Comparison of compact induction with parabolic induction

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Abstract

Let $F$ be any non archimedean locally compact field of residual characteristic $p$, let $G$ be any reductive connected $F$-group and let $K$ be any special parahoric subgroup of $G(F)$. We choose a parabolic $F$-subgroup $P$ of $G$ with Levi decomposition $P = MN$ in good position with respect to $K$. Let $C$ be an algebraically closed field of characteristic $p$. We choose an irreducible smooth $C$-representation $V$ of $K$. We investigate the natural intertwiner from the compact induced representation $\text{c-Ind}^{G(F)}_K V$ to the parabolically induced representation $\text{Ind}^{G(F)}_P (\text{c-Ind}^{M(F)}_{M(F) \cap K} V_{N(F) \cap K})$. Under a regularity condition on $V$, we show that the intertwiner becomes an isomorphism after a localisation at a specific Hecke operator. When $F$ has characteristic 0, $G$ is $F$-split and $K$ is hyperspecial, the result was essentially proved by Herzig. We define the notion of $K$-supersingular irreducible smooth $C$-representation of $G(F)$ which extends Herzig’s definition for admissible irreducible representations and we give a list of $K$-supersingular irreducible representations which are supercuspidal and conversely a list of supercuspidal representations which are $K$-supersingular.

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1 Introduction

Let $F$ be a non archimedean locally compact field of residual characteristic $p$, let $G$ be a reductive connected $F$-group and let $C$ be an algebraically closed field of characteristic
p. We are interested in smooth admissible $C$-representations of $G(F)$. Two induction techniques are available, compact induction $\text{c-Ind}^{G(F)}_{K}$ from a compact open subgroup $K$ of $G(F)$ and parabolic induction $\text{Ind}_{P(F)}^{G(F)}$ from a parabolic subgroup $P(F)$ with Levi decomposition $P(F) = M(F)N(F)$. Here we want to investigate the interaction between the two inductions.

More specifically assume that $G(F) = P(F)K$ and $P(F) \cap K = (M(F) \cap K)(N(F) \cap K)$. We construct (Proposition 2.1) for any finite dimensional smooth $C$-representation $V$ of $K$, a canonical intertwiner

$$I_0 : \text{c-Ind}^{G(F)}_{K} V \to \text{Ind}_{P(F)}^{G(F)}(\text{c-Ind}^{M(F)}_{M(F) \cap K} V_{N(F) \cap K}),$$

where $V_{N(F) \cap K}$ stands for the $N(F) \cap K$-coinvariants in $V$, and a canonical algebra homomorphism

$$S' : \mathcal{H}(G(F), K, V) \to \mathcal{H}(M(F), M(F) \cap K, V_{N(F) \cap K}),$$

where as in [HV], the Hecke algebra $\mathcal{H}(G(F), K, V)$ is $\text{End}_{G(F)}\text{c-Ind}^{G(F)}_{K} V$ seen as an algebra of double cosets of $K$ in $G$, and similarly for $\mathcal{H}(M(F), M(F) \cap K, V_{N(F) \cap K})$. By construction

$$(I_0(\Phi(f)))(g) = S'(\Phi)(I_0(f)(g)),$$

for $f \in \text{c-Ind}^{G(F)}_{K} V, \Phi \in \mathcal{H}(G(F), K, V), g \in G(F)$. Let $V^*$ be the contragredient representation of $V$. We constructed in [HV] a Satake homomorphism

$$S : \mathcal{H}(G(F), K, V^*) \to \mathcal{H}(M(F), M(F) \cap K, (V^*)^{N(F) \cap K}),$$

and we show that $S'$ and $S$ are related by a natural anti-isomorphism of Hecke algebras (Proposition 2.4).

We study further $I_0$ in the particular case where $K$ a special parahoric subgroup and $V$ is irreducible. Such a $V$ is trivial on the pro-$p$-radical $K_+$ of $K$. The quotient $K/K_+$ is the group of $k$-points of a connected reductive $k$-group $G_k$, so that we can use the theory of finite reductive groups in natural characteristic. We write $K/K_+ = G(k)$. The image of $P(F) \cap K = P_0$ in $G(k)$ is the group of $k$-points of a parabolic subgroup of $G_k$. We write $P_0/P_0 \cap K_+ = P(k)$, and we use similar notations for $M$ and $N$ and for the opposite parabolic subgroup $P = MN$ (Section 2.1). We choose a maximal $F$-split torus $S$ in $M$ such that $K$ stabilizes a special vertex in the apartment of $G(F)$ associated to $S$. We choose an element $s \in S(F)$ which is central in $M(F)$ and strictly $N$-positive, in the sense that the conjugation by $s$ strictly contracts the compact subgroups of $N(F)$. There a unique Hecke operator $T_M$ in $\mathcal{H}(M(F), M_0, V_{N(k)})$ with support in $M_0s$ and value at $s$ the identity of $V_{N(k)}$.

**Proposition 1.1.** (Proposition 4.4) The map $S'$ is a localisation at $T_M$.

This means that $S'$ is injective, $T_M$ belongs to the image of $S'$, and is central invertible in $\mathcal{H}(M(F), M_0, V_{N(k)})$, and

$$\mathcal{H}(M(F), M_0, V_{N(k)}) = S'(\mathcal{H}(G(F), K, V))[T_M^{-1}].$$

This comes from an analogous property of $S$ proved in [HV]. We look now at the localisation $\Theta$ of $I_0$ at $T_M$

$$\mathcal{H}(M(F), M_0, V_{N(k)}) \otimes_{\mathcal{H}(G(F), K, V), S'} \text{c-Ind}^{G(F)}_{K} V \to \text{Ind}_{P(F)}^{G(F)}(\text{c-Ind}^{M(F)}_{M(F) \cap K} V_{N(k)}).$$

Our main theorem is

**Theorem 1.2.** (Theorem 4.4) $\Theta$ is injective, and $\Theta$ is surjective if and only if $V$ is $M$-coregular.
This result was essentially proved by Herzig [Herzig, Abe], when $F$ has characteristic 0, $G$ is $F$-split and $K$ is hyperspecial. In the theorem, $P = MN$ is the opposite parabolic subgroup of $P$, and we say that $V$ is $M$-coregular if for $h \in K$ which does not belong to $P_0P_0$, the image of $hV^{N(k)}$ in $V_{N(k)}$ is 0. See Definition 3.6 and Corollary 3.20 for an equivalent definition. As in Herzig and Abe, we define in the last chapter the notion of a $K$-supersingular irreducible smooth $C$-representation of $G(F)$. We see our main theorem as the first step towards the classification of irreducible smooth $C$-representations of $G(F)$ in terms of supersingular ones.

To prove the theorem, we follow the method of Herzig and we decompose $I_0$ as the composite $I_0 = \zeta \circ \xi$ of two $G(F)$-equivariant maps, the natural inclusion $\xi$ of $c\text{-Ind}_K^{G(F)} V$ in $c\text{-Ind}_K^{G(F)} c\text{-Ind}_P^{G(k)} V$, and

$$\zeta : c\text{-Ind}_K^{G(F)} c\text{-Ind}_P^{G(k)} V \to \text{Ind}_{P(F)}^{G(F)}(c\text{-Ind}_{M(F)\cap K}^{M(F)} V_{N(k)}) ,$$

is a natural map associated to the quotient map $c\text{-Ind}_P^{G(k)} V \to N_{N(k)}$ (see 2 below). We write $P$ for the parahoric subgroup inverse image of $P(k)$ in $K$ and $T_P$ for the Hecke operator in $\mathcal{H}(G(F), P, V_{N(k)})$ of support $PS$ and value at $s$ the identity of $V_{N(k)}$. With no regularity assumption on $V$ we prove

$$\zeta \circ T_P = T_M \circ \zeta .$$

Seeing $c\text{-Ind}_K^{G(F)} c\text{-Ind}_P^{G(k)} V = c\text{-Ind}_P^{G(F)} V_{N(k)}$ and $\text{Ind}_{P(F)}^{G(F)}(c\text{-Ind}_{M(F)\cap K}^{M(F)} V_{N(k)})$ as $C[T]$-modules via $T_P$ and $T_M$, the map $\zeta$ is $C[T]$-linear and we prove (Corollary 6.6):

**Theorem 1.3.** The localisation at $T$ of $\zeta$ is an isomorphism.

To study $\xi$, we consider the Hecke operator $T_G$ in $\mathcal{H}(G(F), K, V)$ with support $KsK$ and value at $s$ the natural projector $V \to V^{N(k)}$, and the Hecke operator $T_{K,P}$ from $c\text{-Ind}_P^{G(F)} V_{N(k)}$ to $c\text{-Ind}_K^{G(F)} V$ of support $KsP$ and value at $s$ given by the natural isomorphism $V_{N(k)} \to V^{N(k)}$. With no regularity assumption on $V$ we prove

$$T_{K,P} \circ \xi = T_G .$$

Assuming that $V$ is $M$-coregular we prove:

$$\xi \circ T_{K,P} = T_P$$

$$S'(T_G) = T_M .$$

Seeing $c\text{-Ind}_K^{G(F)} V$ as a $C[T]$-module via $T_G = (S')^{-1}(T_M)$, the map $\xi$ is $C[T]$-linear and

**Theorem 1.4.** The localisation at $T$ of $\xi$ is injective; it is an isomorphism if and only if $V$ is $M$-coregular.

Our main theorem follows.

A motivation for our work is the notion of $K$-supersingularity for an irreducible smooth $C$-representation $\pi$ of $G(F)$ (that we do not suppose admissible).

**Definition 1.5.** We say that $\pi$ is $K$-supersingular when

$$\mathcal{H}(M(F), M_0, V_{N(k)}) \otimes_{\mathcal{H}(G(F), K, V), S'} \text{Hom}_{G(F)}(c\text{-Ind}_K^{G(F)} V, \pi) = 0$$

for any irreducible smooth $C$-representation $V$ of $K$ and any standard Levi subgroup $M \neq G$. 


Hence \( \pi \) is \( K \)-supersingular when the localisations at \( T_M \) of

\[
\text{Hom}_{G(F)}(\text{c-Ind}_K^{G(F)} V, \pi)
\]

are 0 for all \( V \) and all \( M \neq G \).

When \( \pi \) is admissible, this definition is equivalent to: No character of the center \( Z(G(F), K, V) \) of \( \mathcal{H}(G(F), K, V) \) contained in \( \text{Hom}_{G(F)}(\text{c-Ind}_K^{G(F)} V, \pi) \) extends via \( S' \) to a character of \( Z(M(F), M_0, V_N(k)) \) for all \( V \subset \pi|_K, M \neq G \).

Equivalently: The localisations at \( T_M \) of the characters of \( Z(G(F), K, V) \) contained in

\[
\text{Hom}_{G(F)}(\text{c-Ind}_K^{G(F)} V, \pi)
\]

are 0 for all \( V \subset \pi|_K, M \neq G \).

Herzig and Abe when \( G \) is \( F \)-split, \( K \) is hyperspecial and the characteristic of \( F \) is 0 ([Herzig] Lemma 9.9), used this property to define \( K \)-supersingularity.

The properties of \( K \)-supersingularity and of supercuspidality (not being a subquotient of \( \text{Ind}_P^{G(F)} \tau \) for some irreducible smooth \( C \)-representation \( \tau \) of \( M(F) \neq G(F) \)) are equivalent when \( G \) is \( F \)-split, \( K \) is hyperspecial and the characteristic of \( F \) is 0. With the main theorem, we obtain a partial result in this direction in our general case.

**Theorem 1.6.** Let \( \pi \) be an irreducible smooth \( C \)-representation of \( G(F) \).

i. If \( \pi \) is isomorphic to a subrepresentation or is an admissible quotient of \( \text{Ind}_P^{G(F)} \tau \) as above, then \( \pi \) is not \( K \)-supersingular.

ii. If \( \pi \) is admissible and

\[
\mathcal{H}(M(F), M_0, V_N(k)) \otimes_{\mathcal{H}(G(F), K, V), S'} \text{Hom}_{G(F)}(\text{c-Ind}_K^{G(F)} V, \pi) \neq 0
\]

for some \( L \)-coregular irreducible subrepresentation \( V \) of \( \pi|_K \) and some standard Levi subgroups \( M \subset L \neq G \), then \( \pi \) is not supercuspidal.

## 2 Generalities on the Satake homomorphisms

In this first chapter we consider a rather general situation, where \( C \) is any field. We consider a locally profinite group \( G \), an open subgroup \( K \) of \( G \) and a closed subgroup \( P \) of \( G \) satisfying “the Iwasawa decomposition” \( G = K P \). We choose a smooth \( C[K] \)-module \( V \). As in [HV], assume that \( P \) is the semi-direct product of a closed invariant subgroup \( N \) and of a closed subgroup \( M \), and that \( K \) is the semi-direct product of \( K \cap N \) by \( K \cap M \). We also impose the assumptions

(A1) Each double coset \( K g K \) in \( G \) is the union of a finite number of cosets \( K g' \) and the union of a finite number of cosets \( g'' K \) (the first condition is equivalent to the second by taking the inverses).

(A2) \( V \) is a finite dimensional \( C \)-vector space.

The smooth \( C[K] \)-module \( V \) gives rise to a compactly induced representation \( \text{c-Ind}_K^{G(F)} V \) and a smooth \( C[P] \)-module \( W \) gives rise to the full smooth induced representation \( \text{Ind}_P^{G(F)} W \). We consider the space of intertwiners

\[
\mathcal{J} := \text{Hom}_{G}(\text{c-Ind}_K^{G(F)} V, \text{Ind}_P^{G(F)} W).
\]

By Frobenius reciprocity for compact induction (as \( K \) is open in \( G \)), the \( C \)-module \( \mathcal{J} \) is canonically isomorphic to \( \text{Hom}_K(V, \text{Res}_K^{G} \text{Ind}_P^{G} W) \); to an intertwiner \( I \) we associate the function \( v \mapsto I[1, v]_K \) where \( [1, v]_K \) is the function in \( \text{c-Ind}_K^{G(F)} V \) with support \( K \) and value \( v \) at 1. By the Iwasawa decomposition and the hypothesis that \( K \) is open in \( G \), we get by restricting functions to \( K \) an isomorphism of \( C[K] \)-modules from \( \text{Res}_K^{G} \text{Ind}_P^{G} W \)
onto \( \text{Ind}_K^P(\text{Res}_P^G W) \). Using now Frobenius reciprocity for the full smooth induction \( \text{Ind}_K^P \) from \( P \cap K \) to \( K \), we finally get a canonical \( C \)-linear isomorphism

\[
\mathcal{J} \simeq \text{Hom}_{P \cap K}(V, W)
\]

(we now omit mentioning the obvious restriction functors in the notation); this map associates to an intertwiner \( I \) the function \( v \mapsto (I[1, v])_K(1) \).

