Representations of fuzzy torus

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Abstract. A classification of Hermitian representations for the recently introduced fuzzy torus algebra is presented. This is carried out by regarding the fuzzy torus algebra as a $q$-deformation of parafermion. In addition to the known representations, new representations of both finite and infinite dimension are found. Using the infinite dimensional representation, coherent state for the fuzzy torus is constructed. Dirac operator on commutative torus is also discussed.

1. Introduction

Fuzzy spaces provide a method of regularizing the quantum field theory allowing an alternate route to the naive lattice regularization. It has been observed that the use of such spaces has some advantages (see for example [1]). Fuzzy spaces also provide one of the models of noncommutative manifold on which one can develop noncommutative geometry [2]. The basic idea of fuzzy spaces is that the algebras of functions on a smooth manifold, as well as their local coordinates, are replaced with associative algebras of finite dimensional matrices. The size of the matrices is arbitrary. In the limit of infinite dimensional matrices, the geometry on commutative manifold is recovered.

The typical example of the fuzzy spaces and the most extensively studied one may be the fuzzy 2-sphere [3]. Many other examples of fuzzy spaces have been constructed so far. We shall mention some of them below. Regarding the ordinary 2-sphere $S^2$ as a one-dimensional complex projective space $\mathbb{C}P^1$, construction of the fuzzy $S^2$ has been extended to higher dimensional fuzzy projective spaces [4]. The fuzzy $\mathbb{C}P^n$ opens a way to other fuzzy spaces. Fuzzy version of $S^3$ are extracted from fuzzy $\mathbb{C}P^3$[5]. Exploiting the observation that $\mathbb{C}P^3$ is an $S^2$ bundle over $S^4$, the fuzzy $S^4$ has been constructed [6]. Fuzzy version of $S^4$ is also discussed by using tensor product representation of $Spin(5)$ [7] and the Hopf fibration $S^7 \rightarrow S^4$ [8]. Fuzzy $S^5$ has been established by utilizing a $U(1)$ fibration over a fuzzy $\mathbb{C}P^2$[9]. Fuzzy orbifolds [10] and fuzzy toric varieties [11] are obtained as a subset of fuzzy complex projective spaces. It has been shown that fuzzy $S^2$ and fuzzy $\mathbb{C}P^n$ have their supersymmetric extensions [12, 13]. Further examples are found in the investigations of fuzzy unitary Grassmannian spaces [14, 15] and fuzzy orthogonal Grassmannian spaces [16]. Fuzzy orthogonal Grassmanians are used to extract fuzzy spheres of higher dimensions in [16]. Furthermore, fuzzy version of disc [17] and flag manifolds including supersymmetric extension [18] have been discussed.

It should be noted that except some specific cases the matrices representing the coordinates of fuzzy space do not form a closed algebra. In this sense, some of the aforementioned fuzzy spaces
are unsatisfactory. Nevertheless, one can discuss field theory on such spaces by suppressing unwanted terms in functional integral. Recently, it was proposed a general recipe for constructing fuzzy version of compact connected Riemann surfaces of arbitrary genus as closed algebras [19]. The genus one case (torus) was explicitly carried out and the authors of [19] defined the fuzzy torus as a nonlinear C-algebra with three generators. The fuzzy torus algebra \( T^n_F \) has been recast as a \( q \)-deformed Lie algebra in [20]. This may be understood as a linearization of the nonlinear \( T^n_F \) algebra by using the \( q \)-deformed commutators.

In the present work, we are studying the fuzzy torus algebra from a different viewpoint. We regard \( T^n_F \) as a \( q \)-deformed parafermion algebra which is different from the \( q \)-deformations of parafermion or paraboson discussed in the literature [21, 22, 23, 24, 25, 26]. Using this approach, we classify the Hermitian representations of \( T^n_F \). In addition to recovering the known finite dimensional representations [19, 20], families of finite and infinite dimensional representations are newly found. Using the newly found infinite dimensional representations, coherent state for the fuzzy torus is constructed. Coherent states play an important role in noncommutative geometry, since one can define noncommutative star product using coherent states [27, 28]. We also give Dirac operator on the commutative torus in our setting. This is the first step to construct Dirac operator on fuzzy torus. Dirac operator is one of the fundamental building blocks of noncommutative geometry [2] and there is no need to say its physical importance. In many literature discussing Dirac operator on torus, the torus is defined by giving cyclic condition on flat plane. It means that such torus has vanishing curvatures. On the contrary, our torus does have nonvanishing curvatures.

This paper is organized as follows. Next section is a brief review of the construction of fuzzy Riemann surfaces proposed in [19]. The definition of the fuzzy torus is also given there. In §3, the fuzzy torus algebra is reformed into a kind of \( q \)-deformation of parafermion algebra. Using this form, possible Hermitian representations are derived. The results of §3 are also reported in [29]. In §4, coherent state, which is an eigenstate of the annihilation operator of the \( q \)-deformed parafermion, is given. New deformation of the Bessel functions is introduced to formulate the coherent state. We discuss Dirac operator on commutative torus in §5. §6 is devoted to concluding remarks.

2. Fuzzy version of Riemann surfaces

In this section, we briefly review the work of [19] which is a construction of fuzzy version of compact connected Riemann surfaces as a nonlinear algebra. The construction consists of two steps.

