Large time asymptotics of the wave fronts
length II:
surfaces with integrable Hamiltonians

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In the paper [Vi-20], the author proves that the length $|S_t|$ of the wave
front $S_t$ at time $t$ of a wave propagating in an Euclidean disk $\mathbb{D}$ of radius 1,
starting from a source $A$, admits a linear asymptotics as $t \to +\infty$: $|S_t| \sim
(2 \arcsin a) t$ with $a = d(0, A)$. In the paper [Co-Vi-20], we gave a more direct
proof and some improvements of that result.

Here, we will explain that this result is quite general for surfaces with
an integrable Hamiltonian. We discuss only the 2D case for simplicity. The
main idea is to use action-angle coordinates (section 2) in order to get a nice
integral expression for $|S_t|$ (section 4). Integrable systems have in general
singularities, therefore we need to make some genericity assumptions (section
2) and to study what happens to the action-angle coordinates (section 3)
neart these generic singularities. We need then to evaluate some oscillatory
integrals (section 6) using an ergodic lemma (Appendix B).

For the geodesic flow on closed manifolds of negative curvature, Margulis
[Ma-69] proved that the asymptotics of the length is exponential. The generic
behaviour is not known. Here we study the integrable case which is highly
non generic.

Before starting, let us give a rough version of the main theorem 5.1:

**Let** $(X, g)$ **be a 2D-Riemannian manifold. Let** $H : T^*X \to \mathbb{R}$ **be
an integrable Hamiltonian near a given energy** $E$. **Assume that the**

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energy shell $\Sigma := H^{-1}(E)$ is compact and that $dH$ does not vanish on $\Sigma$. We assume also that $H$ satisfies some “generic properties”. The wave front starting from a point $A$ at energy $E$ is the projection onto $X$ of $\phi_t(\Sigma^A)$ where $\phi_t$ is the Hamiltonian flow of $H$ and $\Sigma^A := \{(A, \xi)|H(A, \xi) = E\}$ is assumed to be smooth. If the point $A$ in $X$ is “generic” (see section 4), the g-length of the wave front starting from $A$, at energy $E$, admits a linear asymptotics $|S_t| \sim \lambda(A)t$ as $t \to +\infty$, where $\lambda(A)$ expresses in terms of the action-angle coordinates.

1 Wave fronts

Let us consider a smooth 2D Riemannian manifold $(X, g)$ without boundary and fix a real number $E$. Let $H : T^*X \to \mathbb{R}$ be a smooth Hamiltonian. Assume that $H^{-1}([E - a, E + a])$ is compact for some positive $a$ and that $dH$ does not vanish on $\Sigma := H^{-1}(E)$. Let us fix some point $A \in X$ and put $\Sigma^A := \{(A, \xi) \in \Sigma\}$. We denote by $\omega$ the generic point of $\Sigma^A$. Assume that $d(H|_{T^*_AX})$ does not vanish on $\Sigma^A$. This implies that $\Sigma^A$ is a 1D-compact submanifold of $\Sigma$. We denote by $\pi_X$ the canonical projection of $\Sigma$ onto $X$ and by $\phi_t : \Sigma \to \Sigma$, $t \in \mathbb{R}$, the flow of $\vec{H}$, the Hamiltonian vector field derived from $H$. For any positive $t$, we define the wave front $S_t$ at time $t$ as the set of points of $X$ of the form $\pi_X(\phi_t(\Sigma^A))$. The wave front $S_t$ has a smooth parametrization by $\Sigma^A$. This allows to define its length $|S_t|$ using the Riemannian metric $g$, assumed to be continuous and possibly degenerate:

$$|S_t| = \int_{\Sigma^A} \gamma^\frac{1}{2} \left( \phi_t(\omega); \frac{d}{d\omega} \phi_t(\omega) \right) |d\omega|$$

where $\gamma = \pi_X^*(g)$. Note that $S_t$ admits in general some singular points as a subset of $X$, namely cusps and transversal self-intersections. We are interested in the asymptotic behaviour of $|S_t|$ as $t \to +\infty$.

Examples:

1. Geodesic flows: $H := \frac{1}{2}g^*$ is the Hamiltonian of the geodesic flow of a closed Riemannian manifold $(X, g)$. Let us fix $E = 2$. Then $\Sigma$ is the unit cotangent bundle and, on $\Sigma$, $\phi_t$ is the geodesic flow with speed 1. In this case, $S_t$ is the image by the exponential map at the point $A$ of the circle of radius $t$ in the tangent plane $T_AX$. 

