Two-way quantum cryptography at different wavelengths

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We study the security of two-way quantum cryptography at different wavelengths of the electromagnetic spectrum, from the optical range down to the microwave range. In particular, we consider a two-way quantum communication protocol where Gaussian-modulated thermal states are subject to random Gaussian displacements and finally homodyned. We show how its security threshold (in reverse reconciliation) is extremely robust with respect to the preparation noise and able to outperform the security thresholds of one-way protocols at any wavelength. As a result, improved security distances are now accessible for implementing quantum key distribution at the very challenging regime of infrared frequencies.

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I. INTRODUCTION

Continuous-variable quantum key distribution (QKD) allows the secure distribution of a secret message between two authenticated parties, Alice and Bob, using nothing more than standard optical equipment, such as off-the-shelf lasers and homodyne detectors. Although being conceived and developed many years after the original discrete-variable schemes, continuous-variable (CV) QKD has seen many advances and milestones over the years, both theoretical and experimental. More recent advances include, the security against general attacks in the finite-size regime, analysis of practical imperfections, as well as the role of noiseless amplification and an experimental demonstration over 80 km of optical fiber.

From a practical point of view, there has been recent interest in thermal QKD which essentially considers the realistic effect of unknown preparation noise on Alice’s signal states. One of the applications of thermal QKD is the ability to generate secure keys at different wavelengths of the electromagnetic field, from optical range down to the infrared and microwave regimes. Secure communication at different wavelengths is ubiquitous in today’s communication environment. From the optical telecommunication wavelength of 1550 nm down into the GHz microwave regime, utilized by technologies such as Wi-Fi and cellular phones.

In this paper, we show how to improve the security of thermal QKD at different wavelengths, using two-way quantum communication, a scheme which is already known to tolerate higher levels of loss and noise. Variants of the two-way quantum communication protocol also exist. The idea of using additional preparation noise for two-way quantum communication was preliminarily investigated. Here they showed the ‘fighting noise with noise’ effect, an effect first seen in discrete-variable QKD and later in CV one-way protocols. Essentially what it shows is that if extra noise is added in the appropriate way, then the performance of the protocol can be improved, in terms of secret-key rate and security threshold.

In our paper we go further than this initial analysis by showing the security of the two-way protocol in the presence of considerably large levels of preparation noise, corresponding to the use of different communication wavelengths. We show that two-way QKD is extremely robust in reverse reconciliation, such as to beat one-way protocols for any value of the preparation noise, a feature which allows us to improve the performance of QKD at the infrared regime.

This paper is structured as follows. In Sec. II, we introduce one-way thermal QKD and how this basic setup can be extended to two-way quantum communication. Then, in Sec III, we analyze the security of the two-way thermal QKD protocol by deriving its secret-key rates in direct reconciliation and reverse reconciliation. Finally, Sec. V is used for the conclusion.

II. THERMAL QKD: FROM ONE-WAY TO TWO-WAY QUANTUM COMMUNICATION

A. One-way thermal QKD

A thermal QKD protocol begins with the sender, Alice, randomly displacing thermal states in the phase space according to a bivariate Gaussian distribution. These modulated thermal states are sent to the receiver, Bob, over an insecure quantum channel which is monitored by the eavesdropper, Eve (see Fig. 1). On average, the generic quadrature $\hat{A}$ of Alice’s input mode $A$ can be written as $\hat{A} = \hat{0} + \alpha$, where the real number $\alpha$ is the Gaussian encoding variable with variance $V_\alpha$, and $\hat{0}$ is a quadrature operator accounting for the thermal ‘preparation noise’, with variance $V_0 \geq 1$ (see Ref. [22] for details on our notation). The overall variance of Alice’s average state is therefore given by $V_A = V_0 + V_\alpha$.

The variance $V_0$ can be broken down as $V_0 = 1 + \eta$, where $1$
is the variance of the vacuum noise, and \( \eta \geq 0 \) is the variance of an extra noise which is confined in Alice’s station and not known to either Eve, Alice or Bob. At the output of the channel, Bob homodynes the incoming mode \( B \), randomly switching between position and momentum detections (note that an alternative no-switching protocol based on heterodyne detection can also be considered [23]). In this way, Bob collects an output variable \( b \) which is correlated to Alice’s encoding \( a \).

