ON PROCESSES WHICH CANNOT BE DISTINGUISHED
BY FINITARY OBSERVATION

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Abstract. A function $J$ defined on a family $C$ of stationary processes is finitely observable if there is a sequence of functions $s_n$ such that $s_n(x_1\ldots x_n) \to J(X)$ in probability for every process $X = (x_n) \in C$. Recently, Ornstein and Weiss proved the striking result that if $C$ is the class of aperiodic ergodic finite valued processes, then the only finitely observable isomorphism invariant defined on $C$ is entropy $\mathbb{H}$. We sharpen this in several ways. Our main result is that if $X \to Y$ is a zero-entropy extension of finite entropy ergodic systems and $C$ is the family of processes arising from $X$ and $Y$, then every finitely observable function on $C$ is constant. This implies Ornstein and Weiss’ result, and extends it to many other families of processes, e.g. it shows that there are no non-trivial finitely observable isomorphism invariants for processes arising from Kronecker systems, mild and strong mixing zero entropy systems. It also implies that any finitely observable isomorphism invariant defined on the family of processes arising from irrational rotations must be constant for rotations belonging to a set of full Lebesgue measure.

1. Introduction

Let $(x_n)_{n=-\infty}^{\infty}$ be an aperiodic ergodic process taking on finitely many values; without loss of generality the values are in $\mathbb{N}$. We may assume that $(x_n)$ arises from a generating partition $\mathcal{P} = (P_i)$ of an aperiodic, invertible and ergodic measure preserving system $\mathcal{X} = (X, \mathcal{B}, \mu, T)$; the system $\mathcal{X}$ is unique up to isomorphism. The question we are interested in is: what can we learn about the underlying system $\mathcal{X}$ by observing a sample path $(x_n)$?

In principle, the answer is “everything”, since by the ergodic theorem a typical sample path of $(x_n)_{n=1}^{\infty}$ determines all finite distributions of the process and this determines $\mathcal{X}$ up to isomorphism. However a more realistic scenario is one in which at each time step another output of the process is
revealed, i.e. at time $n$ we have observed the finite sequence $x_1 \ldots x_n$, and are asked to make a guess about the nature of $\mathcal{X}$ based on this data.

We call a scheme for producing such a sequence of guesses an observation scheme. To be precise,

**Definition 1.1.** An observation scheme (or scheme for short) is a metric space $\Delta$ and a sequence of functions $s_n : \mathbb{N}^n \to \Delta$. An observation scheme is said to converge for a family of processes $\mathcal{C}$ if $\lim_{n \to \infty} s_n(x_1 \ldots x_n)$ exists in probability for every process $(x_n) \in \mathcal{C}$. A function $J : \mathcal{C} \to \Delta$ is finitely observable if there is an observation scheme $(s_n)$ which converges to $J((x_n))$ for every $(x_n) \in \mathcal{C}$.

Note that the larger a family of processes is, the harder it is for a scheme to converge for every member of the family, hence large families have fewer finitely observable functions.

Nonetheless, many observation schemes $(s_n)$ are known for which the sequence $s_1(x_1), s_2(x_1, x_2), s_3(x_1, x_2, x_3), \ldots$ converges in probability or even almost surely for every ergodic process $(x_n)$. For example, if $s_n(x_1 \ldots x_n)$ counts the frequencies of 1’s appearing in $x_1 \ldots x_n$, then by the ergodic theorem $\lim_{n \to \infty} s_n(x_1 \ldots x_n)$ exists a.s. and equals the probability of the symbol 1 in the process $(x_n)$. This example and others like it show that some things about a process can be calculated from finite observations; but these are generally not isomorphism invariants, and so tell us nothing about the underlying dynamical system.

For processes $(x_n), (y_n)$ etc. we denote by $\mathcal{X}, \mathcal{Y}$ respectively the dynamical system determined by them. Write $(x_n) \cong (y_n)$ and $\mathcal{X} \cong \mathcal{Y}$ to indicate that $\mathcal{X}, \mathcal{Y}$ are isomorphic as dynamical systems. We will be interested in families of processes $\mathcal{C}$ which are closed under isomorphism, that is, they will have the property that if $(x_n) \in \mathcal{C}$ and $(y_n) \cong (x_n)$ then $(y_n) \in \mathcal{C}$. Such a family is called saturated. Usually we will specify $\mathcal{C}$ by some property of the underlying systems, e.g. $\mathcal{C}$ might be the family of all processes arising from an irrational rotation. In this case we would say for brevity that $\mathcal{C}$ is the class of irrational rotations.

**Definition 1.2.** Let $\mathcal{C}$ be a saturated family of processes, $\Delta$ a metric space and $J : \mathcal{C} \to \Delta$. Then $J$ is an isomorphism invariant for $\mathcal{C}$ (or invariant for short) if for every $(x_n), (y_n) \in \mathcal{C}$,

$$(x_n) \cong (y_n) \Rightarrow J((x_n)) = J((y_n))$$
and \( J \) is a complete invariant for \( C \) if the reverse implication holds. When \( J \) is an invariant we write \( J(X) \) instead of \( J((x_n)) \).

For quite some time it has been known that the entropy \( h((x_n)) = h(X) \) of a process is finitely observable in the class of all ergodic processes. The earliest observation scheme for entropy is due to D. Bailey \[1\]. A number of simpler schemes have been developed, such as the Lempel-Ziv compression algorithm \[12\] and the Ornstein-Weiss estimators \[8, 6\].

D. Ornstein and B. Weiss recently proved a striking converse to this: Every finitely observable invariant for the class of all ergodic processes is a continuous function of entropy \[7\]. They also showed that there are no finitely observable invariants except entropy for any class which contains the Bernoulli processes, for the class of zero entropy processes or for the class of zero entropy weak mixing processes.

However their techniques do not settle what is finitely observable in several other interesting classes of systems. Ornstein and Weiss have asked if there exists a complete finitely observable invariant for the class of irrational rotations (translations by an irrational on the group \( \mathbb{R}/\mathbb{Z} \)); this is not implausible, since for this class there is a complete invariant for isomorphism, namely the spectrum, or equivalently the modulus of rotation (up to sign and mod 1). We remark that there are no known complete invariants in the classes for which Ornstein and Weiss showed that entropy is the only invariant, with the exception of the class of Bernoulli systems, in which entropy is itself a complete invariant.

In an attempt to get a handle on this problem, we came up with the following, which is interesting in its own right:

**Theorem.** Suppose \( \mathcal{X} \to \mathcal{Y} \) is a zero entropy extension of finite entropy dynamical systems, that is \( h(\mathcal{X}) = h(\mathcal{Y}) \). Let \( C \) be the class of processes arising from \( \mathcal{X}, \mathcal{Y} \) (that is, from generating partitions of \( \mathcal{X} \) and \( \mathcal{Y} \)). Then every finitely observable invariant for \( C \) is constant.

This allows us reclaim the results of Ornstein and Weiss, and to settle the following problems:

**Theorem.** If \( J \) is a finitely observable invariant on one of the following classes:

1. **The Kronecker systems (the class of systems with pure point spectrum)**
(2) The zero entropy mild mixing processes
(3) The zero entropy strong mixing processes

Then $J$ is constant.

