A BRIEF INTRODUCTION TO THE ADOMIAN DECOMPOSITION METHOD, WITH APPLICATIONS IN ASTRONOMY AND ASTROPHYSICS

MAN KWONG MAK¹, CHUN SING LEUNG², TIBERIU HARKO³

¹Departamento de Física, Facultad de Ciencias Naturales, Universidad de Atacama, Copiapó 485, Copiapó, Chile, Email: munkwongmak@gmail.com
²Department of Mathematics, Polytechnic University of Hong Kong, Hong Kong, Email: chun-sing-hkpu.leung@polyu.edu.hk
³School of Physics, Sun Yat-Sen University, Xinggang Road, Guangzhou 510275, People’s Republic of China, Email: tiberiu.harko@aira.astro.ro

Abstract. The Adomian Decomposition Method (ADM) is a very effective approach for solving broad classes of nonlinear partial and ordinary differential equations, with important applications in different fields of applied mathematics, engineering, physics and biology. It is the goal of the present paper to provide a clear and pedagogical introduction to the Adomian Decomposition Method and to some of its applications. In particular, we focus our attention to a number of standard first-order ordinary differential equations (the linear, Bernoulli, Riccati, and Abel) with arbitrary coefficients, and present in detail the Adomian method for obtaining their solutions. In each case we compare the Adomian solution with the exact solution of some particular differential equations, and we show their complete equivalence. The second order and the fifth order ordinary differential equations are also considered. An important extension of the standard ADM, the Laplace-Adomian Decomposition Method is also introduced through the investigation of the solutions of a specific second order nonlinear differential equation. We also present the applications of the method to the Fisher-Kolmogorov second order partial nonlinear differential equation, which plays an important role in the description of many physical processes, as well as three important applications in astronomy and astrophysics, related to the determination of the solutions of the Kepler equation, of the Lane-Emden equation, and of the general relativistic equation describing the motion of massive particles in the spherically symmetric and static Schwarzschild geometry.

Key words: Mathematical Methods in Physics – Ordinary Nonlinear Differential Equations – Celestial Mechanics – Astronomy–General Relativity.

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**1. INTRODUCTION**

In recent years, a lot of consideration has been dedicated to the investigations of the Adomian’s Decomposition Method (ADM) (Adomian and Rach, 1983; Adomian, 1988, 1994; Cherrualt et al., 1995; Adomian and Rach, 1996; Duan et al., 2012), which allows us to explore the solutions and properties of a large variety of ordinary and partial differential equations, as well as of integral equations, which describe various mathematical problems, or can be used to mathematically model diverse physical processes. From a historical point of view, the ADM was first introduced, and extensively used in the 1980’s (Adomian and Rach, 1983; Adomian, 1984a,b, 1985, 1986), and ever since many mathematicians and scientists have continuously modified the ADM in an attempt to enhance its accuracy and/or to broaden the applications of the initial method (Cherrualt et al., 1995; Adomian and Rach, 1996; Wazwaz, 1999a,b, 2005; Babolian and Javadi, 2003; Babolian et al., 2004; Jin and Liu, 2005; Jafari and Daftardar-Gejji, 2006a,b; Rach et al., 1992; Wazwaz and El-Sayed, 2001; Biazar et al., 2004, 2003; Sadat, 2010; Bakodah, 2012).

An important benefit of the Adomian Decomposition Method is that it can yield analytical approximations to quite extensive classes of nonlinear (and stochastic) differential equations without resorting to discretization, perturbation, linearization, or closure approximations methods, which could result in the necessity of extensive numerical computations. For most of the mathematical models used for the mathematical description of natural phenomena, in order to obtain the analytical solutions of a nonlinear problem in a closed-form, and thus to make it solvable, it is usually necessary to make some simplifying assumptions, or to impose some restrictive conditions.

It is worth to note that ADM can provide a solution of a differential/integral equation in the form of a series, whose terms are determined individually step by step via a recursive relation using the Adomian polynomials. The main advantage of the Adomian Decomposition Method is that the series solution of the differential/integral equation converges very quickly (Abbaoui and Cherrualt, 1994a,b; Cherrualt et al., 1995), and therefore it saves significant amounts of computing time. On the other hand, it is important to point out again that in the Adomian Decomposition Method there is no need to discretize or linearize the differential and integral equations. One can find reviews of ADM in applied mathematics, and its applications in science in Adomian (1988), Adomian (1994), and Haldar (2016), respectively.

The basic nonlinear ordinary differential equations of mathematics (Riccati and Abel), as well as their physical and engineering applications have continuously attracted the interest of mathematicians and physicists (Mak and Harko, 2012, 2013a; Harko et al., 2016; Mak et al., 2001, Mak and Harko, 2002; Harko and Mak, 2003;
These equations also proved to be a fertile investigation ground from the point of view of the ADM approach. Recently, using the ADM, the Riccati equation was solved in Gbadamosi et al. (2012). The Abel differential equation, having constant coefficients, of the form

\[
\frac{dy}{dt} = \sum_{k=0}^{M} f_k y^k,
\]

(1)

was solved with the help of ADM in Al-Dosary et al. (2008). A modified version of the Adomian Decomposition Method was introduced for solving second order ordinary differential equation in Hassan and Zhu (2008) and Hosseini and Nasabzadeh (2007), respectively. A particular third order ordinary differential equation was investigated by using a modified ADM for solving it in Mak et al. (2018a). The ADM was applied to the third order ordinary differential equation

\[
y''' = y^{-k},
\]

(2)

representing a particular case of a generalized thin film equation describing the flow of a thin film downward of a vertical wall Momoniat et al. (2007). The ADM for solving different classes of differential equations of importance in mathematical physics was studied in Dita and Grama (1997). The fourth order differential equation was solved by ADM in Agom et al. (2016). The biharmonic nonlinear Schrödinger equation, and its standing wave solutions were investigated, via the use of the Laplace-Adomian and Adomian Decomposition Methods, in Mak et al. (2018a).

The Adomian Decomposition Method method was extensively applied in different areas of science and technology, including the study of the dynamics of the population growth models, which can be modelled by single partial or ordinary differential equations, or complex systems of such equations. A few example of specific mathematical systems successfully explored by using the ADM are the shallow water waves (Safari, 2011), the Brusselator model (Wazwaz, 2000), the Lotka-Volterra model (Ruan and Lui, 2007), and the Belousov-Zhabotinsky reduction model (Fatoorehchi et al., 2015), respectively. The Adomian Decomposition Method was applied for the study of the Susceptible-Infected-Recovered (SIR) epidemic model, which is widely applied for the study of the spread of infectious diseases, in Harko and Mak (2020a) and Harko and Mak (2020b), respectively.

The Adomian Decomposition Method did also find some important applications in Physics. Nonlinear matrix differential equations of a new type, which emerge in general relativity as well as other scientific fields, were investigated in Azreg-Aïnou (2010). The solution of the nonlinear Klein-Gordon equation was obtained via the Adomian Decomposition Method in Ghasemi et al. (2014). The obtained semi-analytical solutions are in good accord with the full numerical solutions. The equations of motion of the massive and massless particles in the spherically symmetric and static
Schwarzschild geometry of general relativity were studied extensively in Mak et al. (2018a) by using the Laplace-Adomian Decomposition Method. The physical properties of vortices with arbitrary topological charges arising in weakly interacting Bose-Einstein Condensates, described by differential equations of the form

\[
\frac{d^2 R(x)}{dx^2} + \frac{1}{x} \frac{dR(x)}{dx} - \left[ \frac{l^2}{x^2} + (v(x) - 1) \right] R(x) - R^3(x) = 0,
\]

where \( l \) is a constant, and \( v(x) = 0 \) and \( v(x) = x^2 \), were investigated using the Adomian Decomposition Method in Harko et al. (2020), where the nonlinear Gross-Pitaevskii equation was solved in polar coordinates. Series solutions using the Adomian Decomposition Method have been obtained for the Schrödinger-Newton-\( \Lambda \)-system, described by the system of partial differential equations,

\[
i\hbar \frac{\partial \psi (\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi (\vec{r},t) + m\Phi (\vec{r},t) \psi (\vec{r},t),
\]

\[
\nabla^2 \Phi (\vec{r},t) = 4\pi Gm |\psi (\vec{r},t)|^2 - \frac{1}{2} \Lambda c^2,
\]

where by \( \psi (\vec{r},t) \) we have denoted the particle wave function, \( \Phi (\vec{r},t) \) is the gravitational potential, \( \hbar \), \( G \) and \( \Lambda \) are the Planck, the gravitational and the cosmological constants, respectively, while \( m \) is the mass of the particle, in Mak et al. (2020) and Harko et al. (2020), respectively.

Despite the existence of a large literature on the ADM, to the best knowledge of the authors no clearly written and pedagogical introduction to the method, which would be useful for a large audience of scientists from different fields, does exist presently. It is the purpose of the present paper to give such an introductory review of the Adomian Decomposition Method, and of the Laplace-Adomian Decomposition Method, in which, by means of the detailed and explicit presentation of all the calculations, and by providing a large number of examples, the power and efficiency of the method is clearly outlined. Hopefully, such a presentation would be of interest even for undergraduate students studying sciences and engineering, and will determine them to proceed to the study and investigation of the advanced features of the method.

From the point of view of the applications of the Adomian Decomposition Method in science we have chosen to present the analysis of the Fisher-Kolmogorov equation [Fisher, 1937; Kolmogorov et al., 1937], which plays an essential role in many physical and biological problems. But the main focus of the present paper are the potential astronomical and astrophysical applications of the Adomian Decomposition Method, a field that has yet to be explored in detail. One important astronomical problem that can be handled efficiently and effectively with the Adomian Decomposition Method is obtaining the solution of the Kepler equation, which plays a fundamental role in the determination of the orbits of the celestial orbits.
The hyperbolic and the elliptic Kepler equations were investigated by using ADM in [Ebaid et al., 2017] and [Alshaery and Ebaid, 2017], respectively. One of the basic equations of Newtonian astrophysics is the Lane-Emden equation, which was used, for example, for the study of the white dwarfs, which lead to the fundamental Chandrasekhar mass limit for this type of compact objects (Chandrasekhar, 1967). The Lane-Emden equation was intensively investigated by using the Adomian Decomposition Method, which provides an efficient and computationally powerful procedure to obtain its solutions, in [Adomian et al., 1995; Wazwaz and Rach, 2011; Wazwaz et al., 2013; Hosseini and Abbaspandy, 2015; Rach et al., 2015]. Finally, we will consider the general relativistic motion of massive test particles in the static and spherically symmetric Schwarzschild geometry, and present its Adomian series solution (Mak et al., 2018a). This approach can be used for the extremely precise analytical calculation of the orbit of the planet Mercury, for the study of its perihelion precession, as well as for the computation of the light deflection by the Sun. The solutions of the Kompaneets equation, a nonlinear partial differential equation that plays an important role in astrophysics, describing the spectra of photons in interaction with a rarefied electron gas, were obtained, by using the Laplace-Adomian Decomposition Method, in [González-Gaxiola et al., 2017].

