$S_4$ symmetric four-generation models for quarks and charged leptons

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We propose $S_4$ symmetric four-generation models for quarks and charged leptons. Although an $S_4$ symmetric four-generation model has been already proposed, there are some redundant assumptions and additional parameters in the model. We construct four-generation models with only requirement of exact $S_4$ symmetry. It turned out that at least one of the models is consistent with observations of masses and mixings.

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I. INTRODUCTION

Four-generation models for quarks and leptons are well-motivated extensions of the standard model of the particle physics. These models have been studied extensively in the literature (see Refs. [1,2] for reviews), for examples, especially for quark sector [3-7], for lepton sector [8-14], for interplay between quarks and leptons [15-18], for neutrino sector [19-24], for Higgs sector [25-29] and for dark matter problem [30-31].

In 1989, Ozaki, one of the authors, proposed an $S_4$ symmetric four-generation model for quarks and charged leptons [42] to extend an $S_4$ symmetric three-generation model [43]. For $S_4$ models in its early stages, see references in Ref. [42]. Up to now, many three-generation models based on $S_4$ permutation flavor symmetry have been proposed [44-55]; however, four-generation model based on $S_4$ permutation flavor symmetry has not been proposed yet, aside from Ozaki’s model.

Although, the predicted values of individual Cabbibo-Kobayashi-Maskawa matrix elements in Ozaki’s model were within experimental data in 1989, there are some additional assumptions for the sake of simplicity in calculations. Moreover, there are some additional parameters to fit the model predictions in with observations. These redundant assumptions and additional parameters are disagreeable.

In this paper, we propose exact $S_4$ symmetric four-generation models for quarks and charged leptons. It turns out that at least one of the models is consistent with the current observations of masses and mixings.

The paper is organized as follows. In section I we establish the convention and $S_4$ assignment criteria for model building for later discussions. In section III we show the most viable model which is consistent with observations. In section IV we discuss other candidate models. Finally, we give the summary in section V.

II. MODELS

A. Tensor products

$S_4$ group consists of all permutations among four objects, e.g., $e_1, e_2, e_3,$ and $e_4$. The following tensor products of $S_4$

$$3 \otimes 3 = 3' \otimes 3' = 1 \oplus 3 \oplus 2 \oplus 3',$$
$$3 \otimes 3' = 1' \oplus 3' \oplus 2 \oplus 3,$$
$$3 \otimes 2 = 3' \otimes 2 = 3 \oplus 3',$$
$$2 \otimes 2 = 1 \oplus 2 \oplus 1',$$

(1)

with obvious products $1 \otimes 1 = 1' \otimes 1' = 1, 1 \otimes 1' = 1', 1 \otimes 3 = 1' \otimes 3' = 3, 1 \otimes 3' = 1' \otimes 3 = 3', 1 \otimes 2 = 1' \otimes 2 = 2$ are independent of the basis [53,54], where $1, 1', 2, 3$ and $3'$ denote the dimensions of irreducible representations of $S_4$ (the prime means antisymmetric representation). We use the basis in Refs. [42,43]:

$$x_1 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4),$$
$$x_2 = \frac{1}{\sqrt{2}}(e_1 - e_2),$$
$$x_3 = \frac{1}{\sqrt{2}}(e_3 - e_4),$$
$$x_4 = \frac{1}{2}(e_1 + e_2 - e_3 - e_4),$$

(2)

where $x_1$ is a trivial one-dimensional representation $1$ and \{ $x_2, x_3, x_4$ \} is a three-dimensional representation $3$. The relevant multiplication rules to our study are as follows:

$$\left( \begin{array}{c} a_1 \\ a_2 \end{array} \right)_2 \otimes \left( \begin{array}{c} b_1 \\ b_2 \end{array} \right)_2 = \frac{1}{\sqrt{2}}(a_1 b_1 + a_2 b_2) 1 + \frac{1}{\sqrt{2}} \left( a_1 b_2 + a_2 b_1 \right) 2$$
$$\oplus \frac{1}{\sqrt{2}}(a_1 b_2 - a_2 b_1) 1',$$

(3)
\[
\left( \begin{array}{c}
  a_1 \\
  a_2 \\
  a_3
\end{array} \right)_3 \otimes \left( \begin{array}{c}
  b_1 \\
  b_2 \\
  b_3
\end{array} \right)_3 = \frac{1}{\sqrt{3}} (a_1 b_1 + a_2 b_2 + a_3 b_3)_1
\]
\[
\frac{1}{\sqrt{2}} \left( \begin{array}{c}
  a_1 b_1 + a_2 b_1 \\
  -a_2 b_3 - a_3 b_2 \\
  a_1 b_1 - a_2 b_2
\end{array} \right)_3
\]
\[
\frac{1}{\sqrt{6}} \left( a_1 b_1 + a_2 b_2 - 2a_3 b_3 \right)_2
\]
\[
\frac{1}{\sqrt{2}} \left( \begin{array}{c}
  a_2 b_3 - a_3 b_2 \\
  a_3 b_1 - a_1 b_3 \\
  a_1 b_2 - a_2 b_1
\end{array} \right)_3.
\]

Several bases of representations of $S_4$ group have been used in the literature \[S4\].

