Differentiable Perturbation of Unbounded Operators

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Vienna, Preprint ESI 1154 (2002)  April 5, 2002

Supported by the Austrian Federal Ministry of Education, Science and Culture
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DIFFERENTIABLE PERTURBATION
OF UNBOUNDED OPERATORS

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Abstract. If \( A(t) \) is a \( C^1 \)-curve of unbounded self-adjoint operators on Hilbert space with compact resolvents and common domain of definition, then the eigenvalues can be chosen \( C^1 \) in \( t \).

Theorem. Let \( t \mapsto A(t) \) for \( t \in \mathbb{R} \) be a curve of unbounded self-adjoint operators in a Hilbert space with common domain of definition and with compact resolvent.

1. If \( A(t) \) is real analytic in \( t \in \mathbb{R} \) then the eigenvalues and the eigenvectors of \( A(t) \) may be arranged in such a way that they are real analytic in \( t \).

2. If \( A(t) \) is \( C^\infty \) in \( t \in \mathbb{R} \) then the eigenvalues of \( A(t) \) may be arranged in such a way that they are \( C^1 \) in \( t \). Suppose moreover that no two of the continuously chosen eigenvalues meet of infinite order at any \( t \in \mathbb{R} \) if they are not equal. Then the eigenvalues and the eigenvectors can be chosen smoothly in \( t \), on the whole parameter domain.

3. If \( A(t) \) is \( C^{1,\alpha} \) for some \( \alpha > 0 \) in \( t \in \mathbb{R} \) then the eigenvalues of \( A(t) \) may be arranged in such a way that they are \( C^1 \) in \( t \).

Part (1) is due to Rellich [11] in 1940, see also Kato [8], VII, 3.9. Part (2) has been proved in [1], 7.8, see also [9], 50.16, in 1997; there one finds also a different proof of (1). The purpose of this paper is to prove part (3). It is best possible for the degree of continuous differentiability of the eigenvalues by the first example below.

We thank T. Hoffmann-Ostenhof and T. Kappeler for their interest and hints.

Definitions and remarks. That \( A(t) \) is a real analytic, \( C^\infty \), or \( C^{1,\alpha} \) curve of unbounded operators means the following: There is a dense subspace \( V \) of the Hilbert space \( H \) such that \( V \) is the domain of definition of each \( A(t) \), and such that \( A(t)^* = A(t) \) with the same domains \( V \), where the adjoint operator \( A(t)^* \) is defined by \( \langle A(t)u, v \rangle = \langle u, A(t)^*v \rangle \) for all \( v \) for which the left hand side is bounded as function in \( u \in H \). Moreover we require that \( t \mapsto \langle A(t)u, v \rangle \) is real analytic, \( C^\infty \), or \( C^{1,\alpha} \) for each \( u \in V \) and \( v \in H \). This implies that \( t \mapsto A(t)u \) is of the same class.

1991 Mathematics Subject Classification. 47A55, 47A75.

Key words and phrases. perturbation theory, differentiable choice of eigenvalues.

PWM was supported by FWF, Projekt P 14195 MAT, and WK.

Typeset by \LaTeX\XeX
$\mathbb{R} \to H$ for each $u \in V$ by [9], 2.3 or [7], 2.6.2. Here a function $f$ is $C^{1,\alpha}$ if it is differentiable and for the derivative the expression $\frac{f(t)-f(s)}{|t-s|^\alpha}$ is locally bounded in $t \neq s$.

The first part of the proof will show that $t \mapsto A(t)$ of class $C^{1,\alpha}$ implies that the resolvent $(A(t) - z)^{-1}$ is $C^{1,\alpha}$ in $t$ and $z$ jointly; only $C^1$ is used later in the proof. The reason why the proof works is that $C^{1,\alpha}$ can be described by boundedness conditions only; and for these the uniform boundedness principle is valid.

**Result.** (Rellich [12], see also Kato [8], II, 6.8) Let $A(t)$ be a $C^1$-curve of Hermitian $(n \times n)$-matrices. Then the eigenvalues can be chosen $C^1$ in $t$, on the whole parameter interval.

This result is best possible for the degree of continuous differentiability, as is shown by the following example.

