Networked Constrained Cyber-Physical Systems subject to malicious attacks: a resilient set-theoretic control approach

Walter Lucia* Bruno Sinopoli** Giuseppe Franzè*

Abstract

In this paper a novel set-theoretic control framework for Networked Constrained Cyber-Physical Systems is presented. By resorting to set-theoretic ideas and the physical watermarking concept, an anomaly detector module and a control remediation strategy are formally derived with the aim to contrast severe cyber attacks affecting the communication channels. The resulting scheme ensures Uniformly Ultimate Boundedness and constraints fulfillment regardless of any admissible attack scenario. Simulation results show the effectiveness of the proposed strategy both against Denial of Service and False Data Injection attacks.

I. INTRODUCTION

Cyber-Physical Systems (CPSs) represent the integration of computation, networking, and physical processes that are expected to play a major role in the design and development of future engineering systems equipped with improved capabilities ranging from autonomy to reliability and cyber security, see [15] and references therein. The use of communication infrastructures and heterogeneous IT components have certainly improved scalability and functionality features in several applications (transportation systems, medical technologies, water distributions, smart grids and so on), but on the other hand they have made such systems highly vulnerable to cyber threats, see e.g. the attack on the network power transmission [8] or the Stuxnet worm which infects the Supervision Control and Data Acquisition system used to regulate uranium enrichments [5]. Recently, the analysis of the CPS security from a theoretic perspective has received increasing attention and different solutions to discover cyber attack occurrences have been proposed, see [11], [12], [14], [18] and reference therein for detailed discussions. First, it is important to underline that if the attacker and defender share the same information then a passive anomaly detection system has no chance to identify stealthy attacks [18]. There, the authors propose the introduction of an artificial time-varying model correlated to the CPS dynamics so that any adversary attempting to manipulate the system state is revealed through its effect on such an extraneous time-varying system.

Along these lines, a relevant approach is provided in [12] where the physical watermarking concept is exploited. Specifically, a noisy control signal is superimposed to a given optimal control input in order to authenticate the physical dynamics of the system. In [16], the authors modify the system structure in order to reveal zero dynamic attacks, while in [11] a coding sensor outputs is considered to detect FDI attacks. It is worth to point out that most of works addressing CPSs focus their attention only on the detection problem leaving out the control countermeasures. To the best of the author’s knowledge very few control remediation strategies against cyber attacks have been proposed, see e.g. [6] where a first contribution for dealing with CPS affected by corrupted sensors and actuators has been presented.

In this paper two classes of cyber attacks will be analyzed: i) partial model knowledge attacks and ii) full model knowledge attacks [17]. The former is capable to break encryption algorithms which protect the communication channels and to modify the signals sent to the actuators and to the controller with the aim to cause physical damages. The second class can inject malicious data within the control architecture. Zero-dynamics and FDI attacks fall into such a category [17].

*Walter Lucia and Giuseppe Franzè are with the DIMES, Università degli Studi della Calabria, Via Pietro Bucci, Cubo 42-C, Rende (CS), 87036, ITALY {walter.lucia, giuseppe.franze}@unical.it

**Bruno Sinopoli is with the Electrical and Computer Engineering Department, Carnegie Mellon University, Pittsburgh, PA 15213, USA, brunos@ece.cmu.edu
In the sequel, the main aim is to develop a control architecture capable to manage constrained CPSs subject to malicious data attacks. As one of its main merits, the strategy is able to combine into a unique framework detection/mitigation tasks with control purposes. In fact both the detection and control phases are addressed by using the watermarking approach and the set-theoretic paradigm firstly introduced in [2] and then successfully applied in e.g. [4], [1], [7]. Specifically, the identification module can be viewed as an active detector that, differently from the existing solutions, does not require neither input nor model manipulations. Moreover, a watermarking like behavior can be simply obtained during the on-line computation of the control action. The attack mitigation is achieved by exploiting the concept of one-step controllable set jointly with cyber actions (communication disconnection, channels re-encryption) in order to ensure guaranteed control actions under any admissible attack scenario. Finally, a simulations campaign is provided under several attack scenarios to prove the effectiveness of the proposed methodology.

II. PRELIMINARIES AND NOTATIONS

Let us consider the class of Networked Constrained Cyber-Physical System (NC-CPS) described by the following discrete-time LTI model where we assume w.l.o.g. that the state vector is fully available:

\[ \begin{align*}
    x(t+1) &= Ax(t) + Bu(t) + B_ddx(t) \\
    y(t) &= x(t) + d_y(t)
\end{align*} \tag{1} \]

where \( t \in \mathbb{Z}_+ := \{0, 1, \ldots \} \), \( x(t) \in \mathbb{R}^n \) denotes the plant state, \( u(t) \in \mathbb{R}^m \) the control input, \( y(t) \in \mathbb{R}^p \) the output state measure and \( d_x(t) \in \mathcal{D}_x \subset \mathbb{R}^d_x \), \( d_y(t) \in \mathcal{D}_y \subset \mathbb{R}^d_y \), \( \forall t \in \mathbb{Z}_+ \), exogenous bounded plant and measure disturbances, respectively. Moreover (1) is subject to the following state and input set-membership constraints:

\[ u(t) \in \mathcal{U}, \quad x(t) \in \mathcal{X}, \forall t \geq 0, \tag{2} \]

**Definition 1:** Let \( S \) be a neighborhood of the origin. The closed-loop trajectory of (1)-(2) is said to be Uniformly Ultimate Boundedness (UUB) in \( S \) if for all \( \mu > 0 \) there exists \( T(\mu) > 0 \) and \( u := f(y(t)) \in \mathcal{U} \) such that, for every \( \|x(0)\| \leq \mu \), \( x(t) \in S \) \( \forall d_x(t) \in \mathcal{D}_x, \forall d_y(t) \in \mathcal{D}_y, \forall t \geq T(\mu) \).

