Smoothing effect and asymptotic behavior of solutions to nonlinear elastic wave equations with viscoelastic terms in the framework of $L^p$-Sobolev spaces

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Abstract

The Cauchy problem for nonlinear elastic wave equations with viscoelastic damping terms is investigated in $L^p$ framework. It is proved that the small global solutions constructed in $L^2$-Sobolev spaces in our preceding paper [12] satisfies consistency property corresponding to the additional regularity of the initial data. As a result, sharp estimates in $t$ and approximation formulas by the diffusion waves are established.

Keywords: nonlinear elastic wave equation, damping terms, consistency, smoothing effect, asymptotic profile, the Cauchy problem

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1 Introduction

This paper is a sequel to our recent work [12] and concerned with the Cauchy problem for the system of quasi-linear hyperbolic equations with strong damping

\[
\begin{aligned}
\partial_t^2 u - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u - \nu \Delta \partial_t u &= F(u), \quad t > 0, \quad x \in \mathbb{R}^3, \\
u(0, x) &= f_0(x), \quad \partial_t \nu(0, x) = f_1(x), \quad x \in \mathbb{R}^3,
\end{aligned}
\]

where, throughout the present paper, we follow the notation used in [12]. Namely, \(u = u(t, x) : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3\) is the unknown function; and \(f_j = (f_{j1}, f_{j2}, f_{j3}) (j = 0, 1)\) are initial data, and the superscript \(^t\) means the transpose of the matrix. The Lamé constants are supposed to satisfy

\[\mu > 0, \quad \lambda + 2\mu > 0,\]

and the viscosity parameter \(\nu\) is a positive constant. We assume that the nonlinear term \(F(u)\) is given by \(\nabla u \nabla Du\), where \(\nabla\) is the spatial gradient and \(D\) stands for the \(t, x\) gradient. The equation is a simplified, dimensionless model of viscoelasticity (for the detail see [11], [14] and the references therein).

We are interested in the consistency and smoothing effect of global solutions in \(L^p\) Sobolev spaces. We also establish sharp decay estimates in \(t\) and approximation formulas by the diffusion waves as \(t \to \infty\) in the same framework.

We briefly review related results for (1.1). Ponce [14] proved the existence of global solutions of (1.1) with \(\lambda + \mu = 0\) in \(L^2\)-Sobolev spaces and obtained decay estimates. The method of proof in [14] is based on the energy method for the higher order derivatives of the solution and the \(L^p-L^q\) type estimates for the fundamental solution of the linearized equation. In our preceding paper [12], we considered the Cauchy problem (1.1) with (1.2) in the framework of \(L^2\)-Sobolev spaces; and we established quantitative estimates for smoothing and decay properties which reflect both the damping aspect (i.e., the parabolic aspect) and the dispersive aspect (i.e., the hyperbolic aspect) of the system (1.1). The proof in [12] is based on the detailed analysis of the fundamental solution of the linearized equation that takes the spreading effect of the diffusion waves into account as in [7, 8, 13, 16].

Especially, we proved the asymptotic profile of the solution as \(t \to \infty\) is given by the convolution of the Riesz transform and the diffusion waves \(G_j^{(\beta)}(t) (j = 0, 1)\) depending on the parameter \(\beta > 0\) defined by

\[
G^{(\beta)}_0(t, \xi) := e^{-\nu|\xi|^2 t} \cos(\beta|\xi| t), \quad G^{(\beta)}_1(t, \xi) := e^{-\nu|\xi|^2 t} \frac{\sin(\beta|\xi| t)}{\beta|\xi|}
\]

and

\[
G_j^{(\beta)}(t, x) := \mathcal{F}^{-1}[G_j^{(\beta)}(t, \xi)],
\]

where \(\mathcal{F}^{-1}\) represents the Fourier inverse transform. The results of [12] improve the decay estimates obtained in [14]. For the detail, see Propositions [2.1, 2.2] below. We also mention the results in [11]. For the Cauchy problem (1.1) with \(\lambda + \mu = 0\), Jonov-Sideris [11] dealt with the nonlinear term given by the Klainerman null condition (cf. [17] and [1]) with small perturbation, and obtained the relationship between the lifespan, the coefficients \(\mu, \nu\) and the deviation of the nonlinearity from being null in the weighted \(L^2\)-Sobolev spaces.
As a result, they also proved existence of the global solutions for the initial data which is not necessarily small in the sense of norms.

On the other hand, Hoff and Zumbrun [7] considered the compressible Navier-Stokes system whose linearized system at the motionless state has a fundamental solution similar to that for (1.1). In [7], the fundamental solution for the linearized compressible Navier-Stokes system was investigated in detail; and, in particular, it was shown that a difference between the diffusion waves $G_j^{(\beta)}(t,x)$ and the heat kernel (i.e., $G_j^{(0)}(t,x)$ with $\beta = 0$) appears in quantitative estimates in $L^p$-norms for $p \neq 2$. Based on the linearized analysis, they derived the asymptotic approximation formula in $L^p$ for $1 \leq p \leq \infty$ as $t \to \infty$ which is given by a sum of the heat kernel and diffusion waves. After that they also proved optimal decay estimates of the diffusion waves in $L^p$ for $1 \leq p \leq \infty$ in [8], and then Kobayashi-Shibata [13] improved decay estimates of the fundamental solutions to linearized compressible Navier-Stokes system, which suggest the consistency in $L^p$ Sobolev spaces. See also [16, 10] for the analysis of the diffusion waves and [9, 4] for the consistency of the linearized compressible Navier-Stokes system.

Motivated by these works, in this paper, we investigate the consistency, smoothing effect and asymptotic behavior of global solutions to (1.1) in the framework of $L^p$-Sobolev spaces.

Now we explain our main results of this paper. We will firstly show the consistency of the global solution constructed in [12] under the additional condition $(f_0, f_1) \in \dot{W}^{3,p} \times \dot{W}^{1,p}$, where $1 \leq p \leq \infty$. Based on the estimates derived in the proof of the consistency, we will secondly prove the smoothing effect of the global solutions. Finally, we will show asymptotic profiles of global solutions as $t \to \infty$ in the functions spaces which global solutions belongs to. Here we note that the situation changes corresponding to the cases: $1 < p < \infty$, $p = 1$ and $p = \infty$. More precisely, when $1 < p < \infty$, we conclude the following consistency property

$$u \in \{C([0, \infty); \dot{W}^{3,p} \cap \dot{W}^{1,p}) \cap C^1([0, \infty); W^{1,p})\}^3$$

with

$$\|\nabla^\alpha u(t)\|_p \leq C(1 + t)^{-\frac{\alpha}{2}(1 - \frac{1}{p}) + \frac{1}{p} - \frac{\alpha}{2}}, \quad 1 \leq \alpha \leq 3, \quad (1.3)$$

$$\|\nabla^\alpha \partial_t u(t)\|_p \leq C(1 + t)^{-\frac{\alpha}{2}(1 - \frac{1}{p}) + \frac{1}{p} - \frac{\alpha + 1}{2}}, \quad 0 \leq \alpha \leq 1, \quad (1.4)$$

and the smoothing effect of the global solution

$$u \in \{C^1((0, \infty); \dot{W}^{2,p}) \cap C^2((0, \infty); L^p)\}^3$$

with

$$\|\nabla^2 \partial_t u(t)\|_p \leq C(1 + t)^{-\frac{3}{2}(1 - \frac{1}{p}) + \frac{1}{p} - 1} t^{-\frac{1}{2}}, \quad (1.5)$$

$$\|\partial_t^2 u(t)\|_p \leq C(1 + t)^{-\frac{3}{2}(1 - \frac{1}{p}) + \frac{1}{p} - \frac{1}{2}} t^{-\frac{1}{2}}. \quad (1.6)$$

Our proof for $1 < p < \infty$ relies on the direct estimation of the fundamental solutions and the Gronwall type argument. Especially, more detailed analysis of the fundamental solutions are required to show $u \in \{C([0, \infty); \dot{W}^{3,p} \cap C^2((0, \infty); L^p)\}^3$ with (1.3) and

$$F(u) = \nabla u \nabla^2 u \quad \text{and} \quad u \in \{C^1((0, \infty); \dot{W}^{2,p}) \cap C^2((0, \infty); L^p)\}^3 \quad \text{with} \quad (1.5) \quad \text{and} \quad (1.6)$$

for $F(u) = \nabla u \nabla \partial_t u$. To overcome the difficulty, following the method in [16, 13],
we will show the estimates for the high frequency parts of the fundamental solutions to (1.1) based on the Fourier multiplier theory and the oscillatory integral. More precisely, we will show $L^p$-$L^p$ type estimates by the combination of the parabolic smoothing (i.e. parabolic aspect) and use of the cancellation in the integration by parts by the oscillation integral (i.e. hyperbolic aspect). See Corollary 4.11 later.

On the other hand, when $p = 1, \infty$, we cannot apply the same argument to prove the smoothing effect of the global solution, because of the well-known fact, the absence of $L^p$-$L^p$ boundedness of the Riesz transform, which appears in the fundamental solutions to (1.1). As a result, we will conclude the smoothing effect of the global solutions as $u \in \{C([0, \infty); \dot{W}^{1,1} \cap C^1([0, \infty); W^{1,1}) \cap C^2((0, \infty); L^1)\}^3$ satisfying (1.3) with $\alpha = 1$, (1.4) and (1.6) for $p = 1$, and

$$u \in \{\dot{W}^{2,\infty}(0, \infty; L^\infty)\}^3$$

satisfying (1.6) for $p = \infty$. In the case $p = 1$, we establish estimates by using the difference of propagation speeds of the wave part of the fundamental solutions to overcome the regularity loss, which plays the role of a substitute of the direct estimation of the fundamental solutions. Such kind of estimates are found in [13] for the linearized compressible Navier-Stokes equations, where the situation is slightly different from ours. On the other hand, when $p = \infty$, we take the following two steps, which is the crucial point to obtain the smoothing estimate (1.6) for $p = \infty$. Using the regularity of the initial data and interpolation theory, we firstly apply the smoothing effect of the global solutions for the case $1 < p < \infty$, to have decay properties and regularity of the global solution in $u \in \{C([0, \infty); \dot{W}^{3,q} \cap \dot{W}^{1,q}) \cap C^1([0, \infty); \dot{W}^{1,q}) \cap C^1((0, \infty); \dot{W}^{2,q}) \cap C^2((0, \infty); L^q)\}^3$ for $2 \leq q < \infty$. As a next step, we estimate the solution to show that $u$ belongs to $\{\dot{W}^{2,\infty}(0, \infty; L^\infty)\}^3$ with desirable decay properties.

This paper is organized as follows. Section 2 is devoted to preliminaries, which includes explanation of notation, detailed description of the results of [12] for $L^2$-Sobolev spaces and useful facts used later. We state the main results of this paper precisely in section 3. In section 4, we summarize the estimates of the fundamental solutions to the Cauchy problem (1.1). Sections 5-7 are devoted to the study of the consistency and smoothing effect of the global solutions to (1.1). We deal with the case $1 < p < \infty$ in section 5, $p = 1$ in section 6 and $p = \infty$ in section 7, respectively.

## 2 Preliminaries

In this section, we set up the notation, following [12]. After that, we summarize the estimates for the fundamental solutions of the strongly damped wave equations and wave equations. We also review some of the standard facts on the Riesz transform and the interpolation theory.

### 2.1 Notation

For simplicity, we denote $I_3 \in M(\mathbb{R}; 3)$ is the identity matrix and

$$\mathcal{P} := \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|}$$

(2.1)
We also define the 3-d valued constant vector depending on the initial data and the nonlinear term as follows:

\[ m_j = \ell(m_{j1}, m_{j2}, m_{j3}), \quad M[u] := \ell(M_1[u], M_2[u], M_3[u]), \]

where

\[ m_{0k} := \int_{\mathbb{R}^3} \nabla f_{0k}(x)dx, \quad m_{1k} := \int_{\mathbb{R}^3} f_{1k}(x)dx \]

and

\[ M_k[u] := \int_0^\infty \int_{\mathbb{R}^3} F_k(u)(\tau, y)d\tau d\tau \]

for \( k = 1, 2, 3 \). Using the above notation, we define the functions \( G, H \) and \( \tilde{G} \) by

\[
G(t, x) := \nabla^{-1} \mathcal{F}^{-1} \left[ \left( \mathcal{G}_0^{(\sqrt{\lambda + 2\mu})}(t, \xi) - \mathcal{G}_0^{(\sqrt{\mu})}(t, \xi) \right) \mathcal{P} + \mathcal{G}_0^{(\sqrt{\mu})}(t, \xi) \right] m_0 \\
+ \mathcal{F}^{-1} \left[ \left( \mathcal{G}_1^{(\sqrt{\lambda + 2\mu})}(t, \xi) - \mathcal{G}_0^{(\sqrt{\mu})}(t, \xi) \right) \mathcal{P} + \mathcal{G}_0^{(\sqrt{\mu})}(t, \xi) \right] (m_1 + M[u]),
\]

\[
H(t, x) := \\
\nabla^{-1} \mathcal{F}^{-1} \left[ \left( (\lambda + 2\mu)\mathcal{G}_1^{(\sqrt{\lambda + 2\mu})}(t, \xi) - \mu\mathcal{G}_1^{(\sqrt{\mu})}(t, \xi) \right) \mathcal{P} + \mu\mathcal{G}_1^{(\sqrt{\mu})}(t, \xi) \right] m_0 \\
+ \mathcal{F}^{-1} \left[ \left( \mathcal{G}_0^{(\sqrt{\lambda + 2\mu})}(t, \xi) - \mathcal{G}_0^{(\sqrt{\mu})}(t, \xi) \right) \mathcal{P} + \mathcal{G}_0^{(\sqrt{\mu})}(t, \xi) \right] (m_1 + M[u]),
\]

and

\[
\tilde{G}(t, x) := \\
- \Delta \nabla^{-1} \mathcal{F}^{-1} \left[ \left( (\lambda + 2\mu)\mathcal{G}_0^{(\sqrt{\lambda + 2\mu})}(t, \xi) - \mu\mathcal{G}_0^{(\sqrt{\mu})}(t, \xi) \right) \mathcal{P} + \mu\mathcal{G}_0^{(\sqrt{\mu})}(t, \xi) \right] m_0 \\
- \Delta \mathcal{F}^{-1} \left[ \left( (\lambda + 2\mu)\mathcal{G}_1^{(\sqrt{\lambda + 2\mu})}(t, \xi) - \mu\mathcal{G}_1^{(\sqrt{\mu})}(t, \xi) \right) \mathcal{P} + \mu\mathcal{G}_1^{(\sqrt{\mu})}(t, \xi) \right] (m_1 + M[u]),
\]

respectively.

For function spaces, \( L^p = L^p(\mathbb{R}^3) \) is a usual Lebesgue space equipped with the norm \( \|f\|_p \) for \( 1 \leq p \leq \infty \). The symbol \( W^{k,p}(\mathbb{R}^3) \) stands for the usual Sobolev spaces

\[
W^{k,p}(\mathbb{R}^3) := \left\{ f : \mathbb{R}^3 \to \mathbb{R}; \|f\|_{W^{k,p}(\mathbb{R}^3)} := \|f\|_p + \|\nabla_x f\|_p < \infty \right\}.
\]

When \( p = 2 \), we denote \( W^{k,2}(\mathbb{R}^3) = H^k(\mathbb{R}^3) \). For the notation of the function spaces, the domain \( \mathbb{R}^3 \) is often abbreviated. We write by \( \tilde{W}^{k,p} \) and \( \tilde{H}^k \) the corresponding homogeneous Sobolev spaces, respectively.

Let us denote by \( \hat{f} \) the Fourier transform of \( f \) defined by

\[
\hat{f}(\xi) := (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)dx.
\]

Also, let us denote by \( \mathcal{F}^{-1}[f] \) or \( \hat{f} \) the inverse Fourier transform.
2.2 Consistency and smoothing effect of the global solutions in $L^2$ Sobolev spaces

We recall the basic facts on the global solutions to (1.1), which are shown in [12]. We begin with precise description of the global existence, decay properties and smoothing effect of the $\dot{H}^3$ solutions when $F(u) = \nabla u \nabla^2 u$.

**Proposition 2.1** ([12]). Suppose that $F(u) = \nabla u \nabla^2 u$. Let $(f_0, f_1) \in \{\dot{H}^3 \cap \dot{W}^{1,1}\}^3 \times \{H^1 \cap L^1\}^3$ with sufficiently small norms. Then there exists a unique global solution to (1.1) in the class

$$\{C([0, \infty); \dot{H}^3 \cap \dot{H}^1) \cap C^1([0, \infty); H^1)\}^3$$

satisfying the following time decay properties:

$$\|\nabla^\alpha u(t)\|_2 \leq C(1 + t)^{-\frac{3}{2} - \frac{\alpha}{2}}, \quad 1 \leq \alpha \leq 3,$$

$$\|\partial_t \nabla^\alpha u(t)\|_2 \leq C(1 + t)^{-\frac{3}{2} - \frac{\alpha}{2}}, \quad 0 \leq \alpha \leq 1,$$  \hspace{1cm} (2.2)

for $t \geq 0$. Moreover the solution $u(t)$ satisfies

$$u \in \{C^1((0, \infty); \bigcup_{2 \leq p < 6} \dot{W}^{2, p} \cap W^{1,\infty}(0, \infty; W^{1,\infty}) \cap C^2((0, \infty); \bigcup_{2 \leq p < 6} L^p)\}^3$$

and

$$\|\nabla^\alpha u(t)\|_\infty \leq C(1 + t)^{-\frac{3}{2} - \frac{\alpha}{2}}, \quad 0 \leq \alpha \leq 1 \hspace{1cm} (2.3)$$

for $t \geq 0$ and

$$\|\nabla^2 \partial_t u(t)\|_p \leq C(1 + t)^{-\frac{7}{2} + \frac{1}{2} \frac{3}{p} - \frac{3}{4} + \frac{3}{2p}}, \quad 2 \leq p < 6,$$  \hspace{1cm} (2.4)

$$\|\nabla^\alpha \partial_t u(t)\|_\infty \leq C(1 + t)^{-\frac{3}{2}t^{-\frac{1}{2} + \frac{1}{2} \frac{3}{p}}}, \quad 0 \leq \alpha \leq 1,$$  \hspace{1cm} (2.5)

$$\|\partial_t^2 u(t)\|_p \leq C(1 + t)^{-\frac{3}{2}t^{-\frac{1}{2} + \frac{3}{2p}}}, \quad 2 \leq p < 6$$  \hspace{1cm} (2.6)

for $t > 0$.

We next state the result for the case $F(u) = \nabla u \nabla \partial_t u$.

