Hyperbolicity and Constrained Evolution in Linearized Gravity

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Abstract

Solving the 4-d Einstein equations as evolution in time requires solving equations of two types: the four elliptic initial data (constraint) equations, followed by the six second order evolution equations. Analytically the constraint equations remain solved under the action of the evolution, and one approach is to simply monitor them (unconstrained evolution). Since computational solution of differential equations introduces almost inevitable errors, it is clearly “more correct” to introduce a scheme which actively maintains the constraints by solution (constrained evolution). This has shown promise in computational settings, but the analysis of the resulting mixed elliptic hyperbolic method has not been completely carried out. We present such an analysis for one method of constrained evolution, applied to a simple vacuum system, linearized gravitational waves.

We begin with a study of the hyperbolicity of the unconstrained Einstein equations. (Because the study of hyperbolicity deals only with the highest derivative order in the equations, linearization loses no essential details.) We then give explicit analytical construction of the effect of initial data setting and constrained evolution for linearized gravitational waves. While this is clearly a toy model with regard to constrained evolution, certain interesting features are found which have relevance to the full nonlinear Einstein equations.

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I. INTRODUCTION

Binary black hole systems are expected to be the strongest possible astrophysical gravitational wave sources. In the final moments of stellar mass black hole inspiral, the radiation will be detectable in the current (LIGO-class) detectors. If the total binary mass is of the order of $10M_\odot$, the moment of final plunge to coalescence will emit a signal detectable by the current generation of detectors from very distant (Gpc) sources. The merger of supermassive black holes in the center of galaxies will be the very dominant signal in the spaceborne LISA detector, and detectable out to large redshift. Simulation of these mergers will play an important part in the prediction, detection, and the analysis of their gravitational signals in gravitational wave detectors. To do so requires a correct formalism which does not generate spurious singularities during the attempted simulation.

Recent work at Texas [1], [2] has found that constrained 3-d evolution can produce substantially stabilized long-term single black hole simulations, stimulating interest in constrained evolution. However, it is true that to date there has been no analysis that addresses the behavior of constrained evolution for computational relativity. That is the purpose of this paper. (Note that unlike the situation in electromagnetism, where a discrete simplectic calculus can be written which assures conservation of a discretized version of the E&M constraints, there appears to be no equivalent formulation for general relativity that conserves discretized versions of the gravitational constraints[3].)

II. $3+1$ FORMULATION OF EINSTEIN EQUATIONS

We take a Cauchy formulation (3+1) of the ADM type, after Arnowitt, Deser, and Misner [4]. In such a method the 3-metric $g_{ij}$ and its momentum $K_{ij}$ (the extrinsic curvature) are specified at one initial time on a spacelike hypersurface, and evolved into the future. The ADM metric is

$$ds^2 = - (\alpha^2 - \beta_i \beta^i) \, dt^2 + 2 \beta_i \, dt \, dx^i + g_{ij} \, dx^i \, dx^j$$

(1)

where $\alpha$ is the lapse function and $\beta^i$ is the shift 3-vector; functions that relate the coordinates on each hypersurface to each other. [16]

[16] Latin indices run 1, 2, 3 and are lowered and raised by $g_{ij}$ and its 3-d inverse $g^{ij}$. 

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The Einstein field equations contain both hyperbolic evolution equations, and elliptic constraint equations. The constraint equations for vacuum in the ADM decomposition are:

\[ H = \frac{1}{2}[R - K^i_j K^{ij} + K^2] = 0, \]

\[ H^i = \nabla_j (K^{ij} - g^{ij} K) = 0. \]

Here \( R \) is the 3-d Ricci scalar constructed from the 3-metric, and \( \nabla_j \) is the 3-d covariant derivative compatible with \( g_{ij} \). Initial data must satisfy these constraint equations; one may not freely specify all components of \( g_{ij} \) and \( K_{ij} \).

The evolution equations from the Einstein system are

\[ \partial_t g_{ij} = -2\alpha K_{ij} + \nabla_j \beta_i + \nabla_i \beta_j, \]

and

\[ \partial_t K_{ij} = -\nabla_i \nabla_j \alpha + \alpha [R_{ij} - 2K_{ia}K^a_j + KK_{ij}] + \beta^k K_{ij,k} + K_{kj} \beta_j^k + K_{ik} \beta_j^k, \]

where \( R_{ij} \) is the 3-d Ricci tensor.

