Klimontovich’s S theorem in nonextensive formalism and the problem of constraints

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Ordinary Boltzmann-Gibbs entropy is inadequate to be used in systems depending on a control parameter that yield different mean energy values. Such systems fail to give the correct comparison between the off-equilibrium and equilibrium entropy values. Klimontovich’s S theorem solves this problem by renormalizing energy and making use of escort distributions. Since nonextensive thermostatistics is a generalization of Boltzmann-Gibbs entropy, it too exhibits this same deficiency. In order to remedy this, we present the nonextensive generalization of Klimontovich’s S theorem. We show that this generalization requires the use of ordinary probability and the associated relative entropy in addition to the renormalization of energy. Lastly, we illustrate the generalized S theorem for the Van der Pol oscillator.

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I. INTRODUCTION

Recently, a new measure of complexity called renormalized entropy (RE) has been proposed [1]. This measure is based on Klimontovich’s S theorem which states that the renormalized entropy decreases in the process of self-organization [2-5]. Originally, the S theorem was used by Klimontovich to remedy the failure of Boltzmann-Gibbs (BG) entropy when it is used in open systems. In these cases, BG entropy resulted out of equilibrium entropy values greater than the corresponding equilibrium case. In order to solve this problem, he made use of escort distributions and energy renormalization. By equating the mean energy values, he was able to redefine the intensity of random source in such a way that the equilibrium entropy value was maximum once again. Klimontovich’s S theorem has been used in many fields ranging from logistic map [1], heart rate variability [6, 7] to the analysis of electroencephalograms of epilepsy patients [8]. Later, Quiroga et al. [9] have shown that RE is negative of the Kullback-Leibler (KL) entropy [10] i.e., the relative entropy associated with the ordinary Boltzmann-Gibbs (BG) entropy once the reference distribution is taken to be the renormalized escort distribution [11, 12, 13].

In this paper, we generalize Klimontovich’s S theorem and construct corresponding renormalized entropy measure in nonextensive formalism [14]. This new approach will allow us to understand open systems from a nonextensive point of view. We will illustrate this by solving the model of the Van der Pol Oscillator in the presence of friction and energy pumping. One important aspect of this work is that this generalization can only be achieved using the ordinary probability distribution rather than the escort distribution.

In Section II, we present RE within the context of ordinary statistics and show its relation to KL entropy. In Section III, we study RE within the framework of nonextensive formalism. The relation of NRE and the associated relative entropies is discussed in Section IV. The ordinary and generalized S theorems is applied to the Van der Pol oscillator in Section V. The results are summarized in Section VI.
II. RENORMALIZED ENTROPY AND KULLBACK-LEIBLER ENTROPY

We begin by supposing two different probability distributions i.e. $p=\{p_i\}$ and $r=\{r_i\}$. These distributions refer to the state of a physical system with different control parameters, for example [1]. Both of them are normalized to unity i.e.,

$$\sum_i p_i = \sum_i r_i = 1.$$  \hspace{1cm} (1)

Comparing these two states in order to understand which one is more ordered than the other by using the associated BG entropies is in general not possible since the energies in both states may be different, as in the case of Van der Pol oscillator [2]. However, it will still be possible to calculate them. For example, for a probability distribution $p$, we can calculate the corresponding BG entropy

$$S(p) = -\sum_i p_i \ln p_i,$$  \hspace{1cm} (2)

where units are chosen such that the Boltzmann constant $k$ is taken to be equal to one. Following Ref. [1], we introduce the effective Hamiltonian $H_{eff}$ of the system as

$$H_{eff} = -\ln r.$$  \hspace{1cm} (3)

The escort probability corresponding to the distribution $r=\{r_i\}$ is given by

$$\tilde{r}_i = \frac{r_i^\beta}{C}.$$  \hspace{1cm} (4)

where $\beta$ is a positive integer. Next, we renormalize energies by setting

$$\langle H_{eff} \rangle^{(0)} = \langle H_{eff} \rangle^{(1)},$$  \hspace{1cm} (5)

where superscripts denote the different states. Using Eq. (3), it can be written as

$$\sum_i \tilde{r}_i \ln r_i = \sum_i p_i \ln r_i.$$  \hspace{1cm} (6)

