The competition number of a graph in which any two holes share at most one edge

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Abstract
The competition graph of a digraph $D$ is a (simple undirected) graph which has the same vertex set as $D$ and has an edge between $x$ and $y$ if and only if there exists a vertex $v$ in $D$ such that $(x, v)$ and $(y, v)$ are arcs of $D$. For any graph $G$, $G$ together with sufficiently many isolated vertices is the competition graph of some acyclic digraph. The competition number

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$k(G)$ of $G$ is the smallest number of such isolated vertices. In general, it is hard to compute the competition number $k(G)$ for a graph $G$ and it has been one of important research problems in the study of competition graphs to characterize a graph by its competition number.

A hole of a graph is a cycle of length at least 4 as an induced subgraph. It holds that the competition number of a graph cannot exceed one plus the number of its holes if $G$ satisfies a certain condition. In this paper, we show that the competition number of a graph with exactly $h$ holes any two of which share at most one edge is at most $h + 1$, which generalizes the existing results on this subject.

**Keywords:** competition graph; competition number; hole

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1 Introduction and Preliminaries

1.1 Introduction

Let $D$ be an acyclic digraph. The *competition graph* of $D$, denoted by $C(D)$, is the (simple undirected) graph which has the same vertex set as $D$ and has an edge between two distinct vertices $x$ and $y$ if and only if there exists a vertex $v$ in $D$ such that $(x, v)$ and $(y, v)$ are arcs of $D$. For any graph $G$, $G$ together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. From this observation, Roberts [15] defined the *competition number* $k(G)$ of a graph $G$ to be the smallest number $k$ such that $G$ together with $k$ isolated vertices is the competition graph of an acyclic digraph.

The notion of competition graph was introduced by Cohen [2] as a means of determining the smallest dimension of ecological phase space. Since then, various variations have been defined and studied by many authors (see [4, 12] for surveys). Besides an application to ecology, the concept of competition graph can be applied to a variety of fields, as summarized in [14]. Roberts [15] observed that characterization of competition graph is equivalent to computation of competition number. It does not seem to be easy in general to compute $k(G)$ for a given graph $G$, as Opsut [13] showed that the computation of the competition number of a graph is an NP-hard problem (see [4, 6] for graphs whose competition numbers are known).

It has been one of important research problems in the study of competition graphs to characterize a graph by its competition number. From this point of view, we study the relationship between the competition number and the number of holes of a graph.

A cycle in a graph is called an *induced cycle* (also called a *chordless cycle* or a *simple cycle*) if it is an induced subgraph of the graph. A *hole* in a graph is an induced cycle of length at least 4 in the graph. We denote the number of holes in a graph $G$ by $h(G)$. A graph without holes is called a *chordal graph*.

The competition number of a graph with a few holes has been studied:

**Theorem 1.1** (Roberts [15]). Let $G$ be a chordal graph. Then the competition number of $G$ is at most 1.

**Theorem 1.2** (Cho and Kim [1]). Let $G$ be a graph with exactly one hole. Then the competition number of $G$ is at most 2.

**Theorem 1.3** (Lee, Kim, Kim, and Sano [8], Li and Chang [11]). Let $G$ be a graph with exactly two holes. Then the competition number of $G$ is at most 3.

Recently, it has been shown that the competition number of a graph with exactly $h$ holes is at most $h + 1$ under several assumptions.
Definition (Li and Chang [10]). A hole $C$ of a graph $G$ is called independent if, for any hole $C'$ of $G$,

- $|V(C) \cap V(C')| \leq 2$.
- If $|V(C) \cap V(C')| = 2$, then $|E(C) \cap E(C')| = 1$ and $|V(C)| \geq 5$.

**Theorem 1.4** ([10]). Let $G$ be a graph with exactly $h$ holes satisfying the following property (LC):

(LC) All the holes of $G$ are independent.

Then the competition number of $G$ is at most $h + 1$.

**Theorem 1.5** (Kamibeppu [3]). Let $G$ be a graph with exactly $h$ holes satisfying the following property (K):

(K) For any hole $C$ of $G$, there exists an edge $e_C$ of the hole $C$ such that the edge $e_C$ is not contained in any other induced cycle of $G$.

Then the competition number of $G$ is at most $h + 1$.

**Theorem 1.6** (Kim, Lee, and Sano [7]). Let $G$ be a graph with exactly $h$ holes satisfying the following property (E0):

(E0) Any two distinct holes of $G$ are mutually edge disjoint.

Then the competition number of $G$ is at most $h + 1$.

In this paper, we generalize the above results except Theorem 1.3. Our main result is the following:

**Theorem 1.7.** Let $G$ be a graph with exactly $h$ holes satisfying the following property:

(E1) For any two distinct holes $C_1$ and $C_2$ of $G$, $|E(C_1) \cap E(C_2)| \leq 1$.

Then the competition number of $G$ is at most $h + 1$.

### 1.2 Relationships among conditions

We remark that the class of graphs satisfying the condition (E1) is larger than the class of graphs satisfying one of the conditions (LC), (K), and (E0). (See Figure 1, Figure 2 and Table 1. Give examples which shows that each region $i$ in Figure 1 is not empty.)
Remark 1.8. The disjointness of the class of graphs which satisfy the condition (E0) and do not satisfy the condition (LC) (the region 14 in Figure 1) with the class of graphs satisfying the condition (K) or the class of graphs with \( h(G) = 2 \) follows from \cite[Theorem 1.4]{7} that states a graph which satisfies the condition (E0) and do not satisfy the condition (LC) must have an induced subgraph isomorphic to the complete tripartite graph \( K_{2,2,2} \). Here \( K_{2,2,2} \) has three holes which violate the condition (K).

Proposition 1.9. If a graph \( G \) satisfies the condition (LC), then \( G \) also satisfies the condition (E1).

Proof. If a graph \( G \) satisfies the condition (LC), then it holds that \( |V(C) \cap V(C')| \leq 2 \) for any distinct holes \( C \) and \( C' \). This implies that \( |E(C) \cap E(C')| \leq 1 \) holds for any distinct holes \( C \) and \( C' \). Hence \( G \) satisfies the condition (E1).

The following proposition is rather obvious:

Proposition 1.10. If a graph \( G \) satisfies the hole-edge-disjoint condition (E0), then \( G \) also satisfies the condition (E1).
Figure 2: Examples of graphs where $\Gamma_i$ belongs to the region $i$ in Figure 1

To show that the class of graphs satisfying the condition (K) is contained in the class of graphs satisfying the condition (E1), we prepare the following two lemmas:

**Lemma 1.11.** Let $G$ be a graph and $C$ be a cycle of $G$. Then exactly one of the
Table 1: Graphs $\Gamma_i$ in Figure 2 and the conditions (LC), (K), (E0), (E1)

| Graph | $h(G)$ | (LC) | (K) | (E0) | (E1) |
|-------|--------|------|-----|------|------|
| $\Gamma_1$ | 0 | * | * | * | * |
| $\Gamma_2$ | 1 | * | * | * | * |
| $\Gamma_3$ | 1 | * | * | * | * |
| $\Gamma_4$ | 2 | * | * | * | * |
| $\Gamma_5$ | 3 | * | * | * | * |
| $\Gamma_6$ | 2 | * | * | * | * |
| $\Gamma_7$ | 3 | * | * | * | * |
| $\Gamma_8$ | 2 | * | * | * | * |
| $\Gamma_9$ | 3 | * | * | * | * |
| $\Gamma_{10}$ | 2 | * | * | * | * |
| $\Gamma_{11}$ | 3 | * | * | * | * |
| $\Gamma_{12}$ | 2 | * | * | * | * |
| $\Gamma_{13}$ | 3 | * | * | * | * |
| $\Gamma_{14}$ | 3 | * | * | * | * |
| $\Gamma_{15}$ | 2 | * | * | * | * |
| $\Gamma_{16}$ | 3 | * | * | * | * |
| $\Gamma_{17}$ | 2 | * | * | * | * |
| $\Gamma_{18}$ | 3 | * | * | * | * |

following holds:

(a) $C$ is an induced cycle of $G$,

(b) There exist induced cycles $C_1, \ldots, C_s$ ($s \geq 2$) in $G$ such that

1. $V(C_i) \subsetneq V(C)$ ($i = 1, \ldots, s$),
2. Any edge $e$ of $C$ is an edge of $C_i$ for some $i \in \{1, \ldots, s\}$,
3. For any edge $e$ of $E(C_i) \setminus E(C)$, there exists unique $j \in \{1, \ldots, i - 1, i + 1, \ldots, s\}$ such that $e \in E(C_j)$.

