A NOTE ON THE DIOPHANTINE EQUATION $2^{n-1}(2^n - 1) = x^3 + y^3 + z^3$

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Abstract. Motivated by the recent result of Farhi we show that for each $n \equiv \pm 1 \pmod{6}$ the title Diophantine equation has at least two solutions in integers. As a consequence, we get that each (even) perfect number is a sum of three cubes of integers. Moreover, we present some computational results concerning the considered equation and state some questions and conjectures.

1. Introduction

Let $n \in \mathbb{N}_+$ and put $P_n = 2^{n-1}(2^n - 1)$. We say that $N$ is a perfect number if its the sum of proper divisors. In other words, $\sigma(N) = 2N$, where $\sigma(N) = \sum_{d|N} d$. Up to the date, we do not know whether there is an odd perfect number. On the other side, as was proved by Euclid, if $N$ is an even perfect number then $N = P_p$, where $p$ and $2^p - 1$ are primes. An early state of research on perfect numbers is presented in the first chapter in Dickson classical book [1]. We know that there are at least 49 even perfect numbers. The largest known corresponds with $p = 74207281$. One among many interesting properties of perfect numbers, is the property observed by Heath, that each even perfect number $> 6$ is a sum of consecutive odd cubes of positive integers. This observation motivated Farhi to ask what is the smallest number $r$ such that each even perfect number $> 6$ is the sum of at most $r$ cubes of non-negative integers. In [2], Farhi proved that $r = 5$ does the job. In fact, he observed that if $n \equiv 1 \pmod{6}$, then $M_n$ is the sum of three cubes of positive integers. This is simple consequence of the classical polynomial identity

$$2^6 - 1 = (t^2 + t - 1)^3 + (t^2 - t - 1)^3 + 1.$$ 

Indeed, multiplying it by $t^6$ and then taking $t = 2^n$ we immediately get the representation of $P_{6n+1}$ as sum of three positive cubes. In case of $n = 5 \pmod{6}$ the number $P_n$ is a sum of five positive cubes. It is important to note that $P_n$ is not necessarily perfect in the proof presented by Farhi. Let us also note that perfect numbers corresponding to $p = 3, 5, 7, 13, 17$ can be represented as a sum of three cubes of positive integers. This observation motivated Farhi to state the conjecture saying that each perfect number is such a sum (Conjecture 2 in [2]). Unfortunately, we were unable to prove this statement. This fail is a good motivation to consider the Diophantine equation

$$P_n = x^3 + y^3 + z^3$$

for fixed $n$, and asks about its solutions in (not necessarily positive) integers.

The question about the existence of integer solutions of the equation $N = x^3 + y^3 + z^3$ is a classical one. The equation has no solutions for $N \equiv \pm 4 \pmod{9}$ and it is conjectured that there are infinitely many solutions otherwise. However, this conjecture is proved only for $N$ being a cube or twice a cube (see for example [4]). It is clear that the number $P_n$ is not a cube nor twice a cube and $P_n \not\equiv \pm 4 \pmod{9}$ for all $n \in \mathbb{N}_+$. Thus, the question concerning the existence of integer solutions of the equation (1) is non-trivial.

In Section 2 we prove that for $n \equiv 1, 2, 4, 5 \pmod{6}$ the Diophantine equation (1) has at least one solution in integers. Moreover, in the case of $n \equiv \pm 1 \pmod{6}$ we show the existence of at least two solutions. We also prove that for each $n \in \mathbb{N}_+$ the number $P_n$ can be represented as a sum of four cubes of integers. In Section 3 we present results of our numerical computations concerning the equation (1). In particular, for each $n \leq 40$ a solution of (1) is found and the table of all non-negative solutions is presented. Moreover, we state some questions and conjectures which may stimulate further research.

2. The results

We have the following

Key words and phrases. perfect numbers, sums of three cubes.
**Theorem 2.1.** If \( n \equiv 1 \pmod{3} \) or \( n \equiv 2 \pmod{6} \) then the Diophantine equation \( \Box \) has at least one solution in integers. Moreover, if \( n \equiv \pm 1 \pmod{6} \) then the Diophantine equation \( \Box \) has at least two solutions in integers.

