Non-divisibility and non-Markovianity in a Gaussian dissipative dynamics

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Abstract

We study a stochastic Schrödinger equation that generates a family of Gaussian dynamical maps in one dimension permitting a detailed exam of two different definitions of non-Markovianity: one related to the explicit dependence of the generator on the starting time, the other to the non-divisibility of the time-evolution maps. The model shows instances where one has non-Markovianity in both senses and cases when one has Markovianity in the second sense but not in the first one.

Recent theoretical and experimental advances have aroused a lot of interest in non-Markovian effects when quantum systems interact with an environment which cannot be considered at equilibrium \cite{1-15}. More specifically, consider a system $S$ embedded in an environment $E$, under the hypothesis of an initial factorized state, i.e., a density matrix of the form $\rho \otimes \rho_E$; tracing away the environment degrees of freedom obtains an exact completely positive (CP) reduced dynamics for $S$ that sends an initial state $\rho$ at time $t_0 \geq 0$ into a state $\rho_{t,t_0}$ at time $t \geq t_0$. This irreversible time-evolution is generated by an integro-differential equation of the form

$$\partial_t \rho_{t,t_0} = \int_{t_0}^{t} \mathrm{d}u K_{t,u} [\rho_{u,t_0}], \quad \rho_{t_0,t_0} = \rho,$$

where the operator kernel embodies the dependence on the past history of the system. The previous equation can be cast in the convolution-less form \cite{10}

$$\partial_t \rho_{t,t_0} = \mathbb{I}_{t,t_0} [\rho_{t,t_0}],$$

where the presence of memory effects is now incorporated in the dependence of the generator.
on the initial time $t_0$. Because of this, the CP maps which solve (2),
\begin{equation}
\Gamma_{t,t_0} = \mathcal{T} \exp \left( \int_{t_0}^{t_1} du \mathbb{L}_{u,t_0} \right),
\end{equation}
with $\mathcal{T}$ time-ordering, violate, in general, the (two-parameter) semigroup composition law, namely
\begin{equation}
\Gamma_{t,t_1} \circ \Gamma_{t_1,t_0} \neq \Gamma_{t,t_0}, \quad 0 \leq t_0 \leq t_1 \leq t.
\end{equation}
Indeed, if $\mathbb{L}_{u,t_0} = \mathbb{L}_u$ then (3) yields the equality in (4); vice versa, if in (4) the equality holds, by taking the time derivative of both sides with respect to $t$ one obtains $\mathbb{L}_{t,t_1} = \mathbb{L}_{t,t_0}$ for all $t_1 \geq t_0 \geq 0$. In [10], the dependence of the generator $\mathbb{L}_{t,t_0}$ on $t_0$ and thus (4) is taken as a criterion of non-Markovianity.

On the other hand, in [12]–[14] a different approach is considered whereby, given a one-parameter family of CP maps $\gamma_t$, $t \geq 0$, their non-Markovianity is related to non-divisibility, namely to the fact that no CP map $\Lambda_{t,u}$, $t \geq u \geq 0$, exists that connects the maps $\gamma_t$. In other words, the criterion of non-Markovianity becomes
\begin{equation}
\gamma_t = \Lambda_{t,u} \circ \gamma_u \implies \Lambda_{t,t_0} \text{ not CP}.
\end{equation}
If a CP $\Lambda_{t,u}$ existed, it would follow that certain CP monotone like the trace distance, the fidelity or the relative entropy should be decreasing; then, non-Markovianity is identified by the increase in time of such quantities which can also be taken as a measure of non-Markovianity.

