Quantum Groups and Quantum Field Theory in Rindler Space-Time

Gaetano Lambiase *

Dipartimento di Fisica Teorica e S.M.S.A.
Università di Salerno, 84081 Baronissi (Salerno), Italy.
Istituto Nazionale di Fisica Nucleare, Sezione di Napoli.

Abstract

Quantum Field Theory (QFT) developed in Rindler space-time and its thermal properties are analyzed by means of quantum groups approach. The quantum deformation parameter, labelling the unitarily inequivalent representations, turns out to be related to the acceleration of the Rindler frame.

PACS number(s): 02.20.-a, 04.62.+v.
1 Introduction

In recent years there have been remarkable progresses and growing interest in two fields of research: quantum gravity and quantum groups (q-groups).

Although many attempts have been made to quantize gravity, a satisfactory and definitive theory still does not exist. As well known indeed, one of the most discussed problems is the non-renormalizability of General Relativity (GR), or its various generalizations, when quantized as a local quantum field theory. In the absence of a theory of quantum gravity, one can try to analyze quantum aspects of gravity by studying QFT in curved space-time, namely by studying the quantization of matter fields in the presence of the gravitational field as a classical background described by GR.

An important result in this approach has been the discovery by Hawking that quantum effects can lead to thermal evaporation of black holes \cite{1}. Owing to this result, different background space-times have been investigated, putting special attention to the Rindler space-time, associated with an uniformly accelerated observer in Minkowski space-time. Davies \cite{2} and Unruh \cite{3} have shown that the vacuum state for an inertial observer is a canonical ensemble for an uniformly accelerated observer (Rindler observer). The temperature characterizing this ensemble is related to the acceleration of the observer by the relation

$$T = \frac{a}{2\pi}$$

in units $\hbar = c = k_B = 1$. This result is known as thermalization theorem (for a review, see \cite{4}). We note that the replacement of the acceleration with the surface of gravity of a black hole leads to the Bekenstein-Hawking temperature.

The purpose of this paper is to show the formal connection between thermalization theorem and q-groups. Such a connection will be established by means of the generator of Bogoliubov transformations expressed in terms of quantum deformation operators of the Weyl-Heisenberg (q-WH) algebra.

One feature of physical interest which emerges from our analysis is that the quantum deformation parameter is related to the Rindler acceleration; hence, for the equivalence principle, to the static gravitational field. From this result follows that the quantum deformation can be induced by the gravitational background. Such a conclusion, although in a different context, has been also observed in Ref. \cite{5}. 

\[1\]
Besides, as we will show, the Rindler acceleration labels the unitarily inequivalent representations of the canonical commutation relations. The existence of the unitarily inequivalent representations plays a crucial role in the quantization of matter fields in curved space-time \cite{6} and in our case, in the quantization of a free scalar field in the Rindler manifold.

Here we will not present and discuss the properties of q-groups, being well studied in the literature. We recall only that q-groups are examples of quasi-triangular Hopf algebra \cite{7}. More specifically, q-groups are the deformation of the universal enveloping algebra of a finite-dimensional semi-simple Lie algebra. Due to its richer structure than that of Lie groups, q-groups provide a powerful mathematical tool in different topics of modern physics: quantum optics \cite{8}, quantum dissipation \cite{5}, gauge field theory \cite{9}, quantum gravity \cite{10, 11}, etc.

The layout of this paper is the following. In Section 2 we review the salient points of the thermalization theorem in order to make more transparent its connection with q-groups. In Section 3 we present, for one degree of freedom, a realization of the quantum deformation of Weyl-Heisenberg algebra in term of finite difference operator over the set of entire analytical functions \cite{12}. Finally, in Section 4, we generalize the results of Section 3 to the case of infinite degrees of freedom in order to establish the formal relation between thermalization theorem and q-groups. Section 5 is devoted to the conclusions.

In this paper we do not study the role of the coproduct operation, nor do we investigate the superalgebra features of q-WH algebra in connection with QFT in Rindler space-time. Such an analysis needs further and deeper formal investigation, which goes beyond the task of present paper (concerned mainly to displaying the relation between the Rindler acceleration and the q-deformation parameter) and which we plan for future work.

2 Thermalization Theorem

In this Section, we shall briefly discuss the derivation of the relation between the inertial and accelerated description of free quantum fields in flat space-time. We shall treat the case of a complex massive scalar quantum field $\phi(x)$ in n-dimensional Minkowski space-time.

