PARTITIONS AND COVERINGS OF TREES BY BOUNDED-DEGREE SUBTREES

DAVID R. WOOD

ABSTRACT. This paper addresses the following questions for a given tree $T$ and integer $d \geq 2$: (1) What is the minimum number of degree-$d$ subtrees that partition $E(T)$? (2) What is the minimum number of degree-$d$ subtrees that cover $E(T)$? We answer the first question by providing an explicit formula for the minimum number of subtrees, and we describe a linear time algorithm that finds the corresponding partition. For the second question, we present a polynomial time algorithm that computes a minimum covering. We then establish a tight bound on the number of subtrees in coverings of trees with given maximum degree and pathwidth. Our results show that pathwidth is the right parameter to consider when studying coverings of trees by degree-3 subtrees. We briefly consider coverings of general graphs by connected subgraphs of bounded degree.

CONTENTS

1. Introduction 2
2. Partitioning Trees 3
3. Coverings by Paths 6
4. An Algorithm for Covering Trees 10
5. Coverings of Complete Trees 14
6. Coverings of Caterpillars 16
7. Pathwidth and Rooted Coverings 17
8. Pathwidth and Unrooted Coverings 26
9. Coverings of General Graphs 28
References 31
Appendix A. Complete Multipartite Graphs 34
Appendix B. Integer Linear Programs 36

Key words and phrases. graph, tree, covering, pathwidth.

Supported by a QEII Research Fellowship from the Australian Research Council; research initiated at Universitat Politècnica de Catalunya, where supported by the Marie Curie Fellowship MEIF-CT-2006-023865, and by the projects MEC MTM2006-01267 and DURSI 2005SGR00692.
1. Introduction

This paper addresses the following questions, which are motivated by a recent approach to drawing trees\footnote{We consider graphs $G$ that are simple and finite. A graph with one vertex is trivial. Let $G$ be an (undirected) graph. The degree of a vertex $v$ of $G$, denoted by $\text{deg}_G(v)$, is the number of edges of $G$ incident with $v$. The minimum and maximum degrees of $G$ are respectively denoted by $\delta(G)$ and $\Delta(G)$. We say $G$ is degree-$d$ if $\Delta(G) \leq d$. Now let $G$ be a directed graph. Let $v$ be a vertex of $G$. The indegree of $v$, denoted by $\text{indeg}_G(v)$, is the number of incoming edges incident to $v$. The outdegree of $v$, denoted by $\text{outdeg}_G(v)$, is the number of outgoing edges incident to $v$. We say $G$ is outdegree-$d$ if $\text{outdeg}_G(v) \leq d$ for every vertex $v$ of $G$. A rooted tree is a directed tree such that exactly one vertex, called the root, has indegree $0$. It follows that every vertex except $r$ has indegree $1$, and every edge $vw$ of $T$ is oriented ‘away’ from $r$; that is, if $v$ is closer to $r$ than $w$, then $vw$ is directed from $v$ to $w$. If $r$ is a vertex of a tree $T$, then the pair $(T,r)$ denotes the rooted tree obtained by orienting every edge of $T$ away from $r.$} developed in the companion paper [22]. For a given tree $T$ and integer $d \geq 2$,

- what is the minimum number of degree-$d$ subtrees that partition $E(T)$?
- what is the minimum number of degree-$d$ subtrees that cover $E(T)$?

Here a partition of a graph $G$ is a set of connected subgraphs of $G$ such that every edge of $G$ is in exactly one subgraph. A partition can also be thought of as a (non-proper) edge-colouring, with one colour for each connected subgraph. A covering of $G$ is a set of connected subgraphs of $G$ such that every edge of $G$ is in at least one subgraph. For $d \geq 2$, let $\text{minpart}_d(G)$ be the minimum number of degree-$d$ connected subgraphs that partition $G$, and let $\text{mincover}_d(G)$ be the minimum number of degree-$d$ connected subgraphs that cover $G$. We emphasise that ‘trees’ and ‘subtrees’ are necessarily connected.

In Section 2 we answer the first question above. In particular, we present an explicit formula for $\text{minpart}_d(T)$, and describe a linear time algorithm that finds the corresponding partition (amongst other results).

The remainder of the paper addresses the second question above. Section 3 considers coverings of trees by paths (that is, degree-2 subtrees). A tight bound on the number of paths is obtained, amongst other combinatorial and algorithmic results.

Then Section 4 describes a polynomial time algorithm that computes $\text{mincover}_d(T)$ and the corresponding covering. Section 5 describes an example of this algorithm applied to ‘complete’ trees.

Then Section 6 studies minimum coverings of caterpillars by degree-$d$ subtrees. Again tight bounds on the number of subtrees are obtained. Coverings of caterpillars provide a natural precursor to the results in Sections 7 and 8. These sections establish tight upper bounds on the number of covering subtrees in terms of the pathwidth and maximum
degree of the tree. Essentially, these results show that pathwidth is the right parameter to consider when studying coverings of trees by degree-3 subtrees.

Finally, Section 9 studies coverings of general and planar graphs by connected subgraphs of bounded degree. While this problem has not previously been explicitly studied, in the case \( d = 2 \), a related concept has been extensively studied. Harary defined the pathos or path number of a graph \( G \), denoted by \( p(G) \), to be the minimum number of paths that partition \( E(G) \); see [8, 9, 19, 20, 24, 26, 31, 32]. Harary and Schwenk [20] defined the unrestricted path number of a graph \( G \), denoted by \( p^*(G) \), to be the minimum number of paths that cover \( G \). Since every cycle is the union of two disjoint paths,

\[
\minpart_2(G) \leq p(G) \leq 2 \cdot \minpart_2(G), \quad \text{and}
\mincover_2(G) \leq p^*(G) \leq 2 \cdot \mincover_2(G),
\]

where the lower bounds on \( p(G) \) and \( p^*(G) \) become equalities if \( G \) is a tree. Also concerning the \( d = 2 \) case, Gallai conjectured that \( p(G) \leq \left\lceil \frac{n+1}{2} \right\rceil \) for every connected graph \( G \) with \( n \) vertices. While this conjecture remains unsolved, Lovász [24] proved that \( \minpart_2(G) \leq \left\lfloor \frac{n}{2} \right\rfloor \) for every (not necessarily connected) graph \( G \); also see [8, 9, 13, 14, 27].

2. Partitioning Trees

This section considers partitions of (the edge-set of) a tree into bounded-degree subtrees\(^2\). First we prove a formula for \( \minpart_d(T) \). Interestingly, it only depends on the degrees modulo \( d \).

**Theorem 2.1.** Let \( T \) be a non-trivial tree with \( n \geq 2 \) vertices, and let \( d \geq 2 \). Define

\[
n_i := \left| \left\{ v \in V(T) : \left\lceil \frac{\deg(v)}{d} \right\rceil \frac{\deg(v)}{d} + i \right\} \right|
\]

for \( i \in [0, d-1] \). Then

\[
\minpart_d(T) = 1 + \sum_{v \in V(T)} \left( \left\lceil \frac{\deg(v)}{d} \right\rceil - 1 \right) = 1 + \frac{2(n-1)}{d} - n + \sum_{i=0}^{d-1} \frac{i \cdot n_i}{d}.
\]

Moreover, there is a linear-time algorithm to compute \( \minpart_d(T) \) and a corresponding partition.

---

\(^2\)Note that the question for bounded-degree subforests is easily answered. A straightforward inductive argument proves that for every degree-\( \Delta \) tree \( T \), there is a partition of \( E(T) \) into \( \left\lceil \frac{\Delta}{d} \right\rceil \) degree-\( d \) subforests. This is just an edge-colouring such that every vertex is incident to at most \( d \) monochromatic edges; see [21] for analogous results for general graphs. The bound of \( \left\lceil \frac{\Delta}{d} \right\rceil \) is tight for every tree \( T \), and there is a linear-time algorithm to compute the partition.
Proof. First note that
\[
1 + \sum_{v \in V(T)} \left( \left\lceil \frac{\deg(v)}{d} \right\rceil - 1 \right) = 1 + \sum_{v \in V(T)} \left( \frac{\deg(v)}{d} - 1 \right) + \sum_{i=0}^{d-1} i \cdot n_i \\
= 1 + \frac{2(n-1)}{d} - n + \sum_{i=0}^{d-1} i \cdot n_i .
\]

Thus it suffices to prove the first equality. We proceed by induction. In the base case, \( T = K_2 \) and \( \text{minpart}_d(T) = 1 \). Now assume that \( T \) has at least three vertices. Thus \( T \) has a set of leaves \( S \) with a common neighbour \( w \), such that \( w \) is a leaf in \( T - S \). By induction,
\[
\text{minpart}_d(T - S) = 1 + \sum_{v \in V(T - S)} \left( \left\lceil \frac{\deg_{T-S}(v)}{d} \right\rceil - 1 \right) .
\]

To extend the partition of \( T - S \), the colour that is assigned to the edge in \( T - S \) incident to \( w \) can be assigned to \( d - 1 \) edges incident to \( S \). There are \((\deg(w) - 1) - (d - 1) = \deg(w) - d\) remaining leaf edges incident to \( w \). These leaf edges can be partitioned into \( \left\lfloor \frac{\deg(w) - d}{d} \right\rfloor = \left\lfloor \frac{\deg(w)}{d} \right\rfloor - 1 \) stars rooted at \( w \), each with at most \( d \) edges. This defines a partition of \( T \) into \( \text{minpart}_d(T - S) + \left\lfloor \frac{\deg(w)}{d} \right\rfloor - 1 \) subtrees, which equals the claimed upper bound on \( \text{minpart}_d(T) \) since leaves do not contribute to the summation.

It is easy to convert this proof into a linear-time algorithm. Here is a sketch. Root \( T \) at a vertex \( r \), and partition the vertex sets according to their distance from \( r \) (using BFS). The set \( S \) is simply a maximal set of leaves at maximum distance \( d \) from the root with a common neighbour (at distance \( d - 1 \)). The partition is then easily computed.

This bound on \( \text{minpart}_d(T) \) is optimal since at most \( d - 1 \) edges incident to \( S \) can share the same colour as the edge in \( T - S \) incident to \( w \), and at least \( \left\lfloor \frac{\deg(w) - d}{d} \right\rfloor \) colours not used in \( T - S \) must be introduced on the remaining leaf edges. \( \square \)

Note that Theorem 2.1 with \( d = 2 \) reduces to the following result by Stanton et al. [31]:

**Corollary 2.2 ([31]).** For every non-trivial tree \( T \), \( \text{minpart}_2(T) \) equals half the number of odd-degree vertices.

Theorem 2.1 also implies:

**Corollary 2.3.** For every integer \( d \geq 2 \) and tree \( T \) with \( n \geq 2 \) vertices,
\[
\text{minpart}_d(T) \leq 1 + \frac{n - 2}{d} ,
\]
with equality if and only if \( \deg(v) \equiv 1 \pmod{d} \) for every vertex \( v \).

A degree-\( d \) subtree \( X \) of a tree \( T \) is **degree-\( d \) maximal** if no edge of \( T \) can be added to \( X \) to obtain a new degree-\( d \) subtree. Observe that \( X \) is degree-\( d \) maximal if and only
if \( \deg_X(v) = \min\{d, \deg_T(v)\} \) for every vertex \( v \) of \( X \). In particular, \( v \) is leaf in \( X \) if and only if \( v \) is a leaf in \( T \). Clearly every degree-\( d \) subtree is contained in a maximal degree-\( d \) subtree.

**Proposition 2.4.** Every degree-\( d \) maximal subtree \( S \) of a tree \( T \) is in a minimum partition of \( T \) into degree-\( d \) subtrees.

**Proof.** For each vertex \( v \) of \( S \), let \( T_v \) be the component of \( T - E(S) \) that contains \( v \). Note that \( T_v \) is trivial (that is, \( v \) is the only vertex in \( T_v \)) if every edge incident to \( v \) is in \( S \). If \( T_v \) is non-trivial then \( \deg_S(v) = d \), as otherwise \( S \) is not maximal. Let \( N \) be the set of vertices \( v \) in \( S \) such that \( T_v \) is non-trivial. Taking \( S \) with a minimum partition of each \( T_v \) into degree-\( d \) subtrees (where \( T_v \) is non-trivial) gives a partition of \( T \) into

\[
1 + \sum_{v \in N} \minpart_d(T_v)
\]

parts. By Theorem 2.1,

\[
1 + \sum_{v \in N} \minpart_d(T_v) \\
\leq 1 + \sum_{v \in N} \left( 1 + \sum_{x \in V(T_v)} \left( \left\lceil \frac{\deg_T(x)}{d} \right\rceil - 1 \right) \right) \\
= 1 + \sum_{v \in N} \left( 1 + \left( \left\lceil \frac{\deg_T(v) - d}{d} \right\rceil - 1 \right) + \sum_{x \in V(T_v - v)} \left( \left\lceil \frac{\deg_T(x)}{d} \right\rceil - 1 \right) \right) \\
= 1 + \sum_{v \in N} \left( \left\lceil \frac{\deg_T(v)}{d} \right\rceil - 1 + \sum_{x \in V(T_v - v)} \left( \left\lceil \frac{\deg_T(x)}{d} \rceil - 1 \right) \right) .
\]

Observe that if \( x \) is in \( V(T) - V(S) \), then \( x \) is in exactly one subtree \( T_v \), and this \( T_v \) is non-trivial, and \( \deg_{T_v}(x) = \deg_T(x) \). Thus

\[
1 + \sum_{v \in N} \minpart_d(T_v) \leq 1 + \sum_{x \in V(T) - V(S) \cup N} \left( \left\lceil \frac{\deg_T(x)}{d} \right\rceil - 1 \right) .
\]

If \( T_v \) is trivial then \( \deg_S(v) = \deg_T(v) \leq d \), as otherwise \( S \) is not maximal. Thus \( \left\lceil \frac{\deg_T(v)}{d} \right\rceil - 1 = 0 \). Hence

\[
1 + \sum_{v \in N} \minpart_d(T_v) \leq 1 + \sum_{x \in V(T)} \left( \left\lceil \frac{\deg_T(x)}{d} \right\rceil - 1 \right) .
\]

Hence this partition is minimum by Theorem 2.1. \( \square \)
3. Coverings by Paths

This section studies coverings of trees by degree-2 subtrees. Since a subtree is degree-2 if and only if it is a path, minpart$_2(T)$ is the minimum number of paths that cover $T$. Since each path covers at most two leaves, if $T$ has $\ell$ leaves then at least $\lceil \frac{\ell}{2} \rceil$ paths are required. Harary and Schwenk [20] first proved the converse:

**Theorem 3.1** ([2, 20]). The minimum number of paths that cover a tree with $\ell$ leaves is $\lceil \frac{\ell}{2} \rceil$.

In this section we explore this topic further. First, we investigate the total number of edges in a covering of a tree by the minimum number of paths. Let $P$ be a path in a tree $T$. Then $P$ is leafy if both endpoints of $P$ are leaves in $T$. And $P$ is a pendant path if one endpoint of $P$ is a leaf in $T$, the other endpoint of $P$ has degree at least 3 in $T$, and every internal vertex in $P$ has degree 2 in $T$.

**Theorem 3.2.** Let $T$ be a tree with $n$ vertices and $\ell$ leaves. Then $T$ has a covering by $\lceil \frac{\ell}{2} \rceil$ paths with $2n - 2 - \ell$ edges in total. Moreover, for infinitely many trees $T$, every covering of $T$ by $\lceil \frac{\ell}{2} \rceil$ paths has at least $2n - 2 - \ell$ edges in total.

**Proof.** Let $\mathcal{P}$ be a set of $\lceil \frac{\ell}{2} \rceil$ paths that cover $T$ and minimise the total number of edges. By Theorem 3.1, this is well defined. Each leaf is in exactly one path in $\mathcal{P}$ (by the minimality of $\mathcal{P}$). Each path in $\mathcal{P}$ covers at most two leaves. Thus every path in $\mathcal{P}$ is leafy, except if $\ell$ is odd, in which case, one path in $\mathcal{P}$ is a pendant path, and every other path is leafy. Also note that this pendant path shares no edge in common with another path in $\mathcal{P}$ (again by the minimality of $\mathcal{P}$).

Suppose on the contrary that some edge $e$ of $T$ is in three distinct paths $P_1, P_2, P_3 \in \mathcal{P}$. Thus each $P_i$ is leafy. Let $R$ be an edge-maximal path in $T$ that contains $e$, such that every internal vertex of $R$ has degree exactly 2. (It is possible that $e$ is the only edge in $R$.) Observe that $R$ is contained in each of $P_1, P_2, P_3$. Let $v$ and $w$ be the endpoints of $R$. Let $T_v$ and $T_w$ be the two component subtrees of $T - E(R)$, such that $v$ is in $T_v$ and $w$ is in $T_w$. Let $Q_v := T_v \cap (P_1 \cup P_2)$ and $Q_w := T_w \cap (P_1 \cup P_2)$. Thus $Q_v, Q_w, P_3$ are three paths that cover $P_1 \cup P_2 \cup P_3$, but with $2|E(R)|$ less edges in total. Hence, replacing $P_1$ and $P_2$ by $Q_v$ and $Q_w$ in $\mathcal{P}$, produces a covering of $T$ by $\lceil \frac{\ell}{2} \rceil$ paths with less edges in total. This contradiction proves that every edge in $T$ is in at most two paths in $\mathcal{P}$. Hence $\mathcal{P}$ has at most $2(n - 1)$ edges in total. Moreover, no leaf edge of $T$ is in two paths in $\mathcal{P}$. Thus $\mathcal{P}$ has at most $2n - 2 - \ell$ edges in total.

