The Algebraic Connectivity of a Graph and its Complement

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Abstract

For a graph $G$, let $\lambda_2(G)$ denote its second smallest Laplacian eigenvalue. It was conjectured that $\lambda_2(G) + \lambda_2(\overline{G}) \geq 1$, where $\overline{G}$ is the complement of $G$. In this paper, it is shown that $\max\{\lambda_2(G), \lambda_2(\overline{G})\} \geq \frac{2}{5}$.

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1 Introduction

All graphs considered in this paper are simple (no loops and no multiple edges). If $G$ is a graph and $v \in V(G)$, we denote by $N_G(v)$ the set of vertices adjacent to $v$. We denote the complementary graph of $G$ by $\overline{G}$.

The adjacency matrix $A(G)$ of $G$ is the matrix whose $(u,v)$-entry is equal to 1 if $uv \in E(G)$ and 0 otherwise. If $D(G)$ denotes the diagonal matrix of vertex degrees, then the Laplacian of the graph $G$ is defined as $L(G) = D(G) - A(G)$. We denote the Laplacian eigenvalues of $G$ by

$$0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G).$$

The second smallest eigenvalue $\lambda_2(G)$ is also called the algebraic connectivity of $G$ and is an important indicator related to various properties of the graph. It is well-known that

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the eigenvalues of the complementary graph \( L(G) \) are

\[
0 = \lambda_1(G) \leq n - \lambda_n(G) \leq n - \lambda_{n-1}(G) \leq \ldots \leq n - \lambda_2(G).
\]

The Laplacian spread of a graph \( G \) is defined to be \( \lambda_n(G) - \lambda_2(G) \). Clearly, \( \lambda_n(G) - \lambda_2(G) \leq n \). It was conjectured [10,11] that this quantity is at most \( n - 1 \).

**Conjecture 1.** For any graph \( G \) of order \( n \geq 2 \), the following holds:

\[
\lambda_n(G) - \lambda_2(G) \leq n - 1,
\]

or equivalently \( \lambda_2(G) + \lambda_2(G) \geq 1 \), with equality if and only if \( G \) or \( \overline{G} \) is isomorphic to the join of an isolated vertex and a disconnected graph of order \( n - 1 \).

This conjecture holds for trees [5], unicyclic graphs [1], bicyclic graphs [4,6,8], tricyclic graphs [3], cactus graphs [7], quasi-tree graphs [9], graphs with diameter not equal to 3 [11], bipartite and \( K_3 \)-free graphs [2].

In this paper, we provide a positive constant lower bound for \( \lambda_2(G) + \lambda_2(G) \) by proving the following.

**Theorem 2.** Let \( G \) be a graph of order \( n \geq 2 \). Then

\[
\max \{ \lambda_2(G), \lambda_2(G) \} \geq \frac{2}{5}.
\]

## 2 Routings

If \( P \) is a path in a graph \( G \), we denote its length by \( \|P\| \). Let \( \mathcal{P} \) be a set of paths in \( G \). For any edge \( e \in G \), let \( w_\mathcal{P}(e) \) be the sum of the lengths of all paths in \( \mathcal{P} \) which contain \( e \). We say \( \mathcal{P} \) has weighted congestion \( w \), where \( w = \max_{e \in E(G)} w_\mathcal{P}(e) \). We denote the weighted congestion of \( \mathcal{P} \) by \( w(\mathcal{P}) \).

A set \( \mathcal{P} \) of paths in \( G \) is called a routing if for any distinct vertices \( x, y \in V(G) \), there is exactly one path \( P_{xy} \in \mathcal{P} \) with endpoints \( x \) and \( y \). In particular, this means that \( P_{xy} = P_{yx} \).

**Theorem 3.** Let \( G \) be a graph of order \( n \). If \( G \) has a routing \( \mathcal{P} \) of weighted congestion at most \( w \), then \( \lambda_2(G) \geq \frac{n}{w} \).