**B. $S_4$ assignment criteria**

We denote the left-handed (LH) doublets, right-handed (RH) singlets and Higgs doublets by

\[
Q_i = \begin{pmatrix} u_i \\ d_i \end{pmatrix}_L, \quad U_i = u_{iR}, \quad D_i = d_{iR},
\]
\[
L_i = \begin{pmatrix} \nu_i \\ \ell_i \end{pmatrix}_L, \quad E_i = \ell_{iR}, \quad \phi_i = \begin{pmatrix} \phi_i^+ \\ \phi_i^0 \end{pmatrix}_i,
\]

where $i = 1, 2, 3, 4$. The neutrino masses and mixing are omitted in this study \[S4\].

We assume that these fermions and Higgs scalars are assigned as one of the irreducible representations of $S_4$ according to the following criteria:

1. Higgs doublets are assigned as $\phi_1 : 1$ and $\{\phi_2, \phi_3, \phi_4\} : 3$ in all models.
2. LH doublets $Q_i, L_i$ are assigned as $1, 3, 1'$ or $3'$.
3. RH singlets $U_i, D_i, E_i$ are assigned as $1, 2, 3, 1'$ or $3'$.
4. representation of $Q_i$ and of $L_i$ are the same.
5. representation of $U_i, D_i$ and of $E_i$ are the same.
6. the order of generations is always $\{1, 2, 3, 4\}$, such as $\{Q_1, Q_2, Q_3\}, \{Q_2, Q_3, Q_4\}, \{Q_2, Q_3\}$. For examples, an assignment

\[
Q_1 : 1, \quad \{Q_2, Q_3, Q_4\} : 3,
\]
\[
\{Q_1, Q_2, Q_3\} : 3 \quad Q_4 : 1,
\]
is permitted; however,

\[
Q_2 : 1, \quad \{Q_1, Q_3, Q_4\} : 3,
\]
\[
\{Q_1, Q_2, Q_4\} : 3 \quad Q_3 : 1,
\]
is forbidden.

An abbreviation 13-112 will be used to show the following assignment

\[
Q_1 : 1, \quad \{Q_2, Q_3, Q_4\} : 3,
\]
\[
U_1 : 1, \quad U_2 : 1, \quad \{U_3, U_4\} : 2,
\]
\[
D_1 : 1, \quad D_2 : 1, \quad \{D_3, D_4\} : 2,
\]

and

\[
L_1 : 1, \quad \{L_2, L_3, L_4\} : 3,
\]
\[
E_1 : 1, \quad E_2 : 1, \quad \{E_3, E_4\} : 2,
\]

with

\[
\phi_1 : 1, \quad \{\phi_2, \phi_3, \phi_4\} : 3.
\]

Other abbreviations 13-1’12, 1’3-1’12, etc., will also be used in the same manner.

**C. Candidates**

All candidates of $S_4$ symmetric four-generation model for quarks and leptons satisfied with the $S_4$ assignment criteria in this paper are shown in Table \[S4\]. Table \[S4\] shows the model structure for down-type quarks in each models. The model structures for up-type quarks and charged leptons are exactly same as it for down-type quarks.

For example, in the line for model (a), $Q_1 \phi_1 D_1$ and $Q_1 \phi_1 D_2$ denote the singlet interactions

\[
A_d(u_1, d_1)_L \phi_1 d_1 R + B_d(u_1, d_1)_L \phi_1 d_2 R,
\]

from $1 \otimes 1 \otimes 1, Q_{234} \phi_{234} D_1$ and $Q_{234} \phi_{234} D_2$ denote the singlet interactions

\[
C_d \left[ (u_2, d_2)_L \phi_2 + (u_3, d_3)_L \phi_3 + (u_4, d_4)_L \phi_4 \right] d_1 R
\]
\[
+ D_d \left[ (u_2, d_2)_L \phi_2 + (u_3, d_3)_L \phi_3 + (u_4, d_4)_L \phi_4 \right] d_2 R,
\]

from $3 \otimes 3 \otimes 1$ and $Q_{234} \phi_{234} D_{34}$ denote the singlet interactions

\[
E_d \left\{ \frac{1}{\sqrt{2}} \left[ (u_2, d_2)_L \phi_3 + (u_3, d_3)_L \phi_2 \right] d_3 R + \frac{1}{\sqrt{6}} \left[ (u_2, d_2)_L \phi_2 + (u_3, d_3)_L \phi_3 - 2(u_4, d_4)_L \phi_4 \right] d_4 R \right\}
\]

from $3 \otimes 3 \otimes 2$. We can see that the rank of mass matrix is equal to four (we will see it in section \[S4\] and there are five Yukawa couplings for the down-type quarks, $A_d, B_d, C_d, E_d, E_d$ in the model 13-112.