In order to study the two criteria of non-Markovianity, we consider a stochastic Schrödinger equation originally proposed as a non-Markovian mechanism for the wave function collapse [16]. Specifically, we take a particle in one dimension subjected to a time-dependent random Hamiltonian of the form (for sake of simplicity, in the following, vector and matrix multiplication will be understood)
\begin{equation}
\hat{H}_t^w = \hat{H} - w^T(t) \hat{r},
\end{equation}
where the Hamiltonian $\hat{H}$ is at most quadratic in position and momentum operators $\hat{r}^T = (\hat{r}_1, \hat{r}_2) = (\hat{q}, \hat{p})$, while $w^T(t) = (w_1(t), w_2(t))$ is a Gaussian noise vector with zero mean and $2 \times 2$ correlation matrix $D(t,s)$:
\begin{equation}
\left[ D(t,s) \right]_{ij} = \langle \langle w_i(t) w_j(s) \rangle \rangle,
\end{equation}
where $\langle \langle \cdot \rangle \rangle$ denotes the average over the noise. This latter matrix is real symmetric, $D_{ij}(t,s) = D_{ji}(s,t)$, and of positive-definite type, that is
\begin{equation}
\sum_{i,j,t_a,t_b} \xi_i(t_a) \xi_j(t_b) D_{ij}(t_a,t_b) \geq 0, \forall \xi(t_a) \in \mathbb{R}^2,
\end{equation}
for any choice of times \( \{ t_a \}_{a=1} \). For each realization of the noise, the Schrödinger equation \((\hbar = 1)\)

\[
\frac{d|\psi^w(t)\rangle}{dt} = [\hat{H} - \mathbf{w}^T(t) \hat{r}] |\psi^w(t)\rangle ,
\]

generates unitary maps \( \hat{U}_{t,t_0}^w \) on the system Hilbert space that send an initial vector state \( |\psi\rangle \) at time \( t = t_0 \) into \( |\psi_{t,t_0}^w\rangle \) at time \( t \). Averaging the projector \( |\psi_{t,t_0}^w\rangle \langle \psi_{t,t_0}^w| \) over the noise yields a density matrix

\[
\rho_{t,t_0} = \langle \langle |\psi_{t,t_0}^w\rangle \langle \psi_{t,t_0}^w| \rangle \rangle .
\]

In order to find \( \hat{U}_{t,t_0}^w \), one first goes to the interaction representation and sets:

\[
|\tilde{\psi}_{t,t_0}^w\rangle = \hat{U}_{t-t_0}^{\dagger} |\psi_{t,t_0}^w\rangle ,
\]

\[
\frac{d|\tilde{\psi}_{t,t_0}^w\rangle}{dt} = \mathbf{w}^T(t) \hat{r}(t-t_0) |\tilde{\psi}_{t,t_0}^w\rangle ,
\]

where \( \hat{U}_t = \exp(-i \hat{H} t) \) and:

\[
\hat{r}(t) = \hat{U}_t^{\dagger} \hat{r} \hat{U}_t \equiv S_t \hat{r} ,
\]

\( S_t \) being a suitable symplectic matrix. For a given realization of the noise \( \mathbf{w}(t) \), the solution is of the form \( |\tilde{\psi}_{t,t_0}^w\rangle = \hat{U}_{t,t_0}^w |\psi\rangle \) where, a part for a pure phase,

\[
\tilde{\psi}_{t,t_0}^w = \exp \left\{ -i \int_{t_0}^{t} du \tilde{\mathbf{w}}^T(u) \hat{r}(u-t_0) \right\} |\psi_{t,t_0}^w\rangle ,
\]

\[
|\psi_{t,t_0}^w\rangle = \hat{U}_{t-t_0}^w |\psi\rangle .
\]

By averaging over the noise, the corresponding density matrix \((10)\) satisfies:

\[
i \partial_t \rho_{t,t_0} = [\hat{H}, \rho_{t,t_0}] - \sum_{j=1}^{2} \left[ \hat{r}_j, \langle \langle \mathbf{w}_j(t) |\psi_{t,t_0}^w\rangle \langle \psi_{t,t_0}^w| \rangle \rangle \right] .
\]

This stochastic Liouville equation can be turned into a standard master equation by means of the Furutsu-Novikov-Donsker relation \([17]\):

\[
\langle \langle \mathbf{w}(s) \mathbf{X}[\mathbf{w}] \rangle \rangle = \int_{-\infty}^{+\infty} du \langle \langle \mathbf{w}(s) \mathbf{w}(u) \rangle \rangle \langle \langle \delta R[\mathbf{w}] \rangle \rangle \langle \langle \delta \mathbf{w}(u) \rangle \rangle ,
\]

