1. Introduction.

1.1. Arithmetic quotients. Arithmetic quotients are real analytic manifolds of the form $S_{\Gamma,G,M} := \Gamma \backslash G / M$, for $G$ a connected semi-simple linear algebraic $\mathbb{Q}$-group, $G := G(\mathbb{R})^+$ the real Lie group connected component of the identity of $G(\mathbb{R})$, $M \subset G$ a connected compact subgroup and $\Gamma \subset G(\mathbb{Q})^+ := G(\mathbb{Q}) \cap G$ a neat arithmetic group. By a morphism of arithmetic quotients we mean the real analytic map $f : S_{\Gamma',G',M'} \longrightarrow S_{\Gamma,G,M}$ deduced from a morphism $f : G' \longrightarrow G$ of semi-simple linear algebraic $\mathbb{Q}$-group such that $f(M') \subset M$ and $f(\Gamma') \subset \Gamma$.

Such quotients are ubiquitous in various parts of mathematics. For $M = \{1\}$ the arithmetic quotients $S_{\Gamma,G,\{1\}} = \Gamma \backslash G$ are the main players in homogeneous dynamics, for example Ratner’s theory [Rat91-0], [Rat91-1]. For $K \subset G$ a maximal compact subgroup the arithmetic quotients $S_{\Gamma,G,K}$ are the arithmetic riemannian locally symmetric spaces, for instance the arithmetic hyperbolic manifolds $\Gamma \backslash SO(n,1)^+/SO(n)$. They are intensively studied by differential geometers and group theorists. When $G$ is moreover of Hermitian type then $S_{\Gamma,G,K}$ is a so-called arithmetic variety (also called a connected Shimura variety if $\Gamma$ is a congruence subgroup): this is a smooth complex quasi-projective variety, naturally defined over $\mathbb{Q}$ in the Shimura case. The simplest examples of connected Shimura varieties are the modular curves $\Gamma_0(N) \backslash SL(2,\mathbb{R})/SO(2)$. Connected
Shimura varieties play a paramount role in arithmetic geometry and the Langlands program. Much more generally, the connected Hodge varieties are arithmetic quotients which play a crucial role in Hodge theory as target of period maps.

1.2. Moderate geometry of arithmetic quotients. For $S_{Γ,G,K}$ a connected Shimura variety, the study of the topological tameness properties of the uniformization map $π : G/M → S_{Γ,G,K}$ recently provided a crucial tool for solving longstanding algebraic and arithmetic questions (see [P11], [PT14], [KUY16], [Ts18], [KUY17], [MPT17]). Here tameness is understood in the sense proposed by Grothendieck [Gro, §5 and 6] and developed by model theory under the name “o-minimal structure” (see below). The first goal of this paper is to develop a similar study for a general arithmetic quotient $S_{Γ,G,M}$.

Among real analytic manifolds the ones with the tamest geometry are certainly the complex algebraic ones. However most arithmetic quotients have no complex algebraic structures, as they do not even admit a complex analytic one (for instance for obvious dimensional reasons). What about a real algebraic structure? In [Rag68] Raghunathan proved that any riemannian locally symmetric space is compactifiable, i.e. diffeomorphic to the interior of a compact smooth manifold with boundary; Akbulut and King [AK81] proved that any compactifiable manifold is diffeomorphic to a non-singular real algebraic set (generalizing a result of Tognoli [Tog73] in the compact case, conjectured by Nash [Na52]). Hence any riemannian locally symmetric space is diffeomorphic to a non-singular semi-algebraic set. On the other hand such abstract real algebraic models are useless if they don’t satisfy some basic functorial properties. A crucial feature of the geometry of arithmetic quotients is the existence of infinitely many real-analytic finite self-correspondences: any element $g ∈ G(\mathbb{Q})$ commensurates $Γ$ (meaning that the intersection $gΓg^{-1} \cap Γ$ is of finite index in both $Γ$ and $gΓg^{-1}$) hence defines a Hecke correspondence

$$c_g = (c_1, c_2) : S_{Γ,G,M} → S_{Γ,G,M},$$

where the left and right maps are the natural finite étale projections, and the map in the middle is induced by the left multiplication by $g$ on $G$. We would like these Hecke correspondences to be real algebraic. Such functorial real algebraic models do exist in certain cases: see [Jaf75], [Jaf78], [Le79]; but we don’t know of any general procedure for producing such a nice real algebraic structure on all arithmetic quotients. Hence our need to work with a more general notion of tame geometry.

Recall that a structure $S$ in the sense of model theory is a collection $(S_n)_{n ∈ \mathbb{N}^*}$, where $S_n$ is a set of subsets of $\mathbb{R}^n$ (called the $S$-definable sets), such that: all algebraic subsets of $\mathbb{R}^n$ are in $S_n$; $S_n$ is a boolean subalgebra of the power set of $\mathbb{R}^n$; if $A ∈ S_n$ and $B ∈ S_m$ then $A × B ∈ S_{n+m}$; if $p : \mathbb{R}^{n+1} → \mathbb{R}^n$ is a linear projection and $A ∈ S_{n+1}$ then $p(A) ∈ S_n$. A function $f : A → B$ between $S$-definable sets is said to be $S$-definable if its graph is $S$-definable. The structure $S$ is said in addition to be o-minimal if the definable subsets of $\mathbb{R}$ are precisely the finite unions of points and intervals (i.e. the semi-algebraic subsets of $\mathbb{R}$). This o-minimal axiom guarantees the possibility of doing geometry using definable sets as basic blocks: it excludes infinite countable sets, like $\mathbb{Z} ⊂ \mathbb{R}$, as well as Cantor sets or space-filling curves, to be definable. Intuitively, subsets of $\mathbb{R}^n$ definable in
an o-minimal structure are the ones having at the same time a reasonable local topology and a tame topology at infinity. Given an o-minimal structure \( \mathcal{S} \), there is an obvious notion of \( \mathcal{S} \)-definable manifold: this is a manifold \( S \) admitting a finite atlas of charts \( \varphi_i : U_i \rightarrow \mathbb{R}^n, i \in I \), such that the intersections \( \varphi_i(U_i \cap U_j), i, j \in I \), are \( \mathcal{S} \)-definable subset of \( \mathbb{R}^n \) and the change of coordinates \( \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \rightarrow \varphi_i(U_i \cap U_j) \) are \( \mathcal{S} \)-definable maps.

The simplest o-minimal structure is \( \mathbb{R}_{\text{alg}} \), the definable sets being the semi-algebraic subsets. There exist more general o-minimal structures. A result of Van den Dries based on Gabrielov’s results [Ga68] shows that the structure

\[
\mathbb{R}_{\text{an}} := (\mathbb{R}, +, \times, <, \{f\} \text{ for } f \text{ restricted analytic function})
\]

generated from \( \mathbb{R}_{\text{alg}} \) by adding the restricted analytic functions is o-minimal. Here a real function on \( \mathbb{R}^n \) is restricted analytic if it is zero outside \([0,1]^n\) and coincides on \([0,1]^n\) with a real analytic function \( g \) defined on a neighbourhood of \([0,1]^n\). The \( \mathbb{R}_{\text{an}} \)-definable sets of \( \mathbb{R}^n \) are the globally subanalytic subsets of \( \mathbb{R}^n \) (i.e. the ones which are subanalytic in the compactification \( \mathbb{P}^n \mathbb{R} \) of \( \mathbb{R}^n \)). A deep result of Wilkie [Wil96] states that the structure \( \mathbb{R}_{\text{exp}} := (\mathbb{R}, +, \times, <, \exp : \mathbb{R} \rightarrow \mathbb{R}) \) generated from \( \mathbb{R}_{\text{alg}} \) by making the real exponential function definable is also o-minimal. Finally the structure \( \mathbb{R}_{\text{an,exp}} := (\mathbb{R}, +, \times, <, \exp, \{f\} \text{ for } f \text{ restricted analytic function}) \) generated by \( \mathbb{R}_{\text{an}} \) and \( \mathbb{R}_{\text{exp}} \) is still o-minimal [VdM94].

The first result of this paper is the following:

**Theorem 1.1.** Let \( G \) be a connected linear semi-simple algebraic \( \mathbb{Q} \)-group, \( \Gamma \subset G(\mathbb{Q})^+ \) a torsion-free arithmetic lattice of \( G := G(\mathbb{R})^+ \), and \( M \subset G \) a connected compact subgroup.

1. The arithmetic quotient \( S_{\Gamma, G, M} := \Gamma \backslash G/M \) admits a natural structure of \( \mathbb{R}_{\text{alg}} \)-definable manifold, characterized by the following property. Let \( G/M \) be endowed with its natural semi-algebraic structure (see Lemma 2.1) and \( \mathcal{S} \subset G/M \) be a semi-algebraic Siegel set (see Section 2.2 for the definition of Siegel sets). Then

\[
\pi|_{\mathcal{S}} : \mathcal{S} \rightarrow S_{\Gamma, G, M}
\]

is \( \mathbb{R}_{\text{alg}} \)-definable.

In particular, there exists a semi-algebraic fundamental set \( \mathcal{F} \subset G/M \) for the action of \( \Gamma \) on \( G/M \) such that

\[
\pi|_{\mathcal{F}} : \mathcal{F} \rightarrow S_{\Gamma, G, M}
\]

is \( \mathbb{R}_{\text{alg}} \)-definable.

The structure of \( \mathbb{R}_{\text{an}} \)-definable manifold on \( S_{\Gamma, G, M} \) extending its \( \mathbb{R}_{\text{alg}} \)-structure is the one induced by the real-analytic structure with corners of its Borel-Serre compactification \( \overline{S}_{\Gamma, G, M}^{\text{BS}} \).

2. Any morphism \( f : S_{\Gamma, G, M'} \rightarrow S_{\Gamma, G, M} \) of arithmetic quotients is \( \mathbb{R}_{\text{alg}} \)-definable. In particular the Hecke correspondences \( c_g, g \in G(\mathbb{Q})^+ \), on \( S_{\Gamma, G, M} \) are \( \mathbb{R}_{\text{alg}} \)-definable.