The present paper is organized as follows. We introduce the basics of the Adomian Decomposition Method in Section 2. In Section 3 we discuss the application of the ADM to the case of the first order differential equations. We begin our discussion with the simplest case of the ordinary linear first order differential equation, whose solution can be obtained exactly. The power series solution of the linear equation is obtained by using a power series expansion. We consider then a particular case, and we compare the power series and the exact solutions. Next we proceed to the investigation of the Bernoulli, Riccati and Abel type equations with constant coefficients, by using ADM, and the series solutions of these equations are obtained. In each case the power series solution is compared with the exact solution of a particular differential equation. The case of the second order differential equations is considered in Section 4. Two specific example are also presented, and discussed in detail. The fifth order ordinary differential equation is analyzed in Section 5. As an example of the use of the ADM for solving nonlinear partial differential equations, in Section 6 we consider the case of the Fisher-Kolmogorov equation, a nonlinear differential equation with many applications in biology. We present the Laplace-Adomian Decomposition Method for second order nonlinear differential equations in Section 7. Astronomical and astrophysical applications of the Adomian Decomposition Method (Kepler equation, Lane-Emden equation, and the motion of massive particles in the Schwarzschild geometry) are presented in Section 8. Finally, we discuss and conclude our results in Section 9.
2. THE ADOMIAN DECOMPOSITION METHOD

We illustrate now the basic ideas of the Adomian Decomposition Method by considering the case of a nonlinear partial differential equation written in the general form

\[ \hat{L}_t[y(x,t)] + \hat{R}[y(x,t)] + \hat{N}[y(x,t)] = f(x,t), \]  

(6)

where \( \hat{L}_t[.] = \partial / \partial t [.] \) denotes the partial derivative operator with respect to the time \( t \), while \( \hat{R}[.] \) is the linear operator, generally containing partial derivatives with respect to \( x \). Moreover, \( \hat{N}[.] \) represents a nonlinear analytic operator, and \( f(x,t) \) is a non-homogeneous arbitrary function, assumed to be independent of \( y(x,t) \). Eq. (6) has to be considered together with the initial condition \( y(x,0) = g(x) \). In the following we assume that the operator \( \hat{L}_t \) is invertible, and therefore we can apply \( \hat{L}_t^{-1} \) to both sides of Eq. (6), thus first obtaining

\[
y(x,t) = g(x) + \hat{L}_t^{-1}[f(x,t)] - \hat{L}_t^{-1}\hat{R}[y(x,t)] - \hat{L}_t^{-1}\hat{N}[y(x,t)].
\]

(7)

The ADM postulates the existence of a series solution of Eq. (6) in which \( y(x,t) \) can be represented by

\[
y(x,t) = \sum_{n=0}^{\infty} y_n(x,t).
\]

(8)

Moreover, it is assumed that the nonlinear term \( \hat{N}[y(x,t)] \) can be decomposed according to

\[
\hat{N}[y(x,t)] = \sum_{n=0}^{\infty} A_n(y_0, y_1, \ldots, y_n),
\]

(9)

where \( \{A_n\}_{n=0}^{\infty} \) are called the Adomian polynomials. They can be computed according to the simple rule (Adomian and Rach, 1983; Adomian, 1988, 1994; Cherrualt et al., 1995; Adomian and Rach, 1996)

\[
A_n(y_0, y_1, \ldots, y_n) = \left[ \frac{d^n}{dt^n} \hat{N}\left(t, \sum_{k=0}^{n} \epsilon^k y_k\right) \right]_{\epsilon=0}.
\]

(10)

After the substitution of the series expansions (8) and (9) into Eq. (6), we obtain

\[
\sum_{n=0}^{\infty} y_n(x,t) = g(x) + \hat{L}_t^{-1}[f(x,t)] \hat{L}_t^{-1}\hat{R} \left[ \sum_{n=0}^{\infty} y_n(x,t) \right] - \hat{L}_t^{-1} \left[ \sum_{n=0}^{\infty} A_n(y_0, y_1, \ldots, y_n) \right].
\]

(11)
From the above equation we immediately obtain the following recurrence relation, which gives the series solution of Eq. (6) as

\[ y_0(x,t) = g(x) + L^{-1} f(x,t), \]

\[ y_{k+1}(x,t) = L^{-1} R[y_k(x,t)] - L^{-1} A_k(y_0, y_1, \ldots, y_n), \quad k = 0, 1, 2, \ldots \]

Therefore, an approximate solution of Eq. (6) is obtained as

\[ y(x,t) \approx \sum_{k=0}^{n} y_k(x,t), \]

and

\[ \lim_{n \to \infty} \sum_{k=0}^{n} y_k(x,t) = y(x,t). \]

For an arbitrary nonlinearity \( \hat{N}[y(x,t)] \), the Adomian polynomials can be obtained according to the rule

\[ A_0 = \hat{N}[y_0], \quad A_1 = y_1 \frac{d}{dy_0} \hat{N}[y_0], \]

\[ A_2 = y_2 \frac{d}{dy_0} \hat{N}[y_0] + \frac{y_1^2}{2!} \frac{d^2}{dy_0^2} \hat{N}[y_0], \]

\[ A_3 = y_3 \frac{d}{dy_0} \hat{N}[y_0] + y_1 y_2 \frac{d^2}{dy_0^2} \hat{N}[y_0] + \frac{y_1^3}{3!} \frac{d^3}{dy_0^3} \hat{N}[y_0]. \]

This procedure can be continued indefinitely. The greater the number of considered terms in the Adomian Decomposition Method series expansion, the higher is the numerical accuracy of the semi-analytical solution.

In the following Sections we will present in detail the application of the Adomian Decomposition Method for a large class of nonlinear ordinary and partial differential equations.

### 3. THE ADOMIAN DECOMPOSITION METHOD FOR FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

In the present Section we introduce the application of the ADM to the case of first order differential equations. The linear, Bernoulli, Riccati and Abel differential equations are considered in detail.
3.1. LINEAR DIFFERENTIAL EQUATION \( \frac{dy}{dx} + P(x)y = Q(x) \)

The decomposition method can be used to solve the linear differential equations. Consider that the differential equation takes the standard form of the first order ordinary differential equation,

\[
\frac{dy}{dx} + P(x)y = Q(x),
\]

where \( P(x) \) and \( Q(x) \) are arbitrary function of \( x \). Eq. (19) must be solved together with the initial condition \( y(0) = y_0 \). Assume that the solution of Eq. (19) can be obtained in power series form,

\[
y(x) = \sum_{n=0}^{\infty} y_n(x).
\]

Now integrating Eq. (19) yields the integral equation

\[
y(x) = y(0) + \int_{0}^{x} Q(x) \, dx - \int_{0}^{x} P(x) \, ydx.
\]

Substituting Eq. (20) into Eq. (21) gives the relation

\[
\sum_{n=0}^{\infty} y_n(x) = y_0(x) + \sum_{n=1}^{\infty} y_n(x) = y_0(x) + \sum_{n=0}^{\infty} y_{n+1}(x)
\]

\[
= y(0) + \int_{0}^{x} Q(x) \, dx - \int_{0}^{x} P(x) \sum_{n=0}^{\infty} y_n(x) \, dx.
\]

Next we rewrite Eq. (22) in the recursive forms

\[
y_0(x) = y(0) + \int_{0}^{x} Q(x) \, dx,
\]

\[
y_{k+1}(x) = - \int_{0}^{x} P(x) \, y_k(x) \, dx.
\]

From Eqs. (21) and (22) we can obtain the approximate semi-analytical solution of Eq. (19), as given by

\[
y(x) = \sum_{n=0}^{\infty} y_n(x).
\]

3.1.1. Example: \( \frac{dy}{dx} + 2xy = 4x^3 \)

Consider the differential equation

\[
\frac{dy}{dx} + 2xy = 4x^3,
\]
which we solve with the initial condition $y(0) = 1$. Then its general solution is given by

$$y(x) = 3e^{-x^2} + 2\left(x^2 - 1\right).$$

(27)

In the present case we have $P(x) = 2x$ and $Q(x) = 4x^3$, respectively. Hence the power series of the equation is obtained as

$$y_0(x) = y(0) + \int_0^x Q(x) \, dx = 1 + x^4,$$

(28)

$$y_1(x) = -2\int_0^x xy_0(x) \, dx = -x^2 - \frac{x^6}{3},$$

(29)

$$y_2(x) = -2\int_0^x xy_1(x) \, dx = \frac{x^4}{2} + \frac{x^8}{12},$$

(30)

$$y_3(x) = -2\int_0^x xy_2(x) \, dx = -\frac{x^6}{3} - \frac{x^{10}}{60},$$

(31)

$$y_4(x) = -2\int_0^x xy_3(x) \, dx = \frac{x^8}{24} + \frac{x^{12}}{360}.$$  

(32)

$$y(x) \approx y_0(x) + y_1(x) + y_2(x) + y_3(x) + y_4(x) = 1 - x^2 + \frac{3x^4}{2} - \frac{x^6}{2} + \frac{x^8}{8} \ldots \quad (33)$$

On the other hand by series expanding the exact solution (27) we obtain

$$y(x) = 3e^{-x^2} + 2\left(x^2 - 1\right) = 1 - x^2 + \frac{3x^4}{2} - \frac{x^6}{2} + \frac{x^8}{8} \ldots, \quad (34)$$

Clearly, the solution (33), obtained by the Adomian Decomposition Method is identical to the exact solution (34).

3.2. BERNOULLI DIFFERENTIAL EQUATION $\frac{dy}{dx} + P(x) y = Q(x) y^n$

The Adomian Decomposition Method is very powerful for solving nonlinear ordinary differential equations. Consider that the differential equation takes the Bernoulli equation form

$$\frac{dy}{dx} + P(x) y = Q(x) y^n,$$  

(35)

where $P(x)$ and $Q(x)$ are arbitrary function of $x$, and $n$ is an arbitrary constant. Assume that the solution of Eq. (35) is given by the power series form

$$y(x) = \sum_{n=0}^{\infty} y_n(x).$$

(36)
The nonlinear term $y^n$ can be decomposed in terms of the Adomian polynomials $A_n(x)$, given by

$$
y^n(x) = \sum_{n=0}^{\infty} A_n(x), \quad (37)
$$

Generally, for an arbitrary function $f(t,x)$, the Adomian polynomials are defined as (Adomian, 1994)

$$
A_n = \frac{1}{n!} \frac{d^n}{d\epsilon^n} f \left( t, \sum_{i=0}^{\infty} \epsilon^i y_i \right) \bigg|_{\epsilon=0}. \quad (38)
$$

The first four Adomian polynomials can be obtained in the following form,

$$
A_0 = f(t,y_0), \quad A_1 = y_1 f'(t,y_0), \quad A_2 = y_2 f'(t,y_0) + \frac{1}{2} y_1^2 f''(t,y_0), \quad (39)
$$

$$
A_3 = y_3 f'(t,y_0) + y_1 y_2 f''(t,y_0) + \frac{1}{6} y_1^3 f'''(t,y_0). \quad (40)
$$

For the function $y^n$ a few Adomian polynomials are (Wazwaz, 2005)

$$
A_0 = y_0^n, \quad A_1 = n y_1 y_0^{n-1}, \quad A_2 = n y_2 y_0^{n-1} + n(n-1) \frac{y_1^2}{2} y_0^{n-2}, \quad (41)
$$

$$
A_3 = n y_3 y_0^{n-1} + n(n-1) y_1 y_2 y_0^{n-2} + n(n-1)(n-2) \frac{y_1^3}{3!} y_0^{n-3}. \quad (42)
$$

Now integrating Eq. (35) yields the integral equation

$$
y(x) = y(0) + \int_0^x [Q(x) y^n - P(x) y] \, dx, \quad (43)
$$

where $y(0)$ is the initial condition. Substituting Eqs. (36) and (37) into Eq. (43) gives the relation

$$
\sum_{n=0}^{\infty} y_n(x) = y(0) + \int_0^x Q(x) \sum_{n=0}^{\infty} A_n(x) \, dx - \int_0^x P(x) \sum_{n=0}^{\infty} y_n(x) \, dx. \quad (44)
$$

We rewrite Eq. (44) in the recursive forms

$$
y_0(x) = y(0), \quad (45)
$$

$$
y_{k+1}(x) = \int_0^x [Q(x) A_k(x) - P(x) y_k(x)] \, dx. \quad (46)
$$

From Eqs. (45) and (46), we obtain the semi-analytical solution of Eq. (35), given by

$$
y(x) = \sum_{n=0}^{\infty} y_n(x). \quad (47)
$$
3.2.1. Example: $\frac{dy}{dx} - 2xy = -4x^3y^2$

Consider now the differential equation

$$\frac{dy}{dx} - 2xy = -4x^3y^2,$$  \hspace{1cm} (48)

with initial condition $y(0) = 1$, having the general solution

$$y(x) = \frac{1}{3e^{-x^2} + 2(x^2 - 1)}.$$  \hspace{1cm} (49)

In this case $P(x) = -2x$ and $Q(x) = -4x^3$, respectively, and $n = 2$. Next we compute a few Adomian polynomials for $y^2$.

