The Dyson Game
René Carmona, Mark Cerenziya, Aaron Zeff Palmer

Abstract

We introduce an explicitly solvable $N$ player dynamic game that admits Dyson Brownian motion as a Nash equilibrium and investigate consequences of its many atypical properties. We find that game theoretic symmetry for the naturally ordered players requires selfish behavior and moreover depends on the information available to them. Most significantly, the universality class of the equilibrium depends on this information structure and not just on the form of cost players face, in contrast to the folklore that this dependence should disappear as $N \to \infty$ given mean field interactions. The game theoretic symmetry in turn allows us to establish strong localized convergence of the $N$–Nash system to the expected mean field master equation against locally optimal player ensembles, i.e., those in the Nash–optimal universality class. This convergence notably features a nonlocal–to–local transition in the population dependence.

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1 Introduction

In the random matrix theory (RMT) community, there is a well known study [42] by physicists Krbálek and Šeba arguing that the spacing and arrival statistics of buses on a route in Cuernavaca,
Mexico are well-described by the eigenvalues of a random matrix belonging to the Gaussian unitary ensemble (GUE), which have a repulsive density on \( \mathbb{R}^N \), \( N \geq 2 \), proportional to

\[
\prod_{1 \leq k < \ell < N} |x^\ell - x^k|^\beta e^{-N^{-1} \sum_{i=1}^N (x^i)^2}, \quad \beta = 2.
\]  

Although the repulsion parameter choice \( \beta = 2 \) is related to the very special algebraic structure of determinantal correlations and to “symmetry class” (see Chapter 1 of Forrester [35]), there have been many studies identifying the emergence of such statistics for general \( \beta > 0 \), often where repulsive dynamics are natural: among parked cars [1, 33, 57, 61], pedestrians [11], perched birds [62], and even the New York City subway system [40] (such statistics have also appeared in a geographical study of France, in genetics [53], and perhaps most notably among gaps between zeros of the Riemann zeta function [56], which has already generated much research in number theory).

There have been some direct attempts to explain these observations through rigorous mathematics, such as Baik–Borodin–Deift–Suidan [5] and Baik [4], but a common theme among these real-world studies has largely been ignored: each involves players optimizing in a setting that is readily interpretable as a game theoretic. The recent numerical physics paper of Warchol appears to be the only study of an agent-based model attempting to investigate this connection.

Motivated to prove rigorous theorems on this implicit link, we introduce a dynamic \( N \) player game whose closed and open loop models are explicitly solvable with Nash–optimal trajectories given by Dyson Brownian motion [24], first introduced in [32] by Freeman Dyson; we accordingly call it the Dyson game. More precisely, players in this prototype game aim to minimize a long time average (“ergodic”) cost based on their distance from the origin and on the squared reciprocal distance between one another (similar to Calogero–Moser–Sutherland models; cf. Remark 3.3).

Merely constructing an agent–based model or a genuine player–based game yielding RMT–type statistics is not difficult, but it is significant for us to be able to identify how the solution depends on player information and how the freedom to act individually can achieve “game theoretic symmetry” despite the natural ordering in equilibrium. This latter feature is qualitatively consistent with the motivating example of the buses of Cuernavaca as well as the other observational studies above. It turns out such symmetry fails in the open loop model but is present in the closed loop model of “full information” (Section 4.1) allowing us to pursue strong “localized” convergence (Section 7).

Open loop models are often easier to analyze because opponent reactions may not be considered by players in their search for a Nash equilibrium. The open loop model for the Dyson game is further simplified by its potential structure (Lemma 3.7) that reduces the search to a single auxiliary global problem: a “central planner” can tell every player what to do and they end up not acting selfishly (though a priori they could). This puts us in the realm of classical physics, but to continue the game theoretic interpretation, the open loop Nash equilibrium prescribes higher repulsion to accommodate players densely packed near the origin. In contrast, the closed loop model realizes the option to behave selfishly through consideration of opponent reactions, leading to a less repulsive equilibrium that benefits players near the edge. Thus, the closed loop equilibrium is more “fair” in that all players incur the same cost, but this cost is higher than the average open loop cost; see Figure 1.

Further, both folklore and rigorous results of game theory suggest that the difference between these two models should disappear as \( N \to \infty \) given mean field interactions; see Remark 2.27 and pgs.122, 212 in Carmona–Delarue [21] for discussion of explicit solutions, as well as the forthcoming work of Lacker [45] for a theoretical approach. The intuition is that a single player cannot dramatically influence another through the empirical distribution if \( N \) is large. However, in the Dyson game, the singular dependence on the population allows nearby neighbors to have a large impact on a given player’s cost, and so the difference between the closed and open loop Nash equilibriums does
not disappear in the limit (cf. Section 4.1). Consequently, the universality class of the equilibrium depends on player information, not just on the form of cost they face. We believe this result offers a new perspective on Problem 9 of Deift’s list [29]; see the end of Section 4.1 for a discussion.

By design, the solutions (9), (21) to the ergodic Hamilton–Jacobi–Bellman (HJB) equations (8), (20) associated to the search for a Nash equilibrium can (essentially) each be expressed with a single functional (39), (43) on Wasserstein space, and so we are able to guess the limiting mean-field equations (40), (42). The so–called master equation (40) in particular allows us to formulate a mean field game (MFG) analog of the Dyson game, whose analysis requires navigating nontrivial issues of gradient flows on Wasserstein space. However, the connection between these two regimes remains formal without more work. Thus, we carry out a localized version of the convergence program for fully informed players introduced in Remark (x) following Theorem 2.3 of the seminal work [50] of Lasry and Lions. By “localized”, we mean we consider sequences of equations from the \( N \rightarrow \infty \) system (8) for individual ranked players, and in doing so we rigorize the intuition that players really only care about the population at their mean field “location” for large \( N \). Further, this convergence holds against player ensembles that are locally optimal, i.e., in the Nash–optimal universality class.

For this localized convergence of equations, the main calculation Proposition 7.1 is an extension of a guess in Remark 3.9 of Gorin–Shkolnikov [37], made at the edge, to any sequence of players under a convex potential. To prove it, we do not need to rely on detailed local limit behavior in the bulk, such as sine kernel asymptotics [3] or more generally the Sine\( \beta \) process [68, 69], despite their clear relevance to such calculations (cf. Remark 7.3). This result implies that the control cost term does not converge in \( L^2 \) to 0 in the bulk (as the Euler–Lagrange equation (71) might contrastingly suggest), instead contributing a local term in the limit. Indeed, we believe these considerations improve our understanding of limits that arise naturally in the RMT literature, such as the heuristic limit (1.4.4) of Biane–Speicher [13] for relative free Fisher information (cf. Remarks 5.4 and 7.7). However, the main finding here is that the local contributions in the bulk from the diffusion and drift–interaction terms of the \( N \rightarrow \infty \) system (8) cancel, leaving only the control cost term to contribute, which yields the expected master equation (40) (see Theorem 7.5).

There has been a growing interest in explicit solutions to \( N \) player games, in the mean field convergence problem, and in rank–based systems. As Lacker–Zariphopoulou [46] recently point out, explicit solutions for \( N \) player games are scarce, especially for the setting of full information. A solvable prototype for this setting is the class of Linear–Quadratic (LQ) models, examples of which are reviewed in Section 2.4 of [21]; see [46] and references therein for some non–LQ but explicitly solvable models. The work of Bardi [7] is an informative explicit case–study in the Gaussian ergodic setting, but it is rarely remarked that the model works with “narrow strategies” (see Fischer [34]). For the convergence problem, the early works [7, 34, 44] consider open loop and narrow strategies, but much research on convergence for closed loop models with full information has been generated by the systematic approach of Cardaliaguet–Delarue–Lasry–Lions [19]; see the forthcoming work of Lacker [45] and references therein. Finally, there are many models in the literature that include costs depending on rank, e.g. [9, 10, 22, 55], but the player states are still designed to be exchangeable.

The Dyson game exhibits many interesting properties that are atypical, if not new, for the literature on many–player games. First, the Dyson game is a non–LQ but explicitly solvable \( N \) player model that involves singular convolution transforms of the empirical distribution. Second, the model naturally features nonlocal–to–local transition in the population argument of the \( N \rightarrow \infty \) system as \( N \rightarrow \infty \) (see Cardaliaguet [18] for more general discussion of such transitions). Third, the singular cost term produces a singular drift in equilibrium and a quadratic dependence on the local density in the master equation (40). Fourth, the difference between the closed and open loop models of the Dyson game does not vanish in the limit and thus the models correspond to different
universality classes. Fifth and finally, we believe this paper is the first to work directly with the naturally ordered players and to establish convergence of equations in such a strong localized sense.

Outline

After introducing frequently used notation in Section 2, we articulate the closed and open loop models for the Dyson game in Section 3 and then use the solutions (9), (21) of the ergodic HJBs (8), (20) to prove Verification Theorems 4.2, 4.5 in Section 4. In Section 5, we use the mean field analogs (39), (43) of these solutions to guess the limiting equations (40), (42) on Wasserstein space; most notably, the master equation (40) features a local coupling. Then, using the theory of gradient flows, Section 6 proves a Verification Theorem 6.4 for the associated MFG formulation. Finally, Section 7 justifies the passage to the limit by establishing a strong localized convergence of equations against player ensembles in the Nash–optimal universality class. In addition, Section 8 quickly reviews the analogous results when the players live in the periodic setting of the circle.

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2 Notation

We often make use of the abbreviation “∑_{k:k≠i}” for “∑_{k=1,k≠i}^{N}” when N ≥ 2 is implicitly fixed. We write P(R) for the set of probability measures μ on R, P_{2}(R) for the subset with finite second moment, ∫_{R} |x|^2 μ(dx) < ∞, and P^{p}(R) for the subset with densities m(x) in L^{p}(R), p > 0. We also write P_{2}^{p}(R) := P_{2}(R) ∩ P^{p}(R). If T : R → R is Borel measurable, we denote the push forward of μ ∈ P(R) by T#μ := μ ◦ T^{-1}. We always assume P(R) to be topologized by weak convergence of probability measures and P_{2}(R) to be endowed with the r–Wasserstein distance with r = 2, defined by

\[ d_r(\mu,\nu) := \min_{\gamma \in \Gamma(\mu,\nu)} \left\{ \left( \int_{R \times R} |x-y|^r \gamma(dx,dy) \right)^{1/r} \right\}, \quad r \geq 1, \]

where Γ(μ, ν) is the set of couplings γ for μ, ν ∈ P_{2}(R).

To emphasize the nonlocal–to–local transition when passing to the limit in equations, we use Greek letters “μ(dx), ν(dx), ...” for probability measures and use Latin letters “m(x), n(x), ...” for their densities. We also use bold symbols “x, X, φ, α, ...” to indicate vectors in R^{N}, though the N dependence will remain implicit. Accordingly, for any x = (x^{1},...,x^{N}) ∈ R^{N}, we write

\[ \mu_{x} := \frac{1}{N} \sum_{k=1}^{N} \delta_{x^{k}}, \quad \mu_{x,i} := \frac{1}{N - 1} \sum_{k:k\neq i} \delta_{x^{k}} \]

for the ordinary and i-th empirical distribution of x, 1 ≤ i ≤ N, respectively. For any μ ∈ P(R) and f ∈ L^{1}(μ), we write μ[f] := ∫_{R} f(x)μ(dx) (and the same for higher dimensions). Write also

\[ \mathcal{W}^{N} := \{ x \in R^{N} : x^{1} < \cdots < x^{N} \} \]
for the open Weyl chamber and \( \overline{W^N} \) for its closure. For partial derivatives, we often use the abbreviation \( \partial_i := \frac{\partial}{\partial x^i} \).

For any \( \mu \in \mathcal{P}(\mathbb{R}) \), consider a function \( h : \mathbb{R} \to \mathbb{R} \) such that, for almost every \( x \in \mathbb{R} \), the integral \( \int_{\mathbb{R}} h(x - y)\mu(dy) \) exists or its principal value exists. Denote this quantity \( (h \ast \mu)(x) \), the convolution of \( h \) with \( \mu \). We will find it convenient to set for \( x \in \mathbb{R} \setminus \{0\} \)

\[
h_0(x) := \log |x|, \quad h_1(x) := 1/x, \quad h_2(x) := 1/x^2.
\] (2)

Then we may write the Hilbert Transform as

\[
H\mu(x) := \text{p.v.} \int_{\mathbb{R}} \frac{\mu(dy)}{x - y} = \lim_{\epsilon \to 0} \int_{|x - y| > \epsilon} \frac{\mu(dy)}{x - y} = (h_1 \ast \mu)(x).
\] (3)

Recall there exists \( A_p > 0 \) such that \( \|H\mu\|_p \leq A_p \|\mu\|_p \) for all \( \mu \in \mathcal{P}^p(\mathbb{R}) \), \( 1 < p < \infty \), with density \( m(x) \) (see, e.g., Theorem 1.8.8 of Blower [15]).

Finally, for any \( \beta > 0 \), let \( \mu_\beta \in \mathcal{P}(\mathbb{R}) \) denote the Wigner semicircle law with density \( m_\beta(x) := \frac{1}{\pi\beta} \frac{1}{\sqrt{2\beta - x^2}} \cdot 1_{[-\sqrt{2\beta}, \sqrt{2\beta}]}(x) \). (4)

Throughout the paper, \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) will denote a complete filtered probability space supporting an \( N \)-dimensional Wiener process \((W_t)_{t \geq 0}\) in \( \mathbb{R}^N \) and any other random objects we might need. We often write \( X \sim \mu \) to mean the random variable \( X \) has distribution \( \mu \in \mathcal{P}(\mathbb{R}) \).

### 3 N player formulation

**Closed loop model**

**Definition 3.1.** A function \( \phi : W^N \to \mathbb{R}^N \) is **admissible** if for every \( x_0 \in \overline{W^N} \), there exists a unique strong solution \((X_t)_{t \geq 0} = (X_t^\phi)_{t \geq 0}\) to the stochastic differential equation

\[
dX_t = \phi(X_t)dt + \frac{\sigma}{\sqrt{N - 1}}dW_t, \quad X_0 = x_0, \quad \sigma \geq 0
\] (5)

that remains in \( \overline{W^N} \) and satisfies the integrability condition

\[
\mathbb{E} \int_0^T \left[ \|\phi(X_t)\|^2 + \|X_t\|^2 + \sum_{1 \leq k < \ell \leq N} \frac{1}{(X_t^\ell - X_t^k)^2} \right] dt < \infty, \text{ for all } T > 0.
\] (6)

We denote the class of such feedbacks by \( \mathcal{A}^{(N)} \).

The class \( \mathcal{A}^{(N)} \) is fairly rich; indeed, Theorem 2.2 of Cépa–Lépingle [20] offers general solvability of (5) under the state constraint, while the stronger condition (6) will need to be checked (see the proof of Theorem 4.2). Also, the constraint “\( X_t \in \overline{W^N} \) for all \( t \geq 0 \)” is consistent with the framework suggested by the early work [19] of Lasry–Lions for state-constrained problems, but this condition will be **forced** in most cases by the form of singular cost (see Remark 3.5).

The **closed loop model for the N player Dyson game** can be formulated as follows. Fix any feedback profile \( \phi \in \mathcal{A}^{(N)} \) and \( C \in \mathbb{R} \). Interpreting the components \((X_t^i)_{t \geq 0}\) of (5) as **players**, we...
accordingly define for every $1 \leq i \leq N$ the $i$th player’s admissible class $A^i(\phi^{-i})$ to be the collection of $\psi : \mathbb{R}^N \to \mathbb{R}$ such that $(\psi, \phi^{-i}) := (\phi^1, \ldots, \phi^{i-1}, \psi, \phi^{i+1}, \ldots, \phi^N) \in A^{(N)}$. Define the state-cost

$$F^{N,i}(x) := \frac{(x^i)^2}{8} + C \cdot \frac{(h_2 \ast \mu^N_{x^i})(x^i)}{N - 1} = \frac{(x^i)^2}{8} + \frac{C}{(N - 1)^2} \sum_{k:k \neq i} \frac{1}{(x^i - x^k)^2}, \ x \in \mathcal{W}^N,$$

(recall the definitions in (2)). Then the search for Nash equilibria in the closed loop model requires each player $i, 1 \leq i \leq N$, to minimize the ergodic cost

$$J^{N,i}(\psi|x_0, \phi^{-i}) := \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \int_0^T \left[ \frac{1}{2} \psi(x_t)^2 + F^{N,i}(x_t) \right] dt$$

over deviations $\psi \in A^i(\phi^{-i})$, subject to $(x_t)_{t \geq 0} = (x_t^{\psi,\phi^{-i}})_{t \geq 0}$ satisfying (5) with $X_0 = x_0 \in \mathcal{W}^N$.

