Abstract

We study conformal quantities at generic parameters with respect to the harmonic measure on the boundary of the connectedness loci $M_d$ for unicritical polynomials $f_c(z) = z^d + c$. It is known that these parameters are structurally unstable and have stochastic dynamics. We prove $C^{1+\frac{2}{d}-\epsilon}$-conformality, $\alpha = 2 - \text{HD}(J_{c_0})$, of the parameter-phase space similarity maps $\Upsilon_{c_0}(z) : \mathbb{C} \mapsto \mathbb{C}$ at typical $c_0 \in \partial M_d$ and establish that globally quasiconformal similarity maps $\Upsilon_{c_0}(z)$, $c_0 \in \partial M_d$, are $C^1$-conformal along external rays landing at $c_0$ in $\mathbb{C} \setminus J_{c_0}$ mapping onto the corresponding rays of $M_d$. This conformal equivalence leads to the proof that the $z$-derivative of the similarity map $\Upsilon_{c_0}(z)$ at typical $c_0 \in \partial M_d$ is equal to $1/T'(c_0)$, where $T(c_0) = \sum_{n=0}^{\infty} (D(f^n_{c_0})(c_0))^{-1}$ is the transversality function.

The paper builds analytical tools for a further study of the extremal properties of the harmonic measure on $\partial M_d$, [25]. In particular, we will explain how a non-linear dynamics creates abundance of hedgehog neighborhoods in $\partial M_d$ effectively blocking a good access of $\partial M_d$ from the outside.

1 Introduction

One of the main open problems in dynamical systems is the density of hyperbolic polynomials in the complex plane [17, 55, 49]. Even in the simplest case
of quadratic polynomials $z^2 + c$, the problem is far from being solved despite a great deal of research. The main object of this study is the boundary of connectedness locus $\mathcal{M}_d$, $d \geq 2$, and its relations with the corresponding Julia sets $\mathcal{J}_c$ through local similarity maps. Understanding the fractal structure of $\mathcal{M}_d$ which is both "self-similar" and "chaotic" is one of the most interesting aspects of complex dynamics.

Since $\mathcal{M}_d$ is a full compact [15, 48], Carathéodory’s theorem implies that local connectivity of $\partial \mathcal{M}_d$ is equivalent to the existence of continuous extension of the Riemman map $\Psi : \hat{\mathbb{C}} \setminus \mathbb{D} \mapsto \hat{\mathbb{C}} \setminus \mathcal{M}_d$ tangent to the identity at $\infty$. By [15], the local connectivity of $\partial \mathcal{M}_d$ implies the density of hyperbolicity. This is not known, nor it is known whether every hyperbolic geodesic in $\mathbb{C} \setminus \mathcal{M}_d$ lands.

The Julia set $\mathcal{J}_c$ of a unicritical polynomial $f_c(z) = z^d + c$ is defined as the closure of all repelling periodic points of $f_c$,

$$\mathcal{J}_c = \{ z \in \mathbb{C} : \exists n \in \mathbb{N} \ f^n_c(z) = z \text{ and } |(f^n_c)'(z)| > 1 \}.$$  

Let $\mathcal{M}_d$ be the set of all $c \in \mathbb{C}$ for which $\mathcal{J}_c$ is connected. When $c$ is outside $\mathcal{M}_d$ then Julia sets $\mathcal{J}_c$ are totally disconnected. The boundary of $\mathcal{M}_d$ is the topological bifurcation locus of $\mathcal{J}_c$.

A mathematical interest in $\mathcal{M}_d$ goes beyond unicritical dynamics. C. McMullen proved in [39] that the bifurcation locus of any non-trivial holomorphic family of rational maps over the unit disk contains almost conformal copies of $\mathcal{M}_d$. The distribution of the harmonic measure on $\partial \mathcal{M}_d$ has some extremal properties and it is in the same time computationally accessible. For various relations with classical problems in complex analysis see [4, 8, 30, 41, 24].

Our goal is to develop analytical tools to understand how dynamics unfolds at generic parameters with respect to the harmonic measure on $\partial \mathcal{M}_d$. The current work is based on two ingredients which fit well into a program of J.-C. Yoccoz to study the parameter space $\mathcal{M}_d$ through interaction of analytic and combinatorial structures. The first ingredient is an organization of the parameter space into a Markov system, proposed by J.-C. Yoccoz, which is asymptotically stable with respect to the natural holomorphic motions. The second one stems from [24] and combines an outside combinatorics given by holomorphically moving Böttcher coordinates with some simple probabilistic models of [21]. This “outside” approach of [21, 50] allows to control the behavior of the harmonic measure on $\partial \mathcal{M}_d$ and use the results from harmonic analysis and potential theory to get a further information about the dynamics [22]. Another pertinent examples include [34, 56].

In [24], a system of similarity functions $\Upsilon_{c_0}(z)$ was constructed which is parametrized by typical points $c_0 \in \partial \mathcal{M}_d$ with respect to the harmonic measure. The maps $\Upsilon_{c_0}(z)$ are quasiconformal and become asymptotically conformal. The main result of the paper, stated as Theorem 1 asserts that the similarity maps $\Upsilon_{c_0}(z)$ depend $C^1$-continuously along hyperbolic geodesics landing at $c_0$ even if the geodesics
have a rather complicated geometry, spiralling in both directions infinitely many times [24]. However, the oscillations can be controlled asymptotically by a universal function of Theorem 4. The idea of comparing the phase and parameter spaces dates back to the origins of complex dynamics [15] and the similarity theorem of T. Lei [52] was one of the first results in the area inspired by computer visualisations.

Theorem 1 has several applications but the most interesting direction from the point of view of the dynamical systems is given by the formula which relates the derivative of \( \Upsilon_{c_0}(z) \) with the transversality function \( T(c) \), cf. formula (4), introduced by M. Benedicks and L. Carleson in [5, 6]. In the Misiurewicz case, the formula was proven by J. Riviera-Letelier in [44] by methods exploiting an underlying hyperbolicity. The Misiurewicz parameters are defined by the condition that the critical point is not recurrent and thus of bounded combinatorial complexity. The parameter selection methods of Benedicks-Carleson [5, 6] rely on the fact that \( T(c_0) \neq 0 \) at the parameter \( c_0 \) which undergoes a perturbation. Since \( T(c) \) is analytic for \( c \) from the complement of \( M_d \), it can not take the same value on large sets. The size of these sets will depend on integrability properties of \( T(c) \) [1, 11]. On the other hand, the failure of the transversality condition \( T(c_0) \neq 0 \) endows dynamics with some weak expansion properties. If the series \( T(c) = \sum_{n=0}^{\infty} (Df^n(c))^{-1} \) converges absolutely then \( J_{c_0} \) is locally connected [19] and of Lebesgue measure zero [10]. The perturbative techniques of [5, 6] were already applied in the complex quadratic setting in [7] and proven to work well in various holomorphic instances by M. Aspenberg [3, 2]. A direct relation of the transversality function to the Fatou conjecture for unicritical polynomials was discussed in [33]. The transversality condition \( T(c_0) \neq 0 \) was also intensively studied for real maps in the context of Jakobson’s theorem [31], see for example [53], where the transversality condition was explicitly stated. The real methods are quite different than these adopted in the current paper and will not be further discussed.

1.1 Similarity structures

By Fatou’s theorem, \( \Psi : \hat{C} \setminus D \mapsto \hat{C} \setminus M_d \), tangent to the identity at \( \infty \), extends radially almost everywhere on the unit circle with respect to the normalized 1-dimensional Lebesgue measure \( \lambda_1 \). The harmonic measure \( \omega \) on \( \partial M_d \) is equal to \( \Psi_\ast(\lambda_1) \). If \( c \in \partial M_d \) then \( J_c \) is a full compact. Denote by \( \Psi_c : \hat{C} \setminus D \mapsto \hat{C} \setminus J_c \) the Riemann map tangent to the identity at \( \infty \). We have a one parameter family \( \omega_c \in \partial M_d \) of the harmonic measures supported on the corresponding Julia sets \( J_c \), \( \omega_c = (\Psi_c)_\ast(\lambda_1) \). If \( c \in \partial M_d \) is typical with respect to \( \omega \) then the same is true for the critical orbit \( \{f^n(c)\}_{n \in \mathbb{N}} \) with respect to the harmonic measure on \( J_c \), \( \omega_c \) is also \( f_c \)-invariant, ergodic, and of the maximal entropy \( \log d \) [9].

The multifaceted relation between the harmonic measure \( \omega \) and dynamics can be quantified. Theorem 3 of [24] describes the similarity between \( M_d \) and \( J_c_0 \).
through one-parameter family of asymptotically conformal maps $\Upsilon_{c_0} : \mathbb{C} \to \mathbb{C}$, with $c_0$ typical with respect to the harmonic measure on $\partial M_d$. We state it as Fact 1.1. A certain complexity in the formulation of Fact 1.1 is related to the introduction of a full compact $Z$ which does not have a canonical dynamical meaning. The role of $Z$ is to "enlarge" $\mathcal{J}_{c_0}$ to compensate for the fact that $M_d$ and $\mathcal{J}_{c_0}$ have different topological properties, $M_d$ has a non-empty and dense interior $\mathbb{R}^n$ while the corresponding Julia set $\mathcal{J}_{c_0}$ is a dendrite. The compact $Z$ depends on a construction as the critical orbit $\{f^n_{c_0}(z)\}_{n \in \mathbb{N}}$ is dense in $\mathcal{J}_{c_0}$ and lacks any $c$-stable hyperbolic structure [18]. Since outside $Z$, the similarity map $\Upsilon_{c_0}$ agrees with the natural univalent map $\Psi \circ \Psi^{-1} : \hat{\mathbb{C}} \setminus \mathcal{J}_{c_0} \to \hat{\mathbb{C}} \setminus M_d$, $Z$ can be considered as an asymptotically negligible correction of $\mathcal{J}_{c_0}$ near $c_0$ so that $\Psi \circ \Psi^{-1}$ extends across $Z$ to a global quasiconformal map that fails to be analytic on $Z$ but its distortion can be still controlled through quasiconformal constants.

