Hypercohomologies of truncated twisted holomorphic de Rham complexes

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Abstract
We investigate the hypercohomologies of truncated twisted holomorphic de Rham complexes on (not necessarily compact) complex manifolds. In particular, we generalize Leray–Hirsch, Künneth and Poincaré–Serre duality theorems on them. At last, a blowup formula is given, which affirmatively answers a question posed by Chen and Yang (Ann Global Anal Geom 56:277–290, 2019).

Keywords
Hypercohomology · Truncated twisted holomorphic de Rham complex · Leray–Hirsch theorem · Künneth theorem · Poincaré–Serre duality theorem · Blowup formula

Mathematics Subject Classification
Primary 32C35; Secondary 14F43

1 Introduction
All complex manifolds mentioned in this paper are connected and not necessarily compact unless otherwise specified. Let \( X \) be a complex manifold. The de Rham theorem says that the cohomology \( H^k(X, \mathbb{C}) \) with complex coefficients can be computed by the hypercohomology \( H^k(X, \Omega^\bullet_X) \) of the holomorphic de Rham complex \( \Omega^\bullet_X \). Moreover, if \( X \) is a compact Kähler manifold, the Hodge filtration \( F^p H^k_{dR}(X, \mathbb{C}) = \bigoplus_{r \geq p} H^r, k-r_{dR}(X, \mathbb{C}) \) on the de Rham cohomology \( H^k_{dR}(X, \mathbb{C}) \) is just the hypercohomology \( H^k(X, \Omega_{\geq p}^\bullet_X) \) of the truncated holomorphic de Rham complex \( \Omega_{\geq p}^\bullet_X \), which plays a significant role in the Hodge theory. The Deligne cohomology is an important object in the research of algebraic cycles, since it connects with the intermediate Jacobian and the integral Hodge class group. It has a strong relation with the singular cohomology \( H^*(X, \mathbb{Z}) \) and the hypercohomology \( \mathbb{H}^*(X, \Omega_{\leq p-1}^\bullet_X) \) of the truncated holomorphic de Rham complex \( \Omega_{\leq p-1}^\bullet_X \) via the long exact sequence

\[ \cdots \rightarrow \mathbb{H}^r(X, \Omega_{\leq p-1}^\bullet_X) \rightarrow H^r_{dR}(X, \mathbb{C}) \rightarrow \mathbb{H}^{r+1}(X, \mathbb{C}) \rightarrow \mathbb{H}^{r+1}(X, \Omega_{\leq p-1}^\bullet_X) \rightarrow \cdots \]
\[ \cdots \to H^{k-1}(X, \mathbb{Z}) \to H^{k-2}(X, \Omega^{\leq p-1}_X) \to H^{k-1}(X, \mathbb{Z}(p)) \to H^{k}(X, \mathbb{Z}) \to H^{k+1}(X, \Omega^{\leq p-1}_X) \to \cdots. \]

So it seems natural to investigate the hypercohomology of truncated holomorphic de Rham complexes.

We consider the more general cases—the hypercohomology \( \mathbb{H}^a(X, \Omega^{[s,t]}_X(L)) \) of truncated twisted holomorphic de Rham complexes \( \Omega^{[s,t]}_X(L) \) (see Sect. 3.2 or [3, Section 4.2] for the definition)—and obtain the generalizations of several classical theorems for the de Rham cohomology with values in a local system as follows.

**Theorem 1.1** (Leray–Hirsch theorem) Let \( \pi : E \to X \) be a holomorphic fiber bundle with compact fibers over a complex manifold \( X \) and \( \mathcal{L} \) a local system of \( \mathbb{C}_X \)-modules of finite rank on \( X \). Assume that there exist \( d \)-closed forms \( t_1, \ldots, t_r \) of pure degrees on \( E \) such that the restrictions of their Dolbeault classes \([t_1]_\partial, \ldots, [t_r]_\partial \) to \( E_x \) are a basis of \( H^{r}_{\partial \cdot \cdot}(E_x) = \bigoplus_{p,q \geq 0} H^{p,q}_{\partial \cdot \cdot}(E_x) \) for every \( x \in X \). Then there exists an isomorphism

\[ \bigoplus_{i=1}^r \mathbb{H}^{k-u_i-v_i}(X, \Omega^{[s-u_i,t-u_i]}_X(\mathcal{L})) \cong \mathbb{H}^k(X, \Omega^{[s,t]}_X(\pi^{-1}L)) \]

for any \( k, s, t \), where \( (u_i, v_i) \) is the degree of \( t_i \) for \( 1 \leq i \leq r \).

**Theorem 1.2** (Künneth theorem) Let \( X, Y \) be complex manifolds and \( \mathcal{L}, \mathcal{H} \) local systems of \( \mathbb{C}_X \cdot, \mathbb{C}_Y \cdot \)-modules of finite ranks on \( X, Y \), respectively. If \( X \) or \( Y \) is compact, then there is an isomorphism

\[ \bigoplus_{a+b=c \atop a+u+w=e \atop w+v+t=r} \mathbb{H}^a(X, \Omega^{[u,v]}_X(\mathcal{L})) \otimes \mathbb{H}^b(Y, \Omega^{[w,u]}_Y(\mathcal{H})) \cong \mathbb{H}^c(X \times Y, \Omega^{[s,t]}_{X \times Y}(\mathcal{L} \boxtimes \mathcal{H})) \]

for any \( c, s, t \).

**Theorem 1.3** (Poincaré–Serre duality theorem) Let \( X \) be an \( n \)-dimensional compact complex manifold and \( \mathcal{L} \) a local system of \( \mathbb{C}_X \cdot \)-modules of finite rank on \( X \). Then there exists an isomorphism

\[ \mathbb{H}^k(X, \Omega^{[s,t]}_X(\mathcal{L})) \cong (\mathbb{H}^{2n-k}(X, \Omega^{[n-t,n-s]}_X(\mathcal{L}^\vee)))^* \]

for any \( k, s, t \), where \( * \) denotes the dual of a \( \mathbb{C} \)-vector space.

Recently, the blowup formulas for the de Rham cohomology with values in a local system have been studied with different approaches [3,5,6,8,16,17]. More generally, Chen, Y. and Yang, S. posed a question on the existence of a blowup formula for the hypercohomology of a truncated twisted holomorphic de Rham complex [3, Question 10]. Now, we prove that it truly exists.

**Theorem 1.4** Let \( \pi : \widetilde{X} \to X \) be the blowup of a complex manifold \( X \) along a complex submanifold \( Y \) and \( \mathcal{L} \) a local system of \( \mathbb{C}_X \cdot \)-modules of finite rank on \( X \). Assume that \( i_Y : Y \to X \) is the inclusion and \( r = \text{codim}_\mathbb{C} Y \geq 2 \). Then there exists an isomorphism

\[ \mathbb{H}^k(X, \Omega^{[s,t]}_X(\mathcal{L})) \oplus \bigoplus_{i=1}^{r-1} \mathbb{H}^{k-2i}(Y, \Omega^{[n-t-i,-i]}_Y(i_Y^{-1}L)) \cong \mathbb{H}^k(\widetilde{X}, \Omega^{[s,t]}_{\widetilde{X}}(\pi^{-1}L)), \quad (1.1) \]

for any \( k, s, t \).
In Sect. 2, we recall some basic notions on complexes and give some properties of them, which may be well known for experts. In Sect. 3, the double complexes $S^*_{X}(\mathcal{L}, s, t)$ and $T^*_{X}(\mathcal{L}, s, t)$ are defined and studied, which play important roles in the study of the hypercohomology $H^k(X, \Omega^{[s, t]}_X(\mathcal{L}))$. In Sect. 4, Theorems 1.1–1.4 are proved.

## 2 Complexes

Let $\mathcal{C}$ be an Abelian category. All complexes and morphisms in this section are in $\mathcal{C}$.

### 2.1 Complexes

A complex $(K^\bullet, d)$ consists of objects $K^k$ and morphisms $d^k : K^k \to K^{k+1}$ for all $k \in \mathbb{Z}$ satisfying that $d^{k+1} \circ d^k = 0$. Sometimes, we briefly denote $(K^\bullet, d)$ by $K^\bullet$ and denote its $k$th cohomology by $H^k(K^\bullet)$. A morphism $f : K^\bullet \to L^\bullet$ of complexes means a family of morphisms $f^k : K^k \to L^k$ for all $k \in \mathbb{Z}$ satisfying that $f^{k+1} \circ d^k = d^k \circ f^k$ for all $k \in \mathbb{Z}$. Moreover, if there exists a morphism $g : L^\bullet \to K^\bullet$ satisfying that $f \circ g = \text{id}$ and $g \circ f = \text{id}$, then $f$ is said to be an isomorphism.

