Spectral action on $SU_q(2)$

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Abstract

The spectral action on the equivariant real spectral triple over $A(SU_q(2))$ is computed explicitly. Properties of the differential calculus arising from the Dirac operator are studied and the results are compared to the commutative case of the sphere $S^3$.

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### 1 Introduction

The quantum group $SU_q(2)$ has already a rather long history of studies \[33\] being one of the finest examples of quantum deformation. This includes an approach via the noncommutative notion of spectral triple introduced by Connes \[10,15\] and various notions of Dirac operators were introduced in \[2,4,6,13,28\]. Finally, a real spectral triple, which was exhibited in \[21\], is invariant by left and right action of $U_q(su(2))$ and satisfies almost all postulated axioms of triples except the commutant and first-order properties. These, however, remain valid only up to infinitesimal of arbitrary high order. The last presentation generalizes in a straightforward way all geometric construction details of the spinorial spectral triple for the classical three-sphere. In
particular, both the equivariant representation and the symmetries have a \( q \to 1 \) proper classical limit.

The goal of this article is to obtain the spectral action defined in [7] by

\[
S(D_A, \Phi, \Lambda) := \text{Tr} \left( \Phi \left( D_A / \Lambda \right) \right)
\]

where \( D \) is the Dirac operator, \( A \) is a selfadjoint one-form, \( D_A = D + A + JAJ^{-1} \) and \( J \) is the reality operator. Here, \( \Phi \) is any even positive cut-off function which could be replaced by a step function up to some mathematical difficulties investigated in [23]. This means that \( S \) counts the spectral values of \( |D_A| \) less than the mass scale \( \Lambda \). Actually, as shown in [8]

\[
S(D_A, \Phi, \Lambda) = \sum_{0 < k \in Sd^+} \Phi_k \Lambda^k \int |D_A|^{-k} + \Phi(0) \zeta_{D_A}(0) + \mathcal{O}(\Lambda^{-1}),
\]

where \( D_A = D_A + P_A, P_A \) the projection on \( \text{Ker}D_A \), \( \Phi_k = \frac{1}{2} \int_0^\infty \Phi(t) t^{k/2-1} dt \), \( d \) is the spectral dimension of the triple and \( Sd^+ \) is the strictly positive part of the dimension spectrum \( Sd \) of \( (A, H, D) \). Here, \( Sd^+ = Sd = \{1, 2, 3\} \), so

\[
S(D_A, \Phi, \Lambda) = \sum_{1 \leq k \leq 3} \Phi_k \Lambda^k \int |D_A|^{-k} + \Phi(0) \zeta_{D_A}(0).
\]

Recall that the tadpole of order \( \Lambda^k \) is the linear term in \( A \in \Omega^1_D(A) \) in the \( \Lambda^k \) part of (4).

Note that there are no terms in \( \Lambda^{-k}, k > 0 \) because the dimension spectrum is bounded below by 1. This spectral action has been computed on few examples: [3, 8, 9, 15, 22, 24–26, 30, 34].

Here, we compute (4) with the main difficulty which is to control the differential calculus generated by the Dirac operator. To proceed, we introduce two presentations of one-forms. The main ingredient is \( F = \text{sign} (D) \) which appears to be a one-form up to \( OP^{-\infty} \).

In section 2, we discuss the spectral action of an arbitrary 3-dimensional spectral triple using cocycles.

In sections 3 and 4 we recall the main results on \( SU_q(2) \) of [21] and show that the full spectral action with reality operator given by (4) is completely determined by the terms

\[
\int A^q |D|^{-p}, \quad 1 \leq q \leq p \leq 3.
\]

This question of computation of spectral action was addressed in the epilogue of [37].

In section 5, we establish a differential calculus up to some ideal in pseudodifferential operators and apply these results to the precise computation of previous noncommutative integrals.

Section 6 is devoted to explicit examples, while in next section are given different comparisons with the commutative case of the 3-sphere corresponding to \( SU(2) \).

### 2 Spectral action in 3-dimension

#### 2.1 Tadpole and cocycles

Let \( (A, H, D) \) be a spectral triple of dimension 3. For \( n \in \mathbb{N}^* \) and \( a_i \in A \), define

\[
\phi_n(a_0, \cdots, a_n) := \int a_0[D, a_1]D^{-1} \cdots [D, a_n]D^{-1}.
\]
We also use notational integrals on the universal $n$-forms $\Omega^n_n(A)$ defined by

$$\int_{\phi_n} a_0 da_1 \cdots da_n := \phi_n(a_0, a_1, \ldots, a_n).$$

and the reordering fact that $(da_0)a_1 = d(a_0a_1) - a_0 da_1.$

We use the $b - B$ bicomplex defined in [10]: $b$ is the Hochschild coboundary map (and $b'$ is truncated one) defined on $n$-cochains $\phi$ by

$$b\phi(a_0, \ldots, a_{n+1}) := b'(a_0, \ldots, a_{n+1}) + (-1)^{n+1}\phi(a_{n+1}a_0, a_1, \ldots, a_n),$$

$$b'(a_0, \ldots, a_{n+1}) := \sum_{j=0}^n (-1)^j \phi(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1}).$$

Recall that $B_0$ is defined on the normalized cochains $\phi_n$ by

$$B_0\phi_n(a_0, a_1, \ldots, a_{n-1}) := \phi_n(1, a_0, \ldots, a_{n-1}),$$

thus

$$\int_{B_0\phi_n} d\omega = \int_{\phi_n} \omega$$

for $\omega \in \Omega^{n-1}_n(A).$ Then $B := NB_0,$ where $N := 1 + \lambda + \ldots \lambda^n$ is the cyclic skewsymmetrizer on the $n$-cochains and $\lambda$ is the cyclic permutation $\lambda \phi(a_0, \ldots, a_n) := (-1)^n \phi(a_n, a_0, \ldots, a_{n-1}).$

We will also encounter the cyclic 1-cochain $N\phi_1$:

$$N\phi_1(a_0, a_1) := \phi_1(a_0, a_1) - \phi_1(a_1, a_0)$$

and

$$\int_{N\phi_1} a_0 da_1 := N\phi_1(a_0, a_1).$$

**Remark 2.1.** Assume the integrand of $f$ is in $OP^{-3}.$ Since $[D^{-1}, a] = -D^{-1}[D, a]D^{-1} \in OP^{-2},$ this commutator introduces a integrand in $OP^{-4}$ so has a vanishing integral: under the integral, we can commute $D^{-1}$ with all $a \in A$ and all one-forms.

**Lemma 2.2.** We have

(i) $b\phi_1 = -\phi_2.$

(ii) $b\phi_2 = 0.$

(iii) $b\phi_3 = 0.$

(iv) $B\phi_1 = 0.$

(v) $B_0\phi_2 = -(1 - \lambda)\phi_1.$

(vi) $bB_0\phi_2 = 2\phi_2 + B_0\phi_3.$

(vii) $B\phi_2 = 0.$

(viii) $B_0\phi_3 = Nb'\phi_1.$

(ix) $B\phi_3 = 3B_0\phi_3.$

**Proof.** (i)

$$b\phi_1(a_0, a_1, a_2) = \int a_0 a_1 [D, a_2] D^{-1} - \int a_0 (a_1 [D, a_2] + [D, a_1] a_2) D^{-1} + \int a_2 a_0 [D, a_1] D^{-1}$$

$$= \int a_0 [D, a_1] (D^{-1} a_2 - a_2 D^{-1}) = - \int a_0 [D, a_1] D^{-1} [D, a_2] D^{-1}$$

$$= -\phi_2(a_0, a_1, a_2).$$
where we have used the trace property of the noncommutative integral.

(ii) \( b \phi_2(a_0, a_1, a_2, a_3) \)

\[
= \int a_0 a_1 [D, a_2] D^{-1}[D, a_3] D^{-1} - \int a_0 (a_1 [D, a_2] + [D, a_1] a_2) D^{-1}[D, a_3] D^{-1} \\
+ \int a_0 [D, a_1] D^{-1}(a_2[D, a_3] + [D, a_2] a_3) D^{-1} - \int a_3 a_0 [D, a_1] D^{-1}[D, a_2] D^{-1} \\
= \int a_0 [D, a_1] (D^{-1} a_2 - a_2 D^{-1}) [D, a_3] D^{-1} + \int a_0 [D, a_1] D^{-1}[D, a_2] (a_3 D^{-1} - D^{-1} a_3) \\
= - \int a_0 [D, a_1] D^{-1}[D, a_2] D^{-1}[D, a_3] D^{-1} + \int a_0 [D, a_1] D^{-1}[D, a_2] D^{-1}[D, a_3] D^{-1} \\
= 0.
\]

(iii) Using Remark 2.1 we get \( \phi_3(a_0, a_1, a_2, a_3) = \int a_0 [D, a_1] [D, a_2] [D, a_3] [D]^{-3} \), so similar computations as for \( \phi_2 \) gives \( b \phi_3 = 0 \).

(iv) \( B_0 \phi_1(a_0) = \int [D, a_0] D^{-1} = \int (D a_0 D^{-1} - a_0) = 0. \)

(v) \( B_0 \phi_2(a_0, a_1) = \int [D, a_0] D^{-1}[D, a_1] D^{-1} = \int a_0 D^{-1}[D, a_1] - \int a_0 [D, a_1] D^{-1} \\
= \int a_0 a_1 - \int a_0 D^{-1} a_1 D - \int a_0 [D, a_1] D^{-1} \\
= - \int a_1 [D, a_0] D^{-1} - \int a_0 [D, a_1] D^{-1} = - \phi_1(a_1, a_0) - \phi_1(a_0, a_1). \)

(vi) Since \(-b \lambda \phi_1(a_0, a_1, a_2) = \phi_1(a_2, a_0 a_1) - \phi_1(a_1 a_2, a_0) + \phi_1(a_1, a_2 a_0), \) one obtains that

\[-b \lambda \phi_1(a_0, a_1, a_2) = \int a_0 a_1 D^{-1} a_2 D + a_0 D^{-1} a_1 D a_2 - a_0 D^{-1} a_1 a_2 D - a_0 a_1 a_2. \]

So by direct expansion, this is equal to \(- \int a_0 D^{-1}[D, a_1] D^{-1}[D, a_2] \) which means that

\[-b \lambda \phi_1(a_0, a_1, a_2) = \int [D^{-1}, a_0][D, a_1] D^{-1}[D, a_2] - \int a_0 [D, a_1] D^{-1}[D, a_2] D^{-1} \\
= - B_0 \phi_3(a_0, a_1, a_2) - \phi_2(a_0, a_1, a_2). \]

Now the result follows from (i), (v).

(vii) \( B \phi_2 = N B_0 \phi_2 = - N(1 - \lambda) \phi_1 = 0 \) since \( N(1 - \lambda) = 0. \)

(viii) \( B_0 \phi_3(a_0, a_1, a_2) = \int [D, a_0] D^{-1}[D, a_1] D^{-1}[D, a_2] D^{-1} \\
= \int a_0 D^{-1}[D, a_1] D^{-1}[D, a_2] - \int a_0 [D, a_1] D^{-1}[D, a_2] D^{-1} \\
= \int a_0 a_1 a_2 - \int a_0 a_1 D^{-1} a_2 D - \int a_0 D^{-1} a_1 D a_2 + \int a_0 D^{-1} a_1 a_2 D \\
- \int a_0 [D, a_1] D^{-1}[D, a_2] D^{-1} \\
= \int a_0 a_1 a_2 - a_2 D a_1 a_0 D^{-1} + a_1 a_2 D a_0 D^{-1} + a_2 D a_0 a_1 D^{-1} \\
- (a_0 D a_1 a_2 D^{-1} - a_0 D a_1 D^{-1} - a_0 a_1 D a_2 D^{-1} + a_0 a_1 a_2). \)
Expanding \((id + \lambda + \lambda^2)b'\phi_1(a_0, a_1, a_2)\), we recover previous expression. 

\(\text{(ix)}\) Consequence of \((viii)\).

### 2.2 Scale-invariant term of the spectral action

We know from [8] that the scale-invariant term of the action can be written as

\[
\zeta_{DA}(0) - \zeta_D(0) = -\int AD^{-1} + \frac{1}{2} \int AD^{-1} AD^{-1} - \frac{1}{3} \int AD^{-1} AD^{-1} AD^{-1}. \tag{5}
\]

In fact, this action can be expressed in dimension 3 as contributions corresponding to tadpole and the Yang–Mills and Chern–Simons actions in dimension 4:

**Proposition 2.3.** For any one-form \(A\),

\[
\zeta_{DA}(0) - \zeta_D(0) = -\frac{1}{2} \int_{N\phi_1} A + \frac{1}{2} \int_{\phi_2} (dA + A^2) - \frac{1}{2} \int_{\phi_3} (AdA + \frac{2}{3}A^3). \tag{6}
\]

To prove this, we calculate each terms of the action.

**Lemma 2.4.** For any one-form \(A\), we have

\(\text{(i)}\) \(\int_{\phi_3} dA = \int_{B_0\phi_3} A = -\int_{\phi_1} A - \int_{\phi_2} A\).

\(\text{(ii)}\) \(\int AD^{-1} = \int_{\phi_1} A = \frac{1}{2} \int_{N\phi_1} A - \frac{1}{2} \int_{\phi_2} dA\).

\(\text{(iii)}\) \(\int AD^{-1} AD^{-1} = -\int_{\phi_3} AdA + \int_{\phi_2} A^2\).

\(\text{(iv)}\) \(\int AD^{-1} AD^{-1} AD^{-1} = \int_{\phi_3} A^3\).

**Proof.** \(\text{(i)}\) and \(\text{(ii)}\) follow directly from Lemma 2.2 \((v)\).

\(\text{(iii)}\) With the shorthand \(A = a_i db_i\) (summation on \(i\))

\[
\int AD^{-1} AD^{-1} = \int a_0[D, b_0] D^{-1} a_1[D, b_1] D^{-1} = -\int a_0[D, b_0] AdA + \int a_0[D, b_0]a_1[D, b_1] D^{-1} - \int a_0[D, b_0]a_1 D^{-1} b_1.
\]

We calculate further the remaining terms

\[
\int a_0[D, b_0]a_1[D, b_1] D^{-1} - \int a_0[D, b_0]a_1 D^{-1} b_1 = \int a_0 Db_0 a_1 b_1 D^{-1} - \int a_0 b_0 Da_1 b_1 D^{-1} - \int a_0 [D, b_0]a_1[D, b_1] D^{-1} - \int a_0 [D, b_0]a_1 D^{-1} b_1 + \int a_0 b_0 Da_1 D^{-1} b_1,
\]

which are compared with \(\int_{\phi_2} A^2 = \int_{\phi_2} a_0(d_0a_1)db_1 = \int_{\phi_2} a_0 d_0(b_0a_1)db_1 - a_0 b_0 da_1 db_1\):

\[
\int_{\phi_2} A^2 = \int a_0[D, b_0] a_1[D, b_1] D^{-1} - \int a_0 b_0[D, a_1] D^{-1} [D, b_1] D^{-1} - \int a_0 Db_0 a_1 b_1 D^{-1} - \int a_0 b_0 a_1 D^{-1} b_1 - \int a_0 b_0 Da_1 b_1 D^{-1} + \int a_0 b_0 a_1 b_1
\]

\[
- \int a_0 b_0 D a_1 b_1 D^{-1} + \int a_0 b_0 D a_1 D^{-1} b_1 + \int a_0 b_0 D a_1 b_1 D^{-1} - \int a_0 b_0 a_1 b_1
\]

\[
= \int a_0 Db_0 a_1 b_1 D^{-1} - \int b_1 a_0 Db_0 a_1 D^{-1} - \int a_0 b_0 D a_1 b_1 D^{-1} + \int b_1 a_0 b_0 D a_1 D^{-1}.
\]
(iv) Note that
\[
\int_{\phi_3} A^3 = \int_{\phi_3} a_0(d_b a_1 a_2) d_b - a_0 b_0 d_b a_1 a_2 d_b = \int_{\phi_3} a_0 d_b a_1 a_2 d_b - a_0 b_0 d_b a_1 a_2 d_b
\]
\[
= \int a_0 [D, b_0 a_1] D^{-1}[D, b_1 a_2] D^{-1}[D, b_2] D^{-1} - a_0 b_0 [D, a_1] D^{-1}[D, b_1 a_2] D^{-1}[D, b_2] D^{-1}
\]
\[
- a_0 [D, b_0 a_1] D^{-1}[D, a_2] D^{-1}[D, b_2] D^{-1} + a_0 b_0 [D, a_1 b_1] D^{-1}[D, a_2] D^{-1}[D, b_2] D^{-1}.
\]
Summing up the first two terms and the last two ones gives
\[
\int_{\phi_3} A^3 = \int a_0 [D, b_0 a_1] D^{-1}[D, b_1 a_2] D^{-1}[D, b_2] D^{-1} - a_0 [D, b_0 a_1] D^{-1}[D, a_2] D^{-1}[D, b_2] D^{-1}.
\]
Using Remark 2.3, we can commute under the integral $D^{-1}$ with all $a \in A$ and similarly
\[
\int A D^{-1} A D^{-1} A D^{-1} = \int a_0 [D, b_0] a_1 D^{-1}[D, b_1] a_2 D^{-1}[D, b_2] D^{-1}
\]
which proves (iv).

We deduce Proposition 2.3 from [5] using the previous lemma.

3 The $SU_q(2)$ triple

3.1 The spectral triple

We briefly recall the main facts of the real spectral triple $(\mathcal{A}(SU_q(2)), \mathcal{H}, D)$ introduced in [21], see also [4,5,13].

**The algebra:**

Let $\mathcal{A} := A(SU_q(2))$ be the *-algebra generated polynomially by $a$ and $b$, subject to the following commutation rules with $0 < q < 1$:

\[
ba = q ab, \quad b^* a = q ab^*, \quad bb^* = b^* b, \quad a^* a + q^2 b^* b = 1, \quad aa^* + bb^* = 1. \tag{7}
\]

**Lemma 3.1.** For any representation $\pi$ of $\mathcal{A}$,

\[
\text{Spect}(\pi(bb^*)) = \{ 0, q^{2k} | k \in \mathbb{N} \} \text{ or } \pi(b) = 0,
\]

\[
\text{Spect}(\pi(aa^*)) = \{ 1, 1 - q^{2k} | k \in \mathbb{N} \} \text{ or } \pi(b) = 0 \text{ and } \pi(a) \text{ is a unitary}.
\]

**Proof.** [31] Since $\{ 0 \} \cup \sigma(\pi(aa^*)) = \{ 0 \} \cup \sigma(\pi(a^*a))$, we get

\[
\{ 1 \} \cup B = \{ 1 \} \cup q^2 B \tag{8}
\]

if $B := \sigma(\pi(bb^*))$. Since $0 \leq \pi(bb^*) \leq 1$, so $B$ is a closed subset of $[0, 1]$. Assume $b \neq 0$.

Let $s := \sup(B)$ and suppose $s \neq 1$. Then $s = q^2 x$ where $x \in B$. Thus $s = q^2 x < x \leq s$ gives $s = 0$ and the contradiction $b = 0$, thus $1 \in B$. Similar argument for $\inf(B)$ implies $0 \in B$.

Let $C := \{ 0, q^{2k} | k \in \mathbb{N} \} \subset B$ and assume $B \setminus C \neq 0$. Then $B \setminus C = (q^2 B) \setminus C$ by (8) and this is equal to $q^2(B \setminus C)$ since $q^{-2} > 1$. If $s := \sup(B \setminus C)$, then $s = \lim_n (q^2 x_n)$ where $x_n \in B \setminus C$ and $s = q^2 \lim_n x_n \leq q^2 s$ implying $s = 0$. But $B \setminus C \subset \{ 0 \}$ yields a contradiction, so $B \setminus C = \emptyset$. ∎
This lemma is interesting since it shows the appearance of discreteness for $0 \leq q < 1$ while for $q = 1$, $SU_q(2) = SU(2) \simeq S^3$ and the spectrum of the commuting operator $\pi(aa^*)$ and $\pi(bb^*)$ are equal to $[0,1]$. Moreover, all foregoing results on noncommutative integrals will involve $q^2$ and not $q$.

Any element of $\mathcal{A}$ can be uniquely decomposed as a linear combination of terms of the form $a^\alpha b^\beta c^\gamma$ where $\alpha \in \mathbb{Z}$, $\beta, \gamma \in \mathbb{N}$, with the convention

$$a^{-|\alpha|} := a^{*|\alpha|}.$$  

The spinorial Hilbert space: 

$\mathcal{H} = \mathcal{H}^\dagger \oplus \mathcal{H}^\dagger$ has an orthonormal basis consisting of vectors $|j\mu n\rangle$ with $j = 0, \frac{1}{2}, 1, \ldots$, $\mu = -j, \ldots, j$ and $n = -j^+, \ldots, j^+$, together with $|j\mu n\rangle$ for $j = \frac{1}{2}, 1, \ldots$, $\mu = -j, \ldots, j$ and $n = -j^-, \ldots, j^-$ (here $x^\pm := x \pm \frac{1}{2}$).

It is convenient to use a vector notation, setting:

$$|j\mu n\rangle := \langle (j\mu n) |$$

and with the convention that the lower component is zero when $n = \pm (j + \frac{1}{2})$ or $j = 0$.

