Hölder Continuity of Spectral Measures for the Finitely Differentiable Quasi-Periodic Schrödinger Operators

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Abstract. In the present paper, we prove the $\frac{1}{2}$-Hölder continuity of spectral measures for the $C^k$ Schrödinger operators. This result is based on the quantitative almost reducibility and an estimate for the growth of the Schrödinger cocycles in [5].

Key Words: Schrödinger operator, quasi-periodic, almost reducibility, finitely differentiable.

AMS Subject Classifications: 52B10, 65D18, 68U05, 68U07

1 Introduction

In this paper, we consider the Schrödinger operators defined on $\ell^2(\mathbb{Z})$

$$(H_{V,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n,$$

where $V : \mathbb{T}^d \rightarrow \mathbb{R}$ is the potential, $\theta \in \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ is the phase, and $\alpha \in \mathbb{T}^d$ is the frequency.

These operators have been extensively and thoroughly studied for the deep connection with quasi-crystal and quantum Hall effects [11,18]. This paper concerns the regularity of the spectral measure of the quasi-periodic Schrödinger operators. For the analytic potential $V \in C^\infty(\mathbb{T}^d, \mathbb{R})$, there is some significant progress [5,21,24]. However for the smooth potential $V \in C^k(\mathbb{T}^d, \mathbb{R})$, there is no similar result as far as we know, so we will give a supplementary answer to this situation.

Let us review some results on the Hölder continuity of the integrated density of states (IDS) and the individual spectral measures.

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1.1 Hölder continuity of IDS

Let $\Sigma_{V,\alpha,\beta}$ be the spectrum of $H_{V,\alpha,\beta}$, then $\Sigma_{V,\alpha,\beta} \subset \mathbb{R}$ since $H_{V,\alpha,\beta}$ is the bounded self-adjoint operator in $L^2(\mathbb{Z})$. The spectrum is independent of $\theta$ if $(\alpha, 1)$ is rational independent. For any $f \in L^2(\mathbb{Z})$, the spectral measure $\mu_{V,\alpha,\beta}$ of $H_{V,\alpha,\beta}$ can be defined as

$$\langle (H_{V,\alpha,\beta} - E)^{-1} f, f \rangle = \int_{\mathbb{R}} \frac{1}{E'} d\mu_{V,\alpha,\beta}(E'), \quad \forall E \in \mathbb{C} \setminus \Sigma_{V,\alpha}. \quad (1.1)$$

Let $\mu_{V,\alpha,\beta} = \mu_{V,\alpha,\beta} - \mu_{V,\alpha,\beta}'$, where $\{e_i\}_{i \in \mathbb{Z}}$ is the canonical basis of $L^2(\mathbb{Z})$. Let $N_{V,\alpha}$ be the IDS of $H_{V,\alpha,\beta}$, it is well known that IDS is the average of the spectral measure $\mu_{V,\alpha,\beta}$ with respect to $\theta$, i.e.,

$$N_{V,\alpha}(E) = \int_{\mathbb{T}_d} \mu_{V,\alpha,\beta}(-\infty, E] d\theta.$$

Hence the regularity of IDS is closely related to that of the spectral measure.

Recall that $\alpha \in \mathbb{T}_d$ is Diophantine if there exist $\gamma > 0$ and $\tau > d - 1$ such that $\alpha \in \text{DC}_d(\gamma, \tau)$, where

$$\text{DC}_d(\gamma, \tau) = \left\{ \alpha : \inf_{j \in \mathbb{Z}} | \langle n, \alpha \rangle - j | > \frac{\gamma}{|n|^\tau}, \forall n \in \mathbb{Z}^d \setminus \{0\} \right\}.$$

Let $\text{DC}_d = \bigcup_{\gamma > 0, \tau > d - 1} \text{DC}_d(\gamma, \tau)$. For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, let $p_n / q_n$ be the continued fraction approximants to $\alpha$, then one can define

$$\beta(\alpha) = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n}.$$

Given the operator $H_{V,\alpha,\beta}$, one can define the Lyapunov exponent $L(\alpha, S^V_E)$ (see Section 2.1) of the corresponding Schrödinger cocycle $(\alpha, S^V_E(\theta))$, where $E \in \mathbb{R}$ and

$$S^V_E(\theta) = \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}.$$

Hadj Amor [16] proved the $\frac{1}{2}$-Hölder continuity of IDS if $\alpha \in \text{DC}_d$ and $V \in C^\omega(\mathbb{T}_d, \mathbb{R})$ is small, and her approach is based on the almost reducibility scheme developed by Eliasson [13]. Recall that the cocycle $(\alpha, A)$ is reducible if $(\alpha, A)$ can be conjugated to some constant cocycles and the cocycle $(\alpha, A)$ is almost reducible if the closure of its conjugates contains a constant. Avila and Jitomirskaya [4] proved the $\frac{1}{2}$-Hölder continuity of IDS for $\alpha \in \text{DC}_1$ and the small analytic potential. Their result was non-perturbative, which means that the smallness is independent of $\alpha$. After that, Avila [2,3] generalized the result for the small analytic potential with $\beta(\alpha) = 0$ if there is $\delta > 0$ such that $L(\alpha, S^V_{E,\delta}(\theta)) = 0$ for $|\epsilon| < \delta$. Note that Leguil-You-Zhao-Zhou [20] showed the same result as well by the global theory of the one-frequency Schrödinger operators [1].
The Hölder continuity of $N_{V,\alpha}(E)$ is equivalent to the Hölder continuity of $L(\alpha, S_E^V)$ according to the famous Thouless formula [6], i.e.,

$$L(\alpha, S_E^V) = \int \ln |E - E'| dN_{V,\alpha}(E').$$

Suppose that $L(\alpha, S_E^V) > 0$. Goldstein and Schlag [14] proved the Hölder continuity of $L(\alpha, S_E^V)$ if $V \in C^\omega(T, \mathbb{R})$, and $\alpha \in SDC^1$ by the avalanche principle and sharp large deviation theorem. Later, You and Zhang [22] proved that for analytic potential, the $L(\alpha, S_E^V)$ is Hölder continuous if $\alpha \in DC_1$ or some weaker Liouvillian $\alpha$ by a refined large deviation theorem.

For the special kinds of potentials, there are some other interesting results. If the potential is a small perturbation of a trigonometric polynomial of degree $d$ and $\alpha \in DC_1$, Goldstein-Schlag [15] proved that IDS is $(\frac{1}{2} - \epsilon)$-Hölder continuous for any $\epsilon > 0$ by assuming $L(\alpha, S_E^V) > 0$. In particular, if the potential $V(x) = 2\lambda \cos(x)$, we get the so-called almost Mathieu operators (AMO), i.e.,

$$(H_{\lambda,\alpha,\theta} u)_n = u_{n+1} + u_{n-1} + 2\lambda \cos(\theta + n\alpha) u_n,$$

where $\lambda \in \mathbb{R}$ is the coupling constant. For the large coupling, Bourgain [7] proved that Lyapunov exponent of AMO is $(\frac{1}{2} - \epsilon)$-Hölder continuous for any $\epsilon > 0$. Avila [2] obtained the exact $\frac{1}{2}$-Hölder continuity of IDS for AMO with $|\lambda| < 1$.

### 1.2 Hölder continuity of the spectral measure

In general, the spectral measure is less regular than IDS. In the $\beta(\alpha) = 0$ regime, Avila and Jitomirskaya [5] proved that if $V \in C^\omega(T, \mathbb{R})$ is small enough and $\alpha \in DC_1$, then the spectral measure $\mu_{V,\alpha,\theta}^f$ of one-frequency Schrödinger operators is $\frac{1}{2}$-Hölder continuous for any $f \in \ell^1(\mathbb{Z}) \cap \ell^2(\mathbb{Z})$. They also showed $\frac{1}{2}$-Hölder continuity of absolutely continuous spectral measures for the one-frequency Schrödinger operators with $V \in C^\omega(T, \mathbb{R})$.

In the $\beta(\alpha) > 0$ regime, Liu and Yuan [21] extended the results in [5] to that for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with $\beta(\alpha) < \infty$, whenever the analytic radius of $V$ is sufficiently large.

