Heavy Fermion Non-Decoupling Effects in Triple Gauge Boson Vertices

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Within a spontaneously broken gauge group we carefully analyse and calculate triple gauge boson vertices dominated by triangle one-loop Feynman diagrams involving heavy fermions compared to external momenta and gauge boson masses. We perform our calculation strictly in four dimensions and derive a general formula for the off-shell, one-particle irreducible (1PI) effective vertex which satisfies the relevant Ward Identities and the Goldstone boson equivalence theorem. Our goal is to search for non-decoupling heavy fermion effects highlighting their synergy with gauge chiral anomalies. Particularly in the Standard Model, we find that when the arbitrary anomaly parameters are fixed by gauge invariance and/or Bose symmetry, the heavy fermion contribution cancels its anomaly contribution leaving behind anomaly and mass independent contributions from the light fermions. We apply these results in calculating the corresponding CP-invariant one-loop induced corrections to triple gauge boson vertices in the SM, minimal $Z'$ models as well as their extensions with a fourth generation, and compare with experimental data.

I. INTRODUCTION

In general, the Appelquist-Carazzone theorem states that the effect from a heavy fermion mass $m$ at low energy observables is suppressed by powers of $m$. However, this theorem does not hold for theories with chiral gauge couplings or large mass splitting within gauge multiplets, a situation known to take place in the minimal Standard Model (SM) of particle physics. Failure of the decoupling of a heavy fermion from radiative corrections requires breaking of a local gauge symmetry and, in addition, breaking of a global symmetry by these corrections.

Another aspect of theories with chiral gauge couplings is the Adler-Bell-Jackiw or chiral anomaly. This is the situation where certain classical Ward Identities (WIs) are violated by quantum corrections (for reviews see [11–13]). For a model that is non-anomaly free, anomalous Ward Identities render it non-renormalizable and non-unitary. This problem shows up in every symmetry breaking stage of the model. In order to cancel chiral anomalies associated with axial (AAA) or vector-axial (VAA) currents in gauge theories, we either need to stick to only by-construction anomaly-free gauge groups, or, to introduce additional chiral femionic fields [14, 15].

An energy region of experimental interest corresponds to the case where a fermion mass $m$ is very heavy, $m^2 \ll s < m^2$, so that it cannot be pair-produced at Tevatron, LHC or a future lepton-collider. If this fermion is chiral i.e., it receives its mass from the Higgs mechanism which is also responsible for the gauge boson mass, then the question of the decoupling of this particle would cause a problem in anomaly cancellation and therefore to gauge invariance. This question has been tackled in many papers in the literature most notably by D’Hoker and Farhi in ref. [16, 17]: decoupling of a fermion whose mass is generated by a Yukawa coupling induces an action functional of the Higgs field and gauge boson fields term, analogous to Wess-Zumino-Witten (WZW) term [18, 19] in chiral Lagrangian. Then D’Hoker and Farhi showed that the theory without the decoupled fermion but with the WZW term is gauge invariant.

Applications of this non-decoupling effect has been utilised in many physics projects from hadronic up to electroweak physics of the SM and beyond, see for example refs. [20–26]. However, to our knowledge, the above conclusion has not been drawn in the broken phase of theories with spontaneous gauge symmetry breaking like the SM. It is after all meaningful to discuss non-decoupling effects only in theories where the physical masses appear explicitly.

The problem when discussing decoupling effects or in general physics associated with the fermionic triangle graph is related to the question: what is the correct result for such a graph? The answer depends on the physical set-up in which it arises [27]. For example, as we shall show below in the case of SM, gauge invariance and Bose symmetry are enough to set the triple neutral gauge boson vertices finite and well defined. Only then can we reach the conclusions for the theory at the heavy fermion mass limit.

If the SM gauge group is extended by extra $U(1)'s$ then anomaly cancellation conditions become more involved. Recently, the authors of refs. [28, 29] noted that such cancellations may occur inside a “cluster” of anomaly-free heavy fermion sector which is not accessible by the current colliders, leaving behind non-decoupling effects in trilinear gauge boson vertices of the extra massive gauge boson $Z'$ and those of the SM $Z, Z', W, W, Z, Z, Z, Z$ that may be observable at low energies. These effects are visible in the energy region where $M_{Z'} \sim g v < \sqrt{s} \ll m \sim \lambda v$. For these non-decoupling effects to occur it is necessary for fermions and gauge bosons to receive mass from the same Higgs boson and there must be a hierarchy between Yukawa and gauge coupling, $\lambda \sim O(1) \gg g$. In this paper we also elaborate on this issue categorising conditions among cou-
plings where such a situation occurs. We then present a few toy-model examples with two or three different external gauge bosons.

We note in passing that, within field theory, mixed anomaly cancellations via 4d Green-Shwartz mechanism have been discussed and analysed phenomenologically in many papers e.g. [30–35].

Our goal here is to construct a perturbative, gauge invariant one-loop proper effective vertex for three external gauge bosons that incorporates both chiral anomaly ambiguities together with non-decoupling effects induced by heavy fermions in an explicit manner. We would like to apply this effective vertex in order to:

- investigate the interplay between chiral anomaly effects and non-decoupling effects of individual particles in trilinear gauge boson vertices in the SM and its extensions,
- categorise all possible models of mixed anomaly cancellations and non-decoupling effects of very heavy fermions that are directly unreachable at the LHC,
- search for phenomenological implications at colliders.

General Lorentz-invariant expressions for three gauge boson vertices have been analysed in detail in refs. [36]–[37]. One-loop corrections in the SM for the $VWW, V = Z, \gamma$ using dimensional regularisation were considered in [5] with special emphasis on the non-decoupling effects due to large doublet mass splittings. The first correct calculation for the $Z\gamma\gamma$ vertex was performed in ref. [38] while for $ZZ\gamma$ in ref. [39]. Phenomenological studies including expectations for those interactions at hadron and lepton colliders were studied in detail in refs. [40]–[43].

Finally, a complete 1PI vertex for three off-shell gauge bosons is a useful tool in analysing low energy inelastic scattering processes with a photon in the final state. Dark matter scattering off atomic electrons and nuclei mediated by light gauge boson particles is one application among many (see refs. [44]–[46]).

The outline of our article is as follows: in section II, we first present the 1PI effective action for the three gauge boson vertex and then in section III we discuss all possible and general non-decoupling effects from heavy fermions. These two sections are supplemented by three Appendices A, B and E, which contain all relevant details of our calculation. Section IV contains applications of the general vertex in the SM, in minimal $Z'$ models and their extensions with a fourth sequential generation. Appendices C and D deal with the evaluation of charged external gauge boson triple vertices and with analytical expressions of various integrals, respectively. Section V concludes with a brief discussion of our findings.

II. THE TRILINEAR GAUGE BOSON VERTEX

In this section we briefly present the main results for the three gauge boson 1PI vertex, $\Gamma^{\mu\nu\rho}$. The details of this calculation are given in Appendices A and B. Furthermore, the behaviour of $\Gamma^{\mu\nu\rho}(s)$ at high energies $s$, and issues on gauge invariance and Goldstone boson equivalence theorem are discussed in the subsequent subsections.

A. The construction of $\Gamma^{\mu\nu\rho}$

The relevant diagrams are depicted in Fig. 1 and their evaluation is developed in Appendix B. What we basically need in order to calculate the diagrams in Fig. 1 is the interaction part of the Lagrangian

$$L_{\text{int}} \supset e \bar{\Psi} \gamma^\mu (\alpha + \beta \gamma_5) \Psi A_\mu,$$  \hspace{1cm} (1)

where $\Psi(x)$ is a 4-component spinor consisting of a pair of two Dirac fermions coupled chirally to a vector field $A_\mu(x)$. Flavour or spinor indices are silently implied. We shall assume a model interaction for eq. (1) that arises from a spontaneously broken Abelian gauge theory. A toy model as such is described in Appendix A. Then $\alpha$ and $\beta$ in eq. (1) are real numbers (in units of $e$) related to linear combinations of hypercharges [see for instance eq. (A8)].

The integral representation for this diagram is given in eq. (B1). By naive power counting this integral is linearly divergent. This means that when we make a shift of integration variable, e.g., $p \to p + a$, the result depends upon the choice of the arbitrary vector $a^\mu$. This change is only reflected in the form factors proportional to $k_1$ and $k_2$ in Lorentz invariant expansion of $\Gamma^{\mu\nu\rho}$ [see eq. (2) below]. As a result, the naive Ward Identities (WIs), eqs. (B15), (B16) and (B18) are violated by terms that contain the arbitrary four vector $a^\mu$. It is useful to write this four vector as a linear combination of the two independent external momenta: $a^\mu = z k_1^\mu + w k_2^\mu$, with $z, w$ arbitrary real parameters.

In order to write out an explicit form for the trilinear gauge boson vertex, say for three identical massive gauge bosons, we make use of an explicit expression for the gauge boson vertex, say for three identical massive gauge bosons, we make use of an explicit expression for the $g$-factor $\Gamma^{\mu\nu\rho}$, consistent with Lorentz and parity symmetry, is,

$$\Gamma^{\mu\nu\rho}(k_1, k_2; w, z) = \left[A_1(k_1, k_2; w) \varepsilon^{\mu\nu\rho\sigma} k_2^\sigma + A_2(k_1, k_2; z) \varepsilon^{\mu\nu\rho\sigma} k_1^\sigma + A_3(k_1, k_2) \varepsilon^{\mu
u\rho\delta} k_2^\rho k_2^\sigma + A_3(k_1, k_2) \varepsilon^{\mu\nu\rho\delta} k_2^\rho k_2^\sigma + A_4(k_1, k_2) \varepsilon^{\mu\nu\rho\delta} k_2^\rho k_2^\sigma + A_6(k_1, k_2) \varepsilon^{\mu\nu\rho\delta} k_2^\rho k_2^\sigma \right].$$  \hspace{1cm} (2)
By naive power counting the dimensionless form factors $A_{1,2}$ are infinite. They can be rendered finite by forcing them to obey the relevant, albeit anomalous, Ward Identities. However, $A_{1,2}$ are in general undetermined since they depend on arbitrary parameters $w$ and $z$. This arbitrariness can be fixed by physical requirements like for example conservation of charge. On the other hand, the form factors (or integrals) $A_{3,6}$ are finite having dimension of inverse mass square. The latter can be found independently by direct diagrammatic methods. The whole procedure is described in detail in Appendix B.

Therefore, non-decoupling effects should originate solely from the $A_1$ and $A_2$ parts of $\Gamma^{\mu\nu\rho}$ but without any further physical input they are undetermined. A direct calculation of $A_{1,2}$ with dimensional regularisation [47] or with Pauli-Villars regularisation is not a good choice when shifting integration variables within linearly (and above) divergent Feynman integrals in four dimensions [48 [50]. The outcome for a single external gauge boson ($i = j = k$ in Fig. 1) triangle graph is appended in eqs. (B26), (B27) and (B28). From these expressions and from eq. (2) we obtain $A_1(k_1, k_2; w)$ and $A_2(k_1, k_2; z)$ in terms of the finite integrals $A_{3,6}$. The corresponding results, in the case of three external identical gauge bosons, are given by eqs. (B31) and (B35) while the finite integrals $A_{3,6}$ by eqs. (B33), (B34) and (B35).

Furthermore, although Bose symmetry could constrain the arbitrary numbers $w$ and $z$, it is not enough to eliminate them altogether: a physical condition is needed, e.g., conservation of electric charge for fermions coupled to external photons or vanishing triangle graph for on-shell momenta of massive gauge bosons or, even, a pure theoretical reason, like the decoupling property. It is straightforward, albeit tedious, to generalize $\Gamma^{\mu\nu\rho}$ in eq. (2) to the case of three distinct external, massive or massless, gauge bosons ($i \neq j \neq k$ in Fig. 1).

With the assignments depicted in Fig. 1, the generalised Ward Identities for vertices $\mu, \nu, \rho$ are written respectively as

\[ q_{\mu} \Gamma^{\mu\nu\rho}(k_1, k_2, w, z) = i m_{A_i} \Gamma^{\nu\rho}(k_1, k_2) + \frac{e^3}{4\pi^2} (\alpha_1 \alpha_j + \beta_i \beta_j) \beta_k + (\alpha_i \beta_j + \alpha_j \beta_1) \alpha_k \varepsilon^{\lambda \mu \nu} k_{1\lambda} k_{2\sigma} (w - z), \]  

\[ -k_{1\nu} \tilde{\Gamma}^{\nu\rho\mu}(k_1, k_2, w, z) = i m_{A_i} \Gamma^{\rho\mu}(k_1, k_2) + \frac{e^3}{4\pi^2} (\alpha_j \alpha_k + \beta_j \beta_k) \beta_1 + (\alpha_j \beta_k + \alpha_k \beta_j) \alpha_1 \varepsilon^{\lambda \mu \rho} k_{1\lambda} k_{2\sigma} (w - 1), \]  

\[ -k_{2\rho} \tilde{\Gamma}^{\rho\mu\nu}(k_1, k_2, w, z) = i m_{A_k} \Gamma^{\mu\nu}(k_1, k_2) + \frac{e^3}{4\pi^2} (\alpha_k \alpha_i + \beta_k \beta_i) \beta_j + (\alpha_k \beta_i + \alpha_i \beta_k) \alpha_j \varepsilon^{\lambda \mu \nu} k_{1\lambda} k_{2\sigma} (z + 1), \]

where the corresponding $\Gamma, \tilde{\Gamma}$, and $\tilde{\Gamma}$ are appended in eqs. (B47) and (B48). It is remarkable here to note the i'th gauge boson mass, $m_{A_i} = -2\beta_i e v$, in front of the pseudoscalar 1PI function $\Gamma^{\rho\mu}$. This term and the analogous in eqs. (3b) and (3c) are the source of heavy fermion mass non-decoupling effects since in the formal limit of $m \to \infty$ there is a remaining piece of order $e^3 \varepsilon^{\lambda \mu \rho} k_{1\lambda} k_{2\sigma} / 4\pi^2$ in $\Gamma^{\mu\nu\rho}$ for example. On the other hand, it shows that currents which are associated to unbroken symmetry generators i.e., to massless gauge bosons, do not provide any non-decoupling effect in $\Gamma^{\mu\nu\rho}$. Moreover, $\Gamma^{\nu\mu}, \tilde{\Gamma}^{\rho\mu}, \tilde{\Gamma}^{\mu\nu}$ depend linearly upon the Yukawa coupling $\lambda$, that is responsible for the fermion mass through the Higgs mechanism and vanishes in the limit of $\lambda \to 0$.

1 In order not to clutter the notation we suppress indices $i, j, k$ in the following expressions for $\Gamma$'s.

2 Throughout, we assume chiral fermions that receive mass through Yukawa interactions with the Higgs field.
Using the WI’s for the vertices $\nu$ and $\rho$, i.e., eqs. (3b) and (3c) as well as eq. (2), we obtain the following expressions for the integrals $A_1$ and $A_2$:

$$A_1(k_1, k_2; w) = (k_1 \cdot k_2)A_3 + k_1^2 A_4 - \frac{e^3 m^2 \beta_j}{\pi^2} I_1(k_1, k_2, m) + \frac{e^3 [(\alpha_j \alpha_k + \beta_j \beta_k) \beta_i + (\alpha_j \beta_k + \alpha_k \beta_j) \alpha_i]}{4\pi^2} (w - 1),$$ (4a)

$$A_2(k_1, k_2; z) = (k_1 \cdot k_2)A_6 + k_1^2 A_5 - \frac{e^3 m^2 \beta_k}{\pi^2} I_2(k_1, k_2, m) + \frac{e^3 [(\alpha_i \alpha_k + \beta_i \beta_k) \beta_j + (\alpha_i \beta_k + \alpha_k \beta_i) \alpha_j]}{4\pi^2} (z + 1),$$ (4b)

where the “non-decoupled” integrals are given by

$$I_1(k_1, k_2, m) = \int_0^1 dx \int_0^{1-x} dy \frac{-(\alpha_i \alpha_k + \beta_i \beta_k) + 2x \beta_i \beta_k}{\Delta},$$ (5a)

$$I_2(k_1, k_2, m) = \int_0^1 dx \int_0^{1-x} dy \frac{-(\alpha_i \alpha_j + \beta_i \beta_j) - 2y \beta_i \beta_j}{\Delta},$$ (5b)

with

$$\Delta \equiv \Delta(k_1, k_2) = x(x - 1)k_2^2 + y(y - 1)k_1^2 - 2xyk_1 \cdot k_2 + m^2.$$ (6)

The following limits,

$$\lim_{m \to \infty} m^2 I_1(k_1, k_2, m) = -\frac{1}{6} (3\alpha_i \alpha_k + \beta_i \beta_k),$$ (7a)

$$\lim_{m \to \infty} m^2 I_2(k_1, k_2, m) = \frac{1}{6} (3\alpha_i \alpha_j + \beta_i \beta_j),$$ (7b)

are also useful in simplifying formulae when discussing synergies of anomalous and non-decoupling terms.

We are now ready to complete $\Gamma^{\mu \nu \rho}$ in eq. (2) by reading directly from eq. (B47), the finite (in four dimensions) terms $A_{3,6}$. We find:

$$A_3(k_1, k_2) = -\frac{e^3 [(\alpha_i \alpha_j + \beta_i \beta_j) \beta_k + (\alpha_i \beta_j + \beta_i \alpha_j) \alpha_k]}{\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{xy}{\Delta},$$ (8a)

$$A_4(k_1, k_2) = \frac{e^3 [(\alpha_i \alpha_j + \beta_i \beta_j) \beta_k + (\alpha_i \beta_j + \beta_i \alpha_j) \alpha_k]}{\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{y(y - 1)}{\Delta},$$ (8b)

$$A_5(k_1, k_2) = -\frac{e^3 [(\alpha_i \alpha_j + \beta_i \beta_j) \beta_k + (\alpha_i \beta_j + \beta_i \alpha_j) \alpha_k]}{\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{x(x - 1)}{\Delta},$$ (8c)

$$A_6(k_1, k_2) = -A_3(k_1, k_2).$$ (8d)

One could guess the expressions above with $i \neq j \neq k$ from the ones with a single identical gauge boson $i = j = k$ by exploiting simple combinatoric algebra in eqs. (B33), (B34) and (B35) and eqs. (B37) and (B38). One can check that all the above form factors obey the Bose symmetry specified in eqs. (B39a), (B39b) and (B39c).

In summary, our main result is the trilinear gauge boson vertex $\Gamma^{\mu \nu \rho}$ of eq. (2), supplemented by form factor components $A_{i=1..6}$ read from eqs. (4) and (8). Eq. (2) satisfies the relevant Ward Identities stated in eq. (3) which originate from the partial conservation of vector and axial vector symmetries in (A9).

