Abstract. We prove that for any 1-reduced simplicial set \( X \), Adam s' cobar construction \( CX \) on the normalized chain complex of \( X \) is naturally a strong deformation retract of the normalized chains \( CGX \) on the Kan loop group \( GX \).

In order to prove this result, we extend the definition of the cobar construction and actually obtain the existence of such a strong deformation retract for all 0-reduced simplicial sets.

Introduction

There are two classical differential graded algebra models for the loop space on a 1-reduced simplicial set \( X \): Adam s' cobar construction \( CX \) on the normalized chain complex \( GX \) and the normalized chains \( CGX \) on the Kan loop group \( GX \) [7]. Both of these models are (weakly) equivalent to \( C X \) the chains on the loop space of the realization \( X \).

In this article we show that \( CX \) is actually a strong deformation retract of \( CGX \), opening up the possibility of applying the tools of homological algebra to transferring perturbations of algebraic structure from the latter to the former.

Theorem. For any 1-reduced simplicial set \( X \) there is a natural strong deformation retract of chain complexes

\[
\begin{array}{c}
\text{C}X \\
\downarrow \\
\text{C}G_X
\end{array}
\rightarrow
\begin{array}{c}
\text{C}G_X \\
\downarrow \\
\text{C}f
\end{array}
\rightarrow
\begin{array}{c}
\text{C}X \\
\downarrow \\
\text{C}G_X
\end{array}
\]

Here \( i \) is the identity map on \( CX \) and \( j \) is a chain homotopy from \( i \) to the identity map on \( CGX \). Further, \( i \) and \( j \) are homomorphisms of differential graded algebras.

In particular, \( CX \) is isomorphic to a sub differential graded algebra of \( CGX \), and both \( i \) and \( j \) induce isomorphisms of algebras in homology.

Remark 1. Let \( X \) and \( Y \) be 1-reduced simplicial sets, and \( f : GX \toGY \) be a simplicial map (not necessarily a homomorphism). The theorem above gives us a natural way to construct a chain-level model of \( f \). Indeed, if we set

\[
\text{C}f : \text{C}X \to \text{C}Y;
\]

then

\[
\begin{array}{c}
\text{C}X \\
\downarrow \\
\text{C}G_X
\end{array}
\rightarrow
\begin{array}{c}
\text{C}X \\
\downarrow \\
\text{C}G_X
\end{array}
\]

commutes up to natural chain homotopy.

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Remark 2. It was proved in [6] that if $X$ is a simplicial suspension, then $\text{CX}$ is naturally a primitively generated Hopf algebra, and the chain algebra map $\text{CX} \rightarrow \text{CGX}$ also respects multiplicative structure. In this case the strong deformation retract of the theorem above is actually Eilenberg-Zilber data [3], which implies that the chain algebra map $:\text{CGX} \rightarrow \text{CX}$ is also strongly homotopy multiplicative [ibid.].

In order to prove the theorem above, we extend the definition of the cobar construction and actually obtain the existence of such a strong deformation retract for all 0-reduced simplicial sets.

The homomorphism, which we recall in the first section of this article, was first described by Szczarba [11] in the language of twisting cochains. Given a simplicial set $X$ that is 0-reduced but not necessarily 1-reduced, he gives an explicit, though somewhat complicated, formula for a twisting cochain,

$$:\text{CX} \rightarrow \text{CGX},$$

that is based on the universal twisting function $:X \rightarrow GX$ and that gives rise in usual way to an algebra homomorphism

$$:\text{CX} \rightarrow \text{CGX} :$$

In degree zero the cobar construction is a free associative algebra on symbols given by the non-degenerate 1-simplices of $X$, while the right-hand side is the group ring on the free group on the same symbols. In the first section of the paper we observe that if $X$ is not 1-reduced, then we may perform a change of rings along $\sigma$, obtaining an extended cobar construction $^\sigma \text{CX}$, together with an algebra homomorphism

$$:^\sigma \text{CX} \rightarrow \text{CGX} :$$

that is an isomorphism in degree zero.

In the second section we introduce the retraction map from the chains on the loop group to the extended cobar construction, for which we provide an explicit recursive formula. We prove in fact that is a natural homomorphism of chain algebras and a one-sided inverse of the Szczarba map. It is surprising that such a map has not been previously observed in the literature.

In the third section we complete the strong deformation retraction [11] by defining the natural homotopy. For this, we use the acyclic models for the loop group on 0-reduced simplices studied by M Marm and P Moore [10].

1. Preliminaries

1.1. Sim plicial notions and notation. A simplicial set $X$ is a contravariant functor from the category of finite non-empty ordinals to the category of sets; mone prosai cally it is a sequence of sets $X_n$ $n \geq 0$, and specified face and degeneracy operators

$$d_i: X_n \rightarrow X_{n-1}; \quad s_i: X_n \rightarrow X_{n+1}; \quad (0 \leq i \leq n)$$

satisfying the simplicial identities, see for example [4]. A simplicial set is n-reduced if $X_n = f$ for $k \geq n$. The notions of simplicial group and simplicial objects in other categories are analogous.