We refer to this form of the Einstein equations as of ADM type, referring to the fundamental development [4]; this specific form is called the \( \dot{g} - \dot{K} \) form. Here, Eq. (2)–Eq. (3), the constraint equations, are the vacuum Einstein equations \( 4G_{00} = 0 \) and \( 4G_{0i} = 0 \) respectively, for a 4-metric of form Eq. (1). Eq. (4)–Eq. (5), the evolution equations, are a first order form of the vacuum Einstein equations \( 4R_{ij} = 0 \). The methods described here can be easily extended to linearized versions of some other formulations of Einstein’s equations, and one of these is explicitly addressed below.

III. DATA FORM

In this paper we consider only linearized gravitational waves (but see Appendix A).

We write the spatial metric as \( g_{ab} = \delta_{ab} + h_{ab} \). We work with zero shift vector: \( \beta^i = 0 \) (cf. [13]), but we consider a lapse \( \alpha \) which may differ from the flat background value by a function of first order in the linearization: \( \alpha = 1 + \alpha_1 + O(h_{ab}^2) \). In particular we choose
below a *densitized lapse*:

\[
\alpha = g^b \\
= 1 + b(h_{xx} + h_{yy} + h_{zz}) + O(h_{ab}^2),
\]

where the quantity \( g \) is the determinant of the 3–metric, and \( b \) is a chosen constant. Generally, densitized lapse can involve multiplication by a fixed function of the coordinates, but we assume linearization from flat space, so this function here must be unity.

For our linearized case, Eq. (4) reads

\[
\partial_t h_{ij} = -2K_{ij},
\]

and Eq. (5) is:

\[
\partial_t K_{ij} = \frac{1}{2} \alpha^2 \delta^{lm} \left( \partial_l \partial_m h_{ij} + \partial_i \partial_j h_{lm} - \partial_i \partial_l h_{mj} - \partial_j \partial_l h_{mi} \right) - \partial_i \partial_j \alpha.
\]

Eq. (8) can, with Eq. (7), be read as a second-order in time equation for \( h_{ij} \):

\[
\partial_t^2 h_{ij} = \alpha^2 \delta^{lm} \left( \partial_l \partial_m h_{ij} + \partial_i \partial_j h_{lm} - \partial_i \partial_l h_{mj} - \partial_j \partial_l h_{mi} \right) + 2 \partial_i \partial_j \alpha.
\]

We further assume a 1–dimensional system, so that all quantities are functions of \((x^1, t) = (x, t)\), where \( x \) is a spatial variable. It is then profitable to write out the individual components explicitly. We shall find that the different components \( xx \) (longitudinal-longitudinal: \( LL \)); \( xy \) & \( xz \) (transverse-longitudinal: \( TL \)); \( yy + zz \) (transverse trace); and \( yy - zz \) & \( yz \) (transverse-traceless: \( TT \)) have, as expected, different behavior. This explicit formulation parallels, in a less sophisticated but more transparent manner, the decomposition in \[13\].

The second-order evolution equations, written explicitly in terms of these variables are:

\[
\partial_t^2 h_{xx} = \partial_x^2 (h_{yy} + h_{zz}) + 2 \partial_i \partial_j \alpha
\]

\[
\partial_t^2 h_{xy} = 0
\]

\[
\partial_t^2 h_{xz} = 0
\]

\[
\partial_t^2 (h_{yy} + h_{zz}) = \partial_x^2 (h_{yy} + h_{zz})
\]
\[ \frac{\partial^2}{\partial t^2} (h_{yy} - h_{zz}) = \frac{\partial^2}{\partial x^2} (h_{yy} - h_{zz}) \] (14)

\[ \frac{\partial^2}{\partial t^2} (h_{yz}) = \frac{\partial^2}{\partial x^2} (h_{yz}). \] (15)

The constraint equations become:

\[ H = -\frac{\partial^2}{\partial x^2} (h_{yy} + h_{zz}), \] (16)

\[ H_x = -2\frac{\partial}{\partial x} (K_{yy} + K_{zz}), \] (17)

\[ H_y = 2\frac{\partial}{\partial x} K_{xy}, \] (18)

\[ H_z = 2\frac{\partial}{\partial x} K_{xz}. \] (19)

Note that since the spatial metric is \( \delta_{ab} \), we drop the distinction between raised and lowered indices in these explicit Equations (10)-(19). We relate the constraint equations (17)-(19) to \( \partial_t h_{ij} \) via Eq. (7) above.