In the accelerated frame it is customary to use the Rindler coordinates $(\eta, \xi; \vec{x}_R)$ which are related to Minkowski coordinates $(x^0, x^1; \vec{x}_M)$ by trans-
formations

\[ x^0 = \xi \sinh \eta, \quad x^1 = \xi \cosh \eta, \quad \vec{x}_M = \vec{x}_R = (x^2, \ldots, x^{n-1}). \quad (2.1) \]

The world line of a uniformly accelerated observer is described by \( \xi = \text{const} \). Its acceleration is \((\ddot{x}^\mu \dot{x}_\mu)^{1/2} = \xi = a^{-1} \) and its proper time \( \tau \) is proportional to the Rindler time \( \eta, \eta = a \tau \). The Rindler coordinates cover only two regions of the Minkowski space: the Rindler wedge \( R_+ = \{ x | x^1 > |x^0| \} \) and the wedge \( R_- = \{ x | x^1 < -|x^0| \} \).

The field \( \phi(x) \) is solution of Klein-Gordon equation \((\square + m^2)\phi(x) = 0\), where \( \square = \sqrt{-g}g^{\mu \nu} \partial_\mu \partial_\nu \).

In the Minkowski space, the quantized field \( \phi(x) \) can be decomposed in Minkowski modes \( \{ U_k(x) \} \) \( (k = (k_1, \vec{k})) \)

\[ \phi(x) = \int d^{n-1}k [a_k U_k(x) + \bar{a}_k^\dagger U_k^*(x)]. \quad (2.2) \]

The set \( \{ U_k \} \) forms (with respect to the Klein-Gordon inner product) an orthonormal basis; \( a_k \) and \( \bar{a}_k^\dagger \) operators satisfy the canonical commutation relations

\[ [a_k, a_{k'}^\dagger] = [\bar{a}_k, \bar{a}_{k'}^\dagger] = \delta(k_1 - k'_1)\delta(\vec{k} - \vec{k}'). \quad (2.3) \]

All other commutators are zero.

The Hamiltonian operator reads \( H_M = \int d^{n-1}k \omega_k (a_k^\dagger a_k + \bar{a}_k \bar{a}_k^\dagger) \), where \( \omega_k = \sqrt{k_1^2 + |\vec{k}|^2 + m^2} \). The operators \( a_k, \bar{a}_k \) and \( a_k^\dagger, \bar{a}_k^\dagger \) are interpreted, respectively, as annihilation and creation operators for the states of the Fock space constructed from the Hilbert space of Minkowski modes. The Minkowski vacuum, denoted \( |0_M\rangle \), is defined by

\[ a_k |0_M\rangle = \bar{a}_k |0_M\rangle = 0, \quad \forall k. \quad (2.4) \]

In the Rindler coordinates, the quantization of the scalar field follows the same prescription of the Minkowski case [3]. The quantum field is expanded in terms of Rindler modes \( \{ u_k^{(\sigma)}(x), \sigma = \pm \} \) \( (k = (\Omega, \vec{k})) \)

\[ \phi(x) = \int_0^\infty d\Omega \int d^{n-2}k \sum_\sigma [b_k^{(\sigma)} u_k^{(\sigma)}(x) + \bar{b}_k^{(\sigma)\dagger} u_k^{(\sigma)*}(x)], \quad (2.5) \]

The set \( \{ u_k^{(\sigma)} \} \) is an orthonormal basis in the wedge \( R_\sigma \). The canonical commutation relations are

\[ [b_k^{(\sigma)}, b_{k'}^{(\sigma')\dagger}] = [\bar{b}_k^{(\sigma)}, \bar{b}_{k'}^{(\sigma')\dagger}] = \delta_{\sigma\sigma'}\delta(\Omega - \Omega')\delta(\vec{k} - \vec{k}'). \quad (2.6) \]
All other commutators are zero. The Hamiltonian operator is 
\[ H_R = H_R^{(+)} - H_R^{(-)} \], with
\[ H_R^{(\sigma)} = \int_0^\infty d\Omega \int d^{n-2}k \Omega \left[ b_k^{(\sigma)} b_k^{(\sigma)} + \bar{b}_k^{(\sigma)} \bar{b}_k^{(\sigma)} \right]. \] (2.7)

