Some remarks about disjointly homogeneous symmetric spaces

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Abstract

Let $1 \leq p < \infty$. A symmetric space $X$ on $[0,1]$ is said to be $p$-disjointly homogeneous (resp. restricted $p$-disjointly homogeneous) if every sequence of normalized pairwise disjoint functions from $X$ (resp. characteristic functions) contains a subsequence equivalent in $X$ to the unit vector basis of $l_p$. Answering a question posed in the paper [17], we construct, for each $1 \leq p < \infty$, a restricted $p$-disjointly homogeneous symmetric space, which is not $p$-disjointly homogeneous. Moreover, we prove that the property of $p$-disjoint homogeneity is preserved under Banach isomorphisms.

1 Introduction

A Banach lattice $E$ is called disjointly homogeneous (shortly DH) if two arbitrary sequences of normalized pairwise disjoint elements in $E$ contain equivalent subsequences. In particular, given $1 \leq p \leq \infty$, a Banach lattice is $p$- disjointly homogeneous (shortly p-DH) if each normalized disjoint sequence

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has a subsequence equivalent to the unit vector basis of $l_p$ ($c_0$ when $p = \infty$). These notions were first introduced in [13] and proved to be very useful in studying the general problem of identifying Banach lattices $E$ such that the ideals of strictly singular and compact operators bounded in $E$ coincide [14] (see also survey [16] and references therein). Results obtained there can be treated as a continuation and development of a classical theorem of V. D. Milman [22] which states that every strictly singular operator in $L_p(\mu)$ has compact square.

Recently, in [17], in the setting of symmetric spaces it was introduced a weaker property of restricted 2-DH disjoint homogeneity. A symmetric space $X$ on $[0, 1]$ is said to be restricted 2-DH if every sequence of normalized disjoint characteristic functions contains a subsequence equivalent to the unit vector basis of $l_2$. Clearly, each 2-DH symmetric space is restricted 2-DH. In [17], the authors proved the converse for Orlicz spaces [17, Theorem 5.1] and also asked if a symmetric space $X$ on $[0, 1]$ which is restricted 2-DH, must be 2-DH. This question (repeated also in [16, p. 19]) was motivated by the fact that restricted 2-DH symmetric spaces have rather “good” properties. In particular, they are stable under duality [17, Proposition 3.7] while this is still open problem for 2-DH symmetric spaces, see [16, p. 20]. Moreover, every symmetric space isomorphic (as a Banach space) to a 2-DH symmetric space $Y$ is restricted 2-DH [17, Corollary 3.6]. Observe that analogous result for the 2-DH property was unknown.

In this paper we solve the problem if a restricted 2-DH symmetric space on $[0, 1]$ is also 2-DH in negative. More precisely, given $1 \leq p, q < \infty$ we construct a restricted $p$-DH symmetric space $Z_{p,q}$ on $[0, 1]$, which contains a sequence of pairwise disjoint functions equivalent to the unit vector basis of $l_q$. Clearly, if $p \neq q$ the space $Z_{p,q}$ is not DH. We show also that in the case when $1 < p, q < \infty$ the space $Z_{p,q}$ is reflexive. Moreover, $Z_{p,1}$, $1 \leq p < \infty$, is a disjointly complemented space (i.e., every sequence of pairwise disjoint functions from $Z_{p,1}$ has a subsequence whose span is complemented in $Z_{p,1}$).

At the same time, by using a deep Kalton’s result on uniqueness of rearrangement invariant structures, we prove that in the setting of symmetric spaces on $[0, 1]$ the $p$-DH property is preserved under Banach isomorphisms, i.e., if $1 \leq p < \infty$ then each symmetric space isomorphic to a $p$-DH symmetric space is also $p$-DH.
2 Preliminaries

In this section, we shall briefly list the definitions and notions used throughout this paper. For more detailed information, we refer to the monographs [20, 19, 8].

A Banach space \((X, \| \cdot \|_X)\) of real-valued Lebesgue measurable functions (with identification \(m\)-a.e.) on the interval \([0, 1]\) is called symmetric (or rearrangement invariant) if

(i). \(X\) is an ideal lattice, that is, if \(y \in X\) and \(x\) is any measurable function on \([0, 1]\) with \(|x| \leq |y|\), then \(x \in X\) and \(\|x\|_X \leq \|y\|_X\);

(ii). \(X\) is symmetric in the sense that if \(y \in X\), and if \(x\) is any measurable function on \([0, 1]\) with \(x^* = y^*\), then \(x \in X\) and \(\|x\|_X = \|y\|_X\).

