CHARACTERIZING CHAINABLE, TREE-LIKE, AND CIRCLE-LIKE CONTINUAE

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ABSTRACT. We prove that a continuum $X$ is tree-like (resp. circle-like, chainable) if and only if for each open cover $U_{i} = \{U_{1}, U_{2}, U_{3}, U_{4}\}$ of $X$ there is a $U_{i}$-map $f : X \rightarrow Y$ onto a tree (resp. onto the circle, onto the interval).

Theorem 1 (Hemmingsen). For a compact Hausdorff space $X$ the following conditions are equivalent:

(1) $\dim X \leq n$, which means that any open cover $U$ of $X$ has an open refinement $V$ of order $\leq n + 1$;
(2) each open cover $U$ of $X$ with cardinality $|U| \leq n + 2$ has an open refinement $V$ of order $\leq n + 1$;
(3) each open cover $\{U_{i}\}_{i=1}^{n+2}$ of $X$ has an open refinement $\{V_{i}\}_{i=1}^{n+2}$ with $\bigcap_{i=1}^{n+2} V_{i} = \emptyset$.

We say that a cover $V$ of a cover $U$ is a refinement of a cover $U$ if each set $V \in V$ lies in some set $U \in U$. The order of a cover $U$ is defined as the cardinal

$$\text{ord}(U) = \sup\{|F| : F \subseteq U \text{ with } \bigcap F \neq \emptyset\}.$$ 

An open cover $U$ of $X$ is called

- a chain-like if for $U$ there is an enumeration $U = \{U_{1}, \ldots, U_{n}\}$ such that $U_{i} \cap U_{j} \neq \emptyset$ if and only if $|i - j| \leq 1$ for all $1 \leq i, j \leq n$;
- circle-like if there is an enumeration $U = \{U_{1}, \ldots, U_{n}\}$ such that $U_{i} \cap U_{j} \neq \emptyset$ if and only if $|i - j| \leq 1$ or $\{i, j\} = \{1, n\}$;
- a tree-like if $U$ contains no circle-like subfamily $V \subseteq U$ of cardinality $|V| \geq 3$.

We recall that a continuum $X$ is called chainable (resp. tree-like, circle-like) if each open cover of $X$ has a chain-like (resp. tree-like, circle-like) open refinement. By a continuum we understand a connected compact Hausdorff space.

The following characterization of chainable, tree-like and circle-like continua is the main result of this paper. For chainable and tree-like continua this characterization was announced (but not proved) in [1].

Theorem 2. A continuum $X$ is chainable (resp. tree-like, circle-like) if and only if any open cover $U$ of $X$ of cardinality $|U| \leq 4$ has a chain-like (resp. tree-like, circle-like) open refinement.

In fact, this theorem will be derived from a more general theorem treating $K$-like continua.

Definition 1. Let $K$ be a class of continua and $n$ be a cardinal number. A continuum $X$ is called $K$-like (resp. $n$-$K$-like) if for any open cover $U$ of $X$ (of cardinality $|U| \leq n$) there is a $U$-map $f : X \rightarrow K$ onto some space $K \in K$.

We recall that a map $f : X \rightarrow Y$ between two topological spaces is called a $U$-map, where $U$ is an open cover of $X$, if there is an open cover $\mathcal{V}$ of $Y$ such that the cover $f^{-1}(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\}$ refines the cover $U$. It worth mentioning that a closed map $f : X \rightarrow Y$ is a $U$-map if and only if the family $\{f^{-1}(y) : y \in Y\}$ refines $U$.

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It is clear that a continuum $X$ is tree-like (resp. chainable, circle-like) if and only if it is $K$-like for the class $K$ of all trees (resp. for $K = \{[0,1]\}$, $K = \{S^1\}$). Here $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ stands for the circle.

It turns out that each 4-$K$-like continuum is $K$-like for some extension $\hat{K}$ of the class $K$. This extension is defined with help of locally injective maps.

A map $f : X \to Y$ between topological spaces is called locally injective if each point $x \in X$ has a neighborhood $O(x) \subseteq X$ such that the restriction $f \upharpoonright O(x)$ is injective. For a class of continua $K$ let $\hat{K}$ be the class of all continua $X$ that admit a locally injective map $f : X \to Y$ onto some continuum $Y \in K$.

**Theorem 3.** Let $K$ be a class of 1-dimensional continua. If a continuum $X$ is 4-$K$-like, then $X$ is $\hat{K}$-like.

In Proposition 1 we shall prove that each locally injective map $f : X \to Y$ from a continuum $X$ onto a tree-like continuum $Y$ is a homeomorphism. This implies that $\hat{K} = K$ for any class $K$ of tree-like continua. This fact combined with Theorem 3 implies the following characterization:

**Theorem 4.** Let $K$ be a class of tree-like continua. A continuum $X$ is $K$-like if and only if it is 4-$K$-like.

One may ask if the number 4 in this theorem can be lowered to 3 as in the Hemmingsen’s characterization of 1-dimensional compacta. It turns out that this cannot be done: the 3-$K$-likeness is equivalent to being an acyclic curve. A continuum $X$ is called a curve if $\dim X \leq 1$. It is acyclic if each map $f : X \to S^1$ to the circle is null-homotopic.

**Theorem 5.** Let $K \ni [0,1]$ be a class of tree-like continua. A continuum $X$ is 3-$K$-like if and only if $X$ is an acyclic curve.

