Non-Random Phases in Non-Trivial Topologies

Patrick Dineen1*, Graca Rocha2,3,4 & Peter Coles1

1 School of Physics & Astronomy, University of Nottingham, University Park, Nottingham, NG7 2RD, United Kingdom
2 Astrophysics Group, Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE, United Kingdom
3 Department of Physics, Nuclear & Astrophysics Laboratory, University of Oxford, Keble Road, Oxford OX1 3RH, UK
4 Centro de Astrofísica da Universidade do Porto, R. das Estrelas s/n, 4150-762 Porto, Portugal

ABSTRACT

We present a new technique for constraining the topology of the universe. The method exploits the existence of correlations in the phases of the spherical harmonic coefficients of the CMB temperature pattern associated with matched pairs of circles seen in the sky in universes with non-trivial topology. The method is computationally faster than all other statistics developed to hunt for these matched circles. We applied the method to a range of simulations with topologies of various forms and on different scales. A characteristic form of phase correlation is found in the simulations. We also applied the method to preliminary CMB maps derived from WMAP, but the separation of topological effects from, e.g., foregrounds is not straightforward.

Key words: cosmic microwave background – cosmology: theory – methods: statistical

1 INTRODUCTION

The 1st–year Wilkinson Microwave Anisotropy Probe (WMAP) observations seem in good accord with the emerging standard cosmological model; a flat Λ–dominated universe seeded by scale-invariant adiabatic Gaussian fluctuations (Spergel et al. 2003). However, at large scales there is an unexpected loss of CMB anisotropy power that was also seen in the COBE-DMR data (Bennett et al. 2003a). One possible explanation is that we inhabit a universe that has a non–trivial topology. That is to say, our universe is in fact multi-connected and has a finite volume. If so, then power on scales exceeding the fundamental cell size will be suppressed. Our space may not be large enough to support long-wavelength fluctuations. A number of authors have tried to restrict the topology of the universe using the WMAP data (Luminet et al. 2003; Cornish et al. 2004; de Oliveira-Costa et al. 2004; Roukema et al. 2004; Uzan et al. 2004; Weeks et al. 2004). In particular, Luminet et al. (2003) have shown that the Poincaré dodecahedral space ($Ω_o ≈ 1.013$) accounts for the measured power in the quadrupole and octopole better than an infinite (simply-connected) flat universe. The angular power spectrum, however, is not an effective way to characterise the peculiar form of the anisotropy manifest in small universes of this type (Levin, Scannapieco & Silk 1998).

Whether we can determine the topology of the universe depends on its volume. If the universe is small enough, we should be able to see right around it, since photons can travel across the whole universe. If so, then we may be able to identify ghost images of the same object in different directions in the sky or recognise the topology from signatures in clustering statistics. Luminet & Roukema (1999) provide a review of these methods. However, these methods are hindered by the need to identify good standard candles; objects that can be traced through different eras. This is not a problem when looking at the CMB for signatures of the topology. The CMB photons originate from the same epoch and from a very thin shell, the last scattering surface (LSS), which is the same when viewed from either side. If the physical dimensions of the universe are less than the diameter of the LSS then the sphere self-intersects; the loci of self-intersections are circles (Cornish, Spergel & Starkman 1998). We should therefore be able to match patterns of hot and cold spots around circles. This result holds no matter how complex the topology. A further advantage of using the CMB as an indicator is that the last scattering surface marks the edge of the visible universe, making it a powerful tool for looking for non-trivial topology.

In this paper, we introduce a new method to search for evidence of a finite universe in all-sky CMB maps. Our method is based on the properties of the phases of the (complex) coefficients obtained from a spherical harmonic expansion of all-sky maps, specifically we look for phase correlations arising from matched pairs of circles. In the next section we briefly introduce some basic ideas in topology and discuss the simulations with non-trivial topologies used later on. In Section 3 we sketch the procedure developed to detect phase correlations in the CMB. In Section 4 we discuss
the results of applying the method to the simulations and WMAP data. We draw our conclusions in Section 5.

