The Curie–Weiss model with complex temperature: phase transitions

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Abstract

We study the partition function and free energy of the Curie–Weiss model with complex temperature, and partially describe its phase transitions. As a consequence, we obtain information on the locations of zeros of the partition function.

1 Introduction

An important component of large deviations theory is Varadhan’s lemma, which states that if a sequence of probability measures \( \mu_N \) satisfies the large deviations principle in a (Polish) space \( \mathcal{X} \) with speed \( N \) and rate function \( I \), then for any bounded continuous function \( f : \mathcal{X} \to \mathbb{R} \),

\[
\lim_{N \to \infty} \frac{1}{N} \log \int e^{N f(x)} \mu_N(dx) = \sup_{x \in \mathcal{X}} (f(x) - I(x)).
\] (1.1)

See [2] for a precise statement, relaxed assumptions, and applications.

In many applications, considering real-valued \( f \) is too restrictive, and one may be interested in relaxing it to allow for complex-valued \( f \). Statistical mechanics provides for a rich class of examples; we mention in particular the Yang–Lee theory, where the complex perturbation is in form of a magnetic field, or the quantum spin chain models [5], where quantities of interest such as emptiness formation can be formulated as exponential asymptotics of the type (1.1) with complex integrand. Note that in such examples, because \( f \) is multiplied by \( N \), relatively small changes in phase may lead to sign changes of the integrand in (1.1) and therefore to cancelations.

It seems maybe naive at this point to hope for a general theory, which would consist of an analogue of (1.1). Our goal in this paper is more modest: we consider one simple example, the Curie–Weiss model with complex temperature, and partially develop the asymptotic theory concerning its partition function. While we are not able to give a complete description of

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the associated phase diagram, we will be able to show that the phase diagram is not trivial. As a consequence of our analysis, we will also obtain information on the (complex) zeros of the partition function, of importance in the Yang–Lee theory of phase transitions, see [8, Theorem 2].

We begin by introducing the Curie–Weiss model that we will consider. Let \( \sigma = (\sigma_1, \ldots, \sigma_N) \in \{-1, +1\}^N \). Define the Hamiltonian

\[
H_N(\sigma) = -\frac{1}{2N} \sum_{i,j=1}^{N} \sigma_i \sigma_j = -\frac{N}{2} (m_N(\sigma))^2,
\]

where the magnetization is \( m_N(\sigma) = \frac{1}{N} \sum_{i=1}^{N} \sigma_i \). For \( \beta \in \mathbb{C} \), let \( Z_{\beta,N} \) denote the free energy, i.e.

\[
Z_{\beta,N} = \frac{1}{2N} \sum_{\sigma \in \{-1, +1\}^N} \exp(-\beta H_N(\sigma)) = \int \cdots \int \exp(-\beta H_N(\sigma)) \prod_{i=1}^{N} \mu(d\sigma_i),
\]

where \( \mu(d\sigma) = \frac{1}{2} (\delta_1 + \delta_{-1}) \).

When \( \beta \) is real, it is an easy exercise to apply Varadhan’s lemma (1.1) and Cramer’s theorem concerning the large deviations of \( m_N \) in order to conclude that

\[
F_\beta = \lim_{N \to \infty} \frac{1}{N} \log |Z_{\beta,N}| = \begin{cases} 0, & \beta \in (-\infty, 1] \\ > 0, & \beta \in (1, \infty) \end{cases}
\]

(\( F_\beta \) is referred to as the Free Energy.) More refined analysis (see e.g. [3]) yields that for \( \beta \in \mathbb{R} \setminus \{1\} \),

\[
Z_{\beta,N} = A_\beta e^{NF_\beta}(1 + o(1)),
\]

where \( A_\beta > 0 \) is some constant that depends only on \( \beta \); this is due to the Gaussian nature of the fluctuations of \( \sqrt{N}(m_N - m^*(\beta)) \), where \( m^*(\beta) \) is the asymptotic magnetization, under the measure \( \exp(-\beta H_N(\sigma)) \prod_{i=1}^{N} \mu(d\sigma_i)/Z_{\beta,N} \). Also, \( m^*(\beta) = 0 \) for \( \beta \leq 1 \).

When \( \beta = (1 + \epsilon + iR) \in \mathbb{C} \), one expects to similarly have a separation between a region where \( F_\beta = 0 \) and \( F_\beta \neq 0 \). In particular, one predicts the existence of a critical curve \( \mathcal{C} \) in the complex plane, passing through \( 1 \), that divides the complex plane into a region where \( F_\beta = 0 \) and its complement where \( F_\beta \neq 0 \).

For symmetry reasons, it is enough to consider \( R \geq 0 \). Our first result describes a region where \( F_\beta \) vanishes.

**Theorem 1.1.** There exist constants \( c,c',\epsilon_0 > 0 \) so that, with \( \beta = 1 + \epsilon + iR \), if either \( 0 < \epsilon \leq \epsilon_0 \) and \( c\sqrt{\epsilon} \leq R \leq \frac{c'}{\sqrt{\epsilon}} \) or \( \epsilon < 0 \), then

\[
Z_{\beta,N} = \sqrt{\frac{\beta}{(\beta - \beta^2)(1 + o_c(R)(1))}}, \quad \lim_{N \to \infty} \frac{1}{N} \log |Z_{\beta,N}| = 0.
\]

**Remark 1.1.** One can make the constants \( c,c',\epsilon_0 \) explicit. Our proof gives \( \epsilon_0 = 1/9, c = \sqrt{20}, c' = \pi/\sqrt{32} \), but these are certainly not optimal constants.
Remark 1.2. It is possible to also treat the case of \(\epsilon = 0\), where one may observe a transition as function of \(R\): for \(R = 0\), it is standard, see [7, Theorem 2], that \(Z_{\beta,N}\) is asymptotic to a constant multiple of \(N^{1/4}\), while a local analysis near the saddle point 0 reveals that if \(R > 0\) is small then \(Z_{\beta,N}\) is asymptotic to an (\(R\)-dependent) constant, see Theorem 1.3 below.

Our next result shows that along a particular curve that is asymptotic to 1 + \(i\infty\) and to \(\infty + \pi i\), indeed \(F_\beta > 0\).

Theorem 1.2. For \(\beta = 1 + \epsilon + iR\) on the curve

\[
1 + \epsilon = \frac{R}{2\pi} \log \left(\frac{1 + \frac{\pi}{R}}{1 - \frac{\pi}{R}}\right), \quad \pi < R < \infty,
\]

we have, for some constant \(\tilde{A}_\beta\), that

\[
|Z_{\beta,N}| = Z_{\text{Re} \beta,N} \tilde{A}_\beta(1 + o(1)), \quad \lim_{N \to \infty} \frac{1}{N} \log |Z_{\beta,N}| > 0.
\]

Remark 1.3. The curve in Theorem 1.2 is asymptotic to \(c''/\sqrt{\epsilon}\) as \(\epsilon \to 0\); compare with Theorem 1.1, noting that \(c'' \neq c'\).

In a neighborhood of \(\beta = 1\), we actually can give a complete description of the transition away from \(F_\beta = 0\). Define the even function \(h_\beta(u) = u^2/2\beta - \log \cosh u\) for \(u \in \mathbb{C}, |u| < \pi/2\). With this definition we will see in Proposition 2.1 that

\[
Z_{\beta,N} = \sqrt{\frac{N}{2\pi \beta}} \int_{-\infty}^{\infty} e^{-Nh_\beta(u)} du.
\]

In Claim 5.1 below we show that for some \(c > 0\) small and \(0 < |\beta - 1| \leq c\), \(h'_\beta(u)\) has three zeros in a neighborhood of 0: 0, \(\pm u_{\beta}\).

Theorem 1.3. There exists \(c' \leq c\) such that for \(0 < |\beta - 1| \leq c'\)

1. \(Z_{\beta,N} = \frac{1}{\sqrt{1-\beta}} (1 + O\left(\frac{1}{N}\right))\) when \(\text{Re} \beta \leq 1\),

2. \(Z_{\beta,N} = \frac{1}{\sqrt{1-\beta}} (1 + O\left(\frac{1}{N}\right)) + 2 \sqrt{\frac{\beta}{\beta - \beta^2 + u_{\beta}^2}} e^{-Nh_\beta(u_{\beta})} (1 + O\left(\frac{1}{N}\right))\) when \(\text{Re} \beta \geq 1\),

and for any \(\delta > 0\) the implicit constants are uniform in \(\delta \leq |\beta - 1| \leq c'\).

See Figure 1 for a schematic illustration of our theorems. We remark that on the line \(\text{Re} \beta = 1\) we will see (as a consequence of Claim 1.4 below) that \(\text{Re} h_\beta(u_{\beta}) > 0\) (except for \(\beta = 1\)). In particular, the two statements in Theorem 1.3 coincide on that line.

In Theorem 1.3 an important role is played by those \(\beta\) with \(\text{Re} h_\beta(\pm u_{\beta}) = 0\). These are characterized by the following claim.

Claim 1.4. There exist \(c, C > 0\) and a smooth function \(\epsilon \mapsto R_0(\epsilon)\) on \([-c, c]\) such that \(|R_0(\epsilon) - \epsilon| \leq C\epsilon^2\) and the following holds for \(\beta = 1 + \epsilon + iR\):

- If \(|R| < |R_0(\epsilon)|\), then \(\text{Re} h_\beta(\pm u_{\beta}) < 0\),
Figure 1: Schematic illustration of our results. **Left**: $F_\beta = 0$ to the left of the purple curve on the left (Theorem 1.1); $F_\beta < 0$ on the red curve on the right (Theorem 1.2). We conjecture that the two phases are separated by a curve similar to the one schematically depicted in black. The three curves are asymptotic at infinity to the line Re $\beta = 1$. **Right**: The vicinity of the critical point $\beta = 1$: here $F_\beta > 0$ in the purple region on the right, and $F_\beta = 0$ in the blue region on the left.

- If $|R| = |R_0(\epsilon)|$, then $\text{Re} h_\beta(\pm u_\beta) = 0$,

- If $|R| > |R_0(\epsilon)|$, then $\text{Re} h_\beta(\pm u_\beta) > 0$.

For $c$ as above, define the critical curve $\Gamma = \{\beta = 1 + \epsilon + iR | 0 \leq \epsilon \leq c, R = \pm R_0(\epsilon)\}$. Theorem 1.3 allows us to describe the location of zeros of $Z_{\beta,N}$, and show that in a neighborhood of $\beta = 1$, they are close to the critical curve $\Gamma$. Define

$$\Psi_N(\beta) = \frac{1}{\sqrt{1-\beta}} + 2 \sqrt{\frac{\beta}{(\beta-\beta^2+u_\beta^2)^{1/2}}} e^{-Nh_\beta(u_\beta)}, \text{ Re } \beta \geq 1.$$ 

The zeros of $\Psi_N(\beta)$ near $\beta = 1$ lie near the critical curve $\Gamma$; we will show that the zeros of $Z_{\beta,N}$ are close to those zeros.

