ON THE SUM OF THE TWISTED FOURIER COEFFICIENTS OF MAASS FORMS BY MÖBIUS FUNCTION

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Abstract. In this paper, we study non-trivial upper bounds for the sum $\sum_{n \in S} |\lambda_f(n)|$ where $f$ is a normalized Maass eigencusp form for the full modular group, $\lambda_f(n)$ is the $n$th normalized Fourier coefficient of $f$ and $S$ is a proper subset of positive integers in $[1, x]$ with certain properties.

1. Introduction

One of the important questions for several classical non-negative arithmetical functions is to find how often they are small on average. In this connection, assuming Ramanujan conjecture namely $|\alpha_f(p)| = |\beta_f(p)| = 1$ $\forall$ primes $p$

when $f$ is a normalized Maass eigencusp form, H. Tang and J. Wu in [14] showed that

$$\sum_{n \leq x} |\lambda_f(n)| \ll_f \frac{x}{(\log x)^\theta_1}$$

where $\theta_1 = 0.118\ldots$ and $\lambda_f(n)$ is the $n$th normalized Fourier coefficient of $f$. They have also obtained results on short intervals.

Assuming Sato-Tate conjecture for Hecke eigencusp forms (it is proved for holomorphic case by Barnet-Lamb, Geraghty, Harris, & Taylor in [1], but is still open for Maass cusp forms), it is also established that (see [14]),

$$\sum_{n \leq x} |\lambda_f(n^m)| \sim D_m(f)x(\log x)^{-\delta_m}$$

where $D_m(f)$ is a positive constant depending on $m$ and $f$ and

$$\delta_m := 1 - \frac{4(m+1)}{\pi m(m+2)} \cot \left( \frac{\pi}{2(m+1)} \right).$$

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There are two important questions:

1. Is it possible to improve the upper bound of the inequality in (1) even assuming Ramanujan conjecture?

2. Is it possible to give non-trivial improved upper bounds for the sum \( \sum_{n \in S} |\lambda_f(n)| \) where \( S \) is a proper subset of all the integers in the interval \([1, x]\) such that \(|S| \to \infty\) as \( x \to \infty\)?

Question 1 seems to be very hard, however regarding Question 2, for some proper subsets \( S \), non-trivial upper bounds can be obtained unconditionally. If we obtain \( \sum_{n \in S} |\lambda_f(n)| = o(|S|) \), it ensures that on average \( |\lambda_f(n)| \) is small with respect to the set \( S \). Of course we have compromised on the length of the summand and thus naturally one would expect the upper bound of \( \sum_{n \in S} |\lambda_f(n)| \) to go down. Indeed what we show here is that for certain proper subsets of all the integers in \([1, x]\), the upper bound of \( \sum_{n \in S} |\lambda_f(n)| \) goes down considerably, that too without Ramanujan conjecture.

An important result of J. Hoffstein and P. Lockhart in [7] shows that

\[
\sum_{n \leq x} |\lambda_f(n)|^4 \ll x \log x
\]

and thus on average \( |\lambda_f(n)|^4 \) behaves nicely. This will lead to

\[
\sum_{n \in S} |\lambda_f(n)| \ll_f |S|^{3/4}(x \log x)^{1/4}.
\]

The question of finding a tight upper bound for the sum \( \sum_{n \in S} |\lambda_f(n)| \) becomes more interesting. Thus the main task of this article is to prove:

**Theorem 1.1.** For a normalized Maass eigencusp form \( f \) and \( x \geq x_0 \) with \( x_0 \) sufficiently large, the estimate

\[
\sum_{n \leq x} |\lambda_f(n)\mu(n)| \ll_{f, \epsilon} x (\log \log x)^{5/4} / \sqrt{\log x}
\]

holds unconditionally.
Remark 1.1. We note that
\[
\sum_{n \leq x} |\mu(n)| = \frac{x}{\zeta(2)} + O(x^{1/2}).
\]
From Theorem 1.1, we observe that
\[
\frac{\sum_{n \leq x} |\lambda_f(n)\mu(n)|}{\sum_{n \leq x} |\mu(n)|} \ll_{f,\epsilon} \frac{(\log \log x)^{5/4}}{\sqrt{\log x}}
\]
which tends to zero as \(x \to \infty\) so that the relative density is zero.

Theorem 1.2. Let \(S_k\) be the set of all \(k\)-free integers \((k \geq 3)\) in the interval \([1, x]\), then the inequality
\[
\sum_{n \in S_k} |\lambda_f(n)| \ll_{f, k, \epsilon} \frac{x(\log \log x)^{5/4}}{\sqrt{\log x}} = o \left( |S_k| \right)
\]
holds unconditionally.

