Lower bounds in the quantum cell probe model

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Abstract

We introduce a new model for studying quantum data structure problems — the quantum cell probe model. We prove a lower bound for the static predecessor problem in the address-only version of this model where we allow quantum parallelism only over the ‘address lines’ of the queries. The address-only quantum cell probe model subsumes the classical cell probe model, and many quantum query algorithms like Grover’s algorithm fall into this framework. Our lower bound improves the previous known lower bound for the predecessor problem in the classical cell probe model with randomised query schemes, and matches the classical deterministic upper bound of Beame and Fich [BF99]. Beame and Fich [BF99] have also proved a matching lower bound for the predecessor problem, but only in the classical deterministic setting. Our lower bound has the advantage that it holds for the more general quantum model, and also, its proof is substantially simpler than that of Beame and Fich.

We prove our lower bound by obtaining a round elimination lemma for quantum communication complexity. A similar lemma was proved by Miltersen, Nisan, Safra and Wigderson [MNSW98] for classical communication complexity, but it was not strong enough to prove a lower bound matching the upper bound of Beame and Fich. Our quantum round elimination lemma also allows us to prove rounds versus communication tradeoffs for some quantum communication complexity problems like the ‘greater-than’ problem.

We also study the static membership problem in the quantum cell probe model. Generalising a result of Yao [Yao81], we show that if the storage scheme is implicit, that is it can only store members of the subset and ‘pointers’, then any quantum query scheme must make $\Omega(\log n)$ probes.

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1 Introduction

A static data structure problem consists of a set of data $D$, a set of queries $Q$, a set of answers $A$, and a function $f : D \times Q \to A$. The aim is to store the data efficiently and succinctly, so that any query can be answered with only a few probes to the data structure. In a seminal paper [Yao81], Yao introduced the (classical) cell probe model for studying static data structure problems (in the classical setting). Thereafter, this model has been used extensively to prove (classical) upper and lower bounds for several data structure problems (see e.g. [FKS84, MNSW98, BF99, BMRV00]). A classical $(s, w, t)$ cell probe scheme for $f$ has two components: a storage scheme and a query scheme. Given the data $d \in D$ to be stored, the storage scheme stores it as a table $T_d$ of $s$ cells, each cell $w$ bits long. $w$ is called the word size of the scheme. The query scheme has to answer queries. Given a query $q \in Q$, the query scheme computes the answer $f(d, q)$ to that query by making at most $t$ probes to the stored table $T_d$, where each probe reads one cell at a time. The storage scheme is deterministic whereas the query scheme can be deterministic or randomised. The goal is to study tradeoffs between $s$, $t$ and $w$. For an overview of results in this model, see the survey by Miltersen [Mil99].

In this paper, we study static data structure problems, such as the static membership problem and the static predecessor problem, when the query algorithm is allowed to query the table using a quantum superposition. We formalise this by defining the quantum cell probe model similar to the quantum bit probe model of Radhakrishnan, Sen and Venkatesh [RSV00]. Informally, in the quantum cell probe model, the storage scheme is classical deterministic as before and stores the data $d \in D$ as a table of cells $T_d$; however, the query scheme is quantum and can query the table $T_d$ using a quantum superposition. We show a lower bound for the predecessor problem in a restricted version of this model, which we call the address-only quantum cell probe model. In the predecessor problem, the storage scheme has to store a subset $S$ of size at most $n$ from the universe $[m]$, such that given any query element $x \in [m]$, one can quickly find the predecessor of $x$ in $S$.

Result 1 (Lower bound for predecessor, informal statement) Suppose we have an address-only quantum cell probe solution with constant probability of error for the static predecessor problem, where the universe size is $m$ and the subset size is at most $n$, using $n^{O(1)}$ cells of storage with word size $(\log m)^{O(1)}$ bits. Then the number of queries is at least $\Omega\left(\frac{\log \log m}{\log \log \log m}\right)$ as a function of $m$,

and at least $\Omega\left(\sqrt{\frac{\log n}{\log \log n}}\right)$ as a function of $n$.

We then consider the static membership problem. Here one has to answer membership queries instead of predecessor queries. Yao [Yao81] showed that if the universe is large enough, any classical cell probe solution with an implicit deterministic storage scheme and a deterministic query scheme for the static membership problem must make $\Omega(\log n)$ probes to the table in the worst case. An implicit storage scheme either stores a ‘pointer value’ (viz. a value which is not an element of the universe) or an element of $S$ in a cell. In particular, it is not allowed to store an element of the universe which is not a member of $S$. We generalise Yao’s result to the quantum setting.
Result 2 (Lower bound for membership, informal statement) Suppose we have a quantum cell probe solution with an implicit storage scheme for the static membership problem. Then, if the universe is large enough compared to the number of cells of storage, the size of the universe of ‘pointers’ and the size of the stored subset, the query algorithm must make $\Omega(\log n)$ probes, even if we allow constant probability of error.

Remarks:
1. Our address-only quantum cell probe model subsumes the classical cell probe model with randomised query schemes. Hence, our lower bound for the static predecessor problem also holds in this setting. This improves the previous lower bound $\Omega(\sqrt{\log \log m})$ as a function of $m$ and $\Omega(\log^{1/3} n)$ as a function of $n$ for this setting, shown by Miltersen, Nisan, Safra and Wigderson [MNSW98]. Beame and Fich [BF99] have shown an upper bound matching our lower bound up to constant factors, which uses $n^{O(1)}$ cells of storage of word size $O(\log m)$ bits. In fact, both the storage and the query schemes are classical deterministic in Beame and Fich’s solution. In their paper, Beame and Fich [BF99] also show a lower bound of $t = \Omega\left(\frac{\log \log m}{\log \log \log m}\right)$ as a function of $m$ for $(n^{O(1)}, 2^{(\log m)^{1/3} \cdot O(1)}, t)$ classical deterministic cell probe schemes, and a lower bound of $t = \Omega\left(\frac{\log n}{\log \log n}\right)$ as a function of $n$ for $(n^{O(1)}, (\log m)^{O(1)}, t)$ classical deterministic cell probe schemes. But their lower bound proof breaks down if the query scheme is randomised. Our result thus shows that the upper bound scheme of Beame and Fich is optimal all the way up to the bounded error address-only quantum cell probe model. Also, our proof is substantially simpler than that of Beame and Fich.

2. It is known that querying in superposition gives a speed up over classical algorithms for certain data retrieval problems, the most notable one being Grover’s algorithm [Gro96] for searching an unordered list of $n$ elements using $O(\sqrt{n})$ quantum queries. The power of quantum querying for data structure problems was studied in the context of static membership by Radhakrishnan, Sen and Venkatesh [RSV00]. In their paper, they worked in the quantum bit probe model, which is our quantum cell probe model where the word size is just one bit. They showed, roughly speaking, that quantum querying does not give much advantage over classical schemes for the set membership problem. Our result above seems to suggest that quantum search is perhaps not more powerful than classical search for the predecessor problem as well.

3. In the next section, we formally describe the “address-only” restrictions we impose on the query algorithm. Informally, they amount to this: we allow quantum parallelism over the ‘address lines’ going into the table, but we have a fixed quantum state on the ‘data lines’. This restriction on quantum querying does not make the model trivial. In fact, many non-trivial quantum search algorithms, such as Grover’s algorithm [Gro96], Farhi et al.’s algorithm [FGGS99], and Høyer et al.’s algorithm [HNS01], already satisfy these restrictions.

4. For the static membership problem, Fredman, Kómlos and Szemerédi [FKS84] have shown a classical deterministic cell probe solution where the storage scheme uses $O(n)$
cells of word size $O(\log m)$ bits, and the query scheme makes only a constant number of probes. In this solution, the storage scheme may store elements of the universe in the table which are not members of the subset to be stored. Hence the restriction that the storage scheme be implicit is necessary for any such result. We note that implicit storage schemes include many of the standard storage schemes like sorted array, hash table, search trees etc.

1.1 Techniques

The lower bounds for the static membership problem shown in the quantum bit probe model by Radhakrishnan et al. [RSV00] relied on linear algebraic techniques. Unfortunately, these techniques appear to be powerless in the quantum cell probe model. In fact, to show the lower bound above for the static predecessor problem, we use a connection between quantum data structure problems and two-party quantum communication complexity, similar to what was used by Miltersen, Nisan, Safra and Wigderson [MNSW98], and Beame and Fich [BF99] for showing their (classical) lower bounds. Miltersen et al. [MNSW98] proved a technical lemma in classical communication complexity called the round elimination lemma and derived from it lower bounds for various static data structure problems, including the predecessor problem. But their round elimination lemma was not strong enough to prove a lower bound matching the upper bound of Beame and Fich. In this paper we prove a stronger (!) round elimination lemma for the quantum communication complexity model, which we then use to show a quantum lower bound for the static predecessor problem matching Beame and Fich’s upper bound. Our quantum round elimination lemma is proved using quantum information theoretic techniques. Inspired by these techniques, we prove a still stronger round elimination lemma in classical communication complexity.

We now give an informal description of the round elimination lemma. Suppose $f : E \times F \to G$ is a function. In the communication game corresponding to $f$, Alice gets a string $x \in E$, Bob gets a string $y \in F$, and they have to communicate and compute $f(x, y)$. In the communication game $f^{(n)}$, Alice gets $n$ strings $x_1, \ldots, x_n \in E$; Bob gets an integer $i \in [n]$, a string $y \in F$, and a copy of the strings $x_1, \ldots, x_{i-1}$. Their aim is to communicate and compute $f(x_i, y)$. Suppose a quantum protocol for $f^{(n)}$ is given where Alice starts, and her first message is much smaller than $n$ qubits. Intuitively, it would seem that since Alice does not know $i$, the first round of communication cannot give much information about $x_i$, and thus, would not be very useful to Bob. Hence it should be possible to eliminate the first round of communication, giving a quantum protocol for computing $f(x_i, y)$ where Bob starts, with one less round of communication, and having the same message complexity and similar error probability. The round elimination lemma justifies this intuition. Moreover, we show that this is true even if Bob also gets copies of $x_1, \ldots, x_{i-1}$, a case which is needed in many applications.

**Result 3 (Round elimination lemma, informal statement)** A $t$ round quantum protocol for $f^{(n)}$ with Alice starting, where the first message of Alice is
much smaller than $n$ qubits, gives us a $t-1$ round quantum protocol for $f$ where Bob starts, with the same message complexity and similar error probability. An analogous statement holds for classical randomised protocols.

Round reduction arguments have been given earlier in quantum communication complexity, most notably by Klauck, Nayak, Ta-Shma and Zuckerman [KNTZ01]. However, for technical reasons, the previous arguments do not go far enough to prove lower bounds for the communication games arising from data structure problems like the predecessor problem. We need a technical quantum version of the round elimination lemma of Miltersen et al. [MNSW98], to prove the desired quantum lower bounds.

The round elimination lemma also has applications to other communication complexity problems, which might be interesting on their own. For example, it can be used to prove rounds versus communication tradeoffs for the ‘greater-than’ problem. In the ‘greater-than’ problem $GT_n$, Alice is given $x \in \{0,1\}^n$, Bob is given $y \in \{0,1\}^n$, and they have to communicate and decide whether $x > y$ (treating $x,y$ as integers).

**Result 4** The $t$ round bounded error quantum (classical randomised) communication complexity of $GT_n$ is $\Omega(\frac{n^{1/t}}{t^{3}})$ ($\Omega(\frac{n^{1/t}}{t^{2}})$).

There exists a bounded error classical randomised protocol for $GT_n$ using $t$ rounds of communication and having a complexity of $O(n^{1/t} \log n)$. Hence, for a constant number of rounds, our quantum lower bound matches the classical upper bound to within logarithmic factors. For one round quantum protocols, our result implies an $\Omega(n)$ lower bound for $GT_n$ (which is optimal to within constant factors), improving upon the previous $\Omega(n / \log n)$ lower bound of Klauck [Kla00]. No rounds versus communication tradeoff for this problem, for more than one round, was known earlier in the quantum setting. For classical randomised protocols, Miltersen et al. [MNSW98] showed a lower bound of $\Omega(n^{1/t} 2^{-O(t)})$ using their round elimination lemma. If the number of rounds is unbounded, then there is a classical randomised protocol for $GT_n$ using $O(\log n)$ rounds of communication and having a complexity of $O(\log n)$ [Nis93]. An $\Omega(\log n)$ lower bound for the bounded error quantum communication complexity of $GT_n$ (irrespective of the number of rounds) follows from Kremer’s result [Kre95] that the bounded error quantum communication complexity of a function is lower bounded (up to constant factors) by the logarithm of the one round (classical) deterministic communication complexity.

