Spontaneous symmetry breaking and Nambu-Goldstone modes in dissipative systems

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Abstract

We discuss spontaneous breaking of internal symmetry and its Nambu-Goldstone (NG) modes in dissipative systems. We find that there exist two types of NG modes in dissipative systems corresponding to type-A and type-B NG modes in Hamiltonian systems. To demonstrate the symmetry breaking, we consider a $O(N)$ scalar model obeying a Fokker-Planck equation. We show that the type-A NG modes in the dissipative system are diffusive modes, while they are propagating modes in Hamiltonian systems. We point out that this difference is caused by the existence of two types of Noether charges, $Q^\alpha_R$ and $Q^\alpha_A$: $Q^\alpha_R$ are the symmetry generators of Hamiltonian systems, which are not conserved in dissipative systems. $Q^\alpha_A$ are the symmetry generators of dissipative systems described by a Fokker-Planck equation, which are conserved. We find that the NG modes are propagating modes if $Q^\alpha_R$ are conserved, while those are diffusive modes if they are not conserved. We also consider a $SU(2) \times U(1)$ scalar model with a chemical potential to discuss the type-B NG modes. We show that the type-B NG modes have the different dispersion relation from those in the Hamiltonian systems.
I. INTRODUCTION

Spontaneous symmetry breaking (SSB) is a universal phenomenon and widely observed at various scales in nature, e.g., our vacuum where the electroweak and chiral symmetries are spontaneously broken, superconductors, ferromagnets, solid crystals, and so on [1–4]. Those well known examples are in Hamiltonian systems.

As is the case with the Hamiltonian systems, it is known that the SSB occurs even in dissipative systems such as reaction diffusion [5] and active matter [6, 7] systems. The reaction diffusion system has translational symmetry. The symmetry is spontaneously broken by a pattern structure in space [5] such as the Turing pattern, which is considered as the most basic pattern formation in biology [8]. In the active hydrodynamics, which describes collective motion of biological organisms such as flocks of birds, the rotational symmetry is spontaneously broken. Toner and Tu showed that the Nambu-Goldstone (NG) modes of the active hydrodynamics in $d$ dimensions are given as $d - 2$ diffusive shear modes and a pair of propagating sound modes [6, 7]. Recently, the propagating sound mode seems to be experimentally observed in the situation where the flocks collectively turns [9]. Attanasi et al. also phenomenologically discussed the type of the NG mode, propagating or diffusive modes, based on the SSB of the rotational symmetry and conservation law.

In Hamiltonian systems, NG modes are classified into two types: type-A and type-B [10–13]. When a global symmetry $G$ is broken into its subgroup $H$, the numbers of type-A ($N_A$) and type-B ($N_B$) NG modes are expressed as

$$N_A = N_{BS} - \text{rank} \langle [iQ^\alpha, Q^\beta] \rangle, \quad N_B = \frac{1}{2} \text{rank} \langle [iQ^\alpha, Q^\beta] \rangle,$$

(1)

where $N_{BS} = \text{dim}(G/H)$ is the number of broken symmetries, and $Q^\alpha$ are the Noether charges (generators) of $G$. The total number of NG modes is $N_A + N_B = N_{BS} - \text{rank} \langle [iQ^\alpha, Q^\beta] \rangle / 2$ [10–14]. For the spontaneous breaking of an internal symmetry, both type-A and type-B NG modes are propagating modes. Type-A and type-B NG modes have the linear and quadratic dispersion in momentum, respectively.

Compared to those in Hamiltonian systems, the relation between the SSB, NG modes and their dispersion relations in dissipative systems is not fully understood. We cannot naively apply the above argument to the dissipative systems because it is based on the conservation law. In the dissipative systems, the symmetry does not mean that the generators of the
symmetry are the conserved quantities. For example, a Brownian particle obeying the
Fokker-Planck equation has rotational symmetry; however, the angular momentum is not
conserved due to the dissipation and noise. Nevertheless, we know that the SSB occurs
and there appear gapless modes, in the absence of the conservation law. In the above
example of the active hydrodynamics, the rotational symmetry is spontaneous broken by
the expectation value of the nonvanishing velocity field, where the angular momentum is
not conserved.

To discuss the symmetry breaking in the dissipative systems, it is useful to consider path
integral formulation of Langevin or Fokker Planck equation. The formalism is called the
Martin-Siggia-Rose (MSR) formalism [15, 16], which is successful in the analysis of dynamic
critical phenomena [17–20]. Even in the dissipative system, we can construct the Noether
charge $Q_{\alpha}^A$, which is the conserved quantity, as the generator of the symmetry in the MSR
formalism. This Noether charge is a different charge from that in the Hamiltonian system.
We refer the Noether charge in the Hamiltonian system to $Q_{\alpha}^R$, which is not conserved in
dissipative systems.

In this paper, we show that the SSB in dissipative systems can be discussed by $Q_{\alpha}^A$
instead of $Q_{\alpha}^R$. To this end, we will consider a model with $O(N)$ scalar fields $\phi^A_R(t, \mathbf{x})$.
When $O(N)$ symmetry is spontaneous broken into $O(N-1)$, nonvanishing order parameters
$\langle [iQ_{\alpha}^A, \phi^A_R(t, \mathbf{x})] \rangle$ appear. In this case, the NG modes corresponding to type-A NG modes
become diffusive modes. The existence of diffusive modes is the characteristic feature in
the dissipative systems because the NG modes associated with spontaneous breaking of an
internal symmetry in the Hamiltonian systems are propagating modes. We also consider a
$SU(2) \times U(1)$ scalar model with a chemical potential. We find that a type-B NG appears in
the dissipative system. The mode has the quadratic dispersion as in the Hamiltonian system,
but the damping rate has different order in momentum. In both examples, the absence of
the conservation of $Q_{\alpha}^R$ is essential to determine the patterns of dispersion relations. We
nonperturbatively establish these results using the Ward-Takahashi identity for $Q_{\alpha}^A$ and $Q_{\alpha}^R$.

This paper is organized as follows. In Sec. II, to see symmetry in dissipative systems,
we consider rotational symmetry of Brownian motion as a simple example. We discuss
the mathematical similarity between the Fokker-Planck and the Schrödinger equations. We
also give the MSR formalism for readers who are unfamiliar with. In Sec. III, we consider
$O(N)$ and $SU(2) \times U(1)$ models to discuss the spontaneous breaking of internal symmetries

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and their NG modes in dissipative systems. In Sec. IV, we establish the results in Sec. III in the model independent way using the Ward-Takahashi identity. Section V is devoted to the summary and discussion. In Appendix A, we discuss the action in the real-time formalism, which has two types of Noether charges, $Q^\alpha_R$ and $Q^\alpha_A$. The action reduces to that in a dissipative system by coupling the system to the environment that violates the $Q^\alpha_R$ symmetry.

II. ROTATIONAL SYMMETRY IN DISSIPATIVE SYSTEMS AND MARTIN-SIGGIA-ROSE FORMALISM

We briefly review that symmetry of a Langevin equation can be discussed as in the operator and path integral formalism in quantum mechanics [21–25]. To this end, we consider the rotational symmetry of Brownian motion. The Langevin equation for the Brownian particle $\mathbf{x}(t)$ is given by

$$\frac{d}{dt} \mathbf{u}(t) = -\gamma \mathbf{u}(t) + \mathbf{\xi}(t),$$  \hspace{1cm} (2)

where $\mathbf{u}(t) = d\mathbf{x}(t)/dt$ is the velocity, $\gamma$ the friction coefficient and $\mathbf{\xi}(t)$ the random noise that satisfies the fluctuation-dissipation relation,

$$\langle \mathbf{\xi}_i(t)\mathbf{\xi}_j(t') \rangle = 2\gamma T \delta_{ij} \delta(t - t').$$  \hspace{1cm} (3)

Here, $\langle ... \rangle$ represents the average over the noise, and $T$ is the temperature of a heat bath. In this section, we choose the mass of the Brownian particle unity without loss of generality.

