Bijectons for inversion sequences, ascent sequences and 3-nonnesting set partitions

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Abstract. Set partitions avoiding \( k \)-crossing and \( k \)-nesting have been extensively studied from the aspects of both combinatorics and mathematical biology. By using the generating tree technique, the obstinate kernel method and Zeilberger’s algorithm, Lin confirmed a conjecture due independently to the author and Martinez-Savage that asserts inversion sequences with no weakly decreasing subsequence of length 3 and enhanced 3-nonnesting partitions have the same cardinality. In this paper, we provide a bijective proof of this conjecture. Our bijection also enables us to provide a new bijective proof of a conjecture posed by Duncan and Steingrímsson, which was proved by the author via an intermediate structure of growth diagrams for 01-fillings of Ferrers shapes.

Key words: inversion sequence, ascent sequence, pattern avoiding, 3-nonnesting set partition.

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1 Introduction

Set partitions avoiding \( k \)-crossing and \( k \)-nesting have been extensively studied from the aspects of both combinatorics and mathematical biology; see [5, 6, 11] and the references therein. The objective of this paper is to provide a bijective proof of a conjecture due independently to the author [19] and Martinez-Savage [16], which was recently confirmed by Lin [13] using the generating tree technique, the obstinate kernel method [2] and Zeilberger’s algorithm [17]. Our bijection also enables us to provide a new bijective proof of a conjecture posed by Duncan and Steingrímsson [8], which was proved by the author [19] via an intermediate structure of growth diagrams for 01-fillings of Ferrers shapes [11] and [12]. Let us first give an overview of the notation and terminology.

A sequence \( x = x_1 x_2 \cdots x_n \) is said to be an inversion sequence of length \( n \) if it satisfies \( 0 \leq x_i < i \) for all \( 1 \leq i \leq n \). Inversion sequences of length \( n \) are in easy bijection with permutations of length \( n \). An inversion sequence \( x_1 x_2 \cdots x_n \) can be obtained from any permutation \( \pi = \pi_1 \pi_2 \cdots \pi_n \) by setting \( x_i = \{ j \mid j < i \text{ and } \pi_j > \pi_i \} \).
Given a sequence of integers \( x = x_1 x_2 \cdots x_n \), we say that the sequence \( x \) has an ascent at position \( i \) if \( x_i < x_{i+1} \). The number of ascents of \( x \) is denoted by \( \text{asc}(x) \). A sequence \( x = x_1 x_2 \cdots x_n \) is said to be an ascent sequence of length \( n \) if it satisfies \( x_1 = 0 \) and \( 0 \leq x_i \leq \text{asc}(x_1 x_2 \cdots x_{i-1}) + 1 \) for all \( 2 \leq i \leq n \). Ascent sequences were introduced by Bousquet-Mélou et al. [3] in their study of \((2 + 2)\)-free posets, which are closely connected to unlabeled \((2 + 2)\)-free posets, permutations avoiding a certain pattern, and a class of involutions introduced by Stoimenow [18]. We call an ascent sequence with no two consecutive equal entries a primitive ascent sequence.

Pattern avoiding permutations have been extensively studied over last decade. For a thorough summary of the current status of research, see Bóna’s book [1] and Kitaev’s book [10]. Analogous to pattern avoidance on permutations, Corteel-Martinez-Savage-Weselcouch [7] and Mansour-Shattuck [15] initiated the study of inversion sequences avoiding certain patterns. Pattern avoiding inversion sequences are closely related to Catalan numbers, large Schröder numbers, Euler numbers and Baxter numbers (see [7], [9], [13] and [15]). In their paper [8], Duncan and Steingrímsson studied ascent sequences avoiding certain patterns. Further results on the enumeration of pattern-avoiding ascent sequences could be found in [1, 14, 19]. By using the generating tree technique, the obstinate kernel method and Zeilberger’s algorithm, Lin [13] confirmed the following conjecture proposed by Martinez-Savage [16].