## 1.2 Organisation of the paper

Section 2 contains definitions of various terms that will be used throughout the paper. In Section 3, we discuss some lemmas that will be needed in the proofs of the main theorems. Section 4 contains a proof of the quantum and classical round elimination lemmas. Proofs of some lemmas required to prove the round elimination lemma proper have been relegated to the appendix. In Section 5, we apply our round elimination lemma to prove lower bounds for the query complexity of the static predecessor problem and the communication complexity of the ‘greater-than’ problem. Section 6 contains a proof of our lower bound
for implicit storage quantum cell probe schemes for the static membership problem. We conclude with a few remarks and some open problems in Section 7.

2 Definitions

In this section we define some of the terms which we will be using in this paper.

2.1 The quantum cell probe model

A quantum \((s, w, t)\) cell probe scheme for a static data structure problem \(f : D \times Q \rightarrow A\) has two components: a classical deterministic storage scheme that stores the data \(d \in D\) in a table \(T_d\) using \(s\) cells each containing \(w\) bits, and a quantum query scheme that answers queries by ‘quantumly probing a cell at a time’ \(t\) times. Formally speaking, the table \(T_d\) is made available to the query algorithm in the form of an oracle unitary transform \(O_d\). To define \(O_d\) formally, we represent the basis states of the query algorithm as \(|j, b, z\rangle\), where \(j \in [s - 1]\) is a binary string of length \(\log s\), \(b\) is a binary string of length \(w\), and \(z\) is a binary string of some fixed length. Here, \(j\) denotes the address of a cell in the table \(T_d\), \(b\) denotes the qubits which will hold the contents of a cell and \(z\) stands for the rest of the qubits (‘work qubits’) in the query algorithm. \(O_d\) maps \(|j, b, z\rangle\) to \(|j, b \oplus (T_d)_j, z\rangle\), where \((T_d)_j\) is a bit string of length \(w\) and denotes the contents of the \(j\)th cell in \(T_d\). A quantum query scheme with \(t\) probes is just a sequence of unitary transformations

\[
U_0 \rightarrow O_d \rightarrow U_1 \rightarrow O_d \rightarrow \ldots U_{t-1} \rightarrow O_d \rightarrow U_t
\]

where \(U_j\)’s are arbitrary unitary transformations that do not depend on \(d\) (representing the internal computations of the query algorithm). For a query \(q \in Q\), the computation starts in a computational basis state \(|q\rangle|0\rangle\), where we assume that the ancilla qubits are initially in the basis state \(|0\rangle\). Then we apply in succession, the operators \(U_0, O_d, U_1, \ldots, U_{t-1}, O_d, U_t\), and measure the final state. The answer consists of the values on some of the output wires of the circuit. We say that the scheme has worst case error probability less than \(\epsilon\) if the answer is equal to \(f(d, q)\), for every \((d, q) \in D \times Q\), with probability greater than \(1 - \epsilon\). The term ‘bounded error quantum scheme’ means that \(\epsilon = 1/3\).

We now formally define the address-only quantum cell probe model. Here the storage scheme is as in the general model, but the query scheme is restricted to be ‘address-only’. This means that the state vector before a query to the oracle \(O_d\) is always a tensor product of a state vector on the address and work qubits (the \((j, z)\) part in \((j, b, z)\) above), and a state vector on the data qubits (the \(b\) part in \((j, b, z)\) above). The state vector on the data qubits before a query to the oracle \(O_d\) is independent of the query element \(q\) and the data \(d\) but can vary with the probe number. Intuitively, we are only making use of quantum parallelism over the address lines of a query. This mode of querying a table subsumes classical querying, and also many non-trivial quantum algorithms like Grover’s algorithm [Gro96], Farhi et al.’s algorithm [FGGS99], Hoyer et al.’s algorithm [HNS01] etc. satisfy the ‘address-only’ condition. For classical querying, the state vector on the
data qubits is $|0\rangle$, independent of the probe number. For Grover and Farhi et al., the state vector on the data qubit is $(|0\rangle - |1\rangle)/\sqrt{2}$, independent of the probe number. For Høyer et al., the state vector on the data qubit is $|0\rangle$ for some probe numbers, and $(|0\rangle - |1\rangle)/\sqrt{2}$ for the other probe numbers.

2.2 Quantum communication protocols

We consider two party quantum communication protocols as defined by Yao [Yao93]. Let $E, F, G$ be arbitrary finite sets and $f : E \times F \rightarrow G$ be a function. There are two players Alice and Bob, who hold qubits. When the communication game starts, Alice holds $|x\rangle$ where $x \in E$ together with some ancilla qubits in the state $|0\rangle$, and Bob holds $|y\rangle$ where $y \in F$ together with some ancilla qubits in the state $|0\rangle$. Thus the qubits of Alice and Bob are initially in computational basis states, and the initial superposition is simply $|x\rangle_A |0\rangle_A |y\rangle_B |0\rangle_B$. Here the subscripts denote the ownership of the qubits by Alice and Bob. The players take turns to communicate to compute $f(x, y)$. Suppose it is Alice’s turn. Alice can make an arbitrary unitary transformation on her qubits and then send one or more qubits to Bob. Sending qubits does not change the overall superposition, but rather changes the ownership of the qubits, allowing Bob to apply his next unitary transformation on his original qubits plus the newly received qubits. At the end of the protocol, the last recipient of qubits performs a measurement on the qubits in her possession to output an answer. We say a quantum protocol computes $f$ with $\epsilon$-error in the worst case, if for any input $(x, y) \in E \times F$, the probability that the protocol outputs the correct result $f(x, y)$ is greater than $1 - \epsilon$. The term ‘bounded error quantum protocol’ means that $\epsilon = 1/3$.

We require that Alice and Bob make a secure copy of their inputs before beginning the protocol. This is possible since the inputs to Alice and Bob are in computational basis states. Thus, without loss of generality, the input qubits of Alice and Bob are never sent as messages, their state remains unchanged throughout the protocol, and they are never measured i.e. some work qubits are measured to determine the result of the protocol. We call such protocols secure. We will assume henceforth that all our protocols are secure.

We now define the concept of a safe quantum protocol, which will be used in the statement of the quantum round elimination lemma.

**Definition 1 (Safe quantum protocol)** A $[t, c, l_1, \ldots, l_t]^A ([t, c, l_1, \ldots, l_t]^B)$ safe quantum protocol is a secure quantum protocol where Alice (Bob) starts the communication, the first message is $l_1 + c$ qubits long, the $i$th message, for $i \geq 2$, is $l_i$ qubits long, and the communication goes on for $t$ rounds. We think of the first message as having two parts: the ‘main part’ which is $l_1$ qubits long, and the ‘safe overhead part’ which is $c$ qubits long. The density matrix of the ‘safe overhead’ is independent of the inputs to Alice and Bob.

Later on in the paper, we also use the notation $(t, c, a, b)^A ((t, c, a, b)^B)$ to denote a $[t, c, l_1, \ldots, l_t]^A ([t, c, l_1, \ldots, l_t]^B)$ safe quantum protocol, where the per round message lengths of Alice and Bob are $a$ and $b$ qubits respectively i.e. if Alice (Bob) starts, $l_i = a$ for $i$ odd and $l_i = b$ for $i$ even ($l_i = b$ for $i$ odd and $l_i = a$ for $i$ even).
Remark: The concept of a safe quantum protocol may look strange at first. The reason we need to define it, intuitively speaking, is as follows. The communication games arising from data structure problems often have an asymmetry between the message lengths of Alice and Bob. This asymmetry is crucial to prove lower bounds on the number of rounds of communication. In the previous quantum round reduction arguments (e.g. those of Klauck et al. [KNTZ01]), the complexity of the first message in the protocol increases quickly as the number of rounds is reduced and the asymmetry gets lost. This leads to a problem where the first message soon gets big enough to potentially convey substantial information about the input of one player to the other, destroying any hope of proving strong lower bounds on the number of rounds. The concept of a safe protocol allows us to get around this problem. We show through a careful quantum information theoretic analysis of the round reduction process, that in a safe protocol, though the complexity of the first message increases a lot, this increase is confined to the safe overhead and so, the information content does not increase much. This is the key property which allows us to prove a round elimination lemma for safe quantum protocols.

In this paper we will deal with quantum protocols with public coins. Intuitively, a public coin quantum protocol is a probability distribution over finitely many (coinless) quantum protocols. We shall henceforth call the standard definition of a quantum protocol as coinless. Our definition is similar to the classical scenario, where a randomised protocol with public coins is a probability distribution over finitely many deterministic protocols. We note however, that our definition of a public coin quantum protocol is not the same as that of a quantum protocol with prior entanglement, which has been studied previously (see e.g. [CvDNT98]). Our definition is weaker, in that it does not allow the unitary transformations of Alice and Bob to alter the ‘public coin’.

Definition 2 (Public coin quantum protocol) In a quantum protocol with a public coin, there is, before the start of the protocol, a quantum state called a public coin, of the form $\sum_c \sqrt{p_c} |c\rangle_A |c\rangle_B$, where the subscripts denote ownership of qubits by Alice and Bob, $p_c$ are finitely many non-negative real numbers and $\sum_c p_c = 1$. Alice and Bob make (entangled) copies of their respective halves of the public coin using CNOT gates before commencing the protocol. The unitary transformations of Alice and Bob during the protocol do not touch the public coin. The public coin is never measured, nor is it ever sent as a message.

Hence, one can think of the public coin quantum protocol to be a probability distribution, with probability $p_c$, over finitely many coinless quantum protocols indexed by the coin basis states $|c\rangle$. A safe public coin quantum protocol is similarly defined as a probability distribution over finitely many safe coinless quantum protocols.

Remarks:
1. We need to define public coin quantum protocols in order to make use of the harder direction of Yao’s minimax lemma [Yao77]. The minimax lemma is the main tool which allows us to convert ‘average case’ round reduction arguments to ‘worst case’ arguments.
We need ‘worst case’ round reduction arguments in proving lower bounds for the rounds complexity of communication games arising from data structure problems. This is because many of these lower bound proofs use some notion of “self-reducibility”, arising from the original data structure problem, which fails to hold in the ‘average case’ but holds for the ‘worst case’. The quantum round reduction arguments of Klauck et al. \[KNTZ01\] are ‘average case’ arguments, and this is one of the reasons why they do not suffice to prove lower bounds for the rounds complexity of communication games arising from data structure problems.

2. Parallel repetitions of protocols, as well as constructing new protocols from old ones using both the directions of Yao’s minimax lemma, preserve the “safety” property.

For an input \((x, y) \in E \times F\), we define the error \(\epsilon^P_{x,y}\) of the protocol \(P\) on \((x, y)\), to be the probability that the result of \(P\) on input \((x, y)\) is not equal to \(f(x, y)\). For a protocol \(P\), given a probability distribution \(D\) on \(E \times F\), we define the average error \(\epsilon^P_D\) of \(P\) with respect to \(D\) as the expectation over \(D\) of the error of \(P\) on inputs \((x, y) \in E \times F\). We define \(\epsilon^P\) to be worst case error of \(P\) on inputs \((x, y) \in E \times F\).

3 Preliminaries

In this section we state some facts which will be useful in what follows.

3.1 Yao’s minimax lemma

For completeness, we state Yao’s minimax lemma \[Ya07\] for safe quantum protocols in the (slightly more general) flavour that will be required by us. The proof of this flavour of the lemma is very similar to the standard proof, using the von Neumann minimax theorem.

**Lemma 1 (Yao’s minimax lemma)** Consider \([t, c, l_1, \ldots, l_t]^A\) safe quantum protocols \(P\) for a function \(f : E \times F \to G\). Let \(D\) denote a probability distribution on the inputs \((x, y) \in E \times F\). Then

\[
\inf_{P : \text{public coin}} \epsilon^P = \sup_D \inf_{P : \text{coinless}} \epsilon^P = \sup_D \inf_{P : \text{public coin}} \epsilon^P_D
\]

Analogous properties hold for classical protocols too.

3.2 Quantum cell probe complexity and communication

In this subsection, we describe the connection between the quantum cell probe complexity of a static data structure problem and the quantum communication complexity of an associated communication game. Let \(f : D \times Q \to A\) be a static data structure problem. Consider a two-party communication problem where Alice is given a query \(q \in Q\), Bob is given data \(d \in D\), and they have to communicate and find out the answer \(f(d, q)\). We
have the following lemma, which is a quantum analogue of a lemma of Miltersen [Mil94] relating cell probe complexity to communication complexity in the classical setting.

**Lemma 2** Suppose we have a quantum \((s, w, t)\) cell probe solution to the static data structure problem \(f\). Then we have a \((2^t, 0, \log s + w, \log s + w)^A\) safe coinless quantum protocol for the corresponding communication problem. If the query scheme is address-only, we can get a \((2^t, 0, \log s, \log s + w)^A\) safe coinless quantum protocol. The error probability of the communication protocol is the same as that of the cell probe scheme.