**Proof.** Our result is an immediate consequence of the following identities which hold for all \( n \in \mathbb{N}_+ \):

\[
egin{align*}
P_{3n+1} &= (2^{2n})^3 + (2n)^3 - (2n)^3, \\
P_{6n+2} &= (2^{4n+1})^3 - (2n)^3 - (2n)^3, \\
P_{6n+1} &= (2^{n-2}(2^{3n+2} - 21))^3 + (2n-2(2^{3n+2} + 21))^3 - (11 \cdot 2^{2n-1})^3, \\
P_{6n+5} &= (2n(2^{3(n+1)} + 2^{2(n+1)} + 1))^3 + (2n(2^{3(n+1)} - 2^{2(n+1)} - 1))^3 - (2^{2n+1}(2^{2n+1} + 1))^3 \\
&= (2^{2n+1}(2^{2n+1} - 2n^1 - 1))^3 + (2^{2n+1}(2^{2n+1} + 2n^1 - 1))^3 - (2^{2n+3})^3.
\end{align*}
\]

Replacing \( n \) by \( 2n \) in the first equality we get the second solution of the equation \( P_{6n+1} = x^3 + y^3 + z^3 \). \( \Box \)

**Remark 2.2.** Let us note that the expression for \( P_{6n+1} \) from the proof of Theorem 2.1 can be deduced from the polynomial identity

\[
64t^6(2t^6 - 1) = (4t^3 - 21)^3 + (4t^3 + 21)^3 - (22t^3)
\]

by multiplying both sides by \( \frac{1}{64}t^3 \), and then taking \( t = 2^n \). Moreover, the first expression for \( P_{6n+5} \) follows from the identity

\[
t^3(t^6 - 2) = (t^3 + t^2 + 1)^3 + (t^3 - t^2 - 1)^3 - (t(t^2 + 2))^3
\]

by multiplying both sides by \( \frac{1}{8}t^3 \), and then taking \( t = 2^{n+1} \).

**Corollary 2.3.** For each perfect number \( N \), the number of representations of \( N \) as a sum of three cubes of integers is \( \geq 2 \).

**Proof.** From Theorem 2.1, we know that for each odd prime \( p > 3 \), the number \( N = P_p \) has at least two representations as a sum of three cubes of integers. For \( p = 2, 3 \) we have

\[
P_2 = 2^3 - 1^3 - 1^3 = 65^3 - 43^3 - 58^3, \quad P_3 = 3^3 + 1^3 = 14^3 + 13^3 - 17^3,
\]

and get the result. \( \Box \)

We firmly believe that the equation \( \Box \) has solution in integers for each \( n \in \mathbb{N}_+ \) (see Conjecture \( \Box \)). Unfortunately, we were unable to prove such statement. Instead, we offer the following

**Theorem 2.4.** For each \( n \in \mathbb{N}_+ \), the number \( P_n \) can be represented as a sum of four cubes of integers.

**Proof.** Let us note the classical identity

\[
t^3 - 2(t - 1)^3 + (t - 2)^3 = 6(t - 1),
\]

and observe that \( P_{2n} \equiv 0 \pmod{6} \). Thus, by taking

\[
t = \frac{1}{3}(2^{2n-1} - 2^{(n-1)} + 3)
\]

we get the representation of the number \( P_{2n} \) as a sum of four cubes.

In order to represents \( P_{2n+1} \), we note the identity

\[
(3t - 12)^3 - (3t - 13)^3 - t^3 + (t - 9)^3 = 2(9t - 130).
\]

Using simple induction, we easily get the congruence \( P_{2n+1} \equiv 10 \pmod{18} \) for \( n \in \mathbb{N}_+ \). Thus, by taking

\[
t = \frac{1}{9}(2^{4n} - 2^{2n-1} + 130)
\]

we get the representation of the number \( P_{2n+1}, n \in \mathbb{N} \), as a sum of four cubes. Our theorem is proved. \( \Box \)
3. Numerical results, questions and conjectures

In order to gain more precise insight into the problem we performed a search for solutions of the equation \( (1) \) in integers. Because we are mainly interested in solutions in non-negative integers we use the following procedure. First of all, let us recall that for \( a, b \in \mathbb{Z} \) we have \( a^3 + b^3 \equiv 0, 1, 2, 7, 8 \pmod{9} \). Moreover, we observed that the sequence \((P_n \pmod{9})_{n \in \mathbb{N}_+}\) is periodic of the (pure) period 6. More precisely:

\[
(P_n \pmod{9})_{n \in \mathbb{N}_+} = (1, 6, 1, 3, 1, 0).
\]