**Conjecture 1.1** (Martinez-Savage [16]) Inversion sequences of length \( n \) and with no weakly decreasing subsequence of length 3 are equinumerous with enhanced 3-nonnesting (3-noncrossing) set partitions of \([n]\).

As remarked by Lin [13], this conjecture has already been proposed by Yan [19] in the course of confirming the following conjecture posed by Duncan and Steingrímsson [8].

**Conjecture 1.2** (See [8], Conjecture 3.3) Ascent sequences of length \( n \) and with no decreasing subsequence of length 3 are equinumerous with 3-nonnesting (3-noncrossing) set partitions of \([n]\).

Recall that a subsequence \( x_{i_1} x_{i_2} \ldots x_{i_k} \) of a sequence \( x = x_1 x_2 \ldots x_n \) is said to be decreasing if \( i_1 < i_2 < \ldots < i_k \) and \( x_{i_1} > x_{i_2} > \ldots > x_{i_k} \) and to be weakly decreasing if \( i_1 < i_2 < \ldots < i_k \) and \( x_{i_1} \geq x_{i_2} \geq \ldots \geq x_{i_k} \). Denote by \( \mathcal{A}_k(n) \) and \( \mathcal{PA}_k(n) \) the set of ordinary and primitive ascent sequences of length \( n \) and with no decreasing subsequence of length \( k \), respectively. Let \( \mathcal{I}_k(n) \) denote the set of inversion sequences of length \( n \) and with no weakly decreasing sequences of length \( k \).
A set partition $P$ of $[n] = \{1, 2, \cdots, n\}$ can be represented by a diagram with vertices drawn on a horizontal line in increasing order. For a block $B$ of $P$, we write the elements of $B$ in increasing order. Suppose that $B = \{i_1, i_2, \cdots, i_k\}$. Then we draw an arc from $i_1$ to $i_2$, an arc from $i_2$ to $i_3$, and so on. Such a diagram is called the linear representation of $P$, see Figure 1 for example. The enhanced representation of $P$ is defined to be the union of the standard representation of $P$ and the set of loops $(i, i)$, where $i$ ranges over all the singleton blocks $\{i\}$ of $P$. Then one defines a $k$-crossing of a set partition to be a subset $\{(i_1, j_1), (i_2, j_2), \cdots, (i_k, j_k)\}$ of its linear representation where $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$, and an enhanced $k$-crossing of a set partition to be a subset $\{(i_1, j_1), (i_2, j_2), \cdots, (i_k, j_k)\}$ of its enhanced representation where $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$. A partition without any (enhanced) $k$-crossings is said to be (enhanced) $k$-noncrossing. Similarly, a $k$-nesting is defined to be a subset $\{(i_1, j_1), (i_2, j_2), \cdots, (i_k, j_k)\}$ of its enhanced representation where $i_1 < i_2 < \cdots < i_k < j_k < j_{k-1} < \cdots < j_1$, and an enhanced $k$-nesting is defined to be a subset $\{(i_1, j_1), (i_2, j_2), \cdots, (i_k, j_k)\}$ of its enhanced representation where $i_1 < i_2 < \cdots < i_k \leq j_k < j_{k-1} < \cdots < j_1$. A set partition without any (enhanced) $k$-nestings is said to be (enhanced) $k$-nonnesting. Chen et al. [5] proved that (enhanced) $k$-nonnesting set partitions of $[n]$ are equinumerous with (enhanced) $k$-noncrossing set partitions of $[n]$ bijectively using hesitating tableaux as an intermediate object. Denote by $C_k(n)$ and $E_k(n)$ the set of ordinary and enhanced $k$-nonnesting set partitions of $[n]$, respectively.

![Figure 1: The linear representation of a set partition $\pi = \{\{1, 2, 3, 4, 6, 10\}, \{5, 8\}, \{7, 9\}\}$.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|----|
| 12345678910 |

2 Bijective proof of Conjecture [1.1]

In this section, we shall provide a bijective proof of Conjecture [1.1] by showing that inversion sequences of length $n$ and with no weakly decreasing subsequence of length 3 are in bijection with enhanced 3-nonnesting partitions of $[n]$. To this end, we recall some necessary notation and terminology.