What can we say about the regularity of the spectral measure of Schrödinger operators with finitely differentiable potential? Recall that Cai-Chavaudret-You-Zhou [10] have shown that if $\alpha \in DC_d$ and $V \in C^k(T^d, \mathbb{R})$ is small, then IDS of the Schrödinger operator is $\frac{1}{2}$-Hölder continuous. Recently, Zhao [24] also generalized [5] and [21] to the multi-frequency Schrödinger operators with $V \in C^\omega(T^d, \mathbb{R})$. This paper is motivated by [10] and [24], as a supplemenetary answer, we obtain the $\frac{1}{2}$-Hölder continuity of the spectral measure $\mu_{V,\alpha,\theta}^f$ for $V \in C^k(T^d, \mathbb{R})$.

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1We say $\alpha$ satisfies strong Diophantine condition if there exist some $\gamma > 0$, $\tau > 1$ such that $\alpha \in SDC(\gamma, \tau)$, where

$$SDC(\gamma, \tau) = \left\{ \alpha \in \mathbb{R}^d : \inf_{j \in \mathbb{Z}} |\langle j, n, \alpha \rangle - j| > \frac{\gamma}{|n|(|n| (1 + |n|)^\tau)} \right\},$$

and $SDC = \cup_{\gamma > 0, \tau > 1} SDC(\gamma, \tau)$. 

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**Theorem 1.1.** Let $\alpha \in \text{DC}_d(\gamma, \tau)$, $V \in C^k(T^d, \mathbb{R})$ with $k \geq 5D\tau$ and $D$ is a numerical constant. There exists $\varepsilon = \varepsilon(\gamma, \tau, k)$ such that if $\|V\|_k \leq \varepsilon$, then for any $f \in \ell^2(\mathbb{Z}) \cap \ell^1(\mathbb{Z})$,

$$
\mu_{V,\alpha,\theta}(f) \leq D_{0}|J|^{\frac{1}{2}}\|f\|_{\ell^1}^2,
$$

for all intervals $J$ and all $\theta$, where $D_0 = D_0(V, \alpha) > 0$.

**Remark 1.1.** Note that Theorem 1.1 is perturbative, i.e., the smallness $\varepsilon$ depends not only on the potential $V$, but also on the frequency $\alpha$. From a counterexample of Bourgain [8], one can not expect non-perturbative results in multi-frequency case.

## 2 Preliminaries

For a bounded analytic function $F$ defined on $S_r = \{ \theta : \theta = (\theta_1, \ldots, \theta_d) \in \mathbb{C}^d, |\Im \theta| < r, \forall i = 1, \ldots, d \}$, let $|F|_r = \sup_{\theta \in S_r} \|F(\theta)\|$ and denote by $C^{\mu}_r(T^d, \ast)$ the set of these $\ast$-value functions ($\ast$ will usually denote $\mathbb{R}, sl(2, \mathbb{R})$ or $SL(2, \mathbb{R})$). We also denote the set $C^k(T^d, \ast)$ to be the space of $k$ times differentiable with continuous $k$-th derivatives functions, endowed with the norm

$$
\|F\|_k := \sup_{k \leq k, \theta \in T^d} \|\partial^k F(\theta)\|.
$$

In particular,

$$
\|F\|_0 := \|F\|_{T^d} = \sup_{\theta \in T^d} \|F(\theta)\|.
$$

For $\theta \in \mathbb{R}$, we set $\|\theta\|_T = \inf_{j \in \mathbb{Z}} |\theta - j|$.

### 2.1 Uniform hyperbolicity

Given $A \in C^{\omega}(T^d, SL(2, \mathbb{C}))$ and $\alpha \in \mathbb{R}^d$ rationally independent, one can define the quasi-periodic (Q-P) cocycle $(\alpha, A)$:

$$(\alpha, A) : T^d \times \mathbb{C}^2 \to T^d \times \mathbb{C}^2;$$

$$(\theta, v) \mapsto (\theta + \alpha, A(\theta) \cdot v).$$

The iterations of $(\alpha, A)$ are of form $(\alpha, A)^n = (n\alpha, A_n)$, where

$$A_n(\theta) := \begin{cases} A(\theta + (n - 1)\alpha) \cdots A(\theta + \alpha)A(\theta), & n \geq 0, \\ A^{-1}(\theta + n\alpha)A^{-1}(\theta + (n + 1)\alpha) \cdots A^{-1}(\theta - \alpha), & n < 0. \end{cases}$$

The Lyapunov exponent of the cocycle $(\alpha, A)$ is defined as

$$L(\alpha, A) = \lim_{n \to \infty} \frac{1}{n} \int_{T^d} \|A_n(\theta)\|d\theta.$$
We say the cocycle \((\alpha, A)\) is uniformly hyperbolic if for every \(\theta \in \mathbb{T}^d\), there exists a continuous splitting \(C^2 = E^s(\theta) \oplus E^u(\theta)\) such that for every \(n \geq 0\),
\[
\|A_n(\theta)v\| \leq C e^{-cn} \|v\|, \quad v \in E^s(\theta),
\]
\[
\|A_n(\theta)^{-1}v\| \leq C e^{-cn} \|v\|, \quad v \in E^u(\theta + na),
\]
for some constants \(C, c > 0\). And the splitting is invariant by the dynamics:
\[
A(\theta)E^s(\theta) = E^s(\theta + \alpha), \quad \forall \theta \in \mathbb{T}^d,
\]
\[
A(\theta)E^u(\theta) = E^u(\theta + \alpha), \quad \forall \theta \in \mathbb{T}^d.
\]

### 2.2 Spectral measure and Wely’s m function

Typical examples of \(SL(2, \mathbb{R})\) cocycles are the Schrödinger cocycles \((\alpha, S^V_E)\):
\[
A(\theta) = S^V_E(\theta) = \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}, \quad E \in \mathbb{R}.
\]
Those cocycles come from the eigenvalue equation of one dimensional quasi-periodic Schrödinger operators on \(\ell^2(\mathbb{Z})\):
\[
(H_{V,\alpha,\beta}u)_n = u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n = Eu_n,
\]
and any formal solution \(u = (u_n)_{n \in \mathbb{Z}}\) of \(H_{V,\alpha,\beta}u = Eu\) satisfies
\[
\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = S^V_E(\theta + n\alpha) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}, \quad \forall n \in \mathbb{Z}.
\]

For any \(f \in \ell^2(\mathbb{Z})\), one can define the spectral measure \(\mu_{f, V, \alpha, \beta}\) corresponding to \(f\) as in (1.1). Given \(E + i\epsilon\) with \(E \in \mathbb{R}\) and \(\epsilon > 0\), there exists a non-zero solution \(u^+\) of \(H_{V,\alpha,\beta}u^+ = (E + i\epsilon)u^+\) being square-summable at \(+\infty\). The Wely’s m function is given by \(m^+ = -\frac{u^+}{u^0}\).

Let
\[
M(E + i\epsilon) = \int_{\mathbb{R}} \frac{1}{E' - (E + i\epsilon)} d\mu_{f, V, \alpha, \beta}(E').
\]
From the definition of \(M(\cdot)\), it deduces immediately that \(M(\cdot)\) is a Herglotz function defined on \(\mathbb{H} = \{z : \Im z > 0\}\), and
\[
\Im M(E + i\epsilon) \geq \frac{1}{2\epsilon} \mu_{f, V, \alpha, \beta}(E - \epsilon, E + \epsilon). \quad (2.1)
\]
Recall that in [12], by the usual action of \(SL(2, \mathbb{C})\), we denote
\[
m^+_\beta = R_{-\frac{\beta}{\pi}} \cdot m^+ = \frac{m^+ \cos \beta - \sin \beta}{m^+ \sin \beta + \cos \beta},
\]
then we have the following lemma:
Lemma 2.1 ([5]). Let $\psi(m^+) := \sup_{\beta} |m^+_\beta|$, then $\|M\|_0 \leq \psi(m^+)$. The spectral properties of $H_{V, \alpha, \theta}$ and the dynamics of $(\alpha, S^V_\theta)$ are closely related by the fact: $E \in \Sigma_{V, \alpha}$ if and only if $(\alpha, S^V_\theta)$ is not uniformly hyperbolic [19].

