Spread-out limit of the critical points for lattice trees and lattice animals in dimensions $d > 8$

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Abstract

A spread-out lattice animal is a finite connected set of edges in $\{\{x, y\} \subset \mathbb{Z}^d : 0 < \|x - y\| \leq L\}$. A lattice tree is a lattice animal with no loops. The best estimate on the critical point $p_c$ so far was achieved by Penrose [23]:

$$p_c = \frac{1}{e} + O\left(\frac{L^{-2d/7} \log L}{L} \right)$$

for both models for all $d \geq 1$. In this paper, we show that $p_c = \frac{1}{e} + CL^{-d} + O(L^{-d-1})$ for all $d > 8$, where the model-dependent constant $C$ has the random-walk representation

$$C_{LT} = \sum_{n=2}^{\infty} \frac{n+1}{2e} U^n(o), \quad C_{LA} = C_{LT} - \frac{1}{2e^2} \sum_{n=3}^{\infty} U^n(o),$$

where $U^n$ is the $n$-fold convolution of the uniform distribution on the $d$-dimensional ball $\{x \in \mathbb{R}^d : \|x\| \leq 1\}$. The proof is based on a novel use of the lace expansion for $\tau_p(x)$ and detailed analysis of the 1-point function at a certain value of $p$ that is designed to make the analysis extremely simple.

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Figure 1: A sample $T$ of $\mathcal{T}_o$. Removal of all edges $\{o, y_j\} \in E_T$ leaves disjoint subtrees $R_j$ rooted at $y_j$: $V_T \setminus \{o\} = \bigcup_j V_{R_j}$ and $E_T \setminus \bigcup_j \{o, y_j\} = \bigcup_j E_{R_j}$.

1 Introduction and the main result

Given an $L \in \mathbb{N}$, we consider spread-out lattice animals $A = (V_A, E_A)$, where the vertex set $V_A$ is a finite subset of $\mathbb{Z}^d$ and any pair of vertices in $V_A$ are connected by a path of spread-out edges $E_A \subset \{x, y\} : 0 < \|x - y\| \leq L$; $\| \cdot \|$ is an arbitrary fixed norm on $\mathbb{R}^d$. A lattice tree is a lattice animal with no loops. Both models are statistical-mechanical models for branched polymers.

To investigate their statistical properties, we consider the following generating functions. Let $\Lambda = \{x \in \mathbb{Z}^d : 0 < \|x\| \leq L\}$, $D(x) = \frac{1}{|\Lambda|} \mathbb{1}_{\{x \in \Lambda\}}$, (1.1) where $\mathbb{1}_E$ is the indicator function of $E$, being 1 or 0 depending on whether or not $E$ is true. The function $D$ will be used as a transition probability of the underlying random walk. Then, we define the weight function for a tree $T$ as

$$W_p(T) = \prod_{\{x, y\} \in E_T} pD(x - y) = \left( \frac{p}{|\Lambda|} \right)^{|E_T|},$$

and similarly for a lattice animal $A$ as $W_p(A)$. For a finite set $X \subset \mathbb{Z}^d$, we denote by $\mathcal{T}_X$ (resp., $\mathcal{A}_X$) the set of lattice trees $T$ with $X \subset V_T$ (resp., lattice animals $A$ with $X \subset V_A$); if $X$ consists of a vertex or two, we simply write, e.g., $\mathcal{T}_o$ (for $X = \{o\}$; see Figure 1) or $\mathcal{T}_{o,x}$ (for $X = \{o, x\}$; see Figure 2). The generating functions we want to investigate are the 1-point and 2-point functions, defined respectively as

$$g_p = \sum_{T \in \mathcal{T}_o} W_p(T), \quad \tau_p(x) = \sum_{T \in \mathcal{T}_{o,x}} W_p(T),$$

for lattice trees, and similarly defined for lattice animals. The susceptibility $\chi_p$ is the sum of the 2-point function, defined as

$$\chi_p = \sum_{x \in \mathbb{Z}^d} \tau_p(x) = \sum_{x \in \mathbb{Z}^d} \sum_{T \in \mathcal{T}_o} \mathbb{1}_{\{x \in V_T\}} W_p(T) = \sum_{T \in \mathcal{T}_o} |V_T| W_p(T),$$

(1.4)
for lattice trees, and similarly for lattice animals. It has been known (see [18] and references therein) that there is a model-dependent critical point \( p_c \) such that \( \chi_p \) is finite if and only if \( p < p_c \) and diverges as \( p \uparrow p_c \). The goal of this paper is to reveal the asymptotics of \( p_c \) as \( L \uparrow \infty \) for both models.

The best estimate so far on \( p_c \) for the spread-out model was achieved by Penrose [23]. He investigated the growth constant, which is defined by the \( n \uparrow \infty \) limit of the \( n \)th root of the number \( t_n = \frac{1}{n} \sum_{T \in T_o} 1_{\{|V_T|=n\}} \) of \( n \)-vertex unrooted lattice trees. Since \( \{t_n\}_{n \in \mathbb{N}} \) is a supermultiplicative sequence, i.e., \( t_{n+m} \geq t_n t_m \) (see, e.g., [16]), \( \lim_{n \uparrow \infty} t_n^{1/n} \) exists and is asymptotically \( e|\Lambda| + O(|\Lambda|^{5/7} \log |\Lambda|) \) as \( |\Lambda| \uparrow \infty \) [23]. Since \( |E_T| = |V_T| - 1 \) for each lattice tree, we can rewrite \( \chi_p \) as

\[
\chi_p \overset{1.4}{=} \sum_{n=1}^{\infty} n \sum_{T \in T_o} 1_{\{|V_T|=n\}} W_p(T) = \sum_{n=1}^{\infty} n^2 \left( \frac{p}{|\Lambda|} \right)^{n-1} t_n. \tag{1.5}
\]

Therefore, for large \( |\Lambda| \),

\[
p_c = \lim_{n \uparrow \infty} \left( \frac{n^2}{|\Lambda|^{n-1}} t_n \right)^{-1/n} = \frac{|\Lambda|}{e|\Lambda| + O(|\Lambda|^{5/7} \log |\Lambda|)} = \frac{1}{e} + O(|\Lambda|^{-2/7} \log |\Lambda|), \tag{1.6}
\]

which is true for all dimensions \( d \geq 1 \). Penrose also claimed in [23, Section 3.1] that \( p_c \) for lattice animals obeys the same bound, due to the result of Klarner [15].

A weaker estimate, \( p_c = 1/e + o(1) \) as \( L \uparrow \infty \) for all \( d \) bigger than the critical dimension \( d_c = 8 \), was obtained by Miranda and Slade [20]. In fact, their main concern was to obtain \( 1/d \) expansions of \( p_c \) for the nearest-neighbor models. In [19, 21], they showed that,

\[
p_c = \frac{1}{e} + \frac{3}{2e} |\Lambda|^{-1} + \begin{cases} 
\frac{115}{24e} |\Lambda|^{-2} + o(|\Lambda|^{-2}) & \text{[lattice trees]}, \\
\left( \frac{115}{24e} - \frac{1}{2e^2} \right) |\Lambda|^{-2} + o(|\Lambda|^{-2}) & \text{[lattice animals]},
\end{cases} \tag{1.7}
\]

as \( |\Lambda| = 2d \uparrow \infty \). The proof is based on the lace expansion for the 2-point function \( \tau_p(x) \) and an expansion for the 1-point function \( g_p \) based on inclusion-exclusion. Notice that

Figure 2: A sample tree in \( T_o,x \). Removal of the backbone edges (in blue) yields disjoint subtrees \( \{R_j\} \), called ribs. In this example, \( R_1, R_3 \) and \( R_6 \) are single-vertex trees.
the model-dependence appears only from the $O(|\Lambda|^{-2})$ term, which is due to unit squares contained in $g_p$ for lattice animals, but not in $g_p$ for lattice trees. The lace expansion has been successful in showing mean-field critical behavior in high dimensions for various models, including lattice trees and lattice animals for $d > 8$ (e.g., [6, 8, 10]). The other models are self-avoiding walk for $d > 4$ (e.g., [2, 6, 9]), percolation for $d > 6$ (e.g., [6, 7]), oriented percolation and the contact process for the spatial dimension $d > 4$ (e.g., [22, 24]), and the Ising and $\phi^4$ models for $d > 4$ (e.g., [1, 25, 26, 27]).

For the nearest-neighbor lattice trees and lattice animals, in particular, Hara and Slade showed the mean-field critical behavior for both models in sufficiently high dimensions $d > 8$ in [6, 8, 10], where they did not mention about the specific dimension above which their results can be applied. In contrast, Fitzner and van der Hofstad proved in [5] that the nearest-neighbor lattice trees and lattice animals obey the mean-field critical behavior above $d = 16$ and $d = 17$, respectively. The proof is based on the non-backtracking lace expansion (NoBLE) that is different from the standard lace expansion by Hara and Slade.

In [12], van der Hofstad and the second-named author of the current paper applied the lace expansion to the spread-out models (defined by $D$ in (1.1)) of self-avoiding walk, percolation, oriented percolation and the contact process, and showed that, for all $d$ bigger than the respective critical dimension $d_c$,

$$p_c = 1 + CL^{-d} + O(L^{-d-1}),$$

as $L \uparrow \infty$, where 1 is the mean-field value, and the model-dependent constant $C$ has the following random-walk representation:

$$C = \begin{cases} 
\sum_{n=2}^{\infty} \frac{1}{n!} U^{*n}(o) & [\text{self-avoiding walk, the contact process}], \\
\frac{1}{2} \sum_{n=2}^{\infty} U^{*2n}(o) & [\text{oriented percolation}], \\
U^{*2}(o) + \sum_{n=3}^{\infty} \frac{n + 1}{2} U^{*n}(o) & [\text{percolation}],
\end{cases}$$

(1.9)

where $U^{*n}$ is the $n$-fold convolution in $\mathbb{R}^d$ of the uniform probability distribution $U$ on $\{x \in \mathbb{R}^d : \|x\| \leq 1\}$. For example, if $\|x\| = \|x\|_{\infty} := \max_j |x_j|$, then, for all $n \in \mathbb{N}$,

$$U(x) = \frac{1_{\{\|x\|_{\infty} \leq 1\}}}{2^d}, \quad U^{*(n+1)}(x) = \int_{\mathbb{R}^d} U^{*n}(y) U(x - y) \, d^d y.$$ 

(1.10)

These quantities are the spread-out limit of the underlying random walk generated by $D$ [12, Section 4]. For example, for $d > 4$,

$$\sum_{n=2}^{\infty} \frac{n + 1}{2} D^{*n}(o) = L^{-d} \sum_{n=2}^{\infty} \frac{n + 1}{2} U^{*n}(o) + O(L^{-d-1}),$$

(1.11)

where we have used the same notation $*$ to represent convolutions on $\mathbb{Z}^d$ as well. The error term $O(L^{-d-1})$ is due to Riemann-sum approximation.
We want to achieve a similar result for lattice trees and lattice animals, i.e., a random-walk representation for the difference between \( p_c \) and its mean-field value \( 1/e \), and see how the model-dependence arises in it. In the rest of the paper, we will show the following:

**Theorem 1.1.** For both models with \( d > 8 \) and \( L \uparrow \infty \),

\[
p_c = \frac{1}{e} + CL^{-d} + O(L^{-d-1}),
\]

(1.12)

where the model-dependent constant \( C \) has the following random-walk representation:

\[
C_{LT} = \sum_{n=2}^{\infty} \frac{n+1}{2e} U^{*(n)}(o),
\]

\[
C_{LA} = C_{LT} - \frac{1}{2e^2} \sum_{n=3}^{\infty} U^{*(n)}(o),
\]

(1.13)

The difference in \( p_c \) already shows up in the first error term of order \( L^{-d} \) for the spread-out models, while it appears in (1.7) from the second error term of order \( d^{-2} \) for the nearest-neighbor models, as mentioned earlier. This is due to closed loops of length bigger than 2 in \( g_b \) for lattice animals. The smallest among such loops for the spread-out model is of length 3 and of order \( L^{-d} \), while that for the nearest-neighbor model is of length 4 and of order \( d^{-2} \) (see Lemma 4.1 below). Identifying coefficients of the higher order terms for the spread-out models may need more work since they are absorbed in the error term \( O(L^{-d-1}) \) in (1.12), which is inherent in Riemann-sum approximation, just as mentioned below (1.11).

The proof of the above theorem is based on the lace expansion for the 2-point function and detailed analysis of the 1-point function, similarly to the previous work by Miranda and Slade [21]. The different point of our method from that of them is to introduce a new base point \( p_1 \) defined in (2.1) below, as \( p_1 g_{p_1} = 1 \). It is to estimate various generating functions in terms of massless random walks. For the spread-out models of self-avoiding walk, percolation, oriented percolation and the contact process, van der Hofstad and Sakai [12] simply used the base point \( p_1 = 1 \), because of the unity of the 1-point function for those models. Since the analysis in terms of the underlying random walks is very simple, we do not have to know in detail the lace expansion; the exception is in Lemma 2.1 below, where we investigate the first lace-expansion coefficient \( \hat{\pi}_p^{(1)} \) to prove \( p_c - p_1 = O(L^{-2d}) \). However, the basic facts (summarized in Proposition 2.2 below) and a minimum definition about the lace expansion coefficients should be enough to read the proof, which we hope makes this paper more accessible to wider audience.

Our method can be applied to the nearest-neighbor model as well to identify the coefficient of \( (2d)^{-1} \), as we can use the same method (i.e., Lemma 2.1 below) to conclude \( p_c - p_1 = O(d^{-2}) \), but this limit the accuracy our method can achieve. Therefore, to identify the higher-order coefficients, we may need investigate the lace expansion coefficients at \( p_c \) more carefully as Miranda and Slade do in [21].

The rest of the paper is organized as follows. In Section 2, we show that \( p_c \) is close (up to order \( L^{-2d} \)) to \( p_1 \) that satisfies the identity \( p_1 g_{p_1} = 1 \), which is heavily used in the
analysis in Sections 3 and 4. Section 3 is devoted to evaluating \( g_{p_1} \) for lattice trees. The 1-point function is split into two parts, \( G \) and \( H \), which are investigated in Sections 3.1 and 3.2, respectively. Finally, in Section 4, we demonstrate how to evaluate the difference between lattice trees and lattice animals.