The representation $\pi$ and its approximate $\tilde{\pi}$: 

It is known that representation theory of $SU_q(2)$ is similar to that of $SU(2)$ [39]. The representation $\pi$ given in [21] is:

$$\pi(a) |j\mu n\rangle := \alpha_{j\mu n}^+ |j^+ \mu^+ n^+\rangle + \alpha_{j\mu n}^- |j^- \mu^+ n^+\rangle,$$

$$\pi(b) |j\mu n\rangle := \beta_{j\mu n}^+ |j^+ \mu^- n^+\rangle + \beta_{j\mu n}^- |j^- \mu^- n^+\rangle,$$

$$\pi(a^*) |j\mu n\rangle := \tilde{\alpha}_{j\mu n}^+ |j^+ \mu^- n^-\rangle + \tilde{\alpha}_{j\mu n}^- |j^- \mu^- n^-\rangle,$$

$$\pi(b^*) |j\mu n\rangle := \tilde{\beta}_{j\mu n}^+ |j^+ \mu^- n^-\rangle + \tilde{\beta}_{j\mu n}^- |j^- \mu^- n^-\rangle$$

(10)

where

$$\alpha_{j\mu n}^+ := \sqrt{q^{\mu-n+1/2}[j+\mu+1]} \begin{pmatrix} q^{-j-1/2} q^{-j+1/2} \sqrt{j+n+2} & 0 \\ 1/2 q^{-j+1/2} \sqrt{j+n+2} & q^{-j} \sqrt{j+n+1} \end{pmatrix} \begin{pmatrix} q^{-j-1/2} q^{-j+1/2} \sqrt{j+n+2} & 0 \\ 1/2 q^{-j+1/2} \sqrt{j+n+2} & q^{-j} \sqrt{j+n+1} \end{pmatrix},$$

$$\alpha_{j\mu n}^- := \sqrt{q^{\mu+n+1/2}[j-\mu]} \begin{pmatrix} q^{j+1} & q^{j+1} \sqrt{j+1} \\ 0 & q^{j+1/2} \sqrt{j+1} \end{pmatrix} \begin{pmatrix} q^{j+1} & q^{j+1} \sqrt{j+1} \\ 0 & q^{j+1/2} \sqrt{j+1} \end{pmatrix},$$

$$\beta_{j\mu n}^+ := \sqrt{q^{\mu-n+1/2}[j+\mu+1]} \begin{pmatrix} \sqrt{j+n+2} & 0 \\ q^{-j-1} \sqrt{j+n+2} & q^{-j} \sqrt{j+n+1} \end{pmatrix} \begin{pmatrix} \sqrt{j+n+2} & 0 \\ q^{-j-1} \sqrt{j+n+2} & q^{-j} \sqrt{j+n+1} \end{pmatrix},$$

$$\beta_{j\mu n}^- := \sqrt{q^{\mu+n-1/2}[j-\mu]} \begin{pmatrix} q^{-j-1} \sqrt{j+n+2} & 0 \\ 0 & q^{-j-1/2} \sqrt{j+n+2} \end{pmatrix} \begin{pmatrix} q^{-j-1} \sqrt{j+n+2} & 0 \\ 0 & q^{-j-1/2} \sqrt{j+n+2} \end{pmatrix},$$

with $\tilde{\alpha}_{j\mu n}^+ := (\alpha_{j^+ \mu^- n^-})^*$, $\tilde{\beta}_{j\mu n}^+ := (\beta_{j^+ \mu^- n^-})^*$ and with the q-number of $\alpha \in \mathbb{R}$ be defined as

$$[\alpha] := \frac{q^\alpha - q^{-\alpha}}{q-q^{-1}}.$$
For the purpose of this paper it is sufficient to use the approximate spinorial \( \pi \)-representation \( \pi \) of \( SU_q(2) \) presented in [21, 38] instead of the full spinorial one \( \pi \).

This approximate representation is

\[
\pi(a) := a_+ + a_-, \quad \pi(b) := b_+ + b_-
\]

with the following definitions:

\[
a_+ |j\mu n\rangle := q_{j+\mu}^* \left( \begin{array}{cc} q_{j+n+1} & 0 \\ 0 & q_{j+n} \end{array} \right) |j+\mu + n\rangle, \\
a_- |j\mu n\rangle := q_{j+\mu}^* \left( \begin{array}{cc} 0 & q_{j+n} \\ q_{j+n+1} & 0 \end{array} \right) |j+\mu + n\rangle, \\
b_+ |j\mu n\rangle := q_{j+\mu}^* \left( \begin{array}{cc} q_{j+n} & 0 \\ 0 & q_{j+n+1} \end{array} \right) |j+\mu + n\rangle, \\
b_- |j\mu n\rangle := -q_{j+\mu}^* \left( \begin{array}{cc} 0 & q_{j+n} \\ q_{j+n+1} & 0 \end{array} \right) |j+\mu + n\rangle.
\]

(11)

All disregarded terms are trace-class and do not influence residue calculations. More precisely, \( \pi(x) - \tilde{\pi}(x) \in K_q \), where \( K_q \) is the principal ideal generated by the operators

\[
J_q |j\mu n\rangle := q^j |j\mu n\rangle.
\]

(12)

Actually, \( K_q \) is independent of \( q \) and is contained in all ideals of operators such that \( \mu_n = o(n^{-\alpha}) \) (infinitesimal of order \( \alpha \)) for any \( \alpha > 0 \), and \( K_q \subset OP^{-\infty} \).

We define the alternative orthonormal basis \( v^{j+}_{m,l} \) and \( v^{j-}_{m,l} \) and the vector notation

\[
v^{j+}_{m,l} := \left( \begin{array}{c} v^{j+}_{m,l} \\ v^{j-}_{m,l} \end{array} \right) \text{ where } v^{j+}_{m,l} := |j, m-j, l-j, \uparrow\rangle, \quad v^{j-}_{m,l} := |j, m-j, l-j, \downarrow\rangle.
\]

Here \( j \in \frac{1}{2} \mathbb{N}, \ 0 \leq m \leq 2j, \ 0 \leq l \leq 2j + 1 \) and \( v^{j\pm}_{m,l} \) is zero whenever \( j = 0 \) or \( l = 2j \) or \( 2j + 1 \).

The interest is that now, the operators \( a_\pm \) and \( b_\pm \) satisfy the simpler relations

\[
a_+ v^{j+}_{m,l} = q^{m+l+1} v^{j+}_{m+1,l+1}, \quad a_- v^{j+}_{m,l} = q^{m+l+1} v^{j+}_{m,l}, \\
b_+ v^{j+}_{m,l} = q^l v^{j+}_{m,l+1}, \quad b_- v^{j+}_{m,l} = -q^l q^m v^{j+}_{m,l-1}.
\]

(13)

Thus

\[
a^*_+ v^{j+}_{m,l} = q m q^l v^{j-}_{m-1,l-1}, \quad a^*_+ v^{j+}_{m,l} = q^{m+l+1} v^{j+}_{m,l}, \\
b^*_+ v^{j+}_{m,l} = q^l q^{m+l-1} v^{j+}_{m,l-1}, \quad b^*_+ v^{j+}_{m,l} = -q^m q^{l+1} v^{j+}_{m,l+1}.
\]

(14)

Moreover, we have

\[
a_- a_+ = q^2 a_+ a_-, \quad b_- b_+ = q^2 b_+ b_-, \quad b_+ a_+ = q a_+ b_+, \quad b_- a_- = q a_- b_-,
\]

\[
a^*_+ a_+ = q^2 a^*_+ a_+, \quad a^*_+ a_- = a_+ a^*_-, \quad a^*_+ b_+ = q b_+ a^*_+, \quad a^*_+ b_- = q b^- a^*_-, \\
a^*_+ a_+ = q^2 a_- a^*_+, \quad b^*_+ b_+ = b_+ b^*_-, \quad b^*_+ a_+ = q a_+ b^*_+, \quad a_+ b_+ = q b_+ a_+.
\]

(15)

Note for instance that

\[
a_+ a^*_+ v^{j+}_{m,l} = q^{2m} q^l v^{j+}_{m,l}, \quad a^*_+ a_+ v^{j+}_{m,l} = q^{2m+1} q^l v^{j+}_{m,l}, \\
b_+ b^*_+ v^{j+}_{m,l} = q^{2m} q^l v^{j+}_{m,l}, \quad b^*_+ b_+ v^{j+}_{m,l} = q^{2m+1} q^l v^{j+}_{m,l}.
\]
so applied to \( v^j_{m,l} \), we get the first relation (and similarly for the others)

\[
\begin{align*}
    a_+^*a_+ - q^2 a_+a_+^* + q^2 (b_+^*b_+ - b_+b_+^*) &= 1 - q^2, \\
    a_+a_+^* + a_-^*a_- + b_+b_-^* + b_-b_+ &= 1, \\
    a_+^*a_+ + a_-a_- + q^2 (b_+^*b_+ + b_-b_-) &= 1, \\
    a_-^*a_- - q^2 a_-a_-^* + q^2 b_-^*b_- - q^2 b_-b_-^* &= 0, \\
    a_+a_+^* + b_-b_+ &= 0, \\
    a_-a_-^* + b_-b_+ &= 0, \\
    b_+b_-^* - b_-b_+ + b_-b_-^* - b_-b_- &= 0, \\
    qa_+b_- - b_-a_+ + qa_-a_- - b_+a_- &= 0.
\end{align*}
\]

\( a_j, \mu, n, \uparrow \) denotes the polar decomposition of \( D \) with eigenvalues \( 2j + 1 \) for \( j \in \frac{1}{2}\mathbb{N} \), have multiplicities \((2j + 1)(2j + 2)\),
the eigenvalues \(- (2j + 1)\) for \( j \in \frac{1}{2}\mathbb{N}^* \), have multiplicities \( 2j(2j + 1) \).

The reality operator:
This antilinear operator \( \mathcal{J} \) is defined on the basis of \( \mathcal{H} \) by

\[
\begin{align*}
    \mathcal{J} |j, \mu, n, \uparrow \rangle &= i^{2(2j + \mu + n)} |j, -\mu, -n, \uparrow \rangle, \\
    \mathcal{J} |j, \mu, n, \downarrow \rangle &= i^{2(2j - \mu - n)} |j, -\mu, -n, \downarrow \rangle.
\end{align*}
\]

And two others:
Note that we also use two other infinite dimensional \*-representations \( \pi_{\pm} \) of \( \mathcal{A} \) on \( \ell^2(\mathbb{N}) \) defined as follows on the orthonormal basis \( \{ \varepsilon_n : n \in \mathbb{N} \} \) of \( \ell^2(\mathbb{N}) \) by

\[
\begin{align*}
    \pi_{\pm}(a) \varepsilon_n &:= q_{n+1} \varepsilon_{n+1}, & \pi_{\pm}(b) \varepsilon_n &:= \pm q^n \varepsilon_n, \\
    q_n &:= \sqrt{1 - q^{2n}}.
\end{align*}
\]

These representations are irreducible but not faithful since for instance \( \pi_{\pm}(b - b^*) = 0 \).

The Dirac operator:
It is chosen the same as in the classical case of a 3-sphere with the round metric:

\[
\mathcal{D} |j\mu n \rangle := \left( \begin{array}{c} 2j + 1 \frac{3}{2} \frac{3}{2j} \\ 0 \end{array} \right) |j\mu n \rangle,
\]

which means, with our convention, that \( \mathcal{D} v^j_{ml} = \left( \begin{array}{c} 2j + 1 \frac{3}{2} \frac{3}{2j} \\ 0 \end{array} \right) v^j_{ml} \). Note that this operator is asymptotically diagonal with linear spectrum and

the eigenvalues \( 2j + \frac{3}{2} \) for \( j \in \frac{1}{2}\mathbb{N} \), have multiplicities \((2j + 1)(2j + 2)\),
the eigenvalues \(- (2j + \frac{3}{2})\) for \( j \in \frac{1}{2}\mathbb{N}^* \), have multiplicities \( 2j(2j + 1) \).

So this Dirac operator coincide exactly with the classical one on the 3-sphere (see [1, 32]) with a gap around 0.

Let \( \mathcal{D} = \mathcal{F}^* \mathcal{D} \mathcal{F} \) be the polar decomposition of \( \mathcal{D} \), thus

\[
\begin{align*}
|\mathcal{D}| |j\mu n \rangle &= \left( \begin{array}{c} d_j^* \ 0 \\ 0 \ -d_j \end{array} \right) |j\mu n \rangle, & d_j &:= 2j + \frac{1}{2}, \\
\mathcal{F} |j\mu n \rangle &= \left( \begin{array}{c} 1 \ 0 \\ 0 \ -1 \end{array} \right) |j\mu n \rangle,
\end{align*}
\]

and it follows from (11) and (27) that

\[\mathcal{F} \text{ commutes with } a_\pm, b_\pm.\]
thus it satisfies
\[
J^{-1} = -J = J^* \quad \text{and} \quad DJ = JD,
\]
\[
J v_{m,l}^j = i^{2(m+l)-1} v_{2j-m,2j+1-l}^j, \quad J v_{m,l}^{j+1} = i^{2(m+l)+1} v_{2j-m,2j-1-l}^j.
\]

**The Hopf map $r$**

For the explicit calculations of residues, we need a *-homomorphism $r : \pi_+ (A) \otimes \pi_- (A)$ defined by the tensor product in the sense of Hopf algebras of representations $\pi_+$ and $\pi_-:
\[
\begin{align*}
\delta (\alpha) & := \pi_+ (\alpha) \otimes \pi_- (\alpha), \\
\delta (\beta) & := -\pi_+ (\beta) \otimes \pi_- (\beta),
\end{align*}
\]

These homomorphisms appeared in [39] with the translation $\alpha \leftrightarrow a^*$, $\gamma \leftrightarrow -b$. In particular, if $U := \left( \begin{smallmatrix} a^* & b \\ -a & b^* \end{smallmatrix} \right)$ is the canonical generator of the $K_1 (A)$-group $(\Delta a, \Delta b) = (a, b) \otimes U$ where the last $\otimes$ means the matrix product of tensors of components.

**The grading:**

According to the shift $j \rightarrow j^\pm$ appearing in formulae [13], [14], we get a $\mathbb{Z}$-grading on $X$ defined by the degree $+1$ on $a_+, b_+, a^+, b^+$, and $-1$ on $a_-, b_-, a^-, b^-$. Any operator $T \in X$ can be (uniquely) decomposed as $T = \sum_{j \in \mathbb{Z}} T_j$ where $T_j$ is homogeneous of degree $j$.

For $T \in X$, $T^0$ will denote the 0-degree part of $T$ for this grading and by a slight abuse of notations, we write $r(T)^0$ instead of $r(T^0)$.

**The symbol map:**

We also use the *-homomorphism $\sigma : \pi_\pm (A) \rightarrow C^\infty (S^1)$ defined for $z \in S^1$ on the generators by
\[
\sigma (\pi_\pm (a)) (z) := z, \quad \sigma (\pi_\pm (a^*)) (z) := \bar{z}, \quad \sigma (\pi_\pm (b)) (z) = \sigma (\pi_\pm (b^*)) (z) := 0.
\]

The application $(\sigma \otimes \sigma) \circ r$ is defined on $X$ (and so on $B$) with values in $C^\infty (S^1) \otimes C^\infty (S^1)$.

We define
\[
dT := [\mathcal{D}, T] \quad \text{and} \quad \delta (T) := [[\mathcal{D}], T].
\]

**Lemma 3.2.** $a_\pm, b_\pm$ are bounded operators on $\mathcal{H}$ such that for all $p \in \mathbb{N},$

(i) $\delta (a_\pm) = \pm a_\pm, \quad \delta (b_\pm) = \pm b_\pm,$

(ii) $\delta^p (\pi (a)) = a_\pm + (-1)^p a_\mp, \quad \delta^p (\pi (b)) = b_\pm + (-1)^p b_\mp,$

(iii) $\delta (a_\pm^p) = \pm p a_\pm^p, \quad \delta (b_\pm^p) = \pm p b_\pm^p.
\]

**Proof.** (i) By definition, $a_\pm |j\mu n\rangle = \left( \begin{smallmatrix} 0 & \alpha_\pm \\ 0 & \beta_\pm \end{smallmatrix} \right) |j^\pm \mu ^n \rangle.$ Where the numbers $\alpha_\pm$ and $\beta_\pm$ depend on $j, \mu, \nu,$ and $q$, so we get by [26]
\[
\delta (a_\pm) |j\mu n\rangle = \left( \begin{smallmatrix} 0 & \alpha_\pm \\ 0 & \beta_\pm \end{smallmatrix} \right) |j^\pm \mu ^n \rangle = \left( \begin{smallmatrix} 0 & \alpha_\pm \\ 0 & \beta_\pm \end{smallmatrix} \right) |j^\pm \mu ^n \rangle
\]

and similar proofs for $b_\pm,$

(ii) and (iii) are straightforward consequences of (i) and definition of $\pi.$ \qed

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We note

\( \mathcal{B} \) the \( * \)-subalgebra of \( \mathcal{B}(H) \) generated by the operators in \( \delta^k(\pi(A)) \) for all \( k \in \mathbb{N} \),

\( \Psi_0^0(A) \) the algebra generated by \( \delta^k(\pi(A)) \) and \( \delta^k([D, \pi(A)]) \) for all \( k \in \mathbb{N} \),

\( X \) the \( * \)-subalgebra of \( \mathcal{B}(H) \) algebraically generated by the set \( \{a_\pm, b_\pm\} \).

Remark 3.3. By Lemma [38], we see that, modulo \( OP^{-\infty} \), \( X \) is equal to \( \mathcal{B} \) and in particular

\( \Psi_0^0(A) \) the algebra generated by \( \delta^k(\pi(A)) \) and \( \delta^k([D, \pi(A)]) \) for all \( k \in \mathbb{N} \),

\( X \) the \( * \)-subalgebra of \( \mathcal{B}(H) \) algebraically generated by the set \( \{a_\pm, b_\pm\} \).

3.2 The noncommutative integrals

Recall that for any pseudodifferential operator \( T \), \( \int T := \text{Res}_{s=0} \zeta_D^T(s) \) where \( \zeta_D^T(s) := \text{Tr}(T|D|^{-s}) \).

Theorem 3.4. The dimension spectrum (without reality structure given by \( J \)) of the spectral triple \( (A(SU_q(2)), H, D) \) is simple and equal to \( \{1, 2, 3\} \).

Moreover, the corresponding residues for \( T \in \mathcal{B} \) are

\[
\begin{align*}
\int T|D|^{-3} &= 2(\tau_1 \otimes \tau_1)(r(T)^o), \\
\int T|D|^{-2} &= 2(\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1)(r(T)^o), \\
\int T|D|^{-1} &= (2 \tau_0 \otimes \tau_0 - \frac{1}{2} \tau_1 \otimes \tau_1)(r(T)^o), \\
\int FT|D|^{-3} &= 0, \\
\int FT|D|^{-2} &= 0, \\
\int FT|D|^{-1} &= (\tau_0 \otimes \tau_1 - \tau_1 \otimes \tau_0)(r(T)^o),
\end{align*}
\]

where the functionals \( \tau_0, \tau_1 \) are defined for \( x \in \pi_\pm(A) \) by

\[
\tau_0(x) := \lim_{N \to \infty} \left( \text{Tr}_N x - (N + 1) \tau_1(x) \right), \quad \tau_1(x) := \frac{1}{2\pi} \int_0^{2\pi} \sigma(x)(e^{i\theta}) \, d\theta,
\]

with \( \text{Tr}_N x = \sum_{n=0}^N \varepsilon_n(x, x \varepsilon_n) \).

Proof. Consequence of [38, Theorem 4.1 and (4.3)].

Remark 3.5. Since \( F \) is not in \( \mathcal{B} \), the equation of Theorem [34] are not valid for all \( T \in \Psi_0^0(A) \).

But when \( T \in \Psi_0^0(A) \), \( \int T|D|^{-k} = 0 \) for \( k \not\in \{1, 2, 3\} \) since the dimension spectrum is \( \{1, 2, 3\} \) [38].

Compared to [38] where we had

\[
\begin{align*}
\tau_0^+(x) := \lim_{N \to \infty} \text{Tr}_N x - (N + \frac{3}{2}) \tau_1(x), \\
\tau_0^-(x) := \lim_{N \to \infty} \text{Tr}_N x - (N + \frac{1}{2}) \tau_1(x),
\end{align*}
\]
we replaced them with $\tau_0$:

$$\tau_0^+ = \tau_0 - \frac{1}{2} \tau_1, \quad \tau_0^- = \tau_0 + \frac{1}{2} \tau_1.$$

Note that $\tau_1$ is a trace on $\pi_\pm(A)$ such that $\tau_1(1) = 1$, while $\tau_0$ is not since $\tau_0(1) = 0$ and

$$\tau_0(\pi_\pm(aa^*)) = \lim_{N \to \infty} \sum_{n=0}^{\infty} (1 - q^{2n}) - (N + 1) = -\frac{1}{1-q^2}, \quad \text{(31)}$$

do not vanish on 1-forms since $\tau_0(1) = 0$, we have

so, because of the shift, the replacement $a \leftrightarrow a^*$ gives

$$\tau_0(\pi_\pm(a^*a)) = q^2 \tau_0(\pi_\pm(aa^*)). \quad \text{(32)}$$

### 3.3 The tadpole

**Lemma 3.6.** For $SU_q(2)$, the condition of the vanishing tadpole (see [15]) is not satisfied.

**Proof.** For example, an explicit calculation gives $\int \pi(b)[D, \pi(b^*)]D^{-1} = \frac{2}{1-q^2}$.

Let $x, y \in \pi(A)$. Since $[F, x] = 0$, we have

$$\int x[D, y]D^{-1} = \int x\delta(y)[D]^{-1} = \tau'(r(x\delta(y))^0)$$

where $\tau' := 2\tau_0 \otimes \tau_0 - \frac{1}{2} \tau_1 \otimes \tau_1$.

By Lemma [3.2] $\pi(b)\delta(\pi(b^*)) = (b_+ + b_-)(b_-^* - (b_+)^*) = -b_+ b_+^* + b_- b_-^* + b_+ b_-^* - b_- b_+^*$.

Since only the first two terms have degree 0, we get, using the formulae from Theorem 3.4

$$\tau'(r(-b_+ b_+^*)) = -\tau'(\pi_+ (aa^*) \otimes \pi_-(bb^*)) = -2\tau_0(\pi_+ (aa^*)) \tau_0(\pi_-(bb^*)) + \frac{1}{2} \tau_1 (\pi_+ (aa^*)) \tau_1 (\pi_-(bb^*))$$

and $\tau_1 (\pi_+(aa^*)) = \frac{1}{2\pi} \int_0^{2\pi} 1 d\theta = 1$, $\tau_1 (\pi_-(bb^*)) = 0$. Similarly, using [3.2]

$$\tau'(r(-b_- b_-^*)) = 2\tau_0 (\pi_+ (bb^*)) \tau_0 (\pi_-(a^*a)) = 2q^2 \tau_0 (\pi_-(aa^*)) \tau_0 (\pi_+ (bb^*))$$

Since $\tau_0 (\pi_+(bb^*)) = Tr (\pi_+(bb^*)) = \sum_{n=0}^{\infty} q^{2n} = \frac{1}{1-q^2}$ and [3.1]

$$\int \pi(b)[D, \pi(b^*)]D^{-1} = 2 \frac{1}{1-q^2} \frac{1}{1-q^2} + 2q^2 \frac{1}{1-q^2} \frac{1}{1-q^2} = \frac{2}{1-q^2}. \quad \square$$

In particular the pairing of the tadpole cyclic cocycle $\phi_1$ with the generator of $K_1$-group is nontrivial:

**Remark 3.7.** Other examples: with the shortcut $x$ instead of $\pi(x)$,

$$\begin{align*}
(\tau_1 \otimes \tau_1) r(a\delta(a^*))^0 &= -1, & (\tau_1 \otimes \tau_1) r(a^*\delta(a)^0) &= 1, \\
(\tau_0 \otimes \tau_0) r(a\delta(a^*)) &= \frac{1}{q-1}, & (\tau_0 \otimes \tau_0) r(a^*\delta(a)^0) &= \frac{q^2}{q^2-1}, \\
\int a\delta(a^*)[D]^{-1} &= \frac{q^{2+3}}{2(q^2-1)}, & \int a^*\delta(a)[D]^{-1} &= \frac{3q^2+1}{2(q^2-1)}, \\
\int b\delta(b)[D]^{-1} &= 0, & \int b^*\delta(b^*)[D]^{-1} &= 0, \\
\int b\delta(b^*)[D]^{-1} &= -\frac{2}{q^2-1}, & \int b^*\delta(b)[D]^{-1} &= -\frac{2}{q^2-1}.
\end{align*}$$

In particular, $N\Phi_1$ does not vanish on 1-forms since $\int_{N\Phi_1} ada^* = N\Phi_1(a,a^*) = -1$. 

