Theoretical constraints on reciprocal and non-reciprocal many-body radiative heat transfer

Cheng Guo and Shanhui Fan

Department of Applied Physics, Stanford University, Stanford, California 94305, USA
Ginzton Laboratory and Department of Electrical Engineering, Stanford University, Stanford, California 94305, USA

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We study the constraints on reciprocal and non-reciprocal many-body radiative heat transfer imposed by symmetry and the second law of thermodynamics. We show that the symmetry of such a many-body system in general forms a magnetic group, and the constraints of the magnetic group on the heat transfer can be derived using a generalized reciprocity theorem. We also show that the second law of thermodynamics provides additional constraints in the form of a nodal conservation law of heat flow at equilibrium. As an application, we provide a systematic approach to determine the existence of persistent heat current in arbitrary many-body systems.

I. INTRODUCTION

Thermal radiation is important for both fundamental science and engineering applications [1–6]. The vast majority of the literature on radiative heat transfer assumes Lorentz reciprocity, which imposes strong constraint on radiative heat transfer [7,9]. On the other hand, recently there have been significant progress in studying radiative heat transfer using non-reciprocal materials such as magnetooptical materials [10,17] and magnetic Weyl semimetals [15,20]. These studies have led to the discoveries of novel phenomena in nonreciprocal many-body radiative heat transfer such as persistent heat current [21] and photon thermal Hall effect [20,22,23], which exemplifies the opportunities of exploring novel aspects of radiative heat transfer that can arise in complex reciprocal and nonreciprocal many-body systems [21,25].

In this paper, in order to provide theoretical guidance on the explorations of many-body radiative heat transfer, we consider the general theoretical constraints on such process. Certainly the heat transfer is constrained by the second law of thermodynamics. Moreover, for a non-reciprocal many-body system, where reciprocity is broken with either internal or external bias magnetic fields on at least some of the bodies, its symmetry consists of two classes of operations. The first class is the usual spatial operations, such as rotation and mirror operations, which transforms the magnetic field bias on each of the body in the usual way of pseudovectors. The second class consists of operations that flip all the magnetic field bias in addition to the usual spatial operations. These two classes of operations together form the magnetic group of the many-body system. We show that the properties of many-body radiative heat transfer are strongly constrained by the structure of the magnetic group. The derivation in particular relies upon a generalized reciprocity theorem that relates the properties of two complementary systems. As an illustration of these theoretical constraints, we show these constraints can be used to identify many-body non-reciprocal systems that do not exhibit persistent heat current.

The rest of this paper is organized as follows. In Sec. II we provide the theory. In Sec. III we apply our theory to determine the existence of persistent heat current in arbitrary many-body systems. We conclude in Sec. IV.

II. THEORY

We consider a system consisting of \( N \) bodies that exchange heat via radiation with each other and an environment. We label the environment (env) and the bodies as \( \{0 \equiv \text{env}, 1, 2, \ldots, N\} \). In general, the system is an inhomogeneous dispersive bianisotropic medium, which can be described by a \( 6 \times 6 \) constitutive matrix \( C(\omega, r) \) [20]:

\[
(D \quad B) = C (E \quad H) = \begin{pmatrix} \varepsilon & \zeta \\ \eta & \mu \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix},
\]

where \( \varepsilon, \mu, \zeta, \eta \) are \( 3 \times 3 \) matrices of electric permittivity, magnetic permeability, electric-magnetic coupling strength, and magneto-electric coupling strength, respectively. \( \omega \) and \( r \) denote the frequencies and the spatial coordinates, respectively.

For radiative heat transfer, one considers the spectral heat flux to body \( j \) due to thermal noise sources in body \( i \) of temperature \( T_i \):

\[
S_{i \rightarrow j}(\omega) = \frac{\Theta(\omega, T_i)}{2\pi} F_{i \rightarrow j}(\omega),
\]

where \( \Theta(\omega, T_i) = \hbar\omega/[\exp(\hbar\omega/k_B T_i) - 1] \), and \( F_{i \rightarrow j}(\omega) \) denotes the temperature independent transmission coefficient from body \( i \) to \( j \). A general theory of many-body radiative heat transfer has been developed in Ref. [27] that allows to calculate \( F_{i \rightarrow j}(\omega) \) given \( C(\omega, r) \). All the directional transmission coefficients \( F_{i \rightarrow j}(\omega) \) can be arranged...
FIG. 1. Schematic of a system consisting of four gyrotropic spheres that exchange heat via radiation with each other and an environment. The centers of the spheres form a square on the $x$-$y$ plane. There is a magnetic field along the $z$-direction with alternating signs in its distribution in the $x$-$y$ plane.