Remark 1.2. It is not difficult to see from the arguments of the paper that the proof goes through very well for any proper subset \(S \subset \{1, 2, \ldots, [x]\}\) with the cardinality \(|S|\) satisfying
\[
x \left( \frac{1}{\log x} \right)^{1/2 - \eta} \leq |S| \leq (1 - \eta)x
\]
(\(\eta\) is any small positive constant) so that \(|\lambda_f(n)|\) is small on average with respect to \(S\). However, there are some important proper subsets for which these arguments do not provide the desired results. We exhibit two such examples in the last section of this paper.

Main Idea :

First we study the cognated sum \(\sum_{n \in S} \frac{|\lambda_f(n)|}{n}\) by splitting it into two sums pertaining to \(\mathcal{L}\)-smooth and its compliment to get non-trivial upper bounds. Then by the Lemma 3.6 we pass onto \(\sum_{n \in S} |\lambda_f(n)|\). The whole point here is that we can avoid the Ramanujan conjecture in these situations we consider. We treat the squarefree set case in detail and give the sketch of the proof in the general \(k\)-free set case.
Relation between sums $\sum_{n \leq x} g(n)$ and $\sum_{n \leq x} \frac{g(n)}{n}$:

Let $g(n)$ be a real non-negative arithmetic function. We are interested in the size of the sums

$$S(x) = \sum_{n \leq x} g(n) \quad \text{and} \quad L(x) = \sum_{n \leq x} \frac{g(n)}{n}.$$

Trivially, $S(x) \leq xL(x)$ and Riemann Stieltjes integration gives the relation

$$L(x) = \int_1^x \frac{dS(u)}{u} \leq \frac{S(x)}{x} - \frac{S(1^+)}{1^+} + \int_{1^+}^x \left| \frac{S(u)}{u^2} \right| du.$$

In 1980, Shiu [13] obtained a general upper bound for short sums of functions satisfying certain properties:

Let $\alpha, \beta \in [0, 1]$ and let $x, y$ satisfy $x \geq y \geq x^\alpha$. Then for positive integers $a, q$ with $(a, q) = 1$ we have

$$\sum_{\frac{x}{\omega(m)} \leq n \leq x+y} g_1(n) \ll \frac{y}{\phi(q) \log x} \exp \left\{ \sum_{p \leq x, p \nmid q} g_1(p) \right\}$$

uniformly for $1 \leq q \leq x^\beta$.

Later in 1998, Nair and Tenenbaum in [11] gave an interesting inequality connecting the two sums $S(x)$ and $L(x)$ for a class of non-negative arithmetic functions satisfying some conditions.

Let $F(n)$ be a non-negative arithmetic function such that

$$F(mn) \leq \min \left( D^{\omega(m)}, Em^\epsilon \right) F(n)$$

for all $m, n$ with $(m, n) = 1$ and any $D \geq 1, E \geq 1$. Here $\omega(m)$ denotes the total number of prime factors of $m$, counted with multiplicity. Suppose $Q \in \mathbb{Z}[X]$ is an irreducible polynomial and $\rho(m) = \rho_Q(m)$ denotes the number of roots of $Q$ in $\mathbb{Z}/m\mathbb{Z}$.

Then, a special case of their result gives

$$\sum_{x^\alpha \leq n \leq x} F\left( |Q(n)| \right) \ll y \prod_{p \leq x} \left( 1 - \frac{\rho(p)}{p} \right) \sum_{n \leq x} \frac{F(n)\rho(n)}{n}$$

uniformly for $x^{\alpha} \leq y \leq x$ with $x$ sufficiently large and where $\epsilon$ and $\alpha$ can be arbitrary small positive real numbers satisfying certain conditions.
For some simplified result of the form (2), we refer to Lemma 9.6 of De Koninck and Luca in [3].

2. PRELIMINARIES AND NOTATIONS

Let \( n \geq 2 \), and let \( v = (v_1, v_2, \ldots, v_{n-1}) \in \mathbb{C}^{n-1} \). A Maass form (see [5]) for \( SL(n, \mathbb{Z}) \) of type \( v \) is a smooth function \( f \in L^2(SL(n, \mathbb{Z}) \backslash \mathcal{H}^n) \) which satisfies

1. \( f(\gamma z) = f(z) \), for all \( \gamma \in SL(n, \mathbb{Z}) \), \( z \in \mathcal{H}^n \),
2. \( Df(z) = \lambda_D f(z) \), for all \( D \in \mathcal{D}^n \),
3. \( \int_{(SL(n, \mathbb{Z}) \cap U) \setminus U} f(uz) du = 0 \)

for all upper triangular groups \( U \) of the form

\[
U = \left\{ \begin{pmatrix} I_{r_1} & & * \\ & I_{r_2} & \\ & & \ddots \\ & & & I_{r_b} \end{pmatrix} \right\},
\]

with \( r_1 + r_2 + \ldots + r_b = n \). Here, \( I_r \) denotes the \( r \times r \) identity matrix, and \( * \) denotes arbitrary real entries.

Let \( M^*(\Gamma) \) be the set of normalized Maass eigencusp forms for the full modular group \( \Gamma = SL(2, \mathbb{Z}) \) and \( f \in M^*(\Gamma) \).

