Incompatibility of the tunneling limit with laser fields

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Abstract

The Schwinger limit refers to longitudinal electric fields that are sufficiently strong to “polarize the vacuum” into electron-positron pairs by a tunneling mechanism. Laser fields are transverse electromagnetic fields for which the Schwinger limit has no relevance. Longitudinal and transverse fields are fundamentally different because of the different values of the $F_{\mu\nu}F^{\mu\nu}$ Lorentz invariant that characterizes the fields. One aspect of this difference is the zero-frequency limit, that exists for longitudinal fields, but is ill-defined for transverse fields. The goal of approaching the Schwinger limit with sufficiently strong lasers is thus not a possibility. Tunneling transition rates are characterized by an exponential behavior of the form $\exp \left( -C/E \right)$, where $E$ is the magnitude of the applied electric field and $C$ is a system-dependent constant. Searches for such behavior within a Coulomb-gauge treatment of laser-induced processes are shown to fail.
Laser-induced ionization is widely regarded as a tunneling process. Nevertheless, the tunneling model applies only to a limited range of field parameters, and the use of this model outside that range leads to severe failures of qualitative and quantitative predictions. Among these failures is the prediction of a “tunneling limit” for laser-induced processes and the association of this limit with a simple behavior as the field frequency approaches zero. An example is the concept of a Schwinger limit for laser-induced electron-positron pair production in the vacuum. A single cause underlies these misconceptions: the tunneling model and the associated tunneling limit and Schwinger limit apply to longitudinal fields, whereas lasers produce the fundamentally different transverse fields.

The demonstration of the above-stated conclusions follows two tracks. In one, the theme is elaborated that the tunneling concept has application only to longitudinal fields, whereas laser fields are transverse. The other approach is to show that the characteristic exponential behavior for tunneling-type processes does not arise from the Coulomb-gauge or the velocity-gauge (VG) approaches.

A simple qualitative argument leads to the above conclusions. The transformation by an applied field of an impenetrable barrier to one that is penetrable requires the representation of the applied field by a scalar potential. In atomic ionization, the static Coulomb field responsible for the binding of an electron in an atom has superimposed upon it the electric field envisioned as an oscillating quasistatic electric (QSE) field. The fact that the combined Coulombic and QSE fields can be represented as a superposition of scalar fields is a prerequisite for graphic expression of the tunneling phenomenon. The pictorial view of pair production has the \(-mc^2\) upper boundary for negative energy electrons and the \(+mc^2\) lower boundary for positive energy electrons so tilted by a QSE field as to allow tunneling from the negative energy continuum to the positive energy continuum, which is a pair production mechanism. With the presumed “laser field” represented by a scalar potential \(\phi\), the implication is that there is no vector potential: \(\mathbf{A} = 0\); and thus there is no magnetic field: \(\mathbf{B} = 0\). The absence of a magnetic field means that the fundamental Lorentz invariant is positive: \(E^2 - B^2 > 0\) (Gaussian units). A positive value for this invariant is the hallmark of a QSE field; that is, it is a longitudinal field. This is in contrast to a transverse (plane-wave) field that is characterized by \(E^2 - B^2 = 0\). Strong fields accentuate the mismatch of the Lorentz invariant between longitudinal and transverse fields.

An appropriate value for the Lorentz invariant is related to the fact that when there are no
external sources or currents to sustain a field, as is the case with a laser beam, the Maxwell equations can be satisfied only by a transverse field. A transverse field is configured such that the electric field vector, the magnetic field vector, and the propagation vector form a mutually orthogonal triad. In contrast to this unique situation, a QSE field (and its zero-frequency limit – a constant electric field) requires external sources to generate it, and the sole inherent direction in the process is the direction of the electric field vector. For example, the field between the plates of a capacitor is a constant electric field if there are fixed, unequal charges on the plates, or it will be a QSE field if the capacitor is part of an AC circuit. In both cases, an external source is required to sustain the field.

