On the polarization of fermion in an intermediate state

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We show that calculation of a final fermion polarization (for a pure initial state) is equivalent to the problem of looking for complete polarization axis of bispinor. This gives the method for calculation of polarization applicable both for final and intermediate state fermions. We suggest to use fermion propagator (bare or dressed) in form of spectral representation, which gives the orthogonal off-shell energy projectors. This representation leads to covariant separation of particle and antiparticle contributions and gives a natural definition for polarization of intermediate state fermion. The most evident application is related with consistent description of $t$-quark polarization.

PACS numbers: 13.88.+e

Keywords: fermion polarization; virtual fermion; intermediate fermion polarization
I. INTRODUCTION

It is well known how to calculate polarization of a final electron in framework of quantum field theory, see for example [1], §65. If we are interested in polarization of intermediate fermion, first of all we need to give an accurate definition for this value. However, the concept of polarization in intermediate state is used for a long time in particle physics. One can recall the account of polarization in method of equivalent photons [2]. Another example — experimental and theoretical activity concerning of polarization of $t$-quark produced in hadron collisions, see experimental papers [3–6] and reviews [7, 8]. Note that in the last case the naive definition for polarization is used in analogy with on-mass-shell particle.

The first part of this paper concerns to technical aspects of calculation of a fermion polarization. Namely, we discuss the problem of search for a complete polarization axis (36) for the final bispinor. It can be shown that the found axis (4-vector $z$) coincides with polarization of final fermion $s^{(f)}$ calculated by the standard way (23), (26) for pure initial state. Since the proposed method uses an amplitude but not its square, it can be applied also for intermediate state. In this case one should simply change an energy projector in a problem to the off-shell one.

The other part of this paper is devoted to application of proposed method for intermediate fermion. It is known that the on-mass-shell fermion density matrix consists of two factors: the energy projector and spin density matrix [9]. In order to define the fermion polarization in an intermediate state, first of all one should introduce some off-shell energy projectors. Here may exist different variants, but there is no some commonly accepted definition. For example, in the known method of quasi-real electrons [10, 11] the energy poles are accompanied by rather artificial energy projectors which are not orthogonal to each other.

To give exact meaning for fermion polarization in an intermediate state, we suggest to use the covariant form of propagator Eq. (52), which contains the orthogonal off-shell projection operators. This view of free propagator is a particular case of the spectral representation and can be generalized with account of interactions. In Ref. [12] the spectral representation of fermion propagator in the theory with $P$-parity violation was constructed. Corresponding energy projection operators (see (57)) differ from the standard off-shell projectors and also contain $\gamma^5$. Moreover, in such theory the standard spin projectors do not commute with propagator and have to be modified as well [13]. The use of spectral representation for dressed propagator gives natural definition for polarization of fermion in an intermediate state.

The paper is organized as follows. Section 11 is devoted to discussion of the problem of looking for complete polarization axis for non-relativistic electron and its connection with calculation of a final electron polarization as such. The equivalence of these two problems is shown that allows
to compute final electron polarization using the amplitude but not its square.

In section III similar problem is discussed for bispinor. The equivalence of these two problems is shown for relativistic electron that in fact is consequence of similar property for two-dimensional spinors. As an example the scattering of electron in external field is considered.

In section IV the eigenvalue problem for inverse propagator and corresponding spectral representation are discussed. This representation leads to covariant separation of positive and negative energy poles which are accompanied by the off-shell energy projectors orthogonal to each other. The presence of $\gamma^5$ in vertex leads to modification (dressing) of both standard energy and spin projectors.

Section V is devoted to application of the section III method for fermion in an intermediate state. If to use the spectral representation for propagator, the problem of search for a complete polarization axis allows to find polarization vectors for particle and antiparticle contributions. Let us note that the pure spin density matrices correspond to these contributions.

In Conclusions we briefly discuss the obtained results. Some technical details are grouped in Appendix.

II. SCATTERING OF NON-RELATIVISTIC ELECTRON

A. Polarization of final electron

Assume that electron from initial state characterized by twice the mean spin vector $\zeta$ is turned into final state characterized by vector $\zeta'$ (polarization selected by a detector), then transition amplitude is

$$f_{fi} = w^\dagger(\zeta') \hat{f} w(\zeta).$$

Here $w(\zeta)$ is eigenstate of operator $\sigma \zeta$:

$$(\sigma \zeta) w(\zeta) = w(\zeta)$$

and similarly $w(\zeta')$ is eigenstate of operator $\sigma \zeta'$. The operator $\hat{f}$ in the general case is equal to $\hat{f} = A + \sigma B$, where $A$ and $B$ are complex parameters. For the sake of definiteness we choose $B = B \nu$ where $\nu$ is unit real vector such that

$$\hat{f} = A + B (\sigma \nu), \quad \nu^2 = 1.$$ 

It is easy to calculate the $\zeta^{(f)}$, i.e. the polarization of final state as such (see §140 in [14]):

$$\zeta^{(f)} = \frac{(|A|^2 - |B|^2) \zeta + 2 |B|^2 \nu (\nu \zeta) + 2 \text{Im}(AB^*) \nu \times \zeta + 2 \text{Re}(AB^*) \nu}{|A|^2 + |B|^2 + 2 \text{Re}(AB^*) (\nu \zeta)}.$$
Note that Eqs. (1) and (2) are written for pure quantum states for which $|\zeta| = |\zeta'| = 1$. But the result (4) has exactly the same form both for pure and mixed initial states, when $|\zeta| \leq 1$.

