FROM QUANTUM HYDRODYNAMICS TO QUANTUM GRAVITY

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We discuss some lessons from quantum hydrodynamics to quantum gravity.

1. Introduction

In the presentations at the Session ‘Analog Models of and for General Relativity’ at 11 Marcel Grossmann Meeting, general relativity has been considered as emergent phenomenon. General approaches to emergent relativity have been analyzed. Particular example when gravity is induced in the low-energy corner of quantum condensed matter of the proper universality class has been presented. It was suggested that induced metric for scalar field may lead to superluminal propagation of scalar field and escape from the black hole without violation of Lorentz invariance. On the kinematic level the metric field emerges in many different systems, and this allows us to simulate (at least theoretically) effects of relativistic quantum field theory (QFT) in curved space. At the moment the most promising media for simulations are Bose-Einstein condensate (BEC), where the propagation of phonons is identical to propagations of a massless scalar field on a curved space-time. In particular, it was suggested to use the renormalization techniques developed in QFT to study the depletion of BEC; in other presentation the stability of sonic horizons in BEC and the scattering problems on rotating acoustic black holes have been discussed. Effective metric appears for light propagating in non-linear dispersive dielectric media and in moving media; for surface waves – ripplons – propagating on the surface of quantum liquids or at the interface between two superfluids. The latter allows us to study experimentally the instability of the quantum vacuum in ergoregion.

Probably our experience with superfluids and BEC will give us some hints for solution of the fundamental problems in gravity, such as quantum gravity and gravitating vacuum energy. Here we shall discuss the quantum hydrodynamics of BEC and superfluids. Both hydrodynamics and general relativity are perfect classical theories. General relativity can be viewed as the theory of hydrodynamic type where the collective variables are the metric fields . At the quantum level, quantum hydrodynamics and quantum gravity also share many common features, e.g. both have quadratic divergences. This is the reason why the problem of quantization of hydrodynamics is at least 65 years old (see quantization of the macroscopic dynamics of liquid in the first Landau paper on superfluidity of ); it is almost as old as the problem of quantization of gravity. Thus the lessons from quantum
hydrodynamics could be useful for quantum gravity.

2. Classical hydrodynamics

2.1. Classical hydrodynamics

The first quantization scheme for hydrodynamics was suggested by Landau in 1941 when he developed the theory of superfluidity in liquid $^4$He.\textsuperscript{11} In his approach Landau separated liquid $^4$He into two parts: the ground state (which we now call the vacuum) and quasiparticles – excitations above the ground state (which we call matter). Such separation into vacuum and matter is generic and is applicable to relativistic quantum fields (RQF). The Landau approach was essentially different from that of Tisza,\textsuperscript{13} who suggested to separate liquid $^4$He into the Bose condensate and the non-condensed atoms. Tisza’s approach does make sense, especially for the dilute Bose gases, where the condensed fraction can be easily detected. However, it is important that the dynamics of the Bose condensate and the exchange of energy and atoms between the condensed and non-condensed fractions, belong to high-energy microscopic physics. On the other hand, the low-energy behavior of the superfluid liquids and gases is governed by the Landau hyrodynamics picture. In particular, at zero temperature both condensed and non-condensed atoms participate in the coherent motion of the quantum vacuum with the total mass density $\rho$. This is because at $T = 0$ the whole liquid is in the coherent state described by a single many-body wave function,\textsuperscript{14} and thus the whole liquid is involved in the superfluid motion in agreement with Landau ideas.

According to Landau, the Hamiltonian of quantum hydrodynamics is the classical energy of liquid where the classical fields, velocity $v$ and mass density $\rho$, are substituted by the corresponding quantum operators $\hat{v}$ and $\hat{\rho}$. So let us start with the classical hydrodynamic energy of the liquid:

$$H_{\text{hydro}}(\rho, v) = \int d^3 x \left( \frac{1}{2} \rho v^2 + \tilde{\epsilon}(\rho) \right) , \quad \tilde{\epsilon}(\rho) = \epsilon(\rho) - \mu \rho .$$

(1)

Here $\epsilon(\rho)$ is the energy of static liquid which only depends on $\rho$: since we consider the vacuum of the liquid (i.e. without excitations which will appear after quantization) it is assumed that the temperature $T = 0$. We added here the term with Lagrangian multiplier – the constant chemical potential $\mu$. This term does not change the hydrodynamic equations, but it allows us to study thermodynamics of the liquid. For example, the equilibrium mass density of static liquid is obtained by minimization of the energy with taking into account the conservation of the total mass of the liquid, which gives:

$$\frac{d\epsilon}{d\rho} = \mu , \quad \text{or} \quad \frac{d\tilde{\epsilon}}{d\rho} = 0 .$$

(2)

The pressure of the liquid in equilibrium at $T = 0$ is

$$P = - \frac{d(\nu \epsilon (M/V))}{dV} = - \tilde{\epsilon} ,$$

(3)
where $M$ is the total mass of liquid. This suggests that the relation between the pressure $P$ and energy $\tilde{\epsilon}$ can be considered as the equation of state for vacuum, and this is true. Such equation of state $P = -\tilde{\epsilon}$ is applicable to the ground state (vacuum) of any system, relativistic or non-relativistic; it follows from the general thermodynamic arguments and does not depend on the microscopic physics of the vacuum state. It is also applicable to the vacuum of RQF. So, further on we shall treat the quantities $\epsilon_{\text{vac}} \equiv \tilde{\epsilon}$ and $P_{\text{vac}} \equiv -\tilde{\epsilon}$ as vacuum energy density and vacuum pressure correspondingly.

There is no satisfactory description of classical hydrodynamics in terms of Lagrangian: this requires introduction of either artificial variables or extra dimension. The hydrodynamic equations can be obtained using the Hamiltonian formalism of Poisson brackets. The Poisson brackets between the hydrodynamic variables are universal, they are determined by the symmetry of the system and do not depend on the Hamiltonian (cf.\cite{15}), that is why it is not necessary to use the microscopic quantum theory for their derivation. For classical hydrodynamic variables $\mathbf{v}$ and $\rho$ one has the following Poisson brackets:\cite{15,16}

\begin{align}
\{\rho(\mathbf{r}_1), \rho(\mathbf{r}_2)\} &= 0 , \quad (4) \\
\{\mathbf{v}(\mathbf{r}_1), \rho(\mathbf{r}_2)\} &= -\nabla \delta(\mathbf{r}_1 - \mathbf{r}_2) , \quad (5) \\
\{v_i(\mathbf{r}_1), v_j(\mathbf{r}_2)\} &= -\frac{1}{\rho} \epsilon_{ijk}(\nabla \times \mathbf{v})_k \delta(\mathbf{r}_1 - \mathbf{r}_2) . \quad (6)
\end{align}

