A domain-specific language for the hybridization and static condensation of finite element methods

THOMAS H. GIBSON, LAWRENCE MITCHELL*, DAVID A. HAM, and COLIN J. COTTER, Imperial College London, UK

In this paper, we introduce an embedded domain-specific language (DSL) for concisely expressing localized linear algebra on finite element tensors and its integration within a code-generation framework. This compact framework is general enough to facilitate the automatic generation of element-based dense linear algebra kernels necessary for the implementation of static condensation methods and local solvers for a variety of problems. We demonstrate how this framework can be used to enable the rapid implementation of hybridized mixed and discontinuous Galerkin methods using the Firedrake finite element library. We also describe how this DSL can be used to execute local post-processing procedures to construct superconvergent approximations to mixed problems. This work features high-level implementations conforming to PETSc’s interface for solving linear systems. These preconditioning interfaces provide reduced operators, which are obtained from locally assembled expressions, with the necessary context to specify full solver configurations on the resulting linear systems. We provide examples derived from second order elliptic problems and geophysical fluid dynamics.

CCS Concepts: • Mathematics of computing → Mathematical software; Solvers; Partial differential equations; • Computing methodologies → Symbolic and algebraic manipulation;

Additional Key Words and Phrases: Domain-specific language, automatic code-generation, hybridization, static condensation, finite element methods.

ACM Reference Format:
Thomas H. Gibson, Lawrence Mitchell, David A. Ham, and Colin J. Cotter. 2018. A domain-specific language for the hybridization and static condensation of finite element methods. ACM Trans. Math. Softw. 1, 1 (July 2018), 25 pages. https://doi.org/10.1145/nnnnnn.nnnnnnn

1 INTRODUCTION
The development of simulation software is an increasingly important aspect of modern scientific computing. Such software requires a vast range of knowledge spanning several disciplines, ranging from abstract mathematics to high-performance computing and low-level code optimization. Software projects developing automatic code generation systems have become quite popular in recent years, as such systems help create a separation of concerns which focuses on a particular complexity independent from the rest. Examples of such projects include FreeFEM++ [Hecht 2012], Sundance [Long et al. 2010], the FEniCS Project [Logg et al. 2012], Feel++ [Prud’Homme et al. 2012], and Firedrake [Rathgeber et al. 2016].

The finite element method (FEM) is a mathematically robust framework for computing solutions of partial differential equations (PDEs), with a formulation that is highly amenable to code-generation techniques. A description of the weak formulation of the PDEs, together with appropriate discrete function spaces, is enough to characterize the finite element problem. Both the FEniCS and Firedrake projects employ the Unified Form Language (UFL) [Alnæs et al. 2014] to specify the finite element integral forms and discrete spaces necessary to properly define the finite

* Also with Imperial College London, Department of Computing.

Authors’ address: Thomas H. Gibson, t.gibson15@imperial.ac.uk; Lawrence Mitchell, lawrence.mitchell@imperial.ac.uk; David A. Ham, david.ham@imperial.ac.uk; Colin J. Cotter, colin.cotter@imperial.ac.uk, Imperial College London, Department of Mathematics, South Kensington Campus, London, SW7 2AZ, UK.
element problem. UFL is a highly expressive domain-specific language (DSL) embedded in Python, which provides the necessary abstractions for code generation systems.

There are classes of finite element discretizations resulting in discrete systems that can be solved more efficiently by directly manipulating local tensors. For example, the static condensation technique for the reduction of global finite element systems [Guyan 1965; Irons 1965] produces smaller globally-coupled linear systems by eliminating interior unknowns to arrive at an equation for the facet degrees of freedom only. Alternatively, hybridized finite element methods [Arnold and Brezzi 1985; Brezzi and Fortin 2012; Cockburn et al. 2009a] introduce Lagrange multipliers enforcing certain continuity constraints on finite element functions. Static condensation can then be applied to the augmented system to produce a reduced equation for the multipliers. Methods of this type are often accompanied by local post-processing techniques that produce superconvergent approximations [Bramble and Xu 1989; Cockburn et al. 2010a, 2009b]. These procedures often require invasive manual intervention during the equation assembly process in intricate numerical code.

In this paper, we provide a simple yet effective high-level abstraction for localized dense linear algebra on systems derived from finite element problems. Using embedded DSL technology, we provide a means to enable the rapid development of hybridization and static condensation techniques within an automatic code-generation framework. In other words, the main contribution of this paper is in solving the problem of automatically translating from the mathematics of static condensation and hybridization to compiled code. Our work is implemented in the Firedrake finite element framework [Rathgeber et al. 2016] and the PETSc [Balay et al. 2016, 1997] solver library, accessed via petsc4py [Dalcin et al. 2011].

The rest of the paper is organized as follows. We introduce common notation used throughout the paper in Section 1.1. The embedded DSL, Slate, is introduced in Section 2, which allows concise expression of localized linear algebra operations on finite element tensors. We provide some contextual examples for static condensation and hybridization in Section 3, including a discussion on post-processing. We then outline in Section 4 how, by interpreting static condensation techniques as a preconditioner, we can use Slate to automate solution techniques employing hybridization and static condensation. We demonstrate our implementations for manufactured test problems in Section 5. Section 5.3 illustrates the composability of our implementation of a hybridized mixed method as a preconditioner for the resulting linearized system of a semi-implicit discretization of the rotating shallow water equations. Conclusions follow in Section 6.

1.1 Notation
Let $\mathcal{T}_h$ denote a tessellation of $\Omega \subset \mathbb{R}^n$, the computational domain, consisting of polygonal elements $K$ associated with a mesh size parameter $h$, and $\partial \mathcal{T}_h = \{e \in \partial K : K \in \mathcal{T}_h\}$ the set of facets of $\mathcal{T}_h$. The set of facets interior to the domain $\Omega$ is denoted by $\mathcal{E}^\circ_h \equiv \partial \mathcal{T}_h \setminus \partial \Omega$. Similarly, we denote the set of exterior facets as simply $\mathcal{E}^{\partial}_h = \partial \mathcal{T}_h \cap \partial \Omega$. For brevity, we denote the finite element integral forms over $\mathcal{T}_h$ and any facet set $\Gamma \subset \partial \mathcal{T}_h$ by

$$
(u, v)_K = \int_K u \cdot v \, dx, \quad \langle u, v \rangle_e = \int_e u \cdot v \, ds,
$$

$$
(u, v)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (u, v)_K, \quad \langle u, v \rangle_{\Gamma} = \sum_{e \in \Gamma} \langle u, v \rangle_e,
$$

where $\cdot$ should be interpreted as standard multiplication for scalar functions or a dot product for vector functions.
A DSL for hybridization and static condensation of finite element methods

For any double-valued vector field \( \mathbf{w} \) on a facet \( e \in \partial T_h \), we define the jump of its normal component across \( e \) by

\[
[w]_e = \begin{cases} 
  w^+ \mathbf{n}^+ + w^- \mathbf{n}^-, & e \in \mathcal{E}_h^o \\
  w^+ \mathbf{n}, & e \in \mathcal{E}_h^\partial 
\end{cases}
\] (3)

where + and − denotes the positive and negative sides of the facet respectively. Here, \( \mathbf{n}^+ \) and \( \mathbf{n}^- \) are the unit normal vectors with respect to the positive and negative sides of the facet \( e \). Whenever the facet domain is clear by the context, we omit the subscripts for brevity and simply write \([·]\).

2 A SYSTEM FOR LOCALIZED ALGEBRA ON FINITE ELEMENT TENSORS

We present an expressive language for dense linear algebra on the elemental matrix systems arising from finite element problems. The language, which we call Slate, inherits typical mathematical operations performed on matrices and vectors, hence the input syntax is comparable to high-level linear algebra software such as MATLAB. The Slate language provides basic abstract building blocks which can be used by a specialized compiler for linear algebra to generate low-level code implementations.

Slate is heavily influenced by the Unified Form Language (UFL) [Alnæs et al. 2014; Logg et al. 2012], a DSL embedded in Python which provides symbolic representations of finite element forms. The expressions can be compiled by a form compiler, which translates UFL into low level code for the local assembly of a form over the cells and facets of a mesh. In a similar manner, Slate expressions are compiled to low level code that performs the requested linear algebra element-wise on a mesh.

2.1 An overview of Slate

To clarify conventions and the scope of Slate, we start by considering a general form. Suppose we have a finite element form:

\[
a(c; \mathbf{v}) = \sum_{K \in T_h} \int_K I^c(c; \mathbf{v})dx + \sum_{e \in \mathcal{E}_h^o} \int_e I^{\mathcal{K}^o}(c; \mathbf{v})ds + \sum_{e \in \mathcal{E}_h^\partial} \int_e I^{\mathcal{K}^\partial}(c; \mathbf{v})ds,
\] (4)

where \( dx \) and \( ds \) denote appropriate integration measures. The integral form in (4) is uniquely determined by its lists (possibly of 0-length) of arbitrary coefficient functions \( c = (c_0, \cdots, c_p) \) in the associated finite element spaces, arguments \( \mathbf{v} = (\mathbf{v}_0, \cdots, \mathbf{v}_q) \) describing any test or trial functions, and its integrand expressions for each integral type: \( I^c \), \( I^{\mathcal{K}^o} \), \( I^{\mathcal{K}^\partial} \). The form \( a(c; \mathbf{v}) \) describes a finite element form globally over the entire problem domain. The contribution of (4) in each cell \( K \) of the mesh \( T_h \) is simply

\[
a(c; \mathbf{v})|_K = \int_K I^c(c; \mathbf{v})dx + \sum_{e \in \partial K \cap \partial \Omega} \int_e I^{\mathcal{K}^o}(c; \mathbf{v})ds + \sum_{e \in \partial K \cap \partial \Omega} \int_e I^{\mathcal{K}^\partial}(c; \mathbf{v})ds.
\] (5)

We call (5) the cell-local contribution of \( a(c; \mathbf{v}) \). Equation (5) produces an element tensor which is mapped into a global data structure. However, one may want to produce a new local tensor by algebraically manipulating different element tensors. This is precisely the job of Slate.

