Complementarity of Quantum Correlations in Cloning and Deleting of Quantum State

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We quantify the amount of correlation generated between two different output modes in the process of imperfect cloning and deletion processes. We use three different measures of correlations and study their role in determining the fidelity of the cloning and deletion. We obtain a bound on the total correlation generated in the successive process of cloning and deleting operations. This displays a new kind of complementary relationship between the quantum correlation required in generating a copy of a quantum state and the amount of correlation required in bringing it back to the original state by deleting and vice versa. Our result shows that better we clone (delete) a state, more difficult it will be to bring the state back to its original form by the process of deleting (cloning).

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INTRODUCTION

In the last two decades quantum information processing has emerged as a powerful tool for implementing several tasks that cannot be done using classical means. The information processing tasks such as super dense coding [1], teleportation [2], remote state preparation [3] and key generation [4] are no longer only theoretical possibilities but also have experimentally demonstrated. In all these protocols, quantum entanglement or quantum correlation in a broader sense, plays a pivotal role in making these information processing tasks successful.

Quantifying the amount of quantum correlation present in a pure bipartite system is straightforward as this is given by the amount of entanglement present in the system. There are certain standard measures, more specifically entanglement monotones quantifying the amount of entanglement for both pure as well as mixed states. Two most important measures of them are the negativity [5] and the concurrence [7, 8]. However, there are certain open issues in understanding the nature of correlations present in a mixed state, or a multiqubit state. It is not totally evident whether all the information-processing tasks that can be done more efficiently with quantum systems require entanglement as resource. Several instances have been reported where people have argued that even in the absence or near absence of entanglement one can carry out some information-processing tasks more efficiently in the quantum world. Then, the natural question arises if not entanglement then what is responsible for such a behavior. It has been suggested that the amount of correlation present in the composite system is not entirely captured by the entanglement. The quantity which captures the quantum correlation and gives a meaningful explanation of such behavior is the quantum discord [3, 16]. This aims to capture the non classical correlations present in a system, and those that cannot be witnessed with entanglement. In addition to these standard approaches of quantifying the correlation in quantum mechanical systems there are other approaches of quantifying the quantum correlation and the most important of them are geometric measures [17, 18].

In the quantum information theory the no-cloning theorem plays a fundamental role [19–21]. This theorem tells how nature prevents us from amplifying an unknown quantum state. However, in principle it is always possible to construct a quantum cloning machine that replicates an unknown quantum state approximately [22–27]. These approximate quantum cloning machines can be of two types. One is the state dependent quantum cloning machine, for example, the Wootters-Zurek (WZ) quantum cloning machine, whose copying quality depends on the input state [19]. The other type is the universal quantum copying machine, for example, the Buzek-Hillery (BH) quantum cloning machine [22], whose copying quality remains same for all the input states. In addition, the performance of the universal BH quantum cloning machine is, on the average, better than that of the state dependent WZ cloning machine. The fidelity of cloning of the BH universal quantum copying machine is \( \frac{2}{3} \) which is the optimal fidelity for the universal quantum cloning machines. Although it is impossible to copy a state perfectly, however probabilistically one can clone a quantum state, secretly chosen from a certain set of linearly independent states [20]. Also, it is possible to have linear superposition of multiple clones and obtain the probabilistic cloning machine as a special case of the former [28].

The notion of quantum deletion introduced in the year 2000, tells us about the impossibility of deleting an arbitrary quantum state. More specifically, it states that the linearity of quantum theory precludes deleting of one unknown quantum state against a copy in either a reversible or an irreversible manner. The principle behind the quantum deletion will be
more pellucid to us, if we compare the deletion operation with the Landauer erasure operation [29]. Erasure of classical or quantum information cannot be performed reversibly. The erasure principle says that a single copy of some classical information can be erased at the cost of some energy. Thermodynamically, it is an irreversible operation. In quantum theory the erasure of a single unknown state is considered as swapping it with some standard state and then trashing it into the environment. Unlike the quantum erasure operation, quantum deletion is all together a different concept. The quantum environment. Unlike the quantum erasure operation, quantum deletion is an irreversible operation. In quantum theory information can be erased at the cost of some energy. Thermodynamically then it is possible with a success probability less than unity [36]. It was also shown that using these probabilistic deletion machines one cannot send super luminal signals probabilistically [37].

Since perfect deletion is not possible, it then remains alluring to see whether one can delete an unknown state imperfectly. Researchers have cropped up with various approximate deletion machines. These deletion machines were either state dependent or state independent [38–43]. Further explorations have revealed that one can construct an universal quantum deletion machine [40] and its fidelity can be further enhanced by the application of suitable unitary transformer [41]. These deletion machines can have various applications in quantum information theory [42–44]. However, the optimal quantum deletion machine has not been found yet.

At this point one might ask an important question that whether quantum correlation is responsible for our inability to produce high fidelity states in the approximate cloning or deleting a quantum state. Initially there are no correlations between the input states. This is because they are the individual systems which are in the product state. However, at the output port we always obtain a combined state, which is most of the times correlated. A priori, it is not clear whether this correlation plays an important role in deciding the fidelity of cloning and deleting an unknown quantum state. In order to find an answer to this question, here in this work, we consider a particular type of cloning machine, the BH cloning machine and try to quantify the amount of correlation present in the mixed two qubit output state. Similarly, for the deletion operation we consider a state dependent quantum deleting machine to find out the correlation in the output modes. The basic motivation is to see how the correlation regulates the fidelity of the cloning and deletion process. In accordance with our physical intuition we find that more the output modes are correlated less the fidelity in either cases. In other words, the process of cloning and deletion will be more perfect if the output modes are less correlated. We quantify these correlations with three different kinds of measures and make this more precise.

