Birkhoff’s theorem in Lovelock gravity

Robin Zegers$^{1,2}$

$^1$LPT, Université de Paris-Sud, Bât. 210, 91405 Orsay CEDEX, France
$^2$APC, 11 place Marcelin Berthelot, F-75231 Paris Cedex 05

(Dated: September 23, 2018)

We show that the generic solutions of the Lovelock equations with spherical, planar or hyperbolic symmetry are locally isometric to the corresponding static Lovelock black hole. As a consequence, these solutions are locally static: they admit an additional Killing vector that can either be space-like or time-like, depending on the position. This result also holds in the presence of an abelian gauge field, in which case the solutions are locally isometric to a charged static black hole.

In four dimensional General Relativity, it is well known that the spherically symmetric solutions of Einstein’s field equations in the vacuum are locally isometric to the Schwarzschild solution — this is Birkhoff’s theorem, see for example [1]. As a consequence, these solutions are locally static outside the horizon and a spherically symmetric source does not radiate gravitational waves. In this letter, we extend Birkhoff’s theorem to Lovelock gravity.

Lovelock theory is the most general classical theory of gravity leading to second order field equations and conserved energy-momentum, in $D$ dimensions. The corresponding field equations without matter sources read [2]

$$\sum_{k=0}^{[\frac{D-1}{2}]} \alpha_k \mathcal{E}(k)_a = 0, \quad (1)$$

where the brackets stand for the integer part, the $\alpha_k$ are real constants and the $\mathcal{E}(k)_a$ (with $a = 1 \ldots D$) are given by

$$\mathcal{E}(k)_a = \left( \bigwedge_{l=1}^{k} \Omega_{a_1 \ldots a_{2l-1} a_{2l}} \right) \wedge \theta^{*}_{a_{a_1} \ldots a_{2k}}, \quad (2)$$

which is of order $k$ in the curvature 2-form $\Omega^a_b = \frac{1}{2} R^{a}_{bcd} \theta^c \wedge \theta^d$. Finally, $\theta^{*}_{a_{1} \ldots a_{k}}$ is the Hodge dual of $\theta^{a_1} \wedge \cdots \wedge \theta^{a_k}$, the basis of the space of $k$-forms $\Omega^{(k)}(TM)$, and we thus have

$$\theta^{*}_{a_{1} \ldots a_{k}} = \frac{1}{(D-k)!} \epsilon_{a_1 \ldots a_{k} a_{k+1} \ldots a_{D}} \theta^{a_{k+1}} \wedge \cdots \wedge \theta^{a_{D}}. \quad (3)$$

When $D = 4$, equation (1) reduces to Einstein equations ($k = 1$) with a cosmological constant $\alpha_0$, whilst for $D = 5$, the Gauss-Bonnet term ($k = 2$) must be added. For arbitrary $D$, the static spherically symmetric solutions of (1) were found in [3] and their extension to planar and hyperbolic symmetry is given in [4]. All these solutions belong to a one parameter family and read

$$g = -h(r)dt^2 + \frac{dr^2}{h(r)} + r^2 \tilde{g}_{(D-2,K)}, \quad (4)$$

where $h$ is given as a root of a polynomial that depends on the Lovelock coupling constants $\alpha_k$ and on the mass parameter $\mu$, and $\tilde{g}_{(D-2,K)}$ is the $(D-2)$-dimensional metric with spherical ($K = 1$), planar ($K = 0$) or hyperbolic ($K = -1$) symmetry and hence with isometry groups $SO(D-1)$, $E_{D-2}$ or $SO(1,D-2)$ respectively. Notice that, in addition to these isometries, the solutions (4) also admit $\partial_t$ as a Killing vector and this can either be space-like when $h < 0$ or time-like when $h > 0$, so that $g$ is locally static. This family contains static black holes, with one or more horizons [3]. These horizons can be spherical as in the Schwarzschild case, but also planar or hyperbolic, yielding a much richer topology as for topological black holes [3]. Though solutions of the form (4) do not only describe static black holes, we shall refer to them as static Lovelock black hole solutions. Now, we shall prove the following