A triangular shape of order $n$ is the left-justified array of $\binom{n+1}{2}$ squares in which the $i$th row contains exactly $i$ squares. Let $\Delta_n$ be the triangular shape of order $n$. In a triangular shape, we number rows from top to bottom and columns from left to right and identify squares using matrix coordinate. The $i$th row (column) is
called row (column) \(i\). For example, the square in the first row and second column is numbered \((1, 2)\).

A 01-filling of a triangular shape \(\triangle_n\) is obtained by filling the squares of \(\triangle_n\) with 1’s and 0’s, see Figure 2 for example, where we represent a 1 by a • and suppress the 0’s. A 01-filling of a triangular shape is said to be valid if every row contains at most one 1. A row (column) of a 01-filling is said to a zero if all the squares at this row (column) are filled with 0’s. A NE-chain of a 01-filling is a sequence of 1’s such that any 1 is strictly above and weakly to the right of the preceding 1 in the sequence. For example, in Figure 2 the sequence of 1’s lying in the squares \((6, 3)\), \((5, 4)\) and \((4, 4)\) form a NE-chain of length 3.

![Figure 2: An example of a 01-filling of a triangular shape of order 6.](image)

An inversion sequence \(x_1x_2\ldots x_n\) can be encoded by a 01-filling of \(\triangle_n\) in which the square \((i, x_i + 1)\) is filled with a 1 for all \(1 \leq i \leq n\) and all the other squares are filled with 0’s. It is easily seen that a weakly decreasing sequence of length \(k\) corresponds to a NE-chain of length \(k\). Denote by \(\mathcal{M}_k(n)\) the set of 01-fillings of \(\triangle_n\) with the property that every row contains exactly one 1 and there is no NE-chain of length \(k\).

**Theorem 2.1** There is a one-to-one correspondence between the set \(\mathcal{I}_k(n)\) and the set \(\mathcal{M}_k(n)\).

In his paper [11], Krattenthaler established a bijection between set partitions of \([n]\) and 01-fillings of \(\triangle_n\) in which every row and every column contain at most one 1, and either column \(i\) or row \(i\) contains at least one 1 for all \(1 \leq i \leq n\). For the sake of completeness, we give a brief description of this bijection. Given a set partition \(\pi\) of \([n]\), we can get a 01-filling of \(\triangle_n\) by putting a 1 in the square \((j, i)\) if \((i, j)\) is an arc in its enhanced representation, and, in addition, by putting a 1 in the the square \((i, i)\) if \((i, i)\) is a loop in its enhanced representation. The 01-filling corresponding to the set partition \(\pi = \{\{1, 3, 6\}, \{2, 8\}, \{4\}, \{5, 7, 9\}\}\) is indicated in Figure 3. From the construction of Krattenthaler’s bijection, an enhanced \(k\)-nesting of a set partition corresponds to a NE-chain of length \(k\) in its corresponding 01-filling.

Denote by \(\mathcal{N}_k(n)\) the set of 01-fillings of \(\triangle_n\) satisfying the following properties:
(a1) every row and every column contain at most one 1;
(b1) either column \(i\) or row \(i\) contains at least one 1 for all \(1 \leq i \leq n\);
(c1) there is no NE-chain of length \(k\).

From Krattenthaler’s bijection, we immediately get the following result.

![Figure 3: A set partition \(\pi = \{\{1, 3, 6\}, \{2, 8\}, \{4\}, \{5, 7, 9\}\}\) and its corresponding 01-filling.]

**Theorem 2.2** The 01-fillings of the set \(\mathcal{E}_k(n)\) are in bijection with the 01-fillings of the set \(\mathcal{N}_k(n)\).

In view of Theorems 2.1 and 2.2 in order to prove Conjecture 1.1 it suffices to establish a bijection between the set \(\mathcal{M}_k(n)\) and the set \(\mathcal{N}_k(n)\). To this end, we define two transformations, which will play an essential role in the construction of the bijection.

