THE KAPLANSKY CONDITION AND RINGS
OF ALMOST STABLE RANGE 1

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Abstract. We present some variants of the Kaplansky condition for a K-
Hermite ring $R$ to be an elementary divisor ring. For example, a commutative
K-Hermite ring $R$ is an EDR iff for any elements $x, y, z \in R$ such that $(x, y) = R$
there exists an element $\lambda \in R$ such that $x + \lambda y = uv$, where $(u, z) = (v, 1 - z) = R$.

We present an example of a Bézout domain that is an elementary divisor
ring but does not have almost stable range 1, thus answering a question of
Warren Wm. McGovern.

1. Introduction

First we recall some basic definitions and known results.

All rings here are commutative with unity. A ring $R$ is Bézout if each finitely
generated ideal of $R$ is principal.

Two rectangular matrices $A$ and $B$ in $M_{m,n}(R)$ are equivalent if there exist
invertible matrices $P \in M_{m,m}(R)$ and $Q \in M_{n,n}(R)$ such that $B = PAQ$.

The ring $R$ is K-Hermite if every rectangular matrix $A$ over $R$ is equivalent to
an upper or a lower triangular matrix (following [9, Appendix to §4] we use the
term ‘K-Hermite’ rather than ‘Hermite’ as in [8]). From [8] it follows that this
definition is equivalent to the definition given there. See also [5, Theorem 3]: by
this theorem, a ring is K-Hermite iff for every two elements $a, b \in R$, there are
elements $a_1, b_1, d \in R$ such that $(a, b) = (a_1d + b_1d)$ and $(a_1, b_1) = R$. Parentheses
are used to denote the ideal generated by the specified elements.

A ring $R$ is an elementary divisor ring (EDR) iff every rectangular matrix $A$
over $R$ is equivalent to a diagonal matrix. It follows from [8] that this definition is
equivalent to the definition given there.

An EDR is K-Hermite, and a K-Hermite ring is Bézout. An integral domain is
Bézout iff it is K-Hermite.

By [4, Theorem 6] a ring $R$ is an EDR iff it satisfies the following two conditions:

1) $R$ is K-Hermite;
2) $R$ satisfies Kaplansky’s condition (see §2, condition (K) below).
By [4] Example 4.11, (1) $\Rightarrow$ (2). The question in [6] whether a Bézout domain is an EDR (equivalently, whether it satisfies Kaplansky’s condition) is still open. On the other hand, (2) $\Rightarrow$ (1) (Remark 2.1 below).

In section 2 we elaborate on the Kaplansky condition.

A row $[r_1, \ldots, r_n]$ over a ring $R$ is unimodular if the elements $r_1, \ldots, r_n$ generate the ideal $R$. The stable range $sr(R)$ of a ring $R$ is the least integer $n \geq 1$ (if it exists) such that for any unimodular row $[r_1, ..., r_{n+1}]$ over $R$, there exist $t_1, ..., t_n \in R$ such that the row

$$[r_1 + t_1r_{n+1}, ..., r_n + t_nr_{n+1}]$$

is unimodular (see comments on [9, Theorem 5.2, Ch. VIII]). For background on stable range see [1, §3, Ch. V].

The ring $R$ has almost stable range 1 if every proper homomorphic image of $R$ has stable range 1 (see [10]). By [10] Theorem 3.7 a Bézout ring with almost stable range 1 is an EDR. We elaborate on the almost stable range 1 condition in §3. In particular, we present an elementary divisor domain (so Bézout) that does not have almost stable range 1, thus answering the question of Warren Wm. McGovern in [10] (Example 3.3). By Remark 3.2 below, a ring of stable range 1 is of almost stable range 1. On the other hand, $\mathbb{Z}$ is of almost stable range 1 but is not of stable range 1. Indeed, the stable range of $\mathbb{Z}$ is 2: clearly, there is no integer $m$ such that $2 + 5m = \pm 1$. Thus $sr \mathbb{Z} > 1$. On the other hand, the stable range of any Bézout domain is $\leq 2$; hence $sr \mathbb{Z} = 2$. More generally, the stable range of any K-Hermite ring is $\leq 2$ [11, Proposition 8]. Also by [12, Theorem 1], a Bézout ring is K-Hermite iff it is of stable range $\leq 2$.

For general background see [8], [3, §6, Ch. 3] and [10].

2. ON THE KAPLANSKY CONDITION

By [5] a K-Hermite ring $R$ is an elementary divisor ring iff it satisfies Kaplansky’s condition (see [8, Theorem 5.2]):

(K) For any three elements $a, b, c$ in $R$ that generate the ideal $R$,

there exist elements $p, q \in R$ so that $(pa, pb + qc) = R$.

