A new look at relativity transformations

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Abstract

A free system, considered to be a comparison system, allows for the notion of objective existence and inertial frame. Transformations connecting inertial frames are shown to be either Lorentz or generalized Galilei.

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1 Introduction

When modeling physical systems, the carrier space (space of states or space of events) is usually equipped with some background mathematical structure (for instance, vector space structure and Euclidean metric in elementary mechanics, Hilbert space structure and commutation relations for quantum mechanics, and so on). With the advent of general relativity and the ambition to have a theory of the universe as a whole it is necessary to minimize the use of mathematical structures as given a priori.

Actually the Einstein general approach to physics has this goal:

One of the imperfections of the original relativistic theory of gravitation was that as a field theory it was not complete: it introduced the independent postulate that the law of motion of a particle is given by an equation of geodesic. A complete theory knows only fields and not the concepts of particle and motion. For these must not exist independently of the field, but are to be treated as part of it [1].

In a footnote of the same paper, Einstein and Rosen wrote, on the stress-energy tensor representing the source in the Einstein equations:

It was clear from the very beginning that this was only a provisory complexion of the theory in the sense of a phenomenological interpretation.

In any case, no matter what the motivation may be, it seems desirable to revisit elementary mechanics starting with few assumptions and adding new ones when they are required to resolve ambiguities, if any. In this note we would like to consider the minimal assumptions which may give rise to special relativity and Galilean relativity and make clear which assumptions will discriminate between them and what are the compatibilities with the space structure. It should be remembered that the problem of deriving special relativity without supposing a priori the constancy of the light speed has been discussed by many authors (some of them are quoted in ref. [2]).

To better compare different relativity theories, we shall start with a four-dimensional smooth manifold $M$ as a carrier space for the description of the evolution of point particles (i.e., we start with the space of events). By using a coordinate system say $(y_0, y_1, y_2, y_3 \in \mathbb{R}^4)$, we consider the equations of motion in the form

$$\frac{d^2 y^\mu}{ds^2} = f^\mu(y, \frac{dy}{ds}).$$

(1)

As a simplifying assumption we require our carrier space to be diffeomorphic to $\mathbb{R}^4$.

We are now in a position to define a free system on our space. The
motion is said to be a free motion if there is a (global) coordinate system $x^\mu = x^\mu(y^\nu)$, such that the equations of motion acquire the form

$$\frac{d^2 x^\mu}{ds^2} = 0.$$  \hspace{1cm} (2)

The parameter $s$ is attached to the particular dynamical system we start with and has nothing to do with space-time variables. Any such coordinate system $(x^\mu)$ will define an affine structure on the carrier space inducing the one from $\mathcal{R}^4$. It follows that the notion of straight line, or, more generally, of affine space on the space of events is frame dependent. In addition, in each frame we must select, first of all, a coordinate, say $x^0$, to be associated with a notion of evolution.

Any solution $x^\mu(s)$ of our free equation which represents the world-line of an existing object for the given frame must be such that $\frac{dx^0}{ds} \neq 0$ along the world-line (in what follows we shall use $\frac{dx^0}{ds} > 0$).

After the choice of a single world-line has been made, the translation group can be used to move it and thereby to construct a congruence of world-lines which can be thought of as solutions of a vector field $E$ on the carrier space. Because the translation group is a symmetry group for (2), these world-lines are all solutions of (2).

We now introduce a closed 1-form $\alpha$, invariant under the translation group, such that $\alpha(E) > 0$. This 1-form $\alpha$ defines a family of 3-planes for $\mathcal{R}^4$ (a foliation), transversal to the congruence defined by $E$. As the world-lines associated with $E$ are solutions of (2), we say that $(E, \alpha)$ is an inertial frame. For any such frame, we can define time-like vectors as those $v_m \in T_m \mathcal{R}^4$ which satisfy $\alpha_m(v_m) \neq 0$. They will be future pointing if $\alpha(v_m) > 0$.

We can say that a point $p$ is in the past of $q(p < q)$ if $p$ can be connected to $q$ by a curve whose tangent vectors are future oriented. Similarly we can define a point in the future of $q$. These comments are meant to stress that $\alpha$ and $E$ allow us to define most of the standard non metric structures on space-time (if we had given a metric structure it would have been possible to define $\alpha$ from $E$ and the metric).

Any time-like world-line, equipped with a map associating in a monotonic way a real number to each event on the world-line (clock) will be called an observer. In our approach a clock can be associate to each time-like world-line by considering the pull-back of $\alpha$ to it.

