Wakimoto realization of
the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$

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Abstract

A bosonization of the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$ is presented for an arbitrary level $k \in \mathbb{C}$. The Wakimoto realization is given by using $\xi - \eta$ system. The screening operators that commute with $U_q(\widehat{sl}(M|N))$ are presented for the level $k \neq -M + N$. New bosonization of the affine superalgebra $\widehat{sl}(M|N)$ is obtained in the limit $q \to 1$.

1 Introduction

Bosonization is a powerful method to study representation theory and its application to mathematical physics [1]. Wakimoto realization is the bosonization that provides a bridge between representation theory of affine algebras and the geometry of the semi-infinite flag manifold. The Wakimoto realizations have been constructed for the affine Lie algebra $g = (ADE)^{(r)} (r = 1, 2)$, $(BCFG)^{(1)}$ and $\widehat{sl}(M|N)$, $osp(2|2)^{(2)}$, $D(2, 1, a)^{(1)}$ [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. They have been used to construct correlation functions of WZW models, in the study of Drinfeld-Sokolov reduction and $W$-algebras. It’s nontrivial to give quantum deformation of Wakimoto realization as the same as quantum Drinfeld-Sokolov reduction and quantum $W$-algebras. The quantum Wakimoto realizations have been constructed only for $U_q(\widehat{sl}(N))$ and $U_q(\widehat{sl}(2|1))$ [13, 14]. In this paper we study a higher-rank generalization of the previous works for the quantum affine superalgebra $U_q(\widehat{sl}(2|1))$. We give a bosonization of the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$ for an arbitrary level $k \in \mathbb{C}$, and give the Wakimoto realization using $\xi - \eta$ system. We give the screening operators that commute with $U_q(\widehat{sl}(M|N))$ for the level $k \neq -M + N$. Taking the limit $q \to 1$, we obtain new bosonization of the affine superalgebra $\widehat{sl}(M|N)$. This paper is a shorter review of the papers [18, 19, 20, 21].

2 Quantum affine superalgebra $U_q(\widehat{sl}(M|N))$

In this Section we recall the definition of the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$ for $M, N = 1, 2, 3, \cdots$. Throughout this paper, $q \in \mathbb{C}$ is assumed to be $0 < |q| < 1$. For any integer $n$, define
\[ [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \]

We set \( \nu_i = +1 \) (\( 1 \leq i \leq M \)), \( \nu_i = -1 \) (\( M + 1 \leq i \leq M + N \)) and \( \nu_0 = -1 \). The Cartan matrix \( (A_{i,j})_{0 \leq i,j \leq M+N-1} \) of the affine Lie superalgebra \( \tilde{\mathfrak{sl}}(M|N) \) is given by

\[
A_{i,j} = (\nu_i + \nu_{i+1})\delta_{i,j} - \nu_i\delta_{i,j+1} - \nu_{i+1}\delta_{i+1,j}.
\]

The quantum affine superalgebra \( U_q(\tilde{\mathfrak{sl}}(M|N)) \) is the associative algebra over \( \mathbb{C} \) with the generators \( X^\pm,i \) (\( i = 1, 2, \cdots, M + N - 1, m \in \mathbb{Z} \)), \( H_i^n \) (\( i = 1, 2, \cdots, M + N - 1, n \in \mathbb{Z}_{\geq 0} \)), \( H^i \) (\( i = 1, 2, \cdots, M + N - 1 \)), and \( c \). The \( \mathbb{Z}_2 \)-grading of the generators is given by \( p(X^\pm,m) \equiv 1 \pmod{2} \) for \( m \in \mathbb{Z} \) and zero otherwise. The defining relations of the generators are given as follows.

\[
[\text{central element}], \quad [H^i, H^j]_m = 0, \quad [H^i_m, H^j_n] = \left[ A_{i,j}m \right]_q [cm]_q \delta_{m+n,0},
\]

\[
[H^i, X^\pm,j] = \pm A_{i,j} X^\pm,j(z),
\]

\[
[H^i_m, X^\pm,j(z)] = \pm \left[ A_{i,j}m \right]_q [z^m] \delta_{m,0} X^\pm,j(z),
\]

\[
(z_1 - q^{\pm A_{i,j}} z_2) X^\pm,i(z_1) X^\pm,j(z_2) = (q^{\pm A_{i,j}} - 1) X^\pm,j(z_2) X^\pm,i(z_1), \quad \text{for } |A_{i,j}| \neq 0,
\]

\[
[X^\pm,i(z_1), X^\pm,j(z_2)] = 0 \quad \text{for } |A_{i,j}| = 0,
\]

\[
[X^\pm,i(z_1), X^{-,j}(z_2)] = 0 \quad \text{for } |A_{i,j}| = 1, \quad i \neq M,
\]

\[
[X^\pm,i(z_1), [X^\pm,j(z_2), X^\pm,k(z_3)]_q] = 0 \quad \text{for } |A_{i,j}| = 1, \quad i \neq M,
\]

where we use

\[
[X, Y]_a = XY - (-1)^{p(X)p(Y)} a YX,
\]

for homogeneous elements \( X, Y \in U_q(\tilde{\mathfrak{sl}}(M|N)) \). For simplicity we write \( [X, Y] = [X, Y]_1 \). Here we set \( \delta(z) = \sum_{m \in \mathbb{Z}} z^m \) and the generating functions

\[
X^\pm,j(z) = \sum_{m \in \mathbb{Z}} X^\pm,j_m z^{-m-1},
\]

\[
\Psi_i^q(q^\pm \tilde{h}_i z) = q^{\pm h_i} \exp \left( \pm (q - q^{-1}) \sum_{m > 0} H^i_m \frac{z^m}{z^m} \right).
\]