**Definition 2:** A set \( \mathcal{T} \subseteq \mathbb{R}^n \) is Robustly Positively Invariant (RPI) for (1)-(2) if there exists a control law \( u := f(y(t)) \in \mathcal{U} \) such that, once the closed-loop solution \( x(t+1) = Ax(t) + B_f(y(t)) + B_ddx \) enters inside that set at any given time \( t_0 \), it remains in it for all future instants, i.e. \( x(t_0) \in \mathcal{T} \rightarrow x(t) \in \mathcal{X}, \forall d_x(t) \in \mathcal{D}_x, \forall d_y(t) \in \mathcal{D}_y, \forall t \geq t_0 \).

**Definition 3:** Given the sets \( \mathcal{A}, \mathcal{E} \subseteq \mathbb{R}^n \) \( \mathcal{A} \oplus \mathcal{E} := \{a + e : a \in \mathcal{A}, e \in \mathcal{E}\} \) is the Minkowski Set Sum and \( \mathcal{A} \sim \mathcal{E} := \{a \in \mathcal{A} : a + e \in \mathcal{A}, \forall e \in \mathcal{E}\} \) the Minkowski Set Difference.

A. Set-theoretic receding horizon control scheme (ST-RHC)

In the sequel, the receding horizon control scheme proposed in [1] and based on the philosophy developed in the seminal paper [2] is summarized. Given the constrained LTI system (1)-(2), determine a state-feedback \( u(\cdot) = f(y(\cdot)) \in \mathcal{U} \) capable i) to asymptotically stabilize (1) and ii) to drive the state trajectory \( x(\cdot) \in \mathcal{X} \) within a pre-specified region \( \mathcal{T}^0 \) in a finite number of steps \( N \) regardless of any disturbance realization \( d_x(t) \in \mathcal{D}_x, d_y(t) \in \mathcal{D}_y \).

The latter can be addressed by resorting to the following receding horizon control strategy:

Off-line -
- Compute a stabilizing state-feedback control law \( u^0(\cdot) = f^0(y(\cdot)) \) complying with (2) and the associated RPI region \( \mathcal{T}^0 \).
• Starting from $T^0$, determine a sequence of $N$ robust one-step ahead controllable sets $T^i$ (see [4]):

$$
T^0 := T
$$

$$
T^i := \{ x \in X : \forall d_x(t) \in D_x, d_y(t) \in D_y, \exists u \in U : \\
A(x + d_y(t)) + Bu + B_d d_x(t) \in T^{i-1} \} \\
= \{ x \in X : \exists u \in U : Ax + Bu \in \tilde{T}^{i-1} \}, i = 1, \ldots, N
$$

where $\tilde{T}^{i-1} := T^{i-1} \sim B_d D_x \sim A D_y$.

On-line -

Let $x(0) \in \bigcup_{i=0}^{N} T^i$, the command $u(t)$ is obtained as follows:

• Let $i(t) := \min \{ i : y(t) \in T^i \}$
• If $i(t) = 0$ then $u(t) = f_0(y(t))$ else solve the following semi-definite programming (SDP) problem:

$$
\text{arg min} J_{j(t)}(y(t), u) \quad \text{s.t.} \\
Ax(t) + Bu \in \tilde{T}^{i(t)-1}, \ u \in U
$$

where $J_{j(t)}(y(t), u)$ is a cost function and $j(t)$ a time-dependent selection index.

Remark 1: It is worth noticing that the cost function $J_{j(t)}(y(t), u)$ can be arbitrarily chosen without compromising the final objective of the control strategy and, in principle, it may be changed at each time instant.

III. PROBLEM FORMULATION

In the sequel, we consider CPSs whose physical plant is modeled as (1)-(2), while the controller is spatially distributed and a cyber median is used to build virtual communication channels from the plant to the controller and vice-versa, see Fig. 1. We assume that sensors-to-controller and controller-to-actuators...
communications are executed via Internet by means of encrypted sockets while all the remaining channels are local and externally not accessible. Moreover, malicious agents have the possibility to attack the communication over Internet by breaking the protocol security and may compromise/alter data flows in both the communication channels. Within such a context, two classes of attacks will be taken into account: a) Denials of Service (DoS) and b) False Data Injections (FDI): DoS attacks prescribe that attackers prevent the standard sensor and controller data flows, while FDI occurrences give rise to arbitrary data injection on the relevant system signals, i.e. command inputs and state measurements. Specifically, we shall model attacks on the actuators as

$$\tilde{u}(t) := u^c(t) + u^a(t)$$

where $u^c(t) \in \mathbb{R}^m$ is the command input determined by the controller, $u^a(t) \in \mathbb{R}^m$ the attacker perturbation and $\tilde{u}(t) \in \mathbb{R}^m$ the resulting corrupted signal. Similarly, sensor attacks has the following structure:

$$\tilde{y}(t) := y(t) + y^a(t)$$

where $y^a(t) \in \mathbb{R}^n$ is the attacker signal and $\tilde{y}(t) \in \mathbb{R}^n$ the resulting corrupted measurement.

From now on, the following assumptions are made:

**Assumption 1:** An encrypted socked between controller and plant can be on-demand reestablished in at most $T_{encry}$ time instants.

**Assumption 2:** The minimum amount of time $T_{viol}$ required to violate the cryptography algorithm is not vanishing, i.e $T_{viol} \geq T_{encry}$.

**Assumption 3:** No relevant channel delays are due to the communication medium, i.e. all the induced delays are less than the sampling time $T_c$.

**Remark 2:** Assumption 2 relies on the fact that communication channels are not compromised for at least $T_{encry}$ time instants downline of a new encrypted socked is established. As a consequence, the plant-controller structure is guaranteed w.r.t the sensor/actuator data truthfulness.