**B. Unitarity**

We can make full use of the effective vertex $\Gamma^{\mu \nu \rho}$ in order to calculate, as an example, the matrix element for the process $ZZ \rightarrow ZZ$ with an intermediate massive vector boson $Z'$. We perform the calculation in the center of mass
frame with the following kinematics:

\[
p_1 = (E, 0, 0, p), \quad p_2 = (E, 0, 0, -p), \quad k_1 = (E, p \sin \theta, 0, p \cos \theta), \quad k_2 = (E, -p \sin \theta, 0, -p \cos \theta),
\]

\[
\varepsilon(p_1) = \frac{1}{m_Z} (p, 0, 0, E), \quad \varepsilon(p_2) = \frac{1}{m_Z} (p, 0, 0, -E),
\]

\[
\varepsilon(k_1) = \frac{1}{m_Z} (p, E \sin \theta, 0, E \cos \theta), \quad \varepsilon(k_2) = \frac{1}{m_Z} (p, -E \sin \theta, 0, -E \cos \theta),
\]

where \( p_1 \) and \( p_2 \) are the four-momenta of incoming particles, \( k_1 \) and \( k_2 \) the four-momenta of outgoing particles, \( \varepsilon(p_1), \varepsilon(p_2), \varepsilon(k_1), \varepsilon(k_2) \) are the polarisation vectors of the incoming and outgoing particles respectively and \( \theta \) is the scattering angle of the outgoing \( Z \) boson in the center of mass frame. Non-zero contributions arise only from \( t \) and \( u \)-channels since the \( s \)-channel amplitude vanishes in this frame. Working in the unitary gauge, we find a contribution to \( ZZ \rightarrow ZZ \) due to loop-induced \( \Gamma^{\mu\nu}_{Z\bar{Z}Z} \) of eq. (2) as,

\[
\mathcal{M} = \mathcal{M}_t + \mathcal{M}_u = \left( \frac{E^2 \sin^2 \theta}{t - m_Z^2} \right) \left[ (A_1 - A_2) + \frac{p^2}{u - m_Z^2} \left( 1 - \cos \theta \right) (A_3 - A_6) \right]^2
\]

\[
+ \left( \frac{E^2 \sin^2 \theta}{u - m_Z^2} \right) \left[ (A_1 - A_2) + \frac{p^2}{u - m_Z^2} \left( 1 + \cos \theta \right) (A_3 - A_6) \right]^2,
\]

where \( t = (p_1 - k_1)^2 = -2p^2(1 - \cos \theta) \) and \( u = (k_1 - p_2)^2 = -2p^2(1 + \cos \theta) \). The factors \( A_1 \) and \( A_2 \) in eq. (4) are dimensionless and, in the limit of \( E^2 \rightarrow \infty \), vary at worse as constants while from eq. (8) we have \( A_3 = -A_6 \) which asymptotically goes like \( E^{-2} \). Therefore at high energies \( E^2 \rightarrow \infty \), terms inside the square brackets in eq. (9) behave like constants and so the amplitude does at high energies. This means that unitarity is satisfied as is of course expected for a renormalised theory. It is worthwhile noting that in the limit \( E^2 \rightarrow \infty \) we obtain \( (A_1 - A_2) \propto c(w - z) \), where \( c \) is the anomaly pre-factor present in the second term in the r.h.s. of eq. (3). There is still however a finite and non-vanishing constant contribution from the \( A_{3,6} \) form factors in eq. (9) which for every particle contribution reads,

\[
\lim_{E^2 \rightarrow \infty} \mathcal{M} = -\left( \frac{c}{4\pi^2} \right)^2 \sin^2 \theta \left[ 1 + 2(w - z) + \frac{(w - z)^2}{2 \sin^2 \theta} \right].
\]

We observe that the unknown parameters \( w \) and \( z \) still remain in the amplitude. Only the relation \( w = z \) removes them from the asymptotic limit. We shall come back at this point when discussing the \( Z^* \rightarrow ZZ \)-vertex in section IV C.

C. Goldstone boson Equivalence Theorem and \( R_{\xi} \)-independence

There are literally \( N \)-ways to derive the Ward Identities of eq. (3). A classical method is to demand invariance of the path integral under the combined local vector and axial-vector gauge transformations (A9). We can then represent these WI’s diagrammatically to prove the Goldstone Boson equivalence theorem (11) [52]. This is most clearly explained in Lorentz gauge \( (\xi = 0) \) where the gauge fixing term (A10) does not involve the Goldstone boson field \( \varphi \). Then conservation of the gauge current implies that \( q^\mu \) can be contracted directly with \( \Gamma^{\mu\nu}_\varphi \) and also with

\[
0 = q_\mu \left( \begin{array}{cc}
1 & 1
\end{array} \right) = q_\mu \left( \begin{array}{c}
1PI
\end{array} \right) + \left( \begin{array}{c}
1PI
\end{array} \right) - \left( \begin{array}{c}
1PI
\end{array} \right),
\]

FIG. 2: Graphical representation of the WI in eq. (3a).

the derivatively coupled Goldstone boson to \( \Gamma^{\mu\nu}_\varphi \). In principle there is a third contribution from possible mixings with other gauge bosons, say \( Z' \), that couple to the same fermions in the vertex. This last mixing must necessarily
be proportional to \((g_{\mu} - q_{\mu}/q^2)\) and when contracted with \(q^\mu\), vanishes. Therefore, by using rules from the toy model in Appendix A it is straightforward to see that we recover the classical WI \([3a]\), without the anomalous term. While a possible gauge boson mixing contributes to \(\Gamma^{\mu\nu}\), it does not contribute to WIs in \([3]\). At very high energy, the longitudinal polarization vector is \(\epsilon_{L\mu}(q) \approx q_{\mu}/m_A\), where \(m_A\) is the gauge boson mass. In other words for an anomaly-free model, eq. \((3a)\) or the sum of the diagrams in Fig. 2 can be written as,

\[
\epsilon_{L\mu}(q) \Gamma^{\mu\nu} = i \Gamma^{\nu\rho}.
\]

This equation tells us that at the high energy limit, the physical amplitude with the gauge boson in vertex \(\mu\) will be replaced by the vertex with a Goldstone boson that ‘has been eaten’. However, as is evident from eq. \((3a)\), the relation \((11)\) is broken by possible gauge anomalies. This is another reason why the latter should be absent.

One can easily check by studying for example the fermion-antifermion annihilation process to two gauge bosons with the toy model of Appendix A that Eq. \((11)\) is the required condition for the amplitude to be gauge \(\xi\)-independent. Again the anomalous term \(\textit{must} \) be absent.

### III. NON-DECOUPLING EFFECTS

Heavy fermion non-decoupling effects can be cast in two classes:

A) \(\text{effects that arise from a large mass splitting between particles within an anomaly-free multiplet.}\)

B) \(\text{anomaly driven effects that originate from decoupling a whole anomaly-free multiplet.}\)

In case (A), formal decoupling of the heavy particle that participates in the anomaly cancellation mechanism will leave at low energies an effective Lagrangian \(\Delta \Gamma^{\mu\nu}\) that accounts for the anomaly cancellation missing piece \([10, 17, 21]\). In case (B) the Higgs coupling to fermions will be much larger than the gauge coupling with the latter being approximately zero when the fermion mass is going to infinity \([28, 29]\).

#### A. Non-Decoupling due to large mass splitting

We are going to focus first on the simplest case with three external identical gauge bosons. This means we set \(i = j = k\) in the Ward Identities of eq. \([3]\) or else we look directly at expressions \(\text{(B26)} - \text{(B28)}\). In order to carry out a systematic study of non-decoupling effects and their interplay with chiral anomalies it is essential to keep track of the anomalous terms that depend on the arbitrary parameters \(w\) and \(z\). By exploiting Bose symmetries for on-shell external gauge bosons, and specifically, \([39]\) among legs \(j\) and \(k\) we find \(w = -z\), while with \([40]\) among legs \(i\) and \(j\) we find (after some tedious algebra) \(2w - z - 1 = 0\). The solution of this system,

\[
w = -z = \frac{1}{3}, \tag{12}
\]

finally fixes the arbitrary parameters \(w\) and \(z\). Our observation is that these fixed values for the arbitrary parameters correspond to the case of a particle decoupling from the effective action, \(i.e.,\)

\[
\lim_{m \to \infty} \Gamma^{\mu\nu}(k_1, k_2; w, z) = 0 \Rightarrow w = -z = \frac{1}{3}. \tag{13}
\]

We elaborate this point in what follows. The WIs now take the form:

\[
\begin{align*}
q_{\mu} \Gamma^{\mu\nu}(k_1, k_2; w = 1/3) &= -\frac{e^3 \beta m^2}{\pi^2} \epsilon^{\lambda\nu\rho\sigma} k_{1\lambda} k_{2\sigma} I_0(k_1, k_2; m) + \frac{e^3 (\beta^3 + 3\alpha^2 \beta)}{6\pi^2} \epsilon^{\lambda\nu\rho\sigma} k_{1\lambda} k_{2\sigma}, \tag{14a} \\
-k_{1\nu} \Gamma^{\rho\nu\mu}(k_1, k_2; w = 1/3) &= -\frac{e^3 \beta m^2}{\pi^2} \epsilon^{\lambda\nu\rho\sigma} k_{1\lambda} k_{2\sigma} I_0(k_1, k_2; m) - \frac{e^3 (\beta^3 + 3\alpha^2 \beta)}{6\pi^2} \epsilon^{\lambda\nu\rho\sigma} k_{1\lambda} k_{2\sigma}, \tag{14b} \\
-k_{2\rho} \Gamma^{\rho\nu\mu}(k_1, k_2; w = 1/3) &= -\frac{e^3 \beta m^2}{\pi^2} \epsilon^{\lambda\nu\rho\sigma} k_{1\lambda} k_{2\sigma} I_0(k_1, k_2; m) + \frac{e^3 (\beta^3 + 3\alpha^2 \beta)}{6\pi^2} \epsilon^{\lambda\nu\rho\sigma} k_{1\lambda} k_{2\sigma}. \tag{14c}
\end{align*}
\]

where the integrals \(I_{0, 1, 2}\) are defined in eqs. \(\text{(B5)}, \text{(B22)}\) and \(\text{(B23)}\) respectively. The anomalous terms in \(\text{(14)}\) are then allocated “democratically” in the three legs of \(\Gamma^{\mu\nu}\) as one would have naively expected. Note also that since \(\lim_{m \to \infty} m^2 I_0 = -\lim_{m \to \infty} m^2 I_1 = \lim_{m \to \infty} m^2 I_2 = \frac{1}{6}(\beta^3 + 3\alpha^2 \beta)\) the r.h.s of eqs. \(\text{(14a)}, \text{(14b)}\) and \(\text{(14c)}\)
cancels identically, verifying our statement in eq. (13). Therefore, for a Dirac fermion pair circulating the loop as shown in Fig. 1 and for three identical external gauge bosons, at the formal decoupling limit, the finite contributions are equal and opposite to the anomaly contributions in the vertex. In a Lorentz gauge, terms in $\Gamma^{\mu\nu\rho}$ proportional to $I_{0;1,2}$ arise from the mixing between the Goldstone boson $\varphi$ and the gauge boson as it is shown in Fig. 2. We should notice however, that our calculation of WIs in (14) given in Appendix B contains no reference to a particular gauge choice.

For a Lorentz-invariant and renormalizable chiral gauge theory the anomalous terms i.e., the last terms on the r.h.s of eqs. (14), have to be absent. The only way, consistent with renormalizability, to remove the anomaly terms, is to add a new Dirac fermion pair with opposite $\beta$ i.e., opposite hypercharges $Y_L$ and $Y_R$. A consistent way to describe heavy fermion decoupling effects is to perform the calculation directly in the broken phase of the theory where physical masses appear explicitly. Assuming that the mass of the second (heavy) pair and the energy, $s = (k_1 + k_2)^2$, is much bigger than the first (light) fermion pair, say, $m_2^2 \gg s \gg m_1^2 \approx 0$, there is a non-decoupled term in the 1PI effective action which can be read off from eqs. (B32), (B37) and (B38) or eqs. (2) and (4) for $i = j = k$, to be,

$$\Delta \Gamma^{\mu\nu\rho}(k_1, k_2) \approx \frac{e^3 (\beta^3 + 3\alpha^2\beta)}{6\pi^2} e^{\mu\nu\rho\sigma} (k_1 - k_2)_{\sigma}. \tag{15}$$

This term remains in the 1PI effective function for the light particle. In the heavy mass limit ($m_2 \to \infty$), the form factors $A_{i=3, \ldots, 6}(k_1, k_2)$ vanish as $1/m^2$ leaving only the term (15) in the low energy effective action which has no ‘memory’ anymore from the heavy mass $m_2$. Although, the exact non-kinematic prefactor in eq. (15), depends upon model details, its magnitude (in $e$-units) is approximately, $\alpha/\pi$ and could be observable. Furthermore, the non-decoupling term (15) does not depend on the regularization scheme, i.e., on the parameters $w$ and $z$ in eqs. (B37) and (B38), since the model is by construction anomaly-free.

### B. Anomaly Driven non-decoupling effects

This is a category of possible non-decoupling effects for models possessing an anomaly-free cluster of heavy particles just above those known from the Standard Model.

We systematically then check anomaly cancellation conditions in Ward Identities (3) by demanding the prefactors of $I_{1,2}$ integrals in eqs. (1a) and (1b) to be non-zero. We are seeking for minimal models with up-to three different gauge bosons and up to the least $n$-Dirac fermions.

A model that contains one gauge boson $X$, with V-A couplings as in eq. (1), coupled to only one fermion is impossible to exist because it is anomalous (except the trivial case of a vector-like particle where $\beta = 0$). Adding an extra fermion with the same mass but with opposite axial-vector coupling ($\beta$) renders the model anomaly-free. Such a simple particle content does not lead to non-decoupling effects because all these effects are proportional to an odd power of the axial-vector coupling ($\sim \beta^{2k+1}$) and therefore the sum over the two fermions vanishes. Similar situation arises when more fermions are circulating in the loop.

More interesting is the case where one has two, distinct, external gauge bosons, $X$ and $Y$, either massive or massless. The cancelation of trilinear anomalies requires the existence of at least two fermions with opposite axial-vector couplings but again it is impossible to satisfy instantaneously the mixed anomaly and non-decoupling conditions [see below]. We first obtain the general conditions for an anomaly-free model with two gauge bosons $X$ and $Y$. In notation of eq. (1) these conditions read,

$$\sum_{i=1}^{n} (\beta_X^3 + 3\alpha_X^2\beta_X) = 0, \tag{16a}$$

$$\sum_{i=1}^{n} (\beta_Y^3 + 3\alpha_Y^2\beta_Y) = 0, \tag{16b}$$

$$\sum_{i=1}^{n} (\beta_X^2\beta_Y + 2\alpha_X\alpha_Y\beta_X + \alpha_X^2\beta_Y) = 0, \tag{16c}$$

$$\sum_{i=1}^{n} (\beta_Y^2\beta_X + 2\alpha_X\alpha_Y\beta_Y + \alpha_Y^2\beta_X) = 0, \tag{16d}$$

where $n$ is the total number of fermions. Starting from trilinear anomalies (16a) or (16b) we see that the case $n = 1$ requires only vectorial couplings, $\beta_X = \beta_Y = 0$. Therefore for $n = 1$ there is no non-trivial solution. For $n = 2$ the non-zero couplings must satisfy the following conditions:

$$\beta_{X_2} = -\beta_{X_1}, \quad \alpha_{X_2} = \pm \alpha_{X_1}$$

$$\beta_{Y_2} = -\beta_{Y_1}, \quad \alpha_{Y_2} = \pm \alpha_{Y_1}. \tag{17}$$

Turning to mixed anomalies (16c) and (16d), it is amusing first to note that they are satisfied even with one internal fermion ($n = 1$), if

$$\beta_X = \alpha_X, \quad \beta_Y = -\alpha_Y, \tag{18}$$

or

$$\beta_X = -\alpha_X, \quad \beta_Y = \alpha_Y. \tag{19}$$

---

3 Of course there is the trivial case of vector multiplets i.e., $\beta = 0$.

4 We are not going to consider here the situation of incorporating non-renormalizable counterterms to cancel the anomalies at the expense of introducing a cut-off scale $A \sim 4\pi\nu$. 
Non-decoupling conditions are derived by the requirement that the pre-factors of $I_1$ and $I_2$ integrals in eqs. (4a) and (4b) are non-zero. Hence, in the limit of $k_1^2, k_2^2 \lesssim s \ll m^2$ at least one of the following algebraic expressions,

\[
\sum_{i=1}^{n} (\beta_i^2 \beta_Y + 3\alpha_i \alpha_Y \beta_X)_{i}, \quad \sum_{i=1}^{n} (\beta_i^2 \beta_Y + 3\alpha_i^2 \beta_Y)_{i}, \quad \sum_{i=1}^{n} (\beta_i^2 \beta_X + 3\alpha_i \alpha_Y \beta_X)_{i}, \quad \sum_{i=1}^{n} (\beta_i^2 \beta_X + 3\alpha_i^2 \beta_X)_{i},
\]

must be non-vanishing. For $n = 1$ the choice $[18]$ or (eq. [19]) which eliminates the mixed anomalies sets also eqs. (20) to a non-zero value. However, to cancel the $XXX$ and $YYY$ anomalies one needs at least $n = 2$ fermions to satisfy the conditions [17]. These set the non-decoupling expressions (20) back to zero. The first non-trivial solution of the system eqs. (10) and (20) arises with three pairs of chiral Dirac fermions $(n = 3)$ with an example of quantum numbers given in Table I. Here, we use (18) and (19) to cancel mixed anomalies for $\psi_1$. The other two particles $\psi_2$ and $\psi_3$ are singlets under $U(1)_Y$ and $U(1)_X$, respectively. Plug these into eqs. (B32), (4) and (7), we obtain the non-vanishing operators at the decoupling limit:

\[
\Gamma_{\mu^\nu \rho}^{XXX} = \Gamma_{\mu^\nu \rho}^{YYY} = 0 , \quad \Gamma_{\mu^\nu Y}^{XXX} = \frac{q_1(e_1^2 - e_2^2)}{4\pi^2}(k_2 - k_1)_{\mu} , \quad \Gamma_{\mu^\nu Y}^{YYY} = 0 , \quad \Gamma_{\mu^\nu}^{XXX} = \frac{q_1(e_1^2 - e_2^2)}{4\pi^2}(2k_2 + k_1)_{\mu} , \quad \Gamma_{\mu^\nu}^{YYY} = 0 .
\]

Next is a model example with $n = 4$ Dirac fermions charged under the product of gauge groups $U(1)_X \times U(1)_Y$. This toy model has been examined in ref. [28]. Charge assignments are given in Table II. They are chosen in such a way that triangular anomalies $|U(1)_X|^3$ and $|U(1)_Y|^3$ are canceled separately. The cancelation of mixed anomalies requires the extra condition $q_2 = q_1(e_1^2 - e_2^2)^2$. Charges in Table II follow the general rules of eqs. (17). If we assume that all extra fermions have a common mass $m$ and are all very heavy, then in the low energy limit we find the following expressions for the effective vertices with different combinations of external gauge bosons:

\[
\Gamma_{\mu^\nu \rho}^{XXX} = \frac{e^3}{3\pi^2} (2k_1 + k_2)_{\sigma} \rho_{\mu^\nu \rho} , \quad \Gamma_{\mu^\nu \rho}^{YYY} = \frac{e^3}{3\pi^2} (2k_2 + k_1)_{\sigma} \rho_{\mu^\nu \rho} , \quad \Gamma_{\mu^\nu Y}^{XXX} = \frac{e^3}{3\pi^2} (k_1 - k_2)_{\sigma} \rho_{\mu^\nu \rho} , \quad \Gamma_{\mu^\nu Y}^{YYY} = \frac{e^3}{3\pi^2} (k_2 - k_1)_{\sigma} \rho_{\mu^\nu \rho} , \quad \Gamma_{\mu^\nu \rho}^{XXX} = \frac{e^3}{3\pi^2} (k_1 + k_2)_{\sigma} \rho_{\mu^\nu \rho} , \quad \Gamma_{\mu^\nu \rho}^{YYY} = -\frac{e^3}{3\pi^2} (k_1 - k_2)_{\sigma} \rho_{\mu^\nu \rho} , \quad \Gamma_{\mu^\nu Y}^{XXX} = \frac{e^3}{3\pi^2} (k_1 - k_2)_{\sigma} \rho_{\mu^\nu \rho} , \quad \Gamma_{\mu^\nu Y}^{YYY} = -\frac{e^3}{3\pi^2} (k_2 - k_1)_{\sigma} \rho_{\mu^\nu \rho} , \quad \Gamma_{\mu^\nu \rho}^{XXX} = \frac{e^3}{3\pi^2} (k_2 - k_1)_{\sigma} \rho_{\mu^\nu \rho} , \quad \Gamma_{\mu^\nu \rho}^{YYY} = -\frac{e^3}{3\pi^2} (k_1 - k_2)_{\sigma} \rho_{\mu^\nu \rho} , \quad \Gamma_{\mu^\nu Y}^{XXX} = \frac{e^3}{3\pi^2} (k_2 - k_1)_{\sigma} \rho_{\mu^\nu \rho} , \quad \Gamma_{\mu^\nu Y}^{YYY} = -\frac{e^3}{3\pi^2} (k_1 - k_2)_{\sigma} \rho_{\mu^\nu \rho} , \quad \Gamma_{\mu^\nu Y}^{XXX} = \frac{e^3}{3\pi^2} (k_1 - k_2)_{\sigma} \rho_{\mu^\nu \rho} , \quad \Gamma_{\mu^\nu Y}^{YYY} = -\frac{e^3}{3\pi^2} (k_2 - k_1)_{\sigma} \rho_{\mu^\nu \rho} .
\]
As an example, we observe that heavy fermion non-decoupling effects appear in Eqs. (22c) and (23d). If a model like this with \( X = \gamma', Y = \gamma, Z = Z \) can be embedded in the SM, then it would in principle allow for decays like \( Z' \rightarrow Z\gamma \) that do not depend on the heavy fermion masses.

