Mixed Boundary Problem for the Traversable Wormhole Models

M.Yu. Konstantinov

VNIIMS, 3-1 M. Ulyanovoy str., Moscow, 117313, Russia
e-mail: konst@rgs.phys.msu.su

Abstract

The conditions of the traversable wormhole joining with the exterior space-time are considered in details and the mixed boundary problem for the Einstein equations is formulated. It is shown that, in opposite to some declarations, the conditions of the wormhole joining with the exterior space-time have non-dynamical nature and can not be defined by the physical processes. The role of these conditions in the formation of the causal structure of space-time is analyzed. It is shown that the causal structure of the wormhole-type space-time models is independent from both the interior and exterior energy-momentum tensors. This statement is illustrated in the particular case of the spherical wormhole joining with flat exterior space-time. The same conditions, which define the wormhole joining with the exterior space-time, provide the absence of paradoxes in the models with causality violation. It is pointed out, that the nature and physical interpretation of the conditions of wormhole joining with the exterior space-time and induced boundary conditions for the field variables is one of the fundamental problems, which arise in the models with causality violation.

Moscow 1997
1 Introduction

The investigation of space-time models with causality violation creates much interest in the last years. This interest was stimulated by the series of declarations about possibility of the creation of closed time-like curves (CTCs) in the process of the dynamical evolution of some space-like hypersurface [1, 2, 3, 4]. The statements about unavoidable transformation of the traversable wormhole into the time machine were made also [3], so that classically any given wormhole may be “absurdly easy turned into a time machine” [5]. Since the existence of the closed time-like curves is associated usually with numerous paradoxes [6, 7, 8], the part of the following papers were devoted to finding some physical laws or new principles which may forbid the creation of CTCs [9, 10, 11, 12, 13, 14], while another part of papers were devoted to the discussion of the so-called “self-consistency conditions” [2, 15, 16], which must be added to the usual Cauchy conditions to avoid paradoxes of time travel.

It is clear, that in general case space-time models may contain CTCs. The most known examples of such models are Gödel universe and Taub-NUT space-time [17]. Nevertheless the statements about possibility of the creation of closed time-like curves (CTCs) in the process of the dynamical evolution of some space-like hypersurface and unavoidable transformation of the traversable wormhole into the time machine [1, 2, 3] require more precise definition. Namely, the terms “creation”, “dynamical evolution” and “transformation” (in particular, the term “unavoidable transformation”) are usually applied to the models which may be obtained in a result of development of some initial configuration or, equivalently, to the solutions of the appropriate Cauchy problem. Application of these terms to space-time models with CTCs and time machine contradict to the well known theorems about global hyperbolicity [17], which have pure topological nature and state that globally hyperbolic space-time has topology of the direct product $\mathbb{T} \times M^3$, where $\mathbb{T}$ is a global time axes and $M^3$ - some 3-manifold. It is clear, that globally hyperbolic space-time, which may be considered as a global solution of appropriate Cauchy problem, can not contain CTC [17]. The same conclusion may be derived from the dynamical (3+1)-formalism in general relativity [18, 19, 20] which excludes the existence of CTCs in the hyperbolic region. If space-time is not globally hyperbolic then CTCs may exist in non hyperbolic regions but CTCs in such models can not be considered as the result of transformation, dynamical evolution or development of some initial configuration, because any non globally hyperbolic space-time solves some boundary or mixed boundary problem which can not be reduced to the Cauchy problem. Unfortunately, no attention was paid on such contradiction, because all existing examples of traversable wormholes, including the models with causality violation, were obtained ”by hand” without solution or analysis of the appropriate boundary problem for the Einstein equations.

In the recent papers [21, 22] it was shown that in opposite to the statements of [1, 2] the motion of the wormhole’ mouths and twin paradox do not lead to the transformation of the traversable Lorentzian wormhole into the time machine. It was mentioned also, that the causality violation and the existence of CTCs depend on the boundary conditions which are defined by the manifold structure [21, 22] and provide the absence of any paradoxes in the presence of CTCs. In more details the nature of self-consistency conditions was considered in [23], where it was shown that they are induced by the manifold structure and can not be considered nor as additional conditions nor as the consequence of the principle of minimal
action as it states in [16]. It was stressed out also [21, 22, 23] that in opposite to some declarations, the space-time models with causality violation are the subject of the mixed boundary problem for the Einstein equations. Nevertheless in general form this boundary problem was not considered or formulated explicitly in these papers.