Given an element $x \in X_n$ and any composite of simplicial face and degeneracy operators, represented by a monotonic function $f : \{0; \ldots ; m\} \rightarrow \{0; \ldots ; n\}$, we also write

$$x(0; \ldots ; m) = x_f (0; \ldots ; m)$$
for the corresponding element of \( X_n \). We may write the face and degeneracy maps them selves, for example, as

\[
\begin{align*}
\partial_i(x) &= x(\partial_{i} ; \ldots ; \partial_{i} ; \ldots ; \partial_{n} ) \\
\delta_i(x) &= x(\delta_{i} ; \ldots ; \partial_{i} ; \ldots ; \delta_{n} )
\end{align*}
\]

The simplicial relations imply that any simplicial operator \( X_n \) \( X_n \) has normal form

\[
\delta_i = s_s : s_n d_i : : : d_j
\]

with \( i > j + 1 \) and \( i < j + 1 \) for all \( k \). In this form, the corresponding derived operator \( X_n^{n+1} \) \( X_n^{n+1} \) is

\[
\delta^0 = s_{n+1} : s_n d_{n+1} : : : d_j + 1.
\]

An operator is frontal if it contains no \( \delta_j \); such operators satisfy \( \delta^0 s_0 = s_0 \).

12. The cobar construction. We introduce a slightly extended definition of the cobar construction, which will be better suited for applying to the normal chain complex on a \( 0 \)-reduced simplicial set. Our definition generalizes the classical construction of Adams, with which it agrees for simply connected coalgebras.

Let \((C; \theta)\) be a connected differential graded coalgebra over a commutative ring \( R \), so that \( C_0 = R \). We suppose in them one that \( C \) is \( R \)-linear in each degree. Consider the ring \( B \) given by the free associative \( R \)-algebra on the desuspension of \( C_1 \);

\[
B = T(1 + [1]) = \bigwedge (s^1 C_1)^k ;
\]

Fix an \( R \)-basis \( f_1, f_2, \ldots \) of \( C_1 \), and let \( A \) be the ring obtained from \( B \) by freely adjoining inverses \( j \) of all elements of the form \( 1 + s^1 x_j \), for all \( j \). Explicitly,

\[
A = T_B s^1 j j + 2 j = \bigwedge (1 + s^1 x_j) = 1 = (1 + s^1 x_j) j.
\]

Observe that the relations may also be expressed in the form

\[
j s^1 x_j = 1 j = s^1 x_j j.
\]

The ring \( A \) may be regarded as a differential graded algebra concentrated in degree zero. The graded algebra underlying the extended cobar construction \((\hat{C}; \bar{\theta})\) is then

\[
\hat{C} = T_{\hat{A}} s^1 C_2 \bigwedge (s^1 C_2 A)^k ;
\]

Each \( R \)-module \( \hat{C}_n \) is generated by words

\[
a = a_i \quad r \cdot a \quad j a_j = n_i ; \quad j a_j = n = n_i ;
\]

where either \( a_i = s^1 c \) for some \( C_0 \) element \( c \) \( C_{n+1} \) or \( n_i = 0 \) and \( a_i = j \) for some \( j j + 2 j \). This \( R \)-module is free on those words in which \( j \) does not appear adjacent to \( s^1 x_j \). The algebra multiplication is induced by concatenation of words, extended bilinearly to \( \hat{C}_m \) modulo the relation that \( j \) is inverse to \( 1 + s^1 x_j \).

The differential \( \bar{\theta} \) on \( \hat{C} \) is the derivation of \( A \)-algebras that is specified by

\[
\bar{\theta}_n s^1 c = s^1 d c + (-1)^{c_1} s^1 c_2 s^1 c_1 c^2
\]

for all basis elements \( c \) \( C_{n+1} \) and all \( n \), where \( c_1 = 1 \ \ s + c \ \ 1 + c \ \ c^2 \) (using the Einstein summation convention). Note that \( \bar{\theta} \) is necessarily zero on elements of \( A \).
The unit $1.2 (\tilde{C})_0$ is identified with the empty word via the isomorphism $R = (s_1 C_1)^0$.

Remark 3. If $\tilde{C}$ is simply connected, so that $C_1 = f_0 g$ and therefore $A = B = R$, then $\tilde{C}$ coincides with the usual cobar construction $C$ defined by Adams.

13. The Kan loop group. Let $X$ be a 0-reduced simplicial set and $G$ a simplicial group. A twisting function $f : X \to G$ is a collection of functions of degree 1

$$f : X_n \to G$$

satisfying

$$s_0 x = 1;$$

(2) $$s_1 x = s_0 x;$$

(3) $$d_0 x = d_1 x \quad \text{if } i;$$

(4) $$d_1 x = d_0 x \quad \text{if } i;$$

Let $GX$ denote the Kan loop group on $X$, which is a simplicial group that models the space of based loops on the geometric realization of $X$ (see [7,9]). There is a universal twisting function

$$X_{n+1} (GX)_n = F (X_{n+1}) = \mathbb{Z} (s_0 X_n)$$

sending $x \in X_{n+1}$ to the class of the corresponding generator in quotient of free groups, and the simplicial structure on $GX$ is defined by $X (3)$.

