Strong Coupling Fixed Points of Current Interactions and Disordered Fermions in 2D

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Abstract

The all-orders βeta function is used to study disordered Dirac fermions in 2D. The generic strong coupling fixed ‘points’ of anisotropic current-current interactions at large distances are actually isotropic manifolds corresponding to subalgebras of the maximal current algebra at short distances. We argue that IR fixed point theories are generally current algebra cosets. We illustrate this with the simple example of anisotropic $su(2)$, which is the physics of Kosterlitz-Thouless transitions. We propose a phase diagram for the Chalker-Coddington network model which is in the universality class of the integer Quantum Hall transition. One phase is in the universality class of dense polymers.

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I. INTRODUCTION

Two is the critical dimension for Anderson localization, namely, above 2 dimensions states are localized only above a critical strength of the disorder (impurities) whereas below 2 dimensions states are localized for any strength of the disorder \[1,4\]. For this reason one can expect rich phase structures in two dimensions. It is well understood that a complete understanding of the Quantum Hall transition involves delocalization \[3\] \[4\]. In these localization problems, understanding the quantum phase transition in the conductivity requires finding the right renormalization group fixed point at large distances. This usually occurs at strong coupling, which is why these problems are considered difficult.

The conventional theoretical framework leads to sigma models on various spaces \[5\]. For the quantum Hall transition the sigma model was worked out by Pruisken \[3\]. These results are important, but the sigma models have generally been too difficult to solve. The most recent progress can be found in \[7\].

Critical theories in 2 dimensions are conformally invariant and this imposes strong constraints on the theory \[8\]. Early efforts to use conformal field theory methods in the study of disordered Dirac fermions were made by Bernard \[9\], Mudry et. al. \[10\], and Nersesyan et. al. \[11\]. Perturbed conformal field theory methods were also used for disordered statistical mechanical models \[12\] \[13\]; generally these are more difficult problems since the non-random theories are interacting so that one has to use replicas. Many of the important problems here also are driven to strong coupling under renormalization group (RG) flow.

In this work we describe a general approach to disordered Dirac fermions at strong coupling which uses the all-orders $\beta$eta functions proposed in \[14\]. These models are not integrable, but one can nevertheless sum up all orders in perturbation theory for the $\beta$eta function. In some insightful work, Chalker and Coddington proposed a simple network model which is now believed to be in the universality class of the integer quantum Hall transition \[17\]. This network model has been shown to be equivalent to a certain theory of random Dirac fermions \[18\], and this is the model we analyze using our methods. This model also appeared in the work \[19\]. A sigma model formulation was proposed in \[20\].

This paper is a first attempt at understanding the implications of these all-orders $\beta$eta functions, and for this reason is in part conjectural. The $\beta$eta functions generally do not have non-trivial zeros, but rather have poles. We point out that $\beta$eta functions with poles are also known to occur in supersymmetric gauge theory \[21\]. In the next section we propose a general scheme for the fixed points of marginal symmetry breaking (anisotropic) current-current perturbations for a current algebra $\mathcal{G}$ in 2 dimensions based on these $\beta$eta functions and certain hypotheses. Since there is no small parameter in these theories, one generally does not flow to a single fixed point, but rather to a fixed point manifold in the couplings. In the case of large couplings we show that this manifold generally corresponds to a sub-current-algebra $\mathcal{H}$ of $\mathcal{G}$ and argue that the infra-red fixed point is the current algebra coset $\mathcal{G}/\mathcal{H}$.

The $\beta$eta function shows an interesting duality, namely $g$ flows to $1/g$ as the length scale varies from $r$ to $1/r$. This is a new form of duality since the more well-known manifestations occur in scale-invariant theories. Here, theories at a scale $r$ and coupling $g$ are equivalent to a theory at scale $1/r$ and coupling $1/g$.

Assuming our hypotheses for interpreting the $\beta$eta functions are correct, they turn out to
predict a very tight phase structure; in fact the resulting phase diagram is exact in the sense that the phase boundaries are precisely determined. We are unaware of any other models where the exact beta function allows one to determine so precisely the phase diagram. As an illustration we work out the simplest possible case of an anisotropic $su(2)$ perturbation. This is the subject of Kosterlitz-Thouless transitions which have previously only been studied at weak coupling [16]. The phase diagram is unexpectedly rich. At small coupling one phase exhibits the symmetry restoration recently studied at one loop [22] [23], however it appears this symmetry does not persist at strong coupling. Another phase has a line of fixed points corresponding to a free boson at a certain radius of compactification.

If we extend the physical regime of couplings of the network model, the resulting disordered Dirac theory again has a number of phases. Though many of the features are similar to the anisotropic $su(2)$ case, some features are novel and not completely understood. One phase ($PSL_g$) has a line of fixed points related to $PSL(1|1)$ studied in [24], and appears to be the only massless phase. It is in the same universality class as dense polymers and is also closely related to the disordered XY model [23]. A sigma model version of this theory was proposed by Zirnbauer in connection with the quantum Hall transition [11], based on the work [15] [19]. In the physical regime of the network model we show that this phase is not easily realized since it corresponds to imaginary gauge potential. The other phases do not automatically fall into the category of $G/H$, as explained below.

An unexpected feature of our analysis is that initially positive couplings which are variations of real disordered potentials can flow to negative values, which in turn can be viewed as corresponding to imaginary potentials and thus non-hermitian hamiltonians. This could signify that our approach breaks down for reasons we do not yet comprehend. If this feature turns out to be sensible, it implies that the studies of delocalization transitions in non-hermitian quantum mechanics made by Hatano and Nelson [27] may have some bearing on hermitian systems.

II. GENERAL STRUCTURE OF FIXED POINTS

The general class of quantum field theories we consider are current-current perturbations of a conformally invariant field theory. We assume the conformal field theory possesses left and right conserved currents $J^a(z)$ and $\bar{J}^a(\bar{z})$ in the usual way [28] [29], where $z, \bar{z}$ are euclidean light-cone coordinates, $z = (x + iy)/\sqrt{2}, \bar{z} = (x - iy)/\sqrt{2}$. These currents satisfy the operator product expansion (OPE):

$$J^a(z)J^b(0) = \frac{k}{z^2} \eta^{ab} + \frac{1}{z} f_{bc} J^c(0) + \ldots \quad (2.1)$$

where $k$ is the level and $\eta^{ab}$ the metric on the algebra, and similarly for $\bar{J}$. We will refer to this current algebra as $\mathcal{G}_k$, where $\mathcal{G}$ is a Lie algebra or superalgebra and the formal action for this theory as $\mathcal{S}_{\mathcal{G}_k}$. In the superalgebraic case each current $J^a$ has a grade $[a] = 0$ or 1 corresponding to bosonic versus fermionic. The metric $\eta^{ab}$ generally cannot be diagonalized in the superalgebra case. The metric and structure constants have the following properties:

$$\eta^{ab} = (-)^{|a||b|}\eta^{ba}, \quad f^{abc} = (-)^{|a||b|} f^{bca}, \quad f^{abc} = (-)^{|b||c|} f^{acb} \quad (2.2)$$

where $f^{abc} = f^{abc}_{i} \eta^{ic}$ and $\eta_{ab}\eta^{bc} = \delta^c_a$. 

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The theory $S_{\mathcal{G}_k}$ can be perturbed by marginal operators built out of left-right current bilinears:

$$S = S_{\mathcal{G}_k} + \int \frac{d^2x}{2\pi} \sum_A g_A \mathcal{O}^A, \quad \mathcal{O}^A = \sum_{a,\tilde{a}} d_{ab}^A J^a J^{\tilde{a}}$$  \hspace{1cm} (2.3)

The simplest example is a single coupling $g$ with $d_{a\tilde{a}} = \eta_{a\tilde{a}}$ defining the quadratic Casimir of $\mathcal{G}$. For $su(N)$ at level 1, this is the non-abelian Thirring model, or equivalently the chiral Gross-Neveu model. We are mainly interested in more general situations where the tensors $d_{ab}^A$ break the symmetry $\mathcal{G}$, i.e. are anisotropic.