The ordinary renormalized entropy is defined as

$$R(p||\tilde{r}) \equiv S(p) - S(\tilde{r}).$$  \hspace{1cm} (7)

By explicitly substituting the Shannon entropies given by Eq. (2), we obtain

$$R(p||\tilde{r}) \equiv S(p) - S(\tilde{r})$$

$$= -\sum_i p_i \ln p_i + \sum_i \tilde{r}_i \ln \tilde{r}_i.$$  \hspace{1cm} (8)

Using Eq. (4) for the second term on the right hand side of the equation, we get

$$R(p||\tilde{r}) = -\sum_i p_i \ln p_i + \beta \sum_i \tilde{r}_i \ln r_i - \beta \sum_i \tilde{r}_i \ln C.$$  \hspace{1cm} (9)

Using Eq. (6) for the second term on the right hand side of the equation above,
\[
R(p\|\tilde{r}) = -\sum p_i \ln p_i + \beta \sum p_i \ln r_i - \sum \tilde{r}_i \ln C.
\]  
(11)

After a little algebra and using the normalization condition, we see that

\[
R(p\|\tilde{r}) = -\sum p_i \ln(p_i/\tilde{r}_i).
\]  
(12)

Comparing Eq. (12) with KL entropy [10] which is given by the following equation

\[
K[p\|r] \equiv \sum p_i \ln(p_i/r_i),
\]  
(13)

we see the relation observed by Quiroga et al. [9], i.e.

\[
R(p\|\tilde{r}) = -\sum p_i \ln(p_i/\tilde{r}_i) = -K[p\|\tilde{r}].
\]  
(14)

This final result shows us that RE and K-L entropy is related to one another by a factor of minus one. In other words, one can use RE or K-L entropies in order to study self-organization once one employs the escort distribution and renormalization of mean energy values.

III. RENORMALIZED ENTROPY AND TSALLIS ENTROPY

A nonextensive generalization of the standard Boltzmann-Gibbs (BG) entropy has been proposed by C. Tsallis in 1988 [15-18]. The nonextensive formalism has been used successfully to investigate earthquakes [19], models of fracture roughness [20], entropy production [21], Ising chains [22] and climatological models [23]. This new definition of entropy is given by

\[
S_q(p) = \sum p_i^q - 1 \frac{1}{1-q},
\]  
(15)

where \(p_i\) is the probability of the system in the \(i\)th microstate, \(W\) is the total number of the configurations of the system. The entropic index \(q\) is a real number, which characterizes the degree of nonextensivity as can be seen from the following pseudo-additivity rule:

\[
S_q(A + B)/k = [S_q(A)/k] + [S_q(B)/k] + (1-q)[S_q(A)/k][S_q(B)/k],
\]  
(16)

where \(A\) and \(B\) are two independent systems i.e., \(p_{ij}(A+B)=p_i(A)p_j(B)\). As \(q\to 1\), the nonextensive entropy definition in Eq. (15) becomes

\[
S_{q\to 1} = -\sum p_i \ln p_i,
\]  
(17)

which is the usual BG entropy already given by Eq. (2). This means that the definition of nonextensive entropy contains BG statistics as a special case. The cases \(q<1\), \(q>1\) and \(q=1\) correspond to superextensivity, subextensivity and extensivity, respectively. We define the effective Hamiltonian for this case as

\[
H_{eff} = \frac{p^{q-1} - 1}{1-q} = \ln_q(1/r),
\]  
(18)

where q-logarithm is defined as
\[ \ln_q(x) = \frac{x^{1-q} - 1}{1 - q}. \] (19)

Note that as \( q \) approaches to 1, the effective Hamiltonian given by Eq. (18) becomes identical to the one given by Eq. (3) in BG case since q-logarithm given by Eq. (19) becomes equal to natural logarithm in this limiting case. Setting the mean energy of the two states to be equal to one another as we have done before is tantamount to writing

\[ \sum_k r_k^{q-1} \frac{1}{1 - q} \tilde{r}_k = \sum_k r_k^{q-1} p_k. \] (20)