**Proof:** Given a quantum \((s, w, t)\) cell probe solution to the static data structure problem \(f\), we can get a \((2^t, 0, \log s + w, \log s + w)^A\) safe coinless quantum protocol for the corresponding communication problem by just simulating the cell probe solution. If in addition, the query scheme is address-only, the messages from Alice to Bob need consist only of the ‘address’ part. This can be seen as follows. Let the state vector of the data qubits before the \(i\)th query be \(|\theta_i\rangle\). \(|\theta_i\rangle\) is independent of the query element and the stored data. Bob keeps \(t\) special ancilla registers in states \(|\theta_i\rangle\), \(1 \leq i \leq t\) at the start of the protocol \(P\). These special ancilla registers are in tensor with the rest of the qubits of Alice and Bob at the start of \(P\). Protocol \(P\) simulates the cell probe solution, but with the following modification. To simulate the \(i\)th query of the cell probe solution, Alice prepares her ‘address’ and ‘data’ qubits as in the query scheme, but sends the ‘address’ qubits only. Bob treats those ‘address’ qubits together with \(|\theta_i\rangle\) in the \(i\)th special ancilla register as Alice’s query, and performs the oracle table transformation on them. He then sends these qubits (both the ‘address’ as well as the \(i\)th special register qubits) to Alice. Alice exchanges the contents of the \(i\)th special register with her ‘data’ qubits (i.e. exchanges the basis states), and proceeds with the simulation of the query scheme. This gives us a \((2^t, 0, \log s, \log s + w)^A\) safe coinless quantum protocol with the same error probability as that of the cell probe query scheme.

In many natural data structure problems \(\log s\) is much smaller than \(w\) and thus, in the address-only quantum case, we get a \((2^t, 0, \log s, O(w))^A\) safe protocol. In the classical setting, one gets a \((2^t, 0, \log s, w)^A\) protocol. This asymmetry in message lengths is crucial in proving non-trivial lower bounds on \(t\). The concept of a safe quantum protocol helps us in exploiting this asymmetry.

### 3.3 Background from quantum information theory

In this subsection, we discuss some basic facts from quantum information theory that will be used in the proof of the round elimination lemma. We follow the notation of Klauck, Nayak, Ta-Shma and Zuckerman’s paper [KNTZ01]. For a good account of quantum information theory, see the book by Nielsen and Chuang [NC00].

If \(A\) is a quantum system with density matrix \(\rho\), then \(S(A) \triangleq S(\rho) \triangleq -\text{Tr} \rho \log \rho\) is the von Neumann entropy of \(A\). If \(A, B\) are two disjoint quantum systems, their mutual information is defined as \(I(A : B) \triangleq S(A) + S(B) - S(AB)\). We now state some properties about von Neumann entropy and mutual information which will be useful later. The
proofs follow easily from the definitions, using basic properties of von Neumann entropy like subadditivity and triangle inequality (see e.g. [NC00, Chapter 11]).

Lemma 3 Suppose $A, B, C$ are disjoint quantum systems. Then

$$I(A : BC) = I(A : B) + I(AB : C) - I(B : C)$$

$$0 \leq I(A : B) \leq 2S(A)$$

If the Hilbert space of $A$ has dimension $d$, then

$$0 \leq S(A) \leq \log d$$

Suppose $X, Q$ are disjoint quantum systems with finite dimensional Hilbert spaces $\mathcal{H}, \mathcal{K}$ respectively. For every computational basis state $|x\rangle \in \mathcal{H}$, suppose $\sigma_x$ is a density matrix in $\mathcal{K}$. Suppose the density matrix of $(X, Q)$ is $\sum_x p_x |x\rangle \otimes \sigma_x$, where $p_x > 0$ and $\sum_x p_x = 1$. Thus $X$ is in a mixed state $\{|x\rangle, p_x\}$, and we shall say that $X$ is a classical random variable and that $Q$ is a quantum encoding $|x\rangle \mapsto \sigma_x$ of $X$. Define $\sigma \triangleq \sum_x p_x \sigma_x$. $\sigma$ is the reduced density matrix of $Q$, and we shall say that $\sigma$ is the the density matrix of the average encoding. Then, $S(XQ) = S(X) + \sum_x p_x S(\sigma_x)$, and hence, $I(X : Q) = S(\sigma) - \sum_x p_x S(\sigma_x)$.

Let $X, Y, Q$ be disjoint quantum systems with finite dimensional Hilbert spaces $\mathcal{H}, \mathcal{K}, \mathcal{L}$ respectively. Let $x \in \mathcal{H}$, $y \in \mathcal{K}$ be computational basis vectors. For every $|x\rangle|y\rangle \in \mathcal{H} \otimes \mathcal{K}$, suppose $\sigma_{xy}$ is a density matrix in $\mathcal{L}$. Let $Z$ refer to the quantum system $(X, Y)$. Suppose $(X, Y, Z)$ has density matrix $\sum_{x,y} p_{xy} |x\rangle \langle y| \otimes \sigma_{xy}$, where $p_{xy} > 0$ and $\sum_{x,y} p_{xy} = 1$. Thus, $X$ and $Y$ are classical random variables, and $Z = XY$ is in a mixed state $\{|xy\rangle, p_{xy}\}$. $Q$ is a quantum encoding $|xy\rangle \mapsto \sigma_{xy}$ of $Z$. Define $q_x^y$ to be the (conditional) probability that $Y = y$ given that $X = x$. $|y\rangle \mapsto \sigma_{xy}$ can be thought of as a quantum encoding $Q^x$ of $Y$ given that $X = x$. The joint density matrix of $(Y, Q^x)$ is $\sum_y q_y^x |y\rangle \otimes \sigma_{xy}$. We let $I((Y : Q)|X = x)$ denote the mutual information of this encoding.

We now prove the following propositions.

Proposition 1 Let $M_1, M_2$ be disjoint finite dimensional quantum systems. Suppose $M \triangleq (M_1, M_2)$ is a quantum encoding $|x\rangle \mapsto \sigma_x$ of a classical random variable $X$. Suppose the density matrix of $M_2$ is independent of $X$ i.e. $\text{Tr}_{M_1} \sigma_x$ is the same for all $x$. Let $M_1$ be supported on a qubits. Then, $I(X : M) \leq 2a$.

Proof: By Lemma 3, $I(X : M) = I(X : M_1 M_2) = I(X : M_2) + I(X M_2 : M_1) - I(M_2 : M_1)$. But since the density matrix of $M_2$ is independent of $X$, $I(X : M_2) = 0$. Hence, by again using Lemma 3 we get that $I(X : M) \leq I(X M_2 : M_1) \leq 2S(M_1) \leq 2a$. 

Remarks:

1. This proposition is the key observation allowing us to “ignore” the size of the “safe” overhead $M_2$ in the round elimination lemma. It will be very useful in the applications of the round elimination lemma, where the complexity of the first message in the protocol
increases quickly, but the blow up is confined to the “safe” overhead. Earlier round reduction arguments were unable to handle this large blow up in the complexity of the first message.

2. In the above proposition, if $M$ is a classical encoding of $X$ (i.e. an encoding $|x\rangle \mapsto \sigma_x$, where $\sigma_x$ is a density matrix of a mixture of computational basis vectors), we get the improved inequality $I(X : M) \leq a$.

The next proposition has been observed by Klauck et al. [KNTZ01].

Proposition 2 Suppose $M$ is a quantum encoding of a classical random variable $X$. Suppose $X = X_1 X_2 \ldots X_n$, where the $X_i$ are classical independent random variables. Then, $I(X_1 \ldots X_n : M) = \sum_{i=1}^n I(X_i : MX_1 \ldots X_{i-1})$.

Proof: The proof is by induction on $n$, using Lemma 3 repeatedly. We also use the fact that $I(X_i : X_{i+1} \ldots X_n) = 0$ for $1 \leq i < n$, since $X_1, \ldots, X_n$ are independent classical random variables. ■

Proposition 3 Let $X, Y$ be classical random variables and $M$ be a quantum encoding of $(X, Y)$. Then, $I(Y : MX) = I(X : Y) + E_X[I((Y : M)|X = x)]$.

Proof: Let $\sigma_{xy}$ be the density matrix of $M$ when $X, Y = x, y$. Let $p_x$ be the (marginal) probability that $X = x$ and $q_y^x$ the (conditional) probability that $Y = y$ given $X = x$. Define $\sigma_x = \sum_y q_y^x \sigma_{xy}$. We now have

$$I(Y : MX) = S(Y) + S(MX) - S(MXY)$$

$$= S(Y) + S(X) + \sum_x p_x S(\sigma_x) - (S(XY) + \sum_{x,y} p_x q_y^x S(\sigma_{xy}))$$

$$= I(X : Y) + \sum_x p_x S(\sigma_x) - \sum_y q_y^x S(\sigma_{xy})$$

$$= I(X : Y) + \sum_x p_x I((Y : M)|X = x)$$

$$= I(X : Y) + E_X[I((Y : M)|X = x)]$$

4 The round elimination lemmas

In this section we prove our round elimination lemmas for safe public coin quantum protocols and public coin classical randomised protocols. Since a public coin quantum protocol can be converted to a coinless quantum protocol at the expense of an additional “safe” overhead in the first message, we also get a similar round elimination lemma for coinless protocols. We can decrease the overhead to logarithmic in the total bit size of the inputs by a technique similar to the public to private coins conversion for classical randomised protocols [New91]. But since the statement of the round elimination lemma is cleanest for safe public coin quantum protocols, we give it below for such protocols only. Similar remarks apply to the classical setting.
4.1 The quantum round elimination lemma

In this subsection we prove our round elimination lemma for safe public coin quantum protocols. We first state the following round reduction lemma, which can be proved in a manner similar to the proof of Lemma 4.4 in Klauck et al. [KNTZ01], but with a careful accounting of “safe” overheads in the messages communicated by Alice and Bob. Intuitively speaking, the lemma says that if the first message of Alice carries little information about her input, under some probability distribution on inputs, then it can be eliminated, giving rise to a protocol where Bob starts, with one less round of communication, and the same message complexity and similar error probability, with respect to the same probability distribution on inputs. We observe, in the lemma below, that though there is a overhead of \( l_1 + c \) qubits on the first message of Bob, it is a “safe” overhead.

**Lemma 4** Suppose \( f : E \times F \rightarrow G \) is a function. Let \( D \) be a probability distribution on \( E \times F \), and \( P \) be a \([t, c, l_1, \ldots, l_t]^A\) safe coinless quantum protocol for \( f \). Let \( X \) stand for the classical random variable denoting Alice’s input (under distribution \( D \)), \( M \) be the first message of Alice in the protocol \( P \), and \( I(X : M) \) denote the mutual information between \( X \) and \( M \) under distribution \( D \). Then there exists a \([t-1, c + l_1, l_2, \ldots, l_t]^B\) safe coinless quantum protocol \( Q \) for \( f \), such that

\[
\epsilon_Q^Q \leq \epsilon_D^P + ((2 \ln 2)I(X : M))^{\frac{1}{4}}
\]

A proof of this lemma can be found in the appendix.

We can now prove the quantum round elimination lemma (for the communication game \( f^{(n)} \)).

**Lemma 5 (Quantum round elimination lemma)** Suppose \( f : E \times F \rightarrow G \) is a function. Suppose the communication game \( f^{(n)} \) has a \([t, c, l_1, \ldots, l_t]^A\) safe public coin quantum protocol with worst case error less than \( \delta \). Then there is a \([t-1, c + l_1, l_2, \ldots, l_t]^B\) safe public coin quantum protocol for \( f \) with worst case error less than \( \epsilon \triangleq \delta + (4l_1 \ln 2/n)^{1/4} \).

**Proof:** Suppose the given protocol for \( f^{(n)} \) has worst case error \( \tilde{\delta} < \delta \). Define \( \tilde{\epsilon} \triangleq \tilde{\delta} + (4l_1 \ln 2/n)^{1/4} \). To prove the quantum round elimination lemma it suffices to give, by the harder direction of the minimax lemma (Lemma [1]), for any probability distribution \( D \) on \( E \times F \), a \([t-1, c + l_1, l_2, \ldots, l_t]^B\) safe public coin quantum protocol \( P \) for \( f \) with average distributional error \( \epsilon_D^p \leq \tilde{\epsilon} < \epsilon \). To this end, we will first construct a probability distribution \( D^* \) on \( E^n \times [n] \times F \) as follows. Choose \( i \in [n] \) uniformly at random. Choose independently, for each \( j \in [n] \), \( (x_j, y_j) \in E \times F \) according to distribution \( D \). Set \( y = y_i \) and throw away \( y_j, j \neq i \). By the easier direction of the minimax lemma (Lemma [1]), we get a \([t, c, l_1, \ldots, l_t]^A\) safe coinless quantum protocol \( P^* \) for \( f^{(n)} \) with distributional error, \( \epsilon_D^{p^*} \leq \tilde{\delta} < \delta \). In \( P^* \), Alice gets \( x_1, \ldots, x_n \), Bob gets \( i, y \) and \( x_1, \ldots, x_{i-1} \). We shall construct the desired protocol \( P \) from the protocol \( P^* \).