into an exchange matrix $F$ of dimension $(N+1) \times (N+1)$.

$$F = \begin{pmatrix}
0 & F_{0\rightarrow 1} & \cdots & F_{0\rightarrow N} \\
F_{1\rightarrow 0} & 0 & \cdots & F_{1\rightarrow N} \\
\vdots & \vdots & \ddots & \vdots \\
F_{N\rightarrow 0} & F_{N\rightarrow 1} & \cdots & 0
\end{pmatrix}, \quad (3)
$$

where we have suppressed the parameter $\omega$ for clarity. Our objective is to find the constraints imposed on $F$ by the symmetry of the system as well as the second law of thermodynamics. To illustrate the possible symmetry of these systems, we first consider a concrete example as shown in Fig. 1, which consists of a cluster of four gyrotropic spheres. These spheres are assumed to be made of the same materials, but may subject to different local magnetic field. We use the usual point group symmetry operations, as well as compound symmetry operations which contain an anti-symmetry operator $T$.

The generalization of the discussion above to an arbitrary system is straightforward. We define a local operation of anti-symmetry $T$, which transforms between a general medium as described by $C(\omega, \mathbf{r})$ and its complementary medium as described by $\bar{C}(\omega, \mathbf{r})$:

$$C = \begin{pmatrix}
\varepsilon & \zeta \\
\eta & \mu
\end{pmatrix} \xrightarrow{T} \bar{C} = \begin{pmatrix}
\varepsilon^T & -\zeta^T \\
-\eta^T & \mu^T
\end{pmatrix}, \quad (6)
$$

$T^2 = E$, where $E$ is the identity operation. By this definition, for a gyrotropic plasma under an external DC magnetic field bias, its complementary medium is the same gyrotropic plasma but with the direction of the magnetic field bias reversed. A medium is reciprocal if and only if it is self-complementary, i.e. it is invariant under $T$. Since $T$ acts on the constitutive matrix instead of the ordinary position coordinates, it commutes with all the ordinary spatial operations. The symmetry of a general system consists of ordinary spatial symmetry operations, and their combination with $T$. Following Ref. [29], we refer to the former type of operations as uncolored, and the latter type as colored. The magnetic group of a system as described by $C(\omega, \mathbf{r})$ are the sets of all the symmetry operations that leave the system invariant.

Below we consider the constraints on $F$ as imposed by the two different classes of symmetry operations:

1. Uncolored operation $A_1$:

$A_1$ can be represented by the permutation of the bodies:

$$\mathbb{P}_{A_1} = \begin{pmatrix}
0 & 1 & 2 & \cdots & N \\
0 & P_1 & P_2 & \cdots & P_N
\end{pmatrix}, \quad (7)
$$

We note the environment is invariant under permutation of the bodies ($0 \rightarrow 0$). Such a permutation leaves the system invariant, thus it enforces the constraints

$$F_{P_i \rightarrow P_j} = F_{i \rightarrow j}, \quad i, j = 0, \ldots, N \quad (8)
$$

In matrix form, Eq. (8) can be written as

$$\mathbb{P}_{A_1} F = F \mathbb{P}_{A_1}^T = F, \quad (9)$$

compoundsymmetry $R_a = T A_a$ is a combination of $T$ and a usual spatial operation $A_a$, but $T$ and $A_a$ are not the symmetry by themselves. All $\{A_a\}$ (not $\{R_a\}$) together with the usual point group $D_{2h}$ forms a larger point group $D_{4h}$. The symmetry group of such a gyrotropic cluster is therefore a magnetic group:

$$\mathcal{M} = A_4/mmm = D_{4h} = D_{2h} + T(D_{4h} - D_{2h})$$

$$= \{E, 2C_4(z), 2C_2^y(x), 2C_2^{\perp}(a),$$

$$i, 2S_4(z), \sigma_h(z), 2\sigma_v(x), 2\sigma_d(a)\}, \quad (5)$$

where the underline denotes the compound elements that are combined with $T$. A magnetic group contains the usual point group symmetry operations, as well as compound symmetry operations which contain an anti-symmetry operator $T$ [28–30].
where $\mathbf{P}_{A_i}$ is the permutation matrix corresponding to $\mathcal{P}_{A_i}$.

