Symbol sense characteristics for designing mathematics tasks

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Abstract. Symbol sense, an analogy of number sense in the symbolic world of mathematics, is an ability to give meaning to symbols, expressions, and formulas that involves many characteristics, including friendliness with symbols, an ability to examine symbol meanings, and skills to manipulate symbolic expressions. The characteristics of symbol sense are usually used, for instance, for designing symbolic algebra tasks, but are a bit rare for designing mathematics tasks in general. Therefore, this study aims to use symbol sense characteristics for designing mathematics tasks for secondary school students. To do this, using a design research method, we conducted a literature study to collect characteristics of symbol sense and to see possibilities of designing mathematics tasks in general. Next, we designed mathematics tasks according to the symbol sense characteristics. Finally, we provided possible responses of students for the designed tasks. The results of this study included six types of tasks according to the characteristics of symbol sense, including tasks on symbolic problems and word problems—which require translation into symbolic problems. In addition, we provided predicted responses to examples for each type of the tasks and suggestions for use in the learning and teaching process. For further investigation, we recommend to use the set of designed tasks for assessing student symbol sense ability.

1. Introduction

In educational practice, particularly in mathematics teaching and learning process, designing tasks for assessing student performance is important, but is considered a complex and subtle activity [1]. To do a task design, teachers, researchers or task designers can use an appropriate theoretical framework, such as theory of realistic mathematics education, open-ended approach, problem-based learning, higher-order thinking skills, or symbol sense [1, 2]. The process of designing tasks using one of these theories, such as using the notion of symbol sense, often is not simple but needs guidance and experiences.

The notion of symbol sense has been used in previous studies for studying student algebraic expertise [3] and investigating student algebraic proficiency on the concept of parameter [4]. For the case of Indonesian context, the notion of symbol sense has been used for studying student algebraic thinking on solving substitution problems [5] and for examining student procedural fluency and conceptual understanding in algebra [6]. Whereas the use of symbol sense for designing algebra tasks is common practice [7], the use of this notion for designing mathematics tasks in general is lacking. Taking this into consideration, we conducted a theoretical design study on developing mathematics tasks using characteristics of symbol sense. Therefore, the current study aims to design mathematics tasks for secondary school students using characteristics of symbol sense.

Symbol sense, even if it is not defined precisely and is considered as an analogy of number sense [8], refers to an ability to give meaning to and to see important structures to symbols, expressions, and...
formulas [4, 8, 9]. According to Arcavi [8, 9], symbol sense has at least the following characteristics: Friendliness with symbols; an ability to manipulate and ‘read through’ symbolic expressions in solving problems; the awareness to the capability of creating symbolic relationships that express verbal or graphical information; the ability to select one possible symbolic representation for a problem; the realization of the need to check for symbol meanings during the process of solving problems; and the realization that symbols can play roles as variables or parameters. For the purpose of the current study, each of these characteristics is used in the task design process.

2. Methods
To design mathematics tasks, we used design research method [10], particularly the preliminary design phase, with the following three steps. Firstly, we conducted a literature study in the field of algebra education to collect characteristics of symbol sense, to obtain information about previous studies on symbol sense, and to see possibilities of designing mathematics tasks in general. Secondly, we designed six types of mathematics tasks which correspond to the symbol sense characteristics [8, 9]. In the designing mathematics tasks, we got an inspiration to develop tasks from research articles [2, 3, 8, 9] and from school textbooks as well as preparation test books [11, 12]. Scrutinizing the six types of designed tasks, three of the types concern word problems and the other three types concern symbolic mathematics problems. Finally, we provided possible responses of students, particularly secondary school students, to examples for each type of the designed tasks, and suggested possible strategies for using the tasks in educational practice. Results of these three steps of design process are presented and discussed in the next section.