**IV. INITIAL DATA SETTING AND CONSTRAINED EVOLUTION**

We first give the full, nonlinear data setting (constraint solving) procedure, then specialize in the next section to our linearized plane wave case.

We adopt the conformal transverse-traceless method of York and collaborators [5]-[10] which consists of a conformal decomposition with a scalar \( \phi \) that adjusts the Hamiltonian constraint, and a vector potential \( w^i \) that adjusts the longitudinal components of the extrinsic curvature. The constraint equations are then solved for these new quantities such that the complete solution fully satisfies the constraints.

For initial data setting we pose a trial metric and a trial trace-subtracted extrinsic curvature taken as conformal trial functions \( \tilde{g}_{ij} \) and \( \tilde{A}^{ij} \).

The physical metric, \( g_{ij} \), and the trace-free part of the extrinsic curvature, \( A_{ij} \), are related to the background fields through a conformal factor

\[ g_{ij} = \phi^4 \tilde{g}_{ij}, \] (20)

\[ A_{ij} = \phi^{-10} (\tilde{A}^{ij} + (l^w)_{ij}), \] (21)
where \( \phi \) is the conformal factor, and \( (l\tilde{w})^{ij} \) will be used to cancel any possible longitudinal contribution in \( \tilde{A} \). \( w^i \) is a vector potential, and

\[
(l\tilde{w})^{ij} \equiv \tilde{\nabla}^i w^j + \tilde{\nabla}^j w^i - \frac{2}{3} \tilde{g}^{ij} \tilde{\nabla}^k w^k.
\]  
(22)

(Here \( \tilde{\nabla}_k \) is the covariant derivative in the conformal background space.) The trace \( K \) is not corrected:

\[
K = \tilde{K}.
\]
(23)

Writing the full, nonlinear, Hamiltonian and momentum constraint equations in terms of the quantities in Eqs. (21)–(23), we obtain four coupled elliptic equations for the fields \( \phi \) and \( w^i \):

\[
\tilde{\nabla}^2 \phi = (1/8)(\tilde{R}\phi + \frac{2}{3} \tilde{K}^2 \phi^5 - \phi^{-7}(\tilde{A}^{ij} + (l\tilde{w})^{ij})(\tilde{A}_{ij} + (l\tilde{w})_{ij})),
\]
(24)

\[
\tilde{\nabla}_j (l\tilde{w})^{ij} = \frac{2}{3} \tilde{g}^{ij} \phi^6 \tilde{\nabla}_j K - \tilde{\nabla}_j \tilde{A}^{ij}.
\]
(25)

After the data have been set, the evolution begins. A well designed evolution scheme evolves to the next timestep. In an unconstrained evolution, this completes the time step. In our constrained scheme, the metric and extrinsic curvature so determined are taken as intermediate values in the middle of a timestep and thus as conformal trial functions \( \tilde{g}^{ij} \) and \( \tilde{A}^{ij} \).

The constraint Equations (24), (25) are solved with these trial functions to complete each time update step. The resulting solved \( g_{ij} \) and \( K_{ij} \) are taken as the data for the next time update. Notice that these equations require no specific gauge choice; the constraint solution is independent of the gauge functions \( \alpha \) and \( \beta^i \). A similar approach also can be applied to other formulations which generally have a larger number of constraints.

V. INITIAL DATA SETTING FOR THE LINEARIZED SYSTEM

We assume a choice of \( \tilde{h}_{ab} \) and \( \tilde{K}_{ab} \). We then solve the linearized version of the constraint Equations (24), (25), obtaining \( \phi \) and \( w^i \). As in Section III we assume these variables are functions only of \( x \) (not of time, because this is the intial data problem, which is solved at
one instant of time). Then since the extrinsic curvature vanishes in the (flat) background, the linearized version of the Hamiltonian constraint \(24\) is simply

\[
\partial_x^2 \phi = \frac{1}{8} \tilde{R},
\]

\[
= -\frac{1}{8} \partial_x^2 (\tilde{h}_{yy} + \tilde{h}_{zz}).
\]  

