Anomalous charged fluids in 1+1d from equilibrium partition function

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ABSTRACT: In this note we explore the constraints imposed by the existence of equilibrium partition on parity violating charged fluids in 1+1 dimensions at zero derivative order. We write the equilibrium partition function consistent with 1+1 dimensional CPT invariance and which reproduces the correct anomaly in the charge current. The constraints on constitutive relations obtained in this way matches precisely with those obtained using the second law of thermodynamics.
1. Introduction

Relativistic Fluid dynamics is an effective large wavelength description (at length scales much bigger than the mean free path) of certain phases of matter which at microscopic level are described by relativistic quantum field theories. The basic equations governing dynamics in this description are the conservation laws corresponding to the global symmetries of the underlying theory. More specifically, these are the conservation equations of the stress energy tensor and charge currents. These equations are to be supplemented by constitutive relations which expresses the stress-energy tensor and charge current in terms of the basic fluid variables namely velocity, temperature and chemical potential.

Consistency with second law of thermodynamics has been used as a constraining principle on the constitutive relations in fluid dynamics (see [1, 2, 3, 4, 5, 6] and references therein). This gives two kinds of relations: a) Inequality type relations on dissipative coefficients(which contributes to entropy increase). b) Equality type relations on non dissipative coefficients(which do not contribute to entropy increase). Recently in [8] (see also [9]) it was shown that the requirement of the existence of stationary equilibrium which is generated from a partition function gives all the equality type relations. One of the cases that was studied in [8] was charged fluid dynamics in 3+1 dimensions when the charge current is anomalous. In this case, the results of Son and Surowka [2] on the chiral magnetic and chiral vorticity flows, were recovered without making any reference to an entropy current.
In this note we study the anomalous charged fluid dynamics in 1+1 dimensions using the equilibrium partition function. This system has earlier been studied in [10] using the second law of thermodynamics as well as from an action point of view. In this note we write down the equilibrium partition function for this system at zero derivative order which reproduces the anomalous charge conservation and on comparison with the most general constitutive relations in fluid dynamics, gives the results obtained in [10].

2. 1+1d parity violating charged fluid dynamics

Consider the parity violating charged fluids in 1+1 dimensions with background metric and gauge field

\[ ds^2 = -e^{2\sigma}(dt + a_1 dx)^2 + g_{11} dx^2 \]
\[ A = A_0 dt + A_i dx^i. \]  

(2.1)

The equations of motion are the following anomalous conservation laws

\[ \nabla_\mu T^\mu\nu = F^{\nu\lambda} \mathcal{J}_\lambda \]
\[ \nabla_\mu J^\mu = c \epsilon^{\mu\nu\lambda} F_{\mu\nu} \]
\[ \nabla_\mu J^\mu = \frac{c}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu} \]  

(2.2)

here \( \mathcal{J}, J \) are covariant and consistent currents respectively ([11], see also [8]).

The most general partition function consistent with Kaluza-Klein gauge invariance\(^1\), diffeomorphism along the spatial direction and \( U(1) \) gauge invariance upto anomaly is

\[ \mathcal{W} = \mathcal{W}_{\text{inv}} + \mathcal{W}_{\text{anom}} \]
\[ \mathcal{W}_{\text{inv}} = \int A_1 dx - \int a_1 dx \]
\[ \mathcal{W}_{\text{anom}} = -\frac{C}{T_0} \int A_0 A_1 dx \]  

(2.4)

where \( C, C_1 \) and \( C_2 \) are constants independent of \( \sigma \) and \( A_0 \) and

\[ A_0 = A_0 + \mu_0, \quad A_i = A_i - A_0 a_i. \]  

(2.5)

Equation [2.4] is written in terms of \( A_i \) which unlike \( A_i \), are Kaluza-Klein gauge invariant.

\[ V_i' = V_i - \partial_i \phi V_0, \quad (V')^0 = V^0 + \partial_i \phi V^i. \]  