Specifically, this can be done by defining $c_\alpha : \gamma(\mathcal{R}) \subset M \rightarrow \mathcal{R}$ by setting
\( c_{\alpha}(\gamma(s)) = \int_{s_0}^{s} \gamma^*(\alpha). \) The induced map \( c_{\alpha} : \gamma \mathcal{R} \to \mathcal{R} \) is clearly monotonic because of \( \alpha(E) > 0. \)

The pair \((E, \alpha)\) defines a family of observers with temporal evolution along \( E \) and rest frames associated with \( \ker \alpha \). Two inertial systems \((E_1, \alpha_1)\) and \((E_2, \alpha_2)\) are compatible if \( \alpha_1(E_2) \neq 0 \) and \( \alpha_2(E_1) \neq 0 \). Two compatible systems will perceive the world-lines of each other as representing some physical entities; for this reason the requirement \( \alpha_a(E_b) \neq 0, a, b = 1, 2 \) will be called the *mutual objective existence* condition. We shall make the choice \( \alpha_a(E_b) > 0. \)

For any pair of inertial frames, say \((E_a, \alpha_a), (E_b, \alpha_b)\), we can define relative clocks by setting \( c_{ba} : \mathcal{R} \to \mathcal{R} \) given by

\[
c_{ba}(s) = c_b(\gamma_a(s)) = \int_{s_0}^{s} \gamma^*_a(\alpha_b).
\]

Here \( s_0 \) is determined by the intersection point of the two observers in \((E_a, \alpha_a)\) and \((E_b, \alpha_b)\) respectively. If the two observers belong to the same inertial frame, \( s_0 \) is determined by their intersection with any common rest frame. This view point makes clear that the notion of inertial system is related to the dynamical evolution of some chosen *comparison system* via the selection of a congruence of solutions. It should be stressed that with the assumption that our carrier space is diffeomorphic with \( \mathcal{R}^4 \), all *free systems* are diffeomorphic to each other. By using the linear structure induced by an inertial reference frame, we find that the inhomogeneous general linear group \( \mathcal{IGL}(4, \mathcal{R}) \) acts transitively on the set of solutions of equation (1).

Now we shall look for relativity transformations, *i.e.* transformations connecting pairs of inertial systems, say \((\alpha_1, E_1)\) and \((\alpha_2, E_2)\), and require that they form a subgroup of \( \mathcal{IGL}(4, \mathcal{R}) \). By selecting one fiducial inertial system \((\alpha, E)\) we parametrize all others connected to it by the elements of the relativity group we are searching for.

The restrictions on the accepted relativity transformations means that a *particle* at rest in one inertial frame cannot be perceived by another one as a *particle existing* all over a real line only at a given instant of time, without past and without future.

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From the point of view of the transformation group (*i.e.* transformations connecting physically equivalent systems) we have to exclude the possibility of exchanging *time-* and *space-* axes. For instance, we should exclude from admissible *relativity transformations* those closing on a subgroup of \( \mathcal{IGL}(4, \mathcal{R}) \) which contains rotations in the *time-space* planes.
We shall now describe how to construct these relativity (sub)-groups. The main idea of our procedure consists of looking for the transformation groups which transform an inertial system into another one compatible with the mutual objective existence condition. This is translated in the requirement that a transformed $\alpha$, say $\tilde{\alpha}$ should not contain $E$ in the kernel, i.e. $\tilde{\alpha}(E) \neq 0$ and $\alpha(E) \neq 0$.

Before carrying on this program we shall consider some preliminary aspects (sections 2-4).

2 On the conditions for a system to be free

Given a second order differential equation on space-time, say

$$\frac{d^2y^\mu}{ds^2} = f^\mu(y, \frac{dy}{ds})$$

we ask under which conditions we can find a new coordinate system in which the equation becomes

$$\frac{d^2x^\mu}{ds^2} = 0.$$  \hspace{1cm} (4)

Clearly, by performing a change of coordinates $x^\mu = x^\mu(y)$, we find:

$$0 = \frac{d}{ds} \left( \frac{dx^\mu}{dy^\nu} \frac{dy^\nu}{ds} \right) = \frac{d^2x^\mu}{dy^\rho dy^\nu} \frac{dy^\nu}{ds} \frac{ds}{ds} + \frac{dx^\mu}{dy^\nu} \frac{d^2y^\nu}{ds^2},$$

i.e.

$$\frac{d^2y^\nu}{ds^2} = \left( \left( \frac{dx}{dy} \right)^{-1} \right)^\nu_\mu \frac{d^2x^\mu}{dy^\alpha dy^\beta} \frac{dy^\alpha}{ds} \frac{dy^\beta}{ds} = \Gamma^\nu_{\alpha\beta} \frac{dy^\alpha}{ds} \frac{dy^\beta}{ds}.$$

If we compute the curvature associated with this connection, we find

$$\mathcal{R}^\lambda_{\nu\mu\rho} = \frac{\partial - \frac{\partial}{\partial y^\rho}}{\partial y^\lambda} - \frac{\partial - \frac{\partial}{\partial y^\nu}}{\partial y^\rho} + - \eta^\nu - \frac{\partial}{\partial y^\rho} - \frac{\partial}{\partial y^\eta} = 1;$$

as a matter of fact this condition turns out to be also sufficient to go from (3) to (4).

