An action of the free product $\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$ on the $q$-Onsager algebra and its current algebra

Paul Terwilliger

Abstract

Recently Pascal Baseilhac and Stefan Kolb introduced some automorphisms $T_0, T_1$ of the $q$-Onsager algebra $O_q$, that are roughly analogous to the Lusztig automorphisms of $U_q(\hat{sl}_2)$. We use $T_0, T_1$ and a certain antiautomorphism of $O_q$ to obtain an action of the free product $\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$ on $O_q$ as a group of (auto/antiauto)-morphisms. The action forms a pattern much more symmetric than expected. We show that a similar phenomenon occurs for the associated current algebra $A_q$. We give some conjectures and problems concerning $O_q$ and $A_q$.

Keywords. $q$-Onsager algebra, current algebra, tridiagonal pair.

2010 Mathematics Subject Classification. Primary: 33D80. Secondary 17B40.

1 Introduction

We will be discussing the $q$-Onsager algebra $O_q$ [2, 20]. This algebra is infinite-dimensional and noncommutative, with a presentation involving two generators and two relations called the $q$-Dolan/Grady relations. The algebra appears in a number of contexts which we now summarize. The algebra $O_q$ is a $q$-deformation of the Onsager algebra from mathematical physics [17, 21, Remark 9.1] and is currently being used to investigate statistical mechanical models such as the XXZ open spin chain [1, 2, 4, 6–8]. The algebra $O_q$ appears in the theory of tridiagonal pairs; this is a pair of diagonalizable linear transformations on a finite-dimensional vector space, each acting on the eigenspaces of the other in a block-tridiagonal fashion [13, 19]. A tridiagonal pair of $q$-Racah type [14] is essentially the same thing as a finite-dimensional irreducible $O_q$-module [20, Theorem 3.10]. See [12, 15, 16, 21] for work relating $O_q$ and tridiagonal pairs. The algebra $O_q$ comes up in algebraic combinatorics, in connection with the subconstituent algebra of a $Q$-polynomial distance-regular graph [13, 18]. This topic is where $O_q$ originated; to our knowledge the $q$-Dolan/Grady relations first appeared in [18, Lemma 5.4]. The algebra $O_q$ appears in the theory of quantum groups, as a coideal subalgebra of $U_q(\hat{sl}_2)$ [3, 11, 16]. There exists an injective algebra homomorphism from $O_q$ into the algebra $\square_q$ [23, Proposition 5.6], and a noninjective algebra homomorphism from $O_q$ into the universal Askey-Wilson algebra [22, Sections 9, 10], [25].

We will be discussing some automorphisms and antiautomorphisms of $O_q$. In [9] Pascal Baseilhac and Stefan Kolb introduced two automorphisms $T_0, T_1$ of $O_q$ that are roughly
We now consider some automorphisms of $\mathcal{O}_q$. More information about $T_0, T_1$ is given in [24]. Using $T_0, T_1$ and a certain antiautomorphism of $\mathcal{O}_q$, we will obtain an action of the free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ on $\mathcal{O}_q$ as a group of (auto/antiauto)-morphisms. The action seems remarkable because it forms a pattern much more symmetric than expected. We show that a similar phenomenon occurs for the current algebra $\mathcal{A}_q$ of $\mathcal{O}_q$. Our main results are Theorem 2.11 and Theorem 3.9. At the end of the paper we give some conjectures and problems concerning $\mathcal{O}_q$ and $\mathcal{A}_q$.

2 The $q$-Onsager algebra $\mathcal{O}_q$

We will define the $q$-Onsager algebra after a few comments. Let $\mathbb{F}$ denote a field. All vector spaces discussed in this paper are over $\mathbb{F}$. All algebras discussed in this paper are associative, over $\mathbb{F}$, and have a multiplicative identity. A subalgebra has the same multiplicative identity as the parent algebra. For an algebra $\mathcal{A}$, an automorphism of $\mathcal{A}$ is an algebra isomorphism $\mathcal{A} \to \mathcal{A}$. An antiautomorphism of $\mathcal{A}$ is an $\mathbb{F}$-linear bijection $\sigma : \mathcal{A} \to \mathcal{A}$ such that $(XY)^\sigma = Y^\sigma X^\sigma$ for all $X, Y \in \mathcal{A}$. If $\mathcal{A}$ is commutative, then there is no difference between an automorphism and antiautomorphism of $\mathcal{A}$. If $\mathcal{A}$ is noncommutative, then no map is both an automorphism and antiautomorphism of $\mathcal{A}$. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. Fix $0 \neq q \in \mathbb{F}$ that is not a root of unity. We will use the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n \in \mathbb{N}.$$

**Definition 2.1.** (See [2] Section 2, [20] Definition 3.9.) Define the algebra $\mathcal{O}_q$ by generators $A, B$ and relations

\begin{align*}
A^3 B - [3]_q A^2 B A + [3]_q A B A^2 - B A^3 &= (q^2 - q^{-2})^2(BA - AB), \tag{1} \\
B^3 A - [3]_q B^2 A B + [3]_q B A B^2 - B A^3 &= (q^2 - q^{-2})^2(AB - BA). \tag{2}
\end{align*}

We call $\mathcal{O}_q$ the $q$-Onsager algebra. The relations (1), (2) are called the $q$-Dolan/Grady relations.

We now consider some automorphisms of $\mathcal{O}_q$. By the form of the relations (1), (2) there exists an automorphism of $\mathcal{O}_q$ that swaps $A, B$. The following automorphisms of $\mathcal{O}_q$ are less obvious. In [9] Pascal Baseilhac and Stefan Kolb introduced some automorphisms $T_0, T_1$ of $\mathcal{O}_q$ that satisfy

\begin{align*}
T_0(A) &= A, & T_0(B) &= B + \frac{q A^2 B - (q + q^{-1}) ABA + q^{-1} BA^2}{(q - q^{-1})(q^2 - q^{-2})}, \tag{3} \\
T_1(B) &= B, & T_1(A) &= A + \frac{q B^2 A - (q + q^{-1}) BAB + q^{-1} AB^2}{(q - q^{-1})(q^2 - q^{-2})}. \tag{4}
\end{align*}

The inverse automorphisms satisfy

\begin{align*}
T_0^{-1}(A) &= A, & T_0^{-1}(B) &= B + \frac{q^{-1} A^2 B - (q + q^{-1}) ABA + qBA^2}{(q - q^{-1})(q^2 - q^{-2})}, \tag{5} \\
T_1^{-1}(B) &= B, & T_1^{-1}(A) &= A + \frac{q^{-1} B^2 A - (q + q^{-1}) BAB + qAB^2}{(q - q^{-1})(q^2 - q^{-2})}. \tag{6}
\end{align*}
In [9] the automorphisms $T_0, T_1$ are used to construct a Poincaré-Birkhoff-Witt (or PBW) basis for $\mathcal{O}_q$. In that construction the following result is used.

