Gauge topological nature of the superconductor-insulator transition

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We propose a gauge theory of the superconductor-insulator transition (SIT) in Josephson junction arrays (JJA) and thin superconducting films at very low temperatures. This approach enabled us to unravel the topological nature of the SIT and to construct the phase diagram of the critical vicinity of the SIT, which comprises three competing quantum orders: superconducting, superinsulating, and Bose metal (quantum metal). We derive a criterion defining the conditions for either a direct SIT or for the SIT via the intermediate quantum metallic phase and demonstrate that this quantum metal phase is a topological insulator. Furthermore, we show that the superinsulator realizes Polyakov’s mechanism of linear confinement via instantons.

It has been shown [1–7] that in strongly disordered superconducting films and Josephson junction arrays (JJA) the superconductor-insulator transition (SIT) occurs at the critical self-dual point where vortex- and charge Berezinskii-Kosterlitz-Thouless (BKT) transitions terminate each other [8–12]. Yet, the structure of phases near the SIT, formed by cooperative effects of quantum fluctuations, correlations, and disorder remains mysterious. Here we develop a gauge theory of the SIT and construct the phase diagram of the critical region. We prove that three competing topological quantum orders, the superconductor, the quantum metal [13], which we identify as a topological insulator [14] (TI), and the superinsulator [8–10, 11] join at a quantum tri-critical point. We find that strong quantum fluctuations drive the SIT via the intermediate TI phase, while weak fluctuations result in a direct transition between the superconductor and the superinsulator, and that, at this transition, phase separation into intermixed flickering droplets of coexisting Cooper pair- and vortex condensates occurs. Finally, we reveal that the topological insulator and the superconductor represent topologically massive gauge theories [15], while the superinsulator realizes Polyakov’s instanton-driven linear confinement with neutral mesons as excitations [16].

In JJA the SIT is controlled by the ratio of the Josephson coupling energy $E_J$ and the charging energy $E_C = (2e^2)/2C$ of a single Josephson junction, $C$ being the junction capacitance [1]. At $E_J > E_C$, superconducting correlations win and the system is a superconductor, at $E_J < E_C$, the Coulomb blockade turns the system insulating. In superconducting films disorder suppresses electron diffusion; hence the dynamic screening of Coulomb interactions, which grows stronger with increasing disorder and, eventually, destroys global superconducting coherence [17]. Thus the underlying mechanism of the SIT in both superconducting films and JJA is the competition between Coulomb repulsion and superconducting correlations. For a superconductor, the charge and phase $\varphi$ of the Cooper pair condensate are dual, conjugated quantum variables so that $[N, \varphi] = i\hbar$, where $N$ is the number of particles in the condensate [18]: superconductivity and superinsulation are the fundamental manifestations of this duality [11]. The fixed phase of a superconductor implies, according to the uncertainty principle, an undefined charge. Hence the infinite conductivity, since any scattering would allow to count charges. Conversely, if the charge is pinned, the phase loosely fluctuates (but maintaining the synchronization over the system) to guarantee the maximal quantum mechanical amplitude [10]: hence the finite voltage across the system (due to the Josephson relation) in absence of a current, i.e. infinite resistance [3, 11]. The conceptual existence of superinsulation rests on the finding that strong quantum vortex-antivortex fluctuations prevent the topological screening of the two-dimensional (2D) logarithmic interactions between Cooper pairs [3]. A conclusive step in establishing superinsulation as a paradigmatic foundation of the physics of the SIT came from the realization that it is the divergence of the dielectric constant of a Cooper pair insulator on approach to the SIT which makes the system electrically two-dimensional over macroscopic scales [10, 12]. Accordingly, the Coulomb interaction between charges in the critical vicinity of the SIT is of the 2D logarithmic nature, which established superinsulation as a confined charge BKT state [10].

