Convergent series for polynomial lattice models with complex actions

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Abstract

Lattice models with complex actions are important for the understanding of matter at finite densities, but not accessible by the standard Monte Carlo techniques due to the sign problem. Here we derive a new approach for avoiding the complex action/sign problem, by extending the method of convergent series with a non-Gaussian initial approximation. The main features of the new series are demonstrated on the example of the two dimensional oscillating integral.

1 Introduction

One of the most traditional approaches to computations of the path integral or of its discretized lattice version is the standard perturbation theory (SPT), a perturbative expansion around the Gaussian initial approximation. Significant progress has been made in calculations of the standard quantum field perturbation theory in the last few years. There were developed numerical and analytical methods for calculations of multi-loop Feynman diagrams [1, 2, 3], which led to the evaluation of approximations with a record-breaking number of loops [4, 5, 6, 7]. However, the series of SPT are asymptotic [8, 9] and can be used only for small expansion parameters (coupling constants). The reason is in the incorrect interchange of the summation and integration during the construction of the perturbative expansion. From a physical point of view, the divergence of the perturbative series is related to large fluctuations of fields. Nevertheless, one can avoid the problem of the interchange by choosing a proper non-Gaussian initial approximation.

The convergent series (CS) based on the substitution of the Gaussian initial approximation by a certain interacting theory was initially proposed for the quantum anharmonic oscillator and scalar field theories [16, 17, 18, 19] and later developed in [20, 21, 22, 23]. The main feature of the CS method is that the employed non-Gaussian initial approximation can be reduced by a set of transformations to Gaussian integrals. Consequently, each order of CS is expressed in terms of a linear combination of a finite amount of orders of the

\footnote{For the alternative methods see [10, 11, 12, 13, 14, 15].}
standard perturbation theory. However, the derivation of the convergent series for quantum field theories is based on the assumption of the applicability of the dimensional regularization \[24\] to handle the limit of the infinite number of degrees of freedom. Due to this fact a rigorous mathematical proof of the series convergence is still missing.

Recently, the convergent series similar to \[18, 19\] was derived for the real action models defined on finite lattices \[25, 26\]. There the restriction to a finite amount of degrees of freedom provided the conditions, enough to carry out the construction mathematically rigorously and to prove the possibility to express coefficients of the convergent series for any model on the finite lattice with the real action and an even degree polynomial interaction as linear combinations of SPT-terms. It was also shown in \[26\] that the convergent series has a certain variational invariance, which allows one to improve its convergence significantly.

In the current paper we extend the non-perturbative convergent series method to a class of lattice models with complex actions. For such models the standard Monte Carlo approach is not applicable because of the lack of the Boltzmann weight positivity (complex action problem or sign problem). The sign problem is typical for theories describing matter at non-zero densities and for models with a vacuum term. Different approaches are studied nowadays in attempts to overcome it \[27, 28, 29, 30, 31\]. Nevertheless, all of them have certain limitations and the development of new methods for models with complex actions is still particularly important. Hereafter, we consider sign problems related to the presence of finite chemical potential and, therefore, assume that the complex contributions are contained only in the Gaussian part of the action. We construct the convergent series with a non-Gaussian initial approximation for lattice models with polynomial interactions and complex quadratic part of the action and prove the existence of the variational invariance, highly important for the applicability of the method. We demonstrate the work of the method on the example of the oscillating two dimensional integral.

2 Convergent series

Consider a model defined on the $d$-dimensional cubic lattice, with $N^d$ lattice cites, by the action

$$S[\phi] = \sum_{x,y} \left( \phi_x^T K_{xy} \phi_y + g \sum_x (\phi_x^T \phi_x)^p \right).$$

(1)

Here $\phi_x$ is a real vector of the dimension $\tilde{d}$ (the vector indices are included in $x$) $\phi_x^T$ is a transposition of $\phi_x$, $x$ and $y$ run over all lattice sites and vector indices of fields, $g > 0$ is a coupling constant, $p > 1$ characterizes the degree of the interaction, $K_{xy}$ is some complex matrix with a number of eigenmodes given by $V = N^{d \tilde{d}}$, the latter quantity corresponds to $N^d$. Obviously, a wide class of models is covered by the action (1), for instance, the models of the scalar charged particles can be represented by (1) with $\tilde{d} = 2$.

