Charge Gravastars in $f(T)$ Modified Gravity

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Abstract

In the present work, we have discussed the four dimensional spherically symmetric stellar system in the framework of modified $f(T)$ gravity theory with electro-magnetic field. The field equations have been written for two cases, either $T' = 0$ or $f_{TT} = 0$. Next we have discussed the charged gravastar model which has three regions: interior region, shell region and exterior region. In the interior region, we have found the solutions of all physical quantities like density, pressure, electro-magnetic field and also the metric coefficients for both the cases. For $T' = 0$, gravastar cannot form but it forms only for the case $f_{TT} = 0$. In the exterior region, we have obtained the exterior solution for vacuum model. In the shell region, we have assumed that the interior and exterior regions join together at a place, so the intermediate region must be thin shell with the approximation $h \ll 1$. Under this approximation, we have found the analytical solutions. The proper length of the thin shell, entropy and energy content inside the thin shell have been found and they are directly proportional to the proper thickness of the shell $\epsilon$ under the approximation ($\epsilon \ll 1$). According to the Darmois-Israel formalism, we have studied the matching between the surfaces of interior and exterior regions of the gravastar. The energy density, pressure, equation of state parameter on the surface and mass of the thin shell have been obtained.

1 Introduction

In 2001, Mazur and Mottola [1] [2] have found a solution for the gravitationally collapsing neutral system in the concept of Bose-Einstein condensation to gravitational systems. These describe as super compact, spherically symmetric and singularity free objects, that can be considered to be virtually as compact as the black holes. These gravitationally dark cold vacuum compact star is known as gravastar (“gravitational vacuum condensate stars”). The gravastar is a substitute of black hole i.e., the existence of compact stars minus event horizons, which compresses the matter within the gravitational radius $R = \frac{4GM}{c^2}$, that is very close to the Schwarzschild radius. In this sense, the existence of quantum vacuum fluctuations are predicted near the event horizon. The gravastar consists of an (i) interior de Sitter condensate phase and (ii) exterior Schwarzschild geometry. The gravastar consist of five layers where infinitely two thin shells, apparently, on the regions $r = r_1$ and $r = r_2$ where $r_1$ and $r_2$ are inner and outer radii ($r_1 < r_2$). Also other important regions are (i) Interior region: $0 \leq r < r_1$ with equation of state (EOS) $p = -\rho$ which is defined by the de-Sitter spacetime, (ii) Shell region: $r_1 < r < r_2$ with EOS $p = \rho$ i.e., an intermediary thin layer made of ultra-stiff perfect fluid, (iii) Exterior region: $r_2 < r$ with EOS $p = \rho = 0$ (vacuum) which is described by the Schwarzschild solution. Thus the interior region of the gravastar is surrounded by a thin shell of ultra-relativistic matter whereas the exterior region is completely vacuum. Also, if we replace two infinitely thin stiff shells and the intermediary region with just one infinitely thin shell core [3], then the five layer models can be simplified to the three layer models.

There are lot of works on the gravastar models available in the literature in the framework of Einstein’s general relativity. DeBenedictis et al [4] have found gravastar solutions with continuous pressures and equation of state. Cattoen et al [5] have taken anisotropic pressure in the formation of gravastar. Born-infeld phantom gravastars have been discussed by Bilic et al [6]. Gravastar model in higher dimensional spacetime has been discussed in refs [7] [8] [9] [10]. Gravastar in the frame of conformal motion has also been analyzed by some authors [8] [11] [12]. Physical features of charged gravastar have been investigated [8] [9] [11] [13] [14] [15] [16] [17]. Stable gravastar is discussed by several authors [15] [19] [20] [21] [22].

