Strict entanglement monotonicity under local operations and classical communication

Yu Gu

Institute of Quantum Information Science, School of Mathematics and Statistics,
Shanxi Datong University, Datong, Shanxi 037009, China

Entanglement monotone is defined as a convex measure of entanglement that does not increase on average under local operations and classical communication (LOCC). Here we call an entanglement monotone a strict entanglement monotone (SEM) if it decreases strictly on average under LOCC. We show that, for any convex roof extended entanglement monotone that on pure states is given by some optimal extension, then it is a SEM. Our results imply that entanglement is strictly decreasing on average under LOCC.

Entanglement is one of the most crucial features of quantum theory as compared to classical theory, which is also considered to be a valuable resource for quantum information processing 1, 2. To quantify the amount of entanglement contained in a composite quantum system is a fundamental problem in quantum information science and quantum physics 3, 4. The first significant milestone in this field came from the discovery that entanglement can be as a resource for distributed quantum information processing in the framework of local operations and classical communication (LOCC) 5. Consequently, to identify certain a priori axioms for a good measure of entanglement, Vedral et al. 6 proposed three conditions for a quantity to be such a measure for the first time. Later, Vidal in Ref. 7 explored a more restrictive requirement on LOCC, and an additional demand of convexity is needed, and there the satisfactory measure is called an entanglement monotone.

It is interesting that these constraints on entanglement measures can be easily checked 8. For any convex roof extended entanglement measure, it is an entanglement monotone if it can be defined by both a locally unitary invariant and a concave function on the reduced states of the pure states [see Eqs. (3) and (4) below]. Recently, we found that, for almost all entanglement measures so far, the associated functions are not only concave, but also strictly concave 10. More significantly, this strict concavity guarantees the monogamy of entanglement 10 where the monogamy law is a key feature of entanglement distribution among multiparties (see Refs. 10 12 and references therein for details). This motivates us to investigate entanglement measures deeply. In this paper, we investigate this strict concavity in a more general sense: We show that entanglement is strictly monotonic under LOCC on average for many entanglement monotones. That is, we exploit here a new property of the entanglement monotone.

Let $\mathcal{H}^A \otimes \mathcal{H}^B \equiv \mathcal{H}^{AB}$ be a bipartite Hilbert space with finite dimension, where $A, B$ are subsystems of the composite quantum system, and let $S(\mathcal{H}^{AB}) \equiv S^{AB}$ be the set of density operators acting on $\mathcal{H}^{AB}$. Recall that, a function $E : S^{AB} \to \mathbb{R}_+$ is called a measure of entanglement if it satisfies $\mathcal{E}$: (E1) $E(\rho) = 0$ iff $\rho$ is separable [this condition can be replaced by $E(\rho) = 0$ if $\rho$ is separable]; (E2) $E$ is invariant under local unitary operations, i.e., $E(\rho) = E(U_A \otimes U_B \rho U_A^\dagger \otimes U_B^\dagger)$ for any local unitaries $U_A$ and $U_B$; (E3) $E$ cannot increase under LOCC, i.e., $E(\Phi(\rho)) \leq E(\rho)$ for any LOCC $\Phi$. Note that (E3) implies (E2). The map $\Phi$ is completely positive and trace preserving (CPTP). In general, LOCC can be stochastic in the sense that $\rho$ can be converted to $\sigma_j$ with some probability $p_j$. In this case, the map from $\rho$ to $\sigma_j$ can not be described in general by a CPTP map. More than (E2), $E$ is said to be an entanglement monotone 9 if it is nonincreased on average under stochastic LOCC, i.e.,

$$E(\rho) \geq \sum_j p_j E(\sigma_j), \quad \forall \rho \in S^{AB}. \quad (1)$$

Note that Eq. (1) is more restrictive than $E(\rho) \geq E(\sum_j p_j \sigma_j)$ since in such a case we cannot select subensembles according to a measurement outcome 13. It is possible that $E(\sigma_{j_0}) > E(\rho)$ for some $j_0$. Almost all measures of entanglement studied in literature satisfy (1). The measure is said to be faithful if it is zero only on separable states.

