A UNIFORM STABILITY PRINCIPLE FOR DUAL LATTICES

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Abstract. We prove a highly uniform stability or “almost-near” theorem for dual lattices of lattices \( L \subseteq \mathbb{R}^n \). More precisely, we show that, for a vector \( x \) from the linear span of a lattice \( L \subseteq \mathbb{R}^n \), subject to \( \lambda_1(L) \geq \lambda > 0 \), to be \( \varepsilon \)-close to some vector from the dual lattice \( L' \) of \( L \), it is enough that the inner products \( u \cdot x \) are \( \delta \)-close (with \( \delta < 1/3 \)) to some integers for all vectors \( u \in L \) satisfying \( \|u\| \leq r \), where \( r > 0 \) depends on \( n, \lambda, \delta \) and \( \varepsilon \), only. This generalizes an analogous result proved for integral lattices in [15]. The proof is nonconstructive, using the ultraproduct construction and a slight portion of nonstandard analysis.

Informally, a property of objects of certain kind is “stable” if objects “almost satisfying” this property are already “close” to objects having the property. For that reason results establishing such a stability are frequently referred as a “almost-near” principles or theorems. Making precise the vague notions “almost satisfying” and “close” various rigorous notions of stability can be obtained. The study of stability of functional equations originates from a question about the stability of additive functions \( \mathbb{R} \to \mathbb{R} \) and, more generally, of homomorphisms \( G \to H \) between metrizable topological groups, asked by Ulam, cf. [18], [21], [22]. Since that time Ulam’s type stability, modified in various ways, was studied for various (systems of) functional equations — see, e.g., Rassias [19], Székelyhidi [20]. A systematic and general approach to this topic in the realm of compact Hausdorff topological spaces, using nonstandard analysis was developed by Anderson [1]. The study of stability of the homomorphy property with respect to the compact-open topology was commenced by the second of the present authors [23], [24], [25]. The survey article by Boualem and Brouzet [4] reflects some recent developments.

In the present paper we will prove the stability theorem for dual lattices stated in the abstract, as well as some closely related results. Typically, such a stability result would be formulated in a weaker form, namely that every vector \( x \) from the linear span of \( L \), behaving almost like a vector from the dual lattice \( L' \) of \( L \) in the sense that all its inner products \( u \cdot x = u_1x_1 + \ldots + u_nx_n \) with vectors \( u \) from a “sufficiently big” subset of \( L \) are “sufficiently close” to some integer, is already “arbitrarily close” to a vector \( y \in L' \). Below is the precise formulation arising from this account. In it \( \text{span}(L) \) denotes the linear subspace of \( \mathbb{R}^n \) generated by \( L \), \( |a|_Z = \min_{c \in \mathbb{Z}} |a - c| = \min(a - |a|, |a| - a) \) denotes the distance of the real number \( a \) from the set of all integers \( \mathbb{Z} \), and \( \|x\| = \sqrt{x \cdot x} \) is the euclidean norm, induced by the usual inner (scalar) product \( x \cdot y \) on \( \mathbb{R}^n \).

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Theorem 0.1. Let $L \subseteq \mathbb{R}^n$ be a lattice. Then, for each $\varepsilon > 0$, there exist $\delta > 0$ and $r > 0$ such that for every $x \in \text{span}(L)$, satisfying $|u|x|_L \leq \delta$ for all $u \in L$, $\|u\| \leq r$, there is a $y \in L'$ such that $\|x - y\| \leq \varepsilon$.

Such a statement naturally raises the question how the parameters $\delta$ and $r$ depend on the parameters $n$ and $\varepsilon$ and some properties of the lattice $L$. We, in fact, will prove a stronger and more uniform result, answering partly this question. Namely, we will show that one can pick any $\delta \in (0,1/3)$; then $r$ can be chosen depending on $n$, $\varepsilon$, $\delta$ and, additionally, the Minkowski first successive minimum $\lambda_1(L)$. The precise formulation is given in Theorem 5.2. On the other hand, as the proof of this result uses the ultraproduct construction, it establishes the mere existence of such an $r$, without any estimate of its size.

Theorem 5.2 generalizes an analogous result proved in [15] for integral lattices, replacing the condition $L \subseteq \mathbb{Z}^n$ by introducing an additional parameter $\lambda > 0$ and requiring $\lambda_1(L) \geq \lambda$. The mentioned result in [15] was obtained as a byproduct of a stability result for characters of countable abelian groups the proof of which used Pontryagin-van Kampen duality between discrete and compact groups and the ultraproduct construction. Our present result is based on an intuitively appealing almost-near result (Theorem 1.5) formulated in terms of nonstandard analysis which is linked to its standard counterpart (Theorem 5.2) via the ultraproduct construction. As a consequence, Pontryagin-van Kampen duality is eliminated from the proof. Additionally, the passage from stability of characters to stability of dual lattices in [15] naturally led to a formulation in terms of the pair of mutually dual norms $\|x\|_1 = |x_1| + \ldots + |x_n|$ and $\|x\|_\infty = \max(|x_1|, \ldots, |x_n|)$. In our present work, starting right away from lattices, the (equivalent) formulation in terms of the (selfdual) euclidean norm $\|x\| = \|x\|_2$ seems more natural.

1. LATTICES AND DUAL LATTICES

We assume some basic knowledge of lattices or, more generally, of “geometry of numbers”. The readers can consult, e.g., Cassels [5], Gruber, Lekkerkerker [8] or Lagarias [13]; however, for their convenience we list here the definitions of most notions we use and some facts we build on.

A subgroup $L$ of the additive group $\mathbb{R}^n$, where $n \geq 1$, is called a lattice if it is discrete, i.e., there is a $\lambda > 0$ such that $\|x - y\| \geq \lambda$ for any distinct vectors $x, y \in L$. $\mathbb{R}^n$ is alternatively viewed as a vector space or an affine space and its elements as vectors or points, respectively. The dimension of the linear space span($X$) generated by a set $X \subseteq \mathbb{R}^n$ is called the rank of $X$, i.e., rank($X$) = dim(span($X$)). A full rank lattice is a lattice of rank equal the dimension of the ambient space $\mathbb{R}^n$. A body is a nonempty bounded connected set $C \subseteq \mathbb{R}^n$ which equals the closure of its interior. A body $C$ is called centrally symmetric if $-x \in C$ for any $x \in C$; it is called convex if $ax + (1 - a)y \in C$ for any $x, y \in C$ and $a \in [0,1]$. An example of a centrally symmetric convex body is the euclidean unit ball $B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. The Minkowski successive minima of $L$ (with respect to the unit ball $B$) are defined by

$$
\lambda_k(L) = \inf\{\lambda \in \mathbb{R} : \lambda > 0, \ \text{rank}(L \cap \lambda B) \geq k\}
$$

for $1 \leq k \leq \text{rank}(L)$. In particular, $\lambda_1(L) = \inf\{\|x\| : 0 \neq x \in L\}$. The covering radius of $L$ is defined by

$$
\mu(L) = \inf\{r \in \mathbb{R} : r > 0, \ \text{span}(L) \subseteq L + rB\}.
$$
In all these cases the infima are in fact minima.

A basis of a lattice \( L \subseteq \mathbb{R}^n \) is an ordered \( m \)-tuple \( \beta = (v_1, \ldots, v_m) \) of linearly independent vectors from \( L \) which generate \( L \) as a group, i.e.,
\[
L = \text{grp}(v_1, \ldots, v_m) = \{c_1v_1 + \ldots + c_mv_m : c_1, \ldots, c_m \in \mathbb{Z}\}.
\]

Obviously, in such a case \( \text{rank}(L) = m \). In the proof of the fact that every lattice has a basis the following elementary lemma, to which we will refer within short, plays a key role.

**Lemma 1.1.** Let \( L \subseteq \mathbb{R}^n \) be a lattice of rank \( m \) and \( (v_1, \ldots, v_k) \), with \( k < m \), be an ordered \( k \)-tuple of linearly independent vectors from \( L \) which can be extended to a basis of \( L \). Denote \( V = \text{span}(v_1, \ldots, v_k) \) and assume that the vector \( v_{k+1} \in L \setminus V \) has a minimal (euclidean) distance to the linear subspace \( V \) from among all the vectors in \( L \setminus V \). Then the \( (k+1) \)-tuple \( (v_1, \ldots, v_k, v_{k+1}) \) either is already a basis of \( L \) (if \( k+1 = m \)) or it can be extended to a basis of \( L \) (if \( k+1 < m \)).

