Research Article

Nonpolynomial Spline Interpolation for Solving Fractional Subdiffusion Equations

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Received 6 June 2022; Accepted 3 August 2022; Published 5 September 2022

1. Introduction

In recent years, the derivation of solutions to fractional differential equations has become a hot topic in many areas of applied science and engineering. There are still not enough extremely precise numerical methods, although there have been many works on numerical methods for fractional differential equations. It is also very important to conduct research on fractional-order differential equations in unconstrained fields. A popular approach to solving these problems is to use artificial boundary conditions (ABCs) [1]. There are several applications of spline methods to the numerical solution of differential equations and, in particular, fractional differential equations [2–6]. The derivation of solutions to fractional differential equations has become a first-rate topic in many areas of applied science and engineering. Since fractional derivatives provide more accurate models than integer derivatives for most real-world problems, fractional differential equations are used to formulate a considerable number of applied problems. Applications are diverse and include viscoelastic and viscoelastic flows, control theory, transport problems, tumor evolution, random walks, continuum mechanics, and turbulence [7–13]. Finite differences, Fourier method, orthogonal spline collocation, implicit schemes with alternating direction, and compact finite difference methods have been successfully used in [14–22] to solve fractional subdiffusion equations. The author of [23] has presented an implicit numerical method for the fractional diffusion equation, where the fractional derivative is discretized by spline and the Crank–Nicolson discretization is used for the time variable. The authors of [24] have developed a novel nonpolynomial spline method for solving second-order hyperbolic equations, and their results are numerically more accurate than some finite difference methods, see [25–28]. The authors of [29] presented a quadratic nonpolynomial spline approach to approximate the solution of a system of second-order boundary value problems associated with a one-sided obstacle, and compared to collocation, finite difference, and certain common polynomial spline approaches, we used contact problems and produced approximations that were more accurate.

We consider the fractional differential equation on a bounded domain expressed by [1]

\[ D^\alpha_\delta u(x,t) - u_{xx}(x,t) = f(x,t), (x,t) \in \Omega, \]  
\[ u(x,0) = \psi(x), x \in [0,1], \]
where \( \Omega = \{ (x,t) | 0 \leq x \leq 1, 0 \leq t < +\infty \} \) and \( 0 < a < 1 \).

### 2. Basic Definition

**Definition 1.** Let \( u(x) \) be a function defined on \( (a, b) \), then the Riemann–Liouville fractional derivative is of the following form [12]:

\[
D^\alpha (u(x)) = \frac{1}{\Gamma(m-a)} \frac{d^m}{dx^m} \int_0^x (x-t)^{m-\alpha-1} u(t) dt, \alpha > 0,
\]

where \( m-1 < \alpha < m \).

**Definition 2.** Let \( u(x) \) be a function defined on \( (a, b) \), then the Caputo fractional derivative is of the following form [12]:

\[
D^\alpha (u(x)) = \frac{1}{\Gamma(m-a)} \int_0^x (x-t)^{m-\alpha-1} u^m(t) dt, \alpha > 0,
\]

where \( m-1 < \alpha < m \).

### 3. Nonpolynomial Spline Method

In this section, we present nonpolynomial spline method for solving the problem equations (1)–(4) in the finite interval \([0, T]\). For the positive integers \( n \) and \( k \), we take \( h = 1/n \), \( \tau = T/k \) as the spatial stepsize and temporal stepsize, respectively. We use the notations \( x_i = ih \) (\( 0 \leq i \leq n \)), \( t_j = j\tau \) (\( 0 \leq j \leq k \)), and \( u^j_i = u(x_i, t_j) \). Let us consider a mesh with nodal points \( x_i = ih \) on \([0, 1]\) such that

\[
\Delta: 0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1.
\]

For each segment \( [x_i, x_{i+1}], i = 0, 1, 2, \ldots, n-1 \), the nonpolynomial spline \( Q_i(x, t_j) \) has the following form:

\[
Q_i(x, t_j) = a_i(t_j) \cos(k(x-x_i)) + b_i(t_j) \sin(k(x-x_i)) + c_i(t_j)(x-x_i)^5 + d_i(t_j)(x-x_i)^4 + e_i(t_j)(x-x_i)^3 + f_i(t_j)(x-x_i)^2 + g_i(t_j)(x-x_i) + r_i(t_j), i = 0, \ldots, n,
\]

where \( k \) is the frequency of the trigonometric part of the spline functions which can be real or pure imaginary and which will be used to raise the accuracy of the method [30].

