RESOLUTIONS OF HILBERT MODULES AND SIMILARITY

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Abstract. Let $H_m^2$ be the Drury-Arveson (DA) module which is the reproducing kernel Hilbert space with the kernel function $(z, w) \in B_m \times B_m \to (1 - \sum_{i=1}^{m} z_i \bar{w}_i)^{-1}$. We investigate for which multipliers $\theta : B_m \to \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ the quotient module $\mathcal{H}_\theta$, given by

$$\cdots \to H_m^2 \otimes \mathcal{E} \xrightarrow{M_\theta} H_m^2 \otimes \mathcal{E}_* \xrightarrow{\pi_\theta} \mathcal{H}_\theta \to 0,$$

is similar to $H_m^2 \otimes F$ for some Hilbert space $F$, where $M_\theta$ is the corresponding multiplication operator in $\mathcal{L}(H_m^2 \otimes \mathcal{E}, H_m^2 \otimes \mathcal{E}_*)$ for Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$ and $\mathcal{H}_\theta$ is the quotient module $(H_m^2 \otimes \mathcal{E}_*)/\text{clos}[M_\theta(H_m^2 \otimes \mathcal{E})]$. We show that a necessary condition is the existence of a multiplier $\psi$ in $\mathcal{M}(\mathcal{E}_*, \mathcal{E})$ such that $\theta \psi \theta = \theta$.

Moreover, we show that the converse is equivalent to a structure theorem for complemented submodules of $H_m^2 \otimes \mathcal{E}$ for a Hilbert space $\mathcal{E}$, which is valid for the case of $m = 1$. The latter result generalizes a known theorem on similarity to the unilateral shift, but the above statement is new. Further, we show that a finite resolution of DA-modules of arbitrary multiplicity using partially isometric module maps must be trivial. Finally, we discuss the analogous questions when the underlying operator tuple or algebra is not necessarily commuting. In this case the converse to the similarity result is always valid.

1. Introduction

A well known result in operator theory (see [16] and [17]) states that the contraction operator given by a canonical model is similar to a unilateral shift of some multiplicity if and only if its characteristic function has a left inverse. Various approaches to this one-variable result have been given (cf. [19]) but a new one is given in this paper which uses the commutant lifting theorem (CLT). In particular, the proof does not involve, at least explicitly, the geometry of the dilation space for the contraction.

The Drury-Arveson (DA) space $H_m^2$ has been intensively studied by many researchers over the past few decades. In particular, the CLT has been extended to this space with a few necessary changes. Using the CLT, we extend to the DA space one direction of the one variable result on the similarity of quotient modules of the Hardy space on the unit disk. We show that the converse is equivalent to the assertion that each complemented submodule of $H_m^2 \otimes \mathcal{E}$ for a Hilbert space $\mathcal{E}$ is isomorphic to $H_m^2 \otimes \mathcal{E}_*$ for some Hilbert space $\mathcal{E}_*$. Of course,
this result follows trivially from the Beurling-Lax-Halmos theorem (BLHT) in case $m = 1$. (Actually, for $m = 1$ the submodule is isomorphic to $H^2_m \otimes \mathcal{E}_*$.)

The quotient modules described above are the simplest case of a resolution by DA-modules for which the connecting maps are all partially isometries or inner multipliers in the language of Arveson [2]. In the latter paper, Arveson showed that every pure co-spherical contraction has an inner resolution and suggested that it might not terminate as resolutions do in the algebraic context. In this paper we show that the only isometric inner multiplier, $V : H^2_m \otimes \mathcal{E} \to H^2_m \otimes \mathcal{E}_*$ for Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$, is the trivial one determined by an isometric operator $V_0 : 1 \otimes \mathcal{E} \to 1 \otimes \mathcal{E}_*$. As a consequence, we show that all finite inner resolutions are trivial in a sense that will be explained in Section 4.

A parallel notion of resolution for Hilbert modules was studied by Arveson [3], which is different from the one considered in this paper. For Arveson, the key issue is the behavior of the resolution at $0 \in \mathbb{B}^m$ or the localization of the sequence of connecting maps at $0$. This focus provides a very strong relationship to a resolution of finite dimensional Hilbert spaces, causing the sequence to end in finitely many steps and providing machinery to understand the local behavior of the module. The resolutions considered in ([9], [8]) and this paper are related to dilation theory although the requirement that the connecting maps are partial isometries is relaxed.

In Section 5, we are able to apply essentially the same proofs to the non-commutative case to obtain an analogous result, except here we need the noncommutative analogue of the BLHT due to Popescu ([22], [21]). More precisely, we show that a quotient of the Fock Hilbert space, $F^2_m \otimes \mathcal{E}$, for some Hilbert space $\mathcal{E}$, by the closure of the range of a multi-analytic map $\Theta$ is similar to $F^2_m \otimes \mathcal{F}$ for some Hilbert space $\mathcal{F}$ if and only if $\Theta$ has a multi-analytic regular inverse.

In a concluding section we indicate that many of these results can be extended to complete Nevanlinna-Pick kernel Hilbert spaces and to other Hilbert modules for which the CLT holds.

2. Preliminaries

We consider two cases, the first one in which the operators commute, or for which the algebra is $\mathbb{C}[z_1, \ldots, z_m]$ and hence commutative, and the second in which the operators are not assumed to commute or the algebra is $\mathbb{F}[Z_1, \ldots, Z_m]$. We begin with the commutative case.

Let $\{T_1, \ldots, T_m\}$ be a commuting $m$-tuple of bounded linear operators on a Hilbert space $\mathcal{H}$; that is, $[T_i, T_j] = T_i T_j - T_j T_i = 0$ for $i, j = 1, \ldots, m$. A Hilbert module $\mathcal{H}$ over the polynomial algebra $\mathbb{C}[z_1, \ldots, z_m]$ of $m$ commuting variables is defined so that the module multiplication $\mathbb{C}[z_1, \ldots, z_m] \times \mathcal{H} \to \mathcal{H}$ is defined by

$$p(z_1, \ldots, z_m) \cdot h = p(T_1, \ldots, T_m)h,$$

where $p(z_1, \ldots, z_m) \in \mathbb{C}[z_1, \ldots, z_m]$ and $h \in \mathcal{H}$. We denote by $M_1, \ldots, M_m$ the operators defined to be module multiplication by the coordinate functions. More precisely,

$$M_i h = z_i \cdot h = T_i h, \quad (h \in \mathcal{H}, 1 \leq i \leq m).$$
A Hilbert module over $\mathbb{C}[z_1, \ldots, z_m]$ is said to be \textit{co-spherically contractive}, or define a \textit{row contraction}, if
\[ \| \sum_{i=1}^{m} M_i h_i \|^2 \leq \sum_{i=1}^{m} \| h_i \|^2, \quad (h_1, \ldots, h_m \in \mathcal{H}), \]
or, equivalently, if
\[ \sum_{i=1}^{m} M_i M_i^* \leq I_{\mathcal{H}}. \]