Another useful consequence of the fact that every lattice has a basis is the following.

**Lemma 1.2.** Let \( L \subseteq \mathbb{R}^n \) be a lattice. Then a \( k \)-tuple of vectors \( v_1, \ldots, v_k \in L \) is linearly independent if and only if, for any integers \( c_1, \ldots, c_k \in \mathbb{Z} \), the equality
\[
c_1v_1 + \ldots + c_kv_k = 0
\]
implies \( c_1 = \ldots = c_k = 0 \).

A basis \( (v_1, \ldots, v_m) \) of a lattice \( L \) is Minkowski reduced if, for each \( k \leq m \), \( v_k \) is the shortest vector from \( L \) such that the \( k \)-tuple \( (v_1, \ldots, v_k) \) can be extended to a basis of \( L \). It is known that every lattice has a Minkowski reduced basis.

For any subset \( S \subseteq \mathbb{R}^n \) we denote by
\[
\text{Ann}_\mathbb{Z}(S) = \{x \in \mathbb{R}^n : \forall u \in S : u \cdot x \in \mathbb{Z}\}
\]
the integral annihilator of \( S \). Obviously, \( \text{Ann}_\mathbb{Z}(S) \) is a subgroup of \( \mathbb{R}^n \) for every \( S \subseteq \mathbb{R}^n \), however, even for a lattice \( L \subseteq \mathbb{R}^n \), the integral annihilator \( \text{Ann}_\mathbb{Z}(L) \) need not be a lattice, unless \( \text{rank}(L) = n \). The dual lattice of \( L \) (also called the polar or reciprocal lattice) is defined as the intersection
\[
L' = \text{Ann}_\mathbb{Z}(L) \cap \text{span}(L).
\]

Then \( L' \) is a lattice in \( \mathbb{R}^n \) of the same rank as \( L \) and there is an obvious duality relation \( L'' = L \). The Minkowski successive minima of the original lattice \( L \) and its dual lattice \( L' \) are related through a bound due to Banaszczyk [2]. Similarly, the covering radius of the dual lattice \( L' \) can be estimated in terms of the first Minkowski minimum of \( L \) — see Lagarias, Lenstra, Schnorr [14]. Actually, in the quoted papers these results were stated and proved for full rank lattices, i.e., in case \( m = n \), only. However, introducing an orthonormal basis in the linear subspace \( \text{span}(L) \) and replacing any vector \( x \in \text{span}(L) \) by its coordinates with respect to it, they can be readily generalized as follows.

**Lemma 1.3.** Let \( L \subseteq \mathbb{R}^n \) be a lattice of rank \( m \). Then
\[
\lambda_k(L) \lambda_{m-k+1}(L') \leq m
\]
for each \( k \leq m \), and
\[
\lambda_1(L) \mu(L') \leq \frac{1}{2} m^{3/2}.
\]
2. Ultraproducts of lattices

In order to keep our presentation self-contained, we give a brief account of the ultraproduct construction and some notions of nonstandard analysis here. Nonetheless, the readers are strongly advised to consult some more detailed exposition such as those in Chang-Keisler [6], Davis [7] and Henson [10]. A nonempty system $D$ of subsets of a set $I$ is a called a filter on $I$ if $\emptyset \notin D$, $D$ is closed with respect to intersections, and, for any $X \in D$, $Y \subseteq I$, the inclusion $X \subseteq Y$ implies $Y \in D$. A filter $D$ on $I$ is called an ultrafilter if for any $X \subseteq I$ either $X \in D$ or $I \setminus X \in D$. Ultrafilters of the form $D = \{X \subseteq I : j \in X\}$, where $j \in I$, are called principal. A consequence of the axiom of choice, every filter on $I$ is contained in some ultrafilter; in particular, nonprincipal ultrafilters exist on every infinite set $I$.

Given a set $I$ and a family of first order structures $(A_i)_{i \in I}$ of some first order language $A$, we can form their direct product $\prod_{i \in I} A_i$ with basic operations and relations defined componentwise. If, additionally, $D$ is a filter on $I$, then

$$\alpha \equiv_D \beta \iff \{i \in I : \alpha(i) = \beta(i)\} \in D$$

defines an equivalence relation on $\prod A_i$. Denoting by $\alpha/D$ the coset of a function $\alpha \in \prod A_i$ with respect to $\equiv_D$, the quotient

$$B = \prod A_i/D = \prod A_i/\equiv_D,$$

naturally becomes a $\Lambda$-structure once we define

$$f^B(\alpha_1/D, \ldots, \alpha_p/D) = \beta/D,$$

where $\beta(i) = f^{A_i}(\alpha_1(i), \ldots, \alpha_p(i))$, for any $p$-ary functional symbol $f$, and

$$(\alpha_1/D, \ldots, \alpha_p/D) \in R^B \iff \{i \in I : (\alpha_1(i), \ldots, \alpha_p(i)) \in R^A_i\} \in D$$

for any $p$-ary relational symbol $R$. Then $B$ is called the filtered or reduced product of the family $(A_i)$ with respect to the filter $D$. If $A_i = A$ is the same structure for each $i \in I$, then the reduced product

$$A^I/D = \prod A_i/D$$

is called the filtered or reduced power of the $\Lambda$-structure $A$. If $D$ is an ultrafilter, then we speak of ultraproducts and ultrapowers.

The key property of ultraproducts is the following

**Lemma 2.1.** [Los Theorem] Let $(A_i)_{i \in I}$ be a family of structures of some first order language $A$, $D$ be an ultrafilter on the index set $I$, $\Phi(x_1, \ldots, x_p)$ be a $\Lambda$-formula and $\alpha_1, \ldots, \alpha_p \in \prod A_i$. Then the statement $\Phi(\alpha_1/D, \ldots, \alpha_p/D)$ holds in the ultraproduct $\prod A_i/D$ if and only if

$$\{i \in I : \Phi(\alpha_1(i), \ldots, \alpha_p(i)) \text{ holds in } A_i\} \in D.$$

As a consequence, the canonical embedding of any $\Lambda$-structure $A$ into its ultrapower $^*A = A^I/D$ is elementary. More precisely, identifying every element $a \in A$ with the coset $\bar{a}/D$ of the constant function $\bar{a}(i) = a$, we have

$$\Phi(a_1, \ldots, a_p) \text{ holds in } A \iff \Phi(a_1, \ldots, a_p) \text{ holds in } ^*A$$

for every $\Lambda$-formula $\Phi(x_1, \ldots, x_n)$ and any $a_1, \ldots, a_p \in A$. This equivalence will be referred to as the transfer principle.
The above accounts almost directly apply to many-sorted structures, like modules over rings or vector spaces over fields, as well (see [10]). In particular, if \( (V_i)_{i \in I} \) is a family of vector spaces over a field \( F \), then the ultraproduct \( \prod V_i / D \) becomes a vector space over the ultrapower \( *F = F' / D \), which is a field elementarily extending \( F \). Similarly, if \( (G_i)_{i \in I} \) is a family of abelian groups, viewed as modules over the ring of integers \( \mathbb{Z} \), then the ultraproduct \( \prod G_i / D \) becomes not only an abelian group but also a module over the ring of hyperintegers \( * \mathbb{Z} = \mathbb{Z}' / D \), elementarily extending the ring \( \mathbb{Z} \). And, what is of crucial importance, the Los Theorem is still true for formulas in the corresponding two-sorted language.