Let $E$ be a measure of entanglement on bipartite states. The entanglement of formation $E_F$ associated with $E$ is defined by

$$E_F(\rho) \equiv \min_{\{p_j, \psi_j\}} \sum_{j=1}^n p_j E(\langle \psi_j | \psi_j \rangle), \quad (2)$$

where the minimum is taken over all pure state decompositions of $\rho = \sum_{j=1}^n p_j |\psi_j\rangle \langle \psi_j|$. That is, $E_F$ is the convex roof extension of $E$. Vidal 8, Theorem 2] showed that $E_F$ above is an entanglement monotone on mixed bipartite states if the following concavity condition holds. For a pure state $|\psi\rangle \in \mathcal{H}^{AB}$, $\rho^A = \text{Tr}_B |\psi\rangle \langle \psi|$, define the function $h : S^A \to \mathbb{R}_+$ by

$$h(\rho^A) \equiv E(\langle \psi | \psi \rangle). \quad (3)$$
Note that since $E$ is invariant under local unitaries we must have $h(U \rho U^\dagger) = h(\rho)$ for any unitary operator $U$ acting on $\mathcal{H}^A$. If $h$ is also concave, i.e.

$$h(\lambda \rho_1 + (1 - \lambda) \rho_2) \geq \lambda h(\rho_1) + (1 - \lambda) h(\rho_2) \quad (4)$$

for any states $\rho_1, \rho_2$, and any $0 \leq \lambda \leq 1$, then $E_F$ as defined in (2) is an entanglement monotone.

It was shown in Ref. [10] that for almost all the well-known entanglement measures, the associated function defined as in (3) is strictly concave (from which we proved that $E$ is monogamous on pure tripartite states and $E_F$ is monogamous on both pure and mixed tripartite states, according to our definition in Ref. [11]). Then, in the sense of Vidal [9], what is the corresponding property of $E_F$? We introduce here the concept of strict entanglement monotone in terms of more restriction on the LOCC in (E3). We then show that $E_F$ is a strict entanglement monotone if the associated function $h$ is strict concave. Going further, we will prove that many entanglement measures, such as the negativity [11], the relative entropy of entanglement [8, 15], and the squashed entanglement [16] (if it can be obtained by some optimal extension) are strict entanglement monotones. Our results would demonstrate that entanglement measures are strict in our sense.

For convenience, we fix some terminologies. An entanglement measure $E$ is said to be strictly decreasing on average under LOCC if for any stochastic LOCC,

$$\begin{align*}
\{ \Phi_j : \text{Tr} \Phi_j(\rho) = p_j, \sum_j p_j = 1, \\
\Phi_j(\rho) \neq p_j U_j^A \otimes U_j^B \rho (U_j^A)^\dagger \otimes (U_j^B)^\dagger, \\
(U_j^X)^\dagger U_j^X = I_j^X, \ j = 1, 2, \ldots, d \} \quad (5)
\end{align*}$$

there exists $\rho \in \mathcal{S}^{AB}$ such that

$$E(\rho) \geq \sum_j p_j E(\sigma_j), \quad (6)$$

where $p_j \sigma_j = \Phi_j(\rho)$, $U_j^X$ are unitary operators on $\mathcal{H}^X$. Equivalently, an entanglement measure $E$ decreases strictly on average under LOCC if and only if

$$E(\rho) = \sum_j p_j E(\sigma_j) \quad (7)$$

holds for all states $\rho \in \mathcal{S}^{AB}$ implies that the LOCC is either a local unitary operation (if the LOCC is a map from system $A + B$ to $A' + B'$, then it is a local isometric operation; hereafter, we always assume with no loss of generality that the LOCCs are acting from $A + B$ to itself) or a convex mixture of local unitary operations. If an entanglement monotone $E$ is strictly decreasing on average under LOCC, we call it a strict entanglement monotone (SEM). If an entanglement monotone $E$ is strictly decreasing under LOCC for pure states, we call it a SEM on pure states.