The Slate language consists of two primary abstractions for linear algebra: (1) terminal element tensors corresponding to multi-linear integral forms, or assembled data; and (2) expressions consisting of operations on terminal tensors or existing Slate expressions.

Terminal tensors: In Slate, one associates a tensor with data on an element either by using a form, or assembled coefficient data:
• Tensor\(a(c; v))\)
associates a form, expressed in UFL, with its local element tensor:
\[
A^K \leftarrow a(c; v)|_K, \quad \text{for all } K \in T_h. \tag{6}
\]
The number of arguments \(v\) determine the rank of Tensor, i.e. scalars, vectors, and matrices are produced from 0-forms, 1-forms, and 2-forms\(^1\) respectively.

• AssembledVector\((f)\)
where \(f\) is some finite element function. The result associates a function with its local coefficient vectors.

Symbolic linear algebra: Slate supports typical linear algebra operations with a high-level syntax close to mathematics:

• \(A + B\), the addition of two equal shaped tensors.

• \(A \times B\), a contraction over the last index of \(A\) and the first index of \(B\).

• \(-A\), the additive inverse (negative) of a tensor.

• \(A.\text{T}\), the transpose of a tensor.

• \(A.\text{inv}\), the inverse of a square tensor.

• \(A.\text{solve}(B, \text{decomposition}="\ldots")\), the result of solving \(AX = B\) for \(X\), optionally specifying a factorization strategy.

• \(A.\text{blocks}[\text{indices}]\), where \(A\) is a tensor from a mixed finite element space, allows extraction of subblocks of the tensor indexed by field (slices are allowed). For example, if a matrix \(A\) corresponds to the bilinear form \(a : V \times W \rightarrow \mathbb{R}\), where \(V = V_0 \times \cdots \times V_n\) and \(W = W_0 \times \cdots \times W_m\) are product spaces consisting of finite element spaces \(\{V_i\}_{i=0}^n\), \(\{W_i\}_{i=0}^m\), then the cell-local tensors have the form:
\[
A^K = \begin{bmatrix}
A^K_{00} & A^K_{01} & \cdots & A^K_{0m} \\
A^K_{10} & A^K_{11} & \cdots & A^K_{1m} \\
\vdots & \vdots & \ddots & \vdots \\
A^K_{n0} & A^K_{n1} & \cdots & A^K_{nm}
\end{bmatrix}. \tag{7}
\]
The associated submatrix of (7) with indices \(i = (p, q)\), \(p = \{p_1, \ldots, p_r\}\), \(q = \{q_1, \ldots, q_c\}\), is
\[
A^K_{pq} = \begin{bmatrix}
A^K_{p_1q_1} & \cdots & A^K_{p_1q_c} \\
\vdots & \ddots & \vdots \\
A^K_{p_rq_1} & \cdots & A^K_{p_rq_c}
\end{bmatrix} = A^K.\text{blocks}[p, q], \tag{8}
\]
where \(p \subseteq \{1, \ldots, n\}\), \(q \subseteq \{1, \ldots, m\}\).

These building blocks may be arbitrarily composed, giving us the necessary algebraic framework for a large class of problems, some of which we present in this paper.

Slate expressions are handled by a linear algebra compiler, which employs TSFC to compile kernels for the assembly of terminal tensors and generates a dense linear algebra kernel to be iterated cell-wise. Our compiler generates C++ code, using the templated library Eigen [Guennebaud et al. 2015] for dense linear algebra. During execution, the local computations in each cell are mapped into global data objects via appropriate indirection mappings. Figure 1 provides an illustration of the complete tool-chain.

---

\(^1\)As with UFL, Slate is capable of abstractly representing arbitrary rank tensors. However, only rank \(\leq 2\) tensors are typically used in most finite element applications and therefore we currently only generate code for those ranks.
Fig. 1. The Slate language wraps forms, expressed in UFL, describing the finite element problem. The Slate expressions are handed over to a specialized linear algebra compiler, which produces a single “macro” kernel which assembles the local element contributions, and then performs the dense linear algebra. The resulting kernels are passed to the PyOP2 interface, which wraps the Slate kernel in a mesh-iteration kernel. Parallel scheduling and code generation occurs after the PyOP2 layer.

3 EXAMPLES
We now present a few examples and discuss solution methods which require element-wise manipulations of finite element systems and their specification in Slate. We stress here that Slate is not limited to these model problems; rather these examples were chosen for clarity and to demonstrate key features of the Slate language. In Sections 4 and 5, we discuss more intricate ways the Slate DSL is used in custom preconditioners for linear systems.

3.1 Hybridized discontinuous Galerkin methods
For our model problem, consider the second-order elliptic PDE:

\[-\nabla \cdot (\kappa \nabla p) + cp = f \text{ in } \Omega,\]

\[p = p_0 \text{ on } \partial \Omega_D,\]

\[-\kappa \nabla p \cdot n = g \text{ on } \partial \Omega_N,\]

where \(\partial \Omega_D \cup \partial \Omega_N = \partial \Omega\) and \(\kappa, c : \Omega \rightarrow \mathbb{R}^+\) are positive-valued coefficients. Rewriting as a first order system, we have the following mixed problem:

\[\mu \mathbf{u} + \nabla p = 0 \text{ in } \Omega,\]

\[\nabla \cdot \mathbf{u} + cp = f \text{ in } \Omega,\]

\[p = p_0 \text{ on } \partial \Omega_D,\]

\[\mathbf{u} \cdot n = g \text{ on } \partial \Omega_N,\]

where \(\mu = \kappa^{-1}\) and \(\mathbf{u} = -\kappa \nabla p\).

The hybridized discontinuous Galerkin (HDG) method is a natural extension of discontinuous Galerkin (DG) discretizations. Here, we consider a specific HDG discretization, LDG-H [Cockburn
The LDG-H system has more degrees of freedom than its DG counterpart, but its matrix-form exhibits some immediate advantages. We use this subtle dependency to verify our implementation in Section 5.1.

The matrix system arising from (19)–(22) has the general form:

\[
\begin{bmatrix}
A_{00} & A_{01} & A_{02} \\
A_{10} & A_{11} & A_{12} \\
A_{20} & A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
U \\
P \\
\Lambda
\end{bmatrix}
= \begin{bmatrix}
F_0 \\
F_1 \\
F_2
\end{bmatrix}.
\]

The LDG-H system has more degrees of freedom than its DG counterpart, but its matrix-form exhibits some immediate advantages.

1. By our choice of function spaces, the operator coupling \( U \) and \( P \) is block-sparse. We can therefore simultaneously eliminate the coupled unknowns by performing element-wise static condensation to (23), producing a significantly smaller problem for \( \Lambda \) only:

\[
SL = E,
\]

where \( S \) and \( E \) are given by

\[
S = A_{22} - \begin{bmatrix}
A_{20} & A_{21}
\end{bmatrix}
\begin{bmatrix}
A_{00} & A_{01} \\
A_{10} & A_{11}
\end{bmatrix}^{-1}
\begin{bmatrix}
A_{02} \\
A_{12}
\end{bmatrix},
\]

\[
E = F_2 - \begin{bmatrix}
A_{20} & A_{21}
\end{bmatrix}
\begin{bmatrix}
A_{00} & A_{01} \\
A_{10} & A_{11}
\end{bmatrix}^{-1}
\begin{bmatrix}
F_0 \\
F_1
\end{bmatrix}.
\]

2. The matrix \( S \) is sparse, symmetric, and positive-definite [Cockburn et al. 2009a]. Furthermore, \( S \) is a discrete elliptic operator and therefore we can solve (24) using existing techniques for global elliptic equations.

3. Once \( \Lambda \) is computed, both \( U \) and \( P \) can be recovered locally in each element. This can be accomplished in two stages. First we compute \( P \) by solving:

\[
(A_{11} - A_{10}A_{00}^{-1}A_{01})P = F_1 - A_{10}A_{00}^{-1}F_0 - (A_{12} - A_{10}A_{00}^{-1}A_{02})\Lambda,
\]

Other forms of HDG that involve local lifting operators can also be implemented in this software framework by the introduction of additional local (i.e., discontinuous) variables in the definition of the local solver.
followed by:

\[ A_{00}U = F_0 - A_{01}P - A_{02}\Lambda. \]  

(28)

4. The solution approximations can further be improved via local post-processing. We highlight a few procedures in Section 3.3.

