Degenerate processes killed at the boundary of a domain

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August 21, 2024

Abstract

We investigate quasi-stationarity properties of Feller processes that are killed when exiting a relatively compact set. Our main result provides general conditions ensuring that such a process possesses a (possibly non-unique) quasi stationary distribution. Conditions ensuring uniqueness and exponential convergence are discussed. Our conditions are well-suited to the study of degenerate processes, such as nonelliptic diffusions or piecewise deterministic Markov processes (PDMP). The results are applied to stochastic differential equations and we illustrate the application to PDMPs with an example.

Keywords: Absorbed Markov processes, Quasi-Stationary distributions, Exponential mixing, Hypoellipticity, Stochastic differential equations, Potential theory.

2020 Mathematics Subject Classification: 60J25, 60J60, 60F, 60B10, 35H10, 37A25, 37A30, 47A35, 47A75.

1 Introduction

1.1 Context and objectives

This paper investigates certain properties of a Feller Markov process $(X_t)_{t \geq 0}$ living on some metric space $M$, killed when it exits an open, relatively compact set $D$, at time $\tau_D^{out} = \inf\{t \geq 0 : X_t \notin D\}$. 

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We discuss conditions ensuring:

(a) Existence of a \textit{quasi-stationary distribution} (QSD). That is, a probability \( \mu \) on \( D \) such that

\[
P_\mu(X_t \in \cdot \mid \tau_D^{\text{out}} > t) = \mu(\cdot);
\]

(b) Uniqueness of a QSD;

(c) Convergence of the conditional laws \( P_\nu(X_t \in \cdot \mid \tau_D^{\text{out}} > t) \) for any probability measure \( \nu \) on \( D \) toward this QSD, in total variation.

We refer the reader to the survey paper \[49\] and the monograph \[20\] for a comprehensive introduction to the theory of quasi-stationarity.

Probabilistic criteria (based on Lyapunov functions and minorization conditions) implying (c) are given in \[13, 16\]. These apply to elliptic diffusions, as shown in \[11\] and in \[16\] Section 4. Based on the well-known link between QSDs and eigenproblems for the adjoint Dirichlet operator (see e.g. \[49\]), quasi-stationarity has also been widely studied through spectral-theoretic (in a wide sense) arguments (Sturm-Liouville theory \[10\] \[47\] \[43\] \[44\], compactness or quasi-compactness criteria \[30\] \[35\] \[31\], Hilbert-Schmidt theory \[12\], Krein-Rutman theorem \[26\] \[20\], Tychonov’s fixed point theorem \[19\], \( R \)-positive matrices \[25\], orthogonal polynomials \[42\], to cite few references among many). For diffusion processes (assuming that \( D \) is connected), a lot of works made use of minorization-Lyapunov methods \[15\] \[14\] \[23\] \[11\] \[61\] or spectral methods \[53\] \[29\] \[10\] \[34\] \[43\] \[47\] \[48\] or spectral methods \[53\] \[29\] \[10\] \[34\] \[43\] \[47\] \[48\].

However, the treatment of quasi-stationarity beyond elliptic diffusions i.e. non-elliptic diffusions is a more delicate problem. This topic received only few contributions \[31\] \[45\], based on spectral arguments or two-sided estimates. Concerning the conditions in \[13\] \[16\], they rely on certain global parabolic Harnack type inequalities (condition (A2) in \[13\] or (E3) in \[16\]) that are difficult to verify in general for degenerate (non-elliptic) diffusions, although some results are known in particular cases \[28\]. The two references \[31\] \[45\] study the question (c) for strong Feller diffusions. As we will see, the strong Feller assumption, although natural in certain situations, is unnecessarily limiting. It is interesting to understand under what milder conditions and for which processes more general than diffusions the existence property (a) (respectively the uniqueness property (b)) holds while (b) (respectively (c)) does not, a situation which is typical of degenerate processes, i.e. processes which are Feller but not strong Feller.
Other degenerate processes in the previous sense include piecewise deterministic Markov processes (PDMP), which received contributions on quasi-stationary distributions only for specific examples of processes \[13, 2, 12, 13\]. The main objective of our paper is to propose a general framework and to show how it can be used to obtain new results on general degenerate diffusions, and to illustrate how in can be used for some PDMPs.

The remainder of this introduction is devoted to the presentation of our main results in the specific case where \((X_t)\) is a degenerate diffusion process. Then, Section 2 sets up the general framework and addresses points (a) (Theorem 2.9), (b) (Theorem 2.13) and (c) (Theorem 2.16). These results all assume that the Green kernel sends bounded continuous functions on continuous functions vanishing at the boundary of the domain. Existence (a) is proved under mild accessibility conditions; uniqueness (b) assuming irreducibility and the existence of a positive eigenfunction for the Green operator (a property that can be deduced for example from the compactness of the Green operator and Krein-Rutman’s theorem); and convergence (c) assuming the existence of a positive eigenfunction for the Green operator and a small set condition. These results rely on Tychonov’s fixed point theorem for existence and on a version of Harris theorem based on Lyapunov-minoration criteria applied to the \(Q\)-process (the process conditioned to never be absorbed, cf. e.g. \[49\]). These results are applied in Section 3 to prove (among other things) the results stated in Section 1.2 below for diffusions. The proofs rely on criteria for accessibility, and in particular on Stroock and Varadhan’s support theorem, on Feller’s property and the regularity of point of the boundary to prove that the Green kernel sends bounded continuous functions on continuous functions vanishing at the boundary, on Rothschild and Stein’s Hölder estimates \[56\] for solutions to hypoelliptic PDEs and on the existence of continuous solutions to the Dirichlet problem established by Bony \[8\] to prove that the Green kernel is compact, and on estimates on the Green kernel from \[8\] and on the transition densities by Ichihara and Kunita \[37\] to prove small-set-like properties. Section 4 discusses a simple PDMP example which opens perspectives for other applications of our results.

1.2 Description of the results for SDEs
Let \(\mathcal{D} \subset \mathbb{R}^n\) be an open connected set with compact closure \(\overline{\mathcal{D}}\) and boundary \(\partial \mathcal{D} = \overline{\mathcal{D}} \setminus \mathcal{D}\).
Consider the stochastic differential equation on $\mathbb{R}^n$

$$dX_t = S^0(X_t)dt + \sum_{j=1}^{m} S^j(X_t) \circ dB^j_t,$$

(1)

where $\circ$ refers to the Stratonovich stochastic integral, $S^0, S^j, j = 1,\ldots,m$ are smooth vector fields on $\mathbb{R}^n$ and $B^1,\ldots,B^m$ independent Brownian motions. Equivalently, for the reader familiar with Itô’s calculus,

$$dX_t = \left[S^0(x) + \frac{1}{2} \sum_{j=1}^{m} DS^j(x)S^j(x)\right] dt + \sum_{j=1}^{m} S^j(X_t)dB^j_t,$$

where $DS^j(x)$ stands for the Jacobian matrix of $S^j$ at $x$.

As usual, the law of $(X_t)_{t \geq 0}$ when $X_0 = x$ is denoted $\mathbb{P}_x$. For $x, y \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$, we write $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i, \|x\| = \sqrt{\langle x, x \rangle}$ and $d(x, A) = \inf_{y \in A} \|x - y\|$. If $A$ is Borel, we let $\tau_A = \inf\{t \geq 0 : X_t \in A\}$ and $\tau_{out} = \tau_{\mathbb{R}^n \setminus A}$.

Associated to (1) is the Stroock and Varadhan deterministic control system:

$$\dot{y}(t) = S^0(y(t)) + \sum_{j=1}^{m} u^j(t)S^j(y(t))$$

(2)

where the control function $u = (u^1,\ldots,u^m) : \mathbb{R}_+ \to \mathbb{R}^m$, can be chosen to be piecewise continuous. Given such a control function, we let $y(u,x,\cdot)$ denote the maximal solution to (2) starting from $x$ (i.e. such that $y(u,x,0) = x$).

**Definition 1.1** An open set $U \subset \mathbb{R}^n$ is said to be accessible by $\{S^0, (S^j)\}$ from $x \in \mathbb{R}^n$ if there exist a piecewise continuous control $u$ and $t \geq 0$ such that $y(u,x,t) \in U$. If in addition, $y(u,x,s) \in \mathcal{D}$ for all $0 \leq s \leq t$, (in which case $x \in \mathcal{D}$ and $U \cap \mathcal{D} \neq \emptyset$) we say that $U$ is $\mathcal{D}$-accessible by $\{S^0, (S^j)\}$ from $x$.

By abuse of language, we say that a point $y$ is accessible (respectively $\mathcal{D}$-accessible) by $\{S^0, (S^j)\}$ from $x$, provided every neighborhood $U$ of $y$ is accessible (respectively $\mathcal{D}$-accessible) by $\{S^0, (S^j)\}$ from $x$.

Throughout, we will always assume that the following hypothesis is satisfied (or, like in Corollary 1.9, implied by other assumptions).

**Hypothesis (H1)** The set $\mathbb{R}^n \setminus \mathcal{D}$ is accessible by $\{S^0, (S^j)\}$ from all $x \in \mathcal{D}$.
It easily follows from Stroock and Varadhan support theorem and compactness of \( \overline{D} \) (see Propositions 3.1 and 2.3) that, under (H1),

\[
\mathbb{P}_x(\tau^{\text{out}}_D < \infty) = 1
\]

de for all \( x \in \overline{D} \).

The following definition is classical (see e.g. [4]).

**Definition 1.2** A point \( p \in \partial D \) is called regular for \( \mathbb{R}^n \setminus D \) with respect to (1), provided \( \mathbb{P}_p(\tau^{\text{out}}_D = 0) = 1 \).

Our second standing hypothesis is the regularity of points in \( \partial D \).

**Hypothesis (H2)** Every point \( p \in \partial D \) is regular for \( \mathbb{R}^n \setminus \overline{D} \) with respect to (1).

For the usual Brownian motion in \( \mathbb{R}^n \) (i.e. \( S^0 = 0, m = n \), and \( (S^1, \ldots, S^n) \) is an orthonormal basis in \( \mathbb{R}^n \)), a classical condition on \( D \) ensuring (H2) is the so-called *exterior cone condition*: For all \( p \in \partial D \), there exists an open truncated cone at vertex \( p \) contained in \( \mathbb{R}^n \setminus \overline{D} \) (see e.g. [4], Proposition 21.8). If the diffusion is elliptic at \( p \) (meaning that \( S^1(p), \ldots, S^m(p) \) span \( \mathbb{R}^n \)), the exterior cone condition also ensures that \( p \) is regular. In case the diffusion is in divergence form, this follows directly from Aronson’s estimates [59, 46] which allow to bound from below the law of \((X_t)\) by (up to a constant) the law of a Brownian motion. The general case follows by a straightforward application of Girsanov’s theorem.

For degenerate diffusions, a practical condition is the stronger hypothesis (H2’) given below.

**Definition 1.3** We say that \( D \) satisfies the exterior sphere condition at \( p \in \partial D \), if there exists a vector \( v \in \mathbb{R}^n \), called a unit outward normal vector at \( p \), such that \( \|v\| = 1 \) and \( d(p + rv, \overline{D}) = r \) for some \( r > 0 \).

In other words, the open Euclidean ball with center \( p + rv \) and radius \( r \) is contained in \( \mathbb{R}^n \setminus \overline{D} \) and its closure touches \( \overline{D} \) at \( p \). If the condition holds at every \( p \in \partial D \), we simply say that \( D \) satisfies the exterior sphere condition.

**Remark 1.4** Examples of sets verifying the exterior sphere condition are sets with \( C^2 \) boundary and convex sets. In the first case every \( p \in \partial D \) has a unique unit outward normal vector. In the second case, a point \( p \in \partial D \) may have infinitely many unit outward normal vectors (think of a convex polygon). If \( D \subset \mathbb{R}^2 \) is a nonconvex polygon and \( p \in \partial D \) is a vertex at which the interior angle is \( > \pi \), the exterior sphere condition is not satisfied at \( p \) although the exterior cone is.
The following condition (H2’) implies (H2). This will be proved in Section 3 Proposition 3.12.

**Hypothesis (H2’)** \( \mathcal{D} \) satisfies the exterior sphere condition and for each \( p \in \partial \mathcal{D} \), there exist an outward unit normal vector \( v \) at \( p \), and \( i \in \{1, \ldots, m\} \) such that \( \langle S^i(p), v \rangle \neq 0 \).

The next result is an existence theorem. Its proof, in Sections 3.1 and 3.2, shows that it only requires the Lipschitz continuity of \( S^0 \) and the \( C^2 \) regularity of the \( S^j, j \geq 1 \). Note that this result does not require ellipticity nor hypoellipticity condition.

**Theorem 1.5** Assume that:

(i) Hypotheses (H1) and (H2) hold true;

(ii) For some \( \varepsilon > 0 \), the set \( \mathcal{D}_\varepsilon = \{ x \in \mathcal{D} : d(x, \partial \mathcal{D}) > \varepsilon \} \) is \( \mathcal{D} \)-accessible by \( \{S^0, (S^j)\} \) from all \( x \in \mathcal{D} \setminus \partial \mathcal{D} \).

Then, there exists a QSD for \( (X_t) \) on \( \mathcal{D} \).

Simple criteria ensuring (H1) and condition (ii) of Theorem 1.5 will be discussed in Section 3 (Propositions 3.3 and 3.8). In particular, Proposition 3.8 has the following useful consequence that, when \( \partial \mathcal{D} \) is \( C^2 \), (H2’) implies condition (ii) of Theorem 1.5. Therefore,

**Corollary 1.6** Assume that Hypothesis (H1) holds true, \( \partial \mathcal{D} \) is \( C^2 \), and for each \( p \in \partial \mathcal{D} \) one of the vectors \( S^j(p), j = 1, \ldots, m \), is transverse to \( \partial \mathcal{D} \) (i.e. \( S^j(p) \notin T_p \partial \mathcal{D} \)). Then, there exists a QSD for \( (X_t) \) on \( \mathcal{D} \).

If the SDE (1) enjoys certain hypoellipticity properties, related to classical Hörmander conditions as defined below, more can be said. Given a family \( \mathcal{S} \) of smooth vector fields on \( \mathbb{R}^n \) and \( k \in \mathbb{N} \), we let \( [\mathcal{S}]_k \) denote the set of vector fields recursively defined by \( [\mathcal{S}]_0 = \mathcal{S} \), and

\[
[\mathcal{S}]_{k+1} = [\mathcal{S}]_k \cup \{ [Y,Z] : Y, Z \in [\mathcal{S}]_k \}
\]

where \( [Y,Z] \) stands for the Lie bracket of \( Y \) and \( Z \). Set \( [\mathcal{S}] = \bigcup_k [\mathcal{S}]_k \) and \( [\mathcal{S}](x) = \{Y(x) : Y \in [\mathcal{S}]\} \).

**Definition 1.7** A point \( x^* \in \mathbb{R}^n \) is said to satisfy the weak Hörmander condition (respectively the Hörmander condition, respectively the strong Hörmander condition) if \( \{[S^0, \ldots, S^m](x^*)\} \) (respectively

\[
\{S^1(x^*), \ldots, S^m(x^*)\} \cup \{[Y,Z](x^*) : Y, Z \in \{[S^0, \ldots, S^m]\}\},
\]
respectively
\[ \{S^1, \ldots, S^m\}(x^*) \]
spans \( \mathbb{R}^n \).

