Reflection of light from a moving mirror: derivation of the relativistic Doppler formula without Lorentz transformations

Malik Rakhmanov
Department of Physics, University of Florida, Gainesville, FL 32611

A special case of the relativistic Doppler effect, which occurs when light reflects from a moving mirror, is discussed. The classic formula for the Doppler shift is derived in a fully non-relativistic fashion using basic facts from geometry. The derivation does not involve Lorentz transformations, length contractions and time dilations, and therefore is conceptually simpler than the standard derivations in physics textbooks. This discussion can be useful for teaching introductory physics and also for understanding the foundations of special relativity.

PACS numbers: 03.30.+p, 03.65.Pm, 41.20.Jb

The Doppler effect is commonly known as an apparent change in frequency of a wave when the source of the wave or the observer are moving with respect to each other. Somewhat more complicated version of this effect appears in special relativity in the context of light waves propagating in vacuum. In this case, the usual (nonrelativistic) Doppler contraction of the wavelength of light becomes mixed with the Lorentz contraction. For a source moving away from an observer with velocity \( u \), special relativity predicts

\[
\omega' = \frac{\omega}{\sqrt{1 - u^2/c^2}},
\]

where \( \omega \) is the frequency of the wave measured in the rest-frame of the source, and \( \omega' \) is the frequency of the wave measured in the rest-frame of an observer. This equation is usually derived in physics textbooks with the help of Lorentz transformations applied to the 4-dimensional wavevector \( \mathbf{k} \). Sometimes, a derivation may not include Lorentz transformations explicitly, but then it would rely on relativistic time dilations \( \Delta t' = \Delta t \sqrt{1 - v^2/c^2} \). Even more difficult is the derivation of the frequency change when the wave reflects from a moving object (mirror). In this case, the answer is usually obtained by performing two Lorentz transformations: one from the laboratory frame to the rest frame of the source, and the other in reverse.

An alternative derivation can be obtained by noticing that the mirror forms an image which moves away from an observer. The observer then detects the reflected wave as if it were coming from the image behind the mirror. In classical physics, a mirror moving with velocity \( v \) creates an image moving with velocity \( u = 2v \). In special relativity, the image would be moving with velocity

\[
u = \frac{2v}{1 + v^2/c^2},
\]

which results from the law for relativistic velocity addition. Substituting Eq. (2) into Eq. (1), we obtain the formula for the frequency of the reflected light

\[
\omega' = \frac{1 - v/c}{1 + v/c} \omega.
\]

Although this derivation seems simple enough, it still uses relativistic concepts. Namely, Eq. (2) is usually derived using Lorentz transformations.

It is interesting to note that the law for relativistic velocity addition, of which Eq. (2) is a special case, can be derived directly from the constancy of the speed of light without any use of Lorentz transformations. In this paper we show that even the velocity addition formula is not necessary and give an entirely non-relativistic derivation of Eq. (3).

Consider an electromagnetic wave \( \mathbf{E}(x, t) \) propagating in the positive \( x \)-direction and assume that the wave is incident upon a mirror which is moving along the trajectory: \( x = s(t) \), as shown in Fig. 1. The mirror trajectory can be arbitrary provided that the mirror velocity

\[
v(t) = \frac{ds}{dt},
\]

never exceeds the speed of light. The electric field measured by the observer at time \( t \) must be the same as the field at the time of reflection \( \tau \), when it coincides with input field:

\[
\mathbf{E}_\text{ref}(x, t) = \mathbf{E}_\text{in}[s(\tau), \tau].
\]

The time of reflection can be found from the figure:

\[
\tau = t - \frac{s(\tau) - x}{c}.
\]

This equation defines \( \tau \) as an implicit function of \( x \) and \( t \), which means that in general we cannot solve for \( \tau \). However, we can find its derivatives with respect to \( t \) and \( x \):

\[
\frac{\partial \tau}{\partial t} = c \frac{\partial \tau}{\partial x} = \frac{c}{c + v(\tau)}.
\]