**The transformation \(\alpha\)** Let \(F\) be a valid 01-filling of \(\Delta_n\) without any NE-chain of length 3. If every column of \(F\) contains at most one 1, we simply define \(\alpha(F) = F\). Otherwise, find the leftmost column \(i\) which contains at least two 1’s. Suppose the square \((i, j)\) is filled with a 1 for some \(1 \leq j \leq i\). Assume that the 1’s below and weakly to the left of the square \((i, i)\) are positioned at the squares \((r_1, c_1), (r_2, c_2), \ldots, (r_m, c_m)\) with \(r_1 < r_2 < \ldots < r_m\). Let \(r_0 = i, c_0 = j\). Suppose that the topmost 1 in column \(i\) is at row \(r_s\).

If row \(r_s + 1\) contains a 1 which is to the right of the square \((r_s, c_s)\), then define \(\alpha(F)\) to be the 01-filling of \(\Delta_n\) obtained from \(F\) by the following procedure:

- For all \(0 \leq \ell \leq s\), replace the 1 at the square \((r_\ell, c_\ell)\) with a 0;
- For all \(0 \leq \ell < s\), fill the square \((r_{\ell+1}, c_\ell)\) with a 1;
• Leave all the other squares fixed.

Otherwise, define $\alpha(F)$ to be the 01-filling of $\Delta_n$ obtained from $F$ by the following procedure:

• For all $0 \leq \ell \leq m$, replace the 1 at the square $(r_{\ell}, c_{\ell})$ with a 0;
• For all $0 \leq \ell < m$, fill the square $(r_{\ell+1}, c_{\ell})$ with a 1;
• Leave all the other squares fixed.

Now we proceed to show that the transformation $\alpha$ has the following desired properties.

Lemma 2.3 In $\alpha(F)$, each column to the left of column $i$ contains at most one 1 and column $i$ contains exactly one 1.

Proof. It is obvious from the selection of column $i$ and the construction of the transformation $\alpha$. □

Lemma 2.4 The filling $\alpha(F)$ is a valid 01-filling of $\Delta_n$ containing no NE-chain of length 3.

Proof. According to the construction of the transformation $\alpha$, it is easily seen that $\alpha(F)$ is a valid 01-filling of $\Delta_n$. Now we proceed to show that $\alpha(F)$ contains no NE-chain of length 3. If not, suppose that the 1’s positioned at the squares $(a_1, b_1)$, $(a_2, b_2)$ and $(a_3, b_3)$ form a NE-chain of length 3, where $a_1 < a_2 < a_3$. Since $F$ has no NE-chain of length 3, the square $(a_1, b_1)$ must be positioned below and to the right of the square $(r_s, c_s)$. Suppose that there exists a 1 at row $r_s + 1$ which is to the right of the square $(r_s, c_s)$. From the construction of $\alpha(F)$, it is easy to check that all the squares below row $r_s$ remain the same as those of $F$. This implies that there is no NE-chain of length 3 below row $r_s$ in $\alpha(F)$. This contradicts the fact that $(a_1, b_1)$ is below row $r_s$. Hence, row $r_s + 1$ does not contain a 1 which is to the right of the square $(r_s, c_s)$. This implies that the square $(a_1, b_1)$ is below and to the right of the square $(r_{s+1}, c_s)$. Since $F$ has no NE-chain of length 3, there is no NE-chain of length 2 below and to the left of the square $(r_{s+1}, c_s)$ in $\alpha(F)$. This yields that both the square $(a_1, b_1)$ and the square $(a_2, b_2)$ are positioned to the right of the square $(r_s, c_s)$. From the fact that $F$ contains no NE-chain of length 3, we have $c_s = c_m = i$. Then the 1’s positioned at the squares $(a_1, b_1)$, $(a_2, b_2)$ and $(r_m, c_m)$ would form a NE-chain of length 3 in $F$, which contradicts the hypothesis. This completes the proof. □
Lemma 2.3 states that the column \( i \) that we find in the transformation \( \alpha \) can only go right. Hence, there will be no column containing at least two 1’s in the resulting filling after finitely many iterations of \( \alpha \). Lemma 2.4 tells us that the resulting filling is a valid 01-filling of \( \Delta_n \) containing no NE-chain of length 3. Therefore, we will get a 01-filling in \( N_3(n) \) after finitely applying many iterations of \( \alpha \) to a 01-filling \( F \) in \( M_3(n) \). Define \( \phi(F) \) to be the resulting filling. Figure 4 illustrates an example of two iterations of \( \alpha \) to a 01-filling in \( M_3(9) \).

