Singular tachyon kinks from regular profiles

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Abstract

We demonstrate how Sen’s singular kink solution of the Born-Infeld tachyon action can be constructed by taking the appropriate limit of initially regular profiles. It is shown that the order in which different limits are taken plays an important rôle in determining whether or not such a solution is obtained for a wide class of potentials. Indeed, by introducing a small parameter into the action, we are able circumvent the results of a recent paper which derived two conditions on the asymptotic tachyon potential such that the singular kink could be recovered in the large amplitude limit of periodic solutions. We show that this is explained by the non-commuting nature of two limits, and that Sen’s solution is recovered if the order of the limits is chosen appropriately.

1 Introduction

It is by now well known that the spectrum of objects in string theory includes unstable, uncharged non-BPS D-branes \([1]-[9]\). The dynamics of these unstable D-branes is determined by an action similar to that of D-branes, with the addition of a tachyon scalar that indicates the presence of an instability. Ignoring all fields other than the tachyon \(T\), the action for a single non-BPS Dp-brane is of the Born-Infeld form

\[
S = -\int d^{p+1}x V(T) \sqrt{-\det(\eta_{\mu\nu} + 2\pi\alpha' \partial_\mu T \partial_\nu T)},
\]

where \(\eta_{\mu\nu}\) is the Minkowski metric with signature \((-,+,+,...)\), and \(V(T)\) the tachyon potential which has a global maximum at \(T = 0\). Since a possible endstate is the vacuum, the minimum of

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the potential must be at zero and this is reached when $|T| \to \infty$. Another possible end state is a stable $D(p - 1)$ brane. This is understood \cite{10} to be a static and stable tachyon kink, and enforces a $\mathbb{Z}_2$ symmetry of the potential. The formation of singular kinks as a dynamical process in finite time has been looked at in \cite{10}.

Derrick’s theorem suggests that the kink solutions of this action are singular \cite{11,12,13,14}. (Note that since the vacuua are at $|T| \to \infty$, the topological kink has an infinite amplitude.) If such a singular solution could be shown to exist as a limit of a regular solution, this would give us more confidence in addressing its behaviour and interpretation. In reference \cite{14}, one of us showed how this singular solution could be generated from a sequence of regular solutions as long as the potential $V(T)$ satisfied two restrictions involving its slope as $|T| \to \infty$. In reaching this conclusion, the manifold on which the solution is defined was altered from the line $\mathbb{R}^1$ to a circle $S^1$ thereby guaranteeing the finiteness of the energy, and hence establishing the validity of Derrick’s argument to this case. The idea was then to take the limit where the circle became large and the amplitude of the kink tended to infinity in order to regain the single kink solution of Sen. The two conditions on $V(|T| \to \infty)$ found in \cite{14} arose precisely in order that this regularisation scheme should give the single singular kink solution. However, these two conditions on $V(T)$ contrast somewhat with the original idea of Sen in which the single kink solution was, to a large extent, independent of the form of the potential (having to satisfy simple asymptotic properties only).

In the rest of this short note, we follow the spirit of \cite{14}, but show that if (as opposed to \cite{14}) we start with a slightly modified action we can recover, in a controlled way, the singular kink solution from a series of regular solutions (again compactified on $S^1$) without placing restrictions on the form of the potential. The explanation for the different result to \cite{14} lies in the subtleties associated with two different limits which do not commute. This situation should be contrasted with that which occurs when studying regular solitons such as those in $\lambda \phi^4$ theory: in that case there is no problem with non-commuting limits.

## 2 The kink solution

We look for static solutions to the tachyon equations coming from the action (1.1), so that the energy functional is

$$E[T] = \int d^p x V(T) \left[ \sqrt{1 + \partial_i T \partial^i T} \right], \quad (2 \pi \alpha' = 1).$$

(2.2)

Following Derrick’s theorem \cite{15}, now assume that there exists a kink profile which extremizes the energy and consider the configuration obtained by dilating the coordinates, $x \to \lambda x$. If a solution is to be an extremum of the energy then $(\partial E/\partial \lambda)_{\lambda=1} = 0$, so that

$$\int d^p x V(T) \left[ (1 + \partial_i T \partial^i T)^{-1/2} \left[ p + (p - 1) \partial_i T \partial^i T \right] \right] = 0.$$

(2.3)

For $p = 1$ this forces $V = 0$ or $\partial_i T \partial^i T \to \infty$. This is the reasoning behind the singular kink solution \cite{11}

$$T(x) = \begin{cases} \infty & x > 0 \\ 0 & x = 0 \\ -\infty & x < 0 \end{cases}$$

(2.4)

whose energy is given by

$$E_0 = \int_{-\infty}^{\infty} dT V(T).$$

(2.5)
(\(E_0\) can be obtained by formally writing the solution (2.4) as \(T = \lim_{C \to 0} x/C\) and substituting into (2.2).)