We could have proceeded differently, first applying Frobenius reciprocity to \( \text{Ind}_P^G W \), getting \( \mathcal{J} \simeq \text{Hom}_P(\text{c-Ind}_K^P V, W) \), then identifying \( \text{Res}_P^G \text{c-Ind}_K^P V \) with \( \text{c-Ind}_K^P V \), and finally applying Frobenius reciprocity to \( \text{c-Ind}_K^P V \). In this way we also obtain an isomorphism of \( \mathcal{J} \) onto \( \text{Hom}_{P \cap K}(V, W) \), which is readily checked to be the same as the preceding one.

Assume also that \( W \) is a smooth \( C[M] \)-module, seen as a smooth \( C[P] \)-module by inflation. Then \( \text{Ind}_P^G W \) is the "parabolic induction" of \( W \), and \( \text{Hom}_{P \cap K}(V, W) \) identifies with \( \text{Hom}_{K \cap M}(V_{N \cap K}, W) \), where \( V_{N \cap K} \) is the space of coinvariants of \( N \cap K \) in \( V \). With that identification, an intertwiner \( I \) is sent to the map from \( V_{N \cap K} \) to \( W \) sending the image \( \tau \) of \( v \in V \) in \( V_{N \cap K} \) to \( (I[1, v])_K(1) \). By Frobenius reciprocity again \( \text{Hom}_{K \cap M}(V_{N \cap K}, W) \) is isomorphic to \( \text{Hom}_M(\text{c-Ind}_{K \cap M}^M V_{N \cap K}, W) \), so overall we obtain an isomorphism

\[
j : \mathcal{J} = \text{Hom}_G(\text{c-Ind}_K^G V, \text{Ind}_P^G W) \to \text{Hom}_M(\text{c-Ind}_{K \cap M}^M V_{N \cap K}, W),
\]

which associates to \( I \in \mathcal{J} \) the \( C[M] \)-linear map sending \([1, \tau]_{M \cap K}\) to \((I[1, v])_K(1)\).

The isomorphism \( j \) is natural in \( V \) and \( W \). The functor \( W \to \text{Hom}_G(\text{c-Ind}_K^G V, \text{Ind}_P^G W) \) from the category of smooth \( C[M] \)-modules to the category of \( C[M] \)-modules is representable by \( \text{c-Ind}_{K \cap M}^M V_{N \cap K} \), and \( \text{End}_G(\text{c-Ind}_K^G V) \) embeds naturally in the ring of endomorphisms of the functor. By Yoneda’s Lemma ([HS] Prop. 4.1 and Cor. 4.2), we have an algebra homomorphism

\[
\mathcal{S} : \text{End}_G(\text{c-Ind}_K^G V) \to \text{End}_M(\text{c-Ind}_{K \cap M}^M V_{N \cap K})
\]

such that the diagram

\[
\begin{array}{ccc}
\text{Hom}_G(\text{c-Ind}_K^G V, \text{Ind}_P^G W) & \xrightarrow{j} & \text{Hom}_M(\text{c-Ind}_{K \cap M}^M V_{N \cap K}, W) \\
\downarrow{\delta} & & \downarrow{s'(b)} \\
\text{Hom}_G(\text{c-Ind}_K^G V, \text{Ind}_P^G W) & \xrightarrow{\mathcal{S}'(b)} & \text{Hom}_M(\text{c-Ind}_{K \cap M}^M V_{N \cap K}, W)
\end{array}
\]

is commutative for any \( W \). We have \( j(I \circ b) = j(I) \circ \mathcal{S}'(b) \) for \( b \in \text{End}_G(\text{c-Ind}_K^G V) \).

By the naturality of \( j \) in \( W \), for any homomorphism \( \alpha : W' \to W \) of smooth \( C[M] \)-modules we have a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_G(\text{c-Ind}_K^G V, \text{Ind}_P^G W') & \xrightarrow{j'} & \text{Hom}_M(\text{c-Ind}_{K \cap M}^M V_{N \cap K}, W') \\
\downarrow{\text{Ind}(\alpha)} & & \downarrow{\alpha} \\
\text{Hom}_G(\text{c-Ind}_K^G V, \text{Ind}_P^G W) & \xrightarrow{j} & \text{Hom}_M(\text{c-Ind}_{K \cap M}^M V_{N \cap K}, W)
\end{array}
\]

for any \( V \). For \( W = W' \) we obtain \( j((\text{Ind}_P^G a) \circ I) = a \circ j(I) \) for \( a \in \text{End}_M(W) \).

For \( W' = \text{c-Ind}_{K \cap M}^M V_{N \cap K} \), we write \( j' = j_0 \).

\[
j_0 : \text{Hom}_G(\text{c-Ind}_K^G V, \text{Ind}_P^G(\text{c-Ind}_{K \cap M}^M V_{N \cap K})) \to \text{End}_M(\text{c-Ind}_{K \cap M}^M V_{N \cap K})
\]
We define $I_0$ in $\text{Hom}_G(\text{c-Ind}^G_K V, \text{Ind}_P^G(\text{c-Ind}_K^M V_{N\cap K}))$ such that $j_0(I_0)$ is the unit element of $\text{End}_M(\text{c-Ind}_K^M V_{N\cap K})$. We have

$$j_0((\text{Ind}_P^G a) \circ I_0) = \alpha$$

for all $\alpha$ in $\text{Hom}_M(\text{c-Ind}_K^M V_{N\cap K}, W)$. For $W = W' = \text{c-Ind}_K^M V_{N\cap K}$, we obtain

$$(3) \quad j_0((\text{Ind}_P^G a) \circ I_0) = a .$$

for $a \in \text{End}_M(\text{c-Ind}_K^M V_{N\cap K})$. For $b \in \text{End}_G(\text{c-Ind}_K^G V)$ we have

$$(4) \quad \mathcal{S}'(b) := j_0(I_0 \circ b) .$$

Applying $j_0^{-1}$ to this equality we deduce from [3]

$$(5) \quad I_0 \circ b = (\text{Ind}_P^G \mathcal{S}'(b)) \circ I_0$$

for $b \in \text{End}_G(\text{c-Ind}_K^G V)$. Summarizing we have proved

**Proposition 2.1.** (i) The map

$$\mathcal{S}' : \text{End}_G(\text{c-Ind}_K^G V) \to \text{End}_M(\text{c-Ind}_K^M V_{N\cap K})$$

is an algebra homomorphism such that $I_0 \circ b = (\text{Ind}_P^G \mathcal{S}'(b)) \circ I_0$ for $b \in B$.

(ii) We have for $\alpha$ in $\text{Hom}_M(\text{c-Ind}_K^M V_{N\cap K}, W)$,

$$j((\text{Ind}_P^G \alpha) \circ I_0) = \alpha .$$

(iii) We have $j(I \circ b) = j(I) \circ \mathcal{S}'(b)$ for $b \in B$ and $I$ in $\text{Hom}_G(\text{c-Ind}_K^G V, \text{Ind}_P^G W)$.

**Remark 2.2.** i. An intertwiner $I$ in $\text{Hom}_G(\text{c-Ind}_K^G V, \text{Ind}_P^G W)$ is determined by the values $(I[1,v]_K)(1)$ in $W$, for all $v \in V$, by the Iwasawa decomposition $G = PK$. We have

$$(I_0[1,v]_K)(1) = [1,v]_{M\cap K} .$$

ii. So far we have not used that $V$ is finite dimensional.

We now want to interpret the previous results in terms of actions of Hecke algebras.

By Frobenius reciprocity $B = \text{End}_G(\text{c-Ind}_K^G V)$ identifies with $\text{Hom}_K(V, \text{Res}_G^K \text{c-Ind}_K^G V)$, as a $C$-module; to $\Phi \in B$ we associate the map $v \mapsto \Phi_v := \Phi([1,v]_K)$; from $\Phi$ then, we get a map $G \to \text{End}_C V$, $g \mapsto \{v \mapsto \Phi_v(g)\}$. In this way we identify $B$ with the space $\mathcal{H}(G,K,V)$ of functions $\Phi$ from $G$ to $\text{End}_C V$ such that

(i) $\Phi(kgk') = k \circ \Phi(g) \circ k'$ for $k, k'$ in $K$, $g$ in $G$, where we have written $k, k'$ for the endomorphisms $v \mapsto kv, v \mapsto k'v$ of $V$;

(ii) The support of $\Phi$ is a finite union of double cosets $KgK$.

The algebra structure on $\mathcal{H}(G,K,V)$ obtained from that of $B$ is given by convolution

$$\Phi \ast \Psi(g) = \sum_{h \in G/J} \Phi(h) \Psi(h^{-1} g) = \sum_{h \in J \setminus G} \Phi(gh^{-1}) \Psi(h)$$

(the term $\Phi(h) \Psi(h^{-1} g)(v)$ vanishes, for fixed $g$, outside finitely many cosets $K h$, so that the sum makes sense). Moreover the action of $\mathcal{H}(G,K,V)$ on $\text{c-Ind}_K^G V$ is also given by convolution

$$\Phi \ast f(g) = \sum_{h \in G/J} \Phi(h)(f(h^{-1} g)) = \sum_{h \in J \setminus G} \Phi(gh^{-1})(f(h)) .$$
Proposition 2.3. The homomorphism $S' : \mathcal{H}(G,K,V) \rightarrow \mathcal{H}(M,K \cap M,V_{N \cap K})$ is given by

$$S'(\Phi)(m)(\varpi) = \sum_{n \in (N \cap K) \setminus N} \Phi(nm)(v) \text{ for } m \in M, v \in V,$$

where bars indicate the image in $V_{N \cap K}$ of elements in $V$.

Proof. As $[1,\varpi]_{M \cap K} = I_o[1,v]_K(1)$ we have for $v \in V$,

$$S'\Phi \ast [1,\varpi]_{M \cap K} = S'(\Phi) \ast (I_o[1,v]_K(1)) = (S'(\Phi)I_o([1,v]_K))(1) = I_o(\Phi \ast [1,v]_K)(1).$$

We write the element $I_o(\Phi[1,v]_K)(1)$ of $c$-Ind$^M_{M \cap K} V_{N \cap K}$ as a finite sum of $m^{-1}[1,w_m]_{K \cap M}$ for $m$ running over a system of representatives of $(P \cap K)/P$. As

$$(I_o(h^{-1}[1,v_h])(1)) = (h^{-1}I_o[1,v_h])(1) = (I_o[1,v_h])(h^{-1}) = h^{-1}((I_o[1,v_h])(1)) = m_{h^{-1}}[1,\varpi],$$

where $m_h$ is the image of $h$ in $M$, and $m_{h^{-1}} = m_h^{-1}$, we obtain

$$w_m = \sum_{n \in (N \cap K) \setminus N} [1,\varpi_{mn}] = \sum_{n \in (N \cap K) \setminus N} \Phi(nm)(v).$$

In [HV] we constructed a Satake homomorphism

$$S : \mathcal{H}(G,K,V) \rightarrow \mathcal{H}(M,K \cap M,V_{N \cap K}), \quad S(\Phi)(m)(v) = \sum_{n \in (N \cap K)} \Phi(mn)(v),$$

for $v \in V_{N \cap K}$. To compare $S'$ with $S$ we need to take the dual. Remark that $K$ acts on the dual space $V^* = \text{Hom}_C(V,C)$ of $V$ via the contragredient representation, and that the dual of $V^*$ is isomorphic to $V$ by our finiteness hypothesis on $V$. It is straightforward to verify that the map

$$\iota : \mathcal{H}(G,K,V^*) \rightarrow \mathcal{H}(G,K,V), \quad \iota(\Phi)(g) := (\Phi(g^{-1}))^t,$$

where the upper index $t$ indicates the transpose, is an algebra anti-isomorphism. We denote $A^0$ the opposite ring of a ring $A$. A ring morphism $f : A \rightarrow B$ defines a ring morphism $f^0 : A^0 \rightarrow B^0$ such that $f^0(a) = f(a)$ for $a \in A$. We view $\iota$ as an isomorphism from $\mathcal{H}(G,K,V^*)$ onto $\mathcal{H}(G,K,V)^0$. The linear forms on $V$ which are $(N \cap K)$-fixed identify with the linear forms on $V_{N \cap K}$,

$$(V_{N \cap K})^* \simeq (V^*)^{N \cap K}.$$

This leads to an algebra isomorphism

$$\iota_M : \mathcal{H}(M,M \cap K,(V^*)^{N \cap K}) \rightarrow \mathcal{H}(M,M \cap K,V_{N \cap K})^0.$$

The following proposition describes the relation between the Satake homomorphism $S$ attached to $V^*$ and the homomorphism $S'$ attached to $V$. 


Proposition 2.4. The following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{H}(G,K,V^*) & \xrightarrow{S} & \mathcal{H}(M,M \cap K,(V^*)^N) \\
\downarrow & & \downarrow \\
\mathcal{H}(G,K,V) & \xrightarrow{S^0} & \mathcal{H}(M,M \cap K,V_{N \cap K})
\end{array}
\]

Proof. For \( v \in V \) of image \( \tau \) in \( V_{N \cap K} \) we have:

\[
((\iota_M \circ S)\Phi)(m)(\tau) = (S(\Phi)(m^{-1})^t(\tau) = \sum_{n \in N/(N \cap K)} \Phi(m^{-1}n)^t(v)
\]

\[
= \sum_{n \in (N \cap K) \setminus N} \Phi((nm)^{-1})^t(v) = (S^0 \circ \iota)^t(\tau).
\]

\[\square\]

3 Representations of \( G(k) \)

Let \( C \) be an algebraically closed field of positive characteristic \( p \), let \( k \) be a finite field of the same characteristic \( p \) and of cardinal \( q \), and let \( G \) be a connected reductive group over \( k \). We fix a minimal parabolic \( k \)-subgroup \( B \) of \( G \) with unipotent radical \( U \) and maximal \( k \)-subtorus \( T \). Let \( S \) be the maximal \( k \)-split subtorus of \( T \), let \( W = W_G = W(S,G) \) be the Weyl group, let \( \Phi = \Phi_G \) be the roots of \( S \) with respect to \( U \) (called positive), \( \Delta \subseteq \Phi \) the subset of simple roots. For \( a \in \Phi \), let \( U_a \) be unipotent subgroup denoted in (\cite{BT} 5.1) by \( U(a) \). A parabolic \( k \)-subgroup \( P \) of \( G \) containing \( B \) is called standard, and has a unique Levi decomposition \( P = MN \) with Levi subgroup \( M \) containing \( T \). The standard Levi subgroup \( P = MU = UM \) is determined by \( M \). There exists a unique subset \( \Delta_M \subseteq \Delta \) such that \( M \) is generated by \( T, U_a, U_{-a} \) for \( a \) in the subset of \( \Phi \) generated by \( \Delta_M \). This determines a bijection between the subsets of \( \Delta \) and the standard parabolic \( k \)-subgroups of \( G \).

Let \( \overline{B} = TU \) be the opposite of \( B = TU \), and \( \overline{P} = MN \) the opposite of \( P \). We have \( \overline{B} = w_0 Bu_0^{-1} \) where \( w_0 = u_0^{-1} \) is the longest element of \( W \). The roots of \( S \) with respect to \( \overline{U} \), i.e. the positive roots for \( \overline{U} \), are the negative roots for \( U \). The simple roots for \( \overline{U} \) are \( -a \) for \( a \in \Delta \).

