THE 2-IWASAWA MODULE OVER CERTAIN OICT ELEMENTARY FIELDS

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Abstract. The aim of this paper is to determine the structure of 2-Iwasawa module of some imaginary triquadratic fields.

1. Introduction

Let \( \ell \) denote a prime number. Let \( \mathbb{Z}_\ell \) be the additive group of \( \ell \)-adic integers, \( \Lambda = \mathbb{Z}_\ell[[T]] \) and \( M \) a finitely generated \( \Lambda \)-module. It is known, by the structure theorem, that there exist \( r, s, t, n_i, m_i \in \mathbb{Z} \), and \( f_i \) distinguished and irreducible polynomials of \( \mathbb{Z}_\ell[T] \) such that

\[
M \sim \Lambda^r \oplus \left( \bigoplus_{i=1}^s \Lambda/(\ell^{n_i}) \right) \oplus \left( \bigoplus_{j=1}^t \Lambda/(f_j(T)^{m_j}) \right).
\]

Denote by \( k_\infty/k \) the cyclotomic \( \mathbb{Z}_\ell \)-extension of a number field \( k \). For any integer \( n \geq 1 \), it is known that \( k_\infty \) contains a unique cyclic subfield \( k_n \) of degree \( \ell^n \) over \( k \). The field \( k_n \) is called the \( n \)-th layer of the cyclotomic \( \mathbb{Z}_\ell \)-extension of \( k \). Let \( A_n(k) \) denote the \( \ell \)-class group of \( k_n \) and us consider the Iwasawa module \( A_\infty(k) = \varprojlim A_n(k) \) which is a finitely generated \( \Lambda \)-module.

The determination of the structure of \( A_\infty(k) \), for a given number field \( k \), is a very difficult and interesting problem in Iwasawa theory. In this paper we shall deal with this problem (in the case \( \ell = 2 \)) for some infinite families of number fields of the form \( F = \mathbb{Q}(\sqrt{p}, \sqrt{q}, i) \), where \( p \) and \( q \) are two primes satisfying one of the following conditions:

\[
q \equiv 7 \pmod{8}, \ p \equiv 5 \text{ or } 3 \pmod{8} \text{ and } \left( \frac{p}{q} \right) = 1. \tag{1}
\]

\[
q \equiv 7 \pmod{8}, \ p \equiv 5 \text{ or } 3 \pmod{8} \text{ and } \left( \frac{p}{q} \right) = -1. \tag{2}
\]

Let us adopt the following notations: \( h_2(d) \) the 2-class number of \( \mathbb{Q}(\sqrt{d}) \), \( h(k) \) (resp. \( h_2(k) \)) the class number (resp. 2-class number) of a number field \( k \), \( k^+ \) the

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maximal real subfield of $k$, $\mathrm{Cl}(k)$ the class group of $k$, $q(k)$ the unit index of a multiquadratic number field $k$ and $N_{k'/k}$ the norm map of an extension $k'/k$.

2. The 2-Iwasawa Module

Let us recall some results that will be useful in what follows.

**Lemma 2.1** ([1]). Let $m \geq 1$ and $p$ a prime such that $p \equiv 3$ or 5 (mod 8). Then $p$ decomposes into the product of 2 prime ideals of $K_n = \mathbb{Q}(\zeta_{2n+2})$ while it is inert in $K_n^+$.

**Theorem 2.2** ([7]). Let $F$ and $K$ be CM-fields and $K/F$ a finite 2-extension. Assume that $\mu^-(F) = 0$. Then $\mu^-(K) = 0$ and

$$\lambda^-(K) - \delta(K) = [K_\infty : F_\infty] (\lambda^-(F) - \delta(F)) + \sum (e_\beta - 1) - \sum (e_{\beta+} - 1),$$

where $\delta(k)$ takes the values 1 or 0 according to whether $F_\infty$ contains the fourth roots of unity or not. The $e_\beta$ is the ramification index of a prime $\beta$ in $K_\infty/F_\infty$ coprime to 2 and $e_{\beta+}$ is the ramification index for a prime coprime to 2 in $K_\infty^+/F_\infty^+$.

Let $\text{rank}_\ell(G)$ denote the $\ell$-rank of an abelian group $G$ and $\mu(k)$, $\lambda(k)$ denote the Iwasawa Invariants of $k$. We have:

**Theorem 2.3.** Let $K_\infty/k$ be a $\mathbb{Z}_\ell$-extension of a number field $k$ and assume that any prime which is ramified in $K_\infty/k$ is totally ramified. If $\mu(k) = 0$ and $\lim_{n \to \infty} A_n$ is an elementary $\Lambda$-module, then $\text{rank}_\ell(A_n) = \lambda(k)$ for all $n \geq \lambda(k)$.