One important family of modifications of Einstein-Hilbert action is the $f(R)$ theories of gravity [23] [24] [25] [26]. In such theories, one can use a function of curvature scalar as the Lagrangian density. In the similar line, one can also modify teleparallel equivalent of general relativity where Lagrangian density is equivalent to the torsion scalar $T$ and the field equations of teleparallel gravity [27] [28] [29] [30] [31].
are identical with the Einsteins field equations in any background metric. After that it has been modified teleparallel gravity by having a Lagrangian density equivalent to a function of torsion scalar i.e., $f(T)$ gravity [32]. In theoretical astrophysics, $f(T)$ version of 3-dimensions of BTZ black hole solutions has been calculated as $f(T)$ gravity theory was supported for examining the effects of $f(T)$ gravity models [33]. Recently, static solutions in the spherically symmetric charged source in $f(T)$ gravity have been found [34]. Deliduman et al [35] have investigated the structure of gravastar gravity by having a Lagrangian density equivalent to a function of $f(T)$ gravity with electromagnetic field. In section 3, $f(T)$ gravity have been found [41]. Abbas and his collaborations [42, 43, 44, 45] have discussed the anisotropic gravity theory was modified teleparallel action by replacing $T$ with a function $f(T)$ [40, 50] as follows

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi} f(T) + \mathcal{L}_{\text{Matter}} \right]$$

where we choose $G = c = 1$ and $\mathcal{L}_{\text{Matter}}$ is the matter Lagrangian.

The ordinary matter is an anisotropic fluid so that the energy-momentum tensor is given by

$$T_{\mu\nu}^{\text{matter}} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$$

where $u^\mu$ is the fluid four-velocity satisfying $u_\mu u^\mu = -1$, $\rho$ the energy density of fluid and $p$ is the fluid pressure. Further, the energy momentum tensor for electromagnetic field is given by

$$T_{\mu\nu}^{\text{EM}} = -\frac{1}{4\pi}(g^{\delta\omega} F_{\mu\delta} F_{\nu\omega} - \frac{1}{4}g_{\mu\nu} F^{\delta\omega} F_{\delta\omega})$$

where $F_{\mu\nu}$ is the Maxwell field tensor defined as

$$F_{\mu\nu} = \Phi_{\nu,\mu} - \Phi_{\mu,\nu}$$

and $\Phi_\mu$ is the four potential. The corresponding Maxwell electromagnetic field equations are

$$\left(\sqrt{-g} F^{\mu\nu}\right)_{;\nu} = 4\pi J^\mu \sqrt{-g}, \quad F_{[\mu,\delta]} = 0$$

where $J^\mu$ is the current four-vector satisfying $J^\mu = \sigma u^\mu$, the parameter $\sigma$ is the charge density.

3 Einstein-Maxwell Field equations

We consider the spherically symmetric metric describing the interior space-time as [46]

$$ds^2 = -e^{a(r)} dt^2 + e^{b(r)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

where $a(r)$ and $b(r)$ are functions of $r$. For this metric, we get the torsion scalar $T$ and its derivative $T'$ as in the following forms [38]:

$$T(r) = \frac{2e^{-b}}{r} \left( a' + \frac{1}{r} \right),$$

$$T'(r) = \frac{2e^{-b}}{r} \left[ a'' - \frac{1}{r^2} - \left( a' + \frac{1}{r} \right) \left( b' + \frac{1}{r} \right) \right]$$

where the prime $'$ denotes the derivative with respect to the radial coordinate $r$. 

2 $f(T)$ Modified Gravity and Electromagnetic Field

We consider the torsion and the con-torsion tensor as follows [37]:

$$T^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu} = \epsilon^\alpha_i (\partial_\mu e^i_\nu - \partial_\nu e^i_\mu)$$

$$K^\mu_\alpha = -\frac{1}{2}(T^\mu_\alpha - T^\alpha_\mu - T^\mu_\alpha)$$

and the components of the tensor $S^\mu_\alpha$ are defined as

$$S^\mu_\alpha = \frac{1}{2}(K^\mu_\alpha + \delta^\mu_\alpha T^\beta_\alpha - \delta^\mu_\alpha T^\beta_\alpha)$$

