On \( m \)-quasiinvariants of a Coxeter group

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Abstract

Let \( W \) be a finite Coxeter group in a Euclidean vector space \( V \), and \( m \) a \( W \)-invariant \( \mathbb{Z}_{+} \)-valued function on the set of reflections in \( W \). Chalyh and Veselov introduced in [CV] an interesting algebra \( Q_m \), called the algebra of \( m \)-\textit{quasiinvariants} for \( W \), such that \( \mathbb{C}[V]^W \subseteq Q_m \subseteq \mathbb{C}[V] \), \( Q_0 = \mathbb{C}[V] \), \( Q_m \supseteq Q_m' \) if \( m \leq m' \), and \( \cap_m Q_m = \mathbb{C}[V]^W \). Namely, \( Q_m \) is the algebra of quantum integrals of the rational Calogero-Moser system with coupling constants \( m \). The algebra \( Q_m \) has been studied in [CV], [VSC], [FeV] and [FV]. In particular, in [FV], Feigin and Veselov proposed a number of interesting conjectures concerning the structure of \( Q_m \), and verified them for dihedral groups and constant functions \( m \). Our goal is to prove some of these conjectures in the general case.

1 Definitions and main results

We recall some definitions from [FV].

Consider a real Euclidean space \( V \) of dimension \( n \). We will often identify \( V \) and \( V^* \) using the inner product on \( V \).

Let \( W \) be a finite Coxeter group, i.e. a finite group generated by reflections of \( V \). Let \( N = |W| \). Let \( \Sigma \) be the set of reflections in \( W \), and \( \Pi_s \) be the reflection hyperplane for a reflection \( s \). Let \( m : \Sigma \to \mathbb{Z}_{\geq 0} \), \( s \mapsto m_s \), be a \( W \)-invariant function (called the multiplicity function). A complex polynomial \( q \) on \( V \) is said to be an \( m \)-\textit{quasiinvariant} (under \( W \)) if, for each \( s \in \Sigma \), the function \( x \mapsto q(x) - q(sx) \) vanishes up to order \( 2m_s + 1 \) at the hyperplane \( \Pi_s \). Such polynomials form a graded subalgebra in the graded algebra \( \mathbb{C}[V] = \bigoplus_{i \geq 0} \mathbb{C}[V](i) \), which will be denoted by \( Q_m \). It is obvious that \( Q_m \) contains as a subalgebra the ring \( \mathbb{C}[V]^W \) of invariant polynomials. We denote by \( I_m \) the ideal in \( Q_m \) generated by the augmentation ideal in \( \mathbb{C}[V]^W \). This is a graded ideal in \( Q_m \).

The following two theorems, conjectured in [FV], are two of the main results of this paper.

Let \( T \) be any graded complement of \( I_m \) in \( Q_m \).

**Theorem 1.1.** \( Q_m \) is a free module over \( \mathbb{C}[V]^W \), of rank \( N \). More specifically, the multiplication mapping defines a graded isomorphism \( \mathbb{C}[V]^W \otimes T \to Q_m \). In particular, \( \dim(Q_m/I_m) = \text{codim}(I_m) = N \).

Consider now the \( N \)-dimensional graded algebra \( R_m = Q_m/I_m \). Let \( d = \sum_{s \in \Sigma} (2m_s + 1) \).

**Theorem 1.2.** (i) The space \( R_m(d) \) is one dimensional.

(ii) (Poincare duality). The multiplication mapping \( R_m(j) \times R_m(d - j) \to R_m(d) \) is a nondegenerate pairing for any \( j \). In particular, the Poincare polynomial \( P_{R_m}(t) \) is a palindromic polynomial of degree \( d \) (i.e. \( P_{R_m}(t^{-1}) = t^{-d}P_{R_m}(t) \)), and the algebra \( R_m \) is Gorenstein.

(iii) The algebra \( Q_m \) is Gorenstein.
The proofs of Theorem 1.1 and Theorem 1.2 are given in the next few sections.

Remarks. (i) For \( m = 0 \), the quasiinvariance condition is vacuous, so \( Q_m = \mathbb{C}[V] \).
Thus, for \( m = 0 \) Theorem 1.1 reduces to the Chevalley theorem, which claims that \( \mathbb{C}[V] \) is free over \( \mathbb{C}[V]^W \). Therefore, Theorem 1.1 is an \( m \)-version of the Chevalley theorem. (We note, however, that our proof of Theorem 1.1 makes use of the Chevalley theorem, so we do not obtain a new proof of the Chevalley theorem). Theorem 1.2 for \( m = 0 \) is also well known, and is due to Steinberg.