Recall that the normalized chain complex $CG$ on a simplicial group $G$ also has a differential graded algebra structure, with multiplication given by the shuf map and the multiplication in $G$,

$$m : CG \to CG \cdot C (G \cdot G) \cdot CG;$$

that is,

$$(g \cdot h) = \sum (s_{(1)}) \cdot s_{(0)} \cdot g \cdot s_{(0)} \cdot h; \quad g \cdot 2 G_p \cdot h \cdot 2 G_q;$$

where the summation is over all $(p, q)$ such that $(; ) = 2 Shu (p, q)$.

The following proposition is the motivation for our extension of the cobar construction. Recall that for any simplicial set $X$, the degree $n$ part of its normalized chain complex $C_n X$ is the free abelian group on the set of all nondegenerate $n$-simplices of $X$ and that $CX$ has a multiplication $CX \cdot CX \cdot CX$ given by the Alexander-Whitney diagonal approximation

$$\chi \frac{1}{1} x \cdot x \cdot x \cdot x \quad \text{if } i;$$

for $x \in X_n$, $n \geq 1$, and with $x = x \cdot x$ for $x \in X_0$. In particular, if $X$ is 0-reduced, then $CX$ is a connected, differential graded coalgebra over $Z$.

Proposition 4. Let $X$ be a 0-reduced simplicial set and $GX$ its Kan loop group. Then there is an isomorphism of rings

$$(\tilde{C}X)_0 \xrightarrow{\sim} (CGX)_0$$

determined by

$$\begin{align*}
0 (x) &= x \\
0 (x^1) &= 1 + s_1 x \\
0 (s_1 x) &= x^1 1 \\
0 (x) &= x.
\end{align*}$$

Notice that this isomorphism $\tilde{C}X \iso CGX$ is not natural in $X$.
Proof. The proof is straightforward. Note that if $x$ is the degenerate element $s_0(\cdot)$ then the four equations say $\circ(0) = 1$, $\circ(0) = 0$, $\circ(1) = 1$. In degree 0 the multiplication $(\circ (g \cdot h)) = g \cdot h$, and we have
\[
0(\cdot x_i \cdots x_i') = 0(\cdot x_i') \quad 0(\cdot x_i')
\]
where $x_i^1 X_i$, $i = 0$ and $a_i = s^{-1}x_i$ or $x_i$. This is well defined: $0(g)$ is inverse to $0(g^{-1})$ for all $g \in G(x)$ and $0(x)$ is inverse to $0(1 + s^{-1}x)$ for all $x X_i r f S_0(\cdot) g$. It is also clear that the compositions $00$ and $00$ are the respective identity maps:
\[
00(x) = 0(x) = x \quad 00(x^1) = 0(1 + s^{-1}x) = x^1
\]
\[
00(s^{-1}x) = 0(x^1 1) = s^{-1}x \quad 00(x) = 0(x) = x.
\]

1.4. The Szczarba map. A map of differential graded algebras

$$: CX ! CGX$$

was given explicitly by Szczarba in the language of twisting cochains. The following Definition, Lemma, and Theorem are from sections 2 and 3 of Szczarba’s paper [13], adapted slightly to define a map on the extended cobar construction

$$: C^CX ! CGX$$

that extends the isomorphism $\circ$ of Proposition 3.

Let $S_n$ be the set of $n!$ sequences of integers

$$i = (i_0; \cdots; i_n) \text{ such that } 0 \leq i_k \leq n \text{ for each } k.$$.

In particular, $i_0 = 0$. The sign of such a sequence $i_2 S_n$ is

$$(1)^{\sum i_k} \text{ where } i = i_0 + \cdots + i_k.$$.

Definition 5. Given a twisting function $: X ! G$, the Szczarba operators are the functions

$$Sz_1 : X_{n+1} ! G_n; \quad i = (i_0; \cdots; i_n) 2 S_n;$$

given by the following product in $G_n$,

$$Sz_1 x = D_{ij}^{n+1} x^1 D_{ij}^{n+1} \partial x^1 \partial D_{ij}^{n+1} x^1;$$

Here the operators $D_{ij}^{n+1} G_n j$ $G_n$ for $i_2 S_n$, $j = 0; \cdots; n$, are defined as

$$D_{ij}^1 = \mathrm{Id}_{D_{ij}};$$

\[
D_{ij}^1 = \begin{cases} D_{ij}^0 & \text{if } j < i_k; \\ 0 & \text{if } j = i_k; \\ D_{ij}^0 & \text{if } j > i_k; \\ \end{cases}
\]

As simplicial operators these are all frontal: defining $D_{ij}^{n+1} X_{n+1} j ! X_n$ in the same way, one has $D_{ij}^{n+1} = D_{ij}^{n+1} : X_{n+1} j ! X_n$.

