Two new classes of \(n\)-exangulated categories

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Abstract

Herschend-Liu-Nakaoka introduced the notion of \(n\)-exangulated categories. It is not only a higher dimensional analogue of extriangulated categories defined by Nakaoka-Palu, but also gives a simultaneous generalization of \(n\)-exact categories and \((n+2)\)-angulated categories. Let \(\mathcal{C}\) be an \(n\)-exangulated category and \(\mathcal{X}\) a full subcategory of \(\mathcal{C}\). If \(\mathcal{X}\) satisfies \(\mathcal{X} \subseteq \mathcal{P} \cap \mathcal{I}\), then the ideal quotient \(\mathcal{C}/\mathcal{X}\) is an \(n\)-exangulated category, where \(\mathcal{P}\) (resp. \(\mathcal{I}\)) is the full subcategory of projective (resp. injective) objects in \(\mathcal{C}\). In addition, we define the notion of \(n\)-proper class in \(\mathcal{C}\). If \(\xi\) is an \(n\)-proper class in \(\mathcal{C}\), then we prove that \(\mathcal{C}\) admits a new \(n\)-exangulated structure. These two ways give \(n\)-exangulated categories which are neither \(n\)-exact nor \((n+2)\)-angulated in general.

Key words: \(n\)-exangulated categories; \((n+2)\)-angulated categories; \(n\)-exact categories.

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1 Introduction

Higher-dimensional Auslander-Reiten theory was introduced by Iyama in [I], and it replaces short exact sequences as the basic building blocks for homological algebra, by the longer exact sequences. A typical setting is to consider \(n\)-cluster tilting subcategories of abelian categories (resp. exact categories), where \(n\) is a positive integer. All short exact sequences in such a subcategory are split, but it has nice exact sequences with \(n+2\) objects. This was recently formalized by Jasso [J] in the theory of \(n\)-abelian categories (resp. \(n\)-exact categories). There exists also a derived version of the theory focusing on \(n\)-cluster tilting subcategories of triangulated categories as introduced by Geiss, Keller and Oppermann in the theory of \((n+2)\)-angulated categories in [GKO]. Setting \(n = 1\) recovers the notions of abelian, exact and triangulated categories. We refer to [BJT, BT, L1, L2, LZ, ZW] for a more discussion on this matter.

The class of extriangulated categories, recently introduced in [NP], not only contains exact categories and extension-closed subcategories of triangulated categories as examples, but it is also closed under taking some ideal quotients. This will help to construct an extriangulated category which is not exact nor triangulated, see [NP, Proposition 3.30], [ZZ, Example 4.14]
and [HZZ, Remark 3.3]. The data of such a category is a triplet \((\mathcal{C}, E, s)\), where \(\mathcal{C}\) is an additive category, \(E: \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Ab}\) is an additive bifunctor and \(s\) assigns to each \(\delta \in E(C, A)\) a class of 3-term sequences with end terms \(A\) and \(C\) such that certain axioms hold. Recently, Herschend-Liu-Nakaoka [HLN] introduced an \(n\)-analogue of this notion called \(n\)-exangulated category. Such a category is a similar triplet \((\mathcal{C}, E, s)\), with the main distinction being that the 3-term sequences mentioned above are replaced by \((n+2)\)-term sequences. It should be noted that the case \(n = 1\) corresponds to extriangulated categories. As typical examples we have that \(n\)-exact and \((n+2)\)-angulated categories are \(n\)-exangulated, see [HLN, Propositions 4.5 and 4.34]. However, there are some other examples of \(n\)-exangulated categories which are neither \(n\)-exact nor \((n+2)\)-angulated, see [HLN, Section 6] and [LZ, Remark 4.5]. The main purpose of this paper is to construct more classes of \(n\)-exangulated categories which are neither \(n\)-exact nor \((n+2)\)-angulated.

We now outline the results of the paper. In Section 2, we review some elementary definitions and facts on \(n\)-exangulated categories.

In Section 3, we assume that \((\mathcal{C}, E, s)\) is an \(n\)-exangulated category with enough projectives and enough injectives, and \(\mathcal{P}\) (resp. \(\mathcal{I}\)) is the full subcategory of projective (resp. injective) objects in \((\mathcal{C}, E, s)\). If \(\mathcal{X}\) is a full subcategory of \(\mathcal{C}\) satisfying \(\mathcal{X} \subseteq \mathcal{P} \cap \mathcal{I}\), then we prove that the ideal quotient \(\mathcal{C} / \mathcal{X}\) is an \(n\)-exanglated category, which allows us to construct a new class of \(n\)-exangulated categories which are neither \(n\)-exact nor \((n+2)\)-angulated (see Theorem 3.1 and Example 3.4).

In Section 4, for a given \(n\)-exangulated category \((\mathcal{C}, E, s)\), we define a notion of an \(n\)-proper class of distinguished \(n\)-exangles, denoted by \(\xi\). If \((\mathcal{C}, E, s)\) is equipped with an \(n\)-proper class \(\xi\) of distinguished \(n\)-exangles, then \((\mathcal{C}, E, s)\) admits a new \(n\)-exangulated structure (see Theorem 4.5). It should be noted that the method here is different from the one used in [HZZ, Theorem 3.2] (see Remark 4.6). This construction gives another new class of \(n\)-exangulated categories which are neither \(n\)-exact nor \((n+2)\)-angulated (see Proposition 4.8 and Example 4.9).

\section{Preliminaries}

Let us briefly recall some definitions and basic properties of \(n\)-exangulated categories from [HLN]. Throughout this article, let \(\mathcal{C}\) be an additive category and \(n\) a positive integer.

\begin{definition}[HLN, Definition 2.1] Suppose that \(\mathcal{C}\) is equipped with an additive bifunctor \(E: \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Ab}\), where \(\text{Ab}\) is the category of abelian groups. For any pair of objects \(A, C \in \mathcal{C}\), an element \(\delta \in E(C, A)\) is called an \(E\)-extension or simply an extension. We also write such \(\delta\) as \(A\delta_C\) when we indicate \(A\) and \(C\).

Let \(A\delta_C\) be any extension. Since \(E\) is a bifunctor, for any \(a \in \mathcal{C}(A, A')\) and \(c \in \mathcal{C}(C', C)\), we have extensions

\[
E(C, a)(\delta) \in E(C, A') \quad \text{and} \quad E(c, A)(\delta) \in E(C', A).
\]

We simply denote them by \(a_*\delta\) and \(c^*\delta\). In this terminology, we have

\[
E(c, a)(\delta) = c^*a_*\delta = a_*c^*\delta \in E(C', A').
\]

\end{definition}
We write a morphism $f$.