**Theorem.** The $C^2$-solutions of the generic Lovelock field equations without matter [1], with spherical, planar or hyperbolic symmetry are locally isometric to the corresponding static Lovelock black hole solutions [4].
Before proceeding with the proof, note that by generic we mean that the Lovelock couplings $\alpha_k$ are arbitrary and independent. Indeed, as we shall see, when some fine-tuning conditions hold between the $\alpha_k$, the above theorem can be evaded. In the generic case, the theorem implies that the solutions of \ref{1}, with spherical, planar or hyperbolic symmetry are also locally static: by which we mean that they have an additional Killing vector that is locally time-like. In the specific case of Einstein-Gauss-Bonnet gravity, this theorem was already proven in \ref{7}.

**Proof.** We begin with a $D$-dimensional space-time with spherical, planar or hyperbolic symmetry. We thus consider a Lorentzian manifold $(M, g)$ admitting respectively $SO(D - 1)$, $E_{D - 2}$ or $SO(1, D - 2)$ as an isometry group with $(D - 2)$-dimensional spacelike orbits $\Sigma$. For all point $P \in M$, if $\Sigma_P$ is the orbit of $P$, the tangent space $T_P M$ can be decomposed into $T_P M = T_P \Sigma_P \oplus (T_P \Sigma_P)\perp$. Then, let $\Sigma_P^\perp$ be the set of all the geodesics passing through $P$ that are tangent to $(T_P \Sigma_P)^\perp$. Locally, $\Sigma_P^\perp$ is a 2-dimensional submanifold of $M$ that is perpendicular to the orbit $\Sigma_P$. Thus, on taking $(\partial_t, \partial_z)$ as a coordinate basis of $T\Sigma^\perp$ and making use of the conformal flatness of the 2-dimensional submanifold $\Sigma^\perp$, one can write

$$g = A^2(t, z)(-dt^2 + dz^2) + R^2(t, z)\bar{g}_{(D - 2, K)},$$

where $\bar{g}_{(D - 2, K)}$ is the metric over the orbits of the corresponding isometry group. Since these orbits are invariant under their isometry group, they are homogeneous and have constant induced curvature

$$\bar{\Omega}^i_j = K\bar{\theta}^i \wedge \bar{\theta}_j,$$

where $i, j = 1 \ldots D - 2$, $\bar{\theta}^i$ is the orthonormal frame adapted to $\bar{g}_{(D - 2, K)}$,

$$\bar{g}_{(D - 2, K)} = \delta_{ij}\bar{\theta}^i \otimes \bar{\theta}^j,$$

and $\bar{\theta}^i \wedge \bar{\theta}^j$ is the resulting basis of the space of 2-forms on the orbits, $\Omega^{(2)}(T\Sigma)$. It will be useful to introduce the conformal coordinates

$$u = \frac{z + t}{2} \quad \text{and} \quad v = \frac{z - t}{2},$$

together with the following parametrization of the metric

$$g = e^{2\nu(u,v)}(du \otimes dv + dv \otimes du) + B^2(u, v)\bar{g}_{(D - 2, K)}.$$ 

We define the associated orthonormal frame

$$\theta^u = e^{\nu(u,v)}du \quad (10)$$
$$\theta^v = e^{\nu(u,v)}dv \quad (11)$$
$$\bar{\theta}^i = B(u, v)\bar{\theta}^i \quad (12)$$

so that

$$g = 2\theta^u \otimes \theta^v + \delta_{ij}\theta^i \otimes \theta^j.$$

In this basis, indices are raised and lowered using

$$\eta_{ab} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \delta_{ij} \end{pmatrix}.$$ 

(14)