A double complex $(K^{**}, d_1, d_2)$ consists of objects $K^{p,q}$ and morphisms $d_1^{p,q} : K^{p,q} \to K^{p+1,q}$, $d_2^{p,q} : K^{p,q} \to K^{p,q+1}$ for all $(p,q) \in \mathbb{Z}^2$ satisfying that $d_1^{p+1,q} \circ d_1^{p,q} = 0$, $d_2^{p,q+1} \circ d_2^{p,q} = 0$ and $d_1^{p+1,q} \circ d_2^{p,q} + d_2^{p,q+1} \circ d_1^{p,q} = 0$. Sometimes, it will be shortly written as $K^{**}$. Fixed $p \in \mathbb{Z}$, $(K^{p,\bullet}, d_2^{p,\bullet})$ is a complex. A morphism $f : K^{**} \to L^{**}$ of double complexes means a family of morphisms $f^{p,q} : K^{p,q} \to L^{p,q}$ for all $p,q \in \mathbb{Z}$ satisfying that $f^{p+1,q} \circ d_1^{p,q} = d_1^{p,q} \circ f^{p,q}$ and $f^{p,q+1} \circ d_2^{p,q} = d_2^{p,q} \circ f^{p,q}$ for all $p,q \in \mathbb{Z}$. If there exists a morphism $g : L^{**} \to K^{**}$ satisfying that $f \circ g = \text{id}$ and $g \circ f = \text{id}$, then $f$ is said to be an isomorphism.

Let $K^{**}$ be a double complex. The complex $sK^{**}$ associated with $K^{**}$ is defined as $(sK^{**})^k = \bigoplus_{p+q=k} K^{p,q}$ and $d^k = \sum_{p+q=k} d_1^{p,q} + \sum_{p+q=k} d_2^{p,q}$. Let $(K E_r^{*,*}, F^{*}H^*)$ be the spectral sequence associated with a double complex $K^{**}$. Then the first page $K E_1^{p,q} = H^q(K^{p,*})$ and $K E_1^{p,q} = H^q(K^{*,p})$. A morphism $f : K^{**} \to L^{**}$ of double complexes naturally induces a morphism $sf : sK^{**} \to sL^{**}$ of complexes, where $(sf)^k = \sum_{p+q=k} f^{p,q}$ for any $k \in \mathbb{Z}$.

### 2.2 Shifted complexes

For an integer $m$, denote by $(K^*[m], d[m])$ the $m$-shifted complex of $(K^\bullet, d)$, namely $K^*[m]^k = K^{k+m}$ and $d[m]^k = d^{k+m}$, for all $k \in \mathbb{Z}$. For a pair of integers $(m, n)$, denote by $(K^{**}[m,n], d_1[m,n], d_2[m,n])$ the $(m,n)$-shifted double complex of $(K^{**}, d_1, d_2)$, namely $K^{**}[m,n]^{p,q} = K^{p+m,q+n}$ and $d_1[m,n]^{p,q} = d_1^{p+m,q+n}$, for any $(p,q) \in \mathbb{Z}^2$ and $i = 1, 2$. Clearly, $s(K^{**}[m,n]) = (sK^{**}*[m+n])$.

### 2.3 Dual complexes

In this subsection, assume that $\mathcal{C}$ is the category of $\mathbb{C}$-vector spaces.
For a $\mathbb{C}$-vector space $V$, let $V^* = Hom_{\mathbb{C}}(V, \mathbb{C})$ be its dual space. The dual complex \((K^*)^k\) of \((K^k, d^k)\) is defined as \((K^*)^k = (K^{-k})^*\) and $d^k f^k = (-1)^{k+1} f^k \circ d^{-k-1}$ for any $k \in \mathbb{Z}$.

Let $f^k \in (K^*)^k$ satisfy $d^k f^k = 0$, i.e., $f^k \circ d^{-k-1} = 0$. Then $f^k$ naturally induces a linear map $\bar{f}^k : H^{-k}(K^*) \to \mathbb{C}$. The morphism $f^k \mapsto \bar{f}^k$ gives the following isomorphism, where $[\bar{f}^k]$ denotes the class in $H^k((K^*)^*)$.

**Lemma 2.1** [4, IV (16.5)]. $H^k((K^*)^*) \cong (H^{-k}(K^*))^*$.

The dual double complex \((K^{\bullet \bullet})^k\) of \((K^{\bullet \bullet}, d_1, d_2)\) is defined as \((K^{\bullet \bullet})^{p,q} = (K^{-p,-q})^*\) and $d^{p,q}_{\alpha} f^{p,q} = (-1)^{p+q+1} f^{p,q} \circ d^{1-p,-q}_{\alpha}$ for any $(p, q) \in \mathbb{Z}^2$, where

\[d^{p,q}_{\alpha} f^{p,q} = (-1)^{p+q+1} f^{p,q} \circ d^{1-p,-q}_{\alpha}\]

for any $(p, q) \in \mathbb{Z}^2$. Clearly, \((K^{\bullet \bullet})^{p,q} = 0\) for even $(p, q)$ and \((K^{\bullet \bullet})^{p,q} = \mathbb{C}\) for odd $(p, q)$. Since $K^{\bullet \bullet}$ is bounded, this map gives an isomorphism $(s K^{\bullet \bullet})^* \cong s((K^{\bullet \bullet})^*)$.

**2.4 Tensors of complexes**

In this subsection, assume that $\mathcal{C}$ is the category of $\mathbb{C}$-vector spaces.

For complexes $(K^k, d_k)$, $(L^k, d_k)$ and an integer $m$, the double complex $(K^k \otimes_{\mathbb{C}} L^k)_m$ is defined as $(K^k \otimes_{\mathbb{C}} L^k)_m = K^k \otimes_{\mathbb{C}} L^q$ and $d^{p,q}_{\alpha} = d^{p,q}_{\alpha} \otimes d^{L}_{q}$ for any $(p, q) \in \mathbb{Z}^2$. Clearly, $(K^k \otimes_{\mathbb{C}} L^k)_m$ is equal to $(K^k \otimes_{\mathbb{C}} L^k)_0$ for even $m$ and to $(K^k \otimes_{\mathbb{C}} L^k)_1$ for odd $m$.

**Lemma 2.3** There is an isomorphism $(K^k \otimes_{\mathbb{C}} L^k)_0 \cong (K^k \otimes_{\mathbb{C}} L^k)_1$ of double complexes.

**Proof** Evidently, linear extending $\alpha^p \otimes \beta^q \mapsto (-1)^q \alpha^p \otimes \beta^q$ for $\alpha^p \in K^p$ and $\beta^q \in L^q$ gives an isomorphism $(K^k \otimes_{\mathbb{C}} L^k)_0 \to (K^k \otimes_{\mathbb{C}} L^k)_1$ of double complexes.

Combining [4, IV (15.6)] and Lemma 2.3, we conclude that

**Lemma 2.4** For any integer $m$, $\bigoplus_{p+q=m} H^p(K^*) \otimes_{\mathbb{C}} H^q(L^*) \cong H^k(s((K^k \otimes_{\mathbb{C}} L^k)_m))$.