The same Poisson brackets are obtained from the commutation relations for the corresponding quantum operators $\hat{\rho}$ and $\hat{\mathbf{v}}$ derived by Landau\cite{11} which follow from microscopic physics. Using the Poisson brackets (4)-(6) and the Hamiltonian in Eq.(1), one obtains the hydrodynamic equations:

\begin{align}
\partial_t \rho &= \{H, \rho\} = -\nabla \cdot (\rho \mathbf{v}) , \quad (7) \\
\partial_t \mathbf{v} &= \{H, \mathbf{v}\} = -(\mathbf{v} \cdot \nabla) \mathbf{v} - \nabla \frac{d\epsilon}{d\rho} . \quad (8)
\end{align}

There are no fundamental parameters in classical hydrodynamics. But there are dimensional variables which enter the classical hydrodynamics: mass density $\rho$, and energy density $\epsilon(\rho)$; the speed of sound $c$ is $c^2 = \rho(d^2\epsilon/d\rho^2)$. In principle, in liquids one can construct the “fundamental” parameters, the values of $c = c_0$ and $\rho = \ρ_0$ under two conditions, when the liquid is: (i) static and in equilibrium; and (ii) at zero external pressure. These two conditions give $d\epsilon/d\rho|_{\rho_0} = \mu_0$ and $P = \mu_0\rho_0 - \epsilon(\rho_0) = -\tilde{\epsilon}(\rho) = 0$ correspondingly. At zero external pressure, i.e. in the absence of external environment, one has

$$
\epsilon_{\text{vac}} = -P_{\text{vac}} = 0 .
$$

The nullification of vacuum energy occurs for any non-disturbed equilibrium vacuum.
2.2. Vortex-free classical hydrodynamics

If one is interested in the vortex-free flow only, \( \mathbf{v} = \nabla \theta \), the hydrodynamic equations for \( \theta \) and \( \rho \) can be obtained using the Lagrangian formalism. The corresponding hydrodynamic Lagrangian is:

\[
L_{\text{hydro}}(\rho, \theta) = H_{\text{hydro}} - \rho \partial_t \theta = \frac{1}{2} \rho v^2 + \epsilon(\rho) - \rho \partial_t \theta, \quad \mathbf{v} = \nabla \theta.
\] (10)

The constant chemical potential \( \mu \) is absorbed here by \( \partial_t \theta \).

The Poisson brackets for the vortex degrees of freedom have been discussed in Refs. 16,17.

In linear approximation the Lagrangian (10) describes sound waves. Sound waves propagating over background flow of the inhomogeneous liquid can be obtained from the hydrodynamic equations (7) and (8); the rigorous procedure can be found in Refs. 18,19. As was first found by Unruh 20, the flow of liquid has the same effect on propagation of sound waves as the metric in general relativity on propagation of a massless relativistic particle. The effective metric for sound waves generated by \( \mathbf{v}(r,t) \) and \( \rho(r,t) \) is

\[
g_{00} = -\frac{\rho}{c}(c^2 - \mathbf{v}^2), \quad g_{ij} = \frac{\rho}{c} \delta_{ij}, \quad g_{0i} = -g_{ij} v^j, \quad \sqrt{-g} = \frac{\rho}{c}.
\] (11)

This is the half of general relativity, since the effective metric obeys the hydrodynamic equations rather than Einstein equations. However, this is enough for simulations of aspects of general relativity which do not depend on Einstein equations. For example, effects related to behavior of quantum fields in curved space can be reproduced 20,21.

The full general relativity can be generated in fermionic vacua near the Fermi points. 2,22,23 Fermi point is a generic singularity in the Green’s function which is protected by topology in momentum space. Expansion near the Fermi point leads to chiral fermions, gauge fields and gravity as effective fields in the low-energy corner.

2.3. Extended classical hydrodynamics

The most general classical hydrodynamics is obtained when one introduces corrections to classical hydrodynamics by adding the gradient terms. For static liquids and gases the important modification is the dependence of energy on the gradient of mass density:

\[
H_{\text{extended hydro}}(\rho, \mathbf{v}) = \int d^3x \left( \frac{1}{2} \rho v^2 + \epsilon(\rho) - \mu \rho + \frac{1}{2} K(\nabla \rho)^2 \right),
\] (12)

The other possible terms are \( \propto (\nabla \cdot \mathbf{v})^2 \) and \( \propto (\nabla \times \mathbf{v})^2 \), which we do not discuss here. While the Hamiltonian can be extended, the Poisson brackets for hydrodynamics variables remain intact. It was also stressed by Landau that hydrodynamic equations are less general than the commutation relations for hydrodynamic operators.
2.4. Classical superfluid hydrodynamics

Let us introduce the quantity
\[ \kappa \equiv \frac{\kappa}{2\pi} = 2 \sqrt{K \rho}, \]  
which has dimension of circulation of velocity. Then the classical superfluid hydrodynamics is obtained if one considers within the extended classical hydrodynamics the class of the potential velocity fields:
\[ v = \kappa \nabla \theta. \]  

In this normalization the flow potential \( \theta \) is dimensionless. This allows us to introduce instead of \( \rho \) and \( \theta \) the classical complex field where the dimensionless \( \theta \) plays the role of the phase: \( \Psi = \sqrt{\rho e^{i\theta}} \). In terms of \( \Psi \) the extended version of the hydrodynamic Lagrangian in Eq.(10) becomes:
\[ L_{\text{GP}}(\Psi) = i \kappa^2 (\Psi^* \partial_t \Psi - \Psi \partial_t \Psi^*) + \frac{\kappa^2}{2} \nabla \Psi^* \nabla \Psi + \epsilon(\rho) - \mu \rho, \quad \rho = |\Psi|^2. \]  

The Equation (15) is the Lagrangian of the famous Gross-Pitaevskii (GP) theory generalized to the arbitrary function \( \epsilon(\rho) \). In the original Gross-Pitaevskii theory the non-linear term is quadratic, \( \epsilon(\rho) = (1/2)g\rho^2 \), and the variation of \( L_{\text{GP}}(\Psi) \) leads to the nonlinear Schrödinger equation. Note that this nonlinear Schrödinger equation (or the more general equation obtained using the general form \( \epsilon(\rho) \)) is the classical equation, since the Planck constant \( \hbar \) does not enter Eq.(15). Instead one has the parameter \( \kappa \) (or \( \bar{\kappa} = \kappa/2\pi \)) which has the dimension of circulation of velocity \([\kappa] = [v][r]\). Circulation \( \oint d\mathbf{r} \cdot \mathbf{v} \) is the adiabatic invariant in classical hydrodynamics, and thus should be quantized in quantum theory. Another invariant in hydrodynamics is \( \int d^3x (\mathbf{v} \cdot (\nabla \times \mathbf{v})) \). It is also quantized in quantum theory, see Ref.24