To prove that the lower bound, let $T_0$ be a subdivision of the $p$-leaf star. Say $T_0$ has $q$ vertices. Let $v_1, \ldots, v_p$ be the leaves of $T_0$. Let $T$ be the tree obtained from $T_0$ by adding two leaves $u_i$ and $w_i$ adjacent to $v_i$ for each $i \in \{1, \ldots, p\}$. So $T$ has $n := q + 2p$ vertices and $\ell := 2p$ leaves. Let $\mathcal{P}$ be a set of $p$ paths that cover $T$. Each path in $\mathcal{P}$ connects
two leaves of $T$. But no path $u_iv_iw_i$ is in $\mathcal{P}$ (otherwise $|\mathcal{P}| > p$). Hence each edge in $T_0$ is in at least two paths in $\mathcal{P}$. It follows that $\mathcal{P}$ has at least $2(q - 1) + 2p = 2n - \ell - 2$ edges in total.

We now sharpen Theorem 3.2 for trees with an even number of leaves. Let $L(T)$ be the set of leaves in a tree $T$. Let $vw$ be an edge in $T$. Let $T_v$ and $T_w$ be the components of $T − vw$ that respectively contain $v$ and $w$. Then $vw$ is said to be even–even if $|T_v \cap L(T)|$ and $|T_w \cap L(T)|$ are both even. Let $ee(T)$ be the number of even–even edges in $T$.

**Theorem 3.3.** Let $T$ be a tree with $n$ vertices and $\ell$ leaves, where $\ell$ is even. Then $T$ has a covering by $\left\lfloor \frac{\ell}{2} \right\rfloor$ paths with $n − 1 + ee(T)$ edges in total, and this covering can be computed in $O(n)$ time. Moreover, every covering of $T$ by $\left\lfloor \frac{\ell}{2} \right\rfloor$ paths has at least $n − 1 + ee(T)$ edges.

**Proof.** Let $G$ be the multigraph obtained from $T$ by adding a second copy of each even–even edge in $T$. Consider a non-leaf vertex $v$ of $T$. For each neighbour $w$ of $v$, let $T_w$ be the component of $T − v$ that contains $w$. Since $\ell$ is even, there are an even number of neighbours $w$ of $v$ such that $|V(T_w) \cap L(T)|$ is odd. If $|V(T_w) \cap L(T)|$ is even then $vw$ is doubled in $G$. Hence $v$ has even degree in $G$. Arbitrarily pair the edges incident to $v$ in $G$. By following sequences of paired edges in $G$ we obtain the desired covering of $T$. Since $G$ has $n − 1 + ee(T)$ edges, the total number of edges in the paths is $n − 1 + ee(T)$.

The numbers $|V(T_w) \cap L|$ can be computed in a single traversal of the tree. The pairing step at each vertex $v$ can be implemented in $O(deg(v))$ time, which is $O(n)$ in total. To output the paths, choose a leaf vertex $v$, find the maximal path $P$ starting at $v$ in $G$, delete the edges in $P$ from $G$, and repeat. This algorithm can be easily implemented in $O(n)$ time.

We now prove the ‘moreover’ claim. Let $\mathcal{P}$ be a set of $\left\lfloor \frac{\ell}{2} \right\rfloor$ paths that cover $T$. Each leaf is in some path in $\mathcal{P}$, and each path in $\mathcal{P}$ covers at most two leaves. Thus each leaf is in exactly one path in $\mathcal{P}$, and the endpoints of each path in $\mathcal{P}$ are leaves.

Consider an edge $vw$ that appears in only one path $P \in \mathcal{P}$. Then $P$ connects a leaf in $T_v \cap L(T)$ with a leaf in $T_w \cap L(T)$. Every other path in $\mathcal{P}$ is contained in $T_v$ or in $T_w$. Each such path has both endpoints in $T_v \cap L(T)$ or both endpoints in $T_w \cap L(T)$. Thus $|T_v \cap L(T)|$ and $|T_w \cap L(T)|$ are both odd. Hence each even–even edge is in at least two paths in $\mathcal{P}$. Therefore the total number of edges in $\mathcal{P}$ is at least $n − 1 + ee(T)$ edges.

We now show that there is a minimal covering of $T$ by paths with other properties.

**Proposition 3.4.** Let $T$ be a tree with $\ell$ leaves. Then $T$ has a covering by $\left\lceil \frac{\ell}{2} \right\rceil$ paths that have a vertex in common.

**Proof.** Let $\mathcal{P}$ be a set of $\left\lceil \frac{\ell}{2} \right\rceil$ paths in $T$ that cover every leaf in $T$, and with maximum total size. Suppose that there are disjoint paths $P$ and $Q$ in $\mathcal{P}$. Let $R$ be a minimal
path in $T$ between $P$ and $Q$. Let $v$ and $w$ be the endpoints of $R$, where $v$ is in $P$ and $w$ is in $Q$. Thus $P$ is the union of two paths $P_1$ and $P_2$ whose intersection is $v$. And $Q$ is the union of two paths $Q_1$ and $Q_2$ whose intersection is $w$. Replace $P$ and $Q$ in $\mathcal{P}$ by $P_1 \cup R \cup Q_1$ and $P_2 \cup R \cup Q_2$. We obtain a set of $\left\lceil \frac{\ell}{2} \right\rceil$ paths with greater total size than $\mathcal{P}$. This contradiction proves that the paths in $\mathcal{P}$ are pairwise intersecting. By the Helly property of subtrees of a tree, the paths in $\mathcal{P}$ have a vertex $v$ in common. \hfill \Box

**Lemma 3.5.** If $\mathcal{P}$ is a set of paths in a tree $T$ that cover every leaf, and some vertex $v$ is in every path in $\mathcal{P}$, then $\mathcal{P}$ covers every edge.

**Proof.** Suppose on the contrary that some edge $e$ is not covered by $\mathcal{P}$. Let $T_1$ and $T_2$ be the components of $T - e$. Without loss of generality, $v$ is in $T_1$. Let $P$ be a path in $\mathcal{P}$ that covers some leaf of $T$ contained in $T_2$. Then $v$ is not in $P$. This contradiction proves that $\mathcal{P}$ covers every edge. \hfill \Box

We now characterise those vertices $v$ for which there is a minimal covering by paths all containing $v$.

**Theorem 3.6.** Let $T$ be a tree with $\ell$ leaves. Let $v$ be a vertex of $T$. Then $T$ has a covering by $\left\lceil \frac{\ell}{2} \right\rceil$ paths each containing $v$ if and only if $|V(T') \cap L(T)| \leq \left\lceil \frac{\ell}{2} \right\rceil$ for every component $T'$ of $T - v$.

**Proof.** Let $G$ be the graph with vertex set $L(T)$ where two leaves $x$ and $y$ are adjacent in $G$ if and only if $x$ and $y$ are in distinct components of $T - v$. Thus $G$ is a complete $d$-partite graph, where $d = \text{deg}(v)$. Each colour class of $G$ consists of the leaves in $L(T)$ that are in a single component of $T - v$. Each pair of leaves in distinct components of $T - v$ are the endpoints of a path through $v$. In this way, each edge of $G$ corresponds to a leafy path in $T$. The paths in a covering of $T$ can be assumed to be leafy paths. Thus Lemma 3.5 implies that $T$ has a covering by $\left\lceil \frac{\ell}{2} \right\rceil$ paths each containing $v$ if and only if $G$ contains a matching of $\left\lceil \frac{\ell}{2} \right\rceil$ edges, or equivalently if $G$ contains $\left\lceil \frac{\ell}{2} \right\rceil$ disjoint $(\leq 2)$-cliques. A result of Sitton [29] (see Lemma A.1 for a generalisation) implies that this property holds if and only if each colour class in $G$ has at most $\left\lceil \frac{\ell}{2} \right\rceil$ vertices, which is equivalent to saying that $|V(T') \cap L(T)| \leq \left\lfloor \frac{\ell}{2} \right\rfloor$ for every component $T'$ of $T - v$. \hfill \Box

**Theorem 3.7.** Let $T$ be a tree with $\ell$ leaves. Let $C$ be the set of vertices $v$ in $T$ such that $T$ has a covering by $\left\lfloor \frac{\ell}{2} \right\rfloor$ paths each containing $v$. Then $C$ induces a non-empty path. Moreover, every internal vertex has degree 2 in $T$, unless $\ell$ is odd, in which case $C$ may have exactly one internal vertex $v$ with degree exactly 3, and $v$ is the endpoint of a pendant path.

**Proof.** Let $L$ be the set of leaves in $T$. Let $u, w \in C$. By Theorem 3.6, the number of leaves of $T$ in each component of $T - u$ or of $T - w$ is at most $\left\lceil \frac{\ell}{2} \right\rceil$. Let $v$ be a vertex on
the \( uv \)-path in \( T \). If \( T' \) is a component of \( T - v \), then \( T' \) is contained in a component of \( T - u \) or a component of \( T - w \). Thus the number of leaves of \( T \) in \( T' \) is at most \( \left\lceil \frac{\ell}{2} \right\rceil \).

By Theorem 3.6, \( v \in C \). Hence \( T[C] \) is connected.

Suppose that \( C \) contains a vertex \( v \) with three neighbours \( w_1, w_2, w_3 \) in \( C \). Let \( T_i \) be the component of \( T - v \) containing \( w_i \). Without loss of generality, \( |T_1 \cap L| \leq |T_2 \cap L| \leq |T_3 \cap L| \). Thus \( |T_2 \cup T_3| \cap L| \geq \frac{2}{3} \ell \). But \( T_2 \cup T_3 \) is contained in a component of \( T - w_1 \), implying \( w_1 \notin C \) by Theorem 3.6. This contradiction proves that \( T[C] \) is a path.

Now suppose that \( C \) has an internal vertex \( v \) such that \( \deg_T(v) \geq 3 \). Let \( u \) and \( w \) be the neighbours of \( v \) in \( C \). Let \( L_u \) be the set of leaves of \( T \) in the component of \( T - uw \) that contains \( u \). Let \( L_w \) be the set of leaves of \( T \) in the component of \( T - vw \) that contains \( w \). Let \( L_v \) be the number of leaves of \( T \) in the component of \( T - \{uw, vw\} \) that contains \( v \). Thus \( \ell = |L_u| + |L_v| + |L_w| \). Since \( \deg_T(v) \geq 3 \), we have \( |L_v| \geq 1 \).

Without loss of generality, \( |L_u| \leq |L_w| \). Thus \( |L_v| + |L_w| = \ell - |L_u| \geq \ell - |L_w| \), implying \( \ell \leq 2|L_w| + |L_v| \leq 2|L_w| + 2|L_v| - 1 \). Hence \( \frac{1}{2}(\ell + 1) \leq |L_w| + |L_v| \). Since \( L_v \cup L_w \) is contained in a component of \( T - u \), by Theorem 3.6, \( \left\lceil \frac{\ell}{2} \right\rceil \geq |L_w| + |L_v| \), which is a contradiction if \( \ell \) is even. If \( \ell \) is odd then \( L_v = 1 \), and by a similar argument, \( v \) is the only internal vertex of \( C \) with degree at least 3 in \( T \), and \( v \) is the endpoint of a pendant path.

\[ \square \]

Theorem 3.7 says that the set of vertices \( v \) for which \( T \) has a minimal covering by paths each containing \( v \) is somewhat like the centroid of \( T \), where we measure the ‘weight’ of a component of \( T - v \) by the number of leaves in it rather than the number of vertices.

Finally in this section, we consider the problem of covering a given tree with a small number of subtrees, each with at most \( d \) leaves. Covering by paths corresponds to the \( d = 2 \) case. The next result thus generalises Theorem 3.1 (and with a completely different proof).

**Theorem 3.8.** For every integer \( d \geq 2 \) and for every tree \( T \) with \( \ell \) leaves, the minimum number of subtrees, each with at most \( d \) leaves, that cover \( T \) is \( \left\lceil \frac{\ell}{d} \right\rceil \).

**Proof.** The lower bound is immediate. We prove the upper bound by induction on \( \ell \). Clearly we can assume that \( T \) has no vertex of degree 2. For \( S \subseteq L(T) \), let \( T[S] \) denote the subtree of \( T \) consisting of the union of all leafy paths in \( T \) whose endpoints are both in \( S \). Note that \( T[S] \) has \( |S| \) leaves. Let \( X \) be the set of vertices of \( T \) that have degree at least 3 and are adjacent to at least one leaf.

First suppose that \( |X| \geq d \). For each of \( d \) vertices \( x \in X \), choose one leaf incident to \( x \). We obtain a set \( L_0 \) of \( d \) leaves of \( T \), such that no two vertices in \( L_0 \) have a common neighbour (in \( X \)). Since vertices in \( X \) have degree at least 3, \( T - L_0 \) is a tree with \( \ell - d \) leaves. By induction, there is a covering of \( T - L_0 \) by \( \left\lceil \frac{\ell}{d} \right\rceil - 1 \) subtrees, each with at
most \( d \) leaves. With \( T[L_0] \), we obtain the desired covering of \( T \) (since every edge in \( T - V(T - L_0) \) is adjacent to a vertex in \( L_0 \), and is thus in \( T[L_0] \)).

Now assume that \( |X| \leq d \). If \( \ell < d \), then the result is trivial. Otherwise \( \ell \geq d \). Let \( L_0 \) be a set of \( d \) leaves, such that each vertex in \( X \) is adjacent to at least one leaf in \( L_0 \). Since \( T[L[T]] = T \) and every leaf has a neighbour in common with some leaf in \( L_0 \), every non-leaf edge of \( T \) is in \( T[L_0] \). Arbitrarily partition the \( \ell - d \) leaves in \( S \setminus L_0 \) into sets \( \{ L_i : 1 \leq i \leq \left\lceil \frac{\ell}{d} \right\rceil - 1 \} \) such that each \( |L_i| \leq d \). Hence \( \{ T[L_i] : 0 \leq i \leq \left\lceil \frac{\ell}{d} \right\rceil - 1 \} \) is the desired covering of \( T \).

\[ \square \]

4. An Algorithm for Covering Trees

This section describes a polynomial time algorithm to determine a minimum covering of a tree \( T \) by degree-\( d \) subtrees. Since a subtree is degree-2 if and only if it is a path, the results in this section with \( d = 2 \) generalise some of the results from Section 3.

It will be convenient to consider the following more general scenario. Let \( G \) be a connected graph. A binding function of \( G \) is a function \( f : V(G) \rightarrow \{ 2, 3, 4, \ldots \} \). A subgraph \( X \) of \( G \) is \( f \)-bound if \( \text{deg}(v) \leq f(v) \) for every vertex \( v \) of \( X \). A covering \( C \) of \( G \) is degree-\( f \) if every subgraph \( X \in C \) is \( f \)-bound. For an integer \( d \geq 2 \), a \( d \)-covering of \( G \) is a degree-\( f \) covering of \( G \), where \( f(v) := d \) for each vertex \( v \) of \( G \). Let \( \text{mincover}_f(G) \) be the minimum cardinality of a degree-\( f \) covering of \( G \). An \( f \)-bound subgraph \( X \) of \( G \) is \( f \)-maximal if no edge of \( G - E(X) \) can be added to \( X \) to obtain a new \( f \)-bound subgraph.

This section describes a polynomial time algorithm to determine \( \text{mincover}_f(T) \) and the corresponding degree-\( f \) covering for any given tree \( T \) and binding function \( f \). Observe that a subtree \( X \) of \( T \) is \( f \)-maximal if and only if \( \text{deg}_X(v) = \min\{ f(v), \text{deg}_G(v) \} \) for every vertex \( v \) of \( X \). In particular, \( v \in V(X) \) is a leaf of \( X \) if and only if \( v \) is a leaf of \( T \).

**Lemma 4.1.** Let \( v \) be a vertex of a non-trivial connected graph \( G \). Then \( G \) contains connected subgraphs \( G_1 \) and \( G_2 \) such that \( G_1 \cup G_2 = G \) and \( V(G_1) \cap V(G_2) = \{ v \} \) and \( \text{deg}_{G_1}(v) \leq \max\{ \text{deg}_G(v) - 1, 1 \} \) and \( \text{deg}_{G_2}(v) \leq \max\{ \text{deg}_G(v) - 1, 1 \} \).

**Proof.** First suppose there is a bridge edge \( vw \) incident to \( v \). Let \( A \) be the connected component of \( G - vw \) that contains \( w \). Then \( G_1 := G[V(A) \cup \{ v \}] \) and \( G_2 := G - A \) satisfy the claim. Now assume that no edge incident to \( v \) is a bridge. Let \( vw \) be an edge incident to \( v \). Then \( G_1 := G[\{ v, w \}] \) and \( G_2 := G - vw \) satisfy the claim. \( \square \)

**Lemma 4.2.** Let \( f \) be a binding function of a connected graph \( G \). Let \( s := \text{mincover}_f(G) \). Then \( G \) has a degree-\( f \) covering by \( s \) \( f \)-maximal subgraphs that are pairwise intersecting.

