Two-dimensional $XY$ spin/gauge glasses on periodic and quasiperiodic lattices

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Abstract

Via Monte Carlo studies of the frustrated XY or classical planar model we demonstrate the possibility of a finite (nonzero) temperature spin/gauge glass phase in two dimensions. Examples of both periodic and quasiperiodic two dimensional lattices, where a high temperature paramagnetic phase changes to a spin/gauge glass phase with the lowering of temperature, are presented. The existence of the spin/gauge glass phase is substantiated by our study of the temperature dependence of the Edwards-Anderson order parameter, spin glass susceptibility, linear susceptibility and the specific heat. Finite size scaling analysis of spin glass susceptibility and order parameter yields a nonzero critical temperature and exponents that are in close agreement with those obtained by Bhatt and Young in their random $±J$ Ising model study on a square lattice. These results suggest that certain periodic and quasiperiodic two-dimensional arrays of superconducting grains in suitably chosen transverse magnetic fields should behave as superconducting glasses at low temperatures.
I. INTRODUCTION

In a recent communication [1] in this journal, we reported that the frustrated XY model (see Eq.(1)) on a two-dimensional (2D) Penrose lattice [2] exhibits a low temperature spin/gauge glass phase [3,4]. In this work, we show that quasiperiodicity is not a necessary requirement of the spin glass phase in this model. This conclusion is based on our Monte Carlo (MC) study of the above model on the periodic honeycomb and bathroom tile lattices. When fully frustrated, these lattices appear to undergo a paramagnetic to spin/gauge glass transition at a low, but finite temperature, $T_f$. In addition, we find that the octagonal quasiperiodic lattice [5] exhibits a behavior similar to that of the other three lattices, but with a somewhat higher $T_f$.

These results are unexpected and controversial in the light of the prevalent notion that the critical dimension for a spin glass phase is greater than two [6-8]. The validity of this notion, however, has been put to doubt recently not only by our work [1] but also by the work of Lemke and Campbell [10] in their study of the Ising model with nearest neighbor random $\pm \lambda J$ and next nearest neighbor ferromagnetic interactions on square lattices. An important distinction between our study and that of Lemke and Campbell [10] is that in the latter case frustration is induced via (or accompanied by) disorder with the nearest neighbor interactions varying randomly, whereas in our model frustration is nonrandom: it is determined by the lattice structure and an applied magnetic field. Thus our results provide definite examples of cases where frustration alone (i.e., without disorder) is capable of inducing a spin glass phase. Our results clearly illustrate the role of lattice structure as a relevant variable from the viewpoint of the universality class of the transition. The frustrated XY model has been studied widely on a variety of periodic 2D lattices using mean-field and renormalization group methods, and Monte Carlo simulation [1,2,4,13,14,15,16,17,18,19,20,21,22,23,24]. The results, although sometimes ambiguous and often discrepant among various authors, are certainly lattice-dependent. Two most commonly reported and discussed transitions for the fully frustrated XY model are the Ising type and the Kosterlitz-Thouless(KT) type [25,26].
We provide evidence that rule out the possibility of either of these two types of transitions on the two periodic and the two quasiperiodic lattices considered in this work.

The purpose of this work is two-fold: to show that both periodic and quasiperiodic lattices can exhibit finite temperature spin glass transitions in 2D, and to provide some details of the study on the Penrose lattice, which were left out of our previous communication \[1\]. With the example of the periodic honeycomb and bathroom tile lattices we dispel any misconception we may have inadvertently generated in ref.1 that quasiperiodicity is a necessary condition for the spin glass phase in the model. Both the periodic and the quasiperiodic lattices studied in this work seem to undergo a paramagnetic to spin glass transition at temperatures that are low but certainly nonzero.

We have studied the current model without the frustration term (the $A_{ij}$ term in Eq.(1)), i.e. the standard XY model, on the quasiperiodic 2D Penrose lattice. We find that the transition is the Kosterlitz-Thoules(KT)-type \[25\] with the exponents identical to those obtained by Tobochnik and Chester \[27\] in their study on the square lattice, but the transition temperature is somewhat higher. Thus in the absence of frustration (or variation in the sign of the interactions) quasiperiodicity is an irrelevant variable, as expected. The details of the standard XY model study on 2D and 3D Penrose lattices will be published elsewhere.

The remainder of this paper is divided into sections as follows. In Sec. II we discuss our model and some features of the lattices considered in this work. In Sec. III we present the results of the MC simulations. In Sec. IV we compare our results with those obtained using other lattices. We also suggest what characteristic of our model might be responsible for the spin glass phase. In Sec. V we present our comments and conclusions.

II. THE MODEL

We consider the Hamiltonian for the XY model describing the interaction between 2D spin vectors with orientations $\theta_i$ and $\theta_j$ situated at lattice sites $i$ and $j$ via a nearest neighbor coupling parameter $J$:
\[ H = -J \sum_{[ij]} \cos(\theta_i - \theta_j + A_{ij}) , \]  

(2.1)

where the summation is restricted to nearest neighbor pairs \([ij]\). The parameter \(A_{ij}\) controls the frustration in the model. In the context of an array of superconducting grains, \(\theta_i\) is the phase of the superconducting order parameter at the grain \(i\) and the above Hamiltonian can be seen as describing the resulting Josephson junction of the grains “minimally coupled” to a transverse magnetic field with vector potential \(\mathbf{A}\) with

\[ A_{ij} = \frac{2\pi}{\Phi_0} \int_{r_i}^{r_j} \mathbf{A} \cdot d\mathbf{l} . \]  

(2.2)

\(\Phi_0\) is the elementary flux quantum \(\frac{\hbar c}{2e}\) associated with the Cooper pairs, and \(r_i\) denotes the lattice sites. Here the magnetic field acts as the source of frustration: an \(A_{ij}\) which is an odd multiple of \(\pi\) essentially renders the bond \([ij]\) negative.