Theorem 1.1(1) can be thought as a strengthening and a generalization of the main result of [PS13] (for \( S_{\Gamma, G, K} = \mathcal{A}_g \) the moduli space of principally polarized Abelian
varieties of dimension $g$) and of [KUY16, Theor.1.9] (for a general arithmetic variety), which proved that for any arithmetic variety $S_{G,K}$ endowed with the $\mathbb{R}_{\text{an}}$-definable manifold structure deduced from its complex algebraic Baily-Borel compactification, there exists a semi-algebraic fundamental set $\mathcal{F} \subset G/K$ for the action of $\Gamma$ on $G/K$ such that the map $\pi_{\mathcal{F}} : \mathcal{F} \to S_{G,K}$ is $\mathbb{R}_{\text{an}}$-definable. While these results claim only the definability of $\pi_{\mathcal{F}}$ in $\mathbb{R}_{\text{an,exp}}$ our Theorem 1.1(1) claim it in $\mathbb{R}_{\text{alg}}$. This discrepancy comes from the fact that for $S_{G,K}$ an arithmetic variety, the $\mathbb{R}_{\text{an}}$-definable structure on $S_{G,K}$ extending the natural $\mathbb{R}_{\text{alg}}$-definable structure of Theorem 1.1(1) on $S_{G,K}$ is the one coming from the Borel-Serre compactification $S_{G,K,BS}$ of $S_{G,K}$: it does not coincide with the one coming from the Baily-Borel compactification $S_{G,K,BB}$ but the natural map $S_{G,K,BS} \to S_{G,K,BB}$ is in fact $\mathbb{R}_{\text{an,exp}}$-definable. As we won’t need this result in this paper we just provide the simplest illustration:

**Example 1.2.** Let $\mathcal{H}$ be the Poincaré upper half-plane and $Y_0(1)$ the modular curve $\Gamma \backslash \mathcal{H}$. A semi-algebraic fundamental domain for the action of $\Gamma \backslash \mathcal{H}$ is given by

$$\mathcal{F} := \{(x, y) \in \mathcal{H} \mid x^2 + y^2 \geq 1, -1/2 < x < 1/2 \}.$$  

The Borel-Serre compactification $\overline{Y}_0(1)^{\text{BS}}$ is obtained by adding a circle at infinity to $Y_0(1)$, corresponding to the compactification $\overline{\mathcal{F}}$ of $\mathcal{F}$ obtained by glueing the segment $\{y = \infty, -1/2 < x < 1/2\}$ to $\mathcal{F}$. The Baily-Borel compactification $X_0(1) := \overline{Y}_0(1)^{\text{BB}}$ is the one-point compactification of $Y_0(1)$ and is naturally identified with the complex projective line $\mathbb{P}^1 \mathbb{C}$. The natural map $\overline{Y}_0(1)^{\text{BS}} \to \overline{Y}_0(1)^{\text{BB}}$ contracting the circle at infinity to a point sends a point $(x, t = 1/y) \in [-1/2, 1/2] \times [0, 1)$ to the circle at infinity $t = 0$ to the point $[1, z = \exp(2\pi i x) \exp(-2\pi/t)] \in \mathbb{P}^1 \mathbb{C}$. This map is not globally subanalytic but it is $\mathbb{R}_{\text{an,exp}}$-definable.

It is worth noticing that the proof of the more general Theorem 1.1 is easier than the one in [PS13] (which uses explicit theta functions) or the one in [KUY16] (which uses the delicate toroidal compactifications of [AMRT75]): it relies exclusively on classical properties of Siegel sets (see Section 2.2 for the precise definition of Siegel sets), while the proofs of [PS13] and [KUY16], which apply only to arithmetic varieties, moreover insisted on using only complex analytic maps, thus obscuring to some extent the o-minimality issues.

### 1.3. Moderate geometry of period maps.

Arithmetic quotients of interest to the algebraic geometers arise in Hodge theory as connected Hodge varieties, which are complex analytic quotients of period domains (or more generally Mumford-Tate domains). Let $S$ be a smooth complex quasi-projective variety and let $\mathcal{V} \to S$ be a polarized variation of $\mathbb{Z}$-Hodge structures (PVHS) of weight $k$ on $S$. A typical example of such a PVHS is $\mathcal{V} = R^k \mathcal{F} \to Z$ for $f : X \to S$ a smooth proper morphism; in which case we say that $\mathcal{V}$ is geometric. We refer to [K17] and the references therein for the relevant background in Hodge theory, which we use thereafter. Let $\mathbf{MT}(\mathcal{V})$ be the generic Mumford-Tate group associated to $\mathcal{V}$ (this is a connected reductive $\mathbb{Q}$-group) and $G$ its associated adjoint semi-simple $\mathbb{Q}$-group. The group $G := G(\mathbb{R})^+$ acts by holomorphic transformations and transitively on the Mumford-Tate domain $D = G/M$ associated to
Theorem 1.1. is the following finiteness result on $\Phi(S)$. Theorem 1.3. Theorem 1.3. Section 4.4. So78. Bor72. S Theorem 1.3. Theorem 1.3. So78. Bor72. S is easy in the rare case when the con-

We prove that the period map $\Phi_S$ has a moderate geometry. Let us endow $S$ with the $\mathbb{R}_{an,exp}$-definable manifold structure extending the $\mathbb{R}_{alg}$-definable manifold structure on $S$ coming from its complex algebraic structure; and the connected Hodge manifold $\text{Hod}^0(S, \mathcal{V}) = S_{\Gamma,G,M}$ with the $\mathbb{R}_{an,exp}$-definable manifold structure extending the $\mathbb{R}_{alg}$-definable manifold structure defined in Theorem 1.1).

**Theorem 1.3.** Let $\mathcal{V} \to S$ be a polarized variation of pure Hodge structures of weight $k$ over a smooth complex quasi-projective variety $S$. Let $\Phi_S : S \to \text{Hod}^0(S, \mathcal{V}) = S_{\Gamma,G,M}$ be the holomorphic period map associated to $\mathcal{V}$. Then $\Phi_S$ is $\mathbb{R}_{an,exp}$-definable.

**Remarks 1.4.** (1) Notice that Theorem 1.3 is easy in the rare case when the connected Hodge variety $S_{\Gamma,G,M}$ is compact. In that case, consider $\overline{S}$ a smooth projective compactification of $S$ with normal crossing divisor at infinity. It follows from Borel’s monodromy theorem [Sc73, Lemma (4.5)] and the fact that the cocompact lattice $\Gamma$ does not contain any unipotent element [Rag72, Cor. 11.13] that the monodromy at infinity of $\mathcal{V}$ is finite. Thus, replacing if necessary $S$ by a finite étale cover, the PVHS $\mathcal{V}$ extends to $\overline{S}$. Equivalently the period map $\Phi_S : S \to \text{Hod}^0(S, \mathcal{V}) = S_{\Gamma,G,M}$ extends to a period map $\Phi_{\overline{S}} : \overline{S} \to S_{\Gamma,G,M}$. In particular the period map $\Phi$ is definable in $\mathbb{R}_{an}$ in that case.

(2) When the connected Hodge variety $S_{\Gamma,G,M}$ is an arithmetic variety, Theorem 1.3 implies (see Section 4.4) that $\Phi_S : S \to S_{\Gamma,G,M}$ is an algebraic map, thus recovering a classical result due to Borel [Bor72, Theor. 3.10]. Hence Theorem 1.3 can be thought as an extension of Borel’s result to the general case where the connected Hodge variety $S_{\Gamma,G,M}$ has no algebraic structure. On the other hand, notice that Borel [Bor72, Theor.A] proves in the arithmetic variety case the stronger result that $\Phi_S$ extends to a holomorphic map $\overline{\Phi}_S : \overline{S} \to \overline{S}_{\Gamma,G,M}$, which does not directly follow from Theorem 1.3.

(3) A long standing conjecture of Griffiths, whose proof has recently been announced in [GGLR17], states that $\Phi(S)$ admits a natural completion $\overline{\Phi}(S)$ as a projective algebraic variety (see [So78] for earlier results in this direction) and that the map $\Phi_S : S \to \Phi(S)$ extends to an algebraic map $\overline{\Phi}_S : \overline{S} \to \overline{\Phi}(S)$. This would imply that $\Phi(S)$ has a tame topology, but says nothing about the tameness of $\Phi_S : S \to S_{\Gamma,G,M}$, as the relation between the projective compactification $\overline{\Phi}(S)$ of [GGLR17] and the Hodge variety $S_{\Gamma,G,M}$ is far from clear. Hence Theorem 1.3 seems to go in a direction different from the one followed in [GGLR17].

The main ingredient in the proof of Theorem 1.3 is the following finiteness result on the geometry of Siegel sets:

**Theorem 1.5.** Let $\Phi : (\Delta^*)^n \to S_{\Gamma,G,M}$ be a local period map with unipotent mon-

MT($\mathcal{V}$), with compact isotropy denoted by $M$. If $\Gamma$ is a torsion free arithmetic lattice of $G$ the arithmetic quotient $S_{\Gamma,G,M}$ is a complex analytic manifold called a connected Hodge variety (which carries an algebraic structure in only very few cases). Replacing if necessary $S$ by a finite étale cover, the PVHS $\mathcal{V}$ on $S$ is completely described by its holomorphic period map $\Phi_S : S \to \text{Hod}^0(S, \mathcal{V}) := S_{\Gamma,G,M}$ for a suitable torsion-free arithmetic subgroup $\Gamma \subset G$.

...
universal cover $\mathcal{S}^n$ of $(\Delta^*)^n$. Given constants $R > 0$ and $\eta > 0$ let us define
\[
\mathcal{S}^n_{R,\eta} := \{ z \in \mathcal{S}^n \mid |\Re z| \leq C \text{ and } |\Im z| \geq \eta \}
\]
where $|\Re z| := \sup_{1 \leq j \leq n} |\Re z_i|$ and $|\Im z| := \inf_{1 \leq j \leq n} |\Im z_i|$. 

There exists finitely many Siegel sets $\mathcal{S}_i \subset G/M$, $i \in I$, such that
\[
\tilde{\Phi}(\mathcal{S}^n_{R,\eta}) \subset \bigcup_{i \in I} \mathcal{S}_i.
\]

In the one-variable case ($n = 1$) Theorem 1.5 is due to Schmid (see [Sc73, Cor. 5.29]), with $|I| = 1$. In the multivariable case, Green, Griffiths, Laza and Robles [GGLR17, Claims A.5.8 and A.5.9] show that the result with $|I| = 1$ does not hold.

The main point of the proof of Theorem 1.5 is to show there is a flat frame with respect to which the Hodge form remains Minkowski reduced, up to covering $\mathcal{S}^n_{R,\eta}$ by finitely many sets. We note here that the proof does not use the higher dimensional $SL_2^n$-orbit theorem of [CKS86]. Rather, we deduce the higher-dimensional statement by restricting to curves and using the full power of Schmid’s one-dimensional result, together with the work of [CKS86] and [Ka85] on the asymptotics of Hodge norms.

1.4. Algebraicity of Hodge loci. Recall that the Hodge locus $HL(S, V) \subset S$ associated to the PVHS $V$ is the set of points $s$ in $S$ for which exceptional Hodge tensors for $V_s$ do occur. The locus $HL(S, V)$ is easily seen to be a countable union of irreducible complex analytic subvarieties of $S$, called special subvarieties of $S$ associated to $V$. If $V = R^k f_s^* Q$ for $f : X \to S$ a smooth proper morphism, it follows from the Hodge conjecture that the exceptional Hodge tensors in $V_s$ come from exceptional algebraic cycles in some product $X^N_s$. A Baire category type argument then implies that every special subvariety of $S$ ought to be algebraic. As an immediate corollary of Theorem 1.1, Theorem 1.3, and Peterzil-Starchenko’s Chow o-minimal Theorem 4.9 we obtain an alternative proof of the following result originally proven by Cattani, Deligne and Kaplan [CDK95]:

**Theorem 1.6.** The special subvarieties of $S$ associated to $V$ are algebraic, i.e. the Hodge locus $HL(S, V)$ is a countable union of closed irreducible algebraic subvarieties of $S$.