The multiplication rule for the tensor product is \( \mathbb{Z}_2 \)-graded and is defined for homogeneous elements \( X_1, X_2, Y_1, Y_2 \in U_q(\tilde{\mathfrak{sl}}(M|N)) \) by

\[
(X_1 \otimes Y_1)(X_2 \otimes Y_2) = (-1)^{(p(Y_1)p(X_2))(p(X_1)p(Y_2))} (X_1 X_2 \otimes Y_1 Y_2),
\]

which extends to inhomogeneous elements through linearity.

Let \( \tilde{\alpha}_i, \tilde{\Lambda}_i \) (\( 1 \leq i \leq M + N - 1 \)) be the classical simple roots, the classical fundamental weights, respectively. Let \( \langle \cdot | \cdot \rangle \) be the symmetric bilinear form satisfying \( \langle \tilde{\alpha}_i | \tilde{\alpha}_j \rangle = A_{i,j} \) and \( \langle \tilde{\Lambda}_i | \tilde{\alpha}_j \rangle = \delta_{i,j} \) for \( 1 \leq i, j \leq M + N - 1 \). Let us introduce the affine weight \( \Lambda_0 \) and the null root \( \delta \) satisfying \( \langle \Lambda_0 | \Lambda_0 \rangle = \langle \delta | \delta \rangle = 0 \), \( \langle \Lambda_0 | \delta \rangle = 1 \), and \( \langle \Lambda_0 | \tilde{\alpha}_i \rangle = (\Lambda_0 | \tilde{\Lambda}_i) = 0 \) for \( 1 \leq i \leq M + N - 1 \). The other affine weights and the affine roots are given by \( \Lambda_i = \Lambda_i + \Lambda_0, \alpha_i = \tilde{\alpha}_i \) for \( 1 \leq i \leq M + N - 1 \), and \( \alpha_0 = \delta - \sum_{i=1}^{M+N-1} \alpha_i \). Let \( V(\lambda) \) be the highest-weight module over \( U_q(\tilde{\mathfrak{sl}}(M|N)) \) generated by the highest weight vector \( |\lambda\rangle \neq 0 \) such that

\[
H^i_m |\lambda\rangle = X^\pm,i_m |\lambda\rangle = 0 \quad (m > 0),
\]

\[
X^\pm,i_0 |\lambda\rangle = 0, \quad H^i |\lambda\rangle = l_i |\lambda\rangle,
\]

2
where the classical part of the highest weight is \( \lambda = \sum_{i=1}^{M+N-1} l_i \Lambda_i \).

## 3 Bosonization of \( U_q(\widehat{\mathfrak{sl}}(M|N)) \)

In this Section we give a bosonization of \( U_q(\widehat{\mathfrak{sl}}(M|N)) \) for an arbitrary level \( k \in \mathbb{C} \).

### 3.1 Boson

We introduce bosons \( a^i_m \) \((m \in \mathbb{Z}, 1 \leq i \leq M + N - 1)\), \( b^{i,j}_m \) \((m \in \mathbb{Z}, 1 \leq i < j \leq M + N)\), \( c^{i,j}_m \) \((m \in \mathbb{Z}, 1 \leq i < j \leq M + N)\), and zero mode operators \( Q^{i,j}_b \) \((1 \leq i \leq M + N - 1)\), \( Q^{i,j}_c \) \((1 \leq i < j \leq M + N)\). Their commutation relations are defined as follows.

\[
[a^i_m, a^j_n] = \frac{1}{m} [(k + g)m]_q [A^i_m]_q \delta_{m+n,0}, \quad [a^i_b, Q^{i,j}_b] = (k + g) A^i_m,
\]
\[
[b^{i,j}_m, b^{i',j'}_n] = -\nu_i \nu_j \frac{1}{m} [m]_q^2 \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, \quad [b^{i,j}_b, Q^{i',j'}_b] = -\nu_i \nu_j \delta_{i,i'} \delta_{j,j'},
\]
\[
[c^{i,j}_m, c^{i',j'}_n] = \nu_i \nu_j \frac{1}{m} [m]_q^2 \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, \quad [c^{i,j}_c, Q^{i',j'}_c] = \nu_i \nu_j \delta_{i,i'} \delta_{j,j'},
\]
\[
[Q^{i,j}_b, Q^{i',j'}_c] = \pi \sqrt{-1} \quad (\nu_i \nu_j = \nu_i \nu_{j'} = -1).
\]