Let \( M \) be the first message of Alice in \( P^* \). By the definition of a safe protocol, \( M \) has two parts: \( M_1, l_1 \) qubits long, and the “safe” overhead \( M_2, c \) qubits long. Let the input
to Alice be denoted by the classical random variable $X = X_1 X_2 \ldots X_n$ where $X_i$ is the classical random variable corresponding to the $i$th input to Alice. Let the classical random variable $Y$ denote the input $y$ of Bob. Define $\epsilon_{D^*; i; x_1, \ldots, x_{i-1}}$ to be the average error of $P^*$ under distribution $D^*$ when $i$ is fixed and $X_1, \ldots, X_{i-1}$ are fixed to $x_1, \ldots, x_{i-1}$. Using Propositions 1, 2, 3 and the fact that under distribution $D^*$, $X_1, \ldots, X_n$ are independent classical random variables, we get that

$$\frac{2h}{n} \geq \frac{I(X; M)}{n} = E_i[I(X_i : M X_1, \ldots, X_{i-1})] = E_{i,X}[I((X_i : M) | X_1, \ldots, X_{i-1} = x_1, \ldots, x_{i-1})]$$

(1)

Also

$$\delta \geq \epsilon_{D^*} = E_{i,X} [\epsilon_{D^*; i; x_1, \ldots, x_{i-1}}]$$

(2)

The expectations above are under distribution $D^*$.

For any $i \in [n]$, $x_1, \ldots, x_{i-1} \in E$, define the $[t, c, l_1, \ldots, l_t]^A$ safe coinless quantum protocol $P'_{i; x_1, \ldots, x_{i-1}}$ for the function $f$ as follows. Alice is given $x \in E$ and Bob is given $y \in F$. Bob sets $i$ to the given value, and both Alice and Bob set $X_1, \ldots, X_{i-1}$ to the values $x_1, \ldots, x_{i-1}$. Alice puts an independent copy of a pure state $|\psi\rangle$ (defined below) for each of the inputs $X_{i+1}, \ldots, X_n$. She sets $X_i = x$ and Bob sets $Y = y$. Then they run protocol $P^*$ on these inputs. Here $|\psi\rangle \triangleq \sum_{x \in E} \sqrt{p_x} |x\rangle$, where $p_x$ is the (marginal) probability of $x$ under distribution $D$. Since $P^*$ is a secure coinless quantum protocol, so is $P'_{i; x_1, \ldots, x_{i-1}}$. Because $P^*$ is a secure protocol, the probability that $P'_{i; x_1, \ldots, x_{i-1}}$ makes an error for an input $(x, y)$, $\epsilon_{x,y}^{P'_{i; x_1, \ldots, x_{i-1}}}$, is the average probability of error of $P^*$ under distribution $D^*$ when $i$ is fixed to the given value, $X_1, \ldots, X_{i-1}$ are fixed to $x_1, \ldots, x_{i-1}$, and $X_i, Y$ are fixed to $x, y$. Hence, the average probability of error of $P'_{i; x_1, \ldots, x_{i-1}}$ under distribution $D$ is

$$\epsilon_D^{P'_{i; x_1, \ldots, x_{i-1}}} = \epsilon_{P^*; i; x_1, \ldots, x_{i-1}}$$

(3)

Let $M'$ denote the first message of $P'_{i; x_1, \ldots, x_{i-1}}$ and $X'$ denote the register $X_i$ holding the input $x$ to Alice. Because of the “security” of $P^*$, the density matrix of $(X', M')$ in protocol $P'_{i; x_1, \ldots, x_{i-1}}$ is the same as the density matrix of $(X_i, M)$ in protocol $P^*$ when $X_1, \ldots, X_{i-1}$ are set to $x_1, \ldots, x_{i-1}$. Hence

$I(X' : M') = I((X_i : M) | X_1, \ldots, X_{i-1} = x_1, \ldots, x_{i-1})$

(4)

Using Lemma 4 and equations (3) and (4), we get a $[t - 1, c + l_1, l_2, \ldots, l_t]^B$ safe coinless quantum protocol $P_{i; x_1, \ldots, x_{i-1}}$ for $f$ with

$$\epsilon_D^{P_{i; x_1, \ldots, x_{i-1}}} \leq \epsilon_D^{P'_{i; x_1, \ldots, x_{i-1}}} + (2 \ln 2) I(X' : M')^{1/4} = \epsilon_{P^*; i; x_1, \ldots, x_{i-1}} + (2 \ln 2) I((X_i : M) | X_1, \ldots, X_{i-1} = x_1, \ldots, x_{i-1})^{1/4}$$

(5)

We now construct a $[t - 1, c + l_1, l_2, \ldots, l_t]^B$ safe public coin quantum protocol $P$ for $f$, which is nothing but a probability distribution (under $D^*$) over the safe coinless quantum
protocols \( P; x_1, \ldots, x_{i-1}, i \in [n], x_1, \ldots, x_{i-1} \in E \). For protocol \( P \), we get (note that the expectations below are under distribution \( D^* \))

\[
\epsilon_P^P = E_{i,X}\left[\epsilon_{D^*}^{P; x_1, \ldots, x_{i-1}}\right] \\
\leq E_{i,X}\left[\epsilon_{D^*}^{P; x_1, \ldots, x_{i-1}} + ((2 \ln 2)I((X_i : M)|X_1, \ldots, X_{i-1} = x_1, \ldots, x_{i-1}))^{1/4}\right] \\
\leq E_{i,X}\left[\epsilon_{D^*}^{P; x_1, \ldots, x_{i-1}} + ((2 \ln 2)E_{i,X}[I((X_i : M)|X_1, \ldots, X_{i-1} = x_1, \ldots, x_{i-1})])^{1/4}\right] \\
\leq \tilde{\delta} + \left(\frac{4l_1 \ln 2}{n}\right)^{1/4} \\
= \tilde{\epsilon}
\]

The first inequality follows from (3), the second inequality follows from the concavity of the fourth root function and the last inequality from from (1) and (2).

This completes the proof of the quantum round elimination lemma.

\[\boxed{\blacksquare}\]

4.2 The classical round elimination lemma

The proof of the classical round elimination lemma is similar to that of the quantum round elimination lemma. First, we have the following classical analogue of Lemma 4.

**Lemma 6** Suppose \( f : E \times F \to G \) is a function. Let \( D \) be a probability distribution on \( E \times F \), and \( P \) be a \( [t, 0, l_1, \ldots, l_t]^A \) private coin classical randomised protocol for \( f \). Let \( X \) stand for the classical random variable denoting Alice’s input (under distribution \( D \)), \( M \) be the first message of Alice in the protocol \( P \), and \( I(X : M) \) denote the mutual information between \( X \) and \( M \) under distribution \( D \). Then there exists a \( [t - 1, 0, l_2, \ldots, l_t]^B \) public coin classical randomised protocol \( Q \) for \( f \), such that

\[
\epsilon_Q^Q \leq \epsilon_P^P + \frac{1}{2}(2 \ln 2)I(X : M) \right)^{1/2}
\]

A proof of the lemma is given in the appendix.

We can now prove the classical round elimination lemma (for the communication game \( f^{(n)} \)).

**Lemma 7 (Classical round elimination lemma)** Suppose \( f : E \times F \to G \) is a function. Suppose the communication game \( f^{(n)} \) has a \( [t, 0, l_1, \ldots, l_t]^A \) public coin classical randomised protocol with worst case error less than \( \delta \). Then there is a \( [t - 1, 0, l_2, \ldots, l_t]^B \) public coin classical randomised protocol for \( f \) with worst case error less than \( \epsilon \triangleq \delta + (1/2)(2l_1 \ln 2/n)^{1/2} \).

**Proof:** (Sketch) The proof is similar to that of Lemma 6, but using Lemma 6 instead of Lemma 4. Suppose the given protocol for \( f^{(n)} \) has worst case error \( \tilde{\delta} < \delta \). Define \( \tilde{\epsilon} \triangleq \tilde{\delta} + (1/2)(2l_1 \ln 2/n)^{1/2} \). To prove the classical round elimination lemma it suffices to give, by the harder direction of the minimax lemma (Lemma 1), for any probability
distribution $D$ on $E \times F$, a $[t−1,0,l_{1},\ldots,l_{1}]$ public coin classical randomised protocol $P$ for $f$ with average distributional error $\epsilon_{D}^{P} \leq \tilde{\epsilon} < \epsilon$. To this end, we construct the probability distribution $D^{*}$ on $E^{n} \times [n] \times F$ as before. By the easier direction of the minimax lemma (Lemma 3), we get a $[t,0,l_{1},\ldots,l_{1}]$ classical deterministic protocol $P^{*}$ for $f^{(n)}$ with distributional error, $\epsilon_{D}^{P^{*}} \leq \hat{\epsilon} \leq \delta$. In $P^{*}$, Alice gets $x_{1},\ldots,x_{n} \in E$, Bob gets $i \in [n]$, $y \in F$ and a copy of $x_{1},\ldots,x_{i−1}$. We shall construct the desired protocol $P$ from the protocol $P^{*}$.

Let $M$ be the first message of Alice in $P^{*}$. Let the input to Alice be denoted by the classical random variable $X = X_{1}X_{2}\ldots X_{n}$ where $X_{i}$ is the classical random variable corresponding to the $i$th input to Alice. Let the classical random variable $Y$ denote the input $y$ of Bob. Define $\epsilon_{D^{*};i;x_{1},\ldots,x_{i−1}}^{P^{*}}$ to be the average error of $P^{*}$ under distribution $D^{*}$ when $i$ is fixed and $X_{1},\ldots,X_{i−1}$ are fixed to $x_{1},\ldots,x_{i−1}$. Arguing as before, we get

$$\frac{l_{1}}{n} = E_{i,X}[I((X_{i} : M)|X_{1},\ldots,X_{i−1} = x_{1},\ldots,x_{i−1})]$$

Also

$$\bar{\delta} \geq \epsilon_{D^{*}}^{P^{*}} = E_{i,X}\left[\epsilon_{D^{*};i;x_{1},\ldots,x_{i−1}}^{P^{*}}\right]$$

The expectations above are under distribution $D^{*}$.

For any $i \in [n]$, $x_{1},\ldots,x_{i−1} \in E$, define the $[t,0,l_{1},\ldots,l_{1}]$ private coin classical randomised protocol $P'_{i;x_{1},\ldots,x_{i−1}}$ for the function $f$ as follows. Alice is given $x \in E$ and Bob is given $y \in F$. Bob sets $i$ to the given value, and both Alice and Bob set $X_{1},\ldots,X_{i−1}$ to the values $x_{1},\ldots,x_{i−1}$. Alice tosses her private coin to choose $X_{i+1},\ldots,X_{n} \in E$, where each $X_{j},i+1 \leq j \leq n$ is chosen independently according to the (marginal) distribution on $E$ induced by $D$. Alice sets $X_{i} = x$ and Bob sets $Y = y$. Then they run protocol $P^{*}$ on these inputs. The probability that $P'_{i;x_{1},\ldots,x_{i−1}}$ makes an error for an input $(x,y)$, $\epsilon_{x,y}^{P'_{i;x_{1},\ldots,x_{i−1}}}$, is the average probability of error of $P^{*}$ under distribution $D^{*}$ when $i$ is fixed to the given value, $X_{1},\ldots,X_{i−1}$ are fixed to $x_{1},\ldots,x_{i−1}$, and $X_{i},Y$ are fixed to $x,y$. Hence, the average probability of error of $P'_{i;x_{1},\ldots,x_{i−1}}$ under distribution $D$ is

$$\epsilon_{D}^{P'_{i;x_{1},\ldots,x_{i−1}}} = \epsilon_{D^{*};i;x_{1},\ldots,x_{i−1}}^{P^{*}}$$

Let $M'$ denote the first message of $P'_{i;x_{1},\ldots,x_{i−1}}$ and $X'$ denote the register $X_{i}$ holding the input $x$ to Alice. Then

$$I(X' : M') = I((X_{i} : M)|X_{1},\ldots,X_{i−1} = x_{1},\ldots,x_{i−1})$$

Using Lemma 3 and arguing as before, we can complete the proof of the classical round elimination lemma.