For \( a \in \Delta \) let \( G_a \subseteq G \) be the subgroup generated by the unipotent subgroups \( U_a \) and \( U_{-a} \). Let \( T_a := G_a \cap T \).

Definition 3.1. Let \( \alpha \in \Delta \) be a simple root of \( S \) in \( B \) and let \( \psi : T(k) \to C^* \) be a \( C \)-character of \( T(k) \). We denote by

\[
\Delta_\psi := \{ a \in \Delta \mid \psi(T_a(k)) = 1 \}
\]

the set of simple roots \( \alpha \) such that \( \psi \) is trivial on \( T_a(k) \).

Example 3.2. \( G = GL(n) \). Then \( T = S \) is the diagonal group and the groups \( T_a \) for \( a \in \Delta \) are the subgroups \( T_i \subseteq T \) for \( 1 \leq i \leq n-1 \), with coefficients \( x_i = x_{i+1}^{-1} \) and \( x_j = 1 \) otherwise. When \( k = F_2 \) is the field with 2 elements, \( T(k) \) is the trivial group.

Let \( V \) be an irreducible \( C \)-representation of \( G(k) \). When \( P = MN \) is a standard parabolic subgroup of \( G \), we recall that the natural action of \( M(k) \) on \( V^P(k) \) is irreducible (\cite{CE} Theorem 6.12). In particular, taking the Borel subgroup \( B = TU \), the dimension of the vector space \( V^U(k) \) is 1 and the group \( T(k) \) acts on \( V^U(k) \) by a character \( \psi_V \).
Proposition 3.3. The stabilizer in $G(k)$ of the line $V^{U(k)}$ is $P_V(k)$ where $P_V = M_V N_V$ is a standard parabolic subgroup of $G$ associated to a subset $\Delta_V \subset \Delta_{\psi_V}$.

Proof. [Curtis] Theorem 6.15.

Corollary 3.4. The dimension of $V$ is 1 if and only if $P_V = G$.

Proof. If the dimension of $V$ is 1, then $V = V^{U(k)}$ and $P_V = G$. Conversely if $P_V = G$ the line $V^{U(k)}$ is stable by $G(k)$ hence is equal to the irreducible representation $V$.

Corollary 3.5. When $P = MN$ is a standard parabolic subgroup of $G$, the dimension of $V^{N(k)}$ is equal to 1 if and only if $P \subset P_V$.

Remark 3.6. i. The group $P_V$ measures the irregularity of $V$. A 1-dimensional representation $V$ is as little regular as possible ($P_V = G$), and $V$ is as regular as possible when $P_V = B$.

ii. The group $P_V$ depends on the choice of $B$. Two minimal parabolic $k$-subgroups of $G(k)$ are conjugate in $G(k)$ and for $g \in G(k)$, the stabilizer of $V^{gU(k)g^{-1}} = gV^{U(k)}$ is $gP_Vg^{-1}$. But the inclusion $P \subset P_V$ depends only on $P$ because

$$gB(k)g^{-1} \subset P(k)$$

is equivalent to $g \in P(k)$

([Bki] chapitre IV, §2, 2.5, Prop. 3). The inclusion $P_V \subset P$ depends also only on $P$, for the same reason.

Definition 3.7. We say that

i. $V$ is $M$-regular when the stabilizer $P_V(k)$ in $G(k)$ of the line $V^{U(k)}$ is contained in $P(k)$,

ii. $V$ is $M$-coregular when the stabilizer $\overline{P}_V(k)$ in $G(k)$ of the line $V^{U(k)}$ is contained in $\overline{P}(k)$.

We recall the classification of the $C$-irreducible representations $V$ of $G(k)$.

Theorem 3.8. The isomorphism class of $V$ is characterized by $\psi_V$ and $\Delta_V \subset \Delta_{\psi_V}$. For each $C$-character $\psi$ of $T(k)$ and each subset $J \subset \Delta_{\psi}$ there exists a $C$-irreducible representation $V$ of $G(k)$ such that $\psi_V = \psi, \Delta_V = J$.

Proof. ([Curtis] Theorem 5.7).

Definition 3.9. $(\psi_V, \Delta_V)$ are called the parameters of the irreducible $C$-representation $V$ of $G(k)$.

Example 3.10. The irreducible representations $V$ with $\psi_V = 1$ are classified by the subsets of $\Delta$. They are the special representations called sometimes the generalized Steinberg representations. We denote by $Sp_P$ the special representation $V$ such that $\Delta_V = \Delta_M$ with $P = MN$. The representation $Sp_G$ is the trivial character and $Sp_B$ is the Steinberg representation.

For a standard parabolic subgroup $P = MN$, the irreducible $C$-representation $V^{N(k)}$ of $M(k)$ is associated to $\psi_V$ and to $\Delta_V \cap \Delta_M$.

Proposition 3.11. The $M$-regular irreducible $C$-representations $V$ of $G(k)$ are in bijection with the irreducible representations of $M(k)$ by the map $V \mapsto V^{N(k)}$. Those representations $V$ with $M_V = M$ correspond to the characters of $M(k)$.
Proof. For a given irreducible representation $W$ of $M(k)$ of parameter $(\psi_W, \Delta_W)$ with $\Delta_W \subset \Delta_{\psi_W} \cap \Delta_M$, where $\Delta_{\psi_W} \subset \Delta$ is the set of $a \in \Delta$ with $\psi_W$ trivial on $T_a(k)$, the number of isomorphism classes of irreducible $C$-representations $V$ of $G(k)$ with $V$ isomorphic to $W$, is equal to the number of subsets of $\Delta$ that do not belong to $\Delta_{\psi_W} \cap \Delta_M$. Only one of them satisfies $\Delta_V \subset \Delta_M$. There is a unique (modulo isomorphism) $V$ with $V \simeq W$ if and only if $\psi_W$ is not trivial on $T_a(k)$, for all $a \in \Delta - \Delta_M$. \hfill \Box

The parameters $(\psi_V, \Delta_V)$ depend on the choice of the pair $(T, U)$. The parameters $(\overline{\psi}_V, \overline{\Delta}_V)$ of $V$ for the opposite pair $(T, U)$ are:

**Lemma 3.12.** $\overline{\psi}_V = w_0(\psi_V)$, $\overline{\Delta}_V = w_0(\Delta_V)$.

**Proof.** As $\overline{B} = w_0Bw_0^{-1}$, the torus $T(k)$ acts by the character $w_0(\psi_V)$ on the line $V^{\overline{U}(k)}$ and $\overline{P}_V = w_0P_Vw_0^{-1}$ is the stabilizer of the line $V^{\overline{U}(k)}$. Hence the subset $\overline{\Delta}_V$ of simple roots is equal to $w_0(\Delta_V) \subset -\Delta$. \hfill \Box

The contragredient representation $V^*$ is irreducible and its parameters for the pair $(T, U)$ are:

**Lemma 3.13.** $\psi_{V^*} = w_0(\psi_V)^{-1}$, $\Delta_{V^*} = -w_0(\Delta_V)$.

**Proof.** By Lemma 3.12 it is equivalent to describe the parameters $(\overline{\psi}_V, \overline{\Delta}_V)$ for the opposite pair $(T, U)$. The direct decomposition $V = V^{U(k)} \oplus (1 - \overline{U}(k))V$ implies

$$(V^*)^{\overline{U}(k)} = (V^{\overline{U}(k)})^* \simeq (V^{U(k)})^*.$$  

The group $T(k)$ acts on the line $V^{U(k)}$ by the character $\psi_V$ and on $(V^{U(k)})^*$ by the character $\overline{\psi}_V = \psi_V^{-1}$. Hence $\psi_{V^*} = \psi_V^{-1}$.

The space $(V^*)^{\overline{U}(k)}$ is the subspace of elements on $V^*$ vanishing on $(1 - \overline{U}(k))V$. This space is stable by $M_V(k)$ because the direct decomposition of $V$ for $B$ is the same than for $P_V$ (Remark 3.14). Hence $M_V \overline{U} \subset \overline{P}_V$, equivalently $-\Delta_V \subset \overline{\Delta}_V = w_0(\Delta_{V^*})$. As $V$ is isomorphic to the contragredient of $V^*$ and $-w_0$ is an involution on $\Delta$, we have also the inclusion in the other direction. \hfill \Box

**Remark 3.14.** In general, $-w_0$ does not act by id on $\Delta$ (for example for $G = GL(3)$), hence the stabilizer $\overline{P}_V$ of $V^{\overline{U}(k)}$ in $G(k)$ is not the opposite of $P_V$, the $M$-regularity of $V$ is not equivalent to the $M$-coregularity of $V$. The $M$-regularity of $V$ is equivalent to the $M$-coregularity of $V^*$.

**Proposition 3.15.** We have the $M(k)$-equivariant direct decomposition:

$$V = V^{N(k)} \oplus (1 - \overline{N}(k))V^{N(k)} = V^{N(k)} \oplus (1 - \overline{N}(k))V.$$  

**Proof.** (CE Theorem 6.12). \hfill \Box

**Remark 3.16.** The decomposition is the same for $P = P_V$ than for $P = B$ because $V^{U(k)} = V^{N_U(k)}$ by definition de $P_V$.

**Proposition 3.17.** For $g \in G(k)$, the image of $gV^{U(k)}$ in $V^{\overline{U}(k)}$ is not 0 if and only if $g \in \overline{P}(k)P_V(k)$.

**Proof.** It is clear that the non vanishing condition on $g$ depends only on $\overline{P}(k)gP_V(k)$ and that the image is not 0 when $g = 1$. We prove that the image of $gV^{U(k)}$ in $V^{\overline{U}(k)}$ is 0 when $g$ does not belong to $\overline{P}(k)P_V(k)$.

a) We reduce to the case where $G_{der}$ is simply connected by choosing a $z$-extension defined over $k$,

$$1 \to R \to G_1 \to G \to 1,$$
where $R \subset G_1$ is a central induced $k$-subtorus and $G_1$ is a reductive connected $k$-group with $G_{1,\text{der}}$ simply connected. The sequence of rational points

$$1 \rightarrow R(k) \rightarrow G_1(k) \rightarrow G(k) \rightarrow 1$$

is exact. The parabolic subgroups of $G_1$ inflated from $P, P'$ are $P_1 = M_1N, P'_1 = M'_1N'$ where $1 \rightarrow R \rightarrow M_1 \rightarrow M \rightarrow 1$ and $1 \rightarrow R \rightarrow M'_1 \rightarrow M' \rightarrow 1$ are $z$-extensions defined over $k$. We consider $V$ as an irreducible representation of $G_1(k)$ where $R(k)$ acts trivially. The image of $G_1(k) - \mathcal{P}_1(k)P'_1(k)$ in $G(k)$ is $G(k) - \mathcal{P}(k)P'_1(k)$. For $g_1 \in G_1(k) - \mathcal{P}_1(k)P'_1(k)$ of image $g \in G(k) - \mathcal{P}(k)P'_1(k)$, the image of $g_1V^{N'(k)}$ in $V_{N'(k)}$ is 0 if and only if the image of $gV^{N'(k)}$ in $V_{N(k)}$ is 0.

b) The proposition can be reformulated in terms of Weyl groups because the equality depends only on the image of $g$ in $\mathcal{P}(k)\backslash G(k)/P'(k) = W_M\backslash W/W_{M'}$. We denote $\dot{w}$ a representative of $w \in W$ in $G(k)$. The proposition says that the image of $\dot{w}V^{N'(k)}$ in $V_{N(k)}$ is 0 if $w \in W$ does not belong to $W_MW_{M'}$ under the hypothesis that $W_{\dot{V}} = W_M$ or $W_{\dot{V}} = W_{M'}$ or $W_{\dot{V}} \subset W_M \cap W_{M'}$.

c) We suppose that $G_{\text{der}}$ is simply connected. Then we recall that $V$ is the restriction of an irreducible algebraic representation $F(\nu)$ of $G$ of highest weight $\nu$ equal to a $q$-restricted character of $T$ (?? Appendix 1.3). The stabilizer $W_{\dot{V}}$ of $\nu$ in $W$ is $W_{\dot{V}}$, $F(\nu)^{N}$ is the irreducible algebraic representation $F(\nu)$ of $M$ of highest weight $\nu$, and is equal to the sum of all weight spaces $F(\nu)_{\mu}$ with $\nu - \mu \in \mathbb{Z}\Phi_M$; for $w \in W$, $w\nu$ is a weight of $F(\nu)^{N}$ if and only if $w \in W_MW_{\dot{V}}$. (Herzig Lemma 2.3, and proof of lemma 2.17 in the split case). The space $V^{N'(k)}$ is the restriction of $F(\nu)^{N}$.

We deduce that the decomposition $V = V^{N'(k)}(1 - \mathcal{N}'(k))V$, the weights of $V$ in $V^{N'(k)}$ and the weights in $(1 - \mathcal{N}'(k))V$ are distinct; the weights of $V_{N(k)}$ and of $V^{N'(k)}$ are the same; the image of $wV^{N'(k)}$ in $V_{N(k)}$ is 0 if and only if there exists a weight $\mu$ in $F(\nu)^{N'}$ such that $w(\mu)$ is a weight of $F(\nu)^{N}$.

This implies that, for $g \in G(k)$, the image of $gV^{U(k)}$ in $V_{N(k)}$ is 0 if and only if $g \in \mathcal{P}(k)P_{\dot{V}}(k)$.

\[\mathbb{Q}\]

Corollary 3.18. Let $P' = M'N'$ be another standard parabolic subgroup. The image of $gV^{N'(k)}$ in $V_{N(k)}$ is not 0 if and only if $g \in \mathcal{P}(k)P_{\dot{V}}(k)P'(k)$.

Proof. We have $V^{N'(k)} = \sum_{h \in M'(k)} hV^{U(k)}$ because the right hand side is $N'(k)$-stable and $V^{N'(k)}$ is an irreducible representation of $M'(k)$.

\[\mathbb{Q}\]

Remark 3.19. We have $\mathcal{P}P_{\dot{V}}P' = \mathcal{P}P'$ if and only if $M_V \subset \mathcal{P}P'$. This is true when $V$ is $M$-regular or $M'$-regular. The reverse is true when $P = P'$ but not in general. The property $M_V \subset \mathcal{P}P'$ can be translated into equivalent properties in the Weyl group: $W_{\dot{V}} \subset W_MW_{M'}$, or in the set of simple roots: $\Delta_{\dot{V}} \subset \Delta_M \cup \Delta_{M'}$ and any simple root in $\Delta_{\dot{V}} \cap \Delta_M$ which is not in $\Delta_{M'}$ is orthogonal to any simple root in $\Delta_{\dot{V}} \cap \Delta_{M'}$ which is not in $\Delta_M$.