By making use of a public classical channel, Bob reveals his basis choices and compares a subset of his data with Alice in order to estimate the noise in the channel and, therefore, the maximum information stolen by Eve. If the noise is not too high, compared to a security threshold, Alice and Bob apply classical post-processing procedures of error correction and privacy amplification in order to extract a shared secret-key.

In particular, in standard CV-QKD we consider post-processing procedures based on one-way classical communication, which can be forward from Alice to Bob, also known as direct reconciliation (DR) [24]), or backward from Bob to Alice, also known as reverse reconciliation (RR) [25]). In DR, Bob aims to guess Alice’s encoding \( a \) with the help of classical communication from her, while in RR it is Alice who aims to guess Bob’s decoding \( b \) with the help of classical communication from him. Despite Alice and Bob’s mutual information not depending on the direction of the reconciliation, Eve’s stolen information does. For this reason, there are two different secret-key rates defined for the two different types of reconciliation.

Since the above thermal protocol is a Gaussian protocol [1], its security can be tested against collective Gaussian attacks [26,27], being proven the most powerful attacks allowed by quantum physics, given a suitable symmetrization of the protocols [28]. Collective Gaussian attacks can be characterized and classified in various canonical forms [29], of which the most important for practical purposes is represented by the entangling cloner collective attack [30]. Such an attack consists in Eve interacting an ancilla mode \( E \) with the signal mode \( A \) by means of a beam splitter with transmission \( T \in [0, 1] \). The ancilla mode is part of an Einstein-Podolski-Rosen (EPR) state \( \rho_{EE'} \) [31], which is an entangled Gaussian state with zero mean and covariance matrix (CM)

\[
V_{EE'} = \begin{pmatrix}
W & \sqrt{W^2 - Z^2} \\
\sqrt{W^2 - Z^2} & W
\end{pmatrix},
\]

where \( W \geq 1 \), \( I = \text{diag}(1,1) \) and \( Z = \text{diag}(1,-1) \). Both the kept mode \( E'' \) and the transmitted mode \( E' \) are collected in a quantum memory (QM) which is coherently measured at the end of the protocol.

In a collective entangling-cloner attack, Eve’s stolen information can be over-estimated using the Holevo bound [32]. In particular, we have to consider two Holevo quantities \( I(E : a) \) or \( I(E : b) \) depending if Eve attacks Alice’s variable (DR) or Bob’s variable (RR). Subtracting these quantities from Alice and Bob’s classical mutual information \( I(a : b) \), we get the two secret-key rates of the protocol \( R^\triangleright = I(a : b) - I(E : a) \) and \( R^\blacktriangleright = I(a : b) - I(E : b) \), for DR and RR, respectively. For a thermal QKD protocol based on modulated thermal states and homodyne detection, these key rates are functions of the input parameters, namely the variance of the thermal noise \( V_0 \) and the variance of the classical signal modulation \( V_w \), plus the parameters of the attack, which are the transmission \( T \) of the channel and its thermal variance \( W \).

In the typical limit of high modulation \( V_0 \gg 1 \), we get the two analytical expressions for the asymptotic secret-key rates

\[
R^\triangleright(V_0, W, T) = \frac{1}{2} \log \frac{T \Lambda(W, V_0)}{(1-T) \Lambda(V_0, W)} + h \left[ \sqrt{\frac{W \Lambda(1, W V_0)}{\Lambda(W, V_0)}} - h(W) \right],
\]

(2)

\[
R^\blacktriangleright(V_0, W, T) = \frac{1}{2} \log \frac{W}{(1-T) \Lambda(V_0, W)} - h(W),
\]

(3)

where we have used the two functions

\[
\Lambda(x, y) := T x + (1-T) y,
\]

(4)

\[
h(x) := \log \left( \frac{x + 1}{2} \right) - \log \left( \frac{x - 1}{2} \right).
\]

(5)

(See Appendix A for details on their derivation.) By setting these key rates to zero, we can derive the two security thresholds \( W^\triangleright = W^\triangleright(V_0, T) \) and \( W^\blacktriangleright = W^\blacktriangleright(V_0, T) \) in DR and RR, respectively. Equivalently, these security thresholds can be given in terms of tolerable excess noise \( N^\triangleright = N^\triangleright(V_0, T) \) and \( N^\blacktriangleright = N^\blacktriangleright(V_0, T) \), where \( \Lambda := (W - 1)(1-T)T^{-1} \).