**Corollary 1.5.** For any $\delta > 0$ the following holds for $N \geq N_0(\delta)$. The zeros of $Z_{\beta,N}$ in $\delta < |\beta - 1| < c'$ lie in Re $\beta > 1$, and for any zero $\beta$ of $Z_{\beta,N}$ there exists a unique zero $\beta'$ of $\Psi_N(\beta)$ such that $|\beta - \beta'| < \frac{C_\delta}{N^2}$. Vice versa, for any zero $\beta'$ of $\Psi_N(\beta)$ with $\delta < |\beta' - 1| < c'$ there exists a unique zero $\beta$ of $Z_{\beta,N}$ with $|\beta - \beta'| < \frac{C_\delta}{N^2}$. In particular, the zeros of $Z_{\beta,N}$ lie within $C_\delta N^{-1}$ from the critical curve.

We also obtain information on the empirical measure of zeros of $Z_N$, in a neighborhood of the critical point $\beta = 1$. For $c' > 0$ small, introduce the scaled zero-counting measure

$$\mu_N = \frac{1}{N} \sum_{\beta: |\beta - 1| \leq c', Z_{\beta,N} = 0} \delta_\beta.$$  

(1.7)
Define a positive measure $\mu$ on $\mathbb{C}$, supported on $\Gamma' := \Gamma \cap \{|\beta - 1| \leq \epsilon'\}$ as follows: For a segment $I$ of $\Gamma$ connecting $a_j = 1 + \epsilon_j + iR_0(\epsilon_j) \in \Gamma'$, $j = 1, 2$, with $\epsilon_2 > \epsilon_1 \geq 0$, set

$$
\mu(I) = \frac{1}{2\pi} (\text{Im} \, h_{a_2}(u_{a_2}) - \text{Im} \, h_{a_1}(u_{a_1})),
$$

and extend $\mu$ by symmetry to the lower half plane. One checks that $\mu$ is a finite positive measure on $\mathbb{C}$.

**Corollary 1.6.** $\mu_N \to_{N \to \infty} \mu$ in the weak topology for positive measures on $\mathbb{C}$.

The results above do not completely characterize the phase diagram of the Curie–Weiss model. In Section 6, we discuss this point and present a conjecture for the critical curve separating the region where the free energy vanishes asymptotically from that where it is strictly positive.

## 2 Integral representation and preliminaries

The proofs of all of the theorems are based on the saddle-point analysis of the following integral representation.

**Proposition 2.1.** If $\text{Re} \, \beta > 0$, then

$$
Z_{\beta,N} = \left(\frac{\beta N}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \exp(-N f_\beta(u))du = \left(\frac{N}{2\pi \beta}\right)^{1/2} \int_{-\infty}^{\infty} \exp(-N h_\beta(u))du,
$$

where $f_\beta(u) = \frac{\beta u^2}{2} - \log(\cosh(\beta u))$, $h_\beta(u) = \frac{u^2}{2} - \log \cosh u$, and the branch of the square root is chosen so that $\sqrt{1} = 1$.

**Proof.** Let $X_1, X_2, \ldots, X_N$ be independent identically distributed Bernoulli random variables: $\mathbb{P}\{X_j = 1\} = \mathbb{P}\{X_j = -1\} = \frac{1}{2}$, and let $\beta \in \mathbb{C}$. Then, using $\mathbb{E}$ to denote expectation with respect to these random variables, we have

$$
Z_{\beta,N} = \mathbb{E} \exp \left( \frac{\beta N}{2} \left( \frac{1}{N} \sum_{j=1}^{N} X_j \right)^2 \right) = \mathbb{E} \int \exp \left( -\frac{u^2}{2} + u\sqrt{N} \sum_{j=1}^{N} X_j \right) \frac{du}{\sqrt{2\pi}}
$$

$$
= \sqrt{\frac{\beta N}{2\pi}} \int d\tilde{u} \exp \left( -\frac{\beta N}{2} \tilde{u}^2 \right) \mathbb{E} \exp \left( \beta \tilde{u} \sum_{j=1}^{N} X_j \right),
$$

where the second equality uses the Hubbard-Stratonovich transformation and the last uses the change of variables $\tilde{u} = \frac{u}{\sqrt{\beta N}}$. Since for any $a$ we have $\mathbb{E} \exp(aX) = \cosh a$ and using the assumption that $X_j$ are i.i.d random variables, we obtain

$$
\mathbb{E} \exp \left( \beta \tilde{u} \sum_{j=1}^{N} X_j \right) = \prod_{j=1}^{N} \mathbb{E} \exp \left( \beta \tilde{u} X_j \right) = (\mathbb{E} \exp (\beta \tilde{u} X_j))^N = (\cosh(\beta \tilde{u}))^N.
$$

Combining the last two displays gives

$$
Z_{\beta,N} = \sqrt{\frac{\beta N}{2\pi}} \int d\tilde{u} \exp \left( -\frac{\beta N}{2} \tilde{u}^2 + N \log \cosh(\beta \tilde{u}) \right) = \sqrt{\frac{\beta N}{2\pi}} \int \exp(-N f_\beta(u))du.
$$

$\square$
3 Proof of Theorem 1.1

The proof of Theorem 1.1 for \( \epsilon < 0 \) follows from known asymptotics for the Curie–Weiss model. Indeed, for such \( \epsilon \) and with \( R = 0 \) and \( \sigma_\epsilon^2 = 1/(1 - \epsilon) \), we have by [7, Theorem 2] that \( Z_{1+\epsilon,N} \approx \sigma_\epsilon \) and, under the measure \( e^{-(1+\epsilon)H_N(\sigma)/Z_{1+\epsilon,N}} \), we have by [3] that \( \sqrt{N}m_N \) converges in distribution to a centered Gaussian random variable of variance \( \sigma_\epsilon^2 := -1/\epsilon \).

We then obtain that with \( \epsilon < 0 \),

\[
Z_{\beta,N} \approx \frac{1}{\sqrt{2\pi}} \int e^{-(1-\epsilon)u^2/2-iR u^2/2} du,
\]

which gives the claim.

The proof in case \( \epsilon > 0 \) follows a saddle-point analysis of the integral representation from Proposition 2.1. Throughout, \( C_i \) denote constants that may depend on \( \epsilon \) and \( R \) but not on anything else. The following preliminary claims play an important role in the analysis.

**Claim 3.1.** For any \( \epsilon \leq \frac{1}{9} \), \( u \leq \sqrt{8\epsilon}, \sqrt{24\epsilon} \leq R \leq \frac{\pi}{\sqrt{128\epsilon}} \),

\[
\text{Re} \ f_\beta(u) = (1 + \epsilon) \frac{u^2}{2} - \frac{1}{2} \log(\cosh^2(u(1 + \epsilon)) - \sin^2(uR)) \geq \epsilon \frac{u^2}{2}.
\]

**Proof.** By Taylor expansion,

\[
cosh t \leq 1 + \frac{t^2}{2} + \cosh t \frac{t^4}{24} \leq 1 + \frac{t^2}{2} + \frac{t^4}{12},
\]

where the last inequality used that \( t \leq 3/2 \) and therefore \( \cosh t \leq 2.4 \). Using again \( t < 3/2 \) we obtain

\[
\cosh^2 t \leq 1 + t^2 + \frac{3}{4} t^4.
\]

From the assumptions we get that \( (1 + \epsilon)u \leq (1 + \epsilon)\sqrt{8\epsilon} < 3/2 \). Therefore, using again \( \epsilon < 1/9 \),

\[
\cosh^2((1 + \epsilon)u) \leq 1 + u^2 + \epsilon u^2 \left(2 + \epsilon + \frac{3u^2(1 + \epsilon)^4}{4\epsilon}\right) \leq 1 + u^2 + 12\epsilon u^2.
\]

For \( Ru \leq \frac{\pi}{4} \) we have \( \sin^2(Ru) \geq R^2 u^2/2 \), hence

\[
\cosh^2((1 + \epsilon)u) - \sin^2(Ru) \leq 1 + u^2 + 12\epsilon u^2 - \frac{R^2 u^2}{2}.
\]

Since \( R \geq \sqrt{24\epsilon} \), we have \( R^2 u^2/2 \geq 12\epsilon u^2 \) and therefore

\[
\cosh^2((1 + \epsilon)u) - \sin^2(Ru) \leq 1 + u^2 \leq e^{u^2},
\]

and therefore

\[
\text{Re} \ f_\beta(u) = (1 + \epsilon) \frac{u^2}{2} - \frac{1}{2} \log(\cosh^2(u(1 + \epsilon)) - \sin^2(uR)) \geq \epsilon \frac{u^2}{2}.
\]

\[\Box\]
The next claim handles larger values of the argument \( u \).

**Claim 3.2.** Let \( \epsilon < 1/9 \). For any \( t \geq (1 + \epsilon)\sqrt{8\epsilon} \),

\[
\cosh t \leq \exp \left( (1 - \epsilon) \frac{t^2}{2} \right). \tag{3.1}
\]

In particular, for \( u \geq \sqrt{8\epsilon} \),

\[
\Re f_\beta(u) \geq (1 + \epsilon) \frac{u^2}{2} - \log(\cosh(u(1 + \epsilon))) \geq \frac{\epsilon^2 u^2}{2}. \tag{3.2}
\]

**Proof.** By Taylor expansion we have

\[
\cosh t = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} = 1 + \frac{t^2}{2}(1 - \epsilon) + \left[ \frac{\epsilon}{2} \frac{t^4}{24} + \sum_{k \geq 3} \frac{t^{2k}}{(2k)!} \right],
\]

and

\[
\exp \left( (1 - \epsilon) \frac{t^2}{2} \right) = 1 + \frac{t^2}{2}(1 - \epsilon) + \frac{t^4}{8}(1 - \epsilon)^2 + \sum_{k \geq 3} \frac{t^{2k}(1 - \epsilon)^k}{2^k k!}.
\]

For \( k \geq 3 \) we have, if \( \epsilon \leq 1/2 \),

\[
\frac{(2k)!}{2^k k!} = \frac{(k + 1) \cdot \cdots (2k)}{2^k} \geq 2^k \geq \frac{1}{(1 - \epsilon)^k}.
\]

Therefore

\[
\sum_{k \geq 3} \frac{t^{2k}}{(2k)!} \leq \sum_{k \geq 3} \frac{t^{2k}(1 - \epsilon)^k}{2^k k!},
\]

and, since \( \epsilon \leq \frac{1}{9} \) and \( t \geq (1 + \epsilon)\sqrt{8\epsilon} \),

\[
\frac{t^4}{8}(1 - \epsilon)^2 - \frac{t^4}{24} - \frac{\epsilon t^2}{2} \geq \frac{t^4}{8} \left( (1 - \epsilon)^2 - \frac{1}{3} - \frac{1}{2(1 + \epsilon)^2} \right) \geq 0.
\]

This completes the proof of (3.1).