Here, \(m\) is the usual Lebesgue measure and \(x^*\) denotes the non-increasing, right-continuous rearrangement of a measurable function \(x\) on \([0, 1]\) given by

\[
x^*(t) = \inf \{ s \geq 0 : m\{u \in [0, 1] : |x(u)| > s\} \leq t \}, \quad t > 0.
\]

For any symmetric space \(X\) on \([0, 1]\) we have \(L_\infty[0, 1] \subseteq X \subseteq L_1[0, 1]\). The fundamental function \(\phi_X\) of a symmetric space \(X\) is defined by \(\phi_X(t) := \|\chi_{[0,t]}\|_X\). In what follows \(\chi_A\) is the characteristic function of a set \(A\).

The Köthe dual (or the associated space) \(X'\) of a symmetric space \(X\) consists of all measurable functions \(y\), for which

\[
\|y\|_{X'} := \sup \left\{ \int_0^1 |x(t)y(t)|dt : x \in X, \|x\|_X \leq 1 \right\} < \infty.
\]

If \(X^*\) denotes the Banach dual of \(X\), then \(X' \subset X^*\) and \(X' = X^*\) if and only if the norm \(\| \cdot \|_X\) is order-continuous, i.e., from \(\{x_n\} \subseteq X, x_n \downarrow 0\), it follows that \(\|x_n\|_X \to 0\). Note that the norm \(\| \cdot \|_X\) of the symmetric space \(X\) is order-continuous if and only if \(X\) is separable. We denote by \(X_0\) the closure of \(L_\infty\) in \(X\) (the separable part of \(X\)). The space \(X_0\) is symmetric, and it is separable if \(X \neq L_\infty\).

Let us recall some classical examples of symmetric spaces on \([0, 1]\). Denote by \(\Omega\) the set of all increasing concave functions \(\varphi\) on \([0, 1]\) such that \(\varphi(0) = 0\).
Every \( \varphi \in \Omega \) and \( 1 \leq q < \infty \) generate the Lorentz space \( \Lambda_q(\varphi) \) endowed with the norm
\[
\|x\|_{\Lambda_q(\varphi)} := \left( \int_0^1 x^*(t)^q \, d\varphi(t) \right)^{1/q}.
\]
We set \( \Lambda(\varphi) := \Lambda_1(\varphi) \).

If \( \varphi \in \Omega \) then the Marcinkiewicz space \( M(\varphi) \) consists of all measurable functions \( x \) such that
\[
\|x\|_{M(\varphi)} := \sup_{0 < \tau \leq 1} \frac{1}{\varphi(\tau)} \int_0^\tau x^*(t) \, dt < \infty.
\]
The space \( M(\varphi) \) is not separable provided that \( \lim_{\tau \to 0} \varphi(\tau) = 0 \) (or, equivalently, \( M(\varphi) \neq L_1 \)). At the same time, its subspace \( M_0(\varphi) \), consisting of all \( x \in M(\varphi) \) such that
\[
\lim_{\tau \to 0} \frac{1}{\varphi(\tau)} \int_0^\tau x^*(t) \, dt = 0,
\]
is a separable symmetric space which, in fact, coincides with the separable part \( (M(\varphi))_0 \). We have \( (\Lambda(\varphi))' = M(\varphi) \) and \( (M(\varphi))' = (M_0(\varphi))' = \Lambda(\varphi) \) ([19] Theorems II.5.2 and II.5.4).

For each \( \varphi \in \Omega \) the spaces \( \Lambda(\varphi) \) and \( M(\varphi) \), where \( \bar{\varphi}(t) := t/\varphi(t) \), are the smallest and the largest ones in the class of all symmetric spaces with the fundamental function \( \varphi \), i.e., \( \Lambda(\varphi) \subset X \subset M(\bar{\varphi}) \) whenever \( X \) is a symmetric space such that \( \phi_X = \varphi \) ([19] Theorems II.5.5 and II.5.7).

The behaviour of a function \( \varphi \in \Omega \) is essentially determined by the numbers
\[
\gamma_{\varphi} := \lim_{t \to +0} \frac{\ln M_{\varphi}(t)}{\ln t} \quad \text{and} \quad \delta_{\varphi} := \lim_{t \to \infty} \frac{\ln M_{\varphi}(t)}{\ln t},
\]
where
\[
M_{\varphi}(t) := \sup_{0 < s \leq \min(1, \frac{1}{t})} \frac{\varphi(ts)}{\varphi(s)}.
\]
For each \( \varphi \in \Omega \) the inequalities \( 0 \leq \gamma_{\varphi} \leq \delta_{\varphi} \leq 1 \) hold ([19] § II.1). In the case when \( \gamma_{\varphi} > 0 \) we have
\[
\|x\|_{M(\varphi)} \asymp \sup_{0 < t \leq 1} (x^*(t)\bar{\varphi}(t))
\]
with constants independent of \( x \in M(\varphi) \) (see [19 Theorem 2.5.3]).
Let $1 < q < \infty$ and let $X$ be a symmetric space on $[0, 1]$. Denote by $X^{(q)}$ the $q$-convexification of $X$ defined as $X^{(q)} := \{x$ measurable on $[0, 1] : |x|^q \in X\}$ with the norm $\|x\|_{X^{(q)}} := \| |x|^q \|_X^{1/q}$ (see [20, p. 53]).

