Quasi Band-Limited Coronagraph for Extended Sources

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Abstract

We propose a class of graded coronagraphic image masks for a high throughput Lyot-type coronagraph that transmits light from an annular region around an extended source and suppresses light, with extremely high ratio, elsewhere. The interior radius of the region is comparable with its exterior radius. The masks are designed using an idea inspired by approach due M.J. Kuchner and W.A. Traub (“band-limited” masks) and approach to optimal apodization by D.Slepian. One potential application of our masks is direct high-resolution imaging of exo-planets with the help of the Solar Gravitational Lens, where apparent radius of the “Einstein ring” image of a planet is of the order of an arc-second and is comparable with the apparent radius of the sun and solar corona.

Keywords: Coronagraphy, Optimal Band-Limiting, Exo-Planets, Gravitational Lensing

1 Introduction

Although potential applications of our masks are not restricted only to Solar Gravitational Lens (SGL) imaging, we will demonstrate main principles of the mask design in context of SGL imaging, since the latter was our main motivation.

The idea of using the sun as a powerful telescope goes back to Eshleman [1]: The gravitational field of the sun acts as a spherical lens and magnifies intensity of electromagnetic radiation from distant objects along a semi-infinite focal line with the nearest point of observations being about $Z_{\text{min}}=550$ AU (A good brief, self-contained introduction to the subject and related problems can be found in [3]. For more details, one can e.g. see [4], [5], [7], [8] and references therein). For example, an integral intensity of radiation from an Earth-like exo-planet at distance 30 pc can be pre-magnified by the SGL up to six orders of magnitude (see e.g. [3]). Theoretical angular resolution of the SGL is comparable to that of a telescope with aperture of the order of the sun size. At visible wavelengths this resolution could be as small as $10^{-10}$ arcsec (see e.g. [8]).

Recently, properties of the solar gravitational lens attracted attention both due to discovery of numerous exo-planets and the success of the Voyager-1 spacecraft, presently operating at about 140AU. Possibilities of high resolution (up to mega-pixel) imaging of such planets from the focal line of solar gravitational lens are now being discussed.

Without going into much detail, we recall that an observer at distance $Z$ from the sun sees the image of an exo-planet as the “Einstein ring” of the apparent radius $\alpha_E$

$$\alpha_E(Z) = \alpha_{\text{max}} \sqrt{Z_{\text{min}} / Z}$$
where $\alpha_{\text{max}} = 1.75$ arcsec, is the apparent radius of the sun at $Z = Z_{\text{min}} = 550$ AU. The width (i.e. apparent thickness) of the ring $\delta \alpha$ equals a half of the apparent (angular) diameter of an exo-planet (see Figure 1). For an Earth-like planet at 30pc from the sun $\delta \alpha \sim 10^{-6}$ arcsec. As a consequence, the Einstein ring width is not resolvable by any realistic telescope. In other words, for any practical purpose the ring can be considered as a circle whose brightness is varying along the circle circumference.

A telescope with radius of aperture $\sim 1$ m resolves the circumference of the ring with up to $\sim 10^{2}$ elements. Thus, performing a scan in the observer plane by taking a set of images of the Einstein rings from different $(X,Y)$ - positions, one could then make a tomographic reconstruction of the real image of a planet. Note that amplification of the integral energy flow density by SGL is of order $2\alpha_E/\delta \alpha$ (for details see e.g. [3]).

Apart from formidable technical difficulties of getting to the focal line, there are principal problems of suppression of the diffraction glare from the sun and its corona. Indeed, the apparent radius of the sun at distance $Z$ is

$$\alpha_S(Z) = \alpha_{\text{max}} Z_{\text{min}} / Z.$$ 

The separation between the solar disc and the Einstein ring $\alpha_E(Z) - \alpha_S(Z)$ is zero at $Z = Z_{\text{min}}$ and reaches its maximum at $Z = 4Z_{\text{min}} \approx 2200$ AU. The maximal separation equals $\alpha_{\text{max}} / 4 \approx 0.44$ arcsec. At this distance the apparent radius of the sun $\alpha_S$ equals the separation angle.

The distance $Z = 4Z_{\text{min}} \approx 2200$ AU is considered as minimal practical distance of observation [8]: The image of the ring will be superimposed with that of solar corona. The (angular) surface brightness of corona at the Einstein ring (i.e. at $\alpha_E = \alpha_{\text{max}} / 2 \approx 0.88$ arcsec) equals the surface brightness of the Earth night sky at full moon (astronomical observations of faint objects are not taken during periods of full moon due to the sky brightness). This surface brightness is about $10^{-8} I_0$, where $I_0$ is the surface brightness of the sun at its center (see Figure 1).

More rigorous argument in favour of the above minimal practical distance is the following: In geometrical optics, the surface brightness is invariant. The surface brightness of an Earth-like planet is about $10^{-5}$ times of that of its host star. Therefore, in geometrical optics the surface brightness of the Einstein

1The distribution of linear brightness along such a circle is essentially a Radon transform of the image of the planetary disc.
ring is $I_G \sim 10^{-5} I_0$. Since the width of the ring is not resolved by a telescope, one has to take into account reduction of the surface brightness due to diffraction on the telescope aperture.

Take, for instance, a telescope comparable with the Hubble Space Telescope, i.e. a telescope with the diameter $D \approx 2.5$ m operating at visible wavelengths $\lambda < 750$ nm. The characteristic (diffraction) width of the image of Einstein ring in the focal plane of the telescope is of order $\lambda/D$, which is about $10^{-1}$ arcsec. This is about $10^4 - 10^5$ times bigger than the geometrical optics width of the Einstein ring. Therefore, the surface brightness of the ring in the focal plane of the telescope $I$ will be $10^{-4} - 10^{-5}$ times $I_G$, i.e. about $10^{-9} - 10^{-10}$ times $I_0$. Taking into account that the brightness of corona at $\alpha = \alpha_E (4Z_{\text{min}})$ is of order $10^{-8}$, we see that it exceeds $I$ by one-two orders of magnitude. In other words, even at $Z = 2200$ AU the image of the ring in the focal plane of a realistic space telescope will be one-two orders of magnitude fainter than that of the background corona.