The superfluid hydrodynamics (SH) has three dimensional parameters \( (c, \rho \) and \( \kappa \)), and thus the characteristic length, energy and frequency scales are now determined:
\[ a_{\text{SH}} = \frac{\bar{\kappa}}{c}, \quad \omega_{\text{SH}} = \frac{\bar{\kappa}^3}{c^3}, \quad E_{\text{SH}} = \frac{\bar{\kappa}^3}{c}. \]  

The superfluid hydrodynamics is classical. The corresponding hydrodynamic Hamiltonian is expressed in terms of the classical velocity and mass density fields as in Eq.(12):
\[ H_{\text{GP}}(\rho, v) = \int d^3x C^2 \left( \frac{\rho}{2} |\mathbf{v}|^2 + \epsilon(\rho) - \mu \rho + \frac{\kappa^2}{8\rho}(\nabla \rho)^2 \right). \]  

However, compared to the conventional classical hydrodynamics the classical superfluid hydrodynamics described by Eq.(15) has three modifications:

(i) The so-called quantum pressure term \( (\bar{\kappa}^2/8\rho)(\nabla \rho)^2 \) is added. In principle, this term can be of the classical origin. This term leads to the correction to the linear dispersion relation for sound waves:
\[ \omega(k) = ck \sqrt{1 + \frac{\kappa^2}{\rho}}. \]
(ii) The rotational degrees of freedom are involved in this description. Since the phase \( \theta \) is not single-valued, the superfluid hydrodynamics (SH) contains vortices with quantized circulation \( \oint d\mathbf{r} \cdot \mathbf{v} = n\kappa = 2\pi n\bar{\kappa} \), where \( n \) is integer.

(iii) Outside the vortex cores the velocity field is potential, \( \nabla \times \mathbf{v} = 0 \). The energy required to excite the vortex degrees of freedom is the energy of the vortex loop \( E_{vr} \sim \rho \bar{\kappa}^2 r \) of minimal size \( r \sim a_{SH} = \bar{\kappa}/c \) in Eq. (16). Thus there is the gap for vortex excitations of order

\[ \Delta \sim E_{SH} = \frac{\rho \bar{\kappa}^3}{c}. \]  

(18)

The advantage of Lagrangian Eq. (15) with the general function \( \epsilon(\rho) \) compared to the conventional Ginzburg-Pitaevskii (GP) Lagrangian which describes superfluid hydrodynamics in a dilute Bose condensate is as follows. In a dilute Bose gas almost all the atoms are in the Bose condensate, the depletion – the difference between the total density of atoms and the density of condensate is small and can be neglected in the main approximation. As a result, the equation for the condensate practically coincides with the hydrodynamics equations. It should be mentioned that the Ginzburg-Pitaevskii equation is not applicable to Bose condensate if the depletion is not small, since there is no conservation law for the condensate density.

For strongly interacting liquids the depletion is not small. For example, in superfluid \(^4\)He the condensate comprises only the small fraction of the total density. Nevertheless, even in this case, the Eq. (15) remains reasonable, since the function \( \Psi \) is normalized to the total density: \( |\Psi|^2 = \rho \). This reflects the fact that the superfluid hydrodynamics describes not the dynamics of the condensate density, but the dynamics of the whole superfluid liquid at \( T = 0 \).

The Lagrangian in Eq. (15) leads to correct hydrodynamic equations and to correct energy of quantized vortex lines both in the dilute Bose gases and strongly interacting liquids. This implies that the extended GP Lagrangian gives the reasonable description of the classical hydrodynamics of superfluids at \( T = 0 \), which includes the hydrodynamics of superfluid component at \( T = 0 \) and the classical dynamics of vortices with quantized circulation. The normal component made of quanta of sound waves – phonons – is absent in this approach. It is included at the stage of quantization to obtain the two fluid hydrodynamics at \( T \neq 0 \). The drawback of this description is that as distinct from the GP equation for the dilute Bose gases, the general Lagrangian in Eq. (15) gives only the model description of the vortex core region; however, in many cases such model is sufficient since it allows us to consider the core effects consistently without ambiguous cut-off procedure. The further extension of the model with incorporation of the non-local interaction can be found in Ref. 25.

In conclusion, the model (17) simulates superfluid hydrodynamics not only in weakly interacting Bose gas, but also in real quantum liquids, in which the Bose condensate is either absent or is a small fraction of the total density. It is also important, that as distinct from the Bose gas, liquids can be stable even in the absence
of environment, i.e. at zero external pressure. This is important for the consideration of the problems of vacuum energy and the related problems of cosmological constant\textsuperscript{26,27} using the ground state of an isolated quantum liquid as the physical example of the quantum vacuum in which the nullification of the vacuum energy in equilibrium occurs without any fine tuning.\textsuperscript{28}

3. Quantum hydrodynamics

3.1. Landau quantum hydrodynamics

Landau introduced quantum Hamiltonian expressing the classical energy in Eq.(1) it in terms of the corresponding non-commuting quantum operators $\hat{v}$ and $\hat{\rho}$:

$$\hat{H}_{\text{hydro}}(\hat{\rho}, \hat{v}) = \int d^3x \left( \frac{1}{2} \hat{\rho} \hat{v}^2 + \epsilon(\hat{\rho}) - \mu \hat{\rho} \right).$$

(19)

The commutation relations for the components of velocity field operator $\hat{v}$, and between $\hat{v}$ and $\hat{\rho}$ are

$$[\hat{\rho}(\mathbf{r}_1), \hat{\rho}(\mathbf{r}_2)] = 0,$$

(20)

$$[\hat{v}(\mathbf{r}_1), \hat{\rho}(\mathbf{r}_2)] = \frac{i\hbar}{\epsilon} \nabla \delta(\mathbf{r}_1 - \mathbf{r}_2),$$

(21)

$$[\hat{v}_i(\mathbf{r}_1), \hat{v}_j(\mathbf{r}_2)] = \frac{i\hbar}{\epsilon} \epsilon_{ijk}(\nabla \times \hat{v}) \delta(\mathbf{r}_1 - \mathbf{r}_2),$$

(22)

have been derived by Landau from the microscopics. They can also be obtained from the Poisson brackets (4)-(6) for the classical variables.

Quantum hydrodynamics is characterized by three dimensional quantities. In addition to equilibrium values of $\rho$ and $c$, the really fundamental Planck constant $\hbar$ enters the quantum hydrodynamics through the commutation relations (21) and (22).

