THE COORDINATE ALGEBRA OF A QUANTUM SYMPLECTIC SPHERE
DOES NOT EMBED INTO ANY C*-ALGEBRA

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ABSTRACT. We generalize a result of Mikkelsen–Szymański and show that, for every \( n \geq 2 \), any bounded \(*\)-representation of the quantum symplectic sphere \( S^{4n-1}_q \) annihilates the first \( n-1 \) generators. We then classify irreducible representations of its coordinate algebra \( A(S^{4n-1}_q) \).

1. INTRODUCTION AND MAIN RESULTS

A remarkable and unexpected property of the quantum symplectic sphere \( S^7_q \) has been recently unveiled in [4]. It is shown there that the coordinate algebra of \( S^7_q \) does not embed into its enveloping \( C^* \)-algebra: this results from the discovery that one of its generators is annihilated by every bounded \(*\)-representation. One wonders if this property is an isolated accident or is an example of a more general phenomenon. In the present short note we show that the latter is the case. The sphere \( S^7_q \) was used in [3] as the total space algebra of a quantum \( SU_q(2) \) instanton bundle over a sphere \( S^4_q \). It is one of a family of spheres associated to quantum symplectic groups \( A(\text{Sp}_q(n)) \) with the symplectic \( \mathbb{R} \)-matrix of [1].

Let \( 0 < q < 1 \) be a deformation parameter and \( n \) be a positive integer (this notation will be used throughout the paper). We denote by \( A(S^{4n-1}_q) \) the complex unital \(*\)-algebra generated by elements \( \{x_i, y_i\}_{i=1}^n \) and their adjoints, subject to the relations in Definition 6, in particular the sphere relation (15). This sphere is a comodule algebra for the quantum symplectic group \( A(\text{Sp}_q(n)) \), with coaction \( A(S^{4n-1}_q) \to A(\text{Sp}_q(n)) \otimes A(S^{4n-1}_q) \). The main point of this note is to prove the following theorem, which recovers the result of [4] when \( n = 2 \).

**Theorem 1.** If \( \pi \) is any bounded \(*\)-representation of \( A(S^{4n-1}_q) \), then \( \pi(x_i) = 0 \) for all \( 1 \leq i < n \).

For \( n = 1 \) the statement is vacuous. One may also observe that \( A(S^3_q) \simeq A(SU_q(2)) \). This result singles out the quantum symplectic spheres from the quantum unitary and orthogonal ones.

Next, let \( A(\Sigma^{2n+1}_q) \) be the quotient of \( A(S^{4n-1}_q) \) by the two-sided \(*\)-ideal generated by the \( n-1 \) elements \( \{x_i\}_{i=1}^{n-1} \). As customary, we interpret a quotient algebra as consisting of “functions” on a quantum subspace and think of \( \Sigma^{2n+1}_q \) as a quantum subsphere of \( S^{4n-1}_q \). It follows from Theorem [2] that any bounded \(*\)-representation of \( A(S^{4n-1}_q) \) factors through \( A(\Sigma^{2n+1}_q) \).

Now, if we further quotient by the ideal generated by the coordinate \( x_n \), we get a Vaksman-Soibelman quantum sphere [5], whose representation theory is well known (see e.g. [2]). We

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are then interested in representations of $\mathcal{A}(\Sigma_q^{2n+1})$ that do not annihilate $x_n$. A family of representations, parametrized by $U(1)$, is exhibited in Proposition 7 below. We next show that there are no other irreducible $*$-representations of $\mathcal{A}(\Sigma_q^{2n+1})$.

**Theorem 2.** Any irreducible bounded $*$-representation of $\mathcal{A}(\Sigma_q^{2n+1})$ that does not annihilate $x_n$ is unitarily equivalent to one of the representations in Proposition 7.

The paper is organized as follows. In Section 2 we prove some preliminary lemmas, in Section 3 we prove Theorem 1 while in Section 4 we prove Theorem 2.

2. SOME PRELIMINARY LEMMAS

If $T$ is a bounded operator, we will denote by $\sigma(T)$ its spectrum.

**Lemma 3.** If $T$ is a bounded selfadjoint operator on a Hilbert space $\mathcal{H}$ and $\sigma(T)$ has at most finitely-many accumulation points, then $T$ is diagonalizable.