Few words to clarify the answer (4). In order to calculate a final electron polarization as such one needs the transition probability which is proportional to square of matrix element (1)

$$|f_{fi}|^2 = \text{Sp}\left(\frac{1 + \sigma \zeta'}{2} \hat{f} \frac{1 + \sigma \zeta}{2} \hat{f}^\dagger\right) = a + b\zeta' = a\left(1 + \frac{b}{a}\zeta'\right).$$

(5)

Here the terms containing detector polarization $\zeta'$ and independent of it are presented. The coefficients are calculated as:

$$a = \frac{1}{2} \text{Sp}\left(\hat{f} \frac{1 + \sigma \zeta}{2} \hat{f}^\dagger\right),$$

$$b = \frac{1}{2} \text{Sp}\left(\sigma \hat{f} \frac{1 + \sigma \zeta}{2} \hat{f}^\dagger\right).$$

(6)

The matrix element square (5) represents in fact the projection of spin density matrix of scattered electron (which is characterized by some vector $\zeta^{(f)}$) onto the detector spin density matrix $(1 + \sigma \zeta')/2$. Thus, comparing (5) with

$$\text{Sp}\left(\frac{1 + \sigma \zeta'}{2} \cdot \frac{1 + \sigma \zeta^{(f)}}{2}\right) = \frac{1}{2}(1 + \zeta' \zeta^{(f)})$$

(7)

one should identify the final electron polarization $\zeta^{(f)}$ as following:

$$\zeta^{(f)} = \frac{b}{a}$$

(8)

and we obtain the answer (4).

Let us take a close look on the matrix $\hat{f} \frac{1 + \sigma \zeta}{2} \hat{f}^\dagger$ which will be necessary below. Using $\sigma$-matrix decomposition

$$\hat{f} \frac{1 + \sigma \zeta}{2} \hat{f}^\dagger = x_0 + \sigma x = x_0 \left(1 + \sigma \frac{x}{x_0}\right),$$

$$x_0 = \frac{1}{2} \text{Sp}\left(\hat{f} \frac{1 + \sigma \zeta}{2} \hat{f}^\dagger\right) = a,$$

$$x = \frac{1}{2} \text{Sp}\left(\sigma \hat{f} \frac{1 + \sigma \zeta}{2} \hat{f}^\dagger\right) = b,$$

(9)

one can see that the matrix is determined by the same parameters $a$ and $b$:

$$\hat{f} \frac{1 + \sigma \zeta}{2} \hat{f}^\dagger = a + \sigma b = a(1 + \sigma \zeta^{(f)}).$$

(10)

B. The complete polarization axis of spinor

Let us consider an arbitrary two-dimensional spinor

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix}, \quad |a|^2 + |b|^2 = 1.$$  

(11)
It is known (see §59 in [14]) that for any spinor $\chi$ there exists an axis of complete polarization, i.e. there exists such unit vector $\zeta$ that

$$(\sigma \zeta) \chi = \chi. \quad (12)$$

In scattering process described by the amplitude $\hat{w}$ a new state appears

$$w(\zeta) \rightarrow \hat{f} w(\zeta) = \hat{f} (1 + \sigma \zeta) \chi, \quad (13)$$

where $\chi$ is an arbitrary spinor. Let us consider the problem of search for complete polarization axis $z$ of spinor $\hat{f} w(\zeta)$

$$(\sigma z) \hat{f} (1 + \sigma \zeta) \chi = \hat{f} (1 + \sigma \zeta) \chi. \quad (14)$$

It is convenient to rewrite this equation in an equivalent form

$$\frac{1 + \sigma z}{2} \cdot \hat{f} \frac{1 + \sigma \zeta}{2} \chi = \hat{f} \frac{1 + \sigma \zeta}{2} \chi. \quad (15)$$

Recall that amplitude contains a spinor with definite polarization, so in Eq. (15) vector $\zeta$ is unit one (in contrast to (5)). However, if to use a pure spin density matrix of initial state ($\zeta^2 = 1$) within the spur in (5), then both problems, (5) and (15), are equivalent and two vectors coincide: $\zeta^{(f)} = z$. Let us show this equivalence.

Suppose that we have solved the problem (15), i.e. the vector $z$ is known. Take Hermitian adjoint of the previous equation:

$$\chi^\dagger \frac{1 + \sigma \zeta}{2} \hat{f}^\dagger \cdot \frac{1 + \sigma z}{2} \chi = \chi^\dagger \frac{1 + \sigma \zeta}{2} \hat{f}^\dagger. \quad (16)$$

Multiplying Eqs. (15), (16) by each other one obtains the following matrix relation

$$\frac{1 + \sigma z}{2} \cdot \left( \hat{f} \frac{1 + \sigma \zeta}{2} \hat{f}^\dagger \right) \cdot \frac{1 + \sigma z}{2} = \hat{f} \frac{1 + \sigma \zeta}{2} \hat{f}^\dagger \equiv X. \quad (17)$$

Here the known matrix $X$ (10) has been appeared and previous equation demonstrates its evident property:

$$X \cdot (1 - \sigma z) = (1 - \sigma z) \cdot X = 0. \quad (18)$$

If to recall the matrix $X$ decomposition (10), we get two equations for vector of final polarization $\zeta^{(f)}$ at the given $z$

$$(1 + \sigma \zeta^{(f)}) \cdot (1 - \sigma z) = (1 - \sigma z)(1 + \sigma \zeta^{(f)}) = 0. \quad (19)$$

Writing down two relations (19) in detail:

$$(1 + \sigma \zeta^{(f)}) \cdot (1 - \sigma z) = 1 - \zeta^{(f)} \cdot z + \sigma (\zeta^{(f)} \times z) + i \sigma (\zeta^{(f)} \times z) = 0, \quad (20)$$

$$(1 - \sigma z) \cdot (1 + \sigma \zeta^{(f)}) = 1 - \zeta^{(f)} \cdot z + \sigma (\zeta^{(f)} \times z) - i \sigma (\zeta^{(f)} \times z) = 0,$$

1 The matrix $(1 + \sigma \zeta) \chi \chi^\dagger (1 + \sigma \zeta)$ arisen after multiplication coincides up to a factor with $(1 + \sigma \zeta)$. We come to (17) after factor cancellation (recall that $\chi$ is an arbitrary spinor).
it is easy to see that the only solution is $\zeta^{(f)} = z$.

On the other hand, if we know vector $\zeta^{(f)}$ then from (19) it follows that $z = \zeta^{(f)}$.

Thus we see that for scattering of non-relativistic electron these two problems: calculation of the final electron polarization vector (5) (for pure state) and looking for complete polarization axis (15) are equivalent. So the final electron polarization can be calculated from scattering amplitude instead of its square.