Given a torsionfree metric connection $\nabla$, one can define a connection 1-form $\omega^a_b$, such that $\omega_{ab} = -\omega_{ba}$ and $d\theta^a = -\omega^a_b \wedge \theta^b$. From the latter, it is straightforward to derive the components of the connection 1-form. The curvature 2-form then follows from

$$\Omega^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b,$$

yielding

$$\Omega^u_v = -\Omega^v_u = \frac{2\nu_{uv}}{e^{2\nu}}\theta^v \wedge \theta^u \quad (16)$$
$$\Omega^i_u = \frac{1}{Be^{2\nu}}\left[(B_{uu} - 2B_{uv}u)\theta^u \wedge \theta^i + B_{uv}\theta^v \wedge \theta^i\right] \quad (17)$$
$$\Omega^i_v = \frac{1}{Be^{2\nu}}\left[B_{uv}\theta^u \wedge \theta^i + (B_{vv} - 2B_{uv}v)\theta^v \wedge \theta^i\right] \quad (18)$$
$$\Omega^i_j = \bar{\Omega}^i_j - \frac{2B_{iu}B_{uv}}{B^2e^{2\nu}}\theta^i \wedge \theta^j = \frac{K - 2B_{iu}Be^{-2\nu}}{B^2}\theta^i \wedge \theta^j,$$

(19)
where \( f_{u(v)} = \partial_{u(v)} f \). Now, the projection of the \( u \) (resp. \( v \)) components of \( \mathbf{1} \) onto \( \theta^*_u \) (resp. \( \theta^*_v \)) yields the integrability conditions

\[
P' \left[ \frac{K - 2B_u B_v e^{-2\nu}}{B^2} \right] (B_{u u} - 2B_u \nu_u) = 0 \tag{20}
\]

\[
P' \left[ \frac{K - 2B_u B_v e^{-2\nu}}{B^2} \right] (B_{v v} - 2B_v \nu_v) = 0, \tag{21}
\]

where

\[
P[X] = \sum_{k=0}^{D-1} \frac{\alpha_k}{(D-2k-1)!} X^k \tag{22}
\]

and a prime stands for a derivative with respect to the unique argument of a function. Notice how \( \text{(20)} \) and \( \text{(21)} \) factorize as a product of a polynomial, times the integrability conditions one gets from pure Einstein gravity \([9]\). Up to the possible vanishing of \( P' \), Einstein and Lovelock gravities thus obey the same integrability conditions and the theorem will hold. The projection of the \( u \) (resp. \( v \)) component of \( \mathbf{1} \) onto \( \theta^*_u \) (resp. \( \theta^*_v \)) then yields a further equation

\[
P \left[ \frac{K - 2B_u B_v e^{-2\nu}}{B^2} \right] - \frac{2}{(D-1)} P' \left[ \frac{K - 2B_u B_v e^{-2\nu}}{B^2} \right] \left( \frac{1}{Be^{2\nu}} (B_{u u} - 2B_u B_v) + \frac{K}{B^2} \right) = 0. \tag{23}
\]

Finally, the \( i \) component of \( \mathbf{1} \) only projects onto \( \theta^*_i \) giving

\[
(D-1)(D-2) P \left[ \frac{K - 2B_u B_v e^{-2\nu}}{B^2} \right] - \frac{2}{Be^{2\nu}} P' \left[ \frac{K - 2B_u B_v e^{-2\nu}}{B^2} \right] \left( \frac{(2D-5)}{B} (Ke^{2\nu} - 2B_u B_v) + (2D-6)B_{u u} + 2B_{u u} \right) + \frac{4}{Be^{4\nu}} P'' \left[ \frac{K - 2B_u B_v e^{-2\nu}}{B^2} \right] \left( \left( B_{u u} - 2B_u B_v \frac{K e^{2\nu}}{B} \right)^2 - (B_{u u} - 2B_u B_v) (B_{u u} - 2B_u B_v) \right) = 0. \tag{24}
\]

