An elliptic quantum algebra for $\hat{sl}_2$

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Abstract

An elliptic deformation of $\hat{sl}_2$ is proposed. Our presentation of the algebra is based on the relation $RLL = LLR^*$, where $R$ and $R^*$ are eight-vertex $R$-matrices with the elliptic moduli chosen differently. In the trigonometric limit, this algebra reduces to a quotient of that proposed by Reshetikhin and Semenov-Tian-Shansky. Conjectures concerning highest weight modules and vertex operators are formulated, and the physical interpretation of $R^*$ is discussed.

1 Introduction

In this article, we propose an algebra $A_{q,p}(\hat{sl}_2)$ based on the $R$-matrix of the eight-vertex model, where $p$ is the elliptic nome and $q$ is the crossing parameter of the $R$-matrix.

Following the pioneering work of Sklyanin, several works on elliptic algebras have been published in which finite dimensional representations were discussed. For physical applications it is useful to define an algebra having highest weight modules and vertex operators. Here, therefore, we define an elliptic algebra and conjecture the existence of these structures...
which are natural deformations of their trigonometric counterparts. In addition to the mathematical interest, our motivation in searching for such an algebra is the wish to understand the structure of the space of states for the eight-vertex model \[3\] in similar way as for the six-vertex and related models (see \[4\] for a review).

One way of introducing a quantum algebra is to use the quantum inverse scattering method, or ‘RLL formalism’ \[5,6\]. We follow a similar path here. In the trigonometric case the algebra is presented by two generating series, \(L^+ (\zeta)\) and \(L^- (\zeta)\), which are power series in \(\zeta\) and \(\zeta^{-1}\), respectively. The structure of the elliptic \(R\)-matrix, however, forces us to use \(L^\pm (\zeta)\) which are Laurent series in \(\zeta\), involving both positive and negative powers of \(\zeta\). As a result, the number of generators of the algebra appears to double. We propose to define an elliptic algebra based on a single relation of the form \(RLL = LLR^*\), using a single \(L(\zeta)\). Here, the matrices \(R, R^*\) are elliptic \(R\)-matrices, with elliptic moduli differing by an amount depending on the level \(k\) of the representation on which \(L(\zeta)\) acts. We show that this definition resolves the problem of the said doubling.

Since the matrices \(R\) and \(R^*\) are different, the usual definition of the coproduct fails. Nevertheless, the tensor product of two representations can be defined when one of them has level 0. Assuming the existence of highest weight modules, we formulate the vertex operators as intertwiners, present their commutation relations and motivate their physical interpretation. In particular, the physical \(S\)-matrix is \(S = -R^*\).

Note that in the sine-Gordon model, similar commutation relations for vertex operators, with \(R \neq -S\), were presented by Lukyanov \[7,8\].

The text is organized as follows. We fix the notations for the \(R\)-matrix in section 2. In section 3 the elliptic algebra is defined. The reduction to the trigonometric case is discussed in section 4. In section 5 we formulate conjectures regarding the highest weight modules and vertex operators, in the level one case. Section 6 contains a discussion and remarks.

2 The \(R\)-matrices

The elliptic \(R\)-matrix of the eight-vertex model is a function of three parameters, \(p\), \(q\) and \(\zeta\). In terms of the parameters \(I\), \(I'\), \(\lambda\) and \(u\) introduced in \[3\] (cf. (10.4.23–24) and (10.7.9)),

\[
p = \exp \left( -\frac{\pi I'}{I} \right), \quad q = -\exp \left( -\frac{\pi \lambda}{2I} \right), \quad \zeta = \exp \left( \frac{\pi u}{2I} \right).
\] (1)
We introduce two $R$ matrices,

$$R^\pm(\zeta) = R^\pm(\zeta; p^{1/2}, q^{1/2}),$$

which have the structure

$$R^\pm(\zeta) = \begin{pmatrix} a^\pm(\zeta) & d^\pm(\zeta) \\ b^\pm(\zeta) & c^\pm(\zeta) \end{pmatrix}.$$  \hspace{1cm} (2)

When viewed as holomorphic functions of the complex variables $p^{1/2}$, $q^{1/2}$ and $\zeta$ with $|p^{1/2}| < |q| < 1$, $R^+$ and $R^-$ coincide with the $R$-matrix of [9] up to scalar factors. Here we adopt the formal series point of view, regarding $p^{1/2}$, $q^{1/2}$ and $\zeta$ as indeterminates, and $R^\pm$ as a formal series in $\zeta$.

