A new Model for Solving Three Mixed Integral Equations with Continuous and Discontinuous Kernels

M. A. Abdou¹ and M. I. Youssef¹,²*

¹Department of Mathematics, Faculty of Education, Alexandria University, 21526, Alexandria, Egypt.
²Department of Mathematics, College of Science, Jouf University, 2014, Sakaka, Saudi Arabia.

Authors’ contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, we discuss a new model to obtain the answer to the following question: how can we establish the different types of mixed integral equations from the Fredholm integral equation? For this, we consider three types of mixed integral equations (MIEs), under certain conditions. The existence of a unique solution of such equations is guaranteed. Using analytic and numerical methods, the three MIEs formulas yield the same Fredholm integral equation (FIE) formula of the second kind. For continuous kernel, the solution of these three MIEs, via the FIEs, is discussed analytically. In addition, for a discontinuous kernel, the Toeplitz matrix method (TMM) and Product Nyström method (PNM) are used to obtain, in each method, a linear algebraic system (LAS). Then, the numerical results are obtained, the error is computed in each case, and compared as well.

Keywords: Solving mixed integral equations (MIEs); discontinuous kernels.

MSC (2010): 65R20, 45B05
1 Introduction

A lot of problems arise in applied sciences can be modeled using integral equations. For example, the radiation transfer process is described using the Chandrasekhar H-function, the biological populations are evolving in a way that obeys the delayed Volterra integro-differential equations. Also, there are the integral equations with discontinuous kernels that occur in elasticity and solid mechanics during the formulation of the boundary value problem of the mixed type, see [1-8] and the references therein. The importance of the mixed integral equations (MIEs), in position and time, and the contact problems came from the work of Abdou [1]. The solution of the MIE of the first kind in one, two and three dimensions has been obtained analytically using the separation of variables method in [1]. The MIE of the first kind can be solved analytically using one of the following methods: The Cauchy method, orthogonal polynomial method, potential theory method and Krein’s method, see [9-16] and the references therein for details. The relation between the MIEs and some contact problems can be found in [13-17] and the references therein. This work is a new contribution, to the best of our knowledge, in this active area of scientific research. In this work, we consider the following three models of linear MIEs:

\[
\begin{align*}
Z(x,t) &= f(x,t) + \lambda \int_{-1}^{1} k\left(|x-y|\right) Z(y,t) dy + \lambda \int_{0}^{1} F(t,\tau) Z(x,\tau) d\tau \\
\Phi(x,t) &= f(x,t) + \lambda \int_{0}^{1} \int_{-1}^{1} G(t,\tau) k\left(|x-y|\right) \Phi(y,\tau) dy d\tau + \lambda \int_{0}^{1} F(t,\tau) \Phi(x,\tau) d\tau \\
\Psi(x,t) &= f(x,t) + \lambda \int_{0}^{1} \int_{-1}^{1} G(t,\tau) k\left(|x-y|\right) \Psi(y,\tau) dy d\tau + \lambda \int_{-1}^{1} k\left(|x-y|\right) \Psi(y,t) dy
\end{align*}
\]

in the space \( L_2[-1,1] \times C[0,T], T < 1 \). The kernel of position, \( k\left(|x-y|\right) \), may have a singularity of the weak type, while the kernels of time, \((F(t,\tau), G(t,\tau))\), are positive and continuous functions. The parameter \( \lambda \) may be a complex constant, and possess a physical meaning. The function \( f(x,t) \) is known, while the functions \( Z(x,t), \Phi(x,t) \) and \( \Psi(x,t) \) are unknowns and need to be determined. We guarantee the existence of unique solutions of Eqs. (1) - (3) by assuming the following conditions:

(i) The kernel \( k\left(|x-y|\right) \), in general, satisfies the Fredholm discontinuity condition:

\[
\left[ \int_{-1}^{1} \int_{-1}^{1} k\left(|x-y|\right) dx dy \right]^{\frac{1}{2}} = c, \quad (c \text{ is a constant})
\]

(ii) The two kernels \( G(t,\tau), F(t,\tau) \) are elements in \( C\left([0,T] \times [0,T]\right), \forall 0 \leq \tau \leq t \leq T < 1 \).