For the class of irrational rotations we obtain a slightly weaker result:

**Theorem.** For every finitely observable invariant $J$ on the class of irrational rotations, there is a Borel set $\Theta \subseteq [0,1)$ of full Lebesgue measure such that $J$ assigns the same value to processes arising from rotations by angles in $\Theta$.

In particular there is no complete finitely observable invariant for irrational rotations.

The rest of the paper is organized as follows. Section 2 presents some definitions and background. In section 3 we prove the theorem about zero-entropy extensions. Section 4 contains proofs of the other results, and in section 5 we mention some open problems.

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## 2. Preliminaries

For general background on ergodic theory we refer to [3, 9, 11].

### 2.1. Dynamical systems, partitions and processes.

By an aperiodic ergodic system $\mathcal{X} = (X, \mathcal{B}, \mu, T)$ we mean that $(X, \mathcal{B}, \mu)$ is a standard probability space, $T$ in invertible and acts ergodically, and the set of periodic points is of measure zero. A measure preserving systems $\mathcal{Y} = (Y, \mathcal{C}, \nu, S)$ is a factor of the system $\mathcal{X} = (X, \mathcal{B}, \mu, T)$ if there is a measure-preserving map $f : X \to Y$ defined almost everywhere satisfying $Sf = fT$. If there is such a map which is also invertible and bi-measurable then $\mathcal{X}, \mathcal{Y}$ are isomorphic.

A partition $\mathcal{P}$ of $X$ is a finite ordered collection of pairwise disjoint measurable sets $(P_i)_{i=1}^{\left| \mathcal{P} \right|}$ whose union is $X$ (up to measure zero). If $\mathcal{P}, \mathcal{Q}$ are partitions of $X$ then the partition $\mathcal{P} \vee \mathcal{Q} = (P_i \cap Q_j)_{(i,j)}$ is the join of $P, Q$ (order the pairs $(i,j)$ lexicographically); the join of finitely many partitions is defined similarly. Write $T^n\mathcal{P} = (T^nP_i)$.

A partition $\mathcal{P}$ of $X$ generates $\mathcal{X}$ if $\bigvee_{n=-\infty}^{\infty} T^n\mathcal{P} = \mathcal{B}$ up to measure zero, where $\bigvee_{n=-\infty}^{\infty} T^n\mathcal{P}$ is the $\sigma$-algebra generated by the collection $\bigcup_N \bigvee_{n=-N}^{N} T^n\mathcal{P}$. 
For a partition $\mathcal{P} = (P_i)_{i \in \mathbb{N}}$ and $\omega \in X$ we write $\mathcal{P}(\omega)$ for the index of the set in $\mathcal{P}$ that contains $\omega$. A partition $\mathcal{P}$ determines a stationary ergodic process $(x_n)$ with values in $\mathbb{N}$ by

$$x_n(\omega) = \mathcal{P}(T^n \omega)$$

We say that $x_i(\omega), x_{i+1}(\omega), \ldots, x_j(\omega)$ is the itinerary of $\omega$ (with respect to $\mathcal{P}$) from time $i$ to time $j$. The itinerary of $\omega$ from time 0 to time $N - 1$ is called the $(\mathcal{P}, N)$-name of $\omega$. If $\mathcal{P}$ is a generating partition for $X$ then the system $X$ and the partition $\mathcal{P}$ are determined, up to isomorphism, by the process $(x_n)$. We will say this process arises from $\mathcal{P}$ if $\mathcal{P}$ generates $X$.

The space of ordered partitions of $X$ into $n$ sets comes with a metric $\rho = \rho_n$ defined by

$$\rho(P, Q) = \sum_{i=1}^{n} \mu(P_i \triangle Q_i)$$

for $P = (P_1, \ldots, P_n)$ and $Q = (Q_1, \ldots, Q_n)$ (here $\Delta$ denotes symmetric difference). The metric $\rho_n$ is complete; note however that if $P_i \to P$ in $\rho_n$ it may happen that some of the members of $\mathcal{P}$ are empty.

It is easy to check that if $\rho(\mathcal{P}, Q) < \varepsilon$ then $\rho(\bigvee_{n=1}^{N} T^n P, \bigvee_{n=1}^{N} T^n Q) < N\varepsilon$. It follows that if $\mathcal{P}_k \to \mathcal{P}$ in $\rho$ and $(x_n^{(k)})$, $(x_n)$ denote the processes arising from $\mathcal{P}_k$, $\mathcal{P}$ respectively, then the sequence of processes $(x_n^{(k)})_{n=-\infty}^{\infty}$ converges to $(x_n)_{n=-\infty}^{\infty}$ in probability.

Given a partition $\mathcal{P}$ of $X$ into $r$ sets and an integer $N$ we may consider the distribution that $\mu$ induces on $\{1, \ldots, r\}^N$, where the measure of a word $w \in \{1, \ldots, r\}^N$ is the measure of the set of points whose $(\mathcal{P}, N)$-name is $w$, or in other words $\mu(\cap_{n=1}^{N} T^{-n} P_{w(n)})$. We refer to this as the distribution of $N$-names determined by $\mathcal{P}$.

Since a distribution on $N$-names is just a $r^N$-dimensional probability vector, we can compare these distributions using e.g. the $\ell^1$ metric. When we talk of closeness of $N$-name distributions, we will mean it in this sense. Note that if $\mathcal{P}, Q$ are partitions and $\rho(\mathcal{P}, Q) < \varepsilon$ then the distance between the $N$-name distributions associated with $\mathcal{P}$ and $Q$ is at most $N\varepsilon$.

### 2.2. Entropy.

Let $\mathcal{X} = (X, \mathcal{B}, \mu, T)$ be an invertible ergodic measure preserving system and $\mathcal{P} = (P_i)$ a partition. The entropy of a partition $\mathcal{P}$ is

$$H(\mathcal{P}) = - \sum_i \mu(P_i) \log \mu(P_i)$$
(all logarithms are to base 2 unless specified otherwise). $H(\mathcal{P})$ is non-negative and finite (define $0 \log 0 = 0$). The entropy of the system $\mathcal{X}$ with respect to $\mathcal{P}$ (equivalently, the entropy of the process arising from $\mathcal{P}$) is

$$ h(\mathcal{X}, \mathcal{P}) = \lim_{n \to \infty} \frac{1}{n} H(\mathcal{P} \vee T \mathcal{P} \vee \ldots \vee T^{n-1} \mathcal{P}) $$

the limit above can be shown to exist. The entropy of $\mathcal{X}$ is

$$ h(\mathcal{X}) = \sup\{h(\mathcal{X}, \mathcal{P}) : \mathcal{P} \text{ a finite partition of } X\} $$

If $\mathcal{P}$ is a finite generating partition then $h(\mathcal{X}) = h(\mathcal{X}, \mathcal{P})$, but the relation $h(\mathcal{X}) = h(\mathcal{X}, \mathcal{P})$ is not in itself enough to guarantee that $\mathcal{P}$ generates. However the Krieger generator theorem \[5\] guarantees that if $h(\mathcal{X}) < \log k$ for an integer $k$ then there exists a generating partition $\mathcal{P} = (P_1, \ldots, P_k)$ of $\mathcal{X}$ into $k$ sets.