**Fact 1.1** for almost every $c_0 \in \partial M_d$ with respect to the harmonic measure there exist a full compact $Z$, $c_0 \in \partial Z$, a Jordan disk $U \ni c_0$, and a quasi-conformal map $\Upsilon_{c_0}$ of the plane, $\Upsilon_{c_0}(c_0) = c_0$, with the following properties:

(i) $\mathcal{J}_{c_0} \cap Z$ is connected, $\mathcal{J}_{c_0} \cap U = \mathcal{J}_{c_0} \cap Z$, and every $z \in \partial Z \setminus \{c_0\}$ is non-recurrent,

(ii) $\lim_{r \to 0} \frac{1}{r} d_H(Z \cap \mathbb{D}(c_0, r), \mathcal{J}_{c_0} \cap \mathbb{D}(c_0, r)) = 0$, where $d_H$ stands for the Hausdorff distance,

(iii) $\lim_{r \to 0} \frac{1}{r^2}$ area $(Z \cap \mathbb{D}(c_0, r)) = 0$,

(iv) $\Upsilon_{c_0}(Z \cap U) \supset M_d \cap \Upsilon_{c_0}(U)$, $Z$ is disjoint with the hyperbolic geodesic $\gamma \subset \mathbb{C} \setminus \mathcal{J}_{c_0}$ landing at $c_0$, and $\lim_{r \to c_0} d_H(\xi, \mathcal{J}_{c_0})/d_H(\xi, Z) = 1$,

(v) $\Upsilon_{c_0}$ on $U \setminus Z$ is equal to $\Psi \circ \Psi^{-1}$ where $\Psi_{c_0}$ and $\Psi$ are uniforming maps from $\{\{z \neq 1\} \cap \hat{\mathbb{C}} \setminus \mathcal{J}_{c_0}$ and $\hat{\mathbb{C}} \setminus M_d$, respectively, tangent to the identity at $\infty$,

(vi) the maximal dilation of $\Upsilon_{c_0}$ restricted to $\mathbb{D}(c_0, r)$ tends to 1 when $r$ tends to 0.

(vii) $\Upsilon_{c_0}$ is conformal at $c_0$.

The only claim of Fact 1.1 which is not contained in Theorem 3 of [24] is the limit in (iv). A short proof of (iv) is delegated to Appendix.

Recall that a quasi-conformal mapping $\Upsilon$ is $(1 + \beta)$-conformal at $z_0$, $\beta \geq 0$ if

$$\Upsilon(z) = \Upsilon(z_0) + \Upsilon'(z_0)(z - z_0) + \epsilon(|z - z_0|),$$

with $\Upsilon'(z_0) \neq 0$ and $\lim_{z \to z_0} \frac{\epsilon(|z - z_0|)}{|z - z_0|^{1+\beta}} = 0$. The 1-conformal map is called conformal.

The proof of conformality of $\Upsilon$ in [24] was based on an integral condition of Teichmüller, Wittich, and Belinskiǐ. If the Beltrami coefficient $\mu(z) = \Upsilon_{\overline{z}}(z)/\Upsilon_z(z)$
around \( c \) satisfies
\[
\int_{D(c,r)} \frac{|\mu(z)|}{|z-c|^2} \, dx \, dy < \infty
\]
for some positive \( r \), then \( \Upsilon \) is conformal at \( c \), see \cite{32}.

**Smooth continuity of similarity map along hyperbolic geodesics.**
Let \( \gamma \) denote the hyperbolic geodesic of \( \mathbb{C} \setminus J_{c_0} \) which lands at \( c_0 \), \( Z \) denotes the continuum from Fact \ref{fact1} and \( \chi_Z \) is the indicator function. Theorem \ref{thm7} states that there exist a bound \( o(R) \), \( \lim_{R \to 0^+} o(R) = 0 \), and \( R_0 > 0 \) such that for every \( z_0 \in \gamma \) and if \( |z_0 - c_0| < R_0 \)
\[
\int_{D(z_0,R)} \chi_Z(w) \frac{1}{|z_0 - w|^2} \, d\lambda_2(w) \leq o(R),
\]
where \( \lambda_2 \) is 2-dimensional Lebesgue measure.

We will need the following version of uniform conformality proved in \cite{26} (Theorem 1.4).

**Fact 1.2** Let \( F \) be a quasiconformal self-mapping of the complex plane with complex dilatation \( \mu \) and let \( K \subset \mathbb{C} \) be a compact set. If there are positive constants \( R \) and \( M \) such that
\[
\int_{|z-w| < R} \frac{|\mu(w)|^2}{|z-w|^2} \, d\lambda_2(w) \leq M
\]
holds for every \( z \in K \) and there exists a finite limit
\[
\lim_{r \to 0} \int_{r < |z-w| < R} \frac{\mu(w)}{(z-w)^2} \, d\lambda_2(w)
\]
uniformly for \( z \in K \), then the mapping \( F \) is conformally differentiable on \( K \) and the complex derivative of \( f'(z) \) is continuous on \( K \).

Combining the estimate \ref{estimate2} and Fact \ref{fact1.2}, we obtain a version of uniform similarity along hyperbolic geodesics.

**Theorem 1** The derivative \( D_z \Upsilon_{c_0}(z) \) of the similarity map of Fact \ref{fact1} is continuous along the geodesic of \( \hat{\mathbb{C}} \setminus J_{c_0} \) landing at \( c_0 \) for a typical point \( c_0 \in \partial M_d \) with respect to the harmonic measure \( \omega \).

We will discuss below three applications of Theorem \ref{thm1} and the similarity structures to some known open problems in complex dynamics.
1.2 Deep points

By [21 50 13], HD (J_{c_0}) < 2 for almost all c_0 ∈ M with respect to the harmonic measure. Let α = 2 − HD (J_{c_0}). Theorem 3 states that for any ϵ > 0,

$$\lim_{r \to 0^+} \frac{|Z \cap D(c_0, r)|}{r^{2 + \frac{\alpha}{2} - \epsilon}} = 0,$$

where |·| stands for 2-dimensional Lebesgue measure. Since the similarity map Υ_{c_0}(z) is conformal at z = c_0 and Υ_{c_0}(Z) ⊂ M_d, we can transport the estimate (3) to the parameter space proving that a generic parameter c_0 ∈ ∂M_d with respect to the harmonic measure is \((\frac{\alpha}{2} - \epsilon)\)-measurably deep with respect to C \ M_d according to the definition of [38].

**Theorem 2** For generic parameter c_0 with respect to the harmonic measure on ∂M_d and every ϵ > 0,

$$\lim_{c \to c_0} \frac{|M_d \cap D(c_0, |c - c_0|)|}{|c_0 - c|^{2 + \frac{\alpha}{2} - \epsilon}} = 0.$$

Another consequence of the estimate (3) is an improved integrability in (1). Theorem 2.25 in [38] asserts that if c_0 is a δ-deep point of C \ M_d and Υ : C → C K-quasiconformal map then Υ is \((1 + \beta(\delta, K))\)-conformal at c_0. The dilatation of the similarity map Υ tends to 1 when c approaches c_0.

**Corollary 1.1** The similarity map Υ is \((1 + \frac{\alpha}{2} - \epsilon)\)-conformal for every ϵ > 0 and almost all c_0 ∈ ∂M_d with respect to the harmonic measure.

Theorem 2 and Corollary 1.1 are generalizations of the results of [44] obtained for non-recurrent parameters (Misiurewicz case). Note that Misiurewicz set of parameters is of harmonic measure 0, see [21 50]. The concept of measurable deep points was proposed by C. McMullen in the context of renormalization [38].

The proof of Theorem 2 is based on a global inductive estimate of conformal densities distributed over elements of Yoccoz partitions.

A finite Borel measure ν supported on J_c is called conformal with an exponent κ (or κ-conformal) if for every Borel set B on which f is injective one has

$$\nu(f(B)) = \int_B |f'(z)|^\kappa \, d\nu(z).$$

Of particular importance are conformal measures with the minimal exponents, [51 13]. In [19] it was proved that for a large class of rational maps, including Collet-Eckmann quadratic polynomials, conformal measures with the minimal exponent κ are ergodic (hence unique), non-atomic, and

$$\kappa = \text{HD}(J_c) = \text{HD}(\nu) := \inf_{A: \nu(A) = 1} \text{HD}(A).$$
1.3 Transversality function

M. Benedicks and L. Carleson in their work on unimodal maps $z^2 + c$, $c \in \mathbb{R}$, and the Hénon map, [5, 6], used the transversality function

$$ T(c) = \sum_{n=0}^{\infty} \left( D_z f^n_c (z)_{z=c} \right)^{-1} $$

(4)

to control distortion between the phase and parameter spaces. It was observed in [5, 6] that as long as $T(c) \neq 0$ and $|T(c)| = \sum_{n=0}^{\infty} |D(f^n_c)(c)|^{-1} < \infty$ then the parameter exclusion construction can be initiated. The outcome of the construction is a set of parameters of positive Lebesgue measure with an expanding dynamics. The work [5, 6] generalized an earlier breakthrough due to M. Jakobson on the existence of a set of parameters of positive 1-dimensional Lebesgue measure with a stochastic dynamics. The proof of M. Jakobson was based on very different techniques than that of [5, 6].

**Theorem 3** The sum

$$ T(c_0) = \sum_{n=0}^{\infty} \left( D_z f^n_{c_0} (z)_{z=c_0} \right)^{-1} $$

converges for almost all $c_0 \in \partial M_d$ with respect to the harmonic measure and satisfies

$$ T(c_0) = \frac{1}{D_z \Upsilon_{c_0} (z)_{z=c_0}}, $$

where $\Upsilon_{c_0} (z)$ is the similarity function of Fact 1.1.

For Misiurewicz parameters, Theorem 3 was proven by J. Rivera-Letelier in [44]. The proof in [44] is based on transversality of two different holomorphic motions, the critical value $f_c(c)$ and the postcritical hyperbolic compact $P(c)$ for $c$ from a small neighborhood of $c_0$. Our proof is different as dynamics generic with respect to the harmonic measure does not have an underlying hyperbolic structure. The main idea is to produce uniform estimates for $T(c)$ outside of $M_d$ at some scales and then pass to the limit along the hyperbolic geodesic landing at $c_0$. The main technical ingredient is $C^1$-smoothness of the similarity map $\Upsilon_{c_0} (z)$ along hyperbolic geodesics as stated in Theorem 1.

Fact 4.1 states that for $c \notin M_d$, $T(c) = 0$ iff $D_c \Psi(c) = 0$. Since $\Psi(c)$ is univalent, $T(c)$ can not vanish.

**Corollary 1.2** The transversality function $T(c)$ omits 0 for $c \notin M_d$, is symmetric with respect to the real line and tangent to the constant function 1 at $\infty$.

The transversality condition is closely related to the summability conditions in complex dynamics, $|T|^{\beta}(c) = \sum_{n=0}^{\infty} |D(f^n_c)(c)|^{-\beta} < +\infty$, $\beta \in (0, 1]$, which imply various degrees of metrical or conformal smallness of $J_c$ [19, 10, 46, 33].
Geometric interpretation of the transversality function. For a typical \( c_0 \in \partial \mathcal{M}_d \) with respect to the harmonic measure, the similarity function \( \Upsilon_{c_0} \) maps a hyperbolic geodesic \( \gamma \) of \( \mathbb{C} \setminus \mathcal{J}_d \) landing at \( c_0 \) onto a hyperbolic geodesic \( \Gamma \) of \( \mathbb{C} \setminus \mathcal{M}_d \) landing at \( c_0 = \Upsilon_{c_0}(c_0) \), see Fact 1.1 (v). Let \( \gamma(z) \) denote the subarc of \( \gamma \) between \( z \in \gamma \) and \( c_0 \) and \( |\gamma(z)| \) be the length of \( \gamma(z) \).