Remark 2.1. A local ring $R$ is of stable range 1; thus $R$ satisfies Kaplansky’s condition with $p = 1$. If $R$ is also a Noetherian domain, then $R$ is K-Hermite iff $R$ is a principal ideal ring. Hence a Noetherian local domain that is not a principal ideal ring is of stable range 1 but is not K-Hermite.

In the proof of Lemma 2.3 below, we will use the following well-known fact:

Remark 2.2. Let $R$ be a ring, let $A$ be a matrix in $M_{m,n}(R)$, let $r$ be a row in $M_{1,n}(R)$, and let $1 \leq k \leq n$. Then $r$ belongs to the submodule of $R^n$ generated by the rows of the matrix $A$ iff there exists a matrix $C \in M_{k,m}(R)$ such that $r$ is the first row of the matrix $CA$.

Lemma 2.3. Let $A$ be a $2 \times 2$-matrix over a ring $R$, and let $u$ be a unimodular row of length 2 over $R$. Then $u$ belongs to the submodule of $R^2$ generated by the rows of $A$ $\iff$ there exists an invertible matrix $P$ so that $u$ is the first row of $PA$.

Proof. ($\Rightarrow$) : By Remark 2.2 there exists a 2-row $r$ over $R$ so that $u = rA$. Since the row $u$ is unimodular, the row $r$ is also unimodular. Since $r$ is unimodular of
length 2, there exists an invertible matrix \( P \) with its first row equal to \( r \). Thus \( u \) is the first row of the matrix \( PA \).

\((\Leftarrow\Rightarrow)\) : This follows from Remark 2.2.

**Lemma 2.4.** Let \( A \) be a \( 2 \times 2 \) matrix over a ring \( R \) so that its entries generate the ideal \( R \). Then \( A \) is equivalent to a diagonal matrix \( \iff \) the submodule of \( R^2 \) generated by the rows of \( A \) contains a unimodular row.

In this case \( A \) is equivalent to a matrix of the form \( \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} \).

**Proof.** \((\Rightarrow)\) : By assumption, \( A \) is equivalent to a diagonal matrix \( D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \), where \( d_1, d_2 \in R \). Since the entries of \( A \) generate the ideal \( R \), \( d_1, d_2 \) also generate the ideal \( R \). The sum of the rows of \( D \), namely \([d_1, d_2]\), is unimodular.

\((\Leftarrow)\) : By Lemma 2.3, the matrix \( A \) is equivalent over \( R \) to a matrix \( B \) with first row unimodular. Hence the submodule generated by the columns of \( B \) contains a column of the form \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). By Lemma 2.3 again (for columns), we obtain that \( A \) is equivalent to a matrix \( \begin{pmatrix} 1 & * \\ * & * \end{pmatrix} \) multiplied by \( r \) from its second column and by a similar elementary row transformation, we obtain a diagonal matrix of the form \( \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} \).

**Theorem 2.5** (see [8, Theorem 5.2] and [5, Corollary 5]). Let \( R \) be a ring. Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) a triangular \( 2 \times 2 \)-matrix over \( R \) so that \( (a, b, c) = R \). Then \( A \) is equivalent to a diagonal matrix over \( R \iff \) there exist elements \( p, q \) in \( R \) so that \( (pa, pb + qc) = R \).

**Proof.** Since \( p[a, b] + q[0, c] = [pa, pb + qc] \) for any elements \( p, q \in R \), the theorem follows from Lemma 2.4.

**Remark 2.6.** Let \( R \) be any ring. If Kaplansky’s condition \( (pb + qc, pa) = R \) holds for elements \( a, b, c, p, q \in R \), then

\[(pb + qc, a) = (p, c) = R.

Indeed, Kaplansky’s condition implies that

\[(pb + qc, a) = (pb + qc, p) = R,

so \((p, c) = R \). Cf. the next proposition.

**Proposition 2.7.** Let \( R \) be a \( K \)-Hermite ring, and let \( a, b \) and \( c \) be elements of \( R \) that generate the ideal \( R \). Then the following four conditions are equivalent:

1. The matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is equivalent to a diagonal matrix.
2. There exist elements \( p, q \) in \( R \) so that \( (pa, pb + qc) = R \).
3. There exist elements \( p \) and \( q \) in \( R \) so that \( (pb + qc, a) = (p, c) = R \).
4. For some elements \( \lambda, u, v \in R \) we have \( b + \lambda c = uv \), and \((u, a) = (v, c) = R \).

Moreover, in (4) we may choose the elements \( u \) and \( v \) such that \((u, v) = R \).