So one can write the torsion scalar as in the following form:

$$T = T^\alpha_{\mu\nu} S^{\mu\nu}_\alpha$$

whose importance becomes clear in a moment. Now, define the modified teleparallel action by replacing $T$ with a function $f(T)$ as follows
For the charged fluid source with density $\rho(r)$, pressure $p(r)$ and electromagnetic field $E(r)$, the Einstein-Maxwell (EM) equations reduce to the form ($G = c = 1$) \[36\] \[37\]

\[-2e^{-b}T_f f_{TT} + f \left( T - \frac{1}{r^2} - \frac{e^{-b}}{r} (a' + b') \right) f_T = 8\pi \rho + E^2, \quad (13)\]

\[\left( T - \frac{1}{r^2} \right) f_T - \frac{f}{2} = 8\pi p - E^2, \quad (14)\]

\[e^{-b} \left( \frac{a'}{2} + \frac{1}{r} \right) T_f f_{TT} + \left[ T + e^{-b} \left\{ \frac{a''}{2} + \left( \frac{a'}{4} + \frac{1}{2r} \right) \right\} \right] \phantom{f_T} (15)\]

\[(a' - b') \right] f_T - \frac{f}{2} = 8\pi p + E^2, \quad (16)\]

\[e^{-\frac{b}{2}} \cot \frac{\theta}{2} T_f f_{TT} = 0 \quad (17)\]

Adding equations (13) and (14), we obtain

\[\rho + p = \frac{e^{-b}}{8\pi} (a' + b') f_T \quad (18)\]

Taking the derivative of equation (14) and using the equations (11) and (17), we obtain the energy conservation equation

\[p' + \frac{1}{2} (\rho + p) a' = \frac{1}{8\pi r^4} (r^4 E^2)' \quad (19)\]

The equation for the electric field $E$ is as follows

\[E(r) = \frac{1}{r^2} \int_0^r 4\pi r^2 \sigma(r) e^{\frac{b}{2}} dr = \frac{q(r)}{r^2} \quad (20)\]

where $q(r)$ is the total charge within a sphere of radius $r$. The term $\sigma e^{\frac{b}{2}}$ inside the above integral is known as the volume charge density. The gravitational mass can be written as

\[M(r) = \int_0^r 4\pi r^2 \left( \rho + \frac{E^2}{8\pi} \right) dr \quad (21)\]

From equation (10), we get either $T' = 0$ or $f_{TT} = 0$.

**Case I**: $T' = 0 \Rightarrow T = \text{constant} = T_0$, i.e., $T$ is independent of $r$ and hence $f(T)$, $f_T$, $f_{TT}$, ..., are always constant. Assume $f(T_0) = f_0$ and $f_T(T_0) = f_1$. So equations (13) - (15) reduce to

\[\frac{f_0}{2} - \left( T_0 - \frac{1}{r^2} - \frac{e^{-b}}{r} (a' + b') \right) f_1 = 8\pi \rho + E^2, \quad (22)\]

\[\left( T_0 - \frac{1}{r^2} \right) f_1 - \frac{f_0}{2} = 8\pi p - E^2, \quad (23)\]

\[\left[ \frac{T_0}{2} + e^{-b} \left\{ \frac{a''}{2} + \left( \frac{a'}{4} + \frac{1}{2r} \right) (a' - b') \right\} \right] f_1 - \frac{f_0}{2} = 8\pi p + E^2 \quad (24)\]

**Case II**: $f_{TT} = 0 \Rightarrow f(T) = \alpha T + \beta$, where $\alpha$ and $\beta$ are constants. Equations (13) - (15) reduce to

\[\frac{\alpha e^{-b}}{r} \left( b' - \frac{1}{r} \right) + \frac{\alpha}{r^2} + \beta = 8\pi p + E^2, \quad (25)\]

\[\frac{\alpha e^{-b}}{r} \left( a' + \frac{1}{r} \right) - \frac{\alpha}{r^2} - \beta = 8\pi p - E^2, \quad (26)\]

\[\alpha e^{-b} \left[ \frac{a''}{2} + \left( \frac{a'}{4} + \frac{1}{2r} \right) (a' - b') \right] - \frac{\beta}{2} = 8\pi p + E^2 \quad (27)\]

4 Geometry of Gravastars

In this section we will derive the solutions of the field equations for gravastar and discuss its physical as well as geometrical interpretations. For geometrical regions of the gravastar, it is supposed to be extremely thin having a finite width within the regions $D = r_1 < r < r_2 = D + \epsilon$ where $r_1$ and $r_2$ are radii of the interior and exterior regions of the gravastar and $\epsilon$ is positive small quantity. In these regions, the equation of state (EOS) parameter is structured as follows: (i) Interior region $R_1$: $0 < r < r_1$ with EOS $p = -\rho$, (ii) Shell region $R_2$: $r_1 < r < r_2$ with EOS $p = \rho$, (iii) Exterior region $R_3$: $r_2 < r$ with EOS $p = \rho = 0$.