Equations \( \text{(20)}, \text{(21)}, \text{(23)} \) and \( \text{(24)} \) form the full set of Lovelock equations. From the integrability conditions \( \text{(20)} \) and \( \text{(21)} \), we distinguish two classes of solutions

- class I for which \( P' \left[ \frac{K - 2B_u B_v e^{-2\nu}}{B^2} \right] = 0 \)
- class II for which \( P' \left[ \frac{K - 2B_u B_v e^{-2\nu}}{B^2} \right] \neq 0 \)

For class I,

\[
K - 2B_u B_v e^{-2\nu} = \lambda B^2, \tag{25}
\]

where \( \lambda \) is one of the real roots of \( P' \) and is thus a function of the \( \alpha_k \). On using equation \( \text{(23)} \), it follows that \( P[\lambda] = 0 \), whilst equation \( \text{(24)} \) is trivially satisfied since \( \text{(25)} \) implies that

\[
\left( B_{u u} - 2B_u B_v \frac{K e^{2\nu}}{B} \right)^2 = (B_{u u} - 2B_u B_v) (B_{u u} - 2B_u B_v). \tag{26}
\]

Under the fine-tuning condition \( P[\lambda] = 0 \), the solutions therefore take the form

\[
g = \frac{2B_u B_v}{K - \lambda B^2} (du \otimes dv + dv \otimes du) + B^2 \tilde{g}_{(D-2,K)}, \tag{27}
\]

where \( B \) is an arbitrary function of its two arguments \( u \) and \( v \). Such solutions were already discussed in the case of Einstein-Gauss-Bonnet gravity \([2]\). For class II, the integrability conditions \( \text{(20)} \) and \( \text{(21)} \) yield

\[
B(u, v) = H (F(u) + G(v)) \quad \text{and} \quad e^{2\nu(u,v)} = H' F' G', \tag{28}
\]
where \( H, F \) and \( G \) are three functions that depend only on one argument and a prime denotes the derivative of a function with respect to its single argument. To all functions \( F \) and \( G \), one can associate a new set of coordinates \( \tilde{u} = F(u) \) and \( \tilde{v} = G(v) \) on \( \Sigma^{\perp} \). Furthermore, trading the \((\tilde{u}, \tilde{v})\) conformal coordinates for time-like \( \tilde{t} = \tilde{u} + \tilde{v} \) and space-like \( \tilde{z} = \tilde{u} - \tilde{v} \), the metric can be rewritten as

\[
g = 2H'(\tilde{z})(-dt^2 + dz^2) + H^2(\tilde{z})\tilde{g}_{(D-2,K)},
\]

which has a timelike Killing vector \( \partial_{\tilde{t}} \) in all neighbourhood where \( H'(\tilde{z}) > 0 \) and is thus locally static. In particular, setting \( r = H(\tilde{z}) \), we can put it into the following form

\[
g = -h(r)dt^2 + \frac{dr^2}{h(r)} + r^2\tilde{g}_{(D-2,K)},
\]

where \( h(r) = 2H'(\tilde{z}) \) solves equation (28),

\[
P\left[\frac{K - h}{r^2}\right] = \frac{1}{D - 1}P'\left[\frac{K - h}{r^2}\right]\left(\frac{h'}{r} + \frac{2(K - h)}{r^2}\right) = 0.
\]

The latter can be integrated, yielding

\[
P\left[\frac{K - h}{r^2}\right] = \frac{\mu}{r^{D - 1}},
\]

where \( \mu \) is a real constant. Let \( \Lambda(\alpha_0, \ldots, \alpha_{[(D-1)/2]} \) denote a real root of \( P[X] \), as a function of the Lovelock couplings, so that we can write

\[
h(r) = K - r^2\Lambda\left(\alpha_0 - \frac{\mu}{r^{D-1}}, \alpha_1, \ldots, \alpha_{[(D-1)/2]}\right).
\]

The metric (30), with \( h \) given by (33), is the static Lovelock black hole found in [3, 4]. Class II solutions are the only source-free solutions with spherical, planar or hyperbolic symmetry that are valid for all values of the \( \alpha_k \).