**Proof.** Let $S$ be the set of accumulation points of $\sigma(T)$ and assume that it is finite. Isolated points in the spectrum of a selfadjoint operator are eigenvalues. Thus, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1$ is the $\ell^2$ direct sum of all eigenspaces of $T$ and $\mathcal{H}_2$ is its orthogonal complement. For $i = 1, 2$, let $T_i$ denote the restriction of $T$ to $\mathcal{H}_i$ and note that $T_i(\mathcal{H}_i) \subset \mathcal{H}_i$. Thus $T = T_1 \oplus T_2$. Since $\mathcal{H}_2$ contains no eigenvectors of $T$, the operator $T_2$ has no eigenvectors. Since $\sigma(T_2) \subset \sigma(T)$ and $\sigma(T_2)$ has no accumulation points, no point of $\sigma(T) \setminus S$ lies in $\sigma(T_2)$. Thus $\sigma(T_2) \subset S$. But again, since $\sigma(T_2)$ cannot have any isolated points, it follows that $\sigma(T_2) = \emptyset$. Since every bounded operator on a non-zero Hilbert space has non-empty spectrum, it follows that $\mathcal{H}_2 = \{0\}$. ■

**Lemma 4.** Let $0 < \mu < 1$, $0 < t \leq 1$, $m \in \mathbb{N}$, and let $A \geq 0$ and $B$ be bounded operators on a Hilbert space $\mathcal{H}$ with norm at most $t$ such that:

\[
BA = \mu^2 AB \quad \text{BB}^* = \mu B^* B + (1 - \mu)(t + \mu^m A) \tag{1}
\]

Then $BB^*$ and $B^*B$ are diagonalizable, with respective spectra:

\[
\sigma(BB^*) \subset \{ t(1 - \mu^{k+1}) : k \in \mathbb{N} \} \cup \{ t \},
\]

\[
\sigma(B^*B) \subset \{ t(1 - \mu^k) : k \in \mathbb{N} \} \cup \{ t \}. \tag{2}
\]

Moreover:

(i) if $m \geq 3$, then $A = 0$;

(ii) if $m = 2$ and there exists a bounded operator $C$, such that

\[
CA = \mu AC \quad [C, C^*] = (1 - \mu)A, \tag{3}
\]

then $A = 0$.

**Proof.** We follow [4]. From the first relation in (1) we deduce that $B^*B$ commutes with $A$, so that the second relation implies that $B^*B$ commutes with $BB^*$ as well. Since $BB^* \geq (1 - \mu)t + \mu B^*B$, the joint spectrum $\sigma(BB^*, B^*B)$, i.e. the set $\{ (\chi(BB^*), \chi(B^*B)) : \chi$ is a character of $C^*(BB^*, B^*B) \}$, is contained in the closed triangle displayed in Fig. 1. Note that $(x, y) \in \sigma(BB^*, B^*B)$ implies $x \in \sigma(BB^*)$ and $y \in \sigma(B^*B)$. Note also that:

\[
\sigma(BB^*) \cup \{0\} = \sigma(B^*B) \cup \{0\}.
\]
Since \( \lambda^+ = 0 \), it follows from (1) that, if \( \lambda \in \sigma(B^* B) \), it is in the complement of \( \sigma(B^* B) \) as well. Hence the open segment from \( a \) to \( b \) is in the complement of \( \sigma(B^* B, B^* B) \). This implies that \( \lambda \in \sigma(B^* B, B^* B) \), hence it is in the complement of \( \sigma(B^* B) \) as well and the open segment from \( b \) to \( c \) is in the complement of \( \sigma(B^* B, B^* B) \). It turn, this implies that \( \lambda \in \sigma(B^* B, B^* B) \), etc. Proceeding inductively in this way proves (2).

We will use the convention that \( \lambda^+ = 0 \). It follows from Lemma 3 that both \( B^* B \) and \( B^* B \) are diagonalizable, since each has only one accumulation point in its spectrum. Thus 

\[
\mathcal{H} = \bigoplus_{j,k \in \mathbb{N} \cup \{+\infty\}} V_{j,k}
\]

where 

\[
V_{j,k} := \{ v \in \mathcal{H} : B^* B v = (1 - \mu^{j+1}) t v \wedge B v = (1 - \mu^j) t v \}.
\]

Note that \( V_{j,k} \) may be \( \{0\} \) for some values of \( j \) and \( k \). Let \( A' := t^{-1} A \). From (1) we deduce that \( A' v = \lambda_{j,k} v \) for all \( v \in V_{j,k} \), with:

\[
\lambda_{j,k} = \frac{\mu^{k+1} - \mu^{j+1}}{\mu^m (1 - \mu)},
\]

thus either \( \lambda_{j,k} \) is an eigenvalue of \( A' \), or \( V_{j,k} = \{0\} \).