From now on \( I = \{1, 2, 3, \ldots \} \) denotes the set of all positive integers and \( D \) is some fixed nonprincipal ultrafilter on \( I \). We form the ordered field of hyperreal numbers as the ultrapower \( * \mathbb{R} = \mathbb{R}' / D \) of the ordered field \( \mathbb{R} \). Then
\[
\mathbb{F}' \mathbb{R} = \{ x \in * \mathbb{R} : \exists r \in \mathbb{R}, r > 0 : |x| < r \}, \\
* \mathbb{I} \mathbb{R} = \{ x \in * \mathbb{R} : \forall r \in \mathbb{R}, r > 0 : |x| < r \}
\]
denote the sets of all finite hyperreals and of all infinitesimals, respectively. It can be easily verified that \( \mathbb{F}' \mathbb{R} \) is a subring of \( * \mathbb{R} \) and \( * \mathbb{I} \mathbb{R} \) is an ideal in \( \mathbb{F}' \mathbb{R} \). Hyperreal numbers not belonging to \( \mathbb{F}' \mathbb{R} \) are called infinite. For \( x \in * \mathbb{R} \) we sometimes write \( |x| < \infty \) instead of \( x \in \mathbb{F}' \mathbb{R} \), and \( x \sim \infty \) instead of \( x \notin \mathbb{F}' \mathbb{R} \). Two hyperreals \( x, y \) are said to be infinitesimally close, in notation \( x \approx y \), if \( x - y \in * \mathbb{I} \mathbb{R} \). Moreover, for each \( x \in \mathbb{F}' \mathbb{R} \), there is a unique real number \( {}^o x \in \mathbb{R} \), called the standard part of \( x \), such that \( x \approx {}^o x \). As a consequence, \( \mathbb{F}' \mathbb{R} / * \mathbb{I} \mathbb{R} \cong \mathbb{R} \) as ordered fields.

A hyperreal number \( x = \alpha / D \), where \( \alpha : I \rightarrow \mathbb{R} \), is finite if and only if there is a positive \( r \in \mathbb{R} \) such that \( \{ i \in I : |\alpha(i)| < r \} \subseteq D \); this is equivalent to the convergence of the sequence \( \alpha \) to \( {}^o x \) with respect to the filter \( D \). In particular, \( x \) is infinitesimal if and only if \( {}^o x = 0 \), i.e., if and only if the sequence \( \alpha \) converges to 0 with respect to \( D \). As \( D \) necessarily extends the Frechet filter, \( \lim_{i \rightarrow \infty} \alpha(i) = a \in \mathbb{R} \) in the usual sense implies \( {}^o x = a \), i.e., \( x \approx a \).

The standard part map has the following homomorphy properties with respect to the field operations:
\[
{}^o (x + y) = {}^o x + {}^o y \quad \text{and} \quad {}^o (xy) = {}^o x \cdot {}^o y
\]
for any \( x, y \in \mathbb{F}' \mathbb{R} \), and if additionally \( x \neq 0 \), then also \( {}^o (x^{-1}) = ({}^o x)^{-1} \).

Along with the equivalence relation of infinitesimal nearness \( \approx \), we introduce the relation of archimedean equivalence \( \sim \) or order equality on \( * \mathbb{R} \) as follows:
\[
x \sim 0 \iff x = 0 , \quad \text{and} \quad x \sim y \iff 0 \neq \left| \frac{x}{y} \right| < \infty \quad \text{for} \ x, y \neq 0 .
\]

When \( x \sim y \) we say that \( x \) and \( y \) are of the same (archimedean) order. We also say that \( x \) is of smaller order than \( y \) or that \( y \) is of bigger order than \( x \), in symbols \( x \ll y \), if \( y \neq 0 \) and \( \frac{x}{y} \approx 0 \). Obviously, for \( x, y \neq 0 \), \( x \sim y \) is equivalent to neither \( x \ll y \) nor \( y \ll x \).

According to the transfer principle, we can identify, for any finite integer \( n \geq 1 \), the vector space \( (* \mathbb{R})^n \) over the field \( * \mathbb{R} \) and the ultrapower \( *(* \mathbb{R})^n = (\mathbb{R}^n)' / D \), so that the notation \( * \mathbb{R}^n \) is unambiguous. More generally, for any subset \( S \subseteq \mathbb{R}^n \), we identify the ultrapower \( *S = S' / D \) with the subset
\[
\{ (\alpha_1 / D, \ldots, \alpha_n / D) \in * \mathbb{R}^n : \{ i \in I : (\alpha_1(i), \ldots, \alpha_n(i)) \in S \} \in D \}.
\]
of $\ast\mathbb{R}$. The inner product on $\mathbb{R}^n$ extends to the inner product on $\ast\mathbb{R}^n$, preserving all its first order properties. In order to distinguish the linear spans with respect to the fields $\mathbb{R}$ and $\ast\mathbb{R}$, respectively, we introduce the internal linear span of a set $X \subseteq \ast\mathbb{R}^n$ which, due to the fact that the ambient vector space $\ast\mathbb{R}^n$ has finite internal dimension $n$, can be described as follows:

$$\ast\text{span}(X) = \{a_1 x_1 + \ldots + a_n x_n : x_1, \ldots, x_n \in X, \ a_1, \ldots, a_n \in \ast\mathbb{R}\},$$

We also distinguish the lattice or subgroup, i.e., the $\mathbb{Z}$-submodule $\ast\text{grp}(v_1, \ldots, v_m)$ of $\mathbb{R}^n$ generated by vectors $v_1, \ldots, v_m \in \mathbb{R}^n$, and the internal lattice internally generated by vectors $v_1, \ldots, v_m \in \ast\mathbb{R}^n$, i.e., the $\ast\mathbb{Z}$-submodule

$$\ast\text{grp}(v_1, \ldots, v_m) = \{c_1 v_1 + \ldots + c_m v_m : c_1, \ldots, c_m \in \ast\mathbb{Z}\}$$

of $\ast\mathbb{R}^n$.

Similarly as in $\ast\mathbb{R}$, vectors from $\mathbb{F}^\ast\mathbb{R}^n$ are called finite and vectors from $\mathbb{I}^\ast\mathbb{R}^n$ are called infinitesimal. Obviously,

$$\mathbb{F}^\ast\mathbb{R}^n = \{x \in \ast\mathbb{R}^n : \|x\| < \infty\},$$

$$\mathbb{I}^\ast\mathbb{R}^n = \{x \in \ast\mathbb{R}^n : \|x\| \approx 0\}.$$

Both $\mathbb{F}^\ast\mathbb{R}^n$ and $\mathbb{I}^\ast\mathbb{R}^n$ are vector spaces over $\mathbb{R}$ and even modules over $\mathbb{F}^\ast\mathbb{R}$, but not over $\ast\mathbb{R}$. Vectors $x, y \in \ast\mathbb{R}^n$ are said to be infinitesimally close, in notation $x \approx y$, if $x - y \in \mathbb{I}^\ast\mathbb{R}^n$, i.e., if $\|x - y\| \approx 0$. The standard part of a vector $x = (x_1, \ldots, x_n) \in \mathbb{F}^\ast\mathbb{R}^n$ is the vector $\circ x = (\circ x_1, \ldots, \circ x_n)$; obviously, $\circ x$ is the unique vector in $\mathbb{R}^n$ infinitesimally close to $x$. Then $\mathbb{F}^\ast\mathbb{R}^n/\mathbb{I}^\ast\mathbb{R}^n \cong \mathbb{R}^n$ as vector spaces over $\mathbb{R}$.