As we shall now see, this result still holds in the presence of an abelian gauge field that is invariant under the chosen isometry group. Such a gauge field has a 1-form potential \( A(u, v) = L(u, v)du + M(u, v)dv \) and therefore

\[
F = dA = \frac{M_u - L_v}{e^{2\nu}}\theta^u \wedge \theta^v.
\]

This of course implies that \( dF = 0 \). From \( d \ast F = 0 \), on the other hand, it follows that

\[
F = \frac{Q}{B^{(D-2)}}\theta^u \wedge \theta^v,
\]

where \( Q \) is a real constant. The integrability conditions (20) and (21) are unchanged, but there is no class I solution if \( Q \neq 0 \). We are thus left with class II, which is still free of any fine-tuning, and for which equation (32) becomes

\[
P\left[\frac{K - h}{r^2}\right] = \frac{\mu}{r^{D - 1}} - \frac{Q^2}{r^{2D - 4}},
\]

as it is now sourced by the stress-energy of the gauge field. In the end, the metric is still of the form (30), but now, \( h \) is given by

\[
h(r) = K - r^2\Lambda\left(\alpha_0 - \frac{\mu}{r^{D-1}} + \frac{Q^2}{r^{2D-4}}, \alpha_1, \ldots, \alpha_{[(D-1)/2]}\right).
\]

This is the Lovelock analogue of the Reissner-Nordström black hole of General Relativity [3, 4]. Class II solutions are the only solutions of the Lovelock equations, coupled to a non-vanishing abelian gauge field, with spherical, planar or hyperbolic symmetry and they are locally static.

Though the set of Killing vectors of space-times with spherical, planar or hyperbolic symmetry, \textit{a priori} reduces to the generators of their respective isometry groups \( SO(D - 1) \), \( E_{D-2} \) or \( SO(1, D - 2) \), we have shown that the solutions of the Lovelock equations with these symmetries generically get an additional Killing vector that enlarges their isometry group, so that they reduce to locally static space-times. The static Lovelock black holes therefore span
the whole set of solutions of the generic Lovelock equations without matter or in the presence of an abelian gauge field, with spherical, planar or hyperbolic symmetry.

It is a pleasure to thank Christos Charmousis and Danièle Steer for helpful suggestions and comments.

† Electronic address: robin.zegers@th.u-psud.fr

[1] N. Straumann, *General Relativity*, Springer-Verlag (2004)
[2] D. Lovelock, J.Math.Phys. 12 (1971) 498.
R. C. Myers, Phys. Rev. D 36, 392 (1987).
[3] D. G. Boulware and S. Deser, Phys. Rev. Lett. 55, 2656 (1985).
J. T. Wheeler, Nucl. Phys. B 273, 732 (1986).
J. T. Wheeler, Nucl. Phys. B 268, 73 (1986).
[4] R. G. Cai, Phys. Lett. B 582, 237 (2004) [arXiv:hep-th/0311240].
[5] R. C. Myers and J. Z. Simon, Phys. Rev. D 38, 2434 (1988).
[6] R. B. Mann [arXiv:gr-qc/9709039]
D. Birmingham, Class. Quant. Grav. 16, 1197 (1999) [arXiv:hep-th/9808032].
[7] C. Charmousis and J. F. Dufaux, Class. Quant. Grav. 19, 4671 (2002) [arXiv:hep-th/0202107].
[8] D. L. Wiltshire, Phys. Lett. B 169, 36 (1986).
D. L. Wiltshire, Phys. Rev. D 38, 2445 (1988).
[9] P. Bowcock, C. Charmousis and R. Gregory, Class. Quant. Grav. 17, 4745 (2000) [arXiv:hep-th/0007177].