For double complexes $(K^{\bullet \bullet}, d_{K1}, d_{K2})$ and $(L^{\bullet \bullet}, d_{L1}, d_{L2})$, set

\[A^{p,q,r,s} = K^{p,r} \otimes_{\mathbb{C}} L^{q,s},\]
\[d_1^{p,q,r,s} = d^{K}_{p,r} \otimes_{\mathbb{C}} 1_{L^{q,s}} : A^{p,q,r,s} \to A^{p+1,q,r,s},\]
\[d_2^{p,q,r,s} = (-1)^{p+r} 1_{K^{p,r}} \otimes_{\mathbb{C}} d^{L^{q,s}}_{r,s} : A^{p,q,r,s} \to A^{p,q+1,r,s},\]
\[d_3^{p,q,r,s} = d^{L}_{r,s} \otimes_{\mathbb{C}} A^{p,q,r,s} : A^{p,q,r,s} \to A^{p,q+1,r+1,s},\]
\[d_4^{p,q,r,s} = (-1)^{p+r} 1_{K^{p,r}} \otimes_{\mathbb{C}} d^{L^{q,s}}_{r,s+1} : A^{p,q,r,s} \to A^{p,q+1,r,s+1},\]

for any $p, q, r, s$. We easily see that

\[(A^{p,q,r,s}, d_{K1}^{p,q,r,s}, d_{K2}^{p,q,r,s}) = (K^{p,r} \otimes_{\mathbb{C}} L^{q,s})\]

as double complexes for any $p, q$. Springer
Define \( ss(K^{•,•} \otimes_{\mathcal{C}} L^{•,•}) \) as

\[
ss(K^{•,•} \otimes_{\mathcal{C}} L^{•,•})^{k,l} = \bigoplus_{p+q=k \atop r+s=l} A^{p,q;r,s}
\]

and

\[
D_1^{k,l} = \sum_{p+q=k \atop r+s=l} d_1^{p,q;r,s} + \sum_{p+q=k \atop r+s=l} d_2^{p,q;r,s} : \text{ss}(K^{•,•} \otimes_{\mathcal{C}} L^{•,•})^{k,l} \to \text{ss}(K^{•,•} \otimes_{\mathcal{C}} L^{•,•})^{k+1,l}
\]

\[
D_2^{k,l} = \sum_{p+q=k \atop r+s=l} d_3^{p,q;r,s} + \sum_{p+q=k \atop r+s=l} d_4^{p,q;r,s} : \text{ss}(K^{•,•} \otimes_{\mathcal{C}} L^{•,•})^{k,l} \to \text{ss}(K^{•,•} \otimes_{\mathcal{C}} L^{•,•})^{k,l+1}
\]

for any \( k, l \). It is easily checked that \( ss(K^{•,•} \otimes_{\mathcal{C}} L^{•,•}) \) is a double complex.

By (2.1), the following lemma holds.

**Lemma 2.5** For any \( k \), there is an isomorphism

\[
\text{(ss}(K^{•,•} \otimes_{\mathcal{C}} L^{•,•})^{k,•}, D_2^{k,•}) \cong \bigoplus_{p+q=k} s((K^{p,•} \otimes_{\mathcal{C}} L^{q,•})_p)
\]

of complexes.

By definitions, it follows that

**Lemma 2.6** \( s(ss(K^{•,•} \otimes_{\mathcal{C}} L^{•,•})) \cong s((sK^{•,•} \otimes_{\mathcal{C}} sL^{•,•})_0) \).

Now, we can compute the cohomologies of tensors of double complexes.

**Proposition 2.7** (1) For any \( a \in \mathbb{Z} \),

\[
\bigoplus_{k+l=a} H^k(sK^{•,•}) \otimes_{\mathcal{C}} H^l(sL^{•,•}) \cong H^a(ss(K^{•,•} \otimes_{\mathcal{C}} L^{•,•})).
\]

(2) For any \( k, l \in \mathbb{Z} \),

\[
\bigoplus_{p+q=k \atop r+s=l} H^p(K^{p,•}) \otimes_{\mathcal{C}} H^q(L^{q,•}) \cong H^l(ss(K^{•,•} \otimes_{\mathcal{C}} L^{•,•})^{k,•}).
\]

**Proof** By Lemmas 2.4 and 2.6, we get (1). By Lemmas 2.4 and 2.5, we get (2).

### 3 Truncated twisted holomorphic de Rham complex

Let \( X \) be an \( n \)-dimensional complex manifold. Denote by \( \mathcal{A}^{p,q}_X \) (resp. \( \mathcal{D}^{p,q}_X \), \( \mathcal{D}^{p,q}_X \), \( \Omega^p_X \)) the sheaf of germs of smooth \((p, q)\)-forms (resp. complex-valued smooth \( k \)-forms, \((p, q)\)-currents, holomorphic \( p \)-forms) on \( X \). Denote by \( \mathcal{C}_X \) the constant sheaf with stalk \( \mathbb{C} \) over \( X \). Let \( \mathcal{L} \) be a local system of \( \mathcal{C}_X \)-modules of finite rank on \( X \), namely a locally constant sheaf of finite-dimensional \( \mathbb{C} \)-vector spaces on \( X \). Tensoring \( \otimes_{\mathcal{C}_X} \) between sheaves of \( \mathcal{C}_X \)-modules will be simply written as \( \otimes \). For a sheaf \( \mathcal{F} \) over \( X \) and an open subset \( U \subseteq X \), \( \Gamma(U, \mathcal{F}) \) refers to the group of sections of \( \mathcal{F} \) on \( U \). For a complex \( \mathcal{F}^• \) of sheaves on \( X \), \( H^k(X, \mathcal{F}^•) \) denotes its \( k \)th hypercohomology.
3.1 Twisted Dolbeault cohomology

Since \( \partial : A_X^{p,q} \to A_X^{p+1,q} \) and \( \bar{\partial} : A_X^{p,q} \to A_X^{p,q+1} \) are both morphisms of sheaves of \( \mathbb{C} \)-modules, they naturally induce \( \mathcal{L} \otimes A_X^{p,q} \to \mathcal{L} \otimes A_X^{p+1,q} \) and \( \mathcal{L} \otimes A_X^{p,q} \to \mathcal{L} \otimes A_X^{p,q+1} \), which are still denoted by \( \partial \) and \( \bar{\partial} \), respectively. Then \( \mathcal{L} \otimes \Omega_X^1 \) has a fine resolution

\[
0 \to \mathcal{L} \otimes \Omega_X^1 \to \mathcal{L} \otimes A_X^{0,0} \to \cdots \to \mathcal{L} \otimes A_X^{0,n} \to 0.
\]

These arguments also hold for the sheaves \( \mathcal{D}_X^{\bullet,\bullet} \). So

\[
H^q(X, \mathcal{L} \otimes \Omega_X^1) \cong H^q(\Gamma(X, \mathcal{L} \otimes A_X^{p,\bullet})) \cong H^q(\Gamma(X, \mathcal{L} \otimes \mathcal{D}_X^{\bullet,\bullet})),
\]

which are uniformly called the twisted Dolbeault cohomologies.

3.2 Double complexes \( S_X^{\bullet,\bullet}(\mathcal{L}, s, t) \) and \( T_X^{\bullet,\bullet}(\mathcal{L}, s, t) \)

In [3, Section 4.2], Chen, Y. and Yang, S. introduced the complexes \( \Omega_X^{[s,t]}(\mathcal{L}) \) for \( 0 \leq s \leq t \leq n \) on a compact complex manifold \( X \). For convenience, we generalize this concept a little.

The compactness of \( X \) is not assumed in this paper. Set \( \mathcal{L} \otimes \Omega_X^k = 0 \) for \( k < 0 \) or \( > n \). Given any integers \( s \) and \( t \), the truncated twisted holomorphic de Rham complex \( \Omega_X^{[s,t]}(\mathcal{L}) \) is defined as the zero complex if \( s > t \) and as the complex

\[
0 \to \mathcal{L} \otimes \Omega_X^t \to \mathcal{L} \otimes \Omega_X^{s+1} \to \cdots \to \mathcal{L} \otimes \Omega_X^1 \to 0
\]

(3.1)

if \( s \leq t \), where \( \mathcal{L} \otimes \Omega_X^k \) is placed in degree \( k \) for \( s \leq k \leq t \) and zeros are placed in other degrees. Obviously, \( \Omega_X^{[s,t]}(\mathcal{L}) \) is equal to \( \Omega_X^{[0,t]}(\mathcal{L}) \) for \( s < 0 \) and equal to \( \Omega_X^{[s,n]}(\mathcal{L}) \) for \( t > n \). In particular, \( \Omega_X^{[0,n]}(\mathcal{L}) = \mathcal{L} \otimes \Omega_X^n \) is the twisted holomorphic de Rham complex on \( X \) and \( \Omega_X^{[p,p]}(\mathcal{L}) = (\mathcal{L} \otimes \Omega_X^p)[-p] \), where \( \Omega_X^{p,\bullet} \) denote the complex with \( \Omega_X^p \) in degree 0 and zeros in other degrees.

