Optimal switching problems with an infinite set of modes: an approach by randomization and constrained backward SDEs

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Outline of the talk

I- Preliminaries & motivations
   • The Optimal Switching problem (OSP): primal vs dual formulation.
   • Assumptions for the dual formulation.
   • Why choosing the "dual" approach ?

II- Main results & perspectives
   • The two main results:
     (i): equality between the two value functions;
     (ii): new BSDE characterization.
   • Perspectives
Motivations & preliminaries

I.1 Primal optimal switching problem and value function

On a standard prob. space $(\Omega, \mathcal{F}, \mathbb{P})$, let

- $\mathcal{W}$: standard $d$-dim. Brownian Motion, $\mathcal{W}$ $\mathcal{F}$-adapted. usually: $\mathcal{F} = \mathcal{F}^\mathcal{W} \vee \mathcal{N}$.
- $T$ fixed finite horizon; $A$ set of modes (possibly infinite).
- $\forall (x_0, e) \in \mathbb{R}^n \times A$, let $X^e$ proc. s.t.

\[ \forall t \in [0, T], \quad X_t^e = x_0 + \int_0^t (b^e(s, X^e)ds + \sigma^e(s, X^e)dW_s), \]
Motivations & preliminaries

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- \( \forall (x_0, e) \in \mathbb{R}^n \times A \), let \( X^e \) proc. s.t.

\[
\forall t \in [0, T], \quad X^e_t = x_0 + \int_0^t \left( b^e(s, X^e) \, ds + \sigma^e(s, X^e) \, dW_s \right),
\]

Let \((f^e)_e, (g^e)_e\) and \((c_{e,e'})_{(e,e')}\): 3 families of (possib. random) real-valued data

(i) \( f^e(s, X.) \): instant. profit (when system in mode \( e \))

(ii) \( g^e(X.) \): payoff at time \( T \) when syst. in mode \( e \),

(iii) \( c_{e,e'}(s, X.) \): nonnegative penalty costs incurred at time \( s \) when switching from \( e \) to \( e' \).
Motivations & preliminaries

I.1 Primal optimal switching problem and value function

► Mathematical assumptions:
  • $A$: Borel set (example: any subspace of $\mathbb{R}^d$);
  • Both $(b^e, \sigma^e)_e, (f^e, g^e), (c_{e,e'})_{e,e'}$ may be path-dependent;
  • Let $\mathbb{C}^n$: set of continuous paths $(s \mapsto x(s))_{s \in [0, T]}$
  Topology on $\mathbb{C}^n$: $|x|_* = \sup_{s \in [0, T]} |x(s)|$
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  Topology on $C^n$: $|x|_\ast = \sup_{s \in [0,T]} |x(s)|$

- Measurability
  $(t, \omega, e) \mapsto b^e(t, x(\omega), \omega)$, $\sigma^e(t, \omega, x(\omega), e)$ are $\text{Prog}(C^n) \otimes \mathcal{B}(A)$ meas.; (similar for $f^e, g^e, c_{e,e'}$)
  $\text{Prog}(C^n)$: $\sigma$-algebra of prog. measurable maps on $[0, T] \times \Omega$. 
I.1 Primal optimal switching problem and value function

Math. Assumptions (cont'):

- For every \( t \in [0, T] \),
  
  \( (x, e) \mapsto b_t(x, e) \sigma_t(x, e), f_t(x, e), g(x, e) \) are continuous on \( \mathbb{C}^n \times A \) \( (x, e, e') \mapsto c_t(x, e, e') \) is continuous on \( \mathbb{C}^n \times A \times A \).

- Regularity & growth assumpt (wrt \( x \)):
  \[ \exists K > 0 \text{ s.t. } \forall (t, x, x', e, e') \in [0, T] \times \mathbb{C}^n \times \mathbb{C}^n \times A \times A, \]
  
  \( (i) \ |b_t(x, e) - b_t(x', e)| + |\sigma_t(x, e) - \sigma_t(x', e)| \leq K|x - x'|_t^* \)

  Similar for other data.

