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Weakly compatible mappings with respect to a generalized $c$-distance and common fixed point results

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Abstract: In this paper, we consider weakly compatible mappings with respect to a generalized $c$-distance in cone $b$-metric spaces and obtain new common fixed-point theorems. Our results provide a more general statement, since we need not to nor the continuity of mappings and nor the normality of cone. In particular, we refer to the results of M. Abbas and G. Jungck [Common fixed point results for non-commuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl. 341 (2008) 416–420]. Some corollaries and examples are presented to support the main result proved herein.

Subjects: Analysis - Mathematics; Functional Analysis; Mathematical Analysis; Pure Mathematics; Mathematical Modeling; Foundations & Theorems

Keywords: Generalized $c$-distance; weakly compatible mappings; coincidence point; common fixed point

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1. Introduction

In 1976, Jungck (Jungck, 1976) proved a common fixed point theorem for two commuting mappings. This theorem has many applications but it requires the continuity of one of the two mappings. Then, Sessa (Sessa, 1982) defined the concept of weakly commuting to obtain common fixed point for a pair of mappings. Jungck generalized the idea of commuting mappings, first to compatible mappings (Jungck, 1988) and then to weakly compatible mappings (Jungck, 1996). In the sequel, Jungck and Rhoades (Jungck & Rhoades, 2006) proved some fixed and common fixed-point theorems for noncommuting and compatible mappings in metric spaces (also, see (Rahimi et al., 2015) and references therein).

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PUBLIC INTEREST STATEMENT

Fixed point theory is an important and useful tool for different branches of both mathematical analysis and nonlinear analysis. Accordingly, from when Banach introduced his famous principle in 1929, fixed-point theory and its application in various metrics and different distances have been developed by other scholars. One of these spaces and distances is generalized $S$-$c$-$S$-distance in cone $S$-$b$-$S$-metric spaces introduced by Boo et al.
Ordered normed spaces and cones have many applications in applied mathematics. In particular, the usage of ordered normed spaces in functional analysis date back to 1940s. It seems that Kurepa (Kurepa, 1934) was the first to use ordered normed spaces as the codomain of a metric. Later on, such metric spaces appeared occasionally under names K-metric spaces, abstract metric spaces, and generalized metric spaces (see (Zabrejko, 1997)). In 2007, Huang and Zhang (Huang & Zhang, 2007) reintroduced such spaces under the name of cone metric spaces by substituting an ordered normed space for the real numbers. On the other hands, a new type of spaces which they called b-metric spaces are defined by Bakhtin (Bakhtin, 1989) and Czerwik (Czerwik, 1993). In the sequel, analogously with definition of a b-metric space and a cone metric space, Ćvetković et al. (Ćvetković et al., 2011) and Hussain and Shah (Hussain & Shah, 2011) defined cone b-metric spaces.

In 1996, Kada et al. (Kada et al., 1996) introduced the concept of w-distance in metric spaces, where nonconvex minimization problems were treated. After that, Cho et al. (Cho et al., 2011) defined the concept of c-distance which is a cone version of the w-distance and proved some fixed-point theorems under c-distance in cone metric spaces (also, see (Fallahi et al., 2018; Rahimi & Soleimani Rad, 2014; Rahimi et al., 2015)). In 2014, Hussain et al. (Hussain et al., 2014) introduced the concept of wt-distance on a b-metric space. In the sequel, Bao et al. (Bao et al., 2015) defined generalized c-distance in cone b-metric spaces and obtained several fixed-point results in ordered cone b-metric spaces (also, see (Fadail & Bin Ahmad, 2015; Soleimani Rad et al., 2019)).

**Definition 1.1** ((Deimling, 1985; Huang & Zhang, 2007)). Let E be a real Banach space and P a subset of E. Then P is called a cone if and only if (a) P is closed, non-empty and P ≠ {θ}; (b) a, b ∈ R, a, b ≥ 0, x, y ∈ P implies ax + by ∈ P and (c) if x ∈ P and − x ∈ P, then x = θ.

Given a cone P ⊂ E, a partial ordering ≤ with respect to P is defined by x ≤ y ⇔ y − x ∈ P. We shall write x < y to mean x ≤ y and x ≠ y. Also, we write x ≪ y if and only if y − x ∈ int P (where int P is the interior of P). If int P ≠ ∅, the cone P is called solid. A cone P is called normal if there exists a number K > 0 such that θ ≤ x ≤ y implies that \( \| x \| ≤ K \| y \| \) for all \( x, y ∈ E \).