If $\mathbf{\xi}$ and $\gamma$ vanish, the system reduces to the Hamiltonian system, and the angular momentum $\mathbf{L}_R = \mathbf{x} \times \mathbf{u}$ is conserved. This results from the rotational symmetry of the equation of motion, $\mathbf{x} \rightarrow \mathbf{x}' = R \mathbf{x}$ and $\mathbf{u} \rightarrow \mathbf{u}' = R \mathbf{u}$ with a rotation matrix $R$. The angular momentum plays a role of the generator of rotational symmetry, $\{x_i, L_{Rj}\}_{\text{PB}} = \epsilon_{ijk} x_k$, where $\epsilon_{ijk}$ is the Levi-Civita tensor, and $\{..., ...\}_{\text{PB}}$ represents the Poisson bracket, $\{x_i, u_j\}_{\text{PB}} = \delta_{ij}$. In contrast, when $\mathbf{\xi}$ and $\gamma$ exist, the angular momentum is no longer conserved, $d\mathbf{L}_R/dt = -\gamma \mathbf{x} \times \mathbf{u} + \mathbf{x} \times \mathbf{\xi} \neq 0$ due to the friction and noise. However, this does not mean that the absence of the rotational symmetry. In fact, Eqs. (2) and (3) are still rotationally symmetric under $\mathbf{u} \rightarrow \mathbf{u}' = R \mathbf{u}$, $\mathbf{\xi} \rightarrow \mathbf{\xi}' = R \mathbf{\xi}$. As we will see in the following, this rotational symmetry implies that the existence of another conserved quantity.
For this purpose, it is useful to introduce the probability distribution of the velocity,

\[ P(v, t) \equiv \langle \delta^{(3)}(u(t) - v) \rangle. \]  

(4)

The time evolution of \( P(v, t) \) obeys the Fokker-Planck equation,

\[ \partial_t P(v, t) = \left( \gamma T \frac{\partial^2}{\partial v^2} + \gamma \frac{\partial}{\partial v} v \right) P(v, t). \]  

(5)

A point is here that we can regard the Fokker-Planck equation as a Schrödinger equation [26]. If we rewrite

\[ v \rightarrow q, \quad \frac{\partial}{\partial v} \rightarrow i\mathbf{p}, \]  

(6)

which naturally satisfy the commutation relation \([q_i, p_j] = i\delta_{ij}\), the Fokker-Planck equation is expressed as

\[ \partial_t |P(t)\rangle = -H|P(t)\rangle \]  

with the Hamiltonian,

\[ H = \gamma T \mathbf{p}^2 - i\gamma \mathbf{p} \cdot \mathbf{q}. \]  

(8)

Here, \(|P(t)\rangle\) is the state vector, and \(H\) is the Fokker-Planck Hamiltonian. The Fokker-Planck equation (5) is identified as the coordinate representation of Eq. (7). The important difference from quantum mechanics is that the Hamiltonian (8) is not hermitian, and thus, the left and right eigenstates are generally different.

Obviously, the Hamiltonian \(H\) is invariant under \( \mathbf{x} \rightarrow R\mathbf{x} \) and \( \mathbf{p} \rightarrow R\mathbf{p} \), so that we find the “angular momentum” as the Noether charge of the rotational symmetry,

\[ L_A = \mathbf{q} \times \mathbf{p}, \]  

(9)

which commutes with the Fokker-Planck Hamiltonian \(H\). We emphasize that \(L_A\) is not the actual angular momentum because \(L_A = -i\mathbf{v} \times (\partial / \partial \mathbf{v}) \neq L_R\). Nevertheless, \(L_A\) plays a role of the generator of rotational symmetry; for example, the commutation relation between \(L_{Ai}\) and \(q_j\) gives

\[ [L_{Ai}, q_j] = i\epsilon_{ijk}q_k. \]  

(10)
We can also consider the rotational symmetry of states. Let us consider the following state $|\psi(q_0)\rangle$ as an example:

$$
\langle q|\psi(q_0)\rangle = N \exp\left[-\frac{1}{2}(q - q_0)^2\right],
$$

(11)

where $|q\rangle$ is the left eigenstate of $q$ and $N$ is the normalization constant. By operating $L_A$ to $|\psi(q_0)\rangle$, we obtain

$$
\langle q|L_A|\psi(q_0)\rangle = -iN(q \times q_0) \exp\left[-\frac{1}{2}(q - q_0)^2\right].
$$

(12)

That is, we have

$$
L_A|\psi(q_0)\rangle = 0 \text{ for the symmetric state,}

\neq 0 \text{ for the non-symmetric state,}
$$

as in quantum mechanics. Thus, we can map the Langevin equation to that in “quantum mechanics” and define the “Hamiltonian” and the “Noether charge.” If the stationary solution of the Fokker-Planck equation (7) is a non-symmetric state, the symmetry is spontaneous broken.

By using these, we can discuss the symmetry of dissipative systems and its spontaneous breaking as in quantum field theory.

It is also useful to introduce the path-integral representation of the generating functional $Z[J]$ called the Martin-Siggia-Rose (MSR) formalism [15, 26],

$$
Z[J] \equiv \langle e^{i\int dt J(t) \cdot q(t)} \rangle
$$

$$
= \int Dq Dp e^{iS[q,p] + i\int dt J(t) \cdot q(t)},
$$

(13)

where $J$ represents the source. The action $S[q,p]$ is expressed as

$$
iS[q,p] = \int dt[ip \cdot \partial_t q - H]
$$

$$
= \int dt \left[ip \cdot (\partial_t q + \gamma q) - \gamma Tp^2\right]
$$

$$
= \frac{1}{2} \int dt \begin{pmatrix} q & p \end{pmatrix} \begin{pmatrix} 0 & iD_A^{-1} \\ iD_R^{-1} & -2\gamma T \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}.
$$

(14)

In the last line, we symmetrized $p \cdot \partial_t q$ term as $(p \cdot \partial_t q - q \cdot \partial_t p)/2$, and introduced the inverse of the retarded and advanced Green functions $D_{R,A}^{-1}$ as

$$
D_{R,A}^{-1} = \pm \partial_t + \gamma.
$$

(15)
The MSR action is useful to discuss the SSB in the dissipative systems. Namely, we can use standard techniques for the SSB even in the dissipative systems as we shall see in the next section.

III. SPONTANEOUS SYMMETRY BREAKING IN DISSIPATIVE SYSTEMS

In this section, we consider SSB in dissipative systems. We first provide a toy model of scalar fields, $\phi^a(t, x)$, with $O(N)$ symmetry. We discuss the symmetry and its Noether charge based on the MSR formalism. We shall find that the Noether charge that we call $Q_A^a$ is obtained from the symmetry of the action in the MSR formalism. $Q_A^a$ corresponds to $L_A$ of the Brownian particle in the previous section. In addition, if we drop the dissipation from our model, another Noether charge $Q_R^\alpha$ arises, which corresponds to the actual angular momentum of the Brownian particle, $L_R$.

Next, we discuss the spontaneous breaking of $O(N)$ symmetry in the language of the quantum field theory. The corresponding nonvanishing order parameters are $\langle [Q_A^a, \phi^a(t, x)] \rangle$. In our model, we will have all $\langle [iQ_A^a, Q_B^\beta]\rangle = \langle [iQ_A^a, Q_R^\beta]\rangle = 0$, so that the NG modes belong to type-A NG modes. Furthermore, we show that the NG modes of $O(N)$ model become diffusive modes. This behavior is quite different from those in the Hamiltonian system, in which the NG modes become propagating modes, such as the phonon in superfluids. That is, the diffusive NG mode is characteristic in the dissipative system. In Sec. III C, we also consider a $SU(2) \times U(1)$ model with a chemical potential, which is a simple model for type-B NG mode. In this model, we will have a nonvanishing order parameter, $\langle [iQ_A^a, Q_R^\beta]\rangle$, and thus, there appear a type-B NG mode. We shall find the type-B mode has a different dissipation relation from that in the Hamiltonian system.

A. $O(N)$ scalar model and its symmetry

Let us consider the following Langevin equation as a toy model:

$$\partial_t \phi_R^a(t, x) - \{\phi_R^a(t, x), F\}_{PB} = 0, \quad (16)$$

$$\partial_t \pi_R^a(t, x) - \{\pi_R^a(t, x), F\}_{PB} + \gamma \frac{\delta F}{\delta \pi_R^a} = \xi^a(t, x), \quad (17)$$
where \( \phi_R^a \) and \( \pi_R^a \) are the scalar fields that belong to the fundamental representation of \( O(N) \) symmetry, the subscript \( a \) runs \( 1, 2, \ldots, N \), \( \gamma \) is the dissipation constant, and \( \{\ldots,\ldots\} \) represents the Poisson bracket:

\[
\{X,Y\}_\text{PB} \equiv \int d^3 x \left[ \left( \frac{\partial X}{\partial \phi_R^a(x)} \right) \left( \frac{\partial Y}{\partial \pi_R^a(x)} \right) - \left( \frac{\partial Y}{\partial \phi_R^a(x)} \right) \left( \frac{\partial X}{\partial \pi_R^a(x)} \right) \right].
\] (18)

\( \xi^a(t, x) \) is the random noise satisfying

\[
\langle \xi^a(t, x) \xi^b(t', x') \rangle = A \delta_{ab} \delta(t - t') \delta^3(x - x'),
\] (19)

where \( A \) represents the strength of the noise. If we assume that the fluctuation-dissipation relation, \( A = 2 \gamma T \). \( F \) is the free energy, which has the following form:

\[
F[\phi_R, \pi_R] = \int d^3 x \left[ \frac{1}{2} (\pi_R^a)^2 + \frac{1}{2} (\nabla \phi_R^a)^2 + \frac{1}{2} m^2 (\phi_R^a)^2 + \frac{u^2}{4} ((\phi_R^a)^2)^2 \right].
\] (20)

If we drop \( \gamma \) and \( \xi^a \), our model reduces to the usual Hamiltonian equation whose Hamiltonian is given by Eq. (20). Hence, without the dissipation, our model turns out to be the classical version of the Goldstone model that is used to discuss the SSB in quantum field theory [2].