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2. Schrödinger Hamiltonians: Let $(X, g)$ be a Riemannian manifold without boundary, $V : X \to \mathbb{R}$ a smooth function and $E$ a real number. We take $H := \frac{1}{2} g^* + V$. Our assumptions are satisfied if $V^{-1}([-\infty, E+a])$ is compact for some $a > 0$, $dV$ does not vanish on $V^{-1}(E)$ and $V(A) < E$.

2 Integrable Hamiltonian flows

For this section, one can look at the chapter 4 of [Vu-06] and the section 1 of [Co-Vu-03]. We will assume that the Hamiltonian $H$ is integrable near the energy $E$. “Integrability” means that there exists a positive number $a$ and a smooth map $M = (I, J) : H^{-1}([-a, E+a]) \to \mathbb{R}^2$, called the moment map, so that

- The Poisson bracket $\{I, J\}$ vanishes identically.
- The critical points of $M$ are of measure 0, i.e. the differentials $dI$ and $dJ$ are almost everywhere independent.
- There exists a smooth function $\Phi : M(H^{-1}([-a, E+a])) \to \mathbb{R}$ so that $H = \Phi(I, J)$.

Note that $dI$ and $dJ$ cannot vanish at the same point of $\Sigma$ because $dH$ does not vanish there.

The main examples with the geodesic flows are the surfaces of revolution, the tri-axial ellipsoids ([Ja-39]) and the Liouville metrics on 2D tori (Liouville metrics are of the form $ds^2 = (f(u) + g(v))(du^2 + dv^2)$, see [B-S-K-97] Chap. 7). Usually, integrable systems have singularities. We will make the following “generic” assumption which is already used in [Co-Vu-03]: we assume that the moment map $M$ satisfies the

(A1) Morse-Bott condition: at any point of $\Sigma$ where $dI$ and $dJ$ are linearly dependent, i.e. where $\lambda dI + \mu dJ = 0$ for some pair $(\lambda, \mu) \neq (0,0)$, the function $\lambda I + \mu J$, restricted to $\Sigma$, admits a critical manifold of dimension 1 with a transversally non degenerate Hessian.

This implies that the singular set $Z_0 \subset \Sigma$, i.e. the set of critical points of $M$ located in $\Sigma$, is a finite union of periodic orbits of $\vec{H}$. These periodic orbits are either hyperbolic or elliptic according to the signature of the transversal Hessian. We denote by $Z \subset \Sigma$ the part of the preimage by $M$ of the critical values of $M$ which is the union of $Z_0$ and all the stable and unstable manifolds.
of the hyperbolic periodic orbits. The open set $\Sigma \setminus Z_0$ admits a smooth
Lagrangian foliation given by the level sets of $M$.

The open set $\Sigma \setminus Z$ is foliated by 2D-tori on which the Hamiltonian flow
of $H$ is quasi-periodic. The set of these tori is a smooth 1D-manifold. We
denote it by $L$ and by $\sigma$ the generic point of $L$. The manifold $L$ is a 1D-torus
in the case where there are no singularities, i.e. if $Z$ is empty, and a finite
union of real lines $D_j$, $j = 1, \ldots, N$ if there are some singularities. If $\sigma \in D_j$
tends to one of the infinity of $D_j$, the corresponding torus $T_\sigma$ converges to
a compact connected set $T_{j,\infty}$ of $\Sigma$ which is either an elliptic periodic orbit
of $H$ or the union of a finite set of hyperbolic periodic orbits of $\vec{H}$ and some
cylinders which are connected components of their stable manifolds. In the
last case, $T_{j,\infty}$ is homeomorphic to a 2D torus or to a Klein bottle.

Let us denote by $U_j$ the open connected component of $\Sigma \setminus Z$ which is the
union of the tori associated to the line $D_j$. The projection of $U_j$ onto $D_j$ is a
smooth fibration by 2D-tori which is trivial, because it is a fibration on the
real line. There exist global coordinates $(\theta, \sigma) \in \mathbb{T}^2 \times \sigma$ on $\Sigma \setminus Z$ so that the
torus $\mathbb{T}_\sigma$ is mapped onto $\mathbb{T} \times \{\sigma\}$ and the Hamiltonian flow is mapped on a
vector field $V(\sigma) = A(\sigma)\partial_{\theta_1} + B(\sigma)\partial_{\theta_2}$ on $\mathbb{T}^2$ with some smooth functions $A$ and $B$. Note that $A$ and $B$ have no common zeroes because the Hamiltonian
flow does not vanish on $\Sigma$.