Set

\[
S_X^{p,q}(\mathcal{L}, s, t) = \begin{cases} 
\mathcal{L} \otimes A_X^{p,q}, & s \leq p \leq t \\
0, & \text{others.}
\end{cases}
\]

Then \( (S_X^{\bullet,\bullet}(\mathcal{L}, s, t), d_1, d_2) \) is a double complex of sheaves, where

\[
d_1^{p,q} = \begin{cases} 
\partial, & s \leq p < t \\
0, & \text{others}
\end{cases}
\quad \text{and} \quad
d_2^{p,q} = \begin{cases} 
\bar{\partial}, & s \leq p \leq t \\
0, & \text{others.}
\end{cases}
\]

Denoted it by \( S_X^{\bullet,\bullet}(\mathcal{L}, s, t) \) shortly. Let \( S_X^{\bullet,\bullet}(\mathcal{L}, s, t) = sS_X^{\bullet,\bullet}(\mathcal{L}, s, t) \) be the complex associated with \( S_X^{\bullet,\bullet}(\mathcal{L}, s, t) \). For instance, \( S_X^{\bullet,\bullet}(\mathcal{L}, 0, n) = \mathcal{L} \otimes A_X^{0,\bullet}, \ S_X^{\bullet,\bullet}(\mathcal{L}, 0, n) = \mathcal{L} \otimes A_X^{\bullet,\bullet}, \) and \( S_X^{\bullet,\bullet}(\mathcal{L}, p, p) = \mathcal{L} \otimes A_X^{p,\bullet}[-p] \).

**Lemma 3.1** The inclusion gives a quasi-isomorphism \( \Omega_X^{[s,t]}(\mathcal{L}) \to S_X^{\bullet,\bullet}(\mathcal{L}, s, t) \) of complexes of sheaves.
Proof For any \( p \in \mathbb{Z} \), \( \Omega^{[s,t]}_X(L)^p \rightarrow (S^p_\mathcal{X}, (L, s, t), d^{p,\bullet}_2) \) given by the inclusion is a resolution of \( \Omega^{[s,t]}_X(L)^p \). By [14, Lemma 8.5], the lemma holds. \( \square \)

Set \( S^{p,q}(X, \mathcal{L}, s, t) = \Gamma(X, S^{p,q}_\mathcal{X}(\mathcal{L}, s, t)) \) and \( S^p(X, \mathcal{L}, s, t) = \Gamma(X, S^p_\mathcal{X}(\mathcal{L}, s, t)) \). In particular, the double complex \( S^{\bullet,\bullet}(X, \mathcal{L}, 0, n) \) is just \( (\Gamma(X, \mathcal{L} \otimes \mathcal{A}^{\bullet,\bullet}_X), \partial, \bar{\partial}) \) (see also [3, Section 2.1] or [14, Section 8.2.1]). For any \( p \in \mathbb{Z} \), \( S^p_\mathcal{X}(\mathcal{L}, s, t) \) is \( \Gamma \)-acyclic, since it is a fine sheaf. By [14, Proposition 8.12] and Lemma 3.1,

\[
\mathbb{H}^k(X, \Omega^{[s,t]}_X(L)) \cong H^k(S^{\bullet,\bullet}(X, \mathcal{L}, s, t)) \tag{3.2}
\]

for any \( k \in \mathbb{Z} \). For example, \( \mathbb{H}^k(X, \Omega^{[0,n]}_X(L)) \cong H^k_{dR}(X, \mathcal{L}) \) and \( \mathbb{H}^k(X, \Omega^{[p,p]}_X(L)) \cong H^{k-p}(X, \mathcal{L} \otimes \Omega^p_X) \).

Similarly, we can define \( T^{\bullet,\bullet}_X(\mathcal{L}, s, t), T^{\bullet,\bullet}_X(\mathcal{L}, s, t), T^{\bullet,\bullet}_X(\mathcal{L}, s, t) \) and \( T^\bullet(X, \mathcal{L}, s, t) \), where

\[
T^{p,q}_X(\mathcal{L}, s, t) = \begin{cases} 
\mathcal{L} \otimes \mathcal{P}^{p,q}_X, & s \leq p \leq t \\
0, & \text{others}
\end{cases}
\]

There is an isomorphism

\[
\mathbb{H}^k(X, \Omega^{[s,t]}_X(\mathcal{L})) \cong H^k(T^{\bullet}(X, \mathcal{L}, s, t)) \tag{3.3}
\]

for any \( k \in \mathbb{Z} \).

Notice that all double complexes defined in this section are bounded.

### 3.3 Exact sequences

For integers \( s \leq s' \leq t \leq t' \), there is a natural exact sequence

\[
0 \rightarrow S^{\bullet,\bullet}(X, \mathcal{L}, t + 1, t') \rightarrow S^{\bullet,\bullet}(X, \mathcal{L}, s, t') \rightarrow S^{\bullet,\bullet}(X, \mathcal{L}, s, t)
\]

\[
\rightarrow S^{\bullet,\bullet}(X, \mathcal{L}, s, s' - 1) \rightarrow 0,
\]

where the morphism is the identity or zero at every degree. In particular, we have a short exact sequence

\[
0 \rightarrow S^{\bullet,\bullet}(X, \mathcal{L}, s, t) \rightarrow S^{\bullet,\bullet}(X, \mathcal{L}, r, t) \rightarrow S^{\bullet,\bullet}(X, \mathcal{L}, r, s - 1) \rightarrow 0 \tag{3.4}
\]

for integers \( r \leq s \leq t \). These exact sequences also hold for \( \Omega^{[s,t]}_X, S^{\bullet}(X, \mathcal{L}, s, t), T^{\bullet}(X, \mathcal{L}, s, t) \), etc. By (3.2) and (3.4), there is a long exact sequence

\[
\cdots \rightarrow \mathbb{H}^{k-1}(X, \Omega^{[r,s-1]}_X(L)) \rightarrow \mathbb{H}^k(X, \Omega^{[s,t]}_X(L)) \rightarrow \mathbb{H}^k(X, \Omega^{[r,s]}_X(L)) \rightarrow \cdots
\]

### 3.4 Operations

Suppose that \( X \) is a complex manifold and \( \mathcal{L}, \mathcal{H} \) are local systems of \( \mathcal{C}_X \)-modules of rank \( l \), \( h \) on \( X \), respectively.

For \( \alpha \in \Gamma(X, \mathcal{L} \otimes \mathcal{A}^p_X) \) and \( \beta \in \Gamma(X, \mathcal{H} \otimes \mathcal{A}^q_X) \), we define the wedge product \( \alpha \wedge \beta \in \Gamma(X, \mathcal{L} \otimes \mathcal{H} \otimes \mathcal{A}^{p+q}_X) \) as follows: Let \( U \) be an open subset of \( X \) such that \( \mathcal{L}|_U \) and \( \mathcal{H}|_U \) are trivial. Let \( e_1, \ldots, e_l \) and \( f_1, \ldots, f_h \) be bases of \( \Gamma(U, \mathcal{L}) \) and \( \Gamma(U, \mathcal{H}) \), respectively. Suppose
that $\alpha = \sum_{i=0}^{l} e_i \otimes \alpha_i$ and $\beta = \sum_{j=1}^{h} f_j \otimes \beta_i$ on $U$, respectively, where $\alpha_i \in A^p(U)$ for $1 \leq i \leq l$ and $\beta_i \in A^q(U)$ for $1 \leq j \leq h$. Then $\alpha \wedge \beta$ on $U$ is defined as

$$\sum_{1 \leq i \leq l, 1 \leq j \leq h} e_i \otimes f_j \otimes (\alpha_i \wedge \beta_j).$$

This construction is global. We have

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta,$$

which also holds if using $\partial$ or $\bar{\partial}$ instead of $d$.