The corresponding MSR action and Fokker-Planck Hamiltonian read

\[
iS = \int d^4 x \left[ i \pi_A^a \left( \partial_t \phi_A^a - \pi_A^a \right) - i \phi_A^a \left( \partial_t \pi_A^a + \gamma \pi_A^a + \left( -\nabla^2 + m^2 + u^2 (\phi_R^b)^2 \right) \phi_R^a \right) - \frac{A}{2} (\phi_A^a)^2 \right],
\] (21)

and

\[
H = i \pi_A^a \pi_R^a + i \phi_A^a \left( \gamma \pi_R^a + \left( -\nabla^2 + m^2 + u^2 (\phi_R^b)^2 \right) \phi_R^a \right) + \frac{A}{2} (\phi_A^a)^2,
\] (22)

where \( \pi_A^a \) and \( \phi_A^a \) are the canonical momentum, which satisfy the commutation relations,

\[
[\phi_R^a(t, x), \pi_A^b(t, x')] = [\phi_A^a(t, x), \pi_R^b(t, x')] = i \delta^{ab} \delta^3(x - x'),
\] (23)

and others are zero. The stationary solution of this Fokker-Planck equation is the Gibbs distribution,

\[
\langle \phi_R, \pi_R | P \rangle = P(\phi_R, \pi_R) = \frac{1}{Z} e^{-F[\phi_R, \pi_R]/T},
\] (24)

where \( T \equiv A/(2\gamma) \) is the temperature, and \( Z \) is the normalization constant such that \( \int d\phi_R d\pi_R P(\phi_R, \pi_R) = 1 \). \( |P\rangle \) is the eigenstate of \( H \) with the zero eigenvalue.
The action \((21)\) is invariant under the following infinitesimal \(O(N)\) transformations,

\[
\begin{align*}
\phi^a_R &\rightarrow \phi^a_R + i\epsilon^a [T^\alpha]_{ab} \phi^b_R, \\
\pi^a_R &\rightarrow \pi^a_R + i\epsilon^a [T^\alpha]_{ab} \pi^b_R,
\end{align*}
\]

\[
\phi^a_A \rightarrow \phi^a_A + i\epsilon^a [T^\alpha]_{ab} \phi^b_A, \\
\pi^a_A \rightarrow \pi^a_A + i\epsilon^a [T^\alpha]_{ab} \pi^b_A,
\]

(25)

where \(\epsilon^a\) is an infinitesimal constant, and \([T^\alpha]_{ab}\) is the generator of \(O(N)\) group, which satisfies the Lie algebra, \([T^\alpha, T^\beta] = i f^{\alpha\beta\gamma} T^\gamma\) with the structure constant \(f^{\alpha\beta\gamma}\). For example, the generator of \(O(3)\) symmetry is given as

\[
T^1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -i \\
i & 0 & 0
\end{pmatrix}, \quad T^2 = \begin{pmatrix}
0 & 0 & i \\
0 & 0 & 0 \\
i & 0 & 0
\end{pmatrix}, \quad T^3 = \begin{pmatrix}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

(27)

From this symmetry of the action, we obtain the Noether charges,

\[
Q^a_A = \int d^3x \left[ \pi^a_A i [T^\alpha]_{ab} \phi^b_R + \pi^a_R i [T^\alpha]_{ab} \phi^b_A \right].
\]

(28)

\(Q^a_A\) corresponds to \(L_A\) of the Brownian motion in Sec. II. \(Q^a_A\) satisfies the Lie algebra, \([Q^a_A, Q^b_A] = i f^{\alpha\beta\gamma} Q^\gamma_A\). If \(\gamma = A = 0\), i.e., the Hamiltonian system, the action \((21)\) is invariant under another infinitesimal transformations:

\[
\phi^a_A \rightarrow \phi^a_A + i\epsilon^a [T^\alpha]_{ab} \phi^b_R, \\
\pi^a_A \rightarrow \pi^a_A + i\epsilon^a [T^\alpha]_{ab} \pi^b_R,
\]

(29)

and \(\phi^a_R \rightarrow \phi^a_R, \pi^a_R \rightarrow \pi^a_R\). The Noether charges of this symmetry are given as

\[
Q^a_R = \int d^3x \left[ \pi^a_R i [T^\alpha]_{ab} \phi^b_R \right].
\]

(30)

\(Q^a_R\) corresponds to the actual angular momentum in Sec. II. In fact, their Poisson bracket satisfies the Lie algebra, \(i\{Q^a_R, Q^b_R\}_{PB} = i f^{\alpha\beta\gamma} Q^\gamma_R\), although their commutation relation does not, \([Q^a_R, Q^b_R] = 0\). In the MSR formalism, the doubling of the fields, \(\phi_R\) and \(\phi_A\), occurs and for the Hamiltonian system, it causes the doubling of the symmetry. We will further discuss the doubling in the realtime formalism in Appendix A.

We also give the action by integrating \(\pi^a_R\) and \(\pi^a_A\) out:

\[
iS = \int d^4x \left[ -\frac{1}{2} \begin{pmatrix} \phi^a_R \\ \phi^a_A \end{pmatrix} \begin{pmatrix} 0 & iD_A^{-1} \\ iD_R^{-1} & A \end{pmatrix} \begin{pmatrix} \phi^a_R \\ \phi^a_A \end{pmatrix} - iu^2 (\phi^a_R)^2 \phi^a_A \phi^a_A \right],
\]

(31)

\(^1\) In quantum field theory, \([Q^a_R, Q^b_R]\) may be nonzero. See Appendix A.
where }{\text{d}^4x = dt dB^3 x \text{ and}
\begin{equation}
D_{R,A}^{-1} = \partial_t^2 \pm \gamma \partial_t - \nabla^2 + m^2.
\end{equation}
In the next section, we discuss the SSB in this model and the dispersion relation of the NG modes.

**B. Spontaneous symmetry breaking in a O\((N)\) model**

Let us now discuss the SSB in the \(O(N)\) model with the MSR action (31). When the squared mass is negative, \(-\mu^2 \equiv m^2 < 0\), the symmetric state \(\langle \phi_i^a \rangle = 0\) is disfavored because the propagator obtained from Eq. (32) contains an unstable mode, \(\sim e^{t|\mu|}\). To find the true stable state, we look for the stationary solution of the MSR action (31). For this purpose, we consider the following potential:
\begin{equation}
V_{\text{eff}} = m^2 i \phi_A^a \phi_R^a + \frac{A}{2} (\phi_A^a)^2 + u^2 (\phi_R^b)^2 i \phi_A^a \phi_R^a,
\end{equation}
which is obtained from the action with constant fields. The stationary solutions are given by
\begin{equation}
\frac{\delta V_{\text{eff}}}{\delta \phi_R^a} = \left( -\mu^2 + u^2 (\phi_R^b)^2 \right) i \phi_A^a + 2 u^2 i \phi_A^b \phi_R^b \phi_R^a = 0,
\end{equation}
\begin{equation}
\frac{\delta V_{\text{eff}}}{\delta \phi_A^a} = i \left( -\mu^2 + u^2 (\phi_R^b)^2 \right) \phi_R^a + A \phi_A^a = 0,
\end{equation}
and we obtain the two nontrivial solutions:
\begin{equation}
(\phi_R^a)^2 = \frac{\mu^2}{u^2}, \quad \text{and} \quad \phi_A^a = 0,
\end{equation}
\begin{equation}
(\phi_R^a)^2 = \frac{\mu^2}{3 u^2}, \quad \text{and} \quad \phi_A^a = i \frac{2 \mu^2}{3 A} \phi_R^a,
\end{equation}
in addition to \(\phi_R^a = \phi_A^a = 0\) corresponding to the unstable symmetric state. The second solution (37) is also unstable, whereas the first solution (36) is stable. To see the instability of the second solution, let us suppose \(\phi_R^a\) at \(a = 1\) has the nonzero expectation value:
\begin{equation}
\langle \phi_R^a \rangle = \frac{\phi_0}{\sqrt{3}} \delta^{a1},
\end{equation}
where $\phi_0 = \mu / u$. The squared mass for $\phi_R^b$ for $b = 2, 3, ..., N$ of the second solution is negative:

$$[D_R^{-1}]^{11}(\omega = 0, k = 0) = \frac{\delta^2 V_{\text{eff}}}{i \delta \phi_A^1 \delta \phi_R^1 |_{\phi_0 / \sqrt{3}}} = 0,$$

$$[D_R^{-1}]^{bc}(\omega = 0, k = 0) = \frac{\delta^2 V_{\text{eff}}}{i \delta \phi_A^b \delta \phi_R^c |_{\phi_0 / \sqrt{3}}} = -\frac{\delta_{bc}}{3} \mu^2.$$  