Next, we will make use of the real interpolation method [9]. For a pair of symmetric spaces $(X_0, X_1)$ the Peetre $K$-functional of an element $x \in X_0 + X_1$ is defined for $t > 0$ by

$$K(t, x; X_0, X_1) = \inf \{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}.$$ 

Then, the real Lions-Peetre interpolation spaces are defined as follows

$$(X_0, X_1)_{\theta,p} = \{x \in X_0 + X_1 : \|x\|_{\theta,p} = \left( \int_0^\infty \left[ t^{-\theta} K(t, x; X_0, X_1) \right]^p \frac{dt}{t} \right)^{1/p} < \infty \}$$

if $0 < \theta < 1$ and $1 \leq p < \infty$, and

$$(X_0, X_1)_{\theta,\infty} = \{x \in X_0 + X_1 : \|x\|_{\theta,\infty} = \sup_{t > 0} \frac{K(t, x; X_0, X_1)}{t^\theta} < \infty \}$$

if $0 \leq \theta \leq 1$.

Convergence in measure (resp. in weak topology) of a sequence of measurable functions $\{x_n\}_{n=1}^\infty$ (resp. from a symmetric space $X$) to a measurable function $x$ (resp. from $X$) is denoted by $x_n \overset{m}{\to} x$ (resp. $x_n \overset{w}{\to} x$). The notation $A \asymp B$ will mean that there exist constants $C > 0$ and $c > 0$ not depending on the arguments of $A$ and $B$ such that $cA \leq B \leq CA$. Moreover, throughout the paper $\|f\|_p := \|f\|_{L_p[0,1]}$, $1 \leq p \leq \infty$.

3 Restricted $p$-DH symmetric spaces which are not $p$-DH

We start with the following definitions.

**Theorem 1.** [13] A symmetric space $X$ on $[0,1]$ is disjointly homogeneous (shortly DH) if two arbitrary normalized disjoint sequences from $X$ contain equivalent subsequences.

Given $1 \leq p \leq \infty$, a symmetric space $X$ on $[0,1]$ is called $p$-disjointly homogeneous (shortly $p$-DH) if each normalized disjoint sequence has a subsequence equivalent in $X$ to the unit vector basis of $l_p$ ($c_0$ when $p = \infty$).
For examples and other information related to DH and $p$-DH symmetric spaces and Banach lattices see [13, 14, 15, 16, 17, 4].

**Theorem 2.** [17] Let $1 \leq p \leq \infty$. A symmetric space $X$ on $[0, 1]$ is said to be restricted $p$-DH if for every sequence of pairwise disjoint subsets $\{A_n\}_{n=1}^{\infty}$ of $[0, 1]$ there is a subsequence $\{A_{n_k}\}$ such that $\frac{1}{\|\chi_{A_{n_k}}\|_X} \chi_{A_{n_k}}$ is equivalent to the unit vector basis of $l_p$ ($c_0$ when $p = \infty$).

**Theorem 3.** [15] A symmetric space $X$ on $[0, 1]$ is called disjointly complemented ($X \in DC$) if every disjoint sequence from $X$ has a subsequence whose span is complemented in $X$.

Clearly, each $p$-DH symmetric space is restricted $p$-DH. In [17], there was posed the question if a symmetric space $X$, which is restricted $p$-DH, is $p$-DH (see also [16, p. 19]). The following theorem solves this problem in negative.

**Theorem 4.** Let $1 \leq p,q < \infty$. There exists a restricted $p$-DH symmetric space $Z_{p,q}$ on $[0, 1]$, which contains a sequence of pairwise disjoint functions $\{g_m\}$ equivalent to the unit vector basis of $l_q$ such that the closed linear span $[g_m]$ is complemented in $Z$. If $1 < p,q < \infty$ then the space $Z_{p,q}$ is reflexive. Moreover, $Z_{p,1} \in DC$ for each $1 \leq p < \infty$.