where \( \mathbf{X}[\mathbf{w}] \) is a functional of the noise, \( \delta / \delta \mathbf{w}(u) \) denotes the functional derivative with respect to the noise and \( R[\mathbf{w}] \) is the density operator of the system. With \( R[\mathbf{w}] = |\psi_{t,t_0}^w\rangle \langle \psi_{t,t_0}^w| \), one gets:

\[
\partial_t \rho_{t,t_0} = \mathbb{1}_{t,t_0} [\rho_{t,t_0}] = -i [\hat{H}, \rho_{t,t_0}] + \mathbb{N}_{t,t_0} [\rho_{t,t_0}] .
\]
with:

\[ N_{t,t_0}[\rho] = \sum_{i,j=1}^{2} C_{ij}(t, t_0) \left( \hat{r}_i \rho \hat{r}_j - \frac{1}{2} \{ \hat{r}_j \hat{r}_i, \rho \} \right) \]  

\[ C(t, t_0) = \int_{t_0}^{t} du \left[ D(t, u) S_{u-t} + S_{u-t}^T D^T(t, u) \right] . \]

If \( D(t, u) = \delta(t-u) D \) (i.e., white noise) then one reduces to the Markovian Lindblad type dynamics with a time-independent positive Kossakowski matrix, namely \( C(t, t_0) = D \) [18, 19]. In the time-dependent case, in order that the maps \( \Gamma_{t,t_0} \) generated by \( \mathbb{L}_{t,t_0} \) be CP, the Kossakowski matrix \( C(t, t_0) \) need not to be positive, as we explicitly show in the following. We shall seek a solution of (16) in the form

\[ \rho_{t,t_0} = \Gamma_{t,t_0}[\rho] = \int \frac{d^2r}{2\pi} G_{t,t_0}(r) R(r) \hat{W}(S_{t-t_0}r) , \]  

where we have introduced the Weyl operators:

\[ \hat{W}(r) = e^{i r^T \Omega r} = e^{i(q \hat{p} - p \hat{q})} , \]

with \( r^T = (q, p) \in \mathbb{R}^2 \) and \( \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), and \( R(r) = \text{Tr}[\rho \hat{W}(-r)] \) is related to the initial condition by:

\[ \rho_{t_0,t_0} = \rho = \int \frac{d^2r}{2\pi} R(r) \hat{W}(r) . \]

Because the Hamiltonian \( \hat{H} \) is at most quadratic and the matrix \( S_t \) in (12) is symplectic, one finds:

\[ \hat{U}_t \hat{W}(r) \hat{U}_t^\dagger = \hat{W}(S_t r) . \]

Direct insertion of (19) into (16) yields

\[ \partial_t G_{t,t_0}(r) = -\left[ r^T S_{t-t_0}^T C(t, t_0) S_{t-t_0} r \right] G_{t,t_0}(r) , \]

whence \( G_{t,t_0}(r) = \exp \left[ -\frac{1}{2} r^T g(t, t_0) r \right] \) with

\[ g(t, t_0) = 2 \int_{t_0}^{t} du S_{u-t_0}^T C(u, t_0) S_{u-t_0} \]

\[ = \int_{t_0}^{t} du \int_{t_0}^{t} dv S_{u-t_0}^T D(u, v) S_{v-t_0} . \]

Furthermore, since \( D(u, v) \) is of positive type, the matrix \( g(t, t_0) \) is positive definite and \( G_{t,t_0}(r) \) a real Gaussian function; the solution \( \Gamma_{t,t_0}[\rho] \) can then be cast in a continuous
Kraus-Stinespring decomposition which guarantees the complete positivity of the maps $\Gamma_{t,t_0}$. Let $G_{t,t_0}(r) = \int_{\mathbb{R}^2} d^2x \delta(x-r) G_{t,t_0}(x)$ with

$$\delta(x-r) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} d^2y e^{iy^T\Omega(x-r)}.$$ 

By inserting it into (19) and using $\hat{W}(x)\hat{W}(r)\hat{W}^\dagger(x) = e^{-ix^T\Omega r} \hat{W}(r)$, one rewrites

$$\Gamma_{t,t_0}[\rho] = \int_{\mathbb{R}^2} \frac{d^2y}{2\pi} F_{t,t_0}(y) \hat{U}_{t-t_0} \hat{W}(x) \rho \hat{W}^\dagger(x) \hat{U}_{t-t_0}^\dagger$$

with the Fourier transform

$$F_{t,t_0}(y) = \int_{\mathbb{R}^2} \frac{d^2x}{2\pi} e^{i y^T\Omega x} G_{t,t_0}(x),$$

also a real Gaussian, hence a positive function.