In the space of partitions of $X$ into $n$ sets, the entropy is continuous in the metric $\rho_n$: that is, for a partition $\mathcal{P}$, for every $\delta > 0$ there is an $\varepsilon > 0$ such that if $\rho(\mathcal{P}, \mathcal{Q}) < \delta$ then $|h(\mathcal{X}, \mathcal{P}) - h(\mathcal{X}, \mathcal{Q})| < \varepsilon$.

The main fact about entropy we will use is the following classical theorem:

**Theorem 2.1. (Shannon-McMillan-Breiman theorem)** For any finite partition $\mathcal{P}$ of $\mathcal{X}$ and almost every $x \in X$,

$$ \frac{1}{n} \log \mu(\bigcap_{i=0}^{n-1} \mathcal{P}(T^i x)) \to h(\mathcal{X}, \mathcal{P}) $$

A proof can be found in \[10\] p. 55.

Denote

$$ \mu(u) = \mu(\{x \in X : \text{the } (\mathcal{P}, n)\text{-name of } x \text{ is } u\}) $$

With this notation the Shannon-McMillan-Breiman theorem states that

$$ \frac{1}{n} \log \mu(x_1 \ldots x_n) \to h(\mathcal{X}, \mathcal{P}) $$

almost surely, where $(x_n)$ is the process arising from $\mathcal{P}$.

Also, for partitions $\mathcal{P}, \mathcal{Q}$ and $(u, v) \in \mathbb{N}^n \times \mathbb{N}^n$, we say that $(u, v)$ is the $(\mathcal{P} \times \mathcal{Q}, n)$ name of a point $\omega \in X$ if $u$ is the $(\mathcal{P}, n)$-name of $\omega$ and $v$ is the $(\mathcal{Q}, n)$-name of $\omega$. This is just another way of talking about the partition $\mathcal{P} \vee \mathcal{Q}$. Denote

$$ \mu(v|u) = \frac{\mu(\{x \in X : \text{the } (\mathcal{P} \times \mathcal{Q}, n)\text{-name of } x \text{ is } (u, v)\})}{\mu(\{x \in X : \text{the } (\mathcal{P}, n)\text{-name of } x \text{ is } u\})} $$

We will actually use the following “relative” version of the Shannon-McMillan-Breimann theorem:
Theorem 2.2. (Relative Shannon-McMillan-Breiman) Let $\mathcal{P}, \mathcal{Q}$ be partitions of $\mathcal{X}$ with entropies $h(\mathcal{X}, \mathcal{P}) = s \leq t = h(\mathcal{X}, \mathcal{Q})$. For every $\varepsilon > 0$ there are collections of words $A_n \subseteq \mathbb{N}^n \times \mathbb{N}^n$ for $n = 1, 2, 3, \ldots$ such that

1. $\#\{u \in \mathbb{N}^n : (u, v) \in A_n \text{ for some } v\} < 2^{(s+\varepsilon)n}$ for every $n$.
2. $\#\{v \in \mathbb{N}^n : (u, v) \in A_n\} < 2^{(t-s+\varepsilon)n}$ for every $n$.
3. For almost every point $x \in \mathcal{X}$ the $(\mathcal{P} \times \mathcal{Q}, n)$-name of $x$ is in $A_n$ for all sufficiently large $n$.

Proof. Define

$$A_n = \{(u, v) \in \mathbb{N}^n \times \mathbb{N}^n : \mu(u) > 2^{-(s+\varepsilon)n} \text{ and } \mu(v|u) > 2^{-(t-s+\varepsilon)n}\}$$

The fact that for almost every $x \in \mathcal{X}$ the $(\mathcal{P} \times \mathcal{Q}, n)$-name of $x$ is eventually in $A_n$ follows from the Shannon-McMillan-Breiman theorem, once applied to the partition $\mathcal{P}$ and once to the partition $\mathcal{P} \times \mathcal{Q}$. The estimates on the size of the $u$’s represented in $A_n$ and the $v$’s associated to a given $u$ in $A_n$ follow easily from the definition since the mass of the $u$’s and the mass of the $v$’s relative to a given $u$ must add to at most 1. $\square$

2.3. Towers. A tower of height $n$ in $\mathcal{X}$ is a set of the form $B \cup TB \cup T^2B \cup \ldots \cup T^{n-1}B \subseteq \mathcal{X}$ such that the sets $T^iB$ are measurable and pairwise disjoint for $i = 0, \ldots, n - 1$. The set $B$ is called the base of the tower, and the set $T^iB$ is called the $i$-th level of the tower.

Given a partition $\mathcal{P} = (P_i)$ and a tower $\cup_{i=0}^{n-1} T^iB$, we can partition the base $B$ into disjoint (possibly empty) sets $B_w$ indexed by words $w \in \mathbb{N}^n$, such that

$$B_u = \{\omega \in B : u \text{ is the } (\mathcal{P}, n) \text{- name of } \omega\}$$

This partitions the tower into disjoint subtowers $\cup_{i=0}^{n-1} T^iB_u$ whose base is $B_u$; these subtowers are called columns. Each level $T^iB_u$ is contained entirely in the element $P_{u(i)}$ of $\mathcal{P}$. Put another way, if $(x_n)$ is the process associated with $\mathcal{P}$ then for $\omega \in B_u$ the first $n$ outputs $(x_1(\omega), \ldots, x_n(\omega))$ of the process are equal to $u = (u_1, \ldots, u_n)$.

We will need two tower lemmas.

Lemma 2.3. (Kakutani towers lemma) Let $B$ be a set of positive measure and $N$ an integer. Then the space $\mathcal{X}$ can be partitioned into countably many pairwise disjoint towers all of height no less than $N$, all of whose bases are subsets of $B$. 
Proof. Since \( X \) is aperiodic we can choose a set \( B' \subseteq B \) of positive measure such that if \( x \in B' \) then \( T^i x \notin B' \) for \( 1 \leq i < N \). Partition the base \( B \) according to the first return time to \( B' \), ie let

\[
B^{(n)} = \{ x \in B' : n \text{ is the first positive integer such that } T^n x \in B' \}
\]

Then for each \( n \geq N \) we have a tower \( B^{(n)} \cup TB^{(n)} \cup \ldots \cup T^{(n-1)}B^{(n)} \), these towers are pairwise disjoint, and their union fills \( X \).

A stronger result is a version of the Rohlin lemma whose proof can be found in [9]

**Lemma 2.4. (Strong Rohlin lemma)** Let \( \mathcal{P} = \{P_1, \ldots, P_k\} \) be a partition of \( X \) and \( \varepsilon > 0 \). Then for every \( N \) there is a tower \( B \cup TB \cup \ldots \cup T^{N-1}B \) of height \( N \) whose complement is of measure at most \( \varepsilon \) and such that the partition \( \mathcal{Q} = \{\bigcap_{i=1}^k P_i \} \) induced on \( B \) by \( \mathcal{P} \) has the same distribution relative to \( B \) as \( \mathcal{P} \) has relative to \( X \).

**Corollary 2.5.** Given \( A \subseteq X \) with \( \mu(A) > 1 - \varepsilon \) and any \( N \), there is a tower \( B \cup TB \cup \ldots \cup T^{N-1}B \) in \( X \) filling all but \( 2\varepsilon \) of the space and with \( B \subseteq A \).