That is why $Z = 2200$ AU is considered as a minimal practical distance of observation.

From the above it follows that it is necessary to suppress the diffraction glare of the sun, so that the surface brightness of the glare (in vicinity of the ring image) in the final image plane of the optical system should be about $10^{-10} I_0$. It is important to stress that not only on-axis light, but also off-axis light with incidence angles $\alpha < \alpha_S$ should be suppressed, while the light at $\alpha \approx \alpha_E$ should be transmitted almost entirely (we recall that $\alpha_S = \alpha_E/2$ at $Z = 4Z_{\text{min}} \approx 2200$ AU).

Below we propose a coronagraph that satisfies the above requirements. In addition, it suppresses not only the light from the sun but also the light from most of corona, except the part of corona in a close vicinity of the ring.

In the next section we recall general principles of the Lyot-type coronagraphs as well as those with the “band-limited” masks introduced by M.J. Kuchner and W.A. Traub in [2]. Then we introduce a quasi band-limited mask using approach similar to that of optimal apodization by D. Slepian [11]. Later, we consider problem of suppressing light from the parent star of an exo-planet introducing “one-dimensional” mask based on Slepian’s solution of one-dimensional band-limiting problem. Finally, we introduce the “product” mask which suppresses not only sunlight and the light from a part of corona, but also the light from the parent star. Tolerance to manufacturing errors is discussed in the concluding section of the paper.

## 2 Band-Limited Mask

To establish notations, let us first briefly review general principles of the Lyot-type coronagraph (for more detail see e.g. [9] and references therein). Stages of propagation of light through the coronagraph are depicted in Figure 2.

We consider a telescope with primary of diameter $D$. Let $\vec{R}$ be two dimensional vector defining the coordinates in the entrance pupil plane. Then it is convenient to introduce dimensionless coordinates $\vec{x} = \vec{R}/D$. Similarly, let $\vec{r}$ be two-dimensional coordinates in the first image (focal) plane of the telescope. We re-scale them in the units of the diffraction characteristic scale $\lambda F/D$, where $F$ is the telescope focal length, introducing dimensionless coordinates $\vec{y} = \vec{r}D/(\lambda F)$. It is also convenient to re-scale the incidence angle $\vec{\alpha}$ of the plane wave in the units of the characteristic diffraction angle $\lambda/D$, introducing “dimensionless” incidence $\vec{\beta} = \vec{\alpha}D/\lambda$.

We consider a non-apodized entrance pupil, so, up to an $\vec{x}$-independent common factor, the dimensionless complex field of the plane wave of the unit amplitude immediately after encountering the primary

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2For an optical system, the surface brightness is the energy flux per unit solid angle divided by the pupil area.

3We note that the brightness of corona strongly increases towards the sun disc edge, from $\sim 10^{-8} I_0$ at two solar radii from the center of the sun to $\sim 10^{-5} I_0$ at one solar radius (see Figure 1).

4In what follows we refer to $\vec{\alpha}$ as “incidence angle” and to $\vec{\beta}$ as “incidence”.

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mirror is
\[ A_\beta(x) = P(x)E_\beta(x), \]  
(1)
where
\[ E_\beta(x) = e^{-2\pi i (\vec{\beta} \cdot \vec{x})} \]
and \( P \) is the non-apodized circular aperture function
\[ P(x) = \begin{cases} 1, & x < 1/2 \\ 0, & x > 1/2 \end{cases} , \quad x := |\vec{x}|. \]

From the Fraunhofer theory of diffraction it follows that amplitude of the field in the focal plane of the telescope is the Fourier transform of (1)
\[ a_\beta(y) = \int A_\beta(x)e^{2\pi i (\vec{x} \cdot \vec{y})} d^2x \]
and
\[ a_\beta(y) = p\left(|\vec{y} - \vec{\beta}|\right) , \quad p(y) = \frac{J_1(\pi y)}{2y} , \]  
(2)
where \( J_1 \) stands for the Bessel function. In the Lyot-type coronagraph the image is focused on the occulting mask with amplitude transmission factor \( m(\vec{y}) \), so, immediately after passing the mask the field amplitude becomes
\[ [ma](\vec{y}) = m(\vec{y})a_\beta(\vec{y}). \]  
(3)
Successive optics in the coronagraph transforms this product to the second pupil plane, where the field is \( M \ast A_\beta \). Here, \( \ast \) denotes convolution and \( M(\vec{x}) = \int m(\vec{y})e^{2\pi i (\vec{x} \cdot \vec{y})} d^2y \) is the Fourier transform of \( m \). Note that we consider only symmetric graded masks, i.e. the masks with
\[ m(\vec{y}) = m(-\vec{y}) , \quad m = |m|. \]

As a consequence, \( M \) is real and \( M(\vec{x}) = M(-\vec{x}) \) in our case.

