Unboundedness of triad-like operators in loop quantum gravity

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Abstract
In this paper we deliver the proofs for the claims, made in a companion paper, concerning the avoidance of cosmological curvature singularities in full loop quantum gravity (LQG).

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(Some figures in this article are in colour only in the electronic version)

\section{Introduction}

In this paper, we display a self-contained derivation of the results reported in our companion paper [1]. Basically that paper consisted of three parts: two claims and a scheme.

The first claim was that triad-like operators, which are the direct analogon in full loop quantum gravity (LQG) (see [2, 3] for books and [4, 5] for reviews), of the inverse scale factor in loop quantum cosmology (LQC)\textsuperscript{3} are unbounded even right at the big bang (zero volume eigenstates) and even when restricting to states which are isotropic and homogeneous on large scales. This is in contrast to the LQC result. Note that ‘unbounded’ does not mean ill-defined: in fact, our triad operators are well defined, that is, they are defined on a dense subset of the Hilbert space consisting of the finite linear span of spin network states. This is a well-known result derived almost a decade ago [7–9].

Rather, ‘unboundedness’ means that there are states in the closure of the span on which the operator norm diverges. This means that, in contrast to bounded operators, not every state is in the domain of the operator and therefore it is not granted that the local curvature

\textsuperscript{3} LQC comprises the usual homogeneous minisuperspace models quantized by LQG methods; see, for instance, [6] for a review.
singularity is absent in full LQG because \textit{a priori} any state could be semiclassically relevant in order to describe a collapsing universe.

The second result we claimed was that with respect to a candidate class of semiclassical states\(^4\), describing a collapsing universe which is homogeneous and isotropic on large scales, the analogon of the inverse scale factor is bounded even at zero volume. This is a calculation within full LQG and takes the inhomogeneous and non-isotropic fluctuations into account.

However, this calculation can at best be a first promising hint because, as we showed, reliable conclusions can only be drawn when working with physical operators and physical states. A crude way of implementing such a scheme was also described in [1]. In particular, kinematical boundedness does not imply physical boundedness because a Dirac observable constructed from a partial observable such as the inverse scale factor may have a different spectrum. Secondly, whether or not the quantum evolution with respect to an unphysical time parameter breaks down at zero volume on certain states or not may or may not be an indication of a global initial singularity. On the one hand, if there are restrictions on the allowed set of states that one can evolve with the Hamiltonian constraint, the physical Hilbert space may still be large enough to describe a collapsing universe. On the other hand, if there are no restrictions, it may be the case that an insufficient number of these formal solutions is normalizable with respect to the physical inner product in order to capture a sector of the physical Hilbert space describing a collapsing universe at large scales.

This paper is organized as follows. In section 2 we derive the contribution of the kinetic term of a scalar field to the Hamiltonian constraint operator. This is one of the terms which become singular classically at the big bang.

In section 3 we set up the calculation of the norm of the analogue of the inverse scale factor in LQG, which is a triad-like operator that couples to the quantum scalar matter, with respect to arbitrary spin network states.

In section 4 we specialize the general framework of section 3 to the simplest possible zero volume states: gauge invariant spin network states supported on a trivalent graph. The final result was already displayed in [1] and is given in formula (4.19). We display the unboundedness of this expression graphically by selecting simple configurations. We also display the rather irregular behaviour of the norm of the inverse scale factor on different spin network states.

In section 5 we calculate the norm of the inverse scale factor with respect to kinematical coherent states (for \(U(1)^3\)) and show that the norm remains bounded right at the big bang subject to the constraint that we restrict the valence of the vertex of the graph underlying the coherent states. This seems to be a reasonable restriction on (kinematical) semiclassicality because one would not expect that the Hausdorff dimension of a graph diverges from the classical dimension.

In appendices A, B, C we give a self-contained account of various identities for the volume operator matrix elements and recoupling schemes. We also review the complexifier coherent states [10] needed for our calculation in section 5.

2. Derivation

In this section, we will show how to implement an operator version of \(H_{\text{kin}} = \int_{\Sigma} d^3x \frac{\pi^2(x)}{\det(q)(x)}\) where \(q_{ab}\) is the spatial metric on the hypersurfaces \(\Sigma\) with \(\det(q) \neq 0\ \forall x \in \Sigma\). We will closely follow [8], section 3.3.

\(^4\) Subject to the reservation that currently we can do this only when artificially substituting \(SU(2)\) by \(U(1)^3\), which does not seem to invalidate the qualitative result.
Recall from [8] the following definitions:

\[ V(R) = \int_R d^3 x \sqrt{\det(q)(x)} \]  
(volume of a spatial region \( R \))

\[ \chi_\epsilon(x, y) = \prod_{a=1}^3 \Theta \left( \frac{\epsilon}{2} - |x^a - y^a| \right) \]  
(characteristic function of a cube centred at \( x \) with coordinate volume \( \epsilon^3 \))

\[ V(x, \epsilon) = \int d^3 y \chi_\epsilon(x, y) \sqrt{\det(q)(y)} \]  
(volume of that cube as measured by \( q_{ab} \)).

Here we assume \( \det(q) > 0 \) \( \forall x \in R \) and \( \Theta(z) \) is the usual unit step function with \( \Theta(z) = 0 \) if \( z < 0 \), \( \Theta(z) = \frac{1}{2} \) if \( z = 0 \) and \( \Theta(z) = 1 \) if \( z > 0 \). If we take the limit \( \epsilon \to 0 \) we realize that

\[ \lim_{\epsilon \to 0} \frac{1}{\epsilon^3} \chi_\epsilon(x, y) = \delta(x, y) \]  
and \[ \lim_{\epsilon \to 0} \frac{1}{\epsilon^3} V(x, \epsilon) = \sqrt{\det(q)(x)} \]

such that we have \( \forall x \in R \) and sufficiently small \( \epsilon > 0 \):

\[ \frac{\delta V(R)}{\delta E_\epsilon^i(x)} = \frac{\delta V(x, \epsilon)}{\delta E_\epsilon^i(x)} \]

By using the classical identities

\[ V(R) = \int_R d^3 x \sqrt{\det(q)(x)} \]

\[ = \int_R d^3 x \sqrt{\frac{1}{3!} \epsilon^{ijk} \epsilon_{abc} E_a^i(x) E_b^j(x) E_c^k(x)} \]

and the key identities

\[ E_a^i(x) = \sqrt{\det(q)(x)} e_a^i(x) = \det((e_a^i)) e_a^i(x) \]

where

\[ e_a^i(x) = -\frac{2}{k} \{ A_a^i(x), \mathcal{V}(R) \}. \]

Using (2.1), (2.2) and (2.4) we can now derive a regulated expression:

\[ H_{\text{kin}} = \int_R d^3 x \frac{\pi^2(x)}{\sqrt{\det(q)(x)}} \]

\[ = \int_R d^3 x \pi^2(x) \left[ \frac{\det(e_a^i)}{\det(q)} \right]^2(x) \]
\[
\begin{align*}
&= \lim_{\epsilon \to 0} \int_R d^3 x \, \pi(x) \int_R d^3 y \, \pi(y) \int_R d^3 u \left\{ \det \left( \epsilon_{ij}^a \right) \right\}^2 (u) \\
&\quad \times \int_R d^3 w \left\{ \det \left( \epsilon_{ij}^a \right) \right\}^2 (w) \frac{1}{\epsilon^3} \chi_{(x,y)} \frac{1}{\epsilon^3} \chi_{(u,x)} \frac{1}{\epsilon^3} \chi_{(w,y)} \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon^3} \int_R d^3 x \, \pi(x) \int_R d^3 y \, \pi(y) \int_R d^3 u \frac{1}{3!} \epsilon_{ijk} \epsilon_{abc} \left[ -\frac{8}{\kappa^3} \right] \\
&\quad \times \left\{ A_j^a(u), V(u, \epsilon) \right\} \left\{ A_j^b(u), V(u, \epsilon) \right\} \left\{ A_j^c(u), V(u, \epsilon) \right\} \\
&\quad \times \chi_{(x,y)} \chi_{(u,x)} \chi_{(w,y)} \\
&= \frac{64 \cdot 2^6}{\kappa^6} \lim_{\epsilon \to 0} \int_R d^3 x \, \pi(x) \int_R d^3 y \, \pi(y) \\
&\quad \times \int_R d^3 u \epsilon_{ijk} \epsilon^{abc} \left\{ A_j^a(u), [V(u, \epsilon)]^2 \right\} \left\{ A_j^b(u), [V(u, \epsilon)]^2 \right\} \left\{ A_j^c(u), [V(u, \epsilon)]^2 \right\} \\
&\quad \times \int_R d^3 w \epsilon_{lmn} \epsilon_{def} \left\{ A_j^a(w), [V(w, \epsilon)]^2 \right\} \left\{ A_j^b(w), [V(w, \epsilon)]^2 \right\} \left\{ A_j^c(w), [V(w, \epsilon)]^2 \right\} \\
&\quad \times \chi_{(x,y)} \chi_{(u,x)} \chi_{(w,y)}.
\end{align*}
\]

Now we introduce a triangulation \( T \) of \( R \) adapted to a graph \( \gamma \) as follows: at every vertex \( v \in V(\gamma) \) choose a triple \( e_I, e_J, e_K \) of edges of \( \gamma \) and let a tetrahedron \( \Delta \) be based at \( v \) which is spanned by segments \( s_I, s_J, s_K \) of this triple. Each segment \( s_I \) is given by the part of the corresponding edge \( e_I(t') \) for which the curve parameter \( t' \in [0, \epsilon] \). We have

\[
\int_\Delta d^3 u \epsilon_{ijk} \epsilon^{abc} \left\{ A_j^a(u), [V(u, \epsilon)]^2 \right\} \left\{ A_j^b(u), [V(u, \epsilon)]^2 \right\} \left\{ A_j^c(u), [V(u, \epsilon)]^2 \right\} =
\]

Introduce an adapted coordinate system:

\[
u^a = \sum_{L=1,2,3} s_L^a(t^L) \rightarrow d^3 u = \left| \det \left( \frac{\partial u^a}{\partial t^L} \right) \right| \, dt^I \, dt^J \, dt^K
\]

\[
= \sgn(\det(s_L^a)) \frac{1}{\pi} \epsilon_{def} \epsilon^{ijk} s_L^a(t^I) \hat{s}_j^a(t^J) \hat{s}_k^a(t^K) \, dt^I \, dt^J \, dt^K.
\]

Also note that

\[
\epsilon_{def} \epsilon^{abc} = 3! \, \delta_{[ij]} \delta_{[jk]} \delta_{[kl]}.
\]

\[
= \int_{[0,\epsilon]^3} d^3 t \, \sgn(\det(s_L^a(t))) \epsilon^{ijk} \epsilon_{def} s_L^a(t^I) \hat{s}_j^a(t^J) \hat{s}_k^a(t^K)
\]

\[
\times \left\{ A_j^a(u(t)), [V(u(t), \epsilon)]^2 \right\} \left\{ A_j^b(u(t)), [V(u(t), \epsilon)]^2 \right\} \left\{ A_j^c(u(t)), [V(u(t), \epsilon)]^2 \right\}
\]

5 Of course, this \( \epsilon \) a priori has nothing to do with the \( \epsilon \) we take the limit of. However, one can synchronize both quantities [8], justifying our simplification.
Now we are ready to invoke the relation (valid for small $\epsilon$)

\[ \epsilon \delta_j^i(v) \left\{ A_j^i(v), [V(v, \epsilon)]^\frac{1}{2} \right\} \approx \text{Tr} \left[ \tau_i h_j(\epsilon) \left\{ h_j^{-1}(\epsilon), [V(v, \epsilon)]^\frac{1}{2} \right\} \right] \]

and continue.

\[ \approx \text{sgn}(\det(\delta_j^i))(v) e^{ijk} \text{Tr} \tau_i h_j(\epsilon) \left\{ h_j^{-1}(\epsilon), [V(v, \epsilon)]^\frac{1}{2} \right\} \]

\[ \times \text{Tr} \left[ \tau_i h_j(\epsilon) \left\{ h_j^{-1}(\epsilon), [V(v, \epsilon)]^\frac{1}{2} \right\} \right] \times \text{Tr} \left[ \tau_i h_k(\epsilon) \left\{ h_k^{-1}(\epsilon), [V(v, \epsilon)]^\frac{1}{2} \right\} \right]. \]

The reason for calculating (2.5) is given by the fact that the integration over the spatial region $R$ in (2.5) can be (symbolically) split as follows [3]:

\[ \int_R = \int_{\Gamma'} + \sum_{v \in V(\gamma)} \int_{\Gamma v} \approx \int_{\Gamma'} + \sum_{v \in V(\gamma)} \frac{1}{E(v)} \sum_{e_{\gamma(v,e') \gamma(v,e'')} = v} \left[ \int_{U_{\gamma v}(e_{\gamma(v,e') \gamma(v,e'')} \gamma(v,e''))} + \int_{U_{\gamma v}(e_{\gamma(v,e') \gamma(v,e'')} \gamma(v,e''))} \right] \]

\[ \approx \int_{\Gamma'} + \sum_{v \in V(\gamma)} \frac{1}{E(v)} \sum_{e_{\gamma(v,e') \gamma(v,e'')} = v} \left[ 8 \sum_{\Delta_{\gamma v} (e_{\gamma(v,e') \gamma(v,e'')} \gamma(v,e''))} + \int_{U_{\gamma v}(e_{\gamma(v,e') \gamma(v,e'')} \gamma(v,e''))} \right]. \quad (2.7) \]

Here we have first decomposed $R$ into a region $U_{\gamma v}^1$ not containing the vertices of the graph $\gamma$ and regions $U_{\gamma v}^2$ around each vertex of $v \in V(\gamma)$. Now we choose a triple $e_I, e_J, e_K$ of edges outgoing from $v$ and decompose $U_{\gamma v}^2$ into the region $U_{\gamma v}^2(e_{eIeJ}eK)$ covered by the tetrahedron $\Delta_{\gamma v}^1(e_{eIeJ}eK)$ spanned by $e_I, e_J, e_K$ and its 8 mirror images and the rest $U_{\gamma v}^2(e_{eIeJ}eK)$, not containing $v$. In the last step, we note that the integral over $U_{\gamma v}^2(e_{eIeJ}eK)$ classically converges to 8 times the integral over the original single tetrahedron $\Delta_{\gamma v}^1(e_{eIeJ}eK)$ as we shrink the tetrahedron to zero.

We average over all such triples $e_I \cap e_J \cap e_K = v$ and divide by the number of possible choices of triples for a vertex with $N$ edges $E(v) = \binom{N}{3}$.

We can now decompose the $u$ and $w$ integration over $R$ in (2.5) according to (2.7).

### 2.1. Quantization of the regulated expression

Now we can quantize the thus decomposed expression (2.5) by changing Poisson brackets to commutators times $\frac{1}{\hbar}$. The holonomies act by multiplication only and $V \to \tilde{V}$ becomes the volume operator as defined in [12].

We are now used to spin network function $f_{\gamma}$, we will only have contributions at the vertices $v \in V(\gamma)$ and so only the integration over the tetrahedra $\Delta_{\gamma v}^1(e_{eIeJ}eK)$ has to be considered.

Also the smeared momenta $\pi(R) = \int_R d^3x \pi(x) \chi_\phi(x, v)$ of the scalar field $\phi$ will in the limit $\epsilon \to 0$ turn into $-\hbar k X(v)$ where $X(v) := \frac{1}{2} \left[ X_R(v) + X_L(v) \right]$ is the sum over right and left invariant vector fields acting on the point holonomies $U(v)$ as defined in [14].

So we will work in a Hilbert space $H = H_{\text{gravity}} \otimes H_{\text{matter}}$ as introduced in [14] and [20]. Here we are only interested in the action of $H_{\text{matter}}$ on the gravitational part $H_{\text{gravity}} = H_{\text{AL}}$, the usual Ashtekar Lewandowski Hilbert space of loop quantum gravity.

Note that we can identify the sections $s_I$ used in (2.6) of the edge $e_I$ with edges of the graph $\gamma$ by introducing trivial 2-valent vertices $v_{riv}$ and write $h_{e_I[0,1]} = h_{e_I[0,1]} h_{e_I[e,1]} = h_{e_I[0,1]} h_{e_I[e,1]}$. Now the vertices $v_{riv}$ do not change the gauge behaviour of $f_{\gamma}$ and they are
annihilated by $\hat{V}$. In what follows, we will imply this construction and replace $s_f \rightarrow e_I$.

So we can determine the action of $H_{\text{kin}}$ on a cylindrical function $f_r$, starting from (2.5) and using (2.6), (2.7). We will use the shorthand $\epsilon(IJK) := \text{sgn}(\det(\hat{e}_i \hat{e}_j \hat{e}_k))$. Now we take the limit $\epsilon \rightarrow 0$ and we must have $x = y = v = v'$. For convenience, we introduce

\[
\langle \epsilon \rangle \hat{e}_I^c (v) := \lim_{\epsilon \rightarrow 0} \text{Tr} \left[ \tau_j \hat{h}_I (v) \left[ \hat{h}_I^{-1} (v), \hat{V}^C \right] \right] \bigg|_{v \in \mathcal{V}(y)}. \tag{2.9}
\]

It is easy to see that the limit is trivial, i.e., the operator $\langle \epsilon \rangle \hat{e}_I^c (v)$ is $\epsilon$-independent for sufficiently small $\epsilon$. We will not display the $\epsilon$-dependence any more in what follows. Note that $\langle \epsilon \rangle \hat{e}_I^c (v) = \langle \epsilon \rangle \hat{e}_I^c (v)$. Moreover, we will abbreviate $\sum_{\epsilon_f \epsilon_j \epsilon_k \epsilon_x = u} \rightarrow \sum_{IJK}$. Finally, we find:

\[
\hat{H}_{\text{kin}} f_r = \left[ \frac{P}{\hbar^2 K^3} \sum_{v \in \mathcal{V}(y)} \frac{1}{E(v)} X(v) \sum_{IJK} \epsilon(IJK) e^{IK} \epsilon_{ijk} \hat{e}_I^c (v) \hat{e}_j^c (v) \hat{e}_k^c (v) \right] f_r
\]

\[
\times \left[ \frac{1}{E(v)} X(v) \sum_{LMN} \epsilon(LMN) e^{LMN} \epsilon_{lmn} \hat{e}_L^c (v) \hat{e}_m^c (v) \hat{e}_n^c (v) \right] f_r \tag{2.10}
\]

where $P = \frac{\hbar^2}{2}$ and $X(v) = \lim_{\epsilon \rightarrow 0} \int d^3x \chi(x, y) X(x)$. This reduces the problem to analysing the smeared operator: $\text{Tr} \left[ \tau_j \hat{h}_I \left[ \hat{h}_I^{-1}, \hat{V}^C \right] \right] = \langle \epsilon \rangle \hat{e}_I^c$. This will be done in the following.

**3. Calculation**

In this section, we will calculate the action of the $(\epsilon \hat{e}^c_K)$-operators in a recoupling scheme basis. For definitions, see appendix B.

**3.1. General case**

Let $|T_f\rangle := |\tilde{a} J' M'; \tilde{j} ; \tilde{n}\rangle$ and $|T_J\rangle := |\tilde{a} J M; \tilde{j}; \tilde{n}\rangle$. Then we have

\[
\langle T_f | (\epsilon \hat{e}^c_K) | T_J \rangle = \langle T_f | \text{Tr} \left[ \tau_j \hat{h}_K \left[ \hat{h}_K^{-1}, \hat{V}^C \right] \right] | T_J \rangle
\]

\[
= \langle T_f | \text{Tr} \left[ \tau_j \hat{h}_K \hat{h}_K^{-1} \hat{V}^C - \tau_j \hat{h}_K \hat{V}^C \hat{h}_K^{-1} \right] | T_J \rangle
\]
By linearity of $\text{Tr}$ we have
\[
\text{Tr}\left[\tau_k h_K h_K^{-1} \dot{V} - \tau_k h_K h_K^{-1} K \hat{V} - \tau_k h_K h_K^{-1} h_K^{-1} \right] = \text{Tr}\left[\tau_k \hat{V} - \tau_k h_K h_K^{-1} \hat{V} h_K^{-1} h_K^{-1} \right]
\]
since the $\tau_k$ are trace free.

\[
= -\langle T_J | \text{Tr} \left[ \tau_k h_K h_K^{-1} \hat{V} h_K^{-1} \right] | T_J \rangle
\]

Using the properties of $SU(2)$ given in appendix A we have
\[
h_K = \epsilon^{-1} h_K \epsilon = \epsilon h_K \epsilon^{-1}
\]
\[
h_K^{-1} = \epsilon [h_K]^T \epsilon^{-1}
\]
where the overline denotes complex conjugation.

\[
= -\langle T_J | \text{Tr} \left[ \tau_k \epsilon h_K \epsilon^{-1} \dot{V} \epsilon^{-1} \right] | T_J \rangle
\]

Use cyclicity of $\text{Tr}$ and remove the slashed inner $\epsilon$-terms.

\[
= -\langle T_J | \text{Tr} \left[ \tau_k \epsilon h_K \epsilon^{-1} \dot{V} \epsilon^{-1} \right] | T_J \rangle
\]

In the last line, we have used the expansion coefficients $C_{T_J}^{T_J}(K, A, B)$ defined by

\[
[h_K]_{AB} | T_J \rangle = \sum_{T_J} \langle T_J | [h_K]_{AB} \cdot | T_J \rangle = \sum_{T_J} C_{T_J}^{T_J}(K, A, B) \cdot | T_J \rangle
\]

and explicitly calculated in (B.21). Moreover, we sum over all quantum numbers of the possible intermediate states $| T_J \rangle := | a \dot{J} M; \dot{j}; \dot{\vec{n}} \rangle$ and $| T_J \rangle := | a \dot{J} M; \dot{j}; \dot{\vec{n}} \rangle$.

If we now insert the explicit result of (B.21) we find, as a general expression for the matrix element,

\[
\langle T_J | (\epsilon_{j}^k K) | T_J \rangle = -\sum_{A, B} | T_J \rangle \sum_{T_J \epsilon} \sum_{T_J \epsilon} C_{T_J}^{T_J}(K, A, B) C_{T_J}^{T_J}(K, C, B) \langle T_J | \dot{V} | T_J \rangle
\]
Now the volume operator only affects the intermediate recouplings $\tilde{a}', \tilde{a}$ so we must have (in order to get non-vanishing contributions)

$$\tilde{j}' = \tilde{j} \rightarrow J' = \begin{cases} J - 1 & \text{because } \tilde{j}' = J' \pm \frac{1}{2}, \tilde{j} = J \pm \frac{1}{2} \\ J + 1 & \end{cases}$$

$\tilde{M}' = \tilde{M} \rightarrow M' = M + C - A$

because $\tilde{M}' = M' + A, \tilde{M} = M + C$

$$\tilde{n}' = \tilde{n} \rightarrow n'_L = n_L \forall L \neq K, n'_K = n_K$$

because $\tilde{n}'_K = n'_K + B, \tilde{n}_K = n_K + B$. 

$$\sim \tilde{n} = [n_1, \ldots, n_{K-1}, n_K + B, n_{K+1}, \ldots, n_N]$$

$$\sim \tilde{n} = [n_1, \ldots, n_{K-1}, n_K + B, n_{K+1}, \ldots, n_N]$$

By inspection of (3.3) we can see that the sum over the index $B = \pm \frac{1}{2}$ will only affect the factors $C^{i_{j_k}}_{j_k}(B, n'_K)C^{i_{j_k}}_{j_k}(B, n'_K)$. Since $n'_K = n_K$ and always $\tilde{j}'_K = \tilde{j}_K$ we can explicitly evaluate $\sum_{B = \pm \frac{1}{2}} C^{i_{j_k}}_{j_k}(B, n_K)C^{i_{j_k}}_{j_k}(B, n_K)$. In order to get non-vanishing contributions, we have the following cases:

$$j'_K = j_K - 1 \rightarrow j'_K = j_K - \frac{1}{2}, \tilde{j}'_K = j'_K + \frac{1}{2}, \sum_{B = \pm \frac{1}{2}} C^{i_{j_k}}_{j_k}(B, n_K)C^{i_{j_k}}_{j_k}(B, n_K)$$

$$= \sum_{B = \pm \frac{1}{2}} \left[ \frac{j'_K + 2Bn_K + 1}{2(j'_K + 1)} \cdot \frac{j_K - 2Bn_K}{2j_K} \right]^\frac{1}{2} = 0$$

$$j'_K = j_K \rightarrow j'_K = j_K - \frac{1}{2}, \sum_{B = \pm \frac{1}{2}} C^{i_{j_k}}_{j_k}(B, n_K)C^{i_{j_k}}_{j_k}(B, n_K) = \sum_{B = \pm \frac{1}{2}} 4B^2 \frac{j_K - 2Bn_K}{2j_K} = 1$$
\[
\tilde{j}_K = j_K + \frac{1}{2} \sum_{B=\pm \frac{1}{2}} 4B^2 \frac{j_K + 2Bn_K + 1}{2(j_K + 1)} = 1
\]

So we get
\[
j' = j + 1 \rightarrow \tilde{j}_K = j_K + \frac{1}{2} \tilde{j}_K = j_K + \frac{1}{2} \sum_{B=\pm \frac{1}{2}} C_{j_k}^{j_K}(B, n_K) C_{j_k}^{j_K}(B, n_K) = 0.
\]

So we can carry out the sum over \(B\) and have as additional selection rule \(j_K = \frac{1}{2} \). This is a consistency check for the following reason: during the regularization process of the operator \(\hat{H}_{\text{kin}}\) as defined in (2.10) we had to introduce trivial 2-valent vertices when decomposing the edges \(e \rightarrow s[0, \epsilon] \circ e[\epsilon, 1]\), which do not change the gauge behaviour of the spin network function over the graph \(\chi\). In order to be gauge invariant, the spin \(j\) of the segment \(s\) must not be modified by the \(|^{(s)}e^k_s\)-operators. Moreover, the \(|^{(s)}e^k_s\) transform in the \(J = 1\) representation.