**Proof.** Let \( C \cup TC \cup \ldots \cup T^{N-1}C \) be the tower provided by the strong Rohlin lemma with respect to the partition \( \{A, X \setminus A\} \) and set \( B = C \cap A \).

### 2.4. Approximation methods for partitions.

Often a generating partition with some property is constructed by approximation, that is, a sequence of partitions is defined satisfying more and more of our requirements and which converge in \( \rho \) to a partition with the properties we want. Below we outline some of the tools we use for such constructions.

If \( \mathcal{A} \) is a partition or a algebra of measurable sets and \( B \) is a measurable set then we write \( B \subseteq_{\varepsilon} \mathcal{A} \) to indicate that there is a set \( A \in \mathcal{A} \) such that \( \mu(A \Delta B) < \varepsilon \). Clearly \( B \in \mathcal{A} \) (up to measure zero) iff \( B \subseteq_{\varepsilon} \mathcal{A} \) for every \( \varepsilon > 0 \). For a partition \( \mathcal{P} \) we write \( \mathcal{P} \subseteq_{\varepsilon} \mathcal{A} \) if \( P_i \subseteq_{\varepsilon} \mathcal{A} \) for every \( P_i \in \mathcal{P} \).

Let \( \mathcal{P} \) be a generating partition for \( X \) and suppose that \( \mathcal{Q} \) is a partition such that, for every \( \varepsilon > 0 \), there is an \( N \) such that \( \mathcal{P} \subseteq_{\varepsilon} \bigcap_{n=-N}^{\infty} T^n Q \). It follows that \( P \in \bigcap_{n=-\infty}^{\infty} T^n Q \), and since \( \bigcap_{n=-\infty}^{\infty} T^n Q \) is \( T \)-invariant, \( B = \bigcap_{n=-\infty}^{\infty} T^n P \subseteq \bigcap_{n=-\infty}^{\infty} T^n Q \). Thus \( \mathcal{Q} \) generates.

Suppose \( \mathcal{P}, \mathcal{Q} \) are partitions of \( X \) into \( n \) elements and \( A \subseteq_{\varepsilon} \mathcal{P} \). Then if \( \rho(\mathcal{P}, \mathcal{Q}) < \delta \) we have \( A \subseteq_{\varepsilon+\delta} \mathcal{Q} \). Thus if \( A \subseteq_{\varepsilon} \bigcup_{n=1}^{N} T^n P \) and \( \rho(\mathcal{P}, \mathcal{Q}) < \delta \) then \( A \subseteq_{\varepsilon+N\delta} \bigcup_{n=1}^{N} T^n Q \).
These observations are essentially the proof of the following lemma, see also [9] p.79:

**Lemma 2.6.** Let \((P_k)_{k=1}^\infty\) be a sequence of partitions of \(X\) and \(Q\) a partition of \(X\). Suppose that \(\rho(P_{k-1}, P_k) < \varepsilon(k)\) and \(Q \subseteq \bigvee_{j=-N(k)}^{N(k)} T^{-j} P_k\) for some sequences \(\varepsilon(k) > 0\) and \(N(k) \in \mathbb{N}\) which satisfy \(\sum_{k=1}^{\infty} \varepsilon(k) < \infty\) and \(N(k) \cdot \sum_{j=k+1}^{\infty} \varepsilon(j) \to 0\) as \(k \to \infty\). Then \((P_k)\) converges to a partition \(P\) and \(Q \subseteq \bigvee_{j=-\infty}^{\infty} T^{-j} P\).

The following theorem shows that in order to change a partition \(P\) into a generating partition, you need to perturb \(P\) by an amount of the same order as the difference \(h(\mathcal{X}) - h(P)\). This result is not new but we include a proof for completeness.

**Theorem 2.7.** (Entropy and generating partitions) let \(h \geq 0\) and \(k\) be an integer with \(\log k > h\). Let \(\mathcal{X} = (X, \mathcal{B}, \mu, T)\) be an aperiodic ergodic system with entropy \(h\) and let \(P = (P_1, \ldots, P_k)\) be a partition of \(X\) with \(h(\mathcal{X}, P) = h'\) (so \(h' \leq h\)). Then for every \(\delta > 0\) there is a generating partition \(P' = (P'_1, \ldots, P'_k)\) of \(\mathcal{X}\) such that \(\rho(P, P') < \delta + \frac{h-h'}{\log k - h}\). In particular, the generating partitions are dense in the \(\rho\)-metric among the partitions of maximal entropy.

**Remark.** The parameter \(\delta\) was introduced only in order to deal with the case that \(h = h'\). The fact that the generating partitions are dense among the partitions of maximal entropy is known, but we are unable to find a reference.

**Proof.** Let \(\delta > 0\) be given. Fix a very small \(\varepsilon > 0\) which will determined later. Fix a generating partition \(Q\) of size \(k\), and for \(n = 1, 2, 3\ldots\) let \(A_n \subseteq \mathbb{N}^n \times \mathbb{N}^n\) be as in theorem 2.2 for the partitions \(P, Q\) and parameter \(\varepsilon\). Let \(N \geq \frac{1}{\varepsilon}\) be large enough that the the set \(X_0\) of \(\omega\)'s whose \((P \times Q, n)\)-name in \(A_n\) for all \(n \geq N\) has positive measure. Applying lemma 2.3 we can partition the space \(X\) into disjoint towers of height at least \(\frac{N}{\varepsilon}\) whose bases are contained in \(X_0\), that is for each \(n \geq \frac{N}{\varepsilon}\) we get disjoint towers \(B^{(n)} \cup TB^{(n)} \cup \ldots \cup T^{n-1} B^{(n)}\) of height \(n\) with \(B^{(n)} \subseteq X_0\), and the union of these towers has full measure. Partition the bases \(B^{(n)}\) according to \(A_n\), so for a word \((u, v) \in A_n\) the set \(B^{(n)}_{u,v}\) consists of points whose \((P \times Q, n)\)-name is \((u, v)\).

We construct a partition \(P'\) by modifying the labels of some levels of the columns \(B^{(n)}_{u,v}\). The construction proceeds in three stages.
Marking the base: Fix \( m = \frac{1}{\epsilon} \) (for simplicity we ignore rounding errors and treat \( m \) as an integer, and adopt a similar philosophy later as well). Label the lower \( 2m \) levels of the column \( B_{u,v}^{(n)} \) (i.e. the levels indexed 0 to \( 2m - 1 \)) with 1’s and mark levels \( 2m, 3m, \ldots, \lfloor n/m \rfloor m \) with 0’s.

The result of this procedure is that given any point \( \omega \in \bigcup_{i=0}^{n-1} T^i B^{(n)} \) the base of the column can be identified as the largest index \( i \in \{ -n, -n+1, \ldots, 0 \} \) such that the \( (P', 2m) \)-name of \( T^i \omega \) consists of all 1’s. Thus given the \( P' \) itinerary of \( \omega \) from time \(-n\) to \( n \), we can reconstruct the \( P \)-name of the column to which \( \omega \) belongs. We will preserve this property in the following steps, hence with probability 1 given the \( P' \) itinerary of a point from time \(-\infty\) to \(-\infty\) we can determine the \( n \) corresponding to the column the point belongs to, and the \( P'-\)name of that column.