It is known that each tree-like continuum is an acyclic curve but there are acyclic curves, which are not tree-like $\mathbb{R}$. On the other hand, each locally connected acyclic curve is tree-like (moreover, it is a dendrite $\mathbb{R}$, Chapter X). Therefore, for any continuum $X$ and a class $K \ni [0,1]$ of tree-like continua we get the following chain of equivalences and implications (in which the dotted implication holds under the additional assumption that the continuum $X$ is locally connected):

$$
\begin{align*}
4\text{-chainable} & \iff 4\text{-K-like} \iff 4\text{-tree-like} \iff 3\text{-K-like} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\text{chaining} & \iff \text{K-like} \iff \text{tree-like} \iff \text{acyclic curve}
\end{align*}
$$

Finally, let us present a factorization theorem that reduces the problem of studying $n$-$K$-like continua to the metrizable case. It will play an important role in the proof of the “circle-like” part of Theorem 2.

**Theorem 6.** Let $n \in \mathbb{N} \cup \{\omega\}$ and $K$ be a family of metrizable continua. A continuum $X$ is $n$-$K$-like if and only if any map $f : X \to Y$ to a metrizable compact space $Y$ can be written as the composition $f = g \circ \pi$ of a continuous map $\pi : X \to Z$ onto a metrizable $n$-$K$-like continuum $Z$ and a continuous map $g : Z \to Y$.

### 2. Proof of Theorem 5

Let $K \ni [0,1]$ be a class of tree-like continua. We need to prove that a continuum $X$ is 3-$K$-like if and only if $X$ is an acyclic curve.

To prove the “if” part, assume that $X$ is an acyclic curve. By Theorem 2.1 of $\mathbb{R}$, $X$ is 3-chainable. Since $[0,1] \in K$, the continuum $X$ is 3-$K$-like and we are done.

Now assume conversely, that a continuum $X$ is 3-$K$-like. First, using Hemmingsen’s Theorem $\mathbb{R}$ we shall show that $\dim X \leq 1$. Let $\mathcal{V} = \{V_1, V_2, V_3\}$ be an open cover of $X$. Since the space $X$ is 3-$K$-like, we can find a $\mathcal{V}$-map $f : X \to T$ onto a tree-like continuum $T$. Using the 1-dimensionality of tree-like continua, find an open cover $\mathcal{W}$ of $T$ order $\leq 2$ such that the cover $f^{-1}(\mathcal{W}) = \{f^{-1}(W) : W \in \mathcal{W}\}$ is a refinement of $\mathcal{V}$. The continuum $X$ is 1-dimensional by the implication (2) $\Rightarrow$ (1) of Hemmingsen’s theorem.

It remains to prove that $X$ is acyclic. Let $f : X \to S^1$ be a continuous map. Let $\mathcal{U} = \{U_1, U_2, U_3\}$ be a cover of the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ by three open arcs $U_1, U_2, U_3$, each of length $< \pi$. Such a cover necessarily has $\operatorname{ord}(\mathcal{U}) = 2$. By our assumption there is an open finite cover $\mathcal{V}$ of $X$ inscribed in $\{f^{-1}(U_i) : i = 1, 2, 3\}$. So, there is a tree-like continuum $T \in K$ and $\mathcal{V}$-map $g : X \to T$. We can assume that $T$ is a tree and $\mathcal{V}$ is a tree-open cover of $X$. It is well known (see e.g. $\mathbb{R}$) that there exists a continuous map $h : T \to S^1$ that $h \circ g$ is homotopic to $f$. But each map from a tree to the circle is null-homotopic. Hence $h \circ g$ as well $f$ is null-homotopic too.
3. Proof of Theorem

We shall use some terminology from Graph Theory. So at first we recall some definitions.

By a (combinatorial) graph we understand a pair $G = (V, E)$ consisting of a finite set $V$ of vertices and a set $E \subseteq \{\{a, b\} : a, b \in V, a \neq b\}$ of unordered pairs of vertices, called edges. A graph $G = (V, E)$ is connected if any two distinct vertices $u, v \in V$ can be linked by a path $(v_0, v_1, \ldots, v_n)$ with $v_0 = u, v_n = v$ and $\{v_{i-1}, v_i\} \in E$ for $i \leq n$. The number $n$ is called the length of the path (and is equal to the number of edges involved). Each connected graph possesses a natural path metric on the set of vertices $V$: the distance between two distinct vertices equals the smallest length of a path linking these two vertices.

Two vertices $u, v \in V$ of a graph are adjacent if $\{u, v\} \in E$ is an edge. The degree $\deg(v)$ of a vertex $v \in V$ is the number of vertices $u \in V$ adjacent to $v$ in the graph. The number $\deg(G) = \max_{v \in V} \deg(v)$ is called the degree of a graph. By an $r$-coloring of a graph we understand any map $\chi : V \to r = \{0, \ldots, r - 1\}$. In this case for a vertex $v \in V$ the value $\chi(v)$ is called the color of $v$.

Lemma 1. Let $G = (V, E)$ be a connected graph with $\deg(G) \leq 3$ such that $d(u, v) \geq 6$ for any two vertices $u, v \in V$ of order 3. Then there is a 4-coloring $\chi : V \to 4$ such that no distinct vertices $u, v \in V$ with $d(u, v) \leq 2$ have the same color.