(2.3)
\begin{tabular}{|c|c|c|c|c|}
\hline
Field & C & P & T & CPT \\
\hline
\sigma & + & + & + & + \\
a_1 & + & - & - & + \\
g_{11} & + & + & + & + \\
A_0 & - & + & + & - \\
A_1 & - & - & - & - \\
\hline
\end{tabular}

Table 1: Action of CPT

Under $U(1)$ gauge transformation $A_0 \to A_0$, $A_1 \to A_1 + \partial_1 \phi$, we obtain\(^2\)

\[
\delta W_{\text{inv}} = 0 \\
\delta W_{\text{anom}} = C \frac{T}{T_0} \int \phi \partial_1 A_0 dx = -C_2 \frac{T}{2} \int d^2 x \sqrt{-g_2} \phi \epsilon^\mu_\nu F_{\mu\nu}.
\]

(2.6)

Table (1) lists the action of 2 dimensional C, P and T on various fields. Requiring CPT invariance sets $C_1$ to zero since the term with coefficient $C_1$ is odd under CPT.

Now let us look at the most general constitutive relations allowed by symmetry in the parity violating case at zero derivative order. At this order, there are no gauge invariant parity odd scalar or tensor. But one can construct a gauge invariant vector \(^3\)

\[
\tilde{u}^\mu = \epsilon^\mu_\nu u_\nu.
\]

(2.8)

The most general allowed constitutive relations allowed by symmetry in Landau frame thus take the form

\[
T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + pg^{\mu\nu} \\
\tilde{J}^\mu = qu^\mu + \xi_j \tilde{u}^\mu.
\]

(2.9)

2.1 Equilibrium from Partition Function

In this subsection we will use the equilibrium partition function (2.4) to obtain the stress tensor and charge current at zero derivative order. Setting $C_1$ to zero in (2.4) we have

\[
W = -C \frac{T}{T_0} \int A_0 A_1 dx - C_2 T_0 \int a_1 dx
\]

(2.10)

\(^2\)Since we are interested in time independent background fields, we consider only time independent gauge transformations.

\(^3\)In components the parity odd vector is

\[
\tilde{u}_0 = 0, \quad \tilde{u}^1 = \epsilon^1_0 u_0 = \epsilon^1
\]

(2.7)

where $\epsilon^1 = e^{\sigma} e^{01} = \frac{1}{\sqrt{g_{11}}}$. 

Using the partition function (2.10) it is straightforward to compute the stress tensor and charge current in equilibrium to be

\[ T_{00} = 0, \quad T_{11} = 0, \quad T_{10} = e^{-\sigma} e^1 \left(-T_0^2 C_2 + CA_0^2\right), \]
\[ J_0 = Ce^1 A_1 e^\sigma, \quad J^1 = -Ce^1 e^{-\sigma} A_0. \]

(2.12)

The covariant current \( \tilde{J}^\mu \) can be obtained from the consistent current \( J^\mu \) by an appropriate shift as follows

\[ \tilde{J}^\mu = J^\mu + J^\mu_{sh}, \quad J^\mu_{sh} = Ce^{\mu\nu} A_\nu. \]

(2.13)

In components the covariant current is then

\[ \tilde{J}_0 = 0, \quad \tilde{J}^1 = -2Ce^{-\sigma} e^1 A_0. \]

(2.14)

### 2.2 Equilibrium from Hydrodynamics

We are interested in the stationary equilibrium solutions to conservation equations corresponding to the constitutive relations (2.9). The equilibrium solution in the parity even sector in background (2.1) at zero derivative order is

\[ u^\mu = u^\mu_{(0)} = e^{-\sigma}(1, 0), \quad T = T_0 e^{-\sigma}, \quad \mu = A_0 e^{-\sigma}. \]

(2.15)
Since there are no gauge invariant parity odd scalars in table 3, temperature and chemical potential do not receive any correction. However, the fluid velocity in equilibrium receives correction as

\[ u^\mu = u^\mu_{(0)} + b\epsilon^{\mu\nu}u^{(0)}_\nu. \]  

