Gauge-transformation properties of cosmological observables and its application to the light-cone average

Jaiyul Yoo$^{a,b}$ and Ruth Durrer$^c$

$^a$Center for Theoretical Astrophysics and Cosmology, Institute for Computational Science, University of Zürich, Winterthurerstrasse 190, CH-8057, Zürich, Switzerland
$^b$Physics Institute, University of Zürich, Winterthurerstrasse 190, CH-8057, Zürich, Switzerland
$^c$Département de Physique Théorique & Center for Astroparticle Physics, Université de Genève, Quai E. Ansermet 24, CH-1211 Genève 4, Switzerland

E-mail: jyoo@physik.uzh.ch, ruth.durrer@unige.ch

Received May 18, 2017
Revised July 18, 2017
Accepted August 23, 2017
Published September 11, 2017

Abstract. Theoretical descriptions of observable quantities in cosmological perturbation theory should be independent of coordinate systems. This statement is often referred to as gauge-invariance of observable quantities, and the sanity of their theoretical description is verified by checking its gauge-invariance. We argue that cosmological observables are invariant scalars under diffeomorphisms and their theoretical description is gauge-invariant, only at linear order in perturbations. Beyond linear order, they are usually not gauge-invariant, and we provide the general law for the gauge-transformation that the perturbation part of an observable does obey. We apply this finding to derive the second-order expression for the observational light-cone average in cosmology and demonstrate that our expression is indeed invariant under diffeomorphisms.

Keywords: cosmological perturbation theory, gravitational lensing, supernova type Ia - standard candles

ArXiv ePrint: 1705.05839
1 Introduction

In the past few decades, cosmological observations firmly established that, on sufficiently large scales, our Universe is well approximated by a homogeneous and isotropic expanding Universe described by a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime with small perturbations. In particular, linear-order cosmological perturbation theory has been extensively used to model various cosmological observables beyond the background predictions such as galaxy clustering on large scales and anisotropies of the cosmic microwave background, which play an essential role in establishing the standard cosmological model. In the coming years, cosmology research is poised to take a further leap and enter into an era of truly precision cosmology with numerous large-scale surveys in the offing. Surveys such as the Dark Energy Spectroscopic Instrument, Euclid, WFIRST, and the Large Synoptic Survey Telescope will deliver unprecedented amounts of data with equally unprecedented precision. These will allow measurements of higher-order statistics, which require higher-order perturbation theory for their predictions, and which are expected to provide additional information about the nature of gravity and the initial conditions of the early Universe.

Together with this rapid development in observations and experiments, a large amount of work has been devoted to developing a solid theoretical framework for predictions of cosmological observables. In particular, several higher-order relativistic calculations of cosmological observables have been performed in recent years, see, e.g., [1–4] for the luminosity distance and [5–10] for galaxy clustering. As a consequence of diffeomorphism invariance of the underlying theory, these general relativistic calculations can be performed in any coordinate system at any order. One subtlety in cosmology exists — due to the symmetry in the FLRW metric, it is convenient to split physical quantities into a background contribution and a perturbation. This split is however to some degree arbitrary. Only the full, perturbed spacetime is

1http://desi.lbl.gov/
2http://sci.esa.int/euclid/
3https://wfirst.gsfc.nasa.gov/
4https://www.lsst.org/
physical and not the background neither its perturbation. A gauge transformation is actually a (small) change in this split. To first order, this is equivalent to a coordinate transformation \( x^a \mapsto x^a + \epsilon \xi^a \) under which any arbitrary variable \( O = \bar{O} + \epsilon \delta O \) changes by the Lie derivative,

\[
O \mapsto \bar{O} + \epsilon (\delta O - L_\xi \bar{O}).
\]

This is a first order gauge transformation \([11–13]\) and it can be easily generalized to higher orders, see \([14, 15]\) and section 2.2.

Given diffeomorphism invariance in general relativity, perturbations change (or gauge-transform), while cosmological observables such as galaxy clustering or the cosmic microwave background anisotropies are independent of our coordinate choice. This statement is often phrased as the gauge-invariance of the cosmological observables, and it might be expected to apply to perturbation calculations at all orders. The fact that, at first order, cosmological observables can be expressed in a gauge-invariant way puts tight constraints on their expressions in terms of perturbation variables, providing a useful way to verify the validity of complicated relativistic calculations. This point indeed has been adopted in the past to successfully develop the relativistic formalism for galaxy clustering and gravitational lensing (see, e.g., \([16–24]\)), in which the theoretical expressions for these cosmological observables are expressed in terms of general metric perturbations and are shown to remain invariant under gauge transformations. The caveat is that these gauge-invariant calculations in the past are performed only at the linear-order. As we shall argue here, beyond linear order gauge-invariance is no longer a useful criterion.

Calculations of cosmological observables are much more complicated beyond the linear order in perturbations, and the gauge-transformation properties of second-order (or higher-order) expressions are expected to play a critical role in both verifying the complicated calculations and reaching a consensus among different practitioners in the field. However, as we clarify in this paper, cosmological observables are diffeomorphism invariant scalars, not gauge-invariant. The statement of gauge-invariance of cosmological observables is incorrect beyond linear order and the theoretical expressions for cosmological observables do gauge transform, starting at second order in perturbations.

In section 2, we demonstrate that cosmological observables can be expressed as scalars under diffeomorphisms. Consequently, diffeomorphism invariance of cosmological observables is the condition to be used to test theoretical consistency rather than gauge-invariance. Starting from this condition, we show that the theoretical expressions are in fact gauge-invariant at linear order in perturbations, and we provide the transformation properties beyond linear order. With a few assumptions, the same conclusion was reached \([15]\) (see also \([14]\)), in which the gauge-transformation properties of cosmological observables are derived order by order in perturbations. We comment on this early work in section 4.