In this work, building on the approach put forth in [8], we develop a gauge theory of the Coulomb strength- and magnetic field-driven SIT. We start with the Hamiltonian for the JJA:

$$\mathcal{H} = \frac{1}{2} \sum_x V(C_0 - C \Delta)V + \sum_{x,l} E_J(1 - \cos(2\nabla_l \Phi)),$$

where the sum is taken over all the points $x$ of the 2D grid representing the JJA and $l$ is the vector connecting the superconducting granule at $x$ with its nearest neighbors.
$V$ is the electric field potential in the granule $x$, $\nabla \Phi$ is the superconducting phase difference between adjacent granules, $C_0$ is the granule capacitance to the ground and we will consider the case $C > C_0$. Going over to the coupled Coulomb gas description one arrives at the (continuum) Euclidean action \[ S = \int d^3x \, 4E_C \, \rho_\varnothing \frac{1}{\sqrt{\nabla^2}} \rho_\varnothing + 2\pi^2 E_1 \, \rho_\nu \frac{1}{\sqrt{\nabla^2}} \rho_\nu + \frac{1}{2} \frac{1}{E_3} \hat{\rho}_\varnothing \frac{1}{\sqrt{\nabla^2}} \hat{\rho}_\varnothing + i \int dt \int d^2x \int d^2y \, \rho_\varnothing (t, x) \Theta(x - y) \hat{\rho}_\varnothing (t, y), \quad \tag{2} \]

where we have used

$$\ln|x| = \frac{2\pi}{\sqrt{-\nabla^2}} \delta(x), \quad \tag{3}$$

and

$$\Theta(x) = \arctan \left( \frac{x_2}{x_1} \right). \quad \tag{4}$$

The first two terms in this action represent the two

$$S = \int d^3x \, 4E_C \, \rho_\varnothing \frac{1}{\sqrt{\nabla^2}} \rho_\varnothing + 2\pi^2 E_1 \, \rho_\nu \frac{1}{\sqrt{\nabla^2}} \rho_\nu + \frac{1}{2} \frac{1}{E_3} \hat{\rho}_\varnothing \frac{1}{\sqrt{\nabla^2}} \hat{\rho}_\varnothing + \frac{\pi^2}{4E_C} \rho_\nu \frac{1}{\sqrt{\nabla^2}} \rho_\nu \quad \tag{5}$$

Note that this is a harmless modification, since such a kinetic term for vortices is induced by integration over the charge dynamics \[ S = \frac{\ell^3}{8\pi^2 E_1} f_\mu f_\mu + i \frac{\ell^3}{\pi} a_\mu k_\mu b_\nu + \frac{\ell^3}{16E_C} g_\mu g_\mu + \ell \sqrt{2} a_\mu Q_\mu + i\ell \sqrt{2} b_\mu M_\mu, \quad \tag{6} \]

where the sum is taken over the Euclidean 3D discrete lattice with spacing $\ell$ (Greek indices denote the coordinate axes and repeated indices mean summation). We use natural units $c = 1$, $\hbar = 1$ but we shall occasionally restore physical units in specific formulas for comparison with previous literature. The first and the third terms in the action represent coupling and Coulomb energies in the JJA expressed through gauge fields $a_\mu$ and $b_\mu$ which mediate Coulomb interactions of Cooper pairs and vortex-vortex interactions, respectively, $f_\mu = k_\mu b_\nu$, $g_\mu = k_\mu a_\nu$, and $k_\mu = S_\mu \hat{\epsilon}_\mu \nu d_\nu$, with $S_\mu$ and $d_\nu$ the lattice shift and derivative operators, respectively, is the lattice Chern-Simons operator \[ S = \frac{\ell^3}{8\pi^2 E_1} f_\mu f_\mu + i \frac{\ell^3}{\pi} a_\mu k_\mu b_\nu + \frac{\ell^3}{16E_C} g_\mu g_\mu + \ell \sqrt{2} a_\mu Q_\mu + i\ell \sqrt{2} b_\mu M_\mu, \quad \tag{6} \]

term, the so called mixed Chern-Simons term, describes the Aharonov-Bohm coupling between charges and vortices. Thus, the first three terms represent the doubled topologically massive gauge theory \[ S = \frac{\ell^3}{8\pi^2 E_1} f_\mu f_\mu + i \frac{\ell^3}{\pi} a_\mu k_\mu b_\nu + \frac{\ell^3}{16E_C} g_\mu g_\mu + \ell \sqrt{2} a_\mu Q_\mu + i\ell \sqrt{2} b_\mu M_\mu, \quad \tag{6} \]