In our study of Hecke operators we will use the following particular case:

Corollary 3.20. i. The restriction to $V_{N(k)}$ of the quotient map $V \rightarrow V_{N(k)}$ is an isomorphism.

ii. For $g \in G(k)$, the image of $gV_{N(k)}$ in $V_{N(k)}$ is not 0 if and only if $g \in P(k)\mathcal{P}(k)\mathcal{P}(k)$.
4 Representations of $G(F)$

4.1 Notations

Let $C$ be an algebraically closed field of positive characteristic $p$, let $F$ be a local non-archimedean field of finite residue field $k$ of characteristic $p$ and of ring of integers $o_F$ and uniformizer $p_F$, and let $G$ be a reductive connected group over $F$. We fix a minimal parabolic $F$-subgroup $B$ of $G$ with unipotent radical $U$ and maximal $F$-split $F$-subtorus $S$. The group $B$ has the Levi decomposition $B = ZU$ where $Z$ is the $G$-centralizer of $S$. Let $\Phi(S,U)$ be the set of roots of $S$ in $U$ (called positive for $U$) and $\Delta \subset \Phi(S,U)$ the subset of simple roots. A parabolic $k$-subgroup $P$ of $G$ containing $B$ is called standard (for $U$), and has a unique Levi decomposition $P = MN$ with Levi subgroup $M$ containing $Z$ (called standard), and unipotent radical $N = P \cap U$. The group $(M \cap B) = Z(M \cap U)$ is a minimal parabolic $F$-subgroup of $M$ and $\Delta_M = \Delta \cap \Phi(S,M \cap U)$ are the simple roots of $\Phi(S,M \cap U)$. This determines a bijection between the subsets of $\Delta$, the standard parabolic $k$-subgroups of $G$, and their standard Levi subgroups.

The natural homomorphism $v : S(F) \to \text{Hom}(X^*(S), \mathbb{Z})$, where $X^*(S)$ is the group of $F$-characters of $S$, extends uniquely to an homomorphism $v : Z(F) \to \text{Hom}(X^*(S), \mathbb{Q})$ with kernel the maximal compact subgroup of $Z(F)$. For a standard Levi subgroup $M$, we denote by $Z(F)^{+M}$ the monoid of elements $z$ in $Z(F)$ which are $M$-positive, i.e.

$$a(v_Z(z)) \geq 0 \text{ for all } a \in \Delta - \Delta_M.$$  

When these inequalities are strict, $z$ is called strictly $M$-positive. Analogously we define the monoid $Z(F)^{-M}$ of elements in $Z(F)$ which are $M$-negative, and the strictly $M$-negative elements.

Let $\overline{B} = Z\overline{U}$ be the opposite parabolic subgroup of $B$ of unipotent radical $\overline{U}$. The standard Levi subgroups for $U$ and for $\overline{U}$ are the same. The roots of $S$ in $\overline{U}$ are the positive roots for $\overline{U}$ and the negative roots for $U$; the set $\overline{\Delta}$ of simple positive roots for $\overline{U}$ is the set $-\Delta$ of simple negative roots for $U$. The monoid $Z(F)^{+M}$ of elements in $Z(F)$ which are $M$-positive for $U$ is the set of elements in $Z(F)$ which are $M$-negative for $\overline{U}$.

In the building of the adjoint group $G_{\text{ad}}$ over $F$ we choose a special vertex in the apartment attached to $S$ and we write $K$ for the corresponding special parahoric subgroup, as in [HV] 6.1. The quotient of $K$ by its pro-$p$-radical $K_+$ is the group of $k$-points of a connected reductive $k$-group $G_k$. The group $K/K_+$ is $G_k(k)$. For $H = B, S, U, Z, P, M, N$, the image in $G_k(k)$ of $H(F) \cap K$ is the group of $k$-points of a connected $k$-group $H_k$. Note that $B_k$ is a minimal parabolic subgroup of $G_k$, $S_k$ is a maximal $k$-split torus in $B_k$, $Z_k$ being the centralizer of $S_k$ in $G_k$, is a maximal $k$-subtorus of $B_k$, $B_k = Z_kU_k$ is a Levi decomposition, there is a bijection between $\Delta$ and the set $\Delta_k$ of simple roots of $S_k$ (with respect to $U_k$), $P_k$ is a standard parabolic subgroup of $G_k$, of standard Levi subgroup $M_k$ and unipotent radical $N_k$, the set $\Delta_{k,M_k}$ of simple roots of $S_k$ in $M_k$ is the image of $\Delta_M$ by the bijection above. We shall usually suppress the indices $k$ from the notation, write $H_0 = H(F) \cap K$. With the notations of the chapter on representations of $G(k)$, we have $T(k) = Z(k)$.

We now fix $V$ an irreducible $C$-representation of $G(k)$ of parameters $(\psi_V, \Delta_V)$ (Definition 3.9), a standard parabolic subgroup $P = MN$ different from $G$ and an element $s \in S(F)$ which is central in $M(F)$ and strictly $M$-positive.

4.2 $S'$ is a localisation

We see also $V$ as a smooth $C$-representation of $K$, trivial on $K_+$. We apply Proposition 2.1 to the group $G(F)$, the compact subgroup $K$, and the closed subgroup $P(F) =$
Definition 4.2. A ring morphism $ \phi : A \to B $ is a localisation at $ b \in B $ if $ f $ is injective, $ b \in f(A) $ is central and invertible in $ B $, and $ B = f(A)[b^{-1}] $.

There exists a Hecke operator $ T_Z $ central in $ \mathcal{H}(Z(F)^+ \cap Z_V^+, T_0, V_{U(k)}) $ of support $ Z_0s $ such that $ T_Z(s) = 1 $, because $ s $ is positive and belongs to $ S(F) $ contained in $ Z_V^+ $. The algebra $ \mathcal{H}(Z(F)^+ \cap Z_V^+, T_0, V_{U(k)}) $ is obtained from the algebra $ \mathcal{H}(Z(F)^+ \cap Z_V^+, Z_0, V_{U(k)}) $ by inverting the Hecke operator $ T_Z $ because, for any $ M $-positive element $ z \in Z(F) $ there exists a positive integer $ n $ such that $ s^n z $ belongs to $ Z(F)^+ $, because $ s \in S(F) $ is strictly $ M $-positive.

There exists a unique Hecke operator in $ \mathcal{H}(M(F), M_0, V_{N(k)}) $ of support $ M_0s $ with value $ \text{id}_{V_{N(k)}} $ at $ s $, because $ s $ is central in $ M(F) $ and contained in $ Z_V^+ $.

Definition 4.3. We denote by $ T_M $ the Hecke operator in $ \mathcal{H}(M(F), M_0, V_{N(k)}) $ with support $ M_0s $ and value $ \text{id}_{V_{N(k)}} $ at $ s $. 

$ M(F) \cap N(F) $. As $ K $ is a special parahoric subgroup, the Iwasawa decomposition $ G(F) = P(F) \cap K $ is valid. We get a $ G(F) $-equivariant linear map

$$ I_0 : \text{c-Ind}_{K}^{G(F)} V \to \text{Ind}_{P(F)}^{G(F)} (\text{c-Ind}_{M_0}^{M(F)} V_{N(k)}) $$

which satisfies $ I_0(bf) = S'(b)I_0(f) $ for $ b \in \mathcal{H}(G(F), K, V) $, $ f \in \text{c-Ind}_{K}^{G(F)} V $, for an algebra homomorphism

$$ S' = S'_{M,G} : \mathcal{H}(G(F), K, V) \to \mathcal{H}(M(F), M_0, V_{N(k)}) $$

given by Proposition 2.3. To study the intertwiner homomorphism $ \mathcal{H}(8) $ which satisfies $ \mathcal{H}(S) $, we need to know more about the morphism $ S' $. We use the Satake morphism $ S $ and Proposition 2.4. We denote by $ S'_{S} $ and $ S_{G} $ the morphisms $ S'_{S} $ and $ S_{G} $ in Proposition 2.3 when $ M = Z $. We analogously define $ S'_{M} $ and $ S_{M} $ with a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{H}(M, M_0, (V^*)_0) & \overset{S_{M}}{\longrightarrow} & \mathcal{H}(Z, Z_0, (V^*)_0) \\
\downarrow & & \downarrow \\
\mathcal{H}(M, M_0, V_{N(k)}) & \overset{S'_{M}}{\longrightarrow} & \mathcal{H}(Z, Z_0, V_{U(k)})
\end{array}
$$

By Proposition 2.4 the morphism $ S' $ is injective and

$$ S'_{G} = S'_{M} \circ S' $$

because the Satake morphism $ S $ is injective and satisfies $ S_{G} = S_{M} \circ S $.

We see $ \psi_{V^*} $ as a smooth character of $ Z_0 $ (Lemma 3.13). Let $ Z_{V^*} $ be the stabilizer of $ \psi_{V^*} $ in $ Z(F) $,

$$ Z_{V^*} = \{ z \in Z(F) \} \mid \psi_{V^*}(z x z^{-1}) = \psi_{V^*}(x) \text{ for all } x \in Z_0 \}. $$

Proposition 4.1. The image of the map $ S'_{G} : \mathcal{H}(G(F), K, V) \to \mathcal{H}(Z(F), Z_0, V_{U(k)}) $ is equal to $ \mathcal{H}(Z(F)^+ \cap Z_{V^*}, Z_0, V_{U(k)}) $.

Proof. The image of $ S'_{G} $ is $ \mathcal{H}(Z(F)^- \cap Z_{V^*}, Z_0, (V^*)_0) $ [HV]. Use Proposition 2.3. 

Analogously, the image of $ S'_{M} $ is $ \mathcal{H}(Z(F)^+ \cap Z_{V^*}, Z_0, V_{U(k)}) $.

}\]
The Hecke operator $T_M$ is central and invertible in $\mathcal{H}(M(F), M_0, V_{N(k)})$; it acts on $\text{c-Ind}^{M(F)}_{M_0} V_{N(k)}$ by $T_M([1, \mathfrak{m}]_{M_0}) = s^{-1}[1, \mathfrak{m}]_{M_0}$ for $v \in V$.

We also denote by $T_M$ the $G(F)$-homomorphism of $\text{Ind}^{G(F)}_{P(F)}(\text{c-Ind}^{M(F)}_{M_0} V_{N(k)})$, such that $T_M(f)(g) = T_M(f(g))$ for $f \in \text{Ind}^{G(F)}_{P(F)}(\text{c-Ind}^{M(F)}_{M_0} V_{N(k)})$ and $g \in G(F)$.

Using Proposition 4.4 we see that

\begin{equation}
S'_M(T_M) = T_Z,
\end{equation}

because $(U \cap M)(F)z \cap M_0 s = ((U \cap M)(F)zs^{-1} \cap M_0)s = (U_0 \cap M_0)z s^{-1}$ if $zs^{-1} \in Z_0$ and is 0 otherwise. The Hecke operator $T_M$ belongs to the image of $S'$, because $T_Z$ belongs to the image of $S'_M$ by construction, $S'$ is injective and we have (9), (8). We have shown:

**Proposition 4.4.** The map $S'$ is a localisation at $T_M$.

In (9), we consider the map $I_0$ as a $C[T]$-linear map, $T$ acting on the left side by $(S')^{-1}(T_M)$ and on the right side by $T_M$. By Proposition 4.4 the localisation of $I_0$ at $T$ is the $G(F)$ and $\mathcal{H}(M(F), M_0, V_{N(k)})$-equivariant map

\begin{equation}
\Theta : \mathcal{H}(M(F), M_0, V_{N(k)}) \otimes_{\mathcal{H}(G(F), K, V)}, S' \text{ c-Ind}^{G(F)}_K V \to \text{Ind}^{G(F)}_{P(F)}(\text{c-Ind}^{M(F)}_{M_0} V_{N(k)}) .
\end{equation}

We will prove that the localisation of $I_0$ at $T$ is an isomorphism when $V$ is $M$-coregular. With Proposition 4.4 this implies our main theorem:

**Theorem 4.5.** $\Theta$ is injective, and $\Theta$ is surjective if and only if $V$ is $M$-coregular.

### 4.3 Decomposition of the intertwiner

To go further, following Herzig, we write the intertwiner $I_0$ as a composite of two $G(F)$-equivariant linear maps

\begin{equation}
\xymatrix{ c\text{-Ind}^{G(F)}_K V \ar[rr]^-{\xi} \ar[rd]_-{I_0} & & \text{Ind}^{G(F)}_{P(F)}(\text{c-Ind}^{M(F)}_{M_0} V_{N(k)}) \ar[ld]^-{\zeta} \\
& \text{c-Ind}^{G(F)}_{P(F)} V_{N(k)} & }
\end{equation}

which we now define. In this diagram, $\mathcal{P}$ is the inverse image in $K$ of $P(k)$; it is a parahoric subgroup of $G(F)$ with an Iwahori decomposition with respect to $M$,

\begin{equation}
\mathcal{P} = N_0 M_0 \overline{N}_{0,+} , \quad \overline{N}_{0,+} := \overline{N}(F) \cap K_+ .
\end{equation}

The transitivity of the compact induction implies that

\begin{equation}
c\text{-Ind}^{G(F)}_P V_{N(k)} = c\text{-Ind}^{G(F)}_K (c\text{-Ind}^{G(k)}_{P(k)} V_{N(k)}) .
\end{equation}

**Definition 4.6.** The map $\xi$ is the image by the compact induction functor $c\text{-Ind}^{G(F)}_K$ of the natural embedding $V \to c\text{-Ind}^{G(F)}_{P(F)} V_{N(k)}$. For $v \in V$, $\xi([1, v]_K)$ is the function in $c\text{-Ind}^{G(F)}_P V_{N(k)}$ of support contained in $K$ and value $[1, kv]_P$ at $k \in K$.

The map $\zeta$ sends $[1, \mathfrak{m}]_P$, for $v \in V$, to the function in $\text{Ind}^{G(F)}_{P(F)}(\text{c-Ind}^{M(F)}_{M_0} V_{N(k)})$ of support contained in $P(F)\mathcal{P} = P(F)\overline{N}_{0,+}$ and is the constant function with value $[1, \mathfrak{m}]_{M_0}$ on $\overline{N}_{0,+}$.
Remark 4.7. Later we will use that, for \( g \in G(F) \), \( \zeta(g^{-1}[1,\pi]_p) \) has support in \( P(F)P \) which contains 1 if and only if \( g \in \mathcal{P}P(F) \). Consequently, for \( f \in c\text{-Ind}_p^{G(F)} V_{N(k)} \), the element \( \zeta(f)(1) \) depends only on the restriction of \( f \) to \( \mathcal{P}P(F) \).

Lemma 4.8. \( I_0 = \zeta \circ \xi \).

Proof. This is clear on the definitions of \( I_0, \xi, \zeta \).

Lemma 4.9. The map \( \xi \) is injective.

Proof. As \( V \) is irreducible and \( V_{N(k)} \neq 0 \), the map \( V \to c\text{-Ind}_p^{G(k)} V_{N(k)} \) is injective. As the functor \( c\text{-Ind}_p^{G} \) is exact, the map \( \xi \) is injective.

As \( P \neq G \), we have
\[
c\text{-Ind}_k^{G(F)} V \neq c\text{-Ind}_p^{G(F)} V_{N(k)} ,
\]
hence \( \xi \) is not surjective.

5 Hecke operators

In this chapter we introduce Hecke operators associated to our fixed element \( s \in S(F) \) central in \( M(F) \) and strictly \( M \)-positive, and we show the compatibility of these Hecke operators with the maps \( \xi, \zeta, S' \) (sometimes we need to suppose that \( V \) is \( M \)-coregular).