Using three dimensional quantities one can construct the characteristic ‘Planck’ scales for the energy $E_{\text{QH}}$, mass $M_{\text{QH}}$, length $a_{\text{QH}}$, frequency $\omega_{\text{QH}}$ and energy density $\epsilon_{\text{QH}}$:

$$E_{\text{QH}}^4 = \frac{\hbar^3 \rho}{c}, \quad M_{\text{QH}}^4 = \frac{\hbar^3 \rho}{c^3}, \quad a_{\text{QH}}^4 = \frac{\hbar}{\rho c}, \quad \omega_{\text{QH}} = \left( \frac{\epsilon_{\text{QH}}}{\hbar} \right)^{1/4}, \quad \epsilon_{\text{QH}} \sim \epsilon(\rho) \sim \rho c^2. \quad (23)$$

3.2. Rotational modes

Landau suggested that the only low frequency modes of quantum hydrodynamics are quanta of sound waves – phonons, while the rotational modes (vortex degrees of freedom) are separated by the gap. One may suggest that if the gap exists in quantum hydrodynamics, it is given by the characteristic energy scale $E_{\text{QH}}$ in Eq. (23). However, there are some arguments against that. Since the operators of vorticity
∇ × ℓ and density ˆρ are commuted, the Hamiltonian which governs the rotational
degrees of freedom is

\[ \hat{H}_{\text{transverse}}(\ell_{\perp}) = \frac{1}{2} \int d^3x \rho \ell_{\perp}^2, \tag{24} \]

where \( \ell_{\perp} \) is the transverse (non-potential) part of velocity field. The above vortex
contribution to quantum hydrodynamics contains only two parameters: \( \hbar \) and \( \rho \).
Using these two quantities only one cannot construct the quantity with the dimen-
sion of the energy gap: the “Planck” energy scale \( E_{\text{QH}} \) of quantum hydrodynamics
in Eq.(23) contains \( c \) which is irrelevant for transverse degrees of freedom.

This was probably the reason why Landau proposed different estimate for the
rotational gap which did not contain \( c \), but contained the mass \( m \) of \( ^4 \text{He} \) atom:\(^11\)

\[ \Delta_L = \frac{\hbar^2 \rho^{2/3}}{m^{5/3}}, \tag{25} \]

The atomic mass \( m \) is the microscopic parameter, which is beyond the quantum
hydrodynamics. Incidentally or not, but since in superfluid \(^4 \text{He} \) the atomic mass \( m 
\) and the quantum hydrodynamic mass \( M_{\text{QH}} \) in Eq.(23) are of the same order, the
Landau estimation in Eq.(25) coincides with the estimation for the energy of the
elementary vortex excitation in superfluid \(^4 \text{He} \) – the smallest possible vortex ring –
in Eq.(18).

That quantum hydrodynamics alone cannot describe the superfluid liquid has
been later emphasized by Feynman.\(^14\) The main reason is that the classical hydro-
dynamics lacks (has lost) the information on the important microscopic properties of
the underlying system, such as quantum statistics of atoms. It is the Bose statistics
of atoms which leeds to the gap in the spectrum of quantum vorticity.\(^14\) However,
such gap is not present in Fermi liquids (unless the Cooper pairing occurs). More-
over, Fermi liquids are not described by classical or quantum hydrodynamics.

All this demonstrates that in order to describe the real systems, the quantum
hydrodynamics requires the extension. As the starting point for quantization one can
choose the extended classical hydrodynamics discussed in Sec. 2.4. In this approach
the vortex degrees of freedom would have the gap already at the classical level (see
Eq.(18)). This will be discussed later in Sec. 3.4.

3.3. Quantization of phonon field

If for some reasons the rotational degrees of freedom are separated by the gap, then
the only low-energy degrees of freedom are represented by the vortex-free hydro-
dynamics and sound waves. In linear regime (and in the absence of the rotational
degrees of freedom) the Landau quantum hydrodynamics leads to quantization of
sound waves. Quanta of sound waves are phonons with linear spectrum \( E_k = \hbar ck \).

The nonlinear terms in quantum hydrodynamic Hamiltonian describe interaction of phonon fields, and lead to modification of phonon spectrum at large \( k \). One may expect that the linear dispersion of phonon spectrum (analog of Lorentz in-
variance) is violated at the Planck scale \( k_{\text{QH}} = 1/a_{\text{QH}} \) in Eq.(23), provided that no
microscopic physics intervenes earlier. In principle, the correction to the spectrum of phonons can be computed within the quantum hydrodynamics, however the diverging Feynman diagrams makes this procedure rather ambiguous. In the literature, people used the inverse inter-atomic distance $k_a \sim 1/a$ as the natural ultraviolet cut-off for diverging diagrams (see e.g.\cite{29}). However, such parameter characterizes the microscopic physics beyond the quantum hydrodynamics. Whether it is possible to make regularization in such a way that the natural cut-off is determined by the quantum hydrodynamics itself, i.e. by $k_{QH} = 1/a_{QH}$, is an open question.

If such regularization procedure exists, the first guess would be that the spectrum is modified by the next order term:

$$\omega^2 = c^2 k^2 (1 + \gamma a_{QH}^2 k^2 + ...) \quad , \quad |\gamma| \sim 1.$$  \hspace{1cm} (26)

The quantum hydrodynamic correction in the form of Eq. (26) is naturally obtained in the 1+1 case, when $\rho$ is the one-dimensional mass density and one has $a_{QH}^2 = \hbar/(\rho c)$. In case of the 3+1 quantum hydrodynamics, where according to Eq. (23) one has $a_{QH}^4 = \hbar/(\rho c)$, the correction in Eq. (26) is non-analytic in $\hbar$, and the proper correction would be the higher order term which is linear in $\hbar$:

$$\omega^2 = c^2 k^2 \left( 1 + \gamma a_{QH}^4 k^4 \ln \left( \frac{1}{a_{QH}^4 k^4} \right) + ... \right) \quad , \quad |\gamma| \sim 1.$$  \hspace{1cm} (27)

The above guess is supported by the temperature corrections to the spectrum of phonons\cite{30} after substitution $T \sim \hbar c k$.

The density of zero point energy of quantum phonon field can be estimated using the Planck energy cut-off $k_{QH} = 1/a_{QH}$:

$$\epsilon_{zp} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \hbar c k \sim \hbar c k_{QH} \sim \frac{E_{QH}}{a_{QH}^4} \sim \rho c^2.$$  \hspace{1cm} (28)

This gives the correct estimation of at least the magnitude of the energy density of the liquid (the sign of $\epsilon(\rho) - \mu \rho$ is negative if the external pressure is positive). Note that the result is classical, i.e. it does not depend on $\hbar$. This is not very surprising because the energy density constructed from $\hbar$, $c$ and $\rho$ does not contain $\hbar$ (see Eq. (23)).

