Self-excited relaxation oscillations in networks of modified FitzHugh–Nagumo oscillators

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Abstract. A modification of the well-known FitzHugh–Nagumo model with a fast and a slow variable is proposed. The existence and stability of a nonclassical relaxation cycle in this system are studied. The slow component of the cycle is asymptotically close to a discontinuous function, while the fast component is a \( \delta \)-like function. Self exciting oscillations in a chain of diffusively coupled neurons as well as in one-dimensional ring of unidirectionally coupled neurons are studied. The existence of an arbitrarily large number of traveling waves for this chain has been shown. In order to illustrate the increase in the number of stable traveling waves, numerical methods are involved.

1. Introduction

For an analytical study of self-oscillations in an isolated neuron, various simplified models are used which take into account only external manifestations of neural activity (changes in membrane potential and generation of spikes), but do not reflect intracellular reasons for these phenomena. Two-dimensional singularly perturbed systems are used as simplified neuron models. In the system

\[
\varepsilon \dot{u} = f(u, v), \quad \dot{v} = g(u, v),
\]

\( u = u(t) \) is membrane potential, \( v = v(t) \) is current, and \( \varepsilon > 0 \) is a small positive parameter. In view of the relaxation character of electrical activity in the cell, we shall impose on the right-hand sides \( f, g \in C^\infty \) the standard restrictions \cite{1} ensuring existence of a stable relaxation cycle.

Some models of the form (1) can be deduced from the Hodgkin–Huxley model \cite{2} under several additional assumptions, and others can be expressed independently. One of the simplest mathematical models for the functioning of an individual neuron is the well-known FitzHugh–Nagumo (FHN) model (see \cite{3, 4}), whose underlying system has the form

\[
\varepsilon \dot{u} = v + u - u^3/3 + c, \quad \dot{v} = a - u - bv,
\]

where \( 0 < \varepsilon \ll 1 \), \( a, b = \text{const} > 0 \), and \( c = \text{const} \in \mathbb{R} \). In \cite{3} the system (2) was used as a phenomenological model reproducing the most important properties of the Hodgkin–Huxley model. It was then shown in \cite{4} that the same system of equations describes electrical oscillations in an oscillator with a tunnel diode, provided that current-voltage characteristic of the latter
can be approximated by a cubic parabola. Finally, it should be noted that, for appropriately chosen parameters, the system (2) admits a stable relaxation cycle.

The class of models (1) also contains the well-known Morris–Lecar system [5]. Yet another example of a system (1) is given by the two-dimensional version of the Hindmarsh–Rose system [6], which has the form

\[ \varepsilon \dot{u} = v - au^3 + bu^2 + I, \quad \dot{v} = c - du^2 - v. \]  

(3)

The system (3) is usually taken with \( \varepsilon = 1, a = 1, b = 3, c = 1, d = 5, I = 2.7 \). But if we fix these values \( a, b, c, d, \) and \( I \) and decrease \( \varepsilon \), then the stable cycle in this system becomes a relaxation cycle.

The two-dimensional models (2), (3) have at least two significant drawbacks. The first is that the classical relaxation oscillations arising in these models do not precisely describe the oscillations of the membrane potential of an actual neuron. In principle, there exists a remedy for this. For instance, the papers [7] and [8] proposed a modification

\[ \varepsilon \dot{u} = v - g(u), \quad \dot{v} = a - u - v, \]  

(4)

\( 0 < \varepsilon \ll 1, a = \text{const} > 0 \) of the model (2) in which the component \( u = u(t) \) exhibits \( \delta \)-shaped oscillations, which better correspond to what we have in reality.

Consider the properties of the function \( g(u) \in C^\infty(\mathbb{R}) \) from (4). The nonlinear characteristic \( i = f(u) \) must satisfy the following constraints.

**Condition 1.** There exists \( u = u_* > 0 \) such that

\[ g(0) = 0, \quad g'(u) > 0 \text{ for } u \in (-\infty, u_*), \quad g'(u) < 0 \text{ for } u \in (u_*, +\infty), \]

\[ g'(u_*) = 0, \quad g''(u_*) < 0, \quad a - u_* - g(u_*) > 0. \]  

(5)

**Condition 2.** For \( u \to +\infty \), we have the asymptotic representation

\[ g(u) = a_0 + \sum_{k=1}^{\infty} \frac{a_k}{u^k}, \quad a_0 > 0, \]  

(6)

which remains valid after being differentiated with respect to \( u \) any number of times.