## 2 Results due to the lace expansion

In this section, we approximate \( p_c \) by \( p_1 \) that is defined for both models by the identity

\[
p_1 g_{p_1} = 1.
\]

From now on, we frequently use

\[
\beta = L^{-d}.
\]

**Lemma 2.1.** For both models with \( d > 8 \) and \( L \uparrow \infty \),

\[
0 < p_c - p_1 = O(\beta^2).
\]

The key to the proof is the following collection of the lace-expansion results [6, 17], in which we use

\[
h_p(x) = \begin{cases} 
0 & \text{[lattice trees]}, \\
(1 - \delta_{o,x}) \sum_{A \in \mathcal{A}_o} \text{1}_{\{o \leftrightarrow x\}} W_p(A) & \text{[lattice animals]},
\end{cases}
\]

where \( o \leftrightarrow x \) means that \((o = x \text{ or})\) there is at least one pair of edge-disjoint paths from \( o \) to \( x \) in an animal \( A \). Let \( \hat{h}_p \) denote the sum of \( h_p(x) \) over \( x \in \mathbb{Z}^d \):

\[
\hat{h}_p = \sum_{x \in \mathbb{Z}^d} h_p(x).
\]

**Proposition 2.2 ([6, 17]).** For both models in \( d > 8 \), there is a model-dependent \( L_0 < \infty \) such that, for all \( L \geq L_0 \), the following holds for all \( p \leq p_c \):

1. The 1-point function is bounded away from zero and infinity. In fact,

\[
1 \leq g_p \leq 4.
\]

2. There are nonnegative lace-expansion coefficients \( \pi_p^{(n)}(x) \), \( n \in \mathbb{N} \), such that

\[
\exists K < \infty, \quad \forall x \in \mathbb{Z}^d, \quad \pi_p^{(n)}(x) \leq \frac{KL^{-6}(K\beta)^{n-1}}{\left(\|x\| \vee L\right)^{2d-6}},
\]
and that, by defining $\pi_p(x) = \sum_{n \in \mathbb{N}} (-1)^n \pi_p^{(n)}(x)$, the recursion equation

$$
\tau_p(x) = g_p \delta_{o,x} + h_p(x) + \pi_p(x) \\
+ \sum_{u,v} \left( g_p \delta_{o,u} + h_p(u) + \pi_p(u) \right) pD(v-u) \tau_p(x-v)
$$

(2.8)

holds for all $x \in \mathbb{Z}^d$.

Consequently, there is a $K' < \infty$ such that

$$
\forall x \neq o, \quad \tau_{p_c}(x) \leq \frac{K' L^{-2}}{\|x\| L^{d-1}}, \quad \chi_p \leq \frac{1}{p} (p_c - p)^{-1/2},
$$

(2.9)

where the latter means $\chi_p/(p_c - p)^{-1/2}$ is bounded away from 0 and $\infty$ as $p \uparrow p_c$, and

$$
p_c = \frac{1}{g_{p_c} + h_{p_c} + \hat{\pi}_{p_c}} = \left( g_{p_c} + \sum_{x \neq o} h_{p_c}(x) + \sum_{x} \pi_{p_c}(x) \right)^{-1}.
$$

(2.10)

The above results for lattice trees are proven in [17] by following the same line of proof as in [6] and using the convolution bounds in [4, Lemma 3.2] instead of the weaker ones in [6, Proposition 1.7]. The same strategy applies to lattice animals, and we refrain from showing details.

Consequently, for any $p \leq p_c$,

$$
\hat{\pi}_p^{(n)} = \sum_{x \in \mathbb{Z}^d} \pi_p^{(n)}(x) \leq K(K\beta)^{n-1} \left( \sum_{x: ||x|| \leq L} L^{-2d} + \sum_{x: ||x|| > L} \frac{L^{-6}}{||x||^{2d-6}} \right) = O(\beta)^n.
$$

(2.11)

Moreover, by subadditivity (i.e., forgetting edge-disjointness among paths from $o$ to $x$),

$$
\hat{h}_p \leq \sum_{x \neq o} \tau_p(x)^2 \leq (K' L^{-2})^2 \left( \sum_{x: ||x|| \leq L} L^{2(2-d)} + \sum_{x: ||x|| > L} ||x||^{2(2-d)} \right) = O(\beta).
$$

(2.12)

The identity (2.10) is obtained by summing (2.8) over $x \in \mathbb{Z}^d$, solving the resulting equation for $\chi_p$ and then using the fact that $\chi_p$ diverges as $p \uparrow p_c$. Substituting (2.11)-(2.12) to (2.10) yields

$$
p_c = \frac{1}{g_{p_c}} \left( 1 + \frac{\hat{h}_{p_c} - \hat{\pi}_p^{(1)}_{p_c}}{g_{p_c}} + O(\beta^2) \right)^{-1} = \frac{1}{g_{p_c}} \left( 1 + \frac{\hat{\pi}_p^{(1)}_{p_c} - \hat{h}_{p_c}}{g_{p_c}} \right) + O(\beta^2),
$$

(2.14)

\footnote{In [17], Liang investigated $\hat{\pi}_p^{(1)}$ in (2.14) for lattice trees and showed that, for all $d > 8$, $p_c g_{p_c}$ rather than $p_c$ exhibits

$$
p_c g_{p_c} = 1 + \frac{\beta}{c} \sum_{n=2}^{\infty} \left( \binom{n+1}{2} U^{*n}(o) + O(\beta/L) \right) \quad \text{as} \quad L \uparrow \infty.
$$

(2.13)

This may be a bit of surprise, as the coefficient of $\beta$ is much larger than that in (1.12)–(1.13).}
which is the starting point of the analysis.

**Proof of Lemma 2.1.** First we show \( p_1 < p_c \). Since \( pg_p \) is increasing in \( p \) with \( p_1 g_{p_1} = 1 \), it suffices to show \( p_c g_{p_c} > 1 \). By (2.6) and (2.14), it then suffices to show that \( \hat{\pi}_p^{(1)} - \hat{h}_{p_c} \) is bounded from below by \( \beta \) times a positive constant for large \( L \). Here, and only here, we use the actual definition of the lace-expansion coefficient \( \hat{\pi}_p^{(1)} \) (see, e.g., [8]). We can easily check that \( \hat{\pi}_p^{(1)} \) for both models is larger than the sum of triangles consisting only of three distinct edges: \( \hat{\pi}_p^{(1)} \geq |\Lambda|(|\Lambda| - 1)(p/|\Lambda|)^3 \), which is enough for lattice trees because \( \hat{h}_p \equiv 0 \). For lattice animals, we show below \( \hat{h}_p \leq \frac{1}{4} \hat{\pi}_p^{(1)} + O(\beta^2) \) for \( p \leq p_c \) in high dimensions \( d > 8 \). The aforementioned sufficient condition for \( p_c g_{p_c} > 1 \) is now verified.

Next we show \( p_c - p_1 = O(\beta^2) \) for lattice animals by induction. The same induction also works for lattice trees with \( A = T \) and \( \hat{h}_{p_c} \equiv 0 \). Let \( \{\ell_n\}_{n \in \mathbb{N}} \) be the following increasing sequence bounded above by 2:

\[
\ell_1 = 1, \quad \ell_{j+1} = 1 + \frac{\ell_j}{2} \quad [j \in \mathbb{N}].
\]

(2.15)

Since \( p_c = O(1) \) (see, e.g., (1.6) or [6, Proposition 2.2]) and \( p_1 g_{p_1} = p_c (g_{p_c} + \hat{h}_{p_c} + \hat{\pi}_{p_c}) = 1 \), we have

\[
0 < 1 - \frac{p_1}{p_c} = 1 - \frac{g_{p_c} + \hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1}} = -\frac{g_{p_c} - g_{p_1}}{g_{p_1}} \frac{\hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1}} \quad [g_{p_1} \geq 0],
\]

(2.16)

which is bounded above by \(-\hat{\pi}_{p_c}/g_{p_1} = O(\beta) \) (due to (2.11)), confirming \( p_c - p_1 = O(\beta^{\bar{\tau}}) \).

Now we suppose \( p_c - p_1 = O(\beta^{\bar{\tau}}) \). Notice that \( g_{p_c} - g_{p_1} \) can be rewritten as

\[
g_{p_c} - g_{p_1} = \sum_{A \in A_0} \left( 1 - \left( \frac{p_1}{p_c} \right)^{|E_A|} \right) W_{p_c}(A) = \left( 1 - \frac{p_1}{p_c} \right) \sum_{A \in A_0} \sum_{n=0}^{n=0} \frac{|E_A|^{-1}}{n} \left( \frac{p_1}{p_c} \right)^n W_{p_c}(A) = -\hat{h}_{p_c} + \hat{\pi}_{p_c} F.
\]

(2.17)

Solving this for \( g_{p_c} - g_{p_1} \) yields

\[
g_{p_c} - g_{p_1} = -\frac{\hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1} + F} F,
\]

(2.18)

which is bounded above by \(-\hat{\pi}_{p_c} = O(\beta) \) (due to (2.11)) for both models. By substituting (2.18) to (2.16), we obtain

\[
p_c - p_1 = p_c \left( \frac{1}{g_{p_1}} \frac{\hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1} + F} \right) = -p_c \frac{\hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1} + F} = p_c \frac{\hat{\pi}_{p_c} - \hat{h}_{p_c}}{g_{p_1} + F} + O(\beta^2).
\]

(2.19)
Recall the definition of $F$ in (2.17). Since $(p_1/p_c)^nW_{p_c}(A) = (p_c/p_1)|E_A|^{-n}W_{p_1}(A)$, which is also true for lattice trees, we have

$$F = \sum_{A \in \mathcal{A}_o} \sum_{n=1}^{|E_A|} \left( \frac{p_c}{p_1} \right)^n W_{p_1}(A) \geq \sum_{A \in \mathcal{A}_o} |E_A| W_{p_1}(A) \geq \frac{\lambda p_1}{2}. \quad (2.20)$$

By (2.19) and (2.20), we can estimate $p_c - p_1$ as

$$p_c - p_1 = p_c \frac{\hat{\rho}^{(1)}_{p_c} - \hat{h}_{p_c}}{g_{p_1}F^{-1} + 1} F^{-1} + O(\beta^2) = O(\beta)\chi^{-1}_p + O(\beta^2)$$

$$= O(\beta)(p_c - p_1)^{\frac{1}{2}} + O(\beta^2), \quad (2.21)$$

where, for the last inequality, we use $\chi_p \approx (p_c - p_1)^{-1/2}$ for both models in dimensions $d > 8$. Applying the inductive hypothesis $p_c - p_1 = O(\beta^{\ell_j})$ to (2.21), we obtain $p_c - p_1 = O(\beta^{\ell_j + 1})$. Therefore the induction completed. Since $\lim_{j \uparrow \infty} \ell_j = 2$, this proves $p_c - p_1 = O(\beta^2)$, as required.

\[\square\]

**Proof of $\hat{h}_p \leq \frac{1}{4} \hat{\rho}^{(1)}_p + O(\beta^2)$ for lattice animals.** First we recall that

$$\hat{h}_p = \sum_{x \neq o} \sum_{A \in \mathcal{A}_o} \left( \frac{p}{|A|} \right)^{|E_A|} \mathbb{1}_{\{o \leftrightarrow x\}}. \quad (2.22)$$

We split the sum into two depending on whether or not there are distinct vertices $y, z \in V_A$ such that $o \leftrightarrow y, y \leftrightarrow x, o \leftrightarrow z, z \leftrightarrow x$ and $y \leftrightarrow z$ occur in $A$ edge-disjointly, i.e., those connections occur in distinct sets of $E_A$. (We note that, if $y = o$, for example, then we should interpret this as $o \leftrightarrow x, o \leftrightarrow z$ and $z \leftrightarrow x$ occurring edge-disjointly.) Intuitively,

$$\bigcup_{y,z \in V_A \atop (y \neq z)} o \bigcup_{y,z \in V_A \atop (y \neq z)} z \bigcup_{y,z \in V_A \atop (y \neq z)} y \bigcup_{y,z \in V_A \atop (y \neq z)} x. \quad (2.23)$$

Using submultiplicativity and the $x$-space bound in (2.9), we can show that the contribution from this case is $O(\beta^2)$. On the other hand, if there are no such vertices $y, z \in V_A$, i.e.,

$$\bigcup_{y,z \in V_A \atop (y \neq z)} o \bigcup_{y,z \in V_A \atop (y \neq z)} z \bigcup_{y,z \in V_A \atop (y \neq z)} y \bigcup_{y,z \in V_A \atop (y \neq z)} x, \quad (2.24)$$

then there are exactly two edge-disjoint connections between $o$ and $x$, with two pivotal edges from $o$, say $\{o, u\}, \{o, u'\}$, and two from $x$, say $\{v, x\}, \{v', x\}$, one of which may coincide with either $\{o, u\}$ or $\{o, u'\}$, for the double connection $o \leftrightarrow x$ in $A$. Suppose that there is order among vertices in $\Lambda$. If $u \in \Lambda$ is earlier than $u' \in \Lambda$ in this order, we
Figure 3: Schematic representations of the three terms in (2.25). The black line segments are pivotal for $o \iff x$ in $A$. Removal of those edges results in the animals $\{A_j\}_{j=1}^3$ or $\{A_j\}_{j=1}^4$ that are mutually avoiding, as indicated by the red arrows. The vertices in $\Lambda$ are ordered in an arbitrary way (counter-clockwise in the above 2-dimensional figures).

write $u \prec u'$. Let $\Lambda(x) = \{v \in V : v - x \in \Lambda\}$. Then, the contribution to $\hat{h}_p$ from (2.24) is bounded above by (see Figure 3)

$$
\sum_{x \neq o} \sum_{A_2 \in A_x} W_p(A_1) W_p(A_2) \left( \mathbb{1}_{\{x \in \Lambda\}} \frac{p}{|\Lambda|} \sum_{u \in \Lambda} \sum_{v \in \Lambda(x)} \left( \frac{p}{|\Lambda|} \right)^2 \sum_{A_3 \in A_{u,v}} W_p(A_3) \right)
+ \mathbb{1}_{\{x \in \Lambda\}} \frac{p}{|\Lambda|} \sum_{u \in \Lambda} \sum_{v \in \Lambda(x)} \left( \frac{p}{|\Lambda|} \right)^2 \sum_{A_3 \in A_{u,v}} W_p(A_3)
+ \sum_{u,u' \in \Lambda \; v,v' \in \Lambda(x)} \left( \frac{p}{|\Lambda|} \right)^4 \sum_{A_3 \in A_{u,v}} W_p(A_3) W_p(A_4) \prod_{i \neq j} \mathbb{1}_{\{V_{A_i} \cap V_{A_j} = \emptyset\}}. \tag{2.25}
$$