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Let $U$ be the canonical generator of the $K_1(A)$-group, $U = (\quad a_{q, r} \quad b_{q, r})$ acting on $\mathcal{H} \otimes \mathbb{C}^2$. Then for $A_U := \sum_{k,l=1}^{2} \pi(U_{kl}) d\pi(U^*_{kl})$, using above remark, $\int_{\phi_1} A_U = -2$ as obtained in [38, page 391]: in fact, with $P := \frac{1}{2}(1 + F)$,

$$\psi_1(U, U^*) := 2 \sum_{k,l} \int U_{kl} \delta(U^*_{kl}) P[D]^{-1} - \int U_{kl} \delta^2(U^*_{kl}) P[D]^{-2} + \frac{2}{3} \int U_{kl} \delta^3(U^*_{kl}) P[D]^{-3}$$

satisfies $\psi_1(U, U^*) = 2 \sum_{k,l} \int U_{kl} \delta(U^*_{kl}) P[D]^{-1} = \int_{\phi_1} A_U$.

4 Spectral operator and spectral action on $SU_q(2)$

4.1 Spectral action in dimension 3 with $[F, A] \in OP^{-\infty}$

Let $(A, \mathcal{H}, D)$ a be real spectral triple of dimension 3. Assume that $[F, A] \in OP^{-\infty}$, where $F := D[D]^{-1}$ (we suppose $D$ invertible). Let $A$ be a selfadjoint one form, so $A$ is of the form $A = \sum_i a_i db_i$ where $a_i, b_i \in A$.

Thus, $A \simeq AF \mod OP^{-\infty}$ where $A := \sum_i a_i \delta(b_i)$ is the $\delta$-one-form associated to $A$. Note that $A$ and $F$ commute modulo $OP^{-\infty}$.

We define

$$D_A := D + \tilde{A}, \quad \tilde{A} := A + JAJ^{-1}.$$ 

**Theorem 4.1.** The coefficients of the full spectral action (with reality operator) on any real spectral triple $(A, \mathcal{H}, D)$ of dimension 3 such that $[F, A] \in OP^{-\infty}$ are

(i) $\int |D_A|^{-3} = \int |D|^{-3}$.

(ii) $\int |D_A|^{-2} = \int |D|^{-2} - 4 \int A|D|^{-3}$.

(iii) $\int |D_A|^{-1} = \int |D|^{-1} - 2 \int A|D|^{-3} + 2 \int A^2|D|^{-3} + 2 \int AJAJ^{-1}|D|^{-3}.$

(iv) $\zeta_D(0) = \zeta_D(0) - 2 \int A|D|^{-1} + \int A(A + JAJ^{-1})|D|^{-2} + \int \delta(A)(A + JAJ^{-1})|D|^{-3}$

$$- \frac{2}{3} \int A^3|D|^{-3} - 2 \int A^2JAJ^{-1}|D|^{-3}.$$ 

**Proof.** (i) We apply [22, Proposition 4.9].

(ii) By [22, Lemma 4.10 (i)], we have $\int |D_A|^{-2} = \int |D|^{-2} - \int (\tilde{A}D + D\tilde{A} + \tilde{A}^2)|D|^{-4}$. By the trace property of the noncommutative integral and the fact that $\tilde{A}^2|D|^{-4}$ is trace-class, we get $\int |D_A|^{-2} = \int |D|^{-2} - 2 \int \tilde{A}D|D|^{-4} = \int |D|^{-2} - 4 \int \tilde{A}D|D|^{-4}$. Since $\tilde{A}D \sim A|D| \mod OP^{-\infty}$, we get the result.

(iii) By [22, Lemma 4.10 (ii)], we have

$$\int |D_A|^{-1} = \int |D|^{-1} - \frac{1}{2} \int (\tilde{A}D + D\tilde{A} + \tilde{A}^2)|D|^{-3} + \frac{3}{8} \int (\tilde{A}D + D\tilde{A} + \tilde{A}^2)^2|D|^{-5}.$$
Following arguments of (ii), we get
\[ \int (\bar{\alpha}D + D\bar{\alpha} + \bar{\alpha}^2)|D|^{-3} = 4\int A|D|^{-2} + 2\int A^2|D|^{-3} + 2\int AJAJ^{-1}|D|^{-3}, \]
\[ \int (\bar{\alpha}D + D\bar{\alpha} + \bar{\alpha}^2)|D|^{-5} = 8\int A^2|D|^{-3} + 8\int AJAJ^{-1}|D|^{-3}, \]
and the result follows.

(iv) By [22, Lemma 4.5] gives \( \zeta_{D_\alpha}(0) = \sum_{j=1}^{3} \frac{(-1)^j}{j} \int (\bar{\alpha}D)^{-1}. \)
Moreover, we have \( \int A|D|^{-1} \) and \( \int (\bar{\alpha}D)^{-1} \) are trace-class here, we get
\begin{align*}
\int (\bar{\alpha}D)^{-1} &= 2\int AJAJ^{-1}|D|^{-3} + 2\int A^2|D|^{-3} + 2\int \delta(A)AJAJ^{-1}|D|^{-3}.
\end{align*}
Since \( \delta(A) \in OP^0 \), we can check that \( \int (A|D|^{-1})^2 = \int A^2|D|^{-3} + \int \delta(A)A|D|^{-3} \) and, with the same argument, that \( \int AJAJ^{-1}|D|^{-3} = \int AJAJ^{-1}|D|^{-2} + \int \delta(A)AJAJ^{-1}|D|^{-3} \). Thus, we get
\[ \int (\bar{\alpha}D)^{-1} = 2\int A(A + JAJ^{-1})|D|^{-2} + 2\int \delta(A)(A + JAJ^{-1})|D|^{-3}. \] (33)
The third term to be computed is
\[ \int (\bar{\alpha}D)^{-3} = 2\int (A|D|^{-1})^3 + 4\int (A|D|^{-1})^2JAJ^{-1}|D|^{-1} + 2\int A|D|^{-1}JAJ^{-1}|D|^{-1}A|D|^{-1}. \]
Any operator in \( OP^{-4} \) being trace-class here, we get
\[ \int (\bar{\alpha}D)^{-3} = 2\int A^3|D|^{-3} + 4\int A^2JAJ^{-1}|D|^{-3} + 2\int AJAJ^{-1}A|D|^{-3}. \] (34)
Since \( \int AJAJ^{-1}A|D|^{-3} = \int A^2JAJ^{-1}|D|^{-3} \) by trace property and the fact that \( \delta(A) \in OP^0 \), the result follows then from (33) and (34).

Corollary 4.2. For the spectral action of \( \bar{\alpha} \) without the reality operator (i.e. \( D_\alpha = D + \bar{\alpha} \)), we get
\begin{align*}
\int |D_\alpha|^{-2} &= \int |D|^{-2} - 2\int A|D|^{-3}, \\
\int |D_\alpha|^{-1} &= \int |D|^{-1} - \int A|D|^{-2} + \int A^2|D|^{-3}, \\
\zeta_{D_\alpha}(0) &= \zeta_D(0) - \int A|D|^{-1} + \frac{1}{2} \int A^2|D|^{-2} + \frac{1}{2} \int \delta(A)A|D|^{-3} - \frac{1}{3} \int A^3|D|^{-3}.
\end{align*}

4.2 Spectral action on \( SU_q(2) \): main result

On \( SU_q(2) \), since \( F \) commutes with \( a_\pm \) and \( b_\pm \), the previous lemma can be used for the spectral action computation. Here is the main result of this section

Theorem 4.3. In the full spectral action (iii) (with the reality operator) of \( SU_q(2) \) for a one-form
\[ A \text{ and } A \text{ its associated } \delta\text{-one-form, the coefficients are:} \]
\[
\int |D_A|^{-3} = 2, \\
\int |D_A|^{-2} = -4 \int A|D|^{-3}, \\
\int |D_A|^{-1} = -\frac{1}{2} + 2(\int A^2|D|^{-3} - \int A|D|^{-2}) + |\int A|D|^{-3}|^2, \\
\zeta_{D_A}(0) = -2 \int A|D|^{-1} + \int A^2|D|^{-2} - \frac{2}{3} \int A^3|D|^{-3} + \frac{1}{2} \int A|D|^{-3} \int A|D|^{-2}.
\]

In order to prove this theorem, we will use a decomposition of one-forms in the Poincaré-Birkhoff-Witt basis of \( A \) with an extension of previous representations to operators like \( TJT'J^{-1} \) where \( T \) and \( T' \) are in \( X \).

### 4.3 Balanced components and Poincaré–Birkhoff–Witt basis of \( A \)

Our objective is to compute all integrals in term of \( A \) and the computation will lead to functions of \( A \) which capture certain symmetries on \( A \).

For convenience, let us introduce now these functions:

Let \( \mathcal{A} = \sum_i \pi(x^i)d\pi(y^i) \) on \( SU_q(2) \) be one-form and \( A \) the associated \( \delta \)-one-form. The \( x^i \) and \( y^i \) are in \( \mathcal{A} \) and as such they can be uniquely written as finite sums \( x^i = \sum_\alpha x^i_\alpha m^\alpha \) and \( y^i = \sum_\beta y^i_\beta m^\beta \) where \( m^\alpha := a^{\alpha_1}b^{\alpha_2}b^{\alpha_3} \) is the canonical monomial of \( \mathcal{A} \) with \( \alpha, \beta \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N} \) based on a fixed Poincaré–Birkhoff–Witt type basis of \( \mathcal{A} \).

**Remark 4.4.** Any one-form \( \mathcal{A} = \sum_i \pi(x^i)d\pi(y^i) \) on \( SU_q(2) \) is characterized by a complex valued matrix \( A^\alpha_\beta = \sum_i x^i_\alpha y^i_\beta \) where \( \alpha, \beta \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N} \). This matrix is such that

\[
A = A^\alpha_\beta M^\alpha_\beta
\]

where \( M^\alpha_\beta := \pi(m^\alpha)\delta(\pi(m^\beta)) \).

In the following, we note

\[
\mathcal{A} := \mathcal{A}^\beta_\alpha M^\alpha_\beta
\]

so for any \( p \in \mathbb{N} \), \( \int \mathcal{A}|D|^{-p} = \int \mathcal{A}|D|^{-p} \).

This presentation of one-forms is not unique modulo \( OP^{-\infty} \) since, as we will see in section 5, \( F = \sum_i x_idy_i \) where \( x_i, y_i \in \mathcal{A} \), thus for any generator \( z \), \( [F, z] = \sum_i x_id(y_i z) - x_i dy_i dz - x_i dy_i = 0 \) mod \( OP^{-\infty} \). We do not know however if this presentation is unique when the \( OP^{-\infty} \) part is taken into account.

The \( \delta \)-one-forms \( M^\alpha_\beta \) are said to be canonical. Any product of \( n \) canonical \( \delta \)-one forms, where \( n \in \mathbb{N}^* \), is called a canonical \( \delta^n \)-one-form. Thus, if \( A \) is a \( \delta \)-one-form, \( A^n = (A^n)^{\tilde{\alpha}}_{\tilde{\beta}} \tilde{M}^{\tilde{\beta}} \) where \( \tilde{\alpha} = (\alpha, \alpha', \cdots, \alpha^{(n-1)}) \), \( \tilde{\beta} = (\beta, \beta', \cdots, \beta^{(n-1)}) \) are in \( \mathbb{Z}^n \times \mathbb{N}^n \times \mathbb{N}^n \), \( (A^n)^{\tilde{\beta}}_{\tilde{\alpha}} := A^{\tilde{\beta}}_{\tilde{\alpha}} \cdots A^{\beta_{(n-1)}}_{\alpha_{(n-1)}} \) and \( \tilde{M}^{\tilde{\beta}} \) is the canonical \( \delta^n \)-one form equals to \( M^{\alpha}_{\beta} \cdots M^{\alpha_{(n-1)}}_{\beta_{(n-1)}} \).
Definition 4.5. A canonical $\delta^n$-one-form is $a$-balanced if it is of the form
\[ a^{\alpha_1} \delta(a^{\beta_1}) \cdots a^{\alpha_{n-1}} \delta(a^{\beta_{n-1}}) \]
where \( \sum_{i=0}^{n-1} \alpha_i + \beta_i = 0 \).
For any $\delta$-one-form $A$, the $a$-balanced components of $A^n$ are noted $B_A(A^n)$. Note that
\[ B_A(A^n) = A_{-\beta_1,0} \delta_{\alpha_1+\beta_1,0} \delta_{\alpha_2+\beta_2+\beta_3,0}. \]

Definition 4.6. A canonical $\delta^n$-one-form is balanced if it is of the form
\[ m^\alpha \delta(m^\beta) \cdots m^{\alpha(n-1)} \delta(m^{\beta(n-1)}) \]
where \( \sum_{i=0}^{n-1} \alpha_i + \beta_i = 0 \) and \( \sum_{i=0}^{n-1} \alpha_i + \beta_i = \sum_{i=0}^{n-1} \alpha_i + \beta_i \).
For any $\delta$-one-form $A$, the balanced components of $A^n$ are noted $B(A^n)$. Note that
\[ B(A^n) = A_{-\beta_1,\beta_2,\beta_3} \delta_{\alpha_1+\beta_1,0} \delta_{\alpha_2+\beta_2,\alpha_3+\beta_3}. \]

As we will show, a contribution to the $k$-coefficient in the spectral action, is only brought by one-forms $A$ such that $A^k$ is balanced (and even $a$-balanced in the case $k = 1$).

Note also that if $A$ is balanced, then $A^k$ for $k \geq 1$ is also balanced, whereas the converse is false.

4.4 The reality operator $J$ on $SU_q(2)$

For any $n, p \in \mathbb{N}$,
\[
q_n := \sqrt{1 - q^{2n}}, \quad q_{-n} := 0 \text{ if } n > 0,
q_n^\dagger := q_{n+1} \cdots q_{n+p},
q_n^{-1} := q_n \cdots q_{-(p-1)}.
\]
with the convention $q_{n,0}^{-1} = q_{n,0}^{-1} := 1$. Thus, we have the relations
\[
\pi_+ (a^p) \varepsilon_n = q_n^\dagger \varepsilon_{n+p},
\pi_+ (a^p) \varepsilon_n = q_n^{-1} \varepsilon_{n-p},
\pi_+ (b^p) \varepsilon_n = (\pm q^n)^p \varepsilon_n,
\pi_+ (b^p) \varepsilon_n = (\pm q^n)^p \varepsilon_n,
\]
where $\varepsilon_k := 0$ if $k < 0$.
The sign of $x \in \mathbb{R}$ is noted $\eta_x$. By convention, $a_j := a$, $a_{\pm,j} := a_{\pm}$ if $j \geq 0$ and $a_j := a^*$, $a_{\pm,j} := a^*_{\pm}$ if $j < 0$. Note that, with convention
\[
q_{n,\dagger}^{(\alpha_j)} := q_{n,\dagger}^{\alpha_j} \text{ if } \alpha_j > 0, \quad q_{n,\dagger}^{(\alpha_j)} := q_{n,\dagger}^{\alpha_j} \text{ if } \alpha_j < 0, \quad q_{n,\dagger}^{0} := 1,
\]
we have for any $\alpha_j \in \mathbb{Z}$ and $p \leq \alpha_j$, $\pi_+ (a_{\alpha_j}^p) \varepsilon_n = q_{n,\dagger}^{(\alpha_j)} \varepsilon_{n+\eta_{\alpha_j} p}$.
Recall that the reality operator $J$ is defined by
\[
J v_{m,l}^{(\dagger)} = i^{2(m+1)-1} v_{2j-m,2j+1-l}^{(\dagger)}, \quad J v_{m,l}^{(\ddagger)} = i^{-2(m+1)+1} v_{2j-m,2j-1-l}^{(\ddagger)};
\]
thus the real conjugate operators
\[
\hat{a}_{\pm} := J a_{\pm} J^{-1}, \quad \hat{b}_{\pm} := J b_{\pm} J^{-1}.
\]
satisfy
\[ \hat{a}_+ v_{m,l}^j := -q_{2j+1-m} \left( \begin{smallmatrix} q_{2j+2-l} & 0 \\ 0 & q_{2j-l} \end{smallmatrix} \right) v_{m,l}^j, \quad \hat{a}_- v_{m,l}^j := -q_{2j-m} \left( \begin{smallmatrix} q_{2j+2-l} & 0 \\ 0 & q_{2j-l} \end{smallmatrix} \right) v_{m-1,l-1}^j, \]
\[ \hat{b}_+ v_{m,l}^j := q_{2j+1-m} \left( \begin{smallmatrix} q_{2j+1-l} & 0 \\ 0 & q_{2j-l} \end{smallmatrix} \right) v_{m,l+1}^j, \quad \hat{b}_- v_{m,l}^j := -q_{2j-m} \left( \begin{smallmatrix} q_{2j+1-l} & 0 \\ 0 & q_{2j-l} \end{smallmatrix} \right) v_{m-1,l}^j. \]

So the real conjugate operator behave differently on the up and down part of the Hilbert space. The difference comes from the fact that the index \( l \) is not treated uniformly by \( J \) on up and down parts.

We note \( \hat{X} \) the algebra generated by \( \{ \hat{a}_\pm, \hat{b}_\pm \} \), \( \tilde{X} \) the algebra generated by \( \{ a_\pm, b_\pm, \hat{a}_\pm, \hat{b}_\pm \} \) and \( \mathcal{H}' := \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z}) \) and we construct two \( * \)-representations \( \pi_\pm \) of \( \mathcal{A} \):

The representation \( \pi_+ \) gives bounded operators on \( \mathcal{H}' \) while \( \pi_- \) represents \( \mathcal{A} \) into \( \mathcal{B}(\mathcal{H}' \otimes \mathbb{C}^2) \).

The representation \( \pi_\pm \) is defined on the generators by:
\[ \pi_+(a) \varepsilon_m \otimes \varepsilon_2^j := q_{2j+1-m} \varepsilon_m \otimes \varepsilon_2^{j+1}, \quad \pi_+(b) \varepsilon_m \otimes \varepsilon_2^j := -q_{2j-m} \varepsilon_{m+1} \otimes \varepsilon_2^{j+1} \]
while \( \pi_- \) is defined by:
\[ \pi_-(a) \varepsilon_l \otimes \varepsilon_2^j \otimes \varepsilon_1^l := -q_{2j+1-l} \varepsilon_l \otimes \varepsilon_2^{j+1} \otimes \varepsilon_1^{l+1}, \]
\[ \pi_-(b) \varepsilon_l \otimes \varepsilon_2^j \otimes \varepsilon_1^l := -q_{2j-l} \varepsilon_{l+1} \otimes \varepsilon_2^{j+1} \otimes \varepsilon_1^{l+1}, \]
where \( \varepsilon_1^l \) is the canonical basis of \( \mathbb{C}^2 \) and the + in \( \pm \) corresponds to \( \uparrow \) in \( \uparrow \downarrow \).

The link between \( \pi_\pm \) and \( \pi_\pm \) which explains the notations about these intermediate objects and the fact that \( \pi_\pm \) are representations on different Hilbert spaces, is in the parallel between equations (30), (35) and (36).

Let us give immediately a few properties \((x_\beta = x \text{ if the sign } \beta \text{ is positive and equals } x^* \text{ otherwise}) \)
\[ \pi_+(a_\beta) \varepsilon_m \otimes \varepsilon_2^j = q_{2j-m} \varepsilon_m \otimes \varepsilon_2^{j+1} \]
\[ \pi_-(a_\beta) \varepsilon_l \otimes \varepsilon_2^j \otimes \varepsilon_1^l = (-1)^p q_{2j+1-l-m} \varepsilon_l \otimes \varepsilon_2^{j+1} \otimes \varepsilon_1^{l+1} \]
\[ \pi_+(b_\beta) \varepsilon_m \otimes \varepsilon_2^j = (-1)^p q_{2j-m} \varepsilon_{m+1} \otimes \varepsilon_2^{j+1} \otimes \varepsilon_1^{l+1} \]
\[ \pi_-(b_\beta) \varepsilon_l \otimes \varepsilon_2^j \otimes \varepsilon_1^l = (-1)^p q_{2j-l-m} \varepsilon_{l+1} \otimes \varepsilon_2^{j+1} \otimes \varepsilon_1^{l+1} \]

Note that the \( \pi_\pm \) representations still contain the shift information, contrary to representations \( \pi_\pm \). Moreover, \( \pi_\pm (b) \neq \pi_\pm (b^*) \) while \( \pi_\pm (b) = \pi_\pm (b^*) \).

The operators \( \tilde{a}_\pm, \tilde{b}_\pm \) are coded on \( \mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2 \) as the correspondence
\[ \tilde{a}_+ \longleftrightarrow \pi_+(a) \otimes \pi_-(a), \quad \tilde{a}_- \longleftrightarrow -q \pi_+(b^*) \otimes \pi_-(b^*), \]
\[ \tilde{b}_+ \longleftrightarrow -\pi_+(a) \otimes \pi_-(b), \quad \tilde{b}_- \longleftrightarrow -\pi_+(b^*) \otimes \pi_-(a^*). \] (35)

We now set the following extension to \( \mathcal{B}(\mathcal{H}') \) of \( \pi_+ \) and to \( \mathcal{B}(\mathcal{H}' \otimes \mathbb{C}^2) \) of \( \pi_- \) by
\[ \pi'_+(a) := \pi_+(a) \otimes V, \quad \pi'_+(b) := \pi_+(b) \otimes V \quad (V \text{ is the shift of } \ell^2(\mathbb{Z})), \]
\[ \pi'_-(a) := \pi_-(a) \otimes V \otimes 1_2, \quad \pi'_-(b) := \pi_-(b) \otimes V \otimes 1_2. \]

So, we can define a canonical algebra morphism \( \tilde{\rho} \) from \( \tilde{X} \) into the bounded operators on \( \mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2 \). This morphism is defined on the generators part \( \{ \tilde{a}_\pm, \tilde{b}_\pm \} \) of \( \tilde{X} \) by preceding correspondence and on the generators part \( \{ a_\pm, b_\pm \} \) by \( \text{see (30)} \):
\[ a_+ \longleftrightarrow \pi'_+(a) \otimes \pi'_-(a), \quad a_- \longleftrightarrow -q \pi'_+(b^*) \otimes \pi'_-(b^*), \]
\[ b_+ \longleftrightarrow -\pi'_+(a) \otimes \pi'_-(b), \quad b_- \longleftrightarrow -\pi'_+(b^*) \otimes \pi'_-(a^*). \] (36)
We note $S$ the canonical surjection from $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ onto $\mathcal{H}$. This surjection is associated to the parameters restrictions on $m,j,l,j'$. In particular, the index $j'$ associated to the second $\ell^2(\mathbb{N})$ in $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ is set to be equal to $j$. Any vector in $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ not satisfying these restrictions is sent to 0 in $\mathcal{H}$.