The first step is an embedding of Riemann surfaces in \( \mathbb{R}^3 \). Using the Cartesian coordinates \((x, y, z)\) in \( \mathbb{R}^3 \), the genus zero Riemann surface \( S^2 \) is described by a constrained polynomial \( C(x, y, z) = 0 \), where \( C(x, y, z) = x^2 + y^2 + z^2 - 1 \). It is shown, with the aid of the regular value theorem and Morse theory, that similar polynomial description is possible for any Riemann surfaces of higher genus. Let us consider the following polynomial with a positive constant \( c \):

\[
C(x, y, z) = (P(x) + y^2)^2 + z^2 - c,
\]

\[
P(x) = \sum_{j=0}^{2k} a_j x^j, \quad a_{2k} > 0.
\]

The constrained polynomial \( C(x, y, z) = 0 \) describes a genus \( g \) Riemann surface if the equations \( P(x) \pm \sqrt{c} = 0 \) have the certain number of nondegenerate solutions. The required number of solutions for \( P(x) + \sqrt{c} = 0 \) is \( 2g \) and for \( P(x) - \sqrt{c} = 0 \) is two.

The second step is a Poisson bracket and its quantization. It is known that for arbitrary function \( \ell : \mathbb{R}^3 \to \mathbb{R} \)

\[
\{f, g\}_{\mathbb{R}^3} = \nabla \ell \cdot (\nabla f \times \nabla g),
\]

2
defines a Poisson bracket for functions on \( \mathbb{R}^3 \). In the present case, we take \( \ell = C(x, y, z)/2 \). Then the Poisson brackets between the coordinates read
\[
\{x, y\}_{\mathbb{R}^3} = z, \quad \{y, z\}_{\mathbb{R}^3} = P'(x)(P(x) + y^2), \quad \{z, x\}_{\mathbb{R}^3} = 2y(P(x) + y^2),
\] (2.3)
where \( P'(x) \) denotes the derivative of \( P(x) \). Next we apply Dirac quantization to these Poisson brackets, namely, the Poisson brackets are replaced with commutators. Ordering problem arise in the right hand sides of (2.3). The following is the proposed ordering in [19]:
\[
\{X, Y\} = i\hbar Z,
\]
\[
\{Y, Z\} = i\hbar \sum_{n=1}^{2g} \sum_{j=0}^{n-1} X^j (P(X) + Y^2) X^{n-1-j},
\]
\[
\{Z, X\} = i\hbar \{Y, P(X) + Y^2\},
\] (2.4)
where the quantized coordinates are denoted by capital letters and the parameter \( \hbar \) imparts noncommutativity. Its commutative limit is given by \( \hbar \to 0 \). One can verify that the polynomial algebra (2.4) satisfy the Jacobi identity.

For the case of torus \((g = 1)\), the polynomial (2.1) reads:
\[
C(x, y, z) = (x^2 + y^2 - \mu)^2 + z^2 - c,
\] (2.5)
where \( \mu = -a_0 \) is a real parameter. The relation \( C(x, y, z) = 0 \) describes a torus for \( \sqrt{c} < \mu \) and a squashed sphere for the domain \( -\sqrt{c} < \mu < \sqrt{c} \). The fuzzy torus algebra is obtained from (2.4):
\[
[X, Y] = i\hbar Z, \quad [Y, Z] = i\hbar \{X, \varphi\}, \quad [Z, X] = i\hbar \{Y, \varphi\},
\] (2.6)
and the constraint
\[
C_F \equiv \varphi^2 + Z^2 = c.
\] (2.7)
It is easily observed that the element \( C_F \) is the center of the algebra. This is consistent with the constraint (2.7). In [19], the parameter \( c \) is set equal to unity. Following [20], we, however, keep its value arbitrary throughout this paper.

3. Hermitian representations of fuzzy torus algebra

3.1. Fuzzy torus and \( q \)-Deformed parafermion
The fuzzy torus has been obtained as a nonlinear algebra of three generators in the previous section. Our next question is what kinds of representations are available for the fuzzy torus algebra. Two classes of representations have already been derived in [19]. In this section, we consider the problem more systematic way and try to classify Hermitian representations. To this end, we recast the relations in (2.6) as a form which is interpreted as a \( q \)-deformed parafermion.

Introducing the complex variable \( W = X + iY \) and its Hermitian adjoint \( W^\dagger \), the defining relations (2.6) are reexpressed as
\[
[W, W^\dagger] = 2i\hbar Z, \quad [Z, W] = \hbar \{W, \varphi\}, \quad [Z, W^\dagger] = -\hbar \{W^\dagger, \varphi\},
\] (3.1)
\[
\varphi \equiv \frac{1}{2} \{W, W^\dagger\} - \mu.
\] (3.2)
Eliminating $Z$ via the first equation in (3.1), it is observed [19] that the operators $W, W^\dagger$ obey the following trilinear relation and its Hermitian adjoint:

$$W(W^\dagger)^2 - 2\frac{1 - \hbar^2}{1 + \hbar^2}W^\dagger WW^\dagger + (W^\dagger)^2W = \mu \frac{4\hbar^2}{1 + \hbar^2}W^\dagger.$$  \hspace{1cm} (3.3)