Lemma 6. The Szczarba operators satisfy

$$\partial x_{ij}^{i_0+1} = Sz_1 x_{ij}^{i_0+1};$$

\[
\partial x_{ij}^{i_0+1} = e_{ik} Sz_1 x_{ij}^{i_0+1} x_{i_0+1} \text{ if } i_k > i_k + 1; \\
\partial x_{ij}^{i_0+1} = x_{ij}^{i_0+1} \text{ if } i_k > i_k + 1;
\]

\[
\partial x_{ij}^{i_0+1} = s Sz_1 x_{ij}^{i_0+1} x_{ij}^{i_0+1} \text{ if } i_k > i_k + 1;
\]
In the last equation the sequences $i^0, i^0$, the integer $r$ and the $(r' 1; n - r)$-shuffle $(; )$ are defined by a certain bijection

$$S_{n} = \prod_{r = 1}^{n} \text{Shu}(r' 1; n - r) \quad S_{r+1} S_{n}$$

see [11, Lemma 3.3], which respects parity as follows:

$$n + P_1 = r + \text{sgn}(; ) + P_0 + P_0 \pmod{2}$$

Note that Szczarba's sign conventions differ slightly from ours, and that his inductively-defined parity $(i; n + 1)$ is in fact just $n + i$.

**Theorem 7.** For any twisting function $\mathcal{X} : \mathcal{Y} \to \mathcal{G}$ on a 0-reduced simplicial set $\mathcal{X}$ there is a canonical homomorphism of differential graded algebras

$$\mathcal{C} \mathcal{X} \to \mathcal{C} \mathcal{G}$$

defined by

$$\begin{align*}
\theta_0 & \cdot x_1 \cdot x_1 = x_1 \\
\theta_0 & \cdot s_1 x_1 = x_1 \cdot x_{(1)} \\
n & \cdot s_1 x_{n+1} = x_{n+1} \cdot x_{(1)}
\end{align*}$$

for $x_{n+1} \in \mathcal{X}_{n+1}$.  

**Proof.** The map extends linearly, and multiplicatively via

$$p q (a, b) = m (p(a), q(b)) \quad p + q = p; p = q$$

where $m$ is the multiplication $\otimes$, to all of $\mathcal{C} \mathcal{X}$. We show is a chain map, i.e., that $\theta_n \cdot n = n \cdot \theta_n$. For $x \in \mathcal{X}_2$ we can write

$$\begin{align*}
\theta_1 & \cdot s_1 x = s_1 x_2 + s_1 x_2 + s_1 x_2 + s_1 x_{(1)} + s_1 x_{(2)} \\
& = (1 + s_1 x_2)(1 + s_1 x_{(1)})(1 + s_1 x_{(2)})
\end{align*}$$

and so we have, by Lemma 2,

$$\theta_1 \cdot s_1 x = \theta_1 x_2 = x_2 \cdot s_1 x_2$$

For $x \in \mathcal{X}_{n+1}$ the argument essentially the same. We have

$$\begin{align*}
\theta_n & \cdot s_1 x = x \cdot \sum_{k=0}^{n} (1) s_1 \theta_k x_{(0)} \\
& = x \cdot \sum_{k=0}^{n} (1) s_1 x_{(0)}
\end{align*}$$
where, by Lemma 6, all the terms for $0 < k < n$ cancel, and the terms for $k = 0; n$ may be rewritten as

$$X = (1)^{k+}\sum_i^1s_i d_n+1 x + (1)^{n+}\sum_i^1s_i x_{(i)j\mu i\nu} + sSz\circ x_{(j)j\mu i\nu} + 1$$

We will need one further property of the Szczarba operators.

Lemma 8. For all $x \in X_{n+1}$ and $i \in \mathbb{S}_n$, the following product in $G_n$ is degenerate:

$$D_{n+1}^{n+1} x = D_{i+1}^{n+1} d_{i+1} x + n^{\ast} D_{n+i}^{n+1} d_{i+1} x$$

Proof. For any sequence $i \in \mathbb{S}_n$, we will show by induction that $D_{n+1}^{n+1}$ is $s_{(i)}$-degenerate for all $j > 0$, where $s_{(i)}$ is the least integer such that $i_{(j)} = 0$. If $i_1 = 0$, so that $i = 1$, then by (6)

$$D_{n+1}^{n+1} = D_{0}^{0} x_{i \in \mathbb{S}_n} s_0 = s_0 D_{i \in \mathbb{S}_n} x_{i \in \mathbb{S}_n}$$

for all $j > 0$. If $i_1 > 0$, so that $i > 1$, then $(i_{1}\cdots i_{k}) = (i_{1})$ and we know that $D_{j+1}^{n+1}$ (if $j > 0$) and $D_{j}^{n+1}$ (if $j > 1$) are $s_{(i)}$-degenerate by the inductive hypothesis. The corresponding derived operators are therefore $s_{(i)}$-degenerate and, by (6), so is $D_{n+1}^{n+1}$ for all $j > 0$.

2. The retraction map

2.1. Definition of the map. Let $X$ be a 0-reduced simplicial set. We introduce in this section a map of di ement al graded algebras

$$CGX \rightarrow CX$$

between the chains on the loop group and the extended cobar construction, which is a retraction of the Szczarba map. The map is uniquely determined by the relation

$$n (x \circ g) = n (g) x_{(i)j\mu i\nu} x_{(j)j\mu i\nu} x_{(i)j\mu i\nu} + n \circ i_{0} (d_{i+1} x \circ g)$$

for all $x \in X_{n+1}$ and $g \in \mathbb{G}_n$. Note that the $i = 0$ term on the right-hand side is $s_{(i)} x_{(i)j\mu i\nu} n (x \circ g)$. In fact, may be expressed inductively, on the degree $n$ and the word length in $\mathbb{G}_n$.