Since $\Lambda$ is symmetric with respect to the underlying lattice symmetry, the first and second terms are the same. Due to the same reason, the third term remains unchanged when the restriction $u \prec u'$ is replaced by $u' \prec u$. Therefore, (2.25) equals

$$
\sum_{x \neq o} \sum_{A_2 \in A_x} W_p(A_1) W_p(A_2) \left( \mathbb{1}_{\{x \in \Lambda\}} \frac{p}{|\Lambda|} \sum_{u \in \Lambda} \sum_{v \in \Lambda(x)} \left( \frac{p}{|\Lambda|} \right)^2 \sum_{A_3 \in A_{u,v}} W_p(A_3) \right)
+ \frac{1}{2} \sum_{u,u' \in \Lambda \; v,v' \in \Lambda(x)} \left( \frac{p}{|\Lambda|} \right)^4 \sum_{A_3 \in A_{u,v}} W_p(A_3) W_p(A_4) \prod_{i \neq j} \mathbb{1}_{\{V_{A_i} \cap V_{A_j} = \emptyset\}}. \tag{2.26}
$$

Now we compare (2.26) with $\hat{\pi}_p^{(1)}$ for lattice animals, which is defined as (see Figure 4)

$$
\hat{\pi}_p^{(1)} = \sum_{x} \sum_{\omega = \{(\omega_j, \omega')\}_{j=1}^{\lfloor \omega \rfloor}} \left( \frac{p}{|\Lambda|} \right)^{\lfloor \omega \rfloor} \prod_{j=0}^{\lfloor \omega \rfloor} W_p(B_j) \mathbb{1}_{\{\omega_j \leftrightarrow \omega_{j+1} \text{ in } B_j\}}
\times \prod_{B_0 \cap B_{\omega} \neq \emptyset} \prod_{0 \leq k \leq \lfloor \omega \rfloor \leq \lfloor l \rfloor \neq (0,0)} \mathbb{1}_{\{B_k \cap B_l \neq \emptyset\}}, \tag{2.27}
$$
Figure 4: Schematic representation of $\pi_p^{(1)}(x)$. The sequence of edges $\omega_1, \ldots, \omega_{|\omega|}$ joined by the animals $B_0, \ldots, B_{|\omega|}$ form the backbone from $o$ to $x$ in $A$. The terminal animals $B_0$ and $B_{|\omega|}$ share a vertex (due to $\mathbb{1}_{B_0 \cap B_{|\omega|} \neq \emptyset}$ in (2.27)), otherwise those animals are mutually avoiding (due to the product of indicators in (2.27)). Each animal $B_j$ contains a double connection between $\omega_j$ and $\omega_{j+1}$.

where we have abused the notation $\omega_0 = o$ and $\omega_{|\omega|+1} = x$. This can be bounded below by restricting the sum over $\omega$ to those satisfying $\omega_1 = o$ and $\omega_{|\omega|} = x$ (so that $A_{\omega_0, \omega_{|\omega|}} = A_o$ and $A_{\omega_1, \omega_{|\omega|+1}} = A_x$) and then by restricting the sum over $B_0 \in A_o$ to $B_0 = \{o\}$ (so that $\mathbb{1}_{\{B_0 \cap B_{|\omega|} \neq \emptyset\}} = \mathbb{1}_{\{o \in B_{|\omega|}\}}$) or restricting the sum over $B_{|\omega|} \in A_x$ to $B_{|\omega|} = \{x\}$ (so that $\mathbb{1}_{\{B_0 \cap B_{|\omega|} \neq \emptyset\}} = \mathbb{1}_{\{x \in B_{|\omega|}\}}$):

$$\hat{\pi}_p^{(1)} \geq 2 \sum_{x \neq o} \sum_{B \in A_{o,x}} W_p(B) \left( \mathbb{1}_{\{(o,x) \notin E_B\}} \mathbb{1}_{\{x \in A\}} \frac{p}{|A|} \right) + \sum_{u \in A} \sum_{v \in A(x)} \left( \frac{p}{|A|} \right)^2 \sum_{A_3 \in A_{u,v}} W_p(A_3) \mathbb{1}_{\{V_B \cap V_{A_3} = \emptyset\}} (2.28)$$

Those two terms are basically the same. Splitting the sum over $\omega$ into two depending on whether $|\omega| = 1$ (so that $\omega = \{\omega_1\}$, where $\omega_1 = (o, x)$) or $|\omega| \geq 2$ and then, for the latter, by summing over the animals $B_1, \ldots, B_{|\omega|-1}$ (to form an animal $A_3 \in A_{\omega_1, \omega_{|\omega|-1}}$), we obtain

$$\hat{\pi}_p^{(1)} \geq 2 \sum_{x \neq o} \sum_{B \in A_{o,x}} W_p(B) \left( \mathbb{1}_{\{(o,x) \notin E_B\}} \mathbb{1}_{\{x \in A\}} \frac{p}{|A|} \right) + \sum_{u \in A} \sum_{v \in A(x)} \left( \frac{p}{|A|} \right)^2 \sum_{A_3 \in A_{u,v}} W_p(A_3) \mathbb{1}_{\{V_B \cap V_{A_3} = \emptyset\}} (2.29)$$

We further bound this below by restricting the sum over $B \in A_{o,x}$ to smaller animals $B = (V_B, E_B)$ with either

(i) $V_B = V_{A_1} \cup V_{A_2}$, $E_B = E_{A_1} \cup \{(o, x)\} \cup E_{A_2}$ for some $A_1 \in A_o$, $A_2 \in A_x$ (as in the left and middle figures of Figure 3), or

(ii) $V_B = V_{A_1} \cup V_{A_2} \cup V_{A_4}$, $E_B = E_{A_1} \cup \{(o, u')\} \cup E_{A_4} \cup \{(v', x)\} \cup E_{A_2}$ for some $A_1 \in A_o$, $A_2 \in A_x$, $u' \in \Lambda$, $v' \in \Lambda(x)$, $A_4 \in A_{u', v'}$ (as in the right figure of Figure 3).

The contribution from (i) to the right-hand side of (2.29) is

$$2 \sum_{x \neq o} \sum_{A_1 \in A_o, A_2 \in A_x} W_p(A_1) W_p(A_2) \mathbb{1}_{\{x \in A\}} \frac{p}{|A|} \sum_{u \in A} \sum_{v \in A(x)} \left( \frac{p}{|A|} \right)^2 \sum_{A_3 \in A_{u,v}} W_p(A_3) \prod_{i \neq j} \mathbb{1}_{\{V_{A_i} \cap V_{A_j} = \emptyset\}} (2.30)$$

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while the contribution from (ii) is

\[ 2 \sum_{x \neq o} \sum_{A_1 \in A_o} W_p(A_1) W_p(A_2) \left( \mathbf{1}_{\{x \in \Lambda\}} \frac{p}{|\Lambda|} \sum_{u \in \Lambda} \sum_{v \in \Lambda(x)} \left( \frac{p}{|\Lambda|} \right)^2 \sum_{A_3 \in A_{u,v}} W_p(A_3) \right. \]

\[ + \sum_{u,u' \in \Lambda} \sum_{v,v' \in \Lambda(x)} \left( \frac{p}{|\Lambda|} \right)^4 \sum_{A_3 \in A_{u,v}} W_p(A_3) W_p(A_4) \prod_{i \neq j} \mathbf{1}_{\{V_{A_i \cap V_{A_j} = \emptyset}\}} \right) \tag{2.31} \]

Notice that the sum of (2.30) and (2.31) is four times as large as (2.26). This completes the proof of $\hat{h}_p \leq \frac{1}{4} \pi_p^{(1)} + O(\beta^2)$. \hfill \blacksquare

3 Detailed analysis of the 1-point function for lattice trees

To complete the proof of Theorem 1.1, it remains to investigate $p_1 = 1/g_{p_1}$ (due to (2.1) and (2.3)). In this section, we concentrate our attention to lattice trees and show the following:

**Lemma 3.1.** For lattice trees with $d > 8$ and $L \uparrow \infty$,

\[ g_{p_1} = e \left( 1 - \sum_{n=2}^{\infty} \frac{n+1}{2e} D^n(o) \right) + O(\beta^2). \tag{3.1} \]

Consequently,

\[ p_1 = \frac{1}{e} + \sum_{n=2}^{\infty} \frac{n+1}{2e} D^n(o) + O(\beta^2). \tag{3.2} \]

To prove Lemma 3.1, we first rewrite $g_{p_1}$ by identifying the connected neighbors $Y$ of the origin as

\[ g_{p_1} = \sum_{T \in \mathcal{T}_o} \left( \frac{p_1}{|\Lambda|} \right)^{|E_T|} \]

\[ = 1 + \sum_{Y \subseteq \Lambda \atop \{|Y| \geq 1\}} \sum_{T \in \mathcal{T}_{\Lambda \cup \{o\}}} \left( \frac{p_1}{|\Lambda|} \right)^{|E_T|} \]

\[ = 1 + \sum_{Y \subseteq \Lambda \atop \{|Y| \geq 1\}} \left( \frac{p_1}{|\Lambda|} \right)^{|Y|} \prod_{y \in Y} \sum_{y \in \mathcal{Y}} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{R_y}|} \prod_{u,v \in Y \atop (u \neq v)} \mathbf{1}_{\{V_{R_u} \cap V_{R_v} = \emptyset\}} \right), \tag{3.3} \]
Figure 5: Intuitive explanation of (3.5). The double-headed arrows on the left (= $g_{p_1}$) represents mutual avoidance among subtrees. In the first term on the right (= $G$), those subtrees are independently summed over $T_y \setminus T_o, y \in Y$, where $Y$ is the set of connected neighbors of the origin. In the second term on the right (= $H$), there is at least one pair of subtrees that share vertices.

where, and from now on, $\sum_{Y \subset \Lambda}$ is the sum over sets $Y$ of distinct vertices of $\Lambda$ (we recall that $o$ is not included in $\Lambda$) and

$$
\prod_{y \in Y} \sum_{R_y \in T_y \setminus T_o} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{R_y}|} = \sum_{R_y \in T_y \setminus T_o} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{R_y}|} \ldots \sum_{R_{y_n} \in T_{y_n} \setminus T_o} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{R_{y_n}}|} 
$$

(3.4)

for $Y = \{y_1, \ldots, y_n\}$. By convention, $\prod_{u \neq v} 1_{\{V_{R_u} \cap V_{R_v} = \emptyset\}}$ is regarded as 1 when $|Y| = 1$.

Let (see Figure 5)

$$
g_{p_1} = G - H,
$$

(3.5)

where

$$
G = 1 + \sum_{Y \subset \Lambda, (|Y| \geq 1)} \left( \frac{p_1}{|\Lambda|} \right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in T_y \setminus T_o} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{R_y}|},
$$

(3.6)

$$
H = \sum_{Y \subset \Lambda, (|Y| \geq 2)} \left( \frac{p_1}{|\Lambda|} \right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in T_y \setminus T_o} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{R_y}|} \left( 1 - \prod_{u \neq v \in Y} 1_{\{V_{R_u} \cap V_{R_v} = \emptyset\}} \right).
$$

(3.7)

We investigate those $G$ and $H$ in Sections 3.1 and 3.2, respectively (cf., Lemmas 3.2 and 3.4 below).

### 3.1 Detailed analysis of $G$

From now on, we frequently use

$$
S_{\geq t}(x) = \sum_{n=t}^{\infty} D^n(x),
$$

(3.8)

where $D^0(x) = \delta_{o,x}$ by convention. The following is what we are going to show in this section:
Lemma 3.2. For lattice trees with $d > 8$ and $L \uparrow \infty$,

$$G = e \left( 1 - \frac{1}{2} D^* \right) + O(\beta^2).$$  \hspace{1cm} (3.9)

Proof. Since $p_1 g_{p_1} = 1$, we can rewrite $G$ as

$$G = 1 + \sum_{Y \subset \Lambda \atop (|Y| \geq 1)} \left( \frac{p_1}{|\Lambda|} \right)^{|Y|} \prod_{y \in Y} \left( \sum_{R_y \in T_y} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{R_y}|} \right) - \sum_{R_y \in T_o} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{R_y}|}$$

$$= 1 + \sum_{Y \subset \Lambda \atop (|Y| \geq 1)} \left( \frac{1}{|\Lambda|} \right)^{|Y|} \prod_{y \in Y} \left( 1 - \frac{\tau_{p_1}(y)}{g_{p_1}} \right).$$  \hspace{1cm} (3.10)

If we replace $\prod_{y \in Y} (1 - \tau_{p_1}(y)/g_{p_1})$ by 1, then we obtain

$$G_0 := 1 + \sum_{Y \subset \Lambda \atop (|Y| \geq 1)} \left( \frac{1}{|\Lambda|} \right)^{|Y|} = \left( 1 + \frac{1}{|\Lambda|} \right)^{|\Lambda|}.$$  \hspace{1cm} (3.11)

Since $k \log(1 + 1/k) = 1 - 1/(2k) + O(k^{-2})$ as $k \uparrow \infty$, and since $|\Lambda|^{-1} = D^*(o)$, we obtain

$$G_0 = e \left( 1 - \frac{1}{2|\Lambda|} \right) + O(|\Lambda|^{-2}) = e \left( 1 - \frac{1}{2} D^*(o) \right) + O(\beta^2).$$  \hspace{1cm} (3.12)

Next we consider the remainder, which is

$$G - G_0 = \sum_{Y \subset \Lambda \atop (|Y| \geq 1)} \left( \frac{1}{|\Lambda|} \right)^{|Y|} \left( \prod_{y \in Y} \left( 1 - \frac{\tau_{p_1}(y)}{g_{p_1}} \right) - 1 \right)$$

$$= \sum_{Y \subset \Lambda \atop (|Y| \geq 1)} \left( \frac{1}{|\Lambda|} \right)^{|Y|} \sum_{Z \subset Y \subset \Lambda \atop (|Z| \geq 1)} \prod_{y \in Z} \frac{-\tau_{p_1}(y)}{g_{p_1}}.$$  \hspace{1cm} (3.13)

Changing the order of sums yields

$$G - G_0 = \sum_{Z \subset \Lambda \atop (|Z| \geq 1)} \prod_{y \in Z} \frac{-\tau_{p_1}(y)}{g_{p_1}} \left( \frac{1}{|\Lambda|} \right)^{|Z|} \sum_{Z \subset Y \subset \Lambda \atop (|Y| \geq 1)} \left( \frac{1}{|\Lambda|} \right)^{|Y \setminus Z|}$$

$$= \sum_{Z \subset \Lambda \atop (|Z| \geq 1)} \prod_{y \in Z} \frac{-\tau_{p_1}(y)}{g_{p_1}} \left( \frac{1}{|\Lambda|} \right)^{|Z|} \sum_{Z \subset Y \subset \Lambda \atop (|Y| \geq 1)} \left( \frac{1}{|\Lambda|} \right)^{|Y \setminus Z|}$$

$$= G_1 + G_2.$$  \hspace{1cm} (3.14)
where

\[
G_1 = \sum_{Z \subseteq \Lambda, |Z| = 1} \prod_{y \in Z} \frac{-\tau_{p_1}(y) D(y)}{g_{p_1}} \left(1 + \frac{1}{|\Lambda|}\right)^{|\Lambda \setminus Z|} = \frac{-(\tau_{p_1} * D)(o)}{g_{p_1}} \frac{G_0}{(1 + 1/|\Lambda|)}, \tag{3.15}
\]

\[
G_2 = \sum_{Z \subseteq \Lambda, |Z| \geq 2} \prod_{y \in Z} \frac{-\tau_{p_1}(y) D(y)}{g_{p_1}} \left(1 + \frac{1}{|\Lambda|}\right)^{|\Lambda \setminus Z|}. \tag{3.16}
\]

We make use of reflection symmetry for (3.14)–(3.16), and will use it below frequently. To estimate \(G_1\) and \(G_2\), we use the following lemma, which will be proven after the proof of Lemma 3.2 is completed.