In this paper we mainly demonstrate a new proposal on how to check the highly involved second order calculations, where gauge-invariance does not hold, in the future. We apply it to some simple examples which are not new, they are rather an illustration of our method. As a proof of concept, we apply our findings to derive the second-order expressions of the observational light-cone average of the luminosity distance \(D_L\) in section 3. The light-cone average of the luminosity distance was considered previously in refs. \([25–27]\), where some relativistic corrections are neglected in these works and hence the diffeomorphism invariance of the light-cone average is broken. We show how diffeomorphism invariance can be used to derive the correct expression of the light-cone average at second order in perturbations. Our expression is formulated such that it can easily be generalized to any other observable on the light cone.
The luminosity distance $D_L$ is constructed by measuring the flux of standard candles like supernovae at the observed redshift $z$ and the angular direction $n$. These measurements are often averaged over directions $n$ at the same redshift $z$ to obtain the luminosity-redshift relation as

$$\langle D_L(z, n) \rangle_{\text{obs}} \equiv \frac{1}{N_g} \sum_{i=1}^{N_g} D_L(z, n_i),$$

(1.1)

where $N_g$ is the total number of host galaxies with the luminosity distance measurements in the redshift bin $[z, z + dz]$. Since this observational light-cone average is performed over the angle $n$, it is often referred to as the observational angular average, which is defined with the subscript $\Omega$ in our notation as

$$\langle D_L(z, n) \rangle_{\Omega} \equiv \frac{1}{\Omega} \int d^2 n \; D_L(z, n), \quad \Omega = \int d^2 n,$$

(1.2)

where $\Omega$ is the total angular area the observational angular average is performed over. However, it is well known [25, 26] that beyond linear perturbation theory the two averages (1.1) and (1.2) are related, but not equivalent even in the limit $N_g \to \infty$. Splitting each measurement of $D_L(z, n_i)$ into a background $\bar{D}_L(z)$ and a (dimensionless) perturbation $\delta D_L$ around it, we define the perturbation to the observational light-cone average as

$$\langle \delta D_L \rangle_{\text{obs}} = \langle D_L(z, n) \rangle_{\text{obs}} - \bar{D}_L(z) = \frac{1}{N_g} \sum_{i=1}^{N_g} \delta D_L(z, n_i).$$

(1.3)

The task of theorists is to relate this observational light-cone average to the angular average and eventually to an ensemble average, which can be computed within cosmological perturbation theory at the requested order. In section 3 we compute the observational light-cone average to the second order in perturbations and show that it satisfies the transformation property required to make the observational light-cone average invariant under a general coordinate transformation.

In section 4 we discuss the implications of our findings, and some detailed equations are presented in appendix A. We use $a, b, c \in \{0, 1, 2, 3\}$ to represent the spacetime indices and $i, j, k \in \{1, 2, 3\}$ to represent the spatial indices.

## 2 Observable quantities in cosmology and their gauge-transformation

We show that cosmological observables are expressed in terms of scalars, invariant under coordinate transformations. With this condition for diffeomorphism invariance, we demonstrate that perturbations of cosmological observables are gauge-invariant at linear order in perturbations but they do gauge-transform beyond the linear order.

### 2.1 Observable quantities as scalars under a diffeomorphism

The diffeomorphism symmetry in general relativity provides us with a freedom to choose a global coordinate system. Regardless of how we set up the coordinate system, observable quantities in cosmology should have identical values in any of those coordinates. This requirement implies that scalar $S$, vector $V^a$, or tensor $T_{ab}$ quantities describing cosmological observables transform as

$$\tilde{S}(\tilde{x}) = S(x), \quad \tilde{V}^a(\tilde{x}) = \frac{\partial \tilde{x}^a}{\partial x^b} V^b(x), \quad \tilde{T}_{ab}(\tilde{x}) = \frac{\partial x^c}{\partial \tilde{x}^a} \frac{\partial x^d}{\partial \tilde{x}^b} T_{cd}(x), \quad \cdots,$$

(2.1)
for a given coordinate transformation (or diffeomorphism)

\[ \tilde{x}^a = x^a + \xi^a, \quad \xi^a = (T, L^i), \]

(2.2)
describing the same physical (spacetime) point, while the coordinate values \( x^a \) and \( \tilde{x}^a \) are different in two coordinate systems.\(^5\) We suppress the coordinate indices when used as an argument of functions. However, these quantities describing cosmological observables are not directly observable — they are in fact measured by an observer in her rest frame, in which the four velocity \( u^a = [e_i]^a \) of the observer sets the time direction, a spatial triad \([e_i]^a\) orthogonal to the time direction defines the spatial directions, to that the local metric tensor is Minkowski, \( \eta_{IJ} = g_{ab}[e_I]^a[e_J]^b \) \((I, J = t, x, y, z)\).

As an example, let us consider observing photons emitted from a distant source and the light propagation described by a photon wave vector \( k^a \). This observation is performed in the observer’s rest frame, and the photon wave vector is described by two observable quantities, the photon frequency \( \nu \) \((\omega = 2\pi\nu)\) and the photon propagation direction \( n \) as

\[ k^I_o = \eta^{IJ}[e_J]^b k^b = (\omega, k) = \omega (1, n), \quad n = k/|k|, \quad k^b = \vartheta^b, \]

(2.3)
where we use the subscript \( o \) to represent quantities in the local rest frame of the observer and the photon wave vector is the gradient of the phase \( \vartheta \) of the electromagnetic wave describing our light ray [28]. The photon frequency measured by the observer is a scalar under diffeomorphisms,

\[ -\omega = u \cdot k = \eta_{IJ} u^I_o k^J_o = g_{ab} u^a k^b, \]

(2.4)
and the observed direction of the light propagation is also a scalar under diffeomorphisms

\[ n = \frac{1}{\omega} e_i k^i_o, \quad n^i = \frac{1}{\omega} [e_i]^a k^a, \]

(2.5)
where we use \( e_i \) to represent unit orthonormal vectors in the observer’s rest frame \((i = x, y, z)\). The observed redshift, as another example, is simply the ratio of the photon frequencies (or wavelengths) in the source rest frame and the observer rest frame:

\[ 1 + z = \frac{(u \cdot k)_s}{(u \cdot k)_o}, \]

(2.6)
and it is obviously a scalar under diffeomorphisms.

All vector and tensor quantities describing cosmological observables are measured in the observer’s rest frame with respect to local tetrads, and hence are indeed expressed in terms of scalars under diffeomorphisms. The observations are made in the unique rest frame of the observer (unique up to a global rotation of the triad \([e_i]\)) and they are not affected by diffeomorphisms. The rest-frame of the local observer is fully determined up to a simple spatial rotation (of the spatial triad). This residual symmetry is internal in the sense that it is independent of the FRW coordinate transformation. In this sense, observable quantities are diffeomorphism-invariant scalars.

