COROTATION RESONANCE AND DISKOSEISMOLGY MODES OF BLACK HOLE ACCRETION DISKS

ALEXANDER S. SILBERGLEIT
Gravity Probe B, Stanford University, Stanford, CA 94305-4085; gleit@stanford.edu

AND

ROBERT V. WAGONER
Department of Physics and KIPAC, Stanford University, Stanford, CA 94305-4060; wagoner@stanford.edu

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ABSTRACT

We demonstrate that the corotation resonance affects only some nonaxisymmetric g-mode oscillations of thin accretion disks, since it is located within their capture zones. Using a more general (weaker radial WKB approximation) formulation of the governing equations, such g-modes, treated as perfect fluid perturbations, are shown to formally diverge at the position of the corotation resonance. For the known g-modes with moderate values of the radial mode number and axial mode number (and any vertical mode number), the corotation resonance lies well outside their trapping region (and inside the innermost stable circular orbit), so the observationally relevant modes are unaffected by the resonance. The axisymmetric g-mode has been seen by Reynolds & Miller in a recent inviscid hydrodynamic accretion disk global numerical simulation. We also point out that the g-mode eigenfrequencies approximately obey the harmonic relation $\sigma \propto m$ for axial mode numbers $|m| \geq 1$.

Subject headings: accretion, accretion disks — black hole physics — gravitation — relativity

1. INTRODUCTION

In principle, all adiabatic perturbations of equilibrium models of accretion disks can be analyzed in terms of global normal modes. The pioneering studies of Shoji Kato and his group and the more recent work of our group have focused on accretion disks around black holes, so that no complications from boundary layers are involved. For a recent review of “relativistic diskoseismology,” see Kato (2001). A short summary of observationally relevant results from our recent analyses of the low-lying spectrum, which consists of g-modes (Perez et al. 1997, hereafter Paper I), c-modes (Silbergleit et al. 2001, hereafter Paper II), fundamental $p$-modes (Ortega-Rodriguez et al. 2002, hereafter Paper III), and other $p$-modes (Ortega-Rodriguez et al. 2008, hereafter Paper IV), is given by Wagoner et al. (2001). Local analyses (restricted radial interval) have also played an important role in our understanding of these perturbations (Kato et al. 1998).

One of the first analyses of the corotation resonance was in the series of papers by Papaloizou & Pringle (1984, 1985, 1987) where the stability of rotating compressible tori was studied. They showed that the corotation resonance is a physical singularity for the inviscid perturbation modes. Another motivation for our investigation was the work by Li et al. (2003), who examined the corotation resonance singularity for the inviscid modes of a special class of isothermal accretion disks (constant speed of sound) using an effective potential approximation.

In § 2 we summarize the foundations (assumptions and equations) of our approach to the study of linear inviscid perturbations of fully relativistic geometrically thin, optically thick accretion disk models. In § 3 we examine the corotation resonance location and show that only certain nonaxisymmetric $g$-modes may be affected by the resonance. In § 4 we investigate the behavior of the vertical and radial eigenfunctions near the corotation resonance, exhibiting the local and global divergence, and thus prove that the range of the eigenfrequencies of nonaxisymmetric $g$-modes is reduced to its upper part (specified by the rotational frequency, radial epicyclic frequency, and the azimuthal wavenumber of the mode). In § 5 we discuss the effects of introducing viscosity and buoyancy. We also comment on the possible observational relevance of the spectrum of the $g$-modes.

2. BASIC ASSUMPTIONS AND EQUATIONS

We take $c = 1$ and express all distances in units of $GM/c^2$ and all frequencies in units of $c^3/GM$ (where $M$ is the mass of the black hole) unless otherwise indicated. We employ the Kerr metric to study a thin accretion disk. The equilibrium disk is taken to be described by the standard relativistic thin disk model (Novikov & Thorne 1973; Page & Thorne 1974). The velocity components $v^\prime = v^z = 0$, and the disk semithickness $h(r) \sim c_s(r,0)\Omega \ll r$, where $c_s(r,z)$ is the speed of sound. The key frequencies, associated with free-particle orbits, are

$$\Omega(r) = \left(\frac{r^3}{v^3} + a\right)^{-1},$$
$$\Omega_r(r) = \Omega(r)\left(1 - 4a/r^3 - 3a^2/r^2\right)^{1/2},$$
$$\kappa(r) = \Omega(r)\left(1 - 6/r + 8a/r^3 - 3a^2/r^2\right)^{1/2}$$

the rotational, vertical epicyclic, and radial epicyclic frequencies, respectively. The angular momentum parameter $a = cJ/GM^2$ is less than unity in absolute value. The inner edge of the disk is at approximately the radius of the last stable free-particle circular orbit $r = r_s(a)$, where the epicyclic frequency $\kappa(r_s) = 0$. So all the relations we use are for $r > r_s$, where $\kappa(r) > 0$. The outer disk radius is denoted by $r_o$.