The problem of complementarity or mutually exclusive aspects of quantum phenomenon appeared with the birth of quantum mechanics. Soon after, Heisenberg discovered the uncertainty principle for the momentum and the position [50]. The next year, Bohr came up with the concept of complementarity [51]. Even in the domain of quantum information theory we have seen that the idea of complementarity is not new as some authors showed that the complementarity between the local and non local information of the quantum systems [47]. Here, in this work we observe a new kind of complementarity in terms of successive correlations generated in the system when a state undergoes deletion after cloning or cloning after deletion.

The organization of our paper is as follows. In section II, we provide a short introduction to the relevant measures of quantum correlations. In section III, we analyse the correlation content of the output of the Buzek-Hillery quantum cloning machine. We also analyze how the correlation content of the output modes plays a pivotal role in determining the fidelity of cloning. In section IV, we study the standard approximate deleting machine to obtain a correspondence between the fidelity of deletion with the amount of correlation generated in the process. In section V, we obtain a new kind of complementarity relation between the correlation generated in the system for the process of successive cloning and deletion and also for the case when we clone the state after the deletion. This complementarity gives a new bounds to the total correlation generated in context of various correlation measures.

VARIOUS MEASURES OF QUANTUM CORRELATIONS

For the sake of completeness, in this section, we give a brief description of three different types of measures which we are going to use for quantifying the quantum correlation generated in the process of cloning and deletion operations. These measures are (i) the Negativity, (ii) the Quantum discord and (iii) the Geometric Discord. The basic motivation of choosing these three measures is that each of them represents three different classes of measures and we would like to see how generic is the complimentarity for cloning and deleting if we use different measures of quantum correlations.

Measures in entanglement-separability paradigm: Negativity

The negativity is an entanglement monotone that quantifies how strongly the partial transpose of a density operator fails to become positive [5]. The negativity, $N(\rho_{AB})$, of a bipartite state $\rho_{AB}$ is defined as the absolute value of the sum of the negative eigenvalues of $\rho_{AB}^{T_A}$. Alternatively, we can find
negativity by the following relation
\begin{equation}
N(\rho_{AB}) = \frac{||\rho_{AB}^T||_1 - 1}{2},
\end{equation}
where $||A||_1$ is called trace-norm of $A$ and it is defined as $||A||_1 = \text{Tr}[\sqrt{A^\dagger A}]$. Here, $\rho_{AB}^T$ denotes the partial transpose of $\rho_{AB}$ with respect to the $A$-part \[48\, 49\]. The logarithmic negativity is then defined in terms of the negativity as
\begin{equation}
E_N(\rho_{AB}) = \log_2[2N(\rho_{AB}) + 1].
\end{equation}
For two-qubit states, $\rho_{AB}^T$ has at most one negative eigenvalue \[6\]. It has been seen that for two-qubit states, a positive logarithmic negativity implies that the state is entangled and distillable, whereas a vanishing logarithmic negativity implies that the state is separable \[48\, 49\].

**Information theoretic measure: Quantum Discord**

Information theoretic measures are those category of measures which are constructed from the perspective of defining the notion of quantum correlation from the information theoretic viewpoint (entropic quantities). These measures are not computable in a closed form like the entanglement monotones. In spite of not being computable in a closed form, nevertheless, they can be efficiently computed numerically for two-qubit systems. The most important of them is quantum discord which came into the limelight for showing that the two-qubit systems. The most important of them is quantum discord which came into the limelight for showing that the quantum correlation in the mixed states goes beyond the notion of entanglement.

Quantum discord is defined as the difference between two quantum information-theoretic quantities, whose classical counterparts are equivalent expressions for the classical mutual information. Thus, the quantum discord is the difference between the total correlation and classical correlation present in the bipartite quantum systems, thus quantifying the amount of quantum correlation present in it. It is defined as \[3\, 11\]
\begin{equation}
Q(\rho_{AB}) = I(\rho_{AB}) - J(\rho_{AB}).
\end{equation}
The total correlation, $I(\rho_{AB})$, for a bipartite state $\rho_{AB}$ is given by the expression
\begin{equation}
I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}),
\end{equation}
where $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ is the von Neumann entropy of the quantum state $\rho$, and $\rho_A$ and $\rho_B$ are the reduced subsystems of the bipartite state $\rho_{AB}$. On the other hand, $J(\rho_{AB})$ can be thought of as the amount of classical correlation in $\rho_{AB}$, and is defined as \[10\]
\begin{equation}
J(\rho_{AB}) = S(\rho_A) - S(\rho_{A|B}).
\end{equation}
Here, $S(\rho_{A|B}) = \min_{\{B_i\}} \sum_i p_i S(\rho_{A|i})$, is the average entropy of the entropies of the states $\rho_{A|i}$. These entropies $S(\rho_{A|i})$ are the entropies of the subsystem $A$ conditioned on a measurement performed by $B$ with a rank-one projection-valued measurement $\{B_i\}$. These states are given by $\rho_{A|i} = \frac{1}{p_i} \text{Tr}_B([I_A \otimes B_i](I_A \otimes B_i)]$, with probability $p_i = \text{Tr}_AB([I_A \otimes B_i](I_A \otimes B_i)]$. Here $I$ is the identity operator on the Hilbert space of $A$. It may be noted that the discord is a positive quantity and vanishes on classical-classical and quantum-classical states.

