Counting cliques and clique covers in random graphs*

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Abstract

We study the problem of counting the number of isomorphic copies of a given template graph, say $H$, in the input base graph, say $G$. In general, this is (provably) very hard to solve efficiently. So, a lot of work has gone into designing efficient approximation schemes, especially, when $H$ is a perfect matching. In this work, we present schemes to count $k$-cliques and $k$-clique covers. Our embedding problems are almost always approximable. The decision versions of these problems are fundamental and are well studied. Both of these problems are NP-hard. In particular, the $k$-Clique Cover problem is NP-hard for every $k \geq 3$.

Here, we present fully polynomial time randomized approximation schemes (fpras) to count $k$-cliques when $k = O(\sqrt{\log n})$ and $k$-clique covers when $k$ is a constant. Our work partially resolves an open problem in [Frieze and McDiarmid (1997)], where they ask whether there exists a fpras for counting $k$-cliques in random graphs. As an aside, we also obtain an alternate derivation of the closed form expression for the $k$-th moment of a binomial random variable. Here, we use the equalities that we derive from binomial theorem. The previous derivation [Knoblauch (2008)] was based on the moment generating function of a binomial random variable.

1 Introduction

Given a base graph $G$ and a template graph $H$, the subgraph isomorphism problem is to decide whether an edge preserving injection $\phi$ between the

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vertices of $H$ and $G$ exists. That is, for every edge $\{u,v\}$ in $H$, $\{\phi(u),\phi(v)\}$ is an edge in $G$. Subgraph isomorphism is a generalization of several fundamental NP-complete problems, like Hamiltonian Path and Clique. The problem has applications in many areas, including cheminformatics [24], pattern discovery in databases [17], bioinformatics [19] and social networks [1].

Another widely studied related fundamental problem is that of counting the number of copies of $H$ in $G$. In general, this problem is #P-complete (Valiant [25]). The class #P is defined as $\{f : \text{there exists a non-deterministic polynomial time Turing machine } M, \text{ such that on input } x, \text{ the computation tree of } M \text{ has exactly } f(x) \text{ accepting leaves}\}$. The problems complete in this class are computationally quite difficult, since an oracle access to #P complete problem would make it possible to solve any problem in the polynomial hierarchy in polynomial time (Toda [23]).

The $k$-Clique problem asks whether there exists a $k$-clique in the input graph $G$. A $k$-clique is the complete graph on $k$ vertices. The $k$-Clique problem has numerous applications, particularly in bioinformatics and social networks [19, 1].

The $k$-Clique cover problem asks for the existence of a perfect $k$-clique packing in $G$. More precisely, given base graph $G$ with $n$ vertices and template graph $H$ that is $n/k$ vertex disjoint and edge disjoint copies of $k$ cliques, does there exist an injective mapping from $H$ to $G$. The decision problem $k$-Clique Cover, that is $\{(G, k) : \text{there exists a disjoint cover by } k\text{-cliques } G\}$ is NP-complete on general graphs with clique number 3 [15]. The problem of $k$-clique cover has applications in the orgy problem [5]: Given a group of people with affinities and aversion between them, is it possible to divide them into $k$ members each, such that every person in each group is compatible with every other person in the group. Also, note that a special case of $k$-cover is the $k$ dimensional matching problem, which is also NP-hard for $k \geq 3$. Some of the scheduling problems can also be modeled as an orgy problem. We are given $n$ jobs of length $\leq T$ seconds and $n/k$ machines. Also, for each job $j$, we are given a list of conflicting jobs which can not be scheduled with $j$ on the same machine. The problem is to schedule the jobs on the machines such that the total time to complete all the jobs is minimized.

Counting $k$-cliques in a web-graph has applications in social network analysis. In particular, this gives an estimate of the number of closed communities in the web-graphs. Therefore, fast algorithms for counting $k$-cliques in web-graphs give an insight to the evolution of Internet.

We consider template graphs which are vertex disjoint union of cliques. More specifically, we will be considering problems of counting cliques and
clique covers. We note that our techniques can be extended to counting embeddings of template graphs which are disjoint union of cliques of possibly different sizes. The counting version of the \(k\)-Clique problem is \#P-complete in general. The counting version of the \(k\)-Clique cover problem is \#P-complete even for \(k = 2\) (Valiant (25)), where \(H\) is a perfect matching.

Note that the counting versions of the aforementioned problems are extremely hard even for the simple cases. So, we try to come up with fully polynomial time approximation schemes (abbreviated as fpras) for these problems that work well for almost all graphs. More precisely, fpras must run in time \(\text{poly}(n, \varepsilon^{-1})\) and return an answer within a relative error of \((1 \pm \varepsilon)\) with high probability (i.e., probability tending to 1 as \(n \to \infty\)) for graphs that are uniformly randomly sampled from \(G \in \mathcal{G}(n, p)\). Here, \(\mathcal{G}(n, p)\) denotes the class of graphs in which each edge occurs with probability \(p\). Note that when \(p = \frac{1}{2}\), each graph \(G \in \mathcal{G}(n, p)\) is equiprobable. Another commonly studied model is \(\mathcal{G}(n, m)\) where each graph with \(n\) vertices and \(m\) edges is assigned the same probability, which is \(\binom{n}{m}^{-1}\).

There has been a lot of work in getting an fpras for counting perfect matchings in graphs. For a bipartite graph, it corresponds to calculating the permanent of a \(\{0, 1\}\) matrix. In the seminal paper of Valiant (25), it has been shown to be \#P-complete, even though the decision version of this problem is in P. Various approaches for getting an unbiased estimator with small variance have been explored for this problem. Some of these are determinant based approaches \((10, 14, 4, 18)\), Markov chain Monte Carlo (MCMC) algorithms \((3, 12, 13, 2)\) and search based on Rasmussen’s techniques \((20, 21, 8, 9)\). Chien (4) gives an efficient fpras for counting perfect matchings in random graphs. MCMC algorithms are polynomial time algorithms for all bipartite graphs. It is unclear to us whether both the techniques can be generalized to get unbiased estimators for counting \(k\)-cliques and \(k\)-clique covers. The estimators based on Rasmussen’s approach have also been proved to work well in random graphs, where they lead to simple, polynomial time approximation schemes. In this work, we generalize Rasmussen’s approach to efficiently count \(k\)-cliques and \(k\)-clique covers in random graphs.