To see (3.2), take \( t = (1 + \epsilon)u \), which satisfies the assumptions leading to (3.1). Then,

\[
\cosh((1 + \epsilon)u) \leq \exp \left( (1 - \epsilon)(1 + \epsilon)^2 \frac{u^2}{2} \right) \leq \exp \left( \frac{(1 + \epsilon)(1 - \epsilon)^2 u^2}{2} \right). \tag{3.3}
\]

Hence, we have, using the monotonicity of the logarithm and (3.3)

\[
(1 + \epsilon) \frac{u^2}{2} - \log(\cosh(u(1 + \epsilon))) \geq (1 + \epsilon) \frac{u^2}{2} - \frac{(1 + \epsilon)(1 - \epsilon^2)u^2}{2} \geq \frac{\epsilon^2 u^2}{2}.
\]
We continue with the proof of Theorem 1.1, considering the regime $\epsilon > 0$, and $R$ as in the statement of Claim 3.1. In view of Proposition 2.1, we write
\[
\int_{-\infty}^{\infty} e^{-N f_\beta(u)} du = \left[ \int_{-\infty}^{-\delta} + \int_{-\delta}^{\infty} + \int_{\delta}^{\infty} \right] e^{-N f_\beta(u)} du = I_1' + I_1 + I_2 = 2I_1 + I_2, \tag{3.4}
\]
where the last equality follows from the symmetry. Note that $f_\beta(0) = f'_\beta(0) = 0$. Hence, $u = 0$ is a saddle point. We will show below that the main contribution to the integral comes from a neighborhood of this saddle point. We will choose $\delta = N^{-2/5}$ so that $N\delta^3 \to 0$ and $\delta \sqrt{N} \to \infty$ as $N \to \infty$.

We begin by estimating $I_1$. Using Claim 3.1, we have
\[
I_1 = \int_{\delta}^{\infty} e^{-N f_\beta(u)} du = \left[ \int_{\delta}^{\sqrt{8\epsilon}} + \int_{\sqrt{8\epsilon}}^{\infty} \right] e^{-N f_\beta(u)} du = W_1 + W_2.
\]

Using Claim 3.1, we have
\[
|W_1| \leq \int_{\delta}^{\sqrt{8\epsilon}} e^{-N \text{Re} f_\beta(u)} du \leq \int_{\delta}^{\sqrt{8\epsilon}} e^{-N \frac{u^2}{2}} du \leq e^{-N \frac{\epsilon^2}{2}} (\sqrt{8\epsilon} - \delta) \leq e^{-cN^{1/5}}, \tag{3.5}
\]
for some constant $c > 0$.

To estimate $W_2$, we use (3.1) of Claim 3.2 and obtain
\[
|W_2| \leq \int_{\sqrt{8\epsilon}}^{\infty} e^{-N \text{Re} f_\beta(u)} du \leq \int_{\sqrt{8\epsilon}}^{\infty} e^{-N \frac{u^2}{2}} du \leq e^{-CN}, \tag{3.6}
\]
where $C = C(\epsilon) > 0$ is some constant. Combining (3.5) and (3.6) we get
\[
|I_1| \leq e^{-CN^{1/5}} + e^{-CN} \leq e^{-\tilde{C}N^{1/5}}. \tag{3.7}
\]

We turn to estimating $I_2$. Denote by
\[
P_2(u) = (\beta - \beta^2) \frac{u^2}{2}
\]
the Taylor approximation of $f_\beta(u)$ to second order. Note that, by the assumptions on $R$
\[
\text{Re}(\beta - \beta^2) = -\epsilon - \epsilon^2 + R^2 \geq -2\epsilon + R^2 > 0
\]
For $|u| < \delta$ for our choice of $\delta$ we have $O(u^3) = O(N^{-6/5})$. We get
\[
\int_{-\delta}^{\delta} e^{-N f_\beta(u)} du = \int_{-\delta}^{\delta} e^{-NP_2(u)} e^{-N (f_\beta(u) - P_2(u))} du \tag{3.8}
\]
\[
= \int_{-\delta}^{\delta} e^{-NP_2(u)} du + \int_{-\delta}^{\delta} e^{-NP_2(u)} (e^{-N (f_\beta(u) - P_2(u))} - 1) du.
\]
Since for any $|x| < 1/2$: $|e^x - 1| < 2x$, we obtain
\[
|e^{-N(f_\beta(u) - P_2(u))} - 1| \leq C_1 N |f_\beta(u) - P_2(u)| \leq C_2 N^{-1/5}, \tag{3.9}
\]
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where the constants depend only on $\beta$. Combining (3.8) and (3.9) we obtain

$$\left| \int_{-\delta}^{\delta} e^{-Nf_{\beta}(u)} du - \int_{-\delta}^{\delta} e^{-NP_2(u)} du \right| \leq C_2N^{-1/5} \int_{-\delta}^{\delta} e^{-N\text{Re}\,P_2(u)} du \leq C_3N^{-1/5} \frac{1}{\sqrt{N}}, \quad (3.10)$$

Now we have the following inequality

$$\left| \int_{-\infty}^{\infty} e^{-NP_2(u)} du - \int_{-\delta}^{\delta} e^{-NP_2(u)} du \right| \leq 2 \int_{\delta}^{\infty} e^{-N\text{Re}\,P_2(u)} du \leq 2 \int_{\delta}^{\infty} e^{-N\text{Re}(\beta - \beta^2)\frac{u^2}{2}} du = \frac{1}{C_4N^{3/5}} e^{-C_5N^{1/5}}. \quad (3.11)$$

Since

$$\int_{-\infty}^{\infty} e^{-NP_2(u)} du = \sqrt{\frac{2\pi}{N(\beta - \beta^2)}},$$

we obtain combining (3.10) and (3.11) that

$$\left| I_2 - \sqrt{\frac{2\pi}{N(\beta - \beta^2)}} \right| \leq C_6 \frac{N^{7/10}}{N}.\quad (3.12)$$

Using the estimate (3.7) on $|I_1|$ we obtain

$$\left| \int_{-\infty}^{\infty} e^{-NF_{\beta}(u)} du - \sqrt{\frac{2\pi}{N(\beta - \beta^2)}} \right| \leq C_7 \frac{N^{7/10}}{N}.\quad (3.13)$$

This concludes the proof of the theorem.

**4 Proof of Theorem 1.2**

*Proof.* Observe that

$$f_{1+\epsilon}(u) = (1 + \epsilon) \frac{u^2}{2} - \log(\cosh((1 + \epsilon)u)), \quad f'_{1+\epsilon}(u) = (1 + \epsilon) (u - \tanh((1 + \epsilon)u)).$$

By the assumption (1.5) we get

$$f'_{1+\epsilon} \left( \pm \frac{\pi}{R} \right) = (1 + \epsilon) \left( \pm \frac{\pi}{R} - \tanh \left( \pm \frac{\pi}{R} \frac{R}{2\pi} \log \left( \frac{1 + \frac{\pi}{R}}{1 - \frac{\pi}{R}} \right) \right) \right) = 0, \text{ and } f'_{1+\epsilon}(0) = 0.$$

These are the only real zeros of $f'_{1+\epsilon}$ and $f_{1+\epsilon}(\pm \frac{\pi}{R}) < 0$, in particular the minimum of $f_{1+\epsilon}(u)$ over $\mathbb{R}$ is achieved at $u = \pm \pi/R$. We claim that the same is true of

$$\text{Re} \ f_\beta(u) = (1 + \epsilon) \frac{u^2}{2} - \frac{1}{2} \log(\cosh^2((1 + \epsilon)u) - \sin^2(Ru)).$$

Indeed, by the monotonicity of the logarithm, for any $u \in \mathbb{R}$ we get $\text{Re} \ f_\beta(u) \geq f_{1+\epsilon}(u)$. On the other hand, $\text{Re} \ f_\beta(\pm \frac{\pi}{R}) = f_{1+\epsilon}(\pm \frac{\pi}{R}) < 0$. This yields that the minimum of $\text{Re} \ f_\beta(u)$ over
\( \mathbb{R} \) is achieved at \( \pm \pi/R \), as claimed. We note in passing that \( f_\beta'(\pm \pi/R) = 0 \), i.e., the points \( \pm \pi/R \), which minimize \( \text{Re} \ f_\beta(u) \) over \( u \in \mathbb{R} \), are in fact saddle points.

Now we estimate the integral as before. Let \( \delta = N^{-2/5} \) as in the proof of Theorem 1.1. Similarly to the proof of Theorem 1.1, we divide the integral into pieces and use the symmetry to obtain

\[
\int_{-\infty}^{\infty} e^{-Nf_\beta(u)} du = \left[ 2 \int_{0}^{\pi/\delta} + 2 \right] + 2 \right] e^{-Nf_\beta(u)} du = 2I_1 + 2I_2 + 2I_3.
\]

Also let

\[
\int_{-\infty}^{\infty} e^{-Nf_{1+\epsilon}(u)} du = \left[ 2 \int_{0}^{\pi/\delta} + 2 \right] + 2 \right] e^{-Nf_{1+\epsilon}(u)} du = 2\hat{I}_1 + 2\hat{I}_2 + 2\hat{I}_3.
\]

Then, for any \( j = 1, 2, 3 \) we get \( |I_j| \leq \hat{I}_j \).

We will show that

\[
\hat{I}_1, \hat{I}_3 \leq Ce^{-CN^{1/5}} e^{-Nf_{1+\epsilon}(\pi/N)}, \quad (4.1)
\]

\[
|I_2| \sqrt{|f''(\pi/N/R)|(1 + o(1))} = \hat{I}_2 \sqrt{f''(\pi/N/R)(1 + o(1))} = \sqrt{2\pi N} e^{-Nf_{1+\epsilon}(\pi/N)}, \quad (4.2)
\]

and this, together with the fact that \( \text{Re} \ f_\beta(\pi/R) = f_{1+\epsilon}(\pi/R) < 0 \), will prove the theorem and also (1.4).