**Lemma 2.2.** (See [9, Lemma 3.1].) For the algebra $\mathcal{O}_q$, the map $T_0$ sends
\[ qBA - q^{-1}AB \mapsto qAB - q^{-1}BA, \]
and the map $T_1$ sends
\[ qAB - q^{-1}BA \mapsto qBA - q^{-1}AB. \]

**Proof.** The map $T_0$ is an automorphism of $\mathcal{O}_q$ that fixes $A$. Therefore, $T_0$ sends $qBA - q^{-1}AB \mapsto qT_0(B)A - q^{-1}AT_0(B)$. To check that $qT_0(B)A - q^{-1}AT_0(B) = qAB - q^{-1}BA$, eliminate $T_0(B)$ using (3) and evaluate the result using (1). We have verified the assertion about $T_0$. The assertion about $T_1$ is similarly verified. □

The automorphism group $\text{Aut}(\mathcal{O}_q)$ consists of the automorphisms of the algebra $\mathcal{O}_q$; the group operation is composition.

**Definition 2.3.** Let $N$ denote the subgroup of $\text{Aut}(\mathcal{O}_q)$ generated by $T_0^\pm 1, T_1^\pm 1$.

**Lemma 2.4.** (See [25, Section 1].) The group $N$ is freely generated by $T_0^\pm 1, T_1^\pm 1$.

We have been discussing automorphisms of $\mathcal{O}_q$. We now bring in antiautomorphisms of $\mathcal{O}_q$.

**Lemma 2.5.** There exists an antiautomorphism $S$ of $\mathcal{O}_q$ that fixes $A$ and $B$. Moreover $S^2 = 1$.

**Proof.** By the form of the $q$-Dolan/Grady relations. □

The antiautomorphism $S$ is related to the automorphisms $T_0, T_1$ in the following way.

**Lemma 2.6.** For the algebra $\mathcal{O}_q$,
\[ ST_0S = T_0^{-1}, \quad ST_1S = T_1^{-1}. \]

**Proof.** We verify the equation on the left in (9). In that equation, each side is an automorphism of $\mathcal{O}_q$. These automorphisms agree at $A$ and $B$; this is checked using (5) and (5). These automorphisms are equal since $A, B$ generate $\mathcal{O}_q$. We have verified the equation on the left in (9). The equation on the right in (9) is similarly verified. □

Let $\text{AAut}(\mathcal{O}_q)$ denote the group consisting of the automorphisms and antiautomorphisms of $\mathcal{O}_q$; the group operation is composition. The group $\text{Aut}(\mathcal{O}_q)$ is a normal subgroup of $\text{AAut}(\mathcal{O}_q)$ with index 2. An element of $\text{AAut}(\mathcal{O}_q)$ will be called an (auto/antiauto)-morphism of $\mathcal{O}_q$.

**Definition 2.7.** Let $H$ denote the subgroup of $\text{AAut}(\mathcal{O}_q)$ generated by $S$. Let $G$ denote the subgroup of $\text{AAut}(\mathcal{O}_q)$ generated by $H$ and $N$.

**Lemma 2.8.** The following (i)–(iv) hold.
(i) The group $H$ has order 2 and is not contained in $N$.

(ii) The group $N$ is a normal subgroup of $G$ with index 2.

(iii) $G = N \rtimes H$ (semidirect product).

(iv) $N = \text{Aut}(O_q) \cap G$.

**Proof.** (i) The group $H$ has order 2 by the last assertion of Lemma 2.5. The group $H$ is not contained in $N$, since $\text{Aut}(O_q)$ contains $N$ but not $S$.

(ii) By Lemma 2.6 and part (i) above.

(iii) By parts (i), (ii) above.

(iv) The group $G$ is the union of cosets $N$ and $NS$. The elements of $N$ are in $\text{Aut}(O_q)$, and the elements of $NS$ are not in $\text{Aut}(O_q)$. 

We now consider $G$ from another point of view. Let $Z_2$ denote the group with two elements. The free product $Z_2 \ast Z_2 \ast Z_2$ has a presentation by generators $a, b, c$ and relations $a^2 = b^2 = c^2 = 1$. Shortly we will display a group isomorphism $Z_2 \ast Z_2 \ast Z_2 \rightarrow G$. To motivate this isomorphism we give a second presentation of $Z_2 \ast Z_2 \ast Z_2$ by generators and relations.

**Lemma 2.9.** The group $Z_2 \ast Z_2 \ast Z_2$ is isomorphic to the group defined by generators $s, t_0^{\pm 1}, t_1^{\pm 1}$ and relations

\begin{align*}
    t_0t_0^{-1} &= t_0^{-1}t_0 = 1, \\
    t_1t_1^{-1} &= t_1^{-1}t_1 = 1, \\
    s^2 &= 1, \\
    st_0s &= t_0^{-1}, \\
    st_1s &= t_1^{-1}. 
\end{align*}

An isomorphism sends

\begin{align*}
    a &\mapsto st_1, \\
    b &\mapsto t_0s, \\
    c &\mapsto s. 
\end{align*}

The inverse isomorphism sends

\begin{align*}
    t_0 &\mapsto bc, \\
    t_0^{-1} &\mapsto cb, \\
    t_1 &\mapsto ca, \\
    t_1^{-1} &\mapsto ac, \\
    s &\mapsto c. 
\end{align*}

**Proof.** One checks that each map is a group homomorphism and the maps are inverses. Consequently each map is a group isomorphism.