There are models in which the symmetric representations are replaced with antisymmetric representation and
vice versa (except Higgs sector). For example, (a) 13-112 yields the following \( \Lambda \) 1’3’-1’1’2’:

\[
\begin{align*}
Q_1 : 1', & \quad \{Q_2, Q_3, Q_4\} : 3', \\
U_1 : 1', & \quad U_2 : 1', \quad \{U_3, U_4\} : 2', \\
D_1 : 1', & \quad D_2 : 1', \quad \{D_3, D_4\} : 2', \\
L_1 : 1', & \quad \{L_2, L_3, L_4\} : 3', \\
E_1 : 1', & \quad E_2 : 1', \quad \{E_3, E_4\} : 2',
\end{align*}
\]

(14)

with

\[
\phi_1 : 1, \quad \{\phi_2, \phi_3, \phi_4\} : 3.
\]

The structure of the Yukawa Lagrangian in this model 1’3’-1’1’2’ is exactly same as it for the model 13-112. Similarly, (b) 13-1’12 and (B) 1’3’-11’2 have same structure of Yukawa Lagrangian. We omit models (A),(B),(C),\( \cdots \),(P) related to (a),(b),(c),\( \cdots \),(p) in Table II.

In the next section, we show that at least one of the models (model 13-112) is consistent with observations. Before we go to the next section, we show some details of the Higgs potential because the assignment of the Higgs doublets is same in all models.

D. Higgs sector

The general Higgs potential invariant under \( SU(2) \otimes U(1) \otimes S_4 \) is [42]

\[
V = \mu_2^2(\phi_2\phi_2 + \phi_3\phi_3 + \phi_4\phi_4) + \alpha(\phi_2\phi_2 + \phi_3\phi_3 + \phi_4\phi_4)^2 \\
+ \beta \left[ \frac{1}{2}(\phi_2\phi_3 + \phi_3\phi_2)^2 + \frac{1}{6}(\phi_2\phi_2 + \phi_3\phi_3 - 2\phi_4\phi_4)^2 \right] \\
+ \gamma \left[ \frac{1}{2}(\phi_2\phi_4 + \phi_4\phi_2)^2 + \frac{1}{2}(\phi_3\phi_4 + \phi_4\phi_3)^2 + \frac{1}{2}(\phi_2\phi_2 - \phi_3\phi_3)^2 \right] \\
+ \delta \left[ \frac{1}{2}(\phi_2\phi_3 - \phi_3\phi_2)^2 + \frac{1}{2}(\phi_3\phi_4 - \phi_4\phi_3)^2 + \frac{1}{2}(\phi_2\phi_2 - \phi_3\phi_3)^2 \right] \\
+ \mu_1^2(\phi_1\phi_1 + a\phi_1\phi_1 + b(\phi_1\phi_1)^2 + c \sum_{i=2}^{4} \frac{1}{\phi_i} (\phi_i\phi_i) + h.c.)
\]

(16)

We denote the VEV’s of the neutral components of Higgs doublets as

\[
\langle \phi_1^0 \rangle = v_1 e^{-i\phi_1}, \quad \langle \phi_2^0 \rangle = v_2 e^{-i\phi_2}, \\
\langle \phi_3^0 \rangle = v_3 e^{-i\phi_3}, \quad \langle \phi_4^0 \rangle = v_4 e^{-i\phi_4}
\]

(17)
In terms of the VEV’s we have

\[ V = \mu_2^2 (v_2^2 + v_3^2 + v_4^2) + \alpha (v_2^2 + v_3^2 + v_4^2)^2 \]
\[ + \beta \left[ 2v_2^2 v_3^2 \cos^2 (\theta_2 - \theta_3) + \frac{1}{6} (v_2^2 + v_3^2 - 2v_4^2)^2 \right] \]
\[ + \gamma \left[ 2v_2^2 v_4^2 \cos^2 (\theta_2 - \theta_4) + 2v_3^2 v_4^2 \cos^2 (\theta_3 - \theta_4) + \frac{1}{2} (v_2^2 - v_3^2)^2 \right] \]
\[ - \delta \left[ 2v_3^2 v_4^2 \sin^2 (\theta_3 - \theta_4) + 2v_2^2 v_4^2 \sin^2 (\theta_4 - \theta_2) + 2v_2^2 v_3^2 \sin^2 (\theta_2 - \theta_3) \right] \]
\[ + \{ \mu_1^2 + a(v_2^2 + v_3^2 + v_4^2) \} v_1^2 \]
\[ + b v_1^4. \]  