Proof. We show by induction on $|V(C)|$. If $|V(C)| = 3$, then (a) holds and (b) does not hold since any cycle of length 3 is an induced cycle. Assume that the lemma holds for any cycle $C$ with $|V(C)| = t$. Let $C = v_0v_1v_2 \cdots v_tv_0$ be a cycle with $|V(C)| = t + 1$. Consider the subgraph $H$ of $G$ induced by $V(C)$. If $H$ is a cycle, then $C$ is an induced cycle in $G$ and so (a) holds. If $H$ is not a cycle, then $C$ is not an induced cycle in $G$ and so (a) does not hold. Now we show (b) holds. Note that
any edge in \( E(H) \setminus E(C) \) is a chord for \( C \). Let \( e^* = v_i v_j \) be a minimum chord for \( C \), i.e., \(|i - j| \) is smallest among all the chords for \( C \). Then, the \((v_i, v_j)\)-section \( P_1 \) of the cycle \( C \) and the edge \( e^* \) form an induced cycle \( C^* \) in \( G \) satisfying \( V(C^*) \subseteq V(C) \) and \( e \in E(C^*) \) for \( e \in E(P) \), and the \((v_j, v_i)\)-section \( P_2 \) of the cycle \( C \) and the edge \( e^* \) form a cycle \( C'' \) in \( G \) with \(|V(C'')| < t + 1 \). By the induction hypothesis, one of the following holds: (a)' \( C' \) is an induced cycle of \( G \); (b)' there exist induced cycles \( C_1', \ldots, C_{s'}' \) (\( s' \geq 2 \)) in \( G \) such that the conditions (1)-(3) of \( (b) \) hold. If (a)' holds, then let \( \mathcal{C} = \{C', C''\} \). If (b)' holds, then let \( \mathcal{C} = \{C_1', \ldots, C_{s'}', C''\} \). In each case, the family \( \mathcal{C} \) of induced cycles in \( G \) satisfies the conditions (1)-(3) of \( (b) \). Thus (b) holds. Hence the lemma holds.

The following lemma is well-known:

**Lemma 1.12.** Let \( C \) and \( C' \) be two induced cycles in a graph \( G \). Then, the subgraph of \( G \) induced by the symmetric difference of \( E(C) \) and \( E(C') \) is an edge-disjoint union of cycles of \( G \).

**Proposition 1.13.** If a graph \( G \) satisfies the condition (K), then \( G \) also satisfies the condition (E1).

**Proof.** Suppose that the condition (E1) does not hold, i.e., there are two distinct holes \( C \) and \( C' \) such that \(|E(C) \cap E(C')| \geq 2 \). Consider the subgraph \( H \) of \( G \) induced by \((E(C) \cup E(C')) \setminus (E(C) \cap E(C'))\). By Lemma 1.12, \( H \) is an edge-disjoint union of cycles \( C_1, \ldots, C_k \) (\( k \geq 1 \)) of \( G \). Note that there is no triangle in \( \{C_1, \ldots, C_k\} \) (otherwise, an edge of a triangle would be a chord of the hole \( C \) or the hole \( C' \), which is a contradiction). If there is a hole \( C_i \) in \( \{C_1, \ldots, C_k\} \), then all the edges in \( C_i \) are contained in the hole \( C \) or the hole \( C' \), and so the hole \( C_i \) violates the condition (K). Therefore we may assume that any cycle in \( \{C_1, \ldots, C_k\} \) is not an induced cycle. By Lemma 1.11, there exist induced cycles \( C_{i,s} \) satisfying the conditions (1)-(3) of (b) in Lemma 1.11 for each \( C_i \) (\( 1 \leq i \leq k \)). Note that we can take \( C_{i,s} \) so that every \( C_{i,s} \) is different from the holes \( C \) and \( C' \) since \(|E(C) \cap E(C')| \geq 2 \). Let \( \mathcal{C} := \{C, C'\} \cup \{C_{i,s} \mid i \in \{1, \ldots, k\}, j \in \{1, \ldots, s_i\}\} \). If there is a hole \( C_{i,j} \) in the family \( \mathcal{C} \) other than \( C \) and \( C' \), then the hole \( C_{i,j} \) violates the condition (K) since each edge in \( C_{i,j} \) is contained in an induced cycle in \( \{C, C', C_{i,1}, \ldots, C_{i,s_i}\} \). Therefore, all the induced cycles in \( \mathcal{C} \) other than \( C \) and \( C' \) should be triangles. However, then, each edge in \( E(C) \setminus E(C') \) is contained in a triangle in \( \mathcal{C} \setminus \{C, C'\} \) and each edge in \( E(C) \cap E(C') \) is contained in the hole \( C' \). Thus the hole \( C \) violates the condition (K). Hence the condition (K) does not hold in any case, and so the proposition holds.

\( \square \)
1.3 Preliminaries

A set $S$ of vertices of a graph $G$ is called a clique of $G$ if the subgraph of $G$ induced by $S$ is a complete graph. A set $S$ of vertices of a graph $G$ is called a vertex cut of $G$ if the number of connected components of $G - S$ is greater than that of $G$.

For a hole $C$ in a graph $G$, we denote by $X_C$ the set of vertices which are adjacent to all the vertices of $C$:

$$X_C := \{ v \in V(G) \mid uv \in E(G) \text{ for all } u \in V(C) \}. \quad (1.1)$$

Note that $V(C) \cap X_C = \emptyset$. Given a walk $W$ of a graph $G$, we denote by $W^{-1}$ the walk represented by the reverse of vertex sequence of $W$. For a graph $G$ and a hole $C$ of $G$, we call a walk (resp. path) $W$ a $C$-avoiding walk (resp. $C$-avoiding path) if one of the following holds:

- the length of $W$ is greater than or equal to 2 and none of the internal vertices of $W$ are in $V(C) \cup X_C$;
- the length of $W$ is 1 and one of the two vertices of $W$ is not in $V(C) \cup X_C$.

Throughout this paper, we assume that all subscripts of vertices on a cycle are reduced to modular the length of the cycle.

**Theorem 1.14** ([8, Theorem 2.2]). Let $G$ be a graph and $k$ be a nonnegative integer. Suppose that $G$ has a subgraph $G_1$ with $k(G_1) \leq k$ and a chordal subgraph $G_2$ such that $E(G_1) \cup E(G_2) = E(G)$ and $X := V(G_1) \cap V(G_2)$ is a clique of $G_2$. Then $k(G) \leq k + 1$.

**Lemma 1.15** ([7, Lemma 2.1]). Let $G$ be a graph and $C$ be a hole of $G$. Let $x$ and $y$ be two non-adjacent vertices on $C$. Suppose that there exists a common neighbor $v$ of $x$ and $y$ not on the hole $C$. Then exactly one of the following holds:

(a) $v \in X_C$;

(b) There exists a hole $C^*$ such that $v \in V(C^*)$, $|E(C) \cap E(C^*)| \geq 2$, and all the common edges are contained in exactly one of the $(x,y)$-sections of $C$.

**Lemma 1.16** ([8, Lemma 2.4]). Let $G$ be a graph and $C$ be a hole of $G$. Suppose that there exists a vertex $v$ such that $v$ is adjacent to consecutive vertices $v_i$ and $v_{i+1}$ of $C$, and that $v$ is not in $X_C$ and not on any hole of $G$. Then, $v$ is not adjacent to any vertex in $V(C) \setminus \{v_i, v_{i+1}\}$.
2 Structure of graphs satisfying the condition (E1)

2.1 Properties of graphs satisfying the condition (E1)

Lemma 2.1. Let $G$ be a graph satisfying the condition (E1). Then $G$ is $K_{2,3}$-free.

Proof. Suppose that $G$ has an induced subgraph $H$ isomorphic to $K_{2,3}$. Let $V(H) = \{x_1, x_2, y_1, y_2, y_3\}$ and $E(H) = \{x_iy_j \mid i \in \{1, 2\}, j \in \{1, 2, 3\}\}$. Then $C_1 := x_1y_1x_2y_2x_1$ and $C_2 := x_1y_1x_2y_3x_1$ are holes having two common edges $x_1y_1$ and $x_2y_1$, which is a contradiction to the condition (E1). \qed

Proposition 2.2. Let $G$ be a graph satisfying the condition (E1). Then any two distinct holes of $G$ share at most two vertices.

Proof. By contradiction. Suppose that there exist two distinct holes $C_1$ and $C_2$ in $G$ which share at least three vertices. Let $u$, $v$, and $w$ be three distinct common vertices of $C_1$ and $C_2$. Then they do not induce a triangle in $G$ since $C_1$ and $C_2$ are holes. Without loss of generality, we may assume that $u$ and $v$ are not adjacent. Let $P_1$ be the $(u, v)$-section of $C_1$ containing $w$ and let $P_2$ be the $(u, v)$-section of $C_2$ not containing $w$. (See Figure 3.)