$$A_0 = y_0^2, A_1 = 2y_1y_0, A_2 = 2y_2y_0 + y_1^2, A_3 = 2y_3y_0 + 2y_1y_2,$$  \hspace{1cm} (50)

Hence we obtain

$$y_0(x) = y(0) = 1,$$  \hspace{1cm} (51)

$$y_{k+1}(x) = \int_0^x [Q(x)A_k(x) - P(x)y_k(x)] dx.$$  \hspace{1cm} (52)

Eq. (46) can be written recursively for $k = 0, 1, 2, 3$ in the decomposed solutions

$$y_1(x) = \int_0^x [-4x^3A_0(x) + 2xy_0(x)] dx = x^2 - x^4,$$  \hspace{1cm} (53)

$$y_2(x) = \int_0^x [-4x^3A_1(x) + 2xy_1(x)] dx = \frac{x^4}{2} - \frac{5x^6}{3} + x^8,$$  \hspace{1cm} (54)

$$y_3(x) = \int_0^x [-4x^3A_2(x) + 2xy_2(x)] dx = \frac{x^6}{6} - \frac{17x^8}{12} + \frac{7x^{10}}{3} - x^{12},$$  \hspace{1cm} (55)

$$y_4(x) = \int_0^x [-4x^3A_3(x) + 2xy_3(x)] dx = \frac{x^8}{24} - \frac{49x^{10}}{60} + \frac{25x^{12}}{9} - 3x^{14} + x^{16},$$  \hspace{1cm} (56)

$$y(x) \approx y_0(x) + y_1(x) + y_2(x) + y_3(x) + y_4(x) = 1 + x^2 - \frac{x^4}{2} - \frac{3x^6}{2} - \frac{3x^8}{8} - \ldots.$$  \hspace{1cm} (57)

On the other hand from the exact solution (49), it is easy to obtain

$$y(x) = \frac{1}{3e^{-x^2} + 2(x^2 - 1)} = 1 + x^2 - \frac{x^4}{2} - \frac{3x^6}{2} - \frac{3x^8}{8} - \ldots.$$  \hspace{1cm} (58)

Clearly again, the solution (57) obtained by the Adomian Decomposition Method is identical to the exact solution (58).

3.3. RICCATI DIFFERENTIAL EQUATION $\frac{dy}{dx} = P(x) + Q(x)y^2$

The reduced Riccati differential equation is given by (Kamke, 1959)

$$\frac{dy}{dx} = P(x) + Q(x)y^2,$$  \hspace{1cm} (59)

xi
where $P(x)$ and $Q(x)$ are two arbitrary functions of $x$, and which must be considered together with the initial condition $y_0 = y(0)$. Integrating Eq. (59) yields the equivalent integral equation

$$y(x) = y(0) + \int_0^x P(x) \, dx + \int_0^x Q(x) \, y^2 \, dx,$$

(60)

Substituting $y(x) = \sum_{n=0}^{\infty} y_n(x)$ and $y^2 = \sum_{n=0}^{\infty} A_n(x)$ into Eq. (60) gives the relation

$$\sum_{n=0}^{\infty} y_n(x) = y(0) + \int_0^x P(x) \, dx + \int_0^x Q(x) \sum_{n=0}^{\infty} A_n \, dx.$$

(61)

Next we rewrite Eq. (61) in the recursive forms

$$y_0(x) = y(0) + \int_0^x P(x) \, dx,$$

(62)

$$y_{k+1}(x) = \int_0^x Q(x) A_k(x) \, dx.$$

(63)

From Eqs. (62) and (64), we obtain the semi-analytical solution of Eq. (59), given by $y(x) = \sum_{n=0}^{\infty} y_n(x)$.

### 3.3.1. Example: $\frac{dy}{dx} = 2e^x - e^{-x}y^2$

We consider a particular Riccati equation that has the form

$$\frac{dy}{dx} = 2e^x - e^{-x}y^2,$$

(64)

and which must be solved together with the initial condition $y(0) = 2$. The general solution of the equation is given by

$$y(x) = e^x \left(1 - \frac{3}{1 - 4e^{2x}}\right).$$

(65)

The semi-analytic solution of this particular Riccati equation can be obtained as

$$y_0(x) = y(0) + \int_0^x P(x) \, dx = 2 + 2 \int_0^x e^x \, dx = 2e^x,$$

(66)

$$y_{k+1}(x) = \int_0^x Q(x) A_k(x) \, dx = -\int_0^x e^{-x} A_k \, dx.$$

(67)

In view of Eqs. (66), (67), and (50), we have

$$y_1(x) = \int_0^x Q(x) A_0(x) \, dx = -\int_0^x e^{-x} A_0 \, dx = -4x - 2x^2 - \frac{2x^3}{3} - \frac{x^4}{6} - \sum_{n=5}^{\infty} \frac{x^n}{30 \cdot 180 \cdot 1260} \cdots.$$

(68)
y_2(x) = \int_{0}^{x} Q(x) A_1(x) \, dx = -\int_{0}^{x} e^{-x} A_1 \, dx = 8x^2 + \frac{8x^3}{3} + \frac{2x^4}{3} + \frac{2x^5}{15} + \frac{x^6}{45} + \frac{x^7}{315} \ldots, \quad (69)

y_3(x) = \int_{0}^{x} Q(x) A_2(x) \, dx = -\int_{0}^{x} e^{-x} A_2 \, dx = -16x^3 - \frac{8x^4}{3} - \frac{4x^5}{5} - \frac{4x^6}{45} - \frac{2x^7}{105} \ldots, \quad (70)

y_4(x) = \int_{0}^{x} Q(x) A_3(x) \, dx = -\int_{0}^{x} e^{-x} A_3 \, dx = 32x^4 + \frac{64x^6}{45} - \frac{16x^7}{315} \ldots, \quad (71)

y(x) \approx y_0(x) + y_1(x) + y_2(x) + y_3(x) + y_4(x) = 2 - 2x + 7x^2 - \frac{41x^3}{3} + \frac{359x^4}{12} \ldots, \quad (72)

From the exact solution by series expansion it is easy to obtain

\[ y(x) = e^x \left( 1 - \frac{3}{1 - 4e^{3x}} \right) = 2 - 2x + 7x^2 - \frac{41x^3}{3} + \frac{359x^4}{12} \ldots. \quad (73) \]

Clearly, the solution (72) obtained by the Adomian decomposition method is identical to the exact solution (73).

3.4. ABEL DIFFERENTIAL EQUATION \( \frac{dy}{dx} = M(x) + S(x) y + R(x)y^2 + T(x)y^3 \)

The first kind Abel differential equation takes the form (Kamke, 1959)

\( \frac{dy}{dx} = M(x) + S(x)y + R(x)y^2 + T(x)y^3. \quad (74) \)

Integrating Eq. (74) yields the relation

\[ y(x) = y(0) + \int_{0}^{x} M(x) \, dx + \int_{0}^{x} \left[ S(x)y + R(x)y^2 + T(x)y^3 \right] \, dx. \quad (75) \]

Inserting \( y(x) = \sum_{n=0}^{\infty} y_n(x), y^2 = \sum_{n=0}^{\infty} A_n(x) \) and \( y^3 = \sum_{n=0}^{\infty} B_n(x) \) into Eq. (75) gives the relation

\[ \sum_{n=0}^{\infty} y_n(x) = y(0) + \int_{0}^{x} M(x) \, dx + \int_{0}^{x} \left[ S(x) \sum_{n=0}^{\infty} y_n(x) + R(x) \sum_{n=0}^{\infty} A_n(x) + T(x) \sum_{n=0}^{\infty} B_n(x) \right] \, dx. \quad (76) \]

Then we have
\[ y_0 (x) = y (0) + \int_0^x M (x) \, dx, \quad (77) \]

\[ y_{k+1} (x) = \int_0^x \left[ S (x) y_k (x) + R (x) A_k (x) + T (x) B_k (x) \right] \, dx. \quad (78) \]

From Eqs. (77) and (78), we can obtain the semi-analytical solution of the Abel Eq. (74) as given by \( y (x) = \sum_{n=0}^{\infty} y_n (x) \).

3.4.1. Example: \( \frac{dy}{dx} = x + 3xy + 3xy^2 + xy^3 \)

We consider now a first kind Abel equation that has the form

\[ \frac{dy}{dx} = x + 3xy + 3xy^2 + xy^3 = x(1 + y)^3, \quad (79) \]

which should be solved with initial condition \( y (0) = 0 \), or \( y (0) = -2 \). Its general solution is given by

\[ y (x) = -1 \pm \frac{1}{\sqrt{1 - x^2}}. \quad (80) \]

Now \( M (x) = T (x) = x \) and \( S (x) = R (x) = 3x \), and a few Adomian polynomials of \( y^3 \) are

\[ B_0 = y_0^3, B_1 = 3y_1y_0^2, B_2 = 3y_2y_0^2 + 3y_1^2y_0, B_3 = 3y_3y_0^2 + 6y_1y_2y_0 + y_1^3. \quad (81) \]

With the help of Eqs. (77), (78), (50), and (81), by taking \( y (0) = 0 \), we obtain

\[ y_0 (x) = y (0) + \int_0^x M (x) \, dx = \frac{x^2}{2}, \quad (82) \]

\[ y_1 (x) = \int_0^x \left[ 3xy_0 (x) + 3xA_0 (x) + xB_0 (x) \right] \, dx = \frac{3x^4}{8} + \frac{x^6}{8} + \frac{x^8}{64}, \quad (83) \]

\[ y_2 (x) = \int_0^x \left[ 3xy_1 (x) + 3xA_1 (x) + xB_1 (x) \right] \, dx = \frac{3x^6}{16} + \frac{3x^8}{16} + \frac{9x^{10}}{128} + \frac{3x^{12}}{256} + \frac{3x^{14}}{3584}, \quad (84) \]

\[ y_3 (x) = \int_0^x \left[ 3xy_2 (x) + 3xA_2 (x) + xB_2 (x) \right] \, dx = \frac{9x^8}{128} + \frac{99x^{10}}{640} + \frac{15x^{12}}{128} \ldots, \quad (85) \]

\[ y_4 (x) = \int_0^x \left[ 3xy_3 (x) + 3xA_3 (x) + xB_3 (x) \right] \, dx = \frac{27x^{10}}{1280} + \frac{117x^{12}}{1280} + \frac{2169x^{14}}{17920} \ldots, \quad (86) \]

\[ y (x) \approx y_0 (x) + y_1 (x) + y_2 (x) + y_3 (x) + y_4 (x) = \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16} + \frac{35x^8}{128} + \frac{63x^{10}}{256} \ldots \quad (87) \]
From the exact solution it is easy to obtain
\[ y(x) = -1 + \frac{1}{\sqrt{1 - x^2}} = \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16} + \frac{35x^8}{128} + \frac{63x^{10}}{256} + \ldots \] (88)

It immediately follows that the solution (87) obtained by the Adomian Decomposition Method is identical to the exact solution (88).

