ON THE ESSENTIAL DIMENSION OF AN ALGEBRAIC GROUP WHOSE CONNECTED COMPONENT IS A TORUS

ZINOVY REICHSTEIN AND FEDERICO SCAVIA

Abstract. Let $p$ be a prime integer, $k$ be a $p$-closed field of characteristic $\neq p$, $T$ be a torus defined over $k$, $F$ be a finite $p$-group, and $1 \to T \to G \to F \to 1$ be an exact sequence of algebraic groups. In this paper we study the essential dimension $\text{ed}(G; p)$ of $G$ at $p$. R. Lőtscher, M. MacDonald, A. Meyer, and the first author showed that

$$\min \dim(V) - \dim(G) \leq \text{ed}(G; p) \leq \min \dim(W) - \dim(G),$$

where $V$ and $W$ range over the $p$-faithful and $p$-generically free $k$-representations of $G$, respectively. This generalizes the formulas for the essential dimension at $p$ of a finite $p$-group due to N. Karpenko and A. Merkurjev (here $T = \{1\}$) and of a torus, due to Lőtscher et al. (here $F = \{1\}$). In both of these cases every $p$-generically free representation of $G$ is $p$-faithful, so the upper and lower bounds on $\text{ed}(G; p)$ given above coincide. In general there is a gap between these bounds. Lőtscher et al. conjectured that the upper bound is, in fact, sharp; that is, $\text{ed}(G; p) = \min \dim(W) - \dim(G)$, where $W$ ranges over the $p$-generically free representations, as above. We prove this conjecture in the case, where $F$ is diagonalizable. Moreover, we give an explicit way to compute $\min \dim(W)$ in this case. As an application of our main theorem we compute $\text{ed}(G; p)$, where $G$ is the normalizer of a split maximal torus in a split simple algebraic group, in all previously inaccessible cases.

1. Introduction

Let $p$ be a prime integer and $k$ be a $p$-closed field of characteristic $\neq p$. That is, the degree of every finite extension $l/k$ is a power of $p$. Consider an algebraic group $G$ defined over $k$, which fits into the exact sequence

$$1 \to T \to G \xrightarrow{\pi} F \to 1,$$

where $T$ is a (not necessarily split) torus and $F$ is a (not necessarily constant) finite $p$-group defined over $k$. We say that a representation $G \to \text{GL}(V)$ is $p$-faithful if its kernel is a finite subgroup of $G$ of order prime to $p$ and $p$-generically free if the isotropy subgroup $G_v$ is a finite group of order prime to $p$ for $v \in V(\overline{k})$ in general position. We denote by $\eta(G)$ (respectively, $\rho(G)$) the smallest dimension of a $p$-faithful (respectively, $p$-generically free) representation. R. Lőtscher, M. MacDonald, A. Meyer, and the first author [14] Theorem

---

2010 Mathematics Subject Classification. 20G15, 14L30, 14E05.

Key words and phrases. Essential dimension, algebraic torus, stabilizer in general position.

Zinovy Reichstein was partially supported by National Sciences and Engineering Research Council of Canada Discovery grant 253424-2017.

Federico Scavia was partially supported by a graduate fellowship from the University of British Columbia.
1.1] showed that the essential $p$-dimension $\text{ed}(G; p)$ satisfies the inequalities

\begin{equation}
\eta(G) - \dim(G) \leq \text{ed}_p(G; p) \leq \rho(G) - \dim(G).
\end{equation}

The inequalities (1.2) represent a common generalization of the formulas for the essential $p$-dimension of a finite constant $p$-group, due to N. Karpenko and A. Merkurjev [11, Theorem 4.1] (where $T = \{1\}$), and of an algebraic torus, due to L"otscher et al. [13] (where $F = \{1\}$). In both of these cases, every $p$-faithful representation of $G$ is $p$-generically free, and thus $\eta(G) = \rho(G)$. In general, $\eta(G)$ can be strictly smaller than $\rho(G)$. L"otscher et al. conjectured that the upper bound of (1.2) is, in fact, sharp.

**Conjecture 1.1.** Let $p$ be a prime integer, $k$ be a $p$-closed field of characteristic $\neq p$, and $G$ be an affine algebraic group defined over $k$. Assume that the connected component $G^0 = T$ is a $k$-torus, and the component group $G/G^0 = F$ is a finite $p$-group. Then

$$
\text{ed}(G; p) = \rho(G) - \dim G,
$$

where $\rho(G)$ is the minimal dimension of a $p$-generically free $k$-representation of $G$.

Informally speaking, the lower bound of (1.2) is the strongest lower bound on $\text{ed}(G; p)$ one can hope to prove by the methods of [11], [13], and [14]. In the case, where the upper and lower bounds of (1.2) diverge, Conjecture 1.1 calls for a new approach.

Conjecture 1.1 appeared in print in [20, Section 7.9] on the list of open problems in the theory of essential dimension. The only bit of progress since then has been a proof in the special case, where $G$ is a semi-direct product of a cyclic group $F = \mathbb{Z}/p\mathbb{Z}$ of order $p$, and a split torus $T = G_m^n$, due to M. Huruguen [9]. Huruguen’s argument relies on the classification of integral representations of $\mathbb{Z}/p\mathbb{Z}$ due to F. Diederichsen and I. Reiner [7, Theorem 74.3]. So far this approach has resisted all attempts to generalize it beyond the case, where $G \simeq G_m^n \rtimes (\mathbb{Z}/p\mathbb{Z})$.

Note that $\eta(G)$ is often accessible by cohomological and/or combinatorial techniques; see Section 6 and Lemma 9.3, as well as the remarks after this lemma. Computing $\rho(G)$ is usually a more challenging problem. The purpose of this paper is to establish Conjecture 1.1 in the case, where $F$ is a diagonalizable abelian $p$-group. Moreover, our main result also gives a way of computing $\rho(G)$ in this case.

**Theorem 1.2.** Let $p$ be a prime integer, $k$ be a $p$-closed field of characteristic $\neq p$, and $G$ be an extension of a (not necessarily constant) diagonalizable $p$-group $F$ by a (not necessarily split) torus $T$, as in (1.1). Then

(a) $\text{ed}(G; p) = \rho(G) - \dim G$.

(b) Moreover, suppose $V$ is a $p$-faithful representation of $G$ of minimal dimension, $k$ is the algebraic closure of $k$, and $S \subset G_T$ is a stabilizer in general position for the $G_k$-action on $V_k$. Then $\rho(G) = \eta(G) + \text{rank}_p(S)$.

Here $\text{rank}_p(S)$ is the largest $r$ such that $S$ contains a subgroup isomorphic to $\mu_p^r$. Most of the remainder of this paper (Sections 2-8) will be devoted to proving Theorem 1.2. A key ingredient in the proof is the Resolution Theorem 7.2, which is based, in turn, on an old valuation-theoretic result of M. Artin and O. Zariski [11, Theorem 5.2]. In Section 9 we will use Theorem 1.2 to complete the computation of $\text{ed}(N; p)$ initiated in [18] and [15]. Here $N$ is the normalizer of a split maximal torus in a split simple algebraic group.
2. STABILIZERS IN GENERAL POSITION

In this section we will assume that the base field $k$ is algebraically closed. Let $G$ be a linear algebraic group defined over $k$ and $X$ be a $G$-variety. A $G$-variety $X$ is called primitive if $G$ transitively permutes the irreducible components of $X$.

Let $X$ be a primitive $G$-variety. A subgroup $S \subset G$ is called a stabilizer in general position for the $G$-action on $X$ if there exists an open $G$-invariant subset $U \subset X$ such that $\text{Stab}_G(x)$ is conjugate to $S$ for every $x \in U(k)$. Note that a stabilizer in general position does not always exist. When it exists, it is unique up to conjugacy.

**Lemma 2.1.** Let $G$ be a linear algebraic group over $k$ and $X$ be a primitive quasi-projective $G$-variety. Assume that the connected component $T = G^0$ is a torus and the component group $F = G/G^0$ is finite of order prime to $\text{char}(k)$. Then there exists a stabilizer in general position $S \subset G$.

Proof. After replacing $G$ by $\overline{G} := G/(K \cap T)$, where $K$ is the kernel of the $G$-action on $X$, we may assume that the $T$-action on $X$ is faithful and hence, generically free. In other words, for $x \in X(k)$ in general position, $\text{Stab}_G(x) \cap T = 1$; in particular, $\text{Stab}_G(x)$ is a finite $p$-group. Since $\text{char}(k) \neq p$, Maschke’s theorem tells us that $\text{Stab}_G(x)$ is linearly reductive. Hence, for $x \in X(k)$ in general position, $\text{Stab}_G(x)$ is $G$-completely reducible; see [10, Lemma 11.24]. The lemma now follows from [16, Corollary 1.5].

□

**Remark 2.2.** The condition that $X$ is quasi-projective can be dropped if $k = \mathbb{C}$; see [22, Theorem 9.3.1]. With a bit more effort this condition can also be removed for any algebraically closed base field $k$ of characteristic $\neq p$. Since we shall not need this more general variant of Lemma 2.1, we leave its proof as an exercise for the reader.

We define the (geometric) $p$-rank $\text{rank}_p(G)$ of an algebraic group $G$ to be the largest integer $r$ such that $G$ contains a subgroup isomorphic to $\mu_p^r = \mu_p \times \cdots \times \mu_p$ ($r$ times).

**Lemma 2.3.** Let $X$ be a normal $G$-variety and $Y \subset X$ be a $G$-invariant prime divisor of $X$. Let $S_X$ and $S_Y$ be stabilizers in general position of the $G$-actions on $X$ and $Y$, respectively. Assume that $p$ is a prime and $\text{char}(k) \neq p$. Then:

(a) $\text{rank}_p(S_Y) \leq \text{rank}_p(S_X) + 1$.

(b) Assume the $G$-action on $X$ is $p$-faithful. Denote the kernel of the $G$-action on $Y$ by $N$. Then there is a group homomorphism $\alpha : N \to \mathbb{G}_m$ such that $\text{Ker}(\alpha)$ does not contain a subgroup of order $p$.