Since \( A' \geq 0 \) and \( \lambda_{j,k} < 0 \) for \( j < k \), it follows that \( V_{j,k} = \{0\} \) for all \( j < k \). Thus

\[
\mathcal{H} = \bigoplus_{j,k \in \mathbb{N} \cup \{+\infty\}} V'_{j,k},
\]

where \( V'_{j,k} := V_{j+k,k} \). Set

\[
\lambda'_{j,k} := \lambda_{j+k,k} = \frac{\mu^k - \mu^j}{\mu^{m-1} (1 - \mu)}.
\]

Let \( v \in V'_{j,k} \) and notice that, if \( k \neq 0 \), then \( B v = 0 \) implies \( v = t^{-1} (1 - \mu^k)^{-1} B^* B v = 0 \). It follows from (1) that, if \( v \neq 0 \) and \( k \geq 2 \), then \( B v \) is an eigenvector of \( A' \) with eigenvalue \( \mu^{-1} \lambda'_{j,k} = \lambda'_{j,k-2} \). By induction, if \( \lambda'_{j,k} \) is an eigenvalue of \( A' \) and \( k \) is even, then \( \lambda'_{j,0} \) is an...
eigenvalue of $A'$ as well; if $\lambda'_{j,k}$ is an eigenvalue of $A'$ and $k$ is odd, then $\lambda'_{j,1}$ is an eigenvalue of $A'$ as well. But for all $j \geq 1$:

$$\lambda'_{j,0} \geq \lambda'_{j,1} = \frac{1}{\mu m - 1} > 1 \text{ if } m \geq 2 \quad \text{and} \quad \lambda'_{j,1} \geq \lambda'_{j,1} = \frac{1}{\mu m - 2} > 1 \text{ if } m \geq 3.$$ 

Since $A' \leq 1$, for $n \geq 3$ we deduce that $\lambda'_{j,k}$ is not an eigenvalue of $A'$ for any $j \geq 1$ and $k \geq 0$. Thus $V'_{j,k} = \{0\}$ for all $j \neq 0$ and $H = \bigoplus_{k \in \mathbb{N} \cup \{\infty\}} V'_{0,k}$. Since $\lambda'_{0,k} = 0$, clearly $A' = 0$. This proves (i).

Assume now that $n = 2$ and that there exists $C$ satisfying (3). Let $j \geq 1$. We already proved that $\lambda'_{j,k}$ is not an eigenvalue of $A'$ if $k$ is even. If $k$ is odd, from the commutation relation we deduce that, for all $v \in V'_{j,k}$, $Cv$ is either zero or an eigenvector of $A'$ with eigenvalue $\lambda'_{j,k-1}$, and $C^*v$ is either zero or an eigenvector of $A'$ with eigenvalue $\lambda'_{j,k+1}$. From the argument above, it follows that $Cv = C^*v = 0$. But then

$$0 = [C, C^*]v = (1 - \mu)Av = (1 - \mu)t\lambda'_{j,k}v,$$

which implies that $v = 0$ since $\lambda'_{j,k} \neq 0$. Thus $V'_{j,k} = \{0\}$ for all $j \geq 1$ and any $k$ (even or odd).

Since $H = \bigoplus_{k \in \mathbb{N} \cup \{\infty\}} V'_{0,k}$ and $\lambda'_{0,k} = 0$, clearly $A = 0$. This proves (ii).

**Lemma 5.** Let $A \geq 0$ and $B$ be bounded operators on a Hilbert space $H$ satisfying

$$[B, B^*] = (1 - \mu)A \quad A + B^*B = 1 \quad (4)$$

with $0 < \mu < 1$. Then, $\ker(B) = \{0\}$ if and only if $A = 0$.

**Proof.** If $A = 0$, from (4) it follows that $B$ is unitary, so that $\ker(B) = \{0\}$. We have to prove the opposite implication. From (4) we deduce that $\|A\| \leq 1$ and:

$$BB^* = B^*B + (1 - \mu)A = 1 - \mu A. \quad (5)$$

Since $\|\mu A\| < 1$, the operator $BB^*$ has bounded inverse. Therefore $U := (BB^*)^{-1/2}$ is well defined. Notice that $UU^* = 1$ and $\ker(U) = \ker(U)$. From (5), it follows that

$$\mu A = 1 - BB^* = U(1 - B^*B)U^* = UAU^*.$$ 

Assume that $\ker(U) = \{0\}$. The identity $U(1 - U^*U) = 0$ implies $U(1 - U^*U)v = 0$ and hence $(1 - U^*U)v = 0$ for all $v \in H$. Therefore $U^*U = 1$ and $U$ is a unitary operator.