**Theorem 1.8** Assume that:

(i) Hypotheses (H1) and (H2') hold;

(ii) The weak Hörmander condition is satisfied at every point \( x \in \overline{D} \);

(iii) Every \( y \in D \) is \( D \)-accessible by \( \{S^0, (S^j)\} \) from every \( x \in D \).

Then the QSD \( \mu \) is unique. Its topological support equals \( \overline{D} \) and it has a smooth density with respect to the Lebesgue measure.

Suppose furthermore that there exists a point \( x^* \in D \) at which the weak Hörmander condition is strengthened to the Hörmander condition. Then there exist \( \alpha > 0, C \in [0, +\infty[ \) and a continuous function \( h : D \rightarrow [0, \infty[ \) with \( h(x) \rightarrow 0 \), as \( x \rightarrow \partial D \), satisfying \( \mu(h) = 1 \), such that for all \( \rho \in M_1(D) \) (the set of probability measures over \( D \)),

\[ \| \mathbb{P}_\rho(X_t \in \cdot | \tau^\text{out}_D > t) - \mu(\cdot) \|_{TV} \leq \frac{C}{\rho(h)} e^{-\alpha t} \quad (3) \]

where \( \| \cdot \|_{TV} \) stands for the total variation distance and \( \rho(h) := \int h d\rho \).

Note that Condition (ii) of Theorem 1.5 readily follows from Condition (iii) of Theorem 1.8, so that the assumptions of Theorem 1.8 imply the existence of a QSD.

**Corollary 1.9** Assume that Hypothesis (H2') holds, and that the strong Hörmander condition is satisfied at every point \( x \in \overline{D} \). Then, the conclusions of Theorem 1.8 hold true and the density of \( \mu \) is positive on \( D \).

**Proof:** Let, for \( \varepsilon \geq 0 \), \( y^\varepsilon(x, u, \cdot) \) be defined like \( y(x, u, \cdot) \) when \( S^0 \) is replaced by \( \varepsilon S^0 \). By Chow’s theorem (see e.g. [39], Chapter 2, Theorem 3), the strong Hörmander condition implies that for all \( x, y \in D \) there exists a control \( u \) piecewise continuous with \( u^i(s) \in \{-1, 0, 1\} \) and \( t \geq 0 \) such that \( y^0(x, u, s) \in D \) for all \( 0 \leq s \leq t \) and \( y^0(x, u, t) = y \). By continuity of solutions to an ODE, with respect to some parameter (or by a simple application of Gronwall’s lemma), \( y^\varepsilon(x, u, \cdot) \rightarrow y^0(x, u, \cdot) \) uniformly on \( [0, t] \) as \( \varepsilon \rightarrow 0 \). Let \( u^\varepsilon(s) = \frac{u(e)}{\varepsilon} \) and \( y^\varepsilon(x, u, s) = y(x, u^\varepsilon, \varepsilon s) \). This proves that \( y \) is \( D \)-accessible from \( x \). Because the strong Hörmander condition also holds in a neighborhood of \( \overline{D} \), the
same proof also shows that, for \( y \) in a neighborhood of \( \partial D \) and \( x \in D \), \( y \) is accessible from \( x \), so (H1) is satisfied. The conditions, hence the conclusions, of Theorem 1.8 are then satisfied. Positivity of \( \frac{du}{dx} \) is proved in Lemma 3.16.

\[
\text{Remark 1.10} \quad \text{The results above easily extend to the situation where } \mathbb{R}^n \text{ is replaced by a } n\text{-dimensional manifold, provided } \partial D \text{ is a } (n-1)\text{-dimensional } C^2 \text{ sub-manifold. This is illustrated in Examples 1.12 and 1.13.}
\]

\[
\text{Remark 1.11} \quad \text{The paper [31] considers QSDs and their properties under the assumption that the underlying process is strong Feller. Quasi-stationarity for strong Feller diffusion processes have also been a focus of interest in the recent literature [31, 45, 54]. Observe that none of the conditions of Theorem 1.5, neither the weak Hörmander condition assumed in Theorem 1.8 imply that the semigroup induced by (1) is strong Feller (see Example 1.12 below).}
\]

\[
\text{Example 1.12} \quad \text{We consider the situation where the SDE is defined on the cylinder } \mathbb{R}/\mathbb{Z} \times \mathbb{R} \text{ (instead of } \mathbb{R}^n). \text{ Let } m = 1,
\]

\[S^0(x,y) = \partial_x \text{ and } S^1(x,y) = \partial_y.
\]

Let \( D = \mathbb{R}/\mathbb{Z} \times [0,1]. \) Here the conditions of Theorem 1.8 are easily seen to be satisfied. However, the process is not strong Feller (the dynamics in the } x\text{-variable being deterministic). It is not hard to check that the unique QSD is the measure

\[
\mu(dx dy) = 2\sin(\frac{\pi y}{\pi}) \mathbb{1}_D(x,y) dx dy.
\]

\[
\text{Example 1.13 (Example 1.12, continued) Consider the setting of Example 1.12, but with }
\]

\[S^0(x,y) = a(y)\partial_x \text{ and } S^1(x,y) = \partial_y,
\]

where \( a \) is a smooth function \( \geq 1. \) Like in example 1.12, the unique QSD \( \mu \) is given by (4). Suppose that \( a'(y^*) \neq 0 \) for some \( 0 < y^* < 1. \) Then the Hörmander condition holds at \( (x, y^*) \), so that by Theorem 1.8 \( (P_x(X_t \in \cdot | \tau^+_D > t))_{t \geq 0} \) converges at an exponential rate to \( \mu. \)
Remark 1.14 In absence of condition (iii) in Theorem 1.8 there is no guarantee that the QSD is unique, as shown by the next example. Still every QSD has a smooth density (see Lemma 3.16).

Example 1.15 Let $n = 1$ and $D = ]0, 5[$. We consider smooth functions $\varphi^1, \varphi^2, \psi^1, \psi^2 : D \to [0, 1]$ such that $\psi^1 + \psi^2 = 1$ and

\[
\begin{aligned}
\varphi^1_{|[0,1]} &\equiv 1, \; \varphi^1_{|[1,2]} \leq 1, \; \varphi^1_{|[2,5]} \equiv 0, \\
\varphi^2_{|[0,3]} &\equiv 0, \; \varphi^2_{|[3,4]} \leq 1, \; \varphi^2_{|[4,5]} \equiv 1, \\
\psi^1_{|[0,2]} &\equiv 1, \; 0 < \psi^1_{|[2,3]} \leq 1, \; \psi^1_{|[3,5]} \equiv 0, \\
\psi^2_{|[0,2]} &\equiv 0, \; 0 < \psi^2_{|[2,3]} \leq 1, \; \psi^2_{|[3,5]} \equiv 1.
\end{aligned}
\]

For all $\alpha > 0$, we consider the absorbed diffusion process $X^\alpha$ evolving according to the Itô SDE

\[
dX^\alpha_t = (\varphi^1(X^\alpha_t) + \sqrt{\alpha}\varphi^2(X^\alpha_t)) \, dB_t + (\psi^1(X_t) + \alpha\psi^2(X_t)) \, dt.
\]

and absorbed when it reaches $\partial D = \{0, 5\}$. While $X^\alpha$ satisfies the conditions (i) and (ii) of Theorem 1.8 it does not satisfy condition (iii) and it admits either one or two QSDs, depending on the value of $\alpha$. Indeed, as shown in [6], there exists $\alpha_c > 0$ such that

- for all $\alpha \in ]0, \alpha_c]$, $X^\alpha$ admits a unique QSD supported by $[3, 5]$, \\
- for all $\alpha \in ]\alpha_c, +\infty[$, $X^\alpha$ admits exactly two QSDs, supported respectively by $[3, 5]$ and $[0, 5]$.

2 Killed processes

Throughout, we let $M$ denote a separable and locally compact metric space, and $D \subset M$ a nonempty set with compact closure $K = \overline{D}$ in $M$, such that $D$ is open relative to $K$. That is $D = O \cap K$ for some open set $O \subset M$. Considering such a general setting is important for example for piecewise deterministic Markov processes (see Section 4; see also Example 2.6 below).

2.1 Preliminary results

All the definitions and preliminary results gathered in this section are classical, or adapted from classical ones.
We let \((P_t)_{t \geq 0}\) denote a Markov Feller semigroup\(^1\) on \(M\). By this, we mean (as usual) that \((P_t)_{t \geq 0}\) is a semigroup of Markov operators on \(C_0(M)\) (the closure for the supremum norm of the set of continuous, compactly supported functions from \(M\) to \(\mathbb{R}\)) and that \(P_t f(x) \to f(x)\) as \(t \to 0\) for the uniform norm for all \(f \in C_0(M)\). Observe that since we are interested by the behavior of the process killed outside \(K\), the behavior of \((P_t)_{t \geq 0}\) at infinity is irrelevant (and the reader can think of \(M\) as compact without loss of generality). Feller processes, named after William Feller, forms a large class of processes whose sample path properties have been studied, among others, by Doob, Kinney, Dynkin, Blumenthal, Hunt and Ray. We refer the reader to [17, 40] for properties and historical notes on these processes, see also [44, 41, 55] for expository texts on Feller semi-groups and their sample path properties.

By classical results (see e.g Le Gall [44], Theorem 6.15), there exist a filtered space \((\Omega, \mathcal{F}, (\mathcal{F}_t))\) with \((\mathcal{F}_t)\) right continuous and complete, a family of probabilities \((P_x)_{x \in M}\) on \((\Omega, \mathcal{F})\) and a continuous time adapted process \((X_t)\) on \((\Omega, \mathcal{F}, (\mathcal{F}_t))\) taking values in \(M\), such that:

(i) \((X_t)\) has cad-lag paths,

(ii) \(P_x(X_0 = x) = 1\) and,

(iii) \((X_t)\) is a Markov process with semigroup \((P_t)\), meaning that

\[
\mathbb{E}_x(f(X_{t+s})|\mathcal{F}_t) = P_s f(X_t)
\]

for all \(t, s \geq 0\) and \(f\) measurable bounded (or \(\geq 0\)).

For any Borel set \(A \subset M\) we let \(\tau_A = \inf\{t \geq 0 : X_t \in A\}\) and \(\tau_A^{\text{out}} = \tau_{M \setminus A}\). The assumptions on \((\mathcal{F}_t)\) (right continuous and complete) imply that \(\tau_A\) and \(\tau_A^{\text{out}}\) are stopping times with respect to \((\mathcal{F}_t)\) (see e.g. Bass [3]).

Remark 2.1 For \(X_0 = x \in D, \tau_D^{\text{out}} \leq \tau_K^{\text{out}}\) but it is not true in general that \(\tau_D^{\text{out}} = \tau_K^{\text{out}}\). Consider for example the ODE on \(\mathbb{R}^2\) given by

\[
\begin{cases}
\dot{x} = 1 \\
\dot{y} = 0
\end{cases}
\]

\(^1\)We also mentioned strong Feller semigroups in the introduction. They are defined as Feller semigroups, except for the property that \(P_t\) maps \(C_0(M)\) to itself, which is replaced by the property that \(P_t\) maps bounded measurable functions on \(M\) to bounded continuous functions on \(M\).
Let 
\[ D = \{ (x, y) \in \mathbb{R}^2 \mid -1 < x < 1, -1 < y < x^2 \} \].

For \(-1 < x < 0\) and \(y = 0\) the trajectory \((x(t), y(t)) = (x + t, 0)\) starting at \((x, 0)\) leaves \(D\) at time \(-x\) and \(K\) at time \(-x + 1\).

We now recall the definition of accessibility for general Feller processes.

**Definition 2.2** An open set \(U \subset M\) is said to be accessible (by \((P_t)\)) from \(x \in M\), if there exists \(t \geq 0\) such that 
\[ P_t(x, U) = P_t \mathbb{1}_U(x) > 0. \]

Here also, by abuse of language, we say that \(y \in M\) is accessible from \(x \in M\), if every open neighborhood \(U\) of \(y\) is accessible from \(x\).

We shall assume throughout all this section that the following assumption is satisfied.

**Hypothesis (Standing Hypothesis)** The set \(M \setminus K\) is accessible from all \(x \in K\).

The next result is a basic, but very useful, consequence of this assumption. It is interesting to point out that it only requires Feller continuity, accessibility of \(M \setminus K\) and compactness of \(K\). Note that similar - albeit different- results can be found in the literature (see e.g [4], Proposition 21.2).

**Proposition 2.3** There exist positive constants \(C, \Lambda\) such that 
\[ \mathbb{P}_x (\tau_{\partial K}^\text{out} > t) \leq Ce^{-\Lambda t} \]
for all \(t \geq 0\) and \(x \in K\). In particular \(\tau_{\partial K}^\text{out} < \infty\), and hence \(\tau_D^\text{out} < \infty\), \(\mathbb{P}_x\) almost surely for all \(x \in K\).

**Proof:** Let \(U = M \setminus K\). By Feller continuity, for all \(t \geq 0\), the set 
\[ O_t = \{ x \in M : P_t(x, U) > 0 \} \] is open (possibly empty). We detail the proof since the argument will be used several times in the sequel. We notice that the sequence of functions 
\[ f_n(x) = (nd(x, M \setminus U) \wedge 1) \]
is non-decreasing and converges to \(\mathbb{1}_U(x)\). Therefore, \(x \in O_t\) if and only if \(x \in O^n_t = \{ x \in M : P_t f_n(x) > 0 \}\) for some \(n \geq 1\). Since \(O^n_t\) is open by Feller continuity of the semi-group \(P_t\), we deduce that \(O_t\) is open. By the standing hypothesis, the family \(\{O_t : t \in \mathbb{R}^+\}\) covers \(K\), so that, by compactness, there exist \(t_1, \ldots, t_n\) such that \(K \subset \cup_{i=1}^n O_{t_i}\). In particular, for some \(\delta > 0\) and \(t = \max\{t_1, \ldots, t_n\}\) 
\[ \mathbb{P}_x (\tau_U > t) \leq 1 - \sup_{1 \leq i \leq n} P_{t_i}(x, U) \leq 1 - \delta \] for all \(x \in K\). Thus, by the Markov property, 
\[ \mathbb{P}_x (\tau_U > kt) \leq (1 - \delta)^k. \] This proves the result. \(\square\)
2.2 Green kernel and QSDs

Let $B(D)$ denote the set of bounded measurable functions $f : D \mapsto \mathbb{R}$. For all $f \in B(D)$, $x \in D$ and $t \geq 0$ set

$$P_t^D f(x) = \mathbb{E}_x(f(X_t) \mathbbm{1}_{\tau^\text{out}_D > t}).$$

Then, $(P_t^D)_{t \geq 0}$ is a well defined sub-Markovian semigroup on $B(D)$. The semigroup property is a consequence of the Markov property and the fact that $\tau^\text{out}_D$ is a first exit time and a stopping time [20 Section 2.6].