In [14] an all-orders $\beta$eta function in a certain prescription was proposed and we now summarize this result. The theory is not renormalizable for any choice of $d^A$. The three conditions ensuring renormalizability are:

1. $(-)^{|b|c} d_{ab}^C f_{ij}^a f_{j}^{bd} = C_{C}^{AB} d_{ij}^C$  \hspace{1cm} (2.4)
2. $\eta^{ij} d_{ai}^{A} d_{bj}^{B} = D_{C}^{AB} d_{ba}^{C}$  \hspace{1cm} (2.5)
3. $d_{ij}^{A} f_{k}^{a} f_{j}^{b} = R_{B}^{A} \eta^{ac} d_{cb}^{B}$  \hspace{1cm} (2.6)

The first condition is equivalent to closure of the operator algebra of $\mathcal{O}^A$:

$$\mathcal{O}^A(z, \bar{z}) \mathcal{O}^B(0) \sim \frac{1}{z \bar{z}} \sum_{C} C_{C}^{AB} \mathcal{O}^C(0)$$  \hspace{1cm} (2.7)

and guarantees one-loop renormalizability. The other two conditions are necessary at two loops and higher. The structure constants $D, R$ are also related to an OPE. Define the operator $T^A$ built out of left-moving currents only and $d^A$:

$$T^A(z) = d_{ab}^A J^a(z) J^b(z)$$  \hspace{1cm} (2.8)

The operator $T^A$ is a kind of stress tensor; in the isotropic case it is the affine-Sugawara stress tensor up to a normalization. One finds

$$T^A(z) \mathcal{O}^B(0) \sim \frac{1}{z^2} \left( 2k D_{C}^{AB} + R_{B}^{A} D_{C}^{BD} \right) \mathcal{O}^C(0)$$  \hspace{1cm} (2.9)

Given a particular theory, the above equation is an efficient way to compute $R, D$; the $D$ term is distinguished from the $RD$ term by being proportional to $k$. The conditions required for $T^A$ to satisfy the OPE of a consistent stress-tensor, the so-called master equation studied extensively in [15], involves the same objects $C, D, R$, but appears to be a stronger than our renormalizability conditions.

The renormalization group (RG) structure constants have the following properties:

$$D_{C}^{AB} = D_{C}^{BA}, \quad C_{C}^{AB} = C_{C}^{BA}, \quad D_{C}^{AC} D_{E}^{DB} = D_{C}^{AB} D_{E}^{DC}$$  \hspace{1cm} (2.10)

which can be proven from the defining relations or from the OPE’s.

Let us arrange the couplings into a row vector $g = (g_1, g_2, \ldots)$. Since the higher loop expansion is in $kg$, let us define

$$\tilde{g} = kg/2$$  \hspace{1cm} (2.11)
Let $D(\hat{g})$ be the matrix of couplings

$$D(\hat{g})_B^A = \sum_C D_B^{AC} \hat{g}_C$$

(2.12)

Given two row vectors $v^{1,2}$ we define a new row vector from $C$:

$$C(v^1, v^2)_A = \sum_{B,C} v^1_B v^2_C C_B^{AC}$$

(2.13)

Finally, define

$$\hat{g}' = \frac{1}{1 - D(\hat{g})^2}$$

(2.14)

Then the beta function can be expressed as

$$\beta_{\hat{g}} = \frac{d\hat{g}}{d\log r} = \frac{2}{k} \left( -\frac{1}{2} C(\hat{g}', \hat{g}') (1 + D^2) + C(\hat{g}' D, \hat{g}' D) D - \hat{g}' D R D \right)$$

(2.15)

where $r$ is a length scale and $D = D(\hat{g})$. The flow to the IR corresponds to increasing $r$.

For the purpose of studying fixed points, it will be convenient to define

$$\tilde{C}^{AB}_C = R^A_D D^{BD}_C$$

(2.16)

Note that $\tilde{C}$ appears directly in the OPE (2.9) and the beta function so there is no need to compute $R$ by itself. One can show

$$\tilde{C}(v^1, v^2 D(\tilde{g})) = \tilde{C}(v^1, v^2) D(\tilde{g})$$

(2.17)

using (2.10). The beta function can then be arranged into the form:

$$\beta_{\tilde{g}} = \frac{1}{k} \left[ -C(\tilde{g}', \tilde{g}') (1 - D)^2 + \tilde{C}(\tilde{g}^+, \tilde{g}^-) - \tilde{C}(\tilde{g}^-, \tilde{g}^+) 
- \left( (C + \tilde{C})(\tilde{g}^+, \tilde{g}^-) \right) D - \left( (C + \tilde{C})(\tilde{g}^+, \tilde{g}^-) \right) D \right]$$

(2.18)

where again $D = D(\tilde{g})$ and we have defined

$$\tilde{g}^\pm = \tilde{g} \frac{1}{1 \pm D(\tilde{g})}$$

(2.19)

We now study the possible fixed points based on the above beta function. In the specific examples we have studied $\beta_{\tilde{g}}$ has no non-trivial zeros for any finite values of $g$. The only zeros of $\beta_{\tilde{g}}$ in the two examples below are trivial in the sense that they correspond to $g = 0$ or exactly marginal directions, such as $u(1)$ directions. We believe this is the generic situation but cannot rule out other examples with non-trivial zeros.

The beta function generally has poles as can be seen from (2.14). These arise from summing geometrical series in perturbation theory. Since these series do not converge for $\tilde{g} > 1$ one should question their validity for large $\tilde{g}$. This being the first attempt to make sense of these beta functions, in the sequel we will simply assume the above beta function.
is valid for all \( g \), which amounts to an analytic continuation of the perturbation series. As we will see, the resulting RG flows can be interpreted at large \( g \).

Assuming the \( \beta \eta \) has no non-trivial zeros, then this implies that we must look for fixed point manifolds. Suppose all couplings \( \hat{g} \) have large absolute value. Since \( \hat{g}' \sim 1/\hat{g} \), \( \hat{g}_\pm \sim 1 \) and \( D(\hat{g}) \sim \hat{g} \), the \( \beta \eta \) function is dominated by the \( C + \tilde{C} \) terms which are of order \( \hat{g} \). For large \( g \) we therefore expect the possibility of flowing to manifolds in the coupling constant space satisfying

\[
(C + \tilde{C})(\hat{g}_-, \hat{g}_+) = (C + \tilde{C})(\hat{g}_+, \hat{g}_-) = 0 \tag{2.21}
\]

In the examples below the solutions to the above equations are the same as for

\[
C(g, g) + \tilde{C}(g, g) = 0 \tag{2.22}
\]

The above argument is not strong enough to conclude that the only asymptotic flows are solutions to (2.22). Rather, we expect that solutions to (2.22) represent a generic class of fixed points. This is strongly supported by the numerical analysis of the exact \( \beta \eta \) functions in the examples below; it is found that when couplings flow to infinity, one indeed generally flows to solutions of the equation (2.22).

Solutions to the above fixed point manifold equation are generally couplings that correspond to invariant subalgebras of \( G \). Let \( Q^a \) denote generators of \( G \):

\[
[Q^a, Q^b] = f^{ab}_{\ c} Q^c \tag{2.23}
\]

In the conformal field theory these are realized as conserved charges:

\[
Q^a = \oint \frac{dz}{2\pi i} J^a(z) + \oint \frac{d\z}{2\pi i} \bar{J}^a(\z) \tag{2.24}
\]

Suppose the \( d^A_{\ ab} \) define invariants of \( G \):

\[
[Q^a, d^A_{\ bc} Q^b Q^c] = 0 \tag{2.25}
\]

Then this implies

\[
d^A_{\ ib} f^{ai}_{\ c} = (-)^{[a][c]} d^A_{\ ci} f^{ai}_{\ b} \tag{2.26}
\]

Using this in the renormalizability conditions (2.4) one finds that this implies \( C + \tilde{C} = 0 \).

