Integrable Structure behind WDVV Equations

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Abstract

An integrable structure behind Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations is identified with reduction of a Riemann-Hilbert problem for a homogeneous $\hat{GL}(N,\mathbb{C})$ loop group. Reduction requires the dressing matrices to be fixed points of a loop group automorphism of order two resulting in a sub-hierarchy of $\mathfrak{gl}(N,\mathbb{C})$ hierarchy containing only odd symmetry flows. The model possesses Virasoro symmetry and imposing Virasoro constraints ensures homogeneity property of the Darboux-Egoroff structure. Dressing matrices of the reduced model provide solutions of the WDVV equations.

1. Introduction

Massive topological field theories can be classified locally by the Darboux–Egoroff system of differential equations:

$$\frac{\partial}{\partial u_k} \beta_{ij} = \beta_{ik} \beta_{kj}, \quad i \neq k \neq j, \quad \sum_{k=1}^{N} \frac{\partial}{\partial u_k} \beta_{ij} = 0, \quad i \neq j,$$

$$\beta_{ij} = \beta_{ji},$$

where in addition one also assumes that

$$\sum_{k=1}^{N} u_k \frac{\partial}{\partial u_k} \beta_{ij} = -\beta_{ij}.$$

This was observed by B. Dubrovin in [2], [3]. The $\beta_{ij}, 1 \leq i, j \leq N$ with $i \neq j$ are called the “rotation coefficients”. To be more precise, these Darboux–Egoroff equations are the compatibility conditions of the following linear system of differential equations depending on a spectral parameter $\lambda$:

$$\frac{\partial}{\partial u_i} A_{jk}(u, \lambda) = \beta_{ji}(u) A_{ik}(u, \lambda); \quad i \neq j = 1, \ldots, N$$

$$\sum_{j=1}^{N} \frac{\partial}{\partial u_j} A_{ik}(u, \lambda) = \lambda A_{ik}(u, \lambda); \quad i = 1, \ldots, N$$

for each $k = 1, \ldots, N$. Let $V(u) = (\beta_{ij}(u)(u_j - u_i))_{1 \leq i,j \leq n}$ and assuming that this matrix is diagonalizable, one can construct a local semisimple Frobenius manifold, which provides
an algebraic formulation of a massive topological field theory as follows. Find an invertible power series
\[ A(u, \lambda) = A_0(u) + A_1(u)\lambda + A_2(u)\lambda^2 + \cdots, \] (6)
which is a solution of this system, then
\[ \sum_{k=1}^{N} u_k \frac{\partial}{\partial u_k} A_0(u) = V(u)A_0(u). \] (7)
It is then straightforward to check that \( A_0(u)^{-1}V(u)A_0(u) \) is a constant matrix, hence there exists an invertible complex matrix \( S \) such that
\[ \mu = \sum_{i=1}^{N} \mu_i E_{ii} = S^{-1}A_0(u)^{-1}V(u)A_0(u)S \] (8)
is a diagonal matrix. Now let \( A(u) = A_0(u)S \) with \( A(u) = (a_{ij}(u))_{1 \leq i,j \leq N} \), then
\[ \sum_{k=1}^{N} u_k \frac{\partial}{\partial u_k} A(u) = A(u)\mu. \] (9)
Dubrovin then showed [3] that there exists on the domain \( u_i \neq u_j \) and \( a_{11}a_{21}\cdots a_{N1} \neq 0 \) a local semisimple Frobenius manifold with scaling dimensions \( \mu_{\alpha} - \mu_1 \) (see also [4]). The diagonal metric is given by
\[ ds^2 = \sum_{i=1}^{n} a_{i1}^2(u)(du_i)^2. \] (10)
The flat coordinates \( x^\alpha \) are determined by
\[ \sum_{\beta} \eta_{\alpha\beta} \frac{\partial x^\beta(u)}{\partial u_i} = a_{i1}(u)a_{i\alpha}(u), \] (11)
where
\[ \eta_{\alpha\beta} = \sum_{i=1}^{n} a_{i\alpha}(u)a_{i\beta}(u). \] (12)
Finally, the structure constants
\[ c_{\alpha\beta\gamma}(x(u)) = \frac{\partial^3 F(x(u))}{\partial x^\alpha \partial x^\beta \partial x^\gamma} = \sum_{i=1}^{n} \frac{a_{i\alpha}(u)a_{i\beta}(u)a_{i\gamma}(u)}{a_{i1}(u)} \] (13)
are the third order derivatives of the WDVV prepotential \( F(x) \).

