SPECTRUM OF THE LAMÉ OPERATOR ALONG $\text{Re} \tau = 1/2$ : THE GENUS 3 CASE

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ABSTRACT. In this paper, we study the spectrum $\sigma(L)$ of the Lamé operator
\[ L = \frac{d^2}{dx^2} - 12 \wp(x + z_0; \tau) \text{ in } L^2(\mathbb{R}, \mathbb{C}), \]
where $\wp(z; \tau)$ is the Weierstrass elliptic function with periods 1 and $\tau$, and $z_0 \in \mathbb{C}$ is chosen such that $L$ has no singularities on $\mathbb{R}$. We prove that a point $\lambda \in \sigma(L)$ is an intersection point of different spectral arcs but not a zero of the spectral polynomial if and only if $\lambda$ is a zero of the following cubic polynomial:
\[ \frac{4}{15} \lambda^3 + \frac{8}{5} \eta_1 \lambda^2 - 3 \eta_2 \lambda + 9 \eta_3 - 6 \eta_1 \eta_2 = 0. \]
We also study the deformation of the spectrum as $\tau = \frac{1}{2} + ib$ with $b > 0$ varying. We discover 10 different types of graphs for the spectrum as $b$ varies around the double zeros of the spectral polynomial.

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1. INTRODUCTION

Let $T_\tau := \mathbb{C}/\Lambda_\tau$ be a flat torus with $\Lambda_\tau = \mathbb{Z} + \mathbb{Z} \tau$ and $\tau \in \mathbb{H} = \{ \tau \in \mathbb{C} | \text{Im} \, \tau > 0 \}$. Recall that
\[ \wp(z; \tau) := \frac{1}{z^2} + \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left( \frac{1}{(z - m - n\tau)^2} - \frac{1}{(m + n\tau)^2} \right) \]
is the Weierstrass elliptic function with basic periods $\omega_1 = 1$ and $\omega_2 = \tau$. Denote by $\omega_3 = 1 + \tau$. It is well known that

$$ \frac{d^2}{dx^2} - g(g + 1) \phi(x + z_0; \tau), \quad x \in \mathbb{R} $$

(1.1) $\phi'(z; \tau)^2 = 4 \prod_{k=1}^3 (\phi(z; \tau) - e_k(\tau)) = 4 \phi(z; \tau)^3 - g_2(\tau) \phi(z; \tau) - g_3(\tau),$

where $e_k(\tau) = \phi'(\frac{\omega_k}{2}; \tau), k = 1, 2, 3$, and $g_2(\tau), g_3(\tau)$ are well-known invariants of the elliptic curve.

The Weierstrass zeta function is defined by $\zeta(z) = \zeta(z; \tau) := - \int(z) \phi(z; \tau) d\xi$ with two quasi-periods $\eta_j = \eta_j(\tau), j = 1, 2$:

$$ \eta_j(\tau) = 2 \zeta(\frac{\omega_j}{2}; \tau) = \zeta(z + \omega_j; \tau) - \zeta(z; \tau), \quad j = 1, 2,$$

and the Weierstrass sigma function is defined by $\sigma(z) = \sigma(z; \tau) := \exp \int(z) \zeta(z; \tau) d\xi.$

Notice that $\zeta(z)$ is an odd meromorphic function with simple poles at $\mathbb{Z} + \mathbb{Z} \tau$ and $\sigma(z)$ is an odd entire function with simple zeros at $\mathbb{Z} + \mathbb{Z} \tau$.

In this paper, we study the spectrum $\sigma(L^\xi)$ of the classical Lamé operator

$$ L^\xi = \frac{d^2}{dx^2} - g(g + 1) \phi(x + z_0; \tau), \quad x \in \mathbb{R} $$

in $L^2(\mathbb{R}, \mathbb{C})$, where $g \in \mathbb{N}$ and $z_0 \in \mathbb{C}$ is chosen such that $\phi(x + z_0; \tau)$ has no singularities on $\mathbb{R}$. Remark that $\sigma(L^\xi)$ does not depend on the choice of $z_0$ due to the fact that the Lamé potential $-g(g + 1) \phi(z; \tau)$ is a Picard potential in the sense of Gesztesy and Weikard [14] (i.e. all solutions of the Lamé equation

$$ y''(z) = (g(g + 1) \phi(z; \tau) + \lambda) y(z), \quad z \in \mathbb{C} $$

are meromorphic in $\mathbb{C}$).

The spectral theory of the Schrödinger operator $L$ with complex periodic smooth potentials has attracted significant attention and has been studied widely in the literature; see e.g. [1] [2] [14] [17] [18] [24] and references therein. In this theory, it is known [24] that

$$ \sigma(L) = \Delta^{-1}([-2, 2]) = \{ \lambda \in \mathbb{C} | -2 \leq \Delta(\lambda) \leq 2 \}, $$

where $\Delta(\lambda)$ denotes the Hill’s discriminant which is the trace of the monodromy matrix of $L y = \lambda y$ with respect to $x \rightarrow x + 1$. Furthermore, it was proved in [14] that $\sigma(L)$ consists of finitely many analytic arcs if the potential of $L$ is a Picard potential. In this paper, as in [4] [14] we call the arcs of $\sigma(L)$ as spectral arcs.

For the simplest case $\text{Re} \tau = 0$, since the Lamé potential $-g(g + 1) \phi(x + \omega_3; \tau)$ is real-valued and smooth on $\mathbb{R}$, Ince [19] proved that there are $2g + 1$ distinct real numbers $\lambda_0 > \lambda_1 > \cdots > \lambda_{2g}$ such that the spectrum

(1.2) $\sigma(L^\xi_\alpha) = (-\infty, \lambda_0] \cup [\lambda_2, \lambda_0 - 1, \lambda_2 - 2] \cup \cdots \cup [\lambda_1, \lambda_0] \subseteq \mathbb{R}.$

However, the spectrum $\sigma(L^\xi)$ is no longer of the form (1.2) for general $\tau$’s and becomes very complicated; see e.g. [1] [5] [8] [16] [17] [18] and references therein. Gesztesy-Weikard [17] and Haese-Hill et al. [18] concentrated on
the $g = 1$ case, for which the spectrum $\sigma(L^1_\tau)$ consists of two regular analytic arcs and so there are totally three different types of graphs for different $\tau$'s as shown in the following figure (see also [1][16]).

![Figure 1](image)

In particular, different spectral arcs might intersect as shown in Figure (b)-(c). It was pointed out in [18] Section 5] that the rigorous analysis of $g \geq 2$ cases seems to be difficult since the related explicit formulae quickly become quite complicated as $g$ grows. Recently, [4] Lemma 4.1] inferred that the spectrum $\sigma(L^g_\tau)$ is symmetric with respect to the real line $\mathbb{R}$ if $\tau = \frac{1}{2} + bi$ with $b > 0$. Furthermore, it was proved in [4] that the spectrum $\sigma(L^2_{\frac{1}{2} + bi})$ in $g = 2$ case has exactly 9 different types of graphs for different $b$'s, and the continuous deformation of the spectrum as $b$ increases is as follows.

![Figure 2](image)

In this paper, we will focus on the genus $g = 3$ case. To the best of our knowledge, there seems no more explicit description of $\sigma(L^3_\tau)$ in the literature. It is well known (see [22] p.569]) that the associated spectral polynomial $Q_3(\lambda; \tau)$ is given by

\begin{equation}
Q_3(\lambda; \tau) = \lambda \prod_{k=1}^{3} \left( \lambda^2 - 6e_k \lambda + 15 \left( 3e_k^2 - g_2 \right) \right),
\end{equation}
so the associated hyperelliptic curve \( \{ (\lambda, C) \mid C^2 = Q_3(\lambda; \tau) \} \) is of genus 3.

By applying [14, Theorem 4.1], we see that the spectrum \( \sigma(L^3) \) consists of \( \tilde{g} \leq 3 \) bounded simple analytic arcs \( \sigma_k \) and one semi-infinite simple analytic arc \( \sigma_\infty \) which tends to \(-\infty + \langle q \rangle \), with \( \langle q \rangle = \int_{x_0}^{x_0+1} q(x) dx \), i.e.

\[
\sigma(L) = \sigma_\infty \cup \bigcup_{k=1}^{\tilde{g}} \sigma_k, \quad \tilde{g} \leq 3,
\]

where the finite endpoints of such arcs must be zeros of the spectral polynomial \( Q_3(\lambda; \tau) \) with odd order.

As shown in the above figures, there are two kinds of intersection points of spectral arcs: One is that this intersection point is also an endpoint of some arc (see Figures (b) and (2)), so it is met by 2 \( k \) semi-arcs for some \( k \geq 1 \); the other is that this intersection point is not an endpoint (see Figures (c) and (3)-(6), (9)), so it is met by 2 \( k \) semi-arcs for some \( k \geq 2 \), i.e. it is an inner point of \( k \) arcs; and we call it an inner intersection point.

First, we have to determine completely the inner intersection points of different arcs in order to study the geometry of \( \sigma(L^3) \). Our first result is as follows.

**Theorem 1.1.** Let \( \tau \in \mathbb{H} \) and \( \lambda_0 \in \sigma(L^3) \) with \( Q_3(\lambda_0; \tau) \neq 0 \). Then \( \lambda_0 \) is an inner intersection point if and only if \( \lambda_0 \) satisfies the following cubic equation

\[
f(\lambda) := \frac{4}{15} \lambda^3 + \frac{8}{5} \eta_1 \lambda^2 - 3g_2 \lambda + 9g_3 - 6\eta_1 g_2 = 0.
\]

Now we study the deformation of the spectrum \( \sigma(L^3) \) as \( \tau = \frac{1}{2} + bi \) deforms. This problem is challenging and we can only obtain some partial results. For this purpose, we need to analyze when some endpoint \( \lambda_0 \) (i.e. \( \lambda_0 \) is a zero of \( Q_3(\lambda; \tau) \) with odd order) could be an intersection point.

**Theorem 1.2.** Let \( \tau = \frac{1}{2} + bi \) with \( b > 0 \) and \( d(\lambda) \) denote the number of semi-arcs met at \( \lambda \). Then the zeros of the spectral polynomial \( Q_3(\lambda; \tau) \) are listed as follows

\[
0, \mu, \overline{\mu}, \nu, \overline{\nu}, \theta_+, \theta_-,
\]

where the multiplicities of \( 0, \mu, \overline{\mu}, \nu, \overline{\nu} \) are 1. Furthermore, there is some \( \beta \approx 1.0979 \in \left( \frac{\sqrt{3}}{2}, +\infty \right) \) such that

(z1) \( \theta_- = \theta_+ = 3e_1 \in \mathbb{R} \) if and only if \( b \in \{ \beta, \tilde{\beta} \} \), where \( \tilde{\beta} = \frac{1}{4b} \approx 0.2277 \).

(z2) \( \theta_-, \theta_+ \in \mathbb{R} \) and \( \theta_- < 3e_1 < \theta_+ < 0 \) if and only if \( b \in (0, \beta) \).

(z3) \( \theta_-, \theta_+ \in \mathbb{R} \) and \( 0 < \theta_- < 3e_1 < \theta_+ \) if and only if \( b \in (\beta, +\infty) \).

(z4) \( \theta_- = \overline{\theta_+} \notin \mathbb{R} \) if and only if \( b \in (\tilde{\beta}, \beta) \).

Moreover,

(d1) \( d(0) = 1 + 2d \) for some \( d \geq 1 \) (i.e. the endpoint 0 is an intersection point) if and only if \( b = b_0 \approx 0.47 \); otherwise, \( d(0) = 1 \).

(d2) \( d(\mu) = d(\nu) = d(\overline{\mu}) = d(\overline{\nu}) = 1 \), i.e. the endpoints \( \mu, \nu, \overline{\mu}, \overline{\nu} \) cannot be intersection points.
Note that $Q_3(\lambda; \frac{1}{2} + bi)$ has double zeros at $b \in \{\beta, \tilde{\beta}\}$. Here we mainly study the deformation of the spectrum $\sigma(L^3_\tau)$ along $\tau = \frac{1}{2} + bi$ with $b$ varying around $\tilde{\beta}$ and $\beta$, respectively. We discover 10 different patterns for the spectrum stated in the following theorem. See also the following figure for these 10 rough graphs.

**Theorem 1.3.** Let $\tau = \frac{1}{2} + ib$ with $b > 0$ and denote $L^3_b := L^3_{\tau}$. Then the spectrum $\sigma(L^3_b)$ is symmetric with respect to the real line $\mathbb{R}$. Furthermore, there exist $\varepsilon > 0, \delta > 0$ sufficient small and some $k_1 \in (0, \tilde{\beta})$, $\alpha \approx 0.23217$ such that the rough graphs of the spectra $\sigma(L^3_b)$ for $b \in (0, \alpha + \varepsilon) \cup (\beta - \delta, +\infty)$ are described as follows.

Here, the notations $\sigma_i$ with $i = 1, 2, 3$ denote simple arcs symmetric with respect to $\mathbb{R}$ and they are disjoint with each other. The notation $\sigma_4$ denotes a simple arc in $\{z \in \mathbb{C} \mid \text{Im} z \geq 0\}$ and $\sigma_4$ denotes the conjugate of $\sigma_4$. The notations $\lambda_+$ denote real roots of $f$, where $\lambda_- < 0$ and $\lambda_+ > 0$.

$\sigma(L^3_b)$ for $b$ in different intervals:

1. If $0 < b < k_1$, then $\theta_- < 3\varepsilon_1 < \theta_+ < 0$ and
   \[
   \sigma(L^3_b) = (-\infty, \theta_-] \cup [\theta_+, 0] \cup \sigma_1 \cup \sigma_2
   \]
   with $\sigma_1 \cap (0, +\infty) = \{\text{one point}\}$ and $\sigma_2 \cap (\theta_-, \theta_+) = \{\text{one point}\}$.