In analogy to Minkowski case, \( b_k^{(\sigma)} \), \( \bar{b}_k^{(\sigma)} \) and \( b_k^{(\sigma)} \), \( \bar{b}_k^{(\sigma)} \) are interpreted respectively, as annihilation and creation operators for the states of the Fock space constructed from Hilbert space associated to Rindler modes. The Rindler vacuum, \( |0_R \rangle = |0_+ \rangle \otimes |0_- \rangle \), is defined by
\[ b_k^{(\sigma)} |0_R \rangle = \bar{b}_k^{(\sigma)} |0_R \rangle = 0, \quad \forall \sigma, k. \] (2.8)

Note that the proper energy of Rindler particles seen by an accelerated observer is not \( \Omega \), but \( a\Omega = \tilde{\omega} \), because Rindler modes depend on time as 
\[ e^{-i\Omega \eta} = e^{-i(\tilde{\omega}t)\tau}. \]
Keeping in mind this point and equating the two expressions for the field \( \phi(x) \), Eqs. (2.2) and (2.5), one obtains the Bogoliubov transformations
\[ b_k^{(\sigma)} = \sqrt{1 + N(\tilde{\omega}/a)} d_k^{(\sigma)} + \sqrt{N(\tilde{\omega}/a)} \bar{d}_k^{(-\sigma)}, \] (2.9)
\[ \bar{b}_k^{(-\sigma)} = \sqrt{N(\tilde{\omega}/a)} d_k^{(\sigma)} + \sqrt{1 + N(\tilde{\omega}/a)} \bar{d}_k^{(-\sigma)}, \] (2.10)
where \( k = (\tilde{\omega}, \vec{k}), \tilde{k} = (\tilde{\omega}, -\vec{k}), N(\tilde{\omega}/a) = (e^{2\pi\tilde{\omega}/a} - 1)^{-1} \) and \( -\sigma = - (\pm) = \mp \).

The operators \( d \) and \( \bar{d} \) are related to Minkowski operators \( a \) and \( \bar{a} \) by the relation
\[ d_k^{(\sigma)} = \int_{-\infty}^{+\infty} dk_1 \left[ \frac{1}{\sqrt{2\pi\omega_k}} \left( \frac{\omega_k + k_1}{\omega_k - k_1} \right)^{i\sigma\tilde{\omega}/2} \right] a_{k_1,\tilde{k}}, \] (2.11)
and analogous expression for \( \bar{d}_k^{(-\sigma)} \). These operators annihilate the Minkowski vacuum
\[ d_k^{(\sigma)} |0_M \rangle = \bar{d}_k^{(-\sigma)} |0_M \rangle = 0, \quad \forall \sigma, k, \] (2.12)
and satisfy the canonical commutation relations. Transformations (2.9) and (2.10) will turn out to play a fundamental role in connection with quantum-groups. Bogoliubov transformations and the ansatz \( |0_M \rangle = F(b^\dagger, \bar{b}^\dagger)|0_R \rangle \) (F is a function to be determined) allow to relate the Minkowski vacuum to the Rindler vacuum
\[ |0_M \rangle = Z \exp \left[ \int_0^\infty d\tilde{\omega} \int d^{n-2}k \sum_\sigma e^{-\pi\tilde{\omega}/a} b_k^{(\sigma)} \bar{b}_k^{(-\sigma)} \right] |0_R \rangle, \] (2.13)
where \( Z \) is a normalization constant. The physical meaning of Eq. (2.13) is that the Minkowski vacuum can be expressed as a coherent state of Cooper-like pairs of quanta in the Rindler wedges \( R^+ \) and \( R^- \). For an operator \( O_R \) localized in the Rindler wedge one finds \( \langle 0_M | O_R | 0_M \rangle = \text{tr}(\rho O_R) \), where \( \rho \) is the density matrix \( \rho = e^{-aH_R/T} / \text{tr} e^{-aH_R/T} \). \( H_R \) is the Rindler Hamiltonian and \( T = a/2\pi \). From here the thermalization theorem follows.

Our purpose in this paper is to show that the generator of the Bogoliubov transformations (2.9) and (2.10) can be expressed in terms of operators of the q-WH algebra, thus establishing the link between thermalization theorem and q-groups.