Let \( u(x,t) \) be the exact solution, and let \( S_i^j \) be an approximation to \( u_i^j \) obtained by the segment \( Q_i(x, t_j) \) passing through the points \( (x_i, S_i^j) \) and \( (x_{i+1}, S_{i+1}^j) \).

To derive the coefficients of the equation (8), we first define

\[
Q_i(x_i, t_j) = S_i^j, Q_i(x_i, t_j) = m_i^j, Q_i^{(2)}(x_i, t_j) = M_i^j, Q_i^{(3)}(x_i, t_j) = L_i^j, Q_i^{(4)}(x_i, t_j) = S_{i+1}^j, Q_i^{(5)}(x_i, t_j) = m_{i+1}^j, Q_i^{(6)}(x_i, t_j) = M_{i+1}^j, Q_i^{(7)}(x_i, t_j) = L_{i+1}^j.
\]

By algebraic manipulation, we get

\[
\alpha_i(M_{i-3}^j + M_{i+3}^j) + \alpha_2(M_{i-2}^j + M_{i+2}^j) + \alpha_3(M_{i-1}^j + M_{i+1}^j) + \alpha_4 M_i^j = \frac{1}{h^2} \left[ (u_{i-3}^j + u_{i+3}^j) + \beta_1(u_{i-2}^j + u_{i+2}^j) + \beta_2(u_{i-1}^j + u_{i+1}^j) + \beta_3 u_i^j \right], i = 3, \ldots, n-3.
\]
By expanding equation (10) in Taylor series about \( x_j \) at each fixed \( t = t_j, \ 0 \leq j \leq M \), we obtain the following local truncation error:

\[
t_i = -(2 + 2\beta_1 + 2\beta_2 + \beta_3)u_i + h^2 \left( \frac{d^2u}{dx^2} \right)_i \left( -9 + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\beta_1 - \beta_2 \right) + 1/12 \\
h^3 \left( \frac{d^4u}{dx^4} \right)_i \left( -81 + 108\alpha_1 + 48\alpha_2 + 12\alpha_3 - 16\beta_1 - \beta_2 \right) + \frac{1}{360} \\
h^3 \left( \frac{d^4u}{dx^4} \right)_i \left( -729 + 2430\alpha_1 + 480\alpha_2 + 30\alpha_3 - 64\beta_1 - \beta_2 \right) + \frac{1}{20160} \\
h^3 \left( \frac{d^4u}{dx^4} \right)_i \left( -6561 + 40824\alpha_1 + 3584\alpha_2 + 56\alpha_3 - 256\beta_1 - \beta_2 \right) + \frac{1}{1814400} \\
h^{10} \left( \frac{d^{10}u}{dx^{10}} \right)_i \left( -59049 + 590490\alpha_1 + 230400\alpha_2 + 90\alpha_3 - 1024\beta_1 - \beta_2 \right) + \frac{1}{239500800} \\
h^{12} \left( \frac{d^{12}u}{dx^{12}} \right)_i \left( -531441 + 7794468\alpha_1 + 135168\alpha_2 - 132\alpha_3 - 4096\beta_1 \right) + \frac{1}{43589145600} \\
h^{14} \left( \frac{d^{14}u}{dx^{14}} \right)_i \left( -4782969 + 96722626\alpha_1 + 745742\alpha_2 - 182\alpha_3 - 16384\beta_1 - \beta_2 \right) + \ldots,
\]

where \( \alpha_1, \alpha_2, \alpha_3, \alpha_j, \beta_1, \beta_2, \) and \( \beta_3 \) are given in [1]. In this paper, we use fourth- and sixth-order method. From [30], \( \alpha_1, \alpha_2, \alpha_3, \alpha_j, \beta_1, \beta_2, \) and \( \beta_3 \) are \( \alpha_1 = 50, \alpha_2 = 10, \alpha_3 = -450, \alpha_j = 900, \beta_1 = 24, \beta_2 = 15, \) and \( \beta_3 = -80 \) for fourth-order method and \( \alpha_1 = 1/42, \alpha_2 = 20/7, \alpha_3 = 397/14, \alpha_j = 1208/21, \beta_1 = 24, \beta_2 = 15, \) and \( \beta_3 = -80 \) for sixth-order method.