The entries of (2) are given as follows. Let $B = \mathbb{C}(q^{1/2})[[p^{1/2}]]$ denote the ring of formal power series in $p^{1/2}$ whose coefficients are rational functions in $q^{1/2}$. Introduce the formal power series with coefficients in $B$

$$\rho(\zeta^2) = q^{-1/2} \frac{(q^2 \zeta^2; q^4)_\infty^2}{(\zeta^2; q^4)_\infty (q^4 \zeta^2; q^4)_\infty},$$  \hspace{1cm} (3)

$$\bar{\alpha}(\zeta) = \frac{(p^{1/2} q \zeta; p)_\infty (pq^4 \zeta^2; p, q^4)_\infty (p \zeta^2; p, q^4)_\infty}{(p^{1/2} q^{-1} \zeta; p)_\infty (pq^2 \zeta^2; p, q^4)_\infty},$$  \hspace{1cm} (4)

$$\bar{\beta}(\zeta) = \frac{(-pq \zeta; p)_\infty (pq^4 \zeta^2; p, q^4)_\infty (p \zeta^2; p, q^4)_\infty}{(-pq^{-1} \zeta; p)_\infty (pq^2 \zeta^2; p, q^4)_\infty},$$  \hspace{1cm} (5)

where

$$(z; p_1, \cdots, p_m)_\infty = \prod_{n_1, \cdots, n_m \geq 0} (1 - z p_1^{n_1} \cdots p_m^{n_m})$$

$$= \exp \left( -\sum_{k=1}^{\infty} \frac{1}{(1-p_1^k) \cdots (1-p_m^k)} \frac{z^k}{k} \right).$$

To specify the entries of (2), we demand that $a^\pm(\zeta)$, $b^\pm(\zeta)$ be even in $\zeta$, and $c^\pm(\zeta)$, $d^\pm(\zeta)$ be odd. We set

$$a^\pm(\zeta) + d^\pm(\zeta) = \rho(\zeta^{\pm2})^{\pm1} \frac{\bar{\alpha}(\zeta^{-1})}{\alpha(\zeta)},$$

$$b^\pm(\zeta) + c^\pm(\zeta) = \rho(\zeta^{\pm2})^{\pm1} q^{\pm1} \frac{1 + (q^{-1} \zeta)^{\pm1}}{1 + (q \zeta)^{\pm1}} \frac{\bar{\beta}(\zeta^{-1})}{\beta(\zeta)}.$$  \hspace{1cm} (6)
In the second equation the factor $(1 + (q\zeta)^\pm 1)^{-1}$ should be expanded in powers of $(q\zeta)^\pm 1$.

One can verify that the coefficients of the series $R^\pm(\zeta) = \sum_{n\in\mathbb{Z}} R^\pm_n \zeta^n$ satisfy

$$R^\pm_n \equiv 0 \mod (p^{1/2})^{\max(\mp n, 0)}B \quad \forall n \in \mathbb{Z}.$$ 

Note that when $p = 0$, $R^+(\zeta)$ (resp. $R^-(\zeta)$) contains only non-negative (resp. non-positive) powers in $\zeta$.