Natural examples of co-spherical contractive Hilbert modules over $\mathbb{C}[z_1, \ldots, z_m]$ are the DA-module, the Hardy module and the Bergman module, all defined on the unit ball $B$. These are all reproducing kernel Hilbert spaces over $H$, plays the key role for the class of co-spherically contractive Hilbert modules over $H$. We identify the Hilbert tensor product $H^2_m \otimes \mathcal{E}$ with the kernel $(z, w) \mapsto (1 - \sum_{i=1}^{m} z_i \bar{w}_i)^{-1} I_{\mathcal{E}}$. Consequently, $H^2_m \otimes \mathcal{E}$ is the reproducing kernel Hilbert space with kernel $(z, w) \mapsto (1 - \sum_{i=1}^{m} z_i \bar{w}_i)^{-1} I_{\mathcal{E}}$. The DA-module $H^2_m$ is the reproducing kernel Hilbert space corresponding to the kernel $K : X \times X \rightarrow \mathbb{C}$ which satisfies
\[ \sum_{i,j=1}^{l} c_i c_j K(x_i, x_j) > 0, \]
for $x_1, \ldots, x_l \in X$, $c_1, \ldots, c_l \in \mathbb{C}$ with not all $c_i$ zero and $l \in \mathbb{N}$. The reproducing kernel Hilbert space $\mathcal{H}_K$, corresponding to the kernel $K$, is the Hilbert space of functions defined on $X$ with the following reproducing property
\[ f(x) = \langle f, K_x \rangle, \quad f \in \mathcal{H}_K, \]
where for each $x \in X$, $K_x : X \rightarrow \mathbb{C}$ is the vector in $\mathcal{H}_K$ defined by $K_x(w) = K(w, x)$, $w \in X$. The DA-module $H^2_m$ is the reproducing kernel Hilbert space corresponding to the kernel $K : \mathbb{B}^m \otimes \mathbb{B}^m \rightarrow \mathbb{C}$ defined by
\[ K(z, w) = (1 - \sum_{i=1}^{m} z_i \bar{w}_i)^{-1}, \quad (z, w) \in \mathbb{B}^m \otimes \mathbb{B}^m. \]

We identify the Hilbert tensor product $H^2_m \otimes \mathcal{E}$ with the $\mathcal{E}$-valued $H^2_m$ space $H^2_m(\mathcal{E})$ or the $\mathcal{L}(\mathcal{E})$-valued reproducing kernel Hilbert space with the kernel $(z, w) \mapsto (1 - \sum_{i=1}^{m} z_i \bar{w}_i)^{-1} I_{\mathcal{E}}$. Consequently,
\[ H^2_m \otimes \mathcal{E} = \{ f \in \mathcal{O}(\mathbb{B}^m, \mathcal{E}) : f(z) = \sum_{k \in \mathbb{N}^m} a_k z^k, a_k \in \mathcal{E}, \| f \|^2 := \sum_{k \in \mathbb{N}^m} \frac{\| a_k \|^2}{\gamma_k} < \infty \}, \]
where $\mathcal{O}(\mathbb{B}^m, \mathcal{E})$ is the space of $\mathcal{E}$-valued holomorphic functions on $\mathbb{B}^m$, and $k = (k_1, \ldots, k_m)$ and $\gamma_k = \frac{(k_1 + \cdots + k_m)!}{k_1! \cdots k_m!}$ are the multinomial coefficients. A function $\varphi \in \mathcal{O}(\mathbb{B}^m, \mathcal{L}(\mathcal{E}, \mathcal{E}))$ is said to be a \textit{multiplier} if $\varphi f \in H^2_m \otimes \mathcal{E} \ast = H^2_m(\mathcal{E})$ for all $f \in H^2_m \otimes \mathcal{E} = H^2_m(\mathcal{E})$. By the closed graph theorem, such a multiplier $\varphi$ defines a bounded module map
\[ M_\varphi : H^2_m \otimes \mathcal{E} \rightarrow H^2_m \otimes \mathcal{E}, \quad M_\varphi f = \varphi f, \quad f \in H^2_m \otimes \mathcal{E}. \]
Equivalently, we can consider \( \varphi \in \mathcal{O}(\mathbb{B}^m, \mathcal{L}(\mathcal{E}, \mathcal{E}_*)) \) for which \( M_\varphi \) defines a bounded operator from \( H^2_m \otimes \mathcal{E} \) to \( H^2_m \otimes \mathcal{E}_* \). The set of all such bounded multipliers \( \varphi \in \mathcal{O}(\mathbb{B}^m, \mathcal{L}(\mathcal{E}, \mathcal{E}_*)) \) will be denoted by \( \mathcal{M}(\mathcal{E}, \mathcal{E}_*) \). A multiplier \( \varphi \in \mathcal{M}(\mathcal{E}, \mathcal{E}_*) \) is said to be inner if \( M_\varphi \) is a partial isometry in \( \mathcal{L}(H^2_m \otimes \mathcal{E}, H^2_m \otimes \mathcal{E}_*) \).

We recall an analogue of the CLT due to Ball-Trent-Vinnikov (Theorem 5.1 in [4]) on DA-modules which will be used to prove one of the main results of this paper.

**Theorem 2.1.** (Ball-Trent-Vinnikov) Let \( \mathcal{N} \) and \( \mathcal{N}_* \) be quotient modules of \( H^2_m \otimes \mathcal{E} \) and \( H^2_m \otimes \mathcal{E}_* \) for some Hilbert spaces \( \mathcal{E} \) and \( \mathcal{E}_* \), respectively. If \( X : \mathcal{N} \to \mathcal{N}_* \) is a bounded module map (that is, \( X \) \( \mathcal{N} \to \mathcal{N}_* \)) is a bounded module map (that is, \( X \) \( \mathcal{N} \to \mathcal{N}_* \)), then there exists a multiplier \( \varphi \) in \( \mathcal{M}(\mathcal{E}, \mathcal{E}_*) \) such that

\[
XP_N(M_{z_i} \otimes I_\mathcal{E})|_\mathcal{N} = P_{\mathcal{N}_*}(M_{z_i} \otimes I_{\mathcal{E}_*})|_{\mathcal{N}_*}X,
\]

for \( i = 1, \ldots, m \), then there exists a multiplier \( \varphi \) in \( \mathcal{M}(\mathcal{E}, \mathcal{E}_*) \) such that

(i) \( \|X\| = \|M_\varphi\| \) and

(ii) \( P_{\mathcal{N}_*}M_\varphi \) is a multiplier.