We note here that Fürer and Kasivasanathan (9) have used similar techniques to get fpras for a large class of subgraph isomorphism problems. A fundamental constraint in their analysis was that the template subgraphs had to be triangle-free in most of the cases. In this work, we could avoid the requirement of triangle-freeness in our analysis and hence, get fpras for \(k\)-Clique and \(k\)-Clique cover problems.
The theory of random graphs was initiated by Erdős and Rényi ([6]). We work with the model $G(n, p)$ where we are given a fixed set of $n$ vertices and each of the $\binom{n}{2}$ edges is added with probability $p$.

Our analysis also provides an alternate derivation of the closed form of the $k^{th}$ moment of a binomial random variable $X$ sampled from Binomial($n, p$), which has been derived by Knoblauch ([16]) using moment generating function. We derive the same results using simple binomial equalities that we obtain using the binomial theorem.

1.1 Our results

In this work, we give the new results for $k$-clique counting problems, where $k \geq 4$ in random graphs. Our algorithm is based on the idea of Rasmussen’s unbiased estimator for permanents ([20]). It has been widely used in the context of subgraph isomorphism counting problems ([21, 8, 9]). For counting the number of $k$-cliques in the given random graph $G$, we embed a $k$-clique into $G$, doing so one vertex at a time chosen randomly. If the procedure succeeds, we compute the probability with which the clique is obtained in $G$ and output its inverse. As shown in [9], this is an unbiased estimate of the number of cliques in $G$. We state the results below in Theorem 1.1. Note that Theorem 1.1 partially answers an open question in the survey by Frieze and McDiarmid ([7]). Counting $k$-cliques for $k = \Omega(\sqrt{\log n})$ still remains open for random graphs.

**Theorem 1.1.** Let $H$ be a $k$-clique, where $k = O(\sqrt{\log n})$. Then, there exists an fpras for estimating the number of copies of $H$ in $G \in G(n, p)$ for constant $p$.

For counting $k$-clique cover, we embed one clique at a time, until the whole graph is covered by $k$-cliques. The key observation here is that after embedding a clique, the residual base graph still remains random with edge probability $p$. We obtain the following theorem for counting $k$-clique covers.

**Theorem 1.2.** Let $H$ be a $k$-clique cover, where $k = O(1)$. Then, there exists an fpras for estimating the number of copies of $H$ in $G \in G(n, p)$ for constant $p$.

Our estimators for counting cliques and clique-covers are given in Algorithm 1 and Algorithm 2 respectively in Section 3. As a side result, we obtain an alternate derivation of $\mathbb{E}[X^k]$ for a binomial random variable $X$, for all $k \geq 0$. 

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We note that this has already been obtained in [16] using the moment generating function for binomial random variable.

Outline of the paper: To introduce our techniques to the reader, we give a derivation for the closed form of $k$-th moment for binomial random variables using these techniques Section 2. We move on to describe estimators for counting $k$ cliques and $k$-clique covers in Section 3. We analyze these estimators for counting $k$-cliques and $k$-clique covers for random graphs in Section 4.1 and Section 4.2 respectively, which is the main contribution of this paper.

2 $k^{th}$ moment of a binomial random variable

Consider the binomial random variable $X = \text{binomial}(n, p)$. We are interested in finding the $k^{th}$ moment of $X$, i.e. we want to find $\mathbb{E}[X^k]$. In this section, we give the closed form expression for $\mathbb{E}[X^k]$. We evaluate using new equalities obtained from well known binomial theorem. Note that

$$\mathbb{E}[X^k] = \sum_{i=0}^{n} i^k \binom{n}{i} p^i (1-p)^{n-i}$$

We start with the most fundamental equality known as binomial theorem given below.

$$(1 + x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \quad (1)$$

Suppose we differentiate (1) with respect to $x$ and multiply by $x$ subsequently, we get the following equation.

$$nx(1 + x)^{n-1} = \sum_{i=0}^{n} i \binom{n}{i} x^i \quad (2)$$

Note that substituting $x = \frac{p}{1-p}$ in (2) and multiplying by $(1-p)^n$, we get $np = \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i}$, which is the first moment of $X$. Suppose we differentiate (2) w.r.t. $x$ again and multiply by $x$ subsequently, we get

$$x(1 + x)^{n-1} (n)_1 + x^2 (1 + x)^{n-2} (n)_2 = \sum_{i=0}^{n} i^2 \binom{n}{i} x^i \quad (3)$$

The term $(n)_i$ denotes the falling factorial $n \cdot (n-1) \cdot (n-2) \cdots (n-i+1) = \frac{n!}{(n-i)!}$. Again, substituting $x = \frac{p}{1-p}$ in (3) and multiplying $(1-p)^n$, we get
\[(n)_1 p + (n)_2 p^2 = \sum_{i=0}^{n} i^2 \binom{n}{i} p^i (1-p)^{n-i} = E[X^2].\] The above calculations show an emerging pattern for higher moments, which Lemma 2.1 illustrates.

**Lemma 2.1.**

\[
g(x, k) = \sum_{i=0}^{n} \binom{n}{i} x^i = \sum_{j=1}^{k} \lambda_{k,j} x^j (1+x)^{n-j}(n)_j
\] (4)

Here \(\lambda_{k,j}\) are the coefficients that depend on \(k\) and \(j\) but are independent of \(n\). Here \(0 \leq j \leq k\) \(\lambda_{k,0} = \lambda_{k,k+1} = 0\).