Let $\lambda \in \mathbb{C}$ and suppose $A - \mu^{-1}\lambda$ has bounded inverse. Then

$$A - \lambda = \mu U^* (A - \mu^{-1}\lambda) U$$

has bounded inverse as well. Thus, $\lambda \in \sigma(A)$ implies that $\mu^{-1}\lambda \in \sigma(A)$ and hence, by induction, that $\mu^{-k}\lambda \in \sigma(A)$ for all $k \geq 0$. Since $A$ is bounded and the sequence $(\mu^{-k}\lambda)_{k \geq 0}$ is divergent when $\lambda \neq 0$, it follows that $\sigma(A) = \{0\}$. Hence $A = 0$.

### 3. The Symplectic Quantum Spheres and Their Quotients

**Definition 6.** We denote by $A(S_q^{4n-1})$ the complex unital $*$-algebra generated by elements $\{x_i, y_i\}_{i=1}^n$ and their adjoints, subject to the following relations. Firstly, one has:

$$x_i x_j = q^{-1} x_j x_i \quad (i < j), \quad y_i y_j = q^{-1} y_j y_i \quad (i > j), \quad x_i y_j = q^{-1} y_j x_i \quad (i \neq j), \quad (6)$$
Finally, one has

$$y_i x_i = q^2 x_i y_i + (q^2 - 1) \sum_{k=1}^{i-1} q^{i-k} x_k y_k,$$

(7)

Next, one has

$$x_i x_i^* = x_i^* x_i + (1 - q^2) \sum_{k=1}^{i-1} x_k^* x_k$$

(8)

$$y_i y_i^* = y_i^* y_i + (1 - q^2) \left\{ q^{2(n+1-i)} x_i^* x_i + \sum_{k=1}^{n} x_k^* x_k + \sum_{k=1}^{n} y_k y_k \right\}$$

(9)

$$x_i y_i^* = q^2 y_i^* x_i$$

(10)

$$x_i x_j^* = q x_j^* x_i \quad (i \neq j)$$

(11)

$$y_i y_j^* = q y_j^* y_i - (q^2 - 1) q^{2n+2-i-j} x_i^* x_j \quad (i \neq j)$$

(12)

$$x_i y_j^* = q y_j^* x_i \quad (i < j)$$

(13)

$$x_i y_j^* = q y_j^* x_i + (q^2 - 1) q^{i-j} y_i^* x_j \quad (i > j)$$

(14)

Finally, one has the sphere relation:

$$\sum_{i=1}^{n} (x_i^* x_i + y_i^* y_i) = 1.$$  

(15)

One passes to the notations of [3] by setting $y_i = x_{2n+1-i}$ and replacing $q$ by $q^{-1}$, so that the assumption in [3] that $q > 1$ becomes our assumption that $0 < q < 1$. As already mentioned the sphere comes with a coaction $\mathcal{A}(S_{q^n}^{4n-1}) \to \mathcal{A}(Sp_q(n)) \otimes \mathcal{A}(S_{q^n}^{4n-1})$.

Let $\mathcal{A}(\Sigma_{q}^{2n+1})$ be the quotient of $\mathcal{A}(S_{q^n}^{4n-1})$ by the two-sided $*$-ideal generated by $\{x_i\}_{i=1}^{n-1}$ (thus $\Sigma_{q}^{2n+1}$ is a quantum subsphere of $S_{q^n}^{4n-1}$). Let us write down explicitly the relations in the quotient algebra $\mathcal{A}(\Sigma_{q}^{2n+1})$. If we rename $y_{n+1} := x_n$, it follows from (8) that $y_{n+1}$ is normal. The remaining relations become:

$$y_i y_j = q^{-1} y_j y_i \quad (i > j \land (i, j) \neq (n + 1, n))$$

(16)

$$y_i^* y_j = q^{-1} y_j^* y_i^* \quad (i > j \land (i, j) \neq (n + 1, n))$$

(17)

$$y_{n+1} y_n = q^{-2} y_n y_{n+1} \quad y_{n+1}^* y_n = q^{-2} y_n^* y_{n+1}^*$$