(ii) If \( W \) is a Weyl group, this theorem has a topological interpretation, since in this case \( R_m \) is the cohomology algebra of the flag variety for the corresponding complex semisimple Lie group.

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2 The Calogero-Moser quantum integrals

Let us recall some known facts about the quantum Calogero-Moser systems.

Let \( \mu \) be a complex valued \( W \)-invariant function on \( \Sigma \). Let \( H = H(\mu) \) be the differential operator

\[
H(\mu) = \Delta_V - \sum_{s \in \Sigma} \frac{2\mu_s}{(\alpha_s, x)} \partial_{\alpha_s},
\]

where \( \alpha_s \in V^* \) is an eigenvector for \( s \) with eigenvalue \(-1\), and \( \partial_s, \alpha \in V^* \), is the derivation of \( \mathbb{C}[V] \) defined by \( \partial_s(\beta) = (\alpha, \beta), \beta \in V^* \). The operator \( H(\mu) \) is homogeneous, and has degree \(-2\). It is called the Calogero-Moser quantum hamiltonian.

The following result shows that for any \( \mu \), the Calogero-Moser hamiltonian defines a quantum integrable system.

Let \( p_i, i = 1, ..., n \), be a set of algebraically independent homogeneous generators of \( \mathbb{C}[V]^W \), and \( d_i \) be their degrees.

**Theorem 2.3.** \([OP],[Ch]\) For any \( i \), there exists a differential operator \( L_i = L_i(\mu) \) of homogeneity degree \(-d_i\), with principal symbol \( p_i(\xi) \), such that \([L_i, H] = 0\). The operators \( L_i \) are regular outside of the reflection hyperplanes and algebraically independent.

In fact, the operator \( L_i \) with the above properties is unique. It is obtained by evaluating the polynomial \( p_i \) on the Dunkl operators, and then restricting the resulting operator to invariant functions. The algebra generated by the operators \( L_i \) will be denoted by \( A_{\mu} \).

Let \( D \) be the algebra of differential operators on \( V \) with rational coefficients, which are regular outside of the reflection hyperplanes. For generic \( \mu \), the algebra \( A_{\mu} \) is a maximal
commutative subalgebra of $D$. However, for integer $\mu$, additional quantum integrals turn out to exist. They are described by the theorem below which follows from $[CV],[VSC]$; see also $[FN]$.

**Theorem 2.4.** Let $m$ be a nonnegative integer invariant function on $\Sigma$. There exists a unique algebra homomorphism $\phi: Q_m \rightarrow D$ such that $\phi(p_i) = L_i(m)$, and the symbol of the operator $\phi(q)$ is $q$ for any homogeneous polynomial $q$. This homomorphism maps elements of degree $d$ to elements of degree $-d$. The image of $\phi$ is the centralizer of $A_m$. □

We will denote $\phi(q)$ by $L_q(m)$ or shortly by $L_q$. In particular, $L_i = L_{p_i}$. Thus, if $q$ is $W$-invariant, then $L_q(\mu)$ makes sense not only for positive integer, but actually for any complex $\mu$.

**Remark.** It is worth mentioning an explicit formula for $L_q$, due to Berest: if $q$ is homogeneous of degree $r$ then $L_q = c(\text{ad}H(m))^rq$, where $c$ is a constant.

Now let us introduce the $\psi$-function, which plays the main role in this paper.

**Theorem 2.5.** $[CV],[VSC]$ There exists a unique, up to scaling, function $\psi_m(k,x)$ on $V \times V$, of the form $P(k,x)e^{kx}$, where $P$ is a polynomial, such that for any $q \in Q_m$ one has $L^x_q(m)\psi_m(k,x) = q(k)\psi_m(k,x)$. □

The function $\psi_m$ is called the Baker-Akhiezer function. We will denote it simply by $\psi$, assuming that $m$ is fixed. It is clear that $\psi$ is homogeneous in the sense $\psi(tk,x) = \psi(k,tx)$. One can also show that the highest term of $P(k,x)$ is proportional to $\delta_m(k)\delta_m(x)$, where $\delta_m(x) = \prod_{s \in \Sigma}(\alpha_s,x)^{m_s}$ (up to scaling, it is independent on the choice of $\alpha_s$).