So we get \(j' = j + 1\).

The result is a simplified expression for the matrix element of \(\|^{(s)}e^k_s\):

\[
\langle T' |^{(s)}e^k_s | T \rangle = \langle \tilde{a} J'M'; \tilde{j}; \tilde{n} |^{(s)}e^k_s | \tilde{a} J'M; \tilde{j}; \tilde{n} \rangle = -\sum_{AC}^{[\tau_3]} [\tilde{a} J'M'; \tilde{j}; \tilde{n} |^{(s)}e^k_s | \tilde{a} J'M; \tilde{j}; \tilde{n} \rangle
\]

\[
= -\sum_{AC}^{[\tau_3]} [\tilde{a} J'M'; \tilde{j}; \tilde{n} |^{(s)}e^k_s | \tilde{a} J'M; \tilde{j}; \tilde{n} \rangle
\]

\[
\times \left\{ \sum_{\tilde{a}, A} \frac{\delta_{A 2}}{\delta K} C_{j_K}^{j_k}(A, M', g_{N-1}) C_{j_K}^{j_k}(A, M, g_{N-1}) \delta_{A 2} \delta_{K 2} \delta_{K 2} \right\}.
\]

Here we have used the abbreviations (according to (B.21) and (B.18))

\[
C_{j_K}^{j_k}(A, M, g_{N-1}) = \sum_m \langle g_{N-1} m; j_K M - m | J (g_{N-1} j_K) M \rangle
\]

\[
\times \langle g_{N-1} m; j_K M - m + A_0 | \tilde{J} (g_{N-1} j_K) M + A \rangle
\]

\[
\times \langle j_K M - m; 1/2 A | \tilde{J}(j_K M - m + A) \rangle.
\]

\[
C_{\tilde{a}, j_K}(g_K, \ldots, g_{N-1}) = \langle \tilde{a} K (a_{K-1} j_K) \tilde{a} K+1 (\tilde{a} K j_{K+1}) | g_K (a_{K-1} j_{K+1}) \tilde{a} K+1 (g_K j_K) \rangle
\]

\[
\times \langle a_{K-1} j_K | a_{K+1} (a_{K j_{K+1}) g_K (a_{K-1} j_{K+1}) a_{K+1} (g_K j_K) \rangle
\]

\[
\times \langle \tilde{a} K+1 (g_K j_K) \tilde{a} K+2 (\tilde{a} K+1 j_{K+2}) | g_K+1 (g_K j_{K+2}) \tilde{a} K+2 (g_K+1 j_{K+2}) \rangle
\]

\[
\times \delta_{\tilde{j} j} \delta_{n n}.
\]
where the individual $g_k \ldots g_{N-1}$ can take all values allowed by their arguments due to the Clebsch–Gordan theorem and (B.17)

\[
\tilde{j} = \{j_1, \ldots, j_{K-1}, j_K, j_{K+1}, \ldots, j_N\}
\]

\[
\tilde{a} = \{a_2(j_1j_2)a_3(a_2j_3) \ldots a_{K-1}(a_{K-2}j_{K-1})a_K(a_{K-1}j_{K})a_{K+1}(a_Kj_{K+1}) \ldots a_{N-1}(a_{N-2}j_{N-1})J(a_{N-1}j_{N})\}
\]

\[
\tilde{a}' = \{a_2(j_1j_2)a_3(a_2j_3) \ldots a_{K-1}(a_{K-2}j_{K-1})a_K(a_{K-1}j_{K})a_{K+1}(a_Kj_{K+1}) \ldots a_{N-1}(a_{N-2}j_{N-1})J'(a_{N-1}j_{N})\}
\]

\[
\tilde{a}'' = \{a_2(j_1j_2)a_3(a_2j_3) \ldots a_{K-1}(a_{K-2}j_{K-1})a_K(a_{K-1}j_{K})a_{K+1}(a_Kj_{K+1}) \ldots a_{N-1}(a_{N-2}j_{N-1})\tilde{J}(a_{N-1}j_{N})\}
\]

\[
\tilde{g} = \{g_2(j_1j_2)g_3(g_2j_3) \ldots g_{K-1}(g_{K-2}j_{K-1})g_K(g_{K-1}j_{K})g_{K+1}(g_Kj_{K+2}) \ldots g_{N-1}(g_{N-2}j_{N})J(g_{N-1}j_{K})\}
\]

3.2. Special cases

Furthermore for the special cases $K = 1$, $K = 2$ and $K = N$ we find the following.

3.2.1. $K = 1$

\[
\langle T_j | ^{(\nu)} e_{K=1}^J | T_j \rangle = \langle \tilde{a} J M', \tilde{j}; \tilde{n} | ^{(\nu)} e_{K=1}^J | \tilde{a} J M; \tilde{j}; \tilde{n} \rangle
\]

\[
= - \sum_{AC} | \mathsf{T}_2 | A \delta_{M'M + C - A} \sum_{J_{\tilde{j}} = J_{\tilde{j}} = \frac{1}{2}} \sum_{J'_{\tilde{j}} = J'_{\tilde{j}} = \frac{1}{2}} \delta_{J_{\tilde{j}} J'_{\tilde{j}}} \sum_{\tilde{a} \tilde{a}'} \times \left\{ \sum_{g_2 \cdot g_{N-1}} C_{J_{\tilde{j}} j_{\tilde{j}}}^{j_{\tilde{j}}} (A, M', g'_{N-1}) C_{\tilde{a} j_{\tilde{j}}}^{\tilde{a} j_{\tilde{j}}} (g_2', \ldots, g_{N-1}) \right. \\

\times \sum_{g_2 \cdot g_{N-1}} C_{J_{\tilde{j}} j_{\tilde{j}}}^{j_{\tilde{j}}} (C, M, g_{N-1}) C_{\tilde{a} j_{\tilde{j}}}^{\tilde{a} j_{\tilde{j}}} (g_2', \ldots, g_{N-1}) \\

\times \langle \tilde{a} J M' + A; j; \tilde{n} \rangle \langle \tilde{a} J M + C; j; \tilde{n} \rangle \delta_{J_{\tilde{j}} J'_{\tilde{j}}} \delta_{\tilde{a} \tilde{a}' \tilde{n} n} \right\}.
\]  

(3.5)
Here we have used the same shorthand as in (3.4), but we must use (B.16) instead of (B.15) in the definition of the coefficients $C_{\tilde{a} J_n}^{\tilde{a} J_n}(g_2, \ldots, g_{N-1})$ and the intermediate recoupling scheme $\tilde{g}$.

$$C_{\tilde{a} J_n}^{\tilde{a} J_n}(g_2, \ldots, g_{N-1}) = \langle \tilde{a}_2(\tilde{J}_2) \tilde{a}_3(\tilde{J}_3) \tilde{a}_4(\tilde{J}_4) \rangle (a_2(j_2 a_3(a_3 j_3)) g_2(j_2 j_3) a_3 \langle g_2 j_1 \rangle) (a_3(j_3) a_3(a_3 j_3)) g_3(j_3 j_3) a_3 \langle g_3 j_3 \rangle) \cdots \langle \tilde{a}_{N-3}(\tilde{g}_{N-3}) \tilde{a}_{N-2}(\tilde{g}_{N-2}) \tilde{a}_{N-1}(\tilde{g}_{N-1}) \rangle \langle \tilde{g}_{N-2}(\tilde{g}_{N-1}) \tilde{g}_{N-1}(\tilde{g}_{N-1}) \rangle \langle \tilde{g}_{N-2}(\tilde{g}_{N-1}) \tilde{g}_{N-1}(\tilde{g}_{N-1}) \rangle \langle \tilde{g}_{N-2}(\tilde{g}_{N-1}) \tilde{g}_{N-1}(\tilde{g}_{N-1}) \rangle$$

where the individual $g_2 \ldots g_{N-1}$ can take all values allowed by their arguments due to the Clebsch–Gordan theorem and (B.17),

$$\tilde{g} = \langle g_2(j_2 j_3) g_3(j_3 j_4) \ldots g_{N-2}(j_{N-2} j_{N-1}) \rangle (g_{N-1}(j_{N-1} j_{N-1})).$$

### 3.2.2. $K = 2$

This case can be handled by (3.4). One only has to leave out the prefactor $\prod_{K=2}^{K-1} \delta a \alpha \beta \beta a \alpha \beta$.

### 3.2.3. $K = N$

In this case, the expansion (B.15) into intermediate recoupling schemes is not necessary because the spin the holonomies in $\kappa^k M$ act on is already the last one and we can work directly with (B.20) instead to obtain

$$\langle T_J | \kappa^k M | T_J \rangle = \langle \tilde{a} J M^*; \tilde{j} \tilde{n} | \kappa^k M | \tilde{a} J M; \tilde{j} \tilde{n} \rangle = - \sum_{AC} |T_{k} | C A \delta_{M^* M} \delta_{C - A} \sum_{j = j_2 = \frac{1}{2}}^{j} \sum_{j = j_2 = \frac{1}{2}}^{J} \delta_{j,j} \times \left\{ C_{J n}^{J n} (A, M^*, A_{N-1}) C_{J n}^{J n} (C, M, A_{N-1}) \right\}.$$

### 3.3. Evaluation of the matrix elements $\langle | \tilde{V}^r | \rangle$

In order to evaluate the matrix elements $\langle \tilde{a} J M^* + A; \tilde{j} \tilde{n} | \tilde{V}^r | \tilde{a} J M + C; \tilde{j} \tilde{n} \rangle = \langle \tilde{T}_J | \tilde{V}^r | \tilde{T}_J \rangle$ in (3.4) containing the $r$th power of the volume operator, we have to recall the definition of the volume operator as $\tilde{V} := (\ell P)^r \tilde{V} \tilde{V} = \sqrt{|q|}$ with $\tilde{q}$ being the matrix analysed in [11] and $\ell P$ the Planck length. It is, in particular, a real antisymmetric matrix times $i$. So the eigenvalues $\lambda_3$ of $\tilde{q}$ are real and come in pairs $\pm \lambda_3$ or $\lambda_3 = 0$ if the dimension of $\tilde{q}$ is odd. Moreover, its eigenvectors $| \gamma(\lambda_3) \rangle$ are orthogonal.
The definition of $\hat{V}$ is to be understood as follows. For each of its eigenvalues $\lambda_q$ the matrix $\hat{q}$ has an eigenvector $|\gamma(\lambda_q)\rangle$.

$\hat{V}$ has the same eigenvectors as $\hat{q}$ but its eigenvalues are given by $(\ell_p)^3 \sqrt{|\lambda_q|}$.

We have to insert a $1 = \sum_{\lambda_q} |\gamma(\lambda_q)\rangle\langle\gamma(\lambda_q)|$ in terms of normalized eigenstates $|\gamma(\lambda_q)\rangle$ belonging to each eigenvalue (of course counting multiplicity!) $\lambda_q$ of $\hat{q}$ in order to evaluate $\hat{V}$:

$$
\langle \hat{T}_I | \hat{V} | \hat{T}_J \rangle = \sum_{\lambda_q,\lambda_q'} \langle \hat{T}_I | \gamma(\lambda_q) \rangle \langle \gamma(\lambda_q) | \hat{V} | \gamma(\lambda_q') \rangle \langle \gamma(\lambda_q') | \hat{T}_J \rangle \sqrt{|\lambda_q|} \delta_{\lambda_q,\lambda_q'}
$$

$$
= (\ell_p)^3 \sum_{\lambda_q} |\lambda_q|^{\frac{1}{2}} \cdot \langle \hat{T}_I | \gamma(\lambda_q) \rangle \langle \hat{T}_J | \gamma(\lambda_q) \rangle.
$$

Sometimes it might be more convenient to use the matrix $\hat{q}\hat{q}$ instead, because as a real symmetric matrix its eigenvalues $\lambda_q\lambda_q'$ are real and $\geq 0$ from the beginning, and we do not need to introduce by hand the modulus from the definition of $\hat{V}$ in terms of $\hat{q}$ as required in (3.7), but can write

$$
\langle \hat{T}_I | \hat{V} | \hat{T}_J \rangle = (\ell_p)^3 \sum_{\lambda_q\lambda_q'} (\lambda_q\lambda_q')^{\frac{1}{2}} \cdot \langle \hat{T}_I | \gamma(\lambda_q\lambda_q') \rangle \langle \hat{T}_J | \gamma(\lambda_q\lambda_q') \rangle.
$$

(3.8)

Anyway, we need to know explicitly the spectrum and the eigenstates of $\hat{V}$. This will only be possible to do exactly in very special cases.

4. Gauge invariant 3-vertex

4.1. General properties

As we will see in the simple case of the gauge invariant 3-vertex, it is possible to explicitly evaluate the action of $\hat{H}_{\text{kin}}$ as derived in (2.10). We note that $E(3) = 1$ and look at one 3-vertex only. Then we realize that the sign factors $\epsilon(IJK)\epsilon(LMN) \longrightarrow \epsilon(123)\epsilon(123) = 1$, because every permutation of the edges changing the sign of $\epsilon(123)$ will also change the sign of $\epsilon(IJK)$. So we have to consider only the triple (123). Then (2.10) reads (we get an additional multiplicity factor of $6 \cdot 6$ while summing over all permutations of the edge triple 1, 2, 3)

$$
\hat{H}_{\text{kin}}^{(3)} = \frac{P \cdot 6^3}{h^2 \kappa^4} X(v) X(v) e^{(UK)} e^{(LMN)} \epsilon_{ijk} \epsilon_{lmn} (\epsilon^i_f(v)(\epsilon^j_f(v)(\epsilon^k_f(v)) (\epsilon^k_f(v))(\epsilon^m_L(v))(\epsilon^m_M(v))(\epsilon^m_N(v)) f_{v}'.
$$

Now we again take advantage of the identity

$$
\epsilon_{ijk} \epsilon_{lmn} = 6 \cdot \delta_{ij} [\delta^j_m \delta^k_n]
$$

and introduce the manifestly gauge invariant quantities

$$
\hat{q}_{IL}(r, v) = (\epsilon^i_f(v)(\epsilon^j_f(v)(\epsilon^k_f(v)) \delta_{ij}.
$$

(4.1)

We have introduced the quantity

$$
e^{(v)}_2 = e^{(UK)} e^{(LMN)} q_{IL} \left(\frac{1}{2}\right) q_{JM} \left(\frac{1}{2}\right) q_{KN} \left(\frac{1}{2}\right)
$$

(4.2)
which is the gravitational part of $\hat{H}_\text{kin}$ and up to a constant the corresponding operator version $\frac{1}{\sqrt{\det(q_{IJ})}}$. In what follows, we will explicitly calculate its expectation values w.r.t. gauge invariant states $f^\nu_\flat$ at an unspecified 3-vertex; therefore we drop the argument $\nu$ of $e'(\nu)^2$. At the gauge invariant 3-vertex, we have
\[
\|e^2[0]\|^2 = \langle 0|e'[1]e'[2]|0\rangle = \langle 0|\langle e'[2]|0\rangle\langle e'[1]|0\rangle = \|\langle e'[2]|0\rangle\|^2. \tag{4.3}
\]
Furthermore,
\[
\langle 0|e'[2]|0\rangle = e^{i\epsilon_{L}^{\lambda\mu\nu}}q_{iL}(\frac{1}{2})q_{iM}(\frac{1}{2})q_{KN}(\frac{1}{2})|0\rangle = e^{i\epsilon_{L}^{\lambda\mu\nu}}q_{iL}(\frac{1}{2})|0\rangle(0)|q_{iM}(\frac{1}{2})|0\rangle q_{KN}(\frac{1}{2})|0\rangle. \tag{4.4}
\]
We can evaluate this as follows (the $e'_j$ transform in the $J = 1$ representation of $SU(2)$):
\[
\langle 0|q_{iL}(\frac{1}{2})|0\rangle = \sum_{i} \langle 0|(\frac{1}{2})e'_j|1\rangle\langle 1|(\frac{1}{2})e'_L|0\rangle \delta_{ii} \sum_{J=1}^{J=-1} \sum_{M=0}^{M=-1} a^i_j = \sum_{J=1}^{J=-1} \sum_{M=0}^{M=-1} \sum_{J=1}^{J=-1} \sum_{M=0}^{M=0} \langle J=1^+|e'_j|2^+_2\rangle \langle 2^+_2|e'_j|2^+_1\rangle J(J+1)J(J+1)J(J+1)
\]
\[
\times (a^i_j j_1) J'(a^i_j j_2) = 1 M'|\langle \frac{1}{2}\rangle e'_j|a_2(j_1 j_2) = j_1 J(a_2 j_2) = 0 M = 0 \rangle \times (a^i_j j_1) J'(a^i_j j_2) = 1 M'|\langle \frac{1}{2}\rangle e'_j|a_2(j_1 j_2) = j_1 J(a_2 j_2) = 0 M = 0 \rangle.
\tag{4.5}
\]
Here the overline denotes complex conjugation. As we can see, the remaining task is to calculate matrix elements of the form $\langle 0|\langle e'[2]|1\rangle$ at the gauge invariant 3-vertex. We will apply the general expression (3.4) separately for every $I = 1, 2, 3$.

4.1.1. The coefficients $C_{Jk}^{I_k}(A, M, g_{N-1})$. It is difficult to find a general expression for coefficients of the form $C_{Jk}^{I_k}(A, M, g_{N-1})$ occurring in equation (3.4). However, one can in principle carry out the summation over $-g_{N-1} \leq m \leq g_{N-1}$ where the Clebsch–Gordan coefficients are only contributing for $j_k$ compatible values of $m$.

For the given case of the matrix elements of $(T_{J=1}^{I=1})^k |\langle \bar{V}'| \rangle$ the relevant non-vanishing coefficients $C_{Jk}^{I_k}(A, M, g_{N-1})$ can be calculated to be (MATHEMATICA)

| $I$ | $J$ | $M$ | $A$ | $g_{N-1}$ | $j_k$ |
|-----|-----|-----|-----|-----------|------|
| $0$ | $\frac{1}{2}$ | $0$ | $\pm \frac{1}{2}$ | $g_{N-1} = j_k$ | $j_{k} = j_{k} + \frac{1}{2}$ |
| $1$ | $\frac{1}{2}$ | $-1$ | $\pm \frac{1}{2}$ | $g_{N-1}$ | $-j_{k}$ |
| $1$ | $\frac{1}{2}$ | $0$ | $\pm \frac{1}{2}$ | $g_{N-1}$ | $-j_{k}$ |
| $1$ | $\frac{1}{2}$ | $1$ | $\pm \frac{1}{2}$ | $g_{N-1}$ | $-j_{k}$ |

4.1.2. The matrix elements $\langle \frac{1}{\sqrt{\det(q_{IJ})}}|\bar{V}'|\rangle$. As one can easily see, for $N = 3$, $J' = 1$, $J = 0$ the only allowed intermediate total angular momentum in the matrix elements $\langle \frac{1}{\sqrt{\det(q_{IJ})}}|\bar{V}'|\rangle$ of (3.4) is given by $J = \frac{1}{2}$. Therefore we have to evaluate expressions of the general form:
\[
\langle a_{2}^{J}(j_{1} j_{2}) J(a_{2}^{J} j_{3}) = \frac{1}{2} M'|\bar{V}'|a_{2}(j_{1} j_{2}) J(a_{2}^{J} j_{3}) = \frac{1}{2} M \rangle. \tag{4.6}
\]
Note that we have to perform an eigenvector expansion according to (3.8). In our special case, \( \hat{V} \) takes the form
\[
\hat{V} = (\ell_p)^3 \sqrt{|\bar{Z} \cdot [(J_{12})^2, (J_{23})^2]|} \quad \text{with} \quad J_{12} = J_1 + J_2 \quad \text{and} \quad J_{23} = J_2 + J_3
\]
(4.7)

Here \( \ell_p \) denotes the Planck length, \( \bar{Z} \) is a constant prefactor dependent on the regularization, on the Immirzi parameter \( \beta \) and on the relative orientation of the edges \( e_1, e_2, e_3 \); see [12] for details\(^6\). The matrix elements of \( \hat{q}_{123} = [(J_{12})^2, (J_{23})^2] \) have the general form at the \( N \)-vertex (see [11] for details\(^7\)):
\[
\langle a_2 | \hat{q}_{123} | a_2 - 1 \rangle = -(a_2 - 1) \hat{q}_{123} | a_2 \rangle = \frac{i}{\sqrt{(2a_2 - 1)(2a_2 + 1)}} [(j_1 + j_2 + a_2 + 1) \times (-j_1 + j_2 + a_2)(j_1 - j_2 + a_2)(j_1 + j_2 - a_2 + 1)(j_3 + a_3 + a_2 + 1) \times (-j_3 + a_3 + a_2)(j_3 - a_3 + a_2)(j_3 + a_3 - a_2 + 1)]^2.
\]
(4.8)

For \( N = 3 \) we have to set \( a_3 = a_N = \bar{J} = \frac{1}{2} \). Now for \( \bar{J} = \frac{1}{2} \) we have \( a_2, a'_2 = j_3 \pm \frac{1}{2} \), hence the Hilbert space \( \hat{V} \) is two dimensional and \( \hat{q}_{123} \) is of the form
\[
\hat{q}_{123} = \begin{pmatrix}
0 & -iA_1 \\
iA_1 & 0
\end{pmatrix},
\]
(4.9)

Here \( A_1 \) is (4.8) evaluated at \( a_3 = \bar{J} = \frac{1}{2} \) and \( a_2 + a'_2 = j_3 + \frac{1}{2} \):
\[
A_1 = A_1(j_1, j_2, j_3, \bar{J}, a_2) = MEV2D[\bar{J}] = -i \cdot \langle j_3 + \frac{1}{2} | \hat{q}_{123} | j_3 - \frac{1}{2} \rangle.
\]
(4.10)

So we can immediately see that
\[
\hat{q}^\dagger \hat{q} = \begin{pmatrix}
|A_1|^2 & 0 \\
0 & |A_1|^2
\end{pmatrix}
\]
(4.11)
is already diagonal with eigenvalue \( |A_1|^2 \) and eigenvectors \( (1 \ 0) \) and \( (0 \ 1) \). So
\[
\langle a_2'(j_1, j_2) \bar{J} a_2'(j_1, j_2) | \hat{V}^r | a_2(j_1, j_2) \bar{J} a_2'(j_1, j_2) \rangle = \frac{1}{2} M = (\ell_p)^3 |\bar{Z}|^2 \cdot |A_1|^2 \delta_{a_2 a_2'}.
\]
(4.12)

In the following calculations we will frequently use (4.12). Keep in mind that (4.12) is derived for general \( j_1, j_2, j_3 \), which will be modified due to the action of holonomies in the \( e \)-operators! In order to avoid confusion, we will always write down all the arguments of \( A_1 = MEV2D[\bar{J}] \).