Though the ultraproduct construction can be applied to any family of lattices $L_i \subseteq \mathbb{R}^n$, it is sufficient for our purpose to deal with lattices situated in the same ambient vector space $\mathbb{R}^n$ with $n \geq 1$ fixed. Given a sequence $(L_i)_{i \in I}$ of lattices $L_i \subseteq \mathbb{R}^n$ we can form the ultraproduct $\prod L_i / D$ and identify it with the subset

$$L = \{(\alpha_1/D, \ldots, \alpha_n/D) \in \ast\mathbb{R}^n : \{i \in I : (\alpha_1(i), \ldots, \alpha_n(i)) \in L_i\} \in D\}$$

of the vector space $\ast\mathbb{R}^n$ over $\ast\mathbb{R}$. Then $L$ is an internal discrete additive subgroup of $\ast\mathbb{R}^n$, i.e., it is a module over the ring of hyperintegers $\ast\mathbb{Z}$ and there is a positive $\lambda \in \ast\mathbb{Z}$ such that $\|x - y\| \geq \lambda$ for any distinct $x, y \in L$; however, it should be noticed that $\lambda$ may well be infinitesimal. Moreover, as $D$ is an ultrafilter, there is an $m \leq n$ and a set $J \in D$ such that $\text{rank}(L_i) = m$ for each $i \in J$. We write $\text{rank}(L) = m$ and refer to $L$ as an internal lattice in $\ast\mathbb{R}^n$ of rank $m$. Then we can assume, without loss of generality, that $\text{rank}(L) = m$ for each $i \in I$. The Minkowski successive minima of such an internal lattice $L$ can be defined in two ways which are equivalent by the transfer principle:

$$\lambda_k(L) = \frac{\lambda_k(L_i)}{i \in I}/D = \min\{\lambda \in \ast\mathbb{R} : \lambda > 0, \text{rank}(L \cap \lambda^*B) \geq k\}$$

for $k \leq m$. Then $0 < \lambda_1(L) \leq \ldots \leq \lambda_m(L)$ is a sequence of hyperreal numbers, hence it can contain both infinitesimals as well as infinite hyperreals. Additionally, we put

$$\text{rank}_0(L) = \#\{k : 1 \leq k \leq m, \lambda_k(L) \approx 0\},$$

$$\text{rank}_1(L) = \#\{k : 1 \leq k \leq m, \lambda_k(L) < \infty\},$$

where $\# H$ denotes the number of elements of a finite set $H$. Note that $\text{rank}_0(L) = 0$ if $\lambda_1(L) \neq 0$, as well as $\text{rank}_1(L) = 0$ if $\lambda_1(L) \notin \mathbb{F}^\ast\mathbb{R}$. Obviously, if $\text{rank}_0(L) > 0$,
then it is the biggest $k \leq m$ such that $\lambda_k(L) \approx 0$; similarly, if $\text{rank}_d(L) > 0$, then it is the biggest $k \leq m$ such that $\lambda_k(L) < \infty$.

At the same time, we can assume that $\beta_1, \ldots, \beta_m \in \prod L_i$ are functions such that, for each $i \in I$ (or at least for each $i$ from some set $J \in D$), the $m$-tuple of vectors $\beta(i) = (\beta_1(i), \ldots, \beta_m(i))$ is a Minkowski reduced basis of the lattice $L_i$. Then, due to Los Theorem (Lemma 2.1), the $m$-tuple $\beta/D = (v_1, \ldots, v_m)$, where $v_k = \beta_k/D$ for $k \leq m$, is a Minkowski reduced basis of the internal lattice $L$, i.e., the vectors $v_1, \ldots, v_m$ are linearly independent over $\mathbb{R}$ and generate $L$ as a $\mathbb{Z}$-module.

**Lemma 2.2.** Let $L \subseteq \mathbb{R}^n$ be an internal lattice of rank $m$ and $\beta = (v_1, \ldots, v_m)$ be a Minkowski reduced basis of $L$. Then the following hold true:

(a) If $\|v_k\| \ll \|v_{k+1}\|$ for some $k < m$ and $V = \text{span}\{v_1, \ldots, v_k\}$, then $\|x\| \ll \|v_{k+1}\|$ for every vector $x \in L \setminus V$.

(b) $\|v_k\| \sim \lambda_k(L)$ for each $k \leq m$.

**Proof.** (a) Assume that, under the assumptions of (a), we have $\|x\| \ll \|v_{k+1}\|$ for some $x \in L \setminus V$. We denote the orthogonal projection of a vector $y \in \mathbb{R}^n$ to $V$ by $y_V$. Let $z \in L \setminus V$ be a vector such that its the distance $z - z_V$ to $V$ is minimal from among all the vectors $y \in L \setminus V$. Therefore,

$$\|z - z_V\| \leq \|x - x_V\| \leq \|x\|.$$  

As $z_V \in V$, there are hyperreals $a_1, \ldots, a_k \in \mathbb{R}$ such that $z_V = a_1 v_1 + \ldots + a_k v_k$. Denoting $c_j = \lfloor a_j \rfloor$ their lower integer parts and $z' = z - c_1 v_1 - \ldots - c_k v_k \in L$, we have $z - z' \in V$, hence $\|z' - z_V\| = \|z - z_V\|$, so that the vector $z' \in L$ has the same minimality property as $z$. Then, according to Lemma 1.1 and the transfer principle, the $(k+1)$-tuple $(v_1, \ldots, v_k, z')$ can be extended to a basis of $L$, hence $\|v_{k+1}\| \ll \|z'\|$, as the basis $(v_1, \ldots, v_m)$ is Minkowski reduced. At the same time,

$$z'_V = (a_1 - c_1) v_1 + \ldots + (a_k - c_k) v_k,$$

with $|a_j - c_j| < 1$ for each $j \leq k$. From the triangle inequality we get

$$\|z'\| \leq \|z'_V\| + \|z' - z'_V\|$$

$$= \|(a_1 - c_1) v_1 + \ldots + (a_k - c_k) v_k\| + \|z - z_V\|$$

$$< \|v_1\| + \ldots + \|v_k\| + \|x\|.$$  

Therefore, $\|z'\| \ll \|v_{k+1}\|$, hence $\|z'\| \ll \|v_{k+1}\|$, which is a contradiction.

(b) Because $\|v_1\| = \lambda_1(L)$, the statement of (b) is true for $k = 1$. Assume, toward a contradiction, that $k < m$ for the biggest index satisfying $\|v_k\| \sim \lambda_k(L)$. Then

$$1 \leq \frac{\|v_k\|}{\lambda_k(L)} < \infty \quad \text{and} \quad \frac{\lambda_{k+1}(L)}{\|v_{k+1}\|} \approx 0.$$  

Therefore,

$$\frac{\|v_k\|}{\|v_{k+1}\|} \leq \frac{\lambda_{k+1}(L)}{\lambda_k(L)} \cdot \frac{\|v_k\|}{\|v_{k+1}\|} = \frac{\|v_k\|}{\lambda_k(L)} \cdot \frac{\lambda_{k+1}(L)}{\|v_{k+1}\|} \approx 0.$$  

Then, according to (a), $\frac{\|x\|}{\|v_{k+1}\|} \neq 0$ for every vector $x \in L \setminus \text{span}(v_1, \ldots, v_k)$. In particular, $\frac{\lambda_{k+1}(L)}{\|v_{k+1}\|} \neq 0$. \(\square\)

**Remark 2.3.** (b) of Lemma 2.2 follows immediately, by applying the transfer principle, from the following estimates of the lengths of vectors in any Minkowski
reduced basis \((v_1, \ldots, v_m)\) of a rank \(m\) lattice \(L \subseteq \mathbb{R}^n\) in terms of its Minkowski successive minima:
\[
\lambda_k(L) \leq \|v_k\| \leq 2^k \lambda_k(L)
\]
for all \(k \leq m\) (see Lagarias \[12\]; Mahler \[16\] has even better upper bounds). Then \((a)\) could be proved as an easy consequence of \((b)\). However, it is perhaps worthwhile to notice that, using the internal lattice concept, the purely qualitative estimates \((a), (b)\) follow already from Lemma \[14\] and the existence of Minkowski reduced bases.