**Theorem 2.6.** $[CV],[VSC]$ The function $\psi$ is symmetric under the interchange of $k$ and $x$. In particular, it has the bispectrality property: $L^k_q(\psi(k,x) = q(x)\psi(k,x)$. □

**Theorem 2.7.** $[CV],[VSC]$ The function $\psi(k,x)$ is $m$-quasiinvariant with respect to both variables. □

The following result plays a key role in this paper.

**Theorem 2.8.** $\psi(0,0) \neq 0$.

**Proof.** According to $[DJO]$, integer valued multiplicity functions are nonsingular in the sense of $[DJO]$. This means (see $[DJO]$, p. 247) that there exists a generalized Bessel function $J_m(k,x)$, a holomorphic $W$-invariant in both variables solution of the system of differential equations

$$L^x_i(m)J_m(k,x) = p_i(k)J_m(k,x)$$

This function is unique up to scaling, and can be normalized by the condition $J_m(0,0) = 1$.

Now consider the function $K_m(k,x) = \sum_{w \in W} \psi(k,wx)$. This function is a holomorphic invariant solution of the above system, so it must be proportional to $J_m$. This implies that $\psi(0,0) \neq 0$, as desired. □

In fact, there is an explicit product formula for $\psi(0,0)$, for the normalization of the highest term of $P$ (the polynomial factor in the the $\psi$-function) to be $\delta_m(x)\delta_m(k)$, with all $\alpha_s$ having squared length 2. Such a formula can be deduced from $[DJO]$. However, we will not discuss this formula, and will choose the normalization of $\psi$ such that $\psi(0,0) = 1$. 

3
3 The pairing on $Q_m$

Let us expand $\psi(k, x)$ into a Taylor series. Since $\psi$ is $m$-quasiinvariant with respect to both variables, we can consider $\psi(k, x)$ as an element of $Q_m \hat{\otimes} Q_m$, where $\hat{\otimes}$ is the completed tensor product. Furthermore, because of the homogeneity of $\psi$, we have $\psi = \sum_{j \geq 0} \psi(j)$, where $\psi(j) \in Q_m(j) \otimes Q_m(j)$.

Proposition 3.9. $\psi$ is a nondegenerate tensor, i.e. its left (or right) tensorands span $Q_m$. In other words, the tensorands of $\psi(j)$ span $Q_m(j)$ for all $j \geq 0$.

Proof. Let $Q'_m \subset Q_m$ be the span of left (or right) tensorands of $\psi$. This is a graded subspace of $Q_m$. Let $q_i$ be a homogeneous basis of $Q'_m$. Then we can write $\psi$ in the form $\sum_i q_i(k)q^i(x)$, where $q^i(x)$ is another homogeneous basis of $Q'_m$. Thus, for any $q \in Q_m$, we have

$$q(k)\psi(k, x) = (L_q \psi)(k, x) = \sum_i q_i(k)(L_q q^i)(x).$$

But the function $q(k)\psi(k, x)$ is analytic. Thus, $L_q q_i$ cannot have poles and hence is a polynomial.

Let us now substitute $x = 0$ in the last equality. Since $\psi(k, 0) = \psi(0, 0) = 1$, we get $q(k) = \sum a_i q_i(k)$, where $a_i = (L_q q_i)(0)$. This sum is clearly finite. Thus, $q \in Q'_m$, i.e. $Q'_m = Q_m$, as desired. \qed

This proposition and the fact that $\psi$ is an eigenfunction of $L_q$ has the following corollary, which is also proved in [FV] by another method:

Corollary 3.10. [FV] For any $q \in Q_m$ one has $L_q(Q_m) \subset Q_m$.

Consider now the symmetric bilinear form on $Q_m$ inverse to the element $\psi$. This form is nondegenerate. We will denote it by $(p, q)_m$, or simply by $(p, q)$. The next theorem summarizes the properties of this form.

Theorem 3.11. (i) $(,)$ is $W$-invariant, and $(p, q) = 0$ if $p, q$ are homogeneous of different degrees.

(ii) $(p, q) = (L_q p)(0)$.

(iii) $(p, q) = L_p^{(x)} L_q^{(k)} \psi(x, k)|_{x=k=0}$

(iv) $L_q^* = q$ under $(, )$.

Proof. (i) is clear.

Proof of (ii): Let $q_i$ be a homogeneous basis of $Q_m$, and $q^i$ the dual basis. Then $\psi(k, x) = \sum_i q_i(k)q^i(x)$. Applying $L_q^{(x)}$ to this equation, we get

$$q_i(k)\psi(k, x) = \sum_i q_i(k)(L_q q^i)(x).$$
Substituting \( x = 0 \), we get
\[ q_i(k) = \sum_i q_i(k)(L_q q_i^i)(0), \]
so \( (L_q q^i)(0) = \delta^i_i \), as desired.