**Definition 2.4.** [HLN, Definition 2.3] Let $\delta_C, \delta_C'$ be any pair of $E$-extensions. A morphism $(a, c): \delta \to \delta'$ of extensions is a pair of morphisms $a \in \mathcal{C}(A, A')$ and $c \in \mathcal{C}(C, C')$ in $\mathcal{C}$, satisfying the equality

$$a_*\delta = c^*\delta'.$$

**Definition 2.5.** [HLN, Definition 2.9] Let $\mathcal{C}_{\mathcal{E}}$ be the category of complexes in $\mathcal{C}$. As its full subcategory, define $\mathcal{C}_{\mathcal{E}}^{n+2}$ to be the category of complexes in $\mathcal{C}$ whose components are zero in the degrees outside of $\{0, 1, \ldots, n + 1\}$. Namely, an object in $\mathcal{C}_{\mathcal{E}}^{n+2}$ is a complex $X' = \{X^n, d^n_X\}$ of the form

$$X^0 \xrightarrow{d_0^X} X^1 \xrightarrow{d^1_X} \cdots \xrightarrow{d^{n-1}_X} X^n \xrightarrow{d^n_X} X^{n+1}.$$

We write a morphism $f': X' \to Y'$ simply $f' = (f^0, f^1, \ldots, f^{n+1})$, only indicating the terms of degrees $0, \ldots, n + 1$.

**Definition 2.6.** [HLN, Definition 2.11] By Yoneda lemma, any extension $\delta \in \mathcal{E}(C, A)$ induces natural transformations

$$\delta_X: \mathcal{C}(-, C) \Rightarrow \mathcal{E}(-, A) \quad \text{and} \quad \delta^A: \mathcal{C}(A, -) \Rightarrow \mathcal{E}(C, -).$$

For any $X \in \mathcal{C}$, these $\delta_X$ and $\delta^A_X$ are given as follows.

1. $\delta_X(X) : \mathcal{C}(X, C) \to \mathcal{E}(X, A) ; f \mapsto f^*\delta$.
2. $\delta^A_X : \mathcal{C}(A, X) \to \mathcal{E}(C, X) ; g \mapsto g_*\delta$.

We abbreviately denote $\delta_X(f)$ and $\delta^A_X(g)$ by $\delta_X(f)$ and $\delta^A(g)$, respectively.

**Definition 2.7.** [HLN, Definition 2.9] Let $\mathcal{C}, \mathcal{E}, n$ be as before. Define a category $\mathcal{E}^{n+2} := \mathcal{E}^{n+2}_{(\mathcal{C}, \mathcal{E})}$ as follows.

1. A pair $\langle X', \delta \rangle$ is an object of the category $\mathcal{E}$ with $X' \in \mathcal{C}_{\mathcal{E}}^{n+2}$ and $\delta \in \mathcal{E}(X^{n+1}, X^0)$, called an $E$-attached complex of length $n + 2$, if it satisfies

$$(d^n_X)_*\delta = 0 \quad \text{and} \quad (d^n_X)^*\delta = 0.$$

We also denote it by

$$X^0 \xrightarrow{d^n_0} X^1 \xrightarrow{d^n_1} \cdots \xrightarrow{d^n_{n-1}} X^n \xrightarrow{d^n_n} X^{n+1} \xrightarrow{\delta}.$$

2. For such pairs $\langle X', \delta \rangle$ and $\langle Y', \rho \rangle$, a morphism $f' : \langle X', \delta \rangle \to \langle Y', \rho \rangle$ in $\mathcal{E}$ is defined to be a morphism in $\mathcal{C}_{\mathcal{E}}^{n+2}$ satisfying $(f^0)_*\delta = (f^{n+1})^*\rho$.

**Definition 2.8.** [HLN, Definition 2.13] An $n$-exangle is a pair $\langle X', \delta \rangle$ of $X' \in \mathcal{C}_{\mathcal{E}}^{n+2}$ and $\delta \in \mathcal{E}(X^{n+1}, X^0)$ which satisfies the listed conditions.

1. The following sequence of functors $\mathcal{C}^{\text{op}} \to \text{Ab}$ is exact.

$$\mathcal{C}(-, X^0) \xrightarrow{\mathcal{E}(-, X^0)} \cdots \xrightarrow{\mathcal{E}(-, X^0)} \mathcal{E}(-, X^{n+1}) \xrightarrow{\delta} \mathcal{E}(-, X^0)$$
(2) The following sequence of functors \( \mathcal{E} \to \text{Ab} \) is exact.

\[
\mathcal{E}(X^{n+1}, -) \xrightarrow{\mathcal{E}(d^n_X, -)} \cdots \xrightarrow{\mathcal{E}(d^1_X, -)} \mathcal{E}(X^0, -) \xrightarrow{\delta^1} E(X^{n+1}, -)
\]

In particular any \( n \)-exangle is an object in \( \mathcal{A} \). A \emph{morphism of \( n \)-exangles} simply means a morphism in \( \mathcal{A} \). Thus \( n \)-exangles form a full subcategory of \( \mathcal{A} \).

**Definition 2.7.** [HLN, Definition 2.22] Let \( s \) be a correspondence which associates a homotopic equivalence class \( s(\delta) = [A X^*] \) to each extension \( \delta = A \delta_C \). Such \( s \) is called a \emph{realization} of \( E \) if it satisfies the following condition for any \( s(\delta) = [X^*] \) and any \( s(\rho) = [Y^*] \).

(R0) For any morphism of extensions \((a, c): \delta \to \rho\), there exists a morphism \( f' \in C^{n+2}_e(X^*, Y^*) \) of the form \( f' = (a, f^1, \ldots, f^n, c) \). Such \( f' \) is called a \emph{lift} of \((a, c)\).

In such a case, we simple say that \( "X^* \) realizes \( \delta^*" \) whenever they satisfy \( s(\delta) = [X^*] \).

Moreover, a realization \( s \) of \( E \) is said to be \emph{exact} if it satisfies the following conditions.

(R1) For any \( s(\delta) = [X^*] \), the pair \((X^*, \delta)\) is an \( n \)-exangle.

(R2) For any \( A \in \mathcal{E} \), the zero element \( A0_0 = 0 \in E(0, A) \) satisfies

\[
s(A0_0) = [A \xrightarrow{1_A} A \to 0 \to \cdots \to 0 \to 0].
\]

Dually, \( s(0_A) = [0 \to 0 \to \cdots \to 0 \to A \xrightarrow{1_A} A] \) holds for any \( A \in \mathcal{E} \).

Note that the above condition (R1) does not depend on representatives of the class \([X^*]\).

**Definition 2.8.** [HLN, Definition 2.23] Let \( s \) be an exact realization of \( E \).

(1) An \( n \)-exangle \((X^*, \delta)\) is called an \emph{s-distinguished \( n \)-exangle} if it satisfies \( s(\delta) = [X^*] \). We often simply say \emph{distinguished \( n \)-exangle} when \( s \) is clear from the context.

(2) An object \( X^* \in C^{n+2}_e \) is called an \emph{s-conflation} or simply a \emph{conflation} if it realizes some extension \( \delta \in E(X^{n+1}, X^0) \).

(3) A morphism \( f \) in \( \mathcal{E} \) is called an \emph{s-inflation} or simply an \emph{inflation} if it admits some conflation \( X^* \in C^{n+2}_e \) satisfying \( d^1_X = f \).

(4) A morphism \( g \) in \( \mathcal{E} \) is called an \emph{s-deflation} or simply a \emph{deflation} if it admits some conflation \( X^* \in C^{n+2}_e \) satisfying \( d^n_X = g \).