2. If $b = k_1$, then $\theta_- < 3\varepsilon_1 < \theta_+ < 0$ and
   \[
   \sigma(L^3_{k_1}) = (-\infty, \theta_-] \cup [\theta_+, 0] \cup \sigma_1 \cup \sigma_2
   \]
   with $\sigma_1 \cap (0, +\infty) = \{\text{one point}\}$ and $\sigma_2 \cap (-\infty, \theta_-) = \{\theta_-\}$.

- $\frac{1}{2}$ $b$ $\beta$ $\tilde{\beta}$ $\varepsilon_1$

Figure 3

(1) If $0 < b < k_1$, then $\theta_- < 3\varepsilon_1 < \theta_+ < 0$ and
\[
\sigma(L^3_b) = (-\infty, \theta_-] \cup [\theta_+, 0] \cup \sigma_1 \cup \sigma_2
\]
with $\sigma_1 \cap (0, +\infty) = \{\text{one point}\}$ and $\sigma_2 \cap (\theta_-, \theta_+) = \{\text{one point}\}$.

(2) If $b = k_1$, then $\theta_- < 3\varepsilon_1 < \theta_+ < 0$ and
\[
\sigma(L^3_{k_1}) = (-\infty, \theta_-] \cup [\theta_+, 0] \cup \sigma_1 \cup \sigma_2
\]
with $\sigma_1 \cap (0, +\infty) = \{\text{one point}\}$ and $\sigma_2 \cap (-\infty, \theta_-) = \{\theta_-\}$. 
(3) If $k_1 < b < \hat{\beta}$, then $\vartheta_- < 3\epsilon_1 < \vartheta_+ < 0$ and

$$\sigma(L^3_{\hat{\beta}}) = (-\infty, \vartheta_-] \cup [\vartheta_+, 0] \cup \sigma_1 \cup \sigma_2$$

with $\sigma_1 \cap (0, +\infty) = \{\text{one point}\}$ and $\sigma_2 \cap (-\infty, \vartheta_-) = \{\lambda_-\}$.

(4) If $b = \hat{\beta}$, then $\vartheta_- = \vartheta_+ = 3\epsilon_1 < 0$ and

$$\sigma(L^3_{\hat{\beta}}) = (-\infty, 0] \cup \sigma_1 \cup \sigma_2$$

with $\sigma_1 \cap (0, +\infty) = \{\text{one point}\}$ and $\sigma_2 \cap (-\infty, 0) = \{\lambda_-\}$.

(5) If $\beta < b < \alpha$, then $f(\lambda)$ has three real roots $\lambda_- < \lambda'_- < \lambda_+$ and

$$\sigma(L^3_{\hat{\beta}}) = (-\infty, 0] \cup \sigma_1 \cup \sigma_2 \cup \sigma_3$$

with $\sigma_1 \cap (0, +\infty) = \{\text{one point}\}$, $\sigma_2 \cap (-\infty, 0) = \{\lambda_-\}$ and $\sigma_3 \cap (-\infty, 0) = \{\lambda'_-\}$.

(6) If $b = \alpha$, then

$$\sigma(L^3_{\hat{\beta}}) = (-\infty, 0] \cup \sigma_1 \cup \sigma_4 \cup \overline{\sigma_4}$$

with $\sigma_1 \cap (0, +\infty) = \{\text{one point}\}$ and $\sigma_4 \cap \mathbb{R} = \emptyset$.

(7) If $\alpha < b < \alpha + \epsilon$, then

$$\sigma(L^3_{\hat{\beta}}) = (-\infty, 0] \cup \sigma_1 \cup \sigma_4 \cup \overline{\sigma_4}$$

with $\sigma_1 \cap (0, +\infty) = \{\text{one point}\}$ and $\sigma_4 \cap \mathbb{R} = \emptyset$.

(8) If $\beta - \delta \leq b < \beta$, then

$$\sigma(L^3_{\hat{\beta}}) = (-\infty, 0] \cup \sigma_1 \cup \sigma_4 \cup \overline{\sigma_4}$$

with $\sigma_1 \cap (-\infty, 0) = \{\lambda_-\}$ and $\sigma_4 \cap \mathbb{R} = \emptyset$.

(9) If $b = \beta$, then $\vartheta_- = \vartheta_+ = 3\epsilon_1 > 0$ and

$$\sigma(L^3_{\hat{\beta}}) = (-\infty, 0] \cup \sigma_1 \cup \sigma_2$$

with $\sigma_1 \cap \mathbb{R} = \{3\epsilon_1\}$ and $\sigma_2 \cap (-\infty, 0) = \{\lambda_-\}$.

(10) If $b > \beta$, then $0 < \vartheta_- < 3\epsilon_1 < \vartheta_+$ and

$$\sigma(L^3_{\hat{\beta}}) = (-\infty, 0] \cup [\vartheta_-, \vartheta_+] \cup \sigma_1 \cup \sigma_2$$

with $\sigma_1 \cap (\vartheta_-, \vartheta_+) = \{\lambda_+\}$ and $\sigma_2 \cap (-\infty, 0) = \{\lambda_-\}$.

**Remark 1.4.**

1. The other cases for $b \in [\alpha + \epsilon, \beta - \delta]$ seem difficult and remain open.

2. By comparing Figure 3 with Figure 2, we have a surprising observation. The first 7 graphs in Figure 3 can be obtained from the first 7 graphs in Figure 2 by adding the spectral arc $\sigma_1$ respectively, and the last 3 graphs in Figure 3 can be obtained from the last 3 graphs in Figure 2 by adding the spectral arc $\sigma_2$ respectively.
2. Spectral theory of Lamé operators

In this section, we briefly review some preliminary results about the spectral theory of Lamé operators $L_\tau^\xi$ that are needed in later sections.

Recall the classical Lamé equation

$$L(g, \tau, \lambda) : y''(z) - g(g+1)\wp(z;\tau)y(z) = \lambda y(z), \quad z \in \mathbb{C}.$$ 

Let $Y(z;\tau,\lambda) = (y_1(z;\tau,\lambda), y_2(z;\tau,\lambda))$ be a fundamental matrix of $L(g, \tau, \lambda)$ near a fixed base point $p_0 \in T^*_\tau := T_{\tau} \setminus \{0\}$. In general, $Y(z;\tau,\lambda)$ is multiple-valued with respect to $z$ and might have branch points at 0. Note that $g \in \mathbb{Z}$, it is well-known (cf. [15]) that any solution of $L(g, \tau, \lambda)$ is meromorphic in $\mathbb{C}$. Hence, $Y(z;\tau,\lambda)$ has no branch points at 0. For any loop $\gamma \in \pi_1(T_\tau, p_0)$, denote by $\gamma^* Y(z;\tau,\lambda)$ the analytic continuation of $Y(z;\tau,\lambda)$ along the loop $\gamma$, then there exists a matrix $M(\gamma;\tau,\lambda) \in \text{SL}(2,\mathbb{Z})$ such that $\gamma^* Y(z;\tau,\lambda) = Y(z;\tau,\lambda)M(\gamma;\tau,\lambda)$. This induces a group homomorphism

$$\rho(\tau,\lambda) : \pi_1(T_\tau, p_0) \to \text{SL}(2,\mathbb{Z})$$

which is called the monodromy representation of $L(g, \tau, \lambda)$. By the deck transformations, it is clear to see that the monodromy group is generated by two matrices $M_1(\tau,\lambda), M_2(\tau,\lambda) \in \text{SL}(2,\mathbb{Z})$ satisfying

$$Y(z + \omega;\tau,\lambda) = Y(z;\tau,\lambda)M_i(\tau,\lambda), \quad i = 1, 2,$$

and $M_1M_2 = M_2M_1$ because $\pi_1(T_\tau, p_0) \cong \mathbb{Z}^2$ is abelian. So $\rho(\tau,\lambda)$ is reducible and then $M_1(\tau,\lambda), M_2(\tau,\lambda)$ can be normalized to satisfy one of the following two cases (see [6, Section 2]):

(1) If $\rho(\tau,\lambda)$ is completely reducible, then

$$M_1(\tau,\lambda) = \begin{pmatrix} e^{-2\pi is} & 0 \\ 0 & e^{2\pi is} \end{pmatrix} , \quad M_2(\tau,\lambda) = \begin{pmatrix} e^{2\pi ir} & 0 \\ 0 & e^{-2\pi ir} \end{pmatrix} , \quad (r, s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2;$$

(2) If $\rho(\tau,\lambda)$ is not completely reducible, then

$$M_1(\tau,\lambda) = \epsilon_1 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} , \quad M_2(\tau,\lambda) = \epsilon_2 \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} , \quad \epsilon_j = \pm 1, \quad C \in \mathbb{C} \cup \{\infty\};$$

When $C = \infty$, the monodromy matrices are understood as

$$M_1(\tau,\lambda) = \epsilon_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad M_2(\tau,\lambda) = \epsilon_2 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} .$$

In particular, $y_1(z;\tau,\lambda)$ can be chosen as a common eigenfunction of $M_1(\tau,\lambda)$ and $M_2(\tau,\lambda)$. By a direct computation, the local exponent of $L(g, \tau, \lambda)$ at 0 is either $-g$ or $g+1$, then a classic theorem says that up to a constant, $y_1(z;\tau,\lambda)$ can be written as

$$y_1(z;\xi) = e^{cz} \prod_{i=1}^g \frac{\sigma(z-a_i)}{\sigma(z)^\xi},$$

with $c \in \mathbb{C}$ and $a = \{a_1, \ldots, a_g\} \in \text{Sym}^g(\mathbb{C} \setminus \Lambda_\tau)$. 


Theorem 2.2. [3, Theorem 6.2] Let $a = \{a_1, \ldots, a_g\} \in \text{Sym}^g(C \setminus \Lambda_\tau)$. Then $y_a(z; c)$ is a solution of $\mathcal{L}(g, \tau, \lambda)$ if and only if $a_i - a_j \notin \Lambda_\tau$ for all $i \neq j$ and

$$\sum_{j \neq i}^g (\zeta(a_i - a_j) + \zeta(a_j) - \zeta(a_i)) = 0, \quad 1 \leq i \leq g.$$  

Moreover, if $y_a(z; c)$ is a solution of $\mathcal{L}(g, \tau, \lambda)$, then

\begin{align*}
(2.1) & \quad c = c_a := \sum_{i=1}^g \zeta(a_i), \\
(2.2) & \quad c = c_a := \sum_{i=1}^g \zeta(a_i), \\
(2.3) & \quad \lambda = \lambda_a := (2g - 1) \sum_{i=1}^g \zeta(a_i).
\end{align*}

Remark 2.1. If $y_a(z; c)$ is a solution of $\mathcal{L}(g, \tau, \lambda)$, then $c$ and $\lambda$ are uniquely determined by $a$. Indeed, if both $y_a(z; c_1)$ and $y_a(z; c_2)$ are solutions of $\mathcal{L}(g, \tau, \lambda)$, a direct computation gives us $c_1 = c_2 := c_a$, then $\lambda$ is uniquely determined by $y_a(z; c_a)$ and denoted by $\lambda_a$.

Denote by $[a] = \{[a_1], \ldots, [a_g]\}$ with $[a_i] := a_i \mod \Lambda_\tau \in T^*_\tau = T_\tau \setminus \{0\}$. From Theorem 2.2, if $[a'] = [a]$, we have $\lambda_a = \lambda_a'$, so $\lambda_{[a]} := \lambda_a$ is well defined and then $y_a(z; c_a)$, $y_{a'}(z; c_{a'})$ are linearly dependent solutions of $\mathcal{L}(g, \tau, \lambda_{[a]})$. Therefore, we often write $a$ instead of $[a]$ to simplify notations when no confusion arises. Define

$$Y^g_\tau := \{[a] \in \text{Sym}^g T^*_\tau \mid y_a(z; c) \text{ is a solution of } \mathcal{L}(g, \tau, \lambda) \text{ for some } c \text{ and } \lambda\}$$

$$= \left\{[a] \in \text{Sym}^g T^*_\tau \mid [a_i] \neq [a_j] \text{ for any } i \neq j, \sum_{j \neq i}^g (\zeta(a_i - a_j) + \zeta(a_j) - \zeta(a_i)) = 0, \forall i. \right\}.$$

Clearly, if $[a] \in Y^g_\tau$, then $[-a] := \{[-a_1], \ldots, [-a_g]\} \in Y^g_\tau$ and $\lambda_{[-a]} = \lambda_{[a]}$. Note that the Wronskian of $y_a(z; c_a)$ and $y_{-a}(z; c_{-a})$ is a constant, then either $[a] = [-a]$ or $[a] \cap [-a] = \emptyset$. Moreover, $y_a(z)$ and $y_{-a}(z)$ are linearly independent if and only if $[a] \cap [-a] = \emptyset$. We have the following branched covering map of degree 2 (see [3, Theorem 7.4]):

$$\lambda : Y^g_\tau \to \mathbb{C}$$

$$[a] \mapsto \lambda_{[a]} := \lambda_a.$$

We call $[a] \in Y^g_\tau$ a branch point of $Y^g_\tau$ if $[a] = [-a]$.