Proof. Let $V_3 = \{v \in V : \deg(v) = 3\}$ denote the set of vertices of order 3 in $G$ and let $\bar{B}(v) = \{v\} \cup \{u \in V : \{u, v\} \in E\}$ be the unit ball centered at $v \in V$. It follows from $\deg(G) \leq 3$ that $|\bar{B}(v)| \leq 4$ for each $v \in V$. Moreover, for any distinct vertices $v, u \in V_3$ the balls $\bar{B}(v)$ and $\bar{B}(u)$ are disjoint (because $d(v, u) \geq 6 > 2$). So we can define a 4-coloring $\chi$ on the union $\bigcup_{v \in V_3} \bar{B}(v)$ so that $\chi$ is injective on each ball $\bar{B}(u)$ and $\chi(v) = \chi(w)$ for each $v, w \in V_3$. Next, it remains to color the remaining vertices all of order $\leq 2$ by four colors so that $\chi(x) \neq \chi(y)$ if $d(x, y) \leq 2$. It is easy to check that this always can be done. □

Each graph $G = (V, E)$ can be also thought as a topological object: just embed the set of vertices $V$ as a linearly independent subset into a suitable Euclidean space and consider the union $|G| = \bigcup_{\{u, v\} \in E} [u, v]$ of intervals corresponding to the edges of $G$. Assuming that each interval $[u, v] \subseteq |G|$ is isometric to the unit interval $[0, 1]$, we can extend the path-metric of $G$ to the path-metric $d$ on the geometric realization $|G|$ of $G$. For a point $x \in |G|$ by $B(x) = \{y \in |G| : d(x, y) < 1\}$ and $\bar{B}(x) = \{y \in |G| : d(x, y) \leq 1\}$ denote respectively the open and closed unit balls centered at $x$. More generally, by $B_r(x) = \{y \in |G| : d(x, y) < r\}$ we shall denote the open ball of radius $r$ with center at $x$ in $|G|$.

By a topological graph we shall understand a topological space $\Gamma$ homeomorphic to the geometric realization $|G|$ of some combinatorial graph $G$. In this case $G$ is called the triangulation of $\Gamma$. The degree of $\Gamma = |G|$ will be defined as the degree of the combinatorial graph $G$ (the so-defined degree of $\Gamma$ does not depend on the choice of a triangulation).

It turns out that any graph by a small deformation can be transformed into a graph of degree $\leq 3$.

Lemma 2. For any open cover $\mathcal{U}$ of a topological graph $\Gamma$ there is a $\mathcal{U}$-map $f : \Gamma \to G$ onto a topological graph $G$ of degree $\leq 3$.

This lemma can be easily proved by induction (and we suspect that it is known as a folklore). The following drawing illustrates how to decrease the degree of a selected vertex of a graph.

Now we have all tools for the proof of Theorem. So, take a class $K$ of 1-dimensional continua and assume that $X$ is a 4-$K$-like continuum. We should prove that $X$ is $K$-like.

First, we show that $X$ is 1-dimensional. This will follow from Hemmingsen’s Theorem as soon as we check that each open cover $\mathcal{U}$ of $X$ of cardinality $|\mathcal{U}| \leq 3$ has an open refinement $\mathcal{V}$ of order $\leq 2$. Since $|\mathcal{U}| \leq 4$ and $X$ is 4-$K$-like, there is a $\mathcal{U}$-map $f : X \to K$ onto a continuum $K \in K$. It follows that for some open cover $\mathcal{V}$ of $K$
the cover \( f\inv(\mathcal{V}) \) refines the cover \( \mathcal{U} \). Since the space \( K \) is 1-dimensional, the cover \( \mathcal{V} \) has an open refinement \( \mathcal{W} \) of order \( \leq 2 \). Then the cover \( f\inv(\mathcal{W}) \) is an open refinement of \( \mathcal{U} \) having order \( \leq 2 \).

To prove that \( X \) is \( K \)-like, fix any open cover \( \mathcal{U} \) of \( X \). Because of the compactness of \( X \), we can additionally assume that the cover \( \mathcal{U} \) is finite. Being 1-dimensional, the continuum \( X \) admits a \( \mathcal{U} \)-map \( f : X \to \Gamma \) onto a topological graph \( \Gamma \). By Lemma \( \ref{lem:2} \) we can assume that \( \deg(\Gamma) \leq 3 \). Adding vertices on edges of \( \Gamma \), we can find a triangulation \((V_\Gamma, E_\Gamma)\) of \( \Gamma \) so fine that

- the path-distance between any vertices of degree 3 in the graph \( \Gamma \) is \( \geq 6 \);
- the cover \( \{ f\inv(B_2(v)) : v \in V_\Gamma \} \) of \( X \) is inscribed into \( \mathcal{U} \).

Lemma \( \ref{lem:3} \) yields a 4-coloring \( \chi : V_\Gamma \to 4 \) of \( V_\Gamma \) such that any two distinct vertices \( u, v \in V_\Gamma \) with \( d(u, v) \leq 2 \) have distinct colors. For each color \( i \in 4 \) consider the open 1-neighborhood \( U_i = \bigcup_{v \in \chi^{-1}(i)} B(v) \) of the monochrome set \( \chi^{-1}(i) \subseteq V_\Gamma \) in \( \Gamma \). Since open 1-balls centered at vertices \( v \in V_\Gamma \) cover the graph \( \Gamma \), the 4-element family \( \{U_i : i \in 4\} \) is an open cover of \( \Gamma \). Then for the 4-element cover \( \mathcal{U}_4 = \{f\inv(U_i) : i \in 4\} \) of the 4-\( K \)-like continuum \( X \) we can find a \( \mathcal{U}_4 \)-map \( g : X \to Y \) to a continuum \( Y \in K \). Let \( \mathcal{W} \) be a finite open cover of \( Y \) such that the cover \( g\inv(\mathcal{W}) \) refines the cover \( \mathcal{U}_4 \). Since \( Y \) is 1-dimensional, we can assume that \( \operatorname{ord}(\mathcal{W}) \leq 2 \). For every \( W \in \mathcal{W} \) find a number \( \xi(W) \in 4 \) such that \( g\inv(W) \subseteq f\inv(\{U_{\xi(W)}\}) \).

