Goodwillie calculus in the category of algebras over a chain complex operad

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Abstract. The goal of this paper is to furnish a literature on Goodwillie calculus for functors defined between categories which derive from chain complexes over a ground field $k$. We characterize homogeneous functors $F : C \to D$ where $C, D = Ch$ (chain complexes), $Ch_+$ (non-negatively graded chain complexes) or $Alg_O$ (algebras over a chain complex operad $O$). In the particular case when $D = Alg_O$, our characterization requires $k$ to be of characteristics 0.

We are then extending the results of Walter [Wal06] who studied in characteristics 0 the chain complex cases and when $O$ is the Lie operad.

AMS Classification numbers. Primary: 18D50, 55P65 ; Secondary: 18G55, 55U35, 55U15

Key words. Operads, algebras over an operad, model category, calculus of functors

Introduction

The chain complexes are over a ground field $k$. Let $O$ be a fixed operad on $Ch_+$. In this paper, we give a characterization of homogeneous functors $F : C \to D$, where $C$ and $D$ are either $Alg_O$, $Ch_+$ or $Ch$. This is an "algebraic" version of a couple of publications in Functor Calculus. It starts with Goodwillie [Goo03] when $C$ and $D$ are the category of pointed topological spaces or the associated category of spectra (S-modules of [EKMM97]). He proved that homogeneous functors $F$ are completely described in term of symmetric sequences (of spectra) $\partial_* F$, called "Derivatives". The Functor Calculus for continuous functors was extended by Kuhn, in [Kuh07], to the case the categories $C$ and $D$ respect the following conditions:

1. The categories $C$ and $D$ are simplicial or topological pointed model categories;
2. $C$ and $D$ are proper: the pushout of a weak equivalence by a cofibration is a weak equivalence, and dually for the pullbacks.
3. filtered homotopy colimit in $D$ commutes with finite homotopy limits.

By inspection, one see that under good conditions on the functors, the Kuhn (or Goodwillie) Taylor tower construction can be defined when only the conditions 2. and 3. are satisfied. In our cases the categories $Alg_O, Ch_+$ and $Ch$ satisfy the Kuhn’s requirements 2. and 3. We can therefore follow his lines to develop the approximation of functors. On the other hand, to get a characterization theorem for homogeneous functors similar to Goodwillie’s result, Kuhn needed condition 1. In our case even if the categories $Ch, Ch_+$ and $Alg_O$ (by Hinich in [Hin97 § 4.8]) are simplicial categories, there is not a genuine tensoring of these categories over sSet. Hence, these are not simplicial model categories. At this point, our constructions will deviate a little from the literature and we will replace continuous functors by homotopy functors which is a weaker requirement.
Main results

We develop tools in the category $\text{Alg}_O$ to analyze the Taylor tower of homotopy functors. In fact we give an explicit model of the homotopy pullbacks and we deduce that any loop $O$-algebra has a trivial $O$-algebra structure when the ground field is in characteristics $0$. On the other hand, we give an explicit model of homotopy pushouts. As a consequence, we deduce that the suspension of an $O$-algebra is equivalent to a free $O$-algebra. We apply these constructions to show that a homogeneous functor $F$ is completely described by a symmetric sequences of unbounded chain complexes denoted $\partial^\ast F$. In fact we analyze the Taylor tower $P_n F$ of $F$ and we show that under good conditions, there is a weak equivalence

$$D_n F(X) \simeq \Omega^\infty (\partial, F \otimes_{h \Sigma_n} (\Sigma^\infty X)^{\otimes n}).$$

where $D_n F = \text{hofib}(P_n F \to P_{n-1} F)$. In our construction, the pair $(\Sigma^\infty, \Omega^\infty)$ has a different concept from the Kuhn’s construction. Namely, in Kuhn’s paper [Kuh07], given a simplicial category $D$, the pair

$$\Sigma^\infty : D \rightleftarrows \text{Spectra}(D) : \Omega^\infty$$

is an adjoint pair. As in [BM05], when $D = \text{Alg}_O$, we will identify the category $\text{Spectra}(D)$ with $Ch$. We know that these two categories are related by a zig-zag of Quillen equivalences. This identification will imply a non canonical modification in the construction of these functors. Roughly speaking, $\Sigma^\infty : \text{Alg}_O \to Ch$ becomes the Quillen homology $TQ(-)$, and $\Omega^\infty : Ch \to \text{Alg}_O$ assigns to each chain complex a trivial $O$-algebra structure. The pair $(\Sigma^\infty, \Omega^\infty)$ we get is no more an adjoint pair. We give more detail about these constructions in section 1.3.

When $C$ and $D$ are either $Ch$ or $Ch_+$, or when $O = \text{Lie}$ and the ground field is in characteristics $0$, these constructions and results appear in [Wal06].

Outline of the paper

In the first section 1, we briefly remind the preliminaries on the categories $\text{Alg}_O$, $Ch_+$ and $Ch$. In section 2 we make an explicit construction of homotopy pullbacks and homotopy pushouts in $\text{Alg}_O$. We remind in section 3 the Goodwillie approach in Functor calculus. In section 4 we characterize homogeneous functors. Our method in that section is inspired by the Goodwillie’s approach. Namely we prove that homogeneous functors are infinite loop spaces (as in [Goo03, Thm 2.1]). We use the "stabilization" of the cross effect and at the end of this process, we obtain a similar (with Goodwillie) characterization of homogeneous functors using the Quillen Homology $TQ(-)$ viewed as $\Sigma^\infty$. We compute in section 5 the derivatives of some functors. In the last section 6, we prove that there is a chain rule property associated to the composition of two functors (composed on $Ch$).

Future Work

The work of this paper raises the question of extending the description of homogeneous functors to a classification of Taylor towers. This question was raised by Arone-Ching [AC11] and they investigated the module structure on the collection of derivatives $\partial_n F$ over a certain operad. They built a tower which fails to be the Taylor tower of $F$ up to Tate cohomology. Since Tate cohomologies vanish rationally (see [Kuh04]), this question is studied in our future work ([AN19]) when the ground field $k$ is of characteristics $0$. 

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Acknowledgements. The author is grateful to Pascal Lambrechts and Greg Arone for their suggestions and encouragement during this work.

1 preliminaries

1.1 Background on chain complexes

All chain complexes are over a field \( k \) of any characteristics. The purpose of this section is to fix conventions and review basic properties which are background of our constructions.

In this paper, we denote by \( \text{Ch} \) the category of \( \mathbb{Z} \)-graded chain complexes over \( k \). This category has a symmetric monoidal structure. The tensor product of chain complexes \( V,W \in \text{Ch} \) is defined by:

\[
(V \otimes W)_n := \bigoplus_{p+q=n} V_p \otimes W_q
\]

with the differential such that: \( \forall x \otimes y \in V_p \otimes W_q, \ d(x \otimes y) = d(x) \otimes y + (-1)^p x \otimes d(y) \).

The unit of the monoid \( - \otimes - \), which we denote abusively \( k \), is the chain complex having \( k \) in degree 0 and is trivial in all other degrees. The tensor product \( - \otimes - \) has a right adjoint \( \text{hom}(\ - \ , \ - \) \) given by:

\[
\text{hom}(V,W) := \bigoplus_{i \in \mathbb{Z}} \text{hom}^i(V,W)
\]

where \( \text{hom}^i(V,W) \) denotes the vector space of morphisms \( f : V \to V_{*+i} \) of degree \( i \).

Similarly We denote by \( \text{Ch}_+ \), the sub-category of \( \text{Ch} \) which consist of non negatively graded chain complexes.

Twisted chain complex

Let \( (V,d_V) \) be a chain complex. A twisting homomorphism of degree \(-1\), \( d : V \to V \) is a morphism of graded vector spaces of degree \(-1\) which is added to the internal differential \( d_V \) to produce a new differential \( d_V + d : V \to V \) on \( V \). The equation \((d_V + d)^2 = 0\) is equivalent to the equation \(d_V(d) + d^2 = 0\), with \(d_V(d) := d_Vd + dd_V\).

Model category structure on \( \text{Ch}_+ \)

The category \( \text{Ch}_+ \) is a proper closed model category (for instance see [GJ94, § 4.2]): weak equivalences are quasi-isomorphisms, fibrations are surjections and cofibrations are injections. In this model category, all objects are cofibrant and fibrant.

1.2 Operads

We denote by \( \text{FinSet} \) the category whose objects are finite sets and whose morphisms are bijections. We denote the category of all symmetric sequences in \( \text{Ch}_+ \) by \([\text{FinSet}, \text{Ch}_+]\) (in which morphisms are natural transformations). The composition \( M \circ N \), of the two symmetric sequences \( M \) and \( N \), is defined by:

\[
(M \circ N)(J) := \bigoplus_{J=\bigcup_{j \in J'} J_j} M(J') \otimes \bigotimes_{j \in J'} N(J_j).
\]

The coproduct here is taken over all unordered partitions, \( \{J_j\}_{j \in J'} \), of \( J \). The unit symmetric sequence \( \mathbb{I} \) is given by
I(J) = k, if |J| = 1, and I(J) = 0 otherwise;

**Definition 1.1** (Operads). An operad in $Ch_+$ is a monoid over $([Finset, Ch_+], \circ, \mathbb{I})$. A reduced operad is an operad $\mathcal{O}$ such that $\mathcal{O}(0) = 0$ and $\mathcal{O}(1) = k$. In this paper, we only consider reduced operads in $Ch_+$.

1.3 Algebra over an operad

Let $\mathcal{O}$ be a reduced operad. An $\mathcal{O}$–algebra $X$ is a chain complex together with structure maps, for any $n \geq 0$: $m_n : \mathcal{O}(n) \otimes X^\otimes n \rightarrow X$, satisfying the appropriate compatibility conditions. Maps of $\mathcal{O}$–algebras are given by chain complex morphisms $f : X \rightarrow X'$ which are degree 0 and preserve the $\mathcal{O}$–algebra structures of $X$ and $X'$. The category of $\mathcal{O}$-algebras is denoted $\text{Alg}_\mathcal{O}$.

**Model category structure on $\text{Alg}_\mathcal{O}$**

One use this adjunction $\mathcal{O}(-) : Ch_+ \rightleftarrows \text{Alg}_\mathcal{O} : U$, between the forgetful and the free functors, to define the projective model structure on $\text{Alg}_\mathcal{O}$ (see [GJ94, Thm 4.4]). Namely weak equivalences(resp. fibrations) of $\text{Alg}_\mathcal{O}$ are equivalences (resp. fibrations) in the underlined category $Ch_+$. The cofibrations are morphisms having the right lifting property with respect to acyclic fibrations. In particular, cofibrant $\mathcal{O}$-algebras are retract of quasi-free algebras.

1.4 Cooperad

The notion of cooperad is dual to the notion of operad. The dual notion consists of considering the opposite category $((Ch_+)^{op}, \otimes, \mathbb{I}_{Ch})$. We define the dual composition product $\hat{\circ}$ of two symmetric sequences by replacing the coproduct in the definition ?? with a product. That is

$$(M \hat{\circ} N)(J) := \prod_{J = \bigsqcup J_j} M(J_j) \otimes \bigotimes_{j \in J_j} N(J_j).$$

where the product is taken over all unordered partitions, $\{J_j\}_{j \in J'}$, of $J$.

A cooperad in $\mathcal{C}$ is a triple $(Q, m^c, \eta^c)$, where $Q$ is a symmetric sequence together with maps

$$m^c : Q \rightarrow Q \hat{\circ} Q \quad \text{and} \quad \eta^c : Q \rightarrow \mathbb{I}$$

satisfying the co-associativity, the left and right co-unit condition.

A cooperad $Q$ is connected when $\tilde{Q} := \ker(\eta^c)$ is concentrated strictly in positive degree.

Since (finite) product and direct sum are equivalent in the underlying category $Ch_+$, in the rest of this thesis, the dual composition product $\hat{\circ}$ will simply be denoted $\circ$.

1.5 Coalgebra over a cooperad

Another dual analogy is the notion of the coalgebra over a cooperad. That is, any chain complex $Y$ together with a structure map, $\forall n, m^c_n : Y \rightarrow Q(n) \otimes Y^n$ satisfying the appropriate compatibility conditions. The maps of $Q$-coalgebras are degree 0 chain complex morphisms $f : Y \rightarrow Y'$ which preserves the structures of $Y$ and $Y'$. One denotes the category of $Q$-coalgebras by $\text{coAlg}_Q$.
Model category on coAlg_Q

We use this adjunction $U : \text{coAlg}_Q \rightleftarrows Ch_+ : Q(-)$ between the, forgetful and the cofree functor, to define an injective model structure on coAlg_Q (see [GJ94, Thm 4.7]). Namely weak equivalences(resp. cofibrations) of coAlg_Q are weak equivalences(resp. cofibrations) in the underlined category weak $Ch_+$. The fibrations are morphisms having the left lifting property with respect to acyclic cofibrations.