  \( (ii) \ |b(t, 0, e)| + |\sigma(t, 0, e)| \leq K; \)
I.1 Primal optimal switching problem and value function

- Growth assumpt wrt $x$ (cont’)

$\exists \ r, \ K > 0 \text{ s.t. } \forall (t, x, x', e, e') \in [0, T] \times \mathbb{C}^n \times \mathbb{C}^n \times A \times A,$

$|f(t, x, e) + |g(x, e)| + |c(t, x, e, e')| \leq K(1 + |x|_{t*})$
Motivations & preliminaries

I.1 Primal optimal switching problem and value function

- Growth assumpt wrt $x$ (cont’)

$\exists \ r, \ K > 0 \text{ s.t. } \forall \ (t, x, x', e, e') \in [0, T] \times \mathbb{C}^n \times \mathbb{C}^n \times A \times A,$

(iii) $|f(t, x, e) + |g(x, e)| + |c(t, x, e, e')| \leq K(1 + |x|^r t^*)$

Comment

(i)-(iii) standard to obtain estim.

(a) Estim. (of Hilbertian norm) of process $X^e$ (see Cosso-Confortola-Fuhrman ’18);
(b) Estim. of the value function (well known in Markovian case).
I.1 Primal optimal switching problem and value function

1. Let $\alpha = (\tau^n, \xi^n)_{n \geq 1}$ with $\tau^1 > 0$. $\alpha = \textit{management}$ strategy
2. To $\alpha$, we associate the state proc. $a$ as follows

$$a_s = \xi^1 1_{s < \tau_1} + \sum_{n \geq 1} \xi^{n+1} 1_{\tau^n \leq s < \tau_{n+1}} 1_{\tau^n < T}$$

$a$: piecewise constant proc. $A$-valued
By abuse, one may replace $\alpha$ by $a$. 
I.1 Primal value function: Admissible set $\mathcal{A}$

$a = (\tau^n, \xi^n)$ is said *admissible* ($a$ in $\mathcal{A}$) if

$H_1 \quad (\tau^n(\cdot), \xi^n(\cdot))_{n=1}^{\infty} \times \mathbb{R}^+ \times \mathcal{A}$-valued $\mathbb{F}$-adapt. such that

$\tau_n(\omega) \to +\infty$ and $\tau^n < \tau^{n+1}$, $\mathbb{P}$-a.s

**Simultaneous switchings prohibited**: equivalent to

$$\forall (a_1, a_2, a_2) \in \mathcal{A}^3, \quad c_{a_1,a_2}(t, x) + c_{a_2,a_3}(t, x) > c_{a_1,a_3}(t, x)$$

Stronger than the no-loop property (in finite case).
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Stronger than the no-loop property (in finite case).

$H_2$ $H_1$ implies: $N^a_\tau(\omega) = \text{Card}\{\tau^n(\omega), \tau^n < T\} < \infty$, $\mathbb{P}$-a.s

$H_3$ Impose $\tau^n \neq T$: no switching at terminal time.

In finite case, equivalent to:

$$\forall (i, j) \in \mathcal{A} \times \mathcal{A}, \quad g^i(X^i_T) > g^j(X^j_T) - c_{i,j}(T, X_T).$$
I.1 Primal optimal switching problem and value function

1. For $a$ in $\mathcal{A}$, let $X^a$ (or $X^a$) the controlled proc. s.t.

$$dX^a = b^a(s, X^a)ds + \sigma^a(s, X^a_s)dW_s$$

with $b^a(s, x) = b^{\xi_0}(s, x)1_{s < \tau^1} + \sum_{n \geq 1} b^{\xi_n}(s, x)1_{\tau^n \leq s < \tau^{n+1}}$.

Similar definition for $\sigma^a(s, x)$.

**Remark:** $b$ and $\sigma$ path-dependent $\Rightarrow$ $X^a$ no more Markovian (PDE approach not available).
I.1 Primal control problem and (primal) value function

1. Primal value function $\mathcal{V}$

$$\mathcal{V} = \sup_{\alpha \in \mathcal{A}} (J(\alpha)),$$

where

$$J(\alpha) = \mathbb{E} \left( g^{a_T} (x.) + \int_0^T f^{a_s} (s, x^a) ds - \sum_{n \geq 1, \tau_n < T} c_{\xi_{n-1}, \xi_n} (\tau^n, x^{a_{\tau^n}}) \right)$$

$$= J_1(\alpha) - J_2(\alpha)$$

**Objective:** choose the best $a$ (or $\alpha$) to optimize $J(\alpha)$ and minimize $J_2(\alpha)$. 
Motivations & preliminaries