### III. SCATTERING OF RELATIVISTIC ELECTRON

The scheme outlined above can be applied also for scattering of relativistic electrons. Assume that electron from initial state characterized by momentum $p_{1\mu}$ and polarization vector $s_{1\mu}$ is turned into final state characterized by vectors $p_{2\mu}$ and $s_{2\mu}$ (polarization selected by a detector), then transition amplitude is

$$\mathcal{M} = \bar{u}_2(p_2, s_2)\Gamma u_1(p_1, s_1).$$

(21)

Here $u(p, s)$ is solution of Dirac equation

$$(\hat{p} - m)u(p, s) = 0,$$

(22)

and matrix $\Gamma$ characterizes the transition process.

In order to find the final electron polarization it is necessary to calculate the transition probability which is proportional to square of matrix element (21)

$$|\mathcal{M}|^2 = \text{Sp}(u_2\bar{u}_2\Gamma u_1\bar{u}_1\tilde{\Gamma}) = A + B_{\mu}s_{2\mu} = A\left(1 + \frac{B_{\mu}s_{2\mu}}{A}\right), \quad \tilde{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0.$$  

(23)

Here we write down separately terms dependent on detector polarization $s_2$ and independent on it. Since $(s_2p_2) = 0$, only transverse part of vector $B_\mu$ remains in (23)

$$B_{\mu} = \left(g_{\mu\nu} - \frac{p_{2\mu}p_{2\nu}}{m^2}\right) \cdot B^\nu.$$  

(24)

The matrix element square (23) is in fact the projection of scattered electron density matrix (its spin part is defined by vector $s_{\mu}^{(f)}$) onto the detector density matrix $\rho'$. Thus comparison of (23) with

$$\text{Sp}(\rho'\rho) = \text{Sp}\left(\frac{m + \hat{p}_2}{2m} \cdot \frac{1 + \gamma^5s_2}{2} \cdot \frac{1 + \gamma^5s^{(f)}}{2}\right) = \frac{1}{2}\left(1 - (s_2s^{(f)})\right)$$  

(25)

gives the final electron polarization $s_{\mu}^{(f)}$ as such:

$$s_{\mu}^{(f)} = -\frac{B_{\mu}}{A}.$$  

(26)
Let us introduce short notations for final state projectors

$$\Lambda^\pm_2 = \Lambda^\pm_m(n_2) = \frac{1}{2} (1 \pm \hat{n}_2), \quad \hat{n}_2 = \frac{\hat{p}_2}{m}, \quad n_2^2 = 1,$$

$$\Sigma_2 = \Sigma_0(s_2) = \frac{1}{2} (1 + \gamma^5 \hat{s}_2), \quad \hat{s}_2^2 = -1, \quad (s_2 n_2) = 0$$

and similarly for initial state. The matrix element square in these notations is

$$\langle \mathcal{M} \rangle^2 = \text{Sp}(\Lambda^+_2 \Sigma_2 \Gamma \Lambda^+_1 \Sigma_1 \tilde{\Gamma}) = \text{Sp}(\Sigma_2 X) = \text{Sp}\left(\frac{1}{2} (1 + \gamma^5 \hat{s}_2) X\right).$$

(28)

Here we have introduced necessary for further matrix (analogue of matrix (9) in the case of non-relativistic electron)

$$X = \Lambda^+_2 \Gamma \Lambda^+_1 \Sigma_1 \tilde{\Gamma} \Lambda^+_2.$$  

(29)

The coefficients $A$ and $B_\mu$ in (23) are calculated like

$$A = \frac{1}{2} \text{Sp}(X), \quad B_\mu = \frac{1}{2} \text{Sp}(\gamma^5 \gamma_\mu X),$$

(30)

and the orthogonality property $B_\mu p_2^\mu = 0$ is seen from it.

Let us find the decomposition of the matrix $X$ in $\gamma$-matrix basis. Simple calculations show that all coefficients are easily expressed by means of $p_2$, $A$, $B_\mu$ except of projection on $\sigma^{\mu\nu}$. The decomposition has view

$$X = \frac{A}{2} (1 + \hat{n}_2) - \frac{1}{2} \gamma^5 \hat{B} + \sigma^{\mu\nu} x_{\mu\nu}, \quad \sigma^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu].$$

(31)

However, the coefficient $x_{\mu\nu}$ is related with parameters $A$, $B_\mu$ too. To see it, let us note that $X$ has the following properties

$$(1 - \hat{n}_2)X = X(1 - \hat{n}_2) = 0$$

(32)

and the most general form of such matrix (see Appendix A) is

$$X = (1 + \hat{n}_2)(x_0 + \gamma^5 \hat{x}), \quad x_{\mu} n_2^\mu = 0,$$

(33)

containing an arbitrary parameter $x_0$ and arbitrary 4-vector $x_\mu$ orthogonal to momentum.

Comparison of expression (31) with the most general form (33) fixes $\sigma^{\mu\nu}$ term. As a result the matrix $X$ (29) looks like

$$X = \frac{1}{2} (1 + \hat{n}_2)(A - \gamma^5 \hat{B}) = \frac{A}{2} (1 + \hat{n}_2)(1 - \gamma^5 \hat{B}/A).$$

(34)

Recall that $s^{(f)}_\mu = -B_\mu/A$ is final electron polarization.