Proof. Let $U \subset X$ be a $G$-invariant dense open subset of $X$ such that $\text{Stab}_G(x)$ is conjugate to $S$ for every $x \in U(k)$. If $Y \cap U \neq \emptyset$, then $S_Y = S_X$, and we are done. Thus we may assume that $Y$ is contained in $Z = X \setminus U$. Since $Y$ is a prime divisor in $X$, it is an irreducible component of $Z$. After removing all other irreducible components of $Z$ from $X$, we may assume that $Z = Y$. Since $X$ is normal, $Y$ intersects the smooth locus of $X$ non-trivially. Choose a $k$-point $y \in Y$ such that both $X$ and $Y$ are smooth at $y$ and $\text{Stab}_G(y)$ is conjugate to $S_Y$. After replacing $S_Y$ by a conjugate, we may assume that $\text{Stab}_G(y) = S_Y$. The group $\text{Stab}_G(y)$ acts on the tangent spaces $T_y(X)$ and $T_y(Y)$, hence on the 1-dimensional normal space $T_y(X)/T_y(Y)$. This gives rise to a character $\alpha : S_Y \to \mathbb{G}_m$. 
(a) Assume the contrary: \( S_Y \) contains \( \mu_p^{r+2} \), where \( r = \text{rank}_p(S_X) \). Then the kernel of \( \alpha \) contains a subgroup \( \mu \simeq \mu_p^{r+1} \). By Maschke’s Theorem, the natural projection \( T_y(X) \to T_y(X)/T_y(Y) \) is \( \mu \)-equivariantly split. Equivalently, there exists a \( \mu \)-invariant tangent vector \( v \in T_y(X) \) which does not belong to \( T_y(Y) \). By the Luna Slice Theorem,

\[
T_y(X^\mu) = T_y(X^\mu).
\]

For a proof in characteristic 0, see [19, Section 6.5]. Generally speaking, Luna’s theorem fails in prime characteristic, but (2.1) remains valid, because \( \mu \) is linearly reductive; see [3, Lemma 8.3]. Now observe that since \( \mu \) does not fit into any conjugate of \( S_X \), the subvariety \( X^\mu \) is contained in \( Y = X \setminus U \). Thus \( v \in T_y(X^\mu) = T_y(X^\mu) \subset T_y(Y) \), a contradiction.

(b) Assume the contrary: \( \text{Ker}(\alpha) \) contains a subgroup \( H \) of order \( p \). Then \( H \) (i) fixes a smooth point \( y \) of \( X \) and (ii) acts trivially on both \( T_y(Y) \) and \( T_y(X)/T_y(Y) \) and hence (since \( H \) is linearly reductive) on \( T_y(X) \). It is well known that (i) and (ii) imply that \( H \) acts trivially on \( X \); see, e.g., the proof of [3, Lemma 4.1]. This contradicts our assumption that the \( G \)-action on \( X \) is \( p \)-faithful.

\[\square\]

3. Covers

Let \( k \) be an arbitrary field, and let \( G \) be a linear algebraic group defined over \( k \). As usual, we will denote the algebraic closure of \( k \) by \( \overline{k} \). A \( G \)-variety \( X \) is called primitive if the \( G_{\overline{k}} \)-variety \( X_{\overline{k}} \) is primitive. A dominant \( G \)-equivariant rational map \( X \to Y \) of primitive \( G \)-varieties is called a cover of degree \( d \) if \( [k(X):k(Y)] = d \). Here if \( X_1, \ldots, X_n \) are the irreducible components of \( X \), then \( k(X) \) is defined as \( k(X_1) \oplus \cdots \oplus k(X_n) \).

**Lemma 3.1.** Let \( p \) be a prime integer, \( G \) be a smooth algebraic group such that \( G/G^0 \) is a finite \( p \)-group, \( W \) be an irreducible \( G \)-variety, \( Z \subset W \) be an irreducible divisor in \( W \), and \( \tau: X \dashrightarrow W \) be a \( G \)-equivariant cover of degree prime to \( p \). Then there exists a commutative diagram of \( G \)-equivariant maps

\[
\begin{array}{ccc}
D & \hookrightarrow & X' \\
\downarrow & & \downarrow \alpha \\
Z' & \cong & W \\
\end{array}
\]

such that \( X' \) is normal, \( \alpha \) is a birational isomorphism, \( D \) is an irreducible divisor in \( X' \), and \( \tau' \) is a cover of \( Z \) of degree prime to \( p \).

**Proof.** Let \( X' \) be the normalization of \( W \) in the function field \( k(X) \). Since \( G \) acts on \( W \) and \( X \) compatibly, there is a \( G \)-action on \( X' \) such that the normalization map \( n: X' \to W \) is \( G \)-equivariant. Over the dense open subset of \( W \) where \( \tau \) is finite, \( n \) factors through \( X \). Thus \( n \) factors into a composition of a birational isomorphism \( \alpha: X' \dashrightarrow X \) and \( \tau: X \dashrightarrow W \). This gives us the right column in the diagram.

To construct \( D \), we argue as in the proof of [21, Proposition A.4]. Denote the irreducible components of the preimage of \( Z \) under \( n \) by \( D_1, \ldots, D_r \subset X' \). These components are
permuted by $G$. Denote the orbits of this permutation action by $O_1, \ldots, O_m$. After renumbering $D_1, \ldots, D_m$, we may assume that $D_i \in O_i$ for $i = 1, \ldots, m$. By the ramification formula (see, e.g., [12, Corollary 6.3, p. 490]),

$$d = \sum_{i=1}^{m} |O_i| \cdot [D_i : Z] \cdot e_i,$$

where $[D_i : Z]$ denotes the degree of the cover $n|D_i| : D_i \to Z$, and $e_i$ is the ramification index of $n$ at the generic point of $D_i$. Since $d$ is prime to $p$, and each $|O_i|$ is a power of $p$, we conclude that there exists an $i \in \{1, \ldots, m\}$ such that $|O_i| = 1$ (i.e., $D_i$ is $G$-invariant) and $[D_i : Z]$ is prime to $p$. We now set $D = D_i$ and $\tau = n|D_i|$. \hfill $\Box$

**Lemma 3.2.** Let $G$ be a linear algebraic group over an algebraically closed field $k$, $p \neq \text{char}(k)$ be a prime number and $\tau : X \dashrightarrow W$ be a cover of $G$-varieties of degree $d$. Assume stabilizers in general position for the $G$-actions on $X$ and $W$ exist; denote them by $S_X$ and $S_W$ respectively. Assume $d$ is prime to $p$.

(a) If $H$ is a finite $p$-subgroup of $S_W$, then $S_X$ contains a conjugate of $H$.

(b) $\text{rank}_p(S_X) = \text{rank}_p(S_W)$.

**Proof.** (a) After replacing $W$ by a dense open subvariety, we may assume that the stabilizer of every point in $W$ is a conjugate of $S_W$. Furthermore, after replacing $X$ by the normal closure of $W$ in $k(X)$, we may assume that $\tau$ is a finite morphism. We claim that $W^{S_W} \subset \tau(X^H)$. Indeed, suppose $w \in W^{S_W}$. Then $H$ acts on $\tau^{-1}(w)$, which is a zero cycle on $X$ of degree $d$. Since $H$ is a $p$-group, it fixes a $k$-point in $\tau^{-1}(w)$. Hence, $X^H \cap \tau^{-1}(w) \neq \emptyset$ or equivalently, $w \in \tau(X^H)$. This proves the claim.

Since the stabilizer of every point of $W$ is conjugate to $S_W$, we have $G \cdot W^{S_W} = W$. By the claim, $\tau(G \cdot X^H) = G \cdot \tau(X^H) = W$. Since $G$ acts transitively on the irreducible components of $X$, this implies that $G \cdot X^H$ contains a dense open subset $X_0 \subset X$. In other words, the stabilizer of every point of $X_0$ contains a conjugate of $H$, and part (a) follows.

(b) Clearly $S_X \subset S_W$ and thus $\text{rank}_p(S_X) \leq \text{rank}_p(S_W)$. On the other hand, if $S_W$ contains $H = \mu_p^r$ for some $r \geq 0$, then by part (a), $S_X$ also contains a copy of $\mu_p^r$. This proves the opposite inequality, $\text{rank}_p(S_X) \geq \text{rank}_p(S_W)$. \hfill $\Box$

### 4. Essential $p$-Dimension

Let $X$ and $Y$ be $G$-varieties. By a correspondence $X \rightsquigarrow Y$ of degree $d$ we mean a diagram of rational maps

$$
\begin{array}{ccc}
X' & \overset{f}{\to} & Y \\
\downarrow \text{degree } d \text{ cover} & & \\
Y & & \end{array}
$$

We say that this correspondence is dominant if $f$ is dominant. A rational map may be viewed as a correspondence of degree 1.

The essential dimension $\text{ed}(X)$ of a generically free $G$-variety $X$ is the minimal value of $\dim(Y) - \dim(G)$, where the minimum is taken over all generically free $G$-varieties $Y$ admitting a dominant rational map $X \dashrightarrow Y$. For a prime integer $p$, the essential
Informally speaking, we will show that these groups approximate "p of G" (5.1) particularly interested in the subgroups the quasi-splitting subgroup $F$ the minimum is taken over all generically free $G$-varieties $X$ admitting a $G$-equivariant dominant correspondence $X \rightsquigarrow Y$ of degree prime to $p$.

It follows from [14, Propositions 2.4 and 3.1] that this minimum does not change if we allow the $G$-action on $Y$ to be $p$-generically free, rather than generically free; we shall not need this fact in the sequel. We will, however, need the following lemma.

**Lemma 4.1.** Requiring $Y$ to be projective in the above definitions does not change the values of $\text{ed}(X)$ and $\text{ed}(X; p)$. That is, for any primitive generically free $G$-variety $X$,

(a) there exists a $G$-equivariant dominant rational map $X \rightarrow Z$, where $Z$ is projective, the $G$-action on $Y$ is generically free, and $\dim(Y) = \text{ed}(X; G) + \dim(G)$.

(b) There exists a $G$-equivariant dominant correspondence $X \rightarrow Z'$ of degree prime to $p$, where $Z'$ is projective, the $G$-action on $Z'$ is generically free, and $\dim(Z') = \text{ed}(X; p) + \dim(G)$.

**Proof.** Let $Y$ be a generically free $G$-variety and $V$ be a generically free linear representation of $G$. It is well known that the $G$-action on $V$ is versal; see, e.g., [17, Proposition 3.10]. Consequently, there exists a $G$-invariant subvariety $Y_1 \subset V$ and a $G$-equivariant dominant rational map $Y \rightarrow Y_1$ so that the $G$-action on $Y_1$ is generically free. After replacing $Y_1$ by its Zariski closure $Z$ in $\mathbb{P}(V \oplus k)$, where $G$ acts trivially on $k$, we obtain a $G$-equivariant dominant rational map $\alpha: Y \rightarrow Z$ such that $Z$ is projective and the $G$-action on $Z$ is generically free.

To prove part (a), choose a dominant $G$-equivariant rational map $f: X \rightarrow Y$ such that the $G$-action on $Y$ is generically free and $\dim(Y)$ is the smallest possible, i.e., $\dim(Y) = \text{ed}(X) + \dim(G)$. Now compose $f$ with the map $\alpha: Y \rightarrow Z$ constructed above. By the minimality of $\dim(Y)$, we have $\dim(Z) = \dim(Y)$, and part (a) follows. The proof of part (b) is the same, except that the rational map $f$ is replaced by a correspondence of degree prime to $p$.

The essential dimension $\text{ed}(G)$ (respectively the essential dimension at $p$, $\text{ed}(G; p)$) of the group $G$ is the maximal value of $\text{ed}(X)$ (respectively, of $\text{ed}(X; p)$) taken over all generically free $G$-varieties $X$.