**Proof.** Let \( \{ U_1, \ldots, U_s \} \) be a degree-\( f \) covering such that \( \sum \{ |E(U_i)| \} \) is maximum. Then each \( U_i \) is non-trivial and \( f \)-maximal. Suppose on the contrary that \( V(U_i) \cap V(U_j) = \emptyset \)
for some $i, j$. Let $P$ be a shortest path between $U_i$ and $U_j$ in $G$, where $V(P) \cap V(U_i) = \{v\}$ and $V(P) \cap V(U_j) = \{w\}$.

By Lemma 4.1, $U_i$ contains connected subgraphs $A_1$ and $A_2$ such that $A_1 \cup A_2 = U_i$ and $V(A_1) \cap V(A_2) = \{v\}$, and $\deg_{A_1}(v) \leq \max\{\deg_{U_i}(v) - 1, 1\}$ and $\deg_{A_2}(v) \leq \max\{\deg_{U_i}(v) - 1, 1\}$. Similarly, $U_j$ contains connected subgraphs $B_1$ and $B_2$ such that $B_1 \cup B_2 = U_j$ and $V(B_1) \cap V(B_2) = \{w\}$, and $\deg_{B_1}(v) \leq \max\{\deg_{U_j}(v) - 1, 1\}$ and $\deg_{B_2}(v) \leq \max\{\deg_{U_j}(v) - 1, 1\}$.

Observe that $A_1 \cup P \cup B_1$ and $A_2 \cup P \cup B_2$ are $f$-bound subgraphs of $G$ (since the degree of $v$ in each subgraph is at most $\max\{\deg_{U_i}(v), 2\} \leq f(v)$, the degree of $w$ in each subgraph is at most $\max\{\deg_{U_j}(w), 2\} \leq f(w)$, and for each internal vertex $z$ in $P$, the degree of $z$ in each subgraph is at most $2 \leq f(z)$). Since $A_1 \cup A_2 = U_i$ and $B_1 \cup B_2 = U_j$, replacing $U_i$ and $U_j$ by $A_1 \cup P \cup B_1$ and $A_2 \cup P \cup B_2$ gives a degree-$f$ covering of $G$ with greater total size than $U_1, \ldots, U_s$. This contradiction proves that $V(U_i) \cap V(U_j) \neq \emptyset$ for $i, j \in \{1, \ldots, r\}$. \qed

By the Helly property of subtrees of a tree, Lemma 4.2 implies:

**Lemma 4.3.** Let $f$ be a binding function of a tree $T$. Let $s := \min\text{cover}_{f}(T)$. Then $G$ has a degree-$f$ covering by $s$ $f$-maximal subtrees that have a vertex in common.

Lemma 4.3 implies that to find a minimum degree-$f$ covering of a tree, it suffices to consider degree-$f$ coverings that have a common vertex in every subtree. It is therefore convenient to consider the following more general covering problem. Recall that in a rooted tree $T$ the edges are oriented away from the root vertex $r$. A **rooted covering** of $T$ is a covering $C$ of $T$ such that $r$ is in every subtree in $C$. A **binding function** of $T$ is a function $f : V(T) \rightarrow \mathbb{Z}^+$. A rooted covering $C$ of $T$ is an outdegree-$f$ if $\text{outdeg}_X(v) \leq f(v)$ for every vertex $v$ in every subtree $X \in C$. For an integer $d \geq 1$, a **degree-$d$ rooted covering** of $T$ is an outdegree-$f$ rooted covering of $T$, where $f(v) := d$ for each vertex $v$ of $T$. Let $r\text{mincover}_{f}(T)$ be the minimum cardinality of an outdegree-$f$ rooted covering of $T$. For a vertex $r$ of an unrooted tree $T$, let $r\text{mincover}_{f}(T, r)$ be the minimum cardinality of an outdegree-$f$ rooted covering of the rooted tree $(T, r)$. We now show that the problem of determining a covering of an unrooted tree can be reduced to the case of rooted trees.

**Lemma 4.4.** Let $f$ be a binding function of a (non-rooted) tree $T$. Then

$$\text{mincover}_{f}(T) = \min_{r \in V(T)} r\text{mincover}_{g}(T, r),$$

where $g$ is the binding function of $(T, r)$ defined by $g(r) := f(r)$ and $g(x) := f(x) - 1$ for every vertex $x$ of $T - r$.

**Proof.** First we prove the lower bound on $r\text{mincover}_{f}(T)$. By Lemma 4.3, there is a degree-$f$ covering $C$ of $T$ with some vertex $r$ in every subtree of $C$, and $|C| = \text{mincover}_{f}(T)$. 


Consider a subtree $X \in \mathcal{C}$. Every vertex $v \neq r$ in the rooted subtree $(X, r)$ has outdegree at most $f(v) - 1$ (since the incoming edge incident to $v$ must in $X$). Thus $v$ has outdegree at most $g(v)$ in $(X, r)$. The outdegree of $r$ in $(X, r)$ equals the degree of $r$ in $X$, which is at most $f(r) = g(r)$. Hence $\mathcal{C}$ is a rooted $g$-covering of $(T, r)$, implying

$$\mincover_f(T) = |\mathcal{C}| \geq \min_{r \in V(T)} \mincover_g(T, r).$$

Now we prove the upper bound on $\mincover_f(T)$. Let $r$ be a vertex in $T$ that minimises $\mincover_g(T, r)$. Thus there is a rooted $g$-covering $\mathcal{C}$ of $(T, r)$. Consider a rooted subtree $(X, r)$ of $\mathcal{C}$. Then $r$ is in $X$, and $\deg_X(r) = \outdeg_{(X, r)}(r)$. For every vertex $v$ of $X - r$, we have $\deg_X(v) = 1 + \outdeg_{(X, r)}(v)$. It follows that $\mathcal{C}$ is a degree-$f$ covering of $T$. Hence

$$\mincover_f(T) \leq |\mathcal{C}| = \min_{r \in V(T)} \mincover_g(T, r),$$

as desired. \qed

The next lemma determines $\mincover_f(T)$ precisely.

**Lemma 4.5.** Let $f$ be a binding function of a rooted tree $(T, r)$. Let $v_1, \ldots, v_{\deg(r)}$ be the neighbours of $r$ in $T$. For $i \in \{1, \ldots, \deg(r)\}$, let $T_i$ be the component subtree of $T - r$ that contains $v_i$. Let $c_i := \mincover_f(T_i, v_i)$, where $f$ is restricted to $V(T_i)$. Then

$$\mincover_f(T, r) = \max \left\{ \max_{1 \leq i \leq \deg(r)} c_i, \left[ \frac{1}{f(r)} \sum_{i=1}^{\deg(r)} c_i \right] \right\}.$$

**Proof.** We first prove the upper bound on $\mincover_f(T, r)$. For $i \in \{1, \ldots, \deg(r)\}$, let $\mathcal{C}_i$ be a degree-$f$ covering of $T_i$ with $|\mathcal{C}_i| = c_i$. Let $G$ be the graph with vertex set

$$V(G) := \bigcup_{i=1}^{\deg(r)} \mathcal{C}_i.$$

That is, there is one vertex in $G$ for each subtree in each covering $\mathcal{C}_i$. Two vertices in $G$ are adjacent if and only if they come from distinct $\mathcal{C}_i$. Thus $G$ is isomorphic to the complete $\deg(r)$-partite graph $K\langle c_1, \ldots, c_{\deg(r)} \rangle$.

By Lemma A.1 there is a partition $\mathcal{F}$ of $V(G)$ into

$$\max \left\{ \max_{1 \leq i \leq \deg(r)} c_i, \left[ \frac{1}{f(r)} \sum_{i=1}^{\deg(r)} c_i \right] \right\}$$

$(\leq f(r))$-cliques in $G$. Each $k$-clique $C \in \mathcal{F}$ corresponds to a set of $k$ subtrees from distinct coverings $\mathcal{C}_i$. Let $X_C$ be the subtree of $T$ induced by the union of the subtrees corresponding to $C$ plus the vertex $r$. Thus $r$ has outdegree $|C| \leq f(r)$ in $X_C$. Since each $\mathcal{C}_i$ is outdegree-$f$, every vertex $x \neq r$ in $X_C$ has outdegree at most $f(x)$ in $X_C$. Thus $\{X_C : C \in \mathcal{F}\}$ is an outdegree-$f$ covering of $(T, r)$. Hence $\mincover_f(T, r) \leq |\mathcal{F}|$, as desired.
We now prove the lower bound on $\text{rmincover}_f(T, r)$. Let $\mathcal{C}$ be an outdegree-$f$ covering of $(T, r)$ with $|\mathcal{C}| = \text{rmincover}_f(T, r)$. Let $\mathcal{X}$ be the union, taken over all $X \in \mathcal{C}$, of the set of component subtrees of $X - r$. Each subtree in $\mathcal{X}$ is contained within exactly one component subtree $T_i$ of $T - r$. For $i \in \{1, \ldots, \deg(r)\}$, let $\mathcal{X}_i$ be the set of subtrees in $\mathcal{X}$ that are contained within $T_i$.

We claim that $\mathcal{X}_i$ is an outdegree-$f$ covering of $(T_i, v_i)$. Every edge of $T_i$ is in some subtree of $\mathcal{X}_i$. For every vertex $x$ in $T_i$, we have outdeg$_{T_i}(x) = \text{outdeg}_T(x)$ (since the edge $rv_i$ is incoming at $v_i$). Thus $x$ has outdegree at most $f(x)$ in every subtree in $\mathcal{X}_i$. Thus $\mathcal{X}_i$ is an outdegree-$f$ covering of $(T_i, v_i)$. Hence $|\mathcal{X}_i| \geq \text{rmincover}_f(T_i, v_i) = c_i$.

Let $G$ be the graph with vertex set $V(G) := \mathcal{X}$, where two subtrees in $\mathcal{X}$ are adjacent in $G$ if and only if they are in distinct $\mathcal{X}_i$. Hence $G$ contains the complete $\deg(r)$-partite graph $K\langle c_1, \ldots, c_{\deg(r)}\rangle$ as a subgraph. For each $X \in \mathcal{C}$, distinct components of $X - r$ are in distinct components of $T - r$. Thus the components of $X - r$ are a $k$-clique in $G$, where $k = \text{outdeg}_X(r)$, which is at most $f(r)$. Hence $\mathcal{C}$ defines a partition of $V(G)$ into $|\mathcal{C}|$ cliques each with at most $f(r)$ vertices. By Lemma A.1,

$$\text{rmincover}_f(T, r) = |\mathcal{C}| \geq \max \left\{ \max_{1 \leq i \leq \deg(r)} c_i, \left\lceil \frac{1}{f(r)} \sum_{i=1}^{\deg(r)} c_i \right\rceil \right\} .$$

□

**Theorem 4.6.** There is a $O(n \log n)$-time algorithm that, given a binding function $f$ of a rooted $n$-vertex tree $T$, computes $\text{rmincover}_f(T)$.

**Proof.** Lemma 4.5 gives a recursive algorithm to compute $\text{rmincover}_f(T)$. Let $t(m)$ be the time complexity of this algorithm for a tree $T$ with $m$ edges. We claim that $t(m) \leq \alpha m \log m$ for some constant $\alpha$. (All logarithms are binary.) Let $r$ be the root of $T$. Let $v_1, \ldots, v_{\deg(r)}$ be the neighbours of $r$ in $T$. Let $T_i$ be the component subtree of $T - r$ that contains $v_i$. By induction, $\text{rmincover}_f(T_i, v_i)$ can be computed in $\alpha m_i \log m_i$ time, where $T_i$ has $m_i$ edges. By Lemma 4.5, $\text{rmincover}_f(T)$ can be computed by $2\deg(r)$ arithmetic steps, each operating on integers at most $m$. This computation takes $\alpha \deg(r) \log m$ time. Thus

$$t(m) \leq \alpha \deg(r) \log m + \sum_i \alpha m_i \log m_i \leq \alpha (\log m)(\deg(r)) + \sum_i m_i = \alpha m \log m .$$

The result follows since $m = n - 1$.

□

A proof analogous to that of Theorem 4.6, but also using Lemma A.1, gives:

**Theorem 4.7.** There is a $O(n^2)$-time algorithm that, given a binding function $f$ of a rooted $n$-vertex tree $T$, computes $\text{rmincover}_f(T)$ and the corresponding covering of $T$ by degree-$f$ subtrees.
We now have a polynomial time algorithm for the unrooted covering problem.

**Theorem 4.8.** There is a $O(n^2 \log n)$-time algorithm that, given a binding function $f$ of an $n$-vertex tree $T$, computes $\text{mincover}_f(T)$ and the corresponding covering of $T$ by degree-$f$ subtrees.

**Proof.** For each vertex $r$ of $T$, compute $\text{rmincover}_g(T, r)$, where $g$ is the binding function of $(T, r)$ defined by $g(r) := f(r)$ and $g(x) := f(x) - 1$ for every vertex $x$ of $T - r$. By Theorem 4.6, $\text{rmincover}_g(T, r)$ can be computed to $O(n \log n)$ time. Thus this step takes $O(n^2 \log n)$ time. Let $r$ be the vertex that minimises $\text{rmincover}_g(T, r)$. This computation takes $O(n \log n)$ time. By Lemma 4.4, $\text{mincover}_f(T) = \text{rmincover}_g(T, r)$. By Theorem 4.7, the degree-$g$ covering of $(T, r)$ can be computed in $O(n^2)$ time. By Lemma 4.4, this is an optimal degree-$f$ covering of $T$. The total time complexity is $O(n^2 \log n)$. □

5. Coverings of Complete Trees

This section applies the general methods from the previous section to determine minimum coverings of complete trees. As illustrated in Figure 1 for integers $\Delta \geq 1$ and $h \geq 1$, the **complete $\Delta$-ary rooted tree with height $h$**, denoted by $\Gamma_{\Delta, h}$, is the rooted tree such that every non-leaf vertex has out-degree $\Delta$, and the distance between the root and every leaf equals $h$. For convenience, define $\Gamma_{\Delta, 0} := K_1$. For integers $\Delta \geq 2$ and $h \geq 1$, the (non-rooted) **complete $\Delta$-ary tree with height $h$**, denoted by $\Gamma_{\Delta, h}$, is the (non-rooted) tree in which every non-leaf vertex has degree $\Delta$, and for some vertex $r$, the distance between $r$ and every leaf equals $h$. Define $\Gamma_{\Delta, 0} := K_1$.

![Figure 1. (a) $\Gamma_{2,2}$ and (b) $\Gamma_{3,2}$.](image)

Consider the following recursively defined function. For every real number $x > 0$, let $[x]_0 := 1$, and for every integer $k \geq 1$, let $[x]_k := [x \cdot [x]_{k-1}]$. Thus

$$x^k \leq [x]_k \leq [x]^k,$$

with equality whenever $x$ is an integer. As an example when equality does not hold, observe that $(\frac{3}{2})^2 = \frac{9}{4}$ and $[\frac{3}{2}]_2 = 3$ and $[\frac{3}{2}]^2 = 4$. On the other hand, $[x]_k$ is never far
from $x^k$, since $[x]_k \leq x[x]_{k-1} + 1$ implies that
$$[x]_k \leq \frac{x^{k+1} - 1}{x - 1}.$$  

**Proposition 5.1.** For all integers $\Delta \geq d \geq 1$ and $h \geq 0$,
$$\text{rmincover}_d(\Gamma_{\Delta,h}) = \left\lfloor \frac{\Delta}{d} \right\rfloor_h.$$  

**Proof.** We proceed by induction on $h$. Trivially,
$$\text{rmincover}_d(\Gamma_{\Delta,0}) = 1 = \left\lfloor \frac{\Delta}{d} \right\rfloor_0.$$  

Now assume that $h \geq 1$. Let $r$ be the root of $\Gamma_{\Delta,h}$. Observe that each of the $\Delta$ components of $\Gamma_{\Delta,h} - r$ is isomorphic to $\Gamma_{\Delta,h-1}$, rooted at the neighbour of $r$. By Lemma 4.5
$$\text{rmincover}_d(\Gamma_{\Delta,h}) = \max \left\{ \text{rmincover}_d(\Gamma_{\Delta,h-1}), \left\lfloor \frac{\Delta \cdot \text{rmincover}_d(\Gamma_{\Delta,h-1})}{d} \right\rfloor \right\}.$$  

By induction and since $\Delta \geq d$,
$$\text{rmincover}_d(\Gamma_{\Delta,h}) = \left\lfloor \frac{\Delta}{d} \cdot \left\lfloor \frac{\Delta}{d} \right\rfloor_{h-1} \right\rfloor_h = \left\lfloor \frac{\Delta}{d} \right\rfloor_h,$$  

as desired. □

**Proposition 5.2.** For all integers $\Delta \geq d \geq 1$ and $h \geq 1$,
$$\text{mincover}_d(\Gamma_{\Delta,h}) = \left\lfloor \frac{\Delta}{d} \right\rfloor_h \left\lfloor \frac{\Delta - 1}{d - 1} \right\rfloor_{h-1}.$$  

**Proof.** By Lemma 4.4
(1) $$\text{mincover}_d(\Gamma_{\Delta,h}) = \min_{r \in V(\Gamma_{\Delta,h})} \text{rmincover}_g(\Gamma_{\Delta,h}, r),$$  

where $g$ is the binding function of $\Gamma_{\Delta,h}$ defined by $g(r) := d$ and $g(x) := d - 1$ for every vertex $x \neq r$. Note that $g$ depends on the choice of $r$.