Suppose that $\alpha \in \Gamma(X, \mathcal{H} \otimes A^{p,q}_X)$ satisfies that $\partial \alpha = \bar{\partial} \alpha = 0$, i.e., $d\alpha = 0$. Then wedge products by $\alpha$ give a morphism

$$\bullet \wedge \alpha : S^{*,*}(X, \mathcal{L}, s, t) \to S^{*,*}(X, \mathcal{L} \otimes \mathcal{H}, s + p, t + p)[p, q]$$

of double complexes of sheaves. This define a cup product

$$\cup : \mathbb{H}^k(X, \Omega^{s,t}_X(\mathcal{L})) \times H_{BC}^{p,q}(X, \mathcal{H}) \to \mathbb{H}^{k+p+q}(X, \Omega^{s+p,t+p}_X(\mathcal{L} \otimes \mathcal{H}))$$

for any $k$, where

$$H_{BC}^{p,q}(X, \mathcal{H}) := \frac{\text{Ker}(d : \Gamma(X, \mathcal{H} \otimes A^{p,q}_X) \to \Gamma(X, \mathcal{H} \otimes A^{p+q+1}_X))}{\bar{\partial} \partial \Gamma(X, \mathcal{H} \otimes A^{p-1,q-1}_X)}.$$

$H_{BC}^{p,q}(X, \mathcal{H})$ is the twisted Bott–Chern cohomology of $X$. We can obtain the same cup product by

$$\bullet \wedge \alpha : T^{*,*}(X, \mathcal{L}, s, t) \to T^{*,*}(X, \mathcal{L} \otimes \mathcal{H}, s + p, t + p)[p, q].$$

Let $f : Y \to X$ be a holomorphic map between complex manifolds. The pullbacks give a morphism

$$f^* : S^{*,*}(X, \mathcal{L}, s, t) \to S^{*,*}(Y, f^{-1}\mathcal{L}, s, t)$$

of double complexes, which induces a morphism $f^* : \mathbb{H}^k(X, \Omega^{s,t}_X(\mathcal{L})) \to \mathbb{H}^k(Y, \Omega^{s,t}_Y(f^{-1}\mathcal{L}))$ for any $k$. Moreover, if $f$ is proper and $r = \dim_{\mathbb{C}} Y - \dim_{\mathbb{C}} X$, then the pushforwards give a morphism

$$f_* : T^{*,*}(Y, f^{-1}\mathcal{L}, s, t) \to T^{*,*}(X, \mathcal{L}, s - r, t - r)[-r, -r]$$

of double complexes, which induces a morphism

$$f_* : \mathbb{H}^k(Y, \Omega^{s,t}_Y(f^{-1}\mathcal{L})) \to \mathbb{H}^{k-2r}(X, \Omega^{s-r,t-r}_X(\mathcal{L}))$$

for any $k$. Similarly, these operations can also be defined on complexes $S^*(X, \mathcal{L}, s, t)$ and $T^*(X, \mathcal{L}, s, t)$. By [8, (3.7)], we easily get

**Proposition 3.2** (Projection formula) Let $f : Y \to X$ be a proper holomorphic map between complex manifolds and $\mathcal{L}, \mathcal{H}$ local systems of $\mathcal{O}_X$-modules of finite rank on $X$. Fix integers $k, p, q, s, t$. Then

$$f_*(f^*\alpha \cup \beta) = \alpha \cup f_*\beta$$

for any $\alpha \in \mathbb{H}^k(X, \Omega^{s,t}_X(\mathcal{L}))$ and $\beta \in H_{BC}^{p,q}(Y, f^{-1}\mathcal{H})$.  

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Recall that a complex manifold $X$ is called $p$-Kählerian, if it admits a closed transverse positive $(p, p)$-form $\Omega$ (see [1, Definition 1.1, 1.3]). In such case, $\Omega|_Z$ is a volume form on $Z$, for any complex submanifold $Z$ of pure dimension $p$ of $X$. Any complex manifold is 0-Kählerian, and any Kähler manifold $X$ is $p$-Kählerian for every $0 \leq p \leq \dim_{\mathbb{C}} X$. We generalize [15, Theorem 3.1(a)(b), 4.1(a)(b)] [6, Propositions 2.3, 3.2] [8, Proposition 3.10] as follows.

**Proposition 3.3** Suppose that $f : Y \to X$ is a proper surjective holomorphic map between complex manifolds and $Y$ is $r$-Kählerian, where $r = \dim_{\mathbb{C}} Y - \dim_{\mathbb{C}} X$. Let $L$ be a local system of $\underline{\mathbb{C}}_X$-modules of finite rank on $X$. Then, for any $k, s, t$,

1. $f^* : \mathbb{H}^k(X, \Omega_X^{[s, t]}(L)) \to \mathbb{H}^k(Y, \Omega_Y^{[s, t]}(f^{-1}L))$ is injective,
2. $f_* : \mathbb{H}^k(Y, \Omega_Y^{[s, t]}(f^{-1}L)) \to \mathbb{H}^{k-2r}(X, \Omega_X^{[r-s, r-t]}(L))$ is surjective.

**Proof** Let $\Omega$ be a strictly positive closed $(r, r)$-form on $Y$. Then $c = f_* \Omega$ is a closed current of degree 0, hence a constant. By Sard’s theorem, the set $X_0$ of regular values of $f$ is nonempty. For any $x \in X_0$, $X_x = f^{-1}(x)$ is a compact complex submanifold of pure dimension $r$, so $c = f_{X_x} \Omega|_{Y_x} > 0$. By the projection formula, $f_*(f^* \alpha \cup [\Omega]_{BC}) = c \cdot \alpha$ for any $\alpha \in \mathbb{H}^k(X, \Omega_X^{[s, t]}(L))$, where $[\Omega]_{BC} \in H^{r, r}_{BC}(X)$ denotes the Bott–Chern class of $\Omega$. It is easy to deduce the proposition. \(\square\)

**Corollary 3.4** Let $f : Y \to X$ be a proper surjective holomorphic map between complex manifolds with the same dimensions and $L$ a local system of $\underline{\mathbb{C}}_X$-modules of finite rank on $X$. Then, for any $k, s, t$,

1. $f^* : \mathbb{H}^k(X, \Omega_X^{[s, t]}(L)) \to \mathbb{H}^k(Y, \Omega_Y^{[s, t]}(f^{-1}L))$ is injective,
2. $f_* : \mathbb{H}^k(Y, \Omega_Y^{[s, t]}(f^{-1}L)) \to \mathbb{H}^{k-2r}(X, \Omega_X^{[r-s, r-t]}(L))$ is surjective.

### 3.5 A spectral sequence

Associated with $S^{\bullet, \bullet}(X, L, s, t)$, there is a spectral sequence

$$E_1^{p,q}(X, L, s, t) \Rightarrow \mathbb{H}^{p+q}(X, \Omega_X^{[s, t]}(L)),$$

where

$$E_1^{p,q}(X, L, s, t) = H^q(S^{p, \bullet}(X, L, s, t)) = \begin{cases} H^q(X, L \otimes \Omega_X^p), & s \leq p \leq t \\ 0, & \text{others} \end{cases}$$

It coincides with that associated with $T^{\bullet, \bullet}(X, L, s, t)$ via the natural inclusion $S^{\bullet, \bullet}(X, L, s, t) \to T^{\bullet, \bullet}(X, L, s, t)$. We call it the truncated twisted Frölicher spectral sequence for $(X, L, s, t)$. The truncated twisted Frölicher spectral sequence for $(X, \underline{\mathbb{C}}_X, 0, n)$ is just the classical Frölicher (or Hodge–de Rham) spectral sequence.

If $X$ is compact, one has an inequality

$$b^k(X, \Omega_X^{[s, t]}(L)) \leq \sum_{p+q=k \atop s \leq p \leq t} h^{p, q}(X, L)$$

(3.7)

for any $k$, where $b^k(X, \Omega_X^{[s, t]}(L)) = \dim_{\mathbb{C}} \mathbb{H}^k(X, \Omega_X^{[s, t]}(L))$ and $h^{p, q}(X, L) = \dim_{\mathbb{C}} H^q(X, L \otimes \Omega_X^p)$. The spectral sequence (3.5) degenerates at $E_1$ if and only if equalities (3.7) hold for all $k$. 

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4 Proofs of main theorems

4.1 Leray–Hirsch theorem

A proof of Theorem 1.1 is given as follows.