5. The groups $G_n$

Let $G$ be an algebraic group over $k$ such that the connected component $T = G^0$ is a torus, and the component group $F = G/T$ is a finite $p$-group, as in (1.1). By [14, Lemma 5.3], there exists a finite $p$-subgroup $F' \subset G$ such that $\pi|_{F'}: F' \rightarrow F$ is surjective. We will refer to $F'$ as a “quasi-splitting subgroup” for $G$. We will denote the subgroup generated by $F'$ and $T[n]$ by $G_n$. Here $T[n]$ denotes the $n$-torsion subgroup of $T$, i.e., the kernel of the homomorphism $T \xrightarrow{n} T$. Note that our definition of $G_n$ depends on the choice of the quasi-splitting subgroup $F'$. We will assume that $F'$ is fixed throughout. We will be particularly interested in the subgroups

$$G_1 \subset G_p \subset G_{p^2} \subset G_{p^3} \subset \ldots.$$  

(5.1)

Informally speaking, we will show that these groups approximate “$p$-primary behavior” of $G$ in various ways; see Lemma 5.2 and Proposition 6.2(b) below.
In the sequel we will denote the center of $G$ by $Z(G)$.

\textbf{Lemma 5.1.} (a) Let $z \in Z(G)(\overline{k})$ be a central element of $G$ of order $p^n$ for some $n \geq 0$. Then $z \in G_{p^m}$ for $m \gg 0$.

(b) For every $n \geq 0$, we have $Z(G)[p^n] = Z(G_{p^r})[p^n]$ as group schemes for all $r \gg 0$.

\textit{Proof.} (a) By the definition of $F'$, there exists $g \in F'(-\overline{k})$ and $t \in T(\overline{k})$ such that $g = zt$. Since $F'$ is a $p$-group, $g^N = 1$, where $N$ is a sufficiently high power of $p$. Taking $N \gg p^n$, we also have $z^N = 1$. Since $z$ is central, $1 = g^N = (zt)^N = z^N \cdot t^N$. Thus $t \in T[N](\overline{k}) \subseteq G_r(\overline{k})$ and consequently, $z = gt^{-1}$ is a $\overline{k}$-point of $F' \cdot T[N] = G_N$.

(b) Let $n \geq 0$ be fixed. Since both $Z(G)[p^n]$ and $G_{p^r}$ are finite $p$-groups, and we are assuming that $\text{char}(k) \neq p$, part (a) tells us that there exists $m \geq 0$ such that $Z(G)[p^n] \subseteq Z(G_{p^r})[p^n]$ as group schemes, for all $r \geq m$.

Let $r \geq N$, and let $x \in Z(G_{p^r})[p^n](k_s)$, where $k_s$ is a separable closure of $k$. Let $f_x : T_{k_s} \to T_{k_s}$ be the homomorphism of conjugation by $x$. Passing to character lattices, we obtain a homomorphism $(x) \to \text{GL}_d(\mathbb{Z})$, where $d = \text{rank } X(T_{k_s})$. By a theorem of Jordan, in $\text{GL}_d(\mathbb{Z})$ there are at most finitely many finite subgroups, up to conjugacy. In particular, we may find an integer $N \gg 0$ such that the restriction of $\text{GL}_d(\mathbb{Z}) \to \text{GL}_d(\mathbb{Z}/p^NZ)$ to every finite subgroup is injective.

Thus, if $r \geq N$, $f_x$ is the identity for every $x \in Z(G_{p^r})[p^n](k_s)$. Since $F'$ is contained in $G_{p^r}$, every $x \in Z(G_{p^r})[p^n](k_s)$ commutes with $F'$. Since $G_0$ and $F'$ generate $G$, we deduce that $x \in Z(G)[p^n](k_s)$. This shows that $Z(G_{p^r})[p^n] \subseteq Z(G)[p^n]$ for $r \geq N$. We conclude that for $r \geq \max(N, m)$ we have $Z(G_{p^r})[p^n] = Z(G)[p^n]$.

\textbf{Lemma 5.2.} Let $K$ be a $p$-closed field containing $k$. Then every class $\alpha \in H^1(K, G_{p^r})$ lies in the image of the map $H^1(K, G_{p^r}) \to H^1(K, G)$ for sufficiently high $r$.

\textit{Proof.} Let $\alpha \in H^1(K, G)$. Consider the following commutative diagram with exact rows

\[
\begin{array}{cccccc}
1 & \longrightarrow & T[n] & \longrightarrow & G_n & \longrightarrow & F & \longrightarrow & 1 \\
1 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & F & \longrightarrow & 1
\end{array}
\]

and the associated diagram in Galois cohomology. Let $\overline{\alpha} \in H^1(K, F)$ be the image of $\alpha$ under the natural morphism $H^1(K, G) \to H^1(K, F)$. Since $T$ is abelian, the conjugation actions of $G$ on $T$ and of $G_n$ on $T[n]$ descend to $F$. Twisting the bottom sequence by $\overline{\alpha}$, and setting $U = \overline{\alpha}T$, we see that the fiber of $\overline{\alpha}$ equals the image of $H^1(K, U)$; see [23, Section 1.5.5]. Similarly twisting the top sequence by $\overline{\alpha}$, we see that fiber of $H^1(K, G_n) \to H^1(K, F)$ over $\overline{\alpha}$ equals the image of $H^1(K, U[p^n])$. Hence it suffices to prove the following:

\textbf{Claim:} Let $K$ be a $p$-closed field and $U$ be a torus defined over $K$. Then the natural map $H^1(K, U[p^r]) \to H^1(K, U)$ is surjective for $r$ sufficiently large.

To prove the claim, note that since $K$ is $p$-closed, the torus $U$ is split by an extension $L/K$ of degree $n$, where $n$ is a power of $p$. By a restriction-corestriction argument, it follows that $H^1(K, U)$ is $n$-torsion. Now consider the short exact sequence

\[
\begin{array}{cccccc}
1 & \longrightarrow & U[n] & \longrightarrow & U^\times \overset{n}{\longrightarrow} U & \longrightarrow & 1.
\end{array}
\]
The associated exact cohomology sequence
\[ H^1(K, U[n]) \rightarrow H^1(K, U) \rightarrow H^1(K, U) \times H^1(K, U) \]
shows that \( H^1(K, U[n]) \) surjects onto \( H^1(K, U) \). This completes the proof of the claim and thus of the Lemma \ref{5.2} \qed

6. The index

Let \( \mu \) be a diagonalizable abelian \( p \)-group, and
\[
\begin{array}{c}
1 & \xrightarrow{} & \mu & \xrightarrow{} & G & \xrightarrow{} & \overline{G} & \xrightarrow{} & 1
\end{array}
\]
be a central exact sequence of affine algebraic groups defined over \( k \). This sequence gives rise to the exact sequence of pointed sets
\[ H^1(K, G) \rightarrow H^1(K, \overline{G}) \rightarrow H^2(K, \mu) \]
for any field extension \( K \) of the base field \( k \). Any character \( x : \mu \rightarrow \mathbb{G}_m \), induces a homomorphism \( x_* : H^2(K, \mu) \rightarrow H^2(K, \mathbb{G}_m) \). We define \( \text{ind}^x(G, \mu) \) as the maximal index of \( x_* \circ \partial_K(E) \in H^2(K, \mu) \), where the maximum is taken over all field extensions \( K/k \) and over all \( E \in H^1(K, \overline{G}) \). This number is finite for every \( x \in X(\mu) \); see \cite{17} Theorem 6.1.

Remark 6.1. Since \( \mu \) is a finite \( p \)-group, the index of \( x_* \circ \partial_K(E) \) does not change when \( K \) is replaced by a finite extension \( K'/K \) whose degree is prime to \( p \), and \( E \) is replaced by its image under the natural restriction map \( H^1(K, \overline{G}) \rightarrow H^1(K', \overline{G}) \). Equivalently, we may replace \( K \) by its \( p \)-closure \( K^p \). In other words, the maximal value of \( x_* \circ \partial_K(E) \) will be attained if we only allow \( K \) to range over \( p \)-closed fields extensions of \( k \).

Set \( \text{ind}(G, \mu) := \min \sum_{i=1}^{r} \text{ind}^x_i(G, \mu) \), where the minimum is taken over all generating sets \( x_1, \ldots, x_r \) of the group \( X(\mu) \) of characters of \( \mu \).

Now suppose \( G^0 = T \) is a torus, and \( G/G^0 = F \) is a \( p \)-group, as in \cite{11}. In this case there is a particularly convenient choice of \( \mu \subset G \). Following \cite{14} Section 4 we will denote this central subgroup of \( G \) by \( C(G) \). If \( k \) is algebraically closed, \( C(G) \) is simply the \( p \)-torsion subgroup of the center of \( G \), \( C(G) = Z(G)[p] \). If \( k \) is only assumed to be \( p \)-closed, then we set \( \mu = \text{Split}_k(Z(G)[p]) \) to be the largest \( k \)-split subgroup of \( Z(G)[p] \) in the sense of \cite{13} Section 2.

Proposition 6.2. Let \( G \) be as in \cite{11}. Denote by \( \eta(G) \) the smallest dimension of a \( p \)-faithful \( G \)-representation. Then:
\begin{enumerate}[(a)]
\item \( \text{ind}(G, C(G)) = \eta(G) \).
\item If \( r \) is sufficiently large, then \( \eta(G) = \eta(G[p]) = \text{ed}(G[p]) = \text{ed}(G[p]; p) \).
\end{enumerate}

Proof. (a) Let \( \text{Rep}^x(G) \) be the set of irreducible \( G \)-representations \( \nu : G \rightarrow \text{GL}(V) \) such that \( \nu(z) = x(z) \text{Id}_V \) for every \( z \in \mu(k) \). By the Index Formula \cite{17} Theorem 6.1, \( \text{ind}^x(G) = \gcd \dim(\nu) \), where \( \nu \) ranges over \( \text{Rep}(G) \), and \( \gcd \) stands for the greatest common divisor. By \cite{14} Proposition 4.2, \( \dim(\nu) \) is a power of \( p \) for every irreducible representation \( \nu \) of \( G \) defined over \( k \). Thus one can replace \( \gcd \dim(\nu) \) by \( \min \dim(\nu) \) in the Index Formula. Decomposing an arbitrary representation of \( G \) as a direct sum
of irreducible subrepresentations, we see that \( \text{ind}(G, C(G)) = \) minimal dimension of a \( k \)-representation \( \nu: G \to \text{GL}(V) \) such that the restriction \( \nu|_{C(G)}: C(G) \to \text{GL}(V) \) is faithful. Finally, by [4, Proposition 4.3], \( \nu|_{C(G)} \) is faithful if and only if \( \nu \) is \( p \)-faithful.