Now we consider the subgraph $H$ of $G$ induced by $V(P_1) \cup V(P_2)$. Since $C_1 \neq C_2$, $P_1$ cannot be the other $(u, v)$-section of $C_2$ and $P_2$ cannot be the other section of $C_1$. Thus $H$ is distinct from $C_1$ and $C_2$. If $w$ is adjacent to an internal vertex in $P_2$, then the edge is a chord of $C_2$ and we reach a contradiction. Thus $w$ has degree 2 in $H$. Since $w$ is an internal vertex of $P_1$, $w$ has its neighbors which are also on $P_1$. Let $a$ be a neighbor of $w$ closer to $u$ on $P_1$ and $b$ be the other neighbor of $w$. Then the $(a, u)$-section of $P_1$, $P_2$, and the $(v, b)$-section of $P_1$ form an $(a, b)$-walk in $H$ not containing $w$. Let $P$ be a shortest $(a, b)$-path in $H$. Then, the edge $wa$, the $(a, b)$-path $P$, and the edge $bw$ form a cycle $C$. Since $H$ is an induced subgraph of $G$, $P$ is a shortest $(a, b)$-path in $G$. Therefore the cycle $C$ is a hole in $G$. Since $C$ is also a hole in $H$, $C$ is distinct from the hole $C_1$. Now we reach a contradiction since the holes $C$ and $C_1$ share the two edges $wa$ and $wb$. \qed

2.2 Properties of $X_C$ and $X_K$

Lemma 2.3. Let $G$ be a graph satisfying the condition (E1) and $C$ be a hole of $G$. Let $x$ and $y$ be two non-adjacent vertices on $C$. If there exists a common neighbor $v$ of $x$ and $y$ not on the hole $C$, then $v \in X_C$.

Proof. Since $G$ satisfies the condition (E1), Lemma 1.15 (b) cannot happen and thus the lemma holds. \qed
Lemma 2.4. Let $G$ be a graph satisfying the condition (E1) and $C$ be a hole of length at least 5 in $G$. Then $X_C$ is a clique.

Proof. By contradiction. Suppose that there are two non-adjacent vertices $x_1$ and $x_2$ in $X_C$. Let $v_0v_1 \cdots v_{m-1}v_0$ be the sequence of the vertices of the hole $C$ where $m \geq 5$. Then $C(1) := x_1v_0x_2v_1$ and $C(2) := x_1v_0x_2v_3x_1$ are distinct holes of $G$ sharing the two edges $x_1v_0$ and $x_2v_0$, which is a contradiction.

We denote by $K^m_2$ a complete multipartite with $m$ parts each of which has size 2. If $m = 3$, then we denote $K^3_2$ also by $K_{2,2,2}$. We say that a graph is $K_{2,2,2}$-free if it does not contain a complete tripartite graph $K_{2,2,2}$ as an induced subgraph.

Theorem 2.5. Let $G$ be a graph satisfying the condition (E1). For any hole $C$ in $G$, exactly one of the following holds:

(a) $X_C$ is a clique.

(b) $C$ is contained in an induced subgraph of $G$ which is isomorphic to $K_{2,2,2}$.

Proof. Suppose that (a) does not hold. Then there are two non-adjacent vertices $x_1$ and $x_2$ in $X_C$. By Lemma 2.4 we have $|V(C)| = 4$. Therefore $V(C) \cup \{x_1, x_2\}$ induces $K_{2,2,2}$ and thus (b) holds. If (b) holds, then $|V(C)| = 4$ and we can easily see that there are two non-adjacent vertices which are adjacent to all the vertices of $C$. Thus (a) does not hold.

Corollary 2.6. Let $G$ be a $K_{2,2,2}$-free graph satisfying the condition (E1) and $C$ be a hole in $G$. Then $X_C$ is a clique.

Proof. It immediately follows from Theorem 2.5.

For a vertex $v$ in a graph $G$, we denote by $N_G(v)$ the set of vertices adjacent to $v$ in $G$. We denote the set $N_G(v) \cup \{v\}$ by $N_G[v]$. 

Figure 3: A picture for Proof of Proposition 2.2
For an induced subgraph $K$ of a graph $G$ isomorphic to $K_2^m$ for some $m \geq 2$, we denote by $X_K$ the set of vertices which are adjacent to all the vertices of $K$:

$$X_K := \{v \in V(G) \mid uv \in E(G) \text{ for all } u \in V(K)\}. \quad (2.1)$$

**Lemma 2.7.** Let $G$ be a graph satisfying the condition (E1). Let $m$ be the largest integer such that $G$ has an induced subgraph $K$ isomorphic to $K_2^m$. If $m \geq 2$, then the following hold:

(1) $X_K$ is a clique,

(2) For two non-adjacent vertices $u$, $v$ in $K$, $N_G(u) \cap N_G(v) \subseteq X_K \cup V(K)$.

**Proof.** We show (1) by contradiction. Suppose that there exist two nonadjacent vertices $x_1$ and $x_2$ in $X_K$. Then $V(K) \cup \{x_1, x_2\}$ induces a subgraph isomorphic to $K_2^{m+1}$, which contradicts the choice of $m$.

Now we show (2). Let $u$ and $v$ be two non-adjacent vertices of $K$. If $N_G(u) \cap N_G(v) \subseteq V(K)$, then (2) holds and so we assume that $(N_G(u) \cap N_G(v)) \setminus V(K) \neq \emptyset$. Take a vertex $w \in (N_G(u) \cap N_G(v)) \setminus V(K)$. To show that $w \in X_K$, take any vertex $x$ of $K$. If $x \in \{u, v\}$, then $w$ is adjacent to $x$. Now we assume that $x \in V(K) \setminus \{u, v\}$. By the definition of $K$, $x$ is adjacent to both $u$ and $v$. Since $m \geq 2$, there exists a vertex $y$ of $K$ which is not adjacent to $x$. If $w$ is not adjacent to $x$, then $C_{(1)} := uwvxu$ and $C_{(2)} := uyvxu$ are two distinct holes sharing the two edges $ux$ and $vx$, which is a contradiction. Thus the vertex $w$ is adjacent to $x$. Since $x$ is chosen arbitrarily from $V(K)$, it holds that $w \in X_K$. Hence we have $(N_G(u) \cap N_G(v)) \setminus V(K) \subseteq X_K$, and thus $N_G(u) \cap N_G(v) \subseteq X_K \cup V(K)$. \qed

### 2.3 Properties of $C$-avoiding paths for a hole $C$ of length at least 5

**Lemma 2.8.** Let $G$ be a graph satisfying the condition (E1) and $C$ be a hole of length at least 5 in $G$. Then there is no $C$-avoiding path between two non-adjacent vertices of $C$.

**Proof.** Let $C = v_0v_1 \cdots v_{m-1}v_0$ be a hole of length at least 5 in $G$, where $m \geq 5$. Suppose that there is a $C$-avoiding $(v_i, v_j)$-path for some $i, j \in \{0, 1, \ldots, m-1\}$ satisfying $|i - j| \geq 2$. Let $P$ be a shortest path among all the $C$-avoiding $(v_i, v_j)$-paths in $G$. Then there is no edge joining two non-consecutive vertices on $P$. Let $P_1$ and $P_2$ be the two $(v_i, v_j)$-sections of $C$ containing $v_{i-1}$ and $v_{i+1}$, respectively. Then $P$ and $P_1$ form a cycle $C_{(1)}$ and $P$ and $P_2$ form a cycle $C_{(2)}$ in $G$. Since both $C_{(1)}$ and $C_{(2)}$ share at least two edges with the hole $C$, these cycles cannot be holes of $G$. Since $C_{(1)}$ has a chord, an internal vertex of $P$ is adjacent to an internal vertex on $P_1$. Let $u$ be the first internal vertex on $P$ which is adjacent to an internal vertex...
on \( P_1 \). Then let \( v \) be the first internal vertex on \( P_1 \) which is adjacent to \( u \). (See Figure 4) Then the \((v_i, u)\)-section of \( P \), the edge \( uv \), and the \((v, v_i)\)-section of \( P_1^{-1} \) form a triangle or a hole. In either case, it shares the edge \( v_iv_{i-1} \) with \( C \). Thus, by the condition \((E1)\), \( v = v_{i-1} \) and \( u \) is the vertex immediately following \( v_i \) on \( P \). By applying a similar argument for \( P_2 \), we can show that \( u \) is adjacent to \( v_{i+1} \). Therefore, by Lemma 2.3, we have \( u \in X_C \). However, since \( P \) is a \( C \)-avoiding path, \( u \) does not belong to \( X_C \) and thus we reach a contradiction.

\[ \begin{array}{c}
\text{Figure 4: A picture for Proof of Lemma 2.8} \\
\end{array} \]

**Corollary 2.9.** Let \( G \) be a graph satisfying the condition \((E1)\) and \( C \) be a hole of length at least 5 in \( G \). Given a vertex \( v \) of \( C \), adding new edges joining \( v \) and any other vertices on \( C \) reduces the number of holes.

**Proof.** It is obvious that \( C \) is not a hole in the resulting graph \( G' \). Thus it is sufficient to show that no new hole has been created. We show it by contradiction. Suppose that there is a hole \( C' \) in \( G' \) which is not in \( G \). Then it contains an edge \( vw \), where \( w \) is a vertex on \( C \) which is not adjacent to \( v \) in \( G \). Then \( C' - vw \) is a \( C \)-avoiding \((v, w)\)-path in \( G \). This contradicts Lemma 2.8.