Denote by \( \lambda_f(n) \) the \( n \)th normalized Fourier coefficient of \( f \) and also the eigenvalue of \( f \) under the Hecke operator \( T_n \).

From the Hecke theory, \( \lambda_f(n) \) satisfies the multiplicative relation

\[
\lambda_f(m)\lambda_f(n) = \sum_{d|\gcd(m,n)} \lambda_f\left(\frac{mn}{d^2}\right)
\]

for all integers \( m \geq 1 \) and \( n \geq 1 \).

Thus for each prime number \( p \) there are two complex numbers \( \alpha_f(p) \) and \( \beta_f(p) \) such that

\[
\alpha_f(p)\beta_f(p) = 1
\]

and

\[
\lambda_f(p^n) = \alpha_f(p)^n + \alpha_f(p)^{n-1}\beta_f(p) + \cdots + \beta_f(p)^n
\]
for all integers \( \nu \geq 1 \).

Ramanujan conjecture states that

\[ |\alpha_f(p)| = |\beta_f(p)| = 1 \]

for all primes \( p \).

Unconditionally, we only have

\[
p^{-7/64} \leq |\alpha_f(p)| \leq p^{7/64} \\
p^{-7/64} \leq |\beta_f(p)| \leq p^{7/64}
\]

for all primes \( p \), due to Kim and Sarnak [9].

We use the following standard notations:

1. We write \( \log_1 x := \log x \) and for \( k \geq 2 \), \( \log_k x := \log \log_{k-1} x \).
2. We denote the largest prime factor of an integer \( n \) by \( P^+(n) \) and the smallest prime factor of an integer \( n \) by \( P^-(n) \) (with the convention that \( P^+(1) = P^-(1) = 0 \)).
3. Let \( \mathcal{L} := (\log x)^2 \) where \( x \) is sufficiently large.
4. Constants \( C \) with indices are some fixed positive constants. \( \epsilon \) is any small positive constant.
5. Implied constants might depend at most on \( f \) and \( \epsilon \) in the squarefree set case and on \( f, \epsilon \) and \( k \) in the \( k \)-free set case.

3. SOME LEMMAS

**Lemma 3.1.** Let \( x \) be sufficiently large and \( \frac{1}{2} < a < 1 \). Then there exists a fixed constant \( C_1 > 0 \) such that

\[
\sum_{n \leq x \atop P^+(n) \leq \mathcal{L}} n^{-a} \ll x^{1/2-a} \exp \left( C_1 \frac{\log x}{\log_2 x} \right).
\]

**Proof.** Let \( \psi(x, y) = \sum_{n \leq x \atop P^+(n) \leq y} 1 \) and \( y \geq \mathcal{L} = (\log x)^2 \). Then for \( u = \frac{\log x}{\log y} \),

\[
\psi(x, y) = x \varrho(u) \exp \left( O \left( u \exp \left( -(\log u)^{3/5-\Theta(1)} \right) \right) \right) \quad \text{if} \quad y \geq (\log x)^{1+\epsilon}
\]
where $\rho$ is Dickmann’s function (see [2, 4]) defined by

$$\rho(u) = 1 \quad \text{for} \quad 0 \leq u \leq 1$$

$$\rho'(u) = -\frac{1}{u}\rho(u-1) \quad \text{for} \quad u > 1.$$ 

Norton (see [12]) has shown that (as $u \to \infty$)

$$\rho(u) = \exp\left(-u \left(\log u + \log_2 u - 1 + \frac{\log u}{\log u} + O\left(\frac{1}{\log u}\right)\right)\right).$$

We obtain

$$\psi(x, y) \leq x \exp\left(-u \left(\log u + \log_2 u - 1\right)\right) \quad \text{for} \quad y \geq (\log x)^{1+\epsilon}.$$ 

Using integration by parts (following arguments as in [10]), we obtain

$$S_1 := \sum_{\scriptstyle n \leq x \atop \scriptstyle \rho^+(n) \leq \mathcal{L}} n^{-a}$$

$$= \psi(x, \mathcal{L})x^{-a} + a \int_1^x \psi(w, \mathcal{L})w^{-1-a} \, dw$$

$$\ll 1 + \int_1^x w^{-a} \exp\left(-\left(\frac{\log w}{\log \mathcal{L}}\right)\left\{\log \left(\frac{\log w}{\log \mathcal{L}}\right)ight.\right.$$

$$+ \log_2 \left(\frac{\log w}{\log \mathcal{L}} - 1\right)\left.\right\}\, dw.$$
Partition the interval of integration into subintervals of the form \([xe^{-(k+1)}, xe^{-k})\) for \(0 \leq k \leq \log x\), then

\[
J_k = \int_{xe^{-(k+1)}}^{xe^{-k}} \psi(w, L)w^{-1-a} \, dw
\]

