Accounting for physically nonlinear deformation of the shell under flat loading based on the finite element method

A Sh Dzhabrailov*, Yu V Klochkov and A P Nikolaev
Volgograd State Agrarian University, Volgograd, Russia

E-mail: arsen82@yandex.ru

Abstract. On the basis of the theory of small elastic plastic deformations obtained at the loading step in the curvilinear coordinate system of the relationship between stress increments and strain increments, which are implemented in the matrix of shell stiffness under flat loading in this paper, algorithms for obtaining physically nonlinear relations at the loading step in finite element calculations of shells under flat loading are developed. On the basis of relations of deformation theory of plasticity the equations of connection between increments of stresses and increments of deformations at flat loading in elastic plastic stage of work are received. The coefficients of proportionality are obtained when using the strain diagram of the material. The found defining relations are implemented in the algorithm of forming the stiffness matrix of the finite element of the shell under flat loading. Specific examples show the effectiveness of the developed algorithms for the calculation of the shell with a flat loading beyond the elasticity limits.

Keywords: shell, stiffness matrix, deformation, physical nonlinearity, finite element

1. Introduction

The theory of shells today is at a fairly high level of development [1-6]. Technological progress in the national economy determines the use of shell structures of increasingly complex configuration. The implementation of solving equations of shell theory in practice remains a rather difficult task, in the solution of which the most effective are numerical methods, such as the finite element method [7-10]. Accounting for physical nonlinearity is one of the fundamental problems in the calculation of shell structures. Its solution will allow more efficient use of the strength properties of the materials used in the design of new construction projects, thereby expanding the range of applications of analytical calculations. Among the set of approaches used in finite element calculations of shells taking into account the physical nonlinearity [11-14], the following are distinguished: the preparation of nonlinear equilibrium equations with respect to the initial metric and their solution using an iterative procedure; the method of sequential step loading with a reference metric defined in the initial not deformed state. Each of these approaches has its advantages and disadvantages and none of them can be considered universal. In this paper, the method of step loading is used in the finite element implementation of the calculation of shells of rotation, as it allows to obtain solutions under significant loads. The defining equations at the loading step are obtained by differentiating the basic relations of the deformation theory of plasticity.

2. Materials and Methods

**Basic relation.** Consider a shell structure in the form of an elliptical cylinder under flat loading. The radius-vector of the middle surface, in this case, is determined by the formula

$$R^0 = b \sin(t) \hat{i} + c \cos(t) \hat{k}.$$  \hspace{1cm} (1)
The unit vector of the local basis tangent to the meridian can be obtained by differentiating (1) by the global coordinates

$$a_n^0 = b \cdot \cos(t), b \cdot \sin(t), i \cdot \sin(t).$$

The unit vector of the normal to the middle surface of an elliptic cylinder is determined by the vector product

$$a_n^0 = a_i^0 \times j = c \cdot \sin(t), b \cdot \cos(t), k.$$  

By differentiating (2) and (3) we find the derivatives of the vectors of the local basis

$$a_n^{0,s} = b \cdot \left( \cos(t) \cdot \sin(t) \cdot \sin(t) \cdot \cos(t) \cdot \left( t_{ss} \right)^2 + \sin(t) \cdot t_{ss} \right) i + c \cdot \left( \cos(t) \cdot t_{ss} - \sin(t) \cdot t_{ss} \right) j,$$

$$a_n^{0,s} = c \cdot \left( \cos(t) \cdot \left( t_{ss} \right)^2 + \sin(t) \cdot t_{ss} \right) i + b \cdot \left( \cos(t) \cdot \sin(t) \cdot \sin(t) \cdot \cos(t) \cdot \left( t_{ss} \right)^2 + \sin(t) \cdot t_{ss} \right) k.$$  

For convenience, the expression (2)-(4) write in matrix form

$$\{ a^0 \} = [m^0] \{ i \}; \quad \{ a^0_s \} = [m] \{ i \}.$$  

where \( \{ a^0 \}^T = \{ a_i^0 a_n^0 \}; \quad \{ a^0_s \}^T = \{ a_i^0_s a_n^0_s \}; \quad \{ i \}^T = [i j k].$$