**Definition 3.2.** A feedback profile $\phi^* = (\phi^1, \ldots, \phi^N) \in A^{(N)}$ is a closed loop Nash equilibrium over classes $A^i \subset A^i(\phi^*-i), 1 \leq i \leq N$, if for all $1 \leq i \leq N, \phi^i \in A^i$ and for every $x_0 \in \mathcal{W}^N$, we have

$$\inf_{\psi \in A^i} J^{N,i}(\psi|x_0, \phi^*-i) = J^{N,i}(\phi^*|x_0, \phi^*-i).$$

**Remark 3.3.** A very similar expression to (7) appears in the Hamiltonian for Calogero’s model [17]; see [6, 48, 60] for various perspectives. However, the relationship between the parameters $C$ and $\beta$ (see Proposition 3.4 below) appears to be specific to the game theoretic construction and has some subtle consequences; compare Remarks 3.5 and 4.3 with Remark 4.6.

We now solve explicitly the $N$–Nash system for the closed loop Dyson game, a system of $N$ ergodic Hamilton–Jacobi–Bellman (HJB) equations associated to the search for a Nash equilibrium (compare with equation (1) of [19] and see Section 2.5.3 of [21] for some intuition):

$$-\frac{\sigma^2}{2(N - 1)} \Delta_x \nu^{N,i}(x) + \sum_{k:k \neq i} \partial_k \nu^{N,k}(x) \partial_k \nu^{N,i}(x) + \frac{1}{2} (\partial_i \nu^{N,i}(x))^2 = F^{N,i}(x) - \lambda^{N,i}, \ x \in \mathcal{W}^N, 1 \leq i \leq N.$$  

(8)

**Proposition 3.4.** Assume $\sigma \geq 0$ and that the coefficient $C$ of (7) satisfies $C \geq -\sigma^4/6$, so we may write $C = \frac{\beta}{2} \left( \frac{3}{4} \beta - \sigma^2 \right)$ for some $\beta \in \mathbb{R}$. Define

$$\nu^N_{\beta,i}(x) := \begin{cases} \frac{(x^i)^2}{4} - \frac{\beta}{2} (h_0 \ast \mu^N_{x^i})(x^i) & \ x \in \mathcal{W}^N \\ \infty & \text{otherwise,} \ 1 \leq i \leq N, \end{cases}$$

and write $\lambda^{N,i}_{\beta} := \frac{\sigma^2}{4(N - 1)} + \frac{\beta}{4}$. Then the ergodic value pairs $(\nu^N_{\beta,i}(x), \lambda^{N,i}_{\beta}), 1 \leq i \leq N$, form a classical solution to the $N$–Nash system (8) on $\mathcal{W}^N$.

**Proof.** The proof follows by direct calculation. First we collect some facts:

$$\partial_i \nu^N_{\beta,i}(x) = \frac{x^i}{2} - \frac{\beta}{2} (h_1 \ast \mu^N_{x^i})(x^i), \ \partial^2_i \nu^N_{\beta,i}(x) = \frac{1}{2} + \frac{\beta}{2} (h_2 \ast \mu^N_{x^i})(x^i)$$

and similarly if $k \neq i$

$$\partial_k \nu^N_{\beta,i}(x) = \frac{\beta}{2(N - 1)} x^i - x^k, \ \partial^2_k \nu^N_{\beta,i}(x) = \frac{\beta}{2(N - 1)} \frac{1}{(x^i - x^k)^2}. $$
Then we can compute

\[-\frac{\sigma^2}{2(N-1)} \Delta x_i v_{\beta}^{N,i}(x) = -\frac{\sigma^2}{2(N-1)} \sum_{k=1}^{N} \partial^2_{x} v_{\beta}^{N,i}(x) = -\frac{\sigma^2}{4(N-1)} - \frac{\sigma^2 \beta}{2} \frac{(h_2 \ast \mu_{x}^{N,i})(x)}{N-1}\]

and

\[\frac{1}{2} (\partial_i v_{\beta}^{N,i})^2(x) = \frac{1}{2} \left( x_i \frac{\beta}{2} - \frac{\beta}{2} (h_1 \ast \mu_{x}^{N,i})(x_i) \right)^2 \]

\[= \frac{(x_i)^2}{8} + \frac{\beta^2 (h_2 \ast \mu_{x}^{N,i})(x_i)}{N-1} - \frac{\beta}{4} x_i \cdot (h_1 \ast \mu_{x}^{N,i})(x_i) + \frac{\beta^2}{8(N-1)^2} \sum_{k:k \neq i} \sum_{\ell \neq i,k} \frac{1}{x_i - x_k} \frac{1}{x_i - x_{\ell}}. \tag{11}\]

Similarly, we have

\[\sum_{k:k \neq i} \partial_k v_{\beta}^{N,k}(x) \partial_i v_{\beta}^{N,i}(x) = \frac{\beta}{2(N-1)} \sum_{k:k \neq i} \left( x_k \frac{1}{2} - \frac{\beta}{2} (h_1 \ast \mu_{x}^{N,k})(x_k) \right) \frac{1}{x_i - x_k} \]

\[= \frac{\beta^2 (h_2 \ast \mu_{x}^{N,i})(x_i)}{4(N-1)} + \frac{\beta}{4(N-1)} \sum_{k:k \neq i} x_k - \frac{\beta^2}{4(N-1)^2} \sum_{k:k \neq i} \sum_{\ell \neq i,k} \frac{1}{x_i - x_{k}} \frac{1}{x_i - x_{\ell}}. \tag{12}\]

The two final terms of (11), (12) cancel by the algebra

\[2 \sum_{k:k \neq i} \sum_{\ell \neq i,k} \frac{1}{x_i - x_{\ell}} \frac{1}{x_i - x_{\ell}} = \sum_{k:k \neq i} \sum_{\ell \neq i,k} \frac{1}{x_i - x_{k}} \frac{1}{x_i - x_{\ell}} \]

while the two second–to–last terms of (11), (12) combine to yield the constant \(-\frac{\beta}{4}\). Putting everything together completes the proof. \(\square\)

**Remark 3.5.** If \(C > 0\), the players are forced to be state–constrained in \(W^N\) by the singular costs and not by our notion of admissibility Definition 3.1 (unless the system starts at the boundary). But for the range \(0 \leq \beta \leq \frac{3}{8} \sigma^2\), when \(C = \frac{\beta^2}{2} (1 - \beta - \sigma^2) \leq 0\), it is only admissibility that keeps the players in the chamber \(W^N\); see Lasry–Lions [49] for a discussion of forced boundary conditions. Further we can write \(C = \frac{\beta^2}{2} (1 - \sigma^2)\) with the new parameter \(\kappa := \frac{1}{2} \sigma^2 - \beta\). Thus, there are at least two sets of solutions to the \(N\)-Nash system (8): the pairs \((v_{\beta}^{N,i}(x), \lambda_{\beta}^{N,i})\) and \((v_{\kappa}^{N,i}(x), \lambda_{\kappa}^{N,i})\), \(1 \leq i \leq N\), as defined in (9). Compare with the discussion on pgs.2-4 of Spohn [64].

**Open loop model**

To formulate the open loop model for the \(N\) player Dyson game, we proceed as above.

**Definition 3.6.** A profile \((\alpha_t)_{t \geq 0} = (\alpha_t^1, \ldots, \alpha_t^N)_{t \geq 0}\) of \(\mathbb{R}\)–valued processes is admissible if it is \(\mathbb{F}\)–progressively measurable and for every \(x_0 \in \overline{W}^N\), there exists a unique strong solution \((X_t)_{t \geq 0} = (X_t^\alpha)_{t \geq 0}\) to the stochastic differential equation

\[dX_t = \alpha_t dt + \frac{\sigma}{\sqrt{N-1}} dW_t, \quad X_0 = x_0, \quad \sigma \geq 0 \tag{14}\]

that remains in \(\overline{W}^N\) and satisfies the integrability condition

\[\mathbb{E} \int_0^T \left[ \|\alpha_t\|^2 + \|X_t\|^2 + \sum_{1 \leq k < \ell \leq N} \frac{1}{(X_t^k - X_t^\ell)^2} \right] dt < \infty, \text{ for all } T > 0. \tag{15}\]

We denote the class of such admissible strategies by \(A^{(N)}\).
Fix a strategy profile \((\alpha_t)_{t \geq 0} \in \mathcal{A}^{(N)}\) and \(C \in \mathbb{R}\). Define for every \(1 \leq i \leq N\) the \(i\)th player’s admissible class \(\mathcal{A}^i(\alpha^{-i})\) to be the collection of \(\mathbb{R}\)–valued processes \((\eta_t)_{t \geq 0}\) such that

\[
((\eta_t, \alpha_t^{-i}))_{t \geq 0} := ((\alpha_t^1, \ldots, \alpha_t^{i-1}, \eta_t, \alpha_t^{i+1}, \ldots, \alpha_t^N))_{t \geq 0} \in \mathcal{A}^{(N)},
\]

henceforth abbreviated “\((\eta, \alpha^{-i})\)”. Then the search for Nash equilibria in the open loop model requires each player \(i, 1 \leq i \leq N\), to minimize the ergodic cost (recall the definition \(F^{N,i}\) of the state cost \((7)\))

\[
J^{N,i}(\eta \mid x_0, \alpha^{-i}) := \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \int_0^T \left[ \frac{1}{2} \eta_t^2 + F^{N,i}(X_t) \right] dt.
\]

(16)

over deviations \(\eta \in \mathcal{A}^i(\alpha^{-i})\), subject to \((X_t)_{t \geq 0} = (X_t^{(\eta, \alpha^{-i})})_{t \geq 0}\) satisfying \((14)\) with \(X_0 = x_0 \in \mathcal{W}^N\). Note by a slight abuse, we maintain the same notation despite now working with control processes instead of feedbacks. We omit an explicit definition of open loop Nash equilibrium since it is already indicated by Definition 3.2.

**Lemma 3.7.** Define a global cost function by

\[
F^{N}(x) := \|x\|^2_8 + \frac{C}{2} \sum_{i=1}^N \frac{(h_2 * \mu_{x_{\alpha}^{N,i}})(x^i)}{N-1}, \quad x \in \mathcal{W}^N,
\]

(17)

and corresponding global cost functional

\[
J^{N}(\alpha \mid x_0) := \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \int_0^T \left[ \frac{1}{2} \|\alpha_t\|^2 + F^{N}(X_t) \right] dt
\]

(18)

over \((\alpha)_{t \geq 0} \in \mathcal{A}^{(N)}\), subject to \((X_t)_{t \geq 0} = (X_t^{\alpha})_{t \geq 0}\) satisfying \((14)\) with \(X_0 = x_0 \in \mathcal{W}^N\). Then the open loop model for the Dyson game is a potential game in the following sense: For any profile \((\alpha_t)_{t \geq 0} \in \mathcal{A}^{(N)}\) such that the limit in \((18)\) exists, and for any deviation \((\eta_t)_{t \geq 0} \in \mathcal{A}^i(\alpha^{-i})\), \(1 \leq i \leq N\), such that the limit in \((16)\) exists, we have

\[
J^{N}((\eta, \alpha^{-i}) \mid x_0) - J^{N}(\alpha \mid x_0) = J^{N,i}(\eta \mid x_0, \alpha^{-i}) - J^{N,i}(\alpha^i \mid x_0, \alpha^{-i}).
\]

(19)

**Proof.** First we calculate

\[
\sum_{\ell=1}^N \sum_{k \neq \ell} \frac{1}{(x^\ell - x^k)^2} = 2 \sum_{k \neq i} \frac{1}{(x^i - x^k)^2} + \sum_{\ell \neq k \neq \ell} \frac{1}{(x^\ell - x^k)^2}.
\]

Then under the assumptions of the statement, a straightforward check confirms that condition \((19)\) holds, which exactly meets Definition 2.23 in [21] for potential game.

As indicated by the characterizing condition \((19)\), the potential game structure of Lemma 3.7 allows us to reduce the search for an open loop Nash equilibrium to a single auxiliary global problem. Given this setting of classical optimal control, the optimum will be achieved by strategies in closed loop feedback form and any such candidate is characterized by a solution to the ergodic HJB equation

\[
-\frac{\sigma^2}{2(N-1)} \cdot \frac{1}{N} \Delta x W(x) + \frac{1}{2N} |\nabla_x W(x)|^2 = \frac{1}{N} F^{N}(x) - \lambda, \quad x \in \mathcal{W}^N,
\]

(20)

(the factor of “\(\frac{1}{N}\)” maintains the correct scale for the comparison of Section 4.1 and anticipates taking limits). Compare the next statement with Proposition 3.4.
Proposition 3.8. Assume $\sigma^2 \geq 0$ and that the coefficient $C$ of \((17)\) satisfies $C \geq -\sigma^4/4$, so we may write $C = \frac{\beta}{4}(\beta - 2\sigma^2)$ for some $\beta \in \mathbb{R}$. Define

$$W_\beta(x) := \begin{cases} \sum_{i=1}^{N} \frac{(x_i)^2}{4} - \frac{\beta}{2(N-1)} \sum_{1 \leq k < \ell \leq N} \log(x^\ell - x^k) & x \in \mathcal{W}^N \\ +\infty & \text{otherwise} \end{cases}$$

and write $\lambda^N_\beta := \frac{\sigma^2}{4(N-1)} + \frac{\beta}{8}$. Then the ergodic value pair $(W_\beta(x), \lambda^N_\beta)$ forms a classical solution to the HJB equation \((20)\) on $\mathcal{W}^N$.

Proof. Recall the algebraic identities (see pg.252 of \cite{3})

$$\sum_{i=1}^{N} x^i \cdot (h_1 * \mu^{N,i}_x)(x^i) = \frac{N}{2}. \quad (22)$$

and

$$\sum_{i=1}^{N} \frac{(h_2 * \mu^{N,i}_x)(x^i)}{N-1} = \sum_{i=1}^{N} [(h_1 * \mu^{N,i}_x)(x^i)]^2. \quad (23)$$

Since $\partial_i W_\beta(x) = \partial_i v^{N,i}_\beta(x)$, we have by \((10), \quad (22), \quad \text{and} \quad (23)\)

$$-\frac{\sigma^2}{2N(N-1)} \Delta_x W_\beta(x) = -\frac{\beta\sigma^2}{4(N-1)} - \frac{1}{N} \sum_{i=1}^{N} \frac{(h_2 * \mu^{N,i}_x)(x^i)}{N-1}$$

and

$$\frac{1}{2N} |\nabla_x W_\beta(x)|^2 = \frac{\beta^2}{8} \cdot \frac{1}{N} \sum_{i=1}^{N} \frac{(h_2 * \mu^{N,i}_x)(x^i)}{N-1} + \frac{1}{N} \sum_{i=1}^{N} \frac{(x^i)^2}{8} - \frac{\beta}{8}.$$

Remark 3.9. The same discussion of Remark 3.5 applies here. But note in the range $0 < \beta < 2\sigma^2$ when $C < 0$, the non-uniqueness for bounded below, unbounded above solutions to the ergodic HJB \((20)\) does not contradict Theorem 3.1 of Barles–Meireles \cite{8} because in this range the “righthandside” $F^N(x)$ of the ergodic HJB \((20)\) is no longer bounded below; however, $F^N(x)$ trivially becomes bounded below in the critical case $C = 0$, i.e., $\beta = 0, 2\sigma^2$. This apparent contradiction of non-uniqueness with Theorem 3.1 of \cite{8} evaporates when we recall the state constraint and integrability condition \((15)\) imposed in our notion of admissibility Definition 3.6.