Theorem 3 and \( C^1 \)-smoothness of \( \Upsilon_{c_0} \) along \( \gamma \) yield the following corollary.

**Corollary 1.3** For almost all \( c_0 \in \partial \mathcal{M}_d \) with respect to the harmonic measure,

\[
|\mathcal{T}(c_0)| = \lim_{c \in \Gamma \to c_0} \frac{|\gamma(\Upsilon_{c_0}^{-1}(c))|}{|\Gamma(c)|} \tag{5}
\]

A formula similar to (5) holds for \( \arg \mathcal{T}(c_0) \) but its dynamical meaning seems to be less clear within Benedicks-Carleson perturbation theory. Also, according to [24], the limit \( \lim_{c \in \Gamma \to c_0} \arg(\Gamma(c)) - c_0 \) does not exist for almost all \( c_0 \in \partial \mathcal{M}_d \) with respect to the harmonic measure as \( \Gamma \) twists around \( c_0 \) in both directions infinitely many times. In [50], it was proved that the Collet-Eckmann condition holds for all \( c \in \partial \mathcal{M}_d \) except possibly for a set of harmonic Hausdorff dimension 0. One can ask if the formula (5) holds for all Collet-Eckmann parameters or even for the summability class \( |\mathcal{T}|^1(c) < \infty \).

### 1.4 Geometric applications

**Flat angles.** Let \( \mathcal{K} = \partial \mathcal{K} \) be a continuum. \( \mathcal{K} \) is well-accessible at \( y \in \mathcal{K} \) (or accessible within a twisted angle) if there exist a Jordan curve \( \gamma \subset \mathbb{C} \setminus \mathcal{K} \) terminating at \( y \) and \( C > 0 \) such that for every \( z \in \gamma \),

\[
\text{dist } (z, \mathcal{K}) > C \text{ diam } \gamma(z),
\]

where \( \gamma(z) \) is the subarc of \( \gamma \) between \( z \) and \( y \). If every point from \( \mathcal{K} \) is accessible within a twisted angle of the same aperture then \( \mathbb{C} \setminus \mathcal{K} \) is a John domain. If \( y \) is well-accessible then it is also well-accessible by the hyperbolic geodesic landing at \( y \) [40]. Theorem 3 of [24] states that for almost every \( c \in \partial \mathcal{M}_d \) with respect to the harmonic measure \( \omega \), the parameter \( c \) is a Lebesgue density point of \( \mathbb{C} \setminus \mathcal{M}_d \) but it is not well-accessible.

We say that a point \( c^* \in \partial \mathcal{M}_d \) is iterated log-accessible if a hyperbolic geodesic \( \Gamma \) lands at \( c^* \) and for any \( m > 0 \),

\[
\lim_{\Gamma \ni \gamma \to c^*} \frac{\text{dist } (c, \mathcal{M}_d)}{\text{diam } \Gamma (c)} \log_{[m]} \frac{1}{\text{diam } \Gamma (c)} = +\infty,
\]

where \( \log_{[m]} = \log \circ \cdots \circ \log \) is the \( m \)-th iterate of log function.

**Theorem 4** For almost every \( c^* \in \partial \mathcal{M}_d \) with respect to the harmonic measure, \( c^* \) is iterated log-accessible.
Theorem 4 follows from the existence of the similarity structures and an iterated large deviation estimate for exponential distribution, see [25] for a detailed proof.

**Hedgehogs and porosity in the parameter space.** The concept of porosity has a long history, see [37]. A set $E \subset \mathbb{C}$ is $\beta$-porous, $\beta > 0$, at $z^* \in E$ and scale $r > 0$ if there is $z \in D(z^*, r)$ such that $D(z, \beta r) \cap E = \emptyset$.

By the Makarov law of the iterated logarithm [35], almost every point from $\partial \mathcal{M}_d$ with respect to the harmonic measure is Hölder accessible, and thus $\beta$-porous, $\beta \in (0, 1/2)$ in many scales, see Proposition 2.2 in [23]. The limiting value of $\beta = 1/2$ from [23] falls short of the upper bound 1. It is not known what happens for $\beta$ between 1/2 and 1.

The harmonic measure is supported on a set of points of $\partial \mathcal{M}_d$ that can only be accessed by passing through infinitely many increasingly narrow “tunnels” at scales of positive density. The prevalence of such extremal sets in complex dynamics was shown in [23]. Using the similarity structures from Fact 1.1, one can quantify the lack of porosity and prove that for a typical $c^* \in \partial \mathcal{M}_d$ with respect to the harmonic measure, accessibility within a John angle fails rather badly and an extremal “non-accessibility” in the sense of Makarov theory [35] is observed instead [25].

We will illustrate some of these extremal features of the harmonic measure distribution on $\partial \mathcal{M}_d$, see Figure 1.4. To this aim we will need a concept of hedgehog neighborhoods.

Let $X$ be a planar set. We say that $X$ contains $(m, \epsilon)$-hedgehog layer around $x \in X$ if there exist a ring domain $A$, mod $A \geq m$, and a collection of pairwise disjoint continua $C_k \subset X$, $k = 1, 2 \ldots$, with the property that (i) $x$ belongs to the bounded component of $C \setminus A$, (ii) every $C_k$ intersects both components of $C \setminus A$, (iii) every point from $A$ is at the distance at most $\epsilon/$diam $A$ to some $C_k$ from the

Figure 1: Hedgehog layer at a typical point in the boundary of the Mandelbrot set
collection.

Even though not explicitly stated, the concept of hedgehog layers was introduced by J. Riviera-Letelier in his study of porosity at critical recurrent points for rational functions, see the proof of Theorem C’ in ([45]).

We say that $X$ has hedgehog neighborhood at $x$ if for every $\epsilon, m > 0$ there exists an $(m, \epsilon)$-hedgehog layer around $x \in X$. The phase-parameter space similarity of Fact [1.1] allows to detect hedgehog neighborhoods in the parameter space.

**Theorem 5** The boundary $\partial M_d$ contains hedgehog neighborhood at almost every point $c^* \in \partial M_d$ with respect to the harmonic measure. The corresponding Julia set $J_{c^*}$ has hedgehog neighborhoods at a dense subset of $J_{c^*}$.

Hedgehog neighborhoods are directly related to the concept of ”hairiness” proposed by J. Milnor in the context of renormalization. Theorem 5 indicates that increasingly dense parts of the boundary of the Mandelbrot set is a standard feature of recurrent and non-linear dynamics rather than a staple of the renormalization.

The proof of Theorem 5 explains how the construction of hedgehog neighborhoods in the phase space falls naturally into the setting of box mappings [20] in the unicritical case. Since hedgehog neighborhoods are quasionformal invariants, their abundance in the boundary of the Mandelbrot set follows directly from Fact [1.1].

## 2 Constructions

### 2.1 Preliminaries and the similarity map.

We will follow closely the definitions and notations of [24]. Here is a partial list.

- $f_c(z) = z^d + c$, where $d > 1$ is fixed, $J_c$ is its Julia set, $K_c$ the filled-in Julia set.
- $M_d$ is the locus of connectivity of the family $\{f_c\}_{c \in \mathbb{C}}$.
- $\Psi$ is the Riemann map from the complement of $\overline{D}(0,1)$ onto the complement of $M_d$ tangent to the dentity at $\infty$; analogously, $\Psi_c$ is the Riemann map of the complement of $K_c$ if $c \in M_d$, otherwise $\Psi_c$ can be defined as the Böttker coordinate on a neighborhood of $\infty$ and extended by the dynamics till the Green line $G_c(0)$,

$$G_c(z) = \lim_{n \to \infty} \frac{\log f_c^n(z)}{d^n}.$$ 

There is an explicit formula, $\Psi_c^{-1}(z) = \exp(G_c(z) + 2\pi i \theta)$, $G_c(z) > G_c(0)$, where $\theta \in [0, 1)$ is called external argument or external angle of $z$ [12, 13].
Rays, geodesics, and external angles. When $c \notin \mathcal{K}_c$ then the Green function $G_c$ has critical points at $f_c^i(0)$ for $i = 0, 1, \cdots$. A smooth ray in the phase space is a gradient line of the $G_c$ with closure that intersects both $\infty$ and $\mathcal{K}_c$. We will consider only gradient lines which avoid critical points of $G_c$ and are, therefore, smooth. The closure of some rays intersects $\mathcal{K}_c$ at precisely one point. We say that these rays land at (or converge to) that point. All gradient lines are well defined on the set $\{z : G_c(z) > G_c(0)\}$. They are labeled by the external angles $\theta \in [0, 1)$ at which they enter $\infty$. If $\mathcal{K}_c$ is connected then the ray $\gamma_{\theta,c}$ with an external argument $\theta$ is a hyperbolic geodesic in $\hat{\mathbb{C}} \setminus \mathcal{K}_c$.

Of particular importance is the critical external angle $\theta(c)$, the angle of the gradient line which passes through $c$. Any line in the parameter space of the form $\theta(c) = \omega$ will be named an external ray with angle $\omega$ and denoted by $\Gamma_\omega$ or simply $\Gamma$. The following relation holds,

$$c \in \Gamma_\omega \iff c \in \gamma_{\theta,c}.$$ 

The external rays are hyperbolic geodesics in $\mathbb{C} \setminus \mathcal{M}_d$. The Green function for $\mathcal{M}_d$ satisfies $G_{\mathcal{M}_d}(c) = G_c(c)$ and for every $c \in \mathbb{C} \setminus \mathcal{M}_d$,

$$\Psi^{-1}(c) = \exp(G_c(c) + 2\pi i \theta(c)).$$

Yoccoz puzzle pieces. Again, we refer to the construction in [24]. An initial order 0 Yoccoz puzzle is regarded as fixed and then a Yoccoz puzzle piece of order $k \geq 0$ is one that is mapped into a piece of order 0 by $k$ iterations.

$b_{k,c}$ will denote a piece of order $k$ which contains 0 - it may not exist for all $k$. Then $\beta_{k,c} = f_c(b_{k,c})$. Since $c$ and 0 are in different pieces of order 0, $\beta_{k,c}$ is disjoint from any piece which contains 0.

Nesting for typical parameters. Fix a typical parameter $c_0$ with respect to the harmonic measure. By Proposition 8 of [24], for any $M^*$ we can find a sequence of nesting critical pieces

$$b_{N_0,c_0} \supset b_{N_1,c_0} \supset b_{N_2,c_0} \supset b_{N_3,c_0} \supset b_{N_4,c_0}$$

and a box locus $V_{N_5}$, $N_5 > N_4 > \cdots > N_1 > 10$, such that for every $c \in V_{N_5}$ the nesting condition mentioned above also holds, and

$$\text{mod } (b_{N_j} \setminus \overline{b_{N_{j-1}}}) \geq M^*$$

for $j = 1, 2, 3, 4$.

$M^*$ is a parameter of the construction which in turn defines $N_j$, $j = 0, 1, \cdots, 5$. For brevity, write $Q(M^*)$ for constants which only depend on $M^*, d, N_j$. When dynamical objects depend on $c$, we will supress $c_0$ from the notation, i.e. $b_{N_1,c_0}$ could simply be $b_{N_1}$. 