4.1 Interior Region

The equation of state for interior region $R_1$ ($0 < r < r_1 = D$) of the gravastar is $p = -\rho$. From equation (19), we obtain

\[e^{b(r)} = k \ e^{-a(r)} \quad (28)\]

where $k$ is constant $> 0$.

**Case I**: $T' = 0$

For $T' = 0$, we obtain the solutions as

\[e^{a(r)} = \frac{k T_0 r^3 + C}{6r} \quad (29)\]

and

\[e^{b(r)} = \frac{6k r}{k T_0 r^3 + C} \quad (30)\]

where $C$ is constant. So in this case, the metric becomes

\[ds^2 = -\frac{k T_0 r^3 + C}{6r} dt^2 + \frac{6k r}{k T_0 r^3 + C} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (31)\]

Now we find the expressions of density, pressure and electromagnetic field as

\[\rho = \frac{1}{16\pi} \left[ f_0 + \left( \frac{1}{r^2} - 2T_0 \right) f_1 \right], \quad (32)\]

\[p = \frac{1}{16\pi} \left[ \left( 2T_0 - \frac{1}{r^2} \right) f_1 - f_0 \right] \quad (33)\]
and
\[
E(r) = \frac{\sqrt{f_1}}{\sqrt{2}} r
\]  
(33)

So the charge density for the electric field can be expressed as
\[
\sigma = \sigma_0 r^m \left( \frac{kT_0 r^3 + C}{6r} \right)^{\frac{1}{2}}
\]  
(34)

Also the gravitational mass of the interior region can be found as
\[
M(D) = \int_{r_1}^{D} 4\pi r^2 \left( \rho + \frac{E^2}{8\pi} \right) dr
\]
\[
= \frac{1}{12} (f_0 - 2f_1 T_0) D^3 + \frac{1}{2} f_1 D
\]  
(35)

Since gravastar is singularity free object, but we observe that the central singularity always occurs at \( r = 0 \). So gravastar cannot form in the case of \( T' = 0 \).

**Case II : \( f_{TT} = 0 \)**

For \( f_{TT} = 0 \), there are 4 equations and 5 unknown functions \( a, b, \rho, p, E \). So one function is free. Let us assume \( \sigma e^{\frac{1}{2}} = \sigma_0 r^m \) [9] where \( \sigma_0 \) and \( m \) are constants, so from equation (18) we have
\[
E(r) = E_0 r^{m+1}
\]  
(36)

where \( E_0 = \sqrt{\frac{8\pi}{m+1}} \). Using equation (17), we obtain
\[
p = -\rho = k_1 r^{2m+2} + k_2
\]  
(37)

where \( k_1 = \frac{(m+3)E_0^2}{8\pi (m+1)} \) and \( k_2 \) is constant. From equation (24), we obtain
\[
e^a = ke^{-b} = k \left[ 1 - \frac{k_3}{2\alpha r} \left( \frac{\beta + 16\pi k_2}{6\alpha} \right) r^2 + \frac{(8\pi k_1 - E_0^2)}{\alpha (2m+5)} r^{2m+4} \right]
\]  
(38)

where \( k_3 \) is an integration constant. We see that for \( k_3 \neq 0 \), the central singularity occurs at \( r = 0 \). Since gravastar is singularity free object, so the metric will be non-singular at the center \( r = 0 \). Hence we can choose \( k_3 = 0 \). So the metric becomes (choose \( k = 1 \))
\[
ds^2 = - \left[ 1 + \frac{(\beta + 16\pi k_2)}{6\alpha} r^2 + \frac{(8\pi k_1 - E_0^2)}{\alpha (2m+5)} r^{2m+4} \right] dt^2
\]
\[
+ \left[ 1 + \frac{(\beta + 16\pi k_2)}{6\alpha} r^2 + \frac{(8\pi k_1 - E_0^2)}{\alpha (2m+5)} r^{2m+4} \right]^{-1} dr^2
\]
\[
+ r^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]  
(39)

