Research Article

An SDP Method for Copositivity of Partially Symmetric Tensors

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In this paper, we consider the problem of detecting the copositivity of partially symmetric rectangular tensors. We first propose a semidefinite relaxation algorithm for detecting the copositivity of partially symmetric rectangular tensors. Then, the convergence of the proposed algorithm is given, and it shows that we can always catch the copositivity of given partially symmetric tensors. Several preliminary numerical results confirm our theoretical findings.

1. Introduction

Let $\mathcal{A} = (a_{i_1,\ldots,i_p, j_1,\ldots,j_q})$ be a real $(p,q)$-th order $m \times n$-dimensional rectangular tensor, where $a_{i_1,\ldots,i_p, j_1,\ldots,j_q} \in \mathbb{R}$ for $i_k \in [m], k \in [p]$, $j_q \in [n]$, and $l \in [q]$.

If the entries of the tensor are invariant under any permutation of $i_1, i_2,\ldots,i_p$ and $j_1, j_2,\ldots,j_q$, $\mathcal{A}$ is called a partially symmetric tensor. For the sake of simplicity, let $\mathcal{P}^m_{p,q}$ be the set of all partially symmetric rectangular tensors with order $(p,q)$ and dimension $m \times n$. By the relationship between partially symmetric tensors and homogeneous polynomials, we always use the following notation:

$$f(x,y) = \mathcal{A}x^p y^q = \sum_{i_1,\ldots,i_p, j_1,\ldots,j_q} a_{i_1,\ldots,i_p, j_1,\ldots,j_q} x_{i_1} x_{i_2} \ldots x_{i_p} y_{j_1} y_{j_2} \ldots y_{j_q}$$

(1)

By this notation, we know that $\mathcal{A} = (a_{i_1,\ldots,i_p, j_1,\ldots,j_q}) \in \mathcal{P}^m_{p,q}$ is strictly copositive if and only if

$$\mathcal{A}x^p y^q \geq (>0), \quad \text{for all } x \in \mathbb{R}^m, y \in \mathbb{R}^n \text{ with } \|x\| = 1, \|y\| = 1.$$  

(2)

Particularly, if $m = n$ and $x = y$, then it reduces to the copositivity of symmetric tensors [1–10].

The copositive tensor has attracted many researchers’ attention since it plays an important role in polynomial optimization [11], hypergraph theory [1], vacuum stability of a general scalar potential [12], tensor complementarity problem [13, 14], tensor eigenvalue complementarity problem [15, 16], and so on [17–37]. Kannike proved the vacuum stability conditions for more complicated potentials with the help of the copositive tensor [12]. Ling et al. [16] proposed that the tensor generalized eigenvalue complementarity problem is solvable and has one solution at least under assumptions that the related square tensor is strictly copositive. During the process of application, a challenging problem is how to detect the copositivity of tensors numerically.

Recently, several numerical algorithms are proposed to check the copositivity of symmetric tensors. To the best of our knowledge, the first numerical algorithm was proposed by Chen et al. in [2], where the algorithm is based on the representation of the multivariate form in barycentric coordinates with respect to the standard simplex. Then, by a suitable convex subcone of a copositive tensor cone, an updated numerical algorithm for copositivity of tensors was proposed in [1]. It must be pointed out that the methods of [1, 2] can only capture strictly copositive tensors and noncopositive tensors. To overcome this drawback, in [38], Li et al. proposed an SDP relaxation algorithm to test the
copositivity of higher-order tensors. Very recently, Nie et al. gave a complete semidefinite relaxation algorithm for detecting the copositivity of a matrix or tensor [39]. If the potential tensor is copositive, the algorithm can get a certificate for the copositivity. Otherwise, the algorithm can get a point that refutes the copositivity. Furthermore, it is shown that the detection can be done by solving a finite number of semidefinite relaxations for all matrices and tensors.

For the copositivity of partially symmetric tensors, Gu et al. gave the first two spectral properties in [40], and some necessary or sufficient conditions for a real partially symmetric rectangular tensor to be copositive are further established. Moreover, an equivalent notion of strictly copositive rectangular tensors is presented [40]. In [41], Wang et al. extended the simplicial partition method for symmetric tensors to check the copositivity of partially symmetric tensors. However, as we discussed above, it can only capture all strictly copositive rectangular tensors or noncopositive rectangular tensors. When the input tensor is copositive but not strictly copositive, the algorithm may not stop in general. To solve this, motivated by the algorithm of [38, 39], we propose a new algorithm to check the copositivity of partially symmetric tensors in this paper.