The deformation parameter $q$ is introduced [20] by the relation

$$2\frac{1 - \hbar^2}{1 + \hbar^2} = q + q^{-1}.$$  \hspace{1cm} (3.4)

It follows that the complex parameter $q$ is of unit magnitude: $|q| = 1$. The commutative $\hbar \to 0$ limit now corresponds to $q \to 1$. We now scale the conjugate variables $W$ and $W^\dagger$ as follows

$$a = \left(\frac{2}{|\mu|(2 - q - q^{-1})}\right)^{1/2}W, \quad a^\dagger = \left(\frac{2}{|\mu|(2 - q - q^{-1})}\right)^{1/2}W^\dagger.$$  \hspace{1cm} (3.5)

For a fixed $\mu$ above scaling is well-defined in the noncommutative $q \neq 1$ i.e. $\hbar \neq 0$ regime. The trilinear relation (3.3) now assumes two distinct forms depending on the sign of the parameter $\mu$. For the choice $\mu < 0$, it reads

$$a(a^\dagger)^2 - [2]_q a^\dagger a a^\dagger + (a^\dagger)^2 a = -2a^\dagger,$$  \hspace{1cm} (3.6)

where the $q$-number is defined as usual

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$  \hspace{1cm} (3.7)

For the alternate $\mu > 0$ case, the relation (3.3) may be recast as

$$a(a^\dagger)^2 - [2]_q a^\dagger a a^\dagger + (a^\dagger)^2 a = 2a^\dagger.$$  \hspace{1cm} (3.8)

The trilinear relation (3.6) may be regarded as $q$-deformation of the parafermion [30]. The relation (3.8) has the same form albeit with an opposite sign on the right hand side. One may regard this as a variant of the $q$-deformed parafermion. The transformation (3.5) is singular for a fixed $\mu$ in the limit $q \to 1$. In this scenario the classical $q = 1$ parafermions do not arise. There is, however, an interesting possibility

$$|\mu| \sim (2 - q - q^{-1})^{-1},$$  \hspace{1cm} (3.9)

which allows taking the $q \to 1$ limit in (3.5). This limit is pertinent in understanding some of the representations discussed below.

### 3.2. Representations

Using the $q$-parafermionic form of the fuzzy torus algebra given in the previous subsection, we study possible Hermitian representations. Our strategy is as follows: we start with the smallest number of assumptions and construct the most general representation. Then, further assumptions are imposed to obtain possible subclasses of the representations. As observed in [19], the operators $aa^\dagger$ and $a^\dagger a$ commute, and, therefore, they are simultaneously diagonalizable. We thus assume the existence of a normalized state $|0\rangle$ such that

$$aa^\dagger |0\rangle = p |0\rangle, \quad a^\dagger a |0\rangle = r |0\rangle, \quad \langle 0 | 0 \rangle = 1.$$  \hspace{1cm} (3.10)
Other states may be obtained by repeated actions of $a$ or $a^\dagger$ on $|0\rangle$. The states are required to be normalizable:
\[ |n\rangle = N_n(a^\dagger)^n |0\rangle, \quad |−n\rangle = N_{−n}a^n |0\rangle \quad \forall \quad n > 0, \tag{3.11} \]
where $N_{±n}$ are the normalization constants. Normalization of the $|±1\rangle$ states puts constraints on the values of $p$ and $r$:
\[
\| |1\rangle \|^2 = |N_1|^2 \langle a a^\dagger |0\rangle = |N_1|^2 p, \quad \| |−1\rangle \|^2 = |N_{−1}|^2 \langle a^\dagger a |0\rangle = |N_{−1}|^2 r.
\]
It follows that $p, r > 0$, and we choose $N_1 = p^{−1/2}$, $N_{−1} = r^{−1/2}$. For the states of $|±n\rangle \forall n ≥ 2$, we use the relations (3.6) or (3.8) appropriately for evaluating the norm. For the regime $μ < 0$, the following relations are verified inductively:
\[
\begin{cases}
    a |n\rangle = \left( |n\rangle q p - |n - 1\rangle q r - 2 \left( \frac{n - 1}{2} \right) [n]_q^{1/2} \right) |n - 1\rangle \equiv A_n |n - 1\rangle, \\
    a^\dagger |n\rangle = A_{n+1} |n + 1\rangle,
\end{cases}
\tag{3.12}
\]
and
\[
\begin{cases}
    a |−n\rangle = A_{−n} |−n - 1\rangle, \\
    a^\dagger |−n\rangle = A_{−n+1} |−n + 1\rangle.
\end{cases}
\tag{3.14}
\]
The diagonal operators read
\[
\begin{cases}
    a^\dagger a |±n\rangle = A^2_{±n} |±n\rangle, \\
    a a^\dagger |±n\rangle = A^2_{±n+1} |±n\rangle.
\end{cases}
\tag{3.15}
\]
For the choice $μ > 0$ the construction of states proceeds as before:
\[
\begin{cases}
    a |n\rangle = \left( |n\rangle q p - |n - 1\rangle q r + 2 \left( \frac{n - 1}{2} \right) [n]_q^{1/2} \right) |n - 1\rangle \equiv A_n |n - 1\rangle, \\
    a^\dagger |n\rangle = A_{n+1} |n + 1\rangle,
\end{cases}
\tag{3.16}
\]
while the relations (3.14) and (3.15) hold with the replacement of the normalization constant $A_n$ by $A_n$. The Casimir operator is quartic in the variables $W, W^\dagger$. Its eigenvalues may be computed via the results obtained above:
\[
\frac{4}{\mu^2 (2 - [2]_q)} C_F |±n\rangle = \left\{ \begin{array}{ll}
    \left( p^2 + r^2 - [2]_q pr + 2(p + r) + \frac{4}{2 - [2]_q} \right) |±n\rangle, & \mu < 0, \\
    \left( p^2 + r^2 - [2]_q pr - 2(p + r) + \frac{4}{2 - [2]_q} \right) |±n\rangle, & \mu > 0.
\end{array} \right.
\tag{3.18}
\]
We have formally obtained a double-sided infinite dimensional representation with two parameters: $p, r$. Our representation has close kinship with the symplecton realization [31] of the boson calculus. Reflecting the symmetry of (3.6) (or (3.8)) and its adjoint under the exchange of $a$ and $a^\dagger$, the representation has the symmetry structure $A_{n+1}(p, r) = A_{−n}(r, p)$. The usefulness of the above representations becomes evident below where we impose restrictions for obtaining the subclasses of the above infinite dimensional representation. These restrictions terminate the infinite series of state vectors.