The charge density for electric field will be
\[
\sigma = \sigma_0 r^m \left[ 1 + \frac{(\beta + 16\pi k_2)}{6\alpha} r^2 + \frac{(8\pi k_1 - E_0^2)}{\alpha (2m+5)} r^{2m+4} \right]^{\frac{1}{2}}
\]  
(40)

Also the gravitational mass of the interior region of the gravastar can be found as
\[
M(D) = \int_0^{r_1 = D} 4\pi r^2 \left( \rho + \frac{E^2}{8\pi} \right) dr
\]
\[
= \frac{(E_0^2 - 8\pi k_1)}{2(2m + 5)} r^{2m+5} - \frac{4\pi k_2}{3} r^3
\]  
(41)

Thus gravastar forms in the case of \( f_{TT} = 0 \). In the next subsections, we’ll consider only the case \( f_{TT} = 0 \).

**4.2 Shell Region**

In this region \( R_2 (D = r_1 < r < r_2 = D + \epsilon) \), we assume the thin shell contains stiff perfect fluid which obeys EoS \( p = \rho \). For this non-vacuum region, it is very difficult to obtain the general solutions from the field equations. When two region joins together at a place, the intermediate region must be thin shell. So we shall try to find the analytical solution within the thin shell with limit \( 0 < e^{-b} \equiv \eta \ll 1 \). This thin shell structure suggests that as \( r \) approaches to zero, the corresponding radial parameter generally becomes \( \ll 1 \).

For \( f_{TT} = 0 \) and under the above approximation (we set \( h \) be zero to the leading order), the field equations (24) - (26) reduce to the following forms [9]
\[
- \frac{\alpha h'}{r} + \frac{2\alpha}{r^2} + \beta = 2E^2
\]  
(42)

and
\[
\alpha h' \left( \frac{a'}{4} + \frac{1}{2r} \right) + \frac{\alpha}{r^2} = 2E^2
\]  
(43)

We see that there are two equations but three unknowns \( a, b, E \). Similar to equation (60), let us assume the solution of \( E \) in the form \( E(r) = E_0 r^{m+1} \). Solving equations (42) and (43), we obtain
\[
e^{-b(r)} \equiv h(r) = h_2 + 2r + \frac{\beta r^2}{2\alpha} - \frac{E_0^2}{\alpha (m+2)} r^{2m+4}
\]  
(44)

and
\[
e^{a(r)} = \frac{h_3}{r^2} \exp \left[ \int \frac{8E_0^2 r^{2m+4} - 4\alpha}{2\alpha r^3 + \beta r^2 - 2E_0^2} dr \right]
\]  
(45)

where \( h_2 \) and \( h_3 \) are integration constants and the radius \( r \) corresponds to the shell structure in the region \( R_2 \). In this shell region, the range of \( r \) is \( D < r < D + \epsilon \). Under this assumption \( (h \ll 1), r \ll 1 \), we must have \( h_2 \ll 1 \). From equation (25) we obtain
\[
8\pi p = 8\pi \rho = E_0^2 r^{2m+2} - \frac{\alpha}{r^2} - \frac{\beta}{\alpha}
\]  
(46)

Also the charge density for electric field is given by
\[
\sigma = \sigma_0 r^m \left[ h_2 + 2r + \frac{\beta r^2}{2\alpha} - \frac{E_0^2}{\alpha (m+2)} r^{2m+4} \right]^{\frac{1}{2}}
\]  
(47)
4.3 Exterior Region

For the exterior region $R_3$ ($r > r_2 = D + \epsilon$), the vacuum EoS is given by $(p = \rho = 0)$. In this region, from equation (17), we obtain

$$E(r) = \frac{Q}{r^2}$$

where $Q$ is constant electric charge. From equation (19), we obtain

$$e^{b(r)} = k e^{-a(r)}$$

where $k$ is constant $> 0$.