The remainder of this paper is organized as follows. In Section 2, we recall some preliminaries on polynomials. In Section 3, we formulate the potential problem as a proper polynomial optimization problem which can be efficiently solved by Lasserre-type semidefinite relaxations. Then, a numerical method is proposed to check whether a given partially symmetric tensor is copositive or not, and the convergence of this algorithm is established. Several numerical experiments are listed in Section 4, and final remarks are given in Section 5.

2. Preliminaries

Let $\mathbb{R}[x]$ be the ring of the polynomial with variables $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Let $\mathbb{R}[x]_{\leq d} \subseteq \mathbb{R}[x]$ denote the vector space of polynomials with degree at most $d$. The degree of a polynomial $f$ is denoted as $\deg(f)$. Denote $e_\alpha$ as the vector of all entries which equals one. A polynomial $p$ is called SOS if there exist $p_1, p_2, \ldots, p_j \in \mathbb{R}[x]$ such that $p = p_1^2 + p_2^2 + \ldots + p_j^2$. Denote by $\sum [x]$ the set of all SOS polynomials. For $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$, let $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$. Then, for any polynomial $f \in \mathbb{R}[x]$, it can be denoted by $f(x) = \sum_{\alpha \in \mathbb{N}^n} f_{\alpha} x^\alpha$, and $\text{vec}(f) = (f_{\alpha})_{\alpha \in \mathbb{N}^n}$ denotes its sequence of coefficients in the monomial basis of $\mathbb{R}[x]$. For matrix $A$, its transpose is denoted by $A^T$. For a symmetric matrix $X, X \succeq 0$ means $X$ is positive semidefinite. More details about polynomial optimization can be seen in [42–45].

For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and denote $\mathbb{N}^n_0 = \{ \alpha \in \mathbb{N}^n \mid |\alpha| \leq d \}$. For $t \in \mathbb{R}$, $[t]$ denotes the smallest integer that is not smaller than $t$. If the subset $I \subseteq \mathbb{R}[x]$ satisfies that $I + I \subseteq I$ and $I \cap \mathbb{R}[x] \subseteq \mathbb{R}[x]$, then $I$ is called ideal. For a polynomial tuple $h = (h_1, h_2, \ldots, h_t)$, the ideal generated by $h$ is defined such that

$$\mathcal{I}(h) = h_1 \mathbb{R}[x] + h_2 \mathbb{R}[x] + \ldots + h_t \mathbb{R}[x].$$

The $k$-th truncation ideal generated by $h$ is

$$\mathcal{I}(h)_k = h_1 \mathbb{R}[x]_{\leq \deg(h_1)} + h_2 \mathbb{R}[x]_{\leq \deg(h_2)} + \ldots + h_t \mathbb{R}[x]_{\leq \deg(h_t)}.$$

For complex and real algebraic varieties of polynomial tuple $h$, define

$$\begin{align*}
V_C(h) &= \{ x \in \mathbb{C}^n \mid h(x) = 0 \}, \\
V_R(h) &= V_C(h) \cap \mathbb{R}^n.
\end{align*}$$

The quadratic module generated by $g = (g_1, g_2, \ldots, g_t)$ is (denote $g_0 = 1$)

$$Q(g)_k = \sum [x] + g_1 \sum [x] + \ldots + g_t \sum [x].$$

For $y = (y_1, y_2, \ldots, y_n)$, where $\mathbb{R}^N$ is the space of real vectors indexed by $\alpha \in \mathbb{N}_d^n$, define

$$\left( \sum_{\alpha \in \mathbb{N}_d^n} p_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} y \right) = \sum_{\alpha \in \mathbb{N}_d^n} p_{\alpha} y_{\alpha}.$$  