**Proof.** We will prove the above lemma by induction. For \(i = 1\), this is true as shown in (2). Suppose the lemma is true for \(g(x, 1), g(x, 2), \ldots, g(x, k)\). We prove that it holds for \(g(x, k+1)\). Differentiating (4) w.r.t. \(x\) and subsequently multiplying with \(x\) gives

\[
\sum_{i=0}^{n} i^{k+1} \binom{n}{i} x^i = \sum_{j=1}^{k} \lambda_{k,j} x^j (1+x)^{n-j} + (n-j)x^{j+1}(1+x)^{n-j-1}
\]

\[
= \sum_{j=1}^{k} \lambda_{k,j} x^j (1+x)^{n-j}(n)_j + \sum_{j=1}^{k} \lambda_{k,j} x^{j+1}(1+x)^{n-j-1}(n-j)(n)_j
\]

\[
= \sum_{j=1}^{k} \lambda_{k,j} x^j (1+x)^{n-j}(n)_j + \sum_{j=1}^{k} \lambda_{k,j} x^{j+1}(1+x)^{n-j-1}(n)_j
\]

\[
= \sum_{j=1}^{k+1} (j \lambda_{k,j} + \lambda_{k,j-1}) x^j (1+x)^{n-j}(n)_j
\]

\[
= \sum_{j=1}^{k+1} \lambda_{k+1,j} x^j (1+x)^{n-j}(n)_j
\] (5)

Note that the (5) shows that \(\sum_{i=0}^{n} i^{k+1} \binom{n}{i} x^i = \sum_{j=1}^{k+1} \lambda_{k+1,j} x^j (1+x)^{n-j}(n)_j\)

where \(\lambda_{k+1,j}\) follows the recurrence relation

\[
\lambda_{k+1,j} = j\lambda_{k,j} + \lambda_{k,j-1}.
\]

As given in [16], Stirling numbers of second kind follow this recurrence.

\[
\lambda_{k,j} = \frac{1}{j!} \sum_{j=0}^{i} \binom{i}{j} (-1)^j
\] (6)

\[\square\]
To get the $k^{th}$ moment, we simply substitute $x = \frac{p}{1-p}$ in (4) and multiply by $(1-p)^n$. Hence we have the following theorem.

**Theorem 2.2.**

$$E[X^k] = \sum_{j=1}^{k} \lambda_{k,j} p^j (n)_j$$

where $\lambda_{k,j}$ are as given in (6).

### 3 Estimators for counting $k$-cliques and $k$-clique covers in random graphs

In this section, we formally describe our estimators. The estimator for counting cliques in given in Algorithm 1. Note that it embeds the clique $\{v_1, \ldots, v_k\}$ and outputs the inverse of probability of embedding it in this way into $G$. The estimator embeds one vertex at a time until the whole clique is embedded. If the algorithm gets stuck, it outputs 0. This process can be viewed as decomposing the clique into subgraphs $C_1, C_2, \ldots, C_k$, where each $C_i$ is the subgraph induced by the $i^{th}$ numbered vertex $v_i$ and its lower numbered neighbors. It is denoted by $v_i$.

We denote our randomized estimator by $A$ and let $X$ be the output estimate. To get an fpras, we need that $E[A][X^2]/(E[A][X])^2$, also called the critical ratio, is polynomially bounded. We will bound a related quantity called critical ratio of averages given by $Cr(X) = E[G][E[A][X^2]]/(E[G][E[A][X]])^2$. Here, the outer expectation is over the graphs of $G(n, p)$ and the inner expectation is over the coin tosses of the estimator. Our focus in this work will be to get a bound on critical ratio of averages. As shown in Prop. 3.4, this will also give a polynomial bound on the critical ratio itself. The proof of Prop. 3.4 follows from Corollary 3.3 of Theorem 3.1 from [22].

Consider any induced subgraph $H_v$ of $H$ with $v$ vertices. Let $e_H(v) = \max_{H_v \subseteq H} \{|E(H_v)|\}$ of edge For stating the results, we need to define the following ratio for the template graph $H$.

$$\gamma = \gamma(H) = \max_{3 \leq v \leq n} \{e_H(v)/(v-2)\}.$$ 

Note that $\gamma$ is closely related to the largest possible average degree of an induced subgraph of $H$. In our case, this is $O(\sqrt{\log n})$ for the case of counting cliques and $O(1)$ for counting clique covers. Let $C = C_H(G)$ denote the number of copies of $H$ in $G$. 


Theorem 3.1 ([22]). Let $H$ be a graph on $n$ vertices and $\gamma$ be as defined above. Let $p$ be a constant. Suppose that the following conditions hold: $p \cdot \binom{n}{2} \to \infty, \sqrt{n}(1 - p) \to \infty$ and $np^\gamma / \Delta^4 \to \infty$. Then, with high probability, a random graph $G \in \mathcal{G}(n, p \cdot \binom{n}{2})$ has a spanning subgraph isomorphic to $H$. In general, $C = C_H(G)$ satisfies
\[
\frac{E[C^2]}{E[C]^2} = 1 + o(1).
\]

Remarks. Note that Theorem 3.1 holds for the spanning subgraphs of the random graphs. This assumption can easily be incorporated while embedding a single clique at any step. While embedding each clique, $H$ is considered to be the $n$ vertex graph which is the disjoint union of a clique and the isolated vertices in both the cases. Also, note that $np^\gamma / \Delta^4 \to \infty$ since $\gamma$ and $\Delta$ are both bounded by $O(\sqrt{\log n})$. Therefore, all conditions of Theorem 3.1 are satisfied in our case. So we get the following corollary in our case.

Corollary 3.2. Let $G \in \mathcal{G}(n, \Omega(n^2))$ and $H$ be one of the following graphs
\begin{itemize}
  \item[(a)] a clique of size $O(\sqrt{\log n})$
  \item[(b)] a cover of cliques of constant size,
\end{itemize}
Then $E[C^2] / E[C]^2 = 1 + o(1)$, where $C$ denotes the number of copies of $H$ in $G$.

From the asymptotic equivalence between $\mathcal{G}(n, p)$ and $\mathcal{G}(n, m)$ (see e.g. [11, 20]), we have the following corollary.

