Compact static stars in minimal dilatonic gravity

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In the version of this paper we presented for the first time the basic equations and relations for relativistic static spherically symmetric stars (SSSS) in the model of minimal dilatonic gravity (MDG). This model is locally equivalent to the f(R) theory of gravity and gives an alternative description of the effects of dark matter and dark energy using the Brans-Dicke dilaton Φ. To outline the basic properties of the MDG model of SSSS and to compare them with general relativistic results, in the present paper we use the relativistic equation of state (EOS) of neutron matter as an ideal Fermi neutron gas at zero temperature. We overcome the well-known difficulties of the physics of SSSS in the f(R) theories of gravity applying novel highly nontrivial nonlinear boundary conditions, which depend on the global properties of the solution and on the EOS. We also introduce two pairs of new notions: cosmological-energy-pressure densities and dilatonic-energy-pressure densities, as well as two new EOS for them: cosmological EOS (CEOS) and dilaton EOS (DEOS). Special attention is paid to the dilatonic sphere (in brief – disphere) of SSSS, introduced in this paper for the first time. Using several realistic EOS for neutron star (NS): SLy, BSk19, BSk20 and BSk21, and current observational two-solar-masses-limit, we derive an estimate for scalar-field-mass \( m_\Phi \sim 10^{-13} \text{eV}/c^2 \div 4 \times 10^{-11} \text{eV}/c^2 \). Thus, the present version of the paper reflects some of the recent developments of the topic.

Keywords: Compact Stars; Neutron stars; Minimal Dilatonic Gravity; Dilatonic Sphere.

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1. Introduction

The MDG model was proposed and studied in. Some of its applications and properties can also be found in. It describes a proper generalization of the Einstein general relativity (GR), being based on the following action of the gravi-dilaton sector

\[
A_{g,\Phi} = \frac{c}{2\kappa} \int d^4x \sqrt{|g|} (\Phi R - 2\Lambda U(\Phi)).
\]
Without any relation with astrophysics and cosmology, MDG was studied for the first time by O’Hanlon, as early as in. His goal was to justify the Fujii idea about the "fifth force". There the term "dilaton" for the Branse-Dicke field $\Phi$ was introduced.

To some extent, MDG corresponds to the Branse-Dicke theory with the identically vanishing parameter $\omega$ and additional potential $U(\Phi)$. This property is also used as the name of the MDG model in some publications on the $f(R)$ theories of gravity, see[2-4] and a huge amount of references therein. Since the presence of the potential $U(\Phi)$ yields a radically different physics in comparison with the original Branse-Dicke theory, there is no physical reason to prescribe this name to MDG.

In general, the $f(R)$ theories are only locally equivalent to MDG, see[2] where the critically important, from a physical point of view, class of withholding potentials was introduced for the first time. In the large literature on the $f(R)$ theories one is not able to find examples of $f(R)$-models which own withholding property and are globally equivalent to MDG with the same physical properties.

In Eq. (1), $\kappa = 8\pi G_N/c^2$ is the Einstein constant, $G_N$ is the Newton gravitational constant, $\Lambda$ is the cosmological constant, and $\Phi \in (0, \infty)$ is the dilaton field. The values $\Phi$ must be positive since a change of the sign of $\Phi$ entails a change of the sign of the gravitational factor $G_N/\Phi$ and leads to antigravity instead of gravity. Such a change is physically unacceptable. Besides, the value $\Phi = 0$ must be excluded since it leads to an infinite gravitational factor and makes the Cauchy problem in MDG not well posed[10]. The value $\Phi = \infty$ turns off the gravity and is also physically unacceptable.

The Branse-Dicke field $\Phi$ is introduced to consider a variable gravitational factor $G(\Phi) = G_N/\Phi = G_N g(\Phi)$ instead of the gravitational constant $G_N$. The cosmological potential $U(\Phi)$ is introduced to consider a variable cosmological factor $\Lambda U(\Phi)$ instead of the cosmological constant $\Lambda$. In GR with cosmological constant $\Lambda$ we have $\Phi \equiv 1$, $g(\Phi) \equiv 1$, and $U(1) \equiv 1$. Due to its specific physical meaning, the field $\Phi$ has quite unusual properties.