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2 Up to irrelevant numeric factor.
A. The search for complete polarization axis of bispinor

After scattering described by the amplitude (21) we have a new state
\[ u(p_1, s_1) \rightarrow \Lambda^+_2 \Gamma u(p_1, s_1) = \Lambda^+_2 \Gamma \Lambda^+_1 \Sigma_1 u(p_1, s_1). \] (35)

Let us consider the problem of search for complete polarization axis \( z_\mu \) of bispinor of scattered electron
\[ \gamma^5 \hat{z} \cdot \Lambda^+_2 \Gamma u_1 = \Lambda^+_2 \Gamma u_1, \quad (zn_2) = 0. \] (36)
We know in advance that this problem has a solution (see Appendix[3]). Let us rewrite the equation in equivalent form
\[ \frac{1 + \gamma^5 \hat{z}}{2} \cdot \Lambda^+_2 \Gamma u_1 = \Lambda^+_2 \Gamma u_1. \] (37)

Let us show that (as in the non-relativistic case) the complete polarization axis \( z \) coincides with \( s^{(f)} \). Let us take Hermitian adjoint of previous equation and multiply it by \( \gamma^0 \) from right
\[ \bar{u}_1 \tilde{\Gamma} \Lambda^+_2 \cdot \frac{1 + \gamma^5 \hat{z}}{2} = \bar{u}_1 \tilde{\Gamma} \Lambda^+_2. \] (38)

Multiplying both equations by each other and substituting the density matrix of initial electron
\[ u_1 \bar{u}_1 \equiv X. \] (39)

Note that here the known matrix \( X \) (29), (34) has been appeared. This relation can be transformed into equation connecting the complete polarization axis \( z_\mu \) and final electron polarization \( s^{(f)}_\mu \).

As it follows from previous relation the matrix \( X \) satisfies equations
\[ (1 - \gamma^5 \hat{z}) \cdot X = X \cdot (1 - \gamma^5 \hat{z}) = 0. \] (40)

If to use for \( X \) the expression (34) found above we obtain the following two equations
\[ (1 + \hat{n}_2)(1 - \gamma^5 \hat{z})(1 + \gamma^5 \hat{s}^{(f)}) = (1 + \hat{n}_2)(1 + \gamma^5 \hat{s}^{(f)})(1 - \gamma^5 \hat{z}) = 0. \] (41)

Both expressions are “under the observation” of singular matrix \( 1 + \hat{n}_2 \) but in the relation linking \( z_\mu \) and \( s^{(f)}_\mu \) it does not play any role. Let us multiply spin matrices
\[ (1 - \gamma^5 \hat{z})(1 + \gamma^5 \hat{s}^{(f)}) = 1 + (zs^{(f)}) + \gamma^5 \hat{s}^{(f)} - \gamma^5 \hat{z} + \sigma^{\mu\nu} z_\mu s^{(f)}_\nu = 0, \] (42)
\[ (1 + \gamma^5 \hat{s}^{(f)})(1 - \gamma^5 \hat{z}) = 1 + (zs^{(f)}) + \gamma^5 \hat{s}^{(f)} - \gamma^5 \hat{z} - \sigma^{\mu\nu} z_\mu s^{(f)}_\nu = 0. \]

It immediately follows that these two vectors coincide \( s^{(f)} = z \).

So for relativistic electron there is also the equivalence of these two problems: calculation of the scattered fermion polarization (23), (26) (for pure initial state) and looking for complete polarization axis (37) of bispinor. The distinction of the problem of looking for complete polarization axis is the use of amplitude instead of its square. This makes possible to apply the same method for calculation of fermion polarization both for final and intermediate states.
B. Electron scattering in an external field

By way of simple example let us consider single scattering of electron in an external field. We will check the found above equivalence of two problems and discuss some features.

\[ p_1, s_1 \xrightarrow{\Gamma} p_2, s_2 \]

Matrix element has form (21) in which vertex factor \( \Gamma \) contains Fourier transform of the external field and corresponding \( \gamma \)-matrix.

The problem of looking for axis (37) can be rewritten in equivalent view

\[ \frac{1 + \gamma^5 \hat{z}}{2} \cdot \Lambda_2^+ \Gamma \Lambda_1^+ \Sigma_1 \chi = \Lambda_2^+ \Gamma \Lambda_1^+ \Sigma_1 \chi, \tag{43} \]

where \( \chi \) is an arbitrary bispinor. Since this is an inverse problem (reconstruction of an operator), \( z^\mu \) generally speaking can depend on bispinor \( \chi \). However, the found equivalence \( s^{(f)} = z \) and expression (26) for \( s^{(f)} \) tell that vector \( z \) does not depend on bispinor \( \chi \). It means that (43) can be rewritten as a matrix problem

\[ \frac{1 + \gamma^5 \hat{z}}{2} \cdot \Lambda_2^+ \Gamma \Lambda_1^+ \Sigma_1 = \Lambda_2^+ \Gamma \Lambda_1^+ \Sigma_1. \tag{44} \]

The matrix formulation of the problem gives rather convenient method to find vector \( z \): one should decompose (44) in \( \gamma \)-matrix basis and seek for vector \( z_\mu \) as an expansion over available vectors.\(^3\)

We checked that for external fields of different kinds (\( S, P, V, A \)) the solution of the problem (44) coincides with final fermion polarization calculated according to (26) (for definite polarization of initial fermion, \( s_1^2 = -1 \)).

We will present here solutions of the problem (44) for scalar and vector vertices

- \( \Gamma = 1 \)

\[ z_\mu = s_1 \mu - a_1 p_1 \mu - a_2 p_2 \mu, \quad a_1 = a_2 = \frac{(p_2 s_1)}{(p_1 p_2) + m^2}. \tag{45} \]

\(^3\) For \( S \) and \( P \) vertices these are vectors \( n_{1\mu}, n_{2\mu}, s_{1\mu} \) and expansion looks as

\[ z_\mu = z_1 n_{1\mu} + z_2 n_{2\mu} + z_3 s_{1\mu} + z_4 \epsilon_{\mu\nu\lambda\sigma} n_1^\nu n_2^\lambda s_1^\sigma, \]

for \( V \) and \( A \) vertices there exists also vector \( A_\mu \) and expansion is:

\[ z_\mu = z_1 n_{1\mu} + z_2 n_{2\mu} + z_3 s_{1\mu} + z_4 A_\mu. \]

Case of linear dependence of vectors needs special consideration.
\( \Gamma = \gamma^\mu A_\mu(q) \)

\[
\begin{align*}
  z_\mu &= s_{1\mu} - a_1 p_{1\mu} - a_2 p_{2\mu} - a_3 A_\mu, \\
  a_1 &= -a_2 = \frac{(p_2 s_1)(A A) - 2(p_2 A)(s_1 A)}{D}, \\
  a_3 &= 2 \frac{(p_1 p_2)(s_1 A) - (p_1 A)(p_2 s_1) - (s_1 A)m^2}{D}, \\
  D &= (p_1 p_2)(A A) - 2(p_1 A)(p_2 A) - (A A)m^2.
\end{align*}
\] (46)

Let us emphasize that expressions (45), (46) for polarization of final electron obtained for pure initial state holds for mixed state too.