The proof of Theorem 1.6 in [CDK95] works as follows. Let $S$ be a smooth compactification of $S$ with a simple normal crossing divisor $D$ at infinity. Locally in the analytic topology $S$ identifies with $(\Delta^*)^r \times \Delta^l$ inside $\overline{S} = \Delta^{r+l}$ (where $\Delta$ denotes the unit disk). The $SL_2^n$-orbit theorems of [Sc73] and [CKS86] describes extremely precisely the asymptotic of the period map $\Phi_S$ on $(\Delta^*)^r \times \Delta^l$. Using this description, Cattani, Deligne and Kaplan manage to write sufficiently explicitly the equation of the locus $S(v) \subset (\Delta^*)^r \times \Delta^l$ of the points at which some determination of a given multivalued flat section $v$ of $V$ is a Hodge class to prove that its closure $\overline{S(v)}$ in $\Delta^{r+l}$ is analytic in this polydisk. Our proof via Theorem 1.3 bypasses these delicate local computations, hence seems a worthwhile simplification.

In view of Theorem 1.3, its corollary Theorem 1.6, and the recent proof [BaT17] (using Theorem 1.6 and o-minimal technics) of the Ax-Schanuel conjecture for pure Hodge varieties stated in [K17, Conj. 7.5], we hope to convey the idea that o-minimal
geometry is an important tool in variational Hodge theory. We refer to [K17, section 1.5] for possible applications of these results to the structure of $HL(S, V)$.

Theorem 1.6 has been extended to the case of (graded polarizable, admissible) variation of mixed Hodge structures in [BP09-1], [BP09-2], [BP13], [BPS10], [KNU11] using [CDK95] and the $SL_2$-orbit theorem of [KNU08] which extends [Sc73] and [CKS86] to the mixed case. Our o-minimal proof of Theorem 1.3 should certainly extend to this case, thus giving a simpler proof of the algebraicity of Hodge loci in full generality. We will come back to this problem in a sequel to this paper.

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2. Preliminaries

2.1. Semi-algebraic structure on $G/M$. The existence of a natural semi-algebraic structure on $G/M$, which we use in Theorem 1.1, is classical. We provide a proof for the convenience of the reader.

Lemma 2.1. Let $G$ be a connected semi-simple linear algebraic $\mathbb{Q}$-group, $G := G(\mathbb{R})^+$ the real Lie group connected component of the identity of $G(\mathbb{R})$, and $M \subset G$ a connected subgroup. Then $G/M$ admits a natural structure of a semi-algebraic set, and the projection map $G \rightarrow G/M$ is semi-algebraic.

Remark 2.2. In general $G/M$ does not admit a structure of real algebraic variety. This is already true for $G$: for instance the group $SO(p, q)$ is a real algebraic variety but its connected component $G := SO(p, q)^+$ is only semi-algebraic for $p \geq q > 0$. On the other hand any compact real Lie group $M$ admits a natural structure $M_{\mathbb{R}}$ of real algebraic group, see [OV90, Th. 5, p.133].

Proof. Let $H_{\mathbb{R}}$ be a real reductive algebraic subgroup of $G_{\mathbb{R}}$. Recall the classical:

Sublemma 2.3. (Chevalley) There exists a finite dimensional $G_{\mathbb{R}}$-module $W$ and a line $l \subset W$ such that the stabilizer in $G_{\mathbb{R}}$ of $l$ is precisely $H_{\mathbb{R}}$.

Proof. Consider the action of $G_{\mathbb{R}}$ on itself by left multiplication. Then the stabilizer of the closed subvariety $H_{\mathbb{R}}$ is $H_{\mathbb{R}}$ itself. Thus, $H_{\mathbb{R}}$ is also the stabilizer of the ideal $I(H_{\mathbb{R}}) \subset \mathbb{R}[G_{\mathbb{R}}]$. Let us choose a finite-dimensional space $I \subset I(H_{\mathbb{R}})$ which generates that ideal. As $I(H_{\mathbb{R}})$ is a rational $H_{\mathbb{R}}$-module we can also assume that $I$ is $H_{\mathbb{R}}$-stable. Hence it is contained in a finite dimensional $G_{\mathbb{R}}$-submodule $J \subset \mathbb{R}[G_{\mathbb{R}}]$. As $H_{\mathbb{R}}$ is the stabilizer of $I$, it is also the stabilizer of the line $l := \bigwedge^n I$ of the $G_{\mathbb{R}}$-module $W := \bigwedge^n J$, where $n := \dim(I)$. 

\hfill $\square$
Apply the previous result to $H_{\mathbb{R}} = M_{\mathbb{R}}$, the real algebraic subgroup of $G_{\mathbb{R}}$ such that $M_{\mathbb{R}}(\mathbb{R}) = M$. As the group $M$ is compact connected, it not only stabilizes the line $l$ but fixes any generator $v$ of $l$.

By a classical result of Hilbert (see [W46, Ch. VIII, §14]) the (graded) algebra $\mathbb{R}[W]^{M_{\mathbb{R}}}$ of $M_{\mathbb{R}}$-invariant polynomials on $W$ is finitely generated, say by homogeneous elements $p_1, \ldots, p_d$. Consider the real algebraic map $p : G(\mathbb{R}) \rightarrow W \rightarrow \mathbb{R}^d$ obtained by composing the orbit map of the vector $v \in W$ with $(p_1, \ldots, p_d) : W \rightarrow \mathbb{R}^d$. It identifies $G(\mathbb{R})/M$ with the image $p(G(\mathbb{R}))$, hence $G/M$ with a connected component of $p(G(\mathbb{R}))$. As $p$ is real algebraic the subset $p(G(\mathbb{R}))$, hence its connected component $G/M$, is semi-algebraic. As $p$ is real-algebraic, the projection $G \rightarrow G/M$ is semi-algebraic. □

2.2. Siegel sets. A crucial ingredient in this paper in the classical notion of Siegel sets for $G$, which we recall now. We follow [BJ06a, §2] and refer to [Bor69, §12] for details.

Let $P$ be a $\mathbb{Q}$-parabolic subgroup of $G$. We denote by $N_P$ is unipotent radical and by $L_P$ the Levi quotient $N_P \backslash P$ of $P$. Let $N_P$, $P$, and $L_P$ be the Lie groups of real points of $N_P$, $P$, and $L_P$, respectively. Let $S_P$ be the split center of $L_P$ and $A_P$ the connected component of the identity in $S_P(\mathbb{R})$. Let $M_P := \cap_{\chi \in X^*(L_P)} \ker \chi^2$ and $M_P = M_P(\mathbb{R})$. Then $L_P$ admits a decomposition $L_P = A_PM_P$.

Let $X$ be the symmetric space of maximal compact subgroups of $G := G(\mathbb{R})^+$. Choosing a point $x \in X$ corresponds to choosing a maximal compact subgroup $K_x$ of $G$, or equivalently a Cartan involution $\theta_x$ of $G$. The choice of $x$ defines a unique real Levi subgroup $L_{P,x} \subset P_{\mathbb{R}}$ lifting $(L_P)^\mathbb{R}$ which is $\theta_x$-invariant, see [BS73, 1.9]. Although $L_P$ is defined over $\mathbb{Q}$ this is not necessarily the case for $L_{P,x}$. The parabolic group $P$ decomposes as

$$P = N_PA_{P,x}M_{P,x},$$

inducing a horospherical decomposition of $G$:

$$G = N_PA_{P,x}M_{P,x}K_x. $$

We recall (see [BJ06a, Lemma 2.3]) that the right action of $P$ on itself under the horospherical decomposition is given by

$$ (n_0a_0m_0)(n,a,m) = (n_0 \cdot (a_0m_0)n(a_0m_0)^{-1}, a_0a, m_0m). $$

In the following the reference to the basepoint $x$ in various subscripts is omitted. We let $\Phi(A_P, N_P)$ be the set of characters of $A_P$ on the Lie algebra $n_P$ of $N_P$, “the roots of $P$ with respect to $A_P$”. The value of $\alpha \in \Phi(A_P, N_P)$ on $a \in A_P$ is denoted $a^\alpha$. Notice that the map $a \mapsto a^\alpha$ from $A_P$ to $\mathbb{R}^*$ is semi-algebraic.

There is a unique subset $\Delta(A_P, N_P)$ of $\Phi(A_P, N_P)$ consisting of dim $A_P$ linearly independent roots, such that any element of $\Phi(A_P, N_P)$ is a linear combination with positive integral coefficients of elements of $\Delta(A_P, N_P)$ to be called the simple roots of $P$ with respect to $A_P$.

**Definition 2.4.** (Siegel set) For any $t > 0$, we define $A_{P,t} = \{ a \in A_P \mid a^\alpha > t, \alpha \in \Delta(A_P, N_P) \}$. For any bounded sets $U \subset N_P$ and $W \subset M_PM_{P,x}K_x$ the subset $\mathcal{S} := U \times A_{P,t} \times W \subset G$ is called a Siegel set for $G$ associated to $P$ and $x$.

Given $M \subset K$ a connected compact subgroup, a Siegel set $\mathcal{S}$ for $G/M$ is the image of a Siegel set $\mathcal{S} \subset G$ under the natural projection map $G \rightarrow G/M$. 
Proposition 2.5. [BJ06a, Prop. 2.5]

1. There are only finitely many \( \Gamma \)-conjugacy classes of parabolic \( \mathbb{Q} \)-subgroups. Let \( P_1, \ldots, P_k \) be a set of representatives of the \( \Gamma \)-conjugacy classes of parabolic \( \mathbb{Q} \)-subgroups. There exists Siegel sets \( S_i := U_i \times A_{P_i,t_i} \times W_i \) associated to \( P_i \) and \( x_i \), \( 1 \leq i \leq k \), whose images in \( \Gamma \backslash G/M \) cover the whole space.

2. For any two parabolic subgroups \( P_1 \) and Siegel sets \( S_i \) associated to \( P_i \), \( i = 1, 2 \), the set
\[
\Gamma_{S_1,S_2} := \{ \gamma \in \Gamma \mid \gamma S_1 \cap S_2 \neq \emptyset \}
\]
is finite.

3. Suppose that \( P_1 \) is not \( \Gamma \)-conjugate to \( P_2 \). Fix \( U_i, W_i, i = 1, 2 \). Then \( \gamma S_1 \cap S_2 = \emptyset \) for all \( t_1, t_2 \) sufficiently large.

4. For any fixed \( U, W \), when \( t \gg 0 \), \( \gamma S \cap S = \emptyset \) for all \( \gamma \in \Gamma - \Gamma_P \), where \( \Gamma_P := \Gamma \cap P \).

5. For any two different parabolic subgroups \( P_1 \) and \( P_2 \), when \( t_1, t_2 \gg 0 \) then \( S_1 \cap S_2 = \emptyset \).

When \( U \) and \( W \) are chosen to be relatively compact open semi-algebraic subsets of \( N_P \) and \( M_P K \) respectively then the Siegel set \( S = U \times A_{P,t} \times W \) is semi-algebraic in \( G \). As the projection map \( G \to G/M \) is semi-algebraic, a Siegel set for \( G/M \) image of a semi-algebraic Siegel set for \( G \) is semi-algebraic. We will only consider such semi-algebraic Siegel sets in the rest of the text.