Note that other types of one-way thermal protocols may be considered, e.g., using homodyne detection instead of heterodyne at the output [13]. However, the corresponding security thresholds, \( N^\triangleright \) and \( N^\blacktriangleright \), are numerically very similar to those of the one-way protocol just described, which can be considered as the most representative and practical candidate of one-way thermal QKD.

B. Two-way thermal QKD

The scenario of Fig. 1 can be readily extended to two-way quantum communication [14]. As depicted in Fig. 2, Bob has an input mode \( B_1 \) where a thermal state with preparation variance \( V_0 \) is modulated by a bivariate Gaussian distribution with signal variance \( V_{b_1} := \mu \). On average, we have the input quadrature \( B_1 = 0 + b_1 \), encoding the Gaussian variable \( b_1 \). In
As discussed in Ref. [14], the use of two-mode coherent attacks against the two-way protocol is not advantageous for Eve. In fact, using the OFF configuration against such attacks, Alice and Bob can reach security thresholds which are much higher than those of one-way protocols. Thus, we consider here the analysis of collective one-mode attacks, in particular, those based on entangling cloners, which are the most practical benchmark to test CV-QKD. We show that, using the ON configuration against these attacks, Alice and Bob are able to extract a secret-key in conditions so noisy that any one-way protocol would fail. In particular, this happens in reverse reconciliation which turns out to be extremely robust in the preparation noise, therefore allowing us to improve the performance of CV-QKD in the very noisy regime of infrared frequencies.

### III. CRYPTOANALYSIS OF TWO-WAY THERMAL QKD

Here we study the security performance of the two-way thermal QKD protocol against collective entangling-cloner attacks. Adopting the ON configuration, we derive the analytical expressions of the asymptotic secret-key rates (i.e., for high modulation \( \mu \to +\infty \)), first in DR and then in RR. Such rates are explicitly plotted in the transmission \( T \) for \( W = 1 \) (pure-loss channel) and studied in terms of the preparation noise \( V_0 \). In the specific case of RR, we also analyze the behaviour of the security threshold \( W_{\text{m}} = W_{\text{m}}(V_0, T) \) for different values of \( T \) and \( V_0 \), comparing this threshold with that of the corresponding one-way thermal protocol.

As shown in Fig. [2] a collective entangling-cloner attack against the two-way protocol consists of Eve performing two independent and identical beam-splitter attacks (transmission \( T \)), one for each use of the channel. For each beam splitter \( i = 1 \) or 2, Eve prepares two ancilla modes \( E_i \) and \( E_i' \) in an EPR state with variance \( W \). Eve keeps mode \( E_i' \) while injecting the other mode \( E_i \) into one port of the beam splitter, leading to the transmitted mode \( E_i'' \). These operations are repeated identically and independently for each signal mode sent by Bob as well as the return mode sent back to Bob by Alice. All of Eve’s output modes \( E_i'' \) and \( E_i''' \) are stored in a quantum memory which is coherently detected at the end of the two-way protocol. Eve’s final measurement is optimized based on Alice and Bob’s classical communication.

For such an attack, Bob’s post-processing of his classical variables is just given by \( b = b_2 - b_1 \). This variable is the optimal linear estimator of Alice’s variable \( a \) in the limit of high modulation \( \mu \to +\infty \). Note that this classical post-processing can be equivalently realized by constructing a displaced mode \( B \) with generic quadrature \( B = b_2 - b_1 \) which is then homodyned by Bob. Despite being useful for the theoretical analysis of the protocol in RR, such a physical representation is not practical since it involves the use of a quantum memory to store mode \( B_2 \) whose displacement \( -Tb_1 \) can only be applied once Bob has estimated the channel transmission \( T \). It is also important to remark that this equivalent representation is realized by Gaussian operations, so that the global output state of Bob (\( B \)-mode) and Eve (\( E \)-modes) is Gaussian (this is true...
for both $b_1$ fixed or Gaussian-modulated).

A. Secret-key rate in direct reconciliation

Let us start our security analysis considering DR \[13\]. This type of reconciliation was shown to be very robust for one-way protocols \[12\], which were in principle able to tolerate an infinite amount of preparation noise and still have a finite secret key, albeit very small \[13\]. As we show below, such a behavior is not typical of two-way thermal protocols.