In what follows, we fix some component $U_j$. Let us describe the action-angle coordinates in some neighbourhood of $U_j$ in $T^*X$: there exists a symplectic diffeomorphism $\chi_j$ of some neighbourhood $V_j$ of $U_j$ onto an open set $\mathbb{T}^2 \times \Omega_j$, with $\Omega_j \subset \mathbb{R}^2$, contained in $T^*\mathbb{T}^2 \setminus 0$ with canonical coordinates $(\theta, p)$, so that $H \circ \chi_j^{-1}(\theta, p) = K_j(p)$ with $K_j$ a smooth function from $\Omega_j$ into $\mathbb{R}$. In these coordinates, the vector field $V_j$ is given by $V_j = (\partial K_j/\partial p_1)\partial_{\theta_1} + (\partial K_j/\partial p_2)\partial_{\theta_2}$. We note $\nabla K$ this non vanishing vector field. The vector field $\nabla K_j$ does not vanish and hence the curve $C_j := \{p \in \mathbb{R}^2 | K_j(p) = 1\}$ is a smooth submanifold of $\Omega_j$. The line $D_j$ identifies smoothly to the curve $C_j$. The manifold $L$ can be identified to the disjoint union of the curves $C_j$.

The coordinates $p$ are called the actions: they are given by action integrals $p_j := \int_{\gamma_j} \alpha$, where $d\alpha$ is the symplectic form, and the loops $\gamma_j$, $j = 1, 2$ form a basis of $H_1(\mathbb{T}_\sigma, \mathbb{Z})$ varying continuously in $V_j$. Note that if $\alpha'$ is another primitive of the symplectic form, the difference $\alpha - \alpha'$ is closed, hence the action integrals differ by some constants. There are many choices for the coordinates $\theta$: if $\Lambda \subset U_j$ is a Lagrangian manifold transversal to the foliation by the tori, one can choose $\theta$ vanishing on $\Lambda$. 

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We will need one more “generic” assumption on the Hamiltonian flow:

(A2) For any $j = 1, \cdots, N$, there exists, at any point $p$ of $C_j$, two integers $k \geq 1$ and $l \geq 1$, so that the derivatives of order $k$ and $l$ of the vector field $\nabla K_j$ along $C_j$ are linearly independent.

Note that this condition is independent of the parametrization of $C_j$. For example, $k = 1, l = 2$ means that the curvature of the curve $\{\nabla K_j(p) | p \in C_j\}$ does not vanish while $k = 2, l = 3$ means a generic cusp for that curve.

The assumption (A2) implies that, for any $\nu \in \mathbb{R}^2 \setminus 0$, the map from $C_j$ into $\mathbb{R}$ defined by $p \mapsto \langle \nu | \nabla K_j \rangle$ has only critical points of finite order.

3 The behaviour of $\nabla K$ near $Z$

In this section, we forget about the index $j$: $K$ denotes the expression of $H$ in some of the action-angle coordinates. We are interested at the behaviour of $\nabla K$ near $Z$.

3.1 The Elliptic case

Lemma 3.1 Let $\gamma$ be an elliptic periodic orbit of $H$ (included in $Z_0$), then $K$ is a smooth function of $p_1$ and $p_2$ up to $\gamma$.

Proof.– There exists a symplectic chart of a neighborhood of the elliptic periodic orbit of $H$ so that $H = \Phi (\xi, y^2 + \eta^2)$ in $(T^* \mathbb{T})_{x,\xi} \times (T^* \mathbb{R})_{y,\eta}$ (see [Vu-00]). The invariant tori are the level surfaces of the moment function $M(x,\xi; y,\eta) = (\xi, y^2 + \eta^2)$. Let us choose $\gamma_1 = \{s \rightarrow (s, \xi; y,\eta) | s \in \mathbb{R}/\mathbb{Z}\}$ and $\gamma_2 = \{s \rightarrow (x,\xi, \sqrt{y^2 + \eta^2} \cos 2\pi s, \sqrt{y^2 + \eta^2} \sin 2\pi s) | s \in \mathbb{R}/\mathbb{Z}\}$. If $\alpha = \xi dx + \eta dy$, we get the action integrals $p_1 = \xi$ and $p_2 = \pi(y^2 + \eta^2)$. Hence $K(p_1, p_2) \equiv \Phi(p_1, p_2/\pi)$.