**Definition 2.9.** [HLN, Definition 2.27] For a morphism \( f' \in C^{n+2}_e(X^*, Y^*) \) satisfying \( f^0 = 1_A \) for some \( A = X^0 = Y^0 \), its \emph{mapping cone} \( M_f \in C^{n+2}_e \) is defined to be the complex

\[
X^1 \xrightarrow{d^0_{M_f}} X^2 \oplus Y^1 \xrightarrow{d^1_{M_f}} X^3 \oplus Y^2 \xrightarrow{d^2_{M_f}} \cdots \xrightarrow{d^{n-1}_{M_f}} X^{n+1} \oplus Y^n \xrightarrow{d^n_{M_f}} Y^{n+1}
\]

where \( d^i_{M_f} = \begin{bmatrix} -d_X^i & 0 \\ f^i & d_Y^i \end{bmatrix} \) (\( 1 \leq i \leq n-1 \)), \( d^n_{M_f} = \begin{bmatrix} f^{n+1} & d^n_Y \end{bmatrix} \).

The \emph{mapping cocone} is defined dually, for morphisms \( h' \) in \( C^{n+2}_e \) satisfying \( h^{n+1} = 1 \).
Definition 2.10. [HLN, Definition 2.32] An n-exangulated category is a triplet \((\mathcal{C}, \mathcal{E}, \mathcal{s})\) of additive category \(\mathcal{C}\), additive bifunctor \(\mathcal{E}: \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Ab}\), and its exact realization \(\mathcal{s}\), satisfying the following conditions.

\[(\text{EA1})\] Let \(\xymatrix{A \ar[r]^{f} & B \ar[r]^{g} & C}\) be any sequence of morphisms in \(\mathcal{C}\). If both \(f\) and \(g\) are inflations, then so is \(gf\). Dually, if \(f\) and \(g\) are deflations then so is \(gf\).

\[(\text{EA2})\] For \(\rho \in \mathcal{E}(D, A)\) and \(c \in \mathcal{C}(C, D)\), let \(A(X', c^*\rho)_C\) and \(A(Y', \rho)_D\) be distinguished \(n\)-exangles. Then \((1_A, c)\) has a good lift \(f^*\), in the sense that its mapping cone gives a distinguished \(n\)-exangle \((M^f_1, (d^f_X)_*)\).

\[(\text{EA2}^{\text{op}})\] Dual of (EA2).

Note that the case \(n = 1\), a triplet \((\mathcal{C}, \mathcal{E}, \mathcal{s})\) is a 1-exangulated category if and only if it is an extriangulated category, see [HLN, Proposition 4.3].

Example 2.11. From [HLN, Proposition 4.34] and [HLN, Proposition 4.5], we know that \(n\)-exact categories and \((n+2)\)-angulated categories are \(n\)-exangulated categories. There are some other examples of \(n\)-exangulated categories which are neither \(n\)-exact nor \((n+2)\)-angulated, see [HLN, Section 6] for more details.

Definition 2.12. [ZW, Definitions 3.14 and 3.15] Let \((\mathcal{C}, \mathcal{E}, \mathcal{s})\) be an \(n\)-exangulated category.

(1) An object \(P \in \mathcal{C}\) is called projective if, for any distinguished \(n\)-exangle

\[
\begin{array}{c}
A^0 \xrightarrow{\alpha_0} A^1 \xrightarrow{\alpha_1} A^2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-2}} A^{n-1} \xrightarrow{\alpha_{n-1}} A^n \xrightarrow{\alpha_n} A^{n+1} \xrightarrow{\delta} \\
\end{array}
\]

and any morphism \(c\) in \(\mathcal{C}(P, A^{n+1})\), there exists a morphism \(b \in \mathcal{C}(P, A^n)\) satisfying \(\alpha_n b = c\). We denote the full subcategory of projective objects in \(\mathcal{C}\) by \(\mathcal{P}\). Dually, the full subcategory of injective objects in \(\mathcal{C}\) is denoted by \(\mathcal{I}\).

(2) We say that \(\mathcal{C}\) has enough projectives if for any object \(C \in \mathcal{C}\), there exists a distinguished \(n\)-exangle

\[
\begin{array}{c}
B \xrightarrow{\alpha_0} P^1 \xrightarrow{\alpha_1} P^2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-2}} P^{n-1} \xrightarrow{\alpha_{n-1}} P^n \xrightarrow{\alpha_n} C \xrightarrow{\delta} \\
\end{array}
\]

satisfying \(P^1, P^2, \ldots, P^n \in \mathcal{P}\). We can define the notion of having enough injectives dually.

(3) \(\mathcal{C}\) is called Frobenius if \(\mathcal{C}\) has enough projectives and enough injectives and if moreover the projectives coincide with the injectives.

Remark 2.13.

(1) When \(n = 1\), they agree with the usual definitions [NP, Definitions 3.23, 3.25 and 7.1].

(2) If \((\mathcal{C}, \mathcal{E}, \mathcal{s})\) is an \(n\)-exact category, then they agree with [J, Definitions 3.11, 5.3 and 5.5].

(3) If \((\mathcal{C}, \mathcal{E}, \mathcal{s})\) is an \((n+2)\)-angulated category, then \(\mathcal{P} = \mathcal{I}\) consists of zero objects. Moreover it always has enough projectives and enough injectives.

Lemma 2.14. Let \((\mathcal{C}, \mathcal{E}, \mathcal{s})\) be an \(n\)-exangulated category. Then \(\mathcal{E}(\mathcal{C}, \mathcal{P}) = 0\) and \(\mathcal{E}(\mathcal{I}, \mathcal{C}) = 0\).

Proof. This follows from Lemma 3.4 and its dual in [LZ].
3 Ideal quotients of $n$-exangulated categories

Let $\mathcal{X}$ be a full subcategory of $\mathcal{C}$. For two objects $A, B$ in $\mathcal{X}$ denote by $\mathcal{X}(A, B)$ the subgroup of $\mathcal{C}(A, B)$ consisting of those morphisms which factor through an object in $\mathcal{X}$. Denote by $\mathcal{C}/\mathcal{X}$ the ideal quotient category of $\mathcal{C}$ modulo $\mathcal{X}$: the objects are the same as the ones in $\mathcal{C}$, for two objects $A$ and $B$ the Hom space is given by the quotient group $\mathcal{C}(A, B)/\mathcal{X}(A, B)$. Note that the ideal quotient category $\mathcal{C}/\mathcal{X}$ is an additive category. We denote by $\overline{f}$ the image of $f: A \to B$ of $\mathcal{C}$ in $\mathcal{C}/\mathcal{X}$.

The following construction gives $n$-exangulated categories which are not $n$-exact nor $(n+2)$-angulated in general.

**Theorem 3.1.** Let $(C, E, s)$ be an $n$-exangulated category with enough projectives and enough injectives and $\mathcal{X}$ a full subcategory of $\mathcal{C}$. If $\mathcal{X}$ satisfies $\mathcal{X} \subseteq \mathcal{P} \cap \mathcal{I}$, then the ideal quotient $\mathcal{C}/\mathcal{X}$ is an $n$-exangulated category.