**Example.** This is an elaboration of [1], 7.4. Let $S(2)$ be the vector space of all symmetric real $(2 \times 2)$-matrices. We use the general curve lemma [9], 12.2: There exists a converging sequence of reals $t_n$ with the following property: Let $A_n \in C^\infty(\mathbb{R}, S(2))$ be any sequence of functions which converges fast to 0, i.e., for each $k \in \mathbb{N}$ the sequence $n^k A_n$ is bounded in $C^\infty(\mathbb{R}, S(2))$. Then there exists a smooth curve $A \in C^\infty(\mathbb{R}, S(2))$ such that $A(t_n + s) = A_n(s)$ for $|s| \leq \frac{1}{n^2}$, for all $n$.

We use it for

$$A_n(t) := \begin{pmatrix} \frac{1}{2n^2} & \frac{2\pi}{2n^2} \\ \frac{2\pi}{2n^2} & \frac{1}{2n^2} \end{pmatrix} = \frac{1}{2n^2} \begin{pmatrix} \frac{t}{s_n} & \frac{1}{s_n} \\ \frac{1}{s_n} & -1 \end{pmatrix},$$

where $s_n := 2^{n-1} - \frac{1}{n^2}$.

The eigenvalues of $A_n(t)$ and their derivatives are

$$\lambda_n(t) = \pm \frac{1}{2n^2} \sqrt{1 + \left( \frac{t}{s_n} \right)^2}, \quad \lambda_n'(t) = \pm \frac{2n^2 - 2nt}{\sqrt{1 + \left( \frac{t}{s_n} \right)^2}}.$$

Then

$$\frac{\lambda'(t_n + s_n) - \lambda'(t_n)}{s_n^\alpha} = \frac{\lambda_n'(s_n) - \lambda_n'(0)}{s_n^\alpha} = \pm \frac{2n^2 - 2ns_n}{s_n^\alpha \sqrt{2}} = \pm \frac{2n(\alpha(n-1)^{-1})}{\sqrt{2}} \to \infty \text{ for } \alpha > 0.$$

Note that by [1], 2.1 we may always find a twice differentiable square root, so that we also construct functions $\lambda$ which are twice differentiable but not $C^{1,\alpha}$ for any $\alpha > 0$.

Note that the normed eigenvectors cannot be chosen continuously in this example (see also example [10], §2). Namely, we have

$$A(t_n) = A_n(0) = \frac{1}{2n^2} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \quad A(t_n + s_n) = A_n(s_n) = \frac{1}{2n^2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

**Proof of the Theorem.** By definition the function $t \mapsto \langle A(t)v, u \rangle$ is of class $C^{1,\alpha}$ for each $v \in V$ and $u \in H$. Then by [9], 2.3 (extended from $C^{1,1}$ to $C^{1,\alpha}$ with essentially the same proof) or [6], 5 the curve $t \mapsto A(t)v$ is of class $C^{1,\alpha}$ into $H$. 


For each \( t \) consider the norm \( \|u\|^2 := \|u\|^2 + \|A(t)u\|^2 \) on \( V \). Since \( A(t) = A(t)^* \) is closed, \((V, \| \cdot \|_t)\) is again a Hilbert space with inner product \( \langle u, v \rangle_t := \langle u, v \rangle + \langle A(t)u, A(t)v \rangle \).

(i) Claim. All these norms \( \| \cdot \|_t \) on \( V \) are equivalent, locally uniformly in \( t \). We then equip \( V \) with one of the equivalent Hilbert norms, say \( \| \cdot \|_0 \).