For a plane-monochromatic wave with frequency \( \omega \), the electric field is given by

\[
\mathbf{E}_\text{in}(x, t) = \cos(\omega t - kx),
\]

where \( k \) is the wavenumber: \( k = \omega/c \). In this case, Eq. (5) yields the reflected wave in the form:

\[
\mathbf{E}_\text{ref}(x, t) = \cos[\omega\tau - ks(\tau)].
\]
Here the dependence of the electric field on \( x \) and \( t \) is hidden in \( \tau \). A different, but more familiar representation for the electric field can be found by substituting \( \tau \) from Eq. (6) into Eq. (9). The result can be written as

\[
E_{\text{ref}}(x, t) = \cos \left[ \omega t + kx + \phi(x, t) \right].
\]  
(10)

The phase shift \( \phi(x, t) \) depends on the mirror position at the time of reflection:

\[
\phi(x, t) = -2ks(\tau). \tag{11}
\]

Once the wave is reflected by a moving mirror its frequency is no longer constant; it depends on the position of the observer and the time of the measurement. The instantaneous frequency of the reflected light is defined as a rate at which the total phase of the wave changes in time at a given point:

\[
\omega'(x, t) = \frac{\partial}{\partial t} [\omega t + kx + \phi(x, t)] \tag{12}
\]

\[
= \omega + \frac{\partial \phi}{\partial t}. \tag{13}
\]

Thus, the frequency of the reflected wave is shifted with respect to the frequency of the incident wave by \( \frac{\partial \phi}{\partial t} \). This partial derivative can be found using the chain rule:

\[
\frac{\partial \phi}{\partial t} = -2k \frac{ds}{d\tau} \frac{\partial \tau}{\partial t}. \tag{14}
\]

The first derivative in the right-hand side of this equation, \( ds/d\tau \), is nothing but the mirror velocity at the time of reflection. The second derivative is given by Eq. (14).

We thus find the frequency of the reflected wave as

\[
\omega'(x, t) = \frac{c - v(\tau)}{c + v(\tau)} \omega. \tag{15}
\]

which represents the relativistic Doppler effect and is an extension of Eq. (8) to non-uniform mirror motions.

A natural question which one can ask is: what happens to the wavelength? The wavelength \( \lambda \) can be found from the wavenumber, \( \lambda = 2\pi/k \), whereas the wavenumber is related to the frequency by

\[
\omega = ck. \tag{16}
\]

However, it is not clear that this relationship applies to the wave reflected by the moving mirror because \( \omega \) is no longer constant. Furthermore, \( k \) is not constant either. In this situation, the wavenumber shall be defined as a rate at which the total phase of the wave changes in space, provided that time is frozen:

\[
k'(x, t) = \frac{\partial}{\partial x} [\omega t + kx + \phi(x, t)] \tag{17}
\]

\[
= k + \frac{\partial \phi}{\partial x}. \tag{18}
\]

Expanding \( \frac{\partial \phi}{\partial x} \) as in Eq. (14), we obtain

\[
k'(x, t) = \frac{c - v(\tau)}{c + v(\tau)} k, \tag{19}
\]

which explicitly proves that

\[
\omega'(x, t) = c k'(x, t). \tag{20}
\]

Thus, the standard dispersion relation for electromagnetic waves in vacuum remains the same even if the waves are reflected from a mirror moving along an arbitrary trajectory.

Acknowledgments

The author would like thank N.D. Mermin for illuminating discussions. This research was supported by the National Science Foundation under grant PHY-0070854.

[1] D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics* (John Wiley & Sons, 2000), 6th ed.
[2] J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, 1998), 3rd ed.
[3] P. A. Tipler, *Physics* (W. H. Freeman & Co., 1999).
[4] N. D. Mermin, *Boojums All the Way Through: Communicating Science in a Prosaic Age* (Cambridge University Press, 1990).