4 Verification theorems

Closed loop model

Assume that the coefficient $C$ from \((7)\) can be written $C = \frac{\beta}{4}(\frac{3}{4}\beta - \sigma^2) > -\sigma^4/8$ for some $\beta > \sigma^2 > 0$. Then, by Proposition 3.4, the set of solution pairs \((\ref{9})\) to the $N$-Nash system \((\ref{8})\) furnishes a feedback profile $\phi^*_\beta(x) := (-\partial_i v^{N,i}_\beta(x))_{i=1}^{N}$ with trajectories $(X^*_t)_{t \geq 0}$ given by $\beta/\sigma^2$-Dyson Brownian motion:

$$dX^*_t = -\partial_i v^{N,i}_\beta(X^*_t)dt + \frac{\sigma}{\sqrt{N-1}}dW^i_t = \left[\frac{\beta}{2}(h_1 * \mu^{N,i}_x)(X^*_t) - \frac{X^*_t}{2}\right]dt + \frac{\sigma}{\sqrt{N-1}}dW^i_t. \quad (24)$$

Since $\beta > \sigma^2$, $(X^*_t)_{t \geq 0}$ remains in the interior $\mathcal{W}^N$ for all $t > 0$, even if $X^*_0 = x_0 \in \mathcal{W}^N$ (see \cite{25,59}), but we still need to check that $\phi^*_\beta$ satisfies the integrability condition \((6)\).
Remark 4.1. The appearance of the ratio $\beta/\sigma^2 < \infty$ (the “inverse temperature”) in the name for the dynamics \cite{24} is justified by the form of invariant distribution \cite{28} below. Our notion of admissibility, Definition 3.1, prevents the case of infinite temperature $\beta/\sigma^2 = 0$, or “complete independence”, from occurring for the Dyson game, even in a limiting sense, because the critical case $\beta = \sigma^2$ can only be interpreted through the limit $\beta \downarrow \sigma^2$, as we will see. However, essentially all the results of this paper continue to hold verbatim in the deterministic case $\sigma^2 = 0$, when there is no temperature $\beta/\sigma^2 = \infty$ and thus the equilibrium state is “frozen”; see Remark 4.4.

For $(x, \alpha^i) \in \mathcal{W}^N \times \mathbb{R}$, write the cost as

$$f^N_{\beta, i}(x, \alpha^i) := \frac{(\alpha^i)^2}{2} + \frac{(x^i)^2}{8} + \frac{\beta}{2} \left( \frac{3}{4} \beta - \sigma^2 \right) \cdot \frac{(h_2 * \mu^N_{\beta, i})(x^i)}{N - 1} \tag{25}$$

Note the Hamiltonian of player $i$, $1 \leq i \leq N$, is given for $(x, y^i, \alpha) \in \mathcal{W}^N \times \mathbb{R}^N \times \mathbb{R}^N$ by

$$H^i_{\beta}(x, y^i, \alpha) := \sum_{k=1}^{N} y^i_k \alpha^k + f^N_{\beta, i}(x, \alpha^i). \tag{26}$$

The following Verification Theorem is proven in detail because of the non–uniqueness observed in Remark 3.5 and, somewhat surprisingly, its content supports our proof of Proposition 7.1, the main technical calculation for the localized mean field limit in Section 7.

Theorem 4.2. Fix $\beta > \sigma^2 > 0$ in (25) and recall from Proposition 3.4 the solution pairs $(v^N_{\beta, i}(x), \lambda^N_{\beta, i})$, $1 \leq i \leq N$, to the $N$–Nash system \cite{8}. Fix an interior initial condition $x_0 \in \mathcal{W}^N$. Then for $\beta > \sigma^2$, $\Phi^i_{\beta}(x) := (-\partial_i v^N_{\beta, i}(x))^N_{i=1}$ is a closed loop Nash equilibrium over the classes $A^i \subset A^i(\phi^{-i}_\beta)$ of deviations $\psi^i(x)$, $1 \leq i \leq N$, such that $(X_t)_{t \geq 0} = (X^i_t(\psi^i, \phi^{-i}_\beta))_{t \geq 0}$ of \cite{5} satisfies the stability conditions

$$\lim_{T \to \infty} \frac{1}{T} \mathbb{E}\int_0^T f^N_{\beta, i}(X_t, \psi^i(X_t))dt < \infty, \quad \lim_{T \to \infty} \frac{1}{T} \mathbb{E}[v^N_{\beta, i}(X_t) - v^N_{\beta, i}(x_0)] = 0. \tag{27}$$

Further, the cost to each player $i$ under the equilibrium dynamics $(X^*_t)_{t \geq 0} = (X^i_t)_{t \geq 0}$ satisfies

$$\inf_{\psi \in A^i} J^N_{\beta, i}(\psi|x_0, \phi^{-i}_\beta) = J^N_{\beta, i}(\phi^i_\beta|x_0, \phi^{-i}_\beta) = \lambda^N_{\beta, i} = \frac{\sigma^2}{4(N - 1)} + \frac{\beta}{4}.$$ 

Proof. We check the candidate Nash equilibrium $\phi^i_\beta$ satisfies the integrability condition \cite{6} and that the corresponding dynamics $(X^i_t)_{t \geq 0}$ satisfy the stability conditions in (27). First, we review some facts. Observe $W_\beta(x)$ of (21) is uniformly convex: for any vector $v \in \mathbb{R}^N$ and $x \in \mathcal{W}^N$

$$v^i \nabla \nabla W_\beta(x)v = \frac{1}{2} \sum_{k=1}^{N} v^2_i + \frac{\beta}{2(N - 1)} \sum_{1 \leq k < \ell \leq N} \frac{(v_k - v_\ell)^2}{(x_k - x_\ell)^2} \geq \frac{1}{2} \sum_{k=1}^{N} v^2_i.$$ 

Notice we may write the global dynamics of the system (24) as a gradient flow

$$dX^*_t = -\nabla W_\beta(X^*_t)dt + \frac{\sigma}{\sqrt{N - 1}} dW_t, \quad X^*_0 = x_0,$$

with (unique) globally invariant log–concave probability measure given by the $\beta/\sigma^2$–ensemble:

$$\mu^N_\beta(dx) = m^N_\beta(x)dx := \frac{1}{Z^N_\beta} \cdot \exp \left( -\frac{2(N - 1)}{\sigma^2} W_\beta(x) \right) \mathbb{1}_{\mathcal{W}^N}(x)dx,$$

$$= \frac{1}{Z^N_\beta} \cdot \prod_{1 \leq k < \ell \leq N} (x_k - x_\ell)^{2/\sigma^2} e^{-\frac{(N - 1)}{2\sigma^2} \sum_{i=1}^{N} (x^i)^2} \mathbb{1}_{\mathcal{W}^N}(x)dx, \tag{28}$$
where \( Z_{\beta}^N < \infty \) is the normalization constant (compare these expressions with (13), (14), and (16) of Dyson \(^{32}\)). Write \( P_t f(x) := \mathbb{E}_x f(X_t^\beta) \) for the semigroup and the generator as

\[
\mathcal{L} := \frac{\sigma^2}{2(N-1)} \Delta_x - \nabla_x W_\beta(x) \cdot \nabla_x.
\]

Recall that invariance means \( \mu_\beta^N(\mathcal{L} f) = 0 \) for suitable \( f \). More precisely, we say \( f \) is in the domain of \( \mathcal{L} \) to mean there exists a function \( g \) such that for every \( x \in \mathcal{W}^N \), \( \int_0^T |g(X_t^\beta)| dt < \infty \) and the process \( M_T := f(X_T^\beta) - f(x) - \int_0^T g(X_t^\beta) dt \) is an \( \mathbb{F} \)-martingale; one then writes \( \mathcal{L} f = g \). The dynamics are reversible with respect to \( \mu_\beta^N \) and for any \( f \in L^1(\mu_\beta^N) \), we have (see, e.g., Rey–Bellet’s notes \(^{38}\))

\[
\frac{1}{T} \int_0^T P_t f(x_0) dt \xrightarrow{T \to \infty} \mu_\beta^N[f] = \int f(x) \mu_\beta^N(dx).
\]

(29)

Now we check the integrability condition (0) using ideas from the proof of Lemma 4.3.3 of Anderson–Guionnet–Zeitouni \(^{3}\). Since \( x^2 - 2 \beta \log(1 + |x|) \) is uniformly bounded below and \( \log |x - y| \leq \log(1 + |x|) + \log(1 + |y|) \) for \( x, y \in \mathbb{R} \), we can estimate for any \( 1 \leq k \neq \ell \leq N \)

\[
W_\beta(x) + \frac{\beta}{4(N-1)} \log |x^k - x^\ell| \geq \frac{1}{4} \sum_{i=1}^N [(x^i)^2 - 2 \beta \log(1 + |x^i|)] \geq -M', \tag{30}
\]

for some constant \( M' = M'(N) \in \mathbb{R} \) independent of \( k, \ell \); indeed, we also have \( W_\beta(x) \geq -M' \). Defining \( T_M := \inf \{ t \geq 0 : W_\beta(X_t^\beta) \geq M \} \), the estimate (30) implies that on the event \( T_M > T \), each gap can be controlled: \( |X_t^k - X_t^\ell| \geq \exp \left[ -\frac{4(N-1)}{\beta} (M + M') \right] \) for all \( t \leq T \) and \( k \neq \ell \). Hence, we can use Ito’s formula to compute up until the time \( T_M \)

\[
dW_\beta(X_t^\beta) = \left( -\sum_{i=1}^N \frac{(X_t^{si})^2}{4} + \frac{N}{4} \left( \beta + \frac{\sigma^2}{N-1} \right) + \frac{\beta}{4(\sigma^2 - \beta)} \sum_{i=1}^N \frac{(h_2 + \mu_\beta^{N,i}(X_t^{si}))}{N-1} \right) dt + \nabla W_\beta(X_t^\beta) \cdot dW_t,
\]

(31)

where the local martingale in (31) stopped at time \( T_M \) is a true martingale. Putting everything together and recalling \( \beta > \sigma^2 \), we have

\[
\mathbb{E} \sum_{i=1}^N \int_0^{T \land T_M} (h_2 + \mu_\beta^{N,i}(X_t^{si})) dt \leq \frac{4(N-1)}{\beta(\sigma^2 - \beta)} \left[ \frac{N}{4} \left( \beta + \frac{\sigma^2}{N-1} \right) + M' \right] \mathbb{E}[T \land T_M] \leq C_{\beta,\sigma,N} T < \infty.
\]

(32)

The proof of Lemma 4.3.3 of \(^{3}\) shows \( \lim_{T \to \infty} T_M = +\infty \) almost surely, so an application of Fatou’s lemma implies the expectation of the time integral of the reciprocal gaps squared is finite for every \( T > 0 \). In addition, since \( \mu_\beta^N \) is the global invariant measure, our work also implies that if \( \mathbf{X}^* \sim \mu_\beta^N \), we have \( \mathbb{E} \sum_{i=1}^N (h_2 + \mu_\beta^{N,i}(X_t^{si})) < \infty \). (Notice this argument fails for \( \beta = \sigma^2 \) and in fact these expectations can be shown to diverge by comparison with a 2-dimensional Bessel process.)

Hence, it is now easy to see that the conditions of (6), (27) hold for the candidate Nash equilibrium \( \phi_\beta^* \) when \( \beta > \sigma^2 > 0 \). For example, we now know that \( v_\beta^{N,i} \) is in the domain of \( \mathcal{L} \), so by the fundamental theorem of calculus, the fact that \( \mathcal{L} v_\beta^{N,i} \in L^1(\mu_\beta^N) \), ergodicity (29), and then invariance of \( \mu_\beta^N \), the second condition of (27) follows from

\[
\lim_{T \to \infty} \frac{1}{T} \mathbb{E}[v_\beta^{N,i}(X_T^\beta) - v_\beta^{N,i}(x_0)] = \lim_{T \to \infty} \frac{1}{T} \int_0^T P_t(\mathcal{L} v_\beta^{N,i})(x_0) dt = \mu_\beta^N[\mathcal{L} v_\beta^{N,i}] = 0.
\]

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Now fix $1 \leq i \leq N$ and $\psi \in \mathcal{A}^i$. Using the fact that $(v_{\beta,i}^N(x), \lambda_{\beta,i}^N)$, $1 \leq i \leq N$, solves the $N$–Nash system \((8)\) and that Itô’s formula holds for all time as long as no collisions occur, we can compute (as in Proposition 2.11 of Carmona–Delarue \([21]\))

\[
dv_{\beta,i}^N(X_t) + [f_{\beta,i}^N(X_t, \psi(X_t)) - \lambda_{\beta,i}^N]dt = \frac{\sigma}{\sqrt{N-1}} \sum_{k=1}^{N} \partial_k v_{\beta,i}^N(X_t)dW_t^k \\
+ [H_{\beta,i}^0(X_t, \nabla x v_{\beta,i}^N(X_t), (\psi(X_t), \phi_{\beta,i}^N(X_t)^{-1})) - H_{\beta,i}^0(X_t, \nabla x v_{\beta,i}^N(X_t), \phi_{\beta,i}^N(X_t))]dt
\]

(33)

First, the local martingale on the right of (33) is a true martingale by the integrability condition \((6)\). Second, the Hamiltonian \((26)\) satisfies a strict Isaacs’ condition (see Definition 2.9 of \([21]\)), so the difference of Hamiltonian values is nonnegative. Third, we can use the two conditions of \((27)\) to take expectations and limits to conclude

\[J^N,i(\psi|\mathbf{x}_0, \phi^{\ast,i}) = \lim sup_{T \to \infty} \frac{1}{T} \mathbb{E} \int_0^T f_{\beta,i}^N(X_t, \psi(X_t))dt \geq \lambda_{\beta,i}^N.
\]

with equality if and only if $\psi \equiv \phi^{\ast,i} = -\partial_i v_{\beta,i}^N$. This confirms that the ergodic constant coincides with the associated minimal cost and concludes the proof. \( \square \)

**Remark 4.3.** As we note in Remarks 3.3 and 3.9, we have two candidate solutions of the $N$–Nash system \((8)\) but the proof of Theorem 4.2 shows at most one of them will furnish an admissible feedback, namely, the one of $\phi_{\beta,i}^\ast, \phi_{\kappa}^\ast$ corresponding to whichever of the two parameters $\beta, \kappa := \frac{2}{3}\sigma^2 - \beta$ is strictly larger than $\sigma^2$. Thus, we implicitly chose one by fixing $\beta > \sigma^2$ at this section’s start. Now if $\sigma^2/3 \leq \beta \leq \sigma^2$, when both $\beta, \kappa \leq \sigma^2$ and $-\sigma^4/6 \leq C \leq -\sigma^4/8$, the integrability condition \((6)\) fails for both feedbacks $\phi_{\beta,i}^\ast, \phi_{\kappa}^\ast$ and we can no longer make sense of them as Nash equilibriums according to Definition 3.2.

To summarize, for any $\beta \in \mathbb{R} \setminus [\sigma^2/3, \sigma^2]$, the optimal parameter permitting verification is

\[\beta^\ast := \max \left\{ \beta, \frac{4}{3}\sigma^2 - \beta \right\}, \]

which corresponds to the set of pairs $(v_{\beta,i}^N, \lambda_{\beta,i}^N)$, $1 \leq i \leq N$, with the larger ergodic constant $\lambda_{\beta,i}^N$; further, the strict lower bound of the coefficient $C$ in \((7)\) for the closed loop model to be well–posed is “$-\sigma^4/8$” (a similar lower bound occurs in the Calogero–Sutherland model; see Section 2 of \([48]\)).

**Remark 4.4.** To recover the case $\sigma = 0$, one can formally apply the asymptotic results of Dumitriu–Edelman \([31]\) and Section 4 of Dette–Imhof \([30]\) to conclude the convergence as $\sigma^2 \downarrow 0$ of the long–time equilibrium $X^* \sim \mu_\beta^N$ of \((28)\) to the set of zeros $x^* \in \mathcal{W}^N$ of suitably scaled Hermite polynomials. If we alternatively start with $\sigma = 0$ and fix $N \geq 2$, we can similarly replace the state $X^*$ with the $x^* \in \mathcal{W}^N$ that minimizes the convex function $W_\beta(x)$ of \([21]\). The values in $x^*$ are sometimes referred to as the Fekete points and can be shown to coincide with zeros of orthogonal polynomials as indicated above. See the introduction of Deift–Kriecherbauer–McLaughlin \([28]\) and Section 3.4.1 of Tao \([67]\) for nice discussions of these concepts.

