. It is to be noted that the root analysis for the dispersion relation is presented in the Appendix.

2. MATHEMATICAL FORMULATION

The problem is studied in 2D Cartesian coordinates i.e., in $xy$-plane with the $x$-axis is considered horizontal and $y$-axis is taken positive in the vertically downward direction. A thin elastic
membrane is floating at the free surface $0 < x < \infty$. The water domain is $0 < x < \infty$ and $0 < y < \infty$. The fluid satisfies the potential flow theory properties so velocity potential $\Phi(x, y, t)$ can be written as $\Phi(x, y, t) = Re\{\phi(x, y)e^{-i\omega t}\}$, where $\omega$ is the angular frequency. Therefore, the spatial component of the velocity potential will satisfy the following Laplace equation

$$\nabla^2 \phi = 0,$$

along with the linearized form of membrane covered bc [16] and bottom bcs

$$T \frac{\partial^3 \phi}{\partial y^3} + \frac{\partial \phi}{\partial y} + K' \phi = 0 \text{ on } y = 0,$$

$$\phi \rightarrow 0 \text{ as } y \rightarrow \infty.$$  

In (2), $T = \frac{T_1}{\rho g - m\omega^2}$ and $K' = \frac{\rho \omega^2}{\rho g - m\omega^2}$ are constants associated with the membrane structural parameters as provided in [14]. Here, $T_1$ is the membrane tension, $m$ is the mass per unit length, $g$ is the gravitational constant and $\rho$ is the density of water in SI units. Further, the far field condition is given by

$$\phi \rightarrow \text{constant times } e^{i k_0 x - k_0 y} \text{ as } x \rightarrow \infty, \ y > 0.$$  

Here, $k_0$ is the positive real root of $K' - k(1 + Tk^2) = 0$. The detailed root analysis of the dispersion relation is given in the Appendix.

3. Eigenfunction expansion and its characteristics for infinite water depth

3.1. Eigenfunction expansion

The velocity potential $\phi(x, y)$ satisfying Eqs.(1)-(3) is written as (see [3])

$$\phi(x, y) = \frac{2k_0}{1 + 3Tk_0^2}e^{-k_0 y}\psi_0(x) + \frac{2}{\pi} \int_0^\infty \frac{ky\psi(x, k)N'(k, y) dk}{k^2(1 - Tk^2)^2 + K'^2},$$  

with $N'(k, y) = k(1-Tk^2) \cos ky - K' \sin ky$. Here, $k_0$ is the positive real root of $K' - k(1 + Tk^2) = 0$. The detailed root analysis of the dispersion relation is given in the Appendix. The expressions for $\psi_0(x, k)$ and $\psi(x, k)$ are given by

$$\psi_0(x) = (1 + Tk_0^2) \int_0^\infty \phi(x, y)e^{-k_0 y} dy - T\phi_y(x, 0),$$

$$\psi(x, k) = \frac{1}{k} \int_0^\infty \phi(x, y)N'(k, y) dy - T\phi_y(x, 0).$$

3.2. Construction of Green’s function and its properties

Now, if we want to construct the Green’s function $G(x, y; \xi, \eta)$, then it satisfies the following governing equations and boundary conditions.

$$\nabla^2 G = \delta(x - \xi, y - \eta),$$  

subject to the conditions (2) and (3). In addition, we may take the following conditions also for the construction of Green’s function

$$G \rightarrow \ln r, \text{ as } r = \sqrt{(x - \xi)^2 + (y - \eta)^2} \rightarrow 0,$$
we get the following relation

\[ G \rightarrow \text{const.} \times e^{-k_0 y + i k_0 |x-\xi|}, \quad \text{as } |x - \xi| \rightarrow \infty, \] (10)

It is to be noted that the last condition is the far-field condition. Now, by solving (8) - (10), the form of \( G(x, y; \xi, \eta) \) is obtained as

\[ G(x, y; \xi, \eta) = -2\pi i \frac{(1 + T k_0^2)}{(1 + 3 T k_0^2)} e^{-k_0 (y+\eta) + i k_0 |x-\xi|} - 2 \int_0^\infty e^{-k_0 |x-\xi|} \frac{N'(k, y)N'(k, \eta)}{k (k^2 (1 - Tk^2)^2 + K^2)} \, dk, \] (11)

where \( N'(k, y) \) is defined before. The details procedure of the construction of the Green’s function is skipped as the details are available in [2]. Now, using the identity (see [14]),

\[ \lim_{x \to 0} \frac{\partial G}{\partial x}(x, y; 0, \eta) = \pi \delta(y - \eta), \] (12)

we get the following relation

\[ 2 k_0 \frac{(1 + T k_0^2)}{(1 + 3 T k_0^2)} e^{-k_0 (y+\eta)} + \frac{2}{\pi} \int_0^\infty \frac{N'(k, y)N'(k, \eta)}{k^2 (1 - Tk^2)^2 + K^2} \, dk = \delta(y - \eta), \] (13)

### 3.3. Convergence of eigenfunction expansion to the velocity potential

**Theorem 1** Given the coefficients \( \psi_0(x) \) and \( \psi(x, k) \) as in (6) and (7) exists, where \( \phi(x, y) \) is any function such that its three times differentiable in the water domain \( 0 < x < \infty, 0 < y < \infty \), then the series

\[ \frac{2 k_0}{1 + 3 T k_0^2} e^{-k_0 y} \psi_0(x) + \frac{2}{\pi} \int_0^\infty \frac{k \psi(x, k) N'(k, y)}{k^2 (1 - Tk^2)^2 + K^2} \, dk \]

converges point-wise to the aforementioned function \( \phi(x, y) \) in the domain of definition.