**Proposition 2.10.** There exists a group isomorphism $Z_2 \ast Z_2 \ast Z_2 \rightarrow G$ that sends

\begin{align*}
    a &\mapsto ST_1, \\
    b &\mapsto T_0S, \\
    c &\mapsto S. 
\end{align*}

The inverse isomorphism sends

\begin{align*}
    T_0 &\mapsto bc, \\
    T_0^{-1} &\mapsto cb, \\
    T_1 &\mapsto ca, \\
    T_1^{-1} &\mapsto ac, \\
    S &\mapsto c. 
\end{align*}

**Proof.** For notational convenience we identify the group $Z_2 \ast Z_2 \ast Z_2$ with the group defined in Lemma 2.9 via the isomorphism in Lemma 2.9. Comparing (11) with the relations in Lemmas 2.5, 2.6 we obtain a surjective group homomorphism $\gamma : Z_2 \ast Z_2 \ast Z_2 \rightarrow G$ that sends $s \mapsto S$ and $t_0^{\pm 1} \mapsto T_0^{\pm 1}$ and $t_1^{\pm 1} \mapsto T_1^{\pm 1}$. Using the identification (12) we find that $\gamma$ acts as in (14). We show that $\gamma$ is an isomorphism. Let $\mathcal{N}$ denote the subgroup of $Z_2 \ast Z_2 \ast Z_2$ generated by $t_0^{\pm 1}, t_1^{\pm 1}$. From the relations (11) we see that $Z_2 \ast Z_2 \ast Z_2$ is the union of $\mathcal{N}$ and $\mathcal{N}s$. We have $\gamma(\mathcal{N}) = N$ and $\gamma(\mathcal{N}s) = NS$. The cosets $\mathcal{N}, NS$ are disjoint and $N$ contains the identity, so the kernel of $\gamma$ is contained in $\mathcal{N}$. This kernel is the identity by Lemma 2.4. Therefore $\gamma$ is injective and hence an isomorphism. Line (15) follows from (13).
We now give our first main result. For notational convenience define
\[
C = \frac{q^{-1}BA - qAB}{q^2 - q^{-2}}. \tag{16}
\]

**Theorem 2.11.** The free product \(\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2\) acts on the algebra \(O_q\) as a group of (auto/antiauto)-morphisms in the following way.

(i) The generator \(a\) acts as an antiautomorphism that sends
\[
A \mapsto A + \frac{BC - CB}{q - q^{-1}}, \quad B \mapsto B, \quad C \mapsto C. \tag{17}
\]

(ii) The generator \(b\) acts as an antiautomorphism that sends
\[
B \mapsto B + \frac{CA - AC}{q - q^{-1}}, \quad C \mapsto C, \quad A \mapsto A. \tag{18}
\]

(iii) The generator \(c\) acts as an antiautomorphism that sends
\[
C \mapsto C + \frac{AB - BA}{q - q^{-1}}, \quad A \mapsto A, \quad B \mapsto B. \tag{19}
\]

(iv) On \(O_q\),
\[
a = ST_1 = T_1^{-1}S, \quad b = T_0S = ST_0^{-1}, \quad c = S, \tag{20}
\]
\[
T_0 = bc, \quad T_0^{-1} = cb, \quad T_1 = ca, \quad T_1^{-1} = ac. \tag{21}
\]

The above \(\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2\) action is faithful.

**Proof.** By Lemma 2.6 and Proposition 2.10, the group \(\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2\) acts faithfully on \(O_q\) as a group of (auto/antiauto)-morphisms in a way that satisfies (20), (21). By (20), each of \(a, b, c\) acts on \(O_q\) as an antiautomorphism. Their actions (17)–(19) are routinely obtained using (3)–(6) and (16), along with Lemmas 2.2, 2.5.

**Note 2.12.** Motivated by Theorem 2.11, one might conjecture that there exists an automorphism of \(O_q\) that sends \(A \mapsto B \mapsto C \mapsto A\). This conjecture is false, since \(A, B\) satisfy the \(q\)-Dolan/Grady relations and \(B, C\) do not. This last assertion can be checked by considering the actions of \(B, C\) on the 4-dimensional \(O_q\)-module given in the proof of [22, Lemma 10.8].

### 3 The current algebra \(A_q\)

In the previous section we obtained an action of \(\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2\) on the \(q\)-Onsager algebra \(O_q\). In this section we do something similar for the corresponding current algebra \(A_q\). In [6] Baseilhac and Koizumi introduce \(A_q\) in order to solve boundary integrable systems with hidden symmetries related to a coideal subalgebra of \(U_q(\widehat{sl}_2)\). In [10, Definition 3.1] Baseilhac and K. Shigechi give a presentation of \(A_q\) by generators and relations. The generators are
denoted $\mathcal{W}_k$, $\mathcal{W}_{k+1}$, $\mathcal{G}_{k+1}$, $\tilde{\mathcal{G}}_{k+1}$, where $k \in \mathbb{N}$. In [5, Lemma 2.1], Baseilhac and S. Belliard display some central elements $\{\Delta_{k+1}\}_{k \in \mathbb{N}}$ for $\mathcal{A}_q$. In [5, Corollary 3.1], it is shown that $\mathcal{A}_q$ is generated by these central elements together with $\mathcal{W}_0, \mathcal{W}_1$. The elements $\mathcal{W}_0, \mathcal{W}_1$ satisfy the $q$-Dolan/Grady relations [5, eqn. (3.7)]. In [5, Conjecture 2] it is conjectured that $\mathcal{O}_q$ is a homomorphic image of $\mathcal{A}_q$. We now recall the definition of $\mathcal{A}_q$.

**Definition 3.1.** (See [6], [10, Definition 3.1].) Define the algebra $\mathcal{A}_q$ by generators

$$W_{-k}, \quad W_{k+1}, \quad G_{k+1}, \quad \tilde{G}_{k+1}, \quad k \in \mathbb{N}$$

(22)

and relations

$$[W_0, W_{k+1}] = [W_{-k}, W_1] = (\tilde{G}_{k+1} - G_{k+1})/(q + q^{-1}),$$

(23)

$$[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q = \rho W_{-k-1} - \rho W_{k+1},$$

(24)

$$[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q = \rho W_{k+2} - \rho W_{-k},$$

(25)

$$[W_{-k}, W_{-\ell}] = 0, \quad [W_{k+1}, W_{\ell+1}] = 0,$$

(26)

$$[W_{-k}, W_{\ell+1}] + [W_{k+1}, W_{-\ell}] = 0,$$

(27)

$$[W_{-k}, G_{\ell+1}] + [G_{k+1}, W_{-\ell}] = 0,$$

(28)

$$[W_{-k}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{-\ell}] = 0,$$

(29)

$$[W_{k+1}, G_{\ell+1}] + [G_{k+1}, W_{\ell+1}] = 0,$$

(30)

$$[W_{k+1}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{\ell+1}] = 0,$$

(31)

$$[G_{k+1}, G_{\ell+1}] = 0, \quad [\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0,$$

(32)

$$[\tilde{G}_{k+1}, G_{\ell+1}] + [G_{k+1}, \tilde{G}_{\ell+1}] = 0.$$  

(33)

In the above equations $\ell \in \mathbb{N}$ and $\rho = -(q^2 - q^{-2})^2$. We are using the notation $[X, Y] = XY - YX$ and $[X, Y]_q = qXY - q^{-1}YX$.

