Elementary proof of Jordan-Kronecker theorem

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Abstract

In this paper we prove the Jordan-Kronecker theorem which gives a canonical form for a pair of skew-symmetric bilinear forms on a finite-dimensional vector space over an algebraically closed field.

1 Introduction

The Jordan–Kronecker theorem gives a canonical form for a pair of skew-symmetric bilinear form on a finite-dimensional vector space over an algebraically closed field. This theorem from linear algebra has recently found various applications in various fields of mathematics (see, for example [1], [3]). The proof of the Jordan–Kronecker theorem can be found in [2] and [4]. In this paper we give a simpler proof of this theorem.

Throughout the paper we assume that all vector spaces are finite dimensional and the underlying field has characteristic \( \neq 2 \).

**Theorem 1** (Jordan–Kronecker). Let \( A \) and \( B \) be skew-symmetric bilinear forms on a vector space \( V \) over a field \( K \). If the field \( K \) is algebraically closed, then there exists a basis of the space \( V \) such that the matrices of both forms \( A \) and \( B \) are block-diagonal matrices:

\[
A = \begin{pmatrix}
A_1 & A_2 \\
& \ddots & \ddots \\
& & A_k
\end{pmatrix}, \quad
B = \begin{pmatrix}
B_1 & B_2 \\
& \ddots & \ddots \\
& & B_k
\end{pmatrix}
\]

where each pair of corresponding blocks \( A_i \) and \( B_i \) is one of the following:

1. Jordan block with eigenvalue \( \lambda \in \mathbb{K} \)

\[
A_i = \begin{pmatrix}
0 & & & \lambda & 1 \\
-\lambda & 1 & \ddots & & \ddots \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & -1 & \lambda \\
-1 & -\lambda & \ddots & \ddots & 0
\end{pmatrix}, \quad
B_i = \begin{pmatrix}
0 & & & 1 & 1 \\
-1 & 1 & \ddots & & \ddots \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & -1 \\
-1 & \ddots & \ddots & -1 & 0
\end{pmatrix}
\]
2. Jordan block with eigenvalue $\infty$

$$A_i = \begin{pmatrix}
0 & 1 & 1 \\
-1 & 0 & 0 \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
\end{pmatrix} \quad B_i = \begin{pmatrix}
0 & 0 & 1 \\
-1 & 0 & 0 \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
\end{pmatrix}$$

3. Kronecker block

$$A_i = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
\end{pmatrix} \quad B_i = \begin{pmatrix}
0 & 0 & 1 \\
-1 & 0 & 0 \\
& & \\
& & \\
& & \\
& & \\
& & \\
\end{pmatrix}$$

A Kronecker block is a $(2k + 1) \times (2k + 1)$ block, where $k \geq 0$. In particular, if $k = 0$, then $A_i$ and $B_i$ are two $1 \times 1$ zero matrices

$$A_i = (0) \quad B_i = (0)$$

**Remark 1.** It is easy to prove that the Jordan–Kronecker form of two forms $A$ and $B$ is unique up to the order of blocks.

**Remark 2.** If $e_1, \ldots, e_k, f_0, \ldots, f_{k-1}$ is the basis of a Jordan block with eigenvalue $\infty$, then in the basis $f_0, e_1, f_1, \ldots, f_{k-1}, e_k$ the matrices of the forms are

$$A_i = \begin{pmatrix}
0 & -1 \\
1 & 0 \\
0 & -1 \\
& & \\
& & \\
& & \\
\end{pmatrix} \quad B_i = \begin{pmatrix}
0 & 0 & 1 \\
-1 & 0 & 0 \\
& & \\
& & \\
& & \\
& & \\
\end{pmatrix}$$

If $e_1, \ldots, e_k, f_0, f_1, \ldots, f_k$ is the basis of a Kronecker block, then in the basis $f_0, e_1, f_1, e_2, f_2, \ldots, e_k, f_k$ the matrices of the forms are

$$A_i = \begin{pmatrix}
0 & -1 \\
1 & 0 \\
& & \\
& & \\
& & \\
\end{pmatrix} \quad B_i = \begin{pmatrix}
0 & 0 & 1 \\
-1 & 0 & 0 \\
& & \\
& & \\
& & \\
\end{pmatrix}$$
2 Self-adjoint operator on a symplectic space.

