Asynchronous opinion dynamics on the \( k \)-nearest-neighbors graph

Wilbert Samuel Rossi and Paolo Frasca

Abstract—This paper is about a new model of opinion dynamics with opinion-dependent connectivity. We assume that agents update their opinions asynchronously and that each agent’s new opinion depends on the opinions of the \( k \) agents that are closest to it. We show that the resulting dynamics is substantially different from comparable models in the literature, such as bounded-confidence models. We study the equilibria of the dynamics, observing that they are robust to perturbations caused by the introduction of new agents. We also prove that if the number of agents \( n \) is smaller than \( 2k \), the dynamics converge to consensus. This condition is only sufficient.

I. INTRODUCTION

Driven by the evolution of digital communication, there is an increasing interest for mathematical models of opinion dynamics in social networks. A few such models have become popular in the control community, see the surveys [1], [2]. In the perspective of the control community, opinion dynamics distinguish themselves from consensus dynamics because consensus is prevented by some other dynamical feature. In many popular models, this feature is an opinion dependent limitation of the connectivity. This is the case of bounded confidence (BC) models [3], [4], where social agents influence each other if their opinions are closer than a threshold. This way of defining connectivity, however, seems at odds with several social situations, since it may require an agent to be influenced by an unbounded number of fellow agents. Instead, the number of possible interactions is capped in practice by the limited capability of attention by the individuals. For instance, online social network services are based on recommender systems that select a certain number of news items, those which are closer to the user’s presumed tastes. However, to the best of our knowledge, this important observation has not been incorporated in any suitable model of opinion dynamics, with the partial exception of [5]. The latter paper compares different models of interaction, including one in which each agent is influenced by a fixed number of neighbors.

In a striking contrast, this observation has been made in the field of biology by a number of quantitative studies about flocking in animal groups (these include both theoretical and experimental works) [6], [7], [8], [9]. The importance of this way of defining connectivity has been also captured by graph theorists, who have studied the properties of what they call \( k \)-nearest-neighbors graph. For instance, it is known that \( k \) must be logarithmic in \( n \) to ensure connectivity [10] and flocking behavior [11].

In this paper, we provide the first analysis of the \( k \)-nearest-neighbor opinion dynamics. In this analysis, our contribution is threefold: (1) We describe the equilibria of the dynamics, distinguishing a special type of clustered equilibria that are constituted of separate clusters; (2) We discuss the robustness of clustered equilibria to perturbations consisting in the addition of new agents; (3) We provide a proof of convergence for small groups, that is, groups such that \( n < 2k \).

Our work differs from [5] in several aspects. As per the model, the dynamical model in [5] is synchronous and continuous-time, whereas ours is asynchronous and discrete-time. As per the analysis, [5] focuses on the equilibria and their properties (for instance, the distribution of their clusters’ sizes) are studied by extensive simulations, whereas we study the dynamical properties (robustness to perturbations, convergence) by a mix of simulations and analytical results. Our robustness analysis is based on the approach taken by Blondel, Hendrickx and Tsitsiklis for BC models [12]. Our convergence result is inspired by classical proofs of convergence for randomized consensus dynamics [13, Chapter 3], but its interest and difficulty originate from the lack of reciprocity in the interactions: this feature clearly distinguishes our model from bounded confidence models, where interactions are reciprocal as long as the interaction thresholds are equal for all agents [3], [5], [14], [15], [16], [17].

II. THE DYNAMICAL MODEL

Let \( n \) and \( k \) be two integers with

\[ 1 \leq k \leq n, \]

and let \( V = \{1, \ldots, n\} \) be the set of agents. Each agent is endowed with a scalar opinion \( x_i \in \mathbb{R} \), to be updated asynchronously. The update law

\[ x_i^+ = f(x, i) \quad (1) \]

goes as follows. An agent \( i \) is selected from \( V \); the elements of \( V \) are ordered by increasing values of \( |x_j - x_i| \); then, the first \( k \) elements of the list (i.e. those with smallest distance from \( i \)) form the set \( N_i \) of current neighbors of \( i \). Should a tie between two or more agents arise, priority is given to agents with lower index. Agent \( i \) may but not necessarily does belong to \( N_i \). Once \( N_i \) is determined, agent \( i \) updates his opinion \( x_i \) to