Denote by $I$ the canonical injection of $\mathcal{H}$ into $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ (the index $j$ is doubled). Thus, $\tilde{S} \tilde{\rho}(\cdot)I$ is the identity on $\tilde{X}$.

In the computation of residues of $\zeta^T$ functions, we can therefore replace the operator $T$ by $\tilde{S} \tilde{\rho}(T) I$.

We now extend $\tau_0$ on $\pi'_{\pm}(A)\tilde{\pi}_{\pm}(A)$: For $x,y \in A$, we set

$$\text{Tr}_N (\pi'_+(x)\tilde{\pi}_+(y)) := \sum_{m=0}^{N} \langle \varepsilon_m \otimes \varepsilon_N, \pi'_+(x)\tilde{\pi}_+(y) \varepsilon_m \otimes \varepsilon_N \rangle,$$

$$\text{Tr}_N^1 (\pi'_-(x)\tilde{\pi}_-(y)) := \sum_{l=0}^{N} \langle \varepsilon_l \otimes \varepsilon_{N-l} \otimes \varepsilon_N, \pi'_-(x)\tilde{\pi}_-(y) \varepsilon_l \otimes \varepsilon_{N-l} \otimes \varepsilon_N \rangle,$$

$$\text{Tr}_N^1 (\pi'_-(x)\tilde{\pi}_-(y)) := \sum_{l=0}^{N} \langle \varepsilon_l \otimes \varepsilon_{N+l} \otimes \varepsilon_N, \pi'_-(x)\tilde{\pi}_-(y) \varepsilon_l \otimes \varepsilon_{N+l} \otimes \varepsilon_N \rangle.$$

Actually, a computation on monomials of $A$ shows that $\text{Tr}_N^1 (\pi'_-(x)\tilde{\pi}_-(y)) = \text{Tr}_N (\pi'_-(x)\tilde{\pi}_-(y))$. For convenience, we shall note $\text{Tr}_N (\pi'_-(x)\tilde{\pi}_-(y))$ this functional.

**Lemma 4.7.** Let $x,y \in A$. Then,

(i) $\tau_0(\pi'_+(x)\tilde{\pi}_+(y)) := \lim_{N \to \infty} U_N$ exists where

$$U_N := \text{Tr}_N (\pi'_+(x)\tilde{\pi}_+(y)) - (N+1) \tau_1 (\pi_{\pm}(x)) \tau_1 (\pi_{\pm}(y)).$$

(ii) $U_N = \tau_0(\pi'_+(x)\tilde{\pi}_+(y)) + O(N^{-k})$ for all $k > 0$.

**Proof.** (i) We can suppose that $x$ and $y$ are monomials, since the result will follow by linearity. We will give a proof for the case of the $\pi_+$ representations, the case $\pi_-$ being similar, with minor changes.

We have $\tilde{\pi}_+(y) = (\tilde{\pi}_+ a_{\beta_1})^{[\beta_1]} (\tilde{\pi}_+ b)^{[\beta_2]} (\tilde{\pi}_+ b^*)^{[\beta_3]}$. A computation gives

$$\tilde{\pi}_+(y) \varepsilon_m \otimes \varepsilon_{2j} = \langle \varepsilon_m \otimes \varepsilon_{2j}, \pi'_+(x)\tilde{\pi}_+(y) \varepsilon_m \otimes \varepsilon_{2j} \rangle$$

and with the notation $t_{2j,m} := \langle \varepsilon_m \otimes \varepsilon_{2j}, \pi'_+(x)\tilde{\pi}_+(y) \varepsilon_m \otimes \varepsilon_{2j} \rangle$ and $T_{2j} := \sum_{m=0}^{2j} t_{2j,m}$, we get

$$t_{2j,m} = \langle \varepsilon_m \otimes \varepsilon_{2j}, \pi'_+(x)\tilde{\pi}_+(y) \varepsilon_m \otimes \varepsilon_{2j} \rangle$$

$$= \langle \varepsilon_m \otimes \varepsilon_{2j}, \pi'_+(x)\tilde{\pi}_+(y) \varepsilon_m \otimes \varepsilon_{2j} \rangle$$

$$= f_{\alpha,\beta} q^{2j \lambda} t'_{2j,m} = f_{\alpha,\beta} q^{2j \lambda} t''_{2j,2j-m},$$

where

$$t'_{2j,m} := q^{m(\kappa-\lambda)} \varepsilon_{2j-m} \otimes \varepsilon_m \otimes \varepsilon_N,$$

$$t''_{2j,m} := q^{m(\lambda-\kappa)} \varepsilon_{2j-m} \otimes \varepsilon_m \otimes \varepsilon_N.$$
with $\lambda := \beta_2 + \beta_3 \geq 0$ and $\kappa := \alpha_2 + \alpha_3 \geq 0$. We will now prove that if $\lambda \neq \kappa$, then $(T_{2j})$ is a convergent sequence. Suppose $\kappa > \lambda$. Let us note $U''_{2j} := \sum_{m=0}^{2j} t''_{2j,m}$. Since the $t''_{2j,m}$ are positive and $t''_{2j+1,m} \geq t''_{2j,m}$ for all $j, m$, $U''_{2j}$ is an increasing real sequence. The estimate

$$U''_{2j} \leq \sum_{m=0}^{2j} q^{m(\lambda - \kappa)} \leq \frac{1}{1 - q^{\lambda - \kappa}} < \infty$$

proves then that $U''_{2j}$ is a convergent sequence. With $T_{2j} = f_{\alpha, \beta}^2 \kappa U''_{2j}$, we obtain our result.

Suppose now that $\lambda > \kappa$. Let us note $U''_{2j} := \sum_{m=0}^{2j} t''_{2j,m}$. Since the $t''_{2j,m}$ are positive and $t''_{2j+1,m} \geq t''_{2j,m}$ for all $j, m$, $U''_{2j}$ is an increasing real sequence. The estimate

$$U''_{2j} \leq \sum_{m=0}^{2j} q^{m(\lambda - \kappa)} \leq \frac{1}{1 - q^{\lambda - \kappa}} < \infty$$

proves then that $U''_{2j}$ is a convergent sequence. With $T_{2j} = f_{\alpha, \beta}^2 \kappa U''_{2j}$, we have again our result. Moreover, note that if $\lambda$ and $\kappa$ are both different from zero, the limit of $(T_{2j})$ is zero and more precisely,

$$T_{2j} = O(q^{2j\lambda}) \quad \text{if} \quad \lambda > \kappa > 0, \quad (39)$$

$$T_{2j} = O(q^{2j\kappa}) \quad \text{if} \quad \lambda > \kappa > 0. \quad (40)$$

Suppose now that $\lambda = \kappa \neq 0$. In that case, $(T_{2j})$ also converges rapidly to zero. Indeed, let us fix $q < \varepsilon < 1$. we have $\varepsilon^{-2j\lambda} T_{2j} = \sum_{m=0}^{2j} c_m d_{2j-m} = \varepsilon \delta d_{2j}$ where $c_m := f_{\alpha, \beta} (q/\varepsilon)^\lambda m^{1/\alpha_1} d_{m-\alpha_1,\alpha_1}$ and $d_m := (q/\varepsilon)^\lambda m^{1/\beta_1} d_{m,\beta_1}$. Since both $\sum_{m} c_m$ and $\sum_{m} d_m$ are absolutely convergent series, their Cauchy product $\sum_{j=0}^\infty \varepsilon^{-2j\lambda} T_{2j}$ is convergent. In particular, $\lim_{j \to \infty} \varepsilon^{-2j\lambda} T_{2j} = 0$, and

$$T_{2j} = O(\varepsilon^{2j\lambda}). \quad (41)$$

Finally, $T_{2j}$ has a finite limit in all cases except possibly when $\lambda = \kappa = 0$, which is the case when $1 > \alpha = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0$. In that case, $t_{2j,m} = 1$. A straightforward computation gives $\tau_1(\pi_\pm(x)) \tau_1(\pi_\pm(y)) = \delta_{\alpha_1,0} \delta_{\beta_1,0} \delta_{\alpha_2,0} \delta_{\beta_2,0} \delta_{\alpha_3,0} \delta_{\beta_3,0}$. Thus,

$$U_{2j} = T_{2j} - (2j + 1) \delta_{\alpha_1,0} \delta_{\beta_1,0} \delta_{\alpha_2,0} \delta_{\beta_2,0} \delta_{\alpha_3,0} \delta_{\beta_3,0}$$

has always a finite limit when $j \to \infty$.

(ii) The result is clear if $\lambda = \kappa = 0$ (in that case $U_N = \tau_0 = 0$). Suppose $\lambda = \kappa$ is not zero. In that case $U_{2j} = T_{2j}$. By $(40)$, $(39)$ and $(11)$, we see that if $\lambda > \kappa > 0$ or $\kappa > \lambda > 0$ or $\lambda = \kappa$, $(T_{2j})$ converges to 0 with a rate in $O(\varepsilon^2\alpha)$ where $\alpha > 0$ and $q \leq \varepsilon < 1$. Thus, it only remains to check the cases $(\kappa > 0, \lambda = 0)$ and $(\kappa = 0, \lambda > 0)$. In the first one, we get from $(37)$,

$$U_{2j} = f_{\alpha, \beta} \sum_{m=0}^{2j} q^{m\kappa} t_{2j-\kappa,\kappa|_{\beta_1}}. \quad \text{If} \quad \beta_1 = 0, \quad \text{we are done.}$$

Suppose $\beta_1 > 0$. We have $q^{1/\beta_1} t_{2j-\kappa,\kappa|_{\beta_1}} = \sum_{p=0}^{\infty} l_p q^{|p_1|_{\beta_1} - 1} p_1 q^{2|p_1|_{\beta_1} - 1} p_1 q^{2|p_1|_{\beta_1} - 1} p_1$. Thus, cutting the sum in two, we get, noting $L_{2j} := f_{\alpha, \beta} \sum_{m=0}^{2j} q^{m\kappa}$,

$$U_{2j} - L_{2j} = f_{\alpha, \beta} \sum_{p_1 > \kappa/2} l_p q^{|p_1|_{\beta_1} - 1} q^{2|p_1|_{\beta_1} - 1} q^{2|p_1|_{\beta_1} - 1} p_1 + f_{\alpha, \beta} \sum_{0 \leq p_1 \leq \kappa/2} l_p q^{|p_1|_{\beta_1} - 1} q^{2|p_1|_{\beta_1} - 1} q^{2|p_1|_{\beta_1} - 1} p_1.$$
Since $\sum_{0\leq |p_1| \leq \kappa/2} l_\rho q^{r_\rho} q^{4|p_1|+j} \sum_{m=0}^{2j} q^{m(\kappa-|p_1|)}$ is in $O_{j\to\infty}(j^q 4^j)$, we have, modulo a rapidly decreasing sequence,

$$U_{2j} - L_{2j} \sim f_{\alpha,\beta} \sum_{|p_1| > \kappa/2} l_\rho q^{r_\rho} q^{4|p_1|+j} \frac{2^{4|p_1|+j-q(2j+1-|p_1|)} 1}{q^{\kappa-|p_1|}} =: f_{\alpha,\beta} q^{2\kappa j} V_{2j}$$

with

$$V_{2j} = \sum_{|p_1| > \kappa/2} l_\rho q^{r_\rho} \frac{2^{4|p_1|+j-q(2j+1-|p_1|)} 1}{q^{\kappa-|p_1|}} = \sum_{|p_1| > \kappa/2} \sum_{m=0}^{2j} l_\rho q^{r_\rho} q^{(2|p_1|+\kappa)m}.$$ 

The family $v_{m,p} := (l_\rho q^{r_\rho} q^{(2|p_1|+\kappa)m})_{(p,m)\in I}$, where $I = \{(p,m) \in \mathbb{N}^{\beta_1} \times \mathbb{N} : |p_1| > \kappa/2\}$ is (absolutely) summable. Indeed $|v_{m,p}| \leq |l_\rho|^2 q^{r_\rho} q^{m}$ so $|v_{m,p}|$ is summable as the product of two summable families. As a consequence, $\lim_{j\to\infty} V_{2j}$ exists and is finite, which proves that $\lim_{j\to\infty} V_{2j}$, and thus $(U_{2j} - L_{2j})$ converge rapidly to 0.

Suppose now that $0 < \beta_1 < 0$. In that case, $q^{2j-\kappa_1} q^{j_1} = q^{2j-\kappa_1} q^{j_1}$, we get $U_{2j} = f_{\alpha,\beta} \sum_{m=0}^{2j} q^{m\kappa} q^{j_1} = f_{\alpha,\beta} q^{-\kappa\kappa_1} q^{\kappa_1 \kappa_1} \sum_{m=0}^{2j} q^{m\kappa} q^{j_1} q^{j_1}$, so the same arguments as in case $\beta_1 > 0$ apply here, the summation on $m$ simply shifted of $|j_1|$.

The same proof can be applied for the other case ($\kappa = 0, \lambda > 0$). This time, we only need to use (38) instead of (37) and the preceding arguments follow by replacing $\kappa$ by $\lambda$ and $\beta_1$ by $\alpha_1$.  

**Remark 4.8.** Contrary to the preceding $\tau_0$, the new functional contains the shift information. In particular, it filters the parts of nonzero degree.

If $T \in X\tilde{\mathcal{X}}$, $\tilde{\rho}(T) \in \pi_+ (A) \pi_+ (A) \otimes \pi_- (A) \pi_- (A)$. For notational convenience, we define $\tau_1$ on $\pi_+ (A) \pi_\pm (A)$ as

$$\tau_1 (\pi_\pm (x) \pi_\pm (y)) := \tau_1 (\pi_\pm (x)) \tau_1 (\pi_\pm (y)).$$

In the following, the symbol $\tilde{\sim}$ means equals modulo a entire function.

**Theorem 4.9.** Let $T \in X\tilde{\mathcal{X}}$. Then

(i) $\tilde{\zeta}_T (s) \tilde{\sim} 2 (\tau_1 \otimes \tau_1) (\tilde{\rho}(T)) \zeta(s-2) + 2 (\tau_0 \otimes \tau_1 + \tau_1 \otimes \tau_0) (\tilde{\rho}(T)) \zeta(s-1) + 2 (\tau_0 \otimes \tau_0 - \frac{1}{2} \tau_1 \otimes \tau_1) (\tilde{\rho}(T)) \zeta(s),$

(ii) $\int T|D|^{-3} = 2 (\tau_1 \otimes \tau_1) (\tilde{\rho}(T)),$

(iii) $\int T|D|^{-2} = 2 (\tau_0 \otimes \tau_1 + \tau_1 \otimes \tau_0) (\tilde{\rho}(T)),$

(iv) $\int T|D|^{-1} = 2 (\tau_0 \otimes \tau_0 - \frac{1}{2} \tau_1 \otimes \tau_1) (\tilde{\rho}(T)).$

**Proof.** (i) Since $T \in X\tilde{\mathcal{X}}$, $\tilde{\rho}(T)$ is a linear combination of terms like $\pi_\pm (x) \pi_\pm (y) \otimes \pi_\pm (z) \pi_\pm (t)$, where $x, y, z, t \in A$. Such a term is noted in the following $T_+ \otimes T_-$. Linear combination of these term is implicit. With the shortcut $T_{c_1,\ldots,c_p} := \langle \varepsilon_{c_1} \otimes \cdots \otimes \varepsilon_{c_p}, T \varepsilon_{c_1} \otimes \cdots \otimes \varepsilon_{c_p} \rangle$, recalling that
\( v_{m,j}^{j,1} \) is 0 when \( j = 0 \), or \( l \geq 2j \), we get

\[
\zeta_D^T(s) = \sum_{2j=0}^{\infty} \sum_{m=0}^{2j} \sum_{l=0}^{2j+1} \left( \binom{j+1}{m,l} \cdot S_\rho(T)I \left( \binom{0}{m,l} \right) \right) d_{j-l}^{-s} + \left( \binom{0}{m,1} \cdot S_\rho(T)I \left( \binom{1}{m,1} \right) \right) d_{j-1}^{-s}
\]

\[
= \sum_{2j=0}^{\infty} \sum_{m=0}^{2j} \sum_{l=0}^{2j+1} \tilde{\rho}(T)_{m,2j,l,2j,l} d_{j-l}^{-s} + \sum_{2j=1}^{\infty} \sum_{m=0}^{2j-1} \sum_{l=0}^{2j-1} \tilde{\rho}(T)_{m,2j,l,2j,l} d_{j-l}^{-s}
\]

\[
= \sum_{2j=0}^{\infty} \left( \text{Tr}_{2j}(T_+) \text{Tr}_{2j+1}(T_-) + \text{Tr}_{2j+1}(T_+) \text{Tr}_{2j}(T_-) \right) d_{j-1}^{-s}.
\]

By Lemma 4.7 (ii), for all \( k > 0 \),

\[
\text{Tr}_{2j}(T_+) = (2j + \frac{3}{2}) \tau_1(T_+) + \tau_0(T_+) - \frac{1}{2} \tau_1(T_+) + \mathcal{O}((2j)^{-k}),
\]

\[
\text{Tr}_{2j+1}(T_+) = (2j + \frac{3}{2}) \tau_1(T_+) + \tau_0(T_+) + \frac{1}{2} \tau_1(T_+) + \mathcal{O}((2j)^{-k}).
\]

The result follows by noting that the difference of the Hurwitz zeta function \( \zeta(s, \frac{3}{2}) \) and Riemann zeta function \( \zeta(s) \) is an entire function.

(ii, iii, iv) are direct consequences of (i).

4.5 The smooth algebra \( C^\infty(SU_q(2)) \)

In [13, 38], the smooth algebra \( C^\infty(SU_q(2)) \) is defined by pulling back the smooth structure \( C^\infty(D^2_{q^2}) \) into the \( C^* \)-algebra generated by \( \mathcal{A} \), through the morphism \( \rho \) and the application \( \lambda \) (the compression which gives an operator on \( \mathcal{H} \) from an operator on \( l^2(\mathbb{N}) \otimes l^2(\mathbb{N}) \otimes l^2(\mathbb{Z}) \otimes \mathbb{C}^2 \)). The important point is that with [13, Lemma 2, p. 69], this algebra is stable by holomorphic calculus. By defining \( \rho := \tilde{\rho} \circ c \) and \( \lambda(\cdot) := S(\cdot)I \), the same lemma (with same notation) can be applied to our setting, with \( c := \pi(x) \mapsto \tilde{\pi}(x) \) and

\[
\mathcal{C} := C^\infty(D^2_{q^2}) \otimes C^\infty(S^1) \otimes C^\infty(D^2_{q^2}) \otimes C^\infty(S^1) \otimes \mathcal{M}_2(\mathbb{C})
\]

as algebra stable by holomorphic calculus containing the image of \( \tilde{\rho} \). Here, we use Schwartz sequences to define the smooth structures. We finally obtain \( C^\infty(SU_q(2)) \) with real structure as a subalgebra stable by holomorphic calculus of the \( C^* \)-algebra generated by \( \pi(\mathcal{A}) \cup J_\pi(\mathcal{A})J^{-1} \) and containing \( \pi(\mathcal{A}) \cup J_\pi(\mathcal{A})J^{-1} \).

Corollary 4.10. The dimension spectrum of the real spectral triple \( (C^\infty(SU_q(2)), \mathcal{H}, D) \) is simple and given by \( \{1, 2, 3\} \). Its KO-dimension is 3.

Proof. Since \( F \) commutes with \( \tilde{\pi}(\mathcal{A}) \), the pseudodifferential operators of order 0 (without the real structure and in the sense of [22]) are exactly (modulo \( OP^{-\infty} \)) the operators in \( \mathcal{B} + BF \). From Theorem 3.4 we see that the dimension spectrum of \( SU_q(2) \) without taking into account the reality operator \( J \) is \( \{1, 2, 3\} \), in other words, the possible poles of \( \zeta_D^T : s \mapsto \text{Tr}(bF^s|D|^{-s}) \) (with \( \varepsilon \in \{0, 1\} \), \( b \in \mathcal{B} \)) are in \( \{1, 2, 3\} \). Theorem 4.9 (i) shows that the possible poles are still \( \{1, 2, 3\} \) when we take into account the real structure of \( SU_q(2) \), that is to say, when \( \mathcal{B} \) is enlarged to \( BJB^{-1} \). Indeed, any element of \( BJB^{-1} \) is in \( X\tilde{X} \) and it is clear from the preceding proof that adding \( F \) in the previous zeta function do not add any pole to \( \{1, 2, 3\} \).

All arguments goes true from the polynomial algebra \( \mathcal{A}(SU_q(2)) \) to the smooth pre-\( C^* \)-algebra \( C^\infty(SU_q(2)) \).

KO-dimension refers just to \( J^2 = -1 \) and \( D \cdot J = J \cdot D \) since there is no chirality because spectral dimension is 3.

\( \square \)
4.6 Noncommutative integrals with reality operator and one-forms on $SU_q(2)$

The goal of this section is to obtain the following suppression of $J$:

**Theorem 4.11.** Let $A$ and $B$ be $\delta$-one-forms. Then

(i) $\int A J B J^{-1} |D|^{-3} = \frac{1}{2} \int (A |D|^{-3} B |D|^{-3})$

(ii) $\int A J B J^{-1} |D|^{-2} = \frac{1}{2} \int (A |D|^{-2} B |D|^{-2}) + \frac{1}{2} \int (A |D|^{-3} B |D|^{-2})$

(iii) $\int A^2 J B J^{-1} |D|^{-3} = \frac{1}{2} \int (A^2 |D|^{-3} B |D|^{-3})$

(iv) $\int \delta(A) A |D|^{-3} = \int \delta(A) J A J^{-1} |D|^{-3} = 0.$

We gather at the beginning of this section the main notations for technical lemmas which will follow.

For any pair $(k, p) \in \mathbb{N}^3 \times \mathbb{N}^3$ such that $k_i \leq |\alpha|, p_i \leq |\beta|$, where $\alpha, \beta \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}$, we define

\begin{align*}
v_{k, p} &:= g(p) \begin{pmatrix} |\alpha| \begin{pmatrix} (\alpha_1)_{k_1} \end{pmatrix}^2 \eta \kappa_1 (\alpha_2)_{k_2} \end{pmatrix} \begin{pmatrix} |\beta| \begin{pmatrix} (\beta_1)_{p_1} \end{pmatrix}^2 \eta \kappa_{p_1} (\beta_2)_{p_2} \end{pmatrix} \end{pmatrix} (-1)^{k_1+p_1+\alpha_2+\alpha_3+\beta_2+\beta_3} q^{k, p}, \\
h_{k, p} &:= \alpha_1 + \alpha_2 - \alpha_3 - 2(\eta \kappa_1 k_1 + k_2 - k_3) + g(p), \\
g(p) &:= \beta_1 + \beta_2 - \beta_3 - 2(\eta \kappa_2 p_1 + p_2 - p_3), \\
\sigma_{k, p} &:= k_1 + p_1 + \sigma_{k, p}^1 + \sigma_{k, p}^2, \\
\sigma_{k, p}^1 &:= k_1 \kappa_2 - \kappa_3 (k_1 + k_2) + \eta \kappa_1 \kappa_2 (k_1 + p_1) - \kappa_3 (k_1 + p_1 + p_2), \\
\sigma_{k, p}^2 &:= (k_3 + \eta \kappa_2 p_1 - p_2 + p_3) (k_1 + \kappa_2 + \kappa_3) - k_2 (k_1 + \kappa_3) + (p_1 + \kappa_2) (-p_2 + p_3) + \kappa_3 p_3, \\
t_{k, p} &:= a_{\alpha_1}^1 a_{\alpha_2}^2 a_{\alpha_3}^3 a_{\beta_1}^1 a_{\beta_2}^2 a_{\beta_3}^3 b^{k_1+p_1}, \\
u_{k, p} &:= a_{\alpha_1}^1 a_{\alpha_2}^2 a_{\alpha_3}^3 a_{\beta_1}^1 a_{\beta_2}^2 a_{\beta_3}^3 b^{k_1+p_1}.
\end{align*}

where we used the notation

$\kappa_i := |\alpha| - k_i$, $\kappa_i := |\beta| - p_i$.

so $0 \leq \kappa_i \leq |\alpha|$, $0 \leq \kappa_i \leq |\beta|$. We will also use the shortcut $\kappa := (k_1, k_2, k_3)$.