We apply the general relativistic formalism that Ipser & Lindblom (1992) developed for perturbations of purely rotating perfect fluids, although the effects of viscosity are included in the equilibrium model (and some of our notation differs from theirs, see below). The pressure $p$ is much less than the mass-energy density $\rho$. We neglect the self-gravity of the disk, which is a good approximation since the ratio of disk to black hole mass is usually very small (see Paper IV, § 7). Ipser & Lindblom (1992)
then show that one can express the Eulerian perturbations of all physical quantities through a single function,
\[ \delta V = \frac{\delta p}{\rho \beta \omega}. \]
(2)

Because of the stationary and axisymmetric equilibrium, the angular and time dependences are factored out as \( \delta V = V(r, z) \exp \left[ i (\omega t + \sigma r) \right] \), and the master equation (39) of Ipser & Lindblom (1992) for the function \( V \) (see also Paper I, eq. [2.21]) assumes the form
\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ r^2 Y^2 \rho \left( \frac{\omega^2}{\omega^2 - \kappa^2} \right) \frac{\partial V}{\partial r} \right] + \frac{\partial}{\partial z} \left[ r Y \rho \left( \frac{\omega^2}{\omega^2 - N_z^2} \right) \frac{\partial V}{\partial z} \right] + r Y \beta \omega \Phi V = 0. \quad (3)
\]
The corotation frequency \( \omega \) is related to the eigenfrequency \( \sigma \) by
\[ \omega(r, \sigma) = \sigma + m \Omega \rho \].
(4)
The buoyancy frequency is dominated by its vertical component \( N_z \) (Paper I). The Kerr metric component \( g^\mu_\nu \equiv Y^2 (r) \) and four-velocity component \( u^\mu \equiv \frac{dt}{dr} + \sigma = \beta(\omega) \) are both of order unity (in Boyer-Lindquist coordinates); their expressions can be found in Paper I. Compared with Ipser & Lindblom (1992), the definitions of \( \sigma \) and \( \omega \) are switched, while their \( \gamma \) becomes our \( \beta \). The function \( \Phi(r, z, \omega, m) \) depends on various properties of the unperturbed disk, as well as the mode eigenfrequency, \( \sigma \), and its azimuthal wavenumber, \( m \).

To simplify the analysis, we consider barotropic disks \( [\rho = \rho(\rho)] \), so the buoyancy frequency \( N_z = 0 \). Since this frequency should be lower than other characteristic frequencies and, in general, vanishes on the midplane of symmetry \( z = 0 \), this should be a good approximation except possibly in the neighborhood of the corotation resonance where \( \omega(r) = 0 \). Hydrostatic equilibrium then provides the following unperturbed density and pressure distributions \( (\Gamma > 1) \) is the adiabatic index,
\[ \rho(r, y) = \rho_0(r)(1 - y^2)^{\gamma}, \quad p(r, y) = \rho_0(r)(1 - y^2)^{\gamma+1}, \]
\[ g \equiv 1/(\Gamma - 1) > 0; \quad (5) \]
the disk surfaces are at \( y = \pm 1 \), with the new coordinate \( y \) related to the vertical coordinate \( z \) through the characteristic disk semithickness \( h(r) \),
\[ y = \frac{z}{h(r)} \sqrt{\frac{\Gamma - 1}{2 \Gamma}}, \quad h(r) = \frac{1}{\beta(\rho_0(r))} \sqrt{\rho_0(r)} \quad \text{yields} \]
\[ c_0^2 = \frac{\Gamma p}{\rho} = \Gamma (\beta h \rho_0(r)) (1 - y^2). \quad (6) \]
The speed of sound \( c_0(r, y) \) is specified by
\[ c_0^2 = \frac{\Gamma p}{\rho} = \Gamma (\beta h \rho_0(r)) (1 - y^2). \quad (7) \]

With these simplifications, for the function \( \Phi \) which appears at the end of the master equation (3) we obtain
\[
\Phi(r, z) = \frac{\rho \beta \omega}{c_0^2} + \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{\rho Y r^2}{\beta^2 (\omega^2 - \kappa^2)} \frac{\partial}{\partial r} \left( \beta \omega \right) \right] - \frac{2 \rho Y r^2 (m + \beta \omega \omega_0)}{\beta^3 (\omega^2 - \kappa^2)} \frac{\omega_0 (m + \beta \omega \omega_0)^2}{\beta^2 Y^2 r^2 (\omega^2 - \kappa^2)}. \quad (8)
\]
The quantity \( \Omega^2 \) is a component of the angular velocity four-vector, while \( \omega_0 \) is another component of the fluid four-velocity. Like all other quantities in equation (8) except \( \rho \) and \( c_0 \), they are functions of \( r \) alone.

Because of the structure of the density and speed of sound as functions of the coordinates, equation (3) does not allow for an exact separation of variables in either of their two pairs, \((r, z)\) or \((r, y)\). Therefore, in the past we adopted the (usually realistic) assumption of strong variation of the modes in the radial direction (characteristic radial wavelength \( \lambda_r \ll r \)) and used the asymptotic separation of variables based on it. In particular, we look for a separated solution to the master equation (3) of the form
\[ V = V_r(r)V_z(z, r). \quad (9) \]