We regard $R^\pm(\zeta)$ as linear operators on $V \otimes V$, with $V = B_{\varepsilon_1} \oplus B_{\varepsilon_2}$, and set $R^\pm(\zeta)v_{\varepsilon_1} \otimes v_{\varepsilon_2} = \sum v_{\varepsilon_1} \otimes v_{\varepsilon_2} R^\pm(\zeta)_{\varepsilon_1\varepsilon_2;\varepsilon'_1\varepsilon'_2}$. When written in the matrix form, the entries of $R^\pm$ are arranged in the order $(\varepsilon_1, \varepsilon_2) = (++, (-+), (-+), (--))$. The $R$-matrices have the following properties:

**Yang-Baxter equation**

$$R^\pm_{12}(\zeta_1/\zeta_2) R^\pm_{13}(\zeta_1/\zeta_3) R^\pm_{23}(\zeta_2/\zeta_3) = R^\pm_{23}(\zeta_2/\zeta_3) R^\pm_{13}(\zeta_1/\zeta_3) R^\pm_{12}(\zeta_1/\zeta_2), \quad (7)$$

**Unitarity**

$$R^\pm_{12}(\zeta_1/\zeta_2) R^\mp_{21}(\zeta_2/\zeta_1) = \text{id}, \quad (8)$$

**Crossing symmetry**

$$R^\pm_{21}(\zeta_2/\zeta_1)^{t_1} = \sigma^x R^\pm_{12}((-q^{-1}\zeta_1/\zeta_2)) \sigma^x, \quad (9)$$

**Quasi-periodicity**

$$R^\pm_{12}(\zeta_1) = \sigma^x R^\pm_{12}(\zeta) \sigma^x = \sigma^x R^\mp_{12}(\zeta) \sigma^x, \quad (10)$$

$$R^\pm_{12}(-p^{1/2}\zeta) = \sigma^x R^-_{12}(-\zeta) \sigma^x = \sigma^x R^-_{12}(\zeta) \sigma^x. \quad (11)$$

Here, if $R^\pm(\zeta) = \sum a_i \otimes b_i$, with $a_i, b_i \in \text{End}(V)$, then $R^\pm_{21}(\zeta_1) = \sum b_i \otimes a_i$, $R^\pm_{13}(\zeta) = \sum a_i \otimes \text{id} \otimes b_i$, etc.. The Pauli matrices are chosen to be

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the suffix $j(= 1, 2)$ indicating that they are acting on the $j^{\text{th}}$-component.

The formulas (10)–(11) are the formal series version of the characterizing properties of the elliptic $R$-matrix in Belavin’s approach [10].
3 The algebra $\mathcal{A}_{q,p}(s\mathfrak{l}_2)$

Let us proceed to the definition of the elliptic algebra. Consider a free associative algebra $U''$ over $\mathbb{C}(q^{1/2})$ on the letters $\bar{L}_{\varepsilon\varepsilon',n}$, where $n \in \mathbb{Z}$, $\varepsilon, \varepsilon' = \pm$ and $\varepsilon\varepsilon' = (-1)^n$. By convention we set

$$\bar{L}_{\varepsilon\varepsilon',n} = 0 \quad \text{if } \varepsilon\varepsilon' \neq (-1)^n. \quad (12)$$

Let $U'$ denote the algebra obtained by adjoining an invertible central element $q^{-c/2}$ to $U'' \otimes \mathbb{C}(q^{1/2})B$. Let further $U = \lim_\leftarrow U'/p^{N/2}U'$ be the $p^{1/2}$-adic completion of $U'$.

Consider the formal series with entries in $U$

$$L(\zeta) = \begin{pmatrix} L_{++}(\zeta) & L_{+-}(\zeta) \\ L_{-+}(\zeta) & L_{--}(\zeta) \end{pmatrix} \quad (13)$$

where

$$L_{\varepsilon\varepsilon'}(\zeta) = \sum_{n \in \mathbb{Z}} L_{\varepsilon\varepsilon',n} \zeta^{-n}, \quad L_{\varepsilon\varepsilon',n} = (-p^{1/2})^{\max(0,n)} \bar{L}_{\varepsilon\varepsilon',n}. \quad (14)$$

We define the algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_2)$ by imposing the following relations on $L(\zeta)$:

$$R_{12}^+(\zeta_1/\zeta_2) \quad \frac{1}{2} L(\zeta_1) \frac{2}{2} L(\zeta_2) = \frac{1}{2} L(\zeta_1) R_{12}^+(\zeta_1/\zeta_2), \quad (15)$$

where

$$1 \quad \frac{L(\zeta)}{L(\zeta) \otimes \text{id}}, \quad 2 \quad \frac{L(\zeta)}{\text{id} \otimes L(\zeta)},$$

and

$$R^{+*}(\zeta) = R^+(\zeta; p^{1/2}, q^{1/2}), \quad p^{1/2} = p^{1/2}q^{-c}. \quad (16)$$