In the present paper we consider the boundary conditions which arise due to the wormhole joining with the exterior space-time and discuss the connection of the corresponding mixed boundary problem with the causal properties of space-time. In the next section the general topological structure of the traversable wormhole models is briefly described. The conditions of the wormhole joining with the exterior space-time are considered in section 3 and the mixed boundary problem for such models is considered in section 4. The connection between the conditions of the wormhole joining with the exterior space and causal properties of space-time is discussed in section 5. Simple particular model is considered in section 6. The last section contains short summary and discussion.

2 The topological structure of the traversable wormhole models

The general space-time model with traversable wormhole may be considered as the result of attaching of the interior space-time with the topology $M^4_{int} = T_{int} \times M^3_{int}$ and exterior space-time $M^4_{ext} = T_{ext} \times M^3_{ext}$. Here $T_{int}$ and $T_{ext}$ are the interior and exterior time-like axes, $M^3_{int}$ and $M^3_{ext}$ are the interior and exterior spaces, which satisfy to the following conditions:

- $M^3_{int}$ is a connected compact orientable 3-manifold whose boundary $\partial M^3_{int}$ consists of two compact orientable 2-manifolds $M^2_l$ and $M^2_r$ with empty intersection ($M^2_l \cap M^2_r = \emptyset$);
- the intersection $M^4_{int} \cap M^4_{ext} = T_{int} \times (M^2_l \cup M^2_r) = T_{ext} \times (M^2_l \cup M^2_r)$;
- both $T_{int} \times M^2_l = T_{ext} \times M^2_l$ and $T_{int} \times M^2_r = T_{ext} \times M^2_r$ belong to the same connected component of $M^4_{ext}$.

The interior 3-space $M^3_{int}$ is called a handle of the wormhole and the manifolds $M^2_l$ and $M^2_r$ are often called as the ”left“ and ”right“ mouths of the wormhole. According to well-known theorems [23], $M^2_l$ and $M^2_r$ may be arbitrary 2-manifolds with genera $p_l$ and $p_r$, i.e. they are diffeomorphic to the 2-sphere $S^2$ with $p_l$ and $p_r$ handles respectively, and the manifold $M^3_{int}$ is an interpolating manifold.

In the simplest models [1, 4, 8, 21, 22, 23, 24, 25, 26] the interior space of the wormhole has topological structure of the direct product $M^3_{int} = I \times M^2$ of interval $I = (-L_1, L_2)$ and a compact orientable 2-dimensional manifold $M^2$, so that the ”left“ and ”right“ mouths of the wormhole have the same topology $M^2_l = M^2_r = M^2$. The sum $L = L_1 + L_2$ is called a length (to be exact, a coordinate length) of the wormhole’s handle.

The attaching of the interior and exterior space-times is made by the following standard manner. Let $t \in T_{ext}$ and $\tau \in T_{int}$ are the exterior and interior time coordinates. Then, for fixed $\tau$ one (left) wormhole mouth is attached to the space-like section $M^3_{ext1} = (t_1(\tau), M^3_{ext})$ while another (right) mouth is attached to the section $M^3_{ext2} = (t_2(\tau), M^3_{ext})$. The attaching
of the wormhole mouths to the exterior spaces \( M^3_{\text{ext1}} \) and \( M^3_{\text{ext2}} \) are made as follows: to attach the wormhole along its left mouth \((-L_1, M^2_l)\) to the exterior space \( M^3_{\text{ext1}} \) it is necessary to remove the tubular neighborhood of \( M^2_l \in M^3_{\text{ext1}} \) from the exterior space \( M^3_{\text{ext1}} \) and join the boundary \( M^2_l \) of the rest of \( M^3_{\text{ext1}} \) with the left wormhole mouth \((-L_1, M^2_l)\). The same procedure is used for the joining of the right wormhole mouth \((L_2, M^2_r)\) with the exterior space \( M^3_{\text{ext2}} \). In the result of such joining the wormhole connects the points of exterior space-like hypersurface \((t_1, M^3_{\text{ext1}})\) with the points of exterior space-like hypersurface \((t_2, M^3_{\text{ext2}})\), where \( t_1 \neq t_2 \) in general case.