Now, the field passes through the Lyot stop, which can be described by an aperture function \( L(x) \). After passing the Lyot stop the field becomes
\[ F_\beta(x) = L(\vec{x})[M \ast A_\beta](\vec{x}), \]  
(4)
where \( [M \ast A_\beta](\vec{x}) = \int M(\vec{x} - \vec{x}')A_\beta(\vec{x}')d^2x' \).
Then, the field passes the final pupil and is focused into the final (i.e. detector’s) image plane where the field amplitude is the Fourier transform of \( \Phi(x) \). In other words, the field amplitude in the final image plane equals
\[
\hat{\Phi}(y) = \hat{L} \ast [\text{ma}_\beta],
\]
where \( L \) is the Fourier transform of \( \Phi \). The field \( F \) in \( \Phi(x) \) is usually referred as the “final field” and we refer its Fourier transform \( f \) in \( \hat{\Phi}(y) \) as the “detected field”. More precisely, detector registers intensity of image \( \mu \) in the final (i.e. detector’s) image plane. This intensity is the Point Spread Function (PSF) of the optical system and equals square of the absolute value of the detected field
\[
\mu(\beta, y) = |\hat{\Phi}(y)|^2.
\]
The surface brightness \( \mathcal{I} \) of a final image of a point source with incidence \( \beta \) is the PSF divided by the Lyot stop area
\[
\mathcal{I}(\beta, y) = \mu(\beta, y)/S, \quad S = \int L(x)d^2x.
\]
For an extended incoherent source with surface brightness distribution \( I_s(\xi) \), the surface brightness at the detector plane \( I \) equals
\[
I(y) = \int I_s(\xi)\mathcal{I}(\xi, y)d^2\xi.
\]
Now, we turn our attention to the band-limited masks. For such masks, the Fourier transform \( M \) of the transmission amplitude \( m \) has a finite support. In the rest of this and the next section we will consider only circularly symmetric masks \( m(y) = m(\|y\|) \) (obviously, \( M \) is also circularly symmetric). For such symmetric band limited mask, \( M(x) \) is “concentrated” completely within the disc of diameter \( \epsilon \) and vanishes elsewhere, i.e.
\[
M(x) = 0, \quad x > \epsilon/2.
\]
We also consider only circular non-apodized Lyot stops. Let dimensionless diameter of the stop be \( \sigma < 1 \), then
\[
L(x) = P(x/\sigma), \quad S = \pi\sigma^2/4.
\]
It has been noticed in \( \Phi \) that from \( \Phi(x) \) and \( \hat{\Phi}(y) \) it follows that in the case of Lyot stops, whose dimensionless diameter \( \sigma \) does not exceed \( 1 - \epsilon \), the final field equals
\[
\Phi(x) = L[M \ast A_\beta] = L[M \ast E_\beta] = m(\beta)LE_\beta.
\]
In other words, the final field of the band-limited coronagraph \( \Phi \) coincides with that of the plane wave of amplitude \( m(\beta) \) at incidence \( \beta \) that passed through a pupil of dimensionless diameter \( \sigma \leq 1 - \epsilon \). Thus the detected field equals
\[
f_\beta(y) = m(\beta)l(y - \beta),
\]
where for a circular stop of diameter \( \sigma \)
\[
l(y) = \sigma p(\sigma y) = \sigma J_1(\pi\sigma y)/2y.
\]
It follows that the intensity in the detector image plane equals \( m(\beta)^2 \) times the PSF of the stop \( \Phi \), i.e.
\[
\mu(\beta, y) = m(\beta)^2l^2(|y - \beta|) = m(\beta)^2\mu_L(\beta, y).
\]

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5For a point source (i.e. plane wave) of a dimensionless unit intensity \( I_s(\xi) = \delta(\xi - \beta) \).

6By the “band limited coronagraph” we mean a Lyot-type coronagraph with the band limited mask and dimensionless diameter of the stop not exceeding \( 1 - \epsilon \).

7We recall that in our case \( m, l \) and \( f \) are all real.
Thus, the band-limited coronagraph completely suppresses an on-axis source when \( m(0) = 0 \).

The final throughput of the energy for the incidence \( \beta \) equals

\[
\tau(\beta) = m(\beta)^2 \sigma^2. \tag{11}
\]

Working throughput of the coronagraph equals \( \tau(\beta_W) \), where \( \beta_W \) is the working incidence. Usually one chooses \( m(\beta_W) \) to be close to unity in order to get the maximal useful throughput which is reached when \( \sigma = 1 - \epsilon \).

It is important to note that the band limited coronagraph designed for the wavelength \( \lambda \) will work for smaller wavelengths \( \lambda' < \lambda \) as well, since the change \( \lambda \) to \( \lambda' \) is equivalent to the scalings \( m(y) \to m(\lambda' y/\lambda) \), \( M(x) \to M(\lambda x/\lambda') \) and \( \beta \to \lambda \beta/\lambda' \), while \( \sigma \) remains invariant. As a consequence, the energy transmission at a given incidence angle \( \alpha = \beta \lambda/D \) is the same for \( \lambda' < \lambda \).

### 3 Quasi Band-Limited Mask

Our aim is to construct a coronagraph that transmits the light coming from an annular region containing the Einstein ring and suppress it elsewhere. We recall that the band-limited coronagraph completely suppresses light at incidences \( \beta \) for which \( m(\beta) = 0 \). However, the band-limited coronagraph that completely suppresses light coming from outside of the annular region is impossible to construct due to the uncertainty principle: A function and its Fourier transform cannot both have finite supports.

Below we propose construction of a graded mask that is “almost” band limited: Quasi Band Limited Mask or QBLM. It is band-limited in the sense that the transmission factor \( m(y) \) has a finite support, vanishing outside some annual region of the interior radius \( y_1 \) and the exterior radius \( y_2 \)

\[
m(y) = 0, \quad y \notin y_1 < y < y_2, \tag{12}
\]

while its Fourier conjugate \( M(x) \) is “concentrated” in the disc of diameter \( \epsilon \), “almost” vanishing elsewhere. For instance, in examples that will be presented below, maximum of the “tail” of \( M \) outside the disc is about 5 orders of magnitude smaller than maximum of the “main lobe” of \( M \) on the disc (see Fig. [4]).

In what follows, we refer to functions having the property (12) as “annular-limited”.

Let us now split \( M(x) \) into two parts \( M_\epsilon \) and \( \delta M \):

\[
M = M_\epsilon + \delta M, \tag{13}
\]

where \( M_\epsilon \) is the “main lobe” that vanishes outside the disc of diameter \( \epsilon \)

\[
M_\epsilon(x) = \begin{cases} M(x), & x < \epsilon/2 \\ 0, & x > \epsilon/2 \end{cases}, \tag{14}
\]

and the “tail”, that vanishes on the disc

\[
\delta M = \begin{cases} 0, & x < \epsilon/2 \\ M(x), & x > \epsilon/2 \end{cases}. \tag{15}
\]

Taking a coronagraph with \( \sigma \leq 1 - \epsilon \), from the decomposition (13) and eqs. (4, 8) we get the final field

\[
F_{\tilde{\beta}}(\vec{x}) = m(\tilde{\beta}) L(x) E_{\tilde{\beta}}(\vec{x}) - L(x) \Delta_{\tilde{\beta}}(\vec{x}),
\]

where

\[
\Delta_{\tilde{\beta}}(\vec{x}) = \delta m(\tilde{\beta}) E_{\tilde{\beta}}(\vec{x}) - [\delta M * A_{\tilde{\beta}}](\vec{x}).
\]
Here, $\delta m$ is the Fourier transform of the “tail” $\delta M$.

