On the problem of the dynamical reactions of a rolling wheelset to real track irregularities

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Abstract We investigate numerically the dynamical reactions of a moving wheelset model to real measured track irregularities. The background is to examine whether the dynamics are suitable as the input to the inverse problem: determine the true track geometry from measured wheelset dynamical reactions. It is known that the method works well for the vertical position of the rails but the computed lateral position is often flawed. We find that the lateral motion of the wheelset often may differ from the track geometry. The cases are investigated closely but the reasons remain unknown. While the wheelset dynamics reflect the larger (>4–6 mm) aperiodic track disturbances and single large disturbances quite well, this does not seem to be the case for general smaller or periodic track irregularities or sections behind single large disturbances. The resulting dynamics of a wheelset to lateral track irregularities are in general not sufficiently accurate to be used as the basis for a description of the track irregularities.

Keywords Track monitoring · Vehicle dynamical response · Inverse problem · Numerical analysis · Estimate of errors · Track irregularity

1 Introduction

The railway companies always checked the quality of the track and formulated early limits for the irregularities of the geometrical measures of the track. The control of the track is in the form of in situ visual inspection and by use of inspection cars with instruments that allow accurate measurements of the irregular track geometry on the move. The results are very good, and such measurement cars are today used all over the world. They are, however, expensive in use, and they are most often only used once or twice a year on each railway line. Only a few of them have so high maximum speeds that they are useful on high speed railway systems. The railway companies are therefore interested in other and cheaper ways of track measurements that can detect track irregularities in their early development.

In the last decades, the monitoring of the dynamics of moving railway vehicles for early detection of—among other faults—mechanical faults in the vehicle has grown. The idea then came up to use the results of the dynamical monitoring of the vehicles in normal use also to evaluate the track irregularities. It is now common to measure the accelerations of the wheelsets or bogie frames of in-service vehicles to determine the position and grade of the irregularities on the railway lines. The detection of vertical irregularities works quite well, but the measurements of the intensity of the lateral irregularities are often uncertain. In the worst cases, the standard deviation is of the order of magnitude of the measurement itself. The results depend on the type and the condition of both the monitoring vehicles and the track. On the same geographical position, two different vehicles may yield different dynamical results [1]. Later, applications have led to improvements of the technique [2].

The dynamical theoretical model of a vehicle contains among its many parameters also the track geometry. Then, a new idea came up: to invert the dynamical theoretical model and use the inversion to determine the track geometry based on the measured vehicle dynamics. In theory, such an inversion may be made, but it is VERY difficult if not impossible to deliver a result with the desired accuracy.
The mathematical system that models the dynamics of a rolling railway vehicle is differential-algebraic, nonlinear and non-smooth. Even if an inversion is theoretically possible, will the results of its use deliver a unique solution? That is, do two different configurations of the rail geometry always result in different solutions of the dynamics of the same vehicle? It is a question of the forward problem and the positive answer is a necessary condition for the uniqueness of the solution to the inverse problem.

The inverse mathematical dynamical problem is so complicated that it must be solved numerically, which gives rise to two questions: Is the lateral dynamics of the vehicle a sufficiently accurate response of the lateral position of the track to yield a solution of the inverse problem with the desired accuracy? Christiansen [3] showed in his article by a couple of examples that the nonlinear operator in his vehicle dynamical problem is not in general one-to-one, but is the numerical solver sufficiently accurate? In this paper, we shall use Christiansen’s numerical model [3] of the dynamics of a railway vehicle running on a measured railway track to investigate theoretically the dynamical response of the vehicle to the track irregularities and discuss the applicability of the inverse problem.

Only questions related to the inverse theoretical problem are considered in this article. We choose the dynamics of the front wheelset as the input in the inverse problem. Therefore, we investigate the dynamic response of the leading wheelset to real track disturbances in this article. The independent variable, time $t$, is continuous but the numerical solver discretizes the dynamical system. All inputs and outputs are handled discretely in the system solver. In the solution process, the numerical solver will need values of the variables between the support points, $t_n$, so a smoothing of the variables using an appropriate filter is necessary. In addition to the errors stemming from the choice of the system solver with its numerical system parameters, such as step lengths and tolerances, errors may develop through the application of smoothing of the variables. These errors are investigated in the article.

2 The dynamical model

The model consists of one two-axle Cooperrider bogie and half a car body. This was done in order to significantly reduce the computation time. A realistic rail and wheel profile is used in the model.

The degrees of freedom (DOF) include

- The lateral and vertical translational motions of the wheelsets and the bogie frame.
- The yaw of the wheelsets and the bogie frame.
- The pitch motion of the bogie frame.
- The roll motions of the wheelsets, the bogie frame and the car body.
- The lateral motion of the car body.
- A rolling constraint is modelled for each wheelset.