First, let us consider the case when one of the forms in nondegenerate. (Without loss of generality this is form $B$). Let us restate the problem.

Recall that any bilinear function $B : V \times V \to \mathbb{K}$ defines a map $B : V \to V^*$ given by the formula

$$\langle Bu, v \rangle = B(u, v),$$

where $\langle Bu, v \rangle$ is the value of a covector $Bu$ on a vector $v$. If a bilinear form is nondegenerate $\ker B \neq 0$, then it defines an isomorphism between the given space and its dual $V \cong V^*$.

Put $P = B^{-1} : V \to V$. The operator $P$ is self-adjoint with respect to both forms $A$ and $B$.

$$A(Pu, v) = A(u, Pv), \quad B(Pu, v) = B(u, Pv)$$

In the sequel we need the following simple assertions.

**Assertion 1.** If $P$ is a self-adjoint operator on a symplectic space $V$, then the orthogonal complement of an invariant subspace $W \subset V$ is invariant. That is

$$P W \subset W \Rightarrow PW \subset W^\perp$$

**Proof.** For any $v \in W^\perp$ we have $B(u, Pv) = B(Pu, v) = 0$, since $Pu \in W$. \hfill \Box

**Assertion 2.** Let $P$ be a self-adjoint operator on a symplectic space $(V, \omega)$. Then for any vector $v \in (V, \omega)$ all vectors $v, Pv, \ldots, P^i v, \ldots$ are pairwise orthogonal.

**Proof.** Evidently, $B(P^i v, P^j v) = B(P^{i+j} v, v) = B(v, P^{i+j} v) = 0$. \hfill \Box

Now Theorem 1 can be restated as follows.

**Theorem 2.** For any self-adjoint operator $P : V^{2n} \to V^{2n}$ on a symplectic space $(V^{2n}, B)$ over an algebraically closed field $\mathbb{K}$ there exists a basis of $V^{2n}$ such that the matrix $P$ of the operator $P$ and the matrix $B$ of the form $B$ are block-diagonal matrices

$$P = \begin{pmatrix} P_1 & & & \\ & P_2 & & \\ & & \ddots & \\ & & & P_k \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_k \end{pmatrix}$$

Each pair of blocks $P_i$ and $B_i$ has the form

$$P_i = \begin{pmatrix} \lambda & & & \\ & -\lambda & & \\ & & \ddots & \\ & & & -\lambda \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & 0 & \\ & & & 0 & 1 \end{pmatrix}$$

**Proof.** Theorem 2. The proof is in two steps.
Step 1. First, let us prove the statement when \( P \) is a nilpotent operator \( P^n = 0 \).

Let us show how to extract one block. Suppose that the degree of the operator \( P \) is \( m \), that is
\[
P^m = 0, \quad P^{m-1} \neq 0
\]
Take an arbitrary vector \( e_1 \in V \) such that \( P^{m-1} e_1 \neq 0 \). Let \( e_i = P^{i-1} e_1 \). Then \( \langle e_1, \ldots, e_n \rangle \) is an isotropic subspace.

Since the form \( B \) is nondegenerate on \( V \) there exists a vector \( f_n \in V \) such that
\[
B(e_i, f_n) = \delta^i_n
\]
The existence of the vector \( f_n \) easily follows from the following simple assertions from linear algebra.

Assertion 3. Any isotropic subspace is contained in a Lagrangian subspace.

Assertion 4. Any basis \( e_1, \ldots, e_n \) of a Lagrangian subspace \( L \subset (V, \omega) \) can be extended to a symplectic basis \( e_i, f_j \) of the space \( (V, \omega) \)
\[
\omega(e_i, f_j) = \delta^i_j
\]
Put \( f_i = P^{n-i} f_n \). Then \( e_i, f_j \) is a basis of a Jordan block. It is easy to see that
\[
B(e_i, e_j) = 0, \quad B(f_i, f_j) = 0
\]
(this is assertion 2). It is also easy to see that
\[
B(e_i, f_j) = \delta^i_j
\]
Indeed, \( B(e_i, f_j) = B(e_i, P^{n-i} f_n) = B(P^{n-i} e_i, f_n) = B(e_{n+i-j}, f_n) = \delta^i_{n+i-j} = \delta^j_i \).