1.6 Bar and Cobar constructions

1.6.1 Bar construction

$J$-tree

Let $J$ be a finite set. A $J$–tree is an abstract planar tree with one output edge on the bottom, and input edges on the top whose sources also called leaves are indexed by $J$. These input edges and the edge from the root are the external edges of the tree, and the other edges are called internal edges. The vertices of internal edges are called internal vertices. Given an $J$–tree $T$, we denote by $V(T)$ the set of its internal vertices, and $E(T)$ the set of edges. The set of $J$–trees, denoted by $\beta(J)$, is equipped with a natural groupoid structure. Formally, an isomorphism of $J$–trees $\beta : T' \to T$ is defined by bijections $\beta_v : V(T') \to V(T)$ and $\beta_E : E(T') \to E(T)$ preserving the source and target of edges. In other word, $\beta(J)$ is the groupoid of $J$–labeled trees and non-planar isomorphisms.

Definition 1.2 (Free object). Let $M$ be a symmetric chain complex. The free object, associated to $M$, and denote by $F(M)$ consists of: chain complexes $F(M)(J), \partial_0$, for any finite set $J$, defined as

$$F(M)(J) = \bigoplus_{T \in \beta(J)} T(M)/\equiv$$

where $T(M) = \bigotimes_{v \in V(T)} M(J_v)$, and the equivalence classes are made of non planar isomorphisms of $J$–trees. The differential $\partial_0$ is induced naturally by the differentials of the chain complexes $(M(J_v), \partial_{J_v})$.

A bijection $J \to J'$ gives an isomorphism $F(M)(J) \to F(M)(J')$ by relabeling the leaves of the underlined trees. In this way $F(M)$ becomes a symmetric sequence in chain complexes.

Let $\mathcal{O}$ be an operad, $R$ is a right $\mathcal{O}$-module and $L$ is a left $\mathcal{O}$-module.

Definition 1.3 (Two sided bar construction). The two sided bar construction $B(R, \mathcal{O}, L)$ is the symmetric sequence of chain complexes given by: for any finite set $J$,

$$B(R, \mathcal{O}, L)(J) := (R \circ F(s\mathcal{O}) \circ L(J), \partial_0 + \partial), \text{ with } \mathcal{O} = \ker \varepsilon.$$

The differential $\partial_0$ is induced in the natural way by the differentials of the chain complexes \{(R(J'), d_p)\}_{J' \subseteq J}, \{(\mathcal{O}(J'), d_p)\}_{J' \subseteq J}, \text{ and } \{(L(J'), d_p)\}_{J' \subseteq J}. \text{ The second differential } \partial = \partial_R + \partial_\mathcal{O} + \partial_L \text{ of this complex is the derivation which integrates the structure morphisms: } m_R : R \circ \mathcal{O} \to R, m_L : \mathcal{O} \circ L \to L, \text{ and } m_\mathcal{O} : \mathcal{O} \circ \mathcal{O} \to \mathcal{O} \text{ (for explicit description, see [Fre04] \S 4.4.3.).}$

If $L = R = \mathbb{I}$, then $B(R, \mathcal{O}, L)$ is the usual bar construction $B(\mathcal{O})$. 

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Bar construction with coefficients in $\mathcal{O}$-algebras

Given an $\mathcal{O}$-algebra $X$, there is an associated left $\mathcal{O}$-module $\hat{X}$ defined as follows:

$$\hat{X}(0) = X, \hat{X}(n) = 0 \text{ if } n \geq 1$$

and the left action $\mathcal{O} \circ \hat{X} \rightarrow \hat{X}$ is induced in the obvious way by the $\mathcal{O}$-algebra structure of $X$.

This defines an embedding functor $\hat{\cdot} : \text{Alg}_\mathcal{O} \rightarrow \mathcal{O}\text{-mod}$ of $\mathcal{O}$-algebras to the category of left modules over $\mathcal{O}$.

We use this embedding and the bar construction with coefficients in left and right $\mathcal{O}$-modules described in Definition 1.3 to define the bar construction with coefficients in $\mathcal{O}$-algebras.

**Definition 1.4** (Bar construction on algebras). Let $X$ be an algebra over a reduced operad $\mathcal{O}$. We define the bar construction on $\mathcal{O}$ with coefficient in $X$ as the chain complex:

$$B(\mathcal{O}, X) := \bigoplus_n (B(I, \mathcal{O}, \hat{X})(n), \partial_0 + \partial^*)$$

### 1.6.2 Cobar construction

Let $(Q, Q \xrightarrow{m} Q \circ Q, Q \xrightarrow{\eta} I)$ be a connected cooperad in $\text{Ch}_+$, and denote $\tilde{Q} := \ker(\eta^e)$. The cobar construction of $Q$, denoted $B^c(Q)$ is the dual version of the bar construction (for operads). Namely, this is the quasi-free operad

$$B^c(Q) = (F(s^{-1}\tilde{Q}), \partial_0 + \partial^e),$$

where $\partial_0$ is the internal differential of $F(s^{-1}\tilde{Q})$ induced by that of $Q$, and $\partial^e$ is the differential defined by reversing all the arrows in the definition of $\partial$ on $B(\mathcal{O})$ in Definition 1.3 (when $L = R = I$).

In this definition, the cooperad $Q$ needs to be connected to avoid the case where $B^c(Q)$ has elements in negative degree. We are now ready to state the next theorem (in characteristics 0) which gives a duality between the bar construction and the cobar construction.

**Theorem 1.5.** [GJ94, Theorem 2.17] The functors $B^c$ and $B$ form an adjoint pair between the categories of connected cooperads and augmented operads.

In addition, it is proved in [GK95, Theorem 3.2.16] that the unit $Q \rightarrow BB^c(Q)$ and the counit $B^eB(\mathcal{O}) \rightarrow \mathcal{O}$ of this adjunction are quasi-isomorphisms.

### Cobar construction with coefficient in $Q$-coalgebras

Let $Q$ be a connected cooperad on chain complexes and $Y$ a $Q$-coalgebra. One can follow the same procedure such as in the case of bar construction over algebras to define the cobar construction $B^c(\mathcal{O}, Y)$. In this sense we will get literally an object in the category of $B^c(Q)$-algebra. We do not follow these steps here since we will need further as an application a cobar construction functor which send a $B(\mathcal{O})$-coalgebra (for a given reduced operad $\mathcal{O}$) into an $\mathcal{O}$-algebra.

We consider to have from now a reduced operad $\mathcal{O}$ such that $B^c(Q) \xrightarrow{\sim} \mathcal{O}$. This later morphism induces a degree 0 morphism $s^{-1}\tilde{Q} \rightarrow \mathcal{O}$ which gives a morphism $\theta : \tilde{Q} \rightarrow \mathcal{O}$ of degree $-1$. We use $\theta$ to define the composition
\[ w : Q(Y) \xrightarrow{Q(m_Y)} Q(Q(Y)) \xrightarrow{\theta(1_{QY})} \mathcal{O}(Q(Y)) \]

The derivation \(d_w : \mathcal{O}(Q(Y)) \to \mathcal{O}(Q(Y))\) of degree \(-1\) associated to this \(w\) satisfies the equation of twisting homomorphism \(d(w) + d_w.w = 0\) on \(Q(Y)\). This is equivalent to say that \((\mathcal{O}(Q(Y)), d + d_w)\) is a quasi-free \(\mathcal{O}\)-algebra. The morphism \(\theta\) which is at the base of this construction will be called in the literature twisting cochain (see [GJ94, def 2.16]).

**Definition 1.6** (cobar construction on a \(Q\)-coalgebra). Let \(Y\) be a \(Q\)-coalgebra. The cobar construction on \(Y\), associated to the twisting cochain \(\theta : \tilde{Q} \to \mathcal{O}\), and denoted \(B^c_\theta(Q,Y)\) is the quasi-free \(\mathcal{O}\)-algebra

\[ B^c_\theta(Q,Y) = (\mathcal{O}(Q(Y)), d + d_w) \]

where \(d\) is the internal differential of \(\mathcal{O}(Q(Y))\) induced by the complexes \(\mathcal{O}, Q\) and \(Y\).

When \(Q = B(\mathcal{O})\), the map \(\theta\) that we consider is given by the projection \(B(\mathcal{O}) \to \mathcal{O}\).

In that specific case, we will always drop \(\theta\) and simply write \(B^c(Q,Y)\) to mean as \(\theta\) is given by the projection \(B(\mathcal{O}) \to \mathcal{O}\).

One form the cobar-bar adjoint pair

\[ B^c(B(\mathcal{O}), -) : \text{coAlg}_{B(\mathcal{O})} \rightleftarrows \text{Alg}_{\mathcal{O}} : B(\mathcal{O}, -) \]

whose the unit and co-unit functors have the following property(in characteristics 0):

**Theorem 1.7** ([GJ94, Theorem 2.19]). Given an \(\mathcal{O}\)-algebra \(X\) and a \(B(\mathcal{O})\)-coalgebra \(Y\), the co-unit \(B^c(B(\mathcal{O}), B(\mathcal{O}, X)) \to X\) and the unit \(Y \to B(\mathcal{O}, B^c(B(\mathcal{O}), Y))\) are weak equivalences.

With the model structure defined on \(\text{coAlg}_{B(\mathcal{O})}\) and \(\text{Alg}_{\mathcal{O}}\), we can see that the cobar-bar adjunction is actually a Quillen pair, and Theorem 1.7 completes in proving that this adjunction is a Quillen equivalence.

**Remark 1.8.** Fresse proved in [Fre04, Prop 3.1.12] that the counit \(B^c B(\mathcal{O}) \to \mathcal{O}\) is a quasi-isomorphism when the ground field \(k\) is of any characteristics. Thus one can apply this fact in [Fre04, Thm 4.2.4] to deduce that the co-unit \(B^c(B(\mathcal{O}), B(\mathcal{O}, X)) \to X\) is a cofibrant resolution of \(X\) if \(k\) is a field of any characteristics.

### 1.7 Definition of the functors \(\Omega^\infty\) and \(\Sigma^\infty\)

Let \(X\) be an \(\mathcal{O}\)-algebra. We define the functor \(\tau_1\mathcal{O}_X : \text{Alg}_\mathcal{O} \to \text{Alg}_\mathcal{O}\) as follows:

\[ \tau_1\mathcal{O}_X : \colim_{\text{Alg}_\mathcal{O}} (\tau_1\mathcal{O} \circ \mathcal{O} \circ (X) \Rightarrow \tau_1\mathcal{O} \circ (X)) \]

where \(\tau_1\mathcal{O}\) is the operad: \(\tau_1\mathcal{O}(1) = k\), and \(\tau_1\mathcal{O}(t) = 0\) if \(t \neq 1\); The first map of this colimit is produced by the multiplication \(\tau_1\mathcal{O} \circ \mathcal{O} \to \tau_1\mathcal{O}\) and the second map is given by the algebra structure map \(\mathcal{O}(X) \to X\). Strictly speaking, the algebra \(\tau_1\mathcal{O}_X\) has a trivial \(\mathcal{O}\)-algebra structure, thus we will define the abelianization functor as its composite with the forgetful functor

\[ (-)^{ab} : \text{Alg}_\mathcal{O} \xrightarrow{\tau_1\mathcal{O}_{X}^{\text{ab}}} \text{Alg}_\mathcal{O} \xrightarrow{\mu} \text{Ch}_+ \xrightarrow{\iota} \text{Ch} \]
where \( I \) is the inclusion functor defined by \( I(V)_t := \begin{cases} V_t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \)

The abelianization functor has a right Quillen adjoint functor:

\[ \Omega^\infty : Ch \xrightarrow{red_0} Ch_+ \xrightarrow{(\cdot)_{triv}} \Alg_O \]

where for any chain complex \( C_\ast \), \( red_0(C_\ast)_t := \begin{cases} C_t & \text{if } t > 0 \\ \ker(d_0) & \text{if } t = 0 \end{cases} \)

and \( (\cdot)_{triv} \) is the functor which assigns to any non-negative chain complex the trivial \( O \)-algebra structure.

The functor \( (-)^{ab} \) does not preserve quasi-isomorphisms in general, apart from preserving quasi-isomorphisms between quasi-free algebras (since they are cofibrant objects in \( \Alg_O \)). Its associated derive functor is called in the literature Quillen homology.

**Definition 1.9** (Quillen homology). If \( X \) is an \( O \)-algebra, then the Quillen homology \( TQ(X) \) of \( X \) is the \( O \)-algebra \( \tau_1\Omega^h_{O} X \).

Again since the algebra structure on \( TQ(X) \) is trivial, we will abuse notation and consider it as an object in \( Ch_+ \). We will give in the next lines an explicit model of the functor \( TQ(-) \) which we will need to define \( \Sigma^\infty \).

Let \( X \) be an algebra over \( O \). One associate to \( X \) the symmetric sequence \( X_\ast \) defined as \( X(k) = X \), if \( k = 0 \) and \( X(k) = 0 \), if \( k \neq 0 \). One can see that \( X_\ast \) is a left \( O \)-module with the structure map induced naturally by the algebra structure map. In this sense the category \( \Alg_O \) embeds in \( Lt_O( \text{the category of left } O \text{-modules}) \) as a full subcategory of left modules concentrated at degree 0, via the functor \( \sim : Alg_O \rightarrow Lt_O, X \mapsto \hat{X} \).