Interesting possibilities for non-trivial fixed points thus arise when the \( d^A \) couple commuting subalgebras of \( G \). Let

\[
O^A = \eta_{ab} J^a_A J^b_A \tag{2.27}
\]

where \( J_A \) are the currents for a subalgebra \( H^A \) of \( G \), \( \eta^A \) defines the Casimir for \( H^A \), and these subalgebras commute for \( A \neq B \). The currents \( J_A \) satisfy a current algebra \( H^A_{k_A} \), where the level \( k_A \) is generally not equal to \( k \). This is sometimes referred to as a conformal embedding. Only the currents \( J_A \) matter in (2.27) and the fixed point manifold condition \( C + \tilde{C} = 0 \) is satisfied.

Let us further suppose that each \( H^A \) has a unique quadratic Casimir so that

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\( \eta_{ij} f^c_{j} f^k_d = C^A_{adj} \delta_d^c \) \hspace{1cm} (2.28)

where \( C^A_{adj} \) is the quadratic Casimir in the adjoint representation of \( \mathcal{H}^A \). Not all algebras have this property; in particular the superalgebras with indecomposable representations which we will encounter below do not. Assuming (2.28), one finds that

\[ C^A_{adj} = -C^A_{adj}, \quad R^A_B = C^A_{adj} \delta^A_B, \quad D^A_A = 1 \] \hspace{1cm} (2.29)

The beta functions then obviously decouple and take the simple form:

\[ \beta_{g_A} = \frac{C^A_{adj} \tilde{g}_A^2}{k(1 + \tilde{g}_A)^2} \] \hspace{1cm} (2.30)

The beta function (2.30) has some interesting properties. Let us focus on a single coupling \( \tilde{g}_A = \tilde{g} \). Define a dual coupling

\[ g^* = \frac{1}{\tilde{g}} \] \hspace{1cm} (2.31)

The beta function satisfies

\[ \beta^*(g^*) = -\beta(\tilde{g} \to g^*) \] \hspace{1cm} (2.32)

This implies that

\[ \tilde{g}(r) = \frac{1}{g_A(1/r)} \] \hspace{1cm} (2.33)

This behavior can be seen from the solution:

\[ (\tilde{g} - 1/\tilde{g})/2 + \log \tilde{g} = \frac{C_{adj}}{2k} \log r/r_0 \] \hspace{1cm} (2.34)

where \( r_0 \) is the scale where \( \tilde{g} = 1 \). As shown below, the anisotropic case also exhibits this kind of duality.

There are four cases to consider. First suppose \( C_{adj}/k > 0 \). Then \( \tilde{g} \) always increases. From the duality (2.33) one sees that \( \tilde{g} > 0 \) at short distances (small \( r \)) flows to \( \tilde{g} = \infty \) at large \( r \). On the other hand any \( \tilde{g} < 0 \) flows to 0. Thus \( \tilde{g} > 0 \) is marginally relevant and \( \tilde{g} < 0 \) marginally irrelevant. This confirms the expectations based on one-loop. Note that the double pole in the beta function at \( \tilde{g} = -1 \) has no physical effect whatsoever: near the pole \( \tilde{g} \) grows faster as \( r^{C_{adj}/2k} \) rather than \( \log r \) but simply flows through the pole. When \( C_{adj}/k < 0 \) everything is reversed: \( \tilde{g} > 0 \) flows to 0 and \( \tilde{g} < 0 \) flows to \( -\infty \). Thus in this case of isotropic couplings, the beta function appears to be physically sensible for all \( \tilde{g} \), and this supports the validity of the analytic continuation of the beta function.

We thus propose the following generic scheme for non-trivial infrared (IR) fixed points that can occur when couplings flow to infinity. For every marginally relevant \( g_A \), \( g_A \) flows to either \( \pm \infty \). The degrees of freedom coupled by \( g_A \) are thus massive and should decouple in the flow to the IR. This picture was used in the work [30] on the spin quantum Hall effect. For example consider an \( su(3)_1 \) current algebra perturbed by the marginal operator that isotropically only couples an \( su(2)_1 \) sub-current algebra with coupling \( g \). The \( su(3)_1 \)
current algebra can be bosonized in terms of two free bosons $\phi_1, \phi_2$ and has $c = 2$. The $su(2)_1$ current algebra can be bosonized in terms of a single scalar field $\phi$ which is a linear combination of $\phi_1, \phi_2$, and has $c = 1$. As $g$ goes to $\infty$ under RG flow, correlation functions of $\phi$ tend to zero. Thus the field $\phi$ decouples and the IR fixed point is $su(3)_1/\text{su}(2)_1$. Generally, the fixed point is thus

$$\text{IR fixed point} = \frac{\mathcal{G}_k}{\mathcal{H}_k}, \quad \mathcal{H}_k = \otimes_A \mathcal{H}^A_{kA} \tag{2.35}$$

where $\otimes_A$ is only over marginally relevant $g_A$. The IR theory is non-trivial only if $\mathcal{H} \neq \mathcal{G}$. This situation corresponds to a massless phase and $\mathcal{G}_k/\mathcal{H}_k$ are the massless degrees of freedom. When $\mathcal{G}_k/\mathcal{H}_k$ is empty, e.g. when $\mathcal{G} = \mathcal{H}$, then this is a massive phase. In localization problems a massive phase corresponds to all states being localized. Coset conformal field theories were studied in generality in [31]. The Virasoro central charge is $c_{\mathcal{G}} - c_{\mathcal{H}}$, and the conformal scaling dimension $\Delta$ of an operator at the fixed point is

$$\Delta^{\mathcal{G}/\mathcal{H}} = \Delta^{\mathcal{G}} - \Delta^{\mathcal{H}} \tag{2.36}$$

An interesting open question is whether other models can flow to the more general solutions of the master equation [15] which do not correspond to cosets.

Due to the existence of poles in the beta function, not all flows that we find in the examples below fall neatly into the above scheme. In some cases below the RG flows are attracted to these poles. In most of these cases, as one approaches the pole, some of the other couplings flow off to infinity in such a way that suggests one can still perform a coset.

III. ANISOTROPIC $SU(2)$

In this section we illustrate the scheme of the last section in the simplest possible example of an anisotropic $su(2)$. This is the subject of Kosterlitz-Thouless flows which have previously only been studied at weak coupling [16]. We originally worked out this example to check of the validity of the exact beta functions. The phase diagram is unexpectedly rich.

We normalize the currents as follows:

$$J(z)J(0) \sim \frac{k}{2} \frac{1}{z^2}, \quad J(z)J^\pm(0) \sim \pm \frac{1}{z} J^\pm(0), \quad J^+(z)J^-(0) \sim \frac{k}{2} \frac{1}{z^2} + \frac{1}{z} J(0) \tag{3.1}$$

and consider the action

$$S = S_{su(2)_k} + \int \frac{d^2 x}{2\pi} \left( g_1 (J^+ \bar{J}^- + J^- \bar{J}^+) + g_2 J \bar{J} \right) \tag{3.2}$$

A simple computation using (2.7, 2.9) gives the RG data:

$$C^{12}_1 = C^{21}_1 = -1, \quad C^{11}_2 = -2 \tag{3.3}$$

$$\bar{C}^{21}_1 = \bar{C}^{21}_1 = 1, \quad \bar{C}^{12}_1 = 2 \tag{3.4}$$

$$D^{11}_1 = D^{22}_2 = 1/2 \tag{3.5}$$

The matrix $D(g)$ is diagonal:
\[ D(g) = \begin{pmatrix} g_1/2 & 0 \\ 0 & g_2/2 \end{pmatrix} \]  

(3.6)

The resulting beta functions are

\[ \beta_{g_1} = -\frac{g_1(g_2 - g_1^2k/4)}{(1 - k^2g_1^2/16)(1 + kg_2/4)} \]  

(3.7)

\[ \beta_{g_2} = \frac{g_1^2(1 - kg_2/4)^2}{(1 - k^2g_1^2/16)^2} \]  

(3.8)

There is again an interesting duality in these beta functions. Define the dual couplings

\[ \begin{align*}
  g_1^* &= \frac{16}{k^2g_1}, \\
  g_2^* &= \frac{16}{k^2g_2}
\end{align*} \]  

(3.9)

Then the beta function satisfies

\[ \beta^*(g^*) = -\beta(g \to g^*) \]  

(3.10)

The above beta functions predict a tight but unexpectedly rich phase diagram. The duality (3.11) explains some features of the diagram, namely, the self-dual lines \( g = g^* \) are phase boundaries or lines of attraction. The solutions to \( C + \bar{C} = 0 \) are (i) \( g_1 = g_2 \), which corresponds to the subalgebra \( \mathcal{H}_k = \text{su}(2)_k \) and (ii) \( g_1 = 0 \) corresponding to \( \mathcal{H} = \text{u}(1) \).