Lemma 9. The definition of in (7) may be rewritten as

$$n (x \circ g) = x_{(i)j\mu i\nu} n (g) x_{(i)j\mu i\nu} x_{(j)j\mu i\nu} + n \circ i_{0} (x \circ g)$$

for $i = 1$.
where

\[ !_1(x; g) = \frac{d}{dx} \frac{d}{dg} \begin{pmatrix} 2 & GX \end{pmatrix} ; \]
\[ \mathcal{T}_i(x; h) = !_i(x; x^1 h) = \frac{d}{dx^1} \frac{d}{dh} \begin{pmatrix} x \end{pmatrix} ; \]

Proof. Collecting the terms in involving \( n(x; g) \) and dividing by \( 1 + s^1 x_{(0,1)} \) gives the first equation. The second is obtained by taking \( g = x^1 h \) in the first.

From these formulae it is straightforward to give the map explicitly in low degrees.

Lemma 10. The map \( \delta : GX \rightarrow (CX)_1 \) agrees with that defined in Proposition 7 and the map \( \delta : GX \rightarrow (CX)_1 \) is given for \( x_j \mid x_2 \mid x_1 = 1 \) by

\[ 1 \begin{pmatrix} x \end{pmatrix} = \frac{X^i}{j=1} \frac{d}{dx^i} \frac{d}{dx^i} \begin{pmatrix} x \end{pmatrix} \]

where \( n(x; g) \) is a reduced word in \( GX \). Suppose \( 0 \leq n \leq 1 \) and

\[ w = s_j (x \quad g); \]

where \( n(x; g) \) is a reduced word in \( GX \). For \( n = 1 \) we have

\[ \begin{pmatrix} 1 + s^1 x_{(0,1)} \end{pmatrix} \]

\[ \frac{X^i}{j=1} \frac{d}{dx^i} \frac{d}{dx^i} \begin{pmatrix} x \end{pmatrix} \]

in which the \( i \)th term is zero. Each term in the summation is also zero since \( (s_j \mid x)_{(0,1)} \) is degenerate for \( j \leq i \) and \( !_1(s_j \mid x; s_j g) = s_j !_1(x; g) \) for \( j < i \). Since \( 1 + s^1 x_{(0,1)} \) is invertible, we have \( n(w) = 0 \).

The argument for \( n = 1 \) is similar.

2.2. Properties of the retraction map. We now prove that \( \delta \) is a morphism of differential graded algebras and a retraction of the Szczarba map \( \delta \).

Proposition 12. \( \delta \) is a chain map.

Proof. We will show that for all \( x_2 \mid x_1 \) and \( g_2 \mid GX \),

\[ 0 \leq n \leq 1 \]

by induction on \( n \) and on word length in \( GX \). We first observe that

\[ n (d_0 (x \quad g)) = n (\frac{d}{dx} \frac{d}{dg} x) \]

\[ \mathcal{T}_i \begin{pmatrix} 1 \end{pmatrix} \]

\[ \frac{X^i}{j=1} \frac{d}{dx^i} \frac{d}{dx^i} \begin{pmatrix} x \end{pmatrix} \]

using the second formula in Lemma 3. Now since \( \frac{d}{dx} \frac{d}{dg} x = !_1(x; g) \), and also

\[ \mathcal{T}_i (d_0 x; \quad g) = \mathcal{T}_i (d_1 x; \quad g) = !_1 (d_1 x; d_0 g) = !_{i+1} (x; g), \]

we get

\[ !_1 (d_0 (x \quad g)) = \frac{X^i}{j=1} \frac{d}{dx^i} \frac{d}{dx^i} \begin{pmatrix} x \end{pmatrix} \]

\[ k \leq k \]
and, substituting $d_i^{r−1}x$ and $d_i^{r−1}g$ for $x$ and $g$ respectively, we obtain

\[ (9) \quad n \cdot (d_k! \cdot (x;g)) = \sum_{k=1}^{n} s^{k} x_{(1+\cdots+k+1)} n \cdot k! \cdot (x;g). \]

Now, using Lemma 3, we know that

\[ \theta_n \cdot n \cdot (x;g) = \theta_n \cdot \left( \sum_{i=1}^{n} s^{i} x_{(1+\cdots+i+1)} n \cdot i! \cdot (x;g) + n \cdot g \right) = \sum_{i=1}^{n} \theta_{i} \cdot s^{i} x_{(1+\cdots+i+1)} n \cdot i! \cdot (x;g) \]

\[ = \sum_{i=1}^{n} X^{i} \cdot \left( 1^{i} \sum_{s=1}^{i} x_{(1+\cdots+s+1)} n \cdot s! \cdot (x;g) + \sum_{k=0}^{n-1} (1)^{n-k} \cdot k! \cdot (x;g) \right) \]

\[ = \sum_{i=1}^{n} X^{i} \cdot \left( n \cdot d_{i} \cdot (x;g) + \sum_{k=0}^{n-1} (1)^{n-k} \cdot k! \cdot (x;g) \right) \]