Assuming that the state $|0\rangle$ is annihilated by the operator $a$, we have, $r = 0$. All states $|−n\rangle$ labeled by negative integers are eliminated while the semi-infinite series of states $\{|n\rangle | n = 0, 1, 2, \ldots \infty \}$ are retained. This is a Fock-type representation constructed on the vacuum $|0\rangle$. The number of parameters is reduced to one.
A key requirement for the existence for the classes of representations discussed below is that their Hermiticity needs to be maintained. This, in turn, requires the constants $A_{2n}^2, A_{-2n}^2 \forall n > 0$ to be nonnegative. Keeping this in mind, we study all the possibilities in order.

(1) **Truncated Fock-type representation**

To obtain finite dimensional Fock-type representations, we further assume that there exist a positive integer $N$ such that

$$A_N = 0 \quad \text{for} \quad \mu < 0, \quad A_N = 0 \quad \text{for} \quad \mu > 0.$$  \hspace{1cm} (3.19)

The representation space is spanned by $N$ independent states: $|0\rangle, |1\rangle, \ldots, |N-1\rangle$ with the highest state satisfying $a^\dagger |N-1\rangle = 0$. For instance, the $N$ dimensional representation for $\mu < 0$ reads

$$a^\dagger = \begin{pmatrix}
0 & A_{N-1} & 0 & \cdots & 0 \\
0 & A_{N-2} & \cdots & 0 \\
& \ddots & \ddots & \vdots \\
& & 0 & A_1 \\
& & & 0
\end{pmatrix}. \hspace{1cm} (3.20)$$

The null condition (3.19) requires

$$[N]_q p + 2 \left[ \frac{N-1}{2} \right]_q [N]^{\sqrt{q}} = 0,$$  \hspace{1cm} (3.21)

where the upper (lower) sign refers to $\mu < 0$ ($\mu > 0$). The relation (3.21) holds for two cases:

(i) A generic value of

$$q = \exp(i\theta) \hspace{1cm} (3.22)$$

leads to a $N$-dependent order parameter $p$:

$$p = \pm 2 \left[ \frac{N-1}{2} \right]_q \frac{[N]^{\sqrt{q}}}{[N]_q} = \pm \left( \frac{\tan(N\theta/2)}{\tan(\theta/2)} - 1 \right). \hspace{1cm} (3.23)$$

This case corresponds to the 'string solutions' of [19, 20]. The angle $\theta$ is restricted by the requirement of nonnegativity of the elements $A_{2n}^2 (A_{-2n}^2) \forall n \in (1, \ldots, N-1)$:

$$A_{2n}^2 = -A_{-2n}^2 = \frac{\sin^2(n\theta/2)}{\sin^2(\theta/2)} \left( \frac{\tan(N\theta/2)}{\tan(n\theta/2)} - 1 \right). \hspace{1cm} (3.24)$$

For the regime $\mu < 0$ the positivity $A_{2n}^2 > 0$ holds in the domain $-\pi/N < \theta < \pi/N$, whereas for the choice $\mu > 0$ the positivity $A_{2n}^2 > 0$ is satisfied, for instance, in the restricted region $\pi/N < \theta < \pi/(N-1), -\pi/(N-1) < \theta < -\pi/N$. This class of representation matrices are symmetric with respect to the minor diagonal: $A_{N-k} = A_k, A_{N-k} = A_k$.