The minimization conditions are

\[ \frac{\partial V}{\partial v_1} = 2 \{ \mu_1^2 + a(v_2^2 + v_3^2 + v_4^2) \} v_1 + 4 b v_1^3 = 0, \]

\[ \frac{\partial V}{\partial v_2} = \left( 4 \alpha + \frac{2}{3} \beta + 2 \gamma \right) v_2^3 + \left\{ 2 \mu_2^2 + \left[ 4 \alpha + 4 \beta \cos^2 (\theta_2 - \theta_3) + \frac{2}{3} \beta - 2 \gamma - 4 \delta \sin^2 (\theta_2 - \theta_3) \right] \right\} v_3^2 \]
\[ + \left[ 4 \alpha - \frac{4}{3} \beta + 4 \gamma \cos^2 (\theta_2 - \theta_4) - 4 \delta \sin^2 (\theta_4 - \theta_2) \right] v_4^2 + 2 a v_1^2 + 4 c v_1^2 \cos 2(\theta_2 - \theta_1) \]  

\[ = 0, \]

\[ \frac{\partial V}{\partial v_3} = \left( 4 \alpha + \frac{2}{3} \beta + 2 \gamma \right) v_3^3 + \left\{ 2 \mu_3^2 + \left[ 4 \alpha + 4 \beta \cos^2 (\theta_3 - \theta_4) + \frac{2}{3} \beta - 2 \gamma - 4 \delta \sin^2 (\theta_2 - \theta_3) \right] \right\} v_2^2 \]
\[ + \left[ 4 \alpha - \frac{4}{3} \beta + 4 \gamma \cos^2 (\theta_3 - \theta_4) - 4 \delta \sin^2 (\theta_4 - \theta_3) \right] v_4^2 + 2 a v_1^2 + 4 c v_1^2 \cos 2(\theta_3 - \theta_1) \]  

\[ = 0, \]

\[ \frac{\partial V}{\partial v_4} = \left( 4 \alpha + \frac{8}{3} \beta \right) v_4^3 + \left\{ 2 \mu_4^2 + \left[ 4 \alpha - \frac{4}{3} \beta + 4 \gamma \cos^2 (\theta_4 - \theta_1) - 4 \delta \sin^2 (\theta_4 - \theta_1) \right] \right\} v_2^2 \]
\[ + \left[ 4 \alpha - \frac{4}{3} \beta + 4 \gamma \cos^2 (\theta_3 - \theta_4) - 4 \delta \sin^2 (\theta_4 - \theta_3) \right] v_3^2 + 2 a v_1^2 + 4 c v_1^2 \cos 2(\theta_4 - \theta_1) \]  

\[ = 0, \]

\[ \frac{\partial V}{\partial \vartheta_1} = 4 c \left[ v_2^2 \sin 2(\vartheta_2 - \vartheta_1) + v_3^2 \sin 2(\vartheta_3 - \vartheta_1) + v_4^2 \sin 2(\vartheta_4 - \vartheta_1) \right] v_1^2 = 0, \]

\[ \frac{\partial V}{\partial \vartheta_2} = -2 \beta v_2^2 v_3^2 \sin 2(\vartheta_2 - \vartheta_3) - 2 \gamma v_2^2 v_4^2 \sin 2(\vartheta_2 - \vartheta_4) \]
\[ + 2 \delta \left[ v_2^2 v_3^2 \sin 2(\vartheta_4 - \vartheta_2) - v_2^2 v_4^2 \sin 2(\vartheta_2 - \vartheta_4) \right] - 4 c v_2^2 \sin 2(\vartheta_2 - \vartheta_1) = 0, \]

\[ \frac{\partial V}{\partial \vartheta_3} = -2 \beta v_3^2 v_4^2 \sin 2(\vartheta_3 - \vartheta_2) - 2 \gamma v_3^2 v_4^2 \sin 2(\vartheta_3 - \vartheta_4) \]
\[ + 2 \delta \left[ -v_3^2 v_4^2 \sin 2(\vartheta_4 - \vartheta_3) - v_3^2 v_4^2 \sin 2(\vartheta_3 - \vartheta_4) \right] - 4 c v_3^2 \sin 2(\vartheta_3 - \vartheta_1) = 0, \]

and

\[ \frac{\partial V}{\partial \vartheta_4} = 2 \gamma \left[ v_2^2 v_4^2 \sin 2(\vartheta_2 - \vartheta_4) + v_3^2 v_4^2 \sin 2(\vartheta_3 - \vartheta_4) \right] \]
\[ + 2 \delta \left[ v_2^2 v_4^2 \sin 2(\vartheta_3 - \vartheta_4) - v_2^2 v_4^2 \sin 2(\vartheta_4 - \vartheta_2) \right] - 4 c v_4^2 \sin 2(\vartheta_4 - \vartheta_1) = 0. \]
The Eq. (23) is not independent of Eqs. (24)-(26); this reflects the fact that \( V \) depends only on three angles; we can set \( \vartheta_1 = \vartheta_4 \). When we substitute Eqs. (24)-(26) into Eqs. (19)-(22), we get

\[
\left( 4\alpha - \frac{4}{3}\beta + 2\gamma - 2\delta \right) v_2^2 + \left( 4\alpha + \frac{8}{3}\beta - 2\gamma - 2\delta \right) v_3^2 + \left( 4\alpha - \frac{4}{3}\beta + 2\gamma - 2\delta \right) v_4^2 + 2\mu_2^2 + 2av_1^2 = 0,
\]