(b) Since \( G_{p'} \) is a (not necessarily constant) finite \( p \)-group and \( k \) is \( p \)-closed, the identities \( \eta(G_{p'}) = \text{ed}(G_{p'}) = \text{ed}(G_{p'}; p) \) follow from [3, Theorem 7.1]. It thus remains to show that

\[
\eta(G) = \eta(G_{p'}) \quad \text{for } r \gg 0.
\]

By Lemma 5.1(b), \( Z(G)[p] = Z(G_{p'})[p] \) and thus \( C(G) = C(G_{p'}) \) for \( r \gg 0 \). In view of part (a), (6.2) is thus equivalent to

\[
\text{ind}(G, C(G)) = \text{ind}(G_{p'}, C(G)) \quad \text{for } r \gg 0.
\]

Let \( h \) be the natural projection \( G \to \overline{G} = G/C(G) \). Note that the group \( \overline{G} \) is of the same type as \( G \). That is, the connected component \( \overline{G}^0 \) is the torus \( \overline{T} := h(T) \), and since the homomorphism \( F = G/T \to \overline{G}/\overline{T} \) is surjective, \( F := \overline{G}/\overline{G}^0 \) is a \( p \)-group. Moreover, if \( F' \) is a quasi-splitting subgroup for \( G \) (as defined at the beginning of Section 5), then \( F' := h(F') \) is a quasi-splitting subgroup for \( \overline{G} \). We will use this subgroup to define the finite subgroups \( \overline{G}_n \) of \( \overline{G} \) for every integer \( n \) in the same way as we defined \( G_n \):

\[ \overline{G}_n \text{ is the subgroup of } \overline{G} \text{ generated by } F' \text{ and torsion subgroup } T[n]. \]

Now observe that since \( C(G) \) is \( p \)-torsion in \( G \), \( h(T[n]) \subset T[n] \subset h(T[\text{pn}]) \) and thus

\[
(6.4) \quad h(G_n) \subset \overline{G}_n \subset h(G_{pn}).
\]

for every \( n \). We now proceed with the proof of (6.3). Consider the diagram of natural maps

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & C(G) & \longrightarrow & G & \longrightarrow & \overline{G} & \longrightarrow & 1 \\
\downarrow & & \uparrow i & & \downarrow i & & \uparrow i & & \\
1 & \longrightarrow & C(G) & \longrightarrow & \overline{G}_{p'} & \longrightarrow & h(G_{p'}) & \longrightarrow & 1,
\end{array}
\]

and the induced diagram in Galois cohomology

\[
\begin{array}{ccccccc}
H^1(K, G) & \longrightarrow & H^1(K, \overline{G}) & \longrightarrow & \overline{H}^1(K, C(G)) & \longrightarrow & H^2(K, C(G)) \\
\downarrow i_* & & \downarrow \overline{i}_* & & \downarrow \overline{\partial}_K & & \downarrow \\
H^1(K, G_{p'}) & \longrightarrow & H^1(K, h(G_{p'})) & \longrightarrow & \overline{H}^1(K, C(G)) & \longrightarrow & H^2(K, C(G)).
\end{array}
\]

In view of Remark 6.1 for the purpose of computing \( \text{ind}(G, C(G)) \) and \( \text{ind}(G_{p'}, C(G)) \), we may assume that \( K \) is a \( p \)-closed field. We claim that for \( r \gg 0 \), the vertical map \( \overline{i}_*: H^1(K, h(G_{p'})) \to H^1(K, \overline{G}) \) is surjective for every \( p \)-closed field \( K/k \). If we can prove this claim, then for \( r \gg 0 \), the image of \( \overline{\partial}_K \) in \( H^2(K, C(G)) \) is the same as the image of \( \partial_K \). Thus \( \text{ind}^x(G) \) and \( \text{ind}^x(G_{p'}) \) are the same for every \( x \in X(C(G)) \), and (6.3) will follow.
To prove the claim, note that by (6.4), \( G_{p^r} \subset h(G_{p^{r+1}}) \). Consider the composition
\[
H^1(K, G_{p^r-1}) \longrightarrow H^1(K, h(G_{p^r})) \longrightarrow H^1(K, G).
\]
By Lemma 5.2, the map \( H^1(K, G_{p^r-1}) \rightarrow H^1(K, G) \) is surjective for \( r \gg 0 \). Hence, so is \( \tilde{\iota} \). This completes the proof of the claim and thus of (6.3) and of Proposition 6.2 \( \square \)

7. A resolution theorem for rational maps

The following lemma is a minor variant of [6, Lemma 2.1]. For the sake of completeness, we supply a self-contained proof.

**Lemma 7.1.** Let \( K \subset L \) be a field extension and \( v : L^\times \rightarrow \mathbb{Z} \) be a discrete valuation. Assume that \( v|K^\times \) is non-trivial and denote the residue fields of \( v \) and \( v|K^\times \) by \( L_v \) and \( K_v \), respectively. Then \( \text{trdeg}_K L \geq \text{trdeg}_{K_v} L_v \).

**Proof.** Let \( \bar{x}_1, \ldots, \bar{x}_m \in L_v \). For every \( i \), let \( x_i \) be a preimage of \( \bar{x}_i \) in the valuation ring \( \mathcal{O}_L \). It suffices to show that \( x_1, \ldots, x_m \) are algebraically independent over \( K_v \), then \( x_1, \ldots, x_m \) are algebraically independent over \( K \). To prove this, we argue by contradiction. Suppose there exists a non-zero polynomial \( f \in K[t_1, \ldots, t_m] \) such that \( f(x_1, \ldots, x_m) = 0 \). Multiplying \( f \) by a suitable power of a uniformizing parameter for \( v|K^\times \), we may assume that \( f \in \mathcal{O}_K[x_1, \ldots, x_m] \) and that at least one coefficient of \( f \) has valuation equal to 0. Reducing modulo the maximal ideal of the valuation ring \( \mathcal{O}_K \), we see that \( x_1, \ldots, x_m \) are algebraically dependent over \( K_v \), a contradiction. \( \square \)

Recall that if \( X_1 \) is regular in codimension 1 (e.g. \( X_1 \) is normal) and \( X_2 \) is complete, any rational map \( f : X_1 \dashrightarrow X_2 \) is regular in codimension 1. It follows that if \( D \subset X_1 \) is a prime divisor of \( X_1 \), the closure of the image \( \bar{f}(D) \subset X_2 \) is well-defined.

**Theorem 7.2.** Let \( G \) be a linear algebraic group over \( k \), and \( f : X \dashrightarrow Y \) be a dominant rational map of \( G \)-varieties. Assume that \( Y \) is complete, \( D \subset X \) is a prime divisor, and \( \bar{f}(D) \neq Y \). Then there exist a commutative diagram of \( G \)-equivariant dominant rational maps
\[
\begin{array}{ccc}
Y' & \xrightarrow{\pi} & Y \\
\downarrow f' & \nearrow \downarrow \pi & \\
X & \rightarrow & Y
\end{array}
\]
and a divisor \( E \subset Y' \) such that \( Y' \) is normal and complete, \( \pi : Y' \rightarrow Y \) is a birational morphism, and \( \bar{f}'(D) = E \).

**Proof.** Let \( v : k(X)^\times \rightarrow \mathbb{Z} \) be the valuation given by the order of vanishing or pole along \( D \). Define \( C := \bar{f}(D) \) and let \( w : k(Y)^\times \xrightarrow{f^*} k(X)^\times \xrightarrow{v} \mathbb{Z} \). Let \( \varphi \in k(Y)^\times \) be such that \( f \) is regular in an open neighbourhood \( U \) of the generic point of \( C \), and such that \( \varphi|_{U_{gC}} = 0 \). It follows that \( \varphi \circ f \) is zero on \( D \), hence \( w(f) = v(\varphi \circ f) > 0 \). This shows that \( w \) is non-zero, and so \( w \) is a discrete valuation on \( k(Y) \).

Since \( D \) maps dominantly onto \( C \), we have an inclusion of local rings \( f^* : \mathcal{O}_{Y,C} \hookrightarrow \mathcal{O}_{X,D} \). It follows that if \( \varphi \in \mathcal{O}_{Y,C} \), then \( w(\varphi) = v(\varphi \circ f) \geq 0 \), i.e. \( \mathcal{O}_{Y,C} \) is contained in the valuation ring of \( w \). In other words, \( C \) is the center of \( w \).
Denote by $k(Y)_w$ the residue field of $w$. By Lemma 7.1 we have
\[
\text{trdeg}_k k(X) - \text{trdeg}_k k(Y) \geq \text{trdeg}_k k(D) - \text{trdeg}_k k(Y)_w.
\]
Since $\text{trdeg}_k k(D) = \text{trdeg}_k k(X) - 1$, we obtain that $\text{trdeg}_k k(Y)_w \geq \text{trdeg}_k k(Y) - 1$. By the Zariski-Abhyankar inequality [1, VI, §10.3, Cor 1] we have $\text{trdeg}_k k(Y)_w \leq \text{trdeg}_k k(Y) - 1$, hence
\[
\text{trdeg}_k k(Y)_w = \text{trdeg}_k k(Y) - 1.
\]
By [1] Theorem 5.2, there exists a sequence of proper birational morphisms
\[
Y' = Y_n \to Y_{n-1} \to \cdots \to Y_1 \to Y_0 = Y
\]
such that $Y_{i+1} \to Y_i$ is a blow-up at the center of $w$ on $Y_i$, and such that the center $E'$ of $w$ on $Y'$ is a prime divisor and $Y'$ is normal at the generic point of $E'$. Since $C$ is $G$-invariant, by the universal property of the blow-up, the $G$-action on $Y$ lifts to every $Y_i$, and the maps $Y_{i+1} \to Y_i$ are $G$-equivariant.

We let $\pi : Y' \to Y$ be the composition of the maps $Y_{i+1} \to Y_i$, and $f' : X \to Y'$ be the composition of $f$ with the birational inverse of $\pi$. By construction, $f'$ is $G$-equivariant. It suffices to show that $f'(D) = E$. Since the center of $w$ is the divisor $E \subseteq Y'$, the valuation $w$ is given by the order of vanishing or pole along $E$. If we identify $k(Y')$ with $k(Y)$ via $\pi$, we also have $w = (f')^*v$. It follows that for every $\varphi \in k(Y')^\times$, $\varphi$ is regular and vanishes at the generic point of $E$ if and only if $w(\varphi) > 0$ if and only if $v(\varphi \circ f') = 0$ if and only if $\varphi$ vanishes at the generic point of $f'(D)$. We conclude that $f'(D) = E$, as desired. Finally, after replacing $Y'$ by its normalization, $(Y')^n$ and $E'$ by its preimage in $(Y')^n$, we may assume that $Y'$ is normal everywhere (and not just at a generic point of $E'$). The $G$-action naturally lifts to $(Y')^n$. □

8. Proof of Theorem 1.2

Let $G$ be an algebraic group as in (1.1). Let $V$ be a $p$-faithful representation of $G$ of minimal dimension $\eta(G)$. By Lemma 2.1 there exists a stabilizer in general position $S_V$ for the $G^\times$-action on $V^\times$. Since $V(k)$ is dense in $V$, we may assume without loss of generality that $S_V$ is the stabilizer of a $k$-point of $V$. In particular, we may assume that $S_V$ is a closed subgroup of $G$ defined over $k$. Since $T$ acts $p$-faithfully on $V$, we have $S_V \cap T = \{1\}$.