**Open loop model**

To express the cost of the global functional $J^N$ from \((18)\), we write (compare with \((25)\) above)

\[
f_{\beta}(x, \alpha) := \frac{1}{2}||\alpha||^2 + ||x||^2 + \frac{\beta}{8}(\beta - 2\sigma^2) \sum_{i=1}^{N} \frac{(h_2 * \mu_{x,i})(x^i)}{N-1}. \]

(35)
Theorem 4.5. Fix \( \beta > \sigma^2 > 0 \) in \([35]\) and recall from Proposition 3.8 the solution pair \((W_\beta(x), \lambda^N)\) to the ergodic HJB equation \((20)\). Fix an interior initial condition \(x_0 \in \mathcal{W}^N\). Let \((X_t^\alpha)_{t \geq 0}\) be given by \(\beta/\sigma^2\)-Dyson Brownian motion \([24]\). Then the profile \((\alpha_t^\alpha)_{t \geq 0} := (-\nabla_x W_\beta(X_t^\alpha))_{t \geq 0}\) is a global minimizer over the class \(\mathcal{A} \subset \mathcal{A}^N\) of strategies \((\alpha_t)_{t \geq 0}\) such that \((X_t^\alpha)_{t \geq 0}\) of \((14)\) satisfies the stability conditions

\[
\limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \int_0^T f_N^\alpha(X_t, \alpha_t)dt < \infty, \quad \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}[W_\beta(X_T) - W_\beta(x_0)] = 0. \tag{36}
\]

The global cost under the equilibrium dynamics \((X_t^\alpha)_{t \geq 0} = (X_t^{\alpha*})_{t \geq 0}\) then satisfies

\[
\inf_{\alpha \in \mathcal{A}} J^N(\alpha|x_0) = J^N(\alpha^*|x_0) = N \cdot \lambda^N_\beta = N \cdot \left(\frac{\sigma^2}{4(N-1)} + \frac{\beta}{8}\right).
\]

Further, for the Dyson game, the strategy \((\alpha_t^\alpha)_{t \geq 0} \in \mathcal{A}(\alpha_t^{*\iota})\), \(1 \leq \iota \leq N\), of deviations \((\eta_t)_{t \geq 0}\) such that the limits in \((36)\) exist, i.e.,

\[
\inf_{\eta \in \mathcal{A}^{\iota}} J^{N,\iota}(\eta|x_0, \alpha^{*\iota}) = J^{N,\iota}(\alpha^{*\iota}|x_0, \alpha^{*\iota}), \quad 1 \leq \iota \leq N.
\]

Finally, when \(\beta = \sigma^2\) in the cost \((35)\) of \(J^N\), the global minimization problem admits a minimizing sequence with value \(N \cdot \lambda^N_{\sigma^2}\).

Proof. The proof follows essentially verbatim that of the closed loop Verification Theorem \((4.2)\), except for the last two statements. The first one follows because we have identified in Proposition 3.7 that the potential game structure holds over stable deviations where the limit of the time average cost \((16)\) exists. To sketch the last claim, for \(\beta > \sigma^2\), \(1 \leq \iota \leq N\), and with \(X \sim \mu^N_\beta\) of \((28)\), we have

\[
\mathbb{E}[f_N^\alpha(X, -\nabla_x W_\beta(X))] - \mathbb{E}[f_N^{\alpha\iota}(X, -\nabla_x W_\beta(X))] = \left[\frac{\beta}{8}(\beta - 2\sigma^2) - \left(\frac{-\sigma^4}{8}\right)\right] \sum_{i=1}^N \frac{\mathbb{E}(h_2 \ast \mu^N_\beta)(X^i)}{(N-1)}

= \left[\frac{\beta}{4} - \frac{(\beta + \sigma^2)}{8}\right] (\beta - \sigma^2) \sum_{i=1}^N \frac{\mathbb{E}(h_2 \ast \mu^N_\beta)(X^i)}{(N-1)}

\xrightarrow{\beta, \sigma^2} 0, \tag{37}
\]

where we rely on \((32)\) to know that the factor right of the square brackets is bounded as \(\beta \downarrow \sigma^2\). \(\square\)

Remark 4.6. A similar discussion as in Remark 4.3 applies for Theorem 4.5, but note that we may take any \(\beta \geq \sigma^2\) in the auxiliary global problem; indeed, compared to \((34)\), the analogous open loop optimal parameter that permits verification is now \(\beta^* := \max\{\beta, 2\sigma^2 - \beta\}\) for any \(\beta \in \mathbb{R}\).

4.1 Comparison of closed and open loop models

Keeping \(\sigma > 0\) fixed, we know by Theorems 4.2, 4.5 that the Nash–optimal repulsion parameters are given by the larger roots in the variable “\(\beta^*\)” of the quadratic relationships \(C = \frac{\beta}{2} \left(\frac{3}{4}\beta - \sigma^2\right)\) from Proposition 3.4 for \(C > -\sigma^4/8\) and \(C = \frac{\beta}{4}(\beta - 2\sigma^2)\) from Proposition 3.8 for \(C \geq -\sigma^4/4\):

\[
\beta_{\text{closed}}(C) := \frac{2}{3} \left(\sigma^2 + \sqrt{\sigma^4 + 6C}\right), \quad \beta_{\text{open}}(C) := \sigma^2 + \sqrt{\sigma^4 + 4C}; \tag{38}
\]
Notice the open loop strategy prescribes a higher repulsion to accommodate the players densely packed around the origin, i.e., \( \beta_{\text{closed}}(C) > \beta_{\text{open}}(C) \), and in doing so achieves a lower average cost, i.e., \( \bar{\lambda}_{\text{open}}(C) < \frac{1}{N} \sum_{i=1}^{N} \lambda_{\beta_{\text{closed}}(C)}^{N,i} \approx \frac{\beta_{\text{closed}}(C)}{4} \). However, the closed loop Nash equilibrium is fair to all (no matter their rank) while the open loop Nash equilibrium has greater cost for players away from the origin. Lastly, recall \( \lambda_{\beta_{\text{closed}}(C)}^{N} \) is an auxiliary cost; indeed, \( \beta_{\text{open}}(2C) \) minimizes the actual global average cost with minimum \( \lambda_{\beta_{\text{open}}(2C)}^{N} \), for given \( C > -\sigma^4/8 \) (see red curves in Figure 1).

**Figure 1:** The top left plot compares the Nash–optimal \( \beta_{\text{closed}}(C), \beta_{\text{open}}(C) \) and global average minimizing \( \beta_{\text{open}}(2C) \) parameters. The bottom left compares the average cost \( \bar{\lambda}_{\text{open}}(C) \) under the open loop Nash equilibrium with the optimal global average cost \( \lambda_{\beta_{\text{open}}(2C)}^{N} \), which only coincide at \( C = 0 \); in particular, except at this coincidence, the Nash equilibriums are never global average minimizers. Fixing \( C = 0 \), the top right compares the (approximate) optimal costs by location for individual players of each model. The bottom right compares the equilibrium player densities.

(see top left plot of Figure 1). For the open loop model, we can approximate individual player costs using the mean field equation (42) below (see top right plot of Figure 1). We can also approximate the average cost under the open loop equilibrium, which we denote by \( \bar{\lambda}_{\text{open}}^{N} \); namely, averaging (16) over \( 1 \leq i \leq N \) and using the limit calculation (76) of Remark 7.4, we have for \( N \) large

\[
\bar{\lambda}_{\text{open}}^{N}(C) := \lambda_{\beta_{\text{open}}(C)}^{N} + \frac{C}{2} \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}(h_2 * \mu_{X}^{N,i})(X^{i}) \approx \frac{\beta_{\text{open}}(C)}{8} + \frac{C}{4(\beta_{\text{open}}(C) - \sigma^2)},
\]

where \( X \sim \mu_{\beta_{\text{open}}(C)}^{N} \) (recall definition (28) of \( \mu_{\beta}^{N} \)). Notice the open loop strategy prescribes a higher repulsion to accommodate the players densely packed around the origin, i.e., \( \beta_{\text{open}}(C) > \beta_{\text{closed}}(C) \), and in doing so achieves a lower average cost, i.e., \( \bar{\lambda}_{\text{open}}^{N}(C) < \frac{1}{N} \sum_{i=1}^{N} \lambda_{\beta_{\text{closed}}(C)}^{N,i} \approx \frac{\beta_{\text{closed}}(C)}{4} \). However, the closed loop Nash equilibrium is fair to all (no matter their rank) while the open loop Nash equilibrium has greater cost for players away from the origin. Lastly, recall \( \lambda_{\beta_{\text{closed}}(C)}^{N} \) is an auxiliary cost; indeed, \( \beta_{\text{open}}(2C) \) minimizes the actual global average cost with minimum \( \lambda_{\beta_{\text{open}}(2C)}^{N} \), for given \( C > -\sigma^4/8 \) (see red curves in Figure 1).
For either model of player information, we may write the equilibrium profile in feedback form using the function

\[ \phi^*_\beta(x) = -\nabla_x W_\beta(x) = (-\partial_i v^N_{\beta,i}(x))_{i=1}^N = \left( \frac{\beta}{2} (h_1 * \mu^N_x)(x^i) - \frac{x^i}{2} \right)_{i=1}^N. \]

Our discussion above implies a surprising consequence: we never have equality of the equilibrium strategies \( \phi^*_\text{closed}(C)(x) \), \( \phi^*_\text{open}(C)(x) \), even after taking \( N \to \infty \) (unless \( C = 0 \) and we remove the state–constraint and reciprocal gap integrability conditions in our Definition 3.1 of admissibility). Indeed, the closed and open loop Nash equilibria always yield different mean field behavior. Even though this discrepancy only appears in the radius of the limiting semicircle law \( \mu_\beta \) (see the bottom right plot of Figure 1), the repulsion parameter \( \beta/\sigma^2 \) distinguishes universality class in the local limit behavior. We believe this result offers a new perspective on Problem 9 of Deift’s list [29], which asks to construct a model to explain Šeba’s findings [61] that gaps between parked cars can exhibit GUE statistics (\( \beta = 2\sigma^2 \)) on a two–way street but GOE statistics (\( \beta = \sigma^2 \)) on a one–way street. Although we do not construct a specific model for this observational result, we have shown rigorously that universality class can vary depending only on player information and not just on the cost they face. For example, if \( C = 0, \sigma = 1 \), then \( \beta\text{closed}(0) = \frac{4}{3} \) and \( \beta\text{open}(0) = 2 \) (see the right two plots in Figure 1) note the latter corresponds to GUE [1].

5 Mean field equations

Observe we may write the solution (9) to the \( N \)–Nash system (8) in the form \( v^N_{\beta,i}(x) = U_\beta(x^i, \mu^N_x) \), where

\[ U_\beta(x, \mu) := \frac{x^2}{4} - \frac{\beta}{2} (h_0 * \mu)(x), \quad (x, \mu) \in \mathbb{R} \times \mathcal{P}_2^p(\mathbb{R}), \quad 1 < p < \infty, \]

(39)

(see the estimate (67) below to explain the choice of domain). Contrastingly, the state cost \( F^N_{\beta,i}(x) \) of (7) cannot naively be written as a function with a probability measure argument because the singular cost term \( "(h_2 * \mu)(x)" \) is not well defined for \( x \in \text{supp}(\mu) \). Instead, we expect this ill–behaved transform to be replaced by a local term proportional to the squared density \( "m(x)^2" \).

More precisely, to guess the mean field analog of the \( N \)–Nash system (8) associated to the closed loop model, we introduce the Voiculescu–Wigner master equation on \( \mathbb{R} \times \mathcal{P}_2^p(\mathbb{R}) \), \( 2 \leq p < \infty \):

\[ \int_{\mathbb{R}} \partial_\mu U(x, \mu)(z) \partial_x U(z, \mu) \mu(dx) + \frac{1}{2} |\partial_x U(x, \mu)|^2 = \frac{x^2}{8} + \frac{\pi^2 \beta^2}{8} m(x)^2 - \lambda, \]

(40)

(compare with equation (71) of Cardaliaguet–Porretta [20]). Here, the Wasserstein gradient \( \partial_\mu " \) can be formally interpreted as

\[ \partial_\mu U(x, \mu)(z) = \partial_z \frac{\delta \tilde{U}}{\delta m}(x, m)(z), \]

(41)

for \( \mu \) almost every \( z \in \mathbb{R} \), where we write \( "\tilde{U}(x, m)" \) for the mapping on \( \mathbb{R} \times L^p(\mathbb{R}) \) induced by \( U(x, \mu) \) and \( \frac{\delta}{\delta m} " \) is the Fréchet derivative on the linear space \( L^p(\mathbb{R}) \).

Similarly, to serve as the mean field analog of the ergodic HJB equation [20] for the global minimization problem associated to the open loop model, we introduce the following ergodic Hamilton–Jacobi equation on the Wasserstein space \( \mathcal{P}_2^1(\mathbb{R}) \):

\[ \frac{1}{2} \int_{\mathbb{R}} |\partial_\mu U(\mu)(x)|^2 \mu(dx) = \int_{\mathbb{R}} \left( \frac{x^2}{8} + \frac{\pi^2 \beta^2}{24} m(x)^2 \right) \mu(dx) - \lambda, \]

(42)
Proof. For the first statement, note by item (2) of Theorem 2.2 in Carton–Lebrun [24] that for any
it is enough that we can make sense of each term and establish equality.

(43)

Lemma 5.1. The formal calculations
\begin{align*}
\partial_\mu U_\beta(x, \mu) &= \partial_x U_\beta(x, \mu) = \frac{x}{2} - \frac{\beta}{2} H\mu(x), \\
\partial_\mu U_\beta(x, \mu)(z) &= \frac{\beta}{2} \frac{1}{x - z},
\end{align*}

obtained from the computational expression (41) can be understood rigorously as follows:

For every \( \mu \in \mathcal{P}_2^p(\mathbb{R}) \), the subdifferential of minimum norm is the unique element \( \partial_\mu U_\beta(x, \mu) \in L^2(\mu) \) satisfying, for every \( \phi \in C_0^\infty(\mathbb{R}) \),
\begin{equation}
\int \partial_\mu U_\beta(x, \mu)(z)\phi(z)\mu(dz) = \int \frac{z}{2} \phi(z)\mu(dz) - \frac{\beta}{4} \int \int \frac{\phi(z) - \phi(y)}{z - y} \mu(dz)\mu(dy).
\end{equation}

For every \( \mu \in \mathcal{P}_2^p(\mathbb{R}) \), \( 2 \leq p < \infty \), with density \( m(x) \) and for every \( \phi(x) \in L^2(\mu) \), we define
\begin{equation}
\int \partial_\mu U_\beta(x, \mu)(z)\phi(z)\mu(dz) := \lim_{\delta \downarrow 0} \frac{U_\beta(x, (T_\delta)_#\mu) - U_\beta(x, \mu)}{\delta} = \frac{\beta}{2} H[\phi \cdot m](x),
\end{equation}

where \( T_\delta(z) := z + \delta \phi(z) \).

Proof. Either Lemma 3.7 of Carrillo–Ferreira–Precioso [23] or Lemma 5.3 of Berman–Önnheim [12] establishes the existence of a minimal selection \( \partial_\mu U_\beta(x, \mu) \in L^2(\mu) \) for every \( \mu \in \mathcal{P}_2^p(\mathbb{R}) \), and the weak expression (45) appears as (3.12) of the first source or as (5.3) of the second.