(18)

$$[y_i, y_j^*] = (1 - q^2) \sum_{k=1}^{n+1} y_k^* y_k \quad (i \neq n)$$

(19)

$$[y_n, y_n^*] = (1 - q^4) y_{n+1}^* y_{n+1}$$

(20)

plus the ones obtained by adjunction and the sphere relation:

$$\sum_{i=1}^{n+1} y_i^* y_i = 1.$$  

(21)

It follows from Theorem [5] that any bounded $*$-representation of $\mathcal{A}(S_{q^n}^{4n-1})$ comes from one of $\mathcal{A}(\Sigma_{q}^{2n+1})$. If we further quotient by the ideal generated by $x_n$, we get a Vaksman-Soibelman quantum sphere [5], whose representation theory is well known (see e.g. [2]). We turn then to the representations of $\mathcal{A}(\Sigma_{q}^{2n+1})$ that do not annihilate $x_n$.

It is straightforward to check the following statement.
Proposition 7. For every $\lambda \in \mathbb{U}(1)$, an irreducible bounded $*$-representation $\pi_\lambda$ of $\mathcal{A}(\Sigma_4^{2n+1})$ on $\ell^2(\mathbb{N}^n)$ is given by the formulas:

$$\pi_\lambda(y_i) |k\rangle = q^{k_1 + \ldots + k_{i-1}} \sqrt{1 - q^{2k_i}} |k - e_i\rangle \quad (1 \leq i \leq n - 1)$$

$$\pi_\lambda(y_n) |k\rangle = q^{k_1 + \ldots + k_{n-1}} \sqrt{1 - q^{4k_n}} |k - e_n\rangle ,$$

$$\pi_\lambda(x_n) |k\rangle = \lambda q^{k_1 + k_n} |k\rangle ,$$

where $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$, $|k\rangle := k_1 + \ldots + k_n$, $\{ |k\rangle \}_{k \in \mathbb{N}^n}$ is the canonical orthonormal basis of $\ell^2(\mathbb{N}^n)$ and $e_i$ the $i$-th row of the identity matrix of order $n$.

This proposition will be used in Section 5 for the proof of Theorem 2 showing there are no additional irreducible $*$-representations.

4. PROOF OF THEOREM 1

In this section we shall prove Theorem 1 whose statement we repeat here.

Theorem 1. If $\pi$ is any bounded $*$-representation of $\mathcal{A}(\Sigma_4^{4n-1})$, then $\pi(x_i) = 0$ for all $1 \leq i < n$.

We need the following results.

Lemma 8. If $\pi$ is any bounded $*$-representation of $\mathcal{A}(\Sigma_4^{4n-1})$ then, for all $1 \leq k \leq n - 1$,

(i) $\pi(x_i) = 0 \quad (1 \leq i \leq k)$,

(ii) $\sigma \left( \sum_{i=1}^k \pi(y_i^*y_i) \right) \subset \{1 - q^{2j} : j \in \mathbb{N}\} \cup \{1\}$.

Proof. For $n = 1$ the above statements are vacuous, so we assume $n \geq 2$; note that the case $n = 2$ was proved already in [4]. From (9) and (15) we find that:

$$y_1y_1^* = q^2y_1^*y_1 + (1 - q^2)(1 + q^{2n}x_1^*x_1) . \quad (22)$$

By applying Lemma 4 to $A = \pi(x_1^*x_1)$, $B = \pi(y_1)$ and $\mu := q^2$, we deduce that $A = 0$, which implies (i). The statement (ii) is simply (2), and for $n = 2$ this concludes the proof.

For $n \geq 3$, we will prove the Lemma by induction on $k$, having just proved (i) and (ii). Assume by inductive hypothesis that (i$_{k-1}$) and (ii$_{k-1}$) are true for some fixed $2 \leq k \leq n - 1$. Let $D := \sum_{i=1}^{k-1} \pi(y_i^*y_i)$. It follows from (ii$_{k-1}$) and Lemma 3 that $D$ can be diagonalized. Decompose $\mathcal{H}$ as the orthogonal direct sum

$$\mathcal{H} = \bigoplus_{j \in \mathbb{N} \cup \{+\infty\}} \mathcal{H}_j$$

where

$$\mathcal{H}_j := \{ v \in \mathcal{H} : Dv = (1 - q^{2j})v \}$$

and $q^{+\infty} = 0$ by convention; notice that $\mathcal{H}_m$ may be zero for some values of $m$.