Using (19) one can study the composition properties of the maps $\Gamma_{t,t_0}$, since:

$$\Gamma_{t_2,t_1} \circ \Gamma_{t_1,t_0}[\rho] = \int \frac{d^2r}{2\pi} G_{t_2,t_1}(S_{t_1-t_0}r) G_{t_1,t_0}(r) R(r) \hat{W}(S_{t_2-t_0}r),$$

in order to satisfy the semigroup composition law $\Gamma_{t_2,t_1} \circ \Gamma_{t_1,t_0} = \Gamma_{t_2,t_0}$ one should have

$$G_{t_2,t_1}(S_{t_1-t_0}r) G_{t_1,t_0}(r) = G_{t_2,t_0}(r).$$

Using (22), one instead finds that

$$\left( \int_{t_0}^{t_2} \int_{t_1}^{t_2} + \int_{t_0}^{t_1} \int_{t_0}^{t_1} du \, dv \right) \left( \begin{array}{c} S_{u-t_0}^T D(u,v) S_{v-t_0} \end{array} \right)$$

$$\neq \int_{t_0}^{t_2} \int_{t_0}^{t_2} du \, dv \, S_{u-t_0}^T D(u,v) S_{v-t_0}.$$  

(25)

This fact remains true even when $D(s,u) = D(|s-u|)$ in which case from (22) we have

$$g(t,t_0) = \int_{t_0}^{t-t_0} du \int_{0}^{t-t_0} dv S_{u-t_0}^T D(u,v) S_{v-t_0}$$

and $\Gamma_{t,t_0} = \Gamma_{t-t_0,0}$.

Consider the master equation (16); if $t_0 = 0$ its solutions $\rho_{t,0} = \Gamma_{t,0}[\rho]$ propagate the initial state $\rho$ from $t_0 = 0$ to $t \geq 0$. Because of the above result, $\Gamma_{t,0} \neq \Gamma_{t,0} \circ \Gamma_{t,0}$. However, setting $t_0 = 0$ in (16) and searching a solution $\Lambda_{t,0}[\rho]$ in the form (19), one gets

$$\Lambda_{t,0}[\rho] = \int \frac{d^2r}{2\pi} L_{t,0}(r) R(r) \hat{W}(S_t r)$$

(26)
where $L_{t,t_0}(r) = \exp \left\{ -\frac{1}{2}r^T \ell(t, t_0) r \right\}$ with:

$$\ell(t, t_0) = \int_{t_0}^t du \ S_{u-t_0}^T C(u, 0) \ S_{u-t_0}$$

$$= \int_{t_0}^t du \int_0^u dv \ S_{u-t_0}^T D(u, v) \ S_{v-t_0} \cdot$$

(27)

The function $L_{t,t_0}(r)$ plays the role of $G_{t,t_0}(r)$ in (19) to which it reduces when $t_0 = 0$; that is $\Lambda_{t,0} = \Gamma_{t,0}$. Note however that, in contrast to $g(t, t_0)$ in (21), in $\ell(t, t_0)$ one integrates $C(u, 0)$, not $C(u, t_0)$, from $t_0$ to $t$. As a consequence, $\Gamma_{t,0} = \Lambda_{t,t_0} \circ \Gamma_{t_0,0}$; indeed,

$$\Lambda_{t,t_0} \circ \Gamma_{t_0,0}[\rho] = \int_{\mathbb{R}^2} \frac{d^2r}{2\pi} L_{t,t_0}(S_{t_0} r) L_{t_0,0}(r) R(r) \hat{W}(S_t r) \ ,$$

where now, unlike in (25),

$$S_{t_0}^T \ell(t, t_0) S_{t_0} + \ell(t_0, 0) = \left( \int_{t_0}^t \mathcal{A}_0 \right) du \ S_{u-t_0}^T D(u, v) \ S_v$$

$$= \int_0^t du \int_0^u dv \ S_{u-t_0}^T D(u, v) \ S_v = \ell(t, 0) \ .$$

However, contrary to the maps $\Lambda_{t,0}$ which, as we have seen, are CP, the maps $\Lambda_{t,t_0}$ cannot be CP as this would imply [9] the positive definiteness of the matrix $C(t, t_0)$ in (17). In fact, the maps $\Lambda_{t,t_0}$ are in general not even positive.