As in general we are not requiring the force $f^\mu$ in (11) to be quadratic in the velocities, this sufficiency condition should be stated in the framework of generalized connections associated with any second order vector field."
In the particular case our starting system is in the form of a spray

\[ f^i = -\Gamma^i_{km} u^k u^m, \]

we have

\[ \Gamma^k_{ij} = -\frac{1}{2} \frac{\partial^2 f^k}{\partial u^i \partial u^j}, \quad R^m_{ij} = \tilde{R}^m_{kij} u^k. \]

Thus, relation (5) is necessary and sufficient condition for a second order differential equation to represent a free system on \( R^4 \).

In coming section we shall describe a constructive procedure to find these special reference frames.

3 Natural coordinates for second order equations: Searching for inertial frames

Starting with a second order equation on \( M \), say

\[ \frac{d^2 y^\mu}{ds^2} = f^\mu(y, \frac{dy}{ds}), \]

we can define a natural coordinate system for any neighbourhood \( U \ni m \) in the following way.

We consider a fiducial point \( m \) and the tangent space \( T_m M \). With any vector \( v_m \in T_m M \) we associate a point in \( M \) by looking for the (unique) solution of (1) originating at \( m \) with initial velocity \( v_m \).

The flow \( \phi : R \times TM \to TM \), associated with (1), defines a map by restriction

\[ \phi : R \times \{ m \} \times T_m M \to U \subset M, \]

\[ \phi(s, m, v_m) = m_v(s), \]

where \( m_v(s) \) is the point in \( U \) reached after a time \( s \) via the solution of (1) with initial conditions \( \{ m, v_m \} \). In particular the application \( \phi(s = 1, m, v) = m_v \) may be considered as a generalization to a generic second order equation of the standard exponential map for geodesic equations.
For simplicity, we assume that (1) defines a complete vector field on $TM$. In this hypothesis the particular map we have constructed is defined for any $v_m \in T_m M$ injectively (we recall that $M$ is diffeomorphic to $\mathbb{R}^4$ by assumption and our considerations apply to a neighbourhood $U$ of $m$). In particular we may define an addition rule on $U$ by setting $m_{v_1} + m_{v_2} := m_{v_1 + v_2}$.

The vector space structure induced on $U$ for each, complete, second order vector field on $M$, depends on $m$. When these vector space structures on $U$ are linearly related, i.e. transition functions are linear maps, our starting equation (1) reduces to a *free particle* equation. When moving from a point $m$ to a point $m'$ we go from the vector space structure associated with $T_m M$ to the one associated to $T_{m'} M$, this can be done *transporting* one vector space onto the other along the *solution curve* connecting $m'$ to $m$.

Depending on $m$ and $m'$ this connection map may fail to be linear up to some order depending on the *extension* of the neighbourhood, the set of points we may reach from $m_0$ by using *solution curves* while keeping the *linearity* violated to no more than some preassigned power $k$ in the parameters will be a $k$-order local inertial frame.

When equation (5) is satisfied, this procedure defines a global linear inertial frame.

### 4 Transforming inertial frames

We start with an inertial frame, i.e. a reference inertial frame described by $(E, \alpha)$, giving rise to a global coordinate system $(x^0, x^1, x^2, x^3)$. We notice that a particular form for $\alpha$ would be $\alpha = dx^0$.

It is worth stressing that a different choice of the congruence (i.e. a different choice of $E$) in general may give rise to a different class of inertial systems.

Having found one coordinate system in which the equation of motions have the form (2), how many of them exist?