2.3. The Borel-Serre compactification \( S_{\Gamma,G,M}^{BS} \). In [BS73] Borel and Serre construct a natural compactification \( S_{\Gamma,G,K}^{BS} \) of any arithmetic locally symmetric space \( S_{\Gamma,G,K} \) in the category of real-analytic manifolds with corners, using the notion of geodesic actions and \( S \)-spaces. In [BJ06a, §3] Borel and Ji give a uniform construction of the so-called Borel-Serre compactification \( S_{\Gamma,G,M}^{BS} \) of any arithmetic quotient \( S_{\Gamma,G,M} \) in the category of real-analytic manifolds with corners, simplifying the approach of [BS73] as they do not rely anymore on the notion of \( S \)-spaces and delicate inductions: they construct a partial compactification \( \overline{G}^{BS} \) of \( G \) in the category of real-analytic manifolds with corners [BJ06a, Prop.6.3], such that the left \( G(\mathbb{Q})^\tau \)-action on \( G \) (see [BJ06a, prop. 3.12]) and the commuting right \( K \)-action of a maximal compact subgroup \( K \) (see [BJ06a, Prop.3.17]) both extend to an action by weakly analytic maps to \( \overline{G}^{BS} \) (see proof of [BJ06a, Prop. 6.4]). For any neat arithmetic subgroup \( \Gamma \) of \( G \) and compact subgroup \( M \) of \( G \), the action of \( \Gamma \times M \) on \( \overline{G}^{BS} \) is free and proper. The quotient \( S_{\Gamma,G,M}^{BS} := \Gamma \backslash \overline{G}^{BS} / M \) provides a compactification of the arithmetic quotient \( S_{\Gamma,G,M} \) in the category of real-analytic manifolds with corners.

Let us provide the details of this construction we will need. Let \( P \subset G \) be a parabolic subgroup. Let \( \Delta = \{ \alpha_1, \ldots, \alpha_r \} \) be the set of simple roots in \( \Phi(A_P, N_P) \). Consider the semi-algebraic diffeomorphism \( e_P : A_P \to (\mathbb{R}_{>0})^r \) defined by
\[
e_P(a) = (a^{-\alpha_1}, \ldots, a^{-\alpha_r}) \in (\mathbb{R}_{>0})^r \subset \mathbb{R}^r.
\]
Let \( \overline{A_P} = [0, \infty)^r \subset \mathbb{R}^r \) be the closure of \( e_P(A_P) \) in \( \mathbb{R}^r \). We denote by \( \overline{A_{P,t}} \subset \overline{A_P} \) the closure of \( e_P(A_{P,t}) \).
Let
\[ G^{BS} = G \cup \bigcup_{P \in G} (N_P \times (M_P K)) \]
be the Borel-Serre partial compactification of $G$ constructed in [BJ06a, §3.2]. The topology on $G^{BS}$ is such that an unbounded sequence $(y_j)_{j \in \mathbb{N}}$ in $G$ converges to a point $(n, m) \in N_P \times (M_P K)$ if and only if, in terms of the horospherical decomposition $G = N_P \times A_P \times (M_P K)$, $y_j = (n_j, a_j, m_j)$ with $n_j \in N_P$, $a_j \in A_P$, $m_j \in M_P K$, and the components $n_j$, $a_j$ and $m_j$ satisfy the conditions:

1) For any $\alpha \in \Phi(A_P, N_P)$, $(a_j)^{\alpha} \to +\infty$,
2) $n_j \to n$ in $N_P$ and $m_j \to m$ in $M_P K$.

We refer to [BJ06a, p274-275] for the precise description of the similar glueing between $N_P \times (M_P K)$ and $N_Q \times (M_Q K)$ for two different parabolic subgroups $P \subset Q$.

Then:

**Proposition 2.6.** [BJ06a, Prop.3.3] The embedding $N_P \times A_P \times (M_P K) = G \subset G^{BS}$ extends naturally to an embedding $N_P \times \overline{A_P} \times (M_P K) \to G^{BS}$.

We denote by $G(P)$ the image of $N_P \times \overline{A_P} \times (M_P K)$ under this embedding. It is called the corner associated with $P$. As explained in [BJ06a, Prop. 6.3] $G^{BS}$ has the structure of a real-analytic manifold with corners, a system of real analytic neighbourhood of a point $(n, m) \in N_P \times (M_P K)$ being given by the $G^{BS}$ of $U \times \overline{P_P \times W}$, for $U$ a neighborhood of $n$ in $N_P$, $W$ a neighborhood of $m$ in $M_P K$ and $t > 0$, see [BJ06a, Lemma 3.10, Prop. 6.1 and Prop. 6.3]. As the right action of any compact subgroup $M$ of $K$ on $G$ extends to a proper real analytic action on $G^{BS}$, the quotient $G^{BS}/M$ is a partial compactification of $G/M$ in the category of real-analytic manifolds with corners.

The left $G(\mathbb{Q})$-multiplication on $G$ extends to a real analytic action on $G^{BS}/M$: see [BJ06a, Prop. 3.12] for the extension to a continous action and the proof of [BJ06a, Prop. 6.4] for the proof that the extended action is real analytic. The restriction of this extended action to a neat $\Gamma$ is free and properly discontinuous (see [BJ06a, Prop. 3.13 and Prop. 6.4]). Then $\overline{S_{\Gamma,G,M}}^{BS} := \overline{\Gamma \backslash G^{BS}/M}$ is a compact real analytic manifold with corners compactifying $S_{\Gamma,G,M}$. We denote by $\overline{\pi} : \overline{G^{BS}/M} \to \overline{S_{\Gamma,G,M}^{BS}}$ the extension of $\pi$.

3. Proof of Theorem 1.1

3.1. Proof of Theorem 1.1(1).

By Proposition 2.5(1) there exist finitely many $P_1, \ldots, P_k$ parabolic subgroups of $G$ and Siegel sets $\mathcal{G}_i := U_i \times A_{P_i, k} \times W_i, 1 \leq i \leq k$, with $U_i, W_i$ compact semi-algebraic subsets of $N_{P_i}$ and $M_{P_i} K$ respectively, whose images $V_i := \pi(\mathcal{G}_i), 1 \leq i \leq k$ cover $S_{\Gamma,G,M}$. The set $S_{\Gamma,G,M}$ can thus be obtained as the quotient of the set $\prod_{i=1}^{k} \mathcal{G}_i$ by the equivalence relation $E$ defined by

\[ x_1 \in \mathcal{G}_{i_1} \sim_E x_2 \in \mathcal{G}_{i_2} \iff \exists \gamma \in \Gamma \mid \gamma x_1 = x_2. \]

As each $\mathcal{G}_i, 1 \leq i \leq k$, is semi-algebraic, the set $\prod_{i=1}^{k} \mathcal{G}_i$ is $\mathbb{R}_{alg}$-definable. As the action of $\Gamma$ is real-algebraic on $G$, the equivalence relation $E$ is $\mathbb{R}_{alg}$-definable. Moreover
it follows from Proposition 2.5(2) that $E$ is definably proper in the sense of [VDD98, (2.13) p.166]. Hence by [VDD98, (2.15) p.166] the quotient $S_{\Gamma,G,M}$ of the $\mathbb{R}_{\text{alg}}$-definable set $\coprod_{i=1}^{k} \mathcal{G}_i$ by the $\mathbb{R}_{\text{alg}}$-definably proper equivalence relation $E$ is naturally an $\mathbb{R}_{\text{alg}}$-definable manifold: each $V_i$ is $\mathbb{R}_{\text{alg}}$-definable and the restriction $\pi_i : \mathcal{G}_i \to V_i$ of $\pi$ to $\mathcal{G}_i$ is $\mathbb{R}_{\text{alg}}$-definable. The $V_i$, $1 \leq i \leq k$, form an explicit finite atlas of the $\mathbb{R}_{\text{alg}}$-definable manifold $S_{\Gamma,G,M}$.

Let $\mathcal{S} \subset G/M$ be any Siegel set. By Proposition 2.5(1) there exists $\gamma \in \Gamma$ such that $\gamma \mathcal{S}$ is associated to one of the parabolics $\mathcal{P}_i$ for some $1 \leq i \leq k$. Replacing $\mathcal{S}_i$ by a bigger Siegel set for $\mathcal{P}_i$ if necessary in the previous construction, we can assume without loss of generality that $\gamma \mathcal{S}$ is contained in $\mathcal{S}_i$. Hence $\pi_{iS} : \mathcal{S} \to S_{\Gamma,G,M}$ coincide with the composite

$$\mathcal{S} \xrightarrow{\gamma} \gamma \mathcal{S} \xrightarrow{\pi_i} S_{\Gamma,G,M}$$

hence is $\mathbb{R}_{\text{alg}}$-definable.

With the notations above, the set $\mathcal{F} := \cup_{i=1}^{k} \mathcal{G}_i \subset G/M$ is a semi-algebraic fundamental set for the action of $\Gamma$ on $G/M$. As each $\pi_i : \mathcal{S}_i \to S_{\Gamma,G,M}$ is $\mathbb{R}_{\text{alg}}$-definable, it follows that $\pi_{\mathcal{F}} : \mathcal{F} \to S_{\Gamma,G,M}$ is $\mathbb{R}_{\text{alg}}$-definable.

By Proposition A.2 the compact real analytic manifold with corners $\overline{S_{\Gamma,G,M}}^{BS}$ admits a natural structure of $\mathbb{R}_{\text{an}}$-definable manifold with corner. Explicitly: the images $\overline{V}_i := \pi(\mathcal{G}_i) \subset \overline{S_{\Gamma,G,M}}^{BS}$, $1 \leq i \leq k$, cover $\overline{S_{\Gamma,G,M}}^{BS}$ and form a finite atlas of the $\mathbb{R}_{\text{an}}$-definable manifold with corners $\overline{S_{\Gamma,G,M}}^{BS}$. Let us show that the $\mathbb{R}_{\text{an}}$-definable manifold structure on $S_{\Gamma,G,M}$ obtained by restriction of this structure of $\mathbb{R}_{\text{an}}$-definable manifold with corner on $\overline{S_{\Gamma,G,M}}^{BS}$ coincide with the $\mathbb{R}_{\text{an}}$-definable manifold structure extending the $\mathbb{R}_{\text{alg}}$-definable manifold structure on $S_{\Gamma,G,M}$ we just constructed. We are reduced to showing that for each $i$, $1 \leq i \leq k$, the map $\mathcal{S}_i \xrightarrow{\pi_i} \overline{V}_i \cong V_i$ is $\mathbb{R}_{\text{an}}$-definable. It factorises as

$$\mathcal{S}_i \xrightarrow{1_{\mathcal{V}_i} \times \epsilon_{\mathcal{P}_i} \times 1_{W_i}} \overline{\mathcal{S}_i} \xrightarrow{\pi_i} \overline{V}_i.$$  

On the one hand, it follows from the definition (2.4) of $\epsilon_{\mathcal{P}_i} : A_{\mathcal{P}_i,t_i} \to A_{\mathcal{P}_i,t_i} \subset \mathbb{R}^{t_i}$ that $\epsilon_{\mathcal{P}_i}$, hence also $1_{\mathcal{U}_i} \times \epsilon_{\mathcal{P}_i} \times 1_{W_i}$, is semi-algebraic. On the other hand, the map $\overline{\pi_i} : \mathcal{U}_i \times A_{\mathcal{P}_i,t_i} \times W_i \to \overline{V}_i$ is a real analytic map between compact sets, hence is $\mathbb{R}_{\text{an}}$-definable. This concludes the proof that $\pi_{iS} : \mathcal{S} \to S_{\Gamma,G,M}$ is $\mathbb{R}_{\text{an}}$-definable.