The space of \( G(F) \)-equivariant homomorphisms from \( c\text{-Ind}_k^{G(F)} V \) to \( c\text{-Ind}_p^{G(F)} V_{N(k)} \), is isomorphic to the space \( \mathcal{H}(G(F), P, K, V, V_{N(k)}) \) of functions \( \Phi : G(F) \to \text{End}_C(V, V_{N(k)}) \) satisfying
(i) \( \Phi(jgj') = j \circ \Phi \circ j' \) for \( j \in P, j' \in K \),
(ii) \( \Phi \) vanishes outside finitely many double cosets \( P\backslash K \).

We call \( \Phi \) an Hecke operator. We shall usually use the same notation for the Hecke operator and for the corresponding \( G(F) \)-equivariant homomorphism, defined by: for all \( v \in V \),
\[
[1, v]_K = \sum_{g \in \mathcal{P} \backslash G(F)} g^{-1}[1, \Phi(g)(v)]_P .
\]
The map \( \xi \) corresponds to the Hecke operator of support \( K \) and value at 1 the projection \( v \mapsto \pi : V \to V_{N(k)} \).

In the same way, the space of \( G(F) \)-equivariant homomorphisms \( c\text{-Ind}_p^{G(F)} V_{N(k)} \to c\text{-Ind}_k^{G(F)} V \), corresponds to a space \( \mathcal{H}(G(F), K, \mathcal{P}V_{N(k)}, V) \) of functions \( G(F) \to \text{Hom}_C(V_{N(k)}, V) \).

5.1 Definition of Hecke operators

Definition 5.1. We denote by \( T_G \) the Hecke operator in \( \mathcal{H}(G(F), K, V) \) with support \( K \backslash sK \) such that \( T_G(s) \in \text{End}_C(V) \) is the natural projector of image \( V^S_{(k)} \), factorizing by the quotient map \( V \to V_{N(k)} \) (Proposition \( 5.7 \)).

This Hecke operator exists (\cite{HV} 7.3 Lemma 1), because \( s \in S(F) \) is positive and belongs to \( Z_{\mathcal{L}} \). The Hecke operator \( T_M \) could have been defined in the same way as \( T_G \). We shall prove later that \( S'(T_G) = T_M \) when \( V \) is \( M \)-coregular.

We define now Hecke operators \( T_P \) in \( \mathcal{H}(G(F), P, V_{N(k)}) \) and \( T_{K, P} \) in \( \mathcal{H}(G(F), K, \mathcal{P}V_{N(k)}, V) \) generalizing the Hecke operators \( T_G \) and \( T_M \).
Proposition 5.2. (i) There exists a unique Hecke operator $T_P$ in $\mathcal{H}(G(F), \mathcal{P}, V_{N(k)})$ with support $\mathcal{P}s\mathcal{P}$ and value at $s$ the identity of $V_{N(k)}$.

(ii) There exists a unique Hecke operator $T_{K,P}$ in $\mathcal{H}(G(F), K, \mathcal{P}, V_{N(k)}, V)$ with support $Ks\mathcal{P}$ such that $T_{K,P}(s) : V_{N(k)} \to V$ is given by the isomorphism $\varphi : V_{N(k)} \to V_{\mathcal{N}(k)}$ deduced from Proposition 3.15.

Proof. (i) By the condition (i) for Hecke operators, it suffices to check that for $h, h' \in \mathcal{P}$, the relation $hs = sh'$ implies that the actions of $h$ and of $h'$ on $V_{N(k)}$ are the same. By the Iwahori decomposition, we have

$$sPs^{-1} = sN_0M_0N_0s^{-1} = sN_0s^{-1}M_0sN_0s^{-1}$$

as $s$ is central in $M(F)$, and $h$ and $h'$ have the same component in $M_0$.

(ii) It suffices to check that for $h \in K, h' \in \mathcal{P}$, the relation $hs = sh'$ implies that $h'(\varphi(\mathcal{P})) = \varphi(h(\mathcal{P}))$ for all $v \in V$. As $s$ is central in $M(F)$ and strictly $M$-positive we have

$$sPs^{-1} \subset N_0sN_0s^{-1}$$

and $K \cap sPs^{-1} \subset N_0sM_0N_0$.

The elements $h \in N_0sM_0N_0$ and $h'$ have the same component in $M_0$. \qed

5.2 Compatibilities between Hecke operators

In this section, we prove the following result:

Proposition 5.3. i. The left diagram

$$\begin{array}{ccc}
c-\text{Ind}^{G(F)}_K V & \xrightarrow{\xi} & c-\text{Ind}^{G(F)}_P V_{N(k)} \\
T_G & & T_{K,P} \\
c-\text{Ind}^{G(F)}_K V & & c-\text{Ind}^{G(F)}_K V \\
\end{array}$$

is commutative; the right diagram is commutative when $V$ is $M$-coregular.

ii. The diagram

$$\begin{array}{ccc}
c-\text{Ind}^{G(F)}_P V_{N(k)} & \xrightarrow{\xi} & \text{Ind}^{G(F)}_{P(F)} (c-\text{Ind}^{M(F)}_{M_0} V_{N(k)}) \\
T_P & & T_M \\
c-\text{Ind}^{G(F)}_P V_{N(k)} & & c-\text{Ind}^{G(F)}_P V_{N(k)} \\
\end{array}$$

is commutative.

iii. $S'(T_G) = T_M$ when $V$ is $M$-coregular.

By the $(G(F))$-homomorphisms corresponding to $\xi, T_G, T_P$ and $T_{K,P}$, satisfy for $v \in V$,

$$\xi : [1, v]_K \mapsto \sum_{g \in \mathcal{P}\setminus K} g^{-1}[1, g\mathcal{P}]_\mathcal{P},$$

$$T_G : [1, v]_K \mapsto \sum_{g \in K\setminus KsK} g^{-1}[1, T_G(g)(v)]_K,$$

$$T_P : [1, \mathcal{P}]_\mathcal{P} \mapsto \sum_{g \in \mathcal{P}\setminus \mathcal{P}s\mathcal{P}} g^{-1}[1, T_P(g)(\mathcal{P})]_\mathcal{P},$$

$$T_{K,P} : [1, \mathcal{P}]_\mathcal{P} \mapsto \sum_{g \in K\setminus KsK} g^{-1}[1, T_{K,P}(g)(\mathcal{P})]_K.$$
The formula for $T_P$ and for $T_{K,P}$ simplify, using (12):

$$PsP = P_{sN_0+} \quad \text{and} \quad KsP = KsN_0+,$$
and, for $g$ in $sN_0+$, we have $T_P(g)(\overline{\pi}) = \overline{\pi}$ and $T_{K,P}(g)(\overline{\pi}) = \varphi(\overline{\pi})$ by the property (i) of the Hecke operators, because this is true for $g = s$ and $N_0+$ acts trivially on $V_{N(k)}$.

The formula for $T_G$ also simplifies: clearly the surjective map $h \mapsto sh : K \to sK$ induces a bijection

$$(K \cap s^{-1}Ks) \setminus K \to K \setminus KsK.$$  

We remark that $K \cap s^{-1}Ks$ is contained in $P$ (HV 6.13 Proposition) and that the inclusion $N_{0+} \subset P$ induces a bijection

$$s^{-1}N_0s \setminus N_{0+} \to (K \cap s^{-1}Ks) \setminus P.$$  

This is a consequence of the Iwahori decomposition (12) and of the fact that $s$ is strictly $M$-positive. The group $N_{0+}$ acts trivially on $V$ and $T_G(s)(v) = \varphi(\overline{\pi})$ for $v \in V$.

We deduce that:

$$T_P : [1,\overline{\pi}]_P \mapsto \sum_{\overline{\pi} \in s^{-1}N_0s \setminus N_{0+}} \overline{\pi}^{-1}s^{-1}[1,\overline{\pi}]_P ,$$  

$$T_{K,P} : [1,\overline{\pi}]_P \mapsto \sum_{\overline{\pi} \in s^{-1}N_0s \setminus N_{0+}} \overline{\pi}^{-1}s^{-1}[1,\varphi(\overline{\pi})]_K ,$$  

$$T_G : [1,v]_K \mapsto \sum_{h \in P \setminus K} h^{-1} \sum_{\overline{\pi} \in s^{-1}N_0s \setminus N_{0+}} \overline{\pi}^{-1}s^{-1}[1,\varphi(\overline{hv})]_K .$$

$T_P([1,\overline{\pi}]_P)$ is the function in $c\text{-Ind}_P^{G(F)} V_{N(k)}$ of support $PsP$ equal to $\overline{\pi}$ on $sN_0+$, $T_{K,P}([1,\overline{\pi}]_P)$ is the function in $c\text{-Ind}_K^{G(F)} V$ of support $KsP$ equal to $\varphi(\overline{\pi})$ on $sN_0+$, $T_G([1,v]_K)$ is the function in $c\text{-Ind}_K^{G(F)} V$ of support contained in $KsK$ equal to $\varphi(\overline{hv})$ on $sh$ for all $h \in K$.

We see on these formula that the left diagram in i is commutative:

$$T_G = T_{K,P} \circ \xi .$$

When $v$ lies in $V_{\overline{N}(k)}$, $\varphi$ disappears from the formula of $T_{K,P}([1,\overline{\pi}]_P)$, because $\varphi(\overline{\pi}) = v$, hence:

$$T_{K,P}([1,\overline{\pi}]_P) = \sum_{\overline{\pi} \in s^{-1}N_0s \setminus N_{0+}} \overline{\pi}^{-1}s^{-1}[1,\overline{\pi}]_K .$$

**Remark 5.4.** When $v \in V_{\overline{N}(k)}$ and $g \in G(k)$ we have $\overline{gv} \neq 0$ if and only if $g \in \overline{P}(k)\overline{P}_V(k)$ (Corollary 5.20). We have $\overline{P}(k)\overline{P}_V(k) = M(k)\overline{P}_V(k)$. The inverse image in $K$ of $\overline{P}_V(k)$ is a parahoric subgroup $\overline{P}_V$ acting on $V_{\overline{N}(k)}$ by a character that we still denote $\overline{\psi}_V$. For $h \in \overline{P}_V(k)$ we have $hv = \overline{\psi}_V(h)v$ and $\varphi(hv) = \overline{\psi}_V(h)v$. In the formula for $\xi([1,v]_K)$ or $T_G([1,v]_K)$, we can replace the sum over $P \setminus K$ by a sum over $P \cap \overline{P}_V \setminus \overline{P}_V$, and we obtain for $v \in V_{\overline{N}(k)}$:

$$\xi([1,v]_K) = \sum_{h \in P \cap \overline{P}_V \setminus \overline{P}_V} \overline{\psi}_V(h)h^{-1}[1,\overline{\pi}]_P ,$$

$$T_G([1,v]_K) = \sum_{h \in P \cap \overline{P}_V \setminus \overline{P}_V} \overline{\psi}_V(h)h^{-1} \sum_{\overline{\pi} \in s^{-1}N_0s \setminus N_{0+}} \overline{\pi}^{-1}s^{-1}[1,v]_K .$$
Remark 5.5. When \(v \in V^{\overline{N}(k)}\) and \(V\) is also the function \(\xi([1,v])\) with value \([1,\nu]\_p\), hence under our hypothesis on \((v, V)\):

\[T_G([1,v]_K) = \text{the function in } c\text{-Ind}_K^{G(F)} V \text{ of support contained in } Ks\overline{N}_0 = v \text{ on } s\overline{N}_0,\]

\[(26) \quad \xi([1,v]_K) = \sum_{\nu \in N_{0+} \setminus \overline{N}_0} \nu^{-1}[1,\nu]_p,\]

\[(27) \quad T_G([1,v]_K) = \sum_{\nu \in s^{-1}N_{0+} \setminus \overline{N}_0} \nu^{-1}s^{-1}[1,v]_K.\]

\[(28) \quad (\xi \circ T_K)_p([1,\nu]_p) = \sum_{\nu \in s^{-1}N_{0+} \setminus \overline{N}_0} \nu^{-1} \sum_{\nu \in N_{0+} \setminus \overline{N}_0} \nu^{-1}[1,\nu]_p = \sum_{\nu \in s^{-1}N_{0+} \setminus \overline{N}_0} \nu^{-1}s^{-1}[1,\nu]_p.\]

Comparing (19) and (28) we see that:

\[(29) \quad T_p = \xi \circ T_K, p.\]

When \(V\) is \(M\)-coregular, the right diagram in i is commutative.

Remark 5.5. When \(v \in V^{\overline{N}(k)}\) and \(V\) is \(M\)-coregular, we compute easily:

\[(\xi \circ T_G)([1,v]_K) = \sum_{\nu \in s^{-1}N_{0+} \setminus \overline{N}_0} \nu^{-1}s^{-1} \sum_{\nu \in N_{0+} \setminus \overline{N}_0} \nu^{-1}[1,\nu]_p = \sum_{\nu \in s^{-1}N_{0+} \setminus \overline{N}_0} \nu^{-1}s^{-1}[1,\nu]_p,\]

\[(T_p \circ \xi)([1,v]_K) = \sum_{\nu \in N_{0+} \setminus \overline{N}_0} \nu^{-1} \sum_{\nu \in s^{-1}N_{0+} \setminus \overline{N}_0} \nu^{-1}s^{-1}[1,\nu]_p = \sum_{\nu \in s^{-1}N_{0+} \setminus \overline{N}_0} \nu^{-1}s^{-1}[1,\nu]_p,\]

\[(T_K, p \circ \xi)([1,v]_K) = \sum_{\nu \in N_{0+} \setminus \overline{N}_0} \nu^{-1} \sum_{\nu \in s^{-1}N_{0+} \setminus \overline{N}_0} \nu^{-1}s^{-1}[1,v]_K = \sum_{\nu \in s^{-1}N_{0+} \setminus \overline{N}_0} \nu^{-1}s^{-1}[1,v]_K,\]

We consider now the diagram ii. with \(\zeta\), without restriction on \(V\). We have

\[(30) \quad \zeta \circ T_p = T_M \circ \zeta\]

because:

\[(T_M \circ \zeta)([1,\nu]_p)\] is the function \(f_p\) of support \(P^\nu_{0+}\) and constant on \(\overline{N}_{0+}\) with value \(s^{-1}[1,\nu]_{M_0}\), because \((\xi([1,\nu])_p)\) is the function \(f_p\) of support \(P\overline{N}_{0+}\) and constant on \(\overline{N}_{0+}\) with value \([1,\nu]_{M_0}\) and \(T_M([1,\nu]_{M_0}) = s^{-1}[1,\nu]_{M_0}\).

By (19), \((\zeta \circ T_p)([1,\nu]_p) = \sum_{\nu \in s^{-1}N_{0+} \setminus \overline{N}_{0+}} \nu^{-1}s^{-1} \zeta([1,\nu]_p).\) Hence \((\zeta \circ T_p)([1,\nu]_p)\) is also the function \(f_{\nu}\) of support \(P\overline{N}_{0+}\) and constant on \(\overline{N}_{0+}\) with value \(s^{-1}[1,\nu]_{M_0}\).