Coding the \( Q \)-itinerary into \( P' \): Denote \( A_n(u) = \{ v : (u, v) \in A_u \} \subseteq \mathbb{N}^n \). Fix \((u, v) \in A_n\) and enumerate \( A_n(u) = \{ v_1, \ldots, v_r \} \) in a way depending only on \( u \); by assumption \( |A_n(u)| < 2^{(h-h'+\epsilon)n} \).

We modify the column over \( B_{u,v}^{(n)} \) so as to record the index \( i \) for which \( v = v_i \). We do this by writing the base-\( k \) representation of \( i \) near the bottom of the column. To be precise, we record the base-\( k \) digits of \( i \) starting at level \( 2m + 1 \) and writing consecutively in blocks of \( m - 1 \), skipping levels of height 0 mod \( m \) so as not to overwrite what we did in the previous stage. Since there are at most \( 2^{(h-h'+\epsilon)n} \) possible values for \( i \) we need to overwrite \( n(h - h' + \epsilon) \log_k 2 \) levels of the column.

The result of this procedure is that if we know both the \( (P, n) \)-name (the word \( u \)) and the \( (P', n) \)-name of a point in the base \( B^{(n)} \), we can deduce its \( (Q, n) \)-name (the word \( v \)) by extracting the index \( i \) coded just above the base marker in the \( (P', n) \) name, and looking at the \( i \)-th word in the list \( A_n(u) \).

Re-coding the \( P \)-itinerary: Fix again \((u, v) \in A_n\). The \( P \)-name of the column \( B_{u,v}^{(n)} \) has been partly destroyed by the previous steps. We will fix this by overwriting still more of the \( P \)-name, starting where we stopped at the previous stage, skipping levels which are at height 0 mod \( m \), and stopping at some height \( M = M(n) \) which we will determine. This gives us \( M - (2m + \frac{1}{m} + n(h-h'+\epsilon) \log_k 2) \) symbols in which to store information. In this space we want to record the
portion of the name \( u \) which has been overwritten in all three stages (including the current stage). This consists of the first \( M \) symbols of \( u \) plus at most \( \frac{n}{m} \) additional levels overwritten in the first stage. Assuming as we may that \( M > \varepsilon n \geq N \), we know that the number of possibilities for the first \( M \) symbols of \( u \) is bounded by \( 2^{(h' + \varepsilon)M} \) so using the \( k \) symbols at our disposal we need \( M(h' + \varepsilon) \log_k 2 \) symbols in order to record it, plus another \( \frac{n}{m} \) symbols to record what was erased in the first stage. Thus we require of \( M \) that in addition to \( \varepsilon n < M < n \) it satisfy the inequality

\[
M - (2m + \frac{n}{m} + n(h - h' + \varepsilon) \log_k 2) \geq M(h' + \varepsilon) \log_k 2 + \frac{n}{m}
\]

or equivalently

\[
M \geq \frac{((h - h' + \varepsilon) \log_k 2 + 2(\frac{1}{m} + \frac{m}{n}))n}{1 - (h' + \varepsilon) \log_k 2}
\]

Since \( h' \leq h < \log k \), \( \frac{n}{m} = n \varepsilon n = \frac{1}{\varepsilon} n \leq \frac{1}{\varepsilon} n \leq n \), when \( \varepsilon \) is small enough it suffices that

\[
M \geq \frac{((h - h' + \varepsilon) \log_k 2 + 4\varepsilon) n}{1 - (h' + \varepsilon) \log_k 2}
\]

Denote the coefficient of \( n \) in expression on the right hand side by \( C(\varepsilon) \). Note that \( C(\varepsilon) \to \frac{h - h'}{\log k - h'} \) as \( \varepsilon \to 0 \) and \( 0 \leq C(\varepsilon) < 1 \). Thus if we choose \( \varepsilon > 0 \) small enough (in a manner depending only on \( h, h' \) and \( k \)) we can set \( M = \max\{\varepsilon, C(\varepsilon)\} \cdot n \) and \( M \) will satisfy all the requirements, including \( \varepsilon n < M < n \).

The results of this procedure is that given the \((P', n)\)-name of a point in the base of the tower column \( B_{u,v}^{(n)} \), we can reconstruct its \((P, n)\)-name by looking at the data written in this step, and hence by the previous step its \((Q, n)\) name. Together with the previous stages, this means that for any point in \( X \) if we know the entire \( P' \) itinerary we know can determine the column it is in and the \( P' \) of that column, and hence its \( Q(\omega) \). This means that \( P' \) generates.

It remains to estimate how much \( P \) has changed. We have modified \( M + \frac{n}{m} \) levels of each column \( B_{u,v}^{(n)} \), or a \((C(\varepsilon) + \varepsilon)\)-fraction of the mass of that column. summing over all columns, this is the fraction of \( X \) that has changed. For \( \varepsilon > 0 \) sufficiently small, this is less than \( \delta + \frac{h - h'}{\log k - h'} \), implying that

\[
\rho(\mathcal{P}, \mathcal{P}') < \delta + \frac{h - h'}{\log k - h'}. \]

This completes the proof. \( \square \)
3. Zero-entropy extensions

This section is dedicated to proving our main theorem, theorem 3.1. Before going into the details, we would like to say a few words about the relation of this theorem to the work of Ornstein and Weiss in [7], where it was shown that entropy is the only finitely observable invariant in some classes saturated of processes. Their proof used a diagonalization argument: Assuming to the contrary that for some class \( C \) there exists a finitely observable invariant finer than entropy, choose two non-isomorphic processes \((x_n), (y_n) \in C\) with the same entropy \( h \). A third process \((z_n)\) is then constructed, for which the observation scheme does not converge. This is done by inductively defining the \( N \)-block distributions for the process \((z_n)\) for a sequence of rapidly increasing \( N \)'s, where at each step Rohlin towers and copying lemmas are used to make \((z_n)\) look at different time scales as though it comes from \( \mathcal{X} \) or \( \mathcal{Y} \). However, in order to obtain a contradiction it must be ensured that \((z_n) \in C\), since otherwise the observation scheme is not expected to converge. With some care one can ensure that \((z_n)\) is Bernoulli if \( h > 0 \), or weak mixing and deterministic if \( h = 0 \), but other properties, such as pure point spectrum or non-Bernoulliism in positive entropy, are harder to build into \((z_n)\).

Our results derive from the observation that when \((x_n)\) is a zero-entropy extension of \((y_n)\), one can control the isomorphism class of the diagonal process \((z_n)\) and in fact it can be made isomorphic to \((y_n)\).

**Theorem 3.1.** Suppose \( \mathcal{X} \to \mathcal{Y} \) is a zero entropy extension of finite entropy dynamical systems. Let \( C \) be the family of processes arising from \( \mathcal{X} \) and \( \mathcal{Y} \). Then every finitely observable invariant for \( C \) is constant.

**Proof.** We identify \( \mathcal{Y} \) with the sub-\( \sigma \)-algebra of \( \mathcal{X} \) which is the pull-back of the \( \sigma \)-algebra of \( \mathcal{Y} \) through the factor map. Let \( r \in \mathbb{N} \) with \( \log r > h(\mathcal{X}) \); all partitions in the sequel are partitions into \( r \) sets.