2. Colored operation $R_n = \mathcal{T}A_n$:
$A_n$ permutes the bodies by

$$
\mathbf{P}_{A_n} = \begin{pmatrix} 0 & 1 & 2 & \cdots & N \\ P_1' & P_2' & \cdots & P_N' \end{pmatrix},
$$

(10)

The resultant system is complementary to the original one, and an additional $\mathcal{T}$ operation maps the system back to the original one.

The generalized reciprocity theorem \[31\] of electromagnetism requires that the exchange matrices $\mathcal{F}$ and $\bar{\mathcal{F}}$ of two complementary systems $C(\omega, r)$ and $\bar{C}(\omega, r)$ are transpose of each other (see the proof in the Appendix):

$$
\bar{F} = \mathcal{F}^T, \text{ i.e. } \bar{F}_{i\to j} = F_{j\to i},
$$

(11)

Therefore, $R_n$, being a symmetry of the system, enforces the constraints

$$
F_{i'\to j'} = F_{j'\to i'}, \text{ } i, j = 0, \ldots, N
$$

(12)

In matrix form, Eq. (12) can be written as

$$
\mathbf{P}_{A_n} \mathcal{F} \mathbf{P}^T_{A_n} = \mathcal{F}^T,
$$

(13)

where $\mathbf{P}_{A_n}$ is the permutation matrix corresponding to $\mathcal{P}_{A_n}$.

By considering all the symmetry elements $\{A_i, R_n\}$ in the magnetic group $\mathcal{M}$, we obtain all the constraints imposed by the symmetry on radiative heat transfer.

Magnetic groups, denoted as $\mathcal{M}$, can be classified into three types \[28, 30\]:

1. Colorless group. Here $\mathcal{M}$ is ordinary point group $\mathcal{G}$ with no colored elements.

2. Gray group. Here $\mathcal{M}$ is isomorphic to a direct product $\mathcal{G} \otimes \{E, \mathcal{T}\}$, where $\mathcal{G}$ is a ordinary point group. Being a direct product immediately implies that $\mathcal{T}$ commutes with all elements of the point group $\mathcal{G}$.

3. Black/white group. Here $\mathcal{M} = \{A_i, R_n\}$, where half of the elements are colorless forming the set $\{A_i\}$ and the other half are colored forming the set $\{R_n = \mathcal{T}A_n\}$. Moreover, $\{A_i, A_n\}$ forms an ordinary point group $\mathcal{G}'$ and $\{A_i\} = \mathcal{H}$ forms a subgroup of $\mathcal{G}'$ with index 2. Thus a black/white group is of the form

$$
\mathcal{M} = \mathcal{H} \cup \mathcal{T}(\mathcal{G}' - \mathcal{H})
$$

(14)

We denote $\mathcal{M} = \mathcal{G}'(\mathcal{H})$ following Ref. \[30\].

A reciprocal system is by definition invariant under $\mathcal{T}$. Since $\mathcal{T}$ is an element only of a gray group, a system is reciprocal if and only if its symmetry is a gray group; a system is nonreciprocal if and only its symmetry is a colorless or black/white group.

Finally we consider the constraints of the second law of thermodynamics. Since we consider the bodies exchanging energy only by radiation, in the equilibrium case where all bodies as well as the environment have the same temperature, the energy flow into any body must balances that out of the body:

$$
\sum_{j=0; j \neq i}^N F_{i\to j} = \sum_{j=0; j \neq i}^N F_{j\to i},
$$

(15)

i.e. $\mathcal{F}$ matrix has the same row sum and column sum. This represents a nodal conservation law of heat flow at equilibrium. In matrix form, Eq. (15) can be written as

$$
(\mathcal{F} - \mathcal{F}^T)\bar{\mathbf{j}} = 0,
$$

(16)

where $\bar{\mathbf{j}}$ is an all-one vector. Conversely, if Eq. (16) is satisfied, in equilibrium the net heat flow into any of the subsystem consisting of a few of bodies is zero. Thus Eq. (16) is sufficient to impose the second law of thermodynamics in the many-body system.

The second law of thermodynamics can provide unique constraints beyond those from symmetry. For example, a system where radiative heat transfer occurs entirely between two bodies has an exchange matrix

$$
\mathcal{F} = \begin{pmatrix} 0 & F_{1\to 2} \\ F_{2\to 1} & 0 \end{pmatrix}
$$

(17)

The second law of thermodynamics requires $F_{1\to 2} = F_{2\to 1}$, regardless of any symmetry \[21\].