2 NON-TRIVIAL TOPOLOGY

Topology deals with connectivity. To a topologist, a coffee cup and a doughnut are equivalent, while a coffee cup and a bowl are distinct. In the cosmological setting, we are concerned with the connectivity of space: a manifold is described as simply-connected if any closed path can be contracted to a point. The possibility that our universe maybe multi-connected was first suggested by Schwarzschild (1900), but has often been overlooked in favour of the simplicity offered by trivial topological spaces. Indeed there is a common misconception that the shape of the universe can be determined from Einstein’s field equations, given a value of \( \Omega_c \). General Relativity only specifies the local curvature of space-time, so nothing in it forbids a (global) non-trivial topology. The Cosmological Principle merely restricts us to manifolds with constant curvature. Indeed, any detection of non-trivial topology could determine the sign of the spatial Ricci curvature. The simulations presented here are based on those of Rocha et al. (2004): they have \( \Omega_\Lambda = 0 \) and a Harrison–Zel’dovich Gaussian spectrum. The simulations reflect topologies with equal-sided physical dimensions that lead to a suppression of the quadrupole with respect to the high-order modes (Weeks et al. 2004). We define a dimensionless topological scale \( j \) of a simulation as the ratio between the width of the fundamental cell and the horizon size.

These simulations include only those temperature fluctuations generated by the Sachs-Wolfe effect. Matched circles occur because we are looking at the same point on LSS from different directions. In order for this to be true, the temperature fluctuations need to be generated at the LSS which is true in this case, but the SW effect only dominates at large scales. We therefore limit our investigation to \( l < 20 \).

Even at these scales, however, there are three main factors that could confuse the statistic: (i) velocity perturbations generating anisotropies at the LSS; (ii) the integrated Sachs-Wolfe effect due to time varying potential wells crossed by the photons; and (iii) Galactic foreground contamination of CMB data.

3 TESTING FOR PHASE CORRELATIONS

The temperature fluctuations in the CMB at any point in the celestial sphere can be expressed in spherical harmonics as

\[
\Delta(\theta, \phi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} a_{l,m} Y_{l,m}(\theta, \phi),
\]

where the \( a_{l,m} \) are complex coefficients that can be written

\[
a_{l,m} = |a_{l,m}| \exp(\text{i} \phi_{l,m}).
\]

In orthodox cosmologies the temperature fluctuations constitute a statistically homogeneous and isotropic Gaussian random field. In this case the phases \( \phi_{l,m} \) are independent and uniformly random on the interval \([0, 2\pi]\) and the variance of \( a_{l,m} \) depends only upon \( l \). Departures from orthodoxy lead to differences in behaviour of the \( a_{l,m} \). For example, in anisotropic Gaussian fluctuations the variance of the \( a_{l,m} \) depends on \( m \) (Ferreira & Magueijo 1997; Inoue & Sugiyama 2003). Coles et al. (2004) developed a diagnostic of departures from the standard assumption that involves the randomness of phases. The main component of the technique involved using Kuiper’s statistic from an available set of phase angles. First the phases are sorted into ascending order, to give the set \( \{\theta_1, \ldots, \theta_n\} \). Each angle \( \theta_i \) is divided by \( 2\pi \) to give a set of variables \( X_i \), where \( i = 1 \ldots n \). From the set of \( X_i \) we derive two values \( S_n^h \) and \( S_n^v \) where

\[
S_n^h = \max \left\{ \frac{1}{n} - X_1, \frac{2}{n} - X_2, \ldots, 1 - X_n \right\}
\]

and

\[
S_n^v = \max \left\{ X_1, X_2 - \frac{1}{n}, \ldots, X_n - \frac{n-1}{n} \right\}.
\]

Kuiper’s statistic, \( V \), is then defined as

\[
V = (S_n^h + S_n^v) \cdot \left( \frac{\sqrt{n} + 0.155 + 0.24}{\sqrt{n}} \right).
\]

The form of \( V \) is chosen so that it is approximately independent of sample size for large \( n \). Anomalously large values of \( V \) indicate a distribution that is more clumped than a uniformly random distribution, while low values mean that angles are more regular. In order to remove any artifact from the choice of coordinate frame that the \( a_{l,m} \) are measured

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The rotated coefficients are found by employing the Wigner $D$ function
\[
a_{l,m} = \sum_{m'} a_{l,m'} D_{l,m,m'}^j(\alpha, \beta, \gamma),
\]
where the Euler angles $\alpha, \beta, \gamma$ define the magnitude of successive rotations about the coordinate axes. Consequently, a distribution of $V$ is obtained from one set of phases. This distribution is compared, using a $\chi^2$ test, with distributions obtained in a similar manner from 1,000 Monte Carlo (MC) sets of $\phi_{l,m}$, drawn from a uniformly random distribution. The probability $P$ of obtaining a lower value of $\chi^2$ is calculated from the fraction of $\chi^2_{MC}$ that are less than the $\chi^2$ obtained from the phases. A set of angles is assumed to be non-random if $P(\chi^2) \geq 0.95$.