**Proof.** Put $\overline{\mathcal{C}} = \mathcal{C}/\mathcal{X}$. By Lemma 2.14, we have $\mathbb{E}(\mathcal{C}, \mathcal{P}) = 0$ and $\mathbb{E}(\mathcal{I}, \mathcal{C}) = 0$. Thus one can define the additive bifunctor $\overline{\mathbb{E}}: \overline{\mathcal{C}}^\text{op} \times \overline{\mathcal{C}} \to \text{Ab}$ given by

- $\overline{\mathbb{E}}(C, A) = \mathbb{E}(C, A)$ for any $A, C \in \mathcal{C}$,
- $\overline{\mathbb{E}}(\overline{C}, \overline{A}) = \mathbb{E}(c, a)$ for any $a \in \mathcal{C}(A, A')$, $c \in \mathcal{C}(C, C')$, where $\overline{a}$ and $\overline{c}$ denote the images of $a$ and $c$ in $\mathcal{C}/\mathcal{X}$.

For any $\overline{\mathbb{E}}$-extension $\delta \in \overline{\mathbb{E}}(C, A) = E(C, A)$, define

$$\overline{s}(\delta) = s(\delta) = [A \xrightarrow{\alpha_0} B^1 \xrightarrow{\alpha_1} B^2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} B^n \xrightarrow{\overline{\alpha_{n+1}}} C]$$

using $s(\delta) = [A \xrightarrow{\alpha_0} B^1 \xrightarrow{\alpha_1} B^2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} B^n \xrightarrow{\alpha_{n+1}} C]$.

Now we prove that $\overline{s}$ is an exact realization of $\overline{\mathbb{E}}$.

Let $(\overline{\eta}, \overline{\tau}) : A\delta_C \to A'\delta_{C'}$ be any morphism of $\overline{\mathbb{E}}$-extensions. By definition, this is equivalent to that $(a, c) : A\delta_C \to A'\delta_{C'}$ is a morphism of $\mathbb{E}$-extensions. Put

$$\overline{\eta}(\delta) = [B'] = [A \xrightarrow{\overline{\alpha_0}} B^1 \xrightarrow{\overline{\alpha_1}} B^2 \xrightarrow{\overline{\alpha_2}} \cdots \xrightarrow{\overline{\alpha_n}} B^n \xrightarrow{\overline{\alpha_{n+1}}} C],$$

$$\overline{\tau}(\delta') = [D'] = [A' \xrightarrow{\overline{\beta_0}} D^1 \xrightarrow{\overline{\beta_1}} D^2 \xrightarrow{\overline{\beta_2}} \cdots \xrightarrow{\overline{\beta_n}} D^n \xrightarrow{\overline{\beta_{n+1}}} C'].$$

Since the condition in Definition 2.7 does not depend on the representatives of the homotopic equivalence class, we may assume

$$s(\delta) = [A \xrightarrow{\alpha_0} B^1 \xrightarrow{\alpha_1} B^2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} B^n \xrightarrow{\alpha_{n+1}} C],$$

$$s(\delta') = [A' \xrightarrow{\beta_0} D^1 \xrightarrow{\beta_1} D^2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_n} D^n \xrightarrow{\beta_{n+1}} C'].$$

Since $s$ is a realization of $\mathbb{E}$, there exists a morphism $f^* \in \mathcal{C}_e^{n+2}(B', D')$ of the form $f^* = (a, f^1, \ldots, f^n, c)$ such that $f^*$ a lift of $(a, c)$. Thus $\overline{f}$ a lift of $(\overline{\eta}, \overline{\tau})$. So (R0) is satisfied. The remaining conditions (R1) and (R2) are clearly satisfied. This shows that $\overline{s}$ is an exact realization of $\overline{\mathbb{E}}$. 
Let us confirm conditions (EA1) and (EA2). The remaining condition (EA2$^\text{op}$) can be shown dually.

(EA1) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be any sequence of morphisms in $\mathcal{C}$. Assume that $f$ and $g$ are inflations. Then there are two conflations $U^* \in \mathcal{C}^n$ and $V^* \in \mathcal{C}^n$ satisfying $d^0_U = f$ and $d^0_V = g$, respectively. Thus we assume

$$g(\delta) = [X \xrightarrow{f} Y \xrightarrow{d^1_U} U^2 \xrightarrow{d^2_U} \ldots \xrightarrow{d^n_U} U^n \xrightarrow{d^n_V} V^n],$$

$$g(\eta) = [Y \xrightarrow{g} Z \xrightarrow{d^1_V} V^2 \xrightarrow{d^2_V} \ldots \xrightarrow{d^n_V} V^n].$$

As in the proof of (R0), we may assume

$$g(\delta) = [X \xrightarrow{f} Y \xrightarrow{d^1_U} U^2 \xrightarrow{d^2_U} \ldots \xrightarrow{d^n_U} U^n],$$

$$g(\eta) = [Y \xrightarrow{g} Z \xrightarrow{d^1_V} V^2 \xrightarrow{d^2_V} \ldots \xrightarrow{d^n_V} V^n].$$

Then by (EA1) for $(\mathcal{C}, E, s)$, we know that $gf$ is an inflation. Thus the image of $g \circ f$ in $\mathcal{C}$ is also inflation. Dually, we can show that if $f$ and $g$ are deflations, then so is $g \circ f$.

(EA2) For any $\delta \in \mathcal{E}(D, A)$ and $\tau \in \mathcal{E}(C, D)$, let $A(X^*, c^*\delta)_C$ and $A(Y^*, \delta)_D$ be distinguished $n$-angles. As in the proof of (R0), we may assume

$$s(c^*\delta) = [A \xrightarrow{d^0_X} X^1 \xrightarrow{d^1_X} X^2 \xrightarrow{d^2_X} \ldots \xrightarrow{d^n_X} X^n \xrightarrow{d^n_Y} C],$$

$$s(\delta) = [A \xrightarrow{d^0_Y} Y^1 \xrightarrow{d^1_Y} Y^2 \xrightarrow{d^2_Y} \ldots \xrightarrow{d^n_Y} Y^n \xrightarrow{d^n_D} D].$$

Then by (EA2) for $(\mathcal{C}, E, s)$, $(1_A, c)$ has a good lift $f^*$, in the sense that its mapping cone gives a distinguished $n$-angle $(M^*_f, (d^0_X)_\delta)$. Thus the image of these conditions in $\mathcal{C}$ shows (EA2) for $(\mathcal{C}, \mathcal{E}, \mathcal{S})$. 

**Remark 3.2.** In Theorem 3.1, when $n = 1$, it is just the Proposition 3.30 in [NP].

As a direct consequence of Theorem 3.1, we have the following.

**Corollary 3.3.** Let $\mathcal{C}$ be a Frobenius $n$-exangulated category. Then the ideal quotient $\mathcal{C}/\mathcal{I}$ is an $(n + 2)$-angulated category.

**Proof.** This follows from Theorem 3.1 and [HLN, Proposition 4.8].

Now give an example to explain our main result in this section.