For a polynomial $q \in \mathbb{R}[x]_{\leq d}$, the $k$-th localizing matrix of $q$ is the symmetric matrix $L_k(q)$ satisfying

$$\text{vec}(p_1) \top (L_k(q)) \text{vec}(p_2) = \langle q p_1, p_2, y \rangle,$$

for all $p_1, p_2 \in \mathbb{R}[x]$ with $\deg(p_1), \deg(p_2) \leq k - \deg(q)/2$, where $\text{vec}(p_i)$ denotes the coefficient vector of the polynomial $p_i$. When $q = 1$, $L_k(q)$ is the moment matrix $M_k(y) = L_k^{(q)}(y)$. Let $f = (f_1, f_2, \ldots, f_\alpha)$ be a polynomial tuple; its localizing matrix is defined such that

$$L_j^{(k)}(y) = (L_j^{(k)}(y), L_j^{(k)}(y), \ldots, L_j^{(k)}(y)).$$

3. The SDP Algorithm for Copositivity of Partially Symmetric Tensors

In this section, we establish an equivalent condition for the copositivity of partially symmetric tensors. Then, the concerned problem can be reformulated as a polynomial optimization problem. To continue, recall that a partially symmetric tensor $\mathcal{A} \in \mathbb{P}^{|\mathbb{N}^n_0|}$ is strictly copositive if and only if

$$\mathcal{A} x^2 z^2 \succeq 0 \quad \forall x, z \in \mathbb{R}^n, x \succeq 0, z \succeq 0,$$

which is equivalent with the following optimization problem:

$$\begin{align*}
\min & \quad \mathcal{A} x^2 z^2 \\
\text{s.t.} & \quad e_1^T x = 1, e_2^T z = 1 \\
& \quad x \in \mathbb{R}_+^n, z \in \mathbb{R}_+^n.
\end{align*}$$

Clearly, tensor $\mathcal{A}$ is strictly copositive if and only if $f^* \geq 0 \quad \forall f^* \geq 0$. Problem (11) can be solved by classical Lasserre relaxations [46]. Since the objection function is continuous and the feasible region is compact, problem (11) always has a
solution. Without loss of generality, assume \((x^*, z^*)\) is one of the solutions of (11); then, it satisfies the following KKT-conditions with \(\lambda, \mu \in \mathbb{R}, v \in \mathbb{R}^m\), and \(w \in \mathbb{R}^n\):

\[
\begin{align*}
\mathcal{A}x^* &= \lambda e_1 - v = 0, \\
q^T \mathcal{A}x^* &= \mu e_2 - w = 0, \\
\mathcal{A}z^* &\geq 0, \quad v \geq 0, \quad w \geq 0, \\
x^* - \mathcal{A}z^* &\geq 0, \quad v^T - w^T = 0.
\end{align*}
\]

By (12), we obtain that \(\lambda = p\mathcal{A}x^* + z^*\), \(\mu = q^T \mathcal{A}x^* + z^*\), and

\[
x^* - \mathcal{A}z^* = 0, \quad v^T - w^T = 0.
\]

Combining this with the fact that \(\|x^*\| \leq 1, \|z^*\| \leq 1\), we consider the following optimization problem:

\[
\begin{align*}
\min \quad & \mathcal{A}x^* z^* \\
\text{s.t.} \quad & x^T (\mathcal{A}x^* z^*) - (\mathcal{A}x^* z^*) e_1 = 0, \\
& z^T (\mathcal{A}z^* z^*) - (\mathcal{A}z^* z^*) e_2 = 0, \\
& \mathcal{A}x^* z^* - (\mathcal{A}x^* z^*) e_2 \geq 0, \\
& e^T x = 1, e^T z = 1, 1 - \|x\|^2 \geq 0, 1 - \|z\|^2 \geq 0 \\
& x \in \mathbb{R}^m, \quad z \in \mathbb{R}^n.
\end{align*}
\]

It is clear to see that problems (11) and (14) are equivalent in the sense that they have the same optimal solution. To solve (14), we introduce the following notations:

\[
\begin{align*}
f(x, z) &= \mathcal{A}x z, \\
g(x, z) &= \left\{ \mathcal{A}x^T z - (\mathcal{A}x^T z) e_1, \mathcal{A}x^T z - (\mathcal{A}x^T z) e_2, 1 - \|z\|^2, 1 - \|z\|^2, x, z \right\}, \\
h(x, z) &= \left\{ e^T (x, 0)_{\text{rev}} = 1, e^T (0, z)_{\text{rev}} = 1, x^T (\mathcal{A}x^T z^T) e_2, 1 - \|z\|^2, z, e^T (\mathcal{A}x^T z^T) e_2 \right\}.
\end{align*}
\]

The following theorem shows the convergence of Algorithm 1 for any partially symmetric tensor.