We define as usual the Green kernel $G^D$ as the bounded (by Proposition 2.3) operator defined on $B(D)$ given by

$$G^D f(x) = \int_0^\infty P_t^D f(x) dt = \mathbb{E}_x \left( \int_0^{\tau^\text{out}_D} f(X_t) dt \right).$$

For all $x \in D$ and $A \subset M$, a Borel set, we let

$$G^D(x, A) = G^D \mathbbm{1}_{A \cap D}(x).$$

Quasi-stationary distributions

A quasi-stationary distribution (QSD) for $(P_t^D)$ is a probability measure $\mu$ on $D$ such that, for all $t \geq 0$,

$$\mu P_t^D = e^{-\lambda t} \mu$$

for some $\lambda > 0$. For further reference we call $\lambda$ the absorption rate (or simply the rate) of $\mu$. Equivalently [19],

$$\frac{\mu P_t^D(\cdot)}{\mu P_t^D \mathbbm{1}_D} = \mathbb{P}_\mu(X_t \in \cdot \mid \tau^\text{out}_D > t) = \mu(\cdot).$$

**Lemma 2.4** Equation (5) holds if and only if $\mu G^D = \frac{1}{\lambda} \mu$.

**Proof:** Clearly, by definition of $G^D$, equation (5) implies that $\mu G^D = \frac{1}{\lambda} \mu$. Conversely, assume that $\mu G^D = \frac{1}{\lambda} \mu$. Then for every bounded non-negative measurable map $f : D \mapsto \mathbb{R}$, $\mu(G^D f) = \frac{1}{\lambda} \mu(f)$ and also

$$\mu(G^D P_t^D f) = \frac{1}{\lambda} \mu(P_t^D f).$$

That is

$$\mu(\int_0^\infty P_s^D f ds) = \frac{1}{\lambda} \mu(P_t^D f).$$
Equivalently,
\[ \mu(GD f - \int_0^t P_D^s f ds) = \frac{1}{\lambda} \mu(P_D^t f). \]

This shows that the bounded map \( v(t) = \mu(P_D^t f) \) satisfies the integral equation
\[ v(t) - v(0) = -\lambda \int_0^t v(s)ds, \]
for all \( t \geq 0 \). It follows that \( v(t) = v(0)e^{-\lambda t}. \) \( \square \)

Let \( C_b(D) \subset B(D) \) denote the set of bounded continuous functions on \( D \), and \( C_0(D) \subset C_b(D) \) the closure of the set of continuous, compactly supported functions from \( D \) to \( \mathbb{R} \). Observe that, since \( K \) is compact, \( C_0(D) \) is also the subset of functions \( f \) such that \( f(x) \to 0 \) when \( d(x, \partial K D) \to 0 \), where \( d(x, \partial K D) \) is the distance between \( x \) and \( \partial K D := K \setminus D \). Similarly, we define \( C_b(K) \) as the set of bounded continuous functions on \( K \).

**Remark 2.5** In the recent paper [26], the authors prove the existence and convergence to a QSD under the condition that the sub-Markovian semigroup is strong Feller. Observe that in our case, although \( (P_t) \) is Feller, there is no evidence in general that \( (P^D_t) \) is strong Feller nor that it preserves \( C_b(D) \). On the other hand, under rather weak, reasonable conditions, \( G^D \) maps \( C_b(D) \) into \( C_0(D) \), as illustrated by the following example.

**Example 2.6** Consider the ODE on \( \mathbb{R} \) given by \( \dot{x} = -1 \). For \( D = ]0, 1[ \) \( P^D_t f(x) = f(x-t)1_{x>t} \) is not Feller, but \( G^D f(x) = \int_0^x f(u)du \) is Feller (and even strong Feller). If now \( D = ]0, 1[ \), then \( D \) is open relative to \( M = ]-\infty, 1[ \), with compact closure and \( G^D \) maps \( C_b(D) \) into \( C_0(D) \).

The condition that \( G^D(C_b(D)) \subset C_0(D) \) plays a key role in the next theorems and will be investigated in the subsequent sections for degenerate diffusions and PDMPs.

**Definition 2.7** An open set \( U \subset M \) is said \( D \)-accessible (by \( (P_t) \) from \( x \in D \) if \( P^D_t(x, U) > 0 \) for some \( t \geq 0 \). A point \( y \in K \) is said \( D \)-accessible from \( x \in D \) if every open neighborhood of \( y \) is \( D \)-accessible from \( x \).

**Lemma 2.8** An open set \( U \subset M \) is \( D \)-accessible from \( x \in D \) if and only if \( G^D(x, U) > 0 \). In particular, the set of \( D \)-accessible points from \( x \) coincides with the topological support of \( G^D(x, \cdot) \).
Proof: Suppose that $P_t^D(x,U) > 0$ for some open set $U$ and $t \geq 0$. Fatou Lemma and right continuity of paths imply that

$$\liminf_{s \to t, s > t} P_s^D(x,U) \geq E_x(\liminf_{s \to t, s > t} 1_U(X_s) 1_{D^\circ t > s}) \geq P_t^D(x,U) > 0.$$ 

This proves that $s \to P_s^D(x,U)$ is positive on some interval $[t, t + \epsilon]$, hence $G^D(x,U) > 0$. The converse implication is obvious. □

We now state and prove our first main result.

**Theorem 2.9** Assume that $G^D(C_0(D)) \subset C_0(D)$. Then, the conditions (i),(ii) below are equivalent and imply the existence of a QSD.

(i) There exists an open set $U \subset M$, $D$-accessible from all $x \in D$ and such that $U \cap K \subset D$.

(ii) There exists a function $\phi \in C_0(D)$, positive on $D$ and $\theta > 0$, such that $G^D \phi \geq \theta \phi$.

Proof: We first show that (i) implies (ii). For $\epsilon > 0$, let

$$\overline{U}_\epsilon^K = \{x \in K : d(x, U \cap K) \leq \epsilon\}.$$ 

Choose $\epsilon > 0$ small enough so that $\overline{U}_\epsilon^K \subset D$.

Let $\psi(x) = (1 - \frac{d(x, U \cap K)}{\epsilon})^+$ and $\phi = G^D \psi$. Then, $\phi \in C_0(D)$ (because $G^D(C_0(D)) \subset C_0(D)$) and for all $x \in D$,

$$\phi(x) \geq G^D \mathbb{1}_{U \cap K}(x) = G^D \mathbb{1}_U(x) > 0,$$

where the last inequality holds by Lemma 2.8. Thus, by compactness of $\overline{U}_\epsilon^K$, $\theta = \inf_{x \in \overline{U}_\epsilon^K} \phi(x) > 0$. Since $\psi = 0$ on $K \setminus \overline{U}_\epsilon^K$, it follows that for all $x \in D$,

$$\phi(x) \geq \theta \mathbb{1}_{U \cap K} \geq \theta \psi(x).$$

Therefore, $G^D \phi(x) \geq \theta \phi(x)$.

We now prove that (ii) implies (i). For all $\epsilon \geq 0$ and $x_0 \in D$, set

$$V_\epsilon = \{y \in D : \phi(y) > \epsilon\},$$

and

$$\eta = \eta(x_0) := \inf\{\epsilon \geq 0 : G^D(x_0, V_\epsilon) = 0\}.$$ 

Note that $V_{\|\phi\|} = \emptyset$, so that $\eta < +\infty$. By monotone convergence, $G^D(x_0, V_\eta) = \lim_{n \to \infty} G^D(x_0, V_{\eta + 1/n}) = 0$. In particular, $\eta$ is positive because, by (ii), $G^D(x_0, D)$ is. Therefore, by definition of $\eta$,
Let $B$ be bounded continuous functions over $K$. Lemma 2.8. This shows that there exists a point $y \in U$ and, since $\phi$ is positive on $D$, that

$$\eta \in \{ x \in D, d(x, \partial_K D) \leq \alpha \} \text{ and let } \eta(y) \geq \eta_0 > 0 \text{ such that } D \setminus V_{\eta_0} \subset \{ x \in D, d(x, \partial_K D) \leq \alpha \}. \text{ Therefore, (6) implies that } \eta(y) > \eta_0, \text{ hence } \eta = \eta(x_0) > \eta_0. \text{ Since } x_0 \text{ was arbitrary we deduce that the map } x \mapsto \eta(x) \text{ is bounded from below by } \eta_0. \text{ Since } G^D(x_0, V_{\eta_0/2}) \geq G^D(x_0, V_{\eta/2}) > 0, \text{ Lemma 2.8 proves (i) with } U = V_{\eta_0/2} \text{ (and actually } U \subset D).$

Our last goal is to prove the existence of a QSD under Condition(ii). Let $B(K)$ (respectively $C_b(K)$) be the space of bounded (respectively bounded continuous) functions over $K$. The operator $G^D$ extends to a bounded operator $G^{D,K}$ on $B(K)$ defined as

$$G^{D,K} f(x) = \begin{cases} G^D(f|_D)(x) & \text{for } x \in D \\ 0 & \text{for } x \in \partial_K D \end{cases}$$

The assumption that $G^D(C_b(D)) \subset C_0(D)$ implies that $G^{D,K}(C_b(K)) \subset C_b(K)$.

Let $\mathcal{M}_1(\phi)$ be the set of Borel finite nonnegative measures $\mu$ on $K$ such that $\mu(\phi) = 1$ (extending $\phi$ to $K$ by setting $\phi(x) = 0$ for all $x \in \partial_K D$) and let $T: \mathcal{M}_1(\phi) \rightarrow \mathcal{M}_1(\phi)$ be the map defined by

$$T(\mu) = \frac{\mu G^{D,K}}{\mu G^{D,K} \phi}.$$
We first observe that $T$ is continuous for the weak* topology: if $\mu_n \to \mu$ for the weak* topology in $M_1(\phi)$, then $\mu_n G^{D,K} \to G^{D,K}$ and $\mu_n G^{D,K} \phi \to G^{D,K} \phi$. Since in addition $\mu_n G^{D,K} \phi \geq \theta \mu_n(\phi) = \theta > 0$ by (ii), $T(\mu_n) \to T(\mu)$.

Choose an open neighborhood $N$ of $\partial K$ such that $G^{D,K} / BD_K \leq \theta / 2$ on $N \cap K$ and set $C = \sup_{x \in K \setminus N} \frac{G^{D,K} \phi(x)}{\phi(x)} < \infty$. Then, for all $\mu \in M_1(\phi)$

$$T(\mu)(K) = \frac{\mu(G^{D,K} 1_K)}{\mu G^{D,K} \phi} \leq \frac{\mu(G^{D,K} 1_K)}{\theta} \leq \frac{\theta/2 \mu(K) + C \mu(\phi)}{\theta} = \frac{\mu(K)}{2} + \frac{C}{\theta}.$$

It follows that for any $R \geq 2 \frac{C}{\theta}$ the set

$$M^R_1(\phi) = \{ \mu \in M_1(\phi) : \mu(K) \leq R \}$$

is invariant by $T$. Since $M^R_1(\phi)$ is convex and compact (for the weak* topology), $T$ admits a fixed point by Tychonov’s Theorem. If $\mu$ is such a fixed point, $\mu(D) \leq R$ and $\mu(D) > 0$ since $\mu(\phi) = 1$ and $\phi(x) = 0$ for all $x \in K \setminus D$, so we can define the probability measure $\mu = \frac{\mu(D)}{\mu(D)}$ on $D$. Since, for all $f \in B(K)$, $\mu G^{D,K} f = \mu(1_D G^{D} f|_D)$, we deduce that, for all $f \in B(D)$, $\mu G^{D} f = (\mu G^{D,K} \phi) \bar{\mu}(f)$ with $\mu G^{D,K} \phi > 0$. Then $\mu$ is a QSD for $(P^D_t)$ by Lemma 2.4.

Remark 2.10 The last part of the proof of Theorem 2.9 is reminiscent of the proof of the existence Theorem 4.2 in [19].

2.3 Uniqueness and convergence criteria

Right eigenfunctions

We say that $h \in B(D)$ is a positive right eigenfunction for $G^D$ if $h(x) > 0$ for all $x \in D$ and

$$G^D h = \frac{1}{\lambda} h$$

for some $\lambda > 0$.

Lemma 2.11 If $h$ is a positive right eigenfunction for $G^D$, the parameter $\lambda$ in (7) necessarily equals the absorption rate of any QSD and

$$P^D_t h = e^{-\lambda t} h$$

for all $t \geq 0$. 

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Proof: If µ is a QSD with rate λ’, then µ^D h = \frac{1}{\lambda'} µ h = \frac{1}{\lambda} µ h so that λ = λ’. The proof of the second statement is similar to the proof of Lemma 2.14 and left to the reader. 

Observe that there is no assumption in Lemma 2.11 that \( G^D(C_b(D)) \subset C_0(D) \) but when this is the case, if there exists a positive right eigenfunction for \( G^D \), then there always exists a QSD, by application of Theorem 2.9.

The following result shows that a strengthening in the assumptions of Theorem 2.9 ensures the existence of a positive right eigenfunction for \( G^D \).

**Corollary 2.12** Assume that:

(i) \( G^D(C_b(D)) \subset C_0(D) \) and \( G^D \) is a compact operator from \( C_0(D) \) to itself, where \( C_0(D) \) is endowed with the uniform distance;

(ii) For all \( x, y \in D \) \( y \) is \( D \)-accessible from \( x \).

Then, there exists a positive right eigenfunction \( h \in C_0(D) \) for \( G^D \).

**Proof:** Let \( r = \lim_{n \to \infty} \|(G^D)^n\|^{1/n} \) be the spectral radius of \( G^D \) on \( C_0(D) \). Let \( \mu \) be a QSD (whose existence is given by Theorem 2.9) with rate \( \lambda \). For any \( f \in C_0(D) \) such that \( 0 \leq f \leq 1 \) and \( \mu(f) \neq 0 \), \( \|(G^D)^n\| \geq \mu((G^D)^n f) = \frac{1}{\lambda^n} \mu(f) \). Hence \( r \geq \frac{1}{\lambda} > 0 \).

Let \( C_0^+(D) = \{ f \in C_0(D) : f \geq 0 \} \). It is readily seen that \( C_0^+(D) \) is a reproducing cone in \( C_0(D) \) invariant by \( G^D \), meaning that \( C_0^+(D) \) is a cone, \( C_0(D) = \{ u - v : u, v \in C_0^+(D) \} \) and \( G^D(C_0^+(D)) \subset C_0^+(D) \). Therefore, by Krein Rutman Theorem ([22], Theorem 19.2), there exists \( h \in C_0^+(D) \setminus \{0\} \) such that \( G^D h = rh \). Let \( y \in D \) be such that \( h(y) > 0 \). Then, \( h \geq \frac{h(y)}{2} \mathbb{1}_{U \cap D} \) for some neighborhood \( U \) of \( y \). Therefore, by (ii), for all \( x \in D \),

\[
    rh(x) = (G^D h)(x) \geq \frac{h(y)}{2} (G^D \mathbb{1}_{U \cap D})(x) > 0.
\]

This concludes the proof. 

**Uniqueness and convergence**

Similarly as in the conservative setting, we say that \((P_t^D)\) is irreducible if there exists a nontrivial positive measure \( \xi \) on \( D \) such that for all \( x \in D \) and \( A \) Borel,

\[
    \xi(A) > 0 \Rightarrow G^D(x, A) > 0.
\]
Theorem 2.13 Assume that:
(i) \(G^D(C_0(D)) \subset C_0(D)\);
(ii) There exists a positive right eigenfunction \(h \in B(D)\) for \(G^D\);
(iii) \((P^D_t)\) is irreducible.
Then \((P^D_t)\) has a unique QSD.