We have the following

Corollary 3.1 The manifold $\mathcal{L}$ admits an extension as a manifold with boundary at the elliptic periodic orbits of $H$ and the 1-form $d\nabla K$ is smooth and hence integrable on $\mathcal{L}$ near that boundary.

3.2 The Hyperbolic case

In this section, we will use Section 1 of [Co-Vu-03].
3.2.1 Functions of type \( (L) \)

Let us start with a

**Definition 3.1** A function \( f : [0, c] \to \mathbb{R}, \) with \( c > 0 \), is called of type \( (L) \) if there exists two smooth functions \( \phi, \psi : [0, c] \to \mathbb{R} \) so that

\[
\forall x \in [0, c[, \quad f(x) = \phi(x) \log x + \psi(x)
\]

with \( \phi(0) = 0, \quad \phi'(0) \neq 0. \)

This definition is invariant by any smooth change of variable from \([0, c]\) into \([0, c']\). Hence it extends to 1D-manifolds with boundaries. Such a function is invertible in a small enough subinterval of \([0, c]\) and the inverse \( f^{-1} \) is \( C^1 \) up to the boundary \( \psi(0) = \lim_{x \to 0^+} f(x) \).

Now let us describe the application that we have in mind.

**Lemma 3.2** Let \( \alpha \) be a smooth 1-form so that \( d\alpha \) is a volume form in some neighbourhood \([-d, d]^2\) of the origin in the \((y, \eta)\) plane. Let \( m_j(s) = (y_j(s), \eta_j(s)), \ j = 1, 2, \) be two smooth curves with \( m_1(0) = (e, 0), m_2(0) = (0, f) \) and \( e, f \in ]0, d[, \eta_1(0) > 0, \eta_2(0) > 0 \) which are arcs transverse to each of the coordinates axes. Then consider the integral \( I(t) = \int_{\Gamma_t} \alpha \) where \( \Gamma_t \) is, for \( t \) small enough, the part of the hyperbola \( y\eta = t, \ (t > 0) \) between the curves \( m_1 \) and \( m_2 \) oriented in any of the two possible directions. Then \( I(t) \) is a function of type \( (L) \).

This lemma follows from the Stokes formula: the isochoric Morse lemma (see [Co-Ve-79]) allows to reduce to the case where \( d\alpha = dy \wedge d\eta \) and to the change of variable \( t \to F(t) \).

3.2.2 The lines \( D_j \) as 1D-manifolds with boundary

Let us put a structure of a 1D-manifold with boundary on the line \( D_j \) in the “hyperbolic case”. Let us recall that we denote be \( T_\infty \) the limit of the tori \( T_\sigma \) as \( \sigma \) tends to one of the infinities. We showed that \( T_\infty \) is the union of a finite number of closed hyperbolic orbits and of a finite number of cylinders which are parts of the stable an unstable manifolds of these orbits. Near \( T_\infty \setminus Z_0 \), the foliation by the level sets of the moment map is smooth. We can choose any local transverse arc to that foliation. They are all equivalent up to diffeomorphism along any connected component of \( T_\infty \setminus Z_0 \) and give
local parametrization of $D_j$ near that boundary by intervals $[0, c]$. How do we pass from one component to the next by crossing $Z_0$? We choose a Poincaré section at a point of $Z_0$ and use the Morse lemma which gives local coordinates $(y, \eta)$ in that section so that $(\lambda I + \mu J)(y, \eta) = \text{cte} + y\eta$. The local parameter is then the evaluation of the function $y\eta$ which allows to pass from the transversal $\eta = 1$ to the transversal $y = 1$. Both are locally parametrized by the restriction of the function $y\eta$. This gives to $D_j$ the structure of a 1D compact manifold with boundary. Note that this holds in a smooth way with respect to $E'$ close to $E$.