The directed sum of \(A_{ij}\) about a plaquette in a 2D lattice can be written as \(2\pi f\), where \(f\) is the flux through the plaquette in units of \(\Phi_0\). The 2D Penrose lattice [2] is composed of two (fat and thin) rhombic unit cells (plaquettes). The ratio of the areas of the fat and the thin rhombuses in the Penrose lattice is the Golden Mean (\(\tau\)), which is an irrational number \((\frac{1+\sqrt{5}}{2})\). Thus only one set of plaquettes can be fully frustrated at a time with a suitable choice of the magnetic field giving \(f = 1/2\). The flux \(f\) through the individual plaquettes in the other set will then be an irrational number. The octagonal lattice [3] consists of two unit cells, one square and the other a thin rhombus, with a ratio of \(\sqrt{2}\) in their areas. Similar to the Penrose lattice case, fully frustrating one of the plaquettes results in an irrational flux through the other.

Fig.1 displays the lattices used in this work along with the reference \(x\) and \(y\) directions. The linear dimensions of the clusters used in the simulation along the \(x\) and \(y\) directions, \(L_x\) and \(L_y\), and the corresponding numbers of sites in the clusters are given in TABLE I.

In Figs. 1(a) and 1(b) we show sections of Penrose (decagonal) and octagonal quasilattices. Self-similarity, or equivalently, the inflation-deflation property of these two quasilattices are characterized by two irrational numbers, the Golden Mean \((\tau = \frac{1+\sqrt{5}}{2})\) and the
Silver Mean ($\sigma = 1 + \sqrt{2}$), respectively, which also dictate their decagonal and octagonal bond orientational symmetry. The two quasilattices can be generated via projections of 5D and 4D simple hypercubic (periodic) lattices onto the physical 2D plane. They have similar ring structure, both containing only even order rings. The average number of nearest neighbors for both quasilattices is four, as in a square lattice. But unlike the square lattice, the two quasilattices are characterized by variations in near neighbor environments. For the Penrose (decagonal) lattice the number of nearest neighbors varies between three and seven, whereas for the octagonal lattice the number varies between three and eight. In order to reduce the surface effects in our finite cluster MC calculations we have used periodic boundary conditions. Rational approximants of the two quasilattices, which can be repeated periodically, can be obtained from the rational approximations of the irrational numbers, the Golden and the Silver Means. We follow a systematic way to generate these periodic approximants as given by Lançon and Billard [28].

Figs. 1(c) and (d) display sections of the two periodic lattices, honeycomb and bathroom tile, considered in the present work. Note that both are non-Bravais lattices with the same number of nearest neighbors. The smallest unit cells that provide a Bravais lattice description of these lattices involve two sites for the honeycomb lattice and four for the bathroom tile. The honeycomb lattice consists of only one type of plaquettes (hexagonal), while the bathroom tile lattice has two types, square and octagonal, with an irrational ratio of $4(1 + \sqrt{2})$ in the areas of the two plaquettes (octagonal to square). Both structures contain only even order rings.

Finally, a word about the nomenclature used to describe the model given by Eq. (1). If the $A_{ij}$ are restricted to the values 0 and $\pi$ and randomly assume these values with equal probability, then the model becomes the Edwards-Anderson $\pm J$ XY spin glass, with random ferromagnetic ($A_{ij} = 0$) or antiferromagnetic ($A_{ij} = \pi$) coupling between adjacent spins. If the $A_{ij}$ are independent random variables, assuming all values between 0 and $2\pi$, the model is referred to as the gauge or vortex glass model [29] and is believed to belong
to a different universality class, presumably because it lacks the "reflection" symmetry, \( \theta_i \to -\theta_i \forall i \). In the present case the \( A_{ij} \) are lattice structure-dependent. But they assume many different values in the interval between 0 and \( 2\pi \), depending on the lattice structure. Instead of dwelling on the fine differences between a spin glass and a gauge glass, we will use the term spin glass throughout the remainder of the paper, especially in describing the phase transition itself. The quantities we study show temperature variations similar to those observed experimentally in the so-called spin glasses, hence the choice.

III. RESULTS OF MC SIMULATION

For the lattices which contain two different plaquettes, we present results for the case where the plaquettes with the smaller area are fully frustrated. For the Penrose lattice these are the thin rhombohedral plaquettes. For the octagonal and the bathroom tile lattices these are, respectively, the rhombohedral and the square plaquettes. Results for the other case, where the plaquettes with the larger area are fully frustrated, are qualitatively similar. All our results are obtained via MC simulation based on the Metropolis algorithm \([30]\), using periodic boundary conditions. We have cooled our systems in a quasi-static manner, starting from a high temperature \((T \text{ (in units of } J) > 2.0)\) random configuration and then heated the system in the same quasi-static fashion. Since we performed the simulation in \( n \) blocks, the heating and cooling data are obtained by averaging over these blocks, with the error bars representing the standard deviation, obtained by dividing the square root of the sum of squares of the deviations from the mean by \( \sqrt{n-1} \), instead of \( \sqrt{n} \). We then perform a ‘grand average’ over the heating and cooling data.