The remaining commutators vanish. Here \( g = M - N \) stands for the dual Coxeter number. We define free boson fields \( b^{i,j}_\pm(z) \), \( b^{i,j}(z) \) as follows.

\[
b^{i,j}_\pm(z) = \pm (q - q^{-1}) \sum_{m > 0} b^{i,j}_{\pm m} z^m \pm b^{i,j}_0 \log q,
\]
\[
b^{i,j}(z) = - \sum_{m \neq 0} \frac{b^{i,j}_m}{[m]_q} z^{-m} + Q^{i,j}_b + b^{i,j}_0 \log z.
\]

Free boson fields \( a^{i}_\pm(z) \), \( c^{i,j}(z) \) are defined in the same way. We define free boson fields \( (\Delta_L^\varepsilon b^{i,j}_\pm(z)) \), \( (\Delta_R^\varepsilon b^{i,j}_\pm(z)) \) \((\varepsilon = \pm, 0)\) as follows.

\[
(\Delta_L^\varepsilon b^{i,j}_\pm(z)) = \begin{cases}
\frac{b^{i,j+1}_\pm(z)}{b^{i,j}_\pm(z)} & (\varepsilon = \pm),
\frac{b^{i,j}_\pm(z)}{b^{i,j+1}_\pm(z)} & (\varepsilon = 0),
\end{cases}
\]
\[
(\Delta_R^\varepsilon b^{i,j}_\pm(z)) = \begin{cases}
\frac{b^{i,j+1}_\pm(z)}{b^{i,j}_\pm(z)} & (\varepsilon = \pm),
\frac{b^{i,j}_\pm(z)}{b^{i,j+1}_\pm(z)} & (\varepsilon = 0),
\end{cases}
\]

We define free boson fields with parameters \( L_1, \ldots, L_r, M_1, \ldots, M_r, \alpha \) as follows.

\[
\left( \frac{L_1 \cdots L_r}{M_1 \cdots M_r} a^i \right) (z; \alpha) = - \sum_{m \neq 0} \frac{[L_1 m]_q \cdots [L_r m]_q}{[M_1 m]_q \cdots [M_r m]_q} a^i_m [m]_q^{-a|m|} z^{-m} + \frac{L_1 L_2 \cdots L_r}{M_1 M_2 \cdots M_r} (Q^i_b + a^i_0 \log z).
\]

Normal ordering rules are defined as follows.

\[
: b^{i,j}_m b^{i',j'}_n : = : b^{i',j'}_n b^{i,j}_m : = \begin{cases}
b^{i,j}_m b^{i',j'}_n & (m < 0),
b^{i',j'}_n b^{i,j}_m & (m > 0),
\end{cases}
\]
\[
: Q^{i,j}_b Q^{i',j'}_b : = : Q^{i',j'}_b Q^{i,j}_b : = Q^{i,j}_b Q^{i',j'}_b \quad (i > i' \text{ or } i = i', j > j').
\]
Normal ordering rules of $a^i_m$, $c^i_m$ and $Q^{i,j}_m$ are defined in the same way.

### 3.2 Bosonization

We define bosonic operators $\Psi^\pm_i(z) (1 \leq i \leq M + N - 1)$ as follows.

$$
\Psi^\pm_i (q^{\pm/2} z) = e^{\alpha_i (q^2 / 2 + z)} - \sum_{l=1}^{M} (\Delta^\pm_{L} b^i_{-l}(q^{\pm(1-l)/2} z)) - \sum_{l=1}^{M} (\Delta^\pm_{L} b^i_{l}(q^{\pm(1-l)/2} z))
\times e^{-\sum_{l=M+1}^{M+N} (\Delta^\pm_{L} b^i_{l}(q^{\pm(1-l)/2} z))} : (1 \leq i \leq M - 1),
$$

(3.1)

$$
\Psi^M_\pm (q^{\pm/2} z) = e^{\alpha_i (q^2 / 2 + z)} - \sum_{l=1}^{M-1} (\Delta^\pm_{M} b^i_{-l}(q^{\pm(1-l)/2} z)) + \sum_{l=M+2}^{M+N} (\Delta^\pm_{M} b^i_{l}(q^{\pm(1-l)/2} z))
\times e^{-\sum_{l=M+1}^{M+N} (\Delta^\pm_{M} b^i_{l}(q^{\pm(1-l)/2} z))} : (M + 1 \leq i \leq M + N - 1).
$$

(3.2)

We define bosonic operators $X^\pm,i(z) (1 \leq i \leq M + N - 1)$ as follows.

$$
X^{+,i}(z) = \sum_{j=1}^{i} \frac{c_{i,j}}{z} (E^+_i(z) - E^-_i(z)) (1 \leq i \leq M - 1),
$$

(3.4)

$$
X^{+,M}(z) = \sum_{j=1}^{M} c_{M,j} E_{M,j}(z),
$$

(3.5)

$$
X^{-,i}(z) = \sum_{j=1}^{i} \frac{d_{i,j}}{(q - q^{-1}) z} (F^-_i(z) - F^+_i(z)) + \sum_{j=M+1}^{i} \frac{c_{i,j}}{(q - q^{-1}) z} (E^+_i(z) - E^-_i(z))
\times (1 \leq i \leq M + N - 1),
$$