(27)

\[
\left( 4\alpha + \frac{8}{3}\beta - 2\gamma - 2\delta \right) v_2^2 + \left( 4\alpha - \frac{4}{3}\beta + 2\gamma - 2\delta \right) v_3^2 + \left( 4\alpha - \frac{4}{3}\beta + 2\gamma - 2\delta \right) v_4^2 + 2\mu_2^2 + 2av_1^2 = 0,
\]

(28)

\[
\left( 4\alpha - \frac{4}{3}\beta + 2\gamma - 2\delta \right) v_2^2 + \left( 4\alpha - \frac{4}{3}\beta + 2\gamma - 2\delta \right) v_3^2 + \left( 4\alpha - \frac{4}{3}\beta + 2\gamma - 2\delta \right) v_4^2 + 2\mu_2^2 + 2av_1^2 = 0,
\]

(29)

and

\[
2a \left( v_2^2 + v_3^2 + v_4^2 \right) + 2\mu_1^2 - 4c \left( \frac{2c}{\beta + \delta} v_2^2 - \frac{\beta - \gamma}{\beta + \delta} v_1^2 \right) + 4bv_1^2 = 0.
\]

(30)

Eq. (27) and Eq. (28) imply \( v_2^2 = v_3^2 = v_4^2 \). It should be noticed that \( v_2^2 = v_3^2 = v_4^2 \) is not necessarily satisfied. The relation of \( v_2^2 = v_3^2 \) immediately leads to \( \sin 2(\vartheta_2 - \vartheta_4) + \sin 2(\vartheta_3 - \vartheta_4) = 0 \) or \( \vartheta_2 - \vartheta_4 = \vartheta_3 - \vartheta_4 + n\pi \) \((n = 0, \pm 1, \pm 2, \cdots\) with the help of Eq. (26). We choose the symmetry breaking direction as \( \vartheta_1 = \vartheta_4 = 0 \) and obtain \( \vartheta_3 = -\vartheta_2 + n\pi \). Thus there remains only one phase \( \vartheta_2 \). Substituting \( v_2 = v_3 = \xi/\sqrt{2} \), \( \vartheta_2 = \phi \) (and \( \vartheta_1 = \vartheta_4 = 0 \)) into Eq. (17), the VEV's of the neutral components of Higgs doublets are represent as

\[
\langle \phi_0^0 \rangle = v_1, \quad \langle \phi_2^0 \rangle = \frac{\xi}{\sqrt{2}} e^{i\phi},
\]

\[
\langle \phi_0^0 \rangle = \frac{\xi}{\sqrt{2}} e^{-i\phi}, \quad \langle \phi_4^0 \rangle = v_4.
\]

(31)

We rewrite Eq. (24) as

\[
\sin 4\phi = 2 \left[ -\frac{\gamma + \delta}{\beta + \delta} \left( \frac{v_1}{\xi} \right)^2 - \frac{2c}{\beta + \delta} \left( \frac{v_1}{\xi} \right)^2 \right] \sin 2\phi.
\]

(32)

and divide it by \( \sin 2\phi \neq 0 \), then, the minimization condition of the Higgs potential can be obtained as

\[
\cos 2\phi = -\frac{\gamma + \delta}{\beta + \delta} \left( \frac{v_1}{\xi} \right)^2 - \frac{2c}{\beta + \delta} \left( \frac{v_1}{\xi} \right)^2.
\]

(33)

The minimums is stable if

\[
b > 0, \quad |\gamma + \delta| > |\beta + \delta|, \quad |2c| > |\beta + \delta|,
\]

\[
|v_1| < |\xi|, \quad |v_4| < |\xi|.
\]

(34)

In order to generate masses of up-type quarks, we introduce \( \tilde{\phi}_k = i\tau_2 \phi_k^* \) and

\[
\langle \tilde{\phi}_1^0 \rangle = v_1, \quad \langle \tilde{\phi}_2^0 \rangle = \frac{\xi}{\sqrt{2}} e^{-i\phi},
\]

\[
\langle \tilde{\phi}_0^0 \rangle = \frac{\xi}{\sqrt{2}} e^{i\phi}, \quad \langle \tilde{\phi}_4^0 \rangle = v_4.
\]

(35)

III. MOST VIALBLE MODEL (13-112)

We assign the LH doublets, RH singlets and Higgs doublets as \([12]:\)

\[
Q_1 : 1, \quad Q_2, Q_3, Q_4 : 3,
\]

\[
U_1 : 1, \quad U_2 : 1, \quad U_3, U_4 : 2,
\]

\[
D_1 : 1, \quad D_2 : 1, \quad D_3, D_4 : 2,
\]

\[
L_1 : 1, \quad L_2, L_3, L_4 : 3,
\]

\[
E_1 : 1, \quad E_2 : 1, \quad E_3, E_4 : 2.
\]

(36)