**Lemma 1.** Let \( Q \) is an estimation of the smoothness function at class \( C_2 \), then the error estimation obtained as

\[
|u'' - Q_i| \leq \frac{h^2}{12}Q^{(6)}(h^4), \\
|Q'' - u''| \leq \frac{h^2}{12}u^{(4)} + O(h^4), \\
|u'' - Q_i| \leq \frac{h^4}{180}Q^{(5)}(h^5), \\
|Q'' - u''| \leq \frac{h^4}{180}u^{(5)} + O(h^5).
\]

**Proof.** The above relations are due to the expansion of Taylor sentences \( M_{i+1}, M_{i-1}, m_{i+1}, m_{i-1}, u_{i+1}, u_{i-1}, \) at point \( x_i \) can be obtained as follows:

\[
\begin{align*}
   u''_i &= Q_i'' + \frac{h^2}{12}Q_i^{(4)} + \frac{h^4}{240}Q_i^{(6)} - \frac{h^6}{6048}Q_i^{(8)} + O(h^7), \quad i = 1, \ldots, n-1, \quad (I), \\
   Q''_i &= u''_i - \frac{h^2}{12}u_i^{(4)} + \frac{h^4}{360}u_i^{(6)} + \frac{17h^6}{60480}u_i^{(8)} + O(h^7), \quad i = 1, \ldots, n-1. \quad (II).
\end{align*}
\]

Also, we have the same way.
The spline method is obtained as follows:

\[
\begin{align*}
\left\{ \begin{array}{l}
u_i' = Q_i' + \frac{h^4}{180}Q_i^{(5)} + \frac{h^6}{1512}Q_i^{(7)} - \frac{h^8}{14400}Q_i^{(9)} + O(h^9), \quad i = 1, \ldots, n-1, \text{(III)}, \\
Q_i' = u_i' - \frac{h^4}{180}\mu_i^{(5)} + \frac{h^6}{1512}\mu_i^{(7)} + \frac{h^8}{25920}u_i^{(8)} + O(h^9), \quad i = 1, \ldots, n-1. \text{(IV)}.
\end{array} \right.
\]

(14)

And this is the end of the proof. \(\square\)

The local truncation error for the first- and second-order spline method is obtained as follows:

\( \text{(II) Second – order method: } t_i = 307h^6u_i^{(6)} + O(h^7), \)

\( \text{(III) Sixth – order method: } t_i = \frac{h^8u_i^{(8)}}{252} + O(h^9), \)

\( \text{(IV) Eighth – order method: } t_i = \frac{65629h^8u_i^{(10)}}{27956090} + O(h^{11}). \)

(15)

To obtain unique solution for the nonlinear system equation (10), we need four more equations. We define the following identities:

\[
\begin{align*}
(y_0, y_1, y_2, y_3, y_4) &= (65, -104, 14, 24, 1), \quad (\nu_1 = \nu_{n-1} = 1210210269217/326918592000), \\
(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8, \eta_9, \eta_{10}, \eta_{11}, \eta_{12}) &= (-2248215317/19958400, 4539175/17600, -6055918291/6652800, \\
&\quad 9918918899/4989600, 892256279/285120, \\
&\quad 18019157507/4989600, -30650022317/9979200, 2380569353/1247400, \\
&\quad -16851712321/19958400, 503717713/19958400, -130447723/2851200, \\
&\quad 18976637/4989600) \\
(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5) &= (26, 14, -80, 15, 24, 1), \\
(y_2 = \nu_{n-2} = \begin{pmatrix} 521085679991 \\ 373621248000 \end{pmatrix}) = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}), \\
&\quad \begin{pmatrix} -7712274923 & 259885933 & -19222747601 & 8351791919 \\ 159667200 & 5702400 & 53222400 & 11404800 \end{pmatrix} \\
&= \begin{pmatrix} -3712133107 & 15883047 & -2614370381 & 28466192407 \\ 3193344 & 71280 & 2280960 & 39916800 \end{pmatrix} \\
&= \begin{pmatrix} -7204033313 & 16761737 & -43435489 & 113795873 \\ 2280960 & 177408 & 2534400 & 7933600 \end{pmatrix},
\end{align*}
\]

(17)

where from equation (1), we can write \(M_i^j\) in the form

\[
M_i^j = \frac{\partial^4 u_i^j}{\partial x^4} = \sum \Delta_i^4 u(x_i, t_j) - f(x_i, t_j).
\]

(18)