**Proposition 2.2** ([12]). Suppose that $F(u) = \nabla u \nabla \partial_t u$. Let $(f_0, f_1) \in \{\dot{H}^3 \cap \dot{W}^{1,1}\}^3 \times \{H^2 \cap L^1\}^3$ with sufficiently small norms. Then there exists a unique global solution to (1.1) in the class

$$\{C([0, \infty); \dot{H}^3 \cap \dot{H}^1) \cap C^1([0, \infty); H^2)\}^3$$

satisfying the estimates (2.2) and

$$\|\partial_t \nabla^\alpha u(t)\|_2 \leq C(1 + t)^{-\frac{3}{2} - \frac{\alpha}{2}}, \quad 0 \leq \alpha \leq 2$$ \hspace{1cm} (2.7)

for $t \geq 0$. Moreover the solution $u(t)$ has the following regularity and decay properties:

$$u \in \{C^1((0, \infty); \dot{W}^{2,6} \cap W^{1,\infty}(0, \infty; W^{1,\infty}) \cap C^2((0, \infty); L^2) \cap C^2((0, \infty); L^6)\}^3$$

and the estimates (2.3),

$$\|\partial_t u(t)\|_\infty \leq C(1 + t)^{-2},$$ \hspace{1cm} (2.8)
for \( t \geq 0 \) and

\[
\|\nabla^2 \partial_t u(t)\|_p \leq C(1 + t)^{-\frac{3}{2} + \frac{\nu}{p} t^{-\frac{3}{2} + \frac{\nu}{p}}}, \quad 2 \leq p \leq 6, \tag{2.9}
\]

\[
\|\nabla \partial_t u(t)\|_\infty \leq C(1 + t)^{-\frac{3}{2} + \frac{\nu}{2}}, \tag{2.10}
\]

\[
\|\partial_t^2 u(t)\|_p \leq C(1 + t)^{-\frac{3}{2} + \frac{\nu}{2} t^{-\frac{3}{2} + \frac{\nu}{2}}}, \quad 2 \leq p \leq 6 \tag{2.11}
\]

for \( t \geq 0 \) with \( p = 2 \) and \( t > 0 \) with \( p \neq 2 \).

In Proposition 2.4 the estimates in (2.2) imply the consistency of (1.1) with \( F(u) = \nabla u \nabla^2 u \) in \( \dot{H}^3 \cap H^1 \) × \( H^1 \), while the estimates (2.3)-(2.6) suggest the smoothing effect of the global solution is this framework. A similar interpretation holds for Proposition 2.2.

In [12], the large time behavior of the \( \dot{H}^3 \) solutions for \( F(u) = \nabla u \nabla^2 u \) is studied and is formulated as follows:

**Proposition 2.3** ([12]). The global solution \( u(t) \) of (1.1) constructed in Proposition 2.1 satisfies

\[
\|\nabla^\alpha (u(t) - G(t))\|_2 = o(t^{-\frac{\nu}{2} - \frac{\alpha}{2}}), \quad 1 \leq \alpha \leq 3, \tag{2.12}
\]

\[
\|\nabla^\alpha (u(t) - G(t))\|_\infty = o(t^{-\frac{\nu}{2} - \frac{\alpha}{2}}), \quad 0 \leq \alpha \leq 1, \tag{2.13}
\]

\[
\|\nabla^\alpha (\partial_t u(t) - H(t))\|_2 = o(t^{-\frac{\nu}{2} - \frac{\alpha}{2}}), \quad 0 \leq \alpha \leq 2, \tag{2.14}
\]

\[
\|\nabla^2 (\partial_t u(t) - H(t))\|_p = o(t^{-3 + \frac{\nu}{2}}), \quad 2 \leq p < 6, \tag{2.15}
\]

\[
\|\nabla^\alpha (\partial_t u(t) - H(t))\|_\infty = o(t^{-2 - \frac{\nu}{2}}), \quad 0 \leq \alpha \leq 1, \tag{2.16}
\]

\[
\|\partial_t^2 u(t) - \tilde{G}(t)\|_2 = o(t^{-\frac{\nu}{2}}) \tag{2.17}
\]

as \( t \to \infty \).

The \( \dot{H}^3 \) solutions for \( F(u) = \nabla u \nabla \partial_t u \) is also considered and the correspondence is described as follows:

**Proposition 2.4** ([12]). The global solution \( u(t) \) of (1.1) constructed in Proposition 2.2 satisfies the estimates (2.12)-(2.17) and

\[
\|\nabla^2 (\partial_t u(t) - H(t))\|_6 = o(t^{-\frac{31}{12}}), \tag{2.18}
\]

\[
\|\partial_t^2 u(t) - \tilde{G}(t)\|_6 = o(t^{-\frac{25}{12}}), \tag{2.19}
\]

as \( t \to \infty \).

Proposition 2.3 (resp. Proposition 2.4) shows that estimates in Proposition 2.1 (resp. Proposition 2.2) are sharp in \( t \).

### 2.3 Estimates for linear equations

We begin with the strongly damped wave equations:

\[
\begin{aligned}
&\partial_t^2 w - \beta^2 \Delta w - \nu \Delta \partial_t w = 0, \quad t > 0, \quad x \in \mathbb{R}^3, \\
&w(0, x) = w_0(x), \quad \partial_t w(0, x) = w_1(x), \quad x \in \mathbb{R}^3,
\end{aligned} \tag{2.20}
\]
where \( w = w(t, x) : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R} \) and \( \beta > 0 \). Let us denote the characteristic roots \( \sigma_{\pm}^{(\beta)} \) by

\[
\sigma_{\pm}^{(\beta)} := -\nu |\xi|^2 \pm \sqrt{\nu^2 |\xi|^4 - 4\beta^2 |\xi|^2},
\]

Then the solution of (2.20) is expressed as

\[
w(t) = K_0^{(\beta)}(t)w_0 + K_1^{(\beta)}(t)w_1,
\]

where

\[
K_0^{(\beta)}(t)w_0 := \mathcal{F}^{-1}[\mathcal{K}_0^{(\beta)}(t, \xi)\hat{w}_0], \quad K_1^{(\beta)}(t)w_1 := \mathcal{F}^{-1}[\mathcal{K}_1^{(\beta)}(t, \xi)\hat{w}_1]
\]

and

\[
\mathcal{K}_0^{(\beta)}(t, \xi) := \frac{-\sigma_-^{(\beta)} e^{\sigma_-^{(\beta)} t} + \sigma_+^{(\beta)} e^{\sigma_+^{(\beta)} t}}{\sigma_+^{(\beta)} - \sigma_-^{(\beta)}}, \quad \mathcal{K}_1^{(\beta)}(t, \xi) := \frac{e^{\sigma_+^{(\beta)} t} - e^{\sigma_-^{(\beta)} t}}{\sigma_+^{(\beta)} - \sigma_-^{(\beta)}}.
\]

We also define the smooth cut-off functions \( \chi_j = \chi_j(\xi) \in C^\infty(\mathbb{R}^3) \) \((j = L, M, H)\) as follows:

\[
\chi_L := \begin{cases} 
1 & (|\xi| \leq \frac{c_0}{2}), \\
0 & (|\xi| \geq c_0),
\end{cases}
\]

\[
\chi_H := \begin{cases} 
0 & (|\xi| \leq c_1), \\
1 & (|\xi| \geq 2c_1)
\end{cases}
\]

and

\[
\chi_M = 1 - \chi_L - \chi_H.
\]

Here \( c_0 \) and \( c_1 \) \((0 < c_0 < c_1 < \infty)\) are some constants to be determined later. As in [12], we can then use the evolution operators defined by

\[
K_j^{(\beta)}(t)g := \mathcal{F}^{-1}[\mathcal{K}_j^{(\beta)}(t, \xi)\hat{g}], \\
K_{jk}^{(\beta)}(t)g := \mathcal{F}^{-1}[\mathcal{K}_{jk}^{(\beta)}(t, \xi)\chi_k\hat{g}], \\
G_{jk}^{(\beta)}(t)g := \mathcal{F}^{-1}[\mathcal{G}_{jk}^{(\beta)}(t, \xi)\chi_k\hat{g}]
\]

for \( j = 0, 1 \) and \( k = L, M, H \). Firstly we mention the decay properties of the fundamental solutions [22.21].

**Lemma 2.5** ([14], [16], [13]). (i) Let \( 1 \leq q \leq p \leq \infty, \ell \geq \bar{\ell} \geq 0 \) and \( \alpha \geq \bar{\alpha} \geq 0 \). Then it holds that

\[
\left\| \partial_t^\ell \nabla^\alpha K_{0L}^{(\beta)}(t)g \right\|_p \leq C(1 + t)^{-\frac{3}{2} \left(\frac{1}{q} - \frac{1}{p}\right) - \left(\frac{1}{q} - \frac{1}{p}\right) + 1 - \frac{\ell - \bar{\ell} + \bar{\alpha}}{2}} \left\| \nabla^{\bar{\alpha} + \bar{\ell}} \hat{g} \right\|_q,
\]

\[
\left\| \partial_t^\ell \nabla^\alpha K_{1L}^{(\beta)}(t)g \right\|_p \leq C(1 + t)^{-\frac{3}{2} \left(\frac{1}{q} - \frac{1}{p}\right) - \left(\frac{1}{q} - \frac{1}{p}\right) + 1 - \frac{\ell - \bar{\ell} + \bar{\alpha}}{2}} \left\| \nabla^{\bar{\alpha} + \bar{\ell}} \hat{g} \right\|_q,
\]

\[
\left\| \partial_t^\ell \nabla^\alpha G_{0L}^{(\beta)}(t)g \right\|_p \leq C(1 + t)^{-\frac{3}{2} \left(\frac{1}{q} - \frac{1}{p}\right) - \left(\frac{1}{q} - \frac{1}{p}\right) + 1 - \frac{\ell - \bar{\ell} + \bar{\alpha}}{2}} \left\| \nabla^{\bar{\alpha} + \bar{\ell}} \hat{g} \right\|_q,
\]

\[
\left\| \partial_t^\ell \nabla^\alpha G_{1L}^{(\beta)}(t)g \right\|_p \leq C(1 + t)^{-\frac{3}{2} \left(\frac{1}{q} - \frac{1}{p}\right) - \left(\frac{1}{q} - \frac{1}{p}\right) + 1 - \frac{\ell - \bar{\ell} + \bar{\alpha}}{2}} \left\| \nabla^{\bar{\alpha} + \bar{\ell}} \hat{g} \right\|_q.
\]
for \( t \geq 0 \).

(ii) Let \( \ell \geq \bar{\ell} \geq 0 \), \( \alpha \geq \tilde{\alpha} \geq 0 \) and \( t > 0 \). Then it holds that
\[
\sum_{k=M,H} \| \partial_t^K K^{(\beta)}_{0k}(t)g \|_p \leq Ce^{-ct}(\| \nabla^\alpha g \|_p + t^{-\frac{\alpha+\tilde{\alpha}}{2}}(t^{-\frac{\alpha+\bar{\ell}}{2}})\| \nabla^{\tilde{\alpha}+\bar{\ell}} g \|_p)
\] (2.27)
for \( 1 \leq p \leq \infty \) and
\[
\sum_{k=M,H} \| \partial_t^K K^{(\beta)}_{2k}(t)g \|_p \leq Ce^{-ct}(\| \nabla^{(\alpha-2)+} g \|_p + t^{-\frac{\alpha+\tilde{\alpha}}{2}}(t^{-\frac{\alpha+\bar{\ell}}{2}})\| \nabla^{\tilde{\alpha}+\bar{\ell}} g \|_p)
\] (2.28)
for \( 1 < p < \infty \).

(iii) Let \( \alpha, \ell \geq 0 \), \( 1 \leq p \leq \infty \) and \( t > 0 \). Then it holds that
\[
\sum_{k=M,H} \| \partial_t^K \nabla^\alpha R_k R_k F^{-1}[G_0^{(\beta)}(t, \xi)\chi_k] \|_p \leq Ce^{-ct(t-\frac{3}{2}(1-\frac{1}{p})-\frac{\alpha+\ell}{2})}
\] (2.29)
and
\[
\sum_{k=M,H} \| \partial_t^K \nabla^\alpha R_k R_k F^{-1}[G_1^{(\beta)}(t, \xi)\chi_k] \|_p \leq Ce^{-ct(t-\frac{3}{2}(1-\frac{1}{p})-\frac{\alpha+\ell}{2})}
\] (2.30)

Secondly, we show the estimates for the solution of wave equations. Especially, the estimate (2.33) plays a crucial role to obtain the asymptotic profiles of the solutions as \( t \to \infty \) in \( L^1 \) Sobolev spaces.

**Lemma 2.6.** Let \( 1 \leq p \leq \infty \), \( \alpha, \ell \geq 0 \) and \( \gamma > 0 \). There exists \( C > 0 \) such that
\[
\| \partial_t^k W_0^{(\beta)}(t)g \|_p \leq C(\| \nabla^k g \|_p + t\| \nabla^{k+1} g \|_p)
\] (2.31)
\[
\| \partial_t^k W_1^{(\beta)}(t)g \|_p \leq Ct\| \nabla^k g \|_p,
\] (2.32)
\[
\| \nabla^{(\beta)}(W_0^{(\beta)}(t)g - W_0^{(\gamma)}(t)g) \|_p \leq Ct\| \nabla^{(\alpha+1)} g \|_p
\] (2.33)
for \( t > 0 \), where
\[
W_0^{(\beta)}(t)g := F^{-1}[\cos(t\beta|\xi|)\hat{g}], \quad W_1^{(\beta)}(t)g := F^{-1}\left[\frac{\sin(t\beta|\xi|)}{\beta|\xi|}\hat{g}\right].
\]

**Proof.** We only show the third estimate (2.33), since estimates (2.31)–(2.32) are well-known and the direct consequence of the representation formula of the fundamental solutions. Using the representation formula (cf.\([15]\))
\[
W_0^{(\beta)}(t)g = \frac{1}{4\pi} \int_{S^2} g(x + t\beta y)dS_y + \frac{t}{4\pi} \int_{S^2} y \cdot \nabla g(x + t\beta y)dS_y,
\]
where \( \cdot \) represents the inner product in \( \mathbb{R}^3 \). we see
\[
\nabla^{(\beta)}(W_0^{(\beta)}(t)g - W_0^{(\gamma)}(t)g)
\]
\[
= t(\beta - \gamma) \frac{1}{4\pi} \int_{S^2} y \cdot \nabla^{(\alpha+\ell+1)} g(x + ty(\theta \beta + (1 - \theta)\gamma))dS_y
\]
\[
+ \frac{t}{4\pi} \int_{S^2} y \cdot \nabla^{(\alpha+1)}(g(x + t\beta y) - g(x + t\gamma y))dS_y,
\]
where we apply the mean value theorem to have
\[
g(x + t\beta y) - g(x + t\gamma y) = t(\beta - \gamma)y \cdot \nabla g(x + ty(\theta \beta + (1 - \theta)\gamma))
\]
for some \( \theta \in [0, 1] \). Then we conclude the desired estimate (2.33). \(\square\)
Proposition 2.1 satisfies

\[ \|R_ag\|_p \leq C\|g\|_p, \]  

(2.34)

where

\[ R_ag := F^{-1} \left[ \frac{\xi_a}{|\xi|} \hat{g} \right]. \]

(i) Let \( 1 < p < \infty \). There exists \( C > 0 \) such that

\[ \|R_ag\|_p \leq C\|g\|_p, \]

where

\[ R_ag := F^{-1} \left[ \frac{\xi_a}{|\xi|} \hat{g} \right]. \]

(ii) Let \( \ell \geq 0, \alpha \geq 0 \) and \( \ell + \alpha \geq 1 \). There exists \( C > 0 \) such that

\[ \|\partial_t^\alpha \nabla^\alpha R_ag \|_{L^1_{\xi}(\mathbb{R}^3)} \leq C(1 + t)^{-\frac{3}{2} - \ell} \]

(2.35)

for \( t \geq 0 \).

For the proof of (2.34), see e.g. [2]. The estimate (2.35) is proved in [13]. Finally, we recall well-known embedding results.

Lemma 2.8. There exists a constant \( C > 0 \) such that

\[ \|g\|_{L^1(\mathbb{R}^3)} \leq C\|g\|^\frac{3}{2}_{L^2(\mathbb{R}^3)} \|x^2u\|^\frac{3}{2}_{L^2(\mathbb{R}^3)} = C\|\hat{g}\|^\frac{3}{2}_{L^2(\mathbb{R}^3)} \|\nabla^2 g\|^\frac{3}{2}_{L^2(\mathbb{R}^3)}, \]  

(2.36)

\[ \|g\|_{L^\infty(\mathbb{R}^3)} \leq C\|g\|^\frac{3}{2}_{L^2(\mathbb{R}^3)} \|\nabla^2 g\|^\frac{3}{2}_{L^2(\mathbb{R}^3)}, \]  

(2.37)

\[ \|\nabla g\|_{L^p(\mathbb{R}^3)} \leq C\|g\|^\frac{3}{2}_{L^\infty(\mathbb{R}^3)} \|\nabla^2 g\|^\frac{3}{2}_{L^p(\mathbb{R}^3)}, \quad 1 \leq p < \infty, \]  

(2.38)

\[ \|\nabla g\|_{L^q(\mathbb{R}^3)} \leq C\|g\|^\frac{3}{2}_{L^q(\mathbb{R}^3)} \|\nabla^2 g\|^\frac{3}{2}_{L^q(\mathbb{R}^3)}, \quad 3 < q < \infty, \]  

(2.39)

where \( C \) is independent of \( g \).

For the proof of (2.36), see e.g. [2]. The estimate (2.37)-(2.39) are proved in [3].

3 Main results

In this section, we state the main results of this paper.

Propositions 2.1-2.4 provide the large time behavior of the global solutions in the framework of \( L^2 \)-Sobolev spaces. In contrast to [12], our aim in this paper is to discuss the consistency and smoothing effect of the global solutions in the \( L^p \) framework, where \( 1 \leq p \leq \infty \). We firstly consider the case \( 1 < p < \infty \). We have the following result when \( F(u) = \nabla u \nabla^2 u \).

Theorem 3.1. Let \( 1 < p < \infty \). In addition to the assumption on Proposition 2.1, assume also that \( (f_0, f_1) \in \{W^{3,p}\}^3 \times \{W^{1,p}\}^3 \). Then the global solution to (1.1) constructed in Proposition 2.1 satisfies

\[ u \in \{C([0, \infty); W^{3,p} \cap \dot{W}^{1,p}) \cap C^1([0, \infty); W^{1,p}) \cap C^1((0, \infty); W^2) \cap C^2((0, \infty); L^p)\}^3, \]

together with the following properties:

\[ \|\nabla^\alpha u(t)\|_p \leq C(1 + t)^{-\frac{\alpha}{2}+\frac{3}{2}}, \quad 1 \leq \alpha \leq 3, \]

(3.1)

\[ \|\nabla^\alpha \partial_t u(t)\|_p \leq C(1 + t)^{-\frac{\alpha}{2}+\frac{3}{2}}, \quad 0 \leq \alpha \leq 1 \]

(3.2)
for $t \geq 0$,
\[
\|\nabla^2 \partial_t u(t)\|_p \leq C(1 + t)^{-\frac{5}{2}(1 - \frac{1}{p})} t^{-\frac{1}{2}},
\]
\[
\|\partial_t^2 u(t)\|_p \leq C(1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) + \frac{1}{2}} t^{-\frac{3}{2}}
\]
for $t > 0$, and
\[
\|\nabla^\alpha (u(t) - G(t))\|_p = o(t^{-\frac{5}{2}(1 - \frac{1}{p}) + \frac{1}{2} - \frac{\alpha}{2}}), \quad 1 \leq \alpha \leq 3,
\]
\[
\|\nabla^\alpha (\partial_t u(t) - H(t))\|_p = o(t^{-\frac{5}{2}(1 - \frac{1}{p}) + \frac{1}{2} - \frac{\alpha}{2}}), \quad 0 \leq \alpha \leq 2,
\]
\[
\|\partial_t^2 u(t) - \tilde{G}(t)\|_p = o(t^{-\frac{5}{2}(1 - \frac{1}{p})})
\]
as $t \to \infty$.