**Proof.** \((1) \iff (2)\) : This follows from Theorem 2.5

\((2) \implies (4)\) : Since \( (pa, pb + qc) = R \) we obtain \( R = (p, pb + qc) = (p, qc) \), so \((p, pb + qc) = R \). Let \( v \) be an element of \( R \) so that

\[vp \equiv 1 \pmod{(pb + qc)c};

thus \( vp \equiv 1 \pmod{c} \). We have \( v(pb + qc) \equiv b \pmod{c} \), so \( v(pb + qc) = b + \lambda c \) for some element \( \lambda \in R \). Hence \( b + \lambda c = uv \), where \( u = pb + qc \); thus \((u, a) = (v, c) = (u, v) = R \).
Thus (3) holds.

\[(3) \implies (2) : \text{Since } R \text{ is a K-Hermite ring, we may write } (d) = (p,q) \text{ and } d = p_1 p + q_1 q \text{ with } (p_1,q_1) = R. \text{ Hence } \]

\[(p_1,p_1 b + q_1 c) = (p_1,q_1 c) = R, \]

so \((p_1 a,p_1 b + q_1 c) = (p_1,c) = R. \text{ Condition (2) holds with } p \text{ and } q \text{ replaced by } p_1 \text{ and } q_1, \text{ respectively.} \]

\[\square \]

In the proof of Proposition 2.7, we used the assumption that \( R \) is K-Hermite just for the implication \((3) \implies (2)\).

Remark 2.8. If \( R \) is a Bézout domain, then the following condition is equivalent to the conditions of Proposition 2.7.

\[(*) \text{ For some elements } \lambda,a_1,c_1 \in R \text{ we have } b + \lambda c \mid (1 - a_1 a)(1 - c_1 c). \]

Indeed, assume condition \((*)\). Let \( u \in R \) so that

\[(u) = (b + \lambda c, 1 - a_1 a); \]

thus \((u,a) = R \) and \( \frac{b + \lambda c}{u} \mid \left(1 - \frac{a_1 a}{u}\right)(1 - c_1 c). \) Since \( \left(\frac{b + \lambda c}{u}, \frac{1 - a_1 a}{u}\right) = R, \) we see that \( v := \frac{b + \lambda c}{u} \) divides \( 1 - c_1 c, \) so \((v,c) = R. \) Thus condition \((*)\) implies condition \((4)\) of Proposition 2.7. The converse implication is obvious. \[\square\]

Since a K-Hermite ring is an EDR iff each matrix of the form \( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \) with \((a,b,c) = (1)\) has a diagonal reduction \cite{5}, Proposition 2.7 provides necessary and sufficient conditions for a K-Hermite ring to be an EDR. We present an additional condition in the next proposition.

Theorem 2.9. Let \( R \) be a K-Hermite ring. The following two conditions are equivalent:

\[
(1) \text{ } R \text{ is an elementary divisor ring.} \\
(2) \text{ } \text{For any elements } x,y,z \in R \text{ such that } (x,y) = R, \text{ there exists an element } \lambda \in R \text{ such that } x + \lambda y = uv, \text{ where } (u,z) = (v,1 - z) = R. 

Moreover, the elements } u \text{ and } v \text{ can be chosen such that } (u,v) = R. \]

Proof. \((1) \implies (2) \) [including the requirement that \((u,v) = (1)\): We apply condition \((4)\) of Proposition 2.7 to the elements \( a = z, b = x, c = y(1 - z) \).

\[(2) \implies (1) : \text{ We apply condition } (4) \text{ of Proposition } 2.7 \text{ to the elements } a = z, b = x, c = y(1 - z). \]

Let \((a,b,c) = R. \text{ Let } (d) = (b,c); \text{ thus } (d,a) = (b,c,a) = R. \text{ Hence } a \mid 1 - d_1 d \text{ for some element } d_1 \in R. \text{ Also } b = b_1 d, c = c_1 d, \text{ where } (b_1,c_1) = R. \text{ We apply condition } (2) \text{ of the present proposition to the elements } \]

\[x = b_1, y = c_1, z = d_1 d. \]

Thus there are elements \( \lambda_1, u_1, v \in R \) so that \( b_1 + \lambda_1 c_1 = u_1 v, \text{ where } (u_1,1 - d_1 d) = (v,d_1 d) = R. \text{ Let } u = du_1; \text{ thus } (u,a) = 1. \text{ Let } \lambda = \lambda_1 d. \text{ Hence } b + \lambda c = d(b_1 + \lambda_1 c_1) = uv \text{ and } (u,a) = R. \text{ We have } (v,c) = (v,dc_1) = (v,c_1) \text{ since } (v,d) = R. \text{ Since } v \text{ divides } b_1 + \lambda_1 c_1, \text{ it follows that } (v,c_1) \mid b_1, \text{ so } (v,c_1) = R. \text{ Thus } (v,c) = R, \text{ as required.} \]

\[\square\]
Proposition 2.10. Let $R$ be a Bézout domain. The following two conditions are equivalent:

1. $R$ is an elementary divisor ring.
2. For any nonzero elements $x, y, z \in R$, there exist elements $\lambda, a, b \in R$ such that $x + \lambda y \mid (1 - az)(1 - b(1 - z))$ in $R$.

Proof. (1) $\implies$ (2): Let $(d) = (x, y)$; thus $\frac{x}{d}$ and $\frac{y}{d}$ are comaximal. By Theorem 2.9 there are elements $\lambda, a, b \in R$ so that $(\frac{x}{d} + \lambda \frac{y}{d}) \mid (1 - az)(1 - b(1 - z))$. Hence $x + \lambda y \mid d(1 - az)(1 - b(1 - z))$, so $x + \lambda y \mid y(1 - az)(1 - b(1 - z))$.

(2) $\implies$ (1): Let $x_0$ and $y_0$ be comaximal elements in $R$, and let $z \in R$. Thus $(x_0 + \lambda y_0) \mid y_0(1 - az)(1 - b(1 - z))$ for some elements $\lambda, a, b \in R$. Since the elements $x_0 + \lambda y_0$ and $y_0$ are comaximal, we obtain that $(x_0 + \lambda y_0) \mid (1 - az)(1 - b(1 - z))$, so $R$ is an EDR by Remark 2.8.

3. On rings of almost stable range 1

Proposition 3.1. Let $R$ be any ring. The following conditions are equivalent:

1. $R$ is of almost stable range 1.
2. For each nonzero element $z \in R$, the ring $R/zR$ is of stable range 1.
3. For every three elements $x, y, z \in R$ such that $(x, y) = R$ and $z \neq 0$, there exists an element $\lambda \in R$ such that $(x + \lambda y, z) = R$.

Proof (Cf. [1] Proposition 3.2, Ch. V)).

(1) $\implies$ (2) $\implies$ (3): Clear.

(3) $\implies$ (1): Let $I$ be a nonzero ideal of $R$ and let $z$ be a nonzero element in $I$. Let $x + I, y + I$ be two comaximal elements in $R/I$. Hence there exist elements $r, s \in R$ such that $1 - rx - sy \in I$. By assumption, there exists an element $\lambda \in R$ such that $(x + \lambda(1 - rz), z)R = R$. Thus $x + \lambda sy$ is invertible modulo the ideal $I$. It follows that $R/I$ is of stable range 1, so $R$ is almost of stable range 1.

Remark 3.2. The implication (3) $\implies$ (1) in the previous proposition is clear since if $T$ is a homomorphic image of a ring $R$ with finite stable range, then $sr(T) \leq sr(R)$ [1] Proposition 3.2, Ch. V], although this fact was not used explicitly but rather its proof (in the above proof of the implication (2) $\implies$ (3) we have $sr(R/I) \leq sr(R/(z)) = 1$). This fact implies that if $R$ is an arbitrary ring of stable range 1, then $R$ is of almost stable range 1, thus answering the question in [10] Remark 3.3. See also [10] Proposition 3.2.

As we have seen in [2] the stable range 1 property implies Kaplansky’s condition for an arbitrary ring. The converse is false since even if $R$ is an elementary divisor domain so that $R$ satisfies Kaplansky’s condition, $R$ does not necessarily have almost stable range 1.

Example 3.3. An elementary divisor domain (and so Bézout) that does not have almost stable range 1 (this example answers the question in [10] Remark 4.7)).

We use a well-known example of a Bézout domain, namely, $R = \mathbb{Z} + X\mathbb{Q}[X]$ (for a general theorem on pullbacks of Bézout domains, see [7] Theorem 1.9). $R$ is an elementary divisor ring by [2] Theorem 4.61. However, $R/X\mathbb{Q}[X]$ is isomorphic to $\mathbb{Z}$ and $sr \mathbb{Z} = 2$. Hence $R$ does not have almost stable range 1.
We conjecture that a Bézout domain that is a pullback of type □ (as defined in [7]) of elementary divisor domains is again an EDR. In this case the conditions of [7, Theorem 1.9] must be satisfied. If this conjecture proves to be false, this will yield a negative answer to the question in [6] as to whether a Bézout domain is an EDR.

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