The three sets of constraints Eq. (8), Eq. (12) and Eq. (15, 16) are the main results of this paper. These are all the constraints on radiative heat transfer that can be stated from symmetry and the second law of thermodynamics.

Let us apply the general theory to the concrete example in Fig. 1. The magnetic group of the system is $\mathcal{M} = D_{4h}(D_{2h})$ (Eq. 5). The exchange matrix is:

$$
\mathcal{F} = \begin{pmatrix} 0 & F_{01} & F_{02} & F_{03} & F_{04} \\ F_{10} & 0 & F_{12} & F_{13} & F_{14} \\ F_{20} & F_{21} & 0 & F_{23} & F_{24} \\ F_{30} & F_{31} & F_{32} & 0 & F_{34} \\ F_{40} & F_{41} & F_{42} & F_{43} & 0 \end{pmatrix},
$$

(18)

where $F_{ij} \equiv F_{i\to j}$ for simplicity. We first study the constraints on $\mathcal{F}$ imposed by $\mathcal{M}$ by considering all the elements:

- $2C_4(z)$: $C_4(z)$ permutes the bodies by

$$
\mathbf{P}_{C_4} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 & 1 \end{pmatrix},
$$

(19)
Combining all the constraints Eq. (20) and (22),
\[ F \text{ is satisfied.} \]

\[ F \text{ has only 3 independent components} \]

\begin{align*}
F_{01} &= F_{20} = F_{03} = F_{40}, \\
F_{02} &= F_{30} = F_{04} = F_{10}, \\
F_{12} &= F_{32} = F_{34} = F_{14}, \\
F_{23} &= F_{43} = F_{41} = F_{21}, \\
F_{13} &= F_{42} = F_{31} = F_{24}. \quad (20)
\end{align*}

- \( C_2(z) \): no new constraints, since \( C_2(z) = C_4^2(z) \).
- \( 2C'_2(x) \): permutes the bodies by:

\[ P_{C_2} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 4 & 3 & 2 & 1 \end{pmatrix}, \quad (21)\]

Constraints from Eq. (8) are therefore:
\[ F_{01} = F_{04}, \quad F_{12} = F_{13}. \quad (22)\]

- The remaining elements impose no new constraints.

Combining all the constraints Eq. (20) and (22),
\[ F = \begin{pmatrix} 0 & F_{01} & F_{01} & F_{01} & F_{01} \\ F_{01} & 0 & F_{12} & F_{13} & F_{12} \\ F_{01} & F_{12} & 0 & F_{12} & F_{13} \\ F_{01} & F_{13} & F_{12} & 0 & F_{12} \\ F_{01} & F_{12} & F_{13} & F_{12} & 0 \end{pmatrix}, \quad (23)\]

which has only 3 independent components \( F_{01}, F_{12}, F_{13} \). Also \( F = F^T \), even though this system is nonreciprocal.

The second law of thermodynamics imposes no new constraints, since \( F = F^T \), and Eq. (16) is automatically satisfied.

III. APPLICATIONS

As an application of our theory, we study the persistent heat current in nonreciprocal radiative heat transfer. Persistent heat current is a phenomenon that can exist in some nonreciprocal many-body systems even at thermal equilibrium [21]. By definition, the persistent heat current exists between body \( i \) and \( j \) at equilibrium if and only if \( F_{i \rightarrow j} \neq F_{j \rightarrow i} \). It has been proved that nonreciprocity is a necessary but not sufficient condition for the existence of persistent heat current [21,23]. However, there still lacks a systematic way to determine whether a given system can exhibit persistent heat current. Our theory can provide such a systematic approach.

From the definition, there is persistent heat current in a system between at least one pair of bodies, if and only if \( F \neq F^T \). Since our theory provides all the general constraints on \( F \), we can check whether a system can support persistent heat current by deducing the constrained form of \( F \) and then checking whether \( F = F^T \).