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Returns to a large scale. For a typical $c_0$ we can further construct an increasing sequence $(S_n)_{n \geq 1}$ such that

- for every $n$, there is a critical piece $b_{S_n + N_0}$ which is mapped uni-critically onto $b_{N_0}$ by $f^{S_n}$,
- for every $n$, $f^{S_n}(0) \in b_{N_1}$,
- $S_n > 10N_4$ and $\frac{S_{n+1}}{S_n} < \frac{11}{10}$ for all $n$,
- $\lim_{n \to \infty} (S_{n+1} - S_n) = \infty$,
- $\lim_{n \to \infty} \frac{S_{n+1}}{S_n} = 1$.

First return maps. If $b_{n,c}$ is a critical piece, then $\phi_{n,c}$ will denote the first entry map into $b_{n,c}$ (first entry meaning that it is the identity on $b_{n,c}$ itself).

Let $\Phi(c, z)$ denote the natural holomorphic motion, wherever it is defined.

**Lemma 2.1** For any $n$ and $c \in V_{S_n + N_1}$, the natural holomorphic motion starting at $c_0$ is defined on the complement of the closure of the domain of $\phi_{S_n + N_1}$.

**Proof.** Take a point $z_0$ in the complement of the closure of the domain of $\phi_{S_n + N_1}$. We will show that its natural holomorphic motion extends to $V_{S_n + N_1}$. By Lemma 2.6 of [24], we know that $V_{S_n + N_1}$ is simply connected. It will suffice to prove that for any quasi-disk $D$ compactly contained in $V_{S_n + N_1}$, the holomorphic motion can be extended to an open set which contains $D$.

For any $c \in D$, the orbit of $\Phi(c, z_0)$ under $f_c$ forever avoids $b_{S_n + N_1}$. From Lemma 2.8 of [24], it implies that the distance of that orbit to 0 remains uniformly bounded way from 0 on $D$. By Lemma 2.2 of [24], if $c'$ is now on the boundary of $D$, that means that $\Phi(c, z_0)$ extends to a neighborhood of $c'$.

**Lemma 2.2** There is a natural holomorphic motion defined on

$$V_{S_n + N_1} \times (\partial \beta_{S_n + N_2} \setminus \mathcal{J})$$

**Proof.** By Lemma 2.10 of [24] we need to check that $f^k(\partial b_{S_n + N_2}) \cap b_{S_n + N_1} = \emptyset$ for $0 < k \leq N_2 - N_1$. If for any $k > 0$ that intersection is non-empty, then $f^k(b_{S_n + N_1}) \supset b_{S_n + N_1}$. Since $f^{S_n}$ is uni-critical on $b_{S_n + N_1}$ the smallest $k$ for which it could occur is $k = S_n$. But we assumed $S_n > N_5 > N_2$.

**Lemma 2.3** When $\Phi$ is the holomorphic motion on $\partial \beta_{S_n + N_2}$, natural outside of $\mathcal{J}$, then the equation

$$\Phi(c, z) = c$$

for $z \in \partial \beta_{S_n + N_2}$ has exactly one simple zero on $\partial V_{S_n + N_2}$.
Proof. That follows directly from Lemma 2.11 in [24]. □

The key fact which establishes the existence of the similarity map is the following:

**Proposition 1** There is a $Q(M^*)$-quasiconformal homeomorphism $\Upsilon_n$ defined on a neighborhood of the $V_{S_n+N_3}$ which fixes $c_0$ and coincides with $\Psi \circ \Psi^{-1}$ outside the closure of the domain of $\phi_{S_n+N_1}$. The constant $Q(M^*)$ tends to 1 as $M^*$ tends to $\infty$.

This follows from Proposition 5 of [24], while the modulus claim follows from Lemma 4.3.

**The similarity map.** The similarity map $\Upsilon_n$ allows one to subdivide $V_{S_n+N_3}$ in a way that is homeomorphic to the subdivision of $b_{S_n+N_1}$ into the components of the domain of $\phi_{S_n+1+N_1}$. This subdivision is the best we can do on the annulus $V_{S_n+N_3} \setminus V_{S_n+1+N_3}$ since the inner component can then be subdivided using $\Upsilon_{n+1}$ and they will match along the common boundary.

Let $\Upsilon$ mean the homeomorphism defined on a neighborhood of $c$ which is $\Upsilon_n$ on $A_n$. It is quasi-conformal, since the boundaries of pieces $\beta_{S_n+N_3,c}$ are quasi-circles and therefore removable.

### 2.2 Nesting of Yoccoz pieces.

**Lemma 2.4** For any $n$, pieces $f^j(b_{S_n+N_1})$ are disjoint from $b_{S_n+N_1}$ for $1 \leq j < S_n$.

**Proof.** Since $f^S_n$ is uni-critical on $b_{S_n+N_1}$, we cannot have $f^j(b_{S_n+N_1}) \supset b_{S_n+N_1}$. The opposite inclusion is also impossible, because eventually $b_{S_n+N_1}$ must be mapped on a piece of order 0.

□

**The predecessor function.**

**Definition 2.1** For $n \geq 1$, let $\sigma(n)$ denote the smallest $k \geq 1$ for which $S_k \geq \frac{S_{n+2}}{2}$.

By our hypothesis, for $n > 1$ we get $\sigma(n) \leq n - 1$.

**Lemma 2.5** Let $n \geq 2$ and $b_k$ denote the critical component of the domain of the first return map into $b_{S_{\sigma(n)}+N_1}$. Then $k > S_{n+1} + N_4$. 13
Proof. Since \( b_k \subset b_{\sigma(n)} + N_1 \), then we must have \( k \geq S_{\sigma(n)} \) by Lemma 2.4.

Suppose next that \( b_k \) contains \( b_{S_{n+1}+N_1} \). Then \( S_{n+1} \geq k + S_{\sigma(n)} \), since \( f^k \) first maps \( b_{S_{n+1}+N_1} \) into \( b_{S_{\sigma(n)}+N_1} \) which needs at least \( S_{\sigma(n)} \) more iterates to cover \( b_{N_1} \), while \( f^{S_{n+1}}(b_{S_{n+1}+N_1}) = b_{N_1} \). Hence, \( S_{n+2} > S_{n+1} \geq 2S_{\sigma(n)} \), which contradicts Lemma 2.4.

Consequently, \( b_k \) is strictly contained in \( b_{S_{n+1}+N_1} \). Since \( 0 \in b_k \) is mapped by \( f^{S_{n+1}} \) into \( b_{N_4} \), \( f^{S_{n+1}}(b_k) \cap b_{N_4} \neq \emptyset \). If \( f^{S_{n+1}}(b_k) \supset b_{N_4} \), then recall that \( b_k \) is a domain of the first return map into \( b_{\sigma(n)} \). Since we assumed \( S_n \geq 10N_4 \) for all \( n \), \( b_{\sigma(n)} \subset b_{N_4} \). So, \( f^p(b_k) \) must have covered \( b_{\sigma(n)} \) including \( 0 \) for some \( 0 < p < S_{n+1} \), but this contradicts Lemma 2.4.

The only remaining possibility is \( f^{S_{n+1}}(b_k) \subset b_{N_4} \), in which case \( k > S_{n+1} + N_4 \).

\[\square\]

We will write \( A_n = \beta_{S_{n}+N_3} \setminus b_{S_{n}+N_3} \).

Lemma 2.6 Any component of the domain of \( \phi_{S_{\sigma(n)}+N_1} \) which intersects \( A_n \) is contained in it.

Proof. Let \( \tilde{\zeta} \) be a component of the domain of \( \phi_{S_{\sigma(n)}+N_1} \). Since Yoccoz pieces intersect only if one contains the other, the claim of the Lemma is equivalent to showing that \( \beta_{S_{n+1}+N_3} \not\subset \tilde{\zeta} \). If, to the contrary, the inclusion holds, then \( f^{-1}(\tilde{\zeta}) \) contains a critical piece which is a component of the domain of the first return map into \( b_{S_{\sigma(n)}+N_1} \). That piece cannot contain \( b_{S_{n+1}+N_3} \) by Lemma 2.5.

\[\square\]

Proposition 2 On any component of its domain, the mapping \( \phi_{S_{n}+N_3} \) extends univalently to range \( b_{S_{\sigma(n)}+N_1} \).

Proof. Write \( \zeta \) for the component of the domain of \( \phi_{S_{n}+N_3} \) and let \( \tilde{\zeta} \) be the component of the domain of \( \phi_{S_{\sigma(n)}+N_1} \) which contains \( \zeta \).

For some \( k \), \( f^k \) maps \( \tilde{\zeta} \) univalently onto \( b_{S_{\sigma(n)}+N_1} \) and \( \zeta \) into a subpiece \( f^k(\zeta) \). If \( 0 \in f^k(\zeta) \), then since \( \zeta \) was a component of the first entry map into \( b_{S_{n}+N_3} \), \( f^k(\zeta) \) coincides with \( b_{S_{n}+N_3} \) and the claim of the Proposition follows.

Otherwise, \( f^k(\zeta) \cap b_{S_{n}+N_3} = \emptyset \). Then consider the first return map from \( b_{S_{\sigma(n)}+N_1} \) into itself. \( f^k(\zeta) \) belongs to some component \( \tilde{\zeta}_1 \) of the domain of that map. It cannot be the critical component which must be contained in \( b_{S_{n+1}+N_3} \) by Lemma 2.5. Thus, \( \tilde{\zeta}_1 \) is mapped onto \( b_{S_{\sigma(n)}+N_1} \) univalently by some \( f^{k_1} \) and \( f^{k+k_1}(\zeta) \) is again a subpiece of \( b_{S_{\sigma(n)}+N_1} \). Then we repeat the entire reasoning to conclude that either \( f^{k+k_1}(\zeta) = b_{S_{\sigma(n)}+N_1} \) and the claim of the Proposition follows, or \( f^{k+k_1}(\zeta) \) belongs to a non-critical component \( \tilde{\zeta}_2 \) of the first return map into \( b_{S_{\sigma(n)}+N_1} \) and can be pushed univalently by another \( f^{k_2} \). The process has to end eventually, since \( k + \sum k_j \) cannot exceed the order of \( \zeta \).
Corollary 2.1 If a component of the domain of $\phi_{S_n+N_1}$ is contained in $A$, the domain of univalent extension onto $b_{s_{c_0}+N_1}$ mentioned in Proposition 2 is disjoint from the external ray which lands at $c_0$.

Proof. Since that domain is a Yoccoz puzzle piece and does not contain $c_0$, it is disjoint from the ray.

3 Metric estimates

3.1 Uniform shrinking.

Lyapunov exponent. Let $\lambda$ denote the Lyapunov exponent of $f_{c_0}$ at $c_0$. We know that $\lambda > 0$ by [21; 50] and furthermore, $\lambda = \log d$ by [22].

Roundness of pieces. Let us introduce a definition.