As an example of such a function, we can use

\[ g(u) = c_1 u \exp(-u) + c_2 (1 - \exp(-u)), \quad c_1, c_2 = \text{const} > 0. \]  

(7)

Indeed, Condition 2 is satisfied; moreover, the coefficient \( a_0 \) in (6) is equal to \( c_2 \), while the other \( a_k, k \geq 1 \), vanish. Furthermore, \( u_* \) in (5) is given by \( u_* = 1 + c_2/c_1 \), while the requirement \( a - u_* - g(u_* > 0 \) is equivalent to the condition

\[ \left[ a - u - c_1 u \exp(-u) - c_2 (1 - \exp(-u)) \right]_{u = 1 + c_2/c_1} > 0 \]  

(8)

on the parameters \( a, c_1, \) and \( c_2 \).

Let us analyze the existence and stability of a nonclassical (or impulsive type) relaxation cycle in system (4). According to the terminology used in [9], this is cycle \((u, v) = (u_*(t, \varepsilon), v_*(t, \varepsilon))\) of period \( T_*(\varepsilon) \) in system (4) such that, as \( \varepsilon \to 0 \), the component \( v_*(t, \varepsilon) \) converges pointwise to a discontinuous function, \( T_*(\varepsilon) \) tends to a finite limit \( T_* > 0 \), and the component \( u_*(t, \varepsilon) \) varies in a \( \delta \)-like manner with time.
Before formulating the corresponding rigorous result, we introduce some notation. Define
\[ v^* = g(u^*), \quad v^{**} = 2\alpha_0 - v^*, \quad u^{**} = \phi(v)|_{v = v^{**}}, \quad x^*_{int} = v^* - v^{**}, \quad x^*_{max} = v_0 - \alpha_0, \] (9)
where \( u = \phi(v) \), with \( v \in (-\infty, u_\star] \) being the unique root of the equation \( g(u) = v \) on the interval \((-\infty, u_\star] \). Note that due to properties (5) and (6), we have \( x^*_{int} > 0, \; x^*_{max} > 0 \) and \( u^{**} < u_\star \), and the quantity
\[ T_\star = \int_{u^{**}}^{u_\star} \frac{g'(u)du}{a - u - g(u)}. \] (10)
is positive.

In addition to constants (9) and (10), we will need functions \( u_\star(t) \) and \( v_\star(t) \), where \( u_\star(t) = \phi(v_\star(t)) \) and \( v_\star(t) \) is determined by solving the Cauchy problem
\[ \dot{v} = a - \phi(v) - v, \quad v|_{t=0} = v^{**}. \] (11)

It is easy to see that, when continued from the interval \( 0 \leq t \leq T_\star \) to the entire axis \( t \) according to the \( T_\star \)-periodicity law, these functions are discontinuous at the points \( t = kT_\star, \; k \in \mathbb{Z} \). The following statement holds [7].

**Theorem 1.** Let Conditions 1 and 2 hold. Then there is a sufficiently small \( \varepsilon_0 > 0 \) such that, for all \( 0 < \varepsilon \leq \varepsilon_0 \), system (4) has an exponentially orbitally stable relaxation cycle
\[ \Gamma_\star(\varepsilon) = \{(u, v) : \; u = u_\star(t, \varepsilon), \; v = v_\star(t, \varepsilon), \; 0 \leq t \leq T_\star(\varepsilon)\} \] (12)
of period \( T_\star(\varepsilon) \), where \( u_\star(0, \varepsilon) \equiv u_\star + 1 \). Moreover,
\[ \lim_{\varepsilon \to 0} T_\star(\varepsilon) = T_\star, \] (13)
\[ \lim_{\varepsilon \to 0} \int_0^{t_\star(\varepsilon)} u_\star(t, \varepsilon)dt = x^*_{int}, \quad \lim_{\varepsilon \to 0} \max_t (\sqrt{\varepsilon} u_\star(t, \varepsilon)) = x^*_{max}, \] (14)

![Figure 1. Relaxation oscillations of the system (4)](image-url)
\[
\lim_{\varepsilon \to 0} \max_{\delta_1 \leq t \leq T_s(\varepsilon) - \delta_2} \left( |u_s(t, \varepsilon) - u_s(t)| + |v_s(t, \varepsilon) - v_s(t)| \right) = 0. 
\] (15)

Here, \( t_s(\varepsilon) = O(\sqrt{\varepsilon}) \) is the first positive root of the equation \( u_s(t, \varepsilon) = u_s + 1 \) and \( \delta_1, \delta_2 \in (0, T_s/2) \) are arbitrary fixed constants.