If \( F = F^T \), there is no persistent heat current in such a system. Otherwise, there is no symmetry reason against the existence of persistent heat current.

\begin{figure}[h]
\centering
\includegraphics[width=\linewidth]{fig2}
\caption{(a) A reciprocal system that exhibits no persistent heat current. It consists of four InSb spheres with centers placed at the vertices of a square on the \( x-y \) plane under no external magnetic field. (b) The calculated transmission coefficient spectra \( F_{ij}(\lambda) \) for the system in (a). Only the independent components are plotted. (c) A nonreciprocal system that exhibits no persistent heat current. It consists of the same spheres as (a), but under an external magnetic field along the \( z \) direction with alternating strength in the \( x-y \) plane: \( B_1 = B_3 = -B_2 = -B_4 = 1 \) T. (d) The calculated independent transmission coefficient spectra \( F'_{ij}(\lambda) \) for the system in (c). (e) A nonreciprocal system that exhibits persistent heat current. It consists of the same spheres as (a), but under a uniform external magnetic field along the \( z \) direction: \( B_1 = B_2 = B_3 = B_4 = 1 \) T. (f) The calculated independent transmission coefficient spectra \( F''_{ij}(\lambda) \) for the system in (e).}
\end{figure}

To demonstrate such a procedure, we consider three exemplary systems as shown in Fig. 2(a,c,e). The geometries are similar: all the systems consist of four gyrotropic spheres made of \( n \)-doped InSb. Each sphere has a radius of 100 nm. The centers of the four spheres are placed at the vertices of a square on the \( x-y \) plane. The side length of the square is 320 nm. These systems differ in the magnetic field configurations: the first system is under no magnetic field, the second under spatially alternating fields \( (B_1 = B_3 = -B_2 = -B_4 = 1 \) T), and the last under a uniform field \( (B_1 = B_2 = B_3 = B_4 = 1 \) T). The external magnetic field is perpendicular to the \( x-y \) plane. Under the magnetic field, \( n \)-doped InSb has a
Consequently, the three constrained system has a black/white group \( M \). The second system is reciprocal and has a gray group \( m \). Fig. 1, has a black/white group \( m \) system is reciprocal and has a gray group \( m \), following forms respectively:

\[
\begin{bmatrix}
1 + i\frac{\omega}{\omega_p} & -i\frac{\omega}{\omega_p} & 0 \\
-i\frac{\omega}{\omega_p} & 1 + i\frac{\omega}{\omega_p} & 0 \\
0 & 0 & \frac{\omega(\omega+i\Gamma)^2 - \omega^2}{\omega(\omega+i\Gamma)^2 - \omega^2}
\end{bmatrix}
\]

Here, the first term is the background permittivity as taken from Ref. [32]. The second term takes into account free-carrier contribution, which is sensitive to external magnetic field. \( \Gamma \) is the free-carrier relaxation rate, \( \omega_c = eB/m^* \) is the cyclotron frequency, and \( \omega_p = \sqrt{n_e e^2/(m^* e_0)} \) is the plasma frequency. For calculation, we use \( n_e = 1.36 \times 10^{19} \text{ cm}^{-3} \), \( \Gamma = 10^{12} \text{ s}^{-1} \) and \( m^* = 0.08 m_e \).

The three systems have different symmetries. The first system is reciprocal and has a gray group \( \mathcal{M}_1 = D_{4h} \otimes \{E, T\} \). The second system, which is identical to that in Fig. 1 has a black/white group \( \mathcal{M}_2 = D_{4h}(D_{2h}) \). The last system has a black/white group \( \mathcal{M}_3 = D_{4h}(C_{4h}) \). Consequently, the three constrained \( \mathcal{F} \) matrices have the following forms respectively:

\[
\mathcal{F}_1 = \begin{pmatrix}
0 & F_{01} & F_{01} & F_{01} & F_{01} \\
F_{01} & 0 & F_{12} & F_{13} & F_{12} \\
F_{01} & F_{12} & 0 & F_{12} & F_{13} \\
F_{01} & F_{13} & F_{12} & 0 & F_{12} \\
F_{01} & F_{12} & F_{13} & F_{12} & 0
\end{pmatrix}, \tag{24}
\]

\[
\mathcal{F}_2 = \begin{pmatrix}
0 & F_1 & F_1 & F_1 & F_1 \\
F_1 & 0 & F_2 & F_2 & F_2 \\
F_1 & F_2 & 0 & F_2 & F_2 \\
F_1 & F_2 & F_2 & 0 & F_2 \\
F_1 & F_2 & F_2 & F_2 & 0
\end{pmatrix}, \tag{25}
\]

\[
\mathcal{F}_3 = \begin{pmatrix}
0 & F_1 & F_1 & F_1 & F_1 \\
F_1 & 0 & F_2 & F_2 & F_2 \\
F_1 & F_2 & 0 & F_2 & F_2 \\
F_1 & F_2 & F_2 & 0 & F_2 \\
F_1 & F_2 & F_2 & F_2 & 0
\end{pmatrix}. \tag{26}
\]