The transformation \( \beta \) Let \( F \) be a valid filling of \( \Delta_n \) which verifies property (b1) and contains no NE-chain of length 3. If every row contains a 1 in \( F \), then we simply define \( \beta(F) = F \). Otherwise, find the lowest zero row \( i \). Suppose that the 1’s below and weakly to the left of the square \((i,i)\) are positioned at the squares \((r_1,c_1),(r_2,c_2),\ldots,(r_m,c_m)\) with \( r_1 < r_2 < \ldots < r_m \). Assume that \( r_0 = i \). Suppose that the topmost 1 at column \( i \) is positioned at the square \((r_s,c_s)\).

If there is at least one 1 which is above and to the right of the square \((r_s,c_s)\), then find the topmost square, say \((p,q)\), containing such a 1. Then we have \( p = r_t + 1 \) for some \( 0 \leq t \leq s - 1 \). Define \( \beta(F) \) to be the 01-filling of \( \Delta_n \) obtained from \( F \) by the following procedure:

- For all \( 1 \leq \ell \leq t \), replace the 1 at the square \((r_\ell,c_\ell)\) with a 0;
- For all \( 0 \leq \ell \leq t \), fill the square \((r_\ell,c_{\ell+1})\) with a 1 with the assumption \( c_{t+1} = i \);
- Leave all the other squares fixed.

Otherwise, define \( \beta(F) \) to be the 01-filling of \( \Delta_n \) obtained from \( F \) by the following procedure:

- For all \( 1 \leq \ell \leq m \), replace the 1 at the square \((r_\ell,c_\ell)\) with a 0;
- For all \( 0 \leq \ell \leq m \), fill the square \((r_\ell,c_{\ell+1})\) with a 1 with the assumption \( c_{m+1} = i \);
- Leave all the other squares fixed.

Now we proceed to show that the transformation \( \beta \) has the following analogous properties of \( \alpha \).

Lemma 2.5 In \( \beta(F) \), every row below row \( i \) (including row \( i \)) contains exactly one 1.

Proof. It is obvious from the selection of row \( i \) and the construction of the transformation \( \beta \).
Lemma 2.6 The filling $\beta(F)$ is a valid 01-filling of $\Delta_n$ which verifies property (b1) and contains no NE-chain of length 3.

Proof. It is obvious that $\beta(F)$ is a valid 01-filling of $\Delta_n$ which verifies property (b1). Now we proceed to show that $\beta(F)$ contains no NE-chain of length 3. If not, suppose that the 1’s positioned at the squares $(a_1, b_1)$, $(a_2, b_2)$ and $(a_3, b_3)$ form a NE-chain of length 3, where $a_1 < a_2 < a_3$.

Suppose that there is at least one 1 which is above and to the right of the square $(r_s, c_s)$ in $F$. In this case, all the squares below row $r_t$ in $\beta(F)$ remain the same as those of $F$. Since $F$ contains no NE-chain of length 3, we must have $(a_1, b_1) = (r_t, i)$. Hence, the 1’s positioned at the squares $(p, q)$, $(a_2, b_2)$ and $(a_3, b_3)$ form a NE-chain of length 3, which contradicts the hypothesis. Thus, $F$ does not contain a 1 which is above and to the right of the square $(r_s, c_s)$. According to the construction of $\beta(F)$, one of $(a_1, b_1)$, $(a_2, b_2)$ and $(a_3, b_3)$ must fall in $(r_m, i)$. Since there is no 1 which is below row $r_m$ and to the left of column $i$ in $\beta(F)$, we have $(a_3, b_3) = (r_m, i)$. Then the 1’s positioned at the squares $(a_1, b_1)$, $(a_2, b_2)$ and $(r_m, i)$ would form a NE-chain of length 3 in $F$, which contradicts the hypothesis. This completes the proof. \[
\]