(26)

The linearized version of the momentum constraints \(25\), written explicitly for each component, are:

\[
\frac{4}{3} w_{x,x} = \frac{2}{3} K_{,x} - \tilde{A}_{x,x}. \quad (27)
\]

\[
w_{y,x} = -\tilde{A}_{y,x}. \quad (28)
\]

\[
w_{z,x} = -\tilde{A}_{z,x}. \quad (29)
\]

A. Elliptic Equation Boundary Conditions

A solution of the elliptic constraint equations requires that boundary data be specified. For this demonstration of the technique, analytic integration in our 1-dimensional system will lead to arbitrary integration constants. Specific choices for these constants is equivalent to setting \(\phi\) and \(w_i^x\) to specific boundary values; this will be amplified in the Appendix, where we give a brief discussion of the effect of such boundary condition choices. For instance, we will see that setting the integration constants to zero leads to simplest analytical results. However this is somewhat at variance with the process we use in full 3-dimensional simulations to handle black holes, where we choose conditions \(\phi = 1\) and \(w^i = 0\) at the inner boundaries (the excision boundaries of the black holes); and mixed (Robin) \([11]\) conditions at the outer boundaries for black holes \([9]\).

B. Constraint Solution for Linear Waves

In the linearized regime the Equations \(26\) - \(29\) have decoupled. Each of these equations is an ordinary differential equation, and is straightforwardly integrated. We obtain solutions:

\[
\phi_1 = -\frac{1}{8} (\tilde{h}_{yy} + \tilde{h}_{zz}) + Ax + B.
\]  

(30)
where \( A, B, C_i \) are integration constants. Note that since \( w^i \) is a potential, we need only its first derivatives.

From Eq. (21), the effect of the Hamiltonian constraint solution for \( \phi \) is:

\[
g_{ab} = \delta_{ab} + h_{ab} = \exp 4\phi [\delta_{ab} + \tilde{h}_{ab}] = (1 + 4\phi_1) [\delta_{ab} + \tilde{h}_{ab}] + O(h_{ab}^2). \tag{34}
\]

At the linearized level this conformal factor does not affect the off-diagonal components, so we obtain:

\[
h_{xx} = \tilde{h}_{xx} - \frac{1}{2}(\tilde{h}_{yy} + \tilde{h}_{zz}) + 4Ax + 4B, \tag{35}
\]
\[
h_{xy} = \tilde{h}_{xy}, \tag{36}
\]
\[
h_{xz} = \tilde{h}_{xz}, \tag{37}
\]
\[
h_{yy} + h_{zz} = 0 + 8Ax + 8B, \tag{38}
\]
\[
h_{yy} - h_{zz} = \tilde{h}_{yy} - \tilde{h}_{zz}, \tag{39}
\]
\[
h_{yz} = \tilde{h}_{yz}. \tag{40}
\]

Notice that the \( TT (yy - zz \& yz) \) components are unchanged, but the transverse trace \( (yy + zz) \) is set to zero modulo integration constants, with a similar subtraction of the \( LL (xx) \) component, which, however, generally leaves \( h_{xx} \) nonzero. With only this one variable \( \phi \) it is not surprising that the \( TL \) terms \( (zx \& yx) \) are left unchanged.

The effect of solution of the momentum constraints is to modify the extrinsic curvature by adding \( \tilde{\ell} w^{ab} \). The components \( w^y \) and \( w^z \) set \( A_{xy} \) and \( A_{xz} \) to zero, respectively (modulo integration constants). \( w^x \) modifies all the diagonal components of \( A_{ij} \): \( A_{xx} = \tilde{A}_{xx} + \frac{4}{3}w^{x,x}, \ A_{yy} = \tilde{A}_{yy} - \frac{2}{3}w^{x,x}, \ A_{zz} = \tilde{A}_{zz} - \frac{2}{3}w^{x,x} ; \) note that \( A_{ab} \) is traceless since \( \tilde{A}_{ab} \) is. The results:

\[
A_{xx} = \frac{2}{3}K, \ A_{yy} = \tilde{A}_{yy} + \frac{1}{2}\tilde{A}_{xx} - \frac{1}{3}K, \ A_{zz} = \tilde{A}_{zz} + \frac{1}{2}\tilde{A}_{xx} - \frac{1}{3}K. \tag{31}
\]