The function $U(\Phi)$ defines the cosmological potential which must be a positive single valued function of the dilaton field $\Phi$ by astrophysical reasons[1]. See[2] for all physical requirements on the cosmological potential $U(\Phi)$, which are necessary for a sound MDG model. There the class of withholding potentials was introduced. These confine dynamically the values of the dilaton $\Phi$ in the physical domain. It is hard to formulate such a property for the function $f(R)$ in a simple intuitive way.

In[3][4], one can find a comparison of MDG with observations, observational restrictions on the mass of the dilaton $m_\Phi$, discussions of the MDG-cosmology, and some study of the structure of boson stars in MDG. An important physical comment on possible relation of MDG with quantum gravity may be found in[5][6].

The astrophysical observations show that in the observable Universe the cosmological term in Eq. (1) has a unique negative value, $\Lambda$ being a positive constant.
The f(R) gravity may have difficulties with the existence of stable star configurations and the presence of a different type of singularities inside the stars, see, for example,\textsuperscript{2,3,16–19} There exist a large number of articles on stars which combine different f(R) or other models of gravity with different EOS for star matter, see, for example,\textsuperscript{20–23,26} and the references therein.

On the other hand, by representing \( f(R) = R + \Delta f(R) \) with small perturbation \( \Delta f(R) \) one obtains models of SSSS which deviate not very much from the corresponding GR models.\textsuperscript{24,25}

The main goal of the present article is to construct an example of a physically consistent family of SSSS in MDG using the simplest withholding potential \( U(\Phi) \) and simplest matter EOS (MEOS).

We explicitly show how the dilatonic field \( \Phi \) changes the structure of the compact stars and creates a specific disphere around them.

2. Basic equations and boundary conditions for SSSS in MDG

In units \( G_N = c = 1 \) the field equations of MDG can be written in the form:

\[
\Phi \hat{R}_\alpha^\beta + \nabla_\alpha \nabla^\beta \Phi + 8\pi \hat{T}_\alpha^\beta = 0, \quad \Box \Phi + \Lambda V'(\Phi) = \frac{8\pi}{3} T.
\]  

(2)

Here \( \hat{T}_\alpha^\beta \) is the standard energy-momentum tensor of the matter, \( \hat{X}_\alpha^\beta = X_\alpha^\beta - \frac{1}{4} X \delta_\alpha^\beta \) is the traceless part of any tensor \( X_\alpha^\beta \) in four dimensions, \( X = X_\alpha^\alpha \) is its trace, the relation \( V'(\Phi) = \frac{2}{3} \left( \Phi U'(\Phi) - 2U(\Phi) \right) \) introduces the dilatonic potential \( V(\Phi) \), and the prime denotes differentiation with respect to the variable \( \Phi \). See for conventions and notation.\textsuperscript{21}

In the problems under consideration, the space-time-interval is \( ds^2 = e^\nu(r) dt^2 - e^\lambda(r) dr^2 - r^2 d\Omega^2 \) where \( r \) is the luminosity distance to the center of symmetry, and \( d\Omega^2 \) describes the space-interval on the unit sphere. Then, after some algebra one obtains the following basic results for a SSSS of the luminosity radius \( r^* \).