\[ x_i^+ = \frac{1}{k} \sum_{j \in N_i} x_j, \]
In this section we discuss some properties of the equilibria of system (1). Motivated by the simulations, we introduce the following terminology. Given a configuration $x \in \mathbb{R}^n$, the directed graph that represents the possible interactions (i.e., the opinion dependencies for any possible selection of the node to be updated) is

$$G(x) = (V, E(x)) \quad \text{with} \quad E(x) = \bigcup_{i \in V} \{(i, j), j \in N_i \},$$

where $N_i$ is the set of neighbors of $i$, should $i$ be selected to update his opinion. Clearly, if $k = n$ the graph $G(x) = (V, V \times V)$ is complete. A configuration $x \in \mathbb{R}^n$ is an equilibrium for the asynchronous dynamics if

$$x = f(x, i) \quad \text{for every } i.$$

If $k = 1$, then $G(x)$ contains only links between nodes with the same opinion: in this trivial case, every configuration is an equilibrium because agents cannot change opinion.

A configuration $x$ is called clustered if

$$x_{N_i} = x_i 1_{N_i} \quad \text{for every } i,$$

that is, if for every node all of his neighbors have the same opinion. Furthermore, a clustered configuration $x = c1$ for some $c \in \mathbb{R}$ is called consensus.

It is immediate to see that clustered configurations are equilibria. However, there exist equilibria that are not clustered. It is possible to obtain a simple counterexample with $n = 7$ and $k = 3$ and exploiting the tie break rule. Consider any configuration $x \in \mathbb{R}^7$ of the form

$$x_{\{1,3,5\}} = \alpha 1_{\{1,3,5\}}, \quad x_{\{2,4,6\}} = \beta 1_{\{2,4,6\}}, \quad x_7 = \frac{\alpha + \beta}{2},$$

where $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$. The above is an equilibrium even if $x_{N_7} = x_{\{1,2,7\}} \neq \frac{1}{2}(\alpha + \beta) 1_{\{1,2,7\}}$.

The tie breaking rule is not central for the existence of non-clustered equilibria, as one can see in the following example inspired by Figure 2.

**Example 1:** Consider $x \in \mathbb{R}^{20}$ with

$$x_{\{1,2,\ldots,11\}} = \alpha 1_{\{1,2,\ldots,11\}},$$

$$x_{12} = x_{13} = \frac{3\alpha + 2\beta}{5},$$

$$x_{14} = x_{15} = \frac{2\alpha + 3\beta}{5},$$

$$x_{\{16,17,\ldots,20\}} = \beta 1_{\{16,17,\ldots,20\}},$$

where $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$. For instance, the neighbors of agent 12 are $N_{12} = \{1, 12, 13, 14, 15\}$ because

$$|x_{12} - x_{12}| = |x_{13} - x_{12}| = 0,$$

$$|x_{12} - x_{14}| = |x_{12} - x_{15}| = \frac{1}{5}(\beta - \alpha),$$

$$|x_{12} - x_{11}| = \frac{2}{5}(\beta - \alpha),$$

while the remaining agents are at distance $\frac{2}{5}(\beta - \alpha)$ or larger. Such configuration is an equilibrium with $x_{N_{12}} \neq x_{12} 1_{N_{12}}$.

A simple analysis shows that clustered configuration are those in which the agents form clusters of at least $k$ participants with the same opinion. To make this claim formal, let
\(V_i = \{j : x_j = x_i\}\) be the set of nodes that share the same opinion of \(i\).