Remark 2.3. If $[a] \in Y^g_\tau$ is not a branch point, then it was proved in [3, Proposition 5.8.3] that

$$\sum_{j \neq i}^g (\zeta(a_i - a_j) + \zeta(a_j) - \zeta(a_i)) = 0, \quad \forall 1 \leq i \leq g,$$
is equivalent to

\[(2.6) \quad \sum_{j=1}^{g} \varphi'(a_j) \varphi(a_j)^l = 0, \quad \forall 0 \leq l \leq g - 2.\]

By (2.6), it follows that for \([a] \in Y^g_\tau\), we have

\[g_{[a]}(z) := \sum_{i=1}^{g} \frac{\varphi'(a_i)}{\varphi(z) - \varphi(a_i)} = \frac{C_{[a]}}{\prod_{j \neq i} (\varphi(a_i) - \varphi(a_j))}\]

for a constant

\[C_{[a]} = \varphi'(a_i) \prod_{j \neq i} (\varphi(a_i) - \varphi(a_j)), \quad \text{which is independent of } i.\]

Remark that \(C_{[a]} = 0\) if and only if \([a]\) is a branch point.

**Theorem 2.4.** [3, Theorem 7.4] There is a so-called spectral polynomial \(Q^g_\tau(\lambda; \tau) \in Q[g_2(\tau), g_3(\tau)]\) of degree \(2g + 1\) such that if \([a] \in Y^g_\tau\), then \(C^g_{[a]} = Q^g_\tau(\lambda_{[a]}; \tau)\). Furthermore, \(Q^g_\tau(\lambda; \tau)\) is homogeneous of weight \(2g + 1\) in \(\lambda, g_2, g_3\) when \(\lambda, g_2, g_3\) are given weights 1, 2, 3 respectively.

This spectral polynomial \(Q^g_\tau(\lambda; \tau)\) is called the Lamé polynomial in the literature. Theorem 2.4 implies

\[(2.7) \quad Y^g_\tau \cong \{ (\lambda, C) \mid C^2 = Q^g_\tau(\lambda; \tau) \}\]

Therefore, \(Y^g_\tau\) is a hyperelliptic curve, known as the Lamé curve. And \([a] = [-a]\) is a branch point of \(Y^g_\tau\) if and only if \(Q^g_\tau(\lambda_{[a]}; \tau) = 0\).

Let \([a] \in Y^g_\tau\. The Legendre relation \(\tau \eta_1 - \eta_2 = 2\pi i\) implies that there is a unique \((r, s) \in \mathbb{C}^2\) satisfying

\[r + s\tau = a_1 + \cdots + a_g \quad \text{and} \quad r\eta_1 + s\eta_2 = \zeta(a_1) + \cdots + \zeta(a_g),\]

which is equivalent to

\[(2.8) \quad \zeta(a_1) + \cdots + \zeta(a_g) - \eta_1(a_1 + \cdots + a_g) = -2\pi is,\]

\[\tau(\zeta(a_1) + \cdots + \zeta(a_g)) - \eta_2(a_1 + \cdots + a_g) = 2\pi ir.\]

Furthermore, the transformation law \(\sigma(z + \omega_j) = e^{(z + \omega_j)\eta_j} \sigma(z)\) with \(j = 1, 2\) implies

\[(2.9) \quad y^\pm_a(z + 1; c \pm a) = e^{\pm \sum_{j=1}^{g} (\zeta(a_j) - \eta_1a_j)} y^\pm_a(z; c \pm a) = e^{\mp 2\pi is} y^\pm_a(z; c \pm a),\]

\[y^\pm_a(z + \tau; c \pm a) = e^{\pm \sum_{j=1}^{g} (\zeta(a_j) - \eta_2a_j)} y^\pm_a(z; c \pm a) = e^{\pm 2\pi ir} y^\pm_a(z; c \pm a),\]

namely \(y^\pm_a(z; c \pm a)\) are elliptic of the second kind. Note that \(y^\pm_a(z; c \pm a)\) are solutions of \(L^g_\tau y = \lambda_{[a]} y\), then \(y^\pm_a(x + z_0; c \pm a)\) are solutions of \(L^g_\tau y = \lambda_a y\).

**Case 1.** If \([a]\) is not a branch point, i.e., \([a] \cap [-a] = \emptyset\), then \(y_a(x + z_0; c a)\) and \(y_{-a}(x + z_0; c_{-a})\) are linearly independent solutions of \(L^g_\tau y = \lambda_{[a]} y\) and satisfy

\[(2.10) \quad y^\pm_a(x + z_0 + 1) = e^{\mp 2\pi is} y^\pm_a(x + z_0) \quad \text{with } s \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}.\]
In this case, the monodromy representation $\rho(\tau, \lambda_a)$ is completely reducible.

**Case 2.** If $[a]$ is a branch point, i.e., $[a] = [-a]$, then $y_a(x + \pi; c_a)$ and $y_{-a}(x + \pi; c_{-a})$ are linearly dependent solutions of $L_\tau y = \lambda_a y$. By (2.9), we get $2r, 2s \in \mathbb{Z}$. It was proved in [8, Theorem 2.6] that there is a solution $y_2(z)$ linearly independent with $y_1(z)$ such that (note $e^{2\pi is} = e^{-2\pi is} = \pm 1$)

$$y_1(z + 1) = e^{2\pi is} y_1(z), \quad y_2(z + 1) = e^{2\pi is} y_2(z) + e^{2\pi is} \chi[a] y_1(z),$$

where

$$\chi[a] = -\sum_{i=1}^{n} c_{a_i} (\varphi(a_i) + \eta_i),$$

$$c_{a_i} = 2\varphi''(a_i)^{-1} \prod_{j \neq i} (\varphi(a_i) - \varphi(a_j))^{-1}, \quad \text{if } a_i = -a_i,$$

$$c_{a_i} = c_{-a_i} = \varphi'(a_i)^{-1} \prod_{j \neq i, i^*} (\varphi(a_i) - \varphi(a_j))^{-1}, \quad \text{if } a_i \neq -a_i.$$}

In this case, $\rho(\tau, \lambda_a)$ is not completely reducible.

### 2.1. Spectrum of the Lamé operator

Let $y_1(x)$ and $y_2(x)$ be any two linearly independent solutions of

$$L_\tau y = \lambda y.$$ 

Then so do $y_1(x + 1)$ and $y_2(x + 1)$ and hence there is a monodromy matrix $M(\lambda) \in SL(2, \mathbb{C})$ such that

$$(y_1(x + 1), y_2(x + 1)) = (y_1(x), y_2(x)) M(\lambda).$$

Define the Hill’s discriminant $\Delta(\lambda)$ by

$$\Delta(\lambda) := \text{tr} M(\lambda),$$

which is clearly an invariant of (2.13), i.e. does not depend on the choice of linearly independent solutions. From (2.10) and (2.11), the Hill’s discriminant is given by

$$\Delta(\lambda) = e^{-2\pi is} + e^{2\pi is} = e^{\sum_{j=1}^{g}(\xi(a_j) - \eta a_j)} + e^{-\sum_{j=1}^{g}(\xi(a_j) - \eta a_j)}.$$ 

This entire function $\Delta(\lambda)$ encodes all information of the spectrum $\sigma(L_\tau^g)$; see e.g. [17] and references therein. Indeed, Rofe-Beketov [24] proved that the spectrum $\sigma(L_\tau^g)$ can be described as:

$$\sigma(L_\tau^g) = \Delta^{-1}([-2, 2]) = \{ \lambda \in \mathbb{C} \mid -2 \leq \Delta(\lambda) \leq 2 \}.$$ 

This important fact play a key role in this paper.

Clearly $\lambda$ is a (anti)periodic eigenvalue if and only if $\Delta(\lambda) = \pm 2$. Define

$$d(\lambda) := \text{ord}_\lambda(\Delta(\cdot)^2 - 4).$$

Then it is well known (cf. [23] Section 2.3) that $d(\lambda)$ equals to the algebraic multiplicity of (anti)periodic eigenvalues. Let $c(\lambda, \chi_0)$ and $s(\lambda, \chi_0)$ be
the principal fundamental system of solutions of (2.13) satisfying the initial values
\[ c(\lambda, x_0, x_0) = s'(\lambda, x_0, x_0) = 1, c'(\lambda, x_0, x_0) = s(\lambda, x_0, x_0) = 0. \]
Then we have
\[ \Delta(\lambda) = c(\lambda, x_0 + 1, x_0) + s'(\lambda, x_0 + 1, x_0). \]
Define
\[ p(\lambda, x_0) := \text{ord}_\lambda s(\cdot, x_0 + \omega, x_0), \]
\[ p_1(\lambda) := \min \{ p(\lambda, x_0) : x_0 \in \mathbb{R} \}. \]
It is known that \( p(\lambda, x_0) \) is the algebraic multiplicity of a Dirichlet eigenvalue on the interval \([x_0, x_0 + \omega]\), and \( p_1(\lambda) \) denotes the immovable part of \( p(\lambda, x_0) \) (cf. [14]). It was proved in [14, Theorem 3.2] that
\[ d(\lambda) - 2p_1(\lambda) \geq 0. \]
Applying the general result [14, Theorem 4.1] to \( L^3_\tau \), we obtain

**Theorem 2.5.** [14, Theorem 4.1] Let \( \tau \in \mathbb{H} \), the spectrum \( \sigma(L^3_\tau) \) consists of finitely many bounded simple analytic arcs \( \sigma_k \), \( 1 \leq k \leq g \) for some \( g \leq 3 \) and one semi-infinite simple analytic arc \( \sigma_\infty \) which tends to \( -\infty + \langle q \rangle \), with \( \langle q \rangle = \int_{x_0}^{x_0+1} q(x) dx \), i.e.
\[ \sigma(L^3_\tau) = \sigma_\infty \bigcup \bigcup_{k=1}^{g} \sigma_k. \]
Furthermore,
1. The finite end points of such arcs are exactly zeros of \( Q_3(\cdot; \tau) \) with odd order;
2. There are exactly \( d(\lambda) \)'s semi-arcs of \( \sigma(L^3_\tau) \) meeting at each zero \( \lambda \) of \( Q_3(\cdot; \tau) \) and
3. \[ d(\lambda) = \text{ord}_\lambda Q_3(\cdot; \tau) + 2p_1(\lambda). \]

Furthermore, we need the following conclusions about \( \sigma(L^3_\tau) \).

**Theorem 2.6.** Let \( \tau = \frac{1}{2} + bi \) with \( b > 0 \).
1. [4, Lemma 4.1] The spectrum \( \sigma(L^3_\tau) \) is symmetric with respect to the real line \( \mathbb{R} \). In particular, \( \sigma_\infty \subseteq \mathbb{R} \).
2. [17, Theorem 2.2] The complement \( \mathbb{C} \setminus \sigma(L^3_\tau) \) is path-connected.

Note that the multiplicity of any zero of \( Q_3(\lambda; \tau) \) is at most 2 by Theorem 1.2, we have the following results about the spectrum of \( L^3_\tau \).

**Theorem 2.7.** [9, Theorem 1.3] Let \( \tau \in \mathbb{H} \),
1. for any finite endpoint \( \lambda_a \) of \( \sigma(L^3_\tau) \),
\[ d(\lambda_a) = \text{ord}_{\lambda_a} (\Delta(\lambda)^2 - 4) \geq 3 \text{ if and only if } \chi_a = 0, \]
where \( \chi_a \) is defined in (2.12).
2. \( \sigma(L^3_\tau) \) has at most one endpoint with \( d(\lambda) \geq 3 \).

By Theorem 2.6(1) and Theorem 2.7(2), we obtain the following corollary directly.
Corollary 2.8. Let $\tau = \frac{1}{2} + bi$ with $b > 0$, any simple zero $\lambda \in \mathbb{C} \setminus \mathbb{R}$ of $Q_3(\lambda; \tau)$ satisfies $d(\lambda) = 1$, i.e. such $\lambda$ can not be an intersection point of spectral arcs.

2.2. Mean field equation. The purpose of this section is to study the relation between the spectrum $\sigma(L^g)\tau$ and the number of even axisymmetric solutions of the mean field equation

$$\Delta u + e^u = 8g\pi \delta_0 \quad \text{on} \quad T_\tau.$$  

First of all, we recall the connection between (2.17) and the generalized Lamé equation $L(g, \tau, \lambda)$ which was studied in [3].

Theorem 2.9. [7, Theorem 3.1] The mean field equation (2.17) has an even solution if and only if there exists $\lambda \in \mathbb{C}$ such that the monodromy of $L(g, \tau, \lambda)$ is unitary. Furthermore, the number of even solutions equals the number of those $\lambda$’s such that the monodromy of $L(g, \tau, \lambda)$ is unitary.

For any $\lambda \in \mathbb{C}$, there exists $[a] \in Y^g$ such that $\lambda = \lambda_a$ by the covering map (2.4). From (2.15), we have

$$\Delta(\lambda_a) = e^{2\pi is} + e^{-2\pi is}, \quad 2\pi is = \sum_{i=1}^g (\eta_1a_i - \zeta(a_i)) \in \mathbb{C},$$

then $\sigma(L^g) = \Delta^{-1}([-2, 2]) = \{\lambda_a \mid s \in \mathbb{R}\}$. Since the Lamé potential is doubly periodic, we can also consider its spectrum along the $\tau$ direction.