The standard part \(^*X\) of a set \(X \subseteq \mathbb{R}^n\) consists of the standard parts of all finite vectors from \(X\); alternatively, it can be formed by taking the quotient of the set of finite vectors from \(X\) with respect to the equivalence relation of infinitesimal nearness. Identifying the results of both approaches, we have
\[
\begin{align*}
\text{Proof.} & \quad \text{Let's start with an arbitrary } c_m \in \mathbb{Z} \text{ such that } c_mv_m \in F \ltimes L \text{ (e.g., one can put } c_m = \left[\|v_m\|^{-1}\right] \text{ guaranteeing that } 1 \leq \|c_mv_m\| < 1 + \|v_m\| \approx 1). \text{ Further we proceed by backward recursion. Assuming that } 2 \leq k \leq m \text{ and } c_k \text{ is already defined, we put } c_{k-1} = c_k \text{ if } c_kv_k \neq 0 \text{ (as } \|v_{k-1}\| < \|v_k\|, \text{ so } c_{k-1}v_{k-1} \in FL \text{ is satisfied automatically), otherwise we put } c_{k-1} = bc_k \text{ where } b \in \mathbb{Z} \text{ is any hyperinteger such that } bc_kv_{k-1} \in \mathbb{Z} \ltimes L \text{ (e.g., } b = \left[\|c_kv_{k-1}\|^{-1}\right] \text{ will work). Obviously, } c_k \in \mathbb{Z} \text{ divides } c_{k-1} \in \mathbb{Z} \text{ for any } 2 \leq k \leq m. \text{ Assume that } x \approx 0 \text{ where } x = a_1c_1v_1 + \ldots + a_mv_mv_m \text{ for some } a_1, \ldots, a_m \in \mathbb{Z}, \text{ not all equal to } 0. \text{ Let } q \leq m \text{ be the biggest index such that } a_q \neq 0. \text{ Then } x' = \frac{1}{c_q} x = \sum_{k=1}^{q} \frac{a_kc_k}{c_q} v_k \neq 0 \text{ is a vector from the internal lattice } L. \text{ Moreover, } c_qx' = x \approx 0, \text{ while } c_qv_q \neq 0, \text{ hence } \|x'\| \ll \|v_q\|. \text{ Let } p \leq q \text{ be the smallest index such that } \|x'\| \ll \|v_p\|. \text{ Denote } \lambda = \|x'\| \text{ if } p = 1, \text{ or } \lambda = \max(\|v_{p-1}\|, \|x'\|) \text{ if } p > 1. \text{ Then the hyperball } \lambda^*B...
contains \( p \) linearly independent vectors \( v_1, \ldots, v_{p-1}, x' \) from \( L \), hence \( \lambda_p(L) \leq \lambda \) and, at the same time, \( \lambda \ll \|v_p\| \), contradicting Lemma 2.2(b).

**Proposition 2.5.** Let \( L = \prod L_i / D \subseteq \mathbb{R}^n \) be an internal lattice of rank \( m \) and \( \mathcal{L} \) be its standard part. Then the following hold true:

(a) \( \mathcal{L} \) is a lattice in \( \mathbb{R}^n \) if and only if there is a positive \( \lambda \in \mathbb{R} \) such that the set \( \{ i \in I : \lambda_1(L_i) \geq \lambda \} \) belongs to \( D \). This is equivalent to \( \lambda_1(L) \neq 0 \) as well as to \( \text{rank}_0(L) = 0 \).

(b) \( \mathcal{L} \) is the direct sum of a linear subspace of \( \mathbb{R}^n \) of dimension \( \text{rank}_0(L) \) and a lattice in \( \mathbb{R}^n \) of rank \( \text{rank}_0(L) - \text{rank}_0(L) \).

(c) \( \mathcal{L} \) is a lattice of rank \( q \leq m \) if and only if \( \text{rank}_0(L) = 0 \) and \( \text{rank}_k(L) = q \).

**Proof.** (a) The equivalence of any of the first two conditions to the discreteness of the group \( \mathcal{L} \) is obvious. Similarly, any of the obviously equivalent conditions \( \lambda_1(L) \neq 0 \) and \( \text{rank}_0(L) = 0 \) implies the discreteness of \( \mathcal{L} \). Otherwise, there is at least one nonzero infinitesimal vector \( v \in L \). Then one can find a hyperinteger \( q \in \mathbb{Z} \) such that \( cv \) is finite but not infinitesimal. Obviously, its standard part \( w = q(cv) \neq 0 \) belongs to \( \mathcal{L} \), so that \( \text{span}(w) = \mathbb{R}w \) is a line in \( \mathbb{R}^n \). We prove the inclusion \( \mathbb{R}w \subset \mathcal{L} \). Taking any \( x = aw \in \mathbb{R}w \), with \( a \in \mathbb{R} \), and putting \( b = [ac] \in \mathbb{Z} \), we have \( b \leq ac < b + 1 \) which, by the virtue of \( v \approx 0 \), implies \( bv \approx acv \). Hence

\[
x = aw \approx acv \approx bv \in \mathbb{F}L,
\]

and \( x = q(bv) \in \mathcal{L} \). It follows that \( \mathcal{L} \), containing the line \( \mathbb{R}w \subset \mathbb{R}^n \), is not discrete.

(b) Let \( (v_1, \ldots, v_m) \) be a Minkowski reduced basis of \( L \). Denote \( p = \text{rank}_0(L) \) and \( q = \text{rank}_k(L) \). According to Lemma 2.2(b), a vector \( v_k \) is infinitesimal if and only if \( k \leq p \), and it is finite if and only if \( k \leq q \). For the same reason, if \( x \in L \setminus *\text{grp}(v_1, \ldots, v_q) \) then \( \|x\| \ll v_{q+1} \), hence \( x \notin \mathcal{L} \). Therefore the standard part \( \mathcal{L} \) of the internal lattice \( L \) coincides with the standard part of its internal sublattice \( *\text{grp}(v_1, \ldots, v_q) \). Due to Lemma 2.4 there are hyperintegers \( c_1, \ldots, c_p \in \mathbb{Z} \) such that \( c_k v_k \in \mathbb{F}L \setminus L \) for any \( k \) and \( c_k \) divides \( c_{k-1} \) for \( k \geq 2 \). Then the internal sublattice \( M = *\text{grp}(c_1, \ldots, c_p) \) \( L \) contains no nonzero infinitesimal vector. Let us denote \( w_k = q(c_k v_k) \) for \( k \leq p \), and, additionally, \( c_k = 1 \), \( w_k = q(v_k) = q(v_k) \) for \( p < k \leq q \). As a consequence, \( \mathcal{L} \) coincides with the sum of the linear subspaces \( \text{span}(v_1, \ldots, v_p) \) and the lattice \( \text{grp}(w_{p+1}, \ldots, w_q) \).

The proof of (b) will be complete once we establish the following claim.

**Claim.** The vectors \( w_1, \ldots, w_q \) are linearly independent over \( \mathbb{R} \).

Indeed, let \( b \in \mathbb{N} \) be any infinite hypernatural number. Put \( v'_{k} = b^{-1}v_k, c'_{k} = bc_k \)
for any \( k \leq q \). Then all the vectors \( v_1', \ldots, v_q' \) are infinitesimal and form a Minkowski reduced basis of the lattice \( L' = \{ b^{-1}x : x \in L \} \). Now, all the vectors \( c'_{k} v'_{k} = c_k v_k \), where \( k \leq q \), are finite but not infinitesimal and \( c'_{k} \) divides \( c_{k-1} \) for \( k \geq 2 \). From Lemma 2.4 we infer that the internal lattice

\[
N = *\text{grp}(c_1 v_1, \ldots, c_q v_q) = *\text{grp}(c'_{1} v'_1, \ldots, c'_q v'_q)
\]

satisfies \( \lambda_1(N) \neq 0 \). Then, by (a), its standard part \( \mathcal{N} \) is a lattice in \( \mathbb{R}^n \). According to Lemma 2.2 it suffices to show that \( a_1 w_1 + \ldots + a_q w_q = 0 \) implies \( a_1 = \ldots = a_q = 0 \) for any integers \( a_1, \ldots, a_q \in \mathbb{Z} \). Since the first equality is equivalent to \( a_1 c_1 v_1 + \ldots + a_q c_q v_q \approx 0 \) and the left hand vector belongs to \( N \), which contains no infinitesimal vector except for 0, we have \( a_1 c_1 v_1 + \ldots + a_q c_q v_q = 0 \), and the desired
conclusion follows from the linear independence of the vectors $c_1 v_1, \ldots, c_q v_q$ over $*\mathbb{R}$.

(c) follows directly from (a) and (b).

Let us record the following direct consequence of (b).

**Corollary 2.6.** Let $L$ be an internal lattice in $*\mathbb{R}^n$. Then its standard part $^\circ L$ is a closed subgroup of the additive group $\mathbb{R}^n$.

3. An “almost-near” result for systems of linear equations

We denote by $F_{m \times n}$ the vector space of all $m \times n$ matrices over a field $F$. Unless otherwise said, the vector space $F^n$ consists of column vectors. The transpose of a matrix $A$ is denoted by $A^T$. A matrix $A \in *\mathbb{R}^{m \times n}$ is called finite, in symbols $A \in F^*\mathbb{R}^{m \times n}$, if all its entries $a_{ij}$ are finite. Then the matrix $^\circ A = (^\circ a_{ij}) \in \mathbb{R}^{m \times n}$ is called the standard part of $A$. The preservation of addition and multiplication by the standard part map on $F^*\mathbb{R}$ extends to finite matrices, i.e.,

$$ ^\circ (A + B) = ^\circ A + ^\circ B \quad \text{and} \quad ^\circ (AC) = ^\circ A ^\circ C $$

for any $A, B \in F^*\mathbb{R}^{m \times n}, C \in F^*\mathbb{R}^{n \times p}$.