**Geometric Measure: Geometric Discord**

Apart from these two classes of measures there is another way by which one can quantify the amount of quantum correlation present in a two qubit bipartite state. This is captured by the geometric measures of quantum correlation. It had been argued that the experienced difficulty of calculating quantum discord can be minimized, for a general two-qubit state, by defining its geometrical version \[17\]. It is well known that almost all (entangled or separable) states are disturbed by the measurement; however, there are certain states which are invariant under the measurement performed on the sub-system $A$. These states are the so-called classical–quantum states, whose elements have a density matrix of the form
\begin{equation}
\rho = \sum_i p_i |i\rangle\langle i| \otimes \rho_i,
\end{equation}
where $p_i$ is a probability distribution, $\{|i\rangle\}$ is an orthonormal set of vectors for $A$ and $\rho_i$ are the elements of $B$. A classical-quantum state is not affected by a measurement on $A$ in any case. One can show that the state $\rho$ is of zero-discord if and only if there exists a Von Neumann measurement $\{\Pi_k = |\psi_k\rangle\langle\psi_k|\}$ such that \[13\]
\begin{equation}
\sum_k (\Pi_k \otimes I_B)\rho(\Pi_k \otimes I_B) = \rho.
\end{equation}
It had been seen that in Ref. \[17\], that these two states in equation \[6\] and \[7\] are identical. Let $S$ be the set consisting of all classical–quantum two qubit states, and let us assume that $\chi$ be a generic element of this set. Then the geometric discord $D_G$ of an arbitrary state $\rho$ is then given by the distance between the state $\rho$ and the closest classical-quantum state. The geometric discord had been introduced as
\begin{equation}
D_G(\rho) = 2 \min_{\chi \in S} ||\rho - \chi||^2,
\end{equation}
where the coefficient 2 in the right hand side is the normalization factor. The geometric discord of the state $\rho$ can have a nice closed form for that first one needs to express the state in terms of the Pauli matrices as $\rho = \frac{1}{4} (I_4 + \sum_{i=1}^3 x_i \sigma_i \otimes I_2 + \sum_{j=1}^3 y_j I_2 \otimes \sigma_j + \sum_{i,j=1}^3 t_{ij} \sigma_i \otimes \sigma_j)$, where $t_{ij} =
\[ \text{Tr} [\rho(\sigma_i \otimes \sigma_j)], \sigma_0 = \mathbb{I}_2, \sigma_i (i = 1, 2, 3) \text{ are the Pauli matrices, } \vec{x} = \{x_i\}, \vec{y} = \{y_i\} \text{ represent the three-dimensional Bloch column vectors. Then, we can rewrite the geometric discord as } D_G(\rho) = \frac{1}{2} (\|\vec{x}\|^2 + \|\vec{y}\|^2 - 4k_{\text{max}}) = 2\text{Tr}|S| - 2k_{\text{max}} \text{ with } k_{\text{max}} \text{ being the largest eigenvalue of the matrix } S = \frac{1}{\sqrt{2}} (\vec{x}^T \vec{x} + \vec{y}^T \vec{y}). \]

Though, there are other approaches to define the geometric discord, we are not considering all those here.

**ANALYSIS OF THE CORRELATION CONTENT OF THE OUTPUT OF BUZEK-HILLERY COPYING MACHINE**

In this section we consider the universal Buzek-Hillery cloning machine and quantify the correlation present in the output copies of the Buzek-Hillery cloning machine \[^{[22]}\]. But before that we give a short description of the Buzek-Hillery cloning machine. We recall that the action of the Buzek-Hillery open cloning machine. We recall that the action of the Buzek-Hillery quantum cloning machine \[^{[13]}\] is given by the transformations

\[ |00\rangle_{ab} |Q\rangle_x \rightarrow |00\rangle_{ab} |Q\rangle_x + [01\rangle_{ab} + |10\rangle_{ab}] |Y\rangle_x, \]

\[ |11\rangle_{ab} |Q\rangle_x \rightarrow |11\rangle_{ab} |Q\rangle_x + [01\rangle_{ab} + |10\rangle_{ab}] |Y\rangle_x, \]

where \(a, b\) and \(x\) denote qubits corresponding to input state port, blank state port and the machine state port. The unitarity and the orthogonality of the cloning transformation demand the following conditions to be satisfied

\[ x\langle Y_i | Q_i \rangle_x + x\langle Y_i | Y_i \rangle_x \neq 0, \]

(9)

Here, we assume that the machine state vectors \( |Y_i \rangle_x \) and \( |Q_i \rangle_x \) to be mutually orthogonal. This is also true for the state vectors \( \{ Q_0 \}, \{ Q_1 \} \).