\(^5\) We defined the coordinate transformation in a non-perturbative way: \( \xi^a = c_1 \xi^{a(1)} + c_2 \xi^{a(2)} + \cdots \). There exist other definitions for the transformation in literature using the Lie derivative.
2.2 Cosmological observables and gauge-invariance

In cosmology, a quantity $O$ including cosmological observables is split into the background $\bar{O}$ and the perturbation $\delta O$ around it, and due to the spatial symmetry in a homogeneous and isotropic universe, the background quantities depend only on time $t$:\(^6\)

$$O(x) \equiv \bar{O}(t)(1 + \delta O), \quad \delta O = \delta O(t, x^i) = \delta O^{(1)} + \delta O^{(2)} + \cdots . \quad (2.7)$$

General covariance of general relativity allows any coordinate system to be used to describe physical phenomena, and a coordinate transformation in eq. (2.2) necessarily involves changes in these quantities according to eq. (2.1). A gauge transformation, on the other hand, is a change in the correspondence of our physical universe to a homogeneous and isotropic background introduced to define the perturbations $\delta O$, while the background coordinate $x^a$ and hence the background quantities $\bar{O}(t)$ are fixed [12, 29].

Let us consider the transformation of the perturbation part of cosmological observables under the diffeomorphism given in eq. (2.2). Since the same physical point has two different values for its time coordinate, the correspondence to the background in a homogeneous universe differs in the two coordinates by

$$\tilde{\eta} = \eta + T, \quad \bar{O}(\tilde{\eta}) = \bar{O}(\eta) \left(1 + \frac{1}{2} \frac{d\bar{O}}{d\eta} T + \frac{1}{2} \frac{d^2\bar{O}}{d\eta^2} T^2 + \cdots \right), \quad (2.8)$$

where we use conformal coordinate time $\eta$ instead of the cosmic proper time coordinate $t$ and we expand $\bar{O}(\tilde{\eta})$ in a Taylor series. Observable quantities are invariant under diffeomorphisms, $\bar{O}(\tilde{x}) = O(x)$. This condition implies that the perturbation part $\delta O$ transforms as

$$\tilde{\delta O}(\tilde{x}) = [1 + \delta O(x)] \left(1 + \frac{1}{2} \frac{d\bar{O}}{d\eta} T + \frac{1}{2} \frac{d^2\bar{O}}{d\eta^2} T^2 + \cdots \right)^{-1} - 1. \quad (2.9)$$

It is noted that the above equations are valid at all orders, and the quantities like $T = T^{(1)} + T^{(2)} + \cdots$ are non-perturbative. At linear order in perturbations, this condition for $\delta O$ translates into

$$\tilde{\delta O}^{(1)}(\tilde{x}) = \tilde{\delta O}^{(1)}(x) = \delta O^{(1)}(x) - \frac{d\ln \bar{O}}{d\eta} T^{(1)}, \quad (2.10)$$

where we ignore the difference between $\tilde{x}$ and $x$ when computing the perturbation $\tilde{\delta O}^{(1)}$ at linear order. Evaluated at the same coordinate value in two different coordinates (hence the background correspondence is fixed), this relation is simply the gauge-transformation relation for $\delta O^{(1)}$.

While the relation (2.9) is general, in cosmology we often replace the background time coordinate by the observed redshift $z$, another invariant scalar under the diffeomorphism. Indeed, the observed redshift provides the simplest, physically meaningful way in cosmology to assign a ‘time coordinate’ to an event (provided it emits photons). Let us repeat the above for the perturbation of the redshift $\delta z$. First, we split the observed redshift into a background $\bar{z}$ and a perturbation $\delta z$ as

$$1 + z \equiv (1 + \bar{z})(1 + \delta z), \quad 1 + \bar{z}(\eta) \equiv \frac{a(\bar{\eta}_0)}{a(\eta)}, \quad \bar{\eta}_0 = \int_0^\infty \frac{dz}{H(z)}, \quad (2.11)$$

\(^6\)From now on, we absorb the small parameter $\epsilon$ in the corresponding variable and indicate the order simply by a superscript $(\epsilon)$. 

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where $\bar{\eta}_o$ is the conformal time today in a homogeneous universe, $H(z)$ is a Hubble parameter, and the distortion $\delta z$ encodes the perturbations due to the peculiar velocity and the potential fluctuation (see appendix A). We normalize the scale factor $a$ to unity at $\bar{\eta}_o$ so that the redshift parameter $\bar{z}$ is simply a representation of the time coordinate $\eta$ in a given coordinate system. Though apparent in eq. (2.6), we check whether the observed redshift is indeed invariant under diffeomorphisms. We consider the time coordinates $\eta$ and $\tilde{\eta} = \eta + T$. The background redshift parameters are related in the two coordinate systems by

$$1 + \tilde{\bar{\eta}} = \left( \frac{a(\eta)}{a(\tilde{\eta})} \right) \left( 1 + \frac{H T}{2} + \frac{a''}{a} T^2 + \cdots \right)^{-1} = \left( 1 + \frac{H T}{2} + \frac{a''}{a} T^2 + \cdots \right)^{-1},$$

(2.12)

where $H = a'/a$ is the conformal Hubble parameter. This relation dictates the transformation properties of the perturbation of the observed redshift

$$\tilde{\delta}z(\tilde{x}) = (1 + \delta z(x)) \left( 1 + \frac{H T}{2} + \frac{a''}{a} T^2 + \cdots \right)^{-1} - 1,$$

(2.13)

and at linear order in perturbations this general relation is indeed verified by explicit calculations of its gauge-transformation (see appendix A):

$$\tilde{\delta}z^{(1)}(\tilde{x}) = \delta z^{(1)}(x) + H T^{(1)}.$$

(2.14)

As a working example at second order, we consider the luminosity distance $D_L$. The background quantity $\bar{D}_L(z)$ is the luminosity distance in a homogeneous universe:

$$\bar{D}_L(z) = (1 + z)\bar{r}_z,$$

(2.15)

where $\bar{r}_z$ is the comoving distance to the redshift $z$ in a homogeneous universe. Note that in this case the redshift $z$ is a parameter of the given functions $\bar{r}$ and $H$. However, in real observations, the luminosity distance $D_L$ is constructed from observable quantities, including the observed redshift $z$ and the observed luminosity of standard candles. We split it into a background $\bar{D}_L$ and a perturbation $\delta D_L$ using the observed redshift $z$:

$$D_L(z, n) \equiv \bar{D}_L(z)(1 + \delta D_L).$$

(2.16)