Coulomb gases with densities $\rho_\varnothing$ and $\rho_\nu$ for charges and vortices, respectively, the third is the kinetic term for the charges, and the final term represents the Aharonov-Bohm topological interaction between charges and vortices. The only term which breaks perfect duality between charges and vortices in this expression is the kinetic term for charges, which encodes the Josephson currents. The self-dual approximation \[ S = \frac{\ell^3}{8\pi^2 E_1} f_\mu f_\mu + i \frac{\ell^3}{\pi} a_\mu k_\mu b_\nu + \frac{\ell^3}{16E_C} g_\mu g_\mu + \ell \sqrt{2} a_\mu Q_\mu + i\ell \sqrt{2} b_\mu M_\mu, \quad \tag{6} \]

Now we reformulate the action via gauge fields mediating vortex-vortex and Coulomb interactions. The Euclidean lattice version of this gauge theory is given by

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integrates. Infinitely long strings can also end on electric or magnetic (monopole) instantons, representing tunneling events in the present Euclidean field theory setting.

Integrating out the gauge fields we obtain the free energy for topological strings of length $L = N\ell$:

$$F = \left(\frac{1}{g}Q^2 + gM^2 - \frac{1}{\eta}\right)\mu\eta N,$$

(7)

carrying electric and magnetic quantum numbers $Q^2 = Q_eQ_e$ and $M^2 = M_eM_e$, where $Q_e$ and $M_e$ are electric and magnetic quantum numbers assigned to every bond belonging to the string. We introduced the dimensionless coupling parameter $g = \sqrt{\pi^2E_J/(2E_C)}$, $g = g_c = 1$ corresponding to the SIT. The other dimensionless quantity $\eta = \eta_{mG}G(m\ell)/\mu$, see the numerical plot in Fig. 1, reflects the strength of quantum fluctuations; $G(m\ell)$ is the diagonal element of the lattice Green function $G(x-y)$ representing the inverse of the operator $(\nabla^2)^2$. Further, $m = \sqrt{8E_JE_C}$ is the Chern-Simons mass, corresponding to the JJ.A plasma frequency, and $\mu \approx \ln(5)$ is the string entropy per bond.

If the expression in brackets is negative, i.e. if

$$\left(\frac{1}{g}\right)Q^2 + gM^2 < \frac{1}{\eta},$$

(8)

the proliferation of loops of arbitrary size, infinitely long strings and instantons, i.e. the Bose condensation of charges and/or vortices, becomes energetically advantageous. The formation of vortex and Cooper pair condensates breaks the original $R$ gauge symmetry down to $Z$. If both condensates are allowed, the state with the lowest free energy is realized. Importantly, the state with coexisting electric and magnetic condensates is unstable, because the topological linking number term contributes an imaginary term to the energy. The condition (8) implies that a particular condensate forms if the pair $(Q,M)$ on a square lattice of integer electric and magnetic charges falls within the interior of an ellipse with the semi axes $r_Q = (g/\eta)^{1/2}$ and $r_M = 1/(g\eta)^{1/2}$. Figure 2 illustrates the conditions for formation of the Bose condensate comprising quantum fluctuations of Cooper pairs—“anti”-Cooper pairs, carrying the ‘unit’ charges $\pm 2e$, while vortices are pinned. In this superconducting state a passing charge current does not induce any voltage, hence $R = 0$. One can say that the condensate of the electric strings prevents the formation of vortex loops that cause the finite resistance $R$. In the dual, superinsulating state, see Fig. 2, $M = \pm 1$ and quantum fluctuations of vortices and antivortices form a Bose condensate which blocks the propagation of Cooper pairs. Alternatively one can say that the condensed magnetic strings suppress the formation of the electric ones, mediating the tunneling conductance in the insulating state [21]. The conditions shown in Fig. 2, refer to coexisting Cooper

Figure 2. Zero temperature near the SIT states phase diagram in $g = \sqrt{\pi^2E_J/2E_C} - \eta$ coordinates. The parameter $\eta$ characterizes the strength of quantum fluctuations. When those are strong, $\eta > 1$, an intermediate quantum metal/topological insulator phase opens up between the superconductor and superinsulator.