4.2. Calculation of the different cases

4.2.1. \( K = I \). Due to the special case we start with (3.5). We always have \( \bar{J} = \frac{1}{2} \):

\(^6\) There \( \bar{Z} \) takes the value \( \bar{Z} = \frac{1}{4} \cdot \beta^3 \cdot (\bar{Z})^3 \), \( \beta \) being the Immirzi parameter.

\(^7\) A similar expression was also derived in [13] using a different method.
\[
(a'_2(j_1 j_2))J'(a'_2 j_3) = \begin{cases} 
M' |_{A}^{|A}| a_2(j_1 j_2) J(a_2 j_3) = 0 & M = 0 \\
\sum_{\tilde{g}^2} C_{\tilde{g}^2}^{J=1j_1} (A, M', g_2') (a'_2(j_1 j_2) j'(\tilde{a}'_2 j_3)) |g_2'(j_2 j_3) j(g_2' j_1)) \\
\times (a'_2(j_1 j_2) J'(a'_2 j_3) |g_2'(j_2 j_3) J'(g_2' j_1)) \\
\times \sum_{g_2} C_{g_2}^{J=0j_1} (C, M, g_2) (\tilde{a}_2(j_1 j_2) j(\tilde{a}_2 j_3)) |g_2(j_2 j_3) j(g_2 j_1)) \\
\times (a_2(j_1 j_2) J(a_2 j_3) |g_2(j_2 j_3) J(g_2 j_1)) \\
\times (\tilde{a}'_2(j_1 j_2) j(\tilde{a}'_2 j_3)) = \frac{1}{2} M' + A |V^r| \tilde{a}_2(j_1 j_2) j(\tilde{a}_2 j_3) = \frac{1}{2} M + C \\
\end{cases}
\]

Note:
(i) Due to integer arguments the \((-1)\) prefactors are equal to 1
(ii) With (4.12) we have \(\tilde{a}'_2 = \tilde{a}_2 = j_3 \pm \frac{1}{2}\) and have \(M E V 2 D [j_1, j_2, j_3, \tilde{j} = \frac{1}{2}, a_2 = j_3 + \frac{1}{2}]\), independent of \(\tilde{a}'_2\)
   \(\rightarrow\) we can use the orthogonality relations of the 6j symbols (see [15], p 96) in order to carry out the remaining sum over \(\tilde{a}'_2:\)

\[
\sum_{\tilde{a}'_2} (2\tilde{a}'_2 + 1) \begin{vmatrix} j_1 & j_2 & \tilde{a}'_2 \\
j_3 & j & g_2' \end{vmatrix} \begin{vmatrix} j_1 & j_2 & \tilde{a}'_2 \\
j_3 & j & g_2' \end{vmatrix} = \frac{1}{(2g_2 + 1)} \delta_{\tilde{g}^2 g_2} \sim \begin{vmatrix} j_1 & j_2 & \tilde{a}'_2 \\
j_3 & j & g_2 \end{vmatrix} = \frac{1}{2g_2 + 1}.
\]

(iii) For \(J = 0\) we have
(a) \(a_2 = j_3\),
(b) \(\begin{vmatrix} j_1 & j_2 & a_2 \\
j_3 & j & g_2 \end{vmatrix} = \begin{vmatrix} j_1 & j_2 & j_3 \\
j_3 & 0 & 0 \end{vmatrix} = \begin{vmatrix} j_2 & j_1 & j_3 \\
j_3 & 0 & 1 \end{vmatrix} = \frac{(-1)^{j_1 + j_2 + j_3}}{\sqrt{(2j_1 + 1)(2j_3 + 1)}}.\)
\[
\begin{align*}
&= - \sum_{AC} [\mathcal{T}_k \mathcal{C} \delta_{\mathcal{M}' \mathcal{M} + C - A} (-1)^{j_1 + j_2 + j_3} \sqrt{(2a'_2 + 1)(2a'_2 + 1)} \{ j_2 \quad j_1 \quad a'_2 \}]
\times \sum_{j_3 = j_1 + \frac{1}{2}} C^{J = j_1}_{j_3} (A, M', j_1) C^{J = 0}_{j_2} (C, 0, j_1)
\times (\ell_p)^3 |\tilde{Z}|^2 \cdot MEV2D \left[ j_1, j_2, j_3, \tilde{j} = \frac{1}{2} \cdot j_3 + \frac{1}{2} \right] \\
\text{Now}
C^{J = j_1}_{j_2} (C, 0, j_1) = \begin{cases} 
(-1)^{j_1 + 1} \left[ \frac{j_1}{2j_1 + 1} \right]^4 & j_1 = j_1 - \frac{1}{2} \\
(-1)^{j_1} \left[ \frac{j_1}{2j_1 + 1} \right]^4 & j_1 = j_1 + \frac{1}{2}
\end{cases}
\end{align*}
\] independent of \( C \) and \((-1)^{j_1} = (-1)^{-j_1}.\)

\[
\begin{align*}
&= - (\ell_p)^3 |\tilde{Z}|^2 \cdot \sum_{AC} [\mathcal{T}_k \mathcal{C} \delta_{\mathcal{M}' \mathcal{M} + C - A} (-1)^{j_1 + j_2 + j_3} \sqrt{(2a'_2 + 1)} \{ j_2 \quad j_1 \quad a'_2 \}]
\times \left\{ - \sqrt{j_1} C^{J = -1}_{j_2} (A, M', j_1) MEV2D \left[ j_1 - \frac{1}{2}, j_2, j_3, \frac{1}{2}, \frac{1}{2}, j_3 + \frac{1}{2} \right]^2 \\
&\quad + \sqrt{j_1} + 1 C^{J = 1}_{j_2} (A, M', j_1) MEV2D \left[ j_1 + \frac{1}{2}, j_2, j_3, \frac{1}{2}, \frac{1}{2}, j_3 + \frac{1}{2} \right]^2 \right\}.
\end{align*}
\]

4.2.2. \( K = 2. \) We always have \( \tilde{j} = \frac{1}{2}:\)

\[
\langle a'_2(j_1, j_2) J'(a'_2 j_1) \rangle = 1 M' |(\ell'_p)'_{K = 2} a_2(j_1, j_2) J(a_2, j_2) = 0 M = 0) \]

\[
\begin{align*}
&= - \sum_{AC} [\mathcal{T}_k \mathcal{C} \delta_{\mathcal{M}' \mathcal{M} + C - A} \sum_{j_3 = j_1 + \frac{1}{2}} C^{J = j_1}_{j_3} (A, M', g'_2)(\tilde{a}'_2(j_1, j_2))
\times \tilde{j} (\tilde{a}'_2(j_1, j_2)) g'_2(j_1, j_2) j'(a'_2(j_1, j_2)) j'(a'_2(j_1, j_2)) g'_2(j_1, j_2) J'(a'_2(j_1, j_2))
\times \sum_{g_2(j_1, j_2)} C^{J = j_1}_{j_2} (C, M, g_2) (\tilde{a}_2(j_1, j_2)) g_2(j_1, j_2) j(j_2, j_2))
\times \langle a_2(j_1, j_2) J(a_2, j_2) g_2(j_1, j_2) j(j_2, j_2) (\tilde{a}_2(j_1, j_2)) j(j_2, j_2) \rangle = \frac{1}{2} M' + A |\tilde{V}'|
\times \tilde{a}_2(j_1, j_2) j(j_2, j_2) (\tilde{a}_2(j_1, j_2) j(j_2, j_2) = \frac{1}{2} M + C)
\end{align*}
\]
\[
\begin{align*}
&\times (-1)^{\hat{J} + \hat{j}_2 - \hat{a}_2} (-1)^{\hat{J} + \hat{g}_2 - \hat{j}} (-1)^{\hat{J} + \hat{j}_1 + \hat{j}_2 + \hat{j}_3 + \hat{g}_2} \left\{ \begin{array}{ccc}
\hat{j}_2 & \hat{j}_1 & \hat{a}_2 \\
\hat{j}_3 & \hat{j} & \hat{g}_2 \\
\end{array} \right\} \sqrt{(2a_2 + 1)(2g_2 + 1)} \\
&\times (-1)^{\hat{J} + \hat{a}_1 - \hat{a}_2} (-1)^{\hat{J} + \hat{g}_2 - \hat{j}} (-1)^{\hat{J} + \hat{j}_1 + \hat{j}_2 + \hat{j}_3 + \hat{g}_2} \left\{ \begin{array}{ccc}
\hat{j}_2 & \hat{j}_1 & \hat{a}_2 \\
\hat{j}_3 & \hat{j} & \hat{a}_1 \\
\end{array} \right\} \\
&\times \left\{ \hat{a}_2(j_1 \hat{j}_2 \hat{j})(\hat{a}_2 j_3) = \frac{1}{2}M' + A|\hat{V}| \hat{a}_2(j_1 \hat{j}_2 \hat{j})(\hat{a}_2 j_3) = \frac{1}{2}M + C \right\}
\end{align*}
\]

Note:
(i) Due to integer arguments the \((-1)\) prefactors are equal to \((-1)^{a_2 - a_2}; (-1)^{a_2 - a_2} \).
(ii) With (4.12) we have \(\hat{a}_2^2 = \hat{a}_2 = j_3 \pm \frac{1}{2}\) and have MEV 2D\( [j_1, j_2, j_3, \hat{j} = \frac{1}{2}, j]\)
\(\rightarrow\) we can use the orthogonality relations of the 6j symbols (see [15], p 96) in order to carry out the remaining sum over \(\hat{a}_2:\)
\[
\sum_{\tilde{a}_2} (2\tilde{a}_2 + 1) \left\{ \begin{array}{ccc}
\tilde{j}_2 & \tilde{j}_1 & \tilde{a}_2 \\
\tilde{j}_3 & \tilde{j} & \tilde{g}_2 \\
\end{array} \right\} \left\{ \begin{array}{ccc}
\tilde{j}_2 & \tilde{j}_1 & \tilde{a}_2 \\
\tilde{j}_3 & \tilde{j} & \tilde{g}_2 \\
\end{array} \right\} = \frac{1}{(2g_2 + 1)} \delta_{\tilde{g}_2 g_2} \sim \left\{ \begin{array}{ccc}
\tilde{g}_2 & \tilde{a}_2 = g_2 = j_2 \\
\end{array} \right\}
\]
(iii) For \(J = 0\) we have
(a) \(a_2 = j_1\)
(b) \(\left\{ \begin{array}{ccc}
\tilde{j}_2 & \tilde{j}_1 & \tilde{a}_2 \\
\tilde{j}_3 & \tilde{j} & \tilde{g}_2 \\
\end{array} \right\} = \left\{ \begin{array}{ccc}
\tilde{j}_2 & \tilde{j}_1 & \tilde{a}_2 \\
\tilde{j}_3 & \tilde{j} & \tilde{g}_2 \\
\end{array} \right\} = \frac{(-1)^{\hat{J} + \hat{j}_1}}{(2J_2 + 1)(2j_3 + 1)}.
\]

\[
= - \sum_{AC} |\tilde{\tau}_{AC} \delta_{M', M + C - A} (-1)^{\hat{J} + \hat{j}_3 + \hat{j}_2} (-1)^{a_2 - \hat{a}_2} \sqrt{(2j_2 + 1)(2a_2 + 1)} \left\{ \begin{array}{ccc}
\tilde{j}_1 & \tilde{j}_2 & \tilde{a}_2 \\
\tilde{j}_3 & \tilde{j} & \tilde{g}_2 \\
\end{array} \right\}
\]
\[
\times \sum_{j_1 = j_3 \pm \frac{1}{2}} C_{j_2 = \frac{1}{2} J_2}^{j_1 = \frac{1}{2} J_2}(A, M', j_2) C_{j_2 = \frac{1}{2} J_2}^{j_1 = \frac{1}{2} J_2}(C, 0, j_2) \times (\ell_P)^3 |\hat{Z}| \cdot MEV 2D
\]
\[
\times \left\{ \begin{array}{ccc}
j_1, \tilde{j}_2, j_3, \hat{j} = \frac{1}{2}, j_3 + \frac{1}{2} \end{array} \right\}
\]

Now
\[
C_{j_2 = \frac{1}{2} J_2}^{j_1 = \frac{1}{2} J_2}(C, 0, j_2) = \begin{cases}
(-1)^{2j_1 + 1} \left[ \frac{\tilde{j}_1}{\tilde{j}_2} \right]^\frac{1}{2} & \tilde{j}_2 = j_2 - \frac{1}{2} \\
(-1)^{2j_1} \left[ \frac{\tilde{j}_1}{\tilde{j}_2} \right]^\frac{1}{2} & \tilde{j}_2 = j_2 + \frac{1}{2}
\end{cases}
\]

independent of \(C\) and \((-1)^{3j_2} = (-1)^{-j_2}\).

\[
= -(\ell_P)^3 |\hat{Z}| \cdot \sum_{AC} |\tilde{\tau}_{AC} \delta_{M', M + C - A} (-1)^{\hat{J} + \hat{j}_3 + \hat{j}_2} \sqrt{(2a_2 + 1)} \left\{ \begin{array}{ccc}
j_1 & j_2 & a_2 \\
1 & j_3 & j_2 \\
\end{array} \right\}
\]
\[
\times \left\{ \begin{array}{ccc}
-\sqrt{\tilde{j}_2} C_{j_2 = \frac{1}{2} \tilde{j}_2 - \frac{1}{2}}^{j_1 = \frac{1}{2} \tilde{j}_2 - \frac{1}{2}}(A, M', j_2) MEV 2D \left\{ \begin{array}{ccc}
j_1, j_2 - \frac{1}{2}, j_3, \frac{1}{2}, j_3 + \frac{1}{2} \end{array} \right\} \\
+ \sqrt{\tilde{j}_2 + 1} C_{j_2 = \frac{1}{2} \tilde{j}_2 + \frac{1}{2}}^{j_1 = \frac{1}{2} \tilde{j}_2 + \frac{1}{2}}(A, M', j_2) MEV 2D \left\{ \begin{array}{ccc}
j_1, j_2 + \frac{1}{2}, j_3, \frac{1}{2}, j_3 + \frac{1}{2} \end{array} \right\} \right\}.
\]
4.2.3. $K = 3$. In this special case (since $N = 3$ we have to consider the general expression for $K = N$) we start from (3.6). Again $\tilde{j} = \frac{1}{2}$:

$$
\langle a'_2(j_1,j_2) J'(a'_2j_3) = 1 M'|\epsilon_{K=3}^{(i)} a_2(j_1,j_2) J(a_2,j_3) = 0 M = 0 \rangle = - \sum_{\mathcal{AC}} |\tau_{k}|_{CA} \delta_{M:M'-C-A} \times \sum_{j_3 = j_3 + \frac{1}{2}} \tilde{C}_{j_3 = j_3 + \frac{1}{2}}^{J=j_3}(A, M', a_2) C_{j_3 = j_3 + \frac{1}{2}}^{J=j_3}(C, M, a_2) \langle \tilde{a}_2(j_1,j_2) \tilde{j}(a_2,j_3) | g_2(j_1,j_3) \rangle
$$

$$
\times \tilde{j}(g_2,j_3) \left( a'_2(j_1,j_2) \tilde{j}(a'_2,j_3) = \frac{1}{2} M' + A |\hat{V}'| a_2(j_1,j_2) \tilde{j}(a_2,j_3) = \frac{1}{2} M + C \right)
$$

Note:

(i) With (4.12) we have $a'_2 = a_2 = j_3$

(ii) The Hilbert space structure is as follows. Because we must have $|a_2 - \tilde{j}_3| = \frac{1}{2}$ the matrix element of the volume operator has to be taken according to $\tilde{j}_3$:

$$
\tilde{j}_3 = j_3 - \frac{1}{2} \rightarrow a_2 = \begin{cases} j_3 - 1 & \tilde{q} = \begin{pmatrix} 0 & -iA_1 \\ iA_1 & 0 \end{pmatrix} \\ j_3 & \tilde{j} = \frac{1}{2}, a_2 = j_3 \end{cases}
$$

$$
A_1 = A_1[j_1, j_2, j_3 - \frac{1}{2}, \tilde{j} = \frac{1}{2}, a_2 = j_3]
$$

$$
\tilde{j}_3 = j_3 + \frac{1}{2} \rightarrow a_2 = \begin{cases} j_3 & \tilde{q} = \begin{pmatrix} 0 & -iA_1 \\ iA_1 & 0 \end{pmatrix} \\ j_3 + 1 & \tilde{j} = \frac{1}{2}, a_2 = j_3 + 1 \end{cases}
$$

$$
A_1 = A_1[j_1, j_2, j_3 + \frac{1}{2}, \tilde{j} = \frac{1}{2}, a_2 = j_3 + 1]
$$

(iii) $C_{j_3 = j_3 + \frac{1}{2}}^{J=j_3}(C, 0, j_3) = \begin{cases} (-1)^{j_3+1} [\frac{j_3+1}{2j_3+1}]^{\frac{1}{2}} & \tilde{j}_3 = j_3 - \frac{1}{2} \\ (-1)^{j_3} [\frac{j_3}{2j_3+1}]^{\frac{1}{2}} & \tilde{j}_3 = j_3 + \frac{1}{2} \end{cases}$

$$
= - (\ell_p)\mu \hat{Z} \sum_{\mathcal{AC}} [\tau_{k}]_{CA} \delta_{\alpha_2} \delta_{M:M'-C-A} \frac{(-1)^{j_3} \tilde{j}_3}{\sqrt{2j_3 + 1}} \left\{ - \sqrt{j_3} C_{j_3 = j_3 + \frac{1}{2}}^{J=j_3}(A, M', j_3) \times MEV2D \left[ j_1, j_2, j_3 - \frac{1}{2}, \tilde{j}_3 \right]^{\frac{1}{2}} + \sqrt{j_3 + 1} C_{j_3 = j_3 + \frac{1}{2}}^{J=j_3}(A, M', j_3) \times MEV2D \left[ j_1, j_2, j_3 + \frac{1}{2}, \tilde{j}_3 \right]^{\frac{1}{2}} \right\}.
$$

(4.15)

4.2.4. The expressions $q_{1L}(\frac{1}{2}) = \delta_{i_l} (\frac{1}{2}) \epsilon_{j_l} (\frac{1}{4}) \epsilon_{l_L}$. Given the explicit expressions (4.13), (4.14), (4.15) we can now calculate the matrix elements of $q_{1L}$-operators as prescribed in (4.5):

$$
\langle 0 | q_{1L} (\frac{1}{2}) | 0 \rangle = Q_{IL} = \sum_{[j_1]} \langle 0 | (\frac{1}{2}) \epsilon_{l_1} (\frac{1}{4}) \epsilon_{l_L} | 0 \rangle \delta_{i_l} \sum_{\delta_{i_l} = \frac{1}{2}} \sum_{j_2 = j_2 - 1}^{\frac{1}{2}} \sum_{j_2 = j_2 + 1}^{\frac{1}{2}} \sum_{M' = \frac{1}{4}} \sum_{M' = \frac{1}{4}} \times \langle a'_2(j_1,j_2) J'(a'_2j_3) = 1 M'|(\frac{1}{2}) \epsilon_{l_1} (\frac{1}{4}) \epsilon_{l_L} | a_2(j_1,j_2) = j_3 J(a_2,j_3) = 0 M = 0 \rangle
$$

$$
\times \langle a'_2(j_1,j_2) J'(a'_2j_3) = 1 M'|(\frac{1}{2}) \epsilon_{l_1} (\frac{1}{4}) \epsilon_{l_L} | a_2(j_1,j_2) = j_3 J(a_2,j_3) = 0 M = 0 \rangle.
$$

(4.16)
We have done the calculation with MATHEMATICA. Here we have introduced the shorthand\(^8\): 
\[ V_{1A} = M EV2D[j_1 = j_1 - \frac{1}{2}, j_2, j_3, j = \frac{1}{2}, j_3 + \frac{1}{2}] \]
\[ V_{1B} = M EV2D[j_1 = j_1 + \frac{1}{2}, j_2, j_3, j = \frac{1}{2}, j_3 + \frac{1}{2}] \]
\[ V_{2A} = M EV2D[j_1, j_2 = j_2 - \frac{1}{2}, j_3, j = \frac{1}{2}, j_3 + \frac{1}{2}] \]
\[ V_{2B} = M EV2D[j_1, j_2 = j_2 + \frac{1}{2}, j_3, j = \frac{1}{2}, j_3 + \frac{1}{2}] \]
\[ V_{3A} = M EV2D[j_1, j_2, j_3 = j_3 - \frac{1}{2}, j = \frac{1}{2}, j_3] \]
\[ V_{3B} = M EV2D[j_1, j_2, j_3 = j_3 + \frac{1}{2}, j = \frac{1}{2}, j_3 + 1] \]

Moreover, we use the abbreviation \( A_k = j_k (j_k + 1) \). Note that we now specify to 
\[ r = \frac{1}{2} \] and use the abbreviation \( Q_{1L} := \langle 0 | q_{1L} \left( \frac{j}{2} \right) | 0 \rangle \).

The result is 
\[ Q_{11} = (\ell_p)^3 \left| \tilde{Z} \right|^2 \cdot \frac{4j_1(1 + j_1)(V_{1A}^\dagger - V_{1B}^\dagger)^2}{(1 + 2j_1)^2} \]
\[ Q_{12} = (\ell_p)^3 \left| \tilde{Z} \right|^2 \cdot \frac{2(2 + (-1)^{2(j+\frac{1}{2})})(A_1 + A_2 - A_3)(V_{1A}^\dagger - V_{1B}^\dagger)(V_{2A}^\dagger - V_{2B}^\dagger)}{3(1 + 2j_1)(1 + 2j_2)} \]
\[ Q_{13} = (\ell_p)^3 \left| \tilde{Z} \right|^2 \cdot \frac{2(2 + (-1)^{2(j+\frac{1}{2})})(A_1 - A_2 + A_3)(V_{1A}^\dagger - V_{1B}^\dagger)(V_{3A}^\dagger - V_{3B}^\dagger)}{3(1 + 2j_1)(1 + 2j_3)} \]
\[ Q_{22} = (\ell_p)^3 \left| \tilde{Z} \right|^2 \cdot \frac{4A_1(V_{2A}^\dagger - V_{2B}^\dagger)^2}{(1 + 2j_2)^2} \]
\[ Q_{33} = (\ell_p)^3 \left| \tilde{Z} \right|^2 \cdot \frac{4A_3(V_{3A}^\dagger - V_{3B}^\dagger)^2}{(1 + 2j_3)^2}. \] (4.17)

With the explicit expressions in (4.17) we are now able to evaluate (4.4): 
\[ \langle 0 | e^{-\frac{1}{2}} | 0 \rangle = e^{\ell_p} e^{L^{MN}} \langle 0 | q_{1L} \left( \frac{j}{2} \right) q_{JM} \left( \frac{j}{2} \right) q_{KN} \left( \frac{j}{2} \right) | 0 \rangle = e^{\ell_p} e^{L^{MN}} \langle 0 | q_{1L} \left( \frac{j}{2} \right) | 0 \rangle \langle 0 | q_{JM} \left( \frac{j}{2} \right) | 0 \rangle \langle 0 | q_{KN} \left( \frac{j}{2} \right) | 0 \rangle = e^{\ell_p} e^{L^{MN}} Q_{1L} Q_{JM} Q_{KN}. \] (4.18)

\(^8\) During the calculation, it turned out that it is much faster if we make placeholders for the matrix elements of the volume operator due to the fact that MATHEMATICA tries to simplify expressions during evaluation.
The result is

\[
\langle 0 \mid [\varepsilon']^2 \mid 0 \rangle = \frac{32(\ell_p)^9 |\hat{Z}|^{\frac{j}{2}}}{9(1 + 2j)^2(1 + 2j_2)^2(1 + 2j_3)^2} \left[ 108A_1A_2A_3 - 3(2(-1)^{2j} + (-1)^{2j_2}) \right.
\]
\[
\times A_1(-A_1 + A_2 + A_3)^2 - 3(1 + 2(1)^{2j_2} + (-1)^{2j_2}) \right]
\]
\[
\times A_1(A_1 + A_2 - A_3)^2 - (1 + 2(1)^{2j_2}) (2(-1)^{2j}) \]
\[
\left. + (-1)^{2j} (2(-1)^{3j} + (-1)^{2j})(-A_1 + A_2 + A_3) \right]
\]
\[
\times (A_1 - A_2 + A_3)(A_1 + A_2 - A_3) \{(V_{1A}^{\frac{1}{2}} - V_{1B}^{\frac{1}{2}})^2 \}
\]
\[
\times (V_{2A}^{\frac{1}{2}} - V_{2B}^{\frac{1}{2}})^2 \{V_{1A}^{\frac{1}{2}} - V_{1B}^{\frac{1}{2}}\}. \tag{4.19}
\]

where all quantities are defined as on the last page, and \(\hat{Z}\) is a numerical constant dependent on the regularization of the volume operator and on the Immirzi parameter (see [12] for details). The ninth power of \(\ell_p\) gives together with the \(\hbar^{-6} \kappa^{-6} = (\ell_p)^{-12}\) coming from the quantization of the gravitational part \(\frac{1}{2 \sqrt{16\pi G}}\) of the Hamiltonian in (2.8) the correct \(\frac{1}{(\ell_p)^9}\) prefactor. Note the remarkable symmetry of (4.19). We will illustrate the non-trivial behaviour of (4.19) in the following (neglecting the prefactors \((\ell_p)^9 |\hat{Z}|^{\frac{j}{2}}\) in some examples.