3. Result and Discussion
In this section we present designed mathematics tasks according to characteristics of symbol sense and predicted responses for each task and corresponding suggestions for using it in the learning and teaching process. Table 1 presents six types of designed tasks according to the characteristics of symbol sense. Below we address each type of the tasks and its corresponding possible responses.

| Type of tasks | Characteristics of symbol sense | Example of tasks |
|---------------|---------------------------------|------------------|
| 1             | Friendliness with symbols       | Find two positive numbers such that the addition of the two numbers is 5 and the multiplication of the two numbers is also 5. |
| 2             | An ability to manipulate and ‘read through’ symbolic expressions in solving problems | Solve the equation $\frac{3x+2}{6x+4} = 5$. |
| 3             | The awareness to the capability of creating symbolic relationships that express verbal or graphical information | A rectangle has length 7 more than its width. If the perimeter of the rectangle is 30 cm, find the area of the rectangle. |
| 4             | The ability to select one possible symbolic representation for a problem | Find three consecutive numbers whose sum is 972. |
| 5             | The realization of the need to check for symbol meanings during the process of solving problems | Solve the equation $(4x - 1)^2 + 7 = 32$. |
| 6             | The realization that symbols can play roles as variables or parameters | Solve the following system of equations: $\begin{cases} x + y = a \\ x - y = b \end{cases}$ |
Task type 1 was designed according to the characteristic of friendliness with symbols. This means that the task asks for students to decide how and when symbols can and should be used in solving the task. Let us consider the example of Task Type 1. (For convenience of discussion we rewrite the task here.)

Find two positive numbers such that the addition of the two numbers is 5 and the multiplication of the two numbers is also 5.

For solving the task, we predicted that there would be at least two possible responses that might emerge from students. If a student considers the task as an arithmetic problem and thinks in the world of arithmetic only, then she or he will find it difficult for finding two positive numbers satisfying the condition. However, if the student resorts to symbols in the world of algebra, she or he will find that the task is equivalent to a system of equations \( x + y = xy = 5 \), where \( x \) and \( y \) represents numbers to find. By solving this system, the student would conclude that the two numbers are \( x = (5 + \sqrt{5})/2 \) and \( y = (5 - \sqrt{5})/2 \). The ability to decide how and when to use symbols to solve mathematics tasks shows a symbol sense ability [8, 9]. For use in an educational practice, we suggest teachers to use this type of tasks as a kind of challenge and motivation for students at the beginning of a mathematics lesson on the topic of systems of equations.

Task type 2 was designed according to the characteristic of an ability to manipulate and read through symbolic expressions in solving problems. This means that this type of tasks requires students to understand the meaning of a problem before carrying out a solution process. Let us consider an example of tasks for this type.

Solve the equation \( (3x + 2)/(6x + 4) = 5 \).

We predicted that there would probably be two possible responses to this equation that might emerge from students. If a student directly manipulates this equation without reading through it for gaining understanding, then she or he will end up to \( x = -2/3 \) and conclude it as the ‘solution’ of the equation. However, this ‘solution’ is incorrect. By substituting this ‘solution’ to the equation, it does not satisfy it. If the student reads through the equation first before solving the equation through doing algebraic manipulations, she or he will see that the left side of the equation is \( 1/2 \) which is never equal to 5 on the right side. By this example, we see that the ability to read through a symbolic expression is important in problem solving which characterizes the symbol sense ability. We suggest teachers to use this type of tasks for a formative assessment when students are in the process of learning equations. In this way, a necessary action can be decided to improve student understanding on equations and to improve the learning and teaching process in general [2].

Task type 3 was designed in accordance with the characteristic of the awareness to the capability of creating symbolic relationships from verbal or graphical information. This means that the task invites students to translate verbal or graphical information into symbolic relationships. For example, consider the example of the tasks for this type.

A rectangle has length 7 more than its width. If the perimeter of the rectangle is 30 cm, find the area of the rectangle.

A possible response that we predicted to emerge from students is as follows. Rather than using a trial and error method to find the length and the width of the rectangle—which is possible to do, a student might represent for instance the width as \( x \) and so the length would be \( x + 7 \). As the perimeter of the rectangle is 30 cm, then by translating this verbal information into a symbolic problem, the student can obtain an equation \( 4x + 14 = 30 \) which leads to \( x = 4 \). Therefore, the area of the rectangle is \( x(x + 7) = 4.11 = 44 \text{ cm}^2 \). The ability to translate the problem situation into a symbolic mathematics problem characterizes symbol sense ability. From the perspective of Realistic Mathematics Education theory, this process is called mathematization [13, 14]. For an educational practice in mathematics teaching and learning, we suggest this type of tasks is used for assessing student ability and difficulty in solving word problems.
Task type 4 was designed based on the characteristic of the ability to select one possible symbolic representation for a problem. This means that the task asks for students to choose possible symbolization or even the best representation to solve the problem. For example, consider the example of the tasks for this type.