2.3 Analytic approximation

Assume $f \in C^k(T^d, sl(2, \mathbb{R}))$, according to [23], there exists a sequence $\{f_j\}_{j \geq 1}$ with $f_j \in C^\omega(T^d, sl(2, \mathbb{R}))$ and a universal constant $C' > 0$, such that

\begin{align}
\|f_j - f\|_k &\to 0, \quad j \to +\infty, \quad (2.2a) \\
|f_j|_1 &\leq C'\|f\|_k, \quad (2.2b) \\
|f_{j+1} - f_j|_{1/2} &\leq C'(j)^{-k}\|f\|_k. \quad (2.2c)
\end{align}

2.4 Space decomposition

Given $A \in SL(2, \mathbb{R})$, $\alpha \in \mathbb{R}^d$ and $\eta > 0$, one can obtain the decomposition $B_k = B_k^{(nre)}(\eta) \oplus B_k^{(re)}(\eta)$, where

$$B_k = \left\{ f \in C^k(T^d, sl(2, \mathbb{R})) : \|f\|_k < \infty \right\},$$

and $B_k^{(nre)}(\eta)$ is the subspace of $B_k$ such that for any $Y \in B_k^{(nre)}(\eta)$,

$$A^{-1}Y(\theta + \alpha)A \in B_k^{(nre)}(\eta), \quad \|A^{-1}Y(\theta + \alpha)A - Y(\theta)\|_k > \eta\|Y\|_k.\]

Lemma 2.2 ([10, 17]). Assume that $A \in SL(2, \mathbb{R})$, $\alpha \in \mathbb{R}^d$, $\epsilon' \leq (4\|A\|)^{-4}$, and $\eta \geq 13\|A\|^2\epsilon'^{1/2}$. For any $g \in B_k$ with $\|g\|_k \leq \epsilon'$, there exist $Y \in B_k$ and $g^{re} \in B_k^{(re)}(\eta)$ such that

$$e^{-Y(\theta + \alpha)}Ae^{\theta(\theta)}e^{Y(\theta)} = Ae^{\theta(\theta)},$$

with $\|Y\|_k \leq \epsilon'^{1/2}$, and $\|g^{re}\|_k \leq 2\epsilon'$.

The continuous version ($C^\omega$ linear systems) and discrete version ($C^\omega$ cocycles) of Lemma 2.2 are shown in [17] and [10] respectively. The proof of Lemma 2.2 only depends on $B_k$ being a Banach space, thus one can obtain the result by replacing analytic norm with $C^k$ norm in the Appendix of [10].

3 Dynamical estimates of $C^k$ quasi-periodic cocycles

To deal with the almost reducibility of the $C^k$ cocycles, the strategy used here is treating the analytic cocycles firstly and turning the estimates of analytic cocycles into those of finitely differentiable cocycles by analytic approximation.
More precisely, let $N = \text{\textit{then we have the estimates:}}$

\[(A) \text{ (Non-resonant case).} \quad (\theta, x) \mapsto (\theta + a, Ae^{f(\theta)} \cdot x),\]

where $a \in \text{DC}_d(\gamma, \tau), A \in \text{SL}(2, \mathbb{R})$ and $f(\theta) \in C^\omega(T^d, \text{sl}(2, \mathbb{R}))$ with $r > 0$. Assume $|f|_r \leq \varepsilon$, we are going to show that the perturbation will tend to zero by KAM scheme.

**Proposition 3.1 ([10, 20]).** Let $a \in \text{DC}_d(\gamma, \tau), \gamma, r > 0, \tau > d - 1$ and $\sigma = \frac{1}{10}$. Then for any $r_+ \in (0, r)$, there exist $c = c(\gamma, \tau, d)$ and a numerical constant $D$ such that if

\[\varepsilon \leq \frac{c}{\|A\|^D(r - r_+)^D}, \quad (3.1)\]

then there exist $B(\theta) \in C^\omega(T^d, \text{SL}(2, \mathbb{R}))$, $A_+ \in \text{SL}(2, \mathbb{R})$ and $f_+(\theta) \in C^\omega(T^d, \text{sl}(2, \mathbb{R}))$ such that $(a, Ae^{f(\theta)})$ is conjugated to $(\alpha, A_+e^{f_+(\theta)})$ by $B(\theta)$, i.e.,

\[B(\theta + \alpha)^{-1}Ae^{f(\theta)}B(\theta) = A_+e^{f_+(\theta)}.\]

More precisely, let $N = \frac{2|\ln r|}{r_{f+}^2}$ and $\{e^{2\pi ip}, e^{-2\pi ip}\}$ be the two eigenvalues of $A$, we can distinguish between two cases:

**(A) (Non-resonant case).** Assume that

\[\|2\rho - \langle n, \alpha \rangle\|_T \geq \varepsilon', \quad \forall n \in \mathbb{Z}^d \quad \text{with} \quad 0 < |n| \leq N,\]

then we have the estimates:

\[|f_+(\theta)|_{r_+} \leq 4e^{3-2\sigma}, \quad |B(\theta) - \text{Id}|_{r_+} \leq e^{\frac{1}{2}}, \quad \|A_+ - A\| \leq 2\|A\|\varepsilon.\]

**(B) (Resonant case).** If there exists $n^* \in \mathbb{Z}^d$ with $0 < |n^*| \leq N$ such that

\[\|2\rho - \langle n^*, \alpha \rangle\|_T < \varepsilon',\]

then we have the estimates:

\[|B(\theta)|_{r_+} \leq C_1|n^*|^\frac{3}{2}e^{\pi|n^*|r_+}, \quad \|B(\theta)\|_0 \leq C_1|n^*|^\frac{3}{2}, \quad |f_+(\theta)|_{r_+} \ll \varepsilon^{100}, \quad (3.2)\]

where $C_1 = 4\|A\|^\frac{3}{2}\gamma^{-\frac{1}{2}}$. Moreover, let $A_+ := e^{A''}$ with $A'' \in \text{sl}(2, \mathbb{R})$, then $\|A''\| \leq 16e^{\sigma}$.

### 3.1 Real quantitative estimates for $C^k$ Q-P cocycles

We are going to deal with the almost reducibility of the $C^k$ cocycles by analytic approximation. Let $\{f_j\}_{j \geq 1}, f_j \in C^\omega_{\frac{1}{2}}(T^d, \text{sl}(2, \mathbb{R}))$ be the analytic sequence approximating
Then one can check that for any $k \geq 5D\tau$ and any $m \geq 10$, $m \in \mathbb{Z}$,

$$\frac{c}{(2\|A\|)^{D_m\frac{1}{m^\xi}}} \leq \epsilon_0' \left(\frac{1}{m'} \frac{1}{m^{\xi}}\right).$$

Denote $l_j = M_2^{j-1}$, $\forall j \in \mathbb{Z}^+$, where $M > \max\{10, \frac{(2\|A\|)^{D_m}}{\epsilon}\}$ is an integer.

**Theorem 3.1.** Let $\alpha \in \text{DC}_d(\gamma, \tau)$, $A \in \text{SL}(2, \mathbb{R})$, $\sigma = \frac{1}{2M}$, $f(\theta) \in C^k(\mathbb{T}^d, \text{sl}(2, \mathbb{R}))$ with $k \geq 5D\tau$. Let $\{f_j\}_{j \geq 1}$ be the analytic sequence approximating $f(\theta)$ defined in (2.2). There exists $\bar{c} = \bar{c}(\gamma, \tau, d, k, \|A\|)$ such that $\|f\|_k \leq \bar{c}$, then the following holds:

**A** There exist $B_{ij}(\theta) \in C_{\omega_{\delta}}\left(\mathbb{T}^d, \text{SL}(2, \mathbb{R})\right)$, $A_{ij} \in \text{SL}(2, \mathbb{R})$ and $f_{ij}(\theta) \in C_{\omega_{\delta}}\left(\mathbb{T}^d, \text{sl}(2, \mathbb{R})\right)$ such that

$$B_{ij}(\theta + \alpha)^{-1}A e^{f_{ij}(\theta)} B_{ij}(\theta) = A_{ij} e^{f_{ij}(\theta)},$$

with following estimates

$$\|B_{ij}(\theta)\|_0 \leq (l_j |\ln \epsilon_j|)^{\tau(1+\sigma)}, \quad |f_{ij}(\theta)| \frac{1}{|\ln \epsilon_j|} \leq \frac{1}{2} \epsilon_j^{\xi},$$

$$|B_{ij}(\theta)| \frac{1}{|\ln \epsilon_j|} \leq (l_j |\ln \epsilon_j|)^{\tau(1+\sigma)} \epsilon_j^{-\xi},$$

where $\xi \in \left(\frac{2\pi}{M_2^{j-1}}, \frac{1}{2}\right)$ is a constant.