Using the fact that the probability distributions are normalized, we can write Eq. (20) as

\[ \sum_k r_k^{q-1} \tilde{r}_k = \sum_k r_k^{q-1} p_k. \] (21)

Using the definition of RE given by Eq. (7), we can form the NRE

\[ R_q(p || \tilde{r}) = -\frac{1}{(q - 1)} (\sum_k p_k^q - \sum_k \tilde{r}_k^q). \] (22)

We can rewrite the equation above as

\[ R_q(p || \tilde{r}) = -\frac{1}{(q - 1)} (\sum_k p_k^q - \sum_k \tilde{r}_k^q + (q - 1) \sum_k \tilde{r}_k^q - (q - 1) \sum_k \tilde{r}_k^q). \] (23)

Using the ordinary probability definition by taking \( \beta = 1 \) in Eq. (4) i.e.,

\[ \tilde{r}_k \equiv \frac{r_k}{C} \] (24)

we can write

\[ \sum_k \tilde{r}_k^q = \sum_k r_k^{q-1} \frac{r_k^{q-1}}{C^{q-1}} = \frac{1}{C^{q-1}} \sum_k \tilde{r}_k^{q-1}. \] (25)

Using Eq. (21), we see that

\[ \sum_k \tilde{r}_k^q = \frac{1}{C^{q-1}} \sum_k p_k r_k^{q-1}. \] (26)

which is equal to

\[ \sum_k \tilde{r}_k^q = \sum_k p_k \tilde{r}_k^{q-1}. \] (27)

Using the relation above in Eq. (23) for the last two terms, we obtain

\[ R_q(p || \tilde{r}) = -\left( \frac{\sum_k p_k^q}{q - 1} + \sum_k \tilde{r}_k^q - \frac{1}{q - 1} \sum_k p_k \tilde{r}_k^{q-1} - \sum_k p_k \tilde{r}_k^{q-1} \right). \] (28)

Before proceeding with an analysis of this explicit form of NRE, we need to see the two different expressions of relative entropy in nonextensive formalism.
IV. NONEXTENSIVE RENORMALIZED ENTROPY AND NONEXTENSIVE RELATIVE ENTROPIES

In nonextensive formalism, we have two different expressions of relative entropy. The first one is of Bregman type [24-26] and is given by

$$K_q[p||r] = \frac{\sum_k p_k^q}{q-1} + \sum_k r_k^q - \frac{1}{q-1} \sum_k p_k r_k^{q-1} - \sum_k p_k r_k^{q-1},$$

(29)

whereas the second one is of Csiszár type [27-31] and reads

$$I_q[p||r] = \frac{1}{1-q} [1 - \sum_k p_k r_k^{1-q}].$$

(30)

These two forms of relative entropies have also been used in quantum theoretical framework in order to study second law of thermodynamics and purity of states in quantum information related contexts [32-35]. In order to see the close connection between these relative entropies and K-L entropy, note that it can be written in the following form

$$K[p||r] = \frac{d}{dx} \sum_i (p_i)^x (r_i)^{1-x} |_{x \to 1}.$$  

(31)

This form is preserved exactly if one uses Jackson $q$-differential operator [32, 36-38] instead of ordinary differential operator above i.e.,

$$I_q[p||r] = D_q \sum_i (p_i)^x (r_i)^{1-x} |_{x \to 1},$$

(32)

where Jackson $q$-differential operator [36] is defined as

$$D_q f(x) = [f(qx) - f(x)]/[q(x - 1)].$$

(33)

However, there is no such simple correspondence in the case of relative entropy of Bregman type [39]. Yet, one can still write

$$K[p||r] = \frac{dG(x)}{dx} |_{x \to 1},$$

(34)

where the function $G(x)$ is given by

$$G(x) = \frac{\sum_k p_k^x}{x} - \frac{\sum_k r_k^x}{x} - \sum_k p_k r_k^{x-1} + \sum_k r_k^{x-1}.$$  

(35)

Then, we have the following relation between the relative entropy of Bregman type and the function $G(x)$

$$K_q[p||r] = q D_q G(x) |_{x \to 1}.$$  

(36)

Finally, we note that both forms of nonextensive relative entropies are non-negative and equal to zero if and only if two distributions are equal to one another preserving these properties shared by K-L entropy [26, 39]. Both of these relative entropies become K-L entropy as the parameter $q$ approaches 1.