From the modern point of view, the classical hydrodynamics as well as classical gravity, is the classical output of the quantum system in the low-energy corner. The ‘initial classical’ energy density $\epsilon(\rho)$ is not only the starting point for ‘quantum hydrodynamics’ but also is the final classical macroscopic result: it contains all the quantum contributions to the energy density of the liquid. This also means that the contribution of zero point energy of phonons to vacuum energy has already been included from the very beginning and should not be counted again. Thus the phonon Hamiltonian in the quadratic approximation must be written without zero-point energy of phonons:

$$\hat{H} = E_{vac} + \sum_k \hbar c a_k a_k^\dagger \quad , \quad E_{vac} = V \epsilon_{vac} = V (\epsilon(\rho_0) - \mu \rho_0)$$ \hspace{1cm} (29)
where $a_k^\dagger$ and $a_k$ are operators of creation and annihilation of phonons, and $V$ is the volume of liquid.

### 3.4. Quantum superfluid hydrodynamics

There are several ways of quantization of superfluid hydrodynamics: (i) One can perform the full Landau quantization of Eq. (12), expressing the extended hydrodynamic Hamiltonian in terms of the quantum fields $\hat{\rho}$ and $\hat{\mathbf{v}}$. (ii) One can perform quantization starting with the superfluid hydrodynamics with quantized vortices described by the Lagrangian in Eq. (15) by expressing it in terms of the non-commuting fields $\hat{\Psi}$ and $\hat{\Psi}^\dagger$. Let us consider the last case. The quantum counterpart of classical Lagrangian Eq. (15) is the Hamiltonian

$$H_{GP}(\hat{\Psi}) = \int \! d^3x \left( \frac{\hbar^2}{2} \nabla^2 \hat{\Psi} \hat{\Psi}^\dagger + \epsilon(\hat{\rho}) - \mu \hat{\rho} \right),$$

which is supplemented by commutation relations for quantum fields

$$[\hat{\Psi}(r_1), \hat{\Psi}^\dagger(r_2)] = \frac{\hbar}{\kappa} \delta(r_1 - r_2).$$

If one identifies the parameter $\hbar/\kappa$ with the mass of an atom of the liquid, one obtains that the extended quantum hydrodynamics is nothing but the microscopic quantum mechanics of a system of identical bosonic atoms with mass $m = \hbar/\kappa$ and with a special type of interaction term $\epsilon(\hat{\rho})$ which only depends on density.

The superfluid quantum hydrodynamics (SQH) contains four parameters $\hbar, m$, speed of sound $c$, and equilibrium density $\rho$. One can introduce the dimensionless mass parameter $m_{\text{SQH}}$:

$$m_{\text{SQH}} = \frac{m}{M_{\text{QH}}}. \quad (32)$$

One may suggest that this dimensionless parameter characterizes microscopically different systems, which have the common macroscopic (low-energy, hydrodynamic) properties. In dilute Bose gases one has $m_{\text{SQH}} \ll 1$, while in superfluid liquid $^4\text{He}$ and in superfluid liquid $^3\text{He}$ this parameter is of order unity, $m_{\text{SQH}} \sim 1$.

However, if one compares the quantum hydrodynamic Hamiltonian (30) with the Hamiltonian of exact microscopic theory

$$H_{\text{micro}} = \int \! d^3x \hat{\Psi}^\dagger(x) \left( \frac{\hbar^2}{2} \nabla^2 - \mu \right) \hat{\Psi}(x) + \frac{1}{2} \int \! d^3x \int \! d^3y \hat{\Psi}^\dagger(x) \hat{\Psi}^\dagger(y) U(x - y) \hat{\Psi}(y) \hat{\Psi}(x),$$

one finds that the difference in the interaction term is enormous. In other words, the prescribed down-up route from classical to quantum theory (see Fig. 1) does not lead in general to the true microscopic theory.

And this is not the only drawback of quantum hydrodynamics. One may suggest that inspite of disagreement with exact microscopic theory, the ‘microscopic’ Hamiltonian in Eq. (30) may serve as a relevant microscopic model. In principle, starting
with this Hamiltonian, one may obtain in the long-wave limit (i.e. in the up-down route in Fig. 1) the classical hydrodynamic Hamiltonian for superfluid liquid state. However, the emerging function $\epsilon$ will essentially deviate from $\epsilon$ in the original classical hydrodynamics, i.e. $\epsilon_2(\rho) \neq \epsilon_1(\rho)$. Moreover this function $\epsilon(\rho)$ cannot be expressed in terms of the renormalized coupling $g$. That is why this procedure – down-up (quantization), up-down (emergence of effective theory in the low-energy corner of quantum theory), down-up, etc. in Fig. 1 – in general does not converge.

**Fig. 1.** From classical to quantum hydrodynamics (quantization) and back (to low-energy limit of quantum system).

### 3.5. Weak coupling limit

The only case in which the exact theory and extended quantum hydrodynamics fit each other is when the energy density is quadratic function of $\rho$:

$$
\epsilon(\hat{\rho}) = \frac{g}{2}(\hat{\Psi}^\dagger \hat{\Psi})^2,
$$

(34)

and the parameter $m_{\text{SQH}}$ in Eq. (32) is small: $m_{\text{SQH}} \ll 1$. This corresponds to small coupling $g$

$$
g \frac{g_0}{m_{\text{SQH}}} = m_{\text{SQH}}^{8/3} \ll 1, \quad g_0 = \frac{\hbar^2 a}{m^3} = \frac{\hbar^2}{\rho_{\text{J}}^{1/3} m_{\text{SQH}}^{8/3}},
$$

(35)

where $a = (\rho/m)^{-1/3}$ is the interatomic distance. In liquid $^4\text{He}$ one has $m \sim M_{\text{QH}}$, and thus quantization of the hydrodynamics does not make sense.

The limit of small coupling $g$ corresponds to the model of weakly interacting Bose gas, which has been solved by Bogoliubov. When one starts with the microscopic theory

$$
\hat{H}_{\text{GP}}(\Psi) = \int d^3x \left( \frac{\hbar^2}{2m^2} \nabla \hat{\Psi}^\dagger \nabla \hat{\Psi} + \frac{g}{2} \hat{\rho}^2 - \mu \hat{\rho} \right)
$$

(36)
with small $g$, one obtains in the long-wave-length limit (i.e. on the up-down route in Fig. 1) the classical hydrodynamics, where in the main approximation the function $\epsilon(\rho) = (g/2)\rho^2$ coincides with that in microscopic theory. In the next approximation the function $\epsilon(\rho)$ is modified by the “quantum” correction, as follows from the Bogoliubov theory:

$$\epsilon(\rho) = \frac{g}{2}\rho^2 \left(1 + \frac{16}{15\pi^2} \left(\frac{g}{g_0}\right)^{3/2}\right)$$