**Example 3.4.** Let $\Lambda$ be the path algebra of the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$$

with relation $\alpha\beta\gamma = 0$. Then $\text{mod}\Lambda$ has a unique 2-cluster tilting subcategory $\mathcal{C}$ consisting of all direct sums of projective and injective modules. By [J, Theorem 3.6], we know that $\mathcal{C}$ is a 2-abelian category which can be viewed as a 2-exangulated category. The Auslander-Reiten
That is $\mathcal{C} := \text{add}\{4, \frac{3}{4}, \frac{2}{3}, \frac{1}{2}, 1\}$. Take

$\mathcal{X} := \text{add}\{\frac{2}{3}\} \subseteq \text{add}\{\frac{2}{3}, \frac{1}{3}\} = \mathcal{P} \cap \mathcal{I}$.

By Theorem 3.1, we obtain that $\mathcal{C}/\mathcal{X}$ is a 2-exangulated category, but it is neither a 2-exact category nor a 4-angulated category. Since $4 \rightarrow \frac{2}{3} \rightarrow \frac{1}{3} \rightarrow 1$ is a 2-exact sequence in $\mathcal{C}$, but it induces the sequence

$4 \rightarrow 0 \rightarrow \frac{1}{3} \rightarrow 1$

is not a 2-exact sequence. So $\mathcal{C}/\mathcal{X}$ is not 2-exact. Since $\mathcal{C}/\mathcal{X}$ has non-zero projective and injective objects, hence it is not a 4-angulated category.

4 n-proper classes in n-exangulated categories

In this section, we introduce a notion of n-proper class in an n-exangulated category, and give a new class of an n-exangulated category. Unless otherwise specified, we assume that $(\mathcal{C}, E, s)$ is an n-exangulated category.

A morphism $f^* : \langle X^*, \delta \rangle \rightarrow \langle Y^*, \rho \rangle$ of distinguished n-exangles is called a weak isomorphism if $f^k$ and $f^{k+1}$ are isomorphisms for some $k \in \{0, 1, \ldots, n + 1\}$ with $n + 2 := 0$. The following result is essentially taken from [J, Proposition 2.7], where a variation of it appears. The proof given there carries over to the present situation.

**Proposition 4.1.** If $f^* : \langle X^*, \delta \rangle \rightarrow \langle Y^*, \rho \rangle$ is a weak isomorphism of distinguished n-exangles, then $f^* : X^* \rightarrow Y^*$ is a homotopy equivalence in $\mathcal{C}^{n+2}_{\mathcal{E}}$.

A class $\xi$ of distinguished n-exangles is called closed under base change if for any $\xi$-distinguished n-exangle $A\langle X^*, \delta \rangle_C$ and any morphism $c : C' \rightarrow C$, then any distinguished n-exangle $A\langle Y^*, c^*\delta \rangle_{C'}$ belongs to $\xi$. Dually, $\xi$ is called closed under cobase change if for any $\xi$-distinguished n-exangle $A\langle X^*, \delta \rangle_C$ and any morphism $a : A \rightarrow A'$, then any distinguished n-exangle $A\langle Y^*, a_*\delta \rangle_{C'}$ belongs to $\xi$.

A class $\xi$ of distinguished n-exangles is called saturated if for any commutative diagram in
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where \( a, b \) and \( d \) are deflations. If \( a \) and \( b \) are \( \xi \)-deflations, then so is \( d \).

An \( n \)-exangle \( A(X, \delta)_C \) is called split if \( \delta = 0 \). It follows from [HLN, Claim 2.15] that it is split if and only if \( d^n_X \) is a section or \( d^n_Y \) is a retraction. The full subcategory consisting of the split \( n \)-exangles will be denoted by \( \Delta_0 \).

**Definition 4.2.** A class \( \xi \) of distinguished \( n \)-exangles is called an \( n \)-proper class if the following conditions hold:

1. \( \Delta_0 \subseteq \xi \) and \( \xi \) is closed under finite coproducts and weak isomorphisms.
2. \( \xi \) is closed under base change and cobase change.
3. \( \xi \) is saturated.

The following lemma is very important, which will be used later.

**Lemma 4.3.** Let \( \xi \) be a class of distinguished \( n \)-exangles in \((\mathcal{C}, E, s)\) satisfying the conditions (1) and (2) in Definition 4.2. Then \( \xi \) is saturated if and only if for any commutative diagram in \( \mathcal{C} \)

\[
A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow[d]{} D,
\]

where \( a, b \) and \( c \) are inflations. If \( a \) and \( b \) are \( \xi \)-inflations, then so is \( c \).

**Proof.** We only prove the “only if” part and the “if” part is similar.

Let \( A(X, \delta)_M \) be a \( \xi \)-distinguished \( n \)-exangle with \( d_X^0 = a \), \( X^0 = A \), and \( X^1 = B \), and \( B(Y, \theta)_N \) a \( \xi \)-distinguished \( n \)-exangle with \( d_Y^0 = b \), \( Y^0 = B \) and \( Y^1 = D \). Since \( d_X^0 \) and \( d_Y^0 \) are inflations, their composition \( d_Y^0 d_X^0 \) becomes an inflation by (EA1) in Definition 2.10. Thus there is some distinguished \( n \)-exangle \( A(Z', \tau)_L \) which satisfies \( Z^1 = D \) and \( d_Z^1 = d_Y^0 d_X^0 \) as follows

\[
A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow[d]{} D.
\]

Applying [HLN, Proposition 3.5], we have the following commutative diagram

\[
A \xrightarrow{d_X^0} B \xrightarrow{d_Y^0} X^2 \xrightarrow{d_Y^2} \cdots \xrightarrow{d_Y^{n-1}} X^n \xrightarrow{d_Y^n} B \xrightarrow{f} M \xrightarrow{\delta} \tau.
\]

Thus we have a morphism \( f' : (X', \delta) \to (Z', \tau) \) which satisfies \( f^0 = 1_A \), \( f^1 = d_Y^0 \) and makes \( B(M'_f, (d_X^0)_* \tau)_L \) a distinguished \( n \)-exangle. Applying [HLN, Proposition 3.5] again, we have
the following commutative diagram

$$
\begin{array}{c}
B \xrightarrow{d^0_M} X^2 \oplus D \xrightarrow{[0, 1]} X^3 \oplus Z^2 \xrightarrow{\cdots} M \oplus Z^n \xrightarrow{d^n_M} L \xrightarrow{(d^0_X), \tau} \\
B \xrightarrow{d^0_M} D \xrightarrow{\bar{d}^1} Y^2 \xrightarrow{\cdots} \xrightarrow{Y^n} d^n_M \xrightarrow{\bar{\phi}^n+1} N \xrightarrow{\bar{\phi}^n} \\
\end{array}
$$

where $d^0_M = \left[ -d^1_X \atop d^0_Y \right]$, $d^n_M = [f_{n+1} \atop d^n_Z]$. Hence we have a morphism $g^* : \langle M'_f, (d^0_X), \tau \rangle \to \langle Y^*, \theta \rangle$ which satisfies $g^0 = 1_B, g^1 = [0 \atop 1]$ and makes $X^2 \oplus D(M'_f, [\bar{d}^0_X], \theta)_N$ a distinguished $n$-exangle. In particular, $(d^0_X), \tau = (g^{n+1})^* \theta$. Since $\langle Y^*, \theta \rangle$ is a $\xi$-distinguished $n$-exangle and $\xi$ is closed under base change, $\langle M'_f, (d^0_X), \tau \rangle$ is a $\xi$-distinguished $n$-exangle.