The motions accounted for result in a system of 30 first-order ODEs describing the railway vehicle dynamics. For more details regarding the model used, the reader is referred to Christiansen [3]. The rails are standard UIC60 rails with a cant of 1/40, and the gauge is standard 1435 mm. The wheel profile is the DSB97-1 profile, which is a modified S1002 profile with the flange a little closer to the track centerline. The primary suspension consists of linear springs, which act longitudinally, laterally, and vertically. The secondary suspension consists of linear springs and dampers in parallel, which act in the lateral and vertical directions, and a linear torsion spring, which resists the yaw motion of the bogie frame (see Fig. 1). The track, the wheelsets, the bogie frame and the car body are all rigid, but we allow for elastic deformations in the wheel–rail contact surface. The model is set in a Cartesian inertial frame that moves with the constant speed $V$ of the vehicle along the centerline of the undisturbed track. The $x$-axis (longitudinal) is in the direction of the motion, the $y$-axis (lateral) points to the left, and the $z$-axis is vertically upward. Newton’s laws of motion are used for the mathematical formulation of the vehicle dynamical model. The dynamical model consists of 14 second-order ordinary differential equations plus two first-order ordinary differential equations for the calculation of the differences between the actual speed of rotation of the wheels and the theoretical value $\Omega = V/R_0$, where $R_0$ is the nominal wheel radius [3]. The system of equations is solved using a Runge-Kutta 45 method with variable step size. The wheel-rail contact geometry is calculated numerically using the routine RSXEO [4] and tabulated. Hertz’s theory [5] is used to calculate the depth of penetration and the normal force in the contact plane step by step. The tangential force in the contact plane, the creep force, is calculated step by step using the Shen-Hedrick-Elkins model [6].

3 Preparation of the track data

The German Federal Railways, DB Netz AG contributed the track data. They consist of digital measurements of the position of the rails on a section of a real track in Germany. Table 1 shows an example of the measurements. The maximum speed on the railway line is 120 km/h–33 m/s.

As seen from Table 1, the measured data (in mm) have only a single decimal point of accuracy and they are given on a discrete set of points. The dynamical simulation, however, needs a continuous set of data points. The smoothing is done using a kernel, which is described in Sect. 3.1. The...
The data set contains no missing or otherwise invalid values. When the numerical values of $\text{curv}$ are close to zero, then the curve radius is very large and for our purpose, the vehicle then runs on a tangent track, and we only use the part of the track data from a tangent track. It corresponds to the data from position 88.59160 km to and including position 89.80008 km. This corresponds to the first 7554 observations. The position values will be translated such that the first position value is zero.

It is interesting to see which types of the track displacements that influence the irregularities the most. This may indicate how wear and tear develop over time.

For the lateral disturbances, we define two types of displacements: lateral centerline displacement (alignment) and gauge variation. Two similar types of displacements are defined for the vertical displacements: vertical centerline displacement (longitudinal level) and the height difference between the left and right rail (cross-level). The influence of these types of displacements on the irregularities of the tangent track can be estimated by a calculation of their standard deviations from the tangent track section of the data set. They are presented in Table 2. It is found that for the lateral displacements, the centerline displacement has the largest influence on the track irregularities. For the vertical
displacements, the two types of displacements have almost the same influence on the track irregularities.

The section of the data used for the simulations, simply referenced as the data from this point on, is only 1.2 km long, which is not enough for the simulations. For example, the vehicle will need a transitioning phase to settle on the track. Because of this, the data will be extended by replicating the data such that the motion corresponds to first driving forward and then driving backward. The result is a 2.4 km long section, which may hereafter be replicated repeatedly. For clarity, this is illustrated in Fig. 3. The reason for flipping the data, such that the vehicle drives backward, is to make the ends of the data match up so it becomes more reasonable in regards to the true values. The first and last observations are not replicated in the backward section. Thus, the observations are observation 7552, 7553, 7554, 7553 and then 7552 when reaching the end of the 1.2 km data section, similarly for the first observations.

3.1 The kernel, the need, the choice and the application of the kernel

There are two reasons for implementing a method for the determination of the rail displacements at any given position. First, the distance between two consecutive position values in the provided data is about 16 cm. This difference is large compared to the step sizes that the ODE solver uses. Therefore, we need intermediate rail displacement values for the time steps that determine a position that is not given in the data. Second, the implemented ODE solver uses variable time steps. Therefore, we need a method to provide values for the positions of the rails at any time during the simulation.

The intermediate rail displacement values will be determined by filtering the data with a tricube kernel. Then, the filtered data will be a continuous and twice differentiable set, which corresponds to a welded section of a real tangent track. The tricube kernel is

\[ K(x) = \frac{70}{51(1 - |x|^3)^3}, \ |x| \leq 1, \]

and a plot of the kernel can be seen in Fig. 4.

It will now be explained how the data are filtered using the tricube kernel. Given a position, \( p \), for which a position value, \( \gamma \), is sought, the discrete rail displacement values, which should be included in the filtering process, need to be determined. This is done by defining an interval of size

![Fig. 2](image1.png)

**Fig. 2** The \( \text{curv} \) values plotted against the position values. The section of the track used for the simulations, i.e., the straight section, is marked. The last observation of the straight track is the 7554th observation

![Fig. 3](image2.png)

**Fig. 3** The used values of \( y_{\text{left}} \) plotted against position. The first and the 7554th observation are not replicated when the track is flipped

![Fig. 4](image3.png)

**Fig. 4** The tricube kernel

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1 The length will for brevity be 1.2 km. However, the true length from the first until the 7554th observation is 1.20848 km.

| Lateral centerline | Gauge | Vertical centerline | Cross-level |
|------------------|-------|---------------------|-------------|
| 2.40             | 1.46  | 1.64                | 1.61        |