To simplify notation we assume that \((s_n)\) is an observation scheme whose range is \( \mathbb{R} \); there is no loss of generality here since given some other range we can always compose with continuous functions from the range to \( \mathbb{R} \). Suppose that there are \( \xi, \eta \in \Delta \) such that for every pair of processes \((x_n), (y_n)\) arising from \( \mathcal{X}, \mathcal{Y} \) respectively and generating them,

\[
\lim s_n(x_1 \ldots x_n) = \xi \quad \text{in probability}
\]

\[
\lim s_n(y_1 \ldots y_n) = \eta \quad \text{in probability}
\]
We must show that $\eta = \xi$. In order to do this will construct a generating partition $\mathcal{P}^*$ of $\mathcal{Y}$ and a sequence $N(k)$ such that $s_{N(k)}(y^*_1 \ldots y^*_{N(k)}) \rightarrow \xi$ in probability (here $(y^*_n)$ is the process arising from $\mathcal{P}^*$). This suffices because by assumption, $\lim s_n(y^*_1 \ldots y^*_n) \rightarrow \eta$, so $\eta = \xi$.

The partition $\mathcal{P}^*$ will be obtained as the limit of a sequence of generating partitions $\mathcal{P}^{(k)}$ of $\mathcal{Y}$, which will be constructed inductively. The induction step is provided by the following lemma:

**Lemma 3.2.** For any generating partition $\mathcal{P}$ of $\mathcal{Y}$, and any $\varepsilon > 0$, there is a generating partition $\overline{\mathcal{P}}$ of $\mathcal{Y}$ with $\rho(\mathcal{P}, \overline{\mathcal{P}}) < \varepsilon$, and an integer $N$ so that

$$P(|s_N(y_1 \ldots y_N) - \xi| < \varepsilon) > 1 - \varepsilon$$

where $(y_n)$ is the process arising from $\mathcal{P}$.

Before proving the sequence let us show how it is used to prove the theorem. We construct a sequence $\mathcal{P}^{(k)}$ of generating partitions of $\mathcal{Y}$ and associated processes $(y^{(k)}_n)$, starting with an arbitrary generating partition $\mathcal{P}^{(0)}$ provided by the Krieger generator theorem.

At the induction step, given $\mathcal{P}^{(k-1)}$ we construct $\mathcal{P}^{(k)}$ using the lemma; we choose the parameter $\varepsilon = \varepsilon(k) < 1/k$ in the lemma to be very small with respect to the previous stages of the construction (see below). Thus we have

(3.1) $\rho(\mathcal{P}^{(k-1)}, \mathcal{P}^{(k)}) < \varepsilon(k)$

From the lemma we also get an integer $N(k)$ such that

(3.2) $P(|s_{N(k)}(y_1^{(k)} \ldots y_N^{(k)}) - \xi| < \frac{1}{k}) > 1 - \frac{1}{k}$

and since $\mathcal{P}^{(k)}$ generates $\mathcal{Y}$ there is an integer $L(k)$ such that

(3.3) $\mathcal{P}^{(0)} \subseteq \bigvee_{i=-L(k)}^{L(k)} T^i \mathcal{P}^{(k)}$

During the construction we are free to choose the $\varepsilon(k)$ as small as we like. First of all we will choose them so that $\sum \varepsilon(k) < \infty$. Since the metric $\rho = \rho_r$ is complete (or using the Borel-Cantelli lemma) this guarantees that $\mathcal{P}^{(k)}$ converges to a partition $\mathcal{P}^*$ of $\mathcal{Y}$, with associated process $(y^*_n)$. Second, note that $\rho(\mathcal{P}^*, \mathcal{P}^{(k-1)}) \leq \sum_{m=k}^{\infty} \varepsilon(m)$. Thus at the beginning of step $k$ of the construction, when $\mathcal{P}^{(k-1)}$ is given, we may choose a $\delta = \delta(k) > 0$ depending on all the data defined so far and prescribe that $\rho(\mathcal{P}^*, \mathcal{P}^{(k-1)}) < \delta(k)$ by requiring $\varepsilon(m) \leq 2^{-m} \delta(k)$ for every $m \geq k$. The point is that
the conditions (3.2) and (3.3) remain true for any partition (and associated process) sufficiently close to \( \mathcal{P}^{(k)} \), and hence a prudent choice of \( \delta(k) \) implies that they hold for \( \mathcal{P}^* \) and \( (y_n^*) \), that is,

\[
\forall m \quad P(|s_{N(m)}(y_1^* \ldots y_N^*) - \xi| < \frac{1}{m}) > 1 - \frac{1}{m}
\]

and

\[
\forall k \quad \mathcal{P}(0) \subseteq \bigcup_{i=-L(k)}^{L(k)} T^i \mathcal{P}^* \]

The first of these implies \( \lim_{k \to \infty} s_{N(k)}(y_1^* \ldots y_N^*) = \xi \) in probability, and the second that \( \mathcal{P}(0) \subseteq \bigcup_{i=-\infty}^{\infty} T^i \mathcal{P}^* \), so \( \mathcal{P}^* \) generates \( \mathcal{Y} \). \( \square \)

**Proof.** (of lemma 3.2) We first present a sketch of the proof, and afterwards the details. Since \( \mathcal{P} \) generates \( \mathcal{Y} \) it has full entropy, which by assumption is equal to the entropy of \( \mathcal{X} \). Therefore we can find a generating partition \( \mathcal{Q} \) for \( \mathcal{X} \) with \( \rho(\mathcal{P}, \mathcal{Q}) < \varepsilon / 2 \). Let \( (x_n) \) be the process determined by \( \mathcal{Q} \); then \( s_n(x_1 \ldots x_n) \to \xi \) in probability, so we can choose an \( N \) such that

\[
P(|s_N(x_1 \ldots x_N) - \xi| < \varepsilon) > 1 - \varepsilon
\]

Since \( \mathcal{P}, \mathcal{Q} \) are both defined on \( \mathcal{X} \) we get a joining of the \( \mathcal{P} \)- and \( \mathcal{Q} \)-processes. Choose now a \( \delta > 0 \) and a suitably large \( K \). Now working in \( \mathcal{Y} \) again, we can construct a partition \( \mathcal{R} \) whose joint \( K \)-block distribution with \( \mathcal{P} \) is within \( \delta \) of the joint \( K \)-block distribution of \( \mathcal{P}, \mathcal{Q} \). Thus (assuming we chose \( K \) large enough), the order of magnitude of \( \rho(\mathcal{P}, \mathcal{R}) \) will be of the order of \( \rho(\mathcal{P}, \mathcal{Q}) + \delta \), the \( N \)-block distribution of the \( \mathcal{R} \)-process will be within \( \delta \) of the \( N \)-block distribution of the \( \mathcal{Q} \)-process, and the entropy \( \mathcal{R} \) is \( \delta \)-close to \( h(\mathcal{Y}) \). Thus although \( \mathcal{R} \) doesn’t necessarily generate \( \mathcal{Y} \) we need only make an additional small correction to get a generating partition \( \mathcal{P} \) for \( \mathcal{Y} \), and we can arrange that this doesn’t disturb the \( N \)-block distributions very much.