### 4.3. Different configurations

#### 4.3.1. 'Oscillating' \(j_1 = j_2 = \frac{\hbar}{2}\)

If we set \(j_1 = j_2 = \frac{\hbar}{2}\) where \(j_3 \in \mathbb{N}\), we get

\[
\langle 0 \mid [\varepsilon']^2 \mid 0 \rangle \propto \frac{128 \sqrt{2} (-1 + (-1)^{j_3}) j_3 \{(-3(2 + (-1)^{j_3}) + (1 + (-1)^{3j_3}) j_3)\}}{9(1 + j_3)(1 + 2j_3)^{\frac{3}{2}}}. \tag{4.20}
\]

\(^9\) There \(\hat{Z}\) is found to be \(\hat{Z} = \frac{j}{3} \left( \frac{\hbar}{2} \right)^3\).
Unboundedness of triad-like operators in loop quantum gravity

Figure 3. Plot for \( j_1 = j_2 = \frac{j_3 + 1}{2} \) where \( j_3 \in \mathbb{N} \) with \( 1 \leq j_3 \leq 40 \). The graph oscillates between 0 (if \( j_3 \) odd) and an increasing value (if \( j_3 \) even).

Asymptotically this increases as
\[
< 0 |[e']^2|0 > \propto 5.9 \times 10^{-5} j_3^{\frac{3}{4}}
\]

4.3.2. Increasing. \( j_1 = \frac{3}{2}, j_2 = j_3 + \frac{1}{2} \) If we set \( j_1 = \frac{3}{2}, j_2 = j_3 + \frac{1}{2} \) where \( j_3 \in \mathbb{N} \), we get
\[
\langle 0 | [e']^2 | 0 \rangle \propto \frac{1}{9(2 + j_3)^4(1 + 2j_3)^2} 128\sqrt{2}(1 + (-1)^{j_3})j_3^2(1 + j_3)^3(-3(-7 + 3(-1)^{j_3}) + (-1)^{j_3} + (-1 + (-1)^{j_3})j_3)(-2\frac{1}{2}(j_3(1 + j_3))^{\frac{1}{2}})
\]
\[
+ ((1 + j_3)^2(3 + 2j_3))^\frac{1}{2}((j_3(1 + j_3))^\frac{1}{2} - (j_3(3 + 5j_3 + 2j_3^2))^\frac{1}{2})^4.
\]

4.3.3. Identical 0. If we set \( j_1 = j_2 = j_3 \) and more generally \( j_1, j_2, j_3 \in \mathbb{N} \) (all spins integer numbers) in (4.19) then
\[
\langle 0 | [e']^2 | 0 \rangle = 0.
\]

4.4. General configurations

Using the general result (4.19) (without the prefactors \((\ell_P)^{\frac{3}{2}}|Z|^{\frac{3}{2}}\)) we use the quantity
\[
Q = \begin{cases} 
30 + \ln \left( |[e']^2(j_1, j_2, j_3)| \right) & |j_1 - j_2| \leq j_3 \leq j_1 + j_2 \text{ and } j_1 + j_2 + j_3 \text{ is integer} \\
0 & |[e']^2(j_1, j_2, j_3) = 0
\end{cases}
\]

and
\[
Q = \begin{cases} 
30 + \ln \left( |[e']^2(j_1, j_2, j_3)| \right) & |j_1 - j_2| \leq j_3 \leq j_1 + j_2 \text{ and } j_1 + j_2 + j_3 \text{ is integer} \\
0 & |[e']^2(j_1, j_2, j_3) = 0
\end{cases}
\]
Asymptotically this increases as 

\[ < 0 | [e']^2 | 0 > \propto 1.06 \cdot 10^{-6} j_3^2 \]

Figure 4. Plot for \( j_1 = \frac{3}{2}, j_2 = j_3 + \frac{1}{2} \) where \( j_3 \in \mathbb{N} \) with \( 1 \leq j_3 \leq 40 \). The graph first decreases for \( 1 \leq j_3 < 3 \) and is 0 for \( j_3 = 3 \). It decreases for \( j_3 > 3 \).

and do a three-dimensional (3D) plot\(^{10}\) (in the range \( \frac{1}{2} \leq j_1 \leq j_{\text{max}}, \frac{1}{2} \leq j_2 \leq j_{\text{max}} \) for each fixed value \( 5 \leq j_3 \leq \frac{15}{2} \)). It turns out that the non-zero configurations are grouped symmetrically along lines parallel to the \( j_1 = j_2 \) axis. The reason for this is, of course, the integer requirement \( j_1 + j_2 + j_3 = \text{integer} \). Therefore we will get contributions on the \( j_1 = j_2 \)-axis only if \( j_3 \) is integer. Because (4.19) is symmetric with respect to the interchange of \( j_1 \leftrightarrow j_2 \) we may restrict ourselves to the range \( j_1 \geq j_2 \).\(^{11}\)

Additionally, we extract for each of those 3D plots a two-dimensional plot along the lines 

\( j_2 = j_1 - l \) with \( 0 \leq l \leq \min \left[ j_3, j_{\text{max}} - \frac{1}{2} \right] \), \( l + \frac{1}{2} \leq j_1 \leq j_{\text{max}} = 25 \).

The restriction for the parameter \( l \) is a result of the requirements \( |j_1 - j_2| \leq j_1 \leq j_1 + j_2 \) from which, for \( j_1 > j_2 \), we may remove the modulus. In order to give a better impression, we have joined only non-vanishing values of \( Q \) along the lines described above\(^{12}\).

5. Expectation values in \( U(1)^3 \) CS

Note that for the construction of the \( U(1)^3 \) coherent states and for the definitions concerning the volume operator in \( U(1)^3 \) coherent states, we follow \([20]\). Note, however, that we will use a slightly different construction in order to find (C.25).

5.1. Set-up

Let us have \( M \) edges on the vertex \( v \in V(\gamma) \). Then we find for the expectation value of a polynomial of the operators \( \hat{q}_{jk}^\beta \) in our coherent states (C.25):

\[
\langle \Psi_{m,\gamma}^{(v)}(A) \bigg| \prod_{k=1}^{N} \hat{q}_{jk}^\beta(r) \bigg| \Psi_{m,\gamma}^{(v)}(A) \rangle = \frac{(2\pi)^M}{\|\Psi_{m,\gamma}^{(v)}\|^2} \]

\(^{10}\)The numerical constant 30 is added for technical reasons only.

\(^{11}\)The rapid increase of the curves for small \( j_3 \) is due to the fact that the first non-zero values have been connected by a line to the last 0.

\(^{12}\)For integer \( j_3 \) also every second configuration on the lines gives a 0 due to section 4.3.3 and by joining all the points the plots oscillate between 0 and the plotted curves, and it would be hardly possible to see anything useful from them in this case if we joined all points.
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Figure 5. Plot for $j_3 = 5$.

Figure 6. Plot for $j_3 = 5$.

Figure 7. Plot for $j_1 = \frac{11}{2}$.

Figure 8. Plot for $j_3 = \frac{11}{2}, j_2 = j_1 - l$. The different curves are (bottom to top) $l = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{11}{2}$.

\[ \lambda^e (\{ n^e_h \} ) \overset{\text{(C.34)}}{=} \lambda^e (\{ n^e_h \} ) - \lambda^e (\{ n^e_h + \delta_{ih} \delta_{jk} \} ) \]

\[ \lambda^e (\{ n^e_h \} ) \overset{\text{(C.32)}}{=} (\ell_P)^3 |Z| \sqrt{\sum_{ijk} \epsilon_{ijk} \epsilon (IJK) n^i_h n^j_h n^k_h} \]

where $I \equiv e_f$ now labels the edges $e \in E(v)$ and we recall

\[ \lambda^e (\{ n^e_h \} ) \overset{\text{(5.1)}}{=} \sum_{\{ n^e_h \} \subseteq Z} e^{\sum_{ijkl} \frac{1}{2} \{ n^e_h \} + 2p_{ijkl} \{ n^e_h \} } \prod_{k=1}^N \lambda^{\epsilon (I)} (\{ n^e_h \} ) \]

\[ \lambda^e (\{ n^e_h \} ) \overset{\text{(5.1)}}{=} (\ell_P)^3 |Z| \sqrt{\sum_{ijk} \epsilon_{ijk} \epsilon (IJK) n^i_h n^j_h n^k_h} \]
Figure 9. Plot for $j_3 = 6$.  

Figure 10. Plot for $j_3 = 6$.  

Figure 11. Plot for $j_3 = \frac{11}{2}$.  

Figure 12. Plot for $j_3 = \frac{13}{2}, j_2 = j_1 - 1$. The different curves are (bottom to top) $j = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{13}{2}$.

Note that this expectation value only depends on the pair $m = (A, E)$ the coherent state is peaked on via $p_j'(m) = p_j'(A, E)$ according to (C.16). Using the upper bound (C.39) we can estimate (5.1) as

$$\langle \cdot \rangle \leq \frac{(2\pi)^{3M} (\ell_p)^{3rN} (9M)^N |Z|^\frac{2}{rN}}{\Vert \Psi_{m, y}^{(0)} \Vert} \sum_{\{n'_j\}} \frac{\sum_{I,J} \left[ -t(I) \right] \sum_{\{n'_j\}} \left[ n'_j \right]^2 }{\sum_{J,j} \left[ n'_j \right]^2} .$$

Now define $T_I := \sqrt{t(I)}, x_j := T_I n'_j$ and perform a Poisson resummation according to (22) to arrive at (note that we replace, at the same time, $n'_j \to \frac{1}{T_I} x_j$)

$$\langle \cdot \rangle \leq \frac{(2\pi)^{3M} (\ell_p)^{3rN} (9M)^N |Z|^\frac{2}{rN}}{\Vert \Psi_{m, y}^{(0)} \Vert} \sum_{\{n'_j\}} \frac{\sum_{I,J} \left[ -t(I) \right] \sum_{\{n'_j\}} \left[ n'_j \right]^2 }{\sum_{J,j} \left[ n'_j \right]^2} \int_{\mathbb{R}^3} d^3M x e^{\sum_{I,J} \left[ -t(I) \right] \sum_{\{n'_j\}} \left[ n'_j \right]^2}.$$
As shown in [20] we may complete the square in the exponent of the integrals by introducing $X'_{j} = x'_{j} - \frac{1}{T_{j}}(p_{j}^{(m)} - i\pi N_{j})$ and rewrite

$$\langle \cdot \rangle \leq \frac{(2\pi)^{3M}(\epsilon \rho)^{3\gamma N}(9M)^{N}|Z|^\frac{2}{\gamma}}{\left\| \psi_{m,\gamma}^{(q)} \right\|^{2} \prod_{j=1}^{M}(T_{j})^{3}} \sum_{\left\{ N_{j}' \right\} \in \mathbb{Z}} e^{\sum_{j \neq j} \frac{1}{2}(p_{j}^{(m)})(p_{j}^{(m)} - i\pi N_{j})^{2} - 2i\pi N_{j}'(p_{j}^{(m)} - i\pi N_{j})} \times \int_{\mathbb{R}^{3M}} d^{3M}X e^{-\sum_{j}(|X|)^{2}} \left[ \sum_{j \neq j} \frac{1}{(T_{j})^{2}} \left| X'_{j} + \frac{1}{T_{j}}(p_{j}^{(m)} - i\pi N_{j}) \right|^{2} \right]^{N}.$$ (5.4)
For the norm \( \| \Psi_{m,j}^{(x)} \| \) we find
\[
\| \Psi_{m,j}^{(x)} \|^2 = \prod_{e_j \in \mathcal{E}(x)} \| \Psi_{m,j}^{(x)} \|^2 = \prod_{e_j \in \mathcal{E}(x)} 2\pi \sqrt{\pi} \frac{1}{T_j} \sum_{N_j' \in \mathbb{Z}} e^{-\frac{\pi}{T_j^2} \left[ \pi \left( \frac{N_j'}{N_j} \right)^2 + 2\pi \frac{1}{T_j} \Re \left( p_j^{(m)} \right) \right]} \]
\[
= \left( 2\pi \right)^{3M} \frac{\left( \sqrt{\pi} \right)^{9M}}{\prod_{I=1}^{M} (T_I)^2} e^{\sum_{I,j} \frac{1}{T_I} \Re \left( p_j^{(m)} \right)} \prod_{I,j} \sum_{N_j' \in \mathbb{Z}} e^{-\frac{\pi}{T_I^2} \left[ \pi \left( \frac{N_j'}{N_j} \right)^2 + 2\pi \frac{1}{T_I} \Re \left( p_j^{(m)} \right) \right]} \left[ 1 + \mathcal{K}^{(I,j)} \right] . \tag{5.5}
\]

Here again \( K^{(I,j)} = O(t(I)^{\infty}) \).

If we now divide (5.4) by the norm (5.5) we find for the expectation value in normalized \( U(1)^3 \) coherent states at a \( M \)-valent vertex \( \nu \)
\[
\langle \cdot | \sum_{e_j \in \mathcal{E}(x)} e^{-\frac{\pi}{T_j^2} \left[ \pi \left( \frac{N_j'}{N_j} \right)^2 + 2\pi \frac{1}{T_j} \Re \left( p_j^{(m)} \right) \right]} \right]^{N}
\[
\leq \mathcal{K} \sum_{N_j' \in \mathbb{Z}} \left| e^{-\frac{\pi}{T_j^2} \left[ \pi \left( \frac{N_j'}{N_j} \right)^2 + 2\pi \frac{1}{T_j} \Re \left( p_j^{(m)} \right) \right]} \left[ \sum_{I,j} \frac{1}{T_I^2} \left( X_j' + \frac{1}{T_I} \Re \left( p_j^{(m)} \right) \right) \right]^{N} \right|
\]
\[
\times \int_{\mathbb{R}^{3M}} d^{3M} X e^{-\sum_{I,j} \sum_{I,j} X_j^2} \left[ \sum_{I,j} \frac{1}{T_I^2} \left( X_j' + \frac{1}{T_I} \Re \left( p_j^{(m)} \right) \right) \right]^{N} . \tag{5.6}
\]

Here we have introduced the abbreviation \( \mathcal{K} \) for the prefactor. Now we estimate
\[
\left| X_j' + \frac{1}{T_I} \Re \left( p_j^{(m)} \right) \right|^2 = \left( X_j' + \frac{1}{T_I} \Re \left( p_j^{(m)} \right) \right)^2 + \frac{\pi^2}{(T_I)^2} \left[ N_j' \right]^2
\]
\[
\leq \left[ X_j'^2 + \frac{2}{T_I} \Re \left( p_j^{(m)} \right) \right] \left[ \frac{1}{T_I^2} \left( \Re \left( p_j^{(m)} \right) \right)^2 + \frac{\pi^2}{(T_I)^2} \left[ N_j' \right]^2
\]
\[
\leq \left[ 1 + \frac{2}{T_I} \left( \Re \left( p_j^{(m)} \right) \right) \right] \left[ X_j'^2 \right] + \frac{1}{2} \frac{1}{T_I} \left( \Re \left( p_j^{(m)} \right) \right) \left[ X_j'^2 + \frac{1}{T_I} \left( \Re \left( p_j^{(m)} \right) \right) \right] + \frac{\pi^2}{(T_I)^2} \left[ N_j' \right]^2
\]
\[
A_j' \left[ X_j'^2 \right] + \frac{1}{2} \frac{1}{T_I} \left( \Re \left( p_j^{(m)} \right) \right) A_j' + \frac{\pi^2}{(T_I)^2} \left[ N_j' \right]^2
\]
where we have used that \( x \leq x^2 + \frac{1}{4} \forall x \in \mathbb{R} \). Additionally, taking into account that
\[
| e^{-\frac{\pi}{T_j^2} \left[ \pi \left( \frac{N_j'}{N_j} \right)^2 + 2\pi \frac{1}{T_j} \Re \left( p_j^{(m)} \right) \right]} | \leq 1
\]
and the fact that the remaining two moduli in (5.6) are real numbers.
we continue with (5.6)\(^{13}\):

\[
\langle \cdot \rangle \leq \mathcal{K} \cdot \sum_{[N_j] \in \mathbb{Z}} e^{\sum_{i,j} \frac{\pi^2}{2(T_j)^2}|N_j|^2} \int_{\mathbb{R}^{3M}} d^{3M}X e^{-\sum_{i,j}|X_j|^2} \times \left[ \sum_{i,j} \frac{1}{(T_j)^2} \left( A_j^2 [X_j]^2 + \frac{1}{2T_j^2} |p_j^p(m)| A_j^2 \right) \right]^N. \tag{5.7}
\]

Now let us introduce \(X_j := \sqrt{2} Y_j\) and \(T := \min_{i,j} T_i, p := \max_{i,j} |p_j^p(m)|, A := 1 + \frac{p}{T}\). Then an upper bound for (5.7) is given by

\[
\langle \cdot \rangle \leq \mathcal{K} \cdot \sqrt{2}^{3M} \sum_{[N_j] \in \mathbb{Z}} e^{\sum_{i,j} \frac{\pi^2}{2(T_j)^2}|N_j|^2} \int_{\mathbb{R}^{3M}} d^{3M}Y e^{-2\sum_{i,j}|Y_j|^2} \left\{ \frac{2A}{T^2} \sum_{i,j} |Y_j|^2 + \frac{A}{T^2} 3Mp \right\}
\]

\[
+ \frac{\pi^2}{T^2} \sum_{i,j} [N_j]^2 \right\}^N = \left( \frac{(\ell_p)^3 N (9M)^N |Z|^2}{\mathcal{N}} \right)^N \sum_{[N_j] \in \mathbb{Z}} e^{\sum_{i,j} \frac{\pi^2}{2(T_j)^2}|N_j|^2} \times \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}^{3M}} d^{3M}Y e^{-2\sum_{i,j}|Y_j|^2} \left\{ \sum_{i,j} |Y_j|^2 + \frac{3Mp}{4T} + \frac{\pi^2}{2AT^2} \sum_{i,j} [N_j]^2 \right\}^N. \tag{5.8}
\]

If we now define \(\|Y\|^2 := \sum_{i,j} [Y_j]^2\) we may first rewrite (5.8) and then finish with

\[
\langle \cdot \rangle \leq \mathcal{K} \sum_{[N_j] \neq [0]} e^{-\sum_{i,j} \frac{\pi^2}{2(T_j)^2}|N_j|^2} \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}^{3M}} d^{3M}Y e^{-2\|Y\|^2}(\|Y\|^2 + D)^N
\]

Polynomial theorem: \((x+a)^n = \sum_{n_0=0}^{n_0=0} \binom{H_0}{n} x^n a^{n-n}.

\[
= \mathcal{K} \sum_{[N_j] \neq [0]} e^{-\sum_{i,j} \frac{\pi^2}{2(T_j)^2}|N_j|^2} \sum_{n=0}^{N} \frac{N!}{(N-n)!n!} D^{N-n} \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}^{3M}} d^{3M}Y e^{-2\|Y\|^2} \|Y\|^{2n}.
\]

Now according to (C.43) we have

\[
\frac{\sqrt{2}}{\pi} \int_{\mathbb{R}^{3M}} d^{3M}Y e^{-2\|Y\|^2} \|Y\|^2 = I_0 = \prod_{l=1}^{n} \frac{3M + 2(l-1)}{4}
\]

\[
= \mathcal{K} \sum_{[N_j] \neq [0]} e^{-\sum_{i,j} \frac{\pi^2}{2(T_j)^2}|N_j|^2} \sum_{n=0}^{N} \frac{N!}{(N-n)!n!} D^{N-n} I_n
\]

\[
= \mathcal{K} \sum_{[N_j] \neq [0]} e^{-\sum_{i,j} \frac{\pi^2}{2(T_j)^2}|N_j|^2} \left\{ D^N + \sum_{n=1}^{N} \frac{N!}{(N-n)!n!} D^{N-n} \prod_{l=1}^{n} \frac{3M + 2(l-1)}{4} \right\}. \tag{5.9}
\]

\(^{13}\) As can be seen from (C.16), \(p_j^p\) is a real number. We only take its modulus here in order to be independent of construction conventions (signatures) of the coherent states.
In analogy with [20] we see that all sum terms in (5.9) are due to the prefactors $e^{-\sum \frac{1}{2} T_{ij} N_j^2}$ of order $t^{-\infty}$ as long as $\{N_j^i\} \neq 0$. So one gets (non-negligible) contributions only from the $\{N_j^i\} = \{0\}$ term.

Finally, we find

\[ \langle \Psi_1^{(v)}(A) \mid \prod_{k=1}^N \hat{q}_k^i(r) \mid \Psi_1^{(v)}(A) \rangle \leq \frac{(\ell P)^{3N} (9M)^N |Z|^N [\frac{2A}{T}]^N}{\prod_{I,i}[1 + K_{i(I)}]} \times \left[ \frac{3Mp}{4T} \right]^N + \sum_{n=1}^N \frac{N!}{(N-n)n!} \left[ \frac{3Mp}{4T} \right]^{N-n} \prod_{i=1}^n \left[ \frac{3M + 2(l - 1)}{4} \right] \right] (5.10) \]

with

\[ T := \min\{T_i\} \quad \rightarrow \quad A := 1 + \frac{p}{T} \]
\[ p := \max_{i,j} \{|p_j^i(m)|\}. \]

Concerning the result (5.10) some comments are appropriate.

- The upper bound (5.10) is robust with respect to the classical configuration the coherent state $\Psi_1^{(v)}(A)$ is peaked on. Only $p$ has to be bounded from above. In particular at the classically singular configuration of the big bang where $\sqrt{\det(q)} = 0$ and thus $E_j^p = 0$ which implies $p = 0$ nothing special happens.

- Unfortunately (5.10) scales with the number of edges $M$ as $M^{2N}$. So one could essentially take two points of view:

  1. We are working on the kinematical level only, therefore we would expect some high valent vertex suppression in the physical theory.

  2. Maybe the approximations used here are too rough, but we see that the only possibility in order to avoid a dependence on the edge number $M$ is to have the integrals (C.43) independent of their dimension $3M$. Only the integral $I_0$ is naturally independent of $M$. Hence one would like to derive a better estimate so that the integrand becomes independent of $M$. One could expect that this might be possible by taking the sign factor $\epsilon(I,J,K)$ into account which may lead to cancellations. However, as we will show explicitly in C.5 of the appendices one can construct edge configurations so that (for any $M$) all $\epsilon(I,J,K)$ have equal sign.

  The ultimate answer has to be left open as a future task at the moment. It might provide us with some criteria physical states should fulfill.