**Lemma 1:** A configuration is clustered if and only if \(|V_i| \geq k\) for every \(i\).

**Proof:** By definition, in a clustered configuration \(N_i \subseteq V_i\) for every \(i\). Assume \(|V_i| \geq k\) for every \(i\). For any \(i\) there are at least \(k\) nodes \(j\) (including \(i\)) with \(x_j = x_i\): such nodes have zero distance from \(i\) and hence \(N_i \subseteq V_i\). This holds for every \(i\) so the configuration is clustered. On the other hand, assume that exists \(i\) with \(|V_i| \leq k - 1\). The set \(N_i\) must contain a node \(j\) with \(x_j \neq x_i\), so not in \(V_i\), violating the definition of clustered configuration.

From this result, it follows that a clustered configuration allows up to
\[
\left\lfloor \frac{n}{k} \right\rfloor
\]
distinct sets \(V_i\) (and this bound is tight). For the special case of consensus, this claim becomes the following corollary.

**Corollary 2:** Consensus is the only possible clustered configuration if and only if
\[
n < 2k.
\]

**IV. Robustness of the equilibria**

The clustered equilibria of the dynamics described above have interesting robustness properties regarding the addition of new nodes or the removal of nodes. The model shows different behavior with respect to a standard Asynchronous Bounded Confidence (ABC) model. In this section, we briefly introduce for comparison the ABC model; then we provide a few simulations to motivate the following discussion of the robustness properties.

**A. ABC model**

Given a fixed range of confidence \(d > 0\), we introduce the Asynchronous Bounded Confidence (ABC) update law
\[
x^+ = f_{\text{ABC}}(x, i).
\]
(2)

where \(i\) is the agent that updates his opinion. The neighborhood of \(i\) is \(N_{i, \text{ABC}} = \{j : |x_j - x_i| \leq d\}\) and always contains \(i\) itself. The new opinion of agent \(i\) is
\[
x_i^+ = \frac{1}{|N_{i, \text{ABC}}|} \sum_{j \in N_{i, \text{ABC}}} x_j,
\]
while the remaining agents do not change opinion
\[
x_j^+ = x_j \quad \text{for every} \quad j \neq i.
\]

**B. Simulations**

We present a simulation to show the difference between model (1) and model (2) when a few agents are added to a consensus configuration (which is an equilibrium for both models). We set \(k = 5\) for model (1) and \(d = 0.25\) for model (2). We start with 10 agents sharing opinion 0.4; at steps \(t = 2, 3, 4, 5\) we add a new agent, with opinion chosen uniformly at random in \([0, 1]\). We select the agent that updates his opinion among those present at that time, independently and uniformly at random: the same selection is used in both models. Figure 3 contains the plots of the simulation. The upper plot regards the dynamics of model (1): the four new agents converge to the consensus opinion, which does not change; they are too few to form a new cluster. The lower plot contains the dynamics of the model (2): the consensus configuration is not preserved and the agent added at step \(t = 3\) remains isolated during the dynamics and keeps his opinion. The other three new agents join the original ten; this group of 13 agents converge to the same opinion which however is different from the original consensus value.

**C. Robustness of the equilibria**

We now provide a general discussion that explains the observations from Figure 3. Let \(n, k\) with \(1 \leq k \leq n\) be given and consider a clustered equilibria \(x \in \mathbb{R}^n\) of the model (1). We first discuss the addition of a new agent with opinion \(x_{n+1} = \alpha\) to the configuration \(x\), that becomes \([x; \alpha] \in \mathbb{R}^n\) with \(n' = n + 1\). Before the addition of the new node, clusters have to contain at least \(k\) agents. This fact remains true after the addition and we have that
\[
f([x; \alpha], i) = [x; \alpha']
\]
for every $i$, meaning that the original (clustered) portion of
the configuration $[x; \alpha]$ remains unperturbed. For a generic
value of $\alpha$ the limit of the dynamics has the same cluster
locations of $x$, with one of the clusters getting a new
member. For some specific values, it may happen that the
configuration $[x; \alpha]$ is a non-clustered equilibrium. In any
case, none of the original agents changes opinion. Instead,
in the metric ABC model (2) with uniform visibility radius $d$,
either the new agent is further apart from the original agents
and nothing happens or he falls within the visibility radius
of a cluster of agents. In the latter case both the new agents
and the agents in the cluster change opinions, converging to
an intermediate value.