In contrast, for the first solution, we set $\langle \phi^a_R \rangle = \phi_0 \delta^a_1$, and obtain the positive squared mass

$$\frac{\delta^2 V_{\text{eff}}}{i \delta \phi_A^1 \delta \phi_R^1 |_{\phi_0}} = 2 \mu^2, \quad \text{and} \quad \frac{\delta^2 V_{\text{eff}}}{i \delta \phi_A^b \delta \phi_R^c |_{\phi_0}} = 0.$$ (41)

Therefore, the stable solution is the first solution (36) and the SSB occurs.

As in quantum field theory, the symmetry breaking is characterized by the nonvanishing expectation value of Noether charge and a local operator. In our case, it is

$$\langle [i Q^a_A, \phi^a_R] \rangle = i [T^a]_{a1} \phi_0.$$ (42)

There are $(N - 1)$’s independent nonvanishing components. Thus, the $O(N)$ symmetry is spontaneously broken into the $O(N - 1)$ symmetry. The corresponding $\phi_R^b$ for $b = 2, 3, ..., N$ are the Nambu-Goldstone fields. For example, in the case of $O(3)$, $Q^2_A, Q^3_A$ and $\phi^3_R, \phi^2_R$ are broken Noether charges and NG fields, respectively:

$$\langle [i Q^2_A, \phi^3_R] \rangle = -\langle [i Q^3_A, \phi^2_R] \rangle = \phi_0 \neq 0.$$ (43)

Furthermore, $Q^a_R$ are also spontaneously broken if we take the limit $\gamma \to 0$ and $A \to 0$:

$$\langle [i Q^a_R, \phi^a_R] \rangle = i [T^a]_{a1} \phi_0.$$ (44)

Since $\langle [i Q^a_R, Q^b_A] \rangle = \langle [i Q^a_A, Q^b_A] \rangle = 0$, the NG modes belong to type-A modes.

Now we discuss the dispersion relation of NG modes. To this end, we consider fluctuations around the expectation value (38), in which we parametrize the fields as

$$\phi_R^a(x) = (\phi_0 + \sigma_R(x), \chi_R^b(x)), \quad \phi_A^a(x) = (\sigma_A(x), \chi_A^b(x)),$$ (45)

where the subscript $b$ runs $2, 3, ..., N$. The MSR action in $\sigma$ and $\chi^b$ turns out to be

$$iS = \int d^4x \left[ -\frac{1}{2} \left( \begin{array}{cc} \sigma_R & \sigma_A \\ \sigma_A & \sigma_A \end{array} \right) \left( \begin{array}{c} 0 \\ iD_{\sigma,\sigma}^{-1} \end{array} \right) \left( \begin{array}{c} \sigma_R \\ \sigma_A \end{array} \right) \\ -\frac{1}{2} \left( \begin{array}{cc} \chi_R^b & \chi_A^b \\ \chi_A^b & \chi_A^b \end{array} \right) \left( \begin{array}{c} 0 \\ iD_{\chi,\chi}^{-1} \end{array} \right) \left( \begin{array}{c} \chi_R^b \\ \chi_A^b \end{array} \right) - V_{\text{int}} \right],$$ (46)
where the inverse propagators $D_{\sigma_R}^{-1}$ and the interaction term $V_{\text{int}}$ are given as

$$D_{\sigma,R,A}^{-1} = \partial_t^2 \pm \gamma \partial_t - \nabla^2 + 2u^2 \phi_0^2,$$

$$D_{\chi,R,A}^{-1} = \partial_t^2 \pm \gamma \partial_t - \nabla^2,$$

$$V_{\text{int}} = iu^2 \left[ \sigma_A \left( 3\phi_0^2 \sigma_R^2 + \phi_0 (\chi_R^b)^2 + \sigma_R (\chi_R^b)^2 + \sigma_R^3 \right) + \chi_A \phi_0^2 + (\chi_R^b)^2 + \sigma_R^2 \right].$$

We can see that $\chi^b$ does not have the mass term, i.e., $\chi^b$ is gapless. The dispersion relation of the type-A NG modes is determined by

$$D_{\chi,R}^{-1}(\omega, k) = -\omega^2 - i\gamma \omega + k^2 = 0.$$  

The solutions are given as

$$\omega(k) = -\frac{i}{2}\gamma \pm \frac{i}{2}\sqrt{\gamma^2 - 4k^2}$$

$$\sim -\frac{i}{\gamma}k^2 \text{ and } -i\gamma + \frac{i}{\gamma}k^2,$$

where we have expanded $\omega(k)$ with respect to $k$ up to the second order. Since $\omega(k)$ has no real part, these modes are purely damping modes at small $k$. One is the diffusive mode in which the damping vanishes at $k = 0$. The other has a finite damping even at $k = 0$. In this model, the number of the diffusive NG modes coincides with the number of broken symmetries $N - 1$. We note that NG modes in Hamiltonian systems become propagating mode, such as the spin wave in ferromagnets [13]. In fact, if we set $\gamma = 0$ in Eq. (50), we obtain the propagating mode, $\omega(k) = \pm|k|$. As mentioned in the Introduction, the diffusive NG mode is characteristic in the dissipative system.

C. **Spontaneous symmetry breaking in a $SU(2) \times U(1)$ model**

Here, we discuss the dispersion relation of type-B NG modes in dissipative systems. We consider $SU(2) \times U(1)$ model with a chemical potential, which is known as a simple model for realizing type-B NG modes [27, 28]. Suppose that the MSR action has the form,

$$iS = \int d^4x \left( i\varphi_A^\dagger((-\partial_0 + i\mu)^2 + \nabla^2 - \gamma \partial_0)\varphi_R - 2\lambda(\varphi_A^\dagger \varphi_R)\varphi_R \right)$$

$$+ i\varphi_R^\dagger((-\partial_0 + i\mu)^2 + \nabla^2 + \gamma \partial_0)\varphi_A - 2\lambda(\varphi_R^\dagger \varphi_R)\varphi_A - A\varphi_A^\dagger \varphi_A \right),$$

$$V_{\text{int}} = iu^2 \left[ \sigma_A \left( 3\phi_0^2 \sigma_R^2 + \phi_0 (\chi_R^b)^2 + \sigma_R (\chi_R^b)^2 + \sigma_R^3 \right) + \chi_A \phi_0^2 + (\chi_R^b)^2 + \sigma_R^2 \right].$$
where \( \varphi_i = (\varphi_i^1, \varphi_i^2) \) is the two component complex scalar fields, \( \mu \) the chemical potential, \( \lambda \) the coupling constant, and \( \gamma \) the friction coefficient. This action is invariant under \( SU(2) \times U(1) \) transformation, \( \varphi_i \rightarrow \varphi_i + i\epsilon_a T^a \varphi_i \), where \( T^0 \) is the \( U(1) \) generator, and \( T^a \) (\( a = 1, 2, 3 \)) are the \( SU(2) \) generators satisfying the Lie algebra, \([T^a, T^b] = i\epsilon^{abc}T^c \). We choose the normalization of the generators as \( \text{tr} T^a T^b = \delta^{ab}/2 \). The Noether charges are given as

\[
Q^a_A = \int d^3x \left[ \pi^+_a i T^a \varphi_R + \pi^+_a i T^a \varphi_A - \varphi^+_a i T^a \pi_A - \varphi^+_a i T^a \pi_R \right],
\]

where \( \pi_R = (\partial_0 + i\mu) \varphi_R \) and \( \pi_A = (\partial_0 + i\mu - \gamma) \varphi_A \). We can also define