Since \( Y \) is a continuum, in particular, a normal Hausdorff space, we may find a partition of unity subordinate to the cover \( \mathcal{W} \). This is a family \( \{\lambda_W : W \in \mathcal{W}\} \) of continuous functions \( \lambda_W : Y \to [0, 1] \) such that

\[
\begin{align*}
(\text{a}) & \quad \lambda_W(y) = 0 \quad \text{for } y \in Y \setminus W, \\
(\text{b}) & \quad \sum_{W \in \mathcal{W}} \lambda_W(y) = 1 \quad \text{for all } y \in Y.
\end{align*}
\]

For every \( W \in \mathcal{W} \) consider the “vertical” family of rectangles

\[ R_W = \{W \times B(v) : v \in V_\Gamma, \chi(v) = \xi(W)\} \]

in \( Y \times \Gamma \) and let \( \mathcal{R} = \bigcup_{W \in \mathcal{W}} R_W \). For every rectangle \( R \in \mathcal{R} \) choose a set \( W_R \in \mathcal{W} \) and a vertex \( v_R \in V_\Gamma \) such that \( R = W_R \times B(v_R) \). Also let \( \mathcal{R}_R = \{S \in \mathcal{R} : R \cap S \neq \emptyset\} \).

**Claim 1.** For any rectangle \( R \in \mathcal{R} \) and a point \( y \in W_R \) the set \( \mathcal{R}_{R,y} = \{S \in \mathcal{R}_R : y \in W_S\} \) contains at most two distinct rectangles.

**Proof.** Assume that besides the rectangle \( R \) the set \( \mathcal{R}_{R,y} \) contains two other distinct rectangles \( S_1 = W_{S_1} \times B(v_{S_1}) \) and \( S_2 = W_{S_2} \times B(v_{S_2}) \). Taking into account that \( y \in W_R \cap W_{S_1} \cap W_{S_2} \) and \( \operatorname{ord}(\mathcal{W}) \leq 2 \), we conclude that either \( W_{S_1} = W_{S_2} \) or \( W_R = W_{S_1} \) or \( W_R = W_{S_2} \). If \( W_{S_1} = W_{S_2} \), then

\[ \chi(v_{S_1}) = \xi(W_{S_1}) = \xi(W_{S_2}) = \chi(v_{S_2}). \]

Since \( B(v_R) \cap B(v_{S_1}) \neq \emptyset \neq B(v_R) \cap B(v_{S_2}) \) the property of 4-coloring \( \chi \) implies that \( v_{S_1} = v_{S_2} \) and hence \( S_1 = S_2 \). By analogy we can prove that \( W_R = W_{S_1} \) implies \( R = S_1 \) and \( W_R = W_{S_2} \) implies \( R = S_2 \), which contradicts the choice of \( S_1, S_2 \in \mathcal{R}_{R,y} \setminus \{R\} \). \( \square \)

Claim \( \ref{claim:1} \) implies that for every rectangle \( R = W_R \times B(v_R) \) the function \( \lambda_R : W_R \to \bar{B}(v_R) \subseteq \Gamma \) defined by

\[ \lambda_R(y) = \begin{cases} 
\lambda_{W_R}(y)v_R + \lambda_{W_S}(y)v_S, & \text{if } \mathcal{R}_{R,y} = \{R, S\} \text{ for some } S \neq R, \\
v_R, & \text{if } \mathcal{R}_{R,y} = \{R\}
\end{cases} \]

is well-defined and continuous. Let \( \pi_R : R \to W_R \times B(v_R) \subseteq \bar{R} \) be the map defined by \( \pi_R(y, t) = (y, \lambda_R(y)) \).

The graphs of two functions \( \lambda_R \) and \( \lambda_S \) for two intersecting rectangles \( R, S \in \mathcal{R} \) are drawn on the following picture:
Let us show that \( n \in \mathbb{N} \). Let us choose a set \( Y \) from every \( n \in \mathbb{N} \), which is not possible as \( Y \) belongs to the class \( \hat{G} \). Hence for every \( n \in \mathbb{N} \), the family \( Y \) is locally injective because \( \hat{G} \) is a homomorphism. The projection \( \text{pr}_Y : L \to Y \) is locally injective because \( L \subseteq \bigcup R \) and for every \( R \in \mathcal{R} \) the restriction \( \text{pr}_Y | R \cap L : R \cap L \to Y \) is injective. Taking into account that \( Y \in K \), we conclude that \( L \in \hat{K} \), by the definition of the class \( \hat{K} \).

4. Locally injective maps onto tree-like continua and circle

The following theorem is known for metrizable continua [6].

**Proposition 1.** Each locally injective map \( f : X \to Y \) from a continuum \( X \) onto a tree-like continuum \( Y \) is a homeomorphism.