3.2.3 The asymptotic behaviour of the action integrals

There exists, in a neighborhood $V_j$ of $T_\infty$, invariant by the flows of $\vec{I}$ and $\vec{J}$, an Hamiltonian $P$, Poisson commuting with $I$ and $J$, whose orbits are periodic of period 1 (Theorem 1.6 of [Co-Vu-03]). $P$ is constant on $Z_0$. This gives a smooth action of the group $S^1$ on $V_j$. Note that this action is principal on $(V_j \cap \Sigma) \setminus Z_0$, but can get some non trivial isotropy $\mathbb{Z}/2\mathbb{Z}$ on $Z_0$. Let $\gamma_1(z), z \in V_j$, be the $S^1$-orbits. They are all homotopic. If $z$ lies in some invariant torus, $\gamma_1$ is a homotopically non trivial loop in this torus. We denote by $p_1$ the action integral on $\gamma_1(z)$ which is clearly smooth in $V_j$. Note that $p_1$ is a function of $P$ which is a local diffeomorphism.
We need to choose a loop $\gamma_2$ on the tori in $V_j$ which, with $\gamma_1$, generates a basis of the homology of the invariant tori. Let $R_h := V_j \cap P^{-1}(h)/S^1$ with $h$ close to $P(Z_0)$. The reduced manifolds (see Appendix C) is foliated by the reduction of the integrable foliation restricted to $P^{-1}(h)$. Let us denote by $Z_h$ the singular set of that foliation. As does $V_j \cap \Sigma$, the orbifold $R_h$ consists of a singular part $R_{\text{sing},h}$, the quotient of $Z_h \cap V_j$, which is homeomorphic to a circle, and an open set smoothly foliated by circles which are the reductions of the invariant tori. Together they give a topological foliation of $R_h$ depending smoothly of $h$. The singular leaf $R_{\text{sing},h}$ is smooth outside the finite set of points which are quotients by the $S^1$ action of the hyperbolic periodic orbits of $H$. This foliation is smooth outside these singular points. We take for $\gamma_2$ a lift of the projection of $T_{\sigma}$ depending continuously of $\sigma$.

We have the following crucial Lemma:

**Lemma 3.3** The action integrals $(p_1, p_2)$ on the previously chosen loops $\gamma_1$ and $\gamma_2$ satisfy at the boundary

- the action $p_1$ is smooth up to the boundary
- The action $p_2$ as a function of $\sigma$ is of type $(L)$ at the boundary and depends smoothly of $h$ and hence of $p_1$.
Proof.– We saw already the smoothness of $p_1$. The function $p_2(\sigma, h) - p_2(Z_h)$ is given by the symplectic area in $R_h$ between the reduction of $\gamma_2$ and $Z_h$. Lemma 3.2 implies that $p_2$ is of type (L) depending smoothly of $h$ and hence of $p_1$. □

3.2.4 The asymptotic behaviour of $\nabla K$

We have the following important

Corollary 3.2 The 1-form $d\nabla K$ is integrable at any hyperbolic boundary point of $\mathcal{L}$.

Proof.–

Near a closed orbit of $Z_0$, we have the normal form $H = \Phi(\xi, \eta)$ with $(x, \xi, y, \eta) \in T^*_x T \times T^*_y \mathbb{R}$. We have $p_1 = \xi$ up to a constant. We get $H = K(p_1, p_2) = \Phi(p_1, F(p_1, p_2))$ expressing $H$ in terms of the actions. We get

$$\partial_1 K = \partial_1 \Phi + \partial_2 \Phi \times \partial F/\partial p_1, \quad \partial_2 K = \partial_2 \Phi \times \partial F/\partial p_2$$

which are smooth outside $\Sigma$ and continuous on $\Sigma$. Hence their derivatives are integrable. □

4 An integral formula for $|S_t|$.

One of the difficulties in extending the result for the disk to this case is the fact that the action-angle coordinates only exist outside $Z$. Therefore, we need to make some assumptions on the point $A$.

4.1 Assumptions on the point $A$

If $\Lambda \subset T^*X$ is a Lagrangian manifold, the caustic set of $\Lambda$ is the set of critical points of the projection $\pi_X$ restricted to $\Lambda$. We first need a

Lemma 4.1 Let us take $\omega_0 \in \Sigma^A$ so that $(A, \omega_0) \notin Z_0$ and denote by $E_0$ the 2D-leaf of the invariant foliation of $\Sigma$ containing $(A, \omega_0)$. If $(A, \omega_0)$ does not belong to the caustic set of $E_0$, then $\Sigma^A$ and $E_0$ are transversal at the point $(A, \omega_0)$.
Proof. – $\Sigma^A$ is a 1D-submanifold of $T^*_A X$ and hence $\pi_X(\Sigma^A) = \{A\}$. On the other hand, the fact that $(A, \omega_0)$ is not in the caustic set means that $(\pi_X)|_{F_0}$ is a local diffeomorphism onto $X$ near $(A, \omega_0)$. The conclusion follows. $\square$

We will assume:

(A3) The intersection of $\Sigma^A$ with $Z$ is a countable set.