**Lemma 3.3.** For any \(d > 2\) and \(x \neq o\), the lattice-tree 2-point function satisfies

\[
0 \leq S_{\geq 1}(x) - \frac{\tau_{p_1}(x)}{g_{p_1}} \leq \sum_{y, z \in \mathbb{Z}^d} S^{*2}_{\geq 0}(z - y) S_{\geq 0}(y) S_{\geq 1}(z - y) S_{\geq 0}(x - z). \tag{3.17}
\]

**Remark.** The right-hand side of (3.17) is diagrammatically represented by

\[
\sum_{y, z \in \mathbb{Z}^d} S^{*2}_{\geq 0}(z - y) S_{\geq 0}(y) S_{\geq 1}(z - y) S_{\geq 0}(x - z) = \begin{array}{c}
\triangle
\end{array}_{x}, \tag{3.18}
\]

where an unslashed (resp., slashed) line represents \(S_{\geq 0}\) (resp., \(S_{\geq 1}\)) and an unlabelled vertex is summed over \(\mathbb{Z}^d\). Due to translation invariance, we can change the order of terms in (3.17) for a given \(x \neq o\). Then (3.18) is also equal to

\[
\sum_{y \in \mathbb{Z}^d} S^{*2}_{\geq 0}(x - y) S_{\geq 0}(x - y) S_{\geq 1}(x - y) = \begin{array}{c}
\triangle
\end{array}_{x}. \tag{3.19}
\]

These diagramatic representations will be used in the proof of Lemma 3.4 below.

First we estimate \(G_2\). By the first inequality in (3.17) and the heat-kernel bound (see, e.g., [3, (1.6)]):

\[
\|D^n\|_\infty = O(\beta) n^{-d/2} \quad [n \in \mathbb{N}], \tag{3.20}
\]

we can show

\[
\frac{(\tau_{p_1} * D)(o)}{g_{p_1}} \leq \sup_{x \in \Lambda} \frac{\tau_{p_1}(x)}{g_{p_1}} \leq \sup_{x \in \Lambda} S_{\geq 1}(x) \leq \sum_{n=1}^{\infty} \|D^n\|_\infty d^{\geq 2} O(\beta). \tag{3.21}
\]
Therefore,

$$|G_2| \leq G_0 \sum_{n=2}^{[\Lambda]} \sum_{Z \subseteq \Lambda \atop (|Z|=n)} \prod_{y \in Z} \frac{\tau_{p_1}(y) D(y)}{g_{p_1}} \leq G_0 \sum_{n=2}^{\infty} \left( \frac{(\tau_{p_1} \ast D)(o)}{g_{p_1}} \right)^n = O(\beta^2). \quad (3.22)$$

Next we estimate $G_1$ in (3.15). By using (3.17), we have

$$0 \leq S_{\geq 2}(o) - \frac{(\tau_{p_1} \ast D)(o)}{g_{p_1}} = \sum_{x \in \mathbb{Z}^d} D(x) \left( S_{\geq 1}(x) - \frac{\tau_{p_1}(x)}{g_{p_1}} \right) \leq \sum_{x \in \mathbb{Z}^d} D(x) \sum_{y,z \in \mathbb{Z}^d \atop (y \neq z)} S_{\geq 0}^2(z-y) S_{\geq 0}(y) S_{\geq 1}(z-y) S_{\geq 0}(x-z) \leq \sum_{y,z \in \mathbb{Z}^d \atop (y \neq z)} S_{\geq 0}^2(z-y) S_{\geq 0}(y) S_{\geq 1}(z-y) S_{\geq 1}(z) \leq \sup_{w \neq o} S_{\geq 0}^2(w) \left( S_{\geq 0} \ast S_{\geq 1}^2 \right)(o). \quad (3.23)$$

By the heat-kernel bound (3.20), we can estimate each term as

$$S_{\geq 0}^2(w) = \sum_{s,t=0}^{\infty} D^{*(s+t)}(w) \leq \sum_{n=1}^{\infty} (n+1) D^{*n}(w) \overset{d \geq 4}{=} O(\beta), \quad (3.24)$$

$$(S_{\geq 0} \ast S_{\geq 1}^2)(o) = \sum_{s=0}^{\infty} \sum_{t,u=1}^{\infty} D^{*(s+t+u)}(o) = \sum_{n=2}^{\infty} \left( \frac{n}{2} \right) D^{*n}(o) \overset{d \geq 6}{=} O(\beta), \quad (3.25)$$

so that

$$0 \leq S_{\geq 2}(o) - \frac{(\tau_{p_1} \ast D)(o)}{g_{p_1}} = O(\beta^2). \quad (3.26)$$

Therefore, by (3.12),

$$G_1 = \left( - S_{\geq 2}(o) + O(\beta^2) \right) \left( e - \frac{e}{2} D^{*2}(o) + O(\beta^2) \right) \overset{\text{(3.21)}}{=} O(\beta) \overset{\text{(3.20)}}{=} O(\beta^2) \overset{\text{(3.28)}}{=} -eS_{\geq 2}(o) + O(\beta^2). \quad (3.27)$$

Summarizing (3.12), (3.14), (3.22) and (3.27), we complete the proof of Lemma 3.2.

Proof of Lemma 3.3. First we recall

$$\frac{\tau_{p_1}(x)}{g_{p_1}} = \frac{1}{g_{p_1}} \sum_{T \in \mathcal{T}_o , \Lambda} \left( \frac{p_1}{|\Lambda|} \right)^{|E_T|}. \quad (3.28)$$

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Since a tree \( T \in \mathcal{T}_{0,x} \) can be divided into a unique path \( \omega = (\omega_0, \omega_1, \ldots, \omega_{|\omega|}) \) from \( \omega_0 = 0 \) to \( \omega_{|\omega|} = x \), called a backbone, and disjoint subtrees \( R_j \in \mathcal{T}_{\omega_j} \), called ribs (see Figure 2), we can rewrite the above expression as

\[
\frac{\tau_{p_1}(x)}{g_{p_1}} = \frac{1}{g_{p_1}} \sum_{\omega:o \to x} \left( \frac{p_1}{|\Lambda|} \right)^{|\omega|} \prod_{j=0}^{R_j \in \mathcal{T}_{\omega_j}} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{R_j}|} \prod_{s<t}^{1\{V_{R_s} \cap V_{R_t} = \emptyset\}}. \quad (3.29)
\]

If we replace the indicator \( \prod_{s<t}^{1\{V_{R_s} \cap V_{R_t} = \emptyset\}} \) by 1, then we obtain

\[
\frac{1}{g_{p_1}} \sum_{\omega:o \to x} \left( \frac{p_1}{|\Lambda|} \right)^{|\omega|} \prod_{j=0}^{R_j \in \mathcal{T}_{\omega_j}} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{R_j}|} \leq \sum_{\omega:o \to x} \left( \frac{1}{|\Lambda|} \right)^{|\omega|} \prod_{s<t}^{1\{V_{R_s} \cap V_{R_t} = \emptyset\}}. \quad (3.30)
\]

Next we consider the remainder. Since \( 1 - \prod_{j=1}^{n} (1-a_j) \) as long as \( 0 \leq a_j \leq 1 \) for all \( j \), we can bound the remainder as

\[
S_{\geq 1}(x) = \frac{\tau_{p_1}(x)}{g_{p_1}} \leq \frac{1}{g_{p_1}} \sum_{\omega:o \to x} \left( \frac{p_1}{|\Lambda|} \right)^{|\omega|} \prod_{s<t}^{1\{V_{R_s} \cap V_{R_t} = \emptyset\}} \left( 1 - \prod_{s<t}^{1\{V_{R_s} \cap V_{R_t} = \emptyset\}} \right) \leq \sum_{\omega:o \to x} \left( \frac{1}{|\Lambda|} \right)^{|\omega|} \prod_{s<t}^{1\{V_{R_s} \cap V_{R_t} = \emptyset\}}. \quad (3.31)
\]

If \( V_{R_s} \cap V_{R_t} \neq \emptyset \), then there must be a \( w \in \mathbb{Z}^d \) that is shared by those two ribs. Therefore, the remainder is further bounded above as

\[
S_{\geq 1}(x) = \frac{\tau_{p_1}(x)}{g_{p_1}} \leq \frac{1}{g_{p_1}} \sum_{\omega:o \to x} \left( \frac{p_1}{|\Lambda|} \right)^{|\omega|} \prod_{s<t}^{1\{V_{R_s} \cap V_{R_t} = \emptyset\}} \left( 1 - \prod_{s<t}^{1\{V_{R_s} \cap V_{R_t} = \emptyset\}} \right) \times \sum_{w \in \mathbb{Z}^d} \sum_{R_s \in \mathcal{T}_{\omega_s, w}} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{R_s}|} \sum_{R_t \in \mathcal{T}_{\omega_t, w}} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{R_t}|}.
\]

\[
= \frac{1}{g_{p_1}^2} \sum_{\omega:o \to x} \left( \frac{1}{|\Lambda|} \right)^{|\omega|} \sum_{w \in \mathbb{Z}^d} \tau_{p_1}(w-\omega_s) \tau_{p_1}(\omega_t-w) \sum_{s<t}^{1\{V_{R_s} \cap V_{R_t} = \emptyset\}} \left( \frac{1}{|\Lambda|} \right)^{|\omega|}
\]

\[
= \sum_{y,z \in \mathbb{Z}^d} \tau_{p_1}(y-z) \frac{1}{g_{p_1}^2} \sum_{\omega:o \to x} \left( \frac{1}{|\Lambda|} \right)^{|\omega|} \sum_{z \in \mathbb{Z}^d} \tau_{p_1}(y-z) \frac{1}{g_{p_1}^2} \sum_{\omega:o \to x} \left( \frac{1}{|\Lambda|} \right)^{|\omega|}
\]

\[
= \sum_{y,z \in \mathbb{Z}^d} \tau_{p_1}(y-z) \frac{1}{g_{p_1}^2} S_{\geq 0}(y) S_{\geq 1}(z-y) S_{\geq 0}(x-z). \quad (3.32)
\]

The proof of (3.17) is completed by applying (3.29)–(3.30) to \( (\tau_{p_1}/g_{p_1})^2 \) in the above bound. \( \blacksquare \)
3.2 Detailed analysis of $H$

To complete the proof of Lemma 3.1, it suffices to show the following:

**Lemma 3.4.** For lattice trees with $d > 8$ and $L \uparrow \infty$,

$$H = e \sum_{n=3}^{\infty} \frac{n-1}{2} D^n(o) + O(\beta^2).$$

(3.33)

**Proof.** Recall the definition (3.7) of $H$:

$$H = \sum_{Y \subset \Lambda} \left( \frac{p_1}{|\Lambda|} \right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in T_y \setminus T_o} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{R_y}|} \left( 1 - \prod_{u,v \in Y \atop u \neq v} \mathbb{1}_{\{V_{R_u \cap V_{R_v} = \emptyset}\}} \right).$$

(3.34)

First we split the indicator $1 - \prod_{u \neq v} \mathbb{1}_{\{V_{R_u \cap V_{R_v} = \emptyset}\}}$ in (3.7) by introducing order among pairs of distinct vertices in $\Lambda$, called bonds. If a bond $b$ is earlier than another bond $b'$ in that order, we denote it by $b < b'$. Then we have

$$1 - \prod_{\{u,v\} \subset Y} \mathbb{1}_{\{V_{R_u \cap V_{R_v} = \emptyset}\}} = \sum_{\{u,v\} \subset Y} \prod_{\{u',v'\} \subset Y \atop \{u',v'\} < \{u,v\}} \mathbb{1}_{\{V_{R_{u'} \cap V_{R_{v'}} = \emptyset}\}} - \sum_{\{u,v\} \subset Y} \mathbb{1}_{\{V_{R_u \cap V_{R_v} = \emptyset}\}} \left( 1 - \prod_{\{u',v'\} \subset Y \atop \{u',v'\} < \{u,v\}} \mathbb{1}_{\{V_{R_{u'} \cap V_{R_{v'}} = \emptyset}\}} \right),$$

(3.35)

where the second sum on the right is zero when $|Y| = 2$. Let $H_1$ be the contribution from the first sum on the right:

$$H_1 = \sum_{Y \subset \Lambda} \left( \frac{p_1}{|\Lambda|} \right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in T_y \setminus T_o} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{R_y}|} \sum_{\{u,v\} \subset Y} \mathbb{1}_{\{V_{R_u \cap V_{R_v} = \emptyset}\}}.$$

(3.36)

We will later show (after the derivation of (3.33); see (3.54)) that

$$H_2 := H_1 - H = O(\beta^2).$$

(3.37)

Next we investigate $H_1$. Let $H'_1$ be the contribution from the case of $|Y| = 2$:

$$H'_1 = \sum_{\{u,v\} \subset \Lambda} \left( \frac{p_1}{|\Lambda|} \right)^2 \sum_{R_u \in T_u \setminus T_o} \sum_{R_v \in T_v \setminus T_o} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{R_u}| + |E_{R_v}|} \mathbb{1}_{\{V_{R_u \cap V_{R_v} = \emptyset}\}}.$$