To be precise, consider the difference (LHS)−(RHS) of (14) and take its coefficients for various powers of $\zeta_1, \zeta_2$. Their matrix entries are well-defined elements of $U$. Let $\mathcal{I}$ denote the ideal generated by these matrix elements, and let $\overline{\mathcal{I}}$ denote its closure in the $p^{1/2}$-adic topology. Then we define $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_2) = U/\overline{\mathcal{I}}$.

By a standard argument based on equations (14)–(16) we find that the following quantum determinant belongs to the center of $\mathcal{A}_{q,p}(\widehat{\mathfrak{gl}}_2)$:

$$q^{-\det L(\zeta)} = L_{++}(\zeta/q)L_{--}(\zeta) - L_{-+}(\zeta/q)L_{+-}(\zeta).$$
Imposing further the relation $q^{-\det L(\zeta)} = q^{c/2}$ we define the quotient algebra

$$A_{q,p}(\hat{sl}_2) = A_{q,p}(\hat{gl}_2)/\langle q^{-\det L(\zeta)} - q^{c/2} \rangle.$$  

Unlike in the trigonometric case one cannot define the notion of weights since the $R$-matrix does not have a spin-conservation property, i.e. does not commute with matrices of the form $h \otimes h$ with $h$ an arbitrary diagonal matrix. Nevertheless the algebra $A_{q,p}(\hat{gl}_2)$ admits a $\mathbb{Z}$-grading

$$\deg L_{\pm\epsilon',n} = -n$$

which corresponds to the principal grading for affine Lie algebras. Clearly $A_{q,p}(\hat{sl}_2)$ inherits this grading as well.

4 Reduction to the trigonometric case

The formulation of [6] involves two $L$-operators, $L^\pm(\zeta)$. At $p = 0$, our algebra $A_{q,p}(\hat{gl}_2)$ reduces to a factor algebra of $A(R)$ of [6], in the following sense. The algebra $A(R)$, with the conditions $L^\pm_n = 0$ if $\mp n > 0$ and $\tilde{L}^\pm(\zeta)^{-1}$ (see eq. 3.21 of [11]) imposed, is our $A_q(\hat{gl}_2) \overset{\text{def}}{=} A_{q,0}(\hat{gl}_2)$. To see this, define the auxiliary $L$-operators

$$L^+(\zeta) = L(q^{c/2}\zeta), \quad L^-(\zeta) = \sigma^z L(-p^{1/2}\zeta)\sigma^z.$$  

These are Laurent series of the form

$$L^+(\zeta) = \cdots + p^{1/2}L^+_1\zeta^{-1} + L^+_0\zeta^0 + L^+_1\zeta + \cdots,$$

$$L^-(\zeta) = \cdots + L^-_1\zeta^{-1} + L^-_0\zeta^0 + p^{1/2}L^-_1\zeta + \cdots,$$

and hence at $p = 0$ they become power series in $\zeta^{\pm1}$, respectively. Using [11] one can show

**Proposition 1** The following relations hold in $A_{q,p}(\hat{gl}_2)$.

$$R^\pm_{12}(\zeta_1/\zeta_2) L^\pm_1(\zeta_1) L^\pm_2(\zeta_2) = L^\pm_1(\zeta_2) L^\pm_2(\zeta_1) R^\pm_{12}(\zeta_1/\zeta_2),$$

$$R^+_1(q^{c/2}\zeta_1/\zeta_2) L^+(\zeta_1) L^-(\zeta_2) = L^-(\zeta_2) L^+(\zeta_1) R^+_1(q^{-c/2}\zeta_1/\zeta_2).$$
These are the defining relations used in \[6\]. We remark that it is due to the scaled nome in \(R^*\) that these three equations follow from the single equation \((14)\).