**Proof.** To do the structure of the proof more understandable and transparent, split it into three parts.

**Step 1.** Following an idea of the proof of Theorem 3 from [3], we construct two separable symmetric spaces $E_0$ and $E_1$ on $[0, 1]$ with the fundamental functions $\psi$ and $\varphi$, respectively, such that $E_0 \subset E_1$, $E_0$ contains a sequence of pairwise disjoint functions $\{v_m\} \subset E_0$, which is equivalent to the unit vector basis of $c_0$ both in $E_0$ and in $E_1$, and

$$\lim_{t \to 0} \frac{\varphi(t)}{\psi(t)} = 0.$$  \hspace{1cm} (1)

Take for $E_0$ the space $M_0(\psi)$ (i.e., the separable part of the Marcinkiewicz space $M(\psi)$), where $\psi(t) = t^{1/2} \log_2^{-1/2} \frac{4}{t}$, $0 < t \leq 1$.

Since $\gamma_\psi = 1/2$, then by [19, Theorem 2.5.3] (see also Section 3),

$$\|x\|_{E_0} \asymp \sup_{0 < t \leq 1} (x^*(t)\overline{\psi}(t)),$$ \hspace{1cm} (2)
where \( \bar{\psi}(t) = t/\psi(t) = t^{1/2} \log^{1/2} \frac{1}{t} \). Moreover, from the inequality \( x^*(t) \leq \frac{1}{t} \int_0^t x^*(s) \, ds \), \( 0 < t \leq 1 \), for every \( x \in E_0 \) it follows

\[
\lim_{t \to 0} (x^*(t) \bar{\psi}(t)) = 0. \tag{3}
\]

Let us define the space \( E_1 \). We put \( \alpha_k = (k+2)^{-1/2}2^{k/2} \) and \( z_k(t) = \alpha_k \chi_{(0,2^{-k})}(t) \), \( k = 0, 1, \ldots \). Moreover, we define the sequence of positive integers \( \{n_m\}_{m=0}^{\infty} \) such that

\[
n_m := \max \left\{ n = 1, 2, \ldots : \sum_{k=n_{m-1}}^{n-1} \frac{1}{k+2} \leq 1 \right\}, \quad m = 1, 2, \ldots \tag{4}
\]

Then, we denote

\[
w_m(t) := \max_{n_m \leq k < n_{m+1}} z_k(t), \quad m = 0, 1, \ldots
\]

Since the sequence \( \{\alpha_k\}_{k=0}^{\infty} \) increases, from (4) it follows that the \( L_2 \)-norms of the functions \( w_m \) satisfy the estimates

\[
\|w_m\|_2^2 \geq \sum_{k=n_m}^{n_{m+1}-1} \alpha_k^2 2^{-k-1} = \frac{1}{2} \sum_{k=n_m}^{n_{m+1}-1} \frac{1}{k+2} \geq \frac{1}{4}
\]

and

\[
\|w_m\|_2^2 \leq \sum_{k=n_m}^{n_{m+1}-1} \alpha_k^2 2^{-k} = \sum_{k=n_m}^{n_{m+1}-1} \frac{1}{k+2} \leq 1,
\]

whence

\[
\frac{1}{2} \leq \|w_m\|_2 \leq 1, \quad m = 0, 1, \ldots \tag{5}
\]

Further, we denote \( \chi_b := b^{-1/2} \chi_{(0,b)} \), \( 0 < b \leq 1 \), \( \bar{w}_m := w_m/\|w_m\|_2 \), \( m = 0, 1, \ldots \), and define the set \( V \) as follows

\[
V := \{\chi_b\}_{0 < b \leq 1} \bigcup \{\bar{w}_m\}_{m=0}^{\infty}.
\]

Moreover, let \( E_1 \) consist of all measurable functions \( x(t) \) on \( [0, 1] \) such that

\[
\lim_{s \to +0} \sup_{v \in V} \int_0^s x^*(t)v(t) \, dt = 0. \tag{6}
\]
Then $E_1$ with the norm

$$||x||_{E_1} := \sup_{v \in V} \int_0^1 x^*(t)v(t)\,dt$$

is the separable part of the intersection of Lorentz spaces constructed by the functions $\int_0^t v(s)\,ds$ with $v \in V$ and so $E_1$ is a separable symmetric space on $[0,1]$. In addition, from the definition of the norm of $E_1$ it follows that

$$||x||_{M(t^{1/2})} \leq ||x||_{E_1} \leq ||x||_2.$$

Therefore, $\varphi(t) = t^{1/2}$ is the fundamental function of $E_1$. Thereby the fundamental functions of the spaces $E_0$ and $E_1$ (i.e., the functions $\psi$ and $\varphi$) satisfy condition (1).