Now, we need to use the definition of discrete approximation of fractional derivative in time of any appropriate function \(z2, 10\) as
\[ D^a_i \varphi(x_i, t_j) = \frac{1}{h^2} \left( u_{i+1} - 2u_i + u_{i-1} \right) + O(r^{2-a}), \quad (19) \]

\[ M^j_i = a \sum_{q=0}^{j-1} R_{j,q}(u_{i+1}^q - u_i^q) - f_i^j. \quad (20) \]

where \( a = 1/(2-a) \) and \( R_{j,q} = (j - q)^{1-a} \). By substituting equations (20) in (10), we have following systems:

\[ \alpha_i a \left( u_{i-1} + u_{i+1} \right) + \left( \alpha_2 a - \frac{\beta_1}{h^2} \right) \left( u_{i-2} + u_{i+2} \right) + \left( \alpha_3 a - \frac{\beta_2}{h^2} \right) \left( u_{i-1} + u_{i+1} \right) + \left( \alpha_4 a - \frac{\beta_3}{h^2} \right) u_i \]

\[ = a \left[ \alpha_i (u_{i-3} + u_{i+3}) + \alpha_2 (u_{i-2} + u_{i+2}) + \alpha_3 (u_{i-1} + u_{i+1}) + \alpha_4 u_i \right] \]

\[ + \alpha_i (u_{i-3} + u_{i+3}) + \alpha_2 (u_{i-2} + u_{i+2}) + \alpha_3 (u_{i-1} + u_{i+1}) + \alpha_4 u_i \]

for \( i = 3, 4, \ldots, n - 3 \), if \( j = 1 \), and

\[ \alpha_i a \left( u_{i-1} + u_{i+1} \right) + \left( \alpha_2 a - \frac{\beta_1}{h^2} \right) \left( u_{i-2} + u_{i+2} \right) + \left( \alpha_3 a - \frac{\beta_2}{h^2} \right) \left( u_{i-1} + u_{i+1} \right) + \left( \alpha_4 a - \frac{\beta_3}{h^2} \right) u_i \]

\[ = a \left[ \alpha_i (u_{i-2} + u_{i+2}) + \alpha_2 (f_{i-1} + f_{i+1}) + \alpha_3 (f_{i-1} + f_{i+1}) + \alpha_4 f_i \right] \]

\[ + \alpha_3 (f_{i-2} + f_{i+2}) + \alpha_4 f_i + \mu_i^j \]

for \( i = 3, 4, \ldots, n - 3 \), if \( j = 2, 3, \ldots, k \), with

\[ \mu_i^j = a \sum_{q=0}^{j-2} R_{j,q}(u_{i-3} + u_{i+3}) + \alpha_2 (u_{i-2} + u_{i+2}) + \alpha_3 (u_{i-1} + u_{i+1}) + \alpha_4 u_i \]

\[ - a \sum_{q=0}^{j-2} R_{j,q}(u_{i-1} + u_{i+1}) + \alpha_2 (u_{i-2} + u_{i+2}) + \alpha_3 (u_{i-1} + u_{i+1}) + \alpha_4 u_i. \]

Substituting equations (20) in (16), we get

\[ \sum_{k=0}^{4} y_k u_k^j + h^2 \sum_{k=1}^{12} \eta_k \left( a \sum_{q=0}^{j-1} R_{j,q}(u_{k+1}^q - u_k^q) - f_k^j \right) + \eta^2 h^{14}(u_{(14)})_0^j = 0, \text{at } i = 1, \]

\[ \sum_{k=0}^{5} \mu_k u_k^j + h^2 \sum_{k=1}^{12} \sigma_k \left( a \sum_{q=0}^{j-1} R_{j,q}(u_{k+1}^q - u_k^q) - f_k^j \right) + \sigma_{j-2} h^{14}(u_{(14)})_0^j = 0, \text{at } i = 2, \]

\[ \sum_{k=0}^{5} \mu_k u_k^j + h^2 \sum_{k=1}^{12} \sigma_k \left( a \sum_{q=0}^{j-1} R_{j,q}(u_{k+1}^q - u_k^q) - f_k^j \right) + \sigma_{j-2} h^{14}(u_{(14)})_0^j = 0, \text{at } i = n - 2, \]

\[ \sum_{k=0}^{4} y_k u_k^j + h^2 \sum_{k=1}^{12} \eta_k \left( a \sum_{q=0}^{j-1} R_{j,q}(u_{k+1}^q - u_k^q) - f_k^j \right) + \eta_{n-1} h^{14}(u_{(14)})_0^j = 0, \text{at } i = n - 1. \]
In the above systems, for each \( j = 1, 2, \ldots, k \), clearly, we have \( n \) linear equations in terms of \( n \) unknowns \( u_{j1}, u_{j2}, \ldots, u_{jn} \).