Then, the problem we want to solve can be stated as follows:

**Resilient Control Problem of NC-CPSs subject to cyber attacks (RC-NC-CPS) - Consider the control architecture of Fig. 7** Given the NC-CPS model (7)-(2) subject to DoS and/or FDI (6)-(7) attacks, determine

- (P1) An anomaly detector module $D$ capable to discover cyber attack occurrences;
- (P2) A control strategy $u(\cdot) = f(y(\cdot), D)$ such that Uniformly Ultimate Boundedness is ensured and prescribed constraints fulfilled regardless of the presence of any admissible attack scenario. Moreover, if $u^a(t) \equiv 0, y^a(t) \equiv 0$ (attack free scenario) and $d_x(t), d_y(t) \equiv 0$ (disturbance free scenario) $\forall t \geq t$ then the regulated plant is asymptotically stable.

The RC-NC-CPS problem will be addressed by properly customizing the dual model set-theoretic control scheme described in Section I-A.

**IV. SET-THEORETIC CHARACTERIZATION AND IDENTIFICATION OF ATTACKS**

In this section, an identification attack module will be developed. To this end, the following preliminaries are necessary.

First, notice that according to (4)-(5) the following set-membership conditions hold true:

$$x(t+1) \in Y^+(y(t), u^c(t))$$

$$T^{i(t)-1} \supseteq Y^+(y(t), u^c(t)) := \{z \in \mathbb{R}^n : \exists d_x \in D_x, d_y \in D_y, z = Ay(t) + Bu^c + B_d d_x + d_y\}$$

with $Y^+(y(t), u^c(t))$ the expected output prediction set. Then, by using the classification given in [17], we consider attackers having the following disclosure and disrupt resources:

- **Disclosure:** An attacker can access to the command inputs $u(t)$ and to the sensor measures $y(t)$;
• **Disrupt**: An attacker can inject arbitrary vectors $u^a(t), y^a(t)$ on the actuator and sensor communication channels but it cannot read and write on the same channel in a single time interval.

Finally, we consider attacks belonging to the following categories:

**Definition 4**: Let us denote with $I_a$ and $Y^+_a$ the attacker model knowledge and expected output prediction set, respectively, then an *Attack with full model knowledge* is an attack with full information, $I_{full}$, about the closed-loop dynamics of the physical plant,

$$I_a \equiv I_{full} := \{ [1] - [2], f^0, \{ T^i \}_{i=0}^N, y(t), \text{opt: (4)} - (5) \}$$

and perfect understanding of the expected output set, $Y^+_a \equiv Y^+$.

**Definition 5**: An *Attack with partial model knowledge* is an attack with partial information, $I_a$, about the closed-loop dynamics of (1), e.g.

$$I_a \subset I_{full} \quad \text{and} \quad Y^+_a \neq Y^+$$

### A. Attacks with partial model knowledge

The next result proposition shows that such attacks cannot compromise the system integrity while remaining stealthy.

**Proposition 1**: Given the NC-CPS model (1)-(2) subject to cyber attacks modeled as (6) and (7) and regulated by the state feedback law $u^c(t) = f(\tilde{y}(t))$ obtained via the ST-RHC scheme, then a detector module $D$, capable to reveal attacks with partial model knowledge, $I^a \subset I_{full}$, is achieved as the result of the following set-membership requirement:

$$\tilde{y}(t+1) \in Y^+$$

**Proof**: Under the attack free scenario hypothesis, the current control action $u^c(t)$ guarantees that the one-step ahead state evolution $y^+ := A\tilde{y}(t) + Bu^c(t) + B_d d_x(t) + d_y(t)$ belongs to $Y^+$:

$$y^+ \in Y^+ (\tilde{y}(t), u^c(t)), \quad \forall d_x(t) \in D_x, d_y(t) \in D_y \quad (12)$$

Since cyber attacks can occur, two operative scenarios can arise at the next time instant $t+1$:

(i) $\tilde{y}(t+1) \notin Y^+$,  
(ii) $\tilde{y}(t+1) \in Y^+$

If (i) holds true then the attack is instantaneously detected. Otherwise when (ii) takes place, the following arguments are exploited. First, an attacker could modify the control signal by adding a malicious data $u^a(t)$ and, simultaneously, the detection can be avoided by infecting the effective measurement $y(t)$ as follows:

$$\text{Find } y^a(t) : y(t) + y^a(t) \in Y^+.$$ 

Because the set $Y^+$ is unknown (see **Definition 5**) such a reasoning is not feasible. A second possible scenario could consist in injecting small sized perturbations $u^a(t)$ and $x^a(t)$ such that

$$Bu^a(t) + B_d d_x(t) \in D_x \quad \text{and} \quad x^a(t) + d_y(t) \in D_y,$$

Clearly, in this case by construction the computed command $u^c(t+1)$ remains feasible. ■
B. Attacks with full model knowledge

A simple stealthy attack can be achieved by means of the following steps:

**Stealthy Attack algorithm**

| Step | Description |
|------|-------------|
| 1    | Acquire $y(t)$; |
| 2    | Estimate the control action $\hat{u}(t)$ by emulating the optimization $\{4\}-\{5\}$; |
| 3    | Compute the expected disturbance-free one-step ahead state measurement $\hat{y}^+ = Ay(t) + B\hat{u}(t) \in Y^+_d$ |
| 4    | Corrupt $\hat{u}(t)$ with an arbitrary malicious admissible signal $a \hat{u}(t)$ such that $a \hat{u}(t) + \hat{u}(t) \in \mathcal{U}$ |
| 5    | Corrupt the output vector $y(t+1)$, according to expected one-step state evolution $\hat{y}^+$, i.e., $\hat{y}(t+1) + \hat{y}(t) =: \hat{y}^+$ |
| 6    | $t \leftarrow t + 1$, goto Step 1 |

Note that the above attack can never be identified by the proposed detector $D$ because condition $I_{\text{full}}$ is always satisfied. As a consequence, the only way to detect it is to increase the information available at the defender side $I_{\text{full}}$ so that the partial model knowledge attack structure is re-considered:

$$Y^+_d \neq Y^+$$

The key idea traces the philosophy behind the watermarking approach [12], where the defender superimposes a noise control signal (new information not available at the defender side) in order to authenticate the physical dynamics. In particular, a watermarking-like behavior can be straightforwardly obtained by using the ST-RHC property discussed in Remark 1.