**B** Moreover, $\|A_{ij}\| \leq 2\|A\|$ and there exist some unitary matrices $U_j \in \text{SL}(2, \mathbb{C})$ such that $A_{ij}$ can be written as

$$A_{ij} = U_j \begin{pmatrix} e^{2\pi i \rho_j} & c_j e^{2\pi i \rho_j} \\ 0 & e^{-2\pi i \rho_j} \end{pmatrix} U_j^{-1} \quad \text{with} \quad \rho_j \in i\mathbb{R} \cup \mathbb{R} \quad \text{and} \quad c_j \in \mathbb{C}.$$

Then for any constant $\kappa \in [1, \frac{C'}{C_0}]$, there exists constant $C = C(\|A\|) > 0$ such that

$$\|B_{ij}(\theta)\|^\kappa \cdot |c_j| \leq C.$$

**Proof.** We will prove Theorem 3.1 by the induction process as in [9] and [10].

**First Step.** Suppose that

$$\|f\|_k \leq \frac{c}{C'(2\|A\|)^{D_m\frac{1}{m^\xi}}},$$
where $C'$ is a universal constant defined in (2.2). Thus we have
\[
|f_{i_1}|_{1, \frac{1}{T}} \leq \varepsilon_{i_1} \leq \varepsilon_{i_1}' \left( \frac{1}{i_1}, \frac{1}{l_2} \right).
\]
By Proposition 3.1, one can find $B_{i_1}(\theta) \in C^\omega_{\frac{1}{T}}(2T^d, SL(2, \mathbb{R}))$, $A_{i_1} \in SL(2, \mathbb{R})$ and $f'_{i_1}(\theta) \in C^\omega_{\frac{1}{T}}(T^d, sl(2, \mathbb{R}))$ such that
\[
B_{i_1}(\theta + \alpha)^{-1} A e^{f'_{i_1}(\theta)} B_{i_1}(\theta) = A_{i_1} e^{f'_{i_1}(\theta)}.
\]
More precisely, let $N_{i_1} = \frac{2|\ln \varepsilon_{i_1}|}{\pi}$ and $\{e^{2\pi i \rho}, e^{-2\pi i \rho}\}$ be two eigenvalues of $A$, we can distinguish two cases:

(Non-resonant case). If the first step is obtained by non-resonant case:
\[
\|2\rho - (n, \alpha)\|_T \geq \varepsilon_{i_1}^\rho, \quad \forall n \in \mathbb{Z}^d \quad \text{with} \quad 0 < |n| \leq N_{i_1},
\]
then we have the estimates:
\[
|f'_{i_1}(\theta)|_{1, \frac{1}{2}} \leq 4\varepsilon_{i_1}^{3-2\rho}, \quad |B_{i_1}(\theta)|_{1, \frac{1}{2}} \leq 1 + \varepsilon_{i_1}^{\frac{1}{2}}, \quad \|A_{i_1} - A\| \leq 2\|A\|\varepsilon_{i_1}.
\]

(Resonant case). If the first step is obtained by resonant case: there exists $n_{i_1}^* \in \mathbb{Z}^d$ with $0 < |n_{i_1}^*| \leq N_{i_1}$ such that $\|2\rho - (n_{i_1}^*, \alpha)\|_T < \varepsilon_{i_1}^\rho$, then
\[
\|B_{i_1}(\theta)\|_0 \leq C_2(\gamma, \tau, \|A\|)(l_1|\ln \varepsilon_{i_1}|)^{\frac{1}{2}} \leq (l_1|\ln \varepsilon_{i_1}|)^{\frac{1}{2}},
\]
\[
|B_{i_1}(\theta)|_{1, \frac{1}{2}} \leq (l_1|\ln \varepsilon_{i_1}|)^{\frac{1}{2}} \varepsilon_{i_1}^{\frac{3}{2}}, \quad |f'_{i_1}(\theta)|_{1, \frac{1}{2}} \ll \varepsilon_{i_1}^{100} \leq \frac{1}{2}\varepsilon_{i_1}^2.
\]
Moreover, let $A_{i_1} := e^{A''_{i_1}}$ with $A''_{i_1} \in sl(2, \mathbb{R})$ we have $\|A''_{i_1}\| \leq 16\varepsilon_{i_1}^\rho$.

**Induction Step:** Assume that in $(l_j)$-th step with $j \leq n$, we already have that
\[
B_{i_j}(\theta + \alpha)^{-1} A e^{f'_{i_j}(\theta)} B_{i_j}(\theta) = A_{i_j} e^{f_{i_j}(\theta)}
\]
with the following estimates
\[
|B_{i_j}(\theta)|_{l_j+1} \leq (l_j|\ln \varepsilon_{i_j}|)^{(\sigma+1)}\varepsilon_{i_j}^{-\frac{\sigma}{2}}, \quad \|A_{i_j}\| \leq 2\|A\|,
\]
\[
|B_{i_j}|_0 \leq (l_j|\ln \varepsilon_{i_j}|)^{(\sigma+1)}, \quad |f'_{i_j}(\theta)|_{l_j+1} \leq \frac{1}{2}\varepsilon_{i_j}^\frac{\sigma}{2}.
\]
Moreover, if the $(l_j)$-th step is obtained by the resonant case, we have
\[
A_{i_j} = e^{A''_{i_j}}, \quad \|A''_{i_j}\| \leq 8\varepsilon_{i_j}^\rho, \quad \|A_{i_j}\| \leq 1 + 16\varepsilon_{i_j}^\rho.
\]
If the \((l_j)\text{-th step}\) is obtained by the non-resonant case, we have

\[
\|A_{l_j} - A_{l_{j-1}}\| \leq 2\|A_{l_{j-1}}\| \epsilon_{l_{j-1}}, \quad |B_{l_j}|_{\frac{1}{n+1}} \leq (1 + \epsilon_{l_j}^\frac{1}{2})|B_{l_{j-1}}|_{\frac{1}{n}}. \tag{3.5}
\]

Now let \(j = n + 1\) and focus on the cocycle \((\alpha, A)e^{f_{l_{n+1}}(\theta)}\), it follows that

\[
B_{l_n}(\theta + \alpha)^{-1}A)e^{f_{l_{n+1}}}B_{l_n}(\theta) = A_{l_n}e^{f_{l_n}} + B_{l_n}(\theta + \alpha)^{-1}(A)e^{f_{l_{n+1}} - A)e^{f_{l_n}}B_{l_n}(\theta).
\]

If we rewrite \(A_{l_n}e^{f_{l_n}} + B_{l_n}(\theta + \alpha)^{-1}(A)e^{f_{l_{n+1}} - A)e^{f_{l_n}}B_{l_n}(\theta) = A_{l_n}e^{f_{l_n}}(\theta)\),

by (3.3) and \(\xi \in (\frac{2\pi}{M - 1}, \frac{1}{2})\), we have

\[
|\mathcal{F}_{l_{n+1}}(\theta)|_{\frac{1}{n+1}} \leq |f_{l_{n+1}}'|_{\frac{1}{n+1}} + \|A_{l_n}^{-1}\| \cdot |B_{l_n}(\theta + \alpha)^{-1}(A)e^{f_{l_{n+1}} - A)e^{f_{l_n}}B_{l_n}|_{\frac{1}{n+1}} \leq \frac{1}{2}\epsilon_{l_{n+1}} + 2\|A\|^2 \times (l_n|\ln \epsilon_{l_n}|)^{2\tau(x+1)}\epsilon_{l_n}^{-2\tau} \times \frac{c(2\|A\|)^{Dl_l}}{\epsilon_{l_{n+1}}}.\]

Apply Proposition 3.1 to the cocycle \((\alpha, A)e^{f_{l_{n+1}}(\theta)}\), one can obtain \(\tilde{B}_{l_n}(\theta) \in C_{\frac{1}{n+2}}(2T^d, SL(2, \mathbb{R}))\), \(A_{l_{n+1}} \in SL(2, \mathbb{R})\) and \(f_{l_{n+1}}'(\theta) \in C_{\frac{1}{n+2}}(T^d, sl(2, \mathbb{R}))\) such that

\[
\tilde{B}_{l_n}(\theta + \alpha)^{-1}A_{l_n}e^{f_{l_n}}(\theta)\tilde{B}_{l_n}(\theta) = A_{l_{n+1}}e^{f_{l_{n+1}}}(\theta).
\]