Note also that for electron scattering in external field the initial pure spin state leads to definite polarization of final state. It is not evident in advance, quite the contrary, it would be natural to expect the appearance of a mixed final state in this case but calculations of amplitude square demonstrates this.

Both these facts are consequence of the equivalence of two problems: \( s(f) = z \), discussed above.

**IV. SPECTRAL REPRESENTATION OF PROPAGATOR**

We want to apply the problem of looking for complete polarization axis (37) to the case of fermion in an intermediate state. All aforesaid is easily extended for the case of virtual fermions if to use a spectral representation of propagator.

In order to construct this representation one needs to solve eigenvalue problem for inverse propagator \( S(p) = G^{-1}(p) \). Since there is convenient \( \gamma \)-matrix basis, one can solve a matrix problem: to look for eigenprojectors \( \Pi \) and eigenvalues \( \lambda \)

\[
S \Pi = \lambda \Pi.
\] (47)

Having solved this problem, i.e. having found eigenvalues \( \lambda_i \) and orthogonal system of eigenprojectors \( \Pi_i \)

\[
\Pi_i \Pi_k = \delta_{ik} \Pi_i, \quad i, k = 1, 2,
\] (48)

we can construct the spectral representation of inverse propagator

\[
S(p) = \lambda_1 \Pi_1 + \lambda_2 \Pi_2.
\] (49)

If the system of projectors is complete, then this expression can be easily reversed and propagator looks like this:

\[
G(p) = \frac{1}{\lambda_1} \Pi_1 + \frac{1}{\lambda_2} \Pi_2,
\] (50)
i.e. propagator poles are zeroes of eigenvalues \( \lambda_i \). Let us consider how the propagator spectral representation is shown up in particular cases.

The eigenprojectors \( \Pi_i \) for a bare propagator are the known off-shell projector operators \( \Lambda^\pm_W \)

\[
\Lambda^\pm_W = \frac{1}{2} \left( 1 \pm \frac{\hat{p}}{\hat{W}} \right), \quad p^2 = W^2,
\]

where \( W \) is center-of-mass energy. As a result the bare propagator

\[
G_0(p) = \frac{1}{\hat{p} - m_0} = \frac{1}{W - m_0} \Lambda^+_W + \frac{1}{-W - m_0} \Lambda^-_W,
\]

\[
\lambda_1 = W - m_0, \quad \lambda_2 = -W - m_0,
\]

is shown up as a sum of poles with positive and negative energies. It is necessary to stress that we have covariant separation of poles \(1/(W \pm m_0)\) because \( W \) is invariant mass. Besides that, since fermion and anti-fermion have opposite parities the representation (52) sorts out contributions by parity.

With account of interaction the inverse propagator takes the form

\[
S(p) = \hat{p} - m_0 - \Sigma(p),
\]

where \( \Sigma(p) \) is self-energy contribution. For dressed propagator the spectral representation looks differently depending on interaction.

- If theory does not have \( \gamma^5 \) in a vertex, then \( \Sigma(p) \) contains unit matrix and \( \hat{p} \)

\[
\Sigma(p) = A(p^2) + \hat{p}B(p^2) = \Sigma^+(W)\Lambda^+_W + \Sigma^-(W)\Lambda^-_W,
\]

where \( \Sigma^\pm(W) = A(W^2) \pm WB(W^2) \). In this case eigenprojectors coincide with the such for bare propagator (51) and the propagator spectral representation has form

\[
G(p) = \frac{1}{\hat{p} - m_0 - \Sigma(p)} = \frac{1}{W - m_0 - \Sigma^+(W)} \Lambda^+_W + \frac{1}{-W - m_0 - \Sigma^-(W)} \Lambda^-_W.
\]

- In theory with \( \gamma^5 \) the self-energy contribution also has \( \gamma^5 \) terms

\[
\Sigma(p) = A(p^2) + \hat{p}B(p^2) + \gamma^5C(p^2) + \hat{p}\gamma^5D(p^2),
\]

and eigenprojectors \( \Pi_i \) do not coincide with \( \Lambda^\pm_W \). In this case the eigenprojectors take a more complicated form [12]:

\[
\Pi_{1,2}(p) = \frac{1}{2} \left( 1 \pm \hat{n}\tau \right), \quad \hat{n} = \frac{\hat{p}}{\hat{W}},
\]

\[
\tau = \frac{1}{R} \left( 1 - B - \gamma^5D - \hat{n}\gamma^5C \right), \quad R = \sqrt{(1 - B)^2 - D^2 + C^2/W^2},
\]

\[
4 \text{ In contrast to equivalent electrons method [10, 11] (see also the usage of this method in concrete calculations [15, 13]) the representation [52] is based on the orthogonal projectors and has covariant form. Besides, it is easily generalized for the case of dressed propagator.}
\]
and eigenvalues $\lambda_i(W)$ are

$$
\lambda_{1,2}(W) = -m_0 - A(W^2) \pm WR(W^2).
$$

(58)

An essential aspect related with the completeness of eigenprojectors system is the existence of spin projectors commuting with propagator. The existence of such projectors follows simply from counting of degrees of freedom in the problem (47). However, the standard spin projectors

$$
\Sigma_0(s) = \frac{1 + \gamma^5 \hat{s}}{2}, \quad s^2 = -1, \quad (sp) = 0,
$$

(59)

cease to commute with propagator in the presence of $\gamma^5$ in a vertex.

Nevertheless, there exist the generalized (dressed) spin projectors [13] closely related with eigenprojectors (57) and having all desired properties

$$
\Sigma(s) = \frac{1}{2} (1 + \gamma^5 \hat{s} \tau).
$$

(60)

Note that $\Pi_i(p)$ and $\Sigma(s)$ have the same matrix factor $\tau$ (57) which possesses the following property. Without interaction ($B = C = D = 0$) or in theory with parity conservation ($C = D = 0$) this factor turns into unit matrix $\tau = 1$. As a consequence the projectors $\Pi_i(p)$ and $\Sigma(s)$ return to the standard (bare) form.