We should finally remark that in models considered in Tables \[ \text{III} \] gravitational anomalies cancel out since it is always \( \sum_f \beta^X_f = 0 \) for a given axial vector coupling between a vector boson \( X \) and a fermion \( f \).

### IV. APPLICATIONS

#### A. Standard Model

Focusing first in the Standard Model with neutral, \( Z \) or \( \gamma \) triple gauge boson vertices we need only to consider the interaction Lagrangian with fermions. This reads as

\[
\mathcal{L}_{\text{int}} = \sum_f \alpha^V_f A^\mu \overline{\Psi} f \gamma^\mu \Psi_f + \sum_f \left( Z^\mu \overline{\Psi} f \gamma^\mu (\alpha_f^Z + \beta_f^Z \gamma_5) \overline{\Psi} f \right),
\]

where the factors \( \alpha^V_f, \beta^V_f \) with \( V = \gamma, Z \) are

\[
\alpha_f^V = e Q_f, \quad \beta_f^V = 0,
\]

\[
\alpha_f^Z = \frac{2}{3} e, \quad \alpha_u^Z = \frac{2}{3} \left( 1 - \frac{4}{3} s_w^2 \right), \quad \alpha_d^Z = \frac{2}{3} \left( 1 + \frac{4}{3} s_w^2 \right), \quad \beta_u^Z = -\frac{g_Z}{4}, \quad \beta_d^Z = \frac{g_Z}{4},
\]

and \( T^3_f \) and \( Q_f \) are the third component of weak isospin and charge of the SM Dirac fermions \( f = \nu, e, u, d \), respectively. Explicitly in the SM, the prefactors \( \alpha_f^Z \) and \( \beta_f^Z \) take the form:

\[
\alpha_e^Z = \frac{2}{3} e, \quad \alpha_u^Z = \frac{2}{3} \left( 1 - \frac{4}{3} s_w^2 \right), \quad \beta_u^Z = -\frac{g_Z}{4},
\]

\[
\alpha_d^Z = -\frac{1}{3} e, \quad \alpha_d^Z = \frac{2}{3} \left( 1 + \frac{4}{3} s_w^2 \right), \quad \beta_d^Z = \frac{g_Z}{4},
\]

\[
\alpha_e^Z = -e, \quad \alpha_u^Z = \frac{2}{3} \left( 1 - \frac{4}{3} s_w^2 \right), \quad \beta_u^Z = \frac{g_Z}{4},
\]

\[
\alpha_d^Z = 0, \quad \alpha_d^Z = \frac{2}{3} \left( 1 + \frac{4}{3} s_w^2 \right), \quad \beta_d^Z = \frac{g_Z}{4},
\]

where \( g_Z = e/s_w \) is the weak boson gauge coupling and \( s_w, c_w \) are the sinus and cosinus of the weak mixing angle.

1. \( \gamma^* ZZ \)

Our first application refers to the vertex \( \gamma^* ZZ \) with \( V = \gamma, Z \) being off-shell. This interaction has been searched for at LEP and Tevatron while is currently under scrutiny at the LHC. At one-loop level the only CP-conserving contribution arises from the triangle graph in Fig. 1. Applying our general form of the 1PI vertex in eq. (2) and making use of the Bose symmetry

\[
\Gamma^{\mu\nu\rho\sigma}_{ZZ}(k_1, k_2; w) = \left( e^{\mu\nu\rho\sigma}(k_1 - k_2)_{\sigma} \left( -A_1 + \frac{s}{2} A_3 \right) \right)
\]

\[
+ A_3 q^{\mu} e^{\rho\nu\delta} k_1,\beta_{2} k_{2}, \]

where the polarization vectors \( \epsilon^{\mu}_p(k_1) \epsilon^{*\nu}_p(k_2) \) outside the square brackets have been omitted, and also, we set \( A_1 = A_1(k_1, k_2) \) etc for simplicity. More specifically, \( A_1 \) is ambiguous: it depends on how the momentum is routing the loop \( i.e. \), the parameter \( w \). This arbitrariness (or regularization scheme dependence if you wish) is further fixed by exploiting the fact that the \( ZZZ \) on-shell boson vertex vanishes by Bose symmetry. The latter requires \( w = 1/3 \). On the other hand for the vertex \( \gamma ZZ \), conservation of the vector current and Bose symmetry implies that \( w = z = 0 \).

Having specified the arbitrary parameters \( w \) and \( z \) we apply our general expressions for \( A_1 \) and \( A_3 \) found in eqs. (4a) and (8a), specifically to the vertices \( \gamma^* ZZ \) and \( \gamma^* ZZ \) and sum over all SM fermions. By ignoring (see below however), the last term proportional to \( q^{\mu} \) in eq. (27), we can easily find,
\[ \Gamma_{\gamma ZZ}(k_1, k_2) = \varepsilon^{\mu\nu\rho}(k_1 - k_2) \sum_{f=u,d,c,\nu} \left[ m_Z^2 (A_{3f} - A_{4f}) + m_f^2 \frac{\beta_f^Z}{\pi^2} I_{1f} + \frac{1}{6\pi^2} (\beta_f^Z + 3\beta_f^Z \frac{\alpha_f^Z}{\pi}) \right] \]

\[ \equiv \varepsilon^{\mu\nu\rho}(k_1 - k_2) \gamma_{\gamma ZZ}(s) , \]

\[ \Gamma_{V ZZ}(k_1, k_2) = \varepsilon^{\mu\nu\rho}(k_1 - k_2) \sum_{f=u,d,c,\nu} \left[ m_Z^2 (A_{3f} - A_{4f}) + m_f^2 \frac{\beta_f^Z}{\pi^2} I_{1f} + \frac{1}{2\pi^2} \alpha_f^Z \beta_f^Z \right] \]

\[ \equiv \varepsilon^{\mu\nu\rho}(k_1 - k_2) \gamma_{V ZZ}(s) , \]

where \( s = (k_1+ k_2)^2 \) and \( I_{1f} \) is given by eq. (5a). The last term in eqs. (28) and (29) is the anomaly contribution, while the second term is a non-decoupling one in the limit of heavy fermion mass, \( m_f \to \infty \). Again we should notice here that in this limit and for one fermion contribution, the last two terms mutually cancel while the first term vanishes as \( m_2^3/m_f^2 \). Therefore, the decoupling of heavy fermions in \( V^{*} ZZ \) vertex is operative even if those fermions have vastly different, but always much greater than the EW scale, masses among each other. In the SM for example, what is left behind after the decoupling of the top quark is a theory with an anomalous (sometimes called Chern-Simons) term that is necessary to render the effective low energy theory gauge invariant.

Especially for \( \gamma^{*} ZZ \) one can go one step further and write the whole effective vertex in terms of one integral only, namely

\[ \Gamma_{\gamma ZZ}(s) = \frac{s}{2} \sum_{f=u,d,c,\nu} A_{3f}(s) . \]  

Now bringing back the last term on the r.h.s of eq. (27) we find a perfectly fine and gauge invariant form for \( \gamma^{*} ZZ \)-vertex

\[ \Gamma_{\gamma ZZ}(s) = \sum_{f} \frac{s A_{3f}}{2} \times \left[ \varepsilon^{\mu\nu\rho}(k_1 - k_2) \gamma^{\mu\nu\rho} - \varepsilon^{\mu\nu\rho} \frac{q^\mu q^\nu q^\rho}{s} (k_1 - k_2) \right] . \]

This vertex must be proportional to \( s \) in order to cancel the pole contribution arising at \( s = q^2 = 0 \). This is a generic statement for all \( \gamma^{*}VV \) vertices we address below. One should recall that this expression has been derived only after fixing the anomaly coefficients, \( w \) and \( z \), by symmetry requirements. We could have done the reverse: to fix \( w, z \) from the requirement of no pole contribution in eq. (31). In a way, the anomaly and the non-decoupled terms have been absorbed in the finite integral \( A_3 \). It is now evident from eqs. (30) and (3) that \( \Gamma_{\gamma ZZ}(s \to 0) = 0 \) for every fermion contribution, independently. Furthermore, as expected, for asymptotic values of \( s \) we also observe, \( \Gamma_{\gamma ZZ}(s \to \infty) = 0 \), after summing over all SM fermion contributions.

Within one generation of fermions, the SM is a chiral, gauge, and, anomaly-free Quantum Field Theory (QFT). As a result, contributions to \( \Gamma_{V^{*}ZZ} \) from (approximately) massless generations, vanish identically (recall that form factors \( A_{3,4} \) are proportional to the anomaly factors, see eqs. (8a) and (8b)), and the second term vanishes in the massless case). Therefore to a good approximation, for \( \sqrt{s} \gtrsim 2M_Z \), the only non-negligible contribution to \( \Gamma_{V^{*}ZZ} \) arises from the third generation and is due to the large mass difference between the top quark and all other fermions. The top quark influences mainly the last two terms in the square bracket of \( \Gamma_{V^{*}ZZ} \) and \( \gamma_{V ZZ} \) in eqs. (28) and (29). If we make the (numerically crude) approximation of \( m_2^3 \ll s < m_t^3 \) and exploit eq. (D12c) from the Appendix we find (\( N_c = 3 \) is the color factor),

\[ \frac{m_t^2 \beta_t^Z}{\pi^2} I_{1t} \approx - \frac{N_c}{6\pi^2} (\beta_t^Z + 3\beta_t^Z \alpha_t^Z) - \frac{N_c}{120\pi^2} (\beta_t^Z + 5\beta_t^Z \alpha_t^Z) \frac{s}{m_t^3} . \]

The first term is just the opposite of the top quark anomaly contribution in \( \Gamma_{V^{*}ZZ} \) and they both cancel out in the limit of heavy top quark. One can prove easily this statement for all SM vertices, \( \Gamma_{V^{*}VV} = Z, \gamma \) appearing below in this article and we claim, following the arguments of section III, that this is a general theorem: a heavy particle cancels its own anomaly contribution in a triple gauge boson vertex and at the (non-perturbative) limit of \( m \to \infty \) leaving no trace from itself behind. Of course in the top-less SM the last term in \( \Gamma_{V^{*}ZZ} \) does not vanish since the particle content \( (\tau, \nu, b) \) is now anomalous. It is also evident from eq. (12) that the behaviour of \( \Gamma_{V^{*}ZZ}(s) \) at \( s \approx m_t^2 \) rises approximately linearly with \( s \) as \( s/m_t^3 \). This is also verified from our numerical results shown in Fig. 3. Similar conclusions one can derive for \( \gamma_{V ZZ} \) and Fig. 3, but this is rather obvious now because of eq. (30).

Furthermore, it is also instructive to study the behaviour of the vertices \( \Gamma_{V^{*}ZZ}(s) \) in the asymptotic region, \( s \gg m_t^2 \). By exploiting eq. (D13) and keeping only terms of order \( m_t^2/s \) we arrive at the following
expression,
\[
\Gamma_{ZZZ}(s \gg m_t^2) \approx N_e \frac{m_t^2}{s} \left( \frac{2 \beta^3}{\pi^2} \left( 2 - \ln \frac{s}{m_t^2} - i\pi \right) + \frac{\beta^3 + \alpha^2 \beta^2}{\pi^2} \left( \frac{1}{2} \ln \frac{s}{m_t^2} - \frac{\pi^2}{2} + i\pi \ln \frac{s}{m_t^2} \right) \right),
\]
(33)
in which both real and imaginary parts vanish at asymptotic values of \( s \) as they should following unitarity arguments. The effect of a “heavy” particle (here the top quark) is to just delay the “falling off” of \( |\Gamma_{ZZZ}(s)| \) [see Fig.3a], as \( s \to \infty \). Finally, it is also obvious that the real and imaginary part of \( \Gamma_{ZZZ} \) are of the same order of magnitude, a situation which remains true everywhere after the threshold energy, \( s \geq 4m_t^2 \).

Translating our numerical results for the SM to the notation of ref. [34] that is usually followed by the theoretical and experimental literature, we find for \( m_t = 173 \) GeV and LEP energies, that
\[
\begin{align*}
f_5^Z(\sqrt{s} = 200 \text{ GeV}) &= 1.8 \times 10^{-4}, \quad (34) \\
f_5^Z(\sqrt{s} = 500 \text{ GeV}) &= 2.1 \times 10^{-4}, \quad (35)
\end{align*}
\]
where we have neglected small imaginary part contributions from light quark and lepton mass thresholds. These results agree with those quoted in ref. [32]. Unfortunately, they are too small to have been reached by LEP [33].

Just above the top quark threshold energies \( s \geq 4m_t^2 \), the vertex develops a significant absorptive part. This is apparent from our analytical expressions in Appendix D for integrals \( A_{3,6} \) and \( f_{1,2} \) and the discussion above. For \( \sqrt{s} = 500 \) GeV we find:
\[
\begin{align*}
f_5^Z(\sqrt{s} = 500 \text{ GeV}) &= (0.4 - 0.5i) \times 10^{-4}, \quad (36) \\
f_5^Z(\sqrt{s} = 500 \text{ GeV}) &= (-0.3 + 3i) \times 10^{-4}. \quad (37)
\end{align*}
\]

Note again that the imaginary part of the amplitude is of the same order of magnitude as the real part.

2. \( V^*\gamma Z \)

Another non-trivial class among trilinear neutral gauge boson vertices that have been and being searched for at colliders is the amplitude \( V^*\gamma Z \). In the notation of Fig. 1 we assign \( V^*(q), \gamma (k_1) \) and \( Z (k_2) \) to the 1PI effective vertex \( \Gamma_{V^*\gamma Z}^{\mu\nu\rho} \) of eq. (2) with \( V = Z, \gamma \). When the photon and the \( Z \)-gauge boson are both on-shell we find:
\[
\Gamma_{V^*\gamma Z}^{\mu\nu\rho}(k_1, k_2) = \epsilon^{\mu\nu\rho} k_{1\sigma} \left( A_2 + \frac{s + m_Z^2}{2} A_3 \right)
\]
\[
+ \epsilon^{\mu\nu\rho} q_{\beta} k_{\beta} \left( A_3 + A_6 \right) + \epsilon^{\mu\nu\rho} q_{\beta} k_{\beta} A_6.
\]
(38)
We have seen however in eq. (38) that \( A_3 = -A_6 \) and therefore, the second term in eq. (38) vanishes at one-loop. Furthermore, the last term when coupled to a light quark or lepton vector current, is proportional to the mass of the incoming fermions and for current collider architectures this contribution is negligible. Hence, only the first term remains with potentially visible effects. When all external particles are on-shell, Bose symmetry and gauge invariance require the vertex \( V \gamma Z \) to vanish. Bose symmetry relations among form factors and gauge invariance fix the arbitrary parameters \( w \) and \( z \) to be:
\[
Z\gamma Z : \quad w = 1, \quad z = 0, \quad (39)
\]
\[
\gamma\gamma Z : \quad w = 1, \quad z = 1. \quad (40)
\]
By substituting the form in \( A_2 \) from the general expression of (11) we obtain:

\[
\Gamma_{V^*\gamma Z}^{\mu\nu\rho}(k_1, k_2) = \epsilon^{\mu\nu\rho} k_{1\sigma} \sum_{f = u, d, e, \nu} \left[ \frac{m_Z^2 (A_{3f} + A_{5f})}{\pi^2} - \frac{m_f^2 \beta_f I_{2f}^z}{\pi^2} + \frac{1}{2\pi^2} \alpha_f^2 \alpha_f^2 \beta_f^2 I_{2f}^z \right] \equiv \epsilon^{\mu\nu\rho} k_{1\sigma} \Gamma_{V^*\gamma Z}(s),
\]
(41)
\[
\Gamma_{V^*\gamma Z}^{\mu\nu\rho}(k_1, k_2) = \epsilon^{\mu\nu\rho} k_{1\sigma} \sum_{f = u, d, e, \nu} \left[ \frac{m_Z^2 (A_{3f} + A_{5f})}{\pi^2} - \frac{m_f^2 \beta_f I_{2f}^z}{\pi^2} + \frac{1}{2\pi^2} \alpha_f^2 \alpha_f^2 \beta_f^2 I_{2f}^z \right] \equiv \epsilon^{\mu\nu\rho} k_{1\sigma} \Gamma_{V^*\gamma Z}(s).
\]
(42)

One should notice that the square bracket of \( \Gamma_{V^*\gamma Z} \) is approximately equal to \( \Gamma_{\gamma\gamma ZZ} \) since in this case \( A_5 \simeq -A_4 \) and \( I_1 \simeq -I_2 \).

It is amusing to see how greatly the \( \gamma^*\gamma Z \)-vertex is simplified. Placing back the last term of eq. (38) in order

\[\text{[Specific equations and discussions]}\]

5 We multiply \( \Gamma_{V^*\gamma Z}(s) \) in eqs. [28] and [29] with \( \epsilon m_Z^2 / (s - m_V^2) \).
6 This term however is important for gauge invariance to be preserved, as in eq. [41] before.
FIG. 3: The dependence of $|\Gamma_{VV}|$ with $\sqrt{s}$ for different gauge bosons combinations, $V = \gamma, Z : (a) \, Z^* ZZ, \, (b) \, \gamma^* ZZ, \, (c) \, Z^* \gamma Z, \, (d) \, \gamma^* \gamma Z, \, (e) \, Z^* \gamma \gamma$. The solid curve corresponds to the SM, the dashed curve corresponds to the SM + 4th generation fermion model. Masses for light quarks and leptons are neglected while $m_t = 173$ GeV.

Fourth generation quarks and lepton masses are taken as in (66). 

to restore gauge invariance, we find,

$$\Gamma_{\gamma^* \gamma \gamma}^{\mu \nu \rho}(s) = \sum_f s A_{3f} \times \left[ \epsilon^{\mu \nu \rho \sigma} k_{1 \sigma} - \frac{\epsilon^{\mu \rho \beta \sigma} q^\mu k_{2 \beta} k_{1 \sigma}}{s} \right].$$

(43)

The s-factor outside the vertex is expected because it must cancel the pole behaviour of the second term in the square bracket. Once again, the “physical” choice of $w, z$ in the anomalous terms played a crucial role in eq. (43) like in the case of $\gamma^* ZZ$ vertex. Regarding decoupling effects, eq. (43) is self explained: for every particle contribution, a synergy between anomalous and non-decoupling terms results in a well defined integral $s A_{3f}$ that vanishes asymptotically due to the anomaly-
free condition. If however, the energy $\sqrt{s}$ is between two particle masses which combined render the model anomaly-free then there should be non decoupling effects in this regime. One the other hand, adding to the SM, anomaly-free and heavy chiral fermions, there should be no nondecoupling effects remaining in the low energy regime where $\sqrt{s} \lesssim 2m_t$.

One can go one step further also in the case of $Z^\ast \gamma Z$ of eqs. (41). In fact, we can eliminate $J_f$ and the anomaly factors from eq. (41), leaving only the finite integrals $A_3$ and $A_5$, as

$$
\Gamma_{Z^\ast \gamma Z}(s) = \frac{1}{2} \sum_f \left[ (s + m_Z^2) A_3 f + m_Z^2 A_5 f \right].
$$

(44)

After using few integral tricks, like for example the ones of eq. (B44), it is easy to show that $\Gamma_{Z^\ast \gamma Z}(s)$ behaves like $(s - m_Z^2) A_3 f$ near the $Z$-pole. In general, $\Gamma_{V^\ast \gamma Z} \propto \sum_f (s - m_Z^2) A_3 f$ near the pole, is clearly verified when performing the full numerical evaluation of the integrals as in Figs. 3c, 3d.