Proof of (iii): By (ii), the right hand side is \( \sum_i (p, q^i)(q, q_i) = (p, q) \).

Proof of (iv): by (ii) \( (L_q p_1, p_2) = (L_{q_2} L_q p_1)(0) = (L_{q_2} p_1)(0) = (q p_2, p_1) \). \( \square \)

The above results on the form on \( Q_m \) imply the following. Let \( D_m \) be the algebra generated by \( q, L_q, q \in Q_m \).

**Proposition 3.12.** \( Q_m \) is an irreducible \( D_m \)-module.

**Proof.** First of all, \( D_m \) clearly contains the Euler vector field, so any submodule of \( Q_m \) has to be graded. Thus, it is sufficient to show that for any homogeneous element \( q \in Q_m \), one has \( q \in D_m 1 \) and \( 1 \in D_m q \). But this is clear, since for any homogeneous \( q \in Q_m \) one has \( q = q_1 \), and \( 1 = L_p q \) for \( p \) of degree \( \text{deg}(q) \) such that \( (p, q) = 1 \) (which exists by nondegeneracy of the form). \( \square \)

### 4 Proof of Theorem 1.1

We are ready to prove Theorem 1.1. For this purpose, for any \( k \in V \), define the subspace \( H_m(k) \) of the power series completion \( \mathbb{C}[[V]] \) of \( \mathbb{C}[V] \), which consists of solutions of the differential equations \( L_i f = p_i(k) f, \ i = 1, \ldots, n. \)

**Theorem 4.13.** (essentially, contained in [FV])

(i) \( \dim H_m(k) = N \) for all \( k \).

(ii) \( H_m(k) \) is contained in the power series completion \( \hat{Q}_m \) of \( Q_m \).

(iii) \( H_m(0) \) is a graded subspace of \( Q_m \).

**Proof.** (i) Looking at the symbols of \( L_i \) and using the Chevalley theorem, we conclude that the dimension cannot be more than \( N \) (since this is the dimension of the space of “abstract” solutions of the system, in the sense of differential Galois theory). On the other hand, for generic \( k \), it is easy to see that the functions \( \psi(x, wk), w \in W \), are linearly independent elements of \( H_m(k) \). Thus the dimension is generically (and hence always) greater than or equal to \( N \). Combining the two results, we get that the dimension is exactly equal to \( N \).

(ii) The statement says that elements of \( H_m(k) \) are \( m \)-quasiinvariant. This is clear for generic \( k \) since we showed in the proof of (i) that \( \psi(x, wk) \) is a basis of \( H_m(k) \). Therefore, it is true for all \( k \).

(iii) This is clear, as \( H_m(0) \) is graded and finite dimensional. \( \square \)

**Remark.** The space \( H_m = H_m(0) \) is called in [FV] the space of \( m \)-harmonic polynomials. Now let \( I_m(k) \) be the ideal in \( Q_m \) generated by the polynomials \( p_i - p_i(k) \). In particular, \( I_m(0) = I_m \).
Lemma 4.14. $I_m(k)$ is the orthogonal complement of $H_m(k)$ in $Q_m$ with respect to the form $(,)$.

Proof. This follows from the fact that $L_q^* = q$. □

Corollary 4.15. $Q_m/I_m(k) = H_m(k)^*$ is a flat family of vector spaces, of dimension $N$.

Now let us consider $Q_m$ as a module over $C[V]^W$. The fiber of this module at the point $k$ is $Q_m/I_m(k)$. Since this family is flat, the module $Q_m$ is locally free. But since it is graded, it is freely generated by any local homogeneous generators $t_1, ..., t_N$ at the point $k = 0$. This proves Theorem 1.1.

The proof of Theorem 1.1 we gave also implies the following Corollary, which was conjectured in [FV] :

Corollary 4.16. One has the following identity for Poincare series:

$$P_{Q_m}(t) = \frac{P_{H_m}(t)}{\prod_{i=1}^{m}(1 - t^d_i)}.$$  

In particular, $P_{R_m}(t) = P_{H_m}(t)$.

The polynomial $P_{H_m}$ is calculated in [PeV]. Thus, the Corollary allows one to compute $P_{Q_m}$ and $P_{R_m}$.