From (2.8) and (2.9), it is clear to see that the Hill’s discriminant along the $\tau$ direction denoted by $\Delta_t$ is as follows:

$$\Delta_t(\lambda_a) = e^{2\pi ir} + e^{-2\pi ir}, \quad 2\pi ir = \sum_{i=1}^g (\tau \zeta(a_i) - \eta_2a_i) \in \mathbb{C},$$

then the spectrum along the $\tau$ direction is the following:

$$\sigma_t(L^g) = \Delta_t^{-1}([-2, 2]) = \{\lambda_a \mid r \in \mathbb{R}\}.$$

Let $\tau = \frac{1}{2} + ib$ with $b > 0$. We will see the spectrum along the imaginary axis plays the same role as $\sigma_t(L^g)$. Note that $2\tau - 1 \in i\mathbb{R}$ and

$$y_{\pm a}(z + 2\tau - 1) = e^{\pm 2\pi i(2r+s)}y_{\pm a}(z),$$

the Hill’s discriminant along the imaginary direction is given by

$$\Delta_i(\lambda_a) := e^{2\pi i(2r+s)} + e^{-2\pi i(2r+s)},$$

and then

$$\sigma_i(L^g) := \Delta_i^{-1}([-2, 2]) = \{\lambda_a \mid 2r + s \in \mathbb{R}\}.$$  

Clearly, $\sigma(L^g) \cap \sigma_i(L^g) = \sigma(L^g) \cap \sigma_t(L^g)$. In particular, the spectrum $\sigma_i(L^g)$ can be obtained from $\sigma(L^g)$ by a dilation.
Lemma 2.10. Let $\tau = \frac{1}{2} + bi$ with $b > 0$. We have

$$\sigma_i(\mathcal{L}^S_0) = \frac{1}{4b^2} \sigma(\mathcal{L}^S_{1b})$$

and the endpoints of $\sigma_i(\mathcal{L}^S_0)$ are exactly the endpoints of $\sigma(\mathcal{L}^S_{1b})$.

Proof. Note that

$$\tau = \frac{1}{2} + \frac{i}{4b} \quad \text{for} \quad \tau = \frac{1}{2} + ib.$$ 

Since $y_{\pm a}(z)$ are solutions of $\mathcal{L}(g, \tau, \lambda_a)$, then $\tilde{y}_{\pm a}(z) := y_{\pm a}((2\tau - 1)z)$ satisfies

$$\tilde{y}_{\pm a}''(z) = (2\tau - 1)^2 (g(z + 1)\phi((2\tau - 1)z; \tau) + \lambda_a) \tilde{y}_{\pm a}(z)$$

$$= (g(z + 1)\phi(z; \frac{1}{2} + i\frac{1}{4b}) + (2\tau - 1)^2 \lambda_a) \tilde{y}_{\pm a}(z)$$

$$= (g(z + 1)\phi(z; \frac{1}{2} + i\frac{1}{4b}) - 4b^2 \lambda_a) \tilde{y}_{\pm a}(z),$$

and

$$\tilde{y}_{\pm a}(z + 1) = y_{\pm a}((2\tau - 1)z + (2\tau - 1)) = e^{\pm 2\pi i(2\tau + s)} \tilde{y}_{\pm a}(z).$$

Therefore, we have

$$\Delta(-4b^2 \lambda_a; \frac{1}{4b}) = e^{2\pi i(2\tau + s)} + e^{-2\pi i(2\tau + s)} = \Delta(\lambda_a; b).$$

Consequently, we conclude from (2.18) that

$$\sigma_i(\mathcal{L}^S_0) = \{ \lambda \in \mathbb{C} \mid -2 \leq \Delta(-4b^2 \lambda; \frac{1}{4b}) \leq 2 \}$$

$$= \{ \lambda \in \mathbb{C} \mid -4b^2 \lambda \in \sigma(L^S_{\frac{1}{4b}}) \} = \frac{1}{4b^2} \sigma(L^S_{\frac{1}{4b}}).$$

Theorem 2.4 tells us that $Q_6(\lambda; b) \in Q[Q_2(b), Q_3(b)]$ is homogeneous of weight $2g + 1$ in $\lambda, Q_2, Q_3$ when $\lambda, Q_2, Q_3$ are given weights 1, 2, 3 respectively. By the modular properties (cf. (10)), we have

$$g_2(\frac{1}{4b}) = 16b^g g_2(b), \quad g_3(\frac{1}{4b}) = -64b^g g_3(b),$$

then

$$Q_6(-4b^2 \lambda; \frac{1}{4b}) = (-4b^2)^{2g+1} Q_6(\lambda; b).$$

Denote by

$$Z(Q_6(\lambda; b)) := \{ \lambda \in \mathbb{C} \mid Q_6(\lambda) = 0 \},$$

combining with (2.20), we have

$$Z(Q_6(\lambda; b)) = Z(Q_6(-4b^2 \lambda; \frac{1}{4b})) = \frac{1}{-4b^g} Z(Q_6(\lambda; \frac{1}{4b})) \subset \sigma_i(\mathcal{L}^S_0),$$

thus the endpoints of $\sigma_i(\mathcal{L}^S_0)$ are the same as the endpoints of $\sigma(\mathcal{L}^S_{1b})$. \qed

By the proof of Lemma 2.10, we have

$$Z(Q_6(\lambda; b)) \subset \sigma(\mathcal{L}^S_0) \cap \sigma_i(\mathcal{L}^S_0),$$
which contains all finite endpoints of both $\sigma(L_0^g)$ and $\sigma_i(L_0^g)$. Note that $Q_g(\lambda_a; b) \neq 0$ implies that $(s, r) \notin \frac{1}{2}\mathbb{Z}^2$, we have

$$
\Xi_b := (\sigma(L_0^g) \cap \sigma_i(L_0^g)) \setminus Z(\sigma(Q_g(\lambda; b)))
= \{ \lambda_a \mid (s, r) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2 \}.
$$

The following theorem establishes the precise connection between even solutions of the mean field equation and the spectrum.

**Theorem 2.11.** Let $\tau = \frac{1}{2} + bi$ with $b > 0$. The number of even solutions of the mean field equation (2.17) equals to $\#\Xi_b$. Furthermore, the number of even axisymmetric solutions equals to $\#(\Xi_b \cap \mathbb{R})$.

**Proof.** Note that the monodromy of $L(g, \tau, \lambda_a)$ is completely reducible if and only if $Q_g(\lambda_a; b) \neq 0$ (cf. [6] Theorem 2.7), then the monodromy of $L(g, \tau, \lambda_a)$ is unitary if and only if $\lambda_a \in \Xi_b$. Together with Theorem 2.9, we conclude that the number of even solutions of (2.17) equals to $\#\Xi_b$.

Let $u(z) = u(x, y)$ (here we use complex variable $z = x + iy$) be an even solution of (2.17), then there exists $\lambda_a \in \Xi_b$ (cf. [3]) such that

$$(u_{zz} - \frac{1}{2}u_z^2)(z) = -2(g(g + 1)\wp(z; \tau) + \lambda_a).$$

Clearly $\tilde{u}(z) = \tilde{u}(x, y) := u(x, -y) = u(\overline{z})$ is also an even solution of (2.17) and satisfies (note that $u(z)$ is real-valued as a solution of (2.17))

$$(\tilde{u}_{zz} - \frac{1}{2}\tilde{u}_z^2)(z) = \overline{(u_{zz} - \frac{1}{2}u_z^2)(z)}
= -2(g(g + 1)\wp(z; \tau) + \overline{\lambda_a})
= -2(g(g + 1)\wp(z; \tau) + \overline{\lambda_a}),$$

i.e. $\overline{\lambda_a} \in \Xi_b$ if $\lambda_a \in \Xi_b$. From here and the fact stated in Theorem 2.9 that there is a one-to-one correspondence between $\lambda \in \Xi_b$ and even solutions of (2.17), we conclude that $\lambda_a = \overline{\lambda_a}$ if and only if $u(z) = \tilde{u}(z)$, i.e. $u(z) = u(\overline{z})$ is axisymmetric. Therefore, the number of even axisymmetric solutions equals to $\#(\Xi_b \cap \mathbb{R})$. \hfill \square

By [13] Lemma 2.3, the spectrum $\sigma(L_0^g)$ is a horizontal translation of the spectrum of the following Darboux-Treibich-Verdier operator

$$
\tilde{L}_b^g := \frac{d^2}{dx^2} - g(g + 1) \left( \wp(x + z_0; 2bi) + \wp(x + \frac{1+i}{2} + z_0; 2bi) \right),
$$

with $z_0 \in \mathbb{C}$ is chosen such that the potential has no singularities on $\mathbb{R}$. Note that the mean field equation (2.17) has no solutions for $\tau = \frac{1}{2} + \frac{1}{2}i$ (cf. [11]), combining with Theorem 2.11 and [13] Theorem 1.7, we have the following corollary.
Corollary 2.12. Let \( \tau = \frac{1}{2} + bi \) with \( b > 0 \). There exists two real numbers \( k_1, k_2 \) with \( 0 < k_1 \leq k_2 < \frac{1}{2} \) such that
\[
(2.21) \quad \#(\mathcal{E}_b \cap \mathbb{R}) = \begin{cases} 
2 & \text{if } b \in (0, k_1) \cup (\frac{1}{4k_1}, +\infty) \\
1 & \text{if } b \in [k_1, k_2) \cup (\frac{1}{4k_2}, \frac{1}{k_1}] \\
0 & \text{if } b \in [k_2, \frac{1}{4k_2}].
\end{cases}
\]

In fact, our calculation for the spectrum supplies another way to compute \( k_1 \) and \( k_2 \). In particular, \( k_2 = b^*_\sigma \) (See Section 5).

3. INNER INTERSECTION POINTS

In this and the following sections, we study the spectrum of the \( g = 3 \) Lamé operator:
\[ L_3^g = \frac{d^2}{dx^2} - 12\wp(x + z_0; \tau), \quad x \in \mathbb{R}. \]

We will prove Theorem 1.1 recalled here.

**Theorem 3.1** (=Theorem 1.1). Let \( \tau \in \mathbb{H} \) and \( \lambda_0 \in \sigma(L_3^g) \) with \( Q_3(\lambda_0; \tau) \neq 0 \). Then \( \lambda_0 \) is an inner intersection point if and only if \( \lambda_0 \) satisfies the following cubic equation
\[ f(\lambda) := \frac{4}{15} \lambda^3 + \frac{8}{5} \eta_1 \lambda^2 - 3g_2 \lambda + 9g_3 - 6\eta_1 g_2 = 0. \]

**Proof.** Let \( \lambda_0 \in \sigma(L_3^g) \) with \( Q_3(\lambda_0; \tau) \neq 0 \). By (2.7) there is a small neighborhood \( U \subset \mathbb{C} \) of \( \lambda_0 \) such that \( Q_3(\lambda; \tau) \neq 0 \) for \( \lambda \in U \) and \( \lambda \in U \) can be a local coordinate for the hyperelliptic curve \( Y_3^T \), namely \( a_1 = a_1(\lambda), a_2 = a_2(\lambda) \) and \( a_3 = a_3(\lambda) \) are holomorphic for \( \lambda \in U \). For all \( \lambda \in U \), (2.4) tells us that
\[ \lambda = \lambda_4 = 5(\wp(a_1) + \wp(a_2) + \wp(a_3)) \]
and then
\[ \wp'(a_1)a_1'(\lambda) + \wp'(a_2)a_2'(\lambda) + \wp'(a_3)a_3'(\lambda) = \frac{1}{5} \quad \text{for } \lambda \in U \]
and so
\[ (a_1'(\lambda_0), a_2'(\lambda_0), a_3'(\lambda_0)) \neq (0, 0, 0). \]

Next, note that \( Q_3(\lambda; \tau) \neq 0 \) for \( \lambda \in U \) implies
\[ \{a_1(\lambda), a_2(\lambda), a_3(\lambda)\} \cap \{-a_1(\lambda), -a_2(\lambda), -a_3(\lambda)\} = \emptyset \quad \text{for } \lambda \in U, \]
i.e., \( a(\lambda) = \{a_1(\lambda), a_2(\lambda), a_3(\lambda)\} \) is not a branch point of \( Y_3^T \) for all \( \lambda \in U \.

Hence
\[ (3.2) \quad \wp(a_i(\lambda)) \neq \wp(a_j(\lambda)) \quad \text{for all } \lambda \in U, \quad 1 \leq i < j \leq 3, \]
and (2.6) holds for \( \lambda \in U \), i.e.,
\[ (3.3) \quad \wp'(a_1) + \wp'(a_2) + \wp'(a_3) = 0, \]
\[ \wp'(a_1)\wp(a_1) + \wp'(a_2)\wp(a_2) + \wp'(a_3)\wp(a_3) = 0. \]
Taking derivative with respect to $\lambda$ in (3.3) and evaluating at $\lambda_0$, we obtain from $(\varphi')^2 = 4\varphi^3 - g_2\varphi - g_3$ and $\varphi'' = 6\varphi^2 - \frac{g_2}{2}$ that

\begin{equation}
\sum_{i=1}^{3}(6\varphi_i^2 - \frac{g_2}{2})\varphi_i' = 0,
\end{equation}

\begin{equation}
\sum_{i=1}^{3}(10\varphi_i^3 - \frac{3}{2}g_2\varphi_i - g_3)\varphi_i' = 0,
\end{equation}

where $\varphi_i := \varphi \left( a_i(\lambda_0) \right)$ for $i = 1, 2, 3$.