**Proof.** By the local injectivity of \( f \), there is an open cover \( U' \) such that for every \( U \in U' \) the restriction \( f | U \) is injective. Let \( U \) be an open cover of \( X \) whose second star \( S_2(U) \) refines the cover \( U' \). Here \( S_2(U) = \bigcup \{ U' \subseteq U' : U \cap U' \neq \emptyset \} \), \( S_2(U) = \{ S_2(U, U) : U \in U \} \) and \( S_2(U) = \{ S_2(U, S_2(U)) : U \in U \} \).

For every \( x \in X \) choose a set \( U_x \in U \) that contains \( x \). Observe that for distinct points \( x, x' \in X \) with \( f(x) = f(x') \) the sets \( U_x, U_{x'} \) are disjoint. In the opposite case \( x, x' \in U_x \cup U_{x'} \subseteq S_2(U_x, U) \subseteq U \) for some set \( U \in U' \), which is not possible as \( f | U \) is injective.

Hence for every \( y \in Y \) the family \( U_y = \{ U_x : x \in f^{-1}(y) \} \) is disjoint. Since \( f \) is closed and surjective, the set \( V_y = Y \setminus f(X \setminus \bigcup U_y) \) is an open neighborhood of \( y \) in \( Y \) such that \( f^{-1}(V_y) \subseteq \bigcup U_y \).

Since the continuum \( Y \) is a tree-like, the cover \( V = \{ V_y : y \in Y \} \) has a finite tree-like refinement \( W \). For every \( W \in W \) find a point \( y_W \in Y \) with \( W \subseteq V_{y_W} \) and consider the disjoint family \( U_W = \{ U \cap f^{-1}(W) : U \in U_W \} \). It follows that \( f^{-1}(W) = \bigcup U_W \) and hence \( U_W = \bigcup_{W \in W} U_W \) is an open cover of \( X \).

Now we are able to show that the map \( f \) is injective. Assuming the converse, find a point \( y \in Y \) and two distinct points \( a, b \in f^{-1}(y) \). Since \( X \) is connected, there is a chain of sets \( \{ G_1, G_2, \ldots, G_n \} \subseteq U_W \) such that \( a \in G_1 \) and \( b \in G_n \). We can assume that the length \( n \) of this chain is the smallest possible. In this case all sets \( G_1, \ldots, G_n \) are pairwise distinct.

Let us show that \( n \geq 3 \). In the opposite case \( a \in G_1 = U_1 \cap f^{-1}(W_1) \in U_W \), \( b \in G_2 = U_2 \cap f^{-1}(W_2) \in U_W \) and \( G_1 \cap G_2 \neq \emptyset \). So, \( a, b \in U_1 \cup U_2 \subseteq S_2(U_1, U) \subseteq U \) for some \( U \in U' \) and then the restriction \( f | U \) is not injective. So \( n \geq 3 \).
For every \( i \leq n \) consider the point \( y_i = y_{i\mathbf{y}} \), and find sets \( W_i \in \mathcal{W} \) and \( U_i \in \mathcal{U}_y \) such that \( G_i = U_i \cap f^{-1}(W_i) \in \mathcal{U}_{W_i} \). Then \((W_1, \ldots, W_n)\) is a sequence of elements of the tree-like cover \( \mathcal{W} \) such that \( y = W_1 \cap W_n \) and \( W_i \cap W_{i+1} \neq \emptyset \) for all \( i < n \). Since the tree-like cover \( \mathcal{W} \) does not contain circle-like subfamilies of length \( \geq 3 \) there are two numbers \( 1 \leq i < j \leq n \) such that \( W_i \cap W_j \neq \emptyset \), \( |j-i| > 1 \) and \( \{i, j\} \neq \{1, n\} \). We can assume that the difference \( k = j - i \) is the smallest possible. In this case \( k = 2 \). Otherwise, \( W_i, W_{i+1}, \ldots, W_j \) is a circle-like subfamily of length \( \geq 3 \) in \( \mathcal{W} \), which is forbidden. Therefore, \( j = i+2 \) and the family \( \{W_i, W_{i+1}, W_{i+2}\} \) contains at most two distinct sets (in the opposite this family is circle-like, which is forbidden). If \( W_i = W_{i+1} \), then \( U_i = U_{i+1} \) as the family \( \mathcal{U}_{W_i} \) is disjoint. The assumption \( W_{i+1} = W_{i+2} \) leads to a similar contradiction. It remains to consider the case \( W_i = W_{i+2} \neq W_{i+1} \). Since the sets \( U_i, U_{i+2} \in \mathcal{U}_y \) are distinct, there are distinct points \( x_i, x_{i+2} \in f^{-1}(y_i) \) such that \( x_i \in U_i \) and \( x_{i+2} \in U_{i+2} \). Since \( x_i, x_{i+2} \in U_i \cup U_{i+2} \subset S^{2}(U_i, U) \subset U \) for some \( U \in \mathcal{U} \), the restriction \( f|U \) is not injective. This contradiction completes the proof. \( \Box \)

**Proposition 2.** If \( f : X \to S^{1} \) is a locally injective map from a continuum \( X \) onto the circle \( S^{1} \), then \( X \) is an arc or a circle.

**Proof.** The compact space \( X \) has a finite cover by compact subsets that embed into the circle. Consequently, \( X \) is metrizable and 1-dimensional. We claim that \( X \) is locally connected. Assuming the converse and applying Theorem 1 of [1] \( \mathbb{V} \) [49.II] (or [9] 5.22(b) and 5.12), we could find a convergence continuum \( K \subseteq X \). This a non-trivial continuum \( K \), which is the limit of a sequence of continua \( (K_{n})_{n \in \mathbb{N}} \) that lie in \( X \setminus K \).