Assuming $n$ sufficiently large, the removal of an agent
from a clustered equilibrium presents interesting differences
too. In the metric ABC model (2) the removal of an agent
does not trigger any dynamics in the remaining agents. In
model (1), if the agent is removed from a cluster with $k+1$
agents or more, nothing happens. But if the agent is removed
from a cluster with $k$ agents, the new configuration is not
an equilibrium anymore and the remaining nodes from that
group will evolve towards some new equilibrium.

V. CONVERGENCE TO CONSENSUS

In this section we show that process (1) converges to a
consensus, provided $n < 2k$ and the choice of the agent
that updates his opinion at time $t$ is an i.i.d. uniform random
variable over $V$. We recall from Section III that the consensus
is the unique clustered equilibrium for $n < 2k$.

For $t \geq 0$, let $x(t) \in \mathbb{R}^n$ be the sequence of opinion
vectors and $I(t) \in V$ a sequence of agents. Given an initial
configuration $x(0) = x^0$, we consider the dynamics

$$x(t + 1) = f(x(t), I(t)) \quad \text{for every } t \geq 0,$$

where $I(t)$ is the agent that updates his opinion at time $t$.

We introduce two functions $\mu, M : \mathbb{R}^n \to V$ that, given an
opinion vector $x$, return respectively the index of the smallest
and largest components, with ties sorted

$$\mu(x) = \min(\arg \min_i x_i), \quad M(x) = \min(\arg \max_i x_i).$$

The outer $\min$ sorts possible ties; note that $M(x) = \mu(-x)$.

In the following two lemmas we prove the properties of the dynamics in which the agent with smallest opinion is the
one that updates his opinion.

Lemma 3: Given $n, k$ with $1 \leq k \leq n$ and an initial
configuration $x^0 \in \mathbb{R}^n$ consider dynamics (3) with $I(t) =
\mu(x(t))$ and the scalar sequence $y(t) := \max_{i \in \mu(x(t))} x_i(t)$.

Then:

- the set sequence $\mu(x(t))$ and the scalar sequence $y(t)$
  are constant;
- for every $i \in \mu(x(0))$ the sequences $x_i(t)$ non-decreasing
  and satisfy $x_i(t) \leq y(t)$;
- for every $i \notin \mu(x(0))$ the sequences $x_i(t)$ are constant.

Proof: The proof goes by induction. First, consider the
trivial case with $x_{\mu(x(t))}(t) = y(t)$. This condition means
$x_i(t) = y(t)$ for every $i \in \mu(x(t))$ and thus $x_{\mu(x(t))}(t+1) =
x_{\mu(x(t))}(t)$ so everything remains unchanged.

Next, consider the case with $x_{\mu(x(t))}(t) < y(t)$. We have

$$x_{\mu(x(t))}(t + 1) = \frac{1}{k} \sum_{j \in \mu(x(t))} x_j(t) \in \left(x_{\mu(x(t))}(t), y(t)\right).$$

Therefore,

$$\{i : x_i(t) < y(t)\} = \{i : x_i(t+1) < y(t)\}$$

and

$$\{i : x_i(t) = y(t)\} = \{i : x_i(t+1) = y(t)\}.$$

Moreover, the cardinality of the set $\{i : x_i(t) < y(t)\}$
is strictly smaller than $k$. This implies that $N_{\mu(x(t+1))} =
N_{\mu(x(t))}$ and also $y(t+1) = y(t)$. The claims follow by
induction and by observing that only the agents $i \in N_{\mu(x(0))}$
can update their opinions at some time $t \geq 0$ and the updated
value $x_i(t+1)$ belongs to $[x_i(t), y(t)]$.

Lemma 4: Given $n, k$ with $1 \leq k \leq n$ and an initial
configuration $x^0 \in \mathbb{R}^n$ consider the dynamics (3) with $I(t) =
\mu(x(t))$ and the scalar sequence $y(t) = \max_{i \in \mu(x(t))} x_i(t)$.