We start from the estimate of \( \hat{I}_3 \). We write

\[
\hat{I}_3 = \int_{\pi/\delta}^{\infty} e^{-Nf_{1+\epsilon}(u)} du = \left[ \int_{\pi/\delta}^{3} + \int_{3}^{\infty} \right] e^{-Nf_{1+\epsilon}(u)} du =: W_1 + W_2.
\]

To estimate \( W_2 \), note that since \( \cosh(x) \leq e^x \) for \( x \) real, we have that for \( u \geq 3 \),

\[
f_{1+\epsilon}(u) = (1 + \epsilon)u^2/2 - \log \cosh((1 + \epsilon)u) \geq (1 + \epsilon)u(u/2 - 1) \geq (1 + \epsilon)u/2.
\]

Thus,

\[
W_2 \leq e^{-3N/2}.
\]

(4.3)

To estimate \( W_1 \), note that for any \( 3 \geq u > \pi/R + \delta \) the function \( u \mapsto f_{1+\epsilon}(u) \) is increasing and therefore \( f_{1+\epsilon}(\pi/R + \delta) \geq f_{1+\epsilon}(\pi/R) + c\delta^2 \). Thus,

\[
W_1 \leq Ce^{-N(f_{1+\epsilon}(\pi/R) + c\delta^2)} = Ce^{-Nf_{1+\epsilon}(\pi/N)} e^{-CN^{1/5}}.
\]

(4.4)

Combining (4.3) and (4.4) yields (4.1) for \( \hat{I}_3 \). On the other hand, since \( f_{1+\epsilon}(u) \) is decreasing for any \( 0 \leq u \leq \pi/R - \delta \) with the minimum at \( \pi/R - \delta \), in the same way we obtain

\[
\hat{I}_1 \leq Ce^{-Nf_{1+\epsilon}(\pi/N)} e^{-CN^{1/5}},
\]

which proves (4.1) for \( \hat{I}_1 \).
We turn to the proof of (4.2), which follows a saddle point analysis similar to that done in the proof of Theorem 1.1. Recall that $\pi/R$ is a saddle point of $f_\beta$ and let $\tilde{P}_{2,\beta}$ denote its second order Taylor approximation there, i.e.

$$\tilde{P}_{2,\beta}(u) := f_\beta \left( \frac{\pi}{R} \right) + \left( \beta - \frac{\beta^2}{\cosh^2((1 + \epsilon)\frac{\pi}{R})} \right) \frac{(u - \frac{\pi}{R})^2}{2}. $$

As in (3.8) and (3.9), replacing the domain of integration to $[\pi/R - \delta, \pi/R + \delta]$ and $P_2$ by $\tilde{P}_{2,\beta}$, we obtain the following analog of (3.10):

$$\left| \int_{\pi/R - \delta}^{\pi/R + \delta} e^{-Nf_\beta(u)} du - \int_{\pi/R - \delta}^{\pi/R + \delta} e^{-N\tilde{P}_{2,\beta}(u)} du \right| \leq \exp \left( -N \text{Re} f_\beta \left( \frac{\pi}{R} \right) \right) C_1 N^{-1/5} \frac{1}{\sqrt{N}}. \quad (4.6)$$

Similarly to (3.11), we also have

$$\left| \int_{-\infty}^{\pi/R + \delta} e^{-N\tilde{P}_{2,\beta}(u)} du - \int_{-\infty}^{\pi/R - \delta} e^{-N\tilde{P}_{2,\beta}(u)} du \right| \leq \exp \left( -N \text{Re} f_\beta \left( \frac{\pi}{R} \right) \right) \frac{1}{C_2 N^{3/5}} e^{-C_3 N^{1/5}}. \quad (4.7)$$

Finally, by Gaussian integration we have

$$\int_{-\infty}^{\infty} e^{-N\tilde{P}_{2,\beta}} du = \exp \left( -N f_\beta \left( \frac{\pi}{R} \right) \right) \sqrt{\frac{2\pi}{Nf_\beta'' \left( \frac{\pi}{R} \right)}}. $$

Combining the last display with (4.6) and (4.7) gives (4.2) for $I_2$. The analysis of $\hat{I}_2$ is identical, taking $\beta = 1 + \epsilon$ in $\tilde{P}_{2,\beta}$. \hfill \Box

5 Proof of Theorem 1.3

5.1 Construction of the saddle points for $h_\beta(u)$

We begin with the analysis of the critical points of $h_\beta$. Let $K$ be a large constant (the choice of $K = 241$ will work). Define the discs in the complex plane:

1. $D_0(\beta) = \{ |u| \leq K|\beta - 1|^{3/2} \}$,

2. $D_+(\beta) = \{ |u - \sqrt{3 \left( 1 - \frac{1}{\beta} \right)}| \leq K|\beta - 1|^{3/2} \}$,

3. $D_-(\beta) = \{ |u + \sqrt{3 \left( 1 - \frac{1}{\beta} \right)}| \leq K|\beta - 1|^{3/2} \}$,

where the branch of the square-root is chosen so that $\text{Im} \sqrt{3 \left( 1 - \frac{1}{\beta} \right)}$ is in the upper half plane if $\beta$ is in the upper half plane. For sufficiently small $c$ and for $0 < |\beta - 1| < c$, these circles are disjoint.
Claim 5.1. For any $\beta$ such that $0 < |\beta - 1| < c$, the function $h'_\beta$ has exactly three zeros in $|u| \leq c_1 = 10\sqrt{c}$, one in each of the discs: $0 \in D_0$, $u_\beta \in D_+$, $-u_\beta \in D_-$.  

Proof. Introduce the Taylor approximation of $h_\beta(\cdot)$ up to fourth order,

$$P_4(u) = \left(\frac{1}{\beta} - 1\right) \frac{u^2}{2} + \frac{u^4}{12}.$$  

Then,

$$P'_4(u) = \left(\frac{1}{\beta} - 1\right) u + \frac{u^3}{3},$$

and $P'_4(u)$ has exactly three zeros $u = 0$, $u_\pm = \pm \sqrt[3]{3} \left(1 - \frac{1}{\beta}\right)$. We will show that on the boundary of each disc, namely on $\partial D_0 \cup \partial D_+ \cup \partial D_-,$

$$|P'_4(u) - h'_\beta(u)| < |P'_4(u)|,$$  

which will show by Rouché’s theorem that $h'_\beta(\cdot)$ has a unique zero in each disc.

We check (5.1) on $\partial D_+$, the other two case are similar. Since $h_\beta(u)$ is even, all odd coefficients in its Taylor approximation vanish. Next, for any $u \in \mathbb{C}$ with $|u| \leq \frac{\pi}{4}$ we get $|\tanh u| \leq 1$ and therefore, repeatedly using that $\tanh'(u) = 1 - \tanh^2(u),$  

$$|h^{(6)}_\beta(u)| = | - 16 + 136 \tanh^2 u - 240 \tanh^4 u + 120 \tanh^6 u| \leq 512,$$

thence

$$|h'_\beta(u) - P'_4(u)| \leq \frac{512}{5!} |u|^5 \leq 5|u|^5. \quad (5.2)$$

On the boundary $\partial D_+$ we have by a direct computation

$$|P'_4(u)| = \frac{1}{3} \left|u + \sqrt{3} \left(1 - \frac{1}{\beta}\right)\right| \left|u - \sqrt{3} \left(1 - \frac{1}{\beta}\right)\right| |u|$$

$$\geq \frac{1}{3} K |\beta - 1|^{3/2} \left(\sqrt{12} \left|1 - \frac{1}{\beta}\right| - K |\beta - 1|^{3/2}\right) \left(\sqrt{3} \left|1 - \frac{1}{\beta}\right| - K |\beta - 1|^{3/2}\right).$$

Choosing $c > 0$ small enough, we obtain that if $K |\beta - 1| \leq c$ then

$$|P'_4(u)| \geq \frac{1}{3} K |\beta - 1|^{3/2} 3 |\beta - 1|^{1/2} |\beta - 1|^{1/2} = K |\beta - 1|^{5/2}.$$  

On the other hand, combining the estimate (5.2) and that we are on the boundary of $D_+$ we obtain

$$|h'_\beta(u) - P'_4(u)| \leq 5|u|^5 \leq 5 \left[3 \left|1 - \frac{1}{\beta}\right| + K |\beta - 1|^{3/2}\right]^5 \leq 5 \cdot 2^5 \cdot \frac{3}{2} |\beta - 1|^{5/2} < K |\beta - 1|^{5/2},$$

since we assumed $K > 240$. Therefore, Rouché’s theorem applies and $D_+$ contains exactly one zero of $h'_\beta$.

To see that $h'_\beta$ has no more zeros in $|u| \leq c_1$, note that for such $u$  

$$|h'_\beta(u) - P'_4(u)| \leq 5|u|^5 \leq 5c_1^5, \quad |P'_4(u)| \geq Cc_1^3,$$

and we have the needed estimate by adjusting the constant $c_1$ such that $C > 5c_1^2$. Therefore, by an application of Rouché’s theorem we obtain the claim. \hfill \square
5.2 Proof of Theorem [1.3]

Since $h_{\beta}$ is even, we write $h_{\beta}(u) = \tilde{h}_{\beta}(u^2)$. Note that $\tilde{h}_{1}(0) = \tilde{h}'_{1}(0) = 0$, while $\tilde{h}''_{1}(0) = 1/6$.

We use a change of variables provided by a theorem of Levinson, which reduces $\tilde{h}_{\beta}$ to a polynomial of degree 2. Indeed, by Levinson’s theorem \cite{4} (see also \cite[Theorem 1]{6}, after correcting for typos), there exist $\rho, c' > 0$ and analytic functions

- $V : D(0, \rho) \times D(1, c') \rightarrow \mathbb{C}$, $V(0, \beta) = 0$,
- $U : \{(v, \beta) | \beta \in D(1, c'), v \in V(D(0, \rho), \beta)\} \rightarrow \mathbb{C}$,

such that, for $\beta \in D(1, c')$,

\begin{align*}
U(V(z, \beta), \beta) &= z \text{ on } D(0, \rho), \text{ that is, } V \text{ is the inverse of } U, \quad (5.3) \\
V &\text{ is one to one on } D(0, \rho), \text{ and } 0 < \frac{1}{C} \leq |V'(z, \beta)| \leq C < \infty, \quad (5.4) \\
\tilde{h}_{\beta}(V(z, \beta)) &= \frac{z^2}{2} - \xi(\beta)z, \text{ with } \xi(\beta) \text{ analytic on } D(1, c), \xi(1) = 0, \quad (5.5)
\end{align*}

where, for any function $f = f(z, \beta)$, we write $f'(z, \beta) = \frac{\partial}{\partial z} f(z, \beta)$. From (5.5) we obtain

\begin{align*}
(\tilde{h}_{\beta}(V(z, \beta)))' &= z - \xi(\beta), \quad (5.6)
\end{align*}

and therefore, since $|V'(z, \beta)| \neq 0$, one deduces that $\tilde{h}'_{\beta}(V(\xi(\beta), \beta)) = 0$. In particular, $V(\xi(\beta), \beta)$ is a critical point of $\tilde{h}_{\beta}$. Since for $0 < |\beta - 1| < c$ the point $u_{\beta}^2$ is the unique critical point of $\tilde{h}_{\beta}$ in a neighborhood of zero, we obtain that

\begin{align*}
V(\xi(\beta), \beta) &= u_{\beta}^2, \quad (5.7) \\
\xi(\beta) &= \sqrt{-2h_{\beta}(u_{\beta})}. \quad (5.8)
\end{align*}

Using (5.6) once again and L'Hôpital’s Rule, we obtain

\begin{align*}
V'(\xi(\beta), \beta) &= \sqrt{\frac{1}{\tilde{h}''_{\beta}(V(\xi(\beta), \beta))}} = \frac{2\beta u_{\beta}}{\sqrt{\beta - \beta^2 + u_{\beta}^2}}, \quad (5.9)
\end{align*}

where the last equality follows since, by a direct computation and using (5.7), we obtain

\begin{align*}
\tilde{h}''_{\beta}(V(\xi(\beta), \beta)) &= \frac{1}{4u_{\beta}^3} \left[ \frac{1}{\beta} - 1 + \tanh^2 u_{\beta} \right] = \frac{1}{4u_{\beta}^3} \left[ \frac{1}{\beta} - 1 + \frac{u_{\beta}^2}{\beta^2} \right],
\end{align*}

and the last equality follows since $u_{\beta}$ is the critical point of $h_{\beta}(u)$ obeying $\tanh u_{\beta} = \frac{u_{\beta}}{\beta}$.