Proof Fix two integers $s$ and $t$. Set

$$K^{\bullet, \bullet} = \bigoplus_{i=1}^{r} S^{\bullet, \bullet}(X, L, s - u_i, t - u_i)[-u_i, -v_i]$$

and $L^{\bullet, \bullet} = S^{\bullet, \bullet}(E, \pi^{-1}L, s, t)$. By (3.6), we get the first pages

$$K^{p,q}_{E_1} = \bigoplus_{i=1}^{r} H^{q-v_i}((S^{p-u_i, \bullet}(X, L, s - u_i, t - u_i))$$

$$= \begin{cases} \bigoplus_{i=1}^{r} H^{q-v_i}(X, L \otimes \Omega_X^{p-u_i}), & s \leq p \leq t \\ 0, & \text{others} \end{cases}$$

and

$$L^{p,q}_{E_1} = \begin{cases} H^{q}(E, \pi^{-1}L \otimes \Omega^p_E), & s \leq p \leq t \\ 0, & \text{others} \end{cases}$$

of the spectral sequences associated with $K^{\bullet, \bullet}$ and $L^{\bullet, \bullet}$, respectively. By [8, Theorem 5.6 (2)], the morphism $\sum_{i=1}^{r} \pi^*(\bullet) \wedge t_i : K^{\bullet, \bullet} \to L^{\bullet, \bullet}$ of double complexes induces an isomorphism $K^{p,q}_{E_1} \to L^{p,q}_{E_1}$ at $E_1$-pages and hence induces an isomorphism $H^k(sK^{\bullet, \bullet}) \to H^k(sL^{\bullet, \bullet})$ for any $k$. Notice that

$$sK^{\bullet, \bullet} = \bigoplus_{i=1}^{r} S^{\bullet, \bullet}(X, L, s - u_i, t - u_i)[-u_i, -v_i]$$

and $sL^{\bullet, \bullet} = S^{\bullet}(E, \pi^{-1}L, s, t)$. By (3.2),

$$H^k(sK^{\bullet, \bullet}) = \bigoplus_{i=1}^{r} H^{k-u_i-v_i}(S^{\bullet}(X, L, s - u_i, t - u_i))$$

$$\cong \bigoplus_{i=1}^{r} \mathbb{H}^{k-u_i-v_i}(X, \Omega_X^{[s-u_i, t-u_i]}(L))$$

and

$$H^k(sL^{\bullet, \bullet}) = H^k(S^{\bullet}(E, \pi^{-1}L, s, t)) \cong \mathbb{H}^k(E, \Omega^i_E(\pi^{-1}L)).$$

We complete the proof. \qed

We generalize [10, Proposition 3.3], [7, Corollary 3.2], [13, Proposition 5], [2, Proposition 2], [6, Corollary 4.7], [9, Lemma 3.3] as follows.
Corollary 4.1 Let \( \pi : \mathbb{P}(E) \to X \) be the projective bundle associated with a holomorphic vector bundle \( E \) on a complex manifold \( X \) and \( L \) a local system of \( \mathbb{C}_X \)-modules of finite rank on \( X \). Set \( \text{rank}_C E = r \).

(1) There exists an isomorphism
\[
\bigoplus_{i=0}^{r-1} \mathbb{H}^{k-2i}(X, \mathcal{S}_X^{[s-i,t-i]}(L)) \cong \mathbb{H}^k(\mathbb{P}(E), \Omega^{[s,t]}_{\mathbb{P}(E)}(\pi^{-1}L))
\]
for any \( k, s, t \).

(2) Suppose that \( X \) is compact and \( s \leq t \). Then the truncated twisted Frölicher spectral sequence for \( (\mathbb{P}(E), \pi^{-1}L, s, t) \) degenerates at \( E_1 \)-page if and only if so do those for \((X, L, s-r+1, t-r+1), \ldots, (X, L, s, t)\).

Proof Let \( u \in A^{1,1}(\mathbb{P}(E)) \) be a first Chern form of the universal line bundle \( \mathcal{O}_{\mathbb{P}(E)}(-1) \) on \( \mathbb{P}(E) \) and \( h = [u]_\partial \in H^{1,1}_\partial(\mathbb{P}(E)) \) its Dolbeault class. For every \( x \in X, 1, h, \ldots, h^{r-1} \) restricted to the fiber \( \pi^{-1}(x) = \mathbb{P}(E_x) \) freely linearly generate \( H^{\bullet,\bullet}_\partial(\mathbb{P}(E_x)) \). By Theorem 1.1, we get (1).

In general, we have
\[
b^k(\mathbb{P}(E), \Omega_{\mathbb{P}(E)}^{[s,t]}(\pi^{-1}L)) = \sum_{i=0}^{r-1} b^{k-2i}(X, \mathcal{S}_X^{[s-i,t-i]}(L)) \quad \text{(by Corollary 4.1 (1))}
\]
\[
\leq \sum_{i=0}^{r-1} \sum_{p+q=k-2i} h^{p,q}(X, L) \quad \text{(by (3.7))}
\]
\[
= \sum_{p+q=k} \sum_{s-i \leq p \leq t} h^{p-i,q-i}(X, L)
\]
\[
= \sum_{p+q=k} h^{p,q}(\mathbb{P}(E), \pi^{-1}L) \quad \text{(by [11, Lemma 3.3] or [8, Corollary 5.7(2)]).}
\]

So,
\[
b^k(\mathbb{P}(E), \Omega_{\mathbb{P}(E)}^{[s,t]}(\pi^{-1}L)) = \sum_{p+q=k} h^{p,q}(\mathbb{P}(E), \pi^{-1}L) \text{ for all } k,
\]
if and only if,
\[
b^k(X, \Omega_X^{[s-i,t-i]}(L)) = \sum_{p+q=k} h^{p,q}(X, L) \text{ for all } k \text{ and } 0 \leq i \leq r - 1.
\]

Thus, (2) follows. \( \square \)

Remark 4.2 From the proof of [7, Corollary 5.7], we may similarly obtain the flag bundle formula of the hypercohomology of truncated twisted holomorphic de Rham complexes. Of course, its expression is much more sophisticated.
For an $n$-dimensional complex manifold $X$, the inclusion $\Omega^{[p,n]}_{X}(\mathbb{C}_{X}) \to \Omega_{X}^{[0,n]}(\mathbb{C}_{X})$ induces the Hodge filtration on $H^{k}_{dR}(X, \mathbb{C})$ [14, Definition 8.2] as

\[
F^{p}H^{k}_{dR}(X, \mathbb{C}) = \text{Im}(H^{k}(X, \mathbb{C}_{X}, \Omega^{[p,n]}_{X}(\mathbb{C}_{X}))) \to H^{k}_{dR}(X, \mathbb{C})
\]

for any $k$ and $p$. By (3.2) and (3.3), the inclusions $S^{*}(X, \mathbb{C}_{X}, p, n) \to S^{*}(X, \mathbb{C}_{X}, 0, n)$ and $T^{*}(X, \mathbb{C}_{X}, p, n) \to T^{*}(X, \mathbb{C}_{X}, 0, n)$ induce

\[
F^{p}H^{k}_{dR}(X, \mathbb{C}) = \text{Im}(H^{k}(S^{*}(X, \mathbb{C}_{X}, p, n))) \to H^{k}(S^{*}(X, \mathbb{C}_{X}, 0, n))
\]

and

\[
F^{p}H^{k}_{dR}(X, \mathbb{C}) = \text{Im}(H^{k}(T^{*}(X, \mathbb{C}_{X}, p, n))) \to H^{k}(T^{*}(X, \mathbb{C}_{X}, 0, n)).
\]

respectively.

**Corollary 4.3** [9, Lemma 3.4] Let $\pi : \mathbb{P}(E) \to X$ be the projective bundle associated with a holomorphic vector bundle $E$ on a complex manifold $X$ and $\text{rank}_{\mathbb{C}}E = r$. Suppose that $u \in \mathcal{A}^{1,1}(\mathbb{P}(E))$ is a first Chern form of the universal line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$ on $\mathbb{P}(E)$. Then

\[
\sum_{i=0}^{r-1} \pi^{*}(\bullet) \wedge u^{i}
\]

gives an isomorphism

\[
\bigoplus_{i=0}^{r-1} F^{p-i}H^{k-2i}_{dR}(X, \mathbb{C}) \to F^{p}H^{k}_{dR}(\mathbb{P}(E), \mathbb{C}),
\]

for any $k$, $p$.