**Definition 1.2** ((Čvetković et al., 2011; Hussain & Shah, 2011)). Let X be a nonempty set, E be a real Banach space equipped with the partial ordering ≤ with respect to the cone P ⊂ E and θ be the zero vector of E. Suppose that a mapping \( d : X × X → E \) satisfies the following conditions:

1. \( d(x, y) ≤ d(x, z) + d(z, y) \) for all \( x, y, z ∈ X \).
2. \( d(x, y) = 0 \) if and only if \( x = y \).
3. \( d(x, y) = d(y, x) \) for all \( x, y ∈ X \).
4. \( d(x, z) ≤ s[d(x, y) + d(y, z)] \) for all \( x, y, z ∈ X \).

Then, \( d \) is called a cone b-metric and \( (X, d) \) is called a cone b-metric space (or cone metric type space).

Obviously, for \( s = 1 \), the cone b-metric space is a cone metric space. Moreover, if \( X \) is an nonempty set, \( E = \mathbb{R} \) and \( P = [0, ∞) \), then cone b-metric on \( X \) is a b-metric on \( X \). For notions such as convergent and Cauchy sequences, completeness, continuity, and etc in cone b-metric spaces, we refer to (Čvetković et al., 2011; Hussain & Shah, 2011). Also, we use of the following properties for all \( u, v, w, c ∈ E \) when the cone \( P \) may be non-normal.

1. \( (p_{u}) \) If \( u ≤ v \) and \( v ≪ w \), then \( u ≪ w \).
2. \( (p_{v}) \) If \( θ ≤ v ≤ c \) for each \( c ∈ int P \), then \( v = θ \).
3. \( (p_{n}) \) If \( v ≤ λ v \) where \( v ∈ P \) and \( 0 < λ < 1 \), then \( v = θ \).
4. \( (p_{a}) \) Let \( a_n → θ \) in \( E \), \( θ ≤ a_n \) and \( θ ≪ c \). Then, there exists a positive integer \( n_0 \) such that \( a_n ≪ c \) for each \( n > n_0 \).
Definition 1.3 (Babaei et al., 2015). Let \((X, d)\) be a cone \(b\)-metric space with parameter \(s \geq 1\). A mapping \(q : X \times X \to E\) is said to be a generalized \(c\)-distance on \(X\) if for any \(x, y, z \in X\), the following properties are satisfied:

\[
\begin{align*}
&q_1: \theta \preceq q(x, y); \\
&q_2: q(x, z) \leq s[q(x, y) + q(y, z)]; \\
&q_3: q is \(b\)-lower semi-continuous in its second variable i.e., if \(q(x, y_n) \preceq u\) for all \(n \geq 1\) and for some \(u = u_n\), then \(q(x, y) \preceq su_n\), where \(\{y_n\}\) is a sequence in \(X\) which converges to \(y \in X\); \\
&q_4: for any \(c \in int P\), there exists \(e \in E\) with \(\theta \ll e\) such that \(q(z, x) \ll e\) and \(q(z, y) \ll e\) imply that \(d(x, y) \ll c\).
\]

Let \((X, d)\) be a \(b\)-metric space, \(E = \mathbb{R}\) and \(P = [0, \infty)\). Then, wt-distance (Hussain et al., 2014) on a \(b\)-metric space \(X\) is a generalized \(c\)-distance. But the converse does not hold. Further, if \(s = 1\), the generalized \(c\)-distance is a \(c\)-distance (Cho et al., 2011). Also, set \(s = 1\), \(E = \mathbb{R}\) and \(P = [0, \infty)\) in the above definition. Then, we obtain the definition of \(w\)-distance (Kada et al., 1996) for more details, see (Babaei et al., 2020). Moreover, for any generalized \(c\)-distance \(q\), \(q(x, y) = \theta\) is not necessarily equivalent to \(x = y\) for all \(x, y \in X\) and \(q(x, y) = q(y, x)\) does not necessarily hold for all \(x, y \in X\).

