The semi-infinite cohomology of affine Lie algebras

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Abstract

We study the semi-infinite or BRST cohomology of affine Lie algebras in detail. This cohomology is relevant in the BRST approach to gauged WZNW models. Our main result is to prove necessary and sufficient conditions on ghost numbers and weights for non-trivial elements in the cohomology. In particular we prove the existence of an infinite sequence of elements in the cohomology for non-zero ghost numbers. This will imply that the BRST approach to topological WZNW model admits many more states than a conventional coset construction. This conclusion also applies to some non-topological models.

Our work will also contain results on the structure of Verma modules over affine Lie algebras. In particular, we generalize the results of Verma and Bernstein-Gel’fand-Gel’fand, for finite dimensional Lie algebras, on the structure and multiplicities of Verma modules.

The present work gives the theoretical basis of the explicit construction of the elements in cohomology presented previously. Our analysis proves and makes use of the close relationship between highest weight null-vectors and elements of the cohomology.

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1 Introduction and summary of results

The present work studies the semi-infinite or BRST cohomology of affine Lie algebras. The motivation comes from the quantization of Wess-Zumino-Novikov-Witten (WZNW) models. These models play an essential part in the understanding and classification of conformal field theories. The BRST symmetry arises as a consequence of the gauging of a WZNW model w.r.t. a subgroup \[ \mathfrak{g} = \mathfrak{g}_k \oplus \tilde{\mathfrak{g}}_{\tilde{k}}. \] Here \( g_k \) and \( \tilde{g}_{\tilde{k}} \) correspond to the same finite dimensional Lie algebra, but have different central elements \( k \) and \( \tilde{k} = -k - 2c_{\bar{g}} \) (see section 2 for notation). The latter affine Lie algebra corresponds to an auxiliary, and in general non-unitary, WZNW model that arose in the derivation in [1]. The physical states in the gauged WZNW model are now given by the non-trivial elements of the resulting BRST cohomology.

In [2] it was proved that the BRST approach was equivalent to the conventional coset construction, so that the states were ghost-free and satisfied the usual highest weight conditions w.r.t. the subalgebra \( \mathfrak{g}_k \). The conditions for this proof was that one selected a specific range of representations for the auxiliary WZNW model. For the original ungauged WZNW model the range of representations were assumed to be the integrable ones.

In this work we will consider completely general highest weight representations (an analogous treatment may be given for lowest weight representations). The motivation for this is that it may be that a more general situation than in ref.[2] is the physically relevant one. Our analysis of the cohomology is most straightforwardly applied to the case when the gauged subgroup coincides with the original group i.e. when we have a topological WZNW model. But, as we will show, it also generalizes to the most important class of non-topological models, namely those in which the ungauged WZNW model is unitary.

In [3] the explicit construction of elements in the BRST cohomology was considered. The procedure presented there for obtaining these elements showed that they were intimately related to certain null-vectors. The key to the construction was to make a selection of null-vectors that generated the states in the cohomology. It turned out that these null-vectors are the highest weight vectors. Then by using the explicit form of highest weight null-vectors given by Malikov, Feigin and Fuchs [4], the elements may be constructed. Our work here may be seen as the theoretical basis of this construction. We will here prove that the procedure in [3] will always
generate non-trivial states in the cohomology. We will also prove that the ghost numbers that appeared in the construction are the only possible ones. The ghost numbers will be uniquely determined by the representations of the algebras involved, and for fixed representations only one value (and its negative) will occur. It is still an open question whether the construction provides all the possible states. We also lack a general result on the dimensionality of the cohomology.

The plan of the paper and its main results are the following. In section 2 we give basic definition and facts for affine algebras and associated modules. In section 3 we discuss the structure of Verma modules. This is important since our analysis of the cohomology relies very heavily on this structure, in particular, on the embeddings of Verma modules into Verma modules. We make extensive use of a technique due to Jantzen to perturb the highest weight of a reducible Verma module to obtain an irreducible one. This perturbation gives also a filtration of modules in a given Verma module. Section 3 contains results on the structure of Verma modules, which we have been unable to find in the literature. The main results are Theorem 3.10 and Theorem 3.11. These are generalizations of results of Verma and Bernstein, Gel’fand and Gel’fand, respectively, for finite dimensional Lie algebras and of Rocha-Caridi and Wallach for affine Lie algebras with highest weights on Weyl orbits through dominant weights. The proof of Theorem 3.11 is almost identical to the proof of the finite dimensional case given in Theorem 7.7.7 (which is used also in ). The proof of Theorem 3.10 only partly coincides with , as the latter does not extend to the case of antidominant weights.

In section 4 we proceed to introduce the BRST formalism. Most of the material (except Lemma 4.2) is well-known. In particular, we recapitulate a theorem due to Kugo and Ojima. This theorem will partly be used in the main section, section 5. It is also conceptually important in understanding the basic mechanism behind the appearance of elements in the BRST cohomology for non-zero ghost numbers, which we now explain. The theorem, which applies only to irreducible modules, states that elements in the cohomology form either singlet or doublet (singlet pair) representations w.r.t the BRST algebra. Furthermore, elements that are trivial or outside the cohomology form so-called quartets in the terminology of i.e. sets four states, in which two of the elements are BRST exact. In order to obtain an irreducible module, we use a trick due to Jantzen, to perturb a reducible module into an irreducible one. In the irreducible case one may prove (Corollary 5.2), that only ghost-free highest weight states are BRST non-trivial. As the perturbation
is taken to zero and the module becomes reducible, certain quartets will evolve into singlet pairs in the following way. Two of states of the quartet will remain in the irreducible module and will then form a singlet pair in this module. The two other states will become null-states. One of the main results in this paper (Theorem 5.12) is the determination of the relevant null-states. This theorem gives the necessary and sufficient conditions on the null-states to be part of a quartet, that will contain a singlet pair as the perturbation is set to zero. The implications of the theorem is exploited in Theorems 5.14 and 5.15, which give the necessary and sufficient conditions on the ghost-numbers and weights for which the cohomology is non-trivial. In particular in Theorem 5.15 a sequence of non-trivial BRST invariant states is proved to exist. This sequence is exactly the one for which the construction has been given in [3]. The ghost numbers appearing are $\pm p$, where $p = l(\tilde{\lambda}) - l(\lambda)$ and $l(\lambda)$ is the length of a Weyl transformation associated with $\lambda$ (see section 3). This means that for given highest weights $\lambda$ and $\tilde{\lambda}$ of the original and auxiliary sectors, $|p|$ is fixed to exactly one value. By Theorem 5.14 these ghost numbers and weights are the only non-trivial ones.

Let us also address the question of how the embedding of $g$ into a larger algebra may affect our results. As our approach relies on the use of null-vectors, the crucial question is what happens to the relevant null-vectors as $g$ is embedded. If the null-vector w.r.t. $g$ will cease to be null in the larger algebra, then the entire quartet, to which the vector belongs for non-zero perturbation, will remain a quartet as Jantzen’s perturbation is set to zero. Thus the corresponding elements in the cohomology of $g$ will now be exact. In addition, many more elements may disappear from the cohomology group. This is most evident from the construction in [4], where one used non-trivial states at ghost number $p - 1$ ($p > 0$) to construct a BRST non-trivial element of ghost number $p$. In the extreme case the module over the larger algebra is irreducible and all elements, except the one at zero ghost number, will disappear.

There is one case in which the embedding will be straightforward. This will happen when we select integrable representations of the larger algebra. In this case it is known [4] that the irreducible module over the larger algebra is completely reducible w.r.t. to any subalgebra. Hence, the results given here generalize directly. This was the situation analyzed in [2]. Corollary 5.11 proves that the solutions given in [3] for a selected range of representations of the auxiliary sector, are in fact the unique solutions for zero ghost number for any selection of representations of the auxiliary sector.
The existence of extra elements in the cohomology, which have non-zero ghost numbers, implies that the BRST approach to WZNW models is different from the conventional coset approach. This applies to the topological case, but also to the non-topological case, at least when we take integrable representations of the original algebra. The rôle of these extra states is at this point unclear. It may be that their appearance will lead to inconsistencies. One may avoid the states by selecting an appropriate range of representations for the auxiliary sector. Then only ghost free states will appear in the cohomology. This was the situation treated in ref [2]. It may on the other hand be that the extra states are a new and important part of the quantization of WZNW models. In the latter case one may expect that the extra states will be needed to ensure S-matrix unitarity and hence will appear as poles in scattering amplitudes.

2 Preliminaries

Let \( \bar{g} \) be a simple finite dimensional Lie algebra of rank \( r \). We denote by \( g_k \) the corresponding affine Lie algebra of level \( k \). The set of roots of \( \bar{g} \) and \( g \) are \( \bar{\alpha} \in \bar{\Delta} \) and \( \alpha \in \Delta \), respectively. The highest root of \( \bar{g} \) is denoted \( \bar{\psi} \) and its length is taken to be one. The restriction to positive roots are denoted by \( \bar{\Delta}^+ \), \( \Delta^+ \) and to simple roots by \( \bar{\Delta}_s \), \( \Delta_s \). The weight and root lattices of \( \bar{g} \) and \( g \) are \( \bar{\Gamma}_w \), \( \bar{\Gamma}_r \), \( \Gamma_w \) and \( \Gamma_r \). \( \Gamma_r^+ \) is the lattice generated by positive roots. Let \( \Gamma_w^+ = \{ \lambda \in \Gamma_w | \alpha_i \cdot \lambda \geq 0 \text{ for } \alpha_i \in \Delta^s \} \). Let \( \Gamma_w^\rho = \{ \lambda_i \in \Gamma_w^+ | \frac{2\lambda_i \cdot \alpha_j}{(\alpha_j)^2} = \delta_{ij} \text{ for } \alpha_j \in \Delta^s \} \) be the set of fundamental weights. Here \( \lambda_i \cdot \alpha_j \) denotes the invariant scalar product on \( g \) and \( (\alpha_j)^2 = \alpha_j \cdot \alpha_j \). Define \( \rho \) as twice the sum of fundamental weights of \( g \). \( \bar{\rho} \) is the corresponding sum for \( \bar{g} \). \( \rho \) satisfies \( \rho \cdot \alpha_i = (\alpha_i)^2 \), \( \alpha_i \in \Delta^s \). We define the set of antiderdominant weights \( \Gamma_w^- = \{ \lambda \in \Gamma_w | \alpha_i \cdot (\lambda + \rho/2) \leq 0 \text{ for } \alpha_i \in \Delta^s \} \). A weight \( \mu \in \Gamma_w \) is said to be singular if it is orthogonal to at least one positive root and is said to be regular otherwise.

The Weyl group \( W \) of \( g \) is the set of transformations on \( \Gamma_w \) generated by the simple reflections

\[
\sigma_i(\lambda) = \lambda - \frac{2\lambda \cdot \alpha_i}{(\alpha_i)^2} \alpha_i \quad \alpha_i \in \Delta^s.
\]
between weights. Let \( \mu, \nu \in \Gamma_w \) be such that \( \mu - \nu \in \Gamma^+_w \). We then write \( \mu \geq \nu \). If \( \mu - \nu \in \Gamma^+_w / \{0\} \), then this is denoted by \( \mu > \nu \). Two weights \( \lambda \) and \( \mu \) are said to be on the same Weyl orbit if there exists \( w \in W \) such that \( \mu = w(\lambda) \). Similarly, they are said to be on the same \( \rho \)-centered Weyl orbit if \( \mu = w^\rho(\lambda) \).

We make a triangular decomposition of \( g = g^- \oplus h \oplus g^+ \). We will use the notation \( e_\alpha \) for the generators of \( g^+ \), \( f_\alpha \) for those of \( g^- \) and \( h_i \), \( i = 1, \ldots, r + 2 \) for the generators of the Cartan subalgebra \( h \). \( h_i \), \( i = 2, \ldots, r + 1 \) span \( h \), \( h_1 \) is a central element of \( g \) with eigenvalue \( k/2 \) and \( h_0 \) is a derivation. We have a corresponding decomposition of \( \mathcal{U}(g) \), the universal enveloping algebra of \( g \), as \( \mathcal{U}(g) = \mathcal{U}(n_-) \otimes \mathcal{U}(h) \otimes \mathcal{U}(n_+) \).

Let \( M(\lambda) \) denote the highest weight Verma module over \( g \) of highest weight \( \lambda \). The module is generated by a highest weight primary vector \( v_{0\lambda} \) satisfying

\[
e_\alpha v_{0\lambda} = 0
\]

\[
h_i v_{0\lambda} = \lambda_i v_{0\lambda} \quad h_i \in h.
\]

(2.2)

\( M(\lambda) \) admits a weight decomposition

\[
M(\lambda) = \bigoplus_{\eta \in \Gamma^+_w} M_{\eta}(\lambda).
\]

Vectors in \( M_{\nu}(\lambda) \) will be called weight vectors of degree \( \nu \) and their weights differ from the highest weight by \( \nu \). We consider throughout only vectors \( v \in M_{\eta}(\lambda) \) with \( \dim M_{\eta}(\lambda) < \infty \). The dimension of \( M_{\nu}(\lambda) \) is \( P(\nu) \), which is the number of ways \( \nu \) may be written as a linear combination of positive roots with non-negative coefficients. Let \( M'(\lambda) \) be the proper maximal submodule of \( M(\lambda) \). Then \( M(\lambda)/M'(\lambda) \) is irreducible and isomorphic to the unique irreducible \( g \)-module \( L(\lambda) \).