One can easily see from further working out eqs. (41) and (42) that due to the fact that the SM is an anomaly-free QFT, the whole contribution arises to a very good approximation from particle of the third generation. Numerically, in the conventions of ref. [30] [see also footnote 4], we find for LEP energies

$$
h_3^2(\sqrt{s} = 200 \text{ GeV}) = 2.1 \times 10^{-4},
$$

(45)

$$
h_3^2(\sqrt{s} = 200 \text{ GeV}) = 7.2 \times 10^{-4},
$$

(46)

up to tiny small imaginary parts. These results are in agreement with those presented in ref. [42]. As we have noticed above, it is also confirmed numerically that $|f^2_\gamma| \approx |h^2_3|$. SM predictions of eqs. (47) and (48) are in the best case for $|h^2_3|$. Two orders of magnitude below the published LEP bounds [54].

For comparison, at higher energies the SM predicts:

$$
h_3^2(\sqrt{s} = 500 \text{ GeV}) = (0.3 - 0.6i) \times 10^{-4},
$$

(47)

$$
h_3^2(\sqrt{s} = 500 \text{ GeV}) = (0.9 - 1.8i) \times 10^{-4}.
$$

(48)

Full numerical results for $|\Gamma_{V^\ast \gamma Z}(s)|$ are represented by solid lines in Figs. 3c, 3d. We observe that in the neighborhood of the top threshold, $|\Gamma_{\gamma^0 \gamma Z}(s)|$ is one order of magnitude bigger than $|\Gamma_{Z^\ast \gamma Z}(s)|$. They are both however far below the current Tevatron and LHC sensitivity [55, 56]. Following the projecting sensitivity calculated in ref. [10], observation at the LHC (14 TeV) seems extremely difficult within SM, even for $\gamma^0 \gamma Z$-vertex.

3. $V^\ast \gamma \gamma$

We now turn our discussion to the last SM neutral triple gauge boson vertex, the $V^\ast \gamma \gamma$. Of course, thanks to Furry’s theorem only the case $V = Z$ is valid (for $V = \gamma$ all three currents are vector-like, i.e., $\beta_i = 0$).

However, even in $Z^\ast \gamma \gamma$ there are no non-decoupling effects since there is no would be Goldstone boson associated with the unbroken $U(1)_em$, i.e., the final particles are massless. Nevertheless one can write a simple $Z^\ast \gamma \gamma$ 1PI vertex. We obtain:

$$
\Gamma_{Z^\ast \gamma \gamma}(k_1, k_2) = e^{\mu \nu \beta \delta} q^\mu k_1^\beta k_2^\delta [A_3]
$$

$$
+ \frac{\beta^Z_f (\alpha^Z_f)^2}{4\pi^2} e^{\mu \nu \rho \sigma} [(w - 1) k_2 + (z + 1) k_1]_\rho.
$$

(49)

Landau [57] and Yang [58] say that the on-shell amplitude, $\epsilon_\mu(q) \Gamma_{Z^\ast \gamma \gamma}(k_1, k_2)$ must vanish due selection rules on space inversion and angular momentum conservation. This fixes the arbitrary parameters $w = -z = 1$ for every fermion contribution $f$. One obtains the same values for $w$ and $z$ from $U(1)_em$ gauge invariance, i.e., satisfaction of Ward Identities. Although it is necessary to preserve gauge invariance, this remaining contribution is negligible for light s-channel incoming particles e.g., $e^+e^- \rightarrow \gamma \gamma$, but nevertheless it may be important for heavy external particles like for example dark matter particles or heavy neutrinos annihilating into photons (see related work in refs. [59, 60]).

Defining $\Gamma_{Z^\ast \gamma \gamma}(s) \equiv \sum_{f} m_Z^2 A_{3f}(s)$ and summing over the SM particles, we find numerically,

$$
\Gamma_{Z^\ast \gamma \gamma}(\sqrt{s} = 200 \text{ GeV}) = 2.9 \times 10^{-4},
$$

(50)

$$
\Gamma_{Z^\ast \gamma \gamma}(\sqrt{s} = 500 \text{ GeV}) = (3.2 - 5.6i) \times 10^{-5}.
$$

(51)

For various values of $s$, the function $|\Gamma_{Z^\ast \gamma \gamma}(s)|$ is plotted in Fig. 3f. Notably, at very small $s$ this quantity behaves like $1/s$ and in contrary to the previous $Z^\ast VV$ vertices does not vanish at $s = m_Z^2$. For general values of $s$, and $k_1^2 = k_2^2 = 0$, $\Gamma_{Z^\ast \gamma \gamma}(s)$ is easily written as

$$
\Gamma_{Z^\ast \gamma \gamma}(s) = \sum_{f} \frac{\beta^Z_f (\alpha^Z_f)^2}{2\pi^2} m_Z^2 \frac{s}{\xi_f} J(\xi_f),
$$

(52)

where $\xi_f \equiv 4m_f^2/m_Z^2$ and the function $J(\xi_f)$ is appended in eq. (D2). For energies $(s)$ below the top quark threshold, $\Gamma_{Z^\ast \gamma \gamma}(s)$, approximately takes the form,

$$
\Gamma_{Z^\ast \gamma \gamma}(s) \equiv \sum_{f} m_Z^2 A_{3f} \left( m_Z^2 < s < m_f^2 \right) \approx
$$

$$
- N_c \frac{\alpha^2}{\pi^2} \left[ \frac{m_Z^2}{2s} + \frac{m_Z^2}{m_f^2} \left( \frac{1}{24} + \frac{s}{180m_f^2} \right) \right],
$$

(53)

a behaviour which shows decoupling of a heavy top-quark mass. This follows our general statement just below eq. (42), since the anomalous term in eq. (49) vanishes due to the physical choice of $w$ and $z$, there is no non-decoupled remnants to cancel it. In the asym-
to the mass difference of the two fermions involved; analogously for \(A_{1,2}\). Our main conclusion for a general vertex that contains external charged gauge bosons is given by eqs. (C2) and (C3).

The relevant couplings \(\alpha_{ff'}^W\), and \(\beta_{ff'}^W\), can be read from the charged current part of the SM Lagrangian,

\[
\mathcal{L} \supset g_Z (W^+_{\mu} J^+_{\mu} + W^-_{\mu} J^-_{\mu}) ,
\]

with the \(J^\pm_{W}\)-currents being

\[
J^\pm_{W} = (J^\pm_{\mu})^\dagger = \frac{1}{2\sqrt{2}} \left[ \mp \gamma^\mu (1 - \gamma_5) c + \mp \gamma^\mu (1 - \gamma_5) d \right] .
\]

Hence \(\alpha_{ff'}^W = -\beta_{ff'}^W = \frac{g_5}{2\sqrt{2}}\) for the pairs \((ff') = (\nu, e), (u, d)\), respectively. For simplicity, we ignore quark and lepton mixing effects, but these can easily be included.

| FIG. 4: Standard Model fermion contributions to \((Z, \gamma)WW\) one-loop vertex. |

We therefore set \(\alpha_{j,k} = -\beta_{j,k} = \frac{g_5}{2\sqrt{2}}\) in eqs. (C2) and (C3). The neutral gauge boson-fermion couplings, \(\alpha_f^Y, \beta_f^Y\), are taken from eq. (26). Assuming CP-conservation, the 1PI effective action \(\Gamma_{V_{WW}}^{\mu\rho\sigma}\) with \(V = \gamma, Z\) looks exactly the same as in eq. (27) with the only difference being the form factors being replaced by those given in eqs. (C2) [and the paragraph below (C2)]. Therefore we write,

\[
\Gamma_{V_{WW}}^{\mu\rho\sigma}(k_1, k_2) \equiv \epsilon^{\mu\rho\sigma} (k_1 - k_2)_{\sigma} \Gamma_{V_{WW}}^+(s) ,
\]

where

\[
\Gamma_{V_{WW}}^+(s) = \sum_{\text{doublets}} \left[ m^2_W (A_3 - A_4) + \frac{g_2^2 \alpha_{fd}^V}{16\pi^2} T_1 + \frac{g_2^2 \beta_{fd}^V}{16\pi^2} T_2 + \frac{g_2^2}{32\pi^2} (\alpha_{fd}^V - \beta_{fd}^V) (w - 1) + (f_u \leftrightarrow f_d) \right] .
\]

\[\text{(60)}\]

\[7\] Our notation for \(\Gamma_{V_{WW}}^+(s)\) is related to the standard form factor of ref. [30], as \(\Gamma_{V_{WW}}^+(s) = -g_{VVW} f^Y_{\mu}(s)\).
In this formula we abbreviate \( A_{3,4} = A_{3,4}(m_{f_u}^2, m_{f_d}^2) \) and \( I_{1,2} = I_{1,2}(m_{f_u}^2, m_{f_d}^2) \), with

\[
I_1 = \int_0^1 dx \int_0^{1-x} dy \frac{-(x+y)\Delta m^2 + m_{f_u}^2}{x(x-1)m_W^2 + y(y-1)m_W^2 - xy(s-2m_W^2) - (x+y)\Delta m^2 + m_{f_u}^2},
\]
\[
I_2 = \int_0^1 dx \int_0^{1-x} dy \frac{2x m_{f_d}^2 + (x+y)\Delta m^2 - m_{f_u}^2}{x(x-1)m_W^2 + y(y-1)m_W^2 - xy(s-2m_W^2) - (x+y)\Delta m^2 + m_{f_u}^2},
\]

where \( \Delta m^2 = m_{f_u}^2 - m_{f_d}^2 \). In the limit of heavy masses, \( m^2 = m_{f_u}^2 = m_{f_d}^2 \gg s, m_W^2 \), we obtain,

\[
\lim_{m^2 \to \infty} I_1 = \frac{1}{2}, \quad \lim_{m^2 \to \infty} I_2 = -\frac{1}{6}.
\]

Let us examine the \( \gamma^* W^- W^+ \) case first. We must set \( \beta_{f_u,d} = 0 \). In this case gauge invariance [see eq. (64)] implies \( w = z \) and CP-invariance \( w = -z \), and therefore \( w = z = 0 \). Having fixed the anomalous term the result for this vertex turns out to be simply,

\[
\Gamma_{\gamma^* W^- W^+}(s) = \frac{1}{2} s \sum_{\text{doublets}} \left[ A_3(m_{f_u}^2, m_{f_d}^2) + (f_u \leftrightarrow f_d) \right],
\]

where \( A_3 \) is a form factor defined in the Appendix C. We should note here that \( \Gamma_{\gamma^* W^- W^+}(s = 0) = 0 \) as it should be [36, 37], i.e., there is no pole at \( q^2 = 0 \). This is a special case where the anomaly term conspires with \( I_1 \)-term such that the final result contains no nondecoupling terms. In order for gauge invariance to be non-anomalous, the last terms in the WIs system \([C4]\), must vanish. This implies a relation among fermion charges,

\[
\sum_{f=e,v,d,u} \alpha_f = Q_e + Q_\nu + 3Q_d + 3Q_u = 0,
\]

which is exactly the charge conservation condition. Then, in the asymptotic limit, \( s \gg m_W^2, m_{f_u,d}^2 \), the amplitude for \( \Gamma_{\gamma^* W^- W^+}(s \to \infty) \) vanishes, thanks to eq. [64]. This is obvious from the numerical outcome in Fig. 5. It also shows an enhanced threshold behaviour around \( \sqrt{s} \approx 2m_t \) (solid line). Quantitatively, this can be seen from eq. [63] by expanding \( A_3 \) around the threshold. Compared to \( \Gamma_{\gamma^* Z Z}(s) \), there is an additional contribution due to the large mass difference \( \Delta m^2 = m_{f_u}^2 - m_{f_d}^2 \approx m_t^2 \), in the numerical factor that multiplies \( s/m_t^2 \). Our evaluation of integrals contains one numerical integration and follows the procedure of Appendix B in ref. [5]. Our analytic formulae in Appendix [D] at the limit of \( m_W = 0 \), are in full agreement with these results. Few representative values are,

\[
\Gamma_{\gamma^* W W}(\sqrt{s} = 200 \text{ GeV}) = (6.8 - 6.4i) \times 10^{-4},
\]
\[
\Gamma_{\gamma^* W W}(\sqrt{s} = 500 \text{ GeV}) = (-1.5 + 15i) \times 10^{-4}.
\]

Comparing with \( \gamma^* ZZ \) vertex we see here that the mass splitting generates a sizeable absorptive part that dominates the vertex after \( \sqrt{s} \gtrsim 2m_W \).

We now turn to the \( Z^* W^- W^+ \) vertex. This time we have only CP-symmetry at our disposal which sets only the constraint \( w = -z \). At the broken limit there is no other symmetry remaining in order to fix the parameter \( w \) alone. However, in the exact \( SU(2) \)-limit, where \( [g', s_w \to 0, \alpha_f = -\beta_f] \), this vertex should be exactly the same as the \( Z^* ZZ \)-vertex. There, the arbitrary parameters are fixed by Bose symmetry to be \( w = -z = 1/3 \). For this choice of \( w \) and at the heavy mass limit, \( m^2 = m_{f_u}^2 = m_{f_d}^2 \gg s, m_W^2 \), the vertex is proportional to \( \alpha_f + \beta_f \propto s_W^2 \), for every fermion contribution, which in turn is proportional to \( SU(2) \)-breaking effects. Another, equally good, choice would be \( w = 0 \), for example. The physical requirement here is the decoupling of a particle from the \( \Gamma_{Z^* W W} \)-vertex.

In conclusion, the \( Z^* WW \) vertex is undetermined: there is only CP-symmetry, that is not enough to fix two arbitrary parameters. However, for the anomaly-free SM...
this arbitrariness is irrelevant since it is cancelled when the whole fermion contribution is taken into account. We shall meet this situation again in the $Z'VV$-vertex below.

Our numerical evaluation of the SM $|\Gamma_{Z'WW}(s)|$ is shown in Fig. 6. This time, the top quark threshold destructively adds to the vertex. As in previous cases, we present few representative values,

\[
\begin{align*}
\Gamma_{Z'WW}(\sqrt{s} = 200 \text{ GeV}) &= -(8.5 + 7.6i) \times 10^{-4}, \\
\Gamma_{Z'WW}(\sqrt{s} = 500 \text{ GeV}) &= -(3.8 + 3.5i) \times 10^{-4},
\end{align*}
\]

that show similar order of magnitude values for the real part as in the $Z'ZZ$ vertex but an enhanced absorptive part. The latter is due to custodial symmetry breaking effects i.e., the large mass difference between the top and the bottom quarks. Although there is an intense experimental ongoing analyses at LEP [61], Tevatron [62] and LHC [63, 64] for the first three CP-invariant non-chiral operators, $f_{V}^{I}$ of eq. (56), we are not aware of a similar experimental search on the chiral $f_{V}^{Y}$ of eq. (55).

### B. Models with a sequential fourth fermion generation

In our first departure from the SM we assume a fourth generation matter of quarks and leptons. Apart from the fact that the 4th generation neutrino has to weight more than 45 GeV, a certain tuning to avoid EW constraints is needed. More specifically, one extra doublet of degenerate leptons contributes a piece of approximately $1/6\pi \approx 0.05$ into the S-parameter [65] while the current fit [60] to the EW data gives,

\[ S = 0.04 \pm 0.10 . \]  

(65)

Therefore, a 4th, mass degenerate, fermion generation will contribute a $4/6\pi \approx 0.2$ piece to $S$-parameter which is incompatible with the fit. A certain mass difference or else a certain weak isospin violation is needed which is parameterized by the $T$ parameter [65]. A consistent parameter space with EW precision data and published direct searches is

\[
\begin{align*}
&\frac{m_{t,4}}{m_{t}} = 400 \text{ GeV}, &\frac{m_{t,4}}{m_{t}} = 660 \text{ GeV}, \\
&\frac{m_{t,4}}{m_{t}} = 358 \text{ GeV}, &\frac{m_{b,4}}{m_{b}} = 372 \text{ GeV}.
\end{align*}
\]

(66)

This mass spectrum corresponds to Tevatron experiments allowed region, where the analyses from CDF [67] have excluded $t_{4}$ and $b_{4}$ quarks to have masses smaller than the values quoted above. The leptons mass spectrum is chosen such that it does not contribute significantly to the oblique parameters, e.g., for these values of lepton masses one has $\Delta S_{T} \approx 0$ [66].

Due to the fact that the charges are the same as in the SM, the anomalies are canceled in each generation. It is important to notice here that if all the extra fermions were very heavy and had the same mass, no effect would be left back and the decoupling would work perfectly. The reason is, first of all, that the sum over all extra fermions of expressions that contain the finite integrals $A_{3}, A_{4}$ or $A_{5}$ vanishes because the integrand factors out a term $\sum c_{f}$, where $c_{f}$ is the pre-anomaly factor of each fermion. But this sum is equal to zero for an anomaly-free generation. On the other hand, terms proportional to $I_{1}$ or $I_{2}$ in eq. (1), in the limit of large fermion mass, are canceled exactly by the anomalous term for special values of $w$ and $z$ parameters that are fixed by the Bose symmetry in each case. But this constraint is not necessary, e.g., if an anomaly-free generation of very heavy mass degenerate chiral fermions is added to the SM, it has no effects at low energies, no matter what the values of $w$ and $z$ are. This is guaranteed by the fact that the extra generation is anomaly-free.

The numerical analysis for the three gauge bosons vertices is the same as previously. Using the approximate integral expressions from Appendix D we draw plots for

---

8 Currently, the sequential 4th generation is under siege from LHC [68]. If there exist new heavy SM type quarks, they will contribute a factor of up to $N_{2}^{2} = 9$ into the Higgs production cross section for the (triangle) process $gg \rightarrow H$. The current cross section sensitivity at the LHC is within a few of the SM prediction and therefore it sets an indirect bound over the whole exclusion Higgs area, up to 550-600 GeV. Other direct bounds from the LHC on 4th generation top and bottom quarks involve assumptions about their mass difference to be smaller than the W-mass. These caveats are discussed in some detail with complete references in ref. [69].
the amplitudes $|\Gamma_{V \gamma V}(s)|$ and $|\Gamma_{VWV}(s)|$ versus $\sqrt{s}$ in different combinations of the external gauge bosons $V = \gamma, Z$. These plots are collected in Fig. 3 and Figs. 5, 6, respectively [dashed line].

The extra generation has a significant contribution to $\Gamma$’s, in the region near twice the threshold of each extra fermion, where the amplitude rises until those values (shown as peaks in every combination of external gauge bosons) and drops fast as $1/s$ (apart from $V^* \gamma \gamma$ which drops as $1/s^2$). We see that for small values of energy the two curves (the curve that corresponds to the SM case and the curve that corresponds both to the SM and the 4th generation) have the same form. In this energetic region ($\sqrt{s} \lesssim 600$ GeV) the dominant feature is the first peak that corresponds to the threshold energy for the creation of the top quark ($\sqrt{s} \approx 350$ GeV $\approx 2m_t$). In addition, the contribution from the extra fermionic generation is negligible, because all the extra fermions are heavy compared to the energy, i.e., $(2m_f > \sqrt{s})$. These extra fermions have more or less similar masses. As before with the top-quark mass, there is a cancellation between the anomaly contributions and the $f_1,2$ parts of the amplitude for each fermion separately. As a result, the total contribution from the fourth generation is negligible as we can see from Fig. 3.

The situation is different when $\sqrt{s}$ runs over the mass spectrum of the extra fermionic generation. Firstly for ($\sqrt{s} \gtrsim 600$ GeV) we see different peaks that correspond to the threshold energy for the creation of the extra fermions ($\sqrt{s} \approx 2m_i$). When $(2m_i < \sqrt{s} < 2m_f)$, there is a non-zero contribution to the total amplitude. In this case, fermions whose masses are very heavy compared to $\sqrt{s}$, exhibit the same behaviour as previously i.e., the anomalous term cancels out against the finite contribution.

C. Minimal Z’ models

Grand Unified Theories (GUTs) with rank larger than four could break to the SM gauge group times additional $U(1)'s : SU(3) \times SU(2) \times U(1) \times U(1)'s$. This symmetry is broken down to $U(1)_{em}$ and therefore there is a possibility of additional forces mediated by the $Z'$ gauge bosons associated with the broken $U(1)'$ symmetries (for a review see ref. [20]).