As we know, the secret key rate for DR is given by $R^\bullet := I(a : b) - I(a : E)$. The mutual information between Alice and Bob is derived from the differential Shannon entropy \[34\] and is simply given by

$$I(a : b) = \frac{1}{2} \log_2 \frac{V_b}{V_{b|a}} ,$$

where $V_b$ is the variance of Bob’s post-processed variable $b$, and $V_{b|a}$ its variance conditioned to Alice’s encoding $a$. These variances are easy to compute once we write the Bogoliubov transformations for the quadratures.

The output mode $B_2$ has generic quadrature

$$\hat{B}_2 = T \hat{B}_1 + \sqrt{T} \hat{a} + \sqrt{1 - T} (\sqrt{T} \hat{E}_1 + \hat{E}_2) .$$

Subtracting off the input modulation $b_1$ (known only to Bob), we get the processed quadrature $\hat{B} = \hat{B}_2 - T b_1$ equal to

$$\hat{B} = T \hat{0} + \sqrt{T} \hat{a} + \sqrt{1 - T} (\sqrt{T} \hat{E}_1 + \hat{E}_2) ,$$

with variance $V_B = T^2 V_0 + T V_a + (1 - T^2) W$. Since $V_B = V_b$ and $V_a = \mu$, we get

$$V_b = T^2 V_0 + T \mu + (1 - T^2) W ,$$

which gives $V_b \rightarrow T \mu$ in the limit of high modulation.

In the same limit, the conditional variance $V_{b|a}$ is given by setting $\mu = 0$ in the previous equation for $V_b$, i.e., we have

$$V_{b|a} = V_{b|a} = T^2 V_0 + (1 - T^2) W .$$

Therefore, the mutual information between Alice and Bob is given by

$$I(a : b) = \frac{1}{2} \log_2 \frac{T^2 V_0 + T \mu + (1 - T^2) W}{T^2 V_0 + (1 - T^2) W} .$$

$$\rightarrow \frac{1}{2} \log_2 \frac{T \mu}{T^2 V_0 + (1 - T^2) W} .$$

Eve’s Holevo information on Alice’s encoding variable $a$ is defined as

$$H(a : E) := S(E) - S(E|a) ,$$

where $S(\cdot)$ is the von Neumann entropy of Eve’s multimode output state $\rho_E$ (modes $E'_1 E''_1 E'_2 E''_2$) and $S(E|a)$ the entropy of the conditional state $\rho_E|a$ for fixed values of Alice’s encoding variable $a$. Since these states are Gaussian, their entropies can be computed from the symplectic spectra of their covariance matrices, $V_E$ and $V_{E|a}$, respectively \[1\].

By generalizing the derivation in Ref. \[29\] to include the presence of preparation noise ($V_0 \geq 1$) we get the following expression of Eve’s CM for the Gaussian state $\rho_E$ of modes $E'_1 E''_1 E'_2 E''_2$

$$V_E(V_a, V_a) = \begin{pmatrix} \varepsilon I & \varphi Z & \chi I & 0 \\ \varphi Z & WI & \theta Z & 0 \\ \chi I & \theta Z & \Delta(V_a, V_a) & \varphi Z \\ 0 & 0 & \varphi Z & WI \end{pmatrix} ,$$

where $0 := \text{diag}(0, 0)$ and the parameters are defined as

$$\varepsilon := (1 - T)V_{B_1} + TW ,$$

$$\chi := -\sqrt{T}(1 - T)(W - V_{B_1}) ,$$

$$\theta := -(1 - T)(W^2 - 1) ,$$

$$\gamma := T(1 - T)V_{B_1} + (1 - T + T^2) W ,$$

$$\varphi := \sqrt{T(W^2 - 1)} ,$$

$$\Delta(V_a, V_a) := \gamma I + (1 - T) \text{diag}(V_a, V_a) .$$

In the previous parameters, we set $V_{B_1} = V_0 + \mu$ and $V_0 = \mu$, and we consider the limit of high modulation ($\mu \rightarrow +\infty$). Thus, we are able to compute the asymptotic symplectic spectrum of the CM which is given by the four eigenvalues $\nu_1 \rightarrow W$, $\nu_2 \rightarrow W$ and $\{\nu_3, \nu_4\}$ such that $\nu_3 \nu_4 \rightarrow (1 - T)^2 \mu^2$. Using these eigenvalues, we compute the entropy of Eve’s state $\rho_E$ which is given by \[35\]

$$S(E) = \sum_{k=1}^{4} h(\nu_k) \rightarrow 2 h(W) + \log \left(\frac{\varepsilon}{2}\right)^2 (1 - T)^2 \mu^2 ,$$

where we have used the h-function of Eq. \[5\] and its asymptotic expansion $h(x) \simeq \log(ex) / 2$ for large $x$.