Written in terms of \( K_{ab} \) rather than \( A_{ab} \), and restoring the integration constants, we find:
\( K_{xx} = K + C_x, \) \( (41) \)

\( K_{xy} = 0 + C_y, \) \( (42) \)

\( K_{xz} = 0 + C_z, \) \( (43) \)

\( K_{yy} + K_{zz} = 0 - C_x, \) \( (44) \)

\( K_{yy} - K_{zz} = \bar{K}_{yy} - \bar{K}_{zz}, \) \( (45) \)

\( K_{yz} = \bar{K}_{yz}. \) \( (46) \)

Note, as assumed, that the trace, \( K, \) of the extrinsic curvature is not modified in the conformal constraint solution.

By inserting these results \((35)-(46)\) into Equations \((16)-(19)\), we can verify that the constraints are satisfied.

**VI. FREE EVOLUTION AND HYPERBOLICITY**

We now turn to the hyperbolicity of this system, assuming the data have been correctly set (the constraints are initially satisfied). Note that neither the \( LL \) metric component \( h_{xx} \) nor its initial time derivative is generically zero.

Kreiss and Ortiz \cite{12} have shown that a second-order form, and its equivalent first-order form have the same hyperbolicity classification. In pure second order form (no first derivatives) the classification is especially simple: If all the modes satisfy a wave equation with real nonzero propagation velocity, the system is strongly hyperbolic and well posed for Cauchy data. If any of the modes has zero propagation velocity, the system is at best weakly hyperbolic and is potentially ill posed. If any of the propagation velocities is imaginary the system is not hyperbolic, and is completely ill posed for Cauchy data.

If a unit lapse is chosen so that \( \partial_2^2 \alpha = 0 \) then the \( LL \) equation \((10)\) contains no spatial derivatives of \( h_{xx} \); the propagation velocity for \( h_{xx} \) is zero. Eq. \((10)\) then reflects at best weak hyperbolicity of the system; clearly it admits analytic linear growth in \( h_{xx} \), which can plainly lead to late-time difficulties. If, however, the lapse is densitized, with constant \( b > 0 \) (cf Eq. \((6)\)), then Eq. \((11)\) becomes wavelike for \( h_{xx} \). If this were the only equation on the system this choice of lapse would have made the system strongly hyperbolic.
However, one must investigate the entire set of evolution equations. Note that the \( TL \) components (11), (12) of the field equations have no spatial derivatives, again signaling weak hyperbolicity, and analytic linear-growth solutions. Though the initial data can analytically set the slope to zero (if we choose \( C_y = C_z = 0 \)), errors can arise in numerical evolution that triggers such linear growth.

Thus the \( \dot{g} - \dot{K} \) version of these equations is weakly hyperbolic, and questions can arise as to the well-posedness of the solutions, even with densitized lapse. In ref[13], it is shown that a relatively simple extension that captures one of the main ideas of the BSSN[14][15] approach can cure the nonhyperbolicity of the \( TL \) components. This involves considering the combination

\[
f_j = \delta^{kl} \partial_k h_{lj} - \frac{1}{2} \partial_k h. \tag{47}
\]

In that case one must treat the first order form of the equations because those for \( f_j \) are first order in time. The evolution equation for \( f_i \) is determined by differentiation of its definition. However the result is still a weakly hyperbolic system. By subtracting a multiple of the momentum constraint \( H_i \) to the right hand side of the equation for \( \partial_t f_i \), a strongly hyperbolic system can finally be achieved.

We will take a somewhat different approach here, as we discuss in the next section. The result just noted, that momentum constraint improves the hyperbolicity of the system, is of interest, because we will demonstrate momentum constraint solution, below.

VII. CONSTRAINED EVOLUTION

Constrained evolution as we described in Section IV above modifies the evolution. After each explicit step, the metric and extrinsic curvature (via Eq. (1)) are treated as conformal background quantities and are reset according to Eqs(35)-(46). Suppose in this Section that integration constants \( A, B, C_i \) are all zero.

The transverse trace, which might have grown an error from the discretization of the equation, is reset to zero; cf Eq. (38). Also, the momentum of this component is reset to zero (Eq. (38)).