In the detector image plane we have

$$f_\beta(\vec{y}) = m(\vec{\beta})l(\vec{y} - \vec{\beta}) - \delta f_\beta(\vec{y}),$$

where

$$\delta f_\beta = l * [(\delta_\beta - a_\beta)\delta m],$$

and $\delta_\beta$ stands for the two-dimensional Dirac $\delta$-function:

$$\delta_\beta(\vec{y}) = \delta(\vec{y} - \vec{\beta}).$$

We refer to the first term in the RHS of (16) as the “main field”, while the second term $\delta f$ is referred as the “residue field”.

Since the transmission amplitude vanishes outside the annulus (12), the main field for incidences outside this annular region is completely suppressed and

$$f_\beta(y) = -\delta f_\beta(y), \quad \mu(\vec{\beta}, \vec{y}) = \delta f_\beta(y)^2, \quad \beta \notin y_1 < \beta < y_2.$$

The residue field is determined by the Fourier transform $\delta m$ of the “tail”. In what follows we call $\delta m$ the residue transmission. From (13, 14, 15) it follows that the residue transmission equals

$$\delta m = (1 - \hat{K}_\epsilon)m,$$  

where $\hat{K}_\epsilon$ is the following integral operator

$$\hat{K}_\epsilon[m](\vec{y}) = \int_{y_1 < y' < y_2} K_\epsilon(\vec{y} - \vec{y}') m(\vec{y}')d^2y, \quad K_\epsilon(\vec{y}) = \frac{\epsilon J_1(\pi \epsilon y)}{2y},$$

and the integral is taken over the annulus $y_1 < y < y_2$.

On the other hand, for working incidences, $m$ is of the order of unity and $m(\beta_W) \gg \max|\delta m(y)|$. As a consequence, at these incidences the residue field can be neglected and, similarly to the band limited coronagraph (10), PSF of the optical system is $m(\beta_W)^2$ times the PSF of the Lyot Stop.

4 Optimal Occultation

Now we are going to find the annular-limited $m(y)$, having the “smallest” possible “tail” $\delta M$. The tail is smallest in the sense of the optimal apodization by D.Slepian [11]: that is, it has the smallest possible energy. In more details, one has to maximize the energy of the main lobe, i.e. the ratio

$$\kappa = \frac{\int_{x<\epsilon/2} M(x)^2d^2x}{\int M(x)^2d^2x},$$

where the integral in the numerator is taken over the disc of diameter $\epsilon$, while the integral in the denominator is taken over the whole plane. Obviously, $0 < \kappa < 1$. The above equation can be rewritten in terms of $m$:

$$\kappa = \frac{\int d^2y \int d^2y'K_\epsilon(\vec{y} - \vec{y}') m(\vec{y})m(\vec{y}')} {\int m(y)^2d^2y}.$$
Figure 3: Optimal transmission amplitude $m(y)$ for the mask with $y_1 = 7$, $y_2 = 28$ and $\epsilon = 0.4$. Here $1 - \kappa \approx 4 \times 10^{-7}$. The working incidence $\beta_W = 2y_1 = 14$. The working throughput is 30 percents.

It is not difficult to see that optimal $m(y)$ is an eigenfunction of the integral operator $\hat{K}_\epsilon$ corresponding to its maximal eigenvalue $\kappa$:

$$\kappa m(\vec{y}) = \hat{K}_\epsilon[m](\vec{y}), \quad y_1 < y < y_2.$$  \hspace{1cm} (22)

Since we consider radially symmetric $m(\vec{y}) = m(y)$, the two-dimensional integral operator (19) can be reduced to one-dimensional one by integration in polar coordinates in the $y$-plane (see Appendix 1) and

$$\kappa m(y) = (\pi \epsilon)^2 \int_{y_1}^{y_2} K(\pi \epsilon y, \pi \epsilon y') m(y') dy', \quad y_1 < y < y_2,$$  \hspace{1cm} (23)

where

$$K(y, y') = \frac{y J_1(y) J_0(y') - y' J_1(y') J_0(y)}{y^2 - y'^2}. \hspace{1cm} (24)$$

Our optimization differs from that of the apodization problem considered by D. Slepian\(^8\) in [11]. There, an analog of our function $m(y)$ is “disc”-limited, which allows to reduce the apodization problem to solution of an eigenvalue problem for an ordinary differential operator. We do not know if a reduction to some “sparce” operator is possible in the annular-limited case, but the diagonalization (23) can be easily performed numerically\(^9\) due to symmetry and positive definiteness of the operator.

We call the problem of optimization of $m(y)$ for an arbitrary shaped support on the mask the optimal occultation. Similarly to results of the optimal apodization, $\kappa$ is very close to 1 in our examples of the optimal occultation (see below). There the value of $1 - \kappa$ is of order of $10^{-7}$.

To make estimates of coronagraphic suppression, we will need to find the residue transmission $\delta m$. According to \([12 \hspace{0.2cm} 18 \hspace{0.2cm} 22]\)

$$\delta m(y) = \begin{cases} 
(1 - \kappa)m(y), & y_1 < y < y_2 \\
-\hat{K}_\epsilon[m](y), & \text{otherwise}
\end{cases} \hspace{1cm} (25)$$

\(^8\)General formulation of the problem for arbitrary finite supports has been posed earlier, e.g. in [10]. There general properties of $m$ are given, but solution was presented only for the disc-limited case.

\(^9\)Diagonalization of our integral operator takes seconds of CPU time on modern PC, while at the time when works of D. Slepian et al [11] were published such computation power was unavailable.
Figure 4: \(\log_{10}\frac{M(x)^2}{M(0)^2}\) of the optimal mask from Figure 3 \((y_1 = 7, y_2 = 28\) and \(\epsilon = 0.4\)). The part of the plot to the left from the vertical dashed line corresponds to the “main lobe” of \(M\), while the part on the right corresponds to its “tail” (see eqs. (13, 14, 15)).

