Non-commutative quantum dynamics*

Jakub Rembieliński & Kordian A. Smoliński
Department of Mathematical Physics
University of Łódź
ul. Pomorska 149/153
90–236 Łódź, Poland
April 1993

Abstract
We described a $q$-deformation of a quantum dynamics in one dimension. We prove that there exists only one essential deformation of quantum dynamics.

1 Introduction

Many theoretical physicists try to construct physical models based on non-commutative geometry [4, 7]. A large family of such models there are deformations of standard commutative (classical and quantum) dynamics to dynamics in $q$-deformed phase-space [1, 2, 3, 9, 10, 11]. Other interesting approaches in this field are the formulation of dynamics on non-commutative configuration manifolds [4] or view on standard quantum mechanics as a form of non-commutative geometry [5] and non-commutative differential calculus [4].

In this paper we formulate unitary non-commutative $q$-dynamics on the quantum level. Our starting point is an assumption that a possible deformation of the standard quantum mechanics lies in change of the algebra of observables with consequences on the level of dynamics.

*Supported by KBN grant No. 2 0218 91 01
We start with the well known statement, that probabilistic interpretation of quantum mechanics causes an unitary time evolution of physical system irrespectively of the choice of the algebra of observables (standard or \(q\)-deformed). As a consequence the Heisenberg equations of motion hold in each case (in the Heisenberg picture). In the following we restrict ourselves to the one degree of freedom systems.

The paper is organized as follows. Section 2. is devoted to re-describing of the standard quantum mechanical algebra of observables. In Section 3, we deform the algebra of observables and find equations of motion. In Section 4. and 5, we consider and solve few simple models and explain why in the Aref’eva-Volovich treatment [1] the unitarity of time evolution was lost. In Section 6, is showed that we can reparametrize the deformed theory to a “canonical” form. Finally, Section 7. contains some conclusions and final remarks.

2 Observables in standard QM

Construction of quantum spaces by Manin [7] as quotient of a free algebra by two-sided ideal can be applied also to the Heisenberg algebra case. In fact the Heisenberg algebra can be introduced as the quotient algebra

\[
\mathcal{H} = A(I, x, p)/J(I, x, p),
\]

where \(A(I, x, p)\) is an unital associative algebra freely generated by \(I, x\) and \(p\), while \(J(I, x, p)\) is a two-sided ideal in \(A\) defined by the Heisenberg rule

\[xp = px + i\hbar I.\]

There is an antilinear anti-involution (star operation) in \(A\) defined on generators as below

\[x^* = x, \quad p^* = p.\]

From the above construction it follows that this anti-involution induces in \(\mathcal{H}\) a \(*\)-anti-automorphism defined again by the eqs. (3).

Now, according to the result of [1], confirmed in [8] for the relativistic case, some parameters of the non-commutative dynamics, like inertial mass, do not commute with the generators \(x\) and \(p\). This means that these parameters should be treated themselves as generators of the algebra. Therefore it
is reasonable to treat them analogously on the commutative level too. To be more concrete let us consider a conservative system described by the Hamiltonian

\[ H = p^2 \kappa^2 + V(x, \kappa, \lambda). \]  

(4)

Here \( \kappa \) and \( \lambda \) are assumed to be additional hermitean generators of the extended algebra \( \mathcal{H}' \) satisfying the following re-ordering rules

\[ [x, p] = \i \hbar \lambda^2, \]  

(5)

\[ [x, \lambda] = [p, \lambda] = [x, \kappa] = [p, \kappa] = [\kappa, \lambda] = 0. \]  

(6)

We observe that the generators \( \kappa \) and \( \lambda \) belong to the center of \( \mathcal{H}' \). Thus the irreducibility condition on the representation level implies that \( \lambda \) and \( \kappa \) are multipliers of the identity \( I \). Consequently they can be chosen as follows

\[ \lambda = I, \]  

\[ \kappa = \frac{1}{\sqrt{2m}} I, \]  