### 3 Boundary conditions for the traversable wormhole models

To discuss the causal structure of the wormhole-type models it is more suitable to use local coordinate consideration. Consider for this purpose the curve \( l \in M^3_{\text{int}} \) which pass through the wormhole handle \( M^3_{\text{int}} \) and connects two points \( p_l \in M^2_l \) and \( p_r \in M^2_r \) of the left and right mouths of the wormhole. Consider the tubular neighborhood \( U_{\text{int}} \subset M^3_{\text{int}} \) of the curve \( l \) in \( M^3_{\text{int}} \), which has topology of the direct product \( U_{\text{int}} = l \times D^2 \) of the line \( l \) and an open disk \( D^2 \subset R^2 \). Let \( \{\tau, \xi^1, \xi^2, \xi^3\} \) are the local coordinates in \( T_{\text{int}} \times U_{\text{int}} \), such that \(-\infty < \tau < \infty\) is an interior time-like coordinate, \( \xi^1 \) is the coordinate along the line \( l \), \(-(\sigma_1 + L_1) < \xi^1 < L_2 + \sigma_2 \), where \( \sigma_1, \sigma_2, L_1 \) and \( L_2 \) are some positive constants, values \( \xi^1 = -L_1 \) and \( \xi^1 = L_2 \) correspond to the left and right mouths of the wormhole respectively, the regions \(-(\sigma_1 + L_1) < \xi^1 < L_1 \) and \( L < \xi^1 < L + \sigma_2 \) correspond to the wormhole intersection with the exterior space-time, \( \xi^2 \) and \( \xi^3 \) are the coordinates in the disk \( D^2 \).

In general form the metric of space-time in the wormhole interior may be written in the coordinates \( \{\tau, \xi^1, \xi^2, \xi^3\} \) as follows

\[
\begin{align*}
  ds^2_{\text{int}} &= a^2(\tau, \xi)d\tau^2 - 2b_i(\tau, \xi)d\tau d\xi^i - \tilde{\gamma}_{ij}(\tau, \xi)d\xi^i d\xi_j
\end{align*}
\]  

(1)

where \( \xi = \{\xi^1, \xi^2, \xi^3\} \), \( \tilde{\gamma}_{ij}(\tau, \xi) \) denotes the metric of the interior 3-space \( \tau = \text{const} \), \( a^2(\tau, \xi) > 0 \) because of the supposition about the wormhole traversability and \( b_i(\tau, \xi) \neq 0 \) in general case. The limiting values of the function \( a^2(\tau, \xi) \), 3-vector \( b_i(\tau, \xi) \) and 3-tensor \( \tilde{\gamma}_{ij}(\tau, \xi) \) at \( \xi^1 \rightarrow -L_1 \), \( L_2 \) are defined by the wormhole joining with the exterior space-time as it describes below.

Without loss of generality it may be supposed that in the exterior space-time both wormhole mouths are covered by the same map \( \{t, x^1, x^2, x^3\} \), where \(-\infty < t < \infty\) is an exterior time and \( \{x^1, x^2, x^3\} \) are the coordinates on the exterior space section \( t = \text{const} \). For simplicity it will be supposed additionally, that the coordinates \( \{t, x^1, x^2, x^3\} \) of the exterior space-time are synchronous, so, the exterior space-time metrics may be written as

\[
\begin{align*}
  ds^2_{\text{ext}} &= dt^2 - \gamma_{ij}dx^idx^j
\end{align*}
\]  

(2)

where \( \gamma_{ij} \) is a positive definite metric of the exterior 3-space \( t = \text{const} \).

Now it is necessary to define the wormhole joining with the exterior space-time. Without loss of generality it may be supposed, that the exterior coordinates are comoving to the left mouth of the wormhole, so, for \(-(\sigma_1 + L_1) < \xi^1 < -L_1 \) we have

\[
t_{\text{left}} = \tau,
\]  

(3)
while for the right mouth, i.e. for $L_2 < \xi^1 < L_2 + \sigma_2$, we have in general

$$t_{\text{right}} = t_r(\tau),$$

$$x^i_{\text{right}} = x^i(\tau, \xi^1, \xi^2, \xi^3)$$

These equations define the joining of the interior space-time of the wormhole with the exterior space-time. Equations (3) and (5) show that wormhole connects the space section $t = \tau$ of the exterior space-time with the space section $t = t_r(\tau)$ which, in general, does not coincide with space section $t = \tau$.

It is necessary to note, that in several wormhole models instead of the condition (5) more general condition $t_{\text{right}} = t_r(\tau, \xi^i)$ is used [1, 2]. In this case different points of the right wormhole’s mouth (i.e. the points with coordinates $\{L_2, \xi^2, \xi^3\}$ are placed on different space sections of the exterior space-time. It is easy to see, that such generalization has no influence on the causal structure of space-time and by this reason it will not considered here.