### 5 Applications of the round elimination lemma

In this section, we apply our round elimination lemmas to prove lower bounds for the query complexity of the static predecessor problem, and rounds versus communication tradeoffs for the ‘greater-than’ problem.
5.1 Static predecessor problem

The proof of our lower bound for the static predecessor problem in the address-only quantum cell probe model is similar to the classical proof in Miltersen et al. [MNSW98]. But because we use a stronger round elimination lemma, we can prove stronger lower bounds. We start by some preliminary observations.

**Definition 3** (Rank parity communication games, [MNSW98]) In the rank parity communication game $\text{PAR}_{p,q}$, Alice is given a bit string $x$ of length $p$, Bob is given a set $S$ of bit strings of length $p$, $|S| \leq q$, and they have to communicate and decide whether the rank of $x$ in $S$ (treating the bit strings as integers) is odd or even. By the rank of $x$ in $S$, we mean the cardinality of the set $\{y \in S \mid y \leq x\}$. In the game $\text{PAR}^{(k),A}_{p,q}$, Alice is given $k$ bit strings $x_1, \ldots, x_k$ each of length $p$, Bob is given a set $S$ of bit strings of length $p$, $|S| \leq q$, an index $i \in [k]$, and a copy of $x_1, \ldots, x_{i-1}$; they have to communicate and decide whether the rank of $x_i$ in $S$ is odd or even. In the game $\text{PAR}^{(k),B}_{p,q}$, Alice is given a bit string $x$ of length $p$ and an index $i \in [k]$, Bob is given $k$ sets $S_1, \ldots, S_k$ of bit strings of length $p$, $|S_j| \leq q$; they have to communicate and decide whether the rank of $x$ in $S_i$ is odd or even.

**Proposition 4** Let there be a $(n^{O(1)}, (\log m)^{O(1)}, t)$ address-only quantum cell probe solution to the static predecessor problem, where the universe size is $m$ and the subset size is at most $n$. Then there is a $(2t + O(1), 0, 0(\log n), (\log m)^{O(1)})^A$ safe coinless (and hence, public coin) quantum protocol for the rank parity communication game $\text{PAR}_{\log m, n}$. The error probability of the communication protocol is the same as that of the cell probe scheme.

**Proof:** Consider the static rank parity data structure problem where the storage scheme has to store a set $S \subseteq [m]$, $|S| \leq n$, and the query scheme, given a query $x \in [m]$, has to decide whether the rank of $x$ in $S$ is odd or even. Fredman, Komlós and Szemerédi [FKS83] have shown the existence of two-level perfect hash tables containing, for each member $y$ of the stored subset $S$, $y$’s rank in $S$, and using $O(n)$ cells of word size $O(\log m)$ and requiring only $O(1)$ classical deterministic cell probes. Combining a $(n^{O(1)}, (\log m)^{O(1)}, t)$ address-only quantum cell probe solution to the static predecessor problem with such a perfect hash table, gives us a $(n^{O(1)} + O(n), \max((\log m)^{O(1)}, O(\log m)), t + O(1))$ address-only quantum cell probe solution to the static rank parity problem. The error probability of the cell probe scheme for the rank parity problem is the same as the error probability of the cell probe scheme for the predecessor problem. By Lemma 2, we get a $(2t + O(1), 0, 0(\log n), (\log m)^{O(1)})^A$ safe coinless quantum protocol for the rank parity communication game $\text{PAR}_{\log m, n}$. The error probability of the communication protocol is the same as that of the cell probe scheme for the predecessor problem.

**Proposition 5** ([MNSW98]) Suppose $k$ divides $p$. A communication protocol for $\text{PAR}_{p,q}$ with Alice starting, gives us a communication protocol for $\text{PAR}^{(k),A}_{p/k,q}$ with Alice starting, with the same message complexity, number of rounds and error probability.
Proof: Consider the problem $\text{PAR}_{p/k,q}^{(k),A}$. Alice, who is given $x_1, \ldots, x_k$, computes the concatenation $\hat{x} \triangleq x_1 \cdot x_2 \cdots x_k$. Bob, who is given $S$, $i$ and $x_1, \ldots, x_{i-1}$, computes
\[
\hat{S} \triangleq \left\{ x_1 \cdot x_2 \cdots x_{i-1} \cdot y \cdot 0^{p(1-i/k)} \mid y \in S \right\}
\]
Alice and Bob then run the protocol for $\text{PAR}_{p,q}$ on the inputs $\hat{x}$, $\hat{S}$ to solve the problem $\text{PAR}_{p/k,q}^{(k),A}$. 

Proposition 6 ([MNSW98]) Suppose $k$ divides $q$, and $k$ is a power of 2. A communication protocol for $\text{PAR}_{p,q}$ with Bob starting, gives us a communication protocol for $\text{PAR}_{p-\log k-1,q/k}^{(k),B}$ with Bob starting, with the same message complexity, number of rounds and error probability.

Proof: Consider the problem $\text{PAR}_{p-\log k-1,q/k}^{(k),B}$. Alice, given $x$ and $i$, computes $\hat{x} \triangleq (i-1) \cdot 0 \cdot x$. Bob, given $S_1, \ldots, S_k$, computes the sets $S'_1, \ldots, S'_k$ where
\[
S'_j \triangleq \begin{cases} 
\{(j-1) \cdot 0 \cdot y \mid y \in S_j\} & \text{if } |S_j| \text{ is even} \\
\{(j-1) \cdot 0 \cdot y \mid y \in S_j\} \cup \{(j-1) \cdot 1^{p-\log k}\} & \text{if } |S_j| \text{ is odd}
\end{cases}
\]
Above, the integers $(i-1), (j-1)$ are to be thought of as bit strings of length $\log k$. Bob also computes $\hat{S} \triangleq \bigcup_{j=1}^{k} S'_j$. Alice and Bob then run the protocol for $\text{PAR}_{p,q}$ on inputs $\hat{x}$, $\hat{S}$ to solve the problem $\text{PAR}_{p-\log k-1,q/k}^{(k),B}$. 

We now prove the lower bound on the query complexity of static predecessor in the address-only quantum cell probe model.

Theorem 1 Suppose we have a $(n^{O(1)}, (\log m)^{O(1)}, t)$ bounded error quantum address-only cell probe solution to the static predecessor problem, where the universe size is $m$ and the subset size is at most $n$. Then the number of queries $t$ is at least $\Omega \left( \frac{\log \log log m}{\log log log log m} \right)$ as a function of $m$, and at least $\Omega \left( \sqrt{\frac{\log n}{\log \log n}} \right)$ as a function of $n$.

Proof: We basically imitate the proof of Miltersen et al. [MNSW98], but in our quantum setting. By Proposition 4, it suffices to consider communication protocols for the rank parity communication game $\text{PAR}_{\log m,n}$. Let $n = 2^{(\log \log m)^2/\log \log \log m}$. Let $c_1 \triangleq (4 \ln 2)12^4$. For any given constants $c_2, c_3 \geq 1$, define
\[
a \triangleq c_2 \log n \quad b \triangleq (\log m)^{c_3} \quad t \triangleq \frac{\log \log m}{(c_1 + c_2 + c_3) \log \log \log m}
\]
We shall show that the rank parity communication game $\text{PAR}_{\log m,n}$ does not have bounded error $(2t, 0, a, b)^A$ safe public coin quantum protocols, thus proving the desired lower bounds on the query complexity of static rank parity (and hence, static predecessor) by Lemma 4.

Given a $(2t, 0, a, b)^A$ safe public coin quantum protocol for $\text{PAR}_{\log m,n}$ with error probability $\delta$ ($\delta < 1/3$), we get a $(2t, 0, a, b)^A$ safe public coin quantum protocol for
\[
\text{PAR}_{\log m,n}^{(c_1 a t^4)^A}
\]
\[
\text{PAR}_{\log m,n}^{(c_1 a t^4)^A}
\]
with the same error probability $\delta$, by Proposition 3. Using the quantum round elimination lemma (Lemma 5), we get a $(2t - 1, a, a, b)^B$ safe public coin quantum protocol for

$$\text{PAR}^{\log m,\frac{n}{c_1 b t^4}}_{c_1 b t^4, a}$$

but the error probability increases to at most $\delta + (12t)^{-1}$. Using the reduction of Proposition 3, we get a $(2t - 1, a, a, b)^B$ safe public coin quantum protocol for

$$\text{PAR}^{(c_1 b t^4), B}_{\log m, \frac{n}{2c_1 a t^4}, \frac{n}{c_1 b t^4}}_{\log m, \frac{n}{c_1 b t^4}, \frac{n}{c_1 b t^4}}$$

with error probability at most $\delta + (12t)^{-1}$. From the given values of the parameters, we see that

$$\frac{\log m}{(2c_1 a t^4)^t} \geq \log(c_1 b t^4) + 1$$

This implies that we also have a $(2t - 1, a, a, b)^B$ safe public coin quantum protocol for

$$\text{PAR}^{(c_1 b t^4), B}_{\log m, \frac{n}{2c_1 a t^4}, \frac{n}{c_1 b t^4}}_{\log m, \frac{n}{c_1 b t^4}, \frac{n}{c_1 b t^4}}$$

but the error probability increases to at most $\delta + 2(12t)^{-1}$.

We do the above steps repeatedly. After applying the above steps $i$ times, we get a $(2t - 2i, i(a + b), a, b)^A$ safe public coin quantum protocol for

$$\text{PAR}^{\log m, \frac{n}{(2c_1 a t^4)^i}, \frac{n}{(c_1 b t^4)^i}}_{(2c_1 a t^4)^i, \frac{n}{c_1 b t^4}^i, \frac{n}{c_1 b t^4}^i}$$

with error probability at most $\delta + 2i(12t)^{-1}$.

By applying the above steps $t$ times, we finally get a $(0, t(a + b), a, b)^A$ safe public coin quantum protocol for

$$\text{PAR}^{\log m, \frac{n}{(2c_1 a t^4)^t}, \frac{n}{(c_1 b t^4)^t}}_{(2c_1 a t^4)^t, \frac{n}{c_1 b t^4}^t, \frac{n}{c_1 b t^4}^t}$$

with error probability at most $\delta + 2t(12t)^{-1} < 1/2$. From the given values of the parameters, we see that

$$\frac{\log m}{(2c_1 a t^4)^t} \geq (\log m)^{\Omega(1)} \frac{n}{(c_1 b t^4)^t} \geq n^{\Omega(1)}$$

Thus we get a zero round protocol for a rank parity problem on a non-trivial domain with error probability less than $1/2$, which is a contradiction.

In the above proof, we are tacitly ignoring “rounding off” problems. We remark that this does not affect the correctness of the proof.
5.2 The ‘greater-than’ problem

Theorem 2 The $t$ round bounded error quantum (classical randomised) communication complexity of $\text{GT}_n$ is $\Omega(n^{1/t}t^{-3})$ ($\Omega(n^{1/t}t^{-2})$).

Proof: We recall the following reduction from $\text{GT}_n^{(k)}$ to $\text{GT}_n$ (see [MNSW98]): In $\text{GT}_n^{(k)}$, Alice is given $x_1, \ldots, x_k \in \{0, 1\}^{n/k}$. Bob is given $i \in [k]$, $y \in \{0, 1\}^{n/k}$, and copies of $x_1, \ldots, x_{i-1}$, and they have to communicate and decide if $x_i > y$. To reduce $\text{GT}_n^{(k)}$ to $\text{GT}_n$, Alice constructs $\tilde{x} \in \{0, 1\}^n$ by concatenating $x_1, \ldots, x_{i-1}, y, 1^{n(1-i/k)}$. It is easy to see that $\tilde{x} > \tilde{y}$ iff $x_i > y$.