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Table 2 Standard deviations of different types of track displacements (unit: mm), which are calculated from the straight track section of the provided data
with midpoint $p$, a so-called window of size $s$ centered on $p$. We call $s$ the kernel size. All data points in the window will then be included in the filtering. Then, a measure will be defined, which describes how far away a position value, $q_d$, of an included data point, $d$, is from $p$. If the difference between $p$ and $q_d$ is large, then the absolute value of the measure should be large. Furthermore, this measure shall be used as input to Eq. (1), and therefore, it must return values $ρ(in the interval $[-1, 1]$. We define the measure $ρ(q_d)$, which satisfies the requirements:

$$ρ(q_d) = \frac{p - q_d}{s/2} = \frac{p - q_d}{r},$$

(2)

where $r = s/2$ is called the window radius or the kernel radius. The reader may check that the maximum absolute value of $p−q$ is $r$ and therefore satisfies the requirements. Calculating $K(ρ(q_d))$ then yields a kernel weight, $w_d$, which is large when $q_d$ is close to $p$ and small when $q_d$ is far from $p$. It can be verified by looking at Fig. 4. The weighted displacement value of $d$, denoted by $ω_p$, is then obtained by multiplying $w_d$ with the displacement value of $d$. Doing this $∀d ∈ D$, where $D$ denotes the set of all included data points, yields the sets $W$ and $Ω$, which contain $w_d∀d ∈ D$ and $ω_d∀d ∈ D$, respectively. The sought displacement value $γ$ can now be determined by the following weighted average,

$$γ = \frac{\sum_{ω ∈ Ω}}{\sum_{w ∈ W}} W.$$

(3)

Hereby, a method for determining the rail displacement at any given position has been obtained.

The value of the size parameter $r$ is arbitrary. The results of the filtering of the data may, however, differ a lot in dependence on the chosen value. The filtered data will become more smoothed for large values of $r$ because the applied window then covers more data points. Therefore, the original data and the intended use of the filtered data should be taken into account when the value of $r$ is chosen.

The kernel plays an important role as it estimates the displacement of the track in positions where the displacement is not observed. The following we will look at some behavior of the kernel. This includes the smoothing effect on the track and the kernel radius used.

### 3.1.1 Change of the radius

The observations used to determine the track displacements at the current position are the observations, which lie at most one kernel radius away from the current position (see Sect. 3.1 and Christiansen [3] for more details). Thus, the kernel radius effectively determines the number of observations used to estimate the displacements at the current position. Changing the number of observations used for estimating the displacements will likely alter the displacement estimates for a given position, due to the nature of the kernel. Thus, the choice of kernel radius also affects the result mentioned previously in Sect. 3.1. In order to see this and to evaluate the effect of different kernel radii on the wheelset dynamics, the following simulations use radii of 0.5, 1.0 and 1.5 m. Figure 5 shows that the wheelset dynamics around large deviations, 5400–5450 m, are very similar for all kernel radii. In contrast, the dynamics of the wheelset around areas with small deviations, 5530–5620 m, seem to be highly dependent on the chosen kernel radius as the dynamics are seen to deviate significantly from each other. The small figure in the lower right corner of Fig. 5 shows that the track data obtained using the 0.5, 1.0, and 1.5 m kernel radius, respectively, are very similar. It can be seen, however, from the dynamics of the wheelset that the small differences are large enough to change the dynamics of the wheelset.

#### 3.1.2 Kernel additional times

The kernel smooths the data and in order to see this effect, the kernel was applied to the original track displacement data at each position for which an observation exists.

Doing this again on the results then gives the effect of using the kernel twice, and likewise, the kernel can be applied to the former result to get the result of applying the kernel three times.

A simulation using the original data sees the continuous curve after applying the kernel once; another simulation using the result of applying the kernel once to the original data then sees the continuous curve after applying the kernel twice; likewise for a simulation using the result of applying the kernel three times.

Figure 6 shows that the kernel smooths the track displacement observations slightly, while Fig. 7 shows how these small differences affect the dynamics.

The dynamics of the wheelset around the large dip at 5375–5440 m is not affected much, however, at 5550–5600 m, the dynamics of the wheelset is affected noticeably by small differences in the track displacements. Most noticeably, the phase is affected. In Fig. 6, the curves almost coincide.

### 3.2 The initial transition

When the dynamics of the vehicle is simulated, it is necessary to ensure that the vehicle initially $(t=0)$ is situated on the track and the vehicle dynamics caused by the track irregularities have stabilized. This is done by multiplying the rail displacement values by a scalar, $c$, which increases linearly from zero, at $t=0$, to one over a given time period $T$. In this way, the rails will not be displaced at $t=0$, and since
the positions of the wheelsets initially are 0, the wheelsets will be on the track at \( t = 0 \). The linear growth of \( c \) ensures that the simulation of the dynamics adjusts to the rail displacements and the wheelsets stay on the track.

Compared with the full simulation time, a small value of \( T \) is chosen, since the desired results require the railway vehicle to ride on the measured track, i.e., for \( c = 1 \), for an extended period of time. However, \( T \) must still be so large that a derailment is prevented. In this work, \( T \) was chosen to be the time it took the railway vehicle to drive 1.2 km.

When \( t = T \), the track displacement values will have reached their full values, and another time period is used to give the simulation time to settle and let the transients caused by the growing \( c \) die out. Experimentally, we found that \( T \) was a good choice.

### 4 Tests of the reliability and the accuracy of the simulations

In the following subsections, “the displacement of the wheelset” always refers to the lateral displacement of the front wheelset unless otherwise stated. Furthermore, the plots marked ‘Wheelset’ will show the lateral displacement of the front wheelset. Likewise, “The displacement of the track” is, if not otherwise specified, the displacement using the kernel on the lateral displacement of the left rail. When the data or the modified data are referred, then the plot shows the observations found in the used data file, and they are not affected by the kernel.