By the local injectivity of \( f \), the continuum \( K \) meets some open set \( U \subseteq X \) such that \( f \upharpoonright U : U \to S^{1} \) is a topological embedding. The intersection \( U \cap K \), being a non-empty open subset of the continuum \( K \) is not zero-dimensional. Consequently, its image \( f(U \cap K) \subseteq S^{1} \) also is not zero-dimensional and hence contains a non-empty open subset \( V \subseteq S^{1} \). Choose any point \( x \in U \cap K \) with \( f(x) \in V \). The convergence \( K_{n} \to K \), implies the existence of a sequence of points \( x_{n} \in K_{n}, n \in \omega \), that converge to \( x \). By the continuity of \( f \), the sequence \( (f(x_{n}))_{n \in \omega} \) converges to \( f(x) \in V \). So, we can find a number \( n \) such that \( f(x_{n}) \in V \subseteq f(U \cap K) \) and \( x_{n} \in U \). The injectivity of \( f \upharpoonright U \) guarantees that \( x_{n} \in U \cap K \) which is not possible as \( x_{n} \in K \subseteq X \setminus K \).

Therefore, the continuum \( X \) is locally connected. By the local injectivity, each point \( x \in X \) has an open connected neighborhood \( V \) homeomorphic to a (connected) subset of \( S^{1} \). Now we see that the space \( X \) is a compact 1-dimensional manifold (possibly with boundary). So, \( X \) is homeomorphic either to the arc or to the circle. \( \Box \)

### 5. Proof of Theorem \[6\]

In the proof we shall use the technique of inverse spectra described in [3] \( \mathbb{V} \) [2.5] or [4] Ch.1. Given a continuum \( X \) embed it into a Tychonov cube \([0, 1]^{\kappa}\) of weight \( \kappa \geq \aleph_0 \).

Let \( A \) be the set of all countable subsets of \( \kappa \), partially ordered by the inclusion relation: \( \alpha \leq \beta \) iff \( \alpha \subseteq \beta \).

For a countable subset \( \alpha \subseteq \kappa \) let \( X_{\alpha} = \text{pr}_{\alpha}(X) \) be the projection of \( X \) onto the face \([0, 1]^{\alpha}\) of the cube \([0, 1]^{\kappa}\) and \( p_{\alpha} : X \to X_{\alpha} \) be the projection map. For any countable subsets \( \alpha \subseteq \beta \subseteq \kappa \) let \( p_{\beta} : X_{\beta} \to X_{\alpha} \) be the restriction of the natural projection \([0, 1]^{\beta} \to [0, 1]^{\alpha}\). In such a way we have defined an inverse spectrum \( S = \{X_{\alpha}, p^{\beta}_{\alpha} : \alpha, \beta \in A\} \) over the index set \( A \), which is \( \omega \)-complete in the sense that any countable subset \( B \subseteq A \) has the smallest upper bound \( \sup B = \bigcup B \) and for any increasing sequence \( \{\alpha_{i}\}_{i \in \omega} \subseteq A \) with supremum \( \alpha = \bigcup_{i \in \omega} \alpha_{i} \) the space \( X_{\alpha} \) is the limit of the inverse sequence \( \{X_{\alpha_{i}}, p_{\alpha_{i+1}}^{\alpha}, \omega\} \). The spectrum \( S \) consists of metrizable compacta \( X_{\alpha}, \alpha \in A \), and its inverse limit \( \lim S \) can be written as the composition \( f = f_{\alpha} \circ p_{\alpha} \) for some index \( \alpha \in A \) and some continuous map \( f_{\alpha} : X_{\alpha} \to Y \).

Now we are able to prove the “if” and “only if” parts of Theorem \[6\]. To prove the “if” part, assume that each map \( f : X \to Y \) factorizes through a metrizable \( n \)-K-like continuum. To show that \( X \) is \( n \)-K-like, fix any open cover \( \mathcal{U} = \{U_{1}, \ldots, U_{n}\} \) of \( X \). By Lemma 5.1.6 of [5], there is a closed cover \( \{F_{1}, \ldots, F_{n}\} \) of \( X \) such that \( F_{i} \subseteq U_{i} \) for all \( i \leq n \). Since \( F_{i} \) and \( X \setminus U_{i} \) are disjoint closed subsets of the compact space \( X = \lim S \), there is an index \( \alpha \in A \) such that for every \( i \leq n \) the images \( p_{\alpha}(X \setminus U_{i}) \) and \( p_{\alpha}(F_{i}) \) are disjoint and hence \( W_{i} = X_{\alpha} \setminus p_{\alpha}(X \setminus U_{i}) \) is an open neighborhood of \( p_{\alpha}(F_{i}) \). Then \( \{W_{1}, \ldots, W_{n}\} \) is an open cover of \( X_{\alpha} \) such that \( p_{\alpha}^{-1}(W_{i}) \subseteq U_{i} \) for all \( i \leq n \).
By our assumption the projection \( p_\alpha : X \to X_\alpha \) can be written as the composition \( p_\alpha = g \circ \pi \) of a map \( \pi : X \to Z \) onto a metrizable \( n \)-K-like continuum \( Z \) and a map \( g : Z \to X_\alpha \). For every \( i \leq n \) consider the open subset \( V_i = g^{-1}(W_i) \) of \( Z \). Since \( Z \) is \( n \)-K-like, for the open cover \( \mathcal{V} = \{ V_1, \ldots, V_n \} \) of \( Z \) there is a \( \mathcal{V} \)-map \( h : Z \to K \) onto a space \( K \in K \). Then the composition \( h \circ \pi : X \to K \) is a \( \mathcal{U} \)-map of \( X \) onto the space \( K \in K \) witnessing that \( X \) is an \( n \)-K-like continuum.