Define a Hermitean form \( \langle ..|.. \rangle \) as the mapping from \( M(\lambda) \times M(\lambda) \) to the complex numbers by

\[
\langle v_{0\lambda}|v_{0\lambda} \rangle = 1
\]

\[
\langle w_{\lambda}|uv_{\lambda} \rangle = \langle u^\dagger w_{\lambda}|v_{\lambda} \rangle,
\]

(2.3)

where \( u \in \mathcal{U}(g) \) and \( ( )^\dagger \) denotes the Hermite conjugation defined by \( e_\alpha^\dagger = f_\alpha \), \( f_\alpha^\dagger = e_\alpha \), \( h_i^\dagger = h_i \). For \( v_\eta, w_\mu \in M_{\eta}(\lambda) \) we clearly have \( \langle w_\mu|v_\eta \rangle = 0 \) for \( \eta \neq \mu \). If \( \eta = \mu \), then \( F(\lambda)_\eta = \langle w_\eta|v_\eta \rangle \) may be viewed as a \( P(\eta) \times P(\eta) \) matrix, whose entries are
polynomials in \( \lambda \). The determinant of \( F(\lambda) \) is given by the Kac-Kazhdan formula

\[
\det F(\lambda) = \text{const.} \prod_{\alpha \in \Delta^+} \prod_{n=1}^{\infty} \left[ (\lambda + \rho/2) \cdot \alpha - \frac{n}{2} \alpha^2 \right]^{P(\eta-n\alpha)} \tag{2.4}
\]

where roots \( \alpha \in \Delta^+ \) are taken with their multiplicities and \( P(\eta) = 0 \) if \( \eta \not\in \Gamma^+ \). The zeros of the determinant are associated with highest weight vectors \( v_\mu \) that occur in \( M(\lambda) \) (see the following section). From eq.\((2.4)\) one may infer that \( \mu = \lambda - n\alpha \), which implies that the Verma module \( M(\mu) \) is a submodule of \( M(\lambda) \). \( M(\lambda) \) is irreducible if and only if there does not exist \( n \in \mathbb{Z} \) and \( \alpha \in \Delta^+ \) such that

\[
(\lambda + \rho/2) \cdot \alpha - \frac{n}{2} \alpha^2 = 0. \tag{2.5}
\]

Notice that this equation will for any imaginary root \( \alpha \) (i.e. \( \alpha^2 = 0 \)) be equivalent to the condition \( k = -c_{\bar{g}} \), where \( c_{\bar{g}} \) is the quadratic Casimir of the adjoint representation of \( \bar{g} \).

3 Structure of embeddings of Verma modules

If the Kac-Kazhdan equation \((2.5)\) has non-trivial solutions for a given module \( M(\lambda) \), then there will exist Verma modules \( M(\mu) \) that are submodules of \( M(\lambda) \). This implies the existence of a \( g \)-homomorphism, \( \phi \in \text{Hom}_g(M(\mu), M(\lambda)) \), such that \( M(\mu) \overset{\phi}{\rightarrow} M(\lambda) \). We will in this section and throughout the rest of this paper assume \( k \neq -c_{\bar{g}} \), so that solutions to eq.\((2.5)\) only occur for real roots \( \alpha \). The structure of embeddings is most clearly depicted through a filtration due to Jantzen \cite{5}. Introduce

\[
z = \sum_{\lambda \in \Gamma^+} z_\lambda \lambda \text{, where } z_\lambda \text{ are non-zero complex numbers.}
\]

Consider the one-parameter family of weights \( \lambda_\epsilon = \lambda + \epsilon z \). If \( \lambda \) is a weight of a reducible module \( M(\lambda) \) or \( M^*(\lambda) \) and \( z_i \neq 0 \), then for \( 0 < |\epsilon| \ll 1 \), \( M(\lambda_\epsilon) \) and \( M^*(\lambda_\epsilon) \) are irreducible. We now define a filtration

\[
M(\lambda_\epsilon) \supset M^{(1)}(\lambda_\epsilon) \supset M^{(2)}(\lambda_\epsilon) \supset \ldots \tag{3.1}
\]

by

\[
M^{(n)}(\lambda_\epsilon) = \{ v \in M(\lambda_\epsilon) \mid \langle w^*|v \rangle \text{ is divisible by } \epsilon^n \text{ for any } w^* \in M^*(\lambda_\epsilon) \}. \tag{3.2}
\]

We will often write \( M_\epsilon \) for \( M(\lambda_\epsilon) \) etc. for \( M^{(n)} \). If \( v = uv_{0\lambda}, u \in \mathcal{U}(g) \), then we write \( v_\epsilon = u v_{0\lambda} \). In the dual case one may define a corresponding filtration. In the limit
\( \epsilon \to 0 \) this induces a filtration of modules in \( M(\lambda) \)

\[
M(\lambda) \supset M^{(1)}(\lambda) \supset M^{(2)}(\lambda) \supset \ldots
\]  

(3.3)

Note that Jantzen’s filtration is hereditary: Let \( M(\mu) \in M^{(s)}(\lambda) \) and \( M(\nu) \in M^{(t)}(\mu) \). Then \( M(\nu) \in M^{(s+t)}(\lambda) \).

Any irreducible subquotient of a \( g \)-module \( M(\lambda) \) is isomorphic to an irreducible \( g \)-module \( L(\mu) \), \( \lambda - \mu \in \Gamma^+ \). Denote by \( (M(\lambda) : L(\mu)) \) the multiplicity of \( L(\mu) \) in \( M(\lambda) \). \( M^{(1)}(\lambda) \) is the maximal proper submodule of \( M(\lambda) \) and hence \( M(\lambda)/M^{(1)}(\lambda) \) is isomorphic to the irreducible module \( L(\lambda) \). We will call the vectors in \( M^{(1)} \) null-vectors of \( M(\lambda) \). We define \( \pi_L \) to be the projection \( M(\lambda) \xrightarrow{\pi_L} L(\lambda) \).

The submodules of a given Verma module are generally not all of Verma type. It is convenient to introduce the notion of primitive vectors. Let \( V \) be a \( g \)-module. A vector \( v_\lambda \in V \) is said to be primitive if there exists a submodule \( U \) of \( V \) such that \( v_\lambda \not\in U, xv_\lambda \in U \) for any \( x \in n_+ \).

(3.4)

\( \lambda \) is called a primitive weight. Highest weight vectors are clearly primitive, but in general they do not exhaust all primitive vectors, even in the case of finite dimensional algebras, as was first noted in [11]. In fact, there may be infinitely many more primitive vectors than highest weight vectors (see [12] for an example for finite dimensional algebras). Any module \( V \) is generated by its primitive weights as a \( g \)-module.

We will call a module which is generated by acting freely with \( \mathcal{U}(n_-) \) on a primitive vector, which is not of highest weight type, a Bernstein-Gel’fand (BG) module. The corresponding primitive vector will be called a Bernstein-Gel’fand primitive vector.

Although every zero in the determinant eq.(2.4), i.e. every \( (\alpha, n) \) for which the Kac-Kazhdan eq.(2.5) is satisfied, corresponds to a highest weight vector in \( M(\lambda) \) (cf. Proposition 3.8), the converse is in general not true. For a given \( \lambda \) there are usually more highest weight vectors than solutions \( (\alpha, n) \). Let \( \text{Hom}_g(M(\mu), M(\lambda)) \neq 0 \) for a pair \( (\alpha, n) \) in eq.(2.5) with \( \alpha \) real i.e. \( \mu = \lambda - n\alpha, n \geq 1 \) and \( \alpha \in \Delta^+ \cap \Delta^R \), where \( \Delta^R \) is the set of real roots. Then we may write

\[
\mu = \sigma^\lambda_\alpha < \lambda.
\]  

(3.5)

The inequality ensures that a solution to eq.(2.5) exists. In the form eq.(3.5) it is clear that by iteration, we will find new highest weight vectors not given by solutions to the Kac-Kazhdan equation for \( \lambda \). It also follows that \( M(\lambda) \) is irreducible if and only if \( \lambda \) is antidominant. Notice that this requires \( k < -c_g \).
Let us proceed to give a more precise classification of highest weight vectors in $M(\lambda)$ in terms of Weyl transformations. Define the Bruhat ordering on $W$. Let $w, w' \in W$. We write $w' \rightarrow w$ if there exists $\alpha \in \Delta^+ \cap \Delta^R$, such that $w = \sigma_\alpha w'$ and $l(w) = l(w') + 1$. We write $w' \prec w$ if there are $w_0, w_1, \ldots, w_p \in W$ such that $w' = w_p \rightarrow w_{p-1} \rightarrow \ldots \rightarrow w_1 \rightarrow w_0 = w$. It may be shown that $w' \prec w$ if and only if the reduced expressions $w' = \sigma_j_1 \ldots \sigma_j_p$ and $w = \sigma_i_1 \ldots \sigma_i_q$ are such that $(j_1, \ldots, j_p)$ is obtained by deleting $q - p$ elements from $(i_1, \ldots, i_q)$.

By combining Theorem 4.2 in [14] with eq.(3.5) we have the following.

**Theorem 3.1.** A Verma module $M(\lambda)$ contains an irreducible subquotient $L(\mu)$ if and only if the following condition is satisfied:

\[ (*) \lambda = \mu, \text{ or there exists a sequence of positive roots } \alpha_1, \alpha_2, \ldots, \alpha_k \text{ and } \text{a sequence of weights } \lambda = \mu_1, \mu_2, \ldots, \mu_k, \mu_{k+1} = \mu \text{ such that } \mu_{i+1} = \sigma_{\alpha_i}(\mu_i) < \mu_i \text{ for } i = 1, 2, \ldots, k. \]

**Lemma 3.2.** Let $\mu \in \Gamma_w$. Then there exists $w \in W$ and a unique $\lambda + \rho/2 \in \Gamma_w^+ (k > -c_\gamma)$ or $\lambda \in \Gamma_w^- (k < -c_\gamma)$ such that $\mu = w^\rho(\lambda) = \sigma_{i_1}^\rho \sigma_{i_{n-1}}^\rho \ldots \sigma_{i_1}^\rho (\lambda)$, where $i_1, \ldots, i_n$ denote the simple roots $\alpha_{i_1}, \ldots, \alpha_{i_n}$ with

\[ (***) \mu = \lambda, \text{ or } \mu \neq \lambda \text{ and } \sigma_{i_{p+1}}^\rho \sigma_{i_p}^\rho \ldots \sigma_{i_1}^\rho (\lambda) < \sigma_{i_p}^\rho \ldots \sigma_{i_1}^\rho (\lambda) (k > -c_\gamma) \text{ or } \sigma_{i_{p+1}}^\rho \sigma_{i_p}^\rho \ldots \sigma_{i_1}^\rho (\lambda) > \sigma_{i_p}^\rho \ldots \sigma_{i_1}^\rho (\lambda) (k < -c_\gamma), \ p = 1, 2, \ldots, n - 1. \]

**Proof.** Consider $k < -c_\gamma$. For $\mu \in \Gamma_\gamma^-$ the lemma is trivial ($w = 1$). Let $\mu = \mu_1 \notin \Gamma_\gamma^-$. Then there exists $\alpha_1 \in \Delta^+$ such that $n_1 = (2\mu_1 + \rho) \cdot \alpha_1/\alpha_1^2 \in \mathcal{N} = 1, 2, 3, \ldots$. This implies that $\mu_2 = \sigma_{\alpha_1}^\rho(\mu_1)$ satisfies $\mu_2 < \mu_1$. Let $\lambda + \rho/2 \in \Gamma_\gamma^-$ be such that $(\mu_2 - \lambda)^2 \geq 0$ (which is always possible, as can be seen by an explicit parametrization of the weights). We have $(\mu_2 - \lambda)^2 = (\mu_1 - \lambda)^2 + n_1(2\lambda + \rho) \cdot \alpha_1$ and, therefore, $(\mu_2 - \lambda)^2 < (\mu_1 - \lambda)^2$. If $\mu_2 \notin \Gamma_\gamma^-$ we can continue this process. We get a sequence of weights $\mu_1 = \mu, \mu_2, \ldots, \mu_r$ with $(\mu_{p+1} - \lambda)^2 < (\mu_p - \lambda)^2$ and $\mu_{p+1} = \sigma_{\alpha_p}^\rho(\mu_p) < \mu_p$, $p = 1, \ldots, r - 1$. This sequence must terminate after a finite number of steps, since $(\mu_r - \lambda)^2 \geq 0$ from $(\mu_2 - \lambda)^2 \geq 0$. But this can only happen if the last weight $\mu_r$ of the sequence satisfies $\alpha_i \cdot (2\mu_r + \rho) \leq 0$ for all $\alpha_i \in \Delta^+$ i.e. $\mu_r \in \Gamma_\gamma^+$. We now prove the uniqueness. Assume $w, w' \in W$ and $\lambda, \lambda' \in \Gamma_\gamma^+$ such that $\mu = w^\rho(\lambda) = w'^\rho(\lambda')$. Then $\lambda = w^{-1}w'^\rho(\lambda')$. This implies $\lambda = \lambda'$, as follows by an adaption of [14]. Lemma A in section 13.2, to the present case. The case $k > -c_\gamma$ is proved in a completely
analogous fashion.

**Lemma 3.3.** Let $\mu$ and $\lambda$ be as in Lemma 3.2 and $\mu_0 = \lambda$, $\mu_1 = \sigma_i^\rho (\mu_0)$, $\mu_2 = \sigma_i^\rho (\mu_1)$, $\ldots$, $\mu_n = \sigma_i^\rho (\mu_{n-1}) = \mu$, where $\sigma_i^\rho$, $k = 1, 2, \ldots, n$ are simple reflections satisfying (**). Then for $k > -c_\bar{g}$, $\text{Hom}_g (M(\mu_p), M(\mu_{p-1})) \neq 0$, $p = 1, 2, \ldots, n$ and for $k < -c_\bar{g}$, $\text{Hom}_g (M(\mu_{p-1}), M(\mu_p)) \neq 0$, $p = 1, 2, \ldots, n$.