Note that $0 \to 0 \to \cdots \to 0 \to Z^n \xrightarrow{1_Z} Z^n \xrightarrow{0} 1$ is a split distinguished $n$-exangle. Then it is a distinguished $n$-exangle in $\xi$. Thus $X^0 \xrightarrow{d^0_X} X^1 \xrightarrow{d^1_X} X^2 \xrightarrow{d^2_X} \cdots \xrightarrow{d^{n-1}_X} X^n \oplus Z^n \xrightarrow{[d^n_X 0 \atop 1_Z]_n} M \oplus Z^n \xrightarrow{\delta} L$ is a distinguished $n$-exangle in $\xi$ since $\xi$ is closed under finite coproducts by hypothesis. Consider the following commutative diagram

$$
\begin{array}{c}
X^n \oplus Z^n \xrightarrow{[d^n_X 0 \atop 1_Z]} M \oplus Z^n \\
\xrightarrow{[f^n 1]_n} Z^n \xrightarrow{d^n_Z} L, \\
\end{array}
$$

where $d^n_Z = \left[ d^n_X 0 \atop 0 1_Z \right], \ [f^n 1 \atop d^n_Z]$ are deflations. Note that $[f^n 1 \atop d^n_Z]$ and $[d^n_X 0 \atop 1_Z]$ are $\xi$-deflations. Then $d^n_Z$ is a $\xi$-deflation because $\xi$ is saturated. Assume that $E(W^*, \rho)_L$ is a $\xi$-distinguished $n$-exangle with $d^n_W = d^n_Z$ and $W^n = Z^n$. By the dual of [HLN, Proposition 3.5], we have the following commutative diagram

$$
\begin{array}{c}
E \xrightarrow{d^0_W} W^1 \xrightarrow{d^1_W} W^2 \xrightarrow{\cdots} Z^n \xrightarrow{d^n_Z} L \xrightarrow{\rho} \\
\xrightarrow{\rho^0} \xrightarrow{\rho^1} \xrightarrow{\rho^2} \xrightarrow{\rho^n+1} \\
A \xrightarrow{d^0_Z} D \xrightarrow{d^1_Z} Z^2 \xrightarrow{\cdots} Z^n \xrightarrow{d^n_Z} L \xrightarrow{\tau} .
\end{array}
$$

Thus we have a morphism $h^* : \langle W^*, \rho \rangle \to \langle Z^*, \tau \rangle$ which satisfies $h^n = 1_{Z^n}, h^{n+1} = 1_L$ and $(h^0), \rho = \tau$, and hence $\langle Z^*, \tau \rangle$ is a $\xi$-distinguished $n$-exangle because $\xi$ is closed under cobase change. Since $c : A \to C$ is an inflation, there is a distinguished $n$-exangle $\sigma(V^*, \gamma)_K$ with $d^0_V = c$ and $V^1 = C$. By [HLN, Proposition 3.5], we have the following commutative diagram

$$
\begin{array}{c}
A \xrightarrow{c} C \xrightarrow{d^0_V} V^2 \xrightarrow{\cdots} V^n \xrightarrow{d^n_V} K \xrightarrow{\gamma} \\
\xrightarrow{\gamma^0} \xrightarrow{\gamma^1} \xrightarrow{\gamma^2} \xrightarrow{\gamma^n+1} \\
A \xrightarrow{d^0_Z} D \xrightarrow{d^1_Z} Z^2 \xrightarrow{\cdots} Z^n \xrightarrow{d^n_Z} L \xrightarrow{\tau} .
\end{array}
$$

Thus we have a morphism $s^* : \langle V^*, \gamma \rangle \to \langle Z^*, \tau \rangle$ which satisfies $s^0 = 1_A$ and $s^1 = d$ such that $\gamma = (s^{n+1})^* \tau$, and hence $\gamma$ is a $\xi$-distinguished $n$-exangle because $\xi$ is closed under base change. So $c$ is a $\xi$-inflation, as desired.
By the proof of Lemma 4.3, we have the following corollary.

**Corollary 4.4.** Let $\xi$ be an $n$-proper class in $(\mathcal{C}, E, s)$. Then the class of $\xi$-inflations (resp. $\xi$-deflations) is closed under compositions.

The following is our main result of this section.

**Theorem 4.5.** Let $\xi$ be a class of distinguished $n$-exangles in $(\mathcal{C}, E, s)$ which is closed under weak isomorphisms. Set $\mathcal{E}_\xi := E|_{\xi}$, that is,

$$\mathcal{E}_\xi(C, A) = \{ \delta \in E(C, A) \mid \delta \text{ is realized as a distinguished } n\text{-exangle } A\langle X^*, \delta \rangle_C \text{ in } \xi \}$$

for any $A, C \in \mathcal{C}$, and $s_\xi := s|_{\mathcal{E}_\xi}$. Then $\xi$ is an $n$-proper class of distinguished $n$-exangles if and only if $(\mathcal{C}, \mathcal{E}_\xi, s_\xi)$ is an $n$-exangulated category.

**Proof.** “$\Rightarrow$” It is easy to check that $(\mathcal{C}, \mathcal{E}_\xi, s_\xi)$ satisfies the condition of (EA1) by Corollary 4.4. For $\rho \in \mathcal{E}_\xi(D, A)$ and $c \in \mathcal{C}(C, D)$, let $A\langle X^*, c^*\rho \rangle_C$ and $A\langle Y^*, \rho \rangle_D$ be distinguished $n$-exangles in $(\mathcal{C}, \mathcal{E}_\xi, s_\xi)$. Then $A\langle X^*, c^*\rho \rangle_C$ and $A\langle Y^*, \rho \rangle_D$ are $\xi$-distinguished $n$-exangles in $(\mathcal{C}, E, s)$. Hence $(1, c)$ has a good lift $f^*$, in the sense that its mapping cone gives a distinguished $n$-exangle $(M_f', (d_{X'}^0)^*, \rho)$. Note that $\xi$ is closed under base change by hypothesis. So $(M_f', (d_{X'}^0)^*, \rho)$ is a $\xi$-distinguished $n$-exangle, which implies that (EA2) holds. The proof of (EA2$^\text{op}$) is similar.