In any case, for the special case (2.10) we thus get as upper bound for the expectation value of the gravitational part evaluated at a single vertex $v \in V(\gamma)$

\[ \hat{H}_{\text{grav}}^{(v)} := \frac{1}{\sqrt{\det q}} \sum_{v \in V(\gamma)} \frac{1}{E(v)} \sum_{IJK} \epsilon(IJK) e^{UK} e_{ijk} (\frac{1}{2}) \hat{c}_j^i(v) (\frac{1}{2}) \hat{c}_j^k(v) (\frac{1}{2}) \hat{c}_k^j(v) \]
\[ \times \frac{1}{E(v)} \sum_{LMN} \epsilon(LMN) e^{LMN} e_{mun} (\frac{1}{2}) \hat{c}_l^m(v) (\frac{1}{2}) \hat{c}_m^l(v) (\frac{1}{2}) \hat{c}_n^m(v) \]

14 Neglecting all terms of order $t^{-\infty}$. 

\[ \hat{H}_{\text{kin}} := \frac{P}{\hbar^2} \sum_{v \in V(\gamma)} \frac{1}{E(v)} \sum_{IJK} \epsilon(IJK) e^{UK} e_{ijk} (\frac{1}{2}) \hat{c}_j^i(v) (\frac{1}{2}) \hat{c}_j^k(v) (\frac{1}{2}) \hat{c}_k^j(v) \]
\[ \times \frac{1}{E(v)} \sum_{LMN} \epsilon(LMN) e^{LMN} e_{mun} (\frac{1}{2}) \hat{c}_l^m(v) (\frac{1}{2}) \hat{c}_m^l(v) (\frac{1}{2}) \hat{c}_n^m(v) \]
of $\hat{H}_{\text{kin}}$ with $N = 6$, $r = \frac{1}{2}$, $E(v) = \left(\frac{M}{3}\right)$, $P$ being the prefactor as defined in (2.10)$^{15}$, $(\kappa \hbar = \epsilon_P^2)$ and $\hat{H}_{\text{kin}}^{(\text{grav})}$ evaluated at one vertex $v$ only:

$$\langle \Psi^{(v)}_{m,\gamma}(A) | \frac{1}{\sqrt{\det q}} | \Psi^{(v)}_{m,\gamma}(A) \rangle \leq \frac{P}{\kappa \hbar^N} \sum_{v \in V(\gamma)} (36)^{\frac{1}{2}} (E_P)^{10} (9M)^{\frac{1}{2}} [\frac{2}{3}]^{\frac{1}{2}} \left[ 1 + \frac{K_i}{K_i(T)} \right] \left[ (\frac{3M_P}{4T})^6 \right]$$

$$+ \sum_{n=1}^{6} \frac{6!}{(6-n)!n!} \left[ (\frac{3M_P}{4T})^6-n \right] \prod_{l=1}^{n} \left[ 3M + 2(l-1) \right].$$

(5.11)

Here we again have estimated all sign factors by 1 and also neglected the signs contributed from the $\epsilon_{ijk}$, $\epsilon_{lmn}$ symbols (which gives the $36^2 = (3!)^4$ prefactor).

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Appendix A. $SU(2)$ properties (defining representation)

We have the $\tau$-matrices given by $\tau_k := -i \sigma_k$ with $\sigma_k$ the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{with} \quad [\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k$$

$$\tau_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad \text{with} \quad [\tau_i, \tau_j] = 2 \epsilon_{ijk} \tau_k.$$

Additionally, we use

$$\epsilon = -\tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{with the obvious properties} \quad \epsilon^{-1} = \epsilon^T = -\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In the defining representation of $SU(2)$ we have for a group element $h \in G$

$$h = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad \text{with} \quad \det h = |a|^2 + |b|^2 = 1$$

$$h^{-1} = \epsilon h^T \epsilon^{-1} = \begin{pmatrix} \bar{a} & -\bar{b} \\ b & a \end{pmatrix}$$

$$\overline{h} = [h^{-1}]^T = [\epsilon h^T \epsilon^{-1}]^T = \epsilon h \epsilon^T = \epsilon h \epsilon^{-1}.$$

For the $\tau_k$ we additionally have

$$-\overline{\tau_k}^T = \tau_k.$$

Here the overline always means complex conjugation of the matrix elements and $^T$ means transpose.

$^{15}$ Note that the denominator of the gravitational part is equipped with a prefactor $\frac{1}{\kappa \hbar^N} = \frac{1}{(\epsilon_P^2)}^N$. 

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Appendix B. Action of a holonomy on a recoupling scheme

B.1. Representation matrix elements of SU(2) elements

For the matrix element \([\pi_j(h)]_{mn}\) of an irreducible representation with weight \(j\) of a holonomy \(h \in SU(2)\) we have

\[
[\pi_j(h)]_{mn} = \left[ \frac{\binom{2j}{j+m} \binom{2j}{j+n}}{2^{2j} (j-m)! (j+m)! (j-n)! (j+n)!} \right]^{\frac{1}{2}} c_{jm}^c_{jn} h^{(A_1; B_1; h_{A_2; B_2} \cdots h_{A_N; B_N})}_{AB}
\]

(B.1)

where the round bracket means symmetrization w.r.t. the defining \(\pi_j(h)\). Note that we sum here over one fixed but arbitrary combination \(A_1, \ldots, A_N\), and \(h_{A_1; B_1} \cdots h_{A_N; B_N}\) is a matrix element of the defining \((j = \frac{1}{2})\) representation:

\[
h_{AB} := [\pi_{j=\frac{1}{2}}]_{AB}\quad \text{with} \quad h = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \det h = h_{11}h_{22} - h_{12}h_{21} = |a|^2 + |b|^2 = 1.
\]

(B.2)

Here we use the conventions

\[
\begin{align*}
  h_{11} &= h_{\frac{1}{2}+\frac{1}{2}} = a & h_{12} &= h_{\frac{1}{2}+\frac{1}{2}} = b \\
  h_{21} &= h_{\frac{1}{2}-\frac{1}{2}} = -b & h_{22} &= h_{\frac{1}{2}-\frac{1}{2}} = a.
\end{align*}
\]

(B.3)

We have

\[
N = 2j \quad \begin{align*}
  m &= A_1 + \cdots + A_N \\
  n &= B_1 + \cdots + B_N
\end{align*} \quad \begin{align*}
  j + m : \#A_k &= +\frac{1}{2} \\
  j - m : \#A_k &= -\frac{1}{2} \\
  j + n : \#B_k &= +\frac{1}{2} \\
  j - n : \#B_k &= -\frac{1}{2}.
\end{align*}
\]

(B.4)

We can explicitly check the representation property:

\[
\sum_{l=j}^{j} [\pi_j(h)]_{mn} [\pi_j(g)]_{ln} = c_{jm} c_{jn} \sum_{l=B_1 + \cdots + B_N} \left( c_{jl} \right)^2 \cdot h^{(A_1; B_1; h_{A_2; B_2} \cdots h_{A_N; B_N})}_{AB} \cdot g^{(B_1; C_1; g_{B_2; C_2} \cdots g_{B_N; C_N})}_{BC}
\]

\[
= c_{jm} c_{jn} \sum_{l=B_1 + \cdots + B_N} \left( c_{jl} \right)^2 \cdot h^{(A_1; B_1; h_{A_2; B_2} \cdots h_{A_N; B_N})}_{AB} \cdot g^{(B_1; C_1; g_{B_2; C_2} \cdots g_{B_N; C_N})}_{BC}
\]

\[
= c_{jm} c_{jn} \sum_{l=B_1 + \cdots + B_N} \left( 2j \right)^{j+l} \cdot h^{(A_1; B_1; h_{A_2; B_2} \cdots h_{A_N; B_N})}_{AB} \cdot g^{(B_1; C_1; g_{B_2; C_2} \cdots g_{B_N; C_N})}_{BC}
\]

Note that we sum here over one fixed but arbitrary combination \(B_1, \ldots, B_N\). Because of the symmetrization in the \(A_k\) and the \(C_k\) each such arbitrary combination is equivalent to summing over all index combinations in the tensor space that fulfil \(l = B_1 + \cdots + B_N\) (with \(j + l : \#B_k = +\frac{1}{2}\) \(j - l : \#B_k = -\frac{1}{2}\)). Multiplication in the tensor space requires us to perform a sum \(\sum_{B_1, \ldots, B_N = \pm \frac{1}{2}}^{1} \) over all possible \(B_k\)-combinations. Now this summation contains \(\binom{2j}{j} \binom{j}{l}\) configurations with \(B_1 + \cdots + B_N = l\) each giving the same contribution to the sum due to the symmetrization in the \(A_k\) and the \(C_k\).
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\[ c_{jm} \sum_{B_1, \ldots, B_N} h_{(A_1 B_1) h_{A_2 B_2} \ldots h_{A_N B_N}} \cdot \mathcal{B}(1, \mathcal{B}(2, \ldots, \mathcal{B}(N)) \]

\[ = c_{jm} \frac{1}{2} \sum_{B_1, \ldots, B_N} [h_{A_1 B_1} (h_{A_2 B_2} \ldots h_{A_N B_N})]_{C_N} \]

\[ = c_{jm} [\pi_j (h_{g})]_{C_N} \]

**B.2. Adding one more irreducible representation—tensor product**

\[ h_{A_0 B_0} h_{(A_1 B_1) h_{A_2 B_2} \ldots h_{A_N B_N}} \]

\[ = \frac{1}{N!} \sum_{\pi(0 \ldots N)} h_{A_0 B_0} h_{A_{\pi(0)} B_{\pi(0)}} h_{A_{\pi(1)} B_{\pi(1)}} h_{A_{\pi(2)} B_{\pi(2)}} \ldots h_{A_{\pi(N)} B_N} \]

\[ = \frac{1}{N + 1} \frac{1}{N!} \sum_{\pi(0 \ldots N)} \left( h_{A_0 B_0} h_{A_{\pi(0)} B_{\pi(0)}} h_{A_{\pi(1)} B_{\pi(1)}} h_{A_{\pi(2)} B_{\pi(2)}} \ldots h_{A_{\pi(N)} B_N} \right) \]

\[ + h_{A_{\pi(1)} B_{\pi(1)}} h_{A_{\pi(2)} B_{\pi(2)}} \ldots h_{A_{\pi(N)} B_N} \]

\[ + h_{A_{\pi(1)} B_{\pi(1)}} h_{A_{\pi(2)} B_{\pi(2)}} \ldots h_{A_{\pi(N)} B_N} - h_{A_{\pi(1)} B_{\pi(1)}} h_{A_{\pi(2)} B_{\pi(2)}} \ldots h_{A_{\pi(N)} B_N} \]

\[ + \cdots \]

\[ + h_{A_{\pi(1)} B_{\pi(1)}} h_{A_{\pi(2)} B_{\pi(2)}} \ldots h_{A_{\pi(N)} B_N} - h_{A_{\pi(1)} B_{\pi(1)}} h_{A_{\pi(2)} B_{\pi(2)}} \ldots h_{A_{\pi(N)} B_N} \]

Note that the first terms in this sum are the symmetrization

\[ \frac{1}{(N + 1)!} \sum_{\pi(0 \ldots N)} h_{A_{\pi(0)} B_{\pi(0)}} h_{A_{\pi(1)} B_{\pi(1)}} h_{A_{\pi(2)} B_{\pi(2)}} \ldots h_{A_{\pi(N)} B_N} \]

Furthermore, note that

\[ h_{A_1 B_1} h_{A_2 B_2} - h_{A_2 B_2} h_{A_1 B_1} = \epsilon_{A_1 A_2} \epsilon_{B_1 B_2} \]

for the conventions

\[ \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} a \\ -b \end{pmatrix} \]

since by definition \( \det h = h_{11} h_{22} - h_{21} h_{12} = |a|^2 + |b|^2 = 1 \).
\[ + \epsilon_{A_0A_{(N)}} \epsilon_{B_0B_0} h_{A_{(1)}B_1} h_{A_{(2)}B_2} \ldots h_{A_{(N)}B_N} \Bigr\} \]  

(B.5)

In (B.5) we have the following contributions (using the conventions (B.3), (B.4)):

1. \((j \equiv n)\) rows will contribute for \(B_0 = \pm \frac{1}{2}\) due to \(\epsilon_{B_0B_0}\).
2. If we carry out the sum over all permutations \(\pi(1 \ldots N)\) then in each of the contributing rows we will find \((j \equiv m)\) non-zero combinations \(\epsilon_{A_0A_{(k)}}\) for \(A_0 = \pm \frac{1}{2}\) and fixed \(A_{(k)}\).
3. For each such fixed combination there remain \((2j - 1)! = (N - 1)!\) permutations within the \(R_k\)-terms.
4. The contributing \(R_k\)-terms have \(m^{(i)}_{\pi_{(i)}} = \sum_k A_k(R_k) - m + A_0\), \(n^{(j)}_{\pi_{(j)}} = \sum_k B_k(R_k) = n + B_0\) since the contraction with the \(\epsilon\) ‘annihilates’ one \(A_k = \mp \frac{1}{2}\), respectively \(B_k = \pm \frac{1}{2}\), for \(A_0 = \pm \frac{1}{2}\), respectively \(B_0 = \pm \frac{1}{2}\).
5. Each \(\epsilon\) contributes a ± sign.

Now we can discuss the contribution of the second term in (B.5) dependent on the values of \(A_0, B_0\). Using the conventions (B.3), (B.4) and the observations made, we can write

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
A_0 & B_0 & \epsilon_{A_0B_0} \neq 0 \text{ iff } & \text{Valid for } & \epsilon_{A_0B_0} \neq 0 \text{ iff } & \text{Valid for } & m^{(i)}_{\pi_{(i)}}
\hline
1 \Leftrightarrow \frac{1}{2} & 1 \Leftrightarrow \frac{1}{2} & A_{(1)}B_1 \mp 2 \Leftrightarrow \frac{1}{2} & (j \equiv n) & A_{(2)}B_2 \mp 2 \Leftrightarrow \frac{1}{2} & (j \equiv m) & (2j - 1)!
\hline
1 \Leftrightarrow \frac{1}{2} & 2 \Leftrightarrow \frac{1}{2} & A_{(1)}B_1 \mp 1 \Leftrightarrow \frac{1}{2} & (j \equiv n) & A_{(2)}B_2 \mp 2 \Leftrightarrow \frac{1}{2} & (j \equiv m) & (2j - 1)!
\hline
2 \Leftrightarrow \frac{1}{2} & 1 \Leftrightarrow \frac{1}{2} & A_{(1)}B_1 \pm 2 \Leftrightarrow \frac{1}{2} & (j \equiv n) & A_{(2)}B_2 \mp 1 \Leftrightarrow \frac{1}{2} & (j \equiv m) & (2j - 1)!
\hline
2 \Leftrightarrow \frac{1}{2} & 2 \Leftrightarrow \frac{1}{2} & A_{(1)}B_1 \pm 1 \Leftrightarrow \frac{1}{2} & (j \equiv n) & A_{(2)}B_2 \pm 1 \Leftrightarrow \frac{1}{2} & (j \equiv m) & (2j - 1)!
\hline
\end{array}
\]

Thus the sum in (B.5) can then be rewritten as \((N = 2j, \text{ using } A_0, B_0 = \pm \frac{1}{2})\)

\[h_{A_0B_0} h_{A_{(1)}B_1} h_{A_{(2)}B_2} \ldots h_{A_{(N)}B_N} = h_{A_0B_0} h_{A_{(1)}B_1} h_{A_{(2)}B_2} \ldots h_{A_{(N)}B_N}
+ 4A_0B_0 \frac{1}{2j + 1} \frac{1}{2j!} (2j - 1)! (j - 2A_0m)(j - 2B_0n)
\]

\[\times [h_{A_{(1)}B_1} h_{A_{(2)}B_2} \ldots h_{A_{(N-1)}B_{N-1}}]_{A_1 + \ldots + A_{(N-1)} = m + A_0}
B_1 + \ldots + B_{(N-1)} = n + B_0]
\]

\[= h_{A_0B_0} h_{A_{(1)}B_1} h_{A_{(2)}B_2} \ldots h_{A_{(N)}B_N}
+ 4A_0B_0 \frac{1}{(2j + 1)2j} (j - 2A_0m)(j - 2B_0n)
\]

\[\frac{1}{(2j + 1)2j} (j - 2A_0m)(j - 2B_0n)
\]

Using (B.1) this can be finally rewritten as

\[\left[\pi_{\frac{1}{2}}(h)\right]_{A_0B_0}\left[\pi_{\frac{1}{2}}(h)\right]_{m} = \frac{(2j)!}{\sqrt{(j + m)! (j - m)! (j + n)! (j - n)!}} \times
\]

\[\times \sqrt{\frac{(j + \frac{1}{2} - m + A_0)(j + \frac{1}{2} - m + A_0)(j + \frac{1}{2} + n + B_0)(j + \frac{1}{2} + n + B_0)}{(2j + 1)!}}
\]

\[\times \left[\pi_{\frac{1}{2}}(h)\right]_{m + A_0n + B_0} + \sqrt{(j + m)! (j - m)! (j + n)! (j - n)!} \times \frac{(2j)!}{\sqrt{(j + m)! (j - m)! (j + n)! (j - n)!}} \times
\]
\[ \sqrt{(j - \frac{1}{2} + m + A_0)! (j - \frac{1}{2} - m - A_0)! (j - \frac{1}{2} + n + B_0)! (j - \frac{1}{2} - n - B_0)! } \]
\[ \times 4A_0B_0 \frac{(j - 2A_0m)(j - 2B_0n)}{(2j + 1)2j} [\pi_{j - \frac{1}{2}}(h)]_{m + A_0, n + B_0} \]

Just set \( A_0, B_0 = \pm \frac{1}{2} \) and carefully cancel corresponding terms.

\[ \left[ \pi_{\frac{1}{2}}(h) \right]_{A_0B_0} [\pi(h)]_{mn} = \sqrt{\frac{(j + 2A_0m + 1)(j + 2B_0n + 1)}{(2j + 1)(2j + 2)}} [\pi_{j + \frac{1}{2}}(h)]_{m + A_0, n + B_0} \]
\[ + 4A_0B_0 \sqrt{\frac{(j - 2A_0m)(j - 2B_0n)}{(2j + 1)}} [\pi_{j - \frac{1}{2}}(h)]_{m + A_0, n + B_0}. \]  

(B.6)

Here \(-j \leq m, n \leq j \) and \( A_0, B_0 = \pm \frac{1}{2} \).

### B.3. Action on a SNF

We have to work out the exact action of a holonomy acting on the gauge invariant SNF: we will use the general expression (B.6).

If we use the correspondence for matrix elements of a representation \( \sqrt{2j + 1} [\pi_j(g)]_{mn} = |jm; n\rangle \) we can rewrite (B.6) as

\[ [\pi_{\frac{1}{2}}(g)]_{A_0B_0} |jm; n\rangle = \left[ \frac{(j + 2A_0m + 1)(j + 2B_0n + 1)}{(2j + 1)(2j + 2)} \right]^{\frac{1}{2}} \left| j + \frac{1}{2} m + A_0; n + B_0 \right\rangle \]
\[ \oplus 4A_0B_0 \left[ \frac{(j - 2A_0m)(j - 2B_0n)}{2j(2j + 1)} \right] \left| j - \frac{1}{2} m + A_0; n + B_0 \right\rangle. \]  

(B.7)

We can furthermore rewrite (B.7) by realizing that parts of the coefficients \( C_{(+)} \) and \( C_{(-)} \) correspond to Clebsch–Gordan coefficients:

\[ \left[ \frac{(j + 2A_0m + 1)}{(2j + 1)} \right]^{\frac{1}{2}} = \left| jm; \frac{1}{2} A_0 \right| \left| j + \frac{1}{2} m + A_0 \right\rangle \]
\[ - 2A_0 \left[ \frac{(j - 2A_0m)}{(2j + 1)} \right]^{\frac{1}{2}} = \left| jm; \frac{1}{2} A_0 \right| \left| j - \frac{1}{2} m + A_0 \right\rangle. \]  

(B.8)

16 For clarity of the notation, we will be a bit sloppy in our notation here. Correctly this correspondence has to be written as \( \sqrt{2j + 1} \left[ \pi_j(g) \right]_{mn} = \langle g | jm; n \rangle \) and thus expressing the orthonormality of the spin states:

\[ \langle jm'; n' | jm; n\rangle = \int_{SU(2)} d\mu_H(g) \langle jm'; n' | g \rangle \langle g | jm; n\rangle = (2j + 1) \int_{SU(2)} d\mu_H(g) \left[ \pi_j(g) \right]_{mn} \left[ \pi_j(g) \right]_{mn} = \delta_{jm'; n' n} \]

as stated by the Peter–Weyl theorem. So within our calculations we will use \( \sqrt{2j + 1} \left[ \pi_j(g) \right]_{mn} \) in order to express the function \( \sqrt{2j + 1} \left[ \pi_j(g) \right]_{mn} = : |jm; n\rangle. \)
So we can finally write

\[
\begin{align*}
\left[ \pi_j^1(g) \right]_{B_0} |jm; n \rangle & = \left[ j + 2B_0 \sigma + 1 \right] \left( j + \frac{1}{2}m + A_0 \right) \cdot \left| j + \frac{1}{2}m + A_0; n + B_0 \rightangle \\
& \oplus (-2) \left[ j - 2B_0 \sigma \right] \left( j - \frac{1}{2}m + A_0 \right) \cdot \left| j - \frac{1}{2}m + A_0; n + B_0 \rightangle \\
& = \bigoplus_{j = j_0 + \frac{1}{2}} C_j^1(B_0, n) \left( jm; \frac{1}{2}A_0 \right) \left| jm + A_0; n + B_0 \rightangle
\end{align*}
\]

(B.9)

where we have introduced the coefficients \( C_j^1(B_0, n) \).

So we note a fundamental difference between *internal* (tensor product of representations of the same edge-holonomy) and *external* (tensor product of representations of holonomies of different edges) recoupling!

### B.4. N-vertex

#### B.4.1. Recoupling schemes

In this section, we will briefly review the definitions of recoupling schemes. We will closely follow \([11, 12]\).

In what follows, we will frequently use \( \vec{m}_k := m_1 + m_2 + \cdots + m_k \).