Figure 1. Configurations that minimize the string gas free energy are defined by the integer quantum numbers that fall into the interior of an ellipse with semi-axes, $r_Q = (g/\eta)^{1/2}$ and $r_M = 1/(g\eta)^{1/2}$, depending on the system parameters. a) superconductor: strings with electric quantum numbers condense, b) topological insulator/quantum metal: all strings are suppressed by their high self-energy, c) superinsulator: strings with magnetic quantum numbers condense, d) coexistence of long electric and magnetic strings: this is an unstable configuration near the first-order direct transition from a superinsulator to a superconductor, the strings with the quantum numbers that minimize their self-energy are the stable configurations.
pair and vortex condensates, $Q, M = \pm 1$. Finally, Fig. 11 illustrates the situation where none of the condensates can form, $Q = M = 0$. This is a quantum metal, the conductivity of which is provided by the ballistic motion of both Cooper pairs and vortices.

The phase diagram of the critical region in $g$-$\eta$ coordinates is shown in Fig. 2. The superinsulating phase exists at $g < 1$ and is confined between the $\eta = 1/g$ and $g = 1$ lines. The superconducting state forms at $g > 1$ and is restricted by $\eta = g$ and $g = 1$ lines. Finally, above the $\eta = 1/g$ and $\eta = g$ lines lies the quantum (Bose) metal [12] which, as we show below, is actually a topological insulator by its electronic structure. At $\eta < 1$ the SIT is a first-order direct superconductor-superinsulator transition. At $\eta > 1$, the transition occurs via the intermediate Bose metal phase. The two lines $\eta = 1/g$ and $\eta = g$ separating this intermediate phase from the superinsulator and the superconductor constitute continuous phase transitions [22]. The point $g = 1$ and $\eta = 1$ where they join the first-order direct SIT is a tri-critical point.

The nature of the Bose (quantum metal), intimately connected to the SIT, is the subject of a decades-long continuous interest and intense debate [23–31]. A nice qualitative consideration [4] shows that in its “pure” incarnation, the Bose metal phase possesses quantum sheet resistance at $T = 0$. Using the Josephson relation, one finds that moving vortices generate the voltage $V = (h/2)e\dot{\theta} = (h/2c)n_{\nu}$, where $n_{\nu}$ is the vortex density. At the same time the Cooper pairs’ charge current is $I = 2e\dot{n}_{\nu}, n_{c}$ being the sheet density of the Cooper pairs. This results in the sheet resistance [4]

$$R = \frac{V}{I} = \frac{h}{(2e c)^2} n_{\nu}, \quad (9)$$

where $\dot{n}_{\nu}$ is the flux of vortices passing across the sample perpendicular to the current and $\dot{n}_{c}$ is the flux of the Cooper pairs traversing between the electrodes. Speculating that under complete duality exactly one vortex crosses the system for each passing Cooper pair, one arrives at the quantum resistance $R_{Q}$ for the Bose metal. Yet, in spite of this intense attention, the underlying topological structure of the Bose metal remains a mystery. In order to gain insight into that, we resort to a more refined calculation of the resistance, computing the Cooper pair and vortex induced currents as $j^{\mu} = (1/\ell^{3}) (\delta S^{\text{eff}}(A_{\mu}, B_{\mu})/\delta A_{\mu})$ and $\phi^{\mu} = (1/\ell^{3}) (\delta S^{\text{eff}}(A_{\mu}, B_{\mu})/\delta B_{\mu})$, where $S^{\text{eff}}$ is the effective action for the electromagnetic gauge potential $A_{\mu}$ and the corresponding external probe $B_{\mu}$ for magnetic currents. In the case where neither vortices nor Cooper pairs may condense, one immediately finds that the bulk part of the effective action suppresses electric currents and that the only contribution comes from the edge excitations, reproducing Eq. (9). This kind of edge states is well known in field theory [32] and corresponds to two chiral bosons circulating along the edges in two opposite directions. This identifies the quantum (Bose) metal as a topological insulator [14, 33]. Note that in real experiments the resistance appeared to be less than $R_{Q}$ signaling a deviation from strict duality, most probably due to disorder effects.