3 q-Deformation of the W-H Algebra

In this Section we shall focus our attention on the main features of the q-WH algebra in terms of finite difference operator.

The Weyl-Heisenberg algebra is generated by the set of operators \( \{ \alpha, \alpha^\dagger, I \} \)

\[
[\alpha, \alpha^\dagger] = I, \quad [N, \alpha] = -\alpha, \quad [N, \alpha^\dagger] = \alpha^\dagger,
\]

with \( N = \alpha^\dagger \alpha \) and all other commutators equal to zero. The Fock space \( \mathcal{H} \) is generated by operating with \( \alpha^\dagger \) on the vacuum \( |0\rangle \). The q-WH algebra is generated by operators \( \{ \alpha_q, \hat{\alpha}_q, N_q \equiv N, q \in \mathbb{C} \} \)

\[
[\alpha_q, \hat{\alpha}_q] = q^N, \quad [N, \alpha_q] = -\alpha_q, \quad [N, \hat{\alpha}_q] = \hat{\alpha}_q.
\]

In the limit \( q \to 1, \alpha_q \to \alpha, \hat{\alpha}_q \to \alpha^\dagger \). By following Refs. [5, 12], it is convenient to work in the space \( \mathcal{F} \) of analytic functions because in \( \mathcal{F} \) the analytic properties of the Lie algebra structure are preserved under q-deformation. To this end, we will adopt the Fock-Bargmann representation [16] in which the operators

\[
\alpha^\dagger \to z, \quad \alpha \to \frac{d}{dz}, \quad N \to z \frac{d}{dz}, \quad z \in \mathbb{C},
\]

provide a realization of the WH algebra (3.1). In this context, the space \( \mathcal{F} \) becomes isomorphic to \( \mathcal{H} \) and has a well defined inner product. A generic wave function \( \psi(z) \) is expressed as

\[
\psi(z) = \sum_n c_n u_n(z), \quad u_n(z) = \frac{z^n}{\sqrt{n!}}, \quad z \in \mathbb{C}, n \in \mathbb{N}.
\]
The set \( \{ u_n(z) \} \) forms an orthonormal basis in \( \mathcal{F} \). The conjugation of the operators is defined with respect to the inner product in \( \mathcal{F} \). In order to provide a realization of the q-WH algebra in the Fock-Bargmann representation, let us introduce the finite difference operator \( D_q \) \((q\text{-derivative operator}) \) [12, 17], defined as

\[
D_q f(z) = \frac{f(qz) - f(z)}{(q - 1)z} = \frac{q^z \frac{df}{dz} - 1}{(q - 1)z} f(z),
\]

(3.5)

where \( f(z) \in \mathcal{F} \). \( D_q \) reduces to standard derivative in the limit \( q \to 1 \). The set of operators \( \{ z, \frac{d}{dz}, D_q \} \) satisfies the algebra

\[
[D_q, z] = q^z \frac{d}{dz}, \quad [z \frac{d}{dz}, D_q] = -D_q, \quad [z \frac{d}{dz}, z] = z.
\]

(3.6)

In terms of the operators \( \{ \alpha_q, \hat{\alpha}_q, N_q \equiv N \} \), with

\[
N \to z \frac{d}{dz}, \quad \hat{\alpha}_q \to z, \quad \alpha_q \to D_q,
\]

(3.7)

the finite difference operator algebra (3.6) becomes

\[
[N, \alpha_q] = -\alpha_q, \quad [N, \hat{\alpha}_q] = \hat{\alpha}_q, \quad [\alpha_q, \hat{\alpha}_q] = q^N.
\]

(3.8)

The algebra (3.8) thus provide a realization of the quantum WH algebra. Following Refs. [5, 12], the commutator \( [\alpha_q, \hat{\alpha}_q] = q^N \) can be formally written in \( \mathcal{F} \) as

\[
[\alpha_q, \hat{\alpha}_q] = \sqrt{q} e^{i(\alpha^2 - \alpha^\dagger 2)} \equiv \sqrt{q} S(\epsilon), \quad q \equiv e^\epsilon, \epsilon \text{ real}.
\]

(3.9)

\( S(\epsilon) \) generates the following transformation

\[
\alpha \to \alpha(\epsilon) = S(\epsilon) \alpha S^{-1}(\epsilon) = \alpha \cosh \epsilon + \alpha^\dagger \sinh \epsilon,
\]

(3.10)

and its hermitian conjugate. We finally observe that \( S(\epsilon) \) is an element of the group \( SU(1, 1) \). In fact, by defining \( J_- = \frac{1}{2} \alpha^2, J_+ = \frac{1}{2} \alpha^\dagger 2, J_0 = \frac{1}{2}(\alpha^\dagger \alpha + \frac{1}{2}) \), the \( su(1,1) \) algebra is closed.