The Yukawa Lagrangian relevant for the down-type quark masses and invariant under \( SU(2) \otimes U(1) \otimes S_4 \) symmetry is

\[
L_\text{Y}^d = A_{dL}(u_1, d_1)_{L} \phi_1 d_{1R} + B_{dL}(u_1, d_1)_{L} \phi_1 d_{2R}
\]

\[
+ C_{dL} \left( [u_2, d_2]_{L} \phi_2 + [u_3, d_3]_{L} \phi_3 + [u_4, d_4]_{L} \phi_4 \right) d_{1R}
\]

\[
+ D_{dL} \left( [u_2, d_2]_{L} \phi_2 + [u_3, d_3]_{L} \phi_3 + [u_4, d_4]_{L} \phi_4 \right) d_{2R}
\]

\[
+ E_{dL} \left( \frac{1}{\sqrt{2}} [u_2, d_2]_{L} \phi_2 + [u_3, d_3]_{L} \phi_2 + [u_4, d_4]_{L} \phi_2 \right) d_{3R}
\]

\[
+ \frac{1}{\sqrt{6}} \left( [u_2, d_2]_{L} \phi_2 + [u_3, d_3]_{L} \phi_2 - 2[u_4, d_4]_{L} \phi_2 \right) d_{4R}
\]

\[\text{h.c.} \quad . \]

(37)

Simillarly, the Yukawa Lagrangians relevant for the up-type quarks masses \( L_\text{Y}^u \) and charged lepton masses \( L_\text{Y}^\nu \).
are obtained,
\[
\mathcal{L}_{Y}^u = \{A_d, B_d, C_d, D_d, E_d, d_{iR}, \phi_i\} \text{ in } \mathcal{L}_{Y}^d,
\]
\[
\mathcal{L}_{Y}^f = \{A_d, B_d, C_d, D_d, E_d, u_{iL}, d_{iL}, d_{iR}\} \text{ in } \mathcal{L}_{Y}^d.
\]

At the tree level, the mass terms in the Lagrangian to be
\[
Y^U = (\ell_1, \ell_2, \ell_3, \ell_4)_{L} M_{\ell} \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \end{pmatrix} R
\]
\[
+ (u_1, u_2, u_3, u_4)_{L} M_{u} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} R
\]
\[
+ (d_1, d_2, d_3, d_4)_{L} M_{d} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} + \text{h.c.},
\]
where the down-type quark mass matrix is
\[
M_d = \begin{pmatrix}
A_d v_1^* & B_d v_1 \\
C_d e^{i\phi} & D_d e^{i\phi} \\
E_d e^{-i\phi} & F_d e^{-i\phi} \\
G_d & H_d
\end{pmatrix}
\]
and the up-type quark and charged lepton mass matrices are
\[
M_u = \{A_d, B_d, C_d, D_d, E_d, \phi\} \text{ in } M_d,
\]
\[
M_{\ell} = \{A_d, B_d, C_d, D_d, E_d\} \text{ in } M_d.
\]

These mass matrices are diagonalized by bi-unitary transformations:
\[
U_{d}^\dagger M_u U_{d} = \text{diag.}(m_u, m_c, m_t, m_{\tau}),
\]
\[
U_{d}^\dagger M_d U_{d} = \text{diag.}(m_d, m_s, m_b, m_{\tau}),
\]
\[
U_{d}^\dagger M_{\ell} U_{d} = \text{diag.}(m_\ell, m_{\mu}, m_{\tau}, m_L),
\]
where $m_{\tau}, m_{\nu}$ and $m_L$ denote the masses of up-type quark, down-type quark and charged lepton in fourth generation, respectively. The matrices $U_d$ and $V_d$ for down-type sector are satisfied with
\[
U_{d}^\dagger (M_d U_{d}^\dagger) U_{d} = U_{d}^\dagger M_d (V_d V_d^\dagger) M_d U_{d}
\]
\[
= (V_d^\dagger M_d V_d) (U_d^\dagger M_d U_d)
\]
\[
= (U_d^\dagger M_d V_d) (U_d^\dagger M_d V_d)^\dagger
\]
\[
= DD^\dagger,
\]
and
\[
V_{d}^\dagger (M_d^\dagger M_d)V_{d} = V_{d}^\dagger M_d^\dagger (U_d U_d^\dagger) M_d V_{d}
\]
\[
= (V_d^\dagger M_d U_d) (U_d^\dagger M_d V_d)
\]
\[
= (U_d^\dagger M_d V_d) (U_d^\dagger M_d V_d)^\dagger
\]
\[
= D^\dagger D,
\]
where $D$ denotes a diagonal mass matrix. Similarly, the matrices $U_u$ and $V_u$ for up-type sector (as well as $U_{\ell}$ and $V_{\ell}$ for charged lepton sector) are obtained. The Cabbibo-Kobayashi-Maskawa matrix is given by $V_{CKM} = U_u^\dagger U_d$.