Repeating this computation at $u = 0$ we obtain

\begin{align*}
V'(0, \beta) &= \frac{2\beta \xi(\beta)}{\beta - 1}. \quad (5.10)
\end{align*}
We need to estimate the following integral
\[
\int_{-\infty}^{\infty} e^{-Nh_\beta(u)} du = 2 \int_{0}^{\infty} e^{-Nh_\beta(u)} du. \tag{5.11}
\]
Let \(\nu = |\beta - 1|^{0.1}\) and consider the following change of contour.
\[
T_1 = [0, \sqrt{V(\nu, \beta)}], \quad T_2 = [\sqrt{V(\nu, \beta)}, 6^{1/4} \sqrt{\nu}], \quad T_3 = [6^{1/4} \sqrt{\nu}, \infty],
\]
where \(V(\nu, \beta) \in \mathbb{C}\) and the square-root taken so that \(\text{Im}(V(\nu, \beta)) > 0\) if \(\text{Im} \beta > 0\). (Because \(c\) is small, the region contained between \(T_1 \cup T_2 \cup T_3\) and \(\mathbb{R}_+\) does not contain any pole of \(h_\beta\).) Now we rewrite the integral (5.11) as follows
\[
2 \int_{0}^{\infty} e^{-Nh_\beta(u)} du = 2 \left[ \int_{T_1} + \int_{T_2} + \int_{T_3} \right] e^{-Nh_\beta(u)} du = I + E' + E. \tag{5.12}
\]
The reason for this change of contour is that in order to estimate the term \(I\), we would like to perform a change of variables given by Levinson’s theorem, and we would like for the obtained contour (as a result of this change) to be an interval \([0, \nu] \subset \mathbb{R}\).
First, we estimate the error term \(E'\). We perform the change of variables \(u = \sqrt{v}\) and obtain
\[
E' = 2 \int_{T_2} e^{-Nh_\beta(u)} du = \int_{\tilde{T}_2} e^{-\tilde{N}h_\beta(v)} \frac{dv}{\sqrt{v}},
\]
where \(\tilde{T}_2\) is the push forward of \(T_2\) by the change of variables, and has endpoints \(V(\nu, \beta), \sqrt{6}\nu\).
Now we perform another change of variables \(v = V(z, \beta)\) with \(V(z, \beta)\) given by Levinson’s theorem, and obtain, after another contour modification,
\[
\int_{\tilde{T}_2} e^{-\tilde{N}h_\beta(v)} \frac{dv}{\sqrt{v}} = \int_{\nu}^{U(\sqrt{6}\nu, \beta)} e^{-N(z^2/2 - \xi(\beta)z)} \frac{V'(z, \beta)}{\sqrt{V(z, \beta)}} dz.
\]
Since \(U'(0, \beta) = \frac{1}{V'(0, \beta)}\), using the expression (5.10) for \(V'(0, \beta)\) we get
\[
U(\sqrt{6}\nu, \beta) = U'(0, \beta)\sqrt{6}\nu + O(\nu^2) = \frac{\beta - 1}{2\beta \xi'(\beta)} \sqrt{6}\nu + O(\nu^2)
\]
\[
= \frac{1}{2\xi'(1)} (1 + O(\beta - 1)) \sqrt{6}\nu + O(\nu^2) = \frac{\sqrt{6}\nu}{2\xi'(1)} (1 + O(\nu)) = \frac{\sqrt{6}\nu}{2\sqrt{3/2}} (1 + O(\nu))
\]
\[
= \nu (1 + O(\nu)),
\]
where we used that \(\xi'(1) = \sqrt{\frac{3}{2}}\), see the computation (5.29) below. Therefore, for \(z\) in the segment \([\nu, U(\sqrt{6}\nu, \beta)]\) we obtain
\[
\text{Re} \left( \frac{z^2}{2} - \xi(\beta)z \right) = \frac{\nu^2}{2} + O(\nu^2) \geq \frac{\nu^2}{4}.
\]
Since on this segment \(|V'(z, \beta)| \neq 0\) and \(c \leq |V(z, \beta)| \leq C\), we get
\[
|E'| \leq \left| \int_{\nu}^{U(\sqrt{6}\nu, \beta)} e^{-N(z^2/2 - \xi(\beta)z)} \frac{V'(z, \beta)}{\sqrt{V(z, \beta)}} dz \right| \leq Ce^{-CN\nu^2}. \tag{5.13}
\]
Next, we estimate the error term $E$. Note that for some small $c > 0$

$$\text{Re}\frac{1}{\beta} = \frac{1 + \epsilon}{(1 + \epsilon)^2 + R^2} = 1 - \frac{\epsilon + \epsilon^2 + R^2}{(1 + \epsilon)^2 + R^2} \geq 1 - c|\beta - 1|.$$ 

Set $b = \text{Re}(1/\beta)$. Then, $h'_b(u) = bu - \tanh u$ and it vanishes on $(0, \infty)$ at a single point $u^*$ which is of order $|\beta - 1|^{1/2}$, while $h'_b(u) \to u_\to \infty \infty$. Hence, $h'_b(u) > 0$ for any $u > u^*$, in particular, this holds for any $u \geq 6^{1/4}\sqrt{\nu} > u^*$. Note that $h_b(6^{1/4}\sqrt{\nu}) \geq cv^2 > 0$ and this is the minimum of $h_b(u)$ on the interval $[6^{1/4}\sqrt{\nu}, C]$ for any $C > 6^{1/4}\sqrt{\nu}$. Since $\lim_{u \to \infty} \frac{h_b(u)}{u^*} = \hat{c} > 1/2$, there exists $\hat{C}$ such that $h_b(u) > u^2/2$ for any $u > \hat{C}$. Therefore, we obtain

$$|E| \leq \left[ \int_{6^{1/4}\sqrt{\nu}}^{\hat{C}} + \int_{\hat{C}}^{\infty} \right] e^{-Nh_b(u)} du \leq (\hat{C} - 6^{1/4}\sqrt{\nu})e^{-Nh_b(6^{1/4}\sqrt{\nu})} + \frac{2}{N} e^{-N\hat{C}}. \quad (5.14)$$ 

Now we estimate the main term $I$. First, we perform the change of variables $u = \sqrt{v}$ and obtain

$$I = 2 \int_{T_1} e^{-Nh_\beta(u)} du = \int_{\tilde{T}_1} e^{-\tilde{h}_\beta(v)} \frac{dv}{\sqrt{v}}, \quad (5.15)$$

where $\tilde{T}_1$ is the push forward of $T_1$ by the change of variables. Note that $v = 0$ is not a critical point for $\tilde{h}_\beta(v)$ for $\beta \neq 1$. However, it is the boundary of the integration in (5.15), therefore it may give a non-vanishing contribution to the value of the integral.

We perform one more change of variables $v = V(z, \beta)$ with $V(z, \beta)$ given by Levinson’s theorem, and modify the contour of integration to obtain

$$\int_{\tilde{T}_1} e^{-\tilde{h}_\beta(v)} \frac{dv}{\sqrt{v}} = \int_0^\nu e^{-N(\frac{\xi^2}{2} - \xi(\beta)z)} \frac{V'(z, \beta)}{\sqrt{V(z, \beta)}} dz.$$ 

Note that, around $z = 0$ we obtain

$$\frac{V'(z, \beta)}{\sqrt{V(z, \beta)}} = \frac{V'(0, \beta) + O(z)}{\sqrt{V(0, \beta) + V'(0, \beta)z + O(z^2)}} = \sqrt{\frac{V'(0, \beta)}{z}} (1 + O(z)) = \sqrt{\frac{2\beta\xi(\beta)}{(\beta - 1)z}} (1 + O(z)), \quad (5.16)$$

where in the last equality we used that $V(0, \beta) = 0$, the condition (5.4), and the computation (5.10) of the value $V'(0, \beta)$. In the same way we obtain around $z = \xi(\beta)$

$$\frac{V'(z, \beta)}{\sqrt{V(z, \beta)}} = \frac{V'(\xi(\beta), \beta) + O(z - \xi(\beta))}{\sqrt{V(\xi(\beta), \beta) + V'(\xi(\beta), \beta)(z - \xi(\beta)) + O((z - \xi(\beta))^2)}}$$

$$= \sqrt{\frac{V'(\xi(\beta), \beta)}{V(\xi(\beta), \beta)}} \left[ 1 + C_1(\beta)(z - \xi(\beta)) + O((z - \xi(\beta))^2) \right]$$

$$= \frac{2\beta}{\sqrt{\beta - \beta^2 + u_\beta^2}} \left[ 1 + C_1(\beta)(z - \xi(\beta)) + O((z - \xi(\beta))^2) \right], \quad (5.17)$$
where in the last equality we used the results (5.7) and (5.9) for the values $V(\xi(\beta), \beta)$ and $V'(\xi(\beta), \beta)$. We note that for all $\delta > 0$ the implicit constants and $C_1(\beta)$ in (5.16) and (5.17) are uniform in $\delta < |\beta - 1| \leq c'$.

Now we perform one more change of the contour of integration. We change the contour to $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where $\Gamma_i$ are the following intervals

$$
\Gamma_1 = [0, -\xi(\beta)], \quad \Gamma_2 = [-\xi(\beta), \nu + i \text{Im} \, \xi(\beta)], \quad \Gamma_3 = [\nu + i \text{Im} \, \xi(\beta), \nu].
$$

Denote for $j = 1, 2, 3$,

$$
I_j = \int_{\Gamma_j} e^{-N(\xi(\beta)^2 - \xi(\beta)z)} \frac{V'(z, \beta)}{\sqrt{V(z, \beta)}} (1 + O(z)) dz.
$$

Recall that $\xi(\beta) = -2h_{\beta}(u_{\beta})$, see (5.8). Our main estimate is the following.