Unlike the previous analyses, here we do not neglect the derivatives of functions of \( \omega(r) \) [in addition to \( V_r \) and \((\omega^2 - \kappa^2)^{-1}\)] since it varies much more strongly than \( V_z \) and all other quantities near the corotation resonance (for \( m \neq 0 \)). However, we take all other functions out of the radial derivatives. Then, with the separation (eq. [9]) also used, the master equation assumes the form
\[
\frac{\gamma^2}{\gamma} \frac{\partial}{\partial r} \left[ \left( \frac{\omega^2}{\omega^2 - \kappa^2} \right) \frac{\partial V_z}{\partial r} \right] + G(r) = - \frac{1}{(1 - y^2)^2 V_z} \frac{\partial}{\partial z} \left[ (1 - y^2)^2 \frac{\partial V_z}{\partial z} \right] - \frac{\omega_z^2}{\Gamma h^2 (1 - y^2)}. \quad (10)
\]
We have introduced \( \omega_z(r, \gamma) = \omega_0(r, \gamma) \), and the function \( G(r) \) is defined by
\[ G(r) \equiv G(r, a, \sigma, m) = \frac{\beta \omega}{c_0^2} \left( \frac{\beta \omega}{c_0^2} \right)^2 \]
\[ = \gamma^2 \omega \frac{d}{dr} \left( \frac{1}{\omega^2 - \kappa^2} \right) \frac{d \omega}{dr} - \frac{2 \Omega^2 \omega}{\beta^2 r} \]
\[ \times \frac{d}{dr} \left( m + \beta \omega \omega_0 \right) - \frac{(m + \beta \omega \omega_0)^2 \omega}{\beta^2 r^2 \gamma^2 (\omega^2 - \kappa^2)}. \quad (11) \]

The left-hand side of equation (10) is a rapidly varying function of \( r \), while the right-hand side is a rapidly varying function of \( z \). Therefore, within this (weak) radial WKB approximation both are equal to a slowly varying separation constant \( S(r) \), which (in keeping with our previous convention) we denote instead by
\[ S(r) \equiv \left[ \Omega^2 - \omega_z^2 \right] / (\Gamma h^2). \quad (12) \]

The vertical eigenvalue \( \Psi \) is thus the redefined separation constant. Employing everywhere the new vertical coordinate \( y \) defined by equation (6), this radial WKB approximation then does indeed produce separated equations for \( V_r = V_z(z, r) \) (a slowly varying function of \( r \)) and \( V_z = V_z(z, r) \),
\[
(1 - y^2) \frac{d^2 V_z}{dy^2} - 2 y \frac{d V_z}{dy} + 2 y \omega_z^2 y^2 + \Psi (1 - y^2) \] \( V_r = 0, \quad (13) \)
\[ \frac{d}{dr} \left( \frac{\omega^2}{\omega^2 - \kappa^2} \right) \frac{d V_r}{dr} + \left( G - \frac{\Psi - \omega_z^2}{\Gamma h^2} \right) \] \( V_r = 0. \quad (14) \)

Away from the corotation resonance, \( G \approx 1/r^2 + m^2/r^2 \), which is seen to be of order \((h/r)^2\) smaller than its competing term
\[ \omega^2/(\Gamma h^2) = [\beta \omega/c_r(r, 0)]^2 \] in equations (10) and (14). That is why \( G(r) \) has been neglected in Papers I–IV.

Together with the proper boundary conditions, discussed in detail in the referenced papers, these two equations specify (slowly varying) vertical eigenvalues \( \Psi = \Psi(\sigma, r) \) and eigenfrequencies \( \sigma \), as well as the corresponding vertical \( (V_r) \) and radial \( (V) \) eigenfunctions for modes of all types. Lagrangian displacements are related to \( V \) by the expressions (see eqs. [2.25] and [2.26] of Paper I)

\[
\xi^r \equiv \frac{\omega Y^2}{\beta(\omega^2 - \kappa^2)} \frac{\partial V}{\partial r}, \quad \xi^\tau \equiv \frac{1}{\beta \kappa} \frac{\partial V}{\partial \tau},
\]

\[
\xi^z \equiv \frac{\beta^2 [1 - (2/r)(1 - a\Omega)]}{\kappa \omega} \left[ \frac{d\Omega}{dr} + \frac{r \omega^2}{\beta^2(r^2 - 2r + a^2)} \right] \xi^r,
\]

where \( \omega^2 \) is a component of the vorticity four-vector (Ipser & Lindblom 1992).

### 3. THE COROTATION RESONANCE AND ITS IMPLICATIONS FOR DISKSEISMOLOGY MODES

#### 3.1. Frequency Range and Location of the Corotation Resonance

As always, in view of the \( \{ \sigma \rightarrow -\sigma, m \rightarrow -m \} \) symmetry, we can restrict our consideration to \( m \geq 0 \). For axially symmetric modes \( m = 0 \) and \( \omega \equiv \sigma \neq 0 \). However, for \( m > 0 \) there might exist a point \( r_{\text{cor}} = r_{\text{cor}}(\sigma, m, a) \), where the corotation frequency \( \omega \) goes to zero,

\[ \omega(r_{\text{cor}}) = \sigma + m\Omega(r_{\text{cor}}) = 0. \]  

This specifies the location of the corotation resonance for a given eigenfrequency. Since \( \Omega(r) \) is a decreasing function of the radius, the unique solution \( r_{\text{cor}} = r_{\text{cor}}(\sigma, m, a) \) to equation (16) exists within the disk, \( r_1 < r_{\text{cor}} < r_2 \), for all eigenfrequencies in the range

\[ m\Omega(r_2) < -\sigma < m\Omega(r_1). \]  