From (2.9), (2.15) and (2.16) we get the parity odd correction to the equilibrium stress tensor and charge current, which receive contribution from correction to the constitutive relations as well as from correction to the equilibrium fluid velocity, to be

\[ \delta T^0_0 = \delta J^0_0 = \delta T^{ij} = 0, \]
\[ \delta T^1_0 = -e^\sigma(\epsilon + P)b\epsilon^1, \]
\[ \delta \tilde{J}^1 = (qb + \xi_j)\epsilon^1. \]

2.3 Constraints on Hydrodynamics

Comparing the non trivial components of the equilibrium stress tensor and charge current of (2.12) and (2.17) we find that the coefficient of velocity correction (2.16) is

\[ b = -\frac{T^2}{\epsilon + p} \left( -C_2 + C\nu^2 \right) \]  

and the coefficient in correction to charge current (2.9) is

\[ \xi_j = C \left( \frac{q\mu^2}{\epsilon + p} - 2\mu \right) - C_2 \frac{qT^2}{\epsilon + p}. \]  

where \( \nu = \frac{\mu}{T} = \frac{A_0}{T_0} \).

The expressions (2.19) agree exactly with the results of [10] based on the requirement of positivity of the entropy current and effective action.

2.4 The Entropy Current

The equilibrium entropy can be obtained from the partition function using

\[ S = \frac{\partial}{\partial T_0} (T_0 \log Z) \]
\[ = -2C_2T_0 \int \sqrt{-g_{11}}\epsilon^1 a_1 dx. \]

In this subsection we determine the constraints on the hydrodynamical entropy current \( J_{S}^\mu \) from the requirement that (2.20) agree with the local integral

\[ S = \int dx \sqrt{-g_2} J_{S}^0. \]

The most general form of the entropy current allowed by symmetry \(^5\), at zero derivative order is

\[ J_{S}^\mu = su^\mu + \xi_s\tilde{u}^\mu + he^{\mu\nu}A_\nu, \]  

\(^5\)Let us note that the entropy current need not be gauge invariant, see [8] for more details.
where $h$ is a constant.

The parity odd correction to the entropy current in equilibrium, which receives contributions both from correction to the hydrodynamical entropy current and equilibrium velocity, is given by

$$J^0_{S|\text{correction}} = s\delta u^0 + \xi_s \tilde{u}^0 + h\varepsilon^0 A_1.$$  \hspace{1cm} (2.23)

Now using

$$\nu = \frac{A_0}{T_0}, \quad \delta u^0 = -a_1 \delta u^1 = -b\epsilon^1 a_1, \quad \tilde{u}^0 = -\epsilon^1 a_1$$

the correction to the hydrodynamical entropy in equilibrium is given by

$$\int dx \sqrt{-g_2} J^0_s|_{\text{correction}} = \int dx \, e^\sigma \left((-sb - \xi_s)\epsilon^1 a_1 + h\varepsilon^1 (A_1 + A_0 a_1)\right).$$  \hspace{1cm} (2.24)

Comparing this expression with (2.20) and using (2.19) we find

$$\xi_s = C' \frac{s\mu^2}{\epsilon + p} + C_2 T \left(1 + \frac{\rho\mu}{\epsilon + p}\right), \quad h = 0.$$  \hspace{1cm} (2.25)

This result is in precise agreement with those of [10].

3. Conclusion

To conclude, for 1 + 1d parity violating charged fluid in a time independent background with anomaly one can write down a local equilibrium partition function and the constraints obtained on the constitutive relations by demanding consistency with this partition function are in agreement with those obtained from a local form of entropy increase principle. In [8], by demanding the existence of a partition function it was noted that, for first order 2 + 1d parity violating charged fluid and second order 3 + 1d uncharged fluid, one obtains weaker constraints on the non dissipative part of the entropy current as compared to that obtained by demanding entropy increase. However, for the case of first order 3 + 1d charged fluid with anomaly, the entropy current obtained in both ways agree so is also for 1 + 1d anomalous case, as shown in this note. It would be interesting to check this for an anomalous fluid in arbitrary dimensions.

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