(iii) The given function \( f(x,t) \) is continuous, has continuous partial derivatives with respect to \( x \) and \( t \) in the space \( L_2[-1,1] \times C[0,T] \) and equipped with the norm

\[
\|f(x,t)\| = \max_{0 \leq t \leq T} \left[ \int_{0}^{1} \int_{-1}^{1} |f(x,\tau)|^2 dx d\tau \right]^{\frac{1}{2}} = H, \quad H \text{ is a constant}
\]

(iv) The unknown functions \( Z(x,t), \Phi(x,t) \) and \( \Psi(x,t) \) satisfy the Lipschitz condition with respect to the position and Hölder condition with respect to the time.
In addition, we suppose that the separation of variables method is applicable to the unknown functions, force function and kernels in order to represent the three types of the MIEs (1) – (3) as a one general formula of FIE having different time dependent constants.

Notice that

From condition (ii), it is clear that there exist two constants \(N\) and \(M\) such that 
\[
|G(t, \tau)| \leq N, \quad |F(t, \tau)| \leq M.
\]

The subsequent parts in this work are organized as follows. In section (2), we apply the separation of variables method to prove that Eqs. (1) – (3) can be represented as a one general formula of the FIE type with different constants in each \(t \in [0, T], T < 1\). In section (3), we consider the first application when the kernels of position and time are continuous. Then, we derive the solution of the three MIEs from the solution of the FIE of the second kind. In section (4), we study the second application when the kernel of position has a weak singularity. For this, we use the TMM and PNM, as the best two numerical methods, to solve the FIE numerically. Then, the corresponding solutions of the three MIEs are derived. In section (5), the numerical results are considered, and the error is estimated in each case. Section (6) is devoted to the discussion of the results and conclusion.

2 The Separation of Variables Method and its Physical Meaning

To discuss the solution of Eqs. (1) – (3), we suppose that the sought functions \(Z(x, t)\), \(\Phi(x, t)\), \(\Psi(x, t)\) and the force function \(f(x, t)\) take the form: \(\prod(x, t) = \prod_n(x) t^n\). Also, let \(G(t, \tau) = t^\nu \tau^m\), \(F(t, \tau) = t^\nu \tau^m\), \((n; m; \ell; \xi, \nu = 0, 1, 2 \ldots)\). Hence, we have

\[
Z_n(x) = \gamma(t) f_n(x) + \lambda \gamma(t) \int_{-1}^{1} k\left(|x - y|\right) Z_n(y) dy;
\]

where
\[
\gamma(t) = \frac{\ell + n + 1}{(\ell + n + 1) - \lambda t^{(n+\xi+1)}}, \quad \lambda(t) \neq (\ell + n + 1)t^{-(n+\xi+1)}, \quad \forall 0 \leq t \leq T < 1,
\]

\[
\Phi_n(x) = \gamma(t) f_n(x) + \lambda \delta(t) \gamma(t) \int_{-1}^{1} k\left(|x - y|\right) \Phi_n(y) dy; \quad \delta(t) = \frac{\ell^{(m+n+1)}}{(m+n+1)};
\]

\[
\Psi_n(x) = f_n(x) + \lambda \beta(t) \int_{-1}^{1} k\left(|x - y|\right) \Psi_n(y) dy; \quad \beta(t) = (1 + \delta(t)).
\]

1) The formulas (4)-(6) can be represented in a one formula as the following:

\[
X_n^{(\alpha)}(x) = \rho_\alpha(t) f_n(x) + \lambda \eta_\alpha(t) \int_{-1}^{1} k\left(|x - y|\right) X_n^{(\alpha)}(y) dy
\]

(i). For the formula (4), we let \(\alpha = 1\), \(\rho_1(t) = \eta_1(t) = \gamma(t)\).

(ii). For the formula (5), we let \(\alpha = 2\), \(\rho_2(t) = \gamma(t)\), \(\eta_2(t) = \gamma(t) \delta(t)\).
(iii). Let $\alpha = 3$, $\rho \beta = 1$, $\eta \beta = \beta(t)$, for the formula (6).