Remark 2.14 Conditions (i) and (ii) of Theorem 2.13 are implied by the assumptions of Corollary 2.12.

Proof: Let \(h\) be a positive right eigenfunction and \(\mu\) a QSD with rate \(\lambda\). Let \(Q\) and \(\pi\) respectively denote the Markov kernel and the probability on \(D\) defined by
\[Q(f) = \lambda \frac{G^D(fh)}{h},\]
and
\[\pi(f) = \frac{\mu(fh)}{\mu(h)}\]
for all \(f \in B(D)\). Then, \(\pi\) is invariant by \(Q\). The assumption that \((P^D_t)\) is irreducible makes \(Q\) irreducible, in the sense that \(\xi(A) > 0 \Rightarrow Q(x,A) > 0\) for all \(x \in D\) and \(A\) Borel. Therefore, by a standard result (see e.g [24] or [50]), \(\pi\) is the unique invariant probability of \(Q\). Assume now that \(\nu\) is another QSD with rate \(\alpha\). Then \(\nu(G^Dh) = \frac{1}{\alpha}\nu(h) = \frac{1}{\lambda}\nu(h)\). Since \(h\) is positive on \(D\), we deduce that \(\alpha = \lambda\). It follows that the probability \(\pi'\) defined like \(\pi\) with \(\nu\) in place of \(\mu\) is invariant by \(Q\).

By uniqueness, \(\pi = \pi'\) and consequently \(\mu = \nu\). \(\square\)

A sufficient (and often more tractable than the definition) condition ensuring irreducibility is given by the next lemma.

Lemma 2.15 Suppose that there exists an open set \(U \subset M\), \(D\)-accessible from all \(x \in D\), and a non trivial measure \(\xi\) such that for all \(x \in U\)
\[G^D(x,\cdot) \geq \xi(\cdot).\]
Then \((P^D_t)\) is irreducible.

Proof: For all \(x \in D\), there exists, by \(D\)-accessibility, \(t \geq 0\) such that \(P^D_t(x,U) > 0\). For every Borel set \(A \subset M\),
\[G^D(x,A) \geq \int_0^\infty P^D_{t+s}(x,A)ds = \int_0^\infty \int_M P^D_t(x,dy)P^D_s(y,A)ds\]
\[\geq \int_0^\infty \int_U P^D_t(x,dy)P^D_s(y,A)ds \geq P_t(x,U)\xi(A).\]
If the local minorization $G^D(x, \cdot) \geq \xi(\cdot)$ appearing in Lemma 2.15 can be improved to local minorization involving $(P^D_t)$, we also get the exponential convergence of the conditional semigroup toward $\mu$. More precisely,

**Theorem 2.16** Suppose that:

(i) $G^D(C_0(D)) \subset C_0(D)$;

(ii) There exists a positive right eigenfunction $h \in C_0(D)$ for $G^D$;

(iii) There exist an open set $U \subset M$, $D$-accessible from all $x \in D$, a non trivial measure $\xi$ on $D$, and $T > 0$ such that for all $x \in U$,

$$P^D_T(x, \cdot) \geq \xi(\cdot).$$

Then there exist $C, \alpha > 0$ such that, for all $\rho \in M_1(D)$,

$$\left\| \frac{\rho P^D_t}{\rho P^D_1 \mathbb{1}_D} - \mu(\cdot) \right\|_{TV} \leq \frac{C}{\rho(h)} e^{-\alpha t}$$

where $\| \cdot \|_{TV}$ stands for the total variation distance.

**Proof:** Let $h$ be a positive right eigenfunction and $\mu$ a QSD with rate $\lambda$. For all $f \in B(D)$ and $t \geq 0$, let

$$Q_t(f) = e^{\lambda t} \frac{P^D_t(fh)}{h}$$

and

$$\pi(f) = \frac{\mu(fh)}{\mu(h)}$$

It is readily seen that $(Q_t)$ is a Markov semigroup (usually called the $Q$–process induced by $\mu$) having $\pi$ as invariant probability. In order to prove the theorem we will show that:

**Step 1** There exists a probability $\nu$ on $D$ and $T_0 > 0$ such that for every compact set $\bar{K} \subset D$ there is some integer $n$ and some constant $c$ (both depending on $\bar{K}$) such that

$$Q_{nT_0}(x, \cdot) \geq c \nu(\cdot)$$

for all $x \in \bar{K}$.
\textbf{Step 2} The function \( V = 1/h \) is a continuous and proper Lyapunov function for \( Q_{T_0} \), that is,

\[
\lim_{x \to \partial K} V(x) = \infty,
\]

and

\[
Q_{T_0} V \leq \rho V + C
\]

for some \( 0 \leq \rho < 1 \) and \( C \geq 0 \).

Assume that these two steps are completed. We now explain how the theorem can be deduced. From step 2 we deduce that for all \( n \geq 0 \)

\[
Q_{nT_0} V \leq \rho^n V + \frac{C}{1 - \rho}. \tag{9}
\]

Choose \( R > \frac{2C}{\rho} \) and set \( \tilde{K} = \{ x \in D : V(x) \leq R \} \). Then \( \tilde{K} \) is a compact subset of \( D \) and, by step 1, there is some \( n \geq 1 \) such that

\[
Q_{nT_0}(x, \cdot) \geq c \nu(\cdot) \tag{10}
\]

on \( \tilde{K} \). Now, relying on a version of Harris’s theorem proved by Hairer and Mattingly in [32], (9) and (10) imply that for all \( f : D \mapsto \mathbb{R} \) measurable, and \( k \geq 0 \),

\[
|Q_{kT_0}(f)(x) - \pi(f)| \leq C \gamma^k (1 + V(x)) \| f \|_V
\]

for all \( x \in D \), where \( 0 \leq \gamma < 1 \) and \( \| f \|_V = \sup_{x \in D} \frac{|f(x)|}{1 + V(x)} \). Recalling that \( h \) is bounded (it lies in \( C_0(D) \)) and \( V = 1/h \), one has \( \inf_{x \in D} V(x) > 0 \). This entails the existence of a constant \( C > 0 \) such that, for all measurable function \( f \) such that \( \| f \|_V \leq 1 \), for all \( k \geq 0 \), for all \( x \in D \),

\[
|Q_{kT_0}(f)(x) - \pi(f)| \leq CV(x)\gamma^k.
\]

Integrating this last inequality with respect to \( \rho(dx) \), for any probability measure \( \rho \in \mathcal{M}_1(D) \),

\[
|\rho Q_{kT_0}(f) - \pi(f)| \leq C \rho(V) \gamma^k.
\]

Hence, by the property of semigroup for \( (Q_t) \), for any \( k \in \mathbb{Z}_+ \) and \( s \in [0, nT_0] \), for all \( \rho \in \mathcal{M}_1(D) \),

\[
|\rho Q_{s+kT_0}(f) - \pi(f)| \leq C \rho Q_s(V) \gamma^k.
\]
Now, for all \( s \in [0, nT_0] \) and \( \rho \in \mathcal{M}_1(D) \),
\[
\rho Q_s(V) = \rho Q_s(1/h) = \int_D \rho(dx) \frac{e^{\lambda_s P_s^D(1D)}(x)}{h(x)} \leq e^{\lambda nT_0 \rho(1/h)}.
\]

Thus, setting \( \alpha = -\ln \gamma \), we deduce that, for any \( t \geq 0 \) and \( f \) such that \( \|f\|_1/h \leq 1 \),
\[
|\rho Q_t f - \pi(f)| \leq C e^{\lambda nT_0 \rho(1/h)} \gamma^{-\alpha t}.
\]

In particular, by definition of \( \pi \), for any \( f \) such that \( \|f\|_1 \leq 1 \) and \( \rho \in \mathcal{M}_1(D) \) such that \( \rho(1/h) < +\infty \), up to a change in the constant \( C > 0 \),
\[
\left| \rho Q_t \left[ \frac{f}{h} \right] - \frac{\mu(f)}{\mu(h)} \right| \leq C \rho(1/h) e^{-\alpha t}.
\]

For any \( \rho \in \mathcal{M}_1(D) \), denote \( \rho_h(dx) := \frac{h(x)\rho(dx)}{\rho(h)} \in \mathcal{M}_1(D) \). Remark then that, for all \( \rho \in \mathcal{M}_1(D) \), \( \rho_h(\frac{1}{h}) = \frac{1}{\rho(h)} < +\infty \). Then the last inequality (applied to \( \rho_h \) instead of \( \rho \) ) entails that, for any \( f \) such that \( \|f\|_\infty \leq 1 \) and \( \rho \in \mathcal{M}_1(D) \), for all \( t \geq 0 \),
\[
\frac{\mu(f)}{\mu(h)} - \frac{C}{\rho(h)} e^{-\alpha t} \leq (\rho_h)Q_t \left[ \frac{f}{h} \right] \leq \frac{\mu(f)}{\mu(h)} + \frac{C}{\rho(h)} e^{-\alpha t}. \tag{11}
\]

Moreover, using that \( P^D_t f(x) = e^{-\lambda t h(x)Q_t[f/h]}(x) \),
\[
\frac{\rho P^D_t f}{\rho P^D_t 1} = \frac{(\rho_h)Q_t[f/h]}{(\rho_h)Q_t[1D/h]}.
\]

Thus, fixing \( \rho \in \mathcal{M}_1(D) \) and denoting \( t_\rho := \frac{1}{\alpha} \log \left( \frac{C\mu(h)}{\rho(h)} \right) \), this equality and \( (11) \) entail that, for any \( t > t_\rho \),
\[
\frac{\mu(f)}{\rho(h)} - \frac{C}{\rho(h)} e^{-\alpha t} \leq \frac{\rho P^D_t f}{\rho P^D_t 1} \leq \frac{\mu(f)}{\rho(h)} + \frac{C}{\rho(h)} e^{-\alpha t}. \tag{12}
\]

Therefore, since \( |\mu(f)| \leq \|f\|_\infty \leq 1 \), for all \( t \geq 1 + t_\rho \),
\[
\frac{\mu(f) - 2 C \mu(h)}{\rho(h)} e^{-\alpha t} \leq \frac{\rho P^D_t f}{\rho P^D_t 1} \leq \frac{\mu(f) + 2 C \mu(h)}{1 - C \mu(h)} e^{-\alpha t} \leq \mu(f) + 2 \frac{C \mu(h)}{1 - e^{-\alpha t}}.
\]
Hence, we have proved that, for all $t > 1 + t_{\rho}$,
\[
\left\| \frac{\rho P^D_t}{p P^D_1} - \mu \right\|_{TV} \leq \frac{2C}{1 - e^{-\alpha}} \rho(\mu) e^{-\alpha t}.
\]
If $t \leq 1 + t_{\rho}$,
\[
\left\| \frac{\rho P^D_t}{p P^D_1} - \mu \right\|_{TV} \leq 2 \leq 2 e^{\alpha (1 + t_{\rho})} e^{-\alpha t} = \frac{2 e^{\alpha} C \rho(\mu)}{\rho(\mu)} e^{-\alpha t}.
\]
To sum up, there exists two constants $C, \alpha > 0$ such that, for all $\rho \in \mathcal{M}_1(D)$ and $t \geq 0$,
\[
\left\| \frac{\rho P^D_t}{p P^D_1} - \mu \right\|_{TV} \leq \frac{C \rho(\mu)}{\rho(\mu)} e^{-\alpha t}.
\]
Hence, it remains to prove steps 1 and 2 to conclude the proof.

We start with a preliminary step, ensuring that there exist a non-trivial measure $\zeta$ with $\zeta(U) > 0$, and positive numbers $T_0 > \varepsilon > 0$ such that for all $x \in U, T_0 - \varepsilon \leq t \leq T_0$,
\[
P^D_t(x, \cdot) \geq \zeta(\cdot).
\]
(13)

**Preliminary step:** Since $U$ is $D$-accessible from all $x \in D$, we deduce that
\[
\xi G^D 1_U(x) = \int_0^\infty \xi P^D_t 1_U(x) dt > 0.
\]
Hence there exists $t_\xi > 0$ such that $\xi P^D_{t_\xi} 1_U > 0$. Setting $\zeta' = \xi P^D_{t_\xi}$ and $T' = T + t_\xi$, we then obtain $\zeta'(U) > 0$ and, for all $x \in U$,
\[
P^D_{T'}(x, \cdot) = \delta_p P^D_{T' + t_\xi} \geq \xi P^D_{t_\xi} = \zeta'.
\]
Now, by Fatou Lemma and right continuity of paths (like in the proof of Lemma 2.8), $\liminf_{t \to 0} \zeta' P^D_t(U) \geq \zeta'(U) > 0$. Then, there exist $\varepsilon, \delta > 0$ such that $\zeta' P^D_t 1_U \geq \delta$ for all $t \in [0, \varepsilon]$. Hence, for all $x \in U$ and all $t \in [0, \varepsilon]$,
\[
P^D_{2T' + t}(x, \cdot) \geq \zeta' P^D_{T' + t} \geq \zeta' P^D_{T'}(1_U P^D_t(\cdot)) \geq \delta \zeta'.
\]
Setting $T_0 = 2T' + \varepsilon$ and $\zeta = \delta \zeta'$, we conclude that (13) holds true.

**Proof of Step 1:** For $\delta > 0$ let $O_{\delta} = \{x \in D : G^D(x, U) > \delta\}$. Using a similar argument as for Proposition 2.3 it is not hard to see that $O_{\delta}$ is open in $D$. By (iii), the family $(O_{\delta})_{\delta > 0}$ covers $D$. Thus,
for every compact set $\tilde{K} \subset D$ there exists $\delta > 0$ such that for all $x \in \tilde{K}, G^D(x, U) \geq \delta$. Now, relying on Proposition 2.3 one can choose $S > 0$ large enough so that, for all $x \in \tilde{K}$,

$$\int_0^S P^D_t(x, U)dt > \frac{\delta}{2}.$$ 

Consequently, for all $x \in \tilde{K}$ there is some $0 \leq t_x \leq S$ such that

$$P^D_{t_x}(x, U) \geq \frac{\delta}{2S} =: \delta'.$$

On the other hand, $h$ is bounded below by a positive constant on $U$, because for all $x \in U$ $h(x) = e^{\lambda T_0} P^D_{t_0} h(x) \geq e^{\lambda T_0} \zeta(h) > 0$. Hence,

$$Q_{t_x}(x, U) \geq \delta''$$

for some $\delta'' > 0$ and all $x \in \tilde{K}$. By (13), there exists $c' > 0$ such that

$$Q_t(x, \cdot) \geq c' \zeta'(\cdot)$$

for all $x \in U$ and $T_0 - \varepsilon \leq t \leq T_0$, where $\zeta'(f) = \zeta(fh)$. Choose now $n$ sufficiently large so that $\frac{S}{n} < \varepsilon$. Then for all $x \in \tilde{K}, nT_0 = t_x + n\tau_x$ for some $\tau_x \in [T_0 - \varepsilon, T_0]$. Thus

$$Q_{nT_0}(x, \cdot) \geq \int_U Q_{t_x}(x, dy) Q_{n\tau_x}(y, \cdot) \geq \delta''(c' \zeta'(U))^{n-1}c' \zeta'(\cdot).$$

Proof of Step 2: By Markov inequality,

$$\mathbb{P}_x(\tau^\text{out}_D > T_0) \leq \frac{\mathbb{E}_x(\tau^\text{out}_D)}{T_0} = \frac{G^D(x, D)}{T_0}.$$ 

Let $\theta > 0$ be such that $\rho = e^{\lambda T_0} \theta < 1$. By the assumption that $G^D(C_b(D)) \subset C_0(D)$, there exists $\eta > 0$ and $C' \geq 0$ such that

$$\mathbb{P}_x(\tau^\text{out}_D > T_0) \leq \theta + C' \mathbb{1}_{\{x \in D, d(x, \partial K_D) \geq \eta\}}.$$ 

Then

$$Q_{T_0}(V)(x) = e^{\lambda T_0} \mathbb{P}_x(\tau^\text{out}_D > T_0) \leq \rho V(x) + \frac{C' e^{\lambda T_0}}{\inf_{y \in D, d(y, \partial K_D) \geq \eta} h(y)}.$$ 

$\square$
3 Application to SDEs

The main purpose of this section is to prove Theorems 1.5 and 1.8, to complete the proof of Corollary 1.9, and to provide criteria allowing to check their assumptions. In order to do so, we work in the SDE settings of Section 1.2 and use the results of Section 2 when \(M = \mathbb{R}^n\), \(D = \mathcal{D}\) is a connected, bounded, open subset of \(\mathbb{R}^n\) and \(K = \mathcal{D}\). For simplicity, we denote \(\partial \mathcal{D}\) instead of \(\partial K\) in this section.