Note that there exist three different presentations of the quantum affine algebra \(U_q(\hat{sl}_2)\). The first two, using the Chevalley generators \([12,13]\) and using the Drinfeld generators \([14]\), were shown by Beck \([15]\) to be isomorphic. A third, isomorphic, presentation is that of Ding and Frenkel \([11]\). The algebra \(A_q(\hat{sl}_2)\) is similar to that of \([11]\), the \(R\)-matrices differing by a scalar factor. It appears that the precise relation between \(A_q(\hat{sl}_2)\) and \(U_q(\hat{sl}_2)\) remains an open question.

5 Vertex operators

The simplest representation of \(U = A_{q,p}(\hat{sl}_2)\) is the analog of the spin-1/2 evaluation module of \(A_q(\hat{sl}_2)\). Let \(V_\xi = V \otimes B[\xi, \xi^{-1}]\). Then \(V_\xi\) becomes a \(U\)-module with the assignment

\[
L_{\varepsilon_1 \varepsilon'_1}(\zeta) v_{\varepsilon_2} = \sum_{\varepsilon_2} v_{\varepsilon_2} R^+ (\zeta/\xi)_{\varepsilon_1 \varepsilon_2 \varepsilon'_1 \varepsilon'_2}, \quad q^{c/2} = 1.
\]

In general we say that a \(U\)-module has level \(k \in \mathbb{Z}\) if the central element \(q^{c/2}\) acts as a scalar \(q^{k/2}\). Thus \(V_\xi\) has level 0.

For physical applications we are interested in representations with non-zero level. In what follows we will formulate conjectures concerning highest weight representations of level 1.

We expect the following to hold:

**Highest weight modules** There exist \(U\)-modules \(\mathcal{H}^{(i)} (i = 0, 1)\) generated by a single vector \(|i\rangle\) with the properties

\[
L_{\varepsilon \varepsilon', n} |i\rangle = 0 \quad (n > 0), \quad L_{++, 0} |i\rangle = a q^{(1-i)/2} |i\rangle, \quad L_{--, 0} |i\rangle = a q^{i/2} |i\rangle
\]

where \(a\) is a scalar to be specified below (see \((24)\)). They are \(\mathbb{Z}\)-graded:

\[
\mathcal{H}^{(i)} = \bigoplus_{d=0}^{\infty} \mathcal{H}^{(i)}_d, \quad \text{and have the character}
\]

\[
\sum_{d=0}^{\infty} \dim \mathcal{H}^{(i)}_d t^d = \prod_{j=1}^{\infty} (1 + t^j). \quad (16)
\]

**Vertex operators** There exist intertwiners of \(U\)-modules

\[
\Phi(\xi) : \mathcal{H}^{(i)} \rightarrow \mathcal{H}^{(1-i)} \otimes V_\xi, \quad \Phi(\xi) = \sum \Phi_\varepsilon(\xi) \otimes v_\varepsilon,
\]

\[
\Psi^*(\xi) : V_\xi \otimes \mathcal{H}^{(i)} \rightarrow \mathcal{H}^{(1-i)}, \quad \Psi^*_\varepsilon(\xi) = \Psi^*\varepsilon(\xi) (v_\varepsilon \otimes \cdot),
\]
which we call vertex operators of type I and type II, respectively. To be more precise, the vertex operators are formal series

\[ \Phi_{\varepsilon}(\xi) = \sum_{n \equiv i+(1+\varepsilon)/2 \mod 2} \Phi_{\varepsilon,n} \xi^{-n}, \]

\[ \Psi^*_{\varepsilon}(\xi) = \sum_{n \equiv i+(1-\varepsilon)/2 \mod 2} \Psi^*_{\varepsilon,n} \xi^{-n}, \]

with components \( \Phi_{\varepsilon,n}, \Psi^*_{\varepsilon,n} \):

\[ H(i) d \to H(1-i) d - n, \]

satisfying the intertwining relations

\[ \Phi_{\varepsilon}^2(\zeta_2) L_{\varepsilon,1} \Phi_{\varepsilon}(\zeta_1) = \sum R^+_{\varepsilon,1,1,1}'(\zeta_1/\zeta_2) L_{\varepsilon,1} \Phi_{\varepsilon}(\zeta_1), \]

\[ L_{\varepsilon,2} \Psi^*_{\varepsilon}(\zeta_1) = \sum R^*_{\varepsilon,1,1,1}'(\zeta_1/\zeta_2) L_{\varepsilon,1} \Psi^*_{\varepsilon}(\zeta_1), \]