In this paper, we use a Riemann–Hilbert problem for the loop group of \( GL(N) \) to obtain a similar structure. The symmetry conditions (2) for the rotation coefficients are obtained by using the Cartan involution on the loop group. The model possesses Virasoro symmetry and imposing Virasoro constraints ensures the homogeneity property (3) of the Darboux–Egoroff structure.
2. Symmetry flows, Riemann-Hilbert problem and the \( \tau \) function

We will introduce the \( G = \hat{G}L(N) \) symmetry flows in terms of (extended) Riemann-Hilbert problem involving two subgroups of the Lie loop group \( G \) defined as:

\[
G_- = \left\{ g \in G | g(\lambda) = 1 + \sum_{i<0} g^{(i)} \right\}, \quad G_+ = \left\{ g \in G | g(\lambda) = \sum_{i \geq 0} g^{(i)} \right\},
\]

here \( g_i \) has grading \( i \) with respect to a homogeneous gradation defined by a derivation \( \lambda d/d\lambda \). It also holds that \( G_+ \cap G_- = I \). Let the loop algebra corresponding to \( G \) be \( \hat{G} = gl(N) \). This algebra splits into the direct sum \( \hat{G} = \hat{G}_+ \oplus \hat{G}_- \) with respect to the homogeneous gradation, where \( \hat{G}_\pm \) are Lie algebras associated with the subgroups \( G_\pm \).

Let the “un-dressed” wave (matrix) function be

\[
\Psi_0(u, \lambda) = \exp \left( \sum_{j=1}^{N} \sum_{n=1}^{\infty} E_{jj}^{(n)} u_j^{(n)} \right) = \sum_{j=1}^{N} E_{jj} e^{\sum_{n=1}^{\infty} \lambda^n u_j^{(n)}}
\]

We now define an extended Riemann-Hilbert factorization problem for the homogeneous gradation:

\[
\Psi_0(u, \lambda) g(\lambda) = \Theta^{-1}(u, \lambda) M(u, \lambda)
\]

where \( g : S^1 \to G_- G_+ \) while \( \Theta \in G_- \), \( M \in G_+ \). We use the multi-time notation with \( (u) = (u_1, \ldots, u_N) \) to denote \( N \) multi-flows \( u_j \). Each argument \( u_j \), \( j = 1, \ldots, N \) is an abbreviated notation for the multi-flows \( u_j^{(n)} \) with \( n \) running from 1 to \( \infty \).

For the symmetry flows one derives from (16) expressions:

\[
\frac{\partial}{\partial u_j^{(n)}} \Theta(u, \lambda) = - \left( \Theta E_{jj}^{(n)} \Theta^{-1} \right)_- \Theta(u, \lambda)
\]

\[
\frac{\partial}{\partial u_j^{(n)}} M(u, \lambda) = \left( \Theta E_{jj}^{(n)} \Theta^{-1} \right)_+ M(u, \lambda)
\]

where \( (\ldots)_\pm \) denote the projections onto \( \hat{G}_\pm \). Note, that \( \sum_{j=1}^{N} \partial \Theta / \partial u_j^{(n)} = 0 \) for any \( n > 0 \).

In the following we will denote for brevity \( u_j = u_j^{(1)} \) and \( \partial_j = \partial / \partial u_j^{(1)} \) for \( j = 1, \ldots, N \).

It is natural to associate to the Riemann-Hilbert factorization a one-form:

\[
\mathcal{J} = - \sum_{j=1}^{N} \sum_{n=1}^{\infty} \text{Res}_\lambda \left( \frac{1}{1+d\lambda} E_{jj}^{(n)} \right) du_j^{(n)} = - \text{Res}_\lambda \left( \frac{1}{1+d\lambda} \sum_{n=1}^{\infty} \text{tr} \left( \Theta^{-1} d\Theta \Psi_0^{-1} d\Psi_0 \right) \right)
\]

which due to equation (17) is closed: \( d\mathcal{J} = 0 \) with \( d = \sum_{j=1}^{N} \sum_{n=1}^{\infty} \frac{\partial}{\partial u_j^{(n)}} \) \( du_j^{(n)} \). Hence, locally, \( \mathcal{J} \) can be written as \( d \) of a zero-form being some function of the dynamical variables of the model. Conventionally, this function is denoted as the logarithm of the \( \tau \)-function:

\[
\mathcal{J} = - d \log \tau = - \sum_{j=1}^{N} \sum_{n=1}^{\infty} \frac{\partial \log \tau}{\partial u_j^{(n)}} du_j^{(n)}
\]
This equation can effectively be integrated to yield an explicit expression for the \( \tau \)-function. To do this it is required to extend the formalism from the loop groups to the centrally extended groups. As in case of the AKNS hierarchy with the \( \tau \)-function defined in [8] we embed the framework in the centrally extended group with elements denoted as \( \hat{g} \) and propose:

\[
\tau (u) = \langle 0| \hat{\Psi}_0 \hat{g} |0 \rangle \quad (21)
\]

where \( |0\rangle \) is the highest weight state such that \( X_{\geq 0} |0\rangle = 0 \) and \( \langle 0|X_{\leq 0} = 0 \) and furthermore \( \langle 0|\hat{k}|0\rangle = 1 \), where \( \hat{k} \) denotes the direction associated to the central element within the affine Kac-Moody algebra.