By $(\varphi')^2 = 4\varphi^3 - g_2\varphi - g_3$ and (3.3), we easily obtain

\begin{equation}
\frac{4\varphi_1^3 - g_2\varphi_1 - g_3}{(\varphi_2 - \varphi_3)^2} = \frac{4\varphi_2^3 - g_2\varphi_2 - g_3}{(\varphi_1 - \varphi_3)^2} = \frac{4\varphi_3^3 - g_2\varphi_3 - g_3}{(\varphi_1 - \varphi_2)^2} =: \mathcal{U},
\end{equation}

which is equivalent to

\begin{equation}
\begin{cases}
4\varphi_1^3 - g_2\varphi_1 - g_3 = \mathcal{U}(\varphi_2 - \varphi_3)^2, \\
4\varphi_2^3 - g_2\varphi_2 - g_3 = \mathcal{U}(\varphi_1 - \varphi_3)^2, \\
4\varphi_3^3 - g_2\varphi_3 - g_3 = \mathcal{U}(\varphi_1 - \varphi_2)^2.
\end{cases}
\end{equation}

Denote by

\begin{equation}
\begin{cases}
s_1 = \varphi_1 + \varphi_2 + \varphi_3, \\
s_2 = \varphi_1\varphi_2 + \varphi_1\varphi_3 + \varphi_2\varphi_3, \\
s_3 = \varphi_1\varphi_2\varphi_3,
\end{cases}
\end{equation}

we have

\begin{equation}
(x - \varphi_1)(x - \varphi_2)(x - \varphi_3) = x^3 - s_1x^2 + s_2x - s_3,
\end{equation}

then (3.7) is equivalent to

\begin{equation}
\begin{cases}
4s_1\varphi_1^2 - (4s_2 + g_2)\varphi_1 + 4s_3 - g_3 = \mathcal{U}(\varphi_2 - \varphi_3)^2, \\
4s_1\varphi_2^2 - (4s_2 + g_2)\varphi_2 + 4s_3 - g_3 = \mathcal{U}(\varphi_1 - \varphi_3)^2, \\
4s_1\varphi_3^2 - (4s_2 + g_2)\varphi_3 + 4s_3 - g_3 = \mathcal{U}(\varphi_1 - \varphi_2)^2.
\end{cases}
\end{equation}

First, (3.9) + (3.10) + (3.11) lead to

\begin{equation}
\mathcal{U} = \frac{4(s_1^3 - 3s_1s_2 + 3s_3) - g_2s_1 - 3g_3}{2s_1^2 - 6s_2}.
\end{equation}

Note that $\varphi_i \neq \varphi_j$ for $i \neq j$, then (3.9)-(3.10), (3.9)-(3.11) and (3.10)-(3.11) yield

\begin{equation}
\begin{cases}
4s_2 + g_2 - 4s_1(\varphi_1 + \varphi_2) = \mathcal{U}(\varphi_1 + \varphi_2 - 2\varphi_3), \\
4s_2 + g_2 - 4s_1(\varphi_1 + \varphi_3) = \mathcal{U}(\varphi_1 + \varphi_3 - 2\varphi_2), \\
4s_2 + g_2 - 4s_1(\varphi_2 + \varphi_3) = \mathcal{U}(\varphi_2 + \varphi_3 - 2\varphi_1).
\end{cases}
\end{equation}

Next, (3.13) + (3.14) + (3.15) gives us

\begin{equation}
s_2 = \frac{2}{3}s_1^2 - \frac{1}{4}g_2,
\end{equation}
and (3.13)−(3.14) gives us

\begin{equation}
\tilde{\Omega} = -\frac{4}{3}s_1.
\end{equation}

Combine (3.12), (3.16) and (3.17), we obtain that

\begin{equation}
s_3 = \frac{5}{9}s_1^2 - \frac{1}{3}g_2s_1 + \frac{1}{4}g_3.
\end{equation}

On the other hand, for any \( \lambda \in U \), denote by \( A(\lambda) := \sum_{j=1}^{3}(\xi(a_j) - \eta_1a_j) \).

Since \( a(\lambda) \cap -a(\lambda) = \emptyset \), by (??), we have that for \( \lambda \in U \),

\begin{align*}
\Delta(\lambda) &= e^A + e^{-A}, \\
\Delta'(\lambda) &= (e^A - e^{-A})A', \\
\Delta''(\lambda) &= (e^A - e^{-A})(A'' + (A')^2), \\
\Delta'''(\lambda) &= (e^A - e^{-A})(A''' + (A')^3) + 3\Delta A'A'''.
\end{align*}

**Sufficiency.** Let \( \lambda_0 \in \sigma(L^3_\tau) \) with \( Q_3(\lambda_0; \tau) \neq 0 \) be an inner intersection point, then \( \lambda_0 \) is met by \( 2k \geq 4(k \in \mathbb{Z}) \) semi-arcs of the spectrum.

Consider the local behavior of the spectrum at \( \lambda_0 \in \sigma(L^3_\tau) \):

\begin{equation}
\Delta(\lambda) - \Delta(\lambda_0) = c(\lambda - \lambda_0)^k(1 + O(|\lambda - \lambda_0|)), \quad k \geq 1, \quad c \neq 0.
\end{equation}

If \( \Delta(\lambda_0) \in (-2,2) \), it follows from (3.20) and \( \sigma(L^3_\tau) = \{\lambda \mid -2 \leq \Delta(\lambda) \leq 2\} \) that there are precisely \( 2k \) semi-arcs of \( \sigma(L^3_\tau) \) meeting at \( \lambda_0 \). If \( \Delta(\lambda_0) = \pm2 \), then there are precisely \( k \) semi-arcs of \( \sigma(L^3_\tau) \) meeting at \( \lambda_0 \).

If \( \Delta(\lambda_0) = \pm2 \), then our assumption implies \( k \geq 4 \), i.e. \( \Delta'(\lambda_0) = \Delta''(\lambda_0) = \Delta'''(\lambda_0) = 0 \). Since \( \Delta(\lambda_0) = \pm2 \) implies \( e^A = \pm1 \) at \( \lambda_0 \), we obtain \( A'(\lambda_0) = 0 \).

If \( \Delta(\lambda_0) \in (-2,2) \), then our assumption implies \( 2k \geq 4 \), i.e. \( k \geq 2 \) and so \( \Delta'(\lambda_0) = 0 \). Since \( \Delta(\lambda_0) \neq \pm2 \) implies \( e^A \neq \pm1 \) at \( \lambda_0 \), again we obtain \( A'(\lambda_0) = 0 \).

Therefore, we always have \( A'(\lambda_0) = 0 \), i.e.,

\begin{equation}
(\varphi_1 + \eta_1)a_1'(\lambda_0) + (\varphi_2 + \eta_1)a_2'(\lambda_0) + (\varphi_3 + \eta_1)a_3'(\lambda_0) = 0.
\end{equation}

Noting from (3.1), we conclude from (3.21) that the determinant of the matrix

\[ \Omega := \begin{pmatrix}
\varphi_1 + \eta_1 & \varphi_2 + \eta_1 & \varphi_3 + \eta_1 \\
6\varphi_1^2 - \frac{6\eta_1^2}{2} & 6\varphi_2^2 - \frac{6\eta_1^2}{2} & 6\varphi_3^2 - \frac{6\eta_1^2}{2} \\
10\varphi_1^3 - \frac{3g_2\varphi_1 - g_3}{2} & 10\varphi_2^3 - \frac{3g_2\varphi_2 - g_3}{2} & 10\varphi_3^3 - \frac{3g_2\varphi_3 - g_3}{2}
\end{pmatrix} \]

vanishes, i.e.,

\[ (\varphi_2 - \varphi_1)(\varphi_3 - \varphi_1)(\varphi_3 - \varphi_2)(60s_3 + 60\eta_1s_2 + 5g_2s_1 - 6g_3 - 9g_2\eta_1) = 0. \]

By (3.2), we obtain that

\begin{equation}
60s_3 + 60\eta_1s_2 + 5g_2s_1 - 6g_3 - 9g_2\eta_1 = 0.
\end{equation}
Plug (3.16), (3.18) and (3.23) into (3.22), we finally obtain that
\[
\frac{4}{15} \lambda_0^3 + \frac{8}{5} \eta_1 \lambda_0^2 - 3 \eta_2 \lambda_0 + 9 \eta_3 - 6 \eta_1 \eta_2 = 0.
\]

**Necessity.** Suppose \( \lambda_0 \in \sigma(L^3_\tau) \) satisfies \( Q_3(\lambda_0; \tau) \neq 0 \) and
\[
\frac{4}{15} \lambda_0^3 + \frac{8}{5} \eta_1 \lambda_0^2 - 3 \eta_2 \lambda_0 + 9 \eta_3 - 6 \eta_1 \eta_2 = 0.
\]
This, together with (3.16), (3.18) and (3.23) implies \( \det \Omega = 0 \). Since \( \wp_i \neq \wp_j \) for \( i \neq j \), the second row of \( \Omega \) is nonzero. Suppose that the last two rows of \( \Omega \) are linearly dependent, there is \( c \in \mathbb{C} \) such that
\[
10 \wp_i^3 - \frac{3}{2} g_2 \wp_i - g_3 = c(6 \wp_i^2 - \frac{\wp_i}{2}), \quad i = 1, 2, 3,
\]
then
\[
(x - \wp_1)(x - \wp_2)(x - \wp_3) = x^3 - \frac{3}{5} c x^2 - \frac{3}{20} g_2 x + \frac{g_2}{20} c - \frac{1}{10} g_3.
\]
Compare (3.8) with (3.24), we have
\[
s_1 = \frac{3}{5} c,
\]
\[
s_2 = -\frac{3}{20} g_2,
\]
\[
s_3 = \frac{1}{10} g_3 - \frac{c}{20} g_2.
\]
Combine these with (3.16) and (3.18), we obtain that
\[
27 g_3^2 = 5 g_2^3 \quad \text{and} \quad \lambda_0 = 5 s_1 = \frac{9 g_3}{2 g_2}.
\]
Therefore, if \( \tau \in \mathbb{H} \setminus \{ \tau \in \mathbb{H} \mid 27 g_3^2(\tau) = 5 g_2^3(\tau) \} \), the last two rows of \( \Omega \) are linearly independent, thus the first row can be linearly spanned by the last two rows. Hence (3.4) and (3.5) yields (3.21). Finally, by the continuity of the left hand side in (3.21) with respect to \( \tau \), we obtain (3.21) holds for all \( \tau \in \mathbb{H} \).

If \( \Delta(\lambda_0) \in (-2, 2) \), then we see from (3.21) and (3.19) that \( \Delta'(\lambda_0) = 0 \), i.e. \( k \geq 2 \) in (3.20), and so there are \( 2k \geq 4 \) semi-arcs of \( \sigma(L^3_\tau) \) meeting at this \( \lambda_0 \).

If \( \Delta(\lambda_0) = \pm 2 \), then \( e^A = \pm 1 \) at \( \lambda_0 \). From here and (3.21), (3.19) we see that \( \Delta'(\lambda_0) = \Delta''(\lambda_0) = \Delta'''(\lambda_0) = 0 \). This means \( k \geq 4 \) in (3.20), and so there are \( k \geq 4 \) semi-arcs of \( \sigma(L^3_\tau) \) meeting at this \( \lambda_0 \). Therefore, \( \lambda_0 \) is an inner intersection point. \( \square \)
4. ZEROS OF THE SPECTRAL POLYNOMIAL

In this and the following sections, we always assume $\tau = \frac{1}{2} + bi$ with $b > 0$. In order to emphasize $\tau = \frac{1}{2} + bi$, we will use $b$ instead of $\tau$ in notations. Sometimes, we omit the notation $\tau$ freely. We will first recall some basic properties for the quantities $e_1, e_2, e_3, g_2, g_3$ and $\eta_1$ associated with the Weierstrass elliptic function $\wp(z; \tau)$.

First of all, the second equality in (1.1) gives us

\begin{align}
(4.1) & \quad e_1 + e_2 + e_3 = 0, \\
(4.2) & \quad g_2 = 2(e_1^2 + e_2^2 + e_3^2), \\
(4.3) & \quad g_3 = 4e_1e_2e_3.
\end{align}

Note that $e_1, e_2, e_3 \not\in \mathbb{R}$ and (4.1), in what follows, we set

\begin{align}
(4.4) & \quad e_1 = 2x, \quad e_2 = -x + iy, \quad e_3 = -x - iy \quad \text{with } x, y \in \mathbb{R} \text{ and } y \neq 0,
\end{align}

and then

\begin{align}
(4.5) & \quad g_2 = 4(3x^2 - y^2), \\
& \quad g_3 = 8x(x^2 + y^2) = 4e_1^3 - e_1g_2.
\end{align}

Since $e_1 \neq e_2 \neq e_3 \neq e_1$, it is easy to see that

\begin{align}
(4.6) & \quad g_2 - 3e_k^2 = (e_i - e_j)^2 \neq 0, \quad \text{for } \{i, j, k\} = \{1, 2, 3\}.
\end{align}

In particular,

\begin{align}
(4.7) & \quad g_2 - 3e_1^2 = (e_1 - e_3)^2 = -4y^2 < 0, \quad \text{i.e., } g_2 < 3e_1^2.
\end{align}

The derivatives of $e_1, g_2$ and $\eta_1$ with respect to $b$ are as follows:

\begin{align}
(4.8) & \quad e_1'(b) = \frac{1}{\pi} (e_1^2 - \eta_1e_1 - \frac{1}{6}g_2), \quad \text{(see [8] (2.15))}, \\
& \quad g_2'(b) = \frac{1}{\pi} (3g_3 - 2\eta_1g_2) = \frac{1}{\pi} (12e_1^3 - 3e_1g_2 - 2\eta_1g_2), \quad \text{(see [12])}, \\
& \quad \eta_1'(b) = \frac{1}{4\pi} (g_2 - 12\eta_1^2), \quad \text{(see [9] (1.5))}.
\end{align}

By (4.5), we have

\begin{align}
(4.9) & \quad g_2'(b) = \frac{1}{\pi} \left(\frac{1}{6}g_2^2 - 3\eta_1g_3\right) = \frac{1}{\pi} \left(\frac{1}{6}g_2^2 + 3e_1\eta_1g_2 - 12e_1^3\right).
\end{align}

Moreover, we have the following conclusions about the derivatives.

**Proposition 4.1.** [21] Theorem 1.7] \textbf{We have }$e_1 \left(\frac{1}{2}\right) = 0$ \textbf{and}

\begin{align}
& e_1'(b) > 0 \quad \text{for all } \quad b > 0.
\end{align}

**Proposition 4.2.** [12] Corollary 4.4] \textbf{There exists }$b_8 \approx 0.47 \in (\frac{1}{2\sqrt{3}}, \frac{1}{2})$ \textbf{such that}

\begin{align}
(4.10) & \quad g_2'(b) \begin{cases}
< 0 & \text{for } b \in (0, b_8), \\
= 0 & \text{for } b = b_8, \\
> 0 & \text{for } b \in (b_8, \infty).
\end{cases}
\end{align}

And $g_2(b) = 0$ if and only if $b \in \left\{ \frac{1}{2\sqrt{3}}, \sqrt{2} \right\}$. 