On the basis of (5) we obtain the dependence

$$\{ i \} = [m^0]^{-1} \{ a^0 \}.$$  

Then the derivatives of the vectors of the local basis, taking into account (6), we express through the global coordinates in the original basis

$$\{ a^0_i \} = [m][m^0]^{-1} \{ a^0 \} = [n] \{ a^0 \}.$$  

The position of the midpoint of the surface during deformation is determined by equality

$$R = R^0 + v.$$  

The displacement vector of a point in the middle surface \( v \), part (8), is determined through the nodal unknowns in the following way

$$v = u a_i^0 + w a_n^0.$$  

By performing the operation of differentiation (9) on the curvilinear coordinate \( s \), taking into account (7) it is possible to obtain derivatives of the displacement vector of the point of the middle surface of the shell

$$v_s = t_i^0 a_i^0 + t_n^0 a_n^0; \quad v_{ss} = t_i^0 a_i^0 + t_n^0 a_n^0.$$  

Having differentiated the expression (8) by the global coordinate \( s \), we determine the vectors of the inner point of the middle surface in the deformed state

$$a_1 = R_s = (R^0 + v)_s = a_i^0 + t_i^0 a_i^0 + t_n^0 a_n^0.$$  

The unit normal vector in the deformed state is defined as follows

$$a_n = a_i^0 \times j.$$  

The radius vectors of the point of an arbitrary layer of the shell in the initial and deformed states can be represented by dependencies

$$R^0 = R^0 + \zeta a_i^0; \quad R^0 = R^0 + \delta v.$$  

The displacement vector \( V \) of the point of an arbitrary layer of the shell in the deformed state is determined on the basis of the hypothesis of direct normals

$$V = v + \zeta (a - a^0).$$  

Curvatures and deformations in an arbitrary layer of the shell spaced from the middle surface at a distance \( \zeta \) are determined by the formulas of continuum mechanics

$$\varepsilon^0_{\alpha\beta} = \left( g_{\alpha\beta} + \delta g_{\alpha\beta} \right)/2;$$
where \( g_{\alpha\beta} \) and \( g^0_{\alpha\beta} \) are components of the metric tensor in the deformed and initial states determined by the corresponding scalar products of covariant vectors of the basis of points in an arbitrary shell layer.

The expression (15) for plane loading will have the form

\[
\varepsilon_1 = g_{10} \sigma \to \epsilon^0 \quad \varepsilon_2 = 0
\]

where \( g_{10} = R_{s} \) and the vector \( \mathbf{V} \) is determined by differentiation (14), taking into account (10), (11) and (12).

Finally get

\[
\varepsilon_1 = \varepsilon_1^t + \varepsilon_{11}
\]

where \( \varepsilon_1^t = u, s \); \( \varepsilon_{11} = -w, ss \).

**Binding ratio of the deformation theory** When assembling the plasticity matrix included in the structure of the stiffness matrix, it is necessary to establish the relationship between the strains and stresses at the loading step. For this purpose, the second hypothesis of the deformation theory of plasticity was used in this article, according to which the components of the strain deviator are related to the components of the stress deviator by the ratio

\[
\gamma = \frac{\varepsilon_{11}}{\sigma_{11}} = \frac{\varepsilon_{12}}{\sigma_{12}} = \frac{\varepsilon_{33}}{\sigma_{33}} = \gamma
\]

where the function \( \gamma \) is determined by the ratio of the strain intensity \( \varepsilon_i \) to the intensity of stress \( \sigma_i \); \( \gamma \to \frac{3 \varepsilon_i}{2 \sigma_i} \) and \( \varepsilon_0 \) and \( \sigma_0 \) - the average linear strain and stress, respectively.

Using the data from the tensile test results, it is possible to determine the relationship between the stress intensity and the strain intensity at the point of an arbitrary shell layer

\[
\sigma_i = \sigma_i, \varepsilon_i = \frac{1 - 2 \mu}{E} \sigma_i
\]

where \( \mu \)-ratio of Poisson.

On the basis of (19) we can construct a stress-strain diagram. The nonlinear section of the deformation curve in this paper is approximated as a function

\[
\sigma_i = a \varepsilon_i^2 + b \varepsilon_i + c
\]

where the coefficients \( a, b \) and \( c \) were determined on the basis of experimental data of the curve (19).