By associating a spin \( \vec{S}_i = (\cos \theta_i, \sin \theta_i) \) with every lattice site \( i \), we can define the quantity:

\[
\frac{1}{N} \sum_i^N \langle \vec{S}_i \rangle ,
\]

where \( \langle \ \rangle \) denotes a canonical ensemble average at a temperature \( T \), as the magnetic moment per site at temperature \( T \). Note that this quantity does not represent the actual magnetic
moment of a cluster of superconducting grains in a transverse magnetic field. However, defined as above, magnetic moment per lattice site calculated for all the lattices studied is found to be small ($< 0.02$) over the entire temperature range. The magnitude of the moment decreases steadily with the size of the cluster, suggesting that the magnetization is strictly zero in the thermodynamic limit. This is an indication that the ground state (more appropriately, the lowest temperature ($0.02 \ J$) state studied via our MC simulation) retains the continuous $O(2)$ symmetry of the Hamiltonian, and there is no spontaneous breakdown into the discrete $Z(2)$ symmetry, as would be the case for an Ising transition. However, the vanishing of the magnetic moment at all temperatures does not rule out the possibility of a KT transition. Below we present further analysis in an attempt to determine the existence and nature of the transition(s) for the various lattices.

A. Edwards-Anderson Order parameter

Since the spins appear to be disordered at all temperatures, it is appropriate to explore the possibility of spin freezing over macroscopic time scales. To study the freezing of the spins at the lattice sites we calculate the Edwards-Anderson [31] order parameter. In Fig. 2 we show this order parameter defined by:

$$q_{EA} = 1/N \sum_i (\vec{S}_i)^2. \quad (3.2)$$

In a completely frozen system $q_{EA}$ is unity, while for a completely ergodic system it is zero. This order parameter shows a monotonic decrease with increasing temperature, clearly vanishing at temperatures beyond 0.5 for all lattices. The results shown in Fig. 2 were obtained by averaging over 5 blocks of 60,000 configurations, generated after equilibrium was achieved. In all cases the order parameter is seen to vanish not abruptly, but continuously with a long tail. This is a consequence of the finite system size. It is expected that the tail region will decrease with increasing system size and eventually disappear in the thermodynamic limit. We find that the tail persists, even for our largest systems (e.g. $\sim 11,000$ site cluster for the
Penrose lattice) and, consequently, the transition temperature, at which \( q_{EA} \) goes to zero, cannot be appropriately determined from Fig. 2. Thus, we use other quantities to analyse the transition and provide estimates of the transition temperatures \( T_f \) for the four lattices.

In Fig. 2, in addition to the lattice sizes, we have included our estimates of the transition temperatures for the corresponding lattice, obtained from a finite size scaling analysis of spin glass susceptibility (to be described later).

Although we cannot accurately determine a \( T_f \) from this method, it is clear that all the systems studied have a low temperature spin/gauge phase with a nonzero order parameter \( q_{EA} \), changing into a phase with zero \( q_{EA} \) as the temperature is raised.

**B. Linear Susceptibility**

Linear susceptibilities per spin for the Penrose and honeycomb lattices, calculated from the fluctuations in the magnetization (net magnetic moment \(|m|\) for a lattice of \( N \) sites),

\[
\chi = \frac{\langle m^2 \rangle - \langle |m| \rangle^2}{N k_B T},
\]

are shown in Fig. 3 (a) and (b), respectively. At low temperatures \((0.02 - 0.2)J\), the results are obtained by averaging over 5 blocks of 125,000 MC steps, while 5 blocks of 15,000-45,000 steps were used for higher temperatures. The large hysteresis in the low temperature region indicates a high number of metastable states, which is a characteristic of spin glasses. These metastable states give rise to large error bars in \( \langle m^2 \rangle - \langle |m| \rangle^2 \) at low temperatures, which are further accentuated by a division by \( T \) in Eq.(4). Although we feel that it might be possible to reduce the size of these error bars, this would require very long runs and one must also ensure that the system does not become trapped in one of these metastable states. Nevertheless, despite the large error bars, a cusp-like feature in \( \chi \) is clearly visible at \( T_f \sim 0.15 \) for both the Penrose and honeycomb lattices. We also find a saturation in this cusp with respect to system size, which is consistent with spin glass behavior.