It is important to compare \( r_{\text{cor}} \) with the Lindblad resonances at the radii \( r_1 = r_1(a, \sigma, m) \) \( (r_1 \leq r_{\text{cor}}) \) defined as the roots of \( \omega^2(r) - \kappa^2(r) = 0 \). As pointed out in Paper I (note especially Fig. 3), for \( m > 0 \) they exist in a wider range of the eigenfrequencies than equation (17), namely, when

\[ m\Omega(r_2) < -\sigma < \max_{r_1 < r < r_2} \left[ m\Omega(r) + \kappa(r) \right]. \]  

This is also the maximum possible frequency range of \( g \)-modes, according to their definition specifying their capture zone as the interval between the Lindblad resonances where \( \kappa^2(r) - \omega^2(r) > 0 \).

So, whenever the corotation resonance exists, the Lindblad resonances are also present (but not necessarily vice versa). Moreover, since \( \kappa(r) \) is positive inside the disk, we can write

\[ m\Omega(r_2) = -\sigma + \kappa(r_2) > -\sigma = m\Omega(r_{\text{cor}}) > -\sigma - \kappa(r_2) = m\Omega(r_2). \]

Note that the inequalities here are based on the expressions involving \( \sigma \). By looking at their counterparts with the angular velocity \( \Omega(r) \) and invoking again its monotonic decrease, we conclude that

\[ r_1 < r_{\text{cor}} < r_2. \]

Hence, whenever the corotation resonance occurs, it lies between the Lindblad resonances.

#### 3.2. Corotation Resonance and the Diskseismology Modes

The result from equation (19) shows that the corotation resonance does not significantly affect both \( c \)-modes, captured in a region \( r_1 < r < r_2 \) of the inner disk region (see Paper II), and inner and outer \( p \)-modes, residing respectively in \( r_1 < r < r_2 \) and \( r_+ < r < r_0 \) (see Papers III and IV). The only disk oscillations which could be strongly affected by it are thus the (nonaxisymmetric) \( g \)-modes, since the region \( r_- < r < r_+ \) is their capture domain. So, the first significant result of the paper allows us to discuss only \( g \)-modes in the sequel. It should also be noted that \( g \)-modes are the most robustly determined, since their capture zone does not include either of the disk boundaries, where the physical conditions and validity of our assumptions are more uncertain.

Furthermore, for the effect of the corotation resonance to show up, the resonance itself must be present; this means that only those \( g \)-modes can be influenced by it whose eigenfrequencies are in the corotation resonance range from equation (17). All the \( g \)-mode eigenfrequencies found so far (see Paper I, Tables 1–3) belong, in fact, to the more restricted range

\[ m\Omega(r_2) < -\sigma < \max_{r_1 < r < r_2} \left[ m\Omega(r) + \kappa(r) \right], \]  
i.e., to the upper part of the maximum \( g \)-mode range from equation (18). Section 4 shows that this is not unexpected, since the \( g \)-modes with the eigenfrequencies in the range from equation (17) are simply absent.

Meanwhile, it seems appropriate to describe the result on the corotation resonance location once again, in purely geometric terms. Namely, in radius-frequency coordinates one can plot the following three curves: \( m\Omega(r) - \kappa(r) \) (C1), \( m\Omega(r) \) (C2), and \( m\Omega(r) + \kappa(r) \) (C3), with \( m \geq 1 \). Then a straight horizontal line \( -\sigma = \text{const} > 0 \) with \( \sigma \) in the range from equation (17) intersects once each of the three curves C1, C2, and C3 (in this order for increasing \( r \)) at the radii \( r_- \), \( r_{\text{cor}} \), and \( r_+ \), respectively. However, with \( \sigma \) in the range from equation (20), its straight line intersects only the curve C3, at both \( r_- \) and \( r_+ \). The other two curves remain below the line for all radii within the disk, so no corotation resonance occurs within it.

### 4. COROTATION RESONANCE SINGULARITY AND THE FREQUENCY RANGE OF NONAXISYMMETRIC \( g \)-MODES

#### 4.1. Possible \( g \)-Mode Divergence at the Corotation Resonance

Let us find out what can happen to those \( m > 0 \) \( g \)-modes which have the corotation resonance in their capture domain, that is, eigenfrequencies in the interval from equation (17). The behavior of the eigenfunction, \( V \) and its radial derivative, \( \partial V/\partial r \), at the corotation resonance needs to be examined, because one or both of them may become singular there. This would imply, in the first place, the singularity of the Lagrangian displacements by equations (15), which is hardly acceptable for small perturbations. However, as shown in § 4.3, this is exactly what happens. Unfortunately, an integral divergence turns out also to be involved.

To see this, one multiplies the master equation (3) by \( rV(r, z) \) and integrates the result over the disk, and then integrates by parts, taking into account that the unperturbed density \( \rho \) vanishes at the disk boundary, to obtain

\[ \int_{\text{disk}} r^2 \Phi \beta \omega V^2 \, dr \, dz = \int_{\text{disk}} r^2 \Phi \left( \frac{\partial V}{\partial z} \right)^2 + \frac{\gamma^2 \omega^2}{\omega^2 - \kappa^2} \left( \frac{\partial V}{\partial r} \right)^2 \, dr \, dz. \]  

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The left-hand side of this identity must be finite for any reasonable perturbation, since it represents the relevant norm of it. The two sources of divergences on the right are (1) the corotation resonance and (2) the Lindblad resonances.