**Theorem 1.** Suppose \(\mathcal{A} \in \mathbb{P}^m\) is a partially symmetric tensor. Then, the following properties hold:

(i) For all \(k \geq 0\), problem (16) is feasible and achieves its optimal value \(\rho_k = f^*\) for all \(k\) sufficiently large

(ii) For all \(k \geq 0\), problem (19) has an optimizer if it is feasible

(iii) If \(\mathcal{A}\) is copositive, Algorithm 1 must stop with \(\rho^k \geq 0\) when \(k\) is sufficiently large

(iv) If \(\mathcal{A}\) is not copositive, Algorithm 1 must stop with \(f(x, z) < 0\) for almost all \(\xi \in \mathbb{R}^{m \times n}\) when \(k\) is sufficiently large.

**Proof**

(i) Since the feasible set of (11) is compact, it must have a minimizer \((x^*, z^*)\). On the contrary, \((x^*, z^*)\) is a feasible point for (16), which implies that the semidefinite relaxation (17) is always feasible. Since \(L_{(k)} \geq 0\), let \(X = \{\langle x, 0 \rangle, (0, z) | x \in \mathbb{R}^m, z \in \mathbb{R}^n\} \cap \mathbb{R}^{m \times n}\); then, it holds that \(L_{(k)} \geq 0\). We now show that the feasible set of (17) is compact as follows. First of all, we have

\[
1 \geq y_{2\ell_1} + y_{2\ell_2} + \ldots + y_{2\ell_m}.
\]
Step 0: given an arbitrary vector $\xi \in \mathbb{R}^{m+n}$. Let $k = \max\{[p/2],[q/2]\}$.

Step 1: solve the semidefinite relaxation (17). If $\rho_k \geq 0$, then stop, and $\alpha$ is copositive. If $\rho_k < 0$, go to Step 2.

Step 2: solve the following semidefinite program:

for an optimizer $y'$ if it is feasible. If it is infeasible, let $k = k + 1$ and go to Step 1.

Step 3: let $(x',x') = ((y')_{1m},(y')_{2m},\ldots,(y')_{nm})$. If $\alpha x'x'^{t} < 0$, then $\alpha$ is not copositive and stop. Otherwise, let $k = k + 1$ and go to Step 1.

**Algorithm 1:** An SDP method for copositivity of a partially symmetric tensor $\alpha \in \mathbb{P}^{m+n}_{p,q}$.

Then, $0 \leq y_{2i} \leq 1$; since $M_k(y) \geq 0$, $i \in [m+n]$. Furthermore, for all $0 < |\alpha| \leq k - 1$, the $(\alpha, \alpha)$-th diagonal entry of $L^{(k)}_{1-|\alpha|}$ is nonnegative, which implies that

$$y_{2a} \geq y_{2a}, y_{2a}, y_{2a}, y_{2a} \geq y_{2a}$$

Take $\alpha = e_1, e_2, \ldots, e_{m+n}$ in the following analysis. By the same argument as (21) and repeating $k - 1$ times, we can show that $0 \leq y_{2i} \leq 1$ for all $i \leq k$. By the definition of $M_k(y)$, we know that the diagonal entries $M_k(y)$ are precisely $y_{2i}, |\beta| \leq k$. Since $M_k(y) \geq 0$, all the entries of $M_k(y)$ must be between $-1$ and $1$. So, $y$ is bounded, and the feasible set of (17) is compact. Hence, the optimal value can always be achieved. In the following, we will show that $\rho_k = f^*$ for all $k$ sufficiently large.

$$f(x,z) = \alpha x^{p}z^{q},

\begin{align*}
g(x,z) &= \{(x,z), \alpha x^{p-1}z^{q} - \alpha x^{p}z^{q}e_1, \alpha x^{p}z^{q-1} - \alpha x^{p}z^{q}e_2\}

h(x,z) &= \{e^{T}(x,0)_{m+n}, e^{T}(0,z)_{m+n}, -1, \alpha x^{p-1}z^{q}, \alpha x^{p}z^{q-1}, \alpha x^{p}z^{q}\},