\[
\ll (xe^{-k})^{1-a} \exp \left( - \left( \frac{\log x - (k + 1)}{2 \log_2 x} \right) \right) \left\{ \log \left( \frac{\log x - (k + 1)}{2 \log_2 x} \right) \right. \\
+ \log_2 \left( \frac{\log x - (k + 1)}{2 \log_2 x} \right) - 1 \left\} \right)
\]

\[
\ll x^{1-a}e^{-k(1-a)} \exp \left( - \left( \frac{\log x - (k + 1)}{2 \log_2 x} \right) \right) \left\{ \log \left( \frac{\log x - (k + 1)}{2 \log_2 x} \right) \right. \\
+ \log_2 \left( \frac{\log x - (k + 1)}{2 \log_2 x} \right) - 1 \left\} \right)
\]

\[
\ll x^{1-a}e^{-k(1-a)}x^{-1/2} \exp \left( \frac{\log 2 \log x}{2 \log_2 x} \right)
\]

\[
= x^{1/2-a}e^{-k(1-a)} \exp \left( \frac{\log 2 \log x}{2 \log_2 x} \right).
\]

Now summing over \(k\), we get

\[
S_1 = \sum_{n \leq x, P^+(n) \leq L} n^{-a} \ll x^{1/2-a} \exp \left( C_1 \frac{\log x}{\log_2 x} \right).
\]

\[\square\]

**Lemma 3.2.** For \(x\) sufficiently large, we have

\[
\sum_{n \leq x, P^+(n) \leq L} \frac{|\lambda_f(n)\mu(n)|}{n} \ll \epsilon x^{-\frac{23}{64}+\epsilon} \exp \left( C_1 \frac{\log x}{\log_2 x} \right).
\]
Proof. Using the unconditional bound for $|\lambda_f(n)|$, we get

$$\sum_{n \leq x \atop P^+(n) \leq L} \frac{|\lambda_f(n)\mu(n)|}{n} \leq \sum_{n \leq x \atop P^+(n) \leq L} \frac{|\lambda_f(n)|}{n} \leq \sum_{n \leq x \atop P^+(n) \leq L} \frac{n^{\frac{7}{64}}d(n)}{n} \ll \epsilon \sum_{n \leq x \atop P^+(n) \leq L} n^{-1+\frac{7}{64}+\epsilon} = \sum_{n \leq x \atop P^+(n) \leq L} n^{-(1-\frac{7}{64})}.$$ 

Taking $a = 1 - \frac{7}{64} - \epsilon$ in Lemma 3.1 (note that $\frac{1}{2} < a < 1$), we get

$$\sum_{n \leq x \atop P^+(n) \leq L} \frac{|\lambda_f(n)\mu(n)|}{n} \ll \epsilon x^{\frac{1}{2}-1+\frac{7}{64}+\epsilon} \exp \left( C_1 \frac{\log x}{\log_2 x} \right) \ll \epsilon x^{-\frac{23}{64}+\epsilon} \exp \left( C_1 \frac{\log x}{\log_2 x} \right).$$

\[\square\]

Lemma 3.3. For $x$ sufficiently large, we have

$$\sum_{n \leq x \atop P^-(n) > L} \frac{1}{n} \ll \log x.$$

Proof. Trivially,

$$\sum_{n \leq x \atop P^-(n) > L} \frac{1}{n} \leq \sum_{n \leq x} \frac{1}{n} \leq 1 + \int_1^x \frac{du}{u} \leq 1 + \log x \ll \log x.$$

\[\square\]
Lemma 3.4. We have
\[
\sum_{\substack{n \leq x \\ P^{-}(n) > L}} \frac{|\lambda_f(n)|^4}{n} \ll \log^2 x.
\]

Proof. For \( L < L_2 \leq x \), we get
\[
\sum_{\substack{n \leq x \\ P^{-}(n) > L}} \frac{|\lambda_f(n)|^4}{n} = \int_{L_2}^{x} \frac{1}{u} \left( \sum_{\substack{n \leq x \\ P^{-}(n) > L(u)}} |\lambda_f(n)|^4 \right) du
\]
\[
= \frac{1}{L_2} \left( \sum_{\substack{n \leq u \\ P^{-}(n) > L(u)}} |\lambda_f(n)|^4 \right) \Bigg|_{L_2}^{x} + \int_{L_2}^{x} \left( \sum_{\substack{n \leq u \\ P^{-}(n) > L(u)}} |\lambda_f(n)|^4 \right) \frac{1}{u^2} du.
\]