Let \(B_{l_{n+1}} := \tilde{B}_{l_n}B_{l_n} \in C_{\frac{1}{n+2}}(2T^d, SL(2, \mathbb{R}))\), and note that whether or not the \((l_{n+1})\text{-th step}\) is in the resonant case, the following estimate holds:

\[
|\mathcal{F}_{l_{n+1}}(\theta)|_{\frac{1}{n+1}} \leq \frac{1}{2}\epsilon_{l_{n+1}}. \tag{3.6}
\]

We are going to analyze the structure of \(A_{l_{n+1}}\) and estimate the norm of the conjugation \(B_{l_{n+1}}\) in \((l_{n+1})\text{-th step}\). Let \(A_{l_{n+1}} := e^{A''_{l_{n+1}}}\), and if \(\|A''_{l_{n+1}}\|\) is sufficiently small, one can always find some unitary matrices \(U \in SL(2, \mathbb{C})\) such that

\[
U^{-1}A_{l_{n+1}}U = \begin{pmatrix}
    e^{2\pi i \rho_{l_{n+1}}} & c_{n+1} \\
    0 & e^{-2\pi i \rho_{l_{n+1}}}
\end{pmatrix}
\]

with the estimate \(c_{n+1} \leq 2\|A''_{l_{n+1}}\|\). Let us focus on the cocycle \((\alpha, A)e^{f_{l_n}}(\theta)\) and we need to distinguish between two cases.
We deduce that $\|2\rho_{l_n} - \langle n_{l_{n+1}}^*, a \rangle\| \leq |\varepsilon_{l_{n+1}}''| < 0 < |n_{l_{n+1}}^*| \leq N_{l_{n+1}} := 2\ln |\varepsilon_{l_{n+1}}| \frac{1}{l_{n+1} - \frac{1}{2}}$, where $\{e^{2\pi i\rho_n}, e^{-2\pi i\rho_n}\}$ are two eigenvalues of $A_{l_n}$. Then by Proposition 3.1,

$$\left|\tilde{B}_{l_n}\right|_{l_{n+1}^{-1}} \leq C_3(\gamma, \tau)(l_{n+1} |\ln \varepsilon_{l_{n+1}}|^\frac{1}{2}) \times \varepsilon_{l_{n+1}}^{-\frac{2\gamma}{l_{n+1} - 1}}. \quad (3.7)$$

We also have

$$\|\tilde{B}_{l_n}(\theta)\|_0 \leq C_3(l_{n+1} |\ln \varepsilon_{l_{n+1}}|^\frac{1}{2}).$$

Hence,

$$|B_{l_{n+1}}|_{l_{n+1}^{-1}} \leq C_3(l_{n+1} |\ln \varepsilon_{l_{n+1}}|^\frac{1}{2}) \times \varepsilon_{l_{n+1}}^{-\frac{2\gamma}{l_{n+1} - 1}} \times (l_n |\ln \varepsilon_{l_n}|)^{\tau(\sigma+1)} \varepsilon_{l_n}^{-\frac{\gamma}{l_n}}$$

$$\|B_{l_{n+1}}\|_0 \leq (l_{n+1} |\ln \varepsilon_{l_{n+1}}|)^{\tau(\sigma+1)}. \quad (3.8a)$$

Moreover one can get that $\|A''_{l_{n+1}}\| \leq 16\varepsilon_{l_{n+1}}''$, which gives $|c_{n+1}| \leq 2\|A''_{l_{n+1}}\| \leq 32\varepsilon_{l_{n+1}}''$. Combine with (3.8a), for any $\kappa \in [1, \frac{\pi}{\gamma} \varepsilon_{l_n}'' \ln \varepsilon_{l_{n+1}}]$, we have

$$\|B_{l_{n+1}}\|_0 \cdot |c_{n+1}| \leq 32 |\ln \varepsilon_{l_{n+1}}|^\frac{\kappa}{\Pi} \cdot \varepsilon_{l_{n+1}}'' \times \varepsilon_{l_{n+1}}'' < \infty. \quad (3.9)$$

(Non-resonant case). If the $(l_{n+1})$-th step is obtained by non-resonant case, we track back to the nearest resonant step, says the $(l_m)$-step. If such $m$ does not exist, we deduce that each step is in non-resonant case, thus

$$\|A_{l_{n+1}} - A\| \leq \|A_{l_1} - A\| + \sum_{i=1}^{n} \|A_{l_i+1} - A_{l_1}\| \leq 8\|A\|\varepsilon_{l_1},$$

and $|c_{n+1}| \leq \|A_{l_{n+1}}\| \leq 2\|A\|$. Since $B_{l_{n+1}}$ is close to identity by construction, it follows that

$$\|B_{l_{n+1}}\|_0 \leq |B_{l_{n+1}}|_{l_{n+1}^{-1}} \leq 2. \quad (3.10)$$

We deduce that

$$\|B_{l_{n+1}}\|_0 \cdot |c_{n+1}| \leq 2^{\kappa+1} \|A\| < \infty. \quad (3.11)$$

If such $m < n + 1$ exists, let $A_{l_m} = e^{A''_{l_m}}$, then by (3.3) and (3.4), we have

$$\|A''_{l_m}\| \leq 16\varepsilon_{l_m}'', \quad \|A_{l_m}\| \leq 1 + 32\varepsilon_{l_m}'', \quad |B_{l_m}|_{l_{n+1}^{-1}} \leq (l_m |\ln \varepsilon_{l_m}|)^{\tau(\sigma+1)} \varepsilon_{l_m}^{-\frac{\gamma}{l_m}}.$$
Since each step is in non-resonant case from \((l_m)\)-th step to \((l_{m+1})\)-th step, it deduces that
\[
\|A_{l_{m+1}} - A_{l_m}\| \leq 8\|A\| \varepsilon_{l_{m+1}}, \quad \|A_{l_{m+1}}\| \leq 2\|A\|. \tag{3.12}
\]
Moreover, (3.12) also implies
\[
|c_{l_{m+1}}| \leq 2\|A_{l_{m+1}}''\| \leq 64\varepsilon_{l_m}^2. \tag{3.13}
\]
Note that each conjugation is close to identity from \((l_m)\)-step to \((l_{m+1})\)-step by construction in (3.5), thus one can get that
\[
\|B_{l_{m+1}}\|_0 \leq |B_{l_{m+1}}|_{l_{m+1}} \leq 2\|B_{l_m}\|_{l_{m+1}} \leq \left(\ln \varepsilon_{l_{m+1}}\right)^{\tau(\sigma+1)} \varepsilon_{l_{m+1}}^{-\zeta}. \tag{3.14}
\]
Combine (3.13) with (3.14), it follows that
\[
\|B_{l_{m+1}}\|_{l_{m+1}}^\kappa \cdot |c_{l_{m+1}}| \leq 2^{\kappa+6} \ln \varepsilon_{l_m} \varepsilon_{l_{m+1}}^{\kappa} \varepsilon_{l_{m+1}}^{-\zeta} < \infty. \tag{3.15}
\]
This completes the proof of (A) by (3.6), (3.8a), (3.10) and (3.14). It also finishes the proof for (B) by (3.9), (3.11) and (3.15).

**Remark 3.1.** Theorem 3.1 has been proved in [10] essentially, however for the technical reason, we replace the estimate (3.12) by (3.13) in Proposition 3.2 of [10] by \(|B_{l_j}|_0 \leq (l_j \ln \varepsilon_{l_j})^{\tau(\sigma+1)}\), so that one can get \(|B_{l_j}|_{l_{m+1}}^\kappa \cdot |c_{l_{m+1}}| < \infty\).

### 3.2 Complex almost triangularization for C^k Q-P cocycles

In Theorem 3.1, one can see that the conjugation \(B_j : 2T^d \to SL(2, \mathbb{R})\) is real, which results in \(A_j\) being real. In fact, we can choose the complex conjugation \(B_j : 2T^d \to SL(2, \mathbb{C})\) to make \(A_j\) complex almost triangularization provided that the cocycle is not uniformly hyperbolic.