We shall concentrate here on minimal models with one additional neutral gauge boson, the $Z'$. Minimal here means models that contain no-additional i.e., no exotic, matter particles apart from the SM ones and right handed neutrinos. The latter play a crucial role in cancelling anomalies due to the additional $U(1)'$ and in producing viably small neutrino masses. These models were devised first in ref. [17] and later elaborated in refs. [22] [73]. Following the notation of [72] we can describe these models with three additional parameters: the mass of the new gauge boson, $M_{Z'}$, and the couplings $g_Y$ and $g_{BL}$. The latter enter into the current which couples to the unmixed $Z'_0$ gauge boson as

$$J_{\alpha\beta}^\mu = \sum_{f=1,2} [g_Y Y_f + g_{BL} (B - L)_f] J_{\gamma f}^\mu,$$

From this, it is easy to construct $\mathcal{L}_{int}$ in eq. (24) with

$$\alpha_f^Z = \cos \theta_f^Z \alpha_f^{Z_0} - \sin \theta_f^Z \alpha_f^{Z_0^*},$$
$$\alpha_f^{Z'} = \sin \theta_f^Z \alpha_f^{Z_0} + \cos \theta_f^Z \alpha_f^{Z_0^*},$$
$$\beta_f^Z = \cos \theta_f^Z \beta_f^{Z_0} - \sin \theta_f^Z \beta_f^{Z_0^*},$$
$$\beta_f^{Z'} = \sin \theta_f^Z \beta_f^{Z_0} + \cos \theta_f^Z \beta_f^{Z_0^*},$$

where $\theta_f$ is the mixing angle between $Z$ and $Z'$ gauge bosons given by,

$$\tan \theta_f = \frac{g_Y}{g_{BL}} \frac{M_{Z_0}^2}{M_{Z'_0}^2 - M_{Z_0}^2},$$

with $M_{Z_0}^2 = g_Y^2 v^2 / 4$ the ‘SM’ $Z$-boson mass. Also in eq. (68) we obtain for $\alpha_f^{Z_0}, \beta_f^{Z_0}$,

$$\alpha_u^{Z_0} = \frac{1}{2} \left( \frac{5}{6} g_Y + \frac{2}{3} g_{BL} \right), \quad \beta_u^{Z_0} = \frac{g_Y}{4},$$
$$\alpha_d^{Z_0} = \frac{1}{2} \left( -\frac{1}{6} g_Y + \frac{2}{3} g_{BL} \right), \quad \beta_d^{Z_0} = -\frac{g_Y}{4},$$
$$\alpha_e^{Z_0} = \frac{1}{2} \left( \frac{3}{2} g_Y - 2 g_{BL} \right), \quad \beta_e^{Z_0} = -\frac{g_Y}{4},$$
$$\alpha_{\nu}^{Z_0} = \frac{1}{2} \left( \frac{1}{2} g_Y - 2 g_{BL} \right), \quad \beta_{\nu}^{Z_0} = \frac{g_Y}{4},$$

while the corresponding expressions for $\alpha_f^{Z_0'}, \beta_f^{Z_0'}$ are given by eq. (26). This parameterisation through $g_Y$ and $g_{BL}$ helps us to very easily incorporate several models that have been studied in the literature: $Z_{B-L}$ when the $U(1)_{B-L}$ charges of the SM fermions are proportional to $(B - L)$ quantum numbers, $Z_X$ a GUT inspired $SO(10) \rightarrow SU(5) \times U(1)_X$ model and finally, $Z_{3R}$ where the corresponding $U(1)_{3R}$ charges are proportional to $T_{3R}$ generator of the global $SU(2)_R$ symmetry. We summarise the couplings of these models in the following table:

| $Z_{B-L}$ | $Z_X$ | $Z_{3R}$ |
|-----------|-------|---------|
| $g_Y$ | $0$ | $-\frac{1}{2} g_{BL}$ |
| $g_{BL}$ | $\sqrt{\frac{2}{3}} g_Y$ | $\frac{1}{2} g_{BL}$ |

Here, we wish to calculate the effective vertices $\Gamma_{Z' \gamma Z}$ and $\Gamma_{Z'^* ZZ}$ for those models. Recalling eqs. (38) and (39) with $i = Z'$, $j = \gamma$ or $Z$ and $k = Z$ respectively, we obtain
\[ \Gamma^{\mu\nu}_{Z'\gamma Z}(s) \approx e^{\mu\nu\sigma} k_{1\sigma} \sum_{f=u,d,e,\nu} \left[ m_Z^2 (A_{3f} + A_{5f}) - \frac{m_f^2 \beta_f}{\pi^2} I_{2f} + \frac{(z + 1)}{4\pi^2} (\alpha_f^Z \bar{\beta}_f^Z + \alpha_f^Z \bar{\beta}_f^Z) \alpha^\gamma \right] \]
\[ \equiv e^{\mu\nu\sigma} k_{1\sigma} \Gamma^{\mu\nu\sigma}_{Z'\gamma Z}(s), \]
\[ \Gamma^{\mu\nu}_{Z'ZZ}(s) = e^{\mu\nu\sigma} (k_1 - k_2) \sigma \sum_{f=u,d,e,\nu} \left[ m_Z^2 (A_{3f} - A_{4f}) + \frac{m_f^2 \beta_f^Z}{\pi^2} I_{1f} - \frac{(w - 1)}{4\pi^2} [(\alpha_f^Z)^2 \bar{\beta}_f^Z + (\bar{\beta}_f^Z)^2 \beta_f^Z + 2 \alpha_f^Z \bar{\beta}_f^Z \bar{\beta}_f^Z] \right] \]
\[ \equiv e^{\mu\nu\sigma} (k_1 - k_2) \sigma \Gamma^{\mu\nu\sigma}_{Z'ZZ}(s), \]

with \( \alpha_f \) and \( \beta_f \) given in eqs. (68) and (70). Again the last terms on the r.h.s of eqs. (71) and (72) arise from the chiral anomaly of individual fermion contributions. These anomalous terms cancel out when we sum over all SM fermions (here we also need the right handed neutrino). This also removes the arbitrariness due to the unknown parameters \( w, z \). Contrary to the SM vertices, we cannot use here any physical arguments in order to remove completely both \( w \) and \( z \) parameters. We only have \( U(1)_{em} \) gauge invariance for \( Z'\gamma Z \) and Bose symmetry for \( Z'ZZ \) while in the SM we have two neutral gauge bosons and two symmetries.

But lets for the moment keep the anomalous terms. Obviously they are multiplied by arbitrary parameters \( (z + 1) \) (for \( Z'\gamma Z \)) and \( (w - 1) \) (for \( Z'ZZ \)). Focusing on the \( Z_{B-L} \) model, where the mixing angle \( \theta' \) vanishes, we observe that for any single heavy fermion contribution the 2nd and the 3rd term on the r.h.s of eqs. (71) and (72) mutually cancel and what remains back is the effective theory with the low mass fermion contributions but together with their anomalous terms included. The latter do not depend on particle masses. The choices for the arbitrary parameters are \( w = z = 1 \) for \( Z'\gamma Z \) and \( w = z = 0 \) for \( Z'ZZ \). The last condition can be interpreted as follows: for the amplitude \( ZZ \to ZZ \) to hold for asymptotic values of energies, eq. (10) requires \( w = z \) but Bose symmetry requires \( w = -z \). This conclusion does not stand firm in the case of mixing between \( Z \) and \( Z' \) i.e., in models \( Z_x, Z_{3R} \) of the table above, and the contribution of a heavy mass particle is undetermined. Of course anomalies do cancel when all model fermions are added.

In Fig. 7 we display numerical results for the absolute value of the scalar part of the 1PI effective vertices \( Z'\gamma Z \) and \( Z'ZZ \) in eqs. (71) and (72) for \( M_{Z'} = 1 \) TeV and \( g_{Z'} = \alpha_{em} \). Figs. (7a,b) refer to \( Z_{B-L} \) model, Figs. (7c,d) to \( Z_x \) models and, finally, Figs. (7e,f) to \( Z_{3R} \) models. For the values of \( M_{Z'} \) and \( g_{Z'} \) chosen, fits to electroweak observables and direct searches are satisfied. We also present results when adding a sequential 4th generation of fermions with the same masses (and the reasoning) as we did for the SM case of section IV A. We observe that there is an enhancement of the vertices by a factor of 2 for \( Z_{B-L} \), and a factor of 10-15 for \( Z_x \).

Numerically, we can define analogous quantities \( h_3^{Z'} \) and \( f_5^{Z'} \) by simply replacing \( Z \) with \( Z' \) in the definition given by footnote 5. As an example, for the \( B-L \) model we obtain,

\[
\begin{align*}
\ h_3^{Z'} (\sqrt{s} = 200 \text{ GeV}) &= -2.7 \times 10^{-5} , \\
\ h_3^{Z'} (\sqrt{s} = 500 \text{ GeV}) &= (-2.7 + 5.3i) \times 10^{-4} , \\
\ f_5^{Z'} (\sqrt{s} = 200 \text{ GeV}) &= -7.2 \times 10^{-6} , \\
\ f_5^{Z'} (\sqrt{s} = 500 \text{ GeV}) &= (-7.7 + 18i) \times 10^{-5} .
\end{align*}
\]
FIG. 7: a,b) $|\Gamma_{Z'VV}(s)|$ versus $\sqrt{s}$ for different gauge bosons combinations as they are given by eqs. (71) and (72). The solid curve corresponds to the SM spectrum with an extra $U(1)_{B-L}$, while the dashed curve corresponds to the same but with a 4th sequential fermion generation added as in Fig. 3. We take $M_{Z'} = 1$ TeV and $g_{Z'} = \alpha_{em}$. c,d) The same as (a,b) but with $U(1)_{\chi}$. (e,f) The same as (a,b) but with $U(1)_{3R}$.
\[ \Gamma(Z' \rightarrow \gamma Z) = \frac{1}{48\pi} \sum_{f=u,d,e,\nu} \left[ m_Z^2 (A_{3f} + A_{5f}) - \frac{m_Z^2 \beta_f^2}{\pi^2} I_{1f} + \frac{(z+1)}{4\pi^2} \left( \alpha_f^2 \beta_f^2 + \alpha_f^2 \beta_f^2 \right) \alpha_f^2 \right] \]
\[ \times \frac{m_Z^2}{m_Z^2} (1 - \frac{m_Z^2}{m_Z^2})^3 (1 + \frac{m_Z^2}{m_Z^2}) , \]
\[ \Gamma(Z' \rightarrow ZZ) = \frac{1}{96\pi} \sum_{f=u,d,e,\nu} \left[ m_Z^2 (A_{3f} - A_{4f}) + \frac{m_Z^2 \beta_Z^2}{\pi^2} I_{1f} - \frac{(w-1)}{4\pi^2} \left[ (\alpha_f^2 \beta_f^2 + (\beta_f^2)^2 \beta_f^2 + 2 \alpha_f^2 \alpha_f^2 \beta_f^2 \right] \right] \]
\[ \times \frac{m_Z^2}{m_Z^2} (1 - \frac{4m_Z^2}{m_Z^2})^{5/2} , \]
\[ \Gamma(Z' \rightarrow W^+W^-) = \alpha_{em} m_{Z'} \sin^2 \theta' \left( 1 - 4 \frac{m_W^2}{m_Z^2} \right)^{3/2} \left[ 1 + 20 \frac{m_W^2}{m_Z^2} + 12 \frac{m_W^2}{m_Z^2} \right] \left( \frac{m_W^2}{m_Z^2} \right)^{-2} , \]
\[ \Gamma(Z' \rightarrow \bar{f}f) = \frac{N_c m_{Z'}}{12\pi} \left[ (\alpha_f^2 + \beta_f^2) - \frac{3m_W^2}{m_Z^2} (\alpha_f^2 - \beta_f^2) \right] \sqrt{1 - \frac{4m_W^2}{m_Z^2}} , \]

where \( N_c \) is the color factor (3 for quarks and 1 for leptons) and the tree level decay width for \( Z' \rightarrow WW \) has been taken from ref. [74] and is dominant over the loop-induced ones. For \( g_{Z'} = \alpha_{em} \), \( M_{Z'} = 1 \text{ TeV} \) and SM spectrum with three generations we obtain for the \( B - L \) (\( \chi \)) [3R] models:

\[
\text{Br}(Z' \rightarrow \nu \nu) = 37.7 (42.3) [12.5] \% ,
\text{Br}(Z' \rightarrow \ell \ell) = 37.7 (12.5) [12.6] \% ,
\text{Br}(Z' \rightarrow q\bar{q}) = 24.5 (45.1) [74.8] \% ,
\text{Br}(Z' \rightarrow WW) = 0.03 (3.2) [8.1] \times 10^{-5} ,
\text{Br}(Z' \rightarrow Z\gamma) = 5.8 (\sim 10^{-3}) [3.7] \times 10^{-6} ,
\text{Br}(Z' \rightarrow ZZ) = 3.0 (2.5) [0.9] \times 10^{-7} .
\]

These results are pretty much the same for bigger \( M_{Z'} \) values. As we see, the branching fraction for \( Z' \rightarrow \gamma Z \) is in the region of \( 10^{-5} - 10^{-6} \) while for \( Z' \rightarrow ZZ \) in the region \( \sim 10^{-7} \). These are very challenging numbers even for LHC@14 TeV.

In coordinate space representation, the vertices (71) and (72) arise on-shell from the following operators

\[ \mathcal{O}_{Z'\gamma Z} \sim \varepsilon_{\mu'\nu'\rho} Z'_{\mu'} Z_{\nu'} F_{\rho\sigma} , \]
\[ \mathcal{O}_{Z'ZZ} \sim \varepsilon_{\mu'\nu'\rho} Z'_{\mu'} Z_{\nu'} \partial_{\rho} V_{\sigma} , \]

which are both P-odd but CP-invariant. Although not present in the SM and in the \( Z' \)-models under consideration there may be P-even but CP-violating operators of the form \( \mathcal{O}_{Z'ZZ} \sim Z'^{\mu}(\partial^\nu Z_{\mu}) Z_{\nu} \) induced by a triple scalar loop instead. The latter would interfere with (80) and there is a proposal in ref. [75] on how their effects can be separated at the LHC. However, within minimal \( Z' \)-models considered here this looks very difficult due to tiny \( \text{Br}(Z' \rightarrow VV) \) of eq. (78).

V. CONCLUSIONS

We construct an effective 1PI vertex for triple gauge bosons for every renormalized theory making explicit mentioning to the chiral anomalies and their synergy with heavy fermion decoupling phenomena. Our method for calculating the vertex is based on ref. [38]. It is quite general and can be divided in four steps:

1. Write down the most general, Lorentz (and/or possibly other symmetry) invariant effective vertex \( \Gamma^{\mu\nu\rho} \) [like eq. (2)] with unknown form factors.

2. Isolate the -potentially- infinite form factors and calculate only the finite parts.

3. Derive Ward Identities arising from the underlying spontaneously broken gauge symmetries at the quantum level. Apply them to \( \Gamma^{\mu\nu\rho} \) and calculate the ambiguous form factors, thus forcing them to be finite.

4. If the vertex is still undetermined \( i.e. \), if arbitrary parameters still remain, try to fix them by physical requirements. If nevertheless arbitrariness persists, then the model needs completion, perhaps with new particles or new dynamics.

This method, explained in detail in Appendix [3] and in section [1] does not require dimensional regularisation or other integral regularisation technics. It may require,
however, “shifting momenta” technics like eq. (B1). The above steps can be augmented with additional relations. Instead of WIs, one could use other identities like for example those arising from perturbative unitarity sum rules or the Goldstone boson equivalence theorem e.g., eq. (11).

All the above steps are realized when calculating triple gauge boson vertices in spontaneously broken gauge theories, like for example the SM or its extensions like minimal Z′-models. The anomalous terms are arbitrary and can only be fixed by physics. Only then can we discuss non-decoupling effects in the broken limit. We observe that for $V^*VV, V = \gamma, Z$ and for $\gamma^*WW$ vertices, there are two arbitrary parameters that are completely determined by two physical symmetries: $U(1)_{em}$ and Bose symmetry or CP-invariance. We find that at the limit of heavy fermion masses, non-decoupled terms cancel exactly those that arise from anomalies. For example, in the SM, decoupling of the top quark will leave behind anomalous-terms of light quarks and leptons plus finite parts. On the other hand vertices like $Z\gamma\gamma, Z\gamma Z$ are in general undetermined because there are no enough symmetries to fix the arbitrary parameters. Of course for anomaly-free models this arbitrariness is removed when adding up all fermion contributions.

We made a numerical analysis for SM and minimal $Z'$-model vertices. To this end, we made an effort to calculate finite integrals in terms of standard functions that are easy to handle. For example in Appendix E we solved analytically the integrals for $V^*V$-vertices. We then proceeded to SM predictions for the triple gauge boson vertices. Unfortunately, it turns out that within the SM these are rather small to be discovered even at the LHC with $\sqrt{s} = 14$ TeV. Similar results are obtained in the SM extended by a sequential fourth fermion generation. The difference w.r.t the SM, is that $|\Gamma_{\gamma^*VV}(s)|$ is “delayed” to vanish for large $\sqrt{s}$ due to the heavy, 4th generation thresholds (see Figs. 3). In the best case, the SM + 4th generation predicts a maximum of a few $10^{-3}$ for $|\Gamma_{\gamma^*Z}|$ [see dashed lines in Figs. 3].

We have performed a numerical analysis, shown in Fig. 7 for minimal $Z'$-models with $U(1)_{B-L}$ symmetry, SO(10)-like and $U(1)_{\Delta R}$ also extended with a 4th fermion generation. For a conservative choice of $M_{Z'} = 1$ TeV and $g_z = \alpha_{em}$, we find $|\Gamma_{Z'ZZ}|$ and $|\Gamma_{Z'\gamma Z}|$ in the regime below a few $10^{-5}$. We also briefly discussed $Z'$-decays to $Z\gamma$ and $ZZ$. Adopting the parameters space above, their branching ratio come out to be in the neighborhood of $10^{-5}$ and $10^{-7}$, respectively.

In section III B and Appendix E we calculated non-decoupling effects that arise instantaneously with vanishing anomalies. We constructed several toy models with two or three external gauge bosons and a number of fermions where this situation could take place. In principle, these models can be used as a basis towards realistic extensions of the SM.

Our main result, the effective triple gauge boson vertex obtained in section III can be used in various ways: i) in models with anomalous spectrum, ii) in realistic anomaly driven models of section III B, iii) in MSSM and its extensions, iv) in dark matter or neutrino - nucleon scattering processes with a photon in the final state. We will pursue some of these issues in a forthcoming article.

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**Appendix A: A Set-Up Toy Model for Calculations**

Consider a gauge theory of a complex scalar field $\Phi$ charged under a local $U(1)$ with charge $Y_\Phi$ (in units of $e$), a vector spin-1 abelian gauge boson $A_\mu$ and a pair of Dirac fermions $E_L$ and $e_R$ with $U(1)$-charges $Y_L$ and $Y_R$ respectively. This gauge theory is described by the Lagrangian $\mathcal{L}$

$$\mathcal{L} = \mathcal{L}_g(\Phi, A_\mu) + \mathcal{L}_f(E_L, e_R, A_\mu) + \mathcal{L}_Y(E_L, e_R, \Phi) ,$$

where the gauge boson-scalar interactions are

$$\mathcal{L}_g(\Phi, A_\mu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (G)^2 + |D_\mu \Phi|^2 - V(\Phi) ,$$

---

9 Throughout we follow the notation and conventions of ref. [70].
The local, $U(1)$ gauge-transformation

$$\Phi(x) \rightarrow e^{ieY_\Phi(x)}\Phi(x) , \quad A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \Lambda(x) , \quad (A5)$$

and $D_\mu \Phi = \partial_\mu \Phi + ieY_\Phi A_\mu \Phi$, $D_\mu E_L = \partial_\mu E_L + ieY_L A_\mu E_L$, and $D_\mu e_R = \partial_\mu e_R + ieY_R A_\mu e_R$. $L_g$ is invariant under the local, $U(1)$ gauge-transformation

$$E_L(x) \rightarrow e^{ieY_L(x)}E_L(x) , \quad e_R(x) \rightarrow e^{ieY_R(x)}e_R(x) , \quad (A6)$$

iff $Y_\Phi = Y_L - Y_R$. It is convenient to combine the left and right-handed fermions into a single Dirac four-component spinor $\Psi = (E_L, e_R)^T$. Then the interaction Lagrangian relevant to our study for triangle graphs reads:

$$L_{int} = -\lambda_e \bar{\Psi} \gamma_\mu P_R \Psi - \lambda_e \bar{\Psi} \gamma_\mu P_L \Psi - eA_\mu \bar{\Psi} \gamma^\mu (\alpha + \beta \gamma_5) \Psi , \quad (A7)$$

where

$$\alpha = \frac{Y_L + Y_R}{2} , \quad \beta = \frac{Y_R - Y_L}{2} . \quad (A8)$$

Under gauge transformations the 4-component field $\Psi$ transforms as

$$\Psi(x) \rightarrow e^{ie(\alpha + \beta \gamma_5)\Lambda(x)}\Psi(x) , \quad (A9a)$$

$$\bar{\Psi}(x) \rightarrow \bar{\Psi}(x)e^{-ie(\alpha - \beta \gamma_5)\Lambda(x)} , \quad (A9b)$$

which together with eq. (A6) leave $L$ invariant if $Y_\Phi = -2\beta$.