Proposition 4.3. [21, Theorem 1.7] There exists \( b_\eta \approx 0.24108 < \frac{1}{2\sqrt{3}} \) such that

\[
\eta'(b) \begin{cases} 
> 0 & \text{for } b \in (0, b_\eta), \\
= 0 & \text{for } b = b_\eta, \\
< 0 & \text{for } b \in (b_\eta, +\infty).
\end{cases}
\]

Remark 4.4. All numerical computations in this paper are based on the \( q = e^{2\pi i \tau} = -e^{-2\pi b} \) expansions of \( e_1, g_2, \eta_1 \) which are recalled here for readers’ convenience.

\[
e_1(b) = 16\pi^2 \left( \frac{1}{24} + \sum_{k=1}^{\infty} (-1)^k \sigma^k_0 e^{-2k\pi b} \right), \quad \text{where } \sigma^k_0 = \sum_{1 \leq d \mid k, d \text{ is odd}} d,
\]

\[
g_2(b) = 320\pi^4 \left( \frac{1}{240} + \sum_{k=1}^{\infty} (-1)^k \sigma^k_2 e^{-2k\pi b} \right), \quad \text{where } \sigma^k_2 = \sum_{1 \leq d \mid k} d^3,
\]

\[
\eta_1(b) = 8\pi^2 \left( \frac{1}{24} - \sum_{k=1}^{\infty} (-1)^k \sigma^k_1 e^{-2k\pi b} \right), \quad \text{where } \sigma^k_1 = \sum_{1 \leq d \mid k} d.
\]

By numerical computation, \( \eta_1 \) vanishes at \( b_0 \approx 0.13094 \).

Now we figure out all zeros of the spectral polynomial \( Q_3(\lambda; b) \) and prove Theorem 1.2.

Proof of Theorem 1.2. Recall that [cf. (1.3)]

\[
Q_3(\lambda) := Q_3(\lambda; b) = \lambda \prod_{k=1}^{3} (\lambda^2 - 6e_k \lambda + 15(3e_k^2 - g_2)).
\]

First of all, by (4.6), 0 is not a zero of

\[
R_k(\lambda) := \lambda^2 - 6e_k \lambda + 15(3e_k^2 - g_2), \quad k = 1, 2, 3.
\]

So 0 is a simple zero of \( Q_3(\lambda) \). By a direct computation, the resultant of \( R_i(\lambda) \) and \( R_j(\lambda) \) with \( i \neq j \)

\[
\text{(4.10)} \quad \text{Res} (R_i(\lambda), R_j(\lambda)) = -135(e_i - e_j)^4 \neq 0,
\]

thus \( R_i(\lambda) \) and \( R_j(\lambda) \) cannot have common zeros for any pair \( i \neq j \). Since \( R_2(\lambda) = R_3(\lambda) \), we obtain that the two zeros \( \mu, \nu \) of \( R_2(\lambda) \) are not real, i.e., \( \mu, \nu \in \mathbb{C} \setminus \mathbb{R} \), and then \( \mu, \nu \in \mathbb{C} \setminus \mathbb{R} \) are the zeros of \( R_3(\lambda) \). We claim that \( \mu \neq \nu \), otherwise, the discriminant of \( R_2(\lambda) \) vanishes, i.e.,

\[
(6e_2)^2 - 60(3e_2^2 - g_2) = 0,
\]

which gives us

\[
5g_2 = 12e_2^2 \in \mathbb{R},
\]

then (4.4) yields \( x = 0 \) and thus \( e_1 = 0, e_3 = -e_2 \). Combine with (4.2), we have

\[
10(e_2^2 + e_2^2) = 12e_2^2,
\]

which is a contradiction because \( e_2 \neq 0 \).
By Corollary 2.8, we have \( d(\mu) = d(\nu) = d(\pi) = d(\tau) = 1 \). Therefore, the multiplicity of any zero of \( Q_3(\lambda; b) \) is at most 2, and a zero of \( Q_3(\lambda; b) \) is at multiplicity 2 if and only if it is a zero of \( R_1(\lambda) \). Denote by \( \vartheta_\pm := 3\xi_1 \pm \frac{1}{2} \sqrt{\Delta_R_1} \) to be the zeros of

\[
R_1(\lambda) = \lambda^2 - 6\xi_1 \lambda + 15(3\xi_1^2 - 2g_2),
\]

where \( \Delta_R_1 \) denotes the discriminant of \( R_1(\lambda) \), i.e.,

\[
\Delta_R_1 := 12(5g_2 - 12\xi_1^2) = 48(3\xi^2 - 5y^2).
\]

Note that \( g_2 < 3\xi_1^2 \), then \( \vartheta_+ \) and \( \vartheta_- \) have the same sign when \( \Delta_R_1 > 0 \). Clearly, \( \Delta_R_1 \) is holomorphic with respect to \( b \). By Proposition 4.1 and Proposition 4.2 \( \Delta_R_1 \) is strictly increasing for \( b \in \left[ b_3, \frac{1}{2} \right] \), i.e.,

\[
\frac{d\Delta_R_1}{db} > 0, \quad \text{for } b \in \left[ b_3, \frac{1}{2} \right].
\]

By (4.8), we have

\[
\frac{d\Delta_R_1}{db} = \frac{12}{\pi} \left( 15g_3 - 10\eta_1g_2 - 24\xi_1^3 + 24\xi_1^2\eta_1 + 4\xi_1g_2 \right)
\]

\[
= \frac{12}{\pi} \left( 36\xi_1^3 - 11\xi_1g_2 - 10\eta_1g_2 + 24\xi_1^2\eta_1 \right).
\]

For \( b \in \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \), since \( \xi_1 > 0 \) and \( g_2 \leq 0 \), we have

\[
\frac{d\Delta_R_1}{db} > 0, \quad \text{for } b \in \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right).
\]

For \( b \in \left( \frac{\sqrt{3}}{2}, +\infty \right) \), note that

\[
\frac{d\Delta_R_1}{db} = \frac{12}{\pi} \left( 36\xi_1^3 - 11\xi_1g_2 - 10\eta_1g_2 + 24\xi_1^2\eta_1 \right)
\]

\[
= \frac{96}{\pi} \left( 3\xi^3 + 11\xi y^2 + (5y^2 - 3\xi^2) \eta_1 \right),
\]

and \( g_2(b), g'_2(b) > 0 \) gives us

\[
\eta_1 < \frac{3g_3}{2g_2} = \frac{3x(x^2 + y^2)}{3x^2 - y^2}.
\]

If \( 5y^2 - 3\xi^2 \geq 0 \), it is clear that \( \Delta'_R_1(b) > 0 \). If \( 5y^2 - 3\xi^2 < 0 \), we have

\[
\frac{d\Delta_R_1}{db} = \frac{96}{\pi} \left( 3\xi^3 + 11\xi y^2 + (5y^2 - 3\xi^2) \eta_1 \right),
\]

\[
> \frac{96}{\pi} \left( 3\xi^3 + 11\xi y^2 + (5y^2 - 3\xi^2) \frac{3x(x^2 + y^2)}{3x^2 - y^2} \right)
\]

\[
= \frac{384(9x^2 y^2 + y^4)}{\pi(3x^2 - y^2)} > 0.
\]
Hence
\[
\frac{d\Delta_{R_1}}{db} > 0, \quad \text{for } b \in \left(\frac{\sqrt{3}}{2}, +\infty\right).
\]
Note that
\[
\lim_{b \to +\infty} \Delta_{R_1}(b) = 12\left(\frac{20\pi^4}{3} - \frac{48\pi^2}{9}\right) = 16\pi^4 > 0,
\]
and
\[
\Delta_{R_1}(b) = 12(5g_2 - 12e_1^2) < 0 \quad \text{for } b \in \left[\frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2}\right],
\]
then there exists unique $\beta \in \left[\frac{1}{2\sqrt{3}}, +\infty\right)$ such that
\[
\Delta_{R_1}(b) \begin{cases} < 0, & \text{for } b \in \left(\frac{1}{2\sqrt{3}}, \beta\right), \\ = 0, & \text{for } b = \beta, \\ > 0, & \text{for } b \in (\beta, +\infty). \end{cases}
\]
In fact, $\beta \approx 1.0979 > \frac{\sqrt{3}}{2}$ by numerical computation. On the other hand, since $\tau = \frac{1}{2} + ib$, we have
\[
\frac{\tau - 1}{2\tau - 1} = \frac{1}{2} + i\frac{1}{4b} \quad \text{for } \tau = \frac{1}{2} + ib.
\]
It is well known that [110, p. 346]
\[
e_1\left(\frac{1}{4b}\right) = -4b^2e_1(b),
\]
(4.11)
\[
g_2\left(\frac{1}{4b}\right) = 16b^4g_2(b),
\]
then
\[
\Delta_{R_1}(b) = \frac{1}{16b^4}\Delta_{R_1}\left(\frac{1}{4b}\right),
\]
so there exists unique $\hat{\beta} = \frac{1}{4\beta} \approx 0.2277 \in \left(0, \frac{1}{2\sqrt{3}}\right)$ such that
\[
\Delta_{R_1}(b) \begin{cases} < 0, & \text{for } b \in (\hat{\beta}, \frac{1}{2\sqrt{3}}), \\ = 0, & \text{for } b = \hat{\beta}, \\ > 0, & \text{for } b \in (0, \hat{\beta}). \end{cases}
\]
Moreover,
\[\Delta'_{R_1}(b) < 0 \quad \text{for } b \in \left(0, \frac{1}{2\sqrt{3}}\right).\]
Consequently,
\[
\Delta_{R_1}(b) \begin{cases} > 0, & \text{for } b \in (0, \hat{\beta}) \cup (\beta, +\infty), \\ = 0, & \text{for } b \in \{\hat{\beta}, \beta\}, \\ < 0, & \text{for } b \in (\hat{\beta}, \beta). \end{cases}
\]
If \( b \in (\tilde{b}, \beta) \), then \( \vartheta_+ = \tilde{\vartheta}_+ \not\in \mathbb{R} \). Corollary 2.8 gives us
\[
d(\tilde{\vartheta}_+) = d(\vartheta_-) = 1.
\]

Next, if \( \lambda \in \mathbb{C} \) satisfies \( Q_3(\lambda) = 0 \), by (2.4) and (2.7), there exists unique (cf. [3, Proposition 6.4]) \( a = \{a_1, a_2, a_3\} \in Y_b^3 \) with \( a = -a \) such that
\[
\lambda = \lambda_a = 5(\wp(a_1) + \wp(a_2) + \wp(a_3)).
\]

Note that \( a = -a \) with \( a \in Y_b^3 \) yields that \( a = \{\omega_0/2, \omega_0/2, \omega_0/2\} \) or \( a \in M \) with
\[
M = \left\{ \frac{\omega_k}{2}, a, -a \right\} \mid a \in T_r^* \setminus \{\omega_0/2, \omega_0/2, \omega_0/2\}, \ k = 1, 2, 3. \quad \right\}
\]
If \( a = \{\omega_0/2, \omega_0/2, \omega_0/2\} \), then
\[
\lambda_{\{\omega_0/2, \omega_0/2, \omega_0/2\}} = 5(e_1 + e_2 + e_3) = 0.
\]
By [9, Example 3.4],
\[
\chi_{\{\omega_0/2, \omega_0/2, \omega_0/2\}} = \frac{12(3g_3 - 2\eta_1 g_2)}{g_2^3 - 27g_3^2} = \frac{12\pi g_2^3}{g_2^3 - 27g_3^2},
\]
where \( g_2^3 - 27g_3^2 = 16(e_1 - e_2)^2(e_1 - e_3)^2(e_2 - e_3)^2 \neq 0 \). Proposition 4.2 tells us that \( \chi_{\{\omega_0/2, \omega_0/2, \omega_0/2\}} = 0 \) if and only if \( b = b_8 \). By Theorem 2.7 and the multiplicity of 0 is 1, we obtain that \( d(0) \geq 3 \) if and only if \( b = b_8 \), otherwise, \( d(0) = 1 \).

For \( b \in \{\tilde{b}, \beta\} \), note that \( 3e_1 = \vartheta_+ = \vartheta_- \) is of multiplicity 2 and then (2.16) gives
\[
d(3e_1) = 2 + p_i(3e_1) \geq 2.
\]
By a direct computation, we have \( \{\frac{1}{2}, a, -a\} \in M \) if and only if \( \{\frac{1}{2}, a, -a\} \in M \) and \( \{\omega_0/2, a, -a\} \in M \) if and only if \( \{\omega_0/2, a, -a\} \in M \). Then the zeros of \( R_1(\lambda) \) can be written as \( \lambda_a \) with \( a = \{1/2, a, -a\} \in M \). In particular, there exists \( a \in T_r^* \setminus \{\omega_0/2, \omega_0/2, \omega_0/2\} \) such that
\[
3e_1 = \lambda_{\{\omega_0/2, a, -a\}} = 5(e_1 + 2\wp(a)),
\]
thus
\[
\wp(a) = -\frac{1}{5}e_1.
\]
At the same time, the discriminant \( \Delta_{R_1} = 0 \) gives us
\[
g_2 = \frac{12}{5}e_1^2,
\]
and then
\[
g_3 = 4e_1^3 - e_1g_2 = \frac{8}{5}e_1^3.
\]
Note that \( \psi^2 = 4\psi^3 - g_2\psi - g_3 \) and \( \psi'' = 6\psi^2 - g_2/2 \), by (2.12), we have
\[
-\frac{1}{2} (e_1 - \psi(a)) \chi(\frac{\omega}{2}, a_{-a}) = \frac{e_1 + \eta_1}{(6e_1^2 - g_2/2) (e_1 - \psi(a))} - \frac{\psi(a) + \eta_1}{4\psi(a)^3 - g_2\psi(a) - g_3}.
\]

Combine with (4.15, 4.16, 4.17), we obtain that
\[
-\frac{3}{5} e_1 \chi(\frac{\omega}{2}, a_{-a}) = \frac{25}{24} \eta_1 > 0,
\]
so \( \chi(\frac{\omega}{2}, a_{-a}) \neq 0 \) which means \( d(3e_1) \leq 2 \) by Theorem 2.7. Apply (4.14), we obtain that
\[
d(3e_1) = 2 \quad \text{for} \quad b \in \{\beta, \beta'\}.
\]
The proof of Theorem 1.2 is complete. \( \square \)

Finally, from Theorem 1.2, Theorem 2.6 and Proposition 4.1, we have the following proposition.