In the insets of Fig. 3(a) and (b) we have shown the quantity \( T\chi \), which approaches a constant at high temperatures. Thus the high temperature phase is strictly paramagnetic.
with $\chi$ obeying the Curie law. In Fig. 3 we have shown the susceptibility function for
a quasiperiodic (Penrose) and a periodic (honeycomb) lattice. The general temperature
dependence of $T_\chi$ for the other two lattices, octagonal and bathroom tile, is similar to
that shown in Fig. 3. Note that since the magnetic moment $\langle m \rangle \sim 0$ for all the lattices,
the quantity plotted in the insets of Fig. 3(a) and (b) is proportional to $\langle m^2 \rangle$. For a KT
transition this quantity diverges below the transition temperature. For an Ising transition
it remains finite as the temperature approaches zero, and also is dependent on the system
size. Thus the fact that $\langle m^2 \rangle$ approaches zero for all the lattices studied is an indication
that they do not exhibit either KT or Ising-type transition as the temperature is lowered
from the high temperature paramagnetic phase. The saturation of the susceptibility with
the system size, the appearance of a kink-like feature and the vanishing of the function $\langle m^2 \rangle$
as the temperature approaches zero are all consistent with the scenario of a low temperature
spin glass phase. As mentioned earlier the variation of $\langle m^2 \rangle$ or $T_\chi$ with temperature for the
other two lattices, octagonal and bathroom tile, is similar to that shown in the insets of Fig.
3, although some irregular features seem to appear in the function $\chi$ itself as the division by
$T$, especially at low temperatures, strongly magnifies minor deviations in $\langle m^2 \rangle$ from their
exact values. For all the four lattices studied $\langle m^2 \rangle$ rises, more or less smoothly, from zero
and saturates at a constant value as the temperature is increased. Note that the bump in
the inset of Fig. 3(b) is probably due to a convergence problem, as indicated by the large
error bars.

It should also be noted that we have repeated the susceptibility calculation for the
Penrose lattice without the frustration term $A_{ij}$ in Eq.(1). We obtain a divergence in
susceptibility as the temperature is lowered, consistent with a KT transition. By fitting
the susceptibility to the form proposed by Kosterlitz [26] we have obtained a KT transition
temperature $T_{KT}$ of 1.027±0.002$J$ for this lattice and an exponent in agreement with that
obtained by Tobochnik and Chester [27] for the square lattice. The value of 1.027 for
$T_{KT}$ obtained by us for the Penrose lattice is slightly higher than the value (0.89 − 0.95)$J$
reported in the literature for the square lattice (see Tobochnik and Chester [27] and references

9
therein). Details of the unfrustrated Penrose lattice calculation will be published elsewhere.

C. Specific Heat

Specific heats per site for the two periodic and the two quasiperiodic lattices obtained from the fluctuations in the energy $U$ of the system:

$$\langle U^2 \rangle - \langle U \rangle^2 \quad Nk_B T^2,$$

are shown in Fig. 4. Results for various system sizes, clearly indicating a saturation with respect to system size, are shown. These results are averages between heating and cooling, with the low temperature results having a somewhat larger hysteresis. All averages were obtained after equilibrating, however, the high temperature values were obtained by averaging over 5 blocks of 15,000 steps, whereas 5 blocks of 45,000 steps were used for the low temperatures. The result for the unfrustrated Penrose lattice (without the $A_{ij}$ term in Eq.(1)) is also shown (the inset of the Penrose section of Fig. 4). Like the frustrated case, the unfrustrated model shows a saturation in specific heat with respect to system size. In the unfrustrated case the peak in the specific heat is at 1.10, which is beyond the Kosterlitz-Thouless (KT) transition temperature $T_{KT} \sim 1.027$ obtained by us from the analysis of the susceptibility results. The peaking of the specific heat at a temperature beyond the temperature at which a divergence appears in the linear susceptibility is the expected behavior for a KT transition. The specific heat peak for the frustrated case is more rounded relative to the unfrustrated case and occurs at a temperature lower than $T_{KT}$. This temperature is, however, higher than the temperature $T_f$ at which, we believe, the Edwards-Anderson order parameter goes to zero or a cusp appears in the linear susceptibility. The saturation in the peak height of the specific heat is a consequence of the fact that it appears at a temperature at which the spin glass correlation is finite. Note that in both the unfrustrated and the frustrated cases the zero temperature specific heat approaches a value of $0.5k_B$. This is consistent with the equipartition theorem valid for the Hamiltonian (1) with the cosine function being truncated at the quadratic term.
A few comments regarding our results on the honeycomb lattice are in order at this stage. Shih and Stroud \[12\] carried out a Monte Carlo study of the fully frustrated XY model on the honeycomb lattice and reported the nature of the transition as KT. Their study on the honeycomb lattice was carried out together with the triangular lattice, and the conclusions regarding the nature of the transition, Ising vs. KT, were primarily based on the saturation of the specific heat with respect to system size. For the honeycomb lattice a saturation in the specific heat was obtained, while the triangular lattice did not show any saturation. Consequently, the transitions were classified as KT and Ising-type for the fully frustrated XY model on honeycomb and triangular lattices, respectively. Our work on the honeycomb lattice shows that the conclusion drawn by Shih and Stroud \[12\] was premature, since the susceptibility for the honeycomb lattice shows no divergence characteristic of a KT transition. Our results for the specific heat agree numerically with the results of Shih and Stroud. In addition our results for the system energy at the lowest temperature studied (0.02\(J\)) agree with the ground state energy reported by Shih and Stroud \[12\]. In TABLE I we present the average energy per spin for all the lattices and cluster sizes used in the simulation. We find that with periodic boundary conditions magnetic moments or ferromagnetic correlations decrease with increasing system size at low temperatures, yielding higher energy per spin for larger sizes. For the 256 site honeycomb lattice our value of average energy per site at \(T = 0.02J\), \(-1.2169J\), is lower than the ground state energy \(-1.2071J\) reported by Shih and Stroud \[12\]. Presumably, Shih and Stroud report the value obtained for their largest cluster of 576 sites. For a 1600 site cluster our value of the energy at 0.02\(J\), \(-1.198J\), is slightly higher. In addition to specific heat and energy, our results for \(q_{EA}\) are in good agreement with the local order parameter values reported by Shih and Stroud \[12\]. Note that the local order parameter studied by Shih and Stroud is the square root of the order parameter \(q_{EA}\) studied in this work. Based on these comparisons it appears that the study by Shih and Stroud was correct in terms of the accuracy of the quantities reported, but incomplete with regard to correctly identifying the nature of the transition.
D. Spin Glass Susceptibility