According to Theorem 1.7, \( B^r(B(O), B(O, X)) \) is a cofibrant replacement of \( X \), therefore \( TQ(X) \cong UB(O, X) \), where \( U : \Coalg_{B(O)} \rightarrow Ch_+ \) is the forgetful functor. Under this last quasi-isomorphism, we will consider the functor \( UB(O, -) \) as our explicit model for the functor \( TQ(-) \) and we will denote by \( \Sigma^\infty \) the composition:

\[
\begin{array}{cccc}
\Alg_O & \xrightarrow{B(O, -)} & \Coalg_{B(O)} & \xrightarrow{U} & Ch_+ & \xrightarrow{\Sigma^\infty} & Ch \\
\xrightarrow{I} & & & & & \xrightarrow{\sim} & \\
\end{array}
\]

Before concluding these constructions, we make the following remark:

We consider the following two adjunctions

\[
\begin{array}{cccc}
\coAlg_{B(O)} & \xrightarrow{U} & Ch_+ & \xrightarrow{I} & Ch \\
\xrightarrow{\sim} & & & \xrightarrow{red_0} & \\
\end{array}
\]

where the top functors are each left adjoint and the bottom functors are each right adjoint.

We then observe that the associate comonad is \( IUB(O, (-)_{triv}) red_0 \cong \Sigma^\infty \Omega^\infty \). Therefore even if the functor \( \Omega^\infty \) is not adjoint to \( \Sigma^\infty \), we can say that \( T = \Sigma^\infty \Omega^\infty : Ch \rightarrow Ch \) is a comonad and is a "homotopy good model" of the comonad \( (-)_{triv}^{ab} \).

We extend construction of the functors \( \Sigma^\infty \) and \( \Omega^\infty \) to other categories as follows:

- \( \Sigma^\infty := I : Ch_+ \rightarrow Ch \);
- \( \Sigma^\infty = Id : Ch \rightarrow Ch \);
- \( \Omega^\infty = red_0^- : Ch \rightarrow Ch_+ \);
- \( \Omega^\infty = Id : Ch \rightarrow Ch \).
2 Homotopy limits and colimits in Alg\(\mathcal{O}\)

The purpose of this section is to remind a brief notion of homotopy limits and colimits, and give their explicit description in Alg\(\mathcal{O}\) in terms of holims and hocolims in chain complexes.

Let \(\mathcal{C}\) and \(\mathcal{D}\) be any of the categories Alg\(\mathcal{O}\), coAlg\(\mathcal{B}(\mathcal{O})\) and Ch\(_+\). These categories are complete and cocomplete. The authors of [DHKS04] proved, in a general argument for complete and cocomplete model categories, that holims and hocolims always exists in \(\mathcal{C}\)(see [DHKS04, 19.2]). More explicitly, given a small category \(J\), and an \(J\)-diagram \(D\) in \(\mathcal{C}\), they replace \(D\) through a functor \(D \mapsto D_{vf}\) (resp. \(D \mapsto D_{vc}\)) which associate a so called "virtually fibrant replacement" (resp "virtually cofibrant replacement") \(D_{vf}\) (resp. \(D_{vc}\)) such that there is a map \(D \xrightarrow{\sim} D_{vf}\) (resp. \(D_{vc} \xrightarrow{\sim} D\)) natural in \(D\).

According to this vocabulary we can now set the definition of holims and hocolims:

**Definition 2.1.** Given an \(J\)-diagram \(D\) in \(\mathcal{C}\),

\[
\text{holim}_\mathcal{C}(D) := \lim_{\mathcal{C}}(D)_{vf} \text{ and } \text{hocolim}_\mathcal{C}(D) := \text{colim}_{\mathcal{C}}(D)_{vc}.
\]

2.1 Homotopy pullback in Alg\(\mathcal{O}\)

We assume in this section that the ground field \(k\) is of characteristics 0. The homotopy limit in Alg\(\mathcal{O}\) is calculated using observations in the underlined category Ch\(_+\). Given a diagram \(D : X \xrightarrow{g} Z \xleftarrow{f} Y\) in Ch\(_+\), if either \(f\) or \(g\) is a surjection (to mean fibration), then \(\text{holim}_{\text{Ch}_+}(D) \simeq \lim_{\text{Ch}_+}(D)\). This comes out easily when we apply the homology long exact sequence theorem to the two parallel fibrations of the pullback square associated to \(D\). Our methodology to define an explicit homotopy limit in Alg\(\mathcal{O}\) is to replace the maps of the \(\mathcal{O}\)-algebra diagram by explicit surjections.

Construction of path objects in Alg\(\mathcal{O}\)

Let \(\mathcal{I} = (\wedge(t, dt), d)\) be the free differential graded commutative algebra generated by the element \(t\) in degree 0 and \(dt\) in degree -1, with differential \(d\) given by \(d(t) = dt\) and \(d(dt) = 0\). It is useful to notice that an element \(\alpha\) of \(\mathcal{I}\) has the form \(\alpha = P(t) + Q(t)dt\) with \(P, Q \in k[t]\).

There are natural commutative algebra maps \(s_0 : k \rightarrow \mathcal{I}\) and \(p_0, p_1 : \mathcal{I} \rightarrow k\) defined as: \(\forall(\alpha = P(t) + Q(t)dt \in \mathcal{I})\) and \(k \in k,\)

\[
p_0(\alpha) := P(0), \; p_1(\alpha) := P(1) \text{ and } s_0(k) = k
\]

\(s_0\) is a quasi-isomorphism and \(p_0s_0 = p_1s_0 = 1_k\).

For any \(\mathcal{O}\)-algebra \(X\), there is a natural \(\mathcal{O}\)-algebra structure on \(\mathcal{I} \otimes X\) (see [Liv99, §2.4]) given by: If \(a \in \mathcal{O}(n), \alpha_i \otimes x_i \in \mathcal{I} \otimes X\), for \(1 \leq i \leq n,\)

\[
m(a \otimes \alpha_1 \otimes x_1 \otimes \ldots \otimes \alpha_n \otimes x_n) := \pm \alpha_1 \ldots \alpha_n \otimes m_X(a \otimes x_1 \otimes \ldots \otimes x_n)\]

One then get the factorization in \(\mathcal{O}\)-algebras (unbounded algebras)

\[
X \xrightarrow{s_0 \otimes X} \mathcal{I} \otimes X \xrightarrow{p_0 \otimes X} X
\]

which yield to the diagram in Alg\(\mathcal{O}\):
One can prove that \( p_0^X \) and \( p_1^X \) are trivial surjections.

**Definition 2.2** (path object). A path object associated to an \( \mathcal{O} \)-algebra \( X \) is the \( \mathcal{O} \)-algebra \( X^I := \text{red}_0(\mathcal{I} \otimes X) \) together with the \( \mathcal{O} \)-algebra morphisms \( p_0^X, p_1^X \) and \( s_0^X \).

**Construction of homotopy pullbacks in \( \text{Alg}_\mathcal{O} \)**

Let us consider the commutative diagram in \( \text{Alg}_\mathcal{O} \):

![Diagram](image)

where the square in the middle is a pullback. From the left triangle, we build the following factorization of \( f \):

\[
Y \xrightarrow{(s_0^Y, f, Y)} Z^I \times Y \xrightarrow{(\pi_2, f)} Y
\]

\[
f = p_1^Z \pi_1 f
\]

We use this factorization to replace \( f \) in a diagram \( D : X \to Z \leftarrow Y \) by the fibration \( p_1^Z \pi_1 \).

**Proposition 2.3.** Given an \( \mathcal{O} \)-algebra diagram \( D : X \to Z \leftarrow Y \), a homotopy pullback of \( D \) is the \( \mathcal{O} \)-algebra \( P_D = X \times Z^I \times Y \), namely \( P_D = \lim_{\text{Alg}_\mathcal{O}}(X \to Z \xrightarrow{p_1^Z \pi_1} Z^I \times Y) \).

**Proof.**

1. The morphism \( p_1^Z \pi_1 \) is a surjection, it then follows from our comment in the introduction of this section that in \( \text{Ch}_{\mathcal{O}} \), we have

\[
UP_D \simeq \text{holim}_{\text{Ch}}(UX \to UZ \xrightarrow{p_1^Z \pi_1} UZ^I \times Y)
\]

\[
\simeq \text{holim}_{\text{Ch}} UD
\]

As a consequence we deduce that the functor \( P_{-} \) preserves weak equivalence of diagrams in \( \text{Alg}_\mathcal{O} \). Therefore we retain that \( P_D \xrightarrow{\simeq} P_{D_{\mathcal{O}}} \).

2. We now prove that \( \lim_{\text{Alg}_\mathcal{O}}(D_{\mathcal{O}}) \xrightarrow{\simeq} P_{D_{\mathcal{O}}} \). Let's consider \( D_{\mathcal{O}} : X' \to Z' \leftarrow Y' \) and the cube in \( \text{Ch}_{\mathcal{O}} \):
where we have applied the forgetful functor $U : \text{Alg}_\mathcal{O} \to Ch_\perp$ to the original natural cube in $\text{Alg}_\mathcal{O}$. The square obtained from the homotopy fibers of the morphisms $h_1, h_2, h_3$ and $h_4$ has the following characteristics:

$\xymatrix{ \text{hofibre}(h_1) \ar[r]^{(1)} \ar[d] & \text{hofibre}(h_2) \ar[d]^{(2)} \ar[r] & \text{hofibre}(h_4) \ar[r]^{(3)} \ar[d] & \text{hofibre}(h_3) \ar[d] \ar[r] & \text{hofibre}(h_1) \ar[d] }$

where

1. is a weak equivalence as the top square of the cube is a homotopy pullback;
2. is a weak equivalence because $1_{UZ'}$ and $U(s'_0 f, Y')$ are weak equivalences, and these imply that the right hand square is a homotopy pullback;
3. is a weak equivalence since the bottom square is a homotopy pullback.

We can conclude that $\text{hofibre}(h_4) \to \text{hofibre}(h_1)$ is a weak equivalence and therefore that $\text{lim}_{\text{Alg}_\mathcal{O}}(D_{ef}) \xrightarrow{\sim} P_{D_{ef}}$. We then conclude in conclusion that

$P_D \xrightarrow{\sim} P_{D_{ef}} \xleftarrow{\sim} \text{lim}_{\text{Alg}_\mathcal{O}}(D_{ef})$

In general this construction is extended in the obvious manner in order to define higher dimensional limits in $\text{Alg}_\mathcal{O}$.

**Lemma 2.4.** If $X$ is an $\mathcal{O}$-algebra, then the map

$$\Phi : (\text{red}_0 s^{-1}X)_{\text{triv}} \to \Omega X$$

$$s^{-1}x \mapsto (0, dt \otimes x, 0)$$

is a weak equivalence in $\text{Alg}_\mathcal{O}$.

**Proof.** We first prove that $\Phi$ is a map of $\mathcal{O}$-algebras. Namely let $x_1, ..., x_n \in X$, and $a \in \mathcal{O}(n), (n \geq 2)$, then

$$m_{\Omega X}(a \otimes \Phi(s^{-1}x_1) \otimes ... \otimes \Phi(s^{-1}x_n)) = (0, dt^n \otimes m_X(a \otimes x_1 \otimes ... \otimes x_n), 0) = 0 \ (\text{since } dt^n = 0)$$

This computation proves that $\Phi$ is a map of $\mathcal{O}$-algebras as the $\mathcal{O}$-algebra structure on $(\text{red}_0 s^{-1}X)_{\text{triv}}$ is trivial. It is obvious that the map $\Phi$ commutes with differentials of the two complexes.

Now we prove by hand that $H_*(\Phi)$ is injective and surjective. Let us take
Proof.

One can see that $$\Sigma$$

This implies that $$H$$

One can also see that

If Lemma 2.5.

This last equality implies that $$\forall l \geq 1, a_l = 0$$ and thus $$\Sigma \sum_{l \geq 1} a_l = 0$$. One then get:

$$\pi = \sum_{l \geq 1} a_l + \sum_{k \geq 0} \sum_{l \geq 1} l^k dtb_k \in X$$ such that $$(0, \pi, 0) \in \Omega X \cap Ker d$$

where for each $$l$$ and $$k$$, $$a_l, b_k \in X$$;

$$(0, \pi, 0) \in \Omega X \iff p_1^X(\pi) = 0 = p_0^X(\pi) \iff a_0 = 0 = \sum_{l \geq 1} a_l$$

This last equality implies that $$\forall l \geq 1, a_l = \frac{1}{t}db_{l-1}$$ and thus $$\sum_{l \geq 1} \frac{1}{t}db_{l-1} = 0$$. One then get:

$$\pi = \sum_{l \geq 1} \frac{1}{t}db_{l-1} + \sum_{l \geq 1} l^{l-1} dtb_{l-1}$$

$$= \sum_{l \geq 1} \frac{1}{t}(l^{l-1} dtb_{l-1}) + \frac{1}{t}db_{l-1}$$

$$= d(\sum_{l \geq 1} \frac{1}{t}db_{l-1})$$

$$= d(\sum_{l \geq 1} \frac{1}{t}db_{l-1} - t \sum_{l \geq 1} \frac{1}{t}db_{l-1})$$

$$= d(\sum_{l \geq 1} \frac{1}{t}db_{l-1}) + d(t \sum_{l \geq 1} \frac{1}{t}db_{l-1})$$

One can see that $$\sum_{l \geq 1} \frac{1}{t}db_{l-1} - t \sum_{l \geq 1} \frac{1}{t}db_{l-1} \in \Omega X$$ and that $$d(t \sum_{l \geq 1} \frac{1}{t}db_{l-1}) = dt \otimes \sum_{l \geq 1} \frac{1}{t}db_{l-1}$$, therefore

$$[\pi] = [dt \otimes \sum_{l \geq 1} \frac{1}{t}db_{l-1}] = H_*(\Phi)([s^{-1}\sum_{l \geq 1} \frac{1}{t}db_{l-1}])$$

This implies that $$H_*(\Phi)$$ is surjective.