**Proposition 4.5.** For \( b \in [\beta, +\infty) \), we have \((-\infty, 0] \subseteq \sigma(L^3_b)\).

5. **Proof of the Main Theorem**

Recall that
\[
f(\lambda) = \frac{4}{15} \lambda^3 + \frac{8}{5} \eta_1 \lambda^2 - 3g_2\lambda + 9g_3 - 6\eta_1 g_2.
\]

By Proposition 4.2, we have
\[
f(0) = 3\pi g_2'(b) \begin{cases} < 0 & \text{for} \quad b \in (0, b_g), \\ = 0 & \text{for} \quad b = b_g, \\ > 0 & \text{for} \quad b \in (b_g, +\infty). \end{cases}
\]
From here and Proposition 4.5, we obtain the following lemma.

**Lemma 5.1.** For any \( b \in (b_g, +\infty) \), there exists \( \lambda_- < 0 \) such that \( \lambda_- \) is an inner intersection point of \( \sigma(L^3_b) \). Furthermore, the spectrum \( \sigma(L^3_b) \) always includes the following piece denoted by \( \Sigma \).

\[
\Sigma : \begin{align*}
\lambda_- & \quad \rightarrow \\
-\infty & \quad \lambda_+ \\
0 & \quad \rightarrow
\end{align*}
\]

Consider the derivative of \( f(\lambda) \) denoted by
\[
f'(\lambda) = \frac{4}{5} \lambda^2 + \frac{16}{5} \eta_1 \lambda - 3g_2,
\]
we have the following conclusion about the discriminant of \( f'(\lambda) \).
Lemma 5.2. There exist $\gamma_1 \in \left(\frac{1}{2\sqrt{3}}, b_g\right)$ and $\gamma_2 \in \left(\frac{1}{4g_2}, \frac{\sqrt{3}}{2}\right)$ such that

$$\Delta f' = \frac{16}{25} (16\eta_1^2 + 15g_2) \begin{cases} 
> 0 & \text{if } b \in (0, \gamma_1) \cup (\gamma_2, +\infty), \\
= 0 & \text{if } b \in \{\gamma_1, \gamma_2\}, \\
< 0 & \text{if } b \in (\gamma_1, \gamma_2). 
\end{cases}$$

Consequently, $f$ is strictly increasing over $[\gamma_1, \gamma_2]$ and thus has only one real root for $b \in [\gamma_1, \gamma_2]$.

Proof. First of all, it is clear that $\Delta f'(b) > 0$ for $b \in (0, \frac{1}{2\sqrt{3}}] \cup [\frac{\sqrt{3}}{2}, +\infty)$ by Proposition 4.2. The well-known Fourier expansion of $g_2$ gives us numerically $g_2(\frac{1}{2}) \approx -76.6\pi^2$. This, together with the modular property of $g_2$ (cf. (4.11)) and $\eta_1(\frac{1}{2}) = 2\pi, \eta_1(\frac{1}{2\sqrt{3}}) = 2\sqrt{3}\pi$ (see e.g. (4.14)), implies

$$\max_{b \in [b_g, \frac{1}{2\sqrt{3}}]} \Delta f'(b) < \frac{16}{25} \left(16 \left(\eta_1 \left(\frac{1}{2\sqrt{3}}\right)\right)^2 + 15g_2 \left(\frac{1}{2}\right)\right) < 0,$$

$$\max_{b \in [\frac{1}{2\sqrt{3}}, b_g]} \Delta f'(b) < \frac{16}{25} \left(16 \left(\eta_1 (\frac{1}{2})\right)^2 + 15g_2 \left(\frac{1}{4\sqrt{3}}\right)\right)$$

$$= \frac{16}{25} \left((2\pi)^2 + 15\frac{4}{8g_2} (b_g)\right)$$

$$< \frac{16}{25} \left(4\pi^2 + 15(\frac{1}{2\sqrt{3}})^2 g_2 (\frac{1}{2})\right) < 0.$$

By Proposition 4.2 and 4.3, $\Delta f'$ is strictly decreasing over $[\frac{1}{2\sqrt{3}}, b_g]$, hence, there exists unique $\gamma_1 \in (\frac{1}{2\sqrt{3}}, b_g)$ such that

$$\Delta f'(\gamma_1) \begin{cases} 
> 0, & \text{for } b \in (0, \gamma_1), \\
= 0, & \text{for } b = \gamma_1, \\
< 0, & \text{for } b \in (\gamma_1, b_g]. 
\end{cases}$$

Since $\Delta f'(\frac{1}{4g_2}) < 0$ and $\Delta f'(\frac{\sqrt{3}}{2}) > 0$, there exist at least one $b \in (\frac{1}{4g_2}, \frac{\sqrt{3}}{2})$ such that $\Delta f'(b) = 0$. On the other hand, if $b \in (\frac{1}{4g_2}, \frac{\sqrt{3}}{2})$ satisfying $\Delta f'(b) = \frac{16}{25} (16\eta_1^2 + 15g_2) = 0$, by (4.8), we have

$$\Delta f'(b) = \frac{16}{25\pi} (45g_3 - 16\eta_1^3 - \frac{86}{3}\eta_1g_2)$$

$$= \frac{16}{25\pi} (45g_3 - \frac{41}{3}\eta_1g_2) > 0$$

because $g_3 = 4e_1|e_2|^2 > 0$ for $b > \frac{1}{2}$. Therefore, there exists unique $b \in (\frac{1}{4g_2}, \frac{\sqrt{3}}{2})$ denoted by $\gamma_2$ such that $\Delta f'(\gamma_2) = 0$. The proof is complete. \(\square\)
Denote by \( f'(\lambda) = \frac{4}{5}\lambda^2 + \frac{16}{5}\eta_1\lambda - 3g_2 := \frac{4}{5}(\lambda - \lambda_L)(\lambda - \lambda_R), \) we have

\[
\begin{align*}
\lambda_L + \lambda_R &= -4\eta_1, \\
\lambda_L\lambda_R &= -\frac{15}{4}g_2.
\end{align*}
\]

If \( \lambda = \lambda_L \) or \( \lambda_R \), i.e., \( f'(\lambda) = 0 \), then

\[
f(\lambda) = \frac{\lambda}{3} \left( 3g_2 - \frac{16}{5}\eta_1\lambda \right) + \frac{8}{5}\eta_1\lambda^2 - 3g_2\lambda - 6\eta_1g_2 + 9g_3
\]

\[
= \frac{8}{15}\eta_1\lambda^2 - 2g_2\lambda - 6\eta_1g_2 + 9g_3
\]

\[
= \frac{2}{3}\eta_1 \left( 3g_2 - \frac{16}{5}\eta_1\lambda \right) - 2g_2\lambda - 6\eta_1g_2 + 9g_3
\]

\[
= -2 \left( \frac{16}{15}\eta_1^2 + g_2 \right)\lambda - 4\eta_1g_2 + 9g_3
\]

\[
:= -\frac{5}{24}\Delta_f(\lambda - H),
\]

where

\[
H = \frac{24(9g_3 - 4\eta_1g_2)}{5\Delta_f}.
\]

So

\[
f(\lambda_L)f(\lambda_R) = \frac{25}{24^2}\Delta_f^2(H - \lambda_L)(H - \lambda_R)
\]

\[
= \frac{25}{24^2}\Delta_f^2 \left( H^2 + 4\eta_1H - \frac{15}{4}g_2 \right)
\]

\[
= \frac{25}{24^2}\Delta_f^2 \left( H + 2\eta_1 \right)^2 - \frac{25}{64}\Delta_f
\]

\[
= \frac{25}{24^2}\Delta_f^2 \left( \frac{(64\eta_1^3 + 9g_3)^2}{25^2} - \frac{25^2}{192^2}\Delta_f^3 \right)
\]

\[
= \frac{1}{225} \left( (64\eta_1^3 + 135g_3)^2 - (16\eta_1^2 + 15g_2)^3 \right).
\]

Clearly, all zeros of \( f \) are real if and only if \( f(\lambda_L)f(\lambda_R) \leq 0 \).

Now, let us prove the main Theorem 1.3.

**Proof of Theorem 1.3** First of all, by Theorem 1.2, the zeros of the spectral polynomial \( Q_3(\lambda; b) \) are

\[
0, \mu, \overline{\mu}, \nu, \overline{\nu}, \vartheta_+, \vartheta_-, \theta_+,
\]

and \( d(\mu) = d(\overline{\mu}) = d(\nu) = d(\overline{\nu}) = 1 \). Hence each of \( \mu, \overline{\mu}, \nu, \overline{\nu} \) is met by a single semi-arc of the spectrum, i.e. they are all endpoints but not intersection points. In this following argument, we will use Theorem 2.6 frequently.
that $\mathbb{C} \setminus \sigma(L_b^3)$ is path-connected and $\sigma(L_b^3)$ is symmetric with respect to the real line.

**Step1: The spectrum at $b = \beta$.** By Theorem 1.2, we have $\vartheta_- = \vartheta_+ = 3e_1 > 0$ and $d(0) = 1$, $d(3e_1) = 2$. From Theorem 2.5, Theorem 2.6, Proposition 4.5 and Lemma 5.1, we obtain the rough graph of $\sigma(L_b^3)$ is as follows:

\[
\begin{array}{c}
-\infty \\
\lambda_- \\
0 \\
3e_1 \\
\end{array}
\]

**Step2: The spectrum for $b > \beta$.** By Theorem 1.2, we have $0 < \vartheta_- < 3e_1 < \vartheta_+$ and $d(0) = 1$, which implies again that $0$ is an endpoint but not an intersection point. By (2.16), we have

\[d(\vartheta_{\pm}) = 1 + 2p_i(\vartheta_{\pm})\]

is odd, then Theorem 2.6 gives us the closed interval $[\vartheta_-, \vartheta_+] \subset \sigma(L_b^3)$. Hence, from Lemma 5.1, Theorem 2.5, Theorem 2.6 and above analysis, we obtain the graph of $\sigma(L_b^3)$ consisting of the following two parts:

\[
\begin{array}{c}
-\infty \\
\lambda_- \\
0 \\
\vartheta_- \\
\vartheta_+ \\
\ell \\
\end{array}
\]

(part I) (part II)

and $\ell \cap \mathbb{R} = \{\text{one point}\}$. Next, we need the following lemma.

**Lemma 5.3.** There exists $\lambda_+ \in (\vartheta_-, \vartheta_+)$ such that $f(\lambda_+) = 0$.

**Proof.** Recall that

\[
\begin{cases}
R_1(\lambda) = \lambda^2 - 6e_1\lambda + 15(3e_1^2 - g_2) = (\lambda - \vartheta_-)(\lambda - \vartheta_+), \\
f(\lambda) = \frac{4}{15}\lambda^3 + \frac{8}{5}\eta_1\lambda^2 - 3g_2\lambda + 9g_3 - 6\eta_1g_2,
\end{cases}
\]

and $g_3 = 4e_1^2 - e_1g_2$, we have

\[
(5.3) \begin{cases}
\vartheta_- + \vartheta_+ = 6e_1, \\
\vartheta_- \vartheta_+ = 15(3e_1^2 - g_2),
\end{cases}
\]
From (5.3) and (5.4), we have

\[ f(\vartheta_\pm) = \left( \frac{4}{15} \vartheta_\pm + \frac{8}{5} \eta_1 \right) (6e_1 \vartheta_\pm - 15(3e_1^2 - g_2)) - 3g_2 \vartheta_\pm + 9g_3 - 6\eta_1g_2 \]

\[ = \frac{8}{5}e_1 (6e_1 \vartheta_\pm - 15(3e_1^2 - g_2)) + \left( g_2 + \frac{48}{5} \eta_1e_1 - 12e_1^2 \right) \vartheta_\pm + 9g_3 + 18\eta_1g_2 - 72e_1^2 \eta_1 \]

\[ = \left( g_2 + \frac{48}{5} \eta_1e_1 - \frac{12}{5} e_1^2 \right) \vartheta_\pm + 3 \left( 6g_2 \eta_1 + 5g_2e_1 - 72\eta_1e_1^2 - 36e_1^3 \right) \]

\[ := B(\vartheta_\pm - D), \]

where

\[ D = -3 \frac{6g_2 \eta_1 + 5g_2e_1 - 72\eta_1e_1^2 - 36e_1^3}{g_2 + \frac{48}{5} \eta_1e_1 - \frac{12}{5} e_1^2}. \]  

(5.4)

From (5.3) and (5.4), we have

\[ f(\vartheta_-)f(\vartheta_+) \]

\[ = B^2(\vartheta_- - D)(\vartheta_+ - D) \]

\[ = B^2 \left( D^2 - 6e_1D + 15(3e_1^2 - g_2) \right) \]

\[ = B^2 \left( (D - 3e_1)^2 - 3(5g_2 - 12e_1^2) \right) \]

\[ = B^2 \left( -18e_1 + \frac{15}{2} \eta_1 \left( g_2 - 12e_1^2 \right) \right)^2 - 3(5g_2 - 12e_1^2) \]

\[ = \frac{3B^2 \left( 5g_2 - 12e_1^2 \right)^2}{4 \left( e_1^2 - 4e_1 \eta_1 - \frac{5}{12} g_2 \right)^2} \left( 3(5g_2 - 12e_1^2)(e_1 + \eta_1)^2 - 4(e_1^2 - 4e_1 \eta_1 - \frac{5}{12} g_2)^2 \right) \]

\[ = - \frac{\pi B^2 \Delta \bar{R}_1}{288(e_1^2 - 4e_1 \eta_1 - \frac{5}{12} g_2)^2} \left( 12(e_1 + \eta_1)g_2' - 5(12e_1^2 - g_2) \eta_1' \right) < 0 \]

for all \( b > \beta \) because \( e_1 > 0, \eta_1 > 0, g_2' > 0, \eta_1' < 0, g_2 < 3e_1^2 \) and \( \Delta \bar{R}_1 > 0 \).