**Lemma 2.10 ([1, Lemma 4]).** Suppose that a graph \( G \) has exactly one hole \( C \). If \( G \) has a \( C \)-avoiding \((v_i, v_{i+1})\)-path for two adjacent vertices \( v_i \) and \( v_{i+1} \) on \( C \), then \( X_C \cup \{v_i, v_{i+1}\} \) is a vertex cut of \( G \).

We can extend this lemma as follows:

**Lemma 2.11.** Let \( G \) be a graph satisfying the condition \((E1)\) and \( C \) be a hole of length at least 5 in \( G \). If \( G \) has a \( C \)-avoiding \((v_i, v_{i+1})\)-path for two adjacent vertices \( v_i \) and \( v_{i+1} \) on \( C \), then \( X_C \cup \{v_i, v_{i+1}\} \) is a vertex cut of \( G \).
Proof. We prove by induction on the number \( h \) of holes of a graph. If a graph has exactly one hole, then it immediately follows from Lemma 2.10. Suppose that the lemma holds for any graph satisfying the condition (E1) with at most \( h-1 \) holes for \( h \geq 2 \). Now let \( G \) be a graph satisfying the condition (E1) with \( h \) holes. Suppose that \( G \) has a \( C \)-avoiding \((v_i, v_{i+1})\)-path for some hole \( C \) of \( G \) and two adjacent vertices \( v_i \) and \( v_{i+1} \) on \( C \). Since \( h \geq 2 \), there exists another hole \( C' \). Take a vertex \( w \) of \( C' \) and add new edges between \( w \) and any other vertices on \( C' \) by new edges. Then, by Corollary 2.9, the resulting graph \( G' \) has less than \( h \) holes. Since no new hole has been created, \( G' \) is still a graph satisfying the condition (E1). By the condition (E1), \( C \) and \( C' \) share at most one edge and therefore no chord for \( C \) is created in the process of adding the edges. Thus \( C \) is still a hole of \( G' \). By the induction hypothesis, \( X_C \cup \{v_i, v_{i+1}\} \) is a vertex cut of \( G' \). Since \( G \) is a spanning subgraph of \( G' \), it holds that \( X_C \cup \{v_i, v_{i+1}\} \) is a vertex cut of \( G \). \(\square\)

2.4 Properties of \( C \)-avoiding paths for a hole \( C \) of length 4

Proposition 2.12. Let \( G \) be a graph satisfying the condition (E1). Suppose that \( G \) has a hole \( C = v_0v_1v_2v_3v_0 \) of length 4 and that there exists a \( C \)-avoiding \((v_0, v_2)\)-path of length at least 3. Let \( P = x_0x_1x_2\cdots x_{l-1}x_l \) be a shortest \( C \)-avoiding \((v_0, v_2)\)-path, where \( x_0 = v_0 \), \( x_l = v_2 \), and \( l(\geq 3) \) is the length of \( P \). Then, for any \( i \in \{1, \ldots, l-1\} \), the following hold:

1. \( x_i \) is adjacent to exactly one of the vertices \( v_1, v_3 \);
2. If \( x_iv_1 \notin E(G) \), then \( x_{i+1}v_1 \in E(G) \);
3. If \( x_iv_3 \notin E(G) \), then \( x_{i+1}v_3 \in E(G) \).

Proof. We show (1) by contradiction. Suppose that there is \( i \in \{1, \ldots, l-1\} \) such that \( x_i \) is not adjacent to exactly one of the vertices \( v_1, v_3 \). First suppose that
$x_1v_1 \in E(G)$ and $x_1v_3 \in E(G)$. If $i \neq 1$, then $v_0v_1x_1v_3v_0$ is a hole and shares two edges $v_0v_1$ and $v_0v_3$ with the hole $C$. If $i = 1$, then $v_2v_1x_1v_3v_2$ is a hole and shares two edges $v_2v_1$ and $v_2v_3$ with $C$.

Suppose that $x_1v_1 \notin E(G)$ and $x_1v_3 \notin E(G)$. Let $x_{i_1}$ (resp. $x_{i_3}$) be the last vertex on the $(x_0, x_{i-1})$-section of $P$ that is adjacent to $v_1$ (resp. $v_3$), and let $x_{i_2}$ (resp. $x_{i_4}$) be the first vertex on the $(x_{i+1}, x_i)$-section of $P$ that is adjacent to $v_1$ (resp. $v_3$). Then $(1) = v_1x_{i_1}x_{i_1+1} \cdots x_{i_2}v_1$ and $(2) = v_3x_{i_3}x_{i_3+1} \cdots x_{i_4}v_3$ are holes of $G$, and they share two edges $x_{i-1}x_i$ and $x_ix_{i+1}$, which is a contradiction. Hence (1) holds.

We show (2) by contradiction. Suppose that there is $i \in \{0, \ldots, l-1\}$ such that $x_1v_1 \notin E(G)$ and $x_{i+1}v_1 \notin E(G)$. Since $x_1 = v_2$ and $v_1v_2 \in E(G)$, we have $i \neq l-1$. By (1), $x_1v_3 \in E(G)$ and $x_{i+1}v_3 \in E(G)$. Let $x_i$ be the vertex defined in (1) and let $x_{i_5}$ be the first vertex on the $(x_{i+1}, x_i)$-section of $P$ that is adjacent to $v_3$. Then $(3) = v_1x_{i_1}x_{i+1} \cdots x_{i_5}v_1$ and $(4) = v_1x_{i_1}x_{i+1} \cdots x_{i_5}v_3v_2v_1$ are holes of $G$. The two edges $v_1x_{i_1}$ and $x_{i_1}x_{i+1}$ are contained in both $C(3)$ and $C(4)$, which is a contradiction. Hence it holds that if $x_1v_1 \notin E(G)$, then $x_{i+1}v_1 \in E(G)$.

Statement (3) can be shown by an argument similar to the proof of (2).

We denote by $[x_1y_1|x_2y_2|x_3y_3]$ the graph with vertex set $\{x_1, x_2, x_3\} \cup \{y_1, y_2, y_3\}$ and edge set $\{x_ix_j | 1 \leq i < j \leq 3\} \cup \{y_iy_j | 1 \leq i < j \leq 3\} \cup \{x_iy_i | 1 \leq i \leq 3\}$. A graph isomorphic to $[x_1y_1|x_2y_2|x_3y_3]$ is called a 3-prism graph. In this paper, we call a 3-prism graph just a prism. We say that a graph is prism-free if the graph does not contain a prism as an induced subgraph.

**Proposition 2.13.** Let $G$ be a graph satisfying the condition (E1). Suppose that $G$ has a hole $C = v_0v_1v_2v_3v_0$ of length 4, and that there is a $C$-avoiding $(v_0, v_2)$-path. Let $P$ be a shortest $C$-avoiding $(v_0, v_2)$-path. Then the length of $P$ is equal to 3 and the subgraph of $G$ induced by $V(C) \cup V(P)$ is a prism $[v_0v_3|xy|v_1v_2]$ or a prism $[v_0v_1|xy|v_3v_2]$, where $P = v_0v_2v_1$.

**Proof.** Let $P = x_0x_1x_2 \cdots x_{l-1}x_1$ be a shortest C-avoiding $(v_0, v_2)$-path, where $x_0 = v_0$ and $x_1 = v_2$. Since $v_0v_2 \notin E(G)$, $l \neq 1$. If $l = 2$, then $P = v_0v_1v_2$ and so we have $x_1 \in X_C$ by Lemma 2.3 which contradicts the fact that $P$ is a C-avoiding path. Thus the length $l$ of $P$ is at least 3. Suppose that $l \geq 4$. Then $x_3 \neq v_2$. By Proposition 2.12 (1), exactly one of $x_1v_1$, $x_1v_3$ is an edge of $G$. Without loss of generality, we may assume that $x_1v_1 \notin E(G)$ and $x_1v_3 \notin E(G)$. Then, by Proposition 2.12 (3), $x_2v_3 \in E(G)$. By (1) of the same proposition, $x_2v_1 \notin E(G)$. By (2), $x_3v_1 \in E(G)$. Then, by (1), $x_3v_3 \notin E(G)$. (See Figure 3) Then $C(1) := v_0v_1x_3v_2v_3v_0$ and $C(2) := v_1x_3x_2x_1v_1$ are holes of $G$. The two edges $v_1x_3$ and $x_2x_3$ are contained in both $C(1)$ and $C(2)$, which is a contradiction to the fact that $G$ satisfy the condition (E1). Hence $l = 3$. Furthermore, by Proposition 2.12 Y the subgraph of $G$ induced by $V(C) \cup V(P)$ is either a prism $[v_0v_3|xy|v_1v_2]$ or a prism $[v_0v_1|xy|v_3v_2]$.  