The random-phase hypothesis can be further scrutinised by investigating subsets of the phases. These subsets should also be uniformly random on the interval $[0, 2\pi]$. In this paper, we look at two subsets: (i) the phase differences for fixed values of $m$ ($\phi_{l+1,m} - \phi_{l,m}$) and (ii) the phase differences for fixed values of $l$ ($\phi_{l+1,m} - \phi_{l,m}$) (even $l$-modes only). Both subsets are of particular interest since Cornish et al. (2004) indicate that matched circles are associated with phase correlations. In their paper, the significance level for detection of matched circles in the WMAP data is calculated from 'scrambled' versions of the data. In the scrambled versions, phase correlations are removed by randomly exchanging the $a_{l,m}$ at fixed $l$. Also, as previously mentioned multi-connectedness breaks the global isotropy and sometimes the global homogeneity of the Universe (except the case of the projective space). This will induce correlations between the $a_{l,m}$ of different $l$ and $m$. For instance, due to the symmetries of the hypertorus case, $\langle a_{l,m} a_{l',m'}^\ast \rangle \neq 0 \Rightarrow m - m' \equiv 0 \pmod{2}$ and $l - l' \equiv 0 \pmod{2}$ (Riazuelo et al. 2004). These non-zero off-diagonal terms will induce phase correlations as detected in our study.

Overall, for a given temperature map, we obtain a value of $P(\chi^2)$ for each mode; 18 values for subset (i) and 10 for subset (ii). To improve the presentation of the results, we combine $P(\chi^2)$ for each subset in two ways. First, we count the number of modes with $P(\chi^2) \geq 0.95$ and find the mode that displays the highest value. Second, we perform a Kolmogorov-Smirnov (K-S) test on the distribution of $P(\chi^2)$ over the modes. If the phases are random, the set of $P(\chi^2)$ should be uniform in the interval $[0, 1]$. To quantify the significance of the K-S value obtained, 10,000 MC sets of $P(\chi^2)$ are generated and the probability of obtaining a lower K-S value is calculated. The reason for doing both these things is that one would expect one in twenty modes to yield a value of $P(\chi^2) \geq 0.95$. The second approach gives a (very conservative) idea of the significance of the whole set of modes rather than each individual one.

4 RESULTS AND DISCUSSION

The simulations were generated in HEALPix† format (Gorski, Hivon & Wandelt 1998) with a resolution parameter $N_{side} = 32$. The $a_{l,m}$ were derived using the 'anafast' routine in the HEALPix package. $V$ was binned from 0–2.75 and 0–2.5 for subsets (i) and (ii), respectively. Subset (i) required 10,000 rotations in order to obtain stable results. To analyse one realisation took 18 minutes on a 1.40 GHz CPU desktop. On the other hand, 3,000 rotations produced stable results for subset (ii), resulting in each analysis taking $2\frac{1}{4}$ minutes on the same desktop.

The six flat models listed in Section 2 were studied with $j = 0.5$. In order to see the effect $j$ had on the results, the hypertorus was explored in more detail. A further six simulations with $j < 2$ were scrutinised. Also, six simulations in which matched circles are not anticipated ($j \geq 2$) were studied. The results for subsets (i) and (ii) are shown in Tables 1 and 2 respectively. The tables show the number of modes with $P(\chi^2) \geq 0.95$, the most non-random mode and the K-S fractional probabilities. For subset (i), the method was applied to five realisations (sets of rotations) of each simulation. The average values obtained from these realisations are shown in the tables. For subset (ii), only one realisation was necessary in order to obtain consistent results.

If the phases are random, the number of modes with $P(\chi^2) \geq 0.95$ should be 0.9 for subset (i) and 0.5 for subset (ii). From Table 1 it is clear that we are finding correlations when scanning across fixed $m$ for most of the simulations (we shall refer to these as $m$–correlations). The $m$–correlations are less obvious in terms of the K-S probabilities. However, this test is more general: it does not search specifically for large values of $P(\chi^2)$, so it should be interpreted as a very crude measure of departure from uniformity. The average count for all the simulations displayed in the Table 1 is 2.5. The average value is significantly larger than the expected value of 0.9. This contrasts with the average value of 0.7, from Table 2, that is only slightly higher than the expected value. From the results, it is evident that no particular mode can be chosen to look for signs of non-randomness in the phases. This is unsurprising, as the correlations would manifest themselves across many modes, whose nature would depend on the location and size of the circles with respect to the observer.