The structure of the phase diagram is determined in part by the behavior near the poles at \( g_1, g_2 = \pm 4/k \). Consider the pole at \( (g_1, g_2) = (4, 4)/k \) and let \( g_{1,2} = 4(1 + \epsilon_{1,2})/k \). Near \( \epsilon = 0 \) the behavior is

\[ \begin{align*}
  \beta_{\epsilon_1} &\approx 8 - \frac{4\epsilon_2}{\epsilon_1}, \\
  \beta_{\epsilon_2} &\approx \frac{4\epsilon_2^2}{\epsilon_1^2}
\end{align*} \]  

(3.11)

Around this pole, this leads to the behavior shown in figure 1. In this figure, heavy lines correspond to phase boundaries. The region, or phase, \( A \) is attracted to the line \( g_1 = g_2 \), since this line is stable in region \( A \). Beyond \( (g_1, g_2) = (4, 4)/k \) the line \( g_1 = g_2 \) becomes unstable. Since the line becomes unstable beyond the pole, one has to reach the line exactly in the region \( A \) before flowing off to infinity along it; otherwise one can flow elsewhere (see below). If one is on the line, then one flows to infinity and the IR fixed point is the empty coset \( \text{su}(2)_k/\text{su}(2)_k \).

In region \( D \), \( g_2 \) flows to \( \infty \) whereas \( g_1 \) flows to a constant, but after a finite scale transformation. Because one coupling is blowing up, the flow cannot be continued to larger scales numerically. Since the ratio \( g_1/g_2 \) flows to zero, one possible interpretation of this flow is that \( g_1 \) is effectively zero and \( g_2 = \infty \). The subalgebra coupled by \( g_2 \) is \( \mathcal{H} = \text{u}(1) \), and the IR fixed point would thus be \( \text{su}(2)_k/\text{u}(1) \). This is a well-known conformal embedding [32] and corresponds to the critical theory of \( Z_k \) parafermions with central charge \( c = 2(k - 1)/(k + 2) \), where \( k = 2 \) is the Ising model. However we emphasize that since the RG flow cannot be continued to arbitrarily large length scales, this is not a true fixed point.

In phase \( E \), \( g_1 \) grows to infinity and \( g_2 \) to a non-universal value \( 0 < g_2 < 4 \). Here the flow can be continued to arbitrarily large scales. For \( k = 1 \) we interpret this as a sine-Gordon phase. At \( k = 1 \) we can bosonize the currents.
\[ J^z = \frac{1}{\sqrt{2}} e^{\pm i\sqrt{2} \varphi}, \quad J = \frac{i}{\sqrt{2}} \partial_z \varphi \]  

(3.12)

where \( \varphi(z) \) is the \( z \)-dependent part of a free massless scalar field \( \phi \). Viewing the \( g_2 \) coupling as a perturbation of the kinetic term and rescaling the field \( \phi \) one obtains the sine-Gordon action

\[ S = \frac{1}{4\pi} \int d^2 x \left[ \frac{1}{2} (\partial \phi)^2 + g_1 \cos(\beta \phi) \right] \]  

(3.13)

The region \( A \) is also sine-Gordon like at small coupling. The operator \( \cos \beta \) has scaling dimension \( \Gamma = \beta^2 \). To relate \( \beta \) to \( g_2 \), consider the beta function when \( g_1 \) is small. Then \( \beta_{g_1} = g_1 g_2 / (1 + g_2 / 4) \). Since this is proportional to \( g_1 \), we can identify the dimension of the coupling as \( \text{dim}(g_1) = g_2 / (1 + g_2 / 4) = 2 - \Gamma \). Thus:

\[ \frac{\beta^2}{2} = 1 - \frac{g_2 / 4}{1 + g_2 / 4}, \quad \text{for} \ g_1 \approx 0 \]  

(3.14)

In general, we expect \( \beta \) to be a function of both \( g_1, g_2 \). Note that for small \( g_1 \), values of \( \beta \) between 0 and 4 correspond to \( 0 < \beta^2 < 2 \) which is the expected regime for the sine-Gordon model. (The conventional sine-Gordon coupling \( \beta \) is \( \beta = \beta \sqrt{4\pi} \).) For \( k > 1 \) the appropriate generalization is the fractional super sine-Gordon model [34].

In the \( G \) phase \( g_1 \) flows to zero and \( g_2 \) to a fixed value on the line \(-4 < kg_2 < 0\). For example, as \( r \to \infty \) one finds

\[ (g_1, g_2) = (2, -3) \to (0, -2.337693444..) \]  

(3.15)

when \( k = 1 \). This line of fixed points corresponds to a free boson at some radius of compactification determined by the value of \( g_2 \) on the fixed line.

In region I one is attracted to the isotropic line and then flows toward the poles at \((kg_1, kg_2) = (4, -4)\).
FIG. 1. Phase Diagram for anisotropic $su(2)$. Phase boundaries are solid heavy lines. Heavy dashed lines denote lines of fixed points. Shaded circles are fixed points.

The flows toward the poles in region A and I is rather delicate, and the outcome cannot be studied numerically by solving the differential equations because of the singularity. As stated above, one possibility is to continue to flow off to infinity along the isotropic lines once one reaches the poles. However this appears to be inconsistent with the duality (3.10). The duality implies that if a flow passes through the self-dual point $g = g^*$, then $g$ and $g^*$ are on the same RG trajectory, where $g$ flows to $g^*$ as $r$ goes to $1/r$. We can use this to extend the flows through the poles1. Namely, the region in A below the isotropic line flows to the region D, whereas the region above the line flows to region E. Similarly, the region above the isotropic line in I flows to region G, whereas the region below the line flows to A then to D. This implies that the symmetry restoration seen in region A is destroyed once the flow reaches beyond the pole.

The remaining phases are a mirror image of the above. Though the line $g_1 = -g_2$ does not appear as a solution to $\mathcal{C} + \tilde{\mathcal{C}} = 0$, and it would naively break the $su(2)$ symmetry, it turns out that it does possess an $su(2)$ symmetry. There is an automorphism of $su(2)$ $J^\pm \rightarrow -J^\pm$ that preserves the algebra. We can perform this automorphism on the left currents only. This takes $\mathcal{O}^1 \rightarrow -\mathcal{O}^1$, i.e. flips the sign of $g_1$. The phase diagram then has symmetry with respect to reflections about the $g_2$ axis. Thus the $C, F, B, I$ phases are similar to the $D, E, A, H$ phases.