To describe the magnetic field-driven transition in a JJA and a superconducting film that is already on the verge of the SIT, let us introduce the frustration factor $f = B/B_{c}$, where $B_{c}$ is the magnetic field corresponding to one flux quantum $\Phi_{0} = \pi h c / e$ piercing one plaquette of the JJA. An external magnetic field corresponds to a special case of condensed magnetic strings with non-integer quantum number and results thus in a shift of the magnetic quantum number in the free energy [7] and, accordingly in the condensation condition $\delta S / \delta A = 0$. The SIT along the magnetic axis modifies the condensation conditions, see Fig. 3. Let us assume $g = 1 + \epsilon$, where $\epsilon \ll 1$, so that the system is in a superconducting state but close to the SIT. Then for a direct SIT at $\eta < 1$ one finds $f_{c} = (1/2)(g^{2} - 1) \approx \epsilon$. At $\eta > 1$, but still close to the tri-critical point, we set $g = \eta + \epsilon$ and find that the superconductor transforms into a topological insulator at $f_{c} = \sqrt{\eta^{3/2}}/(g - 1)/\eta^{3/2}$. These results hold as long as $\epsilon$ is small enough so that the next order corrections due to the suppression of the superconducting gap $\Delta$ and the renormalization of $E_{c}$ by the magnetic field are negligible.

Josephson junction arrays are not only an ideal model system to study the quantum SIT itself but are also a perfect model for strongly disordered superconducting films [34], as illustrated by the striking similarity of the SIT behaviours in these two systems. Thus, transcribing the above results, obtained in terms of the JJA parameters $E_{c}$ and $E_{f}$ via material characteristics of the films provides a fair description of the SIT in the latter. The role of the tuning parameter driving the film across the SIT is taken by the resistance per square, $R_{\Sigma}$ (or, equivalently, by the dimensionless conductance $g = 4R_{Q}/R_{\Sigma}$). The SIT in the models for films is expected to occur at $g = g_{c} = 1$, which justifies our notation $g = \sqrt{\alpha^{2}E_{f}/(2E_{c})}$ introduced above. We will describe the film in the vicinity of the quantum tri-critical point, thus we can safely put $\sqrt{2E_{c}/\alpha^{2}E_{f}} \approx 1$. The equivalence of the electromagnetic response of a superconducting film and the corresponding JJA on spatial scales well exceeding the elemental unit size is established by the relation $\lambda_{c} = c\phi_{0}/(8\pi^{2}I_{c}) [35]$, where $\lambda_{c} = \lambda_{c}^{2}/d$ is the Pearl screening length, $\lambda_{c}$ is the bulk London penetration depth, $d$ is the film thickness, and $I_{c}$ is the critical current of a single Josephson junction. Then, the dimensionless Landau parameter $\tilde{\gamma} = m \ell c$ acquires a remarkable form

$$\tilde{\gamma} = \frac{1}{\delta\alpha} \frac{\delta d}{\lambda_{c}^{4}}, \quad (10)$$
where $\alpha = e^2/(\hbar c)$ is the fine structure constant, and $\ell$ now plays the role of the characteristic microscopic cutoff length of order of the superconducting coherence length $\xi$. Hence, the Chern-Simons mass, setting the energy scale of topological excitations in the film is a product of a world constant and the geometrical characteristics of the film encoding its material properties. Note also that, as a consequence of (10), the parameter $\eta$ contains a factor $\hbar$ in the numerator. This parameter describes the strength of quantum fluctuations: an intermediate quantum metal/topological phase occurs when these are strong enough, i.e. $\eta > 1$.

Making use of $\lambda_c^2 = [4\pi n_s e^2/(m c^2)](\tau \Delta/\hbar)$, where $n_s$ is the density of superconducting electrons, which at $T = 0$ is equal to the total electron density $n_e$ and $\tau$ is the transport scattering time, one can estimate $\delta \approx (\alpha \epsilon)(\ell d n_e^{2/3})(\tau \Delta/\hbar)$. In the example shown in Fig. 2, the tri-critical point corresponds to $\delta = \delta_c \approx 0.65$. If we associate the ultraviolet cutoff $\ell$ with the minimal superconducting scale $\xi$, and use the realistic experimental material parameters, we see that the critical value $\delta_c$ is indeed a plausible number separating the direct SIT and the transition via the quantum metal (SMIT) in experiments on thin films. Moreover, the dependence $\delta \sim n_e^{2/3}$ reflects correctly the trend of crossing over from the direct SIT to SMIT upon an increase of the electron density in films.