This means that after the first iteration (down-up and up-down in Fig. 1) the coupling constant is renormalized:

$$\tilde{g} = g \left(1 + \frac{16}{15\pi^2} \left(\frac{g}{g_0}\right)^{3/2}\right), \quad g \ll g_0.$$  \hfill (38)

There is a temptation to consider the correction to $\epsilon(\rho) = (g/2)\rho^2$ in Eq. (37) as the back reaction of the quantum vacuum to quantum fields of phonons. At first glance, one may identify this correction with the properly regularized zero point energy of phonon field:

$$\epsilon_{zp} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \hbar c k \left(\sqrt{1 + \frac{\bar{\kappa}^2 k^2}{4c^2}} - \frac{\bar{\kappa} k}{c} - \frac{c^3}{\bar{\kappa}^3 k^3} \right)$$

$$= \frac{8}{15\pi^2} \frac{\hbar c^5}{\bar{\kappa}^4} = \frac{8}{15\pi^2} g^2 \rho^2 \left(\frac{g}{g_0}\right)^{3/2}. \hfill (40)$$

However, such interpretation is only valid at small $g$ when the microscopic Bogoliubov theory provides three counter-terms in Eq. (39).

Moreover, this correction contains the Planck constant $\hbar$ in the denominator (since $\bar{\kappa} = \hbar/m$). This means that the weakly interacting Bose gas (the system with small $g$) actually corresponds to the ultra-quantum limit, in which the contribution of zero point motion of phonon field is small compared to the main quantum contribution $(1/2)g\rho^2$ to the vacuum energy.

3.5.1. Energy Scales

Because of the dimensionless quantity $g/g_0$ (or $m/M_{QH}$), the extended quantum hydrodynamics in Eq. (36) contains different physically interesting scales for each dimensional quantity. In addition to the hydrodynamic energy scale $E_{QH} = M_{QH}c^2$ in Eq. (23), there is the scale $E_L = mc^2 \ll E_{QH}$, where the Lorentz violation occurs in a dilute Bose condensate.

Another energy scale was introduced by Landau\textsuperscript{11} in Eq. (25). This corresponds to the energy of the smallest vortex ring in superfluid $^4$He: $E_v \sim \rho\bar{\kappa}^2 a \sim \Delta_L$, where $a$ is the interatomic spacing ($\rho \sim m/a^3$) which determines the smallest possible radius of the vortex ring in superfluid $^4$He.
There are several important length scales in dilute Bose gases in addition to the hydrodynamic length $a_H$: interatomic spacing $a$ and the coherence length $\xi = \hbar/mc$:

$$a_H = \left( \frac{g_0}{g} \right)^{1/8}, \quad \xi = \left( \frac{g_0}{g} \right)^{1/2}. \quad (41)$$

4. Back reaction of quantum vacuum

4.1. Depletion of condensate

The term ‘depletion’ means that because of interaction (i.e. for $g \neq 0$) the non-vanishing number of atoms is not in the Bose-condensate. For a weakly interaction dilute Bose gas, the atoms in the condensate are prevailing. The relative values of the condensate and non-condensate mass densities are

$$\frac{\rho_{\text{cond}}}{\rho} = 1 - \frac{1}{3\pi^2} \left( \frac{g}{g_0} \right)^{3/2}, \quad \frac{\rho_{\text{non-cond}}}{\rho} = \frac{1}{3\pi^2} \left( \frac{g}{g_0} \right)^{3/2}. \quad (42)$$

In strongly interacting $^4\text{He}$ liquid the fraction of the non-condensate atoms is prevailing, with $\rho_{\text{cond}}/\rho < 0.1$. Nevertheless, at $T = 0$ the whole liquid is in the coherent superfluid state – the quantum vacuum – with the superfluid component density $\rho_s = \rho$. The same occurs in two dimensional systems, where the condensate is completely depleted even in the absence of interaction, $\rho_{\text{cond}} = 0$.

The depletion of the Bose-condensate is not in the framework of Landau quantum hydrodynamics. It is in the framework of the Tisza description of superfluids and is fully microscopic phenomenon, which is beyond the low-energy hydrodynamics. Let us stress again that the Landau description in terms of vacuum and matter (quasiparticles) is applicable for superfluids in the low-energy regime. In this regime the hydrodynamics with its Euler and continuity equations has no information on the separation of the liquid into the Bose condensate and atoms above the condensate caused by interaction, since at $T = 0$ both these fractions participate in a single coherent flow of the quantum vacuum. The Tisza picture of condensed and non-condensed fractions requires the microscopic description of the particle and energy exchange between the two fractions; this is the high-energy phenomenon which is certainly beyond the responsibility of hydrodynamics.

In general, the depletion of the Bose-condensate is also beyond the quantum superfluid hydrodynamics, except for the limit $g \ll g_0$, where the superfluid quantum hydrodynamics coincides with the microscopic Bogoliubov model, and the depletion can be studied using perturbation theory. This is the reason why the calculations of the depletion using the quantum fluctuations of phonon field (see e.g. Ref.4) or other back reaction effects (see e.g. Ref.31) cannot be considered as generic.

However, there are some problems which are within the responsibility of Landau quantum hydrodynamics. One of them is the depletion of the mass density caused by phonons. This is the back reaction of quanta of sound waves onto the ‘classical’ quantum vacuum (let us stress again that in the low-energy limit the superfluid quantum vacuum behaves as classical liquid).
4.2. Back reaction of vacuum density to quantum matter

At non-zero temperature the liquid consists of the vacuum (the ground state) with density $\rho$ and excitations (quanta of sound waves – phonons) in Eq.(29). Let us find how thermal phonons modify the mass density $\rho$ of the quantum vacuum. This is the back reaction of the vacuum to the quanta of sound waves. We assume that temperature is small, $T \ll E_{QH}$, so that only low-frequency phonons with linear spectrum $\omega = c k$ contribute to the thermal energy, and consider fixed external pressure. The correction can be obtained by minimization of the free energy density of the liquid $F = \epsilon - TS$ over $\rho$. The free energy is the sum of the energy of ground state (quantum vacuum) and the free energy of the phonon gas (matter). For the phonons with linear dispersion relation the free energy density is the radiation pressure with minus sign:

$$F_{\text{mat}} = -P_{\text{mat}} = -(1/3)\epsilon_{\text{mat}} \quad , \quad \epsilon_{\text{mat}} = \frac{\pi^2}{30\hbar^2 c^3} T^4 \quad ,$$

where $\epsilon_{\text{mat}}$ is the energy density of the gas of thermal phonons (radiation energy).