All these various possibilities can be seen in a concrete example; consider a free particle of unit mass, $\hat{H} = \hat{p}^2/2$, so that $S_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, and a diagonal noise with correlation matrix given by

$$D(t, u) = \frac{\gamma e^{-\gamma|t-u|}}{2} \begin{pmatrix} d_q & 0 \\ 0 & d_p \end{pmatrix} \ .$$

(29)

First suppose the noise couples only to the position operator: $d_q = 1, d_p = 0$; then, from (18),

$$C(t, t_0) = \begin{pmatrix} 1 - \frac{e^{-\gamma(t-t_0)}}{e^{-\gamma(t-t_0)} + 1} & \frac{e^{-\gamma(t-t_0)}(t-t_0) - 1}{2\gamma} \\ \frac{e^{-\gamma(t-t_0)}(t-t_0) - 1}{2\gamma} & 0 \end{pmatrix} \ .$$

(30)

has a negative eigenvalue for all $t > t_0 \geq 0$. In spite of the non-positivity of the Kossakowski matrix in (18), the maps $\Gamma_{t,t_0}$ in (23) are nevertheless CP for all $0 \leq t_0 \leq t$.

We consider, as initial condition at $t_0$, a Gaussian state $\rho_\sigma$ with covariance matrix (CM) $\sigma \gamma$ and zero first moments, $\text{Tr} \left[ \rho_\sigma \hat{W}(-r) \right] = \exp \left\{ -\frac{1}{2}r^T (\Omega \Omega^T) r \right\}$. Using (19), $\Gamma_{t,t_0}$ maps $\rho_\sigma$ to the Gaussian state $\text{Tr} \left[ \Gamma_{t,t_0}[\rho_\sigma] \hat{W}(r) \right] = \exp \left\{ -\frac{1}{2}r^T \Omega^T \sigma_{t,t_0} \Omega r \right\}$, where $\sigma_{t,t_0} = S_{t-t_0} \sigma S_{t-t_0}^T + \bar{g}(t, t_0)$ with

$$\bar{g}(t, t_0) = \int_{t_0}^t du \ Omega^T S_{u-t_0}^T C(u, t_0) S_{u-t_0} \Omega \ .$$

(31)
Instead, if the same initial condition is taken for the maps $\Lambda_{t,t_0}$, the matrix $\tilde{g}(t, t_0)$ is to be substituted by

$$\tilde{\ell}(t, t_0) = \int_{t_0}^{t} du \Omega^T S_{u-t} C(u, 0) S_{u-t} \Omega.$$  \hspace{1cm} (32)

If we choose $\sigma = S_{t_0-t}\sigma_0 S_{t_0-t}^T$ and expand $\sigma_{t,t_0} = \sigma_0 + \tilde{\ell}(t, t_0)$ to first order about $t_0$, we have:

$$\sigma_{t,t_0} \simeq \sigma_0 + (t - t_0)\Omega^T C(t_0, 0) \Omega,$$  \hspace{1cm} (33)

where $C(t_0, 0)$ is calculated from Eq. (30). Now, the second matrix at the l.h.s. is real symmetric and has one positive and one negative eigenvalue, $\lambda \geq 0$ and $-\mu < 0$; let $V$ be the symplectic, orthogonal matrix which diagonalizes it. Then, choosing an initial state with CM diagonal in the same basis, i.e., $\sigma_0 = \text{Diag}[\sigma_{qq}, \sigma_{pp}]$, such that $\sigma_0 + \frac{i}{2}\Omega \geq 0$ (positivity of the initial state), one gets:

$$\sigma_{t,t_0} \simeq V^T \begin{pmatrix} \sigma_{qq} + \lambda(t - t_0) & 0 \\ 0 & \sigma_{pp} - \mu(t - t_0) \end{pmatrix} V,$$

and a sufficiently small $\sigma_{pp}$ would yield a non positive-definite CM $\sigma_{t,t_0}$, thus exhibiting the non-positivity of the map $\Lambda_{t,t_0}$. The non-positive preserving character of $\Lambda_{t,t_0}$ is exposed by very specific states; on other states as, for instance, on all those of the form $\Gamma_{t_0} \rho [\rho]$ it acts perfectly well for $\Lambda_{t,t_0} \circ \Gamma_{t_0} = \Gamma_{t_0}$. In addition, starting from $t_0 = 0$, $\Lambda_{t,0} = \Gamma_{t,0}$ is CP.

Therefore, in this case the master equation (16) generates a non-Markovian dynamics both according to the criterion (4), since the generator $L_{t,t_0}$ depends on the initial time $t_0$ and also according to the other criterion (5). In fact, the family of maps $\Gamma_{t,0}$ is non-divisible for $\Lambda_{t,t_0}$ is uniquely defined and non-positive.

Since $\Lambda_{t,t_0}$ is not (completely) positive, certain quantities that exhibit monotonic behavior under CP maps fail to do so when evolving the system from time $t_0$ to time $t$. One of such quantities is the fidelity $\mathcal{F}(t) = \mathcal{F}(\Gamma_{t,0}[\rho_1], \Gamma_{t,0}[\rho_2])$ of two states $\rho_1$ and $\rho_2$ evolving in time according to $\Gamma_{t,0}$. While $\mathcal{F}(t) \geq \mathcal{F}(0)$ for all $t \geq 0$, $\mathcal{F}(t_0 + t)$ may become smaller than $\mathcal{F}(t_0)$ for some $t, t_0 > 0$. This is showed in Fig. 1 for two Gaussian states with zero first moments and “squeezed” CM. As one may expect, the effect disappears when $\gamma$ increases towards the Markovian limit.

On the other hand, if in (9), the noise affects the particle momentum only, namely if $d_q = 0, d_p = 1$, then, from (29),

$$C(t, t_0) = (1 - e^{-\gamma(t-t_0)}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \hspace{1cm} (34)$$

is positive definite. It follows that the intertwining map $\Lambda_{t,t_0}$ is CP, whence the family of maps $\Gamma_{t,0}$ is divisible and Markovian according to the criterion (5). However, it is non-Markovian according to the other criterion (4). Indeed, the generator resulting from (9) depends on the starting time $t_0$. 

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Figure 1: Plot of the time evolution of the fidelity $\mathcal{F}$ between two Gaussian states $\rho_k$, $k = 1, 2$, with zero first moments and CMs $\sigma_k = {\frac{1}{2}} \text{Diag} [\exp(2r_k), \exp(-2r_k)]$, with $r_1 = -r_2 = 1.5$, evolving under the map $\Gamma_{t,0}$ for different values of $\gamma$. The inset refers to the time derivative of the fidelity. The non-monotonic behavior denotes non-Markovian evolution [15]; note that as $\gamma$ increases, $\mathcal{F}$ becomes monotonic.

In conclusion, the analysis of above examples indicates that the criterion identifying non-Markovianity with the explicit dependence of the generator $\mathbb{L}_{t,t_0}$ on the starting time $t_0$ appears stronger than the criterion based on the non-divisibility of the maps $\Gamma_{t,0}$. Indeed, on one hand, we have provided a case where the map $\Gamma_{t,0}$ is divisible, yet the generator of $\Gamma_{t,t_0}$ explicitly depends on the initial time $t_0$; on the other hand, a Markovian evolution according to the first criterion readily implies the semigroup composition law, i.e., (4) with the equality sign, hence divisibility of $\Gamma_{t,0}$. Nevertheless, the non-divisibility criterion is the only one at disposal when one is presented just with the family of maps $\Gamma_{t,0}$: in such a case, one may reconstruct the generator $\mathbb{L}_{t,0}$ starting from $t_0 = 0$, but, in general, no information is available on the full generator $\mathbb{L}_{t,t_0}$ at $t_0 > 0$.

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