For given \( n \) and each \( x \in \{0, \ldots, \lfloor P_n^{1/3} \rfloor\} \) satisfying \((P_n - x^3) \pmod{9} \in \{0, 1, 2, 7, 8\}\), we computed the set

\[
D_n(x) = \{d \in \mathbb{N}_+: P_n - x^3 \equiv 0 \pmod{d}\},
\]

i.e., the set of all positive divisors of the number \( P_n - x^3 \). The congruence condition is useful in some cases because reduce the number of computations which need to be performed. Indeed, if \( n \equiv 2, 4 \pmod{6} \) then \( P_n \equiv 6, 3 \pmod{9} \) respectively, and we need to have \( x \equiv 2 \pmod{3} \) (\( x \equiv 1 \pmod{3} \)). Unfortunately, in remaining cases we need to compute all values of \( x \) in order to find non-negative solutions. Next, for each \( d \in D_n(x) \) such that \( d < (P_n - x^3)/d \), we solved the system of equations

\[
d = y + z, \quad \frac{P_n - x^3}{d} = y^2 - yz + z^2
\]

for \( y, z \) and get

\[
y = \frac{1}{6} \left( 3d \pm \sqrt{3 \left( 4(P_n - x^3)/d - d^2 \right)} \right), \quad z = \frac{1}{6} \left( 3d \mp \sqrt{3 \left( 4(P_n - x^3)/d - d^2 \right)} \right).
\]

In consequence, if the numbers \( y, z \) computed in this way were integers we got the solution of the equation \( (1) \). This procedure was implemented in Magma computational package [3], and allows us to get all solutions in positive integers of the equation \( (1) \) with \( n \leq 40 \). The results of our computations are presented in Table 1 below. We also added the value of \( g := \gcd(x, y, z) \).
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Table 1. All solutions of the Diophantine equation $P_n = x^3 + y^3 + z^3$ in non-negative integers $x, y, z$ and $n \leq 40$.

For given $n$, the time needed to compute solutions with our method was from seconds (for $n \leq 25$) to four days in case of $n = 40$. All computations were performed on typical laptop with generation i7 processor and 16 GB of RAM. Moreover, it should be noted that our procedure also computes (some) solutions satisfying $yz < 0$, which is a consequence of the construction. In consequence, for each $n \in \{2, \ldots, 40\} \setminus \{2, 8, 20\}$, our procedure produce a solution of the equation \(I\) with $yz < 0$, i.e., exactly one among the numbers $y, z$ is negative. In Table 2 below, we present the integer solution of the equation \(I\) without non-negative solutions and with smallest value of $\min\{|x|, |y|, |z|\}$.

Table 2. Certain integer solutions of the Diophantine equation $P_n = x^3 + y^3 + z^3$ for $n \leq 40$ and without non-negative solutions.
Moreover, in Table 3 we present the number of integer solutions which were found by our procedure.

| n  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 0   | 1  | 1  | 3  | 2  | 2  | 0  | 3  | 2  | 2  | 8  | 6  | 1  |
| 15  | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |
| 4   | 1  | 8  | 38 | 17 | 0  | 7  | 3  | 18 | 4  | 18 | 4  | 16 |
| 28  | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 4   | 12 | 11 | 17 | 1  | 4  | 6  | 54 | 14 | 75 | 3  | 10 | 3  |

Table 3. The number of integer solutions of the Diophantine equation \( P_n = x^3 + y^3 + z^3 \), \( n \leq 40 \), found by the described procedure.

The search of solutions for \( n = 2, 8, 20 \) was performed in a similar way, but without the assumption of positivity of \( P_n - x^3 \) and with the replacement of \( P_n - x^3 \) by \( |P_n - x^3| \). In this way, for \( n = 2 \), we found the solutions of the equation (1) presented in the proof of Corollary 2.3. Moreover, we get the equalities

\[
P_8 = 32^3 - 4^3 - 4^3 = 404^3 - 124^3 - 400^3, \\
P_{20} = 8192^3 - 64^3 - 64^3 = 9404^3 - 472^3 - 6556^3,
\]

which fill the gap.