To prove that $$H_*(\Phi)$$ is injective, let’s take $$[s^{-1}x] \in (red_0 s^{-1}X)_{\text{triv}}$$ such that $$H_*(\Phi)([s^{-1}x]) = 0$$. This implies that $$dtx = d\pi$$, for a given $$\pi \in \Omega X$$. As before we set $$\pi = \sum_{l \geq 1} l^a t^{l-1} dtb_k$$, with $$\sum_{l \geq 1} a_l = 0$$. An easy comparison on the degree of the polynomials proves that

$$dtx = \sum_{l \geq 1} l^a t^{l-1} dtb_k \iff x = a_l - db_l$$ and $$\forall l \geq 2, a_l = \frac{1}{l} db_{l-1}$$

$$\implies x = - \sum_{l \geq 2} \frac{1}{l} db_{l-1} - db_0 \iff (\sum_{l \geq 1} \frac{1}{l} db_{l-1})$$

this means that $$[s^{-1}x] = 0$$ and proves that $$H_*(\Phi)$$ is injective.

$$\square$$

Lemma 2.5. If $$Y$$ is an $$\mathcal{O}$$-algebra such that $$Y \simeq \Omega X$$ then $$Y \simeq \Omega^\infty U Y$$.

Proof. From Lemma 2.4, we deduce that $$Y \simeq \Omega^\infty s^{-1}X$$. When we apply the forgetful functor $$U$$, we get the quasi-isomorphism in chain complexes $$UY \simeq U\Omega^\infty s^{-1}X$$. We apply again the functor $$\Omega^\infty$$ and get the $$\mathcal{O}$$-algebra weak equivalences

$$\Omega^\infty U Y \simeq \Omega^\infty U \Omega^\infty s^{-1}X \cong \Omega^\infty s^{-1}X \simeq Y.$$ 

$$\square$$

In this sense, strictly speaking we will just say that any loop space in Alg$$\mathcal{O}$$ has a trivial $$\mathcal{O}$$-algebra structure.
2.2 Homotopy pushouts in $\text{Alg}_O$

We assume in this section that the ground field $\mathbb{k}$ is of characteristic 0 in order to describe an explicit model for homotopy pushouts. In the particular case of describing the suspension of $O$-algebras, the ground field $\mathbb{k}$ can be of any characteristics.

Construction of a cylinder of a quasi-free $O$-algebra

We give in this part the construction of a cylinder of a quasi-free $O$-algebra in the same line that the definition for differential graded Lie algebras in [Tan83 II.5.], and for closed DGL’s in [BFMT16 § 5.].

Let $(O(V), d)$ be a quasi-free $O$-algebra, and let $V'$ be a copy of $V$. We define:

- $O(V) \otimes I := (O(V \oplus V' \oplus sV'), D)$, where: $(sv')_n = v'_{n-1}$, $Dv' = 0$, $Dsv' = v'$, $Dv = dv$.

- $\lambda_0 : (O(V), d) \longrightarrow O(V) \otimes I$ the canonical injection;

- $p : O(V) \otimes I \longrightarrow (O(V), d)$ is the $O$-algebra morphism given by:
  
  $p(v) = v$; $p(v') = p(sv') = 0$; $p$ is a quasi-isomorphism since $O(V' \oplus sV')$ is acyclic.

- $i : O(V) \otimes I \longrightarrow O(V) \otimes I$ is the degree +1 $O$-algebra derivation given by: $i(v) = sv'$; $i(sv') = i(v') = 0$;

- The $O$-algebra derivation of degree 0, $\theta = Di + iD$ verifies $\theta D = D\theta$, $\theta(v') = \theta(sv') = 0$. We have the induced automorphism of $O$-algebras $e^\theta = \sum_{n \geq 0} \frac{\theta^n}{n!}$ (with inverse $e^{-\theta}$).

The automorphism $e^\theta$ is well defined for the following reason: let $v \in V_n$. We write down explicitly the differential $d$ of $(O(V), d)$ by $d = d_1 + d_2 + ...$, where $d_k v \in O(k) \otimes V_{\otimes k}$, for any given $k$. Computation gives that $\theta^2(v) = \theta i(d_2 v + d_3 v + ...) \in O(V_{<n}) \otimes I$. Therefore we deduce inductively that for any $x \in O(V) \otimes I$, there always exist an integer $n_x$ such that $\theta^{n_x}(x) = 0$.

- We define the second injection $\lambda_1 : (O(V), d) \longrightarrow O(V) \otimes I$ by, $\lambda_1(v) = e^\theta(v)$.

The couple $(O(V) \otimes I, \lambda_0, \lambda_1, p)$ forms a cylinder of $(O(V), d)$.

Construction of homotopy pushouts in $\text{Alg}_O$

From now to short expressions we set the cobar-bar functor of $O$-algebras $(-)^c : Z \longrightarrow Z^c := B_c(B(O), B(O, Z))$, and the cylinder object defined above and associated to $Z^c$ will be denoted simply by $Z^c \otimes I := (O(V \oplus V' \oplus sV'), D_1)$, where $V = B(O, Z)$.

Let $Z \rightarrow Y$ in $\text{Alg}_O$, we apply the functor $(-)^c$ to get the weakly equivalent morphism $Z^c \rightarrow Y^c$. Let us consider the commutative diagram in $\text{Alg}_O$:
where the square in the middle is a pushout. From the lower triangle, we can then build the following factorization of $f^c$:

$$
\begin{array}{c}
\pi_1 i_1 \\
\downarrow \quad \downarrow f^c p i_1 i_1 \\
Z^c \quad \longrightarrow \quad Z^c \hat{\otimes} I \coprod \quad Z^c Y^c \\
\end{array}
\quad \vdash \quad (\quad \quad Y^c)
$$

We use this later factorization to replace $f^c$ in the diagram $D^c : X^c \longrightarrow Z^c \quad f^c \longrightarrow \quad Y^c$ by the cofibration $\pi_1 i_1$.

**Proposition 2.6.** Given a $O$-algebra diagram $D : X \quad g^c \longrightarrow \quad Z \quad f^c \longrightarrow \quad Y$, a homotopy pushout of $D$ is given by $C_D = X^c \coprod Z^c \hat{\otimes} I \coprod Z^c Y^c$. Namely

$$
C_D = \text{colim}_{\text{Alg}_O} (X^c \quad g^c \quad Z^c \quad \pi_1 i_1 \quad Z^c \hat{\otimes} I \coprod \quad Z^c Y^c).
$$

**Proof.** This is analogue as the proof of Proposition 2.3. We simply replace holims by hocolims and $Z^I$ by $Z^c \hat{\otimes} I$.

1. Let $D_i : \quad X_i \quad g_i \longrightarrow \quad Z_i \quad f_i \longrightarrow \quad Y_i$, $i \in \{1, 2\}$, be two $O$-algebra diagrams so that $D_1 \xrightarrow{\sim} D_2$. One make the following computations:

$$
\begin{align*}
\Sigma^\infty C_{D_1} \overset{(1)}{=} & \text{colim}_{\text{Alg}_O} (UB(O, X_1) \quad g_{11} \quad UB(O, Z_1) \quad f_{11} \quad UB(O, Z_1^c \hat{\otimes} I \coprod Z_1^c Y_1^c)) \\
\overset{(2)}{=} & \text{hocolim}_{\text{Alg}_O} (UB(O, X_1) \quad g_{11} \quad UB(O, Z_1) \quad f_{11} \quad UB(O, Z_1^c \hat{\otimes} I \coprod Z_1^c Y_1^c)) \\
\simeq & \text{hocolim}_{\text{Alg}_O} (UB(O, X_2) \quad UB(O, Z_2) \quad UB(O, Z_2^c \hat{\otimes} I \coprod Z_2^c Y_2^c)) \\
\simeq & \Sigma^\infty C_{D_2}
\end{align*}
$$

where

- (1) is obtained by applying the left adjoint functor $(-)^{ab}$ (which is equivalent in this case to $\Sigma^\infty$) to the diagram $C_{D_1};$

- (2) is obtained by replacing colim with hocolim since $UB(O, \pi_1 i_1)$ is an injection (cofibration).

One then obtain $C_{D_1} \simeq C_{D_2}$, and deduce that the functor $C_-$ preserves weak equivalences of diagrams of the form $\bullet \xleftarrow{\cdot} \longrightarrow \bullet \longrightarrow \bullet$ in $\text{Alg}_O$. One deduce from this property that $C_{D_{vc}} \xrightarrow{\sim} C_D$, where $D_{vc} \xrightarrow{\sim} D$ is a virtually cofibrant replacement of $D$.

2. Now we prove that $C_{D_{vc}} \xrightarrow{\sim} \text{colim}_{\text{Alg}_O}(D_{vc})$. We consider $D_{vc} : X \xleftarrow{\cdot} \longrightarrow Z \longrightarrow Y$; and we form the diagram:
The square obtained from the homotopy fibers of the horizontal morphisms $h_1, h_2, h_3$ and $h_4$ is described as follows:

\[
\begin{array}{c}
h_1 \quad \text{hocolim}_{\text{Alg}_O} D_{vc} \quad \approx \\
\downarrow \quad \downarrow \quad \downarrow \\
h_2 \quad Z \quad Y \\
\downarrow \quad \downarrow \quad \downarrow \\
h_3 \quad Z^c \quad C_{D_{vc}} \\
\end{array}
\]

where

1. is a weak equivalence as the top square of the cube is a homotopy pushout;
2. is a weak equivalence because the back face of the cube is trivially a homotopy pushout;
3. is a weak equivalence. In fact the functor $\Sigma^\infty(-)$ applied to the diagram at the bottom (of cofibrant algebras) gives a homotopy pushout diagram in $Ch$. One then deduce that

\[
\Sigma^\infty\text{hocolibre}(h_4) \xrightarrow{(1)} \Sigma^\infty\text{hocolibre}(h_2)
\]

and equivalently we get

\[
\text{hocolibre}(h_4) \xrightarrow{(3)} \text{hocolibre}(h_3)
\]

By this we conclude that (4) is a weak equivalence, and therefore that $C_{D_{vc}} \approx \text{colim}_{\text{Alg}_O}(D_{vc})$.

\[\square\]

**Remark 2.7.** If $D : X \xrightarrow{g} Z \xrightarrow{f} Y$ is a diagram of quasi-free $O$-algebras, then we don’t need the cofibrant replacement functor $(-)^c$ in the construction, and we have

\[C_D = X \amalg Z \amalg I \amalg Y.\]

**Lemma 2.8.** Let $(O(V), d)$ be a quasi-free algebra with the notation for the differential: $d = d_1 + d_2 + \ldots$. Then $\Sigma(O(V), d) \simeq (O(sV'), D_1)$, where $D_1(sv') := -sd_1v'$ and $V'$ is a copy of $(V, d)$.
Proof. We set for short $Z = (O, d)$.

In Proposition 2.6 we have proved that $(0 \amalg Z \amalg Z \amalg 0, D) \simeq \Sigma Z$.

Since $(e^a)^{ab}(v) = v' + sd_1v'$, we deduce that in $(0 \amalg Z \amalg Z \amalg 0)^{ab}$,

$$[Dsv'] = [v'] = [v' + sd_1v - sd_1v'] = [-sd_1v']$$

Now we consider the morphism of $O$-algebras

$$\psi : (O(sV'), D_1) \longrightarrow (0 \amalg Z \amalg Z \amalg 0, D)$$

given by $\psi(sv') = [sv']$.

This is a well defined chain complex morphism since $[D\psi(sv')] = \psi(D_1(sv'))$ and in addition $B(O, \psi) \simeq \psi^{ab}$ is a quasi-isomorphism. We deduce that $\psi$ is a quasi-isomorphism.

\[\square\]

**Remark 2.9.** The result of Lemma 2.8 holds in general when the ground field $k$ is of any characteristics. In fact, we have the following pushout diagram

$$
\begin{array}{ccc}
O(V) & \longrightarrow & O(V \oplus sV) \simeq 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & O(sV)
\end{array}
$$

This is also a homotopy pushout diagram, thus we deduce that $\Sigma O(V) \simeq O(sV)$.

**Corollary 2.10.** We assume that the ground field $k$ is of any characteristics. Given an $O$-algebra $Z$, then $\Sigma Z$ is the free $O$-algebra $(O(sU B(O, Z)), d_1)$, where $d_1$ is the internal differential induced by the differential of $(B(O, Z), d)$.