(3.6)

$$
X^{-,M}(z) = \sum_{j=1}^{M-1} \frac{d_{M,j}}{(q - q^{-1}) z} (F^-_{M,j}(z) - F^+_{M,j}(z)) + \sum_{j=M+1}^{M+N} \frac{c_{M,j}}{(q - q^{-1}) z} E^+_M(z) + \sum_{j=M+1}^{M+N} \frac{d_{M,j}}{(q - q^{-1}) z} E^-_M(z),
$$

(3.7)

$$
X^{-,i}(z) = \sum_{j=1}^{i} \frac{d_{i,j}}{(q - q^{-1}) z} (F^-_i(z) - F^+_i(z)) + \sum_{j=M+1}^{i} \frac{c_{i,j}}{(q - q^{-1}) z} (E^+_i(z) - E^-_i(z))
\times (1 \leq i \leq M + N - 1),
$$

(3.8)

We set $E^\pm_{i,j}(z)$ as follows.

$$
E^\pm_{i,j}(z) = e^{(b+c)^{i-1} (q^{1-1} z) + b^{i-1} (q^{1-1} z) - (b+c)^{i-1} (q^{1-1} z) + \sum_{l=1}^{i-1} (\Delta^\pm_{L} b^i_{l})(q^{l} z)}.
$$
We set $E_{t,i}$ as follows.

\[ E_{M,M}(z) = : e^{b_{M}M(q^{j-1}z)-(b+c)j(q^{j-1}z) - b^{i}j(q^{j-1}z)-(b+c)j(q^{j-1}z)} :\]

\[ E_{t,i}(z) = : e^{b_{i}j(q^{j-1}z)-(b+c)j(q^{j-1}z) - b^{i}j(q^{j-1}z)-(b+c)j(q^{j-1}z)} :\]

\[ E_{t,j}(z) = : e^{b_{i}j(q^{j-1}z)-(b+c)j(q^{j-1}z) + b^{i}j(q^{j-1}z) - (b+c)j(q^{j-1}z)} :\]

\[ (M+1 \leq i \leq M + N - 1, 1 \leq j \leq M).\]

We set $F_{i,j}(z)$, $F_{i,j}(z)$ as follows.

\[ F_{1,1}(z) = : e^{a_{i}j(q^{i+j}z)-(b+c)j(q^{i+j}z) - b^{i}j(q^{i+j}z)-(b+c)j(q^{i+j}z)} :\]

\[ F_{1,1}(z) = : e^{a_{i}j(q^{i+j}z)-(b+c)j(q^{i+j}z) - b^{i}j(q^{i+j}z)-(b+c)j(q^{i+j}z)} :\]

\[ (1 \leq j \leq M - 1),\]

\[ (M+1 \leq i \leq M + N - 1, 1 \leq j \leq M).\]
In this Section we introduce the Wakimoto realization.

The coefficients $c_{i,j} \in \mathbb{C}$ and $d_{i,j}^1, d_{i,j}^2, d_{i,j}^3 \in \mathbb{C}$ satisfy the following conditions.

$$d_{i,j}^1 = \nu_{i+1} \frac{1}{c_{i,j}} \times \begin{cases} 1 & (1 \leq i \leq M - 1, 1 \leq j \leq i - 1), \\ q^{i-1} & (i = M, 1 \leq j \leq M - 1), \\ q^{-k-1} & (M + 1 \leq i \leq M + N - 1, 1 \leq j \leq M), \\ 1 & (M + 1 \leq i \leq M + N - 1, M + 1 \leq j \leq i - 1), \end{cases}$$

$$d_{i,i}^2 = \nu_{i+1} \frac{1}{c_{i,i}} \times \begin{cases} 1 & (1 \leq i \neq M \leq M + N - 1), \\ q^{M-1} & (i = M), \end{cases}$$

$$d_{i,j}^3 = \nu_{i+1} \frac{1}{c_{i,j}} \prod_{l=1}^{j-i-1} \frac{c_{i+l,i+1}}{c_{i+l,i}} \times \begin{cases} 1 & (1 \leq i \leq M - 1, i + 2 \leq j \leq M), \\ q^{k+3M+1-2j} & (1 \leq i \leq M - 1, M + 1 \leq j \leq M + N), \\ q^{(M-1)(j-M)} & (i = M, M + 2 \leq j \leq M + N), \\ 1 & (M + 1 \leq i \leq M + N - 1, i + 2 \leq j \leq M + N). \end{cases}$$

**Theorem 3.1** The bosonic operators $\Psi_{\pm}(z)$ defined in (3.1)-(3.3), and $X_{\pm}^i(z)$ defined in (3.2)-(3.3) and (3.4)-(3.7) satisfy the defining relations of the quantum affine superalgebra $U_q(\mathfrak{sl}(M|N))$ with the central element $c = k \in \mathbb{C}$.