(3.38)
By subadditivity, we already know that $H'_1 = O(\beta)$ for $d > 4$. By changing the order of sums, we can rewrite $H_1 - H'_1$ as

$$H_1 - H'_1 = \sum_{Y \subseteq \Lambda \atop |Y| \geq 2} (\frac{p_1}{|\Lambda|})^{|Y|} \prod_{y \in Y} \sum_{R_y \in T_y \setminus T_0} (\frac{p_1}{|\Lambda|})^{|E_{R_y}|} \sum_{\{u,v\} \subseteq Y} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}}$$

$$= \sum_{\{u,v\} \subseteq \Lambda} (\frac{p_1}{|\Lambda|})^{2} \sum_{R_u \in T_u \setminus T_0} (\frac{p_1}{|\Lambda|})^{|E_{R_u}| + |E_{R_v}|} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}}$$

$$\times \sum_{Y' \subseteq \Lambda \setminus \{u,v\} \atop |Y'| \geq 1} (\frac{p_1}{|\Lambda|})^{|Y'|} \prod_{y' \in Y'} \sum_{R_y' \in T_{y'} \setminus T_0} (\frac{p_1}{|\Lambda|})^{|E_{R_y'}|} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}}.$$  

(3.39)

Similarly to the proof of Lemma 3.2, we can show that the last line is estimated as

$$\sum_{Y' \subseteq \Lambda \setminus \{u,v\} \atop |Y'| \geq 1} \left( \frac{1}{|\Lambda|} \right)^{|Y'|} \prod_{y' \in Y'} \left( 1 - \frac{\tau_{p_1}(y')}{g_{p_1}} \right)$$

$$= \left( 1 + \frac{1}{|\Lambda|} \right)^{|\Lambda| - 2} - 1 + \sum_{Y' \subseteq \Lambda \setminus \{u,v\} \atop |Y'| \geq 1} \left( \frac{1}{|\Lambda|} \right)^{|Y'|} \left( \prod_{y' \in Y'} \left( 1 - \frac{\tau_{p_1}(y')}{g_{p_1}} \right) \right) - 1$$

$$= e - 1 + O(\beta).$$  

(3.40)

Therefore,

$$H_1 - H'_1 = H'_1(e - 1 + O(\beta)), $$

(3.41)

or equivalently

$$H_1 = eH'_1 + O(\beta^2).$$

(3.42)

Next we investigate $H'_1$. To do so, we first rewrite $\mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}}$ in (3.38) by introducing order among vertices in $\mathbb{Z}^d$. For a vertex set $V$ and an element $x \in V$, we denote by $V^{<x}$ the set of vertices in $V$ that are earlier than $x$ in that order. By identifying the earliest element $x$ among $V_{R_u}$ that is also in $V_{R_v}$ (so that $V^{<x}_{R_u} \cap V_{R_v} = \emptyset$), we can rewrite $\mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}}$ as

$$\mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}} = \sum_{x \in V_{R_u}} \mathbb{1}_{\{x \in V_{R_v}\}} \mathbb{1}_{\{V^{<x}_{R_u} \cap V_{R_v} = \emptyset\}}$$

$$= \sum_{x \in \mathbb{Z}^d} \mathbb{1}_{\{x \in V_{R_u} \cap V_{R_v}\}} - \sum_{x \in V_{R_u} \cap V_{R_v}} \left( 1 - \mathbb{1}_{\{V^{<x}_{R_u} \cap V_{R_v} = \emptyset\}} \right).$$  

(3.43)
Let $H''_1$ be the contribution from the first sum in the last line:

\[
H''_1 = \sum_{\{u,v\} \subset \Lambda} \left( \frac{p_1}{|\Lambda|} \right)^2 \sum_{x \in \mathbb{Z}^d} \sum_{R_u \in \mathbb{T}_{o,u} \setminus \mathbb{T}_o, R_v \in \mathbb{T}_{o,v} \setminus \mathbb{T}_o} \left( \frac{p_1}{|E_{R_u} \cup E_{R_v}|} \right)
\]

\[
= \sum_{\{u,v\} \subset \Lambda} \left( \frac{p_1}{|\Lambda|} \right)^2 \sum_{x \in \mathbb{Z}^d} \left( \tau_{p_1}(u - x) - \tau_{p_1}^{(3)}(o, u, x) \right) \left( \tau_{p_1}(v - x) - \tau_{p_1}^{(3)}(o, v, x) \right)
\]

\[
(2.1) \quad \sum_{\{u,v\} \subset \Lambda} \left( \frac{1}{|\Lambda|} \right)^2 \sum_{x \in \mathbb{Z}^d} \left( \frac{\tau_{p_1}(u - x)}{g_{p_1}} - \frac{\tau_{p_1}^{(3)}(o, u, x)}{g_{p_1}} \right) \left( \frac{\tau_{p_1}(v - x)}{g_{p_1}} - \frac{\tau_{p_1}^{(3)}(o, v, x)}{g_{p_1}} \right), \quad (3.44)
\]

where $\tau_{p}^{(3)}(o, u, x)$ is a 3-point function, defined as

\[
\tau_{p}^{(3)}(o, u, x) = \sum_{T \in \mathbb{T}_{o,u,v}} W_p(T). \quad (3.45)
\]

We will later show that

\[
H''_2 := H''_1 - H'_1 = O(\beta^2). \quad (3.46)
\]

Finally we investigate $H''_1$. The dominant contribution to $H''_1$ comes from the product of 2-point functions:

\[
H''_1 := \sum_{\{u,v\} \subset \Lambda} \left( \frac{1}{|\Lambda|} \right)^2 \sum_{x \in \mathbb{Z}^d} \tau_{p_1}(u - x) \tau_{p_1}(v - x)
\]

\[
= 2 \sum_{\{u,v\} \subset \Lambda} \left( \frac{1}{|\Lambda|} \right)^2 \tau_{p_1}(u - v) + \sum_{\{u,v\} \subset \Lambda} \left( \frac{1}{|\Lambda|} \right)^2 \sum_{x \neq u, v} \tau_{p_1}(u - x) \tau_{p_1}(v - x), \quad (3.47)
\]

where we have used the identity $\tau_{p}(o) = g_{p}$. We will later show that the other contribution to $H''_1$ which involves 3-point functions is estimated as

\[
H''_2 := H''_1 - H''_1 = O(\beta^2). \quad (3.48)
\]

By Lemma 3.3, the first term in (3.47) is estimated as

\[
2 \sum_{\{u,v\} \subset \Lambda} \left( \frac{1}{|\Lambda|} \right)^2 \tau_{p_1}(u - v)
\]

\[
= \sum_{u,v \in \Lambda \atop u \neq v} \left( \frac{1}{|\Lambda|} \right)^2 S_{\geq 1}(u - v) + \sum_{u,v \in \Lambda \atop u \neq v} \left( \frac{1}{|\Lambda|} \right)^2 \left( \tau_{p_1}(u - v) \frac{1}{g_{p_1}} - S_{\geq 1}(u - v) \right)
\]

\[
= S_{\geq 1}(o) + \sum_{u,v \in \Lambda \atop u \neq v} \left( \frac{1}{|\Lambda|} \right)^2 \left( \tau_{p_1}(u - v) \frac{1}{g_{p_1}} - S_{\geq 1}(u - v) \right), \quad (3.49)
\]

\[
\leq \left(1 - \frac{a}{d} \right)^2 \quad (3.17) - (3.18)
\]

\[
\leq \quad (3.17) - (3.18)
\]

\[
\leq \quad (3.17) - (3.18)
\]
where a gap next to the origin in the last diagram represents $1/|A|$. By translation invariance and (3.24)-(3.25), the last term is bounded above by

\[ \sum_{y \in \mathbb{Z}^d, y \neq o}^a S_{\geq 1}^2 = \sum_{y \in \mathbb{Z}^d, y \neq o}^a \sum_{y \in \mathbb{Z}^d, y \neq o}^a S_{\geq 1}^2 \leq ||S_{\geq 1}^2||_\infty (S_{\geq 0}^2 * S_{\geq 1})(o) = O(\beta^2). \] (3.50)

Similarly, the second term in (3.47) is estimated as

\[ \sum_{\{u,v\} \subseteq \Lambda} \left( \frac{1}{|A|} \right)^2 \sum_{x \neq u,v} \tau_{p_1}(u-x) \tau_{p_1}(v-x) \frac{g_{p_1}}{g_{p_1}} \leq ||S_{\geq 1}^2||_\infty \]

\[ \frac{1}{2} S_{\geq 2}^2(o) - \frac{1}{2 |A|} S_{\geq 1}^2(o) \]

\[ + \frac{1}{2} \sum_{u,v \in \Lambda, u \neq v} \left( \frac{1}{|A|} \right)^2 \sum_{x \neq u,v} \left[ \tau_{p_1}(u-x) \tau_{p_1}(v-x) \frac{g_{p_1}}{g_{p_1}} \right] - S_{\geq 1}(u-x)S_{\geq 1}(v-x) \]

\[ \leq \frac{2}{2} \sum_{v}^a \sum_{x}^a + \frac{2}{2} \sum_{y,z}^a \sum_{x}^a \leq ||S_{\geq 0}^2 * S_{\geq 1}^2||_\infty \]

By Lemma 3.3, (3.19) and the translation invariance, the last term of (3.51) is bounded above by

\[ \frac{1}{2} \sum_{y,z}^a \sum_{x}^a \leq ||S_{\geq 0}^2 * S_{\geq 1}^2||_\infty \]

\[ \leq \frac{1}{2} ||S_{\geq 0}^2 * S_{\geq 1}^2||_\infty \]

\[ + \frac{1}{2} \sum_{y,z}^a \sum_{x}^a \]

\[ \leq ||S_{\geq 0}^2 * S_{\geq 1}^2||_\infty \]

\[ + \frac{1}{2} ||S_{\geq 0}^2 * S_{\geq 1}^2||_\infty (S_{\geq 0}^2 * S_{\geq 1})(o)^2 \]

\[ = O(\beta^2) \] (3.52)

Therefore,

\[ H''_1 = S_{\geq 3}(o) + \frac{1}{2} S_{\geq 2}^2(o) + O(\beta^2) = \sum_{n=3}^{\infty} D^n(o) + \frac{1}{2} \sum_{n,m=2}^{\infty} D^{n+m}(o) + O(\beta^2) \]

\[ = \sum_{n=3}^{\infty} \frac{n-1}{2} D^n(o) + O(\beta^2). \] (3.53)

Summarizing all the above estimates, we arrive at

\[ H \overset{(3.37)}{=} H + O(\beta^2) \overset{(3.42)}{=} eH + O(\beta^2) \overset{(3.46)}{=} eH'' + O(\beta^2) \overset{(3.48)}{=} eH'' + O(\beta^2) \]

\[ \overset{(3.53)}{=} e \sum_{n=3}^{\infty} \frac{n-1}{2} D^n(o) + O(\beta^2), \] (3.54)

as required. It remains to show (3.37), (3.46) and (3.48).

\[ \blacksquare \]
Proof of (3.37): bounding $H_2$. First we recall that $H_2$ is the contribution from the second sum on the right of (3.35):

$$H_2 = \sum_{Y \subseteq A \atop |Y| \geq 3} \left( \frac{p_1}{|A|} \right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in T_y \setminus \mathcal{T}_0} \left( \frac{p_1}{|A|} \right)^{|E_{R_y}|} \sum_{\{u,v\} \subseteq Y} \mathbb{I}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}}$$

which is nonnegative. Since we get an upper bound in a same manner as (3.35), we can bound $H_2$ as

$$1 - \prod_{\{u',v'\} \subseteq Y \atop \{u',v'\} \neq \{u,v\}} \mathbb{I}_{\{V_{R_{u'}} \cap V_{R_{v'}} = \emptyset\}} \leq \sum_{\{u',v'\} \subseteq Y \atop \{u',v'\} \neq \{u,v\}} \mathbb{I}_{\{V_{R_{u'}} \cap V_{R_{v'}} \neq \emptyset\}}$$

(3.56)

in a same manner as (3.35), we can bound $H_2$ as

$$H_2 \leq \sum_{Y \subseteq A \atop |Y| \geq 3} \left( \frac{p_1}{|A|} \right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in T_y \setminus \mathcal{T}_0} \left( \frac{p_1}{|A|} \right)^{|E_{R_y}|} \sum_{\{u,v\} \subseteq Y \atop \{u',v'\} \subseteq Y \atop \{u',v'\} \neq \{u,v\}} \mathbb{I}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}} \mathbb{I}_{\{V_{R_{u'}} \cap V_{R_{v'}} \neq \emptyset\}}$$

$$= \frac{1}{2} \sum_{Y \subseteq A \atop |Y| \geq 3} \left( \frac{p_1}{|A|} \right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in T_y \setminus \mathcal{T}_0} \left( \frac{p_1}{|A|} \right)^{|E_{R_y}|} \sum_{\{u,v\} \subseteq Y \atop \{u',v'\} \subseteq Y \atop \{u',v'\} \neq \{u,v\}} \mathbb{I}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}} \mathbb{I}_{\{V_{R_{u'}} \cap V_{R_{v'}} \neq \emptyset\}}.$$

(3.57)

Since $\{u,v\} \neq \{u',v'\}$, the union $\{u,v\} \cup \{u',v'\}$ consists of either three or four distinct vertices. We denote the contribution from the former by $H_{2,3}$, and that from the latter by $H_{2,4}$ and then we obtain

$$H_2 \leq \frac{1}{2} (H_{2,3} + H_{2,4}).$$

(3.58)

First we investigate $H_{2,4}$, which is bounded as (see Figure 6)

$$H_{2,4} = \sum_{Y \subseteq A \atop |Y| \geq 4} \left( \frac{p_1}{|A|} \right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in T_y \setminus \mathcal{T}_0} \left( \frac{p_1}{|A|} \right)^{|E_{R_y}|} \sum_{\{u,v,u',v'\} \in Y \atop \{u,v,u',v'\} \neq \{u,v\}} \mathbb{I}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}} \mathbb{I}_{\{V_{R_{u'}} \cap V_{R_{v'}} \neq \emptyset\}}$$

$$= \sum_{u,v,u',v' \in A \atop \text{(distinct)}} \left( \frac{p_1}{|A|} \right)^{4} \sum_{R_u \in T_u \setminus \mathcal{T}_0} \left( \frac{p_1}{|A|} \right)^{|E_{R_u}|} \mathbb{I}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}} \mathbb{I}_{\{V_{R_{u'}} \cap V_{R_{v'}} \neq \emptyset\}}$$

$$\times \left( 1 + \sum_{Y' \subseteq A \setminus \{u,v,u',v'\} \atop |Y'| \geq 1} \left( \frac{p_1}{|A|} \right)^{|Y'|} \prod_{y' \in Y'} \sum_{R_{y'} \in T_{y'} \setminus \mathcal{T}_0} \left( \frac{p_1}{|A|} \right)^{|E_{R_{y'}}|} \right).$$