We make several observations. The first system in Fig. 2(a) is reciprocal, thus cannot exhibit persistent heat current as expected \( (\mathcal{F}_1 = \mathcal{F}_1^T) \). Moreover, \( \mathcal{F}_1 \) has only 3 independent components \( F_{01}, F_{12} \) and \( F_{13} \) as required by symmetry. The second system in Fig. 2(c), even though is nonreciprocal, cannot exhibit persistent heat current either since \( \mathcal{F}_2 = \mathcal{F}_2^T \). Interestingly, \( \mathcal{F}_2 \) has exactly the same form as \( \mathcal{F}_1 \). This highlights the possibility that many-body systems with completely different symmetries can exhibit the same qualitative behavior in radiative heat transfer. The last system in Fig. 2(e) is nonreciprocal and can hold persistent heat current as \( \mathcal{F}_3 \neq \mathcal{F}_3^T \). \( \mathcal{F}_3 \) has 4 independent components \( F_{01}, F_{01}, F_{12}, F_{13} \) and \( F_{13} \).

We numerically verify these observations by calculating the transmission coefficient spectra \( F_{ij}(\lambda) \) in Figs. 2(b,d,f) corresponding to the systems in Figs. 2(a,c,e), respectively. We plot all the independent components of the \( \mathcal{F} \) matrices, and verify that the other components indeed obey the relations in Eq. [24][26]. We see there is no persistent heat (e.g. \( F_{12} = F_{21} \) and \( F_{13} = F_{23} \)) in the first and the second systems, while there is persistent heat current in the last system \( (F_{12} = F_{21}) \).

As another application of our theory, we have the following proposition: if a system has a colored symmetry \( R = TA \) where \( \mathbb{P}_A = I \) is an identity permutation, it cannot exhibit persistent heat current. This is because if there is such an element \( R \), its constraint on \( \mathcal{F} \) (Eq. (13)) is:

\[
IFI = FT,
\]

where \( I \) is the identity matrix. Therefore \( \mathcal{F} = FT \), which precludes persistent heat current.

As an example, we consider clusters of gyrotropic spheres with their centers lying on a plane under an external magnetic field parallel to that plane. The spheres can be of different sizes, and the magnetic field can be inhomogeneous. One typical system is depicted in Fig. 3(a). Such systems have the \( Tm \) symmetry, where \( m \) is the mirror operation with respect to the plane passing through the centers, and \( \mathbb{P}_m = I \). Therefore, such systems cannot exhibit persistent heat current. In contrast, clusters of randomly positioned gyrotropic particles subjected to magnetic field with random magnitudes and directions in general do not have \( Tm \) symmetry and can therefore exhibit persistent heat current.

We now provide numerical evidences. For the convenience of simulation, instead of Fig. 3(a), we consider the system in Fig. 3(b), where the four InSb spheres are of the same size with a radius of 100 nm, and the magnetic field is inhomogeneous but along the same \( (y) \) direction. The centers of the spheres are placed on the \( x-y \) plane with randomly chosen coordinates (unit: nm): \((484, -146), (-167, 313), (174, -252), (-303, -41)\). The spheres are under randomly assigned magnetic fields \( B_1 = -0.256 T, B_2 = -1.896 T, B_3 = 0.199 T, B_4 = -0.259 T \). Such a system has a black/white magnetic group \( C_{1h}(C_1) = \{E, Tm\} \). The only constraints that can be deduced are \( F_{ij} = F_{ji} \). Thus there are 10 independent components for the \( \mathcal{F} \) matrix. We numerically calculate the transmission coefficient spectra \( F_{ij}(\lambda) \). Fig. 3(c) plots the transmission coefficients between bodies. Here we use logarithmic scale to accommodate the different magnitudes. Fig. 3(d) plots the transmission coefficients between the bodies and the environment. These two plots confirm that \( F_{ij}(\lambda) = F_{ji}(\lambda), 0 \leq i < j \leq 4 \), and there are indeed 10 independent components.

**IV. DISCUSSION AND CONCLUSION**

Throughout the paper, we have used clusters of spherical particles subjected to local magnetic field as concrete examples to illustrate the theory. Our theory, which is based on symmetry argument alone, is not restricted to either spherical particles or sub-wavelength particles, but
is applicable to arbitrary many-body systems, which can include non-spherical objects, or objects with sizes comparable or larger than the relevant thermal wavelengths.