**Theorem 1.** Using the notations above, if $E$ is a SEM on pure states, then $E_F$ is a SEM as well.

**Proof.** According to the LOCC scenario in Ref. [14], in order to prove that a local unitary invariant function $E : \mathcal{S}^{AB} \to \mathbb{R}_+$ satisfying condition (E1) is an entanglement monotone, we only need to consider a family $\{\Phi_k\}$ consisting of completely positive linear maps such that $\Phi_k(\rho) = p_k \sigma_k$, $\rho \in \mathcal{S}^{AB}$, where $\Phi_k(\rho) = I^A \otimes M_k X I^A \otimes M_k^\dagger$ transforms pure states to some scalar multiple of pure states, $\sum_k M_k^i M_k = I^B$.

Applying $\Phi_k$ to $\rho$, the state becomes

$$\sigma_k = \Phi_k(\rho)/p_k$$

with probability $p_k = \text{Tr} \Phi_k(\rho)$. We assume that $\rho = \ket{\psi}\bra{\psi} \in \mathcal{S}^{AB}$ is an entangled pure state. It yields

$$E(\ket{\psi}\bra{\psi}) \geq \sum_k p_k E(\sigma_k) = \sum_k p_k E_F(\sigma_k). \quad (8)$$

If $E$ is a SEM on pure states and the equality holds in (8) for any pure state $\ket{\psi} \in \mathcal{H}^{AB}$, then either $\Phi_k \equiv \Phi^B$ for some local unitary operation $\Phi^B$ or $\Phi_k(\cdot) = p_k I^A \otimes U_k^B(\cdot) I^A \otimes (U_k^B)^\dagger$.

Now we assume that $\rho$ is mixed. Perform $\Phi_k$ on $\rho$ and denote $\sigma_k = \Phi_k(\rho)/p_k$ with probability $p_k = \text{Tr} \Phi_k(\rho)$. Observe that there exists an ensemble $\{t_j, \ket{\eta_j}\}$ of $\rho$ such that

$$E_F(\rho) = \sum_j t_j E(\ket{\eta_j}).$$

For each $j$, let $\sigma_{jk} = \frac{1}{t_{jk}} \Phi_k(\ket{\eta_j}\bra{\eta_j})$, where $t_{jk} = \text{Tr} \Phi_k(\ket{\eta_j}\bra{\eta_j})$. Then $\sigma_k = \frac{1}{p_k} \sum_j t_{jk} \sigma_{jk}$ and $E(\ket{\eta_j}) \geq \sum_k t_{jk} E(\sigma_{jk})$ by what is proved for pure states above. It follows that

$$E_F(\rho) = \sum_j t_j E(\ket{\eta_j}) \geq \sum_{j,k} t_{jk} E(\sigma_{jk}) \geq \sum_k p_k E_F(\sigma_k). \quad (9)$$

If $E_F(\rho) = \sum_k p_k E(\sigma_k)$ for any $\rho \in \mathcal{S}^{AB}$, then $E(\ket{\eta_j}) = \sum_k t_{jk} E_F(\sigma_{jk})$, which completes the proof by the result of the case for pure states.

**Proposition 2.** $E_F$ as defined in (2) is a SEM if the associated function $h$ in Eq. (3) is strictly concave, i.e.

$$h(\lambda \rho_1 + (1 - \lambda) \rho_2) > \lambda h(\rho_1) + (1 - \lambda) h(\rho_2) \quad \text{whenever} \ \rho_1 \neq \rho_2, \ 0 < \lambda < 1.$$

**Proof.** We only need to check it for pure states by Theorem 1. We use the notations as in the proof of Theorem 1 and we assume without loss of generality that $k = 1, 2$.