Recall that the sets \(B(D)\), \(C_b(D)\) and \(C_0(D)\) denote respectively the space of bounded measurable functions on \(D\), of bounded continuous functions on \(D\) and the closure of the set of compactly supported functions on \(D\) (i.e. the set of bounded continuous functions \(f\) on \(D\) such that \(f(x) \to 0\) when \(d(x, \partial D) \to 0\)). These function spaces are equipped with the norm of uniform convergence.

We let \((P_t)\) denote the Markov semigroup induced by the SDE (1), and by \((P^D_t)\) and \(G^D\) respectively the sub-Markov semigroup and Green kernel of the sub-Markov semi-group induced by the SDE (1) killed when it exits \(D:\) for all \(f \in B(D), x \in \mathcal{D}\) and \(t \geq 0\),

\[
P^D_t f(x) = \mathbb{E}_x \left( f(X_t) \mathbb{1}_{t < \tau^D_{\text{out}}} \right) \quad \text{and} \quad G^D f(x) = \int_0^\infty P^D_u f(x) \, du.
\]

3.1 Accessibility

In this section, we recall that the general notion of accessibility defined in Section 2 reduces to the one of Section 1.2 for diffusions and we provide simple criteria ensuring that Hypothesis (H1) (accessibility of \(\mathbb{R}^n \setminus \mathcal{D}\) from \(\mathcal{D}\)) and condition (ii) of Theorem 1.5 (accessibility of \(D_\varepsilon\) from \(D\)) are satisfied.

Recall that \(y(u, x, \cdot)\) denotes the maximal solution to the control system (2) starting from \(x\) (i.e. \(y(u, x, 0) = x\)).

The following proposition easily follows from the celebrated Stroock and Varadhan’s support theorem [60] (see also Theorem 8.1, Chapter VI in [38]). It justifies the terminology used in Definition 1.1 and makes the link with Definitions 2.2 and 2.7.

**Proposition 3.1** Assume that the \(S^j, j \geq 1\), are bounded with bounded first and second derivatives and \(S^0\) is Lipschitz and bounded. Let \(x \in \mathbb{R}^n\) and \(U \subset \mathbb{R}^n\) open. Then \(U\) is accessible (respectively \(\mathcal{D}\)-accessible) by \((P_t)\) from \(x\) (see Definitions 2.2 and 2.7) if it is accessible (respectively \(\mathcal{D}\)-accessible) by \(\{S^0, (S^j)\}\) from \(x\) (see Definition 1.1).
Remark 3.2 The boundedness assumption is free of charge here, since - by compactness of $\overline{D}$ - we can always modify the $S^j$ outside $\overline{D}$ so that they have compact support.

As an illustration of this latter proposition, we provide an elementary proof of the following result, originally due to Pinsky [52].

**Proposition 3.3** Suppose there exists $\bar{x} \in \mathbb{R}^n \setminus \overline{D}$ and $\delta > 0$ such that for all $x \in D$

$$\sum_{j=1}^{m} \langle S^j(x), x - \bar{x} \rangle^2 \geq \delta \|x - \bar{x}\|^2. \quad (14)$$

Then $\mathbb{R}^n \setminus \overline{D}$ is accessible by $\{S^0, (S^j)\}$ (hence by $(P_t)$) from all $x \in \overline{D}$.

**Proof:** Note that by continuity and compactness (of $\overline{D} \times \{x \in \mathbb{R}^n : \|x\| = 1\}$), one can always assume that (14) holds true on some larger open bounded domain $D'$ with $\overline{D} \subset D'$. For $x \in D$ and $j \in \{1, \ldots, m\}$ set

$$u^j(x) = -\frac{1}{2\delta\varepsilon} \langle S^j(x), x - \bar{x} \rangle$$

where $\varepsilon$ will be chosen later. Consider the ODE

$$\dot{x} = S^0(x) + \sum_j u^j(x)S^j(x)$$

and set $v(t) = \|x(t) - \bar{x}\|^2$. Then, as long as $x(t) \in D'$,

$$\frac{dv(t)}{dt} = -\frac{1}{\varepsilon}v(t) + a$$

where $a = \sup_{x \in D'} 2(\langle S^0(x), x - \bar{x} \rangle)$. Thus

$$v(t) = \|x(t) - \bar{x}\|^2 \leq e^{-t/\varepsilon}(v_0 - a\varepsilon) + a\varepsilon.$$

One can choose $\varepsilon$ small enough so that $x(t)$ meets $D' \setminus \overline{D}$. \qed

The next result (Proposition 3.8) provides a natural condition ensuring that condition (ii) of Theorem 1.5 holds.

Suppose that $D$ satisfies the exterior sphere condition as defined in Definition 1.3. Let $N_p$ denote the set of unit outward normal vectors at $p \in \partial D$ and let

$$N = \{(p, v) : p \in \partial D, v \in N_p\}.$$
Definition 3.4 We say that a vector $w \in \mathbb{R}^n$ points inward $D$ at $p \in \partial D$ if

$$\langle w, v \rangle \leq 0 \text{ for all } v \in N_p, \text{ and } \langle w, v \rangle < 0 \text{ for at least one } v \in N_p.$$  

We say that it points strictly inward $D$ at $p$ if $\langle w, v \rangle < 0$ for all $v \in N_p$.

We say that a vector field $F$ points inward (respectively strictly inward) $D$ if $F(p)$ points inward (strictly inward) $D$ at $p$ for all $p \in \partial D$.

Lemma 3.5 Suppose that $D$ satisfies the exterior sphere condition. Let $F$ be a Lipschitz vector field pointing inward $D$ and let $\Phi = \{\Phi_t\}$ be its flow. Then

(i) $\Phi_t(\overline{D}) \subset D$ for all $t > 0$;

(ii) There exists a compact set $A \subset D$ invariant under $\{\Phi_t\}$ (i.e. $\Phi_t(A) = A$ for all $t \in \mathbb{R}$) such that for all $x \in D \omega_{\Phi}(x) \subset A$, where $\omega_{\Phi}(x)$ stands for the omega limit set of $x$ for $\Phi$.

Proof: We first show that $\Phi_t(\overline{D}) \subset D$ for all $t \geq 0$. Suppose not. Then, for some $p \in \partial D$ and $\varepsilon > 0$, $d(\Phi_t(p), \overline{D}) > 0$ on $[0, \varepsilon]$. The function

$$t \to V(t) := d(\Phi_t(p), \overline{D}),$$

being Lipschitz on $[0, \varepsilon]$ it is absolutely continuous, hence almost everywhere derivable and $V(t) = \int_0^t \dot{V}(u)du$. Let $0 < t_0 \leq \varepsilon$ be a point at which it is derivable, $x_0 = \Phi_{t_0}(p)$ and $p_0 \in \partial D$ be such that $\|x_0 - p_0\| = d(x_0, \overline{D})$. In particular, $\frac{x_0 - p_0}{\|x_0 - p_0\|} \in N_{p_0}$. Then for all $t > 0$,

$$\frac{V(t_0 + t) - V(t_0)}{t} = \frac{d(\Phi_t(x_0), \overline{D}) - d(x_0, \overline{D})}{t} \leq \frac{\|\Phi_t(x_0) - p_0\| - \|x_0 - p_0\|}{t}.$$  

Letting $t \to 0$, using that $F(p_0)$ points inward $D$ at $p_0$, we get

$$\dot{V}(t_0) \leq \frac{\langle F(x_0), x_0 - p_0 \rangle}{\|x_0 - p_0\|} \leq \frac{\langle F(x_0) - F(p_0), x_0 - p_0 \rangle}{\|x_0 - p_0\|} \leq L\|x_0 - p_0\| = LV(t_0),$$

where $L$ is a Lipschitz constant for $F$. By Gronwall’s lemma we then get that for all $0 < s < t \leq \varepsilon$, $V(t) \leq e^{L(t-s)}V(s)$. Since $V(0) = 0$, $V$ cannot be positive. This proves the desired result. Note that this first result doesn’t require that the exterior sphere condition holds everywhere but only that, for all $p \in \partial D$ at which it holds, $\langle F(p), v \rangle \leq 0$ for all $v \in N_p$ (compare with Theorem 2.3 in [S]).
We now show that $\Phi_t(\overline{D}) \subset D$ for $t > 0$. This amounts to show that for all $p \in \partial D$ and all $\varepsilon > 0$ small enough, $\Phi_{-\varepsilon}(p) \in \mathbb{R}^n \setminus \overline{D}$. So let $p \in \partial D$ and choose $v \in N_p$ such that $\delta := -\langle F(p), v \rangle > 0$. By assumption $d(p + rv, \overline{D}) = r$ for some $r > 0$. Thus for all $\varepsilon > 0$ such that $\varepsilon\|F(p)\|^2 < r\delta$,

$$\|(p - \varepsilon F(p)) - (p + rv)\|^2 < r^2 - \varepsilon \delta r.$$  

Since $\|\Phi_{-\varepsilon}(p) - (p - \varepsilon F(p))\| = o(\varepsilon)$, this shows that $\|\Phi_{-\varepsilon}(p) - (p + rv)\| < r$ for $\varepsilon > 0$ small enough, so that $\Phi_{-\varepsilon}(p) \not\in \overline{D}$.

Assertion (ii) is a consequence of (i). It suffices to set

$$A = \cap_{t \geq 0} \Phi_t(\overline{D}).$$

For $(p, v) \in N$, set $R(p, v) = \sup\{r > 0 : d(p + rv, \overline{D}) = r\} \in (0, \infty]$, and let

$$R_{\partial D} = \inf_{(p, v) \in N} R(p, v).$$

**Definition 3.6** We say that $D$ satisfies the **strong exterior sphere condition** if it satisfies the exterior sphere condition and $R_{\partial D} \neq 0$.

**Remark 3.7** If $D$ is convex or $\partial D$ is $C^2$, then $D$ satisfies the strong exterior sphere condition. However, the following example shows that the exterior sphere condition and the strong exterior sphere condition are not equivalent: For $1 \leq \alpha \leq 2$, let $D^\alpha \subset \mathbb{R}^2$ be defined as

$$D^\alpha = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, |y| < x^\alpha\}.$$  

Then $D^\alpha$ satisfies the exterior sphere condition for all $1 \leq \alpha \leq 2$ but not the strong one for $1 < \alpha < 2$. Indeed, for such an $\alpha$, $N_{(0,0)} = \{v : \|v\| = 1, v_1 < 0\}$ and

$$\lim_{v \to (0,1), v \in N_{(0,0)}} R((0,0), v) = 0.$$  

**Proposition 3.8** Assume that $D$ satisfies the **strong exterior sphere condition**. Recall that $N_p$ denotes the set of unit outward normal vectors at $p \in \partial D$.

(i) For each $p \in \partial D$, the two following conditions are equivalent:

(a) $N_p \cap -N_p = \emptyset$ and for each $v \in N_p$

$$\langle S^0(p), v \rangle < 0 \text{ or } \sum_{i=1}^{m} \langle S^i(p), v \rangle^2 \neq 0;$$
(b) There exists a vector $w \in \text{Span}\{S^1(p), \ldots, S^m(p)\}$ such that $S^0(p) + w$ points strictly inward $D$ at $p$.

(ii) If for all $p \in \partial D$ condition $(i) - (a)$ (or $(i) - (b)$) holds, then

$$D_\varepsilon = \{ x \in D : d(x, \partial D) > \varepsilon \}$$

is $D$-accessible by $\{S^0, (S^i)\}$ from all $x \in D$, for some $\varepsilon > 0$.

Proof: We first prove Point (ii). Assume that $(i) - (b)$ holds at every $p \in \partial D$ and prove that $D_\varepsilon$ is accessible for some $\varepsilon > 0$. The assumption $R_{\partial D} > 0$ makes $N$ closed (hence compact). Indeed, if $(p_n, v_n) \to (p, v)$ with $(p_n, v_n) \in N$, then, for any $0 < r < R_{\partial D}$, $d(p_n + rv_n, D) = r$. Thus $d(p + rv, \overline{D}) = r$.

This has the consequence that, if a continuous vector field $F$ points strictly inward $D$ at $p \in \partial D$, it points strictly inward $D$ at $q \in \partial D$ for all $q$ in a neighborhood of $p$. Therefore, by compactness, there exists a covering of $\partial D$ by open sets $U_1, \ldots, U_k$, and vector fields $W_1, \ldots, W_k \in \text{Span}\{S^1, \ldots, S^m\}$ such that for all $p \in \partial D \cap U_i$, $F_i(p) := S^0(p) + W_i(p)$ points strictly inward $D$ at $p$. Set $U_0 = \mathbb{R}^n \setminus \partial D$ and let $\{\rho_i\}_{i=0}^k$ be a partition of unity subordinate to $\{U_i\}_{i=0}^k$. That is $\rho_i \in C^\infty(\mathbb{R}^n), \rho_i \geq 0, \sum_{i=0}^k \rho_i = 1$, and $\text{supp}(\rho_i) \subset U_i$. Define $F = \rho_0 S^0 + \sum_{i=1}^k \rho_i F_i$. Then, $F$ points strictly inward $D$ and writes

$$F = S^0 + \sum_{i=1}^m u^i S^i$$

with $u^i \in C^\infty(\mathbb{R}^n)$. In view of (15), Lemma 3.5 proves the result.

Point (i). We now prove that conditions $(i) - (a)$ and $(i) - (b)$ are equivalent. The implication $(i) - (b) \Rightarrow (i) - (a)$ is straightforward. We focus on the converse implication. Let

$$\text{Cone}(N_p) = \{tv, t \geq 0, v \in N_p\}$$

and $\text{conv}(N_p)$ be the convex hull of $N_p$. We claim that

$$\text{conv}(N_p) \subset \text{Cone}(N_p)$$

and $0 \notin \text{conv}(N_p)$. To prove the first inclusion, it suffices to show that $\text{Cone}(N_p)$ is convex. To shorten notation, assume (without loss of generality) that $p = 0$. Let $x, y \in \text{Cone}(N_0)$ and $0 \leq t \leq 1$. By definition of $N_0$, $\text{Cone}(N_0) = \{z \in \mathbb{R}^n, \exists r > 0 \text{ s.t. } d(rz, \overline{D}) = ||rz||\}$,
so there exists \( r > 0 \) such that \( d(rx, D) = \|rx\| \) and \( d(ry, D) = \|ry\| \).