The \( h_{yy} + h_{zz} \) terms evolve to set the term \( \tilde{h}_{yy} + \tilde{h}_{zz} \) to zero in Eq. (35). In this case the only terms on the right side of the \( \partial^2_t h_{xx} \) equation are the second derivatives of the lapse,
α. In setting data the trace of the extrinsic curvature, $K_{\text{initial}}$, is one of the data functions to be set, and with constrained data (and zero integration constants) $K_{\text{initial}}$ gives $\partial_t h_{xx}$ via Eq. (7).

With densitized lapse, the $h_{xx}$ equation (35) is effectively a wave equation for $h_{xx}$. Thus at least in this analytic discussion of the evolution, the linearized system appears completely controlled in a constrained, densitized-lapse evolution.

If the lapse is chosen constant, but constraint solution Eq. (38) is imposed, then Eq. (10) with Eq. (41) shows the the evolution is constant $K$ ($K = K_{xx}$ is constant in time, though $K_x$ is not necessarily zero). While this causes no trouble in the linearized system, it will eventually violate the $|h_{xx}| << 1$ assumption. In the full nonlinear case this growth will eventually definitely cause problems. As we have noted, the nondensitized case is definitely weakly-hyperbolic.

Finally, notice that in this case that constraint solution sets the $TL$ momenta $K_{xy}$ and $K_{xz}$ to zero and the TL momenta are reset to zero (Eqs. (42)-(43)). Thus the remaining components contributing to weak hyperbolicity are controlled as a consequence of the momentum constraint. This is of interest because the method of [13] required the addition of the momentum constraint to control the behavior of the $TL$ components in the evolution equations.

VIII. RELATION TO STANDARD GAUGE SETTING FOR LINEARIZED WAVES

In (“elementary”) textbook formulations, it is common to say that a gauge can be set to put $h_{yy} + h_{zz} = 0$ and $h_{xi} = 0$ initially, and that this gauge is then maintained by the evolution equations. Those formulations are set in terms of 4−dimensional gauge setting, rather than a 3 + 1 split as we employ. We have seen that $h_{yy} + h_{zz} = 0$ is a requirement of the Hamiltonian constraint, modulo integration constants. In our 3 + 1 context, we then see that the $h_{xx}$ stays zero only if the initial slice is chosen $K = 0$. This is of course a choice of the initial slice, so this is a 4−dimensional gauge choice.

This observation is relevant to the superficially very strange feature that, even just concentrating on the $h_{xx}$ equation, strong hyperbolicity requires densitized lapse. Mathematically this is perfectly well defined and consistent. To a physicist, however, it is extremely strange
that we must warp our concept of $t = \text{constant}$ to achieve controllable solutions, and that this process actually accomplishes mathematically well behaved solutions. However, in the context of the observation that $K$ need not be initially set to zero, the situation becomes more comprehensible. The choice $K \neq 0$ is not consistent with the elementary textbook gauge setting just described. Instead by taking $K \neq 0$ one has taken a peculiarly curved initial spatial slice; the specific choice of densitized lapse is required to correct it. While it may still seem strange that a wavelike (propagating) solution is the best way to accomplish it, this is what happens.

Further, another part of the assertion in elementary treatments is that the $TL$ components $h_{xy}$ and $h_{xz}$ be set to zero as a gauge choice, and will remain zero. The approach from hyperbolicity achieves this by introducing another equation with contribution on the right side proportional to the momentum constraint, which is ideally zero. The resulting system maintains evolution close to satisfaction of the gauge conditions. The constrained evolution approach guarantees exact solution of this gauge condition ($h_{xy}$ and $h_{xz}$ remain constant, and can be set to zero by a simple rotation around the $x$–axis). Because small errors can arise in any computational system, a combination of the two approaches seems the best approach to achieving long term computational relativity code stability.

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**Appendix A: Generality of Study of Hyperbolicity**

The analysis of the hyperbolicity in section V B is written in terms of linearized gravity as defined in section III. However, based on arguments in [13], the results are completely general for ADM, whether posed in first or second order form. We now summarize arguments for several points from [13].
For reasonable shift vectors (shift velocities less than \( c \)), “the role of the shift vector is to displace the value of the eigenvalue... , and the lapse re-scales it. But a change of lapse... and shift can not change a real eigenvalue into an imaginary one. It can not affect the hyperbolicity of the system.” See \([13]\), section III A.