Corollary 3.3. Let $G \in \mathcal{G}(n, p)$ and $H$ be one of the following graphs
\begin{itemize}
  \item[(a)] a clique of size $O(\sqrt{\log n})$
  \item[(b)] a cover of cliques of constant size,
\end{itemize}
Then $C \geq E[C] / \omega$, where $\omega = \omega(n)$ be a real valued function that goes to $\infty$ as $n \to \infty$.

Theorem 3.1 along with Corollary 3.2 and Corollary 3.3 yield the following proposition. The proof is identical to the one given for a similar proposition in [9], but we give it make the write-up self sufficient.

Proposition 3.4. Let $G \in \mathcal{G}(n, p)$ and $H$ be one of the following graphs
\begin{itemize}
  \item[(a)] a clique of size $O(\sqrt{\log n})$
  \item[(b)] a cover of cliques of constant size.
\end{itemize}
Let $X$ be the output of Algorithm Embeddings, and let $p$ be a constant. Then, for a random graph $G \in \mathcal{G}(n, p)$ the critical ratio satisfies
\[
E[X^2] / (E[X])^2 \leq \omega^3 \frac{E[D_4[X^2]]}{(E[D_4[X]])^2},
\]
where $\omega = \omega(n)$ such that $\omega \to \infty$ as $n \to \infty$. 

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Proof. For the unbiasted estimator $A$, we have $C = \mathbb{E}_A[X]$. Therefore, from Corollary 3.3, we have that $C = \mathbb{E}_A[X] \leq \mathbb{E}_G[\mathbb{E}_A[X]]/\omega$ with high probability. Also, from Markov’s inequality we have $\Pr[\mathbb{E}_A[X^2] > \omega \mathbb{E}_G[\mathbb{E}_A[X^2]]] \leq 1/\omega$. Therefore with probability at least $1 - 1/\omega$, we have $\mathbb{E}_A[X^2] \leq \omega \mathbb{E}_G[\mathbb{E}_A[X^2]]$. Our result follows from these inequalities.

In the rest of the paper, we focus on bounding the critical ratio of averages. The estimator for counting $k$-cliques is given in Algorithm 1. It embeds one clique of size $k = O(\sqrt{\log n})$ in $G$ and outputs the inverse of probability of embedding. This is done by the procedure EMBED-Clique, which is called only once in this case.

**Algorithm 1** Count-cliques($G, k$)

1: **procedure** EMBED-Clique($G, k$)
2: $i \leftarrow 0$ \quad \triangleright $i$ denotes the number of nodes already embedded in $G$
3: $v_1 \leftarrow$ ArbitraryNode($G$) \quad \triangleright Arbitrarily assign a node from $G$ to $v_0$
4: **while** $i < k$ **do**
5: $\mathcal{N}_i \leftarrow$ CommonNeighbors($\{v_1, \ldots, v_i\}$)
6: \quad **if** $\mathcal{N}_i = \emptyset$ **then**
7: \quad \quad $X \leftarrow 0$ \quad \triangleright Embedding algorithm has failed; so terminate
8: \quad **end if**
9: $X_i \leftarrow |\mathcal{N}_i|$
10: $v_{i+1} \leftarrow$ RandomNode($\mathcal{N}_i$) \quad \triangleright uniformly randomly assign a node
11: \quad \quad from $\mathcal{N}_i$ to $v_{i+1}$
12: $X \leftarrow X \cdot X_i$
13: $i \leftarrow i + 1$
14: **end while**
15: **return** $X/(k!)$ \quad \triangleright Estimator outputs unbiased estimate of number of $k$-cliques
16: **end procedure**

The estimator for counting $k$-clique covers of $G$ is given in Algorithm 2. It uses the procedure EMBED-Clique described in Algorithm 1 to embed each $k$-clique in the cover. This process is sequentially repeated until all the vertices are covered. In the end, it returns the inverse of probability of finding the cover, if successful. Note that this is the product of the probabilities of embedding the individual cliques in the cover.
Algorithm 2 Count-clique-covers\((G, k)\)

1: \(G_{\text{res}} \leftarrow G\)
2: \(a \leftarrow (k!)^{\frac{n}{k}} \cdot \left(\frac{n}{k}\right)!\) \hspace{1em} \triangleright \text{Size of the automorphism group of } k\text{-clique cover}
3: \(X \leftarrow 1\)
4: \textbf{while} \(G_{\text{res}} \neq \emptyset\) \textbf{do}
5: \hspace{1em} \(X \leftarrow X \cdot \text{Embed-Clique}(G_{\text{res}}, k)\)
6: \hspace{1em} \textbf{if} \ \text{Embed-Clique}(G_{\text{res}}, k) = 0 \ \textbf{then}
7: \hspace{2em} \(X \leftarrow 0\) \hspace{1em} \triangleright \text{Embedding algorithm has failed; so terminate}
8: \hspace{1em} \textbf{end if}
9: \hspace{1em} \(G_{\text{res}} \leftarrow G \setminus \{v_1, \ldots v_k\}\) \hspace{1em} \triangleright \text{Remove the currently embedded clique}
10: \hspace{1em} \{v_1, v_2, \ldots, v_k\} \text{ from } G \text{ to get } G_{\text{res}}\)
11: \textbf{end while}
12: \textbf{return} \(X/a\)

4 Analysis of estimator for counting cliques and clique-covers in random graphs

In this section, we show a polynomial bound on the critical ratio of averages for the estimators in Algorithm 1 and Algorithm 2. Note that from Prop. 3.4, this is sufficient to bound the critical ratio of the estimator and hence get an fpras for counting \(k\)-cliques (for \(k = O(\sqrt{\log n})\)) and \(k\)-clique covers (for \(k = O(1)\)) in random graphs.

4.1 Counting Cliques

In this section, we prove Theorem 1.1. In this case, the estimator embeds a single clique onto the base graph and outputs the inverse of probability of embedding the same. Let \(X\) be the output of the estimator. The estimator selects first vertex in the graph arbitrarily and embeds one edge at a time until the whole clique is embedded. It outputs the inverse of probability of embedding if it goes through, else it outputs 0.