### 4. Stability

In this section, we analyze the stability of equation (21) by means of the von Neumann method. We consider

\[
\xi_j = \xi_j \exp (I\Phi h),
\]

where \( I = \sqrt{-1} \) and \( \Phi \) is a real mode number and \( \xi \) is a complex number. We substitute equations (25) into (21) to obtain

\[
\xi_j = a^k \left[ \begin{array}{c}
\left( \alpha_1 a - \frac{1}{h^2} \right) (\exp (I (i - 3)\theta) + \exp (I (i + 3)\theta)) + \left( \alpha_2 a - \frac{\beta_1}{h^2} \right) (\exp (I (i - 2)\theta) + \exp (I (i + 2)\theta)) + \\
\left( \alpha_3 a - \frac{\beta_2}{h^2} \right) (\exp (I (i - 1)\theta) + \exp (I (i + 1)\theta)) + \left( \alpha_4 a - \frac{\beta_3}{h^2} \right) (\exp (I i\theta))
\end{array} \right] + \mu_j^i,
\]

in which

\[
\mu_j^i = a \left( 2\alpha_1 \cos 3\theta + 2\alpha_2 \cos 2\theta + 2\alpha_3 \cos \theta + \alpha_4 \right) \exp (I i\theta) \sum_{q=0}^{i-2} R_{ij}(\xi_q - \xi_{q+1}).
\]
From the denoted values of $x$ and $y$, the equation (30) can be simplified as

$$y\xi_j = x\xi_{j-1} + x \sum_{q=0}^{j-2} R_{j,q}(\xi_q - \xi_{q+1}).$$

(32)

Then, we obtain

$$\xi_j = \rho\xi_{j-1} + \rho \sum_{q=0}^{j-2} R_{j,q}(\xi_q - \xi_{q+1}), \ j = 2, 3, \ldots, k.$$  (33)

Now, from this equation, it is obtained that $|\xi_j| \geq |\xi_0|$ for all $j = 2, 3, \ldots, k$ (see [3]). Hence, the nonpolynomial spline method is unconditionally stable.

5. Numerical Results

In this section, the wealth of the presented method non-polynomial spline method will be tested on three different examples, which are widely used in the literature. All problems were solved with best error estimations, and their results were compared with analytical or spline solutions.
given in the literature. In the following section, detailed information about this comparison will be presented.

**Example 1.** Consider problem equations (1)–(4) with

\[
\begin{align*}
\psi(x) &= x^4(x-1)^4, x \in [0, 1], \\
\phi(t) &= t^{(3-a)}, \phi(t) = 0, t \in [0, T].
\end{align*}
\]

The exact solution of the problem is as follows [7]:

\[
u(x, t) = (x - 1)^4 \left[ \exp(-x)t^{3 + \alpha} + x^4 \right], (x, t) \in \Omega.
\] (35)

This problem is solved using the methods described in Section 2 for \( T = 1, 2, 3 \). The results are reported in Tables 1 and 2.

**Example 2.** Consider problem equations (1)–(4) with

\[
\begin{align*}
f(x, t) &= \frac{\Gamma(3)}{\Gamma(3 - \alpha)} t^{-\alpha} \sin(2\pi x) \\
&\quad + (2\pi)^2 t^2 \sin(2\pi x), (x, t) \in \Omega
\end{align*}
\]

\[
\begin{align*}
\psi(x) &= 0, x \in [0, 1], \\
\phi(t) &= 0, \phi(t) = 0, t \in [0, T].
\end{align*}
\] (36)

It can be checked that the exact solution is \( u(x, t) = t^2 \sin(2\pi x) \) [10].

This problem is solved using the methods described in Section 2 for \( T = 1/4, 1/2, 1 \). The results are reported in Table 3.

**6. Conclusion**

In this paper, we constructed the application of the non-polynomial spline method to obtain the numerical point solution of fractional partial differential equations. Numerical results showed that these techniques achieve accuracy, in numerical results illustrative graphs, the Figure 1 the comparison of the present results with the result of [10] shows the efficiency of the suggested technique. Also, the stability and error estimates of the methods are investigated and shown some figures of the results. Moreover, illustrations for the stability region of the schemes are derived.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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