5 Some determinants

In this section we will calculate the order of vanishing of some determinants, which will be used later.

Let $\delta_{2m+1}(x) = \prod_s \alpha_s(x)^{2m_s+1}$ be the m-version of the discriminant.

Proposition 5.17. Let $k \in V$. The polyvector $u(k) = \bigwedge_{g \in W} \psi(gk,*) \in \bigwedge^N H_m(k)$ is of the form $\delta_{2m+1}(k)u_*(k)$, where $u_*(k)$ is a nonvanishing section of the line bundle $\Lambda^N H_m(k)$ over $V$.

Proof. It is clear that when $k$ is regular then $u(k) \neq 0$, because in this case $\psi(gk,x)$ form a basis of $H_m(k)$. Thus, it is sufficient to show that $u(k)$ has exactly the prescribed order of vanishing on the reflection hyperplanes.

To show this, let $k_0$ be a generic point of $\Pi_s$, and $v$ be a nonzero vector orthogonal to $\Pi_s$. Define the function

$$\check{\psi}(k_0, x) = \lim_{\epsilon \to 0} \frac{\psi(k_0 + \epsilon v, x) - \psi(k_0 - \epsilon v, x)}{\epsilon^{2m_s+1}}.$$  

This function is obviously nonzero, and is well defined up to normalization (the normalization depends on $v$).
It is easy to see that the functions $\psi(gk_0, x)$ and $\tilde{\psi}(gk_0, x)$, $g \in W/(1, s)$, are linearly independent, and form a basis of $H_m(k_0)$. Therefore, the wedge product

$$u'(k) = \bigwedge_{g \in W/(1, s)} (\psi(gk, x) + \psi(gsk, x)) \wedge \bigwedge_{g \in W/(1, s)} \frac{\psi(gk, x) - \psi(gsk, x)}{\alpha_s(k)^{2m_s+1}}$$

has a nonzero finite limit as $k \to k_0$. But it is clear that $u'(k)$ is a constant multiple of $u(k)/\alpha_s(k)^{N(2m_s+1)/2}$. This implies the required statement. \qed

Let $T$ be a graded linear complement to $I_m$ in $Q_m$. Let $t_i, i = 1, ..., N$, be a homogeneous basis of $T$. Let $k \in V$. Let $A(k)$ be the matrix whose entries are $t_i(gk), i = 1, ..., N, g \in W$.

**Lemma 5.18.**

$$\det A(k) = c\delta_{2m+1}(k)^{N/2},$$

where $c$ is a nonzero constant.

**Proof.** First, note that the pairing $(,) : T \times H_m(k) \to \mathbb{C}$ is nondegenerate for any $k \in V_C$. Indeed, since $T$ is graded, the degeneracy locus of this pairing in the $k$-space is invariant under dilations. Also, this locus is clearly closed. So, if it is nonempty, it must contain zero. But the pairing between $T$ and $H_m(0)$ is nondegenerate by the definition of $T$.

This implies that for any regular point $k \in V_C$, the evaluation map $T \to \mathbb{C}[W \cdot k]$ is an isomorphism (since for $f \in T, f(gk) = (f, \psi(gk, *))$, and $\psi(gk, *)$ is a basis of $H_m(k)$). Thus, $\det A(k)$ is nonzero outside of the reflection hyperplanes in $V_C$. So it suffices to check that $\det A(k)$ has exactly the predicted degree of vanishing on the hyperplanes, i.e. degree $N(2m_s + 1)/2$ on $\Pi_s$.

Let us first check that $\det A(k)$ has degree of vanishing at least $N(2m_s + 1)/2$ on $\Pi_s$. To this end, look at the limit in which $k$ approaches a generic point $k_0$ on a hyperplane $\Pi_s$. Since $t_i$ are quasiinvariants, for any $g \in W/(1, s)$, the difference between $t_i(gk)$ and $t_i(gsk)$ is of the order at least $\alpha_s(k)^{2m_s+1}$ in this limit. This gives the desired lower bound.

Now let us obtain the upper bound. As we mentioned, the pairing $T \times H_m(k) \to \mathbb{C}$ given by $(,)_m$ is nondegenerate. Thus, there exists a basis $f_j = f_j^{(k)}$ of $H_m(k)$ such that $(L_t, f_j)(0) = \delta_t^j$.