Let \( \theta^{(-1)} \) be the grade \(-1\) term of expansion:

\[
\Theta = 1 + \theta^{(-1)} \lambda^{-1} + \theta^{(-2)} \lambda^{-2} + \ldots \quad (22)
\]

From equations (19) and (20) (for \( n = 1 \)) one finds:

\[
(\theta^{(-1)})_{ii} = -\partial_i \log \tau \quad (23)
\]

Introduce now the “rotation coefficients” \( \beta_{ij} \) with \( i \neq j \) as the off-diagonal elements of \( \theta^{(-1)} \):

\[
\beta_{ij} = (\theta^{(-1)})_{ij}, \quad i \neq j = 1, \ldots, N \quad (24)
\]

For \( \theta^{(-1)} \) matrix we find from (17):

\[
\partial_j \theta^{(-1)} = [E_{jj}, \theta^{(-2)}] + [\theta^{(-1)}, E_{jj}] \theta^{(-1)} \quad (25)
\]

from which it follows that:

\[
\partial_j (\theta^{(-1)})_{ik} = (\theta^{(-1)})_{ij} (\theta^{(-1)})_{jk}, \quad j \neq i, \ j \neq k \quad (26)
\]

Accordingly, the “rotation coefficients” \( \beta_{ij} \) given by (24) satisfy (1).

As found in [1], the matrix elements of the dressing matrix \( \Theta = (\theta_{ij})_{1 \leq i,j \leq N} \) are given by:

\[
\theta_{ij}(u, \lambda) = \frac{1}{\tau(u)} \times \left\{ \frac{1}{\tau(u)} \beta_{ij}(u_{\lambda} - [\lambda^{-1}]) \tau(u_{\lambda} - [\lambda^{-1}]) \right\} \quad (27)
\]

where the multi-flow arguments \( u_j \) were shifted according to \( (u_j - [\lambda^{-1}]) = (u_j^{(1)} - 1/\lambda, u_j^{(2)} - 1/2\lambda^2, \ldots) \).

The matrix coefficients of the wave matrix \( \Psi \):

\[
\Psi(u, \lambda) = \Theta(u, \lambda) \Psi_0(u, \lambda) \quad (28)
\]

and the matrix coefficients of \( M(u, \lambda) = \Psi(u, \lambda) g(\lambda) \) satisfy the linear system (4)-(5), which are the compatibility conditions of equations (1). This is the Darboux-Egoroff system provided we also impose the symmetry condition on the rotation coefficients. To do this we reduce the integrable hierarchy associated with the flow equation equations (17),(18) using the automorphism \( \sigma \) :

\[
\sigma (X(\lambda)) = \left( (X(-\lambda))^T \right)^{-1}, \quad X \in G = \hat{GL}(N) \quad (29)
\]
which leaves the evolution equations (17)-(18) invariant for odd flows only (labeled by \( n \) being an odd integer). Accordingly, we define the integrable sub-hierarchy by constraining the dressing matrices \( \Theta(u, \lambda) \) and \( M(u, \lambda) \) to the fixed points of the loop group automorphism \( \sigma \) :

\[
\Theta^{-1}(u, \lambda) = \Theta^T(u, -\lambda), \quad M^{-1}(u, \lambda) = M^T(u, -\lambda)
\]

with \( \Theta(u, \lambda) \) and \( M(u, \lambda) \) depending only on odd coordinates \( u \): ( \( u^{(2k+1)}_j, k = 0, 1, \ldots \) ). The odd flows of the reduced sub-hierarchy preserve the conditions (30). The fixed points of the automorphism \( \sigma \) form a subgroup of \( G = \hat{GL}(N) \), called a twisted loop group of \( GL(N) \) (see e.g. [7]). For the first term \( \theta_{(-1)} \) of the expansion (22) the constraint (30) implies that \( \theta_{(-1)} = \theta_{(-1)}^T \). Accordingly, the “rotation coefficients” \( \beta_{ij} \) associated to the subhierarchy via (24) are symmetric and give rise to the Darboux-Egoroff metrics.

3. Condition of conformal invariance and homogeneity

We propose here the Virasoro constraints in the setting of the Riemann-Hilbert problem approach to the integrable models. In this setting the Virasoro constraints are seen to arise as symmetries of the hierarchy.

The action of the Virasoro symmetry flows on the dressing matrices is given by:

\[
\begin{align*}
\delta^V_k \Theta(u, \lambda) &= (\Theta L_k \Theta^{-1}) \Theta - \sum_{j=1}^{N} \sum_{n=1}^{\infty} n u^{(n)}_j \frac{\partial \Theta}{\partial u_j^{(n+k)}} \\
\delta^V_k M(u, \lambda) &= -(\Theta L_k \Theta^{-1}) M - \sum_{j=1}^{N} \sum_{n=1}^{\infty} n u^{(n)}_j \frac{\partial M}{\partial u_j^{(n+k)}} 
\end{align*}
\]

where

\[
L_k = -\lambda^{k+1} \frac{d}{d\lambda} - \sum_{j=1}^{N} \sum_{n=1}^{\infty} n u^{(n)}_j E^{(k+n)}_{jj}, \quad k \geq -1
\]

\[
U^{(n)} = \text{diag}(u^{(n)}_1, \ldots, u^{(n)}_N) = \sum_{i=1}^{N} u^{(n)}_i E_{ii}
\]

and \( l_k = -\lambda^{k+1} \frac{d}{d\lambda} \). Both \( l_k \) and \( L_k \) for \( k \geq -1 \) generate a subalgebra of the Witt (centerless Virasoro) algebra:

\[
[l_k, l_r] = (k - r) l_{k+r}, \quad [L_k, L_r] = (k - r) L_{k+r}
\]