Example 1.4. (Soleimani Rad et al., 2019) Let \(E = C^1([0, 1], \mathbb{R})\) with the norm \(\|x\| = \|x\|_\infty + \|x\|_1\) and consider the non-normal cone \(P = \{x \in E : x(t) \geq 0 \text{ for all } t \in [0, 1]\}\). Also, let \(X = [0, \infty)\) and define a mapping \(d : X \times X \to E\) by \(d(x, y) = |x - y|^\theta\) for all \(x, y \in X\), where \(\theta : [0, 1] \to \mathbb{R}\) is defined by \(\theta(t) = t^2\) for all \(t \in [0, 1]\). Then \((X, d)\) is a cone \(b\)-metric space with \(s \in \{1, 2\}\). Define a mapping \(q : X \times X \to E\) by \(q(x, y) = y^\theta\) or \(q(x, y) = (x^\theta + y^\theta)^\theta\) for all \(x, y \in X\) and \(s \in \{1, 2\}\). Then \(q\) is a generalized \(c\)-distance.

Lemma 1.5 (Soleimani Rad et al., 2019). Let \((X, d)\) be a cone \(b\)-metric space and \(q\) be a generalized \(c\)-distance on \(X\). Let \(\{x_n\}\) and \(\{y_n\}\) be sequences in \(X\), \(\{u_n\}\) and \(\{v_n\}\) be two convergent sequences in \(P\). For any \(x, y, z \in X\),

\[
\begin{align*}
&q_{(a)}: \text{if for all } n \in \mathbb{N}, \quad q(x_n, y) \preceq u_n \text{ and } q(x_n, z) \preceq v_n, \quad \text{then } y = z. \quad \text{In particular, if } q(x, y) = \theta \text{ and } q(x, z) = \theta, \quad \text{then } y = z; \\
&q_{(b)}: \text{if for all } n \in \mathbb{N}, \quad q(x_n, y_n) \preceq u_n \text{ and } q(x_n, z) \preceq v_n, \quad \text{then } \{y_n\} \text{ converges to } z; \\
&q_{(c)}: \text{if for } m, n \in \mathbb{N}, \quad \text{with } m > n, \quad \text{we have } q(x_n, x_m) \preceq u_n \text{, then } \{x_n\} \text{ is a Cauchy sequence in } X; \\
&q_{(d)}: \text{if for all } n \in \mathbb{N}, \quad q(y_n, x_n) \preceq u_n \text{, then } \{x_n\} \text{ is a Cauchy sequence in } X.
\end{align*}
\]

Definition 1.6 (Jungck & Rhoades, 2006). Let \(f\) and \(g\) be two self-mappings defined on a set \(X\). If \(fw = gw = z\) for some \(w \in X\), then \(w\) is called a coincidence point of \(f\) and \(g\), and \(z\) is called a point of coincidence of \(f\) and \(g\). Also, the mappings \(f\) and \(g\) are said to be weakly compatible if they commute at every coincidence point; that is, if \(f(gw) = g(fw)\) for all coincidence points \(w\).

Lemma 1.7 (Abbas & Jungck, 2008). Let \(f\) and \(g\) be weakly compatible self-mappings on a set \(X\). If \(f\) and \(g\) have a unique point of coincidence \(z = fw = gw\), then \(z\) is the unique common fixed point of \(f\) and \(g\).

2. Main results
Our main result is the following theorem. We prove a common fixed point theorem under the concept of a generalized \(c\)-distance on cone \(b\)-metric spaces without assumption of normality for a cone.

Theorem 2.1. Let \((X, d)\) be a cone \(b\)-metric space over a solid cone \(P\) with given real number \(s \geq 1\). Also, let \(q\) be a generalized \(c\)-distance and \(f, g : X \to X\) be two mappings with \(f(X) \subseteq g(X)\) and \(g(X)\) be
a complete subspace of $X$. Suppose that there exist mappings $a_i : X \rightarrow [0, 1)$ for $i = 1, 2, 3, 4$ such that the following conditions hold:

(i) $a_i(fx) \leq a_i(gx)$ for all $x \in X$;
(ii) $s(a_1 + a_2)(x) + a_3(x) + (s^2 + s)a_4(x) < 1$ for all $x \in X$;
(iii) for all $x, y \in X$,
\[q(fx, fy) \leq a_1(gx)q(gx, gy) + a_2(gx)q(gx, fx) + a_3(gx)q(gy, fy) + a_4(gx)q(gx, fy) .\] \hfill (2.1)