From \([7]\), we observe that there exists non-negative coefficients \( \lambda^*(n) \) such that
\[
\sum_{n \leq x} |\lambda_f(n)|^4 \leq \sum_{n \leq x} \lambda^*(n) \ll_f x \log x
\]
where
\[
\sum_{n=1}^{\infty} \frac{\lambda^*(n)}{n^s} := L(s, \text{sym}^4 f)L^3(s, \text{sym}^2 f)\zeta^2(s)
\]
in \( \Re(s) > 1 \). Hence,
\[
\sum_{\substack{n \leq x \\ P^{-}(n) > L}} \frac{|\lambda_f(n)|^4}{n} = O\left( \log x \right) + O\left( \log_2 x \right) + O\left( \int_{L_2}^{x} \frac{u \log u}{u^2} du \right)
\]
\[
\ll \left( \int_{L_2}^{x} \frac{\log u}{u} du \right) + \log x
\]
\[
\ll \left( \log^2 x \right).
\]

Lemma 3.5. For sufficiently large \( x \), we get
\[
\sum_{\substack{n \leq x \\ P^{-}(n) > L}} \frac{|\lambda_f(n)\mu(n)|}{n} \ll (\log x)^{\frac{5}{4}}.
\]
Proof. We have

\[
\sum_{\substack{n \leq x \\ \pi^{-1}(n) > L}} \left| \frac{\lambda_f(n) \mu(n)}{n} \right| \leq \sum_{\substack{n \leq x \\ \pi^{-1}(n) > L}} \left| \frac{\lambda_f(n)}{n} \right| 
\]

\[
= \sum_{\substack{n \leq x \\ \pi^{-1}(n) > L}} \left| \frac{\lambda_f(n)}{n} \right| + \sum_{\substack{n \leq x \\ \pi^{-1}(n) > L}} \left| \frac{\lambda_f(n)}{n} \right|
\]

\[
= S_1 + S_2
\]

where \( M \) is an open positive quantity which might depend on \( x \).

We get

\[
S_1 = \sum_{\substack{n \leq x \\ \pi^{-1}(n) > L}} \left| \frac{\lambda_f(n)}{n} \right|
\]

\[
\leq \sum_{\substack{n \leq x \\ \pi^{-1}(n) > L}} M \frac{1}{n}
\]

\[
\ll M \log x
\]

using Lemma 3.3

Now,

\[
S_2 = \sum_{\substack{n \leq x \\ \pi^{-1}(n) > L}} \left| \frac{\lambda_f(n)}{n} \right|
\]

\[
= \sum_{\substack{n \leq x \\ \pi^{-1}(n) > L}} \left| \frac{\lambda_f(n)}{n} \right| \frac{1}{\left| \lambda_f(n) \right|^3 n}
\]

\[
\leq \sum_{\substack{n \leq x \\ \pi^{-1}(n) > L}} \frac{1}{M^3} \frac{\left| \lambda_f(n) \right|^4}{n}
\]

So, using Lemma 3.4

\[
S_2 \ll \frac{\log^2 x}{M^3}
\]
Now choose $M$ such that
\[ M \log x \sim \frac{\log^2 x}{M^3} \]
i.e.,
\[ M \sim \{(\log x)\}^{\frac{1}{4}}. \]
Hence,
\[ \sum_{n \leq x} \frac{|\lambda_f(n)|}{n} \ll (\log x)^{\frac{5}{4}}. \]

\[ \square \]

**Lemma 3.6.** Let $g$ be a multiplicative function such that $g(n) \geq 0$ for all $n$, and such that there exists constants $A$ and $B$ such that for all $x > 1$ both inequalities
\[ \sum_{p \leq x} g(p) \log p \leq Ah_1(x) \]
and
\[ \sum_{p} \sum_{\alpha \geq 2} \frac{g(p^\alpha)}{p^\alpha} \log p^\alpha \leq Bh_2(x) \]
hold where $h_1(x)$ (increasing) and $h_2(x)$ are positive functions of $x$ for all $x \geq 1$. Then for $x > 1$, we have
\[ \sum_{n \leq x} g(n) \leq (Ah_1(x) + Bh_2(x) + 1) \frac{x}{\log x} \sum_{n \leq x} \frac{g(n)}{n}. \]

**Proof.** We follow the arguments as in Lemma 9.6 of [3].

Let $S(x) = \sum_{n \leq x} g(n)$ and $L(x) = \sum_{n \leq x} \frac{g(n)}{n}$. Then
\[ L(x) = \sum_{n \leq x} \frac{g(n)}{n} \geq \frac{1}{x} \sum_{n \leq x} g(n) = \frac{1}{x} S(x) \]
i.e., $S(x) \leq xL(x)$.

\[ S(x) \log x = \sum_{n \leq x} g(n) \log x \]
\[ = \sum_{n \leq x} g(n) \log \left( \frac{x}{n} \right) + \sum_{n \leq x} g(n) \sum_{p|n} \log p + \sum_{n \leq x} g(n) \sum_{\alpha \geq 2} \frac{\log p^\alpha}{p^\alpha | n} \]
\[ = S_1 + S_2 + S_3. \]
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\[ S_1 = \sum_{n \leq x} g(n) \log \left( \frac{x}{n} \right) \]
\[ \leq \sum_{n \leq x} g(n) \frac{x}{n} \]
\[ = x \sum_{n \leq x} \frac{g(n)}{n} \]
\[ \leq xL(x). \]