The signs in eqs. (31) and (32) have been chosen in such a way that the transformation \( \delta^V_k \) also satisfies the Virasoro algebra \( [\delta^V_k, \delta^V_r] = (k - r) \delta^V_{k+r} \) when applied on the dressing matrices \( \Theta \) and \( M \). Furthermore, the transformations \( \delta^V_k \) commute with the flows \( \partial/\partial u^{(n)}_j \):

\[
[\delta^V_k, \frac{\partial}{\partial u^{(n)}_j}] \Theta(u, \lambda) = 0
\]

Accordingly, the Virasoro transformations from eqs. (31) and (32) constitute additional symmetries of the integrable model defined by the Riemann-Hilbert factorization problem.
One can show that only the even Virasoro flows will preserve the conditions (30) and define additional symmetries of the reduced sub-hierarchy associated with the twisted loop group of $GL(N)$. We can therefore consistently impose the constraints $\delta^V_k \Theta(u, \lambda) = 0$ with even $k$. For $k = 0$ the equation (31) specializes to:

$$\delta^V_0 \Theta(u, \lambda) = \lambda \frac{d}{d\lambda} \Theta - \sum_{j=1}^{N} \sum_{n=1}^{\infty} n u_j^{(n)} \frac{\partial \Theta}{\partial u_j^{(n)}}$$

(36)

as follows from identity $-(\Theta \lambda d\Theta^{-1}/d\lambda) \cdot \Theta = \lambda d\Theta/d\lambda$. Hence for $k = 0$ the constraint $\delta^V_k \Theta(u, \lambda) = 0$ takes a form

$$\left(\lambda \frac{d}{d\lambda} - E_0\right) \Theta(u, \lambda) = 0$$

(37)

where we have introduced the Euler vector field $E_0$:

$$E_0 = \sum_{j=1}^{N} \sum_{n=1}^{\infty} n u_j^{(n)} \frac{\partial}{\partial u_j^{(n)}}$$

(38)

The homogeneity condition (37) is also a condition for the conformal Frobenius manifold.

By projecting in equation (36) on the terms with $-1$ grade we obtain:

$$\delta^V_0 \theta^{(-1)} = -\theta^{(-1)} - \sum_{j=1}^{N} \sum_{n=1}^{\infty} n u_j^{(n)} \frac{\partial \theta^{(-1)}}{\partial u_j^{(n)}}$$

(39)

where $\theta^{(-1)}$ is the grade $-1$ term of expansion: (22).

Hence, the condition (37) implies that

$$E_0 \theta^{(-1)} = -\theta^{(-1)}$$

(40)

Plugging relations (23) and (24) into equation (39) we obtain:

$$\delta^V_i \partial_i \log \tau = \partial_i \delta^V_0 \log \tau = -\partial_i \sum_{j=1}^{N} \sum_{n=1}^{\infty} n u_j^{(n)} \frac{\partial \log \tau}{\partial u_j^{(n)}}, \quad i = 1, \ldots, N$$

(41)

$$\delta^V_{kl} = -\beta_{kl} - \sum_{j=1}^{N} \sum_{n=1}^{\infty} n u_j^{(n)} \frac{\partial \beta_{kl}}{\partial u_j^{(n)}}$$

(42)

4. Virasoro flows on the $\tau$ function

This subsection studies action of the general Virasoro transformation on the $\tau$-function of the underlying integrable system.

Recall an expression for derivatives of $\log \tau$ given in (19) and (20). Applying $\partial_i$ on both sides of eq. (20) we obtain an identity:

$$\frac{\partial^2 \log \tau}{\partial u_j^{(n)} \partial u_i} = \text{Res}_\lambda \left( \text{tr} \left( \Theta E_j^{(n)} \Theta^{-1} E_i \right) \right) \quad i, j = 1, \ldots, N, \quad n > 0$$

(43)
Combining eqs. (31) and (19)-(20) we find
\[ \delta_k \frac{\partial \log \tau}{\partial u_j^{(n)}} = \text{Res}_\lambda \left( \text{tr} \left( \Theta E_{jj}^{(n)} \Theta^{-1} \frac{d}{d\lambda} (\Theta L_k \Theta^{-1}) \right) \right) \] (44)

Since \( \lambda d(\lambda^{-1})/d\lambda = -\lambda^{-1} \) the last equation becomes
\[ \delta_k \partial_i \log \tau = -\text{Res}_\lambda \left( \text{tr} \left( E_{ii} \Theta l_k \Theta^{-1} \right) \right) - \sum_{j=1}^{N} \sum_{n=1}^{\infty} n u_j^{(n)} \partial_i \text{Res}_\lambda \left( \text{tr} \left( \Theta E_{jj}^{(k+n)} \Theta^{-1} E_{ii} \right) \right) \] (45)