An essential distinction between transverse and longitudinal fields is that a transverse field requires a magnetic component if it is to be a propagating field. There is no such thing as a propagating longitudinal field. It can oscillate with time, but it cannot propagate. (A seeming exception to this statement comes from the introduction by Keldysh of a putative Volkov solution in a problem described within the length gauge (LG), where only a scalar potential exists. However, the Volkov solution, and the gauge transformation employed to transform from the Coulomb gauge to the length gauge, are both dependent on a vector potential that does not exist in the length gauge. A proposed solution to this dilemma is to replace the vector potential by an integral of the electric field over all times prior to the laboratory time. This causes the Volkov solution in the length gauge to be a non-local quantity. This “Volkov solution” in a gauge that does not have a true Volkov solution is treated in the dipole approximation, where it does not possess the propagation property.)

A major distinction between longitudinal fields and transverse fields arises from contrasting properties as the field frequency $\omega$ approaches zero. When $\omega \rightarrow 0$, a longitudinal field approaches a constant electric field, where exact solutions are simple and well-known. This has led some authors to base the soundness of tunneling-type theories on a presumptive “rigorous limit” as $\omega \rightarrow 0$.

Transverse fields do not possess a clear limit as the frequency approaches zero. This is easily seen from the property, fundamental for propagating fields, that there is a ponderomotive energy $U_p$ associated with such fields that behaves as $U_p \sim \omega^{-2}$, making it energetically impossible to reach zero frequency. Furthermore, the dipole approximation is invalid when $U_p \gtrsim O(mc^2)$, so as $\omega \rightarrow 0$ this limit will be surpassed, the Göppert-Mayer (GM) gauge transformation is not applicable, and tunneling concepts lose all meaning for laser fields.
FIG. 1: The region to the left of the line labeled “$\beta_0 = 1$” is a region where magnetic forces become strong enough to render invalid the dipole approximation. To the left of the line $z_f = 1$, conditions are relativistic, where dipole-approximation results are entirely meaningless. These low-frequency limitations on the dipole approximation are frequently overlooked. The figure is from Ref. [5].

The above considerations establish that the tunneling model has limited applicability for the description of atomic ionization, and that the Schwinger limit applies only to longitudinal fields but not to laser fields.

Whenever the magnetic field component of a laser field is strong enough that $B$ is important, and particularly if $U_p \gtrsim O(mc^2)$, then the GM gauge transformation is not valid for a transverse field, and tunneling concepts do not apply to laser fields. These limitations are illustrated graphically in Fig. 1 reproduced from Ref. [5], that shows the onset of magnetic field effects as well as the onset of a fully relativistic regime. Within the parameter space displayed in Fig. 1 it is only the limited regime of frequencies and intensities shown by the shaded area (the “oasis”) where tunneling concepts have any connection with laser phenomena (although tunneling problems can arise even within the oasis).

Figure 1 illustrates that one must specify both the intensity $I$ and the frequency $\omega$ of a laser field in order to establish the physical conditions in which the ionization occurs. Alternatively, two dimensionless intensity parameters may be specified. The notion that this two-dimensional role can be played by a single parameter such as the Keldysh parameter $\gamma_K$ is a seriously flawed concept [5–8].
Transition rates for tunneling processes are characterized by the exponential behavior

$$\exp \left( -\frac{C}{E} \right),$$

where $C$ is a constant dependent on the properties of the system being irradiated, and $E$ is the magnitude of the electric field. Attempts to show the validity of tunneling behavior are often based on efforts to demonstrate that the transition probability exhibits the behavior shown in Eq. (1).

This goal is not to be confused with a quite different process. The process of strong-field photon-multiphoton pair production from the vacuum, which was first investigated in Ref. [9], leads to a particular limit of a form resembling tunneling behavior. The “Toll-Wheeler process”[9, 10] is one in which a constant background electric field produces pairs upon interaction with a perturbative QSE incoming field. The process is governed by a tunneling-like result proportional to

$$\exp \left( -\frac{4}{3\chi} \right), \quad \chi = \frac{\tilde{\omega} E}{m E_{crit}}, \quad E_{crit} = \frac{m^2}{e},$$

where natural units $\hbar = c = 1$ are used, and $E_{crit}$ is the Schwinger critical field. In the Toll-Wheeler process, the constant field replaces the laser field of the photon-multiphoton pair-production process of Ref. [9]. The tunneling exponential form of Eq. (2) is similar to Eq. (1), but $E$ is replaced by $\tilde{\omega}E$, where $\tilde{\omega}$ is the frequency of the incoming field and $E$ is the amplitude of the background field. The goal of the limiting process examined in Ref. [9] is to show that the zero frequency limit for pair production is the Toll-Wheeler result. From [2], this means that $\tilde{\omega}E \to 0$ has to be examined. In contrast, Eq. (1), the usual tunneling limit, would be associated with large values of the denominator quantity $E$. Not only are the limits of opposite type, but the Toll-Wheeler parameter is a product of the frequency of one field with the amplitude of the other. This is unrelated to a tunneling limit.