A visual representation of the properties of the impulsive-type relaxation cycle is given by Fig. 1, which shows its components as functions of time. These plots were obtained with the help of the numerical integration of system (4) in the case of (7) and (8) with \( \varepsilon = 0.005 \), \( a = 12 \), \( c_1 = 10 \), \( c_2 = 3 \) (the solid and broken curves depict \( u_s(t+c, \varepsilon) \) and \( v_s(t+c, \varepsilon) \), respectively, where \( c \in \mathbb{R} \) is a phase shift).

Of great interest is the problem of interaction of oscillators of the form (4). The following paragraphs examine two such problems. In the first of them, a chain of diffusively coupled oscillators of the form (4) is considered. In the second, a ring of unidirectionally coupled oscillators is studied.

2. Self-exiting oscillations in a chain of diffusively coupled neurons

Consider a one-dimensional chain of diffusively coupled neurons with Neumann boundary conditions. Assume that every individual neuron is described by system (4). As a result, we obtain the system

\[
\varepsilon \dot{u}_j = v_j - g(u_j) + d(u_{j+1} - 2u_j + u_{j-1}), \quad \dot{v}_j = a - u_j - v_j, \quad j = 1, \ldots, m, 
\] (16)

where \( u_{m+1} = u_m \), \( u_0 = u_1 \), and \( d = \text{const} > 0 \).

The system (16) admits a homogeneous or synchronous cycle

\[
\dot{u}_1 \equiv \ldots \equiv \dot{u}_m = u_s(t, \varepsilon), \quad \dot{v}_1 \equiv \ldots \equiv \dot{v}_m = v_s(t, \varepsilon), 
\] (17)

where \((u_s(t, \varepsilon), v_s(t, \varepsilon))\) is the periodic solution of system (4) given by Theorem 1. Thus, the question arises as to whether this cycle is stable within the framework of the diffusion model.

To solve this problem, we linearize system (16) on the cycle (17) to obtain the system

\[
\varepsilon \dot{h}_{1,j} = h_{2,j} - g'(u_s(t, \varepsilon))h_{1,j} + d(h_{1,j+1} - 2h_{1,j} + h_{1,j-1}), \quad \dot{h}_{2,j} = -h_{1,j} - h_{2,j}, 
\] (18)

\[ j = 1, \ldots, m, \quad h_{1,m+1} = h_{1,m}, \quad h_{1,0} = h_{1,1}. \]

Applying the Fourier method over the eigenvectors of the difference Laplacian to (18), more precisely, setting

\[
h_{r,j} = \sum_{k=0}^{m-1} g_{r,k}(t) \cos \left( \frac{\pi k}{2m} (2j - 1) \right), \quad j = 1, \ldots, m, \quad r = 1, 2, 
\] (19)

we see that the pairs \((g_{1,k}(t), g_{2,k}(t))\), \( k = 0, 1, \ldots, m - 1 \), in (19) satisfy the system

\[
\varepsilon \dot{g}_1 = g_2 - (g'(u_s(t, \varepsilon)) + z)g_1, \quad \dot{g}_2 = -g_1 - g_2 
\] (20)

at \( z = z_k \), where

\[
z_k = 4d \sin^2 \left( \frac{\pi k}{2m} \right), \quad k = 0, 1, \ldots, m - 1. 
\] (21)

For convenience, the parameter \( z \) in (20) is assumed to vary continuously on a certain interval \( z_1 \leq z \leq z_2 \), where \( z_2 > z_1 > 0 \). Let \( \mu_1(\varepsilon, z) \), and \( \mu_2(\varepsilon, z) \) denote the multipliers of this system.
Lemma 1. The limiting equalities

$$\lim_{\varepsilon \to 0} \mu_1(\varepsilon, z) = \exp \left( - \int_0^{T_*} \left( 1 + \frac{1}{g'(u_*(t))} + z \right) dt \right) < 1, \quad \lim_{\varepsilon \to 0} \mu_2(\varepsilon, z) = 0,$$

hold uniformly in $z \in [z_1, z_2]$. Here, $T_*$ is the constant given by (10) and $u_*(t)$ is the function from (15).