Let us express the solutions $\psi(gk, x)$ via this basis. It is clear that $\psi(gk, x) = \sum_j c_j(g)f_j(x)$, where $c_j(g) = (L_t^{(x)}\psi(gk, x)|_{x=0} = t_j(gk)$. Thus,

$$\psi(gk, x) = \sum_j t_j(gk)f_j(x),$$

and

$$u(k) = \det A(k) \cdot \bigwedge_{j=1}^N f_j^{(k)}.$$ 

The second factor is holomorphic in $k$. Thus, the lower upper bound follows from Proposition 5.17. The Lemma is proved. \qed
Corollary 5.19. (proved also in [FeV]) The number $P'_{H_m}(0) = \sum \deg(t_j)$ equals to $(N/2)\sum s(2m_s + 1)$.

Proof. This is obtained from Lemma 5.18 by comparing the degrees of the two sides. □

6 Linear independence theorem and [FV]-conjectures

Let $T, t_i$ be as in the previous section. Recall [FV] that $\delta_{2m+1}(x) \in H_m$. Hence for any $j$, one has $L_t \delta_{2m+1} \in H_m$.

Theorem 6.20. The elements $L_t \delta_{2m+1}$ are linearly independent, and hence form a basis of $H_m$.

Proof. Let $k \in V$ be regular. Consider the function $\delta^{(k)}_{2m+1}(y) = \sum_{h \in W} (-1)^h \psi(hk, y) \delta_{2m+1}(k)$. It is easy to see from quasiinvariance of $\psi$ that this function (as a function of $k$) extends to a holomorphic function on $V_{\mathbb{C}}$. In particular, there exists a limit $\delta^{(0)}_{2m+1}(y) := \lim_{k \to 0} \delta^{(k)}_{2m+1}(y)$, and $\delta^{(0)}_{2m+1}(y)$ is an antisymmetric m-harmonic polynomial. Hence, $\delta^{(0)}_{2m+1}(y) = b\delta_{2m+1}(y)$, where $b \in \mathbb{C}$.

Consider the polyvector

$$\Delta(k) = \bigwedge_{j=1}^{N} L_t \delta^{(k)}_{2m+1} = \bigwedge_{j=1}^{N} L_t \sum_{h \in W} (-1)^h \psi(hk, *) \delta^{(k)}_{2m+1}(k)$$

(the last expression applies to regular $k$ only). We have $\Delta(0) = b^N \bigwedge_{j=1}^{N} L_t \delta_{2m+1}$. Thus, it is sufficient for us to show that $\Delta(0)$ is nonzero.

For regular $k$, we have

$$\Delta(k) = \bigwedge_{j=1}^{N} \sum_{h \in W} (-1)^h t_j(hk) \psi(hk, *) \delta_{2m+1}(k) = \delta_{2m+1}(k)^{-N} \cdot \det(A(k)J) \cdot u(k),$$

where $J_{hh'} = (-1)^h \delta_{hh'}$. Thus, by the lemmas on determinants,

$$\Delta(k) = \pm \delta_{2m+1}(k)^{-N} \delta_{2m+1}(k)^{N/2} (\delta_{2m+1}(k)^{N/2} u_*(k)) = \pm u_*(k).$$

But we have seen above that $u_*(0) \neq 0$. Hence, $\Delta(0) \neq 0$, as desired. □

Remark. In particular, we have shown that $b \neq 0$.

Corollary 6.21. (Conjecture 1 of [FV]). Consider the linear map $\pi_m : Q_m \to H_m$, given by $\pi_m(q) = L_q \delta_{2m+1}$. Then $\pi_m$ is surjective, and the kernel of $\pi_m$ is $I_m$. 8
Proof. The first statement is clear from Theorem 6.20. The second statement follows from the first one, since \( \text{Ker}(\pi_m) \supset I_m \), and \( \text{codim}(I_m) = \dim(H_m) \).

Consider now the bilinear form \(<,>: Q_m \times Q_m \to \mathbb{C} \) defined by \(< p, q > = (p, \pi_m q) = (L_{pq} \delta_{2m+1})(0) \). It is clear that this form is symmetric, and its kernel contains \( I_m \). Thus, this form induces a form \(<,> \) on the algebra \( R_m = Q_m/I_m \), which has homogeneity degree \( d = \sum_s (2m_s + 1) \).

**Proposition 6.22.** The form \(<,> \) on \( R_m \) is nondegenerate.

Proof. It is sufficient to show that the restriction of \(<,> \) to \( T \) is nondegenerate. But this follows from the fact that \( \pi_m : T \to H_m \) is an isomorphism (Corollary 6.21), and that \( (,): T \times H_m \to \mathbb{C} \) is nondegenerate (definition of \( T \)).