Using the identity (43) we rewrite the last equation as
\[ \delta_k \partial_i \log \tau = -\text{Res}_\lambda \left( \text{tr} \left( E_{ii} \Theta l_k \Theta^{-1} \right) \right) - \sum_{j=1}^{N} \sum_{n=1}^{\infty} n u_j^{(n)} \partial_i \frac{\partial \log \tau}{\partial u_j^{(k+n)}} \] (46)

The cases \( k = 0, 1 \) were calculated and eq. (43) gives :
\[ \partial_i \delta_k \log \tau = -\partial_i \sum_{j=1}^{N} \sum_{n=1}^{\infty} n u_j^{(n)} \frac{\partial \log \tau}{\partial u_j^{(n+k)}} \tag{47} \]

and
\[ \partial_i \delta_{-1} \log \tau = -\partial_i \sum_{j=1}^{N} \sum_{n=2}^{\infty} n u_j^{(n)} \frac{\partial \log \tau}{\partial u_j^{(n-1)}} \] (48)

Only the case \( k = 0 \) applies to the reduced case.

From eqs. (31) and (32) we find:
\[ \delta_k \left( \Theta^{-1} M \right) = -\Theta^{-1} (\Theta L_k \Theta^{-1})_{} M - \Theta^{-1} (\Theta L_k \Theta^{-1})_{} M = -L_k \left( \Theta^{-1} M \right) \]
\[ = \lambda^{k+1} \frac{d}{d\lambda} \left( \Theta^{-1} M \right) - \sum_{j=1}^{N} \sum_{n=1}^{\infty} n u_j^{(n)} E_{jj}^{(n+k)} \left( \Theta^{-1} M \right) \] (49)

Recalling the factorization problem (19) we find
\[ \delta_k \left( \Theta^{-1} M \right) = \Psi_0(u, \lambda) \lambda^{k+1} \frac{dg}{d\lambda} \] (50)

since
\[ \lambda^{k+1} \frac{d}{d\lambda} \Psi_0(u, \lambda) = \sum_{j=1}^{N} \sum_{n=1}^{\infty} n u_j^{(n)} E_{jj}^{(n+k)} \Psi_0(u, \lambda) \] (51)

Recall, the definition (21) of the \( \tau \)-function \( \tau_g(u) = \langle 0 | \tilde{\Psi}_0 g | 0 \rangle \), rewritten here in the form which explicitly shows dependence of \( \tau \) on the loop group element \( g(\lambda) \).

It now appears that
\[ \delta_k \tau_g = \langle 0 | \lambda^{k+1} \tilde{\Psi}_0 g | 0 \rangle = \tau_{\lambda^{k+1} \frac{dg}{d\lambda}} \] (52)
So under infinitesimal $S^1$ diffeomorpism $g \to g + \lambda^{k+1} dg/d\lambda$ the $\tau$-function transforms as
\[
\tau_g \to \tau_{g + \lambda^{k+1} dg/d\lambda} = \tau_g + \delta^V \tau_g
\] (53)

5. Virasoro symmetry and dressing

Let
\[
\delta^{(n)}_j = \frac{\partial}{\partial u^{(n)}_j} - E_{jj}^{(n)}, \quad \delta_j = \delta^{(1)}_j = \partial_j - \lambda E_{jj}, \quad j = 1, \ldots, N
\] (54)
be a set of commuting operators:
\[
[\delta^{(n)}_j, \delta^{(k)}_i] = 0,
\] (55)

The above operators and the Virasoro operators $L_k$ from (33) commute with each other:
\[
[\delta^{(n)}_j, L_k] = 0, \quad i, j = 1, \ldots, N
\] (56)

Both operators (54) and (33) annihilate the bare wave function $\Psi_0$ from (15):
\[
L_k \Psi_0 = 0, \quad \delta^{(n)}_j \Psi_0 = 0, \quad j = 1, \ldots, N.
\] (57)

Similarly the operators:
\[
D^{(n)}_j = \Theta \delta^{(n)}_j \Theta^{-1}, \quad L_k = \Theta L_k \Theta^{-1}
\] (58)
satisfy the commutation relations (56) and (34) and annihilate the wave (matrix) function $\Psi(u, \lambda)$ defined in (28):
\[
L_k \Psi = 0, \quad D^{(n)}_j \Psi = 0, \quad j = 1, \ldots, N.
\] (59)

Since
\[
\Theta \partial_j \Theta^{-1} = \partial_j + \Theta (\partial_j \Theta^{-1}) = \partial_j + \left(\Theta \lambda E_{jj} \Theta^{-1}\right)_-
\] (60)
we obtain
\[
D_j = D^{(1)}_j = \Theta (\partial_j - \lambda E_{jj}) \Theta^{-1} = \partial_j - \left(\Theta \lambda E_{jj} \Theta^{-1}\right)_+ = \partial_j - \lambda E_{jj} - V_j
\] (61)

where
\[
V_j \equiv [\theta^{(-1)}_j, E_{jj}], \quad (V_j)_{kl} = (\delta_{ij} - \delta_{kj}) \beta_{kl}
\] (62)

for $\theta^{(-1)}$ defined in (23).