4. SOLVING SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS VIA ADOMIAN DECOMPOSITION METHOD

Consider a second order non-linear differential equation that takes the form
\[ \frac{d^2 y}{dx^2} + f(x) \frac{dy}{dx} + s(x)y + g(x)y^n = k(x), \] (89)

and which must be solved together with the initial conditions \( y(0) \) and \( y'(0) \), respectively, where \( f(x), s(x), g(x) \) and \( k(x) \) are arbitrary function of \( x \), and \( n \) is a constant. We define the integral operator \( L^{-1} \) as
\[ L^{-1}(\cdot) = \int_0^x e^{-\int f(x)dx} \int_0^x e^{\int f(x)dx} \cdot dxdx. \] (90)

We consider first the action of the integral operator \( L^{-1} \) on the first two terms of the equation, which gives
\[
\begin{align*}
L^{-1} \left[ \frac{d^2 y}{dx^2} + f(x) \frac{dy}{dx} \right] &= \int_0^x e^{-\int f(x)dx} \int_0^x e^{\int f(x)dx} \left[ \frac{d^2 y}{dx^2} + f(x) \frac{dy}{dx} \right] dxdx \\
&= \int_0^x e^{-\int f(x)dx} \left( \int_0^x e^{\int f(x)dx} dy' + \int_0^x e^{\int f(x)dx} f y' dx \right) dxdx \\
&= \int_0^x e^{-\int f(x)dx} \left[ e^{\int f(x)dx} y' \right]_0^x dx \\
&= \int_0^x y' dx - \left[ e^{\int f(x)dx} \right]_0^x y'(0) \int_0^x e^{-\int f(x)dx} dx \\
&= y(x) - y'(0) - y'(0) \int_0^x e^{-\int f(x)dx} dx.
\end{align*}
\] (91)

Then we have
\[
L^{-1} \left[ \frac{d^2 y}{dx^2} + f(x) \frac{dy}{dx} \right] = L^{-1} \left[ k(x) - s(x)y - g(x)y^n \right], \] (92)

\[ y(x) = \phi(x) + \int_0^x e^{-\int f(x)dx} \left\{ \int_0^x e^{\int f(x)dx} \left[ k(x) - s(x)y - g(x)y^n \right] dx \right\} dx, \] (93)
where we have denoted $\phi(x)$ as

$$\phi(x) = y(0) + y'(0) \left[ e^{\int f(x) \, dx} \right]_{x=0}^{x} e^{-\int f(x) \, dx} \, dx.$$  \hspace{1cm} (94)

Hence we obtain

$$\sum_{n=0}^{\infty} y_n(x) = \phi(x) + \int_{0}^{x} e^{-\int f(x) \, dx} \left[ \int_{0}^{x} e^{\int f(x) \, dx} k(x) \, dx \right] \, dx - \int_{0}^{x} e^{-\int f(x) \, dx} \left\{ \int_{0}^{x} e^{\int f(x) \, dx} \left[ s(x) \sum_{n=0}^{\infty} y_n(x) + g(x) \sum_{n=0}^{\infty} A_n(x) \right] \, dx \right\} \, dx.$$  \hspace{1cm} (95)

Then for the solution of the second order nonlinear differential equation we have

$$y_0(x) = \phi(x) + \int_{0}^{x} e^{-\int f(x) \, dx} \left[ \int_{0}^{x} e^{\int f(x) \, dx} k(x) \, dx \right] \, dx,$$  \hspace{1cm} (96)

$$y_{k+1}(x) = - \int_{0}^{x} e^{-\int f(x) \, dx} \left[ \int_{0}^{x} e^{\int f(x) \, dx} [s(x) y_k(x) + g(x) A_k(x)] \, dx \right] \, dx.$$  \hspace{1cm} (97)

From Eqs. (41)-(42), and (96), (97), we obtain the semi-analytical solution of Eq. (89), given by $y = \sum_{n=0}^{\infty} y_n(x)$.

4.1. EXAMPLE: \[ \frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 3y = 3 \]

As an example of the application of the ADM we consider a particular second order differential equation that takes the form

$$\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 3y = 3,$$  \hspace{1cm} (98)

which must be considered together with the initial conditions $y(0) = 1$ and $y'(0) = 2$.

The general solution of the equation is given by

$$y(x) = -e^{-3x} + e^{-x} + 1,$$  \hspace{1cm} (99)

From the equation we easily obtain $e^{\int f(x) \, dx} = e^{4x} = e^{4x}$, $f(x) = 4$, $g(x) = 0$, $k(x) = 3$ and $s(x) = 3$, respectively. Then we have

$$\sum_{n=0}^{\infty} y_n(x) = \phi(x) + \int_{0}^{x} e^{-\int f(x) \, dx} \left[ \int_{0}^{x} e^{\int f(x) \, dx} k(x) \, dx \right] \, dx - \int_{0}^{x} e^{-\int f(x) \, dx} \left[ \int_{0}^{x} e^{\int f(x) \, dx} s(x) \sum_{n=0}^{\infty} y_n(x) \, dx \right] \, dx.$$  \hspace{1cm} (100)

We rewrite Eq. (100) in the recursive forms
\[ y_0 (x) = y(0) + y'(0) \left[ \int_0^x e^{f(x)} \, dx \right]_{x=0}^x - f(x) \, dx + \int_0^x e^{-f(x)} \, dx \int_0^x e^{f(x)} \, dx k(x) \, dx \] (101)

\[ = \frac{21}{16} + \frac{3}{4} x - \frac{5}{16} e^{-4x}, \] (102)

and

\[ y_{k+1} (x) = -3 \int_0^x e^{-4x} \left[ \int_0^x e^{4x} y_k (x) \, dx \right] \, dx. \] (103)

Hence we obtain

\[ y_1 (x) = -3 \int_0^x e^{-4x} \left[ \int_0^x e^{4x} y_0 (x) \, dx \right] \, dx \] (104)

\[ = -3 \int_0^x e^{-4x} \left[ \int_0^x e^{4x} \, dx \right] \, dx \] (105)

\[ = -3 \int_0^x e^{-4x} \left[ \int_0^x e^{4x} \, dx \right] \, dx \] (106)

\[ y_2 (x) = -3 \int_0^x e^{-4x} \left[ \int_0^x e^{4x} y_1 (x) \, dx \right] \, dx \] (107)

\[ = -3 \int_0^x e^{-4x} \left[ \int_0^x e^{4x} y_1 (x) \, dx \right] \, dx \] (108)

\[ y_3 (x) = -3 \int_0^x e^{-4x} \left[ \int_0^x e^{4x} y_2 (x) \, dx \right] \, dx \] (109)

\[ = -3 \int_0^x e^{-4x} \left[ \int_0^x e^{4x} y_2 (x) \, dx \right] \, dx \] (110)

\[ y_4 (x) = -3 \int_0^x e^{-4x} \left[ \int_0^x e^{4x} y_3 (x) \, dx \right] \, dx \] (111)

\[ = -3 \int_0^x e^{-4x} \left[ \int_0^x e^{4x} y_3 (x) \, dx \right] \, dx \] (112)

The semi-analytical solution of Eq. (98) is given by

\[ y(x) \approx y_0 (x) + y_1 (x) + y_2 (x) + y_3 (x) + \ldots \] (113)

\[ = 1 + 2x - 4x^2 + \frac{13x^3}{3} + \frac{10x^4}{3} + \frac{121x^5}{60} - \frac{91x^6}{90} + \frac{1093x^7}{2520} + \ldots \] (114)

On the other hand from the exact solution it is easy to obtain

\[ y(x) = -e^{-3x} + e^{-x} + 1 = 1 + 2x - 4x^2 + \frac{13x^3}{3} - \frac{10x^4}{3} + \frac{121x^5}{60} - \frac{91x^6}{90} + \frac{1093x^7}{2520} + \ldots \] (115)
As one can easily see, the solution (113) of the second order differential Eq. (98) obtained by the Adomian decomposition method is identical to the exact solution (114).

4.2. EXAMPLE: \( \frac{d^2y}{dx^2} + \frac{dy}{dx} + e^x y + e^x y^2 = e^x \)

As a second example of the application of the ADM for solving a nonlinear differential equation we consider that the differential equation takes the form

\[
\frac{d^2y}{dx^2} + \frac{dy}{dx} + e^x y + e^x y^2 = e^x,
\]

and it must be solved together with the initial condition \( y(0) = 1 \) and \( y'(0) = 2 \), respectively. From the equation we obtain easily \( e^\int f(x)dx = e^\int dx = e^x, f(x) = 1, g(x) = k(x) = s(x) = e^x \). Hence we immediately find

\[
y_0(x) = y(0) + y'(0) \left[ e^\int f(x)dx \right]_{x=0}^{x} + \int_{0}^{x} e^{-\int f(x)dx} \left[ \int_{0}^{x} e^{\int f(x)dx} k(x) dx \right] dx = 1 + 2 \int_{0}^{x} e^{-x} dx + \int_{0}^{x} e^{-x} \left[ \int_{0}^{x} e^{2x} dx \right] dx = 2 - \frac{3}{2} e^{-x} + \frac{1}{2} e^x.
\]

and

\[
y_{k+1}(x) = - \int_{0}^{x} e^{-\int f(x)dx} \left[ \int_{0}^{x} e^{\int f(x)dx} [s(x) y_k(x) + g(x) A_k(x)] dx \right] dx,
\]

\[
y_{k+1}(x) = - \int_{0}^{x} e^{-x} \left[ \int_{0}^{x} e^{2x} [y_k(x) + A_k(x)] dx \right] dx,
\]

From Eqs. (50), (117), and (118), we obtain

\[
y_1(x) = - \int_{0}^{x} e^{-x} \left[ \int_{0}^{x} e^{2x} [y_0(x) + A_0(x)] dx \right] dx = -x^2 - x^3 - \frac{13x^4}{24} - \frac{2x^5}{15} - \frac{17x^6}{240} - \frac{23x^7}{1260} - \frac{257x^8}{40320} - \frac{17x^9}{10080} - \frac{67x^{10}}{145152} \ldots
\]

\[
y_2(x) = - \int_{0}^{x} e^{-x} \left[ \int_{0}^{x} e^{2x} [y_1(x) + A_1(x)] dx \right] dx = \frac{x^4}{4} + \frac{9x^5}{20} + \frac{29x^6}{80} + \frac{521x^7}{2520} + \frac{233x^8}{2520} + \frac{155x^9}{4032} + \frac{17293x^{10}}{1209600} \ldots
\]
\[ y_3(x) = - \int_0^x e^{-x} \left[ \int_0^x e^{2x} \left( y_2(x) + A_2(x) \right) dx \right] dx \]
\[ = - \frac{7x^6}{120} - \frac{23x^7}{168} - \frac{719x^8}{4480} - \frac{3797x^9}{30240} - \frac{5143x^{10}}{67200} \ldots, \tag{123} \]

\[ y_4(x) = - \int_0^x e^{-x} \left[ \int_0^x e^{2x} \left( y_3(x) + A_3(x) \right) dx \right] dx = \]
\[ = \frac{27x^8}{2240} + \frac{2203x^9}{60480} + \frac{3149x^{10}}{57600} \ldots. \tag{124} \]

Hence the semi-analytical solution of Eq. (115) is given by
\[ y(x) \approx y_0(x) + y_1(x) + y_2(x) + y_3(x) + \ldots \]
\[ = 1 + 2x - \frac{3x^2}{2} - \frac{2x^3}{3} - \frac{x^4}{3} + \frac{167x^6}{720} + \frac{131x^7}{2520} - \frac{503x^8}{8064} - \frac{9503x^9}{181440} - \frac{7283x^{10}}{907200} \ldots. \tag{125} \]

5. THE FIFTH ORDER ORDINARY DIFFERENTIAL EQUATION VIA THE ADOMIAN DECOMPOSITION METHOD

Consider the following fifth order ordinary differential equation, which takes the form
\[ \frac{d^5 y}{dx^5} + a_4 \frac{d^4 y}{dx^4} + a_3 \frac{d^3 y}{dx^3} + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0, \tag{126} \]
where \( a_i, \ i = 0, 1, 2, 3, 4 \) are constants. Eq. (126) should be integrated with the initial conditions \( y(0) = y_0, \ y'(0) = y_{01}, \ y''(0) = y_{02}, \ y'''(0) = y_{03}, \) and \( y^{(iv)}(0) = y_{04} \), respectively. Now applying the 5 fold integral operator \( L^{-1} \), defined as
\[ L^{-1}(.) = \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x (.) dx dx dx dx dx, \tag{127} \]
to Eq. (126), yields the relation
\[ \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x \left( \frac{d^5 y}{dx^5} + a_4 \frac{d^4 y}{dx^4} + a_3 \frac{d^3 y}{dx^3} + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y \right) dx^5 = 0. \tag{128} \]
From Eq. (129), we obtain

\[ y(x) = y(0) + \left[ y'(0) + a_4 y(0) \right] x + \left[ y''(0) + a_4 y'(0) + a_3 y(0) \right] \frac{x^2}{2} + \]

\[ \left[ y'''(0) + a_4 y''(0) + a_3 y'(0) + a_2 y(0) \right] \frac{x^3}{6} + \]

\[ \left[ y'''(0) + a_4 y''(0) + a_3 y'(0) + a_2 y(0) + a_1 y(0) \right] \frac{x^4}{24} - \]

\[ a_4 \int_0^x y \, dx - a_3 \int_0^x \int_0^x y \, dx \, dx - \]

\[ a_2 \int_0^x \int_0^x \int_0^x y \, dx \, dx \, dx - a_1 \int_0^x \int_0^x \int_0^x \int_0^x y \, dx \, dx \, dx \, dx - \]

\[ a_0 \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x y \, dx \, dx \, dx \, dx \, dx. \] (130)