**Proof of Theorem 1.2.**

(i) Since the form \(<,> \) has degree \( d \) and is nondegenerate, we have \( \dim R_m(j) = \dim R_m(d-j) \). In particular, \( \dim R_m(d) = 1 \).

(ii) It is clear that \( R_m(d) \) is spanned by the image of \( \delta_{2m+1} \). Indeed this image is clearly nonzero (as \( \delta_{2m+1} \) is the lowest degree antisymmetric element in \( Q_m \), see [FV]), which implies that it spans \( R_m(d) \).

Now, it is easy to see that the multiplication mapping in question \( p, q \mapsto pq \) is proportional to \( p, q \mapsto \delta_{2m+1} \). The nondegeneracy conclusion follows, and the Gorenstein property follows from nondegeneracy.

(iii) The algebra \( Q_m \) is graded and is free as a module over \( \mathbb{C}[p_1, ..., p_n] \). By standard results of commutative algebra (see [Eis], Chapter 21), this implies that \( Q_m \) is Gorenstein if and only if so is \( Q_m/(p_1, ..., p_n) \). But \( (p_1, ..., p_n) = I_m \), so \( Q_m/(p_1, ..., p_n) = R_m \), and we know from (ii) that \( R_m \) is Gorenstein. The theorem is proved.

We conclude this section with an

**Alternative proof of Theorem 1.2.** This proof is based on the following remarkable result due to R. Stanley [St]

**Theorem 6.23.** A positively graded Cohen-Macaulay domain is Gorenstein if and only if its Poincare series \( h(t) \) is a rational function which satisfies the equation \( h(t) = (-1)^n t^l h(t^{-1}) \) for some \( l \) and for \( n \) being the (algebra-geometric) dimension of \( A \).

Let us use this result to prove Theorem 1.2. First of all, we note that by Theorem 1.1, the algebra \( Q_m \) is Cohen-Macaulay (since it is a free module over a smooth subalgebra, see [Eis], Corollary 18.17). It is also positively graded and does not have zero divisors (as it is a subring of \( \mathbb{C}[V] \)).

Next, we cite a result of [FeV]:

**Proposition 6.24.** The polynomial \( P_{H_m} \) is a palindromic polynomial of degree \( d \). That is, \( P_{H_m}(t^{-1}) = t^{-d} P_{H_m}(t) \).
Thus, the same is true about $P_{R_m}$, since by Corollary 1.16, $P_{R_m} = P_{H_m}$. Therefore, it is easy to check that Stanley’s criterion is satisfied, and by Theorem 5.23, $Q_m$ and hence $R_m$ are Gorenstein algebras. This proves Theorem 1.2.

The other results of this section follow easily from this. Indeed, since $p \ast q = \langle p, q \rangle \delta_{2m+1}$, $p, q \in R_m$, we get that $\langle \cdot, \cdot \rangle$ is nondegenerate, and since $\langle p, q \rangle = (p, \pi_m q)$, we get that $\pi_m : R_m \to H_m$ is an isomorphism.

Remark. One can define an obvious analog $Q_m(\Sigma)$ of the algebra $Q_m$ for any arrangement of hyperplanes $\Sigma$ in a Euclidean space, and any positive integer function $m$ on $\Sigma$. However, in general this algebra will not be as nice as $Q_m$.

For example, suppose that $\Sigma$ consists of two lines through 0 in the plane, and $m = 1$. If the lines are perpendicular, we have the Coxeter configuration for the group $W = (\mathbb{Z}/2)^2$, so $Q_m$ has Poincaré series $P(t) = (1-t+t^2)^2$ and is Gorenstein. However, if the lines are not perpendicular, it is not difficult to show that the Poincaré series of $Q_m(\Sigma)$ is given by $\hat{P}(t) = P(t) - t^2$, so

$$\hat{P}(t) = \frac{1 - 2t + 2t^2}{(1-t)^2}.$$ 

It is clear that this function does not satisfy Stanley’s criterion. Therefore, we see that $Q_m(\Sigma)$ is not Gorenstein unless the lines are perpendicular, i.e. unless we have a Coxeter configuration.

It would be interesting to know whether this phenomenon occurs for more general classes of configurations.