Proof. We make the following computation

$$\Sigma Z \simeq \Sigma B^e(B(O), B(O, X)) \quad \text{(Using Thm 1.7 and Remark 1.8)}$$

$$\simeq O(sUB(O, Z)) \quad \text{(Using Proposition 2.8 and Remark 2.9)}$$

\[\square\]

Finally, we remind the following relation between holims and hocolims.

**Lemma 2.11.** In $\text{Alg}_O$, filtered homotopy colimits, that are colimits of filtered diagrams, commute with finite homotopy limits.

Proof. This follows from the fact that this property is true in $\text{Ch}_+$, and that the forgetful functor $U : \text{Alg}_O \longrightarrow \text{Ch}_+$ commutes with finite limits and filtered colimits.

\[\square\]
3 Goodwillie approach in functor calculus

We assume in that section that the ground field \( k \) is of any characteristics. Let \( C \) and \( D \) be any of the model categories \( \text{Alg}_O, \text{Ch}_+ \) or \( \text{Ch} \). We remind in this section the theory of functor calculus, for functors \( F : C \to D \). We follow the lines of [Kuh07, § 4 and § 5](and implicitly [Goo03]), except that in our case our functors are not required to be continuous. This was a way for Kuhn to get an assembly map \( F(X) \otimes K \to F(X \otimes K) \), where \( K \in sSets \), and \( X \in C \). In fact, we will see (in Lemma 4.13) that homotopy functors \( F : \text{Ch} \to \text{Ch} \) have a natural assembly map (at least at the level of the homotopy category \( \text{HoCh} \)). However, we will require our functors to be homotopy(preserve weak equivalences), which is a weaker version of being continuous, since continuous implies homotopy.

**Definition 3.1 (Homotopy functor).** Let \( C \) and \( D \) be any of the model categories \( \text{Alg}_O, \text{Ch}_+ \) or \( \text{Ch} \) and \( F : C \to D \) be a functor.

1. The functor \( F \) is reduced if \( F(0) \simeq 0 \);
2. \( F \) is a homotopy functor if it preserves weak equivalences.
3. \( F \) is finitary if it preserves filtered homotopy colimits.

**Definition 3.2 (\( n \)-excisive functor).** Let \( C \) and \( D \) be any of the model categories \( \text{Alg}_O, \text{Ch}_+ \) or \( \text{Ch} \) and \( F : C \to D \) be a homotopy functor.

1. An \( n \)-cube in \( C \) is a functor \( \mathcal{X} : \mathcal{P}(\underline{n}) \to C \), where \( \mathcal{P}(\underline{n}) \) is the poset of subsets of \( \underline{n} := \{1, ..., n\} \).
2. The functor \( F : C \to D \) is called \( n \)-excisive if whenever \( \mathcal{X} \) is a strongly coCartesian \( n + 1 \)-cube in \( C \), \( F(\mathcal{X}) \) is a cartesian cube in \( D \).

**Definition 3.3 ([Kuh07], 4.6).** Let \( C \) be any of the model categories \( \text{Alg}_O, \text{Ch}_+ \) or \( \text{Ch} \). Let \( X \in C \) and \( T \) be a finite set. We define the joint \( X \ast T \), of \( X \) and \( T \), to be the homotopy cofiber of the folding map

\[
X \ast T = \text{hocof} \left( \coprod_T X \xrightarrow{\nabla} X \right)
\]

**Example 3.4.** Using Proposition 2.6, we make the following computation: for \( X \in \text{Alg}_O \),

- \( X \ast \underline{0} = X \ast \emptyset = B^c(B(O), B(O, X)) \);
- \( X \ast 1 = cB^c(B(O), B(O, X)) \);
- \( X \ast 2 = \Sigma B^c(B(O), B(O, X)) \).

Let \( C \) and \( D \) be any of the model categories \( \text{Alg}_O, \text{Ch}_+ \) or \( \text{Ch} \) and let \( F : C \to D \) be a homotopy and reduced functor. For \( X \in C \), define the \( n \)-cube

\[
\chi_n(X) : \mathcal{P}(\underline{n}) \to C \text{ by } \chi_n(X) : T \longmapsto X \ast T.
\]

This is a strongly coCartesian \( \underline{n} \)-cube (see [Wal06, lemma 7.1.4]), the fact is that homotopy colimits commute with themselves. One set

\[
T_{n-1}F(X) := \text{holim}_{T \in \mathcal{P}(\underline{n}) - \{\emptyset\}} F(\chi_n(X)(T))
\]
If \( F \) is \( n \)-excisive, then the natural map \( t_{n-1}F : F(X) = F(\chi_n(X)(\emptyset)) \to T_{n-1}F(X) \) is a weak equivalence. Write \( T_{n-1}^iF \) defined inductively by \( T_{n-1}^{i+1}F := T_{n-1}(T_{n-1}^iF) \) and

\[
P_{n-1}F := \text{hocolim} \left( F \stackrel{t_{n-1}F}{\longrightarrow} T_{n-1}F \stackrel{T_{n-1}(t_{n-1}F)}{\longrightarrow} T_{n-1}(T_{n-1}F) \stackrel{T_{n-1}^2(t_{n-1}F)}{\longrightarrow} \cdots \right)
\]

**Example 3.5.** \( T_1F(X) = \text{holim} \left( F(X \ast 1) \longrightarrow F(X \ast 2) \longleftarrow F(X \ast 1) \right) \); if \( F \) is reduced then \( F(X \ast 1) \simeq 0 \) and we deduce that \( T_1F(X) \simeq \Omega F(\Sigma X) \); Therefore inductively we get

\[
P_iF(X) \simeq \text{holim}_{p \to \infty} \Omega^p F \Sigma^p
\]

**Definition 3.6** (homogeneous functors). Let \( F : \mathcal{C} \to \mathcal{D} \) be a homotopy and reduced functor. \( F \) is called \( n \)-homogeneous if

- \( F \) is \( n \)-excisive and
- \( P_{n-1}F \simeq 0 \).

When \( \mathcal{D} = \text{Alg}_\mathcal{O} \), we make the following remark:

**Remark 3.7.** Let \( \mathcal{C} = \text{Alg}_\mathcal{O} \), \( \text{Ch}_+ \), or \( \text{Ch} \). If a functor \( F : \mathcal{C} \to \text{Alg}_\mathcal{O} \) is \( n \)-homogeneous, then for any \( X \in \mathcal{C} \), \( F(X) \) has a trivial \( \mathcal{O} \)-algebra structure. In fact Goodwillie [Goo03, Lemma 2.2] proves in a completely general argument that there is a homotopy pullback diagram

\[
\begin{array}{ccc}
P_nF & \longrightarrow & P_{n-1}F \\
\downarrow & & \downarrow \\
0 & \longrightarrow & R_nF
\end{array}
\]

, where \( R_nF : \mathcal{C} \to \text{Alg}_\mathcal{O} \) is \( n \)-homogeneous. Thus if \( F \) is \( n \)-homogeneous, then \( F \simeq P_nF \simeq \Omega R_nF \). Therefore, when the ground field is of characteristics 0, we can rewrite \( F \) as \( F \simeq \Omega^\infty UF \), where \( U : \text{Alg}_\mathcal{O} \to \text{Ch}_+ \) is the forgetful functor (see Lemma [2.5]).

Since \( F = T_{n-1}^0F \), the functor \( P_{n-1}F \) is equipped with a map \( F \longrightarrow P_{n-1}F \). In addition the inclusion of categories \( \mathcal{P}(n) \to \mathcal{P}(n+1) \) induces a map \( T_nF \to T_{n-1}F \) which extends formally to give a map \( q_nF : P_nF \to P_{n-1}F \) which is a fibration (see [Goo03, Page 664]). By inspection this map is again a fibration in \( \text{Alg}_\mathcal{O} \) and in \( \text{Ch}_+ \), since the maps \( T_nF \to T_{n-1}F \) will always be a surjection, and filtered colimits of surjections is again a surjection.

**Theorem 3.8.** [Goo03, 1.13] A homotopy functor \( F : \mathcal{C} \to \mathcal{D} \) determines a tower of functors \( \{P_nF : \mathcal{C} \to \mathcal{D}\}_n \), where \( P_nF \) are \( n \)-excisif, \( q_nF : P_nF \to P_{n-1}F \) are fibrations, the functors \( D_nF = \text{hofibre} (q_nF) \) are \( n \)-homogeneous.

**Remark 3.9.** A straight consequence of Lemma [2.11] for this section is that the functor \( P_n \), which is basically a homotopy colimit, commutes with finite homotopy limits. In particular \( P_n \) preserves fiber sequences.
4 Characterization of homogeneous functors

In this section, we characterize homogeneous functors with the cross effect. Before getting to this result, we will make a couple of constructions and provide intermediate results. The characterization itself appears in Corollary 4.12 at the end of this section.

There are two ways to define the cross effect associated to a functor. One can define it as a homotopy fiber (hofib) and we can also define it as a total homotopy fiber (thofib). These definitions are reported here below.

**Definition 4.1 (Cross-effects).** Let \( C \) and \( D \) be any of the model categories \( \text{Alg}_O \), \( \text{Ch}_+ \) or \( \text{Ch} \) and let \( F : C \to D \) be a homotopy and reduced functor. We define the \( n \)th cross-effect of \( F \), to be the functor of \( n \) variables given by

\[
\text{cr}_n F(X_1, \ldots, X_n) = \text{hofib}\{F(\prod_{i \in \underline{n}} X_i) \to \text{holim}_{T \in P(n)} F(\prod_{i \in \underline{n}-T} X_i)\}
\]

This is equivalent to define the \( n \)th cross-effect of \( F \) as:

\[
\text{cr}_n F(X_1, \ldots, X_n) = \text{thofib}(T \mapsto F(\prod_{i \in \underline{n}-T} X_i)).
\]

In the particular cases where \( C = \text{Ch}_+ \) or \( \text{Ch} \) and \( D = \text{Ch} \), we can also describe the cross effect of a functor \( F \) using the total homotopy cofiber (thocofib) of a certain cube. This dual construction, also called the ”co-cross-effect”, was considered by McCarthy [McC01, 1.3] in studying dual calculus, and the equivalence between the cross-effect and co-cross-effect was proved by Ching [Chi10, Lemma 2.2] for functors with values in spectra.

Let \( W_1, \ldots, W_n \in C \), we associate the \( n \)-cube \( \mathcal{X} \) in \( C \) defined as follows:

- \( T \subseteq \underline{n}, \mathcal{X}(T) := \bigoplus_{i \in T} W_i; \)

- For \( T \subseteq \underline{n} \) and \( j \in \underline{n} \setminus T \), the map \( \mathcal{X}(T) \to \mathcal{X}(T \cup \{j\}) \) (in the cube) is induced by the inclusion:

\[
\bigoplus_{i \in T} W_i \to \bigoplus_{i \in T} (\bigoplus_{i \in T} W_i) \oplus W_j
\]

\[
x \mapsto (x, 0)
\]

**Definition 4.2 (Co-cross-effects).** Let \( C = \text{Ch}_+ \) or \( \text{Ch} \) and \( F : C \to \text{Ch} \) be a homotopy functor. The \( n \)th co-cross effect of \( F \) is the functor \( \text{cr}_n F : C^{\times n} \to \text{Ch} \) which computes the homotopy total fiber of \( F(\mathcal{X}) \). That is:

\[
\text{cr}_n F(W_1, \ldots, W_n) := \text{hocofib}\{\text{hocolim}_{T \subseteq \underline{n}} F(\bigoplus_{i \in T} W_i) \to F(W_1 \oplus \ldots \oplus W_n)\}.
\]

**Lemma 4.3.** Let \( C = \text{Ch}_+ \) or \( \text{Ch} \) and \( F : C \to \text{Ch} \) be a homotopy functor. Then the \( n \)th cross-effect of \( F \) is equivalent to the \( n \)th co-cross-effect of \( F \). That is:

\[
\text{cr}_n F(W_1, \ldots, W_n) \cong \text{cr}_n F(W_1, \ldots, W_n)
\]

**Proof.** Since \( \text{Ch} \) is a stable category and that in \( C \) products and coproducts are isomorphic, we simply mimic Ching’s proof. \( \square \)

To understand homogeneous functors, Goodwillie [Goo03] pointed the following proposition for functors with values in spectra. We reformulate it in our algebraic context though the proof follows literally [ [Goo03], proposition 3.4] and [ [Goo92], proposition 2.2].
Proposition 4.4. Let $C$ and $D$ be any of the model categories $\text{Alg}_O$, $\text{Ch}_+$ or $\text{Ch}$. If $H : C \rightarrow D$ is an $n$–excisive and reduced functor such that $\text{cr}_n H \simeq 0$, then $H$ is $(n - 1)$–excisive.