Let \(X_j\) corresponds to the number of ways to embed vertex \(j\) in the residual graph. Note that \(X = X_1 \cdot X_2 \cdots X_k\). Now consider the term \(\text{Cr}(X) = \mathbf{E}_G[|\mathbf{E}_A[X]^2|/\mathbf{E}_G[|\mathbf{E}_A[X]|]]^2\).

To estimate the critical ratio of averages, we need the definition of \(k\)-nesting, denoted by \(N(k, n, p)\), as follows.

Definition 4.1 (\(k\)-nesting). A \(k\)-nesting is a function \(N(k, n, p)\) that can be evaluated in the following recursive way.
(i) The 2-nesting is defined as

\[ N(2, n, p) = n^2 \left( \sum_{i=0}^{n-1} i^2 \binom{n-i}{i} p^i (1 - p)^{n-1-i} \right) \]

(ii) The k-nesting is defined as

\[ N(k, n, p) = n^2 \left( \sum_{i=0}^{n-1} N(k-1, i, p) \binom{n-i}{i} p^i (1 - p)^{n-1-i} \right) \]

Note that the embedding of a k-clique can be thought of as embedding the \( i^{th} \)-vertex to get an \( i \)-clique from \( i-1 \)-clique for each \( i \in \{1, 2, \ldots, k\} \). So, we have the following equality.

\[ E_G[E_A[X_1^2 X_2^2 \cdots X_k^2]] = N(k, n, p) \quad (7) \]

Lemma 4.2 shows the exact structure of \( N(k, \ell, p) \), which we use in getting the bound on the critical ratio.

**Lemma 4.2.**

\[ N(k, \ell, p) = \sum_{j=k}^{2k-1} \ell(\ell)_{f_{k,j}}(p) \]

Here \( f_{k,j}(p) \) is a function in \( k, j, p \) that is independent of \( \ell \) with the following properties.

(i) \( f_{k,i}(p) = 0 \) for all \( i \in \{1, \ldots, k\} \) and \( f_{k,2k+i} = 0 \) for all \( i \geq 0 \)

(ii) \( f_{k+1,j}(p) = p^{j-1} ((j-1)f_{k,j-1}(p) + f_{k,j-2}(p)) \)

**Proof.** We prove this by induction on \( k \). For the base case, i.e. for \( k = 2 \) this is

\[
N(2, \ell, p) = \ell^2 \left( \sum_{i=0}^{\ell-1} i^2 \binom{\ell-1}{i} p^i (1 - p)^{\ell-1-i} \right) \
= \ell^2(\ell - 1)p + \ell^2(\ell - 1)2p^2 \text{ (using [4] with } k = 2) \
= \ell(\ell)_{2p^2(2)} + \ell(\ell)_{3p^2(2)}
\]
Suppose the claim is true for \( N(i, \ell, p) \) for \( i = \{1, 2, \ldots, k\} \). We will show that the claim is true for \( i = k + 1 \). From [Definition 4.1] we have

\[
N(k + 1, \ell, p) = \ell^2 \sum_{m=0}^{\ell-1} N(k, \ell, p) \left( \frac{\ell - 1}{m} \right) p^m (1 - p)^{\ell - 1 - m}
\]

\[
= \ell^2 \sum_{m=0}^{\ell-1} \sum_{j=k}^{2k-1} (m(m)_j f_{k,j}(p)) \left( \frac{\ell - 1}{m} \right) p^m (1 - p)^{\ell - 1 - m}
\]

\[
= \sum_{j=k}^{2k-1} \sum_{m=0}^{\ell-1} \left( \ell^2 (m)_j \left( \frac{\ell - 1}{m} \right) p^m (1 - p)^{\ell - 1 - m} \right) f_{k,j}(p) \quad \text{(interchanging the summations)}
\]

\[
= \sum_{j=k}^{2k-1} (j \cdot \ell^2 (\ell - 1)_j p^j + \ell^2 (\ell - 1)_{j+1} p^{j+1}) f_{k,j}(p) \quad \text{(from Claim 4.3)}
\]

\[
= \sum_{i=k+1}^{2k+1} (j \cdot \ell (\ell - 1)_i p^i + \ell (\ell - 1)_{i+1} p^{i+1}) f_{k,i}(p) \quad \text{(using } \ell (\ell - 1)_i = (\ell)_{i+1}, j + 1 = i, f_{k,2k}(p) = 0)\)
\]

\[
= \sum_{i=k+1}^{2k+1} p^{i-1}((i - 1)f_{k,i-1}(p) + f_{k,i-2}(p))\ell(i)_i \quad \text{(rearranging the terms and using } f_{k,k-1}(p) = 0)\)
\]

\[
= \sum_{i=k+1}^{2k+1} f_{k+1,i}(p)\ell(i)_i \quad \text{(rearranging the terms and using } f_{k,k-1}(p) = 0)\)
\]

(8)

The following claim is used in the proof of [Lemma 4.2]

Claim 4.3.

\[
\sum_{m=j}^{n} m(m)_j \left( \frac{n}{m} \right) x^{m-j} = j(n)_j (1 + x)^{n-j} + (n)_{j+1} x (1 + x)^{n-j-1} \quad \text{(9)}
\]

In particular, if we multiply (9) by \( x^j (1 - p)^n \) and substitute \( x = p/(1 - p) \) we get

\[
\sum_{m=j}^{n} m(m)_j \left( \frac{n}{m} \right) p^m (1 - p)^{n-m} = j(n)_j p^j + (n)_{j+1} p^{j+1}
\]
Proof. We prove the identity in \([9]\) using induction. For \(j = 0\) (base case) we need to show that \(\sum_{m=0}^{n} m \binom{n}{m} x^m = nx(1+x)^{n-1}\), which holds from \([2]\). For hypothesis, assume that \([9]\) holds for \(j\). We prove that it also holds for \(j + 1\) as follows. Differentiating \([9]\) w.r.t. \(x\) gives

\[
\sum_{m=j+1}^{n} m(m-j)(m-j)x^{m-j-1} = \sum_{m=j+1}^{n} m(m-j)x^{m-j} = \sum_{m=j+1}^{n} m(m-j+1)x^{m-j+1}
\]

\[
\sum_{m=j+1}^{n} m(m-j)x^{m-j} = j(n-j)(1+x)^{j-1}(1+x)^{n-j-1} + (n-j-1)(n-j+1)x(n-j)x^{n-j-2}
\]

\[
\sum_{m=j+1}^{n} m(m-j+1)x^{m-j+1} = \sum_{m=j+1}^{n} m(m-j+1)x^{m-j} = j(n-j+1)(1+x)^{j-1} + (n-j+1)x(n-j)x^{n-j-1} + (n-j+2)x(1+x)^{n-j-2}
\]

Hence the identity holds for \(j + 1\).