Now we consider the conditional CM $V_{E|a}$ which is given by $V_{E|a}$ \[29\]. As a result, $V_{E|a}$ is the same as $V_E$ except for the adjustment of the variable $\Delta(\mu, \mu) \rightarrow \Delta(0, \mu)$. In the usual limit ($\mu \rightarrow +\infty$) we compute the conditional spectrum $\nu_1 \rightarrow 1$, $\nu_2 \rightarrow W$ and $\{\nu_3, \nu_4\}$ such that $\nu_3 \nu_4 \rightarrow (1 - T)^2 \sqrt{(1 - T)^2 W \mu^2}$. Such eigenvalues allow us to derive the conditional entropy $S(E|a)$ and, therefore, to compute Eve’s Holevo information

$$I(a : E) \rightarrow h(W) + \frac{1}{2} \log \frac{(1 - T)\mu}{(1 + T)W} .$$

Combining Eqs. \[11\] and \[21\], we get following asymptotic expression for the DR secret-key rate

$$R^\bullet (V_0, T, W) = \frac{1}{2} \log \frac{T(1 + T)W}{(1 - T)(T^2 V_0 + (1 - T^2) W)} - h(W) .$$
V₀. As we can see from the figure, two-way quantum communication with modulated pure states (V₀ = 1) is able to beat the 3 dB loss limit (corresponding to the threshold T = 1/2). However, as the preparation noise is increased, we see a fairly rapid deterioration in the security of the protocol. Such a behavior is different from what happens in DR for the corresponding one-way thermal protocol [12, 13]. In fact, despite one-way thermal QKD being secure only within the 3 dB loss limit, such limit is not affected by the preparation noise V₀, so that high values of V₀ are tolerable in the range 0.5 < T < 1 with the DR secret-key rate remaining positive even if close to zero.

However, contrarily to what happens in DR, we show below that two-way thermal QKD is very robust to the preparation noise in RR, such that its security threshold outperforms both the thresholds (in DR and RR) of the one-way thermal QKD at any value of the preparation noise V₀. This is the feature that we will exploit to improve the security at the infrared regime.

### B. Secret-key rate in reverse reconciliation

Let us derive the RR secret-key rate $R^\bullet := I(a : b) - I(E : b)$ [33]. Here we need to compute Eve’s Holevo information on Bob’s processed variable b, i.e., $I(E : b) = S(E) - S(E|b)$. From the formula, it is clear that we need to compute the entropy $S(E|b)$ of Eve’s output state $\rho_{E|b}$ conditioned to Bob’s variable b. To compute the CM $V_{E|b}$ of this state, we first derive the global CM

$$V_{EB} = \begin{pmatrix} V_E & D \\ D^T & V_b I \end{pmatrix},$$

(23)

describing Eve’s modes $E', E''$, $E_1', E_2'$, with reduced CM $V_E$ given in Eq. (13), plus Bob’s virtual mode $B$, with reduced CM $V_b I$, with the variance $V_b$ computed in Eq. (9).

Then, we apply homodyne detection on mode $B$, which provides

$$V_{E|b} = V_E - \langle 1/V_b \rangle D D I D^T,$$

where $\Pi := \text{diag}(1, 0, 0, 0)$. Here the off-diagonal block $D$ describes the correlations between Eve’s and Bob’s modes, and is given by

$$D^{\bullet} = \begin{pmatrix} \xi_1 I, & \phi_1 Z, & \xi_2 I, & \phi_2 Z \end{pmatrix},$$