**Proposition 2:** Let $\{1\}-\{2\}$ and $\{6\}-\{7\}$ be the plant and the FDI attack models, respectively. Let

$$J = \{j_k(y(t), u)\}_{k=1}^{N_j}, \quad F^0 = \{f_k^0(y(t))\}_{k=1}^{N_j}, \quad N_j > 1$$

be finite sets of cost functions and stabilizing state-feedback control laws compatible with $T^0$, respectively. Let $j(t): \mathbb{Z}_+ \to [1, \ldots, N_j]$ be a random function. If at each time instant $t$ the command input $u^c(t)$ is obtained as the solution of $\{4\}-\{5\}$ with $J_{j(t)}(\hat{y}(t), u)$ and $f^0_{j(t)}(\hat{y}(t))$ randomly chosen, then the anomaly detector module $\{11\}$ is capable to detect complete model knowledge $I_{\text{full}}$ attacks.

**Proof:** Because the additional information $j(t)$ is not available to the attacker, then the following time-varying information flow results:

$$I^\text{full}(t) := \{1\}-\{2\}, F^0, \{\mathcal{T}^i\}_{i=0}^N, j(t) \supset I^\text{full}$$

This implies that $I_a \subset I^\text{full}(t)$ and, as a consequence, a perfect stealthy attack is no longer admissible

$$Y^+(\hat{y}(t), u^c(j(t))) \neq Y^+_d(\hat{y}(t), u^c(j(t)))$$

Therefore, the detection rule $I_{11}$ is effective.

Finally, by collecting the results of Propositions $1|2$, a solution to the $P1$ problem is given by the following detector module:

$$\text{Detector}(D)(\hat{y}(t)) := \begin{cases} \text{attack} & \text{if} \quad \hat{y}(t+1) \notin Y^+ \\ \text{no attack} & \text{if} \quad \hat{y}(t+1) \in Y^+ \end{cases}$$

V. CYBER-PHYSICAL COUNTERMEASURES FOR RESILIENT AND SECURE CONTROL

Once a attack has been physically detected, the following cyber countermeasures can be adopted to recover an attack free scenario:

- Interrupt all the sensor-to-controller and controller-to-actuators communications links;
- Reestablishing new secure encrypted channels.

From a physical point of view, the prescribed actions imply that for an assigned time interval, namely $T_{\text{ency}}$, update measurements and control actions are not available at the controller and actuator sides, respectively. Therefore, the main challenge is:

**How can we ensure that, at least, the minimum safety requirements $x(t) \in X, u(t) \in \mathcal{U}$ are met while the communication are interrupted for $T_{\text{ency}}$ time instants?**

The next section will be devoted to answer to this key question.
A. \(\tau\)-steps feasible sets and associated set-theoretic controller (\(\tau\)-ST-RHC)

Let \(\mathcal{T}\) be a RPI region for the plant model (1)-(2) subject to the induced time-delay \(\tau\), see [7]. Then, a family of \(\tau\)-steps controllable sets, \(\{\mathcal{T}^i(\tau)\}_{i=1}^N\), can be defined as follows

\[
\mathcal{T}^0(\tau) := \mathcal{T} \\
\mathcal{T}^i(\tau) := \left\{ x \in \mathcal{X} : \exists u \in \mathcal{U} : A^kx + B(k)u \subseteq \tilde{\mathcal{T}}^{i-1}(\tau) \right\}
\]

with

\[
\left\{ \begin{align*}
\tilde{\mathcal{T}}^i(\tau) &= \mathcal{T}^i \sim B_dD_x \sim AD_y, \\
\tilde{\mathcal{T}}^i_k(\tau) &= \tilde{\mathcal{T}}_{k-1}^i(\tau) \sim A^{k-1}B_dD_x \sim A^kD_y, \quad k = 2, \ldots, \tau.
\end{align*} \right.
\]

(18)

An equivalent description \(\{\Xi^i(\tau)\}\) of (17) can be given in terms of the extended space \((x, u)\):

\[
\Xi^i(\tau) = \{(x, u) \in \mathcal{X} \times \mathcal{U} : A(k)x + B(k)u \in \tilde{\mathcal{T}}^{i-1}(\tau), \quad \forall k = 1, \ldots, \tau \}
\]

\[
= \bigcap_{k=1}^{\tau} \{(x, u) \in \mathcal{X} \times \mathcal{U} : A(k)x + B(k)u \in \tilde{\mathcal{T}}^{i-1}(\tau) \}
\]

(19)

Hence, the sets of all the admissible state and input vectors are simply determined as follows:

\[
\mathcal{T}^i(\tau) = \text{Proj}_{x}\Xi^i(\tau), \quad \Xi^i(\tau) = \text{Proj}_{u}\Xi^i(\tau)
\]

(20)

where \(\text{Proj}_{i}\) is the projection operator [4].

Proposition 3: Let the set sequences \(\{\Xi^i(T_{\text{encry}})\}_{i=1}^N\), \(\{\Xi^i_{\text{encry}}(T_{\text{encry}})\}_{i=1}^N\), \(\{\mathcal{T}^i(T_{\text{encry}})\}_{i=1}^N\), \(\{\mathcal{T}^i_{\text{encry}}(T_{\text{encry}})\}_{i=1}^N\) be given. Under the attack free scenario hypothesis \((u^{a}(t) \equiv 0, y^{a}(t) \equiv 0)\), the control action \(u^{c}(t)\), computed by means of the following convex optimization problem

\[
u^{c}(t) = \arg \min_{u} J_{j(t)}(y(t), u) \quad \text{s.t.} \quad [y(t), u] \in \Xi^i(T_{\text{encry}}), \quad u \in \Xi^i_{\text{encry}}(T_{\text{encry}})
\]

(21)

and consecutively applied to (1) for \(T_{\text{encry}}\) time instants, ensures: i) constraints fulfillment; ii) state trajectory confinement, i.e. \(x(t+k) \in \mathcal{T}^i_{\text{encry}}(T_{\text{encry}})\), \(\forall k = 1, \ldots, T_{\text{encry}}\), regardless of any \(d_x(\cdot) \in D_x\) and \(d_y(\cdot) \in D_y\) realizations and any cost function \(J_{j(t)}(y(t), u)\).