\[
Q^a_R = \int d^3x \left[ \pi^+_a i T^a \varphi_R - \varphi^+_a i T^a \pi_R \right],
\]

which are the Noether charges associated with the transformation, \( \varphi_A \rightarrow \varphi_A + i\epsilon_a T^a \varphi_R \). Let us find the stationary solution of \( \delta S/\delta \varphi_i = 0 \). Using the same analysis in Sec. III B, we find a stable stationary solution, \( \varphi_R = (0, v) \) with \( v = \mu/\sqrt{2\lambda} \). Since the following expectation values,

\[
\frac{1}{V} \langle [iQ^1_A, Q^2_R] \rangle = -\frac{1}{V} \langle [iQ^2_A, Q^1_R] \rangle = \mu v^2
\]

are nonvanishing, \( Q^1_R \) and \( Q^2_R \) corresponds to the type-B NG fields in the nondissipative limit. Here \( V \) is the volume of the system. To analyze the dispersion relation, we parametrize the fields as \( \varphi_R = (\chi^1_R + i\chi^2_R, v + \psi^1_R + i\psi^2_R) \), and \( \varphi_A = (\chi^1_A + i\chi^2_A, \psi^1_A + i\psi^2_A) \). Then we find the inverse propagator for \( \psi \) and \( \chi \) sectors,

\[
D^{-1}_{\psi R}(\omega, k) = \begin{pmatrix} -\omega^2 - i\gamma \omega + k^2 + 2\mu^2 & 2i\mu \omega \\ -2i\mu \omega & -\omega^2 - i\gamma \omega + k^2 \end{pmatrix},
\]

\[
D^{-1}_{\chi R}(\omega, k) = \begin{pmatrix} -\omega^2 - i\gamma \omega + k^2 & 2i\mu \omega \\ -2i\mu \omega & -\omega^2 - i\gamma \omega + k^2 \end{pmatrix}.
\]

The dispersion relation is obtained from \( \det D^{-1}_{\psi R} = 0 \) and \( \det D^{-1}_{\chi R} = 0 \). At small \( k \), we find the diffusive NG mode, \( \omega = -i|k|^2/\gamma \) in the \( \psi \) sector and type-B mode,

\[
\omega = \frac{|k|^2}{4\mu^2 + \gamma^2} (\pm 2\mu + i\gamma),
\]

in the \( \chi \) sector. In the limit of \( \gamma \rightarrow 0 \), we obtain the dispersion relations \( \omega = \pm |k|/\sqrt{3} \) and \( \omega = \pm |k|^2/(2\mu) \) in the \( \psi \) and \( \chi \) sectors, respectively. Therefore, in this model, the type-B NG mode is the propagating mode with quadratic dispersion, while the type-A NG mode is the diffusive mode.
IV. WARD-TAKAHASHI IDENTITY IN DISSIPATIVE SYSTEMS

In the previous section, we discussed the NG modes and their dispersion relations associated with the spontaneous breaking of $O(N)$ and $SU(2) \times U(1)$ models in the classical approximation. In this section, we nonperturbatively establish the result using the Ward-Takahashi identity. We consider a system describing a Fokker-Planck equation, $\partial_t |P\rangle = -H|P\rangle$. We assume that the Fokker-Planck Hamiltonian does not explicitly depend on time, $\partial_t H = 0$, and the real part of right eigenvalues of $H$ are non-negative and it contains at least one zero eigenvalue. In general, the stationary state with the zero eigenvalue may not be the thermal state, i.e., a steady state is allowed in this formalism. We assume that the stationary state does not break the spacetime symmetries. We consider a continuum symmetry group $G$ with the generator $Q^\alpha_A$ as internal symmetry, which commutes with the Fokker-Planck Hamiltonian, $[H, Q^\alpha_A] = 0$. For fields belonging to a linear representation, $\phi^a_R$ and $\phi^a_A$ transforms

$$[iQ^\alpha_A, \phi^a_R] = i[T^\alpha]_{ab} \phi^b_R, \quad \text{and} \quad [iQ^\alpha_A, \phi^a_A] = i[T^\alpha]_{ab} \phi^b_A.$$  \hspace{1cm} (59)

We assume that the Poisson bracket is defined, and we also introduce $Q^a_R$ such that

$$\{iQ^a_R, \phi^a_R\}_{PB} = i[T^\alpha]_{ab} \phi^b_R.$$  \hspace{1cm} (60)

In general, $Q^a_R$ does not commute with the Hamiltonian, but it does in the Hamiltonian system. When we consider the system in which the Poisson bracket cannot be defined, we need not consider $Q^a_R$ in the following argument.

In the following, we show the following relation from the Ward-Takahashi identity,

$$[D^{-1}]^{ab}_{ij}(\omega = 0, k = 0) \langle [iQ^\alpha_A, \phi^a_j] \rangle = 0,$$  \hspace{1cm} (61)

where $D^{-1}_{ij}$ is the inverse propagator and indices $i$ and $j$ run $R$ and $A$. The derivation in this section is commonly used in quantum field theory [29].

To drive Eq. (61), it is useful to move to the path integral representation of the generating functional:

$$Z[J] = \int D\phi e^{iS[\phi] + i \int d^4x J \cdot \phi},$$  \hspace{1cm} (62)

where we used the vector notation: $\phi = (\phi^a_R, \phi^a_A)$ and $J = (J^a_A, J^a_R)$. We assume that $\phi^a_i$ contain the order parameters. The action $S[\phi]$ and the path integral measure $D\phi$ is invariant
under the infinitesimal transformation, \( \phi \to \phi + \epsilon_\alpha \delta^\alpha_A \phi \), where \( \epsilon_\alpha \) is an infinitesimal constant, and \( \delta^\alpha_A \phi = [iQ_A^\alpha, \phi] \) in the operator formalism. Since the generating functional is invariant under the reparameterization of the fields \( \phi \to \phi' = \phi + \epsilon_\alpha \delta^\alpha_A \phi \), we have

\[
Z[J] = \int D\phi' e^{iS[\phi'] + i \int d^4x J \cdot \phi'}
= \int D\phi e^{iS[\phi] + i \int d^4x J \cdot \phi} \left( 1 + i \epsilon_\alpha \int d^4x J \cdot \delta^\alpha_A \phi \right) + \mathcal{O}(\epsilon^2)
= Z[J] \left( 1 + i \epsilon_\alpha \int d^4x J \cdot \langle \delta^\alpha_A \phi \rangle_J \right) + \mathcal{O}(\epsilon^2),
\]

where and \( \langle \ldots \rangle_J \) is the expectation value in the presence of the source \( J \):

\[
\langle \phi \rangle_J \equiv \frac{1}{Z[J]} \int D\phi e^{iS[\phi] + i \int d^4x J \cdot \phi}.
\]

From Eq. (63), we obtain the identity

\[
\int d^4x J \cdot \langle [iQ_A^\alpha, \phi] \rangle_J = 0,
\]

where we used \( \langle \delta^\alpha_A \phi \rangle_J = \langle [iQ_A^\alpha, \phi] \rangle_J \). Using the effective action

\[
\Gamma[\phi] \equiv -i \ln Z[J] - \int d^4x \phi \cdot J,
\]

we can write the identity as

\[
\int d^4x \frac{\delta \Gamma}{\delta \phi_j^b(t, x)} \langle [iQ_A^\alpha, \phi_j^b(t, x)] \rangle_J = 0,
\]

where we used \( \delta \Gamma / \delta \phi = -J \). Differentiating Eq. (67) with respect to \( \phi_i^a(t', x') \) and taking the limit of \( J \to 0 \), we obtain

\[
\int d^4x [D^{-1}]_{ij}^{ab}(t' - t, x' - x) \langle [iQ_A^\alpha, \phi_j^b(t, x)] \rangle = 0,
\]

where the inverse of propagator is obtained as \( [D^{-1}]_{ij}^{ab}(t' - t, x' - x) = (\delta^2 \Gamma / \delta \phi_i^a(t, x) \delta \phi_j^b(t', x')) \). In momentum space, we obtain Eq. (61). This identity represents the eigenvalue equation with the zero eigenvalue, whose eigenvectors are \( \langle [iQ_A^\alpha, \phi_j^b] \rangle \). The number of independent eigenvectors is equal to the number of broken Noether charges.