The unknown quantum state which is to be cloned is

\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle, \]

where \(\alpha, \beta\) are complex numbers satisfying, \(|\alpha|^2 + |\beta|^2 = 1.\) After using the cloning transformation \[^{[13]}\] on quantum state \[^{[11]}\] and then tracing out the machine state, the reduced density operator describing the two qubit output modes of the original and the cloned state is given by

\[ \rho_{\text{clone}}^{ab} = (1 - 2\xi)(2|\alpha|^2 |00\rangle_{ab} |00\rangle_{ab} + |\alpha|^2 |11\rangle_{ab} |11\rangle_{ab}) + \frac{\alpha^2}{\sqrt{2}} (1 - 2\xi)(2|\alpha|^2 |\psi\rangle_{ab} |\psi\rangle_{ab} + |\psi\rangle_{ab} |\overline{\psi}\rangle_{ab} + |\overline{\psi}\rangle_{ab} |\psi\rangle_{ab} + 2\xi |\overline{\psi}\rangle_{ab} |\overline{\psi}\rangle_{ab}), \]

(12)

with \(\xi\) being the machine parameter determining the nature of the cloning transformations. The output state \(\rho_{\text{out}}^{ab}\) is of our prime importance as we will investigate the amount of correlation present in it. The cloning fidelity is given by the overlap between the real output state \(\rho_{\text{out}}^{ab}\) with the desired output state \(|\psi\rangle\).

It can be seen that that the cloning fidelity \(F_{\text{cl}} = \text{Tr}[\rho_{\text{out}}^{ab} |\psi\rangle \langle \psi|] = 1 - \xi\) is dependent on the machine parameter \(\xi\). It has been shown that the BH cloning machine should satisfy the inequality \(\eta \leq 2(\xi - 2\xi^2)\). The relation \(\eta = 1 - 2\xi\) reduces the inequality \(\eta \leq 2(\xi - 2\xi^2)\) to the inequality \(\frac{1}{2} \leq \xi \leq \frac{1}{2}\). Henceforth, our motivation is to study the different measures of quantum correlation in this range of the machine parameter to see how it behaves with the cloning fidelity. The amount of correlation generated in the process of cloning is given by the difference between the amount of correlation in the output modes and the amount of correlation in the same two modes before the application of cloning operations. We will denote this difference of correlations with the notation \(\Delta_{\text{clone}} = K(\rho_{\text{final}}) - K(\rho_{\text{initial}})\), for a correlation measure \(K(\rho_{\text{ab}})\). Here we compute three different correlation measures namely (i) the negativity (\(N\)), (ii) the discord (\(D\)) and (iii) the geometric discord (\(DG\)) for both the initial input states and final output states. Since in the case of the Buzek-Hillary cloning machine we start with product states, the respective differences \(\Delta_{\text{clone}}^N, \Delta_{\text{clone}}^D\) and \(\Delta_{\text{clone}}^DG\) of correlations are nothing but the amount of correlations \(N(\rho_{\text{final}}), D(\rho_{\text{final}})\) and \(DG(\rho_{\text{final}})\) in the output modes.

Our first motivation is to see how these different measures of correlation behaves with the fidelity of cloning. For this purpose, we first express these different measures of the correlation \(\Delta_{\text{clone}}\) in terms of the fidelity \(F_{\text{cl}}\) of cloning. We rewrite these measures as a function of the variable like the fidelity of cloning \(F_{\text{cl}}\) and input state parameter \(\alpha\). The expression for \(\Delta_{\text{clone}}^N\) is given by

\[ \Delta_{\text{clone}}^N = \frac{1}{2} \left[ 2 \left\{ g_1 + \frac{1}{2} g_2 f_1 \right\}^\frac{3}{2} + \left\{ g_1 + g_2 f_2 \right\}^\frac{3}{2} \right], \]

(13)

where \(f_1 = |\alpha|^2 \beta^2, f_2 = (1 + \frac{1}{2} |\alpha|^2 - \alpha^* \beta)^2, f_3 = |\alpha|^2(|\alpha|^2 + \frac{1}{2} \beta^2)\) (here, \(\{\cdot\}\) denotes absolute value and \(\ast\) the complex conjugation), \(g_1 = (F_{\text{cl}} - 1)^2\) and \(g_2 = (2F_{\text{cl}} - 1)^2\). Similarly the expression for \(\Delta_{\text{clone}}^D\) is given by

\[ \Delta_{\text{clone}}^D = H(F_{\text{cl}}, 1 - F_{\text{cl}}) + mH(X_+, X_-) - nH(Y_+, Y_-) \]

(14)

where \(H(x, y) = -x \log_2 x - y \log_2 y, x = \frac{1}{2} \{ \frac{1}{2} (1 + (x_+ + C_+)) \}, y = \frac{1}{2} \{ \frac{1}{2} (1 + (4F_{\text{cl}}(7 - 5F_{\text{cl}}) + C_- + 2) \}, C_+ = F_{\text{cl}}[-2 - 10\alpha^2 + F_{\text{cl}}(1 + 8\alpha^2)] \pm 3\alpha^2, m = n - 1, \text{ and } n = \alpha^2 + (1 - 2\alpha^2) F_{\text{cl}}\). Lastly, the corresponding expression
for the geometric discord is given by

$$\Delta_{DG}^{\text{clone}} = 2(\lambda + \lambda_+ + \lambda_- - \max[\lambda, \lambda_+, \lambda_-]),$$  \hspace{1cm} (15)

where \(\lambda = (1 - F_{cl})^2, \lambda_\pm = \frac{1}{2}(3.5 - 9F_{cl} + 6F_{cl}^2 \pm \sqrt{p - \alpha^2\beta^2q}), \) (here \(p = 2.25 - 15F_{cl} + 37F_{cl}^2 - 40F_{cl}^3 + 16F_{cl}^4\) and \(q = 5 - 36F_{cl} + 96F_{cl}^2 - 112F_{cl}^3 + 48F_{cl}^4\)).