Then

$$y(k-1) - \min_i x_i(k-1) \leq \left(1 - \frac{1}{k}\right)(y(0) - \min_i x_i(0))$$

Proof: First, compute $x_{\mu(x(t))}(t+1)$ for a generic $t \geq 0$.
We have

$$x_{\mu(x(t))}(t + 1) = \frac{1}{k} \sum_{j \in \mu(x(t))} x_j(t) = \frac{1}{k} \sum_{j \in \mu(x(0))} x_j(t) \geq \frac{1}{k} \sum_{j \in \mu(x(0))} x_j(0)$$

thanks to Lemma 3. Then,

$$x_{\mu(x(t))}(t + 1) \geq \frac{k-1}{k} x_{\mu(x(0))}(0) + \frac{1}{k} y(0) = x_{\mu(x(0))}(0) + \frac{1}{k} (y(0) - x_{\mu(x(0))}(0)).$$

Next, consider the set

$$S(t) = \{i : x_i(t) < x_{\mu(x(0))}(0) + \frac{1}{k} (y(0) - x_{\mu(x(0))}(0))\},$$

and observe that either $S(t) = \emptyset$ or $|S(t+1)| = |S(t)| - 1$
because $\mu(x(t)) \notin S(t+1)$. Since the set $S(0)$ contains at
most $k-1$ elements, the set $S(k-1)$ is empty. Hence,

$$x_i(k-1) \geq x_{\mu(x(0))}(0) + \frac{1}{k} (y(0) - x_{\mu(x(0))}(0))$$

for every $i$, a fact that implies

$$x_{\mu(x(k-1))}(k-1) \geq x_{\mu(x(0))}(0) + \frac{1}{k} (y(0) - x_{\mu(x(0))}(0)).$$

Using Lemma 3 we know that $N_{\mu(x(t))} = N_{\mu(x(0))}$ for
every $t \geq 0$ and that for every $i$ therein, $x_i(t) \leq y(t) = y(0)$.
Therefore

$$y(k-1) - x_{\mu(x(k-1))}(k-1) \leq y(0) - x_{\mu(x(0))}(0) = \frac{1}{k} (y(0) - x_{\mu(x(0))}(0))$$

and the thesis follows because $x_{\mu(x(t))} = \min_i x_i(t)$.

The following lemma follows from Lemma 3 and 4 using the
property $M(x) = \mu(-x)$.

Lemma 5: Given $n, k$ with $1 \leq k \leq n$ and an initial
configuration $x^0 \in \mathbb{R}^n$ consider the dynamics (3)
with \( I(t) = M(x(t)) \) and the scalar sequence \( z(t) := \min_{i \in N_M(x(t))} x_i(t) \). Then:
- the set sequence \( N_M(x(t)) \) and the scalar sequence \( z(t) \) are constant;
- for every \( i \in N_M(x(0)) \) the sequences \( x_i(t) \) are non-increasing and satisfy \( x_i(t) \geq z(0) \);
- for every \( i \notin N_M(x(0)) \) the sequences \( x_i(t) \) are constant.

Moreover,
\[
\max_i x_i(k-1) - z(k-1) \leq (1 - \frac{1}{k}) \left( \max_i x_i(0) - z(0) \right).
\]

The next equivalence will be crucial in the following.

**Lemma 6:** Given \( n, k \) with \( 1 \leq k \leq n \), consider \( x \in \mathbb{R}^n \) and define the quantities
\[
y := \max_{i \in N_M(x)} x_i \quad \text{and} \quad z := \min_{i \in N_M(x)} x_i.
\]
Then, \( z \leq y \) for every \( x \in \mathbb{R}^n \) if and only if \( n < 2k \).