Proof: By construction of (17)-(20), it is always guaranteed that, if \(x(t) \in \mathcal{T}^i(T_{\text{encry}})\), the optimization (21)-(22) is feasible and an admissible \(u^{c}(t)\) there exists. Moreover if for \(T_{\text{encry}}\) time instants the command \(u^{c}(t)\) is consecutively applied to (1), one has that

\[
x(t+k) = A^kx(t) + \sum_{j=0}^{k-1} (A^jB)u^{c}(t) + \sum_{j=0}^{k-1} (A^jBd) d_x(j), \quad k = 1, \ldots, T_{\text{encry}}
\]

Then in virtue of (17), the disturbance-free evolution \(\tilde{x}(t+k)\) is

\[
\tilde{x}(t+k) \in \tilde{\mathcal{T}}^{i-1}_{k}(T_{\text{encry}}), \quad \forall k = 1, \ldots, T_{\text{encry}}
\]

and the following implications hold true

\[
\forall d_x(t+k) \in D_x, d_y(t+k) \in D_y \Rightarrow x(t+k+1) \in \mathcal{T}^{i-1}_{k}(T_{\text{encry}}), \quad k = 0, \ldots, T_{\text{encry}} - 1.
\]

Remark 3: The optimization (21)-(22) is solvable in polynomial time and the required computational burdens are irrespective of the number of steps \(T_{\text{encry}}\). Further details on the computation of the \(\tau\)-steps
In the sequel, the control strategy arising from the solution of (21)-(22) will be named $\tau$-ST-RHC controller. Note that it is not able to address all the attack scenarios, because if the more recent action $u(t)$ has been corrupted, the Proposition 3 statement becomes no longer valid. In such a case, the defender can only use a smart actuator module that locally, by means of simple security checks, is able to understand if the most recent command input is malicious.

B. Pre-Check and Post-Check firewalls modules

In what follows, two complementary modules, hereafter named Pre-Check and Post-Check, are introduced, see Fig. 1. The reasoning behind them is to passively detect attacks before they could harm the plant. In particular, such modules are in charge of checking the following state and input set-membership requirements:

$$\text{Pre-Check}(i(t)) := \begin{cases} \text{true} & \text{if } \tilde{u}(t) \in \{U^i(T_{ency})\}_{i=1}^{i(t)} \\ \text{false} & \text{otherwise} \end{cases} \quad (23)$$

$$\text{Post-Check}(i(t)) := \begin{cases} \text{true} & \text{if } y(t) \in \{T^i(T_{ency})\}_{i=1}^{i(t)} \\ \text{false} & \text{otherwise} \end{cases} \quad (24)$$

Conditions (23)-(24) check if the received $\tilde{u}(t)$ and the measurement $y(t)$ are “coherent” with the expected set level $i(t)$. If one of these tests fails, then a warning flag is sent to the actuator and an attack is locally claimed.

In response to the received flag, different actions are performed by the actuator: if the Pre-Check fails, $\tilde{u}(t) \notin \{U^i(T_{ency})\}_{i=1}^{i(t)}$, then $\tilde{u}(t)$ is discarded and the admissible stored input, hereafter named $u_{-1} := \tilde{u}(t - 1)$, applied; if the Post-Check fails, $y(t) \notin \{T^i(T_{ency})\}_{i=1}^{i(t)}$, then an harmful command $\tilde{u}(t - 1)$ has been applied bypassing the Pre-Check control. As a consequence, $u_{-1}$ cannot be used at the next time instants. In this circumstance, a possible solution consists in applying the zero input $u(t) \equiv 0$ until safe communications are reestablished. The latter gives rise to the following problem:

How can one ensure that the open-loop system subject to $u(t) \equiv 0$ fulfills the prescribed constraints (2) and is $UUB$?

The following developments provide a formal solution. Let denote with $T^{i_{max}}$, $i_{max} \leq N$ the maximum admissible set computed as follows

$$i_{max} = \max_{i \leq N} i \quad \text{s.t.} \quad \begin{align*}
A^k T^i(T_{ency}) & \oplus \sum_{j=0}^{k-1} A^j B D_x \oplus \prod_{j=1}^{\min(N,i+T_{viol})} T^j(T_{ency}) \\
\text{first term} & \quad \text{second term}
\end{align*} \quad (25)$$

Note that the first term represents the autonomous state evolution of (1), whereas the second one takes care of an unknown input $u \in U$. Moreover, the upper bound $\min(N,i_{max} + T_{viol})$ is complying with the Assumption 2 where it is supposed that, after the recovery phase, a new attack could only occurs after $T_{viol}$ time instants.

The reasoning behind the introduction of $T^{i_{max}}$ concerns with the following feasibility retention arguments. When data (state measurements and control actions) flows are interrupt the NC-CPS model (1) evolves in an open-loop fashion under a zero-input action. Therefore the computation (25) guarantees
that, starting from any initial condition belonging to \( \bigcup_{i=1}^{i_{\text{max}}} \mathcal{T}(T_{\text{encry}}) \) the resulting \( T_{\text{encry}} \)-step ahead state predictions of (11) are embedded in the worst case within the DoA \( \bigcup_{i=1}^{N} \mathcal{T}(T_{\text{encry}}) \).

Proposition 4: Let \( \{ \mathcal{T}(T_{\text{encry}}) \}_{i=1}^{N} \) and \( \mathcal{T}_{i_{\text{max}}}(T_{\text{encry}}) \) be the \( \tau \)-step ahead controllable set sequence and the maximum admissible set, respectively. If the plant model (11) is operating under a free attack scenario and the state evolution \( x(\cdot) \) is confined to \( \bigcup_{i=1}^{i_{\text{max}}} \mathcal{T}(T_{\text{encry}}) \), then the zero-input state evolution of (11) will be confined to \( \bigcup_{i=1}^{N} \mathcal{T}(T_{\text{encry}}) \) irrespective of any cyber attack occurrence and disturbance/noise realizations.