Lemma 2.5 states that the row $i$ that we find in the transformation $\beta$ can only go upside. Hence, there will be no zero row after finitely many iterations of $\beta$. Lemma 2.6 tells us that the resulting filling is a valid 01-filling of $\Delta_n$ containing no NE-chain of length 3. Hence, we will get a 01-filling in $M_3(n)$ after finitely applying many iterations of $\beta$ to a 01-filling $F$ in $N_3(n)$. Define $\psi(F)$ to be the resulting filling.

Theorem 2.7 The maps $\phi$ and $\psi$ induce a bijection between the set $M_3(n)$ and the set $N_3(n)$.

Proof. It suffices to show that the maps $\phi$ and $\psi$ are inverses of each other. First, we proceed to show that $\phi$ is the inverse of the map $\psi$, that is, $\phi(\psi(F)) = F$ for any 01-filling $F \in N_3(n)$. To this end, it suffices to show that $\alpha(\beta^k(F)) = \beta^{k-1}(F)$. Suppose that at the $k$th application of $\beta$ to $\beta^{k-1}(F)$, the selected row is row $i$. Suppose that the 1’s below and weakly to the left of the square $(i, i)$ are positioned at the squares $(r_1, c_1)$, $(r_2, c_2)$, \ldots, $(r_m, c_m)$ with $r_1 < r_2 < \ldots < r_m$. Assume that $r_0 = i$. Suppose that the topmost 1 at column $i$ is positioned at the square $(r_s, c_s)$. We have two cases.

If there is at least one 1 which is above and to the right of the square $(r_s, c_s)$ in $\beta^{k-1}(F)$, then find the topmost square $(p, q)$ containing such a 1. Assume that $p = r_t + 1$ for some $0 \leq t < s$. From the construction of the transformation $\beta$, the square $(r_t, c_{t+1})$ of $\beta^k(F)$ is filled with a 1 for all $1 \leq \ell \leq t$ with the assumption $c_{t+1} = i$ and all the other squares remain the same as those of $\beta^{k-1}(F)$. Clearly, in
\(\beta^k(F)\), all the columns to the left of column \(i\) contains at most one 1, and column \(i\) contains exactly two 1’s. Hence, when we apply the transformation \(\alpha\) to \(\beta^k(F)\), the column that we select is just column \(i\) and the topmost 1 at column \(i\) is positioned at the square \((r_t, i)\). Moreover, the 1 positioned at the square \((p, q)\) is to the right of square \((r_t, i)\) and \(p = r_t + 1\). From the construction of \(\alpha\), it is not difficult to check that \(\alpha(\beta^k(F)) = \beta^{k-1}(F)\).

If there does not exist any 1 which is above and to the right of square \((r_s, c_s)\) in \(\beta^{k-1}(F)\), from the construction of the transformation \(\beta\), the square \((r_\ell, c_{\ell+1})\) of \(\beta^k(F)\) is filled with a 1 for all \(1 \leq \ell \leq m\) with the assumption \(r_{m+1} = i\) and all the other squares remain the same as those of \(\beta^{k-1}(F)\). Clearly, in \(\beta^k(F)\), all the columns to the left of column \(i\) contains at most one 1, in which column \(i\) contains exactly two 1’s. Hence, when we apply the transformation \(\alpha\) to \(\beta^k(F)\), the column that we select is just column \(i\) and the topmost 1 at column \(i\) is positioned at the square \((r_s-1, c_s)\). Notice that there does not exist any 1 which is above and to the right of the square \((r_s, c_s)\) in \(\beta^{k-1}(F)\). Hence, there is no 1 at row \(r_{s-1} + 1\) which is to the right of the square \((r_{s-1}, c_s)\). From the construction of \(\alpha\), it is easily seen that \(\alpha(\beta^k(F)) = \beta^{k-1}(F)\).