**Proof:** We prove the equivalent claim that \( x \in \mathbb{R}^n \) with \( z > y \) exists if and only if \( n \geq 2k \). Indeed, if \( n \geq 2k \) consider the vector \( x \in \mathbb{R}^n \) such that
\[
x_1 \leq x_2 \leq \ldots \leq x_k < x_{k+1} \leq \ldots \leq x_{n-k+1} \leq \ldots \leq x_n
\]
where \( n-k+1 > k \). The set \( N_M(x) \) contains the \( k \) smallest elements of \( x \) so \( y = x_k \), while the set \( N_M(x) \) contains the \( k \) largest elements of \( x \), so \( z = x_{n-k+1} > x_k = y \). For the converse, assume that \( x \) with \( z > y \) exists, meaning
\[
(\max_{i \in N_M(x)} x_i) < (\min_{i \in N_M(x)} x_i).
\]
Both sets \( N_M(x) \) and \( N_M(x) \) contain \( k \) elements, so the sets
\[
\{ j : x_j \leq \max_{i \in N_M(x)} x_i \} \quad \text{and} \quad \{ j : x_j \geq \min_{i \in N_M(x)} x_i \}
\]
contain at least \( k \) elements each. These two sets are disjoint, thus the vector \( x \in \mathbb{R}^n \) has at least \( n \geq 2k \) components.

The next lemma describes a “shrinking sequence”.

**Lemma 7:** Given \( n, k \) with \( 1 \leq k \leq n \) and an initial configuration \( x^0 \) consider the dynamics (3) with
\[
I(t) = \begin{cases} 
\mu(x(t)) & \text{for } t \in \{0, \ldots, k-2\} \\
M(x(t)) & \text{for } t \in \{k-1, \ldots, 2k-3\}
\end{cases}
\]
If \( n < 2k \) then
\[
\max_i x_i(T) - \min_i x_i(T) \leq (1 - \frac{1}{k}) \left( \max_i x_i(0) - \min_i x_i(0) \right)
\]
where \( T = 2k-2 \).

**Proof:** For the sake of compactness, we set
\[
\alpha(t) := \min_i x_i(t), \quad \beta(t) := \max_i x_i(t), \quad \gamma := (1 - \frac{1}{k}),
\]
introduce the two sequences
\[
y(t) := \max_{i \in N_M(x(t))} x_i(t) \quad \text{and} \quad z(t) := \min_{i \in N_M(x(t))} x_i(t),
\]
and set \( R := k - 1 \). We have
\[
\beta(T) - \alpha(T) = \beta(T) - z(T) + z(T) - \alpha(T) \\
\leq \gamma (\beta(R) - z(R)) + z(R) - \alpha(R)
\]
using Lemma 5 with initial configuration \( x(R) \). Then
\[
= \gamma (\beta(R) - y(R)) + \gamma (y(R) - z(R)) + z(R) - \alpha(R) \\
\leq \gamma (\beta(R) - y(R)) + (y(R) - z(R)) + z(R) - \alpha(R)
\]
since \( \gamma < 1 \) and since \( y(R) - z(R) \geq 0 \) if \( n < 2k \) by Lemma 6. Then
\[
= \gamma (\beta(R) - y(R)) + \gamma (y(R) - \alpha(R)) \\
= \gamma (\beta(0) - \alpha(0))
\]
using Lemma 3 and 4 with initial configuration \( x(0) \). We have finally obtained \( \beta(T) - \alpha(T) \leq \gamma (\beta(0) - \alpha(0)) \).

If \( n < 2k \) and the agent \( I(t) \) that updates his opinion at time \( t \) is chosen independently and uniformly at random over \( V \), then process (3) converges almost surely to a consensus, from any initial configuration. The almost sure convergence is guaranteed because the finite sequence of updates introduced in the Lemma 7 appears infinitely often with probability one. This fact is proved in the following theorem, which provides the desired converge result.

**Theorem 8:** Let \( n, k \) with \( 1 \leq k \leq n \) be given. Let \( \{I(t), t \geq 0\} \) be a sequence of independent and uniformly distributed random variables over \( \{1, \ldots, n\} \) and consider dynamics (3). If \( n < 2k \), then
\[
\lim_{t \to \infty} x(t) = 1c \quad \text{almost surely}
\]
for any \( x^0 \in \mathbb{R}^n \), with \( c \in [\min_i(x^0), \max_i(x^0)] \).