Definition 3.1 Consider a simply connected bounded domain $U \subset \mathbb{C}$ and $z_0 \in U$. We will say that $U$ is $K$-balanced with respect to $z_0$ if for any $z \in U$, $\theta \in \mathbb{R}$, $z_0 + K^{-1}e^{i\theta}(z - z_0) \in U$.

Domains $\beta_{S_n+N_1}$, $j = 1, \cdots, 4$ are $K(M^*)$ balanced with respect to $c$, while $b_{S_n+N_1}$ are $Q(M^*)$ balanced with respect to 0. They also are $K(M^*)$-quasi-discs. These properties will be referred to as the roundness of critical pieces. The roundness directly follows from the conditions imposed on returns to the large scale.

Lemma 3.1

$$\log \left( \text{diam} \beta_{S_n+N_1} \right)^{-1} = S_n \lambda + o_{M^*}(S_n).$$

Proof. $f^{S_n-1}$ maps $\beta_{S_n+N_1}$ onto $b_{N_1}$ with distortion bounded in terms of $M^*$, since the map extends univalently onto $b_{N_0}$. The estimate follows from the notion of the Lyapunov exponent.

$$\sum_{k=2}^{n} \text{mod} \ (\beta_{S_{k-1}+N_1} \setminus \beta_{S_{k}+N_1}) = S_n \lambda + o_{M^*}(S_n).$$
Proof. Since pieces $\beta S_n + N_1$ are all round,
\[
\sum_{k=2}^{n} \mod (\beta_{S_{k-1}+N_1} \setminus \overline{\beta_{S_k+N_1}}) = -\log \diam \beta S_n + N_1 + O_{M^*}(n) .
\]

Since $\lim_{n \to \infty} S_n/n = \infty$, the term linear in $n$ can be absorbed into the constant $o_{M^*}(S_n)$ and so the Lemma follows from Lemma 3.1.

\[\square\]

Proposition 3 For every component $\zeta$ of the domain of $\phi_{S_n+N_1}$,
\[
\log (\diam(\zeta))^{-1} \geq S_n \frac{\lambda}{d} + o_{M^*}(S_n) .
\]

Proof. From Lemma 3.2,
\[
\sum_{k=2}^{n} \mod (b_{S_{k-1}+N_1} \setminus b_{S_k+N_1}) = S_n \frac{\lambda}{d} + o_{M^*}(S_n) .
\]

From Proposition 2 $\zeta$ is surrounded by nesting annuli which are conformally equivalent to $b_{S_{k-1}+N_1} \setminus b_{S_k+N_1}$. The claim follows by superadditivity of moduli and Teichmüller’s modulus estimates, see [32].

\[\square\]

Additional estimates on the sizes of pieces. Now we denote by $\{\zeta_{n,j}\}_{j=1}^{\infty}$ the components of the domain of $\phi_{S_n+N_1}$ which are contained in $A_n$.

Lemma 3.3 For any $\epsilon > 0$ and $M^*$ there is $n_0$ such that if $n \geq n_0$, then
\[
\sup \{\diam \zeta_{n,j} : j = 1, \cdots\} \leq \diam (\beta_{S_n+N_3})^{1+\frac{1}{d}-\epsilon} .
\]

Proof. $\beta_{S_n+N_3}$ together with any $\zeta_{n,j}$ contained in it are mapped by $f^{S_n-1}$ into $b_{N_3}$. By Proposition 3
\[
\diam f^{S_n-1}(\zeta_{n,j}) \leq \exp \left( -S_n \frac{\log d}{d} + o_{M^*}(S_n) \right) .
\]

Taking into account Lemma 3.1
\[
\diam f^{S_n-1}(\zeta_{n,j}) \leq (\diam \beta_{S_n+N_1})^{1/d} \exp(o_{M*}(S_n)) .
\]

Again by Lemma 3.1
\[
\exp(o_{M*}(S_n)) \leq (\diam \beta_{S_n+N_1})^{-\epsilon}
\]
for any $\epsilon > 0$ provided that $n$ is sufficiently large.

Pulling back by $f^{S_n-1}$ will introduce another factor $\diam \beta_{S_n+N_1}$ on the right-hand side together with an error term depending on $M^*$, which can be be absorbed in $o_{M^*}(S_n)$.
Lemma 3.4 For any $\epsilon > 0$ and $M^*$ there is $n_0$ such that whenever $n \geq n_0$, then
$$
\text{diam } \beta_{S_n + N_1} \geq (\text{diam } \beta_{S_{n-1} + N_1})^{1+\epsilon}.
$$

Proof. By Lemma 3.1
$$
\log \frac{\text{diam } \beta_{S_{n-1} + N_1}}{\text{diam } \beta_{S_n + N_1}} = (S_{n+2} - S_{n-1}) \lambda + o_{M^*}(S_{n+2}) + o_{M^*}(S_{n-1}).
$$
Since $\lim_{n \to \infty} \frac{S_{n+1}}{S_n} = 1$, the right-hand side is $o_{M^*}(S_{n-1})$, which is at least $(\text{diam } \beta_{S_{n-1} + N_1})^{-2}$ provided that $n$ is large enough.

3.2 Estimates based on the conformal measure.

Let $\nu$ denote the conformal measure on $\mathcal{J}$. The exponent of $\nu$ will be denoted with $2 - \alpha$ and is equal to $\text{HD}(\mathcal{J})$. The existence of a unique non-atomic $\nu$ with the minimal exponent $\text{HD}(\mathcal{J})$ was established in [19]. Since $\text{HD}(\mathcal{J}) < 2$ by [21], $\alpha > 0$.

Lemma 3.5 Let $\zeta$ denote a component of the domain of $\phi_{S_n + N_j}$, $n > 1$, $j = 1, 2, 3, 4$. Then
$$
\nu(\zeta) \geq K_1(M^*) (\text{diam } \zeta)^{2-\alpha}.
$$

Proof. By Proposition 2 domain $\zeta$ is mapped onto $b_{S_n + N_j}$ with distortion which is bounded depending on $M^*$. Because of roundness we get
$$
\nu(\zeta) \geq L_1(M^*) \nu(b_{S_n + N_j}) \left( \frac{\text{diam } \zeta}{\text{diam } b_{S_n + N_j}} \right)^{2-\alpha}.
$$
One can further see that
$$
\nu(b_{S_n + N_j}) \geq L_2(M^*) \nu(\beta_{S_n + N_j}) \left( \frac{\text{diam } b_{S_n + N_j}}{\text{diam } \beta_{S_n + N_j}} \right)^{2-\alpha}
$$
and since $\beta_{S_n + N_j}$ is mapped onto $b_{N_j}$ with bounded distortion,
$$
\nu(\beta_{S_n + N_j}) \geq L_3(M^*) (\text{diam } \beta_{S_n + N_j})^{2-\alpha}.
$$
These estimates together yield the claim of the Lemma.
We will use the symbol \( M^* \) to join quantities which are equivalent with positive multiplicative constants which depend on \( M^* \).

**Lemma 3.6** For any \( n > 1 \),

\[
\nu(A_n) \overset{M^*}{\sim} (\text{diam } \beta_{S_n+N_3})^{2-\alpha}.
\]

**Proof.** This follows straight from the definition of the conformal measure given that \( f^{S_n-1} \) maps \( \beta_{S_n+N_3} \) onto \( b_{N_3} \) with distortion bounded in terms of \( M^* \).

□

**Lemma 3.7** For any \( n > 1 \),

\[
\sum_j (\text{diam } \zeta_{n,j})^{2-\alpha} < K_2(M^*) (\text{diam } \beta_{S_n+N_3})^{2-\alpha}.
\]

**Proof.** Summing up over \( j \) and using Lemma 3.5 together the estimate of \( \nu(A_n) \) given by Lemma 3.6 yields the claim.

□

**Lemma 3.8** For any \( \epsilon > 0 \) and \( M^* \) there is \( n_0 \) such that whenever \( n \geq n_0 \)

\[
\nu(b_{S_n+N_3} \setminus \overline{b_{S_{n+1}+N_3}}) \leq (\text{diam } b_{S_{n+1}+N_3})^{2-\alpha-\epsilon}.
\]

**Proof.** The absolute value of the derivative of \( f^{-1} \) on \( A_n \) is bounded above by \( L_1(M^*) (\text{diam } \beta_{S_{n+1}+N_3})^{1-\frac{1}{d}} \). So,

\[
\nu(b_{S_n+N_3} \setminus \overline{b_{S_{n+1}+N_3}}) \leq L_2(M^*) (\text{diam } \beta_{S_{n+1}+N_3})^{(2-\alpha)(\frac{1}{d}-1)} (\text{diam } \beta_{S_n+N_3})^{2-\alpha}
\]

\[
\leq L_2(M^*) \left( \frac{\text{diam } \beta_{S_{n+1}+N_3}}{\text{diam } \beta_{S_n+N_3}} \right)^{2-\alpha} (\text{diam } b_{S_{n+1}+N_3})^{2-\alpha}.
\]

The first factor is bounded by \( (\text{diam } \beta_{S_{n+1}+N_3})^{-\epsilon} \) by Lemma 3.4. The second was obtained from \( \text{diam } b_{S_{n+1}+N_3} \overset{M^*}{\sim} (\text{diam } \beta_{S_{n+1}+N_3})^{1/d} \) by roundness. This concludes the proof.

□

**Lemma 3.9** For every \( \epsilon > 0 \) and \( M^* \) there is \( n_0 \) such that if \( n \geq n_0 \), then

\[
\nu(b_{S_n+N_3}) \leq (\text{diam } b_{S_n+N_3})^{2-\alpha-\epsilon}.
\]
Proof. By Lemma 3.8 for any $\epsilon_1 > 0$ and $n \geq n_0(\epsilon_1)$

$$\nu(b_{S_n+N_3}) \leq \sum_{j=1}^{\infty} \left( \text{diam } b_{S_{n+j}+N_3} \right)^{2-\alpha-\epsilon_1}.$$ 

By Proposition 3, this can be further bounded from above,

$$\nu(b_{S_n+N_3}) \leq \sum_{j=1}^{\infty} \exp \left( \left( -S_{n+j} \frac{\lambda}{d} + o_{M^*}(S_n) \right)(2-\alpha-\epsilon_1) \right) \frac{1}{1-\exp(-\lambda/d)},$$

from estimating the sum of the geometric progression. By Lemma 3.1, we further obtain

$$\nu(b_{S_n+N_3}) \leq (1 - e^{-\lambda/d})^{-1} \left( \text{diam } b_{S_{n+N_3}} \right)^{2-\alpha-\epsilon_1} \exp(o_{M^*}(S_n)).$$

By choosing $\epsilon_1 < \epsilon$ and $n_0$ sufficiently large, we can absorb the constant and factor $\exp(o_{M^*}(S_n))$ into the form of the estimate of Lemma 3.9.