Using the joining equations (3)-(6) near the left wormhole mouth (i.e. for $-(\sigma_1 + L_1) < \xi^1 < -L_1$) the exterior space-time metric (2) may be rewritten in the interior coordinates $\{\tau, \xi^1, \xi^2, \xi^3\}$ as

$$ds^2_{\text{ext}} = d\tau^2 - \gamma'_{ij}d\xi^i d\xi^j$$

where

$$\gamma'_{ij} = \gamma_{kl}x^k_i x^l_j, \quad x^k_i = \partial x^k/\partial \xi^i,$$

while near the right mouth ($L_2 < \xi^1 < L_2 + \sigma_2$) it takes the form

$$ds^2_{\text{ext}} = \alpha^2(1 - v_l v^l)d\tau^2 - 2\beta_i d\tau d\xi^i - \gamma''_{ij}d\xi^i d\xi^j$$

where

$$\alpha = dt_r/d\tau, \quad v_l = \frac{1}{\alpha} \frac{\partial x^k}{\partial \tau}, \quad \beta_i = \gamma_{kl} \frac{\partial x^k}{\partial \tau} \frac{\partial x^l}{\partial \xi^i},$$

$$\gamma''_{ij} = \gamma_{kl}x^k_i x^l_j,$$

$v_i$ is a vector of the ”3-velocity” of the right mouth in the coordinates $\{t, x^i\}$ and $x^k_i = \partial x^k/\partial \xi^i$ with $x^k = x^k(\tau, \xi^1, \xi^2, \xi^3)$. The traversability condition gives the following restrictions on the functions $t_r(\tau)$ and $x^i(\tau, \xi)$:

$$\alpha^2(1 - v_l v^l) > 0,$$

and hence

$$\alpha^2 > \varepsilon > 0,$$

where $\varepsilon = \text{const} > 0$, i.e. $t_r(\tau)$ must be a monotonous function on $\tau$ without stationary points, and

$$v_l v^l < 1.$$  

Equations (7)-(11) with constraints (12)-(13) define the boundary conditions for the interior wormhole metrics (4). Namely, the comparison of (4) with its limiting forms (7) and (9) gives

$$a^2(\tau, \xi) = \begin{cases} 
1 & \text{for } -(\sigma_1 + L_1) < \xi^1 < -L_1; \\
\alpha^2 \cdot (1 - v_l v^l) & \text{for } L_2 < \xi^1 < L_2 + \sigma_2,
\end{cases}$$
\[ \beta_i = \begin{cases} 0 & \text{for } -(\sigma_1 + L_1) < \xi^1 < -L_1 \\ \gamma_{kl} \frac{\partial x^k}{\partial \tau} \frac{\partial x^l}{\partial \xi^i} & \text{for } L_2 < \xi^1 < L_2 + \sigma_2. \end{cases} \] \tag{15}

and

\[ \tilde{\gamma}_{ij} = \begin{cases} \gamma'_{ij} & \text{for } -(\sigma_1 + L_1) < \xi^1 < -L_1 \\ \gamma''_{ij} & \text{for } L_2 < \xi^1 < L_2 + \sigma_2. \end{cases} \] \tag{16}

where \( \gamma_{kl} \) are the metric tensor of the exterior 3-space, \( x^k = x^k(\xi^i) \) near the left mouth \((\xi^1 \to -(\sigma_1 + L_1))\) and \( x^k = x^k(\tau, \xi^i) \) near the right mouth \((\xi^1 \to L_2)\), \( \gamma'_{ij} \) and \( \gamma''_{ij} \) are defined by the equations (8) and (11).

### 4 Mixed boundary problem for the wormhole-type models

Now we may formulate the mixed boundary problem for the traversable wormhole models.

It is clear, that to construct any space-time model it is necessary to have the following data: structure of manifold, field equations, initial and/or boundary conditions.

1. **The structure of manifolds.** According to the above assumption space-time consist of the internal and external parts, which have topologies \( T_{int} \times M^3_{int} \) and \( T_{ext} \times M^3_{ext} \) respectively, where \( T_{int} \) and \( T_{ext} \) are the interior and exterior times and \( M^3_{int} \) and \( M^3_{ext} \) are the interior and exterior spaces. Up to the definition of \( M^3_{int} \) and \( M^3_{ext} \), the structure of space-time, i.e. the joining of interior and exterior parts, is given by the equations (3)-(6) which must satisfy to the constraints (12) and (13).

2. **Field equations.** For simplicity, it will supposed that the metric of space-time must satisfy to the standard Einstein equations

\[ G^0_0 = \kappa T^0_0, \quad G^0_i = \kappa T^0_i \] \tag{17}

and

\[ G^i_j = \kappa T^i_j \] \tag{18}

where \( G^\alpha_\beta \) is Einstein tensor, \( \kappa \) is Einstein gravitational constant and \( T^\alpha_\beta \) is the energy-momentum tensor of matter and non-gravitational fields. Equations (17) are the constraint equations and equations (18) are dynamical. These equations must be supplemented by the matter and non-gravitational fields equations, which will not considered here.