Finally, a few remarks regarding the executed simulations will be stated:

![Fig. 5 Simulations with different kernel radii. The upper figure shows the displacement of the wheelset, while the lower figure shows the track displacement. The figure in the lower right corner shows a zoomed-in segment of the track data](image)

![Fig. 6 The track displacement observations after using the kernel multiple times. Only the lateral displacement of the left rail is shown](image)
4.1 Response to real sine wave track irregularities

Christiansen [3] showed many interesting examples of the wheelset dynamics when the wheelset runs on an ideal track that was modified by periodic gauge or centerline variations. It should be noted that the sine waves used to create gauge and centerline variations in the simulations are only added to the lateral track displacements and not the vertical track displacements.

4.1.1 Gauge variations

Christiansen [3] showed that gauge variations given by a sine function with amplitude 5 mm and a wavelength of 10 m on an otherwise ideal track caused the displacement of the wheelsets to remain at zero (see Fig. 13c in Christiansen [3]). Using the real track measurements, one may try a similar setup. We shall test whether added gauge variations given by a sine wave with amplitude 5 mm and a wavelength of 10 m to the real track data influence the dynamics of the wheelset. Figure 8 shows that the track displacement is not affected much by whether the sine wave is added pre- or post-kernel (see the caption of Fig. 8 for explanation of the terms; pre- and post-kernels). However, the dynamics of the wheelset are affected differently than expected from Christiansen’s work [3]—namely that the sine wave does affect the dynamics. One also sees that the small difference in the track displacement resulting from adding the sine wave either pre- or post-kernel does affect parts of the dynamics, see the dynamics around 4300–4400 m.

Christiansen [3] also showed that gauge variations given by a sine with amplitude 5 mm and a wavelength of 5 m on an otherwise ideal track caused the displacement of
the wheelsets to follow a triangle-like wave. The wave has an amplitude of approximately 9 mm and a wavelength of approximately 10 m (see Christiansen Fig. 13a in [3]) (this oscillation will henceforth be referred to as the triangle wave). The frequency of the triangle wave is about half the frequency of the gauge variations, and the amplitude of the triangle wave is almost twice as large as the amplitude of the gauge variations. A similar setup can be tested on the real track by adding gauge variations given by a sine wave with amplitude 5 mm and a wavelength of 5 m to the real track data. It can then be evaluated whether or not these gauge variations on a real track influence the dynamics of the wheelset in a manner similar to the results obtained in [3] for an ideal track. In the following text and figures, the abbreviations, FW, FL, FR, lat., and irreg stand for front wheelset, front left, front right, lateral, and irregularity, respectively. Irregularity here refers to the left lateral rail displacement at the position of a given wheelset. That is, “FL irreg” corresponds to the lateral displacement of the left rail at the position of the front wheelset.

In Fig. 9, the results of the simulation corresponding to the above described setup have been shown. Notice that data are only plotted up until a position of approximately 1650 m. The reason for this is that the simulated railway vehicle derailed at this position, and the simulation terminated. The simulation is ramping up the amplitude of the track displacements up until about 2400 m (see Christiansen [3] for reference), implying that the results seen in Fig. 9 cannot be compared directly to the results one would have obtained for the full real track. However, the results from the last 800 m still show a pattern, which likely would have continued for the rest of the track if the

![Wheelset and Track Diagonal Plots](image-url)
simulation had not terminated. This pattern is therefore worth to investigate.

Up until approximately 800 m, the displacement of the wheelset has an amplitude, which is less or equal to the amplitude of the FL/FR irregularities. It is hard to see from the plot whether the wheelset displacement generally follows the irregularities closely. It is, however certain that the triangle wave dynamics are not present up until approximately 800 m. From approximately 800 m and onward, the displacement of the wheelset seems to follow a kind of pattern. Why the pattern suddenly appears after around 800 m is unknown.

In order to investigate this pattern more closely, Fig. 10 shows the motion more detailed in the interval 1100–1200 m. It shows that the displacement of the wheelset follows a triangle-like wave, which has a frequency about half the frequency of the gauge variations, and an amplitude, which is almost twice as large as the amplitude of the gauge variations in the interval 1100–1200 m. Thus, the dynamics of the wheelset for these gauge variations behave very similarly to the results obtained by Christiansen for an ideal track. The main difference between the results seen in Christiansen [3] and the results seen in Fig. 10 is that the mean of the triangle wave in Fig. 10 seems to change in accordance with the measured track displacements instead of being constant. The gauge variations thus seem to dominate the dynamics of the wheelset; however, the dynamics are still affected slightly by the underlying measured track displacements. It should be noted that this behavior has only been found in position intervals where no large dips occur (large dips refer to position intervals with extraordinarily large track displacements, e.g., the interval 5400–5450 m in Fig. 7).