(7)

so the extended algebra \( \mathcal{H}' \) reduces to the homomorphic Heisenberg algebra \( \mathcal{H} \) defined by (1) and (2). Notice that \( \mathcal{H}' \) can be interpreted as a quotient of a free unital, associative and involutive algebra \( A(I, x, p, \kappa, \lambda) \) by the two-sided ideal \( J(I, x, p, \kappa, \lambda) \) defined by eqs. (5–6) i.e.

\[ \mathcal{H}' = A(I, x, p, \kappa, \lambda)/J(I, x, p, \kappa, \lambda). \]  

(8)

It is remarkable, that eqs. (5–6) are nothing but the Bethe Ansatz for \( \mathcal{H}' \).

Finally, dynamics defined by the Hamiltonian (4) and the Heisenberg equations lead to the Hamilton form of the equations of motion:

\[ \dot{\lambda} = 0, \]  

(9)

\[ \dot{\kappa} = 0, \]  

(10)

\[ \dot{x} = \frac{1}{m} p, \]  

(11)

\[ \dot{p} = -V'(x). \]  

(12)
3 Observables in $q$-QM

Now, the formulation of the standard quantum mechanics by means of the algebra $\mathcal{H}'$ suggest a natural $q$-deformation of the algebra of observables; namely the $q$-deformed algebra $\mathcal{H}_{q\xi\tau\varepsilon}$ is assumed to be a quotient algebra

$$\mathcal{H}_{q\xi\tau\varepsilon} = A(I, x, p, K, \Lambda) / J_{q\xi\tau\varepsilon}(I, x, p, K, \Lambda),$$

(13)

where the two-sided ideal $J_{q\xi\tau\varepsilon}$ is defined now by the following Bethe Ansatz re-ordering rules

$$xp = q^2 px + i\hbar q A^2,$$

(14)

$$xA = \xi Ax,$$

(15)

$$pA = \xi^{-1} Ap,$$

(16)

$$xK = \tau^2 Kx,$$

(17)

$$pK = \varepsilon^2 Kp,$$

(18)

$$\Lambda K = \tau \varepsilon K \Lambda,$$

(19)

where $K$ and $\Lambda$ are assumed to be invertible and

$$x^* = x, \quad p^* = p, \quad K^* = K, \quad \Lambda^* = \Lambda.$$

(20)

A consistency of the system (14–19) requires

$$|q| = |\xi| = |\tau| = |\varepsilon| = 1.$$  

(21)

The corresponding conservative Hamiltonian has the form

$$H = p^2 K^2 + V(x, K, \Lambda).$$

(22)

Now, similarly to the standard case, $\Lambda$ and $K$ are assumed constant in time:

$$\dot{\Lambda} = \frac{i}{\hbar} [H, \Lambda] \equiv 0,$$

(23)

$$\dot{K} = \frac{i}{\hbar} [H, K] \equiv 0,$$

(24)

which implies, under the requirement of the proper classical limit (5–6),

$$\varepsilon = 1,$$

(25)

$$\tau = \xi^{-1},$$

(26)
and by means of eqs. \((25–26)\)

\[
\begin{align*}
V(x, K, A) &= V(\xi x, \xi K, A), \\
V(x, K, A) &= V(\xi^2 x, K, A).
\end{align*}
\]  

(27) 

(28)

So, our algebra of observables \(H_{q_\xi \tau \varepsilon}\) reduces to the algebra

\[
H_{q_\xi} = A(I, x, p, K, \Lambda)/J_{q_\xi}(I, x, p, K, A),
\]

(29)

where now the ideal \(J_{q_\xi}\) is defined by rules

\[
\begin{align*}
x_p &= q^2 px + i\hbar q^2, \\
x_A &= \xi A x, \\
p_A &= \xi^{-1} A p, \\
x_K &= \xi^{-2} K x, \\
p_K &= K p, \\
\Lambda K &= \xi^{-1} K \Lambda.
\end{align*}
\]

(30) 

(31) 

(32) 

(33) 

(34) 

(35)