Write \( n = mp \) such that \( p \nmid m \) in \( S_2 \).

\[ S_2 = \sum_{n \leq x} g(n) \sum_{p|n} \log p \]
\[ = \sum_{mp \leq x} g(mp) \sum_{p|n} \log p \]
\[ = \sum_{m \leq x} g(m) \sum_{p \leq x/m} g(p) \log p \]
\[ \leq \sum_{m \leq x} g(m) \sum_{p \leq x/m} g(p) \log p \]
\[ \leq \sum_{m \leq x} g(m) \frac{x}{m} h_1 \left( \frac{x}{m} \right) \]
\[ \leq Axh_1(x) \sum_{m \leq x} \frac{g(m)}{m} \]
\[ = Axh_1(x)L(x). \]

Write \( n = mp^\alpha \) such that \( p \nmid m \) in \( S_3 \).
\[ S_3 = \sum_{n \leq x} g(n) \sum_{\alpha \geq 2 \atop p^\alpha | n} \log p^\alpha \]
\[ = \sum_{mp^\alpha \leq x} g(mp^\alpha) \sum_{\alpha \geq 2 \atop p \nmid m} \log p^\alpha \]
\[ = \sum_{p} \sum_{\alpha \geq 2} g(p^\alpha) \log p^\alpha \sum_{\alpha \geq 2 \atop p \mid m} g(m) \]
\[ \leq \sum_{p} \sum_{\alpha \geq 2} g(p^\alpha) \log p^\alpha S \left( \frac{x}{p^\alpha} \right) \]
\[ \leq \sum_{p} \sum_{\alpha \geq 2} g(p^\alpha) \log p^\alpha \frac{x}{p^\alpha} L \left( \frac{x}{p^\alpha} \right) \]
\[ \leq xL(x) \sum_{p} \sum_{\alpha \geq 2} \frac{g(p^\alpha)}{p^\alpha} \log p^\alpha \]
\[ \leq xL(x) Bh_2(x). \]

Finally,

\[ S(x) \log x = S_1 + S_2 + S_3 \leq \left( 1 + Ah_1(x) + Bh_2(x) \right) xL(x). \]

Therefore,

\[ S(x) \leq \left( Ah_1(x) + Bh_2(x) + 1 \right) \frac{x}{\log x} L(x). \]

\[ \square \]

**Lemma 3.7.** For all \( x > 1 \), there exists a positive constant \( A \) such that

\[ \sum_{p \leq x} |\lambda_f(p)\mu(p)| \log p \leq A x \sqrt{\log x} \]

and

\[ \sum_{p} \sum_{\alpha \geq 2} \frac{|\lambda_f(p^\alpha)\mu(p^\alpha)|}{p^\alpha} \log p^\alpha = 0. \]
Proof. We get

$$\sum_{p \leq x} |\lambda_f(p)\mu(p)| \log p \leq \sum_{p \leq x} |\lambda_f(p)| \log p$$

$$= \sum_{p \leq x, |\lambda_f(p)| \leq L} |\lambda_f(p)| \log p + \sum_{p \leq x, |\lambda_f(p)| > L} |\lambda_f(p)|^4 |\log p|$$

$$\leq L \sum_{p \leq x} \log p + \frac{\log x}{L^3} \sum_{p \leq x} |\lambda_f(p)|^4$$

$$\leq L \sum_{p \leq x} \log p + \frac{\log x}{L^3} \sum_{n \leq x} |\lambda_f(n)|^4$$

$$\leq L \sum_{p \leq x} \log p + \frac{\log x}{L^3} \sum_{n \leq x} \lambda^*(n)$$

$$\ll_f xL + x\frac{(\log x)^2}{L^3}$$

by prime number theorem and inequality (3).

Now choose $L$ such that

$$xL \sim x\frac{(\log x)^2}{L^3}$$

i.e.,

$$L \sim \sqrt{\log x}.$$

Hence,

$$\sum_{p \leq x} |\lambda_f(p)\mu(p)| \log p \leq Ax\sqrt{\log x}.$$

Second result follows trivially since $\mu(p^\alpha) = 0$ for $\alpha \geq 2$. \qed

Lemma 3.8. We have

$$\sum_{n \leq x} |\lambda_f(n)\mu(n)| \leq C_2 \frac{x}{\sqrt{\log x}} \sum_{n \leq x} \frac{|\lambda_f(n)\mu(n)|}{n}.$$

Proof. The result follows by taking $g(n) = |\lambda_f(n)\mu(n)|$ in Lemma 3.6 using Lemma 3.7. \qed
4. Proof of Theorem 1.1