If $h$ is strictly concave, we assume that the equality holds in (3), which leads to

$$h(\sigma_k) = h \left( \sum_k p_k \sigma_k^k \right) = \sum_k p_k h(\sigma_k^k). \quad (10)$$
since $E(|\psi\rangle\langle\psi|) = h(\rho^A) = h(\sigma^A)$ and $\sum_k p_k h(\sigma^A_k) = \sum_k p_k E(\sigma_k) = \sum_k p_k E_F(\sigma_k)$, where $\rho^A = Tr_B|\psi\rangle\langle\psi|$, $\sigma^A_k = Tr_{B\bar{k}}\sigma_k$, and $\sigma^A = \sum_k p_k \sigma^A_k$. Then $\sigma^A = \sigma^A_k$ for any $k$ and $l$, which implies that either $\Phi_k = \Phi_{k'}$ for some local unitary operation $\Phi_B$ or $\Phi_k(\cdot) = p_k I^A \otimes U^k_B(\cdot) I^A \otimes (U^k_B)^\dagger$, where $U^k_B$’s are unitary operators on $H^B$; $\sum_k p_k = 1$.

The proof is completed. \qed

Note that, many entanglement measures, such as entanglement of distillation $E_{A\|C}$, entanglement cost $E_{C}$, the squashed entanglement $E_{sq}$, and the relative entropy from condition (11). We recall part

$$\sigma_{j_0}^A \neq \rho^A \quad (11)$$

for some $j_0$ we have (10) holds, where $p_j \sigma_j = \Phi_j(\rho)$, $X^A = Tr_B X$.

Proof. The ‘only if’ part is clear. Conversely, if (11) holds, it is equivalent to say that if $E(\rho) = \sum_j p_j E(\sigma_j)$ then we must have $\sigma_j^A = \rho^A$ for any $j$. Note that $\sigma_j$’s are pure states, it follows that for any pure state $\rho \in S^{AB}$, $h(\rho^A) = \sum_j p_j h(\sigma_j^A)$ if and only if $\sigma_j^A = \rho^A$ for all $j$.

That is, $h$ is strictly concave.

It is interesting that we can give another proof of part 1 in Ref. (10), Theorem 1 from condition (11). We recall part 1 of the Theorem in Ref. (10): Let $E$ be an entanglement monotone for which $h$, as defined in Eq. (3), is strictly concave. If $\rho^{ABC} = |\psi\rangle\langle\psi|^{ABC}$ is pure and $E(\rho^{ABC}) = E(\rho^A)$, then $H^B$ has a subspace isomorphic to $H^{B_1} \otimes H^{B_2}$ and up to local unitary on system $B_1 B_2$,

$$|\psi\rangle^{ABC} = |\phi\rangle^{AB_1} |\eta\rangle^{B_2C}, \quad (12)$$

where $|\phi\rangle^{AB_1} \in H^{AB_1}$ and $|\eta\rangle^{B_2C} \in H^{B_2C}$ are pure states. In particular, $\rho^{AC}$ is a product state and $E(\rho^{AC}) = 0$, so that $E$ is monogamous on pure tripartite states. In order to see this, we let $\{|i_k\rangle\}$ and $\{|j_l\rangle\}$ be orthonormal bases of $H^B$ and $H^C$, respectively. Define

$$V_j |\psi\rangle = \sum_i \langle i_k | j_l | \psi \rangle |i_k\rangle, \quad \forall |\psi\rangle \in H^{BC}. \quad (13)$$

It follows that

$$Tr_C \rho^{ABC} = \sum_j I^A \otimes V_j \rho^{ABC} I^A \otimes V_j^\dagger. \quad (14)$$

Let $\rho^{ABC} = |\psi\rangle\langle\psi|^{ABC}$ and assume that it satisfies $E(\rho^{ABC}) = E(\rho^{AB})$, $\rho^{AB} = Tr_{C} \rho^{ABC}$. Let $\{|k_n\rangle\}$ be an orthonormal basis of $H^A$, then

$$|\psi\rangle^{ABC} = \sum_{k, i, j} a_{kij} |k_n\rangle |i_k\rangle |j_l\rangle.$$