We next state the result for the case $F(u) = \nabla u \nabla \partial_t u$.

**Theorem 3.2.** Let $1 < p < \infty$. In addition to the assumption on Proposition 2.2 assume also that $(f_0, f_1) \in \{\dot{W}^{3,p}\}^3 \times \{\dot{W}^{1,p}\}^3$. Then the global solution to \((1.1)\) constructed in Proposition 2.2 satisfies
\[
u \in \{C([0, \infty); \dot{W}^{3,p} \cap \dot{W}^{1,p}) \cap C^1([0, \infty); \dot{W}^{1,p}) \cap C^1((0, \infty); \dot{W}^{2,p} \cap C^2((0, \infty); L^p)\}^3,
\]
together with the estimates \((3.1), (3.7)\).

It is worth pointing out that although the nonlinear interaction is given by quasi-linear and the linear principal part is hyperbolic, we establish the smoothing effect of the global solution by reducing the problem to the integral equation and applying the smoothing effect of the strong damping, without using the energy method based on the integration by parts. Especially, high frequency estimates stated in Corollary 4.11 below are essential to avoid the regularity-loss of the nonlinear terms in \(L^p\)-Sobolev spaces.

Focusing on the difference of the propagation speed of the hyperbolic parts in the fundamental solutions, we deal with the case $p = 1$. Then the estimate \((2.33)\) plays an essential role in the proof. We first state the result when $F(u) = \nabla u \nabla^2 u$.

**Theorem 3.3.** Under the assumption on Proposition 2.4 assume also that $(f_0, f_1) \in \{\dot{W}^{3,1}\}^3 \times \{\dot{W}^{1,1}\}^3$. Then the global solution to \((1.1)\) constructed in Proposition 2.4 satisfies
\[
u \in \{C([0, \infty); \dot{W}^{1,1}) \cap C^1([0, \infty); \dot{W}^{1,1}) \cap C^2((0, \infty); L^1)\}^3
\]
with the following properties:
\[
\|\nabla u(t)\|_1 \leq C(1 + t)^{\frac{1}{2}},
\]
\[
\|\nabla^\alpha \partial_t u(t)\|_1 \leq C(1 + t)^{\frac{1}{2} - \frac{\alpha}{2}}, \quad 0 \leq \alpha \leq 1
\]
for $t \geq 0$,
\[
\|\partial^2_t u(t)\|_1 \leq C(1 + t)^{\frac{1}{2}} t^{-\frac{1}{2}}
\]
for $t > 0$, and
\[
\|\nabla (u(t) - G(t))\|_1 = o(t^{\frac{1}{4}}),
\]
\[
\|\nabla^\alpha (\partial_t u(t) - H(t))\|_1 = o(t^{\frac{1}{2} - \frac{\alpha}{2}}), \quad 0 \leq \alpha \leq 1
\]
\[
\|\partial^2_t u(t) - \tilde{G}(t)\|_1 = o(1)
\]
as $t \to \infty$.
We have the following result when \( F(u) = \nabla u \nabla \partial_t u \).

**Theorem 3.4.** Under the assumption on Proposition 2.2, assume also that \((f_0, f_1) \in \{\dot{W}^{3,1}\}^3 \times \{\dot{W}^{1,1}\}^3\). Then the global solution to (1.1) constructed in Proposition 2.2 satisfies
\[
    u \in \{C([0, \infty); \dot{W}^{1,1}) \cap C^1([0, \infty); W^{1,1}) \cap C^2([0, \infty); L^1)\}^3
\]
with the estimates (3.8) - (3.13).

Finally we discuss the case \( p = \infty \). In this case, the estimate (2.33) does not work well since we cannot obtain the decay properties directly. Our alternative is to pay attention to the regularity of the initial data, which assures the application of Theorem 3.1 (resp. Theorem 3.2). As a consequences, our estimation for the global solution becomes easier since it has sufficient regularity and we obtain the following results:

**Theorem 3.5.** Under the assumption on Proposition 2.1, assume also that \((f_0, f_1) \in \{\dot{W}^{3,\infty}\}^3 \times \{\dot{W}^{1,\infty}\}^3\). Then the global solution to (1.1) constructed in Proposition 2.1 satisfies
\[
    u(t) \in \{\dot{W}^{2,\infty}(0, \infty; L^\infty)\}^3
\]
with the following time decay properties:
\[
    \|\partial_t^2 u(t)\|_\infty \leq C(1 + t)^{-2} t^{-\frac{1}{2}} \tag{3.14}
\]
for \( t > 0 \), and
\[
    \|\partial_t^2 u(t) - \tilde{G}(t)\|_\infty = o(t^{-\frac{1}{2}}) \tag{3.15}
\]
as \( t \to \infty \).

**Theorem 3.6.** Under the assumption on Proposition 2.2, assume also that \((f_0, f_1) \in \{\dot{W}^{3,\infty}\}^3 \times \{\dot{W}^{1,\infty}\}^3\). Then the global solution to (1.1) constructed in Proposition 2.2 satisfies
\[
    u(t) \in \{\dot{W}^{2,\infty}(0, \infty; L^\infty)\}^3
\]
with the estimates (3.14) - (3.15).

### 4 Basic estimates for the fundamental solutions

In this section, we summarize the results from [12], since our proof of main results deeply depend on them. At first, we mention the estimates for the low frequency parts of the fundamental solutions to (1.1). For this purpose, we introduce the notation
\[
    K_0^{(j)}(t, x) := \mathcal{R}_a \mathcal{R}_b F^{-1}[\mathcal{K}^{(j)}(t, \xi) \chi_L],
\]
\[
    K_1^{(j)}(t, x) := \mathcal{R}_a \mathcal{R}_b F^{-1}[\mathcal{K}^{(j)}(t, \xi) \chi_L],
\]
\[
    G_j^{(j)}(t, x) := \mathcal{R}_a \mathcal{R}_b F^{-1}[\mathcal{G}_j^{(j)}(t, \xi) \chi_L]
\]
for \( j = 0, 1 \) and \( a, b = 1, 2, 3 \). The decay properties of them are described as follows:
Lemma 4.1. Let \( \alpha \geq \tilde{\alpha} \geq 0, \ell \geq \tilde{\ell} \geq 0, m \geq 0, 1 \leq q \leq p \leq \infty \) and \( t \geq 0 \). Then it holds that

\[
\left\| \partial_t^\alpha \nabla^\beta K_{01}^{(1)}(t) * g \right\|_p \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{p}) - (\frac{1}{q} - \frac{1}{p}) + \frac{1}{2} - \frac{\ell + \alpha - \tilde{\alpha}}{2} \left\| \nabla^{\tilde{\alpha} + \tilde{\ell}} g \right\|_q \tag{4.1}
\]

\[
\left\| \partial_t^\alpha \nabla^\beta K_{01}^{(1)}(t) * g \right\|_p \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{p}) - (\frac{1}{q} - \frac{1}{p}) + 1 - \frac{\ell + \alpha - \tilde{\alpha}}{2} \left\| \nabla^{\tilde{\alpha} + \tilde{\ell}} g \right\|_q \tag{4.2}
\]

\[
\left\| \partial_t^\alpha \nabla^\beta G_{0}^{(1)}(t) \right\|_p \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{p}) - (\frac{1}{q} - \frac{1}{p}) + 1 - \frac{\ell + \alpha - \tilde{\alpha}}{2}, \tag{4.3}
\]

\[
\left\| \partial_t^\alpha \nabla^\beta G_{1}^{(1)}(t) \right\|_p \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{p}) - (\frac{1}{q} - \frac{1}{p}) + 1 - \frac{\ell + \alpha - \tilde{\alpha}}{2}. \tag{4.4}
\]

where \( 1 < p \leq \infty \) for \( \ell + \alpha = 0 \) and \( 1 \leq p \leq \infty \) for \( \ell + \alpha \geq 1 \).

Next we recall the expansion formulas of \( K_{0j}^{(1)}(t) * g \) and \( K_{1j}^{(1)}(t) * g \) for \( j = 0, 1 \) as \( t \to \infty \).

Lemma 4.2. Let \( \alpha \geq \tilde{\alpha} \geq 0, \ell \geq \tilde{\ell} \geq 0, m \geq 0, 1 \leq q \leq p \leq \infty \) and \( t \geq 0 \). Then it holds that

\[
\left\| \nabla^\alpha \left( \partial_t^\ell K_{01}^{(1)}(t) * g - (-1)^{\frac{\ell + 1}{2}} \beta^\ell \nabla^\ell \nabla G_{01}^{(1)}(t) * g \right) \right\|_p
\]

\[
+ \left\| \nabla^\alpha \left( \partial_t^\ell K_{00}^{(1)}(t) * g - (-1)^{\frac{\ell + 1}{2}} \beta^\ell \nabla^\ell \nabla G_{01}^{(1)}(t) * g \right) \right\|_p \tag{4.5}
\]

\[
\leq C(1 + t)^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{p}) - (\frac{1}{q} - \frac{1}{p}) - \frac{\ell + \alpha - \tilde{\alpha}}{2} \left\| \nabla^{\tilde{\alpha} + \tilde{\ell}} g \right\|_q
\]

for \( \ell = 2m \),

\[
\left\| \nabla^\alpha \left( \partial_t^\ell K_{01}^{(1)}(t) * g - (-1)^{\frac{\ell + 1}{2}} \beta^\ell \nabla^\ell \nabla G_{11}^{(1)}(t) * g \right) \right\|_p
\]

\[
+ \left\| \nabla^\alpha \left( \partial_t^\ell K_{00}^{(1)}(t) * g - (-1)^{\frac{\ell + 1}{2}} \beta^\ell \nabla^\ell \nabla G_{11}^{(1)}(t) * g \right) \right\|_p \tag{4.6}
\]

\[
\leq C(1 + t)^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{p}) - (\frac{1}{q} - \frac{1}{p}) - \frac{\ell + \alpha - \tilde{\alpha}}{2} \left\| \nabla^{\tilde{\alpha} + \tilde{\ell}} g \right\|_q
\]

for \( \ell = 2m + 1 \),

\[
\left\| \nabla^\alpha \left( \partial_t^\ell K_{11}^{(1)}(t) * g - (1)^{\frac{\ell + 1}{2}} \beta^\ell \nabla^\ell \nabla G_{11}^{(1)}(t) * g \right) \right\|_p
\]

\[
+ \left\| \nabla^\alpha \left( \partial_t^\ell K_{10}^{(1)}(t) * g - (1)^{\frac{\ell + 1}{2}} \beta^\ell \nabla^\ell \nabla G_{11}^{(1)}(t) * g \right) \right\|_p \tag{4.7}
\]

\[
\leq C(1 + t)^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{p}) - (\frac{1}{q} - \frac{1}{p}) + \frac{1}{2} - \frac{\ell + \alpha - \tilde{\alpha}}{2} \left\| \nabla^{\tilde{\alpha} + \tilde{\ell}} g \right\|_q
\]

for \( \ell = 2m \) and

\[
\left\| \nabla^\alpha \left( \partial_t^\ell K_{11}^{(1)}(t) * g - m_g(-1)^{\frac{\ell + 1}{2}} \beta^\ell \nabla^\ell \nabla G_{01}^{(1)}(t) * g \right) \right\|_p
\]

\[
+ \left\| \nabla^\alpha \left( \partial_t^\ell K_{11}^{(1)}(t) * g - m_g(-1)^{\frac{\ell + 1}{2}} \beta^\ell \nabla^\ell \nabla G_{01}^{(1)}(t) * g \right) \right\|_p \tag{4.8}
\]

\[
\leq C(1 + t)^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{p}) - (\frac{1}{q} - \frac{1}{p}) + \frac{1}{2} - \frac{\ell + \alpha - \tilde{\alpha}}{2} \left\| \nabla^{\tilde{\alpha} + \tilde{\ell}} g \right\|_q
\]

for \( \ell = 2m + 1 \).
The following lemma states the large time behavior of \( G_0^{(\beta)}(t) * g \) and \( G_1^{(\beta)}(t) * g \).

**Lemma 4.3.** Let \( \alpha, \ell, m \geq 0 \) and \( g \in L^1 \). Then it holds that

\[
\left\| \nabla^\alpha \left( \partial_t^\ell G_0^{(\beta)}(t) * g - m_g (-1)^{\frac{\ell+1}{2}} \beta^\ell \nabla^{\ell+1} G_0^{(\beta)}(t) \right) \right\|_p = o(t^{-\frac{3}{2}(1-\frac{1}{p})-(1-\frac{1}{p})+\frac{1}{2}-\frac{\ell+\alpha}{2}}) \tag{4.9}
\]

for \( \ell = 2m \),

\[
\left\| \nabla^\alpha \left( \partial_t^\ell G_0^{(\beta)}(t) * g - m_g (-1)^{\frac{\ell+1}{2}} \beta^{\ell+1} \nabla^{\ell+1} G_1^{(\beta)}(t) \right) \right\|_p = o(t^{-\frac{3}{2}(1-\frac{1}{p})-(1-\frac{1}{p})+\frac{1}{2}-\frac{\ell+\alpha}{2}}) \tag{4.10}
\]

for \( \ell = 2m + 1 \),

\[
\left\| \nabla^\alpha \left( \partial_t^\ell G_1^{(\beta)}(t) * g - m_g (-1)^{\frac{\ell+1}{2}} \beta^\ell \nabla^{\ell+1} G_1^{(\beta)}(t) \right) \right\|_p = o(t^{-\frac{3}{2}(1-\frac{1}{p})-(1-\frac{1}{p})+1+\frac{\ell+\alpha}{2}}) \tag{4.11}
\]

for \( \ell = 2m \) and

\[
\left\| \nabla^\alpha \left( \partial_t^\ell G_1^{(\beta)}(t) * g - m_g (-1)^{\frac{\ell+1}{2}} \beta^{\ell-1} \nabla^{\ell-1} G_0^{(\beta)}(t) \right) \right\|_p = o(t^{-\frac{3}{2}(1-\frac{1}{p})-(1-\frac{1}{p})+1+\frac{\ell+\alpha}{2}}) \tag{4.12}
\]

for \( \ell = 2m + 1 \), as \( t \to \infty \), where \( 1 < p \leq \infty \) for \( \ell + \alpha = 0 \) and \( 1 \leq p \leq \infty \) for \( \ell + \alpha \geq 1 \). Here \( m_g \) is defined by

\[
m_g := \int_{\mathbb{R}^3} g(x) dx. \tag{4.13}
\]

By a similar way, we also have the large time behavior of \( G_0^{(\beta)}(t) * g \) and \( G_1^{(\beta)}(t) * g \). We note that in this case, we can deal with \( p = 1 \) and \( \ell + \alpha = 0 \).

**Lemma 4.4.** Let \( \alpha, \ell, m \geq 0, 1 \leq p \leq \infty \) and \( g \in L^1 \). Then it holds that

\[
\left\| \nabla^\alpha \left( \partial_t^\ell G_0^{(\beta)}(t) * g - m_g (-1)^{\frac{\ell+1}{2}} \beta^\ell \nabla^{\ell+1} G_0^{(\beta)}(t) \right) \right\|_p = o(t^{-\frac{3}{2}(1-\frac{1}{p})-(1-\frac{1}{p})+\frac{1}{2}-\frac{\ell+\alpha}{2}}) \tag{4.14}
\]

for \( \ell = 2m \),

\[
\left\| \nabla^\alpha \left( \partial_t^\ell G_0^{(\beta)}(t) * g - m_g (-1)^{\frac{\ell+1}{2}} \beta^{\ell+1} \nabla^{\ell+1} G_1^{(\beta)}(t) \right) \right\|_p = o(t^{-\frac{3}{2}(1-\frac{1}{p})-(1-\frac{1}{p})+\frac{1}{2}-\frac{\ell+\alpha}{2}}) \tag{4.15}
\]

for \( \ell = 2m + 1 \),

\[
\left\| \nabla^\alpha \left( \partial_t^\ell G_1^{(\beta)}(t) * g - m_g (-1)^{\frac{\ell+1}{2}} \beta^\ell \nabla^{\ell+1} G_1^{(\beta)}(t) \right) \right\|_p = o(t^{-\frac{3}{2}(1-\frac{1}{p})-(1-\frac{1}{p})+1+\frac{\ell+\alpha}{2}}) \tag{4.16}
\]

for \( \ell = 2m \) and

\[
\left\| \nabla^\alpha \left( \partial_t^\ell G_1^{(\beta)}(t) * g - m_g (-1)^{\frac{\ell+1}{2}} \beta^{\ell-1} \nabla^{\ell-1} G_0^{(\beta)}(t) \right) \right\|_p = o(t^{-\frac{3}{2}(1-\frac{1}{p})-(1-\frac{1}{p})+1+\frac{\ell+\alpha}{2}}) \tag{4.17}
\]

for \( \ell = 2m + 1 \), as \( t \to \infty \).

As we see from (4.10), we cannot expect the \( L^1-L^1 \) estimates for \( G_0^{(\beta)}(t) * g \), because of the Riesz transform (cf. [13]). On the other hand, the following proposition yields information about the \( L^1 \) estimates for \( (G_0^{(\beta)}(t) - G_0^{(\gamma)}(t)) * g \) with \( \beta, \gamma > 0 \) and \( \beta \neq \gamma \).
Proposition 4.5. Let $\beta, \gamma > 0$ with $\beta \neq \gamma$. Then it holds that
\[
\left\| (G_0^{(\beta)}(t) - G_0^{(\gamma)}(t)) * g \right\|_p \leq Ct^{\frac{1}{2}} \|g\|_p,
\]
for $1 \leq p \leq \infty$ and $t \geq 0$, and
\[
\left\| (G_0^{(\beta)}(t) - G_0^{(\gamma)}(t)) * g - m_g(G_0^{(\beta)}(t) - G_0^{(\gamma)}(t)) \right\|_1 = o(t^{\frac{3}{2}})
\]
as $t \to \infty$ for $g \in L^1$.