The profile (2.4) is clearly singular, complicating the analysis of its properties; it would therefore be useful to describe this solution as a limit of a regular solution. One approach, adopted in [14], was to start with the given action (1.1), compactify the coordinate \(x\) leading to finite energy periodic solutions for \(T(x)\), and then study the behaviour of these solutions in the decompactified limit. A crucial parameter in that analysis was the constant integral of motion \(V_0\), since the topological kink is obtained in the limit \(V_0 \to 0\). (The amplitude of the kink solution depends inversely on \(V_0\).) This approach led to two restrictions on the allowed form of \(V(T)\) such that the singular kink (2.4) with energy (2.5) was obtained in the \(V_0 \to 0\) limit. These conditions were that for \(|T| \to \infty\), \(V'/V \to 0\) and \(V'/V^2 \to \infty\).

Here we propose a different way to regularize the singular solution. It is based on adding a small correction to the action (1.1). With that correction, for all \(V(T)\) we can recover the single kink solution (2.4) with energy (2.5) (provided of course that the integral (2.5) is finite). Then, at the end, we take the limit where the correction vanishes. As can be anticipated when trying to construct such singular kinks, there is an issue of non-commuting limits (see the discussion in the next section).

Before proceeding, note that in the case of regular kinks, such as the tanh-like solutions of \(\lambda \phi^4\) theory in 1+1 dimensions, both the two methods outlined in the previous paragraph can be used to construct the topological kink. They both give the correct result independently of the order in which limits are taken.

Consider therefore the modified action and energy functional

\[
S = \int d^{p+1}x \mathcal{L} = - \int d^{p+1}x V(T) \left[ \sqrt{1 + \partial_\mu T \partial^\mu T} + \epsilon (\partial_\mu T \partial^\mu T)^n \right],
\]

\[
E[T] = \int d^p x V(T) \left[ \sqrt{1 + \partial_i T \partial^i T} + \epsilon (\partial_i T \partial^i T)^n \right],
\]

where \(n > \frac{1}{2}\). Note that taking the small parameter \(\epsilon \to 0\) gives back the original action (1.1). We now follow the prescription in [14], and repeat the scaling argument to find

\[
\int d^p x V(T) \left[ (1 + \partial_i T \partial^i T)^{-1/2} \left[ p + (p-1) \partial_i T \partial^i T \right] + (p-2n) \epsilon (\partial_i T \partial^i T)^n \right] = 0. \tag{2.8}
\]

For \(p = 1\) this again leads to the singular kink (2.4) when \(\epsilon \to 0\). For \(\epsilon \neq 0\), however, equation (2.8) can now be satisfied by

\[
T'^{4n+2} + T'^{4n} = \frac{1}{(2n-1)^2 \epsilon^2}, \tag{2.9}
\]

(the dash denotes a derivative with respect \(x\)), whose solution is a linear tachyon profile \(T(x) = \kappa x\). The singular kink is recovered in the limit \(\epsilon \to 0\), in which case \(\kappa^{2n+1} \to 1/[(2n-1)\epsilon] \to \infty\). Using the same arguments as those which led to (2.5), the energy of this solution is given by

\[
E_{\epsilon \to 0} = E_0 \left\{ 1 + 1/[(2n-1)\kappa^2] \right\} \tag{2.10}
\]

for all values of \(n > \frac{1}{2}\). This reproduces the energy of a single kink as \(1/\kappa^2 \to [(2n-1)\epsilon]^{2/(2n+1)} \to 0\).

While Derrick’s theorem shows that a solution could exist, it is not guaranteed. However, the equations of motion coming from (2.6) are (denoting \(V' = \partial V/\partial T\)),

\[
\left[ (1 + T'^2)^{-3/2} + 2n(2n-1)\epsilon T'^{2(n-1)} \right] T'' = \frac{V'}{V} \left[ (1 + T'^2)^{-1/2} - (2n-1)\epsilon T'^{2n} \right]. \tag{2.11}
\]
showing that for \( p = 1 \) the static, linear profile is indeed a solution. Furthermore by doing a perturbative analysis, one can show that it is a stable solution. For completeness we note that for the static kink \( V_0 \) is given by

\[
V_0 \equiv T' \frac{\partial \mathcal{L}}{\partial T'} - \mathcal{L} = V(T) \left[ (1 + T'^2)^{-1/2} - (2n - 1) \epsilon T'^{2n} \right].
\]

(2.12)

As we shall be studying oscillatory solutions note that at \( T' = 0 \) (the peaks of the wave) we have \( V_0 = V(T_{\text{peak}}) \), and as \( V(T) \) decrease for large \( |T| \) then large amplitude solutions have small \( V_0 \).