In a ferromagnet, the approach to the ferromagnetic phase from temperatures above the Curie temperature $T_C$ is accompanied by a dramatic increase in the range of the spin correlations, which then diverges at $T_C$. A corresponding phenomenon occurs in spin glasses. However, it is not the spin correlation function $\langle \vec{S}_i \cdot \vec{S}_j \rangle$, but rather its square that acquires a long-range. This leads to the divergence, at the spin glass transition temperature $T_f$, of the spin glass susceptibility

$$\chi_{SG} = \frac{1}{N} \sum_{ij} \langle \vec{S}_i \cdot \vec{S}_j \rangle^2 \quad (T > T_f).$$

(3.5)

$\chi_{SG}$ satisfies a finite-size scaling relation of the form

$$\chi_{SG} = L^{2-\eta} \bar{\chi}(L^{1/\nu}(T - T_f)).$$

(3.6)

where $\bar{\chi}$ is the scaling function, $L$ is the system length, $\nu$ is the exponent for spin glass correlation length $\xi$ for $T \geq T_f$, and $\eta$ describes the power law decay of the spin glass correlation at $T_f$. For the quasiperiodic lattices the linear dimensions in the $x$ and $y$ directions, $L_x$ and $L_y$, are different. Thus the scaling relations could be studied using either of these two as a measure of the linear dimension of the cluster. Alternatively, one could use $L = \sqrt{N}$ as a measure of the the linear dimension of a cluster of $N$ sites. All three options give similar results for all the lattices we have studied. In the following, we will present results with $L_x = L$ in Eq. (7). To ensure a proper convergence of $\chi_{SG}$, calculated via Eq.(6), we have averaged over 5 blocks of 40,000-60,000 steps at low temperatures ($< 0.2J$) and 5 blocks of 60,000-80,000 steps at higher temperatures. By examining our results every 5,000 steps, we find, for all lattices, little change in $\chi_{SG}$ over the last 5,000-10,000 steps. Thus, we estimate that these chain lengths produce at least a 95% convergence in $\chi_{SG}$.

In Figs. (5-8) we show the scaling behavior of $\chi_{SG}$ in terms of $L_x$. In order to obtain estimates of $T_f$, we have been guided by the exponents $\nu$ and $\eta$ reported by Bhatt and Young in their study of the $\pm J$ Ising model, and by Jain and Young for the $\pm J$ XY model.
model, on square lattices. We have also examined the Edwards-Anderson parameter in the vicinity of $T_f$, where the relation $q_{EA} \sim (T - T_f)^\beta$ is supposed to hold. Our overall finding is that the $\pm J$ Ising model exponents $\nu=2.6 \pm 0.4$ and $\eta=0.2 \pm 0.05$ reported by Bhatt and Young [6] provide a better fit to our data than the $\pm J$ XY model exponents $\nu=1.08 \pm 0.27$ and $\eta=0.3 \pm 0.3$ given by Jain and Young [7]. Our results are summarized in TABLE II. Below we examine our results case by case.

For the Penrose lattice, the Ising exponents (Fig.5(a)) fit the $\chi_{SG}$ data very well for $T_f = 0.137J$ (henceforth given in units of $J$). In addition the slope of the log-log plot of $q_{EA}$ vs. $(T - T_f)$ (inset of Fig.5(a) and TABLE II) yields a value $\beta = 0.26$ in perfect agreement with the hyperscaling relation $\beta = \eta \nu / 2$. The XY exponents $\eta = 0.3$ and $\nu = 1.08$ also provide a reasonably good fit to the data (for $T_f = 0.11$), but the agreement between the value of $\beta = 0.19$ obtained from the simulation and the hyperscaling value $\eta \nu / 2 = 0.162$ is poorer. In Fig.5(b) we have chosen $\eta = 0.26$, and $\nu = 1.35$, values within the error bars given for the XY model. With $T_f = 0.11$, the fit to the $\chi_{SG}$ data is as good as for $\eta = 0.3$ and $\nu = 1.08$, but these exponents ($\eta = 0.26$, $\nu = 1.35$) yield $\eta \nu / 2 = 0.1755$, in better agreement with the value $\beta = 0.19 \pm 0.01$ obtained from the order parameter results (inset of Fig.5(b) and TABLE II).