When \( Q_R^\alpha \) is conserved, we obtain a similar result for \( Q_R^\alpha \):

\[
[D^{-1}]_{ij}^{ab}(\omega = 0, k = 0) \langle [iQ_R^\alpha, \phi_j^b] \rangle = 0.
\]
For $O(N)$ model with the expectation value $\langle \phi_R^a \rangle = \delta^a_1 \phi_0$, using useful normalization of $Q_A^a$, we can write $\langle [i Q_A^a, \phi_R^b] \rangle = \delta^{ab}$ for $a = 2, 3, \ldots, N$. Then, from Eq. (69), we obtain

$$[D^{-1}]_{iR}^{aa}(\omega = 0, k = 0) = 0. \quad (70)$$

This identity gives the constraint to the dispersion relations.

Now, we expand the inverse of propagators with respect to $\omega$ and $k$ as

$$-i[D^{-1}]_{RR}^{ab} = 0, \quad (71)$$

$$-i[D^{-1}]_{AR}^{ab} = \delta^{ab} C_{(0,2)}^{} k^2 - i \delta^{ab} C_{(1,0)}^{} \omega - \delta^{ab} C_{(2,0)}^{} \omega^2 + \cdots, \quad (72)$$

$$-i[D^{-1}]_{AA}^{ab} = -i \delta^{ab} A_{(0,0)}^{(0,0)} + \cdots, \quad (73)$$

where the coefficients $C_{(n,m)}^{(n,m)}$ and $A_{(n,m)}^{(n,m)}$ are generally nonzero without constraint from another symmetry. Here, $-i[D^{-1}]_{RR}^{ab}$ generally vanishes due to conservation of the probability. Furthermore, $C_{(0,0)}^{(0,0)}$ becomes zero from the constraint Eq. (70). From the equation $[D^{-1}]_{AR}^{ab}(\omega, k) = 0$, we obtained the dispersion relation for the dissipative NG modes $\omega \sim -|k|^2$. In addition, when, $Q_R$ is also conserved, which corresponds to the Hamiltonian system, we find that $A_{(0,0)}^{(0,0)}$ vanishes from Eq. (69). When the system satisfies the fluctuation-dissipation relation, $A_{(0,0)}^{(0,0)}$ is related to $C_{(1,0)}^{(1,0)}$: $A_{(0,0)}^{(0,0)} = 2C_{(1,0)}^{} T$ with the temperature $T$, and thus, $C_{(1,0)}^{(1,0)}$ vanishes. In this case the NG modes are the propagating modes with the dispersion relation $\omega = a_R |k| - ia_I |k|^2$ [13], where $a_R$ and $a_I$ are constants depending on $C_{(n,m)}^{(n,m)}$.

The symmetry breaking pattern discussed in the $O(N)$ model for $N \geq 4$ is relatively a simple case because the broken Noether charges transform as the vector representation under unbroken $O(N - 1)$ symmetry. In other words, the NG modes belong to the vector representation of the $O(N - 1)$. The unbroken symmetry restricts coupling between NG modes and others, and thus, the inverse of the propagators proportional to the Kronecker delta\(^2\). If the diffusive NG field has no internal unbroken charge, the analysis will be more complicated. In particular, the coupling between NG modes to hydrodynamic modes must be taken into account.

A different type of dispersion relations will be found when $\langle [Q_A^a, Q_R^b] \rangle$ is nonzero, which corresponds to Type-B modes. For $SU(2) \times U(1)$ model, the unbroken symmetry is $U(1)$;\(^2\)

\(^2\) When the unbroken symmetry is the antisymmetric tensor $O(2)$, there is a possibility to have $\epsilon^{ab}$ in the inverse of the propagator.
we have two second rank invariant tensors in the real representation: the Kronecker delta $\delta^{ab}$ and the antisymmetric tensor $\epsilon^{ab}$. Then, the inverse propagator is expanded as

$$-i[D^{-1}]^{ab}_{AR} = \delta^{ab}C^{S}_{(0,2)}k^2 + \epsilon^{ab}C^{A}_{(0,2)}k^2 - i\delta^{ab}C^{S}_{(1,0)}\omega - i\epsilon^{ab}C^{A}_{(1,0)}\omega + \cdots.$$  \hspace{1cm} (74)

In this case, the dispersion relation has the form, $\omega = c_R|k|^2 - ic_I|k|^2$, where $c_R$ and $c_I$ are constants. Therefore, the type-B NG mode can propagate. We note that the dispersion relation of type-B NG modes in the Hamiltonian system is $\omega = b_R|k|^2 - ib_I|k|^4$, where $b_R$ and $b_I$ are constants [13].

For more general cases, the coefficients are matrices,

$$-i[D^{-1}]^{ab}_{AR}(\omega, k) = C_{(0,2)}^{ab}k^2 - iC_{(1,0)}^{ab}\omega - C_{(2,0)}^{ab}\omega^2 + \cdots.$$  \hspace{1cm} (75)

If $\det C_{(1,0)}^{ab}$ is nonzero, $C_{(2,0)}^{ab}$ is negligible at small $k$, and we obtain the dispersion relation from the eigenvalue of $i[C^{-1}]^{ac}_{(1,0)}C_{(0,2)}^{ab}k^2$. The eigenvalue is generally complex, and thus, we have the form $\omega = d_R|k|^2 - id_I|k|^2$. We expect the existence of the coefficient $d_R$ is related to the nonvanishing $\langle [Q^a_R, Q^b_R] \rangle$, although we have not given the proof, which is beyond the scope of this paper.

V. SUMMARY AND DISCUSSION

We discussed spontaneous symmetry breaking (SSB) and the Nambu-Goldstone (NG) modes in dissipative systems described by a Langevin or Fokker-Planck equation. For this purpose, we employed the $O(N)$ and $SU(2) \times U(1)$ scalar models as toy models. In the nondissipative limit, which corresponds to a Hamilton system, there exist the Noether charges $Q^a_R$ that are the generators of the internal symmetry by means of the Poisson bracket and they are conserved. In contrast, in the dissipative system, $Q^a_R$ are no longer conserved due to dissipation and noise. Instead, there exist other conserved quantities $Q^a_A$, which are the Noether charges of the internal symmetry in the Fokker-Planck equation.

Symmetry breaking is characterized by existence of a nonvanishing order parameter. In the $O(N)$ model, $O(N)$ symmetry is spontaneously broken into $O(N - 1)$, and the order parameter is $\langle [iQ^a_A, \phi^a_R] \rangle$. Since $\langle [iQ^a_A, Q^b_R] \rangle = 0$, the NG modes belong to the type-A modes. We found that the NG modes are the diffusive modes, $\omega \sim -i|k|^2$. This is the different behavior compared with the Hamiltonian system, where the NG modes are the propagating
modes. This difference is caused by whether $Q^\alpha_A$ is conserved: When both $Q^\alpha_A$ and $Q^\alpha_R$ are conserved and their symmetry is spontaneously broken, there appear the propagating NG modes. In dissipative systems, when $Q^\alpha_A$ is broken, the diffusive NG modes appear. We established this result by using the Ward-Takahashi identity for $Q^\alpha_A$ and $Q^\alpha_R$ symmetries in Sec. IV.

We also discussed type-B NG modes in $SU(2) \times U(1)$ model, where $\langle [iQ^\alpha_A, Q^\beta_R] \rangle \neq 0$. In this case, the dispersion relation of NG modes have the form of $\omega = a|k|^2 - ib|k|^2$, while they are $\omega = a'|k|^2 - b'|k|^4$ in the Hamiltonian system, where $a, b, a'$, and $b'$ are constant parameters. In contrast to type-A NG modes, type-B NG modes are still propagating modes.

In this paper, we focus only classical systems. Generalization to quantum systems is straightforward: We may add higher terms in $\phi_A$ such as $(\phi^b_A)^2 \phi^a_A \phi^a_R$, and take into account the Bose and Fermi statistics.

Our approach can apply to the spontaneous breaking of spacetime symmetries, although our result in this paper is limited to that of internal symmetry, although the situation will be more complicated. Even in the Hamiltonian system, the general counting rule and dispersion relation of their NG modes have not been well-known. An interesting example in a dissipative system is discussed in the active hydrodynamics, where energy and momentum are not conserved, but equations of motion respect spacetime translational and rotational symmetries [6, 7]. In this situation, the velocity fields is the order parameter and it breaks the rotational symmetry. For $d$-spatial dimensions, there appear $d - 2$ diffusive (shear) modes, and one propagating sound mode. This sound mode caused by the mixing between longitudinal NG mode and the hydrodynamic mode associated with the number conservation [6, 7]. The mixing of hydrodynamic mode can change the dispersion relation.