**Proposition 4.1 (A3)** is satisfied as soon as there is only a finite number of $\omega \in \Sigma^A \cap (Z \setminus Z_0)$ so that $(A, \omega)$ is in the caustic set of the Lagrangian leaf in which it lives.

Proof. – The intersection of $\Sigma^A$ with $Z_0$ is a finite set. On the other hand, the points $\omega$ so that $(A, \omega)$ is not a caustic point of the corresponding leaf are isolated inside $\Sigma^A$. Hence there is at most a countable set of such points. $\square$

(A4) The set of critical points of the smooth map $\omega \to \sigma$ from $\Sigma^A \cap (\Sigma \setminus Z)$ into $\mathcal{L}$ is countable.

**Proposition 4.2 (A4)** is satisfied as soon as there is only a finite number of $\omega \in \Sigma^A \cap (\Sigma \setminus Z)$ so that $(A, \omega)$ is a caustic point of the invariant torus containing that point.

The argument is quite similar to that of the proof of Proposition 4.1

4.2 Exact formulae for $|S_t|$

We will compute the lengths of the wave front using the action-angle coordinates.

We will start with the finite covering of $\Sigma \setminus Z$ by the semi-global action-angle charts. This allows a description of $S_t$ as follows: let $\chi : U \to \mathbb{T}^2 \times C$ be one of these charts and let $\Pi_X : \mathbb{T}^2 \times C \to X$ be the map $\pi_X \circ \chi^{-1}$.

This way, if we call $(\theta(\omega), p(\omega))$ the image of $\omega \in \Sigma^A$ by $\chi$, we can assume that $\theta(\omega)$ vanishes identically, because $T^*_A X$ is Lagrangian. We get that the corresponding part of the wave front $S_t$ is defined by

$$S_t = \{\Pi_X \left(t\nabla K(p(\omega)), p(\omega)\right) | \omega \in \Sigma^A\}$$

where $K$ is the Hamiltonian $H$ expressed in the action coordinates and $(\theta(\omega), p(\omega))$ are the action-angle coordinates of $\omega \in \Sigma^A$. We get, using the Assumption (A3), the expression

$$|S_t| = t \int_{\Sigma^A} \gamma^\frac{1}{2} \left(\nabla K(p(\omega)), p(\omega)\right) \cdot \frac{d}{d\omega} \nabla K(p(\omega)) d\omega$$
where $\gamma$ is the pull-back of $g$ by $\Pi_X$. We can make a change of variable: instead of $\omega$, one can use $\sigma \in \mathcal{L}$ thanks to assumption (A4). We get the

**Proposition 4.3** The length of the wave front is given by

$$|S_t| = t \int_{\mathcal{L}} N_A(\sigma) \gamma^{\frac{1}{2}} \left( (t \tilde{\nabla} K(\sigma), \sigma); d\tilde{\nabla} K(\sigma) \right)$$

where $N_A(\sigma) = \# \{ \Sigma^A \cap \mathcal{L} \}$.

### 5 The main result

**Theorem 5.1** Let $(X, g)$ be a Riemannian manifold of dimension 2 with $g$ continuous, possibly degenerate. Let $H$ be an Hamiltonian integrable at energy $E$ and satisfying the assumptions (A1) and (A2). Let $A \in X$ be a point satisfying the assumptions (A3) and (A4). The length for the metric $g$ of the wave front $S_t$ starting from $A$ has a linear asymptotics $|S_t| \sim \lambda(A)t$ as $t \to \infty$.

Let us denote be $\mathcal{L}$ the 1D-manifold of all invariant Lagrangian tori $L_\sigma$, $\sigma \in \mathcal{L}$, filling $\Sigma \setminus \mathcal{Z}$ and consider the continuous density on $\mathcal{L}$ defined by

$$|d\sigma| = \int_{\mathcal{L}} \gamma^{\frac{1}{2}} \left( (\theta, \sigma); d(\tilde{\nabla} K(\sigma)) \right) |d\theta|$$

where $\gamma$ is the pull-back of $g$ by the projection $\Pi_X$. The measure $|d\sigma|$ is independent of $A$. We have

$$\lambda(A) = \int_{\mathcal{L}} N_A(\sigma) |d\sigma|$$

with $N_A(\sigma) := \# \{ \Sigma^A \cap L_\sigma \}$.