Reduction 8.1. To prove Theorem 1.2, it suffices to construct a $G$-representation $V'$ such that $\text{dim}(V') = \text{rank}_p(S)$, $W := V \oplus V'$ is $p$-generically free, and
\[
\text{ed}(W; p) = \text{dim}(W) - \text{dim}(G).
\]

Here when we write $\text{ed}(W; p)$, we are viewing $W$ as a generically free $G/\text{Ker}(\varphi)$-variety, were $\varphi : G \to W$ denotes the representation of $G$ on $W$. The kernel, $\text{Ker}(\varphi)$, of this representation is a finite normal subgroup of $G$ of order prime to $p$.

Proof. Suppose we manage to construct $V'$ so that (8.1) holds. Then
\[
\text{ed}(W; p) \overset{(i)}{=} \text{ed}(G/\text{Ker}(\varphi); p) \overset{(ii)}{=} \text{ed}(G; p) \overset{(iii)}{\leq} \rho(G) - \text{dim}(G) \overset{(iv)}{\leq} \text{dim}(W) - \text{dim}(G),
\]
where
(i) follows from the fact that $W$ is a versal $G/\text{Ker}(\varphi)$-variety; see, e.g., [17, Propositions 3.10 and 3.11],

(ii) by [14, Proposition 2.4],

(iii) is the right hand side of (1.2), and

(iv) is immediate from the definition of $\rho(G)$.

If we know that (8.1) holds, then the inequalities (iii) and (iv) are, in fact, equalities. Equality in (iii) yields Theorem 1.2(a). On the other hand, since $\text{ed}(G) = \eta(G) + \text{rank}_p(S)$, equality in (iv) tells us that $\eta(G) + \text{rank}_p(S) = \rho(G)$, thus proving Theorem 1.2(b). □

To construct $W$, we begin with a $p$-faithful linear representation $\nu : G \to \text{GL}(V)$ of minimal possible dimension $d = \eta(G)$. The kernel of $\nu$ is a finite group of order prime to $p$; it is contained in the maximal torus $T$ of $G$. From now on we will replace $G$ by $G/\text{Ker}(\nu)$. All other $G$-actions we will construct (including the linear $G$-action on $W$) will factor through $G/\text{Ker}(\nu)$. In the end we will show that $\text{ed}(W; p) = \text{ed}(G/\text{Ker}(\nu); p)$; once again, this is enough because $\text{ed}(G; p) = \eta(G) = \eta(G/\text{Ker}(\nu)) = \text{ed}(G/\text{Ker}(\nu); p)$ by [14, Proposition 2.4]. In other words, from now on we may (and will) assume that the $G$-action on $V$ is faithful.

Recall that $S_V$ denotes the stabilizer in general position for the $G$-action on $V$, and that we have chosen $S_V$ (which is a priori a closed subgroup of $T$ defined up to conjugacy), so that it is defined over $k$. Since $T$ is a torus, and $T$ acts faithfully on $V$, this action is automatically generically free. That is $S_V \cap T = 1$ or equivalently, the natural projection $\pi|_{S_V} : S_V \to T$ is injective. In particular, $\pi(S_V)$ is diagonalizable. By our assumption $F$ is isomorphic to $\mu_{p^{i_1}} \times \cdots \times \mu_{p^{i_R}}$ for some integers $R \geq 0$ and $i_1, \ldots, i_R \geq 1$. Moreover, this isomorphism can be chosen so that $\pi(S_V) = \mu_{p^{i_1}} \times \cdots \times \mu_{p^{i_R}}$ for some $0 \leq r \leq R$ and some integers $1 \leq j_t \leq i_t$, for every $t = 1, \ldots, r$. Let $\chi_t$ be the composition of $\pi: G \to F$ with the projection map $F \to \mu_{p^{j_t}}$ to the $t$-th component and $V_t$ be a 1-dimensional vector space on which $G$ acts by $\chi_t$. Set $W_d = V$ and $W_{d+t} = V \oplus V_1 \oplus \cdots \oplus V_t$ for $m = 1, \ldots, r$. A stabilizer in general position for the $G$-action on $W_{d+m}$ is clearly

$$S_{W_{d+m}} = S_V \cap \text{Ker}(\chi_1) \cap \cdots \cap \text{Ker}(\chi_m)$$

and thus

$$S_{W_{d+m}} \simeq \pi(S_{W_{d+m}}) = \{1\} \times \cdots \times \{1\} \times \mu_{p^{j_{m+1}}} \times \cdots \times \mu_{p^{j_{d+r}}}$$

for any $0 \leq m \leq r$. In particular, $S_{W_{d+r}} = \{1\}$, in other words, the $G$-action on $W_{d+r}$ is generically free. We now set

$$W = W_{d+r} = V \oplus V_1 \oplus \cdots \oplus V_r.$$

Having defined $W$, we now proceed with the proof of (8.1). In view of Lemma 4.1(b) it suffices to establish the following.
Proposition 8.2. Let $W$ be as above. Consider a dominant $G$-equivariant correspondence

\[ \xymatrix{ & X & \\
\tau \ar@{-->}[r] & f \ar@{-->}[r] & Y, \\
W \ar@{-->}[u] & W^d_{d+r-1} \ar@{-->}[u]
} \]

of degree prime to $p$, where $Y$ is a $p$-generically free projective $G$-variety. Then $\dim(Y) = \dim(W) = d + r$.

We now proceed with the proof of the proposition. By Lemma 3.1 (with $Z = W^d_{d+r-1}$) there exists a commutative diagram of $G$-equivariant maps

\[ \xymatrix{ D_{d+r-1} \ar[r] & X_{d+r} \\
\tau_{d+r-1} \ar[u] & X_{d+r} \ar[u] \ar[r] & Y_{d+r} \ar[u] \ar[r] & Y, \\
W_{d+r-1} \ar[u] & W_{d+r-1} \ar[u] \ar[r] & W \ar[u]
} \]

such that $X_{d+r}$ is normal, $\alpha_{d+r}$ is a birational isomorphism, $D_{d+r-1}$ is an irreducible divisor in $X_{d+r}$, and $\tau_{d+r-1}$ is a cover of $W_{d+r-1}$ of degree prime to $p$. Let $S_{D_{d+r-1}} \subset G$ be a stabilizer in general position for the $G$-action on $D_{d+r-1}$; it exists by Lemma 2.1. In view of (8.2), Lemma 3.2 tells us that

(8.3) $\text{rank}_p(S_{D_{d+r-1}}) = 1$.

On the other hand, by our assumption the $G$-action on $Y$ is $p$-generically free. Thus the restriction of $f$ (viewed as a dominant rational map $X_{d+r} \to Y$) to $D_{d+r-1}$ cannot be dominant, and Theorem 7.2 applies: there exists a commutative diagram

\[ \xymatrix{ X_{d+r} \ar[r]^{f_{d+r}} & Y_{d+r} \\
\alpha_{d+r} \ar[u] & \sigma_{d+r} \ar[u] \ar[r] & Y_{d+r} \ar[u] \ar[r] & Y, \\
X \ar[u] & X \ar[u] \ar[r] & X \ar[u] \ar[r] & Y,
} \]

of dominant $G$-equivariant rational maps, where $\sigma_{d+r}$ is a birational morphism, $Y_{d+r}$ is normal and complete, and $f_{d+r}$ restricts to a dominant $G$-equivariant rational map $D_{d+r-1} \to E_{d+r-1}$ for some $G$-invariant irreducible divisor $E_{d+r-1}$ of $Y_{d+r}$. We will denote this dominant rational map by $f_{d+r-1} : D_{d+r-1} \to E_{d+r-1}$. We now iterate this construction with $f_{d+r}$ replaced by $f_{d+r-1}$. 
By Lemma 3.3 there exists a commutative diagram of $G$-equivariant maps

\[
\begin{array}{ccc}
D_{d+r-2} & \xrightarrow{\tau_{d+r-2}} & X_{d+r-1} \\
\downarrow & & \downarrow \alpha_{d+r-1} \\
W_{d+r-2} & \xrightarrow{\tau_{d+r-1}} & W_{d+r-1}
\end{array}
\]

such that $X_{d+r-1}$ is normal, $\alpha_{d+r-1}$ is a birational isomorphism, $D_{d+r-2}$ is an irreducible divisor in $X_{d+r-1}$, and $\tau_{d+r-2}$ is a cover of $W_{d+r-2}$ of degree prime to $p$.

Denote a stabilizer in general position for the $G$-action on $E_{d+r-1}$ by $S_{E_{d+r-1}}$. Recall that the $G$-action on $Y$ (and thus $Y_{d+r}$) is $p$-generically free. Since $E_{d+r-1}$ is a $G$-invariant hypersurface in $Y_{d+r}$, Lemma 2.3(a) tells us that $\text{rank}_p(S_{E_{d+r-1}}) \leq 1$. On the other hand, since $X_{d+r-1}$ maps dominantly to $E_{d+r-1}$, $S_{E_{d+r-1}}$ contains a conjugate of $S_{X_{d+r-1}}$ and thus $\text{rank}_p(S_{E_{d+r-1}}) \geq \text{rank}_p(S_{X_{d+r-1}})$, where $\text{rank}_p(S_{X_{d+r-1}}) = 1$ by (8.4). We conclude that $\text{rank}_p(S_{E_{d+r-1}}) = 1$. Now observe that since $\text{rank}_p(S_{E_{d+r-1}}) = 1$ and $\text{rank}_p(S_{X_{d+r-1}}) = 2$ (see (8.2)), $f_{d+r-1}(X_{d+r-2})$ cannot be dense in $E_{d+r-1}$. Consequently, Theorem 7.2 can be applied to $f_{d+r-1}: X_{d+r-1} \to E_{d+r-1}$. It yields a birational morphism $\sigma_{d+r-1}: Y_{d+r-1} \to E_{d+r-1}$ such that $Y_{d+r-1}$ is normal and complete, and the composition $\sigma_{d+r-1}^{-1} \circ f_{d+r-1}$ restricts to a dominant $G$-equivariant rational map $f_{d+r-2}: D_{d+r-2} \to E_{d+r-2}$ for some $G$-invariant prime divisor $E_{d+r-2}$ of $Y_{d+r-1}$. Proceeding recursively, we obtain a commutative diagram of $G$-equivariant maps

\[
\begin{array}{ccc}
X_d & \xrightarrow{f_d} & Y_d \\
\downarrow \alpha_d & & \downarrow \sigma_d \\
D_d & \xrightarrow{f_{d+1}} & Y_{d+1} \\
\downarrow \alpha_{d+1} & & \downarrow \sigma_{d+1} \\
\cdots & \cdots & \cdots \\
\downarrow \tau_d & & \downarrow \sigma_{d+r-2} \\
D_{d+r-2} & \xrightarrow{f_{d+r-1}} & Y_{d+r-1} \\
\downarrow \alpha_{d+r-1} & & \downarrow \sigma_{d+r-1} \\
D_{d+r-1} & \xrightarrow{f_{d+r}} & Y_{d+r} \\
\downarrow \alpha_{d+r} & & \downarrow \sigma_{d+r} \\
\cdots & \cdots & \cdots \\
\downarrow \tau_{d+r-1} & & \downarrow \sigma_{d+r} \\
W_d & \xrightarrow{f} & Y \\
\end{array}
\]

such that for every $m$, we have
(i) $D_{d+m}$ is an irreducible divisor in $D_{d+m}$ and $E_{d+m-1}$ is an irreducible divisor in $Y_{d+m}$,
(ii) the vertical maps $\alpha_{d+m}$ and $\sigma_{d+m}$ are birational isomorphisms,
(iii) $X_{d+m}$ and $Y_{d+m}$ are normal and $Y_{d+m}$ is complete,
(iv) $\text{rank}_p(S_{X_{d+m}}) = \text{rank}_p(S_{Y_{d+m}}) = r - m$,
(v) $\tau_{d+m}$ is a cover of degree prime to $p$.