\end{align*}

\text{Corresponding Lasserre’s relaxations for (22) are}

$$\rho_k = \min \sum_{\alpha \in N^{n}} f_{\alpha} y_{\alpha}

\text{s.t. } L_{g}^{(k)}(y) \geq 0, L_{h}^{(k)}(y) = 0,$$

$$y_0 = 1, M_k(y) \geq 0, y \in \mathbb{R}^{m+n}_{2k}.$$  

For $k = k_0, k_0 + 1, \ldots$, where $k_0 = \max\{[p/2],[q/2]\}$, any feasible solution of (17) is also a feasible solution of (24), so

$$\rho_k^* \leq \rho_k \leq f^*, \ k = k_0, k_0 + 1, \ldots$$

Next, we show that the set of polynomials

By direct computation, the optimization (16) is equivalent with the following problem:

$$\min \alpha x^{p}z^{q}

\text{s.t. } x_1(\alpha x^{p-1}z^{q}), x_1(\alpha x^{p}z^{q})x_1 = 0,

z_1(\alpha x^{p}z^{q-1}), z_1(\alpha x^{p}z^{q})z_1 = 0,

\alpha x^{p-1}z^q - \alpha x^{p}z^q e_1 \geq 0,

\alpha x^{p}z^{q-1} - \alpha x^{p}z^q e_2 \geq 0,

\alpha x^{p}z^{q}, \alpha x^{p}z^{q} \geq 0,

x \in \mathbb{R}^m, z \in \mathbb{R}^n.$$

For simplicity, denote

$$F = \left\{ \left(1 - e^{T}x\right)\phi + \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{q} e_j z_j \phi + \sum_{i=1}^{n} z_i \left(\sum_{j=1}^{q} e_j z_j\right) \right) \right\}.$$  

is Archimedean, i.e., there exists $f \in F$ such that the inequality $f(x) \geq 0$ defines a compact set in $\mathbb{R}^{m+n}$. Let $f = 2 - \|x\|^2$ and $X = (x,z)_{m+n}$, we have

$$2 - \|x\|^2 = \left(1 - e^{T}x\right)(1 + \|x\|^2) + \sum_{i=1}^{n} x_i (1 - x_i)^2$$

$$+ \sum_{i=1}^{m} x_i x_i + \left(1 - e^{T}x\right)(1 + \|z\|^2) + \sum_{j=1}^{n} z_j (1 - z_j)^2$$

$$+ \sum_{i=1}^{m} z_i z_i.$$
So, $F$ is Archimedean by Theorem 3.3 of [47]; we know that $\rho_k = f^*$ when $k$ is sufficiently large. Hence, $\rho_k = f^*$ when all $k$ values are sufficiently large.

(ii) The proof is the same with (i).

(iii) Clearly, $\mathcal{A}$ is copositive if and only if $f^* \geq 0$. By item (i), $\rho_k = f^*$ for all $k$ big enough. Therefore, if $\mathcal{A}$ is copositive, we must have $\rho_k \geq 0$ for all $k$ large enough.

(iv) If $\mathcal{A}$ is not copositive, then $f^* < 0$. By (i), there exists $k_1 \in \mathbb{N}$ such that $\rho_k = f^*$ for all $k \geq k_1$. Hence, for all $k \geq k_1$, problem (19) is equivalent with the following problem:

$$
\begin{align*}
\bar{\rho}_k & = \min \langle \xi \rangle \left( x, y \right)_{m+n, y} \\
\text{s.t.} \quad & y_0 = 1, L_y \left( k \right) \left( x, y \right) \geq 0, L_y \left( k \right) \left( y, 0 \right) \geq 0, M_k \left( y \right) \geq 0, L_y \left( k \right) \left( f^* \left( x, y \right) \right) \geq 0, \quad X = \left( x, z \right)_{m+n}, \quad y \in \mathbb{R}^{\mathbb{N}, k}.
\end{align*}
$$

(28)

It is $k$-th Lasserre's relaxation for the polynomial optimization

$$
\begin{align*}
\min \xi \left( x, z \right)_{m+n} & \geq 0, 1 - \xi \left( 0, 0 \right)_{m+n} \geq 0, x \geq 0, z \geq 0, f^* - f \left( X \right) \geq 0.
\end{align*}
$$