Figure 6: A picture for Proof of Proposition 2.13

Let $G$ a graph satisfying the condition (E1) and $C = v_0v_1v_2v_3v_0$ be a hole of length 4 in $G$. For two distinct prisms $Y_1$ and $Y_2$ containing $C$, we say that $Y_1$ and $Y_2$ are of the same type if a triangle in $Y_1$ has a common edge with one of the two triangles in $Y_2$, and we say that $Y_1$ and $Y_2$ are of different types if both of the two triangles in $Y_1$ have no common edge with the two triangles in $Y_2$. That is, two prisms of the forms $[v_0v_3|xy|v_1v_2]$ and $[v_0v_3|x'y'|v_1v_2]$ are of the same type, two prisms of the forms $[v_0v_1|xy|v_3v_2]$ and $[v_0v_1|x'y'|v_3v_2]$ are of the same type, and two prisms of the forms $[v_0v_1|xy|v_3v_2]$ and $[v_0v_3|x'y'|v_1v_2]$ are of different types. (See Figure 7.)

Figure 7: Prisms $[v_0v_3|xy|v_1v_2]$ and $[v_0v_1|xy|v_3v_2]$

Corollary 2.14. Let $G$ be a graph satisfying the condition (E1). Suppose that $G$ has a hole $C = v_0v_1v_2v_3v_0$ of length 4. Then, there is a $C$-avoiding $(v_0, v_2)$-path if and only if there is a $C$-avoiding $(v_1, v_3)$-path.

Proof. Suppose that there is a $C$-avoiding $(v_0, v_2)$-path. Let $P$ be a shortest path among all $C$-avoiding $(v_0, v_2)$-paths. By Proposition 2.13, the length of $P$ is equal to 3. Let $P = v_0xv_2$. By Proposition 2.13, either $v_1xyv_3$ or $v_1yv_3$ is a $C$-avoiding $(v_1, v_3)$-path. (See Figure 7) We can show the converse similarly.

Lemma 2.15. Let $G$ be a graph satisfying the condition (E1). Suppose that $G$ has a hole $C$ of length 4. Then the prisms containing $C$ must be of the same type.
Proof. Let $C := v_0v_1v_2v_3v_0$. Suppose that $C$ is contained in prisms $Y_1$ and $Y_2$ of different types. Without loss of generality, we may assume that $Y_1 = [v_0v_1|x_1x_2|v_3v_2]$ and $Y_2 = [v_0v_3|y_1y_2|v_1v_2]$ for some $x_1, x_2, y_1, y_2 \in V(G)$. (See Figure 8.) Suppose that one of $x_1, x_2$ and one of $y_1, y_2$ are adjacent. By the symmetry, we may assume that $x_1$ and $y_1$ are adjacent. Then $C_{(1)} := v_1v_2v_3x_1y_1v_1$ is a hole of $G$. But the edges $v_1v_2$ and $v_2v_3$ are contained in both $C$ and $C_{(1)}$, which is a contradiction to the condition (E1). Therefore, there is no edge between $\{x_1, x_2\}$ and $\{y_1, y_2\}$. Then $C_{(2)} := v_0x_1x_2v_2y_1v_0$ and $C_{(3)} := v_0x_1x_2v_1v_0$ are holes of $G$ and they share the two edges $v_0x_1$ and $x_1x_2$, which is a contradiction to the condition (E1). Hence, the prisms containing $C$ must be of the same type. \qed

![Figure 8: A picture for Proof of Lemma 2.15](image-url)

Let $K_t \Box K_2$ be the graph defined by $V(K_t \Box K_2) = \{x_1, \ldots, x_t\} \cup \{y_1, \ldots, y_t\}$ and $E(K_t \Box K_2) = \{x_ix_j \mid 1 \leq i < j \leq t\} \cup \{y_iy_j \mid 1 \leq i < j \leq t\} \cup \{x_iy_i \mid 1 \leq i \leq t\}$. (See Figure 9 for $t = 3, 4$.) Note that $K_3 \Box K_2$ is a prism.

![Figure 9: $K_3 \Box K_2$ and $K_4 \Box K_2$](image-url)

Lemma 2.16. Let $G$ be a graph satisfying the condition (E1) and $C := v_0v_1v_2v_3v_0$ be a hole of length 4 in $G$. Suppose that there exist two distinct $C$-avoiding $(v_0, v_2)$-paths. Let $P_1$ and $P_2$ be two distinct shortest $(v_0, v_2)$-paths. Then the subgraph of $G$ induced by $V(C) \cup V(P_1) \cup V(P_2)$ is $K_4 \Box K_2$. \hfill 17
Proof. By Lemma 2.13, the lengths of $P_1$ and $P_2$ are equal to 3. Let $P_1 := v_0x_1x_2v_2$ and $P_2 := v_0y_1y_2v_2$. By Lemma 2.13, $V(C) \cup V(P_1)$ and $V(C) \cup V(P_2)$ induce two distinct prisms $Y_1$ and $Y_2$. In addition, by Lemma 2.15, $Y_1$ and $Y_2$ are of the same type. Without loss of generality, we may assume that $Y_1 = [v_0v_1|x_1x_2|v_3v_2]$ and $Y_2 = [v_0v_1|y_1y_2|v_3v_2]$ for some $x_1, x_2, y_1, y_2 \in V(G)$.

First we show $x_1y_2, x_2y_1 \notin E(G)$. Suppose that $x_1y_2 \in E(G)$ or $x_2y_1 \in E(G)$.

Without loss of generality, we may assume that $x_1y_2 \in E(G)$. (See Figure 10 (a).) Then $C(1) := v_0x_1x_2v_1v_0$ and $C(2) := v_0x_1y_2v_1v_0$ are holes of $G$. The two edges $v_0x_1$ and $v_0v_1$ are contained in both $C(1)$ and $C(2)$, which is a contradiction to the condition (E1). Thus $x_1y_2, x_2y_1 \notin E(G)$. Second we show $x_1y_1 \in E(G)$. Suppose that $x_1$ and $y_1$ are not adjacent. If $x_2y_2 \in E(G)$, then let $C(3) := v_0x_1x_2y_2y_1v_0$ and $C(4) := v_3x_1x_2y_2y_1v_3$. (See Figure 10 (b).) If $x_2y_2 \notin E(G)$, then let $C(3) := v_0x_1x_2y_2y_1v_0$ and $C(4) := v_3x_1x_2y_2y_1v_3$. (See Figure 10 (c).) Then, in both cases, $C(3)$ and $C(4)$ are holes in $G$. Moreover, the $(x_1, y_1)$-section $P$ of $C(3)$ not containing $v_0$ coincides with the $(x_1, y_1)$-section of $C(4)$ not containing $v_3$. Since $P$ contains at least 2 edges, we reach a contradiction. Thus $x_1$ and $y_1$ are adjacent. The same argument holds for $x_2$ and $y_2$, and so it follows that $x_2$ and $y_2$ are adjacent. Hence, the subgraph of $G$ induced by $V(C) \cup V(P_1) \cup V(P_2)$ is $K_4 \square K_2$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure10.png}
\caption{Pictures for Proof of Lemma 2.16}
\end{figure}

Theorem 2.17. Let $G$ be a graph satisfying the condition (E1). For any hole $C := v_0v_1v_2v_3v_0$ of length 4 in $G$, exactly one of the following holds.

(a) $G$ has no $C$-avoiding $(v_0, v_2)$-path and no $C$-avoiding $(v_1, v_3)$-path.

(b) $C$ is contained in an induced subgraph of $G$ which is isomorphic to $K_t \square K_2$ for some $t \geq 3$.

Proof. Suppose that (a) does not hold. By Corollary 2.14, $G$ has a $C$-avoiding $(v_0, v_2)$-path. Let $P_1, \ldots, P_s$ ($s \geq 1$) be the shortest $C$-avoiding $(v_0, v_2)$-paths. If
s = 1, then $V(C) \cup V(P_1)$ induces a prism $K_3 \Box K_2$ by Proposition 2.13. If $s \geq 2$, then, by Lemma 2.16, $V(C) \cup V(P_1) \cup V(P_s)$ induces $K_4 \Box K_2$ for any $1 \leq i < j \leq s$. Thus the subgraph of $G$ induced by $V(C) \cup V(P_1) \cup \cdots \cup V(P_s)$ is isomorphic to $K_{s+2} \Box K_2$ and contains the hole $C$. Hence (b) holds.

Suppose that (b) holds. Let $H$ be an induced subgraph of $G$ isomorphic to $K_t \Box K_2$ for some $t \geq 3$ containing $C$. Take a vertex $x_1$ of $H$ adjacent to $v_0$ other than $v_1$ and $v_3$. Then there exists a unique vertex $x_2$ which is adjacent to both vertices $x_1$ and $v_2$. Therefore, we can see that $v_0,x_1,x_2,v_2$ is a $C$-avoiding $(v_0,v_2)$-path and so (a) does not hold. Hence the theorem holds. \[\Box\]

2.5 A classification of the holes in a graph satisfying the condition (E1)

**Theorem 2.18.** Let $G$ be a graph satisfying the condition (E1), and let $C$ be a hole in $G$. Then exactly one of the following holds:

(A) There is no $C$-avoiding path between two non-adjacent vertices of $C$, and $X_C$ is a clique.