In the inner domain \( r \in [0, r^*] \) the SSSS structure is determined by the system:

\[
\frac{dm}{dr} = 4\pi r^2 \epsilon_{eff}/\Phi, \quad \frac{d\Phi}{dr} = -4\pi r^2 p_\Phi/\Delta, \quad \frac{dp_\Phi}{dr} = - \frac{p_\Phi}{r^3} \left( \frac{3}{2} \frac{p_\Phi}{3} + \epsilon_{eff}/\Phi \right) - \frac{2}{r^2} \epsilon_{eff}, \quad \frac{dp}{dr} = - \frac{p + \epsilon_m + 4\pi r^3 p_{eff}/\Phi}{\Delta - 2\pi r^3 p_\Phi/\Phi}.
\]  

(3) \( \text{written in dimensionless variables, see the Appendix A.} \)

The four unknown functions are \( m(r), \Phi(r), p_\Phi(r), \) and \( p(r) \). In Eqs. (3) \( \Delta = r - 2m - \frac{1}{3} \Lambda r^3, \) \( \epsilon_{eff} = \epsilon + \epsilon_\Lambda + \epsilon_\Phi, \) \( p_{eff} = p + p_\Lambda + p_\Phi. \) (7)
In addition, we obtain two novel EOS, which are specific for MDG:

\[ \epsilon_\Lambda = -p_\Lambda - \frac{\Lambda}{12\pi} \Phi; \quad (8) \]

\[ \epsilon_\phi = p - \frac{1}{3}\epsilon + \frac{\Lambda}{8\pi} V'(\Phi) + \frac{p_\phi}{2} \frac{m + 4\pi r^3 p_{\phi eff}/\Phi}{\Delta - 2\pi r^3 p_\phi/\Phi}; \quad (9) \]

\[ \epsilon = \epsilon(p). \quad (10) \]

Equation (8) is the CEOS for the cosmological energy density \( \epsilon_\Lambda \) and the cosmological pressure \( p_\Lambda \):

\[ \epsilon_\Lambda = \frac{\Lambda}{8\pi} \left( U(\Phi) - \Phi \right), \quad p_\Lambda = -\frac{\Lambda}{8\pi} \left( U(\Phi) - \frac{1}{3} \Phi \right). \quad (11) \]

These new quantities depend on the values of the dilaton \( \Phi \) and on the values of the cosmological potential \( U(\Phi) \).

Equation (9) follows from Eq. (4) and presents the DEOS for dilatonic energy density \( \epsilon_\phi \) and dilatonic pressure \( p_\phi \):

\[ \epsilon_\phi = \frac{1}{8\pi r^2} (\Delta/r)^{1/2} \frac{d}{dr} \left( \frac{r^2 (\Delta/r)^{1/2}}{d\Phi} \right) \quad \text{and} \quad p_\phi = -\frac{\Delta}{12\pi r^3} \frac{d\Phi}{dr}. \]

These new quantities depend on the gradient of gravitational factor with respect to the luminosity radius \( r \).

Using the area \( A = 4\pi r^2 \) of the surrounding sphere and the true geometrical distance \( dl \) defined by the representation of the four-interval in the form

\[ ds^2 = \alpha(l) dt^2 + dl^2 + A(l)/4\pi d\Omega^2 \Rightarrow dl = \sqrt{1 - \frac{2m}{r} - \frac{1}{3} \Lambda r^2}, \]

we obtain much more compact expressions:

\[ \epsilon_\phi = \frac{1}{8\pi} \frac{1}{A} \frac{dA}{dl} \left( A \frac{d\Phi}{dl} \right), \quad p_\phi = \frac{1}{8\pi} \frac{1}{A} \frac{dA}{dl} \frac{d\Phi}{dl}. \quad (12) \]

Equation (10) presents the usual MEOS of star matter, see, for example, for a modern detailed survey.