It is interesting to clarify the relation between the broken symmetry, the NG modes and their dispersion relations in dissipative systems. Recently, in the Hamiltonian system without the Lorentz invariance, those relations have been made clear [10–13], which does not cover the dissipative system. From our observations, we propose the following conjecture: We suppose that the Fokker-Planck Hamiltonian $H$ commutes with the generator $Q^\alpha_A$ of a Lie group $G$. We also suppose that the Poisson bracket is defined, and $Q^\alpha_R$ exists as the generator of $G$ in the sense of Poisson bracket. In general, $Q^\alpha_R$ does not commute with the Fokker-Planck Hamiltonian. When the $G$ is spontaneously broken into its subgroup $H$, the
number of type-A ($N_A$) and type-B ($N_B$) NG modes will be given as

\[ N_A = N_{BS} - \text{rank}\langle [iQ^\alpha_A, Q^\beta_R]\rangle, \]

\[ N_B = \frac{1}{2} \text{rank}\langle [iQ^\alpha_A, Q^\beta_R]\rangle. \]

where $N_{BS} = \text{dim}(G/H)$ is the number of broken symmetries. These equations correspond to Eq. (1) for the Hamiltonian system. Their dispersion relations will be classified as four types:

Type-A

\[ \omega = c_R|k| - ic_I|k|^2, \quad \text{if } [Q^\alpha_R, H] = 0, \]

\[ \omega = -ic_I|k|^2, \quad \text{if } [Q^\alpha_R, H] \neq 0, \]

Type-B

\[ \omega = c_R|k|^2 - ic_I|k|^4, \quad \text{if } [Q^\alpha_R, H] = 0, \]

\[ \omega = c_R|k|^2 - ic_I|k|^2, \quad \text{if } [Q^\alpha_R, H] \neq 0. \]

Of course, the models discussed in this paper satisfy these relations. We leave the detailed analysis and the proof of this conjecture leave to our future work.

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Appendix A: Symmetry of Hamiltonian and dissipative systems in the real-time formalism

Here, we discuss symmetry of an action in Hamiltonian and dissipative systems from the real-time formalism. By coupling the system to the environment explicitly violates the $Q^\alpha_R$ symmetry, we shall see that our model corresponds to a dissipative system.

We consider the following Lagrangian to discuss the symmetry of the Hamiltonian system,

\[ \mathcal{L} = \frac{1}{2}(\partial_t \phi^o)^2 - \frac{1}{2}(\nabla \phi^o)^2 - \frac{1}{2}m^2_0(\phi^o)^2 - \frac{u_0^2}{4}((\phi^o)^2)^2. \]

The generating functional in the real-time formalism is given as
FIG. 1. The contour on the complex time plane in the real-time formalism. The branch $C_1$ runs on the real axis from $-\infty$ to $\infty$, while $C_2$ runs backward from $\infty$ to $-\infty$. $C_3$ runs from $-\infty$ to $-\infty-i\beta$, where $\beta$ is the inverse temperature.

\[ Z[J_a^1, J_a^2] = \int \mathcal{D}\phi_i^a \mathcal{D}\phi_j^a e^{iS[\phi_i^a, \phi_j^a]} + i \int d^4x (J_i^a \phi_i^a - J_j^a \phi_j^a), \]  

(A2)

with the action,

\[ iS[\phi_i^a, \phi_j^a] = -\frac{1}{2} \int d^4xd^4x' \phi_i^a(t, x) D^{-1}_{ij}(t-t', x-x') \phi_j^a(t', x') \]
\[ -i \frac{u_0^2}{4} \int d^4x \left( (\phi_i^a(t, x))^2 - (\phi_j^a(t, x))^2 \right)^2, \]  

(A3)

where the subscript $i,j$ run 1, 2, and $\phi_{1,2}^a$ represent the fields on the forward and backward branches, respectively (See the caption of Fig. 1.) Here, $D^{-1}_{ij}$ are the inverse of the propagators, which are given as \[30\]

\[ D_{11}(\omega, k) = i P \frac{1}{\omega^2 - k^2 - m_0^2} + \left( \frac{1}{2} + n(\omega) \right) \rho(\omega, k), \]  

(A4)

\[ D_{22}(\omega, k) = (D_{11}(\omega, k))^*, \]  

(A5)

\[ D_{12}(\omega, k) = n(\omega) \rho(\omega, k), \]  

(A6)

\[ D_{21}(\omega, k) = (1 + n(\omega)) \rho(\omega, k), \]  

(A7)

where $P$ denotes the principal value, $n(\omega) = 1/(e^{\beta\omega} - 1)$ the Bose distribution function, and $\rho(\omega, k) = 2\pi \varepsilon(\omega) \delta(\omega^2 - k^2 - m_0^2)$ the spectral function with the sign function $\varepsilon(\omega)$. The important point is that the doubling of the fields occurs: $\phi^a \rightarrow \phi_i^a$ and $\phi_j^a$. This doubling causes the doubling of the symmetry.
Before discussing the symmetry, we change the field variables to
\[ \phi_R^a = \frac{1}{2}(\phi_1^a + \phi_2^a), \quad \phi_A^a = \phi_1^a - \phi_2^a. \] (A8)

The generating functional and the action for \( \phi_{R,A}^a \) is written in the new variables as
\[ Z[J_R^a, J_A^a] = \int D\phi_R^a D\phi_A^a e^{iS[\phi_R, \phi_A] + i\int d^4x (J_R^a \phi_R^a + J_A^a \phi_A^a)}, \] (A9)
with
\[ iS[\phi_R^a, \phi_A^a] = -\frac{1}{2} \int d^4xd^4x' \phi_R^a(t, \mathbf{x}) D^{-1}_{ij}(t - t', \mathbf{x} - \mathbf{x}') \phi_R^a(t', \mathbf{x}') \]
\[ - iu_0^2 \int d^4x \left( (\phi_R^a)^2 \phi_R^b \phi_A^b + \frac{1}{4} (\phi_A^a)^2 \phi_R^b \phi_A^b \right), \] (A10)
where the subscript \( i \) and \( j \) run \( R \) and \( A \), and \( J_R^a \equiv J_1^a - J_2^a \) and \( J_A^a \equiv (J_1^a + J_2^a)/2 \). Here, \( D_{R,A}^{-1} \) is given in momentum space as
\[ D_{ij}^{-1}(\omega, k) = \begin{pmatrix} 0 & iD_A^{-1}(\omega, k) \\ iD_R^{-1}(\omega, k) & i \left( \frac{1}{2} + n(\omega) \right) \left( D_R^{-1}(\omega, k) - D_A^{-1}(\omega, k) \right) \end{pmatrix}, \] (A11)
where \( D_R^{-1}(\omega, k) \equiv -(\omega + i\epsilon)^2 + k^2 + m_0^2 \) and \( D_A^{-1}(\omega, k) = (D_R^{-1}(\omega, k))^* \). \( D_{AA}^{-1} \) is infinitesimally small in this case because
\[ D_{AA}^{-1} = i \left( \frac{1}{2} + n(\omega) \right) (D_R^{-1} - D_A^{-1}) \]
\[ = 2e\omega(1 + 2n(\omega)) \sim \epsilon. \] (A12)

Let us discuss the symmetry of Eq. (A10). We can easily see that the action is invariant under the following transformation:
\[ \phi_R^a \rightarrow \phi_R^a + ie^a[T^a]_{ab} \phi_A^b, \quad \phi_A^a \rightarrow \phi_A^a + ie^a[T^a]_{ab} \phi_R^b. \] (A13)

The Noether charge of this symmetry is given as
\[ Q_A^a = \int d^3x \left[ \pi_A^a i[T^a]_{ab} \phi_R^b + \pi_R^a i[T^a]_{ab} \phi_A^b \right], \] (A14)
where the \( \pi_{R,A}^a \) are defined as
\[ \pi_R^a \equiv (\pi_1^a + \pi_2^a)/2, \quad \pi_A^a \equiv \pi_1^a - \pi_2^a. \] (A15)

Here, \( \pi_{1,2}^a \equiv \partial_t \phi_{1,2}^a \) are the canonical momentum of \( \phi_{1,2}^a \) and satisfy the commutation relations
\[ [\phi_{1}^a(t, \mathbf{x}), \pi_{1}^b(t, \mathbf{y})] = i\delta^{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi_{2}^a(t, \mathbf{x}), \pi_{2}^b(t, \mathbf{y})] = -i\delta^{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}), \] (A16)
We note that \( \phi_2^a \) and \( \pi_2^a \) are the fields on the backward branch \( C_2 \) hence the commutation relation has the negative sign. We can see that the form of the charge (A14) is equal to that of the Langevin equation (28).