Proof. We firstly prove the estimate (4.18). Noting that
\[
(G_0^{(\beta)}(t) - G_0^{(\gamma)}(t)) * g = (W_0^{(\beta)}(t) - W_0^{(\gamma)}(t)) R_{a} R_{b} F^{-1} [e^{-\frac{\|\xi\|^2}{2}} \chi_L] * g,
\]
we apply the estimates (2.32) and (2.34) to see that
\[
\left\| (G_0^{(\beta)}(t) - G_0^{(\gamma)}(t)) * g \right\|_p \leq Ct \|\nabla R_{a} R_{b} F^{-1} [e^{-\frac{\|\xi\|^2}{2}} \chi_L] * g \|_p \leq Ct^{\frac{1}{2}} \|g\|_p,
\]
which is the desired estimate (4.18). Next we show the estimate (4.19). For the simplicity of the notation, we define $H_0^{(\beta,\gamma)}(t, x)$ as
\[
H_0^{(\beta,\gamma)}(t, x) := G_0^{(\beta)}(t, x) - G_0^{(\gamma)}(t, x).
\]
Here we can rephrase (4.19) as
\[
\|H_0^{(\beta,\gamma)}(t)\|_1 \leq Ct^{\frac{1}{2}}.
\]
We also easily have
\[
\|\nabla H_0^{(\beta,\gamma)}(t)\|_1 \leq C\|\nabla G_0^{(\beta)}(t)\|_1 + C\|\nabla G_0^{(\gamma)}(t)\|_1 \leq C
\]
by (4.3). Now we observe that
\[
H(t) * g - m_g H(t, x) = \int_{|y| \leq t^{\frac{1}{4}}} (H(t, x - y) - H(t, x)) g(y) dy + \int_{|y| \geq t^{\frac{1}{4}}} H(t, x - y) g(y) dy - \int_{|y| \geq t^{\frac{1}{4}}} H(t, x) g(y) dy.
\]
Therefore when $t \geq 1$, the mean value theorem gives
\[
|H_0^{(\beta,\gamma)}(t, x) - H_0^{(\beta,\gamma)}(t, y)| \leq C|y|\|\nabla H_0^{(\beta,\gamma)}(t, x - \theta y)\|
\]
for some $\theta \in [0, 1]$ and
\[
\|H(t) * g - m_g H(t, x)\|_1
\leq Ct^{\frac{1}{2}} \int_{|y| \leq t^{\frac{1}{4}}} \|\nabla H_0^{(\beta,\gamma)}(t)\|_1 |g(y)| dy + \int_{|y| \geq t^{\frac{1}{4}}} \|H_0^{(\beta,\gamma)}(t)\|_1 |g(y)| dy + \int_{|y| \geq t^{\frac{1}{4}}} \|H_0^{(\beta,\gamma)}(t)\|_1 |g(y)| dy
\leq Ct^{\frac{1}{2}} \|g\|_1 + Ct^{\frac{1}{2}} \int_{|y| \geq t^{\frac{1}{4}}} |g(y)| dy.
\]
Since $g \in L^1$, we see $\lim_{t \to \infty} \int_{|y| \geq t^{\frac{1}{4}}} |g(y)| dy = 0$. Therefore the estimate (4.20) implies the desired estimate. We complete the proof of the proposition.
Combining Lemmas \[4.2\] and \[4.3\] we can obtain the approximation formulas of \( K_0^{(β)}(t) \star g \) and \( K_1^{(β)}(t) \star g \) by the diffusion waves with the Riesz transform.

**Corollary 4.6.** Under the assumption on Lemma \[4.3\], the following estimates hold.

\[
\left\| \nabla^α \left( \partial_t K_0^{(β)}(t) \star g - m_g(-1)^\frac{β}{2} β^\ell \nabla^\ell \mathcal{R}_α \mathcal{R}_β G_0^{(β)}(t) \right) \right\|_p = o(t^{-\frac{β}{2}(1-\frac{1}{p})+\frac{1}{2}-\frac{ℓ+α}{2}}) \quad (4.21)
\]

for \( ℓ = 2m \),

\[
\left\| \nabla^α \left( \partial_t K_0^{(β)}(t) \star g - m_g(-1)^\frac{ℓ+1}{2} β^{ℓ+1} \nabla^{ℓ+1} \mathcal{R}_α \mathcal{R}_β G_1^{(β)}(t) \right) \right\|_p = o(t^{-\frac{β}{2}(1-\frac{1}{p})+\frac{1}{2}-\frac{ℓ+α}{2}}) \quad (4.22)
\]

for \( ℓ = 2m + 1 \),

\[
\left\| \nabla^α \left( \partial_t K_0^{(β)}(t) \star g - m_g(-1)^\frac{ℓ+1}{2} β^{ℓ+1} \nabla^{ℓ+1} \mathcal{R}_α \mathcal{R}_β G_1^{(β)}(t) \right) \right\|_p = o(t^{-\frac{β}{2}(1-\frac{1}{p})+1-\frac{ℓ+α}{2}}) \quad (4.23)
\]

for \( ℓ = 2m \) and

\[
\left\| \nabla^α \left( \partial_t K_0^{(β)}(t) \star g - m_g(-1)^\frac{ℓ+1}{2} β^{ℓ-1} \nabla^{ℓ-1} \mathcal{R}_α \mathcal{R}_β G_0^{(β)}(t) \right) \right\|_p = o(t^{-\frac{β}{2}(1-\frac{1}{p})+1-\frac{ℓ+α}{2}}) \quad (4.24)
\]

for \( ℓ = 2m + 1 \), as \( t \to \infty \).

As we expect, we have a similar conclusion to \( K_0^{(β)}(t) \) and \( K_1^{(β)}(t)g \), which is formulated as follows.

**Corollary 4.7.** Under the assumption on Lemma \[4.4\], the following estimates hold.

\[
\left\| \nabla^α \left( \partial_t K_0^{(β)}(t) \star g - m_g(-1)^\frac{β}{2} β^\ell G_0^{(β)}(t) \right) \right\|_p = o(t^{-\frac{β}{2}(1-\frac{1}{p})+\frac{1}{2}-\frac{ℓ+α}{2}}) \quad (4.25)
\]

for \( ℓ = 2m \),

\[
\left\| \nabla^α \left( \partial_t K_0^{(β)}(t) \star g - m_g(-1)^\frac{ℓ+1}{2} β^{ℓ+1} G_1^{(β)}(t) \right) \right\|_p = o(t^{-\frac{β}{2}(1-\frac{1}{p})+\frac{1}{2}-\frac{ℓ+α}{2}}) \quad (4.26)
\]

for \( ℓ = 2m + 1 \),

\[
\left\| \nabla^α \left( \partial_t K_0^{(β)}(t) \star g - m_g(-1)^\frac{ℓ+1}{2} β^{ℓ+1} G_1^{(β)}(t) \right) \right\|_p = o(t^{-\frac{β}{2}(1-\frac{1}{p})+1-\frac{ℓ+α}{2}}) \quad (4.27)
\]

for \( ℓ = 2m \) and

\[
\left\| \nabla^α \left( \partial_t K_0^{(β)}(t) \star g - m_g(-1)^\frac{ℓ+1}{2} β^{ℓ-1} G_0^{(β)}(t) \right) \right\|_p = o(t^{-\frac{β}{2}(1-\frac{1}{p})+1-\frac{ℓ+α}{2}}) \quad (4.28)
\]

for \( ℓ = 2m + 1 \), as \( t \to \infty \).

The following lemma plays an essential role to obtain the asymptotic profiles of the nonlinear term as \( t \to \infty \).
Lemma 4.8. Let $\alpha, \ell, m \geq 0$. Suppose that $f \in L^1(0, \infty; L^1(\mathbb{R}^3))$ with $\|f(t)\|_1 \leq C(1 + t)^{-2}$. Then the following estimates hold as $t \to \infty$.

(i) \[
\left\| \nabla^\alpha \left( \int_0^\tau \partial_t^\ell K_1^\beta(t - \tau) * f(\tau) d\tau - (-1)\frac{\ell}{2} \beta^\ell \nabla^\ell \int_0^\tau \int_{\mathbb{R}^n} f(\tau, y) dy d\tau \mathbb{G}_1(t) \right) \right\|_p
= o(t^{-\frac{\alpha}{2}(1 - \frac{1}{p}) + 1 - \frac{\alpha + \ell}{2}})
\]
for $\ell = 2m$ and

\[
\left\| \nabla^\alpha \left( \int_0^\tau \partial_t^{2m} K_1^\beta(t - \tau) * f(\tau) d\tau - (-1)^{\frac{\ell}{2}} \beta^{2m-1} \nabla^{2m-1} \int_0^\tau \int_{\mathbb{R}^n} f(\tau, y) dy d\tau \mathbb{G}_0(t) \right) \right\|_p
= o(t^{-\frac{\alpha}{2}(1 - \frac{1}{p}) + 1 - \frac{\alpha + \ell}{2}})
\]
for $\ell = 2m + 1$, where $1 < p \leq \infty$ for $\ell + \alpha = 0$ and $1 \leq p \leq \infty$ for $\ell + \alpha \geq 1$.

(ii) \[
\left\| \nabla^\alpha \left( \int_0^\tau \partial_t^\ell K_1^\beta(t - \tau) * f(\tau) d\tau - (-1)^{\frac{\ell}{2}} \beta^\ell \nabla^\ell \int_0^\tau \int_{\mathbb{R}^n} f(\tau, y) dy d\tau G_1(t) \right) \right\|_p
= o(t^{-\frac{\alpha}{2}(1 - \frac{1}{p}) + 1 - \frac{\alpha + \ell}{2}})
\]
for $\ell = 2m$ and

\[
\left\| \nabla^\alpha \left( \int_0^\tau \partial_t^{2m} K_1^\beta(t - \tau) * f(\tau) d\tau - (-1)^{\frac{\ell}{2}} \beta^{2m-1} \nabla^{2m-1} \int_0^\tau \int_{\mathbb{R}^n} f(\tau, y) dy d\tau G_0(t) \right) \right\|_p
= o(t^{-\frac{\alpha}{2}(1 - \frac{1}{p}) + 1 - \frac{\alpha + \ell}{2}})
\]
for $\ell = 2m + 1$, where $1 \leq p \leq \infty$.

We conclude this section with the estimates for the middle and high frequency parts. The following results are firstly mentioned in [12], and as announced there, we give the detailed proof. For this purpose, following [16] (see also [13]), we introduce the evolution operators

\[
J_H^\beta(t) g := \mathcal{F}^{-1} \left[ \frac{\sigma^\beta_+ e^{\sigma^\beta_- t} \xi_a \xi_b}{\sigma^\beta_+ - \sigma^\beta_-} \chi_H \frac{\xi^2}{|\xi|^2} \hat{g} \right],
\]

\[
J_{0H^+}^\beta(t) g := \mathcal{F}^{-1} \left[ e^{\sigma^\beta_+ t} \chi_H \frac{\xi_a \xi_b}{|\xi|^2} \hat{g} \right], \quad J_{1H^+}^\beta(t) g := \mathcal{F}^{-1} \left[ \frac{e^{\sigma^\beta_+ t}}{\sigma^\beta_+ - \sigma^\beta_-} \chi_H \frac{\xi_a \xi_b}{|\xi|^2} \hat{g} \right]
\]
for $a, b = 1, 2, 3$. Then we can decompose $K_j^\beta(t) \mathcal{R}_a \mathcal{R}_b g$ ($j = 0, 1$) as follows:

\[
K_{0H}^\beta(t) \mathcal{R}_a \mathcal{R}_b g = J_{0H^+}^\beta(t) g - \partial_t J_{1H^+}^\beta(t) g + J_H^\beta(t) g,
K_{1H}^\beta(t) \mathcal{R}_a \mathcal{R}_b g = J_{1H^+}^\beta(t) g - J_{1H^-}^\beta(t) g.
\]

Now we claim the $L^p$-$L^p$ type estimates for $J_H^\beta(t) g$, $J_{0H^\pm}^\beta(t) g$ and $J_{1H^\pm}^\beta(t) g$. 

Lemma 4.9. Let $\alpha \geq \tilde{\alpha} \geq 0$, $t \geq 2\tilde{t} \geq 0$ and $1 \leq q \leq p \leq \infty$. Then it holds that

$$
\| \partial_t^k \nabla^\alpha J_{0H+}^{(\beta)}(t) g - e^{-\frac{\beta}{2} t} R_a R_b F^{-1} \{ \chi H \hat{g} \} \|_p \leq C e^{-c t} \| \nabla^\alpha g \|_p,
$$

(4.34)

$$
\| \partial_t^k \nabla^\alpha J_{1H+}^{(\beta)}(t) g \|_p \leq C e^{-c t} \| \nabla^\alpha g \|_p,
$$

(4.35)

for $t \geq 0$ and

$$
\| \partial_t^k \nabla^\alpha J_{H+}^{(\beta)}(t) g \|_p \leq C e^{-c t} t^{\frac{1}{2} \frac{1}{2} - \frac{1}{p}} \frac{\gamma^2 \alpha}{\alpha - \tilde{\alpha} - (\tilde{t} - \tilde{t} + 1)} \| \nabla^\alpha g \|_p,
$$

(4.36)

$$
\| \partial_t^k \nabla^\alpha J_{1H-}^{(\beta)}(t) g \|_p \leq C e^{-c t} t^{\frac{1}{2} \frac{1}{2} - \frac{1}{p}} \frac{\gamma^2 \alpha}{\alpha - \tilde{\alpha} - (\tilde{t} - \tilde{t} + 1)} \| \nabla^\alpha g \|_p.
$$

(4.37)

for $t > 0$.

Remark 4.10. We remark that our proof of Lemma 4.9 is easily extended to $n$ dimensional case.

Proof. At first, we show the estimate (4.35). Noting

$$
\frac{1}{i ! \gamma ! x^\gamma} e^{ix \xi} = \partial_\xi^\gamma e^{ix \xi}
$$

for $x \neq 0$ and $\gamma \in \mathbb{Z}_+^3$, we have the following decomposition:

\[
\partial_t^k \nabla^\alpha J_{1H+}^{(\beta)}(t) g = \mathcal{F}^{-1} \left[ \frac{e^{\sigma_+^\beta t}}{\sigma_+^\beta - \sigma_-^\beta} \chi_H \frac{\xi a \xi_b}{|\xi|^2} \right]
\]

\[
= (2\pi)^{-\frac{3}{2}} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} e^{ix \xi - |\xi|^2 / 2} \left( \frac{e^{\sigma_+^\beta t}}{\sigma_+^\beta - \sigma_-^\beta} \chi_H \frac{\xi a \xi_b}{|\xi|^2} \right) \partial_\xi^\alpha g
\]

(4.38)

\[
= \frac{C}{x^\gamma} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} e^{ix \xi - |\xi|^2 / 2} \partial_\xi^\gamma \left( \frac{e^{\sigma_+^\beta t}}{\sigma_+^\beta - \sigma_-^\beta} \chi_H \frac{\xi a \xi_b}{|\xi|^2} \right) d\xi
\]

\[
= J_{11}(t, x) \ast \nabla^\alpha g + J_{12}(t, x) \ast \nabla^\alpha g,
\]

where

\[
J_{11}(t, x) := \frac{C}{x^\gamma} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} e^{ix \xi - |\xi|^2 / 2} \chi_H \partial_\xi^\gamma \left( \frac{e^{\sigma_+^\beta t}}{\sigma_+^\beta - \sigma_-^\beta} \chi_H \frac{\xi a \xi_b}{|\xi|^2} \right) d\xi,
\]

\[
J_{12}(t, x) := \frac{C}{x^\gamma} \int_{\mathbb{R}^n} e^{ix \xi} \sum_{\gamma_1 + \gamma_2 = \gamma \atop |\gamma_1| \geq 1} \partial_\xi^\gamma \left( \frac{e^{\sigma_+^\beta t}}{\sigma_+^\beta - \sigma_-^\beta} \chi_H \frac{\xi a \xi_b}{|\xi|^2} \right) \partial_\xi^\gamma \chi_H d\xi.
\]

Now, direct calculations show

\[
\partial_\xi^\gamma \left( \frac{e^{\sigma_+^\beta t}}{\sigma_+^\beta - \sigma_-^\beta} \chi_H \frac{\xi a \xi_b}{|\xi|^2} \right) = \sum_{\gamma_1 + \gamma_2 = \gamma \atop |\gamma_1| \geq 1} C_{\gamma} \gamma_1 \partial_\xi^\gamma \left( \frac{e^{\sigma_+^\beta t}}{\sigma_+^\beta - \sigma_-^\beta} \chi_H \frac{\xi a \xi_b}{|\xi|^2} \right) \partial_\xi^\gamma e^{\sigma_+^\beta t}
\]

(4.39)

and

\[
\partial_\xi^\gamma \left( \frac{e^{\sigma_+^\beta t}}{\sigma_+^\beta - \sigma_-^\beta} \chi_H \frac{\xi a \xi_b}{|\xi|^2} \right) = O(|\xi|^{-|\gamma_1| - 2})
\]

(4.40)
as $|\xi| \to \infty$. On the other hand, observing that
\[
\left| \prod_{k=1}^{m} \partial_\xi^{\tilde{\gamma}_k} (t \sigma_+^{(\beta)}) \right| \leq C \prod_{k=1}^{m} (t^{|\xi|^{-\tilde{\gamma}_k}}) = Ct^{|\xi|^{-\gamma_2}}
\]
for $\xi \in \text{supp} \chi_H$ and $\sum_{k=1}^{m} \tilde{\gamma}_k = \gamma_2$, we have
\[
\chi_H |\partial_\xi^{\gamma_2} e^{t\sigma_+^{(\beta)}}| = \chi_H \sum_{m=1}^{|\gamma_2|} \sum_{\sum_{k=1}^{m} \tilde{\gamma}_k = \gamma_2} \prod_{k=1}^{m} \partial_\xi^{\tilde{\gamma}_k} (t \sigma_+^{(\beta)}) \leq Ce^{-ct} \chi_H \sum_{m=1}^{|\gamma_2|} t^{|\gamma_2| - |\gamma_2|} \leq Ce^{-ct} \chi_H \xi^{-|\gamma_2|}.
\] (4.41)

Then it follows from the estimates (4.39)-(4.41) that
\[
\left| \partial_\xi^{\gamma_2} \left( (\sigma_+^{(\beta)})^\ell \frac{e^{\sigma_+^{(\beta)} t}}{\sigma_+^{(\beta)} - \sigma_-^{(\beta)} |\xi|^{-2}} \xi_a \xi_b \right) \right| \leq Ce^{-ct} |\xi|^{-|\gamma| - 2}.
\] (4.42)

The estimate (4.42) implies that
\[
|J_{11}(t, x)| = \begin{cases} 
  e^{-ct} O(|x|^{-2}) & \text{for } |x| \leq 1, \\
  e^{-ct} O(|x|^{-4}) & \text{for } |x| \geq 1,
\end{cases}
\]
where we choose $\gamma \in \mathbb{Z}_+^3$ as $|\gamma| = 2$ for $|x| \leq 1$ and $|\gamma| = 4$ for $|x| \geq 1$. Thus we immediately have
\[
\|J_{11}(t)\|_1 \leq e^{-ct}.
\] (4.43)

To obtain the estimate for $J_{12}(t, x)$, we apply the integration by parts to see that
\[
\left| \int_{\mathbb{R}^3} e^{i x \cdot \xi} \sum_{\tilde{\gamma}_1 + \tilde{\gamma}_2 = \gamma, |\gamma_2| \geq 1} \partial_\xi^{\tilde{\gamma}_2} \left( (\sigma_+^{(\beta)})^\ell \frac{e^{\sigma_+^{(\beta)} t}}{\sigma_+^{(\beta)} - \sigma_-^{(\beta)} |\xi|^2} \xi_a \xi_b \right) \partial_\xi^{\gamma_2} \chi_H d\xi \right| \leq Ce^{-ct},
\]
since $\text{supp} \partial_\xi^{\gamma_2} \chi_H$ with $\tilde{\gamma}_2 \neq 0$ is compact in $\mathbb{R}^3$. Therefore we choose $\gamma \in \mathbb{Z}_+^3$ satisfying $|\gamma| = 2$ for $|x| \leq 1$ and $|\gamma| = 4$ for $|x| \geq 1$ again to have
\[
|J_{12}(t, x)| = \begin{cases} 
  e^{-ct} O(|x|^{-2}) & \text{for } |x| \leq 1, \\
  e^{-ct} O(|x|^{-4}) & \text{for } |x| \geq 1,
\end{cases}
\]
which shows
\[
\|J_{12}(t)\|_1 \leq e^{-ct}.
\] (4.44)
Combining the estimates (4.43) and (4.44), we obtain the estimate
\[
\| \partial_t^\ell \nabla^\alpha J_{1H+}^{(\beta)}(t)g \|_p \leq C(\| J_{11}(t) \|_1 + \| J_{12}(t) \|_1) \| \nabla^\alpha g \|_p \leq C e^{-ct} \| \nabla^\alpha \hat{g} \|_p,
\]
which is the desired estimate (4.33).