It is clear we have to look for all coordinate systems $\xi^\mu = \xi^\mu(x, v)$ such that

$$\frac{d^2 \xi^\mu}{ds^2} = 0. $$

As

$$\frac{d \xi^\mu}{ds} = \frac{\partial \xi^\mu}{\partial x^\nu} \frac{dx^\nu}{ds} + \frac{\partial \xi^\mu}{\partial v^\nu} \frac{dv^\nu}{ds},$$
by using

\[ \frac{dx^\mu}{ds} = v^\mu, \quad \frac{dv^\mu}{ds} = 0 \]

we find

\[ \frac{\partial^2 \xi^\mu}{\partial x^\nu \partial x^\rho} v^\rho v^\nu = 0, \]

i.e., \( \xi^\mu \) must be linear in \( x^\nu \) (it can be however any function of constants of the motion for system (2)). We consider therefore

\[ \xi^\mu = A^\mu_\nu(v) x^\nu + a^\mu(v), \quad \frac{d\xi^\mu}{ds} = A^\mu_\nu(v) \frac{dx^\nu}{ds} = w^\mu. \]

In terms of the initial conditions, we have

\[ \xi^\mu(s) = A^\mu_\nu(v(0)) x^\nu(0) + A^\mu_\nu(0) v^\nu(0) s + a^\mu(v(0)) = \]

\[ = A^\mu_\nu(v) x^\nu + A^\mu_\nu(0) v^\nu s + a^\mu(v) = \xi^\mu(0) + w^\mu(v)s, \]

where \( v \) stays for \( v(s) \).

**Remark**

As constants of the motion, say \( C \), for our comparison dynamics satisfy

\[ \frac{d}{ds} C = \tau, \]

it is clear that \( \xi^\mu \) can be any function of them in addition to the explicit dependence on \((x^\mu)\). Here, we only consider the dependence on constants of the motion \((v^\mu)\) and do not consider a possible dependence on other constants of motion. This is a simplifying assumption useful for computations because \((x^\mu, v^\mu)\) parametrize the position-velocity phase space.

Because our approach to inertial frames is a dynamical one, there is no reason to restrict our transformations to be point transformations (i.e. to tangent bundle automorphisms, to use the language from differential geometry). Therefore, new velocities need not be linear functions of the old velocities. The congruence of curves corresponding to

\[ x^\mu(s) = v^\mu s + x^\mu(0), \quad v^0 > 0 \]

will be given by

\[ \xi^\mu(s) = w^\mu s + \xi^\mu(0), \quad w^0 > 0, \]

where \((w^\mu)\) can be any smooth function of \((v^\mu)\).
5 Relativity transformations

5.1 A preliminary lemma

At this point we look for linear transformations on a given pair \((E, \alpha)\) with the requirement that any transformed one still satisfies \(\tilde{\alpha}(E) > 0, \alpha(\tilde{E}) > 0\), i.e. our transformations generate new inertial frames satisfying the mutual objective existence. From here we shall be able to construct the relativity groups which are compatible with our requirement.

As usual, we can dispose of the translation part and concentrate our analysis on the linear homogeneous part.

By using a passive point of view, we can consider our transformations from \(\mathbb{R}^4\) to \(\mathbb{R}^4\), preserving the origin.

We shall therefore consider linear transformations \(x^\mu = A^\mu_\nu \tilde{x}^\nu\), with the additional requirement \(\frac{dx^0}{\tilde{x}^0} > 0\), to implement \(\tilde{\alpha}(E) > 0, \alpha(\tilde{E}) > 0\). Here we think of the choice \(E = \frac{\partial}{\partial x^0}, \alpha = dx^0; \tilde{E} = \frac{\partial}{\partial \tilde{x}^0}, \tilde{\alpha} = d\tilde{x}^0\).

We have a preliminary lemma:

*Any invertible linear transformation, \(\mathbb{R}^4 \rightarrow \mathbb{R}^4\), which, along with its inverse, preserves the time-like character can be decomposed into the product of a linear transformation in the \((0,1)\)-plane and two space-like transformations.*

We denote by \(A\) our generic transformation and apply it to a standard vector:

\[
A \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}
\]

where \(a_0 > 0\) by assumption.

Now, by using a space-transformation \(R\), we can transform

\[
\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \text{into} \quad \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}.
\]

Therefore

\[
RA \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_0 \\ a \\ 0 \\ 0 \end{pmatrix}.
\]
By using a linear transformation $L$ in the $(0,1)$-plane, acting as the identity in the remaining components and preserving the time-like character, we find

$$LRA \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$ 

By using the arbitrariness of the starting time-like vector, we find that $S \equiv LRA$ is a space-transformation and we get the decomposition

$$A = R^{-1}L^{-1}S.$$ 

This decomposition lemma allows us to deal first with transformations in the $(0,1)$ plane to find out which ones are compatible with the requirement on the objective existence condition and then to compose them. This analysis will be done in the following sections.