$\Gamma_{\Delta,h}$ has a vertex $v$ such that each component of $\Gamma_{\Delta,h} - v$, rooted at the neighbour of $v$, is $\Gamma_{\Delta-1,h-1}$. First we compute $\text{rmincover}_g(\Gamma_{\Delta,h}, v)$. Later we prove that $v = r$ minimises $\text{rmincover}_g(\Gamma_{\Delta,h}, r)$ in (1). Each component of $\Gamma_{\Delta,h} - v$, rooted at the neighbour of $v$, is isomorphic to $\Gamma_{\Delta-1,h-1}$. By Lemma 4.5
$$\text{rmincover}_g(\Gamma_{\Delta,h}, v) = \max \left\{ \text{rmincover}_g(\Gamma_{\Delta-1,h-1}), \left\lfloor \frac{\Delta \cdot \text{rmincover}_g(\Gamma_{\Delta-1,h-1})}{g(v)} \right\rfloor \right\}.$$  

Since $\Delta \geq d = g(v)$ and $g(x) = d - 1$ for every vertex $x \neq v$,
$$\text{rmincover}_g(\Gamma_{\Delta,h}, v) = \left\lfloor \frac{\Delta}{d} \cdot \text{rmincover}_{d-1}(\Gamma_{\Delta-1,h-1}) \right\rfloor.$$  

15
We now prove that $r := v$ minimises $r_{\mincover_g}(\Gamma_{\Delta,h}, r)$ in (1). Let $w \neq v$ be a vertex in $\Gamma_{\Delta,h}$. Then some component of $\Gamma_{\Delta,h} - w$, rooted at the neighbour of $w$, contains $\Gamma_{\Delta-1,h}^{\rightarrow}$ rooted at $v$. Thus with $g$ defined with respect to $w$,

\[
\begin{align*}
\mincover_g(\Gamma_{\Delta,h}, w) &\geq \mincover_g(\Gamma_{\Delta-1,h}^{\rightarrow}) = \mincover_{d-1}(\Gamma_{\Delta-1,h}^{\rightarrow}) \\
&\geq \left\lceil \frac{\Delta - 1}{d-1} \cdot \mincover_{d-1}(\Gamma_{\Delta-1,h}^{\rightarrow}) \right\rceil \\
&= \mincover_g(\Gamma_{\Delta,h}, v).
\end{align*}
\]

Hence $r := v$ minimises $r_{\mincover_g}(\Gamma_{\Delta,h}, r)$ in (1). Thus

\[
\mincover_g(\Gamma_{\Delta,h}) = \mincover_g(\Gamma_{\Delta,h}, v) = \left\lceil \frac{\Delta}{d} \cdot \mincover_{d-1}(\Gamma_{\Delta-1,h}^{\rightarrow}) \right\rceil.
\]

By Proposition 5.1,

\[
\mincover_{d-1}(\Gamma_{\Delta-1,h}^{\rightarrow}) = \left\lceil \frac{\Delta - 1}{d-1} \right\rceil_{h-1}.
\]

Thus

\[
\mincover_g(\Gamma_{\Delta,h}) = \left\lceil \frac{\Delta}{d} \left\lceil \frac{\Delta - 1}{d-1} \right\rceil_{h-1} \right\rceil,
\]

as desired. \hfill \Box

### 6. Coverings of Caterpillars

Consider the problem of covering a given tree with subtrees of bounded degree. Since a tree with maximum degree $d$ has at least $d$ leaves, Theorem 3.8 implies that for every integer $d \geq 2$, every tree $T$ with $\ell$ leaves can be covered by $\left\lceil \frac{\ell}{d} \right\rceil$ degree-$d$ subtrees. However, the number of leaves can be very large, even in trees that can be covered by a few subtrees of bounded degree, as we now show for caterpillars.\(^\text{3}\)

**Theorem 6.1.** For all integers $\Delta \geq d \geq 3$, every degree-$\Delta$ caterpillar $T$ has a covering by $\left\lceil \frac{\Delta - 2}{d-2} \right\rceil$ degree-$d$ subtrees. Conversely, for all integers $\Delta \geq d \geq 3$, there are infinitely many degree-$\Delta$ caterpillars $T$ such that at least $\left\lceil \frac{\Delta - 2}{d-2} \right\rceil$ subtrees are needed in every covering of $T$ by degree-$d$ subtrees.

**Proof.** Let $t := \left\lceil \frac{\Delta - 2}{d-2} \right\rceil$. We first prove that every degree-$\Delta$ caterpillar $T$ has a covering by $t$ degree-$d$ subtrees. Let $L$ be the set of leaves of $T$. Let $P$ be the path $T - L$. Consider a vertex $x$ of $P$. Thus $1 \leq \deg_P(x) \leq 2$. As illustrated in Figure 2, partition the at most $\Delta - \deg_P(x)$ leaf edges incident to $x$ into $\left\lceil \frac{\Delta - \deg_P(x)}{d-\deg_P(x)} \right\rceil$ sets each with at most $d - \deg_P(x)$ elements. Since $\left\lceil \frac{\Delta - 2}{d-2} \right\rceil \leq t$, we have partitioned the leaf edges of $T$

\(^3\)A caterpillar is a tree for which a path is obtained by deleting the leaves.
into $t$ sets, such that the union of $P$ and any one set is a degree-$d$ subtree of $T$. Every edge is in at least one such subtree.

![Figure 2. Covering a degree-6 caterpillar by two degree-4 subtrees.](image)

Now we show that this bound is best possible. For $n \geq 2t - 1$, let $T_n$ be the caterpillar obtained from the path $(u, v_1, \ldots, v_n, w)$ by adding $\Delta - 2$ leaves incident to $v_i$ for $i \in \{1, \ldots, n\}$. Thus each such vertex $v_i$ has degree $\Delta$. Every other vertex is a leaf, and $T_n$ is a degree-$\Delta$ caterpillar. Suppose on the contrary that $T_n$ can be covered by $t - 1$ degree-$d$ subtrees $F_1, \ldots, F_{t-1}$. Say a subtree $F_j$ hits a vertex $v_i$ if at least $d - 1$ leaf edges incident to $v_i$ are in $F_j$. If some vertex $v_i$ is not hit, then each subtree contains at most $d - 2$ leaf edges incident to $v_i$, which is not possible since $(t - 1)(d - 2) < \Delta - 2$. Thus each vertex $v_i$ is hit by at least one subtree. Hence the total number of hits is at least $n$. Since $n > 2(t - 1)$, some subtree $F_j$ hits at least three vertices, say $v_a, v_b, v_c$ where $1 \leq a < b < c \leq n$. Since $F_j$ is connected, $F_j$ contains the path $(v_a, v_{a+1}, \ldots, v_c)$. Thus $v_b$ has degree at least $(d - 1) + 2$ in $F_j$, which contradicts the assumption that $F_j$ has maximum degree at most $d$. □

7. Pathwidth and Rooted Coverings

While Section 4 describes an algorithm for computing minimal coverings of a given tree by degree-$d$ subtrees, this section and the next considers the following question: which classes of trees have coverings by a bounded number of degree-$d$ subtree? The results in Section 6 say that caterpillars are such a class. To answer this question more fully, the concept of pathwidth will be important. Pathwidth is an important parameter in structural and algorithmic graph theory, and can be defined in many ways. For forests we have the following recursive definition, which is easily seen to be equivalent to the standard definition in terms of path decompositions:

1. the pathwidth of $K_1$ is 0,
2. the pathwidth of a forest $F$ is the maximum pathwidth of a connected component of $F$,
3. the pathwidth of a tree $T$ is the minimum $k$ such that there exists a path $P$ of $T$ and the pathwidth of $T - V(P)$ is at most $k - 1$. 

17
Caterpillars are precisely the trees of pathwidth 1. In general, the pathwidth of an \( n \)-vertex tree can be computed in \( \mathcal{O}(n) \) time \([4, 11, 30]\), and is at most \( \mathcal{O}(\log n) \) \([5]\). To generalise Theorem 6.1 for graphs of given pathwidth, we first consider a rooted variant of the problem in Section 7, and then we consider the unrooted version in Section 8.

We now consider rooted coverings of rooted trees with given pathwidth, where (for our purposes) the pathwidth of a directed graph is defined to be the pathwidth of the underlying undirected graph. For all integers \( \Delta \geq d \geq 3 \) and \( k \geq 0 \), let \( \pi(\Delta, d, k) \) be the maximum of \( r_{\text{mincover}}(T) \), where \( T \) is an outdegree-\( \Delta \) rooted tree with pathwidth \( k \). That is, \( \pi(\Delta, d, k) \) is the minimum integer such that every outdegree-\( \Delta \) rooted tree with pathwidth \( k \) has an outdegree-\( d \) rooted covering with \( \pi(\Delta, d, k) \) subtrees. Below we show that \( \pi(\Delta, d, k) \) is finite. In particular, we prove that \( \pi(\Delta, d, k) \) satisfies the following recurrence, thus determining \( \pi(\Delta, d, k) \) precisely.

**Theorem 7.1.** For every integer \( \Delta \geq d \geq 3 \),

\[
\pi(\Delta, d, 0) = 1,
\]

and for every integer \( k \geq 1 \), if \( \Delta = d \) then

\[
\pi(\Delta, d, k) = 1,
\]

and if \( \Delta = d + 1 \) and \( t := \pi(\Delta, d, k - 1) \mod d(d - 1) \) then

\[
\pi(d + 1, d, k) = \begin{cases} 
\left\lceil \frac{d - 1}{d - 2} \cdot \pi(\Delta, d, k - 1) - \frac{2}{d - 2} \left\lceil \frac{\pi(\Delta, d, k - 1)}{d(d - 1)} \right\rceil \right\rceil & \text{if } t < (d - 1)(d - 2) \\
\left\lceil \frac{d}{d - 1} \cdot \pi(\Delta, d, k - 1) + \left\lceil \frac{\pi(\Delta, d, k - 1)}{d(d - 1)} \right\rceil \right\rceil & \text{if } t \geq (d - 1)(d - 2)
\end{cases}
\]

and if \( \Delta \geq d + 2 \) then,

\[
\pi(\Delta, d, k) = \left\lceil \frac{\Delta - 2}{d} \cdot \pi(\Delta, d, k - 1) + \frac{2}{d} \left\lceil \frac{\Delta - 1}{d - 1} \cdot \pi(\Delta, d, k - 1) \right\rceil \right\rceil.
\]

First observe that \( \pi(d, d, k) = 1 \) since a tree covers itself, and that \( \pi(\Delta, d, 0) = 1 \) since the only tree with pathwidth 0 is \( K_1 \). We now prove the upper bound on \( \pi(\Delta, d, k) \), which indeed shows that \( \pi(\Delta, d, k) \) is finite and well defined. The next lemma will facilitate our inductive proof.

**Lemma 7.2.** For every vertex \( r \) of a tree \( T \) (with at least one edge) there is a degree-3 subtree \( H \) of \( T \) such that:

- the pathwidth of \( T - V(H) \) is less than the pathwidth of \( T \),
- \( r \) is in \( H \) and \( \deg_H(r) \in \{1, 2\} \),
- there is at most one vertex of \( H \) with degree 3, and
- if there is a vertex of \( H \) with degree 3, then \( \deg_H(r) = 1 \).

**Proof.** By definition, there is a path \( P \) of \( T \), such that the pathwidth of \( T - V(P) \) is less than the pathwidth of \( T \). Extend \( P \) so that it has at least one edge. Let \( Q \) be the
(possibly empty) path from \( r \) to \( P \) in \( T \). It is easily verified that \( H = P \cup Q \) satisfies the lemma.

\[ \square \]

**Proof of Upper Bound in Theorem 7.1** for \( \Delta \geq d + 2 \). To prove an upper bound on \( \pi(\Delta, d, k) \) we construct the desired rooted covering of a given outdegree-\( \Delta \) rooted tree with pathwidth \( k \). We proceed by induction on \( k \geq 1 \). For the base case, suppose that \( k = 1 \). Then \( \pi(\Delta, d, 0) = 1 \) and \( t = 1 < (d - 1)(d - 2) \). Thus the theorem claims that \( \pi(\Delta, d, 1) = \left\lceil \frac{\Delta - 2}{d - 2} \right\rceil \), which is proved in Theorem 6.1. Now assume that \( k \geq 2 \). Let \( T \) be an outdegree-\( \Delta \) rooted tree with pathwidth \( k \). Let \( r \) be the root of \( T \). By Lemma 7.2, there is a degree-3 subtree \( H \) of \( T \) such that the pathwidth of \( T - V(H) \) is at most \( k - 1 \), \( r \) is in \( H \) with degree 1 or 2, there is at most one vertex of \( H \) with degree 3, and if there is a vertex of \( H \) with degree 3, then \( \deg_H(r) = 1 \). Thus, if \( H \) inherits the orientation of \( T \), then \( H \) has outdegree at most 2, and at most one vertex in \( H \) has outdegree 2. As a shorthand, define

\[
\pi := \pi(\Delta, d, k - 1) \quad \text{and} \quad \Lambda := \left\lceil \frac{\Delta - 1}{d - 1} \pi \right\rceil .
\]

For each vertex \( v \) in \( H \), let \( T_v \) be the component of \( T - E(H) \) that contains \( v \). Thus \( T_v \) is rooted at \( v \) (in the orientation of \( T \)), as illustrated in Figure 3.

![Figure 3](image-url)

**Figure 3.** Construction in the proof of the upper bound in Theorem 7.1.

We now determine a rooted covering of \( T_v \). For each neighbour \( w \) of \( v \) in \( T_v \), the component of \( T_v - v \) that contains \( w \) has pathwidth at most \( k - 1 \). Thus by induction this component (rooted at \( w \)) has a rooted covering by \( \pi \) subtrees.
Case 0. \( \text{outdeg}_H(v) = 0 \): Let \( f \) be the binding function of \( T_v \) defined by \( f(x) := d \) for every vertex \( x \) in \( T_v \). By Lemma 4.5, \( T_v \) has an outdegree-\( d \) rooted covering \( C_v \), where

\[
|C_v| \leq \left\lfloor \frac{\Delta + 1}{d} \right\rfloor \leq \Lambda.
\]

Case 1. \( \text{outdeg}_H(v) = 1 \): Let \( f \) be the binding function of \( T_v \) defined by \( f(v) := d \) and \( f(x) := d \) for every vertex \( x \) in \( T_v - v \). Since \( v \) has outdegree at most \( \Delta - 1 \) in \( T_v \), by Lemma 4.5 applied to \( f \), \( T_v \) has an outdegree-\( d \) rooted covering \( C_v \), such that \( v \) has outdegree at most \( d - 1 \) in each subtree in \( C_v \), and

\[
|C_v| \leq \left\lfloor \frac{\Delta - 1 + 1}{d - 1} \right\rfloor = \Lambda.
\]

Case 2. \( \text{outdeg}_H(v) = 2 \): By induction, each component of \( T_v - v \), rooted at the neighbour of \( v \), has an outdegree-\( d \) rooted covering consisting of \( \pi \) subtrees. Let \( G \) be the graph with one vertex for each subtree in the coverings of the components of \( T_v - v \), where two vertices are adjacent if the corresponding subtrees come from distinct components. Since \( v \) has outdegree at most \( \Delta - 2 \) in \( T_v \), \( G \) is isomorphic to a subgraph of the Turan \((\Delta - 2)\)-partite graph with \( \pi \) vertices in each colour class. Apply Corollary A.3 with \( n = (\Delta - 2)\pi \) and \( p = d - 2 \) and \( q = d \) and \( m = \Lambda \). (Observe that \( \Delta \geq d + 2 \) implies that \( \Delta - 2 \geq p, q \), and thus, Corollary A.3 is applicable.) Hence there is a partition of \( V(G) \) into \( \Lambda \) \((d - 2)\)-cliques and

\[
\left\lfloor \frac{\max\{(\Delta - 2)\pi - (d - 2)\Lambda, 0\}}{d} \right\rfloor (\leq d)\text{-cliques}.
\]

Since \( \Delta \geq d \) we have \((d - 1)(\Delta - 2)\pi \geq (d - 2)(\Delta - 1)\pi \). Thus \((d - 1)(\Delta - 2)\pi + d(d - 1) > (d - 2)(\Delta - 1)\pi + (d - 1)(d - 2)\pi \). That is, \((\Delta - 2)\pi + d > (d - 2)(\Delta - 1)\pi + 1 > (d - 2)\pi \). Hence \((\Delta - 2)\pi - (d - 2)\Lambda \geq -d \), implying \( \left\lfloor \frac{1}{d} (\Delta - 2)\pi - (d - 2)\Lambda \right\rfloor \geq 0 \). Hence \( \frac{1}{d} \max\{(\Delta - 2)\pi - (d - 2)\Lambda, 0\} \) is a partition of \( V(G) \) into \( \Lambda \) \((d - 2)\)-cliques and

\[
\left\lfloor \frac{(\Delta - 2)\pi - (d - 2)\Lambda}{d} \right\rfloor (\leq d)\text{-cliques}.
\]

Using the method in Lemma 4.5, it follows that \( T_v \) has an outdegree-\( d \) rooted covering \( C_v \cup D_v \), such that \( |C_v| \leq \Lambda \) and \( v \) has outdegree \( d - 2 \) in each subtree in \( C_v \); and

\[
|D_v| \leq \left\lfloor \frac{(\Delta - 2)\pi - (d - 2)\Lambda}{d} \right\rfloor,
\]

and \( v \) has outdegree at most \( d \) in each subtree in \( D_v \).