To have a better insight, we plot these expressions \(\Delta_N^{\text{clone}}, \Delta_D^{\text{clone}}\) and \(\Delta_{DG}^{\text{clone}}\) of the correlation generated in terms of in terms of the fidelity \(F_{cl}\) of cloning and the input state parameter \(\alpha\) in the figure \(\text{(I)}\). Since \(\xi\) lies in the range \(\frac{1}{6} \leq \xi \leq \frac{1}{2}\), we have the range of fidelity \(\frac{1}{2} \leq F_{cl} \leq \frac{5}{6}\) and the range of the input parameter \(\alpha\) from 0 to 1. In the figure \(\text{(I)}\), we find that when the states are more correlated when the fidelity of cloning is less. In other words, when we have a cloning machine that performs better, the joint output mode will be less correlated. In together all of these plots are the witnesses of the fact that the amount of correlation generated in the process of cloning plays a vital role in determining the fidelity of cloning. It is quite evident from each of these figures that the more the amount of correlation present in the original and the cloned copy in the output more difficult is to copy the information of the original copy in the blank state as the information gets hidden in the correlation between the copies. Though we have considered a particular type of cloning machine to illustrate this phenomenon, we believe that this phenomenon is independent of the transformation we choose and is true in general for the process of imperfect quantum cloning.

![Figure 1](image1.png)

**FIG. 1:** The plot shows how the correlation measures (i) negativity \((\Delta_N^{\text{clone}})\), (ii) discord \((\Delta_D^{\text{clone}})\) and (iii) geometric discord \((\Delta_{DG}^{\text{clone}})\) varies with the input parameter \(\alpha\) and the fidelity of cloning \(F_{cl}\).

### ANALYSIS OF THE CORRELATION CONTENT OF THE OUTPUT OF A STATE DEPENDENT DELETING MACHINE

In this section we analyze the correlation generated in the process of quantum deletion which can be thought of as somewhat opposite procedure of the quantum cloning. As an example, we consider a state dependent quantum deletion machine and study the amount of correlation present in the output modes. Quite similar to the previous section here also our basic motivation is to determine the role of quantum correlation in regulating the fidelity of deletion. In order to do that we consider three different correlation measures and indeed we see that the physical finding is no different from the cloning.

The action of a state dependent deleting machine as mentioned in the reference \[38\] is given by the unitary operation

$$|\psi⟩_A|ψ⟩_B|A⟩_C \rightarrow \alpha^2|0⟩_A|0⟩_B|A₀⟩_C + \beta^2|1⟩_A|0⟩_B|A₁⟩_C + \alpha\beta(|0⟩_A|1⟩_B + |1⟩_A|0⟩_B)|A⟩_C, \hspace{1cm} (16)$$

where we start with two copies of the unknown state \(|\psi⟩ = \alpha|0⟩ + \beta|1⟩\) with the purpose of deleting one copy against the other. In this process we also assume that the input parameters \(\alpha\) and \(\beta\) to be constrained by the condition \(|\alpha|^2 + |\beta|^2 = 1\). Here \(|A⟩_C\) is the initial state of the ancilla, \(|A₀⟩_C\) and \(|A₁⟩_C\) are the final states of the ancilla. Moreover, the unitarity of the transformation demands the states \(|A⟩, |A₀⟩\) and \(|A₁⟩\) to be orthogonal to each other. After application of the deletion machine on two copies of \(|ψ⟩\), the output reduced density matrix of these two modes takes the form

$$\rho_{ab}^{\text{del}} = |\alpha|^4|00⟩⟨00| + |\beta|^4|10⟩⟨10| + 2|\alpha|^2|\beta|^2|ψ^+⟩⟨ψ^+|,$$

(17)
where $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. The fidelity of deletion for this machine is given by $F_{del} = 1 - |\alpha|^2|\beta|^2$. By expressing the input parameter $|\alpha|^2$ in terms of fidelity $F_{del}$ we have $|\alpha|^2 = \frac{1}{2}(1 \pm \sqrt{4F_{del} - 3})$. However the feasible solution for $|\alpha|^2$ is $\frac{1}{2}(1 - \sqrt{4F_{del} - 3})$. Based on the range of $|\alpha|^2$ we find that $F_{del}$ is bounded by the relation $\frac{3}{4} \leq F_{del} < 1$. This is also consistent with the fact that if we are given two copies of an unknown qubit, and we perform optimal measurement on both the copies, then we can estimate the state with a fidelity $3/4$ [24] which is also the lower bound of the deletion machine.