The condition that the observed luminosity distance $D_L$ is an invariant scalar implies that the perturbation $\delta D_L$ should follow the relation

$$\delta \tilde{D}_L(\tilde{x}) = \delta D_L(x),$$

(2.17)

because the observed redshift $z$ is also invariant under diffeomorphisms, the correspondence to the background $\bar{D}_L(z)$ is identical in both coordinate systems. This relation for the perturbation implies that the perturbation $\delta D_L^{(1)}$ is gauge-invariant at linear order in the perturbations,

$$\delta \tilde{D}_L^{(1)}(\tilde{x}) = \delta D_L^{(1)}(x),$$

(2.18)

where we again ignore the difference between $\tilde{x}$ and $x$ since this would be second order. Compared to the general relation derived in eq. (2.10), this is a special case, which is however generally valid: the perturbation of an arbitrary (scalar) cosmological observable, expressed in terms of the observed redshift is gauge invariant at first order.
In summary, the diffeomorphism invariance of observable quantities implies the gauge-invariance of their perturbation at linear order. Beyond linear order, however, perturbations are not gauge-invariant (but still invariant scalars under diffeomorphisms). Given the condition in eq. (2.17), the second-order perturbation part gauge-transforms as

$$\tilde{\delta D}^{(2)}_L(x) = \delta D^{(2)}_L(x) - \xi^{(1)} \partial_\xi \delta D^{(1)}_L(x).$$

(2.19)

This relation replaces gauge-invariance for second order perturbations of any variable which is gauge invariant at first order. It is the ‘sanity check’ we can use to verify our second order calculations. Furthermore, it relates results obtained in different gauges.

2.3 The Stewart-Walker lemma

In this section we connect our discussion to the Stewart-Walker lemma [30], a well known result from linear cosmological perturbation theory: consider an observable $O$. Under a linearized coordinate transformation, $\tilde{x}^a = x^a + \xi^a$, it transforms into

$$\tilde{O}(x) = O(x) - \mathcal{L}_\xi O(x),$$

(2.20)

where $\mathcal{L}_\xi$ denotes the Lie derivative in direction $\xi$. Splitting $O$ into a background and a perturbation, $O = \bar{O} + \delta O$, the first order perturbation $\delta O^{(1)}$ transforms as

$$\tilde{\delta O}^{(1)}(x) = \delta O^{(1)}(x) - \mathcal{L}_{\xi^{(1)}} \bar{O}(x).$$

(2.21)

The first order gauge transformation is determined by the Lie derivative of the background variable in the direction of the (first order) displacement $\xi^{(1)}$. The perturbation variable $\delta O^{(1)}$ is invariant under all gauge transformations if and only if $\bar{O}(x) \equiv 0$ (or constant). This lemma can easily be generalized to higher order perturbations [14, 15]. At order $n$ the perturbation $\delta O^{(n)}(x)$ transforms under $\xi = \xi^{(1)} + \xi^{(2)} + \cdots + \xi^{(n)} + \cdots$ as

$$\tilde{\delta O}^{(n)}(x) = \delta O^{(n)}(x) - \mathcal{L}_{\xi^{(n)}} \bar{O}(x) - \mathcal{L}_{\xi^{(n-1)}} \delta O^{(1)}(x) - \cdots - \mathcal{L}_{\xi^{(1)}} \delta O^{(n-1)}(x) + 1 \frac{1}{2} \mathcal{L}_{\xi^{(1)}} \mathcal{L}_{\xi^{(n-2)}} \delta O^{(1)}(x) + \cdots +$$

$$+ \cdots + \frac{1}{n!} \left( -\mathcal{L}_{\xi^{(1)}} \right)^n \bar{O}. \quad (2.22)$$

For example at second order this gives

$$\tilde{\delta O}^{(2)}(x) = \delta O^{(2)}(x) - \mathcal{L}_{\xi^{(2)}} \bar{O}(x) - \mathcal{L}_{\xi^{(1)}} \delta O^{(1)}(x) + \frac{1}{2} \left( \mathcal{L}_{\xi^{(1)}} \right)^2 \bar{O}(x). \quad (2.23)$$

Hence a second order perturbation variable is gauge invariant only if its background and first order counter parts vanish (or are constant). For a variable which is gauge invariant at first order eq. (2.23) implies again (2.19).

The higher order Steward Walker lemma therefore states:

A perturbation variable is gauge invariant at order $n$ if and only if all its lower order perturbations and its background component vanish (or are constant). As an example let us again consider the redshift. This is clearly an observable and a diffeomorphism invariant scalar. But as is well known, we show this explicitly also in appendix A, it is not gauge-invariant, not even at first order as $\tilde{z} \neq 0$. The split of the observed redshift into a background component and a perturbation is arbitrary and not intrinsic.
The situation is somewhat different if we consider the function $D_L(z, n)$. Strictly speaking this quantity is a bi-scalar, depending on the coordinates $x_0$ of the observer and $x_s$ of the source. Once the observer position is fixed, the fact that the source must lie on the observer’s background light-cone, together with the observed redshift $z$ and the direction of observation $n$ fully determine the source position. Since $D_L$ does not depend on direction, the higher multipoles of an expansion of $D_L(z, n)$ into spherical harmonics are automatically gauge invariant at first order (their background contribution vanishes). But they are not gauge-invariant at second order.

The situation is more complicated for the monopole,

$$\langle \delta D_L(z, n) \rangle_\Omega = \frac{1}{4\pi} \int_{S^2} \delta D_L(z, n) d\Omega_n. \tag{2.24}$$

Without further care this first order perturbation variable is not gauge invariant as its background contribution does not vanish. However, it has been shown in ref. [31], that one can choose a physical time coordinate at the observer such that this distance becomes gauge-invariant.