The dependence (18) in the case of flat loading will take the form

\[
\frac{\varepsilon_1}{\sigma_{11}} = \frac{\varepsilon_2}{\sigma_{22}} = \frac{\varepsilon_3}{\sigma_{33}} = \gamma
\]

(21)

The increments of the strain tensor are determined by equality

\[
\Delta \varepsilon_{11} = \frac{c_1}{2} \sigma_{11}, \Delta \varepsilon_{33} = \frac{c_2}{2} \sigma_{33}
\]

(22)

To determine the differential operators \( \partial \varepsilon_{11} / \partial \sigma_{11} \), included in (22), it is necessary to express the strain of (21) explicitly

\[
\varepsilon_{11} = \frac{\sigma_{11}}{3}, \varepsilon_{33} = \frac{\sigma_{33}}{3}
\]

(23)

By performing the differentiation (23) we obtain

\[
\frac{\partial \varepsilon_{11}}{\partial \sigma_{11}} = \frac{\sigma_{11}}{2}, \frac{\partial \varepsilon_{33}}{\partial \sigma_{33}} = \frac{\sigma_{33}}{2}
\]

(24)
where $E_k$ and $E_c$ - the tangent and the secant modulus deformation diagram.

Finally, the expression (22) can be written in matrix form

$$\{\Delta \varepsilon_{\alpha \beta}^\varepsilon\} = [D] \{\Delta \sigma_{\alpha \beta}\}. \quad (25)$$

where $[D]^{-1}$-matrix plasticity $[C_p]$. 

Discretization element and stiffness matrix at the loading step. A one-dimensional finite element was chosen to implement the numerical integration procedure. This fragment of the shell, bounded by two parallel planes, is approximated to a linear element with the coordinates of the nodes $\eta = -1$ and $\eta = 1$, where $\eta$ is the local coordinate used for the convenience of numerical integration (figure 1).

![Figure1. The discretization element at the loading step.](image)

In local coordinates the global coordinates of the shell are determined by the dependencies

$$s = \frac{(1-\eta)}{2} s_i + \frac{(1+\eta)}{2} s_j ; \quad t = \frac{(1-\eta)}{2} t_i + \frac{(1+\eta)}{2} t_j. \quad (26)$$

In the case of finite element analysis of the shell, the components of the displacement vector and their derivatives were chosen as nodal unknowns at the loading step

$$\{U\}^T = \{\{u\}^T \{w\}^T\} \quad (27)$$

where $\{u\}^T = \{u^i u^j u^\eta\}, \{w\}^T = \{w^i w^j w^\eta\}$.

The relationship between the derivatives in the local and global coordinate systems is determined by dependencies

$$\frac{\partial q}{\partial \eta} = \frac{\partial q}{\partial s} \frac{\partial s}{\partial \eta} ; \quad \frac{\partial^2 w}{\partial \eta^2} = \frac{\partial^2 w}{\partial s^2} \left(\frac{\partial s}{\partial \eta}\right)^2, \quad (28)$$

where $q$ is the meridional ($u$) or normal ($w$) components of the displacement vector at the loading step.

The meridional and normal displacements of the inner point of the finite element at the loading step are expressed through columns of nodal unknowns as follows

$$u = \{\varphi\}^T \{u\}, \quad w = \{\varphi\}^T \{w\}. \quad (29)$$

The functions of the form included in (29) matrix-rows are determined by Hermite polynomials of the third degree by corresponding expressions

$$\{\varphi\} = \{h_1 h_2 h_3 h_4\}^T \quad (30)$$

By differentiating (29) it is possible to obtain increment derivatives at the loading step in the global coordinate system

$$u_s = \{\varphi_{,\eta}\}^T \frac{\partial \eta}{\partial s} \{u\}, \quad w_s = \{\varphi_{,\eta}\}^T \frac{\partial \eta}{\partial s} \{w^\eta\}, \quad w_{ss} = \{\varphi_{,\eta\eta}\}^T \left(\frac{\partial \eta}{\partial s}\right)^2 \{w\}. \quad (31)$$

where $\frac{\partial \eta}{\partial s} = \frac{2}{s_i - s_j}$.

The stiffness matrix and the column of nodal loads of a one-dimensional finite element at the loading step are formed on the basis of the equality of the external and internal forces on the possible displacement [15-17]
\[ \int_V \left\{ \Delta \varepsilon_{\alpha \beta}^\xi \right\}^T \left( \{(\sigma_{\alpha \beta})\} + \{\Delta \sigma_{\alpha \beta}\} \right) dV = \int_F \{A\Upsilon\}^T \left( \{(P)\} + \{\Delta P\} \right) dF. \]  

(32)

In an arbitrary layer of the shell, the increment of deformation can be defined as the matrix product

\[ \{\Delta \varepsilon_{\alpha \beta}^\xi\} = [G][B]\{\Delta \Upsilon_{\alpha \beta}\}, \]

(33)

where matrices \([G]\) and \([B]\) are formed on the basis of (17).