A general (standard-) recoupling scheme. It is defined as follows:

\[
\left[ \vec{a}(12)JM; \vec{m} \right] = |a_2(j_1j_2)a_3(j_2j_3) \cdots a_{K-1}(a_{K-2}j_K) ak(a_{K-1}j_k) \times a_{K+1}(a_Kj_{K-1}) \cdots a_{N-1}(a_{N-2}j_{N-1}) j(a_{N-1}j_N) M; n_1 \cdots n_N \rangle
\]

\[
= \sum_{m_1 + \cdots + m_N = M} \langle j_1m_1; j_2m_2; j_3m_3 | a_2(j_1j_2) \vec{m}_2 \rangle \langle j_2m_2; j_3m_3 | a_3(j_2j_3) \vec{m}_3 \rangle \cdots \langle j_{K-1}m_{K-1}; j_Km_K | a_{K-1}(a_{K-2}j_K) \vec{m}_{K-1} \rangle \langle a_{K-1}(a_{K-2}j_K) \vec{m}_{K-1}; j_Km_K | ak(a_{K-1}j_k) \vec{m}_k \rangle \langle ak(a_{K-1}j_k) \vec{m}_k; j_Km_K | a_{K+1}(a_Kj_{K+1}) \vec{m}_{K+1} \rangle \cdots \langle a_{N-2}(a_{N-1}j_{N-1}) \vec{m}_{N-1}; j_{N-1}m_{N-1} | a_{N-1}(a_{N-2}j_{N-1}) \vec{m}_{N-1} \rangle \langle a_{N-1}(a_{N-2}j_{N-1}) \vec{m}_{N-1}; j_{N-1}m_{N-1} | j(a_{N-1}j_N) M \rangle
\]

(B.10)

*Orthogonality relations between recoupling schemes.* For the scalar product of two recoupling schemes we have \(^1^7\)

\[\langle j_1m_1; n_1 \rangle \otimes \langle j_2m_2; n_2 \rangle \cdots \otimes \langle j_{K-1}m_{K-1}; n_{K-1} \rangle \otimes \langle j_Km_K; n_K \rangle \otimes \langle j_{K+1}m_{K+1}; n_{K+1} \rangle \cdots \otimes \langle j_{N-1}m_{N-1}; n_{N-1} \rangle \otimes \langle j_Nm_N; n_N \rangle.\]

\(^1^7\) We suppress the quantum numbers \( n_1, \ldots, n_N \) (in general we have \( \delta_{\vec{m}_j} \)) and use the reality of the Clebsch–Gordan coefficients.
\[ \langle a'_{\bar{a}}' \cdots a'_{\bar{a}} | a_2 a_3 \cdots a_{N-1} J \rangle \]

\[ = \sum_{m_i + \cdots + m_N = M} \langle j_1 m_1' | j_2 m_2' | a'_{\bar{a}} \rangle \langle j_1 m_1' | j_2 m_2' | j_3 m_3' | a_{\bar{a}} \rangle \]

\[ \vdots \]

\[ = \delta_{MM'} \sum_{m_i + \cdots + m_N = M} \langle j_1 m_1' | j_2 m_2' | a'_{\bar{a}} \rangle \langle j_1 m_1' | j_2 m_2' | j_3 m_3' | a_{\bar{a}} \rangle \]

Introduce \( m_2 = \tilde{m}_2 - m_1 \).

\[ = \delta_{MM'} \delta_{\bar{a}a} \sum_{\tilde{m}_2 + m_1 + \cdots + m_N = M} \langle a_{\bar{a}} \tilde{m}_2' | a_{\bar{a}} \tilde{m}_2' | j_3 m_3' | j_3 m_3' | a_{\bar{a}} \rangle \]

Use unitarity of the Clebsch–Gordan coefficients in order to carry out the sum over \( m_1 \).

\[ = \delta_{MM'} \delta_{\bar{a}a} \sum_{\tilde{m}_2 + m_1 + \cdots + m_N = M} \langle a_{\bar{a}} \tilde{m}_2' | a_{\bar{a}} \tilde{m}_2' | j_3 m_3' | j_3 m_3' | a_{\bar{a}} \rangle \]

Introduce \( m_3 = \tilde{m}_3 - \tilde{m}_2 \).

\[ = \delta_{MM'} \delta_{\bar{a}a} \sum_{\tilde{m}_2 + m_1 + \cdots + m_N = M} \langle a_{\bar{a}} \tilde{m}_2' | a_{\bar{a}} \tilde{m}_2' | j_3 m_3' | j_3 m_3' | a_{\bar{a}} \rangle \]
Use unitarity of the Clebsch–Gordan coefficients in order to carry out the sum over \( \tilde{m}_2 \).

\[
\delta_{MM'}\delta_{a_2a_1}\delta_{d_1d_0} \sum_{\tilde{m}_1+\tilde{m}_2=\tilde{m}_{M}} \langle a_3\tilde{m}_3; j_m|a_4\tilde{m}_4\rangle \langle a_3\tilde{m}_3; j_m|a_4\tilde{m}_4\rangle \\
\vdots \\
\langle a'_N\tilde{m}_{N-2}; j_{N-1}|a'_N\tilde{m}_{N-1}\rangle \\
\langle a_{N-2}\tilde{m}_{N-2}; j_{N-1}|a_{N-1}\tilde{m}_{N-1}\rangle \\
\langle a'_{N-1}\tilde{m}_{N-1}; j_{N}|j_M\rangle \langle a_{N-1}\tilde{m}_{N-1}; j_{N}|j_M\rangle \\
\vdots \\
This process continues \( N-2 \) times.
\]

\[
= \delta_{MM'}\delta_{a_2a_1}\delta_{d_1d_0} \sum_{\tilde{m}_1+\tilde{m}_2=\tilde{m}_{M}} \langle a_{N-1}\tilde{m}_{N-1}; j_{N}|j_M\rangle \\
\times \langle a_{N-1}\tilde{m}_{N-1}; j_{N}|j_M\rangle \\
= \delta_{MM'}\delta_{a_2a_1}\delta_{d_1d_0} \delta_{a_{N-2}a_{N-1}} \delta_{d_{N-2}d_{N-1}} \delta_{j_{N-1}j_{N}} \delta_{j_{N}j_{M}} \delta_{\tilde{m}_{N-1}\tilde{m}_{N} \cdot \tilde{m}_M}.
\]

In the last line we have reintroduced the \( n \)-quantum numbers for completeness. Note that it is also possible to rearrange the Clebsch–Gordan coefficients and to start from the last line to the top (see [15]).

The result is written here only to demonstrate the nice properties of the Clebsch–Gordan coefficients.

It can easily be understood by recalling the definition of a recoupling scheme \( |a_2(j_1j_2)a_3(a_2j_3) \ldots a_{N-1}(a_{N-2}j_{N-1})J(a_{N-1}j_N)\rangle M \) as the simultaneous eigenstate for the operators \( (G_2)^2 = (J_1 + J_2)^2, (G_3)^2 = (G_2 + J_3)^2, \ldots, (G_{N-1})^2 = (G_{N-2} + J_{N-1})^2 \), \( J_2^2 = (G_{N-1} + J_N)^2 = (J_1 + \ldots + J_N)^2 \) with eigenvalues \( a_2(a_2 + 1), a_3(a_3 + 1), \ldots, a_{N-1}(a_{N-1} + 1), J(J+1) \).

Partial orthogonality relations between recoupling schemes. The same argument can also be applied to cases where we have to calculate the scalar product of two recoupling schemes of different recoupling order. For illustration, let us consider two recoupling schemes

\[
|\tilde{a}JM\rangle = |a_2(j_1j_2)a_3(a_2j_3) \ldots a_{K-1}(a_{K-2}j_{K-1}a_K(a_{K-1}j_K) \ldots a_L(a_{L-1}j_L) \\
\times a_{L+1}(a_{L+1}j_{L+1}) \ldots a_{N-1}(a_{N-2}j_{N-1})J(a_{N-1}j_N)\rangle M \\
|\tilde{g}JM\rangle = |g_2(j_1j_2)g_3(a_2j_3) \ldots g_{K-1}(g_{K-2}j_{K-1}g_{K-1}j_{K-1}) \ldots g_L(g_{L-1}j_L) \\
\times g_{L+1}(g_{L+1}j_{L+1}) \ldots g_{N-1}(g_{N-2}j_{N-1})J(g_{N-1}j_N)\rangle M.
\]

Here from 2 \( \ldots K-1 \) the spins \( j_1, j_2 \ldots j_{K-1} \) are coupled in \( \tilde{a} \) and \( \tilde{g} \) in the same order. Then \( j_K \ldots j_L \) are coupled in the standard way to \( \tilde{a} \) but in a different order to \( \tilde{g} \). After that \( j_{L+1} \ldots j_N \) are successively coupled to each scheme again.

Now it is clear that \( \tilde{a} \) and \( \tilde{g} \) simultaneously diagonalize not only \( (G_2)^2 = (J_1 + J_2)^2, (G_3)^2 = (G_2 + J_3)^2, \ldots, (G_{K-1})^2 = (G_{K-2} + J_{K-1})^2 \) but also \( (G_L)^2, (G_{L+1})^2, \ldots, (G_{N-1})^2 \), \( J_2^2 = (J_1 + \ldots + J_N)^2 \). Therefore we can write down immediately

\[
|\tilde{a}JM\rangle \langle \tilde{g}JM| = |a_2(a_2j_2) \ldots a_{K-1}(a_{K-2}j_{K-1})a_K(a_{K-1}j_K) \ldots a_L(a_{L-1}j_L) \\
\times |g_2(a_2j_2) \ldots g_{K-1}(g_{K-2}j_{K-1})a_K(g_{K-1}j_K) \\
\times \delta_{a_2j_2} \ldots \delta_{a_{K-1}j_{K-1}} \times \delta_{a_Lj_L} \ldots \delta_{a_{N-1}j_{N-1}} \times \delta_{j_{K}j_{K}} \times \delta_{MM'}.
\]

Note that (B.12) is according to lemmas 5.1 and 5.2 in [12].
After the action of a holonomy. We have according to (B.9)

\[
\left[ \frac{\pi}{2} (\hbar K) \right]_{jK} |\tilde{g}(12)JM; \tilde{n} \rangle = \sum_{j_{K} = j_{K} \pm \frac{1}{2}} \sum_{m_{1} + \cdots + m_{N} = M} \langle j_{1}m_{1}; j_{2}m_{2}|a_{2}(j_{1}j_{2})\tilde{m}_{2} \rangle \\
\langle a_{1}\tilde{m}_{1}; j_{3}m_{3}|a_{3}(a_{2}j_{3})\tilde{m}_{3} \rangle \\
\vdots \\
\langle a_{K-1}\tilde{m}_{K-1}; j_{K}m_{K}|a_{K}(a_{K-1}j_{K})\tilde{m}_{K} \rangle \\
\left( j_{K}m_{K}; 1A \right) \frac{1}{2} \left( j_{K}m_{K} + A \right) C^{J}_{j_{K}}(B, n_{K})
\]

This will spoil the use of the unitarity trick!

\[
\langle j_{1}m_{1}; j_{2}m_{2}|J(a_{N-1}j_{N})M \rangle \\
|j_{1}m_{1}; n_{1}\rangle \otimes \cdots \otimes |j_{K}m_{K} + A; n_{K} + B \rangle \otimes \cdots \otimes |j_{N}m_{N}; n_{N} \rangle.
\]

(B.13)

Now the action (B.13) of a holonomy clearly prevents us from directly using the unitarity properties of the Clebsch–Gordan coefficients from \( j_{K} \) on, since then we have different Clebsch–Gordan coefficients and the unitarity condition cannot be used any more!

**Cure:** expansion into another recoupling scheme. In order to get rid of all the \( m \)-summations and to obtain a closed expression, we will expand the recoupling scheme (B.10) as follows:

\[
|a_{2}(j_{1}j_{2}) \ldots a_{K-1}(a_{K-2}j_{K-1})a_{K}(a_{K-1}j_{K}) \rangle \\
\times a_{K+1}(a_{K}j_{K+1}) \ldots a_{N-1}(a_{N-2}j_{N-1})J(a_{N-1}j_{N})M\tilde{n} \rangle \\
= \sum_{\tilde{g}} \langle \tilde{g}JM\tilde{n}|\tilde{a}JM\tilde{n} \rangle |g_{2}(j_{1}j_{2}) \ldots g_{K}(g_{K-2}j_{K-1})g_{K}(g_{K-1}j_{K+1}) \ldots g_{N-1}(g_{N-2}j_{N})J(g_{N-1}j_{N})M\tilde{n} \rangle.
\]

(B.14)

In the following calculation we always have \( M' = M \) and \( J' = J \), since, otherwise, the scalar product would vanish. We suppress the quantum numbers \( M \) and \( \tilde{n} = (n_{1} \ldots n_{N}) \).

Note that we will frequently use the reality of the \( 3n \) symbols, that is, \( \langle \tilde{g}JM\tilde{n} | aJM\tilde{n} \rangle = \langle aJM\tilde{n} | \tilde{g}JM\tilde{n} \rangle \) and (B.12). Moreover, \( |\rangle \) means a bra-vector containing the same arguments as its companion ket-vector (\( |\rangle \)). We have

\[
\langle \tilde{a}J|\tilde{g}J = \langle a_{2}(j_{1}j_{2}) \ldots a_{K-1}(a_{K-2}j_{K-1})a_{K}(a_{K-1}j_{K})a_{K+1}(a_{K}j_{K+1}) \ldots a_{N-1} \times (a_{N-2}j_{N-1})J(a_{N-1}j_{N})|g_{2}(j_{1}j_{2}) \ldots g_{K}(g_{K-2}j_{K-1}) \times g_{K}(g_{K-1}j_{K+1})g_{K+1}(g_{K}j_{K+2}) \ldots g_{N-1}(g_{N-2}j_{N})J(g_{N-1}j_{N}) \rangle
\
= \delta_{a_{2}|g_{2} \ldots \delta_{a_{K-1}j_{K-1}} \ldots \delta_{a_{K-1}j_{K-1}}}
\times \langle a_{K}(a_{K-1}j_{K})a_{K+1}(a_{K}j_{K+1}) \ldots a_{N-1}(a_{N-2}j_{N-1})J(a_{N-1}j_{N}) \rangle
\times |g_{K}(g_{K-1}j_{K+1})g_{K+1}(g_{K}j_{K+2}) \ldots g_{N-1}(g_{N-2}j_{N})J(g_{N-1}j_{N}) \rangle
\]

In order to get rid of all the \( m \)-summations and to obtain a closed expression, we will expand the recoupling scheme (B.10) as follows:

\[
1 = \sum_{\tilde{h}} |h_{K}(a_{K-1}j_{K})h_{K+1}(h_{K}j_{K})h_{K+2}(h_{K+1}j_{K+2}) \ldots J(h_{N-1}j_{N})\rangle |\%angle.
\]

\[
= \delta_{a_{2}|g_{2} \ldots \delta_{a_{K-1}j_{K-1}} \ldots \delta_{a_{K-1}j_{K-1}}}
\times \langle a_{K}(a_{K-1}j_{K}) \rangle
\]
\[ \Delta_{a_{K+1}a_{K+1}a_{h_{N+1}}} \]

\[ a_{K+1}(a_{K+1}) \ldots a_{N-1}(a_{N-2}J_{N-1})J(a_{N-1}J_{N})J(h_{N-1}J_{N})h_{K}(a_{K-1}J_{K+1})h_{K+1}(h_{K}J_{K})h_{K+2}(h_{K+1}J_{K+2}) \ldots J(h_{N-1}J_{N})g_{K}(g_{K-1}J_{K+1}) \]

\[ \times g_{K+1}(g_{K}J_{K+2}) \ldots g_{N-1}(g_{N-2}J_{N-1})J(g_{N-1}J_{N}) \]

\[ = \Delta_{a_{H_{K+2}}a_{H_{K+2}}a_{h_{N+1}}} \]

\[ a_{K+1}(a_{K+1}J_{K+2}) \ldots J(a_{N-1}J_{N})h_{K}(a_{K+1}J_{K+2})h_{K+2}(h_{K+1}J_{K+2}) \ldots J(h_{N-1}J_{N}) \]

\[ \times \langle h_{N-1}J_{N} \rangle \langle h_{K}J_{K} \rangle \langle h_{K+1}J_{K+1} \rangle \ldots \langle h_{N-1}J_{N} \rangle \]

\[ \Delta_{a_{H_{K+1}}} \]

\[ a_{K+1}(a_{K+1}) \ldots a_{N-1}(a_{N-2}J_{N-1})J(a_{N-1}J_{N})h_{K}(a_{K-1}J_{K+1})h_{K+1}(h_{K}J_{K})h_{K+2}(h_{K+1}J_{K+2}) \ldots J(h_{N-1}J_{N})g_{K}(g_{K-1}J_{K+1}) \]

\[ \times g_{K+1}(g_{K}J_{K+2}) \ldots g_{N-1}(g_{N-2}J_{N-1})J(g_{N-1}J_{N}) \]

\[ \Delta_{a_{H_{K}}} \]

This can be continued until we finally get the following.

\[ \Delta_{a_{H_{K+2}}a_{H_{K+2}}a_{h_{N+1}}} \]

\[ a_{K+1}(a_{K+1}J_{K+2}) \ldots J(a_{N-1}J_{N})h_{K}(a_{K+1}J_{K+2})h_{K+2}(h_{K+1}J_{K+2}) \ldots J(h_{N-1}J_{N}) \]

\[ \times \langle h_{N-1}J_{N} \rangle \langle h_{K}J_{K} \rangle \langle h_{K+1}J_{K+1} \rangle \ldots \langle h_{N-1}J_{N} \rangle \]

\[ \Delta_{a_{H_{K+1}}} \]

\[ a_{K+1}(a_{K+1}) \ldots a_{N-1}(a_{N-2}J_{N-1})J(a_{N-1}J_{N})h_{K}(a_{K-1}J_{K+1})h_{K+1}(h_{K}J_{K})h_{K+2}(h_{K+1}J_{K+2}) \ldots J(h_{N-1}J_{N})g_{K}(g_{K-1}J_{K+1}) \]

\[ \times g_{K+1}(g_{K}J_{K+2}) \ldots g_{N-1}(g_{N-2}J_{N-1})J(g_{N-1}J_{N}) \]

\[ \Delta_{a_{H_{K}}} \]

This can be continued until we finally get the following.

\[ \Delta_{a_{H_{K+2}}a_{H_{K+2}}a_{h_{N+1}}} \]

\[ a_{K+1}(a_{K+1}J_{K+2}) \ldots J(a_{N-1}J_{N})h_{K}(a_{K+1}J_{K+2})h_{K+2}(h_{K+1}J_{K+2}) \ldots J(h_{N-1}J_{N}) \]

\[ \times \langle h_{N-1}J_{N} \rangle \langle h_{K}J_{K} \rangle \langle h_{K+1}J_{K+1} \rangle \ldots \langle h_{N-1}J_{N} \rangle \]

\[ \Delta_{a_{H_{K+1}}} \]

This can be continued until we finally get the following.

\[ \Delta_{a_{H_{K+2}}a_{H_{K+2}}a_{h_{N+1}}} \]

\[ a_{K+1}(a_{K+1}J_{K+2}) \ldots J(a_{N-1}J_{N})h_{K}(a_{K+1}J_{K+2})h_{K+2}(h_{K+1}J_{K+2}) \ldots J(h_{N-1}J_{N}) \]

\[ \times \langle h_{N-1}J_{N} \rangle \langle h_{K}J_{K} \rangle \langle h_{K+1}J_{K+1} \rangle \ldots \langle h_{N-1}J_{N} \rangle \]

\[ \Delta_{a_{H_{K}}} \]

This can be continued until we finally get the following.

\[ \Delta_{a_{H_{K+2}}a_{H_{K+2}}a_{h_{N+1}}} \]

\[ a_{K+1}(a_{K+1}J_{K+2}) \ldots J(a_{N-1}J_{N})h_{K}(a_{K+1}J_{K+2})h_{K+2}(h_{K+1}J_{K+2}) \ldots J(h_{N-1}J_{N}) \]

\[ \times \langle h_{N-1}J_{N} \rangle \langle h_{K}J_{K} \rangle \langle h_{K+1}J_{K+1} \rangle \ldots \langle h_{N-1}J_{N} \rangle \]

\[ \Delta_{a_{H_{K}}} \]

This can be continued until we finally get the following.

\[ \Delta_{a_{H_{K+2}}a_{H_{K+2}}a_{h_{N+1}}} \]

\[ a_{K+1}(a_{K+1}J_{K+2}) \ldots J(a_{N-1}J_{N})h_{K}(a_{K+1}J_{K+2})h_{K+2}(h_{K+1}J_{K+2}) \ldots J(h_{N-1}J_{N}) \]

\[ \times \langle h_{N-1}J_{N} \rangle \langle h_{K}J_{K} \rangle \langle h_{K+1}J_{K+1} \rangle \ldots \langle h_{N-1}J_{N} \rangle \]

\[ \Delta_{a_{H_{K}}} \]

This can be continued until we finally get the following.

\[ \Delta_{a_{H_{K+2}}a_{H_{K+2}}a_{h_{N+1}}} \]

\[ a_{K+1}(a_{K+1}J_{K+2}) \ldots J(a_{N-1}J_{N})h_{K}(a_{K+1}J_{K+2})h_{K+2}(h_{K+1}J_{K+2}) \ldots J(h_{N-1}J_{N}) \]

\[ \times \langle h_{N-1}J_{N} \rangle \langle h_{K}J_{K} \rangle \langle h_{K+1}J_{K+1} \rangle \ldots \langle h_{N-1}J_{N} \rangle \]

\[ \Delta_{a_{H_{K}}} \]
\[
\langle h_{N-2}(g_{N-3}j_{N-1})h_{N-1}(h_{N-2}j_k)\rangle_J\langle h_{N-1}(j_k)\rangle\langle h_{N-2}(g_{N-3}j_{N-1})g_{N-1}(g_{N-2}j_N)\rangle_J(g_{N-1}j_k)
\]

\[
\delta_{g_{N-2}j_{N-1}}
\]

\[= \delta_{a_2g_2} \cdots \delta_{a_Kj_K} \langle \hat{a} (a_Kj_k) a_{K+1} (a_Kj_k+1) \rangle \langle \hat{a} (a_Kj_k+1) a_{K+1} (g_{K}j_k) \rangle \]
\[
\langle \hat{a} (a_Kj_k) a_{K+1} (a_Kj_k+1) \rangle \langle \hat{a} (a_Kj_k+1) a_{K+1} (g_{K}j_k) \rangle \]
\[
\cdots
\]
\[
\langle (a_{N-3}g_{N-4}j_k) a_{N-2} (a_{N-3}j_{N-2}) \rangle \langle (a_{N-3}g_{N-4}j_k) a_{N-2} (a_{N-3}j_{N-2}) \rangle \]
\[
\langle (a_{N-2}g_{N-3}j_k) a_{N-1} (a_{N-2}j_{N-1}) \rangle \langle (a_{N-2}g_{N-3}j_k) a_{N-1} (a_{N-2}j_{N-1}) \rangle \]
\[
\langle (a_{N-1}g_{N-2}j_k) J(a_{N-1}j_k) \rangle \langle (a_{N-1}g_{N-2}j_k) J(a_{N-1}j_k) \rangle \]
\[
\langle (a_{N-1}g_{N-2}j_k) J(a_{N-1}j_k) \rangle \langle (a_{N-1}g_{N-2}j_k) J(a_{N-1}j_k) \rangle \].
\]

Note that all expressions in (B.15) can be rewritten as \(6j\) symbols! Moreover, for the special case \(K = 1\) we have to start from
\[
\langle \hat{g}J \rangle_M \equiv \langle g_2(j_2j_1)g_3(j_2j_1) \rangle \cdots \langle g_{N-1}(g_{N-2}j_1) \rangle\]
and obtain a result similar to (B.15) which (structurally) differs only by the recoupling order of the first two spins:
\[
\langle \hat{a} J \rangle \langle \hat{g} J \rangle := \langle a_2(j_1j_2)a_3(a_2j_1) \rangle \langle g_2(j_2j_1) \rangle \]
\[
\langle a_3(g_2j_1)a_4(a_3j_1) \rangle \langle g_3(j_2j_1) \rangle \]
\[
\langle a_4(g_3j_1)a_5(a_4j_1) \rangle \langle g_4(j_2j_1) \rangle \]
\[
\cdots
\]
\[
\langle (a_{N-3}g_{N-4}j_k) a_{N-2} (a_{N-3}j_{N-2}) \rangle \langle (a_{N-3}g_{N-4}j_k) a_{N-2} (a_{N-3}j_{N-2}) \rangle \]
\[
\langle (a_{N-2}g_{N-3}j_k) a_{N-1} (a_{N-2}j_{N-1}) \rangle \langle (a_{N-2}g_{N-3}j_k) a_{N-1} (a_{N-2}j_{N-1}) \rangle \]
\[
\langle (a_{N-1}g_{N-2}j_k) J(a_{N-1}j_k) \rangle \langle (a_{N-1}g_{N-2}j_k) J(a_{N-1}j_k) \rangle \].
\]

B.4.2. Discussion: action of a holonomy \(\pi_1(h_{e_\varepsilon})\) on a recoupling scheme. If we use the expansion (B.14) with the coefficients explicitly obtained in (B.15), and the abbreviations
\[
\langle \hat{a} J M ; \hat{n} \rangle := \langle a_2(j_1j_2)a_3(a_2j_1) \cdots a_{K+1}(a_{K+2}j_K) \rangle \times \langle a_{K+1}(a_{K+2}j_K) \cdots a_{N}(a_{N-1}j_N) = J M ; \hat{n} \rangle
\]
\[
\langle \hat{a}' J' M' ; \hat{n}' \rangle := \langle a'_2(j'_1j'_2)a'_3(a'_2j'_1) \cdots a'_K(a'_Kj_K) \rangle \times \langle a'_K(a'_Kj_K) \cdots a'_N(a'_Nj_N) = J' M' ; \hat{n}' \rangle
\]
\[
\langle \hat{g} J M ; \hat{n} \rangle := \langle g_2(j_1j_2)g_3(j_1j_2) \cdots g_{K-1}(g_{K-2}j_K) \rangle \times \langle g_{K-1}(g_{K-2}j_K) \cdots g_{N-1}(g_{N-2}j_N) \rangle \times \langle g_{N-1}(g_{N-2}j_N) = J M ; \hat{n} \rangle
\]
\[
\langle \hat{g}' J' M' ; \hat{n}' \rangle := \langle g'_2(j'_1j'_2)g'_3(j'_1j'_2) \cdots g'_{K-1}(g'_{K-2}j_K) \rangle \times \langle g'_{K-1}(g'_{K-2}j_K) \cdots g'_{N-1}(g'_{N-2}j_N) \rangle \times \langle g'_{N-1}(g'_{N-2}j_N) = J' M' ; \hat{n}' \rangle
\]
we can write
\[
\langle \hat{a} J' M' ; \hat{n}' \rangle [\pi_1(h_{e_\varepsilon})]_{AB} \langle \hat{a} J M ; \hat{n} \rangle = \sum g \langle \hat{g}' J' M' ; \hat{n}' \rangle \langle \hat{g} J M ; \hat{n} \rangle \langle \hat{g} J M ; \hat{n} \rangle.
\]

Let us introduce the shorthand \(\hat{m}_1 = m_1 + \cdots + m_1\), that is, the sum of all \(m\) according to the recoupling order in \(\hat{g}\). Especially, we have
\[
\hat{m}_{1 \leq K} = m_1 + \cdots + m_1
\]
\[
\hat{m}_{1 \geq K} = m_1 + \cdots + m_{K-1} + m_{K+1} + \cdots + m_1
\]
\[
\hat{m}_{N-1} = m_1 + \cdots + m_{K-1} + m_{K+1} + \cdots + m_N
\]
\[
M = \hat{m}_N = m_1 + \cdots + m_{K-1} + m_{K+1} + \cdots + m_N + m_K.
\]
Using these conventions and the coefficient definition of (B.9) we can write
\[ \langle g' J'M'; \tilde{n} | \left[ \frac{1}{2} \Sigma (h_k) \right]_{AB} | g J'M; \tilde{n} \rangle \]
\[ = \sum_{j_k = \pm \frac{1}{2}} \sum_{m'_k = \mp m_k \ldots m'_1 = \pm m_1 = \pm m_M = M} \left( j'_1 m'_1 ; j'_2 m'_2 \right) \delta_{\tilde{n}' \tilde{n}} \delta_{\tilde{n}' \tilde{n} + B} \times \ldots \times \left( j'_N m'_N ; \tilde{n}' \tilde{n} + B \right) \]
\[ \times \delta_{j'_1 j_1} \delta_{j'_2 j_2} \ldots \delta_{j'_N j_N} \]
\[ = \sum_{j_k = \pm \frac{1}{2}} \sum_{m'_k = \mp m_k \ldots m'_1 = \pm m_1 = \pm m_M = M} \left( j_1 m_1 ; j_2 m_2 \right) \delta_{\tilde{n}' \tilde{n}} \delta_{\tilde{n}' \tilde{n} + B} \times \ldots \times \left( j_N m_N ; \tilde{n}' \tilde{n} + B \right) \]
\[ \times \delta_{j'_1 j_1} \delta_{j'_2 j_2} \ldots \delta_{j'_N j_N} \]
\[ \times \prod_{L=1}^{N} \delta_{\tilde{n}' \tilde{n} + B} \delta_{\tilde{n}' \tilde{n} + B} \delta_{j'_L j_L} \times \delta_{j'_1 j_1} \]

Now we can use the unitarity properties of the Clebsch–Gordan coefficients to carry out all but the last sum (starting from the first line) over the \( m \).