(B) The length of $C$ is equal to 4, and $C$ is contained in an induced subgraph of $G$ isomorphic to $K_t \Box K_2$ for some $t \geq 3$.

(C) The length of $C$ is equal to 4, and $C$ is contained in an induced subgraph of $G$ isomorphic to $K_2^m$ for some $m \geq 3$.

Moreover, if (B) happens, then $X_C$ is a clique, and if (C) happens, then there is no $C$-avoiding path between two non-adjacent vertices of $C$.

**Proof.** First, we show by contradiction that (B) and (C) cannot happen at the same time. Suppose that both (B) and (C) hold. Let $C := v_0v_1v_2v_3v_0$ be a hole of length 4 contained in both a prism $Y$ and an induced subgraph $K$ isomorphic to $K_{2,2,2}$. Without loss of generality, we may assume that $Y = [v_0v_1|x_0v_3v_2]$ and $V(K) = \{v_0, v_1, v_2, v_3, u_1, u_2\}$ for some $x, y, u_1, u_2 \in V(G)$. (See Figure 11.) If $u_1x \notin E(G)$ and $u_1y \notin E(G)$, then the hole $u_1v_0x_0v_3u_1$ shares the two edges $u_1v_0$ and $u_1v_2$ with the hole $u_1v_0u_2v_2u_1$, which is a contradiction to the condition (E1). If $u_1x \notin E(G)$ and $u_1y \in E(G)$, then the hole $u_1v_0x_0u_1$ shares the two edges $v_0x$ and $xy$ with the hole $v_0xv_1v_0$, which is a contradiction. If $u_1x \in E(G)$ and $u_1y \notin E(G)$, then the hole $u_1x_0v_3u_1$ shares the two edges $xy$ and $yv_2$ with the hole $v_2yxv_3v_2$, which is a contradiction. Thus $u_1x \in E(G)$ and $u_1y \in E(G)$. By applying the same argument for $u_2$ instead of $u_1$, we can show that $u_2x \in E(G)$ and $u_2y \in E(G)$. Then the hole $u_1v_0u_2yv_1$ shares the two edges $u_1v_0$ and $u_2v_0$ with the hole $u_1v_0u_2v_2u_1$, which is a contradiction. Thus we have shown that (B) and (C) cannot happen at the same time.
Now, we show the theorem. If the length of $C$ is at least 5, then (A) holds by Lemmas 2.4 and 2.8 and neither (B) nor (C) can happen. Therefore, we assume that the length of $C$ is equal to 4. Suppose that (B) holds. Then (A) does not hold since there is a $C$-avoiding path between two non-adjacent vertices of $C$, and (C) does not hold by the previous argument. Next suppose that (B) does not hold. Then it follows from Theorem 2.17 that there is no $C$-avoiding path between two non-adjacent vertices of $C$. If $X_C$ is a clique, then (A) holds and (C) cannot happen by Theorem 2.5. If $X_C$ is not a clique, then (A) does not hold obviously and (C) happen by Theorem 2.5. Hence exactly one of (A), (B), (C) holds.

If (B) happens, then (C) cannot happen and so $X_C$ is a clique by Theorem 2.5. If (C) happens, then (B) cannot happen and so there is no $C$-avoiding path between two non-adjacent vertices of $C$ by Theorem 2.17. \hfill \Box

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure11}
\caption{A picture for Proof of Theorem 2.18}
\end{figure}

3 Operations on graphs satisfying the condition (E1)

3.1 Deleting an edge from a graph

For a graph $G$ and an edge $uv$ in $G$, we denote by $G - uv$ the graph obtained from $G$ by deleting the edge $uv$.

\textbf{Lemma 3.1.} \textit{Let $G$ be a graph satisfying the condition (E1). Suppose that there exists a hole $C$ and two adjacent vertices $u$ and $v$ on the hole $C$ such that there is no $C$-avoiding $(u, v)$-path. If $G$ is $K_{2,2,2}$-free, then the following hold:}

\begin{enumerate}
\item $G - uv$ also satisfies the condition (E1).
\end{enumerate}
Figure 12: Deleting an edge from a graph

(2) If the number of holes of $G$ is $h$, then that of $G - uv$ is at most $h - 1$.

(3) $G - uv$ is also $K_{2,2,2}$-free.

(4) If $G$ is prism-free, then $G - uv$ is still prism-free.

Proof. First we show that there is no new hole is created by deleting the edge $uv$ from $G$. Suppose that there is a hole $C'$ in $G - uv$ which is not a hole in $G$. Now consider the two distinct $(u, v)$-sections $P_1$ and $P_2$ of $C'$. If $|E(P_1)| \geq 3$ or $|E(P_2)| \geq 3$, then $P_1$ or $P_2$ is a $C$-avoiding $(u, v)$-path, which contradicts the hypothesis. Thus $|E(P_1)| = 2$ and $|E(P_2)| = 2$. Then $P_1 = uwv$ and $P_2 = uw'v$ for some vertices $w$ and $w'$ of $G$. Since $G$ does not have a $C$-avoiding $(u, v)$-path by the hypothesis, it holds that $\{w, w'\} \subseteq X_C \cup V(C)$.

However, if $w \in V(C)$, then at least one of $uw$ or $vw$ is a chord of $C$, which is a contradiction. If $w, w' \in X_C$, then $w$ and $w'$ are adjacent by Corollary 2.7 since $G$ is $K_{2,2,2}$-free. Then the edge $ww'$ is a chord of $C'$ in $G - uv$, which is a contradiction. Therefore $G - uv$ has no hole other than the ones of $G$. Since no new hole is created in $G - uv$, the graph $G - uv$ still satisfies the condition (E1). In addition, since $C$ is not a hole in $G - uv$ and the number of holes of $G - uv$ is at most $h - 1$. Suppose that $G - uv$ has an induced subgraph $H$ isomorphic to $K_{2,2,2}$. Then $u$ and $v$ consist in a part of $H$. Let $x$ and $y$ be the vertices of another part of $H$. Obviously neither $x$ nor $v$ is on $C$ and $uxv$ and $uyv$ are paths in $G$. Since $x$ and $y$ are not adjacent, one of them should be a $C$-avoiding path by Corollary 2.6 which is a contradiction. A similar argument can applied to reach a contradiction if $G - uv$ contains a prism. $\square$

3.2 Adding an edge to a graph

For a graph $G$ and two non-adjacent vertices $u$ and $v$ in $G$, we denote by $G + uv$ the graph obtained from $G$ by joining $u$ and $v$ by an edge.
Lemma 3.2. Let $G$ be a graph satisfying the condition (E1) and $C := v_0v_1v_2v_3v_0$ be a hole of length 4 in $G$. Then adding an edge joining two non-adjacent vertices of $C$ does not create a new hole.

Proof. By contradiction. Suppose a new hole $C'$ is created by adding an edge joining two vertices $v_0$ and $v_2$. Since $C' - v_0v_2$ together with edges $v_0v_1$ and $v_2v_1$ form a cycle of length at least 4 in $G$ sharing two edges with $C$, $v_1$ must be adjacent to a vertex of $V(C') \setminus \{v_0, v_2\}$. Similarly, $v_3$ is adjacent to a vertex of $V(C') \setminus \{v_0, v_2\}$. If a vertex $x$ on $C$ is joined to both $v_1$ and $v_3$, then $v_0v_1xv_2v_0$ is a hole of $G$ sharing two edges $v_0v_1$ and $v_0v_2$ with the hole $C$, a contradiction. Let $y$ and $z$ be vertices of $C$ adjacent to $v_1$ and $v_3$, respectively, such that a shorter $(y, z)$-section $P$ of $C'$ is the shortest. Then no interior vertex of $P$ is adjacent to $v_0$ or $v_3$. Then $P$ together with the edges $v_1v_2, v_2v_3, v_1y, v_3z$ form a hole in $G$. However, this hole shares the two edges $v_1v_2, v_2v_3$ with $C$, which is a contradiction. Thus no new hole is created by adding an edge joining two vertices $v_0$ and $v_2$. By symmetry, no new hole is created by adding an edge joining two vertices $v_1$ and $v_3$. Hence adding an edge joining two non-adjacent vertices of $C$ does not create a new hole.

Lemma 3.3. Let $G$ be a graph satisfying the condition (E1). Let $m$ be the maximum integer such that $G$ contains an induced subgraph $K$ isomorphic to $K_2^m$. Let $u, v \in V(K)$ be two non-adjacent vertices of $K$. Then the following hold:

(1) $G + uv$ also satisfies the condition (E1).

(2) If $m \geq 3$ and the number of holes of $G$ is $h$, then that of $G + uv$ is at most $h - 2$.