4.1.2 Centerline variations

Christiansen [3] showed that centerline variations given by a sine function with amplitude 4 mm and a wavelength of 10 m on an otherwise ideal track caused the displacement of the wheelsets to follow a triangle-like wave with
an amplitude of approximately 8 mm and a wavelength of approximately 10 m (see Christiansen Fig. 7d in [3]). Once again, this oscillation will simply be referred to as the triangle wave henceforth for simplicity. Thus, the frequency of the triangle wave seemed identical to the frequency of the centerline variations, and the amplitude of the triangle wave was almost twice as large as the amplitude of the centerline variations. Furthermore, the triangle wave was found to be phase shifted slightly in relation to the FL/FR irregularities such that the peaks in the displacement of the wheelset always occurred shortly after the peaks in the FL/FR irregularities. A similar setup can be tested on the real track by adding centerline variations given by a sine wave with amplitude 4 mm and a wavelength of 10 m to the real track data. It can then be evaluated whether these centerline variations on a real track influence the dynamics of the wheelset in a manner similar to the results obtained by Christiansen [3]. Thus, the dynamics of the wheelset for these centerline variations behave very similarly to the results obtained by Christiansen [3] for an ideal track. Again, the main difference between the results in Christiansen [3] and the ones in Fig. 12 is that the mean of the triangle wave in Fig. 12 seems to change in

In order to investigate this pattern more closely, Fig. 12 shows the results of Fig. 11 more detailed in the interval 4500–4600 m. Figure 12 shows that the displacement of the wheelset follows a triangle-like wave with a frequency, which is nearly identical to the frequency of the centerline variations, and an amplitude, which is generally close to twice as large as the amplitude of the centerline variations. Furthermore, the triangle wave is phase shifted slightly in relation to the FL/FR irregularities in a manner, which corresponds well with the results from Christiansen [3]. Thus, the dynamics of the wheelset for these centerline variations behave very similarly to the results obtained by Christiansen [3] for an ideal track. Again, the main difference between the results in Christiansen [3] and the ones in Fig. 12 is that the mean of the triangle wave in Fig. 12 seems to change in

![Fig. 11](image-url) Simulation with centerline variations with 4 mm amplitude and 10 m wavelength. The red line shows the lateral displacement of the front wheelset, and the blue and green dashed lines show the lateral displacement of the left and right rail, respectively, at the position of the front wheelset.

![Fig. 12](image-url) Simulation with centerline variations with 4 mm amplitude and 10 m wavelength in the interval 4500–4600 m. The figure is a zoomed-in version of Fig. 11.
accordance with the measured track displacements instead of being constant. In addition to this, the triangle wave shape is not as consistent in Fig. 12 as in the results in Christiansen [3] due to the irregularity of the measured track displacements.

4.2 Influence of track displacement amplitude on wheelset dynamics

It can be of interest to investigate how the dynamics of the railway vehicle change when the amplitude of the track displacements is altered.

4.2.1 Full track

First, a simulation is done in which the observed track displacements are amplified by a factor of two in both the lateral and vertical direction. A plot of the original and amplified track displacements is shown in Fig. 13. In Figs. 14 and 15, the dynamics of the wheelset resulting from the simulation is shown alongside the dynamics of an original simulation (a simulation with no modifications). Figures 14 and 15 show that the dynamics of the wheelset are noticeably different when the amplitudes are changed. Interestingly, the dynamics around the large dips (5400–5450 and 5500–5525 m) are nearly related by a linear scaling of two. This can be seen in Fig. 15 where the two curves at the large dips almost lie on top of each other. However, the dynamics outside these regions are noticeably different. This could indicate that for larger track displacements (“larger” being very loosely defined), the displacement of the wheelset yields a more accurate picture of the displacement of the track.

In Fig. 14, the displacement of the wheelset is seen to oscillate quite a lot following the second large dip (5500–5525 m) for both the original and amplified simulation, which could lead one to think that these oscillations exemplified eigen modes of the wheelset. However, it is very difficult to conclude, since the track is not ideal during the oscillations. It implies that the oscillations may be a result of the variations in the track displacement and not a result of the wheelset entering an eigen mode oscillation. Furthermore, when looking at the two curves, it can be seen that the wheelset displacement oscillates faster for the amplified track than for the original track. It makes it less probable that the oscillations are eigen modes.

4.2.2 Large dip

In order to investigate the oscillations properly, two new data files of track displacements were created. They have a track displacement of zero everywhere except at the first
The two data files deviate from each other by how much of the first dip they include. The Data7 file includes the first dip from approximately 5360 to 5480 m, whereas the Data8 file includes data from approximately 5400 to 5450 m.

The lateral left rail track displacement of Data7 and Data8 is shown in Fig. 16a. The track displacements equal zero for all positions not shown in the plot. Figure 16b then shows the wheelset displacements resulting from using Data7 as track data in two simulations: one where the track data were multiplied by one (Amp = 1) and one where the track data were multiplied by two (Amp = 2). Finally, the only difference between Fig. 16b and c is that Data8 was used for Fig. 16c instead of Data7.

Looking at Fig. 16b and c, it is seen clearly that both the frequency and amplitude of the oscillation following the dip depend heavily on the track displacement values at and around the dip. Therefore, it is most probable that the oscillations are simply residual oscillations from the dip rather than damped eigen modes of the wheelset. This also implies that a large dip in the track displacement data cannot be modeled as an impulse to the system—the dip simply does not act as a Dirac delta function.

4.3 The kernel error and its effect on wheelset dynamics

When the kernel is applied to the original discrete track data to create continuous track data, the continuous track displacement data, at the positions of the original discrete data, will often deviate slightly from the original track displacement data. This difference between the discrete and continuous data corresponds to errors induced by the kernel (referred to as kernel errors henceforth). These errors should be investigated since the quality of all results obtained from the real track simulations depends on the assumption that the kernel approximates the true track displacement accurately. The size of the kernel errors and their effect on the dynamics of the wheelset must therefore be investigated.