**Proof.** The proof is by explicit construction. Consider e.g. $k < -c_\bar{g}$ and $\mu_p = \sigma_i^\rho (\mu_{p-1})$. We take the $sl_2$ subalgebra generated by $e_{i_p}$, $f_{i_p}$ and $h_{i_p}$ satisfying $[f_{i_p}, e_{i_p}] = h_{i_p}$ and $[h_{i_p}, f_{i_p}] = f_{i_p}$. Let $v_{\mu_p}$ be the highest weight vector that generates $M(\mu_p)$ and $h_{i_p} v_{\mu_p} = \mu_i v_{\mu_p}$. Then it is straightforward to check that $v_{\mu_{p-1}} = (f_{i_p})^{\mu_{p-1}+1} v_{\mu_p}$ is a highest weight vector and it will generate a submodule isomorphic to $M(\mu_{p-1})$. Hence, $\text{Hom}_g (M(\mu_{p-1}), M(\mu_p)) \neq 0$.

By Theorem 3.1, Lemma 3.2 and Lemma 3.3 and we have the following:

**Proposition 3.4.** Let $\mu \in \Gamma_w$. Then there exists a unique $\lambda + \rho/2 \in \Gamma_w^+$ ($k > -c_\bar{g}$), or $\lambda \in \Gamma_w^-$ ($k < -c_\bar{g}$), such that $\text{Hom}_g (M(\mu), M(\lambda)) \neq 0$ ($k > -c_\bar{g}$), or $\text{Hom}_g (M(\lambda), M(\mu)) \neq 0$ ($k < -c_\bar{g}$). Furthermore, if $\nu \in \Gamma_w$ and $\text{Hom}_g (M(\mu), M(\nu)) \neq 0$, then

$$[\dim \text{Hom}_g (M(\mu), M(\lambda))] [\dim \text{Hom}_g (M(\nu), M(\lambda))] \neq 0 \text{ for } k > -c_\bar{g} \text{ or}$$

$$[\dim \text{Hom}_g (M(\lambda), M(\mu))] [\dim \text{Hom}_g (M(\lambda), M(\nu))] \neq 0 \text{ for } k < -c_\bar{g}.$$

**Lemma 3.5.** Let $\lambda + \rho/2 \in \Gamma_w^+$ ($k > -c_\bar{g}$) or $\lambda \in \Gamma_w^-$ ($k < -c_\bar{g}$), $w \in W$ and $\alpha \in \Delta^+ \cap \Delta_R$. Then

(i) $\sigma_\alpha^\rho w^\rho (\lambda) < w^\rho (\lambda) \Rightarrow l(\sigma_\alpha w) > l(w)$ for $k > -c_\bar{g}$ or $l(\sigma_\alpha w) < l(w)$ for $k < -c_\bar{g}$.

(ii) $l(\sigma_\alpha w) > l(w)$ for $k > -c_\bar{g}$ or $l(\sigma_\alpha w) < l(w)$ for $k < -c_\bar{g} \Rightarrow \sigma_\alpha^\rho w^\rho (\lambda) \leq w^\rho (\lambda)$

**Proof.** The proof of (i) is identical to that of Lemma 7.7.2 (ii) in [14] (cf. [3], Lemma 8.2). Note that in the proof of Lemma 7.7.2 in [14], $\lambda \in \Gamma^+_w$ is assumed. The weaker condition on $\lambda$, assumed in our case, does not affect (i). We prove (ii) for $k > -c_\bar{g}$. 

9
We have
\[ \sigma^\rho_\alpha w^\rho(\lambda) = w^\rho(\lambda) - n\alpha. \]

Here \( n = \frac{(2w^\rho(\lambda) + p)\cdot \alpha}{\alpha^2} \in \mathbb{Z} \) If \( n < 0 \) then \( \sigma^\rho_\alpha w^\rho(\lambda) > w^\rho(\lambda) \). By (i), we get \( l(\sigma_\alpha w) < l(w) \) which is a contradiction. Hence, \( n = 0, 1, 2, \ldots \) and (ii) follows. The proof for \( k < -c_\beta \) is analogous. \( \blacksquare \)

The following two lemmas are direct generalizations of [14], Lemma 7.7.4 and Lemma 7.7.5 (cf. [14], Lemma 8.4 and Lemma 8.5).

**LEMMA 3.6.** Let \( w_1, w_2 \in W, \gamma \in \Delta^+ \cap \Delta^R \) and \( \alpha \in \Delta^s \), with \( \gamma \neq \alpha \). The following conditions are equivalent:

(i) \( \sigma_\alpha w_1 \leftarrow^\alpha w_1 \) and \( \sigma_\alpha w_1 \leftarrow^\gamma w_2 \)

(ii) \( w_2 \leftarrow^\alpha \sigma_\alpha w_2 \) and \( w_1 \leftarrow^\gamma \sigma_\alpha w_2 \).

**LEMMA 3.7.** Let \( w \in W \) and \( \gamma \in \Delta^+ \cap \Delta^R \) be such that \( l(w) > l(\sigma_\gamma w) \). Then \( w \succ \sigma_\gamma w \).

We proceed to obtain results on the \( g \)-homomorphisms \( M(\nu) \rightarrow M(\mu) \). First we have the following:

**PROPOSITION 3.8.** (cf. [14], Lemma 7.6.11). Let \( \nu \in \Gamma_w, \alpha \in \Delta^+, \mu = \sigma^\rho_\alpha(\nu) \). Assume \( \mu \leq \nu \). Then Hom\(_g\) (\( M(\mu), M(\nu) \)) \( \neq 0 \).

**PROOF.** The proof is essentially the same as in [14]. The case \( \mu = \nu \) is trivial, so we assume \( \mu < \nu \). We consider only \( k > -c_\beta \) as the the case \( k < -c_\beta \) is analogous. By Lemma 3.2 there exists \( w \in W \) and \( \lambda' + \rho/2 \in \Gamma^+_w \) such that \( \nu = w^\rho(\lambda') \). Let \( w = \sigma_{\alpha_n} \ldots \sigma_{\alpha_1} \) be a reduced expression of \( w \) in terms of simple reflexions and

\[ \nu_0 = \lambda', \ \nu_1 = \sigma^\rho_{\alpha_1}(\nu_0), \ \nu_2 = \sigma^\rho_{\alpha_2}(\nu_1), \ldots, \nu_n = \sigma^\rho_{\alpha_n}(\nu_{n-1}) = \nu \]

\[ \mu_0 = \lambda, \ \mu_1 = \sigma^\rho_{\alpha_1}(\mu_0), \ \mu_2 = \sigma^\rho_{\alpha_2}(\mu_1), \ldots, \mu_n = \sigma^\rho_{\alpha_n}(\mu_{n-1}) = \mu. \]

Then \( \nu_0 = w^\rho(\mu_0) \) for some \( w' \in W \) (from \( \nu_0 = w^{-1}\rho(\nu) = w^{-1}\rho \sigma^\rho(\mu) \) and \( \mu_0 + \rho/2 \in \Gamma^+_w \), hence \( \mu_0 - \nu_0 \in \Gamma^+_r \). On the other hand, \( \mu_n - \nu_n = -m\alpha, \ m > 0 \). Since the same element of \( W \) transforms \( \mu \) and \( \nu \) into \( \mu_p \) and \( \mu_p \), respectively, \( p = 0, 1, 2, \ldots, n, \mu_p \) is transformed from \( \nu_p \) by a reflexion \( \sigma^\rho_{\gamma_p} (\gamma_p \in \Delta^+) \), hence \( \mu_p - \nu_p \in \Gamma^+_r \) or
\( \nu_p - \mu_p \in \Gamma^+_r \). Hence, there exists a smallest integer \( k \) such that \( \mu_k - \nu_k \in \Gamma^+_r \) and \( \mu_{k+1} - \nu_{k+1} \in -\Gamma^+_r \). Now \( \mu_k - \nu_k = \sigma_{\Delta k+1}^0 (\mu_{k+1} - \nu_{k+1}) \). Since \( \mu_{k+1} - \nu_{k+1} \) is proportional to \( \gamma_{k+1} \), it can be seen that \( \sigma_{\Delta k+1} (\gamma_{k+1}) \in \Delta^- \). Hence, \( \gamma_{k+1} = \alpha_{k+1} \) (since \( \sigma_{\Delta k+1} \) permutes all positive roots except \( \alpha_{k+1} \)). The relations \( \mu_{k+1} - \nu_{k+1} \in -\Gamma^+_r \) and \( \mu_{k+1} = \sigma_{\Delta k+1}^0 (\nu_{k+1}) \) imply \( \text{Hom}_g (M(\mu_{k+1}), M(\nu_{k+1})) \neq 0 \) (Lemma 3.3). On the other hand \( M(\mu_{k+2}) = M(\sigma_{\Delta k+2}^0 (\mu_{k+1})) \) so that \( \text{Hom}_g (M(\mu_{k+2}), M(\mu_{k+1})) \neq 0 \). Hence, \( \text{Hom}_g (M(\mu_{k+2}), M(\mu_{k+1})) \neq 0 \). Continuing this step by step we arrive at \( \text{Hom}_g (M(\mu), M(\nu)) \neq 0 \).

As a corollary to this proposition we can generalize results obtained by [15] and [11] for finite dimensional Lie algebras.

**Corollary 3.9.** A necessary and sufficient condition for \( M(\mu) \) to be a submodule of \( M(\nu) \) is that the condition (*) in Theorem 3.1 is satisfied.

Note here the following. Firstly, Theorem 3.1 and Corollary 3.9 imply that a BG module \( V(\mu) \) is a submodule of \( M(\lambda) \) if and only if \( (M(\lambda) : L(\mu)) \geq 2 \). Secondly, if a BG module \( V(\mu) \subset M(\lambda) \), then there exists a \( g \)-homomorphism \( \phi_{V,M} \) such that \( V(\mu) \overset{\phi_{V,M}}{\longrightarrow} M(\mu) \subset M(\lambda) \).

We are now ready to formulate one of the main results of this section namely the dimension of the \( g \)-homomorphisms \( M(\mu) \rightarrow M(\nu) \). This result generalizes the result of Verma [15] for finite dimensional Lie algebras and Rocha-Caridi, Wallach [R] for representations with highest weights on Weyl orbits through dominant weights.

**Theorem 3.10.** Let \( \mu, \nu \in \Gamma_w \). Then \( \dim \text{Hom}_g (M(\mu), M(\nu)) \leq 1 \).

**Proof.** We consider the cases \( k > -c_g \) and \( k < -c_g \) separately.

\( k > -c_g \): By Proposition 3.4 it is sufficient to prove that \( \dim \text{Hom}_g (M(\mu), M(\lambda)) \leq 1 \), where \( \mu = w^\rho (\lambda), \lambda + \rho / 2 \in \Gamma^+_w \). The proof is then similar to that of [R], Lemma 8.14, using induction on \( l(w) \). We only sketch it. For \( l(w) = 0 \) the theorem is trivial. Assume it to be true for \( l(w) < p \). Consider \( l(w) = p \). Let \( i = 1, 2, \ldots, n \) be such that \( \sigma_i^0 (\mu) > \mu \), where \( \sigma_i \) are reflections corresponding to simple roots \( \alpha_i \). Then \( l(\sigma_i w) < l(w) \) (Lemma 3.5) and \( \dim \text{Hom}_g (M(\mu), M(\sigma_i^0 (\mu))) \neq 0 \) (Proposition 3.7). Consider the \( sl_2 \) subalgebra \( g_i \) corresponding to the simple root \( \alpha_i \), \( i = 1, 2, \ldots, n \). \( M(\sigma_i^0 (\mu)) \) is the so-called completion of \( M(\mu) \) w.r.t \( g_i \) and is unique ([R], Proposition 3.6, and [R].
Proposition 8.11). Then, \( \dim \text{Hom}_g(M(\mu), M(\lambda)) = \dim \text{Hom}_g(M(\sigma^\rho_i(\mu)), M(\lambda)) \). By the induction hypothesis \( \dim \text{Hom}_g(M(\sigma^\rho_i(\mu)), M(\lambda)) = 1 \). This gives the theorem in the case \( k > -c_g \).

\( k < -c_g \): We will prove the theorem using essentially the original argument of Verma [13], Theorem 2. By Proposition 3.4 it is sufficient to prove that \( \dim \text{Hom}_g(M(\lambda), M(\mu)) \leq 1 \), where \( \mu = w^\rho(\lambda), \lambda \in \Gamma_w^- \). As \( M(\lambda) \) is irreducible, we can count the number of states in \( M(\mu) \) and \( M(\lambda) \) to establish that if \( \dim \text{Hom}_g(M(\lambda), M(\mu)) \geq 2 \), then

\[
P(\eta) = \dim \text{Hom}_g(\mu) \geq 2 \dim \text{Hom}_g(\lambda - \mu) = 2P(\eta + \lambda - \mu).
\]

This is, however, a contradiction [13], Lemma 3, as can be seen by considering large \( \eta \).

As Theorem 3.10 shows that an element of \( \text{Hom}_g(M(\mu), M(\nu)) \) is either zero or unique (up to a multiplicative constant), we write \( M(\mu) \subset M(\nu) \) whenever \( \text{Hom}_g(M(\mu), M(\nu)) \neq 0 \). We next generalize a result established for finite dimensional Lie algebras [19] and for \( k > -c_g \) in [8].

**Theorem 3.11.** Let \( \mu, \nu \in \Gamma_w^- \). Then there exist \( w, w' \in W \) and \( \lambda + \rho/2, \lambda' + \rho/2 \in \Gamma_w^+(k > -c_g) \), or \( \lambda, \lambda' \in \Gamma_w^-(k < -c_g) \) such that \( \mu = w^\rho(\lambda') \) and \( \nu = w^\rho(\lambda) \).