Substituting \( y(x) = \sum_{n=0}^{\infty} y_n(x) \) into Eq. (130) yields

\[ \sum_{n=0}^{\infty} y_n(x) = y(0) + \left[ y'(0) + a_4 y(0) \right] x + \left[ y''(0) + a_4 y'(0) + a_3 y(0) \right] \frac{x^2}{2} + \]

\[ \left[ y'''(0) + a_4 y''(0) + a_3 y'(0) + a_2 y(0) \right] \frac{x^3}{6} + \]

\[ \left[ y'''(0) + a_4 y''(0) + a_3 y'(0) + a_2 y(0) + a_1 y(0) \right] \frac{x^4}{24} - \]

\[ a_4 \int_0^x \sum_{n=0}^{\infty} y_n(x) \, dx - a_3 \int_0^x \int_0^x \sum_{n=0}^{\infty} y_n(x) \, dx \, dx - \]

\[ a_2 \int_0^x \int_0^x \int_0^x \sum_{n=0}^{\infty} y_n(x) \, dx \, dx \, dx - \]

\[ a_1 \int_0^x \int_0^x \int_0^x \int_0^x \sum_{n=0}^{\infty} y_n(x) \, dx \, dx \, dx \, dx - \]

\[ a_0 \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x \sum_{n=0}^{\infty} y_n(x) \, dx \, dx \, dx \, dx \, dx. \] (131)

We rewrite Eq. (131) in the recursive forms

\[ y_0(x) = y(0) + \left[ y'(0) + a_4 y(0) \right] x + \left[ y''(0) + a_4 y'(0) + a_3 y(0) \right] \frac{x^2}{2} + \]

\[ \left[ y'''(0) + a_4 y''(0) + a_3 y'(0) + a_2 y(0) \right] \frac{x^3}{6} + \]

\[ \left[ y'''(0) + a_4 y''(0) + a_3 y'(0) + a_2 y(0) + a_1 y(0) \right] \frac{x^4}{24}, \] (132)
and
\[ y_{k+1}(x) = -a_4 \int_0^x y_k(x) \, dx - a_3 \int_0^x \int_0^x y_k(x) \, dx \, dx - a_2 \int_0^x \int_0^x \int_0^x y_k(x) \, dx \, dx \, dx - a_1 \int_0^x \int_0^x \int_0^x \int_0^x y_k(x) \, dx \, dx \, dx \, dx - a_0 \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x y_k(x) \, dx \, dx \, dx \, dx. \] (133)

From Eqs. (132) and (133), we can obtain the semi-analytical solution of Eq. (127), given by
\[ y(x) = \sum_{n=0}^{\infty} y_n(x). \]
For the solution of a particular third-order ordinary differential equation see Pue-on and Viriyapong (2012).

5.1. EXAMPLE: \[ \frac{d^5 y}{dx^5} - 3 \frac{d^4 y}{dx^4} - 5 \frac{d^3 y}{dx^3} + 15 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} - 12 y = 0 \] (134)
In the following we consider a particular fifth order ordinary differential equation that takes the form
\[ \frac{d^5 y}{dx^5} - 3 \frac{d^4 y}{dx^4} - 5 \frac{d^3 y}{dx^3} + 15 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} - 12 y = 0, \] (134)
and which should be solved together with the initial conditions \( y(0) = 1, y'(0) = -1, y''(0) = 2, y'''(0) = -2, y''''(0) = 3. \) The coefficients \( a_i \) of the equation are given by \( a_4 = -3, a_3 = -5, a_2 = 15, a_1 = 4 \) and \( a_0 = -12 \), respectively. The general solution of the equation is given by
\[ y(x) = -\frac{1}{4} e^x + \frac{19}{24} e^{-x} + \frac{1}{3} e^{2x} + \frac{1}{5} e^{-2x} - \frac{3}{40} e^{3x}. \] (15)
By applying the ADM we have
\[ y_0(x) = 1 - 4x + 2x^3 - \frac{x^4}{2}, \] (136)
\[ y_{k+1}(x) = 3 \int_0^x y_k(x) \, dx + 5 \int_0^x \int_0^x y_k(x) \, dx \, dx - 15 \int_0^x \int_0^x \int_0^x y_k(x) \, dx \, dx \, dx - 4 \int_0^x \int_0^x \int_0^x \int_0^x y_k(x) \, dx \, dx \, dx \, dx + 12 \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x y_k(x) \, dx \, dx \, dx \, dx \, dx. \] (137)
From Eqs. (136) and (137), we obtain

\[ y_1(x) = 3x - \frac{7x^2}{2} - \frac{35x^3}{6} + \frac{23x^4}{6} + \frac{13x^5}{30} - \frac{2x^6}{5} \ldots, \] (138)

\[ y_2(x) = \frac{9x^2}{2} - x^3 - \frac{185x^4}{24} + \frac{97x^5}{60} + \frac{241x^6}{144} \ldots, \] (139)

\[ y_3(x) = \frac{9x^3}{2} + \frac{9x^4}{8} - 6x^5 - \frac{289x^6}{720} \ldots, \] (140)

\[ y_4(x) = \frac{27x^4}{8} + \frac{9x^5}{5} - \frac{27x^6}{8} \ldots. \] (141)

Thus we obtain the ADM solution of the equation as

\[ y(x) \approx y_0(x) + y_1(x) + y_2(x) + y_3(x) + y_4(x) = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{8} \ldots \] (142)

On the other hand from the exact solution it is easy to obtain

\[ y(x) = -\frac{1}{4}e^x + \frac{19}{24}e^{-x} + \frac{1}{3}e^{2x} + \frac{1}{5}e^{-2x} - \frac{3}{40}e^{3x} = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{8} \ldots, \] (143)

Hence, the solution (142) of the fifth order differential Eq. (134), obtained by the Adomian decomposition method, is identical to the exact solution (143).

6. SOLVING PARTIAL DIFFERENTIAL EQUATIONS VIA ADM: THE FISHER-KOLMOGOROV EQUATION

The three dimensional Fisher-Kolmogorov equation (Fisher, 1937; Kolmogorov et al., 1937; Harko and Mak, 2015a), has many important applications in physics and biology. In particular, it can be used to describe the growth of glioblastoma (Harko and Mak, 2015b). The Fisher-Kolmogorov equation is given by

\[ \frac{\partial c(t,x,y,z)}{\partial t} = D \Delta c(t,x,y,z) + ac(t,x,y,z) \left[ 1 - \frac{c(t,x,y,z)}{N} \right], \] (144)

where

\[ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \] (145)

and \( a, N \) and \( D \) are constants. Eq. (144) must be considered together with the initial condition \( c(0,x,y,z) = c_0(x,y,z) \). From the point of view of the ADM the Fisher-Kolmogorov equation was studied in Wazwaz and Gorguis (2004) and Bhalekar and Patade (2016), respectively. In order to apply the ADM method we rewrite Eq. (144) as

\[ L_t c = D \Delta c + F(c), \] (146)
where \( L_t = \frac{\partial}{\partial t} \) and \( F(c) = ac\left[1 - \frac{c}{N}\right] \). Now applying the inverse operator \( L_t^{-1} \) defined as \( L_t^{-1}(\cdot) = \int_0^t (\cdot) dt \) to Eq. (146), the general solution of Eq. (146) can be obtained formally as

\[
c(t,x,y,z) = c_0(x,y,z) + DL_t^{-1}\Delta c(t,x,y,x) + L_t^{-1}F(c).
\] (147)

According to the Adomian Decomposition Method we look for series solutions of Eq. (147) of the form

\[
c(t,x,y,z) = \sum_{n=0}^{\infty} c_n(t,x,y,z), F(c) = \sum_{n=0}^{\infty} A_n(t,x,y,z),
\] (148)

where the Adomian polynomials \( A_n(t,x,y,z) \) are defined as

\[
A_n(t,x,y,z) = \frac{1}{n!}\left[\frac{d^n}{dt^n}F(c_\lambda)\right]_{\lambda=0},
\] (149)

where \( c_\lambda = \sum_{i=0}^{\infty} \lambda^i c_i \). The first few Adomian polynomials are given by

\[
A_0 = F(c_0) = ac_0\left(1 - \frac{c_0}{N}\right),
\] (150)

\[
A_1 = c_1F'(c_0) = ac_1\left(1 - \frac{2c_0}{N}\right),
\] (151)

\[
A_2 = c_2F'(c_0) + \frac{1}{2}c_1^2F''(c_0) = ac_2\left(1 - \frac{2c_0}{N}\right) - \frac{a}{N}c_1^2,
\] (152)

\[
A_3 = c_3F'(c_0) + c_1c_2F''(c_0) + \frac{1}{6}c_1^3F'''(c_0)
\] (153)

\[
= ac_3\left(1 - \frac{2c_0}{N}\right) - \frac{2a}{N}c_1c_2.
\] (154)

Therefore, after substituting Eq. (148) into Eq. (147), the latter becomes

\[
\sum_{n=0}^{\infty} c_n(t,x,y,z) = c_0(x,y,z) + DL_t^{-1}\Delta c(t,x,y,x) + L_t^{-1}F(c).
\] (155)

We rewrite Eq. (155) as

\[
c_0(t,x,y,z) = c_0(x,y,z),
\] (156)

\[
c_{k+1}(t,x,y,z) = DL_t^{-1}\Delta [c_k(t,x,y,z)] + L_t^{-1}[A_k(t,x,y,z)].
\] (157)
From Eq. (157), we obtain

\[ c_1 = \int_0^t (D\Delta c_0 + A_0) dt, \quad c_2 = \int_0^t (D\Delta c_1 + A_1) dt, \quad c_3 = \int_0^t (D\Delta c_2 + A_2) dt, \quad (158) \]

\[ \ldots \]

\[ c_{m+1} = \int_0^t (D\Delta c_m + A_m) dt. \quad (159) \]

where \( k = 0, 1, 2 \ldots m \). The approximate solution of the Fisher-Kolmogorov equation can be written as

\[ c(t, x, y, z) = \sum_{i=0}^{m+1} c_i(t, x, y, z). \quad (160) \]

6.1. EXAMPLE: FISHER-KOLMOGOROV EQUATION WITH THE INITIAL CONDITION

\[ c(0, x, y, z) = x^2 + y^2 + z^2 \]

As an example of the application of the Adomian Decomposition Method, we consider the case in which Eq. (144) should be solved together the initial condition \( c(0, x, y, z) = c_0(x, y, z) = x^2 + y^2 + z^2 \). Then we obtain

\[ c_0(x, y, z) = x^2 + y^2 + z^2, \quad (161) \]

\[ c_1(t, x, y, z) = \int_0^t (D\Delta c_0 + A_0) dt = t \left[ 6D + a \left( x^2 + y^2 + z^2 \right) \times \right] \left[ 1 - \frac{x^2 + y^2 + z^2}{N} \right], \quad (162) \]

\[ c_2(t, x, y, z) = \int_0^t (D\Delta c_1 + A_1) dt \quad (163) \]

\[ = \frac{at^2}{2N^2} \left\{ 4DN \left[ 3N - 8 \left( x^2 + y^2 + z^2 \right) \right] + a \left( x^2 + y^2 + z^2 \right) \times \right\} \left[ N^2 - 3N \left( x^2 + y^2 + z^2 \right) + 2 \left( x^2 + y^2 + z^2 \right)^2 \right], \quad (164) \]

\[ c_3(t, x, y, z) = \int_0^t (D\Delta c_2 + A_2) dt, \quad c_4(t, x, y, z) = \int_0^t (D\Delta c_3 + A_3) dt, \quad (165) \]

\[ c(t, x, y, z) \approx c_0 + c_1 + c_2 + c_3 + c_4. \quad (166) \]
Hence it follows that the Adomian method is also a very powerful approach for solving partial differential equations. The solutions converge fast, thus saving a lot of computing time.