Since the vacuum does not contribute to the entropy of the system, the total free energy density of a liquid is

$$F(T, \rho) = \epsilon(\rho) - \mu \rho - \frac{1}{3} \epsilon_{\text{mat}}(\rho) \quad .$$

Let $\rho_0$ be the equilibrium density at $T = 0$ and $\mu = \mu_0$, then considering $\epsilon_{\text{mat}}$ as perturbation one obtains the following expansion in terms of $\delta \rho = \rho - \rho_0$ and $\delta \mu = \mu - \mu_0$:

$$F(T, \rho) = F(T, \rho_0) + \frac{1}{2} \frac{\partial^2 \epsilon_{\text{vac}}}{\partial \rho^2} (\delta \rho)^2 - \frac{1}{3} \frac{\partial \epsilon_{\text{mat}}}{\partial \rho} \delta \rho - \frac{1}{3} \epsilon_{\text{mat}} \rho \quad .$$

For phonon gas the dependence of the radiation energy on $\rho$ in Eq.(43) only comes from the speed of sound,

$$\frac{\partial \epsilon_{\text{mat}}}{\partial \rho} = -3 \frac{\epsilon_{\text{mat}} c}{\rho} \quad .$$

Here we introduced the function $u$

$$u = \frac{\partial \ln c}{\partial \ln \rho} \quad ,$$

which is the fluid-state analogue of Grüneisen parameter, see e.g.30

Then we must take into account that the chemical potential $\mu$ must be changed to support the fixed external pressure. The total change of the pressure of the liquid, which is the sum of the vacuum pressure of the liquid and the radiation pressure of phonons, must be zero, $\delta P_{\text{vac}} + P_{\text{mat}} = 0$. This gives

$$\delta P_{\text{vac}} = -P_{\text{mat}} = -\frac{1}{3} \epsilon_{\text{mat}} \quad .$$

As a result the change in the chemical potential is

$$\delta \mu = \frac{\delta P_{\text{vac}}}{\rho_0} = -\frac{1}{3} \frac{\epsilon_{\text{mat}}}{\rho_0} \quad .$$
Introducing Eqs. (46) and (49) into the free energy (45) and minimizing over \(\delta\rho\) one obtains the response of the density of the liquid to the phonon gas:

\[
\frac{\delta\rho}{\rho} = -\frac{\epsilon_{\text{mat}}}{\rho c^2} \left( \frac{1}{3} + u \right). \tag{50}
\]

The result in Eq. (50) can be also obtained from the analysis of classical hydrodynamic equations made by Stone in Refs. 18, 19. The second term on the rhs of Eq. (50) comes from the second order correction to the density of the liquid induced by the sound wave. This is the Eq. (4.13) of Ref. 18 integrated over thermal quanta of sound waves – phonons. The first term in the rhs of Eq. (50), which is due to the change in the vacuum pressure, can be also obtained using Stone’s formalism.

Note that the depletion of liquid density \(\delta\rho \propto T^4\), while the temperature correction to the depletion of the condensate is \(\propto T^2\) (see e.g. 4). The reason for such difference is that the density \(\rho\) is conserved quantity, while the condensate density is not because of the Josephson coupling between the condensate and non-condensate atoms. In conclusion, the depletion of the mass density is universal and is completely determined by hydrodynamics, while the depletion of the condensate is beyond the quantum hydrodynamics and strongly depends on the microscopic physics.

4.3. **Response of dark (vacuum) energy to matter**

Let us consider the back reaction of vacuum energy to thermal phonons. According to Eq. (3) the analog of the vacuum energy density in liquids is \(\epsilon_{\text{vac}} = \tilde{\epsilon} = \epsilon(\rho) - \mu\rho\). It obeys the correct equation of state for quantum vacuum

\[
P_{\text{vac}} = \mu\rho - \epsilon(\rho) = -\epsilon_{\text{vac}}. \tag{51}
\]

The correction to the ‘vacuum energy’ density due to thermal phonons is

\[
\delta\epsilon_{\text{vac}} = \delta(\epsilon(\rho) - \mu\rho) = \left( \frac{d\epsilon}{d\rho} - \mu_0 \right) \delta\rho - \rho_0 \delta\mu = -\rho_0 \delta\mu = \frac{1}{3} \epsilon_{\text{mat}}, \tag{52}
\]

where we used Eq. (49) for \(\delta\mu\).

Let us consider an equilibrium liquid in the absence of environment, i.e. when the external pressure is zero. Then in the absence of phonons the vacuum energy and pressure are zero according to Eq. (9), \(P_{\text{vac}} = -\epsilon_{\text{vac}} = 0\). At \(T \neq 0\), thermal phonons produce radiation pressure which must be compensated by the pressure of the vacuum. As a result the vacuum energy density becomes non-zero:

\[
\epsilon_{\text{vac}} = \frac{1}{3} \epsilon_{\text{mat}}. \tag{53}
\]

This is the back reaction of the vacuum to relativistic matter. The same relation between the dark energy and hot matter is applicable for such Universes in which gravity is absent, i.e. in which the Newton constant \(G = 0\) (see Refs. 2, 32). Note that in liquids, where the effective gravity obeys hydrodynamic equations rather than Einstein general relativity, the vacuum energy is naturally of the order of matter density. For Universes with gravity, situation is more complicated, since the vacuum
energy responds also to gravitating matter, curvature, expansion and other perturbations of the vacuum state. However, the main result is that the vacuum energy is naturally determined by macroscopic quantities, rather than by huge microscopic Planck energy scale.\textsuperscript{28}

5. Lessons for quantum gravity

5.1. From quantum gravity to quantum hydrodynamics

The results in Eqs. (50) and (52) for the back reaction of the vacuum are expressed completely in terms of quantum hydrodynamics, i.e. in terms of the function $\epsilon(\rho)$ and Planck constant $\hbar$. These results are generic and do not depend on the microscopic physics, so that the extension to quantum superfluid hydrodynamics (with its extra parameter $\kappa = h/m$) is not required. The only role of the microscopic physics is to supply us with the macroscopic function $\epsilon(\rho)$.

In general relativity, there are also examples of the universal behavior of the back reaction, such as universal temperature corrections to Einstein equations and to Newton constant $G$. The temperature correction to the free energy of gravitational field induced by $N_F$ massless fermionic quantum fields and $N_s$ scalar quantum fields is\textsuperscript{33}

$$F = \frac{N_F - 2N_s}{288\pi} \int d^3x \sqrt{-g} T^2[R + 6w^2].$$ (54)

Here $R$ is the Ricci curvature of gravitational field and $w^2 = w^{\mu}w_{\mu}$, where $w_{\mu} = \frac{1}{2} \partial_{\mu} \ln g_{00}$ is 4-acceleration. This result also does not depend on the microscopic Planck physics. It is expressed in terms of the Planck constant $h$ and integral numbers – numbers of species $N_F$ and $N_s$ (actually one should also add contribution of the vector fields and gravitons). Thus the only role of the microscopic Planck physics is to supply us with the definite number of fermionic and bosonic quantum fields in the low energy corner.