Let us show that $E_0 \subset E_1$.\hfill (7)

Firstly, we check that

$$\sup_{v \in V} \int_0^1 \frac{v(t)}{\psi(t)}\,dt < \infty. \quad \text{(8)}$$

Indeed,

$$\int_0^1 \chi_b(t) \frac{dt}{\psi(t)} = 2b^{-1/2} \int_0^b \frac{d(t^{1/2})}{\log_2^{1/2} 4/t} \leq 2 \text{ for all } 0 < b \leq 1,$$

and, by (4),

$$\int_0^1 w_m(t) \frac{dt}{\psi(t)} \leq \sum_{k=n_m}^{n_m+1-1} \alpha_k \int_0^{2^{-k}} \frac{dt}{\psi(t)} = 2 \sum_{k=n_m}^{n_m+1-1} \alpha_k \int_0^{2^{-k}} \frac{d(t^{1/2})}{\log_2^{1/2} 4/t} \leq 2 \sum_{k=n_m}^{n_m+1-1} \frac{1}{k+2} \leq 2.$$

Combining the last inequalities with (8) yields (8). Now, let $x \in E_0$ be arbitrary. Then, we have

$$\sup_{v \in V} \int_0^s x^*(t)v(t)\,dt \leq \sup_{0 \leq t \leq s} (x^*(t)\bar{\psi}(t)) \cdot \sup_{v \in V} \int_0^1 \frac{v(t)}{\psi(t)}\,dt.$$
Hence, from (3) and (8) it follows (6), i.e., $x \in E_1$. Thus, embedding (7) is proved.

Next, we set $D_m := (2^{-n_{m+1}}, 2^{-nm}]$ and

$$v_m(t) := w_m(t) \chi_{D_m}(t) = \sum_{k=n_m}^{n_{m+1}-1} \alpha_k \chi_{(2^{-k-1}, 2^{-k}]}(t), \ m = 0, 1, \ldots$$

Clearly, the functions $v_m$, $m = 0, 1, \ldots$, are pairwise disjoint. We show that the sequence $\{v_m\}$ is equivalent to the unit vector basis of $c_0$ both in $E_0$ and in $E_1$.

Let

$$v(t) = \sum_{m=0}^{r} c_m v_m(t), \ 0 < t \leq 1,$$

where $r \in \mathbb{N}$, $c_m \in \mathbb{R}$, $m = 0, 1, \ldots, r$. Without loss of generality, we can assume that $c_m \geq 0$. Then, $w(t) := \max_{0 \leq m \leq r} c_m w_m(t)$ is a non-increasing function on $(0, 1]$ and $v(t) \leq w(t)$. Therefore, from (2) it follows that

$$||v||_{E_0} \leq ||w||_{E_0} \leq C \max_{0 \leq m \leq r} \left\{ c_m \max_{n_m \leq k \leq n_{m+1}} \alpha_k \bar{\psi}(2^{-k}) \right\}.$$

Hence, in view of the fact that $\alpha_k \bar{\psi}(2^{-k}) = 1$ for $k = 0, 1, 2, \ldots$, we obtain

$$||v||_{E_0} \leq C \max_{0 \leq m \leq r} c_m. \quad (9)$$

Now, let us estimate the norm $||v||_{E_1}$ from below. By (4), for each $m = 0, 1, \ldots, r$ we infer

$$\int_{0}^{1} v_m^*(t) w_m(t) \ dt \geq \int_{D_m} w_m^2(t) \ dt = \int_{0}^{1} w_m^2(t) \ dt$$

$$= \sum_{k=n_m}^{n_{m+1}-1} \alpha_k^2 2^{-k-1} = \frac{1}{2} \sum_{k=n_m}^{n_{m+1}-1} \frac{1}{k+2} \geq \frac{1}{4}.$$ 

Combining this together with (3) and with the definition of the norm in $E_1$, we obtain $||v_m||_{E_1} \geq 1/4$. Therefore,

$$||v||_{E_1} \geq \max_{0 \leq m \leq r} \{ c_m ||v_m||_{E_1} \} \geq \frac{1}{4} \max_{0 \leq m \leq r} c_m,$$
and so, according to (7) and (9), there exists a constant \( B > 0 \) such that, for arbitrary \( r \in \mathbb{N} \) and all \( c_m \in \mathbb{R} \), we have

\[
B^{-1} \max_{0 \leq m \leq r} |c_m| \leq \left\| \sum_{m=0}^{r} c_m v_m \right\|_{E_0} \leq B \max_{0 \leq m \leq r} |c_m| \tag{10}
\]

and

\[
B^{-1} \max_{0 \leq m \leq r} |c_m| \leq \left\| \sum_{m=0}^{r} c_m v_m \right\|_{E_1} \leq B \max_{0 \leq m \leq r} |c_m|.
\]

This completes Step 1.