The action of $\Phi_s(\cdot) \equiv I^A \otimes V_s(\cdot) I^A \otimes V_s^\dagger$ on $\rho^{ABC}$ gives

$$p_s \rho_s^{ABC} = \Phi_s(\rho^{ABC}) = |\psi_s\rangle\langle\psi_s|, \quad (15)$$

where $|\psi_s\rangle = \sum_k a_{kis} |k_n\rangle |i_k\rangle$. That is, $\rho_s^{AB}$ is a pure state for any $s$. On the other hand, $E$ obeys (11), which results in

$$\rho_s^A = \rho^A, \quad \forall s, \quad (16)$$

where $\rho_s^A = Tr_{BC}\rho_s^{ABC}$. Note that $\rho^{AB} = \sum_s |\psi_s\rangle\langle\psi_s|^{AB}$, then following the proof of the Theorem in Ref. (10), we can conclude that $H^B$ has a subspace isomorphic to $H^{B_1} \otimes H^{B_2}$ and up to local unitary on system $B_1 B_2$,

$$|\psi\rangle^{ABC} = |\phi\rangle^{AB_1} |\eta\rangle^{B_2C}, \quad (17)$$

where $|\phi\rangle^{AB_1} \in H^{AB_1}$ and $|\eta\rangle^{B_2C} \in H^{B_2C}$ are pure states.

In what follows, we discuss whether or not the entanglement monotones that are not derived via the convex roof structure are SEMs as well. The well known one is the computable measure of entanglement, negativity, which is defined by

$$N(\rho) = \|\rho^{TA}\|_T - \frac{1}{2}, \quad \rho \in S^{AB}, \quad (18)$$

where $\|X\|_T = Tr\sqrt{XX^\dagger}$ and $\rho^{TA}$ denotes the partial transposition with respect to part $A$ under some given orthonormal bases of $H^A$ and $H^B$. The logarithmic negativity $E_N$ is defined as

$$E_N(\rho) = \log_2 N(\rho). \quad (19)$$

It is known that the negativity $N$ is a SEM on pure states (11) and thus $N_F$ is also a SEM by Proposition 2. In what follows we will show that $N$ is also a SEM on mixed states, and thus it is a SEM.

Theorem 4. The negativity $N$ is a SEM.

Proof. According to the scenario in Ref. (14), we only need to consider a family $\{|\phi_k\rangle\}$ consisting of completely positive linear maps such that $\Phi_k(\rho) = p_k \sigma_k$, where $\Phi_k(\cdot) = I^A \otimes M_k \cdot I^A \otimes M_k^\dagger$ transforms pure states to some scalar multiple of pure states, $\sum_k M_k^\dagger M_k = I^B$. For any $\rho \in S^{AB}$ with $N(\rho) > 0$, we let

$$\rho^{TA} = (1 + a) \rho^+ - \rho^-, \quad (20)$$

where $(1 + a) \rho^+$ and $\rho^-$ are the positive part and the negative part of $\rho^{TA}$, respectively. That is, $N(\rho) = a$, $\rho^+ \rho^- = \rho^- \rho^+ = 0$. It follows that