Listing 1 displays the corresponding Slate code for assembling the trace system, solving (24), and recovering the eliminated unknowns.

Listing 1. Slate code for solving (23), given UFL expressions a, L for (19)–(21). Arguments of the mixed space \( U_h \times V_h \times M_h \) are indexed by 0, 1, and 2 respectively. Lines 8 and 9 correspond with (25) and (26) for building the trace system. Any vanishing conditions on the trace variables should be provided as boundary conditions during operator assembly. Lines 27 and 29 correspond to the expressions given in (27) and (28) (using LU).

```plaintext
# Element tensors defining the local 3-by-3 block system
_A = Tensor(a)
_F = Tensor(L)

# Extracting blocks to form the reduced system
A = _A.blocks
F = _F.blocks
S = A[2, 2] - A[2, :2] * A[:2, :2].inv * A[:2, 2]
E = F[2] - A[2, :2] * A[:2, :2].inv * F[:2]

# Assemble and solve: SA = E
Smat = assemble(S, bcs=[...])
Evec = assemble(E)
lambda_h = Function(M)
solve(Smat, lambda_h, Evec)

p_h = Function(V)  # Function to store the result: P
u_h = Function(U)  # Function to store the result: U

# Intermediate expressions
Sd = A[1, 1] - A[1, 0] * A[0, 0].inv * A[0, 1]
Sl = A[1, 2] - A[1, 0] * A[0, 0].inv * A[0, 2]
Lambda = AssembledVector(lambda_h)  # Coefficient vector for \( \Lambda \)
P = AssembledVector(p_h)  # Coefficient vector for P

# Local solve expressions for P and U
p_sys = Sd.solve(F[1] - A[1, 0] * A[0, 0].inv * F[0] - Sl * Lambda,
                decomposition="PartialPivLu")
u_sys = A[0, 0].solve(F[0] - A[0, 1] * P - A[0, 2] * Lambda,
                    decomposition="PartialPivLu")
assemble(p_sys, p_h)
assemble(u_sys, u_h)
```

3.2 Hybridized mixed methods

To motivate the hybridization of mixed methods, we start by recalling the mixed method for the first order system (12)–(15). Standard \( H(\text{div}) \times L^2 \) methods seek a solution \((u_h, p_h)\) in the finite dimensional spaces:

\[ U_h = \{ w \in H(\text{div}; \Omega) : w|_K \in U(K), \forall K \in T_h, w \cdot n = g \text{ on } \partial \Omega_N \}, \]  

(29)

\[ V_h = \{ \phi \in L^2(\Omega) : \phi|_K \in V(K), \forall K \in T_h \}. \]  

(30)
respectively. The space $U_h$ consists of $H(\text{div})$-conforming piecewise vector polynomials, where choices of $U(K)$ typically include the Raviart-Thomas (RT), Brezzi-Douglas-Marini (BDM), or Brezzi-Douglas-Fortin-Marini (BDFM) elements [Brezzi et al. 1987, 1985; Nédélec 1980; Raviart and Thomas 1977].

The mixed finite element formulation of (12)–(15) reads as follows: find $(u_h, p_h) \in U_h \times V_h$ satisfying

\begin{align}
(w, \mu u_h)|_{T_h} - (\nabla \cdot w, p_h)|_{T_h} &= - (w \cdot n, p_0)_{\partial \Omega_D}, \quad \forall w \in U_{h,0}, \\
(\phi, \nabla \cdot u_h)|_{T_h} + (\phi, c p_h)|_{T_h} &= (\phi, f)|_{T_h}, \quad \forall \phi \in V_h,
\end{align}

(31) (32)

where $U_{h,0}$ is the space of functions in $U_h$ whose normal components vanish on $\partial \Omega_N$. With $U$ and $P$ denoting the vector of degrees of freedom for $u_h$ and $p_h$ respectively, computing the solution of (31)–(32) requires solving the saddle point system:

\begin{equation}
\begin{bmatrix}
A & -B^T \\
B & C
\end{bmatrix} \begin{bmatrix}
U \\
P
\end{bmatrix} = \begin{bmatrix}
F_0 \\
F_1
\end{bmatrix}.
\end{equation}

(33)

Methods to efficiently invert such systems include $H(\text{div})$-multigrid [Arnold et al. 2000] (requiring complex overlapping-Schwarz smoothers), global Schur complement factorizations (which require an approximation to the inverse of the elliptic Schur complement operator), or auxiliary space multigrid [Hiptmair and Xu 2007]. Here, we focus on a solution approach using a hybridized mixed method [Arnold and Brezzi 1985; Brezzi and Fortin 2012].

For hybridized mixed methods, the discrete solution spaces are $V_h$ for $p_h$ and $U_h^d$ for $u_h^d$, where the superscript $d$ denotes a discontinuous approximation to $u$ in the space

\begin{equation}
U_h^d = \{ w \in [L^2(\Omega)]^n : w|_K \in U(K), \forall K \in \mathcal{T}_h \}.
\end{equation}

(34)

The vector finite element space $U_h^d$ is a subspace of $[L^2(\Omega)]^n$ consisting of local $H(\text{div})$ functions, but normal components are no longer continuous on $\partial \mathcal{T}_h$. Lagrange multipliers enforcing normal continuity are introduced as an auxiliary variable in the trace space $M_h$.

Deriving the hybridized mixed system is accomplished through integration by parts over each element $K$. Testing with $w \in U_h^d(K)$ and integrating (12) over $K$ produces:

\begin{equation}
(w, \mu u_h^d)|_{K} - (\nabla \cdot w, p_h)|_{K} + (w \cdot n, \lambda_h)|_{\partial K \cap \partial \Omega_D} = - (w \cdot n p_0)|_{\partial K \cap \partial \Omega_D}.
\end{equation}

(35)

The trace function $\lambda_h$ is introduced in surface integrals approximating $p_h$ on elemental boundaries. An additional constraint equation is added to close the system. The global hybrid-mixed formulation reads: find $(u_h^d, p_h, \lambda_h) \in U_h^d \times V_h \times M_h$ such that

\begin{align}
(w, \mu u_h^d)|_{T_h} - (\nabla \cdot w, p_h)|_{T_h} + (w \cdot n, \lambda_h)|_{\partial T_h \cap \partial \Omega_D} &= - (w \cdot n, p_0)|_{\partial \Omega_D}, \quad \forall w \in U_h^d, \\
(\phi, \nabla \cdot u_h^d)|_{T_h} + (\phi, c p_h)|_{T_h} &= (\phi, f)|_{T_h}, \quad \forall \phi \in V_h,
\end{align}

(36) (37)

\begin{equation}
\langle \gamma, [u_h^d] \rangle_{\partial T_h \cap \partial \Omega_N} = \langle \gamma, g \rangle_{\partial \Omega_N}, \quad \forall \gamma \in M_{h,0}.
\end{equation}

(38)

where $M_{h,0}$ denotes the space of traces vanishing on $\partial \Omega_D$.

Arising from the hybridized-mixed formulation, we have the augmented system of linear equations with the general form:

\begin{equation}
\begin{bmatrix}
\hat{A} & K^T \\
K & 0
\end{bmatrix} \begin{bmatrix}
X \\
\Lambda
\end{bmatrix} = \begin{bmatrix}
F \\
G
\end{bmatrix},
\end{equation}

(39)

where $F$ is a mixed vector of the form $F = \{F_0 \ F_1\}^T$, and $X = \{U^d \ P\}^T$, $\Lambda$ are the vectors of degrees of freedom for the flux, scalar, and trace unknowns. Equation (39) defines a $3 \times 3$ block system which is block-sparse by our choices of $U_h^d, V_h$, and $M_h$. If the space of Lagrange multipliers $M_h$ is chosen appropriately, then the flux $u_h^d$, albeit sought a priori in a discontinuous space, will
A DSL for hybridization and static condensation of finite element methods

coincide with its $H(\text{div})$-conforming counterpart $\mathbf{u}_h$. The constraint in (38) enforces both continuity of the normal components of $\mathbf{u}_h^\perp$ across elemental boundaries, as well as the Neumann condition on $\partial \Omega_N$. As a result, the formulations in (36)–(37) and (31)–(32) are solving equivalent problems \cite{Arnold1985}.

The system in (39) defines a three-field problem similar to that of the HDG method in (23), and hence possesses the same properties previously discussed. The solution approach is nearly identical, including the Slate code. A few simplifications are present in the expressions for building the trace system and local solvers ($A_{12} = A_{21} = A_{22} = 0$). The resulting matrix for the $\Lambda$-system also defines an elliptic operator. For the interested reader, a unifying framework for the hybridization of mixed and DG methods is detailed by Cockburn et al. \cite{Cockburn2009a}.

### 3.3 Local post-processing

Local post-processing techniques for the construction of superconvergent approximations were discussed within the context of mixed methods \cite{Arnold1985, Bramble1989, Stenberg1991}, and discontinuous Galerkin methods \cite{Cockburn2010a, Cockburn2009b}. Here, we present two post-processing techniques for producing local solutions of higher approximation order. The Slate code follows naturally from previous discussions in Sections 3.1 and 3.2, using the standard set of operations on local tensors.