The partition function corresponding to the action (1) is given by

$$Z = \int [d\phi] e^{-S[\phi]}, \quad \int [d\phi] = \prod_x \left( \int [d\phi_x] \right).$$

(2)
To construct the convergent expansion for (2), we choose a non-Gaussian initial approximation

$$N[\phi] = \alpha \|\phi\|^2 + \sigma \|\phi\|^{2p}, \quad (3)$$

where $\|\phi\| = \sqrt{\phi^*_x M_{xy} \phi_y}$ is a norm in $\mathbb{R}^{N \times d}$ determined by a hermitian matrix with positive eigenvalues, $M_{xy}$. The parameter $\alpha \in \mathbb{C}$, $Re\{\alpha\} > 0$ and $\sigma > 0$ is fixed by the inequality

$$\sigma \|\phi\|^{2p} \geq g \sum_x (\phi^*_x \phi_x)^p, \quad (4)$$

which always holds for $\sigma$ big enough. Then, we expand the partition function (2) into a series as

$$Z = \sum_n \frac{1}{n!} \int [d\phi] e^{-N[\phi]} (N[\phi] - S[\phi])^n. \quad (5)$$

When the inequality (4) is valid, the series (5) is absolutely convergent

$$\sum_n \frac{1}{n!} \int [d\phi] e^{-N[\phi]} (N[\phi] - S[\phi])^n \leq \sum_n \frac{1}{n!} \int [d\phi] e^{-Re\{N[\phi]\}} |N[\phi] - S[\phi]|^n \leq \sum_n \frac{1}{n!} \int [d\phi] e^{-Re\{\alpha\}\|\phi\|^2 + |\alpha|\|\phi\|^2 - \sum_{x,y} \phi^*_x K_{xy} \phi_y} |\phi|^{2p} - g \sum_x (\phi^*_x \phi_x)^p)$$

$$= \int [d\phi] e^{-Re\{\alpha\}\|\phi\|^2 + |\alpha|\|\phi\|^2 - \sum_{x,y} \phi^*_x K_{xy} \phi_y} |\phi|^{2p} - g \sum_x (\phi^*_x \phi_x)^p \leq \infty. \quad (6)$$

To evaluate terms of the series (5), we change $\|\phi\|$ to a positive one dimensional variable $t$ by introducing auxiliary integration with the delta function

$$Z = \sum_n \frac{1}{n!} \int [d\phi] \int_0^\infty dt \delta(t - \|\phi\|) e^{-\alpha t^2 - \sigma t^{2p}} (P_2[\phi] + \alpha t^{2p} - g \sum_x (\phi^*_x \phi_x)^p)^n, \quad (7)$$

where $P_2[\phi]$ is the quadratic part of the perturbation,

$$P_2[\phi] = \alpha \|\phi\|^2 - \sum_{x,y} \phi^*_x K_{xy} \phi_y. \quad (8)$$
Then, rescaling fields as $\phi_{x}^{\text{new}} = t\phi_{x}^{\text{old}}$ and denoting $\phi_{x}^{\text{new}}$ as $\phi_{x}$, we obtain

$$Z = \sum_{n} \frac{1}{n!} \int [d\phi] \int_{0}^{\infty} dt \delta(t - t||\phi||) \cdot e^{-\alpha t^2 - \sigma t^p} \left(t^2 P_2[\phi] + t^p (\sigma - g \sum_{x} (\phi_{x}^T \phi_{x})^p)\right)^n =$$

$$\sum_{n} \sum_{k=0}^{n} \frac{1}{k!} \binom{n}{k} I_{2p}[2k + 2p(n - k), \alpha, \sigma, V] \cdot \int [d\phi] \delta(1 - ||\phi||)(P_2[\phi])^k (\sigma - g \sum_{x} (\phi_{x}^T \phi_{x})^p)^{n-k} =$$

$$\sum_{n} \sum_{k=0}^{n} \sum_{l=0}^{k} \sigma^{n-k-l} \frac{\binom{n}{k}}{n!} \binom{n-k}{l} \frac{I_{2p}[2k + 2p(n - k), \alpha, \sigma, V]}{I_{2p}[2k + 2p(l), \alpha, \sigma, V]} \cdot \int [d\phi] \delta(1 - ||\phi||)(P_2[\phi])^k (\sigma - g \sum_{x} (\phi_{x}^T \phi_{x})^p)^{l},$$

(9)

where $I_{2p}[r, \alpha, \sigma, V]$ is the integral over the variable $t$, given by

$$I_{2p}[r, \alpha, \sigma, V] = \int_{0}^{\infty} dt t^{V-1+r} e^{-\alpha t^2 - \sigma t^p}. \quad (10)$$