There is a redundancy among the generators (22), since we could use (23) to eliminate $\{G_{k+1}\}_{k \in \mathbb{N}}$ or $\{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}$ in (24)–(33). These eliminations yield the equations in the next lemma.

**Lemma 3.2.** The following equations hold in $\mathcal{A}_q$. For $k \in \mathbb{N},$

$$[[W_{k+1}, W_0], W_0] = [G_{k+1}, W_0],$$

(34)

$$[W_1, [W_1, W_{-k}]] = [W_1, G_{k+1}],$$

(35)

$$[W_0, [W_0, W_{k+1}]] = [W_0, \tilde{G}_{k+1}],$$

(36)

$$[[W_{-k}, W_1], W_1] = [\tilde{G}_{k+1}, W_1].$$

(37)

For $k, \ell \in \mathbb{N},$

$$[[W_0, W_{k+1}], [W_0, W_{\ell+1}]] = 0,$$

(38)

$$[[W_1, W_{-k}], [W_1, W_{-\ell}]] = 0.$$  

(39)

We now consider some automorphisms of $\mathcal{A}_q$. 

6
Lemma 3.3. (See [5] Remarks 1, 2.) There exists an automorphism $\Omega$ of $A_q$ that sends $\mathcal{W}_{-k} \leftrightarrow \mathcal{W}_{k+1}$ and $\mathcal{G}_{k+1} \leftrightarrow \tilde{\mathcal{G}}_{k+1}$ for $k \in \mathbb{N}$. Moreover $\Omega$ fixes $\Delta_{k+1}$ for $k \in \mathbb{N}$. We have $\Omega^2 = 1$.

Lemma 3.4. (See [24] Proposition 7.4.) There exists an automorphism $T_0$ of the algebra $A_q$ that acts as follows. For $k \in \mathbb{N}$, $T_0$ sends

- $\mathcal{W}_{-k} \mapsto \mathcal{W}_{-k}$,
- $\mathcal{W}_{k+1} \mapsto \mathcal{W}_{k+1} + \frac{q \mathcal{W}_0^2 \mathcal{W}_{k+1} - (q + q^{-1}) \mathcal{W}_0 \mathcal{W}_{k+1} \mathcal{W}_0 + q^{-1} \mathcal{W}_{k+1} \mathcal{W}_0^2}{(q - q^{-1})(q^2 - q^{-2})}$,
- $\mathcal{G}_{k+1} \mapsto \mathcal{G}_{k+1} + \frac{q \mathcal{W}_0^2 \mathcal{G}_{k+1} - (q + q^{-1}) \mathcal{W}_0 \mathcal{G}_{k+1} \mathcal{W}_0 + q^{-1} \mathcal{G}_{k+1} \mathcal{W}_0^2}{(q - q^{-1})(q^2 - q^{-2})} = \tilde{\mathcal{G}}_{k+1}$,
- $\tilde{\mathcal{G}}_{k+1} \mapsto \tilde{\mathcal{G}}_{k+1} + \frac{q \mathcal{W}_0^2 \tilde{\mathcal{G}}_{k+1} - (q + q^{-1}) \mathcal{W}_0 \tilde{\mathcal{G}}_{k+1} \mathcal{W}_0 + q^{-1} \tilde{\mathcal{G}}_{k+1} \mathcal{W}_0^2}{(q - q^{-1})(q^2 - q^{-2})}$,
- $\Delta_{k+1} \mapsto \Delta_{k+1}$.

Moreover $T_0^{-1}$ sends

- $\mathcal{W}_{-k} \mapsto \mathcal{W}_{-k}$,
- $\mathcal{W}_{k+1} \mapsto \mathcal{W}_{k+1} + \frac{q^{-1} \mathcal{W}_0^2 \mathcal{W}_{k+1} - (q + q^{-1}) \mathcal{W}_0 \mathcal{W}_{k+1} \mathcal{W}_0 + q \mathcal{W}_{k+1} \mathcal{W}_0^2}{(q - q^{-1})(q^2 - q^{-2})}$,
- $\mathcal{G}_{k+1} \mapsto \mathcal{G}_{k+1} + \frac{q^{-1} \mathcal{W}_0^2 \mathcal{G}_{k+1} - (q + q^{-1}) \mathcal{W}_0 \mathcal{G}_{k+1} \mathcal{W}_0 + q \mathcal{G}_{k+1} \mathcal{W}_0^2}{(q - q^{-1})(q^2 - q^{-2})}$,
- $\tilde{\mathcal{G}}_{k+1} \mapsto \tilde{\mathcal{G}}_{k+1} + \frac{q^{-1} \mathcal{W}_0^2 \tilde{\mathcal{G}}_{k+1} - (q + q^{-1}) \mathcal{W}_0 \tilde{\mathcal{G}}_{k+1} \mathcal{W}_0 + q \tilde{\mathcal{G}}_{k+1} \mathcal{W}_0^2}{(q - q^{-1})(q^2 - q^{-2})} = \mathcal{G}_{k+1}$,
- $\Delta_{k+1} \mapsto \Delta_{k+1}$.

Definition 3.5. Define $T_1 = \Omega T_0 \Omega$, where $\Omega$ is from Lemma 3.3 and $T_0$ is from Lemma 3.4. By construction $T_1$ is an automorphism of the algebra $A_q$.