 Turning to \( U_\beta(x, \mu) \), the action in (46) makes sense since \( \phi(x)m(x) \in L^{\frac{2p}{p+1}}(\mathbb{R}) \):
\begin{equation}
\int (\phi(x)^2 m(x))^{\frac{p}{p+1}} \cdot m(x)^{\frac{p}{p+1}}dx \leq \left( \int \phi(x)^2 m(x)dx \right)^{p/(p+1)} \left( \int m(x)^pdx \right)^{1/(p+1)} < \infty,
\end{equation}

by Hölder’s inequality. Then we can compute the metric derivative directly:
\begin{equation}
\lim_{\delta \downarrow 0} \frac{U_\beta(x, (T_\delta)_#\mu) - U_\beta(x, \mu)}{\delta} = \lim_{\delta \downarrow 0} \frac{-\beta}{2} \int \int \log(|z + \delta \phi(z) - x| - \log(|z - x|)) \mu(dz) = \frac{\beta}{2} H[\phi \cdot m](x),
\end{equation}

where the last equality follows by the monotone convergence theorem (see also the proof of Lemma 2.45 of Deift–Kriecherbauer–McLaughlin [28]).

Theorem 5.2. Fix \( \beta \in \mathbb{R} \). For every \( \mu \in \mathcal{P}_2^p(\mathbb{R}) \), \( 2 \leq p < \infty \), the pair \((U_\beta(x, \mu), \frac{\partial}{\partial x})\) forms a solution of the Voiculescu–Wigner master equation (40). For every \( \mu \in \mathcal{P}_2^3(\mathbb{R}) \), the pair \((U_\beta(\mu), \frac{\partial}{\partial \mu})\) forms a solution to the ergodic Hamilton–Jacobi equation (42).

Remark 5.3. As Cardaliaguet points out on pg. 3 of [18], the local dependence on the righthandside of the master equation (40) makes it difficult to define a general notion of solution. For our purposes, it is enough that we can make sense of each term and establish equality.

Proof. For the first statement, note by item (2) of Theorem 2.2 in Carton–Lebrun [24] that for any \( m \in L^p(\mathbb{R}) \) with \( 2 \leq p < \infty \), we have the Hilbert transform product rule
\begin{equation}
\pi^2 m^2(x) = (Hm)^2(x) - 2H[mHm](x),
\end{equation}

(49)
for almost every $x \in \mathbb{R}$. Moreover, since $\mu \in \mathcal{P}^p(\mathbb{R})$, we have $xm(x) \in L^{2p/(p+1)}(\mathbb{R})$ by (47). Since $\frac{2p}{p+1} \geq \frac{4}{3}$, the two expressions of (44) from Lemma 5.1 can now be combined rigorously to compute

$$\int \partial_x U_\beta(x, \mu)(z) \partial_x U_\beta(z, \mu) \mu(\text{d}z) = \frac{\beta}{4} H[z m(z)](x) - \frac{\beta^2}{4} H[m H m](x),$$

(50)

Next, we have

$$\frac{1}{2} \left( \frac{x}{2} - \frac{\beta}{2} H \mu(x) \right)^2 = \frac{x^2}{8} + \frac{\beta^2}{8} (H \mu)^2(x) - \frac{\beta}{4} x H \mu(x),$$

and these last two equations lead us to compute for almost every $x \in \mathbb{R}$

$$\frac{\beta}{4} (H[z m(z)](x) - x H m(x)) = -\frac{\beta}{4}.$$ 

The statement then follows by combining terms and using the product rule (49).

For the second claim, Lemma 6.5.4 of Blower [15] or Lemma 3.3 of Voiculescu [70] establishes the identity

$$\int_\mathbb{R} (H \mu)^2(x) \mu(\text{d}x) = \frac{\pi^2}{3} \int_\mathbb{R} m^3(x) \text{d}x$$

(51)

for $\mu \in \mathcal{P}^3(\mathbb{R})$. Further, for $\mu \in \mathcal{P}^3(\mathbb{R})$ we can integrate the identity

$$x(H \mu)(x) = 1 - \text{p.v.} \int_\mathbb{R} \frac{y}{y - x} \mu(\text{d}y)$$

against $\mu$ to compute $\int_\mathbb{R} x(H \mu)(x) \mu(\text{d}x) = \frac{1}{2}$. The proof then follows by direct computation. □

**Remark 5.4.** The notion of free information, defined on $L^3(\mathbb{R})$ by $\int m^3(x) \text{d}x$, was introduced by Voiculescu in [70] and the identity (51) indicates the role of “$H \mu(x)$” as score function. The Hamilton–Jacobi equation (42) essentially appears as the heuristic limit (1.4.4) of Biane–Speicher [13]; indeed, when $\lambda = \frac{\beta}{4}$, the righthand side of (42) can be written as the relative free Fisher information, defined on $\mathcal{P}^3(\mathbb{R})$ by (cf. Section 6.1 of [13])

$$\frac{1}{2} \int \left( \frac{\beta}{2} H \mu(x) - \frac{x}{2} \right)^2 \mu(\text{d}x).$$

(52)

Further, taking the Fréchet derivative of (52) formally yields the righthand side of (40) with $\lambda = \frac{\beta}{4}$. Therefore, finding a measure $\bar{\mu}$ such that $\bar{U}_\beta(x, \bar{\mu})$ is constant corresponds to a first order condition on the relative free Fisher information (52), which is consistent with Voiculescu’s information–minimizing characterization of Wigner’s semicircle law, Proposition 5.2 of [70]. These considerations motivate why we refer to (40) as the Voiculescu–Wigner master equation (although Lions [52] is responsible for the lefthand side). The idea that RMT-type statistics occur when some metric of information is minimized appears repeatedly in the literature; see Chapter 3.6 of Mehta [54] for the origins of this idea for a fixed “symmetry class,” but the attempt to identify a critical point of the repulsion parameter $\beta/\sigma^2$ using mutual information seems to be more recent, e.g., [71, 72].

**Characterizations of the semicircle law**

We close this section with a quick review of how the semicircle law arises from the mean field objects just discussed above. Fix $\beta > 0$. A standard characterization of the semicircle law $\mu_\beta$ with density
\( m_\beta \) defined in (4) is as the unique solution of the integral variational system

\[
U_\beta(x, \mu_\beta) \begin{cases} 
= c_\beta, & x \in \text{supp}(\mu_\beta) \\
\geq c_\beta, & \text{otherwise}
\end{cases}
\]

(53)

for some constant \( c_\beta \) (see Lemma 2.6.2 of [3], Theorem 4.4.1 of [15], Proposition 2.3 of [23], Problem 1 of [28], or Section 2 of [13] for various perspectives). In particular, \( U_\beta(x, \mu_\beta) \) is a constant for \( x \) in the support of \( \mu_\beta \). Similarly, Theorem 3 of the classic work [11] of Ben Arous–Guionnet confirms that the potential function \( U_\beta(\mu) \) of [43] achieves its minimum at \( \mu_\beta \).

Following Remark 5.4, take any measure \( \bar{\mu} \) with density \( \bar{m} \) such that \( U_\beta(x, \bar{\mu}) \) is constant for \( x \in \text{supp}(\bar{\mu}) \). Then the Voiculescu–Wigner master equation (40) reads

\[
0 = \frac{x^2}{8} + \frac{\pi^2 \beta^2}{8} m(x)^2 - \frac{\beta}{4},
\]

(54)

which implies \( \bar{m}(x) = m_\beta(x) \) necessarily (note \( \text{supp}(\bar{\mu}) = \text{supp}(\mu_\beta) \) follows because \( \bar{\mu} \in \mathcal{P}(\mathbb{R}) \)). Now the local, first order, ergodic MFG system associated to the master equation (40) has the form

\[
\begin{align*}
\frac{1}{2} |\partial_x v(x)|^2 + \lambda &= \frac{x^2}{8} + \frac{\pi^2 \beta^2}{8} m(x)^2 \\
-\partial_x (m(x) \partial_x v(x)) &= 0
\end{align*}
\]

(55)

and it follows from (53), (54) that this system is solved by the triple \( (U_\beta(x, \mu_\beta), \lambda_\beta, m_\beta(x)) \) with \( \lambda_\beta := \frac{\beta}{4} \). If there is another solution \( (v^*(x), \lambda^*, m^*(x)) \) with larger ergodic constant \( \lambda^* > \lambda_\beta \), then

\[
\frac{x^2}{8} + \frac{\pi^2 \beta^2}{8} m_\beta(x)^2 = \frac{1}{2} |\partial_x U_\beta(x, \mu_\beta)|^2 + \lambda_\beta \leq \frac{1}{2} |\partial_x v^*(x)|^2 + \lambda^* = \frac{x^2}{8} + \frac{\pi^2 \beta^2}{8} m^*(x)^2.
\]

But this last equation implies \( m_\beta(x) < m^*(x) \) for \( x \) in the support of \( m_\beta \), which is impossible since both are probability measures. Hence, \( \lambda_\beta \) is the largest ergodic constant for which the system (55) admits a solution. In the next section, Theorem 6.4 verifies that the ergodic constant \( \lambda_\beta \) coincides with the infimum of the costs of the associated mean field control problem.

### 6 Mean field game formulation

Fix a curve \( (\mu_t)_{t \geq 0} \in C([0, \infty), \mathcal{P}_2^3(\mathbb{R})) \) of probability measures.

**Definition 6.1.** A feedback functional \( \phi : \mathbb{R} \times \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R} \) is admissible for \( (\mu_t)_{t \geq 0} \) if for any \( \mathcal{F}_0 \)-measurable initial condition \( \xi \) with law in \( \mathcal{P}_2^3(\mathbb{R}) \), there is a unique solution \( (X_t)_{t \geq 0} = (X_t^\phi)_{t \geq 0} \) to the \( \mathbb{R} \)-valued dynamics

\[
dX_t = \phi(X_t, \mu_t) dt, \quad X_0 = \xi,
\]

(56)

such that \( (X_t)_{t \geq 0} \) has laws \( (\nu_t)_{t \geq 0} \in C([0, \infty), \mathcal{P}_2^3(\mathbb{R})) \) and satisfies the integrability condition

\[
\mathbb{E} \int_0^T \phi(X_t, \mu_t)^2 dt < \infty, \quad \text{for all} \quad T > 0.
\]

(57)

We denote the class of such feedbacks by \( \mathcal{A}(\mu) \).
For any \( \mathcal{F}_0 \)-measurable initial condition \( \xi \) with law in \( \mathcal{P}_2^3(\mathbb{R}) \), consider the optimization problem of minimizing the ergodic cost

\[
J(\psi|\xi, \mu_\cdot) := \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \int_0^T \left[ \frac{1}{2} \psi(X_t, \mu_t)^2 + \frac{X_t^2}{8} + \frac{\pi^2 \beta^2}{8} \cdot m_t(X_t)^2 \right] dt
\]  
(58)

over \( \psi \in \mathcal{A}(\mu_\cdot) \), subject to \( (X_t)_{t \geq 0} = (X^\psi_t)_{t \geq 0} \) satisfying (56) with \( X_0 = \xi \).

**Remark 6.2.** The notion of admissibility Definition 6.1 ensures the expression (58) is finite for every \( T > 0 \) (by Hölder’s inequality), which partially explains the choice of space \( \mathcal{P}_2^3(\mathbb{R}) \). Relatedly, this space is a natural domain for the relative free Fisher information (52) of Remark 5.4.

**Definition 6.3.** Let \( (\mu_t)_{t \geq 0} \in C([0, \infty), \mathcal{P}_2^3(\mathbb{R})) \) have densities \( (m_t(x))_{t \geq 0} \) and let \( \phi \in \mathcal{A}(\mu_\cdot) \). Then the pair \( (\phi(x, \mu_\cdot), (\mu_t)_{t \geq 0}) \) is an **ergodic MFG solution over the class** \( \mathcal{A}_\mu \subset \mathcal{A}(\mu) \) if the process \( (X_t)_{t \geq 0} = (X^\phi_t)_{t \geq 0} \) of (56) satisfies the fixed point condition \( X_t \sim \mu_t, \ t \geq 0, \) and the optimality condition

\[
\inf_{\psi \in \mathcal{A}} J(\psi|X_0, \mu_\cdot) = J(\phi|X_0, \mu_\cdot).
\]  
(59)

Recall \( \beta/\sigma^2 \)-Dyson Brownian motion \( (X^\ast_t)_{t \geq 0} \) of Theorem 1 of Rogers–Shi [59] (see also Section 4.3.2 of [3]) provides natural conditions under which, for any \( T > 0, (\mu^N_t)_{0 \leq t \leq T} \) converges a.s. on \( C([0, T]; \mathcal{P}(\mathbb{R})) \) as \( N \to \infty \) to the unique solution \( (\mu^\ast_t)_{0 \leq t \leq T} \) of the McKean–Vlasov equation

\[
\int f(x)\mu_T(dx) = \int f(x)\mu_0(dx) + \frac{1}{2} \int_0^T \left( \frac{\beta}{2} \int \frac{f'(y) - f'(y)}{x-y} \mu_t(dx) \mu_t(dy) - \int f'(x)\mu_t(dx) \right) dt
\]  
(60)

for any twice continuously differentiable test function \( f \) with \( f, f', f'' \) bounded. The flow \( (\mu^\ast_t)_{t \geq 0} \) is thus the natural candidate for an MFG solution in the sense of Definition 6.3. Before stating the next theorem that verifies this candidate, we define for \( (x, \mu, y, \alpha) \in \mathbb{R} \times \mathcal{P}_2^3(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \) the running cost

\[
f(x, \mu, \alpha) := \frac{\alpha^2}{2} + \frac{x^2}{8} + \frac{\pi^2 \beta^2}{8} \cdot m(x)^2
\]  
(61)

and the Hamiltonian

\[
H(x, \mu, y, \alpha) := y \cdot \alpha + f(x, \mu, \alpha).
\]  
(62)

Recall the definitions (39), (43) of \( U_\beta(x, \mu), U_\beta(\mu) \), respectively.

**Theorem 6.4.** Fix \( \beta > 0 \). Assume \( (\mu^\ast_t)_{t \geq 0} \) satisfies (60) with \( \mu^\ast_0 \in \mathcal{P}_2^3(\mathbb{R}) \). Consider the (nonlocal) feedback

\[
\phi^\ast_\beta(x, \mu) := -\partial_x U_\beta(x, \mu) = -\partial_x U_\beta(x, \mu) = \frac{\beta}{2} H(\mu)(x) - \frac{x}{2}.
\]  
(63)

Then the pair \( (\phi^\ast_\beta(x, \mu), (\mu^\ast_t)_{t \geq 0}) \) forms an ergodic MFG solution over the class \( \mathcal{A} \subset \mathcal{A}(\mu^\ast) \) of deviations \( \psi(x, \mu) \) such that \( (X_t)_{t \geq 0} = (X^\psi_t)_{t \geq 0} \) of (56) with \( X_0 \sim \mu^\ast \) satisfies the stability conditions

\[
\limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \int_0^T f(X_t, \mu_t^\ast, \psi(X_t, \mu_t^\ast)) dt < \infty, \quad \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}[U_\beta(X_T, \mu_T^\ast) - U_\beta(X_0, \mu_0^\ast)] = 0.
\]  
(64)

Further, the cost achieved by the optimal trajectories \( (X^\ast_t)_{t \geq 0} \) satisfies

\[
\inf_{\psi \in \mathcal{A}} J(\psi|X^\ast_0, \mu_\cdot) = J(\phi^\ast_\beta|X^\ast_0, \mu_\cdot) = \lambda_\beta = \frac{\beta}{4}.
\]
Proof. Note the curve \((\mu^*_t)_{t \geq 0}\) is in \(C([0, \infty), \mathcal{P}^2_2(\mathbb{R}))\) by known regularity results: first, \(\mu^*_t\) remains in \(\mathcal{P}_2(\mathbb{R})\) for all \(t > 0\) being a gradient flow of the “free energy” functional \(U_\beta(\mu)\) by Theorem 3.2.1(1) of Carrillo–Ferreira–Precioso \[23\] second, \(\mu^*_t\) remains in \(\mathcal{P}^3(\mathbb{R})\) for all \(t > 0\) by Corollary 5.3 of Biane–Speicher \[13\] or Remark 7.7 of Biler–Karch–Monneau \[14\] after rescaling variables as in Section 2.1 of \[23\] (see also Proposition 4.7 of Voiculescu \[70\]). Now Theorem 3.8 of \[23\] implies the densities \((m^*_t(x))_{t \geq 0}\) of \((\mu^*_t)_{t \geq 0}\) satisfy the following nonlinear transport equation (in the distributional sense; cf. (8.1.3) of \[2\]):

\[
\partial_t m^*_t(x) + \partial_x \left( m^*_t(x)\phi^*_\beta(x, \mu^*_t) \right) = 0.
\]

Then to obtain the probabilistic representation \((X_t^*)_{t \geq 0} = (X^*_t)_{t \geq 0}\) as in \((56)\), we have by the free energy identity (see Theorem 11.2.1 of \[2\], Proposition 6.1 of \[13\], or Theorem 3.2.4(4) of \[23\])

\[
\int_0^T \int_{\mathbb{R}} |\phi^*_\beta(x, \mu^*_t)|^2 \mu^*_t(dx)dt = U_\beta(\mu^*_0) - U_\beta(\mu^*_T) < \infty.
\]

Besides confirming \((57)\), this checks the rather weak sufficient condition of Theorem 8.2.1 from Ambrosio–Gigli–Savaré \[2\], which furnishes a probability measure on path space \(C([0, T], \mathbb{R})\) for all \(T > 0\) such that the associated process satisfies \(X_t^* \sim \mu^*_t\) for each \(t \geq 0\) (see equation (8.2.8) of \[2\]). Hence, we have checked the Definition 6.1 of admissibility for \((63)\), i.e., \(\phi^*_\beta(x, \mu) \in \mathcal{A}(\mu^*_t)\).