**Remark 3.1.** Let us also note that the non-negative solutions of the equation (1) for given \( n \) often satisfy the condition \( \gcd(x, y, z) = 2^k \) for certain, not too small, value of \( k \). Having in mind this property, we performed numerical search of positive solutions for certain values of \( n > 40 \). The method employed was the same as in the case \( n \leq 40 \), but instead to work for given \( n \), with \( P_n \) we worked with the (smaller) number \( M_{k,n} = 2^{k,n} \cdot 2^{3k}(2^n - 1) \), where \( k \in \{1, 2, 3, 4, 5\} \) and \( a_n = n - 1 \) (mod 3). Each representation of \( M_{k,n} \) after multiplication by \( 2^{3m} \), where \( m = (n - 1 - a_n - 3k)/3 \), leads to the representation of \( P_n \) as a sum of three cubes. Using this approach we found the following representations

\[
P_{41} = (2^{12} \cdot 441)^3 + (2^{12} \cdot 22063)^3 + (2^{12} \cdot 29022)^3, \\
P_{42} = (2^9 \cdot 183840)^3 + (2^9 \cdot 301469)^3 + (2^9 \cdot 337507)^3, \\
P_{43} = (2^{14})^3 + (2^{14} \cdot 16255)^3 + (2^{14} \cdot 16511)^3, \\
P_{45} = (2^{12} \cdot 18326)^3 + (2^{12} \cdot 144043)^3 + (2^{12} \cdot 181837)^3, \\
P_{47} = (2^{14} \cdot 5835)^3 + (2^{14} \cdot 41149)^3 + (2^{14} \cdot 129702)^3, \\
P_{48} = (2^{14} \cdot 8479)^3 + (2^{14} \cdot 160641)^3 + (2^{14} \cdot 169400)^3, \\
P_{49} = (2^{16})^3 + (2^{16} \cdot 65279)^3 + (2^{16} \cdot 65791)^3, \\
P_{51} = (2^{15} \cdot 91838)^3 + (2^{15} \cdot 252707)^3 + (2^{15} \cdot 380629)^3.
\]

Our numerical search and Theorem 2.1 suggest the following

**Conjecture 3.2.** For each \( n \in \mathbb{N}_+ \) the Diophantine equation (1) has a solution in integers.

From our table we note that the equation (1) has no solutions in non-negative integers \( x, y, z \) for

\[
n = 2, 4, 6, 8, 10, 12, 14, 16, 20, 24, 32, 33.
\]

This numerical observation lead us to the following

**Conjecture 3.3.** For each \( \epsilon \in \{0, 1\} \), there are infinitely many \( n \equiv \epsilon \) (mod 2) such that the equation (1) has no solutions in non-negative integers \( x, y, z \).

Moreover, according to our numerical search, one can also ask whether the conjecture proposed by Farhi is not too optimistic. Indeed, in his proof of the existence of representations of a perfect number \( P_p \) as a sum of five non-negative cubes, with \( p \geq 3 \), he used only the fact that \( p \equiv \pm 1 \) (mod 6) and the well-known polynomial identity

\[
2^{6k} - 1 = (t^2 + t - 1)^3 + (t^2 - t - 1)^3 + 1,
\]

i.e., any special property of perfect numbers was used. We also observed that the smallest odd \( n \in \mathbb{N}_{\geq 3} \), such that the equation (1) has no solutions in positive integers is 33. Due to our limited experimental data \( n \leq 40 \) in our search, there is no strong reason to believe that for all perfect
numbers $P_p$, the equation $P_p = x^3 + y^3 + z^3$ has a solution in non-negative integers. On the other side, the first possible candidate for the counterexample to the conjecture is $p = 89$. The corresponding perfect number $P_{89}$ has 54 digits, and the question about the existence of positive integer solutions of the equation $P_{89} = x^3 + y^3 + z^3$ is rather difficult.

It is also interesting to note the equalities

$$P_3 = 1^3 + 3^3, \quad P_7 = 28^3 - 24^3, \quad P_9 = 60^3 - 44^3,$$

which give all solutions of the equation $P_n = x^3 + y^3, n \leq 140$, in integers. This observation lead us to the following

**Question 3.4. Is the set of integer solutions (in variables $n, x, y$) of the Diophantine equation $P_n = x^3 + y^3$ finite?**

We expect that the answer is YES.

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