**Lemma 5.2.** Let $\delta < |\beta - 1| \leq c'$.

1.  

$$
\left| I_1 - \sqrt{\frac{2\pi \beta}{N(1 - \beta)}} \right| \leq C(\delta) \frac{N^{3/2}}{N},
$$

(5.18)

2.  

For any $\eta > 0$,

$$
\text{for } \operatorname{Re} \xi(\beta) > \eta \left| I_2 - \frac{2\beta e^{N\xi(\beta)^2}}{\sqrt{\beta - \beta^2 + u_{\beta}^2}} \sqrt{\frac{2\pi}{N}} \right| \leq C(\delta, \eta) \frac{e^{N\eta \xi(\beta)^2}}{N^{3/2}},
$$

(5.19)

$$
\text{for } \operatorname{Re} \xi(\beta) \geq 0 \left| I_2 \right| \leq C(\delta) \frac{e^{N\operatorname{Re} \xi(\beta)^2}}{N},
$$

(5.20)

$$
\text{for } \operatorname{Re} \xi(\beta) \leq 0 \left| I_2 \right| \leq C(\delta) e^{-N\left(\frac{\operatorname{Re} \xi(\beta)^2}{2} + \xi(\beta)^2\right)},
$$

(5.21)

3.  

$$
\left| I_3 \right| \leq O(e^{-cN}).
$$

Given Lemma 5.2, we now complete the proof of Theorem 5.3.

**Proof of Theorem 5.3.** The result follows from the combination of estimate (5.13) on $E'$, the estimate (5.14) on $E$, the definition (1.6) of $Z_{\beta, N}$, and Lemma 5.2, when we apply Lemma 5.2 as follows. We consider three cases

1.  

$$
\operatorname{Re} \xi(\beta) \geq \delta^{10},
$$

2.  

$$
0 \leq \operatorname{Re} \xi(\beta) \leq \delta^{10},
$$

3.  

$$
\operatorname{Re} \xi(\beta) \leq 0.
$$
In the first case, we use the asymptotics (5.19) for $I_2$. In the second case, the formula (5.8) linking $\xi(\beta)$ and $h_\beta(u_\beta)$ and the estimate $|\xi(\beta)| \geq C_1|\beta - 1| \geq C_1\delta$ which follows from the computation of $\xi'(1)$ in (5.20) below imply that

$$\text{Re} \, h_\beta(u_\beta) = -\frac{1}{2} \text{Re} \, \xi(\beta)^2 \geq -\frac{1}{2}(\delta^{\alpha_0} - C_2\delta^2) \geq C_3\delta^2.$$ 

Therefore, the second term in the statement (2) of the Theorem is subdominant. In this case we use the estimate (5.20) for $I_2$. In the third case, we are even further to the left of the critical curve $\Gamma$, and we use the rough estimate (5.21) for $I_2$.

In all the three cases, we use the first statement of the Lemma for $I_1$ and the third statement for $I_3$. This finishes the proof. \hfill \Box

**Proof of Lemma 5.2.** We start with the estimate of $I_1$. Assume $\text{Im} \, \xi(\beta) > 0$ (the case $\text{Im} \, \xi(\beta) < 0$ is done in the same way). Define change of variables $z = -\xi(\beta)t$, $t \in \mathbb{R}$. Then, for $z \in [0, -\xi(\beta)]$ we get $t \in [0, 1]$ and

$$I_1 = \sqrt{\frac{2\beta\xi(\beta)}{\beta - 1}} \int_{\Gamma_1} e^{-N(\frac{z^2}{2} - \xi(\beta)z)} \frac{1}{\sqrt{z}}(1 + O(z))dz$$

$$= \sqrt{\frac{2\beta|\xi(\beta)|^2}{\beta - 1}} \int_0^1 e^{-N(\frac{(\xi(\beta))^2t^2 + t|\xi(\beta)|^2)} \frac{1}{\sqrt{t}}(1 + O(t))dt$$

$$= \sqrt{\frac{2\beta|\xi(\beta)|^2}{\beta - 1}} \int_1^{-1} e^{-N(\frac{(\xi(\beta))^2t^2 + y^2|\xi(\beta)|^2)}(1 + O(y^2))dy,}$$

where we used the change of variables $t = y^2$. Note that the unique minimum of $g(y) = \frac{\text{Re} \, \xi(\beta)^2}{2}y^4 + y^2|\xi(\beta)|^2$ is at $y = 0$. Indeed, if $\text{Re} \, \xi(\beta)^2 \geq 0$, then $g(y)$ is a monotone increasing function on $\mathbb{R}_+$ with a unique minimum at $y = 0$. If $\text{Re} \, \xi(\beta)^2 < 0$, then for any $0 \leq y \leq 1$,

$$g'(y) = 2y^3 \text{Re} \, \xi(\beta)^2 + 2y|\xi(\beta)|^2 \geq 2y(|\xi(\beta)|^2 - y^2(\text{Re} \, \xi(\beta)^2)) \geq 0,$$

and the last inequality follows from $|\text{Re} \, \xi(\beta)^2| \leq |\xi(\beta)|^2$. We can now apply the Laplace method (for example, in the form of [1, Theorem 3.5.3], keeping track of the error term in the proof) to the last integral in (5.22), and conclude with (5.18).

Now we treat $I_2$. First, we prove the first two cases (5.19) and (5.20), where $\text{Re} \, \xi(\beta) \geq 0$. Using the expansion (5.17) of the non-exponential term in the integral we obtain

$$I_2 = \frac{2\beta}{\sqrt{\beta - \beta^2 + u^2}} \int_{\Gamma_2} e^{-N(\frac{z^2}{2} - \xi(\beta)z)}[1 + C_1(\beta)(z - \xi(\beta)) + O((z - \xi(\beta))^2)]dz.$$

Define the following change of variables $z = \xi(\beta) + t$, $t \in \mathbb{R}$. Then, $\xi^2 - \xi(\beta)z = -\xi^2 + \frac{t^2}{2}.$ Note that, $\text{Im}(-\xi^2 + \frac{t^2}{2}) = \text{Im}(-\xi^2) = \text{const}$, thus this is a minimal phase contour, and for $\text{Re} \, \xi(\beta) \geq 0$ it passes throughout the critical point $\xi(\beta)$. Therefore, the main contribution
to the integral on this contour comes from the saddle point and the rest is small. With this change of variable, we obtain

\[ I_2 = \frac{2\beta}{\sqrt{\beta - \beta^2 + u_\beta^2}} e^{N\Re(\xi(\beta))^2} \int_{-2\Re(\xi(\beta))}^{\nu - \Re(\xi(\beta))} e^{-N\frac{t^2}{2}} [1 + C_1(\beta)t + O(t^2)] dt \]

= \frac{2\beta}{\sqrt{\beta - \beta^2 + u_\beta^2}} e^{N\Re(\xi(\beta))^2} [J_1 + J_2 + J_3]. \quad (5.24)

We begin with the first case (5.19). In this case, the result is an immediate (elementary) application of the Laplace method, see again [1, Theorem 3.5.3]. The correction of order \(O(N^{-1})\) in (5.19) comes from the estimate on \(J_3\), therefore we have finished with this case. Note that the implicit constant is not uniform in \(\eta \to +0\).

To prove the estimate (5.20) we do the following rough bound

\[ |I_2| \leq C(\delta) e^{N\Re(\xi(\beta))^2} \int_{-\infty}^{\infty} e^{-N\frac{t^2}{2}} dt \leq \frac{\tilde{C}(\delta)}{\sqrt{N}} e^{N\Re(\xi(\beta))^2}. \]

Note that \(C(\delta)\) and \(\tilde{C}(\delta)\) are uniform in \(\Re(\xi(\beta)) \geq 0\).

Now we prove the last case (5.21). If \(\Re(\xi(\beta)) \leq 0\), then \(\Re(-\xi(\beta)) \geq 0\). At the point \(z = -\xi(\beta)\) we obtain

\[ \left[ \frac{z^2}{2} - \xi(\beta)z \right] \bigg|_{z = -\xi(\beta)} = \frac{\xi(\beta)^2}{2} + |\xi(\beta)|^2 > 0. \]

Note that \(\frac{z^2}{2} - \xi(\beta)z\) is a monotone increasing function on the interval \(\Gamma_2\) with a minimum attained at \(z = -\xi(\beta)\). Thus, we obtain

\[ |I_2| \leq \left| \int_{-\xi(\beta)}^{\nu + i\text{Im} \xi(\beta)} e^{-N\frac{t^2}{2} - \xi(\beta)z} \max_{z \in \Gamma_2} \left| V'(z, \beta) \right| dz \right| \leq C(\delta) e^{-N\left(\frac{\Re(\xi(\beta))^2}{2} + |\xi(\beta)|^2\right)}. \]

To estimate \(I_3\), note that on this contour \(z \in i\mathbb{R}\), therefore, we get for \(\tilde{z} = \text{Im} z\)

\[ |I_3| \leq |\sqrt{V'(0, \beta)}| \int_{\nu}^{\nu + i\text{Im} \xi(\beta)} e^{-N\left(-\frac{z^2}{2} + \text{Im} \xi(\beta)\tilde{z}\right)} \frac{d\tilde{z}}{\sqrt{\tilde{z}}} (1 + O(\tilde{z})) \leq C e^{-cN\text{Im} \xi(\beta)}, \quad (5.25) \]

where the last inequality follows since the function \(-\frac{z^2}{2} + \text{Im} \xi(\beta)\tilde{z}\) is monotone decreasing for \(\tilde{z} \geq \text{Im} \xi(\beta)\) with a minimum attained at \(\nu + i\text{Im} \xi(\beta)\). \(\square\)

### 5.3 Construction of the critical curve

**Proof of Claim 5.14.** First, let us note the following

\[ h_\beta(u) = \frac{u^2}{2\beta} - \log \cosh u = \frac{u^2}{2} \left( \frac{1}{\beta} - 1 \right) - \left( \log \cosh u - \frac{u^2}{2} \right) = \frac{u^2}{2} \left( \frac{1}{\beta} - 1 \right) + \frac{u^4}{12} + O(u^6), \]

(5.26)
where the last equality holds since $\log \cosh u - \frac{u^2}{2} = -\frac{u^4}{12} + O(u^6)$. By Claim 5.1 we get
\( u_{\beta}^2 = 3(\beta - 1)(1 + O(\beta - 1)), \) (5.27)

therefore we obtain
\[
\begin{align*}
    h_{\beta}(u_{\beta}) &= \frac{u_{\beta}^2}{2\beta}(1 - \beta) + \frac{u_{\beta}^4}{12} + O(u_{\beta}^6) \\
    &= -\frac{3}{2\beta}(\beta - 1)^2(1 + O(\beta - 1)) + \frac{9}{12}(\beta - 1)^2(1 + O(\beta - 1)) + O((\beta - 1)^3) \\
    &= -\frac{3}{4}(\beta - 1)^2(1 + O(\beta - 1)).
\end{align*}
\] (5.28)

From the equation (5.8) linking $\xi(\beta)$ and $h_{\beta}(u_{\beta})$ we obtain that $\text{Re} h_{\beta}(u_{\beta}) = 0$ if and only if $\xi(\beta) \in e^{i\pi/4}\mathbb{R} \cup e^{-i\pi/4}\mathbb{R}$. Combining (5.8) and (5.28) we conclude that
\[
\xi'(1) = \sqrt{\frac{3}{2}},
\] (5.29)

therefore, $\xi(\beta)$ is one to one in $|\beta - 1| < c$ for $c > 0$ sufficiently small. Then, the curves $\gamma_{\pm} = \{\beta \mid |\beta - 1| < c, \xi(\beta) \in e^{i\pi/4}\mathbb{R} \cup e^{-i\pi/4}\mathbb{R}\}$ are analytic. Note that, $\gamma_- = \gamma_+$.