The function \( t \mapsto \langle u, v \rangle_t = \langle u, v \rangle + \langle A(t)u, A(t)v \rangle \) is \( C^{1, \alpha} \) since \( t \mapsto A(t)u \) is it, for fixed \( u, v \in V \), thus it is also locally Lipschitz \((C^{0,1} = \text{Lip}^0)\). By the multilinear uniform boundedness principle \((9), 5.18\) or \([7], 3.7.4\) the mapping \( t \mapsto \langle u, v \rangle_t \) is \( C^{0,1} \) into the space of bounded bilinear forms on \((V, \| \cdot \|_s)\) for each fixed \( s \). By the exponential law \([7], 4.3.5\) for \( \text{Lip}^0 \) the mapping \( (\, t, u, v ) \mapsto \langle u, v \rangle_t \) is \( C^{0,1} \) from \( \mathbb{R} \times (V, \| \cdot \|_s) \times (V, \| \cdot \|_s) \to \mathbb{R} \) for each fixed \( s \). Thus all Hilbert norms \( \| \cdot \|_t \) are locally uniformly equivalent, since \( \{ ||u||_t : |t| \leq K, ||u||_s \leq 1 \} \) is bounded by \( L_{K,s} \) in \( \mathbb{R} \), so \( ||u||_t \leq L_{K,s} ||u||_s \) for all \( |t| \leq K \), and claim (i) follows.

Then each \( A(t) \) is a globally defined operator \( V \to H \) with closed graph and is thus bounded, and by \([6], 5\) and the linear uniform boundedness theorem we see that \( t \mapsto A(t) \) is a \( C^{1, \alpha} \)-mapping \( \mathbb{R} \to L(V, H) \), and thus \( C^1 \) in the usual sense, again by \([6] \) 5. Alternatively, if reference \([6] \) is not available, one may use \([9], 2.3\), extended from \( C^{1,1} \) to \( C^{1, \alpha} \) with essentially the same proof, and note that it suffices to test with linear mappings which recognize bounded sets, by \([9], 5.18\). Alternatively again, one may use \([7], 3.7.4 + 4.1.12\), extended from \( C^{1,1} \) to \( C^{1, \alpha} \).

If for some \((t, z) \in \mathbb{R} \times \mathbb{C}\) the bounded operator \( A(t) - z : V \to H \) is invertible, then this is true locally and \( (t, z) \mapsto (A(t) - z)^{-1} : H \to V \) is \( C^1 \) since inversion is smooth on Banach spaces, by the chain rule.

Since each \( A(t) \) is Hermitian the global resolvent set \( \{ (t, z) \in \mathbb{R} \times \mathbb{C} : (A(t) - z) : V \to H \text{ is invertible} \} \) is open, contains \( \mathbb{R} \times (\mathbb{C} \setminus \mathbb{R}) \), and hence is connected. Moreover \( (A(t) - z)^{-1} : H \to H \) is a compact operator for some (equivalently any) \( (t, z) \) if and only if the inclusion \( i : V \to H \) is compact, since \( i = (A(t) - z)^{-1} \circ (A(t) - z) : V \to H \to H \).

Let us fix a parameter \( s \). We choose a simple closed smooth curve \( \gamma \) in the resolvent set of \( A(s) \) for fixed \( s \).

(ii) Claim. For \( t \) near \( s \), there are \( C^1 \)-functions \( t \mapsto \lambda_i(t) : 1 \leq i \leq N \) which parametrize all eigenvalues (repeated according to their multiplicity) of \( A(t) \) in the interior of \( \gamma \).

By replacing \( A(s) \) by \( A(s) - z_0 \) if necessary we may assume that \( 0 \) is not an eigenvalue of \( A(s) \). Since the global resolvent set is open, no eigenvalue of \( A(t) \) lies on \( \gamma \) or equals 0, for \( t \) near \( s \). Since

\[
t \mapsto -\frac{1}{2\pi i} \int_{\gamma} (A(t) - z)^{-1} \, dz =: P(t, \gamma)
\]

is a \( C^1 \) curve of projections (on the direct sum of all eigenspaces corresponding to eigenvalues in the interior of \( \gamma \) with finite dimensional ranges, the ranks (i.e. dimension of the ranges) must be constant: it is easy to see that the (finite) rank cannot fall locally, and it cannot increase, since the distance in \( L(H, H) \) of \( P(t) \) to the subset of operators of rank \( \leq N \) is \( \text{rank}(P(s)) \) is continuous in \( t \) and is either 0 or 1. So for \( t \) near \( s \), there are equally many eigenvalues in the interior, and we may call them \( \mu_i(t) : 1 \leq i \leq N \) (repeated with multiplicity).
To see that the eigenvalues $\mu_i(t)$ can be rearranged (for each $t$) in such a way that they become $C^1$ in $t$ note that the images of the projections $P(t, \gamma)$ of constant rank for $t$ near $s$ describe the fibers of a $C^1$ vector bundle. The restriction of $A(t)$ to this bundle, viewed in a $C^1$ framing, becomes a $C^1$ curve of hermitian matrices, for which by Rellich’s result the eigenvalues can be chosen $C^1$. This finishes the proof of claim (ii).