*Proof.* Let \( f : V(G) \to \mathbb{R} \) be an eigenvector of \( L(G) \) corresponding to the eigenvalue \( \lambda_2(G) \). Then \( f \) is orthogonal to the all-one vector \( \mathbf{1} \), since \( \mathbf{1} \) is an eigenvector corresponding to \( \lambda_1(G) \). This means that \( \sum_{x \in V(G)} f(x) = 0 \).
Note that we have
\[ \|f\|^2 = \frac{1}{n} \sum_{\{x,y\} \subseteq V(G)} (f(x) - f(y))^2, \]  
(1)
because
\[ 2 \sum_{\{x,y\} \subseteq V(G)} (f(x) - f(y))^2 = \sum_{x,y \in V(G)} (f(x) - f(y))^2 \]
\[ = \sum_{x,y \in V(G)} (f(x)^2 + f(y)^2 - 2f(x)f(y)) \]
\[ = 2n \sum_{x \in V(G)} f(x)^2 - 2 \left( \sum_{x \in V(G)} f(x) \right)^2 \]
\[ = 2n \|f\|^2. \]

For any distinct vertices \( x, y \in V(G) \), let \( P_{xy} \in \mathcal{P} \) be the path in \( \mathcal{P} \) with end points \( x \) and \( y \). We may assume that \( \|f\| = 1 \). Then,
\[ \lambda_2(G) = \sum_{xy \in E(G)} (f(x) - f(y))^2 \]
\[ = \frac{n \sum_{xy \in E(G)} (f(x) - f(y))^2}{\sum_{\{x,y\} \subseteq V(G)} (f(x) - f(y))^2} \]  
(by (1))
\[ \geq \frac{n \sum_{xy \in E(G)} (f(x) - f(y))^2}{\|P_{xy}\| \sum_{uv \in P_{xy}} (f(u) - f(v))^2} \]  
(by Cauchy-Schwarz inequality)
\[ \geq \frac{n \sum_{xy \in E(G)} (f(x) - f(y))^2}{\sum_{uv \in E(G)} w_P(uv)(f(u) - f(v))^2} \]
\[ \geq \frac{n}{w}. \]

The proof is complete. \( \square \)

3 Main Result

Now, we prove the main result of this paper.

**Theorem 4.** Let \( G \) be a graph of order \( n \geq 2 \). At least one of \( G \) or \( \overline{G} \) has a routing of weighted congestion at most \( 5n/2 \).

**Proof.** The proof is by induction on \( n \). If \( G \) has diameter at most 2, we select for each pair \( \{x, y\} \) a path of length 1 or 2 joining \( x \) and \( y \). For any edge \( e = xy \in E(G) \), the
paths through $e$ are the edge $xy$, some paths from $x$ to $V(G) \setminus (N_G(x) \cup \{x\})$ and some paths from $y$ to $N_G(x) \setminus \{y\}$. Thus the weighted congestion of $e$ is at most $1 + 2(n - 1 - |N_G(x)|) + 2(|N_G(x)| - 1) = 2n - 3$. Thus, this routing has weighted congestion at most $2n - 3$. Hence, we may assume from now on that neither $G$ nor $\overline{G}$ has diameter less than 3. This implies that both, $G$ and $\overline{G}$ have diameter exactly 3 and that $n \geq 4$.

Let $u, v$ be vertices whose distance in $\overline{G}$ is 3, and let $u', v'$ have distance 3 in $G$. Then $uv \in E(G), u'v' \in E(\overline{G})$ and $N_G(u) \cup N_G(v) = N_{\overline{G}}(u') \cup N_{\overline{G}}(v') = V(G)$. This implies in particular that in $G$, every vertex is at distance at most 2 from $u$. Thus $u \notin \{u', v'\}$. By symmetry, we conclude that $\{u, v\} \cap \{u', v'\} = \emptyset$. Let

$X := N_G(u) \setminus (N_G(v) \cup \{v\})$, $Y := N_G(v) \setminus (N_G(u) \cup \{u\})$, $Z := N_G(u) \cap N_G(v)$,

$X' := N_{\overline{G}}(u') \setminus (N_{\overline{G}}(v') \cup \{v'\})$, $Y' := N_{\overline{G}}(v') \setminus (N_{\overline{G}}(u') \cup \{u'\})$, $Z' := N_{\overline{G}}(u') \cap N_{\overline{G}}(v')$.

Without loss of generality assume that $|X| \leq |Y|$ and $|X'| \leq |Y'|$.

Since $u'$ and $v'$ are at distance 3 in $G$, we have

$$|\{u', v'\} \cap X| = |\{u', v'\} \cap Y| = 1. \tag{2}$$

We may thus assume that $u' \in X$ and $v' \in Y$. Similarly, we may assume that $u \in X'$ and $v \in Y'$.

Let $H = G - \{v, v'\}$. By induction, at least one of $H$ or $\overline{H}$ has a routing of weighted congestion at most $5(n - 2)/2$.