(3.59)
By (2.1) (i.e., $p_1 g_{p_1} = 1$), the last line is equal to

$$1 + \sum_{Y' \subseteq \Lambda \setminus \{u,v,u',v'\} \mid |Y'| \geq 1} \left( \frac{1}{|\Lambda|} \right)^{|Y'|} = \left( 1 + \frac{1}{|\Lambda|} \right)^{|\Lambda| - 4} = e + O(\beta).$$

(3.60)

Then, by ignoring the constraint that $\{u, v\}$ and $\{u', v'\}$ are disjoint pairs and using the trivial inequality $1 \{V_R \cap \Lambda \neq \emptyset\} \leq \sum_{x \in \Lambda} \{x \in V_R \cap \Lambda\}$ as well as the relation $T_{u,x} \setminus T_o \subseteq T_{u,x}$, $H_{2,4}$ is further bounded above as

$$H_{2,4} \leq (e + O(\beta)) \left( \sum_{(u,v) \subseteq \Lambda} \left( \frac{p_1}{|\Lambda|} \right)^2 \sum_{R_u \in T_{u,x} \setminus T_o} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{R_u}| + |E_{R_v}|} \right)^2 \leq (e + O(\beta)) \left( \sum_{(u,v) \subseteq \Lambda} \left( \frac{1}{|\Lambda|} \right)^2 \sum_{x \in \mathbb{Z}^d} \tau_{p_1} (u - x) \tau_{p_1} (v - x) \right)^2 \leq (e + O(\beta)) \left( \sum_{(u,v) \subseteq \Lambda} \left( \frac{1}{|\Lambda|} \right)^2 S_{\geq 0}^2 (u - v) \right)^2 \leq (e + O(\beta)) S_{\geq 1}^2 (o)^2 \overset{2.1}{=} O(\beta^2).$$

(3.61)

Finally, by using $\tau_{p_1} (u - x) / g_{p_1} \leq S_{\geq 1} (u - x)$ for $x \neq u$ (cf., the first inequality in (3.17)) and $\tau_p (o) = g_p$ for $x = u$, we arrive at

$$H_{2,4} \leq (e + O(\beta)) \left( \sum_{u,v \in \Lambda} \left( \frac{1}{|\Lambda|} \right)^2 S_{\geq 0}^2 (u - v) \right)^2 \leq (e + O(\beta)) S_{\geq 1}^2 (o)^2 \overset{d \geq 4}{=} O(\beta^2).$$

(3.62)

Next we investigate $H_{2,3}$, which is bounded in a similar way to (3.61) as (see Figure 7)
Due to submultiplicativity, we can bound the sum in (3.63) as

$$H_{2,3} = \sum_{Y \subseteq \Lambda \atop |Y| \geq 3} \left( \frac{p_1}{|\Lambda|} \right)^{|Y|} \prod_{y \in Y} \sum_{R_0 \in T_y \setminus T_o} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{R_0}|} \sum_{u,v,v' \in Y \atop \text{(distinct)}} \mathbb{1}_{\{V_{R_0} \cap V_{R_0'} \neq \emptyset\}} \mathbb{1}_{\{V_{R_0} \cap V_{R_0'} \neq \emptyset\}}$$

$$\leq \left( e + O(\beta) \right) \sum_{u,v,v' \in \Lambda \atop \text{(distinct)}} \left( \frac{p_1}{|\Lambda|} \right)^3 \sum_{x,x' \in \mathbb{Z}^d} \sum_{R_0 \in T_{u,x,x'}} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{R_0}|} \tau_p^{(3)}(u, x, x') \tau_p^{(3)}(v - x) \tau_p^{(3)}(v' - x').$$

Figure 7: Schematic representation of $H_{2,3}$.

$$\tau_p^{(3)}(u, x, x') \leq \sum_{y \in \mathbb{Z}^d} \tau_p(u - y) \tau_p(x - y) \tau_p(x' - y).$$

Then, by using $\tau_p(u - x)/g_{p_1} \leq S_{\geq 1}(u - x)$ for $x \neq u$ and $\tau_p(o) = g_p$ for $x = u$, we can bound the sum in (3.63) as

$$\sum_{u,v,v' \in \Lambda \atop \text{(distinct)}} \left( \frac{1}{|\Lambda|} \right)^3 \sum_{y,x,x' \in \mathbb{Z}^d} \frac{\tau_p(u - y) \tau_p(x - y) \tau_p(x' - y) \tau_p(v - x) \tau_p(v' - x')}{g_{p_1} g_{p_1} g_{p_1} g_{p_1} g_{p_1}}$$

$$\leq \sum_{y,x,x' \in \mathbb{Z}^d} S_{\geq 0}(x - y) S_{\geq 0}(x' - y) \sum_{u,v,v' \in \Lambda \atop \text{(distinct)}} \left( \frac{1}{|\Lambda|} \right)^3 S_{\geq 0}(u - y) S_{\geq 0}(v - x) S_{\geq 0}(v' - x')$$

$$= \sum_{y \in \mathbb{Z}^d} (S_{\geq 0} * S_{\geq 1})(y)^2 S_{\geq 1}(y)$$

$$\leq \|S_{\geq 0} * S_{\geq 1}\|_{\infty} (S_{\geq 0} * S_{\geq 1})(o)^{\frac{d+6}{\delta}} = O(\beta^2).$$

(3.65)
Figure 8: Schematic representation of the bound on $H''_2$ due to (3.68).

This together with (3.58) and (3.62) implies

$$H_2 \leq \frac{1}{2}(H_{2,3} + H_{2,4}) = O(\beta^2),$$

(3.66)
as required.

**Proof of (3.46): bounding $H''_2$.** First we recall that $H''_2$ is the contribution to $H'_1$ from the second sum on the right of (3.43):

\[
H''_2 = \sum_{\{u,v\} \subset \Lambda} \left( \frac{p_1}{|\Lambda|} \right)^2 \sum_{x \in \mathbb{Z}^d} \sum_{\substack{R_u \in \mathcal{T}_{u,x} \setminus \mathcal{T}_o \ R_v \in \mathcal{T}_{v,x} \setminus \mathcal{T}_o \ x \in \mathbb{Z}^d}} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{Ru}| + |E_{Rv}|} \left( 1 - \mathbb{1}_{V_{Ru} \cap V_{Rv} = \emptyset} \right).
\]

(3.67)

Notice that

\[
1 - \mathbb{1}_{V_{Ru} \cap V_{Rv} = \emptyset} \leq \sum_{x' \in \mathbb{Z}^d \setminus \{x\}} \mathbb{1}_{x' \in V_{Ru} \cap V_{Rv}}.
\]

(3.68)

By the inclusion relation $\mathcal{T}_{u,x,x'} \setminus \mathcal{T}_o \subset \mathcal{T}_{u,x,x'}$ and using (3.64), (2.1) and (3.17), we can bound $H''_2$ as (see Figure 8)

\[
H''_2 \leq \sum_{\{u,v\} \subset \Lambda} \left( \frac{p_1}{|\Lambda|} \right)^2 \sum_{x,x' \in \mathbb{Z}^d} \sum_{\substack{R_u \in \mathcal{T}_{u,x,x'} \ (x \neq x') \ R_v \in \mathcal{T}_{v,x,x'} \ (x \neq x')}} \left( \frac{p_1}{|\Lambda|} \right)^{|E_{Ru}| + |E_{Rv}|} \frac{\tau_{(u,x)}^{(3)}(u,x)}{g_{p_1}} \frac{\tau_{(x,x')}^{(3)}(v,x')}{g_{p_1}}
\]

\[
\leq \frac{g_{p_1}^4}{2} \sum_{x,x',y,y' \in \mathbb{Z}^d} \frac{\tau_{p_1}(y - x)}{g_{p_1}} \frac{\tau_{p_1}(y' - x)}{g_{p_1}} \frac{\tau_{p_1}(y' - x)}{g_{p_1}} \frac{\tau_{p_1}(y' - x)}{g_{p_1}}
\]

\[
\times \frac{1}{2} \sum_{u,v \in \Lambda} \left( \frac{1}{|\Lambda|} \right)^2 \frac{\tau_{p_1}(u - y)}{g_{p_1}} \frac{\tau_{p_1}(v - y')}{g_{p_1}}
\]

\[
\leq \frac{g_{p_1}^4}{2} \sum_{x,x',y,y' \in \mathbb{Z}^d} S_{\geq 0}(y - x) S_{\geq 0}(y' - x) S_{\geq 0}(y' - x) S_{\geq 1}(y) S_{\geq 1}(y').
\]

(3.69)
Shifting the variables by \(-x'\) and changing the variables \(x - x', y - x', y' - x'\) to the new ones \(w, z, z'\), respectively, we can rewrite the above sum as

\[
\sum_{w, z, z' \in \mathbb{Z}^d \atop (w \neq o)} S_{\geq 0}(z - w) S_{\geq 0}(z) S_{\geq 0}(z' - w) S_{\geq 0}(z') \sum_{x' \in \mathbb{Z}^d} S_{\geq 1}(z + x') S_{\geq 1}(z' + x'),
\]

(3.70)

which is bounded above by

\[
\|S_{\geq 1}^2\|_\infty \sum_{w \neq o} S_{\geq 0}^2(w)^2 = \|S_{\geq 1}^2\|_\infty \sum_{w \neq o} \left( \sum_{n=1}^\infty (n+1)D^{*n}(w) \right)^2 \leq \|S_{\geq 1}^2\|_\infty \sum_{t=2}^\infty D^t(o) \left( \sum_{n=1}^{t-1} (n+1) (t-n+1) \right) \overset{d \geq 8}{=} O(\beta^2),
\]

(3.71)
as required.

**Proof of (3.48): bounding \(H''_2\).** First we recall that \(H''_2\) is the contribution to \(H'_2\) which involves 3-point functions (cf., (3.44)):

\[
H''_2 = \sum_{\{o, u, v\} \subset \Lambda} \left( \frac{1}{|\Lambda|} \right)^2 \sum_{x \in \mathbb{Z}^d} \frac{\tau^{(3)}_{p_1}(o, u, x)}{g_{p_1}} \left( \frac{\tau^{(3)}_{p_1}(o, v, x)}{g_{p_1}} - 2 \frac{\tau_{p_1}(v-x)}{g_{p_1}} \right).
\]

(3.72)

By (2.1), (3.64) and (3.17), we can readily conclude that

\[
|H''_2| \leq g_{p_1}^2 \sum_{u, v \in \Lambda} \left( \frac{1}{|\Lambda|} \right)^2 \sum_{x, y \in \mathbb{Z}^d} \frac{\tau_{p_1}(y)}{g_{p_1}} \frac{\tau_{p_1}(y-u)}{g_{p_1}} \frac{\tau_{p_1}(y-x)}{g_{p_1}} \times \left( \frac{g_{p_1}^2 \sum_{z \in \mathbb{Z}^d} \tau_{p_1}(z)}{g_{p_1}^2} \frac{\tau_{p_1}(z-v)}{g_{p_1}} \frac{\tau_{p_1}(z-x)}{g_{p_1}} + 2 \frac{\tau_{p_1}(v-x)}{g_{p_1}} \right)
\]

\[
\leq g_{p_1}^2 \sum_{x, y \in \mathbb{Z}^d} S_{\geq 0}(y) S_{\geq 1}(y) S_{\geq 0}(y-x)
\]

\[
\times \left( \frac{g_{p_1}^2 \sum_{z \in \mathbb{Z}^d} S_{\geq 0}(z) S_{\geq 1}(z) S_{\geq 0}(z-x) + S_{\geq 1}(x)}{g_{p_1}^2} \right)
\]

\[
= g_{p_1}^2 \sum_{y \in \mathbb{Z}^d} S_{\geq 0}(y) S_{\geq 1}(y) \left( \frac{g_{p_1}^2}{2} \sum_{z \in \mathbb{Z}^d} S_{\geq 0}(z) S_{\geq 1}(z) S_{\geq 0}(y-z) + (S_{\geq 0} * S_{\geq 1})(y) \right)
\]

\[
\leq g_{p_1}^2 (S_{\geq 0} * S_{\geq 1})(o) \left( \frac{g_{p_1}^2}{2} \sum_{z \in \mathbb{Z}^d} (S_{\geq 0} * S_{\geq 1})(o) \|S_{\geq 0}^2\|_\infty + \|S_{\geq 0} * S_{\geq 1}\|_\infty \right) \overset{O(\beta) \text{ for } d \geq 4}{=} O(\beta^2),
\]

(3.73)
as required.
4 Difference between lattice trees and lattice animals

Finally we prove Theorem 1.1 for lattice animals. Recall that, by Lemma 2.1, it suffices to investigate \( p_1 = 1/gp_1 \) (cf., (2.1)). The following is the key lemma:

**Lemma 4.1.** For lattice animals with \( d > 8 \) and \( L \uparrow \infty \),

\[
gp_1 = e \left( 1 - \sum_{n=2}^{\infty} \frac{n + 1}{2} D^n(o) \right) + \frac{1}{2} S_{\geq 3}(o) + O(\beta^2). \tag{4.1}
\]

Consequently,

\[
p_1 = \frac{1}{e} + \sum_{n=2}^{\infty} \frac{n + 1}{2e} D^n(o) - \frac{1}{2e^2} S_{\geq 3}(o) + O(\beta^2). \tag{4.2}
\]

**Proof.** As a first step, we want a similar decomposition to (3.3) for lattice animals. To do so, we identify the connected neighbors \( Y \) of the origin, just as done in (3.3). Then, we introduce \( \Gamma(Y) \), which is the set of all partitions of \( Y \). For example, if \( Y = \{1, 2, 3\} \),

\[
\Gamma(Y) = \left\{ \{Y\}, \\{\{1, 2\}, \{3\}\}, \\{\{1, 3\}, \{2\}\}, \\{\{1\}, \{2, 3\}\}, \\{\{2\}\}, \\{\{1\}\}, \\{\{2\}\}, \\{\{3\}\} \right\}. \tag{4.3}
\]

For a partition \( \gamma \in \Gamma(Y) \), we denote by \( |\gamma| \) the number of sets in \( \gamma \), so that \( \gamma = \{\gamma_j\}_{j=1}^{\gamma} \).