A slightly different way of achieving similar results is the following. Rather than taking the action modified according to (2.6), consider instead

\[
S = - \int d^{p+1}x V(T)(1 + \partial_\mu T \partial^\mu T)^q.
\]

(2.13)

For \( q > 1/2 \), the equations of motion (or equivalently a Derrick-type argument) show that there is a topological kink solution given by \( T = x/\sqrt{2q - 1} \). Now follow the construction of [14]. Taking \( q = 1/2 \) first followed by the \( V_0 \rightarrow 0 \) limit leads to the two conditions on \( V(T) \) discussed in [14]. On the other hand, notice that the opposite choice of limits, namely \( V_0 \rightarrow 0 \) followed by \( q \rightarrow 1/2^+ \) gives the singular kink (2.4) with energy \( E_0 \) for all \( V(T) \). Hence we conclude that the limits

\[
q \rightarrow 1/2^+ \quad \text{and} \quad V_0 \rightarrow 0
\]

(2.14)
do not commute.

3 A question of limits

We have seen that the single kink can be found from the action Eqn. (2.7) once the appropriate limit for \( \epsilon \) is taken. Similarly it can be obtained from the action (2.13) again being careful with the order of limits. Indeed, as opposed to regular solitons, it seems crucial to start with a modified action in order to be able to reproduce the singular kink for all \( V(T) \). To show explicitly this non-commuting nature, we look for regular kink solutions on a circle using the regularized energy functional Eqn. (2.7). As opposed to [14] we take the decompactified and \( V_0 \rightarrow 0 \) limits followed by \( \epsilon \rightarrow 0 \). The typical solutions are periodic, with the wavelength representing the separation of the kinks and anti-kinks. To reproduce the single kink (2.4) we need a solution with infinite wavelength i.e. infinite kink/anti-kink separation, and infinite amplitude as discussed above.

To be specific, we take the exponential potential \( V = \exp(-T^2) \) as an example, but the results are independent of this specific form. Note that this potential does not satisfy the conditions of [14]. Typical periodic solutions for \( \epsilon = 0 \) are shown in Fig. 1. Here we see the problem which was examined in [14], namely that as the amplitude increases (i.e. \( V_0 \) decreases), the wavelength decreases. This behaviour means that with \( \epsilon = 0 \) one cannot reproduce (2.4) in the large radius (large wavelength) limit. In contrast to this, Fig. 2 shows some typical profiles for \( \epsilon \neq 0 \). What we see here is that the large amplitude solutions track the regular single kink solution as \( T \) increases from zero. In other words increasing amplitude solutions have increasing wavelength — the type of behaviour required to reproduce the result of (2.4) for all \( V(T) \). As the regular single kink solution exists for all non-zero \( \epsilon \) this behaviour persists as \( \epsilon \) is reduced.

From the perspective of the regularized action we now see that the constraints on \( V(T) \) discussed in [14] arise when the \( \epsilon \rightarrow 0 \) limit is taken before the large amplitude (or \( V_0 \rightarrow 0 \)) limit. Here we take the \( V_0 \rightarrow 0 \) limit followed by \( \epsilon \rightarrow 0 \) limit. In fact we can go further and consider the relationship between all three quantities, \( \epsilon \), wavelength and amplitude. Fig. 3 allows us to follow the contours
of constant wavelength as we reduce $\epsilon$. They naturally lead to regions of increasing amplitude, and in the limit $\epsilon \to 0$ following such a contour we arrive at an array of singular kinks/anti-kinks with a separation depending on the contour chosen.

4 Conclusions

In this note, motivated by the results of [13], we have attempted to address the question of whether it is possible to produce the singular kink solution of Sen by taking the appropriate limit of a regular class of solutions. The original work of Sen suggests that the solution exists subject to only mild constraints on the tachyon potential $V(T)$. We have shown how it is possible to reproduce the single kink solution, provided we added a term to the action and later send it to zero. This allows the singular kink to be reached in a controlled manner from regular solutions, without the restrictions found in [13]. This result is important because it lifts the rather tight restrictions on the tachyon potential found in [13] and brings the requirements on $V(T)$ more in line with those originally anticipated by Sen. However, it also underlines the fact that one must treat the action (1.1) with some care.

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Figure 2: Periodic tachyon profiles $T(x)$ for the potential $V = e^{-T^2}$ and $\epsilon = 0.1$. The curves have increasing amplitude. The straight line corresponds to the regular single kink solution given in the text.

Figure 3: The dependence of wavelength on $\epsilon$ and amplitude.
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