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1An earlier version of this paper did not consider this possibility, which will be further reported on in [33].
A check of the above $\beta$eta function is based on comparison with the sine-Gordon $\beta$eta function which is known to two loops. For a recent discussion, see [35]. These perturbative computations were performed around $\hat{\beta} = \sqrt{2}$ and $g_1$ equal to zero. Define new couplings $\delta, \alpha$ as

$$1 + \delta = \frac{\hat{\beta}^2}{2} = \frac{1 - g_2/4}{1 + g_2/4}, \quad \alpha = 4g_1$$  \hspace{1cm} (3.16)$$

Then the $\beta$eta function (3.7) implies the following $\beta$eta functions for $\delta, \alpha$ to 5 loops:

$$\beta_\delta = -\frac{1}{32}\alpha^2 - \frac{1}{16}\alpha^2\delta - \frac{1}{4096}\alpha^4 - \frac{1}{32}\alpha^2\delta^2 - \frac{1}{2048}\alpha^4\delta$$

$$\beta_\alpha = -2\alpha\delta - \frac{1}{64}\alpha^3 - \frac{1}{64}\alpha^3\delta - \frac{1}{16384}\alpha^5$$  \hspace{1cm} (3.17)$$

It is known that with more than one coupling, not all 2 loop contributions are prescription independent. Define new couplings to second order as

$$\delta' = \delta - \frac{1}{32}\alpha^2, \quad \alpha' = \alpha$$  \hspace{1cm} (3.18)$$

Then the $\beta$eta function (3.17) to two loops leads to

$$\beta_\delta' = -\frac{1}{32}\alpha^2 + \frac{1}{16}\alpha^2\delta'$$

$$\beta_\alpha' = -2\alpha'\delta' - \frac{5}{64}\alpha^3$$  \hspace{1cm} (3.19)$$

This agrees with the result presented in [36] [35].

IV. THE NETWORK MODEL

A. $\beta$eta function

The network model can be mapped onto a model of disordered Dirac fermions [18]. The two-component hamiltonian in the two spacial dimensions $x, y$ is

$$H = \frac{1}{\sqrt{2}}(-i\partial_x - A_x)\sigma_x + \frac{1}{\sqrt{2}}(-i\partial_y - A_y)\sigma_y + V + M\sigma_z$$  \hspace{1cm} (4.1)$$

where $\sigma$ are Pauli matrices and $A(x, y), V(x, y), M(x, y)$ are real disordered potentials. Disorder in $A, V$ and $M$ respectively corresponds to randomness in the individual link phases, the total Aharanov-Bohm phase per plaquette, and the tunneling at the nodes. This model was also considered in the work [19].

Introducing 2-component Dirac fermions

$$\Psi = \begin{pmatrix} \psi_+ \\ \psi_+ \end{pmatrix}, \quad \Psi^* = \begin{pmatrix} \bar{\psi}_-, \bar{\psi}_- \end{pmatrix}$$  \hspace{1cm} (4.2)$$

one needs to study the action
\[ S = i \int \frac{d^2x}{2\pi} \Psi^* H \Psi \]  

\[ = \int \frac{d^2x}{2\pi} \left[ \bar{\psi}_-(\partial_z - iA_z)\psi_+ + \psi_-(\partial_z - iA_z)\psi_+ - iV(\bar{\psi}_+\psi_+ + \bar{\psi}_+\psi_+) - iM(\bar{\psi}_+\psi_+ + \bar{\psi}_+\psi_+) \right] \]  

where \( z = (x + iy)/\sqrt{2} \), \( \bar{z} = (x - iy)/\sqrt{2} \) and \( A_z = (A_x - iA_y)/\sqrt{2}, A_{\bar{z}} = (A_x + iA_y)/\sqrt{2} \).

We take the potentials to have the following gaussian distributions:

\[ P[V] = \exp \left( -\frac{1}{2g_V} \int \frac{d^2x}{2\pi} V^2 \right) \]  

\[ P[M] = \exp \left( -\frac{1}{2g_M} \int \frac{d^2x}{2\pi} M^2 \right) \]  

\[ P[A] = \exp \left( -\frac{1}{g_A} \int \frac{d^2x}{2\pi} A_z A_{\bar{z}} \right) \]  

The couplings \( g_v, g_m, g_a \) are positive variances of the potentials. If \( A \to iA \), then \( g_a \to -g_a \), and similarly for \( M, V \). Thus negative \( g \)'s can be interpreted as corresponding to imaginary potentials.

Since we are dealing with a free theory, we can use the supersymmetric method for disorder averaging [37]. In the present context this method was studied in [9] [10]. Introducing bosonic ghost partners \( \beta_+, \beta_- \) of the fermions and performing the gaussian integrals one obtains:

\[ S_{\text{eff}} = S_{\text{free}} + \int \frac{d^2x}{2\pi} \left( g_v O_v + g_m O_m + g_a O_a \right) \]  

\[ S_{\text{free}} = \int \frac{d^2x}{2\pi} \left( \bar{\psi}_-\partial_z \psi_+ + \psi_-\partial_z \psi_+ + \bar{\beta}_-\partial_z \beta_+ + \beta_-\partial_z \bar{\beta}_+ \right) \]  

A treatment of the \( c = -1 \) ghost system can be found in [38].

The maximal conserved currents of \( S_{\text{free}} \) are the 8 possible bilinears in the fermions and ghosts:

\[ H = \psi_+ \psi_-, \quad J = \beta_+ \beta_-, \quad J_{\pm} = \beta_{\pm}^2, \quad S_{\pm} = \pm \psi_\pm \beta_\pm, \quad \hat{S}_{\pm} = \psi_\pm \beta_{\mp} \]  

Using the OPE’s:

\[ \psi_+(z)\psi_-(0) \sim \psi_-(z)\psi_+(0) \sim 1/z \]  

\[ \beta_+(z)\beta_-(0) \sim -\beta_-(z)\beta_+(0) \sim 1/z \]  

one finds that the currents satisfy the \( osp(2|2)_{k=1} \) current algebra:
\[ J(z)J_\pm(0) \sim \pm \frac{2}{z} J_\pm, \quad J_+(z)J_-(0) \sim \frac{2k}{z^2} - \frac{4}{z} J \]
\[ J(z)S_\pm(0) \sim \frac{1}{z} S_\pm, \quad J(z)\hat{S}_\pm(0) \sim \pm \frac{1}{z} \hat{S}_\pm \]
\[ H(z)S_\pm(0) \sim \frac{1}{z} S_\pm, \quad H(z)\hat{S}_\pm(0) \sim \mp \frac{1}{z} \hat{S}_\pm \]
\[ J_\pm(z)S_\mp(0) \sim \frac{2}{z} \hat{S}_\pm, \quad J_\pm(z)\hat{S}_\mp(0) \sim -\frac{2}{z} S_\pm \]

\[ S_\pm(z)\hat{S}_\pm(0) \sim \pm \frac{1}{z} J_\pm \]
\[ S_+(z)S_-(0) \sim \frac{k}{z^2} + \frac{1}{z}(H - J) \]
\[ \hat{S}_+(z)\hat{S}_-(0) \sim -\frac{k}{z^2} + \frac{1}{z}(H + J) \]

To better reveal the algebraic structure, we define new couplings

\[ g_v \mathcal{O}^v + g_m \mathcal{O}^m = g_+ \mathcal{O}^+ + g_- \mathcal{O}^- \]  

(4.14)

with

\[ g_\pm = g_v \pm g_m, \quad \mathcal{O}^\pm = (\mathcal{O}^v \pm \mathcal{O}^m)/2 \]  

(4.15)

The perturbing operators can then be written in terms of currents in the following way:

\[ \mathcal{O}^+ = \hat{S}_+\overline{S}_- - \hat{S}_-\overline{S}_+ + \frac{1}{2} \left( J_+\overline{J}_- + J_-\overline{J}_+ \right) \]
\[ \mathcal{O}^- = J\overline{J} - H\overline{H} + S_+\overline{S}_- - S_-\overline{S}_+ \]
\[ \mathcal{O}^a = (J - H)(\overline{J} - \overline{H}) \]

(4.16)

Using the OPE’s (4.13), it is straightforward to compute the RG data from (2.7, 2.9). One finds the non-zero values:

\[ C^{a+} = C^{+a} = C^{++} = -4 \]
\[ C^{a+} = C^{+a} = C^{++} = -2 \]

and

\[ \tilde{C}^{++} = \tilde{C}^{+a} = \tilde{C}^{++} = -2 \]
\[ \tilde{C}^{+a} = \tilde{C}^{+a} = -4 \]