For $\beta_1 \in \mathbb{Z}$ and $j \in \mathbb{N}$, we define

\begin{align*}
w_1(\beta_1, j) &:= \sum_{n=0}^{\infty} (q^{2jn} (\eta \kappa_{n, |\beta_1|})^2 - \delta_{j, 0}), \\
w_2 \beta_1 &:= 2\beta_1 q^{\beta_1 (2\alpha_3 + \beta_3 - \beta_2)} w_1(\beta_1, \alpha_3 + \beta_3).
\end{align*}

We introduce the following notations:

\begin{align*}
q_{k, p, n}^+ &:= q^{n(|k|+|p|)} q_{n+r_{k, p}^+} \begin{pmatrix} (\alpha_1)_{k_1} \end{pmatrix} q_{n-\kappa_1 k_1} q_{n-\kappa_2 p_1} q_{n-\kappa_3 p_2} q_{n-\kappa_3 p_3} q_{n-\kappa_4 q_1} q_{n-\kappa_4 q_2} q_{n-\kappa_4 q_3} q_{n-\kappa_4 q_4} q_{n-\kappa_4 q_5}, \\
q_{k, p, n}^- &:= q^{n(|k|+|p|)} q_{n+r_{k, p}^-} \begin{pmatrix} (\alpha_1)_{k_1} \end{pmatrix} q_{n-\kappa_1 k_1} q_{n-\kappa_2 p_1} q_{n-\kappa_3 p_2} q_{n-\kappa_3 p_3} q_{n-\kappa_4 q_1} q_{n-\kappa_4 q_2} q_{n-\kappa_4 q_3} q_{n-\kappa_4 q_4} q_{n-\kappa_4 q_5} \times (1) |k|+|p|, \\
r_{k, p}^+ &:= \eta \kappa_1 k_1 + k_2 - k_3 + \eta \kappa_2 p_1 + p_2 - p_3, \\
r_{k, p}^- &:= \eta \kappa_1 k_1 - k_2 + k_3 + \eta \kappa_2 p_1 - p_2 + p_3.
\end{align*}
Thus, \( \pi_+(t_{k,p}) \varepsilon_n = q^+_n \varepsilon_{n+r^+_{k,p}} \) and \( \pi_-(u_{k,p}) \varepsilon_n = q^-_n \varepsilon_{n+r^-_{k,p}} \).

**Lemma 4.12.** We have

\[
r \left((M^n_\beta)'^{\circ}\right) = \sum_{k,p} \delta_{h_{k,p,0}} v_{k,p} \pi_+(t_{k,p}) \otimes \pi_-(u_{k,p})
\]

where the summation is done on \( k_i, p_i \) in \( \mathbb{N} \) such that \( k_i \leq \vert \alpha_i \vert, p_i \leq \vert \beta_i \vert \) for \( i \in \{1, 2, 3\} \).

**Proof.** Since \( \pi(m^{\alpha}) = (a_+ + a_-)^{\alpha_1}(b_+ + b_-)^{\alpha_2}(b_+^* + b_-^*)^{\alpha_3} \), we have

\[
\pi(m^{\alpha}) = \sum_k v_k c_k \text{ where } c_k := \alpha_1^{-k_1} \alpha_2^{-k_2} b_+^{\alpha_3 - k_3} b_-^{k_3}.
\]

By Lemma 3.2 (iii) we see that \( \delta(\pi(m^{\beta})) = \sum_p w_p d_p \) where we introduce

\[
w_p := \left(\begin{array}{c}
\frac{\beta_1}{p_1}\\
\frac{\beta_2}{p_2}\\
\frac{\beta_3}{p_3}
\end{array}\right)
\]

and

\[
d_p := g(p) a_+^{|\beta_1|} a_-^{|\beta_2|} b_+^{|\beta_3| - p_3} b_-^{|\beta_3| - p_3}.
\]

As a consequence, \( (M^n_\beta)'^{\circ} = \sum_{k,p} \delta_{h_{k,p,0}} g(p) v_k w_p c_k \) where

\[
\begin{align*}
c_{k,p} := & \alpha_1 a_+^{-k_1} a_-^{-k_2} b_+^{k_3} b_-^{k_3} a_+^{\alpha_3 - k_3} a_-^{k_3} a_+^{p_1} a_-^{p_1} b_+^{p_2} b_-^{p_2} b_+^{p_3} b_-^{p_3} \quad \text{ (42)}
\end{align*}
\]

With (42), we get \( r(c_{k,p}) = (-1)^{k_1 + p_1 + \alpha_2 + \alpha_3 + \beta_2 + \beta_3} q^{k_1 + p_1} \pi_+(t'_{k,p}) \otimes \pi_-(u'_{k,p}) \) where

\[
\begin{align*}
t'_{k,p} = & \alpha_1 a_+ b_+^{-k_1} b_+^{k_1} b_+^{k_2} b_-^{k_3} b_+^{\alpha_3 - k_3} a_+^{p_1} a_-^{p_1} b_+^{p_2} b_-^{p_2} b_+^{p_3} b_-^{p_3},
\end{align*}
\]

\[
\begin{align*}
u'_{k,p} = & \alpha_1 a_+ b_+^{-k_1} b_+^{k_1} b_+^{k_2} b_-^{k_3} b_+^{\alpha_3 - k_3} a_+^{p_1} a_-^{p_1} b_+^{p_2} b_-^{p_2} b_+^{p_3} b_-^{p_3}.
\end{align*}
\]

A recursive use of relation \( ba_j = q^{\alpha} a_j b \) yields the result. \( \square \)

**Lemma 4.13.** We have

(i) \( \pi_1 \otimes \pi_1 \left(r(M^n_\beta)'^{\circ}\right) = \beta_1 \delta_{\alpha_1, -\beta_1} \delta_{\alpha_2, 0} \delta_{\alpha_3, 0} \delta_{\beta_2, 0} \delta_{\beta_3, 0} \).

(ii) \( \pi_1 \otimes \pi_1 \left(r(M^n_\beta)'^{\circ}\right) = \alpha_1 \delta_{\alpha_2 + \beta_2, \alpha_3 + \beta_3} \pi_1 \).

In particular, if \( A \) is a \( \delta \)-one-form, we have

\[
\int A |D|^{-3} = 2 \beta_1 A_{\beta_1,00}^\beta, \quad \int A |D|^{-2} = 2 w_{\beta}^\beta B(A)_{\alpha}^\beta.
\]

where we implicitly summed on all \( \alpha, \beta \) indices.

**Proof.** (i) Using same notations of Lemma 4.12 we obtain by definition of \( \tau_1 \),

\[
\tau_1(\pi_+(t_{k,p})) = \delta_{k,0} \delta_{p,0} \delta_{\alpha_1 + \alpha_2 - \alpha_3 + \beta_1 + \beta_2 - \beta_3,0} \quad \text{ (43)}
\]

\[
\tau_1(\pi_-(u_{k,p})) = \delta_{k,0} \delta_{p,0} \delta_{\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \beta_3,0} \quad \text{ (44)}
\]

We get \( \tau_1(\pi_+(t_{k,p})) \tau_1(\pi_-(u_{k,p})) = \delta_{k,0} \delta_{p,0} \delta_{\alpha_2,0} \delta_{\alpha_3,0} \delta_{\beta_2,0} \delta_{\beta_3,0} \delta_{\alpha_1, -\beta_1} \), so Lemma 4.12 gives the result.
(ii) Since \( \pi_+(t_{k,p}) \varepsilon_n = q_{k,p,n}^+ \varepsilon_{n+r_{k,p}}^{+} \) and \( \pi_-(u_{k,p}) \varepsilon_n = q_{k,p,n}^- \varepsilon_{n+r_{k,p}}^{-} \), we get,

\[
\tau_0(\pi_+(t_{k,p})) = \delta_{\tau_{k,p}^0} \sum_{n=0}^{\infty} (q_{k,p,n}^+ - \delta_{k,0} \delta_{p,0} \delta_{\alpha_1+\alpha_2+\alpha_3+\beta_1-\beta_3,0}) ,
\]

and

\[
\tau_0(\pi_-(u_{k,p})) = \delta_{\tau_{k,p}^0} \sum_{n=0}^{\infty} (q_{k,p,n}^- - \delta_{k,0} \delta_{p,0} \delta_{\alpha_1+\alpha_2+\alpha_3+\beta_1-\beta_3,0}) .
\]

With (43) and (46) we get

\[
\tau_1(\pi_+(t_{k,p})) \tau_0(\pi_-(u_{k,p})) = \delta_{k,0} \delta_{p,0} \delta_{\alpha_2+\alpha_3+\beta_3} \delta_{\alpha_1-\beta_1} \sum_{n=0}^{\infty} (\delta_{k,0} \delta_{p,0} q_{k,p,n}^- - \delta_{\alpha_3+\beta_3,0})
= \delta_{k,0} \delta_{p,0} \delta_{\alpha_2+\alpha_3+\beta_3} \delta_{\alpha_1-\beta_1} w_1(\beta_1, \alpha_3 + \beta_3),
\]

Using (44) and (45),

\[
\tau_0(\pi_+(t_{k,p})) \tau_1(\pi_-(u_{k,p})) = \delta_{k,0} \delta_{p,0} \delta_{\alpha_2+\alpha_3+\beta_3} \delta_{\alpha_1-\beta_1} \sum_{n=0}^{\infty} (\delta_{k,0} \delta_{p,0} q_{k,p,n}^+ - \delta_{\alpha_3+\beta_3,0})
= \delta_{k,0} \delta_{p,0} \delta_{\alpha_2+\alpha_3+\beta_3} \delta_{\alpha_1-\beta_1} w_1(\beta_1, \alpha_3 + \beta_3).
\]

Lemma 4.12 yields the result.

With notations of Lemma 4.12 it is direct to check that for given \( \bar{\alpha} = (\alpha, \alpha', \ldots, \alpha^{(n-1)}) \) and \( \bar{\beta} = (\beta, \beta', \ldots, \beta^{(n-1)}) \),

\[
r \left( (M_\beta^\alpha)^o \right) = \sum_{K,P} \delta_{h_{k,P},0} v_{K,P} \pi_+(t_{K,P}) \otimes \pi_-(u_{K,P})
\]

(47)

where \( K = (k, k', \ldots, k^{(n-1)}) \), \( P = (p, p', \ldots, p^{(n-1)}) \) with \( 0 \leq k_i^{(j)} \leq |\alpha_i^{(j)}|, 0 \leq p_i^{(j)} \leq |\beta_i^{(j)}|, \)

\[
t_{k,P} := t_{k,p} t_{k',p'} \cdots t_{k^{(n-1)},p^{(n-1)}} , \quad u_{K,P} := u_{k,p} u_{k',p'} \cdots u_{k^{(n-1)},p^{(n-1)}} ,
\]

\[
v_{K,P} := v_{k,p} v_{k',p'} \cdots v_{k^{(n-1)},p^{(n-1)}} , \quad h_{K,P} := h_{k,p} + h_{k',p'} + \cdots h_{k^{(n-1)},p^{(n-1)}} .
\]

In the following, we will use the shortcuts \( A_i := \alpha_i + \alpha'_i + \cdots + \alpha_i^{(n-1)} \), \( B_i := \beta_i + \beta'_i + \cdots + \beta_i^{(n-1)} \).

In the case \( n = 2 \), we also note \( r_{k,P}^\pm := r_{k,p}^\pm + r_{k',p'}^\pm \) and \( q_{K,P,n}^\pm := q_{k,p,n}^\pm q_{k',p',n}^\pm q_{k,p,n+r_{k',p'}}^\pm \).

Thus, we have \( \pi_+(t_{k,p}) \varepsilon_m = q_{K,P,m}^+ \varepsilon_{m+r_{K,P}}^+ \) and \( \pi_-(u_{k,p}) \varepsilon_m = q_{K,P,n}^- \varepsilon_{m+r_{K,P}}^- \).

We also introduce, still for \( n = 2 \),

\[
v_{\beta_1, \alpha_1, \beta'_1}(l, j) := \sum_{n=0}^{\infty} (q_{l+2n}^{l+2n_j} 1_{-\beta'_1-\alpha'_1-\beta_1} q_{n+\beta_1+\beta'_1+\alpha_1+\beta_1}^{1_{\beta_1}} q_{n+\beta_1+\alpha'_1+\beta_1}^{1_{\alpha'_1}} q_{n+\beta_1+\alpha'_1}^{1_{\beta'_1}} q_{n+\beta_1}^{1_{\alpha_1}} q_{n+\beta_1}^{1_{\beta_1}} q_{n}^{1_{\beta_1}} - j_{0,0}) ,
\]

\[
V_{\beta}^{\bar{\alpha}} := 2[\beta_1 \beta'_1 + (\beta_2 - \beta_3)(\beta'_2 - \beta'_3)] q^2 \beta_1 (\alpha_2+\alpha_3+2\beta'_1) (\alpha'_2+\alpha'_3)
\times v_{\beta_1, \alpha_1, \beta'_1}(\alpha_2+\beta_2+\alpha_3+\beta_3)(\alpha'_1+\beta'_1), A_3 + B_3).
\]

**Lemma 4.14.** We have

(i) \( (\tau_1 \otimes \tau_0)(r(M_\beta^\alpha)^o) = \beta_1 \beta'_1 \delta_{A_1-\beta_1,0} \delta_{A_2,0} \delta_{A_3,0} \delta_{B_2,0} \delta_{B_3,0} \).

(ii) \( (\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1)(r(M_\beta^\alpha)^o) = \delta_{A_2+B_2, A_3+B_3} \delta_{A_1-\beta_1,0} V_{\beta}^{\bar{\alpha}} .\)
(iii) \((\tau_1 \otimes \tau_1) (r(M^\alpha_2 M_3^\beta M_4^\gamma)) = \beta_1 \beta_1' \beta_1'' \delta_{A_1,-B_1} \delta_{A_2,0} \delta_{B_2,0} \delta_{B_3,0}\).  

(iv) \((\tau_1 \otimes \tau_1) (r(\delta(M_2^\alpha \delta M_3^\gamma)) = -(\alpha_1' + \beta_1') \beta_1 \delta_{A_1,-B_1} \delta_{A_2,0} \delta_{A_3,0} \delta_{B_2,0} \delta_{B_3,0}\).  

(v) In particular, if \(A\) is a \(\delta\)-one-form,

\[
\int A^2 |D|^{-3} = 2 \beta_1 \beta_1' \delta_{A_1} A^2_3, \\
\int A^2 |D|^{-2} = 2V^3_\beta B(A^3_3), \\
\int A^3 |D|^{-3} = 2 \beta_1 \beta_1' \beta_1'' \delta_{A_1} A^3_3 B_a(A^3_3), \\
\int \delta(A) A |D|^{-3} = \int A \delta(A) |D|^{-3} = 0.
\]

**Proof.** We have

\[
\tau_1(\pi_+(t K, p)) = \delta_{K,0} \delta_{p,0} \delta_{A_1 + A_2 + A_3 + B_1 + B_2 - B_3,0}, \\
\tau_1(\pi_-(u K, p)) = \delta_{K,0} \delta_{p,0} \delta_{A_1 + A_2 + A_3 + B_1 + B_2 + B_3,0}.
\]

and

\[
\tau_0(\pi_+(t K, p)) = \delta_{r,0} \sum_{n=0}^\infty (q_{K,p,n} - \delta_{K,0} \delta_{p,0} \delta_{A_1 + A_2 + A_3 + B_1 + B_2 - B_3,0}), \\
\tau_0(\pi_-(u K, p)) = \delta_{r,0} \sum_{n=0}^\infty (q_{K,p,n} - \delta_{K,0} \delta_{p,0} \delta_{A_1 + A_2 + A_3 + B_1 + B_2 + B_3,0}).
\]

(i) Equations (48) and (49) give \((\tau_1 \otimes \tau_1) r(A^A) = \delta_{A_1,-B_1} \delta_{A_2,0} \delta_{A_3,0} \delta_{B_2,0} \delta_{B_3,0} \lambda_{0,0}\). A computation of \(v_{0,0}\) with \(\delta_{A_1,-B_1} \delta_{A_2,0} \delta_{A_3,0} \delta_{B_2,0} \delta_{B_3,0} = 1\) gives the result.

(ii) Equations (48) and (51) yield

\[
\tau_1(\pi_+(t K, p)) \tau_0(\pi_-(u K, p)) = \delta_{K,0} \delta_{p,0} \delta_{A_2 + B_2, A_3 + B_3} \delta_{A_1,-B_1} \\
\times v_{\beta_1, \alpha_1', \beta_1' \beta_2 + \alpha_3 + \beta_3} (\alpha_1' + \beta_1'), A_3 + B_3)
\]

Equations (50) and (49) yield

\[
\tau_0(\pi_+(t K, p)) \tau_1(\pi_-(u K, p)) = \delta_{K,0} \delta_{p,0} \delta_{A_2 + B_2, A_3 + B_3} \delta_{A_1,-B_1} \\
\times v_{\beta_1, \alpha_1', \beta_1' \beta_2 + \alpha_3 + \beta_3} (\alpha_1' + \beta_1'), A_3 + B_3)
\]

and the result follows.

(iii) With (47) a direct computation gives

\[
\tau_1(\pi_+(t K, p)) = \delta_{K,0} \delta_{p,0} \delta_{A_2 + A_3 + B_1 + B_2 - B_3,0}, \\
\tau_1(\pi_-(u K, p)) = \delta_{K,0} \delta_{p,0} \delta_{A_1 - A_2 + A_3 - B_1 + B_2 + B_3,0}.
\]

Using (52) and (53), \((\tau_1 \otimes \tau_1) (r(A^A A^B)) = \delta_{A_1,-B_1} \delta_{A_2,0} \delta_{A_3,0} \delta_{B_2,0} \delta_{B_3,0} v_{0,0}\). A computation of \(v_{0,0}\) with \(\delta_{A_1,-B_1} \delta_{A_2,0} \delta_{A_3,0} \delta_{B_2,0} \delta_{B_3,0} = 1\) gives the result.

(iv) We have \(\delta(M_3^\alpha M_2^\beta) = \delta(x) \delta(y) x' \delta(y') + x \delta^2(y) x' \delta(y')\) where \(x, x', y, y'\) are monomials (\(\pi\) omitted). Since

\[
\pi(x) = \sum_k (a_k^o) \delta_k x^k,
\]

\[
\pi(x, y, y') = \sum_k (a_k^o) \delta_k x^k y^k y'^k =: \sum_k (a_k) c_k,
\]
we get \( \delta(\pi(x)) = \sum_k g(k) \binom{\alpha}{k} c_k \).

Similarly, \( \delta(\pi(y)) = \sum_p g(p) \binom{\beta}{p} c_p \) and \( \delta^2(\pi(y)) = \sum_r g(r)^2 \binom{\beta}{r} c_r \).

Thus, with \( c_{K,P} := c_k c_p c_{K'} c_{P'} \),

\[
\delta(x) \delta(y) x' \delta(y') = \sum_{K,P} g(k) g(p) g(p') \binom{\alpha}{K} \binom{\beta}{P} c_{K,P},
\]

\[
x \delta^2(y) x' \delta(y') = \sum_{K,P} g(p)^2 g(p') \binom{\alpha}{K} \binom{\beta}{P} c_{K,P},
\]

\[
\tau(\delta(M_{C_2}^\alpha M_{\mathcal{C}_2}^\alpha)^0) = \sum_{K,P} \delta_{h_{K,0,0}} (g(k) + g(p)) g(p) g(p') \binom{\alpha}{K} \binom{\beta}{P} r(c_{K,P}) =: \sum_{K,P} \lambda_{K,P} r(c_{K,P}).
\]

Since \( r(c_k) = (-q)^k (1-\alpha_2+\alpha_3 \pi_+ (t_k) \otimes \pi_- (u_k)) \) with \( t_k, u_k \) defined by

\[
t_k := \tilde{c} \tilde{k}_1 b_1 \tilde{k}_2 b_2 a \tilde{k}_3 b_3 \quad \text{and} \quad u_k := \tilde{c} \tilde{k}_1 b_1 \tilde{k}_2 b_2 a \tilde{k}_3 b_3,
\]

we get

\[
r(\delta(M_{C_2}^\alpha M_{\mathcal{C}_2}^\alpha)^0) = \sum_{K,P} \lambda_{K,P} (-q)^k \pi_+ (t_k, u_k) \otimes \pi_- (u_k, p)^0
\]

where \( t_{K,P} = t_k t_k t_{k'} t_{k'}, \) and \( u_{K,P} = u_k u_p u_{k'} u_{k'} \). Direct computations yield

\[
\tau_1(\pi_+ (t_{K,P})) = \delta_{K,0} \delta_{P,0} \delta_{A_1 + A_2 + A_3 + B_2 + B_3,0},
\]

\[
\tau_1(\pi_- (u_{K,P})) = \delta_{K,0} \delta_{P,0} \delta_{A_1 + A_2 + A_3 + B_2 + B_3,0}.
\]

The result follows.