Note that the subscripts are chosen so that $\dim(X_{d+m}) = \dim(W_{d+m}) = d + m$, for each $m = 0, \ldots, r$. We will eventually show that $\dim(Y_{d+m}) = d + m$ for each $m$ as well, but we do not know it at this point.

**Lemma 8.3.** The $G$-action on $Y_{d+m}$ (or equivalently, on $E_{d+m}$) is $p$-faithful for every $m = 0, \ldots, r$.

Assume, for a moment, that this lemma is established. By our construction $f_d$ may be viewed as a dominant $G$-equivariant correspondence $W_d \sim Y_d$ of degree prime to $p$. Now recall that $W_d = V$ is a $p$-faithful representation of $G$ of minimal possible dimension $\eta(G)$. By Lemma 8.3, the $G$-action on $Y_d$ is $p$-faithful. Restricting to the $p$-subgroup $G_n \subset G$, where $n$ is a power of $p$, we obtain a dominant $G_n$-equivariant correspondence $f_d: V \sim Y_d$ of degree prime to $p$, where the $G_n$-action on $Y$ is faithful. Thus $\dim(Y_d) \geq \text{ed}(G_n; p)$.

When $n$ is a sufficiently high power of $p$, Proposition 8.2 tells us that

$$\text{ed}(G_n; p) = \eta(G_n) = \eta(G) = \dim(V) = d.$$

By conditions (i) and (ii) above, $\dim(Y_{d+m+1}) = \dim(E_{d+m}) + 1 = \dim(Y_{d+m}) + 1$ for each $m = 0, 1, \ldots, r$. Thus $\dim(Y) = \dim(Y_{d+r}) = \dim(Y_d) + r = \dim(V) + r = d + r = \dim(W)$, as desired. This will complete the proof of Proposition 8.2 and thus of Theorem 1.2.

**Proof of Lemma 8.3.** For the purpose of this proof, we may replace $k$ by its algebraic closure $\overline{k}$ and thus assume that $k$ is algebraically closed. We argue by reverse induction on $m$. For the base case, where $m = r$, note that by our assumption the $G$-action on $Y$ is $p$-faithful. Since $Y_{d+r}$ is birationally isomorphic to $Y$, the same is true of the $G$-action on $Y_{d+r}$.

For the induction step, assume that the $G$-action on $Y_{d+m+1}$ is $p$-faithful for some $0 \leq m \leq r - 1$. Our goal is to show that the $G$-action on $Y_{d+m}$ is also $p$-faithful. Let $N$ be the kernel of the $G$-action on $Y_{d+m}$. Recall that by Lemma 2.3(b), there is a homomorphism

$$\alpha: N \to \mathbb{G}_m$$

where $\text{Ker}(\alpha)$ has no elements of order $p$. Since $\text{Ker}(\alpha)$ is a subgroup of $G$, and we are assuming that $G^0 = T$ is a torus and $G/G^0 = F$ is a finite $p$-group, we conclude that

$$\text{Ker}(\alpha) \text{ is a finite subgroup of } T \text{ of order prime to } p.$$

It remains to show $\alpha(N)$ is a finite group of order prime to $p$. Assume the contrary: $\alpha(N)$ contains $\mu_p \subset \mathbb{G}_m$.

**Claim:** There exists a subgroup $\mu_p \simeq N_0 \subset N$ such that $N_0$ is central in $G$.

Since $G^0 = T$ is a torus and $G/G^0 = F$ is a $p$-group, if $N_0 \simeq \mu_p$ is normal in $G$, then the conjugation map $G \to \text{Aut}(\mu_p) \simeq \mathbb{Z}/(p-1)\mathbb{Z}$ is trivial, so $N_0$ is automatically
central. Thus in order to prove the claim, it suffices to show that there exists a subgroup $\mu_p \simeq N_0 \subset N$ such that $N_0$ is normal in $G$. Now consider two cases.

Case 1: $G^0 = T$ does not act $p$-faithfully on $Y_{d+m}$. Then $\mu_p \subset N \cap T \triangleleft G$. In view of (8.4) and (8.5), $N \cap T$ contains exactly one copy of $\mu_p$. This implies that $\mu_p$ is characteristic in $N \cap T$ and hence, normal in $G$, as desired.

Case 2: $N \cap T$ does not contain $\mu_p$, i.e., $N \cap T$ is a finite group of order prime to $p$. Examining the exact sequence

$$1 \rightarrow N \cap T \rightarrow N \rightarrow F = G/T$$

we see that $N$ is a finite group of order $pm$, where $m$ is prime to $p$. Let $\text{Syl}_p(N)$ be the set of Sylow $p$-subgroups of $N$. By Sylow’s theorem $|\text{Syl}_p(N)| \equiv 1 \pmod{p}$. The group $G$ acts on $\text{Syl}_p(N)$ by conjugation. Clearly $T$ acts trivially, and the $p$-group $F = G/T$ fixes a subgroup $N_0 \in \text{Syl}_p$. In other words, $N_0 \simeq \mu_p$ is normal in $G$. This proves the claim.

We are now ready to finish the proof of Lemma 8.3. Let $S_{Y_{d+m}} \subset G$ be a stabilizer in general position for the $G$-action on $Y_{d+m}$. Clearly $N_0 \subset N \subset S_{Y_{d+m}}$. Since $f_{d+m} : X_{d+m} \rightarrow Y_{d+m}$ is a dominant $G$-equivariant rational map, $S_{Y_{d+m}}$ contains (a conjugate of) $S_{X_{d+m}}$. By (iv)

$$\text{rank}_p(S_{Y_{d+m}}) = r - m = \text{rank}_p(S_{X_{d+m}}).$$

(8.6)

In particular, $S_{X_{d+m}}$ contains a subgroup $A$ isomorphic to $\mu_p^{r-m}$. Since $N_0 \simeq \mu_p$ is central in $G$, it has to be contained in $A$; otherwise, $S_{Y_{d+m}}$ would contain a subgroup isomorphic to $A \times \mu_p = (\mu_p)^{r-m+1}$, contradicting (8.6). Thus $\mu_p \simeq N_0 \subset S_{X_{d+m}}$. Moreover, since $N_0$ is normal in $G$, it is contained in every conjugate of $S_{X_{d+m}}$. This implies that $N_0$ stabilizes every point of $X_{d+m}$. We conclude that $N_0$ acts trivially on $X_{d+m}$ and hence on $X_d \subset X_{d+m}$ and on $\tau_d(X_d) = W_d = V$. This contradicts our assumption that $G$ acts $p$-faithfully on $W_d = V$.

This contradiction shows that our assumption that $\alpha(N)$ contains $\mu_p$ was false. Returning to (8.4) and (8.5), we conclude that the kernel $N$ of the $G$-action on $Y_{d+m}$ is a finite group of order prime to $p$. In other words, the $G$-action on $Y_{d+m}$ is $p$-faithful. This completes the proof of Lemma 8.3 and thus of Proposition 8.2 and Theorem 1.2. \[\square\]

Remark 8.4. Our proof of Theorem 1.2 goes through even if $F$ is not abelian, provided that the stabilizer in general position $S_V$ projects isomorphically to $F/[F,F]$. (If $F$ is abelian, this is always the case.)

9. Normalizers of maximal tori in split simple groups

In this section $\Gamma$ will denote a split simple algebraic group over $k$, $T$ will denote a $k$-split maximal torus of $\Gamma$, $N$ will denote the normalizer of $T$ in $\Gamma$, and $W = N/T$ will denote the Weyl group. These groups fit into an exact sequence

$$1 \longrightarrow T \longrightarrow N \underset{\pi}{\longrightarrow} W \longrightarrow 1.$$
A. Meyer and the first author [18] have computed \( \text{ed}(N; p) \) in the case, where \( \Gamma = \text{PGL}_n \), for every prime number \( p \). M. MacDonald [15] subsequently found the exact value of \( \text{ed}(N; p) \) for most other split simple groups \( \Gamma \). One reason this is of interest is that
\[
\text{ed}(N; p) \geq \text{ed}(\Gamma; p);
\]
see, e.g., [17, Section 10a]. Let \( W_p \) denote a Sylow \( p \)-subgroup of \( W \) and \( N_p \) denote the preimage of \( W_p \) in \( N \). Then
\[
\text{ed}(N; p) = \text{ed}(N_p; p);
\]
see [18, Lemma 4.1]. The exact sequence
\[
1 \longrightarrow T \longrightarrow N_p \quad \xrightarrow{\pi} \quad W_p \longrightarrow 1
\]
is of the form of (1.1) and thus the inequalities (1.2) apply to \( N_p \). MacDonald computed the exact value of \( \text{ed}(N; p) = \text{ed}(N_p; p) \) for most split simple linear algebraic groups \( \Gamma \) by showing that the left hand side and the right hand side of the inequalities (1.2) for \( N_p \) coincide. There are two families of groups \( \Gamma \), where the exact value of \( \text{ed}(N; p) \) remained inaccessible by this method, \( \Gamma = \text{SL}_n \) and \( \Gamma = \text{SO}_{4n} \). As an application of Theorem 1.2, we will now compute \( \text{ed}(N; p) \) in these two remaining cases. Our main results are Theorems 9.1 and 9.2 below.