Thus for all \( z \in D \)

\[
\|rx - z\|^2 - \|rx\|^2 = \|z\|^2 - 2\langle rx, z \rangle \geq 0.
\]

Similarly \( \|z\|^2 - 2\langle ry, z \rangle \geq 0 \). Thus

\[
\|r(tx + (1-t)y) - z\|^2 - \|r(tx + (1-t)y)\|^2 = t(\|z\|^2 - 2\langle rx, z \rangle) + (1-t)(\|z\|^2 - 2\langle ry, z \rangle) \geq 0.
\]

This proves that \( tx + (1-t)y \in \text{Cone}(N_0) \), hence convexity of \( \text{Cone}(N_0) \).

The fact that \( 0 \notin \text{conv}(N_p) \) follows from the fact that \( N_p \cap -N_p = \emptyset \).

Indeed, suppose to the contrary that \( 0 = \sum_{i=1}^k t_i x_i \) with \( k \geq 2, x_i \in N_p, t_i > 0 \) and \( \sum_{i=1}^k t_i = 1 \). Then

\[
-\frac{t_1}{1-t_1}x_1 \in \text{conv}(x_2, \ldots, x_k) \subset \text{conv}(N_p) \subset \text{cone}(N_p).
\]

Thus \(-x_1 \in N_p \). A contradiction.

We shall now deduce the implication \((i)-(a) \Rightarrow (i)-(b)\) from the Minimax theorem (see e.g. [57]). For all \( j \in \{1, \ldots, m\} \) set \( S^{-j} = -S^j \).

Let \( J = \{-m, \ldots, 0, \ldots m\} \) and

\[
\Delta(J) = \{ \alpha \in \mathbb{R}^J : \alpha_j \geq 0, \sum_{j \in J} \alpha_j = 1 \}.
\]

By condition \((i)-(a)\) and compactness of \( N_p \) there exists \( \delta > 0 \) such that for all \( v \in N_p \)

\[
\min_{j \in J} \langle S^j(p), v \rangle \leq -\delta.
\]

Thus for all \( v \in \text{Cone}(N_p) \)

\[
\min_{j \in J} \langle S^j(p), v \rangle \leq -\delta \|v\|.
\]

The set \( N_p \) being compact (in finite dimension) its convex hull is also compact by Carathéodory’s theorem. Thus, because \( 0 \notin \text{conv}(N_p) \), \( \|v\| \geq \frac{\delta'}{\delta} \) for some \( \delta' > 0 \) and all \( v \in \text{conv}(N_p) \). It then follows that

\[
\sup_{v \in \text{conv}(N_p)} \inf_{\alpha \in \Delta(J)} \left\langle \sum_{j \in J} \alpha_j S^j(p), v \right\rangle \leq \sup_{v \in \text{conv}(N_p)} \min_{j \in J} \langle S^j(p), v \rangle \leq -\delta'.
\]

By the Minimax theorem, the left hand side also equals

\[
\sup_{\alpha \in \Delta(J)} \inf_{v \in \text{conv}(N_p)} \left\langle \sum_{j \in J} \alpha_j S^j(p), v \right\rangle
\]

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and this infimum is achieved for some $\beta \in \Delta(J)$. If $\beta_0 \neq 0$ this implies that
\[
\sup_{v \in \text{conv}(N_p)} (S^0(p) + \sum_{j \in J, j \neq 0} \frac{\beta_j}{\beta_0} S^j(p), v) \leq -\frac{\delta'}{\beta_0} < 0.
\]
If $\beta_0 = 0$, for $R > 0$ sufficiently large
\[
\sup_{v \in \text{conv}(N_p)} (S^0(p) + R \sum_{j \in J} \beta_j S^j(p), v) \leq -R \delta' + \|S^0(p)\| < 0.
\]
This concludes the proof. \qed

Remark 3.9 It follows from Proposition 3.8 that whenever $\partial D$ is $C^2$, Hypothesis (H2') implies condition (ii) of Theorem 1.5 because at each point $p \in \partial D$ there is a unique outward unit normal. The following example shows that this is not true in general when (H2') is satisfied but $\partial D$ is not $C^2$. Let $D^1$ be as in Remark 3.7 with $\alpha = 1$, and let $(X_t)$ be solution to
\[
dX_t = S^1(X_t) \circ dB_t
\]
where $S^1(x, y) = (1, 2)$. At each point $p \in \partial D$ there is at least one $v \in N_p$ such that $(S^1(p), v) \neq 0$ so that condition (i) - (a) is satisfied. However, $D$ does not satisfy the strong exterior sphere condition and, for $0 < \eta < \varepsilon, D_{\varepsilon}^1$ is not $D$-accessible from $(\eta, 0)$.

Observe also that none of the conditions required in Proposition 3.8 is necessary for $D_{\varepsilon}^1$ to be accessible. Let $D^\alpha$ be as in Remark 3.7 with $1 < \alpha \leq 2$ and let
\[
dX_t = e_1 \circ dB^1_t + e_2 \circ dB^2_t,
\]
with $(e_1, e_2)$ the canonical basis of $\mathbb{R}^2$. As shown in Corollary 1.9 (and its proof), $D_{\varepsilon}^\alpha$ is accessible; while for $1 < \alpha < 2, R_{\partial D} = 0$ and for $\alpha = 2, N_{0,0} \cap -N_{0,0} \neq \emptyset$.

3.2 Proof of Theorem 1.5

In this Section, we start with general path properties in Lemma 3.10 and show that, under Assumptions (H1) and (H2), $G^D(C_b(D)) \subset C_0(D)$ in Lemma 3.11. We then prove Theorem 1.5. We conclude this section with Proposition 3.12 which shows that Assumption (H2') implies Assumption (H2).
Let $C(\mathbb{R}^+_+, \mathbb{R}^n)$ be the set of continuous paths $\eta : \mathbb{R}^+_+ \to \mathbb{R}^n$ equipped with the topology of uniform convergence on compact intervals.

Let $(\eta_n)_{n \geq 0}$ be a sequence converging to $\eta$ in $C(\mathbb{R}^+_+, \mathbb{R}^n)$. Set

$$\tau_D^{n,\text{out}} = \inf\{t \geq 0 : \eta_n(t) \in \mathbb{R}^n \setminus D\}, \tau_D^{\text{out}} = \inf\{t \geq 0 : \eta(t) \in \mathbb{R}^n \setminus D\}$$

Define $\tau_D^{n,\text{out}}$ and $\tau_D^{\text{out}}$ similarly. In the following lemmas, $C_0(\overline{D})$ is the space of bounded continuous functions on $\overline{D}$ vanishing on $\partial D$.

**Lemma 3.10**

(i) Suppose $\eta(0) \in D$. Then for all $f \in C_0(\overline{D})$ such that $f \geq 0$ and all $t \geq 0$,

$$\liminf_{n \to \infty} f(\eta_n(t)) 1_{\{\tau_D^{n,\text{out}} > t\}} \geq f(\eta(t)) 1_{\{\tau_D^{\text{out}} > t\}}.$$

In particular

$$\liminf_{n \to \infty} \tau_D^{n,\text{out}} \geq \tau_D^{\text{out}}.$$

(ii) Suppose $\eta(0) \in \overline{D}$. Then

$$\limsup_{n \to \infty} \tau_D^{n,\text{out}} \leq \tau_D^{\text{out}}$$

and for all $f \in C_0(\overline{D}), f \geq 0$ and all $t \geq 0$,

$$\limsup_{n \to \infty} f(\eta_n(t)) 1_{\{\tau_D^{n,\text{out}} > t\}} \leq f(\eta(t)) 1_{\{\tau_D^{\text{out}} > t\}}.$$

**Proof:**

(i) If $\tau_D^{\text{out}} \leq t$ the statement is obvious. If $\tau_D^{\text{out}} > t$, then $\eta([0, t]) \subset D$ so that, for $n$ large enough, $\eta_n([0, t]) \subset D$. That is $\tau_D^{n,\text{out}} > t$ and the statement follows. The assertion that $\liminf_{n \to \infty} \tau_D^{n,\text{out}} \geq \tau_D^{\text{out}}$ follows by choosing $f = 1_D$.

(ii) Suppose to the contrary that $\tau_D^{n,\text{out}} > \tau_D^{\text{out}} + \epsilon$ for some $\epsilon > 0$ and infinitely many $n$. Then $\eta^n([0, \tau_D^{\text{out}} + \epsilon]) \subset D$. Hence $\eta([0, \tau_D^{\text{out}} + \epsilon]) \subset \overline{D}$.

A contradiction. The last assertion directly follows for $t = \tau_D^{\text{out}}$ and $f \in C_0(\overline{D})$ (where $C_0(\overline{D})$ is the set of bounded continuous functions on $\overline{D}$, hence it contains $C_0(\overline{D})$). If now $t = \tau_D^{\text{out}}$ and $f \in C_0(\overline{D})$, $f(\eta^n(t)) \to f(\eta(t)) = 0$. \qed

The next lemma shows that under (H1) and (H2), $G^D(C_b(\overline{D})) \subset C_0(D)$.

In this lemma, $P^D_t$ denotes the semigroup defined as $P^D_t f(x) = \mathbb{E}_x(f(X_t) 1_{x > \tau_D^{\text{out}}})$ for all $f$ bounded and measurable on $\overline{D}$.

**Lemma 3.11**

Suppose that Hypothesis (H1) holds. Then,
For all $f \geq 0, f \in C_b(D), x \in D$ and $t \geq 0$,
\[
\liminf_{y \to x} P_t^D f(y) \geq P_t^D f(x),
\]
(ii) For all $f \geq 0, f \in C^0(D), x \in D$ and $t \geq 0$
\[
\limsup_{y \to x, y \in \overline{D}} P_t^D f(y) \leq P_t^D f(x)
\]
(iii) Suppose, in addition, that Hypothesis (H2) holds. Then,
(a) For all $x \in D$,
\[
P_x(\tau^\text{out} = \tau^\text{out}_D) = 1;
\]
(b) For all $f \in C^0(D)$ and $t \geq 0$, $P_t^D f \in C_b(D)$;
(c) For all $f \in C_b(D)$ and $t \geq 0$, $G^D f \in C^0(D)$.

Proof: Let $(X_t^x)$ be the (strong) solution to (1) with initial condition $X_0^x = x$. We can always assume that $(X_t^x)_{t \geq 0, x \in \mathbb{R}^n}$ is defined on the Wiener space space $C(\mathbb{R}_+, \mathbb{R}^m)$ equipped with its Borel sigma field and the Wiener measure $P$ (the law of $(B_1, \ldots, B_m)$). That is $P_x(\cdot) = P(X^x \in \cdot)$. Also, for all $\omega \in C(\mathbb{R}_+, \mathbb{R}^m)$, the map $x \in \mathbb{R}^n \to X^x(\omega) \in C(\mathbb{R}_+, \mathbb{R}^n)$ is continuous (see for instance [44], Theorem 8.5).

Assertions (i) and (ii) then follow from Lemma 3.10 and Fatou’s lemma.

We now pass to the proof of (iii).
(a) follows from the strong Markov property, valid since $(P_t)_{t \geq 0}$ is Feller (see e.g. [44], Theorem 6.17), as follows. For all $x \in D$,
\[
P_x(\tau^\text{out}_D = \tau^\text{out}_D) = \mathbb{E}_x(P_{X^x}^{\tau^\text{out}_D}(\tau^\text{out}_D = 0)) = 1.
\]

(b) Let $x \in D$. The property $P_x(\tau^\text{out}_D = \tau^\text{out}_D) = 1$ implies that $P_t^D f(x) = P_t^D \hat{f}(x)$ for all $f \in C_0(D)$, where $\hat{f}(x) = f(x)$ if $x \in D$ and $\hat{f}(x) = 0$ if $x \in \overline{D} \setminus D$. Hence, by (i) and (ii),
\[
\lim_{y \to x} P_t^D f(y) = P_t^D f(x)
\]
for all $f \in C_0(D), f \geq 0$. If now $f \in C_0(D)$ it suffices to write $f = f^+ - f^-$ with $f^+ = \max(f, 0)$ and $f^- = (-f)^+$. 

(c) Write $\tau^{x, \text{out}}_D$ for $\inf\{t \geq 0 : X^x_t \notin D\}$. Again, the property $P_x(\tau^{x, \text{out}}_D = \tau^{x, \text{out}}_D) = 1$ combined with Lemma 3.10 imply that, almost
surely, the maps \( x \in \mathcal{D} \to \tau_{\mathcal{D}}^{x} \) and \( x \in \mathcal{D} \to \int_{0}^{\tau_{\mathcal{D}}^{x}} f(X_s^x)ds \) are continuous for all \( f \in C_0(\mathcal{D}) \). Also,

\[
\sup_{x \in \mathcal{D}} \mathbb{E} \left[ \left( \int_{0}^{\tau_{\mathcal{D}}^{x}} f(X_s^x)ds \right)^2 \right] \leq \|f\|^2 \sup_{x \in \mathcal{D}} \mathbb{E}_x[(\tau_{\mathcal{D}}^{x})^2] < \infty
\]

where the last inequality follows from Proposition 2.3. This shows that the family \( (\int_{0}^{\tau_{\mathcal{D}}^{x}} f(X_s^x)ds)_{x \in \mathcal{D}} \) is uniformly integrable. Thus, \( x \in \mathcal{D} \to \mathbb{E}(\int_{0}^{\tau_{\mathcal{D}}^{x}} f(X_s^x)ds) = G_{\mathcal{D}} f(x) \) is continuous.

To conclude, observe that \( |G_{\mathcal{D}} f(x)| \leq \|f\|_{\mathcal{D}} \) and that \( G_{\mathcal{D}} f(x) = \mathbb{E}(\tau_{\mathcal{D}}^{x}) \) almost surely, and hence, by (ii) of Lemma 3.10, \( \tau_{\mathcal{D}}^{x} \to \tau_{\mathcal{D}}^{x} \). Since \( \tau_{\mathcal{D}}^{x} = 0 \) almost surely by Assumption (H2) and since \( (\tau_{\mathcal{D}}^{x})_{x \in \mathcal{D}} \) is uniformly integrable by Proposition 2.3, we deduce that \( \mathbb{E}(\tau_{\mathcal{D}}^{x}) \) converges to 0 when \( x \to p \).

\[ \square \]

**Proof of Theorem 1.5:** We assume that

(i) Hypotheses (H1) and (H2) hold true;

(ii) For some \( \varepsilon > 0 \), the set \( \mathcal{D}_\varepsilon = \{x \in \mathcal{D} : d(x, \partial \mathcal{D}) > \varepsilon\} \) is \( \mathcal{D} \)-accessible by \( \{S^0, (S^i)\} \) from all \( x \in \mathcal{D} \setminus \mathcal{D}_\varepsilon \).