(iii) For the last equality, note that by (iv)

\[
\int \delta(A) |D|^{-3} = -2 \sum_{\alpha_1, \alpha_2, \beta_1, \beta_2} (\alpha_1' + \beta_1') \beta_1 \beta_2 A_{\alpha_1,0,0} B_{\beta_2,0} \delta_{\alpha_1 + \alpha_2 + \beta_1 + \beta_2,0}.
\]

The following change of variables \( \alpha_1 \leftrightarrow \alpha_1', \beta_1 \leftrightarrow \beta_1' \), implies by symmetry that this is equal to zero. \( \square \)

For a given \( \delta \)-1-form \( A \), we say that \( A \) is homogeneous of degree in \( a \) equal to \( n \in \mathbb{Z} \) if it is a linear combination of \( M_\beta^\alpha \) such that \( \alpha_1 + \beta_1 = n \). From Lemma 4.14 (iv) we get,

**Corollary 4.15.** Let \( A, A' \) be two \( \delta \)-1-forms, then

\[
\int (A|D|^{-1})^2 = \int A^2 |D|^{-2},
\]

\[
\int A|D|^{-1} A'|D|^{-1} = \int A A'|D|^{-2} - n \int A A'|D|^{-3}, \quad \text{when} \ A' \text{ homogenous of degree} \ n.
\]

**Lemma 4.16.** We have

(i) \( (\tau_1 \otimes \pi_1) \tilde{\rho}(M_{C_2}^\alpha J M_{C_2}^\beta J^{-1}) = \beta_1 \beta_2' \delta_{\alpha_1 + \beta_1 + \beta_2', \delta_{A_2 + A_3 + B_2 + B_3,0}} \delta_{A_2,0} \delta_{B_2,0} \delta_{B_3,0} \cdot \)

(ii) \( (\tau_0 \otimes \tau_1 + \tau_1 \otimes \tau_0) \tilde{\rho}(M_{C_2}^\alpha J M_{C_2}^\beta J^{-1}) = \delta_{\alpha_1 + \beta_1 + \beta_2', \delta_{\alpha_2 + \alpha_3 + \beta_2, \alpha_2 + \beta_2 + \alpha_3 + \beta_3}} \delta_{\alpha_2 + \alpha_3 + \beta_3} + \beta_1 |D|^{-1} \delta_{\alpha_2 + \beta_2 + \alpha_3 + \beta_3} \delta_{\alpha_2' + \beta_2' + \alpha_3 + \beta_3}. \cdot \)

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(iii) \( (\tau_1 \otimes \tau_1) \tilde{\rho}(M_{\beta}^\gamma M_{\beta'}^\gamma J M_{\beta}^{\prime\prime} J^{-1}) = \beta_1 \beta_2 \beta_3 \delta_{\alpha_1 + \alpha'_2, -\beta_2' - \beta_1'} \delta_{\alpha_3' - \beta_2'} \delta_{\alpha_1' + \alpha_2'} \delta_{\alpha_3 + \alpha'_1, -\beta_3'} \delta_{\delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0}} \cdot\)

(iv) \( (\tau_1 \otimes \tau_1) \tilde{\rho}(M_{\beta}^\gamma J M_{\beta}^{\prime\prime} J^{-1}) = -(\alpha_1' + \beta_1' \beta_2') \delta_{\alpha_1, -\beta_1'} \delta_{\alpha_1' - \beta_2'} \delta_{\alpha_2, -\beta_3'} \delta_{\alpha_3, 0} \delta_{B_2, 0} \delta_{B_3, 0} \cdot\)

(v) In particular, if \( A \) and \( A' \) are \( \delta \)-one forms,

\[
\int A J A' J^{-1} |D|^{-3} = 2(\beta_1 A - \beta_0)(\beta_1 A - \beta_0),
\]

\[
\int A J A' J^{-1} |D|^{-2} = 2(\beta_1 A - \beta_0)(w^\alpha B (A)^\beta) + 2(\beta_1 A - \beta_0)(w^\alpha B (A)^\beta),
\]

\[
\int A^2 J A' J^{-1} |D|^{-3} = 2(\beta_1 A - \beta_0)(\beta_1 B = (A^2)^\beta),
\]

\[
\int \delta(A) J A J^{-1} = 0.
\]

Proof. (i) Following notations of Lemma [4.12], we have

\[
M_{\beta}^\gamma J M_{\beta'}^\gamma J^{-1} = \sum_{K, P} v_{K, P} c_{k, p} J c_{K', p'} J^{-1}
\]

where \( K = (k, k') \), \( P = (p, p') \), \( \lambda_{K, P} = g(p) g(p') v_k v_p w_p w_p' \). Thus,

\[
\tilde{\rho}(M_{\beta}^\gamma J M_{\beta'}^\gamma J^{-1}) = (-1)^{A_2 + A_3 + B_2 + B_3} \sum_{K, P} (-q)^{k_1 + k_2 + p_1 + p_2} \lambda_{K, P} T_{K, P}^+ \otimes T_{K, P}^-
\]

where \( T_{K, P}^+ := \pi_+(t_k t_p) \pi_+(t_k t_p) \) and \( T_{K, P}^- := \pi_-(u_k u_p) \pi_-(u_k u_p) \) with

\[
t_k := a_{k_1} b_{k_1} a_{k_2} b_{k_2} a_{k_3} b_{k_3},
\]

\[
u_k := a_{k_1} b_{k_1} a_{k_2} b_{k_2} a_{k_3} b_{k_3}.
\]

A direct computation leads to

\[
\tau_1(T_{K, P}^+) = \delta_{K, 0} \delta_{P, 0} \delta_{\alpha_1 + \alpha_2 - \alpha_3 + \beta_1 - \beta_2 - \beta_3, 0} \delta_{\alpha'_1 + \alpha'_2 - \alpha'_3 + \beta'_1 - \beta'_2 - \beta'_3, 0},
\]

\[
\tau_1(T_{K, P}^-) = \delta_{K, 0} \delta_{P, 0} \delta_{\alpha_1 + \alpha_2 - \alpha_3 + \beta_1 - \beta_2 - \beta_3, 0} \delta_{\alpha'_1 + \alpha'_2 + \alpha'_3 + \beta'_1 - \beta'_2 + \beta'_3, 0}
\]

which gives the result.

(ii) Using the commutation relations on \( A \), we see that there are real functions of \( (K, P) \), noted \( \sigma_{K, P}^+ \) and \( \sigma_{K, P}^- \) such that

\[
T_{K, P}^+ = q^\sigma_{K, P}^+ \pi_+(t_k t_p) \pi_+(t_k t_p'),
\]

\[
T_{K, P}^- = q^\sigma_{K, P}^- \pi_-(u_k u_p) \pi_-(u_k u_p'),
\]

\[
t_{k, p} := a_{k_1} a_{k_2} b_{k_1} a_{k_2} b_{k_2} a_{k_3} b_{k_3},
\]

\[
u_{k, p} := a_{k_1} a_{k_2} b_{k_1} a_{k_2} b_{k_2} a_{k_3} b_{k_3}.
\]

We have, under the hypothesis \( \tau_1(T_{K, P}^-) = 1 \),

\[
\tilde{\pi_+}(t_{k', p'}) \varepsilon_{m, 2j} = (-1)^{\lambda'} q^{(2j-m)\lambda'} q_{2j - m - s + \beta'_1} q_{2j - m - s + \beta'_1} q_{2j - m - s + \beta'_1} \varepsilon_{m + s, 2j},
\]

\[
s := -\alpha'_1 + \alpha'_2 + \beta'_2 + \beta'_3 = \alpha'_1 + \beta'_1,
\]

\[
\lambda' := \alpha'_2 + \alpha'_3 + \beta'_2 + \beta'_3,
\]

\[
\lambda := \alpha_2 + \alpha_3 + \beta_2 + \beta_3.
\]

\[
\tau_1(T_{K, P}^-) = \delta_{\lambda, 0} \delta_{\lambda', 0}.
\]
and then,
\[
(T^+_K,p)_{m,2j} = q^{\sigma_{K,p} + s\lambda}(-1)^{\lambda'}q^{(2j-m)\lambda'} \lambda F_m F_{2j-m} \delta_{A_1 + B_1,0},
\]
\[
F'_{2j-m} := q_{2j-m-\alpha_1'|\beta_1'} q_{2j-m-\alpha_1'-\beta_1'|}\beta_1',
\]
\[
F_m := q_{m-\alpha_1,|\alpha_1|} q_{m-\beta_1,|\beta_1|}.
\]

Following the proof of Lemma 4.7, we see that \(\tau_0(T^+_K,p)\) is possibly nonzero only in the two cases \(\lambda' = 0\) or \(\lambda = 0\).

Suppose first \(\lambda = \lambda' = 0\). In that case, we have
\[
\tau_0(T^+_K,p) = \lim_{2j \to \infty} \sum_{m=0}^{2j} \left( (q_{m,|\beta_1|}q_{2j-m,|\beta_1'|})^2 - 1 \right) = \sum_{m=0}^{\infty} \left( (q_{m,|\beta_1|})^2 - 1 \right) + \sum_{m=0}^{\infty} \left( (q_{m,|\beta_1'|})^2 - 1 \right)
\]
where the second equality comes from Lemma 4.17.

In the case \((\lambda = 0, \lambda' > 0)\), we get \(\alpha'_1 = -\beta'_1\) and thus,
\[
(T^+_K,p)_{m,2j} = q^{\sigma_{K,p}} q^{m\lambda} (q_{m,|\beta_1|}q_{2j-m,|\beta_1'|})^2 \delta_{\alpha_1 + \beta_1,0}.
\]

Let us note \(U_{2j} = \sum_{m=0}^{2j} q^{m\lambda} (q_{m,|\beta_1|})^2 q_{2j-m,|\beta_1'|}^2\) and \(L_{2j} = \sum_{m=0}^{2j} q^{m\lambda} (q_{m,|\beta_1|})^2\).

Suppose \(\beta'_1 > 0\). Since \(q_{2j-m,|\beta_1'|}^2 - 1 = \sum_{|\beta_1| \neq 0, |\beta_1'| (\lambda \in \{0,1\}) (-1)^{|\beta_1|} q^{r_\lambda} q^{2(2j-m)|\beta_1'|}\) where we have \(r_\lambda = 2 + \cdots + 2\beta'_1\). As in the proof of Lemma 4.7 (ii), we can conclude that \(U_{2j} - L_{2j}\) converges to 0. The case \(\beta'_1 \leq 0\) is similar.

In the other case \((\lambda > 0, \lambda' = 0)\), the arguments are the same, replacing \(\lambda\) by \(\lambda'\) and \(\alpha_1, \beta_1\) by \(\alpha'_1, \beta'_1\). Finally,
\[
\tau_0(T^+_K,p) \tau_1(T^-_K,p) = \delta_{K,0} \delta_{P,0} \delta_{\alpha_1,-\beta_1} \delta_{\beta_1,-\beta_1'} \left( \delta_{\lambda',0} \delta_{\alpha_2,\alpha_3} s_{\alpha,\beta} + \delta_{\lambda',0} \delta_{\alpha_2',\alpha_3'} s_{\alpha',\beta'} \right),
\]
\[
s_{\alpha,\beta} := q^{\beta_1(\alpha_1 - \alpha_2)} \sum_{m=0}^{\infty} \left( q^{m\lambda} (q_{m,|\beta_1|})^2 - \delta_{\lambda,0} \right).
\]

A similar computation of \(\tau_0(T^-_K,p)\) can be done following the same arguments. We find eventually
\[
\tau_1(T^+_K,p) \tau_0(T^-_K,p) = \delta_{K,0} \delta_{P,0} \delta_{\alpha_1,-\beta_1} \delta_{\beta_1,-\beta_1'} \left( \delta_{\lambda',0} \delta_{\alpha_2,\alpha_3} s_{\alpha,\beta} + \delta_{\lambda',0} \delta_{\alpha_2',\alpha_3'} s_{\alpha',\beta'} \right)
\]
and the result follows.

(iii) The same arguments of (i) apply here with minor changes.

(iv) follows from a slight modification of the proof of Lemma 4.14 (iv).

(v) is a straightforward consequence of (i, ii, iii, iv).

Lemma 4.17. Let \(\beta, \beta' \in \mathbb{Z}\). Then,
\[
\lim_{2j \to \infty} \sum_{m=0}^{2j} \left( (q_{m,|\beta|}q_{2j-m,|\beta'|})^2 - 1 \right) = \sum_{m=0}^{\infty} \left( (q_{m,|\beta|})^2 - 1 \right) + \sum_{m=0}^{\infty} \left( (q_{m,|\beta'|})^2 - 1 \right).
\]
The result follows as we get, with the notations \( \lambda_{p,p'} := (-1)^{|p+p'|} q^{r_p+r_{p'}} \) and \( U_{2j} := \sum_{m=0}^{2j} \sum_{|p|_1=|p'|_1} (q_{m,|p|} q_{2j-m,|p'|})^2 - 1 \),

\[
U_{2j} = \sum_{m=0}^{2j} \sum_{|p|_1=|p'|_1} \lambda_{p,p'} q^{2|p|_1 m + 2|p'|_1(2j-m)} = \sum_{|p|_1=|p'|_1} \lambda_{p,p'} V_{2j,p,p'} + \sum_{|p|_1<|p'|_1} \lambda_{p,p'} V'_{2j,p,p'}
\]

where

\[
V_{2j,p,p'} = q^{4j|p'|_1} \sum_{m=0}^{2j} q^{2(|p|_1 - |p'|_1)m}, \quad V'_{2j,p,p'} = q^{4j|p'|_1} \sum_{m=0}^{2j} q^{2(|p|_1 - |p'|_1)m}.
\]

It is clear that \( V_{2j,p,p'} \) has 0 for limit when \( j \to \infty \) when \( |p'|_1 > 0 \), and \( V'_{2j,p,p'} \) has 0 for limit when \( j \to \infty \) when \( |p|_1 > 0 \). As a consequence,

\[
U_{2j} = \sum_{|p|_1=0} \lambda_{0,p'} V_{2j,0,p'} + \sum_{|p'|_1=0} \lambda_{0,p'} V'_{2j,0,p'} + o(1).
\]

The result follows as \( \sum_{m=0}^{2j} ((q_{m,|p|})^2 - 1) = \sum_{|p|_1=0} \lambda_{p,0} V_{2j,0,p} \) and \( \sum_{m=0}^{2j} ((q_{m,|p'|})^2 - 1) = \sum_{|p'|_1=0} \lambda_{0,p'} V'_{2j,0,p'} \).

Proof of Lemma 4.18. The result follows from Lemmas 4.13, 4.14 (v) and 4.16 (v).

4.7 Proof of Theorem 4.3 and corollaries

Lemma 4.18. We have on \( SU_q(2) \),

(i) \( \int |D|^{-3} = 2 \).
(ii) \( \int |D|^{-2} = 0 \).
(iii) \( \int |D|^{-1} = -\frac{1}{2} \).
(iv) \( \zeta_D(0) = 0 \).

Proof. (iv) We have by definition

\[
\zeta_D(s) := \text{Tr}(|D|^{-s}) = \sum_{2j=0}^{\infty} \sum_{m=0}^{2j} \sum_{l=0}^{2j} \langle v_{m,l}^j, |D|^{-s} v_{m,l}^j \rangle.
\]

Since \( |D|^{-s} v_{m,l}^j = \begin{pmatrix} d_{j-s} & 0 \\ 0 & d_{j-s}^{-1} \end{pmatrix} v_{m,l}^j \) where \( d_j := 2j + \frac{1}{2} \), we get

\[
\zeta_D(s) = \sum_{2j=0}^{\infty} (2j+1)(2j+2) d_{j-s}^2 + \sum_{2j=1}^{\infty} (2j+1)(2j) d_{j-s}^2 = 2 \sum_{2j=0}^{\infty} (2j+1)(2j) d_{j-s}^2.
\]

With the equalities \( (2j+1)(2j) = d_j^2 - \frac{1}{4} \) and \( \zeta(s, \frac{1}{2}) = (2s-1)\zeta(s) \) (here \( \zeta(s,x) := \sum_{n \in \mathbb{N}} \frac{1}{n^s} 
\) is the Hurwitz zeta function and \( \zeta(s) \) is the Riemann zeta function) we get

\[
\zeta_D(s) = 2(2s^2 - 1)\zeta(s-2) - \frac{1}{2}(2^{s-1}-1)\zeta(s)
\]

which entails that \( \zeta_D(0) = 0 \).

(i, ii, iii) are direct consequences of equation \( 54 \).

\( \square \)
Proof of Theorem 4.3. It is a consequence of Lemma 4.18 and Theorems 4.1 and 4.11.

As we have seen, the computation of noncommutative integral on $SU_q(2)$ leads to certain function of $A$ which filter some symmetry on the degree in $a$, $a^*$, $b$, $b^*$ of the canonical decomposition. Precisely, it is the balanced features that appear and the following functions of $A^n$, $n \in \{1, 2, 3\}$:

$$\int A^n|D|^{-p}$$

where $1 \leq n \leq p \leq 3$. We will see in the next section a method for the computation of these integrals.

Corollary 4.19. Let $u$ be a unitary in $C^\infty(SU_q(2))$ and $\gamma_u(A) := \pi(u)A\pi(u^*) + \pi(u)d\pi(u^*)$ be a gauge-variant of $A$. Then the following term of Theorem 4.3 are gauge invariant

$$\int A|D|^{-3}, \quad \int A^2|D|^{-3} - \int A|D|^{-2}, \quad -2\int A|D|^{-1} + \int A^2|D|^{-2} - \frac{2}{3}\int A^3|D|^{-3}.$$ 

Proof. It is sufficient to remark that all terms $\int |D_\gamma|^k$ and $\zeta_D(0)$ in spectral action (4) are gauge invariant. This can also be seen via the computation $D_{\gamma_u(A)} = V_uDV_u^* + V_uP_0V_u^*$ where $P_0$ is the projection on Ker $D$ and $V_u = \pi(u)J\pi(u)J^{-1}$ and $\int |D_\gamma|^n = \text{Res}_{s=n-k} \text{Tr}(|D_\gamma|^{n-k})$ (see [22, Prop. 5.1 (iii) and Prop. 4.8].) 

Corollary 4.20. In the case of the spectral action without the reality operator (i.e. $D_\gamma = D + \gamma$), we get

$$\int |D_\gamma|^3 = 2, \quad \int |D_\gamma|^2 = -2\int A|D|^{-3}, \quad \int |D_\gamma|^1 = -\frac{1}{2}\int A|D|^{-2} + \int A^2|D|^{-3},$$

$$\zeta_{D_\gamma}(0) = -\int A|D|^{-1} + \frac{1}{2}\int A^2|D|^{-2} - \frac{1}{3}\int A^3|D|^{-3}.$$ 

As a consequence, if $A$ is a one-form such that $\int A|D|^{-3} = 0$, then the scale invariant term of the spectral action with or without $J$ is exactly the same modulo a global factor of 2.

5 Differential calculus on $SU_q(2)$ and applications

5.1 The sign of $D$

There are multiple differential calculi on $SU_q(2)$, see [33,39]. Thanks to [36, Theorem 3], the 3D and 4D differential calculi do not coincide with the one considers here: the right multiplication of one-forms by an element in the algebra $A$ is a consequence of the chosen Dirac operator which was introduced according to some equivariance properties with respect to the duality between the two Hopf algebras $SU_q(2)$ and $U_q(su(2))$.

It is known that the Fredholm module associated to $\langle A, H, D \rangle$ is onesummable since $[F, \pi(x)]$ is trace-class for all $x \in A$. In fact, more can be said about $F$.

Proposition 5.1. Since

$$F = \frac{1}{1-q^2} \left( \bar{a}(a^*) d\bar{a}(a) + q^2 \bar{a}(b) d\bar{a}(b^*) + q^2 \bar{a}(a) d\bar{a}(a^*) + q^2 \bar{a}(b^*) d\bar{a}(b) \right),$$

$F$ is a central one-form modulo $OP^{-\infty}$.

1Note that a similar result for a different spectral triple over $SU_q(2)$ when $q = 0$ was obtained in [13, eq. (48)]
Proof. Forgetting \( \pi \), this follows from
\[
a^* \delta a + q^2 b \delta b^* + q^2 a \delta a^* + q^2 b^* \delta b
\]
\[
= (a^*_+ + a^*_-)(a_+ - a_-) + q^2 (b_+ + b_-)(b^*_+ - b^*_-) + q^2 (a_+ + a_-)(a^*_+ - a^*_-)
+ q^2 (b^*_+ + b^*_-)(b_+ - b_-)
\]
\[
[a^*_+ a_+ - q^2 a_+ a^*_+ + q^2 b^*_+ b_+ - q^2 b_+ b^*_+] + R = (1 - q^2) + R
\]  
(57)
by (16) where we check that the remainder \( R \) is zero:
\[
R = - [a^*_+ a_+ + q^2 b^*_+ b_+] + [a^*_+ a_+ + q^2 b^*_+ b_+] - [a^*_+ a_+ - q^2 a_+ a^*_+ + q^2 b^*_+ b_- - q^2 b_+ b^*_-]
+ (q^2 a_+ a^*_+ + q^2 q^*_+ b_+) - (a^*_+ a_+ - q^2 b^*_+ b_-),
\]
thus, applying (19), (20), (21), \( R = + (q^2 a_+ a^*_+ + q^2 q^*_+ b_+) - (a^*_+ a_+ - q^2 b^*_+ b_-) = 0 \) using commutation relations (13).

Now, replacing \( \delta \) by \( d \) in (57) gives (56) since \( F \) commute with \( a_\pm, b_\pm \) and \( F \) is central by (28).

Proposition 5.2. The one-form in (56) is in fact exactly a function of the Dirac operator \( D \):
\[
\pi(a^*) d\pi(a) + q^2 \pi(b) d\pi(b^*) + q^2 \pi(a) d\pi(a^*) + q^2 \pi(b^*) d\pi(b) = \xi_q(D) = F \xi_q(|D|),
\]
(58)
where \( \xi_q(s) := q^{[2s]-2s}_{[s+1/2][s-1/2]} \).
Moreover, \( F = \lim_{q \to 0} \xi_q(D) \).

Proof. First, let us observe that the one-form \( \omega \) in (58) is invariant under the action of the \( \mathcal{U}_q(su(2)) \times \mathcal{U}_q(su(2)) \): \( h \triangleright \omega = \epsilon(h) \omega \) for any \( h \in \mathcal{U}_q(su(2)) \times \mathcal{U}_q(su(2)) \). For instance, using notations of [21]
\[
e \triangleright \omega = q^{1/2} a^*_+ db + q^2 \left( -q^{1/2-1} b da^*_+ + q^{-1/2} b da^*_+ - q^{-1-1/2} a^*_+ db \right) = 0 = \epsilon(e) \omega.
\]
Therefore, since both the representation \( \pi \) as well as the operator \( D \) are equivariant, the image of \( \omega \) must be diagonal in the spinorial base. A tedious computation with the full spinorial representation \( \pi \) given in (10) yields
\[
\langle v_{ml}^j, \omega v_{ml}^{j*} \rangle = \frac{q^{3j+8} - q^{3j+6} - (3j+3) q^{4j+4} + (8j+6) q^{4j+2} - (4j+3) q^{4j+2} - q^{2+1}}{(q^{4j+4} - 1)(q^{4j+2} - 1)} = \xi_q(2j + \frac{3}{2}),
\]
\[
\langle v_{ml}^j, \omega v_{ml}^{j*} \rangle = \frac{q^{3j+1} - q^{3j+2} + (3j+1) q^{3j+4} - (8j+2) q^{3j+2} + (4j+1) q^{3j+2} - q^{2+1}}{(q^{3j+1} - 1)(q^{3j+1} - 1)} = -\xi_q(2j + \frac{1}{2}).
\]
These expressions have a clear \( q = 0 \) limit equal respectively to 1 and -1, so \( \omega \to F \) as \( q \to 0 \).