In the proof of the above theorem, Ball-Trent-Vinnikov [4] made the additional assumption that the submodules \( \mathcal{N}_1^1 \) and \( \mathcal{N}_2^1 \) are invariant under the scalar multipliers. However, that this condition is redundant follows from part (iii) of Theorem 5.1 due to McCollough-Trent [15].

The above statement of the CLT for \( \mathbb{C}[z_1, \ldots, z_m] \) is due to Ball-Trent-Vinnikov as indicated. However, Popescu pointed out that the result follows from its noncommutative analogue established earlier by him in [20, 21]. A more recent paper on this topic is due to Davidson and Le ([6]).

We now consider some preliminaries for the case of noncommuting operators. Let \( \mathbb{F}_m^+ \) denote the free semigroup with the \( m \) generators \( g_1, \ldots, g_m \) and let \( F^2_m \) be the full Fock space of \( m \) variables, which is a Hilbert space. More precisely, if we let \( \{e_1, \ldots, e_m\} \) be the standard orthonormal basis of \( \mathbb{C}^m \), then

\[
F^2_m = \bigoplus_{k \geq 0} (\mathbb{C}^m)^{\otimes k},
\]

where \( (\mathbb{C}^m)^{\otimes 0} = \mathbb{C} \). The creation, or left shift, operators \( S_1, \ldots, S_m \) on \( F^2_m \) are defined by

\[
S_if = e_i \otimes f,
\]

for all \( f \) in \( F^2_m \) and \( i = 1, \ldots, m \).

Given \( m \) bounded linear operators, \( \{T_1, \ldots, T_m\} \), on a Hilbert space \( \mathcal{K} \), which are not necessarily commuting, one can make \( \mathcal{K} \) into a Hilbert module over the algebra of polynomials \( \mathbb{F}[Z_1, \ldots Z_m] \), in \( m \) noncommuting variables, as follows:

\[
\mathbb{F}[Z_1, \ldots Z_m] \times \mathcal{K} \to \mathcal{K}, \ p(Z_1, \ldots, Z_m) : h \mapsto p(T_1, \ldots, T_m)h, \ h \in \mathcal{K}.
\]

The module \( \mathcal{K} \) over \( \mathbb{F}[Z_1, \ldots, Z_m] \) is said to be contractive if the row operator given by module multiplication by the coordinate functions is a contraction.

A bounded linear operator \( \Theta \) in \( \mathcal{L}(F^2_m \otimes \mathcal{E}, F^2_m \otimes \mathcal{E}_*) \), for some Hilbert spaces \( \mathcal{E} \) and \( \mathcal{E}_* \), is said to be a multi-analytic operator if it is a module map; that is, if

\[
\Theta(S_i \otimes I_\mathcal{E}) = (S_i \otimes I_{\mathcal{E}_*})\Theta, \ i = 1, \ldots, m.
\]
Given a multi-analytic operator \( \Theta \) as above, one can define a bounded linear operator \( \theta : \mathcal{E} \to F_m^2 \otimes \mathcal{E}_* \) by

\[
\theta x = \Theta(1 \otimes x) \quad (x \in \mathcal{E}).
\]

In this correspondence of \( \Theta \) and \( \theta \), each uniquely determines the other. Moreover, the operator coefficients \( \theta_\alpha \) in \( \mathcal{L}(\mathcal{E}, \mathcal{E}_*) \) of \( \Theta \) for each \( \alpha \in \mathbb{F}_m^+ \) are defined by

\[
\langle \theta_\alpha x, y \rangle = \langle \theta x, e_\alpha \otimes y \rangle = \langle \Theta(1 \otimes x), e_\alpha \otimes y \rangle \quad (x \in \mathcal{E}, y \in \mathcal{E}_*),
\]

where \( \alpha^t = g_{i_p} \cdots g_{i_1} \) for \( \alpha = g_{i_1} \cdots g_{i_p} \). It was proved by Popescu (cf. [22]) that

\[
\Theta = \text{SOT} - \lim_{r \to 1^-} \sum_{l=0}^\infty \sum_{|\alpha| = l} r^{|\alpha|} R^\alpha \otimes \theta_\alpha,
\]

where \( R_i = U^* S_i U \) for \( i = 1, \ldots, m \), are the right creation operators on \( F_m^2 \), \( R^\alpha = R_{g_{i_1}} \cdots R_{g_{i_p}} \) for \( \alpha = g_{i_1} \cdots g_{i_p} \), and \( U \) is the unitary operator on \( F_m^2 \) defined by \( U e_\alpha = e_{\alpha^t} \) for \( \alpha \in \mathbb{F}_m^+ \).

The set of all multi-analytic operators in \( \mathcal{L}(F_m^2 \otimes \mathcal{E}, F_m^2 \otimes \mathcal{E}_*) \) coincides with \( R_\infty \otimes \mathcal{L}(\mathcal{E}, \mathcal{E}_*) \), the WOT closed algebra generated by the spatial tensor product of \( R_\infty \) and \( \mathcal{L}(\mathcal{E}, \mathcal{E}_*) \), where \( R_\infty = U^* F_\infty U \) and \( F_\infty \) is the WOT closed algebra generated by the left creation operators, \( S_1, \ldots, S_m \), and the identity operator on \( F_m^2 \).

### 3. Hilbert Modules over \( \mathbb{C}[z_1, \ldots, z_m] \)

Let \( \theta \in \mathcal{M}(\mathcal{E}, \mathcal{E}_*) \) be a module map for Hilbert spaces \( \mathcal{E} \) and \( \mathcal{E}_* \) and \( \mathcal{H}_\theta \) be the quotient module defined by the sequence

\[
\cdots \to H_m^2 \otimes \mathcal{E} \xrightarrow{M_\theta} H_m^2 \otimes \mathcal{E}_* \xrightarrow{\pi_\theta} \mathcal{H}_\theta \to 0,
\]

where \( M_\theta \) is the module map defined by \( \theta \) and \( \pi_\theta \) is the quotient map of \( H_m^2 \otimes \mathcal{E}_* \) onto the quotient by the closure of the range of \( M_\theta \). There are several questions we can ask about the relationship between these objects:

**Question 1.** Does the sequence split? That is, does there exist a module map \( \sigma_\theta : \mathcal{H}_\theta \to H_m^2 \otimes \mathcal{E}_* \) satisfying

\[
\pi_\theta \sigma_\theta = I_{\mathcal{H}_\theta},
\]

or does \( \pi_\theta \) have a right inverse?

**Question 2.** Does \( M_\theta \) have a left inverse? That is, does there exist a module map \( \psi \) in \( \mathcal{M}(\mathcal{E}_*, \mathcal{E}) \) satisfying \( \psi(z) \theta(z) = I_\mathcal{E} \) for \( z \) in \( \mathbb{B}^m \)?