It is worthy to note that \(m(y)\) can be formally continued beyond the annular region by dropping the restriction \(y_1 < y < y_2\) in equation (23). Then

\[
\tilde{m}(y) = \kappa^{-1} \tilde{K}_\epsilon[m](y)
\]

equals \(m(y)\) inside the annular region and continues it outside the region where, according to eq.(25), \(\delta m = -\kappa \tilde{m}\). Since in all our examples \(1 - \kappa\) will be extremely small (of the order of \(10^{-7}\)), we can write that

\[
\delta m(y) \approx \begin{cases} 
0, & y_1 < y < y_2 \\
-\tilde{m}(y), & \text{otherwise}
\end{cases}
\]

and at boundaries of the region \(\delta m(y_i) \approx -m(y_i), i = 1, 2\). The residue transmission is an oscillating function with the asymptotics \(\delta m(y) \to -M(0)\frac{\sqrt{2}}{2\pi y^{3/2}}\sin(\pi y - \pi/4)\) for \(y \to \infty\).

To get an idea of magnitude of the coronagraphic suppression, one can roughly estimate maximum of PSF for incidences \(\beta\) outside the annular region and compare it with maximum of PSF with no mask applied (i.e. with maximum of PSF of the Lyot stop).

From (17) it follows that absolute value of the residue field \(|\delta f_\beta|\) is not exceeding the biggest of the two following values

\[
2|\delta m(\beta)| \max |l|, \quad 2 \max |l| * \left|a_\beta \delta m\right|.
\]

One may assume that these two values are of the same order on average and

\[
|\delta f_\beta| < O(\max |\delta m(y)|) \max |l|.
\]

Therefore, for incidences outside the annular region:

\[
\max \mu(\tilde{\beta}, \tilde{y}) = s(\beta) \max \mu_L = s(\beta) \left(\frac{\pi \sigma^2}{4}\right)^2, \quad s(\beta) < O(\max m(\beta)^2), \quad \beta \notin y_1 < \beta < y_2,
\]

where \(\mu_L\) stands for PSF of the Lyot stop. In other words, according to our estimate the suppression coefficient \(s(\beta) = \max \mu(\tilde{\beta}, \tilde{y})/ \max \mu_L\) is of order max \(\delta m^2\) or smaller order. This estimate is confirmed
by direct numerical simulations of coronagraph which are presented in the next section. In fact, the actual suppression coefficient turns to be about two-three orders of magnitude smaller (i.e. suppression is 2-3 orders stronger) than our very rough estimate $\max \delta m^2$ for incidences that are not in vicinity of boundary of the annular region (see Figures 3, 6). In vicinity of the boundary it is of the same order. A more accurate estimate could use some envelope $\tilde{m}_e(\beta)$ of the oscillating function $\tilde{m}(\beta)$. The suppression coefficient and square of this envelope are of the same order $s(\beta) = \mathcal{O}(\tilde{m}_e^2(\beta))$ (see Figure 6).

Now, to be specific, we consider an example from the introduction section: Take $D = 2.5m$ telescope at $Z = 4Z_{\min}$ and $\lambda = 750$ nm. For these parameters the magnitude of the incidence corresponding to the Einstein ring (working incidence) $\beta_W = \alpha_E D/\lambda = 14$. Let the apparent boundary of the sun coincide with the interior boundary of the annular region. Then $y_1 = \beta_W/2 = 7$. We chose the outer boundary of the region to be $y = y_2 = 2\beta_W = 28$.

We choose $\epsilon = 0.4$ which corresponds to the maximal throughput $(1-\epsilon)^2 = 0.36$. The optimal transmission amplitude obtained by numerical solution of (23) is shown on Figure 3. The difference between the “main lobe” and the “tail” of the Fourier transform of $m$ is demonstrated on Figure 4. The formal continuation $\tilde{m}(y)$ of $m$, is shown in separate boxes of Figure 3. For this solution $1-\kappa \approx 4 \times 10^{-7}$. Therefore, outside the annular region, $\delta m \approx -\tilde{m}$ with the relative precision $4 \times 10^{-7}$. The residue transmission reaches maximum by absolute value at $y = y_1$ and max $\delta m^2 \approx 6.4 \times 10^{-7}$. Therefore, one can expect the suppression to be of six or more orders of magnitude.

The transmission amplitude at working angle $m(\beta_W)$ equals 0.925, so, according to (11) the maximal working throughput is $m(\beta_W)^2(1-\epsilon)^2 \approx 0.308$ (i.e. about 30 percents).

5 Results of Direct Numerical Simulations

Direct numerical simulations of a Quasi Band Limited Coronagraph are in agreement with the above estimates. We performed direct simulations of all stages of the field propagation (see eqs. (15) and Figure 2) applying the fast Fourier transform at each pupil. Maximal size of the grid was $4096 \times 4096$, but sizes $2048 \times 2048$ and $1024 \times 1024$ give results which differ only by few percents from those of maximal grid. Simulations are run for the Lyot stop of the diameter $\sigma = 1-\epsilon = 0.6$. The suppression is minimal (i.e. suppression coefficients are maximal) in the neighborhood of interior boundary of the annular region. With a good precision the suppression coefficients equal $\delta m(\beta)^2$ in this neighborhood. The above value has its maximum $6.4 \times 10^{-7}$ at the boundary and falls rapidly by about two orders of magnitude as $\beta$ decreases: The suppression coefficients are of the order of $10^{-8} - 10^{-9}$ at the bigger part of the disc $\beta < y_1$, see Figure 3. The suppression coefficient of maximum of PSF is of the same order as the coefficient of energy suppression $\tau(\beta)/\sigma^2$. The latter coefficient equals the ratio of energy flow passed in the presence of mask to that without mask.

The detector plane images of point sources at different incidences $\beta \leq y_1$ and sections of intensity are shown on Figure 5.