(10)

Collecting together the terms for which either $i = 1, k = 0, r = 0,$ or $q = 1$ gives

\[ (1 + s^{1} x_{(0;\cdots)}) \cdot (1 + s^{1} x_{(0;\cdots)}) \cdot n \cdot i! \cdot (x;g) + n \cdot i! \cdot (x;g) \]

\[ + \sum_{i=1}^{n} (1 + s^{1} x_{(0;\cdots)}) \cdot s^{i} x_{(1+\cdots+i+1)} n \cdot i! \cdot (x;g) \]

\[ = (1 + s^{1} x_{(0;\cdots)}) \cdot n \cdot d_{0} \cdot (x;g) \cdot (1 + s^{1} x_{(0;\cdots)}) \cdot n \cdot i! \cdot (x;g) + n \cdot i! \cdot (x;g) \]

by (3), and by Lemma 3, the last two terms here cancel exactly with the terms for $r = 1$ and $i > 1$ in (10).

Now, collecting the terms for $r = i + 1$ and $i > 1$ in (10), together with all the $(i;g)$-indexed term s not already considered, gives

\[ \sum_{i=2}^{n} \theta_{i} \cdot s^{i} x_{(1+\cdots+i+1)} n \cdot i! \cdot (x;g) \]

\[ = \sum_{i=2}^{n} \theta_{i} \cdot s^{i} x_{(1+\cdots+i+1)} n \cdot i! \cdot (x;g) \]

\[ = \sum_{i=2}^{n} \theta_{i} \cdot s^{i} x_{(1+\cdots+i+1)} n \cdot i! \cdot (x;g) \]

by (3), and this cancels exactly with the terms for $t = 0$ in (10).
Thus expression \(14\) is equal to
\[
(1 + s^1 x_{[0,1]}^n) n_1 d_0 (x \cdot g) + \sum_{k=1}^{\infty} s^k n_1 d_k g
\]
\[
X
(1)^{r+1} s^1 x_{[0,1]}^{n_1 r+1} + n_1 ! n_1 (x;g)
\]
\[
2^{r+1} n_1 \alpha
X
(1)^{r+1} s^1 x_{[0,1]}^{n_1 r+1} + n_1 \cdot 1 (x;g)
\]

Now to complete the proof it remains to show that this is equal to
\[
(1 + s^1 x_{[0,1]}^n) \cdot 1 \cdot n_1 (x \cdot g) = (1 + s^1 x_{[0,1]}^n) n_1 d_0 (x \cdot g)
\]
\[
+ \sum_{i=1}^{\infty} X_i
(1)^i (1 + s^1 x_{[0,1]}^n) n_1 (d_i \cdot x \cdot d_i g):
\]
The first term is as required, and by Lemma \(8\) the summation is
\[
X \sum_{i=1}^{\infty} n_1 d_i g \cdot 1
s^i (d_i \cdot x_{[0,1]}^{n_1 i r+1}) n_1 k ! k (d_i \cdot x \cdot d_i g):
\]
The result therefore follows from the observations that
\[
(d_i \cdot x_{[0,1]}^{n_1 i r+1}) = X
s_i (d_i \cdot x_{[0,1]}^{n_1 i r+1}) n_1 k \cdot k (d_i \cdot x \cdot d_i g):
\]

Proposition 13. The map \( \varphi \) is an algebra homomorphism.

Proof. Let \( v \in 2 (G X)_p \) and \( w \in 2 (G X)_q \) and consider \( v \cdot w \in 2 (G X \cdot G X)_n \), \( n = p + q \). To show that \( \varphi \) is multiplicative we must prove
\[
\varphi \cdot m (v \cdot w) = X \cdot n \cdot (s \cdot g, s \cdot w) = p \cdot v \cdot q \cdot w
\]
in \( G X \), by induction on \( p \) and the word length of \( v \). For \( v = 1 \), \( v = 0 \), there is nothing to prove; suppose inductively that \( p = 1 \) and \( v = x \cdot g \) for \( x \in \chi \), \( x \) is similar. Then by Lemma \( 6\)
\[
(1 + s^1 x_{[0,1]}^n) = (s \cdot (x \cdot g) w) = (1 + s^1 x_{[0,1]}^n) (s^1 x \cdot (s \cdot g) w)
\]
\[
= (s \cdot (s \cdot g) w) X \cdot n \cdot (s \cdot g, s \cdot w) \cdot n \cdot 1 \cdot (s \cdot g, s \cdot w)
\]
The term \( d_i \cdot s^i g \cdot x \) will be generated unless \( p \) and \( s_i ; s \) is of the form
\[
(s_i \cdot g) \cdot a \cdot s_i \cdot b \cdot s_i \cdot c \cdot d \cdot s_i \cdot e \cdot f \cdot (s_i \cdot g)
\]
and we have
\[
(1 + s^1 x_{[0,1]}^n) \cdot m (v \cdot w)
\]
\[
= X \cdot n \cdot (s \cdot g, s \cdot w) \cdot n \cdot (s \cdot g, s \cdot w)
\]
\[
= \varphi \cdot m (g \cdot w) \cdot p \cdot g \cdot d_i \cdot s^i g \cdot p \cdot (s \cdot g, s \cdot w) \cdot q \cdot w;
\]
by the inductive hypothesis,
by Lemma 9. The result follows.