For source 2, the combination \((\omega^2 - \kappa^2)^{-1}(\partial V/\partial r)\) has been shown to be finite at \(r_{\pm}\) in Papers I and III, and this conclusion holds with some new singular terms in the coefficient \(G(r)\); by its definition from equation (11), \(G(r) = O((\omega^2 - \kappa^2)^{-1})\) when \(r \rightarrow r_{\pm}\). Therefore, the second term on the right-hand side of the identity from equation (21) is finite (in fact, vanishes) at the Lindblad resonances. Moreover, \(V'\) and \((\omega^2 - \kappa^2)^{-1}V\) are both finite there as well, so that there is no Lindblad resonance divergence in either the first term on the right-hand side of equation (21) or in the term on the left-hand side stemming from the singularity in \(G(r)\).

However, it turns out that at the corotation resonance, the left-hand side remains finite, but both terms on the right-hand side of the identity from equation (21) generically diverge. To show this, we start with the first term and transform it using the WKB separation, the redefinition from equation (6) of the vertical variable, and equation (5) for the density,

\[
\int_{\text{disk}} r^2 Y \rho \left( \frac{\partial V}{\partial z} \right)^2 \, dr \, dz = \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{r^2 Y \rho_0}{h} \sqrt{\frac{\Gamma - 1}{2\Gamma}} \, V_r^2 \, dr \\
\times \int_{-1}^{1} (1 - y^2)^\sigma \left( \frac{\partial V_y}{\partial y} \right)^2 \, dy = k \int_{r_{\text{in}}}^{r_{\text{out}}} V_r^2 \, I_r \, dr, \tag{22}
\]

where \(k > 0\) is the proper average value of the positive function \(r^2 Y \rho_0 h^{-1}(\Gamma - 1)/(\Gamma - 2)^{1/2}\) of the radius, and we denoted

\[I_r = I_r(r) \equiv \int_{-1}^{1} (1 - y^2)^\sigma \left[ V'_y(r,y) \right]^2 \, dy. \tag{23}\]

(Primes now denote the derivative with respect to \(y\).) Clearly, \(I_r(r) \geq 0\) for any \(r\); if \(I_r(r') = 0\) for some \(r' > r\), then equation (23) implies \(V'_y(r',y) \equiv 0\), so the vertical eigenfunction \(V'_y(r',y) = \text{const}\). However, a constant never satisfies the vertical equation (13) except for the case \(r = r_{\text{cor}}\), \(\Psi(r_{\text{cor}}) = 0\). The last condition is never true for \(g\)-modes; the demonstration nevertheless requires a slightly more sophisticated argument (following immediately).

4.2. Vertical Eigenvalue Problem near the Corotation Resonance

We rewrite the vertical equation (13) in the self-adjoint form,

\[
\left[ (1 - y^2)^\sigma V_y' \right]' + 2g \left[ \Psi(1 - y^2) + \frac{\omega^2}{\Omega_{\perp}} y^2 \right] (1 - y^2)^{\sigma - 1} V_y = 0,
\]

multiply by \(V_y\), and integrate over the interval \((-1, 1)\). Using integration by parts and the fact that \(V_y(\pm 1)\) must be finite, we thus obtain

\[I_r(r) = \int_{-1}^{1} (1 - y^2)^\sigma \left[ V'_y(y) \right]^2 \, dy \]

\[
= 2g \left[ \frac{\omega^2}{\Omega_{\perp}} \right] \int_{-1}^{1} y^2 (1 - y^2)^{\sigma - 1} V_y^2 \, dy + \Psi \int_{-1}^{1} (1 - y^2)^{\sigma - 1} V_y^2 \, dy.
\]

At the corotation resonance, this equality becomes

\[I_r(r_{\text{cor}}) = 2g \Psi(r_{\text{cor}}) \int_{-1}^{1} (1 - y^2)^\sigma \left[ V'_y(r_{\text{cor}},y) \right]^2 \, dy \neq 0, \tag{25}\]

unless \(\Psi(r_{\text{cor}}) = 0\). However, if this is the case, we can expand, in the vicinity of \(r_{\text{cor}}\), the eigenvalue and eigenfunction in the small parameter \(\omega^2 \equiv \omega^2/\Omega_{\perp}\) as

\[\Psi = \Psi_1 \omega^2 + \ldots, \quad V_y = V^{(0)} + \Psi^{(1)} \omega^2 + \ldots,
\]

thus obtaining from equation (24)

\[
\mathcal{L} V^{(0)} = 0,
\]

\[
\mathcal{L} V^{(1)} = -2g \left[ 1 - \Psi_1 (1 - y^2) \right] (1 - y^2)^{\sigma - 1} V^{(0)}, \ldots
\]

