Diagonalization of the metric of a Lorentzian 3-manifold

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Abstract

We study the problem of diagonalization of the metric of 3-dimensional Lorentzian manifold. Applying the technique of moving frames, we prove that every smooth Lorentzian 3-manifold admits an atlas in which the metric assumes a diagonal form.

1 Introduction

A (pseudo)-Riemannian $n$-manifold $(M, g)$ is said to have orthogonal coordinates around a point if in the neighborhood of the point there is a chart such that the metric $g$ with respect to it is in diagonal form, i.e.

$$g = \sum_{i=1}^{n} f_i dx^i \otimes dx^i.$$

If the manifold satisfies this property at every point, then we will say that it admits an orthogonal atlas.

Any surface has an orthogonal atlas since there always are isothermal coordinates (see [1]) and the metric assumes the particular diagonal form

$$g = f(x, y)(dx \otimes dx + dy \otimes dy).$$

In the Riemannian setting D. DeTurck and D. Yang in [2] proved that any 3-dimensional smooth manifold $(M, g)$ has an orthogonal atlas using the technique of moving frames. In this case the metric assumes the more general form

$$g = f_1(x, y, z)dx \otimes dx + f_2(x, y, z)dy \otimes dy + f_3(x, y, z)dz \otimes dz.$$

In their paper they point out that in higher dimension the situation changes because the existence of the orthogonal atlas is subject to a condition on the Weyl tensor. Subsequently, P. Tod in [6] studied the problem in the same setting.

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but in dimension \( n \geq 4 \) where he found necessary algebraic conditions for the existence of the orthogonal atlas. In the paper the cases of dimensions \( n = 4, n = 5 \) and \( n \geq 6 \) are studied separately as they require each a different condition on the Weyl tensor of the manifold; more precisely the restriction involves the third derivatives of the tensor for \( n = 4 \), the first derivative for \( n = 5 \), and the tensor itself for \( n \geq 6 \). However, J. Grant and J. A. Vickers in [4] proved that something can be said even in dimension 4, in particular they showed that in the analytic setting, in the Riemannian and in Lorentzian case, one can find a chart such that the metric \( g \) is block diagonal, i.e.

\[
g_{ij} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}
\]

where \( A \) and \( B \) are \( 2 \times 2 \) block matrix, even when no assumptions are made on the Weyl tensor (or its derivatives).

More recently, O. Kowalski and M. Sekizawa in [5] proved the existence of an orthogonal atlas in the real analytic Lorentzian setting by applying the Cauchy-Kovalevski Theorem. On the other hand, P. Gauduchon and A. Moroianu proved in [3] that one cannot find orthogonal coordinates in the neighborhood of any point for the complex and quaternionic projective spaces \( \mathbb{C}P^m \) and \( \mathbb{H}P^q \).

In this paper we will prove that all smooth Lorentzian 3-manifolds admit an orthogonal atlas by following the same method used by DeTurck and Yang. The technique is the following: the problem is initially shifted from a PDE system about the coordinates to a PDE system about a coframe in the cotangent bundle with respect to a fixed coframe, then one proves that the Cauchy problem associated to the second PDE system admits a solution.

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2 Orthogonal coordinates on Lorentzian manifolds

We proceed to illustrate the proof of the following

**Theorem 2.1.** Let \((M, g)\) be a smooth Lorentzian 3-manifold. Then \(M\) admits an orthogonal atlas.

Let \((\bar{e}_1, \bar{e}_2, \bar{e}_3)\) be an orthonormal frame of \((M, g)\) and \((\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3)\) the corresponding coframe. We want to find a triplet of coordinated functions \((x_1, x_2, x_3)\) such that, if \(e_i = \partial_i\) is the coordinated frame of \((x_1, x_2, x_3)\), then \(g(e_i, e_j) = 0\) every time \(i \neq j\). The first difficulty we find both in the Riemannian and in
the Lorentzian setting is the following. Assume \((y^1, y^2, y^3)\) are fixed coordinates and \(g(y)\) is the metric tensor w.r.t. this chart; then the coordinated frame \(\{e_i\}\) can be written as

\[ e_i = \frac{\partial y^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha} \]

and hence the PDE system to be solved is

\[ 0 = g(\partial_i, \partial_j) = \sum_{\alpha,\beta=1}^3 \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} g_{\alpha\beta}(y) \text{ for } i \neq j. \]

This system is nonlinear, and its linearization is not symmetric hyperbolic, which means that the standard results of existence of the solution do not apply. Furthermore, there is an invariance in the solution if the unknowns are the coordinates: assume \((\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)\) are other coordinates, such that \(\tilde{x}^i = f^i(x^i)\) and each \(f^i\) is a strictly monotone function. Then

\[ 0 = g(\partial_i, \partial_j) = \frac{\partial f^i}{\partial x^i} \frac{\partial f^j}{\partial x^j} g(\partial_{\tilde{x}^1}, \partial_{\tilde{x}^2}) \]

and hence also \((\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)\) are orthogonal coordinates.