In the limit \( \text{Im}\{z\} \to 0 \) the above Fock-Bargmann representation scheme gives the Schrödinger representation [5].

Eq. (3.9) is the key relation for establishing the link between thermalization theorem and q-WH algebra.
4 Thermalization Theorem by q-Groups

In this Section the connection between q-groups and thermalization theorem will be established. Such a connection is obtained by extending the q-WH algebra, discussed in Section 3, from one degree of freedom to infinite degrees of freedom. Therefore, some preliminary considerations will be useful.

As pointed out in Section 2, Rindler coordinates cover two disconnected regions of the Minkowski space-time. Then for each Rindler wedge there are two couples of annihilation and creation operators, one for particles and one for anti-particles.

This suggests to apply results of Section 3 to the set of four operators, \((\alpha^{(\sigma)}, \beta^{(\sigma)})\), \(\sigma = \pm\), such that \(\alpha^{(\sigma)}|0_M >= \beta^{(\sigma)}|0_M >= 0\) and satisfying the canonical commutation relations

\[ [\alpha^{(\sigma)}, \alpha^{(\sigma)\dagger}] = [\beta^{(\sigma)}, \beta^{(\sigma)\dagger}] = \delta_{\sigma\sigma'}, \quad \sigma, \sigma' = \pm. \tag{4.1} \]

Repeating for such operators the procedure shown in Section 3 leading to Eq. (3.9), one gets

\[ [\alpha_q^{(\sigma)}, \alpha_q^{(\sigma)\dagger}] = \sqrt{q} e^{\frac{\pi}{2}(\alpha^{(\sigma)2} - \alpha^{(\sigma)\dagger 2})} \equiv \sqrt{q} S_1(\epsilon), \quad q \equiv e^\epsilon, \tag{4.2} \]

\[ \alpha^{(\sigma)} \to \alpha^{(\sigma)}(\epsilon) = S_1(\epsilon)\alpha^{(\sigma)}S_1^{-1}(\epsilon) = \alpha^{(\sigma)} \cosh \epsilon + \alpha^{(\sigma)\dagger} \sinh \epsilon, \tag{4.3} \]

\[ [\beta_q^{(\sigma)}, \beta_q^{(\sigma)\dagger}] = \sqrt{q} e^{-\frac{\pi}{2}(\beta^{(\sigma)2} - \beta^{(\sigma)\dagger 2})} \equiv \sqrt{q} S_2(\epsilon), \quad q' \equiv e^{-\epsilon} = 1/q, \tag{4.4} \]

\[ \beta^{(\sigma)} \to \beta^{(\sigma)}(\epsilon) = S_2(\epsilon)\beta^{(\sigma)}S_2^{-1}(\epsilon) = \beta^{(\sigma)} \cosh \epsilon - \beta^{(\sigma)\dagger} \sinh \epsilon, \tag{4.5} \]

and hermitian conjugates of Eqs. (4.3) and (4.4). For simplicity in Eqs. (4.2)-(4.5) we dropped the \(\sigma\) index in the \(S_i(\epsilon), i = 1, 2\) generators.

The set of operators, \(\{J_{-}^{(\sigma)} = \frac{1}{2}\alpha^{(\sigma)2}, \quad J_{+}^{(\sigma)} = \frac{1}{2}\alpha^{(\sigma)\dagger 2}, \quad J_{0}^{(\sigma)} = \frac{1}{2}(\alpha^{(\sigma)\dagger} \alpha^{(\sigma)} + \frac{1}{2})\}\) and \(\{K_{-}^{(\sigma)} = \frac{1}{2}\beta^{(\sigma)2}, \quad K_{+}^{(\sigma)} = \frac{1}{2}\beta^{(\sigma)\dagger 2}, \quad K_{0}^{(\sigma)} = \frac{1}{2}(\beta^{(\sigma)\dagger} \beta^{(\sigma)} + \frac{1}{2})\}\) close the \(\oplus(\sigma)su(1,1)_{(\sigma)}\) algebra, indeed.