The 1964 Keldysh paper[3] on atomic ionization led to a focus on tunneling as the mechanism of strong-field ionization. Keldysh employed the LG in his work, where a QSE field is represented by scalar ($\phi$) and vector ($A$) potentials $\phi^{LG} = -r \cdot E(t)$, $A^{LG} = 0$. The GM gauge transformation is employed to relate these potentials to the VG potentials $\phi^{VG} = 0$, $A^{VG} = A^{VG}(t)$, where the VG is a Coulomb gauge in which the strong form of the dipole approximation

$$E = E(t), \quad B = 0$$

(3)
is employed. That is, the laser field is treated as if it were a QSE field. This is explicit in the work of Perelomov, Popov, and Terent’ev (PPT)\[11\]. Nikishov and Ritus\[12\] independently generated a tunneling method equivalent to that of PPT.

By the time the VG theory of atomic ionization was published\[13\], it was regarded as a necessity to relate it to tunneling results. The Coulomb gauge in dipole approximation is not the same thing as a QSE approach, which is the reason that the theory of Ref.\[13\] is not the same as a LG theory. This is the cause of the differences in the two sets of results, both of which are confusingly referred to as the Strong-Field Approximation (SFA). This has led to the unjustified assumption that one theory should be exactly gauge-equivalent to the other. Some of these differences were explored in Ref.\[14\]; they will be explicated further elsewhere. The GM gauge transformation connects a QSE theory to a theory resembling the Coulomb gauge, but where the strong-dipole-approximation conditions (3) are applied. These conditions are excessive, since they eliminate the propagation properties of transverse fields. It is possible to apply a more limited version of the dipole approximation that retains propagation properties and hence the correct Maxwell equations for plane-wave fields, but this is not gauge-equivalent to the LG.

The VG theory of Ref.\[13\] has its provenance in a relativistic formulation\[15, 16\], so its content is not the same as a LG theory. It will be shown here that the closest correspondence to a “tunneling limit” predicted by the VG SFA does not yield a tunneling result. This will be shown directly for circular polarization; for linear polarization, reference will be made to already-published results.

The VG SFA ionization probability theory presented in Ref.\[13\] will be restated here with three changes from the original: (1) atomic units (\(\hbar = m = |e| = 1\)) are used in place of “natural” units (\(\hbar = c = 1\)); (2) the symmetrical formulation of the transformation from configuration representation to momentum representation

\[
\phi_i(p) = \frac{1}{(2\pi)^{3/2}} \int d^3r \exp(-ip\cdot r) \phi_i(r)
\]

is used in place of the asymmetrical form

\[
\phi_i(p) = \int d^3r \exp(-ip\cdot r) \phi_i(r)
\]

that was employed in \[13\]; (3) angles are labeled by subscripts to indicate whether they are momentum-space (subscript \(p\)) or configuration-space (subscript \(r\)) angles.
For circular polarization, the differential transition probability for ionization is
\[
\frac{dW}{d\Omega_p} = 2\pi \sum_{n=n_0}^{\infty} p \left( \frac{p^2}{2} + E_B \right)^2 |\phi_i(p)|^2 |J_n(\varsigma_c)|^2,
\]
where \( \Omega \) is solid angle, \( E_B \) is the initial binding energy (or ionization potential), \( \phi_i(p) \) is the momentum-space wave function of the field-free initial state, \( \lceil \cdot \rceil \) denotes the ceiling function (the smallest integer containing the quantity within the brackets), \( U_p \) is the ponderomotive energy of a free electron in the field of frequency \( \omega \), \( \alpha_0 \) is the radius of the classical circular motion of an electron with ponderomotive potential \( U_p \) in the laser field of frequency \( \omega \), \( z \) is the intensity parameter
\[
z \equiv U_p/\omega.
\]
and the axis of spherical coordinates is in the direction of propagation of the laser field. A second, independent intensity parameter is
\[
z_1 \equiv 2U_p/E_B = 1/\gamma_K^2,
\]
where \( \gamma_K \) is the Keldysh parameter.