Now it suffices to check the optimality condition \((59)\) and the conditions \((64)\) for \((X_t^*)_{t \geq 0}\). Toward this end, consider an admissible \(\psi(x, \mu) \in \mathcal{A}\) and let \((X_t)_{t \geq 0} = (X^*_t)_{t \geq 0}\) solve \((56)\) with \(X_0 \sim \mu^*_0\). Then by Lemma 5.1 and the chain rule of Lemma 6.5 below, we have (compare with \((43)\))

\[
\mathbb{E}[U_\beta(X_T^*, \mu^*_T) - U_\beta(X_0, \mu^*_0)] = \mathbb{E} \int_0^T \left( \partial_x U_\beta(X_t^*, \mu^*_t) \cdot \psi(X_t^*, \mu^*_t) + \int_{\mathbb{R}} \partial_\mu U_\beta(X_t, \mu^*_t)(z) \phi^*_\beta(z, \mu^*_t) m^*_t(z)dz \right) dt,
\]

where the second equality uses that \(U_\beta(x, \mu)\) solves the master equation \((40)\) for \(\mu \in \mathcal{P}^3_2(\mathbb{R})\) by Theorem 5.2. Since the difference of the Hamiltonian values in \((66)\) is nonnegative, we have for \(\psi \in \mathcal{A}\)

\[
J(\psi|X_0, \mu^*) = \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \int_0^T f(X_t^*, \mu^*_t, \psi(X_t^*, \mu^*_t))dt \geq \lambda_\beta = \frac{\beta}{4}.
\]

To establish equality here under the feedback \(\phi^*_\beta(x, \mu)\) of \((63)\), note for this choice the difference of the Hamiltonians in \((66)\) vanishes, so we just need to establish the second condition in \((64)\). But from Theorem 3.2.3(3) of \[23\], we have the second moment convergence as \(T \to \infty\) of \(\mu^*_T\) to the semicircle law \(\mu_\beta\) defined by \[1\] (recall it is the minimum of \(U_\beta\) by Theorem 3 of Ben Arous–Guionnet \[11\]) as well as the relative entropy type estimate

\[
0 \leq U_\beta(\mu^*_T) - U_\beta(\mu_\beta) \leq e^{-c(T-s)}[U_\beta(\mu^*_s) - U_\beta(\mu_\beta)]
\]

\[1\] As pointed out in Remark 5.9 of Berman–Önheim \[12\], there is apparently an error here regarding the domain of \(U_\beta\), but the positive initial density in \(L^3(\mathbb{R})\) ensures sufficient regularity of the flow \((\mu^*_t)_{t \geq 0}\) for our purposes.
for every $0 \leq s \leq T$ and some constant $c > 0$. Hence, we readily have

$$\frac{1}{T}E[U_\beta(X^s_T, \mu^*_T)] = \frac{1}{T} \left[ 2 \cdot U_\beta(\mu^*_T) - \frac{1}{4} \int_\mathbb{R} x^2 \mu^*_T(dx) \right] \xrightarrow{T \to \infty} 0,$$

as required. This completes the proof after checking the application of the chain rule for (66).

Since by Lemma 5.1 we interpret “$\partial_\mu U_\beta(x, \mu)(z)$” through the metric derivative (46) rather than as a subdifferential of minimum norm (the latter characterization does not seem straightforward to realize), we prove the following chain rule to complement Lemma 5.1 and justify (66) above.

**Lemma 6.5.** Assume the same setting as for Theorem 6.4. Let $\psi(x, \mu) \in \mathcal{A}(\mu^*)$ so that the laws $(\nu_t)_{t \geq 0} \in C([0, \infty), \mathcal{P}_2^p(\mathbb{R}))$ of $(X^\psi_t)_{t \geq 0}$ of (56) have densities $n_t(x)$ satisfying (in the distributional sense) the transport equation

$$\partial_t n_t(x) + \partial_x (n_t(x)\psi(x, \mu^*_t)) = 0.$$

Then we have for any $T \geq 0$ the following chain rule:

$$\int_\mathbb{R} U_\beta(x, \mu^*_T) \nu_T(dx) - \int_\mathbb{R} U_\beta(x, \mu^*_0) \nu_0(dx) = \int_0^T \int_\mathbb{R} \left( \partial_x U_\beta(x, \mu^*_t) \psi(x, \mu^*_t) + \partial_\mu U_\beta(x, \mu^*_t)(z) \phi^*_\beta(z, \mu^*_t) \mu_t(dx) \right) n_t(dx) dt.$$

**Proof.** For any $\nu \in \mathcal{P}_2^p(\mathbb{R})$, $1 < p < \infty$, with density $n(x)$, we have by Minkowski’s inequality

$$\left| \int_\mathbb{R} \log(|z - x|) \nu(dx) \right| \leq \int_{|z - x| \leq 1} |\log(|z - x|)| \nu(dx) + \int_{|z - x| > 1} |\log(|z - x|)| \nu(dx) \leq 2 \left( \int_0^1 |\log(x)| \frac{dx}{x^p} \right) \frac{2}{p} \|n\|_{L^p(\mathbb{R})} + \int_\mathbb{R} |z - x| \nu(dx)$$

$$\leq C_p \left( \|n\|_{L^p(\mathbb{R})} + z^2 + \int_\mathbb{R} x^2 \nu(dx) \right).$$

for some constant $C_p > 0$. Hence, the mapping $\mu \mapsto \int_\mathbb{R} U_\beta(x, \mu) \nu(dx)$ is continuous on $\mathcal{P}_2(\mathbb{R})$ with respect to the 2–Wasserstein distance. Now fix $t \geq 0$ and let $p = 3$. Define $T_\delta(x) := x + \delta \psi(x, \mu^*_t)$, $\delta > 0$. We know by Proposition 8.4.6 of [2] that $d_2(\mu^*_t + \delta, (T_\delta)_\# \mu^*_t) = o(\delta)$, and by Lemma 5.1 and the continuity of $\mu \mapsto \int_\mathbb{R} U_\beta(x, \mu) \nu(dx)$, we have for almost every $t \geq 0$

$$\int_\mathbb{R} U_\beta(x, \mu^*_t + \delta) \nu(dx) - \int_\mathbb{R} U_\beta(x, \mu^*_t) \nu(dx) = \int_\mathbb{R} \left[ U_\beta(x, (T_\delta)_\# \mu^*_t) - U_\beta(x, \mu^*_t) \right] \nu(dx)$$

$$+ \int_\mathbb{R} U_\beta(x, \mu^*_t + \delta) \nu(dx) - \int_\mathbb{R} U_\beta(x, (T_\delta)_\# \mu^*_t) \nu(dx)$$

$$= \delta \int_\mathbb{R} \int_\mathbb{R} \partial_\mu U_\beta(x, \mu^*_t)(z) \phi^*_\beta(z, \mu^*_t) \mu^*_t(dz) \nu(dx) + o(\delta).$$

By (47) and (50), the integrand $x \mapsto \int_\mathbb{R} \partial_\mu U_\beta(x, \mu^*_t)(z) \partial_x U_\beta(z, \mu^*_t) \mu^*_t(dz)$ is in $L^{3/2}(\mathbb{R})$, and so its integral with respect to $\nu \in \mathcal{P}_2^3(\mathbb{R})$ is finite by Hölder’s inequality. The statement then follows from the product rule and the distributional equation for $(\nu_t)_{t \geq 0}$. \qed
Remark 6.6. To recover the ranked player perspective in the mean field setting, we can use equations (60), (65) satisfied by \((\mu^*_t)_t\) to formally compute the corresponding flow of distribution functions \((F^*_t(x))_t\) as
\[
\partial_t F^*_t(x) = m^*_t(x) \cdot \left(\frac{x}{2} - \frac{\beta}{2} H m^*_t(x)\right).
\]
From here, we may compute the dynamics for the quantiles \(\gamma^q_t := \inf\{x \in \mathbb{R} : F^*_t(x) \geq q\}, q \in [0,1]:
\[
d\gamma^q_t = -\frac{\partial_t F^*_t(\gamma^q_t)}{m^*_t(\gamma^q_t)} \, dt = \left(\frac{\beta}{2} H m^*_t(\gamma^q_t) - \frac{\gamma^q_t}{2}\right) \, dt = -\partial_x U(\gamma^q_t, \mu^*_t) \, dt.
\]
To make these manipulations rigorous, one can probably use the same line of argument as for Proposition 4 of Crisan–Kurtz–Lee [27]. Note (68) is exactly the expected limiting dynamics of a sequence \(X^{*i}_t\) of player(s) \(i = i(N)\) with \(\lim_{N \to \infty} i/N = q \in [0,1]\), which motivates the discussion on convergence of optimal trajectories in Remark 7.9.

7 Localized convergence of equations

If the reader compares the ergodic \(N\)-Nash system (8) with the Voiculescu–Wigner master equation (40), they might be puzzled how to go from one to the other; in particular, it is unclear what should account for the change in the form of the cost (not only the nonlocal–to–local transition, but also the coefficients). To the point, Proposition 3.4 shows the diffusion parameter \(\sigma^2\) is linked to the state cost \(F^{N,i}(x)\) of (7) through its relationship to the singular cost coefficient, \(C = \frac{\beta}{2} (\frac{3}{4} \beta - \sigma^2)\), but only the coupling parameter \(\beta\) appears (explicitly) in the master equation (40). The astute reader might ask whether we forgot a term in the master equation (40) to account for this discrepancy. It turns out that, despite vanishing, the diffusion term \(\sigma\) does contribute in the limit but its contribution cancels with the local contributions from the drift–interaction terms (Theorem 7.5), leaving only a local contribution from the control cost that accounts for the apparent discrepancy above.

To establish convergence of equations, we generalize in two ways the program outlined in Remark (x) after Theorem 2.3 of Lasry–Lions [50]: first we work with a natural class of player ensembles that are locally optimal for the Dyson game, and second we consider localized convergence. By “locally optimal,” we mean we can recover the master equation (40) by integrating the \(N\)-Nash system (8) against ensembles of the Nash–optimal universality class; by “localized,” we mean instead of working with an exchangeable system, we classify the ranked players by their mean field location.

More precisely, fix \(\beta > \sigma^2 > 0\) and let \(V(x)\) be twice continuously differentiable with \(V''(x) \geq c_V\) for all \(x \in \mathbb{R}\) and some constant \(c_V > 0\). Denote by \(X \in \mathcal{W}^N\) the ranked players in a possible long time equilibrium distributed according to a generalized \(\beta/\sigma^2\)-ensemble:
\[
\mu^N_V(dx) = m^N_V(x) \, dx = \frac{1}{Z^N_V} \cdot \prod_{1 \leq k < \ell \leq N} (x^\ell - x^k)^{\beta/\sigma^2} \cdot e^{-\frac{N}{\sigma^2} \sum_{i=1}^N V(x^i)} \mathbb{1}_{\mathcal{W}^N}(x) \, dx,
\]
where \(Z^N_V < \infty\) is a normalization constant. In particular, we consider player ensembles \(\mu^N_V\) of the same universality class as the Nash–optimal Gaussian ensemble \(\mu^N_\beta\) of (28), whose local behavior is dominated by the repulsion strength \(\beta/\sigma^2\). Note \(\mu^N_\beta\) is the invariant distribution of the gradient flow
\[
\frac{dX^i_t}{dt} = \left[\frac{\beta}{2} (h_1 * \mu^{N,i}_x)(X^i_t) - \frac{V''(X^i_t)}{2}\right] \, dt + \frac{\sigma}{\sqrt{N-1}}dW^i_t, \quad 1 \leq i \leq N.
\]
Indeed, the dynamics (70) are nonexplosive by Corollary 6.9 of Graczyk–Małecki [38], and a calculation similar to (31) and (32) confirms admissibility of (70) for the Dyson game when \(\beta > \sigma^2\).
By Theorems 4.4.1, 4.4.3(i), and 5.4.3 of Blower [15], there exists a unique measure \( \mu_V \in \mathcal{P}^2(\mathbb{R}) \), compactly supported on a single interval with density \( m_V(x) \in L^2(\mathbb{R}) \), that satisfies the 1-Wasserstein convergence \( d_1(\mu_X^N, \mu_V) \to 0 \) almost surely for \( X \sim \mu_V \) and that satisfies the characterizing Euler–Lagrange (or Schwinger–Dyson) equation:

\[
\frac{\beta}{2} H \mu_V(x) - \frac{V'(x)}{2} = 0, \quad \text{for all } x \in \text{supp}(\mu_V).
\] (71)

Write \( \gamma^q = \gamma^q_V := \inf \{ x \in \mathbb{R} : \mu_V((-\infty, x]) \geq q \} \) for \( q \in [0, 1] \).

**Proposition 7.1.** Assume \( \beta > \sigma^2 > 0 \). Consider a sequence \( i = i(N) \) of player(s) such that \( \lim_{N \to \infty} i/N = q \in [0, 1] \). Then we have

\[
\lim_{N \to \infty} \frac{\mathbb{E}(h_2 * \mu_X^{N,i})(X^i)}{N - 1} = \lim_{N \to \infty} \frac{1}{(N - 1)^2} \mathbb{E} \sum_{k:k \neq i} \frac{1}{(X^i - X^k)^2} = \frac{\pi^2 \beta}{3(\beta - \sigma^2)} m_V(\gamma^q)^2.
\] (72)

**Remark 7.2.** Proposition 7.1 is an extension of the guess in Remark 3.9 of Gorin–Shkolnikov [37] to the case of a uniformly convex potential \( V(x) \) and any convergent sequence of indices, i.e., both at the edge (\( q = 0, 1 \)) and in the bulk (\( q \in (0, 1) \)). Indeed, our result implies their guess upon taking \( V(x) = x^2/2, \sigma = 1, q = 1, \lim_{N \to \infty} X^N = \gamma^1 = \sqrt{2}\beta; \) for convenience in referencing Section 3 of that paper, the relationship of our ensemble \( X \) to their notation \( \chi \) is \( \sqrt{\frac{N - 1}{N}} \sqrt{\frac{2\beta}{\beta}} X \to X/N \).

**Proof.** First, as \( N \to \infty \) we have \( \mathbb{E}(X^i - \gamma^q)^2 \to 0 \). To see this, we know by Corollary 6.3.5 of Blower [15] that \( \mu_V \) satisfies a logarithmic Sobolev inequality with constant \( (N - 1)/\sigma^2 \) and thus by Proposition 6.7.3 of the same reference satisfies a Poincaré inequality with the same constant, implying \( \text{Var}(X^i) \to 0 \) as \( N \to \infty \). The fact that \( \mathbb{E} X^i \to \gamma^q \) as \( N \to \infty \) follows from the results of Section 2.6 of [3], which yields the desired \( L^2 \) convergence.