### 5.2 Infinitesimal transformation and associated quadratic forms

Having reduced our problem to a two-dimensional one, it is easy to visualize the situation. A vector $E$ is given and we should consider all those linear transformations in $\mathcal{R}^2$ which never take a vector transversal to $E$ (defined by $\ker \alpha$) into one parallel to $E$. If we think of a one-parameter group of transformations connecting two allowed frames, for any infinitesimal generator $\mathcal{A}$, there will never be a value of the parameter $\sigma$ for which $e^{\sigma\mathcal{A}} \ker \alpha$ is parallel to $E$.

At this point it is convenient to decompose any element of $GL(2, \mathcal{R})$ into the product of an element in $SL(2, \mathcal{R})$ and a dilation. Because dilations are in the center of $GL(2, \mathcal{R})$ they can be dealt with separately. Thus, we may restrict our analysis to $SL(2, \mathcal{R})$.

For this analysis it is very convenient to notice that $SL(2, \mathcal{R})$ is the same as $Sp(2, \mathcal{R})$, i.e. the group of canonical transformations in a two-dimensional phase-space. From this point of view, a matrix $\mathcal{A}$ is associated with some Hamiltonian function whose level sets contain the orbits of the one-parameter group associated with $\mathcal{A}$.

It is now clear that the family of inertial frames we obtain from a given one with the action of the one-parameter group $e^{\sigma\mathcal{A}}$, is associated with a quadratic form, the Hamiltonian function generating $\mathcal{A}$, up to a numerical factor. Thus we are led to analyse quadratic forms on $\mathcal{R}^2$ in connection
with the placement of \((E, \ker \alpha)\) in space-time, keeping in mind that their level sets contain the orbits of the \textit{relativity transformation group} we are searching for.

The infinitesimal transformation \(\mathcal{A}\) associated with a transformation matrix
\[
A = a \begin{pmatrix} 1 - \epsilon & \tilde{\alpha}_{01} \\ \tilde{\alpha}_{10} & 1 + \epsilon \end{pmatrix},
\]
which tends to the unit of the group when its parameters tend to zero, has the form
\[
\left( \frac{dA}{d\sigma} \right)_{\sigma=0} = A = \begin{pmatrix} -\epsilon & \alpha_{01} \\ \alpha_{10} & \epsilon \end{pmatrix}.
\]
The evolution of the vector \((x^0, x)\) is described by
\[
\frac{d}{d\sigma} \begin{pmatrix} x^0 \\ x \end{pmatrix} = \mathcal{A} \begin{pmatrix} x^0 \\ x \end{pmatrix} = \begin{pmatrix} \alpha_{01}x - \epsilon x^0 \\ \alpha_{10}x^0 + \epsilon x \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial x^0} \end{pmatrix},
\]
where
\[
2H = \alpha_{10}(x^0)^2 + 2\epsilon x^0x - \alpha_{01}x^2.
\]
If \(\mathcal{A}\) does not admit real eigenvalues, it will define a rotation-like transformation (i.e. level sets of \(H\) are ellipses) and will violate the \textit{mutual objective existence} (in what follows \textit{m.o.e.}) condition. We assume therefore that \(\mathcal{A}\) has real eigenvalues. We get:
\[
-\epsilon x^0 + \alpha_{01}x = \lambda x^0, \quad \alpha_{10}x^0 + \epsilon x = \lambda x, \quad \lambda^2 \equiv \epsilon^2 + \alpha_{01}\alpha_{10}.
\]
Eigen-directions are defined by
\[
\alpha_{10}x^0 + (\epsilon \pm \lambda)x = 0
\]
and, in terms of them, we may write
\[
2H = \alpha_{10} \left( x^0 + \frac{x}{c_-} \right) \left( x^0 - \frac{x}{c_+} \right), \quad \text{where} \quad \frac{1}{c_{\pm}} = \frac{1}{c_{\mp}} \equiv \frac{1}{c_{\mp}} = \frac{\lambda \pm \epsilon}{\alpha_{10}}.
\]
The solution of eq. (7) corresponding to the initial conditions \(x^0 = x^0_0\), \(x = x_0\) is given by
\[
\begin{pmatrix} x^0 \\ x \end{pmatrix} = \cosh \sigma \lambda \begin{pmatrix} 1 - \frac{\epsilon \tanh \sigma \lambda}{\alpha_{10} \tanh \sigma \lambda} & \alpha_{01} \frac{\tanh \sigma \lambda}{\lambda} \\ \alpha_{10} \frac{\tanh \sigma \lambda}{\lambda} & 1 + \frac{\epsilon \tanh \sigma \lambda}{\lambda} \end{pmatrix} \begin{pmatrix} x^0_0 \\ x_0 \end{pmatrix}.
\]
Moreover, the ratio $x$, when $\sigma \to \pm\infty$, tends to $(-c_-, c_+)$, independently of the value of the initial conditions. As a consequence, these quantities are characteristic invariants of the transformation. The quantities