In order to see the relation between RE and nonextensive formalism, we compare Eqs. (28), (29) and (30), to see that
The nonextensive renormalized entropy cannot be written in terms of relative entropy of Csiszár kind given by Eq. (30). In other words, it is possible to obtain an equation in nonextensive case similar to Eq. (14) in ordinary case, showing that nonextensive renormalized entropy is nothing but the nonextensive relative entropy multiplied by a factor of minus one only by adopting the relative entropy of Bregman type given by Eq. (29). There are three important points to be noted. First, we have used Eq. (27) in order to reach Eq. (28). On the other hand, this equation itself is based on Eq. (21) which is the renormalization relation in nonextensive formalism corresponding to Eq. (6) in ordinary case. This simply shows the necessity of energy renormalization even in nonextensive formalism. Second point is the use of Eq. (24). This is nothing but a definition of an ordinary probability distribution with a normalization constant $C$. This is very different than BG case where we have used Eq. (4) i.e., escort distribution. In other words, we are forced to use ordinary probability in nonextensive formalism in the same way we are forced to use escort distribution within BG statistics (and with its corresponding relative entropy which is K-L entropy). Lastly, we have two different relative entropy definitions but we need to use only one of them in order to define the nonextensive version of renormalized entropy. In order to assess the importance of the relation given by Eq. (37), we need to understand one subtle point: In Ref. [39], it has been shown that relative entropy of Bregman type is associated with the ordinary constraint, whereas relative entropy of Csiszár type is the one associated with the escort distribution. This shows that the use of ordinary probability and the nonextensive relative entropy associated with ordinary constraint is enough to study self-organization in nonextensive formalism if one wants to follow Klimontovich’s recipe.

V. APPLICATIONS

In this Section, we present first the application of the ordinary renormalized entropy and then nonextensive renormalized entropy to the Van der Pol oscillator and show that Klimontovich’s S theorem is satisfied in both cases resulting negative renormalized entropy values for all of the involved parameters.

A. Renormalized Entropy and the Van der Pol Oscillator

Now, we apply the ideas explained in the previous Sections to the case of the Van der Pol oscillator [2]. The equation for this case in the presence of a Langevin source can be given as

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} + (a + bE)v + \omega_0^2 x = y(t),$$

(38)

where $\omega_0$ is the eigenfrequency, $b$ is the nonlinear friction coefficient. The term $a$ can be written in terms of two other parameters i.e.,

$$a = \gamma - a_f,$$

(39)

where $\gamma$ is the coefficient of linear friction and $a_f$ is the feedback coefficient. The term $E$ is nothing but the energy of oscillation where mass term is taken to be equal to unity.

$$E = \frac{1}{2}(v^2 + \omega_0^2 x^2).$$

(40)

The random Langevin source term can be defined by the following equations.

$$\langle y(t) \rangle = 0, \quad \langle y(t)y(t') \rangle = 2D\delta(t - t'),$$

(41)

where the the intensity of random source $D$ is a given positive constant and not connected with the temperature via the Einstein formula. For the case when the following conditions hold

$$\gamma, |a|, b\langle E \rangle \ll \omega_0,$$

(42)
one can write the following Fokker-Planck equation for the distribution function \( f(E, t) \):

\[
\frac{\partial f(E, t)}{\partial t} = \frac{\partial}{\partial E} \left( D \frac{\partial f}{\partial E} \right) + \frac{\partial}{\partial E} \left[ (a + bE)Ef \right].
\] (43)