**Linearization:** The hyperbolicity classification is based on the principal part of the symbol of the differential operator, which involves only the highest derivatives. Hence the terms shown in Eqs. (7), (8), or equivalently Eq. (9), above define the hyperbolicity. This is true essentially because the Einstein equations are quasilinear, i.e. linear in the highest derivatives, and this is the reason that one can consider Fourier transforms of the linearized equations; consideration of hyperbolicity essentially looks at only the highest frequency behavior.

We close a defect in the statement just made by following an argument given by \([13]\), “since the norms \( | \delta | \) and \( | \theta | \) are equivalent and smoothly related, ... the properties of the eigenvalues and eigenvectors of the principal symbol are the same with either norm”. Thus the undifferentiated metric (or inverse metric) coefficients in the Einstein equations can be equivalently replaced by \( \delta_{ij} \) or its inverse, leading to Eqs. (7), (8) or the equivalent second order form Eq. (9).

**Restriction to \( 1 - d \) Spatial Dependence:** With the results just above, the background is isotropic, and nothing is lost in picking a specific direction for the wave vector \( k \), associated with the local gradient.

Note that we do not claim generality for application of constraint solution beyond the linearized wave spacetime.

**Appendix B: Other Boundary Conditions in Constraint Solution**

One of the advantages of writing explicit (if simple) solution of the linearized constraint equations is that one can explicitly investigate the effect of boundary conditions. For instance, in full \( 3 - d \) constraint solution we impose the inner conditions \( \phi = 1 \) and \( w^i = 0 \), set at the excision surface of the black hole data. We take the outer conditions as in \([9]\); they are versions of the Robin \([11]\) condition. Thus for the black hole case the boundary condition for \( \phi \) is taken as \( \partial_r (r (\phi - 1)) = 0 \), and the outer boundary condition for the radial component of \( w^i \), i.e. \( w^r \), is taken as \( \partial_r (rw^r) = 0 \). Define also the transverse components of \( w^i \), \( X^jT = w^i (\delta^i_j - \hat{r}^i \hat{r}^j) \) (where the \( T \) means transverse). The outer boundary conditions
on $X^{iT}$ are $\partial_r(r^2X^{iT}) = 0$. We can model these boundary conditions in our $1 - d$ system by assuming a finite $x-$ domain, say $[0, x_m]$. (As in the $3 - d$ situation, we will consider the outer boundary $x_m$ becoming large, $x_m \to \infty$.) We thus write the boundary conditions in terms of $x-$ derivatives rather than $r-$ derivatives. Since we want to impose conditions on $w^i$ rather than its derivatives, the solution for $w^i$, Eqs. (32) - (33) can be integrated once again by writing the explicit (0 to $x$) integration of the source terms, and introducing new integration constants $D_i$. For instance

$$w^y = -\int_0^x \tilde{A}^{yx}dx + C_y x + D_y \quad (48)$$

It is straightforward to verify that one can consistently set each of the variables $\phi_1$, $w^i$ to zero at $x = 0$, and satisfy the Robin conditions at $x = x_m$ as we now show. (Notice that in our linearized problem, $\phi - 1 = \phi_1$.)

We want $w^y = 0$ at $x = 0$ and $\partial_x(x^2 w^y) = 0$ at $x = x_m$. Thus $D_y = 0$ and

$$3C_y = \frac{2}{x_m} \int_0^{x_m} \tilde{A}^{yx}dx + \tilde{A}^{yx}(x_m). \quad (49)$$

A completely analogous treatment applies to $C_z$.

For $w^x$ we take conditions $w^x = 0$ at $x = 0$ (which gives $D_x = 0$), and $\partial_x(xw^x) = 0$ at $x = x_m$, which gives

$$2C_x = (\tilde{A}^{xx} - \frac{2}{3}K)(x_m) + \frac{1}{x_m} \int_0^{x_m} (\tilde{A}^{xx} - \frac{2}{3}K)dx. \quad (50)$$

The solution for $\phi_1$ directly involves two integration constants, $A$, $B$, cf. Eq. (30). We take inner boundary condition $\phi_1 = 0$ at $x = 0$, which sets $B = \frac{1}{8}(\tilde{h}_{yy} + \tilde{h}_{y0})|_0$. We then set $\partial_x(x\phi_1) = 0$ at $x = x_m$, which gives

$$16A = \partial_x(\tilde{h}_{yy} + \tilde{h}_{zz})(x_m) + \frac{1}{x_m} (\tilde{h}_{yy} + \tilde{h}_{zz})|_0^{x_m}. \quad (51)$$

The ideal computational boundary condition has $x_m \to \infty$. With reasonable assumptions about the boundedness of the data at large $x_m$, Eqs. (49) - (50) lead to $C_i \to 0$ as $x_m \to \infty$. In Eq. (51), similarly $A \to 0$ as $x_m \to \infty$. 