For \( k > -c_g \) we have:

(i) \( M(\mu) \subset M(\nu) \iff w < w', \lambda = \lambda' \iff (M(\nu) : L(\mu)) \neq 0 \)

(ii) If \( M(\mu) \subset M(\nu), \mu \neq \nu \), then there are \( \mu = \mu_0, \mu_1, \ldots, \mu_n = \nu \) such that \( \mu_{i+1} = w_i^\rho(\lambda), i = 0, 1, \ldots, n - 1 \) with \( l(w_{i+1}) = l(w_i) - 1, w_0 = w, w_n = w' \) and

\[
M(\mu_0) \subset M(\mu_1) \subset M(\mu_2) \subset \ldots \subset M(\mu_n).
\]

For \( k < -c_g \) we have:

(iii) \( M(\mu) \subset M(\nu) \iff w' < w, \lambda = \lambda' \iff (M(\nu) : L(\mu)) \neq 0 \)

(iv) If \( M(\mu) \subset M(\nu), \mu \neq \nu \), then there are \( \mu = \mu_0, \mu_1, \ldots, \mu_n = \nu \) such that \( \mu_{i+1} = w_i^\rho(\lambda), i = 0, 1, \ldots, n - 1 \) with \( l(w_{i+1}) = l(w_i) + 1, w_0 = w, w_n = w' \) and

\[
M(\mu_0) \subset M(\mu_1) \subset M(\mu_2) \subset \ldots \subset M(\mu_n).
\]

**Proof.** (cf [13] and [8]) Consider \( k < -c_g \). The existence of \( w, w' \) follows from Lemma 3.2. By Theorem 3.1 and Corollary 3.9 we have \( M(\mu) \subset M(\nu) \iff (M(\nu) : L(\mu)) \neq 0 \).
0. Assume $M(\mu) \subset M(\nu)$. By Corollary 3.9 there exist $\gamma_1, \ldots, \gamma_n \in \Delta^+$ such that
\[
\mu = w^\rho(\lambda) < \sigma^\rho_{\gamma_1} w^\rho(\lambda) < \ldots < \sigma^\rho_{\gamma_n} \sigma^\rho_{\gamma_{n-1}} \ldots \sigma^\rho_{\gamma_1} w^\rho(\lambda) = w'^\rho(\lambda'),
\]
Then $\lambda = \lambda'$ (Lemma 3.2) and by Lemma 3.5 we have $l(w) < l(\sigma_{\gamma_1} w) < \ldots < l(w')$. Hence, $w < w'$ (Lemma 3.7).

We now assume $w' \prec w$, $\lambda = \lambda'$. Then there exist $\gamma_1, \ldots, \gamma_n \in \Delta^+$ such that
\[
w = w_0 \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_2} w_2 \cdots \xrightarrow{\gamma_{n-1}} w_{n-1} \xrightarrow{\gamma_n} w_n = w'.
\]
By Lemma 3.5 we have $\mu = w^\rho_0(\lambda) \leq w^\rho_1(\lambda) \leq \ldots \leq w^\rho_n(\lambda) = \nu$ and, hence,
\[
M(w^\rho_0(\lambda)) \subset M(w^\rho_1(\lambda)) \subset \ldots \subset M(w^\rho_n(\lambda))
\]
(Proposition 3.8). This proves (iii) and (iv). The cases (i) and (ii) are proved analogously. ■

It is convenient to introduce the concept of length of a weight. Let $\mu \in \Gamma_w$. Then we define the length $l(\mu)$ as the smallest integer $l(w)$ such that $\mu = w^\rho(\lambda)$, $w \in W$, $\lambda + \rho/2 \in \Gamma^+_w$ or $\lambda \in \Gamma^-_w$. We now prove some useful results involving this concept. First we have a result similar to Lemma 3.5.

**Lemma 3.12.** Let $\lambda + \rho/2 \in \Gamma^+_w$ ($k > -c_\beta$) or $\lambda \in \Gamma^-_w$ ($k < -c_\beta$), $w \in W$ and $\alpha \in \Delta^+ \cap \Delta^R$. The following conditions are equivalent
\[
\begin{align*}
(i) & \quad \sigma^\rho_\alpha w^\rho(\lambda) < w^\rho(\lambda) \\
(ii) & \quad l(\sigma^\rho_\alpha w^\rho(\lambda)) > l(w^\rho(\lambda)) \text{ for } k > -c_\beta, \text{ or } l(\sigma^\rho_\alpha w^\rho(\lambda)) < l(w^\rho(\lambda)) \text{ for } k < -c_\beta.
\end{align*}
\]
**Proof.** We prove $(i) \implies (ii)$ for the case $k > -c_\beta$. Let $w'^\rho(\lambda) = \sigma^\rho_\alpha w^\rho(\lambda)$ with $l(w') = l(\sigma^\rho_\alpha w^\rho(\lambda))$. We have
\[
\sigma^\rho_\alpha w'^\rho(\lambda) = w'^\rho(\lambda) > \sigma^\rho_\alpha w^\rho(\lambda) = w^\rho(\lambda),
\]
and, thus, $l(w) = l(\sigma_\alpha w') < l(w')$ (Lemma 3.5). By definition, $l(w) \geq l(w^\rho(\lambda))$ and, hence,
\[
l(w^\rho(\lambda)) < l(w') = l(\sigma^\rho_\alpha w^\rho(\lambda)).
\]
The case $k < -c_\beta$ is proved analogously.

We now prove $(ii) \implies (i)$ for $k > -c_\beta$. We have
\[
\sigma^\rho_\alpha w^\rho(\lambda) = w^\rho(\lambda) - n\alpha.
\]
Here \( n = \frac{(2w^r(\lambda) + \rho - \alpha)}{\alpha} \in \mathbb{Z} \). \( n = 0 \) implies \( \sigma^\rho_{\alpha} w^p(\lambda) = w^p(\lambda) \) and thus \( l(\sigma^\rho_{\alpha} w^p(\lambda)) = l(w^p(\lambda)) \). This contradicts (ii) and, therefore, we have \( n \neq 0 \). If \( n < 0 \) then \( \sigma^\rho_{\alpha} w^p(\lambda) > w^p(\lambda) \). By the implication \( (i) \implies (ii) \), we again contradict (ii). Hence, \( n = 1, 2, \ldots \) and (i) follows. The proof for \( k < -c_\bar{g} \) is analogous. \( \blacksquare \)

We may easily generalize [14], Proposition 7.6.8, to obtain:

**Lemma 3.13.** Let \( \lambda + \rho/2 \in \Gamma^+_w (k > -c_\bar{g}) \) or \( \lambda \in \Gamma^-_w (k < -c_\bar{g}) \) and \( w = \sigma_{\alpha_n} \ldots \sigma_{\alpha_1} \) be a reduced decomposition of \( w \in W \), where \( \alpha_1, \ldots, \alpha_n \in \Delta^e \). Let \( \lambda_0 = \lambda \), \( \lambda_1 = \sigma^\rho_{\alpha_1}(\lambda_0) \), \( \lambda_2 = \sigma^\rho_{\alpha_2}(\lambda_1) \), \( \ldots, \lambda_n = \sigma^\rho_{\alpha_n}(\lambda_{n-1}) \). Then for \( k > -c_\bar{g} \)

\[
\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_n \text{ and } \alpha_{i+1} \cdot (\lambda_i + \rho/2) \in \{0,1,2\ldots\}
\]

and for \( k < -c_\bar{g} \)

\[
\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n \text{ and } \alpha_{i+1} \cdot (\lambda_i + \rho/2) \in \{0,-1,-2\ldots\}
\]

**Lemma 3.14.** Let \( \lambda + \rho/2 \in \Gamma^+_w (k > -c_\bar{g}) \) or \( \lambda \in \Gamma^-_w (k < -c_\bar{g}) \). Let \( \mu \in \Gamma_w \) with \( \mu = w^p(\lambda), w \in W \). If \( l(\mu) = l(w) \), then \( w, \lambda, \mu \) satisfy (**) in Lemma 3.2 with \( l(\mu) = n \). In addition, this is the minimal integer \( n \) for which (**) is satisfied.

**Proof.** Consider \( k < -c_\bar{g} \). Let \( w = \sigma_{\alpha_n} \ldots \sigma_{\alpha_1} \) with \( l(w) = l(\mu) \). By Lemma 3.13 we have a sequence

\[
\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n \text{ and } \alpha_{i+1} \cdot (\lambda_i + \rho/2) \in \{0,-1,-2\ldots\}.
\]

Assume \( \lambda_i = \lambda_{i+1} \) for some \( i \in \{0,1,2,\ldots,n\} \). Then clearly \( w' = \sigma_{\alpha_n} \ldots \sigma_{\alpha_{i+1}} \sigma_{\alpha_{i-1}} \ldots \sigma_{\alpha_1} \) satisfies \( \mu = w'^p(\lambda) \) and \( l(w') < l(w) \). This contradicts the assumption \( l(\mu) = l(w) \). The last assertion follows by the definition of \( l(\mu) \). \( k > -c_\bar{g} \) is proved analogously. \( \blacksquare \)

**Proposition 3.15.** Let \( M(\mu) \subset M(\nu) \), where \( \mu, \nu \in \Gamma_w \). Then \( l(\mu) - l(\nu) = n \) for \( k > -c_\bar{g} \), or \( l(\nu) - l(\mu) = n \) for \( k < -c_\bar{g} \), if and only if \( n \) is the largest integer for which \( M(\mu) \subset M^{(n)}(\nu) \).

**Proof.** Consider \( k < -c_\bar{g} \). By Proposition 3.4 and the hereditary nature of Jantzen's filtration it is sufficient to prove the proposition for \( l(\mu) = 0 \) i.e. for \( \mu \in \Gamma^-_w \) and some given \( M(\nu) \). We prove the "only if" case by induction on \( l(\nu) \). For \( l(\nu) = 0 \) the proposition is trivial. Assume it to be true for \( l(\nu) \leq p - 1 \) and
consider \( l(\nu) = p \). As \( p \geq 1 \) there must exist \( \alpha \in \Delta^s \) such that \( \nu' = \sigma^\mu_\alpha(\nu) < \nu \). Then \( M(\nu') \subset M(\nu) \) (Proposition 3.8) and \( l(\nu') < l(\nu) \) (Lemma 3.12). If \( l(\nu') < p - 1 \) then \( l(\nu) < p \), which is a contradiction. Hence, \( l(\nu') = p - 1 \). In addition, \( M(\nu') \subset M^{(1)}(\nu) \) and \( M(\nu') \not\subset M^{(2)}(\nu) \). This follows by an explicit construction of the highest weight vector that generates \( M(\nu') \) (cf. the proof of Lemma 3.3). We now use the induction hypothesis on \( M(\nu') \) together with the hereditary nature of Jantzen’s filtration to conclude that the proposition holds for \( l(\nu) = p \).

We prove the “if” case. Consider \( M(\mu) \subset M^{(p)}(\nu), \mu \in \Gamma^+ \) and use induction on \( p \). The case \( p = 0 \) is trivial. Assume the assertion to be true for \( 0 \leq p \leq n - 1 \) and consider \( p = n \geq 1 \). As \( p \geq 1 \) there must exist \( \alpha \in \Delta^s \) such that \( \nu' = \sigma^\mu_\alpha(\nu) < \nu \) and \( M(\nu') \subset M(\nu) \) (Proposition 3.8) with \( l(\nu') < l(\nu) \) (Lemma 3.12). By explicit construction of the highest weight vector one checks that \( M(\nu') \subset M^{(1)}(\nu) \) and \( M(\nu') \not\subset M^{(2)}(\nu) \). Then the hereditary nature of Jantzen’s filtration implies \( M(\mu) \subset M^{(n-1)}(\nu'), \) which by the induction hypothesis yields \( l(\nu') = n - 1 \). Then \( \nu' = \sigma^\mu_\alpha(\nu) \) implies \( l(\nu) = n \), which concludes the proof. The case \( k > -c_\beta \) is proved in a completely analogous fashion. ■

**Lemma 3.16.** ([14], Lemma 7.7.6; [6], Lemma 8.6). Let \( w_1, w_2 \in W \). The number of elements \( w \in W \) such that \( w_1 \leftarrow w \leftarrow w_2 \) is \( 0 \) or \( 2 \).

**Lemma 3.17.** (cf. [14], Lemma 7.7.7 (iii) and [6], Lemma 8.15 (iii)) Let \( M(\mu_1) \) and \( M(\mu_2) \) be Verma modules with highest weights \( \mu_1 \) and \( \mu_2 \), respectively. Let \( \mu_1 + \rho/2 \) and \( \mu_2 + \rho/2 \) be regular. If \( l(\mu_1) = l(\mu_2) + 2 \) \((k > c_\beta) \) or \( l(\mu_1) = l(\mu_2) - 2 \) \((k < c_\beta) \), then the number of \( \mu \) such that \( M(\mu_1) \subset M(\mu) \subset M(\mu_2) \), \( M(\mu_1) \neq M(\mu) \neq M(\mu_2) \) is either \( 0 \) or \( 2 \).