(ii) The root of unity values of $q = \exp(i2\pi/N)$ restricts $p(> 0)$ over a range. The solutions corresponding to this class is novel. We list some low dimensional Hermitian representations for $\mu > 0$ as examples:

$$a^\dagger = \begin{pmatrix}
0 & \sqrt{2-p} & 0 \\
0 & \sqrt{p} & 0 \\
0 & 0 & 0
\end{pmatrix}, \hspace{1cm} a^\dagger = \begin{pmatrix}
0 & \sqrt{2-p} & 0 & 0 \\
0 & \sqrt{2} & 0 & 0 \\
0 & 0 & \sqrt{p} & 0
\end{pmatrix}.$$
\[
\begin{pmatrix}
0 & \sqrt{2 - p} & 0 & 0 \\
\sqrt{2 - p} & 0 & \sqrt{2(1 + (2 - p) \sin \frac{\pi}{10})} & 0 \\
0 & \sqrt{2(1 + (2 - p) \sin \frac{\pi}{10})} & 0 & \sqrt{p} \\
0 & 0 & \sqrt{p} & 0
\end{pmatrix}.
\] (3.25)

The elements \( A_n^2 \) \( \forall n \in (1, \ldots, N - 1) \) for the representations (3.25) read
\[
A_n^2 = \frac{\sin^2(n \pi/N)}{\sin^2(\pi/N)} (1 + (p - 1) f_n), \quad f_n = \frac{\tan(\pi/N)}{\tan(n \pi/N)}.
\] (3.26)

The sequence \( \{f_n|n = 1, \ldots, N - 1\} \) is bounded as \(-1 \leq f_n \leq 1\). The Hermiticity requirement now restricts the order parameter as \(0 < p < 2\). The value of \(p\) being positive definite, the Hermitian representations of this class do not exist for \(\mu < 0\).

(2) Cyclic representations

The cyclic finite dimensional representations may be ensured by identifying the states \(|N\rangle\) and \(|0\rangle\). In particular, this makes the eigenvalues of the operators \(aa^\dagger\) and \(a^\dagger a\) on the states \(|N\rangle\) and \(|0\rangle\) same. For the \(\mu > 0\) case, we obtain
\[
A_{N+1}^2 = p, \quad A_N^2 = r.
\] (3.27)

Eliminating \(r\) from the relations in (3.27), we obtain
\[
[N]_{\sqrt{q}}^2 p = \frac{2}{2 - [2]_q} [N]_{\sqrt{q}}^2.
\]

For the choice \([N]_{\sqrt{q}} \neq 0\), it follows \(p = 2(2 - [2]_q)^{-1}\). It then turns out that \(r = p\), and \(A_n = \sqrt{p} \quad \forall n\). This leads to \(Z|n\rangle = 0\) [19]. This case is uninteresting. Alternate choice \([N]_{\sqrt{q}} = 0\), i.e. \(q^N = 1\) yields a class of two-parametric \((p, r)\) \(N\)-dimensional representations:

\[
\begin{pmatrix}
0 & \sqrt{2 - p} & 0 & 0 \\
\sqrt{2 - p} & 0 & \sqrt{2(1 + (2 - p) \sin \frac{\pi}{10})} & 0 \\
0 & \sqrt{2(1 + (2 - p) \sin \frac{\pi}{10})} & 0 & \sqrt{p} \\
0 & 0 & \sqrt{p} & 0
\end{pmatrix}.
\] (3.28)

These representations corresponds to the ‘loop solutions’ in [19, 20], and possess the symmetry
\[
A_n(p, r) = A_{N+1-n}(r, p).
\] (3.29)

To examine their nonnegativity the elements \(\{A_n^2|n \in (1, \ldots, N)\}\) may be recast by isolating their symmetric part \(S_n(p + r)\) as follows
\[
A_n^2 = \mathcal{P}_n p + \mathcal{S}_n(p + r), \quad \mathcal{P}_n = [n]_q + [n - 1]_q, \quad \mathcal{S}_n(\lambda) = -[n - 1]_q \lambda + 2 \left[\frac{n-1}{2}\right]_q [n]_{\sqrt{q}}. \quad (3.30)
\]
Due to the symmetry (3.29) we only need to check the nonnegativity of the elements $A_n^2$ for $n = 1, \ldots, N/2 \ (\ (N + 1)/2)$ for an even (odd) $N$. In this domain we have $\mathcal{P}_n \geq 0$, and

$$S_n(p + r) = \frac{\sin^2((n - 1)p/N)}{\sin^2(p/N)} (1 + (1 - p - r)f_{n-1}), \quad (3.31)$$

where the sequence $\{f_n\}$ has been defined in (3.26). In the present context the entries of the sequence $\{f_{n-1}\}$, where $n = 2, \ldots, N/2 \ ((N + 1)/2)$ for an even (odd) $N$, is bounded as $0 < f_{n-1} \leq 1$. This yields the parametric values $0 < p, 0 < r, p + r < 2$ for the required nonnegativity. For the odd-$N$ case the symmetric element reads

$$A_{N+1}^2 = \frac{1}{2 \cos(p/N)} \left( \frac{1}{2 \sin^2(p/2N)} - p - r \right). \quad (3.32)$$

It may be noticed that the solutions obtained in (3.25) may be obtained as the $r \to 0$ limiting case of the present cyclic representations. The corresponding matrix structures are, however, of different ranks, and in that sense inequivalent. Moreover the cyclic property of the solutions (3.28) is lost when $r = 0$ is substituted in them. Hermitian cyclic representations do not exist for the $\mu < 0$ case as the elements $\{A_n^2\} n = 1, \ldots, N$ are not nonnegative. For instance, we have

$$A_{N+1}^2 \bigg|_{\text{even } N} = -r - \frac{1}{\sin^2(p/N)}, \quad A_{N+1}^2 \bigg|_{\text{odd } N} = -\frac{1}{2 \cos(p/N)} \left( \frac{1}{2 \sin^2(p/2N)} + p + r \right). \quad (3.33)$$