#### 3.3.1 Post-processing of the scalar solution

There are number of post-processing techniques for enhancing the accuracy of the scalar approximation of the hybridized mixed method and its DG variant. We present a modified version of the procedure presented by Stenberg \cite{Stenberg1991}, and highlighted within the context of hybridizing eigenproblems by Cockburn et al. \cite{Cockburn2010a}.

Let $\mathcal{P}_k(K)$ denote a polynomial space of degree $\leq k$ on an element $K \in \mathcal{T}_h$. Then for a given pair of computed solutions $\mathbf{u}_h, p_h$ of the hybridized methods, we define the post-processed scalar $p^*_h \in \mathcal{P}_{k+1}(K)$ as the unique solution of the local problem:

\[
(\nabla w, \nabla p^*_h)_K = -(\nabla w, \kappa^{-1} \mathbf{u}_h)_K, \quad \forall w \in \mathcal{P}_{k+1}^{l+1}(K),
\]

\[
(v, p^*_h)_K = (v, p_h)_K, \quad \forall v \in \mathcal{P}_l(K),
\]

where $0 \leq l \leq k$. Here, the space $\mathcal{P}_{k+1}^{l+1}(K)$ denotes the $L^2$-orthogonal complement of $\mathcal{P}_l(K)$. This post-processing method directly uses the definition of the flux $\mathbf{u}_h = -\kappa \nabla p_h$ to construct the local problem above. In practice, the space $\mathcal{P}_{k+1}^{l+1}(K)$ may be constructed using an orthogonal hierarchical basis, and solving (40)–(41) amounts to inverting a local symmetric positive definite system.

At the time of this work, Firedrake does not support the construction of such a finite element basis. However, we can introduce Lagrange multipliers to enforce the orthogonality constraint. The resulting local problem then becomes the following mixed system: find $(p^*_h, \psi) \in \mathcal{P}_{k+1}(K) \times \mathcal{P}_l(K)$ such that

\[
(\nabla w, \nabla p^*_h)_K + (w, \psi)_K = -(\nabla w, \kappa^{-1} \mathbf{u}_h)_K, \quad \forall w \in \mathcal{P}_{k+1}(K),
\]

\[
(\phi, p^*_h)_K = (\phi, p_h)_K, \quad \forall \phi \in \mathcal{P}_l(K),
\]

where $0 \leq l \leq k$. The local problems (42)–(43) and (40)–(41) are equivalent, with the Lagrange multiplier $\psi$ enforcing orthogonality of test functions in $\mathcal{P}_{k+1}(K)$ with functions in $\mathcal{P}_l(K)$.

This post-processing method produces a new approximation which superconverges at a rate of $k + 2$ for hybridized mixed methods \cite{Arnold1985, Cockburn2009b, Stenberg1991}. For the LDG-H method, $k + 2$ superconvergence is achieved when $\tau = O(1)$ and $\tau = O(h)$, but only $k + 1$ convergence is achieved when $\tau = O(1/h)$ \cite{Cockburn2010a, Cockburn2009b}. We use this dependency in the convergence rates to verify our software implementation in Section 5.1.
3.3.2 Post-processing of the flux. To post-process the flux in the LDG-H method, we use the numerical trace $\tilde{u}_h$ from (17). The technique we outline here follows that of Cockburn et al. [2009b], and produces a new flux with better conservation properties.

Let $\mathcal{T}_h$ be a mesh consisting of simplices. On each element $K \in \mathcal{T}_h$, we define a new function $u_h^*$ to be the unique element of the local Raviart-Thomas space $[P_h(K)]^n + xP_h(K)$ satisfying
\[
(r, u_h^*)_K = (r, u_h)_K, \quad \forall r \in [P_{h-1}(K)]^n, \\
\langle \mu, u_h^* \cdot n \rangle_e = \langle \mu, \tilde{u}_h \cdot n \rangle_e, \quad \forall \mu \in P_h(e), \forall e \in \partial K.
\]
This local problem produces a new flux $u_h^*$ with the following properties:

1. $u_h^*$ converges at the same rate as $u_h$ for all choices of $\tau$ producing a solvable system for (19)–(21). However,
2. $u_h^* \in H(\text{div}; \Omega)$. That is,
\[
[u_h^*]_e = 0, \quad \forall e \in \mathcal{E}_h^0.
\]
3. Furthermore, the divergence of $u_h^*$ converges at a rate of $k + 1$.

4 STATIC CONDENSATION AS A PRECONDITIONER

Slate enables static condensation approaches to be expressed very concisely. Nonetheless, application of a particular approach to different variational problems using Slate still produces a certain amount of code repetition. By formulating each form of static condensation as a preconditioner, code can be written once and then applied to any mathematically suitable problem. Rather than writing the static condensation by hand, in many cases, it is sufficient to just select the appropriate, Slate-based, preconditioner.

To understand how we can view static condensation as a preconditioner, it is helpful to frame the problem in the particular context of the solver library. Firedrake uses PETSc to provide linear solvers, and we implement our preconditioners as PETSc PC objects. These are defined to act on the problem residual, and return a correction to the solution. We can think of (left) preconditioning the matrix equation in residual form:
\[
r = r(A, b) \equiv b - Ax = 0
\]
by an operator $P$ (which may not necessarily be linear) as a transformation into an equivalent system of the form
\[
Pr = P(b - Ax) = 0.
\]

Given a current iterate $x_i$, the residual at the $i$-th iteration is simply $r_i \equiv b - Ax_i$, and $P$ acts on the residual to produce an approximation to the error $\epsilon_i \equiv x - x_i$. If $P$ is an application of an exact inverse, the residual is converted into an exact (up to numerical round-off) error.

We will denote the application of particular Krylov method for the linear system (47) as $\mathcal{K}_x(r(A, b))$. Upon preconditioning the system via $P$ as in (48), we write
\[
\mathcal{K}_x(P(A, b)).
\]

If (49) is solved directly via the application of $A^{-1}$, then $P(A, b) = A^{-1}b - x$. So, we have that $\mathcal{K}_x(P(A, b)) = \mathcal{K}_x(r(I, A^{-1}b))$ produces the exact solution of (47) in a single iteration of $\mathcal{K}$.

4.1 Interfacing with PETSc via custom preconditioners

The implementation of preconditioners for these systems requires manipulation not of assembled matrices, but rather their symbolic representation. To do this, we use the preconditioning infrastructure developed by Kirby and Mitchell [2018], which gives preconditioners written in Python...
access to the symbolic problem description. We manipulate this appropriately and provide operators assembled from Slate expressions to PETSc for further algebraic preconditioning. Using this technique, we have developed a static condensation interface for the hybridization of $\mathbf{H}(\text{div}) \times L^2$ mixed problems, and a generic interface for hybridized systems (both hybridized mixed and HDG methods). The advantage of writing even the latter as a preconditioner is the ability to switch out the solution scheme for the system, even when nested inside a larger set of coupled equations.

4.1.1 A static condensation interface for hybridization. One of the main advantages of using a hybridized variant of a DG or mixed method is that such systems permit the use of element-wise condensation and recovery. To facilitate this, we provide a PETSc PC static condensation interface, HybridSCPC. This preconditioner takes the discretized hybridized system as in (23), and performs the local elimination and recovery procedures.

More precisely, the incoming system has the form:

\[
\begin{bmatrix}
\mathbf{A} & \mathbf{K} \\
\mathbf{L} & \mathbf{M}
\end{bmatrix}
\begin{bmatrix}
\mathbf{X} \\
\mathbf{\Lambda}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{F} \\
\mathbf{F}_\Lambda
\end{bmatrix},
\]

(50)

where $\mathbf{X} = \begin{bmatrix} \mathbf{U} & \mathbf{P} \end{bmatrix}^T$. Note that in many applications, $\mathbf{K} = \mathbf{L}^T$, however our implementation does not assume symmetry of the off-diagonal operators. Equation (50) defines a $3 \times 3$ block system of equations. In exact arithmetic, the HybridSCPC preconditioner applies the inverse of the Schur complement factorization:

\[
\mathbf{P} = \begin{bmatrix}
\mathbf{I} & \mathbf{A}^{-1}\mathbf{K} \\
0 & \mathbf{I}
\end{bmatrix}
\begin{bmatrix}
\mathbf{A}^{-1} & 0 \\
0 & \mathbf{S}^{-1}
\end{bmatrix}
\begin{bmatrix}
\mathbf{I} & 0 \\
\mathbf{L}\mathbf{A}^{-1} & \mathbf{I}
\end{bmatrix},
\]

(51)

where $\mathbf{S} = \mathbf{M} - \mathbf{L}\mathbf{A}^{-1}\mathbf{K}$ is the Schur complement operator for the $\mathbf{\Lambda}$ system. The only globally coupled system that needs an iterative solver is the reduced problem for the Lagrange multipliers:

\[
\mathcal{K}_\Lambda(\mathbf{P}_1 r(S, \mathbf{E})),
\]

(52)

where $\mathbf{E}$ is the trace right-hand side, and $\mathbf{P}_1$ is another possible choice of preconditioner. Once $\mathbf{\Lambda}$ is computed, the flux and scalar fields are reconstructed element-wise via inverting the local systems in (27) and (28).