3. **Initial and boundary conditions.** According to the assumptions about the interior and exterior space-time structures and the traversability of the wormhole, the interior metric must have the form (1) with \( a^2(\tau, \xi) > \varepsilon > 0 \) and it is supposed for simplicity that the exterior space-time metric has the form (2). Equations (3)-(6), which define the topology of space-time, induce the correspondence between components of the interior and exterior metric tensors in the intersection of interior and exterior regions of space-time. Equations (14)-(16), which define this correspondence, must be considered as additional boundary conditions for the components of the metric tensor.

Indeed, in the regions \(-(\sigma_1 + L_1) < \xi^1 < -L_1 \) and \( L_2 < \xi^1 < L_2 + \sigma_2 \), where interior space-time intersects with exterior one, both the equations (3)-(6) and induced equations (14)-(16)
have the form of coordinate transformation and have no effect on the energy-momentum tensor. Therefore, in opposition to widespread opinion [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 25], these equations are independent from the field equations and hence, have nondynamical nature.

At last, usual initial conditions for the Einstein equations must be given, namely: the components of the interior metric tensor and their first partial derivatives with respect to the interior time coordinate $\tau$, i.e.

$$ a^2(\tau_0, \xi), \beta_i(\tau_0, \xi), \tilde{\gamma}_{ij}(\tau_0, \xi), \partial a^2(\tau_0, \xi)/\partial \tau, \beta_{i,\tau}(\tau_0, \xi), \tilde{\gamma}_{ij,\tau}(\tau_0, \xi), \quad (19) $$

with

$$ a^2(\tau_0, \xi) > 0, $$

and the components of the exterior metric and their first partial derivatives with respect to the exterior time coordinate $t$, i.e.

$$ \gamma_{ij}(t_0, x), \gamma_{ij,t}(t_0, x). \quad (20) $$

These quantities must satisfy to the boundary conditions (14)-(16) and the corresponding conditions for derivatives, which near the left mouth $-(\sigma_1 + L_1) < \xi^1 < -L_1$ take the form

$$ \frac{\partial a^2(\tau_0, \xi)}{\partial \tau} = 0, \quad \frac{\partial \beta_i(\tau_0, \xi)}{\partial \tau} = 0, \quad (21) $$

and

$$ \tilde{\gamma}_{ij,\tau}(\tau_0, \xi) = \gamma_{kl,t}(t_0, x)x^k_i x^l_j \quad (22) $$

while near the right mouth we have

$$ \frac{\partial a^2(\tau_0, \xi)}{\partial \tau} = 2\alpha \alpha \cdot (1 - v_i v^i) - 2\alpha^2 v_i, \tau v^i \quad (23) $$

$$ \beta_{i,\tau}(\tau_0, \xi) = \alpha \gamma_{kl,t} \frac{\partial x^k}{\partial \tau} \frac{\partial x^l}{\partial \xi^i} + \gamma_{kl} \frac{\partial^2 x^k}{\partial \tau^2} \frac{\partial x^l}{\partial \xi^i} + \gamma_{kl} \frac{\partial x^k}{\partial \tau} \frac{\partial^2 x^l}{\partial \xi^i \partial \tau} \quad (24) $$

and

$$ \tilde{\gamma}_{ij,\tau} = \alpha \gamma_{kl,t} \frac{\partial x^k}{\partial \xi^i} \frac{\partial x^l}{\partial \xi^j} + 2\gamma_{kl} \frac{\partial^2 x^k}{\partial \xi^i \partial \tau} \frac{\partial x^l}{\partial \xi^j}. \quad (25) $$

It is obviously, that these quantities must satisfy also to the constraint equations (17).

The contents of this section may be summarized as follows:

**Statement 1** Any space-time model with the traversable wormhole whose interior and exterior metrics have the forms (4) and (5) respectively, is the subject of the mixed boundary problem for the Einstein equation (17)-(18) which is formed by (i) the manifold structure equations (3)-(6), (ii) the boundary conditions (14)-(16) with the traversability constraints (12)-(13), (iii) interior and exterior initial conditions (19)-(20) which must satisfy to (14)-(16) and (21)-(25).
5 Causality violation in the wormhole models

The above equations make possible to obtain some estimations for the causality violation in the traversable wormhole models. Without loss of generality it may be supposed that the sizes of the mouths are much less than the distance between them in the outer space (the approximation of the thin mouths) and the $x^1$ axis connects the centers of the wormhole’s mouths. Let moreover, $t_r(\tau_1) > \tau_1$ for some $\tau_1 > \tau_0$. Consider the light signal which is sent at the moment $t_r(\tau_1)$ from the right mouth to the left one through the wormhole and then return to the right mouth through the outer space. It is clear, that the time delay between the sending and receiving of the signal is equal to

$$\Delta t = \delta t_1 + \delta t_2 - \delta t_3$$

where $\delta t_1$ and $\delta t_2$ are the times of the signal passing through the wormhole and the exterior space respectively and $\delta t_3 = t(\tau_1) - \tau_1$. It is clear that causality violation appears if $\Delta t \leq 0$.