Proposition 14. The map $i$ is a retraction of $\phi$, that is, the composite $i \circ \phi$ is the identity.

Proof. It is enough to prove this on algebra generators of $^\chi CX$. For $x \cdot X_{n+1}$ and $i = (i_1, \ldots, i_n) \cdot 2^S_n$ we will show that

$$n \cdot S_x i = x \quad \text{if} \quad i_e = 0; \quad 0 \quad \text{otherwise}.$$ 

Denote by $x_{0; i}$ the element of $X$ satisfying

$$D_{0; i}^{n+1} x = x_{0; i}.$$ 

Lemma 3 tells us $n (x_{0; i} \cdot Sx) = 0$, and so by Lemma 9 we have

$$n \cdot S_x x = \sum_{k=1}^{n} d_x \cdot \kappa^{n-1} x_{0; i} \quad n! \kappa (x_{0; i} \cdot Sx):$$

From 9 we see that $D_{0; i}^{n+1} x$ has an $i_k$ degeneracy if $i_k \neq 0$. Thus $d_x \cdot \kappa^{n-1} x_{0; i}$ is degenerate except in the case $i_1 = \cdots = i_k = 0$. In this case we see from 3 and Lemma 9 that

$$! \kappa (x_{0; i} \cdot Sx) = d_x \cdot \kappa^{n-1} x_{0; i} \cdot Sx = D_{0; i}^{n+1} \cdot \kappa^{n-1} x_{0; i} \cdot Sx \quad \text{which is degenerate again by Lemma 9.}$$

The only non-zero term is therefore

$$n \cdot S_x x = x_{0; 0; \ldots, 0} \quad n (x) = x;$$

and hence $x = x$ as required.

3. Deformation retraction of the loop group

Both the Kan functor $G$ and the cobar construction model loop spaces. In the $1$-reduced case it is easy to show that the Szczarba map $: CX \to CGX$ is a weak equivalence, by applying Zeev’s comparison theorem to the map of spectral sequences associated with

$$\begin{array}{ccc}
CX & \to & CX \\
\downarrow \scriptstyle{=} & \downarrow \scriptstyle{=} \quad \downarrow \scriptstyle{=} \\
CGX & \to & CGX \\
\end{array}$$

in which the total spaces are acyclic.

We prove here the following stronger result.

Theorem 15. Let $X$ be a $0$-reduced simplicial set. Let $\phi$ be the Szczarba map and the retraction map defined above.

There is a natural strong deformation retraction of chain complexes

$$\begin{array}{ccc}
\wedge CX & \to & CGX \\
\downarrow \phi & \downarrow \phi \quad \downarrow \phi \\
A & \to & B \\
\end{array}$$

Recall that if $A$ and $B$ are chain complexes, $r : A \to B$ and $f : B \to A$ are chain maps, and $h : B \to A$ is linear map of degree $+1$, then

$$A$
is a strong deformation retract if \( f = \text{Id}_A \) and \( \theta + h \theta = r f \text{Id}_B \). Given a strong deformation retract, one can apply the machinery of homological perturbation theory to transfer perturbations of the structure \( B \) across to \( A \), obtaining a new strong deformation retract \([3], [5], [8]\).

Proof. According to Proposition [14], we need only to prove that there is a natural chain homotopy from the composite to the identity map on \( CGX \). The proof, an acyclic models argument, proceeds by induction on degree.

The base step of the induction is trivial, by Proposition [4]. We can simply set \( \theta_0 = 0 : C_0GX ! C_1GX \) for all 0-reduced simplicial sets \( X \).

Suppose now that \( \alpha : C_kGX \to C_{k+1}GX \) has been defined for all \( 0 \leq k \leq n \) and for all 0-reduced simplicial sets \( X \) so that

1. \( \alpha_k \) is natural in \( X \) for all \( k \), and
2. \( \theta_{k+1} \alpha_k = \text{Id}_{C_kGX} \) for all \( k \) and all \( X \),

where \( n \geq 1 \).

Let \( \alpha \) denote the quotient of the standard simplicial n-simplex by its 0-skeleton. If \( x = (k_0, \ldots, k_m) \) is a j-simplex of \( \alpha \), let \( x \) denote the \((j+1)\)-simplex \((k_0, \ldots, k_n)\). Let

\[
\alpha^n_j : G[\alpha]_{i+1} ! G[\alpha]_{i+1}
\]
denote the group homomorphism specified by \( \alpha^n_j(x) = (x, n) \) for all \( x \in [\alpha]_i \).

Let

\[
\alpha^n_j : C \times G[\alpha]_{i+1} \to C \times G[\alpha]_{i+1}
\]
denote the degree + 1 linear map specified by \( \alpha^n_j(w) = \alpha^n_j(w) \) for all \( w \in [\alpha]_{i+1} \).