(4.17)

(4.18)

The non-zero \( D \)'s are

\[ D^{a-}_a = D^{a-}_a = D^{--} = -D^{++}_+ = -1 \]

(4.19)

The matrix \( D(g) \) is thus non-diagonal. In a basis \( g = (g_+, g_-; g_a) \):

\[ D(g) = \begin{pmatrix} g_+ & 0 & 0 \\ 0 & -g_- & -g_a \\ 0 & 0 & -g_- \end{pmatrix} \]

(4.20)
After some algebra we find the \( \beta \) functions:

\[
\beta_{g+} = \frac{8g_+ \left(g_+^2(2g_a - g_- + 2) + 2g_-(2 - g_-) + 8g_0\right)}{(4 - g_+^2)(2 - g_-)^2}
\]

\[
\beta_{g-} = \frac{8g_-^2(2 + g_-)^2}{(4 - g_-^2)^2}
\]

\[
\beta_{g_a} = \frac{4 \left((g_+^2 - g_-^2)(16 - g_+^2g_-^2) + 4g_0g_+^2(2 + g_-)(2 - g_-)^2\right)}{(4 - g_-^2)^2(2 - g_-)^2}
\]  

(4.21)

B. Phases of the network model

The \( \beta \) functions (4.21) have a precise phase structure that is not readily apparent from their form. We studied the phase structure by analyzing the behavior in the vicinity of the poles combined with some modest numerical work. Namely, the \( \beta \) function differential equations were solved numerically for \( g(r) \) for a variety of points in each phase.

To begin describing the phase diagram we start with the solutions to \( C + \bar{C} = 0 \). There are two solutions: (i) \( g_+ = -g_- \), \( g_a = 0 \) and (ii) \( g_+ = 0 \). The first solution corresponds to \( g_v = 0, g_{\pm} = \pm g_m \) and the perturbation is thus \( g_m \mathcal{O}^m \) with

\[
\mathcal{O}^m = \left(-JJ + HH - S_+\bar{S}_- + S_-\bar{S}_+ + \hat{S}_+\bar{S}_- - \hat{S}_-\bar{S}_+ + \frac{1}{2}(J_+J_- + J_-J_+)\right)
\]  

(4.22)

\( \mathcal{O}^m \) is built on the Casimir of \( osp(2|2) \), so this corresponds to an \( osp(2|2) \) symmetric manifold. The \( \beta \) function is

\[
\beta_{g_m} = \frac{-8g_m^2}{(2 + g_m)^2}
\]  

(4.23)

When \( g_a \neq 0 \) this line is unstable so it does not play a significant role in the phase diagram. However as we will see, there is one regime that may be attracted to it.

For the other solution \( g_+ = 0 \) with \( g_-, g_a \) arbitrary, the only currents in the perturbation are \( J, H, S_{\pm} \) which generate \( gl(1|1) \) at level \( k = 1 \). The two operators \( \mathcal{O}^-, \mathcal{O}^a \) correspond to the two independent Casimirs of \( gl(1|1) \), reflecting the indecomposability of the adjoint representation. The \( \beta \) function reduces to

\[
\beta_{g_a} = -\frac{4g_a^2}{(2 - g_-)^2}, \quad \beta_{g_-} = 0
\]  

(4.24)

This model was studied in [24] in connection with the disordered XY model and the localization problem of electrons randomly hopping on a lattice with \( \pi \) flux per plaquette [26]. The coupling \( g_+ = 0 \) corresponds to \( g_v + g_m = 0 \) which means that one of the couplings \( g_v, g_m \) is negative. In [24] such negative couplings arose naturally in a different hermitian hamiltonian with twice as many degrees of freedom. In the model we are studying, as we will see, a large regime of couplings, including initially all positive couplings, are attracted to this \( gl(1|1) \) invariant manifold. This is rather unexpected: starting from a hermitian hamiltonian with
positive variances $g$, in some regimes the couplings flow to negative values which correspond to imaginary potentials and non-hermitian hamiltonians.

There exists another $osp(2|2)$ invariant line which does not appear as a solution to $C + \tilde{C} = 0$. It arises due to the automorphism of $osp(2|2)$:

$$J_\pm \to -J_\pm, \quad \tilde{S}_\pm \to -\tilde{S}_\pm$$

which does not change the algebra. Performing this automorphism on the left-moving currents only sends $\mathcal{O}^m \to -\mathcal{O}^v$. Therefore the line $g_+ = g_-, g_a = 0$, corresponding to $g_m = 0$, is also $osp(2|2)$ preserving with the beta function:

$$\beta_{g_v} = \frac{8g_v^2}{(2 - g_v)^2}$$

For $g_v > 0$, this is marginally relevant and is thus a massive theory. It’s an integrable theory and the exact S-matrix was proposed in [39]. In the network model, one phase (called $\tilde{O}$ below) appears to be attracted to this line but with $g_v < 0$, and thus flows to zero, and is massless.

As in the $su(2)$ case the global features of the phase diagram largely, but not completely, follow from the behavior near the poles $g_+, g_- = \pm 2$. Consider first the vicinity of the pole $(g_+, g_-) = (2, 2)$. Letting $g_\pm = 2 + \epsilon_\pm$, one has

$$\beta_{\epsilon_+} \approx \frac{-64g_a}{\epsilon_+^2 - 4}, \quad \beta_{\epsilon_-} \approx \frac{32\epsilon_+^2}{\epsilon_-^2}, \quad \beta_{g_a} \approx \frac{16(1 + g_a)}{\epsilon_+^2} - \frac{1}{16\epsilon_-^2}$$

If $g_a > 0$, then $\epsilon_+$ is attracted to zero, whereas $\epsilon_-$ always grows. Here, $g_+$ flows to 2 and $g_a, g_-$ flow to infinity if $g_- > 2$. Since $g_+/g_-$ flows to zero, we interpret this phase as flowing along the $gl(1|1)$ invariant line.

The behavior around $(g_+, g_-) = (\pm 2, -2)$ is different. Here one finds:

$$\beta_{\epsilon_+} \approx \frac{-4g_a}{\epsilon_+}, \quad \beta_{\epsilon_-} \approx \frac{2\epsilon_-^2}{\epsilon_+^2}, \quad \beta_{g_a} \approx \frac{4\epsilon_-}{\epsilon_+}$$

Again when $g_a > 0$ one is attracted to $\epsilon_+ = 0$. However when $\epsilon_- = 0, \beta_{\epsilon_-} = 0$.

A simplifying feature is that there are no poles in $g_a$. For each phase, given initial values for $g_\pm$, we find that for some value $g_{a0}$ which depends non-universally on $g_\pm$, then $g_a > g_{a0}$ and $g_a < g_{a0}$ are distinct phases. Examining $\beta_{g_+}$ near the poles $g_+ = \pm 2$, one finds

$$g_{a0} \approx (g_-^2 - 4)/8 \quad \text{near } g_+ = \pm 2$$

For large $g_+$, $\beta_{g_+} \approx -8g_+(2g_a - g_-)/g_-^2$, thus

$$g_{a0} \approx g_-/2 \quad \text{for } g_+ \gg 1$$

We list the regions with distinct behavior below. The phase diagrams are shown in figures 2, 3. For each phase we determine the density of states exponent. To study the average density of states $\rho(E)$ we shift $H \to H - E$ leading to a coupling $E\Phi_E$ in the effective action with
\[ \Phi_E = \overline{\psi}_- \psi_+ + \psi_- \overline{\psi}_+ + \beta_- \beta_+ + \beta_- \overline{\beta}_+ \]  

(4.31)

The density of states is proportional to the one-point function of \( \Phi_E \):

\[ \rho(E) \propto \langle \Phi_E \rangle \]  

(4.32)

Let \( \Gamma_E \) denote the scaling dimension of the operator \( \Phi_E \) in the IR. Then since \( E \) has scaling dimension \( 2 - \Gamma_E \),

\[ \rho(E) \propto E^{\Gamma_E/(2-\Gamma_E)}, \quad \text{as } E \rightarrow 0 \]  