Proof of iii. We proved that \(\xi \circ T_{p,K} = T_p\) when \(V\) is \(M\)-coregular. As in general \(T_{p,K} \circ \xi = T_G\), one deduces \(\xi \circ T_G = T_p \circ \xi\). As we always have \(\zeta \circ T_p = T_M \circ \zeta\), we obtain

\[\zeta \circ \xi \circ T_G = \zeta \circ T_p \circ \xi = T_M \circ \zeta \circ \xi.\]
i.e. \( I_0 \circ T_G = T_M \circ I_0 \). This implies \( S'(T_G) = T_M \).

This ends the proof of Proposition 5.3.

We can have \( S'(T_G) = T_M \) even when the representation \( V \) is not \( M \)-coregular. The trivial representation \( V \) is never \( M \)-coregular because \( M \neq G \).

**Remark 5.6.** For any choice of \( s \in M(F) \) strictly \( M \)-positive we have \( S'(T_G) = T_M \), when \( G = GL(2, F) \), \( B = P = MN \) the upper triangular subgroup, \( M \) the diagonal subgroup, \( K = GL(2, o_F) \) and \( V \) the trivial representation of \( GL(2, k) \).

**Proof.** For \( t \in M(F) \), the value of \( S'(1_{KsK}) \) at \( t \) is the image in \( C \) of the integer

\[
 n_s(t) := |\{ b \in F/o_F \mid nbt \text{ in } KsK \} , \quad n_b := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} .
\]

The integer \( n_s(t) \) depends only on \( sM_0 \). We claim that \( n_s(s) = 1 \) and \( n_s(t) \equiv 0 \) modulo \( p \) for \( t \) not in \( sM_0 \); this implies \( S'(T_G) = T_M \). It suffices to check that the claim is true for \( s_p^n \) with

\[
 s_p := \begin{pmatrix} p_F & 0 \\ 0 & 1 \end{pmatrix}
\]

and \( n > 1 \), because \( s \) belongs to \( \cup_{n \geq 1} Z(G)M_0s_p^n \) where \( Z(G) \) is the center of \( G(F) \).

It is well known that the double coset \( Ks_pK \) is a disjoint union of the \( p + 1 \) cosets \( Ks_p \) and \( K \begin{pmatrix} 1 & a \\ 0 & p_F \end{pmatrix} \) for \( a \) in system of representatives of \( o_F/p_Fo_F \), and more generally

\( Ks_p^nK \) is a disjoint union of the cosets \( K \begin{pmatrix} p_F^u & a \\ 0 & p_F^r \end{pmatrix} \) for \( a \in o_F/p_F^r o_F \) and for \( u, r \in \mathbb{N} \) with \( u + r = n \). It is more convenient to write

\[
 \begin{pmatrix} p_F^u & a \\ 0 & p_F^r \end{pmatrix} = n_{c, s_p^{u-r}} \text{ with } s_p^{u-r} := \begin{pmatrix} p_F^u & 0 \\ 0 & p_F^r \end{pmatrix}
\]

for \( c = ap_F^{-r} \in p_F^{-r} o_F/o_F \).

As \( nbt \) and the representatives \( n_{c, s_p^{u-r}} \) of the cosets \( K \setminus Ks_pK \) all belong to \( B(F) \), \( n_{s_p^n}(t) \) is also the number of \( b \in F/o_F \) such that \( nbt \in \cup_{c, u, r} M_0n_{c, s_p^{u-r}} \). Hence \( n_{s_p^n}(t) \neq 0 \) is equivalent to \( t \in M_0s_p^n \) and in this case

\[
 n_{s_p^n}(t) = n_{s_p^n}(s_p^{u-r}) = |p_F^{-r} o_F/o_F| = q^r
\]

is equal to 1 if \( t \in M_0s_p^n \) and is divisible by \( p \) otherwise. \( \square \)

# 6 Main theorem

The main theorem is a corollary of the following proposition:

**Proposition 6.1.** The map \( \xi \) is injective; when \( V \) is \( M \)-coregular, the image of \( \xi \) contains \( T_P(c \text{-Ind}^G_F V_{N(k)}) \).

The kernel of the map \( \zeta \) is the \( T_{F'}^{-} \)-torsion part of \( c \text{-Ind}^G_F V_{N(k)} \) and the representation \( c \text{-Ind}^G_F(c \text{-Ind}^M_F V_{N(k)}) \) is generated by

\[
 (T_{M}^{n} \circ \zeta)([1, T])\quad \text{for all } n \in \mathbb{Z}
\]

for any fixed non-zero element \( T \in V_{N(k)} \).
For the map $\xi$, the proposition follows from (Lemma 1.3 and 29). The next three lemma will be used in the proof for the map $\zeta$.

**Lemma 6.2.** The map $\zeta$ is injective on the set of functions $f \in \text{c-Ind}_P^{G(F)} V_{N(k)}$ with support in $PZ(F)^+M K$.

**Proof.** Let $f$ such that $\zeta(f) = 0$ with support in $PZ(F)^+K$. We claim that $f = 0$ on $PP(F)$. This implies that $f = 0$ because $G(F) = P(F)K$ and for $k \in K$ the function $k^{-1}f$ satisfies the same conditions as $f$. To prove the claim, we use only that $\zeta(f)(1) = 0$ in $\text{c-Ind}_{M_0}^{M_0} V_{N(k)}$. As $\zeta(f)(1)$ depends only on the restriction of $f$ to $PP(F)$, we assume as we may, that the support of $f$ is contained in $PP(F)$. The support of $f$ is a finite disjoint union of $Pz_i k_i$ for $z_i \in (F)^+$ and $k_i \in K$, with $z_i k_i \in PP(F)$. We have $PP(F) = N_{0, +} F P(F)$ hence $k_i \in z_i^{-1} N_{0, +} z_i F(P)$. As $z_i$ is positive, $z_i^{-1} N_{0, +} z_i \subset N_{0, +}$. This implies that we can suppose $k_i \in P(F) \cap K$. As $P(F) \cap K = N_0 M_0$ and $z_i$ is positive, we can suppose $k_i \in M_0$. We proved that the support of $f$ is a finite disjoint union of $Pz_i k_i$ for $z_i \in (F)^+$ and $k_i \in M_0$. Taking the intersection with $M(F)$, the sets $M(F) \cap Pz_i k_i$ are also disjoint. Writing

$$f = \sum_i (z_i k_i)^{-1} [1, v_i] p$$

we have $\zeta(f)(1) = \sum_i (z_i k_i)^{-1} [1, v_i] M_0$, and $\zeta(f)(1) = 0$ is equivalent to $v_i = 0$ for all $i$. \[\square\]

**Lemma 6.3.** (i) A basis of the open compact subsets of the compact space $P(F) \backslash G(F)$ is given by the $G(F)$-translates of $P(F) \backslash P(F) N_{0, +} s^n$, for all $n \in \mathbb{N}$.

(ii) For any subset $X \subset G(F)$ with finite image in $P \backslash G(F)$ there exists a large integer $n \in \mathbb{N}$ such that $s^n X \subset PZ(F)^+M K$.

**Proof.** See Herzig [Herz] Lemma 2.20.

(i) The compact space $P(F) \backslash G(F)$ is the union of the right $G(F)$-translates of the big cell $P(F) \backslash P(F) N(F)$ which is open, the $s^{-n} N_{0, +} s^n$ for $n \in \mathbb{N}$ form a decreasing sequence of open subgroups of $N(F)$ converging to 1.

(ii) Let $\mathcal{N}$ be the normalizer of $S$ in $G$ and let $B$ be the inverse image of $B(k)$ in $K$ (an Iwahori subgroup). Then $(G(F), B, \mathcal{N}(F))$ is a generalized Tits system [HV]. We have:

a) $G(F) = BN(F)B$,

b) for $\nu \in \mathcal{N}(F)$ there a finite subset $X_\nu$ in $\mathcal{N}(F)$ such that, for all $\nu' \in \mathcal{N}(F)$, we have

$$\nu' B \nu \subset \bigcup_{x \in X_\nu} B \nu' x B.$$ 

c) As the parahoric group $K$ is special, for any $\nu \in \mathcal{N}(F)$ there exists $z \in Z(F)$ such that $\nu K = z K$ because $K$ contains representatives of the Weyl group.

We deduce from a) and c) that $G(F) = BN(F)K$. We write, as we may, $X$ as a finite union $X = \bigcup_i P z_i k_i$ with $z_i \in Z(F), k_i \in K$. We deduce from b) that, for any index $i$, there are finitely many $n_{i,j} \in \mathcal{N}(F)$ such that $z B z_i \subset \bigcup_j B z n_{i,j} B$ for all $z \in Z(F)$. It follows that

$$z P z_i k_i \subset P_{0, +} N_{0, +} z_i k_i \subset \bigcup_j P z n_{i,j} K$$

as $N_{0, +} \subset B$. We choose $z_{i,j} \in Z(F)$ such that $z_{i,j} K = n_{i,j} K$, as we may by c). There exists $n \in \mathbb{N}$ such that $s^n z_{i,j} \in Z(F)^{+M}$ for all $i, j$. Hence $s^n X \subset \bigcup_j P s^n z_{i,j} K \subset PZ(F)^{+M} K$.

\[\square\]

Let $\sigma$ be a smooth $C$-representation of $M(F)$. For any non-zero $y \in \sigma$, there exists a function $f_y \in \text{Ind}_{P(F)}^{G(F)} \sigma$ of support $P(F) N_{0, +}$ and value $y$ on $N_{0, +}$ because the multiplication $P(F) \times N_{0, +} \to P(F) N_{0, +}$ is an homeomorphism.
Lemma 6.4. Let \( \sigma \) be a smooth \( C \)-representation of \( M(F) \) generated by an element \( x \). Then the representation \( \text{Ind}_{P(F)}^{G(F)} \sigma \) is generated by the functions \( f_{s^{-n}x} \) of support \( P(F)N_{0+,s} \) and value \( s^{-n}x \) on \( N_{0+,s} \), for all \( n \in \mathbb{Z} \).

Proof. By Lemma 6.3 we reduce to show that any function \( f_{n,mx} \in \text{Ind}_{P(F)}^{G(F)} \sigma \) of support contained in \( P(F)N_{0+,s}x \) equal to \( mx \) on \( N_{0+,s}x \), for \( n \in \mathbb{N} \) and \( m \in M(F) \), is contained in the subrepresentation generated by \( f_{s^{-r}x} \) for all \( r \in \mathbb{Z} \). The function \( m^{-1}f_{n,mx} \) has support in \( P(F)P(F)N_{0+,s}^{n} \) and value \( s^{-n}x \) on the compact open subset \( m^{-1}s^{-n}N_{0+,s}^{n}m \) of \( N(F) \); this set is a finite disjoint union of \( s^{-n}N_{0+,s}^{n}\pi \) with \( \pi \in (N(F) \right) \) and \( n' \in \mathbb{N} \). For a non-zero \( y \in \sigma \), the function \( (s^{n}\pi)^{-1}f_{y} \in \text{Ind}_{P(F)}^{G(F)} \sigma \) has support \( P(F)N_{0+,s}^{n}\pi \) and value \( s^{-n'}y \) on \( s^{-n}N_{0+,s}^{n}\pi \). The sum of \( (s^{n}\pi)^{-1}f_{s^{-n}x} \) is equal to \( m^{-1}f_{n,mx} \).

To analyse the image of \( \zeta \), we take in Lemma 6.3 the representation \( \sigma = \text{c-Ind}_{M_{0}}^{M(F)} V_{N(k)} \) generated by \( x = [1,\overline{\pi}]_{M_{0}} \), for any non-zero fixed \( \pi \in V_{N(k)} \), and we note that for \( n \in \mathbb{Z} \), by definition 4.3 and 4.6,

\[
(T_{M}^{n} \circ \zeta)([1,\overline{\pi}]_{\overline{\pi}}) = f_{s^{-n}x}
\]

We obtain that the representation \( \text{Ind}_{P(F)}^{G(F)} (\text{c-Ind}_{M_{0}}^{M(F)} V_{N(k)}) \) is generated by the elements \((T_{M}^{n} \circ \zeta)([1,\overline{\pi}]_{\overline{\pi}})\) for all \( n \in \mathbb{Z} \).

We consider now an element \( f \) in the kernel of \( \zeta \). The function \( f \) vanishes outside of a compact set \( X \) of finite image in \( P \backslash G(F) \). We choose the integer \( n \in \mathbb{N} \) such that \( s^{n}X \subset PZ(F)^{+}K \) (Lemma 6.3 ii). The support of \( T_{M}^{n} \) is \( Ps^{n}P \) by 12 and the positivity of \( s \). The support of \( T_{P}^{n}(f) \) is contained in \( Ps^{n}X \) hence in \( PZ(F)^{+}K \). By Lemma 6.2 we conclude that \( T_{P}^{n}(f) = 0 \). This ends the proof of Proposition 6.1.

Corollary 6.5. The kernel of \( I_{0} = \zeta \circ \xi \) is the space of \( T_{P}^{\infty} \)-torsion elements in \( \text{c-Ind}_{K}^{G(F)} V \) identified via \( \zeta \) to a subspace of \( \text{c-Ind}_{K}^{G(F)} \text{c-Ind}_{P(k)}^{G(k)} V_{N(k)} \).

In the diagram (11) the representations are \( C[T] \)-modules, where \( T \) acts as on the middle space by \( T_{K,P} \), on the right space by \( T_{M} \) and on the left space by \( (S')^{-1}(T_{M}) \). Proposition 6.3 tells us that:

The map \( \zeta \) is \( C[T] \)-linear.

When \( V \) is \( M \)-coregular, the map \( \xi \) is \( C[T] \)-linear and \( (S')^{-1}(T_{M}) = T_{G} \).

Corollary 6.6. i. The \( T \)-localisation \( \zeta_{T} \) of \( \zeta \) is an isomorphism.

ii. When \( V \) is \( M \)-coregular, the \( T \)-localisation \( \zeta_{T} \) of \( \xi \) is an isomorphism.

The map \( \Theta \) is the \( T \)-localisation of \( I_{0} = \zeta \circ \xi \). By i., the map \( \Theta = \zeta_{T} \circ \xi_{T} \) is an isomorphism if and only if \( \zeta_{T} \) is an isomorphism. The map \( \Theta \) is always injective (as \( \xi \) is injective) and surjective if and only if \( \xi_{T} \) is surjective.

We prove now the converse of Corollary 6.6 ii.

Proposition 6.7. When \( \xi_{T} \) is surjective, \( V \) is \( M \)-coregular.

Proof. 1) Set \( \tau_{G} := (S')^{-1}(T_{M}) \). Par definition, \( I_{0} \circ \tau_{G} = T_{M} \circ I_{0} \), hence

\[
\zeta \circ \xi \circ \tau_{G} = T_{M} \circ \zeta \circ \xi = \zeta \circ T_{P} \circ \xi
\]

As the localisation \( T \) of \( \zeta \) is injective, \( \xi \circ \tau_{G} = T_{P} \circ \xi \) modulo \( T_{P}^{\infty} \)-torsion.