The estimations of the times $\delta t_1$ and $\delta t_2$ may be obtained from the equations (1) and (2). Namely, let

$$R = x^1(\tau_1, L_2), \quad C_{ext} = \max_{0 \leq x^2 \leq R} \gamma_{11}(t, x),$$

and

$$a_0^2 = \min_{-L_1 \leq \xi^1 \leq L_2} a^2(\tau, \xi), \quad b = \max_{-L_1 \leq \xi^1 \leq L_2} |b_i(\tau, \xi)|, \quad N = \max_{-L_1 \leq \xi^1 \leq L_2} \widetilde{\gamma}_{ij}(\tau, \xi),$$

then

$$\delta t_1 \leq C_{int} L, \quad \delta t_2 \leq C_{ext} R,$$

where $L = L_1 + L_2$, and

$$C_{int} = \frac{b + \sqrt{b^2 + N}}{a_0^2},$$

so,

$$\Delta t \leq C_{int} L + C_{ext} R - |t_r(\tau_1) - \tau_1|.$$ 

Thus, the sufficient condition for the causality violation may be written in the form

$$|t(\tau_1) - \tau_1| \geq C_{int} L + C_{ext} R \quad (26)$$

It is necessary to note that this estimation is very rough, so the causality violation may occur even if the inequality (26) does not satisfied.

Analogously, if

$$|t_r(\tau_1) - \tau_1| < C_{int} L + C_{ext} R \quad (27)$$

where

$$C_{int} = \frac{b_m + \sqrt{b_m^2 + N_m}}{a_1^2}, \quad C_{ext} = \min_{0 \leq x^2 \leq R} \gamma_{11}(t, x),$$

and

$$a_1^2 = \max_{-L_1 \leq \xi^1 \leq L_2} a^2(\tau, \xi), \quad b_m = \min_{-L_1 \leq \xi^1 \leq L_2} |b_i(\tau, \xi)|, \quad N_m = \min_{-L_1 \leq \xi^1 \leq L_2} \widetilde{\gamma}_{ij}(\tau, \xi),$$

for all $\tau \in (-\infty, \infty)$ then there are no CTCs in the considered model.
The analogous estimations may be applied also to the wormhole with finite sizes of the mouths if we suppose, that the $x^1$ axis connects two points $p_l$ and $p_r$ of the left and right mouths such that $R \geq x^1(p_r) - x^1(p_l) = \min(x^1(p_2) - x^1(p_1))$ where $p_1$ and $p_2$ are the arbitrary points of the left and right mouths.

Thus, the main parameters, which define the causal structure of the wormhole-type models with the given function $t_r(\tau)$ are the interior "coordinate length" $L$ of the wormhole's handle, the exterior "coordinate distance" $R$ between its mouths and the factors $C_{int}$ and $C_{ext}$. It is follows from the above consideration, that parameters $L$ and $R$ are the subject of the boundary conditions for space-time models with traversable wormhole. These parameters are independent on each other, on the field equations and on the function $t_r(\tau)$. So, using the appropriate choice of the parameters $L$ and $R$ (conditions (8) and (9)) both causal and non-causal space-time models with traversable wormhole may be obtained for the same $t_r(\tau) \neq \tau$. Of cause, for the given boundary conditions (8)-(10) both causal and non-causal space-time models with traversable wormhole may be obtained for the same $t_r(\tau) \neq \tau$. Of cause, for the given boundary conditions (8)-(10) with $t_r(\tau) \neq \tau$ the causality violation depends on the factors $C_{int}$ and $C_{ext}$ which are defined by the field equations. On the other hand, if $t_r(\tau) \equiv \tau$ then causality violation is impossible in the considered model in opposite to the statement of [3] about unavoidable wormhole transformation into the time machine.

It confirm our early statements about non-dynamical nature of CTCs and the impossibility of the dynamical wormhole transformation into the time machine [21, 22].

6 Spherical wormhole in Minkowskian space-time

To demonstrate that the causality violation has no direct connection with some physical processes consider particular case of the traversable spherical wormhole with immovable mouths which is joined with flat Minkowskian exterior space-time.