The following “almost-near” result for solutions of systems of linear equations will be used in the proof of our first stability Theorem 4.3 in the next section.

**Proposition 3.1.** Let $A \in F^*\mathbb{R}^{m \times n}$ be any matrix such that its rows are linearly independent over the field $*\mathbb{R}$, the standard parts of its rows are linearly independent over $\mathbb{R}$, and $b \in F^*\mathbb{R}^m$. Then, for any $x \in F^*\mathbb{R}^n$ satisfying $Ax \approx b$, there is a $y \in *\mathbb{R}^n$ such that $y \approx x$ and $Ay = b$.

Notice that the vector $y$, being infinitesimally close to the standard vector $x$, is necessarily finite.

**Proof.** The above assumptions guarantee that $m \leq n$ and both the systems $A \xi = b, ^\circ A \xi = ^\circ b$ indeed have solutions (in $*\mathbb{R}^n, \mathbb{R}^n$, respectively), because the internal rank of $A$ over $*\mathbb{R}$, as well as the rank of $^\circ A$ over $\mathbb{R}$ are both equal to $m$. We denote by $V$ the orthocomplement of the internal linear subspace $\{ \xi \in *\mathbb{R}^n : A \xi = 0 \}$ in $*\mathbb{R}^n$.

Let $x \in F^*\mathbb{R}^n$ satisfy $Ax \approx b$ and $y \in *\mathbb{R}^n$ be the orthogonal projection of $x$ to the affine subspace $\{ \xi \in *\mathbb{R}^n : A \xi = b \}$ of $*\mathbb{R}^n$. Then $x - y \in V$ and, of course, $Ay = b$. It suffices to prove that $x \approx y$.

Let $A = P D Q^T$ be the singular value decomposition of $A$. Thus $P \in *\mathbb{R}^{m \times m}$, $Q \in *\mathbb{R}^{n \times n}$ are singular matrices and $D$ is a diagonal matrix with the diagonal formed by the singular values $d_1 \geq \ldots \geq d_m > 0$ of $A$. Then $^\circ A = ^\circ P ^\circ D ^\circ Q^T$ is the singular value decomposition of $^\circ A$, and from the properties of $A$ it follows that all the singular values $^\circ d_1, \ldots, ^\circ d_m$ of $^\circ A$ are still positive, hence all the $d_i$ are noninfinitesimal. The internal linear subspace $V \subseteq *\mathbb{R}^n$ is spanned by the first $m$ columns of the matrix $Q$, and

$$ d_m \|v\| \leq \|A v\| \leq d_1 \|v\}, $$

holds for each vector $v \in V$ (see, e.g., Han, Neumann [9], §5.6, and Bernstein [2], §§5.6, 9.11). In particular, since $Ax \approx b = Ay$,

$$ d_m \|x - y\| \leq \|A(x - y)\| \approx 0, $$

implying $\|x - y\| \approx 0$, i.e., $x \approx y$. 

□
4. The “almost-near” theorems for dual lattices

NONSTANDARD FORMULATION

Given an internal lattice $L = \prod L_i/D$ in $^\ast \mathbb{R}^n$, its internal integral annihilator can be defined as the ultraproduct of the integral annihilators of the particular lattices $L_i \subseteq \mathbb{R}^n$ or, equivalently, as the annihilator of $L$ with respect to the set of hyperintegers $^\ast \mathbb{Z}$. Then the Los Theorem (Lemma 2.1) assures that both the objects coincide, i.e.,

$$\text{Ann}_{^\ast \mathbb{Z}}(L) = \{ u \in ^\ast \mathbb{R}^n : \forall x \in L : ux \in ^\ast \mathbb{Z} \} = \prod \text{Ann}_\mathbb{Z}(L_i)/D.$$

Similarly, we have a two-fold definition of the internal dual of the internal lattice $L$:

$$L' = \text{Ann}_{^\ast \mathbb{Z}}(L) \cap ^\ast \text{span}(L) = \prod L'_i/D.$$

Using the transfer principle, Lemma 1.3 implies the following transference relations between the successive minima of an internal lattice $L \subseteq ^\ast \mathbb{R}^n$ and the successive minima and the covering radius, respectively, of its internal dual lattice.

**Lemma 4.1.** Let $L \subseteq \mathbb{R}^n$ be an internal lattice of rank $m$. Then

$$\lambda_k(L) \lambda_{m-k+1}(L') < \infty$$

for each $k \leq m$, and

$$\lambda_1(L) \mu(L') < \infty.$$

**Remark 4.2.** The preceding relations follow already from the considerably weaker estimates than those in Lemma 1.3 namely,

$$\lambda_k(L) \lambda_{m-k+1}(L') \leq m!,$$

due to Mahler [17], and the almost obvious observation

$$\mu(L') \leq \frac{1}{2} m \lambda_m(L'),$$

which jointly imply

$$\lambda_1(L) \mu(L') \leq \frac{1}{2} m m!.$$

Yet weaker estimates $\lambda_k(L) \lambda_{m-k+1}(L') \leq (m!)^2$ (see Gruber-Lekkerkerker [8], p. 125) are still sufficient (cf. Remark 2.3).

As first we prove an infinitesimal version of the “almost-near” result for integral annihilators of internal lattices.

**Theorem 4.3.** Let $L \subseteq ^\ast \mathbb{R}^n$ be an internal lattice. Then for each $x \in ^\ast \mathbb{R}^n$, such that $|ux|_2 \approx 0$ for every finite $u \in L$, there is a $y \in \text{Ann}_{^\ast \mathbb{Z}}(L)$ such that $y \approx x$.

**Proof.** Let $\beta = (v_1, \ldots, v_m)$ be a Minkowski reduced basis of $L$, and $0 \leq p \leq q \leq m$ be natural numbers such that $v_1, \ldots, v_p$ are all the infinitesimal vectors in $\beta$ and $v_1, \ldots, v_q$ are all the finite vectors in $\beta$. Recalling Proposition 2.3(b) and its proof, there are hyperintegers $c_1, \ldots, c_p \in ^\ast \mathbb{Z}$ such that the vectors $c_1 v_1, \ldots, c_p v_p \in L$ are finite and noninfiniteresimal and each finite vector $u \in L$ is infinitesimally close to a vector of the form

$$(a_1 c_1 v_1 + \ldots + a_p c_p v_p) + (a_{p+1} v_{p+1} + \ldots + a_q v_q),$$

where $a_1, \ldots, a_p \in \mathbb{R}$ and $a_{p+1}, \ldots, a_q \in \mathbb{Z}$.
Let us form the matrix with columns $c_1v_1,\ldots,c_pv_p, v_{p+1},\ldots,v_q$, and denote by $A \in \mathbb{F}^*\mathbb{R}^{q \times n}$ its transpose. Then an $x \in \mathbb{F}^*\mathbb{R}^n$ satisfies the condition $|u x|_2 \approx 0$ for each finite $u \in L$ if and only if $c_kv_k x \approx 0$ for $k \leq p$, and $|v_k x|_2 \approx 0$ for $p < k \leq q$.

Assume that $u$ satisfies this condition and put $b = (0,\ldots,0,\varphi(v_{p+1}x),\ldots,\varphi(v_qx))^T$. Then $b \in \mathbb{Z}^q$ and $x$ satisfies $Ax \approx b$. By the virtue of Proposition 3.1 there is a $y \in \mathbb{F}^*\mathbb{R}^n$ such that $y \approx x$ and $Ay = b$. Then, however, $v_k y = b_k = 0$ for $k \leq p$, and $v_k y = b_k \in \mathbb{Z}$ for $p < k \leq q$. If $q = m$, we are done. Otherwise there exists a sequence of integers $q = q_0 < q_1 < \ldots < q_t = m$ such that

$$\|v_{q_{s-1}}\| \ll \|v_k\| \sim \|v_{q_s}\|$$

for all $s, k$ satisfying $1 \leq s \leq t, q_{s-1} < k \leq q_s$.