### 3.3 Wakimoto realization

In this Section we introduce the $\xi - \eta$ system and give the Wakimoto realization. We set the boson Fock space $F(p_a, p_b, p_c)$ as follows. The vacuum state $|0\rangle \neq 0$ is defined by $a_m^i |0\rangle = b_m^i |0\rangle = c_m^i |0\rangle = 0 \ (m \geq 0)$. Let $|p_a, p_b, p_c\rangle$ be

$$|p_a, p_b, p_c\rangle = \exp \left( \sum_{i,j=1}^{M+N-1} \frac{(A^{-1})_{i,j}}{k+g} p_{a_i}^i Q_a^i - \sum_{1 \leq i < j \leq M+N} \nu_i \nu_j p_{b_{ij}}^i Q_{b_{ij}}^i + \sum_{1 \leq i < j \leq M+N} \nu_i \nu_j p_{c_{ij}}^i Q_{c_{ij}}^i \right) |0\rangle,$$

then $|p_a, p_b, p_c\rangle$ is the highest weight state of the boson Fock space $F(p_a, p_b, p_c)$. The boson Fock space $F(p_a, p_b, p_c)$ is generated by the bosons $a_m^i, b_m^i, c_m^i$ on the highest weight state $|p_a, p_b, p_c\rangle$. We set the space $F(p_a)$ by

$$F(p_a) = \bigoplus_{p_{b_{ij}}^i, p_{c_{ij}}^i \in \mathbb{Z}, \nu_i \nu_j = \pm 1} F(p_a, p_b, p_c).$$
Here we impose the restriction \( p_{ij}^b = -p_{ji}^c \) (\( \nu_i \nu_j = + \)), because \( X^\pm_{m,i} \) change \( Q_{b}^{i,j} + Q_{c}^{i,j} \). \( F(p_a) \) is \( U_q(\widehat{sl}(M|N)) \)-module. Let \(|\lambda\rangle = |p_a,0,0\rangle\) where \( p_i^a = l_i \) (1 \( \leq \) \( i \leq M+N-1 \)). The generators \( H^i, H^m, X^\pm_{m,i} \) act on \(|\lambda\rangle\) as follows.

\[
H^i_m|\lambda\rangle = X^\pm_{m,i}|\lambda\rangle = 0 \quad (m > 0),
\]

\[
X^+_0|\lambda\rangle = 0, \quad H^i|\lambda\rangle = l_i|\lambda\rangle.
\]

We have the level-\( k \) highest weight module \( V(\lambda) \) of \( U_q(\widehat{sl}(M|N)) \).

\[
V(\lambda) \subset F(p_a).
\]

Here the classical part of the highest weight is \( \lambda = \sum_{i=1}^{M+N-1} l_i \lambda_i \).

We introduce the \( \xi - \eta \) system We set bosonic operators \( \xi_{i,j}^{\pm}, \eta_{i,j}^{\pm} \) (\( \nu_i \nu_j = +1,1 \leq i < j \leq M+N \)) as follows.

\[
\eta_{i,j}^{\pm}(z) = \sum_{m \in \mathbb{Z}} \eta_{m}^{\pm} z^{-m-1} =: e^{\pm \xi_{i,j}^{\pm}(z)}, \quad \xi_{i,j}^{\pm}(z) = \sum_{m \in \mathbb{Z}} \xi_{m}^{\pm} z^{-m} =: e^{\mp \xi_{i,j}^{\pm}(z)}:
\]

Fourier components

\[
\eta_{m}^{\pm} = \oint_{C} \frac{dz}{2\pi \sqrt{-1}} z^{m} \eta_{i,j}^{\pm}(z), \quad \xi_{m}^{\pm} = \oint_{C} \frac{dz}{2\pi \sqrt{-1}} z^{m-1} \xi_{i,j}^{\pm}(z)
\]

are well-defined on the module \( F(p_a) \). The \( \mathbb{Z}_2 \)-grading is given by \( p(\xi_{i,j}^{\pm}) = p(\eta_{i,j}^{\pm}) = +1 \). We have direct sum decomposition.

\[
F(p_a) = \eta_0^{i,j} \xi_0^{i,j} F(p_a) \oplus \xi_0^{i,j} \eta_0^{i,j} F(p_a),
\]

where \( \text{Ker}(\eta_0^{i,j}) = \eta_0^{i,j} \xi_0^{i,j} F(p_a), \text{Coker}(\eta_0^{i,j}) = \xi_0^{i,j} \eta_0^{i,j} F(p_a) \). We set

\[
\eta_0 = \prod_{1 \leq i < j \leq M+N \atop \nu_i \nu_j = +1} \eta_{i,j}^{\pm}, \quad \xi_0 = \prod_{1 \leq i < j \leq M+N \atop \nu_i \nu_j = +1} \xi_{i,j}^{\pm}.
\]

We introduce the subspace \( \mathcal{F}(p_a) \) by

\[
\mathcal{F}(p_a) = \eta_0 \xi_0 F(p_a).
\]

The operators \( \eta_0^{i,j}, \xi_0^{i,j} \) commute with \( X^\pm_{m,i}, \Psi_{i}^{\pm}(z) \) up to sign \( \pm 1 \).