Then $f$ and $g$ have a coincidence point $z \in X$. Moreover, if $w = gz = fz$, then $q(w, w) = \theta$. Also, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Since the range of $g$ contains the range of $f$, there exists an $x_1 \in X$ such that $fx_0 = gx_1$. By induction, a sequence $\{x_n\}$ can be chosen such that $fx_n = gx_{n+1}$ for $n = 0, 1, 2, \cdots$. Now, set $x = x_{n-1}$ and $y = x_n$ in (2.1). Thus, by (q2) and (i), we get

\[q(gx_n, gx_{n+1}) = q(fx_{n-1}, fx_n) \leq a_1(gx_{n-1})q(gx_{n-1}, gx_n) + a_2(gx_{n-1})q(gx_{n-1}, fx_n)\]
\[+ a_3(gx_{n-1})q(gx_{n-1}, fx_n) + a_4(gx_{n-1})q(gx_{n-1}, fx_n)\]
\[= a_1(fx_{n-2})q(gx_{n-1}, gx_n) + a_2(fx_{n-2})q(gx_{n-1}, fx_{n-1})\]
\[+ a_3(fx_{n-2})q(gx_{n-2}, fx_n) + a_4(fx_{n-2})q(gx_{n-2}, fx_n)\]
\[\leq a_1(gx_{n-2})q(gx_{n-1}, gx_n) + a_2(gx_{n-2})q(gx_{n-1}, gx_n)\]
\[+ a_3(gx_{n-2})q(gx_{n-2}, gx_{n+1}) + a_4(gx_{n-2})q(gx_{n-2}, gx_{n+1})\]
\[\leq a_1(fx_{n-3}) + a_2(fx_{n-3}) + sa_n(fx_{n-2})q(gx_{n-1}, gx_n)\]
\[+ a_3(fx_{n-3}) + sa_n(fx_{n-2})q(gx_{n-2}, gx_{n+1})\]
\[\leq \cdots \leq a_1(gx_1) + a_2(gx_1) + sa_n(gx_1)q(gx_{n-1}, gx_n)\]
\[+ a_3(gx_1) + sa_n(gx_1)q(gx_{n-1}, gx_{n+1})\]

for all $n \in \mathbb{N}$. Hence, we have

\[q(gx_n, gx_{n+1}) = q(fx_{n-1}, fx_n) \leq hq(gx_{n-1}, gx_n).\] \hfill (2.2)

where

\[0 \leq h = \frac{a_1(gx_0) + a_2(gx_0) + sa_n(gx_0)}{1 - a_3(gx_0) - sa_n(gx_0)} < \frac{1}{s} .\] \hfill (from (ii))

By repeating the procedure, we get

\[q(gx_n, gx_{n+1}) \leq h^{n-1} q(gx_0, gx_1)\] \hfill (2.3)

for all $n \in \mathbb{N}$. Now, for positive integer $m$ and $n$ with $m > n$, it follows from $sh < 1$ and (2.3) that

\[q(gx_n, gx_m) \leq s q(gx_n, gx_{n+1}) + s^2 q(gx_{n+1}, gx_{n+2}) + \cdots + s^{m-n} q(gx_{m-1}, gx_m)\]
\[\leq (sh^{n-1} + s^2 h^{n-1} + \cdots + s^{m-n} h^{m-1}) q(gx_0, gx_1)\]
\[
\frac{sh^n}{1-sh} q(gx_0, gx_1) \in P.
\]
\[\text{(2.4)}\]

Now, Lemma 1.5.(p3) implies that \(\{gx_n\}\) is a Cauchy sequence in \(X\). Since \(g(X)\) is a complete subspace of \(X\) and \(g\) is a self-mapping, there exists a point \(u = g(z) \in g(X)\) for some \(z \in X\) such that \(gx_n \to gz\) as \(n \to \infty\). Also, from (q3) and (2.4), we get
\[
q(gx_n, gz) \leq \frac{s^2h^n}{1-sh} q(gx_0, gx_1).
\]
\[\text{(2.5)}\]

Moreover, by (2.2) and (2.5), we get
\[
q(gx_n, fz) = q(fx_{n-1}, fz) \leq h q(gx_{n-1}, gz) \leq \frac{s^2h^{n-1}}{1-sh} q(gx_0, gx_1) = \frac{s^2h^n}{1-sh} q(gx_0, gx_1).
\]
\[\text{(2.6)}\]

Using Lemma 1.5.(p3) and relations (2.5) and (2.6), we have \(fz = gz\). Consequently, \(z\) is a coincidence point of \(f\) and \(g\) and \(w\) is a point of coincidence of \(f\) and \(g\); that is, \(w = fz = gz\).