Adopting the widespread assumption that the SSSS-center \( C \) (\( \Rightarrow \) index "c") is at \( r_c = 0 \) (i.e., when \( A = 0 \)), we obtain the boundary conditions:

\[ m(0) = m_c = 0, \quad \Phi(0) = \Phi_c, \quad p(0) = p_c; \]

\[ p_\phi(0) = p_{\phi c} = 2 \left( \frac{\epsilon(p_c)}{3} - p_c \right) - \frac{\Lambda}{12\pi} V'(\Phi_c). \quad (13) \]

Requiring \( m_c = 0 \), we ensure finiteness of pressure \( p_c \) simultaneously for the Newton-, GR- and MDG-SSSS. The condition on \( p_{\phi c} \) \( (= -\frac{2}{3}\epsilon p_c) \) ensures its finiteness, being a specific MDG-centre-values-relation: \( F_\phi(p_{\phi c}, p_c, \Phi_c) = 0 \).

The SSSS-edge \( \Rightarrow \) index "\( * \)" is defined by the condition \( p^* = p(r^*; p_c, \Phi_c) = 0 \) (and typically \( \epsilon^* = 0 \)). Then:

\[ m^* = m(r^*; p_c, \Phi_c), \quad \Phi^* = \Phi(r^*; p_c, \Phi_c), \quad p^*_\phi = p_\phi(r^*; p_c, \Phi_c). \quad (14) \]

The luminosity radius of a compact star may vary to some extent for different physically sound MEOS. In general, the safe limits seem to be \( r^* \in [5, 20] \text{ km} \).

Outside the star \( p \equiv 0 \) and \( \epsilon \equiv 0 \), and we have a dilaton-sphere, in brief – a disphere. Its structure is determined by the shortened system (3)–(6): Eq. (6) has
to be omitted. For the exterior domain \( r \in [r^*, r_U] \) we use Eqs. (14) as left boundary conditions. The right boundary is defined by the cosmological horizon with unknown position \( r_U \), where the de Sitter vacuum is reached. Thus, we obtain a new nonlocal system of equations

\[
\Delta (r_U; p_c, \Phi_c) = 0, \quad \Phi (r_U; p_c, \Phi_c) = 1
\]

which relates the values of \( r \) and \( \Phi \) at different space-time points and is defined by the global structure of the space-time with a single compact SSSS in it.

The elimination of the unknown quantity \( r_U \) from Eqs. (15) leads to the second MDG-centre-values-relation:

\[
F_\Lambda (p_c, \Phi_c) = 0.
\]

In Fig. 1 (Left), we present for the first time a numerical solution of this novel nonlocal physical problem. This highly nontrivial solution strongly depends also on the MEOS and on the star interior structure (compare the result of the present paper with the results of the subsequent ones.\[27–29]\).

The two MEOS-dependent-relations

\[
F_\Phi (p_c, p_c, \Phi_c) = 0, \quad F_\Lambda (p_c, \Phi_c) = 0,
\]

show that in MDG, as well as in the Newton gravity and GR, we have a one-parameter-family of SSSS. This very important specific property of the MDG model of SSSS is not typical of other extended theories of gravity.

In dimension-full variables the observable value \( \Lambda \sim 10^{-44} \text{ km}^{-2} \) is very small. As a result, the luminosity radius of the Universe \( r_U \sim 1/\sqrt{\Lambda} \sim 10^{22} \text{ km} \) is very large in comparison with the typical luminosity radius of the compact star \( r^* \sim 10 \text{ km} \). This circumstance generates hard numerical problems and a need for special computational methods.

Further on, we use the cosmological potential \( U(\Phi) = \Phi^2 + \frac{3}{16p^2} (\Phi - 1/\Phi)^2 \). For useful comments and a more general form of the admissible cosmological potentials \( U(\Phi) \) see\[5,6\] The parameter \( p = \sqrt{\Lambda} \hbar/c m_\Phi \) is the dimensionless Compton length (measured in cosmological units) of the dilaton \( \Phi \). MDG is consistent with observation if \( p \lesssim 10^{-30} \) (i.e., if \( m_\Phi > 10^{-3} \text{ eV/c}^2 \))\[5,6\].