Consider
\[
\lambda \frac{d\Psi}{d\lambda} = \lambda \frac{d\Theta}{d\lambda} \Psi_0 + \Theta \sum_{j=1}^N \sum_{n=1}^\infty nu^{(n)}_j E_{jj}^{(n)} \Psi_0
\] (63)

Recall, now from (36) that the condition $\delta^V_{k=0} \Theta(u, \lambda) = 0$ (37) implies
\[
\lambda \frac{d\Theta}{d\lambda} = -(\Theta \sum_{j=1}^N \sum_{n=1}^\infty nu^{(n)}_j E_{jj}^{(n)} \Theta^{-1})_+ \Theta
\] (64)
and accordingly

\[ \frac{\lambda d\Psi}{d\lambda} = (\Theta \sum_{j=1}^{N} \sum_{n=1}^{\infty} nu_j^{(n)} E_{jj}^{(n)} \Theta^{-1}) \cdot \Psi \]  

(65)

Let us now study the operator \( \mathcal{L}_{k=-1} = \Theta L_{-1} \Theta^{-1} \) from (58) for \( \Theta \) satisfying the homogeneity condition (64).

\[
\mathcal{L}_{-1} = -\frac{d}{d\lambda} + \Theta^{-1} \left[ \Theta^{-1} \sum_{j=1}^{N} \sum_{n=1}^{\infty} nu_j^{(n)} E_{jj}^{(n)} \Theta^{-1} \right] - \sum_{j=1}^{N} \sum_{n=1}^{\infty} nu_j^{(n)} E_{jj}^{(n)} \Theta^{-1} \Theta^{-1} 
\]

(66)

and for \( u_j^{(n)} = 0, n > 1 \) one obtains

\[
\mathcal{L}_{-1} = -\frac{d}{d\lambda} + U + \lambda^{-1}[\theta^{(-1)}, U] = -\frac{d}{d\lambda} + U + \lambda^{-1}V 
\]

(67)

The components of the matrix \( V = [\theta^{(-1)}, U] \) are

\[
V_{ij} = (u_j - u_i) \theta_{ij}^{(-1)} = (u_j - u_i) \beta_{ij}, \quad i, j = 1, \ldots, N 
\]

(68)

It follows from the construction that the wave (matrix) function \( \Psi(u, \lambda) \) defined in (59) is annihilated by \( \mathcal{L}_{-1} \) and \( \mathcal{D}_j \)

\[
\mathcal{L}_{-1} \Psi(u, \lambda) = 0 \rightarrow \frac{d\Psi}{d\lambda} = (U + \lambda^{-1}V)\Psi 
\]

(69)

and

\[
\mathcal{D}_j \Psi(u, \lambda) = 0 \rightarrow \frac{\partial\Psi}{\partial u_j} = (\lambda E_{jj} + V_j)\Psi 
\]

(70)

Compatibility of the above equations amounts to \([\mathcal{D}_j, \mathcal{L}_{-1}] = 0\), which follows by dressing of \([\delta_j, \mathcal{L}_{-1}] = 0\). From commutation relations \([\mathcal{D}_j, \mathcal{L}_{-1}] = 0\) and \([\mathcal{D}_i, \mathcal{D}_j] = 0\) one obtains:

\[
\partial_j V = [V_j, V] 
\]

\[
\partial_j V_i = \partial_i V_j + [V_j, V_i] 
\]

\[
[V, E_{jj}] = [V_j, U] 
\]

(71-73)

6. Isomonodromic Tau function \( \tau_I \)

We now consider the coefficients \(-\partial_j \log \tau\) of the one-form \( \mathcal{J} \) from (19) with the conformal constraint (64) imposed. Recall, that this constraint follows from the homogeneity condition (37) and in view of relations (11) and (12) it holds in the reduced case that \( E_0 \tau = \mu \tau \) and \( E_0 \beta_{kl} = -\beta_{kl} \) where \( E_0 = \sum_{j=1}^{N} u_j \partial/\partial u_j \) and \( \mu \) is independent of variables \( u_j, j = 1, \ldots, N \).
Plugging relation (64) into the defining equations (19) and (20) we obtain for \( n = 1 \):

\[
\partial_j \log \tau = -\text{Res}_\lambda \left( \text{tr} \left( E_{jj}(\Theta \sum_{j=1}^{N} u_n^{(n)} E_{jj}^{(n)} \Theta^{-1}) \right) \right), \quad j = 1, \ldots, N \tag{74}
\]

In the rest of the paper we set all \( u_i^{(k)} = 0 \) for \( k > 1 \) leaving the model defined only in terms of the canonical coordinates \( u_j, j = 1, \ldots, N \) and assume from now on that \( \sigma(g(\lambda)) = g(\lambda) \) and that (39) holds.

The formula (74) simplifies significantly and takes the form of

\[
\partial_j \log \tau = -\text{Res}_\lambda \left( \text{tr} \left( E_{jj}(\Theta u U \Theta^{-1}) \right) \right) = -\text{tr} \left( E_{jj} \left( [\theta^{(-2)}, U] + \frac{1}{2} [\theta^{(-1)}, [\theta^{(-1)}, U]] \right) \right) \tag{75}
\]

Using invariance of the trace \( \text{tr}(A[B, C]) = \text{tr}([A, B]C) \) we obtain expression:

\[
\partial_j \log \tau = \frac{1}{2} \text{tr} (V_i V) \tag{76}
\]

with \( V_j, V \) matrices defined in (62) and (68) and related through the commutation relation (73).