Further, we have
\[
q(w, w) = q(fz, fz) \leq a_1(q)q(gz, gz) + a_2(q)q(gz, fz) + a_3(q)q(gz, fz) + a_4(q)q(gz, fz)
\]
\[= (a_1(w) + a_2(w) + a_3(w) + a_4(w))q(w, w).\]

Since \((a_1 + a_2 + a_3 + a_4)(w) \leq (s(a_1 + a_2) + a_3 + (s^2 + s)a_4)(w) < 1\) from (ii), we get \(q(w, w) = \theta\) by (p3). Now we shall show that \(w\) is the unique point of coincidence. Let \(w'\) be another point of coincidence \(f\) and \(g\); that is, \(w' = fz = gz\) for a point \(z \in X\). Then, we have
\[
q(w, w') = q(fz, fz') \leq a_1(q)q(gz, gz') + a_2(q)q(gz, fz') + a_3(q)q(gz, fz') + a_4(q)q(gz, fz')
\]
\[= a_1(w)q(w, w') + a_2(w)q(w, w') + a_3(w)q(w, w') + a_4(w)q(w, w')
\]
\[= (a_1(w) + a_4(w))q(w, w').\]

Similar to the previous discussion, we get \(q(w, w') = \theta\). Since \(q(w, w) = \theta\) and \(q(w, w') = \theta\), we have \(w = w'\) by Lemma 1.5.(p3). Therefore, \(w = fz = gz\) is the unique point of coincidence \(f\) and \(g\).

Now, by a similar procedure in Lemma 1.7, we can prove \(w\) is a unique common fixed point as follows. Since \(f\) and \(g\) are weakly compatible and \(w = fz = gz\), we obtain \(fw = fgz = gfz = gw\); that is, \(fw = gw\) is a point of coincidence \(f\) and \(g\). Thus, \(w = gw\). Also, if \(z = fz = gz\), then \(z\) is a point of coincidence of \(f\) and \(g\). Therefore, by uniqueness, \(z = w\); i.e., \(w\) is a unique common fixed point of \(f\) and \(g\).

**Example 2.2.** Let \(E = R, P = \{x \in E : x \geq 0\}\) and \(X = [0, 1]\). Define a mapping \(d : X \times X \to E\) by \(d(x, y) = |x - y|^2\) for all \(x, y \in X\). Then \((X, d)\) is a cone b-metric space with \(s = 2\). Define a function \(q : X \times X \to E\) by \(q(x, y) = d(x, y)\) for all \(x, y \in X\). Then \(q\) is a generalized c-distance. Also, let mappings \(f, g : X \to X\) defined by \(f(x) = \frac{x}{4}\) and \(g(x) = x\) for all \(x \in X\). Define the mappings \(a_i : X \to [0, 1]\) with \(a_1(x) = \frac{(x + 1)^2}{8}\) and \(a_2(x) = a_3(x) = a_4(x) = 0\) for all \(x \in X\). Clearly, \(a_i(fx) \leq a_i(gx)\) for all \(x \in X\) and \(i = 1, 2, 3, 4\). Moreover, for all \(x \in X\),
\[
(s(a_1 + a_2) + a_3 + (s^2 + s)a_4)(x) = \frac{(x + 1)^2}{8} < 1.
\]

Also, we have
\[
q(fx, fy) = \left(\frac{x^2}{4} - \frac{y^2}{4}\right)^2
\]
\[
\frac{(x+y)^2(x-y)^2}{16} \leq \frac{(x+1)^2(x-y)^2}{16} \quad (y \in X \Rightarrow y \leq 1)
\]

\[
= \alpha_1(gx)q(gx, gy) + \alpha_2(gx)q(gx, fx) + \alpha_3(gx)q(gy, fy) + \alpha_4(gx)q(gx, fy)
\]

for all \(x, y \in X\). Also, \(f\) and \(g\) are weakly compatible at \(x = 0\). Therefore, all conditions of Theorem 2.1 are satisfied. Hence, \(f\) and \(g\) have a unique common fixed point \(x = 0\). Further, \(q(0, 0) = 0\).

The following corollary can be obtained as consequences of Theorem 2.6 which are the extension of some results of Abbas and Jungck (Abbas & Jungck, 2008) under the concept of a generalized c-distance in cone b-metric spaces over a solid cone and by applying control function instead of constant coefficient.