For the octagonal lattice, the choice of $T_f = 0.33$, $\nu = 2.6$, and $\eta = 0.3$ (Fig.6(a)) gives a good fit to the $\chi_{SG}$ data. Here the exponent $\eta$ is a little higher than the upper limit 0.25, but $\nu$ is exactly the same as reported for the Ising model. The order parameter yields $\beta = 0.38$ (inset of Fig. 6(a) and TABLE II), close to the hyperscaling value 0.39. No reasonably good fit to the $\chi_{SG}$ data can be obtained with $\nu$ and $\eta$ chosen around the values given for the XY model. In Fig.6(b) we show the fit obtained with the XY exponents, $\eta = 0.3$, and $\nu = 1.35$. The best results, obtained for $T_f = 0.3$, are still quite poor in terms of satisfying the scaling realtion. The order parameter plot (inset of Fig.6(b) and TABLE II) yields $\beta = 0.25$, while the hyperscaling value of $\beta$ is 0.2025.

For the two periodic lattices, honeycomb and bathroom tile, the results (Figs.7 and 8) are as follows. With $\nu$ close to the Ising model value 2.6 the best fit (Figs.7(a) and 8(a))
is obtained by choosing $\eta$ at the lower limit 0.15 reported by Bhatt and Young [3]. The corresponding transition temperature $T_f$ in both cases is 0.12. The log-log plots of the order parameter vs. $T - T_f$ yield $\beta = 0.35$ and $\beta = 0.33$ (insets of Figs. 7(a) and 8(a)), respectively, for the honeycomb and bathroom tile lattices. These values are about 40% off the corresponding hyperscaling values. With $\eta = 0.29$ and $\nu = 1.30$, chosen within the permitted values for the XY model, the best fits for both the periodic lattices are obtained for $T_f = 0.13$ (Figs. 7(b) and 8(b)). However, the order parameter data yield $\beta = 0.71$ and $\beta = 0.50$ (insets of Figs. 7(b) and 8(b), and TABLE II) for the honeycomb and the bathroom tile lattices, respectively. These values are 3-4 times larger than the corresponding hyperscaling values $\eta \nu / 2$. Thus considering the $\chi_{SG}$ and the order parameter data together the agreement with the Ising model seems better.

We have further examined the possibility of our spin glass susceptibility data being consistent with $T_f=0$. Here we present results for the Penrose lattice only, the results for the other lattices being qualitatively similar. In Fig.9 we show the scaling plots of $\chi_{SG}$ with $T_f=0$ for the $\pm J$ Ising and XY exponents, and also for the exponents given by our best fit. Figs.9(a) and (b) show that there is a clear breakdown of the scaling relation for $T_f=0$ if the exponents are close to those belonging to either of the $\pm J$ Ising or XY model. Fig.9(c) shows that substantial deviations in the values of $\eta$ and $\nu$ from the above two models are needed to fit the $\chi_{SG}$ data with $T_f=0$. The values $\nu=9.6$ and $\eta=0.01$ are far from the values reported in the literature for any spin glass model in 2D (see reference in Bhatt and Young [3]). Even if we accepted the values $\nu=9.6$ and $\eta=0.01$, our data for susceptibility and Edwards-Anderson order parameter would be in conflict with a zero transition temperature. Thus based on an overall analysis of all our data we conclude that the frustrated XY model on the 2D lattices considered exhibits spin glass transitions at temperatures above zero. It is not clear to us whether the seemingly good agreement with the $\pm J$ Ising model exponents is purely coincidental, or there is a connection with the results of Teitel and Jayaprakash [11] on square lattices, where the nature of the transition is found to change from KT to Ising type as a result of including the frustration (the $A_{ij}$ term in Eq.(1)).
IV. COMPARISON WITH RESULTS ON OTHER LATTICES

As stated in the introduction, the frustrated XY model has been studied on a variety of 2D lattices, the most widely studied ones being the square and the triangular. Although there is some controversy regarding the nature of the transition on these lattices, including the issue of the existence of more than one transitions, most authors report the nature of the transition on these two lattices as being ‘Ising-like’. It will be useful to identify some feature of the model on the lattices studied in the present work that distinguishes these from the square or triangular lattice. The obvious quantity to look at is the distribution of the lattice-dependent vector potentials \( A_{ij} \). Equivalently, we could rewrite Eq. (1) as

\[
H = \sum_{ij} -J \cos(A_{ij}) \cos(\theta_i - \theta_j) + \sum_{ij} J \sin(A_{ij}) \sin(\theta_i - \theta_j),
\]

where the first term is simply the standard XY Hamiltonian, with the lattice-dependent frustration appearing via the variation in the effective nearest neighbor exchange parameters \( \cos(A_{ij}) \). We thus look at the distribution of the effective coupling parameters, or simply the quantity \( \cos(A_{ij}) \) for various lattices.

In Fig. 10 we show this quantity for the lattices studied in this work and in Fig.11 we show the same for the fully frustrated square and the triangular lattices. The square and the triangular lattices, which show the Ising-like transition, have much fewer values of the parameter \( \cos(A_{ij}) \) than all the other lattices exhibiting spin glass behavior. It should be noted that in order to compare the distribution of the quantity \( \cos(A_{ij}) \) for various lattices, we chose, in each case, the reference \( x \) and \( y \) axes along some symmetry direction of the lattice. An arbitrary choice of the reference axes, without any regard to the symmetry of the lattice, may result in a distribution showing spuriously large values of the parameter \( \cos(A_{ij}) \). Such values of \( \cos(A_{ij}) \) give rise to frustration which can be simply gauged away by rotating the reference frame used to describe the lattice sites, and cannot be responsible for spin glass behavior \[132\]. In Fig.11 we also show the distribution of \( \cos(A_{ij}) \) for a square lattice with an irrational flux, \( f = \left(3 - \sqrt{5}\right)/2 \), through the plaquettes. This model was
studied by Halsey [33], and was reported to show a low temperature spin glass phase. We note that the distribution of $\cos(A_{ij})$ for this case is similar to the distribution for the four cases studied by us, showing a large number of possible values.