In the \( q = 1 \) limit, these expressions yields identically 0, which is confirmed by the fact that all one-forms are central, it could be expressed as \( d(aa^* + bb^*) = d1 \).

Note that since the invariant one-form we constructed differs by \( OP^{-\infty} \) from \( F \), hence any commutator with it will be itself in \( OP^{-\infty} \).

We do not know if a central form \( \omega \) is automatically invariant by the action of both \( U_q(su(2)) \), that is: \( h \triangleright \omega = \epsilon(h) \omega \).

Proposition 5.3. The order one calculus up to \( OP^{-\infty} \) is not universal.
Proof. Let us take the one-form $\omega_F$ from (56), which gives $F$. Then, for any $x \in \mathcal{A}(SU_q(2))$ we have $\pi(x\omega_F - \omega_F x) = 0$. \hfill \Box

Corollary 5.4. Still modulo $OP^{-\infty}$, $1 \in \pi\left(\Omega^2_u(A)\right)$.

Proof. $1 = F^2$ is by definition in $\pi\left(\Omega^2_u(A)\right)$. \hfill \Box

In fact, one checks, using (16), (19), (22) that

$$q^2 da da^* - da^* da = 1 - q^2$$

(59)

showing again that $1 \in \pi\left(\Omega^2_u(A)\right)$.

Similarly, using (15) and (17), (22), (23) we get still up to $OP^{-\infty}$

$$q da db = db da, \quad q da db^* = db^* da, \quad da^* db = q da da^*, \quad da^* db^* = q db^* da^*, \quad db db^* = db^* db, \quad da da^* + db db^* = -1.$$  

(60)

The use of the last equality of (60) and (59) gives

Proposition 5.5. Up to $OP^{-\infty}$, $F$ is not a (universal) closed one-form, as

$$da^* da + q^2 da da^* + q^2 db^* db + q^2 db db^* = -1 - q^2.$$  

(61)

5.2 The ideal $\mathcal{R}$

In order to perform explicit calculations of all terms of the spectral action, we observe that each $\delta$-one-form could be expressed in terms of $x \delta(z) y$, where $z$ is one of the generators $a, a^*, b, b^*$ and $x, y$ are some elements of the algebra $\mathcal{A}(SU_q(2))$.

Then, for the computation of $\int x dy/D |D|^{-1}$ we can use the trace property of the noncommutative integral to get:

$$\int x \delta(z) y |D|^{-1} = \int y x \delta(z) |D|^{-1} + \int x \delta(z) |D|^{-1} \delta(y) |D|^{-1}.$$

Therefore, the problem of calculating the tadpole-like integral could be in effect reduced to the calculation of much simpler integrals: $\int x \delta(z)$ for all generators $z$ and the integrals of higher order in $|D|^{-1}$.

However, it appears that the calculations of higher-order terms simplify a lot, when we further restrict the algebra by introducing an ideal, which is invisible to the parts of integral at dimension 2 and 3. For instance, consider the space of pseudodifferential operators $T \in \Psi^0(\mathcal{A})$ of order less or equal to zero (see [16]), which satisfy

$$\int T t |D|^{-2} = \int t T |D|^{-2} = \int T t |D|^{-3} = \int t T |D|^{-3} = 0, \forall t \in \Psi^0(\mathcal{A}).$$  

(62)

The elements $a_-, b_- b_+, b_- b^*$ and their adjoints are in this space up to $OP^{-\infty}$: this is due to the fact that in Theorem 3.4, $\tau_1 \otimes \tau_1 (r(x)) = 0$ when $r(x) \in \pi_\pm(\mathcal{A}) \otimes \pi_\pm(\mathcal{A})$ mod $OP^{-\infty}$ contains tensor products of $\pi_\pm(b)$ or $\pi_\pm(b^*)$ since these elements are in the kernel of the grading $\sigma$. 

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Definition 5.6. Let $R$ be the kernel in $X$ of $(\sigma \otimes \sigma) \circ r$ where $r$ is the Hopf-map defined in (30) and $\sigma$ is the symbol map and let $\mathcal{R}$ be the vector space generated by $R$ and $RF$.

Note that $R$ is a $*$-ideal in $X$ and
\[ a_-, b_- b_+ = q^2 b_+ b_-, \quad b_+ b_+^* \text{ are in } \mathcal{R}. \]

By construction and Theorem 3.4, any $T \in \mathcal{R}$ satisfies (62) and $\mathcal{R}$ is invariant by $F$.

Moreover, by (19), $[b_-, b_-^*] \in \mathcal{R}$, so by (16) and (22), $a_+^* a_+ - q^2 a_+ a_+^* - (1 - q^2) \in R$ and by (23), $q a_+ b_- - b_- a_+ \in R$.

It is interesting to quote, thanks to Theorem 3.4 that if $x \in R$, then $\int F x \vert D \vert^{-1} = 0$ while a priori, $\int x \vert D \vert^{-1} \neq 0$.

Note that $F \in \Psi^0(A)$ also satisfies (62) by Theorem 3.4 while $F \notin \mathcal{R}$ since $F^2 = 1$.

Moreover other elements are in $\mathcal{R}$ like for instance $d(b^* b) = d(bb^*)$:
\[ \delta(bb^*) = -\delta(aa^*) = -\delta a^* - a \delta a^* = -(a_+ - a_-)(a_+^* + a_-^*) - (a_+ + a_-)(a_- - a_+^*) \]
\[ = 2(a_+ a_-^* - a_- a_+^*) \]

is in $R$ since $a_- \in R$ yielding $d(bb^*) \in RF$.

We do not know if $\mathcal{R}$ is equal to the subset of the algebra generated by $B$ and $BF$ satisfying (62).

Lemma 5.7. $\mathcal{R}$ is a $*$-ideal in $\Psi^0(A)$ which is invariant by $F$, $d$, $\delta$.

Proof. Since $R$ is an ideal in $X = B \mod OP^{-\infty}$ (see Remark 3.3), $\mathcal{R}$ appears to be an ideal in $\Psi^0(A)$ algebra generated by $B$ and $BF$. Since $\mathcal{R}$ is invariant by $F$, its invariance by $d$ follows from its invariance by $\delta$ which is true on the generators of $R$. \qed

Note that, according to Theorem 4.13, $\int da \vert D \vert^{-2} = \int da \vert D \vert^{-3} = 0$ while $\int a^* da \vert D \vert^{-3} = 2$ which emphasize the role of "for all $t$" in (62).

Lemma 5.8. For any $t \in \Psi^0_0(A)$ and $T \in \mathcal{R}$, we have $\int t T \vert D \vert^{-1} = \int T t \vert D \vert^{-1}$.

Proof. For any $t \in B$, we have $\int T t \vert D \vert^{-1} = \int t T \vert D \vert^{-1} + \int T \vert D \vert^{-1} \delta(t) \vert D \vert^{-1}$ and moreover $\int T \vert D \vert^{-1} \delta(t) \vert D \vert^{-1} = \int T \delta(t) \vert D \vert^{-2} - \int T \delta^2(t) \vert D \vert^{-3}$ which comes from
\[
\int D^{-1} \delta(t) \vert D \vert^{-1} = \delta(t) \vert D \vert^{-2} + \int \vert D^{-1} \delta(t) \vert D \vert^{-1} = \delta(t) \vert D \vert^{-2} - \vert D^{-1} \delta^2(t) \vert D \vert^{-2} \]
\[= \delta(t) \vert D \vert^{-2} - \delta^2(t) \vert D \vert^{-3} + \int \vert D^{-1} \delta^3(t) \vert D \vert^{-3}. \]

So we get the result because $T$ satisfies (62). \qed

Lemma 5.9. If $\simeq$ means equality up to the ideal $\mathcal{R}$, the following rules with $d(\cdot) = [D, \cdot]$ of the first-order differential calculus hold (forgetting $\pi$)
\[
\begin{align*}
    a da & \simeq da a, & a^* da & \simeq -da^* a, & b da & \simeq q da b, & b^* da & \simeq q da b^*, \\
    a da^* & \simeq -da a^*, & a^* da^* & \simeq da^* a^*, & b da^* & \simeq q^{-1} da a b, & b^* da^* & \simeq q^{-1} da^* b^*, \\
    a db & \simeq q^{-1} db a, & a^* db & \simeq q db a^*, & b db & \simeq db b, & b^* db & \simeq db b^* \simeq -b db^*, \\
    a db^* & \simeq q^{-1} db^* a, & a^* db^* & \simeq q db^* a^*, & b db^* & \simeq db^* b \simeq -b^* db, & b^* db^* & \simeq db^* b^*.
\end{align*}
\]

Moreover
\[ a^* da - q^2 da a^* \simeq (1 - q^2) F, \quad q^2 a da^* - da^* a \simeq (1 - q^2) F. \] (63)
Proof. The table follows from relations (7) and Lemma 3.2 with (28) (one can also use (15).) For instance, since $a_\in R$, using the fact that $R$ is invariant by $F$,

$$bda = (b_+ + b_-)(a_+ - a_-)F \simeq (b_+ + b_-)(a_+ + a_-)F = baF = qabF \simeq q(a_+ - a_-)Fb$$

$$= qda b$$

or similarly, $a^*da = (a^*_+ + a^*_*)(a_+ - a_-)F \simeq (a^*_+ - a^*_*)(a_+ + a_-)F = -da^*a$.

The second of equivalence of (53) is just the adjoint of the first one that we prove now:

$$a^*da - q^2da a^* = (a^*_+ + a^*_*)(a_+ - a_-)F - q^2(a_+ - a_-)F(a^*_+ + a^*_+) \simeq (a^*_+ + a^*_*)(a_+ + a_-)F - q^2(a_+ + a_-)(a^*_+ + a^*_+)F = (a^*a - q^2 aa^*)F$$

$$= (1 - q^2)F. \quad \square$$

Remark 5.10. The above written rules remain valid if $dx$ is replaced by $\delta(x)$ and $F$ by 1.

Working modulo $R$ simplifies the writing of a one-form:

**Lemma 5.11.** (i) Every one-form $A$ can be, up to elements from $R$, presented as

$$A \simeq x_\alpha da + da^* x_\alpha^* + x_b db + db^* x_b^*,$$

where all $x_\alpha$ are the elements of $A$.

(ii) When $A$ is selfadjoint, $A$ can be written up to $R$ (not in a unique way, though) as

$$A \simeq x_\alpha da - da^* (x_\alpha)^* + x_b db - db^* (x_b)^*,$$

where $x_\alpha, x_b$ are arbitrary elements of $A$.

Proof. (i) A basis for one-forms consists of the following forms: $a^\alpha b^\beta (b^*)^\gamma d(a^\alpha' b^\beta' (b^*)^\gamma')$, where $\alpha, \alpha' \in \mathbb{Z}$ and $\beta, \gamma, \beta', \gamma' \in \mathbb{N}$.

Using the Leibniz rule and the commutation rules within the algebra (up to the $R$ according to Lemma 5.9), we reduce the problem to the case of the forms: $(a^\alpha b^\beta (b^*)^\gamma)dx (a^\alpha' b^\beta' (b^*)^\gamma')$, where $x$ can be either of the generators $a, a^*, b, b^*$. If $x = b$ or $x = b^*$, the straightforward application of the rules of the differential calculus leads to the answer that the one-form could be expressed as: $a^\alpha b^\beta (b^*)^\gamma db$ and $db^* a^\alpha b^\beta (b^*)^\gamma$.

Similar considerations for the case $x = a, a^*$ lead to the remaining terms.

Note that the presentation is not unique, since there still might remain terms, which are in $R$, for example: $b^* db + db^* b = d(bb^*) \in R$.

(ii) is direct. \quad \square

Next we can start explicit calculation of the integrals, beginning with the tadpole terms.

Application of the Leibniz rule yields to a presentation of one-forms which is different from the one of previous lemma. Each $\delta$-one-form could be expressed, as a finite sum of the terms $x^\delta(z)y$, where $z$ is one of the generators $a, a^*, b, b^*$ and $x, y$ are some elements of the algebra $A(SU_q(2))$.

**Proposition 5.12.** For all $x, y \in A(SU_q(2))$ and $z \in \{a, a^*, b, b^*\}$ we have

$$\int x^\delta(z)y |D|^{-1} = \int yx^\delta(z) |D|^{-1} + \int x^\delta(z)y^\delta(y) |D|^{-2} - \int x^\delta(z)y^\delta^2(y) |D|^{-3}.$$
Proof. This is just the application of the trace property of the noncommutative integral, together with the identity: $|D|^{-1} \delta(z)|D|^{-1} = -[[D]^{-1}, z]$. \hfill \qed

Remark 5.13. The computation of tadpole-like integrals is reduced to the integrals $\oint x\delta(z)|D|^{-1}$ for all generators $z$ and the integrals of higher order in $|D|^{-2}$. However, the calculations of higher-order terms simplify a lot after we use the relations which hold up to the ideal $\mathcal{R}$: this erases parts of integral depending on $|D|^{-2}$ and $|D|^{-3}$. Thus, beside $\oint x\delta(z)|D|^{-1}$, we only need to compute $\oint x\delta(z)\delta(z')|D|^{-2}$ where $z$ and $z'$ are generators, since all the $|D|^{-3}$ integrals have already been explicitly computed in section 4.6 (these integrals do not depend on $q$.) Besides the tadpole, the only integrals that need to be computed are $\oint A|D|^{-2}$ and $\oint A^2|D|^{-2}$ where $A$ is a $\delta$-1-form. Working modulo $\mathcal{R}$ and using again Leibniz rule, we only need to compute $\oint x\delta(z)|D|^{-2}$ and the previous integrals $\oint x\delta(z)\delta(z')|D|^{-2}$.

5.2.1 Operators $L_q$ and $M_q$

In the notation $v_{m,t}^j$ of $\mathcal{H}$, we have already use the $j$ dependence in $|D|^{12}$ with $J_q v_{m,t}^j := q^j v_{m,t}^j$. Let $L_q$ and $M_q$ be the similar diagonal operators

$$L_q v_{m,t}^j := q^{2j} v_{m,t}^j,$$
$$M_q v_{m,t}^j := q^{2m} v_{m,t}^j.$$

We immediately get

Lemma 5.14. For $n \in \mathbb{N}^*$, $\oint (L_q)^n |D|^{-2} |D|^{-2} = \oint (M_q)^n |D|^{-2} = \frac{2}{1-q^{4n}}$.

Proof. We have

$$\text{Tr} \left( L_q^n |D|^{-2s} \right) = \sum_{2j=0}^{\infty} \sum_{m=0}^{2j+1} \sum_{l=0}^{2j+1} (v_{m,l}^j, L_q^n |D|^{-2s} v_{m,l}^j)$$

$$= \sum_{2j=0}^{\infty} (2j+1) \frac{1-q^{2n(2j+2)}}{1-q^{2n}} d_{2j}^{-2-s} + \sum_{2j=0}^{\infty} (2j+1) \frac{1-q^{2n(2j+2)+2}}{1-q^{2n}} d_{2j}^{-2-s}$$

$$\sim_0 \frac{1}{1-q^n} \left( \zeta(s+1, \frac{3}{2}) + \zeta(s+1, \frac{1}{2}) \right) \sim \frac{2}{1-q^n} \zeta(s+1).$$

where $\sim_0$ means modulo a function holomorphic at 0. This gives the result for $L_q^n$ and a similar computation can be done for $M_q^n$. \hfill \qed

The interest of these operators stems in

Lemma 5.15. We have $L_q M_q \in \mathcal{R}$. Moreover,

$$b \delta b^* \simeq M_q - L_q, \quad b^* \delta b \simeq L_q - M_q, \quad b b^* \simeq L_q + M_q,$$
$$a \delta (a^*) \simeq a a^* \simeq L_q + M_q - 1, \quad a^* \delta a \simeq a^* a \simeq 1 - q^2 (L_q + M_q),$$
$$d a d a^* \simeq L_q + M_q - 1, \quad d a^* d a \simeq q^2 (L_q + M_q) - 1,$$
$$b^{n-2} (b^*)^n d b d b \simeq (L_q)^n + (M_q)^n,$$
$$b^{n-1} (b^*)^{n-1} d b d b^* \simeq -(L_q)^n - (M_q)^n,$$
$$b^n (b^*)^{n-2} d b^* d b \simeq (L_q)^n + (M_q)^n.$$
Proof. Since $L_q M_q = q^2 a - a^*_-$, we compute up to the ideal $\mathcal{R}$

$$b \delta b^* = (b_+ + b_-)(b_-^* - b_+^*) \simeq -b_+ b^*_+ + b_- b^*_- = M_q - L_q + L_q M_q (1 - q^2) \simeq M_q - L_q$$

and similarly for the other relations. □

5.2.2 Automorphisms of the algebra and symmetries of integrals

Proposition 5.16. For any $n \in \mathbb{N}^*$,

$$\int (bb^*)^n |D|^{-1} = -\frac{2(1 + q^{2n})}{(1 - q^{2n})^2},$$

$$\int (bb^*)^n b^* \delta b |D|^{-1} = \int (bb^*)^n b \delta b^* |D|^{-1} = \frac{2}{1 - q^{2n+2}},$$

$$\int (bb^*)^n a \, da^* D^{-1} = \frac{-2q^{4n+2} - 2q^{2n} - 2q^2 + 6q^{2n}}{(1 - q^{2n})^2(1 - q^{2n+2})},$$

$$\int (bb^*)^n a^* \, da D^{-1} = \frac{6q^{4n+2} - 2q^{2n} - 2q^2 - 2}{(1 - q^{2n})^2(1 - q^{2n+2})}.$$  

Note that the knowledge of these integral is enough for the computation of any term of the form \( \int x \delta(z)|D|^{-1} \), where $z$ is a generator, since any other $\delta$-one-form will be unbalanced.

To show this proposition, we will use few symmetries, properties of the ideal $\mathcal{R}$ and replacement of $\delta$-one-forms in terms of $L_q$, $M_q$ as above.

Let $U$ be the following unitary operator on the Hilbert space:

$$U v^{j\uparrow}_m = (-1)^{m+l} v^{j\downarrow}_m, \quad U v^{j\downarrow}_m = (-1)^{m+l} v^{j\uparrow}_m.$$  

Then, by explicit computations we have

$$U^* a U = a, \quad U^* a^* U = a^*, \quad U^* b U = b^*, \quad U^* b^* U = b, \quad \text{and} \quad U^* D U = -D.$$

Lemma 5.17. Each noncommutative integral (55) of an element of the algebra or differential forms is (up to sign) invariant under the algebra automorphism $\rho$ defined by

$$\rho(a) := a, \quad \rho(a^*) := a^*, \quad \rho(b) := b^*, \quad \rho(b^*) := b.$$  

Proof. For any homogeneous polynomial $p$ and any $k \in \mathbb{N}$,

$$\int p(a, a^*, b, b^*, D) D^{-k} = \int U^* p(a, a^*, b, b^*, D) D^{-n} U$$

$$= (-1)^k \int p(U^* a U, U^* a^* U, U^* b U, U^* b^* U, U^* D U) D^{-k}$$

$$= (-1)^{k+d} \int p(\rho(a), \rho(a^*), \rho(b), \rho(b^*), D) D^{-k},$$

where $d$ is the degree of $p$ with respect to $D$. □

Corollary 5.18. For any $n \in \mathbb{N}$, \( \int (bb^*)^n b \, db D^{-1} = \int (bb^*)^n b \, db^* D^{-1} \).
Lemma 5.19. For any \( x, y \in \Psi^0(A) \),

\[
(i) \quad \int xy|D|^{-1} = \int xy|D|^{-1} + \int x\delta(y)|D|^{-2} - \int x\delta^2(y) |D|^{-2}.
\]

\[
(ii) \quad \int z x D^{-1} y D^{-1} = \int z xy D^{-2}, \text{ where } z \in A \text{ contains } b \text{ or } b^*.
\]

Proof. (i) is direct consequence of the trace property of \( \mathcal{F} \) and the fact that \( OP^{-4} \) operators are trace-class.

(ii) We calculate:

\[
\int z x D^{-1} y D^{-1} = \int z x \left( y D^{-1} - D^{-1}[D, y] D^{-1} \right) D^{-1} = \int z xy D^{-2} - \int z x D^{-1}[D, y] D^{-2}
\]

\[
= \int z xy D^{-2}.
\]

The last step is based on the observation that any integral with \( D^{-3} \) vanishes if the expression integrated contains \( b \) or \( b^* \).

\[\square\]

Lemma 5.20. For any \( n \in \mathbb{N} \),

\[
(i) \quad \int (bb^*)^n b^* db D^{-1} = \frac{2}{1-q^{n+1}}.
\]

\[
(ii) \quad \int (bb^*)^n d(bb^*) D^{-1} = 0.
\]

\[
(iii) \quad \int (bb^*)^n |D|^{-1} = \frac{-2(1+q^{2n})}{(1-q^{2n})^2}.
\]

Proof. (i) With \( n > 1 \), we begin with \( \int d((bb^*)^n) D^{-1} = 0 \), which follows directly from the trace property of the noncommutative integral. Expanding the expression using Leibniz rule and the commutation

\[xD^{-1} = D^{-1} x + D^{-1}[D, x] D^{-1}, \quad (65)\]

we obtain

\[
0 = \sum_{k=0}^{n-1} \int b^k db b^{n-k-1}(b^*)^n D^{-1} + \sum_{k=0}^{n-1} \int b^n(b^*)^k db^* (b^*)^{n-k-1} D^{-1}
\]

\[
= n \left( \int b^{n-1}(b^*)^n db D^{-1} + \int b^n(b^*)^{n-1} db^* D^{-1} \right)
\]

\[
+ \sum_{k=0}^{n-1} \int \left( b^k db D^{-1} d(b^{n-k-1}(b^*)^n) D^{-1} + b^n(b^*)^k db^* D^{-1} d((b^*)^{n-k-1}) D^{-1} \right).
\]

Using Lemma 5.19

\[
0 = n \int (bb^*)^{n-1}(b^* db + b db^*) D^{-1}
\]

\[
+ \int \left( \frac{1}{2}n(n-1)b^{n-2}(b^*)^n db db + n^2 b^{n-1}(b^*)^{n-1} db db^* + \frac{1}{2}n(n-1)b^n(b^*)^{n-2} db^* db^* \right) D^{-2}.
\]

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The integrals with $D^{-2}$ could be easily calculated when we take restrict ourselves to calculations modulo ideal $R$:

$$n \int (bb^*)^{n-1}(b^* db + b db^*) \, D^{-1} = -2 \left( n(n - 1) - 2n^2 + n(n - 1) \right) \frac{1}{1-q^{2n}} = 4n \frac{1}{1-q^{2n}}.$$  

Hence $\int (bb^*)^{n-1}(b^* db + b db^*) \, D^{-1} = \frac{4}{1-q^{2n}}$, which together with Corollary 5.18 proves (i).