(3) If $G$ is prism-free, then $G + uv$ is still prism-free.

Proof. Since $u$ and $v$ are non-adjacent vertices of a hole of length 4, by Lemma 3.2 $G + uv$ has no hole other than the ones of $G$ and so (1) holds.
If \( m \geq 3 \), then \( u \) and \( v \) belong to at least two distinct holes of length 4 in \( K \) and these holes in \( G \) are not holes anymore in \( G + uv \) as they become 4-cycles with chord \( uv \). By the previous argument, no hole is created by joining \( u \) and \( v \) and so \( G + uv \) has at most \( h - 2 \) holes. Thus (2) holds.

To show (3), suppose that \( G + uv \) contains a prism \([xy|uv|zw]\). Then \( x \) is not adjacent to \( v \), and \( y \) is not adjacent to \( u \) in \( G \). Therefore \( x \) and \( y \) cannot belong to \( K \). Let \( C^* \) be a hole of \( K \) containing \( u \) and \( v \). Then \( uxyv \) is a \( C^* \)-avoiding path, a contradiction. Thus (3) holds.

### 3.3 Breaking prisms in a graph

Suppose that a graph \( G \) has an induced subgraph \( H \) isomorphic to \( K_t \square K_2 \) for \( t \geq 3 \). Let \( V(H) = V_x \cup V_y = \{x_1, \ldots, x_t\} \cup \{y_1, \ldots, y_t\} \) where \( V_x \) and \( V_y \) are isomorphic to \( K_t \) and \( x_iy_i \in E(H) \). Define a graph \( G/H/ \) by \( V(G/H/) = V(G) \) and \( E(G/H/) = E(G) \cup \{x_1y_2, x_2y_3, \ldots, x_{t-1}y_t, x_ty_1\} \).

![Figure 14: Breaking prisms in a graph](image)

**Lemma 3.4.** Let \( G \) be a graph satisfying the condition (E1). Suppose that \( G \) has an induced subgraph \( H \) isomorphic to \( K_t \square K_2 \) for \( t \geq 3 \). Then the following hold:

1. \( G/H/ \) also satisfies the condition (E1).

2. If the number of holes of \( G \) is \( h \), then that of \( G/H/ \) is at most \( h - t \).

**Proof.** By Lemma 3.2, \( G/H/ \) contains no new hole other than the ones in \( G \). Moreover, adding an edge between two non-adjacent vertices of a hole of length 4 in \( H \) breaks the hole. Thus at least \( t \) holes of \( G \) are broken in \( G/H/ \). Hence (1) and (2) hold.

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4 Proof of Theorem 1.7

4.1 Outline of the proof

Let $G$ be a graph satisfying the condition (E1). Then exactly one of the following three cases happens:

(Case A): $G$ is prism-free and $K_{2,2,2}$-free.

(Case B): $G$ is prism-free and $G$ has an induced subgraph $K$ isomorphic to $K_{2,2,2}$.

(Case C): $G$ has an induced subgraph $Y$ isomorphic to a prism.

Suppose that $G$ has exactly $h$ holes $C_1, C_2, \ldots, C_h$. For each $t \in [h] := \{1, \ldots, h\}$, we let

$$C_t = v_{t,0}v_{t,1} \ldots v_{t,m_t}v_{t,0},$$

where $m_t$ is the length of the hole $C_t$. For $t \in [h]$ and $i \in \{0, \ldots, m_t - 1\}$, let $S_{t,i}$ be the set of vertices each of which is an internal vertex of a $C_t$-avoiding walk from $v_{t,i}$ to $v_{t,i+1}$, i.e.,

$$S_{t,i} := \bigcup_{W \in W_{t,i}} V(W) \setminus \{v_{t,i}, v_{t,i+1}\},$$

where $W_{t,i}$ denotes the set of all $C_t$-avoiding $(v_{t,i}, v_{t,i+1})$-walks in $G$.

We will prove Theorem 1.7 by induction on the number of holes by taking the following steps:

**Step 1:** We prove (Case A) by using the operation “Deleting an edge from a graph”.

**Step 2:** (Case B) is reduced to (Case A) by using the operation “Adding an edge to a graph”.

**Step 3:** (Case C) is reduced to (Case B) by using the operation “Breaking prisms in a graph”.

4.2 Proof for (Case A)

Consider (Case A). If there are no holes of length 4 in $G$, then all the holes are independent. Therefore it holds that $k(G) \leq h(G) + 1$ by Theorem 1.4. Suppose that there is a hole of length 4 in $G$. Since $G$ is prism-free and $K_{2,2,2}$-free, the condition (a) in Theorem 2.5 and the condition (a) in Theorem 2.17 hold for each of the holes of length 4 in $G$.

**Theorem 4.1.** Let $G$ be a prism-free $K_{2,2,2}$-free graph satisfying the condition (E1) with exactly $h$ holes and let $Q$ be a clique of $G$ containing an edge of a hole. Then there exists an acyclic digraph $D$ such that $C(D) = G \cup I_{h+1}$ and the vertices of $Q$ have no in-neighbors in $D$. Consequently, the competition number of $G$ is at most $h + 1$. 

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Proof. We shall prove the theorem by induction on $h$. The competition number of $G$ is at most 1 if $h = 0$ by Theorem 1.1. Since there is no hole, there is no clique containing an edge of a hole and so the theorem is true if $h = 0$.

We assume that $h \geq 1$ and the theorem is true for any prism-free $K_{2,2,2}$-free graph satisfying the condition (E1) with less than $h$ holes. Suppose that $\{(v_{t,i}, v_{t,i+1}) \cup S_{t,i}) \cap Q \neq \emptyset$ for some $t \in [h]$ and $i \in \{0, \ldots, m_t - 1\}$. Take $x \in (\{v_{t,i}, v_{t,i+1}\} \cup S_{t,i}) \cap Q$. If $x = v_{t,i}$, then $v_{t,j} \notin Q$ for any $j \neq i - 1, i + 1$. Since $m_t \geq 4$, $i + 3 \neq i$ (mod $m_t$).

Suppose that $x' \in (\{v_{t,i+2}, v_{t,i+3}\} \cup S_{t,i+2}) \cap Q$. Then $x' \in S_{t,i+2} \cap Q$. Since $Q$ is a clique, $x'$ is adjacent to $x$. By the definitions of $S_{t,i}$ and $S_{t,i+2}$, there exists a $C_t$-avoiding $(v_{t,i}, v_{t,i+2})$-walk, which contradicts Theorem 2.18. Thus $\{(v_{t,i+2}, v_{t,i+3}) \cup S_{t,i+2}) \cap Q = \emptyset$. If $x = v_{t,i+1}$, we can show that $(\{v_{t,i-2}, v_{t,i-1}\} \cup S_{t,i-2}) \cap Q = \emptyset$ by a similar argument. If $x \in S_{t,i} \cap Q$ and $x' \in (\{v_{t,i-2}, v_{t,i-1}\} \cup S_{t,i-2}) \cap Q$, then $x$ and $x'$ are adjacent and there exists a $C_t$-avoiding $(v_{t,i-2}, v_{t,i})$-walk. Now Theorem 2.18 is violated. Thus $\{(v_{t,i-2}, v_{t,i-1}) \cup S_{t,i-2}) \cap Q = \emptyset$. We have just shown that there exists $j \in \{0, \ldots, m_t - 1\}$ such that $\{(v_{t,j}, v_{t,j+1}) \cup S_{t,j}) \cap Q = \emptyset$.

We claim that no vertex of $S_{t,j}$ is adjacent to a vertex of $V(G) - (X_t \cup V(C_t) \cup S_{t,j})$. Suppose otherwise, there is a $C_t$-avoiding $(v_{t,j}, v_{t,j+1})$-walk $W$ that contains an internal vertex $x$ adjacent to a vertex $y \in V(G) - (X_t \cup V(C_t) \cup S_{t,j})$. The walk $W'$ obtained by replacing the term $x$ with $xyx$ in the sequence of $W$ is a $C_t$-avoiding $(v_{t,j}, v_{t,j+1})$-walk, which contradicts the assumption that $y \notin S_{t,j}$. On the other hand, there is no $C_t$-avoiding path connecting two non-adjacent vertices of $C_t$ by Theorem 2.18 and so no vertex of $S_{t,j}$ is adjacent to a vertex of $V(C_t) \setminus \{v_{t,j}, v_{t,j+1}\}$. Hence, $X_t \cup \{v_{t,j}, v_{t,j+1}\}$ is a vertex cut of $G$ and no vertex in $S_{t,j}$ belongs to the component that contains $V(C_t) \setminus \{v_{t,j}, v_{t,j+1}\}$.