**Theorem 9.1.** Let \( n \geq 1 \) be an integer, and let \( N \) be the normalizer of a \( k \)-split maximal torus \( T \) in \( \text{SL}_n \). Then
\[
\text{(a) } \text{ed}(N; p) = n/p + 1, \text{ if } p \geq 3 \text{ and } n \text{ is divisible by } p, \\
\text{(b) } \text{ed}(N; p) = n/2 + 1, \text{ if } p = 2 \text{ and } n \text{ is divisible by } 4, \\
\text{(c) } \text{ed}(N; p) = \lfloor n/p \rfloor, \text{ if } p \geq 3 \text{ and } n \text{ is not divisible by } p, \\
\text{(d) } \text{ed}(N; p) = \lfloor n/2 \rfloor, \text{ if } p = 2 \text{ and } n \text{ is not divisible by } 4.
\]

**Theorem 9.2.** Let \( k \) be a field of characteristic \( \neq 2 \) and \( n \geq 1 \) be an integer. Let \( N \) be the normalizer of a \( k \)-split maximal torus of \( \text{SO}_{4n} \). Then \( \text{ed}_k(N; 2) = 4n \).

Our proofs of these theorems will rely on the following simple lemma, which is implicit in [18] and [15]. Let \( F \) be a finite discrete \( p \)-group, and let \( M \) be an \( F \)-lattice. The symmetric \( p \)-rank of \( M \) is the minimal cardinality \( d \) of a finite \( H \)-invariant \( p \)-spanning subset \( \{x_1, \ldots, x_d\} \subset M \). Here “\( p \)-spanning” means that the index of the \( \mathbb{Z} \)-module spanned by \( x_1, \ldots, x_d \) in \( M \) is finite and prime to \( p \). Following MacDonald, we will denote the symmetric \( p \)-rank of \( M \) by \( \text{SymRank}(M; p) \).

**Lemma 9.3.** Consider an exact sequence \( 1 \rightarrow T \rightarrow G \rightarrow F \rightarrow 1 \) of algebraic groups over \( k \), as in (1.1). Assume further that \( T \) is a split torus and \( F \) is a constant finite \( p \)-group. Denote the character lattice of \( T \) by \( X(T) \), we will view it as an \( F \)-lattice. Then \( \eta(G) \geq \text{SymRank}(X(T); p) \).

Here \( \eta(G) \) denotes the minimal dimension of a \( p \)-faithful representation of \( G \), as defined in the Introduction, and \( X(T) \) is viewed as an \( F \)-lattice. If we further assume that the sequence (1.1) in Lemma 9.3 is split, then, in fact, \( \eta(G) = \text{SymRank}(X(T); p) \). We shall not need this equality in the sequel, so we leave its proof as an exercise for the reader.

\(^2\)The omission of \( \text{SL}_n \) from [15, Remark 5.11] is an oversight; we are grateful to Mark MacDonald for clarifying this point for us.
Proof. Let $V$ be a $p$-faithful representation of $G$, of minimal dimension $r = \eta(G)$. As a $T$-representation, $V$ decomposes as the direct sum of characters $\chi_1, \ldots, \chi_r$. A simple calculation shows that the $F$-action permutes the $\chi_i$. Let $S \subseteq G$ be the torus generated by the images of the $\chi_i$. By construction, we have an $F$-equivariant homomorphism whose kernel is finite and of order prime to $p$. Passing to character lattices, we obtain an $F$-equivariant homomorphism $X(S) \to X(T)$ whose cokernel is finite and of order prime to $p$. The images of the $\chi_i$ in $X(T)$ form a $p$-spanning subset of $X(T)$ of size $\eta(G)$. □

For the proof of Theorem 9.1 we will also need the following lemma. Let $\Gamma = \SL_n$, $T$ be the diagonal maximal torus, $N$ be the normalizer of $T$ in $\SL_n$, $H$ be a subgroup of the Weyl group $W = N/T \simeq S_n$, and $N'$ be the preimage of $H$ in $N$. Restricting (9.1) to $N'$, we obtain an exact sequence

$$1 \longrightarrow T \longrightarrow N' \longrightarrow \pi H \longrightarrow 1.$$  

Lemma 9.4. Let $V_n$ be the natural $n$-dimensional representation of $\SL_n$ and $S$ be the stabilizer in general position for the restriction of this representation to $N'$. Then (a) $S \cap T = 1$ and (b) $\pi(S) = H \cap A_n$.

Here, as usual, $A_n$ denotes the alternating group.

Proof. (a) follows from the fact that the $T$-action on $V_n$ is generically free. To prove (b), note that $\pi(S)$ is the kernel of the action of $H$ on $V_n/T$, where $V_n/T$ is the rational quotient of $V_n$ by the action of $T$; see, e.g., the proof of [14 Proposition 7.2]. Consider the dense open subset $G_m^n \subset V_n$ consisting of vectors of the form $(x_1, x_2, \ldots, x_n)$, where $x_i \neq 0$ for any $i = 1, \ldots, n$. We can identify $G_m^n$ with the diagonal maximal torus in $\GL_n$. Now

$$V_n/T \simeq (G_m^n/T) \overset{\simeq}{\longrightarrow} G_m$$

where $S_n$ acts on $G_m$ by $\sigma \cdot t = \text{sign}(\sigma)t$. Thus the kernel of the $H$-action on $V_n/T$ is $H \cap A_n$, as claimed. □

Proof of Theorem 9.1. We will assume that $\Gamma = \SL_n$ and $T$ is the diagonal torus in $\Gamma$. The inequalities

$$\lfloor \frac{n}{p} \rfloor \leq \text{ed}(N; p) \leq \lfloor \frac{n}{p} \rfloor + 1;$$

are known for every $n$ and $p$; see [15 Section 5.4]. We will write $V_n$ for the natural $n$-dimensional representation of $\SL_n$ (which we will sometimes restrict to $N$ or subgroups of $N$).

(a) Suppose $n$ is divisible by $p$. Let $H \simeq (\mathbb{Z}/p\mathbb{Z})^{n/p}$ be the subgroup of $W = N/T \simeq S_n$ generated by the commuting $p$-cycles $(1 \ 2 \ \ldots \ \ p)$, $(p+1 \ p+2 \ \ldots \ 2p), \ldots, (n-p+1 \ \ldots \ n)$. Since $H$ is a $p$-group, it lies in a Sylow $p$-subgroup $W_p$ of $S_n$. Denote the preimage of $H$ in $N$ by $N'$. Then $N'$ is a subgroup of $N$ of finite index, so

$$\text{ed}(N; p) \geq \text{ed}(N'; p);$$

see [5 Lemma 2.2]. It thus suffices to show that $\text{ed}(N'; p) = \frac{n}{p} + 1$.

Claim: $\eta(N') = n$. 

Suppose the claim is established. Then $V_n$ is a $p$-faithful representation of $N'$ of minimal dimension. Since $p$ is odd, $H$ lies in the alternating group $A_n$. By Lemma 9.4(a), the stabilizer in general position for the $N'$-action on $V$ is isomorphic to $H$. By Theorem 1.2

$$\text{ed}(N'; p) = \dim(V_n) + \rank(H) - \dim(N') = n + \frac{n}{p} - (n - 1) = \frac{n}{p} + 1,$$

and we are done.

To prove the claim, note that $N'$ has a faithful representation $V_n$ of dimension $n$. Hence, $\eta(N') \leq n$. To prove the opposite inequality, $\eta(N') \geq n$, it suffices to show that

$$\text{SymRank}(X(T); p) \geq n;$$

see Lemma 9.3. Here we view $X(T)$ as an $H$-lattice. By definition, $\text{SymRank}(X(T); p)$ is the minimal cardinality of a finite $H$-invariant $p$-spanning subset $\{x_1, \ldots, x_d\} \subset X(T)$. The $H$-action on $\{x_1, \ldots, x_d\}$ gives rise to a permutation representation $\varphi : H \to S_d$.

The permutation representation $\varphi$ is necessarily faithful. Indeed, assume the contrary: $1 \neq h$ lies in the kernel of $\varphi$. Then $x_1, \ldots, x_d$ lie in $X(T)^h$. On the other hand, it is easy to see that $X(T)^h$ is of infinite index in $X(T)$. Hence, $\{x_1, \ldots, x_d\}$ cannot be a $p$-spanning subset of $X(T)$. This contradiction shows that $\varphi$ is faithful.

Now [2, Theorem 2.3(b)] tells us that the order of any abelian $p$-subgroup of $S_d$ is $\leq p^{d/p}$. In particular, $|H| \leq p^{d/p}$. In other words, $p^{n/p} \leq p^{d/p}$ or equivalently, $n \leq d$. This completes the proof of (9.4) and thus of the claim and of part (a).

(b) When $p = 2$, the argument in part (a) does not work as stated because it is no longer true that $H$ lies in the alternating group $A_n$. However, when $n$ is divisible by 4, we can redefine $H$ as follows:

$$H = H_1 \times \cdots \times H_{n/4} \hookrightarrow A_4 \times \cdots \times A_4 \ (n/4 \text{ times}) \hookrightarrow A_n,$$

where $H_i \simeq (\mathbb{Z}/2\mathbb{Z})^2$ is the unique normal subgroup of order 4 in the $i$th copy of $A_4$. Now $H \simeq (\mathbb{Z}/2\mathbb{Z})^{n/p}$ is a subgroup of $A_n$, and the rest of the proof of part (a) goes through unchanged.

(c) Write $n = pq + r$, where $1 \leq r \leq p - 1$. The subgroup of $S_n$ consisting of permutations $\sigma$ such that $\sigma(i) = i$ for any $i > pq$, is naturally identified with $S_{pq}$. Let $P_{pq}$ be a $p$-Sylow subgroup of $S_{pq}$, and let $N'$ be the preimage of $P_{pq}$ in $N$. Then $[N : N'] = [S_n : P_{pq}]$ is prime to $p$; hence, it suffices to show that $\text{ed}(N'; p) = [n/p]$. In view of (9.2), it is enough to show that $\text{ed}(N'; p) \leq [n/p]$. Since $r \geq 1$, as an $N'$-representation, $V_n$ splits as $k^{pq} \oplus k^r$ in the natural way. Let us now write $k^r$ as $k^{r-1} \oplus k$ and combine $k^{r-1}$ with $k^{pq}$. This yields a decomposition

$$V_n = k^{n-1} \oplus k$$

where the action of $N'$ on $k^{n-1}$ is faithful. Now recall that $P_{pq}$ has a faithful $q$-dimensional representation; see, e.g., the proof of [18, Lemma 4.2]. Denote this representation by $V'$. Viewing $V'$ as a $q$-dimensional representation of $N'$ via the natural projection $N' \to P_{pq}$, we obtain a generically free representation $k^{n-1} \oplus V'$ of $N'$. Thus

$$\text{ed}(N'; p) \leq \dim(k^{n-1} \oplus V') - \dim(N') = (n - 1) + q - (n - 1) = q = \left\lfloor \frac{n}{p} \right\rfloor,$$

as desired.
(d) The argument of part (c) is valid for any prime. In particular, if \( p = 2 \), it proves part (d) in the case, where \( n \) is odd. Thus we may assume without loss of generality that \( n \equiv 2 \pmod{4} \). Let \( N' \) be the preimage of \( P_n \) in \( N \), where \( P_n \) is a Sylow 2-subgroup of \( S_n \). Then the index \([N : N'] = [S_n : P_n]\) is finite and odd; hence, \( \text{ed}(N; 2) = \text{ed}(N'; 2) \). In view of \([15, \text{Section 5.7}]\), it suffices to show that \( \text{ed}(N'; 2) \leq n/2 \).