The exterior flat region of such model is described by Cartesian coordinates $\{t, x, y, z\}$, which vary from $-\infty$ up to $\infty$, and metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2,$$

while the interior region is described by the coordinates $\{\tau, l, \theta, \phi\}$, $-\infty < \tau < \infty$, $-(\sigma_1 + L_1) < l < L_2 + \sigma_2$, and $(\theta, \phi)$ are the polar coordinates on 2-sphere $S^2$. If the mouths of the wormhole are placed on the $x$-axis with the centers at $x_{left} = 0$ and $x_{right} = R = const$ then the joining conditions (11)-(13) reads

$$t_{left} = \tau, \; x_l = l \cos(\phi) \cos(\theta), \; y_l = l \cos(\phi) \sin(\theta), \; z_l = l \sin(\phi)$$

and

$$t_{right} = t_r(\tau), \; x_r = R + l \cos(\phi) \cos(\theta), \; y_l = l \cos(\phi) \sin(\theta), \; z_l = l \sin(\phi).$$

So, the simplest interior metric which satisfies to the boundary conditions (11)-(13) has the form

$$ds^2 = a^2(\tau, l)d\tau^2 - dl^2 - r^2(l)(d\theta^2 + \sin^2(\theta)d\phi^2), \quad (28)$$

with $a(\tau, l) = 1$ for $l < L_1$ and $a(\tau, l) = dt_r/d\tau$ for $l > L_2$ and $r(l) = l$ for $l < -L_1$ or $l > L_2$. In the particular case then $a = a(l)$ this metric coincide with the static metric, considered in [1, 2, 26].
Direct calculation gives the following values of non-zero components of the Einstein tensor in the wormhole interior:

\[ G^0_0 = -\frac{2rr'' + r'^2 - 1}{r^2} \]
\[ G^1_1 = -\frac{ar'^2 + 2rad'r' - a}{ar^2} \]
\[ G^2_2 = G^3_3 = -\left(\frac{r'' + \frac{a'r'}{a} + \frac{a''}{a}}{r} + \frac{a'r'}{a} \right) \]

where the prime (') denote the partial derivative with respect to \( l \).

It is easy to see, that Einstein tensor \( G^\alpha_\beta \) and hence the energy-momentum tensor for the interior space in this model in non-static case (\( t_{\text{left}} = \tau, t_{\text{right}} = t_r(\tau) \neq \tau \)) have the same structure and properties as in static case (\( a(\tau, l) = a(l), t_{\text{left}} = t_{\text{right}} = \tau \)) which was considered in [26]. In particular, the matter in the wormhole interior must have the same "exotic" properties both in static and non-static cases. Further, the Einstein equations impose no restrictions on the dependence of \( a(\tau, l) \) on the interior time \( \tau \). Taking into consideration that \( t_r(\tau) \) is an arbitrary monotonous function (\( dt_r(\tau)/d\tau \neq 0 \)) it may conclude that Einstein equations (and hence, the physical processes in space-time) have no influence on the causal structure of the considered model.

It is easy to see, that the same result may be obtained for any models with arbitrary static exterior space-time, in particular, for the so-called ringhole model, which was considered recently in [25].

### 7 Conclusion

Any space-time model in classical general relativity solves some initial values, boundary or mixed boundary problem for the Einstein equations. Exact formulation of such problem is very important for the physical interpretation of the model. For the most models, which are usually considered in the cosmological or astrophysical context, formulation of the corresponding problem is trivial enough, because such models, as a rule, have rather simple topology and may be considered in the framework of the Cauchy problem with appropriate asymptotic conditions on spatial infinity. In more complicated cases, in particular, for the topologically nontrivial models with nontrivial causal structure, such consideration is impossible. These models are described by means of finite or countable set of coordinate maps [17, 24, 27] which are joined with each other. The conditions of the maps joining, which define the topological structure of space-time, induce the appearance of the additional boundary conditions (or additional constraints) for the field variables.

In this paper such conditions were considered for the space-time models with traversable wormhole. It was shown that the conditions, which define wormhole joining with the outer space, induce the boundary (or consistency) conditions for the fields variables. Although these conditions in some cases may have the form of the motion equations, they have non-dynamical nature and are independent on the fields equations. These conditions, together with appropriate initial conditions, compose the mixed boundary problem for the field equations. In the present paper this problem was formulated in explicit form for the Einstein equations and its generalization on the other fields is straightforward. The estimated conditions for the
causality violations in the traversable wormhole models show that in opposite to wide spread opinion \cite{1, 2, 3, 11, 15, 16} there is no direct connection between the causal properties in space-time and the physical processes in it. In particular, example of the spherical wormhole connected with exterior flat Minkowskian space-time shows that the energy-momentum tensor in the wormhole interior may have the same properties both in causal and noncausal cases. Moreover, if wormhole connects the events on the same exterior space-like section, i.e. if conditions (3) and (5) have the form \( t_{\text{left}} = t_{\text{right}} = \tau \), causality violation is impossible independently on motion of the wormhole’s mouths and on the other physical processes in space-time.