Furthermore, it can be seen that “under the observation” of energy eigenprojector $\Pi_i(p)$ the spin projector (60) is significantly simplified

$$
\Pi_i(p) \Sigma(s) = \Pi_i(p) \frac{1}{2} (1 + \gamma^5 \hat{s} \hat{n}).
$$

(61)

The problem of looking for complete polarization axis of bispinor (37) is naturally generalized for the case of virtual fermion.

Using free propagator (51) one needs only to change projector $\Lambda_\pm^\dagger$ in (37) to one of off-shell projectors $\Lambda_\pm^\dagger(p_2) = (1 \pm \hat{p}_2/W)/2, \hat{p}_2^2 = W^2$. The problem of looking for complete polarization axis (37) is turned into that:

$$
\frac{1}{2} (1 + \gamma^5 \hat{z}^\pm) \cdot \Lambda_\pm^\dagger(p_2) \Gamma \Lambda_\pm^\dagger \Sigma_1 u_1 = \Lambda_\pm^\dagger(p_2) \Gamma \Lambda_\pm^\dagger \Sigma_1 u_1, \quad (z^\pm p_2) = 0,
$$

(62)

and (see Appendix [13]) such problem also has solution: for any bispinor $u_1$ there exists a vector $z^\pm_\mu, (z^\pm)^2 = -1$.

The aforesaid is also true for dressed propagator in theory with $\gamma^5$ as well. Let us write down the problem of looking for complete polarization axis for dressed energy and spin projectors (57), (60):

$$
\Sigma(z^\pm) \cdot \Pi^\pm(p_2) \Gamma \Lambda_\pm^\dagger \Sigma_1 u_1 = \Pi^\pm(p_2) \Gamma \Lambda_\pm^\dagger \Sigma_1 u_1, \quad (z^\pm p_2) = 0.
$$

(63)
The dressed energy projectors $\Pi^\pm$ in the case of CP conservation can be rewritten in form (it may be considered as $\gamma$-matrices transformation $\gamma_\mu \rightarrow \gamma_\mu'$, see details in Appendix C)

$$\Pi^\pm = \frac{1}{2} (1 \pm \hat{n}\tau) = \frac{1}{2} (1 \pm \hat{n} e^{2\kappa\gamma^5}) = e^{-\kappa\gamma^5} \frac{1}{2} (1 \pm \hat{n}) e^{\kappa\gamma^5}. \quad (64)$$

The dressed spin projector can be presented by the same way

$$\Sigma(z) = \frac{1}{2} (1 + \gamma^5 \hat{z}\tau) = e^{-\kappa\gamma^5} \frac{1}{2} (1 + \gamma^5 \hat{z}) e^{\kappa\gamma^5}. \quad (65)$$

Thereafter the problem of looking for axis (63) takes form:

$$\frac{1}{2} (1 + \gamma^5 \hat{z}^\pm) \cdot \left( \Lambda^\pm_W(p_2) e^{\kappa\gamma^5} \Gamma \Lambda^+_1 \Sigma_1 u_1 \right) = \left( \Lambda^\pm_W(p_2) e^{\kappa\gamma^5} \Gamma \Lambda^+_1 \Sigma_1 u_1 \right), \quad (z^\pm p_2) = 0. \quad (66)$$

So the problem (63) for dressed energy projectors $\Pi^\pm(p_2)$ is reduced to the problem (62), involving the bare off-shell projectors $\Lambda^\pm_W(p_2)$ (51). As a result one can conclude that for the problem (63) also exists the axis of complete polarization $z^\pm$.

V. POLARIZATION OF FERMION IN AN INTERMEDIATE STATE

The spectral representation of propagator, where the orthogonal off-shell projectors are arisen allows to give an accurate definition of fermion polarization in an intermediate state. Thereafter the generalization of the problem of looking for axis (37) (replacing projectors by the off-shell ones) yields the recipe of polarization calculation in an intermediate state.

Consider some process with intermediate fermion is born in $s$-channel. Here the external boson lines correspond either to some on-mass-shell particles or to external field.

$$M = \bar{u}_2(p_2, s_2) \Gamma_1 G(p) \Gamma_2 u_1(p_1, s_1). \quad (67)$$

For the case of bare propagator or theory without $\gamma^5$ the fermion propagator in the intermediate state has form

$$G(p) = \frac{1}{\lambda_1} \Lambda^+_1 + \frac{1}{\lambda_2} \Lambda^-_W, \quad (68)$$

where $\Lambda^-_W$ are off-shell energy projectors (51).
If to recall the problem of looking for complete polarization axis involving $\Lambda_\pm W$ (62), the propagator in the amplitude (67) can be rewritten as following

$$G = \frac{1}{\lambda_1} \Lambda_+^+ \Sigma_0 (z^+) + \frac{1}{\lambda_2} \Lambda_-^+ \Sigma_0 (z^-). \quad (69)$$

This gives a natural definition for polarization of fermion in an intermediate state. Polarization vectors $z^\pm$ are different for poles with positive and negative energies. It is not so evident that spin density matrices in (69) are pure ones: $(z^\pm)^2 = -1$, however, it follows from the problem of looking for axis (62).

It is possible to calculate polarization vector either with the help of the problem of looking for axis (62) or by analogy with (23) with the use of spur:

$$\text{Sp}(\Lambda^+_W \Sigma_0 (z^+) \Gamma u_1 \bar{u}_1 \tilde{\Gamma}) = A \left(1 + \frac{B_\mu}{A} z^\pm \mu \right), \quad z^\pm \mu = -\frac{B^\perp}{A}. \quad (70)$$

It seems that the second way is technically simpler. As indicated above the formula for polarization $z^\pm$ calculated for pure state holds for mixed initial state also.