**Proof.** Consider \( k > -c_\beta \). The definition of \( l(\mu_1) \) and \( l(\mu_2) \) implies together with Lemma 3.2 that there exists \( w_1, w_2 \in W \) such that \( \mu_1 = w^\mu_1(\lambda), \mu_2 = w^\mu_2(\lambda), \lambda + \rho/2 \in \Gamma^+_w \) and \( l(w_1) = l(w_2) + 2 \). In addition, \( \mu_1 + \rho/2 \) and \( \mu_2 + \rho/2 \) regular imply that \( \lambda \in \Gamma^+_w \). Then the number of \( w \in W \) such that \( M(w^\mu_1(\lambda)) \subset M(w^\mu(\lambda)) \subset M(w^\mu_2(\lambda)) \) and \( M(w^\mu_1(\lambda)) \neq M(w^\mu(\lambda)) \neq M(w^\mu_2(\lambda)) \) is \( 0 \) or \( 2 \), as can be seen from combining Lemma 3.16 and Theorem 3.11. This proves the assertion of the lemma for \( k > -c_\beta \). The case \( k < -c_\beta \) is proved analogously. ■
4 The BRST formalism

Define the algebra \( g' = g_k \oplus g_{-k-2c_g} \), where \( c_g \) is the quadratic Casimir of the adjoint representation. This algebra is invariant under the exchange \( k \rightarrow -k - 2c_g \) and, consequently, we may restrict to \( k > c_g \). The singular case \( k = -c_g \) will not be treated here. We will denote \( g_{-k-2c_g} \) by \( \tilde{g} \) and the Verma module over \( \tilde{g} \) will be denoted \( \tilde{M}(\tilde{\lambda}) \), where \( \tilde{\lambda} \) is its highest weight. Let \( B_{n'\dagger}, B_{n'}, B_{h'}, B_{g'} \) be bases of \( n'_\uparrow, n'_\downarrow, h', \tilde{g} \) and \( g' \), respectively. The generators \( \tilde{e}_\alpha, \tilde{f}_\alpha \) and \( \tilde{h}_i \) is a realization of \( B_{\tilde{g}} \) and \( e'_\alpha, f'_\alpha \) and \( h'_i \) a realization of \( B_{g'} \). Define \( M'_{\lambda\lambda} = M(\lambda) \otimes \tilde{M}(\tilde{\lambda}) \) and similarly \( L'_{\lambda\lambda} = L(\lambda) \otimes \tilde{L}(\tilde{\lambda}) \). \( \pi_{L'} \) denotes the projection \( M' \rightarrow L' \).

We define the anticommuting ghost and antighost operators \( c(x) \) and \( b(x) \), respectively, where \( x \in B_{g'} \), with the following properties

\[
(i) \quad \{ c(x), b(y) \} = \delta_{x,y} \\
(ii) \quad c^\dagger(x) = c(x^\dagger), \quad b^\dagger(x) = b(x^\dagger) \\
(iii) \quad b(a_1 x + a_2 y) = a_1 b(x) + a_2 b(y) \quad a_1, a_2 \in \mathcal{C}.
\]

Here \( \delta_{x,y} = 1 \) if \( x = y \) and 0 otherwise. Introduce a normalordering

\[
: c(x)b(y) : = \begin{cases} 
\frac{1}{2}(c(x)b(y) - b(y)c(x)) & \text{otherwise} \\
\text{if either } x \in B_{n'\dagger} \text{ or } y \in B_{n'} \\
-c(x)b(y) & \text{if either } x \in B_{n}\dagger \text{ or } y \in B_{n}\downarrow.
\end{cases}
\]

Define the BRST operator

\[
d = \sum_{x \in B_{g'}} c(x^\dagger)x + \sum_{x \in B_{h'}} c(x^\dagger)\rho(x) - \frac{1}{2} \sum_{x, y \in B_{g'}} b([x, y])c(x^\dagger)c(y^\dagger),
\]

where \( \rho(x) \) is the component of \( \rho \) corresponding to the element \( x \in B_{h'} \). The BRST operator has the following two fundamental properties: \( d^2 = 0 \) and \( d^\dagger = d \). The first property implies that \( x^{tot} = \{ d, b(x) \} = x + \rho + \sum y b([x, y])c(y^\dagger) \) generates an algebra \( g_0 \) which is centerless.

Define a ghost module \( \mathcal{F}^{gh} \). It is generated by the ghost operators acting on a vacuum vector \( v_0^{gh} \) satisfying

\[
c(x)v_0^{gh} = b(y)v_0^{gh} = 0 \quad \text{for } x \in B_{n'} \text{ and } y \in B_{n'} \cup B_{h'}.
\]

We also define a restricted module \( \tilde{\mathcal{F}}^{gh} = \{ v^{gh} \in \mathcal{F}^{gh} \mid b(x)v^{gh} = 0 \text{ for } x \in B_{h'} \} \).

The dual \( \mathcal{F}^{*gh} \) of \( \mathcal{F}^{gh} \) has a vacuum vector \( v_0^{*gh} \) satisfying

\[
c^\dagger(x)v_0^{*gh} = b^\dagger(y)v_0^{*gh} = 0 \quad \text{for } x \in B_{n'} \cup B_{h'} \text{ and } y \in B_{n'_\dagger}.
\]
The restricted dual is \( \hat{F}^{gh} = \{ v^{gh} \in F^{gh} \mid c(x)v^{gh} = 0 \text{ for } x \in B_{h'} \} \). Define a Hermitian form for the ghost sector by

\[
\langle v_0^{gh} | v_0^{gh} \rangle = 1
\]

\[
\langle v^{gh} | uv^{gh} \rangle = \langle u^t v^{gh} | v^{gh} \rangle,
\]

for a polynomial \( u \) in the ghost operators and \( v^{gh} \in \mathcal{F}^{gh}, v^{gh} \in \mathcal{F}^{gh} \). If \( v^{gh} = uv_0^{gh} \) then we denote by \( v^{gh} \) the vector \( u^t v_0^{gh} \).

The ghost number \( N^{gh} \) of any vector \( v^{gh} \in \mathcal{F}^{gh} \) is defined by \( N^{gh}(v_0^{gh}) = 0 \) and \( N^{gh}(c(x)v) = N^{gh}(v) + 1, \ N^{gh}(b(x)v) = N^{gh}(v) - 1 \). The ghost numbers of vectors in the dual module is similarly defined with \( N^{gh}(v_0^{gh}) = 0 \). It is easily seen that \( \langle u^t | v \rangle = 0 \) if \( N^{gh}(u^t) + N^{gh}(v) \neq 0 \). Let \( C(g', V) \) be the complex \( V \otimes \mathcal{F}^{gh} \) for a \( g' \)-module \( V \). We define the relative subcomplex \( \hat{C}(g', V) = \{ \omega \in C(g', V) \mid b(x)\omega = 0, \ x^{tot}\omega = 0 \text{ for } x \in B_{h'} \} \) and \( \hat{C}(g', V^*) \) is the dual complex. If \( \omega = v \otimes v^{gh} \) for \( v \in V, v^{gh} \in \mathcal{F}^{gh} \), then we denote by \( \omega^* \) the vector \( v \otimes v^{gh} \). We decompose \( d \) as

\[
d = \hat{d} + \sum_{x \in B_{h'}} (x^{tot}c(x) + \mathcal{M}(x)b(x)).
\]

We have \( d\omega = \hat{d}\omega \) for \( \omega \in \hat{C}(g', V) \) and consequently on the relative subcomplex we may analyze the cohomology of \( \hat{d} \) in place of \( d \). The cohomology associated with \( \hat{d} \), the semi-infinite or BRST relative cohomology is sometimes denoted by \( \hat{H}^{\infty/2+1}(g', V) \) to distinguish it from the conventional Lie algebra cohomology. We will, however, here for simplicity write \( \hat{H}^p(g', V) \), where \( p \) refers to elements \( \omega \) with \( N^{gh}(\omega) = p \). Our primary interest here will be for \( V = L'_{\lambda, \bar{\lambda}} \). But in order to gain knowledge of this case we will also study \( V = M'_{\lambda, \bar{\lambda}} \) and its submodules.

It will be convenient to make a classification of vectors in the complex \( C(g', V) \) using the BRST operator. A central result due to Kugo and Ojima states the following.

**Theorem 4.1.** Let \( V \) be an irreducible module. Then a basis of \( C(g', V) \) may be chosen so that for an element \( \omega \) in this basis one of the following will be true.

(i) **Singlet case:** \( \omega \in H^*(g', V) \) and \( \langle \omega^* | \omega \rangle \neq 0, N^{gh}(\omega) = 0 \).

(ii) **Singlet pair case:** \( \omega \in H^*(g', V) \) and there exists an element \( \psi \neq \omega \) such that \( \psi \in H^*(g', V) \), \( \langle \psi^* | \omega \rangle \neq 0 \) and \( N^{gh}(\psi) = -N^{gh}(\omega) \).

(iii) **Quartet case:** \( \omega \notin H^*(g', V) \). There will exist four elements \( \omega_1, \omega_2, \psi_1, \psi_2 \in C(g', V) \), where either \( \omega = \omega_1 \) or \( \omega = \omega_2 \), such that \( \langle \psi_1^* | \omega_1 \rangle \neq 0 \) and \( \langle \psi_2^* | \omega_2 \rangle \neq 0 \).
\[ \omega_2 = d\omega_1 \text{ and } \psi_1 = d\psi_2 \text{ and } N^{gh}(\omega_1) = N^{gh}(\omega_2) - 1 = -N^{gh}(\psi_1) = -N^{gh}(\psi_2) - 1. \]

There will exist an analogous classification on the relative subcomplex \( \hat{C}(g', V) \) w.r.t. \( \hat{d} \). In this classification all non-trivial elements in the cohomology will be singlets or singlet pairs. It should be remarked that the condition of irreducibility is essential for the theorem. In the following section, we will find that for \( V \) being a reducible Verma module the above classification does not hold. In particular, the non-trivial elements of the cohomology for non-zero ghost numbers will for this case not be members of singlet pairs. Define Jantzen’s filtration for \( \xi \in \hat{C}(g', M') \) as follows. Let \( \lambda_\epsilon = \lambda + \epsilon \varepsilon \) and \( \tilde{\lambda}_\epsilon = \tilde{\lambda} - \epsilon \varepsilon \). Then \( M^{(n)}(\lambda_\epsilon) = \{ v' \in M(\lambda_\epsilon) \otimes \tilde{M}(\tilde{\lambda}_\epsilon) \mid \langle w'^*|v' \rangle \text{ is divisable by } \epsilon^n \} \). We denote by \( \xi_\epsilon \) the vector \( v_\epsilon \otimes \tilde{v}_\epsilon \otimes v^{gh} \). An element \( \xi_\epsilon \) is always assumed to be finite as \( \epsilon \to 0 \). We denote by \( f(\epsilon) \sim \epsilon^n \) the leading order of a function \( f(\epsilon) \) in the limit \( \epsilon \to 0 \). Note that our definition of Jantzen’s filtration for \( g' \) implies that \( \lambda_\epsilon + \tilde{\lambda}_\epsilon \) is independent of \( \epsilon \). This is required if the cohomology should have at least one non-trivial element for \( \epsilon \neq 0 \), namely the vacuum solution \( \xi_{0\epsilon} = v_{0\epsilon} \otimes \tilde{v}_{0\epsilon} \otimes v^{gh}_0 \).

In the next section the following result will be needed.

**Lemma 4.2.** Let \( \xi_{1\epsilon}, \xi_{2\epsilon} \in \hat{C}(g', M'_\epsilon) \) be non-zero for \( \epsilon = 0 \) and \( \hat{d}_i \xi_{2\epsilon} = g(\epsilon)\xi_{1\epsilon} \), where \( g(\epsilon) \sim 1 \) or \( \epsilon \). Let \( n_1 \) and \( n_2 \) be the largest integers for which \( \xi_i \in \hat{C}(g', M^{(n_i)}) \), \( i = 1, 2 \). Then there exist \( \zeta_{1\epsilon}, \zeta_{2\epsilon} \in \hat{C}(g', M'_\epsilon) \) which are non-zero for \( \epsilon = 0 \) and satisfy

(i) \( \langle \xi_{1\epsilon}^* | \xi_{1\epsilon} \rangle \sim \epsilon^{n_i} \delta_{1,2} \neq 0 \) for \( \epsilon \neq 0 \), \( i = j, 1, 2 \).

(ii) \( \hat{d}_i \zeta_{1\epsilon} = f(\epsilon)\zeta_{2\epsilon} \), where \( f(\epsilon) \sim 1 \) or \( \sim \epsilon \).

(iii) \( n_1, n_2 \) are the largest integers for which \( \zeta_i \in \hat{C}(g', M^{(n_i)}) \).

In addition, for \( g(\epsilon) \sim 1 \): \( n_1 = n_2 \) if and only if \( f(\epsilon) \sim 1 \), \( n_1 = n_2 + 1 \) if and only if \( f(\epsilon) \sim \epsilon \). For \( g(\epsilon) \sim \epsilon \) we have: \( n_1 = n_2 - 1 \) if and only if \( f(\epsilon) \sim 1 \), \( n_1 = n_2 \) if and only if \( f(\epsilon) \sim \epsilon \).

**Proof.** Since \( \xi_{1\epsilon} \in \hat{C}(g', M^{(n_1)}) \) for a largest integer \( n_1 \) and \( M^{(n_1)} \) is irreducible for \( 0 \ll |\epsilon| \ll 1 \) there must exist one vector \( \zeta_{1\epsilon} \in \hat{C}(g', M^{(n_1)}) \) with \( \langle \xi_{1\epsilon}^* | \zeta_{1\epsilon} \rangle \sim \epsilon^{n_1} \).