Let us first assume that $H$ has such a routing $\mathcal{P}_H$. Let $\mathcal{A}$ be the set of following paths taken for every $z \in V(H)$:

$$P_{vz} = \begin{cases} vz & \text{if } z \in Y \cup Z \cup \{u\}, \\ vuz & \text{if } z \in X. \end{cases}$$

We have

$$w_{\mathcal{A}}(e) = \begin{cases} 1 & \text{if } e = vz \text{ and } z \in Y \cup Z, \\ 2|X| + 1 & \text{if } e = vu, \\ 2 & \text{if } e = uz \text{ and } z \in X, \\ 0 & \text{otherwise}. \end{cases}$$

Let $\mathcal{B}$ be the following set of paths taken for each $z \in V(H) \cup \{v\}$:

$$P_{v'z} = \begin{cases} v'z & \text{if } z = v, \\ v'vz & \text{if } z \in Y \cup Z \cup \{u\}, \\ v'vuz & \text{if } z \in X. \end{cases}$$
We have 

\[ w_B(e) = \begin{cases} 
2n - 3 + |X| & \text{if } e = v'v, \\
2 & \text{if } e = vz \text{ and } z \in (Y \cup Z) \setminus \{v'\}, \\
3|X| + 2 & \text{if } e = vu, \\
3 & \text{if } e = uz \text{ and } z \in X, \\
0 & \text{otherwise.} 
\end{cases} \]

The set of paths \( \mathcal{P} = \mathcal{P}_H \cup \mathcal{A} \cup \mathcal{B} \) is a routing in \( G \) with weighted congestion at most \( 5n/2 \) since \( |X| \leq \frac{n-2}{2} \) and \( w(\mathcal{P}_H) \leq 5(n-2)/2 \). Note that all we needed for this conclusion was that we had a routing in \( H \) and that \( v' \in Y \) (since we used the fact that \( |X| \leq |Y| \) to conclude that \( |X| \leq (n-2)/2 \)).

Suppose now that the requested routing exists in \( \overline{\Pi} \). Since \( v \in Y' \), the same proof as above shows that we can obtain a routing in \( \overline{G} \) with weighted congestion at most \( 5n/2 \). This completes the proof.

**Proof of Theorem 2** By Theorems 3 and 4, every graph \( G \) of order \( n \geq 2 \) satisfies:

\[ \max \{ \lambda_2(G), \lambda_2(\overline{G}) \} \geq \frac{n}{5n/2} = \frac{2}{5} \]

which is what we were to prove.

At the end, we pose the following question:

**Question.** What is the supremum of all real numbers \( c \) such that for any graph \( G \) of order at least 2,

\[ \max \{ \lambda_2(G), \lambda_2(\overline{G}) \} \geq c. \]

The path \( P_4 \) (which is self-complementary) has \( \lambda_2(P_4) = 2 - \sqrt{2} < 0.5858 \). This shows that the supremum will be smaller than 0.5858.

**References**

[1] Y.-H. Bao, Y.-Y. Tan, and Y.-Z. Fan, The Laplacian spread of unicyclic graphs, *Appl. Math. Lett.* **22** (2009), 1011–1015.

[2] X. Chen and K.C. Das, Some results on the Laplacian spread of a graph, *Linear Algebra Appl.* **505** (2016), 245–260.
[3] Y. Chen and L. Wang, The Laplacian spread of tricyclic graphs, *Electron. J. Combin.* **16** (2009), Research Paper 80, 18 pp.

[4] Y.Z. Fan, S.D. Li, and Y.Y. Tan, The Laplacian spread of bicyclic graphs, *J. Math. Res. Exposition* **30** (2010), 17–28.

[5] Y.-Z. Fan, J. Xu, Y. Wang, and D. Liang, The Laplacian spread of a tree, *Discrete Math. Theor. Comput. Sci.* **10** (2008), 79–86.

[6] P. Li, J.S. Shi, and R.L. Li, Laplacian spread of bicyclic graphs, (Chinese) *J. East China Norm. Univ. Natur. Sci. Ed.* (2010), 6–9.

[7] Y. Liu, The Laplacian spread of cactuses, *Discrete Math. Theor. Comput. Sci.* **12** (2010), 35–40.

[8] Y. Liu and L. Wang, The Laplacian spread of bicyclic graphs, *Advances in Mathematics (China)* **40** (2011), 759–764.

[9] Y. Xu and J. Meng, The Laplacian spread of quasi-tree graphs, *Linear Algebra Appl.* **435** (2011), 60–66.

[10] Z. You and B. Liu, The Laplacian spread of graphs, *Czechoslovak Math. J.* **62** (137) (2012), 155–168.

[11] M. Zhai, J. Shu, and Y. Hong, On the Laplacian spread of graphs, *Appl. Math. Lett.* **24** (2011), 2097–2101.