$$p_k \sigma_k^{TA} = \Phi_k(\rho^TA) = \Phi_k(\rho^{TA}) = (1 + a) \Phi_k(\rho^+) - a \Phi_k(\rho^-). \quad (21)$$
It is clear that $N(\sigma_k) \leq q_k a/p_k$, $q_k = \text{Tr}\Phi_k(\rho^-)$. Thus, if Eq. \(7\) holds, then $N(\sigma_k) = q_k a/p_k$, and thus $\Phi_k(\rho^+)\Phi_k(\rho^-) = \Phi_k(\rho^-)\Phi_k(\rho^+) = 0$. Take $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \sum_j \lambda_j |j\rangle_k$ as the Schmidt decomposition of $|\psi\rangle$. Then $|\psi\rangle\langle\psi^T_\alpha| = \sum_j \lambda_j^2 |j\rangle_{\alpha} \otimes |j\rangle_{\beta} \langle j|_\alpha + \sum_{i<j} \lambda_i \lambda_j |i\rangle_{\alpha} |j\rangle_{\alpha} \langle i|_\alpha \langle j|_\beta$, where $|\psi^T_\alpha\rangle = \frac{1}{\sqrt{2}} (|j\rangle + |j\rangle_\beta)$. Denoting by $M_k |j\rangle_k \equiv |j\rangle_\beta$, it follows that $\langle i|_\alpha \langle j|_\beta$ holds for any $i$ and $j$, from which we can conclude that $M_k$ is a scalar multiple of some unitary operator. This guarantees that $\Phi_k$ is either a local unitary operation or $\Phi_k(\cdot) = q_k I^A \otimes U_k^B(\cdot) I^A \otimes (U_k^B)^\dagger$ with $\sum_k q_k = 1$ provided that $N(\rho) = \sum_k p_k N(\sigma_k)$. Therefore, $N$ decreases strictly on average under LOCC.

**Proposition 5.** The logarithmic negativity $E_N$ decreases strictly under LOCC on average, but it is not a SEM.

**Proof.** It is clear that $E_N$ decreases strictly under stochastic LOCC on average since the logarithm is strictly concave. But $E_N$ is not convex \[13\], namely, it is not an entanglement monotone, therefore it is not a SEM.

Another important entanglement monotone that is not derived from the convex roof extension is the relative entropy of entanglement \[8, 15\]:

$$E_r(\rho^{AB}) \equiv \min_{\sigma^{AB}} S(\rho^{AB}||\sigma^{AB}), \tag{22}$$

where $S(\rho^{AB}||\sigma^{AB}) \equiv \text{Tr}[\rho^{AB} \ln \rho^{AB} - \ln \sigma^{AB}]$ is the quantum relative entropy and the minimum is taken over all separable states $\sigma^{AB}$ in $S^{AB}$. This measure, as one might expect, is a SEM.

**Theorem 6.** $E_r$ is a SEM.

**Proof.** Let $\mathcal{H}^C$ be an extended Hilbert space of $\mathcal{H}^{AB}$, let $\{|i_{\alpha}\rangle\}$ be an orthonormal basis in $\mathcal{H}^C$, and let $|\alpha\rangle$ be a unit vector. For any CPTP map $\Phi(\rho^{AB}) = \sum_i V_i^{\rho^{AB}}V_i^{\dagger}$, there exists a unitary operator $U$ acting on $\mathcal{H}^{ABC}$ such that \[20, 21\]

$$U(A \otimes P_\alpha)U^\dagger = \sum_{i,j} V_i A V_j^{\dagger} \otimes |i_{\alpha}\rangle \langle j_{\alpha}|. \tag{23}$$

It is clear that

$$\text{Tr}_C \left[ I^{AB} \otimes P_{\alpha} U(\rho^{AB} \otimes P_\alpha) U^\dagger I^{AB} \otimes P_{\alpha} \right] = V_i^{\rho^{AB}} V_i^{\dagger} = \Phi_i(\rho^{AB}) = p_i \rho_i^{AB}. \tag{24}$$

According to the proof of Theorem 2 in Ref. \[15\], we only need to verify that if

$$\sum_i p_i S(\rho_i^{AB}/p_i || \sigma_i^{AB}/q_i) = S(\rho^{AB}/|| \sigma^{AB}) \tag{24}$$

holds for any $\rho^{AB}$ and $\sigma^{AB}$, then

$$\Phi_i(\lambda X) \equiv V_i X V_i^{\dagger} = p_i U X U^\dagger \tag{25}$$

for some unitary operator $U$, where $q_i \sigma_i^{AB} = \Phi_i(\sigma^{AB})$. Note that