Moreover, the temperature correction to the gravitational action in Eq. (54) is applicable not only to general relativity but also to the effective gravity emerging in quantum liquids.\textsuperscript{34} In quantum liquids, the dominating contribution to the ‘gravitational action’ is provided by hydrodynamics, while the subdominant corrections are within responsibility of the QFT in curved space. For superfluid $^4$He and for Bose condensate of single atomic species the microscopic physics gives us $N_F = 0$ and $N_s = 1$. Expressing $R$, $g$ and $w$ in Eq. (54) in terms of the effective metric $g_{\mu\nu}$ experienced by phonon field in Eq. (11), one obtains the correct subdominant contribution to the hydrodynamic free energy of the liquid $^4$He or Bose gas. In case of effective gravity in superfluid $^3$He-A with gapless fermions, the microscopic physics gives us $N_F = 2$ and $N_s = 0$, and using Eq. (54) one obtains the correct subdominant contribution to the gradient energy. These are examples when general relativity helps us to solve some problems in superfluids.

Another example is provided by the universal quantum correction to Newton law (see e.g.\textsuperscript{35}). It has exact analog in quantum hydrodynamics and gives rise to the uni-
versal quantum correction to the classical hydrodynamic action caused by effective QFT in effective curved space of acoustic metric (see Refs.36). As an illustration let us write one of the typical terms generated by the quantum hydrodynamics – the contribution to the quantum pressure caused by quantum fluctuations of phonon field in effective curved acoustic space obtained by Seeley-De Witt expansion:36

\[
P_{\text{quantum}} \sim \hbar c (\nabla^2 \ln \rho)^2 \ln \frac{E_{\text{QH}}}{E_{\text{IR}}}. \tag{55}
\]

This leads to the quantum correction to the spectrum of phonons in Eq. (27) which is proportional to \(\hbar\). The infra-red (IR) logarithmic divergence of the quantum hydrodynamic corrections suggests that they may describe the creation of phonons (matter) by the time dependent flow (gravitational field) in exact analogy with particle production in gravitational field (see e.g. Ref.37). In a similar way, in superfluid \(^3\)He-A the logarithmically divergent action for the effective electromagnetic field leads to the Schwinger-type production of fermionic quasiparticles by the time-dependent order parameter.38

The quantum pressure produced by quantum fluctuations of phonon field in effective curved acoustic space-time is more pronounced in the 1+1 quantum hydrodynamics. The effect is related to the gravitational trace anomaly in the 1+1 space-time,39 and leads to the quantum correction to the phonon spectrum which is also proportional to \(\hbar\): the factor in Eq. (26) is \(\gamma a_{\text{QH}}^2 = -\hbar/(48\pi\rho c)\).

The Hawking radiation also does not distinguish between gravity obeying the general relativity and effective gravity in liquids obeying hydrodynamic equations.9,20 In both cases, Hawking radiation from an astronomical or acoustic black hole is described as the process of semi-classical tunneling between (quasi)particle trajectories inside and outside the horizon.40,41

5.2. From quantum hydrodynamics to quantum gravity

We considered some cases when the quantum hydrodynamics and quantum gravity allow us to obtain the true corrections to hydrodynamics or/and to general relativity. There are some other examples of such kind, when the quantum hydrodynamics and quantum gravity work. However, it is not the general case. Quantum hydrodynamics and quantum gravity reproduce only those (mostly subdominant) terms in the action or in free energy which do not contain dimensional parameters, such as Eqs. (54) and (55). In general, the down-up route from classical to quantum hydrodynamics (see Fig. 1) leads to the theory which does not coincide with the true microscopic theory. This reflects the main property of the emergent physics: there are only very few up-down ways, i.e. from the high energy microscopic theory to the low-energy macroscopic hydrodynamic theory. The way depends on the universality class and is unique for given universality class. But there are infinitely many down-up routes from macroscopics to microscopics. This is the main message for those who would like to quantize gravity and hydrodynamics.
One can quantize sound waves in hydrodynamics to obtain quanta of sound waves – phonons. But one should not use the low-energy quantization for calculation of the radiative corrections which contain Feynman diagrams with integration over high momenta. In particular, the effective field theory is not appropriate for calculations of the vacuum energy in terms of the zero-point energy of quantum fields. Such attempts lead to the cosmological constant problem in gravity, and to the similar paradox for the vacuum energy in quantum hydrodynamics: in both cases the vacuum energy estimated using the effective theory is by many orders of magnitude too big. We know how this paradox is solved in quantum liquids, and we may expect that the same general arguments based on the thermodynamic stability of the ground state of the quantum liquid are applicable to the vacuum of relativistic quantum fields.

Another hint from hydrodynamics is that the underlying microscopic theory of quantum gravity must contain additional parameter to $\hbar$, $c$, and $G$. Then one has the dimensionless parameter, which distinguishes between different microscopic theories with the same macroscopic phenomenology. Example of such parameter in quantum hydrodynamics is $m_{\text{SQH}}$ in Eq. (32). It appears that properly formulated quantum hydrodynamics makes sense only in the limit when this parameter is small, i.e. for the case of dilute Bose gases. The necessity of the small parameter for the emergent general relativity and/or gauge fields is emphasized by Bjorken: ‘the emergence can only work if there is an extremely small expansion parameter in the game’. The role of the small parameter could be played by the ratio $E_{\text{Planck}}/E_{\text{Lorentz}}$ between the Planck energy scale and the energy scale above which the Lorentz invariance is violated (see e.g. discussion in Ref. 43).

As follows from the experience with different quantum condensed matter systems, the metric field $g_{\mu\nu}$ may naturally emerge in the low-energy corner of quantum vacuum. It is important that in some systems gravity emerges as effective geometry, rather than the spin-2 field. Even in such caricature gravity as the effective gravity for sound waves propagating in inhomogeneous moving liquids, the acoustic metric $g_{\mu\nu}$ in Eq. (11) is the emerging geometrical object, which has nothing to do with the spin-2 field. Depending on the hierarchy of parameters of the underlying microscopic system (quantum vacuum), the geometry (metric field) may obey the nonlinear hydrodynamic equations, or the nonlinear equations of general relativity, or Gross-Pitaevskii equations, etc.

In some vacua gravity emerges together with all the ingredients of Standard Model: relativistic chiral fermions and quantum gauge fields. This is the general low-energy property of vacua with the so-called Fermi point in momentum space, which demonstrates that gravity is the natural part of physics, and it should not be separated from the other fermionic and bosonic classical and quantum fields. The separation only occurs at low energy, because of the difference between the running couplings for gauge fields and gravity. This means that if gravity is the emergent phenomenon, it should naturally emerge together and simultaneously with the other...
physical fields and physical laws. This is the main requirement for the future theory of quantum gravity.

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