It means that vectors \( e_i, f_j \) are linearly independent. In the basis \( e_i, f_j \) the restrictions of the form \( B \) and the operator \( P \) to the space \( \langle e_i, f_j \rangle \) have matrices
\[
P = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0 & 1 \\
0 & \cdots & 0 & \cdots & 0
\end{pmatrix}, \quad Q = \begin{pmatrix}
0 & E \\
-E & 0
\end{pmatrix}
\]

Step 2. General case. The space \( V \) decomposes into the sum of generalized eigenspaces of the operator \( P \)
\[
V = \bigoplus_{\lambda \in \mathbb{K}} V^\lambda
\]
Recall that a generalized eigenspace with eigenvalue \( \lambda \) consists of all vectors \( v \in V \) such that \( (P - \lambda E)^m = 0 \) for some natural \( m \in \mathbb{N} \).
Assertion 5. Generalized eigenspaces are orthogonal with respect to the form $B$

\[ V^\lambda \perp_{B} V^\mu, \quad \lambda \neq \mu \]

Proof. If $\lambda \neq \mu$, then the restriction of the operator $(P - \lambda E)$ to $V^\mu$ is nondegenerate. Hence for any vector $v \in V^\mu$ the vector $w = (P - \lambda E)^{-1}v$ such that $(P - \lambda E)w = v$ is well-defined.

For any $e_\lambda \in V^\lambda, e_\mu \in V^\mu$ we have

\[
B(e_\lambda, e_\mu) = B(e_\lambda, (P - \lambda E)(P - \lambda E)^{-1}e_\mu) = B((P - \lambda E)e_\lambda, (P - \lambda E)^{-1}e_\mu) \\
= \cdots = B((P - \lambda E)^m e_\lambda, (P - \lambda E)^{-m}e_\mu) = 0
\]

To conclude the proof, it remains to apply step 1 to the restriction of the operator $P - \lambda E$ to the corresponding generalized eigenspace for each eigenvalue $\lambda$.

Theorem 2 is proved.

3 Proof of Jordan–Kronecker theorem.

Proof. Jordan–Kronecker theorem. If the form $B$ is nondegenerate, then everything is proved (see Theorem 2). Suppose that $\text{Ker} B \neq 0$.

Let us show how to extract one block. That block will be either a Kronecker block or a Jordan block with eigenvalue $\infty$. We need to do the following:

1. To decompose the space into a sum of subspaces orthogonal w.r.t. $A$ and $B$

\[ V = V_m \oplus W_m, \quad V_m \perp_{A,B} W_m \]

2. To find a basis $e_i, f_j$ of the space $V_m$ such that

\[ A(e_i, f_j) = \delta_{j+1}^i, \quad B(e_i, f_j) = \delta_j^i, \]

and all other pairs of basic vectors are orthogonal w.r.t. $A$ and $B$.

We construct a block in several steps. On odd steps we search for vectors $f_i$ and on even steps we try to find vectors $e_j$. If we can not find a vector, then we have found a block.

Step 1. Take an arbitrary vector $f_0 \in \text{Ker} B$ and any additional subspace $W_1 \subset V$

\[ \langle f_0 \rangle + W_1 = V \]

Step 2. Take a vector $e_1$ such that

\[ A(e_1, f_0) = 1 \]

Put $V_2 = \langle e_1, f_0 \rangle$ and $W_2 = V_2^\perp$
Step $2k + 1$. After $2k$ steps we have constructed subspaces $V_{2k}, W_{2k}$ and a basis $f_0, e_1, f_1, \ldots, f_{k-1}, e_k$ of the space $V_{2k}$ such that