The spread in the number count from one realisation to another is quite small (roughly ±1 for larger values) and an exact value for each simulation can be obtained by increasing the number of realisations. To see if there is any worth in doing this, we looked at five further simulations for each model with $j = 0.5$, the results of which are shown in Table 3. The standard deviation is very large, with a count of 1.0 being consistent with all the models. This indicates that it would be very difficult to use the method to distinguish between models or to determine the exact topological scale, you could merely indicate the most probable case. These results confirm that the hypertorus and the $\pi/3$ hexagon models, at this scale, do not display any $m$–correlations in the phases. However, the triple twist torus is detectable from $m$–correlations. This fact is less clear in Table 1, again indicating that the test results vary from simulation to simulation.

Looking at results for the hypertorus displayed in Table 1, it is hard to perceive any trend between the number count and the topological scale of the simulation. The $m$–correlations are seen both with values of $j < 2$ and $j \geq 2$. Intuitively, one would expect phase correlations when matched circles are present ($j < 2$). In such cases, the number of re-

† http://www.eso.org/science/healpix/
peated patterns (circles in the sky) increases with decreasing $j$, and hence we would expect an increase in number count with decreasing $j$. This is not seen in the results, although, this may be masked by the number count varying from simulation to simulation for fixed values of $j$. Applying the statistic to further simulations may reveal such a trend.

On the other hand, the high number count seen at $j \geq 2$ is less easy to explain. Unearthing signs of non--trivial topology beyond the horizon size, suggests the technique is potentially a powerful tool. Diagnostics that hunt for circles in the sky are limited to a maximum value of $j=2$. Our detections beyond the horizon size are by no means serendipitous. Phillips & Kogut (2004) compute the covariance matrix of the $a_{l,m}$ for the hypertorus at various topological scales. They find the off--diagonal terms, that incorporate the phase information, remain prominent even when the width of the fundamental cell is greater than the diameter of the LSS.

Apart from the hypertorus, each of the topologies addressed are not only anisotropic, but also inhomogeneous. The question therefore arises whether the results are affected greatly by changes in the position of the observer. We generated simulations with the observer’s position shifted within the fundamental cell. For models 2 to 6, five simulations with $j=0.5$ were generated with the observer position shifted by $(0.1x,0.1x,0.1x)$ where $x$ is the width of the fundamental cell. The number count for each model was found to be consistent with those of the centrally located observer displayed in Table 3.

A positive detection of $m$-correlations in CMB data would therefore be indicative of the universe having a non-trivial topology. In order to seek evidence of these correlations in the CMB data, we turned to four WMAP-derived maps. The temperature maps were all constructed in a manner that minimises foreground contamination and detector noise, leaving a pure CMB signal. The ultimate goal of these maps is to build an accurate image of the LSS that captures the detailed morphology. Following the release of the WMAP 1 yr data, the WMAP team (Bennett et al. 2003b), Tegmark, de Oliveira-Costa & Hamilton (2003), Naselsky et al. (2003) and Eriksen et al. (2004a) have released CMB-only sky maps (see papers for details). We shall refer to these as the ILC, TOH, Naselsky and Eriksen maps respectively. It is worth pointing out that even though we limit ourselves to full sky maps in this paper, the method can be adapted to cut maps following a suitable adjustment of the MC simulations. Nevertheless, the current WMAP data is preliminary, so we reserve this treatment for future data releases. Again, the results are shown in Tables 1 and 2. The method was applied to five realisations of each CMB map for subset (i).

In three of the four maps, evidence was found for the sort of phase correlations seen in the simulations. However, all the maps, bar the Naselsky map, display $l$--correlations that were discussed in Coles et al. (2004) which can be explained, at least partially, by foreground contamination. If no $m$--correlations were seen then this would suggest there is no evidence for non--trivial topology. However, a positive detection leaves open the possibility of a non--trivial topology.

| Map        | Count (Mode) | K-S Probability |
|------------|--------------|-----------------|
| Model 1 $j=0.5$ | 0.6          | 0.68            |
| Model 2 $j=0.5$ | 1.2 (17)     | 0.35            |
| Model 3 $j=0.5$ | 4.6 (4.5)    | 0.99            |
| Model 4 $j=0.5$ | 0.6          | 0.22            |
| Model 5 $j=0.5$ | 1.8 (14)     | 0.68            |
| Model 6 $j=0.5$ | 0.2          | 0.42            |
| Model 1 $j=0.4$ | 5.4 (1,4,9,12)| 0.99            |
| Model 1 $j=0.6$ | 0.4          | 0.63            |
| Model 1 $j=0.8$ | 0.8          | 0.35            |
| Model 1 $j=0.9$ | 0.0          | 0.56            |
| Model 1 $j=1.0$ | 0.4          | 0.47            |
| Model 1 $j=1.6$ | 3.8 (8,9,10) | 0.77            |
| Model 1 $j=2.0$ | 1.4 (16)     | 0.43            |
| Model 1 $j=4.0$ | 0.0          | 0.66            |
| Model 1 $j=5.0$ | 2.8 (12)     | 0.48            |
| Model 1 $j=6.0$ | 0.0          | 0.62            |
| Model 1 $j=8.0$ | 0.6          | 0.23            |
| Model 1 $j=10.0$ | 2.1 (12)     | 0.88            |