Now for fixed \( N \), we can compute

\[
\sum_{k:k \neq i} \frac{1}{x^i - x^k} \cdot \prod_{1 \leq k < \ell \leq N} (x^\ell - x^k)^{\beta/\sigma^2} = \frac{\sigma^2}{\beta} \partial_i \left[ \prod_{1 \leq k < \ell \leq N} (x^\ell - x^k)^{\beta/\sigma^2} \right]
\]

and

\[
\left( \sum_{k:k \neq i} \frac{1}{x^i - x^k} \right)^2 - \frac{\sigma^2}{\beta} \sum_{k:k \neq i} \frac{1}{(x^i - x^k)^2} \cdot \prod_{1 \leq k < \ell \leq N} (x^\ell - x^k)^{\beta/\sigma^2} = \left( \frac{\sigma^2}{\beta} \right)^2 \partial_i \left[ \prod_{1 \leq k < \ell \leq N} (x^\ell - x^k)^{\beta/\sigma^2} \right].
\]

Hence, we have by integration by parts (see the proof of Lemma 4.3.17 of [3] and Section 3 of [37])

\[
\mathbb{E} \left[ V'(X^i) \cdot (h_1 * \mu_X^{N,i})(X^i) \right] = \frac{\mathbb{E} V'(X^i)^2}{\beta} - \frac{\sigma^2 \mathbb{E} V''(X^i)}{\beta(N - 1)} \to \frac{V'(\gamma^q)^2}{\beta}
\] (73)

and similarly

\[
\mathbb{E} \left[ [(h_1 * \mu_X^{N,i})(X^i)]^2 - \frac{\sigma^2 (h_2 * \mu_X^{N,i})(X^i)}{N - 1} \right] \to \frac{V'(\gamma^q)^2}{\beta^2}
\] (74)

Now let \( \mathcal{L}_V \) be the generator of (70) and take \( f(x) := \frac{V(x^i)}{2} - \frac{\beta}{2} (h_0 * \mu_X^{N,i})(x^i), x \in \mathcal{W}^N \), in the invariance identity \( \mu_V^N[\mathcal{L}_V f] = 0 \) (this application of integration by parts along with Proposition
3.4 recovers the calculation of the optimal cost in Theorem 4.2 for the Gaussian case). Calculating as in the proof of Proposition 3.4 and letting \( N \to \infty \) in this invariance identity, we have
\[
\frac{\beta}{4} \int_{\mathbb{R}} \frac{V'(\gamma z) - V'(z)}{\gamma - z} m_{V}(z) \, dz = \lim_{N \to \infty} \frac{\beta^2}{8} E \left[ \left[ (h_1 * \mu_{\mathbf{X}}^{N,i})(X^i) \right]^2 - \frac{\sigma^2 (h_2 * \mu_{\mathbf{X}}^{N,i})(X^i)}{N - 1} \right]
\]
\[
+ \lim_{N \to \infty} E \left[ \frac{V'(X^i)^2}{4} - \frac{\beta}{4} V'(X^i) \cdot (h_1 * \mu_{\mathbf{X}}^{N,i})(X^i) + \frac{3\beta}{8} \frac{V'(\gamma z)}{\beta^2} \right] m_{V}(z) \, dz - \frac{V'(\gamma z)^2}{\gamma - z} m_{V}(z) \, dz
\]
\[
= \frac{\beta^2}{8} \cdot \frac{V'(\gamma z)^2}{\beta^2} + \frac{V'(\gamma z)^2}{4} - \frac{\beta}{4} \cdot \frac{V'(\gamma z)^2}{\beta} + \frac{3\beta}{8} \frac{V'(\gamma z)}{\beta^2} \lim_{N \to \infty} \frac{E(h_2 * \mu_{\mathbf{X}}^{N,i})(X^i)}{N - 1}.
\]
Rearranging this last expression, we arrive at
\[
\lim_{N \to \infty} \frac{E(h_2 * \mu_{\mathbf{X}}^{N,i})(X^i)}{N - 1} = \frac{1}{3\beta(\beta - \sigma^2)} \cdot \left[ 2\beta \int_{\mathbb{R}} \frac{V'(\gamma z) - V'(z)}{\gamma - z} m_{V}(z) \, dz - \frac{V'(\gamma z)^2}{\gamma - z} m_{V}(z) \, dz \right].
\]
Since \( m_{V}(x) \in L^2(\mathbb{R}) \), we may apply the Hilbert transform product rule \([1]\) by item (2) of Theorem 2.2 in Carton–Lebrun \([24]\) together with some applications of Euler–Lagrange \((71)\) to get
\[
\pi^2 \beta^2 m_{V}(x) = \beta^2 (HM_{V})^2(x) - 2 \beta^2 HM_{V}\{x\} = [2\beta V'(x)HM_{V}(x) - V'(x)^2] - 2\beta HM_{V}[V'(x)].
\]
This is exactly the term in square brackets of \((75)\) when \( x = \gamma z \), which completes the proof. \( \square \)

**Remark 7.3.** For some choices of parameters, one can compute expressions like \((72)\) directly. To sketch this for the archetype choice \( X = x^2/2, \beta = 2, \sigma = 1 \) corresponding to the GUE ensemble \((1)\), we can use the asymptotic formula for the sine kernel (see, e.g., Section 3.5 of \([3]\)) to get
\[
\frac{E(h_2 * \mu_{\mathbf{X}}^{N,i})(X^i)}{N - 1} \approx m_2(\gamma z)^2 \cdot \int_{\mathbb{R}} \frac{1}{y^2} \left( 1 - \left( \frac{\sin(\pi y)}{\pi y} \right)^2 \right) \, dy = 2\pi^2 \frac{m_2(\gamma z)^2}{3}.
\]
Thus, an interesting aspect of the proof of Proposition 7.3 is that we did not need to rely on such detailed local limit behavior in the bulk, \( q \in (0, 1) \); it turns out that the content of the Verification Theorem 4.2 is the right structure to combine with the integration by parts trick \((73), (74)\).

**Remark 7.4.** One can sometimes symmetrize a ranked particle system to ensure exchangeability and tightness while preserving the empirical distribution; see Sections I.3.(e) and II.5 of Sznitman \([66]\). Correspondingly, if we average the optimal cost calculation of the Verification Theorem 4.2 over all indices, we can make use of the identities \((22), (23)\) to conclude the integrated version of the convergence \((72)\):
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \frac{E(h_2 * \mu_{\mathbf{X}}^{N,i})(X^i)}{N - 1} = \frac{1}{2(\beta - \sigma^2)} \frac{\pi^2 \beta}{3(\beta - \sigma^2)} E[\mu_{\beta}(X)^2],
\]
where \( \mathbf{X} \sim \mu_{\beta}^{N} \) and \( \bar{X} \sim \mu_{\beta} \). Recall in Section 4.1 the calculation \((76)\) aided in producing Figure 1.

Observe the control term has vanishing expectation in optimal equilibrium, i.e., \( \mu_{\beta}^{N} \left[ \partial_{x} v_{\beta}^{N,i} \right] = 0, \) and by Euler–Lagrange \((71)\), the analogous mean field identity holds exactly for the semicircle law \( \mu_{\beta} \), i.e., \( \partial_{x} U_{\beta}(x, \mu_{\beta}) = 0 \) for \( x \in \text{supp}(\mu_{\beta}) \). Nevertheless, the next theorem shows that as \( N \to \infty \) the control cost term \( \mu_{\beta}^{N} \left[ \frac{1}{2} (\partial_{x} v_{\beta}^{N,i})^2 \right] \) still contributes a local term in the bulk, i.e., at a location \( \gamma z \) with \( q \in (0, 1) \). The calculation of this contribution helps us establish one of the main results of this paper: the **localized convergence of equations**, from the \( N \)–Nash system \((8)\) to the Voiculescu–Wigner master equation \((40)\), against **locally optimal** player ensembles.
Theorem 7.5. Fix \( \beta > \sigma^2 > 0 \) and let \( V : \mathbb{R} \to \mathbb{R} \) be twice continuously differentiable satisfying \( V''(x) \geq c_V > 0 \) for all \( x \in \mathbb{R} \) for some constant \( c_V \). Recall the definitions (69) of \( \mu_N^V \) and (11) of \( \mu_V \). Note \( \mu_N^V \) is an admissible player ensemble for the closed loop Dyson game. Then for any sequence \( i = i(N) \) of indices with \( \lim_{N \to \infty} i/N = q \in [0, 1] \), we have the following asymptotic contributions:

1. The self-interaction term contributes
\[
\lim_{N \to \infty} \mu_V^N \left[ (\partial_i v_{\beta}^{N,i})^2 \right] = \frac{\pi^2 \beta^2 \sigma^2}{12(\beta - \sigma^2)} m_V(\gamma^q)^2 + |\partial_z U_\beta(\gamma^q, \mu_V)|^2. \tag{77}
\]

2. The drift–interaction term contributes
\[
\lim_{N \to \infty} \mu_V^N \left[ \sum_{k: k \neq i} \partial_k v_{\beta}^{N,k} \partial_k v_{\beta}^{N,i} \right] = \frac{\pi^2 \beta^2 \sigma^2}{12(\beta - \sigma^2)} m_V(\gamma^q)^2 + \int_\mathbb{R} \partial_\mu U_\beta(\gamma^q, \mu_V)(z) \partial_z U_\beta(z, \mu_V) \mu_V(dz).
\]

3. The diffusion term contributes
\[
\lim_{N \to \infty} \mu_V^N \left[ -\frac{\sigma^2}{2(N-1)} \Delta x v_{\beta}^{N,i} \right] = -\frac{\pi^2 \beta^2 \sigma^2}{6(\beta - \sigma^2)} m_V(\gamma^q)^2.
\]

Evidently, the local contributions cancel and so the \( N \)–Nash system (8) with \((v_{\beta}^{N,k}, \lambda_{\beta}^{N,k})_{k=1}^N\) converges against \((\mu_V^N)_{N \geq 2}\) along such index sequences to the master equation (40) at \((\gamma^q, \mu_V) \in \mathbb{R} \times \mathcal{P}_2(\mathbb{R})\).

Proof. First note by the Euler–Lagrange equation (71), we have
\[
|\partial_z U_\beta(x, \mu_V)|^2 = \left( \frac{\beta}{2} H \mu_V(x) - \frac{x^2}{2} \right)^2 = \left( \frac{V'(x)}{2} - \frac{x^2}{2} \right)^2 = \frac{V'(x)^2}{4} + \frac{x^2}{4} - \frac{x V'(x)}{2}, \ x \in \text{supp}(\mu_V),
\]
Let \( X \sim \mu_V^N \). Then we can expand the singular cost term to compute
\[
\mu_V^N \left[ (\partial_i v_{\beta}^{N,i})^2 \right] = \frac{\beta^2}{4} \mathbb{E} \left[ (h_1 * \mu_X^i)(X^i) \right] - \frac{\sigma^2}{\beta} \frac{(h_2 * \mu_X^i)(X^i)}{N-1}
\]
\[
+ \mathbb{E} \left[ \frac{(X^i)^2}{4} - \frac{\beta}{2} X^i \cdot (h_1 * \mu_X^i)(X^i) + \frac{\beta^2 \sigma^2}{4} \frac{(h_2 * \mu_X^i)(X^i)}{N-1} \right]_{N \to \infty}
\]
\[
\cdot \left. \frac{\beta^2}{4} \cdot \frac{V'(\gamma^q)^2}{\beta^2} + \frac{(\gamma^q)^2}{4} - \frac{\beta}{2} \cdot \frac{\gamma^q V'(\gamma^q)}{\beta} + \frac{\pi^2 \beta^2 \sigma^2}{12(\beta - \sigma^2)} m_\beta(\gamma^q)^2 \right]
\]
\[
= |\partial_z U_\beta(\gamma^q, \mu_V)|^2 + \frac{\pi^2 \beta^2 \sigma^2}{12(\beta - \sigma^2)} m_\beta(\gamma^q)^2,
\]
where we have used (72), (74) and the analog of the calculation for (73). The limiting expression for the diffusion term is immediate by (72). Finally, using these two limit calculations and the \( N \)–Nash system (8) itself, we have
\[
\mu_V^N \left[ \sum_{k: k \neq i} \partial_k v_{\beta}^{N,k} \partial_k v_{\beta}^{N,i} \right]
\]
\[
= \mu_V^N \left[ -\frac{\sigma^2}{2(N-1)} \Delta x v_{\beta}^{N,i} \right] - \frac{1}{2} (\partial_x v_{\beta}^{N,i})^2 + \frac{(x_i)^2}{8} + \frac{\beta}{2} \frac{(h_2 * \mu_X^i)(x_i)}{N-1} \right] - \frac{\beta}{4}
\]
\[
\cdot \left. \cdot \frac{\pi^2 \beta^2 \sigma^2}{8(\beta - \sigma^2)} m_V(\gamma^q)^2 - \frac{1}{2} |\partial_z U_\beta(\gamma^q, \mu_V)|^2 + \frac{(\gamma^q)^2}{8} + \frac{\beta}{2} \frac{(h_2 * \mu_X^i)(\gamma^q)}{3(\beta - \sigma^2)} \right] - \frac{\beta}{4}
\]
\[
= \frac{\pi^2 \beta^2 \sigma^2}{12(\beta - \sigma^2)} m_V(\gamma^q)^2 + \frac{(\gamma^q)^2}{8} + \frac{\pi^2 \beta^2 \sigma^2}{8} m_V(\gamma^q)^2 - \frac{1}{2} |\partial_z U_\beta(\gamma^q, \mu_V)|^2 - \frac{\beta}{4}.
\]
Since compactness of support implies $\mu_V$ has finite second moment by Hölder’s inequality, we can use Theorem 5.2 which confirms the pair $(U_\beta(x, \mu), \beta \frac{2}{4})$ satisfies the master equation (40) on $\mathbb{R} \times \mathcal{P}_2(\mathbb{R})$. Hence, the terms in the parentheses become the desired expression, completing the proof.

Remark 7.6. Besides rigorously connecting the $N$–Nash system (8) to the Voiculescu–Wigner master equation (40), Theorem 7.5 is nontrivial because it establishes convergence at the location $\gamma^q$. This fact confirms to some degree the intuition that players in the Dyson game really only care about the population at their mean field “location” as $N \to \infty$.

Remark 7.7. We view the convergence in (77) as a localized version of the heuristic limit (1.4.4) of Biane–Speicher 13 for relative free Fisher information (52) (cf. Remark 5.4). Their limit (1.4.4) under the (closed loop Nash–optimal) Gaussian ensemble $\mu_{\beta}^N$ of (28) is actually not difficult to compute directly using the argument of Remark 7.4, but we provide a more general computation as the first item of the following corollary of our work, which confirms the analogous convergence of equations for the auxiliary global problem associated to the open loop model.

Corollary 7.8. Assume $\beta > \sigma^2 > 0$ and recall the definitions (69) of $\mu_{\beta}^N$ and (71) of $\mu_V$. Let $\bar{X} \sim \mu_V$. Then we have the following asymptotic contributions:

1. The drift term contributes

$$\lim_{N \to \infty} \mu_{\beta}^N \left[ \frac{1}{N} |\nabla_{x} \bar{X}|^2 \right] = \frac{\pi^2 \beta^2 \sigma^2}{12(\beta - \sigma^2)} \mathbb{E}_m \bar{X}^2 + \frac{1}{2} \int_{\mathbb{R}} |\partial_\mu U_\beta(\mu)(x)|^2 \mu_V(dx)$$

2. The diffusion term contributes

$$\lim_{N \to \infty} \mu_{\beta}^N \left[ -\frac{\sigma^2}{2(N - 1)} \cdot \frac{1}{N} \Delta_x \bar{X} \right] = -\frac{\pi^2 \beta^2 \sigma^2}{12(\beta - \sigma^2)} \mathbb{E}_m \bar{X}^2.$$

Consequently, the ergodic HJB equation (20) on $W_N$ associated to the open loop game converges against $(\mu_{\beta}^N)_{N \geq 2}$ to the Hamilton–Jacobi equation (42) on Wasserstein space $\mathcal{P}_2(\mathbb{R})$.