**Proof** Set $\dim_{\mathbb{C}}X = n$. Let $u \in \mathcal{A}^{1,1}(\mathbb{P}(E))$ be a first Chern form of the universal line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$ on $\mathbb{P}(E)$. Then $\partial u = \bar{\partial}u = 0$. Consider the commutative diagram of complexes

\[
\begin{array}{ccc}
\Bigoplus_{i=0}^{r-1} S^{*}(X, \mathbb{C}_{X}, p-i, n+r-1-i)[-2i] & \to & S^{*}(\mathbb{P}(E), \mathbb{C}_{\mathbb{P}(E)}, p+n+r-1) \\
\downarrow & & \downarrow \\
\Bigoplus_{i=0}^{r-1} S^{*}(X, \mathbb{C}_{X}, -i, n+r-1-i)[-2i] & \to & S^{*}(\mathbb{P}(E), \mathbb{C}_{\mathbb{P}(E)}, 0+n+r-1).
\end{array}
\]

Notice that $S^{*}(X, \mathbb{C}_{X}, p-i, n+r-1-i) = S^{*}(X, \mathbb{C}_{X}, p-i, n)$ and $S^{*}(X, \mathbb{C}_{X}, -i, n+r-1-i) = S^{*}(X, \mathbb{C}_{X}, 0, n)$ for $0 \leq i \leq r-1$. Thus, we have the commutative diagram

\[
\begin{array}{ccc}
\bigoplus_{i=0}^{r-1} H^{k-2i}(S^{*}(X, \mathbb{C}_{X}, p-i, n)) & \xrightarrow{\cong} & H^{k}(S^{*}(\mathbb{P}(E), \mathbb{C}_{\mathbb{P}(E)}, p+n+r-1)) \\
\downarrow & & \downarrow \\
\bigoplus_{i=0}^{r-1} H^{k-2i}(S^{*}(X, \mathbb{C}_{X}, 0, n)) & \xrightarrow{\cong} & H^{k}(S^{*}(\mathbb{P}(E), \mathbb{C}_{\mathbb{P}(E)}, 0+n+r-1)).
\end{array}
\]

where the two horizontal maps are isomorphisms by Corollary 4.1 (1). Consequently, we get the corollary by (4.1). □
4.2 Künneth theorem

Let $X$, $Y$ be complex manifolds and let $pr_1$, $pr_2$ be projections from $X \times Y$ onto $X$, $Y$, respectively. For sheaves $\mathcal{L}$ and $\mathcal{H}$ of $\mathbb{C}_X^-$ and $\mathbb{C}_Y^-$-modules on $X$ and $Y$, respectively, the external tensor product of $\mathcal{L}$ and $\mathcal{H}$ on $X \times Y$ is defined as

$$\mathcal{L} \boxtimes \mathcal{H} = pr_1^{-1}\mathcal{L} \otimes_{\mathbb{C}_X \times Y} pr_2^{-1}\mathcal{H}.$$

For coherent analytic sheaves $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{O}_X$- and $\mathcal{O}_Y$-modules on $X$ and $Y$, respectively, the analytic external tensor product of $\mathcal{F}$ and $\mathcal{G}$ on $X \times Y$ is defined as

$$\mathcal{F} \boxtimes \mathcal{G} = pr_1^*\mathcal{F} \otimes_{\mathcal{O}_X \times Y} pr_2^*\mathcal{G}.$$

Now, we verify Theorem 1.2.

**Proof** Fix two integers $s$ and $t$. Consider the double complexes

$$K^{*,*} = \bigoplus_{u+w=s, v+w=t} ss(S^{*,*}(X, \mathcal{L}, u, v)) \otimes_{\mathbb{C}} S^{*,*}(Y, \mathcal{H}, w, w),$$

$$L^{*,*} = S^{*,*}(X \times Y, \mathcal{L} \boxtimes \mathcal{H}, s, t)$$

and a morphism $f = pr_1^*(\bullet) \wedge pr_2^*(\bullet) : K^{*,*} \to L^{*,*}$. The first pages of the spectral sequences associated with $K^{*,*}$ and $L^{*,*}$ are calculated as follows

$$K E_1^{a,b} = \bigoplus_{u+w=s, v+w=t} H^k(S^{p,q}(X, \mathcal{L}, u, v)) \otimes_{\mathbb{C}} H^l(S^{q,p}(Y, \mathcal{H}, w, w)) \quad \text{(by Proposition 2.7 (2))}$$

$$= \bigoplus_{p+q=a, k+l=b} H^k(S^{p,q}(X, \mathcal{L}, u, v)) \otimes_{\mathbb{C}} H^l(S^{q,p}(Y, \mathcal{H}, w, w)) \quad \text{(by (3.6))}$$

$$= \bigoplus_{p+q=a, s-q \leq p \leq t-q, k+l=b} H^k(X, \mathcal{L} \otimes \Omega_X^p) \otimes_{\mathbb{C}} H^l(Y, \mathcal{H} \otimes \Omega_Y^q) \quad \text{(by (3.6))}$$

$$= \begin{cases} 
H^k(X, \mathcal{L} \otimes \Omega_X^p) \otimes_{\mathbb{C}} H^l(Y, \mathcal{H} \otimes \Omega_Y^q), & s \leq a \leq t \\
0, & \text{others}
\end{cases}$$

and

$$L E_1^{a,b} = \begin{cases} 
H^b(X \times Y, (\mathcal{L} \boxtimes \mathcal{H}) \otimes \Omega_{X \times Y}^a), & s \leq a \leq t \\
0, & \text{others}
\end{cases} \quad \text{(by Proposition 2.7 (2) and (3.6)).}$$

The morphism $E_1^{a,b}(f) : K E_1^{a,b} \to L E_1^{a,b}$ at $E_1$-pages induced by $f$ is just the Cartesian product

$$\bigoplus_{p+q=a, k+l=b} H^k(X, \mathcal{L} \otimes \Omega_X^p) \otimes_{\mathbb{C}} H^l(Y, \mathcal{H} \otimes \Omega_Y^q) \to H^b(X \times Y, (\mathcal{L} \boxtimes \mathcal{H}) \otimes \Omega_{X \times Y}^a)$$

for $s \leq a \leq t$ and the identity between zero spaces for other cases. Notice that $(\mathcal{L} \otimes \Omega_X^p) \boxtimes (\mathcal{H} \otimes \Omega_Y^q) = (\mathcal{L} \boxtimes \mathcal{H}) \otimes (\Omega_X^p \boxtimes \Omega_Y^q)$ and $\Omega_{X \times Y}^a = \bigoplus_{p+q=a} (\Omega_X^p \boxtimes \Omega_Y^q)$. By [4, IX (5.23)]
\[ H^c(s K^{\bullet, \bullet}) \cong \bigoplus_{a+b=c \atop u+w=s \atop v+w=t} \mathbb{H}^a(X, \Omega^{[u,v]}_X(L)) \otimes \mathbb{H}^b(Y, \Omega^{[w,u]}_Y(\mathcal{H})) \]

and \[ H^c(s L^{\bullet, \bullet}) \cong \mathbb{H}^c(X \times Y, \Omega^{[s,t]}_{X \times Y}(L \boxtimes \mathcal{H})). \]

We conclude this theorem. \qed

### 4.3 Poincaré–Serre duality theorem

Let \( X \) be a complex manifold and \( L \) a local system of \( \mathbb{C}_X \)-modules of rank \( l \) on \( X \). Denote by \( L^\vee = \mathcal{H}om_{\mathbb{C}_X}(L, \mathbb{C}_X) \) the dual of \( L \). For \( \gamma \in \Gamma(X, L^\otimes L^\vee \otimes A^p_X) \), we construct \( tr(\gamma) \) as follows: Suppose that \( U \) is an open subset of \( X \) such that \( L|_U \) is trivial. Let \( e_1, \ldots, e_l \) and \( f_1, \ldots, f_l \) be bases of \( \Gamma(U, L) \) and \( \Gamma(U, L^\vee) \), respectively. Set \( \gamma = \sum_{1 \leq i \leq l} e_i \otimes f_i \otimes \gamma_{ij} \) on \( U \), where \( \gamma_{ij} \in A^p(U) \) for any \( 1 \leq i, j \leq l \). Then \( tr(\gamma) \) on \( U \) is defined as \( \sum_{1 \leq i \leq l} (e_i, f_j) \gamma_{ij} \), where \( (\cdot, \cdot) \) is the contraction between \( L \) and \( L^\vee \). This construction is global. By the construction, the action \( tr \) on \( \Gamma(X, L \otimes L^\vee \otimes A^p_X) \) is just contracting the parts of \( L \) and \( L^\vee \) and preserving the part of the form of \( \gamma \), which was essentially defined in [12] (even in more general cases).

Assume that \( \mathcal{H} \) is a local system of \( \mathbb{C}_X \)-modules of finite rank on \( X \). For \( \alpha \in \Gamma(X, L \otimes A^q_X) \) and \( \beta \in \Gamma(X, \mathcal{H} \otimes A^q_X) \),

\[
d(tr(\alpha \wedge \beta)) = tr(d\alpha \wedge \beta) + (-1)^p tr(\alpha \wedge d\beta),
\]

which also holds if using \( \partial \) or \( \bar{\partial} \) instead of \( d \).