Now we shall prove the “only if” part of the theorem. Assume that the continuum \( X \) is \( n \)-K-like. We shall need the following lemma.

**Lemma 3.** For any index \( \alpha \in A \) there is an index \( \beta \geq \alpha \) in \( A \) such that for any open cover \( \mathcal{V} = \{ V_1, \ldots, V_n \} \) of \( X_\alpha \) there is a map \( f : X_\beta \to K \) onto a space \( K \in K \) such that \( f \circ \pi : X \to K \) is a \( p_\alpha^{-1}(V) \)-map.

**Proof.** Let \( \mathcal{B} \) be a countable base of the topology of the compact metrizable space \( X_\alpha \) such that \( \mathcal{B} \) is closed under unions. Denote by \( \mathcal{U} \) the family of all possible \( n \)-set covers \( \{ B_1, \ldots, B_n \} \subseteq \mathcal{B} \) of \( X_\alpha \). It is clear that the family \( \mathcal{U} \) is countable.

Each cover \( \mathcal{U} = \{ B_1, \ldots, B_n \} \in \mathcal{U} \) induces the open cover \( p_\alpha^{-1}(\mathcal{U}) = \{ p_\alpha^{-1}(B_i) : 1 \leq i \leq n \} \) of \( X \). Since the continuum \( X \) is \( n \)-K-like, there is a \( p_\alpha^{-1}(\mathcal{U}) \)-map \( f_\mathcal{U} : X \to K_\mathcal{U} \) onto a space \( K_\mathcal{U} \in K \). By the metrizability of \( K_\mathcal{U} \) and the factorizing property of the spectrum \( S \), for some index \( \alpha \mathcal{U} \geq \alpha \) in \( A \) there is a map \( f_\mathcal{U} : X_\alpha \to K_\mathcal{U} \) such that \( f_\mathcal{U} = f_{\alpha \mathcal{U}} \circ p_\alpha \). Consider the countable set \( \beta = \bigcup_{\mathcal{U} \in \mathcal{U}} \alpha \mathcal{U} \), which is the smallest lower bound of the set \( \{ \alpha \mathcal{U} : \mathcal{U} \in \mathcal{U} \} \) in \( A \). We claim that this index \( \beta \) has the required property.

Let \( \mathcal{V} = \{ V_1, \ldots, V_n \} \) be any open cover of \( X_\alpha \). By Lemma 5.1.6 of [3], there is a closed cover \( \{ F_1, \ldots, F_n \} \) of \( X_\alpha \) such that \( F_i \subseteq V_i \) for all \( i \leq n \). Since \( \mathcal{B} \) is the base of the topology of \( X_\alpha \) and \( \mathcal{B} \) is closed under finite unions, for every \( i \leq n \) there is a basic set \( B_i \in \mathcal{B} \) such that \( F_i \subseteq B_i \subseteq V_i \). Then the cover \( \mathcal{U} = \{ B_1, \ldots, B_n \} \) belongs to the family \( \mathcal{U} \) and refines the cover \( \mathcal{V} \). Consider the map \( f = f_{\alpha \mathcal{U}} \circ p_\alpha^{-1} : X_\beta \to K \) and observe that \( f \circ \pi = f_{\alpha \mathcal{U}} \circ p_\alpha \) is a \( p_\alpha^{-1}(\mathcal{U}) \)-map and a \( p_\alpha^{-1}(\mathcal{V}) \)-map. \( \square \)

Now let us return back to the proof of the theorem. Given a map \( f : X \to Y \) to a second countable space, we need to find a map \( \pi : X \to Z \) onto a metrizable \( n \)-K-like continuum \( Z \) and a map \( g : Z \to Y \) such that \( f = g \circ \pi \). Since the spectrum \( S \) is factorizing, there are an index \( \alpha_0 \in A \) and a map \( f_0 : X_{\alpha_0} \to Y \) such that \( f = f_0 \circ p_{\alpha_0} \). Using Lemma 3 by induction construct an increasing sequence \( (\alpha_n)_{n \in \omega} \) in \( A \) such that for every \( i \in \omega \) and any open cover \( \mathcal{V} = \{ V_1, \ldots, V_n \} \) of \( X_{\alpha_i} \) there is a map \( f : X_{\alpha_{i+1}} \to K \) onto a space \( K \in K \) such that \( f \circ p_{\alpha_{i+1}} \) is a \( p_{\alpha_i}^{-1}(\mathcal{V}) \)-map.