$$\frac{\alpha_{01}}{\alpha_{10}} = \frac{1}{c^2} \equiv \frac{1}{c_- c_+} \quad \text{and} \quad \frac{c}{\alpha_{10}} = \frac{1}{c_1} = \frac{1}{2} \left( \frac{1}{c_-} - \frac{1}{c_+} \right)$$

are also invariant. Finally, with the substitution $\tanh(\sigma \lambda) = \tilde{\alpha}_{10} \left( \frac{1}{c_-} + \frac{1}{c_+} \right)$, the transformation matrix in (9) takes the form

$$A = \frac{1}{\sqrt{1 - \tilde{\alpha}_{10}^2 \left( \frac{1}{c_-} + \frac{1}{c_+} \right)}} \left( \begin{array}{cc} 1 - \frac{\tilde{\alpha}_{10}}{c_1} & \frac{\tilde{\alpha}_{10}}{c_1} \\ \frac{\tilde{\alpha}_{10}}{c_1} & 1 + \frac{\tilde{\alpha}_{10}}{c_1} \end{array} \right).$$

Here $\tilde{\alpha}_{10}$ plays the role of parameter of the transformation; no ambiguity arises if we continue to indicate it with $\alpha_{10}$.

The level set corresponding to $H = \overline{\Pi} = 0$ determines the asymptotes of the branches of the hyperbolas defined by $H = \overline{\Pi} > 0$ and $H = \overline{\Pi} < 0$. The asymptotes coincide with the eigen-directions of $A$.

It is now clear that $A$, associated to $H$, will define an acceptable relativity transformation only if $\ker \alpha$ intersects the branches corresponding to $\overline{\Pi} < 0$ and $E$ the branches corresponding to $\overline{\Pi} \geq 0$ (or vice versa, and then we redefine the group parameter so that $\ker \alpha$ always intersects the negative branch). This implies that $A$ (see eq. (6)) cannot transform a vector whose second component is zero in a vector whose first component is zero.

A priori, for a given pair $(E, \alpha)$, we shall find several quadratic functions $H$ satisfying previous requirements, therefore we may think that a combination of them would be also acceptable. Which combinations may be admissible will be discussed in next subsection.

### 5.3 Generic Hamiltonians and compatible transformations

We analyse the trajectories associated with a generic Hamiltonian function (8) ($\alpha_{10} \geq 0$) in the $(x^0, x)$ plane, with reference to Fig. 1.

- **a)** $\alpha_{10} \alpha_{01} < -\epsilon^2$:
  
  no compatibility with the m.o.e. condition; this is the rotation-like case.

- **b)** $\alpha_{10} \alpha_{01} = -\epsilon^2$:
the Hamiltonian reduces to $\alpha_{10}(x^0 + \frac{2}{c_1^2})^2$: no compatibility with the m.o.e. condition;

c) $-\epsilon^2 < \alpha_{10}\alpha_{01} < 0$; both eigen-directions are contained between $E$ and $\ker \alpha$; no compatibility with the m.o.e. condition;

d) $\alpha_{10}\alpha_{01} > 0$; this condition selects candidates to be acceptable Lorentz transformations.

i) Two transformations $A$ and $A'$ of the form (10) belong to the same transformation group only if they have identical invariants; if this is not the case, the m.o.e. existence is violated by the transformation we get composing some powers of them; in fact, if, e.g., $\frac{1}{c_1} > \frac{1}{c_2}$, a world-line, admissible for $A'$, may be rotated by some power of $A$ into $\ker \alpha$ (for the same reason Lorentz-type transformations will not be compatible with Carroll and/or Galilei transformations).

ii) In addition, the product is commutative if and only if $\alpha_{01} \alpha_{10}' = \alpha_{10}' \alpha_{01}$ and $\epsilon' \alpha_{01} = \epsilon \alpha_{01}'$, that is to say if and only if the m.o.e. condition is satisfied.