Step 2. We apply a simple duality argument. Since the spaces \( E_0 \) and \( E_1 \) are separable, the (Banach) dual spaces \( E_0^* \) and \( E_1^* \) coincide (isometrically) with their Köthe duals \( E_0' = (M_0(\psi))' = \Lambda(\psi) \) and \( E_1' \), which have the fundamental functions \( t^{1/2} \log_2^{-1/2} \frac{4}{t} \) and \( t^{1/2} \), respectively [19 §II.4]. Clearly, \( E_1' \subset E_0' \).

Let \( \{u_m\} \subset E_1' \) be the sequence of pairwise disjoint functions such that \( \|u_m\|_{E_1'} = 1 \), \( m = 0, 1, 2, \ldots \), \( \int_0^1 v_m u_m \, dt = 1 \) and \( \int_0^1 v_m u_n \, dt = 0 \) if \( m \neq n \). Let us show that \( \{u_m\} \) is equivalent to the unit vector basis of \( l_1 \) both in \( E_0' \) and in \( E_1' \). Applying (10), we have

\[
\left\| \sum_{m=0}^{\infty} c_m u_m \right\|_{E_0'} \geq \sup \left\{ \int_0^1 \left( \sum_{m=0}^{\infty} c_m u_m \right) \left( \sum_{m=0}^{\infty} d_m v_m \right) \, dt : \left\| \sum_{m=0}^{\infty} d_m v_m \right\|_{E_0} \leq 1 \right\} \\
\geq \sup \left\{ \sum_{m=0}^{\infty} c_m d_m : \max_{0 \leq m \leq r} |d_m| \leq B^{-1} \right\} \geq B^{-1} \sum_{m=0}^{\infty} |c_m|.
\]

Therefore, since \( E_1' \subset E_0' \), \( \|u_m\|_{E_1'} = 1 \) and \( \|u_m\|_{E_0'} \leq C \) for all \( m = 0, 1, 2, \ldots \), where \( C \) is the constant of the embedding \( E_0 \subset E_1 \), we have

\[
D^{-1} \sum_{m=0}^{\infty} |c_m| \leq \left\| \sum_{m=0}^{\infty} c_m u_m \right\|_{E_0'} \leq D \sum_{m=0}^{\infty} |c_m| \tag{11}
\]

and

\[
D^{-1} \sum_{m=0}^{\infty} |c_m| \leq \left\| \sum_{m=0}^{\infty} c_m u_m \right\|_{E_1'} \leq D \sum_{m=0}^{\infty} |c_m| \tag{12}
\]

for some \( D > 0 \) and all \( c_m \in \mathbb{R} \).

Step 3. Given \( 1 \leq q < \infty \), we denote by \( F_0 \) and \( F_1 \) the \( q \)-convexification of the space \( E_0 \) and \( E_1' \), respectively (if \( q = 1 \) we set \( F_0 = E_0' \) and \( F_1 = E_1' \)).
Clearly, $F_1 \subset F_0$. Further, since the functions $u_m, m = 0, 1, 2, \ldots$, are pairwise disjoint, then so are the functions $g_m := |u_m|^{1/q}, m = 0, 1, 2, \ldots$. Furthermore, by the definition of the $q$-convexification of a space combined with (11) and (12), for all $c_m \in \mathbb{R}$ we infer

$$D\left(\sum_{m=0}^{\infty} |c_m|^q\right)^{1/q} \leq \left\| \sum_{m=0}^{\infty} c_m g_m \right\|_{F_0} \leq D\left(\sum_{m=0}^{\infty} |c_m|^q\right)^{1/q} \quad (13)$$

and

$$D^{-1}\left(\sum_{m=0}^{\infty} |c_m|^q\right)^{1/q} \leq \left\| \sum_{m=0}^{\infty} c_m g_m \right\|_{F_1} \leq D\left(\sum_{m=0}^{\infty} |c_m|^q\right)^{1/q} \quad (14)$$

Moreover, since $F_0 = \Lambda_q(\psi)$ (see Section 2), then passing to a subsequence if it is necessary, we may assume that the closed linear span $[g_m]$ is complemented in $F_0$ (see e.g. [12, Theorem 5.1]). Let $P$ be a linear projection bounded in $F_0$ and in $F_1$ (see [13] and [14]), $P$ is also bounded in $F_1$, and so the subspace $[g_m]$ is complemented in $F_1$ with the same projection.

Now, a given $1 \leq p < \infty$, we denote by $Z_{p,q}$ the real Lions-Peetre interpolation space $(F_0, F_1)_{1/2,p}$ (see Section 2).