Importantly, when describing the real superconducting films, one has to take into account the finite dielectric constant $\varepsilon$ of the film. In this case the relativistic invariance is lost and the action given by Eq. (8) ceases to be symmetric. Accordingly, the relation $\eta(\delta)$ ceases to be universal since $\varepsilon$ depends on the proximity of the system to the SIT and diverges as $g \rightarrow 1$ [90]. One can show that $\eta$ decreases with the increasing $\varepsilon$. Therefore, the scenario of the SIT, the direct vs. the SIT via the quantum metal phase, depends on the magnitude of $\eta(\varepsilon_{\text{max}})$ where $\varepsilon_{\text{max}}$ is the maximal dielectric effect achieved at the value of $g < 1$ where the correlation length associated with the SIT compares to the lateral dimension of the film. Taking an estimate of $\varepsilon_{\text{max}} \approx 10^4$ as a characteristic value for NbTiN film, where the divergent $\varepsilon$ was observed on approach to the magnetic field-driven SIT [77], and using the corresponding estimate of $\delta$ for this material one obtains $\eta \approx 0.26 < 1$ and hence one expects this NbTiN film to exhibit the direct SIT. Analogously, one can observe that near the SIT $\eta < 1$ for TiN, which thus also follows the direct SIT scenario. Instead, for NbSi, one does not expect a $\varepsilon$ divergence and, correspondingly, the estimate of $\delta$ gives $\eta > 1$ confirming a transition via an intermediate quantum metal phase.

Now we address the emergent, self-induced electronic granularity that has become a paradigmatic attribute of the critical vicinity of the SIT [38]. Based on experimental observations [59–61], it was conjectured [10, 41] that the emergence of a network of weakly coupled superconducting droplets immersed into the normal matrix is an inherent property of the critical region of the SIT in films and that it is this network, similar by its properties to a JJA, that constitutes a material platform for the superinsulating state. Our gauge theory approach enables us to put this hypothesis on a firm theoretical ground. Considering the region of weak quantum fluctuations, $\eta < 1$, i.e. the direct SIT, we note that there is a region around the critical point $g_c = 1$, where Cooper pairs and vortex condensates coexist. An energy balance consideration at $g > 1$ and $\eta < 1$ for a droplet of perimeter $L$ immersed into a superinsulating matrix of area $A$ yields an energy

$$E \propto A - \frac{1}{4\pi} \left(1 - \frac{1}{g^2}\right) L^2 + \sigma L,$$

where the boundary contribution arises due to fluctuation-induced charge and vortex excitations within the opposite condensates, respectively and $\sigma$ is a numerical coefficient. The energy $E$ has a global minimum $E \propto A/g^2$ when the superconductor droplet fills the whole area $A$, but also a local minimum $E \propto A$ at $L = 0$. These two minima are separated by a maximum at $L \propto g^2/(g^2 - 1)$. If close enough to the SIT, this maximum spreads over a scale that is larger than the radius of the droplet. Hence, it becomes energetically advantageous to fragment a superconducting droplet into smaller ones. The fragmentation process would stop at the minimal achievable dimension, of order $\xi$, which sets the scale of a self-induced granular structure. Whether this granular network forms a regular array or maintains only a short-range order is subject of further study. Note that the droplet picture of a superinsulator agrees with the conclusion of [42] about the formation of a microemulsion of two competing phases at the first order transition in the presence of the 3D Coulomb interactions.

It is worth mentioning a deep connection between the structure of the superinsulating state and field theory models of confinement. To illustrate this, we consider

\begin{figure}[ht]
\centering
\includegraphics[width=0.8\textwidth]{figure3.png}
\caption{The magnetic field-driven SIT. Frustration parameter $0 \leq f < 1$ displaces the original ellipse along the magnetic axis: a) transition from a superconductor to a topological insulator/quantum metal for $\eta > 1$, b) the direct transition from a superconductor to a superinsulator for $\eta < 1$.}
\end{figure}
Figure 4. Generic $\eta(\mathcal{B})$ dependence ($\mathcal{B} = m\ell$) for $\mu = 0.25 \ln(5)$. The critical value $\mathcal{B}_c \approx 0.65$ separates the region of weak quantum fluctuations. In this case, for $\mathcal{B} < \mathcal{B}_c$, where $\eta < 1$ and the direct SIT realizes, from the region of strong quantum fluctuations, $\mathcal{B} > \mathcal{B}_c$, corresponding to $\eta > 1$ where an intermediate topological insulator/quantum metal phase between the superinsulator and the superconductor opens up.

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