To evaluate the sun glare one has to compute integral (7) of the product of distribution of the surface brightness of the sun with the PSF in the detector plane divided by area of the Lyot stop. We recall that we choose the parameters in such a way that apparent radius of the sun equals $y_1$ in units of characteristic diffraction angles, i.e. edge of the sun disc coincides with interior boundary of the annulus. In our computations we presented the sun as a disc of uniform surface brightness $I_0$, so that the limb darkening is not taken into account. Accounting for limb darkening will give better results (i.e. the glare brightness will be smaller). With the optimal mask applied, the glare brightness at working angle

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10 The surface brightness of the edge of the sun disc is less than half of that of the disc center. Since the boundary regions contribute most to the glare, the surface brightness of the glare obtained with account of the limb darkening will be approximately half of that we obtained for the uniform disc.
Figure 5: Bottom: Normalized images of point sources. Top: log$_{10}$ of PSF in the detector plane (i.e. log$_{10}$ $\mu$) along the central sections (i.e. along the dashed lines at corresponding bottom images). Panel A: No mask is applied. Panels B to D: optimal mask is applied. Corresponding incidences are: $\beta = 0$ (Panel B, on-axis source), $\beta = y_1/2 = 3.5$ (Panel C) and $\beta = y_1 = 7$ (Panel D, source at interior boundary of annular region).

Figure 6: Thick solid curve: log$_{10}$ of energy suppression coefficient $\tau(\beta)/\sigma^2$ (the latter is the ratio of energy passed with mask to that passed without mask). Thin solid curve: log$_{10}$ of suppression of maximum of PSF at given $\beta$ (i.e. log$_{10}$ $s(\beta)$, see eq. (27)). Dashed curve: log$_{10}$ of square of continuation $\tilde{m}(\beta)$ (recall that $\tilde{m}(\beta)^2 \approx \delta m(\beta)^2$ for $\beta \notin y_1 < \beta < y_2$).
Figure 7: Left: Detector plane image of the solar disc. Optimal mask is applied. No limb darkening is taken into account (the sun is presented as a disc of uniform surface brightness $I_0$). Center: Radial dependence of the relative surface brightness in the detector plane. Right: Radial dependence without mask applied.

$\beta_W = 2\gamma_1 = 14$ (i.e. at position of the Einstein ring) is about $10^{-10}I_0$ (see Figure. 7). Therefore, the goal set in the introductory section can be achieved with help of our mask.

6 Suppressing the Parent Star, One-dimensional Slepian’s Mask, Product Mask

The circularly symmetric mask can suppress the diffraction glare of the sun to the level of brightness of the detector plane image of an Earth-like exo-planet at about 30pc. However, there remains a problem of suppressing glare from the planet’s parent star. The SGL produces two images of the parent star on the line that passes through the center of the sun (see Figure 1). They appear at opposite sides wrt the center, one inside the circle $\alpha = \alpha_E$ (i.e. inside circle $\beta = \beta_W$) and the other outside it. Minimal angular distance between images and the circle is about a half of the apparent separation between the planet and its parent star. The size of the star is not resolvable by the telescope, so, for any practical purpose the star can be considered as a point source.

Since the SGL amplifies intensity of radiation from a planet much more stronger than it does from its parent star and apparent diameter of the Einstein ring is relatively big, an acceptable suppression level for observations with SGL is several orders of magnitude weaker than that required for observations without it. Indeed, the ratio of the amplification of radiation from an exo-planet to that from its parent star is of the order of the ratio of planet’s orbit radius to its own radius. This is about $3 \times 10^4$. Taking into account that the circumference of the Einstein ring in the detector plane is about 30 resolution elements, reduction of the flux ratio $^{11}\text{ by the SGL is } \sim 10^3$. Since, for an Earth-like planet the unreduced flux ratio is about $\sim 10^{10}$, it will be $\sim 10^7$ in presence of the SGL. Also, since the apparent distance between the bigger part of the Einstein ring and images of the parent star is of the order of 10 diffraction angles, acceptable magnitude of the suppression coefficients can be several orders bigger than $10^{-7}$ (we recall that, according to our definitions, smaller suppression coefficients mean bigger suppression and vice versa).

To solve the problem of simultaneous suppression of the sun and the parent star glare, we first introduce

$^{11}$We define flux ratio as the ratio between the energy flow from the parent star and the energy flow per resolution element from an exo-planet.
Figure 8: Left: Circularly symmetric mask. Center: One-dimensional “Slepian’s” mask. Right: Product mask. Gray-scale values of image pixels are proportional to the mask transparency \( t = m^2 \) (also called intensity transmission factor). Black color corresponds to \( t = 0 \), while white color corresponds to \( t = 1 \).

The interior/exterior radii of the annular support are \( y_1 = 7 \) and \( y_2 = 28 \) correspondingly. The half-width of the Slepian’s mask equals \( y_1 \). For both masks \( \epsilon = 0.4 \).

4. Einstein ring radius \( \beta_W = 2y_1 = 14 \).

an effectively one-dimensional mask that eliminates light from sources located on the line passing through two images of the parent star. The transmission amplitude of this mask is essentially Slepian’s solution to the one-dimensional optimal apodization problem [11]. We call the corresponding mask Slepian’s mask.

Then we will test the mask that is the product of circularly symmetric mask (introduced in previous sections) and the Slepian’s mask (see Figure 8).

Let \( y_\perp \) and \( y_\parallel \) be coordinates in the \( y \)-plane, i.e. \( \vec{y} = (y_\perp, y_\parallel) \). We consider a mask with the transmission amplitude changing only in one direction \( m = m(y_\perp) \). Our aim is to suppress point sources at the line \( y_\perp = 0 \), so that \( m(0) = 0 \). Also

\[
m(\vec{y}) = 1 - u(y_\perp),
\]

where \( u \) is an even function \( u(y_\perp) = u(-y_\perp) \), such that \( u(0) = 1 \) and \( u(y_\perp) \) has a finite support \( y_1 < y_\perp < -y_1 \):

\[
u(y_\perp) = 0, \quad |y_\perp| > y_1.
\]

In other words, the Slepian’s mask is completely transparent outside the strip \( |y_\perp| < y_1 \) (\( m = 1 \) for \( |y_\perp| > y_1 \)). The Fourier transform \( M \) of \( m \) is

\[
M(\vec{x}) = [\delta(x_\perp) - U(x_\perp)] \delta(x_\parallel),
\]

where \( U(x_\perp) = \int e^{2\pi i x_\perp y_\perp} u(y_\perp)dy_\perp \) is one-dimensional Fourier transform of \( u(y_\perp) \) and \( \vec{x} = (x_\perp, x_\parallel) \).