7 A counterexample

We have only proved a part of conjectures from [FV]. The rest of the conjectures claim that

(i) (Conjecture 2) the restriction of the map $\pi_m$ to $H_m$ is an isomorphism $H_m \to H_m$.
(ii) (Conjecture 3) $Q_m$ is generated by $H_m$ over $\mathbb{C}[V]^W$.
(iii) (Conjecture 2*) The restriction of the pairing $\langle \cdot, \cdot \rangle$ to $H_m$ is nondegenerate.

These conjectures were proved in [FV] for dihedral groups and constant functions $m$. Unfortunately, it turned out that these conjectures do not hold for general $W$ and $m$. This is demonstrated by the following example.

Let $W$ be of type $B_6 = C_6$, so the roots are $\pm e_i$, $\pm e_i \pm e_j$, and the basic invariant polynomials are $p_j = \sum_{i=1}^6 x_i^{2j}$, $j = 1, \ldots, 6$. Let $m = 1$ for the short roots $\pm e_i$, and $m = 0$ for the long roots $\pm e_i \pm e_j$.

We claim that (i),(ii), and (iii) are not satisfied for these $W$ and $m$. To see this, we let $M$ be the operator $\frac{d^2}{dx^2} - \frac{2}{x} \frac{d}{dx}$. Then the operators $L_{p_i}$ are

$$L_{p_1} = M_1 + \ldots + M_6, \ldots, L_{p_g} = M_1^6 + \ldots + M_6^6$$
(where $M_i$ is $M$ acting in the variable $x_i$)

It is clear that the polynomial $u = x_1^3$ is m-harmonic. It is easy to check that the polynomial

$$v = x_1^3(x_1^2 + \ldots + x_6^2)$$

is also m-harmonic.

But $u, v$ are linearly dependent over $\mathbb{C}[V]^W$. Thus, since $Q_m$ is free over $\mathbb{C}[V]^W$ of rank $N$, and $H_m$ has dimension $N$, it is impossible that $H_m$ generates $Q_m$ over $\mathbb{C}[V]^W$, which disproves (ii).

We also have $H_m \cap I_m \neq 0$. Indeed, consider a lowest degree element $q$ in the orthogonal complement to $\mathbb{C}[V]^W H_m$ in $Q_m$ with respect to $(, )$. The elements $L_p q$ must also be in this complement, so $L_p q = 0$ and $q \in H_m$. On the other hand, $(H_m, q) = 0$, so $q \in I_m$. This disproves (i).

Finally, since $< p, q > = (p, \pi_m q)$ on $H_m$, (iii) fails since $\pi_m$ must have nonzero kernel in $H_m$.

8 The shift operator

It is interesting to point out the relation of the above with the shift operator.

The following theorem is due to Opdam (see [Op]).

Let $\mu : \Sigma \to \mathbb{C}$ be an invariant function.

**Theorem 8.25.** There exists a unique, up to scaling, differential operator $S(m, \mu)$ (called the shift operator) of order $\sum m_s$, such that $S(m, \mu)H(\mu) = H(m + \mu)S(m, \mu)$. One has $S(m, \mu)\mathit{L}_q(\mu) = L_q(m + \mu)S(m, \mu)$ for $W$-invariant polynomials $q$. The operator $S(m, \mu)$ is of degree 0, has polynomial coefficients, and has symbol proportional to $\delta_m(x)\delta_m(\xi)$.

**Example.** Let $W = \mathbb{Z}/2$ acting on $V = \mathbb{R}$ by $x \to -x$. In this case, there is only one number $m$, and one has: $S(1, \mu) = x\partial - (2\mu + 1)$,

$$S(m, \mu) = S(1, \mu + m - 1)\ldots S(1, \mu) = c(x\partial - (2\mu + 2m - 1))\ldots (x\partial - (2\mu + 1)).$$

The connection between the shift operator and the $\psi$-function is given by the following theorem, in which ‘:’ , the normal ordering sign, means that $x$ stands to the left of $\partial$.

**Theorem 8.26.** (see [VSC]) One has $\psi(k,x) = S(m, 0)(x)e^{(k, x)}$. In other words, we have $S(m, 0) = P(x, \partial) :$

The theorem follows from Theorem 8.25 and the fact that the function $\psi(k, x)$ of the form $P(k, x)e^{(k, x)}$ is uniquely determined already by the equations $L_q^{(x)} \psi = q(k)\psi$ for invariant polynomials $q$.

**Corollary 8.27.** $S(m, 0)(\mathbb{C}[V]) = Q_m$. Thus, $P_{Q_m}(t) + P_{\ker S(m, 0)}(t) = (1 - t)^{-n}$.
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