From the above discussion it appears that frustrated XY models with a wide and more or less uniform distribution of the effective coupling parameters (more appropriately with a large number of values of the parameter $\cos(A_{ij})$ distributed over the interval between -1 and 1) belong to a different universality class than those with only a few possible values of $\cos(A_{ij})$. That the lattice structure is a relevant variable for the frustrated XY model has been known for a long time. Here we have attempted to identify a common feature, dependent on the lattice structure as well as the magnetic field, that might account for the spin/gauge glass phase in the frustrated XY model.

It is of interest to compare our results with some work on 3D models. Huse and Seung [34] have studied the so-called gauge or vortex glass model on simple cubic lattices. The model is the same as that given by Eq.(1) with the parameters $A_{ij}$ varying randomly and uniformly between 0 and $2\pi$. These authors find a spin glass behavior with exponents that agree with the $\pm J$ Ising spin glass exponents rather than the ones for the $\pm J$ 3D XY model. These results are remarkably similar to ours.

V. COMMENTS AND CONCLUSIONS

In summary, we have shown, via the Edwards-Anderson order parameter, spin glass susceptibility and linear susceptibility, the existence of a low temperature spin/gauge glass phase for the frustrated XY model on two quasiperiodic (Penrose and octagonal) and two periodic (honeycomb and bathroom tile) lattices. Our results for magnetization and specific heat also support this picture. We have also carried out a detailed study of the unfrustrated ferromagnetic XY model on the Penrose lattice. The results are similar to that of a square lattice [27], with a slightly higher KT transition temperature.

Our results for the quasiperiodic lattices are consistent with those of Halsey [33], who
finds a spin glass phase for the frustrated XY model on a square lattice with an irrational flux through the plaquettes. Note that for the Penrose, octagonal and the bathroom tile lattices we can fully frustrate only one of the two elementary plaquettes at one time, the corresponding flux through the other plaquette being irrational. However, with the example of the honeycomb lattice we have shown that the irrationality of the flux through the plaquettes is not a necessary condition for the existence of the spin/gauge glass phase. It may, however, be a sufficient condition. The common feature of all the cases studied is that the lattice structure and the transverse magnetic field induce a large number of possible values of the effective coupling parameters $J \cos(A_{ij})$.

The experimental implication of our study is that an array of Josephson junctions, forming any of the lattice structures discussed in this work, in a suitably chosen transverse magnetic field should behave as a superconducting glass at low temperatures. Advanced microfabrication techniques [35] should be capable of generating such periodic/quasiperiodic arrays of superconducting grains. Experimental work of this kind has been reported [36] on 2D fractal (Sierpinski-gasket) networks. Halsey [33] has pointed out that for superconducting arrays with low normal-state resistivities the glass transition should basically appear as a mean-field transition, with fluctuation effects being barely observable. For arrays with high normal-state resistivities the fluctuation effects will cause the glass transition to deviate substantially from a mean-field transition, with noticeable system-dependent details. As discussed by Ebner and Stroud [37], an important property of such glassy superconductors is a large difference between their dc and ac susceptibilities.

It is interesting to note that some 3D quasicrystals (both icosahedral and decagonal) are known to exhibit spin glass phase [38,39]. Of particular relevance to us are the decagonal quasicrystals, periodic in the z-direction and quasiperiodic in the xy-plane. A variation of the model studied in the present work may aptly describe the magnetic properties of some of these decagonal quasicrystals.
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FIGURES

FIG. 1. Lattices used in the present study, along with their reference frames.

FIG. 2. Edwards-Anderson order parameter as a function of temperature. $N$ denotes the number of lattice sites and $T_f$ is the estimate of the transition temperature for the corresponding lattice, obtained from the analysis of spin glass susceptibility.

FIG. 3. Linear susceptibility as a function of temperature for the Penrose and the honeycomb lattices. The inset shows the product $T\chi$, clearly indicating that the high temperature phase is paramagnetic, with a Curie-like behavior for $\chi$.

FIG. 4. Specific heat as a function of temperature. The inset in the Penrose tile shows the specific heat for the unfrustrated Penrose lattice reaching its peak value slightly beyond the KT transition temperature.

FIG. 5. Scaling plot of spin glass susceptibility for the Penrose lattice using the exponents (a) $\eta=0.2$ and $\nu=2.6$ reported by Bhatt and Young in their study of the $\pm J$ Ising model; (b) $\eta=0.26$ and $\nu=1.35$, chosen within the intervals $\nu=1.08\pm 0.27$ and $\eta=0.3\pm 0.3$ reported by Jain and Young in their study of $\pm J$ XY model. The insets show the log-log plot of the Edwards-Anderson order parameter versus $(T_f-T)$ with $T_f=0.137$ for case (a) and $T_f=0.11$ for (b), obtained from the fit to the scaling relation, Eq. 3.4. The slope of this plot in (a) yields a value of 0.26 for the exponent $\beta$, in perfect agreement with the hyperscaling relation $\beta = \nu\eta/2$. In (b) the slope is 0.19, close to the value 0.1755 given by the hyperscaling relation. See text and TABLE II for details.