We are going to construct a sequence of vectors $y^{(0)} = y, y^{(1)},\ldots,y^{(t)} \in \mathbb{F}^*\mathbb{R}^n$, such that $y^{(s)} \approx x$ and $v_k y^{(s)} \in *\mathbb{Z}$ for any $s \leq t, k \leq q_s$. Then already $v y^{(t)} \in *\mathbb{Z}$ for every $v \in L$, as required. This will be achieved by an inductive argument. Obviously, to this end it is enough to prove the following

**Claim.** Let $0 \leq s < t$ and $z \in \mathbb{F}^*\mathbb{R}^n$ be a vector such that $v_k z \in *\mathbb{Z}$ for any $k \leq q_s$. Then there is a $z' \in \mathbb{F}^*\mathbb{R}^n$ such that $z' \approx z$ and $v_k z' \in *\mathbb{Z}$ for any $k \leq q_{s+1}$.

Let us denote $q' = q_s, q'' = q_{s+1}, d = q'' - q' > 0$, for typographical reasons, and form the internal lattice $M = *\text{grp}(v_1,\ldots,v_{q''}) \subseteq L$, as well as the internal linear subspace $V = *\text{span}(M) = *\text{span}(v_1,\ldots,v_{q''}) \subseteq \mathbb{R}^n$. According to Lemma 3.2(b) and Lemma 4.1 we know that

$$\|v_k\| \sim \lambda_k(M) \quad \text{and} \quad \lambda_k(M) \lambda_{q''-k+1}(M') < \infty$$

whenever $q' < k \leq q''$. Putting both the relations together, for $k = q' + 1$ we particularly get

$$\|v_{q'+1}\| \lambda_d(M') < \infty.$$  

Realizing that the vectors $v_k$, for $q < k \leq m$, are infinite, we see that $\lambda_d(M') = 0$. Thus there are vectors $w_1,\ldots,w_d \in M'$, linearly independent over $*\mathbb{R}$ such that $|w_j| \leq \lambda_d(M')$ for $j \leq d$; in particular, all the vectors $w_j$ are infinitesimal.

We will search for the vector $z'$ in the form

$$z' = z + \alpha_1w_1 + \ldots + \alpha_dw_d$$

with unknown coefficients $\alpha_1,\ldots,\alpha_d \in \mathbb{F}^*\mathbb{R}$. This will automatically guarantee that $z' \approx z$.

As $\|v_k\| \ll \|v_{q'+1}\|$, for any $k \leq q', j \leq d$, we have $\|v_k\| \ll \|v_{q'+1}\|$ and

$$|v_k w_j| \leq \|v_k\| \|w_j\| \leq \frac{\|v_k\|}{\|v_{q'+1}\|} \|v_{q'+1}\| \lambda_d(M') \approx 0.$$  

At the same time, $v_k w_j \in *\mathbb{Z}$, hence $v_k w_j = 0$, and

$$v_k z' = v_k z + \sum_{j=1}^d \alpha_j v_k w_j = v_k z \in *\mathbb{Z},$$

regardless of the choice of $\alpha_1,\ldots,\alpha_d$. Moreover, denoting $h: *\mathbb{R}^n \to *\mathbb{R}^d$ the $*\mathbb{R}$-linear mapping given by $H(\xi) = (\xi w_1,\ldots,\xi w_d)^T$ for $\xi \in *\mathbb{R}^n$, we can conclude that the vectors $v_1,\ldots,v_{q''}$ form a basis of the linear subspace $V \cap \text{Ker } h \subseteq *\mathbb{R}^n$. Indeed, as the vectors $w_1,\ldots,w_d$ are linearly independent, $\text{Ker } h$ has dimension $n - d$ and it equals the direct sum of the orthocomplement $V^\perp$ with dimension $n - q''$ and $V \cap \text{Ker } h$. Then the latter necessarily has dimension $(n - d) - (n - q'') = q'$.
On the other hand, for \( q' < k < q'' \), \( j \leq d \), we still have \( \|v_k\| \sim \|v_{q'+1}\| \) and
\[
|v_k w_j| \leq \|v_k\| \|w_j\| \leq \frac{\|v_k\|}{\|v_{q'+1}\|} \lambda_d(M') < \infty,
\]
hence each \( v_k w_j \) is a finite integer, and \( h(v_k) \in \mathbb{Z}^d \) for any \( k \). Since the vectors \( v_1, \ldots, v_{q'}, v_{q'+1}, \ldots, v_{q''} \) are linearly independent over \( \ast \mathbb{R} \) and the first \( q' \) from among them form a basis of \( \mathbb{V} \cap \text{Ker} \psi \), the vectors \( h(v_{q'+1}), \ldots, h(v_{q''}) \) are linearly independent over \( \ast \mathbb{R} \), as well. Then the matrix
\[
B = (b_{ij}) \in \ast \mathbb{R}^{d \times d}
\]
with entries \( b_{ij} = v_{q'+i} w_j \in \mathbb{Z} \) satisfies \( 0 \neq \det B \in \mathbb{Z} \). It follows that \( B \) is strongly regular and \( B^{-1} \) is finite. Thus denoting \( \omega = (\omega_1, \ldots, \omega_d)^T \in \mathbb{F}^d \mathbb{R}^d \) the vector with coordinates \( \omega_j = v_{q'+i} z - [v_{q'+i} z] \) (i.e., the fractional parts of the inner products \( v_{q'+i} z \)), for \( i \leq d \), the system \( B \eta = -\omega \) has a unique solution \( \alpha = (\alpha_1, \ldots, \alpha_d)^T = -B^{-1} \omega \in \mathbb{F}^{d \times d} \), which means that
\[
\sum_{j=1}^d v_{q'+i} w_j \alpha_j = -\omega_i
\]
for each \( i \leq d \). Taking any \( q' < k < q'' \) and putting \( i = k - q' \), now, the following computation
\[
v_k z' = v_k z + \sum_{j=1}^d \alpha_j v_{q'+i} w_j = v_k z + \sum_{j=1}^d b_{ij} \alpha_j
\]
concludes the proof of the Claim, henceforth of the Theorem, too. \( \square \)

Corollary 4.4. Let \( L \subseteq \ast \mathbb{R}^n \) be an internal lattice. Then
\[
\circ (\text{Ann}_Z L) = \text{Ann}_Z (\circ L),
\]
in other words, the standard part of the internal integral annihilator \( \text{Ann}_Z L \) of \( L \) equals the integral annihilator of the standard part \( \circ L \) of \( L \).

Proof. The inclusion \( \text{Ann}_Z (\circ L) \subseteq \circ (\text{Ann}_Z L) \) is a direct consequence of the last Theorem. Indeed, if \( x \in \text{Ann}_Z (\circ L) \) then \( \circ x u \in \mathbb{Z} \) for every finite \( u \in L \). Then \( |x u|_Z \approx 0 \), for any such a \( u \), and, by Theorem \( 13 \) there is a \( y \in \text{Ann}_Z (L) \), such that \( y \approx x \), hence \( x \in \circ (\text{Ann}_Z L) \).

The reversed inclusion \( \circ (\text{Ann}_Z L) \subseteq \text{Ann}_Z (\circ L) \) is easy anyway. It suffices to show that \( \circ x \in \text{Ann}_Z (\circ L) \) for any finite \( x \in \text{Ann}_Z (L) \). Taking any finite \( u \in L \), the inner product \( u x \) is finite and belongs to \( \ast \mathbb{Z} \), hence
\[
\circ u \circ x = \circ (u x) = u x \in \mathbb{Z},
\]
so that \( \circ x \in \text{Ann}_Z (\circ L) \), as required. \( \square \)

The following is the nonstandard formulation of the announced “almost-near” result for dual lattices.

Theorem 4.5. Let \( L \subseteq \ast \mathbb{R}^n \) be an internal lattice. Then for each finite vector \( x \in \ast \text{span}(L) \), such that \( |u x|_Z \approx 0 \) for every finite \( u \in L \), there is a \( y \in L' \) such that \( y \approx x \).