(3) Infinite dimensional representations

Turning towards the infinite dimensional representations we discuss the special cases where we notice the existence of such Hermitian representations. The identity

$$2 \left[\frac{n - 1}{2}\right]_q [n]_{\sqrt{q}} = [n]_{\sqrt{q}}^2 - [n]_q$$

allows us to rewrite the elements $A_n^2$ for the Fock states $(r = 0)$ in the $\mu > 0$ case as

$$A_n^2 = [n]_{\sqrt{q}}^2 + (p - 1) [n]_q. \quad (3.34)$$

For the choice $p = 1$, therefore, the Hermiticity of the infinite set of basis states with $n = 1, 2, \ldots \infty$ is guaranteed. To establish a range of values of the order parameter $p$ around $p = 1$ that preserves the positivity $A_n^2 > 0$, we use (3.22) to obtain

$$A_n^2 = \frac{\sin^2(n\theta/2)}{\sin^2(\theta/2)} (1 + (p - 1)f_n), \quad f_n = \frac{\tan(\theta/2)}{\tan(n\theta/2)}. \quad (3.35)$$

The entries of the sequence $\{f_n| n = 1, 2, \ldots \infty\}$ for any positive integral value of $n$ remain finite for the choice $\theta = 2\pi/N$, where $N$ is an irrational number. Employing the finite maximum and minimum entries of the said sequence a range of values of $p$ may now be easily established for which Fock-state representations are Hermitian and well-defined.

Another possible scenario is to turn to the quasi-classical states mentioned in the context of (3.9). Using the classical limit $q \to 1$ in the context of the $(p, r)$ two-parametric states for the regime $\mu > 0$ we obtain

$$A_n^2 = n^2 + (p - r - 1) n + r \quad \forall n = 1, 2, \ldots \infty. \quad (3.36)$$
For the choice \( p \geq r + 1 \), these two-parametric states maintain \( A^2_n > 0 \) ensuring the Hermiticity of the representations. These infinite dimensional quasi-classical states reflect, in some sense, noncommutative spaces bordering on the commutative regime. Hermitian infinite dimensional representations for the \( \mu < 0 \) case do not exist.

We have studied so far the case of \( \mu \neq 0 \). To complete the study of Hermitian representations, we next investigate the case of \( \mu = 0 \). Although the algebra of this case does not correspond to a deformed parafermion, one can repeat the analysis used earlier. We use the trilinear relation

\[
W(W^\dagger)^2 - [2]_q W^\dagger WW^\dagger + (W^\dagger)^2 W = 0, \tag{3.37}
\]

and assume a state \(|0\rangle\) subject to the conditions

\[
WW^\dagger |0\rangle = p |0\rangle, \quad W^\dagger W |0\rangle = r |0\rangle, \quad \langle 0|0 \rangle = 1. \tag{3.38}
\]

Using the same arguments applied earlier the positivity \( p, r > 0 \) follows, and the states are given by

\[
W |n\rangle = \left( [n]_q p - [n - 1]_q r \right)^{1/2} |n - 1\rangle \equiv w_n |n - 1\rangle, \tag{3.39}
\]

\[
W^\dagger |n\rangle = w_{n+1} |n + 1\rangle. \tag{3.40}
\]

This leads to

\[
W^\dagger W |n\rangle = w_n^2 |n\rangle, \quad WW^\dagger |n\rangle = w_{n+1}^2 |n\rangle. \tag{3.41}
\]

The eigenvalue of the Casimir operator reads

\[
C_F |n\rangle = \frac{p^2 + r^2 - [2]_q pr}{2 - [2]_q} |n\rangle. \tag{3.42}
\]

Relations parallel to (3.39) - (3.42) also hold for the state \(|-n\rangle\).

Fock-type representation is obtained by requiring \( W |0\rangle = 0 \ i.e. \ r = 0 \). Implementing \( w_N = 0 \) we may truncate the Fock-type representation and obtain a finite \( N \) dimensional one. There is a slight difference from the case of \( \mu \neq 0 \). The equation

\[
w_N = \sqrt{[N]_q p} = 0, \quad p > 0
\]

may be solved only for the choice \( [N]_q = 0 \). Namely, it is possible to obtain finite dimensional Hermitian representations from the Fock-type one for \( q^{2N} = 1 \ i.e. \ q = \exp(i\pi/N) \). The expression for \( w_n^2 \) is found to be nonnegative:

\[
w_n^2 = \frac{\sin(n\pi/N)}{\sin(\pi/N)} \ p > 0 \ \forall n \in (1, \ldots, N - 1). \tag{3.43}
\]

Such a truncation for a generic \( q \) does not exist.