Then

\[ g(\epsilon)\langle \xi_{1\epsilon}^* | \xi_{1\epsilon} \rangle = \langle \xi_{1\epsilon}^* | \hat{d}_i \xi_{2\epsilon} \rangle = \langle \hat{d}_i \xi_{1\epsilon}^* | \xi_{2\epsilon} \rangle \]

imply \( \hat{d}_i \xi_{1\epsilon} = f(\epsilon)\xi_{2\epsilon} \), for some vector \( \zeta_{2\epsilon} \) satisfying \( \langle \xi_{2\epsilon}^* | \xi_{2\epsilon} \rangle \neq 0 \) and which is non-
zero for $\epsilon = 0$. In addition, $f(\epsilon)$ is a non-singular function of $\epsilon$. From the fact that $\hat{d}$ is linear in the generators of $g'$ we can conclude that $f(\epsilon) \sim 1$ or $\epsilon$. Pick a basis of $\hat{C}(g', M'_e)$ such that $\xi_1$, $\xi_2$ are two of its elements. Denote the elements of the basis by $\xi_i$, $i = 1, 2, 3, \ldots$. Similarly we pick a basis of $\hat{C}(g', M'_e)$, $\zeta_i$, $i = 1, 2, 3, \ldots$. We choose it such that $\langle \zeta_i | \xi_j \rangle$ is non-zero only for $i = j$. Now since $\langle \zeta_i | \zeta_j \rangle = 0$ for $i \neq 2$ and $\zeta_2 \in \hat{C}(g', M'_e)$ we must have $\langle \zeta_2 | \zeta_2 \rangle \sim \epsilon^{n_2}$. This in turn implies, using $\langle \zeta_2 | \zeta_2 \rangle = 0$ for $j \neq 2$ and the definition of Jantzen's filtration, that $\zeta_2 \in \hat{C}(g', M'_e)$. We now conclude from $\langle \zeta_2 | \zeta_2 \rangle \sim \epsilon^{n_2}$, $\langle \zeta_2 | \zeta_2 \rangle \sim \epsilon^{n_2}$, eq.(4.10) and $f(\epsilon) \sim 1$ or $\epsilon$ that for $g(\epsilon) \sim 1$ we have $\epsilon^{n_1} \sim \epsilon^{n_2} f(\epsilon) \sim \epsilon^{n_2}$ or $\epsilon^{n_2+1}$, while for $g(\epsilon) \sim \epsilon$ we have $\epsilon^{n_1} \sim \epsilon^{n_2-1} f(\epsilon) \sim \epsilon^{n_2-1}$ or $\epsilon^{n_2}$. □

A standard tool in the analysis of the cohomology is a contracting homotopy operator. Let $\omega_0$ be a vacuum vector of $\hat{C}(g', M')$ i.e. $\omega_0 = \nu_0 \otimes v_0^{gh}$, where $\nu_0 = v_0 \otimes \nu_0$ and $v_0, \nu_0$ are primary highest weight vectors of $M$ and $M$, respectively. Consider an element $\omega \in \hat{C}(g, M')$ of the form $\omega = \nu' \otimes v_0^{gh}$ with $\nu' = u \nu_0 \otimes \tilde{u} v_0, u \in U(n_-, \tilde{u}) \in U(n_-)$ and $N^{gh}(v_0^{gh}) = n$. We write $u = u_m + u_{m-1} + \ldots + u_0$, where $u_i \in U(n_-)$ is a monomial of order $i$. Introduce a gradation $N_{gr}$. We define $N_{gr}(\omega_0) = 0$. Furthermore, $N_{gr}(\omega) = m - n$. We will get a filtration $\hat{C}(g', M') = \bigoplus N_{gr} \hat{C}(g', M')_{N_{gr}}$. We now decompose $\hat{d}$ as $\hat{d} = \hat{d}_0 + \hat{d}_{-1}$, where $\hat{d}_0 = \sum_{\alpha} c(\alpha') f_\alpha$. We have $d\omega = d_0 \omega + (\text{lower order terms})$. Let $\omega_{p,q} \in \hat{C}(g', M')_{p+q-r}$ be of the form

$$
\omega_{p,q} = f_{\alpha_1} \ldots f_{\alpha_p} v_0 \otimes \tilde{v} \otimes b(f'_{\beta_1}) \ldots b(f'_{\beta_q}) c(f'_{\gamma_1}) \ldots c(f'_{\gamma_p}) v_0^{gh},
$$

where $\alpha, \beta, \gamma \in \Delta^+$. The homotopy operator $\kappa_0$ is now defined by

$$
\kappa_0 \omega_{p,q} = \frac{1}{p+q} \sum_{i=1}^{p} f_{\alpha_i} \ldots \tilde{f}_{\alpha_i} \ldots f_{\alpha_p} v_0 \otimes \tilde{v} \otimes b(f'_{\alpha_i}) b(f'_{\beta_{i1}}) \ldots b(f'_{\beta_{iq}}) c(f'_{\gamma_1}) \ldots c(f'_{\gamma_p}) v_0^{gh},
$$

$$
\kappa_0 \omega_{0,q} = 0,
$$

where capped factors are omitted. It is now straightforward to verify

$$
(d_0 \kappa_0 + \kappa_0 d_0) \omega_{p,q} = (1 - \delta_{p+q,0}) \omega_{p,q} + (\text{lower order terms}).
$$

One may also define a gradation $\tilde{N}_{gr}$ using the elements of $U(\tilde{n}_-)$ in place of $U(n_-)$. We then have a corresponding decomposition $\hat{d} = \hat{d}_0 + \hat{d}_{-1}$ with $\tilde{d}_0 = \sum_{\alpha} c(\alpha') \tilde{f}_\alpha$ and a homotopy operator $\tilde{\kappa}_0$. 19
5 The BRST cohomology

We will now in detail study the semi-infinite relative cohomology associated with the BRST operator. The notation follows that of previous sections. \(\omega, \xi, \ldots\) always denote elements of \(\hat{C}(g', \ldots)\) that are finite in the limit \(\epsilon \to 0\). Our starting point is Proposition 5.1 concerning the cohomology of Verma modules. This proposition was to our knowledge first given in [20], Proposition 2.29.

**Proposition 5.1.** Let \(M'\) be a highest weight Verma module over \(g'\). Then \(\hat{H}^p(g', M') = 0\) for \(p < 0\).

**Proof.** ([2]) Let \(\omega \in \hat{H}^p(g', M')\) and have a highest order term \(\omega_n\) in the gradation \(N_{gr}\). Then \(0 = \hat{d}\omega = d_0\omega_n + (\text{lower order terms})\) and hence \(d_0\omega_n = 0\) to leading order. Using eq. (4.13) we conclude that \(\omega_n = d_0(\kappa_0\omega_n) + (\text{lower order terms})\). Thus \(\omega\) is a trivial element of \(\hat{H}^p(g', M')\) to highest order. This may be iterated to lower orders and we find that \(\omega \in \hat{H}^p(g', M')\) will be non-trivial only for \(N_{gh}(\omega) \leq p\), which is impossible if \(N_{gh}(\omega) < 0\). ■

**Corollary 5.2.** ([2]) Let \(M'\) in Proposition 5.1 be irreducible. Then \(\hat{H}^p(g', M') = 0\) for \(p \neq 0\). Furthermore, \(\omega \in \hat{H}^0(g', M')\) is the element \(\omega = v_0 \otimes \tilde{v}_0 \otimes v_0^{gh}\), where \(v_0\) and \(\tilde{v}_0\) are primary highest weight vectors of weights \(\lambda\) and \(\tilde{\lambda}\), respectively, satisfying \(\lambda + \tilde{\lambda} + \rho = 0\).

**Corollary 5.3.** Let \(L'\) be the irreducible \(g'\)-module of \(M'\). Let \(\omega \in \hat{C}(g', M')\) be such that \(0 \neq \pi_{L'}(\omega) \in \hat{H}^p(g', L')\), \(p < 0\). Then

\[
\hat{d}\omega = \nu,
\]

where \(\nu \in \hat{H}^{p+1}(g', M'(1))\) and is non-zero.

**Proof.** ([2]) Assume first \(\hat{d}\omega = 0\) in \(\hat{C}(g', M')\). Then Proposition 5.1 implies \(\omega = \hat{d}\eta, \eta \in \hat{C}(g', M')\). Since \(\omega \in \hat{C}(g', M'/M'(1))\) we must have \(\eta \in \hat{C}(g', M'/M'(1))\), which implies that \(\omega\) is cohomologically trivial. Therefore, \(\hat{d}\omega = \nu \neq 0\) and so \(\hat{d}\nu = 0\). If \(\nu \notin \hat{H}^{p+1}(g', M'(1))\), then \(\nu = \hat{d}\nu'\) for some \(\nu' \in \hat{C}(g', M'(1))\) and \(\hat{d}(\omega - \nu') = 0\).
Proposition 5.1 then implies that \( \pi_{L'}(\omega) \) is a trivial element of \( \hat{H}^p(g', L') \).

The following lemma is partly the converse of Corollary 5.3.

**Lemma 5.4.** Let \( \omega \in \hat{C}(g', M') \), \( \hat{d}\omega = \nu \) in \( \hat{C}(g', M') \) with \( \nu \in \hat{H}^{p+1}(g', M'^{(1)}) \) and \( \pi_{L'}(\omega) \neq 0 \), then \( \pi_{L'}(\omega) \in \hat{H}^p(g', L') \).

**Proof.** First, \( \hat{d}\omega = \nu \) with \( \nu \in \hat{H}^{p+1}(g', M'^{(1)}) \) implies that \( \hat{d}\pi_{L'}(\omega) = 0 \). Secondly, assume \( \pi_{L'}(\omega) \) to be trivial i.e. \( \pi_{L'}(\omega) = \hat{d}\pi_{L'}(\psi) \), \( \psi \in \hat{C}(g', M') \). Then \( \omega = \hat{d}\psi + \nu' \) in \( \hat{C}(g', M') \), with \( \nu' \in \hat{C}(g', M'^{(1)}) \), and so \( \nu = \hat{d}\omega = \hat{d}\nu' \). This is a contradiction to the assumption \( \nu \in \hat{H}^{p+1}(g', M'^{(1)}) \).

**Lemma 5.5.** \( \dim(\hat{H}^{p+1}(g', M'^{(1)})) = \dim(\hat{H}^p(g', L')) \) for \( p \leq -2 \).

**Proof.** Let \( \nu \in \hat{H}^{p+1}(g', M'^{(1)}) \) with \( p \leq -2 \), then by Proposition 5.1 \( \nu = \hat{d}\omega \), \( \omega \in \hat{C}(g', M') \) and \( \pi_{L'}(\omega) \neq 0 \). Lemma 5.4 then implies \( \pi_{L'}(\omega) \in \hat{H}^p(g', L') \). We have thus proved that \( \dim(\hat{H}^{p+1}(g', M'^{(1)})) = \dim(\hat{H}^p(g', L')) \). We now prove that the dimensionalities are in fact the same. Assume two elements \( \omega_1, \omega_2 \in \hat{C}(g', M') \) with \( \pi_{L'}(\omega_1), \pi_{L'}(\omega_2) \in \hat{H}^p(g', L') \), corresponding to the same element \( \nu \). By Corollary 5.3 we have in \( \hat{C}(g', M') \): \( \hat{d}\omega_1 = \nu_1 \) and \( \hat{d}\omega_2 = \nu_2 \), where \( \nu_1 = \nu_2 \) as elements in \( \hat{H}^{p+1}(g', M'^{(1)}) \). Subtracting the equations yields \( \hat{d}(\omega_1 - \omega_2) = \nu_1 - \nu_2 = \hat{d}\nu' \), \( \nu' \in \hat{C}(g', M'^{(1)}) \), which by Proposition 5.1 implies \( \omega_1 - \omega_2 - \nu' = \hat{d}(\ldots) \). \( \pi_{L'}(\omega_1) \) and \( \pi_{L'}(\omega_2) \) are therefore identical elements in \( \hat{H}^p(g', L') \).

The results obtained so far are of importance for negative ghost numbers. We now turn to results relevant for positive ghost numbers. We will connect the two cases by the use of Jantzen's perturbation, Theorem 4.1 and Lemma 4.2.

**Lemma 5.6.** Let \( \omega \in \hat{C}(g', M') \) with \( \pi_{L'}(\omega) \in \hat{H}^p(g', L') \), satisfying \( \hat{d}\omega = \nu, \nu \in \hat{C}(g', M'^{(1)}) \). Then:

(i) There exists \( \psi \in \hat{C}(g', M') \) with \( \pi_{L'}(\psi) \in \hat{H}^{-p}(g', L') \) and \( \langle \psi^*|\omega \rangle \neq 0 \).

(ii) With \( \psi \) as in (i): There exists \( \chi \in \hat{C}(g', M'^{(1)}) \) of opposite ghost number of \( \nu \),
satisfying \( \hat{d} \chi_\epsilon = \epsilon \psi_\epsilon \).

(iii) \( \chi, \nu \notin \hat{C}(g', M^{(2)}) \), where \( \chi \) is defined as in (ii).

(iv) With \( \psi \) as in (i): \( \hat{d} \psi = 0 \).

(v) \( p \leq 0 \).

**Proof.** (i) follows directly from Theorem 4.1. (ii) and (iii) follow from Theorem 4.1 and Lemma 4.2 using \( \omega = \xi_2 \), \( \nu = \xi_1 \) and \( g(\epsilon) \sim 1 \). (iv) follows by applying \( \hat{d} \) to the equation \( \hat{d} \chi_\epsilon = \epsilon \psi_\epsilon \), using \( \hat{d}^2 = 0 \) and taking the limit \( \epsilon \to 0 \). Finally (v) may be proved by contradiction. If \( p > 0 \), then by (iv) and Corollary 5.3 \( \psi \) is \( \hat{d} \)-exact and, hence, so is \( \omega \).

**Lemma 5.7.** Let \( \psi \in \hat{C}(g', M') \), \( \pi_{L'}(\psi) \neq 0 \), and \( \chi \in \hat{C}(g', M^{(1)}) \) be such that \( \hat{d} \chi_\epsilon = \epsilon \psi_\epsilon \). Then \( \chi \in \hat{C}(g', M^{(1)}/M^{(2)}) \), \( \pi_{L'}(\psi) \in \hat{H}^p(g', L') \) and \( p \geq 0 \). Conversely, let \( \pi_{L'}(\psi) \in \hat{H}^p(g', L') \), \( p \geq 1 \), then there exists \( \chi \in \hat{C}(g', M^{(1)}) \) such that \( \hat{d} \chi_\epsilon = \epsilon \psi_\epsilon \).