2) The surjectivity of \( \xi_{T} \) means that for all \( f \in \text{c-Ind}_{P(F)}^{G(F)} V_{N(k)} \) there exists an \( n \in \mathbb{N} \) such that \( T_{P}^{n}(f) \) belongs in the image of \( \xi \) (one can change \( n \) by any \( n' \geq n \)). As the representation is generated by \( [1,x]_{P} \) for \( x \in V_{N(k)} \), the hypothesis is that there exists an \( n \in \mathbb{N} \) such that \( T_{P}^{n}([1,x]_{P}) \) belongs in the image of \( \xi \) for all \( x \in V_{N(k)} \). The Hecke operator \( T_{P}^{n} \) is analogous to the Hecke operator \( T_{P} \) but associated to \( s^{n} \) instead of \( s \). Replacing \( s \) by \( s^{n} \) we can work under the hypothesis: \( T_{P}([1,x]_{P}) \) belongs in the image of \( \xi \) for all \( x \in V_{N(k)} \).
3) The support of $T_{p}(1, x|p)$ is contained in $P_{s}P = P_{s}N_{0+}$ and if

$$T_{p}(1, x|p) = \xi(f)$$

for some $f \in c\text{-Ind}^{G(F)}_{K_{F}} V$, the support of $f$ must be contained in $K_{s}P = K_{s}N_{0+}$. Writing $K_{s}P$ as a disjoint union of cosets $K_{s}m_{i}$ with $m_{i} \in N_{0+}$, and $f = \sum_{i} c_{i} [1, v_{i}] K$ for a choice of non-zero $v_{i} \in V$ and a finite set of indices $i$. The equality $(31)$ means that, for each index $i$, $v_{i}$ satisfies the two conditions a) and b): for any $k$ in $K$,

a) if $k s m_{i} \in P_{s}P$, i.e., $k s m_{i} = h s m$ with $h \in P$ and $m \in N_{0+}$, then $h v_{i} = h x$,

b) if $k s m_{i} \notin P_{s}P$ then $h v_{i} = 0$.

4) We show that the condition a) implies that $v_{i} = \phi(x)$ where $\phi(x) \in V_{N(k)}$ lifts $x$.

We have $k = h s m^{-1}_{i} s^{-1}$ and $s m^{-1}_{i} s^{-1} \in \overline{N}(F) \cap K = \overline{N}_{0}$, hence $h \in P_{s} N_{0}$. Conversely if $k = h \nu$ with $h \in P$ and $\nu \in \overline{N}_{0}$, then $k s m^{-1}_{i} = h s s^{-1} \nu s m^{-1}_{i}$ and $s^{-1} \nu s \in N_{0+}$ because $s$ is strictly $M$-positive. The condition a) means that for any $h \in P$ and any $\nu \in \overline{N}_{0}$ we have $h \nu v_{i} = h x$. As $h \in P$ we have $h \nu v_{i} = h m v_{i}$ and the condition a) is equivalent to $m v_{i} = x$ for all $\nu \in \overline{N}_{0}$. Writing $v_{i} = \phi(x) + w_{i}$, the $\overline{N}(k)$-submodule $W$ of $V$ generated by $w_{i}$ is contained in the kernel of $v \mapsto m$. If $W \neq 0$ then $W \overline{N(k)} \neq 0$ and we get a contradiction. Hence $W = 0$ and $v_{i} = \phi(x)$.

5) We interpret now the condition b) which says that if $k$ does not belong to $P_{s}N_{0}$, then $k \phi(x) = 0$, and this for all $x \in V_{N(k)}$. Hence the image of $g \overline{N(k)}$ in $V_{N(k)}$ is 0 for all $g$ not belonging to $P(k) \overline{N(k)}$. By Corollary 3.20 this implies

$$P(k) \overline{N(k)} \subset P(k) \overline{N(k)}$$

hence the $M$-coregularity of $V$ by Corollary 3.19.

This ends the proof of our main theorem (Theorem 4.6).

Remark 6.8. When $V$ has dimension 1 and is given by a character $\epsilon$ of $K$, the map $\Theta$ is not surjective because $V$ is not $M$-coregular as $\overline{P_{V}} = G \neq \overline{P}$. If there exists a character $\epsilon_{M}$ of $M(F)$ equal to $\epsilon$ on $M_{0}$ (such a character $\epsilon_{M}$ does not always exist), one can consider the composite of $I_{0}$ with the surjective natural map

$$\psi : \text{Ind}^{G(F)}_{P(F)}(c\text{-Ind}^{M(F)}_{M_{0}} \epsilon) \to \text{Ind}^{G(F)}_{P(F)} \epsilon_{M}.$$ 

In the case where $\epsilon$ extends to a character $\epsilon_{G}$ of $G(F)$, the image of $\psi \circ I_{0}$ is the subrepresentation $\epsilon_{G}$ of dimension 1 of $\text{Ind}^{G(F)}_{P(F)} \epsilon_{M}$. The map $\psi \circ \Theta$ is also non surjective.

But in the case where $\epsilon$ does not extend to a character $\epsilon_{G}$ of $G(F)$, the map $\psi \circ \Theta$ can be surjective. For example, $\psi \circ \Theta$ is surjective when $\text{Ind}^{G(F)}_{P(F)} \epsilon_{M}$ is irreducible. This is the case, for any choice of $\epsilon_{M}$, when $G = U(2, 1)$ with respect to an unramified quadratic extension of $F$, $B$ is a Borel subgroup and $K$ is a special non hyperspecial parahoric subgroup Remark; this is also the case when $G(F) = GL(2, D)$ with a quaternion skew field over $F$, $B$ is the upper triangular subgroup and $K = GL(2, O_{D})$ [LY].

7 Supersingular representations of $G(F)$

We introduce first the notion of $K$-supersingularity for an irreducible smooth representation $\pi$ of $G(F)$. Then we recall the notion of supercuspidality. We expect that supercuspidality is equivalent to $K$-supersingularity, at least for admissible representations. We will give some partial results in this direction. Finally, when $\pi$ is admissible we give an equivalent definition of $K$-supersingularity which coincides with the definition given by Herzig and Abe when $G$ is $F$-split, $K$ is hyperspecial and the characteristic of $F$ is 0.
Let \( \pi \) be an irreducible smooth \( C \)-representation of \( G(F) \). For any smooth irreducible \( C \)-representation \( V \) of \( K \), we consider
\[
\text{Hom}_{G(F)}(\text{c-Ind}^{G(F)}_{K} V, \pi)
\]
as a right module for the Hecke algebra \( \mathcal{H}(G(F), K, V) \).

**Remark 7.1.** The representation \( \pi|_K \) contains an irreducible subrepresentation \( V \), i.e. by adjunction and the irreducibility of \( \pi \),
\[
\text{Hom}_{G(F)}(\text{c-Ind}^{G(F)}_{K} V, \pi) \neq 0,
\]
because a non-zero element \( v \in \pi \) being fixed by an open subgroup of \( K \), generates a \( K \)-stable subspace of finite dimension, and any finite dimensional smooth \( C \)-representation of \( K \) contains an irreducible subrepresentation.

We recall some elementary facts on localisation.

Let \( f : A \to B \) be an injective ring morphism which is a localisation at \( b \in f(A) \) central and invertible in \( B = f(A)[b^{-1}] \) (Def. 4.2).

A right \( B \)-module \( V \) considered as a right \( A \)-module via \( f \), is called the restriction of \( V \). An homomorphism \( \varphi \) of right \( B \)-modules considered as an homomorphism of right \( A \)-modules is called the restriction of \( \varphi \).

A right \( A \)-module \( V \) induces a right \( B \)-module \( V \otimes_{A,B} B \), called the localisation of \( V \) at \( b \). An homomorphism \( \varphi \) of right \( A \)-modules induces an homomorphism \( \varphi \otimes \text{id} \) of \( B \)-modules called the localisation of \( \varphi \) at \( b \).

A right \( A \)-module where the action of \( f^{-1}(b) \) is invertible is canonically a right \( B \)-module and the homomorphisms \( \text{Hom}_{A}(V, V') \) and \( \text{Hom}_{B}(V, V') \) are the same for such \( A \)-modules \( V \) and \( V' \).

**Lemma 7.2.** The restriction and the localisation at \( b \) are equivalence of categories, inverse to each other, between the category of right \( B \)-modules and the category of right \( A \)-modules where the action of \( f^{-1}(b) \) is invertible.

**Proof.** Clear. \( \square \)

We consider now the localisation
\[
S' = S'_{M,G} : \mathcal{H}(G(F), K, V) \to \mathcal{H}(M(F), M_0, V_{N(k)})
\]
at \( T_M \) (Proposition 4.4).

By Theorem 4.5, the localisation of the left \( \mathcal{H}(G(F), K, V) \)-module \( \text{c-Ind}^{G(F)}_{K} V \) at \( T_M \) is isomorphic to \( \text{Ind}^{G(F)}_{P(F)}(\text{c-Ind}^{M(F)}_{M_0} V_{N(k)}) \) when \( V \) is \( M \)-coregular.

**Definition 7.3.** An irreducible smooth \( C \)-representation \( \pi \) of \( G(F) \) is called \( K \)-supersingular when the localisations of the right \( \mathcal{H}(G(F), K, V) \)-module
\[
\text{Hom}_{G(F)}(\text{c-Ind}^{G(F)}_{K} V, \pi)
\]
at \( T_M \) are 0, for all irreducible smooth \( C \)-representations \( V \) of \( K \) and all standard Levi subgroup \( M \neq G \).

For a given \( M \), the condition means that, for any non-zero \( f \in \text{Hom}_{G(F)}(\text{c-Ind}^{G(F)}_{K} V, \pi) \) there exists \( n \in \mathbb{N} \) such that \( S'^{-1}(T^M_M)(f) = 0 \). The condition does not depend on the choice of \( T_M \), as it is equivalent to :
\[
\mathcal{H}(M(F), M_0, V_{N(k)}) \otimes_{\mathcal{H}(G(F), K, V), S'} \text{Hom}_{G(F)}(\text{c-Ind}^{G(F)}_{K} V, \pi) = 0.
\]

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Definition 7.4. An irreducible smooth $C$-representation $\pi$ of $G(F)$ is called supercuspidal, if $\pi$ is not isomorphic to a subquotient of $c\text{-Ind}^{G(F)}_{P(F)} \tau$ for irreducible smooth $C$-representation $\tau$ of $M(F)$ where $M \neq G$.

The definition does not depend on the minimal parabolic $F$-subgroup $B$ of $G$ used to define the standard parabolic subgroups, as all such $B$’s are conjugate in $G(F)$.

Let $V$ be an irreducible smooth $C$-representation of $K$ and let $\sigma$ be a smooth $C$-representation of $M(F)$ for some standard Levi subgroup $M \neq G$. Our first result concerns the $T_M$-localisation of the right $\mathcal{H}(G(F), K, V)$-module

$$\text{Hom}_{G(F)}(c\text{-Ind}^{G(F)}_K V, \text{Ind}^{G(F)}_{P(F)} \sigma) .$$

Proposition 7.5. i. $V \subset (\text{Ind}^{G(F)}_{P(F)} \sigma)|_K$ if and only if $V_{N(k)} \subset \sigma|_{M_0}$.

ii. In this case, the action of $S'^{-1}(T_M)$ on $\text{Hom}_{G(F)}(c\text{-Ind}^{G(F)}_K V, \text{Ind}^{G(F)}_{P(F)} \sigma)$ is invertible.

Proof. i follows from the Frobenius adjunction isomorphism

$$\text{Hom}_K(V, \text{Ind}^K_{P_0} \sigma) \to \text{Hom}_{M_0}(V_{N(k)}, \sigma) .$$

ii follows from Proposition 2.4.

Our results on the comparison between $K$-supersingular and supercuspidal irreducible smooth $C$-representations of $G(F)$ are:

Theorem 7.6. Let $M \neq G$ be a standard Levi $F$-subgroup and let $\tau$ be an irreducible smooth $C$-representation of $M(F)$.

i. An irreducible subrepresentation of $\text{Ind}^{G(F)}_{P(F)} \tau$ is not $K$-supersingular.

ii. An admissible irreducible quotient of $\text{Ind}^{G(F)}_{P(F)} \tau$ is not $K$-supersingular.

iii. An admissible irreducible smooth $C$-representation $\pi$ of $G(F)$ such that the localisation of the right $\mathcal{H}(G(F), K, V)$-module

$$\text{Hom}_{G(F)}(c\text{-Ind}^{G(F)}_K V, \pi)$$

at $T_M$ is not 0 for some $L$-coregular irreducible subrepresentation $V$ of $\pi|_K$ and some standard Levi subgroup $M \subset L \neq G$, is not supercuspidal.

Proof. i. The last proposition implies that an irreducible subrepresentation of $\text{Ind}^{G(F)}_{P(F)} \tau$ is not $K$-supersingular.

ii. Let $\pi$ be an irreducible quotient of $\text{Ind}^{G(F)}_{P(F)} \tau$. We choose an irreducible smooth $C$-representation $W$ of $M_0$ such that the irreducible representation $\tau$ is a quotient of $c\text{-Ind}^{M(F)}_{M_0} W$. Then $\pi$ is a quotient of $\text{Ind}^{G(F)}_{P(F)}(c\text{-Ind}^{M(F)}_{M_0} W)$. We consider the unique irreducible $M$-coregular representation $V$ of $G(k)$ such that $V_{N(k)} \simeq W$ (Proposition 3.11). By our main theorem (Theorem 4.5):

$$\text{Ind}^{G(F)}_{P(F)}(c\text{-Ind}^{M(F)}_{M_0} W) \simeq \mathcal{H}(M(F), M_0, V_{N(k)}) \otimes_{\mathcal{H}(G(F), K, V), S'} c\text{-Ind}_K^{G(F)} V .$$

we deduce:

$$\text{Hom}_{G(F)}(\mathcal{H}(M(F), M_0, V_{N(k)}) \otimes_{\mathcal{H}(G(F), K, V), S'} c\text{-Ind}_K^{G(F)} V, \pi) \neq 0 .$$

Claim: If $\pi$ is admissible, this implies

$$\mathcal{H}(M(F), M_0, V_{N(k)}) \otimes_{\mathcal{H}(G(F), K, V), S'} \text{Hom}_{G(F)}(c\text{-Ind}_K^{G(F)} V, \pi) \neq 0 .$$
Hence \( \pi \) is not \( K \)-supersingular. The claim follows from elementary algebra and will be proved later 7.7.

iii. The localisation of \( \text{Hom}_{G(F)}(\text{c-Ind}_K^{G(F)}V, \pi) \) at \( T_L \) is not 0 because the localisation of \( \text{Hom}_{G(F)}(\text{c-Ind}_K^{G(F)}V, \pi) \) at \( T_M \) is not 0, by transitivity of the localisation: the localisation at \( T_M \) is equal to the localisation at \( T_M \) of the localisation at \( T_L \). Equivalently

\[
\mathcal{H}_{L,V,\pi} := \mathcal{H}(L(F), L_0, V_{N'(k)}) \otimes_{\mathcal{H}(G(F), K,V), S_{L,G}} \text{Hom}_{G(F)}(\text{c-Ind}_K^{G(F)}V, \pi)
\]

is not 0 because \( \mathcal{H}_{M,V,\pi} \neq 0 \). This follows from the transitivity relation

\[
\mathcal{H}_{M,V,\pi} = \mathcal{H}(M(F), M_0, V_{N(k)}) \otimes_{\mathcal{H}(L(F), L_0, V_{N'(k)}), S'_{M,L}} \mathcal{H}_{M,V,\pi}
\]

which is deduced from the transitivity \( S'_{M,G} = S'_{M,L} \circ S'_{L,G} \).