4.3.1 Leave-one-out

A leave-one-out (LOO) test [7] is performed on the track displacement data to determine the type and size of the kernel errors. This is done in the following way where Pos\textsubscript{i} is the position along the track of the \textit{i}th observation and Dis\textsubscript{ij}, \text{\textit{i}}\in\{1, 2, 3, 4\} are the four displacements of the rails: lateral
left, lateral right, vertical left and vertical right, respectively, and observed at the \( i \)th position.

The kernel is then used to estimate the track displacements at all the observed positions. That is, the kernel estimates the displacement, \( \text{Dis}_{i,j} \), at \( \text{Pos}_i \) for all \( i, j \). Normally, the observations given by an index that is in the set \( S \) given by \( S = \{ j | |\text{Pos}_i - \text{Pos}_j| < R \} \), where \( R \) is the kernel radius, are used to estimate \( \text{Dis}_{i,j} \), \( j \in \{1, 2, 3, 4\} \). In words, \( S \) usually contains all the observations whose position is at most the kernel radius from the position of the \( i \)th observation. Since \( R > 0 \) then \( i \in S \), however in LOO, the \( i \)th observation is omitted during the estimation of the deviation at the \( i \)th position. The estimation is performed using all the observations whose position is at most the kernel radius from \( \text{Pos}_i \) except the \( i \)th observation. Since this removes the most influential information from each estimation, the estimates obtained using LOO should be worse than the estimates obtained when the \( i \)th observation is included. LOO should, therefore, yield larger kernel errors than the kernel errors obtained in a simulation. Thus, LOO should provide an upper bound for the size of the kernel errors.

The results are shown in Table 3. Furthermore, histograms of the kernel errors for the four deviations are shown in Fig. 17. The curves show the normal approximations, and from the histograms, these normal approximations seem appropriate. It, therefore, seems reasonable to model the kernel errors as independent and identically

| Position         | Sum  | Max (×10^{-4}) | Min | Mean (×10^{-5}) | Median (×10^{-5}) | SD (×10^{-5}) | Signed SD (×10^{-5}) |
|------------------|------|----------------|-----|-----------------|---------------------|--------------|----------------------|
| Lateral-left     | 0.433| 12.50          | 0   | 5.74            | 4.63                | 5.52         | 7.97                 |
| Lateral-right    | 0.406| 9.29           | 0   | 5.38            | 4.44                | 4.66         | 7.12                 |
| Vertical-left    | 0.366| 7.83           | 0   | 4.85            | 3.96                | 4.40         | 6.55                 |
| Vertical-right   | 0.351| 3.38           | 0   | 4.65            | 3.87                | 3.75         | 5.98                 |

Table 3  Statistics of the errors from LOO

![Fig. 17](https://example.com/fig17.png)

Fig. 17 Histograms of LOO errors: a lateral-left; b lateral-right; c vertical-left; d vertical-right
distributed random variables with a normal distribution for each of the four displacements.

4.3.2 Added noise

To test the influence of the kernel errors on the dynamics of the wheelset, 200 simulations were made where white noise was added to all the track displacement observations. The noise was added by drawing random samples from four independent normal distributions with zero mean and standard deviations (SD) given by the signed SD found using LOO, which can be seen in Table 4. The noise added to the lateral displacement of the left rail was given by the normal distribution, \( N(0, (7.97 \times 10^{-5})^2) \). The stochastic irregularities do not model any real track.

This ensures that the added noise is proportional in size to the kernel errors estimated using LOO. Thus, the results from these simulations can indicate how large an influence kernel errors can have on the dynamics of the wheelset. In Fig. 18, the results of five of the

| Noise type              | Lateral-left | Lateral-right | Vertical-left | Vertical-right |
|-------------------------|--------------|---------------|---------------|---------------|
| LOO                     | 0            | 0             | 0             | 0             |
| LOO post-kernel         | \(-3.1210 \times 10^{-8}\)  | \(-2.0818 \times 10^{-8}\)  | \(-2.6794 \times 10^{-8}\)  | \(-4.8930 \times 10^{-8}\)  |
| ALOO post-kernel        | \(1.0800 \times 10^{-7}\)  | \(1.2465 \times 10^{-7}\)  | \(-6.0895 \times 10^{-7}\)  | \(8.5444 \times 10^{-7}\)  |

Row one shows the means of the normal distributions used to model the kernel errors estimated from LOO. Row two shows the means of the post-kernel noise where the SD of the added noise was the SD estimated from LOO. Row three shows the same as row two; however, the SD of the added noise was here the SD estimated from LOO amplified by 3. In order to determine the means of the post-kernel noise, the track displacement differences between the original simulation (the orange line in Figs. 18–21) and the 200 simulations with normal noise and amplified noise, respectively, were determined. The means of these differences were then equivalent to the means of the post-kernel noise.

**Fig. 18** Results from five simulations where noise was added (the non-orange lines) compared with the original simulation (no noise added). The noise used the standard deviations from LOO: a post-kernel displacement of the left rail at the position of the front wheelset for the original simulation (orange line) and five of the simulations with noise added to the system; b post-kernel displacement of the right rail at the position of the front wheelset for the original simulation (orange line) and five of the simulations with noise added to the system; c displacement of the front wheelset for the original simulation (orange line) and five of the simulations with noise added to the system. Some deviations from the original simulation can be seen, e.g., around 5250 m
Simulations with added noise are shown alongside an original simulation (a simulation with no noise added to the track displacement data). In order to simplify the upcoming explanations, tables and figures, the terms “pre-kernel” and “post-kernel” will refer to how the track data are before and after applying the kernel, respectively. Figure 18a and b show that the noise only affects the post-kernel track displacements very slightly. Figure 18c shows, however, that the wheelset displacement in some intervals, but never at larger dips, can be affected quite a bit by this slight change in the post-kernel track displacements.