Therefore, there exists \( \lambda_+ \in (\vartheta_-, \vartheta_+) \) such that \( f(\lambda_+) = 0 \). \( \square \)

From Theorem [1,1] and this lemma, \( \lambda_+ \) is an inner intersection point and then

\[ \lambda_+ = \ell \cap \mathbb{R} \in (\vartheta_-, \vartheta_+), \]

so the rough graph of spectrum \( \sigma(L^3_b) \) for \( b > \beta \) is as follows.

\[ \begin{array}{c}
-\infty \end{array} \]

\[ \lambda_- \ 0 \ \lambda_+ \]

\[ \vartheta_- \ \vartheta_+ \]
**Step 3: The spectrum for \( b \) around \( \hat{\beta} \).**

First of all, if \( b = \hat{\beta} \), by Theorem [1.2](#) we have \( \theta_- = \theta_+ = 3\epsilon_1 < 0 \) and then \( \Delta_{R_1}(\hat{\beta}) = 0 \), i.e., \( 5g_2 = 12\epsilon_1^2 \), thus \( 5g_3 = 8\epsilon_1^3 \). By direct computations, we have \( f(3\epsilon_1) = 0 \) and \( f'(3\epsilon_1) = 4\epsilon_1\eta_1/5 < 0 \) because \( e_1(\hat{\beta}) < 0 \) and \( \eta_1(\hat{\beta}) > 0 \). Note that \( f(0) < 0 \), then \( f \) has two negative real zeros \( \lambda_+^1 < \lambda_+^2 = 3\epsilon_1 \) and one positive real zero \( \lambda_+ \), thus \( f(\lambda_L)f(\lambda_R)|_{b=\hat{\beta}} < 0 \). Note that \( f(\lambda_L)f(\lambda_R)|_{b=b_1} > 0 \) by numerical computation, we can define

\[
\alpha := \text{sup}\left\{ b > \hat{\beta} \mid f(\lambda_L)f(\lambda_R)|_{b=b'} < 0 \text{ for all } b' \in [\hat{\beta}, \hat{b}] \right\}.
\]

Clearly, \( \hat{\beta} < \alpha < b_\eta \) and \( f(\lambda_L)f(\lambda_R)|_{b=\alpha} = 0 \). In fact, \( \alpha \approx 0.23217 \) by numerical computation and \( f(\lambda_L)f(\lambda_R)|_{\alpha < b < \alpha + \epsilon} > 0 \) with \( \epsilon > 0 \) sufficient small.

For \( b \in [\hat{\beta}, \alpha) \), all zeros of \( f \) are real because \( f(\lambda_L)f(\lambda_R)|_{[\hat{\beta}, \alpha)} < 0 \). Since \( f(0) = 3\pi g_2(b) < 0 \) for \( b \in (0, b_\beta) \subset [\hat{\beta}, \alpha) \) and \( f \) has two negative real zeros at \( b = \hat{\beta} \), by continuity, \( f \) has two negative real zeros \( \lambda_1, \lambda_2 \) for \( b \in [\hat{\beta}, \alpha) \) and \( \lambda_- := \lambda_1 \) is a negative real zero of \( f \) with multiplicity 2 at \( b = \alpha \).

For \( b \in (\hat{\beta}, b_\beta) \), we have \( d(0) = 1 \) by Theorem [1.2](#). Since all other zeros of \( Q_3(\lambda; b) \) are not real, we obtain that \( (-\infty, 0] \subseteq \sigma(L^3_b) \) by Theorem [2.6](#) and there are exactly 7 finite endpoints:

\[
0, \mu, \bar{\mu}, \nu, \bar{\nu}, \theta_+, \theta_-
\]

which are not intersection points by Corollary [2.8](#).

For \( b \in (\hat{\beta}, \alpha) \subset [\hat{\beta}, \alpha) \subset (\hat{\beta}, b_\beta) \), we have \( (-\infty, 0] \subseteq \sigma(L^3_b) \) and \( \lambda_1 < \lambda_2 < 0 \), thus \( \lambda_1 \) and \( \lambda_2 \) are two inner intersection points by Theorem [1.1](#). Consequently, the spectrum \( \sigma(L^3_b) \) is one of the following graph.

![Graph showing three cases: (Fa), (Fb), and (Fc).](image)

When \( b \) goes to \( \alpha \) from the left hand side, we always have \( (-\infty, 0] \subseteq \sigma(L^3_b) \) and \( d(0) = 1 \), i.e., 0 is always an endpoint of the arc \( (-\infty, 0] \) and cannot be an intersection point. At the same time, \( \lambda_1, \lambda_2 \) will come together to be \( \lambda_- \). Also, the graphs are symmetric with respect to \( \mathbb{R} \) by Theorem [2.6](#) and there is no non-real inner intersection point because all zeros of \( f \) are real. Hence, (Fa) will goes to (Ga), (Fb) and (Fc) will goes to either (Gb) or (Gc).
Since \( f(\lambda_L)f(\lambda_R)|_{\lambda < b < \alpha + \epsilon} > 0 \) and \( f(0) < 0 \), \( f \) has no negative real zero for \( b \in (\alpha, \alpha + \epsilon) \), i.e., there is no negative inner intersection point for \( b \in (\alpha, \alpha + \epsilon) \). By Theorem 2.5 and Theorem 2.6, \( \lambda_\_ \) can not disappear in the deformation of (Gb) and (Gc) when \( b \) moves to the right hand side of \( \alpha \), this is a contradiction! Therefore, the rough graph of \( \sigma(L_\beta^3) \) at \( b = \alpha \) is (Ga), and then the rough graph of \( \sigma(L_\beta^3) \) for \( b \in (\hat{\beta}, \alpha) \) is (Fa). Consequently, the rough graph of \( \sigma(L_\beta^3) \) for \( b \in (\alpha, \alpha + \epsilon) \) with \( \epsilon > 0 \) sufficient small is the following:

\[
\begin{array}{c}
-\infty \\
\circ \\
\lambda_1 \\
\circ \\
0 \\
\end{array}
\]

Now let us go back to \( b = \hat{\beta} \), recall that \( f \) has two negative real zeros \( \lambda_1^- < \lambda_2^- = 3e_1 \) and \( d(3e_1) = 2 \) by Theorem 1.2, we obtain the rough graph of \( \sigma(L_\beta^3) \) at \( b = \hat{\beta} \) from (Fa) as follows:

\[
\begin{array}{c}
-\infty \\
\lambda_1^\_ \\
3e_1 \circ \\
0 \\
\end{array}
\]

Note that \( \sigma(L_\beta^3) = -4\hat{\beta}^2\sigma(L_\beta^3) \), we draw the rough graphs of \( \sigma(L_\beta^3) \) using blue dashed lines and \( \sigma(L_\beta^3) \) using dark solid lines in the following:

\[
\begin{array}{c}
-\infty \\
\lambda_1 \\
3e_1 \circ \\
0 \\
\end{array}
\]

From this graph, it is clear to see that the red point is the only element of \( \Xi_\beta \cup \mathbb{R} \), so \( \#(\Xi_\beta \cup \mathbb{R}) = 1 \).

Furthermore, if \( b \in (\hat{\beta} - \delta, \hat{\beta}) \) with \( \delta > 0 \) sufficient small, then \( \theta^- < 3e_1 < \theta^+ < 0 \) by Theorem 1.2 and \( f \) still has two negative real zeros \( \lambda_1^- < \lambda_2^- \) with \( \lambda_\_^- \) still lying on \( \sigma(L_\beta^3) \). Hence, the rough graph of \( \sigma(L_\beta^3) \) is the following:

\[
\begin{array}{c}
-\infty \\
\lambda_1 \\
3e_1 \\
0 \\
\end{array}
\]
where $\sigma_1, \sigma_2$ denote the spectral arcs which do not lie on $\mathbb{R}$. In particular, $\sigma_1, \sigma_2$ are symmetric with respect to $\mathbb{R}$ and disjoint with each other by Theorem 2.6. Similarly, we put the rough graphs of $\sigma(L^3_b)$ in blue dashed lines and $\sigma(L^3_b)$ in dark solid lines together for $b \in (\hat{\beta} - \delta, \hat{\beta})$.

From this graph, it is clear to see that the red point is the only element of $\Xi_b \cup \mathbb{R}$, so $\#(\Xi_b \cup \mathbb{R}) = 1$ for $b \in (\hat{\beta} - \delta, \hat{\beta})$. By Corollary 2.12, $k_1 \in (0, \hat{\beta})$ and $k_2 \in (\hat{\beta}, \frac{1}{4})$, i.e., $\#(\Xi_b \cup \mathbb{R}) = 2$ for $b \in (0, k_1)$ and $\#(\Xi_b \cup \mathbb{R}) = 1$ for $b \in [k_1, k_2)$. Note that the topological graphs of $\sigma_1(L^3_b)$ are the same for all $b \in (0, \hat{\beta})$, i.e., they are the blue dashed lines in the above picture. Since $d(0) = 1$ for any $b \in (0, \hat{\beta})$, the intersection of $\sigma_1$ and $\mathbb{R}$ can not pass through 0, so $\sigma_1 \cap \mathbb{R} = \{r_+\}$ with $r_+ > 0$. Hence $r_+ \in \Xi_b \cap \mathbb{R}$ for all $b \in (0, \hat{\beta})$. By the above figure, we know $\sigma_2$ will pass $\theta_-$ once and cannot pass through $\theta_+$. Denote by $\sigma_2 \cap \mathbb{R} = \{r_-\}$, we have $r_- = \lambda_1^\prime < \theta_-$ for $b \in (k_1, \hat{\beta})$ and $r_- = \lambda_1^\prime \in (\theta_-, \theta_+)$ for $b \in (0, k_1)$. In particular, $r_- = \theta_-$ for $b = k_1$, equivalently, $d(\theta_-) > 1$ for $b = k_1$. Consequently, the rough graphs of $\sigma(L^3_b)$ with $b \in (0, \hat{\beta})$ are as follows.

Before we end this step, note that $d(0) > 1$ if and only if $b = b_\delta$, we get the following conclusions.

**Proposition 5.4.** For any $b \in (0, b_\delta)$, the spectrum $\sigma(L^3_b)$ always includes a simple spectral arc denoted by $\sigma_1$ which is symmetric with respect to $\mathbb{R}$ and $\sigma_1 \cap \mathbb{R} = \{r_+\}$ with $r_+ > 0$.

Furthermore, combining Proposition 4.5, we obtain that $k_2 = b_\delta$.

**Step4: The spectrum for $b < \hat{\beta}$ and close to $\beta$.**

First, if $b = \hat{\beta}$, then $\Delta_R(\beta) = 0$, i.e., $5g_2 = 12e_1^2$, thus $5g_3 = 8e_3^2$. A direct computation gives us $f(\lambda_L)f(\lambda_R)|_{b = \beta} < 0$. For $b \in (\hat{\beta} - \delta, \beta)$ with
δ > 0 sufficient small, we have \( f(\lambda_L) f(\lambda_R) < 0 \), then all roots of \( f \) are real, i.e., all inner intersection points of \( \sigma(L^3_b) \) are real. By Theorem 1.2 we have \( \sigma_- = \sigma_+ \notin \mathbb{R} \). Similar to \( b > \beta \) case, we know the rough graph of \( \sigma(L^3_b) \) consists of \( \Sigma, \sigma \) and \( \sigma' \), where \( \sigma, \sigma' \) are two simple spectral arcs and \( \sigma \cup \sigma' \) are symmetric with respect to \( \mathbb{R} \). Furthermore, we have \((\sigma \cup \sigma') \cap \Sigma = \emptyset \) by the rough graph of \( \sigma(L^3_b) \). Draw the rough graphs of \( \sigma_i(L^3_b) = -\frac{1}{4\delta} \sigma_i(L^3_{1/4}) \) using blue dashed lines and \( \Sigma \) using dark solid lines in the following:

It is clear to see that \( \xi_b \in \Xi_b \cap \mathbb{R} \). Since \( \frac{1}{4\delta_1} > \beta \), we have \( \#(\Xi_b \cap \mathbb{R}) \leq 1 \), thus \( \Xi_b \cap \mathbb{R} = \{\xi_b\} \subseteq \Sigma \). So \((\sigma_1 \cup \sigma_2) \cap \mathbb{R} = \emptyset \). Note that all inner intersection points of \( \sigma(L^3_b) \) are real. Therefore, the rough graph of \( \sigma(L^3_b) \) with \( b \in (\beta - \delta, \beta) \) is the following:

The proof is complete.

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