(29)

The feasible region of (29) is clearly compact. When $\xi \in \mathbb{R}^{\mathbb{N}, m+n}$ is arbitrary, (29) has a unique optimizer $X^* = (x^*, z^*)$. Hence, for almost all $\xi \in \mathbb{R}^{\mathbb{N}, m+n}$, $X^*$ is the unique optimizer. For notation convenience, denote by $\hat{y}^k$ the optimizer of (19) with the relaxation order $k$. Let $X^k = ((\hat{y}_1)^k, \ldots, (\hat{y}_k)^k)$. By Corollary 3.5 of [48] or Theorem 3.3 of [49], the sequence $\{X^k\}_{k=0}^\infty$ must converge to $X^*$. Since $f^* \leq \rho_k^* < 0$, we must have $f \left( X_k \right) < 0$ when $k$ is sufficiently large.

4. Numerical Examples

In this section, we give several numerical examples to show the efficiency of Algorithm 1. Let $S_{\pi \left( i_1, \ldots, i_m \right)}$ denote the set of all permutations of $i_1, i_2, \ldots, i_m$ and let $\rho^* = 0$ when $|\rho^*| > 10^{-5}$. All experiments are done in Matlab2014b on a desktop computer with Intel (R) Core (TM)i7-6500 CPU @ 2.50 GHz 2.60 GHz and 16 GB of RAM.

Example 1. Suppose that $\mathcal{A} \in \mathbb{P}^2 \mathbb{S}^{2,2}$ is given by

$$
\begin{align*}
a_{1111} = 1, & \quad a_{1122} = 1, \quad a_{1133} = 1, \quad a_{1144} = 1, \quad a_{2211} = 1, \quad a_{2222} = 1, \quad a_{2233} = 1, \quad a_{2244} = 1, \\
\sum_{i_1, i_2} a_{i_1, i_2} & = 2, \sum_{i_1, i_2} a_{i_1, i_2} = 2, \sum_{i_1, i_2} a_{i_1, i_2} = 2, \sum_{i_1, i_2} a_{i_1, i_2} = 2,
\end{align*}
$$

(30)

The corresponding polynomial for tensor $\mathcal{A}$ is

$$
\begin{align*}
f \left( x, y \right) & = (x_1 - x_2)^3 (y_1 + y_2 - y_3 - y_4)^2, x = (x_1, x_2), \quad y = (y_1, y_2, y_3, y_4).
\end{align*}
$$

(31)

By Algorithm 1, we know that $f^* = 0$ with $x = (0.5000, 0.5000), \quad y = (0.2500, 0.2500, 0.2500, 0.2500)$, which implies that rectangular tensor $\mathcal{A}$ is copositive.

Example 2. Suppose that $\mathcal{A} \in \mathbb{P}^2 \mathbb{S}^{1,2}$ with entries such that

$$
\begin{align*}
a_{1111} = 1, & \quad a_{1122} = 1, \quad \sum_{i_1, i_2} a_{i_1, i_2} = -2.
\end{align*}
$$

(32)

The corresponding polynomial of $\mathcal{A}$ is
\( f(x, y) = x_1^2y_1^2 - 2x_1^2y_1y_2 + x_1^2y_2^2, \quad x = (x_1), \quad y = (y_1, y_2). \) 

(33)

By Algorithm 1, we obtain that \( f^* = 0 \) with optimal solution \((x, y) = (1.0000, 0.7071, 0.7071)\), which implies that \( \mathcal{A} \) is copositive but not strictly copositive.

Example 3. Suppose that \( \mathcal{A} \in \mathbb{P}S_{2,2}^{2\times 2} \) is given by

\[
\begin{aligned}
    a_{1111} &= 1, \\
    a_{1122} &= 1, \\
    a_{2211} &= 1, \\
    a_{2222} &= 1, \\
    \sum_{i,j,k,l \in S_{(1,1,1,1,2,2)}} a_{ijkl} &= -2, \\
    \sum_{i,j,k,l \in S_{(1,1,2,1,2,2)}} a_{ijkl} &= -2, \\
    \sum_{i,j,k,l \in S_{(1,2,2,2,2,2)}} a_{ijkl} &= 2, \\
    \sum_{i,j,k,l \in S_{(2,2,2,2,2,2)}} a_{ijkl} &= 2, \\
    \sum_{i,j,k,l \in S_{(1,2,2,2,2,2)}} a_{ijkl} &= -4. \\
\end{aligned}
\]