Tunneling behavior would be associated with a field so intense that \( z_1 \gg 1 \) (or \( \gamma_K \ll 1 \)). It also requires that an innumerably large number of photons are required to supply the ionization potential \( E_B \), or \( E_B \gg \omega \). These conditions,
\[
z_1 \gg 1, \quad E_B \gg \omega,
\]
can also be stated entirely in terms of the \( z \) and \( z_1 \) intensity parameters, as
\[
z_1 \gg 1, \quad 2z \gg z_1.
\]

The nature of Fig. [1] can be examined directly with respect to the conditions [8]. Since \( \omega \) is one of the coordinate axes, the second of the conditions [8] is just the vertical line at \( \omega = 0.5a.u. \), with ground-state hydrogen used as an example. The condition \( z_1 \gg 1 \) (or \( \gamma_K \ll 1 \)) is revealing when lines of constant \( z_1 \) (or constant \( \gamma_K \)) are plotted [8] as in Fig. [2]. There it is seen that one cannot employ very large values of \( z_1 \) (or small values of \( \gamma_K \)) without getting into a domain that is unquestionably relativistic, and hence where the dipole approximation is invalid. Actually, the domain marking the failure of the dipole approximation...
FIG. 2: This figure shows that the Keldysh parameter $\gamma_K$ cannot serve as a single scaling parameter since each line of constant $\gamma_K$ connects regions with completely different physical properties. It also shows that as $\gamma_K \to 0$, laser field conditions must become relativistic, where the dipole approximation is invalid and a tunneling theory cannot describe a laser field. The figure is adapted from Ref. [8].

approximation ends where it is no longer possible to ignore magnetic field effects, and it is the magnetic field that determines the low-frequency boundary of the oasis region in Fig. [1].

The literature on the tunneling limit largely ignores the low-frequency constraints revealed in Figs. [1] and [2] and examines only the effects of $\gamma_K \ll 1$, so that step will be taken here. It is a simple matter to just employ an asymptotic form of the Bessel functions in Eq. (4) when the order $n$ is very large. The appropriate asymptotic form depends on the relative magnitudes of the order $n$ and the argument $\zeta_c$. From the definition of $\zeta_c$ in Eq. (5) and the symmetry of (4), the angle $\theta_p$ can be confined to the first quadrant where $\zeta_c$ is always positive. Starting with the inequality $(n - 2z)^2 \geq 0$, it follows that

$$n^2 \geq 4nz - 4z^2 = 4z(n - z) > 0,$$

(10)

where $n - z > 0$ is a consequence of $n \geq n_0$, and $n_0$ given in (5) is rewritten as $n_0 = [z + (E_B/\omega)]$. When $n_0 \gg 1$, the ceiling function symbol in Eq. (5) can be ignored, so that (5) and (10) give

$$n \geq 2z^{1/2}(n - z)^{1/2}.$$

(11)
The argument of the Bessel function is
\[ \zeta_c = \alpha_0 p \sin \theta_p \leq \alpha_0 p = \left( \frac{2z}{\omega} \right)^{1/2} p. \] (12)