Lemma 3.6. For $k \in \mathbb{N}$, $T_1$ sends

- $\mathcal{W}_{k+1} \mapsto \mathcal{W}_{k+1}$,
- $\mathcal{W}_{-k} \mapsto \mathcal{W}_{-k} + \frac{q \mathcal{W}_0^2 \mathcal{W}_{-k} - (q + q^{-1}) \mathcal{W}_0 \mathcal{W}_{-k} \mathcal{W}_1 + q^{-1} \mathcal{W}_{-k} \mathcal{W}_1^2}{(q - q^{-1})(q^2 - q^{-2})}$,
- $\mathcal{G}_{k+1} \mapsto \mathcal{G}_{k+1} + \frac{q \mathcal{W}_0^2 \mathcal{G}_{k+1} - (q + q^{-1}) \mathcal{W}_0 \mathcal{G}_{k+1} \mathcal{W}_1 + q^{-1} \mathcal{G}_{k+1} \mathcal{W}_1^2}{(q - q^{-1})(q^2 - q^{-2})}$,
- $\tilde{\mathcal{G}}_{k+1} \mapsto \tilde{\mathcal{G}}_{k+1} + \frac{q \mathcal{W}_0^2 \tilde{\mathcal{G}}_{k+1} - (q + q^{-1}) \mathcal{W}_0 \tilde{\mathcal{G}}_{k+1} \mathcal{W}_1 + q^{-1} \tilde{\mathcal{G}}_{k+1} \mathcal{W}_1^2}{(q - q^{-1})(q^2 - q^{-2})} = \mathcal{G}_{k+1}$,
- $\Delta_{k+1} \mapsto \Delta_{k+1}$.
Moreover $T_1^{-1}$ sends
\[ \mathcal{W}_{k+1} \mapsto \mathcal{W}_{k+1}, \]
\[ \mathcal{W}_{-k} \mapsto \mathcal{W}_{-k} + \frac{q^{-1}\mathcal{W}_0^2\mathcal{W}_{-k} - (q + q^{-1})\mathcal{W}_1\mathcal{W}_{-k}\mathcal{W}_1 + q\mathcal{W}_{-k}\mathcal{W}_1^2}{(q - q^{-1})(q^2 - q^{-2})}, \]
\[ \mathcal{G}_{k+1} \mapsto \mathcal{G}_{k+1} + \frac{q^{-1}\mathcal{W}_0^2\mathcal{G}_{k+1} - (q + q^{-1})\mathcal{W}_1\mathcal{G}_{k+1}\mathcal{W}_1 + q\mathcal{G}_{k+1}\mathcal{W}_1^2}{(q - q^{-1})(q^2 - q^{-2})} = \mathcal{G}_{k+1}, \]
\[ \tilde{\mathcal{G}}_{k+1} \mapsto \tilde{\mathcal{G}}_{k+1} + \frac{q^{-1}\mathcal{W}_0^2\tilde{\mathcal{G}}_{k+1} - (q + q^{-1})\mathcal{W}_1\tilde{\mathcal{G}}_{k+1}\mathcal{W}_1 + q\tilde{\mathcal{G}}_{k+1}\mathcal{W}_1^2}{(q - q^{-1})(q^2 - q^{-2})}, \]
\[ \Delta_{k+1} \mapsto \Delta_{k+1}. \]

**Proof.** Use Lemma 3.4 and Definition 3.5. \(\square\)

We have been discussing automorphisms of $A_q$. We now consider antiautomorphisms of $A_q$.

**Lemma 3.7.** There exists an antiautomorphism $S$ of $A_q$ that sends
\[ \mathcal{W}_{-k} \mapsto \mathcal{W}_{-k}, \quad \mathcal{W}_{k+1} \mapsto \mathcal{W}_{k+1}, \quad \mathcal{G}_{k+1} \mapsto \tilde{\mathcal{G}}_{k+1}, \quad \tilde{\mathcal{G}}_{k+1} \mapsto \mathcal{G}_{k+1} \]
For $k \in \mathbb{N}$. Moreover $S$ fixes $\Delta_{k+1}$ for $k \in \mathbb{N}$. We have $S^2 = 1$.

**Proof.** The antiautomorphism $S$ exists by the form of the defining relations (23)–(33) for $A_q$. The map $S^2$ is an automorphism of $A_q$ that fixes $\mathcal{W}_{-k}, \mathcal{W}_{k+1}, \mathcal{G}_{k+1}, \tilde{\mathcal{G}}_{k+1}$ for $k \in \mathbb{N}$. These elements generate $A_q$, so $S^2 = 1$. For $k \in \mathbb{N}$ the map $S$ fixes $\Delta_{k+1}$ by the form of $\Delta_{k+1}$ given in [5, Lemma 2.1]. \(\square\)

**Lemma 3.8.** For the algebra $A_q$,
\[ ST_0 S = T_0^{-1}, \quad ST_1 S = T_1^{-1}. \] (40)

**Proof.** Similar to the proof of Lemma 2.6. \(\square\)

We now obtain our second main result. Recall the free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ from above Lemma 2.9 For $k \in \mathbb{N}$ define
\[ \mathcal{W}_{-k}' = \mathcal{W}_{k+1}, \quad \mathcal{W}_{-k}'' = -\frac{\tilde{\mathcal{G}}_{k+1}}{q - q^{-1}}. \] (41)

Note by (23), (29), (31) that
\[ [\mathcal{W}_{-k}, \mathcal{W}_0] = [\mathcal{W}_0, \mathcal{W}_{-k}'], \quad [\mathcal{W}_{-k}, \mathcal{W}_0'] = [\mathcal{W}_0', \mathcal{W}_{-k}'], \quad [\mathcal{W}_{-k}', \mathcal{W}_0] = [\mathcal{W}_0'', \mathcal{W}_{-k}]. \] (42)

**Theorem 3.9.** The free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ acts on the algebra $A_q$ as a group of (auto/antiauto)-morphisms in the following way.

(i) The generator $a$ acts as an antiautomorphism that sends
\[ \mathcal{W}_{-k} \mapsto \mathcal{W}_{-k} + \frac{[\mathcal{W}_{-k}', \mathcal{W}_0']}{q - q^{-1}}, \]
\[ \mathcal{W}_{-k}' \mapsto \mathcal{W}_{-k}', \quad \mathcal{W}_{-k}'' \mapsto \mathcal{W}_{-k}'', \quad \Delta_{k+1} \mapsto \Delta_{k+1}. \]
(ii) The generator $b$ acts as an antiautomorphism that sends
\[ W'_{-k} \mapsto W'_{-k} + \frac{[W''_{-k}, W_0]}{q - q^{-1}}, \]
\[ W''_{-k} \mapsto W''_{-k}, \]
\[ W_{-k} \mapsto W_{-k}, \]
\[ \Delta_{k+1} \mapsto \Delta_{k+1}. \]

(iii) The generator $c$ acts as an antiautomorphism that sends
\[ W''_{-k} \mapsto W''_{-k} + \frac{[W_{-k}, W_0]}{q - q^{-1}}, \]
\[ W_{-k} \mapsto W_{-k}, \]
\[ W'_{-k} \mapsto W'_{-k}, \]
\[ \Delta_{k+1} \mapsto \Delta_{k+1}. \]

(iv) On $A_q$,
\[ a = ST_1 = T_1^{-1}S, \quad b = T_0S = ST_0^{-1}, \quad c = S, \quad (43) \]
\[ T_0 = bc, \quad T_0^{-1} = cb, \quad T_1 = ca, \quad T_1^{-1} = ac. \quad (44) \]