A (non exhaustive) review of the literature

(1) **OSP with finite set of modes:**
   (i) Using PDE approaches: Ishii-Koike ’91, Yong-Zhou ’99, Ludkowski ’05, Carmona-Ludkovski ’07-08, ...
   (ii) Using BSDE and analyt. tools: Hamadène-Jeanblanc ’02, Djehiche-Hamadene-Popier ’08, Hu-Tang ’07 Chassagneux-Elie Kharroubi; Elie-Kharroubi ’08 ’11, ...
   (iii) **Standard OSP with refinements:** infinite horizon, partial information, non positive costs: Lundstrom -Olofsson, R. Martyr, B. El Asri, ..

(2) **Connection between "finite" OSP & constrained BSDE:**
   (a) Ma-Pham-Kharroubi ’08 (Markovian setting)
   (b) Elie-Kharroubi (’14) (Non Markovian case)
1.2 Randomized set-up & dual formulation

1. On $(\Omega', \mathcal{F}', \mathbb{P}')$ let $\mu = \sum_{n \geq 0} \delta_{\sigma^m, \zeta^m}$ be a Poisson random meas. s.t.

   (i) Random dates & marks $(\sigma^m, \zeta^m)_m \mathbb{R}^+ \times A$-valued;
   (ii) $\mu$ indep. of $W$ with $\hat{\mu}(de, ds) = \lambda(de)ds$ s.t

      (a) $\tilde{\mu} = \mu - \hat{\mu}$ is a martingale measure;
      (b) $\lambda(de)$ has full support and $\lambda(A) < +\infty$. 
Motivations & preliminaries

1.2 Randomized set-up & dual formulation

1. On \((\Omega', \mathbb{F}', \mathbb{P}')\) let \(\mu = \sum_{n \geq 0} \delta_{\sigma^m, \zeta^m}\) be a Poisson random meas. s.t.

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   (ii) \(\mu\) indep. of \(W\) with \(\hat{\mu}(de, ds) = \lambda(de)ds\) s.t
      (a) \(\tilde{\mu} = \mu - \hat{\mu}\) is a martingale measure;
      (b) \(\lambda(de)\) has **full support** and \(\lambda(A) < +\infty\).

2. The randomized dual set up := \((\overline{\Omega}, \overline{\mathbb{P}}, \overline{\mathbb{F}}, \overline{W}, \overline{\mu})\):

   (2.i) Let \(\overline{\Omega} := \Omega \times \Omega', \overline{\mathbb{P}} = \mathbb{P} \otimes \hat{\mathbb{P}}'\) and \(\overline{\mathbb{F}} = \mathbb{F}^{W, \mu}\), with

   \[
   \mathbb{F}^{W, \mu} := (\mathbb{F}^W \vee \mathbb{F}^\mu) \vee \mathcal{N}
   \]

   (2.ii) \(\overline{W}(\omega, \omega') = W(\omega)\) remains a \(\mathbb{F}^{W, \mu}\)- Brownian motion;
   \(\overline{\mu} := (\overline{\sigma}^m, \overline{\zeta}^m)_m\) Poisson r.m. with \(\mathbb{F}^{W, \mu}\)-prog. meas random marks and **same determ. compensator** \(\hat{\mu} = \tilde{\mu} = \hat{\mu}\).
I.2. The randomized set-up and dual formulation

1. Let $\mathcal{I}$ (resp. $\mathcal{I}^\prime$) the Poisson point proc. assoc. with $\mu$ (resp. $\mu^\prime$) as follows

\[ \forall \ t \in [0, T], \ I_t = \zeta^0 \mathbf{1}_{t<\sigma^1} + \sum_{m \geq 1} \zeta^m \mathbf{1}_{\sigma^m \leq t < \sigma^{m+1}}. \]

Note that $N^I_T := \text{Card}\{m \geq 1, \ \sigma^m(\omega') < T\} < \infty$, $\mathbb{P}'$-a.s.