Proof: Constraints fulfillment and UUB trivially follow because \( 0 \in \mathcal{U} \) and \( \bigcup_{i=1}^{\min(N,i_{\text{max}}+T_{\text{viol}})} \{ \mathcal{T}(T_{\text{encry}}) \} \subseteq \mathcal{X} \).

C. The RHC algorithm

The above developments allow to write down the following computable scheme.

---

**Actuators Algorithm**

**Input:** \( \hat{u}(t) \), Pre-Check, Post-Check

**Output:** The applied control input \( u \), the expected set-level \( \hat{i} \)

**Initialization:** \( \hat{i} = i(0) \), \( u_{-1} = u^{c}(0) \)

1: if Pre-Check(\( \hat{i} \))=true & Post-Check(\( \hat{i} \))=true then
2: if \( \hat{u}(t) = \emptyset \) then
3: \( u(t) = u^{-1} \); \( \triangleright \) Command not received
4: else
5: \( u(t) = \hat{u}(t) \), \( \hat{i} = \hat{i} - 1 \); \( \triangleright \) Apply previous command
6: end if
7: else
8: if Pre-Check(\( \hat{i} \))=false then \( u(t) = u_{-1} \)
9: else \( u(t) = 0 \); \( \triangleright \) estimated set-level update
10: end if
11: end if
12: \( u^{-1} \leftarrow u(t) \)
13: \( t \leftarrow t + 1 \), goto Step 1

---

**\( \tau \)-ST-RHC Controller Algorithm (Off-line)**

**Input:** \( T_{\text{encry}} \)

**Output:** \( \{ \mathcal{E} \}_{i=0}^{N}(T_{\text{encry}}) \), \( \{ \mathcal{T} \}_{i=0}^{N}(T_{\text{encry}}) \), \( \{ \mathcal{Q}_{c} \}_{i=0}^{N}(T_{\text{encry}}) \), \( i_{\text{max}} \)

1: Compute a RPI region \( T^{0} \)
2: Compute the families of \( T_{\text{encry}} \)-steps controllable sets \( \{ \mathcal{E} \}_{i=0}^{N}(T_{\text{encry}}) \), \( \{ \mathcal{Q}_{c} \}_{i=0}^{N}(T_{\text{encry}}) \), \( \{ \mathcal{T} \}_{i=0}^{N}(T_{\text{encry}}) \) by resorting to recursion (12) and to the projection (20)
3: Determine the maximum index \( i_{\text{max}} \) satisfying (23)
4: Collect \( N_{j} > 1 \) cost functions (17) and terminal control law \( f_{0}^{j}(\tilde{x}(t)) \)

---

**\( \tau \)-ST-RHC Controller Algorithm (On-line)**

**Input:** \( \hat{y}(t) \), \( \{ \mathcal{E} \}_{i=0}^{N}(T_{\text{encry}}) \), \( \{ \mathcal{T} \}_{i=0}^{N}(T_{\text{encry}}) \), \( \{ \mathcal{Q}_{c} \}_{i=0}^{N}(T_{\text{encry}}) \), \( \text{Detector}(\hat{y}(t)) \), \( i_{\text{max}} \), \( J \)

**Output:** Computed command \( u^{c}(t) \)

**Initialization:** status=no attack, timer=0, encrypted communication channels, initialize Detector, Pre-Check, Post-Check, Actuator modules

**Feasibility start condition:** \( \exists i < (i_{\text{max}} + T_{\text{viol}}) \leq N : x(0) \in \bigcup_{i=0}^{\max(N,i_{\text{max}}+T_{\text{viol}})} \{ \mathcal{T}(T_{\text{encry}}) \} \)
1: if status==no attack then \hspace{1cm} \triangleright \text{Start Automa}
2: \hspace{1cm} \text{if Detector}(\tilde{s}(t)) == attack then status==attack
3: \hspace{1cm} \text{end if}
4: \text{else}
5: \hspace{1cm} if status==attack then \hspace{1cm} \triangleright \text{Wait channel encryption}
6: \hspace{1cm} if timer< T_{encry} then timer=timer+1;
7: \hspace{1cm} \text{else}
8: \hspace{1cm} Re-initialize all modules; status==no attack; timer=0;
9: \hspace{1cm} \text{end if}
10: \text{end if}
11: if status==no attack then \hspace{1cm} \triangleright \text{End Automa}
12: \hspace{1cm} \text{Send u}
13: \hspace{1cm} \text{Randomly choose the selection index } \tilde{j} = j(t);
14: \hspace{1cm} \text{if } i(t) == 0 \text{ then } u'(t) = f_{\tilde{j}}(\tilde{y}(t))
15: \hspace{1cm} \text{else}
16: \hspace{1cm} \text{Compute } u'(t) \text{ by solving (21)-(22) with cost function } J_{\tilde{j}}(\tilde{y}(t),u);
17: \hspace{1cm} \text{end if}
18: \hspace{1cm} \text{Send } u'(t) \text{ to the actuators;}
19: \text{else}
20: \hspace{1cm} \text{Interrupt all the communications}
21: \hspace{1cm} \text{Reestablish encrypted communication channels}
22: \hspace{1cm} \text{end if}
23: \hspace{1cm} \text{end if}
24: \hspace{1cm} \text{t} \leftarrow \text{t} + 1, \text{goto Step 1}

Remark 4: It is important to underline that Pre-Check and Post-Check modules need the current set-level \( i(t) \). Unfortunately, this information cannot be transmitted because it could be modified by some attackers. To overcome such a difficulty, the estimate \( \hat{i}(t) \) provided by the Actuator unit is used. Note that \( i(t) \) and \( \hat{i}(t) \) are synchronized at the initial \( t = 0 \) and at each recovery phase time instants, while in all the other situations it is ensured that such signals are compatible, i.e. \( \hat{i}(t) \geq i(t), \forall t \geq 0 \).