For this reason it works best if one does not set the unknowns to be the coordinated functions \((x_1, x_2, x_3)\), but the normalized coframe \((\omega^1, \omega^2, \omega^3)\), where \(\omega^i = dx^i\) (no sum intended) and \(f_i = 1/|dx^i|\). Applying Frobenius Theorem it is easy to get an equivalent condition to the existence of the coordinated charts depending on the coframe, that is

\[ \omega^i \wedge d\omega^i = 0 \]  \hspace{1cm} (2)

must hold, when \(i = 1, 2, 3\). Now, as the coframe has to be orthonormal, it has to satisfy the first Cartan structure equation

\[ d\omega^i = \sum_j \omega^j \wedge \omega^i_j \]

where \((\omega^i_j)\) is the connection matrix 1-form. Here appears the first difference between the Riemannian and Lorentzian case, although it does not yield any actual change in the proof: in the first case \(\omega^j_i = -\omega^j_i\) for any \(i, j\), but in the second we have

\[ \omega^2_1 = -\omega^1_2, \quad \omega^3_1 = \omega^1_3, \quad \omega^2_3 = \omega^3_2. \]

Hence (2) becomes

\[ \omega^1 \wedge \omega^2 \wedge \omega^1_2 + \omega^1 \wedge \omega^3 \wedge \omega^1_3 = 0 \]

\[ \omega^1 \wedge \omega^2 \wedge \omega^1_2 + \omega^2 \wedge \omega^3 \wedge \omega^2_3 = 0 \]

\[ \omega^1 \wedge \omega^3 \wedge \omega^1_3 + \omega^2 \wedge \omega^3 \wedge \omega^2_3 = 0, \]
thus, by alternatively subtracting one and adding the other we get the system
\[ \begin{align*}
\omega^1 \wedge \omega^2 \wedge \omega^1_0 &= 0, \\
\omega^2 \wedge \omega^3 \wedge \omega^2_0 &= 0, \\
\omega^1 \wedge \omega^3 \wedge \omega^1_0 &= 0.
\end{align*} \tag{4} \]

We now write \( \omega^i \) with respect to \( \bar{\omega}^i \) and vice-versa as
\[ \omega^i = b_j^i \bar{\omega}^j, \quad \bar{\omega}^i = \tilde{b}_j^i \omega^j. \]

and we will solve for the \( b_j^i \). We will solve (4), hence we need \( \omega^i_j \) and we start by noting that
\[
\omega^i \wedge \omega^i = d\omega^i = d \sum_j b_j^i \bar{\omega}^j = \sum_j \left( \sum_k \bar{e}_k(b_j^i) \bar{\omega}^k \wedge \bar{\omega}^j + b_k^i \bar{\omega}^k \wedge \omega^i \right) = \sum_{j,k} \omega^k \wedge (\bar{e}_k(b_j^i) \omega^j + b_k^i \bar{\omega}^j) = \sum_{j,k,l} \omega^j \wedge (b_k^l \bar{e}_k(b_j^i) \bar{\omega}^j + b_k^l b_j^i \omega^i).
\]

As a consequence of the first difference we find a second one here: while
\[ \omega_2^1 = \sum_{j,k} \frac{1}{2} \left\{ b_k^2 \bar{e}_k(b_j^1) - b_k^2 b_j^2 \bar{\omega}^k \right\} \bar{\omega}^j + b_k^2 b_j^i \bar{\omega}^i \]
remains as in (2), the other two differ due to (3) as follows:
\[ \omega_3^1 = \sum_{j,k} \frac{1}{2} \left\{ b_k^3 \bar{e}_k(b_j^1) + b_k^3 b_j^2 \bar{\omega}^k \right\} \bar{\omega}^j + b_k^3 b_j^i \bar{\omega}^i \] and
\[ \omega_3^2 = \sum_{j,k} \frac{1}{2} \left\{ b_k^3 \bar{e}_k(b_j^2) + b_k^3 b_j^3 \bar{\omega}^k \right\} \bar{\omega}^j + b_k^3 b_j^i \bar{\omega}^i. \]