iii) The addition rule for the unique parameter is:

$$
\alpha_{10}'' = \frac{\alpha_{10} + \alpha_{10}'}{1 + \alpha_{10} \alpha_{10}' \left( \frac{1}{c_1^2} + \frac{1}{c_2^2} \right)}
$$

e) $\alpha_{10} = 0$, $2H = x(2\epsilon x^0 - \alpha_{01} x)$; in this Carroll case

$$
A = \frac{1}{\sqrt{1 - \epsilon^2}} \begin{pmatrix} 1 - \epsilon & \alpha_{01} \\ 0 & 1 + \epsilon \end{pmatrix};
$$

i) $E$ is an invariant asymptote; no transformation can move $E$ into $\ker \alpha$; ii) equi-locality is absolute ($dx_0 = 0 \rightarrow dx = 0$).

f) $\alpha_{01} = 0$, $2H = x^0(\alpha_{10} x^0 + 2\epsilon x)$; in this Galileian case

$$
A = \frac{1}{\sqrt{1 - \epsilon^2}} \begin{pmatrix} 1 - \epsilon & 0 \\ \alpha_{10} & 1 + \epsilon \end{pmatrix};
$$
i) ker $\alpha$ is an invariant asymptote,

ii) the product of two transformations of the same type, (with different values for $\epsilon$ and $\alpha_{10}$) satisfies also the m.o.e. condition;

it iii) simultaneity is absolute ($dx^0_0 = 0 \rightarrow dx^0 = 0$).

g) $\alpha_{01} = \alpha_{10} = 0$; $H = \epsilon x^0 x$; this Aristotelian transformation preserves both equi-locality and simultaneity and is compatible with Lorentz, Galilei and Carroll.

5.4 Requiring the identity of relative clocks

Inertial frames connected by a relativity transformation should be considered to be equivalent; therefore we make a further requirement. We impose that inertial frames connected by an allowed relativity transformation should have identical relative clocks, i.e. we require that the two maps $c_{ab}$ and $c_{ba}$ coincide for any two inertial frames we obtain starting with $(E, \alpha)$. This requirement will impose $\epsilon = 0$. Therefore our analysis is now greatly simplified.

The argument for $\epsilon = 0$ is simple; eq. (9) implies that previous condition is satisfied if

$$\frac{\partial x^0}{\partial x^0} = A_{00} = \frac{\partial x^0}{\partial x^0} = (A^{-1})_{00},$$

that is to say if $c_1 \rightarrow \infty$, $\epsilon = 0$. This equation, which is independent of the restriction to a bidimensional space-time, implies that dilation along the time-axis should be excluded from our relativity transformations, i.e. our infinitesimal transformations should not contain $x^0 \frac{\partial}{\partial x^0}$.

Going back to a bi-dimensional space-time, we find that the Hamiltonian (8) assumes the form

$$2H = \alpha_{10} \left( (x^0)^2 - \frac{x^2}{c^2} \right) = \alpha_{10} \left( x^0 + \frac{x}{c} \right) \left( x^0 - \frac{x}{c} \right),$$

while $A$ reduces to

$$A = \frac{1}{\sqrt{1 - \frac{\alpha_{10}^2}{c^2}}} \left( \begin{array}{cc} 1 & \alpha_{10} \\ \alpha_{10} & 1 \end{array} \right).$$

We conclude that our $(1,1)$ space-time may be equipped with an invariant quadratic form given by $dx^0 \otimes dx^0 - \frac{1}{c^2} dx^0 \otimes dx$ in the (generic) Lorentz case and by $dx^0 \otimes dx^0$ in the Galileian case.
The physical interpretation of these transformations and the related problems, like the clock synchronization one, have been extensively discussed in the literature [7], [8], [9].

We conclude this section by remembering that if space and time coordinates of the universe are deformed in such a manner that all the space-time coincidences are conserved, then the universe remains unchanged [10]. Notice that this requirement is satisfied already at the level in which only the m.o.e. is supposed.

6 Back to four-dimensional space-time

We have seen that, by using the decomposition lemma in section 5, we have been able to select transformations compatible with our requirement of equality of relative clocks and mutual objective existence in the (0, 1)-space time.

Here we would like to find out which are the implications in four dimensions, if we compose it with space transformations.

We first consider the (1, 1)-Galilei group. It is clear that because our transformations preserve a space-slicing (i.e. an absolute notion of simultaneity), any linear transformation along the space part will be an acceptable relativity transformation. We find for the homogeneous generalized Galilei group the semidirect product

\[ G_0 := \mathcal{V} \times \rho \text{GL}(3, \mathcal{R}), \]  

(14)

where \( \mathcal{V} \) stays for the three dimensional space of velocities.