Theorem 1: Let \( \{\Xi^i(T_{encry})\}_{i=1}^N \), \( \{\Upsilon^i(T_{encry})\}_{i=1}^N \), \( \{T^i(T_{encry})\}_{i=1}^N \), be non empty controllable set sequences and
\[
\max(N,i_{\max}+T_{viol})\quad x(0) \in \bigcup_{i=0}^{\max(N,i_{\max}+T_{viol})} \{T^i(T_{encry})\}
\]

Then, the proposed set-theoretic control architecture (\( \tau \)-ST-RHC Controller, Detector, Pre-Check and Post-Check) always guarantees constraints satisfaction and Uniformly Ultimate Boundedness for all admissible attack scenarios and disturbance/noise realizations.

Proof: The proof straightforwardly follows by ensuring that under any admissible attack scenario the following requirements hold true:
- The on-line optimization problem (21)-(22) is feasible and the state trajectory \( x(t) \) is confined to \( \bigcup_{i=0}^{N} \{T^i(T_{encry})\} \);
- any attack free scenario can be recovered in at most \( T_{encry} \) time instants.

As shown in Section V-B, the worst case scenario arises when the attacker can successful inject a malicious input that simulates a stealthy attack. First, in virtue of the actions of the Pre-Check and Actuator modules, the input constraints \( u(t) \in \Upsilon \) are always fulfilled. Then, the Post-Check module ensures that, whenever the state trajectory diverges within the \( T_{encry} \) steps ahead controllable set sequence, a recovery procedure starts and an admissible zero-input state evolution takes place, see Proposition 4.

VI. NUMERICAL EXAMPLE

We consider the continuous-time model [3]
\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
1 & 4 \\
0.8 & 0.5
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} u(t) +
\begin{bmatrix}
1 \\
1
\end{bmatrix} d_x(t)
\]
subject to
\[ |u(t)| \leq 5, |x_1(t)| \leq 2.5, |x_2(t)| \leq 10, |d_1(t)| \leq 0.05 \]

The continuous time system has been discretized by means of Forward Euler-method with sampling time \( T_s = 0.02 \text{ sec} \). According to Assumption 1-3, we consider a reliable encrypted communication medium where \( T_{\text{encry}} = 4 \) time steps (0.08 sec) and \( T_{\text{viol}} = 5 \) time steps (0.1 sec).

First, the following polyhedral families of \( T_{\text{encry}} \)-steps controllable sets are computed (see Fig. 2):

\[
\{ \Xi_i \}_{i=0}^{60} (T_{\text{encry}}), \quad \{ \mathcal{U}_i \}_{i=0}^{60} (T_{\text{encry}}), \quad \{ T_i \}_{i=0}^{60} (T_{\text{encry}})
\]

and the maximum safe index set \( i_{\text{max}} = 45 \) has been determined.

The following simulation scenario is considered:

Starting from the initial condition \( x(0) = [-1.09, 5.11]^T \in T_{\text{encry}} \), regulate the state trajectory to zero regardless of any admissible attack and disturbance realization and satisfy the prescribed constraints.

In the sequel, the following sequence of attacks is considered:

- Partial model knowledge attacks - (Attack 1) DoS attack on the controller-to-actuator channel; (Attack 2) DoS attack on the sensor-to-controller channel; (Attack 3) FDI attack on the controller-to-actuator channel.
- FDI Full model knowledge attacks - (Attack 4). By following the Stealthy Attack Algorithm of Section IV-B, the attacker, tries to impose the malicious control action

\[
\hat{u}(t) = \arg \max_u |Ax + Bu|, \quad s.t. \quad Ax + Bu \in \hat{T}_0, \quad u \in \mathcal{U}_0
\]

with the aim to keep the state trajectory as far as possible from the equilibrium and to avoid the Post-Check detection by embedding the dynamical plant behavior within the terminal region.

First, it is interesting to underline that the state trajectory is confined within \( \{ T_i \}_{i=0}^{60} (T_{\text{encry}}) \) and asymptotically converges to the origin when an attack free scenario is recovered \( (i = 4.32 \text{ sec}) \).
**Attack 1** Starting from \( t = 0.14 sec \), the actuator do not receive new packets. According to the Actuators algorithm (Step 3) the most recent available command \( (u(t) = u^c(0.12) = 4.95) \) can be applied since both Pre-Check and Post-Check conditions are satisfied. At \( t = 0.16 sec \), the Detector identifies the attack (see Fig. 5) because

\[
\hat{y}(0.16) \notin Y^+ = \{ z \in \mathbb{R}^n : \exists d \in D, z = A\hat{x}(0.14) + Bu^c(0.14) + B_d d \}
\]

As prescribed in Steps 4-10 of the \( \tau\)-ST-RHC algorithm, the existing communications are interrupted and the procedure to reestablish new encrypted channels started. At \( t = 0.24 sec \), the encryption procedure ends and all the modules re-initialized. It is worth to notice that neither Pre-Check or Post-Check modules trigger a False Input or False output events. This is due to the fact that the most recent control
action was not corrupted and, by construction, it ensures that the state trajectory remains confined within the current controllable set for the successive $T_{encry}$ time instants, see Fig. 4.

-(Attack 2) Starting from $t = 0.34sec$ the Controller does not receive update state measurements and the Detector triggers an attack event. As a consequence, the network is disconnected and the actuator does not receive new control actions and the available command $u(t) = u^c(0.32) = -4.19$ is applied, see Fig. 5. At $t = 0.22sec$, the attack free scenario is recovered.