In the 200 simulations, the kernel is applied to the noisy track displacement data, which effectively smooths the data. Then, it is probable that the SD of the added noise has become smaller post-kernel than pre-kernel. This post-kernel noise will in the upcoming tables be referred to as “LOO post-kernel” since the SD of the added noise is based on the SD estimated from LOO. Similarly, “LOO” will be used in the tables to denote the normal distribution used to approximate the errors estimated from LOO.

Since the noise was added to test the influence of the kernel errors, the SD and mean of the added noise post-kernel should be nearly identical to the SD and mean estimated from LOO. In Table 4, rows one and two show that the mean of the added noise post-kernel is approximately zero and thus nearly identical to the mean estimated from LOO (see the caption of Table 4 for further explanation of the table). However, in Table 5, rows one and two show that the SD of the added noise post-kernel is only about 1/3 of the SD estimated from LOO. The SD of the added noise pre-kernel, therefore, was multiplied with three in an attempt to make the SD of the added noise post-kernel nearly identical to the SD estimated from LOO, and 200 new simulations were made. This “amplified” post-kernel noise will in the upcoming tables be referred to as “ALOO post-kernel” since the SD of the added noise is the SD estimated from

Table 5 Kernel error standard deviations of the various noise types and track displacement directions

| Noise type   | Lateral-left ($\times 10^{-5}$) | Lateral-right ($\times 10^{-5}$) | Vertical-left ($\times 10^{-5}$) | Vertical-right ($\times 10^{-5}$) |
|--------------|-------------------------------|-------------------------------|---------------------------------|----------------------------------|
| LOO          | 7.9652                         | 7.1156                         | 6.5483                          | 5.9771                           |
| LOO post-kernel | 2.5424                        | 2.2722                         | 2.0937                          | 1.9068                           |
| ALOO post-kernel | 7.6284                        | 6.8259                         | 6.2737                          | 5.7212                           |

The explanation of the table is identical to the one presented in Table 4 except that every occurrence of “mean” should be replaced by “standard deviation.”

RESULTS FROM FIVE SIMULATIONS WHERE NOISE USING THREE TIMES THE STANDARD DEVIATION FROM LOO (ALOO) WAS ADDED AND THE ORIGINAL SIMULATION (NO NOISE ADDED), WITH THE SAME COLOR SCHEME AND a, b AND c AS IN FIG. 18. SOME DEVIATIONS FROM THE ORIGINAL SIMULATION CAN BE SEEN, E.G., AROUND 5250 AND 6750 M

Fig. 19
LOO amplified by 3 (i.e., the “A” in “ALOO” stands for “amplified”). These simulations yielded an acceptable mean and SD of the added noise post-kernel as seen in the last row of Tables 4 and 5, respectively. Thus, the results obtained from the simulations with amplified noise will reflect the influence of the kernel errors on the dynamics of the wheelset most accurately. However, the results of the simulations with normal noise will generally still be shown for comparison purposes.

In Fig. 19, the results of five of the simulations with amplified noise have been shown alongside an original simulation. Even though the SD of the noise in these results has been amplified by 3, the results visually seem nearly identical to the results seen in Fig. 18. Therefore, the analysis of Fig. 18 stated previously also applies to Fig. 19.

In order to further investigate the influence of the kernel errors on the dynamics of the wheelset, position intervals from Figs. 18 and 19 where the wheel displacement behaves in an interesting manner will be investigated in more detail. Figure 20 shows the wheelset displacement in the interval from approximately 5050 to 5350 m, which is an interval of small track displacements as seen from Fig. 18a and Fig. 19a. Figure 20a is a zoomed-in version of Fig. 18c and Fig. 20b is a zoomed-in version of Fig. 19c.

Figure 20 shows that the wheelset dynamics in the position around 5250 m vary a lot due to the post-kernel noise. Interestingly enough, the amount of deviation in the wheelset displacement seems to be almost identical for Fig. 20a and b despite the amplitude difference of the noise used. This could imply that the size of the wheelset displacements is robust toward variations in the size of the noise added to the track displacement. Adding noise to the track displacement will alter the wheelset dynamics in this position interval; however, the change of the amplification of the noise only has a small influence on the dynamics. It should be noted that the size of the wheelset displacements seen in Fig. 20 generally is relatively small-around 2 mm with a maximum of approximately 6 mm—which is to be expected due to the small track displacements in the position interval.

Thus, in relation to determining the position of large flaws/damages on the track using the wheelset dynamics, the variations in the wheelset dynamics in Fig. 20 are not of great concern since the size of the wheelset displacements remains relatively small when adding noise to the track. The variations in the wheelset dynamics do show clearly, however, that it is practically impossible to use the wheelset dynamics to determine the position of small- or medium-sized flaws/damages on the track accurately, and in areas of small/medium-sized track displacements, the wheelset
dynamics are simply too sensitive to slight changes in the track displacements. In fact, measurement errors alone could probably be large enough to cause variations in the wheelset dynamics similar to those seen in Fig. 20.