The expression (9) can be reduced to standard Gaussian integrals. Employing the identity

$$\int [d\phi] \delta(1 - ||\phi||)(P_2[\phi])^k (\sigma - g \sum_{x} (\phi_{x}^T \phi_{x})^p)^{l} =$$

$$\frac{1}{I_{2}[r+q, \alpha, V]} \int [d\phi] \exp\{-\alpha ||\phi||^2\} \phi_{x_1}...\phi_{x_r} \phi_{y_1}^T...\phi_{y_q}^T,$$

(11)

where

$$I_{2}[r+q, \alpha, V] = \int_{0}^{\infty} dt t^{V-1+r+q} e^{-\alpha t^2}, \quad (12)$$

one obtains

$$Z = \sum_{n} \sum_{k=0}^{n} \sum_{l=0}^{k} \sigma^{n-k-l} \frac{\binom{n}{k}}{n!} \binom{n-k}{l} \frac{I_{2p}[2k + 2p(n - k), \alpha, \sigma, V]}{I_{2p}[2k + 2p(l), \alpha, \sigma, V]} \cdot \int [d\phi] e^{-\alpha ||\phi||^2} (P_2[\phi])^k (\sigma - g \sum_{x} (\phi_{x}^T \phi_{x})^p)^{l}.$$

(13)

The explicit dependence on the number of degrees of freedom $V$ in (9), (11) and (13) appears from the Jacobian of the rescaling of fields. As it was numerically demonstrated for the real lattice $\phi^4$-model in [23] and [26], this explicit dependence causes the dramatical slowing down of the convergence at large lattice volumes. However, in [26] it was proved that for the real $\phi^4$-model the formula, analogous to (13), is invariant under the substitution of $V$ by a variational parameter $\tau \in [1; \infty)$. Here we prove that a similar invariance holds for
the complex case. Let $m = n - k - l$, then, \[13\], with $V$ changed by $\tau$, becomes

\[
Z = \sum_{m,k,l=0}^{\infty} \frac{\sigma^m}{(m+k+l)!} \binom{m+k+l}{k} (\binom{m+l}{l}) \frac{I_{2p}[2k+2p(m+l), \alpha, \sigma, \tau]}{I_{2}[2k+2p, \alpha, \tau]} \int [d\phi] e^{-\alpha \|\phi\|^2} \left( P_2[\phi] \right)^k \left( -g \sum_x (\phi_x^T \phi_x)^p \right)^l \]

\[
\approx \sum_{m,k,l=0}^{\infty} \frac{1}{k! l!} \int_0^\infty dt \sum_{m=0}^{\infty} \frac{(\sigma t^2)^m}{m!} e^{-\alpha t^2 - \sigma t^2 p} \frac{I_{2}[2k+2p, \alpha, \tau]}{I_{2}[2k+2p, \alpha, \tau]} \int [d\phi] e^{-\alpha \|\phi\|^2} \left( P_2[\phi] \right)^k \left( -g \sum_x (\phi_x^T \phi_x)^p \right)^l \]

\[
= \sum_{m,k,l=0}^{\infty} \frac{1}{k! l!} \int [d\phi] e^{-\alpha \|\phi\|^2} \left( P_2[\phi] \right)^k \left( -g \sum_x (\phi_x^T \phi_x)^p \right)^l. \quad (14)
\]

The sign $\approx$ in \[14\] stands to indicate, that two sides of the equation are only perturbatively equivalent, i.e. have the same standard perturbative expansions, but may differ by some non-analytic contributions like $e^{-\frac{1}{2} z}$. To prove the invariance of \[13\] under the substitution of $V$ by $\tau$ non-perturbatively, we may represent the expansion \[6\] as a result of consequent expansions of the quadratic and interacting parts of the exponent $e^{N[\phi]-S[\phi]}$. Performing the first of them, we obtain an absolutely convergent series

\[
Z = \sum_n \frac{Z^{(n)}}{n!} = \sum_n \frac{1}{n!} \int [d\phi], (P_2[\phi])^n e^{-\alpha \|\phi\|^2 - \sigma \|\phi\|^2 p + (\sigma \|\phi\|^2 - g \sum_x (\phi_x^T \phi_x)^p) \phi \phi^T} \quad (15)
\]

Each term $Z^{(n)}$ can be considered separately and interpreted as an average of the operator $(P_2[\phi])^n$. Then,

\[
Z^{(n)} = \sum_k \frac{1}{k!} \int [d\phi], (P_2[\phi])^n e^{-\alpha \|\phi\|^2 - \sigma \|\phi\|^2 p} (\sigma \|\phi\|^2 p - g \sum_x (\phi_x^T \phi_x)^p)^k \quad (16)
\]

and all other steps of the CS construction can be done. The non-perturbative independence on $\tau$ of each $Z^{(n)}$ can be proved exactly, as in \[20\] for the lattice $\phi^4$-model with the real action.