For $f_{TT} = 0$, we obtain the solutions as

$$e^a = k e^{-b} = k \left(1 - \frac{2M}{r} + \frac{\beta}{6\alpha} r^2 + \frac{Q^2}{4\alpha r^2}\right)$$

where $M$ is the mass of the gravastar. Also the charge density for electric field may be written as

$$\sigma = \sigma_0 r^m \left(1 - \frac{2M}{r} + \frac{\beta}{6\alpha} r^2 + \frac{Q^2}{4\alpha r^2}\right)^{1/2}$$

So in the exterior region, the metric becomes (choose $k = 1$)

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{\beta}{6\alpha} r^2 + \frac{Q^2}{4\alpha r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{\beta}{6\alpha} r^2 + \frac{Q^2}{4\alpha r^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

For $\alpha = 1$ and $\beta = 0$, we get usual Reissner-Nordstrom spacetime metric and also for $\alpha = 1$, $\beta = 0$ and $Q = 0$, we get back to static Schwarzschild metric.

5 Physical Features

Now we shall discuss the physical features of the parameters of the gravastar shell region like proper length of the shell, energy and entropy within the shell.

5.1 Proper Length

Since the stiff perfect fluid propagates between two boundaries of the shell region of the gravastar, so the inner boundary of the shell is located at the surface $r = D$ and outer boundary of the shell is located at the surface $r = D + \epsilon$, where the proper thickness of the shell is assumed to be very small, i.e., $\epsilon \ll 1$. So, the proper thickness of the shell is determined by

$$\ell = \int_D^{D+\epsilon} \sqrt{e^{b(r)}} \, dr$$

Since in the shell region, the expressions of $e^{b(r)}$ is lengthy, so we cannot found the analytical form of the above integral.

So let us assume $\sqrt{e^{b(r)}} = \frac{dg(r)}{dr}$, so from the above integral we can get

$$\ell = \int_D^{D+\epsilon} \frac{dg(r)}{dr} \, dr = g(D+\epsilon) - g(D) \approx \epsilon \frac{dg(r)}{dr} \bigg|_D = \epsilon \sqrt{e^{b(D)}}$$

where we have taken only the first order term of $\epsilon$ (since $\epsilon \ll 1$, so $O(\epsilon^2) \approx 0$). For $f_{TT} = 0$, the proper length will be (from equation (14))

$$\ell = \epsilon \left[ h_2 - 2D - \frac{\beta D}{2} + \frac{E_0}{\alpha(m + 2)} D^{2m+4} \right]^{-1/2}$$

which implies proper length of the shell is proportional to the thickness ($\epsilon$) of the shell.

5.2 Energy

The energy within the shell region of the gravastar is

$$\mathcal{E} = \int_D^{D+\epsilon} 4\pi r^2 \left[ \rho + \frac{E^2}{8\pi} \right] \, dr$$

For $f_{TT} = 0$, we can obtain the energy as

$$\mathcal{E} = \int_D^{D+\epsilon} E_0^2 r^{2m+4} - \frac{\alpha}{2} - \frac{\beta r^2}{4} \, dr$$

$$= E_0^2 \left[ (D + \epsilon)^{2m+4} - D^{2m+4} \right]^{1/2} - \frac{\epsilon \alpha}{2} - \frac{\beta}{12} [(D + \epsilon)^3 - D^3]$$

$$\approx \epsilon E_0^2 D^{2m+4} - \frac{\epsilon \alpha}{2} - \epsilon \beta D^2$$

For this approximation, we see that the energy content in the shell is proportional to the thickness ($\epsilon$) of the shell.

5.3 Entropy

Mazur and Mottola have shown that the entropy density is zero in the interior region $R_1$ of the gravastar. However, the entropy within the shell can be defined by

$$S = \int_D^{D+\epsilon} 4\pi r^2 s(r) \sqrt{e^{b(r)}} \, dr$$

where $s(r)$ is the entropy density for the local temperature $T(r)$ and which can be written as

$$s(r) = \frac{\gamma k_B^2 T(r)}{4\pi \hbar} = \frac{\gamma k_B}{\hbar} \sqrt{\frac{p(r)}{2\pi}}$$

where $\gamma$ is dimensionless constant. So entropy can be written as

$$S = \frac{\gamma k_B}{\hbar} \int_D^{D+\epsilon} r^2 \sqrt{8\pi p(r) \, e^{b(r)}} \, dr$$