FIG. 6. Scaling plot of spin glass susceptibility for the octagonal lattice using the exponents (a) $\eta=0.3$ and $\nu=2.6$; and (b) $\eta=0.3$ and $\nu=1.35$. The slope of the log-log plot in the inset of (a) is 0.38, in agreement with the hyperscaling relation $\beta = \nu\eta/2$. The slope of the log-log plot in the inset in (b) is 0.25, whereas the hyperscaling value is 0.2025. See text and TABLE II for details.
FIG. 7. Scaling plot of spin glass susceptibility for the honeycomb lattice. See text and TABLE II for details with regard to agreement with $\pm J$ Ising and XY spin glass models.

FIG. 8. Scaling plot of spin glass susceptibility for the bathroom tile lattice. See text and TABLE II for details with regard to agreement with $\pm J$ Ising and XY spin glass models.

FIG. 9. Scaling plot of spin glass susceptibility for the Penrose lattice with $T_f=0$, using (a) $\pm J$ Ising, (b) $\pm J$ XY exponents, and (c) exponents obtained from the best fit to Eq. 3.6.

FIG. 10. Distribution of the effective nearest neighbor coupling parameter $\cos(A_{ij})$ for the lattices used in this work.

FIG. 11. Distribution of the effective nearest neighbor coupling parameter $\cos(A_{ij})$ for the fully frustrated triangular and square lattices, and for square lattice with an irrational flux, $f = \left(3 - \sqrt{5}\right)/2$, through the plaquettes.
TABLES

TABLE I. Average energy per spin/site \( \langle u \rangle \) at the lowest temperature, 0.02\( J \), used in the simulation for various lattices and cluster sizes, \( N \). \( L_x \) and \( L_y \) denote the linear dimensions of the clusters in the \( x \) and \( y \) directions, respectively.

| Lattice        | \( N \) | \( L_x \) | \( L_y \) | \( \langle u \rangle \) |
|---------------|--------|--------|--------|-----------------|
| Penrose       | 644    | 24.80  | 21.09  | -1.4938         |
|               | 1686   | 40.12  | 34.13  | -1.4893         |
|               | 4414   | 64.92  | 55.23  | -1.4889         |
|               | 11556  | 105.05 | 89.36  | -1.4884         |
| Octagonal     | 239    | 14.07  | 14.07  | -1.3878         |
|               | 1393   | 33.97  | 33.97  | -1.3926         |
|               | 8119   | 82.01  | 82.01  | -1.3947         |
| Honeycomb     | 242    | 19.05  | 19.05  | -1.2169         |
|               | 1682   | 50.23  | 50.23  | -1.1980         |
|               | 4050   | 77.94  | 77.94  | -1.1926         |
|               | 8192   | 110.85 | 110.85 | -1.1868         |
| Bathroom Tile | 256    | 19.31  | 19.31  | -1.1842         |
|               | 1600   | 48.28  | 48.28  | -1.1846         |
|               | 4096   | 77.25  | 77.25  | -1.1843         |
|               | 8100   | 108.64 | 108.64 | -1.1847         |
TABLE II. Spin glass transition temperature $T_f$ and the exponents $\eta$ and $\nu$ obtained for the various lattices from finite size scaling analysis of the spin glass susceptibility. The exponent $\beta$, obtained from the log-log plot of the Edwards Anderson order parameter $q_{EA}$ and $T - T_f$ is also shown. The division into the categories "Ising" and "XY" is based on the proximity of the exponents $\eta$ and $\nu$ to the values reported by Bhatt and Young $^6$ and Jain and Young $^7$ in their study of $\pm J$ Ising and XY models, respectively, on square lattices.

| Lattice       | $T_f$ | $\eta$ | $\nu$ | $\beta$ | $T_f$ | $\eta$ | $\nu$ | $\beta$ |
|---------------|-------|--------|-------|---------|-------|--------|-------|---------|
| Penrose       | 0.137 | 0.20   | 2.6   | 0.26    | 0.11  | 0.26   | 1.35  | 0.19    |
| Octagonal     | 0.33  | 0.3    | 2.6   | 0.38    | 0.3   | 0.3    | 1.35  | 0.25    |
| Honeycomb     | 0.12  | 0.15   | 2.8   | 0.35    | 0.13  | 0.29   | 1.30  | 0.71    |
| Bathroom Tile | 0.12  | 0.15   | 2.6   | 0.33    | 0.13  | 0.29   | 1.30  | 0.50    |

$^a\pm J$ Ising Exponents in Bhatt and Young$^6$: $\eta = 0.2 \pm 0.05$, $\nu = 2.6 \pm 0.4$, $\beta = \eta\nu/2 = 0.26$.

$^b\pm J$ XY Exponents in Jain and Young$^7$ are $\eta = 0.3 \pm 0.3$, $\nu = 1.08 \pm 0.27$, $\beta = \eta\nu/2 = 0.162$. 