Observe that the fundamental functions of the spaces $F_0$ and $F_1$ are $\phi_{F_0}(t) = t^{1/(2q)} \log_2^{-1/(2q)} \frac{4}{t}$ and $\phi_{F_1}(t) = t^{1/(2q)}$, respectively. Also, for an arbitrary symmetric space $X$ we have $\phi_X(t) = t/\phi_X(t), 0 < t \leq 1$ [19, §II.4]. Thus, applying Formula (1) from [9, §3.5] two times together with the duality theorem (see e.g. [9, Theorem 3.7.1]), we can identify the fundamental function $\phi_{Z_{p,q}}$ as follows

$$\phi_{Z_{p,q}}(t) = \phi_{F_0}(t)^{1/2} \phi_{F_1}(t)^{1/2} = t^{1/(2q)} \log_2^{-1/(4q)} \frac{4}{t}.$$ 

Hence,

$$\lim_{t \to +0} \frac{\phi_{F_0}(t)}{\phi_{Z_{p,q}}(t)} = \lim_{t \to +0} \log_2^{-1/(4q)} \frac{4}{t} = 0,$$

and so, according to [4, Theorem 4], every sequence of the form $\left\{ \frac{\chi_k}{\phi_{Z_{p,q}}(m(A_k))} \right\}$, where $A_k, k = 1, 2, \ldots$, are pairwise disjoint subsets of $[0, 1]$, contains a subsequence equivalent in $Z_{p,q}$ to the unit vector basis of $l_p$. As a result, we conclude that $Z_{p,q}$ is a restricted $p$-DH symmetric space.
On the other hand, since the above projection $P$ is bounded in $F_0$ and $F_1$, then by inequalities (13) and (14) combined together with Baoendi-Goulaouic theorem (see e.g. [25, Theorem 1.17.1]), the sequence $\{g_m\}$ is equivalent in $Z_{p,q}$ to the unit vector basis of $l_q$ and the subspace $[g_m]$ is complemented in $Z_{p,q}$.

Now, let $1 < p, q < \infty$. Recall that $F_0 = \Lambda_q(\psi)$. Taking into account that $\gamma_\psi = 1/2$, we conclude that $F_0$ is $q$-convex and $r$-concave for each $r > 2q$ (see e.g. [23, Corollary 2]). Hence, neither $c_0$ nor $l_1$ is lattice embeddable in $F_0$ and so, by Lozanovsky theorem (see [21] or [2, Theorem 4.71]), $F_0$ is reflexive. Therefore, the canonical embedding of $F_1$ into $F_0$ is weakly compact and, by Beauzamy theorem [7], the space $Z_{p,q}$ is also reflexive.

It remains to prove the last assertion of the theorem.

Let $1 \leq p < \infty$ and let $\{x_n\}_{n=1}^{\infty}$ be an arbitrary sequence of pairwise disjoint functions from the space $Z_{p,1}$, $\|x_n\|_{Z_{p,1}} = 1$, $n = 1, 2, \ldots$ If $\liminf_{n \to \infty} \|x_n\|_{F_0} = 0$, then from [4, Theorem 4 and Remark 2] it follows at once that there is a subsequence $\{x_{n_k}\} \subset \{x_n\}$, which spans a complemented subspace in $Z_{p,1}$.

Now, we consider the case when $\liminf_{n \to \infty} \|x_n\|_{F_0} > 0$, that is,

$$\|x_n\|_{F_0} \asymp \|x_n\|_{Z_{p,1}} = 1, \quad n = 1, 2, \ldots \quad (15)$$

Since $q = 1$, we have $F_0 = E_0' = \Lambda(\psi)$. Therefore, applying [12, Theorem 5.1] once more (see also [21]), we can select a subsequence $\{x_{n_k}\} \subset \{x_n\}$, equivalent in $F_0$ to the unit vector basis of $l_1$, such that the subspace $[x_{n_k}]$ is complemented in $F_0$. Since $Z_{p,1} \subset F_0$, then, by (15), $\{x_{n_k}\}$ is equivalent to the unit vector basis of $l_1$ also in $Z_{p,1}$. Hence, if $Q$ is a bounded projection in $F_0$ with the image $[x_{n_k}]$, for all $x \in Z_{p,1}$ we have

$$\|Q\|_{Z_{p,1}} \leq C\|Q x\|_{F_0} \leq C\|Q\|_{F_0 \to F_0} \|x\|_{F_0} \leq C\|Q\|_{F_0 \to F_0} \|x\|_{Z_{p,1}},$$

that is, $Q$ is bounded in $Z_{p,1}$. Summing up, we conclude that the subspace $[x_{n_k}]$ is complemented in $Z_{p,1}$, and the proof is completed.  