Let us take a look now on the case of the theory with $\gamma^5$ in vertex. In this case dressed fermion propagator in intermediate state may be represented as

$$G(p) = \frac{1}{\lambda_1} \Pi_1 (p) + \frac{1}{\lambda_2} \Pi_2 (p), \quad (71)$$

where $\Pi_{1,2}(p)$ are the energy projectors (57). Using the problem (63) one can see that the dressed propagator inside the diagram acquires spin projectors

$$G \to \tilde{G} = \frac{1}{\lambda_1} \Pi_1 (p) \Sigma (z^+) + \frac{1}{\lambda_2} \Pi_2 (p) \Sigma (z^-), \quad (z^\pm)^2 = -1. \quad (72)$$

It should be pointed out that dressed energy projectors $\Pi_i (p)$ presented here contain self-energy contributions and should be renormalized.

VI. CONCLUSIONS

The main statement of the work is that the final fermion polarization as such (26) can be calculated by a non-standard way, namely by using the problem of search for complete polarization axis of bispinor (36). We have shown that these two problems are equivalent.

It is unlikely that the problem (36) leads to more simple calculations, rather opposite, the calculation of spurs in the standard approach is a routine operation which is successfully performed by a computer. However, in the problem of looking for axis (36) the amplitude is used instead of its square. This allows to apply this problem to find polarization of fermion in an intermediate state.
The essential moment is the use of propagator spectral representation \( (52), (55), (57) \). This approach leads to the system of orthogonal off-shell projectors and allows to give clear definition for fermion polarization in an intermediate state. Thus it is possible to use either free propagator or dressed one, including the theory with \( \gamma^5 \) in a vertex.

When a propagator is written in form of the spectral representation it is possible to use the problem of looking for axis to calculate the off-mass-shell polarization. Indeed, in the problem \( (36) \) of final electron polarization (on-mass-shell) the energy projector is changed to the off-shell one. As a result the propagator inside of a diagram takes view \( (72) \) where spin density matrices correspond to complete polarization.

The fact that, for example, the projector \( \Pi_1 \) of positive energy state in \( (72) \) is accompanied by completely polarized spin density matrix \( \Sigma(z^+) \) \( (z^+)^2 = -1 \) is not evident in advance and is the result of the problem of looking for axis.

The evident application of suggested approach is concerned with \( t \)-quark polarization but it needs single consideration.

We are grateful to V.G. Serbo for fruitful discussions and participation in the beginning of the work.

**Appendix A: Auxiliary matrix problem**

We want to find the most general form of matrix \( X \) obeying equations

\[
X(1 - \hat{n}) = (1 - \hat{n})X = 0, \quad n^\mu = p^\mu / m, \quad n^2 = 1. \tag{A1}
\]

For this purpose we decompose the matrix \( X \)

\[
X = x_0 + x_\mu \gamma^\mu + \bar{x}_0 \gamma^5 \gamma^\mu + \bar{x}_\mu \gamma^5 \gamma^\mu + x_{\mu\nu} \sigma^{\mu\nu}, \tag{A2}
\]

and determine its coefficients.

Computing coefficients at \( \gamma^5 \) in relations \( (A1) \) one obtain pair of equations

\[
\bar{x}_0 + (n\bar{x}) = 0, \quad \bar{x}_0 - (n\bar{x}) = 0, \tag{A3}
\]

from which it immediately follows that \( \bar{x}_0 = 0 \) and \( (n\bar{x}) = 0 \).

Next, determining coefficients at \( \gamma^5\gamma^\mu \) in these relations we come to equations

\[
x_\mu - x_0 n_\mu + 2x_{\mu\nu} n^\nu = 0, \quad x_\mu - x_0 n_\mu - 2x_{\mu\nu} n^\nu = 0. \tag{A4}
\]

Hence it follows that \( x_\mu = x_0 n_\mu \).
Finally, calculating coefficients at $\sigma^{\mu\nu}$ in any of relations \((A1)\) we determine $x_{\mu\nu}$:

$$x_{\mu\nu} = -\frac{i}{2} \epsilon_{\mu\nu\alpha\beta} n^\alpha \bar{x}^\beta.$$ \hfill (A5)

Thus the matrix $X$ has view

$$X = x_0 (1 + \hat{n}) + \bar{x}_{\mu} \gamma^5 \gamma^\mu - \frac{i}{2} \epsilon_{\mu\nu\alpha\beta} n^\alpha \bar{x}^\beta \sigma^{\mu\nu},$$ \hfill (A6)

with arbitrary parameter $x_0$ and vector $\bar{x}_\mu$: $(n \bar{x}) = 0$. Let us rewrite this expression in more compact form

$$X = x_0 (1 + \hat{n}) + \bar{x}_\mu \gamma^5 \gamma^\mu(1 + \hat{n}) = (1 + \hat{n})(x_0 + \bar{x}_\mu \gamma^5 \gamma^\mu), \quad (n \bar{x}) = 0.$$ \hfill (A7)

\textbf{Appendix B: Complete polarization axis of bispinor}

It is known that any spinor has complete polarization axis. Bispinor also has similar property: if $\Psi(p)$ is an arbitrary solution of Dirac equation then there exists such vector $s_\mu$ that (cp. with \(12\))

$$\gamma^5 \hat{s} \cdot \Psi(p) = \Psi(p), \quad s^2 = -1, \quad (sp) = 0, \quad p^2 = m^2. \hfill (B1)$$

Let us show this for solution of Dirac equation with positive energy $u(p, s)$. Write down equation \((B1)\) in split form

$$\gamma^5 \hat{s} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \hfill (B2)$$

In the standard representation of the $\gamma$-matrices we have

$$\gamma^5 \hat{s} = \begin{pmatrix} \sigma \cdot s \quad -s_0 \\ s_0 \quad -\sigma \cdot s \end{pmatrix}, \quad u = \begin{pmatrix} (n_0 + 1) \phi \\ n \cdot \sigma \phi \end{pmatrix},$$

where $n^\mu = p^\mu/m$, $n^2 = 1$, $n^0 > 0$ and $\phi$ is an arbitrary two-component spinor. Substituting these formulas into \((B2)\) we come to the problem of determination of complete polarization axis for two-dimensional spinor

$$\sigma \cdot \left(s - \frac{s_0 n}{n_0 + 1}\right) \phi = \phi. \hfill (B3)$$

It is easy to check that vector appeared here

$$S = s - \frac{s_0 n}{n_0 + 1} \hfill (B4)$$

has unit length $S^2 = 1$ if to recall that $s^2 = -1$, $(ns) = 0$ and $n^2 = 1$.