By defining the action on a vector space-time \((b, \vec{x})\) we find

\[ (\vec{v}, A)[(b, \vec{x})] = [(b, A\vec{x} + b\vec{v})] \]

(15)

along with the composition rule

\[ (\vec{v}_1, A_1) \cdot (\vec{v}_2, A_2) = (A_1\vec{v}_2 + \vec{v}_1, A_1 \cdot A_2). \]

(16)

The rotation group of the standard Galilei group is being replaced by the General Linear group in three dimensions, i.e. the m.o.e. condition does not require any preferred notion of distance along the space part of space-time. A symmetric (0, 2)—tensor of the type \(dx^b \otimes dx^a\) is invariant under the action of \(G_0\). Now we consider the other compatible group, i.e. the
Lorentz type transformations. Here, to carry on computations, we find more convenient to use infinitesimal transformations in terms of vector fields.

For each plane involving a time coordinate and a space coordinate, our (1,1) analysis provides us with the following vector fields:

\[
B_1 = \alpha_{01}^{-1} \alpha_{10} x_1 \frac{\partial}{\partial x_0} + x_0 \frac{\partial}{\partial x_1},
\]
\[
B_2 = \alpha_{02}^{-1} \alpha_{20} x_2 \frac{\partial}{\partial x_0} + x_0 \frac{\partial}{\partial x_2},
\]
\[
B_3 = \alpha_{03}^{-1} \alpha_{30} x_3 \frac{\partial}{\partial x_0} + x_0 \frac{\partial}{\partial x_3}.
\]

It is convenient to redefine coordinates by setting

\[
y_1 = \sqrt{\alpha_{01} \alpha_{10}} x_1, y_2 = \sqrt{\alpha_{02} \alpha_{20}} x_2, y_3 = \sqrt{\alpha_{03} \alpha_{30}} x_3, y_0 = x_0;
\]

we find then

\[
\tilde{B}_1 = y_1 \frac{\partial}{\partial y_0} + y_0 \frac{\partial}{\partial y_1},
\]
\[
\tilde{B}_2 = y_2 \frac{\partial}{\partial y_0} + y_0 \frac{\partial}{\partial y_2},
\]
\[
\tilde{B}_3 = y_3 \frac{\partial}{\partial y_0} + y_0 \frac{\partial}{\partial y_3}.
\]

Now we look for space transformations whose commutator with \( \tilde{B} \)'s does not violate the m.o.e. condition.

We find

\[
\left[ A_j^i y^j \frac{\partial}{\partial y}, \tilde{B}_k \right] = A_k^j y^j \frac{\partial}{\partial y_0} - y_0 A_k^j \frac{\partial}{\partial y_0} =
\]
\[
= A_k^j \left( y^j \frac{\partial}{\partial y_0} + y_0 \frac{\partial}{\partial y^j} \right) - (A_k^j + A_j^k) y_0 \frac{\partial}{\partial y_0},
\]

i.e. we find a combination of boosts and Carrol transformations. To preserve the m.o.e. condition we have to require \( A_j^k = -A_j^k \); in conclusion, the most general space transformations compatible with boosts to preserve the objective conditions are just rotations.

We find that our allowed relativity group is the Lorentz group. Therefore our space time gets equipped with a generalized Minkowskian metric. The associated symmetric \((0,2)\)-tensor has the form

\[
dx^0 \otimes dx^0 - \beta_1^2 dx^1 \otimes dx^1 - \beta_2^2 dx^2 \otimes dx^2 - \beta_3^2 dx^3 \otimes dx^3.
\]

Some further intermediate situations are possible. They correspond to the use of Galilei-type transformations in some one time-one space planes and Lorentz type in the remaining ones. These situationns are obtained when some of the coefficients of the rprevious quadratic form are being set equal to zero, say \( \beta_1 = 0, \beta_2 \) and \( \beta_3 \) being different from zero, or \( \beta_1 = \beta_2 = 0, \beta_3 \) being different from zero.
7 Conclusions

We have found that the notion of *mutual objective existence* along with the identity of relative clocks is enough to select only the Galilei and Lorentz transformations in one space and one time setting. These transformations preserve a quadratic form which is degenerate in the Galileian case and not degenerate in the Lorentz case.

When going to $R^\triangle$, we find that the *mutual objective existence* condition determines a Minkowski-type metric in the case of the Lorentz group, while the Galilei-type group does not impose restrictions on the space structure.

Some intermediate situations are also possible. It is possible to have Lorentz-type behaviour in some directions and Galilei-type behaviour in the complementary directions. These mixed situations cannot be ruled out only on the basis of the *mutual objective existence* condition and some additional physical insight is needed.

We hope we have made clear how a *general relativity ideology* may be useful in dealing also with *special relativity*, where only an affine structure for space-time is needed.

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Figure 1: Geometrical representation of possible special linear transformations in $(1 - 1)$ space time.