Proof. (i) One define the $n$-cube $\mathcal{X} = S^*(X_1, ..., X_n)$, for objects $X_1, ..., X_n$ in $C$, as follows: $\forall T \subseteq [n], \mathcal{X}(T) := \bigoplus_{i \in T} X_i$ and $\mathcal{X}(\emptyset) = 0$. The maps in the cube $\mathcal{X}$ are inclusions. We associate to this cube $\mathcal{X}$ the $n$-cube $S(X_1, ..., X_n)$ which has the same objects with $\mathcal{X}$, but where the inclusions are reversed to the projections. Let $U : D \rightarrow \text{Ch}$ be the forgetful functor when $D = \text{Alg}_O$ and be the identity functor when $D = \text{Ch}_+$. We make the following computations:

$$U \text{thofib } H(\mathcal{X}) \cong \text{thofib } U H(\mathcal{X}) = \Omega^n \text{thofib } U H(S^*(X_1, ..., X_n))$$

$$= \Omega^n \text{thofib } U H(S(X_1, ..., X_n))$$

$$= \Omega^n \text{cr}_n(U H)(X_1, ..., X_n)$$

$$= \Omega^n U \text{cr}_n H(X_1, ..., X_n) \simeq 0,$$

One will then conclude from these that $\text{thofib } H(\mathcal{X}) \simeq 0$ (or equivalently that $H(\mathcal{X})$ is cartesian) for all strongly coCartesian cubes $\mathcal{X}$ in which $\mathcal{X}(\emptyset) = 0$, since any such cube $\mathcal{X}$ is naturally equivalent to $S^*(\mathcal{X}({1}), ..., \mathcal{X}({n}))(\text{see }[\text{Goo92}, \text{proposition } 2.2]).$

(ii) Let $\forall T \subseteq [n]$, and $a, b \in [n]$. Given an arbitrary strongly coCartesian $n$-cube $\mathcal{X}$ in $C$, put $\mathcal{X}'(T) = \text{hocolim}(0 \leftarrow \mathcal{X}(\emptyset) \rightarrow \mathcal{X}(T))$.

We have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{X}(\emptyset) & \rightarrow & \mathcal{X}(T) \\
\downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{X}'(T)
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{X}(T) & \rightarrow & \mathcal{X}(T \cup \{a\}) \\
\downarrow & & \downarrow \\
\mathcal{X}'(T) & \rightarrow & \mathcal{X}'(T \cup \{a\})
\end{array}
\]

where the largest square is a homotopy pushout along with the most left square. It then follows that the right square is also a homotopy pushout and therefore that the following square is a homotopy pushout:

\[
\begin{array}{ccc}
\mathcal{X}(T) & \rightarrow & \mathcal{X}(T \cup \{a\}) \\
\downarrow & & \downarrow \\
\mathcal{X}'(T) & \rightarrow & \mathcal{X}'(T \cup \{a\})
\end{array}
\]

and therefore it follows that

\[
\begin{array}{ccc}
\mathcal{X}'(T) & \rightarrow & \mathcal{X}'(T \cup \{a\}) \\
\downarrow & & \downarrow \\
\mathcal{X}'(T \cup \{b\}) & \rightarrow & \mathcal{X}'(T \cup \{a, b\})
\end{array}
\]

is a homotopy pushout diagram. This proves that the $n$-cube $\mathcal{X}'$ is strongly coCartesian and that the map $\mathcal{X} \rightarrow \mathcal{X}'$ is a strongly cocartesian $n+1$-cube. $H$ is $n$-excisive, thus $H(\mathcal{X}) \rightarrow H(\mathcal{X}')$ is cartesian. In addition since $\mathcal{X}'(\emptyset) = 0$, we deduce from (i) that $H(\mathcal{X}')$ is cartesian and conclude that $H(\mathcal{X})$ is also cartesian.

□
We get the following consequence:

**Corollary 4.5.** Let $F$ and $G$ be two $n$–homogeneous functors $\mathcal{C} \rightarrow \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are any of the model categories $\text{Alg}_O$, $\text{Ch}_+$ or $\text{Ch}$, and a natural transformation $F \xrightarrow{J} G$. If $\text{cr}_n(J) : \text{cr}_nF \rightarrow \text{cr}_nG$ is an equivalence, then so is $J$.

**Proof.** Let $H = \text{hofib}(F \xrightarrow{J} G)$. $H$ is $n$–homogeneous and then $n$–excisive. By hypothesis $\text{cr}_nH \cong \text{hofib}(\text{cr}_nF \xrightarrow{\text{cr}_nJ} \text{cr}_nG) \cong 0$. The functor $H$ gathers then the hypothesis of Proposition 4.4, thus $H$ is $n-1$–excisive. Hence we get

$$H \cong P_{n-1}H = \text{hofib}(P_{n-1}F \rightarrow P_{n-1}G) = 0$$

One deduce from the long exact sequence obtained from the homotopy fiber sequence of $J$ that $J$ is a weak equivalence. 

**Definition 4.6.** Let $\mathcal{C}$ and $\mathcal{D}$ be any of the model categories $\text{Alg}_O$, $\text{Ch}_+$ or $\text{Ch}$, and $F : \mathcal{C} \rightarrow \mathcal{D}$ be a homotopy and reduced functor.

1. The functor $L_nF : \mathcal{C}^n \rightarrow \mathcal{D}$ is obtained from $\text{cr}_nF$ by

   $$L_nF(X_1, \ldots, X_n) \cong \text{hocolim}_{p_i \rightarrow \infty} \Omega^{p_1 + \ldots + p_n} \text{cr}_nF(\Sigma^{p_1} X_1, \ldots, \Sigma^{p_n} X_n)$$

   In the case that $\mathcal{D} = \text{Alg}_O$, this filtered homotopy colimit can be seen as a homotopy colimit in the underlying category of chain complexes.

2. The functor $\Delta_n F : \mathcal{C} \rightarrow \mathcal{D}$ is obtained from $L_nF$ by:

   $$\Delta_n F = (L_nF) \circ \Delta$$

   where $\Delta : \mathcal{C} \rightarrow \mathcal{C}^n$ is the diagonal map. The symmetric group $\Sigma_n$ acts on $\Delta_n F$ by permuting its $n$ entries of the cross effect $\text{cr}_nF$.

3. The functor $\hat{\Delta}_nF(X) : \mathcal{C} \rightarrow \text{Ch}$ is obtained from $\Delta_nF$ by dropping the functor $\text{red}_0$. Namely,

   $$\hat{\Delta}_nF(X) := \text{hocolim}_{p_i \rightarrow \infty} s^{-p_1 \ldots - p_n} \text{cr}_n(U F)(\Sigma^{p_1} X, \ldots, \Sigma^{p_n} X)$$

   where $U : \text{Alg}_O \rightarrow \text{Ch}$ is the forgetful functor and this colimit is taken in the category $\text{Ch}$. The symmetric group $\Sigma_n$ acts on $\hat{\Delta}_nF(X)$ by permuting its $n$ entries of the cross effect $\text{cr}_nUF$.

**Remark 4.7.** The functor $L_nF$ of Definition 4.6 can also be seen as the stabilization of the cross effect, that is the functor obtained by applying the first Taylor approximation functor $P_1$ to each variable position of the multi-variable functor $\text{cr}_nF$. For instance,

1. $L_1F = P_1F$ (see Example 3.3);

2. $L_2F(X,Y) = P_1(Y \rightarrow P_1(X \rightarrow \text{cr}_2(X,Y)))$;

3. and so on.
We assume from now, when it is not specified, that the ground field \( k \) is of characteristic 0.

**Lemma 4.8.** Let \( C \) be any of the model categories \( \text{Alg}_O, Ch_+ \) or \( Ch_0 \), and \( F : C \to \text{Alg}_O \) be a homotopy and reduced functor. Then for any \( X \in C \), there is a weak equivalence of \( O \)-algebras

\[
\Delta_n F(X) \simeq (\text{red}_0 \hat{\Delta}_n F(X))_{\text{triv}}.
\]

**Proof.** If \( U : \text{Alg}_O \to Ch \) denotes the forgetful functor, we make the following computation:

\[
U \Delta_n F(X) \simeq \text{holim}_C [\text{red}_0 s^{-p_1 \cdots -p_n} \text{cr}_n UF(\Sigma^{p_1} X, \ldots, \Sigma^{p_n} X)]
\]

\[
\simeq \text{red}_0 \text{holim}_Ch [s^{-p_1 \cdots -p_n} \text{cr}_n UF(\Sigma^{p_1} X, \ldots, \Sigma^{p_n} X)]
\]

This last equivalence is justified by the fact that the functor \( \text{red}_0 \) commutes with filtered colimits. Now by applying the functor \((-)_{\text{triv}}\), we get the weak equivalence of \( O \)-algebras

\[
(U \Delta_n F(X))_{\text{triv}} \simeq (\text{red}_0 \hat{\Delta}_n F(X))_{\text{triv}}.
\]

In addition since the functor \( \Delta_n F \) is \( n \)-homogeneous, we know from Remark 3.7 that \( \Delta_n F(X) \simeq (U \Delta_n F(X))_{\text{triv}} \), therefore the result follows. \( \square \)

We are now ready to state the next theorem which was inspired by [Kuh07, Thm 5.12] for functors with values in stable model categories.

**Theorem 4.9.** Let \( C \) and \( D \) be any of the categories \( \text{Alg}_O, Ch_+ \) and \( Ch_0 \), and \( F : C \to D \) be a homotopy and reduced functor. Then there is a weak equivalence

\[
D_n F(X) \simeq \Omega^\infty(\hat{\Delta}_n F(X))_{h\Sigma_n},
\]

where \((-)_{h\Sigma_n}\) denotes the homotopy orbits. When \( D = Ch_+ \) or \( Ch \) then this result holds when the ground field \( k \) is of any characteristics.

To prove this, we need the following lemma.

**Lemma 4.10.** Let \( C \) be either \( \text{Alg}_O, Ch_+ \) or \( Ch_0 \), and \( F : C \to D \) be a homotopy and reduced functor. Then we have a weak equivalence

\[
P_n (L_n F \circ \Delta) \simeq L_n (P_n F) \circ \Delta
\]

**Proof.** One make the following observation:

\[
T_n (L_n F \circ \Delta)(X) := \text{holim}_{p_i \to \infty} \text{holim}_{p_i \to \infty} \Omega^{p_1 + \cdots + p_n} \text{cr}_n F(\Sigma^{p_1} (X \ast T), \ldots, \Sigma^{p_n} (X \ast T))
\]

\[
\simeq \text{holim}_{T \in P_{(n+1)}} \text{holim}_{p_i \to \infty} \Omega^{p_1 + \cdots + p_n} \text{cr}_n F((\Sigma^{p_1} X) \ast T, \ldots, (\Sigma^{p_n} X) \ast T)
\]

\[
= \text{holim}_{T \in P_{(n+1)}} \text{holim}_{p_i \to \infty} \Omega^{p_1 + \cdots + p_n} \text{thofib}(A \supseteq \emptyset \mapsto F(\bigamalg_{A} (\Sigma^{p_1} X) \ast T))
\]

\[
\simeq \text{holim}_{T \in P_{(n+1)}} \text{holim}_{p_i \to \infty} \Omega^{p_1 + \cdots + p_n} \text{thofib}(A \supseteq \emptyset \mapsto F(\bigamalg_{A} (\Sigma^{p_1} X) \ast T))
\]

\[
\simeq \text{holim}_{p_i \to \infty} \Omega^{p_1 + \cdots + p_n} \text{thofib}(A \supseteq \emptyset \mapsto T_n F(\bigamalg_{A} (\Sigma^{p_1} X)))
\]

\[
= \text{holim}_{p_i \to \infty} \Omega^{p_1 + \cdots + p_n} \text{cr}_n (T_n F)(\Sigma^{p_1} X, \ldots, \Sigma^{p_n} X)
\]

\[
= L_n (T_n F) \circ \Delta(X)
\]

where
(1) is due to the isomorphism $\Sigma^p (X \ast T) \cong (\Sigma^p X) \ast T$, for each $j$;

(2) is due to the isomorphism $\Pi_{\underline{n}-T} (\Sigma^p X \ast T) \cong (\Pi_{\underline{n}-T} \Sigma^p X) \ast T$, for each $T \subseteq \underline{n}$;

(3) is because finite holims commute with filtered colimits (see Lemma 2.11), and holims commute with loops $\Omega$ and total fibers.

One also deduce from this observation steps that the following square is commutative

\[
\begin{array}{c}
L_n F \circ \Delta \\
\downarrow_{t_n L_n F \circ \Delta} \quad \downarrow_{L_n t_n F \circ \Delta}
\end{array}
\]

\[
T_n (L_n F \circ \Delta) \cong L_n (T_n F) \circ \Delta
\]

Thus we can deduce by induction on the iterations from this square that

\[
P_n (L_n F \circ \Delta) \cong (L_n P_n F) \circ \Delta.
\]

Proof of Theorem 4.9. Let $F : C \to D$ be a homotopy and reduced functor. Let $J$ be the composition in $Ch$:

\[
((cr_n UD_n F) \circ \Delta(X))_{h \Sigma_n} \to (UD_n F(\Pi X))_{h \Sigma_n} \to UD_n F(X)
\]

where the first map is the projection, the second map is induced by the folding map $\Pi X \xrightarrow{\Delta} X$, and if $D = Alg_{\Omega}$, then $U : Alg_{\Omega} \to Ch$ is the forgetful functor, and $U$ is simply the identity functor where $D = Ch$; Since we want to prove that $J$ is a quasi-isomorphism, we will simply show that $cr_n J$ is a quasi-isomorphism and conclude using Corollary 7.3.