Let \( \alpha = \sup_{i \in \omega} \alpha_i = \bigcup_{i \in \omega} \alpha_i \). We claim that the metrizable continuum \( X_\alpha \) is \( n \)-K-like. Given any open cover \( \mathcal{U} = \{ U_1, \ldots, U_n \} \) of \( X_\alpha = \lim_{i \in \omega} X_{\alpha_i} \), we can find \( i \in \omega \) such that the sets \( W_i = X_{\alpha_i} \setminus p_{\alpha_i}^{-1}(\mathcal{U} \setminus U_i) \), \( i \leq n \), form an open cover \( W = \{ W_1, \ldots, W_n \} \) of \( X_{\alpha_i} \) such that the cover \( (p_{\alpha_i}^{-1}(W)) \) refines the cover \( \mathcal{U} \). By the choice of the index \( \alpha_{i+1} \), there is a map \( g : X_{\alpha_{i+1}} \to K \) onto a space \( K \in K \) such that \( g \circ p_{\alpha_{i+1}} : X \to K \) is a \( p_{\alpha_i}^{-1}(\mathcal{V}) \)-map. It follows that \( g \circ p_{\alpha_{i+1}} : X_{\alpha} \to K \) is a \( (p_{\alpha_i}^{-1}(W)) \)-map and hence a \( \mathcal{U} \)-map, witnessing that the continuum \( X_\alpha \) is \( n \)-K-like.

Now we see that the metrizable \( n \)-K-like continuum \( X_{\alpha} \) and the maps \( \pi = p_{\alpha} : X \to X_{\alpha} \) and \( g = f_0 \circ p_{\alpha_0} : X_{\alpha} \to Y \) satisfy our requirements.

6. **Proof of Theorem**

The “chainable and tree-like” parts of Theorem follow immediately from the characterization Theorem 3. So, it remains to prove the “circle-like” part. Let \( K = \{ S^1 \} \). We need to prove that each 4-K-like continuum \( X \) is K-like. Given an open cover \( \mathcal{U} \) of \( X \) we need to construct a \( \mathcal{U} \)-map of \( X \) onto the circle. By Theorem 3 there is a \( \mathcal{U} \)-map onto a metrizable 4-K-like continuum \( Y \). It follows that for some open cover \( \mathcal{V} \) of \( Y \) the cover \( \mathcal{V}^{-1} \) refines \( \mathcal{U} \). The proof will be complete as soon as we prove that the continuum \( Y \) is circle-like. In this case there is a \( \mathcal{V} \)-map \( g : Y \to S^1 \) and the composition \( g \circ f : X \to S^1 \) is a required \( \mathcal{U} \)-map witnessing that \( X \) is circle-like.

By Theorem 3 the metrizable continuum \( Y \) is \( \tilde{K} \)-like. By Proposition each continuum \( K \in \tilde{K} \) is homeomorphic to \( S^1 \) or \( [0, 1] \). Consequently, the continuum \( Y \) is circle-like or chainable. In the first case we are done. So, we assume that \( Y \) is chainable.

By Theorem 12.5, the continuum \( Y \) is irreducible between some points \( p, q \in Y \). The latter means that each subcontinuum of \( X \) that contains the points \( p, q \) coincides with \( Y \). We claim that \( Y \) is either indecomposable
or $Y$ is the union of two indecomposable subcontinua. For the proof of this fact we will use the argument of [9, Exercise 12.50] (cf. also [7, Theorem 3.3]).

Suppose that $Y$ is not indecomposable. It means that there are two proper subcontinua $A, B$ of $Y$ such that $Y = A \cup B$. By the choice of the points $p, q$, they cannot simultaneously lie in $A$ or in $B$. So, we can assume that $p \in A$ and $q \in B$.

We claim that the closure of the set $Y \setminus A$ is connected. Assuming that $\overline{Y \setminus A}$ is disconnected, we can find a proper closed-and-open subset $F \subseteq \overline{Y \setminus A}$ that contains the point $q$ and conclude that $F \cup A$ is a proper subcontinuum of $Y$ that contains both points $p, q$, which is not possible. Replacing $B$ by the closure of $Y \setminus A$, we can assume that $Y \setminus A$ is dense in $B$. Then $Y \setminus B$ is dense in $A$.

We claim that the sets $A$ and $B$ are indecomposable. Assuming that $A$ is decomposable, find two proper subcontinua $C, D$ such that $C \cup D = A$. We can assume that $p \in D$. Then $B \cap D = \emptyset$ (as $Y$ is irreducible between $p$ and $q$). By Theorem 11.8 of [9], the set $Y \setminus (B \cup D)$ is connected. Let $Z_1$ and $Z_2$ be open disjoint subsets of $X$ such that $B \subseteq Z_1$ and $D \subseteq Z_2$. Since $Y$ is 4-$\{S^1\}$-like, for the open cover $Z = \{Z_1, Z_2, Y \setminus (B \cup D)\}$ of $Y$ there exists a $Z$-map $h: Y \to S^1$. Thus $h(B) \cap h(D) = \emptyset$ and $S^1 \setminus (h(B) \cup h(D))$ is the union of two disjoint open intervals $W_1, W_2$. Since $h$ is a $Z$-map, $Y \setminus (B \cup D) = h^{-1}(W_1) \cup h^{-1}(W_2)$ which contradicts the connectedness of the set $Y \setminus (B \cup D)$.

Now we know that $Y$ is either indecomposable or is the union of two indecomposable subcontinua. Applying Theorem 7 of [2], we conclude that the metrizable chainable continuum $Y$ is circle-like.

7. Open Problems

**Problem 1.** For which families $K$ of connected topological graphs every 4-$K$-like continuum is $K$-like? Is it true for the family $K = \{8\}$ that contains 8, the bouquet of two circles?

Also we do not know if Theorem 4 can be generalized to classes of higher-dimensional continua.

**Problem 2.** Let $k \in \mathbb{N}$ and $K$ be a class of $k$-dimensional (contractible) continua. Is there a finite number $n$ such that a continuum $X$ is $K$-like if and only if it is $n$-$K$-like?

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