Finally the Heisenberg equations of motion imply its Hamiltonian form

\[
\dot{x} = \frac{i}{\hbar} [H, x] = \left[ \frac{i}{\hbar} (\xi^4 - q^4) p_x + q(q^2 + \xi^2) p \Lambda \right] K^2,
\]

(36)

and

\[
\dot{p} = \frac{i}{\hbar} [H, p] = \frac{i}{\hbar} p [V((\xi^2 x, K, A) - V(x, K, A)] + \\
- \frac{q}{(\xi^2)^2 - 1} \frac{1}{x} [V((\xi^2 x, K, A) - V(x, K, A)] \Lambda^2.
\]

(37)

Note that the last term in (37) is the quantum (Gauss-Jackson) derivative (gradient) of \(V(x, K, \Lambda) \Lambda^2\).

If we do not assume that \(\dot{x}\) is linear in \(p\) (as it was \textit{implicit} done in [9]) we cannot reduce number of deformation parameters.

### 4 Simple models (I)

Now, let us consider two simple dynamical models.
4.1 Free particle

We choose the potential $V = 0$ so $H = \frac{1}{2\xi^2}M^{-1}p^2\Lambda^{-2}$ and consequently

\[
\dot{x} = \frac{i}{2\hbar}((\frac{q}{\xi})^4 - 1)M^{-1}p^2x\Lambda^{-2} + \frac{q}{2}(\frac{q}{\xi}^2 + 1)pM^{-1},
\]

\[
\dot{p} = 0,
\]

where $M = \frac{\xi}{2}(KL)^{-2}$ and obeys following algebra

\[
xM = \xi^2Mx,
\]

\[
pM = \xi^2Mp,
\]

\[
\Lambda M = \xi^2M\Lambda.
\]

Putting $\xi = q$ and $\Lambda = I$ we obtain Aref’eva-Volovich model [1]. However the choice $\Lambda = I$ leads to contradiction with the algebra (40–42) (especially with (42)) and causes non-unitarity of time evolution (Heisenberg equations do not hold with this choice).

4.2 Harmonic oscillator

We start with the Hamiltonian:

\[
H = \frac{1}{2\xi^2}M^{-1}p^2\Lambda^{-2} + \frac{\omega^2}{2\xi^2}x^2MA^{-2}.
\]

Consequently

\[
\dot{x} = -\frac{i}{2\hbar}((\frac{q}{\xi})^4 - 1)M^{-1}p^2x\Lambda^{-2} + \frac{q}{2}(\frac{q}{\xi}^2 + 1)pM^{-1},
\]

\[
\dot{p} = \frac{i\omega^2}{2\xi^2\hbar}((\frac{q}{\xi})^4 - 1)px^2MA^{-2} - \frac{q\omega^2}{2\xi^2}((\frac{q}{\xi}^2 + 1)xM.
\]

We get Aref’eva-Volovich model choosing $\xi = q$ and $\Lambda = I$ again, but there is no unitary time evolution for the same reason like above.

So it is evident that if we choose the inertial mass as a non-commutative generator the algebra of observables it is necessary to append another generator to this algebra.
5 Reparametrisation

It is easy to see that the conditions (27–28) are fulfilled if we choose the potential \( V \) as a function of the one variable

\[
V(x, K, \Lambda) = V(\frac{1}{\sqrt{\xi}}x\mu\Lambda^{-1})
\]

where we replace \( K \) by a new generator

\[
\mu = \sqrt{\frac{\xi}{2m}(K\Lambda)^{-1}},
\]

where \( m \in \mathbb{R}_+ \) is a parameter describing value of inertial mass of particle. Re-ordering rules for \( \mu \) and the other generators are following

\[
x\mu = \xi\mu x, \quad (48)
\]
\[
p\mu = \xi\mu p, \quad (49)
\]
\[
\Lambda\mu = \xi\mu\Lambda. \quad (50)
\]

Then let us reparametrize the phase-space coordinates. We replace \( x \) and \( p \) by new variables \( X \) and \( P \) respectively:

\[
X = \frac{1}{\sqrt{\xi}}x\mu\Lambda^{-1}, \quad (51)
\]
\[
P = \sqrt{\xi}p(\Lambda\mu)^{-1}. \quad (52)
\]

Note that the transformation (51–52) is not an unitary (canonical) transformation, so physical meaning of \( x, p \) and \( X, P \) are different.