**Proof:** Let \( \delta(t) = \max_i x_i(t) - \min_i x_i(t) \) and observe that, for any \( x(0) = x^0 \) and \( \{I(t), t \geq 0\} \),
\[
\delta(0) \geq 0 \quad \text{and} \quad 0 \leq \delta(t+1) \leq \delta(t) \text{ for every } t \geq 0 \quad \text{because the updates in the dynamics (3), based on model (1), involve convex combinations: the element with highest opinion cannot increase it and the element with lowest opinion cannot decrease it.}
\]
We introduce the sequence of events \( \{A_t, t \geq \infty\} \) with
\[
A_t = \{ I(s) = \mu(x(s)) \text{ for } s \in \{t-2k+3, \ldots, t+k-1\} \} \quad \text{and} \quad I(s) = M(x(s)) \text{ for } s \in \{t-k+2, \ldots, t\},
\]
i.e. the event \( A_t \) is the occurrence of the finite sequence introduced in Lemma 7 in the time window \( \{t-(2k-3), \ldots, t\} \).
In the same lemma we proved that, given the occurrence of \( A_t \), we have \( \delta(t+1) \leq (1 - \frac{1}{k}) \delta(t-2k+3) \). Observe that
\[
0 \leq \lim_{t \to \infty} \delta(t) \leq \lim_{t \to \infty} (1 - \frac{1}{k})^{n_t} \delta(0)
\]
where \( n_t \) is the number of times \( A_t \) occurred up to time \( t \). If \( \mathbb{P}(A_t \text{ infinitely often}) = 1 \) then \( n_t \to \infty \) for \( t \to \infty \) and the rightmost limit above is zero almost surely. Hence, \( \lim_{t \to \infty} \delta(t) \) almost surely, which implies the convergence to consensus. Moreover, \( c \in [\min_i(x^0), \max_i(x^0)] \) because every update in (3) is a convex combination of a subset of the current opinions.

It remains to prove \( \mathbb{P}(A_t \text{ infinitely often}) = 1 \). The events of the sequence \( \{A_t, t \geq 2k - 3\} \) are not independent but
the events in the subsequence \( \{A_{th}, h \geq 1\} \) where \( t_h = h(2k - 2) - 1 \) are. Each of these events has probability

\[
P(A_{th}) = \left( \frac{1}{n} \right)^{2k-2},
\]

thus \( \sum_{h=1}^{\infty} P(A_{th}) = \infty \). Hence, \( \{A_t \text{ i.o.}\} \supset \{A_{2k} \text{ i.o.}\} \). From the second Borel-Cantelli lemma [18, Ch. 2, Thm 18.2]

\[
P(A_t \text{ infinitely often}) \geq P(A_{2k} \text{ infinitely often}) = 1.
\]

The result continues to hold for dynamics where \( I(t) \) is not uniformly distributed over \( \{1, \ldots, n\} \), as long as the probability to sample each agent is constant and positive. The proof has been based on exhibiting one suitable “shrinking sequence”: however, it is clear that plenty of other sequences could do the job and actually play a role in inducing convergence of the dynamics. Therefore, the proof does not imply any good estimate of the convergence time.

VI. CONCLUSION

In this paper we have introduced a new model of opinion dynamics with opinion-dependent connectivity following the \( k \)-nearest-neighbors graph. The model is motivated by the rise of online social network services, where recommender systems select a certain number of news items to present to users, reducing the number of possible interactions to those which are closer to the user’s presumed tastes. The resulting dynamics is substantially different from comparable models in the literature, such as bounded-confidence models. One key difference is the inherent lack of reciprocity of the interactions, which makes all convergence analysis challenging. Another key difference is the robustness of the formed clusters, whose opinions are hard to sway by external leader nodes. This feature makes control approaches based on leadership, like [19], unsuitable to \( k \)-nearest-neighbors dynamics.

REFERENCES

[1] A. V. Proskurnikov and R. Tempo, “A tutorial on modeling and analysis of dynamic social networks. Part I,” Annual Reviews in Control, vol. 43, pp. 65–79, Mar. 2017.