The only zeroth-order solution bounded together with its derivative at \(y = \pm 1\) is, of course, \(V^{(0)}(y) \equiv 1\), and the solvability criterion of the problem for the first correction,

\[0 = \int_{-1}^{1} V^{(0)} \mathcal{L} V^{(1)} \, dy = -2g \int_{-1}^{1} \left[ 1 - \Psi_1 (1 - y^2) \right] (1 - y^2)^{\sigma - 1} \, dy,
\]

provides \(\Psi_1 = -(\Gamma - 1)/2\) by means of an easy calculation.\(^1\)

Hence,

\[\Psi(r)/\omega^2(r) = -(\Gamma - 1)/2 + O(\omega^2(r))\]

\[= -(\Gamma - 1)/2 + O((r - r_{\text{cor}})^2) < 0, \tag{26}\]

for \(r \) near \(r_{\text{cor}}\), which, by definition, corresponds to some \(p\)-mode and not a \(g\)-mode (characterized by \(\Psi/\omega^2 \gg 1\)). Note that the last term in the equalities from equation (26) holds due to the fact that \(\omega\), and hence \(\omega^2\), is to lowest order linear near \(r = r_{\text{cor}}\),

\[\omega(r) = \frac{d\Omega(r_{\text{cor}})}{dr} (r - r_{\text{cor}}) + O((r - r_{\text{cor}})^2). \tag{27}\]

So, \(\Psi(r_{\text{cor}}) \neq 0\) for any \(g\)-mode, and therefore, \(I_r(r_{\text{cor}}) \neq 0\). Thus, from equation (22), the question of whether the first term on the right-hand side of equation (21) diverges or not at the corotation resonance reduces to the same question about the norm of the radial eigenfunction (whose square we denote \(I_r\)),

\[
\int_{\text{disk}} r^2 Y \rho \left( \frac{\partial V}{\partial z} \right)^2 \, dr \, dz \propto I_r, \quad I_r \equiv \int_{r_{\text{in}}}^{r_{\text{out}}} V_r^2 \, dr. \tag{28}
\]

To obtain the answer, it remains only to investigate the behavior of \(V_r(r)\) near \(r = r_{\text{cor}}\).

4.3. Behavior of the Radial Eigenfunction near the Corotation Resonance

Using the Taylor expansion from equation (27), we see that the radial equation (14) near \(r_{\text{cor}}\) can be written as

\[
\frac{d^2 V_r}{dr^2} + \left[ \frac{2}{r - r_{\text{cor}}} + O(1) \right] \frac{dV_r}{dr} + \left[ \frac{Q_r^2}{(r - r_{\text{cor}})^2} + O((r - r_{\text{cor}})^{-1}) \right] V_r = 0, \tag{29}\]

\(^1\) The same derivation of this result was carried out in Paper III, \S\ 3, for \(m = 0\) and \(\omega_r = \sigma \Omega_{\perp}\); the specifics of which play, in fact, no role in it.
where

\[ Q_c^2 = \frac{\beta^2 \kappa^2 \Omega^2}{m^2 c_s^2(r, 0)} \left( \frac{d \Omega}{dr} \right)^2 \bigg|_{r=r_{cor}} \quad (30) \]

and all the higher order terms are integer powers of \((r - r_{cor})\). According to the analytical theory of second-order ordinary differential equations (e.g., Olver 1982), the general solution to equation (29) near \(r_{cor}\) has the form

\[ V_{r}(r) = C_+(r - r_{cor})^{\nu_+} + C_-(r - r_{cor})^{\nu_-} \quad (31) \]

where \(C_\pm\) are some constants and \(\nu_\pm\) are the roots of the characteristic equation

\[ \nu^2 + \nu + Q_c^2 = 0, \quad \nu_\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - Q_c^2}. \quad (32) \]

Note that for an isothermal thin disk model, Li et al. (2003) have found different behavior for traveling waves. Therefore, the estimates from equation (33) always hold, as well as

\[ V_r^2 = O((r - r_{cor})^{-1}), \quad \omega^2 (d V_r/dr)^2 = O((r - r_{cor})^{-1}), \quad (34) \]

for \(r \to r_{cor}\), implied by them.

4.4. Local and Global Corotation Resonance Divergences: Range of Nonaxisymmetric g-Modes

Equations (33) combined with equations (15) show that all three components of the Lagrangian displacement are singular at the corotation resonance,

\[ \xi^r \sim (r - r_{cor})^{-1/2}, \quad \xi^\phi \sim (r - r_{cor})^{-3/2}. \]

However, note from equation (2) that the perturbation \( \delta p/\rho \propto (r - r_{cor})^{1/2} \) vanishes there.

In addition to this local singularity of the physical quantities, a global one also exists. According to the first of the equalities in equation (34), the integral \( I_r \) defined in equations (28) always diverges and, hence, so does the first term in equation (21) for the norm of the eigenfunction. The second term there, in fact, also diverges via the second of the equalities in equation (34). Such a combination of local and global perturbation singularities is definitely unacceptable. (The norm of an eigenfunction belonging to the discrete spectrum must be finite.)