The non-zero space

\[
\text{Hom}_{G(F)}(\text{c-Ind}_K^{G(F)}V, \pi)
\]

contains a simple right \( \mathcal{H}(G(F), K, V) \)-submodule \( \mathcal{N} \) because \( \pi \) is admissible.

The irreducible representation \( \pi \) is a quotient of

\[
(32) \quad \mathcal{N} \otimes_{\mathcal{H}(G(F), K, V)} \text{c-Ind}_K^{G(F)}V
\]

As \( V \) is \( L \)-coregular, \( \mathcal{N} \) is the restriction of a simple \( \mathcal{H}(L(F), L_0, V_{N'(k)}) \)-module, still denoted by \( \mathcal{N} \), and the representation \((32)\) is isomorphic to

\[
(33) \quad \mathcal{N} \otimes_{\mathcal{H}(L(F), L_0, V_{N'(k)})} \text{Ind}_{Q(F)}^{G(F)}(\text{c-Ind}_{L_0}^{L(F)}V_{N'(k)})
\]

by Theorem 4.5. This last representation is isomorphic to \( \text{Ind}_{Q(F)}^{G(F)}\sigma \) where

\[
(34) \quad \sigma := \mathcal{N} \otimes_{\mathcal{H}(L(F), L_0, V_{N'(k)})} \text{c-Ind}_{L_0}^{L(F)}V_{N'(k)}
\]

is a smooth representation of \( L(F) \). The center of \( L(F) \) embeds naturally in the center of the Hecke algebra \( \mathcal{H}(L(F), L_0, V_{N'(k)}) \) and acts by a character on the simple \( \mathcal{H}(L(F), L_0, V_{N'(k)}) \)-module \( \mathcal{N} \) [VigD]. Hence \( \sigma \) has a central character.

The admissible irreducible representation \( \pi \) is a quotient of \( \text{Ind}_{Q(F)}^{G(F)}\sigma \) where \( \sigma \) has a central character. By Proposition 7.8 below, \( \pi \) is a quotient of \( \text{Ind}_{Q(F)}^{G(F)}\tau \) for an admissible irreducible smooth \( C \)-representation \( \tau \) of \( L(F) \). As \( Q \neq G \), the representation \( \pi \) is not supercuspidal.

\[\square\]

Remark 7.7. Proof of the claim.

Proof. We denote \( A = \mathcal{H}(G(F), K, V), T = T_M \in A, B = A[T^{-1}], X = \text{c-Ind}_K^{G(F)}V \). We suppose

\[
\text{Hom}_G(B \otimes_A X, \pi) \neq 0,
\]

and we want to prove that \( B \otimes_A \text{Hom}_G(X, \pi) \neq 0 \) provided that \( \text{Hom}_G(X, \pi) \) is finite dimensional (which is the case if \( \pi \) is admissible).

We consider the natural linear map

\[
r : \text{Hom}_G(B \otimes_A X, \pi) \to \text{Hom}_G(X, \pi), \quad \varphi \mapsto (x \mapsto \varphi(1 \otimes x)).
\]

The space \( \text{Hom}_G(B \otimes_A X, \pi) \) is naturally a right \( B \)-module hence a right \( A \)-module by restriction. The map \( r \) is \( A \)-linear:

\[
r(\varphi a)(x) = (\varphi a)(1 \otimes x) = \varphi(a \otimes x) = \varphi(1 \otimes ax) = r(\varphi)(ax) = (r(\varphi)a)(x),
\]
for $a \in A, x \in X, \varphi \in \text{Hom}_G(B \otimes_A X, \pi)$. Consequently, the image $\text{Im}(r)$ is an $A$-submodule of $\text{Hom}_G(X, \pi)$. We remark that $T \text{Im}(r) = \text{Im}(r)$ because $r(\varphi) = r(\varphi T^{-1})T$ for $\varphi \in \text{Hom}_G(B \otimes_A X, \pi)$.

We show now that our hypothesis implies that $\text{Im}(r)$ is not 0. Indeed, let $\varphi \neq 0$ in $\text{Hom}_G(B \otimes_A X, \pi)$. There exists $b \in B$ and $x \in X$ such that $\varphi(b \otimes x) \neq 0$. Writing $b = T^{-n}a$ with $n \in \mathbb{N}$ and $a \in A$ we get $\varphi(T^{-n}a \otimes x) = \varphi T^{-n}(1 \otimes ax) \neq 0$ so that $r(\varphi T^{-n}) \neq 0$.

We assume now that $\text{Hom}_G(X, \pi)$ is finite dimensional. Then $\text{Im}(r)$ is also finite dimensional then $T$ induces an automorphism of $\text{Im}(r)$ so that $B \otimes_A \text{Im}(r) \neq 0$. The localisation being an exact functor, $B \otimes_A \text{Hom}_G(X, \pi) \neq 0$.

\[ \square \]

**Proposition 7.8.** Let $\pi$ be an admissible irreducible smooth $C$-representation of $G(F)$ which is a quotient of $\text{Ind}_{P(F)}^{G(F)} \sigma$ for a smooth $C$-representation $\sigma$ of $M(F)$ with a central character. Then there exists an admissible irreducible smooth $C$-representation $\tau$ of $M(F)$ such that $\pi$ is a quotient of $\text{Ind}_{P(F)}^{G(F)} \tau$.

When the characteristic of $F$ is 0, Herzig ([Herzig](#)) Lemma 9.9) proved this proposition using the $\mathcal{P}$-ordinary functor $\text{Ord}_{\mathcal{P}}$ introduced by Emerton ([Emerton](#)). His proof contains four steps:

1. As $\sigma$ is locally $Z_M$-finite, we have

   $$\text{Hom}(\text{Ind}_{P(F)}^{G(F)} \sigma, \pi) \simeq \text{Hom}_{M(F)}(\sigma, \text{Ord}_{\mathcal{P}} \tau).$$

2. As $\pi$ is admissible, $\text{Ord}_{\mathcal{P}} \tau$ is admissible.

3. As $\text{Ord}_{\mathcal{P}} \tau$ is admissible and non-zero, it contains an admissible irreducible subrepresentation $\tau$.

4. As $\text{Ord}_{\mathcal{P}}$ is the right adjoint of $\text{Ind}_{P(F)}^{G(F)}$ in the category of admissible representations, we obtain that $\pi$ is a quotient of $\text{Ind}_{P(F)}^{G(F)} \tau$.

The proof is valid without hypothesis on the characteristic of $F$; we checked carefully that the Emerton’s proof of the steps 1, 2, 4 never uses the characteristic of $F$. Only the proof of step 3 given by Herzig has to be replaced by a characteristic-free proof.

**Lemma 7.9.** An admissible smooth $C$-representation of $G(F)$ contains an admissible irreducible subrepresentation.

**Proof.** For any admissible smooth $C$-representation of $G(F)$, the dimension of $\pi^H$ is a positive finite integer for any open pro-$p$-subgroup $H$. In a subrepresentation $\pi_1$ of $\pi$ such that the right $H(G(F), H, \text{id})$-module $\pi_1^H$ has minimal length, the subrepresentation generated by $\pi_1^H$ is irreducible. \[ \square \]

This ends the proof of Proposition hence of the theorem.

**Remark 7.10.** When $\pi$ is an admissible smooth $C$-representation of $G$, then

$$\text{Hom}_{G(F)}(\text{c-Ind}_K^{G(F)} V, \pi)$$

is finite dimensional hence it is 0 or contains a simple $\mathcal{H}(G(F), K, V)$-module.

An irreducible smooth $C$-representation $\pi$ of $G(F)$ such that $\text{Hom}_{G(F)}(\text{c-Ind}_K^{G(F)} V, \pi)$ contains a simple $\mathcal{H}(G(F), K, V)$-module $\mathcal{N}$, has a central character. This follows from:

1. The center of $\mathcal{H}(G(F), K, V)$ acts on $\mathcal{N}$ by a character $\text{VigD}.r$
2. $\pi$ is quotient of $\mathcal{N} \otimes_{\mathcal{H}(G(F), K, V)} \text{c-Ind}_K^{G(F)} V$.

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We want now to show that the $K$-supersingularity of an admissible irreducible representation of $G(F)$ can also be defined using the characters of the center $\mathcal{Z}(G(F), K, V)$ of $\mathcal{H}(G(F), K, V)$ appearing in $\text{Hom}_{G(F)}(\text{c-Ind}_K^G(F)V, \pi)$.

We consider the localisation

$$\mathcal{Z}(G(F), K, V) \to \mathcal{Z}(M(F), M_0, V_{N(k)}) .$$

at $T_M$ obtained by restriction to the centers of the localisation $S'$ at $T_M$ (Proposition 4.4).

**Proposition 7.11.** Let $\pi$ be an admissible irreducible smooth $C$-representation of $G(F)$. The following properties are equivalent:

i. $\pi$ is $K$-supersingular,

ii. The localisation at $T_M$ of any simple $\mathcal{H}(G(F), K, V)$-submodule of

$$\text{Hom}_{G(F)}(\text{c-Ind}_K^G(F)V, \pi)$$

is 0, for all standard Levi subgroups $M \neq G$.

iii. The localisation at $T_M$ of any character of $\mathcal{Z}(G(F), K, V)$ contained in

$$\text{Hom}_{G(F)}(\text{c-Ind}_K^G(F)V, \pi)$$

is 0, for all standard Levi subgroups $M \neq G$.

**Proof.** We suppose first $\pi$ only irreducible and we denote $H_V := \text{Hom}_{G(F)}(\text{c-Ind}_K^G(F)V, \pi)$ for simplicity; we suppose $H_V \neq 0$.

We note that the localisation of $H_V$ at $T_M$ as a $\mathcal{H}(G(F), K, V)$-module, and as a $\mathcal{Z}(G(F), K, V)$-module, are isomorphic to $\mathcal{Z}(M(F), M_0, V_{N(k)})$-modules.

The localisation at $T_M$ is an exact functor hence if the localisation of $H_V$ at $T_M$ is 0, the same is true for the simple $\mathcal{H}(G(F), K, V)$-submodules of $H_V$ and the characters of $\mathcal{Z}(G(F), K, V)$ contained in $H_V$.

We suppose now $\pi$ admissible. Then $H_V$ is finite dimensional and admits a finite Jordan-Hölder filtration as a $\mathcal{H}(G(F), K, V)$-module (or as a $\mathcal{Z}(G(F), K, V)$-module).

The localisation of $H_V$ at $T_M$ is not 0 if and only if the localisation at $T_M$ of one of the simple quotients of $H_V$ as a $\mathcal{H}(G(F), K, V)$-module (or as a $\mathcal{H}(G(F), K, V)$-module) is not 0.

Each character of $\mathcal{Z}(G(F), K, V)$ appearing as a subquotient of $H_V$ also embeds in $H_V$ because $\mathcal{Z}(G(F), K, V)$ is a finitely generated commutative algebra over the algebraically closed field $C$. The finite dimensional space $H_V$ is the direct sum of its generalized eigenspaces $(H_V)_\chi$ with eigenvalue an algebra homomorphism $\chi : \mathcal{Z}(G(F), K, V) \to C$.

Hence the localisation of $H_V$ at $T_M$ is not 0 if and only if the localisation at $T_M$ of a character of $\mathcal{Z}(G(F), K, V)$ contained in $H_V$ is not 0.

The characters of $\mathcal{Z}(G(F), K, V)$ contained in $H_V$ are the central characters of the simple $\mathcal{H}(G(F), K, V)$-submodules of $H_V$.

The localisation at $T_M$ of a simple $\mathcal{H}(G(F), K, V)$-submodule is not 0 if and only if the localisation at $T_M$ of its central character is not 0.

\[\square\]

Herzig and Abe when $G$ is $F$-split, $K$ is hyperspecial and the characteristic of $F$ is 0 ([Herzig] Lemma 9.9), used the property iii to define the $K$-supersingularity of $\pi$ irreducible and admissible.
References

[Abe] Abe Noriyuki: *On a classification of admissible irreducible modulo \( p \) representations of a \( p \)-adic split reductive group*. Preprint 2011.

[Ramla] Abdellatif Ramla: *Autour des représentations modulo \( p \) des groupes réductifs \( p \)-adiques de rang 1*. Thesis in preparation.

[BL] Barthel Laure and Livne Ron: *Irreducible modular representations of \( GL_2 \) of a local field*. Duke Math. J. Volume 75, Number 2 (1994), 261-292.

[Bki] Bourbaki Nicolas: *Groupes et algèbres de Lie, chapitres 4,5 et 6*. Hermann 1968.

[BTII] Bruhat F. et Tits J.: *Groupes réductifs sur un corps local*. Inst. Hautes Études Scient. Publications Mathématiques Vol. 60 (1984), part II, pp. 197-376.

[CE] Cabanes Marc and Enguehard Michel: *Representation theory of finite reductive groups*. Cambridge University Press 2004.

[Curtis] Curtis C. W.: *Modular representations of finite groups with split \((B,N)\)-pairs*. In *Seminar on Algebraic groups and related finite groups* Lecture Notes in Math, 131, Springer-Verlag 1970, Chapter B.

[Emerton] Emerton Matthew: *Ordinary parts of admissible representations of \( p \)-adic reductive groups I*. Astérisque 331, 2010, p. 355-402.

[HV] Henniart Guy and Vigneras Marie-France: *A Satake isomorphism for representations modulo \( p \) of reductive groups over local fields*. Preprint 2011.

[Herzig] Herzig Florian: *The classification of admissible irreducible modulo \( p \) representations of a \( p \)-adic \( GL_n \)*. To appear in Inventiones Math.

[HerzigW] Herzig Florian: *The weight in a Serre-type conjecture for tame \( n \)-dimensional Galois representations*. Duke Math. J. 149 (1): 37-116, 2009.

[HS] Hilton P.J., Stammbach U.: *A Course in Homological Algebra*. GTM 4 , 1971. Springer-Verlag

[Ly] Ly Tony: *Irreducible representations modulo \( p \) representations of \( GL(2,D) \)*. In preparation.

[VigD] Vigneras Marie-France: *Représentations irréductibles de \( GL(2,F) \) modulo \( p \)*. In L-functions and Galois representations, ed. Burns, Buzzard, Nekovar, LMS Lecture Notes 320 (2007)

[VLivre] Vignéras Marie-France: *Représentations \( \ell \)-modulaires d’un groupe réductif \( p \)-adique avec \( \ell \neq p \)*. PM 137. Birkhauser (1996).

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