(a) All vectors $e_i, f_j$ are orthogonal to the space $W_{2k}$ w.r.t. the form $A$

$$V_{2k} \perp_A W_{2k}$$

(b) All vectors except maybe for $e_k$ are orthogonal to the space $W_{2k}$ w.r.t. the form $B$

$$V_{2k-1} \perp_B W_{2k}$$

(c) In the basis $f_0, e_1, f_1, \ldots, f_{k-1}, e_k$ the restrictions of forms have matrices

$$A|_{V_{2k}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ & & 0 & -1 \\ & & 1 & 0 \\ & & & \ddots \\ & & & & 0 & -1 \\ & & & & 1 & 0 \end{pmatrix}, \quad B|_{V_{2k}} = \begin{pmatrix} 0 & 0 & 1 \\ & \ddots & \ddots \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}$$

Take a vector $f_k \in W_{2k}$ such that

$$B(e_k, f_k) = 1.$$

Put $V_{2k+1} = V_{2k} \oplus \langle f_k \rangle$ and $W_{2k+1} = V_{2k+1}^{\perp_B} \cap W_{2k}$

Step $2k + 2$. After previous steps we have found subspaces $V_{2k+1}, W_{2k+1}$ and a basis $f_0, e_1, f_1, \ldots, e_{k-1}, f_k$ of $V_{2k+1}$ such that

(a) All vectors $e_i, f_j$ are orthogonal to the subspace $W_{2k+1}$ w.r.t. the form $B$

$$V_{2k+1} \perp_B W_{2k+1}$$

(b) All vector except maybe for $f_k$ are orthogonal to the subspace $W_{2k+1}$ w.r.t. the form $A$

$$V_{2k} \perp_A W_{2k+1}$$

(c) In the basis $f_0, e_1, f_1, \ldots, e_k, f_k$ the restrictions of forms have matrices

$$A_{V_{2k+1}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ & \ddots \\ & & 0 & -1 \\ & & 1 & 0 \\ & & & 0 \end{pmatrix}, \quad B_{V_{2k+1}} = \begin{pmatrix} 0 & 0 & 1 \\ & \ddots & \ddots \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}$$
Take a vector $e_{k+1} \in W_{2k}$ such that

$$A(e_{k+1}, f_k) = 1.$$ 

Put $V_{2k+2} = V_{2k+1} \oplus \langle e_{k+1} \rangle$ and $W_{2k+2} = V_{2k+1}^* \cap W_{2k+1}$.

If the algorithm stopped on the $2k$-th step (we could not find a vector $f_{k+1}$), then $V_{2k}$ is a Jordan block with eigenvalue $\infty$ and if we stopped on the $(2k + 1)$-th step (there is no vector $e_{k+1}$), then $V_{2k+1}$ is a Kronecker $(2k + 1) \times (2k + 1)$ block.

Indeed, it is not hard to see that $V_i$ and $W_i$ form two sets of nested subspaces

$$V_1 \subset V_2 \subset V_3 \subset \ldots$$

$$W_1 \supset W_2 \supset W_3 \supset \ldots$$

After $2k$ steps in the basis $f_0, e_1, f_1, \ldots, f_{k-1}, e_k$ (and any additional basis of the space $W_{2k}$) the matrix of the form $A$ is

$$A = \begin{pmatrix} 0 & -1 & \ldots & 0 \\ 1 & 0 & & \\ \vdots & & & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

where $A_{2k}$ is the matrix of restriction of the form $A$ to $W_{2k}$.

Analogously, after $(2k + 1)$ steps in basis $f_0, e_1, f_1, \ldots, e_k, f_k$ (and an arbitrary basis of $W_{2k+1}$) the matrix of the form $B$ is

$$B = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ -1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 1 & \ldots & 0 \\ 0 & -1 & \ldots & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

where $B_{2k+1}$ is the matrix of restriction of the form $B$ to $W_{2k+1}$.

This concludes the proof.

**Remark 3.** Actually there is no need to prove Theorem 2. If the field $K$ is algebraically closed, then there always exists a degenerate linear combination of forms $A + \lambda B$ for some $\lambda \in K \cup \{\infty\}$. If the field $K$ is not algebraically closed, then Theorem 2 remains true if and only if all eigenvalues of $P$ lie in $K$, or equivalently if the characteristic polynomial of the operator $P$ splits into linear factors over $K$ (compare to the Jordan normal form theorem). Kronecker blocks and Jordan blocks with eigenvalue $\infty$ can be extracted over any field with characteristic $\neq 2$ (we did not use algebraic closeness of the field in that part of the proof).
References

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