**Proof.** For notational convenience we identify the group $\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$ with the group defined in Lemma 2.9 via the isomorphism in Lemma 2.9. Comparing (11) with the relations in Lemmas 3.7, 3.8, we obtain a group homomorphism $\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2 \to \text{AAut}(A_q)$ that sends $s \mapsto S$ and $t_0^{\pm 1} \mapsto T_0^{\pm 1}$ and $t_1^{\pm 1} \mapsto T_1^{\pm 1}$. This group homomorphism gives an action of $\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$ on the algebra $A_q$ as a group of (auto/antiauto)-morphisms such that $s$, $t_0^{\pm 1}$, $t_1^{\pm 1}$ act as $S$, $T_0^{\pm 1}$, $T_1^{\pm 1}$, respectively. Using the identifications (12), (13) we find that this action satisfies condition (iv) in the theorem statement. By (43) each of $a$, $b$, $c$ acts on $A_q$ as an antiautomorphism. For these elements the action on $W_{-k}$, $W'_{-k}$, $W''_{-k}$, $\Delta_{k+1}$ is routinely obtained using Lemmas 3.4, 3.6, 3.7 along with Lemma 3.2 and (42). \qed

4 Suggestions for future research

In this section we give some conjectures and problems concerning $O_q$ and $A_q$.

Earlier in this paper we gave a $\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$ action on $O_q$ and $A_q$. It is natural to ask whether these algebras are characterized by this sort of $\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$ action. As we pursue this question, let us begin with the simpler case of $O_q$. The following concept is motivated by Theorem 2.11.

**Definition 4.1.** Let $\mathcal{A}$ denote an algebra. A sequence $A, B, C$ of elements in $\mathcal{A}$ is called a **flipping triple** whenever:

(i) there exists an antiautomorphism of $\mathcal{A}$ that sends
\[ A \mapsto A + BC - CB, \quad B \mapsto B, \quad C \mapsto C; \]
(ii) there exists an antiautomorphism of \( \mathcal{A} \) that sends
\[
B \mapsto B + CA - AC, \quad C \mapsto C, \quad A \mapsto A;
\]
(iii) there exists an antiautomorphism of \( \mathcal{A} \) that sends
\[
C \mapsto C + AB - BA, \quad A \mapsto A, \quad B \mapsto B;
\]
(iv) the algebra \( \mathcal{A} \) is generated by \( A, B, C \).

**Example 4.2.** Recall from Definition [2.1] the generators \( A, B \) for the \( q \)-Onsager algebra \( \mathcal{O}_q \). Recall the element \( C \) from (16). By Theorem [2.11] the sequence \( A/(q - q^{-1}), B/(q - q^{-1}), C/(q - q^{-1}) \) is a flipping triple for \( \mathcal{O}_q \).

**Example 4.3.** Assume that \( A, B, C \) freely generate \( \mathcal{A} \). One routinely checks that \( A, B, C \) is a flipping triple for \( \mathcal{A} \).

**Problem 4.4.** Find all the sequences \( A, B, C, \mathcal{A} \) such that \( A, B, C \) is a flipping triple in the algebra \( \mathcal{A} \).

We define some notation. Let \( \lambda_1, \lambda_2, \ldots \) denote mutually commuting indeterminates. Let \( \mathbb{F}[\lambda_1, \lambda_2, \ldots] \) denote the algebra of polynomials in \( \lambda_1, \lambda_2, \ldots \) that have all coefficients in \( \mathbb{F} \). For a subset \( Y \subseteq \mathcal{A}_q \) let \( \langle Y \rangle \) denote the subalgebra of \( \mathcal{A}_q \) generated by \( Y \). Shortly we will encounter some tensor products. All tensor products in this paper are understood to be over \( \mathbb{F} \).

The following conjecture about \( \mathcal{A}_q \) is a variation on [5, Conjecture 1].

**Conjecture 4.5.** The following (i)–(v) hold:

(i) there exists an algebra isomorphism \( \mathbb{F}[\lambda_1, \lambda_2, \ldots] \rightarrow \langle \mathcal{W}_0, \mathcal{W}_{-1}, \ldots \rangle \) that sends \( \lambda_{k+1} \mapsto \mathcal{W}_{-k} \) for \( k \in \mathbb{N} \);

(ii) there exists an algebra isomorphism \( \mathbb{F}[\lambda_1, \lambda_2, \ldots] \rightarrow \langle \mathcal{W}_1, \mathcal{W}_2, \ldots \rangle \) that sends \( \lambda_{k+1} \mapsto \mathcal{W}_{k+1} \) for \( k \in \mathbb{N} \);

(iii) there exists an algebra isomorphism \( \mathbb{F}[\lambda_1, \lambda_2, \ldots] \rightarrow \langle \mathcal{G}_1, \mathcal{G}_2, \ldots \rangle \) that sends \( \lambda_{k+1} \mapsto \mathcal{G}_{k+1} \) for \( k \in \mathbb{N} \);

(iv) there exists an algebra isomorphism \( \mathbb{F}[\lambda_1, \lambda_2, \ldots] \rightarrow \langle \tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2, \ldots \rangle \) that sends \( \lambda_{k+1} \mapsto \tilde{\mathcal{G}}_{k+1} \) for \( k \in \mathbb{N} \);

(v) the multiplication map
\[
\langle \mathcal{W}_0, \mathcal{W}_{-1}, \ldots \rangle \otimes \langle \mathcal{G}_1, \mathcal{G}_2, \ldots \rangle \otimes \langle \tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2, \ldots \rangle \otimes \langle \mathcal{W}_1, \mathcal{W}_2, \ldots \rangle \rightarrow \mathcal{A}_q
\]
\[
u \otimes v \otimes w \otimes x \mapsto uvvx
\]
is an isomorphism of vector spaces.
A proof of Conjecture 4.5 would yield a PBW basis for $\mathcal{A}_q$.

The next conjecture concerns the center $Z(\mathcal{A}_q)$.

**Conjecture 4.6.** The following (i)–(iii) hold:

(i) there exists an algebra isomorphism $F[\lambda_1, \lambda_2, \ldots] \to Z(\mathcal{A}_q)$ that sends $\lambda_{k+1} \mapsto \Delta_{k+1}$ for $k \in \mathbb{N}$;

(ii) there exists an algebra isomorphism $O_q \to \langle W_0, W_1 \rangle$ that sends $A \mapsto W_0$ and $B \mapsto W_1$;

(iii) the multiplication map

$$\langle W_0, W_1 \rangle \otimes Z(\mathcal{A}_q) \to \mathcal{A}_q$$

$$u \otimes v \mapsto uv$$

is an isomorphism of algebras.