**Proof** Fix two integers \( s \) and \( t \). Set

\[ K^{\bullet, \bullet} = S^{\bullet, \bullet}(X, L, s, t) \text{ and } L^{\bullet, \bullet} = (S^{\bullet, \bullet}(X, L^\vee, n-t, n-s))^* [-n, -n]. \]

By (3.6) and Lemma 2.1, we have

\[
\kappa E^{p,q}_1 = \begin{cases} H^q(X, L \otimes \Omega^p_X), & s \leq p \leq t \\ 0, & \text{otherwise} \end{cases}
\]

and

\[
L E^{p,q}_1 = H^{q-n}(S^{\bullet, \bullet}(X, L^\vee, n-t, n-s))^* \otimes A^p_X \\
= H^{q-n}(S^{n-p, \bullet}(X, L^\vee, n-t, n-s))^* \\
= \begin{cases} (H^{r-q}(X, L^\vee \otimes \Omega^{n-p}_X))^*, & s \leq p \leq t \\ 0, & \text{otherwise.} \end{cases}
\]

The map

\[
\mathcal{D} : K^{\bullet, \bullet} \to L^{\bullet, \bullet}, \alpha \mapsto \int_X tr(\alpha \wedge \bullet)
\]

is a morphism of double complexes by (4.3). The morphism \( E^{p,q}_1(D) : \kappa E^{p,q}_1 \to L E^{p,q}_1 \) at \( E_1 \)-pages induced by \( D \) is just the Serre duality map \( H^q(X, L \otimes \Omega^p_X) \to (H^{n-q}(X, L^\vee \otimes \Omega^p_X))^* \). Springer
theorem, by (3.2) and (3.3), the left-hand side and the right-hand side of (1.1) are naturally viewed as the projective bundle $P$ from which Theorem 1.3 follows.

By (3.2) and Lemma 2.1, we have

\[ H^k(sK^{••}) \cong \mathbb{H}^k(X, \mathbb{L}_{X}^{[s,t]}(\mathcal{L})) \quad \text{and} \quad H^k(sL^{••}) \cong \left( \mathbb{H}^{2n-k}(X, \mathbb{L}_{X}^{[n-t,n-s]}(\mathcal{L}^{\vee})) \right)^{•}, \]

from which Theorem 1.3 follows. □

Remark 4.4 Theorem 1.3 is the Poincare duality theorem if $s = 0$, $t = n$ and a special case of the Serre duality theorem if $s = t$.

### 4.4 Blowup formula

Let $\pi : \widetilde{X} \to X$ be the blowup of a complex manifold $X$ along a complex submanifold $Y$ and $\mathcal{L}$ a local system of $\mathbb{C}$-modules of finite rank on $X$. As we know, $\pi|_E : E = \pi^{-1}(Y) \to Y$ can be naturally viewed as the projective bundle $\mathbb{P}(N_Y/X)$ associated with the normal bundle $N_Y/X$ of $Y$ in $X$. Let $u \in \mathcal{A}^{1,1}(E)$ be a first Chern form of the universal line bundle $\mathcal{O}_E(-1)$ on $E = \mathbb{P}(N_Y/X)$. Suppose that $i_Y : Y \to X$ and $i_E : E \to \widetilde{X}$ are the inclusions. Set $r = \text{codim}_Y X \geq 2$.

By [8, Theorem 1.2], Theorem 1.4 can be similarly proved with Theorem 1.1. Now, we provide an alternative proof as follows.

**Proof** Consider the complexes

\[ K^{•}(s,t) = S^{•}(X, \mathcal{L}, s,t) \oplus \bigoplus_{i=1}^{r-1} S^{•}(Y, i_Y^{-1}\mathcal{L}, s-i, t-i)[-2i] \]

and $L^{•}(s,t) = T^{•}(\widetilde{X}, \pi^{-1}\mathcal{L}, s,t)$ and the morphism

\[ f(s,t) = \pi^{••} + \sum_{i=1}^{r-1} i_{E*} \circ (u^{i-1} \wedge) \circ (\pi|_E)^{•} : K^{•}(s,t) \to L^{•}(s,t). \]

By (3.2) and (3.3), the left-hand side and the right-hand side of (1.1) are $H^k(K^{•}(s,t))$ and $H^k(L^{•}(s,t))$, respectively. Our goal is to show that $H^k(f(s,t)) : H^k(K^{•}(s,t)) \to H^k(L^{•}(s,t))$ is an isomorphism for any $s,t$.

We may assume that $0 \leq s \leq t$. By (3.4), there is the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & K^{•}(s,t) & \longrightarrow & K^{•}(0,t) & \longrightarrow & K^{•}(0,s-1) & \longrightarrow & 0 \\
\downarrow {f(s,t)} & & \downarrow {f(0,t)} & & \downarrow {f(0,s-1)} & & & & \\
0 & \longrightarrow & L^{•}(s,t) & \longrightarrow & L^{•}(0,t) & \longrightarrow & L^{•}(0,s-1) & \longrightarrow & 0 \\
\end{array}
\]
of short exact sequences of complexes, which induces the commutative diagram

\[
\begin{array}{ccccccc}
H^{k-1}(K^\bullet(0, i)) & \rightarrow & H^{k-1}(K^\bullet(0, s-1)) & \rightarrow & H^k(K^\bullet(s, t)) & \rightarrow & H^k(K^\bullet(0, i)) & \rightarrow & H^k(K^\bullet(0, s-1)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{k-1}(f^\bullet(0, i)) & \rightarrow & H^{k-1}(f^\bullet(0, s-1)) & \rightarrow & H^k(f^\bullet(s, t)) & \rightarrow & H^k(f^\bullet(0, i)) & \rightarrow & H^k(f^\bullet(0, s-1)) \\
H^{k-1}(L^\bullet(0, i)) & \rightarrow & H^{k-1}(L^\bullet(0, s-1)) & \rightarrow & H^k(L^\bullet(s, t)) & \rightarrow & H^k(L^\bullet(0, i)) & \rightarrow & H^k(L^\bullet(0, s-1)) \\
\end{array}
\]

(4.4)

of long exact sequences. By [8, Theorem 1.2], \(H^\bullet(f(t, t))\) is an isomorphism for any \(t\). First, consider (4.4) for \(s = t\). By the induction on \(t \geq 0\) and the five-lemma, \(H^\bullet(f(0, t))\) is an isomorphism for any \(t \geq 0\). Furthermore, \(H^\bullet(f(s, t))\) is an isomorphism for any \(s, t\), by the five-lemma again in (4.4) for general cases.

\[\square\]

**Remark 4.5** Theorem 1.4 for \(s = 0, t = n\) is just [3, Theorem 1] and the first formula of [8, Theorem 1.2], while the two proofs here are quite different from those there.

Following the proof of Corollary 4.1 (2), we generalize [10, Theorem 1.6] [3, Corollary 2] as follows.

**Corollary 4.6** Under the hypotheses of Theorem 1.4, suppose that \(X\) is compact and \(s \leq t\). Then the truncated twisted Frölicher spectral sequence for \((\tilde{X}, \pi^{-1}L, s, t)\) degenerates at \(E_1\)-page if and only if so do those for \((X, L, s, t)\), \((Y, i_{Y^{-1}}L, s - r + 1, t - r + 1)\), \ldots, \((Y, i_{Y^{-1}}L, s - 1, t - 1)\).

With the similar proof of Corollary 4.3, we easily get the Hodge filtration of a blowup as follows.

**Corollary 4.7** [9, Lemma 3.7] Under the hypotheses of Theorem 1.4,

\[
\pi^* + \sum_{i=1}^{r-1} i_{E^*} \circ (u^{i-1} \wedge) \circ (\pi|_E)^*
\]

gives an isomorphism

\[
\bigoplus_{i=1}^{r-1} F^p H^k_{dR}(Y, \mathbb{C}) \rightarrow \bigoplus_{i=1}^{r-1} F^p H^{k-2i}_{dR}(Y, \mathbb{C}) \rightarrow F^p H^k_{dR}(\tilde{X}, \mathbb{C}),
\]

for any \(k, p\).

### 4.5 A remark

In the proofs of Theorems 1.1–1.3 and the first proof of Theorem 1.4, we use almost the same steps, that is, first proving that the morphisms between double complexes induce isomorphisms at \(E_1\)-pages and then obtaining their induced isomorphisms at \(H\)-pages, where we need the corresponding results (respectively, the Leray–Hirsch, Künneth, Serre duality theorems and the blowup formula) on the twisted Dolbeault cohomology in the former steps. In the second proof of Theorem 1.4, we use the blowup formula on the twisted Dolbeault cohomology and the five-lemma. So the twisted Dolbeault cohomology and some algebraic machines play significant roles in the research of the hypercohomology of truncated twisted holomorphic de Rham complexes. For the twisted Dolbeault cohomology, we refer to [8,11,15].

\[\square\] Springer
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