(iii) Claim. Let $t \mapsto \lambda_i(t)$ be a differentiable eigenvalue of $A(t)$, defined on some interval. Then

$$|\lambda_i(t_1) - \lambda_i(t_2)| \leq (1 + |\lambda_i(t_2)|)(e^{a|t_1-t_2|} - 1)$$

holds for a continuous positive function $a = a(t_1, t_2)$ which is independent of the choice of the eigenvalue.

For fixed $t$ near $s$ take all roots $\lambda_j$ which meet $\lambda_i$ at $t$, order them differentiably near $t$, and consider the projector $P(t, \gamma)$ onto the joint eigenspaces for only those roots (where $\gamma$ is a simple $C^1$ curve containing only $\lambda_i(t)$ in its interior, of all the eigenvalues at $t$). Then the image of $u \mapsto P(u, \gamma)$, for $u$ near $t$, describes a $C^1$ finite dimensional vector subbundle of $\mathbb{R} \times H$, since its rank is constant. For each $u$ choose an orthonormal system of eigenvectors $v_j(u)$ of $A(u)$ corresponding to these $\lambda_j(u)$. They form a (not necessarily continuous) framing of this bundle. For any sequence $t_k \to t$ there is a subsequence such that each $v_j(t_k) \to w_j(t)$ where $w_j(t)$ is again an orthonormal system of eigenvectors of $A(t)$ for the eigenspace of $\lambda_i(t)$.

Now consider

$$\frac{A(t) - \lambda_i(t)}{t_k - t} v_i(t_k) + \frac{A(t_k) - A(t)}{t_k - t} v_i(t_k) - \frac{\lambda_i(t_k) - \lambda_i(t)}{t_k - t} v_i(t_k) = 0,$$

take the inner product of this with $w_i(t)$, note that then the first summand vanishes, and let $t_k \to t$ to obtain

$$\lambda'_i(t) = (A'(t)w_i(t), w_i(t))$$

for an eigenvector $w_i(t)$ of $A(t)$ with eigenvalue $\lambda_i(t)$. This implies, where $V_i = (V, \| \cdot \|_t)$,

$$|\lambda'_i(t)| \leq \|A'(t)\|_{L(V_i, H)} \|w_i(t)\|_{V_i} \|w_i(t)\|_H$$

$$= \|A'(t)\|_{L(V_i, H)} \sqrt{\|w_i(t)\|^2_H + \|A(t)w_i(t)\|^2_H}$$

$$= \|A'(t)\|_{L(V_i, H)} \sqrt{1 + \lambda_i(t)^2} \leq a + a|\lambda_i(t)|,$$

for a constant $a$ which is valid for a compact interval of $t$'s since all norms $\| \cdot \|_t$ are locally in $t$ uniformly equivalent, see above. By Gronwall’s lemma (see e.g. [3], (10.5.1.3)) this implies claim (iii).

By the following arguments we can conclude that all eigenvalues may be numbered as $\lambda_i(t)$ for $i$ in $\mathbb{N}$ or $\mathbb{Z}$ in such a way that they are $C^1$ in $t \in \mathbb{R}$. Note first that by claim (iii) no eigenvalue can go off to infinity in finite time since it may increase at most exponentially. Let us first number all eigenvalues of $A(0)$ increasingly.

We claim that for one eigenvalue (say $\lambda_0(0)$) there exists a $C^1$ extension to all of $\mathbb{R}$; namely the set of all $t \in \mathbb{R}$ with a $C^1$ extension of $\lambda_0$ on the segment from 0 to
$t$ is open and closed. Open follows from claim (ii). If this interval does not reach infinity, from claim (iii) it follows that $(t, \lambda_0(t))$ has an accumulation point $(s, x)$ at the end $s$. Clearly $x$ is an eigenvalue of $A(s)$, and by claim (i) the eigenvalues passing through $(s, x)$ can be arranged $C^1$, and thus $\lambda_0(t)$ converges to $x$ and can be extended $C^1$ beyond $s$.