Remark 7.9. Although this section returns to a perspective found in the original MFG work 50 of Lasry–Lions, the more modern account of Cardaliaguet–Delarue–Lasry–Lions 19 suggests other ways to approach the mean field convergence problem. The two convergence notions in their Theorem 2.13 are actually trivial for our prototype game: by design, we have $v_{\beta,i}^N(x) = U_\beta(x^i, \mu_x^N,i)$ for any $N \geq 2$ and $1 \leq i \leq N$, so in their terminology the solutions $v_{\beta,i}^N(x)$ simply coincide with the finite–dimensional projections of the so–called master field $U_\beta(x, \mu)$. In fact, this discussion suggests it is worth emphasizing an obvious, yet important, point: even if one establishes convergence of the solutions of an $N$–Nash system to the solution of a master equation in some sense, one may still need to check convergence of equations, as we have in Theorem 7.5 in order to confirm that this mean field equation is indeed the correct analog of the $N$–Nash system.

To justify the passage to the limit even further, the notion of convergence of optimal trajectories as in Theorem 2.15 of 19 is still an interesting question to pursue for the Dyson game. More precisely, recall the quantile trajectories $(\gamma^q_t)_{t \geq 0}$ of (68) from Remark 6.6. Then one can endeavor to identify the correct order of magnitude of $\mathbb{E} \sup_{t \in [0,T]} |X_i^t - \gamma^q_t|”$ for any $0 \leq t_0 \leq T$. We suspect recent results on rigidity estimates and local relaxation time for Dyson Brownian motion will be relevant; see, e.g., Theorem 2.4 of Bourgade–Erdős–Yau 16 and Corollary 3.2 of Huang–Landon 39 for the former, and Landon–Yau 47 for the latter.
8 Dyson game on the circle

The insights into the Dyson game we identified above extend to similar classes of optimization problems. To confirm this and to further emphasize the local nature of the Dyson game, we quickly review the analogous results of this paper when players live in the periodic setting of the circle. We will only review the governing equations and results, rather than articulate the probabilistic representation of the game. To emphasize the analogy with the nonperiodic setting above, we freely redefine notations for this section; for example, we now define for \( x \neq 2\pi k, k \in \mathbb{Z} \), (compare with the definitions in (2))

\[
h_0(x) := \log |\sin \left( \frac{x}{2} \right)|, \quad h_1(x) := \frac{1}{2} \cot \left( \frac{x}{2} \right), \quad h_2(x) := \frac{1}{4} \sin^{-2} \left( \frac{x}{2} \right),
\]

but we will write \( W^N := \{ x \in \mathbb{R}^N | x^1 < \cdots < x^N < x^1 + 2\pi \} \).

Closed loop model

The closed loop model for Dyson Game on the circle will have the same ergodic \( N \)-Nash System \((8)\) but with state cost given by

\[
F_{N,i}^N(x) := C \cdot \frac{(h_2 * \mu_{N,i}^N)(x^i)}{N - 1} \sum_{k:k \neq i} \frac{1}{\sin^2 \left( \frac{X^i - X^k}{2} \right)}, \quad x \in W^N,
\]

(78)

(compare with Sutherland’s model [65] and Spohn [64]). The next theorem gives the analog of Proposition 3.4.

**Proposition 8.1.** Assume \( \sigma \geq 0 \) and that the coefficient \( C \) of (78) satisfies \( C \geq -\sigma^4/6 \), so we may write \( C = \beta^2 (3/4 \beta - \sigma^2) \) for some \( \beta \in \mathbb{R} \). Define

\[
v_{\beta}^{N,i}(x) := \begin{cases} -\frac{\beta}{2} (h_0 * \mu_{N,i}^N)(x^i) & x \in W^N, \\ +\infty & \text{otherwise} \end{cases}, \quad 1 \leq i \leq N,
\]

(79)

and write \( \lambda_{\beta}^{N,i} := \frac{\beta^2}{32} \left[ 1 + \frac{2}{N-1} \right] \). Then the pairs \((v_{\beta}^{N,i}(x), \lambda_{\beta}^{N,i})\) form a classical solution to the associated \( N \)-Nash system on \( W^N \).

**Proof.** The proof follows by direct calculation just as for Proposition 3.4, but instead of (13), one uses

\[
2 \sum_{k:k \neq i} \sum_{\ell:\ell \neq k,i} \cot^k \cot^\ell = \sum_{k:k \neq i} \sum_{\ell:\ell \neq k,i} \cot^k \left[ \cot^i - \cot^\ell \right] = \sum_{k:k \neq i} \sum_{\ell:\ell \neq i,k} \cot^i \cot^\ell + \frac{(N-1)(N-2)}{4},
\]

(80)

where we have abbreviated \( \cot^k := h_1(X^k - X^\ell) = \frac{1}{2} \cot \left[ \frac{x^k - x^\ell}{2} \right] \) for \( k \neq \ell \).

Open loop model

By the same argument as for Lemma 3.7, the open loop model of the Dyson game on the circle is a potential game with the same ergodic HJB (20) but now with global cost function

\[
F_{N}^N(x) := C \cdot \frac{1}{2} \sum_{i=1}^{N} \frac{(h_2 * \mu_{x,i}^N)(x^i)}{N - 1}, \quad x \in W^N.
\]

(81)

The next theorem gives the analog of Proposition 3.8.
Proposition 8.2. Assume \( \sigma^2 \geq 0 \) and that the coefficient \( C \) of (81) satisfies \( C \geq -\sigma^4/4 \), so we may write \( C = \beta^2/(\beta - 2\sigma^2) \) for some \( \beta \in \mathbb{R} \). Define

\[
W_\beta(x) := \begin{cases} 
-\frac{\beta}{2(N-1)} \sum_{1 \leq k < \ell \leq N} \log \left| \frac{x^k - x^\ell}{2} \right| & x \in \mathbb{W}^N \\
\infty & \text{otherwise}
\end{cases}
\]

and write \( \lambda^N_\beta := \frac{\beta^2 N^4}{96 N^2 - 1} \). Then the pair \((W_\beta(x), \lambda^N_\beta)\) forms a classical solution to the associated open loop ergodic HJB equation on \( \mathbb{W}^N \).

Proof. The proof follows by direct calculation just as for Theorem 3.8, but instead of (22), one uses

\[
\sum_{i=1}^N \sum_{k:k \neq i} \sum_{\ell: \ell \neq i, k} \cot^{ik} \cot^{i\ell} = -\frac{N(N-1)(N-2)}{12},
\]

which follows by summing over \( i \) in the identity (80) (recall \( \cot^{ik} := \frac{1}{2} \cot \left( \frac{x^k - x^i}{2} \right) \) for \( k \neq \ell \)). \( \square \)

Comparison of closed and open loop models

For \( \beta > \sigma^2 \), the equilibrium dynamics \((X_t)_{t \geq 0}\) will have components given by

\[
dX^i_t = \frac{\beta}{2} (h_1 \ast \mu^N_i)(X^i_t) dt + \frac{\sigma}{\sqrt{N-1}} dW^i_t = \frac{\beta}{4(N-1)} \sum_{k:k \neq i} \cot \frac{X^i_t - X^k_t}{2} dt + \frac{\sigma}{\sqrt{N-1}} dW^i_t,
\]

which by Theorem 3.1 of [26] remains in the interior \( \mathbb{W}^N \) for all \( t > 0 \). Let \( \Pi : \mathbb{R} \to \mathbb{T} := \mathbb{R}/(2\pi \mathbb{Z}) \approx [0, 2\pi], x \mapsto z = \Pi(x) = [x] \), be the quotient map to the equivalence class \([x]\) of \( x \). Then the trajectories \((Z_t)_{t \geq 0} := (\Pi(X_t))_{t \geq 0}\) are ergodic on the compact space \( \Pi \mathbb{W}^N(\mathbb{W}^N) \subset \mathbb{T}^N \).

Now the Verification Theorems 4.2, 4.5 for \((Z_t)_{t \geq 0}\) carry over essentially verbatim, and further the optimal repulsions \( \beta_{\text{closed}}(C), \beta_{\text{open}}(C) \) of each model coincide exactly with the expressions (38) of the nonperiodic Dyson game, which emphasizes the local nature of the model. We can also compare and take limits of the costs:

\[
\lambda^{N,i}_{\beta_{\text{closed}}(C)} = \frac{\beta^2_{\text{closed}}(C)}{32} \left[ 1 + \frac{2}{N-1} \right] N \to \infty \frac{\beta_{\text{closed}}(C)^2}{32}, \quad \lambda^{N}_{\beta_{\text{open}}(C)} = \frac{\beta^2_{\text{open}}(C) N^2}{96} N \to \infty \frac{\beta_{\text{open}}(C)^2}{96}.
\]

Mean–field equations

We now consider the mean field picture of the system \((Z_t)_{t \geq 0} = (\Pi(X_t))_{t \geq 0}\) in the compact space \( \mathbb{T} = [0, 2\pi] \). For analogs of the mean field functionals (39), (43), define

\[
U_\beta(x, \mu) := -\frac{\beta}{2} (h_0 \ast \mu)(x) = -\frac{\beta}{2} \int_0^{2\pi} \log \left| \frac{x - y}{2} \right| \mu(dy), \quad (x, \mu) \in \mathbb{T} \times \mathcal{P}^p(\mathbb{T}), \quad 1 < p < \infty,
\]

and similarly define

\[
U_\beta(\mu) := -\frac{\beta}{4} \int_0^{2\pi} \int_0^{2\pi} \log \left| \frac{x - y}{2} \right| \mu(dy) \mu(dx), \quad \mu \in \mathcal{P}^p(\mathbb{T}), \quad 1 < p < \infty.
\]
For analogs of the mean field equations (40), (42), we introduce the master equation

\[ \int_0^{2\pi} \partial_\mu U_\beta(x, \mu)(z) \partial_x U_\beta(z, \mu) \mu(dz) + \frac{1}{2} |\partial_x U_\beta(x, \mu)|^2 = \frac{\pi^2 \beta^2}{8} m(x)^2 - \lambda, \quad (86) \]

and the Hamilton–Jacobi equation

\[ \frac{1}{2} \int_0^{2\pi} |\partial_\mu U_\beta(\mu)(x)|^2 \mu(dx) = \int_0^{2\pi} \frac{\pi^2 \beta^2}{24} m(x)^2 \mu(dx) - \lambda. \quad (87) \]

Notice in particular that the local population cost indicated by these equations is exactly the same as for the nonperiodic versions (40), (42)!

**Remark 8.3.** The Hamilton–Jacobi equations (42), (87) have connections to quantum mechanics. Indeed, the expression “\( \pi^2 \beta^2 m^3(x) \)” is related to the ground state energy per unit volume of Sutherland’s model [65] and appears in Spohn’s works [63], [64] (in equation (1.5) and on pg.6, respectively). It is also proportional to the integrand of free information; see Remark 5.4.

To sketch the proof that \( U_\beta(x, \mu) \) satisfies (86), we first define the Hilbert transform on the circle by

\[ \mathbb{H} \mu(x) := \text{p.v.} \int_0^{2\pi} \frac{1}{2} \cot \left( \frac{x - y}{2} \right) \mu(dy), \quad x \in \mathbb{T}. \]

We can formally compute using (41)

\[ \partial_x U_\beta(x, \mu) = -\frac{\beta}{2} \mathbb{H} \mu(x), \quad \partial_\mu U_\beta(x, \mu)(z) = \frac{\beta}{4} \cot \left[ \frac{x - z}{2} \right]. \]

and so

\[ \int_0^{2\pi} \partial_\mu U_\beta(x, \mu)(z) \partial_x U_\beta(z, \mu) \mu(dz) = -\frac{\beta^2}{4} \mathbb{H}[m \mathbb{H} \mu](x), \quad \frac{1}{2} |\partial_x U_\beta(x, \mu)|^2 = \frac{\beta^2}{8} (\mathbb{H} \mu)^2(x). \]

The result then follows upon using the product formula for the Hilbert transform on the circle (compare with (49)):

\[ \pi^2 m(x)^2 - \frac{1}{4} = (\mathbb{H} m)^2(x) - 2 \mathbb{H}[m (\mathbb{H} m)](x), \quad (88) \]

for any \( \mu \in \mathbb{P}^p(\mathbb{T}), \ 2 \leq p < \infty \), with density \( m(x) \). To confirm \( U_\beta(\mu) \) solves the Hamilton–Jacobi equation (87), we can formally compute

\[ \partial_\mu U_\beta(\mu)(x) = \partial_x U_\beta(x, \mu) = -\frac{\beta}{2} \mathbb{H} \mu(x) \]

Then we can integrate the product rule (88) to get

\[ \int_0^{2\pi} \left( \pi^2 m(x)^2 - \frac{1}{4} \right) \mu(dx) = \int_0^{2\pi} (\mathbb{H} m)^2(x) \mu(dx) - 2 \int_0^{2\pi} \mathbb{H}[m (\mathbb{H} m)](x) \mu(dx) = 3 \int_0^{2\pi} (\mathbb{H} m)^2(x) \mu(dx), \]

which finally gives us

\[ \frac{1}{2} \int_0^{2\pi} |\partial_\mu U_\beta(\mu)(x)|^2 \mu(dx) = \frac{\beta^2}{8} \int_0^{2\pi} (\mathbb{H} \mu)^2(x) \mu(dx) = \frac{\beta^2}{24} \left( \int_0^{2\pi} \pi^2 m(x)^3 dx - \frac{1}{4} \right), \]
as required.

The invariant distribution in the mean field limit is the uniform distribution \( \bar{\mu} \) with density

\[
\bar{m}(x) := \frac{1}{2\pi}1_{[0,2\pi]}(x)
\]

in both cases. The total cost indicated by the mean field equations can be computed as expected by (85):

\[
\pi^2 \beta_{\text{closed}}(C)^2 \cdot \bar{m}(x)^2 = \frac{\pi^2 \beta_{\text{closed}}(C)^2}{8} \left( \frac{1}{(2\pi)^2} \right) = \beta_{\text{closed}}(C)^2 = \lim_{N \to \infty} \lambda_{\beta_{\text{closed}}(C)}^N, \quad x \in \mathbb{T},
\]

and

\[
\int_0^{2\pi} \frac{\pi^2 \beta_{\text{open}}(C)^2}{24} \bar{m}(z)^3 \, dz = \frac{1}{2\pi} \int_0^{2\pi} \frac{\pi^2 \beta_{\text{open}}(C)^2}{24} \left( \frac{1}{(2\pi)^2} \right) \, dz = \frac{\beta_{\text{open}}(C)^2}{96} = \lim_{N \to \infty} \lambda_{\beta_{\text{open}}(C)}^N.
\]

**Convergence of equations**

Let \( Z \) be distributed as the invariant distribution of the periodic equilibrium dynamics \( (Z_t)_{t \geq 0} \) that are the image of the system \( (X_t)_{t \geq 0} \) in \( \mathbb{W}^N \) of (84) by the quotient map \( \Pi^{\otimes N} : \mathbb{R}^N \to \mathbb{T}^N \). Then analogously to Proposition 7.1 but more simply as in the exchangeable setting of Remark 7.4, one can combine the verification theorem for the open loop model with the identity (83) to compute

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}(h_2 * \mu_{Z,i}^N)(Z^i) = \frac{\beta}{12(\beta - \sigma^2)} = \frac{\pi^2 \beta}{3(\beta - \sigma^2)} \mathbb{E}\bar{m}(\bar{Z})^2,
\]

where \( Z \sim \bar{m}(z) \, dz \) of (89). Using the identity (83), we can further use symmetry to compute for any sequence of indices \( i = i(N) \)

\[
\lim_{N \to \infty} \mathbb{E} \left( \frac{1}{(N-1)^2} \sum_{i \neq k} \sum_{\ell \neq i, k} \frac{1}{2} \cot \left( \frac{Z^i - Z^k}{2} \right) \cdot \frac{1}{2} \cot \left( \frac{Z^i - Z^\ell}{2} \right) \right) = -\frac{1}{12}.
\]

The analog of Corollary 7.8 then follows in the same manner.

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