**Corollary 5.1** Let $H$ be the Hamiltonian of the geodesic flow of a smooth metric $G$ on a closed manifold $X$. If $D$ is a smooth domain with boundary in $X$, the $g$–length of $S_t \cap D$, is given by

$$|S_t \cap D| \sim t \int_D d\mu_A$$

where $d\mu_A$ is an absolutely continuous density $d\mu_A = F|dx|$ with $F \in L^1(X, |dx|)$. whose integral is $\lambda(A)$. 

Proof of the Corollary. - Let $\psi$ be a positive continuous function on $X$. We can apply the previous theorem with $g' = \psi^2g$. This way, we see that the asymptotics of the $g'$-length of $S_t$ is given by replacing the measure $|d\sigma|$ by the measure

$$|d\sigma|' = \int_{\mathcal{T}_\sigma} \psi(\pi_X(\theta, \sigma)) \gamma^\frac{1}{2}(\theta, \sigma; d(\tilde{\nabla}K(\sigma))) |d\theta|$$

This says that

$$|S_t|_{g'} \sim t \int_X \psi d\mu_A$$

where $d\mu_A$ is the pushforward by $\pi_X$ of the absolutely continuous (a.c. in short) finite measure $dM_A := N_A(\sigma)\gamma^\frac{1}{2}(\theta, \sigma; d(\tilde{\nabla}K(\sigma))) |d\theta|$, supported by $\Sigma$.

We need to show that we can apply this when $\psi$ is the characteristic function of a smooth domain. In our situation $\Sigma$ is the unit cotangent bundle and $\pi_X : \Sigma \to X$ is a submersion. It follows that that $d\mu_A$ is a.c. w.r. to $|dx|$.

6 Proof of Theorem 5.1

We start from the expression of $|S_t|$ given in Proposition 4.3. Let us show that we can apply Lemma B.1 to the integral giving $|S_t|/t$. In the notations of that lemma, we have $V(\sigma) = \nabla K(\sigma)$. The Assumption (A2) implies that the assumption on $V$ of the Lemma is satisfied. The function $F$ is given by $F(\sigma, \theta) = N_A(\sigma)\gamma^\frac{1}{2}(\theta; d\tilde{\nabla}K(\sigma)/d\sigma)$. The integrability assumption follows from the Corollaries 3.1 and 3.2 and the upper bound

$$|N_A(\sigma)\gamma^\frac{1}{2}(\theta; W)| \leq C\|W\|$$

The continuity with respect to $\theta$ follows from the continuity of $g$ and the smoothness of the projection of any $\mathbb{T}_\sigma$ onto $X$. It is shown using Lebesgue’s dominated convergence Theorem.
7 Examples

7.1 Surfaces of revolution

Surfaces with a non trivial action of $S^1$ are tori or spheres. In both case, the metric is given by $g = a(s)^2 d\theta^2 + ds^2$ where $s \in \mathbb{R}/L\mathbb{Z}$ in the first case and $s \in [0, L]$ in the second (in this case $s = 0$ and $s = L$ are the poles).

The assumption (A1) is satisfied if and only if $a$ is a Morse function. The assumption (A2) is satisfied for a generic $a$. Assuming (A1) and (A2), the assumption (A3) is satisfied for any point of the torus and for $A$ not a pole in the case of the sphere. while (A4) is always satisfied. If $A$ is a pole, $|S_t|$ is periodic of period $2L$.

7.2 Tri-axial ellipsoids

The integrability was found by C. Jacobi ([Ja-39], see also [Ki-82] and section 3.2 of [Co-Vu-03]). Assumption (A1) and (A2) are satisfied. (A3) is satisfied for $A$ not an ombilical point while (A4) is always satisfied. If $A$ is an ombilical point, $|S_t|$ is periodic.

A Stationary phase

For this section, one can look at [Gu-St-77], chap. 1.

We want to evaluate the asymptotics as $t \to +\infty$ of integrals of the form

$$I(t) := \int_{\mathbb{R}} e^{itS(x)} a(x) dx$$

where $S$ is a real valued smooth function and $a \in C^\infty_\circ (\mathbb{R})$. We have the

**Proposition A.1** Let us assume that the critical points of $S$, i.e. the zeroes of $S'$, are non degenerate, i.e. $S''(x) \neq 0$ if $S'(x) = 0$. Then, if $x_1, \ldots, x_N$ are the critical points of $S$ in the support of $a$, $I(t)$ admits an asymptotic expansion given by

$$I(t) = \sum_{j=1}^{N} \frac{\sqrt{2\pi} e^{i\varepsilon_j \pi/4}}{|tS''(x_j)|^{1/2}} e^{itS(x_j)} (a(x_j) + O(t))$$

with $\varepsilon_j = \pm 1$ depending on the sign of $S''(0)$. 