For the sake of simplicity we set $L(X) = cr_n (UD_n F)(X, \ldots, X)_{h \Sigma_n}$.

\[
cr_n L(X_1, \ldots, X_n) = thofib(L \circ S(X_1, \ldots, X_n))
\]

\[
= thofib(\underline{n} - T \mapsto L(\Pi X_i))
\]

\[
= thofib(\underline{n} - T \mapsto cr_n D_n F(\Pi X_i, \ldots, \Pi X_i)_{h \Sigma_n})
\]

\[
= thofib(\chi)_{h \Sigma_n},
\]

where $\chi : \underline{n} - T \mapsto cr_n (UD_n F)(\Pi X_i, \ldots, \Pi X_i)$. Since $cr_n (UD_n F)$ is multilinear, we deduce the weak equivalence (natural in $T$)

\[
\chi(\underline{n} - T) \cong \prod_{\pi : \underline{n} \to T} cr_n (UD_n F)(X_{\pi(1)}, \ldots, X_{\pi(n)}).
\]

\[(1)\]

Let’s consider the map $\pi : \underline{n} \to \underline{n}$ and consider the cube $\mathcal{Y}_{\pi}$ defined by:

\[
\mathcal{Y}_{\pi}(\underline{n} - T) = \begin{cases} 
  cr_n (UD_n F)(X_{\pi(1)}, \ldots, X_{\pi(n)}) & \text{if } \pi(\underline{n}) \subseteq T \\
  0 & \text{otherwise}
\end{cases}
\]

The morphism (1) is equivalent to $\chi(\underline{n} - T) \cong \prod_{\pi : \underline{n} \to \underline{n}} \mathcal{Y}_{\pi}(\underline{n} - T)$.

- If $\pi$ is not a permutation and then not surjective, we can find an element $s \notin \pi(\underline{n})$.

All the maps $\mathcal{Y}_{\pi}(\underline{n} - T) \to \mathcal{Y}_{\pi}(\underline{n} - T \cup \{s\})$ are isomorphisms, so $\mathcal{Y}_{\pi}$ is cartesian.

- If $\pi$ is a permutation, $thofib(\mathcal{Y}_{\pi}) \cong \mathcal{Y}_{\pi}(\underline{n}) = cr_n UD_n F(X_{\pi(1)}, \ldots, X_{\pi(n)})$.  

\[23\]
Therefore \( \text{thofib}(\chi) \xrightarrow{\sim} \prod_{\pi \in \Sigma_n} cr_n(U D_n F)(X_{\pi(1)}, \ldots, X_{\pi(n)}). \) Thus
\[
\text{thofib}(\chi)_{h \Sigma_n} \xrightarrow{\sim} \left( \prod_{\pi \in \Sigma_n} cr_n(U D_n F)(X_{\pi(1)}, \ldots, X_{\pi(n)}) \right)_{h \Sigma_n} \xrightarrow{\sim} cr_n(U D_n F)(X_1, \ldots, X_n).
\]

Now that we have showed that \( J \) is a quasi-isomorphism, let us consider the chain map \( \alpha \) using Lemma \[4.10\]
\[
\alpha : L_n(U F) \circ \triangle \xrightarrow{p_n(L_n(U F) \circ \triangle)} P_n(L_n(U F) \circ \triangle) \xrightarrow{\sim} (L_n(U P_n F)) \circ \triangle
\]
which is a weak equivalence since \( L_n(U F) \circ \triangle \) is \( n \)-excisive. Putting all these together we form the following diagram:
\[
\begin{array}{c}
\begin{array}{c}
\left( L_n(U F) \circ \triangle \right)_{h \Sigma_n} = (\triangle(U F(X))_{h \Sigma_n} \xrightarrow{\alpha} \left( (L_n(U P_n F)) \circ \triangle \right)_{h \Sigma_n} \\
\end{array}
\end{array}
\]
\[
\xrightarrow{p_1 \ldots p_1 cr_n F} \xrightarrow{\sim} (cr_n(U P_n F) \circ \triangle)_{h \Sigma_n} \xrightarrow{\sim} (cr_n(U D_n F) \circ \triangle)_{h \Sigma_n} \xrightarrow{J} U D_n F
\]
Note that \( (\triangle(U F(X))_{h \Sigma_n} \simeq \text{redo}(\widehat{\Delta}(U F(X)))_{h \Sigma_n}. \) One then deduce the following specializations for \( D, \)

- When \( D = Ch_+ \), we have : \( \text{redo}(\widehat{\Delta}(F(X))_{h \Sigma_n} \simeq D_n F(X) \)

- When \( D = \text{Alg}_O, \) we apply the functor \( \Omega^\infty(\_ \_) \) to the above diagram and since \( \Omega^\infty(U D_n F) \simeq D_n F \) (by Remark \[3.17\]), we get the weak equivalence
\[
\Omega^\infty(\widehat{\Delta}(F(X))_{h \Sigma_n}) \simeq D_n F(X)
\]

- When \( D = Ch, \) the diagram itself gives the proof.

**Corollary 4.11.** Let \( D = \text{Alg}_O, Ch_+ \) or \( Ch, \) and \( F : \text{Alg}_O \rightarrow D \) be a homotopy and reduced functor. Then there is a weak equivalence
\[
D_n F(X) \simeq \Omega^\infty H(\Sigma^\infty X)
\]
where \( H : Ch_+ \rightarrow Ch_k \) is the \( n \)-homogeneous functor given by : \( H(V) := \widehat{\Delta}_n(F(\mathcal{O}(\_ \_)))(V)_{h \Sigma_n}. \)
In particular when \( D = Ch_+ \) or \( Ch \) then this result holds when the ground field \( k \) is of any characteristics.

**Proof.** The functor \( H \) is \( n \)-homogeneous since it is the \( n \)-th stabilization of the cross effect of \( F(\mathcal{O}(\_ \_)). \)

Let \( X \) be an algebra over the operad \( \mathcal{O}, \) and \( F : \text{Alg}_O \rightarrow D \) be a homotopy and reduced functor. We observe that
\[
\widehat{\Delta}_n F(X) \simeq \Omega^n(\widehat{\Delta}_n F)(\Sigma X) \quad (\text{since } L_n UF \text{ is } n\text{-multilinear})
\]
\[
\simeq \Omega^n(\widehat{\Delta}_n F)(\mathcal{O}(s\Sigma^\infty X)) \quad (\text{since } \Sigma X \simeq \mathcal{O}(s\Sigma^\infty X) \text{ from Corollary } \[2.10\])
\]
\[
\simeq \Omega^n(\widehat{\Delta}_n(F(\mathcal{O}(\_ \_)))(s\Sigma^\infty X) \quad (\text{since } \mathcal{O}(\_ \_) \text{ commutes with coproducts})
\]
\[
\simeq \widehat{\Delta}_n(F(\mathcal{O}(\_ \__)))(\Sigma^\infty X) \quad (\text{since } L_n(U F(\mathcal{O}(\_ \_))) \text{ is } n\text{-multilinear})
\]
One deduce from this observation that \( \widehat{\Delta}_n F(X)_{h \Sigma_n} \simeq \widehat{\Delta}_n(F(\mathcal{O}(\_ \__)))(\Sigma^\infty X)_{h \Sigma_n}. \) Using Theorem \[4.9\] we obtain the quasi-isomorphism
\[ D_n F(X) \simeq \Omega^n(\tilde{\Delta}_n F(O)(-))(\Sigma^\infty X)_{h\Sigma_n} \].

\[ \square \]

**Corollary 4.12.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be any of the categories \( \text{Alg}_O, \text{Ch}_+ \) or \( \text{Ch} \), and \( F : \mathcal{C} \to \mathcal{D} \) be a homotopy and reduced functor. If either \( F \) is finitary or if \( X \) is finite we have the following quasi isomorphisms:

1. If \( \mathcal{C} = \text{Alg}_O \), then there is a weak equivalence
\[ D_n F(X) \simeq \Omega^n(\Delta_n F(\mathcal{O}(\kappa)) \otimes (\Sigma^\infty X)^{\otimes n})_{h\Sigma_n} \]

2. If \( \mathcal{C} = \text{Ch}_+ \) or \( \text{Ch} \), then there is weak equivalence
\[ D_n F(X) \simeq \Omega^n(\Delta_n F(k) \otimes (\Sigma^\infty X)^{\otimes n})_{h\Sigma_n} \]

In particular when \( \mathcal{D} = \text{Ch}_+ \) or \( \text{Ch} \) then this result holds when the ground field \( \kappa \) is of any characteristics.

To prove this result which classifies homogeneous functors in our algebraic point of view, we will need the following lemma.

**Lemma 4.13.** Let \( L_r : (\text{Ch}_+)^{\times r} \to \text{Ch} \) be a \( r \)-reduced-multilinear functor. Then for any chain complexes \( V_1, \ldots, V_r \) and finite chain complexes \( W_1, \ldots, W_r \), there is a zig-zag of quasi-isomorphisms
\[ W_1 \otimes \cdots \otimes W_r \otimes L_r(V_1, \ldots, V_r) \simeq L_r(W_1 \otimes V_1, \ldots, W_r \otimes V_r) \]

**Proof.** 1. We first consider the case \( r = 1 \) and we want to construct a zig-zag of quasi-isomorphisms \( W \otimes L_1(V) \simeq L_1(W \otimes V) \), for a given chain complex \( V \) and a finite one \( W \).

Let us consider the following commutative diagram

\[
\begin{array}{ccccccccc}
L_1(sV \oplus V) & \xleftarrow{\simeq} & 0 & \xrightarrow{\simeq} & L_1(sV) \oplus s^{-1}L_1(sV) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
L_1(sV) & \xleftarrow{=} & L_1(sV) & \xrightarrow{=} & L_1(sV) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
L_1(0) & \xleftarrow{\simeq} & 0 & \xrightarrow{=} & 0 \\
\end{array}
\]

A homotopy limit functor applied on each column gives the zig-zag of quasi-isomorphisms
\[ L_1(V) \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} s^{-1}L_1(sV) \]

where the homotopy limit result of the first column in due to the fact that the functor \( L_1 \) is linear. This later zig-zag can also be re-written as:
\[ sL_1(V) \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} L_1(sV) \]

and thus equivalent to: \( \kappa u \otimes L_1(V) \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} L_1(\kappa u \otimes V) \),

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for a given homogeneous element \( u \) of degree 1. One deduce inductively from this construction that, for any \( n \geq 0 \), we have \((k^u)^n \otimes L_1(V) \xleftarrow{\sim} \cdots \xrightarrow{\sim} L_1((k^u)^n \otimes V)\) and therefore, given any homogeneous element \( u \) of arbitrary degree, we have a zig-zag of quasi-isomorphisms: \( \alpha_n : k^u \otimes L_1(V) \xleftarrow{\sim} \cdots \xrightarrow{\sim} L_1(k^u \otimes V) \).

If \( W = (k^u \oplus k^v, d) \) is a chain complex with 2 generators, we set \( \alpha_u + \alpha_v \) to be the composition

\[
\alpha_u + \alpha_v : W \otimes L_1(V) \xrightarrow{\sim} L_1(k^u \otimes V) \oplus L_1(k^v \otimes V) \xrightarrow{\sim} L_1(W \otimes V),
\]

where the last quasi-isomorphism is due to the fact that \( L_1 \) is linear. We generalize this construction inductively on the number of generators to any arbitrary finite chain complex \( W \).

2. In the case that \( r = 2 \), let \( L_{2, V_i} \) be the linear functor \( V_2 \mapsto L_2(V_1, V_2) \); One have:

\[
W_1 \otimes W_2 \otimes L_2(V_1, V_2) \xrightarrow{\sim} W_1 \otimes (W_2 \otimes L_{2, V_i}(V_2)) = W_1 \otimes L_{2, V_i}(W_2 \otimes V_2) \xrightarrow{\sim} W_1 \otimes L_{2, V_i, V_j}(V_1, V_2) = L_2(W_1 \otimes V_1, W_2 \otimes V_2)
\]

Again, we generalize this argument inductively to any arbitrary \( r \).

\[
\square
\]

**Proof of corollary 4.12.** We give the proof using the result of Theorem 4.9 and Corollary 4.11. Namely, let \( F : C \to D \) be a homotopy and reduced functor with \( C \) and \( D \) be any of the categories \( \text{Alg}_D \) or \( \text{Ch}_+ \). Then \( D_n F(X) \simeq (\Omega^s \text{red}_0 H(\Sigma^s X), \text{where } H : \text{Ch}_+ \to \text{Ch} \) is a special case of a \( n \)-multilinear functor \( (V_1, ..., V_n) \mapsto H(V_1, ..., V_n) \).

1. When \( C = D = \text{Alg}_D \), \( H(V) = \hat{\Delta}_n(F \Omega(-))(V)_{\Delta \Sigma_n} = L_n(F \Omega(-))(V, ..., V)_{\Delta \Sigma_n} \).