For the approximation ($\epsilon \ll 1$), we can obtain

$$S \approx \frac{\gamma k_B}{\hbar} D^2 \sqrt{8\pi p(D) \, e^{b(D)}}$$
For \( f_{TT} = 0 \), we get the entropy as

\[
S \approx \frac{c^2 k_B}{h} \left[ E_0^2 D^{2m+3} - \frac{\alpha}{D} - \frac{\beta D}{2} \right]^{\frac{1}{2}} \times \left[ h_2 - 2D - \frac{\beta D}{2} + \frac{E_0^2}{\alpha(m + 2)} D^{2m+1} \right]^{-\frac{1}{2}} \tag{63}
\]

That means entropy in the shell is proportional to the thickness \( (\epsilon) \) of the shell.

### 6 Junction Conditions between Interior and Exterior Regions

Since gravastar consist of three regions: interior region, shell region and exterior region, so it is necessary to matching between the surfaces interior and exterior regions according to the Darmois-Israel formalism \[51\] \[52\] \[53\]. At \( r = D \), the junction surface is denoted by \( \Sigma \). We consider the metric on the junction surface as in the form

\[
ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{64}
\]

where the metric co-efficients are continuous at \( \Sigma \), though their derivatives may not be continuous at \( \Sigma \). With the help of Darmois-Israel formalism, we want to find the expression for the stress-energy surface \( S_{ij} \) from the Lanczos equation \[54\]

\[
S_{ij} = -\frac{1}{8\pi} \left( \eta^j_k - \delta^j_k \eta^i_k \right) \tag{65}
\]

where \( \eta_{ij} = K^+_i - K^-_j \). Here \( K_{ij} \) is the extrinsic curvature. So \( \eta_{ij} \) gives the discontinuous surfaces in the extrinsic curvatures (second fundamental forms). Here the signs “+” and “−” correspond to the interior and the exterior regions of the gravastar respectively. The extrinsic curvatures associated with the both surfaces of the shell region can be written as

\[
K^\pm_{ij} = \left[ -n^\pm_{\nu} \left\{ \frac{\partial^2 x_{\nu}}{\partial \xi^i \partial \xi^j} + \Gamma_{\nu}^{\alpha \beta} \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} \right\} \right]_\Sigma \tag{66}
\]

where \( \xi^i \) are the intrinsic coordinates on the shell, \( n^\pm_{\nu} \) are the unit normals to the surface \( \Sigma \), defined by \( n_{\nu} n^\nu = -1 \). For the above metric, we can obtain

\[
n^\pm_{\nu} = \pm \left[ g^{\alpha \beta} \frac{\partial f}{\partial x^\alpha} \frac{\partial f}{\partial x^\beta} \right]^{-\frac{1}{2}} \frac{\partial f}{\partial x^\nu} \tag{67}
\]

Using the Lanczos equation, we can obtain the stress-energy surface tensor as \( S^j_i = diag(-\varphi, \varphi, \varphi, \varphi) \) where \( \varphi \) is the surface energy density and \( \varphi \) is the surface pressure given by \[10\]

\[
\varphi = -\frac{1}{4\pi D} \left[ \sqrt{f} \right]_+ - \frac{1}{16\pi} \left[ \sqrt{f} \right]^- \tag{68}
\]

and

\[
\varphi = \frac{1}{8\pi D} \left[ \sqrt{f} \right]_+ + \frac{1}{16\pi} \left[ \sqrt{f} \right]^- \tag{69}
\]

For \( f_{TT} = 0 \), we obtain

\[
\varphi = -\frac{1}{4\pi D} \sqrt{1 - \frac{2M}{D} + \frac{\beta D^2}{6\alpha} + \frac{Q^2}{\alpha D^2}} \tag{70}
\]

and

\[
\varphi = \frac{1}{8\pi D} \sqrt{1 - \frac{2M}{D} + \frac{\beta D^2}{6\alpha} + \frac{Q^2}{\alpha D^2}} \tag{71}
\]

#### 6.1 Equation of State

The equation of state parameter \( w(D) \) can be written as

\[
w(D) = \frac{\varphi}{\varphi} \tag{72}
\]