Furthermore, in the limit \( \epsilon \to 0 \), Eq. (A10) is invariant under the transformation,

\[
\phi_A^a \to \phi_A^a + i\epsilon^a [T^\alpha]_{ab} \phi_R^b, \quad \phi_R^a \to \phi_R^a + \frac{i}{4} \epsilon^a [T^\alpha]_{ab} \phi_A^b,
\]

(A17)

and its Noether charge is given as

\[
Q_R^a = \int d^3x \left[ \pi_R^a i[T^\alpha]_{ab} \phi_R^b + \frac{1}{4} \pi_A^a i[T^\alpha]_{ab} \phi_A^b \right].
\]

(A18)

We note that this charge corresponds to the charge (30) if we drop \((1/4)\pi_A^a [T^\alpha]_{ab} \phi_A^b\) term in Eq. (A18). Therefore, the action of the closed system (A10) is invariant under the transformations by \( Q_R^a \) and \( Q_A^a \). That is, the doubling of the symmetry occurs. In the original Lagrangian (A1) is invariant under the transformation \( \phi^a \to \phi^a + i\epsilon^a [T^\alpha]_{ab} \phi^b \). Meanwhile, the action of the real-time formalism (A10) is invariant under the two transformations, Eqs. (A14) and (A18).

Next, we consider the system of \( \phi_i^a \) couples with other environment scalar fields \( \Phi_i^{l,a} \) with mass \( M^l \). We assume that the interaction between \( \phi_i^a \) and \( \Phi_i^{l,a} \) has the following form:

\[
iS_{\text{int}}[\phi_i^a, \Phi_i^{l,a}] = ig \int dt d^3x \sum_l (\phi_A^a \Phi_R^{l,a} + \phi_R^a \Phi_A^{l,a}),
\]

(A19)

where \( g \) is the coupling constant. Then the total action is \( S_{\text{open}} = S[\phi_i^a] + S[\Phi_i^{l,a}] + S_{\text{int}}[\phi_i^a, \Phi_i^{l,a}] \) with

\[
iS[\Phi_i^{l,a}] = -\frac{1}{2} \int d^4x \sum l \left( \Phi_R^{l,a} \Phi_A^{l,a} \right) \begin{pmatrix} 0 & iG_{A}^{l-1} \\ iG_{A}^{l-1} & G_{A}^{l-1} \end{pmatrix} \begin{pmatrix} \Phi_R^{l,a} \\ \Phi_A^{l,a} \end{pmatrix},
\]

(A20)

where

\[
G_{R,A}^{l-1}(\omega, k) = -(\omega \pm i\epsilon)^2 + k^2 + M_i^2, \quad G_{A}^{l-1}(\omega, k) = i \left( \frac{1}{2} + n(\omega) \right) \left( G_{R}^{l-1} - G_{A}^{l-1} \right)
\]

(A21)

in momentum space.

By integrating the environment fields \( \Phi_i^{l,a} \) out, we obtain the effective action for \( \phi_i^a \):

\[
iS_{\text{eff}} = -\frac{1}{2} \int \frac{d\omega}{2\pi} \sum l \left( \phi_A^a \phi_R^a \right) \begin{pmatrix} 0 & iD_{A}^{-1} - i\epsilon^2 G_{A} \\ iD_{A}^{-1} - i\epsilon^2 G_{R} & D_{A}^{-1} - i\epsilon^2 G_{K} \end{pmatrix} \begin{pmatrix} \phi_R^a \\ \phi_A^a \end{pmatrix} - \frac{i}{2} \int \frac{d\omega}{2\pi} \sum l \left( \phi_R^a \phi_R^b \phi_R^b + \frac{1}{4} \phi_A^a \phi_R^b \phi_R^b \right),
\]

(A22)
where \( G_{R,A} \equiv \sum_l G_{R,A}^l \), and \( G_K \) is
\[
G_K(\omega, k) = \left( \frac{1}{2} + n(\omega) \right) \left( G_R(k) - G_A(k) \right).
\] (A23)

We take continuum limit with respect to the index \( l \) of \( G_{R,A}^l \), and replace
\[
G_{R,A}(\omega, k) = \sum_l \frac{1}{-(\omega \pm i\epsilon)^2 + k^2 + M_l^2}
\rightarrow \int_0^\infty dM^2 \frac{\rho_M(M^2)}{-(\omega \pm i\epsilon)^2 + k^2 + M^2},
\] (A24)
where \( \rho_M(M^2) \) is a weight function. Hence, \( G_K \) turns out to be
\[
G_K(\omega, k) = \left( \frac{1}{2} + n(\omega) \right) \left( G_R(\omega, k) - G_A(\omega, k) \right)
= i \left( \frac{1}{2} + n(\omega) \right) (2\pi)\epsilon(\omega)\rho_M(\omega^2 - k^2),
\] (A25)

Now, we discuss the symmetry of the action Eq. (A22). The important point is that it is not symmetric under Eq. (A17) because \( G_K(k) \neq 0 \); see Eqs. (A22) and (A25). That is, the \( Q^a_R \) symmetry Eq. (A17) is explicitly broken. Conversely, \( Q^a_A \) symmetry remains even in the dissipative system. The original action \( S_{\text{open}} \) is invariant under the simultaneous rotation \( \phi^a \) and \( \Phi^a_i \) whereas the integrated action Eq. (A22) does not have \( \Phi^a_i \) fields. As a result, \( Q^a_R \) symmetry is broken, while \( Q^a_A \) symmetry remains as the remnant of the original symmetry.

[1] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961).
[2] J. Goldstone, Il Nuovo Cimento 19, 154 (1961).
[3] J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).
[4] P. W. Higgs, Physics Letters 12, 132 (1964).
[5] G. H. Gunaratne, Q. Ouyang, and H. L. Swinney, Phys. Rev. E 50, 2802 (1994).
[6] J. Toner and Y. Tu, Phys. Rev. Lett. 75, 4326 (1995).
[7] J. Toner and Y. Tu, Phys. Rev. E 58, 4828 (1998).
[8] A. M. Turing, Philosophical Transactions of the Royal Society of London B: Biological Sciences 237, 37 (1952).
[9] A. Attanasi, A. Cavagna, L. Del Castello, I. Giardina, T. S. Grigera, A. Jelić, S. Melillo, L. Parisi, O. Pohl, E. Shen, et al., Nature physics 10, 691 (2014).
[10] H. Watanabe and H. Murayama, Phys. Rev. Lett. 108, 251602 (2012).
[11] Y. Hidaka, Phys. Rev. Lett. 110, 091601 (2013).
[12] H. Watanabe and H. Murayama, Phys. Rev. X 4, 031057 (2014).
[13] T. Hayata and Y. Hidaka, Phys. Rev. D 91, 056006 (2015).
[14] D. A. Takahashi and M. Nitta, Ann. Phys. 354, 101 (2015), arXiv:1404.7696 [cond-mat.quant-gas].
[15] P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A 8, 423 (1973).
[16] D. Hochberg, C. Molina-París, J. Pérez-Mercader, and M. Visser, Phys. Rev. E 60, 6343 (1999).
[17] E. Siggia, B. Halperin, and P. Hohenberg, Phys. Rev. B 13, 2110 (1976).
[18] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. 49, 435 (1977).
[19] C. De Dominicis and L. Peliti, Phys. Rev. B 18, 353 (1978).
[20] U. C. Tuber, Nuclear Physics B - Proceedings Supplements 228, 7 (2012), Physics at all scales: The Renormalization Group Proceedings of the 49th Internationale Universittswochen fr Theoretische Physik Proceedings of the 49th Internationale Universittswochen fr Theoretische Physik.
[21] B. Fong et al., Journal of Mathematical Physics 54 (2013).
[22] R. Graham, D. Roekaerts, and T. Tél, Physical Review A 31, 3364 (1985).
[23] I. V. Ovchinnikov, arXiv preprint arXiv:1308.4222 (2013).
[24] T. Misawa, Journal of Physics A: Mathematical and General 27, L777 (1994).
[25] G. Gaeta and N. R. g. Quintero, Journal of Physics A: Mathematical, Nuclear and General 32, 8485 (1999).
[26] J. Z. Justin, Quantum field theory and critical phenomena (Clarendon, Oxford, 1989).
[27] V. A. Miransky and I. A. Shovkovy, Phys. Rev. Lett. 88, 111601 (2002), arXiv:hep-ph/0108178.
[28] T. Schafer, D. T. Son, M. A. Stephanov, D. Toublan, and J. J. M. Verbaarschot, Phys. Lett. B522, 67 (2001), arXiv:hep-ph/0108210.
[29] S. Weinberg, The Quantum Theory of Fields, Vol. II (Cambridge University Press, Cambridge, UK, 1996).
[30] M. Le Bellac, Thermal field theory (Cambridge University Press, 2000).