**Question 3.** Does \( \theta \) have a regular inverse? That is, does there exist a multiplier \( \psi \) in \( \mathcal{M}(\mathcal{E}, \mathcal{E}_*) \) satisfying

\[
\theta(z) \psi(z) \theta(z) = \theta(z),
\]

for \( z \) in \( \mathbb{B}^m \)?

**Question 4.** Is \( \mathcal{H}_\theta \) similar to \( H_m^2 \otimes \mathcal{F} \) for some Hilbert space \( \mathcal{F} \)?

**Question 5.** Suppose \( H_m^2 \otimes \mathcal{E} \) is a skew direct sum \( S_1 + S_2 \), where \( S_1 \) and \( S_2 \) are submodules such that \( S_1 \) is isomorphic to \( H_m^2 \otimes \mathcal{E}_* \) for some Hilbert space \( \mathcal{E}_* \). Does it follow that \( S_2 \) is also isomorphic to \( H_m^2 \otimes \mathcal{F} \) for some Hilbert space \( \mathcal{F} \)?
QUESTION 6. If $\mathcal{S}$ is a complemented submodule of $H^2_m \otimes \mathcal{E}$ for some Hilbert space $\mathcal{E}$, does it follow that $\mathcal{S}$ is isomorphic to $H^2_m \otimes \mathcal{F}$ for some Hilbert space $\mathcal{F}$?

We will show, following argument from commutative algebra, that Questions 1 and 3 are equivalent and that 2 and 3 are equivalent if $\ker M = \{0\}$. Using the CLT, we show that 1 implies 1 and that 1 and 3 are equivalent. An affirmative answer to Question 6 obviously implies one to Question 3 in general and both have affirmative answers for the case $m = 1$ by the BLHT. However, we are unable to decide if an affirmative answer to Question 1 implies one for Question 5. We will have more to say about that later in this section.

One can reformulate Question 6 in the following equivalent form.

QUESTION 7. Is every complemented submodule $\mathcal{S}$ of $H^2_m \otimes \mathcal{E}$, for some Hilbert space $\mathcal{E}$, the range of $M_\psi$ for a multiplier $\psi \in \mathcal{M}(\mathcal{E}, \mathcal{F})$ with $\ker M_\psi = \{0\}$ for some Hilbert space $\mathcal{F}$?

Note that one could view an affirmative answer as a weak form of the BLHT for DA-modules.

PROPOSITION 3.1. Let $\theta \in \mathcal{M}(\mathcal{E}, \mathcal{E}_*)$ be a multiplier for Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$, and $\mathcal{H}_\theta$ be the quotient module defined by

$$
\cdots \rightarrow H^2_m \otimes \mathcal{E} \xrightarrow{M_\theta} H^2_m \otimes \mathcal{E}_* \xrightarrow{\pi_\theta} \mathcal{H}_\theta \rightarrow 0,
$$

where $\mathcal{H}_\theta = (H^2_m \otimes \mathcal{E}_*) / \operatorname{clos} [\operatorname{ran} M_\theta]$. Then there exists a multiplier $\psi \in \mathcal{M}(\mathcal{E}_*, \mathcal{E})$ satisfying

$$
\theta(z) \psi(z) \theta(z) = \theta(z),
$$

for $z \in \mathbb{B}^m$ if and only if there exists a module map $\sigma_\theta : \mathcal{H}_\theta \rightarrow H^2_m \otimes \mathcal{E}_*$ such that

$$
\pi_\theta \sigma_\theta = I_{\mathcal{H}_\theta}.
$$

Proof. Suppose there exists a multiplier $\psi \in \mathcal{M}(\mathcal{E}, \mathcal{E}_*)$ such that $\theta(z) \psi(z) \theta(z) = \theta(z)$ for $z \in \mathbb{B}^m$. Then it is straightforward to show that

$$
Q = M_\theta M_\psi
$$

is a module map on $H^2_m \otimes \mathcal{E}_*$ such that the range of $Q$ is the closure of the range of $M_\theta$ and $Q^2 = Q$. If we set $S_1 = Q(H^2_m \otimes \mathcal{E}_*)$ and $S_2 = (I - Q)(H^2_m \otimes \mathcal{E}_*)$, then $H^2_m \otimes \mathcal{E}_* = S_1 + S_2$, a skew direct sum. Define $\sigma_\theta : \mathcal{H}_\theta \rightarrow H^2_m \otimes \mathcal{E}_*$ so that $\sigma_\theta = (I - Q) \pi_\theta^{-1}$, which is well defined since $\operatorname{ran} Q = \operatorname{clos} [\operatorname{ran} M_\theta] = \ker \pi_\theta$. Moreover, $\pi_\theta \sigma_\theta = I_{\mathcal{H}_\theta}$, which completes the argument in one direction.

Now suppose there exists $\sigma_\theta : \mathcal{H}_\theta \rightarrow H^2_m \otimes \mathcal{E}_*$ so that $\pi_\theta \sigma_\theta = I_{\mathcal{H}_\theta}$. Then $P = \sigma_\theta \pi_\theta$ is a module idempotent on $H^2_m \otimes \mathcal{E}_*$ such that $\operatorname{ran} (I - P) = \ker \pi_\theta$. Thus, if we consider the map

$$
X = M_\theta^{-1} (I - P),
$$

then it is well-defined since $M_\theta^{-1}$ is one-to-one on $\operatorname{ran} (I - P)$. Moreover, $X : H^2_m \otimes \mathcal{E}_* \rightarrow H^2_m \otimes \mathcal{E}$ is a module map which determines a multiplier $\psi \in \mathcal{M}(\mathcal{E}_*, \mathcal{E})$ satisfying $\theta(z) \psi(z) \theta(z) = \theta(z)$ for $z \in \mathbb{B}^m$.

The above proof is essentially taken from commutative algebra in which one shows that an exact sequence of modules

$$
\cdots \rightarrow A \xrightarrow{\varphi_1} B \xrightarrow{\varphi_2} C \rightarrow 0,
$$

and
splits or $\varphi_2$ has a right inverse if and only if $\varphi_1$ has a regular inverse and a left inverse if and only if, in addition, $\ker \varphi_1 = \{0\}$.

Note that we do not have to assume that the range of $M_\theta$ is closed. If we use its closure to define the quotient, then if the sequence splits, then the range of $M_\theta$ is closed.

**Theorem 3.2.** Given $\theta \in \mathcal{M}(\mathcal{E}, \mathcal{E}_*)$ and $\mathcal{H}_\theta$ defined as above, if $\mathcal{H}_\theta$ is similar to $H^2_m \otimes \mathcal{F}$ for some Hilbert space $\mathcal{F}$, then the sequence splits.

**Proof.** First, assume that there exists an invertible module map $X : H^2_m \otimes \mathcal{F} \to \mathcal{H}_\theta$, and let $\varphi$ be defined by the CLT so that $P_{\mathcal{H}_\theta} M_\varphi = X$. Since $X$ is invertible we have $H^2_m \otimes \mathcal{E}_* = \text{ran} M_\varphi + \text{ran} M_\theta$.