-(Attack 3) At $t = 0.52sec$ the attacker injects a signal $u^d(0.52) = 2$ that corrupts the current input $u^c(0.52) = -4.91$ as indicated in (6). Therefore, the actuator receives $\bar{u}(0.52) = -2.91$ that is still admissible as testified by the Pre-Check unit. The main consequence of the latter is that $x(0.54) \in T^{20}$ while the expected set-membership condition should have to be $T^{18}$: the Post-Check module and the Detector trigger an attack event and the Controller blocks all the communications. From now on, the Actuator logic imposes a zero-input state evolution, see Step 9. Although during the channel encryption phase (the encryption procedure ends at $t = 0.60sec$) the set-membership index increases (see Fig 4), this does not compromise feasibility retention because $x(0.52) \in \{T^i(T_{encry})\}_{i=0}^{\text{max}}$ and the zero-input state evolution will remain confined within the domain of attraction, i.e. $x(0.60) = [-0.58, 3.61] \in T^{42}(T_{encry})$, see Fig. 4.

-(Attack 4) At $t = 3.84sec$, with $x(3.83) \in T^0(T_{encry})$, an FDI attack corrupts both the communication channels, see Fig. 214.

The attacker is capable to remain stealthy until $t = 4.24sec$ and to manipulate the plant input and outputs. This unfavorable phenomenon is due to the fact that it is not possible to discriminate between the attack and the disturbance/noise realizations $d_x(t)$ and $d_y(t)$, i.e. $\bar{y}(t) \in \mathcal{Y}^+, \forall t \in [3.84, 4.22]sec$.

At $t = 4.24sec$, a different behavior occurs in response to $\bar{u}(4.24) = -5.025 \notin \mathcal{U}^0$ with the Pre-Check module that triggers an anomalous event arising when the attacker tries to impose $\bar{u}(4.24) = -4.993$ as the current input. Specifically, the attacker determines the estimate $\hat{u}^c(4.24)$ and modifies the control action as follows $\hat{u}^c(4.24) + u^d(4.24) = \hat{u}(4.24)$. Since a time-varying index is exploited in (21)-(22), the estimate $\hat{u}^c(4.24) = -0.032$ is numerically different from the effective control action $u^c(4.24) = -0.059$. Therefore $\hat{u}(4.24) \notin \mathcal{U}^0$ and, as a consequence, the attack is detected.

For the interested reader, simulation demos are available at the following two web links: J fixed: https://goo.gl/8CQ4b8, J random: https://goo.gl/DQhOxB
VII. CONCLUSIONS

In this paper a control architecture devoted to detect and mitigate cyber attacks affecting networked constrained CPS is presented. The resulting control scheme, which takes mainly advantage from set-theoretic concepts, provides a formal and robust cyber-physical approach against the DoS and FDI attack classes. Constraints satisfaction and Uniformly Ultimate Boundndeness are formally proved regardless of any admissible attack scenario occurrence. Finally, the simulation section allows to show the effectiveness and applicability of the proposed strategy under severe cyber attack scenarios.

REFERENCES

[1] D. Angeli, A. Casavola, G. Franzê, and E. Mosca, “An ellipsoidal off-line MPC scheme for uncertain polytopic discrete-time systems”, Automatica, Vol. 44, No. 12, pp. 3113–3119, 2008.
[2] D. P. Bertsekas and I. B. Rhodes, “On the minimax reachability of target sets and target tubes”, Automatica, Vol. 7, pp. 233–247, 1971.
[3] F. Blanchini and S. Miani, “Any domain of attraction for a linear constrained system is a tracking domain of attraction”, SIAM Journal on Control and Optimization, Vol. 38, No. 3, pp. 971–994, 2000.
[4] F. Blanchini and S. Miani, “Set-Theoretic Methods in Control”, Birkäuser, Boston, 2008.
[5] T.M. Chen, “Stuxnet, the real start of cyber warfare?[editor’s note]”. IEEE Network, Vol. 24, No. 6, pp. 2–3, 2010.
[6] H. Fawzi, P. Tabuada and S. Diggavi, “Secure estimation and control for cyber-physical systems under adversarial attacks”. IEEE Transactions on Automatic Control, Vol. 59, No. 6, pp. 1454–1467, 2014.
[7] G. Franzê, F. Tedesco and D. Famularo, “Model predictive control for constrained networked systems subject to data losses”, Automatica, Vol. 54, pp. 272-278, 2015.
[8] S. Gorman, “Electricity grid in US penetrated by spies”, The wall street journal, A1, 2009.
[9] A. A. Kurzhanskiy and P. Varaiya, “Ellipsoidal toolbox (et)”, 45rd IEEE CDC, pp. 1498–1503, 2006.
[10] A. Kurzhanskii and I. Vályi. “Ellipsoidal calculus for estimation and control”, Berlin: Birkhauser, 1997.
[11] F. Miao, M. Pajic and G.J. Pappas, “Stochastic game approach for replay attack detection”, 52nd IEEE CDC, pp. 1854–1859, 2013.
[12] Y. Mo, S. Weerakkody and B. Sinopoli, “Physical authentication of control systems: designing watermarked control inputs to detect counterfeit sensor outputs”. IEEE Control Systems, Vol. 35, No. 1, pp. 93–109, 2015.
[13] M. Herceg, M. Kvasnica, C. Jones, M. Morari, “Multi-parametric toolbox 3.0”, European Control Conference (ECC), 2013.
[14] F. Pasqualetti, F. Dorfler and F. Bullo, “Attack detection and identification in cyber-physical systems”, IEEE Transaction on Automatic Control Vol. 58 No. 11, pp. 2715-2729, 2013.
[15] T. Samad and A.M. Annaswamy, “The Impact of Control Technology”, IEEE Control Systems Society, 2011.
[16] A. Teixeira, I. Shames, H. Sandberg and K.H. Johansson, ‘Revealing stealthy attacks in control systems’, 50th IEEE Allerton Conference on Communication, Control, and Computing, pp. 1806–1813, 2012.
[17] A. Teixeira, I. Shames, H. Sandberg and K.H. Johansson, “A secure control framework for resource-limited adversaries”, Automatica, Vol. 51, pp. 135–148, 2015.
[18] S. Weerakkody, and B. Sinopoli. “Detecting integrity attacks on control systems using a moving target approach”, 54rd CDC, 2015.