Again by following (2) we rewrite (4) substituting \( \omega^i_j \) and obtaining
\[ \begin{align*}
0 &= \sum_{i,l,j,k} b_{i,j}^1 b_{i,l}^2 \bar{\omega}^i \wedge \bar{\omega}^l \wedge \left[ \frac{1}{2} \left\{ b_k^2 \bar{e}_k(b_j^1) - b_k^2 b_j^2 \bar{\omega}^k \right\} \bar{\omega}^j + b_k^2 b_j^i \bar{\omega}^i \right] \\
0 &= \sum_{i,l,j,k} b_{i,j}^1 b_{i,l}^2 \bar{\omega}^i \wedge \bar{\omega}^l \wedge \left[ \frac{1}{2} \left\{ b_k^3 \bar{e}_k(b_j^1) + b_k^3 b_j^2 \bar{\omega}^k \right\} \bar{\omega}^j + b_k^3 b_j^i \bar{\omega}^i \right] \\
0 &= \sum_{i,l,j,k} b_{i,j}^2 b_{i,l}^3 \bar{\omega}^i \wedge \bar{\omega}^l \wedge \left[ \frac{1}{2} \left\{ b_k^3 \bar{e}_k(b_j^2) + b_k^3 b_j^3 \bar{\omega}^k \right\} \bar{\omega}^j + b_k^3 b_j^i \bar{\omega}^i \right]
\end{align*} \]

The unknowns of the system are \( (b_j^i) \in C^\infty(M, SO(2,1)) \).
We are going to prove that the linearization of this system is diagonal hyperbolic. Consider the linearization \( \beta_j^i = (\delta b_j^i) \) and notice that we can assume
that \( \{ \omega^i \} = \{ \omega^i \} \) when we linearize around \( \{ \omega^i \} \), as such \( b^i_j(x) = \delta^i_j \). Thus, the linearized system is

\[
0 = \delta^i_1 \delta^j_2 \frac{1}{2} (\delta^2_k \bar{e}_k (\beta^1_j) - \delta^k_1 \bar{e}_k (\beta^2_j)) \bar{\omega}^i \wedge \bar{\omega}^j + \text{lower order terms in } \beta
\]

\[
0 = \delta^i_1 \delta^j_3 \frac{1}{2} (\delta^3_k \bar{e}_k (\beta^1_j) + \delta^k_1 \bar{e}_k (\beta^3_j)) \bar{\omega}^i \wedge \bar{\omega}^j + \text{lower order terms in } \beta
\]

\[
0 = \delta^i_2 \delta^j_3 \frac{1}{2} (\delta^3_k \bar{e}_k (\beta^2_j) + \delta^k_2 \bar{e}_k (\beta^3_j)) \bar{\omega}^i \wedge \bar{\omega}^j + \text{lower order terms in } \beta
\]

in which the only non-zero elements are

\[
\frac{1}{2} (\bar{e}_2 (\beta^3_1) - \bar{e}_1 (\beta^2_1)) = \text{terms of order 0 in } \beta,
\]

\[
\frac{1}{2} (\bar{e}_3 (\beta^2_1) + \bar{e}_1 (\beta^3_1)) = \text{terms of order 0 in } \beta,
\]

\[
\frac{1}{2} (\bar{e}_3 (\beta^2_2) + \bar{e}_2 (\beta^3_1)) = \text{terms of order 0 in } \beta.
\]

As \( (b^i_j(x)) \in \text{SO}(2,1) \) we have that \( (\beta^i_j) \in \text{so}(2,1) \), hence we can rewrite everything as

\[
\bar{e}_1 (\beta^2_1) = \text{terms of order 0 in } \beta
\]

\[
\bar{e}_2 (\beta^3_1) = \text{terms of order 0 in } \beta
\]

\[
\bar{e}_3 (\beta^3_2) = \text{terms of order 0 in } \beta.
\]

The differential operator is thus

\[
A(u) = \bar{e}_1 (u) + \bar{e}_2 (u) + \bar{e}_3 (u)
\]

that is in diagonal form, and its symbol is

\[
\sigma(\xi) = \sum_{i=1}^{3} \xi^i.
\]

To finally prove that the metric is diagonalizable we have to find a solution to the Cauchy problem given by the differential operator \( A \) and a set of initial data to be chosen. To do so, we need these data to not be characteristic of the operator. By the form of \( A \) we deduce that the characteristics of the system are the covectors that annihilate \( e_1, e_2 \) and \( e_3 \). Hence, the initial data for the Cauchy problem associated to the system can be given as the coframe \( \{ \omega^i \} \) on a surface \( \Sigma \subset M \) with \( e_i \notin T\Sigma \) for \( i = 1, 2, 3 \) since we need \( \omega^i (v) \neq 0 \) for all \( v \in T\Sigma \). This concludes the proof of Theorem 2.1.

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