Note that \( |C_v| \leq \Lambda \) for every vertex \( v \) of \( H \). For \( i \in \{1, \ldots, \Lambda\} \), let \( X_i \) be the union, taken over every vertex \( v \) in \( H \), of the \( i \)-th subtree in \( C_v \). Observe that in the construction in Case \((j)\), \( \text{outdeg}_{H(v)} = j \) and \( v \) has outdegree at most \( d - j \) in each subtree in \( C_v \). Thus \( X_i \cup H \) has outdegree at most \( d \), and \( X_i \cup H \) contains \( r \).
Suppose that \( \text{outdeg}_H(v) = 2 \) for some vertex \( v \) in \( H \). Let \( Q \) be the directed path from \( r \) to \( v \) in \( H \). (Note that it is possible that \( v = r \).) Then for every subtree \( Y \in \mathcal{D}_v \), \( Y \cup Q \) has outdegree at most \( d \) (since no outgoing edges incident to \( v \) are in \( Q \), and every other vertex in \( Q \) has outdegree 1 in \( Y \cup Q \), and \( 1 < d \).)

Observe that every edge of \( T \) is in some subtree \( X_i \cup H \) (where \( 1 \leq i \leq \Lambda \)) or some subtree \( Y \cup Q \) (where \( Y \in \mathcal{D}_v \) and \( \text{outdeg}_H(v) = 2 \)). Hence

\[
\{ X_i \cup H : 1 \leq i \leq \Lambda \} \cup \{ Y \cup Q : Y \in D_v \}
\]

is an outdegree-\( d \) rooted covering of \( T \). Therefore

\[
\pi(\Delta, d, k) \leq \Lambda + \left\lceil \frac{(\Delta - 2)\pi - (d - 2)\Lambda}{d} \right\rceil = \left\lceil \frac{2\Lambda + (\Delta - 2)\pi}{d} \right\rceil,
\]
as desired. \( \square \)

**Proof of Upper Bound in Theorem 7.1 for \( \Delta = d + 1 \)** We proceed by induction on \( k \geq 1 \). Let \( \pi := \pi(\Delta, d, k - 1) \). For the base case, suppose that \( k = 1 \). Then \( \pi = 1 \) and \( t = 1 < (d - 1)(d - 2) \). Thus the theorem claims that \( \pi(d + 1, d, 1) = \left\lceil \frac{\Delta - 2}{d - 2} \right\rceil \), which is proved in Theorem 6.1. Now assume that \( k \geq 2 \). Let \( T \) be an outdegree-\( \Delta \) rooted tree with pathwidth \( k \). Let \( r \) be the root of \( T \). By definition, there is an (undirected) path \( P \) in \( T \), such that the pathwidth of \( T - V(P) \) is less than \( k \). Let \( Q \) be the shortest path in \( T \) from \( r \) to a vertex in \( P \). Let \( H := P \cup Q \). Let \( s \) be the vertex in \( P \cap Q \). Note that it is possible that \( r \in P \), in which case \( s = r \). Let \( P_1 \) and \( P_2 \) be the subpaths of \( P \) such that \( P_1 \cap P_2 = \{ s \} \) and \( P_1 \cup P_2 = P \). Each \( P_i \) is a directed path starting at \( s \). Let \( Q_i := Q \cup P_i \) for \( i \in \{1, 2\} \). Each \( Q_i \) is a directed path starting at \( r \).

For each vertex \( v \) in \( H \), let \( T_v \) be the component of \( T - E(H) \) that contains \( v \). Thus \( T_v \) is rooted at \( v \) (in the orientation of \( T \)). We now determine a rooted covering of \( T_v \). For each neighbour \( w \) of \( v \) in \( T_v \), the component of \( T_v - v \) that contains \( w \) has pathwidth at most \( k - 1 \). Thus by induction this component (rooted at \( w \)) has a rooted covering by \( \pi \) subpaths.

Let \( t := \pi \mod (d - 1) \). Define

\[
y := \begin{cases} 
\frac{\pi}{d(d - 1)} & \text{if } t < (d - 1)(d - 2) \\
\frac{\pi}{d - 1} & \text{if } t \geq (d - 1)(d - 2)
\end{cases},
\]

and

\[
x := \begin{cases} 
\frac{d - 1}{d - 2} \pi - \frac{2d - 2}{d - 2} y & \text{if } t < (d - 1)(d - 2) \\
\frac{d - 1}{d - 1} \pi - y & \text{if } t \geq (d - 1)(d - 2)
\end{cases}.
\]

It is easily verified that \( (d - 2)x + 2(d - 1)y \geq (\Delta - 2)\pi \) and \( (d - 1)(x + y) \geq (\Delta - 1)\pi \) and \( d(x + y) \geq \Delta \pi \) and \( (d - 1)\pi + d - 2 > 2y(d - 1) \).
Case 0. outdeg$_H(v) = 0$: Then $v \in Q_i$ for some $i \in \{1, 2\}$, and $v$ has outdegree at most $\Delta$ in $T_v$. By Lemma 4.5, $T_v$ has an outdegree-$d$ rooted covering $C_v \cup D^i_v$, where

$$|C_v \cup D^i_v| \leq \left\lceil \frac{\Delta}{d} \right\rceil \leq x + y.$$ 

Hence we can choose $C_v$ and $D^i_v$ so that $|C_v| \leq x$ and $|D^i_v| \leq y$.

Case 1. outdeg$_H(v) = 1$: Then $v \in Q_i$ for some $i \in \{1, 2\}$, and $v$ has outdegree at most $\Delta - 1$ in $T_v$. By Lemma 4.5, $T_v$ has an outdegree-$d$ rooted covering $C_v \cup D^i_v$, such that $v$ has outdegree at most $d - 1$ in each subtree in $C_v \cup D^i_v$, and

$$|C_v \cup D^i_v| \leq \left\lceil \frac{\Delta - 1}{d - 1} \right\rceil \leq x + y.$$ 

Hence we can choose $C_v$ and $D^i_v$ so that $|C_v| \leq x$ and $|D^i_v| \leq y$.

Case 2. outdeg$_H(s) = 2$: By induction, each component of $T_s - s$, rooted at the neighbour of $s$, has an outdegree-$d$ rooted covering consisting of $\pi$ subtrees. Let $G$ be the graph with one vertex for each subtree in the coverings of the components of $T_s - s$, where two vertices are adjacent if the corresponding subtrees come from distinct components. Since $s$ has outdegree at most $\Delta - 2$ in $T_s$, $G$ is isomorphic to a subgraph of the Turan $(\Delta - 2)$-partite graph with $\pi$ vertices in each colour class.

Apply Corollary A.3 with $n = (\Delta - 2)\pi = (d - 1)\pi$ and $p = d - 1$ and $q = d - 2$ and $m = 2y$. (Observe that $\Delta = d + 1$ implies that $\Delta - 2 \geq p, q$, and thus, Corollary A.3 is applicable.) Thus there is a partition of $V(G)$ into $2y$ ($\leq d - 1$)-cliques and

$$\left\lceil \frac{\max\{(d - 1)\pi - (d - 1)2y, 0\}}{d - 2} \right\rceil \leq 1,$$

($\leq d - 2$)-cliques. Since $(d - 1)\pi - 2y(d - 1) > 2 - d$, we have $\frac{(d - 1)\pi - 2y(d - 1)}{d - 2} > 1$, implying

$$\left\lceil \frac{(d - 1)\pi - 2y(d - 1)}{d - 2} \right\rceil \geq 0.$$

Thus

$$\left\lceil \frac{\max\{(d - 1)\pi - (d - 1)2y, 0\}}{d - 2} \right\rceil = \left\lceil \frac{(d - 1)\pi - (d - 1)2y}{d - 2} \right\rceil \leq x.$$ 

Hence there is a partition of $V(G)$ into $2y$ ($\leq d - 1$)-cliques and $x$ ($\leq d - 2$)-cliques. Using the method in Lemma 4.5, it follows that $T_s$ has an outdegree-$d$ rooted covering $C_s \cup D^1_s \cup D^2_s$, such that $|D^1_s| \leq y$ and $s$ has outdegree at most $d - 1$ in each subtree in $D^1_s$; and $|D^2_s| \leq y$ and $s$ has outdegree at most $d - 1$ in each subtree in $D^2_s$; and $|C_s| \leq x$ and $s$ has outdegree at most $d - 2$ in each subtree in $C_s$.

Observe that $|C_v| \leq x$ for every vertex $v$ of $H$. For $j \in \{1, \ldots, x\}$, let $C_j$ be the union, taken over every vertex $v$ in $H$, of the $j$-th subtree in $C_v$. Observe that in the construction in Case ($\ell$), outdeg$_H(v) = \ell$ and $v$ has outdegree at most $d - \ell$ in each subtree in $C_v$. Thus $C_j \cup H$ has outdegree at most $d$. By construction, $C_j \cup H$ is connected and contains $r$. 

22
For $i \in \{1, 2\}$, observe that $|D^i_v| \leq y$ for every vertex $v$ of $Q_i$. For $j \in \{1, \ldots, y\}$, let $D^j_i$ be the union, taken over every vertex $v$ in $Q_i$, of the $j$-th subtree in $D^i_v$. Observe that for $\ell \in \{0, 1\}$, in the construction in Case $(\ell)$, outdeg$_Q(v) = \ell$ and $v$ has outdegree at most $d - \ell$ in each subtree in $D^i_v$. In the construction in Case 2, outdeg$_Q(s) = 1$ and $v$ has outdegree at most $d - 1$ in each subtree in $D^i_v$. Thus $D^j_i \cup Q_i$ has outdegree at most $d$. By construction, $D^j_i \cup Q_i$ is connected and contains $r$.

Observe that every edge of $T$ is in some subtree $C_j \cup H$ (where $1 \leq j \leq x$) or some subtree $D^j_i \cup Q_i$ (where $i \in \{1, 2\}$ and $1 \leq j \leq y$). Hence

$$\{C_j \cup H : 1 \leq j \leq x\} \cup \{D^j_i \cup Q_i : i \in \{1, 2\}, 1 \leq j \leq y\}$$

is an outdegree-$d$ rooted covering of $T$. Therefore

$$\pi(\Delta, d, k) \leq x + 2y = \left\lfloor d \frac{\pi}{d-1} + \left\lfloor \frac{\pi}{d(d-1)} \right\rfloor \right\rfloor \quad \text{if } t < (d - 1)(d - 2)$$

$$\left\lceil d \frac{\pi}{d-1} - \frac{2}{d-2} \left\lfloor \frac{\pi}{d(d-1)} \right\rfloor \right\rceil \quad \text{if } t \geq (d - 1)(d - 2),$$

as desired. 

**Proof of Lower Bound in Theorem 7.1.** To prove a lower bound on $\pi(\Delta, d, k)$ we construct an outdegree-$\Delta$ rooted tree with pathwidth $k$ that requires many subtrees in every outdegree-$d$ covering. For all integers $n_1, \ldots, n_k$, where each $n_i \geq \pi(\Delta, d, i) + 1$, we construct a tree $T\langle n_1, \ldots, n_k \rangle$ with the desired property. (This statement is well-defined since we have already proved that $\pi(\Delta, d, i)$ is bounded from above.) The number of vertices in $T\langle n_1, \ldots, n_k \rangle$ increases with the $n_i$. Hence there are, in fact, infinitely many such trees. Each tree has a nominated root vertex $r$, which has out-degree $\Delta$. Every non-leaf vertex has outdegree $\Delta$ or $\Delta - 1$.

The tree $T\langle n_1, \ldots, n_k \rangle$ is constructed recursively as follows, starting from the path

$$P := (v_{-n}, \ldots, v_{-1}, v_0, v_1, \ldots, v_n),$$

where $n := n_k$. If $k = 1$ then, add $\Delta - 2$ leaf vertices adjacent to $v_0$, and for each vertex $v_i$ in $P$ with $1 \leq |i| \leq n$, add $\Delta - 1$ leaf vertices adjacent to $v_i$. If $k \geq 2$ then, connect $v_0$ to the root vertex in each of $\Delta - 2$ copies of $T\langle n_1, \ldots, n_{k-1} \rangle$, and for each vertex $v_i$ in $P$ with $1 \leq |i| \leq n$, connect $v_i$ to the root vertex in each of $\Delta - 1$ copies of $T\langle n_1, \ldots, n_{k-1} \rangle$. Root $T\langle n_1, \ldots, n_k \rangle$ at $r := v_0$. Thus $v_{-n}$ and $v_n$ have outdegree $\Delta - 1$, and every other vertex $v_i$ has outdegree $\Delta$. By construction, $T\langle n_1, \ldots, n_k \rangle$ has pathwidth $k$.

By the definition of $\pi$, $T\langle n_1, \ldots, n_k \rangle$ has an outdegree-$d$ rooted covering $C = \{F_1, \ldots, F_{\pi(\Delta, d, k)}\}$, and $r$ is in each $F_i$. We classify these subtrees depending on which edges in $F_i \cap P$ are incident to $r$. Let $C^{++}$ be the set of subtrees $F_i \in C$ such that the edges $rv_{-1}$ and $rv_1$ are both in $F_i \cap P$. Let $C^{+-}$ be the set of subtrees $F_i \in C$ such that the edge $rv_{-1}$ is the only edge incident to $r$ in $F_i \cap P$. Let $C^{-+}$ be the set of subtrees $F_i \in C$ such that the
edge \( rv_1 \) is the only edge incident to \( r \) in \( F_i \cap P \). Let \( C^- \) be the set of subtrees \( F_i \in C \) such that the edges \( rv_{-1} \) and \( rv_1 \) are both not in \( F_i \cap P \). Hence

\[
|C^++| + |C^-| + |C^+| + |C^-| = \pi(\Delta, d, k) \tag{2}
\]

For each subtree \( F_i \in C \), let \( F_{i,1}, \ldots, F_{i,s_i} \) be the component subtrees of \( F_i - V(P) \). Let

\[
\mathcal{F} := \{F_{i,j} : 1 \leq i \leq \pi(\Delta, d, k), 1 \leq j \leq s_i \}.
\]

Each subtree \( F_{i,j} \in \mathcal{F} \) is contained in exactly one copy of \( T\langle n_1, \ldots, n_{k-1} \rangle \) in \( T\langle n_1, \ldots, n_k \rangle \). Consider a copy \( T' \) of \( T\langle n_1, \ldots, n_{k-1} \rangle \). Say \( x \) is the root of \( T' \), and \( v_t \) is the neighbour of \( x \) in \( P \). We say that \( v_t \) is the attachment point of \( T' \) and of each subtree \( F_{i,j} \in \mathcal{F} \) that is contained in \( T' \). Since \( r \) is in \( F_i \), the path between every vertex in \( F_{i,j} \) and \( r \) is in \( F_i \). This path includes \( x \), which is thus in each \( F_{i,j} \). Since \( F_i \) has outdegree at most \( d \), each \( F_{i,j} \) has outdegree at most \( d \). Thus the set of subtrees in \( \mathcal{F} \) that are contained in \( T' \) form an outdegree-\( d \) rooted covering of \( T' \). Let \( \pi := \pi(\Delta, d, k-1) \). By induction, at least \( \pi \) subtrees in \( \mathcal{F} \) are contained in \( T' \).

Now partition the subtrees in \( \mathcal{F} \) according to their attachment point in \( P \). Let \( \mathcal{F}^0 \) be the set of subtrees in \( \mathcal{F} \) whose attachment point is \( v_0 \). Let \( \mathcal{F}^+ \) be the set of subtrees in \( \mathcal{F} \) whose attachment point is \( v_i \) for some \( i \in \{1, \ldots, n\} \). Let \( \mathcal{F}^- \) be the set of subtrees in \( \mathcal{F} \) whose attachment point is \( v_i \) for some \( i \in \{-n, \ldots, -1\} \).