The product of the q-deformed algebras (4.2) and (4.4) yields

\[ \prod_{\sigma} [\alpha_q^{(\sigma)}, \alpha_q^{(\sigma)\dagger}] [\beta_q^{(\sigma)}, \beta_q^{(\sigma)\dagger}] = e^{\frac{\pi}{2} \sum_{\sigma} \frac{1}{2}[(\alpha^{(\sigma)2} - \alpha^{(\sigma)\dagger 2}) - (\beta^{(\sigma)2} - \beta^{(\sigma)\dagger 2})]} \tag{4.6} \]

The formal relation between thermalization theorem and q-groups is established by generalizing Eqs. (4.1)-(4.6) to infinite degrees of freedom.
The Bogoliubov transformations (4.3) and (4.5) can be implemented for any \( k \), where \( k \) label the field degrees of freedom (i.e. momentum), as inner automorphisms of \( \mathfrak{s}u(1,1)_{(k,\sigma)} \). To every value of the parameter \( \epsilon \) corresponds a copy \( \{ \alpha_k^{(\sigma)}(\epsilon), \alpha_k^{(\sigma)\dagger}(\epsilon), \beta_k^{(\sigma)}(\epsilon), \beta_k^{(\sigma)\dagger}(\epsilon) \} \) of the original algebra \( \{ \alpha_k^{(\sigma)}, \alpha_k^{(\sigma)\dagger}, \beta_k^{(\sigma)}, \beta_k^{(\sigma)\dagger} \} \), \( \sigma = \pm; |0(\epsilon) > \forall k \} \); the Bogoliubov generator can be thought of as the generator of the group of automorphisms of \( \bigoplus_{(k,\sigma)} \mathfrak{s}u(1,1)_{(k,\sigma)} \) parameterized by \( \epsilon \). In this way, in the limit of infinite degrees of freedom, relations (4.1) and (4.6) become (for a n-dimensional space)

\[
[\alpha_k^{(\sigma)}, \alpha_{k'}^{(\sigma')\dagger}] = [\beta_k^{(\sigma)}, \beta_{k'}^{(\sigma')\dagger}] = \delta_{\sigma\sigma'} \delta(k-k'), \quad \sigma, \sigma' = \pm , \quad (4.7)
\]

\[
\prod_{k,\sigma}[\alpha_{k,q}^{(\sigma)}, \hat{\alpha}_{k,q}^{(\sigma)}][\beta_{k,q}^{(\sigma)}, \beta_{k,q}^{(\sigma)\dagger}] \rightarrow \exp \left[ \frac{1}{2} \frac{V}{(2\pi)^{n-1}} \sum_{\sigma} \int d^{n-1}p \epsilon(p)[(\alpha_p^{(\sigma)})^2 - (\beta_p^{(\sigma)})^2 - (\beta_{p}^{(\sigma)})^2)] \right] \equiv G(\epsilon). \quad (4.8)
\]

Note that for simplicity \( q \) and \( q' \) denote \( q_k \) and \( q_k' \).

In order to write the Bogoliubov generator \( G(\epsilon) \) in (4.8) in a more convenient form and to establish the connection with the results of Section 2, let us write it in terms of the following independent operators \( d_k^{(\sigma)}, \bar{d}_k^{(-\sigma)} \), related to the operators \( \alpha_k^{(\sigma)}, \beta_k^{(\sigma)} \):

\[
\alpha_k^{(\sigma)} = \frac{1}{\sqrt{2}}(d_k^{(\sigma)} + \bar{d}_k^{(-\sigma)}), \quad \beta_k^{(\sigma)} = \frac{1}{\sqrt{2}}(d_k^{(\sigma)} - \bar{d}_k^{(-\sigma)}). \quad (4.9)
\]