4.1.2 Preconditioning mixed methods via hybridization. The preconditioner HybridizationPC expands on the previous one, this time taking an $\mathbf{H}(\text{div}) \times L^2$ system and automatically forming the hybridized problem. This is accomplished through manipulating the UFL objects representing the discretized PDE. This includes replacing argument spaces with their discontinuous counterparts, introducing test functions on an appropriate trace space, and providing operators assembled from Slate expressions.

More precisely, let $\mathbf{AX} = \mathbf{F}$, where $\mathbf{F} = \{\mathbf{F}_U \quad \mathbf{F}_P\}^T$, $\mathbf{X} = \{\mathbf{U} \quad \mathbf{P}\}^T$, and $\mathbf{U}$ and $\mathbf{P}$ are the flux and scalar unknowns respectively, be the mixed saddle point problem. Then this preconditioner replaces $\mathbf{AX} = \mathbf{F}$ with the augmented system:

\[
\begin{bmatrix}
\mathbf{\widehat{A}} & \mathbf{K}^T \\
\mathbf{K} & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{\widehat{X}} \\
\mathbf{\Lambda}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{\widehat{F}} \\
\mathbf{F}_\Lambda
\end{bmatrix},
\]

(53)

where $\mathbf{\widehat{F}} = \{\mathbf{\widehat{F}}_U \quad \mathbf{F}_P\}^T$, $\mathbf{\widehat{F}}_U, \mathbf{F}_P$ are the right-hand sides for the flux and scalar equations respectively, and $^\widehat{\cdot}$ indicates modified matrices and co-vectors with discontinuous functions. Here, $\mathbf{\widehat{X}} = \{\mathbf{U}^d \quad \mathbf{P}\}^T$ are the new discontinuous unknowns.
The preconditioning operator for the hybrid-mixed system (53) has the form:

\[ \widehat{P} = \begin{bmatrix} I & \widetilde{A}^{-1}K^T \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ K\widetilde{A}^{-1} & I \end{bmatrix}, \]

(54)

where S is the Schur complement matrix \( S = -K\widetilde{A}^{-1}K^T \). As before, a single globally coupled system for \( \Lambda \) is required. The recovery of \( U^d \) and \( P \) happens in the same manner as the HybridSCPC.

Since the flux is constructed in a discontinuous space \( U^d \), we must project the computed solution into \( U^h \subset H(\text{div}) \). This can be done cheaply via local facet averaging. The resulting solution is then updated via \( U \leftarrow \Pi \Pi d U^d \), where \( \Pi d : U^d \times V_h \rightarrow U^h \times V_h \) is the projection mapping. This ensures the residual for the original mixed problem is properly evaluated to test for convergence.

With \( \widehat{P} \) as in (54), the preconditioning operator for the original \( AX = F \) system is:

\[ P = \Pi^T \Pi, \quad \Pi = \begin{bmatrix} \Pi d & 0 \\ 0 & I \end{bmatrix}. \]

(55)

We note here that assembly of the right-hand for the \( \Lambda \) system requires special attention. The situation we are given is that we have \( F_U = F_U(w) \) for \( w \in U_h \), but require \( \widehat{F_U}(w^d) \) for \( w^d \in U^d_h \). For consistency, we also require for any \( w \in U_h \) that

\[ \widehat{F_U}(w) = F_U(w). \]

(56)

We can construct such a \( \widehat{F_U} \) satisfying (56) using the local definition:

\[ \widehat{F_U}(\psi^d_i) = \frac{F_U(\psi^d_i)}{N_i}, \quad \psi^d_i \in U^d_i, \]

(57)

where \( N_i \) is the number of cells that the degree of freedom corresponding to the basis function \( \psi_i \in U_h \) touches. For \( H(\text{div}) \)-elements, \( N_i = 2 \) for facet nodes, and 1 otherwise. By construction of the space \( U^d_h \), we have for \( \psi_i \in U_h \):

\[ \psi_i = \begin{cases} \psi_{i}^{d,+} + \psi_{i}^{d,-} & \psi_i \text{ associated with a facet node,} \\ \psi_{i}^{d} & \psi_i \text{ associated with an interior node,} \end{cases} \]

(58)

where \( \psi_i^d, \psi_i^{d,+}, \psi_i^{d,-} \in U^d_i \), and \( \psi_i^{d,\pm} \) are functions corresponding to the positive and negative restrictions associated with the \( i \)-th facet node\(^2\). Using (57) and (58), we can verify that our construction of \( \widehat{F_U} \) satisfies (56).

5 NUMERICAL RESULTS

We now present results utilizing the Slate DSL and our static condensation interfaces for a set of test problems. All parallel results were obtained on a single fully-loaded compute node of dual-socket Intel E5-2630v4 (Xeon) processors with \( 2 \times 10 \) cores (2 threads per core) running at 2.2GHz. In order to avoid potential memory effects due to the operating system migrating processes between sockets, we pin MPI processes to cores.

\(^2\) These are the two “broken” parts of \( \psi_i \) on a particular facet connecting two elements. That is, for two adjacent cells, a basis function in \( U_h \) for a particular facet node can be decomposed into two basis functions in \( U^d_h \) defined on their respective sides of the facet.
5.1 Hybridized methods and effects of post-processing

To verify our computed results, we now perform a simple convergence study for a model Dirichlet problem. We seek a solution to the Poisson equation as a first-order system:

\[
\begin{align*}
\mathbf{u} + \nabla p &= 0 \quad \text{in } \Omega = [0,1]^2, \\
\nabla \cdot \mathbf{u} &= f \quad \text{in } \Omega, \\
p &= p_0 \quad \text{on } \partial \Omega_D,
\end{align*}
\]

where \( f \) and \( p_0 \) are chosen so that the analytic solution is the sinusoid \( p(x, y) = \sin(\pi x) \sin(\pi y) \) and its negative gradient. We solve this problem by hybridizing the mixed formulation of (59), and employ our static condensation preconditioner described in Section 4.1.1. All results were obtained in serial, with MUMPS providing the LU factorization algorithms for the condensed trace system [Amestoy et al. 2000].

Each mesh in our convergence study is obtained by generating a quadrilateral mesh with \( 2^r \) cells in each spatial direction, and dividing each quadrilateral cell into two equal simplicial elements. Once the solutions are obtained, we compute a post-processed scalar solution using the method described in (42)–(43) via Slate-generated kernels. Figure 2 displays the results for the hybridized RT method. Our computations are in full agreement with the theory.

Fig. 2. Error convergence rates for our implementation of the hybridized RT method of orders 0, 1, 2, and 3. We observe the expected rates for the scalar and flux solutions of the standard RT method: \( k + 1 \) in the \( L^2 \)-error for both the scalar and flux approximations. Additionally, we see the effects of post-processing the scalar solution, yielding superconvergent \( k + 2 \) rates.

We repeat this experiment for the LDG-H method with varying choices of \( \tau \) in order to verify how \( \tau \) influences the convergence rates, comparing with the expected rates for the LDG-H method given a particular order of \( \tau \) (see Table 1 for a summary). In all our experiments, we use the post-processing methods described in Section 3.3 to produce approximations \( p_h^* \) and \( u_h^* \). Error convergence plots from our tests are shown in Figure 3 that confirm the expected rates. This rather sensitive test verifies that our software framework is generating correct code.

5.2 Relative performance of the HDG method

Having verified our generated code for the HDG method in the previous section, we now analyze the quality of said code for a model three-dimensional elliptic system:

\[
\begin{align*}
-\nabla \cdot \nabla p + p &= f \quad \text{in } \Omega = [0,1]^3, \\
p &= g \quad \text{on } \partial \Omega_D
\end{align*}
\]

ACM Transactions on Mathematical Software, Vol. 1, No. 1, Article I. Publication date: July 2018.
Table 1. The expected convergence rates of the LDG-H method with a stability parameter \( \tau \) of a particular order.

| parameter   | expected rates of convergence \((k \geq 1)\) |
|-------------|---------------------------------------------|
| \( O(1) \)  | \( k + 1 \)  | \( k + 1 \)  | \( k + 2 \)  | \( k + 1 \)  |
| \( O(h) \)  | \( k \)  | \( k + 1 \)  | \( k + 2 \)  | \( k + 1 \)  |
| \( O(h^{-1}) \) | \( k + 1 \)  | \( k \)  | \( k + 2 \)  | \( k + 1 \)  |

Fig. 3. Error convergence rates for our implementation of the LDG-H method with \( \tau = h \) and \( \tau = \frac{1}{h} \). The expected sensitivity of this discretization subject to appropriate choices of stabilization parameter \( \tau \) is verified. We see no change in the convergence rates between the scalar and post-processed scalar solutions when \( \tau = \frac{1}{h} \). Superconvergence is achieved when taking \( \tau = h \). The post-processed flux rates in both cases match the rates of the unprocessed flux.

where \( f \) and \( g \) are chosen such that the analytic solution is \( p = \exp\{\sin(\pi x) \sin(\pi y) \sin(\pi z)\} \). We use a regular mesh consisting \( 6 \cdot N^3 \) tetrahedral elements \((N \in \{4, 8, 16, 32, 64\})\). At the time of this work, Firedrake does not support sum-factorized algorithms on simplices. We therefore restrict ourselves to “low” polynomial order, only considering approximation degrees \( k = 1, 2, 3 \) for the HDG method. Additionally, we compute a post-processed scalar approximation \( p_h^* \) of the HDG
solution. In all numerical studies here, we set the HDG parameter $\tau = 1$. All results were computed in parallel, utilizing a single compute node (described previously).