With this remarks the main difference between gauge-invariance and general diffeomorphism invariance becomes clear. The former is related to our ability to split a variable into a background component and perturbations in a way which is independent of the coordinate system. While this is a very useful concept at the level of first order perturbation theory, it becomes cumbersome and not very helpful at higher orders. For an arbitrary quantity $O$, any spatial gradients

$$\nabla_i O \equiv \epsilon^i_{\phantom{i}a} \partial_a O, \tag{2.25}$$

are pure perturbations and therefore gauge invariant at first order, providing physical significance to gauge-invariance for linear perturbation theory. Most common gauge-invariant perturbation variables are actually of this form. For the sake of simplicity, we will restrict ourselves to the linear-order perturbations in constructing the gauge-invariant combinations in this section, and we will omit the superscript for the perturbation order. Well known examples are the different gauge invariant combinations for the density perturbation,

$$D_s = \delta + 3(1 + w)H\chi, \tag{2.26}$$
$$D_g = \delta + 3(1 + w)\phi, \tag{2.27}$$
$$D = \delta + 3(1 + w)H(v - \beta). \tag{2.28}$$

Here $\delta$ is the density fluctuation, $v$ is the peculiar velocity potential, $\chi$ and $\phi$ are metric perturbations. The detailed definitions of all variables are given in appendix A.

Denoting the projection operator into the 3-space normal to the cosmic velocity field by

$$P^b_a = \delta^b_a + u^a u^b, \tag{2.29}$$

a short computation gives [32]

$$D_{i} = P^a_i \rho_a \tag{2.30}$$
$$\left(D_s\right)_{ij} + 3(1 + w)\Psi_{ij} = P^a_j P^a_i \rho_{ab} \tag{2.31}$$
$$\left(D_g\right)_{ij} + 3(1 + w)(\Psi + \Phi)_{ij} = P^a_j P^a_i \rho_{ab}, \tag{2.32}$$

where $\Psi$ and $\Phi$ are the Bardeen potentials. Hence $D$ and $D_s + 3(1 + w)\Psi$ are potentials for the first and second gradient of the density projected onto the 3-space normal to $u^a$. 

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Similarly, the sum $\Phi + \Psi$ is the potential for the Weyl curvature and $\Phi - \Psi$ is a potential for the fluid anisotropic stress which both vanish in a FLRW spacetime.

As such, gauge invariance can play an important role for first order perturbation theory but much less so for second and higher order. For this reason, at higher orders we advocate to use diffeomorphism invariance which is of course maintained order by order.

3 Application to the light-cone average

As an application of our findings of the previous section, we derive the expression for the observed light-cone average and identify a few relativistic corrections that are absent in previous work. We show that our expression is also an invariant scalar. ‘Light cone average’ means the average over directions for objects on the light-cone at fixed observed redshift, i.e., it is an angular average over the two dimensional hypersurface of constant redshift on the observer’s background light-cone. This approximates the average done in actual observations. Here we relate it to the ensemble average because this is the quantity which we can calculate. As an example, we use again the luminosity distance but for any other quantity defined on the background light-cone the procedure is equivalent.

3.1 Neglecting volume and source fluctuations

While observers perform the light-cone average on the uniform surface of a sphere with radius $\tilde{r}_z$, the true radius depends on directions leading to an inhomogeneous surface, and this surface depends on the choice of coordinate systems (see appendix A). The observational angular average needs to be expressed in terms of an average of objects over this inhomogeneous surface.

In a given coordinate system, we parametrize the position of the objects in terms of their observed redshift $z$ as

$$x^a_z = [\eta_z + \Delta \eta, \tilde{x}^i_z + \Delta x^i], \quad 1 + z = a(\eta_o) / a(\eta_z), \quad \tilde{x}^i_z = \tilde{r}_z n^i. \quad (3.1)$$

Here $\eta_z$ is defined by the middle identity and $\tilde{r}_z$ is the comoving distance out to redshift $z$ in the background. The spatial displacement $\Delta x^i$ of the source position with respect to the position inferred based on the observed redshift $z$ and the observed direction $n$ are often expressed in spherical coordinates as $(\delta r, \delta \theta, \delta \phi)$. Since the background correspondence is fixed by using the observed redshift (invariant scalar), the displacements in two different coordinate systems are non-perturbatively related by

$$\tilde{\Delta} \eta(\tilde{x}) = \Delta \eta(x) + T(x), \quad \tilde{\Delta} x^i(\tilde{x}) = \Delta x^i(x) + L^i(x). \quad (3.2)$$

At linear order these relations are the gauge-transformation equations and indeed when the displacements are expressed in terms of metric perturbations, they do gauge-transform exactly as expected (see appendix A)

$$\tilde{\Delta} \eta^{(1)}(x) = \Delta \eta^{(1)}(x) + T^{(1)}(x), \quad \tilde{\Delta} x^{i(1)}(x) = \Delta x^{i(1)}(x) + L^{i(1)}(x). \quad (3.3)$$

Suppose we measure the luminosity distances from infinitely many supernovae in the sky and they are all at the same observed redshift. In a first step, we assume that these objects are uniformly distributed on observer directions $n$, such that our observational angular average
of the luminosity distance at the same observed redshift is equally weighted for each object (in reality, objects are weighted differently due to volume distortions and due to source fluctuations, which we discuss in section 3.2). In this case, we need to express the real source position in eq. (3.1) around the uniform sphere and average the luminosity distance over the uniform sphere, accounting for its distortion. This procedure amounts to averaging the luminosity distance over the inhomogeneous sphere. Up to the second order in perturbations, the luminosity distance is written as

\[ D_L(z, n; x^a_s) = \bar{D}_L(z) \left( 1 + \delta D_L^{(1,2)}(\bar{x}_z^a) + \Delta x^{(1)} \partial_a \delta D_L^{(1)}(\bar{x}_z^a) + O(3) \right), \]

and when averaged over the uniform sphere \( \bar{x}_z \) (or a spatial coordinate) we can relate it to the ensemble average in the given coordinate,

\[ \langle D_L(z, n) \rangle_\Omega = \bar{D}_L(z) \left( 1 + \langle \delta D_L^{(1)}(\bar{x}_z^a) \rangle + \langle \delta D_L^{(2)}(\bar{x}_z^a) \rangle + \langle \Delta x^{(1)} \partial_a \delta D_L^{(1)}(\bar{x}_z^a) \rangle + O(3) \right). \]

To ensure that this expression is correct, we check diffeomorphism invariance of the observational angular average to the second order in perturbations, as this observational procedure is again independent of the coordinates used to compute it. In the previous section we derived that the second-order perturbation \( \delta D_L^{(2)} \) of the luminosity distance transforms as

\[ x^a \rightarrow \tilde{x}^a = x^a + \xi^a, \quad \delta D_L^{(2)}(x) \rightarrow \tilde{\delta} D_L^{(2)}(x) = \delta D_L^{(2)}(x) - \xi^{(1)} \partial_a \delta D_L^{(1)}. \]

The correction terms in eq. (3.5) due to the deviation of the source position gauge-transform as

\[ \tilde{\delta} x^{(1)} \partial_a \tilde{\delta} D_L^{(1)} = (\Delta x^a + \xi^{(1)} \partial_a \delta D_L^{(1)}, \]

where \( \delta D_L \) is gauge-invariant at the linear order. Adding these two contributions, we readily derive that the sum of the two terms in our angular average is indeed invariant under diffeomorphisms.