Secondly, we prove the estimate (4.34). Here we only give the proof for the case \( \ell \geq 1 \), since the proof for \( \ell = 0 \) is slightly easier. Now we assume \( \ell \geq 1 \). Then we have
\[
\partial_t^\ell \nabla^\alpha (\xi_0^{(\beta)}H+g) - e^{-\frac{\beta^2}{\nu}t} R_a R_b F^{-1}[\chi H g])
\]
\[
= \sum_{m=0}^{\ell} C_t \partial_t^m e^{-\frac{\beta^2}{\nu}t} F^{-1} \left[ \partial_t^\ell - m (e^{(\sigma_+ + \frac{\beta^2}{\nu})t} - 1) \frac{\xi_a \xi_b}{|\xi|^2} \frac{\chi H}{|\xi|^2} \right] * \nabla^\alpha g
\]
\[
= \sum_{m=0}^{\ell-1} C_t \left( -\frac{\beta^2}{\nu} \right)^m e^{-\frac{\beta^2}{\nu}t} F^{-1} \left[ (\sigma_+ + \frac{\beta^2}{\nu})^{\ell-m} e^{(\sigma_+ + \frac{\beta^2}{\nu})t} \frac{\xi_a \xi_b}{|\xi|^2} \frac{\chi H}{|\xi|^2} \right] * \nabla^\alpha g
\]
\[
+ \left( -\frac{\beta^2}{\nu} \right)^\ell e^{-\frac{\beta^2}{\nu}t} F^{-1} \left[ (e^{(\sigma_+ + \frac{\beta^2}{\nu})t} - 1) \frac{\xi_a \xi_b}{|\xi|^2} \frac{\chi H}{|\xi|^2} \right] * \nabla^\alpha g.
\]
An easy computation shows
\[
\sigma_+ + \frac{\beta^2}{\nu} = O(|\xi|^{-2})
\]
and then
\[
\partial^{\gamma}_\xi \left( \sigma_+ + \frac{\beta^2}{\nu} \right) = O(|\xi|^{-\gamma-2})
\]
as \( |\xi| \to \infty \) for \( \gamma \in \mathbb{Z}_+^\star \). Therefore, it follows from
\[
\left| \prod_{k=1}^{m} t \partial^{\tilde{\gamma}_k}_\xi \left( \sigma_+ + \frac{\beta^2}{\nu} \right) \right| \leq C \prod_{k=1}^{m} (t|\xi|^{-2-|\tilde{\gamma}_k|}) = Ct^m |\xi|^{-2m-|\gamma|}
\]
for \( \xi \in \text{supp} \chi_H \) and \( \sum_{k=1}^{m} \tilde{\gamma}_k = \gamma \) that
\[
|\partial^{\gamma}_\xi e^{(\sigma_+ + \frac{\beta^2}{\nu})t} | = \sum_{m=1}^{\gamma} e^{(\sigma_+ + \frac{\beta^2}{\nu})t} \left| \prod_{k=1}^{m} t \partial^{\tilde{\gamma}_k}_\xi \left( \sigma_+ + \frac{\beta^2}{\nu} \right) \right| \leq C e^{-ct} |\xi|^{-2m-|\gamma|}
\]
by (4.47). We can apply the same argument as that for the proof of (4.34) to estimate the first factor in the right hand side of (4.45). Namely we have
\[
\left| \sum_{m=0}^{\ell-1} C_t \left( -\frac{\beta^2}{\nu} \right)^m e^{-\frac{\beta^2}{\nu}t} F^{-1} \left[ (\sigma_+ + \frac{\beta^2}{\nu})^{\ell-m} e^{(\sigma_+ + \frac{\beta^2}{\nu})t} \frac{\xi_a \xi_b}{|\xi|^2} \frac{\chi H}{|\xi|^2} \right] * \nabla^\alpha g \right|_p
\]
\[
\leq C \sum_{m=0}^{\ell-1} \left| F^{-1} \left[ (\sigma_+ + \frac{\beta^2}{\nu})^{\ell-m} e^{(\sigma_+ + \frac{\beta^2}{\nu})t} \frac{\xi_a \xi_b}{|\xi|^2} \frac{\chi H}{|\xi|^2} \right] \right|_1 \| \nabla^\alpha g \|_p
\]
\[
\leq C e^{-ct} \| \nabla^\alpha g \|_p.
\]
It remains to show the estimate for the term \( F^{-1} \left[ \left( e^{(\sigma_+^{(\beta)} + \frac{\beta^2}{\nu})t} - 1 \right) \frac{\xi_0 \xi_b}{|\xi|^2} \chi_H \right] \). Based on the fact that

\[
\partial_\xi^2 \left( \left( e^{(\sigma_+^{(\beta)} + \frac{\beta^2}{\nu})t} - 1 \right) \frac{\xi_0 \xi_b}{|\xi|^2} \chi_H \right) \\
= \chi_H \left( e^{(\sigma_+^{(\beta)} + \frac{\beta^2}{\nu})t} - 1 \right) \partial_\xi^2 \left( \frac{\xi_0 \xi_b}{|\xi|^2} \right) + \chi_H \sum_{\gamma_1 + \gamma_2 = \gamma, |\gamma_2| \geq 1} \partial_\xi^\gamma \left( \left( e^{(\sigma_+^{(\beta)} + \frac{\beta^2}{\nu})t} - 1 \right) \frac{\xi_0 \xi_b}{|\xi|^2} \right) \partial_\xi^{\gamma_2} e^{(\sigma_+^{(\beta)} + \frac{\beta^2}{\nu})t} \\
+ \sum_{\gamma_1 + \gamma_2 = \gamma, |\gamma_2| \geq 1} \partial_\xi^\gamma \left( \left( e^{(\sigma_+^{(\beta)} + \frac{\beta^2}{\nu})t} - 1 \right) \frac{\xi_0 \xi_b}{|\xi|^2} \right) \partial_\xi^{\gamma_2} \chi_H,
\]

we have the following decomposition

\[
e^{-\frac{\beta^2}{\nu}t} F^{-1} \left[ \left( e^{(\sigma_+^{(\beta)} + \frac{\beta^2}{\nu})t} - 1 \right) \frac{\xi_0 \xi_b}{|\xi|^2} \chi_H \right] \\
= \frac{c_n}{x^\gamma} e^{-\frac{\beta^2}{\nu}t} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} e^{i\xi \cdot \xi - \varepsilon |\xi|^2} \partial_\xi^\gamma \left( \left( e^{(\sigma_+^{(\beta)} + \frac{\beta^2}{\nu})t} - 1 \right) \frac{\xi_0 \xi_b}{|\xi|^2} \right) \chi_H d\xi \\
= J_01(t, x) + J_02(t, x) + J_03(t, x),
\]

where

\[
J_01(t, x) := \frac{c_3}{x^\gamma} e^{-\frac{\beta^2}{\nu}t} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} e^{i\xi \cdot \xi - \varepsilon |\xi|^2} \left( e^{(\sigma_+^{(\beta)} + \frac{\beta^2}{\nu})t} - 1 \right) \chi_H \partial_\xi^\gamma \left( \frac{\xi_0 \xi_b}{|\xi|^2} \right) d\xi,
J_02(t, x) := \frac{c_3}{x^\gamma} e^{-\frac{\beta^2}{\nu}t} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} e^{i\xi \cdot \xi - \varepsilon |\xi|^2} C_\gamma \chi_H \sum_{\gamma_1 + \gamma_2 = \gamma, |\gamma_2| \geq 1} \partial_\xi^\gamma \left( \frac{\xi_0 \xi_b}{|\xi|^2} \right) \partial_\xi^{\gamma_2} e^{(\sigma_+^{(\beta)} + \frac{\beta^2}{\nu})t} d\xi,
J_03(t, x) := \frac{c_3}{x^\gamma} e^{-\frac{\beta^2}{\nu}t} \int_{\mathbb{R}^n} e^{i\xi \cdot \xi} \sum_{\gamma_1 + \gamma_2 = \gamma, |\gamma_2| \geq 1} C_\gamma \partial_\xi^\gamma \left( \left( e^{(\sigma_+^{(\beta)} + \frac{\beta^2}{\nu})t} - 1 \right) \frac{\xi_0 \xi_b}{|\xi|^2} \right) \partial_\xi^{\gamma_2} \chi_H d\xi.
\]

Now we estimate \( J_01(t, x) \). The mean value theorem gives

\[
e^{-\frac{\beta^2}{\nu}t} \left( e^{(\sigma_+^{(\beta)} + \frac{\beta^2}{\nu})t} - 1 \right) = \left( \sigma_+^{(\beta)} + \frac{\beta^2}{\nu} \right) e^{\theta_+^{(\beta)}(1-\theta) \frac{\beta^2}{\nu}} t
\]

for some \( \theta \in [0, 1] \). Namely we see

\[
e^{-\frac{\beta^2}{\nu}t} \left( e^{(\sigma_+^{(\beta)} + \frac{\beta^2}{\nu})t} - 1 \right) = e^{-ct} O(|\xi|^{-2})
\]

as \( |\xi| \to \infty \) by (4.46). Combining with

\[
\partial_\xi^\gamma \left( \frac{\xi_0 \xi_b}{|\xi|^2} \right) = O(|\xi|^{-|\gamma|})
\]

as \( |\xi| \to \infty \) for \( \gamma \in \mathbb{Z}_+^4 \), we obtain

\[
|J_01(t, x)| \leq \frac{C}{|x|^\gamma} e^{-ct} \left| \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} e^{i\xi \cdot \xi - \varepsilon |\xi|^2} |\xi|^{-|\gamma| - 2} \chi_H d\xi \right| = \begin{cases} e^{-ct} O(|x|^{-2}) & \text{for } |x| \leq 1, \\
 e^{-ct} O(|x|^{-4}) & \text{for } |x| \geq 1, \end{cases}
\]

for some \( C \geq 0 \).
where we choose $\gamma \in \mathbb{Z}_+^3$ satisfying $|\gamma| = 2$ for $|x| \leq 1$ and $|\gamma| = 4$ for $|x| \geq 1$. Therefore we have

$$\|J_{01}(t)\|_1 \leq e^{-ct}. \quad (4.52)$$

Similarly we see that

$$J_{02}(t, x) = \frac{C_3}{x^{7\varepsilon}} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} e^{ix - \varepsilon |\xi|^2} \chi_H e^{(\varepsilon + 2)t} \sum_{\gamma_1 + \gamma_2 = \gamma \atop |\gamma| \geq 1} \sum_{m=1}^{\sup{\gamma_2}} t^m O(|\xi|^{2m-|\gamma|}) \, d\xi$$

by (4.48) and (4.51). Thus, with the choice of $\gamma \in \mathbb{Z}_+^3$ as $|\gamma| = 2$ for $|x| \leq 1$ and $|\gamma| = 4$ for $|x| \geq 1$, we have

$$|J_{02}(t, x)| \leq \frac{C}{|x^{7\varepsilon}|} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} e^{ix - \varepsilon |\xi|^2} |\xi|^{-|\gamma|-2} \chi_H \, d\xi = \begin{cases} e^{-ct} O(|x|^{-2}) & \text{for } |x| \leq 1, \\ e^{-ct} O(|x|^{-4}) & \text{for } |x| \geq 1, \end{cases}$$

which implies

$$\|J_{02}(t)\|_1 \leq e^{-ct}. \quad (4.53)$$

Now, we note that $\cup_{1 \leq |\gamma_1|, |\gamma_2| \leq |\gamma|} \supp \partial_{\xi}^{\gamma_2} \chi_H$ is compact in $\mathbb{R}^3$. Then it is easy to see that

$$|J_{03}(t, x)| \leq \frac{C}{|x^{7\varepsilon}|} \int_{\mathbb{R}^3} \sum_{\gamma_1 + \gamma_2 = \gamma \atop |\gamma_2| \geq 1} C_{\gamma_2} \partial_{\xi}^{\gamma_2} \left( \left( e^{(\varepsilon \sigma_+ + 2t)} - 1 \right) \frac{\xi_a \xi_b}{|\xi|^2} \right) \partial_{\xi}^{\gamma_2} \chi_H \, d\xi$$

$$= e^{-ct} O(|x|^{-|\gamma|}) \quad \text{as } |x| \to \infty$$

for all $\gamma \in \mathbb{Z}_+^3$, which implies

$$\|J_{03}(t)\|_1 \leq e^{-ct}. \quad (4.54)$$

Summing up (4.45), (4.49) and (4.52)-(4.54), we obtain the estimate (4.34).

In the third, we prove the estimate (4.36). For the simplicity of the notation, we denote

$$J_{H}^{(\ell, \alpha)}(t, \xi) := \frac{(\sigma_+^{(b)} - (i\xi)^a \sigma_+^{(b)} e^{\sigma_+^{(b)} t})}{\sigma_+^{(b)} - \sigma_+^{(b)}} \chi_H \frac{\xi_a \xi_b}{|\xi|^2}.$$ 

It is easy to see

$$|J_{H}^{(\ell, \alpha)}(t, \xi)| \leq C e^{-ct} e^{-c|\xi|^2 t} |\xi|^{2(\ell-1)+|\alpha|} \chi_H$$

and then

$$\|J_{H}^{(\ell, \alpha)}(t)\|_p \leq C e^{-ct} t^{-\frac{3}{2p} - (\ell-1) - \frac{|\alpha|}{2}} \quad (4.55)$$
for \(1 \leq p \leq 2\). On the other hand, noting that
\[
\left| \prod_{k=1}^{m} \partial_{\xi}^{\gamma_k} (t\sigma_{\beta}^{-}) \right| \leq C \prod_{k=1}^{m} |t\xi|^{2-|\gamma_k|} = Ct^m |\xi|^{2m-|\gamma|}
\]
for \(\xi \in \text{supp } \chi\) and \(\sum_{k=1}^{m} \gamma_k = \gamma_2\), we have
\[
|\partial_{\xi}^\gamma \mathcal{J}_H^{(\beta)} (t, \xi)| = \left\{ \begin{array}{lcl}
\sum_{m=1}^{|\gamma_2|} |\partial_{\xi}^\gamma \mathcal{J}_H^{(\beta)} (t, \xi)| & \leq & Ce^{-ct} e^{-c|\xi|^{2t}} \sum_{m=1}^{|\gamma_2|} t^m |\xi|^{2m-|\gamma|} \leq Ce^{-ct} e^{-c|\xi|^{2t}} |\xi|^{-|\gamma|}.
\end{array} \right.
\]

Thus a direct calculation shows
\[
|\partial_{\xi}^\gamma \mathcal{J}_H^{(\beta)} (t, \xi)| \leq \sum_{\gamma_1+\gamma_2=\gamma} C_{\gamma_1, \gamma_2} \left| \partial_{\xi}^\gamma \left( \frac{(\sigma_-(\beta))^t (i\xi)^\alpha \sigma_+(\beta) \xi_a \xi_b}{\sigma_+(\beta) - \sigma_-(\beta)} \right) \right| \partial_{\xi}^2 \chi_H \]
Recalling the identities (4.33), we can formulate the estimates for the high frequency parts of the fundamental solutions to (1.1).

**Corollary 4.11.** Let \( \alpha \geq \tilde{\alpha} \geq 0, \ell \geq 2\tilde{\ell} \geq 0 \) and \( t > 0 \). Then, the following estimates hold:

\[
\sum_{k=M,H} \left( \| \partial_t^\alpha \nabla^{\alpha} K^{(\beta)}_{0k}(t) g \|_p + \| \partial_t^\ell \nabla^{\alpha} K^{(\beta)}_{0k}(t) \mathcal{R}_a \mathcal{R}_b g \|_p \right) \leq Ce^{-ct} \left( \| \nabla^{\alpha} g \|_p + t^{-\frac{3}{2} \left( \frac{1}{2} - \frac{1}{p} \right)} \sum_{\ell=1}^{\tilde{\alpha}+1} t^{-\frac{\ell}{2} + \frac{1}{2} + \frac{1}{2} - \frac{\ell}{2} + \frac{1}{2}} \| \nabla^{\tilde{\alpha}+\tilde{\ell}} g \|_q \right) \quad (4.57)
\]

for \( 1 < p < \infty \) and \( 1 \leq q \leq p \) and

\[
\sum_{k=M,H} \left( \| \partial_t^\alpha \nabla^{\alpha} K^{(\beta)}_{1k}(t) g \|_p + \| \partial_t^\ell \nabla^{\alpha} K^{(\beta)}_{1k}(t) \mathcal{R}_a \mathcal{R}_b g \|_p \right) \leq Ce^{-ct} \left( \| \nabla^{\alpha} g \|_p + t^{-\frac{3}{2} \left( \frac{1}{2} - \frac{1}{p} \right)} \sum_{\ell=1}^{\tilde{\alpha}+1} t^{-\frac{\ell}{2} + \frac{1}{2} + \frac{1}{2} - \frac{\ell}{2} + \frac{1}{2}} \| \nabla^{\tilde{\alpha}+\tilde{\ell}} g \|_q \right) \quad (4.58)
\]

for \( 1 \leq q \leq p \leq \infty \).