(24)

where

$$\xi_1 = -T \sqrt{1 - T} (V_0 - W), \quad (25)$$

$$\phi_1 = \sqrt{T (1 - T) V^2 - 1}, \quad (26)$$

$$\xi_2 = -\sqrt{T (1 - T) (V_0 + V_0) + TW \sqrt{1 - T}}, \quad (27)$$

$$\phi_2 = \sqrt{1 - T \sqrt{V^2 - 1}}. \quad (28)$$

In the previous formulas, we set $V_0 = V_b = \mu$ and we consider the limit of high modulation $\mu \rightarrow +\infty$. In this limit, we derive the asymptotic expression of the conditional symplectic spectrum $\{\nu_1, \nu_2, \nu_3, \nu_4\}$, which is given by $\nu_1 \rightarrow W$,

$$\nu_2 \rightarrow \sqrt{\frac{W (1 + T^2 V_0 W + T^3 (1 - V_0 W))}{T^2 V_0 + W + T^3 (W - V_0)}},$$

(29)

and

$$\nu_3 \nu_4 \rightarrow \sqrt{\frac{(1 - T)^3 (T^2 V_0 + W + T^3 (W - V_0)) \mu^3}{T}}.$$  (30)

Using this spectrum we compute the conditional entropy $S(E|b)$ and, therefore, the RR secret-key rate, whose asymptotic expression is equal to

$$R^\bullet(V_0, T, W) = \frac{1}{2} \log \frac{T^2 V_0 + W + T^3 (W - V_0)}{(V_0 T^2 + (1 - T^2) W) (1 - T)} + h(\nu_2) - h(W),$$

(31)

![FIG. 3: Plot of the DR secret-key rate of the two-way thermal protocol for a pure-loss channel (W = 1) as a function of the channel transmissivity T, for different values of the preparation noise V₀ = 1, 5, 10 and 100 (from left to right).](image)

![FIG. 4: Plot of the RR secret-key rate of the two-way thermal protocol for a pure-loss channel (W = 1) as a function of the transmissivity T for different values of the preparation noise V₀ = 1, 5, 10, 100 and 1000 (from top to bottom). As the preparation noise is increased, the rate decreases but remains positive for any T > 0.](image)

In Fig. 4, we plot the RR secret-key rate $R^\bullet$ in the presence of a pure-loss channel (W = 1) as a function of the channel transmissivity T for values of the preparation noise from $V_0 = 1$ to $V_0 = 10$. As we can see, there is no reduction.
in the security of the protocol, in the sense that all the curves originate from the common threshold $T = 0$ for any value of the preparation noise $V_0$, even if the rate is decreasing for increasing $V_0$. This is clearly in contrast to what happened before for DR (with the transmission threshold $T$ approaching 1 for high values of $V_0$).

Next, we analyze the security of the two-way thermal protocol against an arbitrary entangling-cloner attack (with $W \geq 1$). In Fig. 5 we plot its RR security threshold, expressed as tolerable excess noise $N^{\bullet} = N^{\bullet}(V_0, T)$ as a function of the transmissivity $T$, for a wide range of values of the preparation noise $V_0$. As we can see, $N^{\bullet}$ is very robust with respect to the preparation noise $V_0$, with all the curves, from $V_0 = 1$ up to $V_0 = 10^6$, being included in the region shown in the figure. Thus, despite the RR secret-key rate $R^{\bullet}$ being decreasing for increasing $V_0$, it remains positive up to the excess noise $N^{\bullet}$ shown in Fig. 5 (Furthermore, the threshold value $N^{\bullet}$ turns out to be slightly increasing in $V_0$, as a result of the ‘fighting noise with noise’ effect of QKD).

From the same figure, we can see that the two-way thermal protocol outperforms the one-way thermal protocol in the transmission range $0 < T < 1$ for any value of the preparation noise $V_0$ (up to $10^6$). In fact, the one-way protocol is not robust in RR (see dashed curves in the figure), and its DR security threshold $N^{\bullet\text{-1-way}}$ is well below the two-way RR threshold $N^{\bullet\text{-2-way}}$, apart from a small overlapping region very close to $T = 1$. By exploiting this robustness and better performance of the two-way thermal protocol, we can improve the security of CV-QKD at the infrared regime as discussed in the following section.

![Graph](image)

**FIG. 5:** (Color online). Two-way thermal protocol in the presence of an arbitrary entangling-cloner attack ($W \geq 1$). We plot the RR security threshold, expressed as tolerable excess noise $N^{\bullet}$ as a function of the channel transmissivity $T$ for different values of the preparation noise from $V_0 = 1$ to $10^6$ (illustrated by the shaded regions). This threshold is compared with the DR threshold of the one-way thermal protocol for $V_0 = 1$ to $10^6$. The plot also shows the RR threshold of the one-way protocol (dashed curves) for $V_0 = 1, 5, 10$ (from left to right).