---

\[ ^{18} \text{If we consider the gauge behaviour of a recoupling state of total angular momentum} J \text{ after the action of a holonomy of weight} \frac{1}{2}, \text{we can easily see that the resulting state transforms under gauge transformations according to} J \otimes \frac{1}{2} \]
\[
\begin{align*}
&= \sum_{j_K=\pm \frac{1}{2}} C_{fK}^{JN}(A, n_K) \sum_{m_N= \pm M} (g_{N-1}M M - \bar{m}_{N-1} + A) (g_{N-1}M M - \bar{m}_{N-1} + A) \\
&\times \left[ \prod_{L=1}^{N} \delta_{L, m_L} \delta_{J, J_L} \right] \times \delta_{m, n_N + B \delta_{J, j_K}} \\
&\times \delta_{M, M + A \delta_{J, j_K}} \prod_{L=1}^{N-1} \delta_{L, j_L}
\end{align*}
\]

Here in the last line we have introduced \( C_{fK}^{JN}(A, M, n_N) \) as can be seen from the context or explicitly in (B.22).

Using (B.15) and (B.20) we can now complete the expansion (B.18)
\[
\langle \bar{a}' J M'; \bar{n}'| \{ \pi_{\frac{1}{2}} (h_K) \}_{AB} | \bar{a} J M; \bar{n} \rangle = \langle T_{J'} | \{ \pi_{\frac{1}{2}} (h_K) \}_{AB} | T_J \rangle
\]
\[
= \sum_{\bar{g} \bar{g}'} \langle \bar{g}' J M'; \bar{n}'| \bar{g} J M; \bar{n} \rangle \langle \bar{g} J M; \bar{n} | \bar{a} J M; \bar{n} \rangle \\
\times \langle \bar{g} J M; \bar{n} | \bar{a} J M; \bar{n} \rangle = \sum_{j_{K}=\pm \frac{1}{2}} \sum_{j_{J}=\pm \frac{1}{2}} \sum_{j_{K}=-\frac{1}{2}} \sum_{j_{J}=-\frac{1}{2}} \sum_{j_{K}=-\frac{1}{2}} \sum_{j_{J}=-\frac{1}{2}} \sum_{m_{N}= \pm M} (g_{N-1}M M - \bar{m}_{N-1} + A) (g_{N-1}M M - \bar{m}_{N-1} + A) \\
\times \left[ \prod_{L=1}^{N} \delta_{L, m_L} \delta_{J, J_L} \right] \times \delta_{m, n_N + B \delta_{J, j_K}} \\
\times \delta_{M, M + A \delta_{J, j_K}} \prod_{L=1}^{N-1} \delta_{L, j_L}
\]

(B.20)
\[
(\alpha_{K+1}'(g_{K}j'_{K})\alpha_{K+2}'(\alpha_{K+1}'j'_{K+2})) = g_{K}j'_{K}(\alpha_{K+1}j_{K+2})(\alpha_{K+1}j_{K+2})^{-1}
\]
\[
= \sum_{\text{even}} \sum_{L=1}^{N-1} \left\{ \frac{\delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}}{\prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}} \right\}
\]
\[
= \sum_{\text{even}} \sum_{L=1}^{N-1} \left\{ \frac{\delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}}{\prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}} \right\}
\]
\[
= \sum_{\text{even}} \sum_{L=1}^{N-1} \left\{ \frac{\delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}}{\prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}} \right\}
\]
\[
C_{j_{K}}^{j_{K}'}(B, n_{K})C_{j_{K}'}^{j_{K}''}(A, M, g_{N-1}) \times \prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}} \times \delta_{M+\alpha_{K}j_{K}} \times \prod_{R=2}^{K-1} \delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}
\]
\[
= \sum_{\text{even}} \sum_{L=1}^{N-1} \left\{ \frac{\delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}}{\prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}} \right\}
\]
\[
= \sum_{\text{even}} \sum_{L=1}^{N-1} \left\{ \frac{\delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}}{\prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}} \right\}
\]
\[
= \sum_{\text{even}} \sum_{L=1}^{N-1} \left\{ \frac{\delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}}{\prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}} \right\}
\]
\[
= \sum_{\text{even}} \sum_{L=1}^{N-1} \left\{ \frac{\delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}}{\prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}} \right\}
\]
\[
= \sum_{\text{even}} \sum_{L=1}^{N-1} \left\{ \frac{\delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}}{\prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}} \right\}
\]
\[
= \sum_{\text{even}} \sum_{L=1}^{N-1} \left\{ \frac{\delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}}{\prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}} \right\}
\]
\[
= \sum_{\text{even}} \sum_{L=1}^{N-1} \left\{ \frac{\delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}}{\prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}} \right\}
\]
\[
= \sum_{\text{even}} \sum_{L=1}^{N-1} \left\{ \frac{\delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}}{\prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}} \right\}
\]
\[
= \sum_{\text{even}} \sum_{L=1}^{N-1} \left\{ \frac{\delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}}{\prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}} \right\}
\]
\[
= \sum_{\text{even}} \sum_{L=1}^{N-1} \left\{ \frac{\delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}}{\prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}} \right\}
\]
\[
= \sum_{\text{even}} \sum_{L=1}^{N-1} \left\{ \frac{\delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}}{\prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}} \right\}
\]
\[
= \sum_{\text{even}} \sum_{L=1}^{N-1} \left\{ \frac{\delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}}{\prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}} \right\}
\]
\[
= \sum_{\text{even}} \sum_{L=1}^{N-1} \left\{ \frac{\delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}}{\prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}} \right\}
\]
\[
= \sum_{\text{even}} \sum_{L=1}^{N-1} \left\{ \frac{\delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}}{\prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}} \right\}
\]
\[
= \sum_{\text{even}} \sum_{L=1}^{N-1} \left\{ \frac{\delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}}{\prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}} \right\}
\]
\[
= \sum_{\text{even}} \sum_{L=1}^{N-1} \left\{ \frac{\delta_{\alpha_{K}n_{K}+B}\delta_{j_{K}j'_{K}}}{\prod_{L=1}^{N-1} \delta_{\alpha_{K}n_{K}+B}} \right\}
\]
\[
\times \left[ \prod_{L=1}^{N} \delta_{n_L n_L} \delta_{j_L j_L} \right] \times \delta_{n_K n_K + B} \delta_{j_K j_K} \times \delta_{M M + A} \delta_{J J} \times \prod_{R=2}^{K-1} \delta_{a a R} = C^{T_j}_{T_j'} (K, A, B) \tag{B.21}
\]

We have used the following abbreviations \((A, B = \pm \frac{1}{2})\):

\[
C^{j_{k}}_{j_{k}} (B, n_K) = \begin{cases} 
-2B \left[ \frac{j_K - 2Bn_K}{2j_K} \right]^{\frac{1}{2}} & \text{if } j_K = j_K - \frac{1}{2} \\
\left[ \frac{j_K + 2Bn_K + 1}{2(j_K + 1)} \right]^{\frac{1}{2}} & \text{if } j_K = j_K + \frac{1}{2}
\end{cases}
\]

\[
C^{j_{k}}_{j_{k}} (A, M, g_{N-1}) = \sum_{m} \langle g_{N-1} m; j_K M - m \mid J(g_{N-1} j_K) M \rangle \\
\times \langle g_{N-1} m; \tilde{j}_K M - m + A_{0} \mid J'(g_{N-1} \tilde{j}_K) M + A \rangle \\
\times \langle j_K M - m; \frac{1}{2} A \mid \tilde{j}_K M - m + A \rangle 
\tag{B.22}
\]

Now the reason for our expansions becomes clear: for arbitrary \(K\) the action of a holonomy \(\pi_{j}(h_K)\) on a standard recoupling scheme can be expressed as a sum of standard recoupling schemes, where the expansion coefficients have a modular structure, which enables us to give explicit general equations since all the expressions in (B.21) can be calculated separately.

Finally, we have achieved our goal to express the state resulting from the action of a holonomy of an edge on a (standard) recoupling scheme as a sum over recoupling schemes,

\[
[\pi_{j}(h_K)]_{AB} |T_{j} \rangle = \sum_{T_{j}'} C^{T_j}_{T_j'} (K, A, B)|T_{j}' \rangle 
\tag{B.23}
\]

where \(T_{j}'\) is a multi-label containing all the quantum numbers \(\vec{a}', J', \tilde{j}', \vec{m}', \tilde{n}'\).\(^{19}\)

Note again the range of the variables involved in order to get a non-vanishing expansion coefficient:

\[
\tilde{j}_K = j_K \pm \frac{1}{2} \quad m_K' = m_K + A \quad n_K' = n_K + B \\
J_j = j_j \quad m_j' = m_j \quad n_j' = n_j \quad \forall l \neq k \quad \tilde{h}_K := [\pi_{j}(h_K)]_{AB} \\
M' = M + A \quad J' = J \pm \frac{1}{2}
\]

Appendix C. Definitions and conventions for U(1)\(^3\) coherent state calculation

In this section, we will summarize the construction of complexifier coherent states for loop quantum gravity. For a more detailed introduction, we refer to [3, 18–20].

\(^{19}\)For example \(\tilde{n} = \{n_1, \ldots, n_N\}\).
C.1. General construction

Formally, a complexifier coherent state can be constructed by

\[ \Psi_m(A) = \left[ e^{-\frac{i}{\hbar} C} \right] A_{A'\to A'^c(m)}. \]  

(C.1)

Here \( \delta_A(A) = \sum_{\gamma \in S} T_\gamma(A')T_\gamma(A) \) denotes the \( \delta \)-distribution on the (quantum-)configuration space \( \mathcal{A} \) with respect to the Ashtekar–Lewandowski measure \( \mu_0 \) in the thus built kinematical Hilbert space \( \mathcal{H}^0_{km} = L_2(\mathcal{A}, d\mu_0) \). The sum has to be extended over all spin network labels \( s \) and the \( T \) are the corresponding spin network functions which provide an orthonormal basis. \( C \) is called a complexifier, and \( A' \to A'^c \) indicates that after the action of \( e^{-\frac{i}{\hbar} C} \) the whole expression has to be analytically continued to all values of the complexification \( A'^c \) of the connection \( A' \) given by

\[ A'^c = \sum_{n=0}^{\infty} \frac{1}{n!} [A, C]_n \]  

(C.2)

can take (with the iterated Poisson bracket given by \( \{F, G\}_0 = F \) and \( \{F, G\}_n = \{\{F, G\}_0, G\}_n \)). This construction works for every compact gauge group \( G \).

C.2. \( G = SU(2) \)

The Hilbert space \( \mathcal{H}^0_{km} = L_2(\mathcal{A}, d\mu_0) \) is constructed as an inductive limit of subspaces \( \mathcal{H}^0_{km} = L_2(\mathcal{A}, d\mu_0) \) consisting of square integrable functions \( T_e \) cylindrical with respect to graphs \( \gamma \) consisting of analytical embedded edges \( e \in E(\gamma) \) which intersect at the vertices \( v \in V(\gamma) \). The coherent states are restricted to an arbitrary but fixed graph \( \gamma \), because due to the uncountability of the set \( s \) in (C.1) the thus constructed coherent state would not be normalizable. In order to allow distributional connections \( A \in \mathcal{A} \) one regularizes the classical Poisson algebra of connections \( A^r_e(x) \) and electric fields \( E^r_e(y) \)

\[ \{A^r_e(x), A^r_e(y)\} = \{E^r_e(x), E^r_e(y)\} = 0 \quad \{A^r_e(x), E^r_e(y)\} = \kappa \delta^r_e \delta(x, y) \]  

(C.3)

by smearing the connection over the one-dimensional edges \( e \) of a graph in order to obtain holonomies \( h_e(A) := \mathcal{P} e^\int A \) and integrating the electric fields over surfaces \( S \) in order to get electric fluxes \( E_j(S) = \int_f \mathcal{P} e^\int E_j \). One finds

\[ \{h_e, h_e\} = \{E_j(S), E_k(S')\} = 0 \quad \text{if } S, S' \text{ do not intersect (as it will be the case in our later considerations)} \]

\[ \{E_j(S), h_e\} = \begin{cases} 0 & e \cap S = \emptyset \quad \text{or} \quad e \cap S = e \\ \kappa \sigma(e, S) \frac{\tau_j}{2} h_e & e \cap S = u \quad u, \ldots, \text{beginning point of } e. \end{cases} \]  

(C.4)

Here the edge \( e \) is adapted to the surface \( S \), that is, \( e \) is outgoing from \( S \). The orientation of the tangent \( \mathcal{e}(u) \) of \( e \) compared to the surface normal at the intersecting point \( u \) is denoted by \( \sigma(e, S) \). For the surface normal pointing up \( \sigma(e, S) = 1 \) if \( \mathcal{e}(u) \) points up, \( \sigma(e, S) = -1 \) if \( \mathcal{e}(u) \) points down. The Peter–Weyl theorem is then employed in order to go to the spin network representation, built from the matrix-element functions of the representation matrices of the holonomies, whose closed linear span provides a basis of \( \mathcal{H}^0_{km} = L_2(\mathcal{A}, d\mu_0) \).

\[ h_e(A) \mapsto \sqrt{\text{dim } \pi_j}[\pi_j(h_e(A))]_{mn} = \sqrt{\text{dim } \pi_j}[\pi_j]_{mn}(h_e(A)) =: \langle h_e(A) | jm; n \rangle \]

\[ T_\gamma \delta_{jmn} := \prod_{e \in E(\gamma)} \sqrt{\text{dim } \pi_j}[\pi_j(h_e(A))]_{mn}. \]  

(C.5)
Here $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ is the weight of the representation and $m, n = -j, -j+1, \ldots, j-1, j$ denote its matrix element. The Poisson algebra (C.4) is then represented on $\mathcal{H}_\text{un}^\dagger$ as

$$\hat{\pi}_j^a [h_e]_{m \rho a} T_{\gamma j ab} = \pi_j [h_e]_{m \rho a} \cdot T_{\gamma j ab}$$
$$\hat{E}_k(S_e) T_{\gamma j ab} = \{h_e, h_e \} = X_k^\#(S_e) T_{\gamma j ab}$$

$$\frac{\text{Tr}_\epsilon}{\pi_j} \left( \pi_j [\epsilon a h_e] T_{\gamma j ab} \right) = \sum_{p,q=-j}^{j} \pi_j, \{[\epsilon a h_e] T_{\gamma j ab} \} \frac{\partial}{\partial \pi_j} T_{\gamma j ab}$$

where $S_e$ denotes that the smearing surface is transversal to the edge $e$, $X_k^\#(S_e)$ is the right invariant vector field on $G$ and $T$ means transpose.

C.3. $G = U(1)^3$

C.3.1. Charge networks We will specify now $G = U(1)^3$ instead of $G = SU(2)$ as an Abelianized model of general relativity. Instead of the three quantum numbers $j, m, n$ we will have three copies of $U(1)$ for each edge with three charges $n_1, n_2, n_3 \in \mathbb{Z}$. The point is now that the two index sets possess the same cardinality and therefore calculations can be (as a proof of concept) first carried out in $U(1)^3$ theory which, as we will show, is much simpler. The justification for this is the fact that the results seem to be qualitatively the same.

Let $A^\#_e$ be a $U(1)^3$ connection on a three-dimensional manifold $\sigma$ and $E$ is a conjugate electric field satisfying the canonical Poisson brackets (C.4). Again the indices $a, b, c, \ldots = 1, 2, 3$ are spatial indices whereas $i, j, k, \ldots = 1, 2, 3$ now indicate the copy of $U(1)$.

In order to regularize (C.3) we again introduce holonomy and flux variables:

$$h^\#_e(A) := e^{\int \gamma_a(x) d\sigma^a} = e^{\int A^a(x) \gamma_a}$$

$$E_j^\#(S) := \int_S \equiv E = \int_S \epsilon_{abc} E^a_j(x) d\sigma^a d\sigma^b = \int_U d\sigma^1 d\sigma^2 \epsilon_{abc} X^a_{S,\sigma} X^b_{S,\sigma} E^a_j(X_{\sigma}(u))$$

Again the indices $a, b, c, \ldots = 1, 2, 3$ are spatial indices whereas $i, j, k, \ldots = 1, 2, 3$ now indicate the copy of $U(1)$.

$$\{h^\#_e, h^\#_k\} = \{E_j^\#(S), E_k^\#(S')\} = 0$$

$$\{E_j^\#(S), h^\#_k\} = \begin{cases} 0 & \text{if } e \cap S = \emptyset \text{ or } e \cap S = e \subseteq S \text{ or } e \cap S = u \end{cases}$$

The prefactor $K = \frac{1}{2}$ if $u$ is the beginning or end point of $e$ (which can always be achieved by adapting the edge $e$ to the surface $S$) or $K = 1$ if $u$ is an internal point of $e$. This raises the following:

Definition C.1.

(i) A charge network $c$ is a pair $(\gamma(c), n(c))$ consisting of a graph $\gamma(c)$ together with a colouring of each of its edges $e \in E(\gamma)$ with three charges $n^j_e(c) \in \mathbb{Z}$. A charge network state is the following function on the space $\mathcal{C} = \mathcal{A}$ of smooth connections:

$$T_c : \mathcal{A} \mapsto \prod_{e \in \gamma(c)} \left[ h^\#_e(A) \right]_{n^j_e(c)}.$$
Figure C1. Foliation $\mathcal{F}^I (I = 3)$ direction into surfaces $\sigma^I_t$ of constant foliation parameter $t$. The foliation varies smoothly with $t$.

Figure C2. Choosing a parquette $P^I_t \Leftrightarrow$ partition of the $t_I(x) = \text{const}$ surface $\sigma^I_t$ into small surfaces $S$. Also $P^I_t$ varies smoothly with $t_I$ Note that for each $I, x$ we get a unique surface $S^I_{x,t}$ such that $x \in S^I_{x,t}$.

Note that if we were to work at the gauge invariant level, then at each $v \in V(\gamma)$ for each $j$ the sum of the charges of the edges would have to add up to 0. However, we will not use gauge invariant states to begin with.

(ii) The Hilbert space $\mathcal{H}^\text{kin}_{\gamma}$ is defined as the closed linear span of the charge network functions which form an orthonormal basis, that is,

$$
\langle T_c | T_{c'} \rangle^\text{kin} = \delta_{c,c'}, \quad (C.10)
$$

and their finite linear span is dense.

(iii) The representation of $(C.8)$ on $\mathcal{H}^\text{kin}_{\gamma}$ is then defined by

$$
\hat{h}^I_e T_e = h^I_e T_e \quad \hat{E}_j(S) T_e = i\hbar \{ E_j(S), T_e \}. \quad (C.11)
$$

C.3.2. Construction of the coherent states. Although our coherent state calculations closely follow [19, 20] we will use a somewhat more general and flexible construction principle here.

In order to give a regularized explicit expression for a $U(1)^3$ group coherent state over an arbitrary graph $\gamma$ we introduce three foliations $\mathcal{F}_I (I = 1, 2, 3)$ of $\sigma$ into two-dimensional hypersurfaces $\sigma^I_t$ such that two leaves $\sigma^I_t, \sigma^J_t$ for $I \neq J$ intersect transversally if they intersect at all (see figure C1).

In addition, choose a parquette $P^I_t$ (see figure C2), that is, we partition each of the $\sigma^I_t$ for fixed $(I, t)$ in small surfaces $S \subset \sigma^I_t$.

Note that each $S^I_t$ is defined by its embedding $X_{S(t)} : \mathbb{R}^2 \supset U \ni \sigma, [-\frac{1}{2}, \frac{1}{2}]^2 \ni u \mapsto X_{S(t)}(u)$, which is $t$-dependent due to the $t$-dependence of $P^I_t$.

By construction there is a bijection $Y_I : \sigma \mapsto \bigcup_{x \in \mathbb{R}} (t_I(x), u_I(x))$, with $u_I(x) = (u^1_I(x), u^2_I(x))$.

Now we can write down a complexifier $C$, which depends, of course, on the foliation $\mathcal{F}$:

$$
C_\mathcal{F} = \frac{1}{2\kappa L^3} \sum_{I=1}^{3} \int_{\mathbb{R}} \sum_{S \in P^I_t} \delta^{\mathcal{F}} E_j(S) E_k(S). \quad (C.12)
$$

Here $L$ is a length parameter that we keep unspecified at the moment.
For the complexifier (C.12) we note that only for \( n = 0, 1 \) we get non-vanishing iterated Poisson brackets in (C.2). This gives rise to the specific result

\[
Z_j^l(x) = A_j^l(x) - \frac{i}{L} \sum_{I=1}^{3} E_j^I(S^I_x) n^{g^I}_a(u^I_t) k^I(x) \quad \text{with} \quad k^I(x)
\]

\[
= \left[ \det \left( \frac{\partial X^I_{S^I_x}(u)}{\partial (t, u^1, u^2)} \right) \right]^{-1} \int_{t=I(t(x))}^{0} \left. \frac{d}{dt} \right|_{u=I(t(x))}.
\]  \tag{C.13}

Note that by (C.3) we only get a non-vanishing contribution for each foliation direction \( I \) if \( x \) is an element of a per construction unique subsurface \( S^I_x \). In this surface \( x \) coincides with a unique pair \( u^I_x = (u^1(x), u^2(x)) \) of embedding parameters. So the positive number \( k^I(x) \) is the inverse of the Jacobian of the embedding \( X^I_{S^I_x}(t) \) with respect to the foliation parameter \( t \) and the embedding parameters \( u = (u^1, u^2) \) at the point \( t = t^I_x \) and \( u = u^I_x \). According to (C.7) the ‘normal’ of the surface \( S^I_x \) is denoted by \( n^{g^I}_{S^I_x}(u) \), and \( E_j^I(S^I_x) \) is defined accordingly.