Similarly to the two-dimensional optimal occultation problem, we split \( U(x) \) into the “main lobe” \( U_\epsilon(x) \) with support on the interval \(-\epsilon/2 < x < \epsilon/2\) and the “tail” \( \delta U(x) \):

\[
U = U_\epsilon + \delta U, \quad U_\epsilon(x) = \begin{cases} U(x), & |x| < \epsilon/2 \\ 0, & |x| > \epsilon/2 \end{cases}, \quad \delta U(x) = \begin{cases} 0, & |x| < \epsilon/2 \\ U(x), & |x| > \epsilon/2 \end{cases}
\]

Energy of the “tail” is minimal when \( u \) is an eigenfunction corresponding to the highest eigenvalue \( \kappa \) of the one-dimensional integral operator:

\[
\kappa u(y) = \hat{K}_\epsilon[u](y) = \int_{-y_1}^{y_1} \frac{\sin \pi \epsilon (y - y')}{\pi(y - y')} u(y') dy'.
\]
Solution of the above equation for $\epsilon = 0.4$ and $y_1 = 7$ (same values as in the circularly symmetric case) is shown on Figure 9. Here the deviation of $\kappa$ from unity equals $5 \times 10^{-7}$ and is of the same order as that of the example considered in previous section for the circularly symmetric case. The maximal residue of the transmission amplitude $\max |\delta m| \approx u(y_1)$ and is approximately of the same order as in the circularly symmetric example ($u(y_1) \approx 1.5 \times 10^{-3}$, compare Figures 3 and 9). Therefore, one can expect the same level of suppression for a point source at the line $\beta_\perp = 0$. And, indeed, direct numerical simulations of a coronagraph with the Slepian’s mask and a circular stop of the diameter $1 - \epsilon = 0.6$ give the result $\max_{\mu} \mu \approx 2 \times 10^{-10}$ for PSF and the energy suppression coefficient is equal $\approx 2 \times 10^{-9}$ when the point source is on the line $\beta_\perp = 0$. Obviously, the suppression coefficients do not depend on $\beta_\parallel$ for this one-dimensional mask.

Consider now the product mask, i.e. the mask whose amplitude transmission factor is the product of factors of the circularly symmetric and Slepian’s mask (see Figure 8). The idea of using the product mask comes from the pure band limited case: A product of two band-limited functions $f(\vec{y})$ and $g(\vec{y})$ is also a band limited function, since the support of the convolution $[F * G](\vec{x})$ is a cartesian sum of supports of $F$ and $G$. In the case of product of the circularly symmetric and one dimensional masks, this is the cartesian sum of a (two-dimensional) disc and a (one-dimensional) segment. However, in the case of the quasi band limited masks, the support of new “main lobe” can be smaller than the cartesian sum of supports of the “main lobes” of $F$ and $G$. Indeed, the definition of support of the “main lobe” as a region where most of the energy is concentrated is somehow arbitrary. Also, the convolution of “main lobes” can be of the same order as the “tail” on a substantial part of a cartesian product. That is why one can try to apply the product mask without changing the size of the Lyot stop.

We have performed direct numerical simulations of the coronagraph with the product mask without changing diameter of the stop $\sigma = 1 - \epsilon = 0.6$, preserving 30 percent working throughput. The results of simulations do not show substantial degradation in the suppression coefficients: The intensity of glare over the bigger part of the working region is of the same order as in the circularly symmetric case. The results of suppression of the glare from the sun are shown on Figure 10. Although, in difference from the circularly symmetric case, the working space is reduced, the glare at the bigger part of the Einstein ring (in total, more than 180 degrees of the working space along the ring circumference) is of the same order as in the circularly symmetric case.

Finally, the suppression coefficient for the point sources located at the line $\beta_\perp = 0$ also do not show...
Figure 10: Center: Detector plane image of the solar disc. Product mask is applied. No limb darkening is taken into account (the sun is presented as a disc of uniform surface brightness $I_0$). Left: $\log_{10}$ of relative surface brightness $I/I_0$ along the circumference of the image of the Einstein ring $y = 2y_1 = 14$. Right: Level contours of $\log_{10}$ of the relative surface brightness.

Figure 11: $\log_{10}$ of the detector plane PSF for the product mask and different position of point sources along the line $\beta_\perp = 0$. Corresponding positions of the sources and the Einstein ring are shown at the rightmost panel ($\beta_\parallel = 12$, $\beta_\parallel = 14$ and $\beta_\parallel = 16$ for the point sources at A, B and C correspondingly).

Figure 12: $\log_{10}$ of suppression coefficients of the product mask as functions of $\beta_\parallel$ for $\beta_\perp = 0$, $\beta_\perp = 0.05$ and $\beta_\perp = 0.1$ (from left to right correspondingly). Thick line corresponds to the energy suppression coefficient, while thin line stands for the PSF maximum suppression coefficient.
Figure 13: $\log_{10}$ of suppression coefficients of the circularly symmetric mask designed for wavelength $\lambda$ and operating at $\lambda' = \lambda/2$. Thick curve corresponds to the energy suppression coefficient, while thin line corresponds to the coefficient for maximum of PSF.

a substantial degradation: The level of suppression is more than sufficient for reducing glare from the parent star to an acceptable level. We also performed simulations of cases with off-alignment of the parent stars and the mask, i.e. the cases when $\beta_\perp \neq 0$ (see Figure 12). The acceptable deviation from the central line is at least of the order of $0.1$, (i.e. $\sim 1/10$ of the characteristic diffraction angle).