One can conclude that for solutions of Dirac equation with positive energy the relation \((B1)\) can be reduced to equation \((B3)\) which has solution for any two-dimensional column (cp. with \(12\)). Of course all the same holds for solutions with negative energy.
The off-shell bispinors are defined as solutions of equations $\Lambda^- W\Psi(p) = 0$ and $\Lambda^+ W\Psi(p) = 0$ at $p^2 = W^2$ (positive and negative frequency solutions). It is easy to check that there exists complete polarization axis $s_\mu$ for such bispinors, in other words

$$\gamma^5 \hat{s} \Psi(p) = \Psi(p), \quad s^2 = -1, \quad (ps) = 0, \quad p^2 = W^2.$$  \hspace{1cm} (B5)

The proof repeats exactly the similar consideration for Dirac equation solution.

**Appendix C: Dressed energy and spin projectors**

Let us show that dressed energy projector (57) (in the theory with $\gamma^5$) can be transformed into the standard one (51) by transition to another representation of $\gamma$-matrices. At first, let us consider the case of CP conservation ($C = 0$ in (57)) and show that in such case

$$\Pi_i = \frac{1}{2} (1 + \hat{n}\tau) = \frac{1}{2} \left( 1 + \hat{n} \frac{(1 - B) - D\gamma^5}{\sqrt{(1 - B)^2 - D^2}} \right) = \frac{1}{2} (1 + n_{\mu} \gamma^\mu).$$  \hspace{1cm} (C1)

with some transformed matrices $\gamma^\mu$.

Firstly, note that matrix $\tau$ can be written in exponential view

$$\tau = \frac{(1 - B) - D\gamma^5}{\sqrt{(1 - B)^2 - D^2}} = \cosh 2\kappa + \gamma^5 \sinh 2\kappa = e^{2\kappa \gamma^5}. $$  \hspace{1cm} (C2)

Next, performing transformation of form

$$\hat{n}\tau = \hat{n} e^{2\kappa \gamma^5} = \hat{n} e^{\kappa \gamma^5} \cdot e^{\kappa \gamma^5} = e^{-\kappa \gamma^5} \hat{n} e^{\kappa \gamma^5} = n_{\mu} \gamma^\prime_{\mu},$$

we see that $\gamma$-matrices in another representation are appeared

$$\gamma^\prime_{\mu} = T^{-1} \gamma_{\mu} T, \quad T = e^{\kappa \gamma^5}, \quad \gamma^{\prime 5} = \gamma^5.$$  \hspace{1cm} (C3)

In the general case when CP is not conserved the $\tau$ can contain matrix $\hat{n}\gamma^5$ as well. In that case in order to reduce projector into standard view a one more transformation of $\gamma$-matrices is required

$$\gamma_{\mu} \rightarrow \gamma^\prime_{\mu} \rightarrow \gamma^{\prime\prime}_{\mu}, \quad \gamma^{\prime\prime}_{\mu} = e^{-\beta \hat{n} \gamma^5} \gamma^\prime_{\mu} e^{\beta \hat{n} \gamma^5}. $$  \hspace{1cm} (C4)

Finally, let us note that transformations (C3), (C4) are not unitary.

[1] V. Berestetskii, L. Pitaevskii, and E. Lifshitz, *Quantum Electrodynamics*, Vol. 4 (Elsevier Science, 2012).

[2] V. M. Budnev, I. F. Ginzburg, G. V. Meledin, and V. G. Serbo, Phys. Rept. 15, 181 (1975).
[3] T. Aaltonen et al. (CDF), Phys. Rev. D83, 031104 (2011), arXiv:1012.3093 [hep-ex].
[4] V. M. Abazov et al. (D0), Phys. Rev. Lett. 108, 032004 (2012), arXiv:1110.4194 [hep-ex].
[5] S. Chatrchyan et al. (CMS), Phys. Rev. Lett. 112, 182001 (2014), arXiv:1311.3924 [hep-ex].
[6] G. Aad et al. (ATLAS), Phys. Rev. Lett. 111, 232002 (2013), arXiv:1307.6511 [hep-ex].
[7] F.-P. Schilling, Int. J. Mod. Phys. A27, 1230016 (2012), arXiv:1206.4484 [hep-ex].
[8] W. Bernreuther and P. Uwer, Proceedings, Advances in Computational Particle Physics: Final Meeting (SFB-TR-9), Nucl. Part. Phys. Proc. 261-262, 414 (2015).
[9] J. D. Bjorken and S. D. Drell, Relativistic quantum mechanics (McGraw-Hill, 1964).
[10] V. Baier, V. S. Fadin, and V. A. Khoze, Nucl.Phys. B65, 381 (1973).
[11] V. N. Baier, E. A. Kuraev, V. S. Fadin, and V. A. Khoze, Phys. Rept. 78, 293 (1981).
[12] A. Kaloshin and V. Lomov, Eur.Phys.J. C72, 2094 (2012), arXiv:1111.1284 [hep-ph].
[13] A. E. Kaloshin and V. P. Lomov, Int. J. Mod. Phys. A31, 1650031 (2016), arXiv:1501.06337 [hep-ph].
[14] L. Landau and E. Lifshitz, Quantum Mechanics: Non-Relativistic Theory, Teoreticheskaia fizika, Vol. 3 (Elsevier Science, 2013).
[15] I. F. Ginzburg and V. G. Serbo, Phys. Rev. D49, 2623 (1994).
[16] M.-S. Chen and P. M. Zerwas, Phys. Rev. D12, 187 (1975).
[17] V. S. Fadin, V. A. Khoze, and A. D. Martin, Phys. Rev. D56, 484 (1997), arXiv:hep-ph/9703402 [hep-ph].