2) The formula (7) represents a class (system) of the Fredholm integral equations (SFIEs) of the second kind that occurs during the time domain $0 \leq t \leq T < 1$, and this class of equations (7) has a unique solution at every time $t \in [0, T]$, $T < 1$, in the space $L_2[-1, 1]$, provided that $\|k(|x-y|)\| \leq \frac{1}{\lambda \eta \nu(t)}$, $\forall 0 \leq t \leq T < 1$.

3) From Eq. (7), we can establish that the time and the two kernels of time, $G(t, \tau)$, $F(t, \tau)$, are transformed to time dependent constants that play an important role in the uniqueness of solution.

4) If, in Eq. (4), we assume that $c(t) = \frac{\lambda t^{\ell(n+1)}}{(\ell + n + 1)}$; $0 < c(t) < 1$, $\forall 0 \leq t \leq T < 1$. Then, we have $\gamma(t) = 1 + c(t) + c^2(t) + \ldots$. So, from Eqs. (4) and (5) respectively and $\forall 0 \leq t \leq T < 1$ we have

$$\mu(t)Z_n(x) = f_n(x) + \lambda \int_{-1}^{1} k(|x-y|)Z_n(y) dy; \quad 0 < \mu(t) = (1 - c(t)) < 1,$$

$$\mu(t)\Phi_n(x) = f_n(x) + \lambda \delta(t) \int_{-1}^{1} k(|x-y|)\Phi_n(y) dy; \quad \delta(t) = \frac{t^{m+\nu+1}}{(m + n + 1)}.$$

5) It is clear that $c(t) = \frac{\lambda t^{\ell(n+1)}}{(\ell + n + 1)}$ should be less than one, because if we suppose that $c(t)$ is greater than one, then the value of $\gamma(t)$ will be negative $\forall 0 \leq t \leq T < 1$, and therefore, the integral equations (4) and (5) have no meaning.

6) Let $\lambda = \lambda(t) = (\ell + n + 1)t^{\ell(n+1)}$ in Eqs. (4) and (5). Then, we have two SFIEs of the first kind. Moreover, the value of $\delta(t)$ becomes $\delta(t) = \frac{(\ell + n + 1)}{(m + n + 1)} t^{(m+\nu+1)}$; $\lambda = 1$.

7) If we set $F(t, \tau) = G(t, \tau)$ and $\lambda = 1$ in Eqs. (2) and (3), then we will get $\delta(t) = c(t)$ and $\beta(t) = (1 + c(t))$ in Eqs. (5) and (6) respectively.

8) For all values of $t$ satisfying $t = \left(\frac{m+\nu+1}{\sqrt{m+n+1}}\right) \land (0 \leq t \leq T < 1)$, the formulas (4) and (5) are equivalent.

3 Applications for Continuous Kernels

Consider the integral equation with degenerate kernel

$$\varphi(x) = f(x) + \int_{a}^{b} k(x, y)\varphi(y) dy; \quad k(x, y) = \sum_{i=0}^{K} a^{(i)}(x)b^{(i)}(y).$$

\[ (10) \]
The functions $a^{(i)}(x)$, $1 \leq i \leq k$, are assumed to be linearly independent. Then the integral equation (10) has a solution in the following form

$$\varphi(x) = f(x) + \lambda \sum_{i=0}^{K} a^{(i)}(x)c^{(i)}; \quad c^{(i)} = \int_{a}^{b} b^{(i)}(y)\varphi(y)dy$$

(11)

where

$$c^{(i)} = f^{(i)} + \lambda \sum_{j=0}^{K} D^{(j)}(x)c^{(j)};$$

$$f^{(i)} = \int_{a}^{b} b^{(i)}(y)f(y)dy;$$

$$D^{(j)} = \int_{a}^{b} b^{(j)}(y)a^{(j)}(y)dy.$$

The IE (10) is considered the zero approximation of the three kinds of the MIEs (1)–(3) at $t = 0$. The corresponding solutions of Eqs. (1)–(3) take respectively the forms

$$Z_n(x) = \gamma(t)f_n(x) + \lambda \gamma(t)\sum_{j=0}^{K} D^{(j)}(x)c_n^{(j)}; \quad \gamma(t) = \frac{\ell + n + 1}{(\ell + n + 1) - \lambda \ell^{(n+1)}}, \ell, n, \zeta = 0, 1, \ldots$$