Suppose $\text{GT}_n$ has a $[t, 0, l_1, \ldots, l_t]^A$ safe public coin quantum protocol with worst case error probability less than $1/3$. Suppose

$$n \geq \left(Ct^3(l_1 + \cdots + l_t)\right)^t$$

where $C = (4 \ln 2)^6$. For $1 \leq i \leq t$, define

$$k_i \overset{\Delta}{=} Ct^4 l_i \quad n_i \overset{\Delta}{=} \frac{n}{\prod_{j=1}^{t} k_j} \quad \epsilon_i \overset{\Delta}{=} \frac{1}{3} + \sum_{j=1}^{i} \left(\frac{(4 \ln 2)l_j}{k_j}\right)^{1/4}$$

Also define $n_0 \overset{\Delta}{=} n$ and $\epsilon_0 \overset{\Delta}{=} 1/3$. Then

$$\epsilon_i \overset{\Delta}{=} \frac{1}{3} + \sum_{j=1}^{t} \left(\frac{(4 \ln 2)l_j}{k_j}\right)^{1/4} = \frac{1}{3} + \frac{t}{6t} = 1/2$$

and

$$n_t = \frac{n}{\prod_{j=1}^{t} k_j} = \frac{n}{(Ct^4)^{l_1} \cdots l_t} \geq \frac{nt^t}{C^{t^4}(l_1 + \cdots + l_t)^{t}} \geq 1$$

We now apply the above self-reduction and the quantum round elimination lemma (Lemma 3) alternately. Before the $i$th stage, we have a $[t - i + 1, \sum_{j=1}^{i-1} l_j, l_i, \ldots, l_t]^Z$ safe public coin quantum protocol for $\text{GT}_{n_{i-1}}$ with worst case error probability less than $\epsilon_{i-1}$. Here $Z = A$ if $i$ is odd, $Z = B$ otherwise. For the $i$th stage, we apply the self-reduction with $k = k_i$. This gives us a $[t - i + 1, \sum_{j=1}^{i-1} l_j, l_i, \ldots, l_t]^Z$ safe public coin quantum protocol for $\text{GT}_{n_i}^{(k_i)}$ with the same error probability. We now apply the quantum round elimination lemma (Lemma 3) to get a $[t - i, \sum_{j=1}^{i} l_j, l_{i+1}, \ldots, l_t]^Z$ safe public coin quantum protocol for $\text{GT}_{n_i}$ with worst case error probability less than $\epsilon_i$. Here $Z' = B$ if $Z = A$ and $Z' = A$ if $Z = B$. This completes the $i$th stage.

Applying the self-reduction and the round elimination lemma alternately $t$ times gives us a zero round quantum protocol for the ‘greater-than’ problem on a domain of size $n_t > 1$ with worst case error probability less than $\epsilon_t = 1/2$, which is a contradiction.

In the above proof, we are tacitly ignoring “rounding off” problems. We remark that this does not affect the correctness of the proof.

This proves the quantum lower bound of $\Omega(n^{1/t}t^{-3})$ on the message complexity.
Using the classical round elimination lemma (Lemma 7) instead of the quantum one, and treating a classical randomised protocol with complexity $l$ as a $[t, 0, l, \ldots, l]^4$ protocol, we get the stronger classical lower bound of $\Omega(n^{1/t^2})$.

Miltersen et al. [MNSW98] also apply their round elimination lemma to prove (classical) lower bounds for other data structure problems and communication complexity problems. We remark that we can extend all those results in a similar fashion to the quantum world.

6 Lower bounds for static membership

Consider the problem of storing a subset $S$ of size at most $n$ of the universe $[m]$ in a table with $q$ cells, so that membership queries can be answered efficiently. We restrict the storage scheme to be implicit, using at most $p$ ‘pointer values’. A ‘pointer value’ is a member of a set of size $p$ (the set of ‘pointers’) disjoint from the universe. The term implicit means that the storage scheme can store either a ‘pointer value’ or a member of $S$ in a cell. In particular, the storage scheme is not allowed to store an element of the universe which is not a member of $S$. The query algorithm answers membership queries by performing $t$ (general) quantum cell probes. We call such schemes $(p, q, t)$ implicit storage quantum cell probe schemes. For universe sizes $m$ that are ‘large’ compared to $n, p, q$, we can prove an $\Omega(\log n)$ lower bound on the number of quantum probes $t$ required to solve the static membership problem with $(p, q, t)$ implicit storage quantum cell probe schemes. We start with the following lemma.

Lemma 8 Suppose $S$ is an $n$ element subset of the universe $[m]$, where $m \geq 2n + 2$. If the storage scheme is implicit, always stores the same ‘pointer’ values in the same locations, and in the remaining locations, stores the elements of $S$ in a fixed order (repetitions of an element are allowed, but all elements have to be stored) based on their relative ranking in $S$, then $\Omega(\log n)$ probes are needed by any bounded error quantum cell query strategy to answer membership queries.

Proof: (Sketch) The proof follows by modifying Ambainis’s lower bound proof for quantum ordered searching [Amb99]. There, it was shown that if $S$ is stored in sorted order in a table $T$ then, given any query element $q$, $\Omega(\log n)$ probes are required by any quantum search strategy to find out the smallest index $i$, $1 \leq i \leq n$, such that $q \leq T(i)$. We observe that the lemma above does not follow directly from the result of Ambainis, since we only need to decide if $q$ is present in the table or not, and this is a weaker requirement. To prove the lemma, we follow the adversary strategy of [Amb99] with some minor changes. We study the behaviour of the quantum query scheme with query element $n + 1$. The proof of Ambainis is based on a clever strategy of subdividing “intervals” (an interval is a contiguous set of locations in the sorted table). We work instead with “logical intervals”, where a logical interval denotes the set of locations in the table where elements contiguous in the natural ordering are stored (as determined by the fixed storing order). After this definition, one can easily show that the same subdivision strategy as in [Amb99] goes through. In Ambainis’s proof, the adversary constructs inputs by padding with zeros from...
the beginning up to the left of an interval, and with ones from the end up to the right of the interval. Instead, we pad with small numbers \((1, 2, \ldots)\) from the logical beginning up to the logical left of a logical interval, and with large numbers \((m, m - 1, \ldots)\) from the logical end up to the logical right of the logical interval. We store the appropriate ‘pointer values’ in the ‘pointer locations’ (predetermined by the storing strategy). After doing this, one can easily show that the same error analysis of [Amb99] goes through. Thus, the adversary finally can produce two inputs, one of them containing \(n + 1\) and the other not, such that the behaviour of the query scheme is very similar on both. This is a contradiction.

**Remark:** Høyer et al. also prove an \(\Omega(\log n)\) lower bound for quantum ordered searching [HNS01]. But their approach, which is based on “distinguishing oracles”, does not seem to be suitable for proving lower bounds for boolean valued functions. Hence to prove Lemma 8, we modify the older \(\Omega(\log n)\) lower bound of Ambainis for quantum ordered searching.

**Theorem 3** For every \(n, p, q\), there exists an \(N(n, p, q)\) such that for all \(m \geq N(n, p, q)\), the following holds: Consider any bounded error \((p, q, t)\) implicit storage quantum cell probe scheme for the static membership problem with universe size \(m\) and size of the stored subset at most \(n\). Then the quantum query scheme must make \(t = \Omega(\log n)\) probes.

**Proof:** (Sketch) Our proof follows from the Ramsey theoretic arguments of Yao [Yao81] together with Lemma 8. The details are omitted.

### 7 Conclusions and open problems

In this paper we introduce the quantum cell probe model, a model for studying static data structure problems in the quantum world. We show that the additional power of quantum querying does not help for the static membership problem when the storage scheme is restricted to be implicit, generalising a result of Yao. We also explore the possibility of using quantum communication complexity to prove lower bounds in the quantum cell probe model. We prove a round elimination lemma for quantum communication complexity and use it to prove lower bounds for the static predecessor problem in a restricted version of the quantum cell probe model, the address-only version. Extending this result to the general model remains an important open problem. We also use the quantum round elimination lemma to prove rounds versus communication tradeoffs for the ‘greater-than’ problem. It would be interesting to find other applications of the round elimination lemma to quantum communication complexity.

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References

[AKN98] D. Aharonov, A. Kitaev, and N. Nisan. Quantum circuits with mixed states. In Proceedings of the 30th Annual ACM Symposium on Theory of Computing, pages 20–30, 1998. Also quant-ph/9806029.

[Amb99] A. Ambainis. A better lower bound for quantum algorithms searching an ordered list. In Proceedings of the 40th IEEE Symposium on Foundations of Computer Science, pages 352–357, 1999. Also quant-ph/9902053.

[BF99] P. Beame and F. Fich. Optimal bounds for the predecessor problem. In Proceedings of the 31st Annual ACM Symposium on Theory of Computing, pages 295–304, 1999.

[BMRV00] H. Buhrman, P. B. Miltersen, J. Radhakrishnan, and S. Venkatesh. Are bitvectors optimal? In Proceedings of the 32nd Annual ACM Symposium on Theory of Computing, pages 449–458, 2000.

[CvDNT98] R. Cleve, W. van Dam, M. Nielsen, and A. Tapp. Quantum entanglement and the communication complexity of the inner product function. In Proceedings of the 1st NASA International Conference on Quantum Computing and Quantum Communications, Lecture Notes in Computer Science, vol. 1509, pages 61–74. Springer-Verlag, 1998. Also quant-ph/9708019.

[FGGS99] E. Farhi, J. Goldstone, S. Gutmann, and M. Sipser. Invariant quantum algorithms for insertion into an ordered list. Manuscript at quant-ph/9901059, January 1999.

[FKS84] M. Fredman, J. Komlós, and E. Szemerédi. Storing a sparse table with $O(1)$ worst case access time. Journal of the Association for Computing Machinery, 31(3):538–544, 1984.

[Gro96] L. Grover. A fast quantum mechanical algorithm for database search. In Proceedings of the 28th Annual ACM Symposium on Theory of Computing, pages 212–219, 1996. Also quant-ph/9605043.

[HNS01] P. Høyer, J. Neerbek, and Y. Shi. Quantum complexities of ordered searching, sorting, and element distinctness. In Proceedings of the 28th International Colloquium on Automata, Languages and Programming, pages 346–357, 2001. Also quant-ph/0102078.
[Kla00] H. Klauck. Quantum communication complexity. In Proceedings of the Satellite Workshops at the 27th International Colloquium on Automata, Languages and Programming, Workshop on Boolean Functions and Applications (invited lecture), pages 241–252. Carleton Scientific, Waterloo, Ontario, Canada, 2000. Also quant-ph/0005032.

[KNTZ01] H. Klauck, A. Nayak, A. Ta-Shma, and D. Zuckerman. Interaction in quantum communication and the complexity of set disjointness. In Proceedings of the 33rd Annual ACM Symposium on Theory of Computing, pages 124–133, 2001.

[Kre95] I. Kremer. Quantum communication. Master’s thesis, Hebrew University, 1995.

[Mil94] P. B. Miltersen. Lower bounds for union-split-find related problems on random access machines. In Proceedings of the 26th Annual ACM Symposium on Theory of Computing, pages 625–634, 1994.

[Mil99] P. B. Miltersen. Cell probe complexity — a survey. In Pre-conference workshop on Advances in Data Structures at the 19th conference on Foundations of Software Technology and Theoretical Computer Science (invited lecture), 1999. Also available from http://www.daimi.au.dk/~bromille/Papers/survey3.ps.

[MNSW98] P. B. Miltersen, N. Nisan, S. Safra, and A. Wigderson. On data structures and asymmetric communication complexity. Journal of Computer and System Sciences, 57(1):37–49, 1998.

[NC00] M. Nielsen and I. Chuang. Quantum Computation and Quantum Information. Cambridge University Press, 2000.

[New91] I. Newman. Private vs common random bits in communication complexity. Information Processing Letters, 39:67–71, 1991.

[Nis93] N. Nisan. The communication complexity of threshold gates. In Combinatorics, Paul Erdős is Eighty (Vol. 1), pages 301–315. Janos Bolyai Mathematical Society, Budapest, Hungary, 1993.

[RSV00] J. Radhakrishnan, P. Sen, and S. Venkatesh. The quantum complexity of set membership. In Proceedings of the 41st Annual IEEE Symposium on Foundations of Computer Science, pages 554–562, 2000. Full version to appear in Special issue of Algorithmica on Quantum Computation and Quantum Cryptography. Also quant-ph/0007021.

[Yao77] A. C-C. Yao. Probabilistic computations: towards a unified measure of complexity. In Proceedings of the 18th Annual IEEE Symposium on Foundations of Computer Science, pages 222–227, 1977.
Appendix

A Proof of Lemma 6

In this section, we prove Lemma 6. The proof is somewhat similar to the proof of Lemma 4.4 in Klauck et al. [KNTZ01], but much simpler since we are in the classical setting. We first state a theorem which will be required in the proof of Lemma 6. The quantum version of this theorem, called the “average encoding theorem”, has been proved by Klauck et al. [KNTZ01], who also use it in the proof of Lemma 4.4 in their paper. Intuitively speaking, the theorem says that if the mutual information between a (classical) random variable and its (classical) encoding is small, then the various probability distributions on the codewords are close to the average probability distribution on the codewords. Below, the notation $\|\sigma - \rho\|_1$ stands for the total variation distance ($\ell_1$ distance) between probability distributions $\sigma$ and $\rho$ over the same sample space.