Now for the details:

**Choosing \( \mathcal{Q} \):** Since \( h(\mathcal{X}, \mathcal{P}) = h(\mathcal{Y}) = h(\mathcal{X}) \), by theorem 2.7 we can find a generating partition \( \mathcal{Q} \) for \( \mathcal{X} \) with

\[
\rho(\mathcal{P}, \mathcal{Q}) < \frac{\varepsilon}{2}
\]

**Choosing \( N \) and \( \delta \):** Denote by \( (x_n) \) the process arising from \( \mathcal{Q} \). Then \( s_n(x_1 \ldots x_n) \to \xi \) in probability, so there is an integer \( N \) such that

\[
\mu(|s_N(x_1 \ldots x_N) - \xi| < \varepsilon) > 1 - \varepsilon
\]
Note that condition above is a property of the $N$-block distribution of $(x_n)$. Thus there is a $\delta \in (0, \frac{\varepsilon}{2})$ with the property that if $(z_n)$ is a process arising from a partition $R$ and the $N$-block distribution induced by $R$ is within $\delta$ in $L^1$ of the $N$-block distribution of $Q$, then $\mu(\{s_N(z_1 \ldots z_N) - \xi < \varepsilon\}) > 1 - \varepsilon$. Note also that if $R, R'$ are two partitions of $Y$ and if $\rho(R, R') < \delta/N$ then the $N$-block distributions of the processes arising from $R, R'$ differ by at most $\delta$.

**Choosing $\alpha, \beta$ and $M$:** Invoking theorem 2.4 choose $\alpha > 0$ such that if $R$ is a partition of $Y$ with entropy $h - \alpha$ then there is a generating partition $R'$ of $Y$ with $\rho(R, R') < \delta/2N$. Let $\beta > 0$ be such that for any partition $S$ of $Y$, if $P \subseteq \beta S$ then $h(S) > h - \alpha$. We may assume that $\beta < \delta/N$.

Since $Q$ generates $X$ and $P$ is measurable in $X$ there is an $M > N$ such that

$$P \subseteq \beta/2 \bigvee_{i=-M}^{M} T^i Q$$

Note that this property depends only on the distribution of $(P \times Q, 2M + 1)$-names, and if $R$ is a partition of $Y$ such that the distribution of $(P \times Q, 2M + 1)$-names is within $\tau$ of the distribution $(P \times R, 2M + 1)$-names (in $L^1(R^2M+1)$) then $P \subseteq \beta/2 + \tau \bigvee_{i=-M}^{M} T^i R$.

**Choosing $L, B$ and $R$:** Fix an integer $L$ with $\max\{M, N\}/L < \beta/8$ and choose a tower $B \cup TB \cup \ldots \cup T^L B$ of height $L$ in $Y$, filling all but $\beta/4$ of the space. We will define a partition $R$ of $Y$ by modifying $P$ at some of the points in the tower.

Let $(B_u)$ be the partition of the base $B$ according to $(P, L)$-names. This partition is measurable in $Y$. We can further partition each $B_u$ according to the $(Q, L)$-names as $B_u = \cup_v B_{u,v}$. The $B_{u,v}$'s are measurable in $X$ but may not be measurable in $Y$. However since $Y$ is non-atomic we can partition the sets $B_u$ into sets $B'_{u,v}$ in $Y$ such that $\mu(B'_{u,v}) = \mu(B_{u,v})$. For each $B'_{u,v}$, modify the column over $B'_{u,v}$ so that it is labeled by $v$ (instead of $u$). Call the resulting partition $R$.

Since

$$\rho(P, R) = 2\mu(\{x \in X : P(x) \neq R(x)\})$$

and on the tower $\bigcup_{i=0}^{L-1} T^i B$ we have

$$\mu\{x \in \bigcup_{i=0}^{L-1} T^i B : P(x) \neq R(x)\} = \mu\{x \in \bigcup_{i=0}^{L-1} T^i B : P(x) \neq Q(x)\}$$
and the tower fills all but $\beta/4$ of the mass, it follows that

$$\rho(P, R) \leq \rho(P, Q) + \frac{\beta}{4} < \frac{\varepsilon}{2} + \frac{\beta}{4}$$

**Choosing $\overline{P}$:** Consider now the difference between the distributions of $(P \times Q, 2M + 1)$-names and the distributions of $(P \times R, 2M + 1)$-names. The only difference between them is incurred at the top and bottom $M$ levels of the tower, which have total mass $< 2M/L < \beta/4$, and the exceptional set outside the tower whose mass is $< \beta/4$. Therefore the distributions of $(P \times Q, 2M + 1)$- and $(P \times R, 2M + 1)$-names differ by at most $\tau = \beta/2$ so

$$\mathcal{P} \subseteq_{\beta/2 + \beta/2} \bigvee_{i=-M}^{M} T^i \mathcal{R}$$

Since the entropy of $\bigvee_{i=-M}^{M} T^i \mathcal{R}$ is the same as the entropy of $\mathcal{R}$, we conclude by the choice of $\beta$ that $\mathcal{R}$ has entropy $> h - \alpha$. We can therefore choose a generating partition $\overline{\mathcal{P}}$ of $\mathcal{Y}$ with $\rho(\overline{\mathcal{P}}, \mathcal{R}) < \delta/2N$. We conclude that

$$\rho(P, \overline{\mathcal{P}}) < \rho(\mathcal{P}, \mathcal{R}) + \rho(\mathcal{R}, \overline{\mathcal{P}}) < \frac{\varepsilon}{2} + \frac{\beta}{4} + \frac{\delta}{2N} < \varepsilon$$

Finally, note that from the construction of $\mathcal{R}$, the $N$-block distribution is the same as the $N$-block distribution of $Q$ except for an error introduced by the top $N$ levels of the tower, which have mass $< \beta/4$, and the exceptional set also of measure $\beta/4$, which means that the $N$-block distribution of $\mathcal{R}$ and $Q$ differ by less than $\delta/2$. Since $\rho(\mathcal{R}, \overline{\mathcal{P}}) < \delta/2N$ we see that the $N$-block distributions of the $\mathcal{R}$-process and the $\overline{\mathcal{P}}$-process differ by at most $\delta/2$, so the $N$-block distributions of the $\overline{\mathcal{P}}$-process and the $Q$-process differ by at most $\delta$; by the definition of $\delta$ this implies

$$\mu(|s_N(\overline{y}_1 \ldots \overline{y}_N) - \xi| < \varepsilon) > 1 - \varepsilon$$

where $(\overline{y}_n)$ is the process defined by $\overline{P}$.

This completes the proof. \hfill \square

4. **Some Applications**

An immediate consequence of theorem 3.1 is:
Proposition 4.1. Let $\mathcal{C}$ be a saturated class of processes with entropy $h$. Suppose that every $\mathcal{X}, \mathcal{Y} \in \mathcal{C}$ either have a common factor or a common extension in $\mathcal{C}$. Then every finitely observable invariant is constant on $\mathcal{C}$.

Proof. If $\mathcal{X}, \mathcal{Y}$ have a common factor $\mathcal{Z}$, then no scheme can distinguish $\mathcal{X}$ and $\mathcal{Z}$, and no scheme can distinguish $\mathcal{Y}$ and $\mathcal{Z}$; so every scheme must give the same value to $\mathcal{X}$ and $\mathcal{Y}$. The case of a common extension is similar. □

We turn now to some specific classes of processes. We begin by recovering some of the results of [7] using the techniques of the last section.