Note that in the case of ‘small’ surfaces \( S^I_x \) we get the bijection

\[
Z_j^l(x) \approx A_j^l(x) - \frac{i}{L} E_j^I(x) q^0_{ab}(x) \quad \text{with} \quad q^0_{ab}(x) = \sum_{I=1}^{3} k^I(x) \int_{S^I_x} d^2 u n^{g^I}_{S^I_x}(u) n^{g^I}_{S^I_x}(u).
\]  \tag{C.14}

Hence \( m = (A, E) \mapsto Z(A, E) = Z(m) \) becomes a bijection because \( q^0_{ab} \) is non-degenerate.

Now we want to construct the holonomies of the complexified connection \( A \) according to (C.8). The holonomies \( h^e_j(A) \) here are simple complex numbers since we work in \( U(1)^3 \) instead of \( SU(2) \):

\[
h^e_j(A) = e^{\int \left. Z^l_j(e(t)) \right|_m}\quad \text{with} \quad \int \left. Z^l_j(e(t)) \right|_m = \int \left. A^l_j(e(t)) \right|_m e^{\varphi(t)} dt.
\]  \tag{C.15}

By using (C.13) specified to a pair \( m = (A, E) \) we define

\[
p^e_j(m) = i \int \left. [Z^l_j(e(t))]_m - A^l_j(e(t))]_m\right] e^{\varphi(t)} dt.
\]  \tag{C.16}

So the complexified holonomy \( h^e_j(Z(m)) \) is given by

\[
h^e_j(Z(m)) = e^{\int \left. Z^l_j(e(t)) \right|_m} - A^l_j(e(t)]_m + A^l_j(e(t))]_m e^{\varphi(t)} dt
\]

\[
= e^{p^e_j(m)} e^{\int \left. A^l_j(e(t))]_m e^{\varphi(t)} dt}
\]

\[
= e^{p^e_j(m)} e^{\varphi(t) (A_m)}
\]

\[
= e^{p^e_j(m)} h^e_j(A | m).
\]  \tag{C.17}

Using (C.8), (C.9), (C.11) we can explicitly evaluate the action of the electric fluxes \( E_j(S) \) on a charge network function \( T_c \):

\[
\hat{E}_j(S) T_c = \hat{E}_j(S) \prod_{e \in \gamma(c)} \left[ h^l_j(A) \right]^{n^l_j(c)} = \frac{-\hbar}{2} \sum_{e \in \gamma(c)} \sigma(e, S) n^l_j(c) T_c
\]

\[
= -\frac{\hbar}{2} \lambda^l_j(S) T_c
\]  \tag{C.18}
which turn out to be diagonal with eigenvalue $\lambda_j(S)$. Next we want to evaluate the action of the operator version of the complexifier (C.12) itself:

$$\hat{C}_F T_c = \frac{1}{2\kappa L^3} \left[ \sum_{I=1}^3 \int_{\mathbb{R}} \left( \sum_{S \in \mathcal{P}^I} \delta^{(3)}(S) E_I(S) E_k(S) \right) \right] T_c$$

$$= \frac{\hbar \kappa}{8 L^3} \left[ \sum_{I=1}^3 \int_{\mathbb{R}} \sum_{S \in \mathcal{P}^I} \delta_{jk} \lambda_j(S) \lambda_k(S) \right] T_c$$

$$= \frac{\hbar \kappa}{8 L^3} \lambda_c(F) T_c.$$ (C.19)

Since $\hat{C}_F$ only consists of electric field operators and our theory is Abelian anyway, it is diagonal with eigenvalues $\lambda_c(F)$, which depend on the choice of the foliation $F$.

The $\delta$-distribution on $H_{\text{kin}}$ is (formally) given by

$$\delta_A(A) = \sum_c T_c(A') \overline{T_c(A)}.$$ (C.20)

By invoking the definition (C.1) of a complexifier coherent state together with (C.20) we are now able to evaluate down coherent states centred at the phase space pair $m = (A, E)$ on the kinematical Hilbert space $H^0_{\text{kin}}$:

$$\Psi_m(A) = e^{-\frac{\hbar}{\kappa} \lambda_c(F)} \sum_c T_c(A') e^{-\frac{\hbar}{\kappa} \lambda_c(F)} \overline{T_c(A)}.$$ (C.21)

Here the sum extends over all possible graphs and edge charges. Again, to make (C.21) normalizable we have to restrict ourselves to an arbitrary but fixed graph $\gamma(c)$. Then we have

$$\Psi_{m,\gamma}(A) = \sum_c e^{-\frac{\hbar}{\kappa} \lambda_c(F)} \prod_{e \in E(\gamma)} \left[ h_I^2(Z(m)) h_I^2(A)^{-1} \right]$$ (C.22)

In the last line, the symbol $\tilde{n}$ denotes the sum over all possible charge configurations the individual copies of $U(1)$ at every edge $E \in E(\gamma)$ can take. In the last step, we will bring (C.21) to the simpler form\footnote{This will be the case when the parquettes $\mathcal{P}^I$ are fine enough, such that to a good approximation $\sigma(e, S) \cdot \sigma(e', S) = 0$ iff $e \neq e'$ and the $(\lambda_j(S))^2$ in (C.18)-terms decompose.}:

$$\lambda_{\tilde{n}}(P) = \sum_{e \in E(\gamma)} f_e \left[ h_e^2 \right]^2$$ (C.23)

with $f_e = \sum_{I=1}^3 \int_{\mathbb{R}} \left( \sum_{S \in \mathcal{P}^I} \sigma(e, S) \right)^2$ being a dimensionful edge specific function, $f_e > 0 \forall e \in E(\gamma)$, $[f_e] = cm^1$. We can thus introduce the (edge specific) dimensionless classicality parameter

$$t(e) := \frac{\hbar \kappa}{4 L^3} f_e \quad (\text{C.24})$$

with $L$ being an (at the moment unspecified) parameter of dimension metre.
Then (C.22) can be simplified to
\[
\Psi_{m,v}(A) = \sum_{n} \prod_{j=1,2,3} e^{-\frac{\pi}{n} |n_j|^2} \left[ h_j^v(Z(m)) h_j^v(A)^{-1} \right]^{n_j}
\]
\[
= \prod_{j=1,2,3} \sum_{n_j \in \mathbb{Z}} e^{-\frac{\pi}{n} |n_j|^2} \left[ h_j^v(Z(m)) h_j^v(A)^{-1} \right]^{n_j}
\]
\[
= \prod_{j=1,2,3} \sum_{n_j \in \mathbb{Z}} e^{-\frac{\pi}{n} |n_j|^2} \left[ e^{p_j^v(m)} e^{i \theta_j^v(m)} e^{-i \theta_j^v(A)} \right]^{n_j}
\]
\[
= \prod_{j=1,2,3} \sum_{n_j \in \mathbb{Z}} e^{-\frac{\pi}{n} |n_j|^2 + p_j^v(m)} \left[ e^{i \theta_j^v(m)} e^{-i \theta_j^v(A)} \right]^{n_j}.
\] (C.25)

Note that the coherent state as defined in (C.25) is not yet normalized. We can perform the normalization by employing the Poisson resummation formula (appendix C.1) in order to obtain its norm. The result is
\[
\|\Psi_{m,v}(A)\|^2 = \prod_{j=1,2,3} \|\Psi_{j,v}\|^2
\]
\[
= \prod_{j=1,2,3} 2\pi \sqrt{\frac{\pi}{\tau(e)}} e^{\frac{\pi}{\tau(e)} (p_j^v(m))^2} \sum_{N_j \in \mathbb{Z}} e^{-\frac{\pi}{\tau(e)} (N_j^2 + 2i N_j p_j^v(m))}. (C.26)
\]

Here the $N_j^v$ are new summation variables\(^{21}\), and $K_{t(e)} = O(t(e)^\infty)$ denotes a function of order $t(e)^\infty$ symbolizing that $\lim_{t(e) \to 0} K_{t(e)} t(e)^b = 0 \forall b < \infty$ such that for small $t(e)$ this quantity can be neglected because then $K_{t(e)} \ll 1$.

C.4. The volume operator $\hat{V}$ acting on $U(1)^3$-charge networks

C.4.1. Set-up
Following [20] we consider polynomials of the operator
\[
\hat{q}_j^v(v, r) = \text{Tr} \left[ \tau_j \hat{h}_j^v(\hat{h}_e)^{-1}, (\hat{V}_r)^v \right] \quad \text{for} \; SU(2)
\] (C.27)
\[
\hat{q}_j^v(v, r) = \hat{h}_j^v(\hat{h}_e)^{-1}, (\hat{V}_r)^v \quad \text{for} \; U(1)^3 \quad r \in \mathbb{Q}.
\] (C.28)

Now the action of the $(U(1)^3$ version (C.28) of $\hat{q}_j^v(v, r)$ on a charge network state
\[
T_v = \prod_{j=1,2,3} \left[ h_j^v(A) \right]^{n_j(v)}
\] (C.29)

at the vertex $v \in V(\gamma)$ is
\[
\hat{q}_j^v(v, r) T_v = \left( \hat{V}_r^v - \hat{h}_j^v \hat{V}_r^v \hat{h}_j^v \right) T_v
\]
\[
= \left\{ \lambda^v \left( |n_j^v| \right) - \lambda^v \left( |n_j^v| - \delta^{j h_j^v} \delta_{1 h_j^v} \right) \right\} T_v.
\] (C.30)

\(^{21}\) Especially, they have nothing to do with the charge labels $n_j^v$ as can be seen from appendix (C.1).
C.4.2. Upper bound for the eigenvalues

With respect to $U(1)^3$ charge networks $T_c$, the volume operator $\hat{V}(R)$ according to the classical expression of the volume of a spatial region $R$, 

$$\text{Vol}(R) = \int_R d^3x \sqrt{\det(E)},$$

is already diagonal and its action on charge network states $T_c$ is given by

$$\hat{V}(R)T_c = \sum_{v \in V(R)} \hat{V}_v T_c$$

$$= \sum_{v \in V(R)} (\lambda_P)^3 \left[ Z \cdot \sum_{\epsilon_I \epsilon_J \epsilon_K} \epsilon_{ijk} \epsilon(I,J,K) n_I^I n_J^J n_K^K \right] T_c$$

$$= \sum_{v \in V(R)} \lambda(T_c) T_c.$$  \hspace{1cm} (C.32)

Here the sum runs over all triples of edges at the vertex $v$, and $\epsilon(I,J,K) = \text{sgn}(\epsilon_{abc} e_I^a(v) e_J^b(v) e_K^c(v))$ gives the sign of the determinant of the tangents $\hat{e}_I(v)$ of the edges $e_I$ evaluated at the vertex $v$ and zero if the tangents are linearly dependent, which is the case in particular if at least two of the labels $I, J, K$ are equal or if one edge in the triple is the analytic continuation of another. Furthermore, $\lambda_P$ is the Planck length, and $Z$ a constant prefactor dependent on the regularization of the volume operator and the Immirzi parameter (see [11, 12] for details). Note that due to the construction of the coherent states (C.25) we will evaluate $\hat{V}$ on the conjugated charge network states

$$\tilde{T}_c = \prod_{e_I \in E(v)} \left[ (h_I^I(A))^{-1} \right]^{n_e^I} = \prod_{e_I \in E(v)} [h_I^I(A)]^{-n_e^I}$$

according to

$$\hat{q}_{c_o}^{e_I}(v, r) \tilde{T}_c = (\hat{V}_v - \hat{h}_o^{e_I} \hat{h}_o^{e_I}) \tilde{T}_c$$

$$= \left\{ \lambda' \left( \{ n_l^l \} \right) - \lambda' \left( \{ n_l^l + \delta^{ij} \delta_{1h} \} \right) \right\} \tilde{T}_c$$

$$= \lambda' \left( \{ n_e^h \} \right) \tilde{T}_c.$$  \hspace{1cm} (C.34)

C.4.2. Upper bound for the eigenvalues $\lambda' \left( \{ n_e^h \} \right)$ of $\hat{q}_{c_o}^{e_I}(v, r)$. We will derive an upper bound for the modulus of the eigenvalues $| \lambda' \left( \{ n_e^h \} \right) | = | \lambda' \left( \{ n_l^l \} \right) - \lambda' \left( \{ n_l^l + \delta^{ij} \delta_{1h} \} \right) |$ of the operators $\hat{q}_{c_o}^{e_I}(v, r)$ as evaluated on a (conjugated) charge network on a single vertex $v$ with an arbitrary number $M$ of outgoing edges $e \in E(v)$, $E(v)$ being the set of edges of $v$. Here $\lambda(T_c) = \lambda_P Z \cdot \sum_{I,J,K} \epsilon_{ijk} \epsilon(I,J,K) n_I^I n_J^J n_K^K$ are the eigenvalues of the volume operator as defined in (C.32), (C.34) where now the edge labels $I, J, K = 1, 2, \ldots, M$ and again $i, j, k = 1, 2, 3$ for the three copies of $U(1)$. Recall that the charges $n_I^I \in \mathbb{Z}$ are integer numbers.

Let us first consider the eigenvalues of the volume operator:

$$\lambda' \left( \{ n_l^l \} \right) = (\lambda_P)^3 |Z|^{\frac{3}{2}} \sum_{I,J,K} \epsilon(I,J,K) \epsilon_{ijk} n_I^I n_J^J n_K^K$$

$$=: (\lambda_P)^3 |Z|^{\frac{3}{2}} |\rho|^3$$

$$=: (\lambda_P)^3 |Z|^{\frac{3}{2}} a^3.$$  \hspace{1cm} (C.35)

22 In [12] $Z$ is found to be $Z = \frac{1}{3} \left( \frac{1}{3} \right)^3$. 


\[
\lambda'(\{n^i_j + \delta^i_j \delta_{jk}\}) = (\epsilon_\rho)^\nu |\mathcal{Z}|^{\frac{\nu}{2}} \left( \sum_{I,J,K} \epsilon(I,J,K)\epsilon_{ijk}n^i_j n^j_k + \sum_{I,J} \epsilon(I,J)\epsilon_{ijk}n^i_j n^j_k \right)
+ \sum_{I,J,K} \epsilon(I,J,K)\epsilon_{ijk}n^i_j n^j_k + \sum_{I,J} \epsilon(I,J)\epsilon_{ijk}n^i_j n^j_k \right|^{\frac{\nu}{2}}
\]
\[
= (\epsilon_\rho)^\nu |\mathcal{Z}|^{\frac{\nu}{2}} \left( \sum_{I,J,K} \epsilon(I,J,K)\epsilon_{ijk}n^i_j n^j_k + 3 \cdot \sum_{I,J,K} \epsilon(I,J,K)\epsilon_{ijk}n^i_j n^j_k \right)
= (\epsilon_\rho)^\nu |\mathcal{Z}|^{\frac{\nu}{2}} |\rho + \sigma|^\frac{\nu}{2}
= (\epsilon_\rho)^\nu |\mathcal{Z}|^{\frac{\nu}{2}} b^\frac{\nu}{2}. \quad (C.35)
\]

Here in the definition of \(b\) we have explicitly decomposed the contribution from the modification \(n^i_j \rightarrow n^i_{j0} + 1\) of the charge label \(n^i_j\) caused by the action of holonomies in the \(\hat{q}_{i0}(v,\ell)\) operator. Due to the (double) antisymmetry of the \(\epsilon(I,J,K)\epsilon_{ijk}\) prefactors in the sum terms and the fact that the remaining summation variables always run over \(J, K = 1, \ldots, M\) we can factor out the multiplicity factor 3.

We can therefore write
\[
\lambda'(\{n^i_{j0}\}) = \lambda'(\{n^i_j\}) - \lambda'(\{n^i_j + \delta^i_j \delta_{jk}\}) = (\epsilon_\rho)^\nu |\mathcal{Z}|^{\frac{\nu}{2}} (a^2 - b^2)
= (\epsilon_\rho)^\nu |\mathcal{Z}|^{\frac{\nu}{2}} (|\rho|^\frac{\nu}{2} - |\rho + \sigma|^\frac{\nu}{2}). \quad (C.36)
\]

Now let \(\frac{\nu}{2} = \frac{\beta}{L} < 1\) be a rational number with \(K < L\) and \(K, L \in \mathbb{N}\). By invoking the (generalized) binomial theorem \(\sum_{k=0}^{N} x^{k} = \frac{x^{N+1} - 1}{x - 1}\) \((x \neq 1 \in \mathbb{C}, N \in \mathbb{N})\) we may write
\[
a^\frac{\beta}{L} - b^\frac{\beta}{L} = \left[ a^\frac{\beta}{L} \right]^{\frac{L}{\beta}} - \left[ b^\frac{\beta}{L} \right]^{\frac{L}{\beta}}
= \left( a^\frac{\beta}{L} - b^\frac{\beta}{L} \right) \sum_{k=0}^{L-1} a^\frac{k}{L} b^\frac{L-k}{L}
= \left( a^\frac{\beta}{L} - b^\frac{\beta}{L} \right) \sum_{l=0}^{L-1} a^\frac{l}{L} b^\frac{L-l}{L}
= (a - b) b^\frac{L-1}{L} \sum_{k=0}^{L-1} \left( a^\frac{1}{L} \right)^k \sum_{l=0}^{L-1} \left( b^\frac{1}{L} \right)^l.
\]

With \(\beta := \left( \frac{\nu}{2} \right)^L, K < L\), this results in
\[
a^\frac{\beta}{L} - b^\frac{\beta}{L} = (a - b) \frac{1 - \beta^L}{b^{1-\frac{1}{L}} - 1 - \beta^L}. \quad (C.37)
\]

In order to give an upper bound for the modulus of (C.37) first note that \(|x| - |y| \leq |x - y|\) \(\forall x, y \in \mathbb{C}\). Secondly, \(\frac{1-\beta^L}{b^{1-\frac{1}{L}} - 1 - \beta^L} \leq 1\) \(\forall \beta \in \mathbb{R}\) if \(K < L\). Thirdly, \(\frac{1}{b^{1-\frac{1}{L}}} \leq 1\) \(\forall b \geq 1\).

Now let us discuss the special cases:
\[
\begin{align*}
\beta &= 1 \quad \text{but then } a = b \quad \text{and thus } a^\frac{\beta}{L} - b^\frac{\beta}{L} = 0 \\
b &< 1 \quad \text{By definition, } a, b \geq 0 \text{ may only vary in integer steps, since the edge charges } \quad \text{n}^i_j \in \mathbb{Z} \text{ are integer numbers, therefore } [\text{the only possible value } b < 1 \text{ is } b = 0]. \quad \text{But if } \\
b = 0 \quad \text{we have } \rho = -\sigma \text{ in (C.35)} \text{ and therefore } a = |\rho| \geq 0. \quad \text{Then either } a = 0 \quad \text{and thus } a^\frac{\beta}{L} - b^\frac{\beta}{L} = 0 \text{ or } a \geq 1 \text{ and thus } a^\frac{\beta}{L} - b^\frac{\beta}{L} = a^\frac{\beta}{L} \leq a = |\rho| = |\sigma|. \quad \text{If } 
\end{align*}
\]
So we can give the general upper bound (note that we sum over the indices $j,k$, and $M$ is the number of edges at the vertex):

$$
|a^T - b^T| \leq (a - b) = \|\rho\| - \|\rho + \sigma\|
$$

$$
\leq |\rho - \rho - \sigma| = |\sigma| = 3 \cdot \sum_{J,K} e(J, K) e_{i,j_k} n_j^k n_k^l
$$

$$
\leq 3 \cdot \sum_{j,k} \sum_{j=1,2,3} |n_j^k| n_k^l
$$

$$
\leq 3 \cdot \sum_{j,k=1,2,3} \left(\sum_{j=1,2,3} |n_j^k|^2 + \sum_{k=1,2,3} |n_k^l|^2\right)
$$

$$
= \frac{3}{2} \cdot 3M \cdot \left(\sum_{j=1,2,3} |n_j^k|^2 + \sum_{j=1,2,3} |n_j^l|^2\right)
$$

$$
= 9M \cdot \sum_{j=1,2,3} |n_j^k|^2.
$$

The final result for an upper bound of the modulus of the eigenvalue $\lambda_r\left\{\{n_j^k\}\right\}$ in (C.36) then reads

$$
|\lambda_r\left\{\{n_j^k\}\right\}| = (\ell_p)^3 |Z|^2 \cdot |a^T - b^T|
$$

$$
\leq (\ell_p)^3 |Z|^2 9M \sum_{j=1,2,3} |n_j^k|^2.
$$

(C.38)

C.5. Explicit construction of a 'pathological' edge configuration at a vertex $v$

Here we will give an explicit construction of a configuration of $M$ edges outgoing from a vertex $v$ where each ordered edge triple $e_I, e_J, e_K, I < J < K$, contributes with a negative sign factor $e(I, J, K)$ of its tangents.

Since we are only interested in the sign factor we can make simplifying assumptions; especially, we may choose certain numerical values.

Consider the vertex $v$ as the origin of a three-dimensional coordinate system with axes $x, y, z$. Now consider a circle with radius $r = 1$ centred at $y = 1$, parallel to the $x$–$z$-plane. Let every edge tangent $\dot{e}_k$ end on a point on the circle with coordinates $(\cos \phi_K, 1, \sin \phi_K)$ and $\phi_K = 2\pi \frac{K}{M}$. Now one may check that for each ordered triple $e_I, e_J, e_K$ with $I < J < K \leq M$ we have

$$
\det(\dot{e}_I, \dot{e}_J, \dot{e}_K) = -4 \cdot \sin \left[ \pi \frac{K - I}{M} \right] \sin \left[ \pi \frac{K - J}{M} \right] \sin \left[ \pi \frac{J - I}{M} \right].
$$

(C.40)

Since for all arguments $x$ of the sin-functions we have $0 < x < \pi$, all of these functions are $\geq 0$ and therefore we get $e(I, J, K) = \text{sgn}(\det(\dot{e}_I, \dot{e}_J, \dot{e}_K)) = -1$ for all ordered edge triples $e_I, e_J, e_K$ with $I < J < K \leq M$ we have

As one can see, this edge configuration is quite special—it is like all edges lying in one octant only (if we rotate the coordinate system). Such an edge configuration would appear
rather one than three dimensional. This shows that the sign factor cannot be used in general to achieve an $M$-independent bound of the expectation values (5.10).

C.6. Theorems needed

C.6.1. Spherical coordinates. For integrals of the form

$$I_k := \frac{2^{-m}}{\pi} \int_{\mathbb{R}^m} d^m x \, e^{-2\|x\|^2} \|x\|^k$$

with $k, m \in \mathbb{N}$ (C.41)

there is a recursion relation

$$I_k = \frac{m + 2(k - 1)}{4} I_{k-1} \quad \text{with} \quad I_0 = 1$$

(C.42)

such that we can write for $k \geq 1$, $I_0 = 1$

$$I_k = \prod_{l=1}^{k} \frac{m + 2(l - 1)}{4}$$

(C.43)

or explicitly

$$m \text{ even} \quad I_k = \frac{\left(\frac{m}{2} + k - 1\right)!}{2^k \left(\frac{m}{2}\right)!}$$

(C.44)

$$m \text{ odd} \quad I_k = \frac{(m - 1 + 2k)!(\frac{m-1}{2})!}{8^k (m - 1)!(\frac{m-1}{2} + k)!}$$

(C.45)

C.6.2. Poisson resummation formula.

Theorem C.1 (Poisson summation formula). Let $f$ be an $L_1(\mathbb{R}, dx)$ function such that the series

$$\phi(y) = \sum_{n=-\infty}^{\infty} f(y + ns)$$

Figure C3. ‘Pathological’ edge configuration.
is absolutely and uniformly convergent for \( y \in [0, s] \), \( s > 0 \). Then

\[
\sum_{n=\infty}^{\infty} f(ns) = \frac{2\pi}{s} \sum_{N=\infty}^{\infty} \hat{f}\left(\frac{2\pi N}{s}\right)
\]

(\text{C.46})

where \( \hat{f}\left(\frac{2\pi N}{s}\right) := \int_{\mathbb{R}} f(x)e^{-\frac{2\pi i N x}{s}} \, dx \) is the Fourier transform of \( f \) and \( x = s \cdot k \).

The proof of this theorem can be found in any textbook on Fourier series; see, for example, the classical book by Bochner [21].

The importance of this remarkable theorem for our purposes is that it converts a slowly converging series \( \sum f(ns) \) as \( s \to 0 \) into a possibly rapidly converging series \( \frac{1}{s} \sum N \hat{f}(2\pi N/s) \), of which in our case almost only the term with \( N = 0 \) will be relevant.

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