On randomized prob. space, $(\mathcal{I}, X^\mathcal{I})$ is a **forward uncontrolled proc.** with

\[ X^\mathcal{I}_t = x_0 + \int_0^t \left( b^\mathcal{I}(s, X^\mathcal{I})ds + \sigma^\mathcal{I}(s, X^\mathcal{I})dW_s \right) \]
I.2. The randomized set-up and dual formulation

1. To any proc. \( \nu \) \( \mathcal{F}, \mu \)-meas., associate process \( \kappa_{\nu} \)

\[
\kappa_{\nu} = \mathcal{E}_T((\nu - 1) \ast \mu) = e^{-\int_0^T \int_A (\nu_s(e)-1) \lambda(de)ds} \prod_{m \geq 1} \nu_{\sigma m}(\zeta^m)
\]

2. Let \( \bar{P}^{\nu} \) with density \( \kappa_{\nu} \), i.e. \( \frac{d\bar{P}^{\nu}}{d\bar{P}} = \kappa_{\nu} \)
then, under \( \bar{P}^{\nu} \),
(a) \( \bar{I} \) remains Poisson point proc.;
(b) New compensated meas. \( \bar{\nu}_s(e) \lambda(de)ds \)

3. Set of dual controls

\[
\mathcal{A}^R := \{ \nu : \Omega \times [0, T] \times A \mapsto ]0; \infty[ \text{ meas. and essentially bounded} \}
\]
1.2 The randomized set-up: dual formulation

1. Let \( \nu_0^R = \sup_{\nu \in A^R} J^R(\nu) \) be the dual value function with

\[
J^R(\nu) = \mathbb{E}^\nu \left( g(X^I, I_T) + \int_t^T f(s, X^I, I_s)ds \right)
\]

\[
= J_1^R(\nu)
\]

\[
- \mathbb{E}^\nu \left( \sum_{m \geq 1} c_{\zeta_{m-1}, \zeta_m}(\sigma^m, X_{\sigma^m}) \right)
\]

\[
= J_2^R(\nu)
\]

\( \mathbb{E}^\nu \) stands for expectation under meas. \( \mathbb{P}^\nu \).
I.2 The randomized set-up: Major comments

1. Unique assumption on $A$: it is a **Borel** space
   No compactness assumption.
   Desirable properties: $A$ both **metric and separable**.

2. Exogeneous proc. $X$ (resp. $\bar{X}$) not necess. Markovian

3. The controlled volatility process may be degenerate
   (contrary to papers using PDE approaches).

4. If $b, \sigma$ only depends on $(x, a)$ not on $\omega$, then the pair $(I, X^I)$
   is a Markov process.
Il First main result & comments

Under all previous assumptions on the primal & dual version of the OSP, one claims

$$\mathcal{V}_0 = \mathcal{V}_0^R = \nu_0(x_0, a_0).$$

This deterministic common value function only depends on $X_0 = x_0$ and initial mode $a_0$ and not of the choice of the randomized set up:
(i.e. neither on the construction of the extended dual set-up nor on the choice of intensity measure $\lambda$).
II. Second main result: BSDE characterization

Let $Y^R_t$ be the *minimal* solution of following BSDE

\[
\begin{align*}
Y^R_t &= g(X_t, I_T) + \int_t^T f_s(X_s, I_s) \, ds + K_T - K_t \\
&\quad - \int_t^T Z_s \, dW_s - \int_{(t,T]} \int_A U_s(a) \tilde{\mu}(ds \, da), \\
U_t(a) &\leq c_t(X_t, I_{t-}, a), \quad \lambda(da)ds \, \mathbb{P} \text{- a.s.}
\end{align*}
\]

then it holds

\[Y^R_0 = V^R_0.\]

**Remark:** (1) is a BSDE with constrained jumps & non decreas. proc $K$: $K$ *only càdlàg*.

$Y^R_t \in \mathcal{F}_t^W, \mu$-adapted.
I.3. Why choosing randomization to study the OSP?

1. when $A$ infinite (even uncountable), the *infinite* system of RBSDEs does not seem well posed (at least to us..)
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2. For the primal OSP, many ingredients deeply use the finiteness of $A$. 
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2. For the primal OSP, many ingredients *deeply* use the finiteness of $A$.

3. the randomized set up allows to tackle general cases: path-dependency, possibly degenerate diffusions, case of an infinite set of modes.
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3. the randomized set up allows to tackle general cases: path-dependency, possibly degenerate diffusions, case of an infinite set of modes.