**Proof.** We first show the estimate (4.57). Noting that

\[
\partial_t^\alpha \nabla^{\alpha} K^{(\beta)}_{0k}(t) \mathcal{R}_a \mathcal{R}_b g = \mathcal{R}_a \mathcal{R}_b \partial_t^\alpha \nabla^{\alpha} K^{(\beta)}_{0k}(t) g
\]

for \( k = M, H \), we apply the estimates (2.34) and (2.27) to have (4.57). We can obtain the estimate (4.58) by a similar way.

We next prove the estimate (4.59). Noting the decomposition (4.33), we have

\[
\partial_t^\alpha \nabla^{\alpha} K^{(\beta)}_{1k}(t) \mathcal{R}_a \mathcal{R}_b g = \partial_t^\alpha \nabla^{\alpha} J^{(\beta)}_{0H}(t) g - \partial_t^{\ell+1} \nabla^{\alpha} J^{(\beta)}_{1H}(t) g + \partial_t^\ell \nabla^{\alpha} J^{(\beta)}_{H}(t) g.
\]

Then, we apply the estimates (4.34)-(4.36) to have

\[
\| \partial_t^\alpha \nabla^{\alpha} K^{(\beta)}_{0H}(t) \mathcal{R}_a \mathcal{R}_b g \|_p \leq Ce^{-ct} \| \nabla^{\alpha} g \|_p + Ce^{-ct} \| \nabla^{\alpha} \mathcal{R}_a \mathcal{R}_b \mathcal{F}^{-1} [\chi_H \hat{g}] \|_p + Ce^{-ct} t^{-\frac{3}{2} \left( \frac{1}{2} - \frac{1}{p} \right)} \| \nabla^{\tilde{\alpha}+\tilde{\ell}} g \|_q \quad (4.61)
\]

On the other hand, using the fact that \( \chi_H = 1 - \chi_L - \chi_M \), we have

\[
e^{-\frac{\beta^2}{\tau} \| \nabla^{\alpha} \mathcal{R}_a \mathcal{R}_b \mathcal{F}^{-1} [\chi_H \hat{g}] \|_p \leq Ce^{-ct} \| \mathcal{F}^{-1} [\chi_H |\xi|^{-2}] (\xi^\alpha \xi_a \xi_b \hat{g}) \|_p \leq Ce^{-ct} \| \nabla^{\alpha+2} g \|_p \quad (4.62)
\]
for $1 \leq p \leq \infty$, since the Fourier multiplier $\chi_H|\xi|^{-2}$ is a bounded operator on $L^p$ for $1 \leq p \leq \infty$. It is also well-known that the middle frequency part is smooth enough and decays exponentially (cf. [13]). Namely we have

$$
\|\partial_t^\alpha \nabla^a K_{0,3}^{(\beta)}(t)\mathcal{R}_a \mathcal{R}_b g\|_p + \|\partial_t^\alpha \nabla^a K_{1,3}^{(\beta)}(t)\mathcal{R}_a \mathcal{R}_b g\|_p
$$

(4.63)

$$
+ \|\partial_t^\alpha \nabla^a K_{0,3}^{(\beta)}(t)g\|_p + \|\partial_t^\alpha \nabla^a K_{1,3}^{(\beta)}(t)g\|_p \leq C e^{-ct} \|\nabla \tilde{g}\|_q
$$

for $1 \leq q \leq p \leq \infty$ and $\alpha, \tilde{\alpha} \geq 0$. Therefore we can conclude the estimate (4.59) by the combination of (4.61)-(4.63). The estimate (4.60) is easily proved by (4.33), (4.35) and (4.57). We complete the proof of Corollary 4.11. \hfill \Box

5 Proof of main results ($1 < p < \infty$)

For the proof of main results, we firstly reformulate (1.1) into the integral equation.

**Proposition 5.1** ([12], [13]). Let $u$ be a solution of (1.1). Then it holds that

$$
\hat{u}(t, \xi) = K_0^{(\sqrt{\lambda+2}\mu)}(t, \xi)\mathcal{P} \hat{f}_0(\xi) + K_0^{(\sqrt{\mu})}(t, \xi)(\mathcal{I}_3 - \mathcal{P}) \hat{f}_0(\xi)
$$

$$
+ K_1^{(\sqrt{\lambda+2}\mu)}(t, \xi)\mathcal{P} \hat{f}_1(\xi) + K_1^{(\sqrt{\mu})}(t, \xi)(\mathcal{I}_3 - \mathcal{P}) \hat{f}_1(\xi)
$$

$$
+ \int_0^t \left\{ K_1^{(\sqrt{\lambda+2}\mu)}(t-\tau, \xi)\mathcal{P} + K_1^{(\sqrt{\mu})}(t-\tau, \xi)(\mathcal{I}_3 - \mathcal{P}) \right\} \hat{F}(u)(\tau, \xi)d\tau.
$$

From Proposition 5.1 we have the expression of the solution $u(t)$ by

$$
u(t) = u_{lin}(t) + u_N[u](t),
$$

(5.1)

where

$$
u_{lin}(t) := (K_0^{(\sqrt{\lambda+2}\mu)}(t) - K_0^{(\sqrt{\mu})}(t))\mathcal{F}^{-1}[\mathcal{P} \hat{f}_0] + K_0^{(\sqrt{\mu})}(t)f_0
$$

$$
+ (K_1^{(\sqrt{\lambda+2}\mu)}(t) - K_1^{(\sqrt{\mu})}(t))\mathcal{F}^{-1}[\mathcal{P} \hat{f}_1] + K_1^{(\sqrt{\mu})}(t)f_1,
$$

and

$$u_N(t) := \int_0^t (K_1^{(\sqrt{\lambda+2}\mu)}(t-\tau) - K_1^{(\sqrt{\mu})}(t-\tau))\mathcal{F}^{-1}[\mathcal{P} \hat{F}(u)(\tau)]d\tau
$$

$$
+ \int_0^t K_1^{(\sqrt{\mu})}(t-\tau)\mathcal{F}^{-1}[\mathcal{P} \hat{F}(u)(\tau)]d\tau.
$$

We also recall the estimates for the nonlinear term, which are obtained in [12] under the assumption on Proposition 2.1 for $F(u) = \nabla u \nabla^2 u$:

$$
\|F(u)(t)\|_1 \leq C(1 + t)^{-2},
$$

(5.2)

$$
\|F(u)(t)\|_p \leq C(1 + t)^{-\frac{3}{2}+\frac{3}{p}}, \quad 2 \leq p \leq 6,
$$

(5.3)

$$
\|\nabla F(u)(t)\|_2 \leq C(1 + t)^{-\frac{1}{4}}.
$$

(5.4)

Here we note that we can obtain the estimates (5.2)-(5.4) under the assumption of Proposition 2.2 which will be used in the proof of Theorems 3.2, 3.4 and 3.6.
5.1 Proof of Theorem 3.1

We firstly collect the estimates for the linear part, which are easy consequences of the estimates (2.23), (2.24), (2.27), (2.28), (4.1), (4.2), (4.57) and (4.58) with the notation

\[ \| f_0, f_1 \|_{Y^p} := \| f_0 \|_{H^3 \cap W^{1, r}_1 \cap W^{3, p}} + \| f_1 \|_{H^1 \cap L^1 \cap W^{1, p}}. \]

Namely, we have

\[
\sup_{t \geq 0} \left\{ \sum_{\alpha=1,3} (1 + t)^{2(1 - \frac{1}{p}) - 1 + \frac{2}{p}} \| \nabla^\alpha u_{lin}(t) \|_p + \sum_{\alpha=0,1} (1 + t)^{2(1 - \frac{1}{p}) - \frac{1}{2}} \| \nabla^{\alpha} \partial_t u_{lin}(t) \|_p \right. \\
\left. + (1 + t)^{2(1 - \frac{1}{p})} t^{\frac{1}{2}} \| \nabla^2 \partial_t u_{lin}(t) \|_p + (1 + t)^{2(1 - \frac{1}{p}) - \frac{1}{2}} t^{\frac{1}{2}} \| \partial_t^2 u_{lin}(t) \|_p \right\} \leq C \| f_0, f_1 \|_{Y^p}. \tag{5.5}\]

Now we turn to the proof of the estimates in Theorem 3.1. Let \( F(u) = \nabla u \nabla^2 u \). We prove the estimate (3.1) with \( \alpha = 3 \). At first, we decompose the nonlinear term as follows:

\[
\left\| \nabla^3 \int_0^t K_1^{(3)}(t - \tau) R_a R_b F_j(u)(\tau) d\tau \right\|_p + \left\| \nabla^3 \int_0^t K_1^{(3)}(t - \tau) F_j(u)(\tau) d\tau \right\|_p \\
\leq \sum_{k=L, M, H} \int_0^t \left\| \nabla^3 K_{1k}^{(3)}(t - \tau) F_j(u)(\tau) \right\|_p d\tau, \tag{5.6}\]

since \( 1 < p < \infty \) and (2.34). When \( 1 < p < 2 \), we see that

\[
\| F(u)(t) \|_p \leq \| F(u) \|^{\frac{1}{2} - 1} \| F(u) \|^{\frac{1}{2}} \leq C (1 + t)^{-\frac{3}{2} + \frac{2}{p}} \leq C (1 + t)^{-\frac{3}{2} + \frac{1}{p}} \tag{5.7}\]

by the Hölder inequality, where \( \frac{2}{2 - p} > 2 \) for \( 1 < p < 2 \), (5.2) and (5.3). Noting that

\[
\| \nabla F(u)(t) \|_1 \leq C \| \nabla^2 u \|_2 + C \| \nabla u \|_1 \| \nabla u \|_2 \leq C (1 + t)^{-\frac{1}{2}} \tag{5.8}\]

by (2.2) and (2.3), we apply the same argument with (5.4) to have

\[
\| \nabla F(u)(t) \|_p \leq \| \nabla F(u) \|^{\frac{1}{2} - 1} \| \nabla F(u) \|^{\frac{1}{2}} \leq C (1 + t)^{-\frac{1}{2} + \frac{1}{p}} - \frac{1}{2} \tag{5.9}\]

for \( 1 < p < 2 \). Then when \( 1 < p < 2 \), we employ the estimates (2.24), (5.2) and (5.7) to obtain

\[
\int_0^t \left\| \nabla^3 K_{1L}^{(3)}(t - \tau) F_j(u)(\tau) \right\|_p d\tau \\
\leq C \int_0^t (1 + t - \tau)^{-\frac{3}{2} + \frac{2}{p}} \| F_j(u)(\tau) \|_1 d\tau + C \int_0^t (1 + t - \tau)^{-\frac{1}{2}} \| F_j(u)(\tau) \|_p d\tau \\
\leq C \int_0^t (1 + t - \tau)^{-\frac{1}{2} + \frac{1}{p} - \frac{1}{2}} (1 + \tau)^{-2} d\tau \\
+ C \int_0^t (1 + t - \tau)^{-\frac{1}{2} + \frac{1}{p} - \frac{1}{2}} (1 + \tau)^{-\frac{1}{2} + \frac{1}{p} - \frac{1}{p} - 1} d\tau \\
\leq C (1 + t)^{-\frac{1}{2} + \frac{1}{p} - \frac{1}{2}}, \tag{5.10}\]

\( 5.5 \)
where we used the fact that $-2 < -\frac{7}{4} < -\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}$. When $p \geq 2$, we apply (2.24), (5.1) and (5.3) again to see that

$$
\int_0^t \left\| \nabla^3 K_{1L}^{(\beta)}(t - \tau) F_j(u)(\tau) \right\|_p d\tau
\leq C \int_0^t (1 + t - \tau)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} \| F_j(u)(\tau) \|_1 d\tau
+ C \int_0^t (1 + t - \tau)^{-\frac{5}{2}(1 - \frac{1}{p})} \| \nabla F_j(u)(\tau) \|_2 d\tau
\leq C \int_0^t (1 + t - \tau)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} d\tau + C \int_0^t (1 + t - \tau)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} (1 + \tau)^{-\frac{1}{2}} d\tau
\leq C (1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}}.
$$

(5.11)

We thus obtain

$$
\int_0^t \left\| \nabla^3 K_{1L}^{(\beta)}(t - \tau) F_j(u)(\tau) \right\|_p d\tau \leq C (1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}}
$$

(5.12)

for $1 < p < \infty$ by (5.10) and (5.11).

For the middle and high frequency parts, using the estimates (4.58) and

$$
\| \nabla F(u)(t) \|_p \leq C \| \nabla^2 u(t) \|_p^2 + C \| \nabla u \|_\infty \| \nabla^3 u \|_p
\leq C \| \nabla u \|_\infty \| \nabla^3 u \|_p \leq C (1 + t)^{-2} \| \nabla^3 u(t) \|_p.
$$

by (2.38) and (2.3), with $\alpha = 1$, we see that

$$
\sum_{k=M, H} \int_0^t \left\| \nabla^3 K_{1k}^{(\beta)}(t - \tau) F_j(u)(\tau) \right\|_p d\tau
\leq C \int_0^t e^{-c(t-\tau)} \| \nabla F(u)(\tau) \|_p d\tau \leq C \int_0^t e^{-c(t-\tau)} (1 + \tau)^{-2} \| \nabla^3 u(\tau) \|_p d\tau.
$$

(5.13)

Taking $\nabla^3$ to the both sides of (5.11) and combining the estimates (5.5), (5.12) and (5.13), we arrive at the estimate

$$
\| \nabla^3 u(t) \|_p \leq C_0 (1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} + C_1 \int_0^t e^{-c_2(t-\tau)} (1 + \tau)^{-2} \| \nabla^3 u(\tau) \|_p d\tau,
$$

(5.14)

where $C_0$, $C_1$ and $c_2$ are positive constants. Applying the Gronwall type argument to (5.14), we can obtain (3.11) with $\alpha = 3$. Indeed, denoting $A(t) := e^{c_2 t} \| \nabla^3 u(t) \|_p$ and

$$
F(t) := C_1 \int_0^t e^{c_2 \tau} (1 + \tau)^{-2} \| \nabla^3 u(\tau) \|_p d\tau,
$$

we can rephrase (5.14) as

$$
A(t) \leq C_0 (1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} e^{c_2 t} + F(t).
$$

(5.15)
Noting that
\[ F'(t) = C_1 e^{c_2 t} (1 + t)^{-2} \| \nabla^2 u(t) \|_p \leq C_1 (1 + t)^{-2} (C_0 (1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{7}{2}} e^{c_2 t} + F(t)) \]
by (5.15), we have
\[
\frac{d}{dt} \left( F(t) e^{\frac{c_1}{1 + t}} \right) \leq C e^{c_2 t + \frac{c_1}{1 + t}} (1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{7}{2}},
\]
and thus,
\[
F(t) e^{\frac{c_1}{1 + t}} \leq C \int_0^t e^{c_2 \tau + \frac{c_1}{1 + \tau}} (1 + \tau)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{7}{2}} d\tau
\]
\[
\leq C e^{c_2 \frac{t}{2}} \int_0^{\frac{t}{2}} (1 + \tau)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{7}{2}} d\tau + C e^{c_2 t} \int_{\frac{t}{2}}^t (1 + \tau)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{7}{2}} d\tau
\]
\[
\leq C e^{c_2 \frac{t}{2}} + C e^{c_2 t} (1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{7}{2}},
\]
where we used the fact that \( F(0) = 0 \). Namely, we get
\[
F(t) \leq C e^{c_2 \frac{t}{2}} + C e^{c_2 t} (1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{7}{2}}
\]
and
\[
A(t) = e^{c_2 t} \| \nabla^3 u(t) \|_p \leq C(1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{7}{2}} e^{c_2 t} + C e^{c_2 t} + C e^{c_2 t} (1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{7}{2}}
\]
by (5.15) again. Therefore we conclude that
\[
\| \nabla^3 u(t) \|_p \leq C(1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{7}{2}} + C e^{-\frac{c_2 t}{2}} + C(1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{7}{2}},
\]
which implies the desired estimate (3.1) with \( \alpha = 3 \).

Similar arguments apply to obtain the estimate (3.1) with \( \alpha = 1 \) and \( 1 < p < 2 \). On the other hand, we immediately have the estimate (3.1) for \( p \geq 2 \) by the estimates (2.2), (2.3) and the interpolation \( \| \nabla u(t) \|_p \leq \| \nabla u(t) \|_\infty^{1 - \frac{1}{p}} \| \nabla u(t) \|_2^{\frac{1}{p}} \). Therefore we can conclude the proof of the estimate (3.1) with \( \alpha = 1 \).

Once we obtain (3.1) with \( 1 \leq \alpha \leq 3 \), we easily have
\[
\| F_j(u)(t) \|_p \leq C \| \nabla u(t) \|_\infty \| \nabla^2 u(t) \|_p \leq C(1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - 2}
\]
and
\[
\| \nabla F(u)(t) \|_p \leq C(1 + t)^{-2} \| \nabla^3 u(t) \|_p \leq C(1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{7}{2}}
\]
for \( 1 < p < \infty \). Thus we can apply the same procedure of the estimate (3.1) again to obtain the estimates (3.2) and (3.3), using (5.1) and (5.5).

Next, we show the estimate (3.4). Now, noting that \( \partial_t K_1^{(3)}(0, \xi) = 1 \),
\[
\partial_t^2 \int_0^t K_1^{(3)}(t - \tau) \mathcal{R}_a \mathcal{R}_b F_j(u)(\tau) d\tau
\]
\[
= \mathcal{R}_a \mathcal{R}_b F_j(u)(t) + \int_0^t \partial_t^2 K_1^{(3)}(t - \tau) \mathcal{R}_a \mathcal{R}_b F_j(u)(\tau) d\tau
\]
(5.18)
and
\[ \partial_t^2 \int_0^t K_1^{(3)}(t - \tau) F_j(u)(\tau) d\tau = F_j(u)(t) + \int_0^t \partial_t^2 K_1^{(3)}(t - \tau) F_j(u)(\tau) d\tau, \] (5.19)
we use the estimates (2.34), (2.24), (4.2), (4.58), (5.2), (5.16) and (5.17) to obtain
\[ \left\| \partial_t^2 \int_0^t K_1^{(3)}(t - \tau) \mathcal{R}_a \mathcal{R}_b F_j(u)(\tau) d\tau \right\|_p + \left\| \partial_t^2 \int_0^t K_1^{(3)}(t - \tau) F_j(u)(\tau) d\tau \right\|_p \]
\[ \leq C \| F_j(u)(t) \|_p + C \int_0^t \left( 1 + (t - \tau)^{-\frac{7}{2}(1 - \frac{1}{p})} \| F(u)(\tau) \|_1 d\tau \right) + C \int_0^t \left( 1 + \tau \right)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{7}{2}} d\tau \]
\[ \leq C \left( 1 + t \right)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{7}{2}}. \]
Therefore we get the estimate (3.41) by (5.1), (5.5) and (5.20).