A continuous Galerkin (CG) discretization serves as a reference for this experiment. Due to the superconvergence in the post-processed solution for the HDG method, we use CG discretizations of polynomial order 2, 3, and 4. This takes into account the enhanced accuracy of the HDG solution, despite being initially computed as a lower-order approximation. We therefore expect both methods to produce equally accurate solutions to the model problem.

Our aim here is not to compare the performance of HDG and CG, which has been investigated elsewhere (for example, see Kirby et al. [2012]; Yakovlev et al. [2016]). Instead, we provide a reference that the reader might be more familiar with in order to evaluate whether our software framework produces a sufficiently performant HDG implementation relative to what might be expected.

For the CG discretization, we use a matrix-explicit iterative solver consisting of the conjugate gradient method preconditioned with hypre’s boomerAMG implementation of algebraic multigrid (AMG) [Falgout et al. 2006]. We use the preconditioner described in Section 4.1.1 to statically condense the LDG-H system, using the same iterative solver as the CG method for the Lagrange multipliers. This is sensible, as the global trace operator defines a symmetric positive-definite operator. To avoid over-solving, we iterate to a relative tolerance such that the discretization error is minimal for a given mesh.

5.2.1 Error versus execution time. The total execution time is recorded for the CG and HDG solvers, which includes the setup time for the AMG preconditioner, matrix-assembly, and the time-to-solution for the Krylov method. In the HDG case, we include the time spent building the Schur-complement for the traces, local recovery of the scalar and flux approximations, and post-processing. The $L^2$-error against execution time is summarized in Figure 4.

![Figure 4. Error against execution time for the CG and HDG ($\tau = 1$, with post-processing) methods.](image-url)
The HDG method of order $k - 1$ ($\text{HDG}_{k-1}$) with post-processing, as expected, produces a solution which is as accurate as the CG method of order $k$ ($\text{CG}_k$). While the full HDG system is never explicitly assembled, the larger execution time is a result of several factors. The total number of trace unknowns for the $\text{HDG}_1$, $\text{HDG}_2$, and $\text{HDG}_3$ discretizations is roughly four, three, and two times larger (resp.) than the corresponding number of CG unknowns. Therefore, each iteration is more expensive. Moreover, we also observe that the trace system requires more Krylov iterations to reach discretization error. The gap in total number of iterations starts to close as the approximation degree increases (see Figure 5). The extra cost of HDG due to the larger degree-of-freedom count and the need to perform local tensor inversion is offset by the local conservation and stabilization properties which are useful for fluid dynamics applications, for example.

![Figure 5. Krylov iterations of the AMG-preconditioned conjugate gradient algorithm (to reach discretization error) against number of cells.](image)

5.2.2 **Break down of solver time.** The HDG method introduces far more additional degrees of freedom than CG or native DG methods. This is largely due to the fact that the HDG method simultaneously approximates the primal solution and its flux. The global matrix for the traces is larger than the one for the CG system at low polynomial order. The execution time for HDG is then compounded by a more expensive global solve. We remind the reader that our goal here is not to compare CG and HDG but to verify that the solve time for HDG is as might be expected given the problem size.

Figure 6 displays the execution times on a simplicial mesh consisting of 1.5 million elements. The execution times have been normalized by the CG total time, showing that the HDG method is roughly 3 times the execution time of the CG method. This is expected given the larger degree-of-freedom count. The raw numerical breakdown of the HDG and CG solvers are shown in Table 2. We isolate each component of the HDG method contributing to the total execution time. Local operations include static condensation (trace operator assembly), forward elimination (right-hand side assembly for the trace system), backwards substitution to recover the scalar and flux unknowns, and local post-processing of the primal solution. For all $k$, our HDG implementation is solver-dominated, indicating that our software framework is producing a relatively performant implementation.
Both operator and right-hand side assembly are dominated by the costs of inverting a local square mixed matrix coupling the primal and dual variables, which is performed directly via an LU factorization. They should therefore be of the same magnitude in time spent. We observe that this is the case across all degrees (ranging between approximately 6—11% of total execution time for both
processes). We observe that the cost of HDG assembly takes approximately the same percentage of time as the CG method.

Back-substitution takes slightly more time (between 8—15% of execution time across all $k$). This is a result of splitting the reconstruction stage into two local solves: one pass to reconstruct $p$, followed by an additional pass to reconstruct $u$. A further performance optimization could be to simultaneously solve for both $p$ and $u$ in each element. However, at the time of this work, the PyOP2 layer of Firedrake does not support direct insertion into mixed data structures. Finally, the additional cost of post-processing accrues negligible time (roughly 2% of execution time across all degrees).

We note that caching of local tensors does not occur. Each pass to perform the local eliminations and backwards reconstructions rebuilds the local element tensors. It is not clear at this time whether the performance gained from avoiding rebuilding the local operators will offset the memory costs of storing the local matrices. Moreover, in time-dependent problems where the operators contain state-dependent variables, rebuilding local matrices will be necessary in each time-step regardless.

5.3 A hybridized finite element solver for the shallow water equations

A primary motivator for our interest in hybridized methods revolves around developing efficient solvers for problems in geophysical flows. In this last section, we present some results integrating the shallow water equations on the sphere using test case 5 (flow past a mountain) from [Williamson et al. 1992]. We use the framework of compatible finite elements [Cotter and Shipton 2012; Cotter and Thuburn 2014].

5.3.1 Semi-implicit solver. We start with the vector-invariant rotating nonlinear shallow water system defined on a two-dimensional spherical surface $\Omega$ embedded in $\mathbb{R}^{3}$:

$$\frac{\partial u}{\partial t} + (\nabla \cdot u + f) u \perp + \nabla \left( g(D + b) + \frac{1}{2}|u|^2 \right) = 0,$$

$$\frac{\partial D}{\partial t} + \nabla \cdot (uD) = 0,$$

where $u$ is the fluid velocity, $D$ is the depth field, $f$ is the Coriolis parameter, $g$ is the acceleration due to gravity, $b$ is the bottom topography, and $(\cdot)\perp \equiv \hat{k} \times \cdot$, with $\hat{k}$ being the unit normal to the surface $\Omega$.

After discretising in space and time using a semi-implicit scheme and Picard linearization, following Natale and Cotter [2017], we must solve an indefinite saddle point system at each step:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Delta U \\ \Delta D \end{bmatrix} = \begin{bmatrix} R_u \\ R_D \end{bmatrix}.$$

(63)

In staggered finite difference models, the standard approach for solving (63) is to neglect the Coriolis term and eliminate the velocity unknowns $\Delta U$ to obtain a discrete elliptic problem for which smoother like Richardson or relaxation methods are convergent. This is more problematic in the compatible finite element framework, since $A$ is not diagonal. Instead, we use the preconditioner described in Section 4.1.2 to form the hybridized problem and eliminate both $\Delta U$ and $\Delta D$ locally.

5.3.2 Atmospheric flow over a mountain. As a test problem, we solve test case 5 of Williamson et al. [1992], on the surface of an Earth-sized sphere. We refer the reader to Cotter and Shipton [2012]; Shipton and Cotter [2017] for a more comprehensive study on mixed finite elements for shallow water systems of this type.

We use the mixed finite element pairs $(RT_1, DG_0)$ (lowest-order RT method) and $(BDM_2, DG_1)$ (next-to-lowest order BDM method) for the velocity and depth spaces. The sphere mesh is generated
A DSL for hybridization and static condensation of finite element methods

from 8 refinements of an octahedron, resulting in a triangulation consisting of 524,288 elements. The grid information, including time-step size, and degree of freedom count is summarized in Table 3.

Table 3. Grid properties for the isolated mountain test case. The implicit Courant number corresponding to the choice of $\Delta t$ is approximately 0.22. The number of unknowns to be determined are summarized for each mixed method (not hybridized).

| Refinements | Number of cells | $\Delta x_{\text{min}}$ (km) | $\Delta x_{\text{max}}$ (km) | $\Delta t$ (s) |
|-------------|-----------------|-----------------------------|-----------------------------|---------------|
| 8           | 524,288         | 28.396                      | 33.052                      | 28.125        |

| Mixed method | Velocity unknowns | Depth unknowns | Total unknowns |
|--------------|-------------------|----------------|---------------|
| $RT_1 \times DG_0$ | 786,432          | 524,288        | 1,310,720     |
| $BDM_2 \times DG_1$ | 3,932,160        | 1,572,864      | 5,505,024     |

We run for a total of 100 time-steps, with a fixed number of 4 Picard iterations in each time-step. We compare the overall simulation time using two different solver configurations for the implicit linear system. First, we use an approximate Schur complement preconditioner for GMRES of the form:

$$
\begin{bmatrix}
I & 0 \\
CA^{-1} & I
\end{bmatrix}
\begin{bmatrix}
A^{-1} & 0 \\
0 & \tilde{S}^{-1}
\end{bmatrix}
\begin{bmatrix}
I & -A^{-1}B \\
0 & I
\end{bmatrix},
$$

(64)

where $\tilde{S} = D - C\text{diag}(A)^{-1}B$, and $\text{diag}(A)$ is a diagonal approximation to the velocity mass matrix. The approximate Schur complement $\tilde{S}$ is inverted using conjugate gradients preconditioned by PETSc’s smoothed aggregation multigrid (GAMG). The Krylov method is set to terminate once the preconditioned residual norm is reduced by a factor of $10^8$. $A^{-1}$ is computed approximately using a single application of incomplete LU (zero fill-in).