(13)

$$\Phi_n(x) = \gamma(t)f_n(x) + \lambda \gamma(t)\delta(t)\sum_{j=0}^{K} D^{(j)}(x)c_n^{(j)}; \quad \delta(t) = \frac{\ell^{(m+n+1)}}{(m+n+1)}, (n, m, \nu = 0, 1, 2, \ldots)$$

(14)

$$\Psi_n(x) = f_n(x) + \lambda \beta(t)\sum_{j=0}^{K} D^{(j)}(x)c_n^{(j)}; \quad \beta(t) = (1 + \delta(t)).$$

(15)

For example, the integral equation $\varphi(x) = f(x) + \lambda \int_{0}^{1} xy \varphi(y)dy$ has a solution given by

$$\varphi(x) = f(x) + \lambda \int_{0}^{1} \frac{3xy}{3 - \lambda} f(y)dy, \lambda \neq 3. \quad \text{Also, } \varphi(x) = f(x) + \lambda \int_{0}^{\pi/2} \frac{\sin 2y}{2 - \lambda} f(y)dy, \lambda \neq 2 \text{ is a solution for } \varphi(x) = f(x) + \frac{\pi}{2} \int_{0}^{\pi/2} \sin 2y \varphi(y)dy.$$

4 Applications for Discontinuous Kernels

4.1 The Toeplitz matrix method (TMM), see [14-16]

In this section, we apply the TMM to represent the SFIEs of Eqs. (4)–(6) as a linear algebraic system (LAS). The essence of this method is to get a linear algebraic system with $2N+1$ equations, where $2N+1$ is the whole number of the discretization points. In the light of the TMM, the SFIEs can be adapted in the following form:
\[ \varphi_{\xi} = f_{\xi} + \lambda \sum_{\eta=-N}^{N} D_{\eta,\xi} \varphi_{\eta}, \quad \eta, \xi \in [-N, N]. \]  

Here, in (15) we assume

\[
D_{\eta,\xi} = D_{\eta}(x) = D_{\eta}(\xi h) = \begin{cases} A_{-N}(x), & \eta = -N \\ A_{\eta}(x) + B_{\eta-1}(x), & -N < \eta < N \\ B_{N-1}(x), & \eta = N \end{cases}
\]  

\[
A_{\eta}(x) = \frac{1}{h} \left[ (\eta h + h) I_{\eta}(x) - J_{\eta}(x) \right], \quad B_{\eta}(x) = \frac{1}{h} \left[ J_{\eta}(x) - \eta h I_{\eta}(x) \right].
\]  

\[
I_{\eta}(x) = \int_{\eta h}^{\eta h + h} k\left( |x-y| \right) dy, \quad J_{\eta}(x) = \int_{\eta h}^{\eta h + h} y k\left( |x-y| \right) dy.
\]  

where \( x = \xi h, h = \frac{1}{N} \), \( \eta, \xi \in [-N, N] \). If \( N \) is large enough, and there exists a constant \( D > 0 \) that is independent of the value of \( N \) such that \( \| \varphi(x) - \varphi_N(x) \| \leq DN^{-r}, \ x \in [-1,1] \), then the TMM is said to be convergent of order \( r \) on \([-1,1]\). The error term \( R \) is estimated using the following formula

\[
R = \left| \int_{\eta h}^{\eta h + h} y^2 k\left( |x-y| \right) dy - A_{\eta}(x) \left( \eta h \right)^2 - B_{\eta}(x) \left( \eta h + h \right)^2 \right| = O(h^3)
\]  