We recall that \( d_k^{(\sigma)} \) and \( \bar{d}_k^{(-\sigma)} \) are linear combinations of the Minkowski annihilation operators alone (cf. Eq. (2.11)), annihilate the Minkowski vacuum \( |0_M > \) (cf. Eq. (2.12)) and satisfy the canonical commutation relations. Moreover \( k = (\tilde{\omega}, \tilde{k}) \) and \( \bar{k} = (\tilde{\omega}, -\tilde{k}) \). In terms of them, the Bogoliubov generator (4.8) reads

\[
G(\epsilon) = \exp \left[ \frac{V}{(2\pi)^{n-1}} \sum_{\sigma} \int d^{n-1}p \epsilon(p)[(d_p^{(\sigma)})^2 - (\bar{d}_p^{(-\sigma)})^2] \right] , \quad (4.10)
\]

where \( d^{n-1}p = d\tilde{\omega}d\tilde{p}, p = (\tilde{\omega}, \tilde{p}) \) and \( \tilde{p} = (\tilde{\omega}, -\tilde{p}) \). \( G(\epsilon) \) is an unitary operator:

\[
G^{-1}(\epsilon) = G(-\epsilon) = G^\dagger(\epsilon)
\]

and induces the following Bogoliubov transformations

\[
d_k^{(\sigma)} \rightarrow d_k^{(\sigma)}(\epsilon) = G(\epsilon)d_k^{(\sigma)}G^{-1}(\epsilon) = d_k^{(\sigma)} \cosh \epsilon(k) + \bar{d}_k^{(-\sigma)\dagger} \sinh \epsilon(k), \quad (4.11)
\]
\[ d_k^{-(-\sigma)\dagger} \rightarrow \tilde{d}_k^{-(-\sigma)\dagger}(\epsilon) = G(\epsilon) d_k^{-(-\sigma)\dagger} G^{-1}(\epsilon) = d_k^{(-\sigma)} \sinh \epsilon(k) + \tilde{d}_k^{-(-\sigma)} \cosh \epsilon(k). \tag{4.12} \]

\( d_k^{(\sigma)}(\epsilon) \) and \( \tilde{d}_k^{(-\sigma)}(\epsilon) \) satisfy for any \( \epsilon \) the canonical commutation relations, i.e. satisfy the same algebra of the operators \( d_k^{(\sigma)} \) and \( \tilde{d}_k^{(-\sigma)} \).

Transformations (4.11) and (4.12) are recognized to be the transformations (2.9) and (2.10) provided

\[ d_k^{(\sigma)}(\epsilon) \equiv b_k^{(\sigma)}, \quad \tilde{d}_k^{(-\sigma)}(\epsilon) \equiv \tilde{b}_k^{(-\sigma)}, \quad \sinh \epsilon(k) = \left( \frac{1}{e^{2\pi/\bar{\omega}} - 1} \right)^{1/2}, \tag{4.13} \]

with \( k = (\bar{\omega}, \vec{k}) \). Eqs. (4.13) and (4.14) are the wanted result: they express the relation between the deformation parameter \( q_k = e^{\epsilon(k)} \) and the accelerated frame operators \( b_k^{(\sigma)} \) and \( \tilde{b}_k^{(-\sigma)} \) (Eq. (4.13)), and the coefficient of the Bogoliubov transformations (Eq. (4.14)) relating inertial and accelerated frame operators (cfr. Eqs. (2.9) and (2.10)). Moreover, Eq. (4.14) shows that the Rindler acceleration is related to the deformation parameter so that we can write now \( q_{k,a} = e^{\epsilon(\bar{\omega},a)} \), and the link between thermalization theorem and q-WH algebra is thus established. Note that since, for a given \( \bar{\omega}, q_{k,a} \rightarrow 1 \) for \( a \rightarrow 0 \), the Rindler acceleration \( a \) plays in fact the role of deformation parameter.

In conclusion, our results may be summarized as follows.

The Hilbert space \( \mathcal{H} \) of the basis vectors associated to the Minkowski space (inertial frame) is build by repeated action of \( (d_k^{(\sigma)\dagger}, \tilde{d}_k^{(-\sigma)\dagger}) \) on the vacuum state \( |0_M> \). Bogoliubov transformations, Eqs. (4.11) and (4.12), relate vectors of \( \mathcal{H} \) to vectors of another Hilbert space labeled by \( \epsilon = \epsilon(\bar{\omega},a) \), \( \mathcal{H}_\epsilon \), for a fixed value of the acceleration \( a \). The relation between these spaces is established by the generator \( G(\epsilon) \) that maps \( \mathcal{H} \) in \( \mathcal{H}_\epsilon \), \( G(\epsilon) : \mathcal{H} \rightarrow \mathcal{H}_\epsilon \) (for fixed \( a \)). In particular, for the vacuum state \( |0_M> \) one has

\[ |0(\epsilon)> = G(\epsilon) |0_M> , \tag{4.15} \]

where \( |0(\epsilon)> \) is the vacuum state of the Hilbert space \( \mathcal{H}_\epsilon \) associated to the accelerated frame. In other words, \( |0(\epsilon)> \equiv |0_R> \).