At present, the experimental determination of the mass of the dilaton is the most significant physical problem of MDG. For more complicated cosmological potentials it is possible to have several values of this mass which correspond to different local minima of the dilaton potential \( V(\Phi)\). If one considers as a dilaton the only currently known fundamental scalar particle, the Higgs boson with mass \( m_H \approx 125 \text{ GeV/c}^2 \), then \( p \approx 1.8 \times 10^{-43} \).

In our numerical calculations in Section 3, we use a maybe non-realistic large value \( p = 10^{-21} \) (the Compton length of the dilaton is \( \sim 9 \text{ km} \) being comparable with a typical value of the NS star luminosity radius \( r^* \)) just for a more transparent graphical representation of our qualitatively new results.
3. The results for the simplest EOS for neutron matter

We do not present here in detail a model of a neutron star with realistic MEOS. The most idealized relativistic MEOS of neutron matter is the well-known one used in the Tolman-Oppenheimer-Volkov (TOV) model\[^3\]. In dimensionless variables it reads

\[
p = \frac{1}{12\pi} (\sinh t - 8\sinh(t/2) + 3t), \quad \epsilon = \frac{1}{4\pi} (\sinh t - t). \tag{17}
\]

It describes the ideal Fermi neutron gas at zero temperature and is free from the difficulties related with unknown properties of neutron matter. The possible values of \(\epsilon > 0\) and \(p > 0\) are not a priori constrained from the above.

The analytic form of MEOS (17) facilitates our study of the new pure-MDG-effects in SSSS, shown in Figs. 1–6. For units see Appendix A. When applicable, we use different line styles for star’s interior and for star’s exterior, as well as for GR-results and for MDG-results.

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**Fig. 1.** Left: The specific MDG-curve \(F_{\Lambda}(p_{c}, \phi_{c}) = 0\). Right: The MDG-SSSS interior for MEOS (17): dimensionless energy density \(\epsilon(r)\) and pressure \(p(r)\).

**Fig. 2.** Left: The MDG-SSSS mass distribution \(m(r)\) in the star interior for MEOS (17). Right: The SSSS-disphere-mass-dependence on \(r\).
4. Comments of the numerical results

4.1. Comments of the numerical results for the MEOS

The interior structure of MDG-SSSS is shown in Fig. 2 (Left). It differs from the structure of GR-SSSS only quantitatively.
Figures 3 (Left) and 6 (Left) confirm our expectation that the $\Lambda$-term in action (1) yields variable positive $\epsilon_\Lambda$ and variable negative pressure $p_\Lambda$ inside, as well as outside MDG-SSSS, as it should be for description of dark energy effects. The changes of $\epsilon_\Lambda(r)$ and $p_\Lambda(r)$ are slightly asymmetric in accord with CEOS (8).

As seen in Fig. 4 (Right), MDG-SSSS with MEOS (17) are lighter and more compact than GR-SSSS.

Precisely as in GR, MDG-SSSS may be stable only until maximal mass is reached, see Figs. 4 (Left) and 4 (Right). As well as in GR, the full investigation of the stability problem of MDG-SSSS requires a study of their oscillations.

One of the most interesting novel physical results is shown in Fig. 2 (Right): The mass of the disphere $m_{\text{disp}}(r)$ outside MDG-SSSS as a function of the luminosity radius $r \in [r^*, r_U]$. As expected, it exponentially goes to a constant. The total mass of the object exceeds the mass of the very star. In our illustrative example the MDG-SSSS mass $m^* \approx 0.5073 M_\odot$ is close to the extremal one. We found the total mass of the disphere $m_{\text{disphere}} \approx 0.1638 M_\odot$, i.e. $\approx 32$ per cent of the mass of the star. In the case under consideration, the total mass of the object $m_{\text{total}} \approx 0.6710 M_\odot$ is reached with good precision at the distance of about several hundred of star radii from the center. The total mass is quite close to the mass of GR-SSSS.