Figures 18c and 19c show that the analysis of Fig. 20 generally applies to position intervals of small- to medium-sized track displacements. Thus, only position intervals containing large track displacements, i.e., large dips, remain to be investigated. Figure 21 shows the wheelset displacement in the position interval from approximately 5390 to 5440 m, which is an interval containing a large dip as seen from Figs. 18a and 19a. Figure 21 shows that the wheelset dynamics are barely affected by the post-kernel noise in this position interval, and again, the amount of deviation in the wheelset displacement seems to be almost identical for Fig. 21a and b despite the amplitude difference of the noise used. This yields further evidence to the hypothesis that the size of the wheelset displacements is robust toward variations in the size of the noise added to the track displacement. However, in contrast to the analysis of Fig. 20, adding noise to large track displacements, i.e., dips, does not seem to alter the wheelset dynamics significantly. Thus, the influence of kernel errors on the wheelset dynamics in areas containing large dips is so small that it can be neglected in relation to determining the position of large flaws/damages on the track using the wheelset dynamics.

So far, the analysis of the influence of the kernel errors on the dynamics of the wheelset has been based solely on visual inspection of data, which can cause the analysis to be slightly biased.

Therefore, in order to make the analysis a bit more rigid, statistics quantifying the influence of the kernel errors on the dynamics of the wheelset will now be presented.
In order to quantify the influence of the kernel errors on the wheelset dynamics, the wheelset displacement differences between the original simulation (the orange line in Figs. 18–21) and the 200 simulations with normal noise and amplified noise, respectively, were determined. Summary statistics of these differences can be seen in Table 6. The statistics are based on the differences over the entire track or over the large dip seen in Fig. 21, where the noise type “LOO post-kernel full track” means that the statistics in this row relate to the wheelset displacement differences between the original simulation and the 200 simulations with normal noise over the entire track.

The SD of ALOO post-kernel full track and ALOO post-kernel dip is of most interest since these SD directly quantify how much influence the kernel errors have on the wheelset dynamics. It can be seen that the SD of ALOO post-kernel dip is about half the size of the SD of ALOO post-kernel full track, implying that the wheelset displacement over large dips is significantly less affected by the kernel errors than the wheelset displacement over the entire track. Furthermore, it can be concluded from the SD of ALOO post-kernel full track and ALOO post-kernel dip that the average deviation of the wheelset displacement caused by the kernel errors is 1 mm. In particular, the average deviation is below 0.5 mm over large dips, which is so small that it can be neglected in relation to determining the position of large flaws/damages on the track using the wheelset dynamics. Finally, it can be seen by comparing the SD of ALOO (i.e., the last column of the two last rows in Table 6) with the SD of LOO (i.e., the last column of the first two rows in Table 6) that the SD of ALOO is not much larger than the SD of LOO, despite the amplitude difference of the noise used.

Based on the above analysis of Table 6, it can be concluded that the statistics support the results of the visual inspection analysis performed previously.

5 Conclusion

The result of finding the lateral geometry of the track through an application of the measured dynamics of a railway vehicle depends on the dynamical reaction of the vehicle to the track geometry. We have presented an investigation of the dynamical reaction of a wheelset of one simplified model of a railway vehicle at one constant speed on one section of a realistic tangent track given by digital data from DB Netz. We have investigated the influence of the digitalization of the measurements, the necessary smoothing (filtering) of the data, the disturbances of the track gauge and the track center line, the errors caused by the numerical handling and the influence of noisy data. Each topic will be commented in the following sections.

The digital data must be smoothed for the numerical handling. We examined the results of smoothing of the digital data and the smoothing of the smoothed digitalization of the original data in order to compare the influence of the digitalization of the measurements. We found no changes in the resulting track data, but the wheelset behavior changed the amplitude and the frequency as well as the phase of the oscillations on the track sections with small disturbances. On the track sections with disturbances larger than approximately 4–5 mm, the wheelset response was unchanged. The amplitude of the wheelset response was often larger than the amplitude of the track disturbance.

We also investigated the effect of a change of the radius of the kernel that as applied for the smoothing (filtering). We again found that the effect on the track data was negligible, and the resulting lateral behavior of the wheelset was very similar to the behavior described above.

Christiansen [3] found examples of deterministic sine variations of the gauge or the centerline of an ideal track that resulted in completely different dynamic responses of the wheelset. We have chosen three interesting examples and superposed each of the different sine variations on the measurements of our digitally given realistic track. The wheelset reacted strongly to the deterministic sine variations of the track. The reactions followed neither the amplitudes nor the frequencies of the track disturbances. The reactions might even depend on whether the sine wave is added to the measurements before or after the smoothing of the track geometry. The total reactions of the wheelset do not reflect the resulting track disturbances at all.

We found from the standard deviations of the wheelset displacements that over large dips, the wheelset displacements are less affected by the kernel errors than the wheelset displacements over the entire track. The average deviation of the wheelset displacement caused by the kernel errors is 1 mm. Over the large dips, it is even below 0.5 mm. It is so small that it can be neglected in connection with the determination of the position of large irregularities of the track by use of the wheelset dynamics. After passing the large irregularity, the wheelset usually enters a transient motion, which does not reflect the position of the rails. Periodic track disturbances may give rise to larger amplitudes and a different wavelength of the wheelset oscillations than the amplitude and wavelength of the periodic disturbance.

While the wheelset dynamics reflect the larger (> 4–6 mm) aperiodic track disturbances and single large disturbances quite well, this does not seem to be the case for general smaller or periodic track irregularities or sections behind single large disturbances. The resulting dynamics of a wheelset to lateral track irregularities are in general not sufficiently accurate to be used as the basis for a description of the track irregularities.
On the problem of the dynamical reactions of a rolling wheelset to real track irregularities

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