From (12), (14) and the inductive hypothesis we deduce that

$$\pi(x_k)\pi(y_i^*) = q\pi(y_i^*)\pi(x_k) \quad \pi(y_k)\pi(y_i^*) = q\pi(y_i^*)\pi(y_k)$$

for all $i \leq k - 1$. Combined with (6) we find that:

$$[\pi(x_k), \pi(y_i^*y_i)] = [\pi(y_k), \pi(y_i^*y_i)] = 0 \quad \forall i \leq k - 1.$$
These relations (and their adjoints) imply that \( \pi(x_k) \), \( \pi(y_k) \) and their adjoints all map each space \( \mathcal{H}_j \) to itself.

On \( \mathcal{H}_{+\infty} \), since \( D \) restricts to the identity, it follows from (15) that \( \pi(x_k^*x_k) \) and \( \pi(y_k) \) must vanish. Thus, on \( \mathcal{H}_{+\infty} \), the operator \( \pi(x_k) \) restricts to the zero map and \( \sum_{i=1}^{k} \pi(y_i^*y_i) = \pi(y_k^*y_k) + D \) restricts to the identity. It remains to see what happens on \( \mathcal{H}_j \) for \( j \in \mathbb{N} \).

Fix \( j \in \mathbb{N} \), let \( \mu := q^2 \), let \( t := q^{2j} \), let \( A \) be the restriction of \( \pi(x_k^*x_k) \) to \( \mathcal{H}_j \), and let \( B \) be the restriction of \( \pi(y_k) \) to \( \mathcal{H}_j \). Call \( m := n + 1 - k \) and notice that \( m \geq 2 \). From (9) and (15) we deduce that

\[
BB^* = \mu B^*B + (1 - \mu)(1 + \mu^m A - D|_{\mathcal{H}_j}) .
\]

But \( D|_{\mathcal{H}_j} = 1 - t \), hence the second relation in (1) is satisfied. The first relation follows from (7), (10) and the inductive hypothesis. Finally, from (15) it follows \( A + B^*B \leq 1 - D|_{\mathcal{H}_j} = t \), thus \( A \) and \( B \) both have norm at most \( t \) and we can use Lemma 4.

From (2) it follows that on \( \mathcal{H}_j \) the operator \( \sum_{i=1}^{k} \pi(y_i^*y_i) = B^*B + D \) has spectrum contained in the set

\[
\{ t(1 - \mu^r) + (1 - t) = 1 - t\mu^r : r \in \mathbb{N} \cup \{+\infty\} \} = \{ 1 - q^{2s} : s = j, j+1, \ldots, +\infty \} .
\]

This proves (ii).k. If \( m \geq 3 \), from Lemma 4(ii) we deduce (i).k. If \( m = 2 \), which means \( k = n - 1 \), we set \( C := \pi(x_n) \) and from Lemma 4(ii) we deduce (i).n−1.

\[\text{Proof of Theorem 7} \text{ The statement (i).n−1 in Lemma 8 is exactly Theorem 1} \]

5. PROOF OF THEOREM 2

We are now ready to prove Theorem 2.

**Theorem 2.** Any irreducible bounded \(*\)-representation of \( \mathcal{A}(\Sigma^2_{q^{n+1}}) \) that does not annihilate \( x_n \) is unitarily equivalent to one of the representations in Proposition 7.

Recall the relations (17)-(21) for the generators \( \{y_i\} \) of the algebra \( \mathcal{A}(\Sigma^2_{q^{n+1}}) \). Then,

**Lemma 9.** For all \( m \geq 1 \) and all \( 1 \leq i < n \):

\[
y_i(y_i^*)^m = q^{2m}(y_i^*)^my_i + (1 - q^{2m})(y_i^*)^{m-1}(1 - \sum_{k<i} y_k^*y_k)
\]

\[
y_n(y_n^*)^m = q^{4m}(y_n^*)^my_n + (1 - q^{4m})(y_n^*)^{m-1}(1 - \sum_{k<n} y_k^*y_k)
\]

**Proof.** When \( m = 1 \), these identities follow from (19) and (20), and can be rewritten using (21) as:

\[
y_i y_i^* = q^2 y_i^* y_i + (1 - q^2) \left(1 - \sum_{k<i} y_k^* y_k \right) \quad \text{if } i < n
\]

\[
y_n y_n^* = q^4 y_n^* y_n + (1 - q^4) \left(1 - \sum_{k<n} y_k^* y_k \right)
\]