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Thus the $w^i$ have precisely the form assumed in the main text with $C_i = 0$. For $\phi$ the situation is only slightly more complicated. We have

$$\phi_1 = -\frac{1}{8}(\tilde{h}_{yy} + \tilde{h}_{zz})|_0^x. \quad (52)$$

From Eq. (38) this gives the result for the metric components after the constraint solve:

$$h_{yy} + h_{zz} = (\tilde{h}_{yy} + \tilde{h}_{zz})|_0. \quad (53)$$

Such a constant value can be removed by a gauge (coordinate) transformation transverse to the $x$ propagation direction. Hence we may take $(\tilde{h}_{yy} + \tilde{h}_{zz}) = 0$ in setting the data.

The result of the analysis in this Appendix is thus that the analysis in the main text accurately models the procedures used in our $3-d$ simulations. This has the important corollary that the boundary conditions in the $3-d$ case are completely internally consistent; we have not set too restrictive conditions (which might have only isolated solutions, i.e. might define an eigenvalue problem).

It is not necessary to consider infinite domains $x_m \to \infty$. To treat cosmologies for instance, it is more appropriate to consider a specific finite value of $x_m$. In this case the integration constants do not vanish. As an example, using Eq. (25) we find that the solution for $A^{xx}$ is $A^{xx} = \frac{2}{3}K + C_x$. Assuming that $K$ changes little from timestep to timestep, the constrained evolution reproduces this same value of $C_x$. Clearly $C_x$ can be consistently set to zero by setting $A^{xx} = \frac{2}{3}K$ exactly, regardless of the value of $x_m$.

As above, we can interpret the integration constants from the Hamiltonian constraint as a kind of gauge choice in the initial data, which is re-imposed in the constrained evolution. For instance, the constants $A, B$, which arise from the Hamiltonian constraint, were set to zero in the body of the paper, implying that the transverse trace $h_{yy} + h_{zz}$ is set to zero. The terms $8Ax + 8B$ in Eq. (35) give constants to $h_{yy} + h_{zz}$, and the constant $C_x$ appearing in Eq. (14) gives a time derivative to this transverse trace. Note that while $h_{yy} + h_{zz}$ satisfies a wave equation, Eq. (13), none of the integration constant terms has a second spatial derivative. Hence they will evolve trivially:

$$h_{yy} + h_{zz} = 8Ax + 8B + 2C_xt. \quad (55)$$
With this form, it can be seen that repeated application of the constraints, as in constrained evolution, consistently leads to the same value of the constant $A$. The form Eq. (55) identifies the terms associated with the integration constants as gauge terms (the gauge vector $\xi_x = 2A(y^2 + z^2)$, $\xi_y = -4Ax$, $\xi_z = -4Axz$ removes the time independent terms). The same method can be used to remove the time dependent term at any one time, but this secular $C_x t$ term can eventually destroy the linearization assumption (unless $C_x$ is forced to zero as suggested above). The secular term can be eliminated by an outgoing wave boundary condition, which forces $C_x = 0$.

Similar considerations apply to the terms $C_y$ and $C_z$ in Eqs. (42), (43).

If the constraints are repeatedly applied, as in our form of constrained evolution, the value of $C_y$ is consistent. That is, from Eq. (42), $A_{xy} = K_{xy}$ has the constant value $C_y$. The evolution equation Eq. (11) shows that $K_{xy}$ remains constant (ideally) during an integration step. If $K_{xy}$ has the value $C_y$ as integrated in Eq. (42), then Eq. (49) consistently returns the same value of $C_y$. Clearly the behavior of $K_{xz}$ is similar. Again, such terms can cause secular growth, but such growth can be controlled by, for instance, putting an outgoing wave boundary condition on $h_{xy}$ and $h_{xz}$.
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