By the same argument we can extend iteratively all eigenvalues $C^1$ to all $t \in \mathbb{R}$; if it meets an already chosen one, the proof of [1], 4.3, see [9], 50.11 shows that we may pass through it coherently. □

**Example.** The first part of the proof cannot be improved. We describe a curve $A(t)$ of unbounded operators with compact resolvent and common domain $V$ of definition on $\ell^2$ such that $t \mapsto \langle A(t)v, u \rangle$ is $C^1$ for all $v \in V$ and $u \in \ell^2$, but $t \mapsto A(t)$ is not differentiable at $0$ into $L(V, \ell^2)$.

Let $\lambda_1 \in C^\infty(\mathbb{R}, \mathbb{R})$ be nonnegative with compact support and $\lambda_1'(0) = 0$. We consider the multiplication operator $B(t)$ on $\ell^2$ given on the standard basis $e_n$ by $B(t)e_n := (1 + \frac{1}{n}\lambda_1(nt))e_n =: \lambda_n(t)e_n$ which is bounded with bounded inverse. Then the function $t \mapsto \langle B(t)x, y \rangle$ is $C^1$ with derivative $\langle B_1(t)x, y \rangle$, where $B_1(t)$ is given by $B_1(t)e_n = \lambda_n'(t)e_n$, since

$$
\mu_n(t) := \frac{\lambda_n(t + s) - \lambda_n(t)}{s} - \lambda_n'(t) = \frac{\lambda_1(nt + ns) - \lambda_1(nt)}{ns} - \lambda_1'(nt)
$$

converges to $0$ for $s \to 0$ pointwise in $n$ and is bounded uniformly in $n$:

$$
\left| \sum_n \mu_n(t)x_n y_n \right| \leq \sum_{n=N+1}^{\infty} |\mu_n(t)x_n y_n| + \sum_{n=1}^{N} |\mu_n(t)x_n y_n| \leq \sup_n |\mu_n(t)| \epsilon + \epsilon \|x\| \|y\|.
$$

Moreover, $\langle B_1(t)x, y \rangle$ is continuous in $t$ since $\mu_n(s) = \lambda_n'(t + s) - \lambda_n'(t) = \lambda_1'(nt + ns) - \lambda_1'(nt)$ also converges to $0$ for $s \to 0$ pointwise in $n$ and is bounded uniformly in $n$.

But $t \mapsto B(t)$ is not differentiable at $0$ into $L(\ell^2, \ell^2)$ since

$$
\left\| \frac{B(t) - B(0)}{t} - B_1(0) \right\| = \sup_n \left| \lambda_n(t) - \lambda_n(0) \right| = \sup_n \left| \frac{\lambda_1(nt) - \lambda_1(0)}{nt} - \lambda_1'(0) \right|
$$

is bounded away from $0$, for $t \to 0$.

Finally, let $C : \ell^2 \to \ell^2$ be given by $C e_n = \frac{1}{n} e_n$, a compact operator, let $V = C(\ell^2)$, and $A(t) = B(t) \circ C^{-1}$.

**Open problem.** Construct a $C^1$-curve of unbounded self-adjoint operators with common domain and compact resolvent such that the eigenvalues cannot be arranged $C^1$.

**Applications.** Let $M$ be a compact manifold and let $t \mapsto g_t$ be a $C^{1,\alpha}$-curve of smooth Riemannian metrics on $M$. Then we get the corresponding $C^{1,\alpha}$-curve $t \mapsto \Delta(g_t)$ of Laplace-Beltrami operators on $L^2(M)$. By the theorem the eigenvalues can be arranged $C^1$.

Let $\Omega$ be a bounded region in $\mathbb{R}^n$ with smooth boundary, and let $H(t) = -\Delta + V(t)$ be a $C^{1,\alpha}$-curve of Schrödinger operators with varying potential, with Dirichlet boundary conditions. Then the eigenvalues can be arranged $C^1$. 

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