**Theorem 3.2.** Suppose that all the conditions in Theorem 3.1 hold. Further assume that 
\((\alpha, Ae^{f(\theta)})\) is not uniformly hyperbolic. There exists \(\varepsilon_* = \varepsilon_*(\gamma, \tau, d, k, \|A\|)\) such that if \(\|f\|_k \leq \varepsilon_*\), then there exist \(\tilde{A}_j \in SL(2, \mathbb{C}), \tilde{F}_j \in C^{k_0}(T^d, SL(2, \mathbb{C}))\) with
\[
k_0 = \left[\frac{k}{20}\right] \quad \text{and} \quad \Phi_j(\theta) \in C^{k_0}_{l_{j+1}}(T^d, SL(2, \mathbb{C})),
\]
\[
\Phi_j(\theta + \alpha)^{-1}Ae^{f(\theta)}\Phi_j(\theta) = \tilde{A}_j + \tilde{F}_j(\theta),
\]
where
\[
\tilde{A}_j = \begin{pmatrix} e^{2\pi i \beta_j} & c_j \\ 0 & e^{-2\pi i \beta_j} \end{pmatrix}
\]
with \( \rho_j \in \mathbb{R} \) and \( c_j \in \mathbb{C} \), also with estimates
\[
\| \mathcal{F}_j(\theta) \|_0 \leq 2\varepsilon_i^j, \quad \| \Phi_j(\theta) \|_0 \leq (l_j \ln \varepsilon_i)_{\sigma+1}^{\tau}, \quad \| \Phi_j(\theta) \|_{\sigma} \cdot |c_j| \leq C, \tag{3.16}
\]
where \( C \) is a constant defined in Theorem 3.1 and \( \kappa \in [1, \frac{\omega}{87}] \).

**Proof.** Recall that in Theorem 3.1, we already have
\[
B_j(\theta + \alpha)^{-1} A e^{\ell_j(\theta)} B_j(\theta) = A_j e^{\ell_j(\theta)}, \quad \forall j \in \mathbb{Z}.
\]
Denote
\[
A_j + \tilde{F}_j = A_j e^{\ell_j(\theta)} + B_j(\theta + \alpha)^{-1} (A e^{\ell_j(\theta)} - A e^{\ell_j(\theta)}) B_j(\theta),
\]
then
\[
B_j(\theta + \alpha)^{-1} A e^{\ell_j(\theta)} B_j(\theta) = A_j + \tilde{F}_j(\theta).
\]
By the estimates of \( (A) \) in Theorem 3.1, we have
\[
\| \tilde{F}_j \|_0 \leq 2 \| A_j f_j(\theta) \|_0 + 2 \| B_j(\theta + \alpha)^{-1} A (f(\theta) - f_j(\theta)) B_j(\theta) \|_0
\leq 2 \| A \varepsilon_j^j + 2 \| A \| (l_j \ln \varepsilon_i)_{\sigma+1}^{2\tau} \times \frac{c}{(2\|A\|)_{\sigma}^{\tau}}
\leq (2\|A\|)^{-1} \varepsilon_j^j,
\]
where the second step uses the fact
\[
\sum_{i \geq j} \| f_{i+1} - f_i \| \leq \frac{c}{(2\|A\|)_{\sigma}^{\tau}}
\]
by (2.2).
Assume that \( \{e^{2\pi i \rho_j}, e^{-2\pi i \rho_j}\} \) are two eigenvalues of \( A_{i,j} \), then there exist unitary \( U \in SL(2, \mathbb{C}) \), such that
\[
U^{-1} A_j U = \begin{pmatrix} e^{2\pi i \rho_j} & c_j \\ 0 & e^{-2\pi i \rho_j} \end{pmatrix}
\]
with \( |c_j| \leq \| A_j \| \leq 2\| A \| \), where \( \rho_j \in i\mathbb{R} \cup \mathbb{R} \) and \( c_j \in \mathbb{C} \). In the following, we need to rule out that \( i \rho_j \in \mathbb{R} \setminus \{0\} \). Suppose that \( \lambda_j = i \rho_j \in \mathbb{R} \setminus \{0\} \). If \( 2\pi|\rho_j| > \varepsilon_i^j \), let \( P := \text{diag}\{2A, \frac{1}{2} \varepsilon_j^j, 2A, -\frac{1}{2} \varepsilon_j^j\} \), then we have
\[
P^{-1} U^{-1} (A_j + \tilde{F}_j) U P = \begin{pmatrix} e^{2\pi \lambda_j} & 0 \\ 0 & e^{-2\pi \lambda_j} \end{pmatrix} + F(\theta)
\]
with \( F \|_0 \leq 2\varepsilon_i^j \). We rewrite
\[
\begin{pmatrix} e^{2\pi \lambda_j} & 0 \\ 0 & e^{-2\pi \lambda_j} \end{pmatrix} + F(\theta) = \begin{pmatrix} e^{2\pi \lambda_j} & 0 \\ 0 & e^{-2\pi \lambda_j} \end{pmatrix} e^{\ell(\theta)} \tag{3.17}
\]
with \( \| \tilde{f}(\theta) \|_0 \leq 4 \| A \| \varepsilon_i. \)

By Lemma 2.2 and Corollary 3.1 of [17], one can conjugate (3.17) to
\[
\begin{pmatrix}
e^{2\pi i \lambda_j} & 0 \\
0 & e^{-2\pi i \lambda_j}
\end{pmatrix}
\begin{pmatrix}
e^{\tilde{f}_r(\theta)} & 0 \\
0 & e^{-\tilde{f}_r(\theta)}
\end{pmatrix}
\begin{pmatrix}
e^{2\pi i \lambda_j} e^{\tilde{f}_r(\theta)} & 0 \\
0 & e^{-2\pi i \lambda_j} e^{-\tilde{f}_r(\theta)}
\end{pmatrix}
\]
with \( \| \tilde{f}_r(\theta) \|_0 \leq 8 \| A \| \varepsilon_i, \) thus \( (a, Ae^{\tilde{f}(\theta)}) \) is uniformly hyperbolic, which contradicts to the assumption. Hence we only need to consider
\[2\pi |\rho_j| \leq \varepsilon_i.\]

In this case, we put \( \rho_j \) into the perturbation so that the new perturbation satisfies
\[\| \tilde{F}_j \|_0 \leq 2\varepsilon_i \quad \text{and} \quad A_{ij} = \begin{pmatrix} 1 & c_j \\ 0 & 1 \end{pmatrix}.\]

Denote
\[\Phi_{ij}(\theta) = B_{ij}(\theta)U \in C_{\frac{1}{1+\varepsilon}}^\omega \left( 2T^d, SL(2, \mathbb{C}) \right),\]
we always have
\[\Phi_{ij}(\theta + \alpha)^{-1} A e^{\tilde{f}(\theta)} \Phi_{ij}(\theta) = \begin{pmatrix} e^{2\pi i \rho_j} & c_j \\ 0 & e^{-2\pi i \rho_j} \end{pmatrix} + \tilde{F}_j(\theta)\]
with \( \rho_j \in \mathbb{R}, c_j \in \mathbb{C} \) and \( \| \tilde{F}_j \|_0 \leq 2\varepsilon_i. \) Since \( U \) is unitary, by Theorem 3.1(A), (3.16) holds.

\[\square\]

4 Sharp Hölder continuity of the spectral measure

Consider the following discrete \( C^k \) Q-P Schrödinger operators:
\[(H_{V,\alpha, \theta} u)_n = u_{n+1} + u_{n-1} + V(\theta + na)u_n, \quad \forall n \in \mathbb{Z}, \quad (4.1)\]
where \( \alpha \in D \mathcal{C}_d(\gamma, \tau) \) and \( V \in C^k(T^d, \mathbb{R}). \) Let \( \mu_{V,\alpha, \theta} \) be the spectral measure of \( H_{V,\alpha, \theta}. \)

Based on the dynamical estimates of corresponding Schrödinger cocycles, we are able to show the \( \frac{1}{4} \)-Hölder continuity of \( \mu_{V,\alpha, \theta}. \)

Let \( A(\theta) := S^V_F(\theta) \) and for any \( n \geq 1, \) we define
\[P_n(\theta) = \sum_{s=1}^n A_{2s-1}(\theta + \alpha) A_{2s-1}(\theta + \alpha),\]
where \( A_s(\theta) = A(\theta + (s-1)\alpha) \cdots A(\theta + \alpha) A(\theta). \) \( P_n \) is an increasing family of positive self-adjoint operators. \( \| P_n \| \) is unbounded since \( \text{tr} P_n \geq 2n. \) Moreover, \( \text{det} P_n \) is also unbounded. For simplicity, we will use the notation \( a \approx b \) which denotes that there exist some constants \( C > 0 \) such that \( C^{-1}a \leq b \leq Ca \) and also use the notation \( a \lesssim b \) which denotes \( a \leq Cb \) for some constants \( C > 0. \)
Theorem 4.1. Let $\alpha \in DC_{\delta}(\gamma, \tau)$, $V \in C^{k}(\mathbb{T}^d, \mathbb{R})$ with $k \geq 5D\tau$ and $D$ is a numerical constant. There exists $\delta = \delta(\gamma, \tau, k)$ such that if $\|V\|_k \leq \delta$, then for any $f \in \ell^2(\mathbb{Z}) \cap \ell^1(\mathbb{Z})$,

$$\mu_{V, \alpha, \theta}^f(I) \leq D_0\|f\|^2_{\ell^1} |I|^\frac{1}{2},$$

for all intervals $I$ and all $\theta$, where $D_0 = D_0(V, \alpha) > 0$.