We have thus proved that nonaxisymmetric \( g \)-modes with eigenfrequencies in the corotation resonance range (eq. [17]) cannot exist within the framework of inviscid perturbations, so their actual frequency range (eq. [20]),

\[ m \Omega(r_s) < -\sigma < \max_{r_{cor} < r < r_s} [m \Omega(r) + \kappa(r)] \equiv \kappa(r_m) + m \Omega(r_m), \quad (35) \]

for \(m = 1, 2, \ldots\), is the upper part of the maximum possible range (eq. [18]), for which the corotation resonance is absent within the disk. In comparison with this, the range of the axially symmetric \( g \)-modes,

\[ \kappa(r_s) < |\sigma| < \max_{r_{cor} < r < r_s} \kappa(r) \equiv \kappa_0(a), \quad (36) \]

includes low frequencies as well.

5. DISCUSSION AND CONCLUSIONS

5.1. Nonaxisymmetric g-Modes with Moderate Radial and Azimuthal Wavenumbers

The established corotation resonance divergence could cast a shadow of doubt on the known results for nonaxisymmetric \( g \)-modes, in particular, on those found in Paper I, especially since the WKB technique used there (see also Paper IV) to calculate eigenfrequencies is not fully sensitive to the presence of the corotation resonance. However, as pointed out in \( \S 2.2\), all the found eigenfrequencies belong to the proper range (eq. [35]), \( |\sigma_{mnj}| > m \Omega(r) \) (\(m, n, j\) are the azimuthal, radial, and vertical mode numbers, respectively). That means that the corotation resonance lies inside the inner radius of the disk (where the gas is spiraling into the black hole). We now indicate why the \( g \)-modes (with moderate values of \( m \) and \( n \)) are the most robust modes and, therefore, astrophysically the most relevant.

For any azimuthal mode number \( m \), we denote the largest eigenfrequency \( |\sigma_{mnj}| \) by \( \sigma_m \equiv \kappa(r_m) + m \Omega(r_m) \). In the limit \( j \to \infty \), the trapping zone \( \Delta r = r_s - r_\ast \to 0 \), with \( r_s > r_\ast > r_\ast \). We now consider moderate values of \( m \) and \( n \), but any value of the vertical mode number \( j \). Then from equation (5.3) and Tables 1–3 of Paper I, it is seen that these \( g \)-modes occupy a very small frequency range below their maximum,

\[ \sigma_m - |\sigma_{mnj}| \leq (\kappa c_s/\Omega)_{r_s}. \quad (37) \]

Thus, as also seen from Figures 3 and 5 of Paper I, it follows that

\[ |\sigma_{mnj}| \approx m \Omega(r_1) \quad (37) \]

for moderate values of \( n \) and \( m \geq 1 \), with the approximate equality becoming quickly more and more exact as \( m \) grows. Therefore, for a given black hole the largest frequency splitting should be due to the azimuthal mode number \( m = 0, 1, 2, \ldots \), with the eigenfrequencies \( |\sigma| \propto m \) for \( m \geq 1 \). We also note that with increasing values of \( m \), the mode location \( r_m \to r_1 \) and its extent \( \Delta r \to 0 \). Then the mode leakage into the flow onto the black hole and the uncertainties in the physical conditions near \( r_1 \) become more important while the fractional modulation of the luminosity decreases. Therefore, the low-\( m \) modes should be the most robust and observable.

5.2. Quasi-periodic Oscillation Features: 3/2 and Other Integer Frequency Ratios

In this connection, one should note the observational claims that some of the quasi-stable high-frequency quasi-periodic oscillations
(QPOs) in black hole X-ray binary sources have frequency ratios close to 3/2 (McClintock & Remillard 2004; Remillard & McClintock 2006). This can be, in principle, explained by excitation of two (groups of) g-modes with \( m = 2 \) and 3, as suggested in Paper IV.

Any such explanation for any QPO frequency implies a relation between the mass and angular momentum of the black hole in the corresponding X-ray binary. Indeed, using the general relation for the dimensional frequency \( f = 3.23 \times 10^4 (M_\odot/M) |\sigma| \text{ Hz} \), from equation (37) one obtains

\[
M/M_\odot \approx 3.23 \times 10^4 m \Omega(r_i)/f_m \text{ Hz}, \quad m \geq 1. \tag{38}
\]

Since \( r_i = r_0(a) \), \( \Omega(r_i) = (r_0^3 + a)^{-1} \equiv \mathcal{F}(a) \) is a universal function of black hole angular momentum only. It is a monotonically increasing function, with \( \mathcal{F}(-1) = 0.038 \) (extreme counterrotation), \( \mathcal{F}(0) = 0.068 \) (nonrotating black hole), and \( \mathcal{F}(1) = 0.5 \) (extreme rotation).

Let us compare the prediction of equation (38) with the data (McClintock & Remillard 2004) for the three binary black holes with 3/2 or 3/2/1 frequency ratios and measured mass. Using the lower limit \( \mathcal{F}(a) > 0.038 \), we obtain \( M > 8.2 M_\odot \) for GRO J1655–40 (\( f = 300, 450 \) Hz), observed to have \( M = 6.3 \pm 0.3 M_\odot \). Similarly for XTE J1550–564 (\( f = 92, 184, 276 \) Hz), we obtain \( M > 13.3 M_\odot \), compared to the observed \( M = 9.6 \pm 1.2 M_\odot \). There are two pairs of 3/2 QPOs observed in GRB 1915+105 (\( f = 41, 67, 113, 168 \) Hz), whose black hole has a measured mass of \( M = 14 \pm 4 M_\odot \). For the higher frequency pair, we obtain \( M > 22 M_\odot \), while for the lower frequency pair we obtain \( M > 58 M_\odot \). If we use no rotation rather than counterrotation to provide the lower mass limits, they would increase by a factor 1.8.