The solution to the equation above for the stationary case is given by

\[
f_0(E) = C \exp\left(-\frac{aE + \frac{1}{2}bE^2}{D}\right).
\] (44)

where \( C \) is the normalization constant. The state of equilibrium corresponds to the case when the feedback parameter \( a_f \) is equal to zero. Then, the corresponding distribution function, adopting the same notation in previous Section, becomes

\[
r(E) = C \exp\left(-\frac{\gamma E + \frac{1}{2}bE^2}{D}\right).
\] (45)

Assuming the good oscillator condition [14] i.e.,

\[
b \langle E \rangle / \gamma \sim Db/\gamma^2 \ll 1,
\] (46)

we obtain the following equilibrium distribution function

\[
r(E) = \frac{\gamma}{D} \exp\left(-\frac{\gamma E}{D}\right).
\] (47)

Using the formula \( \langle E \rangle = \int dEf(E, t)E \) in order to calculate the average energy and Eq. (2) in the continuous case, we obtain for the entropy

\[
S(r) = \ln \left(\frac{D}{\gamma}\right) + 1.
\] (48)

and the energy

\[
\langle E \rangle^{(1)} = \frac{D}{\gamma}.
\] (49)

respectively. The threshold of generation is defined as the state when feedback parameter \( a_f \) is equal to \( \gamma \). Then, according to Eq. (39), \( a=0 \). Therefore, the distribution function for this case can be written as

\[
p(E) = \sqrt{\frac{2b}{\pi D}} \exp\left(-\frac{bE^2}{2D}\right).
\] (50)

The corresponding entropy and energy values are calculated as

\[
S(p) = \ln \left(\sqrt{\frac{\pi D}{2b}}\right) + \frac{1}{2}
\] (51)

and

\[
\langle E \rangle^{(2)} = \sqrt{\frac{2D}{\pi b}}.
\] (52)

respectively. So, we have two entropy values corresponding to two distinct values of the control parameter \( a_f \). Although we would expect the equilibrium entropy to be greater than the off-equilibrium case which is characterized
by nonzero control parameter, we see that this is not the case since the entropy given by Eq. (48) is less than the value given by Eq. (51) if we take into account also Eq. (46). In order to solve this problem, we renormalize these two states so that their energies will be taken to be equal to one another so that we will have new intensity for the random force of the equilibrium state. Therefore, writing

\[ \langle \tilde{E} \rangle^{(1)} = \langle \tilde{E} \rangle^{(2)} = \sqrt{\frac{2D}{\pi b}}, \]  

we obtain the new intensity of the random force as

\[ \tilde{D}^{(1)} = \gamma \sqrt{\frac{2D}{\pi b}}. \]  

Substitution of this new expression into Eq. (48) gives us the new renormalized entropy

\[ S(\tilde{r}) = \ln(\sqrt{\frac{2D}{\pi b}}) + 1. \]  

It is easily seen that the renormalized equilibrium entropy in Eq. (55) is now greater than the off-equilibrium entropy given by Eq. (51).

Note that another way to see this is directly to use Klimontovich’s S theorem which states that renormalized entropy defined by Eq. (7) decreases in this case. This is tantamount to say that the difference of the entropy given by Eq. (51) and Eq. (55) is less than zero. If we calculate it explicitly using Eqs. (51) and (55), we see that

\[ R_q(p|\tilde{r}) = S(p) - S(\tilde{r}) = -0.05 < 0. \]  

This simple observation will be important when we look at the same problem from the nonextensive point of view.

B. Nonextensive Renormalized Entropy and the Van der Pol Oscillator

We now apply the abstract formalism, which has been developed in Sections IV and V to the problem of the Van der Pol oscillator. We have already studied this example in the context of ordinary renormalized entropy. The only difference in treatment will then be the adoption of Tsallis entropy instead of ordinary Boltzmann-Gibbs entropy. We will assume the underlying mechanics does not change so that we will use the same distribution functions. Therefore, we begin by combining Eqs. (29) and (37) in order to write

\[ R_q(p|\tilde{r}) = S_q(p) - S_q(\tilde{r}) = \frac{1}{1 - q} \int_0^\infty dE \rho^q - \int_0^\infty dE \tilde{\rho}^q + \frac{1}{q - 1} \int_0^\infty dE p^{q-1} + \int_0^\infty dE p^{q-1}. \]  