With use of re-ordering rules (14–19) and (48–50) we can find the commutation rules for the new set of observables

\[
XP = (\frac{\xi}{2})^2 PX + i\hbar(\frac{\xi}{2})I, \quad (53)
\]
\[
[X, \mu] = [X, \Lambda] = [P, \mu] = [P, \Lambda], \quad (54)
\]
\[
\Lambda\mu = \xi\mu\Lambda. \quad (55)
\]

The energy takes the form

\[
H = \frac{P^2}{2m} + V(X) \equiv T + V. \quad (56)
\]
Note that \( H \) does not contain either \( \mu \) or \( \Lambda \).

The Heisenberg equations of motion are the following

\[
\dot{X} = \frac{i}{\hbar}(1 - (\frac{q}{\xi})^4)\frac{P^2 X}{2m} + (\frac{q}{\xi})(1 + (\frac{q}{\xi})^2)\frac{P}{2m},
\]

(57)

\[
\dot{P} = \frac{i}{\hbar}P[V((\frac{q}{\xi})^2 X) - V(X)] + \frac{1}{(\frac{q}{\xi})^2 - 1}\frac{1}{X}[V((\frac{q}{\xi})^2 X) - V(X)],
\]

(58)

\[
\dot{\mu} = 0,
\]

(59)

\[
\dot{\Lambda} = 0.
\]

(60)

Note, that eqs. (53) and (57) describing the same deformation as in [10] (under the replacement \((\frac{q}{\xi})^2 \rightarrow q\)).

We see that after the reparametrization (51–52) our algebra of observables \( \mathcal{H}_{q/\xi} \) is a direct sum of the algebra \( \mathcal{H}_{q/\xi} \) given by relation (53) and of the real Manin’s plane \( \mathcal{M}_2 \xi \) (generated by \( \mu \) and \( \Lambda \)). Therefore it is evident that in the above parametrisation under the irreducibility condition we should restrict ourselves to the algebra generated by \( X \) and \( P \). It is obvious that the “static” coordinates \( \mu \) and \( \Lambda \) cannot be treated as true dynamical variables, because they do not appear in the Hamiltonian (56).

Moreover, for \( q = \xi \) the theory in fact reduces to the commutative one. There is essentially non-commutative one if we choose \( \xi \neq q \); in this case velocity is not linear but rather squared in \( P \) (see eq. (57)).

An existence of classical limit \( \hbar \rightarrow 0, (q/\xi) \rightarrow 1 \) requires that first terms in (57, 58) have to vanish, so denoting \( q/\xi = e^{i\theta} \) it leads to the condition on dependence of \( \theta \) on \( \hbar \), namely it must be \( \lim_{\hbar \rightarrow 0} \frac{d\theta(h)}{d\hbar} = 0 \)

6 Simple models (II)

Let us turn back to the models considered in Section 4. Now we show them after the reparametrisation.
6.1 Free particle

Now, Hamiltonian is of the form

\[ H = \frac{P^2}{2m} \quad (61) \]

and Heisenberg equations are the following

\[ \dot{X} = \frac{i}{\hbar}(1 - \left(\frac{q}{\xi}\right)^4)\frac{P^2X}{2m} + \left(\frac{q}{\xi}\right)(1 + \left(\frac{q}{\xi}\right)^2)\frac{P}{2m}, \]
\[ \dot{P} = 0. \quad (62) \]

(63)

We can observe that while the momentum \( P \) is all time constant, the velocity \( \dot{X} \) depends on coordinate \( X \) (for \( \xi \neq q \)). The explicit solution of the eqs. (62–63) is given in [10].