(4.33)

\[ g_a > g_{a0} \] phases

(i) \( dXY^\pm \) phases. Here \( g_+ \) flows to \( \pm 2 \) after a finite scale transformation, and \( g_-, g_a \) flow to \( +\infty \). This is similar to the D region of anisotropic \( su(2) \). The ratio \( g_+/g_- \rightarrow 0 \), so considering \( g_+ \) as effectively zero, one interpretation of the flow is to the coset \( osp(2|2)_1/gl(1|1)_1 \). As in the \( su(2) \) case this is not a true fixed point since the flow cannot be extended to arbitrarily large length scales. The stress tensor for the \( gl(1|1)_1 \) current algebra conformal field theory has a structure that parallels the structure of \( O^-, O^a \). Namely, the stress tensor is an affine-Sugawara construction built on the sum of the two independent Casimirs \([40]\)

\[ T_{gl(1|1)} = -\frac{1}{2k} (J^2 - H^2 + S_+ S_- - S_- S_+) + \frac{1}{2k^2} (J - H)^2 \]  

(4.34)

This stress tensor has \( c = 0 \) so the coset \( osp(2|2)_1/gl(1|1)_1 \) also has \( c = 0 \). The fermions \( \psi_\pm \) have conformal dimension \( \Delta = 1/2 \) with respect to the \( gl(1|1) \) thus the \( gl(1|1)_1 \) dimension of \( \Phi_E \) is 1. This implies \( \Gamma_E = 0 \) and a constant density of states near \( E = 0 \). Though the algebra \( gl(1|1)_1 \) is smaller than \( osp(2|2)_1 \), the \( gl(1|1)_1 \) dimensions of fields are the same as for \( osp(2|2)_1 \). One can in fact construct a level-1 representation for both of these current algebras using the same number of fields: two bosons and a complex fermionic scalar. (See e.g. [24].) Thus \( osp(2|2)_1/gl(1|1)_1 \) is empty and this would be a massive phase.

(ii) \( O^\pm \) phases. This phase is characterized by \( (g_+, g_-) \) flowing to the pole \( (\pm 2, -2) \) while \( g_a \) flows to zero. Unlike previous examples, here one is not attracted to the isotropic lines \( g_+ = \pm g_- \) before reaching the pole since the line is unstable. The fate of the flow once it reaches the pole is again delicate. Based on the example of \( su(2) \) it seems most likely that this flow spills into the \( Q^\pm \) phases. However here we do not have the duality arguments to support this.

(iii) \( Q^\pm \) phases. In this region one flows to the poles at \( g_+ = \pm 2 \) after a finite RG time but unlike the \( dXY^\pm \) phases, numerical integration indicates that the other couplings \( g_-, g_a \) do not flow to infinity but rather to some finite non-universal values as one approaches the pole. Since none of the couplings are flowing to infinity, we cannot interpret this as a coset. This phase clearly requires further investigation.
FIG. 2. Phase Diagram of the network model with \( g_a > g_a^0 \). Phase boundaries are solid heavy lines. Shaded circles are fixed points. The cone opening to the right is the physical regime of the network model.

\( g_a < g_a^0 \) phases

(i) \( PSL_g \) phase. Here \( g_+ \to 0 \), but \( g_- \) flows to a fixed, non-universal value. \( g_a \) flows very slowly to \(-\infty\). In this phase \( g_{a0} \approx 0 \). Unlike the flows to the poles, in this case the flows can be run to arbitrarily large length scales. For example

\[
(g_+, g_-, g_a) = (1, 1, -1) \rightarrow (0, 1.1508434\ldots, -\infty)
\]

(4.35) as \( r \to \infty \). In this phase we are again on the \( gl(1|1) \) invariant manifold but in contrast to the \( dXY \) phase there is a line of fixed points corresponding to the value of \( g_- \). This is consistent with the \( \beta \)eta function (4.24) since \( \beta g_- = 0 \).
FIG. 3. Phase Diagram of the network model with $g_a < g_{a0}$. Phase boundaries are solid heavy lines. The heavy dashed line is a line of fixed points.

The massive decoupled subalgebra $\mathcal{H}$ is then generated by only $H - J$. However it appears inconsistent to divide only by $H - J$. The current $H - J$ is not primary with respect to the stress tensor $T = (J - H)^2/4$, i.e. $T(z)(H - J)(0) \sim 0$. A consistent conformal embedding is based on the subalgebra $\mathcal{H} = u(1) \otimes u(1)$ with stress-tensor

$$T_{\mathcal{H}} = \frac{1}{2} (H^2 - J^2)$$

The fixed point is then:

$$PSL_g \text{ phase : IR fixed point } = \frac{osp(2|2)}{u(1) \otimes u(1)}$$

with $c = -2$. This is a massless phase, since the above coset is not empty. In the formal limit $g_- \to \infty$ the above fixed point is the same as for the $dXY$ phases. As described in [24], dividing by $u(1) \otimes u(1)$ leaves the free field theory of a complex fermionic scalar, with action

$$S = \int \partial_\mu \chi^\dagger \partial_\mu \chi$$

This theory is equivalent to a $(\xi, \eta)$ fermionic system of conformal scaling dimension $(0, 1)$ [38]. It is perhaps the simplest logarithmic conformal field theory [42]. Interestingly the
latter theory was used to describe the dense phase of polymers by Saleur [13]. This theory is also closely related to the $PSL(1|1)$ sigma model introduced into the context of quantum Hall transitions by Zirnbauer [41]. The coupling $g_-$ naively corresponds to the exactly marginal direction $\delta S = g_- S$ where $S$ is given in (4.38), so that $g_-$ is similar to a radius of compactification for a free boson. However the modular invariant partition functions at $c = -2$ do not have a continuous parameter, but rather are related to Coulomb gas partition functions at certain discrete radii [44]. This suggests that as $g_a \to \infty$, $g_-$ can only take discrete values in order to lead to a modular invariant partition function. The spectrum of anomalous dimensions is then the same as in [13]. In particular the twist fields of this theory have fixed dimension $-1/4$ and $3/4$ [43] [44].

(ii) $gX^\pm$ phase. Here $g_+ \to \infty$, $g_a \to -\infty$, and $g_-$ flows to a finite non-universal constant as $r$ goes to $\infty$. For example:

$$(g_+, g_-, g_a) = (10, 5, 1) \to (\infty, 5.8829077.., -\infty)$$

(4.39)

Though $g_a$ flows to $-\infty$, the ratio $g_a/g_+$ flows to zero. Since $g_-/g_+$ also flows to zero, we interpret this as $g_a = g_- = 0$.

The coupling that goes to infinity, $g_+$, couples the currents $\hat{S}_\pm, J_\pm$. These do not form a closed subalgebra, and for this reason this phase is more difficult to comprehend and requires some speculation. It could simply be a massive phase corresponding to a closed subalgebra, and for this reason this phase is more difficult to comprehend and requires some speculation. It could simply be a massive phase corresponding to $osp(2|2)_1/osp(2|2)_1$ since the above currents close on the whole of $osp(2|2)$, as in the massive sine-Gordon phase of $su(2)$ (region E). Alternatively let us suppose that only some of the currents are set to zero by $g_+$ going to $\infty$. Since the commutator of $J_\pm$ with $\hat{S}_\pm$ closes on $gl(1|1)$ currents $S_\pm$, and the $gl(1|1)$ coupling of $g_-$ is not flowing to infinity, we cannot consistently set both $\hat{S}_\pm$ and $J_\pm$ to zero. To distinguish $\hat{S}$ and $J_\pm$ one may need some further $osp(2|2)$ symmetry breaking; this could come from the fact that the operator $\hat{\Phi}_E$ breaks $osp(2|2)$. Two subalgebras of $osp(2|2)$ involving the above currents correspond to another $gl(1|1)$ generated by $(\hat{S}_\pm, J, H)$ and $su(2) \otimes u(1)$ generated by $(J_\pm, J, H)$. Let us suppose the $gl(1|1)$ is set to zero by $g_+ \to \infty$. As for the $dXY$ phase, the coset $osp(2|2)_1/gl(1|1)_1$ is empty and this would be a massive phase.