We choose a renormalizable and gauge invariant potential $V(\Phi)$ such that the field $\Phi$ acquires a non-vanishing vacuum expectation value, $\langle \Phi \rangle = v/\sqrt{2}$, which breaks the local $U(1)$ symmetry spontaneously. We expand eq. (A1) around the minimum, $\Phi = v/\sqrt{2}(v + h + i\varphi)$ and choose a gauge-fixing function in eq. (A2),

$$G = \frac{1}{\sqrt{2}} (\partial_\mu A^\mu - \xi ev\varphi) , \quad (A10)$$

which eliminates the Goldstone boson - gauge boson mixing term. The mass of the vector boson $A_\mu$ and of the unphysical Goldstone boson $\varphi$ in this $R_\xi$-gauge become

$$m_A = evY_\Phi , \quad m_\varphi^2 = \xi m_A^2 . \quad (A11)$$

The ghost part of $L$ is not relevant to our discussion for the one-loop triangle graphs and is not presented. In terms of $\Psi$ and $\bar{\Psi}$, $L_f + L_Y$ becomes

$$L_f(\Psi, A_\mu) + L_Y(\Psi, h, \varphi) = \bar{\Psi} i\not{D} \Psi - eA_\mu \bar{\Psi} \gamma^\mu (\alpha + \beta \gamma_5) \Psi - m\bar{\Psi} \Psi - \bar{\beta}\bar{\Psi} h\Psi - i\bar{\beta}\bar{\Psi} \gamma_5 \varphi \Psi , \quad (A12)$$

where $m = v\hat{\beta}$ and $\hat{\beta} = \frac{\Lambda_0}{\sqrt{2}}$.

This model, albeit very simple, captures the most important non-decoupling heavy fermion effects in the trilinear gauge boson vertices in the Standard Model and its extensions. In the context of chiral anomalies it has been exploited in ref. [15]. With a light language deform it imitates the Standard Model with the difference that its WI’s for the currents corresponding to the gauge symmetry in eq. (A6) are anomalous as we shall see below.

**Appendix B: Calculation of the Three Point Gauge Boson Vertex**

In this Appendix we explicitly evaluate the three external gauge boson, fermionic one-loop amplitude of Fig. 1. The loop function is calculated directly in four dimensions using standard methods studied in refs. [7, 38, 77–79]. Here, we review this calculation in detail for the toy model of Appendix A. At the end we generalise our results to the case of three different external (massive or massless) gauge bosons.
By naive power counting we observe that the two diagrams in Fig. 1 are linearly divergent. This means that their quantum amplitudes depend on the routing of the internal momenta circulating in the loop. In each of the two diagrams we shift the internal momenta with arbitrary four vectors \( a^\mu \) and \( b^\mu \), respectively. By reading Feynman rules from eq. \( (A7) \), the graphs in Fig. 1 become

\[
\Gamma^{\mu\nu\rho}(k_1, k_2; a, b) = (-1) e^3 \times 
\begin{align*}
& \text{Tr} \left\{ \int \frac{d^4p}{(2\pi)^4} \frac{\gamma^\mu(\alpha + \beta \gamma^5)(\not{p} - k_2 + \not{a} + m)\gamma^\nu(\alpha + \beta \gamma^5)(\not{p} + \not{a} + m)\gamma^\rho(\alpha + \beta \gamma^5)(\not{p} + k_1 + \not{a} + m)}{[(p - k_2 + a)^2 - m^2][(p + a)^2 - m^2][(p + k_1 + a)^2 - m^2]} \right. \\
& \left. + \int \frac{d^4p}{(2\pi)^4} \frac{\gamma^\mu(\alpha + \beta \gamma^5)(\not{p} - k_1 + \not{b} + m)\gamma^\nu(\alpha + \beta \gamma^5)(\not{p} + \not{b} + m)\gamma^\rho(\alpha + \beta \gamma^5)(\not{p} + k_2 + \not{b} + m)}{[(p - k_1 + b)^2 - m^2][(p + b)^2 - m^2][(p + k_2 + b)^2 - m^2]} \right\},
\end{align*}
\]

(B1)

where \( m \) is the fermion mass and \(-1\) is a fermionic loop factor. The integral in the second line is the same as the first with only the difference that the upper two external legs in Fig. 1 are interchanged, \( \{\nu, \rho\} \leftrightarrow \{\rho, \nu\} \) and \( k_1 \leftrightarrow k_2 \). Dimensional regularization is a scheme not well suited in calculating \( (B1) \) due to the problems in defining \( \gamma_5 \) and \( \epsilon^{\mu\nu\rho\sigma} \) in \( d > 4 \) spacetime dimensions. We here follow a method for calculating \( (B1) \) first presented by Rosenberg in ref. \[38\] and later used by Adler in his classic paper on chiral anomaly \[7\]. Basically, this method relies on the fact that the abigous part of the integral is stored in two form factors in \( \Gamma^{\mu\nu\rho} \) expansion, \( A_2 \) and \( A_1 \), that multiply the external momenta \( k_1 \) and \( k_2 \), respectively. We then exploit physical arguments like for example conservation of charge, in order to determine the form factors \( A_1, A_2 \) - all others, \( A_3...A_6 \) are finite and can be calculated directly in 4-dimensions.

Our next step is to write down the WIs. This can be done in many ways, probably the most insightful is the use of functional methods (see for instance Chapter 9.6 in the textbook of ref. \[76\]). One finds the classical WIs of eq. \( (3) \), but not the last term on the r.h.s. We show below how to calculate this last term. We need first to calculate the divergence of the 1PI vertex: \( q_\mu \Gamma^{\mu\nu\rho} = (k_1 + k_2)_\mu \Gamma^{\mu\nu\rho} \). It is useful to employ the following algebraic identity:

\[
\hat{q}(\alpha + \beta \gamma^5) = - (\alpha - \beta \gamma^5)(\not{p} - k_2 + \not{a} + m) + 2\beta \gamma^5 m + (\not{p} + k_1 + \not{a} + m)(\alpha + \beta \gamma^5),
\]

(B2)
in the first integral of \( (B1) \) and a similar identity with \( a \rightarrow b \) and \( k_1 \rightarrow k_2 \) in the second one. These identities split \( q_\mu \Gamma^{\mu\nu\rho} \) into two parts,

\[
q_\mu \Gamma^{\mu\nu\rho}(k_1, k_2; a, b) = - \frac{2m\beta e^3}{\hat{\beta}} \Gamma^{\nu\rho}(k_1, k_2; a, b) + \Pi^{\nu\rho}(k_1, k_2; a, b),
\]

(B3)
a part that is proportional to the fermion mass \( m \) and a part which contains divergent two-point functions that \textit{would had been zero if shifting of the momenta variable was allowed}. The latter integrals will be responsible for the failure of the axial vector WI's. Explicitly \( \Gamma^{\nu\rho} \) and \( \Pi^{\nu\rho} \) in eq. \( (B3) \) read,

\[
\Gamma^{\nu\rho}(k_1, k_2; a, b) = - i e^2 \hat{\beta} \times 
\begin{align*}
& \text{Tr} \left\{ \int \frac{d^4p}{(2\pi)^4} \frac{\gamma^\nu(\not{p} - k_2 + \not{a} + m)\gamma^\rho(\alpha + \beta \gamma^5)(\not{p} + \not{a} + m)\gamma^\nu(\alpha + \beta \gamma^5)(\not{p} + k_1 + \not{a} + m)}{[(p - k_2 + a)^2 - m^2][(p + a)^2 - m^2][(p + k_1 + a)^2 - m^2]} \right. \\
& \left. + \int \frac{d^4p}{(2\pi)^4} \frac{\gamma^\nu(\not{p} - k_1 + \not{b} + m)\gamma^\rho(\alpha + \beta \gamma^5)(\not{p} + \not{b} + m)\gamma^\nu(\alpha + \beta \gamma^5)(\not{p} + k_2 + \not{b} + m)}{[(p - k_1 + b)^2 - m^2][(p + b)^2 - m^2][(p + k_2 + b)^2 - m^2]} \right\} = - \frac{i e^2 m}{2\pi^2} \hat{\beta} \varepsilon^{\lambda\mu\nu\rho} k_1\lambda k_2\sigma I_0(k_1, k_2, m),
\end{align*}
\]

(B4)

where

\[
I_0(k_1, k_2, m) = \left. \int \frac{d^4k}{m^4} \left( \frac{\alpha^2 - \beta^2}{x(x - 1)k_2^2} + \frac{2(x + y)\beta^2}{y(y - 1)k_1^2 - 2xyk_1 \cdot k_2} \right) \right|_{x = 0}^{1}.
\]

(B5)

Obviously, the integral in \( \Gamma^{\nu\rho} \) in eq. \( (B4) \) is obtained from \( \Gamma^{\mu\nu\rho} \) in eq. \( (B1) \) with the replacement \( \gamma^\mu(\alpha + \beta \gamma^5) \rightarrow \gamma^5 \), that is a replacement of a vector-axial vector coupling with a pseudoscalar. This validates the PCAC relation in eq. \( (B3) \). Note that \( \Gamma^{\nu\rho} \) is finite and independent on the arbitrary vectors \( a^\mu \) and \( b^\mu \) : \( \Gamma^{\nu\rho}(k_1, k_2; a, b) = \Gamma^{\nu\rho}(k_1, k_2) \).
The divergent part $\Pi^{\nu\rho}$ in the WI of eq. (B3) contains, among others, the anomalous term. It is written explicitly as,

$$
\Pi^{\nu\rho}(k_1, k_2; a, b) = (-e^3) \text{Tr} \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{-(\alpha - \beta \gamma^5)(\alpha - \beta \gamma^5)\gamma^\rho(\phi + \phi + m)\gamma^\nu(\alpha + \beta \gamma^5)(\phi + k_2 + \phi + m)}{[(p + a)^2 - m^2][(p + k_1 + a)^2 - m^2]} + \frac{(\phi - k_2 + \phi + m)\gamma^\rho(\alpha + \beta \gamma^5)(\phi + \phi + m)\gamma^\nu(\alpha + \beta \gamma^5)(\phi + k_2 + \phi + m)}{[(p + b)^2 - m^2][(p + k_2 + b)^2 - m^2]} - \frac{(\alpha - \beta \gamma^5)(\alpha - \beta \gamma^5)\gamma^\rho(\phi + \phi + m)\gamma^\nu(\alpha + \beta \gamma^5)(\phi + k_2 + \phi + m)}{[(p + b)^2 - m^2][(p + k_2 + b)^2 - m^2]} + \frac{(\phi - k_2 + \phi + m)\gamma^\rho(\alpha + \beta \gamma^5)(\phi + \phi + m)\gamma^\nu(\alpha + \beta \gamma^5)(\phi + k_2 + \phi + m)}{[(p + b)^2 - m^2][(p + k_2 + b)^2 - m^2]} \right\}.
$$

(B6)

This is an integral that is divided into two parts: a chiral expression \textit{i.e.}, the one that contains $\gamma^5$ and a non-chiral expression that does not contain $\gamma^5$. Since the anomalous term is originated from the chiral part we start from there. Hence,

$$
\Pi^{\nu\rho}_{\text{chiral}}(k_1, k_2; a, b) = (\beta^3 + 3\alpha^2 \beta)e^3 \times \text{Tr} \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{(\phi + k_2 + \phi)\gamma^\rho(\phi + \phi + m)\gamma^\nu(\alpha + \beta \gamma^5)(\phi + k_2 + \phi + m)}{[(p + k_1 + a)^2 - m^2][(p + a)^2 - m^2]} - \frac{(\phi + \phi)\gamma^\rho(\phi - k_2 + \phi)\gamma^\nu(\alpha + \beta \gamma^5)(\phi - k_2 + \phi + m)}{[(p + b)^2 - m^2][(p - k_2 + a)^2 - m^2]} + \frac{(\phi + \phi)\gamma^\rho(\phi - k_2 + \phi)\gamma^\nu(\alpha + \beta \gamma^5)(\phi - k_2 + \phi + m)}{[(p + b)^2 - m^2][(p - k_2 + b)^2 - m^2]} \right\}.
$$

(B7)

Grouping together the first and the fourth as well as the third and the second terms in the integrand of eq. (B7), we arrive at,

$$
\Pi^{\nu\rho}_{\text{chiral}}(k_1, k_2; a, b) = (\beta^3 + 3\alpha^2 \beta)e^3 \times \text{Tr} \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{(p + k_2 + b)\gamma^\rho(\alpha + \beta \gamma^5)(\phi + \phi + m)\gamma^\nu(\alpha + \beta \gamma^5)(\phi + k_2 + \phi + m)}{[(p + k_2 + b)^2 - m^2][(p + b)^2 - m^2] - \frac{(p + a)\gamma^\rho(\alpha - \beta \gamma^5)(\phi - k_2 + \phi)\gamma^\nu(\alpha + \beta \gamma^5)(\phi - k_2 + \phi + m)}{[(p + a)^2 - m^2][(p - k_2 + a)^2 - m^2]} \right\}.
$$

(B8)

Following the steps described in ref. [78], we first define a function and an integral,

$$
f_{\kappa,\lambda}(p; c, d) = \frac{(p + c)\gamma^\nu(\alpha + \beta \gamma^5)(\phi + \phi + m)\gamma^\rho(\alpha + \beta \gamma^5)(\phi + k_2 + \phi + m)}{[(p + c)^2 - m^2][(p + d)^2 - m^2]},
$$

(B9)

and

$$
I_{\kappa,\lambda}(k; c, d) = \int \frac{d^4p}{(2\pi)^4} \left[ f_{\kappa,\lambda}(p + k; c, d) - f_{\kappa,\lambda}(p; c, d) \right].
$$

(B10)

where $c, d$ are arbitrary four vectors. By exploiting the following “momentum shift” integral relation (see the lecture by R. Jackiw in ref. [77] and refs. [78, 19, 79])

$$
\int \frac{d^4p}{(2\pi)^4} \left[ f(p + a) - f(p) \right] = \frac{i}{(2\pi)^4} \left[ 2\pi^2 a_\mu \lim_{p \to \infty} p^\mu p^2 f_\rho(p) + \pi^2 a_\mu a_\nu \lim_{p \to \infty} p^\mu p^2 \frac{\partial f_\rho(p)}{\partial p_\nu} \right],
$$

(B11)

where only the first term on the r.h.s is relevant to linearly divergent diagrams, and,

$$
f_\rho(p) = \frac{1}{2} f(p) - f(-p), \quad f_\epsilon(p) = \frac{1}{2} \left[ f(p) + f(-p) \right],
$$

(B12)
are the odd and even parts of $f(p)$ respectively, we obtain\(^\text{10}\)

$$I_{\lambda\lambda}(k; c, d) = \frac{i}{96\pi^2} \left[ 2k_\lambda c_\kappa + 2k_\kappa d_\lambda - k_\lambda d_\kappa - k_\kappa c_\lambda - g_{\kappa\lambda} k \cdot (k + c + d) + k_\lambda k_\kappa \right]. \quad (B13)$$

Now we have all the necessary machinery to calculate $\Pi^{\nu\rho}$ in eq. (B8) by applying to it eqs. (B11) and (B13). For the non-chiral part of $\Pi^{\nu\rho}$ the choice $b = -a$ results in $\Pi^{\nu\rho}_{\text{non-chiral}} = 0$ as we expect, since there should be no non-chiral anomalies. With this assignment for vector $b$ we finally obtain for the chiral part:

$$\Pi^{\nu\rho}_{\text{chiral}}(k_1, k_2; a, -a) = \frac{e^3(\beta^3 + 3\alpha^2\beta)}{4\pi^2} \epsilon^{\kappa\nu\alpha\rho} a_\kappa(k_1 + k_2)_\lambda. \quad (B14)$$

Plugging in eqs. (B4) and (B14) into eq. (B3), the WI associated to the leg $-\mu-$ becomes:

$$q_\mu \Gamma^{\mu\nu}(k_1, k_2; a, -a) = -\frac{2m\epsilon_i}{\beta} \Gamma^{\nu\mu}(k_1, k_2) + \frac{e^3(\beta^3 + 3\alpha^2\beta)}{4\pi^2} \epsilon^{\kappa\nu\alpha\rho} a_\kappa(k_1 + k_2)_\lambda. \quad (B15)$$

Along the same lines we can build in the WIs for the other vertices. For example, the WI referring to the conservation of current in vertex $-\nu-$ (see Fig.\[\text{I}\]) reads:

$$-k_{1\nu} \tilde{\Gamma}^{\mu\nu\rho}(k_1, k_2; a, -a) = -\frac{2m\epsilon_i}{\beta} \Gamma^{\nu\mu}(k_1, k_2) - \frac{e^3(\beta^3 + 3\alpha^2\beta)}{4\pi^2} \epsilon^{\kappa\mu\alpha\rho} a_\kappa(k_1 + k_2)_\lambda. \quad (B16)$$

Vertices $\tilde{\Gamma}^{\mu\nu\rho}(k_1, k_2; a, b)$ and $\tilde{\Gamma}^{\nu\mu}(k_1, k_2)$ are obtained from $\Gamma^{\mu\nu\rho}(k_1, k_2; a, b)$ and $\Gamma^{\nu\mu}(k_1, k_2)$ in eqs. (B1) and (B4), respectively, after the following replacements:

$$
\begin{align*}
\mu &\to \nu, \quad \nu \to \rho, \quad \rho \to \mu, \quad a \to a - k_2, \quad b \to b + k_2, \quad k_1 \to k_2, \\
k_2 &\to -k_1 - k_2, \quad q = k_1 + k_2 \to -k_1 - k_2 - k_1 = -k_1 \Rightarrow q \to -k_1.
\end{align*}
\quad (B17)
$$

It is straightforward to see from eq. (B17) that the non-chiral part of $-k_{1\nu} \tilde{\Gamma}^{\mu\nu\rho}(k_1, k_2; a, b)$ vanishes again for the choice $b = -a$. Similarly the WI for the current conservation in the $-\rho-$ vertex,

$$-k_{2\rho} \tilde{\Gamma}^{\mu\nu\rho}(k_1, k_2; a, -a) = -\frac{2m\epsilon_i}{\beta} \Gamma^{\rho\mu}(k_1, k_2) - \frac{e^3(\beta^3 + 3\alpha^2\beta)}{4\pi^2} \epsilon^{\kappa\mu\alpha\rho} a_\kappa(k_1 + k_2)_\lambda. \quad (B18)$$

As previously, $\tilde{\Gamma}^{\nu\mu\rho}(k_1, k_2; a, b)$ and $\tilde{\Gamma}^{\mu\nu}(k_1, k_2)$ can be obtained from eqs. (B1) and (B4) by making the following replacements:

$$
\begin{align*}
\mu &\to \rho, \quad \nu \to \mu, \quad \rho \to \nu, \quad a \to a + k_1, \quad b \to b - k_1, \quad k_1 \to -k_2 - k_1, \\
k_2 &\to k_1, \quad q = k_1 + k_2 \to -k_2 - k_1 + k_1 \Rightarrow q \to -k_2.
\end{align*}
\quad (B19)
$$