Similar conclusions have been reached by Tassev & Bertschinger (2007), who looked for persistent patterns in free-particle orbits, corresponding to vanishing speed of sound (see also Amin & Frolov 2006).

It is perhaps not surprising that these QPOs cannot be explained as \( m \geq 1 \) g-modes, since they occupy a much smaller region of the disk nearer to its inner edge (where our accretion disk model is most suspect; Paper I). In addition, their \( \phi \)-dependence reduces the observed modulation.

One would thus wonder why the \( m = 0 \) g-mode was not seen. It provides a relation \( M/M_\odot = F_0(a)/f_0 \) similar to equation (38) (Paper I; Wagoner et al. 2001). Indeed, all but two of the QPOs in these sources could be fundamental g-modes (for some value of \( a < 1 \)), but of course only one in each source. The value of \( a \) required for some of these is close to that estimated from the spectroscopic method [temperature and luminosity determine \( \rho \)]. For instance, Shafee et al. (2006) obtain \( a = 0.65–0.75 \) for GRO J1655–40, whereas identifying its 300 Hz QPO as an \( m = 0 \) g-mode requires \( a = 0.9–1.0 \). McClintock et al. (2006) obtain \( a > 0.98 \) for GRB 1915+105, whereas identifying its 168 Hz QPO as an \( m = 0 \) g-mode requires \( a = 0.8–1.0 \).

From their MHD simulations, Tagger & Varnière (2006) claim that the \( m = 2, 3, \ldots \) g-modes can grow to dominance over those of \( m = 0, 1 \) via the Rossby wave instability. However, this required a large concentration of magnetic field between the black hole and the accretion disk. In addition, their simulation neglected the vertical structure of the disk. We should note that Arras et al. (2006) found no g-modes in their shearing box MHD simulations of a limited radial region of an accretion disk. There were some indications of the generation of \( p \)-modes within the magnetorotational instability–induced turbulence, however.

Very recently, Reynolds & Miller (2008) reported results of ideal hydrodynamic (2D and 3D) and MHD (3D) global numerical simulations of accretion disks. As in Tagger & Varnière (2006), the nonrotating black hole was represented by a modified Newtonian gravitational potential. The evolution was typically followed for about 108 orbital periods of the inner disk. From power spectra at many radii, the \( m = 0 \) g-mode was seen in the hydrodynamic simulations at the predicted frequency and radial extent. It was not seen in the MHD simulations. However, because of the induced magnetorotational instability (MRI) turbulence, it would not be expected to be seen if it was at the same amplitude as in the hydrodynamic simulations. Because of the limited range of \( \phi \) (with periodic boundary conditions) and frequency in the 3D simulations, the higher \( m \) modes could not have been seen.

5.3. Perturbative Effects of Buoyancy and Viscosity

We need to say a few words about a nonzero buoyancy manifested by a Brunt–Väisälä frequency \( N_z = N_z(r, z) > 0 \). This frequency is involved in all the equations only via the expression \( \omega^2 - N_z^2 \), so the corotation resonance equation (16) becomes

\[
\omega^2 - N_z^2 = 0.
\]

One might expect that the buoyancy would be small in realistic accretion disks, since the MRI produces strong turbulence (Hawley & Krolik 2001), which should locally homogenize the specific entropy. If so, \( N_z \) should be treated as a perturbation in the above equation. Because of that, it leads only to a small and generally \( z \)-dependent change in the corotation resonance position. In fact, the corotation resonance point splits into two nearby ones,

\[
r_{\text{cor}}^\pm \approx r_{\text{cor}} \pm \frac{N_z(r_{\text{cor}}, z)}{m} \left[ \frac{d\Omega(r_{\text{cor}})}{dr} \right]^{-1}.
\]

Our arguments and results regarding the mode divergence apply to both of them.

In the presence of viscosity which acts hydrodynamically (via the \( \alpha \)-model) and perturbatively, Ortega-Rodriguez & Wagoner (2000) found that for most (including these \( g \))-modes, a viscous instability is induced. Such accretion disks are thus secularly unstable. An effective viscosity (in particular, generated by the MRI) should be present in these (thin) accretion disks, but it is not known in what ways it acts like a hydrodynamic viscosity. To lowest order in its magnitude, the viscosity does not change the values of the eigenfrequencies, but introduces a small imaginary part in them. As usual, this imaginary part removes the divergences found above (Nowak & Wagoner 1991), since the corotation resonance equality from equation (16) no longer holds at any radius (a well-known effect of a complex pole near the real axis).

One is naturally tempted to contemplate the outcome of the mode growth. Will enough nonlinearity be induced to lead to significant mode-mode coupling and related effects? One would like to extend the local resonance analysis of Abramowicz & Kluzniak (2001) to this problem. However, one must also investigate how much the spatial extent of the modes averages over the strong MRI-induced turbulence. Could it be sufficient to provide a stochastic driving force (producing mode excitation) within a time-independent “unperturbed” state?

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