**Proposition 3.2** \( \mathcal{F}(p_a) \) is \( U_q(\widehat{sl}(M|N)) \)-module.

We call \( \mathcal{F}(p_a) \) the Wakimoto realization of \( U_q(\widehat{sl}(M|N)) \).

## 4 Screening operator

In this Section we give the screening operators \( Q_i \) (1 \( \leq \) \( i \leq M+N-1 \)) that commute with \( U_q(\widehat{sl}(M|N)) \) for the level \( c = k \neq -g \). We define bosonic operators \( S_i(z) \) (1 \( \leq \) \( i \leq M+N-1 \)) that we call the
screening currents as follows.

\[ S_i(z) = \sum_{j=1}^{M} \frac{e_{i,j}}{(q - q^{-1})z}(S_{i,j}^-(z) - S_{i,j}^+(z)) + \sum_{j=M+1}^{M+N} e_{i,j}S_i(z) \]

for \( 1 \leq i \leq M - 1 \),

\[ S_M(z) = \sum_{j=M+1}^{M+N} e_{M,j}S_{M,j}(z), \]

\[ S_i(z) = \sum_{j=1}^{M} \frac{e_{i,j}}{(q - q^{-1})z}(S_{i,j}^-(z) - S_{i,j}^+(z)) \]

for \( M + 1 \leq i \leq M + N - 1 \).

We set \( S_{i,j}^\pm \) as follows.

\[ S_{i,j}^\pm(z) = e^{-(q - q^{-1})(z;\frac{b_M}{d_M}) + (b + c)^{i+1} - j_z(q^{M - N} - z) - b_{i,j}^+(q^{M - N} - z) - (b + c)^{i+1} - j_z(q^{M - N} - z)} \]

\[ \times e^{\sum_{l=i+1}^{M+N}(\Delta b_{l+1}^{i,j})(q^{-M - N - l}z) + \sum_{l=i+1}^{M+N}(\Delta b_{l-1}^{i,j})(q^{-M - N + l}z)} \]

\[ \times e^{\sum_{l=j+1}^{M+N}(\Delta b_{l+1}^{j,i})(q^{-M - N - l}z) + \sum_{l=j+1}^{M+N}(\Delta b_{l-1}^{j,i})(q^{-M - N + l}z)} \]

\[ S_{i,j}(z) = e^{-(q - q^{-1})(z;\frac{b_M}{d_M}) + (b + c)^{i+1} - j_z(q^{M - N} - z) - b_{i,j}^+(q^{M - N} - z) - (b + c)^{i+1} - j_z(q^{M - N} - z)} \]

\[ \times e^{\sum_{l=i+1}^{M+N}(\Delta b_{l+1}^{i,j})(q^{-M - N - l}z) + \sum_{l=i+1}^{M+N}(\Delta b_{l-1}^{i,j})(q^{-M - N + l}z)} \]

\[ \times e^{\sum_{l=j+1}^{M+N}(\Delta b_{l+1}^{j,i})(q^{-M - N - l}z) + \sum_{l=j+1}^{M+N}(\Delta b_{l-1}^{j,i})(q^{-M - N + l}z)} \]

\[ S_M(z) = e^{-(q - q^{-1})(z;\frac{b_M}{d_M}) + (b + c)^{M+N} - j_z(q^{M - N} - z) + b_{M,j}^+(q^{M - N} - z) - (b + c)^{M+N} - j_z(q^{M - N} - z)} \]

\[ \times e^{\sum_{l=M+1}^{M+N}(\Delta b_{M+1}^{M,j})(q^{-M - N - l}z) + \sum_{l=M+1}^{M+N}(\Delta b_{M-1}^{M,j})(q^{-M - N + l}z)} \]

We set \( e_{i,j} \) as follows.

\[ e_{i,i+1} = \begin{cases} 
1/d_{i,i}^2 & (1 \leq i \leq M - 1), \\
-q^{-N+1}/d_{i,i}^2 & (i = M), \\
-1/d_{i,i}^2 & (M + 1 \leq i \leq M + N - 1),
\end{cases} \]

\[ e_{i,j} = \begin{cases} 
1/d_{i,i}^3 & (1 \leq i \leq M - 1, i + 2 \leq j \leq M), \\
q^{k+1-M-N}/d_{i,j}^3 & (1 \leq i \leq M - 1, M + 1 \leq j \leq M + N), \\
-q^{j-M-N}/d_{M,j}^3 & (i = M, M + 2 \leq j \leq M + N), \\
-1/d_{i,j}^3 & (M + 1 \leq i \leq M + N - 1, i + 2 \leq j \leq M + N).
\end{cases} \]

The \( \mathbb{Z}_2 \)-grading of the screening current is given by \( p(S_{M,j}(z)) \equiv 1 \pmod{2} \) for \( M + 1 \leq j \leq M + N \) and zero otherwise. The Jackson integral with parameters \( q \in \mathbb{C} \) and \( s \in \mathbb{C}^* \) is defined by

\[ \int_0^{\infty} f(w)dw = s(1 - q) \sum_{n \in \mathbb{Z}} f(sq^n)q^n. \]