Now we investigate the possible realizations of the cyclic representations for the \( \mu = 0 \) case. Conditions for cyclic representation are given by

\[
w_N^2 = r, \quad w_{N+1}^2 = p. \tag{3.44}
\]

Above equations may be recast as

\[
[N]_q p = ([N - 1]_q + 1)r, \quad ([N + 1]_q - 1)p = [N]_q r. \tag{3.45}
\]
Table 1. Hermitian representations of $T_F^2$

|                   | $\mu < 0$ | $\mu = 0$ | $\mu > 0$ |
|-------------------|-----------|-----------|-----------|
| infinite          | no        | no        | yes       |
| dimensional       |           |           |           |
| truncated         | $q^N \neq 1$ | yes      | no        | yes       |
| $q^N = 1$         | no        | yes       | yes       |
| cyclic            | no        | no        | yes       |

Eliminating $r$ from (3.45) we obtain a relation for $p$:

$$q^{-N}(1 - q^N)^2p = 0.$$  

Therefore the cyclic representations may exist only if $q^N = 1$. By computation, however, it may be seen that negative values of $w_n^2$ exist:

$$w_n^2 \bigg|_{\text{even}} N = -r, \quad w_n^2 \bigg|_{\text{odd}} N = -\frac{p + r}{2\cos(\pi/N)}.  \quad(3.46)$$

This eliminates the possibility of having Hermitian cyclic representations for the $\mu = 0$ case.

Lastly, paralleling the $\mu > 0$ case the infinite dimensional representations for the choice $\mu = 0$ exist for the said semi-classical states in the $q \to 1$ limit as the elements

$$w_n^2 \bigg|_{q \to 1} = n(p - r) + r \quad \forall n = 1, 2, \ldots \infty  \quad(3.47)$$

are nonnegative for $p \geq r$.

The Hermitian representations of the fuzzy torus algebra $T_F^2$ are summarized in Table 1.

4. Coherent state for infinite dimensional representation

The existence of the infinite dimensional representations for $\mu > 0$ encourages us to define a coherent state for fuzzy torus as an eigenstate of the annihilation operator $a$. We consider the $r = 0$ case here. The coherent state $|\zeta\rangle, \zeta \in \mathbb{C}$ is defined as follows:

$$a |\zeta\rangle = \zeta |\zeta\rangle, \quad |\zeta\rangle = \sum_{n=0}^{\infty} c_n(\zeta) |n\rangle.  \quad(4.1)$$

The coefficients $c_n$ satisfy the recurrence relation:

$$c_{n+1} = \frac{\zeta}{A_n} c_n,  \quad(4.2)$$

which is easily solved to give the expression:

$$c_n = \zeta^n \left(\prod_{k=1}^{n} A_k\right)^{-1} c_0.  \quad(4.3)$$
The coefficient $c_0$ is determined by the normalization condition ($\zeta|\zeta\rangle = 1$:

$$\frac{1}{c_0^2} = \sum_{n=0}^{\infty} \frac{|\zeta|^{2n}}{\prod_{k=1}^{\infty} A_k^2} = \sum_{n=0}^{\infty} \frac{|\zeta|^{2n}}{[n]_q! \prod_{k=1}^{\infty} \left( \frac{[k]_q}{q \sqrt{q}} + p - 1 \right)} , \quad \{n\}_q = \frac{q^n + q^{-n}}{q + q^{-1}} \to 1. \quad (4.4)$$

Comparing the expression (4.4) with the definition of modified Bessel function of the first kind:

$$I_m(2x) = \sum_{n=0}^{\infty} \frac{x^{m+2n}}{n!(n+m)!}, \quad (4.5)$$

the following deformation of $I_m(2x)$ is inspired:

$$J_m^{(q)}(2x) = \sum_{n=0}^{\infty} \frac{x^{m+2n}}{[n]_q! \prod_{k=1}^{\infty} \left( \frac{[k]_q}{q \sqrt{q}} + m \right)} . \quad (4.6)$$

In terms of (4.6), the coherent state reads:

$$|\zeta\rangle = \left( \frac{|\zeta|^{p-1}}{J_p^{(q)}(2|\zeta\rangle)} \right)^{1/2} \sum_{n=0}^{\infty} \frac{\zeta^n}{\prod_{k=1}^{\infty} A_k} |n\rangle. \quad (4.7)$$

The overlap of the coherent states is also given in terms of (4.6):

$$\langle \xi|\eta\rangle = \frac{J_{p-1}^{(q)}(2\sqrt{\xi\zeta})}{\sqrt{J_{p-1}^{(q)}(2|\xi\rangle)J_{p-1}^{(q)}(2|\zeta\rangle)}} e^{-i\phi(p-1)/2}, \quad (4.8)$$

where $\phi = \text{arg}(\zeta) - \text{arg}(\xi)$. If we set $\xi = -\zeta$, then the overlap (4.8) is a real quantity having a simple expression:

$$\langle -\zeta|\zeta\rangle = \frac{J_{p-1}^{(q)}(2|\zeta\rangle)}{J_{p-1}^{(q)}(2|\zeta\rangle)}, \quad (4.9)$$

where $J_{p-1}^{(q)}(2x)$ is a deformation of Bessel function of the first kind defined by

$$J_m^{(q)}(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{m+2n}}{[n]_q! \prod_{k=1}^{\infty} \left( \frac{[k]_q}{q \sqrt{q}} + m \right)} . \quad (4.10)$$

Since the overlap (4.9) is real, one can define orthogonal cat-type states for the fuzzy torus:

$$|\pm\rangle = \frac{1}{\sqrt{2(1 \pm \langle -\zeta|\zeta\rangle)}} (|\zeta\rangle \pm |\zeta\rangle). \quad (4.11)$$