This concludes the proof of Theorem 1.1(1). \hfill $\square$

Remark 3.1. Although the $\mathcal{S}_i$, $1 \leq i \leq k$, are semi-algebraic, it is not true in general that $\overline{S_{\Gamma,G,M}}^{BS}$ admits a natural structure of $\mathbb{R}_{\text{alg}}$-definable manifolds with corners: if $\mathcal{P} \subset \mathcal{Q}$ are two parabolics of $G$ it follows from the proof of [BJ06a, Prop. 6.2] that the inclusion of the corner $G(\mathcal{Q}) \subset G(\mathcal{P})$ is real-analytic but not semi-algebraic in general.

3.2. Morphisms of arithmetic quotients are definable: proof of Theorem 1.1(2).

Let $f : S_{\Gamma',G',M'} \to S_{\Gamma,G,M}$ be a morphism of arithmetic quotients. Hence $f$ is deduced from a morphism $\tilde{f} : G' \to G$ of semi-simple linear algebraic $\mathbb{Q}$-group such that $f(M') \subset M$ and $f(\Gamma') \subset \Gamma$.  

Notice that the statement of Theorem 1.1(2) is non-trivial even in $\mathbb{R}_{\mathrm{an}}$. For instance in the case where $f : G' \to G$ is a strict inclusion the morphism $f$ does not usually extend to a real analytic morphism (or even a continuous one) $\overline{f}$ between the Borel-Serre compactifications (in other words the Borel-Serre compactification is not functorial). The problem is that two parabolic subgroups $P_i \subset G$, $i = 1, 2$, can be non conjugate under $\Gamma$ while their intersections $P_i \cap G'$ are $\Gamma'$-conjugate parabolic subgroups of $G'$. However Theorem 1.1(2) will follow from a finiteness result for Siegel sets due to Orr (see Theorem 3.2).

Let $(V'_i)_{1 \leq i \leq k}$ be an $\mathbb{R}_{\mathrm{alg}}$-atlas for $S_{\Gamma',G',M'}$ as in Section 3.1. Showing that $f : S_{\Gamma',G',M'} \to S_{\Gamma,G,M}$ is $\mathbb{R}_{\mathrm{alg}}$-definable is equivalent to showing that for each $i$, $1 \leq i \leq k$, the restriction $f : V'_i \to S_{\Gamma,G,M}$ is $\mathbb{R}_{\mathrm{alg}}$-definable. As the diagram

$$
\begin{array}{ccc}
\mathcal{G}'_i := U'_i \times A'_{P'_i,t'_i} \times W'_i & \xrightarrow{f} & G \\
\pi_i' \downarrow & & \pi \\
V'_i & \xrightarrow{f} & S_{\Gamma,G,M}
\end{array}
$$

is commutative, it is enough to show that the composite

$$
\mathcal{G}'_i \xrightarrow{f} G \xrightarrow{\pi} S_{\Gamma,G,M}
$$

is $\mathbb{R}_{\mathrm{alg}}$-definable.

The case where $G' = G$ is clear from Theorem 1.1(1) (notice that in that case Goresky and MacPherson show in [GM03, Lemma 6.3] (and its proof) that $f$ extends uniquely to a real analytic morphism $\overline{f} : S_{\Gamma',G',M'}^{\Gamma} \to S_{\Gamma,G,M}^{\Gamma}$).

Suppose that $f : G' \to G$ is surjective. Without loss of generality we can assume that $G$, and then $G'$, are adjoint. Then $G' = G \times H$, $S_{\Gamma',G',M'} = S_{\Gamma \cap H,H,M' \cap H}$, the map $f$ coincides with the projection onto the first factor (this projection is obviously $\mathbb{R}_{\mathrm{alg}}$-definable) composed with the morphism of arithmetic quotients $i : S_{\Gamma \cap G,G,M' \cap G} \to S_{\Gamma,G,M}$, which is $\mathbb{R}_{\mathrm{alg}}$-definable from the case $G' = G$. This proves Theorem 1.1(2) when $f : G' \to G$ is surjective.

We are thus reduced to proving Theorem 1.1(2) in the case where $f : G' \to G$ is a strict inclusion. We use the following:

**Theorem 3.2.** ([O17, Theor.1.2]) Let $G$ and $H$ be reductive linear $\mathbb{Q}$-algebraic groups, with $H \subset G$. Let $\mathcal{S}_H := U_H \times A_{P_{H,t}} \times W_H \subset H(\mathbb{R})$ be a Siegel set for $H$.

Then there exists a finite set $C \subset G(\mathbb{Q})$ and a Siegel set $\mathcal{S} := U \times A_{P,t} \times W \subset G(\mathbb{R})$ such that $\mathcal{S}_H \subset C \cdot \mathcal{S}$.

Applying this result to $G' \subset G$ and the Siegel set $\mathcal{S}'_i$ of $G'$, there exists a finite set $C_i \subset G(\mathbb{Q})$ and a Siegel set $\mathcal{S}_i := U_i \times A_{P_{t_i},t_i} \times W_i$ such that the composition (3.1) factorizes as

$$
\mathcal{G}'_i \to C_i \cdot \mathcal{S}_i \xrightarrow{\pi} S_{\Gamma,G,M}.
$$

The inclusion $\mathcal{G}'_i \to C_i \cdot \mathcal{S}_i$ is semi-algebraic. The map $C_i \cdot \mathcal{S}_i \xrightarrow{\pi} S_{\Gamma,G,M}$ is $\mathbb{R}_{\mathrm{alg}}$-definable by Theorem 1.1(1).

This finishes the proof of Theorem 1.1(2). \qed
4. Definability of the period map

4.1. Reduction of Theorem 1.3 to a local statement. In the situation of Theorem 1.3, let $S \subset \overline{S}$ be a smooth compactification such that $\overline{S} - S$ is a normal crossing divisor. Let $(\overline{S}_i)_{1 \leq i \leq m}$ be a finite open cover of $\overline{S}$ such that the pair $(\overline{S}_i, S_i := S \cap \overline{S}_i)$ is biholomorphic to $(\Delta^n, (\Delta^*)^r_i \times \Delta^{|c-i-r_i|})$. To show that the period map $\Phi_S : S \to \text{Hod}^0(S, \mathbb{V}) = S_{\Gamma, G, M}$ is $\mathbb{R}_{\text{an, exp}}$-definable, it is enough to show that for each $i, 1 \leq i \leq m$, the restricted period map

\[
\Phi_{S|S_i} : S_i = (\Delta^*)^{r_i} \times \Delta^{l_i} \to S_{\Gamma, G, M}
\]

is $\mathbb{R}_{\text{an, exp}}$-definable. Without loss of generality we can assume that $r_i = n$ and $l_i = 0$ by allowing some factors with trivial monodromies. Finally we are reduced to proving:

**Theorem 4.1.** Let $\mathbb{V} \to (\Delta^*)^n$ be a polarized variation of pure Hodge structures of weight $k$ over the punctured polydisk $(\Delta^*)^n$, with period map $\Phi : (\Delta^*)^n \to S_{\Gamma, G, M}$. Then $\Phi$ is $\mathbb{R}_{\text{an, exp}}$-definable.

4.2. Proof of Theorem 4.1 assuming Theorem 1.5.

Let us fix $x_0$ a basepoint in $(\Delta^*)^n$. We denote by $V_\mathbb{Z}$ the fiber $V_{x_0}$ of $V$ at $x_0$ (modulo torsion) and define $V := V_\mathbb{Z} \otimes \mathbb{Q}$. It follows from Borel’s monodromy theorem [Sc73, Lemma (4.5)] that the monodromy transformation $T_i \in G(\mathbb{Z}) \subset G/L(V_\mathbb{Z})$, $1 \leq i \leq n$, of the local system $V$, corresponding to counterclockwise simple circuits around the various punctures, are quasi-unipotent. Replacing $(\Delta^*)^n$ by a finite étale cover if necessary we can assume without loss of generality that all the $T_i$’s are unipotent. Let $N_i \in \mathfrak{g}_\mathbb{Q}$, $1 \leq i \leq n$, be the logarithm of $T_i$; this is a nilpotent element in $\mathfrak{g}_\mathbb{Q}$.

Let $\mathfrak{H}$ denote the Poincaré upper-half plane and $\exp(2\pi i \cdot) : \mathfrak{H} \to \Delta^*$ the uniformizing map of $\Delta^*$. Let $\mathfrak{S}_\mathbb{Q} \subset \mathfrak{H}$ be the usual Siegel fundamental set $\{(x, y) \in \mathfrak{H} \mid y > 1, -1/2 < x < 1/2\}$. Consider the commutative diagram:

\[
\begin{array}{ccc}
\mathfrak{S}_\mathbb{Q}^n & \xrightarrow{\Phi} & D = G/M \\
p = \exp(2\pi i \cdot) & \downarrow & \pi \\
(\Delta^*)^n & \xrightarrow{\Phi} & S_{\Gamma, G, M},
\end{array}
\]

where $\tilde{\Phi}$ is the lifting of $\Phi$ to the universal cover. As the restriction $\exp(2\pi i \cdot)|_{\mathfrak{S}_\mathbb{Q}}$ is $\mathbb{R}_{\text{an, exp}}$-definable, the map $p|_{\mathfrak{S}_\mathbb{Q}} : \mathfrak{S}_\mathbb{Q}^n \to (\Delta^*)^n$ is $\mathbb{R}_{\text{an, exp}}$-definable. We are reduced to proving that the composition $\tilde{\Phi} : \mathfrak{S}_\mathbb{Q}^n \xrightarrow{\Phi} G/M \xrightarrow{\pi} S_{\Gamma, G, M}$ is $\mathbb{R}_{\text{an, exp}}$-definable.

**Lemma 4.2.** The map $\tilde{\Phi} : \mathfrak{S}_\mathbb{Q}^n \to G/M$ is $\mathbb{R}_{\text{an, exp}}$-definable.

**Proof.** The nilpotent orbit theorem [Sc73, (4.12)] states that (after maybe shrinking the polydisk) the map $\Phi : \mathfrak{S}_\mathbb{Q}^n \to G/M$ is of the form

\[
\Phi(z) = \exp \left( \sum_{j=1}^{n} z_j N_j \right) \cdot \Psi(p(z))
\]
for \( \Psi : \Delta^n \rightarrow \hat{D} \) a holomorphic map and \( \hat{D} \supset D \) the compact dual of \( D \). The map \( \Psi \) is the restriction to a relatively compact set of a real analytic map. As \( p_{|\mathfrak{S}_n} : \mathfrak{S}_n^n \rightarrow \Delta^n \) is \( \mathbb{R}_{an,exp} \)-definable, it follows that \( (z \mapsto \Psi(p(z)) \) is \( \mathbb{R}_{an,exp} \)-definable.

The action of \( G(\mathbb{C}) \) on \( \hat{D} \) is algebraic, hence \( \mathbb{R}_{an,exp} \)-definable; \( D \) is a semi-algebraic subset of \( \hat{D} \) and \( \exp(\sum_{j=1}^n z_j N_j) \) is a polynomial in the \( z_j \)'s as the \( N_j \)'s are nilpotent.