Proof. Since the spectral measure $\mu_{V, \alpha, \theta}$ vanishes on $\mathbb{R} \setminus \Sigma_{V, \alpha}$, we only need to consider the case $E \in \Sigma_{V, \alpha}$.

Rewrite the Schrödinger cocycle $(\alpha, S^V_E)$ as $(\alpha, A_Ee^{f(\theta)})$, where

$$A_E = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}, \quad f(\theta) = \ln(Id + A^{-1}_E F(\theta)), \quad F(\theta) = \begin{pmatrix} -V(\theta + na) & 0 \\ 0 & 0 \end{pmatrix}.$$

By the assumption on $\|V\|_k$ and Theorem 3.2, for $\kappa \in [0, \frac{c_k}{\delta_0}]$, there exist $\Phi_{t_j} \in C^{k}(2\mathbb{T}^d, SL(2, \mathbb{C}))$ with $\|\Phi_{t_j}\|_0 \leq (l_j |\ln c_{t_j}|)^{(\sigma + 1)}$ and $\beta_i \in C^{k_0}(\mathbb{T}^d, \mathbb{R})$, $i = 1, 2, 3, 4$ with $k_0 = \lfloor \frac{k}{50} \rfloor$ such that

$$\Phi_{t_j}(\theta + \alpha)^{-1} S^V_E(\theta) \Phi_{t_j}(\theta) = T_{t_j} + \begin{pmatrix} \beta_1(\theta) & \beta_2(\theta) \\ \beta_3(\theta) & \beta_4(\theta) \end{pmatrix},$$

where

$$T_{t_j} = \begin{pmatrix} e^{2\pi i \rho_j} & c_j \\ 0 & e^{-2\pi i \rho_j} \end{pmatrix}$$

with $\rho_j \in \mathbb{R}$ and $c_j \in \mathbb{C}$, also with estimates

$$\|\Phi_{t_j}\|_0 \cdot |c_j| \leq C, \quad \|\beta_i\|_0 \leq 2\varepsilon_{t_j}^{\frac{1}{2}}, \quad i = 1, 2, 3, 4. \quad (4.2)$$

For simplicity of notations, in the following $\Phi_{t_j}, T_{t_j}, \rho_j$ are written as $\Phi, T, \rho$ respectively. Let $\overline{T}(\theta) = \Phi(\theta + \alpha)^{-1} S^V_E(\theta) \Phi(\theta)$, then we have

$$\|\overline{T} - T\|_0 \leq 2\varepsilon_{t_j}^{\frac{1}{2}}. \quad (4.3)$$

We need to compare the dynamics between $(\alpha, S^V_E)$ and $(\alpha, T)$. For this purpose, let $X = \sum_{j=1}^n T_{2j-1}^s \bar{T}_{2j-1}$ and $\bar{X}(\theta) = \sum_{j=1}^n \bar{T}_{2j-1}(\theta) \bar{T}_{2j-1}(\theta)$. The following estimates on $\|X\|_0$ and $\|X^{-1}\|_0^{-1}$ are crucial.

Lemma 4.1 (Lemma 4.3 of [5]). Let

$$K(\theta) = \begin{pmatrix} e^{2\pi i \rho} & t(\theta) \\ 0 & e^{-2\pi i \rho} \end{pmatrix},$$

where $t(\theta) = \bar{t}(r)e^{2\pi i r\theta}$. Let $G = \sum_{j=1}^n K_{2j-1}^s K_{2j-1}$, then

$$\|G\|_0 \approx n(1 + |\bar{t}(r)|^2 \min\{n^2, \|2\rho - \langle r, \alpha \rangle\|^2 \}),$$

$$\|G^{-1}\|_0^{-1} \approx n.$$
Lemma 4.2 (Lemma 4.4 of [5]). Let $G$ and $\tilde{t}(r)$ be as above. Define $\tilde{G} = \sum_{j=1}^{n} \tilde{K}_{2j-1}^{+} \tilde{K}_{2j-1}^{-}$. Then there exists $D_1 > 0$ such that if

$$\| \tilde{K} - K \|_0 \leq D_1 n^{-2} (1 + 2n |\tilde{t}(r)|)^{-2},$$

we have $\| \tilde{G} - G \|_0 \leq 1$.

Apply Lemma 4.1, one can get that

$$\| X \|_0 \approx n (1 + |c|)^2 \min \{ n^2, \| \rho \|_T^2 \}, \quad \| X^{-1} \|_0^{-1} \approx n. \quad (4.4)$$

Let $n^+$ be maximal such that $\| \tilde{X} - X \|_0 \leq 1$ for $1 \leq n < n^+$. So by Lemma 4.2 and (4.3), one can get that

$$2\epsilon_j^2 \geq D_1 (n^+)^{-2} (1 + 2n^+ |c|)^{-2} \geq (n^+)^{-4}.$$  

It follows that

$$n^+ \geq \epsilon_j^{-\frac{1}{6}}. \quad (4.5)$$

Since $\| \tilde{X} \|_0 \leq \| X \|_0 + 1$ and $\| \tilde{X}^{-1} \|_0 \geq \| X^{-1} \|_0 - 1$ for $1 \leq n < n^+$, also we notice that

$$\| P \|_n \leq \| \Phi(\theta) \|_0^4 \cdot \| \tilde{X}(\theta + \alpha) \|_0,$$

$$\| P^{-1} \|_0 \geq \| \Phi(\theta) \|_0^{-4} \cdot \| \tilde{X}(\theta + \alpha)^{-1} \|_0^{-1}.$$  

(4.4) implies

$$\| P \|_0 \leq D_2 n (1 + |c|)^2 n^2 \| \Phi \|_0^4,$$

$$\| P^{-1} \|_0^{-1} \geq D_3 n \| \Phi \|_0^{-4}.$$  

Hence by direct calculation,

$$\frac{\| P \|_0}{\| P^{-1} \|_0^{-3}} \leq D_4 |c| \cdot \| \Phi \|_0^{16} + D_4 \frac{1}{n^2} \| \Phi \|_0^{16}.$$  

From Theorem 3.2, we know that $\| \Phi \|_0^5 \cdot |c| < \infty$ and $\| \Phi \|_0 \leq \epsilon_j^{-\frac{1}{5}}$, then

$$\| P \|_0 \lesssim \| P^{-1} \|_0^{-3}, \quad \forall n \in (n^-, n^+),$$

where

$$n^- := \epsilon_j^{-\frac{1}{5}}. \quad (4.6)$$

Denote the interval $I_j := [D_{5j}^{1/10}, D_{5j+1}^{1/10}], j \in \mathbb{Z}^+$, then by (4.5) and (4.6), for any $n \in I_j$, we have $n^- < n < n^+$. Since $I_j \cap I_{j+1} \neq \emptyset, \forall j \in \mathbb{Z}^+$, thus $\cup_{j \geq 1} I_j$ cover all the $n$ tending to
infinity. Hence \( \|P_n\|_0 \lesssim \|P_n^{-1}\|_0^{-3} \) for any \( n \geq D_5 l_{150}^{\frac{1}{4}} \). On the other hand, the number of \( n \) satisfying \( n < D_5 l_{150}^{\frac{1}{4}} \) is finite, then

\[
\sup_{n < D_5 l_{150}^{\frac{1}{4}}} \frac{\|P_n\|_0}{\|P_n^{-1}\|_0^{-3}} < \infty.
\]

Thus

\[
\|P_n\| \lesssim \|P_n^{-1}\|^{-3}, \quad \forall n \in \mathbb{Z}^+.
\]  \hfill (4.7)

**Lemma 4.3** (Lemma 4.2 of [5]). Let \( \epsilon_n = \frac{1}{2\sqrt{\det P_n}} \), then

\[
D_6^{-1} < \frac{\psi(m^+(E + i\epsilon_n))}{2\epsilon_n\|P_n\|_0} < D_6,
\]

where \( D_6 > 0 \) is a constant and \( \psi(m^+) \) is defined in Lemma 2.1.