A proof of Conjecture 4.6 would yield an algebra isomorphism $O_q \otimes F[\lambda_1, \lambda_2, \ldots] \to \mathcal{A}_q$.

Above Lemma 3.2 we mentioned a redundancy among the generators (22) of $\mathcal{A}_q$. We now pursue this theme more deeply. Using (23) we eliminate the generators $\{G_{k+1}\}_{k \in \mathbb{N}}$:

$$G_{k+1} = \tilde{G}_{k+1} + (q + q^{-1})[W_1, W_{-k}] \quad (k \in \mathbb{N}).$$

Next we use (24), (25) to recursively eliminate $W_{-k}, W_{k+1}$ for $k \geq 1$:

$$W_{-1} = W_1 - \frac{[\tilde{G}_1, W_0]_q}{(q^2 - q^{-2})^2},$$

$$W_3 = W_1 - \frac{[\tilde{G}_1, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_2]_q}{(q^2 - q^{-2})^2},$$

$$W_{-3} = W_1 - \frac{[\tilde{G}_1, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_2]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_3, W_0]_q}{(q^2 - q^{-2})^2},$$

$$W_5 = W_1 - \frac{[\tilde{G}_1, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_2]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_3, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_4]_q}{(q^2 - q^{-2})^2},$$

$$W_{-5} = W_1 - \frac{[\tilde{G}_1, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_2]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_3, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_4]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_5, W_0]_q}{(q^2 - q^{-2})^2},$$

$$\cdots$$
\[ W_2 = W_0 - \frac{[W_1, \tilde{G}_1]}{(q^2 - q^{-2})^2}, \]
\[ W_{-2} = W_0 - \frac{[W_1, \tilde{G}_1]}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_2, W_0]}{(q^2 - q^{-2})^2}, \]
\[ W_4 = W_0 - \frac{[W_1, \tilde{G}_1]}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_2, W_0]}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_3]}{(q^2 - q^{-2})^2}, \]
\[ W_{-4} = W_0 - \frac{[W_1, \tilde{G}_1]}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_2, W_0]}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_3]}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_4, W_0]}{(q^2 - q^{-2})^2}, \]
\[ W_6 = W_0 - \frac{[W_1, \tilde{G}_1]}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_2, W_0]}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_3]}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_4, W_0]}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_5]}{(q^2 - q^{-2})^2}, \]

For any integer \( k \geq 1 \), the generators \( W_{-k}, W_{k+1} \) are given as follows.
For odd \( k = 2r + 1 \),
\[ W_{-k} = W_1 - \sum_{\ell=0}^{r} \frac{[\tilde{G}_{2\ell+1}, W_0]}{(q^2 - q^{-2})^2} - \sum_{\ell=1}^{r} \frac{[W_1, \tilde{G}_{2\ell}]}{(q^2 - q^{-2})^2}, \]
\[ W_{k+1} = W_0 - \sum_{\ell=0}^{r} \frac{[W_1, \tilde{G}_{2\ell+1}]}{(q^2 - q^{-2})^2} - \sum_{\ell=1}^{r} \frac{[\tilde{G}_{2\ell}, W_0]}{(q^2 - q^{-2})^2}. \]
For even \( k = 2r \),
\[ W_{-k} = W_0 - \sum_{\ell=0}^{r-1} \frac{[W_1, \tilde{G}_{2\ell+1}]}{(q^2 - q^{-2})^2} - \sum_{\ell=1}^{r} \frac{[\tilde{G}_{2\ell}, W_0]}{(q^2 - q^{-2})^2}, \]
\[ W_{k+1} = W_1 - \sum_{\ell=0}^{r-1} \frac{[\tilde{G}_{2\ell+1}, W_0]}{(q^2 - q^{-2})^2} - \sum_{\ell=1}^{r} \frac{[W_1, \tilde{G}_{2\ell}]}{(q^2 - q^{-2})^2}. \]

So far, we have expressed the generators \( \tilde{W}_k \) in terms of \( W_0, W_1, \{ \tilde{G}_k \}_{k \in \mathbb{N}} \). We now consider how these remaining generators are related to each other.

**Lemma 4.7.** The following relations hold in the algebra \( \mathcal{A}_q \):
\[ [W_0, \tilde{G}_1] = [W_0, [W_0, W_1]], \]
\[ [\tilde{G}_1, W_1] = [[W_0, W_1], W_1] \]
and for \( k \geq 1 \),
\[ [\tilde{G}_{k+1}, W_0] = \frac{[W_0, [W_0, [W_0, \tilde{G}_k]]]}{(q^2 - q^{-2})^2}, \]
\[ [W_1, \tilde{G}_{k+1}] = \frac{[[[W_0, W_0], W_1], W_1]}{(q^2 - q^{-2})^2}. \]
Proof. The first two relations are (36), (37) with $k = 0$. To obtain the third relation, use (36) and (25), (26) to obtain

\[
[\tilde{G}_{k+1}, W_0] = -[W_0, [W_0, [W_k, W_{k+1}]]_q]
\]
\[
= -[W_0, [W_0, W_{k+1}]]_q
\]
\[
= -[W_0, [W_0, W_{k+1} - W_{1-k}]]_q
\]
\[
= \frac{[W_0, [W_0, [W_1, \tilde{G}_k]]_q]}{(q^2 - q^{-2})^2}
\]
\[
= \frac{[W_0, [W_0, [W_1, \tilde{G}_k]]_q]}{(q^2 - q^{-2})^2}.
\]

The last relation is similarly obtained.

\[\square\]

Conjecture 4.8. The algebra $A_q$ has a presentation by generators $W_0, W_1, \{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$ and relations

\[
[W_0, [W_0, [W_0, W_1]]_q] = (q^2 - q^{-2})^2[W_1, W_0],
\]
\[
[W_1, [W_1, [W_1, W_0]]_q] = (q^2 - q^{-2})^2[W_0, W_1],
\]
\[
[W_0, \tilde{G}_1] = [W_0, [W_0, W_1]]_q,
\]
\[
[\tilde{G}_1, W_1] = [[W_0, W_1]]_q,
\]
\[
[\tilde{G}_{k+1}, W_0] = \frac{[W_0, [W_0, [W_1, \tilde{G}_k]]_q]}{(q^2 - q^{-2})^2} \quad (k \geq 1),
\]
\[
[W_1, \tilde{G}_{k+1}] = \frac{[[[\tilde{G}_k, W_0]]_q, [W_1]]_q, W_1]}{(q^2 - q^{-2})^2} \quad (k \geq 1),
\]
\[
[\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0 \quad (k, \ell \in \mathbb{N}).
\]

5 Acknowledgment

The author thanks Pascal Baseilhac and Samuel Belliard for many discussions about the $q$-Onsager algebra and its current algebra.