“$\Leftarrow$” Note that $(\mathcal{C}, \mathcal{E}_\xi, s_\xi)$ is an $n$-exangulated category by hypothesis. It is easy to check that $\xi$ satisfies the conditions (1) and (2) in Definition 4.2. Next we claim that $\xi$ is saturated. Consider the following commutative diagram in $\mathcal{C}$

$$\begin{array}{ccc}
A & \xrightarrow{a} & B \\
\downarrow{c} & & \downarrow{b} \\
C & \xrightarrow{d} & D,
\end{array}$$

where $a, b$ and $c$ are inflations. Assume that $a$ and $b$ are $\xi$-inflations. Then $f = ba$ is a $\xi$-inflation. Thus there is a $\xi$-distinguished $n$-exangle $A\langle Z^*, \theta \rangle_M$ with $d^1_Z = f$. Note that $c$ is an inflation. Then there is a distinguished $n$-exangle $A\langle X^*, \rho \rangle_N$ with $d^0_X = c$. By [HLN, Proposition 3.5], we have the following commutative diagram

$$\begin{array}{ccccccc}
\begin{array}{cccccccc}
A & \xrightarrow{c} & C & \xrightarrow{d^1_C} & X^2 & \cdots & X^n & \xrightarrow{d^2_X} & N & \xrightarrow{\rho} & \\
\downarrow{f} & & \downarrow{d} & & \downarrow{g^2} & & \downarrow{g^n} & & \downarrow{g^{n+1}} & & \\
A & \xrightarrow{d^1_Z} & D & \xrightarrow{d^2_D} & Z^2 & \cdots & Z^n & \xrightarrow{d^2_M} & M & \xrightarrow{\theta} & .
\end{array}
\end{array}$$

Thus we have a morphism $g^* : (X^*, \rho) \to (Z^*, \theta)$ which satisfies $g_0 = 1_A$ and $g_1 = d$ and $(g^{n+1})^* \theta = \rho$. Since $\xi$ is closed under base change by the proof above, $(X^*, \rho)$ is a $\xi$-distinguished $n$-exangle. So $\xi$ is saturated by Lemma 4.3, as desired. □

**Remark 4.6.** (1) Assume that $\xi$ is a class of distinguished $n$-exangles in $(\mathcal{C}, E, s)$ which is closed under weak isomorphisms. By Theorem 4.5 and [HLN, Proposition 3.14], one can check...
that $\xi$ is an $n$-proper class if and only if $E_\xi$ is a closed additive subfunctor of $E$ defined by Herschend-Liu-Nakaoka in [HLN, Definition 3.9 and Lemma 3.13].

(2) In Theorem 4.5, when $n = 1$, it is just the Theorem 3.2 in [HZZ]. Note that, in [HZZ], one of the key arguments in the proof is that any extriangulated category has shifted octahedrons, while in our general context we do not have this fact and therefore must avoid this kind of arguments. So, the idea of proving Theorem 4.5 will be different from the one in [HZZ].

If we choose $(\mathcal{C}, E, s)$ to be an $n$-exact category or an $(n + 2)$-angulated category, then we have the following corollary which is a consequence of Theorem 4.5.

**Corollary 4.7.** The following are true for any $n$-exangulated category $(\mathcal{C}, E, s)$:

1. If $(\mathcal{C}, E, s)$ is an $n$-exact category and $\xi$ is a class of $n$-exact sequences which is closed under weak isomorphisms, then $\xi$ is an $n$-proper class if and only if $(\mathcal{C}, E_\xi, s_\xi)$ is an $n$-exact category.

2. If $(\mathcal{C}, E, s)$ is an $(n + 2)$-angulated category and $\xi$ is a class of $(n + 2)$-angles which is closed under weak isomorphisms, then $\xi$ is an $n$-proper class if and only if $(\mathcal{C}, E_\xi, s_\xi)$ is an $n$-exangulated category.

Let $\mathcal{C}$ be an additive category and $\mathcal{H}$ a subcategory of $\mathcal{C}$. Recall that a morphism $f : A \to B$ in $\mathcal{C}$ is called a left $\mathcal{H}$-approximation of $A$ if $B \in \mathcal{H}$ and

$$\mathcal{C}(f, H) : \mathcal{C}(B, H) \to \mathcal{C}(A, H)$$

is an epimorphism for any $H \in \mathcal{H}$. Moreover, if $(\mathcal{C}, \Sigma, \Theta)$ is an $(n + 2)$-angulated category, then we say that a subcategory $\mathcal{H}$ of $\mathcal{C}$ is called strongly covariantly finite if for any object $B \in \mathcal{C}$, there exists an $(n + 2)$-angle

$$B \xrightarrow{f} H^1 \to H^2 \to \cdots \to H^{n-1} \to H^n \to C \to \Sigma B$$

where $f$ is a left $\mathcal{H}$-approximation of $B$ and $H^1, H^2, \cdots, H^n \in \mathcal{H}$.

The following construction gives $n$-exangulated categories which are neither $n$-exact nor $(n + 2)$-angulated.

**Proposition 4.8.** Let $(\mathcal{C}, \Sigma, \Theta)$ be an $(n + 2)$-angulated category and $\mathcal{H}$ a full subcategory of $\mathcal{C}$. Denote by $\xi$ the class of $(n + 2)$-angles

$$X^0 \xrightarrow{d_0^X} X^1 \xrightarrow{d_1^X} X^2 \xrightarrow{d_2^X} \cdots \xrightarrow{d_{n-1}^X} X^n \xrightarrow{d_n^X} X^{n+1} \xrightarrow{\delta} \Sigma X^0$$

such that $\mathcal{C}(d_0^X, H) : \mathcal{C}(X^1, H) \to \mathcal{C}(X^0, H)$ is an epimorphism for any $H \in \mathcal{H}$. Then the following statements hold.

1. $\xi$ is an $n$-proper class in $(\mathcal{C}, \Sigma, \Theta)$ which induces an $n$-exangulated category $(\mathcal{C}, E_\xi, s_\xi)$.

2. If $(\mathcal{C}, E_\xi, s_\xi)$ is an $n$-exact category, then $\xi$ is the class of split $(n + 2)$-angles.

3. If $(\mathcal{C}, E_\xi, s_\xi)$ is an $(n + 2)$-angulated category, then $\Theta = \xi$. 
(4) Assume that $\mathcal{H}$ is strongly covariantly finite of $\mathcal{C}$ which is closed under direct summands. If $\{0\} \neq \mathcal{H} \subseteq \mathcal{C}$, then $(H, E_{\xi}, s_{\xi})$ is neither n-exact nor $(n+2)$-angulated.

Proof. (1) By Corollary 4.7, it suffices to show that $\xi$ is closed under base change and cobase change, and $\xi$ is saturated. Let $X^0 \xrightarrow{d^0_X} X^1 \xrightarrow{d^1_X} X^2 \xrightarrow{d^2_X} \cdots \xrightarrow{d^{n-1}_X} X^n \xrightarrow{d^n_X} X^{n+1} \xrightarrow{\delta} \Sigma X^0$ be an $(n+2)$-angle in $\xi$. For any morphism $q : M \to X^{n+1}$, we have the following commutative diagram

$$
\begin{array}{cccccccccccccccc}
X^0 & d^0_X & Y^1 & d^1_Y & Y^2 & \cdots & Y^n & d^n_Y & M & \delta & \Sigma X^0 \\
\| & & l & f^1 & l & f^2 & l & f^n & q & & \\
X^0 & d^0_X & X^1 & d^1_X & X^2 & \cdots & X^n & d^n_X & X^{n+1} & \delta & \Sigma X^0.
\end{array}
$$

Let $H$ be any object in $\mathcal{H}$. Then we have the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{C}(X^1, H) & \xrightarrow{\mathcal{C}(d^1_X, H)} & \mathcal{C}(X^0, H) \\
\| & & | \\
\mathcal{C}(Y^1, H) & \xrightarrow{\mathcal{C}(d^1_Y, H)} & \mathcal{C}(X^0, H).
\end{array}
$$