4. Another motivation: in the Markovian setting, connection already proved by R.Elie & I.Kharroubi ('09, ’10).
Main results: comments

Connection with BSDE associated with the OSP (finite set of modes)

Let $\mathcal{J}$ set of modes and let $(Y^i)_{i \in \mathcal{J}}$ solving

$$
\begin{align*}
Y^i_t &= g(X_T, i) + \int_t^T f_s(X_s, i) \, ds + K^i_T - K^i_t \\
&\quad - \int_t^T Z^i_s \, dW_s,
Y^i_s &\geq \max\{j \in \mathcal{J} \setminus \{i\} \} \left( Y^j_s - c_{i,j}(s, X_s) \right) \quad \text{and} \\
\int_0^T (Y^i_s - \max\{j \in \mathcal{J} \setminus \{i\} \} \left( Y^j_s - c_{i,j}(s, X_s) \right) \, dK^i_s &= 0
\end{align*}
$$

If BSDE system (2) has a solution, the \textit{minimal} solution of dual BSDE (1) is s.t.

$$
Y^R_t = Y^l_t \quad \text{and} \quad U_t(i) = Y^i_t - Y^l_t^{-}.
$$
Main results: the BSDE characterization

The minimal BSDE

Let $Y$ the *minimal* solution of following BSDE

$$
\begin{align*}
Y^R_t &= g(X, I_T) + \int_t^T f_s(X, I_s) \, ds + K_T - K_t \\
& \quad - \int_t^T Z_s \, dW_s - \int_{(t,T]} \int_A U_s(a) \, \tilde{\mu}(ds \, da), \\
U_s(a) &\leq c_s(X, I_{s-}, a), \quad \lambda(da) \, ds \quad \mathbb{P} - \text{a.s.}
\end{align*}
$$

(3)

then it holds: $Y^R_0 = \mathcal{V}^R_0$. Combined with first main result

$$
Y^R_0 = \mathcal{V}^R_0 = \mathcal{V}_0 = \sup_{\alpha \in \mathcal{A}} \mathcal{J}(\alpha).
$$

$Y^R$: obtained as the increasing limit of penalized scheme.
Main results: the BSDE characterization

Probabilistic representation

Let \((Y^n)\) solving

\[
\begin{align*}
Y^n_t &= g(X_t, I_T) + \int_t^T f_s(X_s, I_s) \, ds + K^n_T - K^n_t \\
&\quad - \int_t^T Z^n_s \, dW_s - \int_{(t,T]} \int_A U^n_s(a) \tilde{\mu}(ds \, da), \\
\text{with } dK_s^n &= n \int_A (U^n_s(a) - c_s(X_s, I_s, a))^+ \lambda(da) \, ds.
\end{align*}
\]
Main results: the BSDE characterization

Probabilistic representation

Let \((Y^n)\) solving

\[
\begin{cases}
Y^n_t = g(X, I_T) + \int_t^T f_s(X, I_s) \, ds + K^n_T - K^n_t \\
- \int_t^T Z^n_s \, dW_s - \int_{(t, T]} \int_A U^n_s(a) \, \tilde{\mu}(ds \, da),
\end{cases}
\]

with \(dK^n_s = n \int_A (U^n_s(a) - c_s(X, I_s, a))^+ \lambda(da) \, ds\).

It holds

\[
Y^n_t = \text{ess sup}_{\nu \in \mathcal{A}^{\mathbb{R}}} \left\{ \mathbb{E}^{\nu} \left[ g(X_T, I_T) + \int_t^T f_r(X, I_r) \, dr \right. \right.
\]

\[
\left. - \int_t^T \int_A c_r(X_r, I_{r-}, a) \mu(da, ds) \bigg| \mathcal{F}_t^{W,\mu} \right\}.
\]

(5)
Concluding remarks

Some perspectives: theoretical & numerical

1. Stability results: Approximating the general OSP by the OSP with finite number of modes
   **Objective:** explicit rate of convergence

2. Refinements in Markovian setting ($(I, X^I)$ Markov process)

3. Numerical perspectives
   Numerical solving of the "dual" BSDE.

   **Note** when Card(A) $< \infty$ but too large, simulating the solution of multidim BSDE system becomes unfeasible.
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   Numerical solving of the "dual" BSDE.
   **Note** when $\text{Card}(A) < \infty$ but too large, simulating the solution of multidim BSDE system becomes unfeasible.

Thanks for your attention!