Since \( n \equiv 2 \pmod{4} \), \( P_n = P_{n-2} \times P_2 \), where \( P_2 \simeq S_2 \) is the subgroup of \( S_n \) of order 2 generated by the 2-cycle \((n-1, n)\). Let \( V' \) be a faithful representation of \( P_{n-2} \) of dimension \((n-2)/2 \). We may view \( V' \) as a representation of \( N' \) via the projection \( N' \to P_n \to P_{n-2} \).

Claim: \( V_n \oplus V' \) is a generically free representation of \( N' \).

If this claim is established, then

\[
\text{ed}(N') \leq \dim(V_n \oplus V') - \dim(N') = n + \frac{n-2}{2} - (n-1) = \frac{n}{2},
\]

and we are done.

To prove the claim, let \( S \) be the stabilizer in general position for the action of \( N' \) on \( V_n \). Denote the natural projection \( N' \to P_n \) by \( \pi \). By Lemma \([9, \text{Lemma 9.4 (a)}]\), \( S \cap T = 1 \). In other words, \( \pi \) is an isomorphism between \( S \) and \( \pi(S) \). Since \( P_n = P_{n-2} \times P_2 \), the kernel of the \( P_n \)-action on \( V' \) is \( P_2 \). It now suffices to show that \( S \) acts faithfully on \( V' \), i.e., \( \pi(S) \cap P_2 = 1 \).

By Lemma \([9, \text{Lemma 9.4 (a)}]\), \( \pi(S) \subset A_n \); i.e., every permutation in \( \pi(S) \) is even. On the other hand, the non-trivial element of \( P_2 \), namely the transposition \((n-1, n)\), is odd. This shows that \( \pi(S) \cap P_2 = 1 \), as desired.

**Proof of Theorem \([9, \text{Lemma 9.2}]\).** By \([15, \text{Section 5.7}]\), \( \text{ed}(N; 2) \leq 4n \). Thus it suffices to show that \( \text{ed}(N; 2) \geq 4n \).

Let

\[
(Z/2Z)^{2n}_0 := \{ (\gamma_1, \gamma_2, \ldots, \gamma_{2n}) \in (Z/2Z)^{2n} : \sum_{i=1}^{2n} \gamma_i = 0 \}.
\]

Recall that a split maximal torus \( T \) of \( SO_{4n} \) is isomorphic to \((\mathbb{G}_m)^{2n}\), and the Weyl group \( W \) is a semidirect product \( A \rtimes S_{2n} \), where \( A \) is an elementary abelian 2-group \( A \simeq (Z/2Z)^{2n-1} \). Here \( A \) is the multiplicative group of \( 2n \)-tuples \( \epsilon = (\epsilon_1, \ldots, \epsilon_{2n}) \), where each \( \epsilon_i \) is \pm 1, and \( \epsilon_1 \epsilon_2 \ldots \epsilon_{2n} = 1 \). \( S_{2n} \) acts on \( A \) by permuting \( \epsilon_1, \ldots, \epsilon_{2n} \). The action of \( W \) on \((t_1, \ldots, t_{2n}) \in T \) is as follows: \( S_{2n} \) permutes \( t_1, \ldots, t_{2n} \), and \( \epsilon \) takes each \( t_i \) to \( t_i^{\epsilon_i} \).

Let \( H \) be the subgroup of \( W \) generated by elements \( (\epsilon_1, \ldots, \epsilon_{2n}) \in A \), with \( \epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4, \ldots, \epsilon_{2n-1} = \epsilon_{2n} \), and the \( n \) disjoint 2-cycles \((1, 2), (3, 4), \ldots, (2n-1, 2n)\) in \( S_{2n} \). It is easy to see that these generators are of order 2 and commute with each other, so that \( H \simeq (Z/2Z)^n \). Let \( N' \) be the preimage of \( H \) in \( N \).

Note that \( H \) arises as a stabilizer in general position of the natural \( 4n \)-representation \( V_{4n} \) of \( N \) (restricted from \( SO_{4n} \)). Here \((t_1, \ldots, t_{2n}) \in T \) acts on \((x_1, \ldots, x_{2n}, y_1, \ldots, y_{2n}) \in V_{4n}\) by \( x_i \mapsto t_i x_i \) and \( y_i \mapsto t_i^{-1} y_i \) for each \( i \). The symmetric group \( S_{2n} \) simultaneously permutes \( x_1, \ldots, x_{2n} \) and \( y_1, \ldots, y_{2n} \); \( \epsilon \in A \) leaves \( x_i \) and \( y_i \) invariant if \( \epsilon_i = 1 \) and switches them if \( \epsilon_i = -1 \).

Note that \( N' \) is a subgroup of finite index in \( N \). Hence, \( \text{ed}(N; 2) \geq \text{ed}(N'; 2) \), and it suffices to show that \( \text{ed}(N'; 2) \geq 4n \).

Claim: \( \eta(N') = 4n \).
Suppose for a moment that the claim is established. Then $V_{4n}$ is a 2-faithful representation of $N'$ of minimal dimension. As we mentioned above, a stabilizer in general position for this representation is isomorphic to $H$. By Theorem 1.2,

$$\text{ed}(N'; 2) = \dim(V_{4n}) + \text{rank}(H) - \dim(N') = 4n + 2n - 2n = 4n,$$

and we are done.

To prove the claim, note that $\eta(N') \leq 4n$, since $N'$ has a faithful representation $V_{4n}$ of dimension $4n$. By Lemma 9.4, in order to establish the opposite inequality, $\eta(N') \geq 4n$, it suffices to show that $\text{SymRank}(X(T); 2) \geq 4n$. To prove this last inequality, we will use the same argument as in the proof of Theorem 9.1(a). Recall that $\text{SymRank}(X(T); 2)$ is the minimal size of an $H$-invariant 2-generating set $x_1, \ldots, x_d$ of $X(T)$. The $H$-action on $x_1, \ldots, x_d$ induces a permutation representation $\varphi: H \to S_d$. Once again, this representation has to be faithful. By [2, Theorem 2.3(b)], $|H| \leq 2^{d/2}$. In other words, $2^{2n} \leq 2^{d/2}$, or equivalently, $d \geq 4n$, as claimed.

\[\square\]

**Acknowledgements**

We are grateful to Jerome Lefebvre, Mathieu Huruguen and Mikhail Borovoi for stimulating discussions related to Conjecture 1.1.

**References**

[1] M. Artin. Nérond models. In *Arithmetic geometry (Storrs, Conn., 1984)*, pages 213–230. Springer, New York, 1986.

[2] Michael Aschbacher and Robert M. Guralnick. On abelian quotients of primitive groups. *Proc. Amer. Math. Soc.*, 107(1):89–95, 1989.

[3] Peter Bardsley and R. W. Richardson. Étale slices for algebraic transformation groups in characteristic $p$. *Proc. London Math. Soc. (3)*, 51(2):295–317, 1985.

[4] Nicolas Bourbaki. *Commutative algebra. Chapters 1–7*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1989. Translated from the French, Reprint of the 1972 edition.

[5] Patrick Brosnan, Zinovy Reichstein, and Angelo Vistoli. Essential dimension, spinor groups, and quadratic forms. *Ann. of Math. (2)*, 171(1):533–544, 2010.

[6] Patrick Brosnan, Zinovy Reichstein, and Angelo Vistoli. Essential dimension in mixed characteristic. *Doc. Math.*, 23:1587–1600, 2018.

[7] Charles W. Curtis and Irving Reiner. *Representation theory of finite groups and associative algebras*. Pure and Applied Mathematics, Vol. XI. Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.

[8] Philippe Gille and Zinovy Reichstein. A lower bound on the essential dimension of a connected linear group. *Comment. Math. Helv.*, 84(1):189–212, 2009.

[9] M. Huruguen. On the essential $p$-dimension of a semidirect product $G_m^n \rtimes \mathbb{Z}/p\mathbb{Z}$. preprint, 2015.

[10] Jens Carsten Jantzen. Nilpotent orbits in representation theory. In *Lie theory*, volume 228 of *Progr. Math.*, pages 1–211. Birkhäuser Boston, Boston, MA, 2004.

[11] Nikita A. Karpenko and Alexander S. Merkurjev. Essential dimension of finite $p$-groups. *Invent. Math.*, 172(3):491–508, 2008.

[12] Serge Lang. *Algebra, volume 211 of Graduate texts in mathematics*. Springer-Verlag, New York, 2002.

[13] Roland Lütscher, Mark MacDonald, Aurel Meyer, and Zinovy Reichstein. Essential dimension of algebraic tori. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2013(677):1–13, 2013.
[14] Roland L"otscher, Mark MacDonald, Aurel Meyer, and Zinovy Reichstein. Essential $p$-dimension of algebraic groups whose connected component is a torus. *Algebra Number Theory*, 7(8):1817–1840, 2013.

[15] Mark L. MacDonald. Essential $p$-dimension of the normalizer of a maximal torus. *Transform. Groups*, 16(4):1143–1171, 2011.

[16] Benjamin Martin. Generic stabilisers for actions of reductive groups. *Pacific J. Math.*, 279(1-2):397–422, 2015.

[17] Alexander S. Merkurjev. Essential dimension: a survey. *Transformation groups*, 18(2):415–481, 2013.

[18] Aurel Meyer and Zinovy Reichstein. The essential dimension of the normalizer of a maximal torus in the projective linear group. *Algebra Number Theory*, 3(4):467–487, 2009.

[19] V. L. Popov and E. B. Vinberg. *Invariant Theory*, pages 123–278. Springer Berlin Heidelberg, Berlin, Heidelberg, 1994.

[20] Zinovy Reichstein. Essential dimension. *Proceedings of the International Congress of Mathematicians (Vol. 2)*, 2010.

[21] Zinovy Reichstein and Boris Youssin. Essential dimensions of algebraic groups and a resolution theorem for $G$-varieties. *Canad. J. Math.*, 52(5):1018–1056, 2000. With an appendix by János Kollár and Endre Szabó.

[22] R. W. Richardson, Jr. Deformations of Lie subgroups and the variation of isotropy subgroups. *Acta Math.*, 129:35–73, 1972.

[23] Jean-Pierre Serre. *Galois cohomology*. Springer-Verlag, Berlin, 1997. Translated from the French by Patrick Ion and revised by the author.

**Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada**

*E-mail address*: reichst@math.ubc.ca

*E-mail address*: scavia@math.ubc.ca