We define the screening operators \( Q_i \) \( 1 \leq i \leq M + N - 1 \) as follows, when the Jackson integrals are convergent.

\[ Q_i = \int_0^{\infty} S_i(w)dw_{q^{2(k+q)}w}. \]
Theorem 4.1  The screening operators $Q_i$ ($1 \leq i \leq M + N - 1$) defined in (4.1), (4.2), (4.3), (4.4) commute with the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$.

$$[Q_i, U_q(\widehat{sl}(M|N))] = 0.$$  

5  Limit $q \rightarrow 1$

Bosonization of the affine superalgebra $\widehat{sl}(M|N)$ for an arbitrary level $k$ have been studied in [9, 10, 11]. We obtain new bosonization of the affine superalgebra $\widehat{sl}(M|N)$ in the limit $q \rightarrow 1$.

In what follows we set

$$H^i(z) = \sum_{m \in \mathbb{Z}} H^i_m z^{-m-1} \quad (1 \leq i \leq M + N - 1).$$

We set the parameters $c_{i,j} = 1$ in (3.4)-(3.6), (3.7)-(3.9), (4.1)-(4.3) for simplicity. In the limit $q \rightarrow 1$ we introduce operators $\alpha_i(z)$ ($1 \leq i \leq M + N - 1$), $\beta_{i,j}(z)$, $\tilde{\beta}_{i,j}(z)$, $\gamma_{i,j}(z)$ ($1 \leq i < j \leq M + N, \nu_i \nu_j = +$), and $\psi_{i,j}(z), \psi^\dagger_{i,j}(z)$ ($1 \leq i < j \leq M + N, \nu_i \nu_j = -$) as follows.

$$\alpha_i(z) = \partial_z \left( a^i(z) \right), \quad \gamma_{i,j}(z) =: e^{(b+c)z} i_{i,j}(z);$$

$$\beta_{i,j}(z) =: \partial_z \left( e^{-c_{i,j}z} \right) e^{-b_{i,j}z}; \quad \tilde{\beta}_{i,j}(z) =: \partial_z \left( e^{-b_{i,j}z} \right) e^{c_{i,j}z};$$

$$\psi_{i,j}(z) =: e^{b_{i,j}z}; \quad \psi^\dagger_{i,j}(z) =: e^{-b_{i,j}z}.$$  

They satisfy the following relations.

$$\alpha_i(z) \alpha_j(w) = \frac{(k + g) A_{i,j}}{(z - w)^2} + \cdots,$$

$$\beta_{i,j}(z) \gamma_{i',j'}(w) = \frac{\delta_{i,i'} \delta_{j,j'}}{z - w} + \cdots, \quad \gamma_{i,j}(z) \beta_{i',j'}(w) = -\frac{\delta_{i,i'} \delta_{j,j'}}{z - w} + \cdots,$$

$$\tilde{\beta}_{i,j}(z) \gamma_{i',j'}(w) = \frac{-\delta_{i,i'} \delta_{j,j'}}{z - w} + \cdots, \quad \gamma_{i,j}(z) \tilde{\beta}_{i',j'}(w) = \frac{\delta_{i,i'} \delta_{j,j'}}{z - w} + \cdots,$$

$$\psi_{i,j}(z) \psi^\dagger_{i',j'}(w) = \frac{-\delta_{i,i'} \delta_{j,j'}}{z - w} + \cdots, \quad \psi^\dagger_{i,j}(z) \psi_{i',j'}(w) = \frac{\delta_{i,i'} \delta_{j,j'}}{z - w} + \cdots.$$  

In the limit $q \rightarrow 1$ the operators $a^i_{\pm}(z), b^i_{\pm}(z), (\Delta_L^i b^i_{\pm})(z)$ and $(\Delta_H b^i_{\pm})(z)$ disappear. We obtain the following.

$$H^i(z) = \alpha_i(z) + \sum_{j=1}^i : (\tilde{\beta}_{j,i}(z) \gamma_{j,i}(z) - \tilde{\beta}_{j,i+1}(z) \gamma_{j,i+1}(z)) :$$

$$+ \sum_{j=i+1}^M : (\tilde{\beta}_{i+1,j}(z) \gamma_{i+1,j}(z) - \tilde{\beta}_{i,j}(z) \gamma_{i,j}(z)) :$$

$$+ \sum_{j=M+1}^{M+N} : ((\partial_z \psi_{i+1,j})(z) \psi^\dagger_{i+1,j}(z) - (\partial_z \psi_{i,j})(z) \psi^\dagger_{i,j}(z)) : \quad (1 \leq i \leq M - 1),$$