**Proof.** Assume \( \hat{d} \chi_\epsilon = \epsilon \psi_\epsilon \). We apply Lemma 4.2 with \( \xi_1 = \psi \), \( \xi_2 = \chi \) and \( g(\epsilon) \sim \epsilon \).

Then \( n_1 = 0 \), \( n_2 \geq 1 \) and by the lemma there exist two vectors \( \omega \) and \( \nu \) such that \( \langle \omega^* | \psi_\epsilon \rangle \sim 1 \), \( \langle \nu^* | \chi_\epsilon \rangle \sim \epsilon \) and \( \hat{d} \omega_\epsilon = f(\epsilon) \nu_\epsilon \), with \( f(\epsilon) \sim 1 \) or \( \epsilon \). Furthermore \( f(\epsilon) \sim 1 \), since otherwise \( n_1 = n_2 \). This in turn implies \( n_2 = 1 \), by Lemma 4.2 (iii), and \( \chi, \nu \in \hat{C}(g', M^{(1)}/M^{(2)}) \). We now show that \( \nu \) is not exact in \( \hat{C}(g', M^{(1)}) \).

Assume the contrary i.e. \( \nu = \hat{d} \eta \) with \( \eta \in \hat{C}(g', M^{(1)}/M^{(2)}) \). Then \( \nu_\epsilon = \hat{d} \eta_\epsilon + h(\epsilon) \nu_\epsilon' \), where \( \nu_\epsilon' \in \hat{C}(g', M^{(1)}) \) and \( h(\epsilon) \) is a polynomial in \( \epsilon \) such that \( h(0) = 0 \). This implies that \( \omega_\epsilon' = \omega_\epsilon - f(\epsilon) \eta_\epsilon \) satisfies \( \hat{d} \omega_\epsilon' = f(\epsilon) h(\epsilon) \nu_\epsilon' \). Now \( \lim_{\epsilon \to 0} \langle \omega^*_\epsilon | \psi_\epsilon \rangle = 0 \) since \( \omega' \) and \( \omega \) differ by an element in \( \hat{C}(g', M^{(1)}) \). This is a contradiction as can be seen from

\[
\langle \omega^*_\epsilon | \psi_\epsilon \rangle = \langle \omega^*_\epsilon | \frac{1}{\epsilon} \hat{d} \chi_\epsilon \rangle = \langle \hat{d} \omega^*_\epsilon | \frac{1}{\epsilon} \chi_\epsilon \rangle = \langle f(\epsilon) h(\epsilon) \nu^*_\epsilon | \frac{1}{\epsilon} \chi_\epsilon \rangle \longrightarrow 0 \quad \text{for} \quad \epsilon \to 0.
\]

Thus \( \nu \in \hat{H}^{-p+1}(g', M^{(1)}) \). Lemma 5.4 then gives \( \pi_{L'}(\omega) \in \hat{H}^{-p}(g', L') \), which implies \( \pi_{L'}(\psi) \in \hat{H}^p(g', L') \). The condition \( p \geq 0 \) follows from Corollary 5.3 and the fact that \( \hat{d} \psi = 0 \) in \( \hat{C}(g', M') \).

We now prove the converse statement. Let \( \pi_{L'}(\psi) \in \hat{H}^p(g', L') \), \( p \geq 1 \). Pick a basis as in Theorem 4.1 so that \( \psi \) is one of its elements and \( \omega \in \hat{C}(g', M') \), \( \pi_{L'}(\omega) \in \hat{H}^{-p}(g', L') \), \( \langle \omega^*_\epsilon | \psi \rangle \neq 0 \), is another. Corollary 5.3 implies \( \hat{d} \omega = \nu \) and then Lemma 5.6 gives the assertion.

**Lemma 5.8.** Let \( \psi \) and \( \chi \in \hat{C}(g', M') \) be such that \( N^{gh}(\psi) \geq 1 \), \( \hat{d} \chi_\epsilon = \epsilon \psi_\epsilon \) and
\[ \chi \in \hat{C}(g', M'(1)/M'(2)) \]. Then \( \pi_{L'}(\psi) \in \hat{H}^p(g', L') \).

**Proof.** By Lemma 5.7 it is sufficient to prove that \( \pi_{L'}(\psi) \neq 0 \). Assume the contrary i.e. \( \psi \in \hat{C}(g', M'(1)) \). Then Lemma 4.2 implies that there exist two vectors \( \omega \) and \( \nu \) satisfying \( d\omega_e = f(\epsilon)\nu_e \) in \( \hat{C}(g', M') \), where \( f(\epsilon) \sim \epsilon \). In addition, \( \psi, \omega, \nu \in \hat{C}(g', M'(1)/M'(2)) \) and \( \langle \omega_e^*|\psi_e \rangle \sim \epsilon \), \( \langle \nu_e^*|\chi_e \rangle \sim \epsilon \). Now \( N^{gh}(\omega) \leq -1 \), so that by proposition 5.1, \( \omega = \hat{d}\omega' \) for some vector \( \omega' \). We then have \( \omega_e = \hat{d}\omega'_e + h(\epsilon)\nu'_e \) for some vector \( \nu'_e \), which is non-singular for \( \epsilon = 0 \) and \( h(\epsilon) \) is a polynomial of \( \epsilon \) such that \( h(0) = 0 \). This implies \( \hat{d}\omega_e = h(\epsilon)\hat{d}\nu'_e \), which by compairing with \( \hat{d}\omega_e = f(\epsilon)\nu_e \) yields \( h(\epsilon) \sim \epsilon \) and \( \nu_e \sim \hat{d}\nu'_e \). Then

\[
\epsilon \sim \langle \nu_e^*|\chi_e \rangle \sim \langle \hat{d}\nu'_e^*|\chi_e \rangle = \langle \nu_e^*|\hat{d}\chi_e \rangle = \epsilon \langle \nu_e^*|\psi_e \rangle,
\]

so that \( \langle \nu_e^*|\psi_e \rangle \sim 1 \), which contradicts \( \psi \in \hat{C}(g', M'(1)) \).

**Proposition 5.9.** \( \hat{H}^p(g', L') \) for \( p \geq 1 \) are represented by elements of the form \( v \otimes \bar{v}_0 \otimes v^{gh} \), or equivalently of the form \( v_0 \otimes \bar{v} \otimes v^{gh} \), where \( v \in L, \bar{v} \in \bar{L}, v_0 \) is a primary highest weight vector w.r.t. \( g, \bar{v}_0 \) is a primary highest weight vector w.r.t. \( \bar{g} \) and \( v^{gh} \) satisfies \( c(x)v^{gh} = 0, x \in n_+ \).

**Proof.** Let \( \hat{H}^p(g', L') \), \( p \geq 1 \) be non-zero. Then by Theorem 4.1 there exists \( \omega \in \hat{C}(g', M') \) such that \( \pi_{L'}(\omega) \in \hat{H}^{-p}(g', L') \). We have \( \hat{d}\omega = \nu \) (Corollary 5.3) with \( \nu \in \hat{C}(g', M'(1)) \). It follows by Lemma 5.6 (iv) that \( \psi \in \hat{C}(g', M') \) with \( \pi_{L'}(\psi) \in \hat{H}^p(g', L') \), will satisfy \( \hat{d}\psi = 0 \) in \( \hat{C}(g', M') \). We can now use the gradation \( \tilde{N}_{gr} \) introduced in the previous section to decompose \( \hat{d} = \hat{d}_0 + \hat{d}_1 \) and use the homotopy operator \( \tilde{\kappa}_0 \) to successively eliminate highest order terms of \( \psi \) in this gradation. Since \( p \geq 1 \) we will finally get an element of the form \( v \otimes \bar{v}_0 \otimes v^{gh} \). The alternative form is found by using the gradation \( N_{gr} \).

**Proposition 5.10.** \( \hat{H}^0(g', M') \) are represented by elements \( v \otimes \bar{v}_0 \otimes v^{gh} \), or equivalently by the elements \( v_0 \otimes \bar{v} \otimes v^{gh}_0 \), where \( v, v_0 \) and \( \bar{v}, \bar{v}_0 \) are highest weight vectors w.r.t. \( g \) and \( \bar{g} \), respectively, with \( v_0 \) and \( \bar{v}_0 \) being primary, and \( v^{gh}_0 \) is the ghost vacuum. Furthermore, the weights \( \mu \) and \( \bar{\mu} \) of the primary highest weight vectors \( v_0 \) and \( \bar{v}_0 \), respectively, satisfy \( \mu + \bar{\mu} + \rho = 0 \).

**Proof.** Let \( \psi \in \hat{H}^0(g', M') \). Then \( \hat{d}\psi = 0 \) and we can use the gradation \( \tilde{N}_{gr} \) and the homotopy operator as in the proof of Proposition 5.9 to conclude that since \( N^{gh}(\psi) = 0 \) we must have \( \psi = v \otimes \bar{v}_0 \otimes v^{gh}_0 \). By using the gradation \( N_{gr} \) we get the al-
ternative form. The condition on the weights is a consequence of $h^{tot}(v \otimes \tilde{v} \otimes \tau_0^{gh}) = 0.$

\[\text{Proof.}\]

Let respectively, satisfy

Furthermore, the weights $\mu$ and $\tilde{\mu}$ of the primary highest weight vectors $v_0$ and $\tilde{v}_0$, respectively, satisfy $\mu + \tilde{\mu} + \rho = 0.$

\[\text{Proof.}\]

Let $\psi \in \hat{C}(g', M')$ and $\pi_{L'}(\psi) \in \hat{H}^0(g', L').$ Assume first $d\hat{\psi} = 0$ in $\hat{C}(g', M').$ Then the corollary follows directly from Proposition 5.10. Consider now $\hat{d}\psi = \nu \neq 0,$ where $\pi_{L'}(\nu) = 0.$ Then by Lemma 5.6 (iv) there exists $\omega \in \hat{C}(g', M')$ such that $\hat{d}\omega = 0,$ $\omega \notin \hat{C}(g', M'(1))$ and $\langle \omega^* | \psi \rangle \neq 0.$ We may then apply Proposition 5.10 to $\omega,$ so that $\omega$ is of the form claimed in the corollary. As $\langle \omega^* | \omega \rangle \neq 0,$ $\omega$ is a singleton representation of the BRST cohomology (cf. Theorem 4.1) and, hence, $\psi$ and $\omega$ yield equivalent elements in $\hat{H}^0(g', L').$

\[\text{Theorem 5.12.}\]

A necessary and sufficient condition for $\hat{H}^{\pm p}(g', L'),$ $p \geq 1,$ to be non-zero is either one of the following:

(i) There exists a vector $\nu \in \hat{C}(g', M'(1))$ satisfying $\nu \notin \hat{C}(g', M'(2))$ and $\nu \in \hat{H}^{-p+1}(g', M'(1)).$

(ii) There exists a vector $\chi \in \hat{C}(g', M'(1))$ satisfying $\chi \notin \hat{C}(g', M'(2))$, $N^{gh}(\chi) = p - 1,$ $\hat{d}\chi = 0$ and $\hat{d}\chi_\epsilon \neq 0.$

In addition, $\dim \left(\hat{H}^{-p+1}(g', M'(1))\right) = \dim \left(\hat{H}^{\pm p}(g', L')\right),$ $p \geq 1.$

\[\text{Proof.}\]

\[\text{Necessary:}\]

(i) follows by Corollary 5.3 and Lemma 5.6 (iii). (ii) follows from (i) together with Lemma 5.6 (ii) and (iii).

\[\text{Sufficient:}\]

(i) For $p > 1$ we use Lemma 5.5. This also gives the last assertion of dimensionalities for these cases. For $p = 1$ we have $\nu \in \hat{H}^0(g', M'(1)).$ We have two possibilities. Either $\nu \in \hat{H}^0(g', M')$ or $\nu = \hat{d}\psi$ for some $\psi \in \hat{C}(g', M'),$ $\pi_{L'}(\psi) \neq 0.$ In the first case we have $\hat{d}\nu_\epsilon \neq 0$ from Proposition 5.10, so that we get case (ii) of the theorem, which is proved below. For the second possibility we use Lemma 5.4.

(ii) $\hat{d}\chi = 0$ and $\hat{d}\chi_\epsilon \neq 0$ implies, using that $\hat{d}$ is linear in the generators of $g',$ $\hat{d}\chi_\epsilon = \epsilon \psi_\epsilon$ for some $\psi_\epsilon$ satisfying $\lim_{\epsilon \to 0} \psi_\epsilon \neq 0.$ Proposition 5.8 then gives $\pi_{L'}(\psi) \in \hat{H}^0(g', L').$

We finally prove the assertion concerning dimensionalities for the case $p = 1.$ Assume first that there exist $\omega_1, \omega_2 \in \hat{C}(g', M'),$ with $\pi_{L'}(\omega_1),$ $\pi_{L'}(\omega_2) \in \hat{H}^{-p}(g', L')$ and $\nu_1, \nu_2 \in \hat{H}^{-p+1}(g', M'(1)),$ satisfying $\nu_1 = \hat{d}\omega_1, \nu_2 = \hat{d}\omega_2$ (which is necessary by
Corollary 5.3, where $\nu_1 = \nu_2 \mod$ exact terms. This implies $\hat{d}(\omega_1 - \omega_2) = \nu_1 - \nu_2 = \hat{d}(\ldots)$, so that by Proposition 5.1, $\pi_{L'}(\omega_1) = \pi_{L'}(\omega_2)$ mod exact terms. Consider the opposite case i.e. two different vectors $\nu_1$ and $\nu_2$ give the same element in $\hat{H}^{-1}(g', L')$. Write $\hat{d}\omega_1 = \nu_1$ and $\hat{d}\omega_2 = \nu_2$. The requirement that $\pi_{L'}(\omega_1)$ and $\pi_{L'}(\omega_2)$ are equivalent elements in $\hat{H}^{\pm 1}(g', L')$ now implies $\omega_1 = \omega_2 + \nu' + \hat{d}(\ldots)$, where $\nu' \in \hat{C}(g', M'^{(1)})$. Then $\nu_1 = \nu_2 + \hat{d}\nu'$.