The kinetic energy available to a photoelectron is \( p^2/2 = (n - n_0) \omega \), and energy conservation gives \( p = (2\omega)^{1/2} (n - z - E_B)^{1/2} \). This gives an upper limit on the argument of the Bessel function as
\[ \zeta_c \leq \left( \frac{2z}{\omega} \right)^{1/2} (2\omega)^{1/2} (n - z - E_B)^{1/2} = 2z^{1/2} (n - z - E_B)^{1/2} < 2z^{1/2} (n - z)^{1/2}. \] (13)
Equations (11) and (13) establish that
\[ n > \zeta_c. \] (14)
Equation (14) identifies the appropriate asymptotic Bessel function to be
\[ J_n(x) = J_n \left( \frac{n}{\cosh \alpha} \right) \approx \frac{\exp(n \tanh \alpha - n\alpha)}{\sqrt{2\pi n \tanh \alpha}}, \quad \cosh \alpha \equiv \frac{n}{x}, \] (15)
where the parameter \( \alpha \) is such that
\[ \tanh \alpha = \frac{1}{n} \sqrt{n^2 - x^2}, \]
\[ \alpha = \ln \left( \frac{n}{x} + \frac{1}{x} \sqrt{n^2 - x^2} \right), \]
\[ \exp(-n\alpha) = \frac{x^n}{(n + \sqrt{n^2 - x^2})^n}. \]
The squared asymptotic Bessel function is
\[ [J_n(x)]^2 \approx \frac{1}{2\pi \sqrt{n^2 - x^2}} \frac{x^{2n} \exp(n^2 - x^2)}{(n + \sqrt{n^2 - x^2})^{2n}}. \] (16)
Field intensity dependence occurs within the parameter \( x^2 \) in Eq. (16), where, from Eq. (12), one has
\[ x^2 = \zeta_c^2 = \frac{2z}{\omega} p^2 \sin^2 \theta_p = 4z (n - z - E_B/\omega) \sin^2 \theta_p. \] (17)
The parameter \( z \) is proportional to \( U_p \), and hence to field intensity \( I \), where \( I \) follows from the square of the electric field strength.
The conclusion is that there nothing in the rate \( dW/d\Omega_p \) in Eq. (4) that can exhibit tunneling dependence as in Eq. (1). The VG theory of Ref. [13] is thus never of tunneling form for strong fields of circular polarization.
This is not a surprising result in view of the fact that a tunneling theory would predict that the photoelectron should emerge primarily in the direction of the electric field, which is radial for circular polarization. The observed direction as well as that predicted in the VG SFA, is azimuthal. That is, ionization caused by a circularly polarized laser field is not even approximately a tunneling process.

For linear polarization, the differential transition rate is of a form identical to Eq. (4) apart from a replacement of the ordinary Bessel function by the generalized Bessel function \[ J_n \left( \alpha_0 l \cos \theta_p, -\beta_0 c \right), \]

where \( \alpha_0 \) is the amplitude of motion of the electron in the direction parallel to the electric field when executing the well-known “figure-8” orbit, and \( \beta_0 \) is the amplitude of motion parallel to the propagation vector. These amplitudes are

\[
\alpha_0 = 2 \left( \frac{z}{\omega} \right)^{1/2}, \quad \beta_0 = \frac{z}{2c}.
\]

Using the tunneling conditions (8), it was shown in Ref. [19] that the generalized Bessel function could indeed be placed in the tunneling form, leading to an ionization rate proportional to

\[
\exp \left[ -\frac{2}{3} \left( \frac{2E_B}{E} \right)^{3/2} \right]. \tag{18}
\]

This exactly matches the form anticipated from Eq. (1).

Despite the above result, it was shown in [20] that only a portion of the overall ionization rate possessed the form (18). Depending on the particular ionization event considered, that part of the total rate showing tunneling behavior was generally only a small portion of the complete rate, and furthermore lacked many of the essential features of the total rate. An example of this behavior is shown in Fig. 3, duplicating Fig. 2 of Ref. [20], giving the momentum distribution of photoelectrons in the ionization of ground-state neon.

An earlier attempt to exhibit tunneling behavior in a VG theory was done in Section V of Ref. [13]. It was shown there that it is possible to expand a linear polarization ionization rate as a power series in the parameter \( \beta = \frac{(n - n_0)}{n_0} \). The zero-order term has the form of a tunneling exponential. However, as the field intensity increases, the value of \( \beta \) increases, so that the zero-order term is no longer representative of the actual rate.

The final conclusions reached here are twofold:
The so-called tunneling limit for laser-induced ionization has no relevance since such a limit does not exist for transverse electromagnetic fields, as shown graphically in Figs. 1 and 2. A tunneling limit does exist for longitudinal fields, but laser fields are transverse; fundamentally a different species of electromagnetic phenomena.

Attempts to exhibit the algebraic behavior of tunneling rates of the type of Eq. (1) from velocity-gauge or Coulomb-gauge calculations have led to outright failure (as for circular polarization) or to unrepresentative segments of the complete rate.

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