**Theorem 4 (Average encoding, classical version, [KNTZ01])** Let $X$ be a classical random variable, which takes value $x$ with probability $p_x$, and $M$ be a classical randomised encoding $x \mapsto \sigma_x$ of $X$, where $\sigma_x$ is a probability distribution over the sample space of codewords. The probability distribution of the average encoding is $\sigma_{\Delta} = \sum_x p_x \sigma_x$. Then

$$\sum_x p_x \|\sigma_x - \sigma\|_1 \leq \sqrt{(2 \ln 2) I(X : M)}$$

We now proceed to the proof of Lemma 6.

**Lemma 6** Suppose $f : E \times F \to G$ is a function. Let $D$ be a probability distribution on $E \times F$, and $P$ be a $[t, 0, l_1, \ldots, l_t]^A$ private coin classical randomised protocol for $f$. Let $X$ stand for the classical random variable denoting Alice’s input (under distribution $D$), $M$ be the first message of Alice in the protocol $P$, and $I(X : M)$ denote the mutual information between $X$ and $M$ under distribution $D$. Then there exists a $[t-1, 0, l_2, \ldots, l_t]^B$ public coin classical randomised protocol $Q$ for $f$, such that

$$\epsilon_Q^D \leq \epsilon_P^D + \frac{1}{2}((2 \ln 2) I(X : M))^{1/2}$$

**Proof:** We first give an overview of the plan of the proof, before getting down to the details. The proof proceeds in stages.
Stage 1: Starting from \( P \), we construct a \([t, l_1, \ldots, l_t]^A\) private coin protocol \( P' \), where the first message is independent of Alice’s input, and \( \epsilon_D^{P'} \leq \epsilon_D^P + (1/2)((2 \ln 2)I(X : M))^{1/2} \). The important idea in this step is to first generate Alice’s message using a new private coin without “looking” at her input, and after that, to adjust Alice’s old private coin in a suitable manner so as to be consistent with her message and input.

Stage 2: Suppose the coin tosses in \( P' \) were done in public. Then Bob can generate the first message of \( P' \) himself, as it is independent of Alice’s input. Doing this gives us a \([t-1, l_2, \ldots, l_t]^B\) public coin protocol \( Q \), such that \( \epsilon_{x,y}^{Q} = \epsilon_{x,y}^{Q'} \) for every \((x, y) \in E \times F\).

The protocol \( Q \) of Stage 2 is our desired \([t-1, l_2, \ldots, l_t]^B\) public coin classical randomised protocol for \( f \). We have
\[
\epsilon_D^{Q} = \epsilon_D^{P'} \leq \epsilon_D^P + \frac{1}{2}((2 \ln 2)I(X : M))^{1/2}
\]

We now give the details of the proof. Let \( \sigma_x \) be the probability distribution of the first message \( M \) of protocol \( P \) when Alice’s input \( X = x \). Let \( Y \) denote Bob’s input register. Define \( \sigma \triangleq \sum_x p_x \sigma_x \), where \( p_x \) is the (marginal) probability of \( x \) under distribution \( D \). \( \sigma \) is the probability distribution of the average first message under distribution \( D \). By Theorem 4, we get that
\[
\sum_x p_x \| \sigma_x - \sigma \|_1 \leq \sqrt{(2 \ln 2)I(X : M)}
\]

For \( x \in E \) and an instance \( m \) of the first message of Alice, let \( q_r^{xm} \) denote the (conditional) probability that the private coin toss of Alice results in \( r \), given that Alice’s input is \( x \) and her first message in protocol \( P \) is \( m \). Let \( \sigma(m | x) \) denote the probability that the first message of Alice in \( P \) is \( m \), given that her input is \( x \). Let \( \sigma(m) \) denote the probability of \( m \) occurring in the average first message of Alice. Then, \( \sigma(m) = \sum_x p_x \sigma(m | x) \).

Stage 1: We construct a \([t, 0, l_1, \ldots, l_t]^A\) private coin classical randomised protocol \( P' \) for \( f \) with average error under distribution \( D \), \( \epsilon_D^{P'} \leq \epsilon_D^P + (1/2)((2 \ln 2)I(X : M))^{1/2} \), and where the probability distribution of the first message is independent of the input to Alice. Suppose Alice is given \( x \in E \) and Bob is given \( y \in F \). Alice tosses a fresh private coin to pick \( m \) with probability \( \sigma(m) \). She then sets her old private coin to \( r \) with probability \( q_r^{xm} \). (If in \( P \), message \( m \) cannot occur when Alice’s input is \( x \), we say that protocol \( P' \) gives an error if such a thing happens.) After this, Alice and Bob behave as in protocol \( P \) (henceforth, Alice ignores the new private coin which she had tossed to generate her first message \( m \)). Hence in \( P' \), the probability distribution of the first message is independent of Alice’s input.

Let us now compare the situations in protocols \( P \) and \( P' \) when Alice’s input is \( x \), Bob’s input is \( y \), Alice has finished tossing her private coins, but no communication has taken place as yet. In protocol \( P \), the probability that Alice’s private coin toss results in \( r \) is
\[
\sum_m \sigma(m | x) q_r^{xm}
\]
In protocol $P'$, the probability that Alice’s (old) private coin toss results in $r$ is
\[ \sum_m \sigma(m) q_r^m. \]
Thus, the $\ell_1$ distance between the probability distributions on Alice’s (old) private coin toss is
\[
\sum_r \left| \sum_m q_r^m (\sigma(m | x) - \sigma(m)) \right| \\
\leq \sum_r \sum_m q_r^m |\sigma(m | x) - \sigma(m)| \\
= \sum_m \left( |\sigma(m | x) - \sigma(m)| \sum_r q_r^m \right) \\
= \sum_m |\sigma(m | x) - \sigma(m)| \\
= \|\sigma_x - \sigma\|_1
\]
Hence, the error probability of $P'$ on input $x, y$
\[ \epsilon_{x, y}^{P'} \leq \epsilon_{x, y}^P + \frac{1}{2} \|\sigma_x - \sigma\|_1 \]
Let $q_{xy}$ be the probability that $(X, Y) = (x, y)$ under distribution $D$. Then, the average error of $P'$ under distribution $D$, $\epsilon_{D}^{P'}$, is bounded by
\[
\epsilon_{D}^{P'} = \sum_{x, y} q_{xy} \epsilon_{x, y}^{P'} \\
\leq \sum_{x, y} q_{xy} \left( \epsilon_{x, y}^P + \frac{1}{2} \|\sigma_x - \sigma\|_1 \right) \\
= \epsilon_{D}^P + \frac{1}{2} \sum_x p_x \|\sigma_x - \sigma\|_1 \\
\leq \epsilon_{D}^P + \frac{1}{2} ((2\ln 2) I(X : M))^{1/2}
\]
The last inequality follows from the “average encoding theorem” (Theorem 4).

**Stage 2:** We now construct our desired $[t - 1, 0, l_2, \ldots, l_t]^B$ public coin classical randomised protocol $Q$ for $f$ with $\epsilon_{D}^{Q} = \epsilon_{D}^{P'}$. Suppose all the coin tosses of Alice and Bob in $P'$ were done publicly before any communication takes place. Now there is no need for the first message from Alice to Bob, because Bob can reconstruct the message by looking at the public coin tosses. This gives us the protocol $Q$, and trivially
\[ \epsilon_{D}^{Q} = \epsilon_{D}^{P'} \leq \epsilon_{D}^{P} + \frac{1}{2} ((2\ln 2) I(X : M))^{1/2} \]
This completes the proof of Lemma 6. ■
B Proof of Lemma 4

In this section, we prove Lemma 4. We first start with the definition of the trace norm of linear operators, then state three theorems which will be required in the proof of Lemma 4, and after that, we finally present the proof of Lemma 4.

For a linear operator $A$ on a finite dimensional Hilbert space, the trace norm of $A$ is defined as $\|A\|_t = \operatorname{Tr} \sqrt{A^* A}$. The following fundamental theorem (see [AKN98]) shows that the trace distance between two density matrices $\rho_1, \rho_2$, $\|\rho_1 - \rho_2\|_t$, bounds how well one can distinguish between $\rho_1, \rho_2$ by a measurement.

**Theorem 5 ([AKN98])** Let $\rho_1, \rho_2$ be two density matrices on the same Hilbert space. Let $M$ be a general measurement (i.e. a POVM), and $M\rho_i$ denote the probability distributions on the (classical) outcomes of $M$ got by performing measurement $M$ on $\rho_i$. Let the $\ell_1$ distance between $M\rho_1$ and $M\rho_2$ be denoted by $\|M\rho_1 - M\rho_2\|_1$. Then

$$\|M\rho_1 - M\rho_2\|_1 \leq \|\rho_1 - \rho_2\|_t$$

In the proof of Lemma 4, we will need the following “average encoding theorem” of Klauck et al. [KNTZ01]. Intuitively speaking, it says that if the mutual information between a classical random variable and its quantum encoding is small, then the various quantum “codewords” are close to the “average codeword”.

**Theorem 6 (Average encoding, quantum version, [KNTZ01])** Suppose $X, Q$ are two disjoint quantum systems, where $X$ is a classical random variable, which takes value $x$ with probability $p_x$, and $Q$ is a quantum encoding $x \mapsto \sigma_x$ of $X$. Let the density matrix of the average encoding be $\sigma = \sum_x p_x \sigma_x$. Then

$$\sum_x p_x \|\sigma_x - \sigma\|_t \leq \sqrt{(2 \ln 2) I(X : Q)}$$

We will also need the following “local transition theorem” of Klauck et al. [KNTZ01].

**Theorem 7 (Local transition, [KNTZ01])** Let $\rho_1, \rho_2$ be two mixed states with support in a Hilbert space $\mathcal{H}$, $\mathcal{K}$ any Hilbert space of dimension at least the dimension of $\mathcal{H}$, and $|\phi_i\rangle$ any purifications of $\rho_i$ in $\mathcal{H} \otimes \mathcal{K}$. Then, there is a local unitary transformation $U$ on $\mathcal{K}$ that maps $|\phi_2\rangle$ to $|\phi'_2\rangle = (I \otimes U)|\phi_2\rangle$ ($I$ is the identity operator on $\mathcal{H}$) such that

$$\|\phi_1 \langle \phi_1| - |\phi'_2\rangle \langle \phi'_2|\|_t \leq 2\sqrt{\|\rho_1 - \rho_2\|_t}$$

We now proceed to the proof of Lemma 4. The proof is similar to the proof of Lemma 4.4 in [KNTZ01], but with a careful accounting of “safe” overheads in the messages communicated by Alice and Bob.

**Lemma 4** Suppose $f : E \times F \to G$ is a function. Let $D$ be a probability distribution on $E \times F$, and $P$ be a $[t,c,l_1,\ldots,l_t]^A$ safe coinless quantum protocol for $f$. Let $X$ stand for
the classical random variable denoting Alice’s input (under distribution $D$), $M$ be the first message of Alice in the protocol $P$, and $I(X : M)$ denote the mutual information between $X$ and $M$ under distribution $D$. Then there exists a $[t − 1, c + l_1, l_2, . . . , l_l]^{B}$ safe coinless quantum protocol $Q$ for $f$, such that

$$
\epsilon_D^Q \leq \epsilon_D^P + ((2\ln 2)I(X : M))^{1/4}
$$

**Proof:** We first give an overview of the plan of the proof, before getting down to the details. The proof proceeds in stages. We remark on the similarities between the stages in the quantum proof, and the stages in the classical proof (Lemma [3]). Stages 1A and 1B of the quantum proof together correspond to Stage 1 of the classical proof, and Stages 2A and 2B of the quantum proof together correspond to Stage 2 of the classical proof.

**Stage 1A:** Starting from the $[t, c, l_1, . . . , l_l]^A$ safe coinless protocol $P$, we construct a $[t, c, l_1, . . . , l_l]^A$ safe coinless protocol $\tilde{P}$ with $\epsilon_{x,y}^{\tilde{P}} = \epsilon_{x,y}^{P}$ for every $(x, y) \in E \times F$. $\tilde{P}$ contains an extra “secure” copy of Alice’s input $x \in E$, but is otherwise the same as $P$.

**Stage 1B:** Starting from $\tilde{P}$, we construct a $[t, c, l_1, . . . , l_l]^A$ safe coinless protocol $P'$, where the first message is independent of Alice’s input, and $\epsilon_D^{P'} \leq \epsilon_D^{P} + ((2\ln 2)I(X : M))^{1/4}$. The important idea in this step is to first generate Alice’s average message (which is independent of her input), and after that, use the extra “secure” copy of Alice’s input $x$ to apply a unitary transformation $U_x$ on some of her qubits without touching her message. $U_x$ is used to adjust Alice’s state in a suitable manner so as to be consistent with her input and message. This “adjustment” step requires the use of the “local transition theorem” (Theorem [4]).