### IV. TWO-WAY THERMAL QKD AT DIFFERENT WAVELENGTHS

By exploiting its robustness to the preparation noise, we can use the two-way thermal protocol to improve the security of CV-QKD at longer wavelengths. Given a bosonic mode with frequency $f$ in a thermal bath with temperature $t$, it is described by a thermal state with mean number of photons $\bar{n} = [\exp(hf/k_Bt) - 1]^{-1}$, where $h$ is Planck’s constant and $k_B$ is Boltzmann’s constant. This number gives the noise-variability of the thermal state $V_0 = 2\bar{n} + 1$, i.e., the preparation noise, which is therefore function of the frequency and the temperature, i.e., $V_0 = V_0(f, t)$. In our study we consider a fixed value of the temperature $t = 15 \, ^\circ C$, so that $V_0$ is one-to-one with the frequency $f$.

Eve’s attack is a collective entangling-cloner attack (as before) which is thought to be performed inside a cryostat. The purpose of this is to remove Eve from the background preparation noise at any wavelength, therefore making her ancillary modes pure. In order to cover her tracks, Eve uses an entangling cloner with channel noise equal to the preparation noise, i.e., $W = V_0$. For more information on how to implement an entangling-cloner attack in thermal QKD, see the details given in [13].

Thus, at fixed temperature ($t = 15 \, ^\circ C$), we can express the RR secret-key rate $R^{\bullet}(V_0, T, W)$ as a function of $f$ and $T$, i.e., $R^{\bullet} = R^{\bullet}(f, T)$. By setting $R^{\bullet} = 0$, we get the security threshold $f^{\bullet}(T)$, giving the minimum tolerable frequency $f^{\bullet}$ which can be used at any channel transmission $T$ or, equivalently, the maximum tolerable wavelength $\lambda^{\bullet} = c/f^{\bullet}$, with $c$ being the speed of light. The threshold $f^{\bullet}(T)$ is plotted in Fig. 6 and compared with the thresholds of the one-way thermal protocol in DR and RR [12, 13]. As we can see from the figure, two-way QKD allows us to use a wider range of frequencies than one-way QKD. In particular, this happens for $0.2 \lesssim T \lesssim 0.8$, where the two-way threshold is well below the other thresholds in the infrared regime.

From Fig. 6 we see a crossing point between the DR and RR thresholds of the one-way protocol, for $T \approx 0.6$ and $f \approx 1.2 \times 10^{13} \, Hz$. This point identifies the maximum gap from the two-way configuration, which remains secure for channel transmissions as low as $T \approx 0.4$ at the same crossing frequency $f \approx 1.2 \times 10^{13} \, Hz$. Such a frequency corresponds to a wavelength of about $\lambda = 24 \, \mu m$, an infrared region where quantum communication is very demanding, with free-space losses around 9.7 dB/m under ideal atmospheric conditions (with humidity equal to 1mm of water vapor column and temperature of $15 \, ^\circ C$).

As a result, one-way QKD ($T = 0.4$) is secure up to a distance of 22 cm, while two-way QKD ($T = 0.6$) remains secure up to 41 cm. Despite being a very short distance, this represents an improvement close to 100%, which could be extremely useful in short-range cryptography, e.g., for connecting close computers through infrared ports or interfacing mobile devices with ATM machines.

Note that the infrared regime is less challenging in the $10 \, \mu m$ window, for instance at $\lambda = 12 \, \mu m$. At this wave-
length, the atmospheric absorption is dominated by carbon dioxide, methane, and ozone, with an attenuation which is much smaller (about 0.53 dB/km). In this case, one-way QKD is secure up to 14.6 Km, while two-way QKD allows the parties to distribute secret-keys up to 15.8 Km, corresponding to an 8% improvement in the distance.

V. CONCLUSION

In conclusion, we have shown how the use of two-way quantum communication enables two parties to improve the security of thermal QKD, where (inaccessible) preparation noise is added to the signal states. Considering both types of reconciliation procedures (direct and reverse reconciliation), we have analyzed the secret-key rates and the security thresholds of a two-way protocol which is based on Gaussian-modulated thermal states, random Gaussian displacements and homodyne detections. We have tested its security against collective Gaussian (entangling-cloner) attacks, showing how the security threshold in reverse reconciliation is very robust with respect to the preparation noise, and is able to outperform the security thresholds (in direct and reverse reconciliation) of one-way thermal QKD.