Thus there exists a module idempotent $Q$ on $H^2_m \otimes \mathcal{E}_*$ such that $QM_\theta = M_\theta$, $\text{ran} Q = \text{ran} M_\theta$, and $\text{ran}(I - Q) = \text{ran} M_\varphi$.

Define a bounded linear operator $\hat{Q} : H^2_m \otimes \mathcal{E}_* \to (H^2_m \otimes \mathcal{E})/\ker M_\theta \cong (\ker M_\theta)^\perp \subseteq H^2_m \otimes \mathcal{E}$ by $\hat{Q}(\varphi f + \theta g) = \pi_\theta g$, where $f \in H^2_m \otimes \mathcal{F}$, $g \in H^2_m \otimes \mathcal{E}$ and $\pi_\theta : H^2_m \otimes \mathcal{E} \to (H^2_m \otimes \mathcal{E})/\ker M_\theta$ is the quotient module map and $\varphi f + \theta g$ is in $H^2_m \otimes \mathcal{E}_*$. Note that $Q = M_\theta \hat{Q}$.

Moreover, it is easy to see that $\hat{Q}$ is a module map in $L(H^2_m \otimes \mathcal{E}_*, H^2_m \otimes \mathcal{E}/\ker M_\theta)$. Therefore, another use of the CLT (Theorem 2.1) yields $\hat{Q} = \pi_\theta M_\psi$, for some $\psi \in \mathcal{M}(\mathcal{E}_*, \mathcal{E})$. Therefore,

$$Q = M_\theta \hat{Q} = M_\theta \pi_\theta M_\psi = M_\theta M_\psi$$

and hence

$$M_\theta M_\psi M_\theta = M_\theta.$$

We appeal to Proposition 3.1 to complete the proof.

Combining the proposition and the theorem yields our main result in the commutative setting.

**Corollary 3.3.** For $m > 1$, if $\mathcal{H}_\theta$ is similar to $H^2_m \otimes \mathcal{F}$ for some Hilbert space $\mathcal{F}$, then there exists a multiplier $\psi \in \mathcal{M}(\mathcal{E}_*, \mathcal{E})$ satisfying

$$\theta(z) \psi(z) \theta(z) = \theta(z), \quad \text{for } z \in \mathbb{B}^m.$$  

Question 6 raises an extremely important issue for Hilbert modules: what are the complemented submodules $S$ of $\mathcal{R} \otimes \mathbb{C}^n$ in the sense of whether or not $S$ is always isomorphic to $\mathcal{R} \otimes \mathbb{C}^k$ for some $0 < k < n$. This is certainly not true for a general Hilbert module $\mathcal{R}$. However what if $\mathcal{R}$ belongs to the class of “locally-free” Hilbert modules of multiplicity one which is the case for the DA-module $H^2_m$. For $m = 1$, an affirmative answer follows trivially from the BLHT. A less obvious argument shows that the result holds for more general “locally-free”
Hilbert modules over the unit disk such as the Bergman module. (Although the language is different, this result was proved by J. S. Fang, C. L. Jiang, X. Z. Guo, K. Ti and H. He. The relation of the seven questions in the one-variable case is closely related to the theme of the book by C. L. Jiang and F. Wang [14], where details can be found.) Further, one can establish an affirmative answer if one assume that the multiplier \( \theta \in M(E, E^*) \) is holomorphic on a neighborhood of the closure of \( \mathbb{B}^m \), at least if \( E \) and \( E^* \) are finite dimensional. However, what happens in general for “locally-free” Hilbert modules over \( \mathbb{B}^m \), such as the DA-module, is not clear at this point.

Note that a necessary condition for \( S_1 \) and \( S_2 \) with \( H^2_m \otimes \mathbb{C}^n = S_1 \oplus S_2 \) to be isomorphic to \( H^2_m \otimes \mathbb{C}^k \) and \( H^2_m \otimes \mathbb{C}^{n-k} \), respectively, is the existence of a generating set \( \{f_1, \ldots, f_n\} \) for \( H^2_m \otimes \mathbb{C}^n \) with \( \{f_1, \ldots, f_k\} \) in \( S_1 \) and \( \{f_{k+1}, \ldots, f_n\} \). If one assumes in addition the vectors are in \( M(\mathbb{C}^n, \mathbb{C}^m) \), that is also sufficient.

Note that for a complemented submodule \( S \) of \( H^2_m \otimes E \) with \( S \oplus \tilde{S} = H^2_m \otimes E \), there are generally many choices for \( \tilde{S} \) so that the skew direct sum is isomorphic to \( H^2_m \otimes E^* \) for some Hilbert space \( E^* \). (Here we allow different spaces \( E^* \).) It is not clear, but seems unlikely that there exists a canonical choice of \( E^* \) and \( \tilde{S} \) in some sense or what the “simplest” choice might be. Such ideas are related to the \( K \)-theory group introduced in [14].

4. Resolutions of Hilbert modules over \( \mathbb{C}[z_1, \ldots, z_m] \)

We now recall the notion of purity for a co-spherically contractive Hilbert module \( \mathcal{H} \) over \( \mathbb{C}[z_1, \ldots, z_m] \). Define the completely positive map

\[
P_\mathcal{H} : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})
\]

by

\[
P_\mathcal{H}(A) = \sum_{i=1}^{m} M_i A M_i^*, \quad (A \in \mathcal{L}(\mathcal{H})).
\]

Now

\[
I_\mathcal{H} \geq P_\mathcal{H}(I_\mathcal{H}) \geq P_\mathcal{H}^2(I_\mathcal{H}) \geq \cdots \geq P_\mathcal{H}^l(I_\mathcal{H}) \geq \cdots \geq 0,
\]

so that

\[
P_\infty = \text{SOT} - \lim_{l \to \infty} P_\mathcal{H}^l(I_\mathcal{H})
\]

exists and \( 0 \leq P_\infty \leq I_\mathcal{H} \). The Hilbert module \( \mathcal{H} \) is said to be pure if

\[
P_\infty = 0.
\]