Hence the result. \( \square \)

**Remark 4.3.** Notice that Lemma 4.2 appears also in [BaT17, Lemma 3.1].

It moreover follows from Theorem 1.5, proven in the next section, that there exist finitely many Siegel sets \( \mathfrak{S}_i \ (i \in I) \) for \( G/M \) such that \( \mathfrak{S}_i \) is \( \mathbb{R}_{an,exp} \)-definable by Theorem 1.1(2), and the set \( I \) is finite, we deduce from Lemma 4.2 that \( \pi \circ \tilde{\Phi} : \mathfrak{S}_n^n \rightarrow S_{G,M} \) is \( \mathbb{R}_{an,exp} \)-definable. This concludes the proof of Theorem 4.1, hence of Theorem 1.3, assuming Theorem 1.5. \( \square \)

### 4.3. Proof of Theorem 1.5.

Before the proof we make some preliminary remarks. Let \( C \subset \mathfrak{g}_\mathbb{R} \) be the convex cone generated by the monodromy logarithms \( N_i \). Recall from [CKS86, p.468] that for each \( M \in C \) we have a weight filtration \( W(M) \) on \( V_z \) (our \( W(M) \) coincide with \( W(M)[-k] \) in loc. cit., where \( W(M) \) denotes the monodromy filtration). Let \( M_j = \sum_{i=1}^j N_i \) and let \( F \in \hat{D} \) be the \( \mathbb{R} \)-split point associated to the limit mixed Hodge structure \( (\Psi(0), W(M_n)) \) as in [CKS86, (2.20)]. Write \( \tilde{\Phi}(z) = \gamma(z)F \) where \( \gamma : \hat{S}_n \rightarrow G(\mathbb{C}) \) is the holomorphic lift considered in [CKS86, (5.2)].

The details of the lift \( \gamma(z) \) will be largely irrelevant here, but for concreteness we briefly recall its definition. Writing \( J^{p,q} \) for the Deligne splitting of the mixed Hodge structure on \( \mathfrak{g}_\mathbb{R} \) induced by \( (F, W(M_n)) \), there is an operator \( \delta \in \mathfrak{g}_\mathbb{R} \) which commutes with all of the \( N_i \) and decreases each of the \( (p,q) \) gradings; we have \( F = e^{-i\delta} \Psi(0) \). We can then write \( \gamma(z) = e^{z \cdot N} e^{i\delta e^{v(p(z))}} \) for a unique lift \( v(p(z)) = \bigoplus_{p<0}^{J^{p,q}} \).

**Definition 4.4.** Let \( \Sigma_n \) be the region \( 0 < x_1 < 1 \) and \( y_1 > \cdots > y_n > 1 \) of \( \mathbb{C}^n \). Let \( \mathcal{O} \) be the ring of real restricted analytic functions on \( \Delta^n \) pulled back to \( \Sigma_n \) via \( p : \Delta^n \rightarrow (\Delta^n)^\ast \), \( \mathcal{O}[x, y, y^{-1}] \) the ring of polynomials in \( x_1, \ldots, x_n, y_1, \ldots, y_n, y_1^{-1}, \ldots, y_n^{-1} \) with coefficients in \( \mathcal{O} \), and \( \mathcal{O}(x, y) \) the fraction field. We say a function \( f \in \mathcal{O}(x, y) \) is roughly monomial if it is within a multiplicatively bounded constant of a monomial on \( \Sigma_n \). We say that \( f \in \mathcal{O}(x, y) \) is roughly polynomial if it is of the form \( \frac{h}{g} \) where \( g \in \mathcal{O}[y] \) and \( h \) is roughly monomial. Note that roughly polynomial functions form a ring which we denote \( \mathcal{T}_n \).

Recall that there is a splitting

\[
V_\mathbb{C} = \bigoplus_{p, q_1, \ldots, q_n} I^{p, q_1, \ldots, q_n}
\]
such that

\[
F^k = \bigoplus_{p \geq k} I^{p, q_1, \ldots, q_n} \quad \quad \quad \quad \quad \quad \quad \quad \quad W(M_j)_k = \bigoplus_{p+q \leq k} I^{p, q_1, \ldots, q_n}.
\]

For simplicity we denote \( \alpha = (p, q_1, \ldots, q_n) \). For \( u, v \in V_\mathbb{C} \) we denote by \( h(u, v) \) the function \( \Sigma_n \rightarrow \mathbb{C} \) mapping \( z \in \Sigma_n \) to the Hodge inner product \( h_z(u, v) \) of \( u \) and \( v \) at \( \Phi(z) \). Likewise we denote \( h(u) = h(u, u) \).
Lemma 4.5. Let \( u \in I^a \) and \( v \in I^b \).

(1) \( h(u) \) is roughly monomial.

(2) \( h(γ(z)u) \) is roughly monomial.

(3) \( h(u, v) \) is roughly polynomial.

Remark 4.6. Note that part (3) of the lemma actually implies that \( h(u, v) \) is roughly polynomial for any pair of elements \( u, v \in V_\mathbb{C} \).

Before the proof, observe that the relevant asymptotics for parts (1) and (2) are from [CKS86, Theorem 5.21]. The asymptotics for part (1) are also proven in [Ka85, Theorem 2.4.2]. To make clear the independence from the higher-dimensional SL\(^2\)-orbit theorem of [CKS86], we observe that the asymptotics for part (2) can also be derived by combining the proof of [Ka85, Corollary 2.4.3] with that of [Ka85, Proposition 4.1.1]. We leave the details to the interested reader.

Proof of Lemma 4.5. We first show that the Hodge inner products in the statement of the proposition are in \( O(y) \). Choose a basis \( w_i \) of \( V_\mathbb{C} \) such that each \( w_i \in I^0 \) for some \( α_i = (p_i, q_i^1, ..., q_i^d) \) and the sequence \( p_i \) is non-increasing. Define an increasing filtration \( K^\bullet \) as follows: \( K^j \) is the span of \( w_1, ..., w_j \). Define \( B(x, y) = Q(x, y) \) and for simplicity call \( γ = γ(z) \). An orthogonal basis of the Hodge filtration at \( γF \) is obtained from the \( γw_i \) by the Gram-Schmidt procedure; call this basis \( \tilde{w}_i \). Let \( u = \sum \tilde{u}_i \) with \( \tilde{u}_i \) a multiple of \( \tilde{w}_i \) and likewise for \( v \). We then have

\[
B(\tilde{w}_i) = \frac{B(γ \det K^i)}{B(γ \det K^{i-1})}
\]

where we have abusively denoted \( \det K^i = w_1 \wedge ... \wedge w_i \). We also have

\[
B(u, \tilde{w}_i) = \frac{B((γ \det K^{i-1}) \wedge u, γ \det K^i)}{B(γ \det K^{i-1})}
\]

\[
h(\tilde{u}_i, \tilde{v}_i) = |B(\tilde{u}_i, \tilde{v}_i)| = \frac{|B(u, \tilde{w}_i)B(\tilde{w}_i, v)|}{B(\tilde{w}_i)}
\]

Thus, \( h(u, v) \) is in \( O(y) \), and part (1) follows. That \( h(γu) \) is in \( O(y) \) similarly follows from the fact that the norm of the projection of \( γu \) to \( \tilde{w}_i \) is

\[
B(γu, \tilde{w}_i) = \frac{B(γ(\det K^{i-1} \wedge u) γ \det K^i)}{B(γ \det K^{i-1})}
\]

Now, \( h(γ \det K^i) = |B(γ \det K^i)| \) is roughly monomial by part (2). It then follows that the same is true for (4.2), and thus that \( h(\tilde{u}_i, \tilde{v}_i) \) is roughly polynomial. \( \square \)

Proof of Theorem 1.5. It is sufficient to show that \( |h_z(u, v)| ≪ h_z(u) \) for \( u \in I^a, v \in I^b \) and \( z \in Σ_n \). Now, by Schmid [Sc73] we know that this result is true in the \( n = 1 \) case. Thus, taking Lemma 4.5 into account, the theorem follows from the following lemma:

Lemma 4.7. Let \( f, g \in O(x, y) \) with \( f \) roughly polynomial and \( g \) roughly monomial. Assume that \( f ≪ g \) when restricted to any set of the form \( \{α_1z_1 + β_1 = α_2z_2 + β_2 = ... = α_kz_k + β_k, z_{k+1} = ζ_{k+1}, ..., z_n = ζ_n \} \) for \( ζ_{k+1}, ..., ζ_n \in S, α_1, ..., α_k ∈ Q^k_+, \) and \( β_1, ..., β_k \in \mathbb{R} \). Then \( f ≪ g \) on all of \( Σ_n \).
implies Borel’s algebraicity theorem.

Theorem 1.3

Proof. By clearing denominators it is clearly sufficient to handle the case where \( g \) is a monomial, and in fact where \( g = 1 \) since \( y_i \) is a unit in \( \mathcal{O}[x, y, y^{-1}] \). We proceed by induction on \( n \), with the case \( n = 1 \) being immediate from the assumptions. Separating out the powers of \( x_1 \) and \( y_1 \) we may write \( f = \sum_j a_J x_1^j y_1^{j_1} \in \mathcal{O}[y] \), where the sum is over finitely many pairs \( J = (j_1, j_2) \). Now, the \( a_J \) are real analytic functions in \( x_2, y_2, y_1^{-1} \), \( t_2 := e(z_2), \ldots, n, y_n y_n^{-1}, t_n := e(z_n) \) up to an error (on \( \Sigma_n \)) of \( \mathcal{O}(y_1^{1-\epsilon} e^{-\epsilon y_1}) \) for some positive \( \epsilon > 0 \). Since this decays faster than any monomial, we may restrict to the case where the \( a_J \) are independent of \( t_1 \).

The lemma will follow immediately once we prove the following claim:

Claim: for each element \( J \) occurring in \( f \), we have that \( a_J y_1^{j_1} \ll 1 \) on \( \Sigma_n \).

First, we claim that all the powers of \( y_1 \) are non-positive. If this is not the case, we can fix the other variables at a point where the coefficient of a positive power of \( y_1 \) is non-zero, and get a contradiction as \( y_1 \to \infty \).

Now, since the powers of \( y_1 \) are non-positive and \( y_1 > y_2 \), it is sufficient to prove the claim when we restrict to \( y_1 = y_2 \). Consider for each integer \( m \) and real number \( c \) the function \( f_{m,c} \) which we obtain from \( f \) by setting \( z_1 = mx_2 + c \). Note that our assumptions still apply to \( f_{m,c} \), it follows that \( f_{m,c} \ll 1 \) on all of \( \Sigma_n \). Thus, we have that \( \sum_j a_J (mx_2 + c)^{j_1} (my_2)^{j_2} \ll 1 \) on all of \( \Sigma_n \). Let \( r \) be the highest power of \( x_1 \) that occurs and \( s \) the highest power of \( y_1 \) that occurs subject to the power of \( x_1 \) being \( r \).

Now let \( F_{m,r} := \frac{1}{r!} \sum_{i=0}^r \binom{r}{i} (-1)^i f_{m,i} \). Note that \( F_{m,r} \ll 1 \) and also that \( F_{m,r} = \sum_{J=(r,s)} m^{j_2} a_J y_2^{j_2} \). By taking a finite linear combination of \( F_{m,r} \) with distinct values of \( m \) we may isolate the \( a_{(r,s)} y_2^{j_2} \) term, from which it follows that \( a_{(r,s)} y_2^{j_2} \ll 1 \), thus proving the claim for this monomial. Subtracting it off and proceeding inductively, the claim follows.