Exploiting such a robustness, we have successfully extended two-way thermal QKD to longer wavelengths, where thermal background naturally provides very high values of preparation noise. In particular, we have shown the superiority of two-way quantum communication in the infrared regime, improving the security distances which can be reached by the use of thermal sources at such frequencies.

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Appendix A: Asymptotic secret-key rates for one-way thermal QKD

In this appendix, we provide the main steps for deriving the asymptotic secret-key rates of Eqs. 2 and 3, for the one-way thermal protocol in DR and RR, respectively. These formulas are the explicit analytical expressions of the key rates already studied in Refs. 12 13.

One can easily verify that the CM of Eve’s output modes $E’ E”$ is given by

$$V_E = \left( \begin{array}{c} W_1 \sqrt{T(W^2 − 1)Z} \\ \sqrt{T(W^2 − 1)Z} TW + (1 − T)(V_a + V_0)I \end{array} \right).$$

(A1)

For large modulation ($V_a \gg 1$), we can easily compute its symplectic spectrum [11], which is given by $\nu_1 \rightarrow W$ and $\nu_2 \rightarrow (1 − T)V_a$. Then, Eve’s entropy is equal to

$$S(E) = h(\nu_1) + h(\nu_2) \rightarrow h(W) + \log \frac{e}{2}(1 − T)V_a,$$

(A2)

where $h(\cdot)$ is defined in Eq. 5. In the same limit of large modulation, Eve’s conditional CM $V_{E|a}$ can be computed from $V_E$ by setting $V_a = 0$ for one of the two quadratures. One can then check that $V_{E|a}$ has the following asymptotic eigenvalues

$$\nu_1 \rightarrow \sqrt{\frac{W\Lambda(1, W V_0)}{\Lambda(W, V_0)}}, \quad \nu_2 \rightarrow \sqrt{(1 − T)\Lambda(W, V_0)V_a},$$

(A3)

where the function $\Lambda(x, y)$ is defined in Eq. 4. As a result, Eve’s conditional entropy is given by

$$S(E|a) \rightarrow h(\nu_1) + \log \frac{e}{2}\sqrt{(1 − T)\Lambda(W, V_0)V_a}.$$

(A4)

Combining Eqs. (A2) and (A3), we get Eve’s Holevo information

$$I(E : a) = S(E) − S(E|a)$$

$$\rightarrow h(W) − h(\nu_1) + \frac{1}{2} \log \frac{(1 − T)V_a}{\Lambda(W, V_0)}.$$

(A5)

Very easily, we can also compute Alice and Bob’s mutual information, which is given by

$$I(a : b) = \frac{1}{2} \log \left[1 + \frac{TV_a}{\Lambda(V_0, W)}\right].$$

(A6)

Using Eqs. (A3) and (A5), we derive the asymptotic secret-key rate in DR, equal to Eq. 2.
Let us now compute the key rate in RR. Note that Eve and Bob’s joint CM is given by

\[ V_{EB} = \begin{pmatrix} V_E & C \\ C^T & B \end{pmatrix}, \tag{A7} \]

where \( B = [A(V_0, W) + TV_a]I \), and

\[ C = \begin{pmatrix} \sqrt{(1 - T)(W^2 - 1)}Z \\ \sqrt{T(1 - T)(W - (V_0 + V))I} \end{pmatrix}. \tag{A8} \]

As a result of Bob’s homodyne detection, Eve’s conditional CM is given by

\[ V_{E|b} = V_E - C(\Pi B \Pi)^{-1}C^T, \]

with \( \Pi := \text{diag}(1, 0) \). For large modulation, this CM has asymptotic spectrum \( \tilde{\nu}_1 \to 1 \), and

\[ \tilde{\nu}_2 \to \sqrt{\frac{1 - T}{T} WV_a}, \tag{A9} \]

so that Eve’s conditional entropy is given by

\[ S(E|b) \to \log \frac{e}{2} \sqrt{\frac{1 - T}{T} WV_a}. \tag{A10} \]

Using the expressions for \( S(E) \) and \( I(a : b) \) previously computed, we can derive the asymptotic secret-key rate in RR which is given in Eq. (3).

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