For \( f_{TT} = 0 \), the equation of state parameter can be written in the following form

\[
w(D) = \frac{1}{2} - \frac{1}{4} \left[ \frac{2M}{D} + \frac{\beta D^2}{6\alpha} + \frac{Q^2}{\alpha D^2} \right] \times \left[ \sqrt{1 - \frac{2M}{D} + \frac{\beta D^2}{6\alpha} + \frac{Q^2}{\alpha D^2}} - \frac{(\beta + 16\pi k_2)D}{3\alpha} - \frac{(2m + 4)(8\pi k_1 - E_0^2)}{\alpha(2m + 5)} D^{2m+3} \right] \tag{73}
\]

#### 6.2 Mass

The mass \( M \) of the thin shell can be obtained from the following formula

\[
M = 4\pi D^2 \varphi \tag{74}
\]

For \( f_{TT} = 0 \), the mass of the thin shell can be expressed as

\[
M = -D \left[ \sqrt{1 - \frac{2M}{D} + \frac{\beta D^2}{6\alpha} + \frac{Q^2}{\alpha D^2}} - \frac{(\beta + 16\pi k_2)D^2}{6\alpha} + \frac{(2m + 4)(8\pi k_1 - E_0^2)}{\alpha(2m + 5)} D^{2m+4} \right] \tag{75}
\]
So the total mass $M$ of the gravastar in term of the thin shell can be expressed as

$$M = \frac{D}{2} + \frac{\beta D^3}{12\alpha} + \frac{Q^2}{2\alpha D} - \frac{D}{2} \left[ \sqrt{1 + \left( \frac{\beta + 16\pi k_2}{6\alpha} D^2 \right)^2 + \frac{8\pi k_1 - E_0^2}{\alpha(2m + 5)} D^{2m+4} - \frac{M}{D} } \right]^2$$

We see that the total mass $M$ of the gravastar will be less than $\frac{D}{2} + \frac{\beta D^3}{12\alpha} + \frac{Q^2}{2\alpha D}$.

7 Discussions

In the present work, we have discussed the four dimensional spherically symmetric stellar system in the framework of modified $f(T)$ gravity theory with electro-magnetic field. The field equations have been found for two cases, either $T' = 0$ or $f_{TT} = 0$. For $T' = 0$, $T$ must be constant and all the derivatives of $f(T)$ with respect to $T$ must be constants. Also for $f_{TT} = 0$, we have obtained $f(T)$ is a linear function of $T$. Next we have discussed the charged gravastar model where the equation of state in the three regions of the gravastar satisfies as follows: interior region ($p = -\rho$), shell region ($p = \rho$) and exterior region ($p = \rho = 0$). In the interior region, we have found the solutions of all physical quantities like density, pressure, electro-magnetic field and also metric coefficients for both cases. For $T' = 0$, we have found $E(r) \propto \frac{1}{r}$. Since for $T' = 0$, the central singularity occurs at $r = 0$, so gravastar cannot form in this case. For $f_{TT} = 0$, $E(r) \propto r^{m+1}$ and gravastar forms in this case. In the exterior region, we have obtained the exterior solution for vacuum model. For $f_{TT} = 0$, we found that the metric is generalization of Reissner-Nordstrom spacetime. In the shell region, we have assumed that the interior and exterior regions join together at a place, so the intermediate region must be thin shell with limit $0 < e^{-\beta} = \epsilon < 1$. This thin shell structure suggests that as $r$ approaches to zero, the corresponding radial parameter generally becomes $\epsilon < 1$. Under this approximation, we have found the solutions for $f_{TT} = 0$. The electromagnetic field becomes in the form $E \propto \frac{\beta D^3}{12\alpha} + \frac{Q^2}{2\alpha D}$. The proper length of the thin shell, entropy and energy content inside the thin shell have been found and they are directly proportional to the proper thickness of the shell $\epsilon$ under the approximation $(\epsilon \ll 1)$. According to the Darmois-Israel formalism, we have studied the matching between the surfaces interior and exterior regions of the gravastar. The energy density and pressure on the surface have been obtained. Also the equation of state parameter $w(D)$ have been found. Moreover, the mass $M$ of the thin shell have been obtained and the total mass of the gravastar have been expressed in terms of the thin shell mass.

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