This lemma allows us to prove the statement about the stability of a homogeneous cycle. The stability spectrum of homogeneous cycle (17) of system (16) consists of the multipliers of
system (20) at the discrete values of $z$ given by (21). Among these multipliers, there is only one that equals to one, while the others, due to (22), are less than one in absolute value. Thus, the following result is true.

**Theorem 2.** For any fixed $d > 0$ and all sufficiently small $\varepsilon > 0$, the homogeneous cycle (17) of system (16) is exponentially orbitally stable.

![Image](image-url)

**Figure 4.** Spatial patterns in the evolution of the spikes for $u = u_j(t)$.

Interestingly, with a suitable choice of parameters in system (16), along with the stable homogeneous cycle, there are other attractors. For example, a numerical analysis of system (16) with conditions (7) and (8) for $m = 20$, $a = 5$, $c_1 = 3$, $c_2 = 2$, $\varepsilon = 0.005$, and $d = 0.01$ has revealed that it has 24 coexisting stable inhomogeneous cycles. Figures 2 and 3 show the projections of some of these cycles onto the plane $(u_1, u_{18})$ (the other cycles are obtained from these ones by making the substitutions $u_{m+1-j} \rightarrow u_j$, $v_{m+1-j} \rightarrow v_j$, $j = 1, \ldots, m$).

The evolution of the spikes in $u = u_j(t)$ with increasing $j$, i.e., the propagation of excitation waves along the neural chain deserves a special investigation. For the cycle presented in Fig. 3c, the corresponding spatial pattern is shown in Fig. 4. More specifically, this figure depicts the graphs of the curves $\{(t, u) : u = u_j(t)\}$, $j = 1, \ldots, m$ in parallel planes. The relative positions of these graphs suggest that the excitation waves are divided into groups, so-called packets of pulses. The other stable inhomogeneous cycles found have a similar spatial structure.

### 3. Self-exciting oscillations in a continual ring of unidirectionally coupled neurons

Let us now consider a discrete ring chain of unidirectionally coupled neurons assuming that each separate neuron is described by system (4). As a result we have system

$$
\varepsilon \dot{u}_j = v_j - g(u_j) + \frac{\mu m}{2\pi} (u_{j+1} - u_j), \quad \dot{v}_j = a - u_j - v_j, \quad j = 1, \ldots, m, \quad (23)
$$

where $u_{m+1} = u_1$, $\mu > 0$ is the coupling parameter. Then, given $m \gg 1$, we approximate magnitude $2\pi j/m$ by continuous index $x \in [0, 2\pi)$ (mod $2\pi$) and replace component $m(u_{j+1} -$
\[ u_j / 2\pi \] in (23) with derivative \( \partial u / \partial x \). As a result, we get the boundary value problem

\[
\begin{align*}
\varepsilon \frac{\partial u}{\partial t} &= v - g(u) + \mu \frac{\partial u}{\partial x}, \\
\frac{\partial v}{\partial t} &= a - u - v,
\end{align*}
\]

(24)

which represents a mathematical model of continual ring of unidirectionally coupled neurons.

We will consider special periodic solutions of problem (24), where the so-called traveling waves are denoted as

\[
\begin{align*}
u &= u_n(\xi, \varepsilon, \mu), \\
v &= v_n(\xi, \varepsilon, \mu), \\
\xi &= \omega_n(\varepsilon, \mu)t - nx.
\end{align*}
\]

(25)

Here \( \omega_n(\varepsilon, \mu) > 0, n \in \mathbb{N} \) and \( 2\pi \)-periodic by \( \xi \) functions \( u_n(\xi, \varepsilon, \mu) \), and \( v_n(\xi, \varepsilon, \mu) \) satisfy system

\[
\begin{align*}
(\varepsilon \omega_n(\varepsilon, \mu) + n\mu) \frac{du}{d\xi} &= v - g(u), \\
\omega_n(\varepsilon, \mu) \frac{dv}{d\xi} &= a - u - v.
\end{align*}
\]

(26)

Established theorem 1 allows one to deal with the issue of the existence of traveling waves (25), (26). For this purpose, let us consider the auxiliary system

\[
\begin{align*}
\tilde{\varepsilon} \frac{du}{d\xi} &= v - g(u), \\
\omega \frac{dv}{d\xi} &= a - u - v,
\end{align*}
\]