The nonextensive generalization of the renormalized entropy requires the use of ordinary probability. This means that the average energy values we have already obtained in the context of ordinary renormalized entropy are still valid in the nonextensive context. Therefore, we will still use Eqs. (49) and (52) even though we will adopt Tsallis entropy. Using the previously obtained distribution functions i.e., Eqs. (47) and (50) in Eq. (57) above, we can calculate the first integral on the right hand side as follows

\[ \int_0^\infty dE \rho^q = \int_0^\infty dE \left( \frac{2b}{\pi D} \right)^{q/2} e^{-b q E^2} = \left( \frac{2b}{\pi D} \right)^{q/2} \left( \frac{D \pi}{2 b q} \right)^{1/2}. \]  

The integral appearing in the second term can be calculated in a similar manner but by using the renormalized equilibrium distribution

\[ \int_0^\infty dE \tilde{\rho}^q = \int_0^\infty dE \left( \frac{\pi b}{2 D} \right)^{q/2} e^{-\sqrt{\pi b q} E} = \frac{1}{q} \left( \frac{\pi b}{2 D} \right)^{(q-1)/2}. \]
The only integral to be solved in Eq. (57) is then the following
\[ \int_0^\infty dE \rho^{q-1} = \int_0^\infty dE \left( \frac{2b}{\pi D} \right)^{1/2} \left( \frac{\pi b}{2D} \right)^{(q-1)/2} e^{-\left( \frac{q}{\pi b} \right)^2} e^{-\frac{bE^2}{2D}}. \]  

This integral can be rearranged as
\[ \int_0^\infty dE \left( \frac{2b}{\pi D} \right)^{1/2} \left( \frac{\pi b}{2D} \right)^{(q-1)/2} e^{-\left( \frac{q}{\pi b} \right)^2} e^{-\frac{bE^2}{2D}} = \frac{2}{\pi} \left( \frac{\pi b}{2D} \right)^{q/2} \int_0^\infty dE \exp[-(q-1)\left( \frac{\pi b}{2D} \right)^{1/2} E - \frac{bE^2}{2D}], \]  

The integral above can be solved by the method of completing the squares. Writing
\[ \exp[-(q-1)\left( \frac{\pi b}{2D} \right)^{1/2} E - \frac{bE^2}{2D}] = -\frac{b}{2D} [E + (q-1)\left( \frac{\pi D}{2b} \right)^{1/2} + (q-1)^2 \frac{\pi}{4}] \]  

and doing the following substitution
\[ \sqrt{\frac{b}{2D}} [E + (q-1)\left( \frac{\pi D}{2b} \right)^{1/2}] = z, \]  

we get
\[ \int_0^\infty dE \exp[-(q-1)\left( \frac{\pi b}{2D} \right)^{1/2} E - \frac{bE^2}{2D}] = e^{(q-1)^2 \frac{\pi}{4}} \sqrt{\frac{\pi D}{2b}} \text{erfc}(\sqrt{\pi (q-1)^2 z}). \]  

where \( \alpha \) is given by
\[ \alpha = \frac{\sqrt{\pi}}{2} (q-1) \]  

The complementary error function \( \text{erfc}(x) \) is defined as
\[ \text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt, \]  

where the error function \( \text{erf}(x) \) is
\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \]  

Combining Eqs. (61), (64) and (65), we can write
\[ \int_0^\infty dE \left( \frac{2b}{\pi D} \right)^{1/2} \left( \frac{\pi b}{2D} \right)^{(q-1)/2} e^{-\left( \frac{q}{\pi b} \right)^2} e^{-\frac{bE^2}{2D}} = \frac{2}{\pi} \left( \frac{\pi b}{2D} \right)^{q/2} e^{(q-1)^2 \frac{\pi}{4}} \sqrt{\frac{\pi D}{2b}} \text{erfc}(\frac{\sqrt{\pi}}{2} (q-1)). \]  