4.2. Numerical results for realistic MEOS SLY, BSk19, BSk20 and BSk21 and restrictions on the mass of dilaton from two-solar-masses limit for NS

The comparison of the above results with the results of articles on MDG-NS with realistic MEOS (27-29) shows an interesting new phenomenon. In the last cases the maximal masses $m^*$, as well maximal total masses $m_{\text{tot}}$ are larger than the maximal masses of NS in GR with the same MEOS and depend on dilaton mass $m_\Phi$, see Fig. 7.

In contrast, the maximal GR-mass for MEOS (17) is larger than the maximal MDG-mass. A curious problem is to find a realistic MEOS (maybe a soft one) with the same maximal mass of NS in both GR and MDG.

Another interesting physical possibility is to use the MDG-model of NS to increase the maximal total mass of soft MEOS in a way that will make it possible
to reach the famous two-solar-masses limit for PSRs J1614-2230: $(1.97 \pm 0.04) M_\odot$ and J0348+0432 $(2.01 \pm 0.04) M_\odot$ [33,34]. Then, the rejected at present types of soft MEOS, including the MEOS for quark and hybrid stars [35] may turn out to be compatible with observations due to the use of MDG.

Considering in detail the right Fig. 7 which shows mass-radius relations of MDG-NS with realistic MEOS BSk19, SLy, BSk20 and BSk21 (see for more detail [22]), we conclude that the two-solar-masses limit allows MDG-NS with: i) MEOS BSk19 and $m_\Phi \sim 10^{-11} eV/c^2$; ii) MEOS SLy and $m_\Phi \sim 4 \times 10^{-13} eV/c^2$; and iii) MEOS BSk20 and $m_\Phi \in (10^{-11}, 4 \times 10^{-11}) eV/c^2$; but excludes MDG-NS with MEOS BSk21 under the hypothesis that NS of larger masses does not exist in Nature. In contrast, for GR-NS the two-solar-masses limit allows only NS with MEOS SLy, see our left Fig. 7 and [35,40].
The above novel estimate: \( m_\Phi \sim 4 \times 10^{-13} \div 10^{-11} eV/c^2 \) obtained from the two-solar-masses limit for NS with realistic MEOS, is about 8 \div 10 orders of magnitude smaller than the earlier estimate: \( m_\Phi \gtrsim 10^{-5} eV \), obtained from modern experiments on the Earth surface, see the references in 5 as well as the latest data in 41–43. In addition, in 15, 44, 45 it was shown that from inflationary cosmology one obtains huge mass of dilaton: \( m_\Phi \sim 1.5 \times 10^{-5} M_{pl} \approx 3.65 \times 10^{22}eV/c^2, \quad M_{pl} \approx 2.435 \times 10^{18}GeV/c^2 \) being the reduced Planck mass. The recent observational results of Planck mission 46 yield slightly lower \( m_\Phi \sim 1.3 \times 10^{-5} M_{pl} \approx 3.17 \times 10^{22}eV/c^2 \).

A natural way to avoid the obvious strong tension between these very different estimates in the framework of MDG seems to be the assumption that in Nature we have a withholding dilatonic potential \( V(\Phi) \) with several minima that show up at different scales as different values of the dilaton mass \( m_\Phi \).

5. Concluding remarks

The above consideration shows that the MDG enriches essentially the analysis of the realistic MEOS and the criteria for their acceptance or exclusion. Much more work has to be done in this direction.

It will be quite interesting to work out a model of moving and rotating stars in MDG. This is a much more complicated issue than the same problem in GR because of the nonlinear boundary condition with the moving boundary which must be consistent with the global structure of the MDG Universe. In this case, one can expect not only a nonspherically symmetric configuration but also the appearance of different centers of the star and its disphere, or even detachment of parts of the disphere. Such a nonstatic model may support strongly the possible interpretation of the dilaton field \( \Phi \) as dark matter. For similar effects, observed at scales of galactic clusters, see 47.

Another problem of current importance is to find solution of the problem of binary-NS in MDG and the corresponding radiation of gravitational waves during the binary merger. The presence of the disphere may also have an interesting additional contribution during such processes.