We show that the model 13-112 is a good candidate of the four-generation model for quarks and charged leptons. We have performed a parameter search and found that, in the model 13-112, the parameter set
\[
v_1 = 246\text{GeV}, \quad v_4 = 1.77 \times 10^{-2}\text{GeV},
\]
\[
\xi = 695.8\text{GeV}, \quad \phi = 5 \times 10^{-3}\text{rad},
\]
\[
A_u = 4.01 \times 10^{-3}, \quad B_u = 3.25 \times 10^{-3},
\]
\[
C_u = 3.40 \times 10^{-1}, \quad D_u = 3.30 \times 10^{-1},
\]
\[
E_u = 70.8,
\]
\[
A_d = 1.01 \times 10^{-4}, \quad B_d = 4.76 \times 10^{-4},
\]
\[
C_d = 1.84 \times 10^{-1}, \quad D_d = 4.25 \times 10^{-1},
\]
\[
E_d = 1.43,
\]
and
\[
A_{\ell} = 2.65 \times 10^{-4}, \quad B_{\ell} = 3.42 \times 10^{-4},
\]
\[
C_{\ell} = 2.04 \times 10^{-4}, \quad D_{\ell} = 1.59 \times 10^{-2},
\]
\[
E_{\ell} = 7.21 \times 10^{-1},
\]
yields the following masses of three-generation quarks and charged leptons,
\[
m_c = 0.51\text{MeV}, \quad m_\mu = 106\text{MeV}, \quad m_\tau = 1.776\text{GeV},
\]
\[
m_u = 2.2\text{MeV}, \quad m_c = 1.27\text{GeV}, \quad m_t = 174.2\text{GeV},
\]
\[
m_d = 4.7\text{MeV}, \quad m_s = 96.8\text{MeV}, \quad m_b = 4.18\text{GeV},
\]
these masses are exactly same of those in PDG\textsuperscript{58}. The predicted masses of fourth generation particles
\[
m_L = 410\text{GeV},
\]
\[
m_{\tau} = 40.2\text{TeV},
\]
\[
m_{\nu} = 875\text{GeV},
\]
are consistent with the lower bound in PDG\textsuperscript{58}
\[
m_L^{PDG} \gtrsim 100\text{GeV},
\]
\[
m_{\tau}^{PDG} \gtrsim 800\text{GeV},
\]
\[
m_{\nu}^{PDG} \gtrsim 755\text{GeV}.
\]
The magnitudes of the CKM matrix elements are calculated as

\[ V_{\text{CKM}} = \begin{pmatrix} 0.985 & 0.174 & 0.00751 & 0.0000245 \\ 0.174 & 0.985 & 0.0140 & 0.0000492 \\ 0.00983 & 0.0125 & 0.999 & 0.00398 \\ 0.0000236 & 0.000102 & 0.00398 & 0.999 \end{pmatrix}. \] (55)

From the experiments, we know [58]

\[ V_{\text{PDG}}^\text{CKM} = \begin{pmatrix} 0.97434 & 0.22506 & 0.00375 \\ 0.22492 & 0.97351 & 0.0411 \\ 0.00875 & 0.0403 & 0.9915 \end{pmatrix}, \] (56)

for three-generation. The order of magnitude of the predicted CKM matrix elements for three generations are consistent with the experimental data. Thus, this model is a viable four-generation model for quarks and charged leptons based on $S_4$ permutation symmetry.

IV. OTHER MODELS

A. Model 13-1’12

We assign the LH doublets and RH singlets as

- $Q_1 : 1'$, $\{Q_2, Q_3, Q_4\} : 3$,
- $U_1 : 1'$, $U_2 : 1'$, $\{U_3, U_4\} : 2$,
- $D_1 : 1'$, $D_2 : 1'$, $\{D_3, D_4\} : 2$,
- $L_1 : 1'$, $L_2, L_3, L_4$ : 3,
- $E_1 : 1'$, $E_2 : 1'$, $\{E_3, E_4\} : 2$. \hspace{1cm} (60)

The assignment of the Higgs doublets is same throughout the paper. We obtain the Yukawa Lagrangian

\[ \mathcal{L}_Y^d = A_d (u_1, d_1)_{L} \phi_1 d_{1R} \\
+ B_d \left[ (u_2, d_2)_{L} \phi_2 + (u_3, d_3)_{L} \phi_3 + (u_4, d_4)_{L} \phi_4 \right] d_{2R} \\
+ C_d \left\{ \frac{1}{\sqrt{2}} \left[ (u_2, d_2)_{L} \phi_3 + (u_3, d_3)_{L} \phi_2 \right] d_{3R} \\
+ \frac{1}{\sqrt{6}} \left[ (u_2, d_2)_{L} \phi_3 + (u_3, d_3)_{L} \phi_3 \right] d_{4R} \\
- 2 (u_4, d_4)_{L} \phi_4 \right\} d_{4R} \\
+ \text{h.c.}, \] (58)

and the mass matrix

\[ M_d = \begin{pmatrix} A_d v_1 & 0 & 0 & 0 \\ 0 & \frac{B_d}{\sqrt{2}} e^{i\phi} & \frac{C_d}{\sqrt{2}} e^{-i\phi} & \frac{D_d}{\sqrt{2}} e^{i\phi} \\ 0 & \frac{B_d}{\sqrt{2}} e^{-i\phi} & \frac{C_d}{\sqrt{2}} e^{i\phi} & \frac{D_d}{\sqrt{2}} e^{-i\phi} \\ 0 & \frac{B_d}{\sqrt{2}} e^{-i\phi} & \frac{C_d}{\sqrt{2}} e^{-i\phi} & \frac{D_d}{\sqrt{2}} e^{i\phi} \end{pmatrix}, \] (62)

for down-type quarks. Similarly, we obtain the Yukawa Lagrangian and mass matrix for up-type quarks and charged leptons.