Using Eqs. (58), (59) and (68), we finally obtain the nonextensive renormalized entropy as follows
\[ R_q(p||\tilde{p}) = S_q(p) - S_q(\tilde{p}) = \frac{1}{1-q} \left( \frac{2b}{\pi D} \right)^{q/2} \sqrt{\frac{\pi D}{2bq}} - \frac{1}{q} \left( \frac{\pi b}{2D} \right)^{(q-1)/2} + \left( \frac{q}{q-1} \right)^2 \frac{2}{\pi} \left( \frac{\pi b}{2D} \right)^{q/2} e^{(q-1)^2 \frac{\pi}{4}} \sqrt{\frac{\pi D}{2b}} \text{erfc}(\frac{\sqrt{\pi}}{2} (q-1)). \]
The relation given by Eq. (37) guarantees that the NRE like its ordinary counterpart will take only negative values since the relative entropy of Bregman type on the right hand side of Eq. (37) is positive definite (it only becomes zero when the two probability distributions are equal to one another) [26]. Due to the minus sign in front of it, the NRE is seen to be negative definite for all values of the parameters \(D\), \(b\) and \(q\). In Figs. 1 and 2, we plot NRE for some particular values of the intensity of the random source \(D\) and nonlinear friction coefficient \(b\) as a function of the nonextensivity parameter \(q\). In all these cases, NRE takes only negative values thereby ordering the entropies. The NRE \(R_q(p||\tilde{p})\) attains the value \(-0.05\) as the nonextensivity index \(q\) becomes 1 independent of the values of \(D\) and \(b\). Note that this is exactly the value one obtains by using ordinary RE.

VI. CONCLUSIONS

It is known that BG entropy is inadequate to handle the systems which depend on a control parameter [1-4]. In these types of systems, one has different entropy and energy values corresponding to different values of the control parameter and the maximum entropy value is not necessarily attained by the equilibrium distribution. The solution for this problem has been provided by Klimontovich’s S theorem which is based on the joint use of energy renormalization and escort distribution. This defect is shared by Tsallis entropy since BG entropy is a particular case of Tsallis entropy. In this work, we have generalized S theorem in the context of nonextensive formalism and showed that this is possible only when one adopts ordinary probability distribution. If one uses ordinary distribution and the associated relative entropy together with the energy renormalization condition, one obtains a nonextensive renormalized entropy that is a generalization of Klimontovich’s S theorem. Therefore, what can be achieved in the ordinary BG entropy with energy renormalization and escort distribution in the context of S theorem can be achieved through the use of Tsallis entropy, renormalization of energy and ordinary probability distribution. As a result, we conclude that the use of ordinary probability in nonextensive formalism must not be underestimated since it can be the only form of probability distribution needed in some cases such as the generalization of Klimontovich’s S theorem. We have also applied the nonextensive generalization of S theorem to the Van der Pol oscillator and have shown that the value of nonextensive renormalized entropy is negative definite independent of all the parameters involved and attains \(-0.05\) as the nonextensivity index \(q\) becomes 1 exactly giving the numerical value which one would obtain by using ordinary renormalized entropy. This new nonextensive measure of complexity could be used in the analysis of logistic maps and heart rates as it has been the case with the ordinary renormalized entropy [1, 6, 7].