The final result easily follows using the latter relations, induction on \( m \) and the fact that \( y_k^* y_k \) commutes with \( y_i \) for all \( k < i \leq n \).

\[\text{Lemma 10. Let } \pi \text{ be an irreducible bounded } *\text{-representation of } \mathcal{A}(\Sigma^2_{q^{n+1}}) \text{ with } \pi(y_{n+1}) \neq 0. \text{ Then:}
\]

(i) \( \pi(y_{n+1}) \) is injective;

(ii) there exists a vector \( \xi \neq 0 \) such that \( \pi(y_i)\xi = 0 \) for all \( i \neq n + 1 \).
Proof. (i) If \( a \) is any generator other than \( y_{n+1} \), since \( y_{n+1} a \) is a scalar multiple of \( a y_{n+1} \), the operator \( \pi(a) \) maps the kernel of \( \pi(y_{n+1}) \) to itself. Hence, \( \ker \pi(y_{n+1}) \) carries a subrepresentation of the irreducible representation \( \pi \), so that either \( \ker \pi(y_{n+1}) = \{0\} \) or \( \ker \pi(y_{n+1}) = \mathcal{H} \). The latter implies \( \pi(y_{n+1}) = 0 \), contradicting the hypothesis, so that the former must hold.

(ii) Given \( 1 \leq k \leq n \), let \( \mathcal{H}_k := \bigcap_{i=1}^k \ker \pi(y_i) \). We prove by induction on \( k \) that \( \mathcal{H}_k \neq \{0\} \). When \( k = 1 \), this follows from Lemma 5 applied to the operators \( A = \sum_{i=2}^{n+1} \pi(y_i^* y_i) \) and \( B = \pi(y_1) \). Since \( \pi(y_{n+1}) \neq 0 \), it follows that \( A \neq 0 \), and hence \( \ker(B) \neq \{0\} \).

Now assume that \( \mathcal{H}_{k-1} \neq \{0\} \). Let \( A = \sum_{i=k+1}^{n+1} \pi(y_i^* y_i) \) and \( B = \pi(y_k) \), and note that the operators \( \pi(y_{n+1}), A, B, B^* \) map \( \mathcal{H}_{k-1} \) to itself, since \( y_i y_j \) is a scalar multiple of \( y_i^* y_i \) and \( y_i^* y_j \) is a scalar multiple of \( y_j^* y_i \) for all \( i \neq j \). It follows from point (i) that \( \pi(y_{n+1})|_{\mathcal{H}_{k-1}} \neq 0 \), so that \( A \) is non-zero on \( \mathcal{H}_{k-1} \). The operator \( A + B^* B \) restricts to the identity on \( \mathcal{H}_{k-1} \) and \( [B, B^*] = (1 - \mu)A \) with \( \mu = q^2 \) if \( k < n \) and \( \mu = q^4 \) if \( k = n \). From Lemma 5 applied to the restrictions of \( A, B, B^* \) to \( \mathcal{H}_{k-1} \), it follows that \( \ker(B) \cap \mathcal{H}_{k-1} = \mathcal{H}_k \neq \{0\} \).

Proof of Theorem 2. Let \( \pi \) be a bounded irreducible \( * \)-representation of \( \mathcal{A}(\Sigma^2_{q^{n+1}}) \) on a Hilbert space \( \mathcal{H} \) such that \( \pi(y_{n+1}) \neq 0 \). By abuse of notation, we suppress the map \( \pi \). We know from Lemma 10(ii) that \( V := \bigcap_{i=1}^n \ker(y_i) \neq \{0\} \). From the commutation relations we deduce that \( y_{n+1} V \subset V \), so that \( V \) carries a bounded \( * \)-representation of the commutative C*-algebra \( C^* \langle y_{n+1}, y_{n+1}^* \rangle \) generated by \( y_{n+1} \) and \( y_{n+1}^* \).