A canonical example of a pure co-spherically contractive Hilbert module over \( \mathbb{C}[z_1, \ldots, z_m] \) is the DA-module \( H^2_m \otimes \mathcal{F} \), where \( \mathcal{F} \) is a Hilbert space. Moreover, for a pure co-spherically contractive Hilbert module \( \mathcal{H} \), \( \mathcal{H} \) is a quotient of \( H^2_m \otimes \mathcal{E}_s \) for some Hilbert space \( \mathcal{E}_s \), where the kernel is the range of an inner multiplier \( \theta \in M(\mathcal{E}, \mathcal{E}_s) \) for some Hilbert space \( \mathcal{E} \) (see Theorem 8.5 in [1]). Consideration of resolutions such as those in the preceding section and the ones given by this result raises the question of what kind of resolutions exist for pure co-spherically contractive Hilbert modules over \( \mathbb{C}[z_1, \ldots, z_m] \). In particular, the above result of Arveson yields a unique resolution of an arbitrary pure co-spherically contractive Hilbert
module $\mathcal{M}$ over $\mathbb{C}[z_1, \ldots, z_m]$ in terms of the DA-modules $\{H^2_m \otimes \mathcal{E}_k\}$ for Hilbert spaces $\{\mathcal{E}_k\}$ and inner multipliers $\varphi_k$ in $\mathcal{M}(\mathcal{E}_k, \mathcal{E}_{k-1})$ or partially isometric module maps $\{M_{\varphi_k}\}$ with

$$M_{\varphi_k} : H^2_m \otimes \mathcal{E}_k \rightarrow H^2_m \otimes \mathcal{E}_{k-1}, \quad k \geq 1$$

$$M_{\varphi_0} : H^2_m \otimes \mathcal{E}_0 \rightarrow \mathcal{M},$$

which is exact; that is, $\text{ran } M_{\varphi_k-1} = \ker M_{\varphi_k}$ for $k \geq 0$. Here $k = 0, 1, \ldots, N$, with the possibility of $N = +\infty$. A basic question is whether such a resolution is finite or, equivalently, whether we can take $\mathcal{E}_N = \{0\}$ for some finite $N$. That will be the case if and only if some $M_{\varphi_k}$ is an isometry or, equivalently, $\ker M_{\varphi_k} = \{0\}$. Unfortunately, the following result shows that this is not possible when $m > 1$, unless $\mathcal{M}$ is a DA-module and the resolution is a trivial one.

**Theorem 4.1.** For $m > 1$, if $V : H^2_m \otimes \mathcal{E} \rightarrow H^2_m \otimes \mathcal{E}_*$ is an isometric module map for Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$, then there exists an isometry $V_0 : \mathcal{E} \rightarrow \mathcal{E}_*$ such that

$$V(z^k \otimes x) = z^k \otimes V_0 x, \quad \text{for } k \in \mathbb{N}, x \in \mathcal{E}.$$

Moreover, ran $V$ is a reducing submodule of $H^2_m \otimes \mathcal{E}_*$ of the form $H^2_m \otimes (\text{ran } V_0)$.

**Proof.** For $x \in \mathcal{E}$, $\|x\| = 1$, we have

$$V(1 \otimes x) = f(z) = \sum_{k \in \mathbb{N}} a_k z^k,$$

for $\{a_k\} \subseteq \mathcal{E}$. Then

$$V(z_1 \otimes x) = VM_1(1 \otimes x) = M_1 V(1 \otimes x) = M_1 f = z_1 f,$$

and

$$\|z_1 f\|^2 = \|z_1 V(1 \otimes x)\|^2 = \|z_1 \otimes x\|^2 = 1 = \|f\|^2.$$

Therefore, we have

$$\sum_{k \in \mathbb{N}} \|a_k\|_{\mathcal{E}_*}^2 \|z^k\|^2 = \sum_{k \in \mathbb{N}} \|a_k\|_{\mathcal{E}_*}^2 \|z^{k+e_1}\|^2, \quad \text{where } k + e_1 = (k_1 + 1, \ldots, k_m),$$

or

$$\sum_{k \in \mathbb{N}} \|a_k\|_{\mathcal{E}_*}^2 \{\|z^{k+e_1}\|^2 - \|z^k\|^2\} = 0.$$

If $k = (k_1, \ldots, k_m)$, then

$$\|z^{k+e_1}\|^2 = \frac{(k_1 + 1)! \cdots k_m!}{(k_1 + \cdots + k_m + 1)!} \geq \frac{k_1! \cdots k_m!}{(k_1 + \cdots + k_m)!}$$

and hence, $a_k \neq 0$ implies $\|z^{k+e_1}\| = \|z^k\|$. But $\|z^{k+e_1}\| = \|z^k\|$ if and only if $k_2 = \cdots = k_m = 0$. Repeating this argument using $i = 2, \ldots, m$, we see that $a_k = 0$ unless $k = (0, \ldots, 0)$ and therefore, $f(z) = 1 \otimes y$ for some $y \in \mathcal{E}_*$. Set $V_0 x = y$ to complete the first part of the proof.

Finally, since ran $V = H^2_m \otimes (\text{ran } V_0)$, we see that ran $V$ is a reducing submodule, which completes the proof. \hfill \blacksquare

Note that this result generalizes Corollary 3.3 of [10] and is related to an earlier result of Guo, Hu and Xu [13].
The theorem implies that all resolutions by DA-modules with partially isometric maps are trivial in a sense we will make precise. We start with a definition.

**Definition 4.2.** An inner resolution of length $N$, for $N = 1, 2, 3, \ldots, \infty$, for a pure co-spherical contractive Hilbert module $\mathcal{M}$ is given by a collection of Hilbert spaces $\{E_k\}_{k=0}^N$, inner multipliers $\varphi_k \in M(E_k, E_{k-1})$ for $k = 1, \ldots, N$ and a co-isometric module map $\varphi_0 : H_m^2 \otimes E_0 \to \mathcal{M}$ so that

$$\text{ran} M_{\varphi_k} = \ker M_{\varphi_{k-1}},$$

for $k = 1, \ldots, N$. To be more precise, for $N < \infty$ one has the finite resolution

$$0 \to H_m^2 \otimes E_N \xrightarrow{M_{\varphi_N}} H_m^2 \otimes E_{N-1} \to \cdots \to H_m^2 \otimes E_1 \xrightarrow{M_{\varphi_1}} H_m^2 \otimes E_0 \xrightarrow{M_{\varphi_0}} \mathcal{M} \to 0,$$

and for $N = \infty$, the infinite resolution

$$\cdots \to H_m^2 \otimes E_N \xrightarrow{M_{\varphi_N}} H_m^2 \otimes E_{N-1} \to \cdots \to H_m^2 \otimes E_1 \xrightarrow{M_{\varphi_1}} H_m^2 \otimes E_0 \xrightarrow{M_{\varphi_0}} \mathcal{M} \to 0.$$

**Theorem 4.3.** If the pure, co-spherical contractive Hilbert module $\mathcal{M}$ possesses a finite inner resolution, then $\mathcal{M}$ is isometrically isomorphic to $H_m^2 \otimes \mathcal{F}$ for some Hilbert space $\mathcal{F}$.

**Proof.** Applying the previous theorem to $M_{\varphi_N}$, we decompose $E_N = E_N^1 \oplus E_N^2$ so that $M_{\psi_{N-1}} = M_{\psi_{N-1}} \mid_{H_m^2 \otimes E_{N-1}^1} \in M(E_{N}^2, E_{N-1})$ is an isometry onto $\text{ran} M_{\varphi_{N-1}}$. Hence, we can apply the theorem to $M_{\psi_{N-1}}$. Therefore, by induction we obtain the desired conclusion.