(27)

assuming that \( 0 < \tilde{\varepsilon} \ll 1 \) and parameter \( \omega > 0 \) is of order of one and changes within a finite segment. Applying the mentioned theorem to system (27), we make sure that, for all small enough values of \( \tilde{\varepsilon} > 0 \), it permits the periodic solution

\[
\begin{align*}
u &= u(\xi, \tilde{\varepsilon}, \omega), \\
v &= v(\xi, \tilde{\varepsilon}, \omega), \\
u(0, \tilde{\varepsilon}, \omega) &\equiv u_* + 1
\end{align*}
\]

(28)

of period \( T(\tilde{\varepsilon}, \omega) \). At the same time,

\[
T(\tilde{\varepsilon}, \omega) = \omega T_* + O(\tilde{\varepsilon}^{1/4}), \quad \tilde{\varepsilon} \to 0,
\]

(29)

where \( T_* \) is the constant (10). In turn, asymptotic representation (29) implies that equation

\[
T(\tilde{\varepsilon}, \omega)|_{\tilde{\varepsilon} = \varepsilon \omega + n\mu} = 2\pi
\]

(30)

for defining frequency \( \omega \) has at least one solution

\[
\omega = \omega_n(\varepsilon, \mu), \quad \omega_n(\varepsilon, \mu) = 2\pi / T_* + O((\varepsilon + \mu)^{1/4}), \quad \varepsilon, \mu \to 0.
\]

It is necessary to add that the triplet of functions \( \omega_n(\varepsilon, \mu), u_n(\xi, \varepsilon, \mu), \) and \( v_n(\xi, \varepsilon, \mu) \), where

\[
\begin{align*}
u_n(\xi, \varepsilon, \mu) &= u(\xi, \tilde{\varepsilon}, \omega)|_{\tilde{\varepsilon} = \varepsilon \omega_n(\varepsilon, \mu) + n\mu, \varepsilon = \omega_n(\varepsilon, \mu)}, \\
v_n(\xi, \varepsilon, \mu) &= v(\xi, \tilde{\varepsilon}, \omega)|_{\tilde{\varepsilon} = \varepsilon \omega_n(\varepsilon, \mu) + n\mu, \varepsilon = \omega_n(\varepsilon, \mu)},
\end{align*}
\]

is the sought value; i.e., it turns equations in (26) into correct equalities. The analysis performed above leads to the following statement.

**Theorem 3.** For any natural \( N \) small enough values \( \varepsilon_N > 0 \) and \( \mu_N > 0 \) can be found such that, for all \( 0 < \varepsilon \leq \varepsilon_N, 0 < \mu \leq \mu_N \), boundary value problem (24) permits traveling waves (25), (26) with numbers \( n = 1, \ldots, N \).

It is worth noting that, in the case

\[
\mu = \varepsilon d, \quad d = \text{const} > 0,
\]

(31)
guarantees the existence of a periodic solution of system (34) similar to (28) above. Namely, let us consider auxiliary system similar to (27)

\[
\varepsilon \omega_n(\varepsilon) \frac{du}{d\xi} = v - g(u), \quad \omega_n(\varepsilon) \frac{dv}{d\xi} = a - u - v. \tag{33}
\]

Functions \( \omega_n(\varepsilon) \), \( u_n(\xi, \varepsilon) \), and \( v_n(\xi, \varepsilon) \) from (32) are defined by the same scheme as described above. Namely, let us consider auxiliary system similar to (27)

\[
\varepsilon (\omega - nd) \frac{du}{d\xi} = v - g(u), \quad \omega \frac{dv}{d\xi} = a - u - v, \tag{34}
\]

where parameters \( \omega \), and \( d > 0 \) are of the order of one and inequality \( nd < \omega \) holds. Theorem 1 guarantees the existence of a periodic solution of system (34) similar to (28)

\[
u = u_n(\xi, \varepsilon, \omega), \quad v = v_n(\xi, \varepsilon, \omega), \quad u_n(0, \varepsilon, \omega) \equiv u_* + 1 \tag{35}\]

of period

\[T_n(\varepsilon, \omega) = \omega T_\ast + O(\varepsilon^{1/4}), \quad \varepsilon \to 0. \tag{36}\]

for any small enough value of \( \varepsilon > 0 \). Then (36) implies that the equation \( T_n(\varepsilon, \omega) = 2\pi \), which is similar to (30), has at least one solution as follows:

\[
\omega = \omega_n(\varepsilon) = 2\pi/T_\ast + O(\varepsilon^{1/4}), \quad \varepsilon \to 0. \tag{37}
\]

As for functions \( u_n(\xi, \varepsilon) \), and \( v_n(\xi, \varepsilon) \) in (32), (33), they are obtained from (35) after the substitution of equation (37). In this way, the following statement is established.