Next, we use a preonly application of our hybridization preconditioner, which replaces the original linearized mixed system with its hybrid-mixed equivalent. After hybridization, we have the problem: find $(\Delta \tilde{u}_h, \Delta D_h, \lambda_h) \in \tilde{U}_h \times V_h \times M_h$ such that

$$
(w, \Delta \tilde{u}_h)_{\partial T_h} + \frac{\Delta t}{2} \left( w, f \Delta \tilde{u}_h \right)_{\partial T_h} - \frac{\Delta t}{2} \left( \nabla \cdot w, g \Delta D_h \right)_{\partial T_h} + \left( [w] , \lambda_h \right)_{\partial T_h} = \tilde{R}_u, \quad \forall w \in \tilde{U}_h, \quad (65)
$$

$$
(\phi, \Delta D_h)_{\partial T_h} + \frac{\Delta t}{2} \left( \phi, H \nabla \cdot \Delta \tilde{u}_h \right)_{\partial T_h} = R_D, \quad \forall \phi \in V_h, \quad (66)
$$

$$
\gamma, \left[ \Delta \tilde{u}_h \right] \right)_{\partial T_h} = 0, \quad \forall \gamma \in M_h. \quad (67)
$$

Note that the space $M_h$ is chosen such that these trace functions when restricted to a facet $e \in \partial T_h$ are from the same polynomial space as $\Delta \tilde{u}_h \cdot n$ restricted to that same facet. Additionally, it can be shown that $\lambda_h$ is an approximation to $\Delta t gD/2$.

The resulting three-field problem for (65)–(67) has the matrix form:

$$
\begin{bmatrix}
\tilde{A} & K^T \\
K & 0
\end{bmatrix}
\begin{bmatrix}
\Delta X \\
\Lambda
\end{bmatrix}
= \begin{bmatrix}
\tilde{R}_{AX} \\
0
\end{bmatrix},
$$

(68)

where $\tilde{A}$ is the discontinuous operator coupling $\Delta X = \begin{bmatrix} \Delta \tilde{U} & \Delta D \end{bmatrix}^T$, and $R_{AX} = \begin{bmatrix} \tilde{R}_u & R_D \end{bmatrix}^T$ are the problem residuals. An exact Schur-complement factorization is performed on (68), using Slate.
to generate the local elimination kernels. We use the same set of solver options for the inversion of \( \tilde{S} \) in (64) to invert the Lagrange multiplier system. The increments \( \Delta U \) and \( \Delta D \) are recovered locally, using Slate-generated kernels. Once recovery is complete, \( \Delta U \) is injected back into the conforming \( H(\text{div}) \) finite element space.

Table 4. Preconditioner setup and solve times. These are cumulative times in each stage of the two preconditioners throughout the entire profile run. We display the average iteration count (rounded to the nearest integer) for both the outer GMRES and the inner CG+GAMG Krylov solves.

| Mixed method | Preconditioner | \( t_{\text{total}} \) (s) | \( t_{\text{setup}} \) (s) | Avg. outer its. | Avg. inner its. | \( \frac{t_{\text{approx. Schur}}}{t_{\text{total}}} \) | \( \frac{t_{\text{hybrid}}}{t_{\text{total}}} \) |
|--------------|----------------|------------------|------------------|-----------------|-----------------|-----------------|-------------------|
| \( RT_1 \times DG_0 \) | approx. Schur. | 564.939 | 0.747 | 10 | 5 | 6.987 |
|              | hybridization  | 80.859 | 1.384 | None | 5 |  |
| \( BDM_2 \times DG_1 \) | approx. Schur. | 7856.152 | 2.027 | 17 | 6 | 18.569 |
|              | hybridization  | 423.077 | 4.519 | None | 5 |  |

Table 5. Breakdown of the cost (average) of a single outer linear solve iteration. The hybridized solver takes around the same time per iteration, but is an exact solve.

| Preconditioner | Stage | \( RT_1 \times DG_0 \) | \( BDM_2 \times DG_1 \) |
|----------------|-------|------------------|------------------|
| approx. Schur  | Schur solve | 0.139 | 0.107 |
|                | invert velocity mass: \( A \) | 0.001 | 0.012 |
|                | apply inverse: \( A^{-1} \) | 0.001 | 0.012 |
|                | other | 0.004 | 0.039 |
|                | Total | 0.145 | 1.129 |
| hybridization  | Transfer: \( R_{\Delta X} \rightarrow \tilde{R}_{\Delta X} \) | 0.001 | 0.002 |
|                | Forward elim.: \( -K\tilde{A}^{-1}\tilde{R}_{\Delta X} \) | 0.041 | 0.258 |
|                | Trace solve | 0.092 | 0.502 |
|                | Back sub. | 0.065 | 0.287 |
|                | Projection: \( \Pi_{\text{div}}\Delta U \) | 0.003 | 0.008 |
|                | Total | 0.20 | 1.06 |

Table 4 displays a summary of our findings. When using hybridization, we observe a significant reduction in time spent during the implicit solve stage compared to the preconditioned GMRES approach. This is primarily because we reduce the number of required “outer” iterations to zero; the hybridization preconditioner is performing an exact Schur complement factorization. This is further supported when considering the per-iteration solve times, summarized in Table 5. The hybridized and approximate Schur complement solver times are comparable, demonstrating that the primary cause for the larger execution time of the former is a direct result of the additional outer iterations induced from using an approximate factorization. The setup time for the hybridization preconditioner is roughly twice the setup time of the approximate Schur complement approach. While setup for both includes the AMG preconditioner for the inner elliptic solver, setup for hybridization also includes building the statically condensed trace operator. Both of these costs are a one-time cost for our semi-implicit scheme, as the operator is state-independent.
We also measure the reductions in the true-residual of the linear system (63). Our hybridized method reduces the residual by a factor of $10^8$ on average, which coincides with the specified relative tolerance for the Krylov method on the trace system. Snapshots of a (coarser) 15 day simulation are provided in Figure 7 using the semi-implicit scheme described in this paper. We refer the reader to Shipton and Cotter [2017] for an exposition of shallow water test cases featuring the use of a hybridized implicit solver (as described in Section 4.1.2).

6 CONCLUSIONS AND FUTURE OUTLOOKS

We have presented Slate, and shown how this language can be used to create concise mathematical representations of localized linear algebra on the tensors corresponding to finite element forms. We have shown how this DSL can be used in tandem with UFL in Firedrake to implement solution approaches making use of automated code generation for static condensation, hybridization, and localized post-processing. In particular, this framework alleviates much of the difficulty in implementing such methods within intricate numerical, and paves the way for future low-level optimizations such as vectorization. In this way, Slate can be used to help enable the rapid development and exploration of new hybridization and static condensation techniques. We remark here that the reduction of global matrices via element-wise algebraic static condensation, as described in [Guyan 1965; Irons 1965] is also possible using Slate, including other more general static condensation procedures outside the context of hybridization.

Our approach to preconditioner design revolves around its composable nature, in that these Slate-based implementations can be seamlessly incorporated into complicated solution schemes. In particular, there is current research in the design of dynamical cores for numerical weather prediction using implementations of hybridization and static condensation with Slate [Bauer and Cotter 2018; Shipton and Cotter 2017]. The performance of such methods for geophysical flows is a subject of ongoing research. Slate is a developing project and continues to expand in its features, including extending compiler technology to generate code suitable for different computer architectures.
CODE AVAILABILITY

We cite archives of the exact software versions used to produce the results in this paper. The Slate-based preconditioners from Section 4.1 can be found in [Zenodo/SCPC 2018]. For all components of the Firedrake project, we used the recent versions: COFFEE [Zenodo/COFFEE 2018], FIAT [Zenodo/FIAT 2018], FlnAT [Zenodo/FlnAT 2018], Firedrake [Zenodo/Firedrake 2018], PETSc [Zenodo/PETSc 2018], petsc4py [Zenodo/petsc4py 2018], PyOP2 [Zenodo/PyOP2 2018], TSFC [Zenodo/TSFC 2018], and UFL [Zenodo/UFL 2018]. The numerical experiments, documenting full solver configurations and including raw data and plotting scripts, are available as [Zenodo/Tabula-Rasa 2018].

ACKNOWLEDGMENTS

The authors would like to thank Andrew T. T. McRae for proof-reading this manuscript and Miklós Homolya for his helpful input on developing Slate as a language and suggestions for designing its compiler.

This work was supported by the Engineering and Physical Sciences Research Council, under grant numbers: EP/M011054/1, EP/L000407/1, and EP/L016613/1; and the Natural Environment Research Council under grant number NE/K008951/1.