7. THE LAPLACE-ADOMIAN DECOMPOSITION METHOD

A very powerful version of the Adomian Decomposition Method is represented by the so-called Laplace-Adomian Decomposition Method (LADM) [Khuri, 2001; Wazwaz, 2010; Manafianheris, 2012; Hamoud and Ghadle, 2017]. We will introduce this method by considering the particular example of a second order nonlinear differential equation of the form

$$\frac{d^2y}{dx^2} + \omega^2 y + b^2 + f(y) = 0,$$  \hspace{1cm} (168)

where $\omega$ and $b$ are arbitrary constants, while $f(y)$ is a nonlinear arbitrary function of the dependent variable $y$. We will consider Eq. (168) together with the initial conditions $y(0) = y_0 = a$, and $y'(0) = 0$, respectively.

We define the Laplace transform operator $L$ of an arbitrary function $f(x)$, as

$$L[f(x)](s) = \int_0^{\infty} f(x)e^{-sx}dx.$$  \hspace{1cm} (169)

By using the basic properties of the Laplace transform we straightforwardly obtain

$$(s^2 + \omega^2) L[y] - sy(0) - y'(0) + \frac{b^2}{s} + L[f(y)] = 0.$$  \hspace{1cm} (170)

After explicitly taking into account the initial conditions for our problem we obtain the relation

$$L[y] = \frac{as}{s^2 + \omega^2} - \frac{b^2}{s(s^2 + \omega^2)} - \frac{1}{s^2 + \omega^2} L[f(y)].$$  \hspace{1cm} (171)

We assume now that the solution of Eq. (168) can be represented in the form of an infinite series given by

$$y(x) = \sum_{n=0}^{\infty} y_n(x),$$  \hspace{1cm} (172)

where each term $y_n(x)$ can be calculated recursively. With respect to the nonlinear
operator \( f(y) \), we assume that it can be decomposed according to
\[
f(y) = \sum_{n=0}^{\infty} A_n, \tag{173}
\]
where the functions \( A_n \) are the Adomian polynomials, which can be obtained from the general algorithm (Adomian and Rach, 1983; Adomian, 1994)
\[
A_n = \frac{1}{n!} \frac{d^n}{d\epsilon^n} f \left( \sum_{i=0}^{\infty} \epsilon^i y_i \right) \bigg|_{\epsilon=0}. \tag{174}
\]

The first few Adomian polynomials are given by,
\[
A_0 = f(y_0), A_1 = y_1 f'(y_0), A_2 = y_2 f'(y_0) + \frac{1}{2} y_1^2 f''(y_0), \tag{175}
\]
\[
A_3 = y_3 f'(y_0) + y_1 y_2 f''(y_0) + \frac{1}{6} y_1^3 f'''(y_0), \tag{176}
\]
\[
A_4 = y_4 f'(y_0) + \left[ \frac{1}{2!} y_2^2 + y_1 y_3 \right] f''(y_0) + \frac{1}{2!} y_1 y_2^2 f'''(y_0) + \frac{1}{4!} y_1^4 f^{(iv)}(y_0). \tag{177}
\]

After substituting Eqs. (172) and (173) into Eq. (171) we find
\[
\mathcal{L}_x \left[ \sum_{n=0}^{\infty} y_n(x) \right] = \frac{a s}{s^2 + \omega^2} - \frac{b^2}{s (s^2 + \omega^2)} - \frac{1}{s^2 + \omega^2} \mathcal{L}_x \left[ \sum_{n=0}^{\infty} A_n \right]. \tag{178}
\]

By matching both sides of Eq. (178) gives an iterative algorithm for obtaining the power series solution of Eq. (168), which can be formulated as
\[
\mathcal{L}_x [y_0] = \frac{a s}{s^2 + \omega^2} - \frac{b^2}{s (s^2 + \omega^2)}, \tag{179}
\]
\[
\mathcal{L}_x [y_1] = - \frac{1}{s^2 + \omega^2} \mathcal{L}_x [A_0], \tag{180}
\]
\[
\mathcal{L}_x [y_2] = - \frac{1}{s^2 + \omega^2} \mathcal{L}_x [A_1], \tag{181}
\]
\[...
\]
\[
\mathcal{L}_x [y_{k+1}] = - \frac{1}{s^2 + \omega^2} \mathcal{L}_x [A_k]. \tag{182}
\]

To obtain the value of \( y_0 \) we apply the inverse Laplace transform to Eq. (179). After substituting \( y_0 \) into the first of Eqs. (175) we find the first Adomian polynomial \( A_0 \). The obtained expression of \( A_0 \) is then substituted into Eq. (180), which allows to compute the Laplace transforms of the quantities of its right-hand. Then the further application of the inverse Laplace transform gives the functional expressions of \( y_1 \). All the other terms \( y_2, y_3, \ldots, y_{k+1}, \ldots \) of the series solution can be similarly calculated recursively by using a step by step procedure.
7.1. EXAMPLE: \( f(y) = \sum_{l=0}^{m} a_{l+2} y^{l+2} \)

We will illustrate the applications of the Laplace-Adomian Decomposition Method by considering the case of a second order nonlinear differential equation having the form [Mak et al., 2018a]

\[
\frac{d^2 y}{dx^2} + \omega^2 y + b^2 + \sum_{l=0}^{m} a_{l+2} y^{l+2} = 0,
\]

(183)

where \( \omega, b, \) and \( a_{l+2}, l = 0, \ldots, m \) are arbitrary constants. As usual, we consider Eq. (183) together with the set of initial conditions \( y(0) = y_0 = a, \) and \( y'(0) = 0, \) respectively. We investigate Eq. (183) by using the Laplace-Adomian Decomposition Method. Hence, as a first step, we apply the Laplace transform to Eq. (183), thus finding

\[
L_x \left( \frac{d^2 y}{dx^2} \right) + \omega^2 L_x(y) + b^2 L_x(1) + \sum_{l=0}^{m} a_{l+2} L_x \left[ y^{l+2} \right] = 0.
\]

(184)

Next, by the use of the properties of the Laplace transform, we immediately obtain

\[
L_x(y) \left( s^2 + \omega^2 \right) = sy(0) + y'(0) - \frac{b^2}{s} - \sum_{l=0}^{m} a_{l+2} L_x \left[ y^{l+2} \right] = 0,
\]

(185)

and thus

\[
L_x(y) = \frac{sy(0) + y'(0)}{s^2 + \omega^2} - \frac{b^2}{s(s^2 + \omega^2)} - \frac{1}{s^2 + \omega^2} \sum_{l=0}^{m} a_{l+2} L_x \left[ y^{l+2} \right].
\]

(186)

Hence, from Eq. (186) \( y(x) \) in obtained in the form

\[
y(x) = L_x^{-1} \left[ \frac{sy(0) + y'(0)}{s^2 + \omega^2} - \frac{b^2}{s(s^2 + \omega^2)} - \frac{1}{s^2 + \omega^2} \sum_{l=0}^{m} a_{l+2} L_x \left[ y^{l+2} \right] \right].
\]

(187)

We assume now that the solution \( y(x) \) of Eq. (185) can be represented as \( y(x) = \sum_{n=0}^{\infty} y_n(x) \). Moreover, we decompose the nonlinear terms according to

\[
y^{l+2} = \sum_{n=0}^{\infty} A_{n,l+2}(x),
\]

(188)
where $A_{n,l+2}$ are the Adomian polynomials determining $y^{l+2}$. Then we obtain

$$\sum_{n=0}^{\infty} y_n(x) = L_x^{-1} \left[ \frac{sy(0) + y'(0)}{s^2 + \omega^2} - \frac{b^2}{s(s^2 + \omega^2)} \right] - L_x^{-1} \left\{ \frac{1}{s^2 + \omega^2} \sum_{l=0}^{m} a_{l+2} L \left( \sum_{n=0}^{\infty} A_{n,l+2}(x) \right) \right\}. \quad (189)$$

We reformulate now Eq. (189) in the form

$$y_0(x) + \sum_{n=0}^{\infty} y_{n+1}(x) = L_x^{-1} \left[ \frac{sy(0) + y'(0)}{s^2 + \omega^2} - \frac{b^2}{s(s^2 + \omega^2)} \right] - \sum_{n=0}^{\infty} L_x^{-1} \left\{ \frac{1}{s^2 + \omega^2} \sum_{l=0}^{m} a_{l+2} L_x [A_{n,l+2}(x)] \right\}. \quad (190)$$

Hence from Eq. (190) we obtain the set of recursive relations

$$y_0(x) = L_x^{-1} \left[ \frac{sy(0) + y'(0)}{s^2 + \omega^2} - \frac{b^2}{s(s^2 + \omega^2)} \right], \quad (191)$$

$$y_{k+1}(x) = -L_x^{-1} \left\{ \frac{1}{s^2 + \omega^2} \sum_{l=0}^{m} a_{l+2} L_x [A_{k,l+2}(x)] \right\}. \quad (192)$$

A few Adomian polynomials for the function $y^{l+2}$ are given by

$$A_{0,l+2} = y_0^{l+2}, \quad A_{1,l+2} = (l+2)y_1y_0^{l+1}, \quad (193)$$

$$A_{2,l+2} = (l+2)y_2y_0^{l+1} + (l+1)(l+2)\frac{y_1^2}{2}y_0, \quad (194)$$

$$A_{3,l+2} = (l+2)y_3y_0^{l+1} + (l+1)(l+2)y_1y_2y_0^{l} + l(l+1)(l+2)\frac{y_1^3}{3!}y_0^{l-1}. \quad (195)$$

We find the first order approximation of the solution by taking $k = 0$, thus obtaining

$$y_1(x) = -L_x^{-1} \left\{ \frac{L_x \left[ \sum_{l=0}^{m} a_{l+2} A_{0,l+2} \right]}{s^2 + \omega^2} \right\} = -L_x^{-1} \left\{ \frac{L_x \left[ a_2y_0^2 + a_3y_0^3 + a_4y_0^4 + ... \right]}{s^2 + \omega^2} \right\}. \quad (196)$$
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For $k = 1$ we find $y_2(x)$ as given by

\[ y_2(x) = -L_x^{-1} \left\{ \frac{L_x \left[ \sum_{l=0}^{m} a_{l+2} A_{1,l+2} \right]}{s^2 + \omega^2} \right\} = -L_x^{-1} \left\{ \frac{L_x \left( 2a_2 y_1 y_0 + 3a_3 y_1 y_0^2 + 4a_4 y_1 y_0^3 + \ldots \right)}{s^2 + \omega^2} \right\}. \] (197)

By fixing $k$ as $k = 2$ yields

\[ y_3(x) = -L_x^{-1} \left\{ \frac{L_x \left[ \sum_{l=0}^{m} a_{l+2} A_{2,l+2} \right]}{s^2 + \omega^2} \right\} = -L_x^{-1} \left\{ \frac{L_x \left[ a_2 \left( 2y_2 y_0 + y_1^2 \right) + 3a_3 \left( y_2 y_0^2 + y_1^2 y_0 \right) + a_4 \left( 4y_2 y_0^3 + 6y_1^2 y_0^2 \right) + \ldots \right]}{s^2 + \omega^2} \right\}. \] (198)

As a last case we take $k = 3$, and thus

\[ y_4(x) = -L_x^{-1} \left\{ \frac{L_x \left[ \sum_{l=0}^{m} a_{l+2} A_{3,l+2} \right]}{s^2 + \omega^2} \right\} = -L_x^{-1} \left\{ \frac{1}{s^2 + \omega^2} \left[ 2a_2 \left( y_3 y_0 + y_1 y_2 \right) + a_3 \left( 3y_3 y_0^2 + 6y_1 y_2 y_0 + y_1^3 \right) + a_4 \left( 4y_3 y_0^3 + 12y_1 y_2 y_0^2 + 4y_1^3 y_0 \right) + \ldots \right] \right\}. \] (199)

Hence the truncated power series solution of Eq. (183) is given by

\[ y(x) = \sum_{n=0}^{\infty} y_n(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + y_4(x) + \ldots. \] (200)

8. ASTRONOMICAL AND ASTROPHYSICAL APPLICATIONS

In the present Section we consider some astronomical and astrophysical applications of the ADM. In particular, we will consider the solutions of the Kepler equation via ADM, the solutions of the Lane-Emden equation, and the study of the motion of massive particles in the Schwarzschild geometry.