If the mass \( m_\Phi \) around NS is small enough to yield a disphere of a large effective radius \( \sim \hbar/(m_\Phi c) \), then the disphere around NS can be observed via the Shapiro delay by using the method of 34.

The basic property of the MDG dilaton \( \Phi \) is that it does not interact directly with matter 5–8, 11. Such interaction is possible only in quantum field theory as a perturbative effect of the second order with respect to the small gravitational constant \( G_N \). Hence, the corresponding cross-sections will be extremely small in accordance with the recent observational data, see, for example, 48 and the references therein.

All of the above results will remain valid also in the \( f(R) \) theories of gravity, which correspond to withholding MDG-potentials. 7 Unfortunately, one cannot find

\(^b\)The author is thankful to the unknown referee for this remark
such $f(R)$ theories of gravity in the large existing literature.

As a result of the above statements and the previous work on MDG, a novel basic physical conjecture arises: to look simultaneously for realistic MEO S and for withholding cosmological potentials which are able to describe a variety of cosmological, astrophysical, star, planet and laboratory phenomena at different scales.

Appendix A. Used units

In full physical dimensions the system (3)–(6) reads

$$\frac{dm}{dr} = 4\pi r^2 c^{-2} \epsilon_{eff}/\Phi, \quad (A.1)$$

$$\frac{d\Phi}{dr} = -\frac{G_N c^4}{4}\frac{p_\Phi}{\Delta}, \quad (A.2)$$

$$\frac{dp_\Phi}{dr} = \frac{p_\Phi}{\Delta} \left(3r - \frac{7G_N c^2}{r}m - \frac{2}{3}\Lambda r^3 + \frac{G_N c^2}{r^3}4\pi r^3 \epsilon_{eff}/\Phi\right) - \frac{2}{r} \epsilon_\Phi, \quad (A.3)$$

$$\frac{dp}{dr} = -\frac{G_N c^2}{r} p + \epsilon m + \frac{4\pi r^3 c^{-2} \epsilon_{eff}/\Phi}{\Delta} - \frac{G_N c^2}{2\pi r^3} p_\Phi/\Phi, \quad (A.4)$$

$$\Delta = r - \frac{G_N c^2}{r} \approx 5.35 \times 10^{-3} km^2. \quad (A.5)$$

The dimension-full form of (17) reads

$$p = \frac{1}{4} K \left( \sinh t - 8 \sinh(t/2) + 3t \right), \quad \epsilon = K \left( \sinh t - t \right), \quad (A.7)$$

with $K = m_n c^5/(32\pi^2 h^3)$, $m_n \approx 1.675 \times 10^{-24} g$ being the mass of the neutron.

We obtain the dimensionless form of Eqs. (3)–(6) and (17) making transition to dimensionless variables $r, m, \epsilon, \phi, p_\Phi, \epsilon_\Lambda, p_\Lambda, \Lambda$ and $K = 1/4\pi$ according to

$$r \rightarrow R_0 r, \quad m \rightarrow M_0 m, \quad \epsilon \rightarrow \epsilon_0 \epsilon, \quad p \rightarrow \epsilon_0 p, \quad \Lambda \rightarrow \Lambda_0 \Lambda \quad (A.8)$$

without changing the notation of variables. In Eqs. (A.8)

$$R_0 = \frac{G_N c^2}{m_n} \left( \frac{8\pi}{m_n} \right)^2 M_P^2 \approx 13.676 km, \quad M_0 = \left( \frac{8\pi}{m_n} \right)^2 M_P^2 \approx 9.264 M_\odot, \quad (A.9)$$

$$\epsilon_0 = m_n c^5/(8\pi h^3) \approx 6.47 \times 10^{36} g cm^{-1} s^{-2}, \quad \Lambda_0 = R_0^{-2} \approx 5.35 \times 10^{-3} km^{-2}.$$
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