In conclusion, we have studied the constraints on many-body radiative heat transfer imposed by symmetry and the second law of thermodynamics. We show that the symmetry of these systems in general can be described by a magnetic group. And the constraints of the magnetic group on heat transfer can be derived using the generalized reciprocity theorem. We also show that the second law of thermodynamics provides additional constraints in the form of a nodal conservation law of heat flow at equilibrium. As an application of the theory, we provide a systematic approach to determine the existence of persistent heat current in arbitrary many-body systems. Our work should be useful in providing theoretical guidance for exploring novel effects of radiative heat transfer in complex many-body systems and networks.

### Appendix: Proof of Eq. (11)

We prove Eq. (11) using the generalized reciprocity theorem [31]. First we briefly review Lorentz reciprocity [33]. Consider two sources $J_a$ and $J_b$, which produce fields $E_a$ and $E_b$, respectively. The Lorentz reciprocity theorem states that for a reciprocal medium that satisfies $C = \tilde{C}$,

$$\iiint_V dV E_a \cdot J_b = \iiint_V dV E_b \cdot J_a, \quad (A.1)$$

where the integration is over the volume that contains the sources $a$ and $b$.

The reciprocity theorem stated above can be generalized to arbitrary media [31]. Consider two sources $J_a$ and $J_b$, producing fields $E_a$ and $E_b$ in the original medium $C$, and fields $\tilde{E}_a$ and $\tilde{E}_b$ in the complementary medium $\tilde{C}$, respectively. Then the generalized reciprocity theorem states

$$\iiint_V dV E_a \cdot J_b = \iiint_V dV \tilde{E}_b \cdot J_a, \quad (A.2)$$

integrating over the volume that contains the sources $a$ and $b$. Note it reduces to the ordinary reciprocity theorem for reciprocal medium.

The generalized reciprocity theorem requires the dyadic Green’s functions for the corresponding bodies in the original and complementary systems to be transpose of each other:

$$\tilde{G}_\alpha(r, r') = G^T_\alpha(r', r), \quad (A.3)$$

where body $\alpha$ can be a composite consisting of multiple bodies.

Consequently, the $T$ operators for the corresponding bodies in the two systems are also transpose of each other:

$$\tilde{T}_\alpha(r, r') = T^T_\alpha(r', r). \quad (A.4)$$

This follows from the definition of $T$ [27, 35],

$$\tilde{G}_\alpha = G_0 + G_0 T_\alpha G_0, \quad (A.5)$$

$$\tilde{G}_\alpha = G_0 + G_0 \tilde{T}_\alpha G_0, \quad (A.6)$$

where $G_0$, being the free-space Green’s function, is symmetric, i.e. $G_0(r, r') = G^T_0(r', r)$.

Now we proceed to prove that the exchange matrices $F$ and $F'$ of the original and complementary systems are transpose of each other (Eq. (11)). Without loss of generality, we consider heat exchange between bodies 1 and 2 in the original many-body system, while body 3 includes all other bodies except 1, 2. We label the corresponding bodies in the complementary system as $1'$, $2'$, $3'$. In the derivation below, we follow a similar procedure in the supplement of Ref. [21], and suppress the parameters of the operators for clarity.

The spectral transmission coefficient for heat transfer to body 2 due to thermal noise of body 1 is:

$$F_{1\rightarrow 2}(\omega) = 4 \text{Tr}[Q_2 W_{21} R_3 W_{21}], \quad (A.7)$$

while that to body $2'$ due to thermal noise of body 1' is:

$$\tilde{F}_{2'\rightarrow 1'}(\omega) = 4 \text{Tr}[\tilde{Q}_2 \tilde{W}_{12} \tilde{R}_{2'} \tilde{W}_{12}] = 4 \text{Tr}[\tilde{W}_{12} \tilde{R}_{2'} \tilde{W}_{12} \tilde{Q}_{1'}] = 4 \text{Tr}[\tilde{W}_{12} \tilde{W}_{12} \tilde{Q}_{1'}], \quad (A.8)$$
where we have performed transposition of the matrix product in the second line, and cyclic permutation in the third line. Here,

\[ R_\alpha = G_0 \frac{T_\alpha - T_\alpha^\dagger}{2i} - T_\alpha \text{Im}(G_0 T_\alpha^\dagger) G_0^\dagger, \quad \text{(A.9)} \]

\[ \bar{R}_\alpha^T = \{ G_0 \frac{T_\alpha - T_\alpha^\dagger}{2i} - T_\alpha \text{Im}(G_0 T_\alpha^\dagger) G_0^\dagger \}^T \]

\[ = G_0^\dagger \frac{T_\alpha - T_\alpha^\dagger}{2i} - T_\alpha^\dagger \text{Im}(G_0 T_\alpha) G_0, \quad \text{(A.10)} \]