8.1. THE KEPLER EQUATION

In celestial mechanics, Kepler’s equation plays an essential role in the determination of the orbit of an object evolving under the action of a central force. The Kepler equation for the hyperbolic case is \( \text{[Eibaïd et al., 2017]} \)

\[ e \sin H(t) - H(t) = M(t), \] (201)

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where $e$ is the eccentricity of the orbit, $H(t)$ is the eccentric anomaly, $M(t) = \sqrt{\mu/a^3}(t - \tau)$ represents the mean anomaly, $\mu = GM$, while $a$ is the semi-major axis of the orbit. Moreover, the time interval for the passage through the closest point of approach to the focus of the orbit is denoted by $\tau$. The Kepler equation (201) can be transformed into the forms

$$ey(t) - \arcsinh y(t) = M,$$

under the assumption $y(t) = \sinh H(t)$, and

$$y(t) = \alpha + \beta \arcsinh y(t),$$

where $0 \leq \alpha = M/e < \infty$, and $0 \leq \beta = 1/e \leq 1$, respectively. In the Adomian Decomposition Method approach to the Kepler equation one assumes that $y$ and $\arcsinh y$ can be decomposed as $y(t) = \sum_{n=0}^{\infty} y_n(t)$, and $\arcsinh y = \sum_{n=0}^{\infty} A_n(t)$, where $A_n(t)$ are the Adomian polynomials corresponding to $\arcsinh y$. After substituting the series expansions into Eq. (203) one arrives to the following recursion relations,

$$y_0 = \alpha,$$

$$y_{n+1} = \beta A_n, n = 0, 1, 2, ...,$$

The Adomian polynomials for the function $\arcsinh y$ can be obtained as follows (Ebaid et al., 2017),

$$A_0 = \arcsinh y_0(t), A_1 = \frac{y_1}{(1 + y_0^2)^{1/2}}, A_2 = \frac{2(1 + y_0^2)y_2 - y_0y_1^2}{2(1 + y_0^2)^{3/2}},$$

$$A_3 = \frac{6(1 + y_0^2)y_3 - 6y_0(1 + y_0^2)y_1y_2 + (2y_0^2 - 1)y_1^3}{6(1 + y_0^2)^{5/2}}.$$  

Then we obtain the following solution of the Kepler equation $\Phi(t) = \sum_{i=0}^{n-1} y_i(t)$ (Ebaid et al., 2017),

$$\Phi_2(t) = \alpha + \beta \arcsinh \alpha,$$

$$\Phi_3(t) = \alpha + \beta \arcsinh \alpha + \frac{\beta^2 \arcsinh \alpha}{(1 + \alpha^2)^{12}},$$

$$\Phi_4(t) = \alpha + \beta \arcsinh \alpha + \frac{\beta^2 \arcsinh \alpha}{(1 + \alpha^2)^{12}} + \frac{2(1 + \alpha^2)^{1/2}\beta^3 \arcsinh \alpha - \alpha \beta^3 (\arcsinh \alpha)^2}{2(1 + \alpha^2)^{3/2}}.$$  

For the higher order terms in the Adomian expansion, the convergence of the series and the comparison with the full numerical solution see Ebaid et al. (2017).
For the study of the elliptical Kepler problem via the ADM see Alshaery and Ebaid (2017).

8.2. THE LANE-EMDEN EQUATION

The astrophysical properties of the static Newtonian stars can be fully characterized by the two gravitational structure equations, which are represented by the mass continuity equation, and the equation of the hydrostatic equilibrium, respectively, given by (Chandrasekhar, 1967; Horedt, 2004; Blaga, 2005; Böhmer and Harko, 2010)

\[
\frac{dm(r)}{dr} = 4\pi\rho(r)r^2, \tag{211}
\]

\[
\frac{dp(r)}{dr} = -\frac{Gm(r)}{r^2}\rho(r), \tag{212}
\]

where \(\rho(r) \geq 0\) is the matter density inside the star, \(p(r) \geq 0\) is the thermodynamic pressure, while \(m(r) \geq 0, \forall r \geq 0\) denotes the mass inside radius \(r\), respectively. To close the system of structure equations one should assume an equation of state for the interior stellar matter, \(p = p(\rho)\), which is a functional relation between the thermodynamic pressure and the density of the matter inside the star. An important equation of state is the polytropic equation of state, for which the pressure can be expressed as a power law of the density,

\[
p = K\rho^{1+1/n}, \tag{213}
\]

where \(K \geq 0\) and \(n\) are constants, and \(n \neq 0\). After eliminating the mass function \(m(r)\) between the two structure equations (211) and (212), respectively, we obtain a single second order non-linear differential equation given by

\[
\frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2 dp}{\rho} \right) = -4\pi G\rho, \tag{214}
\]

which describes the global properties of the Newtonian star. By introducing for the density a new dimensionless variable \(\theta\), so that

\[
\rho = \rho_c \theta^n, \tag{215}
\]

where \(\rho_c\) is the central density, and \(n\) is the polytropic index, we obtain for the pressure the expression \(p = K\rho_c^{1+1/n} \theta^{n+1}\). Next we introduce the dimensionless form of the radial coordinate \(\xi\), defined as

\[
r = \alpha \xi; \quad \alpha = \sqrt{\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}}, \quad n \neq -1. \tag{216}
\]
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In these dimensionless variables Eq. (214) takes the form of the Lane-Emden equation of index $n$,

$$\frac{\theta''}{\xi} + \frac{2}{\xi} \theta' + \theta^n = 0. \quad (217)$$

To solve the Lane-Emden equation we adopt the initial conditions $\theta(0) = 1$ and $\theta'(0) = 0$, respectively, where the prime represents the derivative with respect to the dimensionless independent variable $\xi$.

In the limit $n \to 0$, the Lane-Emden equation has the solution $\theta(\xi)|_{n=0} = 1 - \xi^2/6$. For $n = 1$, the Lane-Emden equation (217) reduces to a linear ordinary differential equation, and it has the solution $\theta(\xi)|_{n=1} = \sin(\xi)/\xi$. The nonlinear Lane-Emden equation has only one known exact solution when $n = 5$, given by $\theta(\xi)|_{n=5} = 1/\sqrt{1 + \xi^2/3}$. For series solutions of the mass continuity and of the general relativistic hydrostatic equilibrium equation (the Tolman-Oppenheimer-Volkoff equation), describing the interior properties of high density compact objects, see Mak and Harko (2013b) and Harko and Mak (2016), respectively.

8.2.1. Solving the Lane-Emden equation via ADM

The second order nonlinear ordinary differential equation of the form

$$\frac{d^2 y}{dx^2} + k \frac{dy}{dx} + y^m = 0, \quad (218)$$

where $k > 0$, is called the Lane–Emden equation of the first kind (Rach et al., 2015). It has to be integrated together with the initial conditions $y(0) = 1$ and $y'(0) = 0$, respectively. The Lane-Emden equation of the second kind is given by

$$\frac{d^2 y}{dx^2} + k \frac{dy}{dx} + e^y = 0, \quad (219)$$

where $k > 0$, and the equation is considered together with the initial conditions $y(0) = y'(0) = 0$. However, in the following we will consider the generalized Lane-Emden equation, given by (Rach et al., 2015)

$$\frac{d^2 y}{dx^2} + k \frac{dy}{dx} + f(y) = 0, \quad (220)$$

where $k > 0$, $f(y)$ is an arbitrary analytic function of $y$, and which should be integrated with the initial conditions $y(0) = \alpha$, and $y'(0) = 0$, respectively.

8.2.2. The integral formulation of the Lane-Emden equation

Eq. (220) can be reformulated as an integral equation as follows (Rach et al., 2015). For $k > 0$, and $k \neq 1$, Eq. (220) can be reformulated as

$$\left(x^k y'\right)' = -x^k f(y), \quad (221)$$

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where a prime denotes the differentiation with respect to $x$. Integrating once we obtain

$$y'(x) = \frac{-1}{x^k} \int_0^x t^k f(y(t)) \, dt.$$  \hfill (222)

Integrating again we find

$$y(x) = \alpha + \frac{1}{k-1} \int_0^x \int_0^x t (\frac{t^{k-1}}{x^{k-1}} - 1) f(y(t)) \, dt \, dx,$$  \hfill (223)

By using the Cauchy formula for repeated integration,

$$\int_a^x \int_a^{x_1} \ldots \int_a^{x_{n-1}} f(x_n) \, dx_{n-1} \ldots dx_1 = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) \, dt,$$  \hfill (224)

we immediately obtain the integral equation formulation of the Lane-Emden equation for $k \neq 1$ as \cite{Rach et al. 2015}

$$y(x) = \alpha + \frac{1}{k-1} \int_0^x t \left( \frac{t^{k-1}}{x^{k-1}} - 1 \right) f(y(t)) \, dt, \quad k > 0, k \neq 1.$$  \hfill (225)

For the case $k = 1$ we find \cite{Rach et al. 2015}

$$y(x) = \alpha + \int_0^x t \ln \left( \frac{t}{x} \right) f(y(t)) \, dt, \quad k = 1.$$  \hfill (226)

These two cases can be unified in a single formulation once we introduce the integral kernel $K(x,t;k)$, defined as \cite{Rach et al. 2015},

$$K(x,t;k) = \begin{cases} \frac{1}{k-1} \left( \frac{t^{k-1}}{x^{k-1}} - 1 \right), & k > 0, k \neq 1, \\ t \ln \left( \frac{t}{x} \right), & k = 1 \end{cases}.$$  \hfill (227)

Then the Lane-Emden equation can be formulated generally in an integral form as \cite{Rach et al. 2015}

$$y(x) = \alpha + \int_0^x K(x,t;k) f(y(t)) \, dt.$$  \hfill (228)

\textbf{8.2.3. The Adomian Decomposition Method}

As usual in the Adomian Decomposition Method, we assume that the solution $y(x)$ of the Lane-Emden equation can be represented in the form of an infinite series, $y(x) = \sum_{n=0}^{\infty} y_n(x)$, while the nonlinear term $f(y)$ is decomposed by using the Adomian polynomials, $f(y(x)) = \sum_{n=0}^{\infty} A_n(y_0(x), y_1(x), \ldots, y_n(x))$. Then by substituting these expressions into Eq. \cite{Rach et al. 2015} we find

$$\sum_{n=0}^{\infty} y_n(x) = \alpha + \int_0^x K(x,t;k) \sum_{n=0}^{\infty} A_n(y_0(t), y_1(t), \ldots, y_n(t)) \, dt, \quad k > 0.$$  \hfill (229)
By choosing \( y_0 = \alpha \), we find the following set of recursive relations for the terms in the series solution of the Lane-Emden equation,

\[
y_0 = \alpha, \quad (230)
\]

\[
y_{m+1} = \int_0^x K(x,t;k)A_m(y_0(t),y_1(t),...,y_n(t)) \, dt, \quad m \geq 0. \quad (231)
\]

The above set of relations will lead to the complete determination of each of the components \( y_n(x) \) of the solution \( y(x) \). As a simple application of the Adomian Decomposition Method to the nonlinear Lane-Emden type equations we consider, following [Rach et al. (2015)], the case of the equation

\[
\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} - 16x^2 e^{-2y} = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad (232)
\]

which has the exact solution

\[
y(x) = \ln(1 + x^4). \quad (233)
\]

The recursive Adomian relation is obtained as

\[
y_0(x) = 0, \quad (234)
\]

\[
y_{m+1}(x) = \int_0^x t \ln\left(\frac{t}{x}\right) \left(-16t^2 A_m(t)\right) \, dt, \quad m \geq 0. \quad (235)
\]

After computing the Adomian polynomials for the nonlinear term \( e^{-2y} \), we obtain

\[
y_0(x) = 0, \quad y_1(x) = x^4, \quad y_2(x) = -\frac{1}{2} x^8, \quad y_3(x) = \frac{1}{3} x^{12}, \quad y_4(x) = -\frac{1}{4} x^{16}, \ldots \quad (236)
\]

It is easy to see by series expanding the exact solution (233) that the Adomian series solution

\[
y(x) = x^4 - \frac{1}{2} x^8 + \frac{1}{3} x^{12} - \frac{1}{4} x^{16} + \ldots \quad (237)
\]

coincides with the exact solution.