The proof proceeds as in [10] that for all \( i \geq 1 \)

\[
\theta_{i+1}(\alpha^n_i \theta_i) = \text{Id};
\]

i.e., that \( \alpha^n_i \) is a contraction in positive degrees. It follows that \( H_i(C \times G[\alpha]) = 0 \) for all \( i \geq 1 \).

Consider the infinite wedge

\[
W(n + 1) = \bigvee_{n \geq 0} [\alpha]_n[n + 1];
\]

there is a chain homotopy

\[
\alpha^{n+1} : C \times GW(n + 1) ! C \times GW(n + 1)
\]

that is a contraction in positive degrees and that generalizes M-cycles and P-duality's construction.

Let

\[
w = m_1 \cdots m_n 2 GW(n + 1)n;
\]

where \( m_i \) denotes the unique nondegenerate \((n + 1)\)-simplex in the \( m_i \)-th copy of \([\alpha]_{n+1} \) in \( W(n) \), and \( i = 1 \). Set

\[
\eta_i(w) = \alpha^{n+1} \eta_i(w)_{w \in n_1([\alpha]_i)};
\]

The induction hypothesis implies that \( \eta_i(w)_{w \in n_1([\alpha]_i)} \) is a cycle and that

\[
\eta_i(w) = \alpha^{n+1} \eta_i(w)_{w \in n_1([\alpha]_i)} + \alpha^{n+1} \eta_i(w)_{w \in n_1([\alpha]_i)};
\]

Adding \( n_1([\alpha]_i) \) to both sides of this equation, we obtain

\[
\eta_i(w) + n_1([\alpha]_i) = \eta_i(w)_{w \in n_1([\alpha]_i)};
\]

i.e., \( 2n \) holds for all such \( w \).
Let $X$ be any 0-reduced simplicial set, and let
\[ w = x_1 \cdots x_n \]
be any nondegenerate $n$-simplex in $G X$, where $i = 1$ for all $i$. Let $\iota: [n+1]!$.

Let $[n+1]! X$ denote the simplicial map collapsing everything to the basepoint. Consider the morphism of simplicial groups
\[ \omega = G(1 \cdots)_{\omega} : G W (n+1)! G X : \]

Observe that
\[ \omega(1 \cdots k) = \omega; \]
where $i$ denotes the unique nondegenerate $n$-simplex in the $i$th copy of $[n+1]$ in $W (n+1)$.

Using the map $\omega$ constructed above for any generator $w$ of $C_n G X$, we define $n : C_n G X ! C_{n+1} G X$ for any 0-reduced simplicial set $X$ by
\[ n(w) = C_{n+1} w n(1 \cdots k); \]
Note that if $X = W (n+1)$ and $w = 1 \cdots k$, then
\[ \omega : G W (n+1)! G W (n+1) \]
is a homomorphism of simplicial groups given simplicial maps by permuting generators. It follows from the construction of the chain homotopy $h^{n+1}$ and therefore of the chain homotopy $H^{n+1}$ that $h^{n+1}$ is natural with respect to homomorphisms that simply permute generators, so that
\[ C_{n+1} w H_n = H_n C_n w : \]
Consequently, $\omega$ is indeed well-defined, since $\omega$, and, by the induction hypothesis, $n$ are all natural with respect to simplicial maps.

Moreover,
\[ @ n (w) = C_{n+1} w @ n (1 \cdots k) \]
\[ = C_{n+1} w \{ \text{Id}_{G X} \circ n \circ @ \}(1 \cdots k) \]
\[ = (\text{Id}_{G X} \circ n \circ @)(\omega); \]
where the equality (?) follows from naturality of $\omega$, and $n$. In other words,
\[ @ n + n @ = \text{Id}_{G X}; \]
for all $X$, i.e., condition (2n) holds.

To conclude, observe that condition (1n) holds as well, since for all simplicial maps $g : X ! Y$ between 0-reduced spaces and all $w \in G X$,
\[ G g w = G g(\omega) : G W (n+1)! G Y; \]

Remark 16. It is in order to be able to apply the chain homotopy of $M$ or $P$ route that we work with 0-reduced simplicial sets. There is no such chain homotopy in the 1-reduced case, so it seems we are obliged to prove the existence of the strong deformation retract in the 0-reduced case in order to conclude that it exists in the 1-reduced case as well.

Remark 17. As defined in the proof above, $n$ is almost certainly not a derivation homotopy, since, as easy computations show, $n$ is not a derivation homotopy.
Remark 18. Morace and Prouse showed that $h_{n+1}^i \sim h_n^i = 0$ for all $i$ and $n$, from which it follows that $h_{n+1}^i \sim h_n^i = 0$ as well and therefore that

$$h_{n+1}^i \sim h_n^i = 0$$

for all $i; \cdots ;i$ and $1; \cdots ;k$.

Remark 19. The results in this paper generalise from chain complexes to crossed com plexes. There is a crossed cobar construction $X$ on the fundamental crossed complex $X$, see [2], and we may de ne a ‘crossed’ Szczarba map of crossed chain algebras $X ! GX$ that forms part of a deformation retraction

$$X \rightarrow GX.$$

The classical argument that is a weak equivalence, using [2], does not go through in this slightly non-abelian situation, since it seems there is no good notion of twisted tensor product of crossed com plexes.

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