Comparing Eq. (A.18) and Eq. (A.7), we get

\[ \text{Using Eq. (A.13) and Eq. (A.17), Eq. (A.8) becomes} \]

Using Eq. (A.13) and Eq. (A.17), Eq. (A.8) becomes

\[ \tilde{F}_{2' \to 1'}(\omega) = 4 \text{Tr}[\bar{R}_2^T \bar{W}_{12}^T G_0^\dagger \bar{W}_{12}^T] \]

\[ = 4 \text{Tr}[Q_2 W_{21} R_1 W_{21}^T], \quad \text{(A.18)} \]

Comparing Eq. (A.18) and Eq. (A.7), we get

\[ \tilde{F}_{2' \to 1'}(\omega) = F_{1 \to 2}(\omega) \quad \text{(A.19)} \]

\[ Q_\alpha = G_0^\dagger \frac{T_\alpha - T_\alpha^\dagger}{2i} - T_\alpha^\dagger \text{Im}(G_0) T_\alpha G_0, \quad \text{(A.11)} \]

\[ \tilde{Q}_\alpha = \{ G_0 \frac{T_\alpha - T_\alpha^\dagger}{2i} - T_\alpha^\dagger \text{Im}(G_0 T_\alpha) G_0^\dagger \}^T \]

\[ = G_0 \frac{T_\alpha - T_\alpha^\dagger}{2i} - T_\alpha \text{Im}(G_0) T_\alpha^\dagger G_0^\dagger, \quad \text{(A.12)} \]

where \( \alpha = 1, 2 \), and we have used Eq. (A.4) to simplify Eq. (A.10), (A.12). Therefore,

\[ \bar{R}_\alpha = Q_\alpha, \quad \bar{Q}_\alpha^T = R_\alpha \quad \text{(A.13)} \]

And

\[ W_{21} = G_0^{-1} \frac{1}{1 - G_0 T_3 G_0 T_2 (1 + G_0 T_3)} \frac{1}{1 - G_0 T_1 [(1 + G_0 T_2) - (1 + G_0 T_3)] - 1}, \quad \text{(A.14)} \]

\[ \bar{W}_{12} = G_0^{-1} \frac{1}{1 - G_0 T_3 G_0^\dagger T_1} (1 + G_0 T_3) \frac{1}{1 - G_0 T_1 [(1 + G_0 T_2) - 1] G_0 T_1 - 1 G_0 T_3 - 1} \]

\[ = G_0^{-1} \frac{1}{1 - [(1 + G_0 T_3) - 1] G_0 T_1 - 1] G_0 T_1 - 1 G_0 T_3 - 1} \]

\[ = \frac{1}{1 - [(1 + T_3 G_0) - 1] T_1 G_0 - 1} \frac{1}{1 - T_2 G_0 T_3} G_0^{-1}, \quad \text{(A.15)} \]

\[ \bar{W}_{12}^T = G_0^{-1} \frac{1}{1 - G_0 T_3 G_0^\dagger T_2} (1 + G_0 T_3) \frac{1}{1 - G_0 T_1 [(1 + G_0 T_2) - 1] G_0 T_1 - 1} \]

\[ = G_0^{-1} \frac{1}{1 - G_0 T_3 G_0^\dagger T_2 (1 + G_0 T_3)} \frac{1}{1 - G_0 T_1 [(1 + G_0 T_2) - 1] G_0 T_1 - 1} \]

\[ \text{where in Eq. (A.15) we have used Eq. (11) in the Supplement of [21] to get the third line. We transpose Eq. (A.15) to obtain Eq. (A.16). Therefore,} \]

\[ \bar{W}_{12}^T = W_{21}, \quad \bar{W}_{12} = W_{21}^T, \quad \text{(A.17)} \]

Using Eq. (A.13) and Eq. (A.17), Eq. (A.8) becomes

\[ \text{Since bodies 1, 2 are arbitrarily chosen, we have proved} \]

\[ \tilde{F} = F^T. \quad \text{(A.20)} \]

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