By inverting Eq. (4.15) and using the Gaussian decomposition \([16] \), the vacuum state of the inertial frame can be written as

\[ |0_M> = Z \exp \left[ \sum_{\sigma} \int d\bar{\omega} \int d^{n-2}p \ \tanh \epsilon(\bar{\omega},a) b_p^{(\sigma)\dagger} \tilde{b}_p^{(-\sigma)\dagger} \right] |0_R> , \tag{4.16} \]
where \( \tanh \epsilon(\bar{\omega}, a) = e^{-\pi \bar{\omega}/a} \) and \( Z = \exp\left[ \int d\bar{\omega} \int d^{n-2}p \ln \cosh \epsilon(\bar{\omega}, a) \right] \). As pointed out in Section 2, the relation between the two vacua, \( |0_M> \) and \( |0_R> \), follows by the ansatz \( |0_M> = F(b, \bar{b})|0_R> \). In the q-groups approach such a relation, Eq. (4.16), is directly established by the Bogoliubov generator, i.e. by Eq. (4.15).

The number of modes of type \( b_k^{(\sigma)} \) is given, for each fixed value of \( a \), by
\[
<0_M|b_k^{(\sigma)}|0_M> = \sinh^2 \epsilon(\bar{\omega}, a) = \frac{1}{e^{2\pi \bar{\omega}/a} - 1} \tag{4.17}
\]
and similarly for the modes of type \( \bar{b}_k^{(-\sigma)} \). Moreover, \( <0(\epsilon)|0(\epsilon')> = 1, \forall \epsilon \).

We note that in the infinite-volume limit, we have
\[
<0(\epsilon)|0_M> \to 0 \text{ as } V \to \infty, \forall \epsilon, \tag{4.18}
\]
\[
<0(\epsilon)|0(\epsilon')> \to 0 \text{ as } V \to \infty, \forall \epsilon, \epsilon' \neq \epsilon', \tag{4.19}
\]
i.e., the Hilbert spaces \( \mathcal{H} \) and \( \mathcal{H}_\epsilon \) become orthogonal in the infinite volume limit. In this limit, as \( \epsilon \) evolves by varying the Rindler acceleration \( a \), one runs over a variety of infinitely many unitarily inequivalent representations of the canonical commutation relations. \( \epsilon' \) in Eq. (4.19) corresponds to the Rindler acceleration \( a' \).

Because the quantum deformation parameter acts as a label for the unitarily inequivalent representations in QFT, the mapping between different (i.e. labeled by different values of \( q \)) representations being performed by the Bogoliubov transformations, at finite \( V \) the Rindler accelerations may be taken as a label for such unitarily inequivalent representations, as well.

5 Conclusions

In this paper, n-dimensional QFT developed for accelerated coordinates in flat space-time and its thermal properties have been analyzed in terms of q-groups. We have shown that the Bogoliubov generator relating the annihilation and creation operators in the accelerated frame to the ones in the inertial frame, can be expressed in terms of commutators of the q-WH algebra. Then, quantum deformation parameter turns out to related to the Rindler acceleration, which acts as a label for the unitarily inequivalent representations.
of the canonical commutation relations. The link between thermalization theorem and q-WH algebra thus emerges.

A possible application of these results to different space-time, for instance to Schwarzschild space-time, is certainly of interest. In this case, the procedure of quantization of a free scalar field follows the same prescription of the one in the Rindler manifold and the surface of gravity characterizing the black hole plays the role of the Rindler acceleration. Then, one can conclude that the deformation parameter is related to the surface of gravity.

As our analysis shows, quantum deformations are mathematical structures underlying the QFT in ”curved” space-time. This strongly suggests a deep connection between q-groups and quantum gravity, and encouraging results in this direction have been obtained in different contexts [10, 11]. However, much work is still needed to obtain a full understanding of this subject.

Acknowledgments
The author thanks G. Scarpetta and G. Vitiello for fruitful discussions. This work has been supported by Ministero dell’Universitá e della Ricerca Scientifica.

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