The following statement follows directly from the theorem.

**Corollary 4.4.** If $\theta \in M(E, E_\ast)$ defines an isometric multiplier for the Hilbert spaces $E$ and $E_\ast$, then the quotient module $\mathcal{H}_\theta = (H_m^2 \otimes E_\ast) / \text{ran} M_\theta$ is isometrically isomorphic to $H_m^2 \otimes \mathcal{F}$ for a Hilbert space $\mathcal{F}$. Moreover, $\mathcal{F}$ can be identified with $(\text{ran} V_0)^\perp$, where $V_0$ is the isometry from $E$ to $E_\ast$ given in the theorem.

A resolution of $\mathcal{M}$ can always be made longer in a trivial way. Suppose we have the resolution

$$0 \to H_m^2 \otimes E_N \xrightarrow{X_N} H_m^2 \otimes E_{N-1} \to \cdots \to H_m^2 \otimes E_0 \xrightarrow{X_0} \mathcal{M} \to 0.$$

If $E_{N+1}$ is a nontrivial Hilbert space, then define $X_{N+1}$ as the inclusion map of $H_m^2 \otimes E_{N+1} \subseteq H_m^2 \otimes (E_N \oplus E_{N+1})$ and $X_N$ equal to $X_N$ on $H_m^2 \otimes E_N \subseteq H_m^2 \otimes (E_{N+1} \oplus E_N)$ and equal to 0 on $H_m^2 \otimes E_{N+1} \subseteq H_m^2 \otimes (E_N \oplus E_{N+1})$. Then we obtain a longer resolution essentially equivalent to the original one

$$0 \to H_m^2 \otimes E_{N+1} \xrightarrow{X_{N+1}} H_m^2 \otimes (E_{N+1} \oplus E_N) \xrightarrow{X_N} \cdots \to \mathcal{M} \to 0.$$

Moreover, the new resolution will be inner if the original one is.

The proof of the preceding theorem shows that any finite inner resolution by DA-modules is equivalent to a series of such trivial extensions of the resolution

$$0 \to H_m^2 \otimes E_0 \xrightarrow{M_{\theta_0}} H_m^2 \otimes E_0 \to 0,$$

where $M_{\theta_0} = I_{H_m^2 \otimes E_0}$. We will refer to such resolution as a trivial resolution. We use that terminology to sumnerize this supplement to the theorem in the following statement. (Note that
every inner resolution for a pure co-spherically contractive Hilbert module can be obtained from the minimal resolution (cf. [12]) by adding such zero length resolutions.)

**Corollary 4.5.** All finite inner resolutions for a pure co-spherically contractive Hilbert module $\mathcal{M}$ are trivial inner resolutions.

What happens when we relax the conditions on the multipliers so that $\text{ran} M_{\phi_k} = \ker M_{\phi_{k+1}}$ for all $k$ but do not require them to be partial isometries? In this case, finite non-trivial resolutions can exist, completely analogous to what happens for the case of the Hardy or Bergman modules over $\mathbb{C}[z_1, \ldots, z_m]$ for $m > 1$. We describe an example.

Consider the module $\mathbb{C}(0, 0)$ over $\mathbb{C}[z_1, z_2]$ defined as follows:

$$p(z_1, z_2) \cdot \lambda = p(0, 0)\lambda,$$

where $p \in \mathbb{C}[z_1, z_2]$ and $\lambda \in \mathbb{C}$, and the following resolution:

$$0 \rightarrow H^2_2 \xrightarrow{X_2} H^2_1 \oplus H^2_2 \xrightarrow{X_1} H^2_0 \xrightarrow{X_0} \mathbb{C}(0, 0) \rightarrow 0,$$

where $X_0f = f(0, 0)$ for $f \in H^2_2$, $X_1(f_1 \oplus f_2) = M_{z_1}f_1 + M_{z_2}f_2$ for $f_1 \oplus f_2 \in H^2_1 \oplus H^2_2$, and $X_2f = M_{z_2}f \oplus (-M_{z_1}f)$ for $f \in H^2_2$. One can show that this sequence, which is closely related to the Koszul complex, is exact and non-trivial; that is, it does not split.

Another question one can ask is the relationship between the inner resolution for a pure co-spherically contractive Hilbert module given by the result of Arveson and more general not necessarily inner resolutions by DA-modules. In particular, is there any relation between the shortest length of a not necessarily inner resolution to the inner resolution.

**5. Hilbert modules over $\mathbb{F}[Z_1, \ldots, Z_m]$**

We begin by noting that the definition of a pure co-spherically contractive Hilbert module does not depend upon the underlying algebra; that is, with appropriate change of notation, the concept of a pure contractive Hilbert module $\mathcal{K}$ over $\mathbb{F}[Z_1, \ldots, Z_m]$ can be defined in a similar way. Popescu proved that any pure contractive Hilbert module over $\mathbb{F}[Z_1, \ldots, Z_m]$ can be realized as a quotient module of $\mathbb{F}^2_m \otimes \mathcal{E}$ for some Hilbert space $\mathcal{E}$ (see Theorem 2.10 and references in [22]).

More precisely, given a pure contractive Hilbert module $\mathcal{K}$ over $\mathbb{F}[Z_1, \ldots, Z_m]$, one can associate a multi-analytic operator $\Theta$ in $\mathcal{L}(\mathbb{F}^2_m \otimes \mathcal{E}, \mathbb{F}^2_m \otimes \mathcal{E}_s)$ for some Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_s$, the characteristic function of $\mathcal{K}$ which is isometric and a complete unitary invariant for $\mathcal{K}$ (see [21] and [22]).

When the Hilbert module $\mathcal{H}$ is defined over $\mathbb{C}[z_1, \ldots, z_m]$, one can also define the characteristic function of $\mathcal{H}$. It is in $\mathcal{M}(\mathcal{E}, \mathcal{E}_s)$ (see Theorem 3.7 in [5] and Theorem 4.3 in [21]), but is not isometric in this case. In fact, it can only be isometric when $\mathcal{H} = H^2_m \otimes \mathcal{E}$ for some Hilbert space $\mathcal{E}$.

We now consider the analogous results to those in Section 3 for the noncommutative case. First we need to recall an analogue of the BLHT in this setting due to Popescu ([21], [22]).

**Theorem 5.1. (Popescu)** If $\mathcal{S}$ is a closed subspace of $\mathbb{F}^2_m \otimes \mathcal{F}$ for some Hilbert space $\mathcal{F}$, then the following are equivalent:
(i) $S$ is a submodule of $F^2_m \otimes F$.
(ii) There exists an auxiliary Hilbert space $\mathcal{E}$ and an (isometric) inner multi-analytic operator $\Phi : F^2_m \otimes \mathcal{E} \rightarrow F^2_m \otimes F$ such that

$$S = \Phi(F^2_m \otimes \mathcal{E}).$$

Although we use the following lemma only in the non-commutative case, it holds also in the commutative case as indicated.