$$\sum_i p_i S(\rho_i^{AB}/p_i || \sigma_i^{AB}/q_i) \leq \sum_i p_i S(\rho_i^{AB}/p_i || \sigma_i^{AB}/q_i) + \sum_i p_i \ln \frac{p_i}{q_i} = \sum_i S(p_i \rho_i^{AB} || q_i \sigma_i^{AB}) \leq \sum_i S[I^{AB} \otimes P_{\alpha} U(\rho^{AB}) \otimes P_{\alpha} U(\sigma^{AB}) \otimes P_{\alpha} U(\rho^{AB}) \otimes P_{\alpha}]$$

$$= S \left[ U(\rho^{AB} \otimes P_{\alpha}) U^\dagger \right] = S \left( \rho^{AB} || \sigma^{AB} \right), \tag{26}$$

thus \[20\] holds and leads to $\sum_i p_i \ln \frac{p_i}{q_i} = 0$, which is equivalent to $p_i = q_i$ for any $i$. Therefore $\Phi_i$ has the form as in \[23\]. Taking $V_j = V_j^{A} \otimes V_j^{B}$, the proof is completed.

The squashed entanglement $E_{sq}$ \[16\] is an additive entanglement monotone and has a nice operational meaning. For any state $\rho^{AB} \in S^{AB}$, $E_{sq}$ is defined by \[16\]

$$E_{sq}(\rho^{AB}) \equiv \inf_E \left\{ \frac{1}{2} I(A;B|E) : \text{Tr}_E \rho^{AB} = \rho^{AB} \right\}, \tag{27}$$

where $I(A;B|E) = S(\rho^A) + S(\rho^B) - S(\rho^{AB}) - S(\rho^E)$, $S(\cdot)$ denotes the von Neumann entropy and the infimum is taken over all extensions of $\rho^{AB}$ of $\rho^{AB}$. We show below that $E_{sq}$ is also a SEM with the assumption that it can be attained by some optimal extension [i.e., $E_{sq}(\rho^{AB}) = \frac{1}{2} I(A;B|E)$ for some extension $\rho^{AB}$]. Note that, if there does not exist some optimal extension, whether or not $E_{sq}$ is a SEM remains open since it is defined in terms of the infimum process over all states extension which cannot give an accurate equality between the state and its extension state for the conditional mutual information. However, we still do not know such an extension exists or not for any state \[16\].

**Theorem 7.** If $E_{sq}(\rho^{AB})$ can be attained by optimal extension for any state $\rho^{AB} \in S^{AB}$, then $E_{sq}$ is a SEM.

**Proof.** From the proof of Proposition 3 in Ref. \[16\], if $E_{sq}(\rho) = \sum_k p_k E_{sq}(\rho_k)$ and the associated LOCC is stochastic, then we must have $I(A;B|E) = 0$ (we use the same notations as in Ref. \[16\]), it follows that $\rho^{AB}$ is a Markov state according to the structure of states that satisfying the strong subadditivity of entropy \[22\], a contradiction. Thus the LOCC is a local unitary operation or a convex mixture of local unitary operations.
At last, we present a list of the properties of all entanglement measures that are well-known by now for convenience of readers (see Table I). As one might expect, almost all the entanglement measures are decreasing strictly under LOCC on average for pure states. In addition, one can see from the table that, apart from the strict concavity of the associated function \( h \), monogamy is another property that is also closely related with the strict monotonicity of LOCC. We also found that other properties, such as additivity, convexity and faithfulness, seem not to be the nature of the entanglement measures so far.

To summarize, we explored the action of entanglement under LOCC for many entanglement measures so far, and we showed that the axiomatic definition of entanglement monotone can be improved: \( E \) is defined to be an entanglement monotone if it is convex, vanishes on separable states, and decreases strictly on average under LOCC in the sense of \([9]\). Together with the result in Ref. \([10]\), our results here support the conclusion that entanglement is monogamous. But we still can not prove whether the squashed entanglement (it is defined via the infimum over all extensions), entanglement of distillation, and the entanglement cost are strict entanglement monotones or not.

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