7 SUPPLEMENTARY MATERIALS

7.1 Semi-implicit method for the shallow water system

For some tessellation, $\mathcal{T}_h$, our semi-discrete mixed method for (61)–(62) seeks approximations $(u^h, D_h) \in U_h \times V_h \subset H(\text{div}) \times L^2$ satisfying:

$$\left( w, \frac{\partial u^h}{\partial t} \right)_{\mathcal{T}_h} - \left( \nabla \cdot (w \cdot u^h), u^h \right)_{\mathcal{T}_h} + \left( w, f u^h \right)_{\mathcal{T}_h} + \left( [n \cdot w \cdot \tilde{u}], \tilde{u} \right)_{\partial \mathcal{T}_h}$$

$$- \left( \nabla \cdot w, g(D_h + b) + \frac{1}{2} |u^h|^2 \right)_{\mathcal{T}_h} = 0, \quad \forall w \in U_h, \quad (69)$$

$$\left( \phi, \frac{\partial D_h}{\partial t} \right)_{\mathcal{T}_h} - \left( \nabla \phi, u^h D_h \right)_{\mathcal{T}_h} + \left( [\phi u^h], \tilde{D} \right)_{\partial \mathcal{T}_h} = 0, \quad \forall \phi \in V_h, \quad (70)$$

where $\tilde{\cdot}$ indicates that the value of the function should be taken from the upwind side of each facet. The discretisation of the velocity advection operator is an extension of the energy-conserving scheme of Natale and Cotter [2017] to the shallow-water equations.

The time-stepping scheme follows a Picard iteration semi-implicit approach, where predictive values of the relevant fields are determined via an explicit step of the advection equations, and corrective updates are generated by solving an implicit linear system (linearized about a state of rest) for $(\Delta u_h, \Delta D_h) \in U_h \times V_h$, given by

$$\left( w, \Delta u^h \right)_{\mathcal{T}_h} + \frac{\Delta t}{2} \left( w, f \Delta u^h \right)_{\mathcal{T}_h} - \frac{\Delta t}{2} \left( \nabla \cdot w, g \Delta D_h \right)_{\mathcal{T}_h} = -R_u[u^{n+1}_h, D^{n+1}_h; w], \quad \forall w \in U_h, \quad (71)$$

$$\left( \phi, \Delta D_h \right)_{\mathcal{T}_h} + \frac{H \Delta t}{2} \left( \phi, \nabla \cdot \Delta u^h \right)_{\mathcal{T}_h} = -R_D[u^{n+1}_h, D^{n+1}_h; \phi], \quad \forall \phi \in V_h, \quad (72)$$

where $H$ is the mean layer depth, and $R_u$ and $R_D$ are residual linear forms that vanish when $u^{n+1}_h$ and $D^{n+1}_h$ are solutions to the implicit midpoint rule time discretization of (69)–(70). The residuals are evaluated using the predictive values of $u^{n+1}_h$ and $D^{n+1}_h$. 

ACM Transactions on Mathematical Software, Vol. 1, No. 1, Article . Publication date: July 2018.
The implicit midpoint rule time discretization of the non-linear rotating shallow water equations (69)–(70) is:

\[
(w, u_h^{n+1} - u_h^n)_{\mathcal{T}_h} - \Delta t \left( \nabla \cdot (w \cdot u_h^{n+1}) , u_h^{n+1} \right)_{\mathcal{T}_h} + \Delta t \left( w, f u_h^{n+1} \right)_{\mathcal{T}_h} + \Delta t \left( \| n \cdot w \cdot u_h^{n+1} \|, \tilde{u}_h^{n+1} \right)_{\partial \mathcal{T}_h} = 0, \quad \forall w \in U_h, \quad (73)
\]

Similarly, for \( R_h \) we define residuals as follows. For \( R_h \), we first solve for \( \psi_h \) such that

\[
(w, \psi_h - u_h^n)_{\mathcal{T}_h} - \Delta t \left( \nabla \cdot (w \cdot u_h^n) , \psi_h^n \right)_{\mathcal{T}_h} + \Delta t \left( w, f \psi_h^n \right)_{\mathcal{T}_h} + \Delta t \left( \| n \cdot w \cdot u_h^n \|, \psi_h^n \right)_{\partial \mathcal{T}_h} = 0, \quad \forall w \in U_h, \quad (74)
\]

where \( \psi_h^n = (u_h^{n+1} + u_h^n)/2 \). This is a linear variational problem. Then,

\[
R_h[u_h^{n+1}, D_h^{n+1}; w] = (w, \psi_h - u_h^{n+1})_{\mathcal{T}_h}. \quad (76)
\]

Similarly, for \( R_D \) we first solve for \( E_h \) such that

\[
(\phi, E_h - D_h^n)_{\mathcal{T}_h} - \Delta t \left( \nabla \phi, u_h^n \right)_{\mathcal{T}_h} + \Delta t \left( \| \phi \|, E_h \right)_{\partial \mathcal{T}_h} = 0, \quad \forall \phi \in V_h, \quad (77)
\]

where \( E_h = (E_h + D_h^n)/2 \). This is also a linear problem. Then,

\[
R_D[u_h^{n+1}, D_h^{n+1}; \phi] = (\phi, E_h - D_h^{n+1})_{\mathcal{T}_h}. \quad (78)
\]

The predictive fields \( \psi_h \) and \( E_h \) are then used to construct the right-hand side for the implicit linearized system:

\[
(w, \Delta u_h)_{\mathcal{T}_h} + \frac{\Delta t}{2} (w, D_h^{n+1})_{\mathcal{T}_h} = -R_h[u_h^{n+1}, D_h^{n+1}; w], \quad \forall w \in U_h, \quad (79)
\]

\[
(\phi, \Delta D_h)_{\mathcal{T}_h} + \frac{H \Delta t}{2} (\phi, \nabla \cdot \Delta u_h)_{\mathcal{T}_h} = -R_D[u_h^{n+1}, D_h^{n+1}; \phi], \quad \forall \phi \in V_h, \quad (80)
\]

where \( H \) is the mean layer depth. Once (79)–(80) is solved, the solution \( (\Delta u_h, \Delta D_h) \) is then used to update the iterative values of \( u_h^{n+1} \) and \( D_h^{n+1} \) according to \( (u_h^{n+1}, D_h^{n+1}) \leftarrow (u_h^{n+1} + \Delta u_h, D_h^{n+1} + \Delta D_h) \), having initially chosen \( (u_h^{n+1}, D_h^{n+1}) = (u_h^n, D_h^n) \).

REFERENCES

Martin S. Alnæs, Anders Logg, Kristian B. Ølgaard, Marie E. Rognes, and Garth N. Wells. 2014. Unified form language: A domain-specific language for weak formulations of partial differential equations. ACM Transactions on Mathematical Software (TOMS) 40, 2 (2014), 9.

Patrick R. Amestoy, Iain S. Duff, and J.-Y. L’excellent. 2000. Multifrontal parallel distributed symmetric and unsymmetric solvers. Computer methods in applied mechanics and engineering 184, 2 (2000), 501–520.

Douglas N. Arnold and Franco Brezzi. 1985. Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates. ESAIM: Mathematical Modelling and Numerical Analysis 19, 1 (1985), 7–32.
Pierre-Arnaud Raviart and Jean-Marie Thomas. 1977. A mixed finite element method for 2-nd order elliptic problems. In Mathematical aspects of finite element methods. Springer, 292–315.

Jemma Shipton and C. J. Cotter. 2017. Higher-order compatible finite element schemes for the nonlinear rotating shallow water equations on the sphere. arXiv preprint arXiv:1707.00855 (2017).

Rolf Stenberg. 1991. Postprocessing schemes for some mixed finite elements. ESAIM: Mathematical Modelling and Numerical Analysis 25, 1 (1991), 151–167.

David L. Williamson, John B. Drake, James J. Hack, Rüdiger Jakob, and Paul N. Swarztrauber. 1992. A standard test set for numerical approximations to the shallow water equations in spherical geometry. J. Comput. Phys. 102, 1 (1992), 211–224.

Sergey Yakovlev, David Moxey, Robert M Kirby, and Spencer J Sherwin. 2016. To CG or to HDG: a comparative study in 3D. Journal of Scientific Computing 67, 1 (2016), 192–220.

Zenodo/COFFEE. 2018. COFFEE: A Compiler for Fast Expression Evaluation. (Jul. 2018). https://doi.org/10.5281/zenodo.1306572

Zenodo/FlnAT. 2018. FlnAT: a smarter library of finite elements. (Jul. 2018). https://doi.org/10.5281/zenodo.1306565

Zenodo/Firedrake. 2018. Firedrake: an automated finite element system. (Jul. 2018). https://doi.org/10.5281/zenodo.1306051

Zenodo/SCPC. 2018. SCPC: Static condensation and hybridization in Firedrake & PETSc. (Jul. 2018). https://doi.org/10.5281/zenodo.1306389

Zenodo/TSFC. 2018. TSFC: The Two Stage Form Compiler. (Jul. 2018). https://doi.org/10.5281/zenodo.1306560

Zenodo/UFL. 2018. UFL: The Unified Form Language. (Jul. 2018). https://doi.org/10.5281/zenodo.1306566