Consider the subgraph $G_1$ of the graph $G$ induced by $V(G) - S_{t,j}$ and the subgraph $G_2$ of $G$ induced by $X_t \cup \{v_{t,j}, v_{t,j+1}\} \cup S_{t,j}$. Since $V(G_1) \cap V(G_2) = X_t \cup \{v_{t,j}, v_{t,j+1}\}$ is a vertex cut of $G$ which is a clique, the vertex set of a hole is contained in either $V(G_1) \setminus V(G_2)$ or $V(G_2) \setminus V(G_1)$. Thus, if $h_1$ is the number of holes of $G_1$, then $h_2 := h - h_1$ is the number of holes of $G_2$. It is obvious that $E(G_1) \cup E(G_2) = E(G)$ and that both $G_1$ and $G_2$ are prism-free, $K_{2,2,2}$-free, and satisfy the condition (E1).

Since the hole $C_t$ is not in $G_2$, we have $h_2 < h$. By the induction hypothesis, there exists an acyclic digraph $D_2$ such that $C(D_2) = G_2 \cup I_{h_2+1}$ and the vertices of $X_t \cup \{v_{t,j}, v_{t,j+1}\}$ have only outgoing arcs in $D_2$. Notice that $C_t$ is a hole in $G_1$ which has no $C_t$-avoiding walk from $v_{t,j}$ to $v_{t,j+1}$. By Lemma 3.1, $G_1 - v_{t,j}v_{t,j+1}$ has exactly $h_1 - 1$ holes and satisfy the condition (E1). By the choice of $j$, $Q$ is a clique in $G_1 - v_{t,j}v_{t,j+1}$ and, by the induction hypothesis, there exists an acyclic digraph $D_1$ such that $C(D_1) = (G_1 - v_{t,j}v_{t,j+1}) \cup I_{h_1}$ and the vertices of $Q$ have only
adjacent vertices of \( K = m \).

Theorem 4.1, the theorem holds. Suppose that other than the ones in an induced subgraph \( D \), hypothesis, there exists an acyclic digraph \( uv \) prey of cliques containing the edge \( h \) satisfying the condition (E1) and has at most \( A \). Assume that the theorem is true for any prism-free graph satisfying the condition (E1) with less than \( h \). By induction on \( h \).

Proof. If \( G \) is prism-free, then the competition number of \( G \) is at most \( h + 1 \).

4.3 Reducing (Case B) to (Case A)

Theorem 4.2. Let \( G \) be a graph satisfying the condition (E1) with exactly \( h \) holes. If \( G \) is prism-free, then the competition number of \( G \) is at most \( h + 1 \).

Proof. By induction on \( h \). If \( h = 0 \), then the theorem follows from Theorem 4.1. Assume that the theorem is true for any prism-free graph satisfying the condition (E1) with less than \( h \) holes. Let \( m \) be the maximum integer such that \( G \) contains an induced subgraph \( K \) isomorphic to \( K_m^2 \). If \( m \leq 2 \), then \( G \) is \( K_{2,2,2} \)-free. By Theorem 4.1, the theorem holds. Suppose that \( m \geq 3 \). Let \( u \) and \( v \) be two non-adjacent vertices of \( K \). By Lemma 4.3, the graph \( G' := G + uv \) is a prism-free graph satisfying the condition (E1) and has at most \( h - 2 \) holes. Therefore, by induction hypothesis, there exists an acyclic digraph \( D' \) such that \( C(D') = G' \cup I_{h-1} \).

In the following, we shall construct an acyclic digraph \( D \) such that \( C(D) = G \cup I_{h+1} \) from \( D' \). We first look at the vertices in \( N^+_D(u) \cap N^+_D(v) \) which play as prey of cliques containing the edge \( uv \) in \( G' \). Let \( N^+_D(u) \cap N^+_D(v) = \{w_1, \ldots, w_p\} \) for some integer \( p \geq 1 \). Let \( H_i \) be the subgraph of \( G \) induced by \( N^-_D(w_i) \). In \( G \), the edges of \( H_i \) are covered by exactly two cliques \( N^-_D(w_i) \setminus \{u\} \) and \( N^-_D(w_i) \setminus \{v\} \) unless \( N^-_D(w_i) = \{u, v\} \). Furthermore, since \( w_i \) is a common out-neighbor of \( u \) and \( v \),

\[
\bigcup_{i=1}^{p} N^-_D(w_i) \subseteq (N_G[u] \cap N_G[v]) \subseteq X_K \cup V(K),
\]

where \( X_K \) is defined by (2.1) and the last inclusion follows from Lemma 2.7 (2). Thus

\[
N_G[v] \cap \bigcup_{i=1}^{p} N^-_D(w_i) \subseteq N_G[v] \cap (X_K \cup V(K)).
\]

The vertices in \( N_G[v] \cap (X_K \cup V(K)) \) are covered by exactly two cliques in \( G \). We denote those cliques by \( Z_1 \) and \( Z_2 \). We define a digraph \( D \) as follows:

\[
V(D) = V(D') \cup \{z_1, z_2\};
\]

\[
A(D) = A(D') \setminus \bigcup_{i=1}^{p} \{(v, w_i)\} \cup \{(x, z_1) \mid x \in Z_1\} \cup \{(x, z_2) \mid x \in Z_2\}.
\]
Then it is obvious that $D$ is acyclic and $E(C(D)) \subset E(G)$. By removing the arcs in $\bigcup_{i=1}^{t} \{ (v, w_i) \}$ from $D'$, the competition graph of the new digraph loses the edges joining $v$ and the vertices in $\bigcup_{i=1}^{h} N_{\overline{D'}}(w_i)$. Those edges are contained in the subgraph induced by $N_G[v] \cap (X_K \cup V(K))$ as we argued above. Thus those edges are contained in the cliques formed by $Z_1$ or $Z_2$. Hence $C(D) = G \cup I_{h+1}$ and so $k(G) \leq h + 1$. \qed

4.4 Reducing (Case C) to (Case B)

Now we complete the proof of Theorem 1.7.

Proof of Theorem 1.7. By induction on $h$. If $h = 0$, then the theorem follows from Theorem 1.2. Assume that the theorem is true for any graph satisfying the condition (E1) and the number of holes is less than $h$. Let $t$ be the maximum integer such that $G$ contains an induced subgraph $H$ isomorphic to $K_t \square K_2$. If $t \leq 2$, then $G$ is prism-free. By Theorem 1.2 the theorem holds. Suppose that $t \geq 3$. Consider the graph $G_{/H/}$. Then, by Lemma 3.4, $G_{/H/}$ satisfies the condition (E1) and the number of holes in $G_{/H/}$ is $h - t$ which is less than $h$. By the induction hypothesis, there exists an acyclic digraph $D'$ such that $C(D') = G_{/H/} \cup I_{h-t+1}$. Take $i \in \{1, \ldots, t\}$ and $w \in N_{D'}^{+}(x_i) \cap N_{D'}^{+}(y_{i+1})$. Then $N_{D'}^{-}(w) \subset X_C \cup \{ x_i, x_{i+1}, y_{i+1} \}$ or $N_{D'}^{-}(w) \subset X_C \cup \{ x_i, y_i, y_{i+1} \}$ where $C_i := x_i y_i y_{i+1} x_{i+1} x_i$ (identify $x_{t+1}$ and $y_{t+1}$ with $x_1$ and $y_1$, respectively). By Theorem 2.18 $X_C$ is a clique in $G$ and so $N_{D'}^{-}(w) \setminus \{ x_i \}$ is a clique in $G$. Now we define a digraph $D$ by

$$V(D) = V(D') \cup \{ z_1, \ldots, z_t \},$$

$$A(D) = \left( A(D') \setminus \bigcup_{i=1}^{t} \{ (x_i, w) \mid w \in N_{D'}^{+}(x_i) \cap N_{D'}^{+}(y_{i+1}) \} \right) \cup \bigcup_{i=1}^{t} \{ (x, z_i) \mid x \in \{ x_i, x_{i+1} \} \cup X_C \}.$$ 

Obviously $D$ is acyclic. Note that

$$N_D^{-}(w) = \begin{cases} N_{D'}^{-}(w) \setminus \{ x_i \} & \text{if } w \in N_{D'}^{+}(x_i) \cap N_{D'}^{+}(y_{i+1}) \text{ for some } i \in \{ 1, \ldots, t \}; \\ N_{D'}^{-}(w) & \text{otherwise.} \end{cases}$$

Also notice that deleting the arcs in $\bigcup_{i=1}^{t} \{ (x_i, w) \mid w \in N_{D'}^{+}(x_i) \cap N_{D'}^{+}(y_{i+1}) \}$ from $D'$ may remove edges only in the clique $\{ x_i, x_{i+1} \} \cup X_C$ for some $i \in \{ 1, \ldots, t \}$ from $C(D')$. From these observations, we can conclude that $C(D) = G \cup I_{h+1}$. Hence $k(G) \leq h + 1$. \qed

Corollary 4.3. Let $G$ be a graph with exactly $h$ holes satisfying the following property:
• For any two distinct holes $C$ and $C'$, $|V(C) \cap V(C')| \leq 2$.

Then the competition number of $G$ is at most $h + 1$.

Proof. It follows from Theorem 1.7 and Proposition 2.2.

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