□

Consequences for first entry maps. We will now use these results to obtain an estimate for the domains of first entry maps. Recall how, by Proposition 2, on every component of its domain the map $\phi_{S_n+N_3}$ has a univalent extension onto $b_{S_{\sigma(n)}+N_1}$. By composing that with the first entry map into $b_{S_{\sigma(n)}+N_3}$, one get a univalent extension onto $b_{S_{\sigma(n)}+N_3}$ with a further continuation onto $b_{S_{\sigma(n)}+N_1}$.

Recall that $|\cdot|$ is used to denote 2-dimensional Lebesgue measure of sets in $\mathbb{C}$.

Proposition 4 Suppose that $X$ is a component of the domain of the first entry map $f^r$ into $b_{S_{\sigma(n)}+N_3}$ such that $f^r$ from $X$ continues univalently to map onto $b_{S_{\sigma(n)}+N_1}$. Let $X_n$ be the intersection of $X$ with the domain of $\phi_{S_n+N_1}$. Then for every $n > 1$,

$$\frac{|X_n|}{|X|} \leq \exp \left( -\frac{S_n \lambda \alpha}{2d} + o_{M^*}(S_n) \right).$$

Proof. By Lemma 3.1 and from the roundness of pieces,

$$\log \left( \text{diam } b_{S_{\sigma(n)}+N_3} \right)^{-1} = S_{\sigma(n)} \frac{\lambda}{d} + o_{M^*}(S_{\sigma(n)})$$

while by Proposition 3, any component $\zeta$ of the first entry map $\phi_{S_n+N_1}$ satisfies

$$\log (\text{diam } \zeta)^{-1} \geq S_n \frac{\lambda}{d} + o_{M^*}(S_n).$$
Hence
\[
\log \frac{\text{diam } b_{S_{\sigma(n)} + N_3}}{\text{diam } \zeta} \geq (S_n - S_{\sigma(n)}) \frac{\lambda}{d} + o_{M^*}(S_n) = \frac{S_n \lambda}{2d} + o_{M^*}(S_n) \tag{8}
\]
since from Definition 2.1, \(S_{\sigma(n)} = \frac{S_n}{2} + o_{M^*}(S_n)\).

Now observe that \(X\) is mapped onto \(b_{S_{\sigma(n)} + N_3}\) with distortion bounded in terms of \(M^*\), since the mapping extends univalently onto \(b_{S_{\sigma(n)} + N_1}\).

Thus, if \(\zeta \subset X\), estimate (8) yields
\[
\log \frac{\text{diam } X}{\text{diam } \zeta} \geq \frac{S_n \lambda}{2d} + o_{M^*}(S_n) \tag{9}
\]

Using the same mapping of \(X\) onto \(b_{S_{\sigma(n)} + N_3}\) with bounded distortion, we conclude from Lemma 3.9 that
\[
\nu(X) \leq L_1(M^*) (\text{diam } X)^{2-\alpha} \left(\text{diam } b_{S_{\sigma(n)} + N_3}\right)^{-\epsilon_1}
\]
for every \(\epsilon_1 > 0\) provided that \(\sigma_n\) is large enough.

From Lemma 3.5
\[
\sum_{\zeta \subset X_n} (\text{diam } \zeta)^{2-\alpha} \tag{10}
\]
and \(|\zeta| \leq L_4 (\text{diam } \zeta)^2\), from estimates (10) and (8) one obtains
\[
\frac{|X_n|}{|X|} \leq \frac{L_4}{L_3(M^*)} \frac{\sum_{\zeta \subset X_n} (\text{diam } \zeta)^{2-\alpha}}{(\text{diam } X)^{2-\alpha}} \sup \{((\text{diam } \zeta)^{\alpha} : \zeta \subset X_n\}
\]
\[
\leq L_4(M^*) \left(\text{diam } b_{S_{\sigma(n)} + N_3}\right)^{-\epsilon_1} \exp \left(-\alpha \frac{S_n \lambda}{2d} + o_{M^*}(S_n)\right).
\]

By Lemma 3.1
\[
\text{diam } b_{S_{\sigma(n)} + N_3} = \exp \left(-\sigma(n) \frac{\lambda}{d} + o_{M^*}(S_{\sigma(n)})\right)
\]
which leads to
\[
\frac{|X_n|}{|X|} \leq L_4(M^*) \exp \left(-\alpha \frac{S_n \lambda}{2d} + \epsilon_1 S_{\sigma(n)} \frac{\lambda}{d} + o_{M^*}(S_n)\right).
\]

Since the constants and \(\epsilon_1 S_{\sigma(n)} \frac{\lambda}{d}\) can be rolled into \(o_{M^*}(S_n)\), Proposition 4 follows.

\[\Box\]
3.3 Deep point

Recall that $\zeta_{n,j}$ denoted the connected components of the domain of $\phi_{S_n + N_1}$. Define $Z_n = \bigcup_j \zeta_{n,j}$ and $Z = \bigcup_n Z_n$.

**Theorem 6** For any $\epsilon > 0$ and if $M^*$ is sufficiently large

$$\lim_{r \to 0^+} \frac{|Z \cap \mathbb{D}(c_0, r)|}{r^{2 + \frac{\alpha}{d} - \epsilon}} = 0.$$ 

**Step I.** We show that for any $M^*$ and $\epsilon > 0$ there is $n_0$ such that if $n \geq n_0$ then

$$|Z_n| \leq L \sum_j (\text{diam } \zeta_{n,j})^2 \leq \left[ \sum_j (\text{diam } \zeta_{n,j})^{2-\alpha} \right] \cdot \left[ \sup_j \{\text{diam } \zeta_{n,j}\} \right]^{\alpha}.$$ 

The first factor on the right-hand side can be bounded by Lemma 3.7 and the second by Lemma 3.3 leading to

$$|Z_n| \leq (\text{diam } b_{S_n + N_3})^{2-\alpha} (\text{diam } b_{S_n + N_3})^{\alpha(1+\frac{1}{d}-\epsilon)}$$

provided $n$ is large enough. Finally,

$$|Z_n| \leq (\text{diam } b_{S_n + N_3})^{2+\frac{\alpha}{d}-\alpha \epsilon}.$$ 

By dividing both sides by $(\text{diam } \beta_{S_n + N_3})^2$ and taking into account $\alpha \leq 1$, we get the claim of Step I.

**Step II.** We will prove that for every $\epsilon > 0$ and if $M^*$ is large enough, there is $r_0$ so that if $0 < r < r_0$, then

$$\frac{|Z \cap \mathbb{D}(c_0, r)|}{|\mathbb{D}(c_0, r)|} < r^{\frac{\alpha}{d} - \epsilon}.$$ 

When $M^*$ is sufficiently large, any circle centered at $c_0$ intersects at most two of the annuli $A_n$. So pick $r > 0$ and choose the largest $n(r)$ for which $A_{n(r)} \subset \mathbb{D}(c_0, r)$. There are at most two annuli $A_{n(r) - 1}$ and $A_{n(r) - 2}$ which also intersect $\mathbb{D}(c_0, r)$. By Lemma 3.4 for any $\epsilon_1 > 0$ and if $n(r)$ is large enough depending on $\epsilon_1, M^*$, $\text{diam } \beta_{S_n(r-2) + N_3} \leq r^{1-\epsilon_1}$. Inserting this into the estimate of Step I stated for some $\epsilon_2 > 0$ we get that for $n(r)$ large enough and for $j = 1, 2$
\[
\frac{|Z_{n(r)} - j|}{r^2} \leq r^{(1-\epsilon_1)}(\frac{a}{n} - \epsilon_2) + 2\epsilon_1.
\]  

(11)

By the Step I, for every \( n \geq n(r) \) and any \( \epsilon_3 > 0 \)
\[
\frac{|Z_n|}{|D(c_0, r)|} \leq L \frac{(\text{diam} \beta_{S_n+N_3})^2}{r^2} (\text{diam} \beta_{S_n+N_3})^{\frac{a}{n} - \epsilon_3},
\]

(12)

where \( L \) is a geometric constant. By Lemma 3.1
\[
\frac{(\text{diam} \beta_{S_n+N_3})^2}{r^2} \leq d^{S_n(r)-S_n} (\text{diam} \beta_{S_n+N_3})^{-\epsilon_4}
\]

for any \( \epsilon_4 > 0 \) provided \( n(r) \) is large enough. By summing up the estimates \( (12) \) for \( n \geq n(r) \), we get
\[
\left| \bigcup_{n \geq n(r)} Z_n \right| \leq \frac{Ld}{\epsilon_3 - \epsilon_4}.
\]

By picking \( \epsilon_3, \epsilon_4 \) as well as \( \epsilon_1, \epsilon_2 \) in estimate \( (11) \) suitably small for the desired \( \epsilon \) and making \( r \) small enough to produce \( n(r) \) correspondingly large to absorb the constants, we get the claim of Step II.

Theorem 6 follows directly from Step II.

3.4 Estimates on the ray.

Let \( \gamma \) denote the external ray of \( J_{c_0} \) which lands at \( c_0 \). Use notations \( Z \) and \( Z_n \) from the previous section and let \( \chi_Z \), etc, by the indicator functions. \( \lambda_2 \) is the 2-dimensional Lebesgue measure of the plane.

Theorem 7 There exist a bound \( o(R) \), \( \lim_{R \to 0^+} o(R) = 0 \), and \( R_0 > 0 \) such that for every \( z_0 \in \gamma \) and if \( |z_0 - c_0| < R_0 \)
\[
\int_{D(z_0, R)} \frac{\chi_Z(w) \, d\lambda_2(w)}{|z_0 - w|^2} \leq o(R).
\]

Let us start with a lemma.

Lemma 3.10 Consider a domain \( X \) as in the statement of Proposition 4 for \( n > 1 \) and such that it intersects the domain of \( \phi_{S_n+N_1} \). There is a constant \( 0 < K(M^*) \) so that for every such \( X, n \) and \( z_0 \in \gamma \)
\[
\text{dist} (z_0, \overline{X}) \geq K(M^*) \text{ diam } X.
\]
Proof. By hypothesis, \( X \) is surrounded by an extension domain which is mapped univalently onto \( b_{S_{(n)}+N_1} \) and by Corollary 2.1, the extension domain does not contain \( z_0 \). Hence, we have an annulus \( A \) which is conformally equivalent to \( b_{S_{(n)}+N_1} \setminus \overline{b_{S_{(n)}+N_3}} \) which contains \( X \) in the bounded component of its complement leaving \( z_0 \) in the unbounded one. From our construction, \( \mod A \geq \frac{2}{\delta} M^* \) so the claim follows by Teichmüller’s estimates, see [32]. □

We will now present the proof in a sequence of steps.

Step I. Recall that for \( n > 1 \) the first entry mapping \( \phi_{S_{n}+N_1} \) has a univalent extension from every component of its domain which maps onto \( b_{S_{(n)}+N_3} \) and whose domain is contained in \( A_n \). Let us denote the union of the domains of such extensions by \( \tilde{Z}_n \). We have \( Z_n \subset \tilde{Z}_n \subset A_n \) for each \( n \).