| Map    | Count (Mode) | K-S Probability |
|--------|--------------|-----------------|
| ILC    | 1.2 (4)      | 0.55            |
| TOH    | 0.6          | 0.28            |
| Naselsky | 2.2 (2.4)    | 0.50            |
| Eriksen | 2.0 (1.4)    | 0.33            |

**Table 1.** Phase correlations when scanning across fixed $m$. The column labelled Count shows the average number of modes exceeding 95 per cent significance. The column labelled Mode gives the mode with greatest significance. The last column shows a rough measure of significance for all modes obtained using a K-S test as described in the text. 18 modes studied in total.

| Map    | Count (Mode) | K-S Probability |
|--------|--------------|-----------------|
| Model 1 $j=0.5$ | 0            | 0.17            |
| Model 2 $j=0.5$ | 1            | 0.66            |
| Model 3 $j=0.5$ | 0            | 0.52            |
| Model 4 $j=0.5$ | 1            | 0.45            |
| Model 5 $j=0.5$ | 0            | 0.46            |
| Model 6 $j=0.5$ | 2 (2,12)     | 0.13            |
| Model 1 $j=0.4$ | 2 (8,20)     | 0.32            |
| Model 1 $j=0.6$ | 0            | 0.21            |
| Model 1 $j=0.8$ | 0            | 0.68            |
| Model 1 $j=0.9$ | 1            | 0.97            |
| Model 1 $j=1.0$ | 1            | 0.98            |
| Model 1 $j=1.6$ | 0            | 0.09            |
| Model 1 $j=2.0$ | 1            | 0.49            |
| Model 1 $j=4.0$ | 1            | 0.47            |
| Model 1 $j=5.0$ | 1            | 0.16            |
| Model 1 $j=6.0$ | 1            | 0.54            |
| Model 1 $j=8.0$ | 1            | 0.73            |
| Model 1 $j=10.0$ | 0           | 0.43            |
| ILC    | 2 (14,16)    | 0.50            |
| TOH    | 3 (6,14,16)  | 0.77            |
| Naselsky | 1 (14)       | 0.81            |
| Eriksen | 2 (6,16)     | 0.80            |

**Table 2.** Phase correlations when scanning across fixed $l$. Columns are as in the previous table. 10 even modes studied in total.
5 CONCLUSIONS

In this paper, we have presented a new method for constraining the topology of the universe. The method relies on utilising phase correlations associated with matched pairs of circles in the CMB sky. We applied the method to various simulations with non-trivial topologies. The method appears to detect non-trivial topologies beyond the horizon size. However, the method fails to estimate the scale of features it detects and is not good at discriminating between different models. This can potentially be overcome by studying more simulations as the results were shown to vary from simulation to simulation. The method is simple, computationally fast and does deliver a clear signature: a positive detection of these $m$-correlations is clear evidence for non-trivial topology.

With this in mind, the method was applied to four CMB-only sky maps; we found evidence for $m$-correlations in three of them. However, it would be premature to conclude that there is evidence for non-trivial topology in any of the available CMB temperature maps. The WMAP data is preliminary and has already been shown to have a number of unusual properties that are not yet fully understood (Chiang et al. 2003; Dineen & Coles 2004; Eriksen et al. 2004b). Indeed, Eriksen et al. (2004a) have pointed out that the techniques used in producing these maps result in a poor reconstruction of the cosmological phases which may interfere with the possibility of detecting correlations of the type discussed here. Nevertheless, with improved data, we believe the method will be a useful tool in determining the shape of the universe. Moreover, our method is based on only the simplest possible measure of randomness in the phase distribution. More sophisticated combinations may allow us to improve the method substantially, perhaps unearthing further signs of non-trivial topology beyond the horizon size.

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Table 3. The variation in number count from simulation to simulation. The columns show the average number of modes (along with the variation) and the most significant mode. We evaluated 6 simulations with $j=0.5$ for each model.