There are \( \Delta - 2 \) copies of \( T\langle n_1, \ldots, n_{k-1} \rangle \) that attach at \( v_0 \), each of which contain at least \( \pi \) subtrees in \( \mathcal{F} \). Thus \( |\mathcal{F}^0| \geq \pi \cdot (\Delta - 2) \). For each \( F_i \in C^+ \), since \( v_0 \) has outdegree 2 in \( F_i \cap P \) and outdegree at most \( d \) in \( F_i \), there are at most \( d \cdot 2 \) component subtrees of \( F_i - P \) that are in \( \mathcal{F}^0 \). Similarly, for each \( F_i \in C^- \cup C^+ \), since \( v_0 \) has outdegree 1 in \( F_i \cap P \), there are at most \( d \cdot 1 \) component subtrees of \( F_i - P \) that are in \( \mathcal{F}^0 \). Finally, for each \( F_i \in C^- \), there are at most \( d \) component subtrees of \( F_i - P \) that are in \( \mathcal{F}^0 \). Hence

\[
|\mathcal{F}^0| \leq \pi \cdot (\Delta - 2) \leq (d - 2) \cdot |C^+| + (d - 1) \cdot (|C^-| + |C^+|) + d \cdot |C^-| \tag{3}
\]

There are \( \Delta - 1 \) copies of \( T\langle n_1, \ldots, n_{k-1} \rangle \) that attach at \( v_i \) for \( i \in \{1, \ldots, n\} \), each of which contain at least \( \pi \) subtrees in \( \mathcal{F} \). Thus

\[
|\mathcal{F}^+| \geq \pi \cdot (\Delta - 1)n \tag{4}
\]

For each \( F_i \in C^+ \cup C^- \), the subtree consisting of those edges in \( F_i \) whose source endpoint is in \( \{v_1, \ldots, v_n\} \) is an outdegree-\( d \) caterpillar rooted at \( v_1 \) whose spine is contained in \( \{v_1, \ldots, v_n\} \). Every outdegree-\( d \) caterpillar rooted at the endpoint of its spine and whose spine has at most \( n \) vertices has at most \( (d - 1)n + 1 \) leaves. Thus there are at most \( (d - 1)n + 1 \) component subtrees of \( F_i - P \) that are in \( \mathcal{F}^+ \). For each \( F_i \in C^- \cup C^+ \), no component subtrees of \( F_i - P \) are in \( \mathcal{F}^+ \). Thus by \([1]\),

\[
\pi \cdot (\Delta - 1)n \leq |\mathcal{F}^+| \leq ((d - 1)n + 1) \cdot (|C^+| + |C^-|) \tag{5}
\]

24
Hence
\[ \pi \cdot (\Delta - 1) - (d - 1) \cdot (|C^{++}| + |C^{-+}|) \leq \frac{1}{n} \cdot (|C^{++}| + |C^{-+}|) . \]

By (2) and since \( n = n_k > \pi(\Delta, d, k) \),
\[ \pi \cdot (\Delta - 1) - (d - 1) \cdot (|C^{++}| + |C^{-+}|) \leq \left\lfloor \frac{\pi(\Delta, d, k)}{n} \right\rfloor = 0 . \]

Thus
\[(5) \quad \pi \cdot (\Delta - 1) \leq (d - 1) \cdot (|C^{++}| + |C^{-+}|) . \]

By symmetry,
\[(6) \quad \pi \cdot (\Delta - 1) \leq (d - 1) \cdot (|C^{++}| + |C^{-+}|) . \]

Observe that (3), (5) and (6) define an integer linear program with unknowns \(|C^{++}|, |C^{-+}|, |C^{+-}|, |C^{--}|\). The solution of this integer linear program is given in Lemma B.1, where
\[
x = |C^{++}|, \quad y_1 = |C^{-+}|, \quad y_2 = |C^{+-}|, \quad z = |C^{--}|, \quad A = \pi \cdot (\Delta - 2), \text{ and } \quad B = \pi \cdot (\Delta - 1) .
\]

Since \( \Delta \geq d \) we have \((d - 2)B \leq (d - 1)A\), and Lemma B.1 is applicable. Equation (2) and Lemma B.1 imply that
\[
\pi(\Delta, d, k) = |C^{++}| + |C^{+-}| + |C^{-+}| + |C^{--}|
\geq \left\lfloor \frac{\Delta - 2}{d} \cdot \pi + 2 \left\lfloor \frac{\Delta - 1}{d - 1} \cdot \pi \right\rfloor \right\rfloor .
\]

This complete the proof of the lower bound when \( \Delta \geq d + 2 \).

For \( \Delta = d + 1 \) the above analysis can be slightly improved as follows. Observe that for each \( F_i \in C^{-+} \), there are at most \( d - 1 \) component subtrees of \( F_i - P \) that are in \( \mathcal{F}_0 \) (rather than \( d \) component subtrees in the general case). Hence (3) can be strengthened to:
\[(7) \quad \pi \cdot (\Delta - 2) \leq |\mathcal{F}_0| \leq (d - 2) \cdot |C^{++}| + (d - 1) \cdot (|C^{-+}| + |C^{+-}| + |C^{--}|) . \]

Now consider the integer linear program with unknowns \(|C^{++}|, |C^{+-}|, |C^{-+}|, |C^{--}|\) that is defined in (5), (6) and (7). The solution of this integer linear program is given in Lemma B.2, where
\[
x = |C^{++}|, \quad y_1 = |C^{-+}|, \quad y_2 = |C^{+-}|, \quad z = |C^{--}|, \quad \text{and } A = \pi .
\]
Equation (2) and Lemma B.2 imply that
\[
\pi(\Delta, d, k) = |C^{+}| + |C^{-}| + |C^{+}| + |C^{-}|
\]
\[
\geq \begin{cases} 
\left\lfloor \frac{d-1}{d-2} \cdot \pi - \frac{2}{d-2} \left\lceil \frac{\pi}{d(d-1)} \right\rceil \right\rfloor & \text{if } t < (d - 1)(d - 2) \\
\left\lfloor \frac{d}{d-1} \cdot \pi + \left\lceil \frac{\pi}{d(d-1)} \right\rceil \right\rfloor & \text{if } t \geq (d - 1)(d - 2) 
\end{cases}
\]
as desired in the case that \(\Delta \geq d + 1\). \qed

This completes the proof of Theorem 7.1. We can estimate the recurrence in Theorem 7.1 as follows.

**Corollary 7.3.** For all integers \(\Delta \geq d \geq 2\) and \(k \geq 0\),
\[
\pi(\Delta, d, k) \leq \left\lfloor \frac{\Delta - 2}{d} + \frac{2}{d} \left\lceil \frac{\Delta - 1}{d - 1} \right\rceil \right\rfloor^k,
\]
with equality whenever \(\Delta \equiv d^2 - 2d + 2 \pmod{d^2 - d}\).

**Proof.** It is easily proved that \(\left\lfloor \frac{a}{c} \right\rfloor \leq a \left\lceil \frac{b}{c} \right\rceil\) for all integers \(a, b, c \geq 1\). Applying this observation twice, Theorem 7.1 implies that
\[
\pi(\Delta, d, k) \leq \pi(\Delta, d, k - 1) \cdot \left\lfloor \frac{\Delta - 2}{d} + \frac{2}{d} \left\lceil \frac{\Delta - 1}{d - 1} \right\rceil \right\rfloor.
\]
Since \(\pi(\Delta, d, 0) = 1\),
\[
(8) \quad \pi(\Delta, d, k) \leq \left\lfloor \frac{\Delta - 2}{d} + \frac{2}{d} \left\lceil \frac{\Delta - 1}{d - 1} \right\rceil \right\rfloor^k.
\]
Now assume that \(\Delta \equiv d^2 - 2d + 2 \pmod{d^2 - d}\). Then \(\frac{\Delta - 1}{d - 1} \in \mathbb{Z}\) and \(\frac{\Delta - 2}{d} \in \mathbb{Z}\). (In fact, the converse holds.) Thus Theorem 7.1 implies that
\[
\pi(\Delta, d, k) = \pi(\Delta, d, k - 1) \cdot \left( \frac{\Delta - 2}{d} + \frac{2}{d} \cdot \frac{\Delta - 1}{d - 1} \right).
\]
Thus equality in (8) holds since \(\pi(\Delta, d, 0) = 1\). \qed

### 8. Pathwidth and Unrooted Coverings

This section extends the results in Section 7 to the unrooted setting.

**Theorem 8.1.** For all integers \(\Delta \geq d \geq 3\), every degree-\(\Delta\) tree \(T\) with pathwidth \(k\) satisfies \(\text{mincover}_d(T) \leq t\), where
\[
t := \left\lfloor \frac{\Delta - 2}{d - 2} \cdot \pi \right\rfloor \quad \text{and} \quad \pi := \pi(\Delta - 1, d - 1, k - 1).
\]

Moreover, there are infinitely many degree-\(\Delta\) trees \(T\) with pathwidth \(k\) such that \(\text{mincover}_d(T) = t\).
**Proof.** First we prove the upper bound. \( T \) has a path \( P \) such that \( T - V(P) \) has pathwidth \( k - 1 \). Consider a vertex \( v \) of \( P \). Let \( T_v \) be the subtree of \( T - E(P) \) that contains \( v \), where \( T_v \) is rooted at \( v \). Let \( f \) be the binding function of \( T_v \) defined by \( f(v) := d - \deg_P(v) \) and \( f(x) := d - 1 \) for every other vertex \( x \). Each component \( U \) of \( T_v - v \), rooted at the neighbour of \( v \), has outdegree at most \( \Delta - 1 \) and pathwidth at most \( k - 1 \). Thus \( \text{rmincover}_{d-1}(U) \leq \pi \). Since \( v \) has outdegree at most \( \Delta - \deg_P(v) \) in \( T_v \), by Lemma 4.5 applied to \( f \), \( T_v \) has an outdegree-(\( d - 1 \)) rooted covering \( C_v \), such that \( v \) has outdegree at most \( d - \deg_P(v) \) in each subtree in \( C_v \), and

\[
|C_v| \leq \left\lfloor \frac{\Delta - \deg_P(v)}{d - \deg_P(v)} \cdot \pi \right\rfloor \leq t ,
\]

where the last inequality holds since \( \deg_P(v) \in \{0, 1, 2\} \). For \( i \in \{1, \ldots, t\} \), let \( X_i \) be the union, taken over every vertex \( v \) in \( P \), of the \( i \)-th subtree in \( C_v \) (if it exists). Thus every vertex \( v \) in \( P \) has degree at most \( d - \deg_P(v) \) in \( X_i \), and \( v \) has degree at most \( d \) in \( X_i \cup P \). Since every vertex not in \( P \) has outdegree at most \( d - 1 \) in each \( X_i \), every vertex not in \( P \) has degree at most \( d \) in each \( X_i \). Every edge of \( T \) is in some \( X_i \cup P \). Hence \( \{X_i \cup P : 1 \leq i \leq t\} \) is the desired degree-\( d \) covering of \( T \).

Now we prove the lower bound. Let \( X \) be the outdegree-(\( \Delta - 1 \)) rooted tree with pathwidth \( k - 1 \), such that \( \text{rmincover}_{d-1}(X) = \pi \). (See the proof of the lower bound in Theorem 7.1 for the construction of \( X \).) Let \( n \geq t - 1 \). Let \( T \) be the tree obtained from the path \( P = (v_{-n-1}, v_{-n}, \ldots, v_n, v_{n+1}) \) as follows. For each \( i \in \{-n, \ldots, n\} \), add \( \Delta - 2 \) copies of \( X \) whose roots are adjacent to \( v_i \); thus \( v_i \) has degree \( \Delta \). Hence \( T \) is a degree-\( \Delta \) tree with pathwidth \( k \).

Suppose on the contrary that \( T \) can be covered by \( t - 1 \) degree-\( d \) subtrees. By Lemma 4.3, \( T \) has a degree-\( d \) covering by \( t - 1 \) degree-\( d \) maximal subtrees \( F_1, \ldots, F_{t-1} \) that have a vertex \( r \) in common. Root \( T \) at \( r \). Define \( f(r) := d \) and \( f(x) := d - 1 \) for every other vertex \( x \). Thus \( F_1, \ldots, F_{t-1} \) is a degree-\( f \) covering of the rooted tree \((T, r)\), and \( \text{rmincover}_d(T) = \text{rmincover}_f(T, r) \). Lemma 4.5 provides a recursive formula for \( \text{rmincover}_f(T, r) \), which implies (by the symmetry of \( T \)) that without loss of generality, \( r = v_0 \). In particular, for each copy of \( X \) rooted at some vertex \( w \), every subtree in the induced covering of \( X \) contains \( w \).

Fix \( i \in \{-n, \ldots, n\} \). Let \( E_i \) be the set of \( \Delta - 2 \) edges in \( T - E(P) \) incident to \( v_i \). For each edge \( v_i w \in E_i \), at least \( \pi \) of the subtrees \( F_1, \ldots, F_{t-1} \) intersect the copy of \( X \) rooted at \( w \), and each such subtree contains \( w \). Since \( f(w) = d - 1 \) and each such subtree \( F_j \) is maximal, the edge \( vw_i \) is also in \( F_j \). Thus \( \sum_j |F_j \cap E_i| \geq (\Delta - 2) \pi \). Say a subtree \( F_j \) hits \( v_i \) if \( |F_j \cap E_i| \geq d - 1 \). If \( v_i \) is hit by no subtree, then \( |F_j \cap E_i| \leq d - 2 \) for all \( j \), implying

\[
(t-1)(d-2) \geq \sum_{j=1}^{t-1} |F_j \cap E_i| \geq (\Delta - 2) \pi .
\]
This is a contradiction since $t < \frac{\Delta - 2}{d - 2} \pi + 1$. Thus $v_i$ is hit by at least one subtree.

Hence the total number of hits is at least $2n + 1$. Since $2n + 1 > 2(t - 1)$, some subtree $F_j$ hits at least three vertices, say $v_a, v_b, v_c$ where $-n < a < b < c < n$. Since $F_j$ is connected, $F_j$ contains the path $(v_a, v_{a+1}, \ldots, v_c)$. Thus $v_b$ has degree at least $(d - 1) + 2$ in $F_j$, which contradicts the assumption that $F_j$ has maximum degree at most $d$. Therefore at least $t$ subtrees are needed in every covering of $T$ by degree-$d$ subtrees.

Theorem 8.1 says that trees with bounded maximum degree and bounded pathwidth admit coverings by a bounded number of degree-$d$ subtrees. In the case of $d = 3$, we now prove a converse result for a large class of trees.

**Proposition 8.2.** Let $T$ be a tree in which every non-leaf vertex has degree at least 4. Then $T$ has pathwidth at most $\mincover_3(T)$.

**Proof.** We proceed by induction on $c := \mincover_3(T)$. If $c = 1$ then no vertex has degree at least 4, and every non-leaf vertex has degree at least 4, implying $T \cong K_2$, which has pathwidth 1. Now assume that $c \geq 2$. Fix a covering of $T$ by $c$ degree-3 subtrees $T_1, \ldots, T_c$. Let $S := \bigcap_{i=1}^c T_i$. Since $T_1$ has maximum degree at most 3, $S$ has maximum degree at most 3. Suppose that $\deg_S(v) = 3$ for some vertex $v$. By assumption, $\deg_T(v) \geq 4$, implying there is an edge $vw \not\in S$. Since $vw \in E(T_i)$ for some $i$, we have $\deg_{T_i}(v) \geq 4$, and $T_i$ is not degree-3. This contradiction proves that $\deg_S(v) \leq 2$ for every vertex $v$. Since $T_1$ is connected, $S$ is connected. Thus $S$ is a path. For each edge $vw$ of $T$ such that $v \in V(S)$ and $w \not\in V(S)$, let $T_w$ be the subtree of $T - V(S)$ that contains $w$. Since $vw \not\in E(S)$, at most $c - 1$ of the subtrees $T_1, \ldots, T_c$ contain $vw$. Since each such subtree is connected, $T_w$ is covered by at most $c - 1$ subtrees. That is, $\mincover_3(T_w) \leq c - 1$. By induction, the pathwidth of $T_w$ is at most $c - 1$. Therefore the pathwidth of $T$ is at most $c$. $\square$

Theorem 8.1 and Proposition 8.2 together say that pathwidth is the right parameter to study when considering coverings of trees by a bounded number of degree-3 subtrees.

**9. Coverings of General Graphs**

This section considers coverings of general graphs by connected subgraphs of bounded degree.

A *connected vertex cover* of a graph $G$ is a connected subgraph $H$ of $G$ such that every edge of $G$ has at least one endpoint in $H$; that is, $E(G - V(H)) = \emptyset$. For algorithmic aspects of connected vertex covers, see [12, 15, 18, 25].

**Lemma 9.1.** Let $H$ be a connected vertex cover of a graph $G$. Let $$k := \Delta(G - E(H)) .$$
Then for every integer \( d \geq \Delta(H) + 1 \), there is a covering of \( G \) by

\[
\left\lfloor \frac{k + 1}{d - \Delta(H)} \right\rfloor
\]

connected degree-\( d \) subgraphs.

**Proof.** By Vizing’s Theorem \[35\] applied to \( G - E(H) \), there is a partition \( \{E_i : 1 \leq i \leq k + 1\} \) of \( E(G) - E(H) \), such that each \( E_i \) is a matching in \( G - E(H) \). Grouping the matchings gives a partition \( \{F_j : 1 \leq j \leq \left\lceil \frac{k + 1}{d - \Delta(H)} \right\rceil\} \) of \( E(G) - E(H) \), such that each \( F_j \) is a degree-\((d - \Delta(H))\) subgraph of \( G - E(H) \). Thus \( H \cup F_j \) is a connected degree-\( d \) subgraph of \( G \), and \( \{H \cup F_j : 1 \leq j \leq \left\lceil \frac{k + 1}{d - \Delta(H)} \right\rceil\} \) is the desired covering of \( G \). \( \square \)

**Corollary 9.2.** Let \( H \) be a connected spanning subgraph of a graph \( G \). Then for every integer \( d \geq \Delta(H) + 1 \), there is a covering of \( G \) by

\[
\left\lfloor \frac{\Delta(G) - \delta(H) + 1}{d - \Delta(H)} \right\rfloor
\]

connected degree-\( d \) subgraphs.

**Proof.** The result follows from Lemma \[9.1\] with \( k \leq \Delta(G) - \delta(H) \). \( \square \)

**Corollary 9.3.** For every integer \( d \geq 3 \), every Hamiltonian graph \( G \) has a covering by

\[
\left\lfloor \frac{\Delta(G) - 1}{d - 2} \right\rfloor
\]

connected degree-\( d \) subgraphs.

**Proof.** Apply Corollary \[9.2\] with a Hamiltonian cycle \( H \) of \( G \). Then \( \Delta(H) = \delta(H) = 2 \). The result follows. \( \square \)

This result can be slightly strengthened for \( d = 4 \).