Lemma 3.11 There exist a positive sequence \( \epsilon(n) \) with \( \lim_{n \to \infty} \epsilon(n) = 0 \) and \( n_0 \) such that for every \( n \geq n_0 \) and \( z_0 \in \gamma \) there are \( 0 < \rho_1 < \rho_2 \) so that

\[
\tilde{Z}_n \subset \{ z : \rho_1 < |z - z_0| < \rho_2 \}
\]

with

\[
\log \frac{\rho_2}{\rho_1} \leq \epsilon(n) \log (\text{diam } \beta_{S_{n}+N_3})^{-1}.
\]

The remaining part of Step I is devoted to the proof of Lemma 3.11 which will be divided into several geometric cases.

The case of \( z_0 \) far away. The first case is when \( |c_0 - z_0| \geq 2\text{diam } \beta_{S_{n}+N_3} \). The \( \rho_2/\rho_1 \leq 3 \), while in view of Lemma 3.1

\[
\log (\text{diam } \beta_{S_{n}+N_3})^{-1} \geq L_1 S_n
\]

with positive \( L_1 \) provided \( n \) is large enough. Hence, in this case to satisfy the claim we just need \( \epsilon(n) \geq \frac{\log 3}{L_1 S_n} \).

So, from now on, suppose \( |c_0 - z_0| < 2\text{diam } \beta_{S_{n}+N_3} \). Then we can put

\[
\rho_2 = 3 \cdot \text{diam } \beta_{S_{n}+N_3}.
\]

(13)

\( z_0 \) not too deep. In this case we assume additionally that \( z_0 \) is outside \( \beta_{S_{n+2}+N_3} \).

In order to estimate \( \rho_1 \), let us quote the following

Fact 3.1 For every \( z_0 \in \gamma \), \( D(z_0, |z_0 - c_0|^{1+o(|z_0 - c_0|)}) \cap J = \emptyset \).
Proof. This is a statement of asymptotic Lipschitz accessibility, see [23]. A much stronger claim is provided by Theorem 4.

Thus, for \( \rho'_1 = |z_0 - c_0|^{1 + o(2 \text{diam } \beta_{S_{n+2} + N_3})} \), the ball \( \mathcal{D}(z_0, \rho_1) \) misses \( J \). With our extra hypothesis \( |z_0 - c_0| \geq \text{diam } \beta_{S_{n+2} + N_3} \) and so

\[
\rho'_1 \geq \left( \text{diam } \beta_{S_{n+2} + N_3} \right)^{1 + o(2 \text{diam } \beta_{S_{n+2} + N_3})}.
\] (14)

Now suppose \( X \) is any component of \( \tilde{Z}_n \). By definition, it intersects \( J \). So by Lemma 3.10, \( \rho_1 \geq L_2(M^*)\rho'_1 \) where \( L_2(M^*) := \frac{K(M^*)}{1 + K(M^*)} \) with \( K(M^*) \) from that Lemma.

Taking into account estimates (13) and (14), we arrive at

\[
\frac{\rho_2}{\rho_1} \leq \frac{3}{L_2(M^*)} \frac{\text{diam } \beta_{S_{n+3}}}{\text{diam } \beta_{S_{n+2} + N_3}} \left( \text{diam } \beta_{S_{n+3}} \right)^{o(2 \text{diam } \beta_{S_{n+2} + N_3})}.
\]

Taking logarithms and using Lemma 3.1, we get

\[
\log \frac{\rho_2}{\rho_1} \leq \frac{\log 3}{L_2(M^*)} \left( S_{n+2} - S_n \right) \lambda + o_{M^*}(S_{n+2}) + S_n \lambda o_{M^*}(2 \text{diam } \beta_{S_{n+3}}) \left( \text{diam } \beta_{S_{n+3}} \right)^{o(2 \text{diam } \beta_{S_{n+2} + N_3})}.
\]

We use that \( \lim_{n \to \infty} \frac{S_{n+1}}{S_n} = 1 \). All terms in the numerator can be rolled into \( o_{M^*}(S_n) \) and so the claim follows.

The case of \( z_0 \in \beta_{S_{n+2} + N_3} \). For \( n \) sufficiently large \( S_{n+2} + N_1 \geq S_{n+1} + N_3 \) and so \( \beta_{S_{n+2} + N_3} \) is surrounded inside \( \beta_{S_{n+1} + N_3} \) by an annulus with modulus \( M^*/d \). Hence,

\[ \rho_1 \geq L_3(M^*) \text{diam } \beta_{S_{n+1} + N_3} \]

with \( L_3(M^*) \) and for \( \rho_2 \) we can still take estimate (13). Hence,

\[
\frac{\log \rho_2}{\log(\text{diam } \beta_{S_{n} + N_3})^{-1}} \leq \frac{\log \frac{3}{L_3(M^*)} + (S_{n+1} - S_n) \lambda + o_{M^*}(S_{n+1})}{S_n \lambda + o_{M^*}(S_n)}
\]

which tends to 0 with \( n \) as in the preceding case.

This completes the proof of Lemma 3.11 and Step I.
Step II. Recall set $\tilde{Z}_n$ introduced in Step I. Let $X_n$ be a connected component of $\tilde{Z}_n$. Then for every $n > 1$ and $z_0 \in \gamma$

$$\int_{X_n} \frac{\chi_{Z_n}(w) \, d\lambda_2(w)}{|w-z_0|^2} \leq \exp \left( -\frac{S_n \lambda\alpha}{2} + o_{M^*}(S_n) \right).$$

By Lemma 3.10, $|w_1 - z_0| \geq K(M^*)\log (\frac{d\beta_n}{\rho_2})$ for any $w_1, w_2 \in X_n$.

Then, if $X_n = Z_n \cap \tilde{X}_n$,

$$\int_{X_n} \frac{\chi_{Z_n}(w) \, d\lambda_2(w)}{|w-z_0|^2} \leq \frac{(1 + K(M^*)^{-1})^2 |X_n|}{|X|}$$

and the estimate then follows directly from Proposition 4, since the constant can be rolled into $o_{M^*}(S_n)$.

Step III. For every $n > 1$ and $z_0 \in \gamma$

$$\int_{C} \frac{\chi_{Z_n}(w) \, d\lambda_2(w)}{|w-z_0|^2} \leq \epsilon(n) \exp \left( -\frac{S_n \lambda\alpha}{2} + o_{M^*}(S_n) \right)$$

where $\epsilon(n)$ is the sequence from Step I.

Since every component of $Z_n$ is contained in some $X_n$, Step II and Proposition 4 imply

$$\int_{C} \frac{\chi_{Z_n}(w) \, d\lambda_2(w)}{|w-z_0|^2} \leq \exp \left( -\frac{S_n \lambda\alpha}{2} + o_{M^*}(S_n) \right) \int_{\tilde{Z}_n} \frac{d\lambda_2(w)}{|w-z_0|^2}.$$

By Step I,

$$\int_{\tilde{Z}_n} \frac{d\lambda_2(w)}{|w-z_0|^2} \leq \int_{\rho_1 < |u| < \rho_2} \frac{d\lambda_2(u)}{|u|^2} = \log \frac{\rho_2}{\rho_1} \leq \epsilon(n) \log (\text{diam } \beta_{S_n+N_3})^{-1}.$$

By Lemma 3.1

$$\log (\text{diam } \beta_{S_n+N_3})^{-1} = \lambda S_n + o_{M^*}(S_n).$$

Taking all these estimates together yields

$$\int_{C} \frac{\chi_{Z_n}(w) \, d\lambda_2(w)}{|w-z_0|^2} \leq \epsilon(n) (\lambda S_n + o_{M^*}(S_n)) \exp \left( -\frac{S_n \lambda\alpha}{2} + o_{M^*}(S_n) \right)$$

which gives the claim of Step III, since the factor before the exp involving $S_n$ can be included in the $o_{M^*}(S_n)$ in the exponent.
Step IV. For any $n > 1$, dist $(\tilde{Z}_n, \gamma) > 0$.

Clearly, dist $(J \setminus \beta_{S_{n+1}+N_3}, \gamma) := D > 0$. By Lemma 3.10 applied to each component of $\tilde{Z}_n$,

$$\text{dist} (\tilde{Z}_n, \gamma) \geq \frac{K(M^*)D}{1 + K(M^*)}.$$  

Conclusion of the proof of Theorem 7. Choose $R_0 < \text{dist} (c_0, A_1)$. Then the claim of Step IV also holds for $n = 1$. We conclude that there is a function $n(R)$, $\lim_{R \to 0^+} n(R) = \infty$ such that for any $z_0 \in \gamma \cap \mathbb{D}(c_0, R_0)$, the disk $\mathbb{D}(z_0, R)$ is disjoint from $Z_n$ for all $n < n(R)$.

Then by Step III,

$$\int_{D(z_0, R)} \chi_Z(w) d\lambda_2(w) \leq \sum_{n=n(R)}^\infty \exp \left( - \frac{S_n \lambda \alpha}{2} + o_{M^*}(S_n) \right) := o(R).$$

Since the series

$$\sum_{n=1}^\infty \exp \left( - \frac{S_n \lambda \alpha}{2} + o_{M^*}(S_n) \right) < \infty,$$

the bound $o(R)$ tends to 0 with $R \to 0$.

4 Distortion estimates

The transversality function. The transversality function is defined by

$$\mathcal{T}(c) = \sum_{n=0}^\infty (Df^n_\mathcal{E}(c))^{-1}$$

wherever the series is convergent, which is at least for $c \notin \mathcal{M}_d$.

Fact 4.1 follows from calculus and the definition of $\Psi$ and $\Psi_c$.

Fact 4.1 For $c \notin \mathcal{M}_d$

$$\frac{D_c \Psi^{-1}(c)}{D_c \Psi^{-1}(z)|_{z=c}} = \mathcal{T}(c).$$

The main estimate. Now let $u_0$ be the point on $\partial \mathbb{D}$ with the external argument of $c_0$. $u_n$ is chosen with the same argument as $u_0$ so that $\Psi(u_n) = c_n$ is on the boundary of $V_{S_{n}+N_4}$. Hence,

$$|u_n - u_0| \leq K(M^*)d^{-S_n-N_4}. \quad (15)$$
Choose $u$ on the segment between $u_0$ and $u_n$ and write $c(u) := \Psi(u), z(u) = \Psi_c(u)(u)$ and $z_n(u) = \Psi_{c(u)}(u_n)$. Observe also that $|u_n|_{S_n + N_4}$ is a fixed number corresponding to the equipotential which bounds the initial Yoccoz piece.

Our goal is proving the following.

**Proposition 5** For almost every $c_0 \in \partial M_d$ with respect to the harmonic measure

$$\lim_{n \to \infty} \log \frac{D_z \Psi_{c_n}(z)_{|z = u_n}}{D_z \Psi_{c_0}(z)_{|z = u_n}} = 0.$$  

The idea of the proof is to estimate the derivative of $D_u \log D_z \Psi_{c(u)}(z)_{|z = u_n}$ for $u$ between $u_0$ and $u_n$. It is more convenient to take the derivatives with respect to $c$ instead at $c = c(u)$, so begin by estimating $D_u c(u) = D \Psi(u)$.