References

[1] P. Baseilhac. An integrable structure related with tridiagonal algebras. Nuclear Phys. B 705 (2005) 605–619; \texttt{arXiv:math-ph/0408025}.

[2] P. Baseilhac. Deformed Dolan-Grady relations in quantum integrable models. Nuclear Phys. B 709 (2005) 491–521; \texttt{arXiv:hep-th/0404149}.

[3] P. Baseilhac and S. Belliard. Generalized $q$-Onsager algebras and boundary affine Toda field theories. Lett. Math. Phys. 93 (2010) 213–228; \texttt{arXiv:0906.1215}.
[4] P. Baseilhac and S. Belliard. The half-infinite XXZ chain in Onsager’s approach. *Nuclear Phys. B* 873 (2013) 550–584; [arXiv:1211.6304](https://arxiv.org/abs/1211.6304).

[5] P. Baseilhac and S. Belliard. An attractive basis for the $q$-Onsager algebra. [arXiv:1704.02950](https://arxiv.org/abs/1704.02950).

[6] P. Baseilhac and K. Koizumi. A new (in)finite dimensional algebra for quantum integrable models. *Nuclear Phys. B* 720 (2005) 325–347; [arXiv:math-ph/0503036](https://arxiv.org/abs/math-ph/0503036).

[7] P. Baseilhac and K. Koizumi. A deformed analogue of Onsager’s symmetry in the XXZ open spin chain. *J. Stat. Mech. Theory Exp.* 2005, no. 10, P10005, 15 pp. (electronic); [arXiv:hep-th/0507053](https://arxiv.org/abs/hep-th/0507053).

[8] P. Baseilhac and K. Koizumi. Exact spectrum of the XXZ open spin chain from the $q$-Onsager algebra representation theory. *J. Stat. Mech. Theory Exp.* 2007, no. 9, P09006, 27 pp. (electronic); [arXiv:hep-th/0703106](https://arxiv.org/abs/hep-th/0703106).

[9] P. Baseilhac and S. Kolb. Braid group action and root vectors for the $q$-Onsager algebra. [arXiv:1706.08747](https://arxiv.org/abs/1706.08747).

[10] P. Baseilhac and K. Shigechi. A new current algebra and the reflection equation. *Lett. Math. Phys.* 92 (2010) 47–65; [arXiv:0906.1482v2](https://arxiv.org/abs/0906.1482v2).

[11] S. Belliard and N. Crampe. Coideal algebras from twisted Manin triples. *J. Geom. Phys.* 62 (2012) 2009–2023; [arXiv:1202.2312](https://arxiv.org/abs/1202.2312).

[12] T. Ito, K. Nomura, P. Terwilliger. A classification of sharp tridiagonal pairs. *Linear Algebra Appl.* 435 (2011) 1857–1884; [arXiv:1001.1812](https://arxiv.org/abs/1001.1812).

[13] T. Ito, K. Tanabe, P. Terwilliger. Some algebra related to $P$- and $Q$-polynomial association schemes, in: *Codes and Association Schemes (Piscataway NJ, 1999)*, Amer. Math. Soc., Providence RI, 2001, pp. 167–192; [arXiv:math.CO/0406556](https://arxiv.org/abs/math.CO/0406556).

[14] T. Ito and P. Terwilliger. Tridiagonal pairs of $q$-Racah type. *J. Algebra* 322 (2009), 68–93; [arXiv:0807.0271](https://arxiv.org/abs/0807.0271).

[15] T. Ito and P. Terwilliger. The augmented tridiagonal algebra. *Kyushu J. Math.* 64 (2010) 81–144; [arXiv:0807.3990](https://arxiv.org/abs/0807.3990).

[16] S. Kolb. Quantum symmetric Kac-Moody pairs. *Adv. Math.* 267 (2014) 395-469; [arXiv:1207.6036](https://arxiv.org/abs/1207.6036).

[17] L. Onsager. Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Phys. Rev. (2)* 65 (1944) 117–149.

[18] P. Terwilliger. The subconstituent algebra of an association scheme III. *J. Algebraic Combin.* 2 (1993) 177–210.

[19] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other. *Linear Algebra Appl.* 330 (2001) 149–203; [arXiv:math.RA/0406555](https://arxiv.org/abs/math.RA/0406555).
[20] P. Terwilliger. Two relations that generalize the $q$-Serre relations and the Dolan-Grady relations. In *Physics and Combinatorics 1999 (Nagoya)*, 377–398, World Scientific Publishing, River Edge, NJ, 2001; [arXiv:math.QA/0307016](http://arxiv.org/abs/math.QA/0307016).

[21] P. Terwilliger. An algebraic approach to the Askey scheme of orthogonal polynomials. Orthogonal polynomials and special functions, 255–330, Lecture Notes in Math., 1883, Springer, Berlin, 2006; [arXiv:math.QA/0408390](http://arxiv.org/abs/math.QA/0408390).

[22] P. Terwilliger. The universal Askey-Wilson algebra. *SIGMA Symmetry Integrability Geom. Methods Appl.* 7 (2011) Paper 069, 22pp.

[23] P. Terwilliger. The $q$-Onsager algebra and the positive part of $U_q(\widehat{\mathfrak{sl}_2})$. *Linear Algebra Appl.* 521 (2017) 19–56; [arXiv:1506.08666](http://arxiv.org/abs/1506.08666).

[24] P. Terwilliger. The Lusztig automorphism of the $q$-Onsager algebra. *J. Algebra*. 506 (2018) 56–75; [arXiv:1706.05546](http://arxiv.org/abs/1706.05546).

[25] P. Terwilliger. The $q$-Onsager algebra and the universal Askey-Wilson algebra. *SIGMA Symmetry Integrability Geom. Methods Appl.* 14 (2018) Paper No. 044, 18 pp.

[26] P. Terwilliger and R. Vidunas. Leonard pairs and the Askey-Wilson relations. *J. Algebra Appl.* 3 (2004) 411–426; [arXiv:math.QA/0305356](http://arxiv.org/abs/math.QA/0305356).

Paul Terwilliger
Department of Mathematics
University of Wisconsin
480 Lincoln Drive
Madison, WI 53706-1388 USA
email: terwilli@math.wisc.edu