(ii) In a similar way, $\int (bb^*)^{n-1}d(bb^*) \, D^{-1} = 0 = \int (bb^*)^{n-1}d(aa^*) \, D^{-1}$ implies:

$$0 = \sum_{k=0}^{n-1} (bb^*)^{n-k-1}d(bb^*)(bb^*)^k \, D^{-1}$$

$$= n \int (bb^*)^{n-1}d(bb^*) \, D^{-1} + \frac{1}{2}n(n - 1) \int (bb^*)^{n-2}d(bb^*) \, d(bb^*) \, D^{-2}$$

$$= n \int (bb^*)^{n-1}d(bb^*) \, D^{-1},$$

where in the last step we used that $d(bb^*) \in R$. The identity (ii) now follows from the equality $aa^* = 1 - bb^*$.

(iii) Using Lemma 5.19, we get

$$A_n := \int (bb^*)^n |D|^{-1} = \int (bb^*)^n(aa^* + bb^*)|D|^{-1}$$

and we push now $a^*$ through $|D|^{-1}$ and from cyclicity of the trace through $(bb^*)^n$,

$$= A_{n+1} + \int (bb^*)^n q^{2n} a^* a |D|^{-1} + \int (bb^*)^n q^{2n} a \delta(a^*) |D|^{-2}$$

the last term being calculated explicitly, since up to ideal $R$, $a \delta(a^*) \simeq L_q + M_q - 1,$

$$= A_{n+1}(1 - q^{2n+2}) + q^{2n} A_n + 4 \left( \frac{1}{1-q^{2n+2}} - \frac{1}{1-q^{2n}} \right),$$

which leads to

$$A_n(1 - q^{2n}) + \frac{4}{1-q^{2n}} = A_{n+1}(1 - q^{2n+2}) + \frac{4}{1-q^{2n+2}}.$$

Assuming $A_n = \frac{f_n}{(1-q^{2n})^2}$ we have $\frac{f_{n+1} + 4}{1-q^{2n}} = \frac{f_{n+1} + 4}{1-q^{2n+2}}$, and taking into account that $A_0 = -2 \frac{1+q^2}{(1-q^2)^2}$, we obtain $A_n = -2 \frac{1+q^2}{(1-q^{2n})^2}$.

Finally, to get Proposition 5.18 it remains to prove

**Lemma 5.21.** For $n \geq 1,$

$$\int (bb^*)^n a \, da^* \, D^{-1} = \frac{2q^{4n+2} - 2q^{4n} - 2q^{2n+2} + 6q^{2n}}{(1-q^{2n})(1-q^{2n+2})},$$

$$\int (bb^*)^n a^* \, da \, D^{-1} = \frac{6q^{2n+2} - 2q^{2n} - 2q^{2} - 2}{(1-q^{2n})(1-q^{2n+2})}.$$  

**Proof.** First, using Leibniz rule, (65) and Lemma 5.19, we have (for $n \geq 1$)

$$\int (bb^*)^n a \, da^* \, D^{-1} = -q^{2n} \int (bb^*)^n a^* \, da - \int (bb^*)^n \, da \, da^* \, D^{-2}.$$
Further, we use the identity (56):

\[ \int (bb^* n a^* da + q^2 a da^* + q^2 b db^* + q^2 b^* db) \, D^{-1} = (1 - q^2) \int (bb)^n |D|^{-1}. \]

taking into account that \( F \mathcal{D} = \mathcal{D} \).

These equations give together a system of linear equations

\[
\int (bb^* n a^* da D^{-1} + q^{2n} \int (bb^*)^n a^* da D^{-1} = -4 \left( \frac{1}{1-q^{2n+2}} - \frac{1}{1-q^{2n}} \right),
\qquad q^2 \int (bb^* n a^* da D^{-1} + \int (bb^*)^n a^* da D^{-1} = -2(1 - q^2) \frac{1+q^{2n}}{(1-q^{2n+2})^2} - \frac{4q^2}{1-q^{2n+2}}
\]

which is solved by the expressions stated in the lemma.

\[ \square \]

5.2.3 The noncommutative integrals at \( |D|^{-2} \)

We need to separate this task into two problems. First, we shall to calculate all integrals \( \int x \delta(z) |D|^{-2} \), with \( x \in \mathcal{A}(SU_q(2)) \) and \( z \) being one of the generators. The second problem is to calculate \( \int x \delta(y) \delta(z) |D|^{-2} \), with both \( y \) and \( z \) being the generators \( \{a, a^*, b, b^*\} \).

**Lemma 5.22.** The only a priori non-vanishing integrals of the type \( \int x \delta(z) |D|^{-2} \) are for \( n \in \mathbb{N} \):

\[
\int (bb^* b^* \delta(b) |D|^{-2} = \int (bb^*)^n b^* \delta(b^*) |D|^{-2} = 0,
\int (bb^* a^* \delta(a^*) |D|^{-2} = \frac{4q^{2n}(1-q^2)}{(1-q^{2n+2})^2(1-q^{2n})}, \quad n > 0
\int (bb^* a^* \delta(a) |D|^{-2} = \frac{4(1-q^2)}{(1-q^{2n+2})^2(1-q^{2n})}.
\]

**Proof.** Since \( a \delta(a^*) \simeq L_q + M_q - 1 \) and \( (bb^*)^n \simeq L_q^n + M_q^n \), we get

\[
(bb^* a^* \delta(a^*) \simeq L_q^{n+1} + M_q^{n+1} - L_q^n - M_q^n
\]

and the second result is obtained from Lemma 5.14. The other integrals are computed in a similar way.

\[ \square \]
Lemma 5.23. The only a priori non-vanishing integrals of the type \( \int x \, dy \, dz \, |D|^{-2} \) are for \( n \in \mathbb{N} : \)

\[
\begin{align*}
\int (bb^*)^n (b^*)^2 db \, |D|^{-2} &= \frac{4}{1-q^{2n+1}}, \\
\int (bb^*)^n \, db \, |D|^{-2} &= \frac{4}{1-q^{2n+2}}, \\
\int (bb^*)^n (a^* b^*) (da \, db) \, |D|^{-2} &= 0, \\
\int (bb^*)^n (ab^*) (da \, db) \, |D|^{-2} &= 0, \\
\int (bb^*)^n (a^* b) (da \, db^*) \, |D|^{-2} &= 0, \\
\int (bb^*)^n (ab) (da^* \, db^*) \, |D|^{-2} &= 0, \\
\int (bb^*)^n (da \, da^*) \, |D|^{-2} &= \frac{4(q^{2n+2}-q^{2n})}{(1-q^{2n+2})(1-q^{2n})}, \quad n > 0 \\
\int (bb^*)^n (da^* \, da) \, |D|^{-2} &= \frac{4(q^2 - 1)}{(1-q^{2n+2})(1-q^{2n})}.
\end{align*}
\]

Proof. This follows from Lemma [5.14] with the equivalences up to \( \mathcal{R} \) gathered in Lemma [5.15]. □

6 Examples of spectral action

It is clear from Theorem 4.3 that any one-form of the form \( ada, bdb, adb, a^* db \), etc... do not contribute to the spectral action. Indeed, only the balanced parts of one-forms give a possibly nonzero term in the coefficients. Let us now give the values of the terms \( \int A^p |D|^{-p} \) and the full \( \zeta_D(0) \) for few examples.

| \( A \) | \( \int A |D|^{-3} \) | \( \int A^2 |D|^{-3} \) | \( \int A^3 |D|^{-3} \) | \( \int A |D|^{-2} \) | \( \int A^2 |D|^{-2} \) | \( \int A |D|^{-1} \) | \( \zeta_D(0) \) |
|---|---|---|---|---|---|---|---|
| \( a^* da \) | 2 | 2 | 2 | \( \frac{4q^2}{q^2-1} \) | \( \frac{4q^2(q^2+2)}{q^4-1} \) | \( \frac{3q^2+1}{2(q^2-1)} \) | \( \frac{11q^4+36q^2+13}{3(q^4-1)} \) |
| \( b^* db \) | 0 | 0 | 0 | 0 | \( -\frac{4}{q^2-1} \) | \( -\frac{2}{q^2-1} \) | \( \frac{-q^2}{q^4-1} \) |
| \( ada^* \) | -2 | 2 | -2 | \( \frac{-4}{q^2-1} \) | \( \frac{4(q^2+1)}{q^4-1} \) | \( \frac{2(q^2-1)}{q^2-1} \) | \( \frac{13q^4+36q^2+11}{3(q^4-1)} \) |
| \( bdb^* \) | 0 | 0 | 0 | 0 | \( \frac{-4}{q^2-1} \) | \( -\frac{2}{q^2-1} \) | \( \frac{-q^2}{q^4-1} \) |

1) Clearly the spectral action depends on \( q \): for instance,

\[
S(\mathcal{D}_{a^* da}, \Phi, \Lambda) = 2 \Phi_3 \Lambda^3 - 8 \Phi_2 \Lambda^2 + \frac{q^2+15}{2(1-q^2)} \Phi_1 \Lambda^1 + \frac{11q^4+36q^2+13}{3(q^4-1)} \Phi(0).
\]

2) Moreover, for \( B := a \delta a^* \) and \( A := B + B^* \), we get since \( B \simeq B^* \) mod \( \mathcal{R} \),

\[
\int A^p |D|^{-k} = 2^p \int B^p |D|^{-k}, \quad 1 \leq p \leq k \leq 3.
\]
Thus the spectral action of the selfadjoint one-form $\mathcal{A} := ada^* + (ada^*)^*$ is
\[
S(\mathcal{D}_\mathcal{A}, \Phi, \Lambda) = 2 \Phi_3 \Lambda^3 + 16 \Phi_2 \Lambda^2 + \frac{\sqrt{q+1}(\sqrt{q}^2 - 1)}{2(1-q)} \Phi_1 \Lambda^1 + \frac{12\sqrt{q+1}(\sqrt{q}^2 - 1)}{2(1-q)} \Phi(0).
\]

3) When $B_n := (bb^*)^n b b^* b^*$, then by Lemma (5.15), $B_n \simeq B_n^*$, so for $A_n := B_n + B_n^*$, the equation (66) is still valid and $\int B_n^k |\mathcal{D}|^{-k}$ are all zero but $\int B_n |\mathcal{D}|^{-1} = \frac{2}{1-q^{2n+2}}$ and $\int B_n^2 |\mathcal{D}|^{-2} = \frac{4}{1-q^{4n+4}}$, so
\[
S(\mathcal{D}_\mathcal{A_n}, \Phi, \Lambda) = 2 \Phi_3 \Lambda^3 - \frac{1}{2} \Phi_1 \Lambda^1 + \frac{8}{1+q^{2n+2}} \Phi(0).
\]

Remark however that this spectral action still exists as $q \to 1$!

Note however that the symmetrization process (66) is not true in general, for instance if $B := a \delta b$ and $A := B + B^*$, then $\int A^2 |\mathcal{D}|^{-1} = \frac{8(q^2 - q^4 - 1)}{(1-q^4)^2}$ while $\int B^2 |\mathcal{D}|^{-1} = 0$ or $\int [B, B^*] |\mathcal{D}|^{-1} = \frac{4}{1-q^4}$.

4) The spectral action can be also independent of $q$: for instance, if $\mathcal{A} = \frac{1}{1-q} \xi(\mathcal{D})$ is the $q$-dependent selfadjoint one-form given in (5.5), then,
\[
S(\mathcal{D}_\mathcal{A}, \Phi, \Lambda) = 2 \Phi_3 \Lambda^3 - 8 \Phi_2 \Lambda^2 + \frac{15}{2} \Phi_1 \Lambda^1 - \frac{13}{3}.
\]

7 The commutative sphere $\mathbb{S}^3$

Since $SU(2) \simeq \mathbb{S}^3$, we get a concrete spinorial representation of the algebra $\mathcal{A} := C^\infty(\mathbb{S}^3)$ on the same Hilbert space $\mathcal{H}$ and same Dirac operator $\mathcal{D}$ with (10) where $q = 1$ which means that $q$-numbers are trivial: $[\alpha] = \alpha$. So
\[
\pi(a) |j\mu n\rangle := \alpha_{j\mu n}^+ |j^+ \mu^+ n^+\rangle + \alpha_{j\mu n}^- |j^- \mu^+ n^+\rangle,
\]
\[
\pi(b) |j\mu n\rangle := \beta_{j\mu n}^+ |j^+ \mu^- n^-\rangle + \beta_{j\mu n}^- |j^- \mu^- n^-\rangle,
\]
\[
\pi(a^*) |j\mu n\rangle := \tilde{\alpha}_{j\mu n}^+ |j^+ \mu^- n^-\rangle + \tilde{\alpha}_{j\mu n}^- |j^- \mu^- n^-\rangle,
\]
\[
\pi(b^*) |j\mu n\rangle := \tilde{\beta}_{j\mu n}^+ |j^+ \mu^- n^-\rangle + \tilde{\beta}_{j\mu n}^- |j^- \mu^- n^-\rangle
\]

where
\[
\alpha_{j\mu n}^+ := \sqrt{j + \mu + 1} \begin{pmatrix}
\frac{\sqrt{j+n+3/2}}{2j+2} & 0 \\
\frac{\sqrt{j-n+1/2}}{(2j+1)(2j+2)} & \sqrt{j+n+1/2}
\end{pmatrix},
\]
\[
\alpha_{j\mu n}^- := \sqrt{j - \mu} \begin{pmatrix}
\frac{\sqrt{j-n+1/2}}{2j+1} & -\frac{\sqrt{j+n+1/2}}{2j(2j+1)} \\
0 & \sqrt{j-n-1/2}
\end{pmatrix},
\]
\[
\beta_{j\mu n}^+ := \sqrt{j + \mu + 1} \begin{pmatrix}
\frac{\sqrt{j+n+3/2}}{2j+2} & 0 \\
\frac{\sqrt{j-n+1/2}}{(2j+1)(2j+2)} & \sqrt{j+n+1/2}
\end{pmatrix},
\]
\[
\beta_{j\mu n}^- := \sqrt{j - \mu} \begin{pmatrix}
\frac{\sqrt{j+n+1/2}}{2j+1} & -\frac{\sqrt{j-n+1/2}}{2j(2j+1)} \\
0 & \sqrt{j+n-1/2}
\end{pmatrix}
\]

with $\alpha_{j\mu n}^\pm := (\alpha_{j\pm\mu}^\pm)^*$, $\beta_{j\mu n}^\pm := (\beta_{j\pm\mu}^\pm)^*$.

Note that the representation on the vectors $x^i_{m, \ell}$ is not as convenient as in (11).

One can check that the generators $\pi(a)$, $\pi(b)$ and their adjoint commute and that $[x, [\mathcal{D}', y]] = 0$ for any $x, y \in \mathcal{A}$.
7.1 Translation of Dirac operator

In general the Dirac operator is defined in a more symmetric way than that we did. So, although not absolutely necessary here, we define for the interested reader the unbounded self-adjoint translated operator $D'$ on $\mathcal{H}$ by the constant $\lambda$ as

$$D' := D + \lambda.$$ 

For instance, this gives for $\lambda = -\frac{1}{2}$ in the case of $S^3$, see [32], $D'v_{m,t} = (2j + 1)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)v_{m,t}$ so $v_{m,t}$ is an eigenvector of $|D'|$.

As the following lemma shows, the computation of noncommutative integrals involving $D$ can be reduced to the computation of certain noncommutative integrals involving $D'$:

**Lemma 7.1.** If $\int T := \text{Res}_{s=0} \text{Tr} (T |D'|^{-s})$, then for any 1-form $A$ on a spectral triple of dimension $n$,

$$\int A |D|^{-(n-2)} = \int A |D'|^{-(n-2)} + \lambda (n - 2) \int A D'|D'|^{-n} + \lambda^2 \frac{(n-1)(n-2)}{2} \int A |D'|^{-n},$$

$$\int A D^{-(n-2)} = \int A D'|^{-(n-2)} + \lambda (n - 2) \int A D'|D'|^{-n} + \lambda^2 \frac{(n-1)(n-2)}{2} \int A D'^{-n}.$$ 

**Proof.** Recall from [22, Proposition 4.8] that for any pseudodifferential operator $P$,

$$\int P |D|^{-r} = \text{Res}_{s=0} \text{Tr} (P |D|^{-r} |D'|^{-s}).$$

Moreover, by [22, Lemma 4.3], for any $s \in \mathbb{C}$ and $N \in \mathbb{N}^*$

$$|D|^{-s} = |D'|^{-s} + \sum_{p=1}^{N} K_{p,s} Y^p |D'|^{-s} \mod OP^{-N-1-\Re(s)}$$

where $Y = \sum_{k=1}^{N} (-1)^{k+1}(-2\lambda + \lambda^2)^{k} |D'|^{-2k} \mod OP^{-N-1}$ and $K_{p,s}$ are complex numbers that can be explicitly computed. Precisely, we find $K_{p,s} = (-\frac{1}{2})^p V(p)$ where $V(p)$ is the volume of the $p$-simplex. Since the spectral dimension is $n$, we work modulo $OP^{-(n+1)}$, and for $s = n - 2$, we get from (69): $|D|^{-(n-2)} = |D'|^{-(n-2)} + \lambda (n - 2) D'|D'|^{-n} + \lambda^2 \frac{(n-1)(n-2)}{2} |D'|^{-n} \mod OP^{-(n+1)}$.

As a consequence, we have for $P \in OP^0$ (the $OP^0$ spaces are the same for $D$ or $D'$),

$$\int P |D|^{-(n-2)} = \int P |D'|^{-(n-2)} + \lambda (n - 2) \int P D'|D'|^{-n} + \lambda^2 \frac{(n-1)(n-2)}{2} \int P |D'|^{-n}.$$ 

Since $A$ and $AF$ are in $OP^0$, we get both formulae. \hfill $\square$

7.2 Tadpole and spectral action on $S^3$

We consider now the commutative spectral triple $(C^\infty(S^3), \mathcal{H}, D)$. It is 1-summable since $\langle j \mu n s | [F, \pi(x)] | j \mu n s \rangle = 0$ when $x = a, a^*, b, b^*$ for any $j, \mu, n, s = \uparrow, \downarrow$.

All integrals of above lemma are zero for $S^3$:

**Proposition 7.2.** There is no tadpole of any order on the commutative real spectral triple $(C^\infty(S^3), \mathcal{H}, D)$. In fact, for any one-form or $\delta$-one-form $A$, $\frac{1}{2} \int AD^{-p} = 0$ for $p \in \{1, 2, 3\}$.
Proof. We first want to prove $\oint AD^{-p} = 0$ for $p \in \{1, 2, 3\}$ and any one-form $A$. Since the representation is real, that is any matrix elements of the generators are real, so must be the trace of $AD^{-p}$. Hence $\oint AD^{-p} = \oint A^*D^{-p}$.

The reality operator $J$ introduced in (20) satisfies, when $q = 1$, the commutative relation $JxJ^{-1} = x^*$ for $x \in A$. Thus $JAJ^{-1} = -A^*$, so $\oint AD^{-p} = \oint J(A^*D^{-p})J^{-1} = -\oint A^*D^{-p}$ and $\oint AD^{-p} = 0$. Similar proof for the $\delta$-form $AF$.

For any selfadjoint one-form $A$, $D_A := D + A = D$. Thus, the spectral action for the real spectral triple $(C^\infty(S^3), \mathcal{H}, D)$ for $D_A$ is trivialized by

$$S(D_A, \Phi, \Lambda) = 2 \Phi_3 \Lambda^3 - \frac{1}{2} \Phi_1 \Lambda^1 + \mathcal{O}(\Lambda^{-1}). \quad (70)$$

But it is more natural to compare with the spectral action of $D + A$. This is obtained respectively from Lemma 4.18 and general heat kernel approach [27]:

$$S(D + A, \Phi, \Lambda) = 2 \Phi_3 \Lambda^3 + \int [D + A]^{-1} \Phi_1 \Lambda^1 + \mathcal{O}(\Lambda^{-1})$$

since all terms of (22) in $\Lambda^{n-k}$ are zero for $k$ odd and $\zeta_{D+A}(0) = 0$ when $n$ is odd: as a verification, $\oint [D + A]^{-2} = 0$ according to [22, Lemma 4.10], Lemmas 4.18 and Proposition 7.2. Similarly, $\zeta_{D+A}(0) = 0$ because in (3), all terms with $k$ odd are zero (same proof as in Proposition 7.2). For any selfadjoint one-form $A := D + A = D$.

Proof. Follows from [23, first formula page 511] with $\rho := A = A^*$, $N(\rho) = \rho$ (the constraint $J\rho J^{-1} = \pm \rho$ is not used.).

One can also use [15, Proposition 1.149].

From [22, Lemma 4.10] $\oint [D + A]^{-1} = \oint [D]^{-1}$ using $X := AD + DA + A^2$ and $[D, A] \in OT^0$, but again, it is not that easy to show that the last two terms cancelled: for instance here, for $B = b[D, b^+]$, we obtain by direct computation (using the easiest translated Dirac operator $D'$)

$$\text{Tr} \left( B^2 |D'|^{-3-s} \right) = \text{Tr} \left( (B^*)^2 |D'|^{-3-s} \right) = \frac{1}{2} \text{Tr} \left( BB^* |D'|^{-3-s} \right) = \frac{1}{2} \text{Tr} \left( B^*B |D'|^{-3-s} \right)$$

so $\oint B^2 |D'|^{-3} = \frac{3}{2}$. Similarly, one checks that $\oint (BF)^2 |D|^{-3} = \frac{1}{2} \oint BFB^*F |D|^{-3} = -\frac{2}{3}$. Thus if $A := B + B^*$, $\oint A^2 |D|^{-3} = \oint A^2 |D'|^{-3} = 4$ and $(AF)^2 |D|^{-3} = -\frac{2}{3}$ which yields to (71). Thus for any one-form $A$ on the 3-sphere,

$$S(D + A, \Phi, \Lambda) = 2 \Phi_3 \Lambda^3 - \frac{1}{2} \Phi_1 \Lambda^1 + \mathcal{O}(\Lambda^{-1}, A)$$

which as (70) is not identical to (67) which contains a nonzero constant term $\Lambda^0$ for $q = 1.$

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8 Conclusion

We computed in this paper the full spectral action on the \( SU_q(2) \) spectral triple of [21] with the reality operator \( J \) (notice the change of definition for pseudodifferential operators.) The dimension spectrum being a finite set, there is only a finite number of terms in the spectral action expansion. The tadpole hypothesis is not satisfied on \( SU_q(2) \). We saw that that the action depends on \( q \) and the limit \( q \to 1 \) does not exist automatically. When it exists, such limit does not lead to the associated action on the commutative sphere \( S^3 \). The sign \( F \) of the Dirac operator has special properties: first, it commutes modulo \( OP^{-\infty} \) with elements of the algebra, and second, it can be seen as a one-form, giving terms independent of \( q \) in the spectral action. Here, we were interested in the computation of the spectral action of a quantum group. Naturally, it would be interesting to investigate other related cases like the Podleś spheres [17, 19] or the Euclidean quantum spheres [20, 35], especially the 4-sphere [18].

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