We adopt the similar way of defining the amount of correlation generated in the process of deletion. This is given by the difference between the amount of correlation in the output modes after the process of deletion and the amount of correlation in those two modes before the application of deletion operation. We denote this difference of correlations for any correlation measure $K(\rho)$ with the notation $\Delta_{del}^K = K(\rho_{del}^{\text{final}}) - K(\rho_{del}^{\text{initial}})$. We compute various correlation measures for both the initial input states and the final output states. Since we start with product states having no initial correlation, the amount of correlation generated in the process of deletion is same as the amount of correlation between the output modes. We denote these correlations for three different measures (i) the negativity ($N$), (ii) the discord ($D$) and (iii) the geometric discord ($DG$) by the notations $\Delta_{del}^N$, $\Delta_{del}^D$ and $\Delta_{del}^{DG}$, respectively. The expression for $\Delta_{del}^N$ is given by

$$\Delta_{del}^N = \frac{1}{2} \left[ \left( \frac{1 - a}{4} \right) \{(1 + a)^2 + 1\}^{\frac{3}{2}} + (2 - a)(1 + a), \right.$$ \[ \left. - 1 \right]$$ \tag{18}$$

where $a = \sqrt{4F_{del} - 3}$. Similarly, the expression for $\Delta_{del}^D$ is given by

$$\Delta_{del}^D = 1.44 \left[ H(c, d) - H(\frac{(1 - a)^2}{4}) + H(T_+, T_-) \right.$$

$$\left. - H(S_+, S_-) \right], \tag{19}$$

where $H(x, y) = -x \log_2 x - y \log_2 y$, $H(x) = -x \log_2 x$, $c = \frac{1}{2} F_{del} (a + 1)$, $d = 1 - c$, $S_\pm = \frac{1}{2} (3 - 2F_{del} + a \pm \{14 - 2a + 4F_{del}(a + 5F_{del} - 8)\}^{\frac{1}{2}})$ and $T_\pm = \frac{1}{2} (1 \pm a)$. Lastly, the corresponding expression for the geometric discord is given by

$$\Delta_{del}^{DG} = 2(\lambda_0 + 2\lambda_1 - \max[\lambda_0, \lambda_1]), \tag{20}$$

where $\lambda_0 = \frac{1}{2} l_+^2 + l_+^2$, $\lambda_1 = K_+^2$, $a = \sqrt{4F_{del} - 3}$, $l_\pm = K_\pm - K_+, l_\pm = \frac{1}{2} (1 \pm a) \{(1 - a) \{1 \pm \frac{1}{2} (a - 1)\}\}$.

Now, our aim is to see how correlation generated in the process of cloning also and our conjecture is that this is independent of the machine we select. This also goes with our physical intuition that the amount of information not available for the deletion process is hidden in the correlation between the two modes.

FIG. 2: The figure shows how the correlation measures (i) negativity ($\Delta_{del}^N$), (ii) discord ($\Delta_{del}^D$) and (iii) geometric discord ($\Delta_{del}^{DG}$) varies with the fidelity of deletion ($F_{del}$).

CONCATENATION OF CLONING AND DELETION AND CORRELATION COMPLEMENTARITY

In this section, we consider the successive action of cloning and deletion and its impact on various measures of correlation. We let $K(\rho)$ be a measure of correlation and consider the composition $\rho_{del}^{\text{final}} = \rho_{del}^{\text{initial}} \rho_{clone}^{\text{final}}$. The process of cloning is represented by $\rho_{clone}^{\text{final}}$ and the process of deleting is represented by $\rho_{del}^{\text{final}}$. We calculate the amount of correlation generated as a result of these two
processes is bounded. Here, also we find that similar thing happens even in the opposite case where cloning is followed by the deletion. These bounds actually show a new aspect of quantum correlation, i.e., “the complementarity”. We analytically obtain these bounds for different measures and exemplify for a particular measure with the help of a particular cloning and deletion machine.

Deleting imperfect cloned copies

In this subsection, we consider the case where we start with the state to be cloned along with a blank state. These two states (say $|\psi\rangle$ and $|\Sigma\rangle$ respectively) together are product states having no correlation at all. After the cloning operation these two states are no longer uncorrelated and they are given by joint density matrix $\rho_{ab}^{\text{final}}$. The amount of correlation generated in the process of cloning for a given correlation measure $K$ is given by, $\Delta_{K}^{\text{clone}} = K(\rho_{ab}^{\text{final}}) - K(|\psi\rangle \otimes |\Sigma\rangle)$. Since the initial states are product states we have $K(|\psi\rangle \otimes |\Sigma\rangle) = 0$ and consequently, $\Delta_{K}^{\text{clone}} = K(\rho_{ab}^{\text{final}})$. $K$ being any correlation measure, is bounded by its maximum and minimum values $K_{\text{max}}$ and $K_{\text{min}}$ respectively. Now if we delete these imperfect cloned copies in order to get back to its original product form $|\psi\rangle \otimes |\Sigma\rangle$, we will get a new combined state $\rho'_{ab}$ at the output mode. Then the amount of correlation generated in the process is given by $\Delta_{K}^{\text{del}} = K(\rho'_{ab}) - K(\rho_{ab}^{\text{final}})$ for a particular correlation measure $K$. It can be seen that by combining the correlations generated in the cloning and deleting process we have

$$\Delta_{K}^{\text{clone}} + \Delta_{K}^{\text{del}} = K(\rho'_{ab}). \tag{21}$$

Since the correlation measure $K$ is always bounded by its maximum value $K_{\text{max}}$ for any arbitrary state, we have

$$\Delta_{K}^{\text{clone}} + \Delta_{K}^{\text{del}} \leq K_{\text{max}}. \tag{22}$$

