Brief note on high-multipole Kerr tails

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In this note we reconsider the late-time, power-law decay rate of scalar fields in a Kerr space-time background. We implement a number of mathematical and computational enhancements to our time-domain, (2+1)D Teukolsky evolution code and are able to obtain reliable decay rates for multipoles as high as $\ell = 16$. Our numerical results suggest full agreement with the proposed decay expressions in recent work \cite{6, 9} for both the finite distance and null infinity cases.

I. BACKGROUND & SUMMARY

The asymptotic late-time, power-law decay ($t^n$) rates of matter fields in Kerr black hole space-time has been a matter of some debate for several decades. However, significant progress has been made on this question over the past few years through the development of sophisticated techniques \cite{1–8} to tackle the computational challenges inherent to this problem. Recent work \cite{6, 9} proposes the following late-time decay rate expressions for the scalar field case:

\[ n = \begin{cases} 
- (\ell' + \ell + 3) & \text{for } \ell' = 0, 1 \\
- (\ell' + \ell + 1) & \text{otherwise} \end{cases} \]

and

\[ n^{+\ell} = \begin{cases} 
- \ell' & \text{for } \ell \leq \ell' - 2 \\
- (\ell + 2) & \text{for } \ell \geq \ell' \end{cases} \]

by carefully studying the “inter-mode coupling” effects that are present in Kerr space-time due to frame-dragging. Note that these expressions above are for the axisymmetric multipoles, $\ell'$ refers to the initial field multipole and $\ell$ is the multipole of interest under study.

In this note, we borrow and implement the “best-practice” lessons from all previous numerical work on Kerr tails (especially \cite{5, 7, 10}) and perform very high-accuracy computations for the axisymmetric multipoles up to $\ell = \ell' = 16$. Our results are in full agreement with the expressions \cite{1, 2} above. It should be noted that we are not presenting any new physical results in this work or developing a deeper understanding of known expressions; we are simply demonstrating a proposed result in the literature \cite{6, 9} to be accurate for a large range of parameters.

II. METHODOLOGY

We begin this section by briefly commenting on why numerical computations are so challenging in the context of Kerr tails: (i) These simulations are required to be rather long duration – this is because typically the observed field exhibits an exponentially decaying oscillatory behavior in the initial part of the evolution i.e. quasi-normal ringing, and only much later does this transition over to a clean power-law decay. Therefore, one needs to evolve past the point that the initial oscillations decay away. Moreover, as clearly shown in Refs. \cite{6, 9} there is an “intermediate” tail regime, wherein one observes tails with various decay rates that are not necessarily the late-time asymptotic rates that we are after. Indeed, this intermediate regime is largely responsible for much of the confusion on this topic, in the literature. These intermediate tails decay faster than the asymptotic rate, but typically have dominant amplitudes for a period of time. We must evolve past this regime as well, in order to obtain the true asymptotic rate; (ii) Because each multipole has its own decay rate (which increases with an increase in $\ell$) at late times one obtains numerical data in which different multipoles have widely different amplitudes (often 30 – 40 orders of magnitude apart!). For this reason, the numerical solution scheme is required to have high-order convergence (to reduce the discretization errors to low enough levels, to be able to track the fast decaying multipoles); (iii) Moreover, these computations also require high-precision floating-point numerics, due to the very large range of amplitudes involved, and also to reduce round-off error which can easily overwhelm the fast decaying modes.

To address the challenges mentioned above, we perform the following mathematical and computational advancements to our time-domain, (2+1)D Teukolsky equation evolution code:

A. Mathematical Enhancements

The main advance we make in this context is to recast the problem using the technique of hyperboloidal compactification \cite{11} for the Teukolsky equation in Kerr space-time. This allows one to include null infinity $\mathcal{I}^+$ on the computational grid by mapping the entire space-time onto a compact domain. This technique also allows us to use a rather modest sized grid to sample the entire domain, thus delivering tremendous savings towards the total computational cost. In short, hyperboloidal com-

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10–51

10–35

10–27

10–19

Kutta method. The time-step we use in all our computations in this work is

called spectral domain in this work, with the angular and radial directions expanded in Legendre and Chebyszhev polynomials, respectively, using the Gauss-Lobatto collocation points. We use 245 collocation points in the radial directions and 51 in the angular directions.

We use the method-of-lines approach to evolve forward in time, using a fourth-order Runge-Kutta method. The time-step we use in all our computations is $\Delta t/M = 0.05$. Secondly, we implement high-precision, floating-point arithmetic throughout our code. In particular, this enhancement includes full support for octal-precision numerics (256-bit or ~60 decimal digits). This is required to keep the round-off error in our simulations at acceptable levels. Finally, we also implement algorithmic parallelism in order to speed up the computations, so they complete in a reasonable amount of time.

In particular, we use a small cluster of 16 Sony PlayStation 3 gaming consoles (PS3)¹ to perform all the simulations in this work. Each PS3 works independently on an initial multipole $\ell'$ case (an “embarrassing” coarse-grain parallelism). In addition, we also implement a fine-grain parallelism at the level of the high-precision computations themselves (see Ref. [5, 10] for details) utilizing the parallel architecture of the PS3’s Cell Broadband Engine. Overall, we obtain over two orders-of-magnitude speed-up via this parallel computing approach.