**Proof.** We have $\text{rank}_\ell(A_n) = \lambda(k)$, for $n$ large enough and we have $\text{rank}_\ell(A_n) \leq \lambda(k)$, for all $n$. Assume that $\text{rank}_\ell(A_{m_0}) \leq \lambda(k) - 1$ for some $m_0 \geq \lambda(k)$. This implies that necessary there is $r \leq \lambda(k)$ such that $\text{rank}_\ell(A_r) = \text{rank}_\ell(A_{r+1})$ and therefore by [6, Theorem 1], $\text{rank}_\ell(A_n) = \text{rank}_\ell(A_{k+1}) \leq \lambda(k) - 1$, for all $n \geq r$. Which is absurd. \qed

**Lemma 2.4.** Let $p$ and $q$ be two primes satisfying conditions (1) or (2). Set $\pi_1 = 2$, $\pi_2 = 2 + \sqrt{2}$, ..., $\pi_n = 2 + \sqrt{\pi_{n-1}}$. Then, the 2-class group of $\mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{\pi_n})$ is trivial for all $n \geq 1$.

**Proof.** Assume that $p \equiv 5$ (mod 8). Put $k = \mathbb{Q}(\sqrt{p}, \sqrt{q})$. If $\left(\frac{p}{q}\right) = -1$, then by [3, Lemma 3.2] we easily deduce that $q(k) = 2$ (we similarly show that we have the same equality for other cases). Note that by class number formula (cf. [9]) and the fact that we have $h_2(p) = h_2(q) = 1$ and $h_2(pq) = 2$ (cf. [5, Corollaries 18.4, 19.7 and 19.8]). we have

$$h_2(k) = \frac{1}{4}q(k)h_2(p)h_2(q)h_2(pq) = \frac{1}{4} \cdot 2 \cdot 1 \cdot 1 \cdot 2 = 1.$$

By [3, Theorem 4.4] and [4, Theorem 3.14] the class number of $k_1 = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{2})$ is odd. Therefore the result follows by [6, Theorem 1]. We prove similarly the result for the case $p \equiv 3$ (mod 8). \qed
Theorem 2.5. Let $p$ and $q$ be two primes satisfying conditions (1) or (2). Put $F = \mathbb{Q}(\sqrt[p]{q}, \sqrt[q]{p}, i)$. Let $A_n(F)$ denote the 2-class group of the $n$-th layer of the cyclotomic $\mathbb{Z}_2$-extension of $F$. Then the Iwasawa module $A_\infty(F) = \lim_{n \to \infty} A_n(F)$ of $F$ is a $\Lambda$-module without finite part. If moreover $q \equiv 7 \pmod{16}$, then

- We have $A_\infty(F) \cong \mathbb{Z}_2^n$.
- $\text{rank}_2(A_n(F)) = 3$, for all $n \geq 3$.

Proof. Assume that $p \equiv 5 \pmod{8}$. Let $F_n$ denote the $n$-th ($n \geq 1$) layer of the cyclotomic $\mathbb{Z}_2$-extension of $F$ and $A_n(F)$ denote the 2-class group of $F_n$. Let us define $A_n^+$ as the group of strongly ambiguous classes with respect to the extension $F_n/F^n_+$ and $A_n^- = A_n/A_n^+$ (cf. [8] for more details about this new definition).

Since the class number of $F_n^+$ is odd (Lemma 2.4), then $A_n^+$ is generated by the ramified primes above 2. Since $F_1/F_1^+$ is unramified, then $F_n/F_n^+$ is also unramified. Therefore $A_n^+$ is trivial. So $A_n = A_n^-$. By [8, Theorem 2.5] there is no finite submodule in $A_n^-$. Hence $A_\infty(F) = A_n^-$ is a $\Lambda$-module without finite part.

- Assume now that $q \equiv 7 \pmod{16}$.

Set $K_n = \mathbb{Q}(\zeta_{2n+2})$ and $K = \mathbb{Q}(\sqrt{q}, i)$. Note that $p$ splits into 2 prime ideals in $\mathbb{Q}(\sqrt{q})$ or $\mathbb{Q}(\sqrt{2q})$. Since by Lemma 2.1, $p$ splits into 2 primes of $K_n$ and inert in $K_n^+$, for $n$ large enough, then $p$ splits into 4 primes in $F_n$ while it splits into 2 primes in $F_n^+$. By [2, Theorem 4] we have $\lambda^-(K) = 1$. Since $[F_\infty : K_\infty] = [F^+_\infty : K^+_\infty] = 2$, then by Kida’s formula (Theorem 2.2) applied on $F/K$ we have

$$\lambda^-(F) - 1 = 2(1 - 1) + 4 - 2.$$  

Thus by Lemma 2.4 $\lambda(F) = \lambda^-(F) = 3$. It follows that

$$A_\infty(F) \cong \bigoplus \Lambda/(f_i(T)),$$

for some distinguished polynomials $f_i$ such that $\sum \deg(f_i) = 3$. Since by the division algorithm we have $\Lambda/(f_i(T)) \cong \mathbb{Z}_2^{\deg(f_i)}$, then $A_\infty(F) \cong \mathbb{Z}_2^3$. Therefore, Theorem 2.3 completes the proof of the result for $p \equiv 5 \pmod{8}$. We proceed similarly for the other cases. 

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