Thus, for the different correlation measures like the negativity $(N)$, the discord $(D)$ and the geometric discord $(DG)$ we have various bounds for the correlation as given below

$$\Delta_{N}^{\text{clone}} + \Delta_{N}^{\text{del}} \leq \frac{1}{2}, \tag{23} \quad \Delta_{D}^{\text{clone}} + \Delta_{D}^{\text{del}} \leq 1, \quad \Delta_{DG}^{\text{clone}} + \Delta_{DG}^{\text{del}} \leq 1,$$

respectively. These bounds together tell us about a intriguing property of quantum correlation which is “the Complementarity”. The amount of correlation generated in the process of cloning is complementary to the amount of correlation generated in the process of deletion. Thus, we can say that when the amount of correlation generated in the cloning process is more (less), the amount of correlation for the deletion process is less (more). The above result can be stated differently: it tells us that better we clone then worse we delete. Thus, our conjecture is that this complementarity is not only true for the correlation generated but also true for the fidelity of achieving the cloning and deletion process successively.

Next we exemplify our result with the help of a particular cloning and deleting transformation in the context of a specific correlation measure such as the geometric discord $(DG)$. We start with an arbitrary quantum state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle (|\alpha|^2 + |\beta|^2 = 1)$ and a blank state $|\Sigma\rangle$ initially in the form of a product state (i.e $DG(|\psi\rangle \otimes |\Sigma\rangle) = 0$). Then, we apply the universal Buzek-Hillery quantum cloning machine defined by the transformations (9) on $|\psi\rangle$ and on the output of BH copying machine we apply the deletion operations defined by

$$|0\rangle|0\rangle|Q_0\rangle \rightarrow |0\rangle|0\rangle|A_0\rangle,$$

$$|0\rangle|1\rangle|Y_i\rangle \rightarrow |0\rangle|1\rangle|1\rangle|0\rangle|Y_i\rangle,$$

$$|1\rangle|0\rangle|Q_1\rangle \rightarrow |1\rangle|0\rangle|A_1\rangle, \quad (i = 0, 1) \tag{24}$$

to obtain the final output state (39)

$$\rho_f = \frac{1}{1 + 2\xi}(\alpha^2|00\rangle\langle 00| + \beta^2|10\rangle\langle 10| + 2\xi|\psi^+\rangle\langle \psi^+|) \tag{25}$$

where $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ and $\langle A_i|Y_i \rangle = 0$. The fidelity of deleting imperfect cloned copies is given by $F_3 = \frac{1 + \xi}{1 + 2\xi}$ (39) and it ranges from $\frac{1}{3}$ to $\frac{2}{3}$. The total correlation generated in the successive process of cloning and deletion is given by the sum of the respective correlations

$$\Delta_{DG}^T = \Delta_{DG}^{\text{clone}} + \Delta_{DG}^{\text{del}} = DG(\rho_f). \tag{26}$$

The expression for the $\Delta_{DG}^T$, i.e., $DG(\rho_f)$ is given by

$$\Delta_{DG}^T = 2(\lambda_0 + 2\lambda_1 - \max[\lambda_0, \lambda_1]), \tag{27}$$

where $\lambda_0 = \frac{1}{2} + \sqrt{2}\alpha^4(1 - 2F_3)^2 + 2\alpha^2F_3(1 - 2F_3) - F_3(1 - F_3)$ and $\lambda_1 = (1 - F_3)^2$.

Thus, we see that the total correlation generated in the process is given by the correlation content of the final state and is bounded by its maximum value. Since, we adopt the geometric discord as a measure of correlation, the total correlation content is bounded by one, i.e., $\Delta_{DG}^T < 1$. In the figure (34) we plot the total correlation with respect to the machine parameter $\xi$ and the input state parameter $\alpha$ and clearly find that this is always bounded by its maximum value one.
Cloning of Imperfect deleted Copies

In this subsection we carry out the reverse process where we delete first and then clone the imperfect cloned copies. We start with two identical copies of an unknown quantum state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ ($\alpha, \beta$ are real and $\alpha^2 + \beta^2 = 1$). Initially there is no correlation between these two states as they are in the product form. Consequently, we can write the correlation content of these states for a given correlation measure $K$ as $K(|\psi\rangle \otimes |\psi\rangle) = 0$. However, after the deletion operation they are no longer uncorrelated. Instead, we obtain a correlated two qubit state $\rho_{2} = \rho_{del}$. The amount of the correlation generated in the process of deletion is given by the difference of the correlation of the final and the initial states, i.e., $\Delta^K_{del} = K(\rho_{2}) - K(|\psi\rangle \otimes |\psi\rangle)$.

Next, we apply the cloning transformations on the combined state $\rho_{2}$ in order to get back to the initial identical copies of the state $|\psi\rangle$. However, due to the imperfection of the process we get a mixed state $\rho_{3}$ (say) at the output port. The amount of correlation generated in the process is given by the difference of the correlation of the states $\rho_{2}$ and $\rho_{3}$, i.e., $\Delta^{clone} = K(\rho_{3}) - K(\rho_{2})$. The total correlations generated in the process of cloning and deletion is given by

$$\Delta^K_{del} + \Delta^{clone} = K(\rho_{3}).$$

(28)

Since for a given correlation measure $K$ the correlation of a particular state is always bounded by its maximum and minimum value $K_{max}$ and $K_{min}$, we will get back the same bound on the total correlation generated, i.e.,

$$\Delta^K_{del} + \Delta^{clone} \leq K_{max},$$

(29)

irrespective of the fact that we delete and then clone or we clone it first and then delete. This once again establishes the same complementarity in terms of the correlation generated in the process of cloning and deletion. The complementarity of quantum correlation is independent of whether we apply cloning and deletion first.