The following claim upper bounds \(f_{k,k+i}(p)\) for \(0 \leq i \leq k - 1\)

**Claim 4.4.** For \(k \geq 2\) \(f_{k,2k-i-1}(p) \leq k^{2i}p^{(k)}(\frac{k}{2})^{i} \) where \(0 \leq i \leq k - 1\).

**Proof.** We will prove this claim using induction on \(k\). Consider \(k = 2\) for the base case. From **Definition 4.1** we have \(N(2, n, p) = n(n)p + n(n)^2p^2\). So, the claim holds. Now assume that the claim holds for all clique sizes up to \(k - 1\). Now, from **(8)** we have the following recurrence relation.

\[
f_{k,i}(p) = p^{i-1}(i-1)f_{k-1,i-1}(p) + f_{k-1,i-2}(p)
\]

First we prove for \(i \geq 1\). Using **(10)** we have

\[
f_{k,2k-i-1}(p) = p^{2k-i-2}(2(k-1) - i)f_{k-1,2(k-1)-(i-1)-1}(p) + f_{k-2,2(k-1)-i-1}(p)
\]

\[
\leq p^{2k-i-2}(2(k-1) - i)(k-1)^{2(i-1)}p^{(k-1)}(\frac{k-1}{2})^{i-1} + (k-1)^{2i}p^{(k-1)}(\frac{k-1}{2})^{i-1}
\]

\[
= (k-1)^{2i}p^{(k-1)}(\frac{k-1}{2})^{i-1} \left( 1 + p^{k-i-1} \left( \frac{2}{k-1} - \frac{i}{(k-1)^2} \right) \right)
\]

\[
\leq (k-1)^{2i}p^{(k-1)}(\frac{k-1}{2})^{i-1} \left( 1 + \frac{2}{k-1} \right)
\]

\[
= \left( (k-1)^{2i} + 2(k-1)^{2i-1} \right) p^{(k-1)}(\frac{k-1}{2})^{i} \leq k^{2i}p^{(k)}(\frac{k-1}{2})^{i} \quad \text{(for } i \geq 1)\]

Now we show that \(f_{k,2k-1} = p^{(k)(2)}\). From **(10)** we have \(f_{k,2k-1}(p) = p^{2(k-1)}(2k-2)f_{k-1,2k-2}(p) + f_{k-1,2k-3}(p) = p^{2(k-1)}f_{k-1,2k-3}(p)\) since \(f_{k-1,2k-2}(p) = 0\). Applying the recurrence repeatedly, we get the desired relation. ☐
Now we bound $Cr(X)$ which is same as \( \frac{N(k,n,p)}{(n)_{k,p}(\frac{1}{2})^2} \). We have

\[
Cr(X) = \frac{N(k,n,p)}{((\ell)_{k,p}(\frac{1}{2}))^2} = \sum_{j=k}^{2k-1} \frac{n(n)_{i,f,j}(p)}{(n)_{k,p}(\frac{1}{2})^2} = \sum_{i=0}^{k-1} \frac{n(n)_{2k-i-1,f,2k-1-i}(p)}{(\ell)_{k,p}(\frac{1}{2})^2}.
\]

Lemma 4.5 immediately proves Theorem 1.1.

**Lemma 4.5.** For $k = O(\sqrt{\log n})$, $Cr(X) = \sum_{i=0}^{k-1} \frac{n(n)_{2k-i-1,f,2k-1-i}(p)}{(\ell)_{k,p}(\frac{1}{2})^2}$ is upper bounded by $\text{poly}(n)$.

**Proof.** Consider the ratio $\frac{\ell(\ell)_{2k-i-1,f,2k-i-1}(p)}{(\ell)_{k,p}(\frac{1}{2})^2}$ for a fixed $i$. For $i = 0$, this is $\frac{\ell(\ell)_{2k-i}(p)}{(\ell)_{k,p}(\frac{1}{2})^2}$ since $f_{2k-1} = p^2(\ell)$. Note that $\frac{\ell(\ell)_{2k-i}(p)}{(\ell)_{k,p}(\frac{1}{2})^2} \leq 1$. Now we consider $i \geq 1$.

\[
\frac{\ell(\ell)_{2k-1-i,f,2k-1-i-1}(p)}{(\ell)_{k,p}(\frac{1}{2})^2} = \left( \prod_{j=1}^{k-i-1} \frac{(\ell - (k - 1))}{(\ell - j)} \right) \left( \frac{f_{2k-i-1}(p)}{\prod_{r=1}^{i} (\ell - k - r)} \right) \frac{1}{p^{2(\ell)}}
\]

\[
\leq \left( \frac{\ell - k}{\ell - 1} \right)^{k-i-1} \left( \frac{k^2}{(\ell - k + 1)^i} \right) \frac{1}{p^{2(\ell)}}
\]

\[
= \left( \frac{\ell - k}{\ell - 1} \right)^{k-i-1} \left( \frac{k^2}{\ell - k + 1} \right)^i \frac{1}{p^{(k^2) - (k - 1)^2}}
\]

\[
= \left( \frac{\ell - k}{\ell - 1} \right)^{k-i-1} \left( \frac{k^2}{\ell - k + 1} \right) \left( \frac{1}{p} \right)^{k-(\frac{k+1}{2})} = h(i) \quad (12)
\]