**Proposition 9.4.** For all \( \epsilon > 0 \) there is an integer \( \Delta_0 \) such that every Hamiltonian graph \( G \) with \( \Delta(G) \geq \Delta_0 \) has a covering by

\[
\left\lceil (\frac{1}{2} + \epsilon)(\Delta(G) - 2) \right\rceil
\]

connected degree-4 subgraphs.

**Proof.** A forest is linear if each component is path. The linear arboricity of a graph \( G \) is the minimum number of linear forests that partition \( E(G) \). Alon \[1\] proved that for all \( \epsilon > 0 \) there is an integer \( \Delta_0 \) such that every graph \( G \) with \( \Delta(G) \geq \Delta_0 \) has linear arboricity at most \( \left\lceil (\frac{1}{2} + \epsilon)(\Delta(G)) \right\rceil \). Apply this result to \( G - E(C) \) where \( C \) is a Hamiltonian cycle in \( G \). We obtain a partition \( \mathcal{F} \) of \( E(G) - E(C) \) into \( \left\lceil (\frac{1}{2} + \epsilon)(\Delta(G) - 2) \right\rceil \) linear forests. Thus \( \{C \cup F : F \in \mathcal{F}\} \) is a covering of \( G \) by degree-4 subtrees. \( \square \)
Now consider coverings of planar graphs by connected subgraphs of bounded degree. Tutte [34] proved that every 4-connected planar graph is Hamiltonian. Thus Corollary 9.3 implies the next result.

**Corollary 9.5.** For every integer \( d \geq 3 \), every 4-connected planar graph \( G \) has a covering by

\[
\left\lceil \frac{\Delta(G) - 1}{d - 2} \right\rceil
\]

connected degree-\( d \) subgraphs.

**Proof.** Barnette [3] proved that \( G \) has a degree-3 spanning tree \( H \). The result follows from Corollary 9.2 with \( \Delta(H) = 3 \) and \( \delta(H) = 1 \).

Note that various generalisations of the above-mentioned result by Barnette [3] for graphs embedded on surfaces [6, 10, 16, 23, 28, 33, 36] can be applied to obtain similar results to Corollary 9.6. We omit the details.

We conclude with an open problem: Is there a function \( f \) and constants \( c \) and \( d \) such that every \( c \)-connected graph \( G \) has a covering by \( f(\Delta(G)) \) connected degree-\( d \) subgraphs? We now show that the answer is negative for \( c = 2 \) and \( d = 2 \) (even for outerplanar graphs).

**Proposition 9.7.** For all \( k \geq 2 \) there is a 2-connected outerplanar graph with maximum degree 3 that requires at least \( k \) subgraphs in every covering by degree-2 connected subgraphs.

**Proof.** Let \( m := 2k \) and \( n := 4k \). Let \( H \) be the graph obtained from disjoint paths \((a_1, \ldots, a_m)\) and \((b_1, \ldots, b_m)\) by adding the edge \( a_ib_i \) for all \( i \in [1, m] \). Each edge \( a_ib_i \) is called a cross edge, and \( a_1b_1 \) is called the base edge. As illustrated in Figure 4 let \( G \) be the graph obtained from a cycle \((v_1, \ldots, v_{2n})\) and \( n \) copies \( H_1, \ldots, H_n \) of \( H \) by identifying the base edge of \( H_j \) with the edge \( v_{2j-1}v_{2j} \) for each \( j \in [1, n] \). Observe that \( G \) is 2-connected and outerplanar, and has maximum degree 3. Let \( X_1, \ldots, X_t \) be a covering of \( G \) by connected degree-2 subgraphs (that is, paths and cycles). To complete the proof we now show that \( t \geq k \). Say \( X_i \) occupies \( H_j \) if \( X_i \) contains at least two cross edges in \( H_j \). Observe that if \( X_i \) occupies \( H_j \), then either \( X_i \) is a cycle contained in \( H_j \), or \( X_i \) is a path and it has an endpoint in \( H_j \). Thus each \( X_i \) occupies at most two \( H_j \) subgraphs, implying \( X_i \) contains less than \( n + 2m \) cross edges. Since there are \( nm \) cross edges in total, \( t > \frac{nm}{n+2m} = k \). \( \square \)
This question seems related to a result by Chen et al. [7], who proved that every 3-connected graph $G$ with $n \geq 4$ vertices and maximum degree at most $d \geq 3$ contains a cycle of length at least $n^{\log_2 b} + 2$, where $b = 2(d-1)^2 + 1$.

REFERENCES

[1] Noga Alon. The linear arboricity of graphs. *Israel J. Math.*, 62(3):311–325, 1988. [http://dx.doi.org/10.1007/BF02783300](http://dx.doi.org/10.1007/BF02783300).

[2] Kiyoshi Ando, Atsusi Kaneko, and Severino Gervacio. The bandwidth of a tree with $k$ leaves is at most $\lceil k/2 \rceil$. *Discrete Math.*, 150(1-3):403–406, 1996. [http://dx.doi.org/10.1016/0012-365X(96)00205-1](http://dx.doi.org/10.1016/0012-365X(96)00205-1).

[3] David W. Barnette. Trees in polyhedral graphs. *Canad. J. Math.*, 18:731–736, 1966.

[4] Hans L. Bodlaender. A linear-time algorithm for finding tree-decompositions of small treewidth. *SIAM J. Comput.*, 25(6):1305–1317, 1996. [http://dx.doi.org/10.1137/S0097539795292119](http://dx.doi.org/10.1137/S0097539795292119).

[5] Hans L. Bodlaender. A partial $k$-arboretum of graphs with bounded treewidth. *Theoret. Comput. Sci.*, 209(1-2):1–45, 1998. [http://dx.doi.org/10.1016/S0304-3975(97)00228-4](http://dx.doi.org/10.1016/S0304-3975(97)00228-4).

[6] Richard Brunet, Mark N. Ellingham, Zhicheng Gao, Alice Metzlar, and R. Bruce Richter. Spanning planar subgraphs of graphs in the torus and Klein bottle. *J. Combin. Theory Ser. B*, 65(1):7–22, 1995. [http://dx.doi.org/10.1006/jctb.1995.1041](http://dx.doi.org/10.1006/jctb.1995.1041).

[7] Guantao Chen, Jun Xu, and Xingxing Yu. Circumference of graphs with bounded degree. *SIAM J. Comput.*, 33(5):1136–1170, 2004. [http://dx.doi.org/10.1137/S0097539703436473](http://dx.doi.org/10.1137/S0097539703436473).
[8] Nathaniel Dean and Mekkia Kouider. Gallai’s conjecture for disconnected graphs. *Discrete Math.*, 213(1-3):43–54, 2000. [http://dx.doi.org/10.1016/S0012-365X(99)00167-3](http://dx.doi.org/10.1016/S0012-365X(99)00167-3).

[9] Alan Donald. An upper bound for the path number of a graph. *J. Graph Theory*, 4(2):189–201, 1980. [http://dx.doi.org/10.1002/jgt.3190040207](http://dx.doi.org/10.1002/jgt.3190040207).

[10] Mark N. Ellingham and Zhicheng Gao. Spanning trees in locally planar triangulations. *J. Combin. Theory Ser. B*, 61(2):178–198, 1994. [http://dx.doi.org/10.1006/jctb.1994.1043](http://dx.doi.org/10.1006/jctb.1994.1043).

[11] John A. Ellis, I. Hal Sudborough, and Jonathan S. Turner. The vertex separation and search number of a graph. *Inform. and Comput.*, 113(1):50–79, 1994. [http://dx.doi.org/10.1006/inco.1994.1064](http://dx.doi.org/10.1006/inco.1994.1064).

[12] Bruno Escoffier, Laurent Gourves, and Jérôme Monnot. Complexity and approximation results for the connected vertex cover problem in graphs and hypergraphs. *J. Discrete Algorithms*, 8(1):36–49, 2010. [http://dx.doi.org/10.1016/j.jda.2009.01.005](http://dx.doi.org/10.1016/j.jda.2009.01.005).

[13] Genghua Fan. Path covers of weighted graphs. *J. Graph Theory*, 19(1):131–136, 1995. [http://dx.doi.org/10.1002/jgt.3190190114](http://dx.doi.org/10.1002/jgt.3190190114).

[14] Genghua Fan. Path decompositions and Gallai’s conjecture. *J. Combin. Theory Ser. B*, 93(2):117–125, 2005. [http://dx.doi.org/10.1016/j.jctb.2004.09.008](http://dx.doi.org/10.1016/j.jctb.2004.09.008).

[15] Toshihiro Fujito and Takashi Doi. A 2-approximation NC algorithm for connected vertex cover and tree cover. *Inform. Process. Lett.*, 90(2):59–63, 2004. [http://dx.doi.org/10.1016/j.ipl.2004.01.011](http://dx.doi.org/10.1016/j.ipl.2004.01.011).

[16] Zhicheng Gao and Nicholas C. Wormald. Spanning Eulerian subgraphs of bounded degree in triangulations. *Graphs Combin.*, 10(2):123–131, 1994. [http://dx.doi.org/10.1007/BF02986656](http://dx.doi.org/10.1007/BF02986656).

[17] Jiong Guo, Rolf Niedermeier, and Sebastian Wernicke. Parameterized complexity of generalized vertex cover problems. In *Algorithms and data structures*, vol. 3608 of *Lecture Notes in Comput. Sci.*, pp. 36–48. Springer, 2005.

[18] Jiong Guo, Rolf Niedermeier, and Sebastian Wernicke. Parameterized complexity of vertex cover variants. *Theory Comput. Syst.*, 41(3):501–520, 2007. [http://dx.doi.org/10.1007/s00224-007-1309-3](http://dx.doi.org/10.1007/s00224-007-1309-3).

[19] Frank Harary. Covering and packing in graphs. I. *Ann. New York Acad. Sci.*, 175:198–205, 1970.

[20] Frank Harary and Allen J. Schwenk. Evolution of the path number of a graph: Covering and packing in graphs. II. In *Graph theory and computing*, pp. 39–45. Academic Press, New York, USA, 1972.

[21] Anthony J. W. Hilton and Dominique de Werra. A sufficient condition for equitable edge-colourings of simple graphs. *Discrete Math.*, 128(1–3):179–201,
[22] Ferran Hurtado, Guiseppe Liotta, and David R. Wood. Proximity drawings of high-degree trees. 2010.

[23] Ken-ichi Kawarabayashi, Atsuhiro Nakamoto, and Katsuhiro Ota. Subgraphs of graphs on surfaces with high representativity. *J. Combin. Theory Ser. B*, 89(2):207–229, 2003. http://dx.doi.org/10.1016/S0095-8956(03)00072-8

[24] László Lovász. On covering of graphs. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pp. 231–236. Academic Press, New York, USA, 1968.

[25] Daniel Mölle, Stefan Richter, and Peter Rossmanith. Enumerate and expand: improved algorithms for connected vertex cover and tree cover. *Theory Comput. Syst.*, 43(2):234–253, 2008. http://dx.doi.org/10.1007/s00224-007-9089-3

[26] Juhani Nieminen. Some observations on coverings of graphs. *Glasnik Mat. Ser. III*, 10(30)(1):3–8, 1975.

[27] László Pyber. Covering the edges of a connected graph by paths. *J. Combin. Theory Ser. B*, 66(1):152–159, 1996. http://dx.doi.org/10.1006/jctb.1996.0012

[28] Daniel P. Sanders and Yue Zhao. On 2-connected spanning subgraphs with low maximum degree. *J. Combin. Theory Ser. B*, 74(1):64–86, 1998. http://dx.doi.org/10.1006/jctb.1998.1836

[29] David Sitton. Maximum matchings in complete multipartite graphs. *Furman University Electronic J. Undergraduate Math.*, 2:6–16, 1996.

[30] Konstantin Skodinis. Construction of linear tree-layouts which are optimal with respect to vertex separation in linear time. *J. Algorithms*, 47(1):40–59, 2003. http://dx.doi.org/10.1016/S0196-6774(02)00225-0

[31] Ralph G. Stanton, D. D. Cowan, and Lee O. James. Some results on path numbers. In *Proc. Louisiana Conf. on Combinatorics, Graph Theory and Computing*, pp. 112–135. Louisiana State Univ., Baton Rouge, USA, 1970.

[32] Ralph G. Stanton, Lee O. James, and D. D. Cowan. Tripartite path numbers. In *Graph theory and computing*, pp. 285–294. Academic Press, New York, USA, 1972.

[33] Carsten Thomassen. Trees in triangulations. *J. Combin. Theory Ser. B*, 60(1):56–62, 1994. http://dx.doi.org/10.1006/jctb.1994.1005

[34] William T. Tutte. A theorem on planar graphs. *Trans. Amer. Math. Soc.*, 82:99–116, 1956. http://dx.doi.org/10.2307/1992980

[35] Vadim G. Vizing. On an estimate of the chromatic class of a p-graph. *Diskret. Analiz No.*, 3:25–30, 1964.
APPENDIX A. COMPLETE MULTIPARTITE GRAPHS

For a graph $G$ and integer $d \geq 1$, let $\text{numcliques}_d(G)$ be the minimum number of disjoint $(\leq d)$-cliques of $G$ that partition $V(G)$. For example, $\text{numcliques}_1(G) = |V(G)|$, and $\text{numcliques}_2(G) = |V(G)| - p$, where $p$ is the number of edges in a maximum matching in $G$. Let $K\langle n_1, \ldots, n_k \rangle$ be the complete $k$-partite graph with $n_i$ vertices in the $i$-th colour class. We now determine $\text{numcliques}_d(K\langle n_1, \ldots, n_k \rangle)$.

**Lemma A.1.** For all integers $k \geq d \geq 1$ and $n_1, \ldots, n_k \geq 0$,

$$\text{numcliques}_d(K\langle n_1, \ldots, n_k \rangle) = \max \left\{ \max_{1 \leq i \leq k} n_i, \left\lfloor \frac{1}{d} \sum_{i=1}^k n_i \right\rfloor \right\}.$$

Moreover, there is an $O(\sum_{i=1}^k n_i)$ time algorithm to compute a partition of $K\langle n_1, \ldots, n_k \rangle$ into this many $(\leq d)$-cliques.

**Proof.** Since each vertex in the $i$-th colour class is in a distinct clique of the partition, $\text{numcliques}_d(K\langle n_1, \ldots, n_k \rangle) \geq n_i$. Since every vertex is in some clique of the partition, $d \cdot \text{numcliques}_d(K\langle n_1, \ldots, n_k \rangle) \geq \sum_{i=1}^k n_i$. Thus proves the lower bound on $\text{numcliques}_d(K\langle n_1, \ldots, n_k \rangle)$.

It remains to prove the upper bound. We proceed by induction on $d + \sum_{i=1}^k n_i$. Assume that $n_1 \geq \cdots \geq n_k \geq 0$. If $d = 1$ then

$$\text{numcliques}_1(K\langle n_1, \ldots, n_k \rangle) = \sum_{i=1}^k n_i = \max \left\{ n_1, \sum_{i=1}^k n_i \right\},$$

as desired. Now assume that $d \geq 2$. First suppose that $n_d = 0$. Then

$$\text{numcliques}_d(K\langle n_1, \ldots, n_k \rangle) = \text{numcliques}_{d-1}(K\langle n_1, \ldots, n_{d-1} \rangle),$$

and by induction

$$\text{numcliques}_d(K\langle n_1, \ldots, n_k \rangle) \leq \max \left\{ n_1, \left\lfloor \frac{1}{d-1} \sum_{i=1}^{d-1} n_i \right\rfloor \right\} = n_1 \leq \max \left\{ n_1, \left\lfloor \frac{1}{d} \sum_{i=1}^k n_i \right\rfloor \right\}.$$

Now assume that $n_d \geq 1$. Let $C$ be a set with exactly one vertex from each of the $d$ largest colour classes. So $C$ is a $d$-clique, and

$$\text{numcliques}_d(K\langle n_1, \ldots, n_k \rangle) \leq 1 + \text{numcliques}_d(K\langle n_1-1, \ldots, n_{d-1}, n_{d+1}, \ldots, n_k \rangle).$$

[36] XINGXING YU. Disjoint paths, planarizing cycles, and spanning walks. Trans. Amer. Math. Soc., 349(4):1333–1358, 1997. [http://dx.doi.org/10.1090/S0002-9947-97-01830-8]
Suppose that $n_1 = \cdots = n_{d+1} \geq 1$. Thus by (9) and induction

$$\text{numcliques}_d(K\langle n_1, \ldots, n_k \rangle) \leq 1 + \max \left\{ n_{d+1}, \left\lfloor \frac{1}{d} \left( \sum_{i=1}^{k} n_i \right) - d \right\rfloor \right\}$$

$$= \max \left\{ 1 + n_1, \left\lfloor \frac{1}{d} \sum_{i=1}^{k} n_i \right\rfloor \right\} .$$

Observe that

$$\left\lfloor \frac{1}{d} \sum_{i=1}^{k} n_i \right\rfloor \geq \left\lfloor \frac{(d+1)n_1}{d} \right\rfloor = n_1 + \left\lceil \frac{n_1}{d} \right\rceil \geq n_1 + 1 .$$

Thus

$$\text{numcliques}_d(K\langle n_1, \ldots, n_k \rangle) \leq \left\lfloor \frac{1}{d} \sum_{i=1}^{k} n_i \right\rfloor = \max \left\{ n_1, \left\lfloor \frac{1}{d} \sum_{i=1}^{k} n_i \right\rfloor \right\} ,$$

as desired. Now assume that $n_{d+1} < n_d$. Hence by (9) and induction,

$$\text{numcliques}_d(K\langle n_1, \ldots, n_k \rangle) \leq 1 + \max \left\{ n_1 - 1, \left\lfloor \frac{1}{d} \left( \sum_{i=1}^{k} n_i \right) - d \right\rfloor \right\}$$

$$= \max \left\{ n_1, \left\lfloor \frac{1}{d} \sum_{i=1}^{k} n_i \right\rfloor \right\} ,$$

as desired. It is easily seen that this proof can be adapted to give a greedy linear-time algorithm to compute the partition, where at each stage, a $d$-clique is repeatedly chosen from the $d$ largest colour classes. $\square$

Note that the case $d = 2$ in Lemma A.1 also follows from a result by Sitton [29], who determined the size of the largest matching in $K\langle n_1, \ldots, n_k \rangle$. The Turan graph $K\langle n; k \rangle$ is the complete $k$-partite graph $K\langle n_1, \ldots, n_k \rangle$ where $n = \sum_{i=1}^{k} n_i$ and $|n_i - n_j| \leq 1$ for $i, j \in \{1, \ldots, k\}$.