Next, we give an example of the complementarity phenomenon in this case with the help of a particular deleting and cloning machine in the context of a specific correlation measure, namely the geometric discord ($DG$). Here we start with two identical copies of the state $|\psi\rangle$ and we apply the quantum deletion machine defined in equation (16) which results in two qubit state $\rho_{del}^{ab}$ (see equation (17)). Then, we apply BH cloning operation on the state $\rho_{del}^{ab}$ which will give us two output states as $\rho_{aa} = Tr_{a}[U_{BH}\otimes I](\rho_{del}^{ab}|0\rangle\langle 0|)(U_{BH}\otimes I)^{\dagger}$ and $\rho_{bb} = Tr_{a}[I\otimes U_{BH}](\rho_{del}^{ab}|0\rangle\langle 0|)(I\otimes U_{BH})^{\dagger}$. These states are given by

$$\rho_{aa} = (1 - 2\xi)(\alpha^2|00\rangle\langle 00| + \beta^2|11\rangle\langle 11|) + 2\xi|\psi^{+}\rangle\langle \psi^{+}|,$$

and

$$\rho_{bb} = (1 - 2\xi)(1 - \alpha^2\beta^2)|00\rangle\langle 00| + \alpha^2\beta^2|11\rangle\langle 11|) + 2\xi|\psi^{+}\rangle\langle \psi^{+}|.\quad (30)$$

The total correlation generated in the successive process of deletion and cloning is given by the sum of the respective correlations. Here, we obtain the total correlation in terms of the measure geometric discord ($DG$) as

$$\Delta_{DG}^{T} = \Delta_{DG}^{del} + \Delta_{DG}^{clone} = DG(\rho_{3}).\quad (31)$$

In this case $\rho_{3}$ are $\{|\psi\rangle, \rho_{aa}, \rho_{bb}\}$. Hence, the total correlation for the state $\rho_{aa}$ is given by

$$\Delta_{DG}^{T} = 2(\lambda_{0} + 2\lambda_{1} - \max[\lambda_{0}, \lambda_{1}]),\quad (32)$$

FIG. 3: The figure shows how the total correlation ($\Delta_{DG}^{T}$) for (i) the scheme “deleting imperfect cloned copies”, which varies with the input parameter $\alpha$ and the fidelity of deletion $F_{3}$, and for (ii) the scheme “cloning of imperfect deleted copies”, (iiia) $\Delta_{DG}^{T}$ of equation (32) and (iib) $\Delta_{DG}^{T}$ of equation (33) varies with input parameter $\alpha$ and the cloning machine parameter $\xi$. 

where $\lambda_0 = \frac{1}{4}[L^2 + (L - 2\xi)^2]$, $\lambda_1 = \xi^2$ and $L = (1 - 2\xi)(\alpha^2 - \beta^2)$. And similarly, for $\rho_{bb'}$ we find
\begin{equation}
\Delta_{DG}^T = 2(\lambda_0 + 2\lambda_1 - \max[\lambda_0, \lambda_1]),
\end{equation}
where $\lambda_0 = \frac{1}{4}[J^2 + (1 - 4\xi)^2]$, $\lambda_1 = \xi^2$ and $J = (1 - 2\xi)(1 - 2\alpha^2\beta^2)$. Similar to the previous process we see that here also the total correlation is given by the correlation of the final state. In figure (3(a, b)), we plot the total correlation $\Delta_{DG}^T$ against the input parameter $\alpha$ to find that this is always bounded by its maximum value one, i.e., $\Delta_{DG}^T \leq 1$.

**CONCLUSIONS**

Complimentarity is a fundamental feature of quantum world which manifests for dual physical nature of quantum particles. In this paper, we have shown a different kind of complimentarity between two different physical processes such as the approximate quantum cloning and the deleting. We have shown that there is a relationship between the correlation generated in the process of cloning and deletion to the fidelity of the process in question. This has been illustrated using various measures of quantum correlations such as the geometric discord ($DG$), the entropic quantum discord ($D$) and the negativity ($N$). To bring out the generic nature of the coplimentarity, we have chosen three different classes of measure and irrespective of these measures we find that the fidelity decreases with the increase of correlation for both the process of cloning and deletion. This is well exhibited in terms of the amount of correlation generated in the successive process of cloning and deletion (vice versa). Moreover, we have witnessed an important property of quantum correlation which is called as the “complementarity” property in dual physical processes. We have shown that the sum of total correlation change in the cloning and the deleting is bounded by the maximum value of the measure of quantum correlation. We have illustrated the complimentarity for a particular choice of cloning and deleting machine as well as for a particular measure of correlation. We believe that this phenomenon is true for all classes of correlation measures and is independent of the choice of measure. In future, it will be interesting to see if other quantum correlations display some complimentary behavior in dual physical processes.

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