4.2 The product Nyström method, see [4-6]

In this section, we discuss the solution of the SFIEs using the PNM. For this aim, and in the light of the PNM, the formula (2) yields

\[
\mu_k \varphi_k(x_i) = H_k(x_i) + \lambda \sum_{j=0}^{k-1} w_{k,j} G_{k,k} \bar{k}(x_i, y_j) \varphi_k(y_j),
\]

\[
H_k(x_i) = f_k(x_i) + \lambda \sum_{j=0}^{k-1} \sum_{m=0}^{N} w_{k,m} u_j G_{k,j} \bar{k}(x_i, y_m) \varphi_j(y_m) + \lambda \sum_{j=0}^{k-1} u_j F_{k,j} \varphi_j(x_i),
\]  

\[
\mu_k = 1 - \lambda u_k F_{k,k}.
\]

Where \( x_i = y_i = 1 + ih, i = 0, 1, \cdots, N \) with \( h = \frac{2}{N} \), \( N \) is even. The kernel \( \bar{k} \) is a well-behaved function. Also, we define the function \( p(x, y) \) to be the bad-behaved function. The weights \( w_{i,j} \) are determined in [13] on the form
\[ w_{i,0} = \beta_i(y_i) , \]
\[ w_{i,2j+1} = 2\gamma_{j+1}(y_i) , \]
\[ w_{i,2j} = \alpha_j(y_i) + \beta_{j+1}(y_i) , \]
\[ w_{i,N} = \alpha_N(y_i) , \]

(22)

Where:
\[ \alpha_j(y_i) = \frac{h}{2} \int_0^2 \xi (\xi - 1) p(y_{2j-2} + \xi h, y_i) d\xi , \]
\[ \beta_j(y_i) = \frac{h}{2} \int_0^2 (\xi - 1)(\xi - 2) p(y_{2j-2} + \xi h, y_i) d\xi , \]
\[ \gamma_j(y_i) = \frac{h}{2} \int_0^2 \xi (2 - \xi) p(y_{2j-2} + \xi h, y_i) d\xi . \]

(23)

The PNM is said to have a convergence of order \( r \) on the interval \([a, b]\) if and only if for \( N \) adequately large, there exists a constant \( C > 0 \) that is independent of the value of \( N \) such that \( \| \phi(x) - \phi_N(x) \|_{L_\infty} \leq CN^{-r} \).

5 Numerical Examples

Example 1. Consider the following mixed Volterra-Fredholm integral equation
\[ \Phi(x, t) = f(x, t) + \lambda \int_0^{1} \ln | x - y | \tau^2 \Phi(y, \tau) dy d\tau + \lambda \int_0^{t} \tau^2 \Phi(x, \tau) d\tau ; \quad \Phi(x, t) = x^2 t \]

Using the Maple software with \( \mu = 1, \lambda = 0.02, N = 60 \), we obtain Table 1.

| \( t \)  | \( \Phi_{exact} \) | \( \Phi_N \) | \( E_N \) | \( \Phi_T \) | \( E_T \) |
|---|---|---|---|---|---|
| 0.01 | -0.00001000 | -0.000099936 | 0.963657E-7 | -0.000099966 | 0.38898E-7 |
| 0.02 | -0.0000076 | -0.000007654 | 0.120455E-7 | -0.00000765 | 0.108296E-7 |
| 0.03 | -0.00003000 | -0.000002989 | 0.107503E-7 | -0.000002989 | 0.53917E-7 |
| 0.04 | -0.667x10^{-7} | -0.65690x10^{-7} | 0.976745E-7 | -0.6654x10^{-7} | 0.123618E-7 |
| 0.05 | -0.01000000 | -0.0000999036 | 0.95640E-6 | -0.01001732 | 0.17316E-6 |
| 0.06 | -0.0000766 | -0.0000765462 | 0.12043E-5 | -0.000000767 | 0.53956E-5 |
| 0.07 | -0.000533 | -0.000532177 | 0.115597E-5 | -0.00005353 | 0.24316E-5 |
| 0.08 | -0.00300000 | -0.0029892504 | 0.107495E-5 | -0.000300095 | 0.95243E-6 |
| 0.09 | -0.16000000 | -0.001598484 | 0.151598E-5 | -0.00001625 | 0.259703E-5 |
| 0.1 | -0.12260000 | -0.1224759049 | 0.19076E-5 | -0.12452798 | 0.186131E-5 |
| 0.11 | -0.04800000 | -0.0487287031 | 0.171296E-5 | -0.04870926 | 0.709258E-5 |
| 0.12 | -0.0106667 | -0.0151044294 | 0.156223E-5 | -0.01082365 | 0.156982E-5 |
| 0.13 | -0.64000000 | -0.6400685944 | 0.531405E-4 | -0.64007095 | 0.347309E-4 |
| 0.14 | -0.4906667 | -0.4996608086 | 0.705858E-4 | -0.51646029 | 0.257936E-4 |
| 0.15 | -0.19200000 | -0.1913358428 | 0.664157E-4 | -0.20201104 | 0.100110E-4 |
| 0.16 | -0.0426667 | -0.0420441589 | 0.622507E-4 | -0.04488829 | 0.222162E-4 |
Example 2. Consider the following mixed Volterra-Fredholm integral equation