**Corollary 5.13.** $\dim \hat{H}^{\pm 1}(g', L'_{\mu\bar{\mu}}) = 1$ if $l(\bar{\mu}) - l(\mu) = 1$ and $\mu$ and $-\bar{\mu} - \rho$ are on the same $\rho$-centered Weyl orbit, and $\dim \hat{H}^{\pm 1}(g', L'_{\mu\bar{\mu}}) = 0$ otherwise.

**Proof.** By Proposition 5.10 and Theorem 3.11 we have $\dim \hat{H}^{0}(g', M'^{(1)}) = 1$ if $l(\bar{\mu}) - l(\mu) = 0$ and $\mu$ and $-\bar{\mu} - \rho$ are on the same $\rho$-centered Weyl orbit and $\dim \hat{H}^{0}(g', M'^{(1)}) = 0$ otherwise. Then $\dim \hat{H}^{\pm 1}(g', L'_{\mu\bar{\mu}}) = \dim \hat{H}^{0}(g', M'^{(1)}) = 1$ (Theorem 5.12). With the help of Proposition 5.10 we can easily construct $\nu$ as in Theorem 5.12 (i), which gives the corollary.

**Theorem 5.14.** $\hat{H}^{\pm p}(g', L'_{\mu\bar{\mu}}) = 0$, $p \geq 0$, if $l(\bar{\mu}) - l(\mu) \neq p$, or if $l(\bar{\mu}) - l(\mu) = p$ and $\mu$ and $-\bar{\mu} - \rho$ are not on the same $\rho$-centered Weyl orbit.

**Proof.** The theorem is true for $p = 0$ by Corollary 5.11 and for $p = 1$ by Corollary 5.13. Assume the theorem to be true for $\hat{H}^{\pm q}(g', L'_{\mu\bar{\mu}})$, $0 \leq q \leq p - 1$ and consider $q = p$.

Assume there exists $\omega \in M'_{\mu\bar{\mu}}$ such that $\pi(\omega) \in \hat{H}^{\pm q}(g', L'_{\mu\bar{\mu}}) \neq 0$. Let $\text{grad}(\sigma) = s$ if $s$ is the largest integer for which $\sigma \in \hat{C}(g', M'^{(s)})$. Then there exists $\nu \in \hat{H}^{\pm q+1}(g', M'^{(1)})$, $\text{grad}(\nu) = 1$ (Theorem 5.12). Write $\nu = \nu_1 + \nu_2 + \ldots \nu_n$, where $\nu_i \in V_i$, $i = 1, 2, \ldots, n$, $\text{grad}(\nu_i) = 1$ and $V_i$ are Verma or BG modules of primitive weights $(\mu_i, \bar{\mu}_i)$. We have $l(\bar{\mu}_i) - l(\mu_i) = l(\bar{\mu}_i) - l(\mu_i) - 1$ (Proposition 3.15). We may assume that $\nu$ cannot be written as a sum $\nu' + \nu''$ where $\nu', \nu'' \in \hat{H}^{\pm q+1}(g', M'^{(1)})$ and unequal, $\text{grad}(\nu') = \text{grad}(\nu'') = 1$, as this would yield two different elements in $\hat{H}^{\pm q}(g', L'_{\mu\bar{\mu}})$. If $\hat{d}\nu_i = 0$ for some value of $i$, then $\nu_i = \hat{d}(\ldots)$ (Proposition 5.10) and clearly $\nu - \nu_i$ will correspond to the same element $\omega$. Hence, we may restrict to $\nu_i$ with $\hat{d}\nu_i \neq 0, i = 1, \ldots, n$.

Consider now the equation $\hat{d}\nu = 0$ using the gradation $N_{gr}$. Let $\hat{v}_i$ be the highest order term of $\nu_i$ and $N_{gr}(\hat{v}_i) = N_i$, $i = 1, \ldots, n$. Let $\hat{N} = \max\{N_i\}_{i=1}^n$ and order such that $N_i = \hat{N}$ for $i \in \{1, 2, \ldots, m\}$, $m \leq n$. Then $d_0(\sum_{i=1}^m N_i) = 0$. As $d_0\nu_i \in V_i$, this
being regular. Introduce the following notation. For the Verma module orbit (Theorem 3.11 and Proposition 3.15).

Proof. Let $\phi_{VMi}$ be the $g$-homomorphism $V_i \rightarrow M_i'$, $i = 1, \ldots, n$, where $M_i$ are Verma modules of the same primitive weight as $V_i$. $\phi_{VMi}$ exists for all $i$ (see the note after Corollary 3.9). Then $d_0\phi_{VMi}(\hat{\nu}_i) \in M_i' \cap M_j'$. This is only possible if $d_0\phi_{VMi}(\hat{\nu}_i) \in M_i^{(1)}$ for all $i$. This implies $d\phi_{VMi}(\hat{\nu}_i) \in M_i^{(1)}$ to highest order in $N_{gr}$ and that there exists $\eta_i \in M_i'$ such that $\eta_i = d\phi_{VMi}(\nu_i)$. If there exists $\eta_i' \in M_i^{(1)}$ such that $\eta = d\eta_i'$ to leading order then $d(\phi_{VMi}(\nu_i) - \eta_i')$ to leading order, which contradicts $d\phi_{VMi}(\nu_i) = \xi_i$, where $\xi_i$ is non-exact in $M_i^{(1)}$. Hence, $\eta_i \in \hat{H}^{-p+2}(g', M_{\mu_i}^{(1)})$ to highest order. Theorem 5.12 now asserts that $\pi_{L_i}\phi_{VMi}(\hat{\nu}_i) \in \hat{H}^{-p+1}(g', L_i')$ to highest order. The induction hypothesis implies $l(\bar{\mu}_i) - l(\mu_i) = p - 1$ and that $\mu_i$ and $-\bar{\mu}_i - \rho$ lie on the same $\rho$-centered Weyl orbit. Then $l(\bar{\mu}) - l(\mu) = p$, $\mu$ and $-\bar{\mu} - \rho$ lie on the same Weyl orbit (Theorem 3.11 and Proposition 3.15).

**Theorem 5.15.** Let $\mu, \bar{\mu} \in \Gamma_w$ be such that $\mu + \rho/2$ and $\bar{\mu} + \rho/2$ are regular and $\mu$ and $-\bar{\mu} - \rho$ are on the same $\rho$-centered Weyl orbit. Then $\hat{H}^{\pm p}(g', L_{\mu\bar{\mu}}) \neq 0$, where $p = l(\bar{\mu}) - l(\mu) \geq 0$.

**Proof.** For $p = 0$ the theorem is given by Corollary 5.11 (cf. the proof of Theorem 5.14, where it is shown that $\mu + \bar{\mu} + \rho = 0$ implies $l(\bar{\mu}) - l(\mu) = 0$). For $p = 1$ the theorem follows from Corollary 5.13. We proceed by induction on $p$. Assume the theorem to be true for $0 \leq l(\bar{\mu}) - l(\mu) \leq p - 1$. We will also assume the following to hold to this order of $p$. Let $\omega \in M_{\mu\bar{\mu}}$ such that $\pi_L(\omega) \in \hat{H}^{-q}(g', L_{\mu\bar{\mu}})$ for $0 \leq q \leq p - 1$. We then assume $d\omega = \nu_1 + \nu_2 + \ldots + \nu_n$ with $\nu_i \in M_{\mu_i\bar{\mu}_i}$ and grad($\nu_i$) = 1, $i = 1, \ldots, n$ (with grad(...) defined as in Theorem 5.14). This assumption clearly holds for $p = 1$.

We now consider $\mu, \bar{\mu}$ such that $l(\bar{\mu}) - l(\mu) = p \geq 2$ with $\bar{\mu} + \rho/2$ and $\mu + \rho/2$ being regular. Introduce the following notation. For the Verma module $M_{\mu\bar{\mu}}$ we let $M_1, \ldots, M_n$ denote all submodules such that $M_i \subset M_{\mu_i\bar{\mu}_i}^{(1)}$, $M_i \nsubseteq M_{\mu_i\bar{\mu}_i}^{(2)}$, $i = 1, \ldots, n$. Denote by $(\mu_i \bar{\mu}_i), \ldots, (\mu_n \bar{\mu}_n)$ their respective highest weights. Let $\phi_i$ be a non-zero element of $\text{Hom}_g(M_i, M_{\mu_i\bar{\mu}_i})$, $i = 1, \ldots, n$. Let $M_{i_1 \ldots i_k} = M_{i_1} \cup \ldots \cup M_{i_k}$, $i_1, \ldots, i_k = 1, \ldots, n$ and $\phi_{i_1 \ldots i_k}$ be a non-zero element of $\text{Hom}_g(M_{i_1 \ldots i_k}, M_{\mu_i\bar{\mu}_i})$. Consider now $\omega_1 \in M_1$ with $\pi_L(\omega_1) \in \hat{H}^{-p+1}(g', L_{\mu_1\bar{\mu}_1})$. By Theorem 3.11 and the induction hypothesis $\omega_1$ exists. Then $d\omega_1 = \nu_1 + \ldots + \nu_{s_1}$, where $\nu_1 \in M_{1, i} \subset M_1^{(1)}$ (induction hypothesis). As grad($\nu_i$) = 1 we have grad($\phi_i(\nu_i)$) = 2. Therefore, there will exist a union of Verma modules, $M_{2 \ldots k}$ say, such that $\phi_1(\nu_1 + \ldots + \nu_{s_1}) = \phi_{2 \ldots k}(\nu'_1 + \ldots + \nu'_l)$, $\nu'_1 + \ldots + \nu'_l \in M_{2 \ldots k}$. By Lemma 3.17 $M_{2 \ldots k}$ is non-zero and different from $M_1$. Thus,
\( \phi_1(\nu_1 + \ldots + \nu_s) \) may either be viewed as originating from an element \( \nu_1 + \ldots + \nu_s \) in \( M_1 \) or from an element \( \nu'_1 + \ldots + \nu'_t \) in \( M_{2 \ldots k} \). As \( d(\nu'_1 + \ldots + \nu'_t) = 0 \), there exists an element \( \omega_{2 \ldots k} \in M_{2 \ldots k} \) such that \( \nu'_1 + \ldots + \nu'_t = d\omega_{2 \ldots k} \) (Proposition 5.1). From this it follows that \( d(\phi_{2 \ldots k}(\omega_{2 \ldots k}) - \phi_1(\omega_1)) = 0 \). Define \( \xi = \phi_{2 \ldots k}(\omega_{2 \ldots k}) - \phi_1(\omega_1) \), which must be non-zero as \( M_1 \neq M_{2 \ldots k} \). We now prove that \( \xi \) is a non-trivial element of \( \hat{H}^{-p+1}(g', M''(1)) \), which by Theorem 5.12 proves our assertions for \( l(\tilde{\mu}) - l(\mu) = p \) (including the additional induction assumption).

Assume the contrary, \( \xi = d\eta \) with \( \eta \in M^{(1)}_{\mu\tilde{\mu}} \). Let \( \xi = \xi_1 + \ldots + \xi_k \), where \( \xi_i \in M_i \), \( i = 1, \ldots, k \). By construction \( d\xi_i \in M^{(2)}_{\mu\tilde{\mu}} \), \( i = 1, \ldots, k \). If \( d\xi_i = 0 \) then the corresponding Verma module \( M_i \) may be deleted from \( M_{2 \ldots k} \) without affecting the construction of \( \xi \) (as \( d\xi_i = 0 \) implies \( \xi_i = d(\ldots) \)). Hence, we may consider \( d\xi_i \neq 0 \), \( i = 1, \ldots, k \). To highest order in the gradation \( \tilde{N}_{gr} \) the equation \( \xi = d\eta \) yields \( \xi^{(N)} = \xi_1^{(N)} + \ldots + \xi_k^{(N)} = d\eta^{(N)} \), where \( \xi^{(N)} \) and \( \eta^{(N)} \) are the leading terms in \( \xi \) and \( \eta \), respectively, and \( \xi_i^{(N)} \) denotes the \( N \)’th order term of \( \xi_i \) (which is non-zero for at least one value of \( i \)). Generally, \( d\eta \gamma \in V \) only if \( \gamma \in V \) for any Verma or BG module \( V \). Therefore, \( \eta_1^{(N)} = \eta_{1i}^{(N)} + \ldots + \eta_k^{(N)} \) and \( \xi_i^{(N)} = d\eta_i^{(N)} \), \( \eta_i^{(N)} \in M_i \), \( i = 1, \ldots, k \). This implies \( d\eta \xi^{(N)} = 0 \) and in turn \( d\xi_i = 0 \), which is a contradiction. \( \blacksquare \)

**Remark 1:** Results similar to Theorem 5.15 may be obtained for weights \( \mu + \rho/2 \) and \( \tilde{\mu} + \rho/2 \) being singular, provided the corresponding Verma modules satisfy the multiplicity condition of Lemma 3.17. It is clear, however, that this generalization does not hold for all singular cases.

**Remark 2:** The proof of Theorem 5.15 provides also an explicit method for finding the elements of the cohomology for negative ghost numbers. It is the same method as was presented in ref [3].

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