Combining the two above cases, we have deduced that \(\alpha(\beta^k(F)) = \beta^{k-1}(F)\). By similar arguments, one can verify that \(\beta(\alpha^k(F)) = \alpha^{k-1}(F)\) for any 01-filling \(F\) of \(\mathcal{M}_3(n)\). The details are omitted here. Hence, the maps \(\phi\) and \(\psi\) are inverses of each other. Thus, the maps \(\phi\) and \(\psi\) induce a bijection between the set \(\mathcal{M}_3(n)\) and the set \(\mathcal{N}_3(n)\) as claimed.

![Figure 4: An example of two iterations of \(\alpha\) to a 01-filling in \(\mathcal{M}_3(9)\).](image)

Combining Theorems 2.1, 2.2 and 2.7 we are led to a bijective proof of Conjecture 1.1.
3 Bijective proof of Conjecture 1.2

In this section, we shall give a new bijective proof of Conjecture 1.2 relying on the bijection $\phi$.

In the following, a 01-filling in $M_3(n)$ will be identified with a sequence $\{(1, a_1), (2, a_2), \ldots, (n, a_n)\}$, where $1 \leq a_i \leq i$ and $a_i = k$ if and only if there is a 1 in the $i$th row and $k$th column. In the course of proving Conjecture 1.2, Yan [19] provided a bijection $\gamma$ between the set $PA_3(n+1)$ and the set $M_3(n)$. Let $x = x_1x_2\cdots x_{n+1} \in PA_3(n+1)$. Define $\gamma(x) = \{(1, a_1), (2, a_2), \ldots, (n, a_n)\}$ where $a_i = i + x_{i+1} - asc(x_1x_2\cdots x_{i+1})$ for all $i = 1, 2, \ldots, n$. For example, let $x = 012340415 \in PA_3(9)$. Then we have $\gamma(x) = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 1), (6, 5), (7, 3), (8, 7)\}$.

The inverse of the map $\gamma$ is defined as follows. Let $F = \{(1, a_1), (2, a_2), \ldots, (n, a_n)\}$. Define $\gamma^{-1}(F) = (x_1, x_2, \ldots, x_{n+1})$ inductively as follows:

- $x_1 = 0$ and $x_2 = 1$;
- if $a_{i-1} < a_i$, then $x_{i+1} = asc(x_1x_2\cdots x_i) + 1 + a_i - i$ for all $2 \leq i \leq n$;
- if $a_{i-1} \geq a_i$, then $x_{i+1} = asc(x_1x_2\cdots x_i) + a_i - i$ for all $2 \leq i \leq n$.

Recall that Krattenthaler [11] also established a bijection between set partitions of $[n+1]$ and 01-fillings of $\Delta_n$ in which every row and every column contain at most one 1. Given a set partition $\pi$ of $[n]$, we can get a 01-filling of $\Delta_n$ by putting a 1 in the square $(j-1, i)$ if $(i, j)$ is an arc in its linear representation. From the construction of Krattenthaler’s bijection, a $k$-nesting of a set partition corresponds to a NE-chain of length $k$ in its corresponding 01-filling.

Denote by $P_k(n)$ the set of 01-fillings of $\Delta_n$ in which every row and every column contain at most one 1, and there is no NE-chain of length $k$.

The following result follows immediately from Krattenthaler’s bijection [11].

Theorem 3.1 There is a one-to-one correspondence between the set $C_k(n+1)$ and the set $P_k(n)$.

By Theorem 3.1 in order to provide a bijection between $A_3(n)$ and $C_3(n)$, it suffices to establish a bijection between the set $A_3(n+1)$ and the set $P_3(n)$.

In a 01-filling, if both row $i$ and column $i$ are zero, then row (column) $i$ is said to be critical.
Theorem 3.2 There is a bijection between the set $A_3(n+1)$ and the set $P_3(n)$.