We can rewrite \( gp_1 \) as

\[
gp_1 = 1 + \sum_{Y \subset \Lambda} \left( \frac{p_1}{|A|} \right)^{|Y|} \sum_{|\gamma|} \prod_{j=1}^{\gamma} \sum_{R_j \in A_{\gamma_j} \setminus A_o} \left( \frac{p_1}{|A|} \right)^{|E_{R_j}|} \prod_{i<j} 1_{\{V_{R_i} \cap V_{R_j} = \emptyset\}}. \tag{4.4}
\]

The contribution from the maximum partition \( \bar{\gamma} = \{\{y\}\}_{y \in Y} \) (i.e., \( |\gamma| = |Y| \)) is equal to (3.3) (with \( T \) replaced by \( A \)) and can be decomposed into \( G \) and \( H \) as in (3.5) (with \( R_y \) regarded as animals instead of trees). Let \( I \) be the contribution from the remaining partitions \( \gamma \in \Gamma(Y) \) with \( |\gamma| < |Y| \), which is zero for lattice trees:

\[
I = gp_1 - (G - H) = \sum_{Y \subset \Lambda} \left( \frac{p_1}{|A|} \right)^{|Y|} \sum_{|\gamma| < |Y|} \prod_{j=1}^{\gamma} \sum_{R_j \in A_{\gamma_j} \setminus A_o} \left( \frac{p_1}{|A|} \right)^{|E_{R_j}|} \prod_{i<j} 1_{\{V_{R_i} \cap V_{R_j} = \emptyset\}}. \tag{4.5}
\]

To evaluate \( G, H \) and \( I \) for lattice animals, we cannot apply Lemma 3.3, which is a powerful tool for lattice trees to identify the coefficients of \( \beta \) as well as to estimate the error terms of \( O(\beta^2) \). For the latter purpose for lattice animals, we will use the infrared
bound (2.9) (and monotonicity in $p$, i.e., $\tau_{p_1} \leq \tau_{p_c}$); for the former purpose, we will use the following bounds that correspond to (3.26), (3.49) and (3.51), respectively:

**Lemma 4.2.** For lattice animals with $d > 8$ and $L \uparrow \infty$,

\[
\left| \sum_{u \in \Lambda} \frac{1}{|\Lambda|} \tau_{p_1}(u) - S_{\geq 2}(o) \right| = O(\beta^2),
\]

\[
\left| \sum_{\{u,v\} \subseteq \Lambda} \left( \frac{1}{|\Lambda|} \right)^2 \frac{\tau_{p_1}(u-v)}{g_{p_1}} - \frac{1}{2} S_{\geq 3}(o) \right| = O(\beta^2),
\]

\[
\left| \sum_{\{u,v\} \subseteq \Lambda} \left( \frac{1}{|\Lambda|} \right)^2 \tau_{p_1}(u-x) \tau_{p_1}(x-v) - \frac{1}{2} S_{\geq 2}^*(o) \right| = O(\beta^2).
\]

We will prove Lemma 4.2 after the proof of Lemma 4.1 is completed.

Now we resume the proof of Lemma 4.1 assuming the bounds in Lemma 4.2. First we recall $G = G_0 + G_1 + G_2$ (cf., (3.14)), where $G_0$ is independent of the models and estimated as (3.12); $G_1$ is defined as (3.15) and here we use (4.6) to show (3.27); $G_2$ is defined as (3.16) and obeys the same bound as (3.22). As a result, Lemma 3.2 also holds for lattice animals. Similarly, we can show $H = e(H''_1 + H''_2 - H''_2 - H_2) + O(\beta^2)$ (cf., (3.37), (3.42), (3.46) and (3.48)), where $H''_1$ is defined in (3.47) and here we use (4.7)–(4.8) to show (3.53); $H_2$ is bounded by $H_{2,3} + H_{2,4}$, and $H_{2,3}$ and $H_{2,4}$ are further bounded as (3.59)–(3.61) and (3.63)–(3.65) (with $T$ replaced by $A$), and here we use the infrared bound (2.9) and the convolution bound on power functions [4, Lemma 3.2(i)] to show $H_2 = O(\beta^2)$, such as

\[
\sum_{\{u,v\} \subseteq \Lambda} \left( \frac{1}{|\Lambda|} \right)^2 \sum_{x \in \mathbb{Z}^d} \tau_{p_1}(u-x) \tau_{p_1}(v-x)
\]

\[
\leq \sum_{\{u,v\} \subseteq \Lambda} \left( \frac{1}{|\Lambda|} \right)^2 \sum_{x \in \mathbb{Z}^d} \frac{O(L^{-2})}{\|u-x\| \vee L} \frac{O(L^{-2})}{\|v-x\| \vee L} = O(\beta^2).
\]

Similarly we can show that $H''_2$ and $H''_2$ are both $O(\beta^2)$ by using the infrared bound and the convolution bound, instead of bounding $\tau_{p_1}/g_{p_1}$ by $S_{\geq 0}$ or $S_{\geq 1}$, just as done for lattice trees. As a result, Lemma 3.4 also holds for lattice animals.

Next we investigate $I$, which is unique for lattice animals. Let $I_1$ be the contribution from $\gamma \in \Gamma(Y)$ with $|\gamma| = |Y| - 1$, i.e., consisting of a pair $\{u, v\}$ and $|Y| - 2$ singletons.
Figure 9: Schematic representation of the dominant contribution to $I_1'$.

\[
\{y\}_{y \in Y \setminus \{u,v\}}:
\]

\[
I_1 = \sum_{Y \subset \Lambda \atop (|Y| \geq 2)} \left( \frac{p_1}{|A|} \right)^{|Y|} \sum_{\{u,v\} \subset Y} \sum_{R \in A_{u,v} \setminus A_o} \left( \frac{p_1}{|A|} \right)^{|E_R|} \prod_{y \in Y \setminus \{u,v\}} \sum_{R_y \in A_y \setminus A_o} \left( \frac{p_1}{|A|} \right)^{|E_{R_y}|} \prod_{y \neq z} \mathbb{1}_{\{V_{R_y} \cap V_{R_z} = \emptyset\}} (4.10)
\]

where an empty product is regarded as 1. The dominant contribution to $I_1$, denoted $I_1'$, comes from when the last line is replaced by 1. By the tree-graph inequality (3.64), which is also true for lattice animals due to subadditivity, and then using the infrared bound (2.9), it is estimated as (see Figure 9)

\[
I_1' = \sum_{\{u,v\} \subset \Lambda} \left( \frac{p_1}{|A|} \right)^2 \tau_{p_1}(u-v) \sum_{Y \subset \Lambda \setminus \{u,v\}} \left( \frac{1}{|A|} \right)^{|Y'|} \prod_{y \in Y'} \left( 1 - \frac{\tau_{p_1}(y)}{g_{p_1}} \right) (4.7)
\]

\[
\leq \frac{1}{2} S_{\geq 3}(o) + O(\beta^2) (4.11)
\]

On the other hand, by using $1 - ab \leq (1 - a) + (1 - b)$ for any $a, b \in \{0, 1\}$, we can bound the difference $I_1' - I_1 \geq 0$ as

\[
I_1' - I_1 \leq \sum_{\{u,v\} \subset \Lambda} \left( \frac{p_1}{|A|} \right)^2 \sum_{R \in A_{u,v} \setminus A_o} \left( \frac{p_1}{|A|} \right)^{|E_R|} \sum_{Y \subset \Lambda \setminus \{u,v\}} \left( \frac{p_1}{|A|} \right)^{|Y'|} \prod_{y \in Y'} \sum_{R_y \in A_y \setminus A_o} \left( \frac{p_1}{|A|} \right)^{|E_{R_y}|}
\]
This is $O(\beta^2)$, as the contribution from the former (resp., latter) sum in the last line can be estimated in a similar way to showing $H_{2,3} = O(\beta^2)$ (resp., $H_{2,4} = O(\beta^2)$); see Figure 10. As a result,

$$I_1 = \frac{1}{2} S_{\geq 3}(o) + O(\beta^2).$$

Finally we estimate the difference $I - I_1$:

$$I - I_1 = \sum_{Y \subseteq A \atop (|Y| \geq 3)} \left( \frac{p_1}{|A|} \right)^{|Y|} \sum_{\gamma \in C(Y)} \prod_{j=1}^{\gamma} \sum_{R_j \in A_{\gamma} \setminus A_o} \left( \frac{p_1}{|A|} \right)^{|E_{R_j}|} \prod_{i<j} \mathbb{1}_{\{V_{R_i} \cap V_{R_j} = \varnothing\}}.$$ (4.14)

Since $|\gamma| \leq |Y| - 2$, there are two possibilities: (i) there is a set in $\gamma$ which includes at least 3 distinct neighbors of the origin, or (ii) there are at least two disjoint sets in $\gamma$ both of which include exactly two distinct neighbors of the origin. Therefore,

$$I - I_1 \leq \sum_{Y \subseteq A \atop (|Y| \geq 3)} \left( \frac{p_1}{|A|} \right)^{|Y|} \sum_{\gamma \in C(Y)} \prod_{j=1}^{\gamma} \sum_{R_j \in A_{\gamma} \setminus A_o} \left( \frac{p_1}{|A|} \right)^{|E_{R_j}|} \prod_{i<j} \mathbb{1}_{\{V_{R_i} \cap V_{R_j} = \varnothing\}}$$

$$\times \left( \mathbb{1}_{\{\exists j, |\gamma_j| \geq 3\}} + \mathbb{1}_{\{\exists i \neq j, |\gamma_i| = |\gamma_j| = 2\}} \right)$$

$$= I_3 + I_2,$$ (4.15)

where $I_3$ and $I_2$ are the contributions from $\mathbb{1}_{\{\exists j, |\gamma_j| \geq 3\}}$ and $\mathbb{1}_{\{\exists i \neq j, |\gamma_i| = |\gamma_j| = 2\}}$, respectively.
For $I_2$, we split the set $Y$ of neighbors of the origin into $U$, $V$ and $Y' = Y \setminus (U \cup V)$, where $U \cap V = \emptyset$ and $|U| = |V| = 2$. Partially ignoring the avoidance constraint among animals, we can bound $I_2$ as

$$I_2 \leq \sum_{U \subseteq A \atop (|U|=2)} \left( \frac{p_1}{|A|} \right)^2 \sum_{R \in A_U \setminus A_o} \left( \frac{p_1}{|A|} \right)^{|E_R|} \sum_{V \subseteq A \setminus U \atop (|V|=2)} \left( \frac{p_1}{|A|} \right)^2 \sum_{R' \in A_V \setminus A_o} \left( \frac{p_1}{|A|} \right)^{|E_{R'}|} \times \sum_{Y' \subseteq A \setminus (U \cup V)} \left( \frac{p_1}{|A|} \right)^{|Y'|} \prod_{\gamma \in \Gamma(Y')} \sum_{j=1}^{\gamma} \left( \frac{p_1}{|A|} \right)^{|E_{R_j}|} \prod_{i<j} \mathbb{1}_{\{V_{R_i} \cap V_{R_j} = \emptyset\}}. \tag{4.16}$$

Notice that the second line is almost identical to $g_{p_1}$; the only difference is the domain of summation over $Y'$, and therefore it is bounded above by $g_{p_1}$. Since $p_1 g_{p_1} = 1$, we obtain

$$I_2 \leq g_{p_1} \left( \sum_{\{u,v\} \subseteq A} \left( \frac{p_1}{|A|} \right)^2 \sum_{R \in A_{u,v}} \left( \frac{p_1}{|A|} \right)^{|E_R|} \right)^2 \tag{4.17} \leq p_1 \left( \frac{1}{2} S_{\geq 3}(\theta) + O(\beta^2) \right)^2 = O(\beta^2).$$

For $I_3$, we split the set $Y$ into $X$ and $Y' = Y \setminus X$, where $X$ includes at least 3 distinct vertices $x, y, z \in \Lambda$. Again, by partially ignoring the avoidance constraint among animals, we can bound $I_3$ as

$$I_3 \leq \sum_{\{x,y,z\} \subseteq A \atop (x \notin X, y, z \in X)} \sum_{X \subseteq \Lambda} \left( \frac{p_1}{|A|} \right)^{|X|} \sum_{R \in A_X \setminus A_o} \left( \frac{p_1}{|A|} \right)^{|E_R|} \times \sum_{Y' \subseteq A \setminus X} \left( \frac{p_1}{|A|} \right)^{|Y'|} \prod_{\gamma \in \Gamma(Y')} \sum_{j=1}^{\gamma} \left( \frac{p_1}{|A|} \right)^{|E_{R_j}|} \prod_{i<j} \mathbb{1}_{\{V_{R_i} \cap V_{R_j} = \emptyset\}}. \tag{4.18}$$

Notice again that the second line is bounded above by $g_{p_1}$. Using the relation $A_X \setminus A_o \subset A_{x,y,z}$ and splitting $X$ into $\{x, y, z\}$ and $X' = X \setminus \{x, y, z\}$, we obtain

$$I_3 \leq g_{p_1} \sum_{\{x,y,z\} \subseteq A} \left( \frac{p_1}{|A|} \right)^3 \sum_{R \in A_{x,y,z}} \left( \frac{p_1}{|A|} \right)^{|E_R|} \sum_{X' \subseteq \Lambda \setminus \{x,y,z\}} \left( \frac{p_1}{|A|} \right)^{|X'|}. \tag{4.19}$$

By the tree-graph inequality (3.64), we can show that

$$\sum_{\{x,y,z\} \subseteq A} \left( \frac{p_1}{|A|} \right)^3 \tau_{p_1}^{(3)}(x, y, z) \leq \sum_{\{x,y,z\} \subseteq A} \left( \frac{1}{|A|} \right)^3 \sum_{w \in \mathbb{Z}^d} \tau_{p_1}(x-w) \tau_{p_1}(y-w) \tau_{p_1}(z-w) g_{p_1} g_{p_1} g_{p_1}$$

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\[
\sum_{\{x,y,z\} \subseteq \Lambda} \left(\frac{1}{|\Lambda|}\right)^3 \left( \sum_{\{w\neq x,y,z\}} \frac{\tau_{p_1}(x-w) \tau_{p_1}(y-w) \tau_{p_1}(z-w)}{g_{p_1}} \frac{3 \tau_{p_1}(x-z) \tau_{p_1}(y-z)}{g_{p_1}} \right)
\leq \frac{3}{g_{p_1}} \left( \|D \tau_{p_1}\|_\infty + \|D\|_\infty \right) \sum_{\{x,y\} \subseteq \Lambda} \left(\frac{1}{|\Lambda|}\right)^2 \sum_{w \neq x,y} \frac{\tau_{p_1}(x-w) \tau_{p_1}(y-w)}{g_{p_1}},
\]

hence \(I_3 = O(\beta^2)\). This completes the proof of \(I = \frac{1}{2}S_{\geq 2}(o) + O(\beta^2)\), hence the proof of Lemma 4.1.