### 3.2 Including volume and source fluctuations

As briefly discussed in the previous section, objects on an inhomogeneous sphere are in fact not equally weighted. In the observational light-cone average, equal weights are given to objects within the same background volume element \( d\bar{V} \), determined by the observed redshift \( z \) and the observed direction \( n = (\theta, \phi) \):

\[ d\bar{V}(z) \equiv \frac{\bar{r}^2(z)}{H(z)(1+z)^3} \, dz \, d\Omega, \quad d\Omega = \sin \theta \, d\theta d\phi. \]

In other words, objects within the same observed redshift bin \( dz \) are equally counted over the observed angle. Due to the inhomogeneities in our universe, the physical volume \( dV_{\text{phy}} \) that appears as the observed volume \( d\bar{V} \) is different at each point, and hence each point obtains different weights. This physical volume in the source rest frame is given by [33]

\[ dV_{\text{phy}} = \sqrt{-g} \varepsilon_{abcd} \partial_s \partial_x^a \partial_x^b \partial_x^c \partial_x^d dz \, d\theta \, d\phi \equiv (1 + \delta V) d\bar{V}(z), \]

where \( g \) is the determinant of the metric tensor in the source rest frame. The correction term \( \delta V \) accounts for the deviation of the physical volume from the homogeneous case.
where we define the volume distortion $\delta V$, $\varepsilon_{abcd}$ is the Levi-Civita symbol ($\varepsilon_{0123} = 1$), the metric determinant is $g$, and the source velocity is $u_x^a$. As is apparent from the definition, the volume distortion is another invariant scalar at all orders,

$$\tilde{\delta}V(x) = \delta V(x),$$

(3.10)

and it is gauge-invariant at linear order $\tilde{\delta}V^{(1)}(x) = \delta V^{(1)}(x)$. This covariant expression for the physical volume is essential in deriving the correct relativistic formula for galaxy clustering [16–19, 21]. The dominant contributions to the volume distortion $\delta V$ are redshift-space distortions and gravitational lensing, but there exist other relativistic effects [19]. Accounting for the different weight due to the volume distortion, the observational light-cone average can be expressed in terms of the observational angular average as

$$\langle \delta D_L \rangle_{\text{obs}(V)}(z) = \frac{1}{\int dV_{\text{phys}}} \int dV_{\text{phys}} \delta D_L(z, n) = \frac{\langle \delta D_L(z, n)(1 + \delta V) \rangle_\Omega}{(1 + \delta V)_\Omega},$$

(3.11)

where we use the fact that the observational average is performed over the same observed redshift bin $dz$. Note that we used the superscript $V$ to indicate that we account for the volume weight.

One last ingredient for the complete description of $\langle \delta D_L \rangle_{\text{obs}}$ is to consider the clustering of sources [34]. Apart from the volume distortion $\delta V$, fluctuations in the source number density also affect our observable. In our example, supernova hosts are remote galaxies, and these galaxies are clustered. Furthermore, observational selection such as the magnitude threshold can further bias the observed sample, and all these effects associated with the source galaxies are collectively called the source effect. Similar arguments for the source effect in light-cone averaging are discussed in [26, 35]. Together with the volume distortion, the observational light-cone average indeed gives equal weight to the observed counts $dN_{\text{obs}}^g$ of sources (or galaxies) in the observed volume $dV$. This number of observed galaxies is used to define the observed galaxy number density $n_{\text{g}}^{\text{obs}}$ and is related to the physical galaxy number density $n_{\text{g}}^{\text{phys}}$ and the volume distortion $\delta V$ as

$$dN_{\text{g}}^{\text{obs}}(z, n) = n_{\text{g}}^{\text{obs}}(z, n) d\tilde{V} = n_{\text{g}}^{\text{phys}} dV_{\text{phys}} = \tilde{n}_g(z)(1 + \delta_g)(1 + \delta V) d\tilde{V},$$

(3.12)

where we define the source fluctuation by splitting the physical galaxy number density $n_{\text{g}}^{\text{phys}}$ into the background $\bar{n}_g(z)$ at the observed redshift and the remaining fluctuation $\delta_g$ around it. For the same reason, the intrinsic source fluctuation is an invariant scalar at all orders,

$$\tilde{\delta}_g(x) = \delta_g(x),$$

(3.13)

and it is gauge-invariant at linear order in perturbations $\tilde{\delta}_g^{(1)}(x) = \delta_g^{(1)}(x)$. The intrinsic source fluctuation is the density fluctuation in comoving gauge $D^{(1)}$ in eq. (2.28) up to some bias factor and the perturbation $\delta z$ to compensate the difference between the observed redshift and the proper time of the source galaxies [18]. Therefore, the observational light-cone average $\langle \delta D_L \rangle_{\text{obs}}$ remains unaffected under a diffeomorphism when adding the additional corrections due to the volume distortion $\delta V$ and the source fluctuation $\delta_g$.

Including both, the volume and source fluctuations, the observational light-cone average is given by

$$\langle \delta D_L \rangle_{\text{obs}} = \frac{1}{\int dN_{\text{g}}^{\text{obs}}} \int dN_{\text{g}}^{\text{obs}} \delta D_L(z, n) = \frac{\langle \delta D_L(z, n)(1 + \delta_g)(1 + \delta V) \rangle_\Omega}{(1 + \delta_g)(1 + \delta V)_\Omega},$$

(3.14)
where the denominator in the first equality is just the total number of galaxies within the survey volume \((\Omega, dz)\). This expression is equivalent to that derived in [26], while only the dominant terms are considered in [35] where the relativistic contributions to \(\delta_g\) and \(\delta V\) are ignored. Finally, we relate the observed angular average to the ensemble average by perturbatively expanding the contributions. To second order in perturbations, the observational light-cone average is

\[
\langle \delta D_L \rangle_{\text{obs}} = \langle \delta D_L \rangle_{\Omega} + \langle \delta D_L (\delta_g + \delta V) \rangle_{\Omega} + \mathcal{O}(3) \tag{3.15}
\]

\[
= \langle \delta D_L^{(2)} (\bar{x}_e) \rangle + \langle \Delta x^{(1)} \partial_a \delta D_L^{(1)} (\bar{x}_e) \rangle + \langle \delta D_L^{(1)} (\delta_g^{(1)} + \delta V^{(1)}) \rangle + \mathcal{O}(3),
\]

where we use the relation of the observational angular average to the ensemble average in eq. (3.5).