7 Discussion and Conclusions

In the present article we proposed graded coronagraph masks that can substantially suppress the diffraction glare from extended sources. In examples considered, the diffraction glare from a source in the form of disc of the apparent diameter of the order of one arc-second and of the uniform surface brightness $I_0$ can be reduced by about $10^{8}$ times (example of 2.5m telescope operating at $\lambda = 750$nm). Surface brightness of the glare at the apparent distance twice the radius of the disc from the disc center is about $10^{-10} I_0$.

It is important to note that, similarly to the band-limited masks, the quasi band-limited mask designed for a wavelength $\lambda$ will work for $\lambda' < \lambda$ with the same working throughput and better suppression rates. Figure 13 illustrates the above statement for our example of the circularly symmetric mask designed for $\lambda = 750$nm and operating at $\lambda' = \lambda/2 = 375$nm (compare with Figure 6).

Examples considered in this paper are centered around the particular application: imaging of exoplanets with the help of the solar gravitation lens (SGL). Such imaging requires not only suppressing sunlight, but also reducing glare from the parent star of an exo-planet. The product mask, introduced in the previous section, can perform both of the above tasks.

Another problem related to the SGL imaging is to suppress light from the solar corona: Our mask reduces significantly the glare from almost all the corona, except immediate neighborhood of the Einstein ring. Appendix 2 presents the related results.

There are two important problems related to effectiveness of our coronagraph in realistic conditions: the one of tolerance to manufacturing errors and a tolerance to wavefront imperfections (wavefront control).

To test the tolerance of mask to manufacturing errors we divided transparency (also called intensity transmission factor) of the circularly symmetric mask

$$t(y) = m(y)^2, \quad 0 \leq t \leq 1,$$
into $10^4$ discrete gray-scale levels with spacing $\delta t = 10^{-4}$ between the adjacent levels. Such a discretization leads to about one order of magnitude degradation in the level of suppression which is acceptable for our purposes. The results of the corresponding simulations are shown on Figure 14. It is worthwhile to mention that our numeric computations were performed on a square spatial grid with spacing $\delta y \approx 0.1$, and, therefore $\delta t$ is, in fact, the minimal spacing between levels. The actual spacing at given $y$ is the biggest of the two values $\delta t$ and $|\nabla t(y) \cdot \delta \vec{y}|$. The second of the above values is $\approx 3 \times 10^{-3}$ at points where $t(y)$ is steepest.

Concluding the paper we would like to note that the level of suppression in our examples is comparable with the level of suppression by the WFIRST coronagraph instrument [12], where the active wavefront control is used. In our examples, the working angle is much bigger than the minimal working angle of the WFIRST coronagraph instrument, so one can assume that the wavefront control would require less sophisticated technology in our case.

8 Appendix 1

Below we derive (23, 24): From (20, 21) it follows that

$$\hat{K}_\epsilon[m](\vec{y}) = \int_{y_1 < y' < y_2} K_\epsilon(\vec{y} - \vec{y}')m(y')d^2y' = \int_{y_1 < y' < y_2} d^2y' \int_{x < \epsilon/2} d^2x e^{2\pi i \vec{x} \cdot (\vec{y} - \vec{y}')} m(\vec{y}).$$

After integrating in polar coordinates in the $x$-plane only, one can express $K_\epsilon$ in terms of the Bessel function as in eq. (19), but instead we rewrite all variables in polar coordinates as

$$\vec{x} = (x \cos \theta, x \sin \theta), \quad \vec{y} = (y \cos \varphi, y \sin \varphi), \quad \vec{y}' = (y' \cos \varphi', y' \sin \varphi').$$

Then, taking into account that $m$ is circularly symmetric, i.e. $m(\vec{y}') = m(y')$, and integrating first in $\varphi'$ and then in $\theta$, we get

$$\hat{K}_\epsilon[m](y) = \int_{y_1}^{y_2} \left[ (2\pi)^2 \int_0^{\epsilon/2} J_0(2\pi xy)J_0(2\pi xy')xdx \right] m(y')y'dy'.$$

Taking integral in the square brackets, we obtain [23, 24].

Figure 14: Comparison of the original mask with discretized (degraded) mask. Dashed curves show quantities related to the original mask, while solid curves correspond to degraded mask. Left: transparency $t(y)$ across the annular region $y_1 < y < y_2$. Center: Zoom of transparency in neighborhood of interior boundary of annulus. Right: Comparison of the degraded energy suppression coefficient with the original one.
Figure 15: Dependence of effective distribution of the relative surface brightness of corona on $\beta$, i.e. $I_{\text{eff}}(\beta)/I_0$.

9 Appendix 2

With great precision eq. (10) holds for incidences corresponding to the light from corona $\beta > y_1$. Taking into account (6) and (7) we obtain surface brightness of the corona’s glare in the detector plane

$$I(\vec{y}) = \int I_c(\beta)m^2(\beta)\mathcal{I}_L(\vec{\beta}, \vec{y})d^2\beta,$$

(28)

where $I_c(\beta)$ is the distribution of the surface brightness in corona and $\mathcal{I}_L(\vec{\beta}, \vec{y}) = \mu_L(\vec{\beta}, \vec{y})/S$. From (28) it follows that effect of mask is equivalent to reducing distribution of the surface brightness of corona by the factor $m(\beta)^2$. In other words, corona is seen in the detector plane as if there were no mask, but instead the surface brightness distribution of the corona were $I_{\text{eff}}(\beta) = m(\beta)^2I_c(\beta)$.

According to [6], dependence of the distribution $I_c$ on $\beta$ is

$$I_c = \left(\frac{3.67}{\rho^{18}} + \frac{1.939}{\rho^{5.8}} + \frac{0.0551}{\rho^{2.5}}\right) \times 10^{-6}I_0,$$

(29)

where

$$\rho = \beta/\beta_S > 1$$

is the distance from the center of the sun in units of angular radius of the sun. In our example $\beta_S = y_1$. Multiplying (29) and $m(\beta)^2$ from our example of the circularly symmetric mask (corresponding $m(\beta)$ is shown on Figure 3), we get effective distribution of brightness $I_{\text{eff}}$ shown on Figure 15. Action of the mask results in significant effective reduction of the corona brightness almost everywhere, except close neighbourhood of the Einstein ring. (compare Figure 15 with Figure 4)

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