The stress component increments at the loading step can be determined using the ratio

\[ \{\Delta \sigma_{\alpha \beta}\} = [C_{pl}]\{\Delta \varepsilon_{\alpha \beta}^\xi\}, \]

(34)

where \([C_{pl}]\) - matrix plasticity.

The string of increments of the components of the displacement vector at the loading step can be expressed in a generalized column of increments of the nodal variable parameters of the finite element

\[ \{\Delta \Upsilon\} = [A]\{\Delta \Upsilon_{\alpha \beta}\}, \]

(35)

where \([A]\) is a diagonal matrix containing polynomials of the third degree.

Equality (32), taking into account (33)-(35) can be written in matrix form as follows

\[ [K]\{\Delta \Upsilon_{\alpha \beta}\} = \{F\} + \{f\}, \]

(36)

where \([K] = [I]^T \int_V \{B\}^T\{C_{pl}\}\{I\}[B] dV\) is the stiffness matrix of a quadrangular finite element at the loading step; \([F]^T = [I]^T \int_F \{A\}^T\{\Delta P\} dF\) is the column of nodal loads at the loading step; \([I] = [I]^T \left( \int_F \{A\}^T\{P\} dF - \int_V \{B\}^T\{I\}^T\{\sigma\} dV \right)\) - Newton-Raphson correction.

3. Results

As an example, the problem of determining the stress-strain state of an arbitrary shell in the form of a fragment of an elliptical cylinder was solved (figure 2) loaded by internal pressure of intensity \(q\). The following initial data were taken: \(q=0.18 \text{ MPa}, \mu=0.32; E=7.5 \cdot 10^5 \text{ MPa}; \) shell thickness \(t=0.005 \text{ m};\) the first parameter of the cylinder \(b=1 \text{ m};\) the second parameter of the cylinder \(c=0.5 \text{ m}.\) Shell in the reference section has a rigid clamping, and the right edge is free.

The calculation results are presented in table 1, which shows the numerical values of the meridional stresses on the inner, outer and middle surfaces of the shell, as well as in the fibers of the reference section.

Table 1. The results of calculations of stresses across the shell thickness by varying the number of steps of loading in the reference section

| No. | 1   | 2   | 3   | 4   | 5   | 6   | 7   |
|-----|-----|-----|-----|-----|-----|-----|-----|
|     |     |     |     |     |     |     |     |

Figure 2. Fragment of an elliptical cylinder.
When varying the number of finite elements, it was found their optimal to achieve a satisfactory convergence of the computational process quantitative value equal to 50.

To find an analytical solution, the calculation scheme presented in figure 3 was used

![Design model](image)

**Figure3.** Design model.

The total bending moment relative to point A (figure 3) is found as follows:

\[
M_A = M_1 + M_2 = \frac{qc^2}{2} + \frac{qb^2}{2} = 112.5 (\text{H} \cdot \text{m}).
\]  

(37)

To find the calculated value of the bending moment in the reference section, we present schematically the data in table 1 with the number of loading steps equal to 100 in the form of a diagram (Figure 4).

The calculated moment in the reference section is found as the sum of all moments to the neutral axis:

\[
M_c = \sum M_i (i=1...12) = 109.98 (\text{H} \cdot \text{m})
\]

(38)

As can be seen from table 1, the increase in the number of loading steps does not lead to a significant change in the numerical values of the controlled parameters of the stress-strain state. In addition, the error of calculation of the bending moment in the reference section was 2.24% compared to the calculated analytically.
Figure 4. Plot of meridional stresses in the reference section along the shell thickness, MPa.

Conclusions

Thus, it can be concluded that the developed algorithm for taking into account possible plastic deformations in the calculations of arbitrary shells under flat loading has shown sufficient efficiency and can be recommended in the design engineering practice for the analysis of the stress-strain state of such structures. The resulting defining equations at the loading step by differentiating the relations of the deformation theory of plasticity between stresses and strains allow for the finite element implementation to take into account at the j-th loading step the residual equal to the difference between the work of internal and external forces for the previous loading steps.

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