The first inequality above uses Claim 4.4: $\frac{k-j}{\ell-j} \geq \frac{k-j}{\ell-1-j}$ and $\ell - k + j \geq \ell - k + 1$ for all $1 \leq j \leq k - 1$. Note that $\left( \frac{\ell-j}{\ell-1-j} \right)^{k-i-1} \leq 1$. So, we have $h(i) \leq \left( \frac{k^2}{\ell-k+1} \left( \frac{1}{p} \right)^{k-(\frac{k+1}{2})} \right)^i$, where $h(i)$ is as defined in (12). Note that for $i = O(\sqrt{\log n})$, \( \left( \frac{k^2}{\ell-k+1} \left( \frac{1}{p} \right)^{k-(\frac{k+1}{2})} \right)^i \) is polynomially bounded for all $2 \leq i \leq k - 1$. Therefore $Cr(X)$ is polynomially bounded. \( \Box \)
4.2 Clique cover counting

As noted earlier in [Prop. 3.4] we focus on bounding the critical ratio of averages given by \( \text{Cr}(X) = \frac{\mathbb{E}_{G}[E_A[X]^2]}{(\mathbb{E}_{G}[E_A[X]])^2} \) for Algorithm 1.

The estimator embeds one clique at a time, by selecting a vertex at random at first and then embedding each edge till \( k \) vertices of the clique are embedded. A crucial observation is that the residual graph, after embedding a clique still remains random with edge probability \( p \). Now, the estimator sequentially embeds \( n/k \) cliques to get the clique cover and outputs the inverse of probability of getting this clique cover, if the embedding procedure goes through, otherwise it outputs 0. Note that this is the product of the inverse of the probabilities for embedding each clique. Let \( K_i \) denote the random variable corresponding to the estimate of the number of embeddings of the \( i \)th clique in the residual graph, which is a random graph from \( \mathcal{G}(n - ki - k, p) \). Note that \( K_i \) is independent from \( K_j \) for \( i \neq j \) and \( X = K_1 \cdot K_2 \cdots K_{n/k} \). Therefore we have the following equation.

\[
(\mathbb{E}_{G}[E_A[X]])^2 = \prod_{i=1}^{n} (E[K_i])^2
\]

(13)

Note that the equality follows from the fact that after embedding each \( k \)-clique, the residual graph still remains random with edge probability \( p \). Now, we bound the numerator, i.e., \( \mathbb{E}_{G}[E_A[X^2]] \).

\[
\mathbb{E}_{G}[E_A[X^2]] = \mathbb{E}_{G}[E_A[K_1^2 K_2^2 \cdots K_{n/k}^2]] = \mathbb{E}_{G}[E_A[K_1^2]] \cdot \mathbb{E}_{G}[E_A[K_2^2]] \cdots \mathbb{E}_{G}[E_A[K_{n/k}^2]]
\]

(14)

Let \( X_j \) corresponds to the number of ways to embed vertex \( j \) in the residual graph. Note that \( K_i = X_{ki-k+1} \cdot X_{ki-k+2} \cdots X_{ki} \).

Now consider the term \( \mathbb{E}_{G}[E_A[K_1^2]] = \mathbb{E}_{G}[E_A[X_{k(i-1)+1}^2 X_{k(i-1)+2}^2 \cdots X_{ki}^2]] \).

Note that in this case, we have

\[
\text{Cr}(K_i) = \frac{\mathbb{E}_{G}[E_A[X_{k(i-1)+1}^2 X_{k(i-1)+2}^2 \cdots X_{ki}^2]]}{\mathbb{E}_{G}[E_A[X_{ki-k+1} X_{ki-k+2} \cdots X_{ki-k+k}]]} = N(k, n - ki + k, p)
\]

(15)

We show in Lemma 4.6 that \( \frac{N(k, \ell, p)}{\binom{n}{\ell}} \) is bounded by \( 1 + O\left(\frac{1}{n - ki + 1}\right) \) for all \( i \in \{1, 2, \ldots, n/k\} \), where \( \ell = n - ki + k \).

**Lemma 4.6.** For large \( \ell \), constant \( k \) and constant \( p \) we have

\[
\text{Cr}(K_i) = \sum_{j=0}^{k-1} \frac{\ell(\ell)_{2k-j-1} j_{k, 2k-j-1}^2(p)}{(\ell)_{k}^2} \leq 1 + O\left(\frac{1}{\ell - k + 1}\right)
\]
Proof. Consider the ratio \( \frac{\ell(\ell)_{2k-j-1}f_{k,2k-j-1}(p)}{(\ell)_{k}p_{\ell}^{(k)}} \) for a fixed \( j \). For \( j = 0 \), this is \( \frac{\ell(\ell)_{2k-1}}{(\ell)_{k}^{2}} \) since \( f_{k,2k-1} = p^{2(\ell)} \). Note that \( \frac{\ell(\ell)_{2k-1}}{(\ell)_{k}^{2}} \leq 1 \). Now we consider \( j \geq 1 \). As shown in \([12]\), we have

\[
\ell(\ell)_{2k-1-j}f_{k,2k-j-1}(p) \leq h(j) = \left( \frac{\ell - k}{\ell - 1} \right)^{k-j-1} \left( \frac{k^2}{\ell - k + 1} \left( \frac{1}{p} \right)^{k-(\frac{j+1}{2})} \right)^{j}
\]

To prove the lemma, we handle the cases of \( j \leq 2 \) and \( j \geq 2 \) separately. First we handle the latter case. For \( j \geq 2 \), we prove that \( h(j) \leq \frac{1}{(k-2)!(\ell-k+1)} \). In other words, we prove that \( \log h(j) + \log(k-2) + \log(\ell - k + 1) < 0 \) for constant \( k \).