Let us consider the other possibility. The currents $J, J_\pm$ generate an $su(2)$ at level $k = -1/2$. To see this, let $J \to 2J$, $J_\pm \to \pm 2\sqrt{2}J^\pm$. Then the new currents satisfy the OPE (3.11) with $k = -1/2$. The coset in this case is $osp(2|2)/su(2)_{-1/2} \otimes u(1)$. The central charge of $su(2)_k$ is $c = 3k/(k + 2)$ which gives $c = -1$ for $k = -1/2$. Since $c = 1$ for the $u(1)$, the above coset has $c = 0$. The left-moving conformal dimension of primary fields of spin $j$ in $su(2)_k$ current algebra is

$$\Delta^{(j)}_k = \frac{j(j + 1)}{k + 2}$$

which equals $1/2, 4/3, 5/2, ..$ for $k = -1/2$. The ghost fields $\beta_\pm$ transform in the spin $1/2$ representation. Thus in the IR the operator $\hat{\beta}_+\hat{\beta}_-$ has dimension $\Delta^{osp(2|2)} - \Delta^{su(2)} = 0$. Dividing by $u(1)$ also leads to dimension zero for the fermion part of $\Phi_E$. Thus again $\Gamma_E = 0$ and $\rho(E)$ is a constant. Since the $su(2)_{-1/2}$ is built directly from the ghost fields with the ghost fields in the spin $1/2$ representation, the $osp(2|2)_1$ theory appears to be equivalent to $su(2)_{-1/2} \otimes u(1)$ if the only primary field of $su(2)_{-1/2}$ is $j = 1/2$. The coset $osp(2|2)_1/su(2)_{-1/2} \otimes u(1)$ is thus empty and this is also a massive phase.
C. Physical regime of the network model

The physical regime of the network model is $g_v, g_m, g_a$ all positive. In the $(g_+, g_-)$ plane this regime of couplings corresponds to the intersection of $g_+ > g_-$ and $g_+ > -g_-$, which is the $90^\circ$ cone symmetric about the $g_+$ axis. For reasons we do not yet understand, couplings can flow in or out of this cone. Let us suppose that the initial couplings are in the cone, and furthermore that initially $g_a > 0$. The phases $dXY^+, Q^+$ and $O^+$ are easily accessible since $g_a$ is positive and can easily be chosen greater than $g_a^0$.

Consider now the $g_a < g_a^0$ phases. For $PSL_g$, $g_a^0$ is $(g_2 - 4)/8$ near $g_+ = 2$. But since $g_2 < 4$ in the cone, this phase appears inaccessible to the network model. If a negative $g_a$ turns out to be physically sensible, perhaps for the reasons described in [47], then this phase can be realized. On the other hand, $g_a < (g_2 - 4)/8$ is easily satisfied with a positive $g_a$ in the cone for the $gX^+$ phase. An example is equation (4.39).

In summary, the physical regime of the network model can flow to one of 4 different phases $dXY^+$, $Q^+$, $gX^+$, and $O^+$. Initially weak couplings flow to the phase $Q^+$, which, as stated above, cannot be interpreted with our hypotheses since none of the couplings are flowing to infinity.

In the above phases $dXY^+$ and $gX^+$, the density of states is not critical, i.e. $\rho(E) \propto E^0$ since $\Gamma_E = 0$. This is in accordance with the expectation that the disordered 1-copy theory is not critical. The conventional wisdom is that one needs to study the 2-copy theory and compute disorder averages of the product of retarded and advanced Green functions. This can be done using the methods of this paper. Since the 1-copy theory is contained in the N-copy version, one expects on physical grounds that the beta functions are the same for all $N$. We checked that this turns out to be the case and is a consequence of the zero super-dimension of $osp(2N|2N)$. Though the beta functions are the same as we described for $N = 1$, the fixed points are different since the current algebras involved are different, in particular, $osp(2|2)_1$ is replaced by $osp(2N|2N)_1$. This will be described in a separate publication.

Though the localization length exponent $\nu$ is currently beyond our understanding, let us examine the possible exponents for the dense polymer phase $(PSL_g)$. Tuning through the critical point generally corresponds to a perturbation

$$\delta S = \epsilon \int d^2x \Phi_\epsilon$$

with $\epsilon \approx 0$ for some operator $\Phi_\epsilon$. Let $\Gamma_\epsilon$ denote the scaling dimension of $\Phi_\epsilon$ at the infra-red fixed point. The mass dimension of $\epsilon$ is then $2 - \Gamma_\epsilon$. As $\epsilon \to 0$, there is thus a diverging length scale

$$\xi \propto \epsilon^{-\nu}, \quad \nu = 1/(2 - \Gamma_\epsilon) \quad (4.40)$$

The fields in the dense polymer theory all have dimension $n/16$ plus an integer. In particular there is a field of dimension $\Gamma = 25/16$, corresponding the conformal dimension $h = 25/32$, which is a descendent $(1+)$ of the field which is the dense polymer 1-leg operator ($h = -3/32$) times a twist field ($h = -1/8$) from the $\chi$-theory sector of another copy. Note that this field does not exist in the 1-copy theory. Such a field leads to $\nu = 16/7$. Within this scheme, this appears to be the closest one can come to the numerical value $2.35 \pm 0.03$ [19]. Unfortunately we have no further arguments supporting the significance of this operator.
V. CONCLUSION

Under the hypotheses outlined above we have interpreted most of the RG flows based on the all-orders $\beta$eta functions proposed in [14]. Much remains to be further clarified, in particular the flows that are attracted to poles in the $\beta$eta function after a finite scale transformation need further investigation. In any case, this work shows that disordered fermions in $2D$ at strong coupling can have a rich phase structure. In the physical regime of the network model the important phases that we could understand as cosets appear to have a constant density of states, in accordance with the conventional understanding. In addition, our analysis of anisotropic $su(2)$ could have implications for Kosterlitz-Thouless transition physics at strong coupling.

The only certainly massless phase we found for the network model is the $c = -2$ conformal field theory $osp(2|2)_1/u(1) \otimes u(1)$, which is known to correspond to dense polymers. There exists a classical percolation picture for the quantum Hall transition [48], and it is known that percolation and dilute polymers are closely related $c = 0$ conformal field theories [13]. Our work seems to suggest that with strong disorder the classical dilute polymer theory flows to a dense polymer phase.

We considered the simplest case of one copy of Dirac fermion. In the theory of disorder, for the same model one needs to study more copies to compute averages of products of correlation functions, and this generally has multifractal behavior. For $N$ copies the short distance unperturbed theory has $osp(2N|2N)_1$ current algebra symmetry. The $N$-copy version of the network model, where one expects non-trivial exponents, will be studied in a forthcoming publication. The scheme described in this paper can lead to a classification of disordered critical points that parallels the classification of sub-current-algebras $H_k$ of $osp(2N|2N)_1$. For this, the dictionary in [50] is useful. It would be interesting to compare this classification with the classification based on sigma models [51]. The latter classification is based on discrete symmetries such as time-reversal, so it is not as strong as our classification of the actual critical points.

It would be very interesting to perform more extensive numerical simulations of the network model that vary the relative strengths of the types of disorder and thereby see the phases predicted in this paper. For instance the $gX^+$ phase is characterized by the randomness in the flux per plaquette and the tunneling dominating over the randomness in the individual link phases.

Though the models we discussed are not integrable for general anisotropic couplings, under the RG they can flow to isotropic current-current interactions and these are generally thought to be integrable$^2$.

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$^2$Note added in proof: The $g \rightarrow 1/g$ duality of the $su(2)$ model can be extended to the network model and this leads to a resolution of the flows toward the poles. This will be reported on in [33].
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