If \( \Sigma^s X \) is finite, then using Lemma 4.13, we have an \( n \)-equivariant zig-zag of quasi-isomorphisms:

\[
(\Sigma^s X)^{\otimes n} \otimes \hat{\Delta}_n(F \Omega(-))(k) \xleftarrow{\sim} \cdots \xrightarrow{\sim} \hat{\Delta}_n(F \Omega(-))(\Sigma^s X)
\]

and therefore we deduce the quasi-isomorphism

\[
(\Sigma^s X)^{\otimes n} \otimes \hat{\Delta}_n(F \Omega(-))(k) \xleftarrow{\sim} \cdots \xrightarrow{\sim} \hat{\Delta}_n(F \Omega(-))(\Sigma^s X)_{\Delta \Sigma_n}.
\]

In addition, if \( F \) is finitary then \( L_n(F \Omega(-)) \) is finitary on each variable. In this case for any arbitrary algebra \( X \), we rewrite \( \Sigma^s X \) as a filtered colimit of its finite subcomplexes and then apply again Lemma 4.13 to these finite subcomplexes as above and recover a quasi-isomorphism

\[
(\Sigma^s X)^{\otimes n} \otimes \hat{\Delta}_n(F \Omega(-))(k) \xleftarrow{\sim} \cdots \xrightarrow{\sim} \hat{\Delta}_n(F \Omega(-))(\Sigma^s X)_{\Delta \Sigma_n}.
\]

2. In all other cases of the categories \( C \) and \( D \), we refer again to Theorem 4.9 and Corollary 4.11 to chose the appropriate \( H \) and follows an analogue road map as in 1.
Definition 4.14 (Goodwillie derivatives). Let $D$ be either $\text{Alg}_O$ or $\text{Ch}_+$.

1. If $F : \text{Alg}_O \to D$ is a homotopy and reduced functor, then $\hat{\Delta}_n F(\mathcal{O}(k))$ is called the $n^{th}$ derivative (or $n^{th}$ Goodwillie derivative) of $F$ and is denoted $\partial_n F$.

2. If $F : \text{Ch}_+ \to D$ is a homotopy and reduced functor, then $\hat{\Delta}_n F(k)$ is called the $n^{th}$ derivative (or $n^{th}$ Goodwillie derivative) of $F$ and is denoted $\partial_n F$.

This definition holds when $D = \text{Ch}_+$ or $\text{Ch}$ and the ground field $k$ is of any characteristics.

5 Examples: Computing the Goodwillie derivatives

In this section, we show how one can compute the Goodwillie derivatives for a couple of functors.

Example 5.1. The computation below shows that the Goodwillie derivatives of the identity functor $\text{Id} : \text{Alg}_O \to \text{Alg}_O$ is given by:

$$\partial_n \text{Id} \simeq \text{holim} \sum_{p_1,\ldots,p_n} \mathcal{O}(\Sigma p_1 k) \otimes \cdots \otimes \mathcal{O}(\Sigma p_n k) \simeq \mathcal{O}(n).$$

We use an analogue computation to obtain $\partial_1 \Omega_\infty \simeq B(\mathcal{O})$.

The next example is transformed into a lemma. Let $V$ be a finite non negatively graded chain complex. By finite, we mean of finite dimension in each degree and bounded above.

We define the functor

$$Nk\text{Hom}_{\text{Ch}}(V \otimes Nk\Delta^\bullet, -) : \text{Ch}_+ \to \text{Ch}_+$$

where, $N : s\text{Ab} \to \text{Ch}_+$ is the normalization functor and $k\text{Hom}_{\text{Ch}}(V \otimes Nk\Delta^\bullet, W)$ denotes the free simplicial $k$-vector space generated by the simplicial set $\text{Hom}_{\text{Ch}}(V \otimes Nk\Delta^\bullet, W)$;

Lemma 5.2. Let $V \in \text{Ch}_+$. Then we have the quasi-isomorphism (in Ch)

$$\partial_n Nk\text{Hom}_{\text{Ch}}(V \otimes Nk\Delta^\bullet, -) \simeq \text{hom}(V, k) \otimes \Sigma^n \simeq \mathcal{O}(n).$$

Before we give the proof of this quasi-isomorphism, we remind the following fact which seem to be a classical construction: Let $p \in \mathbb{N}$, $A$ be a simplicial $k$-vector space and consider the following notations:

- We write $A[p]$ to mean the simplicial $k$-vector space given levelwise by $A[p]_n := A_n \otimes k s^p$.

- If $(X_\bullet, *)$ is a pointed simplicial set then $\tilde{k}X_\bullet := kX_\bullet/k*$

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A[p] is a $p$-connected Kan complex (as any simplicial abelian group), thus the Hurewicz map $A[p] \xrightarrow{h} kA[p]$, which is in fact the unit of the adjoint pair $k(-) : sVect_k \rightleftarrows sSet : U$, is $2p$-connected. The Hurewicz theorem stated on this current form appears in [GJ99] Thm 3.7 for abelian groups.

In addition, considering the natural projection $l : kA[p] \rightarrow A[p]$, $\oplus x_i \mapsto \Sigma x_i$.

since the composite $A[p] \xrightarrow{h} kA[p] \xrightarrow{l} A[p]$ is the identity on $A[p] \setminus \{0\}$, we deduce that the map $l$ is also $2p$-connected. Therefore the map $\Omega^p kA[p] \xrightarrow{\Omega^p(l)} \Omega^p A[p]$ is $p$-connected and the map

$$\hocolim_{p \rightarrow \infty} \Omega^p kA[p] \rightarrow \hocolim_{p \rightarrow \infty} \Omega^p A[p]$$

is a weak equivalence of simplicial abelian groups. Now using the fact the the functor $N$ is a left and right Quillen functor of the Dold Kan correspondence we deduce the quasi-isomorphism

$$\hocolim_{p \rightarrow \infty} \Omega^p NkA[p] \rightarrow \hocolim_{p \rightarrow \infty} \Omega^p NA[p] \tag{2}$$

Proof of Lemma 5.2. We use Lemma 4.3 to obtain the quasi-isomorphism:

$$cr_n(Nk\text{Hom}_{Ch^+}(V \otimes Nk\Delta^\bullet, -))(W_1, \ldots, W_n) \simeq \text{thcofib}(\bigoplus_{i \in T} Nk\text{Hom}_{Ch^+}(V \otimes Nk\Delta^\bullet, \oplus W_i))$$

On the other hand the functors $N : sAb \rightarrow Ch^+$ and $k(-) : sSet \rightarrow sAb$ are left Quillen functors, we therefore have the equivalences

$$\text{thcofib}(Nk\text{Hom}_{Ch^+}(V \otimes Nk\Delta^\bullet, \oplus W_i)) \simeq Nk \text{thcofib}(\text{Hom}_{Ch^+}(V \otimes Nk\Delta^\bullet, \oplus W_i))$$

$$\simeq Nk \text{thcofib}(\bigoplus_{i \in T} \text{Hom}_{Ch^+}(V \otimes Nk\Delta^\bullet, W_i))$$

Since the maps in the $n$-cube of pointed simplicial sets: $T \mapsto \bigoplus_{i \in T} \text{Hom}_{Ch^+}(V \otimes Nk\Delta^\bullet, W_i)$ are inclusions, the total homotopy colimit is the strict total cofiber (thcofib), and computation shows (inductively) that

$$\text{thcofib}(\bigoplus_{i \in T} \text{Hom}_{Ch^+}(V \otimes Nk\Delta^\bullet, W_i)) \cong Nk(\text{Hom}_{Ch^+}(V \otimes Nk\Delta^\bullet, W_1) \wedge \ldots \wedge \text{Hom}_{Ch^+}(V \otimes Nk\Delta^\bullet, W_n)) \cong N(\tilde{k}\text{Hom}_{Ch^+}(V \otimes Nk\Delta^\bullet, W_1) \otimes \ldots \otimes \tilde{k}\text{Hom}_{Ch^+}(V \otimes Nk\Delta^\bullet, W_n))$$

We then conclude the quasi-isomorphism:

$$cr_n(\tilde{Ch}^+(V, -))(W_1, \ldots, W_n) \simeq N\tilde{k}\text{Hom}_{Ch^+}(V \otimes Nk\Delta^\bullet, W_1) \otimes \ldots \otimes N\tilde{k}\text{Hom}_{Ch^+}(V \otimes Nk\Delta^\bullet, W_n) \tag{3}$$

If $V$ is bounded below degree $k$, we have

$$\text{Hom}_{Ch^+}(V \otimes Nk\Delta^\bullet, s^{p+k}k) \cong \text{Hom}_{Ch^+}(Nk\Delta^\bullet, \text{hom}(V, s^{p+k}k))$$

$$\cong \text{Hom}_{Ch^+}(Nk\Delta^\bullet, \text{hom}(V, k) \otimes s^{p+k}k)$$

$$\cong (1) \text{Hom}_{Ch^+}(Nk\Delta^\bullet, \text{hom}(V, k) \otimes s^k[k])[p]$$

where the weak equivalence (1) is given by the weak equivalence of simplicial vector spaces.
\[\text{Hom}_{\text{Ch}_+}(Nk\Delta^\bullet, \text{hom}(V,k) \otimes s^k k) \otimes \text{Hom}_{\text{Ch}_+}(Nk\Delta^\bullet, s^p k) \simeq \text{Hom}_{\text{Ch}_+}(Nk\Delta^\bullet, \text{hom}(V,k) \otimes s^{p+k} k)\]

defined in [SS03] (2.8), p 295.

Now, when we replace \(A\) in the map \(\Box\) with \(\text{Hom}_{\text{Ch}_+}(Nk\Delta^\bullet, \text{hom}(V,k) \otimes s^k k)\) and compose it with \(\Omega^k(-)\), we get the quasi-isomorphism

\[\text{hocolim}_{p \to \infty} \Omega^p Nk \text{Hom}_{\text{Ch}_+}(Nk\Delta^\bullet, \text{hom}(V,k) \otimes s^k k) \simeq \text{hocolim}_{p \to \infty} \Omega^{p+k} Nk \text{Hom}_{\text{Ch}_+}(Nk\Delta^\bullet, \text{hom}(V,k) \otimes s^k k)[p] \simeq \text{hocolim}_{p \to \infty} \Omega^{p+k} \text{Hom}_{\text{Ch}_+}(Nk\Delta^\bullet, \text{hom}(V,k) \otimes s^k k)[p] \simeq \text{hocolim}_{p \to \infty} \Omega^{p+k} \text{hom}(V,k) \otimes s^{p+k} \simeq \text{hom}(V,k)\]

Using this above equivalence, we consider the specific case \(W_i = s^p k\) in Equation \(\Box\) and apply the functor hocolim to the left-hand and right-hand side of this same equation, we get the quasi-isomorphism \(\partial_n Nk \text{Hom}_{\text{Ch}_+}(V \otimes Nk\Delta^\bullet, -) \simeq \text{hom}(V,k)^{\otimes n}\).

6 Chain rule on the composition of two functors

This section intends to describe the chain rule property that have the Goodwillie derivatives when we compose two functors.

**Theorem 6.1.** [Ch110, Thm 1.15] Let \(F, G : \text{Ch} \to \text{Ch}\) be homotopy functors, and suppose that \(F\) is finitary, then

\[\partial_*(FG) \simeq \partial_*(F) \circ \partial_*(G)\]

**Corollary 6.2.** Let \(\mathcal{C}\) or \(\mathcal{D}\) be any of the model categories \(\text{Alg}_\mathcal{O}, \text{Ch},\) or \(\text{Ch}_+\), and \(F, G\) be homotopy and reduced functors: \(\mathcal{C} \xrightarrow{G} \text{Ch} \xrightarrow{F} \mathcal{D}\), and suppose that \(F\) is finitary, then

\[\partial_*(FG) \simeq \partial_*(F) \circ \partial_*(G)\]

**Proof.** We prove this result only when \(\mathcal{C} = \text{Alg}_\mathcal{O}\) since the other cases are obtained from a slight adaptation and following the same idea. If \(U : \text{Alg}_\mathcal{O} \to \text{Ch}\) denotes the forgetful functor, then we make the following computation:

\[\partial_*(FG) \simeq \partial_*(\text{GF})(\mathcal{O}(k))\]

\[= \text{hocolim}_{p_i \to \infty} s^{-p_1} \cdots s^{-p_n} \text{cr}_n(\text{GF})(\mathcal{O}(\Sigma^{p_1} k), \ldots, \mathcal{O}(\Sigma^{p_n} k))\]

\[\cong \text{hocolim}_{p_i \to \infty} s^{-p_1} \cdots s^{-p_n} \text{cr}_n(UFG)(\mathcal{O}(\Sigma^{p_1} k), \ldots, \mathcal{O}(\Sigma^{p_n} k))\]

\[\cong \text{hocolim}_{p_i \to \infty} s^{-p_1} \cdots s^{-p_n} \text{cr}_n(UFG\mathcal{O}(-))(\Sigma^{p_1} k, \ldots, \Sigma^{p_n} k)\]

\[\simeq \partial_*(UFG\mathcal{O}(-)\text{red}_0)\]

The two functors \(\text{Ch} \xrightarrow{G\mathcal{O}(-)\text{red}_0} \text{Ch} \xrightarrow{UF} \text{Ch}\) are homotopy functors and \(UF\) preserves filtered homotopy colimits. Therefore, we use Theorem 0.1 to claim that

\[\partial_*(FG) \simeq \partial_*(UF) \circ \partial_*(G\mathcal{O}(-)\text{red}_0)\]

\[\simeq \partial_*(F) \circ \partial_*(G).\]
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