Given the complete description of the observational light-cone average, we also clarify its relation to the directional average in [25], where the observational angular average is related to the ensemble average. In this discussion, volume distortion \(\delta V\) and source fluctuations \(\delta_g\) have been ignored, focusing on the angular average in eq. (3.5). The difference between the ensemble average and the angular average is exactly due to the deviation of the source position from the uniform sphere:

\[
\langle \delta D_L \rangle_{\Omega} - \langle \delta D_L \rangle = \langle \Delta x^{(1)} \partial_a \delta D_L^{(1)} (\bar{x}_e) \rangle + \mathcal{O}(2) \tag{3.16}
\]

\[
= \langle \Delta \eta^{(1)} \delta D_L^{(1)} \rangle + \langle \delta r^{(1)} \frac{\partial}{\partial r} \delta D_L^{(1)} \rangle + \langle \left( \theta^{(1)} \frac{\partial}{\partial \theta} + \phi^{(1)} \frac{\partial}{\partial \phi} \right) \delta D_L^{(1)} \rangle + \mathcal{O}(3).
\]

The angular distortions in the round bracket can be re-arranged as a total derivative and the divergence of the angular distortions as

\[
\left( \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi} \right) \delta D_L^{(1)} = \hat{\nabla} \cdot \left[ (\delta \theta, \delta \phi)^{(1)} \delta D_L^{(1)} \right] - \delta D_L^{(1)} \hat{\nabla} \cdot (\delta \theta, \delta \phi)^{(1)}. \tag{3.17}
\]

The integral over the total divergence vanishes and the second term is the gravitational lensing convergence

\[
-2\kappa \equiv \hat{\nabla} \cdot (\delta \theta, \delta \phi)^{(1)} = \left( \cot \theta + \frac{\partial}{\partial \theta} \right) \delta \theta^{(1)} + \frac{\partial}{\partial \phi} \delta \phi^{(1)}. \tag{3.18}
\]

Averaging over angles, we find that the last term in eq. (3.17) is indeed the expression found in ref. [25], in which only the dominant contributions to the average were considered. In addition to this lensing correction to the angular average, there exist the radial distortion \(\delta r\) and the displacement \(\Delta \eta\) in the time coordinate due to the mismatch between the background source position inferred from its redshift and the real source position. However, these contributions are in general smaller than the gravitational lensing convergence as noted in [25].

4 Discussion

Cosmological information is carried to us by electromagnetic waves (or gravitational waves) from distant sources, and this information is measured by an observer in her rest frame. The observer four velocity sets the time direction, and an orthonormal triad establishes the spatial directions in the observer’s rest frame. This frame is unique up to a rotation \(R \in \text{SO}(3)\) of the
This unique setting specifies the frame in which cosmological information is decoded and stored. For example, photon frequencies are measured in the observer’s rest frame, and photon propagation directions are measured against the spatial triad in the observer’s rest frame. Therefore, cosmological observables that encode physical properties of the sources, the geometry of spacetime and the nature of gravity are expressed in terms of scalars (under diffeomorphisms) defined along the world line of the observer.

In contrast to the observer’s rest frame, theoretical predictions for cosmological observables are described in an arbitrary coordinate system, and general covariance ensures that any choice of coordinates can be used to describe cosmological observables. Changes in the coordinates have no impact on the unique frame of the observer for cosmological observations. Therefore, cosmological observables have identical values, regardless of our choice of coordinates. At linear order, this redundancy is often phrased as gauge freedom and is removed by demanding that theoretical predictions for cosmological observables be gauge-invariant. However, this statement is valid only for linear perturbations, because the non-perturbative condition for cosmological observables is that they remain invariant under diffeomorphisms. Consequently, theoretical predictions beyond the linear order do gauge-transform in a unique way (i.e., they are in general not gauge-invariant), and their transformation property, given in eq. (2.19) for second order perturbations, can be used to check the validity of theoretical predictions beyond linear order in perturbations. Very few cosmological observables actually do have vanishing first order contribution and therefore are gauge invariant at second order. An example is the rotation angle in the lens map for the case of vanishing first order vector and tensor perturbations [36].

We have applied our findings to the observational light-cone average of the luminosity distance. As one of the cosmological observables, the observational light-cone average is an invariant scalar. In previous work [25–27] on the observational light-cone average, even though correctly accounting for the dominant contributions to the second order in perturbations, diffeomorphism-invariance is violated. Improving upon this work, we have provided a complete description of the observational light-cone average, including the relativistic corrections and checking the invariance of the observational light-cone average under diffeomorphisms in FLRW coordinates.

Similar considerations have been performed almost two decades ago [15] (see also [14]). Noting that while perturbations are in general gauge-dependent, the perturbation part of observable quantities such as the cosmic microwave background anisotropies is gauge-independent, they have adopted the view that “an observable quantity in general relativity is simply represented by a scalar field on spacetime,” and proceeded to find that once the background is gauge-independently defined, the perturbation part is gauge-invariant at first order, but transforms at higher orders in perturbations. In our language, they assumed that cosmological observables and the background quantities of such observables are invariant scalars under diffeomorphisms. Here we explicitly prove that cosmological observables measured by the observer in her rest frame are indeed scalars under diffeomorphisms when the background quantities are expressed in terms of the observed redshift. Given this identification, the work by [15] naturally leads to the conclusions of the present paper.

Acknowledgments

We acknowledge useful discussions with Camille Bonvin, Chris Clarkson, Pierre Fleury, Jinn-Ouk Gong, and Roy Maartens. We thank Ermis Mitsou for clarifying the global and the local
symmetries involved in the calculations and Marco Bruni for pointing us to his early work. We acknowledge support by the Swiss National Science Foundation. J.Y. is also supported by a Consolidator Grant of the European Research Council (ERC-2015-CoG grant 680886).