Note that $\mathcal{C}(d^1_X, H) : \mathcal{C}(X^1, H) \to \mathcal{C}(X^0, H)$ is an epimorphism. It follows that $\mathcal{C}(d^1_Y, H) : \mathcal{C}(Y^1, H) \to \mathcal{C}(X^0, H)$ is an epimorphism, which implies that $\xi$ is closed under base change. To prove that $\xi$ is closed under cobase change, we assume that $l : X^0 \to N$ is a morphism in $\mathcal{C}$. Then we have the following commutative diagram

$$
\begin{array}{cccccccccccccccc}
X^0 & d^0_X & X^1 & d^1_X & X^2 & \cdots & X^n & d^n_X & X^{n+1} & \delta & \Sigma X^0 \\
\| & & l & g^1 & l & g^2 & l & g^n & & \\
N & d^0_Z & Z^1 & d^1_Z & Z^2 & \cdots & Z^n & d^n_Z & X^{n+1} & (\Sigma) & (\Sigma).N.
\end{array}
$$

Let $H$ be any object in $\mathcal{H}$. Then we have the following commutative diagram with exact rows

$$
\begin{array}{cccccccccccccccc}
\mathcal{C}(Z^1, H) & \xrightarrow{\mathcal{C}(d^1_Z, H)} & \mathcal{C}(N, H) & \xrightarrow{\mathcal{C}((-1)^n(l\Sigma^{-1}\delta), H)} & \mathcal{C}(\Sigma^{-1}X^{n+1}, H) \\
\| & \| & | & | & \\
\mathcal{C}(X^1, H) & \xrightarrow{\mathcal{C}(d^1_X, H)} & \mathcal{C}(X^0, H) & \xrightarrow{\mathcal{C}((-1)^n\Sigma^{-1}\delta, H)} & \mathcal{C}(\Sigma^{-1}X^{n+1}, H).
\end{array}
$$

Since $\mathcal{C}(d^1_Z, H) : \mathcal{C}(X^1, H) \to \mathcal{C}(X^0, H)$ is an epimorphism, it follows that

$$
\mathcal{C}((-1)^n\Sigma^{-1}\delta, H) = 0.
$$

Thus $\mathcal{C}((-1)^n(l\Sigma^{-1}\delta), H) = 0$, and hence $\mathcal{C}(d^1_Z, H) : \mathcal{C}(Z^1, H) \to \mathcal{C}(N, H)$ is an epimorphism, as desired.

Finally, for any commutative diagram in $\mathcal{C}$

$$
\begin{array}{cccc}
A & & B \\
\| & c & & \| \\
C & & D, & b
\end{array}
$$
where $a, b$ and $c$ are inflations in $(\mathcal{C}, \Sigma, \Theta)$, we have the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{C}(D, H) & \xrightarrow{\mathcal{C}(d, H)} & \mathcal{C}(c, H) \\
\mathcal{C}(b, H) & \xrightarrow{} & \mathcal{C}(c, H) \\
\mathcal{C}(B, H) & \xrightarrow{\mathcal{C}(a, H)} & \mathcal{C}(A, H),
\end{array}
$$

where $H$ is any object in $\mathcal{H}$. Let $a$ and $b$ be $\xi$-inflations. Then $\mathcal{C}(a, H)$ and $\mathcal{C}(b, H)$ are epimorphisms. Thus $\mathcal{C}(c, H)\mathcal{C}(d, H)$ is an epimorphism, and hence $\mathcal{C}(c, H)$ is an epimorphism. So $\xi$ is saturated by Lemma 4.3.

(2) Assume that $(\mathcal{C}, \mathcal{E}_\xi, \sigma_\xi)$ is an $n$-exact category. Then for any $n + 2$-angle

$$
X^0 \xrightarrow{d^0_X} X^1 \xrightarrow{d^1_X} X^2 \xrightarrow{d^2_X} \ldots \xrightarrow{d^{n-1}_X} X^n \xrightarrow{d^n_X} X^{n+1} \xrightarrow{\delta} \Sigma X^0
$$

in $\xi$, one has $X^0 \xrightarrow{d^0_X} X^1 \xrightarrow{d^1_X} X^2 \xrightarrow{d^2_X} \ldots \xrightarrow{d^{n-1}_X} X^n \xrightarrow{d^n_X} X^{n+1}$ is an $n$-exact sequence in $\mathcal{C}$. Thus $d^0_X$ is a split monomorphism by [L1, Lemma 2.4], and hence $\delta = 0$ by [L1, Lemma 2.3]. So $\xi$ is the class of split $(n + 2)$-angles.

(3) The result holds by [GKO, Proposition 2.5].

(4) Assume that $\mathcal{H}$ is strongly covariantly finite of $(\mathcal{C}, \Sigma, \Theta)$ which is closed under direct summands. It is easy to check that the class of injective objects in $(\mathcal{C}, \mathcal{E}_\xi, \sigma_\xi)$ equals $\mathcal{H}$. If $(\mathcal{C}, \mathcal{E}_\xi, \sigma_\xi)$ is an $n$-exact category, then $\xi$ is the class of split $(n + 2)$-angles by (2). Thus $\mathcal{H} = \mathcal{C}$, a contradiction. On the other hand, if $(\mathcal{C}, \mathcal{E}_\xi, \sigma_\xi)$ is an $(n + 2)$-angulated category, then $\mathcal{H}$ consists of zero objects by [LZ, Remark 3.3]. This yields a contradiction. \hfill \Box

Now we give a concrete example to explain our main result in this section.

**Example 4.9.** This example comes from [L2]. Let $\mathcal{T} = D^1(kQ)/\tau^{-1}[1]$ be the cluster category of type $A_3$, where $Q$ is the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. $D^1(kQ)$ is the bounded derived category of finite generated modules over $kQ$, $\tau$ is the Auslander-Reiten translation and $[1]$ is the shift functor of $D^1(kQ)$. Then $\mathcal{T}$ is a 2-Calabi-Yau triangulated category. Its shift functor is also denoted by $[1]$.

We describe the Auslander-Reiten quiver of $\mathcal{T}$ in the following:

$$
\begin{array}{cccc}
P_1 & S_3[1] & P_2 & \vdots \\
\downarrow & \downarrow & \downarrow & \vdots \\
P_2 & I_2 & P_2[1] & \vdots \\
\downarrow & \downarrow & \downarrow & \vdots \\
S_3 & S_2 & S_1 & P_1[1]
\end{array}
$$

It is straightforward to verify that $\mathcal{C} := \text{add}(S_3 \oplus P_1 \oplus S_1)$ is a 2-cluster tilting subcategory of $\mathcal{T}$. Moreover, $\mathcal{C}[2] = \mathcal{C}$. By [GKO, Theorem 1], we know that $(\mathcal{C}, [2])$ is a 4-angulated category. Let $\mathcal{H} = \text{add}(S_3 \oplus S_1)$. Then the 4-angle

$$
P_1 \rightarrow S_1 \rightarrow S_3 \rightarrow P_1 \rightarrow P_1[2]
$$

shows that $\mathcal{H}$ is strongly covariantly finite subcategory of $\mathcal{C}$. Note that $\{0\} \neq \mathcal{H} \subsetneq \mathcal{C}$. So $(\mathcal{C}, \mathcal{E}_\xi, \sigma_\xi)$ is neither $n$-exact nor $(n + 2)$-angulated by Proposition 4.8.
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