The equation (74) can be cast into the Hamiltonian form (see also [3])

\[
\partial_j V = \{ V, H_j \} \tag{77}
\]

with respect to the standard Poisson bracket \( \{ \cdot, \cdot \} \) on \( \mathfrak{so}(N) \) and with the Hamiltonians

\[
H_i = \frac{1}{2} \sum_{j \neq i} \frac{V_{ij}^2}{u_i - u_j}. \tag{78}
\]

The commutation relation (73) in components gives a relation

\[
(u_k - u_j)(V_i)_{kj} = (\delta_{ik} - \delta_{ij}) V_{kj}, \quad k \neq j = 1, \ldots, N \tag{79}
\]

which can be used to prove that

\[
\frac{1}{2} \text{tr} (V_i V) = \sum_{j \neq i} \frac{V_{ij} V_{ji}}{u_i - u_j} = -\sum_{j \neq i} \frac{V_{ij}^2}{u_i - u_j} = -2H_i \tag{80}
\]

From equations (71) and (72) it follows that \( d \text{tr} \sum_{i=1}^{N} (V_i V) du_i = 0 \) and the one-form \( \sum_{i=1}^{N} H_i du_i \) is closed. Locally, this one-form can be represented by the derivative of the logarithm of the so-called isomonodromy tau-function \( \tau_I \)

\[
d \log \tau_I = \sum_{i=1}^{N} H_i du_i = -\frac{1}{2} d \log \tau.
\]

which implies that upto a multiplicative constant:

\[
\tau_I = 1/\sqrt{\tau}. \tag{81}
\]
This formula was also obtained in a different way in [1].

7. Frobenius structure

Expand, the dressing matrix \( M(u, \lambda) \) according to the grading:

\[
M(u, \lambda) = M_0(u) + M_1(u)\lambda + M_2(u)\lambda^2 + \ldots
\]  
(82)

Then, according to the condition (30) the zero grade term \( M_0(u) = M(u, 0) = (m_{ij}(u))_{1 \leq i, j \leq N} \) is an orthogonal matrix \( M_0^T(u) = M_0^{-1}(u) \) which as seen from eqs. (18) and (32) also satisfies the flow equations:

\[
\partial_j M_0(u) = [\theta(u), E_{jj}] M_0(u), \quad \delta_0^V M_0(u) = -V(u) M_0(u)
\]  
(83)

and hence also

\[
\sum_{j=1}^{N} u_j \partial_j M_0(u) = V(u) M_0(u).
\]  
(84)

Define

\[
\mathcal{V} = M_0^{-1} V M_0,
\]  
(85)

it is easy to see that \( \mathcal{V} \) satisfies:

\[
\partial_j \mathcal{V} = 0, \quad \mathcal{V}^T = -\mathcal{V}, \quad \delta_0^V \mathcal{V} = 0, \quad j = 1, \ldots, N
\]  
(86)

and accordingly \( \mathcal{V} \) is a constant. We can bring \( \mathcal{V} \) into Jordan normal form \( \mu = \hat{\mu} + R_0 \), with \( \hat{\mu} \) its semisimple part and \( R_0 \) its nilpotent part, i.e. there exists a complex invertible matrix \( S \), such that

\[
\mathcal{V} = S \mu S^{-1},
\]  
(87)

Now define

\[
M(u) = M_0(u) S = (m_{ij}(u))_{1 \leq i, j \leq N},
\]  
(88)

then using (83) we obtain:

\[
\partial_j M(u) = [\theta^{-1}(u), E_{jj}] M(u) = V_j(u) M(u), \quad \delta_0^V M(u) = -M(u) \mu
\]  
(89)

and hence also

\[
\sum_{j=1}^{N} u_j \partial_j M(u) = M(u) \mu.
\]  
(90)

Now assume that \( \mathcal{V} \) is diagonalizable, i.e. the matrix \( V(u) \) has no nilpotent part, then

\[
\mu = \sum_{i=1}^{N} \mu_i E_{ii}, \quad \text{with} \quad \mu_i = -\mu_{N+1-i}
\]  
(91)

and the columns of \( M(u) \) are eigenvectors for the Euler vectorfield \( E_0 \). Hence, on the domain \( u_i \neq u_j \) and \( m_{11} m_{21} \cdots m_{N1} \neq 0 \) we have a local semisimple Frobenius manifold (see e.g. [1]) with Lamé coefficients