(17)

(18)

We normalize the vertex operators by setting \( \Phi_{\varepsilon,0} |i\rangle = |1-i\rangle, \Psi^*_{\varepsilon,0} |i\rangle = |1-i\rangle. \)

**Commutation relations** The vertex operators satisfy the commutation relations

\[ \sum R^+_{\varepsilon,1,1,1}'(\zeta_1/\zeta_2) \Phi_{\varepsilon}(\zeta_1) \Phi_{\varepsilon}(\zeta_2) = \Phi_{\varepsilon,2}(\zeta_2) \Phi_{\varepsilon,1}(\zeta_1), \]

\[ \tau(\zeta_1/\zeta_2) \Psi^*_{\varepsilon,1}(\zeta_1) \Phi_{\varepsilon,1}(\zeta_1) = \Phi_{\varepsilon,1}(\zeta_1) \Psi^*_{\varepsilon,1}(\zeta_1), \]

\[ -\sum \Psi^*_{\varepsilon,2}(\zeta_2) \Psi^*_{\varepsilon,1}(\zeta_1) R^+_{\varepsilon,1,1,1}'(\zeta_1/\zeta_2) = \Psi^*_{\varepsilon,1}(\zeta_1) \Psi^*_{\varepsilon,2}(\zeta_2). \]

(19)

(20)

(21)

Here

\[ \tau(\zeta) = \zeta^{-1} \frac{(q^2; q^4)_{\infty}}{(q^2 \zeta^2; q^4)_{\infty}} \frac{(q^3 \zeta^{-2}; q^4)_{\infty}}{(q^3 \zeta; q^4)_{\infty}}, \]

\[ R(\zeta) = \tau(q^{1/2} \zeta^{-1})^{-1} R^+(\zeta). \]

(22)

Further, the inversion property

\[ g \sum_{\varepsilon} \Phi_{-\varepsilon}(-\zeta/q) \Phi_{\varepsilon}(\zeta) = \text{id} \]

holds for type I operators where the scalar \( g \) is defined in (25) below.

These conjectures are direct generalizations of the known structures for the trigonometric case [23,16]. We expect in particular the character (16)
to remain the same. The existence of such spaces $H^{(i)}$ and vertex operators have been proposed earlier in \[3, 17\], by physical arguments based on the corner transfer matrix method. The elliptic algebra and the type II operators were not discussed there. Commutation relations of the form (19)–(21), with $R \neq R^\ast$, appeared in the sine-Gordon case in \[\mathcal{F}\], equation (7.2).

One can rewrite the commutation relations (14) for the components as a ‘normal ordering rule’ of the form

$$L_{\varepsilon_1 \varepsilon_1', m} L_{\varepsilon_2 \varepsilon_2', n} = \sum_{\sigma_1, \ldots, \sigma_2' j \geq 0} C_j L_{\sigma_2 \sigma_2', (m+n+s)/2-j} L_{\sigma_1 \sigma_1', (m-n-s)/2+j},$$

where $s = 0, 1$ according to whether $m \equiv n \mod 2$ or not. The coefficients $C_j$ (depending on all $\varepsilon, \sigma, m, n$) are written in terms of the Taylor coefficients of known functions such as $\bar{\alpha}(\zeta)$ and $\bar{\beta}(\zeta)$. This makes it possible in principle to calculate matrix elements for products of $L$-operators. For instance we have

$$\langle i \mid (L_{++}(\zeta_1) L_{--}(\zeta_2) + L_{++}(\zeta_1) L_{--}(\zeta_2)) \mid i \rangle = a^2 q^{1/2} \frac{\beta^*(\zeta_2/\zeta_1)}{\beta(\zeta_2/\zeta_1)}$$

where $\beta^*(\zeta)$ denotes $\beta(\zeta)$ in (3) with $p$ replaced by $p^* = pq^{-2}$. We find that in order to satisfy the quantum determinant relation one must choose $a$ to be

$$a^2 = \frac{\beta(-q)}{\beta^*(-q)}.$$  \hspace{1cm} (24)