Find three consecutive numbers whose sum is 972.

To solve this task, we predicted that some students would like to represent the three consecutive numbers into n, (n + 1), and (n + 2); and other students would like to represent it into (n – 1), n, and (n + 1). A student who has a better feeling on symbols, which characterizes symbol sense ability, might choose the second representation, such that she or he will obtain (n – 1) + n + (n + 1) = 3n = 972, and conclude that n = 324. Therefore, the three consecutive numbers to find are 323, 324, and 325. In our view, the ability to choose the best symbolic representation or mathematical model from the given information in the task concerns mathematization [13, 14], which is similar to the case of the third type of symbol sense tasks. Considering this, we suggest teachers to use this type of tasks for assessing student ability in developing mathematical models from word problems in mathematics.

Task type 5 was designed according to the characteristic of the realization of the need to check for symbol meanings during the process of solving problems. This means that the task calls for students to check meanings of symbols or mathematical expressions during the process of solving problems. For example, consider the example of the tasks for this type.

Solve the equation \((4x – 1)^2 + 7 = 32\).

We predicted that some students would like to directly expand the expression \((4x – 1)^2\) into \(16x^2 – 8x + 1\). So, they will obtain an equation \(16x^2 – 8x – 24 = 0\). Sometimes this quadratic equation, however, is difficult to solve for many students. Other students, who check the equation to gain the meaning of it—which indicates symbol sense ability, will see that the equation is similar to an arithmetic problem of \(\ldots + 7 = 32\). This understanding will lead easily to a conclusion \((4x – 1)^2 = 25\). Next, the students will see that \(4x – 1 = 5\) or \(4x – 1 = -5\). They will eventually obtain \(x = 3/2\) or \(x = -1\) as solutions of the equation. The ability to see the meaning of an algebraic expression concerns symbol sense characteristics [2, 3], which is in line with the ability for the task type 2. For the purpose of using this type of tasks in the learning and teaching process, we suggest to use this type of tasks to assess student conceptual and relational understanding [15] on equations.

Finally, the task type 6 was designed according to the characteristic of the realization that symbols can play roles as variables or parameters. This means that this type of tasks requires students to get used to work with symbols in mathematics that can play different roles as variables or parameters. For example, consider the example of tasks for this type.

Solve the system of equations \(x + y = a\) and \(x – y = b\).

For this task, students are expected to be able to distinguish between variables and parameters for finding solutions. In this case, students should consider \(x\) and \(y\) as variables, \(a\) and \(b\) as parameters. Therefore, they should solve the system by finding the value of \(x\) and \(y\) in terms of \(a\) and \(b\). If students are able to do this, we predicted that they will be able to solve the system. Otherwise, they will probably find difficulties to solve the system. By applying, for instance, an elimination method, we predicted students will find the solution of the system, namely \(x = (a + b)/2\) and \(y = (a – b)/2\). A relevant and thorough study on the investigation of student understanding on the concept of parameter can be seen in [4]. For the learning and teaching process, we suggest teachers to give students for either formative or summative assessment not only algebra tasks containing variables and constants as usual, but also tasks containing variables and parameters. In this way, students will get an experience on the use of variables that play roles as both variables and parameters.
4. Conclusion
From the description in the previous section we draw the following conclusions. Based on the characteristics of symbol sense, we designed six types of mathematics tasks. From these six types, three types concern symbolic mathematics tasks directly, and the other three types concerns word problems in mathematics. For the case of word problems, students should be able to translate them into symbolic tasks in the solution process. The process of translating a word problem into a symbolic mathematics problem in the literature is called mathematization. In addition, we have provided predictions of responses to examples for each type of the tasks and have provided suggestions for using the tasks in the learning and teaching of mathematics. For further investigation, we recommend to use the designed tasks for assessing student symbol sense ability in junior high school or senior high school levels depending on relevant contents of mathematics.

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