**Stage 2A:** Since in $P'$ the first message is independent of Alice’s input, Bob can generate it himself. But it is also necessary to achieve the correct entanglement between Alice’s qubits and the first message. Bob does this by first sending a safe message of $l_1 + c$ qubits. Alice then applies a unitary transformation $V_x$ on some of her qubits, using the extra “secure” copy of her input $x$, to achieve the correct entanglement. The existence of such a $V_x$ follows from Theorem [7]. Doing all this gives us a $[t + 1, c + l_1, 0, 0, l_2, . . . , l_l]^B$ safe coinless protocol $Q'$, such that $\epsilon_{x,y}^{Q'} = \epsilon_{x,y}^{P'}$ for every $(x, y) \in E \times F$.

**Stage 2B:** Since the first message of Alice in $Q'$ is zero qubits long, Bob can concatenate his first two messages, giving us a $[t − 1, c + l_1, l_2, . . . , l_l]^B$ safe coinless protocol $Q$, such that $\epsilon_{x,y}^Q = \epsilon_{x,y}^{Q'}$ for every $(x, y) \in E \times F$. The technical reason behind this is that unitary transformations on disjoint sets of qubits commute.

The protocol $Q$ of Stage 2B is our desired $[t − 1, c + l_1, l_2, . . . , l_l]^B$ safe coinless quantum protocol for $f$. We have

$$
\epsilon_D^Q = \epsilon_D^{Q'} = \epsilon_D^{P'} \leq \epsilon_D^{P} + ((2\ln 2)I(X : M))^{1/4} = \epsilon_D^{P} + ((2\ln 2)I(X : M))^{1/4}
$$
We now give the details of the proof. Let \( \sigma_x \) be the density matrix of the first message \( M \) of protocol \( P \) when Alice’s input \( X = x \). Let \( Y \) denote Bob’s input register. Define \( \sigma \triangleq \sum_x p_x \sigma_x \), where \( p_x \) is the (marginal) probability of \( x \) under distribution \( D \). \( \sigma \) is the density matrix of the average first message under distribution \( D \). By the “security” of \( P \), \( \sigma \) is also the density matrix of the first message when \( |\psi\rangle \) is fed to Alice’s input register \( X \), where \( |\psi\rangle \triangleq \sum_x \sqrt{p_x} |x\rangle \). By Theorem \([1]\), we get that
\[
\sum_x p_x \|\sigma_x - \sigma\|_1 \leq \sqrt{(2 \ln 2) I(X : M)}
\]

Stage 1A: We first construct a \([t, c, l_1, \ldots, l_t]^A\) safe coinless quantum protocol \( \tilde{P} \) for \( f \) such that \( \varepsilon_{x}^{\tilde{P}} = \varepsilon_{x}^{P} \), for every \((x, y) \in E \times F\). Let \( X \) be Alice’s input register in \( P \). In \( \tilde{P} \), Alice has an additional register \( C \), and the input \( x \) to Alice is fed to register \( C \), instead of \( X \). \( X \) is initialised to \( |0\rangle \) in \( \tilde{P} \). In protocol \( \tilde{P} \), Alice first copies the contents of \( C \) to \( X \). After that, things in \( \tilde{P} \) proceed as in \( P \). Register \( C \) is not touched henceforth, and thus, \( C \) holds an extra “secure” copy of \( x \) throughout the run of protocol \( \tilde{P} \).

Stage 1B: We now construct a \([t, c, l_1, \ldots, l_t]^A\) safe coinless quantum protocol \( P' \) for \( f \) with average error under distribution \( D \), \( \varepsilon_{D}^{P'} \leq \varepsilon_{D}^{P} + ((2 \ln 2) I(X : M))^{1/4} \), and where the density matrix of the first message is independent of the input \( x \) to Alice. Alice is given \( x \in E \) and Bob is given \( y \in F \). Consider the situation in \( \tilde{P} \) after the first message has been prepared by Alice, but before it is sent to Bob. Let register \( A \) denote Alice’s qubits excluding the message qubits \( M \) and the qubits of the “secure” copy \( C \) (in particular, \( A \) includes the qubits of register \( X \)). Without loss of generality, one can assume that register \( A \) has at least \( l_1 + c \) qubits, because one can initially pad up \( A \) with ancilla qubits set to \( |0\rangle \). Let \( |x\rangle_C \otimes |\theta_x\rangle_{AM} \) be the state vector of \( CAM \) in \( \tilde{P} \) at this point, where the subscripts denote the registers. \(|\theta_x\rangle_{AM} \) is a purification of \( \sigma_x \). We note that \(|\theta_x\rangle \) is also the state vector of \( AM \) in protocol \( P \) at this point. \( P' \) is similar to \( \tilde{P} \) except for the following. Alice puts \( |\psi\rangle \) in register \( X \) (instead of copying \( C \) to \( X \) as in \( \tilde{P} \)) to create the first message in register \( M \) with density matrix \( \sigma \). \( AM \) now contains a purification \(|\theta\rangle \) of \( \sigma \). Then Alice applies a unitary transformation \( U_x \) depending upon \( x \) (which is available “securely” in register \( C \)) on \( A \), so that \(|\theta'_x\rangle_{AM} \triangleq (U_x \otimes I)|\theta_x\rangle_{AM} \) is “close” to \(|\theta_x\rangle_{AM} \). Here \( I \) stands for the identity transformation on \( M \). Theorem \([2]\) tells us that there exists a unitary transformation \( U_x \) on \( A \) such that
\[
\|\langle \theta_x | \theta_x \rangle - \langle \theta'_x | \theta'_x \rangle\|_1 \leq 2\sqrt{\|\sigma_x - \sigma\|_1}
\]
Thus, \(|x\rangle_C \otimes |\theta'_x\rangle_{AM} \) is the state vector of \( CAM \) in \( P' \) after the application of \( U_x \). Alice then sends register \( M \) to Bob and after this, Alice and Bob behave as in \( \tilde{P} \). Application of \( U_x \) does not affect the density matrix of register \( M \), which continues to be \( \sigma \). Hence in \( P' \), the density matrix of the first message is independent of Alice’s input.

Let us now compare the situations in protocols \( \tilde{P} \) and \( P' \) when Alice’s input is \( x \), Bob’s input is \( y \), Alice has prepared her first message, but no communication has taken place as yet. At this point, in both protocols \( \tilde{P} \) and \( P' \), the state vector of Bob’s qubits is the
same, and in tensor with the state vector of Alice’s qubits. Let $B$ denote the register of Bob’s qubits (including his input qubits $Y$) and let $|\eta\rangle_B$ denote the state vector of $B$ at this point. Hence the global state of protocol $\tilde{P}$ at this point is $|x\rangle_C \otimes |\theta_x\rangle_{AM} \otimes |\eta\rangle_B$, and the global state of $P'$ is $|x\rangle_C \otimes |\theta'_x\rangle_{AM} \otimes |\eta\rangle_B$. Therefore, the global states of protocols $\tilde{P}$ and $P'$ at this point differ in trace distance by the quantity

$$
\| |x\rangle \langle x| \otimes |\theta_x\rangle \langle \theta_x| \otimes |\eta\rangle \langle \eta| - |x\rangle \langle x| \otimes |\theta'_x\rangle \langle \theta'_x| \otimes |\eta\rangle \langle \eta| \|_t = \| |\theta_x\rangle \langle \theta_x| - |\theta'_x\rangle \langle \theta'_x| \|_t \leq 2\sqrt{\|\sigma_x - \sigma\|_t}
$$

Using Theorem 5, we see that the error probability of $P'$ on input $x, y$

$$
\epsilon_{x,y}^{P'} \leq \epsilon_{x,y}^{\tilde{P}} + \frac{1}{2}\| |x\rangle \langle x| \otimes |\theta_x\rangle \langle \theta_x| \otimes |\eta\rangle \langle \eta| - |x\rangle \langle x| \otimes |\theta'_x\rangle \langle \theta'_x| \otimes |\eta\rangle \langle \eta| \|_t \leq \epsilon_{x,y}^{\tilde{P}} + \sqrt{\|\sigma_x - \sigma\|_t}
$$

Let $q_{xy}$ be the probability that $(X, Y) = (x, y)$ under distribution $D$. Then, the average error of $P'$ under distribution $D$, $\epsilon_D^{P'}$, is bounded by

$$
\epsilon_D^{P'} = \sum_{x,y} q_{xy} \epsilon_{x,y}^{P'} 
\leq \sum_{x,y} q_{xy} \left( \epsilon_{x,y}^{\tilde{P}} + \sqrt{\|\sigma_x - \sigma\|_t} \right) 
\leq \epsilon_D^{\tilde{P}} + \sqrt{\sum_{x,y} q_{xy} \|\sigma_x - \sigma\|_t} 
= \epsilon_D^{\tilde{P}} + \sqrt{\sum_x p_x \|\sigma_x - \sigma\|_t} 
\leq \epsilon_D^{\tilde{P}} + ((2\ln 2)I(X : M))^{1/4}
$$

For the second inequality above, we use the concavity of the square root function. The last inequality follows from the “average encoding theorem” (Theorem 5).

**Stage 2A:** We now construct a $[t+1, c+l_1, 0, 0, l_2, \ldots, l_t]^B$ safe coinless quantum protocol $Q'$ for $f$ with $\epsilon_{x,y}^{Q'} = \epsilon_{x,y}^{P'}$, for all $(x, y) \in E \times F$. Alice is given $x \in E$ and Bob is given $y \in F$. The protocol $Q'$ will be constructed from $P'$. The input $x$ is fed to register $C$ of Alice, and the input $y$ is fed to register $Y$ of Bob. Let register $G$ denote all the qubits of register $A$, except the last $l_1 + c$ qubits. In protocol $Q'$ the registers initially in Alice’s possession are $C$ and $G$, and the registers initially in Bob’s possession are $B$, $M$, and a new register $R$, where $R$ is $l_1 + c$ qubits long. The qubits of $G$ are initially set to $|0\rangle$. Bob first prepares the state vector $|\eta\rangle$ in register $B$ as in protocol $P'$. He then constructs a canonical purification of $\sigma$ in registers $MR$. The density matrix of $M$ is $\sigma$. Bob then sends $R$ to Alice. The density matrix of $R$ is independent of the inputs $x, y$ (in fact, if the canonical purification in $MR$ is the Schmidt purification, then the density matrix of $R$ is also $\sigma$). After receiving $R$, Alice treats $GR$ as the register $A$ in the remainder of the protocol. $AM$ now contains a purification of $\sigma$. Alice applies a unitary transformation $V_x$ depending upon $x$ (which is available “securely” in register $C$) on $A$, so that the state
vector of \( AM \) becomes \( |\theta'_x\rangle_{AM} \). The existence of such a \( V_x \) follows from Theorem 7. At this point, the global state vector (over all the qubits of Alice and Bob) in \( Q' \) is the same as the global state vector in \( P' \) viz. \( |x\rangle_C \otimes |\theta'_x\rangle_{AM} \otimes |\eta\rangle_B \). Bob now treats register \( M \) as if it were the first message of Alice in \( P' \), and proceeds to compute his response \( N \) of length \( l_2 \). Bob sends \( N \) to Alice and after this protocol \( Q' \) proceeds as in \( P' \). In \( Q' \) Bob starts the communication, the communication goes on for \( t + 1 \) rounds, the first message of Bob of length \( l_1 + c \) (i.e. register \( R \)) is a safe message, and the first message of Alice is zero qubits long.

**Stage 2B:** We finally construct a \([t-1, c+l_1, l_2, \ldots, l_t]^B\) safe coinless quantum protocol \( Q \) for \( f \) with \( \epsilon^Q_{x,y} = \epsilon^{Q'}_{x,y} \), for all \((x, y)\) \(\in E \times F\). In protocol \( Q \), Bob (after doing the same computations as in \( Q' \)) first sends as a single message register \( RN \) of length \((l_1 + c) + l_2\), and after that Alice applies \( V_x \) on \( A \) followed by her appropriate unitary transformation on \( AN \) (the unitary transformation of Alice in \( Q' \) on her qubits \( AN \) after she has received the first two messages of Bob). At this point, the global state vector (over all the qubits of Alice and Bob) in \( Q \) is the same as the global state vector in \( Q' \), since unitary transformations on disjoint sets of qubits commute. After this, things in \( Q \) proceed as in \( Q' \). In protocol \( Q \) Bob starts the communication, the communication goes on for \( t - 1 \) rounds, and the first message of Bob of length \((l_1 + c) + l_2\) contains a safe overhead (the register \( R \)) of \( l_1 + c \) qubits.

This completes the proof of Lemma 4. \(\blacksquare\)