**Estimate of $D_u \Psi(u)$.**

**Lemma 4.1** For almost every $c_0 \in \partial M_d$ in the sense of the harmonic measure that exists $Q(c_0)$ so that for all $n \geq n_0(c_0)$ and $M^* \geq M_0^*$

$$|u_n - u_0||D_u \Psi(u)| \leq \frac{|Q(c_0)|}{D_z f_{c(u)}^N(z_n(u))}.$$  

**Proof.** Calculate

$$D_u \Psi(u) = (D_c \Psi^{-1}(c)_{|c = c(u)})^{-1} = \frac{D_z \Psi_{c(u)}(z)_{|z = u}}{T(c(u))}$$

from Fact 4.1

Then,

$$D_z \Psi_{c(u)}(z)_{|z = u} = d_{S_n}^N \cdot D_z \Psi_{c(u)}(z)_{|z = u} \cdot \left(D_z f_{c(u)}^N(z)_{|z = c(u)}\right)^{-1}.$$  

By Koebe’s one-quarter lemma, for $n$ large enough,

$$|D_z \Psi_{c(u)}(z)_{|z = u} d_{S_n}^N| \leq K_1 d_{N_4} \cdot \left(J(c(u)), f_{S_n}^N(c(u)) \right) \leq K_2 d_{N_4}^{N_4}$$

(16)

since $f_{S_n}^N(c(u))$ belongs to a bounded set fixed by the Yoccoz puzzle construction.

From estimates $[15, 16]$ and the fact that $|\log T|$ is bounded on almost every external ray of $M_d$, see Theorem 1.2 [21] or apply the Abel theorem,

$$|u_n - u_0||D_u \Psi(u)| \leq \frac{Q_1(c_0)}{D_z f_{c(u)}^N(z)_{|z = c(u)}}.$$  

Finally, the point $z = c(u)$ can be replaced by $z = z_n(u)$ by the bounded distortion of $f_{c(u)}^N$ on $\beta_{S_n + N_4, c(u)}$ which holds if $M^*$ is sufficiently large.

$\square$
Proof of the Proposition. Start with $\Psi_c(z) = f_c^{-S_n} \circ \Psi_c \left(z^{d^{S_n}}\right)$ which leads to

$$\log D_z \Psi_c(z)_{z=u_n} = \log D_z f_c^{-S_n} \circ \Psi_c \left( u_n^{d^{S_n}} \right) + \log D_z \Psi_c \left( u_n^{d^{S_n}} \right) + \log d^{S_n} u_n^{d^{S_n-1}}.$$ 

Under differentiation with respect to $c$ the last term drops out. The derivative of the second one is bounded independently of $n$, thus after multiplying by $D \Psi(u)$ and integrating from $u_0$ to $u_n$ it goes to 0 with $n$ by Lemma 4.1. Hence, only the first term requires closer attention. Recall the nonlinearity $n f = D^2 f$. 

$$D_c \log D_z f_c^{-S_n} \circ \Psi_c \left( u_n^{d^{S_n}} \right) = (n f_c^{-S_n} \left( u_n^{d^{S_n}} \right)) D_c \Psi_c \left( u_n^{d^{S_n}} \right) - D_c \log D_z f_c^{S_n}(z)_{z=z_n(u)}.$$ 

Here the first term is bounded independently of $n$ by the bounded distortion of $f_c^{S_n}$ and goes to 0 after integrating from $u_0$ to $u_n$ so we concentrate on the last one.

$$D_c \log D_z f_c^{S_n}(z)_{z=z_n(u)} = \sum_{k=1}^{S_n-1} (n f_S^{S_n-k} \left( f_c^{k} (z_n(u)) \right).$$

Since the distortion of $f_c^{S_n-k}$ is bounded and in fact can be made as small as needed by adjusting construction parameters $M^*$, for every $\varepsilon > 0$ and $n$ large enough

$$\left| D_c \log D_z f_c^{S_n}(z)_{z=z_n(u)} \right| \leq \varepsilon \sum_{k=1}^{S_n-1} \left| D_z f_c^{S_n-k} \left( f_c^{k} (z_n(u)) \right) \right|.$$ 

By Lemma 4.1, the integral of this term when $u$ changes from $u_0$ to $u_n$ is bounded by

$$Q(c_0) \varepsilon \sum_{k=1}^{S_n-1} \left| D_z f_c^{k}(z)_{z=z_n(u)} \right|^{-1}.$$ 

Since the sum in this formula is uniformly bounded by Theorem 1.2 of [21] and $\varepsilon$ can be arbitrarily small, Proposition 5 is proved.

Proof of Theorem 3. We write using one after another Theorem 1, then Proposition 5 and Fact 4.1.
\[ \frac{1}{D_z Y_{c_0}(z)|_{z=c_0}} = \lim_{n \to \infty} \frac{1}{D_z Y_{c_0}(z)|_{z=\Psi_{c_0}(u_n)}} = \lim_{n \to \infty} \frac{D_z \Psi_{c_0}(z)|_{z=u_n}}{D \Psi(u_n)} = \lim_{n \to \infty} \frac{D_z \Psi_{c_0}(z)|_{z=u_n}}{D \Psi(u_n)} = \lim_{n \to \infty} \frac{D_z \Psi_{c_0}(z)|_{z=u_n}}{D \Psi(u_n)} = \frac{D_z \Psi^{-1}(c_n)}{D_z \Psi^{-1}(c_n)|_{z=c_n}} = \lim_{n \to \infty} T(c_n). \] 

Let us recall now Theorem 1.2 of [21] which says that for Lebesgue almost every \( \alpha \in [0,1] \) there exist \( K > 0 \) and \( \lambda > 1 \) such that for every \( n > 0 \) and \( c \in \Gamma(\alpha) \) the estimate \( |D_z f^n(c)| \geq K \lambda^n \) holds.

Hence, for any such \( \alpha \) and \( c_0 = \lim_{r \to 1^+} \Psi(re^{2\pi i \alpha}) \)

\[
\lim_{c \to c_0, c \in \Gamma(\alpha)} T(c) = T(c_0)
\]

by the Lebesgue dominated convergence theorem applied the sum in formula (17).

Taking into account equation (17) the proof of Theorem 3 is complete.

5 Appendix

5.1 Proof of claim (iv) from Fact 1.1.

By Proposition 4 of [21], every point \( z \) from \( \partial Z \setminus J_{c_0} \) is contained in Yoccoz piece \( Y_z \) so that \( z \) is separated from the boundary of \( Y_z \) by an annulus of the modulus \( m \).

By the construction of box domains in [21], the boundary of \( Y_z \) consists of a finite number of pieces of the fixed Green equipotential line and hyperbolic geodesics of \( \mathbb{C} \setminus J_{c_0} \). If \( \gamma \) from (iv) of Fact 1.1 intersected \( Y_z \) then, from the first part of (iv), it would have to land at a point of \( J_{c_0} \) which does not belong to the interior of \( Z \). This would mean that the landing point \( c_0 \) of \( \gamma \) is non-recurrent point, a contradiction. Therefore, \( \gamma \) is disjoint from \( Y_z \) and by Teichmuller’s module theorem, for every \( \xi \in \gamma \) close enough to \( c_0 \),

\[ d_H(\xi, J_{c_0}) \leq (1 + Ce^{-m}) d_H(\xi, Z), \]

\( C > 0 \) is a universal constant if \( m > 5 \). Since \( m \to \infty \) when \( \xi \in \gamma \) tends to \( c_0 \), the limit from (iv) must be 1.

5.2 Proof of Theorem 5

Non-hyperbolic systems are often studied by taking piecewise defined iterates of the map which have some expansion and bounded distortion properties. In the
case of uni-critical polynomials this leads to the construction of induced sequences of \textit{box mappings}, [20]. We will follow the description of induced box dynamics for generic parameters $c \in \partial \mathcal{M}_d$ with respect to the harmonic measure obtained in [24]. The picture is largely simplified due to the fact that almost all returns are \textit{non-close} and that only dynamics of the central branches is needed to prove the existence of hedgehogs. Fact 5.1 follows directly from the definition of the induced box mappings and Proposition 7 from [24].

\textbf{Fact 5.1} If $c^* \in \mathcal{M}_d$ is typical with respect to the harmonic mesure then there is an infinite induced sequence of proper analytic maps $(\psi_{p,c^*})_{p=0}^\infty$ of degree $d$ with only one critical point at 0, with their ranges $B_{p,c^*}$ and domains $B_{p+1,c^*}$ which are Jordan disks for all $p$ and satisfy $B_{p+1,c^*} \subset B_{p,c^*}$. Every $\psi_{p,c^*}$ is an iterate of $f_{c^*}$.

Moreover,

- the sequence of $(\psi_{p,c^*})_{p=0}^\infty$ shows an exponential decay of geometry,
  
  $$m_p(c^*) := \text{mod} (B_{p,c^*} \setminus B_{p+1,c^*}) \leq \lambda_c^p,$$

  where $\lambda_c > 1$,

- every $\psi_{p,c^*}$ has a proper analytic extension of the degree $d$ to $D_{p,c^*}$, $B_{p,c^*} \supset D_{p,c^*} \supset B_{p+1,c^*}$, with the range $B_{p-1,c^*}$.

We are ready to prove Theorem 5. Let $c^* \in \partial \mathcal{M}_d$ be a typical parameter with respect to the harmonic measure and $(\psi_{p,c^*})_{p=0}^\infty$ the corresponding induced sequence. Since $c^*$ is fixed, we will drop it from the notation whenever there is no confusion.

Let $\epsilon$ and $m$ be the parameters from the definition of hedgehog neighborhoods, Theorem 5. We choose a large $k < p$ so that $1/d^k < \epsilon/10$ and $m_p(c^*) \geq 10md^k$. Since $\mathcal{J}_{c^*}$ is connected, there is a continuum $C_p \subset \mathcal{J}_{c^*}$ which traverses $B_{p-1} \setminus B_p$ for $p$ large enough. Let us put

$$\Phi_{k,p} = \psi_p \circ \cdots \circ \psi_{p+k-1}$$

which is a proper analytic of map of degree $d^k$ on the annulus $A = \Phi_{k,p}^{-1}(B_{p-1} \setminus B_p) \subset D_{p+k}$. Therefore, the modulus of $A$ is at least $10m$.

Since for every $p > 0$, $\psi_p(0) \in B_{p-1}$ and $\psi_p$ is the composition of $z^d$ with a univalent map of a vanishing distortion when $p$ tends to $\infty$, the preimages $\Phi_{p,k}^{-1}(C_p)$ form a $5/d^k$-net relative to the size of the annulus $A$. We have constructed a $(m,\epsilon)$-hedgehog layer around $c^* \in \mathcal{J}_{c^*}$. Since hedgehog neighborhoods are quasi-conformal invariants, their existence at $c^* \in \partial \mathcal{M}_d$ follows from Fact 1.1.

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