By (5.28) we have $h_{\beta}(u_{\beta}) = -\frac{3}{4}(\beta - 1)^2 + O((\beta - 1)^3)$, therefore, for $\beta = 1 + \epsilon + iR \in \gamma_{\pm}$ we get
\[
0 = \text{Re} h_{\beta}(u_{\beta}) = -\frac{3}{4}(\epsilon^2 - R^2) + O(\epsilon^3 + R^3),
\]

namely, $R^2 = \epsilon^2 + O(\epsilon^3)$ and we get
\[
R = \pm \epsilon(1 + O(\epsilon)).
\]

\[\square\]

5.4 Proof of Corollary 1.5

We consider the zeros of
\[
\Psi_N(\beta) = \frac{1}{\sqrt{1 - \beta}} + 2\sqrt{\frac{\beta}{\beta - \beta^2 + u_{\beta}^2}}e^{N(-h_{\beta}(u_{\beta}))}.
\]

We will work with $\text{Re} \beta \geq 1$ in the domain $D_\delta = \{\text{Re} \beta \geq 1\} \cap \{\delta < |\beta - 1| < c'\}$. First, we need the following estimate.

Claim 5.3. In the domain $D_\delta$
\[
\frac{1}{C(\delta)} \text{dist}(\beta, \Gamma) \leq |\text{Re} h_{\beta}(u_{\beta})| \leq C(\delta) \text{dist}(\beta, \Gamma). \] (5.30)
Proof of Claim 5.3. We note that $\beta \mapsto h_\beta(u_\beta)$ is Lipschitz with constant $C$ (independent of $\delta$) on $D_\delta$. This follows from the analyticity of $\xi(\beta)$ and (5.8).

We begin with the proof of the upper bound in (5.30). If $\beta'$ is the point on the critical curve $\Gamma$ closest to $\beta$, then, since $\text{Re } h_{\beta'}(u_{\beta'}) = 0$, we get from the Lipschitz property,

$$|\text{Re } h_\beta(u_\beta)| = |\text{Re } h_\beta(u_\beta) - \text{Re } h_{\beta'}(u_{\beta'})| \leq C|\beta - \beta'| = C\text{dist}(\beta, \Gamma).$$

We turn to the proof of the lower bound in (5.30). We have

$$\frac{d}{d\beta} h_\beta(u_\beta) = -\frac{u_\beta^2}{2\beta^2} + \frac{du_\beta}{d\beta} \left( \frac{u_\beta}{\beta} - \tanh u_\beta \right) = -\frac{u_\beta^2}{2\beta^2},$$

where the last equality holds since $u_\beta$ is a saddle point of $h_\beta(u)$. By Claim 5.1 we get on $D_\delta$,

$$|u_\beta|^2 = |3(\beta - 1)(1 + O(\beta - 1))| \geq C(\delta).$$

and therefore, on $D_\delta$,

$$\left|\frac{d}{d\beta} \text{Re } h_\beta(u_\beta)\right| \geq C'(\delta).$$

(5.32)

Connect $u_\beta$ to some $\beta' \in \Gamma$ by a curve following the gradient $\frac{d}{d\beta} \text{Re } h_\beta(u_\beta)$. The length of this curve is bounded by a constant times the Euclidean distance between $\beta$ and $\beta' \in \Gamma$. Applying (5.32) then yields the lower bound, since $\text{Re } h_{\beta'}(u_{\beta'}) = 0$. \hfill \Box

Now we observe the following

Claim 5.4.

• The zeros of $\Psi_N(\beta)$ in $D_\delta$ lie within $\frac{K(\delta)}{N}$ from $\Gamma$,

• For any $\delta, K > 0$, there exists $C_{K, \delta}$ such that for $\beta \in D_\delta$ with $\text{dist}(\beta, \Gamma) \leq \frac{K(\delta)}{N}$ we have $|\Psi'_N(\beta)| \geq C_{K, \delta}N$, $|\Psi''_N(\beta)| \leq CN^2$, where $C > 0$ does not depend on $K, \delta$.

Proof of Claim 5.4. We start with the first statement. If $\text{Re } h_\beta(u_\beta) > 0$ and the distance $\text{dist}(\beta, \Gamma) > \frac{K}{N}$, then, using the lower bound of Claim 5.3 we obtain

$$|\Psi_N(\beta)| \geq \frac{1}{|\beta - 1|^{1/2}} - 2 \left| \frac{\beta}{\beta - \beta^2 + u_\beta^2} \right|^{1/2} e^{-\frac{K}{C(\delta)}}.$$

When $K = K(\delta)$ is sufficiently large, the right hand side is strictly greater than 0.

Similarly, if $\text{Re } h_\beta(u_\beta) < 0$ and $\text{dist}(\beta, \Gamma) > \frac{K}{N}$, then, using again the lower bound of Claim 5.3 we obtain

$$|\Psi_N(\beta)| \geq -\frac{1}{|\beta - 1|^{1/2}} + 2 \left| \frac{\beta}{\beta - \beta^2 + u_\beta^2} \right|^{1/2} e^{-\frac{K}{C(\delta)}} > 0,$$

for sufficiently large $K = K(\delta)$. Therefore, the zeros of $\Psi_N(\beta)$ in $D_\delta$ lie in $\{\text{dist}(\beta, \Gamma) \leq \frac{K}{N}\}$. 

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Now we prove the second statement. By a direct computation we get

\[
\Psi_N'(\beta) = \frac{1}{2(\beta - 1)^{3/2}} e^{-Nh_\beta(u_\beta)} \left[ -N \left\{ \frac{\partial}{\partial u} h_\beta(u)|_{u=u_\beta} \frac{\partial}{\partial \beta} u_\beta + \frac{\partial}{\partial \beta} h_\beta(u)|_{u=u_\beta} \frac{\partial}{\partial \beta} \right\} \right]
\]

Since \(u_\beta\) is a saddle point of \(h_\beta(u)\) we get \(\frac{\partial}{\partial u} h_\beta(u)|_{u=u_\beta} = 0\) and by (5.31) we get \(\frac{\partial}{\partial \beta} h_\beta(u)|_{u=u_\beta} = -\frac{u_\beta^2}{2\beta^2}\), therefore

\[
\Psi_N'(\beta) = \frac{1}{2(\beta - 1)^{3/2}} + \sqrt{\beta - \beta^2 + u_\beta^2} e^{-Nh_\beta(u_\beta)} \left[ N \frac{u_\beta^2}{2\beta^2} + \frac{\beta + u_\beta^2 - 2u_\beta \frac{\partial}{\partial \beta} u_\beta}{\beta - \beta^2 + u_\beta^2} \right].
\]

Using the upper bound of Claim 5.3 we obtain for sufficiently large \(N\)

\[
|\Psi_N'(\beta)| \geq -C_1(\delta) + e^{-NC(\delta)} \frac{K(\delta)}{N} C_2(\delta) \left[ NC_3(\delta) - C_4(\delta) \right] \geq C_{K,\delta} N.
\]

The bound \(|\Psi_N'(\beta)| \leq CN^2\) is obtained in the same way. \(\square\)

Let \(C_\delta > 0\). The properties of \(\Psi_N(\beta)\) listed in Claim 5.4 imply that the distance between any two zeros of \(\Psi_N(\beta)\) in \(D_\delta\) at least \(\geq \frac{C_\delta^2}{N}\). Indeed, let \(\beta_0\) be a zero of \(\Psi_N(\beta)\). Then,

\[
\Psi_N(\beta) = \Psi_N(\beta_0) + \Psi_N'(\beta_0)(\beta - \beta_0) + O(N^2)(\beta - \beta_0)^2.
\]

Since \(|\Psi_N'(\beta_0)| \geq C_{K,\delta} N\), we obtain

\[
|\Psi_N(\beta)| \geq C_{K,\delta} N|\beta - \beta_0| - CN^2|\beta - \beta_0|^2,
\]

for every \(|\beta - \beta_0| \leq cN^{-1}\). In particular, \(|\Psi_N(\beta)| = 0\) implies \(|\beta - \beta_0| \geq \frac{C_\delta}{N}\).

Now we look at the discs of radius \(C_\delta N^{-2}\) around each zero of \(\Psi_N(\beta)\) near \(\Gamma\) and we claim that there is exactly one zero of \(Z_{\beta,N}\) in each disc. By an additional application of Rouché\'s theorem it is sufficient to show that for sufficiently large \(C_\delta\) we have on the boundary of each disc

\[
|Z_{\beta,N} - \Psi_N(\beta)| \leq \frac{|\Psi_N(\beta)|}{2}. \tag{5.33}
\]

The estimate (5.33) follows since \(|Z_{\beta,N} - \Psi_N(\beta)| \leq O(N^{-1})\) uniformly in \(\{\delta < |\beta - 1| \leq c'\}, \text{ dist}(\beta, \Gamma) \leq \frac{2K(\delta)}{N}\) and on the boundary of each disc of radius \(C_\delta N^{-2}\) we get \(|\Psi_N(\beta)| \geq C_{K,\delta} N \frac{C_\delta}{N^2} = \frac{C_\delta^2}{N}\), where \(C_0\) may be made arbitrarily large by adjusting \(C_\delta\). Therefore, there is exactly one zero of \(Z_{\beta,N}\) in each of these discs.

To show that there are no additional zeros of \(Z_{\beta,N}\) in \(\delta < |\beta - 1| < c'\), first we observe that, by Theorem 1.3, the zeros of \(Z_{\beta,N}\) in \(\delta < |\beta - 1| < c'\) lie in \(\text{Re } \beta \geq 1\).
Consider the domain \( \tilde{D}_\delta = \{ \tilde{\delta} < |\beta - 1| \leq \tilde{c}, \ \text{Re} \beta \geq 1 \} \), where \( \tilde{\delta} \) and \( \tilde{c} \) are such that
\[
\delta - \frac{C_\delta}{N} \leq \tilde{\delta} \leq \delta, \quad \delta' \leq \tilde{c} \leq \delta' + \frac{C_\delta}{N},
\]
and the distance \( \text{dist}(\beta, \partial \tilde{D}_\delta) \geq \frac{C_\delta}{N} \) for any zero \( \beta \) of \( \Psi_N(\beta) \). We will check that the inequality \( (5.33) \) holds on the boundary \( \partial \tilde{D}_\delta \), then by Rouché’s theorem the zeros of \( Z_{\beta,N} \) in \( \tilde{D}_\delta \) are exactly those constructed in the first part of the proof.