A Metric convention and gauge-transformation

Here we briefly summarize our metric conventions and present equations for the fluctuations of the observed redshift and the source position (see [18] for detailed derivations). The metric of an inhomogeneous universe close to a FLRW universe is modelled by four scalar perturbations ($\alpha, \beta, \varphi, \gamma$)

$$ds^2 = -a^2(1 + 2\alpha)d\eta^2 - 2a^2\beta_i d\eta dx^i + a^2[(1 + 2\varphi)\delta_{ij} + 2\gamma_{ij}] dx^i dx^j,$$

(A.1)

where $a(\eta)$ is the scale factor, $\delta_{ij}$ is the Kronecker delta, and we ignore vector and tensor perturbations as well as spatial curvature for simplicity. This metric representation is non-perturbative and fully general, e.g., $\alpha = \alpha^{(1)} + \alpha^{(2)} + \cdots$. However, in this appendix we will only deal with the linear perturbations. Hence we will omit the superscript in the appendix. Under the general coordinate transformation in eq. (2.2) these linear scalar perturbations transform as

$$\tilde{\alpha} = \alpha - 1 \frac{a}{a(\eta)} \left( aT \right)', \quad \tilde{\beta} = \beta - T + L', \quad \tilde{\varphi} = \varphi - \mathcal{H}T, \quad \tilde{\gamma} = \gamma - L,$$

(A.2)

where the prime is the derivative with respect to the conformal time $\eta$ and the scalar $L$ is defined by $L^i \equiv \delta^{ij} \partial_j L$. For a later convenience, we introduce

$$\chi \equiv a(\beta + \gamma'),$$

and the velocity potential $v$ such that the observer four velocity $u^a$ is given by

$$u_i \equiv -v_i.$$

These transform as

$$\tilde{\chi} = \chi - aT, \quad \tilde{v} = v - T.$$

(A.3)

Given our metric convention and the gauge transformation properties, we can derive the expressions for the linear fluctuations of the redshift and the coordinates $\Delta x^a$ introduced in eq. (3.1) by solving the photon geodesic equation. The perturbation of the observed redshift defined in eq. (2.11) is

$$\delta z = -H\chi + (H_o\chi_o + H_o\delta \eta_o) + [V - \alpha\chi]_s - [V - \alpha\chi]_o - \int_0^{\bar{r}_s} d\bar{r} (\alpha\chi - \varphi\chi)', \quad \tilde{\delta} z = \delta z + \mathcal{H}T,$$

(A.4)

where the subscripts $o$ and $s$ represent the observer and the source positions, $\delta \eta_o$ is the coordinate lapse of the observer time coordinate,

$$\delta \eta_o = -\int_0^{\bar{\eta}_o} d\eta a \alpha, \quad \tilde{\delta} \eta_o = \delta \eta_o + T_o,$$

and the equation is re-arranged to isolate the gauge-dependence by using the gauge-invariant quantities:

$$\alpha\chi \equiv \alpha - \frac{1}{a} \chi' \equiv \Psi, \quad \varphi\chi \equiv \varphi - H\chi \equiv -\Phi, \quad V \equiv -\frac{\partial}{\partial r} v\chi, \quad v\chi \equiv v - \frac{1}{a} \chi.$$

(A.6)
Here $\Psi$ and $\Phi$ are the Bardeen potentials. Compared to the inferred position of the source at the observed redshift, the fluctuation of conformal time $\Delta \eta$ is related to the redshift perturbation as

$$\Delta \eta = \frac{\delta z}{H}, \quad \widetilde{\Delta \eta} = \Delta \eta + T.$$  \hfill (A.7)

The fluctuation of the source position $(\delta r, \delta \theta, \delta \phi)$ in eq. (3.1) can be expressed as

$$\delta r = -\frac{\partial \gamma}{\partial \bar{r}} + n_i (\delta x^i + \gamma^i) + (\chi_o + \delta \eta_o) - \frac{1}{H} (\delta z + H\chi) + \int_0^{\bar{r}_z} d\bar{r} \left( \alpha \chi - \varphi \chi \right), \quad \bar{r}_z \delta \theta = -\frac{1}{\bar{r}_z} \frac{\partial \gamma}{\partial \theta} + (e_\theta i) (\delta x^i + \gamma^i) - \int_0^{\bar{r}_z} d\bar{r} \left( \frac{\bar{r}_z - \bar{r}}{\bar{r}} \right) \frac{\partial}{\partial \theta} (\alpha \chi - \varphi \chi), \quad \bar{r}_z \delta \phi = \bar{r}_z \delta \phi,$$

where $e_\theta$ is the unit vector in direction of $\theta$ and $\delta x^i_o$ is the spatial coordinate lapse of the observer,

$$\delta x^i_o = \int_0^{\bar{r}_o} d\eta a u^i_o, \quad \bar{r}_z \delta x^i_o = \delta x^i_o + L^i_o.$$ \hfill (A.10)

A similar equation is obtained for the azimuthal fluctuation $\bar{r}_z \sin \theta \delta \phi$, see e.g. [19]. As is easily checked, the terms in round brackets are gauge-invariant combinations, and consequently all the perturbations transform as expected,

$$\bar{r}_z \delta r = \delta r + \frac{\partial L}{\partial \bar{r}}, \quad \bar{r}_z \delta \theta = \bar{r}_z \delta \theta + \frac{1}{\bar{r}_z} \frac{\partial L}{\partial \theta}.$$ \hfill (A.11)

Finally, the volume distortion defined in (3.9) can be derived as

$$\delta V = 3 (\delta z + H\chi) + \alpha \chi + 2 \varphi \chi + \frac{2}{\bar{r}_z^2} \left( \delta z + \frac{\partial z}{\partial \bar{r}} \right) - 2 \left( \kappa - \frac{1}{2\bar{r}_z^2} \nabla^2 \gamma \right) - H \frac{\partial}{\partial \bar{z}} \left( \frac{\delta z + H\chi}{H} \right) + V,$$

which is indeed gauge-invariant. To see this one has to verify that the convergence $\kappa$ gauge transforms as

$$\tilde{\kappa} = \kappa - \frac{\nabla^2 L}{2\bar{r}_z^2}.$$

More details are found e.g. in [18, 19].

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