In order to prove Theorem 1.5 we show that the assumptions of Theorem 2.9 are satisfied with \( M = \mathbb{R}^n \), \( D = \mathcal{D} \), \( K = \overline{\mathcal{D}} \), \( X \) following the dynamic defined by (1), \( P^D = P^D \) and \( G^D = G^D \). According to Lemma 3.11 (iii) - (c), Hypothesis (H1) entails that \( G^D(C_0(\mathcal{D})) \subset C_0(\mathcal{D}) \). In addition, Condition (ii) of Theorem 1.5 and Proposition 3.1 entail that Assumption (ii) of Theorem 2.9 holds true. In particular, we deduce that Theorem 2.9 applies and hence that there exists a QSD for \( (X_t) \) on \( \mathcal{D} \). \[ \square \]

In the next result, we prove that (H2') implies (H2).

**Proposition 3.12** Let \( p \in \partial \mathcal{D} \). Assume that there exist a unit outward vector \( v \) at \( p \) and \( j \in \{1, \ldots, m\} \) such that \( \langle v, S^j(p) \rangle \neq 0 \). Then, \( p \) is regular with respect to \( \mathbb{R}^n \setminus \overline{\mathcal{D}} \) for (1). In particular, Hypothesis (H2') implies (H2).

**Proof:** Without loss of generality we can assume that \( j = 1 \). By Definition 1.3 there is \( r > 0 \) such that \( d(p+rv, \overline{\mathcal{D}}) = r \). Let \( \Psi : \mathbb{R}^n \to \mathbb{R} \)
be defined as
\[ \Psi(x) = r^2 - \|x - (p + rv)\|^2. \]
Then
\[ \Psi(x) > 0 \Rightarrow x \in \mathbb{R}^n \setminus \overline{D}, \]
and \( \langle \nabla \Psi(p), S^1(p) \rangle \neq 0 \). Hence, for some neighborhood \( U \) of \( p \),
\[ (\langle \nabla \Psi(x), S^1(x) \rangle)^2 \geq a > 0 \]
on \( \overline{U} \).
By Ito’s formula, since \( \Psi(p) = 0 \),
\[ \Psi(X^p_{t \wedge \tau_{\overline{U}}^{\text{out}}}) = \int_0^{t \wedge \tau_{\overline{U}}^{\text{out}}} L\Psi(X^p_s)ds + M_{t \wedge \tau_{\overline{U}}^{\text{out}}} \]
where
\[ L\Psi = S^0(\Psi) + \frac{1}{2} \sum_{j=1}^m (S^j)^2(\Psi), \]
\[ M_t = \sum_{j=1}^m \int_0^t \sigma^j(X^p_s)dB^j_s, \]
and \( \sigma^j(x) \) is any bounded measurable function coinciding with \( \langle \nabla \Psi(x), S^j(x) \rangle \) on \( \overline{U} \). In the definition of \( L \) above, we used the standard notation for differential operators defined from vector fields: \( S^i(f)(x) = \langle S^i(x), \nabla f(x) \rangle \).
For convenience, we set \( \sigma^1(x) = \sqrt{a} \) for \( x \notin \overline{U} \). Therefore,
\[ \Psi(X^p_{t \wedge \tau_{\overline{U}}^{\text{out}}}) \geq M_{t \wedge \tau_{\overline{U}}^{\text{out}}} - b(t \wedge \tau_{\overline{U}}^{\text{out}}) \]
with \( b = \sup_{x \in \overline{U}} |L\Psi(x)| \), and
\[ (M)_t = \sum_{j=1}^m \int_0^t \sigma^j(X^p_s)^2ds \geq at. \]
By Dubins-Schwarz Theorem (see e.g. [44], Theorem 5.13) there exists a Brownian motion \( (\beta) \) such that \( M_t = \beta_{(M)_t} \) for all \( t \geq 0 \). Thus, for all \( \varepsilon > 0 \)
\[ \sup_{0 \leq t \leq \varepsilon} (M_t - bt) \geq \sup_{0 \leq t \leq \varepsilon} (\beta_{(M)_t} - \frac{b}{a}(M)_t) \geq \sup_{0 \leq t \leq \alpha \varepsilon} (\beta_t - \frac{b}{a}t). \]
By the law of the iterated logarithm for the Brownian motion, the right hand term is almost surely positive.
Now, using (16), it follows that \( \tau_{\overline{U}}^{\text{out}} \leq \varepsilon \) almost surely on the event \( \{ \tau_{\overline{U}}^{\text{out}} > \varepsilon \} \). Thus \( \mathbb{P}_p(\tau_{\overline{U}}^{\text{out}} = 0) = 1 \) because \( \mathbb{P}_p(\tau_{\overline{U}}^{\text{out}} > 0) = 1 \). \( \Box \)

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3.3 Proof of Theorem 1.8

Consequences of Hörmander conditions and Bony’s results [8]

We assume throughout all this subsection, as in Theorem 1.8, that:

(i) Hypotheses (H1) and (H2') holds true;

(ii) The weak Hörmander condition (see Definition 1.7) is satisfied at every $x \in \mathcal{D}$.

Lemma 3.13 The operator $G^D$ is a compact operator on $C_0(\mathcal{D})$.

Proof: By Lemma 3.11 (iii) (c) and Proposition 2.3, the operator $G^D$ is a bounded operator on $C_0(\mathcal{D})$ whose image is in $C_0(\mathcal{D})$. It then defines a bounded operator on $C_0(\mathcal{D})$. To prove compactness, we rely on the following (easy) consequence of Ascoli’s theorem (see e.g. [51], exercise 6, chapter 7): A family $F \subset C_0(\mathcal{D})$ is relatively compact if (and only if) it is bounded, equicontinuous at every point $x \in \mathcal{D}$ and vanishes uniformly at $\partial \mathcal{D}$ (regardless of the smoothness of $\partial \mathcal{D}$); that is for all $\varepsilon > 0$ there exists a compact set $W \subset \mathcal{D}$ such that $|f(x)| \leq \varepsilon$ for all $f \in F$ and $x \in \mathcal{D} \setminus W$.

Let $F = \{G^D(f) : f \in C_0(\mathcal{D}), \|f\| \leq 1\}$. Then $F$ is bounded and vanishes uniformly at $\partial \mathcal{D}$ because for all $f$ in the unit ball of $C_0(\mathcal{D})$ $|G^D(f)| \leq G^D \mathbb{1}_\mathcal{D} \in C_0(\mathcal{D})$.

We now prove the equicontinuity property. Let $L$ be the differential operator defined as $L = S^0 + \frac{1}{2} \sum_{j=1}^{m} (S^j)^2$.

We claim that for every $f \in C_0(\mathcal{D})$, the map $g = G^D f$ satisfies $Lg = -f$ on $\mathcal{D}$, in the sense of distributions. We proceed in two steps: first we will show how the result can be proved from the claim and second we will prove the claim.

Step 1. First, assume that the claim is proved. Then, by a theorem due to Rothschild and Stein (Theorem 18 in [56]) there exists $0 < \alpha < 1$ depending only on the family $S^0, S^1, \ldots, S^m$ such that $G^D(C_0(\mathcal{D})) \subset \Lambda^\alpha(\mathcal{D})$. Here (see [56], p. 301 and [58], p. 141), $\Lambda^\alpha(\mathcal{D})$ denotes the set of locally $\alpha$-Hölder functions, that is the set of $f : \mathcal{D} \mapsto \mathbb{R}$ such that for every compact $W \subset \mathcal{D}$,

$$\|f\|_{0,W} := \sup_{x \in W} |f(x)| + \sup_{x \neq y \in W} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.$$ 

Let now $x_0 \in \mathcal{D}$, $W \subset \mathcal{D}$ be a compact neighborhood of $x_0$, $C(W)$ the space of continuous functions on $W$ equipped with the
uniform norm, $C^\alpha(W)$ the Banach space of $\alpha$-Hölder functions on $W$ equipped with the corresponding Hölder norm $\| \cdot \|_{\alpha,W}$ and $i_W : C_0(\mathcal{D}) \to C(W)$, $f \mapsto f|_W$. The operator $i_W \circ G^D : C_0(\mathcal{D}) \to C(W)$ is bounded, where $\circ$ denotes here the composition of functions. Its graph is then closed in $C_0(\mathcal{D}) \times C(W)$ and consequently also in $C_0(\mathcal{D}) \times C^\alpha(W)$. Hence, by the closed graph theorem, it is a bounded operator from $C_0(\mathcal{D})$ into $C^\alpha(W)$. In particular, for all $f$ in the unit ball of $C_0(\mathcal{D})$ and $x,y \in W$, $|G^D(f)(x) - G^D f(y)| \leq \kappa \|x - y\|^\alpha$ for some $\kappa$ depending only on $W$. This proves that $F$ is equicontinuous at $x_0$.

Step 2. We now pass to the proof of the claim.

By a theorem of Bony ([8], Théorème 5.2), for any $a > 0$ and $f \in C_b(\overline{\mathcal{D}})$ (the set of bounded continuous functions on $\overline{\mathcal{D}}$), the Dirichlet problem:

$$\begin{cases}
Lg - ag = -f \text{ on } \mathcal{D} \text{ (in the sense of distributions)} \\
g|_{\partial \mathcal{D}} = 0;
\end{cases} \tag{17}$$

has a unique solution, call it $g_a$, continuous on $\overline{\mathcal{D}}$. Furthermore, if $f$ is smooth ($C^\infty$) on $\mathcal{D}$ so is $g_a$. Note that the assumptions required for this theorem are implied by the properties of $\partial \mathcal{D}$ and of the vector fields $S^j$ near $\partial \mathcal{D}$ given in Assumption (H2') and the weak Hörmander condition assumed in (ii).

Suppose that $f$ is smooth on $\mathcal{D}$, meaning that $f \in C_b(\overline{\mathcal{D}}) \cap C^\infty(\mathcal{D})$ (so that $g_a$ is smooth on $\mathcal{D}$). Then by Ito’s formula

$$\left( e^{-at^{\tau^{\text{out}}}} g_a(X_{t^{\tau^{\text{out}}}}) + \int_0^{t^{\tau^{\text{out}}}} e^{-as} f(X_s) ds \right)_{t \geq 0}$$

is a local martingale. Being bounded, it is a uniformly integrable martingale. Thus, taking the expectation and letting $t \to \infty$, we get that

$$\int_0^\infty e^{-as} P^D_s f(x) ds = \mathbb{E}_x \left( \int_0^{\tau^{\text{out}}} e^{-as} f(X_s) ds \right) = g_a(x). \tag{18}$$

In particular, $G^D f(x) = \lim_{a \to 0} g_a(x)$, where the convergence is uniform by Proposition 2.3.

For every smooth test function $\Phi$ with compact support in $\mathcal{D}$

$$\langle g_a, L^* \Phi \rangle - a \langle g_a, \Phi \rangle = -\langle f, \Phi \rangle$$
\[ \langle h, \Phi \rangle = \int h(x)\Phi(x)dx. \]

Letting \( a \to 0 \), we get that \( \langle G^D f, L^*\Phi \rangle = -\langle f, \Phi \rangle \), that is
\[ LG^D(f) = -f, \]
in the sense of distributions. This proves the claim whenever \( f \) is smooth on \( \mathcal{D} \). If now \( f \in C_0(\mathcal{D}) \), let \((f_n)_{n \in \mathbb{N}}\) be smooth with compact support in \( \mathcal{D} \) with \( f_n \to f \) uniformly on \( \overline{\mathcal{D}} \). Then \( G^D(f_n) \to G^D(f) \) uniformly; and, by the same argument as above, \( LG^D f = -f \) in the sense of distributions. 

\[ \square \]

The proofs of the next two lemmas are similar to the proof of Corollary 5.4 in [5]. For convenience we provide details.

**Lemma 3.14** Let \( p \in \mathcal{D} \) be such that \( S^i(p) \neq 0 \) for some \( i \in \{1, \ldots, m\} \). Then, there exist disjoint open sets \( U, V \subset \mathcal{D} \) with \( p \in U \) and a non-trivial measure \( \xi \) on \( V \) (i.e. s.t. \( \xi(V) > 0, \xi(\mathbb{R}^n \setminus V) = 0 \)) such that for all \( x \in U \),
\[ G^D(x, \cdot) \geq \xi(\cdot). \]

**Proof**: We first assume that for all \( x \in \mathcal{D} \) there is some \( i \in \{1, \ldots, m\} \) such that \( S^i(x) \neq 0 \). Then, by Theorem 6.1 in [8], for all \( a > 0 \), there exists a map \( K_a : \mathcal{D}^2 \to \mathbb{R}_+ \) smooth on \( \mathcal{D}^2 \setminus \{(x, x) : x \in \mathcal{D}\} \) such that for all \( f \in C_b(\mathcal{D}) \), and \( x \in \mathcal{D} \)
\[ \int_0^\infty e^{-as}P^D_s f(x)ds = \int_\mathcal{D} K_a(x, y)f(y)dy. \]

To be more precise, Theorem 6.1 in [8] asserts that the solution to the Dirichlet problem \[17\] can be written under the form given by the right hand side of the last equality. On the other hand, we have shown in the proof of Lemma 3.13 that this solution is given by the left hand side.

Fix \( a = 1 \) and choose \( q \neq p \) such that \( K_1(p, q) > 0 \). Such a \( q \) exists, for otherwise, we would have \( \int_0^\infty e^{-sP^D_s}1_\mathcal{D}(p)ds = 0 \). That is \( \tau^\text{out}_p = 0, \mathbb{P}_p \) almost surely. By continuity of \( K_1 \) off the diagonal, there exist disjoint neighborhoods \( U, V \) of \( p \) and \( q \) and some \( c > 0 \) such that \( K_1(x, y) \geq c \) for all \( x \in U, y \in V \). Thus, for all \( x \in U \),
\[ G^D(x, \cdot) \geq \int_0^\infty e^{-sP^D_s} (x, \cdot) \geq cLeb(V \cap \cdot) \]
where \( Leb \) stands for the Lebesgue measure on \( \mathbb{R}^n \).
We now pass to the proof of the Lemma. Using a local chart around $p$ we can assume without loss of generality that $p = 0$, $S^1(0) \neq 0$ and $\frac{S^1(0)}{\|S^1(0)\|} = e_1$, where $(e_1, \ldots, e_n)$ stands for the canonical basis on $\mathbb{R}^n$. Let $B_\varepsilon = \{ x \in \mathbb{R}^n : \sum_{i=1}^n |x_i| < \varepsilon \}$. For $\varepsilon > 0$ small enough $S^1(x) \neq 0$ for all $x \in B_\varepsilon$ and for all $x \in \overline{B}_\varepsilon$ there is a vector $u$ normal to $\overline{B}_\varepsilon \setminus B_\varepsilon$ such that $\langle S^1(x), u \rangle \neq 0$. The preceding reasoning can then be applied with $B_\varepsilon$ in place of $D$. Thus, for some disjoint open sets $U, V \subset B_\varepsilon$ with $p \in U$ and for all $x \in U$, \[ G^D(x, \cdot) \geq G^{B_\varepsilon}(x, \cdot) \geq c \text{Leb}(V \cap \cdot). \]

\[ \square \]

**Lemma 3.15** Let $p \in \mathcal{D}$ be such that the Hörmander condition holds at $p$. Then, there exist a neighborhood $U$ of $p$, a non trivial measure $\xi$ on $\mathcal{D}$ and $T > 0$ such that for all $x \in U$, $P^D_T(x, \cdot) \geq \xi(\cdot)$.