**Corollary 2.3.** Let \((X, d)\) be a cone b-metric space over a solid cone \(P\) with given real number \(s \geq 1\). Also, let \(q\) be a generalized c-distance and \(f, g : X \to X\) be two mappings with \(f(X) \subseteq g(X)\) and \(g(X)\) be a complete subspace of \(X\). Suppose that there exists \(a : X \to [0, 1)\) such that \(a(fx) \leq a(gx)\) for all \(x \in X\) and \(q(fx, fy) \leq a(gx)q(gx, gy)\) for all \(x, y \in X\), where \(sa(x) < 1\). Then \(f\) and \(g\) have a coincidence point \(z \in X\). Moreover, if \(w = gz = fz\), then \(q(w, w) = \theta\). Also, if \(f\) and \(g\) are weakly compatible, then \(f\) and \(g\) have a unique common fixed point.

In Theorem 2.1 and Corollary 2.3, set \(s = 1\). Then, we obtain the following result in the framework of cone metric spaces associated with a c-distance.

**Theorem 2.4.** Let \((X, d)\) be a cone metric space over a solid cone \(P\). Also, let \(q\) be a c-distance and \(f, g : X \to X\) be two mappings with \(f(X) \subseteq g(X)\) and \(g(X)\) be a complete subspace of \(X\). Suppose that there exist mappings \(a_i : X \to [0, 1)\) for \(i = 1, 2, 3, 4\) such that the following conditions hold:

1. \(a_i(fx) \leq a_i(gx)\) for all \(x \in X\);
2. \((a_1 + a_2 + a_3 + 2a_4)(x) < 1\) for all \(x \in X\);
3. for all \(x, y \in X\),

\[
q(fx, fy) \leq a_1(gx)q(gx, gy) + a_2(gx)q(gx, fx) + a_3(gx)q(gy, fy) + a_4(gx)q(gx, fy).
\]

Then \(f\) and \(g\) have a coincidence point \(z \in X\). Moreover, if \(w = gz = fz\), then \(q(w, w) = \theta\). Also, if \(f\) and \(g\) are weakly compatible, then \(f\) and \(g\) have a unique common fixed point.

**Corollary 2.5.** Let \((X, d)\) be a cone metric space over a solid cone \(P\). Also, let \(q\) be a c-distance and \(f, g : X \to X\) be two mappings with \(f(X) \subseteq g(X)\) and \(g(X)\) be a complete subspace of \(X\). Suppose that there exists \(a : X \to [0, 1)\) such that \(a(fx) \leq a(gx)\) for all \(x \in X\) and \(q(fx, fy) \leq a(gx)q(gx, gy)\) for all \(x, y \in X\) where \(a(x) < 1\). Then \(f\) and \(g\) have a coincidence point \(z \in X\). Moreover, if \(w = gz = fz\), then \(q(w, w) = \theta\). Also, if \(f\) and \(g\) are weakly compatible, then \(f\) and \(g\) have a unique common fixed point.

**Theorem 2.6.** Let \((X, d)\) be a cone b-metric space over a solid cone \(P\) with given real number \(s \geq 1\) and \(q\) be a generalized c-distance. Suppose that there exist two mappings \(f, g : X \to X\) such that

\[
q(fx, fy) \leq a_1(q(gx, gy) + a_2q(gx, fx) + a_3q(gy, fy) + a_4q(gx, fy)
\]

for all \(x, y \in X\), where \(a_i\) are nonnegative coefficients for \(i = 1, 2, 3, 4\) with

\[s(a_1 + a_2 + a_3 + (s^2 + s)a_4 < 1].
\]
If \( f(X) \subseteq g(X) \) and \( g(X) \) be a complete subspace of \( X \), then \( f \) and \( g \) have a coincidence point \( z \in X \). Moreover, if \( w = g(z) = fz \), then \( q(w, w) = 0 \). Also, if \( f \) and \( g \) are weakly compatible, then \( f \) and \( g \) have a unique common fixed point.

**Proof.** We can prove this result by applying Theorem 2.1 with \( \alpha_i(x) = \alpha_i \) for \( i = 1, 2, 3, 4 \).

The following corollaries can be obtained as consequences of Theorem 2.6 which are the extension of some results of Abbas and Jungck (Abbas & Jungck, 2008), and Shi and Xu (Shi & Xu, 2013) under the concept of a generalized \( c \)-distance in cone \( b \)-metric spaces over a solid cone. These are same Corollary 4.4 and Corollary 4.5 of Fadail and Bin Ahmad (Fadail & Bin Ahmad, 2015).