Proposition 4.2. ([7]) There are no nontrivial finitely observable invariants for the class of zero entropy systems or for the class of zero entropy weakly mixing processes.

Proof. Any zero-entropy ergodic systems $\mathcal{X}, \mathcal{Y}$ have an ergodic zero entropy joining (take a typical ergodic component of $\mathcal{X} \times \mathcal{Y}$), and if $\mathcal{X}, \mathcal{Y}$ are zero entropy weakly mixing systems then so is the joining $\mathcal{X} \times \mathcal{Y}$. □

Proposition 4.3. ([7]) If $\mathcal{C}$ is a saturated family of processes which contains the Bernoulli processes (e.g., $\mathcal{C}$ = all aperiodic finite valued ergodic processes) then entropy is the only finitely observable invariant.

Proof. For $h \geq 0$ let $\mathcal{C}_h = \{\mathcal{X} \in \mathcal{C} : h(\mathcal{X}) = h\}$. We must show that every finitely observable invariant scheme on $\mathcal{C}$ is constant on each $\mathcal{C}_h$. For $h = 0$ this is the previous proposition. For $h > 0$, we use Sinai’s theorem, which states that every $\mathcal{X}, \mathcal{Y} \in \mathcal{C}_h$ has a Bernoulli factor with entropy $h$. By Ornstein’s isomorphism theorem, these factors are isomorphic. Since the Bernoulli processes are in $\mathcal{C}$ we conclude that every $\mathcal{X}, \mathcal{Y} \in \mathcal{C}_h$ have a common factor in $\mathcal{C}_h$, so every scheme is constant on $\mathcal{C}_h$. □

Now for something new:

Theorem 4.4. (1) Every finitely observable invariant for the class of Kronecker systems is constant

(2) Every finitely observable invariant for the class of mildly mixing zero entropy systems is constant.

(3) Every finitely observable invariant for the class of strong mixing zero entropy systems is constant.

Proof. Again, we need only note that in these classes every two systems have a joining in the same class. □
An elementary class of systems is the class $\mathcal{R}$ of irrational rotations. A delicate and perplexing question is whether there exist nonconstant finitely observable invariants on this class.

To fix notation, let $([0, 1], \mathcal{B}, \lambda)$ be the probability space of the unit interval with Lebesgue measure. For $\alpha \in [0, 1) \setminus \mathbb{Q}$ let $X_\alpha = ([0, 1], \mathcal{B}, \lambda, T_\alpha)$ where $T_\alpha : [0, 1) \to [0, 1)$ is translation by $\alpha$, that is, $T_\alpha(x) = x + \alpha \pmod{1}$. Let $\mathcal{R} = \bigcup \{X_\alpha : \alpha \in [0, 1) \setminus \mathbb{Q}\}$ be these systems (note that $X_\alpha \cong X_{-\alpha}$). Thus an invariant $J : \mathcal{R} \to \Delta$ induces a map $\tilde{J} : [0, 1) \setminus \mathbb{Q} \to \Delta$ by $\tilde{J}(\alpha) = J(X_\alpha)$.

**Lemma 4.5.** If $J$ is a finitely observable invariant on $\mathcal{R}$ then $\tilde{J}$ is Lebesgue measurable.

**Proof.** We may assume that $\Delta = \mathbb{R}$ by composing continuous real-valued functions on $s_n$. Let $(s_n)$ be an observation scheme which calculates $J$. Fix the partition $\mathcal{P} = ([0, \frac{1}{2}), [\frac{1}{2}, 1))$ of the interval into two equal halves, and note that $\mathcal{P}$ generates for every $X_\alpha \in \mathcal{R}$. Thus denoting by $(x_1^{(\alpha)}, \ldots, x_n^{(\alpha)})$ the process arising from $\mathcal{P}$ and the system $X_\alpha$, we have

$$\tilde{J}(\alpha) = J(X_\alpha) = \lim_{n \to \infty} s_n(x_1^{(\alpha)}, \ldots, x_n^{(\alpha)})$$

where the limit exists in probability and is constant $\lambda$-a.e. in $X_\alpha$.

Define $f_n : [0, 1) \times [0, 1) \to \Delta$ by

$$f_n(\alpha, \omega) = s_n(x_1^{(\alpha)}(\omega), \ldots, x_n^{(\alpha)}(\omega))$$

and $f : [0, 1) \times [0, 1) \to \Delta$ by

$$f(\alpha, y) = \tilde{J}(\alpha)$$

To show that $\tilde{J}$ is measurable it suffices to show that $f$ is measurable. And in fact, the $f_n$ are measurable with respect to the product $\sigma$-algebra and since $f_n$ converges in probability on every fibre $\{\alpha\} \times [0, 1)$ (with respect to $\lambda$), and the limit is the constant function $J(\alpha)$, it follows that $f_n$ converges to $f$ in probability on $[0, 1) \times [0, 1)$ with respect to $\lambda \times \lambda$. \qed

**Theorem 4.6.** Let $J : \mathbb{R} \to \Delta$ be a finitely observable invariant for $\mathcal{R}$. Then $\tilde{J}$ is constant on a set of full measure. In particular, no finitely observable invariant on $\mathcal{R}$ is complete.

**Proof.** If $\alpha, \beta \in [0, 1) \setminus \mathbb{Q}$ are rationally dependent then $\gamma = m\alpha = n\beta \in \mathbb{R} \setminus \mathbb{Q}$ for some $m, n \in \mathbb{N}$. Thus $\mathcal{R}_\gamma$ is a factor both of $\mathcal{R}_\alpha$ and of $\mathcal{R}_\beta$, so $J(\mathcal{R}_\alpha) = J(\mathcal{R}_\beta)$. We conclude that $\tilde{J}$ is a Lebesgue-measurable function on
Classes which cannot be distinguished by finitary invariants

\[ 0,1 \setminus \mathbb{Q} \] which is constant on \( \mathbb{Q} \)-cosets. Any such map is constant on a set of full measure. \( \square \)

5. Remarks and problems

Let us mention two problems which we have not been able to resolve:

**Question.** Let \( \mathcal{R} \) denote as before the class of irrational rotations. Is every finitely observable scheme on \( \mathcal{R} \) constant?

**Question.** Let \( \mathcal{K} \) be the class of non-Bernoulli \( K \)-processes. Are there any finitely observable invariants on \( \mathcal{K} \) finer than entropy?

It has been known for some time that there are no complete Borel invariants on \( \mathcal{K} \) (the Boral structure comes from one of the natural topologies on \( \mathcal{K} \) - see Feldman’s paper [2]). It also follows from work of Hoffman [4] that there exist non-isomorphic \( K \)-systems \( \mathcal{X}, \mathcal{Y} \) of the same entropy such that \( \mathcal{X} \to \mathcal{Y} \) is an extension. This implies by proposition 4.1 that there are no complete finitely observable invariants on \( \mathcal{K} \); but this is not new in view of Feldman’s work.

If it were true that every two processes \( \mathcal{X}, \mathcal{Y} \in \mathcal{K} \) had a common zero-entropy non-Bernoulli \( K \)-extension then proposition 4.1 would imply that there are no finitely observable invariants but entropy on \( \mathcal{K} \). However, the existence of such a joining is an open problem.

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