Proof of Lemma 4.2. First we prove (4.6). By the inverse Fourier transform, we have the rewrite

\[
\sum_{u \in \Lambda} \frac{1}{|\Lambda|} \frac{\tau_{p_1}(u)}{g_{p_1}} = \left(D \ast \frac{\tau_{p_1}(u)}{g_{p_1}}\right)(o) = \int_{[-\pi,\pi]^d} \hat{D}(k) \frac{\hat{\tau}_{p_1}(k)}{g_{p_1}} \frac{d^d k}{(2\pi)^d}.
\]

Notice that the Fourier transform of the recursion equation (2.8) yields

\[
\hat{\tau}_p(k) = \frac{g_p + \hat{h}_p(k) + \hat{\pi}_p(k)}{1 - (g_p + \hat{h}_p(k) + \hat{\pi}_p(k))p\hat{D}(k)}.
\]

We use this identity at \(p_1 = 1/g_{p_1}\). Let

\[
H(x) = \frac{h_{p_1}(x) + \pi_{p_1}(x)}{g_{p_1}}.
\]

Thanks to the symmetry, the Fourier transform \(\hat{H}(k)\) is real. Moreover, by (2.11)–(2.12), we can show that, for \(d > 8\) and \(L \gg 1\), \(|\hat{H}(k)| = O(\beta)\) uniformly in \(k\). Then, we can rewrite \(\hat{\tau}_{p_1}(k)/g_{p_1}\) as

\[
\frac{\hat{\tau}_{p_1}(k)}{g_{p_1}} = \frac{1 + \hat{H}(k)}{1 - (1 + \hat{H}(k))\hat{D}(k)} = \frac{1}{1 - \hat{D}(k)} + \frac{\hat{H}(k)}{1 - \hat{D}(k)} \frac{1}{1 - (1 + \hat{H}(k))\hat{D}(k)}.
\]

Applying this to (4.21) yields the main term \(S_{\geq 2}(o)\) as

\[
\sum_{u \in \Lambda} \frac{1}{|\Lambda|} \frac{\tau_{p_1}(u)}{g_{p_1}} = \int_{[-\pi,\pi]^d} \frac{\hat{D}(k) d^d k}{1 - \hat{D}(k)} + \int_{[-\pi,\pi]^d} \frac{\hat{D}(k)\hat{H}(k)\hat{F}(k) d^d k}{1 - \hat{D}(k)}.
\]

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It remains to show that the second term on the right is $O(\beta^2)$. To do so, we want an effective bound on $\hat{F}(k)$. We will show at the end of the proof that, for $d > 8$ and $L \gg 1$, there is an $L$-independent constant $C < \infty$ such that

$$0 < \hat{F}(k) \leq \frac{C}{1 - \hat{D}(k)},$$

(4.26)

uniformly in $k$. However, to use (4.26) for the second term of (4.25), we have to bound $\hat{D}(k)$ above by the absolute value of it, which makes it difficult to show the error being $O(\beta^2)$. Instead, we first rewrite $\hat{F}(k)$ as

$$\hat{F}(k) = \frac{1}{1 - \hat{D}(k)} + \frac{\hat{D}(k)\hat{H}(k)}{1 - \hat{D}(k)} \hat{F}(k).$$

(4.27)

Then, the second term on the right of (4.25) equals

$$\int_{[-\pi,\pi]^d} \frac{\hat{D}(k)\hat{H}(k)}{1 - \hat{D}(k)} \frac{d^dk}{(2\pi)^d} + \int_{[-\pi,\pi]^d} \left( \frac{\hat{D}(k)\hat{H}(k)}{1 - \hat{D}(k)} \right)^2 \frac{\hat{F}(k)}{(2\pi)^d} d^dk. \quad (4.28)$$

Notice that, due to the identity (4.27), we can make $\hat{D}(k)^2$, which is always positive, in the second term of (4.28). The first term is readily bounded by $\|D*S_{\geq 3}^2\|_\infty \hat{H}(0) = O(\beta^2)$. For the second term, we use $|\hat{H}(k)| = O(\beta)$ and (4.26) to obtain that

$$\int_{[-\pi,\pi]^d} \left( \frac{\hat{D}(k)\hat{H}(k)}{1 - \hat{D}(k)} \right)^2 \frac{\hat{F}(k)}{(2\pi)^d} d^dk \leq O(\beta^2) \int_{[-\pi,\pi]^d} \frac{\hat{D}(k)^2}{(1 - \hat{D}(k))^3} \frac{d^dk}{(2\pi)^d} = O(\beta^2). \quad (4.29)$$

This completes the proof of (4.6).

We can also prove (4.7)–(4.8) in a similar manner by assuming (4.26). Hence we here prove only (4.7). By the inverse Fourier transform, we can rewrite the sum in (4.7) as

$$\sum_{\{u,v\} \subset A} \left( \frac{1}{|A|} \right)^2 \frac{\tau_{p_1}(u - v)}{g_{p_1}} = \frac{1}{2} \left( D^{*2} * \frac{\tau_{p_1}}{g_{p_1}} \right)(o) - \frac{1}{2} D^{*2}(o)$$

$$= \frac{1}{2} \int_{[-\pi,\pi]^d} \hat{D}(k)^2 \frac{\tau_{p_1}(k)}{g_{p_1}} \frac{d^dk}{(2\pi)^d} - \frac{1}{2} D^{*2}(o). \quad (4.30)$$

Then, by the identity (4.24), we can extract the main term $\frac{1}{2} S_{\geq 3}(o)$ as

$$\frac{1}{2} \int_{[-\pi,\pi]^d} \frac{\hat{D}(k)^2}{1 - \hat{D}(k)} \frac{d^dk}{(2\pi)^d} - \frac{1}{2} D^{*2}(o) + \frac{1}{2} \int_{[-\pi,\pi]^d} \frac{\hat{D}(k)^2 \hat{H}(k)}{1 - \hat{D}(k)} \hat{F}(k) \frac{d^dk}{(2\pi)^d}. \quad (4.31)$$
Similarly to (4.29), the second term is bounded as
\[
\left| \frac{1}{2} \int_{[-\pi,\pi]^d} \frac{\hat{D}(k)^2 \hat{H}(k)}{1 - \hat{D}(k)} \frac{d^d k}{(2\pi)^d} \right| \leq O(\beta) \int_{[-\pi,\pi]^d} \frac{\hat{D}(k)^2}{1 - \hat{D}(k)} \frac{d^d k}{(2\pi)^d} = O(\beta^2),
\]
and hence the completion of the proof of Lemma 4.2.

Finally we prove the inequality (4.26), for \( \|k\| \geq \frac{1}{L} \) and \( \|k\| \leq \frac{1}{L} \) separately. We begin with the former case. It is known (cf., e.g., [12]) that our \( D \) satisfies [14, Assumption D]; in particular, there is an \( L \)-independent constant \( \eta \in (0, 1) \) such that
\[
1 + \eta \leq \hat{D}(k) \leq 1 - \eta.
\]
Since \( |\hat{H}(k)| = O(\beta) \), we obtain that, for \( L \gg 1 \),
\[
-1 + \frac{\eta}{2} \leq \left(1 + \hat{H}(k) \right) \hat{D}(k) \leq 1 - \frac{\eta}{2},
\]
hence
\[
0 < \frac{1}{2 - \eta/2} \leq \hat{F}(k) \leq \frac{2}{\eta} \leq \frac{2}{\eta} \frac{2 - \eta}{1 - \hat{D}(k)}.
\]

It remains to show that \( \hat{F}(k) \) is bounded above by a multiple of \( (1 - \hat{D}(k))^{-1} \) uniformly in \( \|k\| \leq \frac{1}{L} \). We note that
\[
\hat{F}(k)^{-1} = 1 - \left(1 + \hat{H}(0) \right) (1 - \hat{D}(k)) + (\hat{H}(0) - \hat{H}(k)) \hat{D}(k)
\]
\[
= -\hat{H}(0) + \left(1 + \hat{H}(0) + \frac{\hat{H}(0) - \hat{H}(k)}{1 - \hat{D}(k)} \hat{D}(k) \right) (1 - \hat{D}(k)).
\]

Since \( -\hat{H}(0) \) is bounded below by a positive multiple of \( \beta \) (as explained in the beginning of the proof of Lemma 2.1), ignoring this term yields a lower bound on \( \hat{F}(k)^{-1} \). Moreover, since \( |k \cdot x| \leq \|k\| \|x\| \leq 1 \) for \( x \in \Lambda \) and \( \|k\| \leq \frac{1}{L} \), and since \( 1 - \cos t \geq \frac{2}{\pi^2} t^2 \) for \( |t| \leq 1 \), there is a \( c > 0 \) such that
\[
1 - \hat{D}(k) = \sum_{x \in \Lambda} \frac{1 - \cos(k \cdot x)}{|\Lambda|} \geq \frac{2}{\pi^2} \sum_{x \in \Lambda} \frac{(k \cdot x)^2}{|\Lambda|} = \frac{2\|k\|^2}{d\pi^2} \sum_{x \in \Lambda} \frac{|x|^2}{|\Lambda|} \geq cL^2.
\]

On the other hand, by \( 1 - \cos t \leq \frac{1}{2} t^2 \) for any \( t \) and using the \( x \)-space bounds (2.7) and (2.12), we have
\[
|\hat{H}(0) - \hat{H}(k)| \leq \sum_x \frac{(k \cdot x)^2}{2} |H(x)| \leq \frac{\|k\|^2}{2d} \sum_x \|x\|^2 |H(x)|.
\]

Therefore, by taking \( L \) sufficiently large, \( \hat{F}(k)^{-1} \) is bounded below by a positive multiple of \( 1 - \hat{D}(k) \), uniformly in \( \|k\| \leq \frac{1}{L} \). Combined with (4.35), this completes the proof of the inequality (4.26), hence the completion of the proof of Lemma 4.2. \qed
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References

[1] D.C. Brydges, T. Helmuth and M. Holmes. The continuous-time lace expansion. Commun. Pure Appl. Math., 74 (2021):2251–2309.

[2] D.C. Brydges and T. Spencer. Self-avoiding walk in 5 or more dimensions. Commun. Math. Phys., 97 (1985): 125–148.

[3] L.-C. Chen and A. Sakai. Critical behavior and the limit distribution for long-range oriented percolation. I. Probab. Theory Relat. Fields, 142 (2008): 151–188.

[4] L.-C. Chen and A. Sakai. Critical two-point functions for long-range statistical-mechanical models in high dimensions. Ann. Probab., 43 (2015): 639–681.

[5] R. Fitzner and R. van der Hofstad. NoBLE for lattice trees and lattice animals. J. Stat. Phys, 185 (2021): No.13.

[6] T. Hara, R.v.d. Hofstad and G. Slade. Critical two-point functions and the lace expansion for spread-out high-dimensional percolation and related models. Ann. Probab., 31 (2003): 349–408.

[7] T. Hara and G. Slade. Mean-field critical behaviour for percolation in high dimensions. Commun. Math. Phys., 128 (1990): 333–391.

[8] T. Hara and G. Slade. On the upper critical dimension of lattice trees and lattice animals. J. Stat. Phys, 59 (1990): 1469–1510.

[9] T. Hara and G. Slade. Self-avoiding walk in five or more dimensions. I. The critical behaviour. Commun. Math. Phys., 147 (1992): 101–136.

[10] T. Hara and G. Slade. The number and size of branched polymers in high dimensions. J. Stat. Phys, 67 (1992): 438–470.

[11] R. van der Hofstad and A. Sakai. Gaussian scaling for the critical spread-out contact process above the upper critical dimension. Electron. J. Probab., 9 (2004): 710–769.

[12] R.v.d. Hofstad and A. Sakai. Critical points for spread-out self-avoiding walk, percolation and the contact process above the upper critical dimensions. Probab. Theory Relat. Fields, 132 (2005): 438–470.
[13] R. van der Hofstad and A. Sakai. Convergence of the critical finite-range contact process to super-Brownian motion above the upper critical dimension: The higher-point functions. Electron. J. Probab., 15 (2010): 801–894.

[14] R. van der Hofstad and G. Slade. A generalised inductive approach to the lace expansion. Probab. Theory Relat. Fields, 122 (2002): 389–430.

[15] D.A. Klarner. Cell growth problems. Canad. J. Math., 19 (1967): 851–863.

[16] D.J. Klein. Rigorous results for branched polymer models with excluded volume. J. Chem. Phys., 75 (1981): 5186–5189.

[17] Y. Liang. Critical point for spread-out lattice trees in dimensions \( d > 8 \). Master thesis, Hokkaido University (2022).

[18] N. Madras and G. Slade. The Self-Avoiding Walk (Birkhäuser, 1993).

[19] Y.M. Miranda. The critical points of lattice trees and lattice animals in high dimensions. Ph.D thesis, University of British Columbia (2012).

[20] Y.M. Miranda and G. Slade. The growth constants of lattice trees and lattice animals in high dimensions. Electron. Commun. Probab., 16 (2011): 129–136

[21] Y.M. Miranda and G. Slade. Expansion in high dimension for the growth constants of lattice trees and lattice animals. Combin. Probab. Comput., 22 (2013): 527–565.

[22] B.G. Nguyen and W.-S. Yang. Triangle condition for oriented percolation in high dimensions. Ann. Probab., 21 (1993): 1809–1844.

[23] M.D. Penrose. Self-avoiding walks and trees in spread-out lattices. J. Stat. Phys., 77 (1994): 3–15.

[24] A. Sakai. Mean-field critical behavior for the contact process. J. Stat. Phys., 104 (2001): 111–143.

[25] A. Sakai. Lace expansion for the Ising model. Commun. Math. Phys., 272 (2007): 283–344.

[26] A. Sakai. Application of the lace expansion to the \( \varphi^4 \) model. Commun. Math. Phys., 336 (2015): 619–648.

[27] A. Sakai. Correct bounds on the Ising lace-expansion coefficients. Commun. Math. Phys., 392 (2022): 783–823.