Proof. Let \( V = \ast \text{span}(L) \subseteq \ast \mathbb{R}^n \) and \( z_V \) denote the orthogonal projection of any \( z \in \ast \mathbb{R}^n \) to \( V \). Then \( \|z_V\| \approx \|z\| \) for any \( z \). According to Theorem \( 4.3 \) under the
above assumptions there is a $y \in \text{Ann}_Z(L)$ such that $y \approx x$. Then $v y v = v y \in \ast Z$ for every $v \in L$, i.e., $g v \in L'$. As $x_v = x$ and $z \mapsto z_v$ is a linear map,
\[
\|x - y v\| = \|x v - y v\| = \|(x - y) v\| \leq \|x - y\| \approx 0,
\]
hence $y v \approx x$. \qed

The last stability Theorem is equivalent to the inclusion $(\ast L)' \subseteq \ast (L')$ for internal lattices $L \subseteq \ast \mathbb{R}^n$. In view of Corollary 5.4 the reader might expect that also the reversed inclusion $\ast (L') \subseteq (\ast L)'$ is satisfied (and even easy to prove). However, as shown by following example, this is not true in general.

Example 4.6. Let $c \in \mathbb{R}$ be positive and $d \in \ast \mathbb{R}$ be positive and infinite. Consider the full rank internal lattice
\[
L = c^\ast \mathbb{Z} \times d^\ast \mathbb{Z} = \{(ac, bd)^T : a, b \in \ast \mathbb{Z}\}
\]
in $\ast \mathbb{R}^2$. Then, as easily seen, its standard part is a rank 1 lattice $\ast L = c \mathbb{Z} \times \{0\}$ in $\mathbb{R}^2$, while its internal dual is the full rank internal lattice $L' = c^{-1} \mathbb{Z} \times d^{-1} \ast \mathbb{Z}$ in $\ast \mathbb{R}^2$. Then $(\ast L)' = c^{-1} \mathbb{Z} \times \{0\}$ is a rank 1 lattice in $\mathbb{R}^2$ while $\ast (L') = c^{-1} \mathbb{Z} \times \mathbb{R}$ is not even a lattice.

5. The “almost-near” theorem for dual lattices

standard formulation

In this final section we state and prove the announced standard version of the stability theorem for dual lattices, strengthening the preliminary Theorem 0.1. It is in fact a standard equivalent of Theorem 4.5. In its proof we will need the following last nonstandard lemma.

Lemma 5.1. Let $L \subseteq \ast \mathbb{R}^n$ be an internal lattice and $G \subseteq L$ be any additive subgroup of $L$. Let further $\delta < \frac{1}{3}$ be a positive real number and $x \in \ast \mathbb{R}^n$ be a vector such that $|u x|_{\mathbb{Z}} \leq \delta$ for every $u \in G$. Then $|u x|_{\mathbb{Z}} \approx 0$ for every $u \in G$.

Proof. As the mapping $u \mapsto u x$ is an additive group homomorphism $L \to \ast \mathbb{R}$, the image $G x = \{u x : u \in G\}$ of the subgroup $G \subseteq L$ under this map must be a subgroup of $\ast \mathbb{R}$. However, if $0 < \delta < \frac{1}{3}$ is a (standard) real number, then $\ast \mathbb{Z} + \ast \mathbb{R}$ is the biggest subgroup of $\ast \mathbb{R}$ satisfying the inclusion $\ast \mathbb{Z} + \ast \mathbb{R} \subseteq \ast \mathbb{Z} + \ast [-\delta, \delta]$. \qed

Recall that $B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ denotes the (euclidean) unit ball in $\mathbb{R}^n$.

Theorem 5.2. Let $n \geq 1$ be an integer and $\delta < \frac{1}{3}$, $\varepsilon$, $\lambda$ be positive reals. Then there exists a real number $r > 0$, depending just on $n$, $\delta$, $\varepsilon$ and $\lambda$, such that every lattice $L \subseteq \mathbb{R}^n$, subject to $\lambda_1(L) \geq \lambda$, satisfies the following condition:

For any $x \in \text{span}(L)$, such that $|u x|_{\mathbb{Z}} \leq \delta$ for all $u \in L \cap r B$, there is a $y \in L'$ such that $\|x - y\| \leq \varepsilon$.

Proof. Assume that the conclusion of the Theorem fails for some fixed quadruple of admissible parameters $n$, $\delta$, $\varepsilon$, $\lambda$. This is to say that for each real number $r > 0$ there is a lattice $L_r \subseteq \mathbb{R}^n$, satisfying $\lambda_1(L_r) \geq \lambda$, and an $x_r \in \text{span}(L)$ such that $|u x|_{\mathbb{Z}} \leq \delta$ for every $u \in L_r \cap r B$, however $\|x_r - y\| > \varepsilon$ for any $y \in L'$, i.e., $(x_r + \varepsilon B) \cap L' = \emptyset$. Let us confine to the values of $r$ from the set $I = \{1, 2, 3, \ldots\}$ of all positive integers.

Let us pick any nonprincipal ultrafilter $D$ on the set $I$ and form the ultraproduct $L = \prod_{r \in I} L_r / D$, as well as the vector $x = (x_r)_{r \in I} / D \in L$ and the infinite
positive hyperinteger $\rho = (1, 2, 3, \ldots)/D$. Then, by the virtue of the Los Theorem (Lemma 2.1), $L \subseteq \ast \mathbb{R}^n$ is an internal lattice satisfying $\lambda_1(L) \geq \lambda$. For the same reason we have $x \in \ast \text{span}(L)$, $|ux|_z \leq \delta$ for every $u \in L \cap \rho \ast B$, as well as $(x + \varepsilon \ast B) \cap L' = \emptyset$. As $\mathcal{F}L = L \cap \mathbb{F} \ast \mathbb{R} \subseteq \rho \ast B$ and it is a subgroup of $L$, in view of Lemma 5.1, the second of the above three conditions implies that $|ux|_z \approx 0$ for every $u \in \mathcal{F}L$.

As a consequence of Lemma 5.1, the covering radius $\mu = \mu(L')$ is a finite positive hyperreal. (In fact, Lemma 1.3 and the /suppress Los Theorem imply that $\mu \leq n^{3/2}/(2\lambda)$, however, this is not important for the moment.) Thus there is a $z \in L'$ such that $\|x - z\| \leq \mu$. Then $x - z \in \ast \text{span}(L)$ and

$$u(x - z) - ux = -uz \in \ast \mathbb{Z},$$

hence $|u(x - z)|_z = |ux|_z$ for any $u \in L$. At the same time,

$$(x - z + \varepsilon \ast B) \cap L' = (x + \varepsilon \ast B) \cap L' = \emptyset.$$

We can conclude, that $x' = x - z \in \ast \text{span}(L)$ is a finite vector satisfying $|ux|_z \approx 0$ for every finite $u \in L$, and $\|x' - y\| > \varepsilon$ for any $y \in L'$. This, however, contradicts Theorem 4.5.

**Final remark.** Theorem 4.5 is rather robust in the sense that it does not explicitly involve any norm on $\mathbb{R}^n$ in its formulation. Moreover,

$$\mathcal{F}^* \mathbb{R}^n = \{x \in \ast \mathbb{R}^n : \|x\| < \infty\} \quad \text{and} \quad \mathcal{I}^* \mathbb{R}^n = \{x \in \ast \mathbb{R}^n : \|x\| \approx 0\}$$

for (the canonic extension to $\ast \mathbb{R}^n$ of) any norm $\|x\|$ on $\mathbb{R}^n$ and not just for the euclidean one. As a consequence, Theorem 4.2 which is its corollary, remains true even if $B$ denotes any centrally symmetric convex body in $\mathbb{R}^n$, $\lambda_1(L)$ is replaced by the first Minkowski successive minimum

$$\lambda_1(C, L) = \min\{s \in \mathbb{R} : s > 0, \ L \cap s \ C \neq \{0\}\}$$

of another centrally symmetric convex body $C \subseteq \mathbb{R}^n$ with respect to $L$, and $\|x\|$ is an arbitrary norm on $\mathbb{R}^n$ (possibly without any direct relation either to $B$ or to $C$).

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