It remains to prove estimates (3.5)-(3.7). We only prove the estimate (3.5), since we can obtain estimates (3.5)-(3.7) by a similar way. We denote
\[ G(t, x) = G_{0,\text{lin}}(t, x) + G_{1,\text{lin}}(t, x) + G_N(t, x), \] (5.21)
where
\[ G_{0,\text{lin}}(t, x) := \nabla^{-1} \mathcal{F}^{-1} \left[ \left( G_0^{(\sqrt{\lambda + 2})}(t, \xi) - G_0^{(\sqrt{\lambda})}(t, \xi) \right) \mathcal{P} m_0 + G_0^{(\sqrt{\lambda})}(t, \xi) m_0 \right], \]
\[ G_{1,\text{lin}}(t, x) := \mathcal{F}^{-1} \left[ \left( G_1^{(\sqrt{\lambda + 2})}(t, \xi) - G_1^{(\sqrt{\lambda})}(t, \xi) \right) \mathcal{P} m_1 + G_1^{(\sqrt{\lambda})}(t, \xi) m_1 \right], \]
\[ G_N(t, x) := \mathcal{F}^{-1} \left[ \left( G_N^{(\sqrt{\lambda + 2})}(t, \xi) - G_N^{(\sqrt{\lambda})}(t, \xi) \right) \mathcal{P} M[u] + G_N^{(\sqrt{\lambda})}(t, \xi) M[u] \right], \]
and \( \mathcal{P} \) is defined by (2.1). Using the decomposition (5.21), we claim that
\[ \| \nabla^\alpha (u_{\text{lin}}(t) - G_{\text{lin}}(t)) \|_p = o(t^{-\frac{5}{2}(1 - \frac{1}{p}) + 1 - \frac{2}{4}}), \quad 1 \leq \alpha \leq 3, \] (5.22)
\[ \| \nabla^\alpha (u_N(t) - G_N(t)) \|_p = o(t^{-\frac{5}{2}(1 - \frac{1}{p}) + 1 - \frac{2}{4}}), \quad 1 \leq \alpha \leq 3, \] (5.23)
as \( t \to \infty \). The estimate (5.22) is shown by the combination of the linear estimates (4.21), (4.23), (4.25), (4.27), (4.57), and (4.58). The proof is completed by showing the estimate (5.23). For this purpose, we recall the useful estimates from [12] and the decomposition of \( u_N(t) - G_N(t) \):
\[ \| \partial_t^\alpha \nabla^\alpha \left( G_{1M}(t) + G_{1H}(t) \right) \|_p \left| \int_0^t \int_{\mathbb{R}^3} F(u)(\tau, y) dy d\tau \right| \]
\[ + \| \partial_t^\alpha \nabla^\alpha \mathcal{F}^{-1} \left[ G_1^{(\beta)}(t, \xi) \mathcal{P}(\chi_M + \chi_H) \right] \|_p \left| \int_0^t \int_{\mathbb{R}^3} F(u)(\tau, y) dy d\tau \right| \]
\[ \leq C e^{-ct} t^{-\frac{7}{2}(1 - \frac{1}{p}) - \frac{11}{4}}. \]
for $1 \leq p \leq \infty$ and $\alpha, \ell \geq 0$ and

$$
\|\partial_t^\ell \nabla^\alpha G_1(t)\|_p \left| \int_\frac{1}{2}^\infty \int_{\mathbb{R}^3} F(u)(\tau, y) dy d\tau \right| \\
+ \|\partial_t^\ell \nabla^\alpha \mathcal{F}^{-1}[G_1^{(\beta)}(t, \xi)\mathcal{P}]\|_p \left| \int_\frac{1}{2}^\infty \int_{\mathbb{R}^3} F(u)(\tau, y) dy d\tau \right| \leq Ct^{-\frac{\ell}{2}(1-\frac{1}{p})-\frac{\alpha+\ell}{2}},
$$

(5.25)

where $1 < p \leq \infty$ for $\ell + \alpha = 0$ and $1 \leq p \leq \infty$ for $\ell + \alpha \geq 1$.

The integrand $u_N(t) - G_N(t)$ is decomposed into 3 parts;

$$
u_N(t) - G_N(t) = J_{0,p,\alpha+2\mu} - J_{0,p,\mu} + J_{0,\phi,\mu},
$$

where

$$
J_{0,p,\beta} := \\
\int_0^t K^{(\sqrt{3})}_{ll}(t - \tau)\mathcal{F}^{-1}[\mathcal{P} \hat{F}(u)(\tau)] d\tau - \mathcal{F}^{-1} \left[ G_1^{(\sqrt{3})}(t, \xi)(\chi_M + \chi_H)\mathcal{P} \right] \int_0^t \int_{\mathbb{R}^3} F(u)(\tau, y) dy d\tau \\
- \mathcal{F}^{-1} \left[ G_1^{(\sqrt{3})}(t, \xi)(\chi_M + \chi_H)\mathcal{P} \right] \int_0^t \int_{\mathbb{R}^3} F(u)(\tau, y) dy d\tau \\
+ \int_\frac{t}{2}^t K^{(\sqrt{3})}_{ll}(t - \tau)\mathcal{F}^{-1}[\mathcal{P} \hat{F}(u)(\tau)] d\tau + \sum_{k=M,H} \int_0^t K^{(\sqrt{3})}_{1k}(t - \tau)\mathcal{F}^{-1}[\mathcal{P} \hat{F}(u)(\tau)] d\tau \\
- \mathcal{F}^{-1} \left[ G_1^{(\sqrt{3})}(t, \xi)\mathcal{P} \right] \int_\frac{1}{2}^\infty \int_{\mathbb{R}^3} F(u)(\tau, y) dy d\tau
$$

and

$$
J_{0,\phi,\beta} := \\
\int_0^t K^{(\sqrt{3})}_{ll}(t - \tau)F(u)(\tau)d\tau - G_1^{(\sqrt{3})}(t) \int_0^t \int_{\mathbb{R}^3} F(u)(\tau, y) dy d\tau \\
- (G^{(\sqrt{3})}_{1M}(t) + G^{(\sqrt{3})}_{1M}(t)) \int_0^t \int_{\mathbb{R}^3} F(u)(\tau, y) dy d\tau \\
+ \int_\frac{t}{2}^t K^{(\sqrt{3})}_{1l}(t - \tau)F(u)(\tau)d\tau + \sum_{k=M,H} \int_0^t K^{(\sqrt{3})}_{1k}(t - \tau)F(u)(\tau)d\tau \\
- G_1^{(\sqrt{3})}(t) \int_\frac{1}{2}^\infty \int_{\mathbb{R}^3} F(u)(\tau, y) dy d\tau.
$$

Then we see that

$$
\|\nabla^\alpha J_{0,p,\beta}\|_p \leq o(t^{-\frac{\ell}{2}(1-\frac{1}{p})+1-\frac{\alpha}{p}}) + C \int_\frac{1}{2}^t (1 + t - \tau)^{1-\frac{\alpha}{p}} \|F(u)(\tau)\|_p d\tau \\
+ \sum_{k=M,H} \int_0^t e^{-c(t-\tau)}(1 + (t - \tau)^{\frac{\alpha}{2}})\|\nabla F(u)(\tau)\|_p d\tau + Ct^{-\frac{\ell}{2}(1-\frac{1}{p})-\frac{\alpha}{p}}
$$

(5.26)
as \( t \to \infty \) for \( 1 \leq \alpha \leq 3 \), by (4.2), (4.29), (4.58), (5.10), (5.24) and (5.25). Likewise, we can get

\[
\| \nabla^\alpha J_{0, \phi, \beta} \|_2 = o(t^{-\frac{3}{2} (1 - \frac{1}{p}) + \frac{1}{2}}) \tag{5.27}
\]
as \( t \to \infty \) for \( 1 \leq \alpha \leq 3 \). Combining (5.26) and (5.27), we have

\[
\| \nabla^\alpha (u_N(t) - G_N(t)) \|_p \leq \| \nabla^\alpha J_{0, \rho, \lambda, 2\mu} \|_p + \| \nabla^\alpha J_{0, \rho, \mu} \|_p + \| \nabla^\alpha J_{0, \phi, \lambda, 2\mu} \|_p \\
= o(t^{-\frac{3}{2} (1 - \frac{1}{p}) + \frac{1}{2}})
\]
as \( t \to \infty \) for \( 1 \leq \alpha \leq 3 \), which is the desired estimate (5.23). We complete the proof of Theorem 3.1.

### 5.2 Proof of Theorem 3.2

Let \( F(u) = \nabla u \nabla \partial_t u \). At first, we note that the linear estimate (5.5) obtained in the proof of Theorem 3.1 is still valid. Then it suffices to show the estimates for nonlinear term. Now we observe that

\[
\| \nabla^3 \int_0^t K^{(3)}(t - \tau) R_a R_b F_j(u)(\tau) d\tau \|_p + \| \nabla^3 \int_0^t K^{(3)}(t - \tau) F_j(u)(\tau) d\tau \|_p \\
\| \nabla^2 \partial_t \int_0^t K^{(3)}(t - \tau) R_a R_b F_j(u)(\tau) d\tau \|_p + \| \nabla^2 \partial_t \int_0^t K^{(3)}(t - \tau) F_j(u)(\tau) d\tau \|_p \\
\leq \sum_{k=L, M, H} \int_0^t \left( \| \nabla^3 K^{(3)}_{1k}(t - \tau) F_j(u)(\tau) \|_p + \| \nabla^2 \partial_t K^{(3)}_{1k}(t - \tau) F_j(u)(\tau) \|_p \right) d\tau \tag{5.28}
\]

by (2.31), (2.21), (4.2) and (4.58), and

\[
\| \nabla F(u) \|_p \leq C \| \nabla^2 u \partial_t \nabla u \|_p + C \| \nabla u \|_\infty \| \partial_t \nabla^2 u \|_p \\
\leq C \| \nabla^2 u \|_p \| \nabla \partial_t u \|_2 + C \| \nabla u \|_\infty \| \nabla^2 \partial_t u \|_p \tag{5.29}
\]

by the Hölder inequality, and (2.38), (2.3) with \( \alpha = 1 \) and (2.8) for \( 1 < p < \infty \).

On the other hand, when \( 1 \leq p \leq 2 \), we see that

\[
\| \nabla F(u)(t) \|_p \leq \| \nabla^2 u \|_2 \| \nabla^2 F(u) \|_2 \leq C(1 + t)^{-\frac{3}{2} (1 - \frac{1}{p}) + \frac{1}{2}}, \tag{5.30}
\]

where we used the fact that

\[
\| \nabla F(u)(t) \|_1 \leq \| \nabla^2 u \|_2 \| \nabla \partial_t u \|_2 + \| \nabla u \|_2 \| \nabla^2 \partial_t u \|_2 \leq C(1 + t)^{-\frac{3}{2}}
\]
and \((5.31)\). Then observing that \(-2 < -\frac{7}{12} < -\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}\) for \(1 < p < 2\), we can obtain the estimate of the RHS of \((5.28)\) as follows:

\[
\text{(RHS of } (5.28)\text{)}
\]

\[
\leq C \int_{0}^{\frac{t}{2}} (1 + t - \tau)^{-\frac{5}{6}(1 - \frac{1}{p}) - \frac{1}{2}} \int_{\tau}^{t} \|\nabla F_j(u)(\tau)\|_p d\tau + C \int_{\frac{t}{2}}^{t} \|\nabla F_j(u)(\tau)\|_p d\tau
\]

\[
+ C \int_{0}^{\frac{t}{2}} e^{-c(t-\tau)}(1 + (t - \tau)^{-\frac{1}{2}}) \|\nabla F_j(u)(\tau)\|_p d\tau
\]

\[
\leq C \int_{0}^{\frac{t}{2}} (1 + t - \tau)^{-\frac{5}{6}(1 - \frac{1}{p}) - \frac{1}{2}} \int_{\tau}^{t} \|\nabla F_j(u)(\tau)\|_p d\tau + C \int_{\frac{t}{2}}^{t} (1 + \tau)^{-\frac{5}{6}(1 - \frac{1}{p}) - \frac{1}{2} - \frac{1}{4}} d\tau
\]

\[
+ C \int_{0}^{\frac{t}{2}} e^{-c(t-\tau)}(1 + (t - \tau)^{-\frac{1}{2}})(1 + \tau)^{-\frac{5}{6}(1 - \frac{1}{p}) - \frac{1}{2}} d\tau
\]

\[
\leq C (1 + t)^{-\frac{5}{6}(1 - \frac{1}{p}) - \frac{1}{4}}
\]

by \((2.21)\), \((4.58)\), \((5.2)\) and \((5.30)\). Therefore we conclude the estimates \((3.1)\) with \(\alpha = 3\) and \((3.3)\) for \(1 < p < 2\) by \((5.31)\) and \((5.3)\).

When \(2 < p < 6\), noting that \(-1 < -\frac{5}{12}(1 - \frac{1}{p}) - \frac{1}{2}\) \(0\), we also have

\[
\text{(RHS of } (5.28)\text{)}
\]

\[
\leq C \int_{0}^{\frac{t}{2}} (1 + t - \tau)^{-\frac{5}{6}(1 - \frac{1}{p}) - \frac{1}{2}} \int_{\tau}^{t} \|\nabla F_j(u)(\tau)\|_p d\tau + C \int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\frac{5}{6}(1 - \frac{1}{p}) - \frac{1}{2} - \frac{1}{4}} d\tau
\]

\[
+ C \int_{0}^{\frac{t}{2}} e^{-c(t-\tau)}(1 + (t - \tau)^{-\frac{1}{2}}) \|\nabla F_j(u)(\tau)\|_p d\tau
\]

\[
+ C \int_{0}^{\frac{t}{2}} e^{-c(t-\tau)}(1 + (t - \tau)^{-\frac{1}{2}})(1 + \tau)^{-\frac{5}{6}(1 - \frac{1}{p}) - \frac{1}{2}} d\tau
\]

\[
\leq C (1 + t)^{-\frac{5}{6}(1 - \frac{1}{p}) - \frac{1}{4}} + C \int_{0}^{t} e^{-c(t-\tau)}(1 + \tau)^{-\frac{5}{6}(1 - \frac{1}{p}) - \frac{1}{2}} \|\nabla F_j(u)(\tau)\|_p d\tau
\]

by \((2.21)\), \((5.2)\), \((5.4)\) and \((5.29)\). Combining the estimates \((5.3)\) and \((5.32)\), we arrive at the estimate

\[
\|\nabla^3 u(t)\|_p + \|\nabla^2 \partial_t u(t)\|_p
\]

\[
\leq C (1 + t)^{-\frac{5}{6}(1 - \frac{1}{p}) t^{-\frac{1}{2}}} + C \int_{0}^{t} e^{-c(t-\tau)}(1 + \tau)^{-\frac{5}{6}(1 - \frac{1}{p}) - \frac{1}{4}} \|\nabla^3 u(\tau)\|_p + \|\nabla^2 \partial_t u(\tau)\|_p d\tau.
\]

Then we have

\[
\|\nabla^3 u(t)\|_p + \|\nabla^2 \partial_t u(t)\|_p \leq C (1 + t)^{-\frac{5}{6}(1 - \frac{1}{p}) t^{-\frac{1}{2}}}
\]
by the same argument as that for the proof of \((3.1)\) with \(\alpha = 3\) in Theorem 3.1. The estimate (5.34) shows (3.3) for \(2 < p < 6\). Moreover we apply the same argument as that to \(\|\nabla^3 u(t)\|_p\) with (5.34) to have

\[
\|\nabla^3 u(t)\|_p \\
\leq \|\nabla^3 u_{lin}(t)\|_p + \|\nabla^3 u_N(t)\|_p \\
\leq C(1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} + C \int_0^t e^{-(t-\tau)}(1 + \tau)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} \|\nabla^3 u(\tau)\|_p + \|\nabla^2 \partial_t u(\tau)\|_p d\tau \\
(5.35)
\]

\[
\leq C(1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} + C \int_0^t e^{-(t-\tau)}(1 + \tau)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} \tau^{-\frac{1}{2}} d\tau \\
\leq C(1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}},
\]

which is the desired estimate (3.1) with \(\alpha = 3\) for \(2 < p < 6\). Here we used the fact that

\[
\int_0^t e^{-(t-\tau)}(1 + \tau)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} \tau^{-\frac{1}{2}} d\tau \\
\leq C e^{-ct} \int_0^{\frac{t}{2}} \tau^{-\frac{1}{2}} d\tau + C(1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} \int_{\frac{t}{2}}^t d\tau \\
\leq C(1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{3}{2}}.
\]

Next we deal with the case \(6 \leq p < \infty\). In this case, we firstly claim that

\[
\|\nabla^3 u(t)\|_5 \leq C(1 + t)^{-\frac{5}{2}}, \\
\|\nabla^2 \partial_t u(t)\|_5 \leq C(1 + t)^{-2} t^{-\frac{1}{2}}. \tag{5.36}
\]

Indeed, the linear solution is estimated as

\[
\|\nabla^3 u_{lin}(t)\|_5 \leq C(1 + t)^{-\frac{5}{2}}, \\
\|\nabla^2 \partial_t u_{lin}(t)\|_5 \leq C(1 + t)^{-2} t^{-\frac{1}{2}}
\]

by (5.5) and \(\|\nabla f\|_5 \leq \|\nabla f\|_2 \|\nabla f\|_{\frac{3p}{p-2}} \|\nabla f\|_{\frac{3p}{3p-2}}\). Then we can apply the estimates (5.34), (5.35) with \(p = 5\) to have (5.36). Moreover we have

\[
\|\nabla F(u)(t)\|_5 \leq C(1 + t)^{-4} t^{-\frac{1}{2}} \tag{5.37}
\]

by (5.29) and (5.36). We now turn to the proof of \((3.1)\) with \(\alpha = 3\) and \(3.3\) for
6 \leq p < \infty. By (5.24), (4.58), (5.41), (5.29) and (5.37), we get

\begin{align*}
\text{(RHS of (5.28))} \\
&\leq C \int_0^t (1 + t - \tau)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} (1 + \tau)^{-\frac{5}{2}} d\tau \\
&\quad + C \int_0^t (1 + t - \tau)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} \|\nabla F_j(u)(\tau)\|_2 d\tau \\
&\quad + C \int_0^t e^{-c(t-\tau)} (1 + \tau)^{-\frac{5}{2}} (\|\nabla^2 u(\tau)\|_p + \|\nabla^2 \partial_t u(\tau)\|_p) d\tau \\
&\quad + C \int_0^t e^{-c(t-\tau)} (t - \tau)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} \|\nabla F_j(u)(\tau)\|_5 d\tau \\
&\leq C(1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} + C \int_0^t (1 + t - \tau)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} (1 + \tau)^{-\frac{5}{2}} d\tau \\
&\quad + C \int_0^t e^{-c(t-\tau)} (1 + \tau)^{-\frac{5}{2}} (\|\nabla^2 u(\tau)\|_p + \|\nabla^2 \partial_t u(\tau)\|_p) d\tau \\
&\quad + C \int_0^t e^{-c(t-\tau)} (t - \tau)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} (1 + \tau)^{-4} d\tau \\
&\quad \leq C(1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) + \frac{1}{2}} + C \int_0^t e^{-c(t-\tau)} (1 + \tau)^{-\frac{5}{2}} (\|\nabla^2 u(\tau)\|_p + \|\nabla^2 \partial_t u(\tau)\|_p) d\tau,
\end{align*}

where we used the facts that $-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2} > -1$ and

\begin{align*}
\int_0^t e^{-c(t-\tau)} (t - \tau)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} (1 + \tau)^{-\frac{5}{2}} d\tau \\
&\leq Ce^{-ct} t^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} \int_0^t \tau^{-\frac{5}{2}} d\tau + C(1 + t)^{-4} t^{-\frac{1}{2}} \int_0^t (t - \tau)^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} d\tau \\
&\leq C(1 + t)^{-4} t^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2}} \leq C(1 + t)^{-\frac{5}{2}(1 - \frac{1}{p}) + \frac{1}{2}}
\end{align*}

for $6 \leq p < \infty$. Therefore we arrive at (5.33) again and have (3.34) for $6 \leq p < \infty$. As in (5.35), we also have (3.1) with $\alpha = 3$ for $6 \leq p < \infty$. Summing up the above estimates, we obtain (3.3) with $\alpha = 3$ and (3.3) for $1 < p < \infty$.