First we note that (with $L = (\log x)^2$),

$$\sum_{n \leq L} \left| \frac{\lambda_f(n) \mu(n)}{n} \right| \leq \sum_{n \leq L} \left| \frac{\lambda_f(n)}{n} \right| \leq \left( \sum_{n \leq L} \left| \frac{\lambda_f(n)}{n} \right|^4 \right)^{1/4} \left( \sum_{n \leq L} \frac{1}{n} \right)^{3/4} \leq C_3 (\log L)^{1/2} (\log L)^{3/4} \leq C_3 (\log L)^{5/4} \leq C_4 (\log \log x)^{5/4}.$$  

For $n > L$, we write $n = m_1 m_2$ where $m_1 = \prod_{p \mid n} p$. Thus,

$$\sum_{L < n \leq x} \left| \frac{\lambda_f(n) \mu(n)}{n} \right| = \sum_{m_1 \leq x} \left| \frac{\lambda_f(m_1) \mu(m_1)}{m_1} \right| \sum_{m_2 \leq \frac{x}{m_1}} \left| \frac{\lambda_f(m_2) \mu(m_2)}{m_2} \right|$$

$$\ll \sum_{m_1 \leq x} \left| \frac{\lambda_f(m_1) \mu(m_1)}{m_1} \right| \left( \log \left( \frac{x}{m_1} \right) \right)^{\frac{5}{4}}$$

$$\ll x^{-\frac{25}{64}+\epsilon} \exp \left( C_5 \frac{\log x}{\log_2 x} \right) \left( \log x \right)^{\frac{5}{4}}$$

using Lemmas 3.2 and 3.5.

By Lemma 3.8, we have

$$\sum_{n \leq x} |\lambda_f(n) \mu(n)| \leq C_2 \frac{x}{\sqrt{\log x}} \sum_{n \leq x} \left| \frac{\lambda_f(n) \mu(n)}{n} \right| \leq C_5 \frac{x}{\sqrt{\log x}} \left\{ (\log \log x)^{5/4} + x^{-25/64+\epsilon} \exp \left( C_1 \frac{\log x}{\log_2 x} \right) (\log x)^{5/4} \right\}$$

$$\ll x (\log \log x)^{5/4} \frac{1}{\sqrt{\log x}}.$$  

This completes the proof of Theorem 1.1.
5. Sketch of proof of Theorem 1.2

A natural number $n = p_1^{a_1}p_2^{a_2} \ldots p_r^{a_r}$ is called $k$-free if $a_i \leq k - 1 \; \forall \; i = 1, 2, \ldots, r$. Let

$$h_k(n) = \begin{cases} 1 & \text{if } n \text{ is } k\text{-free,} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{h_k(n)}{n^s} = \prod_p \left( 1 + p^{-s} + \cdots + p^{-(k-1)s} \right)$$

$$= \prod_p \frac{1 - p^{-ks}}{1 - p^{-s}}$$

$$= \frac{\zeta(s)}{\zeta(ks)}.$$

We know that (see [8]),

$$\sum_{n \leq x} h_k(n) = \frac{x}{\zeta(k)} + O \left( x^{1/k} \exp \left( -C_6 (\log x)^{3/5} (\log \log x)^{-1/5} \right) \right).$$

Similar to the proof of Theorem 1.1, we have

$$\sum_{n \leq L} \left| \lambda_f(n) \right| h_k(n) \ll_{f,k} (\log \log x)^{5/4}.$$

For $n > L$, we write $n = m_1m_2$ where $m_1 = p_1^{a_1}p_2^{a_2} \ldots p_r^{a_r}$ and $m_2 = q_1^{b_1}q_2^{b_2} \ldots q_s^{b_s}$ with $a_i \leq k - 1$ for all $i = 1, 2, \ldots, r$ and $b_j \leq k - 1$ for all $j = 1, 2, \ldots, s$ such that $p_i \leq L$ and $q_j > L$. Hence, $(m_1, m_2) = 1$. Thus,

$$\sum_{L < n \leq x} \frac{\left| \lambda_f(n) \right| h_k(n)}{n} \leq \sum_{m_1 \leq x, p^+(m_1) \leq L} \frac{\left| \lambda_f(m_1) \right| h_k(m_1)}{m_1} \sum_{m_2 \leq m_1} \frac{\left| \lambda_f(m_2) \right| h_k(m_2)}{m_2}$$

$$\ll_{f,k} \sum_{m_1 \leq x, p^+(m_1) \leq L} \frac{\left| \lambda_f(m_1) \right| h_k(m_1)}{m_1} \left( \log \left( \frac{x}{m_1} \right) \right)^{\frac{5}{4}}$$

$$\ll_{f,k, \epsilon} x^{-\frac{29}{64} + \epsilon} \exp \left( C_1 \frac{\log x}{\log_2 x} (\log x)^{\frac{5}{4}} \right).$$
using arguments similar to that of Lemmas 3.2 and 3.5.