Proof. First we shall describe a map $\delta$ from the set $A_3(n+1)$ to the set $P_3(n)$. Let $x \in A_3(n+1)$.

First we shall describe a map $\phi$ from the set $A_3(n+1)$ to the set $P_3(n)$. Let $x \in A_3(n+1)$. It is apparent that the ascent sequence $x$ can be written as $x_1^c_1x_2^c_2\cdots x_{k+1}^c_{k+1}$, where $x_i \neq x_{i+1}$ and $c_i \geq 1$ for all $i \geq 1$. Let $x' = x_1x_2\cdots x_{k+1}$. Obviously, $x'$ is a primitive ascent sequence in $P_A_3(k+1)$. Let $F = \gamma(x')$ and $F' = \phi(F)$. Clearly, we have $F \in M_3(k)$ and $F' \in N_3(k)$. Now we can generate a $01$-filling $F''$ of $\Delta_n$ from $F'$ by inserting $c_1 - 1$ consecutive zero rows immediately above row 1 and $c_1 - 1$ consecutive zero columns immediately to the left of column 1, and inserting $c_i$ consecutive zero rows immediately below row $i$ and $c_i$ consecutive zero columns immediately to the right of column $i$ for all $1 \leq i \leq k$. Define $\delta(x) = F''$. It is not difficult to see that the resulting filling $F''$ is an element of $P_3(n)$. This implies that the map $\delta$ is well defined.

In order to prove that $\delta$ is a bijection, we construct a map $\delta'$ from the set $P_3(n)$ to the set $A_3(n+1)$. Given a $01$-filling $F \in P_3(n)$, we can recover an ascent sequence $\delta'(F)$ as follows. Suppose that there are $k$ non-critical rows in $F$. Let rows $i_1$, $i_2$, $\ldots$, $i_k$ be the non-critical rows of $F$. Assume that there are $c_1$ critical rows immediately above row $i_1$, and $c_{i+1}$ critical rows immediately below row $i_i$ for all $1 \leq i \leq k$. Denote $F'$ the $01$-filling obtained from $F$ by removing all the critical rows and columns from $F$. Moreover, let $F'' = \psi(F')$ and $x = x_1x_2\cdots x_{k+1} = \gamma^{-1}(F'')$. It is easily seen that $F' \in N_3(k)$, $F'' \in M_3(k)$ and $x \in PA_3(k+1)$. Let $\delta'(F) = x_1^{c_1}x_2^{c_2}\cdots x_{k+1}^{c_{k+1}}$. It is apparent that we have $\delta'(F) \in A_3(n+1)$.

Property (b1) ensures that the inserted rows and columns in the construction of $\delta$ are exactly the removed rows and columns in the construction of $\delta'$. Thus the map $\delta'$ is the inverse of the map $\delta$. This implies that $\delta$ is bijection. \qed

For example, let $x = 001234345664$ be an ascent sequence in $A_3(13)$. Then $x$ can be written as $0^21^32^43^44^51^62^4$. Let $x' = 0123434564$, which is an element of $P_A_3(10)$. By applying the map $\gamma$ to $x'$, we get a $01$-filling

$$F = \gamma(x') = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 4), (6, 5), (7, 6), (8, 7), (9, 6)\} \in M_3(9)$$

illustrated in Figure 5. Then by applying the map $\phi$ to $F$, we get a $01$-filling $F'$ as shown in Figure 5. Finally, we obtain a $01$-filling $F'' \in P_3(12)$ by adding one zero row immediately above row 1, one zero column immediately to the left of column 1, two consecutive zero rows immediately below row 8, and two consecutive zero columns immediately to the right of column 8, see Figure 5.

Combining Theorems 3.1 and 3.2 we get a new bijective proof of Conjecture 1.2.

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Figure 5: A 01-filling $F \in M_3(9)$, a 01-filling $F' \in N_3(9)$ and a 01-filling $F'' \in P_3(12)$.

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