These replacements leave invariant the choice $b = -a$ so that finally, the non-chiral part of $-k_{2\rho} \tilde{\Gamma}^{\nu\mu\rho}(k_1, k_2; a, -a)$ vanishes identically everywhere. Furthermore, by direct calculation the vertices $\tilde{\Gamma}^{\mu\nu}$ and $\tilde{\Gamma}^{\mu\nu}$ are found to be,

$$\tilde{\Gamma}^{\nu\mu}(k_1, k_2) = \frac{ie^2 m^2}{2\pi^2} \epsilon^{\rho\kappa\lambda} k_1\kappa k_2\lambda I_1(k_1, k_2, m), \quad (B20)$$

and

$$\tilde{\Gamma}^{\mu\nu}(k_1, k_2) = \frac{ie^2 m^2}{2\pi^2} \epsilon^{\rho\kappa\lambda} k_1\kappa k_2\lambda I_2(k_1, k_2, m), \quad (B21)$$

respectively, where the corresponding integrals $I_{1,2}$ are written explicitly as,

$$I_1(k_1, k_2, m) = \int_0^1 dx \int_0^{1-x} dy \frac{-(\alpha^2 + \beta^2) + 2x\beta^2}{x(x - 1)k_2^2 + y(y - 1)k_1^2 - 2xyk_1k_2 + m^2}, \quad (B22)$$

\(^{10}\) There is a typographical error in the corresponding expression of a classic textbook written by S. Weinberg in ref. [78]. We thank Steve Martin and Howie Haber for communication related to this point.
and

\[ I_2(k_1, k_2, m) = \int_0^1 dx \int_0^{1-x} dy \frac{(\alpha^2 + \beta^2) - 2y\beta^2}{x(x-1)k_1^2 + y(y-1)k_2^2 - 2xyk_1 \cdot k_2 + m^2}. \]  

(B23)

The three-point vertex obeys the following equality,

\[ \Gamma^{\mu\nu\rho} = \tilde{\Gamma}^{\nu\rho\mu} = \tilde{\Gamma}^{\rho\mu\nu}, \]

(B24)

as the property of trace to remain invariant under cyclic permutations. It is instructive to write the arbitrary vector \( a^\mu \), appearing in the WIs, as a linear combination of the two independent momenta \( k_1 \) and \( k_2 \),

\[ a^\mu = z k_1^\mu + w k_2^\mu, \]

(B25)

with \( z, w \) arbitrary real numbers. Then the WIs in eqs. (B15), (B16) and (B18) can be written explicitly in terms of the three integrals \( I_0, I_1, \) and \( I_2 \) and the real numbers \( w \) and \( z \) as,

\[ g_\mu \Gamma^{\mu\nu\rho}(k_1, k_2; w, z) = -\frac{e^3\beta m^2}{\pi^2} \epsilon^{\nu\rho\sigma} k_1 k_2 I_0(k_1, k_2; m) + \frac{e^3(\beta^3 + 3\alpha^2 \beta)}{4\pi^2} \epsilon^{\lambda\nu\rho\sigma} k_1 \cdot k_2 (w - z). \]  

(B26)

\[ -k_1 \mu \Gamma^{\nu\rho\mu}(k_1, k_2; w) = -\frac{e^3\beta m^2}{\pi^2} \epsilon^{\lambda\nu\rho\sigma} k_1 k_2 I_1(k_1, k_2; m) + \frac{e^3(\beta^3 + 3\alpha^2 \beta)}{4\pi^2} \epsilon^{\lambda\mu\nu\rho}(w - 1) k_1 \cdot k_2. \]  

(B27)

\[ -k_2 \mu \Gamma^{\nu\rho\mu}(k_1, k_2; z) = -\frac{e^3\beta m^2}{\pi^2} \epsilon^{\lambda\mu\nu\rho} k_1 k_2 I_2(k_1, k_2; m) + \frac{e^3(\beta^3 + 3\alpha^2 \beta)}{4\pi^2} \epsilon^{\lambda\mu\nu\rho} (z + 1) k_1 \cdot k_2. \]  

(B28)

Obviously, even if we choose \( w = 1 \) and \( z = -1 \) so that the second and third anomalous terms vanish it cannot be done so for the first one. The second term on the r.h.s. of eq. (B26), remains. It is quite interesting to note that in the limit where \( k_1^2, k_2^2, k_1 \cdot k_2 \ll m \to \infty \), there is a choice for \( w = -z = 1/3 \) such that the right hand side of eqs. (B26), (B27) and (B28) vanishes identically. For this choice the fermions get decoupled completely.

Our goal is still to calculate the three gauge boson vertex \( \Gamma^{\mu\nu\rho}(k_1, k_2; a, -a) \). The idea is to first write down the most general, Lorentz invariant vertex, and then:

\[ \Gamma^{\mu\nu\rho}(k_1, k_2; a, -a) = \left[ A_1(k_1, k_2; a, -a) \epsilon^{\mu\nu\rho\sigma} k_2 \sigma + A_2(k_1, k_2; a, -a) \epsilon^{\mu\rho\sigma\nu} k_1 \sigma + A_3(k_1, k_2) \epsilon^{\mu\rho\beta\delta} k_1^\beta k_2^\delta \right. 

\[ + A_4(k_1, k_2) \epsilon^{\mu\beta\delta \gamma} k_1^\beta k_2^\delta + A_5(k_1, k_2) \epsilon^{\mu\beta\delta \gamma} k_2^\beta k_1^\delta + A_6(k_1, k_2) \epsilon^{\mu\beta\delta \gamma} k_1^\beta k_2^\delta \]. \]

(B32)

The form factors \( A_1 \) and \( A_2 \) are dimensionless and, by naive power counting, at most linearly divergent while all the rest, \( A_3...A_6 \) possess dimension of \( m^{-2} \) and are finite. The latter can be calculated directly in four dimensions from eq. (B1). We find explicitly:

\[ A_3(k_1, k_2) = -A_6(k_1, k_2) = -\frac{e^3(\beta^3 + 3\alpha^2 \beta)}{\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{xy}{\Delta}, \]

(B33)

\[ A_4(k_1, k_2) = \frac{e^3(\beta^3 + 3\alpha^2 \beta)}{\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{y(y-1)}{\Delta}, \]

(B34)

\[ A_5(k_1, k_2) = -\frac{e^3(\beta^3 + 3\alpha^2 \beta)}{\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{x(x-1)}{\Delta}. \]

(B35)

\[ A_7(k_1, k_2) \epsilon^{\mu\rho\beta\delta} k_1^\alpha k_2^\beta k_2^\delta + A_8(k_1, k_2) \epsilon^{\mu\rho\beta\delta} k_1^\alpha k_2^\delta k_2^\beta. \]  

(B29)

However, by exploiting the following, very useful, identities

\[ k_1^\mu \epsilon^{\mu\rho\beta\delta} k_1^\beta k_2^\delta = -\epsilon^{\mu\rho\beta\delta} k_1^\beta k_2^\delta + \epsilon^{\mu\rho\beta\delta} k_1^\beta k_2^\delta k_1^\alpha - \epsilon^{\mu\rho\beta\delta} k_1^\beta k_2^\delta k_2^\alpha, \]

(B30)

\[ k_2^\mu \epsilon^{\mu\rho\beta\delta} k_1^\beta k_2^\delta = -\epsilon^{\mu\rho\beta\delta} k_2^\beta k_1^\delta k_2^\alpha + \epsilon^{\mu\rho\beta\delta} k_2^\beta k_1^\delta k_2^\alpha k_1^\alpha - \epsilon^{\mu\rho\beta\delta} k_2^\beta k_1^\delta k_2^\alpha k_1^\alpha, \]

(B31)

We arrive at the six form factors given in eq. (B32).
where the integrand denominator is common for all $A_3\ldots A_6$ and reads:

$$\Delta \equiv x(x-1)k_2^2 + y(y-1)k_1^2 - 2xyk_1 \cdot k_2 + m^2. \quad (B36)$$

To estimate the two divergent integrals, $A_1$ and $A_2$, we apply the Ward Identities for the vertices $\nu$ and $\rho$, i.e., eqs. \([B27]\) and \([B28]\) in the expansion \([B32]\) and obtain,

$$A_1(k_1, k_2; w) = (k_1 \cdot k_2) A_3(k_1, k_2) + k_1^2 A_4(k_1, k_2) - \frac{m^2e^3\beta}{\pi^2} I_1(k_1, k_2, m) + \frac{e^3(\beta^3 + 3\alpha^2\beta)}{4\pi^2}(w-1), \quad (B37)$$

and,

$$A_2(k_1, k_2; z) = (k_1 \cdot k_2) A_6(k_1, k_2) + k_2^2 A_5(k_1, k_2) - \frac{m^2e^3\beta}{\pi^2} I_2(k_1, k_2, m) + \frac{e^3(\beta^3 + 3\alpha^2\beta)}{4\pi^2}(z+1). \quad (B38)$$

Equations \([B22\ B23\ B33\ B38]\) complete the evaluation of the vertex $\Gamma^{\mu\nu\rho}(k_1, k_2, w, z)$ in eq. \([B32]\). In Appendix D we present analytical expressions of the integrals $A_{3,6}$ and $I_{0,1,2}$ in various limits.

Even if the form factors $A_{i=1\ldots6}$ had not been calculated explicitly there is much to say about their structure by exploiting possible Bose symmetries. Hence, referring to the notation of Fig. I Bose symmetry among $j$ and $k$ legs implies,

$$A_1(k_1, k_2) = -A_2(k_2, k_1), \quad (B39a)$$

$$A_3(k_1, k_2) = -A_6(k_2, k_1), \quad (B39b)$$

$$A_4(k_1, k_2) = -A_5(k_2, k_1), \quad (B39c)$$

while in $i$ and $j$ legs,

$$A_1(k_1, k_2) = -A_1(-q, k_2) + A_2(-q, k_2) - (k_1 \cdot k_2) [A_1(-q, k_2) - A_4(-q, k_2)] + k_1^2 A_4(-q, k_2), \quad (B40a)$$

$$A_2(k_1, k_2) = A_2(-q, k_2) + k_2^2 [A_3(-q, k_2) - A_4(-q, k_2)] - (k_1 \cdot k_2) A_4(-q, k_2), \quad (B40b)$$

$$A_3(k_1, k_2) = A_4(-q, k_2) - A_5(-q, k_2), \quad (B40c)$$

$$A_4(k_1, k_2) = A_4(-q, k_2), \quad (B40d)$$

$$A_5(k_1, k_2) = A_5(-q, k_2) - A_6(-q, k_2) + A_3(-q, k_2) - A_4(-q, k_2), \quad (B40e)$$

$$A_6(k_1, k_2) = -A_4(-q, k_2) - A_6(-q, k_2), \quad (B40f)$$

and, finally, in $i$ and $k$ legs we find,

$$A_1(k_1, k_2) = A_1(k_1, -q) - k_1^2 [(A_1(k_1, -q) - A_6(k_1, -q)] - (k_1 \cdot k_2) A_5(k_1, -q), \quad (B41a)$$

$$A_2(k_1, k_2) = A_1(k_1, -q) - A_2(k_1, -q) + (k_1 \cdot k_2) [A_5(k_1, -q) - A_6(k_1, -q)] + k_2^2 A_5(k_1, -q), \quad (B41b)$$

$$A_3(k_1, k_2) = -A_3(k_1, -q) - A_5(k_1, -q), \quad (B41c)$$

$$A_4(k_1, k_2) = A_4(k_1, -q) - A_3(k_1, -q) - A_5(k_1, -q) + A_6(k_1, -q), \quad (B41d)$$

$$A_5(k_1, k_2) = A_5(k_1, -q), \quad (B41e)$$

$$A_6(k_1, k_2) = A_5(k_1, -q) - A_6(k_1, -q). \quad (B41f)$$

The above relations have been repeatedly used in section IV when determining the anomaly parameters $w$ and $z$. The reader should notice that in addition to relations due to Bose symmetry, there are few more relations originated solely from fermionic triangle:

$$A_3(k_1, k_2) = A_3(k_2, k_1), \quad A_6(k_1, k_2) = A_6(k_2, k_1). \quad (B42)$$

We can now exploit Bose symmetry to set constraints on the arbitrary parameters $w$ and $z$. For example, if the gauge bosons associated with legs $j$ and $k$ in Fig. I are identical then eq. \([B39]\) impose the following relation,

$$w + z = 0, \quad (B43)$$
among the undefined (momentum route dependent) parameters. One last remark is that we can rediscover Bose symmetries by using one of the following equivalent representations (i.e., they leave the double integral measure invariant) of the integrals $A_3...A_6$ by noting that

\[
\Delta(k_1, k_2) \xrightarrow{x+y} \Delta(k_2, k_1), \tag{B44}
\]

\[
\Delta(k_1, k_2) \xrightarrow{x-y} \Delta(k_1, -q), \tag{B45}
\]

\[
\Delta(k_1, k_2) \xrightarrow{y-x} \Delta(-q, k_2), \tag{B46}
\]

where $\Delta(k_1, k_2)$ is a function defined in eq. (B36).

As a generalisation of eqs. (B15), (B16) and (B18) we can proceed to the situation where there are three, in general different, external gauge bosons with different couplings to fermions. As in (B11), we write the general three point vertex in Fig. 1 as:

\[
\Gamma^{\mu \nu \rho}(k_1, k_2; a, b) = \Gamma^{\nu \rho \mu}(k_1, k_2; a, b) = \hat{\Gamma}^{\rho \mu \nu}(k_1, k_2; a, b) = -e^2 \int \frac{d^4p}{(2\pi)^4} \times
\]

\[
\text{Tr} \left\{ \frac{\gamma^\mu (\alpha_i + \beta_i \gamma^5)(p - k_2 + \frac{1}{2}m_\mu) \gamma^\rho (\alpha_j + \beta_j \gamma^5)(\beta + \frac{1}{2}m_\mu) \gamma^\nu (\alpha_k + \beta_k \gamma^5)(p + k_1 + \frac{1}{2}m_\mu) }{[16 \pi^2][(p - k_2 + a)^2 - m^2][16 \pi^2][(p + a)^2 - m^2][16 \pi^2][(p + k_1 + a)^2 - m^2]}ight\}, \tag{B47}
\]

and the corresponding two point vertex functions as:

\[
\Gamma^{\mu \rho}(k_1, k_2) = \frac{-ie^2m_\beta}{2\pi^2} \varepsilon^{\lambda \rho \nu \sigma}k_1k_2x_0 \int_0^1 dx \int_0^{1-x} dy \frac{(\alpha_j \alpha_k - \beta_j \beta_k)}{\Delta},
\]

\[
\hat{\Gamma}^{\rho \mu}(k_1, k_2) = \frac{ie^2m_\beta}{2\pi^2} \varepsilon^{\lambda \mu \rho \nu}k_1k_2x_0 \int_0^1 dx \int_0^{1-x} dy \frac{-(\alpha_i \alpha_k + \beta_i \beta_k)}{\Delta}, \tag{B48}
\]

\[
\hat{\Gamma}^{\nu \mu}(k_1, k_2) = \frac{ie^2m_\beta}{2\pi^2} \varepsilon^{\lambda \mu \nu \rho}k_1k_2x_0 \int_0^1 dx \int_0^{1-x} dy \frac{(\alpha_i \alpha_j + \beta_i \beta_j)}{\Delta},
\]

where as before $\Delta \equiv \Delta(k_1, k_2)$ is given by eq. (B36). The complete $\Gamma^{\mu \nu \rho}(k_1, k_2, w, z)$ in this general case is presented in section II.

Appendix C: Charged Gauge boson Vertex

The calculation for $V^*W^-W^+, V = \gamma, Z$ is slightly more complicated than the one for neutral triple gauge boson vertices for two reasons: first, the appearance in the loop of two, in general, different fermion masses and second, the appearance of different $Vf \bar{f}$ vertex for each particle contribution (see Fig. 8). Although the first complication leads to only technical difficulties the latter one is more serious: it does not allow for an obvious exploitation of the master 4D “momentum shift” equation (B11).

Our method for calculating this vertex follows exactly the same steps as described in detail in Appendix B and in section II The chiral part of the $V^*WW$ vertex is still given by eq. (2). The finite form factors $A_3...A_6$ for the first diagram in Fig. 8 are exactly the half of the corresponding ones in (8) but with the replacement of $\Delta(k_1, k_2)$ into

\[
\Delta(k_1, k_2; m_{f_u}^2, m_{f_d}^2) \equiv x(x - 1)k_2^2 + y(y - 1)k_1^2 - 2xyk_1 \cdot k_2 - (x + y) \Delta m^2 + m_{f_u}^2, \tag{C1}
\]

with the mass squared difference being $\Delta m^2 \equiv m_{f_u}^2 - m_{f_d}^2$. $f_u$ and $f_d$ here denote each of the fermion pair $(u, v)$ and $(d, e)$ for leptons and quarks, respectively. Obviously, the contribution of the crossed diagram i.e., the second diagram in Fig. 8 requires the replacement, $f_u \leftrightarrow f_d$. Our calculation here is quite general and is not confined only in to $V^*WW$ vertex. For example, it could be used for the vertex $VW_LW_R$ in an $SU(2)_L \times SU(2)_R \times U(1)$ gauge model.
As before, the “infinite” form factors, $A_{1,2}$ are fixed by the Ward Identities. The calculation of the first diagram of Fig. 8 results in,

\[
A_1(k_1,k_2) = (k_1 \cdot k_2) A_3 + k_1^2 A_4 - \frac{\alpha_j (m_{f_j} - m_{f_d})}{4\pi^2} I_{11}(m_{f_j}^2, m_{f_d}^2) - \frac{\beta_j (m_{f_j} + m_{f_d})}{4\pi^2} I_{12}(m_{f_j}^2, m_{f_d}^2) + \frac{c}{8\pi^2} (w - 1),
\]

\[
A_2(k_1,k_2) = (k_1 \cdot k_2) A_6 + k_2^2 A_5 + \frac{\alpha_k (m_{f_j} - m_{f_d})}{4\pi^2} I_{21}(m_{f_j}^2, m_{f_d}^2) - \frac{\beta_k (m_{f_j} + m_{f_d})}{4\pi^2} I_{22}(m_{f_j}^2, m_{f_d}^2) + \frac{c}{8\pi^2} (z + 1),
\]

where $c \equiv (\alpha_i \alpha_j + \beta_i \beta_j) \beta_k + (\alpha_i \beta_j + \alpha_j \beta_i) \alpha_k$ is the usual anomaly factor. Again, the result depends upon two arbitrary real parameters. As before, we write $a^\mu$ as a linear combination of independent four vectors as $a^\mu = z k_1^\mu + w k_2^\mu$, with $z, w$ arbitrary real parameters. This includes $\gamma, Z, W$-self energy corrections. The latter depend on their own routing momenta arbitrary vectors that can be taken as such in order to eliminate their anomalous contributions. One then expects that this relation renders the non-chiral part independent of $a^\mu$ as it does for the neutral vertices $VVV$, for $V = \gamma, Z$ [see Appendix B]. However, for $VVW$-vertices there are additional contributions to the non-chiral part of $\Gamma^{\mu\nu\rho}$ from $Z, \gamma, W$-self energy corrections that depend on routing momentum arbitrary vectors. When all these corrections are added one expects the result to be independent on these arbitrary vectors.