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In the case where the critical points are degenerate, we have the following result:

**Proposition A.2** If the zeroes of $S'$ in the support of $a$ are of finite order, we have $I(t) \to 0$ as $t \to \infty$.

Note also that in equation (2), the remainders “$O(t)$” are uniform if $S'$ (resp. $a'$) is close to $S$ (resp. close to $a$) in the $C^\infty$ topology and the support of $a$ stays in some fixed bounded intervall.

**B  An “ergodic” lemma**

This section could be of independent interest.

**Lemma B.1** For $s \in J$ where $J$ is an interval of the real line, let $V(s) = A_1(s)\partial_1 + A_2(s)\partial_2$ be a family of constant vector fields on $\mathbb{T}^2$ depending smoothly of $s$. Assume that, for any $s \in J$, there exists two derivatives $V^{(k)}(s)$ and $V^{(0)}(s)$ which are linearly independent.

Let $F$ is a function on $J \times \mathbb{T}^2$ with $F \in C^0(\mathbb{T}^2, L^1(J, ds))$ satisfying the following condition: there exists a function $\psi \in L^1(J, ds)$ so that

$$\forall (s, \theta) \in J \times \mathbb{T}^2 \quad |F(s, \theta)| \leq \psi(s).$$

Then

$$\lim_{t \to +\infty} \int_J F(s, [tV(s)])|ds| = \int_{J \times \mathbb{T}^2} F|dsd\theta|$$

The assumption on the derivatives of $V$ have the following geometrical meaning: if $V'(s_0) = 0$, we get a cusp point which is of finite order; if $V'(s_0) \neq 0$, the curvature of the curve $V$ vanishes at a finite order. In particular the points where $V'$ and $V''$ are linearly dependent are isolated.

**Proof.** It follows from Lebesgue’s dominated convergence theorem, that the map $f : \theta \to F(., \theta)$ is continuous from $\mathbb{T}^2$ into $L^1(J, |ds|)$. Let us choose a finite covering of $\mathbb{T}^2$ by balls of centers $\theta_j$, $1 \leq j \leq N$, so that the $L^1$-oscillation of $f$ is each ball is smaller than $\varepsilon/2$ and a smooth finite partition $(\psi_j)$ of unity subordinated to that covering. Let $F_j \in C^\infty_o(J)$ satisfying $\|F(., \theta_j) - F_j\|_{L^1} \leq \varepsilon/2$. Such functions do exist (see [Fo-99] Prop. 8.17). If $G(s, \theta) = \sum_j \psi_j(\theta)F_j(s)$, we have

$$\int_J |F(s, \theta) - G(s, \theta)|ds| \leq \varepsilon$$
This allows to reduce to prove the result for such a function $G$. We can again approximate $G$ by a function

$$L(s, \theta) = \sum_{n \in \mathbb{Z}^2, \|n\| \leq N} a_n(s) \exp(2\pi i < n|\theta>)$$

uniformly in $L^1(|ds|)$. We have $a_0(s) = \int_{-\pi}^{\pi} L(s, \theta)|d\theta|$. We are left with the integrals

$$\int_J a_n(s) e^{2i\pi t < n|V(s)>}|ds|$$

It follows from the assumption on $V$ that such integrals tend to 0 as $t \to \infty$ for $n \neq 0$.

\[\square\]

### C Symplectic $S^1$-reduction

Let $P : M \to \mathbb{R}$ be an Hamiltonian on a symplectic manifold $(M, \omega)$ so that the vector field $\vec{P}$ is complete and generates an action of $T$ onto $M$. Let us assume that this action is almost free: it is free on an open dense subset of $M$ and all the isotropy subgroups are finite. Let us look at an energy shell $S_h := P^{-1}(h)$ for some $h \in \mathbb{R}$. The quotient of $S_h$ by the $T$-action is an orbifold $R_h$. Let us denote by $\pi_h$ the canonical projection of $S_h$ onto $R_h$. The orbifold $R_h$ admits an unique symplectic structure $\Omega$ so that $\pi^*(\Omega) = \omega$.

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