\[
h_i = m_{i1}.
\]  
(92)
Define,
\[
\eta = (\eta_{\alpha \beta})_{1 \leq \alpha, \beta \leq N} = M^T M = S^T S, \quad \text{and denote} \quad \eta^{-1} = (\eta^{\alpha \beta})_{1 \leq \alpha, \beta \leq N},
\]
then \( \mu \eta + \eta \mu = 0 \) and the derivatives with respect to the flat coordinates \( x^\alpha, \alpha = 1, \ldots, N \) are given by
\[
\frac{\partial}{\partial x^\alpha} = \sum_{i=1}^N m_i \frac{\partial}{\partial u_i}
\]
and
\[
\frac{\partial}{\partial u_i} = \sum_{\alpha, \beta = 1}^N \eta^{\alpha \beta} m_i m_i \frac{\partial}{\partial x^\alpha}.
\]
The metric equals
\[
ds^2 = \sum_{i=1}^N h_i^2 (du_i)^2 = \sum_{\alpha, \beta = 1}^N \eta_{\alpha \beta} dx^\alpha dx^\beta, \quad \text{with} \quad \eta_{\alpha \beta} = \sum_{i=1}^N m_i m_i \eta_{\alpha \beta}.
\]
and the structure constants are given by
\[
c_{\alpha \beta \gamma} = \sum_{i=1}^N \frac{m_i m_i m_i m_i}{m_i m_i}.
\]
Since we have constructed everything in such a way that the columns of \( M(u) \) are eigenvectors, our Frobenius manifold has scaling dimensions \( \mu_\alpha - \mu_1 \).

Now, multiply the wave (matrix) function \( \Psi(u, \lambda) \), defined in (28), from the left by
\[
M^{-1}(u) = S^{-1} M_0^{-1}(u) = \eta^{-1} S^T M_0^T(u) = \eta^{-1} M^T(u)
\]
and from the right by \( S \) to obtain the wave function \( \mathcal{Y}(u, \lambda) \):
\[
\mathcal{Y}(u, \lambda) = M^{-1}(u) \Psi(u, \lambda) S = S^{-1} M_0^{-1}(u) M(u, \lambda) g^{-1}(\lambda) S.
\]
\( \mathcal{Y}(u, \lambda) \) satisfies, as in [5]:
\[
\begin{align*}
\mathcal{Y}(u, -\lambda)^T \eta \mathcal{Y}(u, \lambda) &= \eta \\
\frac{\partial \mathcal{Y}(u, \lambda)}{\partial u_i} &= \lambda \mathcal{E}_i(u) \mathcal{Y}(u, \lambda) \\
\frac{\partial \mathcal{Y}(u, \lambda)}{\partial x^\alpha} &= \lambda \mathcal{C}_\alpha(u) \mathcal{Y}(u, \lambda) \\
\frac{\partial \mathcal{Y}(u, \lambda)}{\partial \lambda} &= \left( \mathcal{U}(u) + \lambda^{-1} \mu \right) \mathcal{Y}(u, \lambda)
\end{align*}
\]
where
\[
\begin{align*}
\mathcal{E}_i(u) &= M^{-1}(u) E_i M(u) \\
\mathcal{U}(u) &= M^{-1}(u) U M(u) \\
\mu &= M^{-1}(u) V(u) M(u) = S^{-1} V S
\end{align*}
\]
and
\[ C_\alpha(u) = \sum_{\beta,\gamma=1}^{N} c^{\gamma}_{\alpha\beta} E^{\beta\gamma} = \sum_{\beta,\gamma=1}^{N} c^{\gamma}_{\alpha\beta\delta} E^{\beta\gamma}. \] (107)

For general \( V(u) \), which may contain a nilpotent part, we have \( \mu^T \eta + \eta \mu = 0 \). Define again \( \mathcal{Y}(u, \lambda) \) by (99) then also
\[ \mathcal{M}(u, \lambda) = M^{-1}(u)M(u, \lambda)S = I + O(\lambda) \] (108)
such that \( \mathcal{Y}(u, \lambda) = \mathcal{M}(u, \lambda)S^{-1}g^{-1}(\lambda)S \) and
\[ \mathcal{M}(u, -\lambda)^T \eta \mathcal{M}(u, \lambda) = \eta. \] (109)

From equation (103) one obtains:
\[ g^{-1}\frac{\partial g}{\partial \lambda} = SM^{-1}\left(\frac{\partial M}{\partial \lambda} - \mathcal{U}M - \frac{\mu}{\lambda}M\right)S^{-1} \] (110)
Assume, that \( g = g_+g_- \) then
\[ g^{-1}\frac{\partial g_-}{\partial \lambda} + \frac{\partial g_+}{\partial \lambda}g_+^{-1} = g_+S^{-1}(\frac{\partial M}{\partial \lambda} - \mathcal{U}M - \frac{\mu}{\lambda}M)S^{-1}g_+^{-1} \] (111)
Define \( G = g_+S\mathcal{M}^{-1}S^{-1} \) then
\[ g_+^{-1}\frac{\partial g_-}{\partial \lambda} = -\frac{1}{\lambda}G_0\mu S^{-1}G_0^{-1} = -\frac{1}{\lambda}G_0\mathcal{V}G_0^{-1} \] (112)
with matrix \( G_0 = (g_+S\mathcal{M}^{-1}S^{-1})_0 = (g_+)_0 \) being orthogonal for a fixed point \( g \) of the automorphism \( \sigma \).

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