$$H^M(z) = \alpha_M(z) + \sum_{j=1}^{M-1} : ((\partial_z \psi_{j,M+1})(z) \psi^\dagger_{j,M+1}(z) + \tilde{\beta}_{j,M}(z) \gamma_{j,M}(z)) :$$
\[ H^i(z) = \alpha_i(z) + \sum_{j=1}^{M} :((\partial_2 \psi_{j,i+1}) (z) \psi_{j,i+1}^\dagger(z) - (\partial_2 \psi_{j,i})(z) \psi_{j,i}(z)) : \\
+ \sum_{j=M+1}^{i} :((\hat{\beta}_{j,i+1}(z) \gamma_{j,i+1}(z) - \hat{\beta}_{j,i}(z) \gamma_{j,i}(z)) : \\
+ \sum_{j=i+1}^{M+N} :((\hat{\beta}_{i,j}(z) \gamma_{i,j}(z) - \hat{\beta}_{i+1,j}(z) \gamma_{i+1,j}(z)) : (M + 1 \leq i \leq M + N - 1). \]

\[ X^{+,i}(z) = \sum_{j=1}^{i} :\beta_{j,i+1}(z) \gamma_{j,i}(z) : (1 \leq i \leq M - 1), \]

\[ X^{+,M}(z) = \sum_{j=1}^{M} :\gamma_{j,M}(z) \psi_{j,M+1}(z) ;, \]

\[ X^{+,i}(z) = \sum_{j=1}^{M} :\psi_{j,i+1}(z) \psi_{j,i}^\dagger(z) : - \sum_{j=M+1}^{i} :\beta_{j,i+1}(z) \gamma_{j,i}(z) : (M + 1 \leq i \leq M + N - 1). \]

\[ X^{-,i}(z) = - :\alpha_i(z) \gamma_{i,i+1}(z) : - \kappa_i :\partial_2 \gamma_{i,i+1}(z) : \\
+ \sum_{j=1}^{i-1} :\beta_{j,i}(z) \gamma_{j,i+1}(z) : - \sum_{j=i+2}^{M} :\beta_{i+1,j}(z) \gamma_{i,j}(z) : - \sum_{j=M+1}^{M+N} :\psi_{i+1,j}(z) \psi_{i,j}^\dagger(z) : \\
+ \sum_{j=i+1}^{M+N} :((\partial_2 \psi_{i,j})(z) \psi_{i,j}^\dagger(z) - (\partial_2 \psi_{i+1,j})(z) \psi_{i+1,j}^\dagger(z)) \gamma_{i,i+1}(z) : (1 \leq i \leq M - 1), \]

\[ X^{-,M}(z) = :\alpha_M(z) \psi_{M,M+1}^\dagger(z) ; + \kappa_M :\partial_2 \psi_{M,M+1}^\dagger(z) ; \\
- \sum_{j=1}^{M-1} :\beta_{j,M}(z) \psi_{j,M+1}^\dagger(z) : - \sum_{j=M+2}^{M+N} :\beta_{M+1,j}(z) \psi_{M+1,j}^\dagger(z) : \\
- \sum_{j=M+2}^{M+N} :((\partial_2 \psi_{M,j})(z) \psi_{M,j}^\dagger(z) + (\partial_2 \psi_{M,j})(z) \psi_{M+1,j}^\dagger(z)) \psi_{M,M+1}^\dagger(z) ;, \]

\[ X^{-,i}(z) = :\alpha_i(z) \gamma_{i,i+1}(z) ; + \kappa_i :\partial_2 \gamma_{i,i+1}(z) ; \\
- \sum_{j=1}^{M} :\psi_{j,i}(z) \psi_{j,i+1}^\dagger(z) : + \sum_{j=M+1}^{i-1} :\beta_{j,i}(z) \gamma_{j,i+1}(z) : - \sum_{j=i+2}^{M+N} :\beta_{i+1,j}(z) \gamma_{i,j}(z) : \\
+ \sum_{j=i+1}^{M+N} :((\hat{\beta}_{i,j}(z) \gamma_{i,j}(z) - \hat{\beta}_{i+1,j}(z) \gamma_{i+1,j}(z)) \gamma_{i,j}(z) : (M + 1 \leq i \leq M + N - 1). \]
Here we have set the coefficients $\kappa_i$ by

$$
\kappa_i = \begin{cases}
  k + i & (1 \leq i \leq M - 1) \\
  k + M - 1 & (i = M) \\
  k + 2M - i & (M + 1 \leq i \leq M + N - 1)
\end{cases}.
$$

In what follows we assume $k \neq -g$. In the limit $q \to 1$ we have the following.

$$
S_i(z) = \sum_{j=i+1}^{M} :\tilde{s}_i(z)\gamma_{i,j}(z)\psi_{i+1,j}(z) + \sum_{j=M+1}^{M+N} :\tilde{s}_i(z)\gamma_{i,j}(z)\psi_{i+1,j}(z) + \\
\quad (1 \leq i \leq M - 1),
$$

$$
S_M(z) = \sum_{j=M+1}^{M+N} :\tilde{s}_M(z)\gamma_{M+1,j}(z)\psi_{M,j}(z),
$$

$$
S_i(z) = \sum_{j=i+1}^{M+N} :\tilde{s}_i(z)\beta_{i,j}(z)\gamma_{i+1,j}(z) : (M + 1 \leq i \leq M + N - 1).
$$

Here we have set the boson operator

$$
\tilde{s}_i(z) = e^{-\left(\frac{1}{\kappa_i}\right)(z;0)}.
$$

Our bosonization is different from [9, 10, 11].

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