We divide the boundary \( \partial \tilde{D}_\delta \) of the domain \( \tilde{D}_\delta \) as follows: \( \partial \tilde{D}_\delta = A \cup B \), where \( A = \partial \tilde{D}_\delta \cap \{ \text{dist}(\beta, \Gamma) \leq \frac{K(\delta)}{N} \} \) and \( B = \partial \tilde{D}_\delta \setminus A \). For sufficiently large \( K(\delta) \) the inequality \( (5.33) \) is valid on \( B \) by Theorem 1.3. We now show that \( (5.33) \) also holds on \( A \). The set \( \{ \delta < |\beta - 1| \leq \tilde{c}, \ \text{dist}(\beta, \Gamma) \leq \frac{K(\delta)}{N} \} \) contains \( A \), therefore we have uniformly in \( A \)
\[
|Z_{\beta,N} - \Psi_N(\beta)| \leq O(N^{-1}).
\]

Also, as before, we have \( |\Psi_N(\beta)| \geq C_K\delta N \frac{C_\delta}{N} = C_0 \) on \( A \). Thus, the inequality \( (5.33) \) holds on \( A \), and we conclude the proof.

### 5.5 Proof of Corollary 1.6

Define
\[
\tilde{\mu}_N = \frac{1}{N} \sum_{\beta: |\beta - 1| \leq \delta'} \delta \beta_{\beta_{\delta_1} \leq 1}, \quad \text{Re} \beta \geq 1, \ \Psi_N(\beta) = 0.
\]

We will show that
\[
\tilde{\mu}_N \to_{N \to \infty} \mu, \quad (5.34)
\]
\[
\mu\{ |\beta - 1| < \delta \} \to_{\delta \to 0} 0, \quad (5.35)
\]
\[
\limsup_{N \to \infty} \mu\{ |\beta - 1| < \delta \} \to_{\delta \to 0} 0. \quad (5.36)
\]

Choose \( \delta > 0 \). Then, by \( (5.34) \) and Corollary 1.5 we obtain
\[
\mu_N \{ |\beta - 1| \leq \delta' \} \to_{\delta \to 0} \mu \{ |\beta - 1| \leq \delta' \}.
\]

Using \( (5.35) \) and \( (5.36) \), and letting \( N \to \infty \) and then \( \delta \to 0 \) we obtain \( \mu_N \to \mu \). It remains to show \( (5.34), (5.35) \) and \( (5.36) \).

Toward this end, note that since \( \xi'(1) = \sqrt{3/2} \neq 0 \), see \( (5.29) \), it follows that \( \xi(\beta) \) is one-to-one in a neighborhood of \( \beta = 1 \), and in fact it maps a neighborhood of \( \beta = 1 \) biconformally onto a neighborhood of \( 0 \). In particular, with \( I \) denoting the line segment \( [0, c' e^{i \pi/2}] \) with \( c' > 0 \) small, we have by Claim 1.4 that \( \xi^{-1}(I) \) is a segment of \( \Gamma \cap \{ \text{Im} \beta \geq 0 \} \) containing \( \beta = 1 \). Therefore, by \( (5.8) \), \( h_\beta(u_{\beta}) \) maps \( \xi^{-1}(I) \) bijectively onto \( (0, ic') \) for some \( c > 0 \). A similar argument applies with \( I_- = [0, -c'' e^{i \pi/2}] \) replacing \( I \) and \( \Gamma \cap \{ \text{Im} \beta \leq 0 \} \)

Replacing \( \Gamma \cap \{ \text{Im} \beta \geq 0 \} \) replacing \( \Gamma \cap \{ \text{Im} \beta \leq 0 \} \)

Let \( \beta_k \in \Gamma, \ k \in \mathbb{Z} \), be such that \( h_{\beta_k}(u_{\beta_k}) = \frac{2\pi i k}{N} \) is smaller in absolute value than \( c \). It follows from the above considerations that
\[
\frac{1}{N} \sum_{|\beta_k - 1| \leq \delta'} \delta_{\beta_k} \to_{N \to \infty} \mu.
\]
since the left-hand side and the right-hand side assign the same value to each half-open
curved segment of $\Gamma$ connecting two points $\beta_l$ and $\beta_l'$; this value is $\frac{\ell - \ell'}{N}$.

Next, let $\tilde{\beta}_k$ be such that $\text{Re} \tilde{\beta}_k \geq 1$ and

$$h_{\tilde{\beta}_k}(u_{\tilde{\beta}_k}) = -\frac{1}{N} \log \left[ -\frac{1}{2} \sqrt{\frac{\tilde{\beta}_k - \tilde{\beta}_k^2 + u_{\tilde{\beta}_k}^2}{\tilde{\beta}_k - \tilde{\beta}_k^2}} \right] + \frac{2\pi i k}{N}.$$  

Then, using the relation (5.8) and that $\xi(\beta)$ is one to one, we get

$$\tilde{\mu}_N = \frac{1}{N} \sum_{|\tilde{\beta}_k - 1| \leq c} \delta_{\tilde{\beta}_k}, \quad \text{sup}_{k \leq N} |\tilde{\beta}_k - \beta_k| = o(1).$$

Hence, $\tilde{\mu}_N \to \mu$, and this finishes the proof of (5.34).

Next, since $\xi(\cdot)$ is Lipschitz in a neighborhood of $\beta = 1$, we get from (5.8) that

$$|h_{\beta}(u_{\beta})| \leq C|\beta - 1|^2.$$  

(5.37)

The relation (5.35) follows since for $b(\delta)$ which is the intersection of the critical curve $\Gamma$ with $|\beta - 1| = \delta$ we obtain

$$\mu\{|\beta - 1| < \delta\} \leq 2|h_{b(\delta)}(u_{b(\delta)})| \leq 2C|\beta - 1|^2 \leq C\delta^2,$$

where the two last inequalities follow from the estimate (5.37) and from the fact that $|\beta - 1| = \delta$.

To prove the relation (5.36), denote by $n_N(\delta)$ the number of zeros of $Z_{\beta,N}$ in $|\beta - 1| \leq \delta$. Then, by Jensen’s formula we obtain

$$n_{\delta}(\delta) = \#\{|\beta - 1| \leq \delta, Z_{\beta,N} = 0\} \leq \frac{1}{\log \frac{2\delta}{\delta}} \log \max_{|\beta - 1| = 2\delta} |Z_{\beta,N}| |Z_{N}(1)|.$$

From the case for real $\beta$ the denominator is bounded from below by $N^{-C}$, for some $C > 0$, and we need to bound the numerator from above. We get

$$\frac{1}{\log 2} \log \frac{\max_{|\beta - 1| = 2\delta} |Z_{\beta,N}|}{|Z_{N}(1)|} \leq C[N \max_{|\beta - 1| = 2\delta} |h_{\beta}(u_{\beta})| + A_{\delta} + \log N] \leq C'[N\delta^2 + A_{\delta} + \log N],$$

where the first inequality follows from Theorem 1.3, and the last one follows from the estimate (5.37) and since $|\beta - 1| = 2\delta$. Thus, using the last estimate, we obtain

$$\mu_N\{|\beta - 1| \leq \delta\} \leq C'\delta^2 + \frac{C' A_{\delta}}{N} + \frac{C' \log N}{N}.$$

Letting first $N \to \infty$ and then $\delta \searrow 0$ we obtain (5.36) and thus conclude the proof. \hfill $\square$
6 Conjecture: the critical curve

We conjecture that there exists a curve

\[ \{1 + \epsilon_0(R) + iR\}_{-\infty < R < \infty} \]

such that

\[ \epsilon_0(0) = 0, \quad \epsilon_0(R) > 0 \text{ for } R \neq 0, \quad \epsilon_0(R) \approx \frac{b}{R}, \text{ as } R \to \infty, \]

and an auxiliary function \( \delta_0(R) \geq 0 \) with equality only at 0, so that the following holds

\[
\lim_{N \to \infty} \frac{1}{N} \log |Z_{1+\epsilon+iR,N}| \begin{cases} = 0, & 0 \leq \epsilon \leq \epsilon_0(R), \\ > 0, & \epsilon_0(R) < \epsilon \leq \epsilon_0(R) + \delta_0(R). \end{cases} \tag{6.1}
\]

Moreover, we conjecture that the curve is described by one branch of the saddle point equation, as follows.

For \( f_\beta(u) \) from Proposition 2.1, consider the saddle point equation

\[
f_\beta'(u) = \beta u - \beta \tanh(\beta u) = 0. \tag{6.2}
\]

It defines a multivalued function \( u(\beta) \). We claim that there exists a branch \( u^*(\beta) \) in \( 0 < \Re \beta \leq C \approx 1.3 \), such that \( u^*(1) = 0 \). Indeed, the equation (6.2) is equivalent to

\[
\beta = \frac{1}{2u} \left[ \log \frac{1+u}{1-u} + 2\pi i k \right], \ k \in \mathbb{Z}, \tag{6.3}
\]

where we take the principal branch of the logarithm. For \( k = 0 \) the equation (6.3) defines a bijection between the first and the fourth quadrants in the \( u \)-plane and the domains depicted in Figure 2 (left). For \( k = 1 \) the function from the right hand side of the equation (6.3) maps the first and the fourth quadrants onto the domains in Figure 2 (right).

\[
\begin{array}{c}
\text{Figure 2: The images of the I-st and IV-th quadrants in the } u\text{-plane under (6.3), with } k = 0 \\
\text{(left) and } k = 1 \text{ (right). The vertical lines in both plots and the large semi-circle at the right} \\
\text{lie at infinity.}
\end{array}
\]
Consequently, one can define a branch \( u^*(\beta) \) in \( \{0 < \text{Re} \beta \leq C, \text{Im} \beta > 0\} \), where \( C \approx 1.3 \) is the real part of the intersection point between the two curves on Figure 2 (right), which corresponds to \( k = 0 \) in the intersection with the domain in Figure 2 (left) and to \( k = 1 \) outside it. Similarly, we define \( u^*(\beta) \) for \( \text{Im} \beta < 0 \).

Conjecture 6.1. The relation (6.1) holds with \( \epsilon_0(R) \) defined by the equation

\[
\text{Re} f_{1+\epsilon_0(R)+iR}(u^*(1 + \epsilon_0(R) + iR)) = 0.
\]

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