**Corollary 2.7.** Let \((X, d)\) be a cone \( b \)-metric space over a solid cone \( P \) with given real number \( s \geq 1 \) and \( q \) be a generalized \( c \)-distance. Suppose that there exist two mappings \( f, g : X \rightarrow X \) such that \( q(fx, fy) \leq \alpha q(gx, gy) \) for all \( x, y \in X \), where \( \alpha \in [0, \frac{1}{2}) \). If \( f(X) \subseteq g(X) \) and \( g(X) \) be a complete subspace of \( X \), then \( f \) and \( g \) have a coincidence point \( z \in X \). Moreover, if \( w = g(z) = fz \), then \( q(w, w) = 0 \). Also, if \( f \) and \( g \) are weakly compatible, then \( f \) and \( g \) have a unique common fixed point.

**Proof.** We can prove this result by applying Theorem 2.6 with \( \alpha_1 = \alpha \) and \( \alpha_i = 0 \) for \( i = 2, 3, 4 \) or by applying Corollary 2.3 with \( \alpha(x) = \alpha \).

**Example 2.8.** Let \( E = C^1_4[0, 1], \mathbb{R} \) with the norm \( \| x \| = \| x \|_{\infty} + \| x' \|_{\infty} \) and consider the non-normal cone \( P = \{ x \in E : x(t) \geq 0 \text{ for all } t \in [0, 1] \} \). Also, let \( X = [0, 2] \) and define a mapping \( d : X \times X \rightarrow E \) by \( d(x, y)(t) = (x - y)^2 \varphi(t) \) for all \( x, y \in X \), where \( \varphi : [0, 1] \rightarrow \mathbb{R} \) is defined by \( \varphi(t) = 2^t \) for all \( t \in [0, 1] \). Then \((X, d)\) is a complete cone \( b \)-metric space with \( s = 2 \). Also, let mappings \( f, g : X \rightarrow X \) be defined by \( fx = \frac{x}{4} \) and \( gx = \frac{x}{2} \) for all \( x \in X \). Then

\[
d(fx, fy)(t) = \left( \frac{x^2}{16} - \frac{y^2}{16} \right)^2 \varphi(t)
\]

\[
= \left( \frac{x - y}{4} \right)^2 \left( \frac{x + y}{4} \right)^2 \varphi(t) \quad (x, y \in [0, 2] \Rightarrow \left( \frac{x + y}{4} \right)^2 \leq 1)
\]

\[
\leq d(gx, gy)(t).
\]

Thus, there is no \( \alpha \in [0, \frac{1}{2}) \) such that \( d(fx, fy)(t) \leq \alpha d(gx, gy)(t) \). Hence, we can not apply Theorem 2.1 of Shi and Xu (Shi & Xu, 2013) for this example on a cone \( b \)-metric space.

Now, define a function \( q : X \times X \rightarrow E \) by \( q(x, y)(t) = y^2 \varphi(t) \) for all \( x, y \in X \) and \( t \in [0, 1] \). Then, \( q \) is a generalized \( c \)-distance. Also,

\[
q(fx, fy)(t) = (fy)^2 \varphi(t)
\]

\[
= \left( \frac{y^2}{16} \right)^2 \varphi(t)
\]

\[
= \left( \frac{y^2}{16} \right) \left( \frac{y^2}{16} \right) \varphi(t) \quad (y \in [0, 2] \Rightarrow y^2 \leq \frac{1}{4})
\]
\[ \frac{1}{4} q(gx, gy)(t). \]

where \( \alpha = \frac{1}{2} \in [0, \frac{1}{2}) \). Also, \( f \) and \( g \) are weakly compatible at \( x = 0 \). Therefore, all conditions of Corollary 2.7 are satisfied. Hence, \( f \) and \( g \) have a unique common fixed point \( x = 0 \). Further, \( q(0, 0) = 0 \).

**Corollary 2.9.** Let \( (X, d) \) be a cone \( b \)-metric space over a solid cone \( P \) with given real number \( s \geq 1 \) and \( q \) be a generalized \( c \)-distance. Suppose that there exist two mappings \( f, g : X \to X \) such that

\[ q(fx, fy) \leq \alpha_1 q(gx, fy) + \alpha_2 q(gx, fx) + \alpha_3 q(gy, fy) + \alpha_4 q(gx, fy) \]

for all \( x, y \in X \), where \( \alpha_i \) are nonnegative coefficients for \( i = 1, 2, 3, 4 \) with \( \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 < 1 \). If \( f(X) \subseteq g(X) \) and \( g(X) \) be a complete subspace of \( X \), then \( f \) and \( g \) have a coincidence point \( z \in X \). Moreover, if \( w = gz = fz \), then \( q(w, w) = 0 \). Also, if \( f \) and \( g \) are weakly compatible, then \( f \) and \( g \) have a unique common fixed point.