(34)

So, the corresponding polynomial of \( \mathcal{A} \) is that

\[
f(x, y) = x_1^2y_1^2 + x_1^2y_2^2 - 2x_1^2y_1y_2 + x_2^2y_1^2 + x_2^2y_2^2 - 2x_2^2y_1y_2 + 2x_1x_2y_1^2 + 2x_1x_2y_2^2 - 4x_1x_2y_1y_2,
\]

(35)

where \( x = (x_1, x_2), \ y = (y_1, y_2) \). By Algorithm 1, we have \( f^* = 0 \) with \( x^* = (0.5126, 0.4874), \ y^* = (0.5000, 5000) \), which implies that the rectangular tensor is copositive.

Example 4. Suppose that \( \mathcal{A} \in \mathbb{P}S_{3,2}^{3\times 2} \) is given by

\[
\begin{aligned}
    a_{11122} &= 1, \\
    a_{22222} &= 1, \\
    a_{33311} &= 1, \\
    \sum_{i,j,k,l \in S_{(3,2,2,2,2,2)}} a_{ijkl} &= -3. \\
\end{aligned}
\]

(36)

The corresponding polynomial of the partially symmetric rectangular tensor \( \mathcal{A} \) is

\[
f(x, y) = x_1^3y_1^2 + x_2^3y_2^2 + x_3^3y_3^2 - 4x_1x_2x_3y_1y_2y_3,
\]

(37)

where \( x = (x_1, x_2, x_3), \ y = (y_1, y_3) \). By Algorithm 1, we know that \( f^* = -0.0639 \) with \( x^* = (0.7652, 0.4702, 0.7652), \ y^* = (0.3572, 0.8724) \), which implies that the rectangular tensor is not copositive.

Example 5. Suppose \( \mathcal{A} \in \mathbb{P}S_{2,2}^{2\times 2} \) is a tensor with entries such that

\[
\begin{aligned}
    a_{1111} &= 1, \\
    a_{1122} &= -1, \\
    a_{2211} &= 1, \\
    a_{2222} &= 1, \\
    \sum_{i,j,k,l \in S_{(1,1,1,1,2,2)}} a_{ijkl} &= 2, \\
    \sum_{i,j,k,l \in S_{(1,1,2,1,2,2)}} a_{ijkl} &= 2, \\
    \sum_{i,j,k,l \in S_{(1,2,2,2,2,2)}} a_{ijkl} &= 2, \\
    \sum_{i,j,k,l \in S_{(2,2,2,2,2,2)}} a_{ijkl} &= 2, \\
    \sum_{i,j,k,l \in S_{(1,2,2,2,2,2)}} a_{ijkl} &= -4. \\
\end{aligned}
\]

(38)

The corresponding polynomial of the partially symmetric rectangular tensor \( \mathcal{A} \) is

\[
f(x, y) = x_1^2y_1^2 - x_1^2y_2^2 + 2x_1^2y_1y_2 + x_2^2y_1^2 + x_2^2y_2^2 + 2x_2^2y_1y_2 + 2x_1x_2y_1^2 + 2x_1x_2y_2^2 - 4x_1x_2y_1y_2,
\]

(39)

where \( x = (x_1, x_2), \ y = (y_1, y_2) \). By Algorithm 1, we know that \( f^* = 0.3333 \) with \( x^* = (0.6666, 0.3334), \ y^* = (0.5000, 5000) \), which implies that the rectangular tensor is strictly copositive.

5. Conclusions

In this paper, based on Lasserre’s hierarchy of semidefinite relaxations, we propose a new criterion to judge whether a given partially symmetric rectangular tensor is copositive or not. The convergence for the proposed algorithm is established. Furthermore, numerical examples demonstrate that the proposed algorithm is effective when the input rectangular tensor has lower dimension and orders, and it is difficult for the case with higher order or higher dimension. We will continue to study this problem in the future.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors’ Contributions

Each author contributed equally to this paper and read and approved the final manuscript.

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