6.2 Harmonic oscillator

Hamiltonian is the following

\[ H = \frac{P^2}{2m} + \frac{\omega^2X^2}{2} \quad (64) \]

and Heisenberg equations of motion:

\[ \dot{X} = \frac{i}{\hbar}(1 - \left(\frac{q}{\xi}\right)^4)\frac{P^2X}{2m} + \left(\frac{q}{\xi}\right)(1 + \left(\frac{q}{\xi}\right)^2)\frac{P}{2m}, \]
\[ \dot{P} = \frac{i\omega^2}{2\hbar}((\frac{q}{\xi})^4 - 1)PX^2 - \frac{\omega^2}{2}(\frac{q}{\xi})(\frac{q}{\xi}^2 + 1)X. \quad (65) \]
\[ \dot{P} = \frac{i\omega^2}{2\hbar}((\frac{q}{\xi})^4 - 1)PX^2 - \frac{\omega^2}{2}(\frac{q}{\xi})(\frac{q}{\xi}^2 + 1)X. \quad (66) \]

In addition to the non-proportionality of the velocity \( \dot{X} \) to the momentum \( P \), we obtain a dependence of force \( \dot{P} \) on the momentum \( P \). However, the above two terms vanish in the commutative limit \( \frac{q}{\xi} \to 1 \).

Note that all the above equations in the commutative case have the standard form.

7 Conclusions

We described the \( q \)-deformation of a quantum dynamics in one dimension. To obtain the unitary time development of observables we had to deform
only the algebra of observables leaving Heisenberg equations as equations of motion unchanged. We were able to reduce number of deformation parameters requiring consistency of the algebra with Heisenberg equations of motion and finally by the decomposition of the full algebra of observables to the direct sum of the dynamical and internal part of this algebra. This last step is done with help of a reparametrisation of the generators. Our final claim is that an essential $q$-deformation of the quantum dynamics is given by the eqs. (53, 57, 58) with only one deformation parameter $q/\xi$. Moreover, an essential deformation ($\xi \neq q$) leads necessarily to quantum corrections to the velocity and force.

Acknowledgements

We are grateful to T. Brzeziński for interesting discussions.

References

[1] I.Ya. Aref’eva and I.V. Volovich. Quantum groups particles and non-Archimedean geometry. *Phys. Lett.* B, 268:179, 1991.

[2] T. Brzeziński, J. Rembieliński, and K.A. Smoliński. Quantum particle on a quantum circle. *Mod. Phys. Lett.* A, 1993. in print.

[3] P. Caban, A. Dobrosielski, A. Krajewska, and Z. Walczak. On the $q$-deformed Hamiltonian mechanics. Lódź Univ. preprint KFT UL 1/93, Jan 1993.

[4] A. Connes. *Géométrie non-commutative*. Inter Editions, Paris, 1990.

[5] A. Dimakis and F. Müller-Hoissen. Quantum mechanics as non-commutative symplectic geometry. *J. Phys. A: Math. Gen.*, 25:5625, 1992.

[6] A. Lukin, A. Stern, and I. Yakushin. Lagrangian and hamiltonian formalism on a quantum plane. Alabama Univ. preprint UAHEP–931, Jan 1993.
[7] Yu.I. Manin. *Quantum Groups and Non-Commutative Geometry*. RIMS, Montreal, 1988.

[8] J. Rembieliński. Non-commutative relativistic kinematics. *Phys. Lett.* B, 287:145.

[9] J. Rembieliński. Noncommutative dynamics. In B. Gruber, editor, *Symmetries in Science VI: From the Rotation Group to Quantum Algebras*. Plenum Press Co., 1993. Talk given at the Symposium Symmtries in Science IV: From the Rotation Group to Quantum Algebras, Bregenz, Austria, 1992.

[10] M.R. Ubriaco. Quantum deformations of quantum mechanics. *Mod. Phys. Lett.* A, 8:89, 1993.

[11] J. Wess and J. Schwenk. A $q$-deformed quantum mechanical toy model. *Phys. Lett.* B, 291:273, 1992.