In Theorem 2.6 and its corollaries, set \( s = 1 \). Then, we obtain the following results in the framework of cone metric spaces associated with a \( c \)-distance. These are same Theorem 3.1, Corollary 3.1 and Corollary 3.2 of Fadail et al. (Fadail et al., 2013).

**Theorem 2.10.** Let \( (X, d) \) be a cone metric space over a solid cone \( P \) and \( q \) be a \( c \)-distance. Suppose that there exist two mappings \( f, g : X \to X \) such that

\[ q(fx, fy) \leq \alpha_1 q(gx, gy) + \alpha_2 q(gx, fx) + \alpha_3 q(gy, fy) + \alpha_4 q(gx, fy) \]

for all \( x, y \in X \), where \( \alpha_i \) are nonnegative coefficients for \( i = 1, 2, 3, 4 \) with \( \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 < 1 \). If \( f(X) \subseteq g(X) \) and \( g(X) \) be a complete subspace of \( X \), then \( f \) and \( g \) have a coincidence point \( z \in X \). Moreover, if \( w = gz = fz \), then \( q(w, w) = 0 \). Also, if \( f \) and \( g \) are weakly compatible, then \( f \) and \( g \) have a unique common fixed point.

The following corollaries can be obtained as consequences of Theorem 2.10 which are the extension of some results of Abbas and Jungck (Abbas & Jungck, 2008) under the concept of a \( c \)-distance.

**Corollary 2.11.** Let \( (X, d) \) be a cone metric space over a solid cone \( P \) and \( q \) be a \( c \)-distance. Suppose that there exist two mappings \( f, g : X \to X \) such that \( q(fx, fy) \leq \alpha q(gx, gy) \) for all \( x, y \in X \), where \( \alpha \in [0, 1) \). If \( f(X) \subseteq g(X) \) and \( g(X) \) be a complete subspace of \( X \), then \( f \) and \( g \) have a coincidence point \( z \in X \). Moreover, if \( w = gz = fz \), then \( q(w, w) = 0 \). Also, if \( f \) and \( g \) are weakly compatible, then \( f \) and \( g \) have a unique common fixed point.

**Corollary 2.12.** Let \( (X, d) \) be a cone metric space over a solid cone \( P \) and \( q \) be a \( c \)-distance. Suppose that there exist two mappings \( f, g : X \to X \) such that \( q(fx, fy) \leq \delta q(gx, fx) + \gamma q(gy, fy) \) for all \( x, y \in X \), where \( \delta \) and \( \gamma \) are nonnegative coefficients with \( \delta + \gamma < 1 \). If \( f(X) \subseteq g(X) \) and \( g(X) \) be a complete subspace of \( X \), then \( f \) and \( g \) have a coincidence point \( z \in X \). Moreover, if \( w = gz = fz \), then \( q(w, w) = 0 \). Also, if \( f \) and \( g \) are weakly compatible, then \( f \) and \( g \) have a unique common fixed point.

Note that the results of our paper contain many papers about common fixed point for weakly compatible mappings in various abstract spaces such as: Abbas and Jungck (Abbas & Jungck, 2008), Fadail et al. (Fadail & Bin Ahmad, 2015; Fadail et al., 2013), Jungck and Rhoades (Jungck & Rhoades, 2006), Rahimi et al. (Rahimi, Soleimani Rad et al., 2015), Sessa (Sessa, 1982), Shi and Xu (Shi & Xu, 2013), and Wang and Gu (Wang & Guo, 2011).

### 3. Conclusion and suggestion

Here, we considered the concept of weakly compatible mappings with respect to a generalized \( c \)-distance in cone \( b \)-metric spaces and proved several fixed-point theorems. Our results are significant, since
The class of generalized c-distance in cone b-metric spaces is bigger than the class of usual c-distance in cone metric spaces. Hence, the authors can prove their results with respect to a c-distance without complete and repetitive proof (by considering $s = 1$ in generalized c-distance).

2. The class of generalized c-distance in cone b-metric spaces is bigger than the class of usual wt-distance in b-metric spaces. Hence, the authors can prove their results with respect to a wt-distance without complete and repetitive proof (by considering $E = \mathbb{R}$ and $P = [0, +\infty)$ in generalized c-distance).

3. We need not to nor the continuity of mapping and nor the normality of cone in the procedure the proof of main results.

To continue this paper, the readers can consider some former researches from 2007 until now and can obtain new results with respect to this distance with its application.

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