For any bounded potential and any solution \( u \) satisfying (4.1), we have

\[
\|u\|_{L^1+} \lesssim \|u\|_L, \quad \text{where} \quad \|u\|_L = \left( \sum_{j=1}^L |u_j|^2 \right)^{\frac{1}{2}}.
\]

In particular, for the solution \( u^\beta \) with \( u_0^\beta \cos \beta + u_1^\beta \sin \beta = 0 \) and \( |u_0^\beta|^2 + |u_1^\beta|^2 = 1 \), we have

\[
det P_n = \inf_{\beta} \|u^\beta\|_2^2 \|u^{\beta + \pi/2}\|_L^2. \]  \hfill (4.8)

By (4.7) and \( P_n : \mathbb{T}^d \rightarrow \mathfrak{gl}(2, \mathbb{R}) \), we have

\[
\|P_n\|_0 = \det P_n \|P_n^{-1}\|_0 \lesssim \epsilon_n^{-2} \|P_n\|^{-\frac{1}{2}},
\]

where \( \epsilon_n \) is defined in Lemma 4.3. Thus \( \|P_n\|_0 \lesssim \epsilon_n^{-\frac{3}{2}} \). According to Lemma 4.3, we deduce that

\[
\psi(m^+(E + i\epsilon_n)) \lesssim \epsilon_n \|P_n\|_0 \lesssim \epsilon_n^{-\frac{3}{2}}.
\]

Since \( \lim_{n \to \infty} \epsilon_n = 0 \), and we also have \( \epsilon_n \lesssim \epsilon_{n+1} \) by (4.8), we only need to consider the case of fixing \( \epsilon_n = \epsilon \). Combine (2.1) with Lemma 2.1, we have

\[
\mu_{V, \alpha, \beta}(E - \epsilon, E + \epsilon) \leq 2\epsilon \Im M(E + i\epsilon) \lesssim \epsilon^\frac{1}{2}.
\]

Since \( \mu_{V, \alpha, \beta} = 0 \) on \( \mathbb{R} \setminus \Sigma_{V, \alpha} \), then there exists \( D_0 = D_0(V, \alpha) > 0 \) such that

\[
\mu_{V, \alpha, \beta}(J) \leq D_0 |J|^{\frac{1}{2}}, \quad \forall J \subset \mathbb{R}.
\]
Let $\sigma : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the shift $f(i + 1) = \sigma f(i)$. Then $\sigma H_{V, \alpha, \theta} \sigma^{-1} = H_{V, \alpha, \theta + \alpha}$. Thus $\mu_{\sigma f, \alpha} = \mu_{\sigma}$ and $\mu_{\sigma}^{\alpha} = \mu_{\sigma + k\alpha}^{\alpha} \leq \mu_{\theta + k\alpha}$. Let $\mathcal{E}(J)$ be the spectral projection of $H_{V, \alpha, \theta}$ on $J$, then

$$
\mu_{V, \alpha, \theta}^f(J) = \langle \mathcal{E}(J) f, f \rangle = \| \mathcal{E}(J) \sum_k f(k) e_k \|^2 \leq \left( \sum_k |f(k)| \cdot \| \mathcal{E}(J) e_k \| \right)^2
$$

$$
= \left( \sum_k |f(k)|\mu_{V, \alpha, \theta}^{\alpha}(J)^2 \right)^2 \leq \left( \sum_k |f(k)|\mu_{V, \alpha, \theta + k\alpha}^{\alpha}(J)^2 \right)^2
$$

$$
\leq D_0 \| f \|_2^2 \cdot |J|^2.
$$

Thus, we complete the proof.

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References

[1] A. Avila, Global theory of one-frequency Schrödinger operators, Acta. Math., 215 (2015), 1–54.
[2] A. Avila, The absolutely continuous spectrum of the almost Mathieu operator, arXiv: 0810.2965.
[3] A. Avila, KAM, Lyapunov exponents and the spectral dichotomy for one-frequency Schrödinger operators, preparation.
[4] A. Avila, and S. Jitomirskaya, Almost localization and almost reducibility, J. Euro. Math. Soc., 12 (2010), 93–131.
[5] A. Avila, and S. Jitomirskaya, Hölder continuity of absolutely continuous spectral measures for one-frequency Schrödinger operators, Commun. Math. Phys., 301 (2011), 563–581.
[6] J. Avron, and B. Simon, Almost periodic Schrödinger operators II, the integrated density of states, Duke. Math. J., 506 (1983), 369–390.
[7] J. Bourgain, Hölder regularity of integrated density of states for the almost Mathieu operator in a perturbative regime, Lett. Math. Phys., 51 (2000), 83–118.
[8] J. Bourgain, On the spectrum of lattice Schrödinger operators with deterministic potential. II, J. Anal. Math., 88 (2002), 221–254.
[9] A. Cai, and L. Ge, Reducibility of finitely differentiable quasi-periodic cocycles and its spectral applications, arXiv:1712.09041.
[10] A. Cai, C. Chavaudret, J. You, and Q. Zhou, Sharp Hölder continuity of the Lyapunov exponent of finitely differentiable quasi-periodic cocycles, Math. Z., 291 (2019), 931–958.
[11] D. Damanik, Schrödinger operators with dynamically defined potentials, Erg. Theory. Dyn. Syst., 37 (2017), 1681–1764.
[12] D. Damanik, R. Killip, and D. Lenz, Uniform spectral properties of one-dimensional quasicrystals. iii: $\alpha$-continuity, Commun. Math. Phys., 212 (2000), 191–204.
[13] L. H. Eliasson, Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation, Commun. Math. Phys., 146 (1992), 447–482.
[14] M. Goldstein, and W. Schlag, Hölder continuity of the integrated density of states for quasi-periodic Schrödinger equations and averages of shifts of subharmonic functions, Ann. Math., 154 (2001), 155–203.

[15] M. Goldstein, and W. Schlag, Fine properties of the integrated density of states and a quantitative separation property of the Dirichlet eigenvalues, Geom. Funct. Anal., 18 (2008), 755–869.

[16] S. Hadj Amor, Hölder continuity of the rotation number for quasi-periodic cocycles in $\text{SL}(2,\mathbb{R})$, Commun. Math. Phys., 187 (2009), 565–588.

[17] X. Hou, and J. You, Almost reducibility and non-perturbative reducibility of quasi-periodic linear systems, Invent. math., 190 (2012), 209–260.

[18] S. Jitomirskaya, Almost Everything About the Almost Mathieu Operator, II, Proceedings of XI International Congress of Mathematical Physics, Int. Press, (1995), 373–382.

[19] R. Johnson, Exponential dichotomy, rotation number, and linear differential operators with bounded coefficients, J. Differential Equations, 61 (1986), 54–78.

[20] M. Leguil, J. You, Z. Zhao, and Q. Zhou, Asymptotics of spectral gaps of quasi-periodic Schrödinger operators, arXiv:1712.04700.

[21] W. Liu, and X. Yuan, Hölder continuity of the spectral measures for one-dimensional Schrödinger operator in exponential regime, J. Math. Phys., 56 (2015), 012701.

[22] J. You, and S. Zhang, Hölder continuity of the Lyapunov exponent for analytic quasiperiodic Schrödinger cocycle with weak Liouville frequency, Erg. Theory. Dyn. Syst., 34 (2014), 1395–1408.

[23] E. Zehnder, Generalized implicit function theorems with application to some small divisor problems, I, Commun. Pure. Math., XXVIII (1975), 91–140.

[24] X. Zhao, Hölder continuity of absolutely continuous spectral measure for multi-frequency Schrödinger operators, J. Funct. Anal., (2020), https://doi.org/10.1016/j.jfa.2020.108508.