Let \( y(j) = \log(h(j)) = (k-1-j)(\log(\ell-k) - \log(\ell-1)) + j(2\log k - \log(\ell-k + 1)) + j \left( k - \frac{(j+1)}{2} \right) \log \frac{1}{p} \). Consider the continuous function \( y(x) = (k - 1 - x)(\log(\ell-k) - \log(\ell-1)) + x(2\log k - \log(\ell-x + 1)) + x \left( k - \frac{(x+1)}{2} \right) \log \frac{1}{p} \). Therefore we have

\[
y'(x) = \frac{\partial y(x)}{\partial x} = \log(\ell - 1) - (\log(\ell-k) + \log(\ell-k + 1)) + 2\log k + \left( k - x + \frac{1}{2} \right) \log \frac{1}{p} \leq \log(\ell - 1) - 2(\log(\ell-k)) + 2\log k + \left( k - x + \frac{1}{2} \right) \log \frac{1}{p}
\]

Observe that for large \( \ell \) and for constant \( k \), the term \(-2(\log(\ell-k))\) dominates all the other terms, so \( y'(x) < 0 \) for \( 1 \leq x \leq k - 1 \). Therefore \( y(x) \) is a decreasing function. We analyze cases \( j = 1 \) and \( j \geq 2 \) separately. First we analyze latter case. We prove that \( y(2) \leq \log \left( \frac{1}{(k-2)!(\ell-k+1)} \right) \), which implies that \( y(j) = \log \left( \frac{1}{(k-2)!(\ell-k+1)} \right) \) for \( 2 \leq j \leq k - 1 \). This proves that \( h(i) = \frac{1}{(k-2)!(\ell-k+1)} \) for \( j \geq 2 \), eventually proving that

\[
\sum_{j=2}^{k-1} \frac{\ell(\ell)_{2k-j-1}f_{k,2k-j-1}(p)}{(\ell)_{k}p_{\ell}^{(k)}} \leq \frac{1}{\ell - k + 1}
\]

Consider the function \( g(\ell) = y(2) + \log(k-2) + \log(\ell - k + 1) \). So we
have
\[ g(\ell) = (k - 3)(\log(\ell - k) - \log(\ell - 1)) + 2(2\log k - \log(\ell - k + 1)) \\
+ (2k - 3) \log \frac{1}{p} + \log(\ell - k + 1) + \log(k - 2) \leq 4\log k + (2k - 3) \log \frac{1}{p} + \log(k - 2) - \log(\ell - k + 1) \]

Note that for constant \( k \), this is smaller than 0 for large enough \( \ell \). Therefore \( g(\ell) < 0 \), hence the claim.

Now we do the analysis for \( j = 1 \). We calculate \( f_{k,2k-2}(p) \) using the recurrence.

\[
f_{k,2k-2}(p) = p^{2k-3} \left( (2k - 3)f_{k-1,2(k-1)-1}(p) + f_{k-1,2(k-1)-2}(p) \right) \\
= (2k - 3)p^{2k-3+2^{(k-1)}} + p^{2k-3}f_{k-1,2(k-1)-2}(p) \text{ (using } f_{k-1,2(k-1)-1} = p^{2^{(k-1)}}) \\
= (2k - 3)p^{2^{(k-1)}-1} + p^{2k-3+2k-5} \left( (2k - 5)f_{k-2,2(k-2)-1}(p) + f_{k-2,2(k-2)-2}(p) \right) \\
= (2k - 3)p^{2^{(k-1)}-1} + (2k - 5)p^{2^{(k-2)}} + p^{2k-3+2k-5}f_{k-2,2(k-2)-2}(p)
\]

Going on as shown in the above equation, we get \( f_{k,2k-2}(p) = \sum_{m=1}^{k-1} (2(k - m) - 1)p^{2^{(k-2)}-m} \). Therefore we have

\[
\frac{\ell(\ell)_{2k-2}}{(\ell)_{k}p^{(k)}} = \frac{\ell(\ell)_{2k-2}}{(\ell)_{k}} \left( \sum_{m=1}^{k-1} \frac{2(k - m) - 1}{p^{m}} \right)
\]

Note that

\[
\left( \sum_{m=1}^{k-1} \frac{2(k - m) - 1}{p^{m}} \right) = \frac{1}{p - 1} \left( 2 \left( \frac{1}{p} - 1 \right) + \frac{1}{p^{k}} - \left( \frac{2k + 2p - 1}{p} \right) \right) \\
\leq \frac{C}{p^{k}} \text{ (for large enough constant } C) \quad (17)
\]

Note that for constant \( k \), \( \frac{C}{p^{k}} = C' \) is a constant. Therefore, using (16) and (17) we have

\[
\sum_{j=0}^{k-1} \frac{\ell(\ell)_{2k-j-1}f_{k,2k-j-1}(p)}{(\ell)_{k}p^{(k)}} \leq 1 + \left( \frac{C' + 1}{\ell - k + 1} \right)
\]

Hence the lemma. \( \Box \)
Note that Lemma 4.6 shows that $\text{Cr}(K_i) = \mathbb{E}_G[\mathbb{E}_{A}[K_i^2]]/\mathbb{E}_G[\mathbb{E}_{A}[K]]^2 = 1 + O\left(\frac{1}{n-k+i}\right)$. Note that Theorem 1.2 follow from Lemma 4.6 since $\prod_{i=1}^{\tilde{k}} \text{Cr}(K_i) = \text{poly}(n)$ in this case.

5 Conclusion and open problems

In this work, we show the first fpras for counting $k$-cliques, where $k = O(\sqrt{\log n})$ and $k$-clique covers (for constant $k$) in random graphs, using the unbiased estimators that are very simple to describe. Both problems are $\#P$-complete in general for the respective values of $k$. Getting a fpras for these problems over general graphs is a long standing open problem. Here are some specific open problems that we think are worth investigating.

1. The problem counting clique is still open for counting cliques of size $\Omega(\sqrt{\log n})$. Solving this will resolve the open problem of Frieze and McDiarmid (\cite{7}) completely.

2. Another specific problem to resolve here is to count clique covers of superconstant sized cliques.

3. The determinant based estimators usually have smaller worst case running times in fpras (e.g. \cite{4}) for random graphs. It is unclear to us how to obtain any determinant based unbiased estimators for the clique and clique cover counting problems.

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