8.3. SOLVING THE EQUATION OF MOTION OF CELESTIAL OBJECTS IN SCHWARZSCHILD GEOMETRY

The equation of motion describing the general relativistic motion of a massive celestial body (for example, a planet) in the spherically symmetric and static Schwarzschild geometry, written in spherical coordinates \((r, \varphi, \theta)\), is given by

\[
\frac{d^2 u}{d\varphi^2} + u = \frac{M}{L^2} + 3Mu^2, \quad (238)
\]

where \( u = 1/r \). For the details of Schwarzschild geometry and of the derivation of Eq. (238) see [Mak et al. (2018a)] and [Harko and Lobo (2018)]. In the following we use the natural system of units with \( G = c = 1 \).
To obtain a simpler mathematical formalism we rescale the function \( u = 1/r \) according to
\[
u = \frac{1}{3M} U. \tag{239}\]
Thus Eq. (238) takes the form
\[
d^2U \over d\phi^2 + U = b^2 + U^2, \tag{240}\]
where we have denoted \( b^2 = 3M^2/L^2 \). We will consider Eq. (240) together with the initial conditions \( U(0) = 3Mu(0) = a \), and \( U'(0) = 0 \), respectively. In the following we will obtain semi-analytical solutions of Eq. (240) by using the Laplace-Adomian Method.

8.3.1. Power series solution of the equation of motion

We assume that the solution of Eq. (240) can be obtained in the form of a power series, so that
\[
U(\phi) = \sum_{n=0}^{\infty} U_n(\phi). \tag{241}\]
We apply now the Laplace transform operator \( \mathcal{L}_\phi \) to Eq. (240), thus obtaining
\[
\mathcal{L}_\phi \left[ \frac{d^2U}{d\phi^2} \right] + \mathcal{L}_\phi [U] = b^2 \mathcal{L}_\phi [1] + \mathcal{L}_\phi [U^2]. \tag{242}\]
By using the properties of the Laplace transform we find
\[
s^2\mathcal{L}_\phi (U) - sU(0) - U'(0) + \mathcal{L}_\phi (U) = \frac{b^2}{s} + \mathcal{L}_\phi [U^2], \tag{243}\]
and
\[
\mathcal{L}_\phi (U) = \frac{sU(0) + U'(0)}{s^2 + 1} + \frac{b^2}{s(s^2 + 1)} + \frac{1}{s^2 + 1} \mathcal{L}_\phi [U^2], \tag{244}\]
respectively. The first four Adomian polynomials for \( U^2 \) are given by
\[
A_0 = U_0^2, A_1 = 2U_1U_0, A_2 = 2U_2U_0 + U_1^2, A_3 = 2U_3U_0 + 2U_1U_2. \tag{245}\]
Now we substitute Eq. (241) and \( U^2 = \sum_{n=0}^{\infty} A_n(\phi) \) into Eq. (244), and thus we obtain the relation
\[
\mathcal{L}_\phi \left[ \sum_{n=0}^{\infty} U_n(\phi) \right] = \frac{sU(0) + U'(0)}{s^2 + 1} + \frac{b^2}{s(s^2 + 1)} + \frac{1}{s^2 + 1} \mathcal{L}_\phi \left[ \sum_{n=0}^{\infty} A_n(\phi) \right], \tag{246}\]
which can be written explicitly as

\[ U_0(\phi) + \sum_{n=1}^{\infty} U_n(\phi) = U_0(\phi) + \sum_{n=0}^{\infty} U_{n+1}(\phi) = \]

\[ \mathcal{L}_\phi^{-1} \left[ \frac{sU(0) + U'(0)}{s^2 + 1} + \frac{b^2}{s(s^2 + 1)} \right] + \sum_{n=0}^{\infty} \mathcal{L}_\phi^{-1} \left[ \mathcal{L}_\phi[A_n(\phi)] \right]. \quad (247) \]

Now we can rewrite Eq. (247) in the following recursive forms

\[ U_0(\phi) = \mathcal{L}_\phi^{-1} \left[ \frac{sU(0) + U'(0)}{s^2 + 1} + \frac{b^2}{s(s^2 + 1)} \right], \quad (248) \]

\[ \ldots, \]

\[ U_{k+1}(\phi) = \mathcal{L}_\phi^{-1} \left[ \mathcal{L}_\phi[A_k(\phi)] \right]. \quad (249) \]

### 8.3.2. The explicit terms of the Adomian expansion

By using the explicit expressions of the Adomian polynomials, we can find the analytical forms of the successive terms in the Adomian series expansion of the solution of the general relativistic equation of motion of a planet in the spherically symmetric and static Schwarzschild geometry as follows. First of all, by neglecting the nonlinear term in Eq. (240), we obtain the zeroth order approximation of the solution as given by

\[ U_0(\phi) = (a - b^2) \cos \phi + b^2. \quad (250) \]

Then for the first Adomian polynomial we obtain

\[ A_0 = U_0^2 = \left[ (a - b^2) \cos \phi + b^2 \right]^2, \quad (251) \]

Once \( A_0 \) is known, for the first order approximation of the solution we find

\[ U_1(\phi) = \mathcal{L}_\phi^{-1} \left[ \frac{\mathcal{L}_\phi[A_0(\phi)]}{s^2 + 1} \right] = \mathcal{L}_\phi^{-1} \left[ \frac{\mathcal{L}_\phi[U_0^2]}{s^2 + 1} \right], \quad (252) \]

or, explicitly,

\[ U_1(\phi) = \frac{1}{6} \left\{ -2 \left( a^2 - 2ab^2 + 4b^4 \right) \cos \phi + 3 \left( a^2 - 2ab^2 + 3b^4 \right) + \right. \]

\[ \left. (a - b^2) \left[ (b^2 - a) \cos(2\phi) + 6b^2 \phi \sin \phi \right] \right\}, \quad (253) \]
The Adomian polynomial $A_1$ can be then obtained as

$$A_1 = 2U_1U_0 = \frac{1}{3} \left[ (a-b^2) \cos \varphi + b^2 \right] \left\{ -2 \left( a^2 - 2ab^2 + 4b^4 \right) \cos \varphi + 
3 \left( a^2 - 2ab^2 + 3b^4 \right) + (a-b^2) \left[ (b^2-a) \cos(2\varphi) + 6b^2 \varphi \sin \varphi \right] \right\},$$

(254)

giving for the second order approximation the expression

$$U_2(\varphi) = L^{-1}_\varphi \left[ \frac{L_\varphi [A_1(\varphi)]}{s^2+1} \right] = 2L^{-1}_\varphi \left[ \frac{L_\varphi [U_1U_0]}{s^2+1} \right],$$

(255)
or, explicitly,

$$U_2(\varphi) = \frac{1}{144} \left\{ 16 \left( a^2 - 5ab^2 + 7b^4 \right) (a-b^2) \cos(2\varphi) + \cos \varphi \left[ 29a^3 - 183a^2b^2 + 
3ab^4 (125 - 24\varphi^2) + b^6 (72\varphi^2 - 509) \right] + 12\varphi \left( 5a^3 - 19a^2b^2 + 41ab^4 - 39b^6 \right) \times 
\sin \varphi - 48 \left( a^3 - 6a^2b^2 + 12ab^4 - 13b^6 \right) - 48\varphi \left( b^3 - ab \right)^2 \sin(2\varphi) + 
3 \left( a - b^2 \right)^3 \cos(3\varphi) \right\},$$

(256)

For the higher terms expansions of the solutions of the general relativistic equation of motion of a massive celestial object in Schwarzschild geometry see [Mak et al. (2018a)], where astrophysical applications of the method (motion of the planet Mercury, perihelion precession, and light deflection) are also presented, and discussed in detail. The study of the deflection of light can be done in a similar manner. Generally, by using LADM we can obtain the power series representation of the solution of the general relativistic equation of motion of planets in Schwarzschild geometry up to an arbitrary precision level as

$$U(\varphi) = U_0(\varphi) + U_1(\varphi) + U_2(\varphi) + U_3(\varphi) + U_4(\varphi) + \ldots$$

9. DISCUSSIONS AND CONCLUDING REMARKS

In the present paper we have presented, at an introductory level, some aspects of the powerful method introduced by G. Adomian to solve nonlinear differential, stochastic and functional equations. Usually this method is known as the Adomian Decomposition Method, or ADM for short. The mathematical technique is essentially based on the decomposition of the solution of the nonlinear operator equation into a series of analytic functions. Each term of the Adomian decomposition series is
computed from a polynomial obtained from the power series expansion of an analytic function. The Adomian technique is very simple, efficient, and effective, but, on the other hand, it may raise the necessity of the in depth investigations of the convergence of the series of functions representing the solution of the given nonlinear equation Abbaoui and Cherruault (1994a,b). The Adomian Decomposition Method has been used very successfully to obtain semianalytical solutions for many important classes of functional, differential, and integral equations, respectively, with important applications in many fields of fundamental and applied sciences, and engineering, respectively. The key to the success of the method relies in the decomposition of the nonlinear term in the differential or integral equations into a series of polynomials of the form \( \sum_{n=1}^{\infty} A_n \), where \( A_n \) are polynomials known as the Adomian polynomials. A large number of algorithms and formulas that can calculate the Adomian polynomials for all expressions of nonlinearity were introduced in Adomian (1988, 1994).

Even that the Adomian method is discussed in many articles, a systematic, simple and pedagogical introduction to the subject is still missing. It is the main goal of the present paper to provide such an introduction, which may be useful for scientists who would like to learn about this method by investigating its simplest applications, before proceeding to more advanced topics. After introducing the basics of the method, we have discussed in detail the ADM for the standard differential equations of mathematics, including the linear ordinary differential equation, and the Bernoulli, Riccati and Abel equations, respectively. In each case we have described in detail the general formalism and the particular method, and we have written down explicitly the Adomian form of the solution. For each type of considered equations we have also analyzed a concrete example, and we have shown that the Adomian solution exactly coincides with the analytic solution that can be obtained by using standard mathematical methods. This full agreement explicitly indicates the power of the Adomian Decomposition Method, which could lead to obtaining even the exact solution of a given complicated nonlinear ordinary differential equation, or of an integral equation. We have performed a similar analysis for the second order and the fifth order ordinary differential equations, by explicitly formulating the full process of obtaining the series solution. Specific example have been analyzed for each case. A very powerful extension of the ADM, the Laplace-Adomian Decomposition Method was also introduced through the study of a particular example of a second order nonlinear differential equation.

Finally, we have briefly considered the applications of the Adomian Decomposition Method to some important cases of differential equations that play an essential role in physics and astronomy. Thus, we have presented in detail the important case of the Fisher-Kolmogorov equation, a fundamental equation in several fields of biology, medicine and population dynamics. In this case, after presenting the general al-
Algorithm for the solution, a particular example has been investigated in detail. We have also described the applications of the ADM in three important fields of astronomy and astrophysics, namely, the determination of the orbits of celestial objects from the Kepler equation, obtaining the solutions of the nonlinear Lane-Emden equation, which plays a fundamental role in the study of the stellar structure, and for the investigation of the general relativistic motion of celestial objects in the Schwarzschild geometry. In all these fields the Adomian Decomposition Method has proven to be a computationally efficient and a highly precise theoretical tool for solving the complicated nonlinear equations describing astronomical and astrophysical phenomena.

Certainly the Adomian decomposition method represents a valuable tool for physicists and engineers working with real physical problems. Hopefully the present introduction to this subject will determine scientists working in various fields to become more involved in this interesting and fertile field of investigation, which is very efficient and productive in dealing with large classes of differential/integral equations and complicated mathematical models describing natural phenomena.

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