\[ \Phi(x,t) = f(x,t) + \lambda \int_{0}^{1} \int_{0}^{1} \frac{t^2 \tau^2}{|x-y|^3} \phi(y, \tau) \, dy \, d\tau + \lambda \int_{0}^{1} \Phi(x, \tau) \, d\tau; \quad \Phi(x,t) = x^2t \]

Using the Maple software with \( \mu = 1 \), \( \lambda = 0.1 \), \( N = 60 \), \( t = 0.6 \), we get Table 2.

| \( \alpha \) | \( \Phi_{\text{Exact}} \) | \( \Phi_N \) | \( E_N \) | \( \Phi_T \) | \( E_T \) |
|------------|----------------|-----|-----|-----|-----|
| \( \alpha = 0.01 \) | -0.0400000 | -0.0399630415 | 0.36958E-6 | -0.04050583 | 0.50582E-6 |
| \( \alpha = 0.22 \) | -0.0400000 | -0.0399630415 | 0.36958E-5 | -0.04050583 | 0.50582E-5 |
| \( \alpha = 0.32 \) | -0.0400000 | -0.0399630415 | 0.36958E-4 | -0.04050583 | 0.50582E-4 |

### 6 Discussion of Results and Conclusion

Through the process of determining the results using the PNM and TMM, we can notice the following points.

1) The estimated error evaluated using the TMM is smaller than the error evaluated utilizing the PNM. Furthermore, in many cases, the maximum error obtained by using the TMM is compared to the minimum one in the PNM.

2) When the function is symmetric with respect to \( x \), the approximate solution obtained by the TMM is also symmetric to the sixth decimal. Whereas, the approximate solution by the PNM may be variant from the first decimal.

3) The error function takes one form in the TMM which is maximum at the ends, \( i.e. \) when \( x \in \{-1,1\} \) and minimum at the middle point of the interval \( [-1,1] \), \( i.e. \) when \( x = 0 \), while the error function in the PNM decreases at \( x \in \{-1,1\} \) and attains the minimum value at \( x = 0 \). So, the TMM is more stable than the PNM and hence, it is better than the PNM in evaluating approximate solutions for the considered problems in this work.

Finally, we want to show that the previous methods used by many researchers were based on following techniques.

1) Dividing time into periods and accordingly the mixed integral equation in position and time turns into a system of integral equations in position, see [2-4] and the references therein for more explanations.

2) Also, the series method can be used to separate position from time in the mixed integral equations, and then it is possible to obtain three Volterra integral equations of the second type that depend on time. The other part takes the form of Fredholm's integral equation, see [2-4,11] and the references therein for more details.

3) In this article, our method depends on deriving all the integral equations of different types in position and time from the Fredholm integral equation so that the parameter is related in time. This parameter, whenever it changes in a certain way, can derive a new type of mixed integral equations. This model of
solutions will play a major role in solving genetic and engineering problems. For future work, we suggest developing a suitable technique to derive the mixed integral equations from the Fredholm integral equation when the nucleus has the form of a generalized potential function.

Competing Interests

Authors have declared that no competing interests exist.

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