Since the space-time models with traversable wormhole in general case is the subject of the mixed boundary problem, the physical processes in such space-time must be analyzed in the framework of the analogous problem. As it was shown in \cite{23}, such consideration does not lead to any paradoxes in the case of causality violation.

It is necessary to note that the serious problem which arise in the wormhole models is the nature and physical interpretation of the conditions of wormhole joining with exterior space-time (3)-(6) and the induced conditions for the field variables (14)-(16) in general case \( t_{\text{right}} \neq t_{\text{left}} \). Really, the mixed boundary problem for the traversable wormhole reduces to the usual Cauchy problem only for the globally hyperbolic models without causality violations. However, excluding the trivial case \( t_{\text{left}} = t_{\text{right}} = \tau \), the noncausal globally hyperbolic nature of the model may be determine, in general, only after solution of the mixed boundary problem which is formulated above. To have the full set of initial and boundary conditions for this problem the local observer must receive information both from his past and future. The last is very problematic, because all known physical processes make possible to send information only from past to future, but not reverse.

Acknowledgments

This work was supported by the Russian Ministry of Science and the Russian Fund of Basic Research (grant N 95-02-05785-a).

References

[1] M.S.Morris, K.S. Thorne, U. Yourtsever, *Phys. Rev. Lett.*, (1988), 61, 1446.
[2] I.D. Novikov, *JETF* (1989), 95, 769 (in Russian).
[3] V.P. Frolov, I.D. Novikov, *Phys. Rev. D.*, (1990), 42, 1057.
[4] J.R. Gott, *Phys. Rev. Lett.*, (1990), 66, 1126.
[5] M. Visser, *Phys. Rev. D.*, (1997), 55, 5212.
[6] D. Deutsch, *Phys. Rev. D.*, (1991), 44, 3197.
[7] S. Kalyana Rama, S. Sen, *Inconsistent Physics in the Presence of Time Machines*, Preprint MRY-PHY/14/94/, TCD-7-94 (hep-th/9410031).
[8] S.V. Krasnikov, *Phys. Rev. D.*, (1997), 55, N 6, 3427-3430.
[9] S.-W. Kim, K.S. Thorne, *Phys. Rev. D.*, (1990), 43, 3929.
[10] V.P. Frolov, *Phys. Rev. D.*, (1991), 43, 3878.
[11] S.W. Hawking, *Phys. Rev. D.*, (1992), 46, 603.
[12] M. Visser, *Phys. Rev. D.*, (1993), 47, 554.
[13] D.S. Goldwirth, M.J. Perry, T. Piran, K.S. Thorn, *Phys. Rev. D.*, (1994), 49, 3951.
[14] T. Tanaka, W.A. Hiscock, *Phys. Rev. D.*, (1994) 49, 5240.
[15] J. Fridman, M.S. Morris, I.D. Novikov, F. Echeverra, G. Klinkhammer, K.S. Thorne, U. Yourtsever, *Phys. Rev. D.*, (1990), 42, 1915.
[16] A. Carlini, V.P. Frolov, M.B. Mensky, I.D. Novikov, H.H. Soleng, *Int. J. Mod. Phys. D.*, (1995), 4, 557.
[17] G.F.R. Ellis, S.W. Hawking, The large-scale structure of space-time, Cambridge U.P., 1973.
[18] C. W. Misner, K. Thorne, J. A. Wheeler, *Gravitation*, v. 2, W.H. Freeman, San Francisco, 1974.
[19] J.W. York - in: Sources of Gravitational Radiation, ed. L. Smarr, Cambridge, Cambridge U.P., 1979, p. 83.
[20] A. E. Fisher and J. E. Marsden, *Initial values problem and dynamical formulation of general relativity*, in: General Relativity. An Einstein Centenary survey, Eds. S.W. Hawking and W. Israel, Cambridge U.P., 1979.
[21] M.Yu. Konstantinov, *Isvestiya vuzov.Fizika*, (1992), N 12, p. 83 (in Russian).
[22] M.Yu. Konstantinov, *Int. J. Mod. Phys. D.*, (1995), v. 4, N 2, p. 247.
[23] M.Yu. Konstantinov, Causality Properties of Topologically Nontrivial Space-Time Models. II. The Nature of ”Self-Consistency” Conditions., submitted to *Int. J. Mod. Phys. D*.
[24] J. Milnor, Morse Theory, Annals of Mathematical Studies N 51, Princeton UP, Princeton, 1963.
[25] Pedro F. Gonzales-Diaz, *Phys. Rev. D.*, (1996), 54, 6122.
[26] M.S. Morris, K.S. Thorne, *Amer. J. Phys.*., (1988), 59, 395.
[27] S. Kobayashi, K. Nomizu, Foundation of differential geometry, v. 1-2, New-York - London, 1963.