In the same way one can calculate the matrix elements of products of $\Phi$ operators to obtain

$$g = \beta(-q) g_0, \quad g_0 = \frac{(q^2; q^4)_\infty}{(q^2; q^4)_\infty}.$$  \hspace{1cm} (25)

In fact the type II operator in the trigonometric case can be expressed as

$$\Psi_{\varepsilon'}(\zeta) = a^{-1} \beta(-q) \sum_{\varepsilon} L_{\varepsilon \varepsilon'}(q^{1/2} \zeta) \Phi_{-\varepsilon}(-q^{3/2} \zeta),$$  \hspace{1cm} (26)

since both sides have the same intertwining property. Using the same definition in the elliptic case, the commutation relations \[33\], \[30\], \[31\] involving $\Psi^*$ are consequences of \[14\], \[17\] and \[19\]. We remark that conversely the inversion property \[23\] enables us to write $L(\zeta)$ as

$$L_{\varepsilon \varepsilon'}(\zeta) = a g_0 \Psi_{\varepsilon'}(q^{-1/2} \zeta) \Phi_{\varepsilon}(\zeta).$$
This formula has been pointed out in the trigonometric case by Miki [18].

The highest weight modules for generic level can be formulated in the same way, and proper generalizations of the relations in this section hold. Similarly, it is straightforward to extend the present formulation to the case of $sl_n$.

6 Discussion

As discussed in [4], type I vertex operators correspond to the half-column transfer matrix of the eight-vertex model. On the other hand the type II operators create the eigenstates of the transfer matrix, the factor $\tau(\zeta)$ being the corresponding single-particle eigenvalue. Such an interpretation was established in the trigonometric case in [13] (see also [4]). It is known [19] that the single-particle excitation energy for the eight-vertex model does not depend on the elliptic modulus, and therefore is the same as in the trigonometric case. Our commutation relation (20) agrees with that picture.

In the trigonometric case the $R$-matrix entering the $\Psi^{\ast}\Psi^{\ast}$ commutation relation is interpreted as the $S$-matrix for the excitations. For the eight-vertex model, we could not find the calculation of the $S$-matrix in the literature. However, it is known that in the limit $I \to \infty$ with $\lambda$, $u$ fixed, in terms of the parameters of (1), the eight-vertex model scales to massive Thirring/sine-Gordon field theory. Considering the $S$-matrix of this theory, Smirnov has suggested to us that the $S$-matrix of the eight-vertex model is given by $-R^{\ast}(\zeta)$ in the region $0 < p \leq q^4$, where there are no bound states [19].

There is another special case worth mentioning. As is well known, at $p = q^4$ the eight-vertex model decouples to two non-interacting Ising models. An independent construction of $\mathcal{H}^{(i)}$ and $\Phi(\zeta)$ was given in [17] using free fermions. In this case $-R^{\ast}(\zeta) = -\sigma^y \otimes \sigma^y$. Passing to the basis in which $\sigma^y$ is diagonal, we see that there are two species of excitations, those of the same kind mutually anti-commute and those of different kind commute. This agrees with the well known fact that for the single Ising model the $S$-matrix is $-1$.

There are several questions which arise: Is it possible to find an analog of the Chevalley generators which satisfy relations involving a finite number of terms, as do Uglov’s generators [20] for level 0? What is the relation between our algebra and the Sklyanin algebra, or Baxter’s intertwining vectors [21, 22]? How should one understand the lack of a Hopf algebra structure?
Is there an analog of the universal $R$-matrix? Do the matrix elements of products of vertex operators satisfy difference equations analogous to the $q$-KZ equations [23]? Can one find an integral formula for them by bosonizing the level one modules?

These remain interesting open problems.

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