[2] A. Proskurnikov and R. Tempo, “A tutorial on modeling and analysis of dynamic social networks. Part II,” Annual Reviews in Control, vol. 45, pp. 166–190, 2018.

[3] U. Krause, “A discrete nonlinear and non-autonomous model of consensus formation,” Communications in Difference Equations, pp. 227–236, 2000.

[4] G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch, “Mixing beliefs among interacting agents,” Advances in Complex Systems, vol. 03, no. 01n04, pp. 87–98, 2000.

[5] A. Aydoğlu, M. Caponigro, S. McQuade, B. Piccoli, N. Pouradier Duteil, F. Rossi, and E. Trélat, “Interaction network, state space, and control in social dynamics,” in Active Particles, Volume 1: Advances in Theory, Models, and Applications, N. Bellomo, P. Degond, and E. Tadmor, Eds. Springer, 2017, pp. 99–140.

[6] M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, M. Viale, and V. Zdravkovic, “Interaction ruling animal collective behavior depends on topological rather than metric distance: Evidence from a field study,” Proceedings of the National Academy of Sciences, vol. 105, no. 4, pp. 1232–1237, 2008.

[7] I. Giardina, “Collective behavior in animal groups: Theoretical models and empirical studies,” HFSP Journal, vol. 2, no. 4, pp. 205–219, 2008, PMID: 19404431.

[8] E. Cristiani, P. Frasca, and B. Piccoli, “Effects of anisotropic interactions on the structure of animal groups,” Journal of Mathematical Biology, vol. 62, no. 4, pp. 569–588, 2011.

[9] A. Aydoğdu, P. Frasca, C. D’Apice, R. Manzo, J. Thornton, B. Gachon, T. Wilson, B. Cheung, U. Tariq, W. Sailer, and B. Piccoli, “Modeling & biology of wires,” Journal of Theoretical Biology, vol. 415, pp. 102–112, 2017.

[10] P. Balister, B. Bollobás, A. Sarkar, and M. Walters, “Connectivity of random k-nearest-neighbour graphs,” Advances in Applied Probability, vol. 37, no. 1, pp. 1–24, 2005.

[11] C. Chen, G. Chen, and L. Guo, “On the minimum number of neighbors needed for consensus of flocks,” Control Theory and Technology, vol. 15, no. 4, pp. 327–339, 2017.

[12] V. D. Blondel, J. M. Hendrickx, and J. N. Tsitsiklis, “On Krause’s multi-agent consensus model with state-dependent connectivity,” IEEE Transactions on Automatic Control, vol. 54, no. 11, pp. 2586–2597, 2009.

[13] F. Fagnani and P. Frasca, Introduction to Averaging Dynamics over Networks, ser. Lecture Notes in Control and Information Sciences. Springer Nature, 2017.

[14] V. D. Blondel, J. M. Hendrickx, and J. N. Tsitsiklis, “Continuous-time average-preserving opinion dynamics with opinion-dependent communications,” SIAM Journal on Control and Optimization, vol. 48, no. 8, pp. 5214–5240, 2010.

[15] A. Mirtabatabaei and F. Bullo, “Opinion dynamics in heterogeneous networks: Convergence conjectures and theorems,” SIAM Journal on Control and Optimization, vol. 50, no. 5, pp. 2763–2785, 2012.

[16] C. Canuto, F. Fagnani, and P. Tilli, “An Eulerian approach to the analysis of Krause’s consensus models,” SIAM Journal on Control and Optimization, vol. 50, no. 1, pp. 243–265, 2012.

[17] F. Ceragioli and P. Frasca, “Continuous and discontinuous opinion dynamics with bounded confidence,” NonLinear Analysis and its Applications B, vol. 13, no. 3, pp. 1239–1251, 2012.

[18] A. Gut, Probability: A Graduate Course, 2nd ed. Springer, 2013.

[19] F. Dietrich, S. Martin, and M. Jungers, “Control via leadership of opinion dynamics with state and time-dependent interactions,” IEEE Transactions on Automatic Control, vol. 63, no. 4, pp. 1200–1207, 2018.