Once we have them, the other estimates in Theorem 3.2 are obtained by the same way as in the proof of Theorem 3.1. We omit the detail. We complete the proof of Theorem 3.2.

6 Proof of main results ($p = 1$)

\textit{Proof of Theorems 3.3-3.4} Theorems 3.3-3.4 are shown in a similar way. We only prove Theorem 3.3.

We begin with the linear estimates. Denoting

$$
\|f_0, f_1\|_{Y_1} := \|f_0\|_{H^3 \cap W^{1,1} \cap W^{3,1}} + \|f_1\|_{H^3 \cap W^{1,1}},
$$

\textit{6.1 Proof of main results (p = 1)}
we simply apply the estimates \(2.23, 2.24, 4.1, 4.2, 4.59\) and \(4.60\) to have
\[
\sup_{t \geq 0} \left\{ \sum_{\alpha=0,1} (1 + t)^{-\frac{1}{2} + \frac{\alpha}{2}} \| \nabla^\alpha \partial_t u_{\text{lin}}(t) \|_1 + \| \partial^2_t u_{\text{lin}}(t) \|_1 \right\} \leq C \| f_0, f_1 \|_{Y_1}.
\] (6.1)

We also have
\[
\sup_{t \geq 0} (1 + t)^{-\frac{1}{2}} \| \nabla u_{\text{lin}}(t) \|_1 \leq C \| f_0, f_1 \|_{Y_1}
\] (6.2)
by \(2.23, 2.24, 4.2\) and \(4.60\), and the fact that
\[
\| \nabla (K_0^\beta(t) - K_0^\gamma(t)) R_a R_b g \|_1
\]
\[
\leq \sum_{k=L,M,H} \| \nabla (K_{0k}^\beta(t) - K_{0k}^\gamma(t)) R_a R_b g \|_1
\]
\[
\leq \| \nabla (K_0^\beta(t) - G_0^\beta(t)) R_a R_b g \|_1 + \| \nabla (K_0^\gamma(t) - G_0^\gamma(t)) R_a R_b g \|_1
\] (6.3)
\[
+ \| \nabla (G_0^\beta(t) - G_0^\gamma(t)) * R_a R_b g \|_1
\]
\[
+ \sum_{k=M,H} \left( \| \nabla K_{0k}^\beta(t) R_a R_b g \|_1 + \| \nabla K_{0k}^\gamma(t) R_a R_b g \|_1 \right)
\]
\[
\leq C (1 + t)^{\frac{1}{2}} \| \nabla g \|_1 + C e^{-\epsilon t} \| \nabla^3 g \|_1
\]
by \(4.5, 4.9, 4.11\) and \(4.59\).

For the nonlinear terms, we again use the estimates \(2.24, 2.28, 4.2, 4.60\) and \(5.2\) to obtain
\[
\left\| \nabla \int_0^t K_1^\beta(t - \tau) R_a R_b F_j(u)(\tau) d\tau \right\|_1 + \left\| \nabla \int_0^t K_1^\beta(t - \tau) F_j(u)(\tau) d\tau \right\|_1
\]
\[
\leq \sum_{k=L,M,H} \int_0^t \left( \| \nabla K_{1k}^\beta(t - \tau) R_a R_b F_j(u)(\tau) \|_1 + \| \nabla K_{1k}^\beta(t - \tau) F_j(u)(\tau) \|_1 \right) d\tau
\]
\[
\leq C \int_0^t (1 + t - \tau)^{\frac{1}{2}} \| F_j(u)(\tau) \|_1 d\tau + C \int_0^t e^{-\epsilon(t-\tau)} \| F_j(u)(\tau) \|_1 d\tau
\] (6.4)
\[
\leq C \int_0^t (1 + t - \tau)^{\frac{1}{2}} (1 + \tau)^{-2} d\tau + C \int_0^t e^{-\epsilon(t-\tau)} (1 + \tau)^{-2} d\tau
\]
\[
\leq C (1 + t)^{\frac{3}{2}}.
\]

By the same way we have
\[
\left\| \nabla^\alpha \partial_t \int_0^t K_1^\beta(t - \tau) R_a R_b F_j(u)(\tau) d\tau \right\|_1 + \left\| \nabla^\alpha \partial_t \int_0^t K_1^\beta(t - \tau) F_j(u)(\tau) d\tau \right\|_1
\]
\[
\leq C (1 + t)^{\frac{3}{2} - \frac{\alpha}{2}}
\] (6.5)
for \(0 \leq \alpha \leq 1\) and
\[
\left\| \partial^2_t \int_0^t (K_1^\beta(t - \tau) - K_1^\gamma(t - \tau)) R_a R_b F_j(u)(\tau) d\tau \right\|_1 \leq C
\] (6.6)
where we used (5.20) and (5.19) again for (6.6). Therefore by (6.1), (6.2) and (6.4), we have the estimate (3.8). We also have (3.9) by (6.1) and (6.3). Likewise, it follows from (6.1) and (6.6) that the estimate (3.10), which is the desired conclusion. Once we have (3.8)-(3.10), the proof of the estimates (3.11)-(3.13) is shown by the same argument as in (3.5)-(3.7). The only point remaining concerns the estimates for \(K_0^\langle p\rangle(t)\mathcal{R}_a\mathcal{R}_b f_0\) in \(\hat{W}^{1,1}\), since \(L^1-L^1\) estimate is not valid.

The estimates (4.5), (4.18), (4.19) and (4.59) yield

\[
\|\nabla(K_0^\langle p\rangle(t) - K_0^\langle \gamma\rangle(t))\mathcal{R}_a\mathcal{R}_b g - m\nabla g\nabla^{-1}\mathcal{R}_a\mathcal{R}_b(G_0^\langle p\rangle(t) - G_0^\langle \gamma\rangle(t))\|_1 \\
\leq \|\nabla(K_0^\langle p\rangle(t) - G_0^\langle p\rangle(t))\mathcal{R}_a\mathcal{R}_b g\|_1 + \|\nabla(K_0^\langle \gamma\rangle(t) - G_0^\langle \gamma\rangle(t))\mathcal{R}_a\mathcal{R}_b g\|_1 \\
+ \|\nabla K_0^\langle \gamma\rangle(t)\mathcal{R}_a\mathcal{R}_b g\|_1
\]

\[
+ C\|\mathcal{R}_a\mathcal{R}_b F^{-1}[G^\langle p\rangle(t,\xi)(\chi_M + \chi_H)]\|_1 + C\|\mathcal{R}_a\mathcal{R}_b F^{-1}[G^\langle \gamma\rangle(t,\xi)(\chi_M + \chi_H)]\|_1 \\
= o(t^{\frac{1}{2}})
\]

as \(t \to \infty\), which implies

\[
\|\nabla(u_{lin}(t) - G_{lin}(t))\|_2 = o(t^{\frac{1}{2}}),
\]

as \(t \to \infty\). We complete the proof of Theorem 3.3.

7 Proof of main results \((p = \infty)\)

Proof of Theorem 3.3. At first, we note that once we have the estimate (3.14), we can apply the same argument as in (3.5)-(3.7) again to get (3.15). Then it suffices to show the estimate (3.14) for \(\mathcal{F}(u) = \nabla u \nabla^2 u\).

We firstly claim that, under the assumption on Theorem 3.3 it holds that

\[
u(t) \in \{C([0, \infty); \hat{W}^{3,q}) \cap C^1((0, \infty); \hat{W}^{2,q})\}^3,
\]

\[
\|\nabla^3 u(t)\|_q \leq C(1 + t)^{-\frac{3}{2}(1 - \frac{1}{q}) - \frac{1}{2}}
\]

for \(t \geq 0\), and

\[
\|\nabla^2 \partial_t u(t)\|_q \leq C(1 + t)^{-\frac{3}{2}(1 - \frac{1}{q})} t^{-\frac{1}{2}}
\]

for \(t > 0\), where \(2 \leq q < \infty\). Indeed, noting the facts that \(\|g\|_q \leq \|g\|^{1-\frac{2}{q}}_\infty \|g\|^\frac{2}{q}_2\), we have

\[
\sup_{t \geq 0} \left\{(1 + t)^{\frac{3}{2}(1 - \frac{1}{q}) + \frac{1}{2}}\|\nabla^3 u_{lin}(t)\|_q + (1 + t)^{\frac{3}{2}(1 - \frac{1}{q})} t^\frac{1}{2} \|\nabla^2 \partial_t u_{lin}(t)\|_q \right\} \leq C\|f_0, f_1\|_\infty
\]

for \(2 \leq q < \infty\) as in (5.5), where

\[
\|f_0, f_1\|_\infty := \|f_0\|_{H^3\cap \hat{W}^{1,1}} + \|f_1\|_{H^1\cap L^1\cap \hat{W}^{1,\infty}}
\]
Therefore we can apply Theorem 3.1 for each $q \in [2, \infty)$ to obtain (7.1) and (7.2). As a consequence, we see that
\[
\| \nabla F(u)(t) \|_q \leq C (1 + t)^{-\frac{5}{2}(1 - \frac{1}{q}) - \frac{5}{2}}
\] (7.3)
for $2 \leq q < \infty$ by (5.17) and (7.1) and,
\[
\| \nabla^2 u(t) \|_\infty \leq C \| \nabla^3 u \|_q^{\frac{q-3}{2q-3}} \leq C (1 + t)^{-\frac{10q^2-23q+15}{2q(2q-3)}}
\]
for $3 < q < \infty$ by (2.39), (2.3) and (7.1). Choosing $q$ sufficiently large (formally $q = \infty - \bar{\varepsilon}$ for sufficiently small $\bar{\varepsilon} > 0$), we arrive at the estimate
\[
\| \nabla^2 u(t) \|_\infty \leq C (1 + t)^{-\frac{5}{4} + \epsilon}
\] (7.4)
with small enough $\epsilon > 0$. Moreover as in (5.5), we easily have
\[
\| \partial_t^2 u_{lin}(t) \|_\infty \leq C (1 + t)^{-2} t^{-\frac{1}{2}} \| f_0, f_1 \|_{Y_\infty}.
\] (7.5)

Now we estimate the nonlinear terms. We observe that
\[
\partial_t^2 \int_0^t (K^{(\beta)}_1(t - \tau) - K^{(\gamma)}_1(t - \tau)) R_a R_b F_j(u)(\tau) d\tau
\]
\[= \int_0^t \partial_t^2 (K^{(\beta)}_1(t - \tau) - K^{(\gamma)}_1(t - \tau)) R_a R_b F_j(u)(\tau) d\tau
\]
by (5.18) and (5.19) again, and
\[
\| F_j(u)(t) \|_\infty \leq \| \nabla u \|_\infty \| \nabla^2 u \|_\infty \leq C (1 + t)^{-\frac{13}{4} + \epsilon}
\]
by (2.3) and (7.1). Using these facts, together with (2.24), (4.2), (4.59), (4.60), (5.2) and
applying the same argument as in the proof of Theorem 3.5, we have
shall have established the theorem if we prove the nonlinear estimate. For this purpose,
(7.1) and (7.2) by Theorem 3.2. The linear estimates (7.5) are also still valid. Then we
for $F$ 3.6 is completed by showing the estimate (3.14) for $F$. As mentioned in the proof of Theorem 3.5, the proof of Theorem 3.6 is completed by showing the estimate (3.14) for $F$. Combining (7.5) and (7.6), we obtain the desired estimate (3.14), which proves Theorem 3.5. \hfill \square

Proof of Theorem 3.6. As mentioned in the proof of Theorem 3.5, the proof of Theorem 3.6 is completed by showing the estimate (3.14) for $F(u) = \nabla u \nabla \partial_1 u$. We firstly obtain (7.1) and (7.2) by Theorem 3.2. The linear estimates (7.5) are also still valid. Then we shall have established the theorem if we prove the nonlinear estimate. For this purpose, applying the same argument as in the proof of Theorem 3.5, we have

$$
\|\nabla F(u)(t)\|_q \leq C(1 + t)^{-\frac{\alpha}{2} + \epsilon} t^{-1} \tag{7.7}
$$

for $F(u) = \nabla u \nabla \partial_1 u$ and $2 \leq q < \infty$ by (2.3), (7.1) and (7.2). We also have

$$
\|F_j(u)(t)\|_\infty \leq \|\nabla u\|_\infty \|\nabla \partial_1 u\|_\infty \leq C(1 + t)^{-\frac{17}{4} t^{-\frac{1}{4}}} \tag{7.8}
$$

by (2.3) and (2.10). Therefore as in (7.6), with the choice of $3 < q < \infty$, we can obtain

$$
\left\| \partial^2_t \int_0^t (K_1^{(\beta)}(t - \tau) - K_1^{(\gamma)}(t - \tau)) \mathcal{R}_a \mathcal{R}_b F_j(u)(\tau) d\tau \right\|_\infty
\leq C(1 + t)^{-\frac{\alpha}{2} - t^{-\frac{1}{2}}} \tag{7.9}
$$

where we choose $3 < q < \infty$ and used the fact that $-\frac{3}{2q} - \frac{1}{2} > -1$. Combining (7.5) and (7.6), we obtain the desired estimate (3.14), which proves Theorem 3.5. \hfill \square
where we used (7.7), (7.8) and the facts that
\[
\int_0^t e^{-c(t-\tau)}(1 + \tau)^{-\frac{3}{2}} (t - \tau)^{-\frac{1}{2}} d\tau \leq Ce^{-ct} \int_0^{\frac{t}{2}} \tau^{-\frac{1}{2}} d\tau + C(1 + t)^{-\frac{3}{2}} t^{-\frac{1}{2}} \int_{\frac{t}{2}}^t d\tau \\
\leq C(1 + t)^{-\frac{7}{2}}
\]
and
\[
\int_0^t e^{-c(t-\tau)}(t - \tau)^{-\frac{3}{2}} (1 + \tau)^{-\frac{5}{8}} (1 - \frac{1}{q})^{-2} \tau^{-\frac{1}{8}} d\tau \\
\leq Ce^{-ct} t^{-\frac{3}{2q}} \int_0^{\frac{t}{2}} \tau^{-\frac{1}{8}} d\tau + (1 + t)^{-\frac{5}{8}(1 - \frac{1}{q})^{-2}} t^{-\frac{1}{8}} \int_{\frac{t}{2}}^t (t - \tau)^{-\frac{3}{2q}} \frac{1}{8} d\tau \\
\leq Ce^{-ct} t^{-\frac{3}{2q}} + C(1 + t)^{-\frac{5}{8}(1 - \frac{1}{q})^{-2}} t^{-\frac{3}{2q}} \leq C(1 + t)^{-2} t^{-\frac{1}{2}}.
\]
Thus we have the estimate (3.14) by (7.5) and (7.9). We complete the proof of Theorem 3.6.

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References

[1] Agemi, R., Global existence of nonlinear elastic waves, Invent. Math. 142 (2000), 225-250.
[2] Brenner, P., Thomée, V. and Wahlbin, L., Besov spaces and applications to difference methods for initial value problems, Lecture Notes in Mathematics, Vol. 434. Springer-Verlag, Berlin-New York, 1975.
[3] Cazenave, T., Semilinear Schrödinger equations. Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.
[4] Charão C. R. and Ikehata, R., Note on asymptotic profile of solutions to the linearized compressible Navier-Stokes flow. Hokkaido Math. J. 48 (2019), 357-383.
[5] Ghisi, M., Gobbino, M. and Harraux, A., Local and global smoothing effects for some linear hyperbolic equations with a strong dissipation. Trans. Amer. Math. Soc. 368 (2016), 2039-2079.
[6] Grafakos, L., Classical Fourier analysis. Second edition. Graduate Texts in Mathematics, 249. Springer, New York, 2008.
[7] Hoff, D. and Zumbrun, K., Multi-dimensional diffusion waves for the Navier-Stokes equations of compressible flow, Indiana Univ. Math. J. 44 (1995), 603-676.
[8] Hoff, D. and Zumbrun, K., Pointwise decay estimates for multidimensional Navier-Stokes diffusion waves, Z. Angew. Math. Phys. 48 (1997), 597-614.

[9] Ikehata, R., Kobayashi, T. and Matsuyama, T., Remark on the $L^2$ estimates of the density for the compressible Navier-Stokes flow in $\mathbb{R}^3$, Proceedings of the Third World Congress of Nonlinear Analysts, Part 4 (Catania, 2000). Nonlinear Anal. 47 (2001), 2519-2526.

[10] Ikehata, R., Asymptotic profiles for wave equations with strong damping, J. Differential Equations 257 (2014), 2159-2177.

[11] Jonov, B. and Sideris, T. C., Global and almost global existence of small solutions to a dissipative wave equation in 3D with nearly null nonlinear terms, Commun. Pure Appl. Anal. 14 (2015), 1407-1442.

[12] Kagei, Y. and Takeda, H., Smoothing effect and large time behavior of solutions to nonlinear elastic wave equations with viscoelastic term, preprint

[13] Kobayashi, T. and Shibata, Y., Remark on the rate of decay of solutions to linearized compressible Navier-Stokes equations. Pacific J. Math. 207 (2002), 199-234.

[14] Ponce, G., Global existence of small solutions to a class of nonlinear evolution equations, Nonlinear Anal. 9 (1985), 399-418.

[15] Shatah, J. and Struwe, M., Geometric wave equations, Courant Lecture Notes in Mathematics, 2. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1998.

[16] Shibata, Y., On the rate of decay of solutions to linear viscoelastic equation, Math. Methods Appl. Sci. 23 (2000), 203-226.

[17] Sideris, T. C., The null condition and global existence of nonlinear elastic waves, Invent. Math. 123 (1996), 323-342.

[18] Takeda, H., Large time behavior of solutions to elastic wave with structural damping, submitted.