Then the “non-decoupling” integrals, $I_{ij} \equiv I_{ij}(m_{f_j}^2, m_{f_d}^2)$ with $i, j = 1, 2$, appearing in eq. (C2) are given by

\[
I_{11} = \int_{0}^{1} \int_{0}^{1-x} dx \int_{0}^{1-x} dy \frac{\alpha_i \beta_k + \alpha_k \beta_i}{\Delta(k_1, k_2; m_{f_j}^2, m_{f_d}^2)} m_{f_d} y + (\alpha_i \beta_k + \alpha_k \beta_i) m_{f_d} x (x + y - 1) + (\alpha_i \beta_k - \alpha_k \beta_i) m_{f_d} x,
\]

\[
I_{12} = \int_{0}^{1} \int_{0}^{1-x} dx \int_{0}^{1-x} dy \frac{(-\alpha_i \beta_k + \alpha_k \beta_i)}{\Delta(k_1, k_2; m_{f_j}^2, m_{f_d}^2)} m_{f_d} y + (\alpha_i \beta_k + \alpha_k \beta_i) m_{f_d} x (x + y - 1) - (\alpha_i \beta_k - \alpha_k \beta_i) m_{f_d} x,
\]

\[
I_{21} = \int_{0}^{1} \int_{0}^{1-x} dx \int_{0}^{1-x} dy \frac{\alpha_i \beta_j - \alpha_j \beta_i}{\Delta(k_1, k_2; m_{f_j}^2, m_{f_d}^2)} m_{f_d} y + (\alpha_i \beta_j + \alpha_j \beta_i) m_{f_d} x (x + y - 1) + (\alpha_i \beta_j + \alpha_j \beta_i) m_{f_d} x,
\]

\[
I_{22} = \int_{0}^{1} \int_{0}^{1-x} dx \int_{0}^{1-x} dy \frac{(\alpha_i \beta_j - \alpha_j \beta_i)}{\Delta(k_1, k_2; m_{f_j}^2, m_{f_d}^2)} m_{f_d} y - (\alpha_i \beta_j + \alpha_j \beta_i) m_{f_d} x (x + y - 1) + (\alpha_i \beta_j + \alpha_j \beta_i) m_{f_d} x,
\]

where $\alpha_i \equiv \alpha_{f_j}, \beta_i \equiv \beta_{f_d}, \ldots$ follow the first diagram of Fig. 8. The corresponding expressions for the crossed diagram are easily obtained from those in eqs. (C2) and (C3) with the replacement $f_u \leftrightarrow f_d$. Note that CP-invariance is maintained since $A_1(k_1, k_2) = -A_2(k_2, k_1)$.

For reasons we explained at the beginning of this Appendix, finding the anomalous terms i.e., the last terms in eq. (C2), is not a straightforward task. The trick here is to add a Lorentz invariant but vanishing integral that generates exactly the anomaly integrals by momentum shift. It is then straightforward to use the 4-D expression (B11).
To complete our analysis for the chiral fermionic triangle with general external charged and neutral gauge bosons, we append here the relevant WI’s analogous to those presented in eq. (3) for neutral external gauge bosons:

\[ q_{\mu}^{\ast} \Gamma^{\mu\nu}(k_1, k_2) = \frac{\beta_i}{2\pi^2} m_{f_\mu} e^{\nu\rho\lambda\sigma} k_1 \lambda k_2 \sigma I_{01}(m_{f_\mu}^2, m_{f_\sigma}^2) + \frac{c}{8\pi^2} e^{\nu\rho\lambda\sigma} k_1 \lambda k_2 \sigma (w - z), \]

(C4a)

\[ -k_{1\mu}^{\ast} \Gamma^{\mu\nu}(k_1, k_2) = \frac{\alpha_{\mu}}{4\pi^2} (m_{f_\mu} - m_{f_\nu}) e^{\nu\rho\lambda\sigma} k_1 \lambda k_2 \sigma I_{12}(m_{f_\mu}^2, m_{f_\sigma}^2) - \frac{\beta_j}{4\pi^2} (m_{f_\mu} + m_{f_\sigma}) e^{\nu\rho\lambda\sigma} k_1 \lambda k_2 \sigma I_{12}(m_{f_\mu}^2, m_{f_\sigma}^2) + \frac{c}{8\pi^2} e^{\nu\rho\lambda\sigma} k_1 \lambda k_2 \sigma (w - 1), \]

(C4b)

\[ -k_{2\mu}^{\ast} \Gamma^{\mu\nu}(k_1, k_2) = \frac{\alpha_{k}}{4\pi^2} (m_{f_\mu} - m_{f_\nu}) e^{\nu\rho\lambda\sigma} k_1 \lambda k_2 \sigma I_{12}(m_{f_\mu}^2, m_{f_\sigma}^2) - \frac{\beta_k}{4\pi^2} (m_{f_\mu} + m_{f_\sigma}) e^{\nu\rho\lambda\sigma} k_1 \lambda k_2 \sigma I_{12}(m_{f_\mu}^2, m_{f_\sigma}^2) + \frac{c}{8\pi^2} e^{\nu\rho\lambda\sigma} k_1 \lambda k_2 \sigma (z + 1). \]

(C4c)

Again, the corresponding expressions for the crossed diagram in Fig. 8 are obtained from eq. (C4) after the replacement \( f_\mu \leftrightarrow f_d \). The integral \( I_{01} \equiv I_{01}(m_{f_\mu}^2, m_{f_\sigma}^2) \) is given by

\[ I_{01} = \int_0^1 dx \int_0^{1-x} dy \frac{(\alpha_{\mu} \alpha_k + \beta_j \beta_k) m_{f_\mu} y - (\alpha_{\mu} \alpha_k - \beta_j \beta_k) m_{f_\mu} (x + y - 1) + (\alpha_{\mu} \alpha_k + \beta_j \beta_k) m_{f_\sigma} x}{\Delta(k_1, k_2; m_{f_\mu}^2, m_{f_\sigma}^2)}. \]

(C5)

As a check, note that in the limit of equal masses \( m_{f_\mu}^2 = m_{f_\sigma}^2 \) all the above integral expressions reduce to the corresponding ones in eqs. (4), (5) and (8) for the neural gauge boson vertex.

**Appendix D: Some useful analytical integral expressions**

In this Appendix we present analytical expressions for integrals related to \( A_{3,6} \) and, \( I_{1,2} \) in the limit where \( k_1^2, k_2^2 \to 0 \) as well as their approximate expressions in various limits. We make an effort to write the latter in terms of standard functions i.e., not dilogarithms, which are easy to handle both symbolically and numerically. We start out with integrals related to eq. (8),

\[ \widetilde{A}_3(\xi) = \int_0^1 dx \int_0^{1-x} dy \frac{xy}{xy - \xi/4} = \frac{1}{2} [1 + \xi J(\xi)], \]

(D1)

where \( \xi \equiv \frac{4m^2}{s} \), \( m \) is the loop fermion mass, and \( s = (k_1 + k_2)^2 \), while,

\[ J(\xi) = - \arctan^2 \left( \frac{1}{\sqrt{\xi - 1}} \right), \quad \xi \geq 1, \]

(D2a)

\[ = \frac{1}{4} \left[ \ln \left( \frac{1 - \sqrt{1 - \xi}}{1 + \sqrt{1 - \xi}} \right) - i\pi \right]^2, \quad \xi \leq 1. \]

(D2b)

This integral has also been calculated in ref. [10] and we find agreement. In the same limit the integral related to \( A_4 \) and \( A_5 \) is:

\[ \widetilde{A}_4(\xi) = \widetilde{A}_5(\xi) = \int_0^1 dx \int_0^{1-x} dy \frac{x(x - 1)}{xy - \xi/4} = \int_0^1 dx \int_0^{1-x} dy \frac{y(y - 1)}{xy - \xi/4}, \]

(D3)

with its exact answer written like

\[ \widetilde{A}_4(\xi) = 1 - \sqrt{\xi - 1} \arctan \left( \frac{1}{\sqrt{\xi - 1}} \right), \quad \xi \geq 1, \]

(D4)

\[ = 1 + \frac{\sqrt{1 - \xi}}{2} \left[ \ln \left( \frac{1 - \sqrt{1 - \xi}}{1 + \sqrt{1 - \xi}} \right) - i\pi \right], \quad \xi \leq 1. \]

(D5)
Integrals that are related to \( I_1 \) and \( I_2 \) of eq. (5) are:

\[
\tilde{I}_1(\xi) = \int_0^1 dx \int_0^{1-x} dy \frac{1}{xy - \xi/4}
\]

\[
= -2 \arctan^2 \left( \frac{1}{\sqrt{\xi - 1}} \right), \quad \xi \geq 1
\]

\[
= \frac{1}{2} \ln \left( \frac{1 - \sqrt{1 - \xi}}{1 + \sqrt{1 - \xi}} \right) - i\pi, \quad \xi \leq 1,
\]

(D6)

and

\[
\tilde{I}_1'(\xi) = \int_0^1 dx \int_0^{1-x} dy \frac{x}{xy - \xi/4} = \int_0^1 dx \int_0^{1-x} dy \frac{y}{xy - \xi/4}
\]

\[
= 2 \left[ \sqrt{\xi - 1} \arctan \left( \frac{1}{\sqrt{\xi - 1}} \right) - 1 \right], \quad \xi \geq 1
\]

\[
= -2 \sqrt{1 - \xi} \left[ \ln \left( \frac{1 - \sqrt{1 - \xi}}{1 + \sqrt{1 - \xi}} \right) - i\pi \right], \quad \xi \leq 1.
\]

(D7)

(D8)

These integrals are related to standard ones, \( A_3..A_6, \) \( I_{1,2} \), and in the limit where \( m_2^2 \ll s < m^2 \), become

\[
A_3(s; m^2) = -A_6(s; m^2) = \frac{c}{s} \tilde{A}_3(\frac{4m^2}{s}) = -\frac{c}{m^2} \left[ \frac{1}{24} + \frac{1}{180} \frac{s}{m^2} + O(s^2/m^4) \right],
\]

(D12a)

\[
A_4(s; m^2) = -A_5(s; m^2) = -\frac{c}{s} \tilde{A}_4(\frac{4m^2}{s}) = -\frac{c}{m^2} \left[ \frac{1}{12} + \frac{1}{120} \frac{s}{m^2} + O(s^2/m^4) \right],
\]

(D12b)

\[
I_1(s; m^2) = \frac{\alpha_i \alpha_k + \beta_i \beta_k}{s} \tilde{I}_1(\frac{4m^2}{s}) - \frac{2 \beta_i \beta_k}{s} \tilde{I}_1'(\frac{4m^2}{s})
\]

\[= -\frac{1}{m^2} \left[ \beta_i \beta_k + 3\alpha_i \alpha_k \right] \frac{s}{6} + \frac{\beta_i \beta_k + 5\alpha_i \alpha_k}{120} \frac{s}{m^2} + O(s^2/m^4) \],

(D12c)

\[
I_2(s; m^2) = -\frac{\alpha_i \alpha_j + \beta_i \beta_j}{s} \tilde{I}_1(\frac{4m^2}{s}) + \frac{2 \beta_i \beta_j}{s} \tilde{I}_1'(\frac{4m^2}{s})
\]

\[= \frac{1}{m^2} \left[ \beta_i \beta_j + 3\alpha_i \alpha_j \right] \frac{s}{6} + \frac{\beta_i \beta_j + 5\alpha_i \alpha_j}{120} \frac{s}{m^2} + O(s^2/m^4) \],

(D12d)
where $c = \frac{e^2}{4\pi}\left(\alpha_1\alpha_2 + \beta_1\beta_2\right)\delta_4 + \frac{e^2}{4\pi}\left(\alpha_1\beta_2 + \beta_1\alpha_2\right)\delta_4$, is the anomaly factor. These expressions are in agreement with the corresponding ones presented in ref. [5]. In the high energy limit $m^2 \ll s$, we obtain,

$$A_3(s; m^2) = -A_6(s; m^2) \simeq c \left\{ \frac{1}{2s} + \frac{m^2}{2s^2} \ln^2 \frac{s}{m^2} - \pi^2 \right\} + i\pi \frac{m^2}{s^2} \ln \frac{s}{m^2} + O(m^4/s^3) \right\}, \quad (D13a)$$

$$A_4(s; m^2) = -A_5(s; m^2) \simeq c \left\{ \frac{1}{2s} \left[ -1 + \frac{1}{2} \ln \frac{s}{m^2} \right] - \frac{m^2}{s^2} \ln \frac{s}{m^2} + 1 \right\} + i\pi \left[ \frac{1}{2s} - \frac{m^2}{s^2} \right] + O(m^4/s^3) \right\}, \quad (D13b)$$

$$I_1(s; m^2) \simeq \left( \frac{\alpha_i\alpha_j + \beta_i\beta_j}{s} \right) \left[ \frac{1}{2} \left( \ln^2 \frac{s}{m^2} - \pi^2 \right) - 2 \frac{m^2}{s} \ln \frac{s}{m^2} \right] - 2 \beta_i\beta_k \left[ \ln \frac{s}{m^2} - 2 - \frac{m^2}{s} \left( \ln \frac{s}{m^2} + 1 \right) \right] + O(m^4/s^3), \quad (D13c)$$

$$I_2(s; m^2) \simeq -\left( \frac{\alpha_i\alpha_j + \beta_i\beta_j}{s} \right) \left[ \frac{1}{2} \left( \ln^2 \frac{s}{m^2} - \pi^2 \right) - 2 \frac{m^2}{s} \ln \frac{s}{m^2} \right] + 2 \beta_i\beta_j \left[ \ln \frac{s}{m^2} - 2 - \frac{m^2}{s} \left( \ln \frac{s}{m^2} + 1 \right) \right] - i\pi \left( \frac{\alpha_i\alpha_j + \beta_i\beta_j}{s} \right) \left[ \ln \frac{s}{m^2} - 2 - \frac{m^2}{s} \left( \ln \frac{s}{m^2} + 1 \right) \right] + O(m^4/s^3). \quad (D13d)$$

Only the real parts of these expressions have been presented in ref. [5] and we find agreement with other useful identities among $A_1$ that have been used in our numerical code for calculating the $V^*ZZ$-vertex are,

$$(A_3 - A_4)(k_1 = m_Z, k_2 = m_Z, s, m = 0) = -\frac{1}{4m_Z^2} + \frac{s}{2m_Z^2} A_3(k_1 = m_Z, k_2 = m_Z, s, m = 0), \quad (D14)$$

and for the $V^*\gamma Z$-vertex,

$$A_3(k_1 = 0, k_2 = m_Z, s, m = 0) = \frac{1}{2(s - m_Z^2)} - \frac{m_Z^2}{2(s - m_Z^2)^2} \ln \left( \frac{s}{m_Z^2} \right), \quad (D15)$$

$$A_3(k_1 = 0, k_2 = m_Z, s, m = 0) = -\frac{1}{2(s - m_Z^2)} \ln \left( \frac{s}{m_Z^2} \right). \quad (D16)$$

Finally, we derive full analytical expressions in the case $k_2^2 = 0$, where one of the external gauge bosons is massless e.g., the $V^*\gamma Z$-vertex. To this end it is useful to define an auxiliary function,

$$F(m_Z, s, m) \equiv \int_0^1 dx \int_0^{1-x} dy \ln[x(x - 1)m_Z^2 - xy(s - m_Z^2) + m^2], \quad (D17)$$

out of which we read $A_3..A_6, I_1, I_2$ by simply taking appropriate derivatives w.r.t $s, k_2^2 = m_Z^2$ or $m^2$. Depending on the region of parameters $s, m^2, m_Z^2$ we have found the function $F$ to be,

$$F(m_Z, s, m) = -\frac{3}{2} + \frac{\ln(m^2)}{2} - \left( \frac{1}{m_Z^2 - s} \right) \left\{ s \sqrt{\frac{4m^2}{s}} - 1 \arctan \left( \frac{1}{\sqrt{4m^2/s}} \right) - m_Z^2 \sqrt{\frac{4m^2}{m_Z^2}} - 1 \arctan \left( \frac{1}{\sqrt{4m^2/m_Z^2} - 1} \right) + 2m^2 \left[ \arctan^2 \left( \frac{1}{\sqrt{4m^2/s}} \right) - \arctan^2 \left( \frac{1}{\sqrt{4m^2/m_Z^2} - 1} \right) \right] \right\}, \quad \frac{4m^2}{s} > 1, \frac{4m^2}{m_Z^2} > 1, \quad (D18)$$

12 For notational matter, our integrals are related to those in ref. [5] like $A_3 = -c_6, \quad A_4 = \frac{1}{2}(c_4 - c_3 - 2c_6)$, where for example $A_3 \equiv A_3(k_1^2 = k_2^2 = m_W^2, s, m_W^2, m_2^2)\ldots$ etc.
In eq. (D21), the plus sign corresponds to $s < m_Z^2$ while the minus sign to $s > m_Z^2$. As an example the full analytical expressions for $A_3$ and $A_5$ can be obtained by taking appropriate derivatives of function $\mathcal{F}$ like, $A_3 = c \frac{\partial \mathcal{F}}{\partial s}$ and $A_5 = -c (\frac{\partial \mathcal{F}}{\partial m_Z^2} + \frac{\partial \mathcal{F}}{\partial m_Z^2})$, where, as above, $c$ is a factor related to the couplings in the corresponding vertex. As a cross check, taking the limit $m \to 0$ in eq. (D21) we arrive at,

$$\mathcal{F}(m_Z, s, 0) = -\frac{3}{2} - \frac{1}{2(m_Z^2 - s)} \left[ s \ln(s) - m_Z^2 \ln(m_Z^2) \right] + \frac{i\pi}{2},$$

and differentiating w.r.t $s$ and $m_Z^2$ we reproduce the expressions eqs. (D15) and (D16).

**Appendix E: Conditions for non-decoupling effects in $X, Y, Z$ model**

In this appendix we present necessary conditions for anomaly cancellation and non-decoupling heavy fermion effects in a model with three different $U(1)$’s corresponding to three distinct massive or massless gauge bosons $X, Y,$ and
Z. For this model to be anomaly-free, the following conditions among couplings [see eq. (1)]:

\[
\sum_{i=1}^{n} (\beta_i^3 + 3\alpha_i^2 \beta_i) = \sum_{i=1}^{n} (\beta_i^3 + 3\alpha_i^2 \beta_i) = \sum_{i=1}^{n} (\beta_i^3 + 3\alpha_i^2 \beta_i) = 0, \\
\sum_{i=1}^{n} (\beta_i^2 \beta_i + 2\alpha_i \alpha_i \beta_i + \alpha_i^2 \beta_i) = \sum_{i=1}^{n} (\beta_i^2 \beta_i + 2\alpha_i \alpha_i \beta_i + \alpha_i^2 \beta_i) = \sum_{i=1}^{n} (\beta_i^2 \beta_i + 2\alpha_i \alpha_i \beta_i + \alpha_i^2 \beta_i) = 0, \\
\sum_{i=1}^{n} (\beta_i^2 \beta_i + 2\alpha_i \alpha_i \beta_i + \alpha_i^2 \beta_i) = \sum_{i=1}^{n} (\beta_i^2 \beta_i + 2\alpha_i \alpha_i \beta_i + \alpha_i^2 \beta_i) = \sum_{i=1}^{n} (\beta_i^2 \beta_i + 2\alpha_i \alpha_i \beta_i + \alpha_i^2 \beta_i) = 0, \\
\sum_{i=1}^{n} (\beta_i \beta_i \beta_i + \alpha_i \alpha_i \beta_i \beta_i + \alpha_i \alpha_i \beta_i \beta_i) = \sum_{i=1}^{n} (\beta_i \beta_i \beta_i + \alpha_i \alpha_i \beta_i \beta_i + \alpha_i \alpha_i \beta_i \beta_i) = \sum_{i=1}^{n} (\beta_i \beta_i \beta_i + \alpha_i \alpha_i \beta_i \beta_i + \alpha_i \alpha_i \beta_i \beta_i) = 0.
\]

(E1)

must hold. Non-decoupling effects in \(XYZ\)-vertex are activated if, in addition to the requirements in eq. (E1), at least one of the following expressions is non-zero:

\[
\sum_{i=1}^{n} (\beta_i^2 \beta_i + 3\alpha_i \alpha_i \beta_i), \\
\sum_{i=1}^{n} (\beta_i^2 \beta_i + 3\alpha_i \alpha_i \beta_i), \\
\sum_{i=1}^{n} (\beta_i^2 \beta_i + 3\alpha_i \alpha_i \beta_i), \\
\sum_{i=1}^{n} (\beta_i^2 \beta_i + 3\alpha_i \alpha_i \beta_i), \\
\sum_{i=1}^{n} (\beta_i^2 \beta_i + 3\alpha_i \alpha_i \beta_i), \\
\sum_{i=1}^{n} (\beta_i \beta_i \beta_i + \alpha_i \alpha_i \beta_i \beta_i). 
\]

(E2)
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