Given \( k \in \mathbb{N}^n \) and \( \xi, \eta \in V \) a unit vector, define:

\[
|k\rangle_\xi := \frac{1}{\sqrt{(q^2; q^2)_k \cdots (q^4; q^4)_{k_{n-1}} (q^4; q^4)_k}} (y_1^* k_1 \cdots (y_n^* k_n) \xi, \eta),
\]

where the q-shifted factorial is given by

\[
(a; b)_\xi := \prod_{i=0}^{\xi-1} (1 - ab^i).
\]

Given \( k \in \mathbb{Z}^n \), set \( |k\rangle_\xi := 0 \) if one of the components of \( k \) is negative. From the commutation relations we deduce:

\[
y_i^* |k\rangle_\xi = q^{k_1 + \cdots + k_{i-1}} \sqrt{1 - q^{2k_i+2}} |k + e_i\rangle_\xi \quad (i < n),
\]

\[
y_n^* |k\rangle_\xi = q^{k_1 + \cdots + k_{n-1}} \sqrt{1 - q^{4k_n+4}} |k + e_n\rangle_\xi,
\]

\[
y_{n+1} |k\rangle_\xi = q^{k_1 + k_n} |k\rangle_{y_{n+1} \xi}.
\]

If \( W \subset V \) carries a subrepresentation of \( C^* \langle y_{n+1}, y_{n+1}^* \rangle \), the Hilbert subspace of \( \mathcal{H} \) spanned by \( |k\rangle_\xi \) for \( \xi \in W \) and \( k \in \mathbb{N}^n \) carries a subrepresentation of \( \mathcal{A}(\Sigma^2_{q^{n+1}}) \). Since \( \mathcal{H} \) is irreducible, \( V \) carries an irreducible representation of \( C^* \langle y_{n+1}, y_{n+1}^* \rangle \). This means that \( V \) is one-dimensional. Let us fix a unit vector \( \xi, \eta \in V \). Observe that the vectors

\[
\langle |k\rangle_\xi |k\rangle_\eta \in \mathbb{C}^n
\]

span \( \mathcal{H} \). Moreover \( y_{n+1} \xi = \lambda \xi \), for some \( \lambda \in \mathbb{R} \), and

\[
y_{n+1} |k\rangle_\xi = \lambda q^{k_1 + k_n} |k\rangle_\xi.
\]
From now on, we omit the subscript “ξ”. By (21), it follows that
\[ 1 = \langle 0 | 0 \rangle = \left\langle 0 \mid \sum_{i=1}^{n+1} y_i^* y_i \right\rangle = |\lambda|^2, \]
hence \( \lambda \in \mathbb{U}(1) \). It remains to prove that the set (24) is orthonormal, so that by adjunction from (23) we get the formulas in Prop. 7.

Let \( W_i \) be the span of vectors \([k]\) with \( k_1 = \ldots = k_i = 0 \). It follows from the commutation relations that \( y_k \) is zero on \( W_i \) for all \( k \leq i \).

Applying the identities in Lemma 9 to a vector \([k]\) we find that
\[
y_i (y_i^*)^m |0, \ldots, 0, k_{i+1}, \ldots, k_n\rangle = (1 - q^{2m}) (y_i^*)^{m-1} |0, \ldots, 0, k_{i+1}, \ldots, k_n\rangle \]
\[
y_n (y_n^*)^m |0\rangle = (1 - q^{4m}) (y_n^*)^{m-1} |0\rangle \]
for all \( m \geq 1 \) and all \( 1 \leq i < n \). Using (23) we find that
\[
y_i |0, \ldots, 0, m - 1, k_{i+1}, \ldots, k_n\rangle = \sqrt{1 - q^{2m}} |0, \ldots, 0, m - 1, k_{i+1}, \ldots, k_n\rangle \]
\[
y_n |0, \ldots, 0, m - 1\rangle = \sqrt{1 - q^{4m}} |0, \ldots, 0, m - 1\rangle \]

Multiplying from the left by \((j - e_i|\) and using (23) again, we find that
\[
\sqrt{1 - q^{2j}} \langle j | k \rangle = \sqrt{1 - q^{2k_i}} \langle j - e_i, k - e_i \rangle, \]
\[
\sqrt{1 - q^{4j}} \langle j_n e_n | k_n e_n \rangle = \sqrt{1 - q^{4k_n}} \langle (j_n - 1) e_n, (k_n - 1) e_n \rangle, \]
where the former is valid whenever \( j_1, \ldots, j_i = k_1 = \ldots = k_l = 0 \). From these relations, an obvious induction proves that the set (24) is orthonormal provided every vector \([k]\) with \( k \neq 0 \) is orthogonal to \([0]\). But this is obvious. If \( k_i \neq 0 \) for some \( i \), then
\[
\langle k | 0 \rangle \propto \langle k - e_i | y_i | 0 \rangle = 0 \]
since \( y_i \) annihilates \( \xi \). ■

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