**Proof 1** : Multiplying both sides of Eq. (13) by \( \phi(x, \eta) \) and integrating w.r.t \( \eta \) from 0 to \( \infty \), we get the following

\[ \int_0^\infty \phi(x, \eta) \delta(y - \eta) \, d\eta = 2 k_0 \frac{(1 + T k_0^2)}{(1 + 3 T k_0^2)} \int_0^\infty \phi(x, \eta) e^{-k_0 (y+\eta)} \, d\eta \]

\[ = 2 k_0 \frac{(1 + T k_0^2)}{(1 + 3 T k_0^2)} e^{-k_0 y} \int_0^\infty \phi(x, \eta) e^{-k_0 \eta} \, d\eta \]

\[ + \frac{2}{\pi} \int_0^\infty \phi(x, \eta) \left\{ \int_0^\infty \frac{N'(k, \eta)N'(k, \eta)}{k^2 (1 - Tk^2)^2 + K^2} \, dk \right\} \, d\eta, \]

\[ = 2 k_0 \frac{(1 + T k_0^2)}{(1 + 3 T k_0^2)} e^{-k_0 y} \int_0^\infty \phi(x, \eta) e^{-k_0 \eta} \, d\eta \]

\[ + \frac{2}{\pi} \int_0^\infty \frac{N'(k, \eta)}{k^2 (1 - Tk^2)^2 + K^2} \left\{ \int_0^\infty \phi(x, \eta) N'(k, \eta) \, d\eta \right\} \, dk, \]

using (6) and (7),

\[ = \frac{2 k_0}{1 + 3 T k_0^2} e^{-k_0 y} \psi_0(x) + \frac{2}{\pi} \int_0^\infty \frac{k N'(k, y) \psi(x, k)}{k^2 (1 - Tk^2)^2 + K^2} \, dk \]

\[ + \left\{ \frac{2 k_0}{1 + 3 T k_0^2} e^{-k_0 y} + \frac{2}{\pi} \int_0^\infty \frac{k N'(k, \eta)}{k^2 (1 - Tk^2)^2 + K^2} \, d\eta \right\} T \psi_0(x, 0), \]

\[ \phi(x, y) = \frac{2 k_0}{1 + 3 T k_0^2} e^{-k_0 y} \psi_0(x) + \frac{2}{\pi} \int_0^\infty \frac{k N'(k, y) \psi(x, k)}{k^2 (1 - Tk^2)^2 + K^2} \, dk. \] (14)
It is to be noted that in the last step, we have used the following identity (see [15] for details)

\[
\int_0^\infty k N'(k,y) \frac{dk}{k^2(1 - Tk'^2 + K'^2)} = -\frac{\pi k_0 e^{-k_0 y}}{1 + 3Tk_0^2}.
\]  

(15)

Hence the convergence of the eigenfunction expansion to the velocity potential is established.

4. Conclusion
The proof of point-wise convergence of the eigenfunction expansion to the velocity potential in membrane coupled gravity waves for infinite water depth is established. A similar type of convergence proof for other non Sturm Liouville type problems will be done in future work.

Appendix

Remark 1
The dispersion relation

\[ G(k) = k(1 + Tk^2) - K', \]

has zero one positive real zero \( k = k_0 \) located on the real axis and pair of complex conjugate zeros \( k = k_I, k_{II}(= \bar{k}_I) \) distributed in II and III quadrants respectively.

Proof 2
As \( G(k) \) is a cubic polynomial in \( k \) with real coefficients and so there is always one real root \( k = k_0 \). Using Descartes’ rule of signs, we get \( k_0 \) is positive and also there are no negative real roots. So, the remaining two roots are complex conjugate in nature. Our remaining task is to locate the complex roots. See, if \( k_0(1 + Tk_0^2) = K' \), then the complex conjugate roots \( x \) and \( \bar{x} \) will satisfy the quadratic equation

\[ Tk^2 + Tk_0k + (1 + Tk_0^2) = 0. \]

Therefore, \( x + \bar{x} = -k_0 \) i.e., \( \Re(x) = -k_0/2 < 0 \). Therefore, the complex conjugate roots of the dispersion relation lie on the second and third quadrants. This can be done using an alternative approach based on argument principle in complex analysis.

Consider the contour as shown in the Fig. OABO with circular arc of radius \( R \). Along OA,

\[ \text{Arg}[G(iy)] = \tan^{-1} \left[ \frac{y(1 - Ty^2)}{-K'} \right]. \]

As \( y \) differ from 0 to \( \infty \), the change in argument is \( \frac{\pi}{2} \). Now, along the arc \( AB \), \( k = Re^{i\theta} \) and \( \theta \in [\frac{\pi}{2}, \pi] \). Therefore, we get

\[ G(Re^{i\theta}) \approx TR^3e^{3i\theta} \quad \text{as} \quad R \to \infty. \]

So, \( \text{Arg}[G(Re^{i\theta})] = 3\theta \) and therefore the change in argument is \( \frac{3\pi}{2} \). In a similar way, the change in argument along \( BO \) when \( x \) varies from \( -\infty \) to 0 will be 0. Therefore, the total change in argument is \( 2\pi \) and thus from the argument principle, the number of zeros in the II quadrant is one and it’s complex conjugate is in III quadrant.
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