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[1] P. Saparin, A. Witt, J. Kurths, V. Anischenko, Chaos, Solitons and Fractals 4, 1907 (1994).
[2] Yu. L. Klimontovich, Physica A 142, 390 (1987).
[3] Yu. L. Klimontovich, Chaos, Solitons and Fractals 5, 1985 (1994).
[4] Yu. L. Klimontovich, Turbulent Motion and the Structure of Chaos: A New Approach to the Statistical Theory of Open System, Kluwer Academic Publishers, Dordrecht, 1991.
[5] Yu. L. Klimontovich, Z. Phys. B 66, 125 (1987).
[6] J. Kurths et al., Chaos 5, 88 (1995).
[7] A. Voss et al., Cardiovasc. Res. 31, 419 (1996).
[8] K. Kopitzki, P. C. Warnke, J. Timmer, Phys. Rev. E 58, 4859 (1998).
[9] R. Q. Quiroga, J. Arnold, K. Lehnertz, P. Grassberger, Phys. Rev. E 62, 8380 (2000).
[10] R. Gray, Entropy and Information Entropy, Springer-Verlag, New York, 1990.
[11] C. Beck, F. Schlögl, Thermodynamics of Chaotic Systems, Cambridge University Press, Cambridge, 1993.
[12] K. Kopitzki et al., Phys. Rev. E 66, 043902 (2002).
[13] R. Q. Quiroga, J. Arnold, K. Lehnertz, P. Grassberger, Phys. Rev. E 66, 043903 (2002).
[14] C. Tsallis, J.Stat. Phys. 52, 479 (1988).
[15] Yu. L. Klimontovich, Statistical Physics, Harwood Academic Publishers, New York, 1986.
[16] C. Tsallis, in: New Trends in Magnetism, Magnetic Materials and their Applications, eds. J.L.Morán-Lopez and J.M. Sánchez (Plenum Press, New York, 1994), p.451.
[17] C. Tsallis, Some comments on Boltzmann-Gibbs statistical mechanics, Chaos, Solitons and Fractals 6, 539 (1995).
[18] E.M.F. Curado and C. Tsallis, J. Phys. A 24 (1991) L69; Corrigenda: J. Phys. A 24 (1991) 3187; 25, 1019 (1992).
[19] R. Silva, G. S. França, C. S. Vilar, J. S. Alcaniz, Phys. Rev. E 73, 026102 (2006).
[20] S. Nadarajah, Samuel Kotz, Physics Letters A 359, 577 (2006).
[21] Sumiyoshi Abe, Yutaka Nakada, Phys. Rev. E 74, 021120 (2006).
[22] R. F. S. Andrade, S. T. R. Pinho, Phys. Rev. E 71, 026126 (2005).
[23] M. Ausloos, F. Betroni, Physica A 373, 721 (2007).
[24] L. M. Bregman, USSR Comput. Math. Math. Phys. 7, 200 (1967).
[25] T. D. Frank, Nonlinear Fokker-Planck Equations: Fundamentals and Applications, Springer Verlag, Berlin, 2005.
[26] Jan Naudts, Rev. Math. Phys. 16, 809 (2004).
[27] I. Csiszár, Period. Math. Hung. 2, 191 (1972).
[28] C. Tsallis, Phys. Rev. E 58, 1442 (1998).
[29] M. Shiino, J. Phys. Soc. Japan 67, 3658 (1998).
[30] L. Borland, A. R. Plastino and C. Tsallis, J. Math. Phys. 39, 6490 (1998); Erratum: 40, 2196 (1999).
[31] S. Furuichi, K. Yanagi, K. Kuriyama, J. of Math. Phys. 45, 4868-4877 (2004).
[32] S. Abe, Phys. Rev. A 68, 032302 (2003).
[33] S. Abe and A. K. Rajagopal, Phys. Rev. Lett. 91, 120601 (2003).
[34] G. B. Bagci, A. Arda, R. Sever, Int. J. Mod. Phys. 20, 2085 (2006).
[35] G. B. Bagci, A. Arda, R. Sever, Mod. Phys. Lett. B, in press.
[36] F. Jackson, Mess. Math. 38, 57 (1909); Quart. J. Pure Appl. Math. 41, 193 (1910).
[37] S. Abe, Phys. Lett. A 224, 326 (1997).
[38] S. Abe, Phys. Lett. A 244, 229 (1998).
[39] S. Abe, G. B. Bagci, Phys. Rev. E 71, 016139 (2005).
FIG. 1: The renormalized nonextensive entropy versus nonextensivity parameter $q$ where the intensity of the random source $D=50$ and nonlinear friction coefficient $b=0.05$

FIG. 2: The renormalized nonextensive entropy versus nonextensivity parameter $q$ where the intensity of the random source $D=5$ and nonlinear friction coefficient $b=20$