We have,

\[
\sum_{p} \sum_{\alpha \geq 2} \frac{\left| \lambda_f(p^\alpha) \right| h_k(p^\alpha) \log p^\alpha}{p^\alpha} \leq \sum_{p} \sum_{\alpha \geq 2} \frac{\left| \lambda_f(p^\alpha) \right| \log p^\alpha}{p^\alpha} \\
\leq \sum_{p} \sum_{\alpha \geq 2} \frac{p^{5\alpha} (\alpha + 1) \alpha \log p}{p^\alpha} \\
\leq \sum_{p} \sum_{\alpha \geq 2} \frac{2\alpha^2 \log p}{p^{5\alpha}} \\
= \sum_{p} \sum_{\alpha \geq 2} \frac{2\alpha^2 \log p}{p^{5\alpha} \cdot e^{\frac{\alpha}{5\log p}}} \\
\leq \sum_{p} \sum_{\alpha \geq 2} \frac{2\alpha^2 \log p}{p^{5\alpha} \cdot \left(\frac{\alpha}{5\log p}\right)^2} \\
\ll \sum_{p} \sum_{\alpha \geq 2} \frac{1}{p^{\frac{5\alpha}{2} \log p}} \\
\ll \sum_{p} \sum_{\alpha \geq 2} \frac{1}{p^{\frac{5\alpha}{2}}} \\
= \sum_{p} \frac{1}{p^{7/8}} \\
= \sum_{p} \frac{1}{p^{7/8} (p^{7/8} - 1)} \\
\ll B.
\]

Using similar result as in Lemma 3.8 we get

\[
\sum_{n \in S_k} |\lambda_f(n)| = \sum_{n \leq x} |\lambda_f(n)| h_k(n) \\
\ll_{f,k} \frac{x}{\sqrt{\log x}} \sum_{n \leq x} \frac{|\lambda_f(n)| h_k(n)}{n} \\
\ll_{f,k,\epsilon} \frac{x}{\sqrt{\log x}} \left\{ \left(\log \log x\right)^{5/4} + x^{-25/64 + \epsilon} \exp \left( C_1 \frac{\log x}{\log^2 x} \right) (\log x)^{5/4} \right\} \\
\ll_{f,k,\epsilon} \frac{x (\log \log x)^{5/4}}{\sqrt{\log x}}
\]

which completes the proof.
6. Concluding remarks

In this section, we discuss two examples where a similar analysis gives an upper bound for $\sum_{n \in S} |\lambda_f(n)|$ which is not of the order $o(|S|)$. Hence, we are not able to ensure that $|\lambda_f(n)|$ assumes smaller values on average on these sets. First is the primes set and second is the squarefull numbers set.

Define
\[
\chi_1(n) = \begin{cases} 
1 & \text{if } n \text{ is a prime}, \\
0 & \text{otherwise}.
\end{cases}
\]

Note that
\[
\sum_{n \leq x} \chi_1(n) = \pi(x) \ll \frac{x}{\log x}.
\]

Then,
\[
\sum_{n \in P} |\lambda_f(n)| = \sum_{1 \leq n \leq x} |\lambda_f(n)| \chi_1(n)
\leq \left( \sum_{n \leq x} |\lambda_f(n)|^4 \right)^{1/4} \left( \sum_{n \leq x} (\chi_1(n))^{4/3} \right)^{3/4}
\ll f \left( \sum_{n \leq x} \lambda^*(n) \right)^{1/4} \left( \sum_{p \leq x} \zeta(3/2)(\log p) \right)^{3/4}
\ll f \left( x \log x \right)^{1/4} \left( \frac{x}{\log x} \right)^{3/4}
\ll f \sqrt{\frac{x}{\log x}}.
\]

A number $n = p_1^{a_1}p_2^{a_2} \ldots p_r^{a_r}$ is called a squarefull number if $a_i \geq 2$ for all $i = 1, 2, \ldots, r$. Let
\[
\chi_2(n) = \begin{cases} 
1 & \text{if } n \text{ is squarefull}, \\
0 & \text{otherwise}.
\end{cases}
\]

From [8], we have
\[
\sum_{n \leq x} \chi_2(n) = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + O(x^{1/3}).
\]
Let $S^*$ denotes the set of squarefull numbers. Then,

$$\sum_{n \in S^*} |\lambda_f(n)| = \sum_{1 \leq n \leq x} |\lambda_f(n)| \chi_2(n)$$

\[ \leq \left( \sum_{n \leq x} |\lambda_f(n)|^2 \right)^{1/4} \left( \sum_{n \leq x} (\chi_2(n))^{1/3} \right)^{3/4} \]

\[ \ll_f \left( \sum_{n \leq x} \lambda^*(n) \right)^{1/4} \left( x^{1/2} \right)^{3/4} \]

\[ \ll_f (x \log x)^{1/4} x^{3/8} \]

\[ \ll_f x^{5/8} (\log x)^{1/4}. \]

It is important to note that in these two cases, the study of cognated sums $S(x)$ and $L(x)$ will actually lead to weaker estimates than what Hölder’s inequality would give. Thus we observe that the averaging result in (3) and Lemma 3.6 have certain limitations which we had already mentioned in Remark 1.2.

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