**Corollary A.2.** For all integers $k \geq d \geq 1$,

$$\text{numcliques}_d(K\langle n; k \rangle) = \left\lceil \frac{n}{d} \right\rceil .$$

Moreover, there is $O(n)$ time algorithm to compute a partition of $K\langle n; k \rangle$ into $\left\lceil \frac{n}{d} \right\rceil \leq d$-cliques.

**Proof.** Let $x$ and $y$ be integers such that $n = xk + y$ where $0 \leq y \leq k - 1$. Then

$$K\langle n; k \rangle \cong K\langle x, \ldots, x, x+1, \ldots, x+1 \rangle .$$

By Lemma A.1

$$\text{numcliques}_d(K\langle n; k \rangle) = \begin{cases} \max\{x, \left\lceil \frac{n}{d} \right\rceil \} & \text{if } y = 0 , \\ \max\{x + 1, \left\lceil \frac{n}{d} \right\rceil \} & \text{if } y \geq 1 . \end{cases}$$
If \( y = 0 \) then \( n = xk \geq xd \) and \( \frac{n}{d} \geq x \). If \( y \geq 1 \) then \( n \geq xk + 1 \geq xd + 1 \) and \( \left\lceil \frac{n}{d} \right\rceil \geq x + 1 \). In both cases, \( \text{numcliques}_d(K(n; k)) = \left\lceil \frac{n}{d} \right\rceil \).

In the proof of the upper bound in Theorem 7.1, we need the following result about partitioning Turan graphs into cliques of two specified sizes.

**Corollary A.3.** For all integers \( n, k, p, q, m \), such that \( n, k \geq 1 \) and \( k \geq p, q \geq 0 \) and \( m \geq 0 \), there is a vertex partition of the Turan graph \( K(n; k) \) into \( m (\leq p) \)-cliques and \( \left\lceil \frac{\max(n-mp,0)}{q} \right\rceil (\leq q) \)-cliques.

**Proof.** We proceed by induction on \( m \). If \( m = 0 \) then by Corollary A.2 \( K(n; k) \) has a partition into \( \left\lceil \frac{n}{q} \right\rceil (\leq q) \)-cliques, as desired. Now assume that \( m \geq 1 \). If \( n \leq p \) then \( K(n; k) \cong K_n \) has a partition into one \((\leq p)\)-clique and zero \((\leq q)\)-cliques, as desired. Now assume that \( n \geq p + 1 \). Let \( C \) be a \( p \)-clique with exactly one vertex from each of the \( p \) largest colour classes of \( K(n; k) \). This is well-defined since \( p \leq k \). Then \( K(n; k) - C \cong K(n-p; k) \). By induction, there is a vertex partition of \( K(n-p; k) \) into \( m-1 (\leq p) \)-cliques and \( \left\lceil \frac{n-p-(m-1)p}{q} \right\rceil (\leq q) \)-cliques. Since \( \left\lceil \frac{n-p-(m-1)p}{q} \right\rceil = \left\lceil \frac{n-mp}{q} \right\rceil \), with \( C \), we have a vertex partition of \( K(n; k) \) into \( m (\leq p) \)-cliques and \( \left\lceil \frac{n-mp}{q} \right\rceil (\leq q) \)-cliques. □

**Appendix B. Integer Linear Programs**

This appendix contains a solution to the integer linear program that arose in the proof of the lower bound in Theorem 7.1.

**Lemma B.1.** Fix integers \( A, B \geq 1 \) and \( d \geq 2 \) such that \( A \leq B \) and \( (d-2)B \leq (d-1)A \). Suppose that some non-negative integers \( x, y_1, y_2, z \) satisfy

\[
\begin{align*}
(10) & \quad (d-2)x + (d-1)(y_1 + y_2) + dz \geq A \\
(11) & \quad (d-1)(x + y_1) \geq B \\
(12) & \quad (d-1)(x + y_2) \geq B.
\end{align*}
\]

Then

\[
 x + y_1 + y_2 + z \geq \left\lceil \frac{A}{d} + \frac{2B}{d(d-1)} \right\rceil.
\]

This bound is achievable, for example by

\[
x := \left\lceil \frac{B}{d-1} \right\rceil, \quad y_1 := y_2 := 0, \quad z := \left\lceil \frac{A-x(d-2)}{d} \right\rceil.
\]

**Proof.** Say \((x, y_1, y_2, z)\) is a solution if (10), (11) and (12) are satisfied. A solution is optimal if it minimises \( x + y_1 + y_2 + z \). Suppose that \((x, y_1, y_2, z)\) is a solution, where
y_1 \geq y_2$. We claim that \((x + y_2, 0, 0, z + y_1)\) is also a solution. By (10) and since \(y_1 \geq y_2\),

\[
(d - 2)(x + y_2) + (d - 1)(0 + 0) + d(z + y_1)
\]

\[
= (d - 2)x + (d - 2)y_2 + dy_1 + dz
\]

\[
\geq (d - 2)x + (d - 1)(y_1 + y_2) + dz
\]

\[
\geq A .
\]

Thus \((x + y_2, 0, 0, z + y_1)\) satisfies (10). By (12) and since \(y_1 \geq y_2\),

\[
(d - 1)(x + y_1) \geq (d - 1)(x + y_2) \geq B .
\]

Thus \((x + y_2, 0, 0, z + y_1)\) satisfies (11) and (12). Hence \((x + y_2, 0, 0, z + y_1)\) is also a solution, as claimed. Since

\[
x + y_1 + y_2 + z = (x + y_2) + 0 + 0 + (z + y_1),
\]

there is an optimal solution \((x, 0, 0, z)\). By (11) or (12),

\[
x = x_i := \left\lfloor \frac{B}{d - 1} \right\rfloor + i
\]

for some integer \(i \geq 0\). By (10),

\[
z \geq z_i := \left\lfloor \frac{A - (d - 2)x_i}{d} \right\rfloor .
\]

Now,

\[
x + z \geq x_i + z_i = \left\lfloor \frac{A + 2x_i}{d} \right\rfloor \geq \left\lfloor \frac{A + 2x_0}{d} \right\rfloor = x_0 + z_0 .
\]

Thus if \((x_0, 0, 0, z_0)\) is a solution, then it is optimal. Thus it suffices to prove that \((x_0, 0, 0, z_0)\) is a solution. Clearly \(x_0 \geq 0\) and (10), (11) and (12) are satisfied. It remains to prove that \(z_0 \geq 0\). We have

\[
(d - 2)x_0 = (d - 2)\left\lfloor \frac{B}{d - 1} \right\rfloor \leq \frac{(d - 2)B}{d - 1} + (d - 2) \leq A + d - 2 .
\]

Thus \(A - (d - 2)x_0 \geq 2 - d\) and

\[
z_0 = \left\lfloor \frac{A - (d - 2)x_0}{d} \right\rfloor \geq \left\lfloor \frac{2 - d}{d} \right\rfloor = \left\lfloor \frac{2}{d} \right\rfloor - 1 = 0 ,
\]

as desired. Hence \((x_0, 0, 0, z_0)\) is an optimal solution. The claimed lower bound on \(x + y_1 + y_2 + z\) follows by substitution.

Now we solve another integer program that is needed in the proof of the lower bound in Theorem 7.1 with \(\Delta = d + 1\).
Lemma B.2. Fix integers $A \geq 1$ and $d \geq 2$. Let $r := A \mod d(d-1)$. Thus $0 \leq r \leq d(d-1) - 1$. Suppose that $x, y_1, y_2, z \in \mathbb{Z}$ satisfy

\begin{align*}
(13) & \quad (d-2)x + (d-1)(y_1 + y_2 + z) \geq (d-1)A \\
(14) & \quad (d-1)(x + y_1) \geq dA \\
(15) & \quad (d-1)(x + y_2) \geq dA.
\end{align*}

Then

$$x + y_1 + y_2 + z \geq \begin{cases} \left\lceil \frac{d-1}{2}A - \frac{2}{d-2}\left\lceil \frac{A}{d(d-1)} \right\rceil \right\rceil & \text{if } r < (d-1)(d-2) \\
\left\lceil \frac{d}{2}A - \left\lceil \frac{A}{d(d-1)} \right\rceil \right\rceil & \text{if } r \geq (d-1)(d-2) \end{cases}. $$

This bound is achievable, for example by

$$x := \begin{cases} \left\lceil \frac{d-1}{2}A - \frac{2d-2}{d-2}\left\lceil \frac{A}{d(d-1)} \right\rceil \right\rceil & \text{if } r < (d-1)(d-2) \\
\left\lceil \frac{d}{2}A - \left\lceil \frac{A}{d(d-1)} \right\rceil \right\rceil & \text{if } r \geq (d-1)(d-2) \end{cases}, $$

$$y_1 := y_2 := \begin{cases} \frac{A}{d(d-1)} & \text{if } r < (d-1)(d-2) \\
\frac{A}{d(d-1)} & \text{if } r \geq (d-1)(d-2) \end{cases}, $$

$$z := 0. $$

Proof. Say $(x, y_1, y_2, z)$ is a solution if [13], [14] and [15] are satisfied. A solution is optimal if it minimises $x + y_1 + y_2 + z$. Observe that if $(x, y_1, y_2, z)$ is an optimal solution, then $(x, y_1 + z, y_2, 0)$ also is an optimal solution. Thus $(x, y_1, y_2, 0)$ is an optimal solution for some $y_1 \geq y_2$.

Let $(x, y_1, y_2, 0)$ be an optimal solution with $y_1 \geq y_2$, such that $y_1 - y_2$ is minimised. Suppose on the contrary that $y_1 - y_2 \geq 1$. Then $(x - 1, y_1, y_2 + 1, 0)$ is a solution since

\begin{align*}
(d-2)(x - 1) + (d-1)(y_1 + y_2 + 1) & = (d-2)x + (d-1)(y_1 + y_2) - (d-2) + (d-1) \\
& = (d-2)x + (d-1)(y_1 + y_2) + 1 \\
& \geq (d-1)A + 1,
\end{align*}

and by [15],

\begin{align*}
(d-1)(x - 1 + y_1) & \geq (d-1)(x + y_2) \geq dA, \text{ and} \\
(d-1)(x - 1 + y_2 + 1) & = (d-1)(x + y_2) \geq dA.
\end{align*}

Moreover, $(x - 1, y_1, y_2 + 1, 0)$ is optimal since $x + y_1 + y_2 = (x - 1) + y_1 + (y_2 + 1)$. This proves that $(x, y_1, y_2, 0)$ does not minimise $y_1 - y_2$, which is a contradiction. Thus $y_1 = y_2$. 

38
Hence \((x^*, y^*, y^*, 0)\) is an optimal solution for some \(x^*, y^* \in \mathbb{Z}\). That is, \(x^*\) and \(y^*\) minimise \(x^* + 2y^*\) such that
\[
(d - 2)x^* + 2(d - 1)y^* \geq (d - 1)A \tag{16}
\]
\[
(d - 1)(x^* + y^*) \geq dA \tag{17}
\]

First consider the case when \(d = 2\). Then (16) and (17) hold if and only if \(y^* \geq \left\lceil \frac{A}{2} \right\rceil\) and \(x^* \geq 2A - y^*\). Thus \(x^* + 2y^*\) is minimised by \(y^* = \left\lceil \frac{A}{2} \right\rceil\) and \(x^* = 2A - y^* = \left\lceil \frac{3A}{2} \right\rceil\). Hence \(x^* + 2y^* = \left\lceil \frac{3A}{2} \right\rceil + 2 \left\lceil \frac{A}{2} \right\rceil = \left\lceil \frac{5A}{2} \right\rceil\). This result matches the claimed bounds since \(r \geq (d - 1)(d - 2)\) when \(d = 2\).

Now assume that \(d \geq 3\). Thus (16) and (17) hold if and only if
\[
x^* \geq \max \left\{ \frac{d - 1}{d - 2} A - \frac{2d - 2}{d - 2} y^* , \frac{d}{d - 1} A - y^* \right\}
\]
That is,
\[
x^* + 2y^* \geq \left\lceil \frac{d - 1}{d - 2} A - \frac{2d - 2}{d - 2} y^* , \frac{d}{d - 1} A + y^* \right\rceil
\]
Define
\[
f(y) := \max \left\{ \frac{d - 1}{d - 2} A - \frac{2d - 2}{d - 2} y , \frac{d}{d - 1} A + y \right\} = \begin{cases} \frac{d - 1}{d - 2} A - \frac{2d - 2}{d - 2} y & \text{if } y \leq \frac{A}{d(d - 1)} \\ \frac{d}{d - 1} A + y & \text{if } y \geq \frac{A}{d(d - 1)} \end{cases}
\]

For a given value of \(y^*\), setting
\[
x^* := \max \left\{ \frac{d - 1}{d - 2} A - \frac{2d - 2}{d - 2} y^* , \frac{d}{d - 1} A - y^* \right\}
\]
implies that that \(x^* + 2y^* = \lfloor f(y^*)\rfloor\). Since \(x^*\) and \(y^*\) minimise \(x^* + 2y^*\),
\[
x^* + 2y^* = \min_{y^* \in \mathbb{Z}} f(y) = \min_{y^* \in \mathbb{Z}} \begin{cases} \left\lceil \frac{d - 1}{2} A - \frac{2d - 2}{d - 2} y^* \right\rceil & \text{if } y \leq \frac{A}{d(d - 1)} \\ \left\lceil \frac{d}{d - 1} A + y^* \right\rceil & \text{if } y \geq \frac{A}{d(d - 1)} \end{cases}
\]
Observe that \(\frac{A}{d(d - 1)}\) is the only local minimum (and thus the global minimum) of \(f\). Hence
\[
y^* = \left\lfloor \frac{A}{d(d - 1)} \right\rfloor \text{ or } y^* = \left\lceil \frac{A}{d(d - 1)} \right\rceil.
\]
If \(r = 0\) then \(y^* = \frac{A}{d(d - 1)}\) is an integer, and we are done with \(x^* = \frac{d + 1}{d} A\) and \(x^* + 2y^* = \frac{d^2 + 1}{d^2 - d} A\). Now assume that \(r \geq 1\). Thus
\[
f \left( \left\lfloor \frac{A}{d(d - 1)} \right\rfloor \right) < f \left( \left\lceil \frac{A}{d(d - 1)} \right\rceil \right)
\]
\[
\iff \quad \frac{d - 1}{d - 2} A - \frac{2}{d - 2} \left\lfloor \frac{A}{d(d - 1)} \right\rfloor < \frac{d}{d - 1} A + \left\lfloor \frac{A}{d(d - 1)} \right\rceil
\]
\[
\iff \quad \frac{d - 1}{d - 2} A - \frac{2}{d - 2} \left( \frac{A - r}{d(d - 1)} \right) < \frac{d}{d - 1} A + \frac{A - r}{d(d - 1)} + 1
\]
\[
\iff \quad r < (d - 1)(d - 2).
\]
Thus
\[ x^* + 2y^* = \lceil f(y^*) \rceil = \begin{cases} \left\lfloor d^{\frac{1}{2}} A - d^{\frac{2}{2}} \frac{A}{d(d-1)} \right\rfloor & \text{if } r < (d-1)(d-2) \\ d \frac{d}{d-1} A + \left\lceil \frac{A}{d(d-1)} \right\rceil & \text{if } r \geq (d-1)(d-2) \end{cases} \]

where
\[ y^* = \begin{cases} \frac{A}{d(d-1)} & \text{if } r < (d-1)(d-2) \\ \frac{A}{d(d-1)} & \text{if } r \geq (d-1)(d-2) \end{cases} \]

and
\[ x^* = \begin{cases} \left\lfloor d^{\frac{1}{2}} A - \frac{2d^{\frac{2}{2}}}{d^{\frac{2}{2}}} \frac{A}{d(d-1)} \right\rfloor & \text{if } r < (d-1)(d-2) \\ d \frac{d}{d-1} A - \left\lceil \frac{A}{d(d-1)} \right\rceil & \text{if } r \geq (d-1)(d-2) \end{cases} \]

\[ \square \]

DEPARTMENT OF MATHEMATICS AND STATISTICS
THE UNIVERSITY OF MELBOURNE
MELBOURNE, AUSTRALIA

E-mail address: woodd@unimelb.edu.au