We defined the coherent states (4.7) as an eigenstate of the annihilation operator $a$. If the states have the property of decomposition of unity, they may be used to define a star product on the fuzzy torus along the line developed in [28]. Detailed analysis of the coherent states will be published elsewhere.
5. Dirac operator on commutative torus

We have obtained algebraic relations and representations for fuzzy torus. What we need now is Dirac operator on fuzzy torus. Physical motivation of this is, of course, introduction of fermions into noncommutative space. Mathematically, Dirac operator together with algebra and Hilbert space on which the algebra acts consists the so-called spectral triple, that is a basic ingredient of noncommutative geometry [2]. Dirac operator on fuzzy space may have some different approaches. This is the case of fuzzy 2-sphere [32, 33, 34, 35]. In any approaches one will take, the fuzzy Dirac operator has to be reduced to the ordinary one in the commutative limit.

As a preliminary to Dirac operator on fuzzy torus, we derive Dirac operator on our commutative torus (2.5). We use the following parameterization of the torus:

\[
x = (\mu + r \sin \theta)^{1/2} \cos \varphi, \quad y = (\mu + r \sin \theta)^{1/2} \sin \varphi, \quad z = r \cos \theta.
\]  

(5.1)

It is easy to read off the metric tensor from the parameterization (5.1):

\[
g_{\mu\nu} = \begin{pmatrix}
r^2 \cos^2 \theta + r^2 \sin^2 \theta & 0 \\
0 & \mu + r \sin \theta
\end{pmatrix}.
\]

(5.2)

The suffices 1 and 2 stand for \(\theta\) and \(\varphi\), respectively. The zweibein which connects curved torus and flat tangent plane on it is given by

\[
\sigma^a_{\mu} = \begin{pmatrix}
\sqrt{g_{11}} & 0 \\
0 & \sqrt{g_{22}}
\end{pmatrix}.
\]

(5.3)

The zweibein satisfies the required relation \(e^a_{\mu} e^b_{\nu} \delta_{ab} = g_{\mu\nu}\), where and in the sequel summation over repeated indices is understood. In two dimensional space, Clifford algebra which provides the Dirac gamma matrices in flat space is realized by the Pauli matrices. Then the gamma matrices on torus are given by

\[
\gamma^1 = \frac{1}{\sqrt{g_{11}}} \sigma_x, \quad \gamma^2 = \frac{1}{\sqrt{g_{22}}} \sigma_y.
\]

(5.4)

where \(\sigma_x, \sigma_y\) are Pauli matrices. These curved gamma matrices satisfy the relation : \(\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}\). We now have enough data to write down the covariant derivative for spinors on the torus. After some algebra, we obtain the followings:

\[
D_1 = \partial_\theta, \quad D_2 = \partial_\varphi + \Gamma_2, \quad \Gamma_2 = -\frac{i}{4\sqrt{g_{11}g_{22}}} \partial_\theta g_{22} \sigma_z.
\]

(5.5)

Thus the Dirac operator on the torus (2.5) reads:

\[
i\gamma^\mu D_\mu = \frac{i}{\sqrt{g_{11}}} \sigma_x \left( \partial_\theta + \frac{\partial_\theta g_{22}}{4g_{22}} \right) + \frac{i}{\sqrt{g_{22}}} \sigma_y \partial_\varphi.
\]

(5.6)

If we solve spectral problem of the operator (5.6), it may be useful in the investigation of Dirac operator on fuzzy torus. However, eigenvalues and eigenfunctions of (5.6) may have its own interest. This work is under progress.
6. Concluding remarks

We introduced a q–parafermionic viewpoint to fuzzy tours. This helps us to do systematic study of Hermitian representations of the fuzzy torus algebra. It was observed that the fuzzy torus algebra have both finite and infinite dimensional representations. Remarkably, depending on the region of \( \mu \), there is a substantial difference in possible representations. The infinite dimensional representations existing for \( \mu > 0 \) were used to define coherent states. The coherent states lead to the introduction of new deformation of the Bessel functions. Investigation of the deformed Bessel functions may have its own interest in the context of possible generalization of the known special functions and their relations to algebraic structures. We want to define Dirac operator on the fuzzy torus. The task, however, seems to have some difficulty, since Dirac operator in commutative limit has enough complicated form as seen in §5. This reflects the structure of the torus polynomial (2.5). One can take another polynomial description for torus which admits Dirac operator of simpler form. For instance, one may take as follows:

\[
(x^2 + y^2 + z^2 + \mu^2 - c)^2 = 4\mu^2 (x^2 + y^2^2).
\]

(6.1)

Applying the quantization procedure of §2 to (6.1), we obtain an another fuzzy torus algebra. However, the algebra is very messy in form and its representations may not be easily obtained.

Similar difficulty arise when we go up to higher genus algebras. Even the genus two case, the fuzzy analogue is defined by the algebra of high complexity. What we learnt from the present study is that in order to study properties and representations of fuzzy algebras for higher genus, it may be helpful to map the algebras to something else such as higher order parafermions.

Acknowledgments

The work of N.A. is partially supported by the grants-in-aid from JSPS, Japan (Contract No. 18540380). Other author (R.C.) is partially supported by DAE (BRNS), Government of India.

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