In summary, pseudo-spectral collocation method enables us to drastically reduce the discretization error, high-precision numerics helps us accurately track amplitudes down to the $10^{-60}$ scale, and the parallelism and compactification help to keep the total runtime to stay reasonable (and also obtain rates at null infinity directly).

| $\ell'$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 |
|--------|---|---|---|---|---|----|----|----|----|
| $\ell$ 0 | -5 | -7 | -9 | -11 | -13 | -15 | -17 | -19 | $X$ |
| 2 | -5 | -7 | -9 | -11 | -13 | -15 | -17 | -19 | $X$ |
| 4 | -7 | -9 | -11 | -13 | -15 | -17 | -19 | $X$ | $X$ |
| 6 | -9 | -11 | -13 | -15 | -17 | -19 | $X$ | $X$ | $X$ |
| 8 | -11 | -13 | -15 | -17 | -19 | $X$ | $X$ | $X$ | $X$ |
| 10 | -13 | -15 | -17 | -19 | -21 | -23 | -25 | $X$ | $X$ |
| 12 | -15 | -17 | -19 | -21 | -23 | -25 | -27 | $X$ | $X$ |
| 14 | -17 | -19 | -21 | -23 | -25 | -27 | $X$ | $X$ | $X$ |

| $\ell'$ | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 |
|--------|---|---|---|---|---|----|----|----|
| $\ell$ 1 | -5 | -7 | -9 | -11 | -13 | -15 | -17 | -19 |
| 3 | -5 | -7 | -9 | -11 | -13 | -15 | -17 | -19 |
| 5 | -7 | -9 | -11 | -13 | -15 | -17 | -19 | $X$ |
| 7 | -9 | -11 | -13 | -15 | -17 | -19 | -21 | $X$ |
| 9 | -11 | -13 | -15 | -17 | -19 | -21 | $X$ | $X$ |
| 11 | -13 | -15 | -17 | -19 | -21 | $X$ | $X$ | $X$ |
| 13 | -15 | -17 | -19 | -21 | -23 | -25 | -27 | $X$ |
| 15 | -17 | -19 | -21 | -23 | -25 | -27 | $X$ | $X$ |

TABLE I: Asymptotic late-time scalar field tails in Kerr space-time at finite distances for even multipoles.

TABLE II: Asymptotic late-time scalar field tails in Kerr space-time at finite distances for odd multipoles.

¹ http://gravity.phy.umassd.edu/ps3.html
III. NUMERICAL RESULTS

In this section, we present the outcome of the enhancements implemented in the previous section. The initial data is a smooth Gaussian wave-packet centered at \( \rho/M = 3.0 \) and of width \( \sigma/M = 2.0 \) and the time evolution terminates at \( t/M = 2000 \). The results are detailed in four tables: two for the finite-distance rates (odd and even multipole cases separately) and another two tables for the null infinity rates. Except for a few cases (close to \( \ell = 16 \)) where even octal-precision numerics prove to be insufficient to produce reliable tail solutions, all results agree precisely with expressions [1][2] above. The tables are self-explanatory. All depicted numerical results are accurate within one percent or less. Figure 1 depicts the actual simulation data for a sample \( \ell' = 5 \) case.

| \( \ell' \backslash \ell \) | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 |
|-----------------------|---|---|---|---|---|----|----|----|----|
| 0                     | -2 | -4 | -6 | -8 | -10 | -12 | -14 | -16 | X   |
| 2                     | -2 | -4 | -6 | -8 | -10 | -12 | -14 | -16 | X   |
| 4                     | -4 | -4 | -6 | -8 | -10 | -12 | -14 | -16 | -18 |
| 6                     | -6 | -6 | -6 | -8 | -10 | -12 | -14 | -16 | -18 |
| 8                     | -8 | -8 | -8 | -8 | -10 | -12 | -14 | -16 | -18 |
| 10                    | -10| -10| -10| -10| -10| -12| -14| -16| -18 |
| 12                    | -12| -12| -12| -12| -12| -12| -14| -16| -18 |
| 14                    | -14| -14| -14| -14| -14| -14| -14| -16| -18 |
| 16                    | -16| -16| -16| -16| -16| -16| -16| -16| -18 |

TABLE III: Asymptotic late-time scalar field tails in Kerr space-time at null infinity for even multipoles.

| \( \ell' \backslash \ell \) | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 |
|-----------------------|---|---|---|---|---|----|----|----|
| 1                     | -3 | -5 | -7 | -9 | -11| -13| -15| -17 |
| 3                     | -3 | -5 | -7 | -9 | -11| -13| -15| -17 |
| 5                     | -5 | -5 | -7 | -9 | -11| -13| -15| -17 |
| 7                     | -7 | -7 | -7 | -9 | -11| -13| -15| -17 |
| 9                     | -9 | -9 | -9 | -9| -11| -13| -15| -17 |
| 11                    | -11| -11| -11| -11| -11| -13| -15| -17 |
| 13                    | -13| -13| -13| -13| -13| -13| -15| -17 |
| 15                    | -15| -15| -15| -15| -15| -15| -15| -17 |

TABLE IV: Asymptotic late-time scalar field tails in Kerr space-time at null infinity and for odd multipoles.

It took approximately two days to perform all the computations we have presented in this section, using the 16 PS3 cluster we mentioned before.

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