Concluding this section, we would like to note that, the expansion \[15\] is similar to the Taylor expansion in chemical potential (TE) \[29\]. It is well known, that TE for the connected correlation functions necessarily brakes down at the chemical potential corresponding to the first singularity of connected functions. Such singularities in connected functions are related to zeros of the partition function (generation functional of full correlation functions) and for the models with complex actions can exist even at finite lattice volumes. However, the expansion \[15\] is different from TE, due to the flexibility of the CS method, providing a wide range of possible choices for the initial approximation. The initial approximation of CS can even contain oscillating factors regulated by the value of $Im\{\alpha\}$. The latter fact significantly increases the potential applicability of CS.
3 Example

We demonstrate the application of the convergent series considering the two dimensional oscillating integral

\[ Z_2 = \int dx \int dy \; e^{-x^2 - y^2 + \mu x y - g(x^4 + y^4)}. \]  

(17)

To calculate the integral we choose the initial approximation as

\[ N = (x^2 + y^2) + g(x^2 + y^2)^2, \]  

(18)

which corresponds to \( \alpha = 1 \) and \( \sigma = g \) (and is not the only one possibility).

In Fig. 1 we present a comparison of the numerically integrated \( Z_2 \) with the results obtained by the convergent series with 10 terms (as well as for plots from Fig. 2 and Fig. 3), depending on \( g \) at several values of \( \mu \). Numerical and CS computations in Fig. 1 are in the excellent agreement. In Fig. 2 in a region of small \( g \) a deviation of the CS results from the correct answer is observed. The reason is that at small \( g \) the expansion (13) is more sensitive to large \( \mu \) and one needs more terms of the series (or alternatively, one has to choose a more appropriate initial approximation) to reproduce the correct result. The dependences of \( Z_2 \) on \( \mu \) at \( g = 0.1 \) and \( g = 1 \), presented in Fig. 3, support the same observations. In Fig. 4 we study the non-perturbative independence on the variational parameter \( \tau \). Three lines corresponding to 5, 10 and 20 computed CS terms show, that the sum of the series (15), in which \( V \) is substituted by \( \tau \), tends to be a constant with respect to the variations of \( \tau \) with increasing number of terms, taken into account. The evolution of the value of \( Z_2 \) depending on number of CS terms at different \( \tau \) is shown in Fig. 5.

![Figure 1: The integral \( Z_2 \) depending on \( g \) at \( \mu = 0.5 \) (black), \( \mu = 1 \) (red) and \( \mu = 2 \) (green). Solid lines correspond to the numerical integration and bars, triangles and filled circles to results of CS.](image-url)
Figure 2: The integral $Z_2$ depending on $g$ at $\mu = 3$ (red) and $\mu = 5$ (blue). Solid lines correspond to the numerical integration, bars and triangles to results of CS.

Figure 3: The integral $Z_2$ depending on $\mu$ at $g = 0.1$ (red) and $g = 1$ (blue). Solid lines correspond to the numerical integration, bars and triangles to results of CS.
Figure 4: The integral $Z_2$ at $g = 1$ and $\mu = 1$ depending on the variational parameter $\tau$ evaluated by $n = 5$ (solid black line) $n = 10$ (red dashed line) and $n = 20$ (green dotted line) terms of CS.

Figure 5: The integral $Z_2$ at $g = 1$ and $\mu = 1$ depending on the number of computed terms of CS at variational parameters $\tau = 1$ (black line with filled circles), $\tau = 2$ (red line with bars) and $\tau = 6$ (green line with triangles).
4 Summary

We have constructed the convergent series for lattice models with polynomial interaction and complex actions. Within the CS framework the observables are expressed as sums of Feynman diagrams, what reduces the computations to standard techniques. The final expression for the series provides a possibility to introduce a variational parameter, allowing one to exclude an explicit dependence on the number of degrees of freedom and to improve the convergence of the method.

The main objects of computations and measurements on the experiment are the connected correlation functions. In the current paper we presented the construction of CS for the partition function. The series for corresponding full correlation functions are derived exactly in the same way and CS for the connected functions can be obtained by keeping only connected Feynman diagrams in the sum of the series.

The numerical results obtained for the simple example together with a variational invariance, excluding explicit dependence on the number of degrees of freedom, open a new potential way to avoid the sign problem.

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