Applying Theorem 4 in the case when $1 \leq p \neq q < \infty$, we obtain

**Corollary 1.** For every $1 \leq p < \infty$ there exists a restricted $p$-DH symmetric space which is not DH.
4 $p$-DH property is preserved under isomorphisms

Theorem 5. Let $1 \leq p < \infty$ and let $X$ be a symmetric space on $[0, 1]$, which is isomorphic to a complemented subspace of a $p$-DH symmetric space $Y$. Then either $X = L_2$ (with equivalence of norms) or $X$ is a $p$-DH space.

Proof. Since the spaces $Y$ and $X$ are separable [13, Proposition 2.1], then by a deep Kalton’s result on uniqueness of rearrangement invariant structures [18, Theorem 7.4] (see also [26, Theorem 5.7]), we can assume that the sequence of Haar functions $\{h_n\}_{n=1}^{\infty}$ is equivalent in $X$ to some sequence $\{u_n\}_{n=1}^{\infty}$ of pairwise disjoint functions in $Y$ (let $h_n$ be normalized in $L_\infty$). Show that each normalized block basis of the Haar system contains a subsequence equivalent in $X$ to the unit vector basis $\{e_i\}_{i=1}^{\infty}$ of $l_p$.

Indeed, let $x_m := \sum_{k=j_k+1}^{j_{k+1}} a_k^m h_k$, $p_1 = 0 < p_1 < p_2 < \ldots$, $\|x_m\|_X = 1$, $m = 1, 2, \ldots$ Then the sequence $\{x_m\}_{m=1}^{\infty}$ is equivalent in $X$ to the sequence $\{y_m\}_{m=1}^{\infty} \subset Y$, consisting of pairwise disjoint functions $y_m := \sum_{k=j_k+1}^{j_{k+1}} a_k^m u_k$, $m = 1, 2, \ldots$ Since $\|y_m\|_Y \asymp 1$, $m = 1, 2, \ldots$, by hypothesis, there is a subsequence $\{y_{m_i}\}$, which is equivalent in $Y$ to $\{e_i\}$. Therefore, the corresponding subsequence $\{x_{m_i}\}$ is also equivalent (in $X$) to $\{e_i\}$.

Now, suppose that $\{f_m\}_{m=1}^{\infty}$ is an arbitrary sequence of pairwise disjoint functions in $X$, $\|f_m\|_X = 1$, $m = 1, 2, \ldots$. It is well known that the Haar system is a basis in every separable symmetric space [20, Proposition 2.e.1], and so, in particular, in $X$. Moreover, since for each $n = 1, 2, \ldots$ we have

$$\left| \int_0^1 f_m(t)h_n(t) \, dt \right| \leq \|f_m\|_X \cdot m(\text{supp } f_m) = m(\text{supp } f_m) \to 0 \text{ as } m \to \infty,$$

then, applying the Bessaga-Pelczyński Selection Principle [11, Proposition 1.3.10], we can find a subsequence $\{f_{m_i}\}$ equivalent in $X$ to some block basis of the Haar system. Passing once more to a subsequence (and keeping the notation), by the observation from the beginning of the proof, we may assume that $\{f_{m_i}\}$ is equivalent in $X$ to the unit vector basis $\{e_i\}$ of $l_p$. Hence, the proof is completed.

Corollary 2. (i) If a symmetric space $X$ is isomorphic to a complemented subspace of a 2-DH symmetric space $Y$, then $X$ is also a 2-DH space.

(ii) Let $1 \leq p < \infty$. If a symmetric space $X$ is isomorphic to a $p$-DH symmetric space $Y$, then $X$ is also a $p$-DH space.
Following to the famous description of relatively weakly compact subsets in $L_1(\mu)$ as equi-integrable sets due to N. Dunford and B. J. Pettis (see [11]; [10] Ch. VII or [1, Theorem 5.2.9]), we say that a symmetric space $X$ on $[0,1]$ satisfies Dunford-Pettis criterion of weak compactness (shortly $X \in (WDP)$) if for each relatively weakly compact set $K \subset X$ we have

$$\lim_{\delta \to 0} \sup_{m(A) < \delta} \sup_{x \in K} \|x \chi_A\|_X = 0.$$ 

Observe that $X \in (WDP)$ if and only if the conditions $\{x_n\}_{n=1}^\infty \subset X$, $x_n^{\text{w}} \to 0$, and $x_n \overset{m}{\to} 0$ imply $\|x_n\|_X \to 0$ [6]. On the other hand, this is equivalent to the assumption that $X$ is a 1-DH symmetric space [15, Proposition 4.9] (see also [5, Theorem 3.3]). Therefore, Theorem 5 implies

**Corollary 3.** If $X$ is a symmetric space, which is isomorphic to a complemented subspace of a symmetric space $Y$ such that $Y \in (WDP)$, then $X \in (WDP)$. 

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