**Proof:** Let $V \subset \mathcal{D}$ be a neighborhood of $p$ such that the Hörmander condition holds at every $x \in V$. Let $P^V_T(x, \cdot)$ stands for the law of the stopped process $X^x_{t\wedge \tau^V_{\mathcal{D}}}$. By [27, Theorem 4.3 (ii)], there exists a nonnegative map $p^V_T(x, y)$ smooth in the variables $(t, x, y) \in \mathbb{R}^*_+ \times V \times V$ such that
\[ P^V_T(x, A) = \int_A p^V_T(x, y) dy \]
for every Borel subset $A$ of $V$. Choose $q \in V$ and $T > 0$ such that $p^V_T(p, q) := c > 0$. Then, by continuity, there exist neighborhoods $U, W \subset V$ of $p$ and $q$ such that $p^V_T(x, y) \geq c/2 > 0$ for all $(x, y) \in U \times W$. Thus, for all $x \in U$,
\[ P^D_T(x, \cdot) \geq P^D_T(x, \cdot \cap W) \geq P^V_T(x, \cdot \cap W) \geq \frac{c}{2} \text{Leb}(W \cap \cdot). \]

\[ \square \]

Note that the continuity properties of the kernels $K_a$ and $p^V_T$ used in the proof of the previous lemmas 3.14 and 3.15 could also be derived from the results obtained by P. Cattiaux ([9], Theorems 4.14 and 3.35) via Malliavin’s calculus.

We conclude this section with the statement and proof of Lemma 3.16 used to complete the proofs of Theorem 1.8 (given below) and Corollary 1.9 (given in Section 1.2).
Lemma 3.16 Let $\mu$ be a QSD for $(X_t)$ on $\mathcal{D}$. Then $\mu$ has a smooth density with respect to the Lebesgue measure on $\mathcal{D}$. If furthermore the strong Hörmander conditions holds at every $x \in \overline{\mathcal{D}}$, this density is positive.

Proof: Let $\Phi$ be a smooth function with compact support in $\mathcal{D}$. By Ito’s formula $(\Phi(X_{t\wedge \tau_{\mathcal{D}}}^\text{out}) - \int_0^{t\wedge \tau_{\mathcal{D}}} L\Phi(X_s)ds)_{t \geq 0}$ is a bounded local martingale, hence a true martingale. Thus, taking the expectation, letting $t \to \infty$ and using Proposition 2.3, it comes that $G^\mathcal{D}(L\Phi)(x) = -\Phi(x)$ for all $x \in \mathcal{D}$. Let $\mu$ be a QSD with rate $\lambda$. Then $-\mu(\Phi) = \mu G^\mathcal{D}(L\Phi) = \frac{1}{\lambda} \mu(L\Phi)$. This shows that $L^*\mu + \lambda \mu = 0$ on $\mathcal{D}$ in the sense of distributions. Now, for all distribution $f$ on $\mathcal{D}$,

$$L^*f = S^0f + \frac{1}{2} \sum_{j=1}^{m} (S^j)^2f + Tf$$

where $T$ is a smooth function and $S^0 + S^0 \in \text{Span}(S^1, \ldots, S^m)$. Therefore $L^*$ satisfies the weak Hörmander property. By Hörmander Theorem [36], it is hypoelliptic. This implies that $\mu$ has a smooth density.

For $a > 0$, set $L^*_a f = L^* f - af$ and choose $a$ sufficiently large so that $L^*_a 1 = T - a < 0$. If the strong Hörmander condition is satisfied at every point $x \in \overline{\mathcal{D}}$, the same is true for $L^*_a$. Now, $L^*_a(-\mu) = (\lambda + a)\mu \geq 0$. Therefore, by application of Bony’s maximum principle ([8], Corollary 3.1), if the density of $-\mu$ vanishes at some $x \in \mathcal{D}$, it has to be zero on $\mathcal{D}$. This is impossible because $\mu$ is a probability measure. $\Box$

Proof of Theorem 1.8

We now assume that Conditions (i), (ii), (iii) of Theorem 1.8 hold. By Lemma 3.13 and Corollary 2.12 there exists a positive right eigenfunction for $G^\mathcal{D}$. By Conditions (i) and (iii) of Theorem 1.8 there exists a point $p \in \mathcal{D}$ near $\partial \mathcal{D}$, $\mathcal{D}$-accessible, at which $S^j(p) \neq 0$ for some $j \geq 1$. Then, by Lemma 3.14 and Lemma 2.15 ($P_t^\mathcal{D}$) is irreducible. Thus, according to Theorem 2.13 it has a unique QSD. Such a QSD has a smooth density by Lemma 3.16. This proves all the statements in Theorem 1.8 except (3) under the additional assumption that there exists $x^* \in \mathcal{D}$ where the Hörmander condition is satisfied. This last result follows from Lemma 3.15 and Theorem 2.16.
4 Beyond diffusions

This section briefly discusses a simple example of piecewise deterministic Markov process. This illustrates the applicability of our results beyond diffusions and justifies some of the abstract conditions made in Section 2. In particular, the facts that $M$ can be chosen to be a general metric space rather than $\mathbb{R}^n$ or sub-space of $\mathbb{R}^n$, and that $D$ can be chosen to be open relative to $K = \overline{D}$ rather than open.

Let $M = \mathbb{R} \times \{0, 1\}$ endowed with the metric $d((x, i), (y, j)) = |i - j| + |x - y|$ and $(Z_t) = ((X_t, I_t))$ be the Markov process on $M$ defined by

$$\dot{X}_t = 2I_t - 1,$$

and $(I_t)$ is a continuous time Markov chain on $\{0, 1\}$ having jump rates $\lambda_{0,1} = \lambda_{1,0} = \lambda > 0$. In words, $(X_t)$ moves at velocity 1 either to the left or to the right and changes direction at a constant rate $\lambda$. It is easy to check that the semi-group associated to $(Z_t)$ is Feller.

Suppose now that $(Z_t)$ is killed when $(X_t)$ exits $]0, 1[$. At the exit time, one clearly has $X_t = I_t$ almost surely, so that it is natural to set $D = [0, 1] \times \{0\} \cup [0, 1] \times \{1\}$.

Defining $K = \overline{D} = [0, 1] \times \{0, 1\}$, $D$ is open relative to $K$ and

$$\partial_K D = \{(0,0)\} \cup \{(1,1)\}.$$

It is not hard to show that $M \setminus K$ is accessible from all $z \in K$ (in the sense of definition 2.2), and that $G^D(C_b(D)) \subset C_0(D)$. The verification is left to the reader.

For any function $f : M \to \mathbb{R}$ and $i \in \{0, 1\}$ we write $f_i(x) = f(x, i)$.

Lemma 4.1 Let $H : [0, 1] \to \mathbb{R}_+$ and $\omega > 0$ be defined as follows:

- If $\lambda > 1$, $H(x) = \frac{\sin(\theta x)}{\theta}, \omega = \lambda(1 + \cos(\theta))$, where $0 < \theta < \pi$ is the unique solution to $\lambda \sin(\theta) = \theta$;
- If $\lambda = 1$, $H(x) = x, \omega = 2$;
- If $\lambda < 1$, $H(x) = \frac{\sinh(\theta x)}{\theta}, \omega = \lambda(1 + \cosh(\theta))$, where $\theta > 0$ is the unique solution to $\lambda \sinh(\theta) = \theta$.

Then,

(i) The map $h \in C_0(D)$ defined by

$$h_i(x) = (1 - i)H(x) + iH(1 - x)$$

is a positive right eigenfunction for $G^D$ with eigenvalue $\frac{1}{\omega}$.
The probability on $D$ defined by

$$
\mu = \frac{1}{2} \int_0^1 H(s) ds (h_1(x) dx \otimes \delta_0 + h_0(x) dx \otimes \delta_1)
$$

is a QSD for $(Z_t)$ on $D$ with absorption rate $\omega$.

**Proof:** Let $L$ denote the infinitesimal generator of the process $(Z_t)$ and $D(L)$ its domain (defined in the usual sense for Feller processes, see e.g. [44]; see also [21] more specifically on piecewise deterministic Markov processes). It is easy to verify that if $f : M \mapsto \mathbb{R}$ is $C^1$ in $x$, with compact support, then $f \in D(L)$ and

$$
Lf(x, i) = \begin{cases}
-f'_0(x) + \lambda (f_1(x) - f_0(x)) & \text{for } i = 0 \\
 f'_1(x) + \lambda (f_0(x) - f_1(x)) & \text{for } i = 1.
\end{cases}
$$

Now, observe that the map $H$ satisfies the identity

$$
-H'(x) + \lambda (H(1-x) - H(x)) = -\omega H(x)
$$

for all $0 \leq x \leq 1$. Extending $H$ to a smooth function on $\mathbb{R}$ having compact support, formula (19) defines a map $h \in D(L)$ such that, for all $z \in D$

$$
Lh(z) = -\omega h(z).
$$

Therefore, $((h(Z_{t \wedge \tau^D_{out}}) - h(z) - \int_0^{t \wedge \tau^D_{out}} Lh(Z_s) ds)_t$ is a $\mathbb{P}_z$ martingale (because $h \in D(L)$) bounded in $L^1$, and by the stopping time theorem

$$
-h(z) + \omega \mathbb{E}_z \left( \int_0^{\tau^D_{out}} h(Z_s) ds \right) = \mathbb{E}_z \left( h(Z_{\tau^D_{out}}) - h(z) - \int_0^{\tau^D_{out}} Lh(Z_s) ds \right) = 0.
$$

That is $G^D h(z) = \frac{1}{\omega} h(z)$. This prove $(i)$.

We now prove $(ii)$. In order to prove that $\mu$ is a QSD with rate $\omega$, it suffices to show that for all $f \in C_0(D)$, $\mu G^D(f) = \frac{1}{\omega} \mu(f)$. Given such a $f$, one can easily solve the problem $Lg = -f$ on $\hat{D}$ with $g \in C_0(D)$. Namely,

$$
g_0(x) = \int_0^x [f_0(u) - \lambda (x-u)(f_0(u) + f_1(u))] du + a \lambda x
$$

and

$$
g_1(x) = \int_0^x [-f_1(u) - \lambda (x-u)(f_0(u) + f_1(u))] du + a (\lambda x + 1),
$$

41
where $a$ is determined by $g_1(1) = 0$. These formulae show that $g_0$ and $g_1$ can be extended to $C^1$ maps on $\mathbb{R}$ with compact support. Reasoning exactly like in (i) this shows that $G^D(f) = g$ on $D$. The problem of showing that $\mu G^D(f) = \frac{1}{\omega} \mu(f)$ then reduces to show that $\mu(Lg) = -\omega \mu(g)$. This latter identity can be easily checked, using (20), (22) and integration by parts.

**Remark 4.2** Observe that $(P^D_t)$ is not a semigroup on $C_0(D)$ because for any $0 < t < 1/2$, $P^D_t \mathbb{1}_D$ is discontinuous. Indeed, $P^D_t \mathbb{1}_D(x, 0) = 1$ for $t < x < 1/2$, while $P^D_t \mathbb{1}_D(t, 0) = 1 - e^{-\lambda t}$. It is, however, a strongly continuous semigroup on $C_0(D)$. Let $L^D$ be its generator and $\mathcal{D}(L^D)$ its domain. A by product of the proof above is that $\mathcal{D}(L^D)$ consists of the maps $f \in C^1_0(D)$ such that

$$
  f_0(0) = f_1(1) = -f'_0(0) + \lambda f_1(0) = f'_1(1) + \lambda f_0(1) = 0,
$$

and $L^D f(x) = Lf(x)$ for all $x \in D$ and $f \in \mathcal{D}(L^D)$. Here $L$ is the operator defined by (21) and $C^1_0(D)$ stands for the set of maps $f \in C_0(D)$ such that $f_0, f_1$ can be extended to $C^1$ maps on $\mathbb{R}$.

**Remark 4.3** The formula defining $g = G^D(f)$ in the previous proof is also valid for $f \in C_0(D) + \mathbb{R} \mathbb{1}_D$. In particular,

$$
  G^D(\mathbb{1}_D)(x, 0) = \mathbb{E}_{x,0}(\tau^\text{out}_D) = x(1 + \lambda - \lambda x).
$$

Observe that the map $x \mapsto \mathbb{E}_{x,0}(\tau^\text{out}_D)$ achieves its maximum at $x = 1$ for $\lambda \leq 1$ and at $x = \frac{1 + \lambda}{2\lambda} < 1$ for $\lambda \geq 1$ (this last fact can be surprising at first sight).

**Proposition 4.4** Let $\mu$ and $h$ be as in Lemma 4.1. There exist $C, \alpha > 0$ such that for all probability $\rho$, on $D$

$$
  \left\| \frac{\rho P^D_t}{\rho P^D_t \mathbb{1}_D} - \mu(\cdot) \right\|_{TV} \leq \frac{C}{\rho(h)} e^{-\alpha t}
$$

**Proof:** We rely on Theorem 2.16. Assertions (i) and (ii) are satisfied. For (iii), fix $\varepsilon < 1/10$ and let $U = \left[ \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon \right] \times \{0\}$. Then $U$ is clearly $D$-accessible from all $z \in D$.

Set $T = \frac{1}{2} - 2\varepsilon$. For $z = (x, 0) \in U$, and $A$ a Borel subset of $\left] \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon \right]$, $P^D_T(z, A \times \{0\}) \geq \mathbb{P}(x - \tau_1 + (\tau_2 - \tau_1) - (T - \tau_2) \in A, \tau_2 < T < \tau_3)$,
where $0 < \tau_1 < \tau_2 < \tau_3$ are the three first jump times of $I_t$. Thus, using the fact that, given $\tau_2 < t < \tau_3$, $(\tau_1, \tau_2)$ is distributed as the order statistics of $(U_1, U_2)$, where $U_1$ and $U_2$ are i.i.d. uniform r.v. on $[0, t]$,

$$P^D_T(z, A \times \{0\}) \geq \mathbb{P}(x - T + 2(\tau_2 - \tau_1) \in A | \tau_2 < T < \tau_3) \frac{(\lambda T)^2}{2} e^{-\lambda T}$$

$$= \frac{(\lambda T)^2}{2} e^{-\lambda T} \int_A \frac{(x + T - u)}{2T^2} du \geq c \text{Leb}(A)$$

for some $c > 0$. This shows condition $(iii)$ of Theorem 2.16 with $d\xi = c1_U du \otimes \delta_0$. \hfill \Box

**Remark 4.5** The minorization of $P^D_T(z, \cdot)$ proved in Proposition 4.4 could also be obtained for general PDMPs under appropriate bracket conditions on the vector fields defining the PDMP, along the lines of the general results proved in [1] or [7]. Hence Theorem 2.16 applies to such processes provided one can prove that $G^D(C_b(D)) \subset C_0(D)$ and $G^D$ is a compact operator on $C_0(D)$, by Corollary 2.12.

**Acknowledgment**

We thank Emmanuel Trelat for fruitful discussions and his suggestion to use reference [56], in Lemma 3.13. We thank Josef Hofbauer for his valuable suggestions on Lemma 3.5. We also thank 4 anonymous referees for their careful reading and valuable suggestions to improve this paper. This research is supported by the Swiss National Foundation grants 200020 196999 and 200020 219913.

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