Since $\text{rank}(M_d) = 4$ (as well as $\text{rank}(M_u) = \text{rank}(M_t) = 4$), all fermions, except neutrinos, acquire masses. However, we can not find the parameter set which is consistent with observations. There are five Yukawa couplings $A_d, B_d, C_d, D_d, E_d$ for the down-type quark sector (as well as up-type quark sector and charged lepton sector) in the succeed model 13-112. On the contrary, there are only three Yukawa couplings $A_d, B_d, C_d$ for the down-type quark sector (as well as up-type quark sector and charged lepton sector) in the model 1’3-1’12. Models with a few parameters are favorable than models with a lots of parameters, but, unfortunately, it seems that the number of couplings is not enough to reproduce the all observed masses and mixings in the model 1’3-1’12.
C. Model 13-13

We assign the LH doublets and RH singlets as

\[
\begin{align*}
Q_1 : & 1, \quad \{Q_2, Q_3, Q_4\} : 3, \\
U_1 : & 1, \quad \{U_2, U_3, U_4\} : 3, \\
D_1 : & 1, \quad \{D_2, D_3, D_4\} : 3, \\
L_1 : & 1, \quad \{L_2, L_3, L_4\} : 3, \\
E_1 : & 1, \quad \{E_2, E_3, E_4\} : 3.
\end{align*}
\]

We obtain the Yukawa Lagrangian

\[
L_Y^d = A_d (u_1, d_1) L \phi_1 d_1 R + B_d (u_1, d_1) L \phi_2 d_2 R + C_d (u_2, d_2) L \phi_3 d_3 R + (u_4, d_4) L \phi_4 d_4 R
\]

for down-type quarks. Similarly, we obtain the Yukawa Lagrangian and mass matrix for up-type quarks and charged leptons.

There are five Yukawa couplings \(A_d, B_d, C_d, D_d, E_d\) for the down-type quark sector (as well as up-type quark sector and charged lepton sector) such as in the succeed model 13-112. However, we can not find the parameter set which is consistent with observations.

D. Others

The results of the numerical analysis for models (a) 13-112, (b) 13-1’12, (f) 1’3-1’12 and (i) 13-13 in Table I are shown. The numerical calculations for other models have not performed yet; however, we can conclude that models with rank\((M_d) < 4\) such as (b)13-1’12, are excluded from observations (see section IV A). Moreover, it seems that the number of couplings is not enough to reproduce the all observed masses and mixing if “# of couplings < 5” (see section IV B). The conditions of rank\((M_d) = 4\) and “# of couplings ≥ 5” may be favorable. Only two models (a)13-112 and (i)13-13 (as well as (A)1’3’-1’1’2 and (I) 1’3’-1’3’) satisfy these conditions. From our numerical calculations, we have success only with the model 13-112 (see section IV B).

V. SUMMARY

We have proposed the \(S_4\) symmetric four-generation models for quarks and charged leptons. Although an \(S_4\) symmetric four-generation model has been already proposed, there are some redundant assumptions and additional parameters in the model. In this paper, we have constructed models with only requirement of exact \(S_4\) symmetry. Thus, proposed models are simplest models for four generations under the exact \(S_4\) symmetry. We have shown that at least one of the models (model 13-112) is consistent with observations.

We comment that there are three crucial problems in the model 13-112. The first problem is the flavor changing neutral current (FCNC) problem. We expect that the FCNC problem may be solved under an assumption that the neutral members are superheavy; however, we should consider carefully the tree-level unitarity violation.

The second problem is the \(\rho\) parameter problem. The \(\rho\) parameter problem may be avoided in some models with different assignment of representation of \(S_4\). The third problem is related to the assignment of the singlet representations. We have assigned \(D_1 : 1, D_2 : 1, \{D_3, D_4\} : 2\) for down type quarks (similarly for up-type quarks charged leptons); however, we have no reason to distinguish \(D_1\) and \(D_2\) with same representation. We expect that this problem may be solved if we consider an \(S_3\) group as a subgroup of \(S_4\), e.g., we may require that \(D_2, D_3, D_4\) obey \(S_3\) symmetry.

Finally, we note that, apart from crucial problems, there are interesting but unsolved matters in the models in this paper, such as (1) numerical calculations for all models in Table I, (2) construction of other \(S_4\) assigned models with different assignment criteria, (3) including neutrino masses and mixings, (4) considerations for neutral fourth generation particle as a dark matter, (5) including neutrino masses and candidates for dark matter simultaneously, e.g., the scotogenic model, (6) consequences of \(S_4\) symmetry breaking, (7) effects on collider phenomenology, etc. More details of these topics will be found in our future study.