4.4. Theorem 1.1 implies Borel’s algebraicity theorem.

Theorem 4.8. [Bor72, Theor. 3.10] Let \( S \) be a complex algebraic variety and \( f : S \to S_{T,\Gamma,K} \) a complex analytic map to an arithmetic variety \( S_{T,\Gamma,K} \). Then \( f \) is algebraic.

Proof. The map \( f \) is a period map, hence is \( \mathbb{R}_{an,exp} \)-definable by Theorem 1.3. The graph of \( f \) is thus a complex analytic, \( \mathbb{R}_{an,exp} \)-definable, subset of the smooth complex algebraic manifold \( S \times S_{T,\Gamma,K} \). Recall the following o-minimal Chow theorem of Peterzil-Starchenko [PS09, Theor. 4.4 and Corollary 4.5] (see also [MPT17, Theor. 2.2] and [Sc18, Theor. 2.11] for more precise versions), generalizing a result of Fortuna-Lojasiewicz [FL81] in the semi-algebraic case:

Theorem 4.9. (Peterzil-Starchenko) Let \( S \) be a smooth complex algebraic manifold (hence endowing the \( \mathbb{C} \)-analytic manifold \( S \) with a canonical \( \mathbb{R}_{alg} \)-definable manifold structure). Let \( W \subset S \) be a complex analytic subset which is also an \( \mathcal{S} \)-definable subset for some o-minimal structure \( \mathcal{S} \) expanding \( \mathbb{R}_{an} \). Then \( W \) is an algebraic subset of \( S \).

It follows that the graph of \( f \) is an algebraic subvariety of \( S \times S_{T,\Gamma,K} \), hence that \( f \) is algebraic (see [Se56, Prop. 8]).
5. Algebraicity of Hodge loci: proof of Theorem 1.6

We refer to [K17, Section 3.1] for the notions of (connected) Hodge datum and morphism of (connected) Hodge data, connected Hodge varieties and Hodge morphisms of connected Hodge varieties. Notice that any connected Hodge variety is in particular an arithmetic quotient and that any Hodge morphism of connected Hodge varieties is in particular a morphism of arithmetic quotients.

A special subvariety $Y$ of the connected Hodge variety $S_{Γ,G,M}$ is by definition the image $Y := f(S_{Γ,G,M}')$ of some Hodge morphism $f : S_{Γ,G,M}' \rightarrow S_{Γ,G,M}$. It follows from Theorem 1.1(3) and the remark above that any special subvariety of $S_{Γ,G,M}$ is an $\mathbb{R}_{an}$-definable subset of $S_{Γ,G,M}$ (endowed with its $\mathbb{R}_{an}$-structure of Theorem 1.1(1)).

The Hodge locus $HL(S_{Γ,G,M})$ is defined as the (countable) union of special subvarieties of $S_{Γ,G,M}$.

The Hodge locus $HL(S,V)$ coincides with the preimage $Φ_S^{-1}(HL(S_{Γ,G,M}))$. Hence to prove Theorem 1.6 we are reduced to proving that the preimage $W := Φ^{-1}(Y)$ of any special subvariety $Y \subset S_{Γ,G,M}$ is an algebraic subvariety of $S$. By Theorem 1.1 the period map $Φ_S : S \rightarrow S_{Γ,G,M}$ is $\mathbb{R}_{an}$-definable. As $Y \subset S_{Γ,G,M}$ is an $\mathbb{R}_{an}$-definable subset of $S_{Γ,G,M}$ it follows that $W = Φ_S^{-1}(Y)$ is an $\mathbb{R}_{an}$-definable subset of $S$ (in particular has finitely many connected components). As $W$ is also a complex analytic subvariety, the o-minimal Chow Theorem 4.9 of Peterzil-Starchenko implies that $W$ is an algebraic subvariety of $S$, which finishes the proof of Theorem 1.6.

□

Appendix A. Real analytic manifolds with corners and definability

A.1. Real analytic manifolds with corners. From the analytic point of view, the class of real analytic manifolds with corners is natural: a compact real-analytic manifold with corners is the real version of the compactification of a complex analytic manifold by a normal crossing divisor. However this class of manifolds has been poorly studied and even their definition is not universally agreed. We use the one given by [Dou61], which has been clarified and developed in [Joy12]. For the convenience of the reader we recall the basic definitions but we refer to [Joy12] for more details. Notice that Joyce works in the $C^∞$ context, but all the definitions we need translate literally to the real-analytic setting by replacing “smooth” with “real-analytic”.

Let $X$ be a paracompact Hausdorff topological space $X$ and $n ≥ 1$ an integer. An $n$-dimensional chart with corners on $X$ is a pair $(U, ϕ)$ where $U$ is an open subset in $\mathbb{R}^k := \mathbb{R}^k_{>0} \times \mathbb{R}^{n-k}$ for some $0 ≤ k ≤ n$ and $ϕ : U \rightarrow X$ is a homeomorphism with a non-empty open set $ϕ(U)$.

Given $A ⊂ \mathbb{R}^m$ and $B ⊂ \mathbb{R}^n$ and $α : A \rightarrow B$ continuous, we say that $α$ is real-analytic if it extends to a real-analytic map between open neighborhoods of $A$, $B$.

Two $n$-dimensional charts with corners $(U, ϕ)$, $(V, ψ)$ on $X$ are said real-analytically compatible if $ψ^{-1} ∘ ϕ : ϕ(U) \cap ψ(V) \rightarrow ψ^{-1}(ϕ(U) \cap ψ(V))$ is a homeomorphism and $ψ^{-1} ∘ ϕ$ (resp. its inverse) are real-analytic in the sense above.

An $n$-dimensional real analytic atlas with corners for $X$ is a system $\{(U_i, ϕ_i) : i ∈ I\}$ of pairwise real-analytically compatible charts with corners on $X$ with $X = \bigcup_{i ∈ I} ϕ_i(U_i)$. We call such an atlas maximal if is not a proper subset of any other atlas. Any atlas
is contained in a unique maximal atlas: the set of all charts with corners \((U, \varphi)\) on \(X\) compatible with \((U_i, \varphi_i)\) for all \(i \in I\).

A real-analytic manifold with corners of dimension \(n\) is a paracompact Hausdorff topological \(X\) equipped with a maximal \(n\)-dimensional real-analytic atlas with corners. Weakly real-analytic maps between real-analytic manifolds with corners are the continuous maps which are real-analytic in charts (cf. [Joy12, def. 3.1], where a stronger notion of real-analytic map is also defined; we won’t need this strengthened notion).

Given \(X\) a real-analytic \(n\)-manifold with corners, one defines its boundary \(\partial X\) (cf. [Joy12, def. 2.6]). This is a real-analytic \(n\)-manifold with corners for \(n > 0\), endowed with an immersion (not necessarily injective) \(i_X : \partial X \longrightarrow X\) (cf. [Joy12, prop.2.7]) which is real-analytic (cf. [Joy12, Theor. 3.4.(iv)]) in particular weakly real-analytic.

A.2. \(\mathcal{R}\)-definable manifolds with corners. Let \(\mathcal{R}\) be any fixed o-minimal expansion of \(\mathbb{R}\). The notion of \(\mathcal{R}\)-definable manifold is given in [VDD98, chap.10] and in [VdM96, p.507]. We will need the extended notion of \(\mathcal{R}\)-definable manifold with corners.

Let \(X\) be a paracompact Hausdorff topological space \(X\). An \(n\)-dimensional chart with corners \((U, \varphi)\) on \(X\) is said to be \(\mathcal{R}\)-definable if \(U\) is an \(\mathcal{R}\)-definable subset of \(\mathbb{R}^n\) (equivalently: of \(\mathbb{R}^n_\mathbb{R}\)).

Two \(n\)-dimensional \(\mathcal{R}\)-definable charts with corners \((U, \varphi), (V, \psi)\) on \(X\) are said \(\mathcal{R}\)-compatible if \(\psi^{-1} \circ \varphi : \varphi^{-1}(\varphi(U) \cap \psi(V)) \longrightarrow \psi^{-1}(\varphi(U) \cap \psi(V))\) is an \(\mathcal{R}\)-definable homeomorphism between \(\mathcal{R}\)-definable subsets \(\varphi^{-1}(\varphi(U) \cap \psi(V))\) and \(\psi^{-1}(\varphi(U) \cap \psi(V))\) of \(\mathbb{R}^n\).

An \(n\)-dimensional \(\mathcal{R}\)-definable atlas with corners for \(X\) is a system \(\{(U_i, \varphi_i) : i \in I\}\), \(I\) finite, of pairwise \(\mathcal{R}\)-compatible \(\mathcal{R}\)-definable charts with corners on \(X\) with \(X = \bigcup_{i \in I} \varphi_i(U_i)\). Two such atlases \(\{(U_i, \varphi_i) : i \in I\}\) and \(\{(V_j, \psi_j) : j \in J\}\) are said \(\mathcal{R}\)-equivalent if all the “mixed” transition maps \(\psi_j \circ \varphi_i^{-1}\) are \(\mathcal{R}\)-definable.

An \(\mathcal{R}\)-definable manifold with corners of dimension \(n\) is a paracompact Hausdorff topological \(X\) equipped with an \(\mathcal{R}\)-equivalence class of \(n\)-dimensional \(\mathcal{R}\)-definable atlas with corners.

Remark A.1. Notice that the definitions of real-analytic manifold with corners and \(\mathcal{R}\)-definable manifold with corners are parallel, except the crucial fact that we work in a strictly finite setting for \(\mathcal{R}\)-definable manifolds: the set \(I\) of charts has to be finite. This finiteness condition, in addition to the definability condition, ensures the tameness at infinity of the \(\mathcal{R}\)-definable manifolds with corners.

We say that a subset \(Z \subset X\) is \(\mathcal{R}\)-definable (resp. open or closed) if \(\varphi_i^{-1}(Z \cap \varphi_i(U_i))\) is an \(\mathcal{R}\)-definable (resp. open or closed) subset of \(U_i\) for all \(i \in I\). An \(\mathcal{R}\)-definable map between \(\mathcal{R}\)-definable manifolds (with corners) is a map whose graph is an \(\mathcal{R}\)-definable subset of the \(\mathcal{R}\)-definable product manifold (with corners).

A.3. Compact real-analytic manifolds with corners are \(\mathbb{R}_{\text{an}}\)-definable.

Proposition A.2. Let \(X\) be a compact real-analytic \(n\)-manifold with corners. Then \(X\) has a natural structure of \(\mathbb{R}_{\text{an}}\)-definable manifold with corners. Moreover the map \(i_X : \partial X \longrightarrow X\) is \(\mathbb{R}_{\text{an}}\)-definable. In particular the interior \(X \setminus i_X(\partial X)\) is an \(\mathbb{R}_{\text{an}}\)-definable manifold.
Proof. For each point $x$ of $X$ choose $\varphi_x : U_x \rightarrow (X, x)$ a real-analytic chart with corners whose image $\varphi(U_x)$ is a neighborhood of $x$. Without loss of generality we can assume that $U_x \subset \mathbb{R}^n_x$ is relatively compact and semi-analytic, hence $\mathbb{R}_{\text{an}}$-definable. Hence $(U_x, \varphi_x)$ is a real-analytic chart with corners for $X$ which is also an $\mathbb{R}_{\text{an}}$-definable chart with corners for $X$.