**Theorem 4.** Assume that for some natural \( N \) inequality \( Nd < 2\pi/T_\ast \) is fulfilled. Then under condition (31) and for all small enough \( \varepsilon > 0 \) boundary value problem (24) permits traveling waves (32), (33) with numbers \( n = 1, \ldots, N \).

The problem of stability of traveling waves in metrics of the phase space \((u, v) \in W^1_2 \times W^1_2 \) \((W^1_2 \text{ – Sobolev space of } 2\pi\text{-periodic functions})\) is reduced to studying the spectrum of some boundary value problems. In the case of traveling waves (25), (26), the procedure for deriving corresponding spectral problem is the following. Let us proceed to traveling spatial variable \( \xi = \omega_n(\varepsilon, \mu)t - nx \) in problem (24), then linearize it at the equilibrium state \( u = u_n(\xi, \varepsilon, \mu) \), \( v = v_n(\xi, \varepsilon, \mu) \). Then, let us substitute equalities \( u = h_1(\xi) \exp(\lambda t), \quad v = h_2(\xi) \exp(\lambda t), \lambda \in \mathbb{C} \) into the obtained linear system. As a result, in order to define \( h_1(\xi), h_2(\xi) \), and \( \lambda \) we have

\[
\varepsilon \lambda h_1 + (\varepsilon \omega_n(\varepsilon, \mu) + n\mu) \frac{dh_1}{d\xi} = h_2 - g'(u_n(\xi, \varepsilon, \mu))h_1,
\]

\[
\lambda h_2 + \omega_n(\varepsilon, \mu) \frac{dh_2}{d\xi} = -h_1 - h_2, \quad h_j(\xi + 2\pi n) \equiv h_j(\xi), \quad j = 1, 2. \tag{38}
\]

A study of location of Eigenvalues \( \lambda \) in problem (38), and a similar boundary value problem for traveling waves (32), (33), represents a separate and still unsolved problem. Therefore, we will confine with results of numeric analysis of boundary value problem (24), which show the principal possibility of the existence of stable traveling waves.
When performing the corresponding numeric experiment, in contrast to (23), we approximate the first derivative by \( \frac{\partial u}{\partial x} \) with symmetric difference operator, i.e., we assume that

\[
\frac{\partial u}{\partial x}(t, x)|_{x=2\pi j/m} \approx \frac{m}{4\pi} (u_{j+1}(t) - u_{j-1}(t)), \quad j = 1, \ldots, m.
\]

Finally, we obtain the following system for variables \( u_j(t) \), and \( v_j(t) \) after the replacement \( \mu/2\pi \rightarrow \mu \):

\[
\varepsilon \dot{u}_j = v_j - g(u_j) + \frac{m}{2} \mu (u_{j+1} - u_{j-1}), \quad \dot{v}_j = a - u_j - v_j, \quad j = 1, \ldots, m, \tag{39}
\]

where \( u_0 = u_m \), and \( u_{m+1} = u_1 \).

\[\text{Figure 5. Traveling-wave type solutions of the system (39)}\]
Numerical analysis of system (39) was performed by the fourth order Runge–Kutta method with a constant step of $h = 10^{-4}$ under conditions (6), (7) and, at values of parameters $m = 21$, $a = 15$, $c_1 = 3$, $c_2 = 1$, $\varepsilon = \mu = 0.01$, (software package Tracer 3.70 developed by D. S. Glyzin was applied). It has been established that, at the given set of parameters, it has seven stable periodic solutions of traveling wave type and four stable two-dimensional invariant tori. Figures 5 show projections of the given attractors to plane ($u_1, u_{18}$) (the first seven images correspond to cycles and the remaining four correspond to invariant tori).

4. Conclusions

To conclude, we note that the new mathematical model proposed for an individual neuron has rich contents. Systems with nonclassical relaxation oscillations give us oscillations of membrane potential of $\delta$-like shape. This feature corresponds to the real biological objects. Moreover, the diffusion chain (16) and one-dimensional ring (24) of unidirectionally coupled neurons corresponding to system (4) exhibits nontrivial dynamics, namely, the buffer phenomenon. In this context, we recall that the buffer phenomenon (i.e., the coexistence of any prescribed finite number of attractors) is typical for neuron systems, as suggested by the results of [10] – [15].

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