Fix $x, y$ two points in $X$. The fact that the two real-analytic charts $(U_x, \varphi_x)$ and $(U_y, \varphi_y)$ are real-analytically compatible implies immediately that they are $\mathbb{R}_{\text{an}}$-compatible.

The space $X$ is compact hence one can extract from the covering family $\{(U_x, \varphi_x), x \in X\}$ a finite subfamily $\{(U_i, \varphi_i), i \in I\}$, such that $X = \bigcup_{i \in I} \varphi_i(U_i)$: this is an $n$-dimensional $\mathbb{R}_{\text{an}}$-definable atlas with corners for $X$, which defines a structure of $\mathbb{R}_{\text{an}}$-definable manifold with corners on $X$.

One easily checks that this structure is independent of the choice of the finite extraction $\{(U_i, \varphi_i), i \in I\}$ of $\{(U_x, \varphi_x), x \in X\}$, and also of the choice of the relatively compact and semi-analytic subsets $U_x$.

Hence $X$ has a natural structure of $\mathbb{R}_{\text{an}}$-definable manifold with corners. The same procedure endows the compact real-analytic $(n - 1)$-manifold with corners $\partial X$ with a natural $\mathbb{R}_{\text{an}}$-definable structure. The fact that $i_X : \partial X \rightarrow X$ is weakly real-analytic implies immediately that $i_X$ is $\mathbb{R}_{\text{an}}$-definable and that the manifold $X \setminus i_X(\partial X)$ is $\mathbb{R}_{\text{an}}$-definable. \qed

References

[AK81] S. Akbulut, H.C. King, The topology of real algebraic sets with isolated singularities, Ann. of Math. 113 (1981), no. 3, 425-446

[AMRT75] A. Ash, D. Mumford, M. Rapoport, Y. Tai, Smooth Compactification of Locally Symmetric varieties, Lie Groups: History Frontiers and applications vol. 4, Math Sci Press. (1975).

[BB66] W.L. Baily, A. Borel, Compactification of arithmetic quotients of bounded symmetric domains, Annals of Math., 84 (1966), 442-528

[BaT17] B. Bakker, J. Tsimerman, The Ax-Schanuel conjecture for variations of Hodge structures, arxiv.org/abs/1712.05088v1

[Bor69] A. Borel, Introduction aux groupes arithmétiques, Publications de l’Institut de Mathématique de l’Université de Strasbourg, XV. Actualités Scientifiques et Industrielles, No. 1341 Hermann, Paris (1969)

[Bor72] A. Borel, Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem, J. Differential Geometry 6 (1972), 543-560

[BJ06a] A. Borel, L.Ji, Compactifications of locally symmetric spaces, J. Differential Geometry 73, (2006), 263-317

[BJ06b] A. Borel, L.Ji, Compactifications of symmetric and locally symmetric spaces, Mathematics: Theory & Applications, Birkhäuser Boston (2006)

[BS73] A. Borel, J.P. Serre, Corners and arithmetic groups, Comm. Math. Helv. 4 (1973), 436-491

[BT65] A. Borel, J.Tits, Groupes réductifs, Publ.Math. IHES 27 (1965), 55-150

[BP09-1] P. Brosnan, G. Pearlstein, Zero loci of admissible normal functions with torsion singularities, Duke Math. J. 150 (2009) no.1, 77-100

[BP09-2] P. Brosnan, G. Pearlstein, The zero locus of an admissible normal function, Ann. of Math. (2) 170 (2009), no. 2, 883-897

[BP13] P. Brosnan, G. Pearlstein, On the algebraicity of the zero locus of an admissible normal function, Compos. Math. 149 (2013), no. 11, 1913-1962.

[BPS10] P.Brosnan, G. Pearlstein, C. Schnell, The locus of Hodge classes in an admissible variation of mixed Hodge structure, C.R. Math. Acad. Sci. Paris 348 (2010), no. 11-12, 657-660
[CK82] E. Cattani, A. Kaplan, Polarized Mixed Hodge structures and the monodromy of a variation of Hodge structure, Invent. Math. 67, 101-115 (1982)

[CDK05] E. Cattani, P. Deligne, A. Kaplan, On the locus of Hodge classes. J. of AMS, 8 (1995), 483-506

[CKS86] E. Cattani, A. Kaplan, W. Schmid, Degeneration of Hodge structures, Ann. of Math. (2) 123 (1986), no. 3, 457-535.

[CKS87] E. Cattani, A. Kaplan, W. Schmid, Variations of polarized Hodge structure: asymptotics and monodromy, Hodge theory (Sant Cugat 1985), 16-31, Lecture Notes in Math. 1246 Springer (1987)

[De74] P. Deligne, La conjecture de Weil II, Publ. Math. IHES 44 (1974), 5-77

[Dou61] A. Douady, Variétés à bord anguleux et voisinages tubulaires, Séminaire Henri Cartan 1961/1962, Exp. 1.

[VDD98] L. van den Dries, Tame Topology and o-minimal structures. LMS lecture note series, 248. Cambridge University Press, 1998.

[VdM94] L. van den Dries, C. Miller On the real exponential field with restricted analytic functions, Israel J. Math. 85 (1994), 19-56.

[VdM96] L. van den Dries, C. Miller Geometric categories and o-minimal structures, Duke Math. J. 84 no.2, (1996), 497-540.

[FL81] E. Fortuna, S. Lojasiewicz, Sur l’algébricité des ensembles analytiques complexes, J. Reine Angew. Math. 320 (1981) 215-220

[Gao15] Z. Gao, Towards the André-Oort conjecture for mixed Shimura varieties: the Ax-Lindemann theorem and lower bounds for Galois orbits of special points, to appear in J. Reine Angew. Math.

[GM03] M. Goresky, R. Macpherson, The topological trace formula, J. Reine Angew. Math. 560 (2003), 77-150

[GGLR17] M. Green, P. Griffiths, R. Laza, C. Robles, Completion of period mappings and ampleness of the Hodge bundle, arXiv:1708.09523

[Gro] A. Grothendieck, Esquisse d’un programme, published in London Math. Soc. Lecture Note Ser. 242, Geometric Galois actions, 1, 5-48, Cambridge Univ. Press, Cambridge

[Ja75] H. A. Jaffe, Real forms of hermitian symmetric spaces, Bull. Amer. Math. Soc. 81 (1975), 456-458

[Ja78] H.A. Jaffe, Anti-holomorphic automorphisms of the exceptional symmetric domains, J. Differential Geom. 13 (1978), no.1, 79-86

[Joy12] D. Joyce, On manifolds with corners, Advances in geometric analysis, Adv. Lect. Math. 21, 225-258, Int. Press, Somerville, MA, 201

[Ka85] M. Kashiwara, The asymptotic behavior of a variation of polarized Hodge structure, Publ. Res. Inst. Math. Sci. 21 (1985), no. 4, 853-875

[KNU08] K. Kato, C. Nakayama, S. Usui, SL(2)-orbit theorem for degeneration of mixed Hodge structure, J. of Algebraic Geom. 17 (2008), no. 3, 401-479

[KNUR] K. Kato, C. Nakayama, S. Usui, Analyticity of the closures of some Hodge theoretic subspaces, Proc. Japan Acad. Ser. A Math. Sci. 87 (2011), no. 9, 167-172

[KU00] K. Kato, S. Usui, Borel-Serre spaces and spaces of SL(2)-orbits, in Algebraic geometry 2000, Adv. Stud. Pure Math., 36, Math. Soc. Japan, Tokyo (2002)

[KUY16] B. Klingler, E. Ullmo, A. Yafaev, The hyperbolic Ax-Lindemann-Weierstraß conjecture, Publ.Math. IHES 123 (2016), 333-360

[KUY17] B. Klingler, E. Ullmo, A. Yafaev, bi-algebraic geometry and the André-Oort conjecture, to appear in Proceedings of 2015 AMS Summer Institute in Algebraic Geometry

[K17] B. Klingler, Hodge loci and atypical intersections: conjectures, to appear in Motives and complex multiplication.

[Le79] D. S. P. Leung, Reflective submanifolds. IV. Classification of real forms of Hermitian symmetric spaces, J. Differential Geom. 14 (1979), no.2., 179-185.

[MPT17] N. Mok, J. Pila, J. Tsimerman, Ax-Schanuel for Shimura varieties, arXiv:1711.02189

[Na52] J. Nash, Real algebraic manifolds, Ann. of Math. 56 (1952), 405-421

[OV] A.L Onishchick, E.B. Vinberg, Lie groups and algebraic groups, Springer Series in Soviet Mathematics, Springer-Verlag (1990)
[O17] M. Orr, *Height bounds and the Siegel property*, https://arxiv.org/abs/1609.01315
[PS13] Y. Peterzil, S. Starchenko, *Definability of restricted theta functions and families of abelian varieties*, Duke Math. J. **162**, (2013), 731-765
[PS09] Y. Peterzil, S. Starchenko, *Complex analytic geometry and analytic-geometric categories*, J. reine angew. Math. 626 (2009), 39-74
[P11] J. Pila, *O-minimality and the Andre-Oort conjecture for C^n*, Annals Math. **173** (2011), 1779-1840.
[PT14] J. Pila, J. Tsimerman, *Ax-Lindemann for \( \mathbb{A}^g \)*, Ann. of Math. **179** (2014), 659-681
[Rag68] M.S. Raghunathan, *A note on quotients of real algebraic groups by arithmetic subgroups*, Invent. Math. **4**, (1967/1968), 318-335
[Rag72] M.S. Raghunathan, *Discrete subgroups of Lie groups*, Ergebnisse der Math. und Ihrer Grenzgebiete **68**, Springer-Verlag (1972)
[Rat91-0] M. Ratner, *On Raghunathan’s measure conjecture*, Ann. of Math. **134** (1991), no.3, 545-607
[Rat91-1] M. Ratner, *Raghunathan’s topological conjecture and distributions of unipotent flows*, Duke Math. J., **63** (1991), no.1, 235-280
[Sc18] T. Scanlon, *Algebraic differential equations from covering maps*, Adv. Math. **330** (2018), 1071-1100
[Sc73] W. Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973), 211-319
[Sc56] J.P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier **6** (1955-56), 1-42
[So78] A.J. Sommese, *On the rationality of the period mapping*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **5** (1978), 683-717
[Ts18] J. Tsimerman, *The André-Oort conjecture for \( \mathbb{A}_g \)*, Annals of Math. (2) **187** (2018), no. 2, 379-390
[UY14] E. Ullmo, A. Yafaev, *Hyperbolic Ax-Lindemann theorem in the cocompact case*, Duke Math. J. **163** (2014) 433-463
[Tog73] A. Tognoli, *Su una congettura di Nash*, Ann. Scuola Norm. Sup. Pisa **27** (1973), 167-185
[W46] H. Weyl, *Classical Groups*, Princeton University Press, Princeton, N.J., 1946.
[Wil96] J. Wilkie, *Model completeness results for expansions of the ordered field or real numbers by restricted Pfaffian functions and the exponential function*, J. Amer. Math. Soc. **9** (1996), no.4, 1051-1094

Benjamin Bakker : University of Georgia
email : bakker@math.uga.edu.

Bruno Klingler : Humboldt Universität zu Berlin
e-mail : bruno.klingler@hu-berlin.de.

Jacob Tsimerman : University of Toronto
e-mail : jacobot@math.toronto.edu.