**Lemma 5.2.** If $\mathcal{H}$ is a co-spherically contractive Hilbert module over $\mathbb{C}[z_1, \ldots, z_m]$ or $\mathbb{F}[Z_1, \ldots, Z_m]$, respectively, which is similar to $H^2_m \otimes F$, or $F^2_m \otimes F$, respectively, for some Hilbert space $F$, then $\mathcal{H}$ is pure.

**Proof.** We use the notation for the commutative case but the proof in both cases is the same. Let $X : \mathcal{H} \rightarrow H^2_m \otimes F$ be an invertible module map. Then $M_i = X^{-1}M_zX$ for all $i = 1, \ldots, m$. Since $\{P_l(I_H)\}_{l=0}^\infty$ is a decreasing sequence of positive operators, it suffices to show that

$$\text{WOT} - \lim_{l \rightarrow \infty} P_l^\prime(I_{H^2_m}) = 0.$$

To see that this is the case, let $f_1$ and $g_1$ be vectors in $\mathcal{H}$ and set $f = X^{-1}f_1$ and $g = X^{-1}g_1$. Then

$$|\sum_{|k|=l} M^k M^*k f_1, g_1| = |\sum_{|k|=l} X^{-1} M^k X X^* M^*k X^{*-1} f_1, g_1| = |\sum_{|k|=l} M^k X X^* M^*k f, g|$$

$$\leq \sum_{|k|=l} |\langle M^k X X^* M^*k f, g \rangle| = \sum_{|k|=l} |\langle X^* M^*k f, X^* M^*k g \rangle|$$

$$\leq \|X\|^2 \left( \sum_{|k|=l} \|M^*k f\|^2 \right)^\frac{1}{2} \left( \sum_{|k|=l} \|M^*k g\|^2 \right)^\frac{1}{2}.$$

Letting $l \rightarrow \infty$ in the last expression, we conclude that the required limit is zero, which completes the proof.

Actually, the proof shows that two similar co-spherically contractive Hilbert modules over $\mathbb{C}[z_1, \ldots, z_m]$, or two similar contractive Hilbert modules over $\mathbb{F}[Z_1, \ldots, Z_m]$, are either both pure or both not pure.

**Theorem 5.3.** Let $\mathcal{H}$ be a pure contractive Hilbert module over $\mathbb{F}[Z_1, \ldots, Z_m]$. Then $\mathcal{H}$ is similar to $F^2_m \otimes F$ for some Hilbert space $F$ if and only if the characteristic operator $\Theta$ of $\mathcal{H}$ in $\mathcal{L}(F^2_m \otimes \mathcal{E}, F^2_m \otimes \mathcal{E}_*)$, for some Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$, is left invertible; that is, if and only if there exists a multi-analytic operator $\Psi : F^2_m \otimes \mathcal{E}_* \rightarrow F^2_m \otimes \mathcal{E}$ such that

$$\Psi \Theta = I_{F^2_m \otimes \mathcal{E}}.$$

**Proof.** First, we realize the pure contractive Hilbert module $\mathcal{H}$ as the quotient module given by its characteristic function, which is an isometric inner multi-analytic map $\Theta$ as

$$\mathcal{H} \cong \mathcal{H}_\Theta = (F^2_m \otimes \mathcal{E}_*)/\Theta(F^2_m \otimes \mathcal{E}).$$
Now given a module map \( X : F^2_m \otimes \mathcal{F} \to \mathcal{H}_\Theta \), we appeal to the noncommutative analogue of the CLT (see Theorem 6.1 in [21] or Theorem 5.1 in [22]) to obtain a multi-analytic operator \( \Phi : F^2_m \otimes \mathcal{F} \to F^2_m \otimes \mathcal{E}_* \) such that
\[
P_{\mathcal{H}_\Theta} \Phi = X.
\]
Therefore, the bounded module map
\[
Z : (F^2_m \otimes \mathcal{F}) \oplus (F^2_m \otimes \mathcal{E}) \to F^2_m \otimes \mathcal{E}_*
\]
defined by
\[
Z(f \oplus g) = \Phi f + \Theta g,
\]
for all \( f \oplus g \in (F^2_m \otimes \mathcal{F}) \oplus (F^2_m \otimes \mathcal{E}) \) is invertible if and only if \( X \) is invertible. This follows by noting that \( X \) is invertible if and only if the range of \( Z \), which is the span of \( \mathcal{H}_\Theta \) and ran \( \Theta \) is \( F^2_m \otimes \mathcal{E}_* \), and \( X \) is one-to-one if and only if \( Z \) is.

To prove the necessary part of the theorem, assume that \( X \) is invertible or, equivalently, \( Z \) is invertible. Consequently, we can define a module idempotent \( Q \) on \( F^2_m \otimes \mathcal{E}_* \), such that
\[
Q \Theta = \Theta
\]
and
\[
\text{ran } Q = \text{ran } \Theta.
\]
Then the bounded module map \( \hat{Q} : F^2_m \otimes \mathcal{E}_* \to F^2_m \otimes \mathcal{E} \) defined by
\[
\hat{Q}(\Phi f + \Theta g) = g, \quad \Phi f + \Theta g \in (F^2_m \otimes \mathcal{E}_*)
\]
satisfies
\[
Q = \Theta \hat{Q}.
\]
Since \( \hat{Q} \) is a module map, there exists a multi-analytic operator \( \Psi : F^2_m \otimes \mathcal{E}_* \to F^2_m \otimes \mathcal{E} \) such that
\[
\hat{Q} = \Psi.
\]
Hence
\[
\Theta = Q \Theta = \Theta \hat{Q} \Theta = \Theta \Psi \Theta.
\]
Since \( \Theta \) is an isometry, the necessary part follows; that is, \( \Theta \) has a left inverse.

To prove the sufficiency part, let \( \Psi : F^2_m \otimes \mathcal{E}_* \to F^2_m \otimes \mathcal{E} \) be a multi-analytic operator such that
\[
\Psi \Theta = I_{F^2_m \otimes \mathcal{E}}.
\]
Then \( Q = \Theta \Psi \) is an idempotent and any \( f \) in \( F^2_m \otimes \mathcal{E}_* \) can be expressed as
\[
f = (f - \Theta \Psi f) + \Theta \Psi f,
\]
where \( f - \Theta \Psi f \) is in ker \( \Psi \) and \( \Theta \Psi f \) is in ran \( \Theta \). Thus,
\[
\text{ran } Q = \text{ran } \Theta, \quad \text{and} \quad \ker \Psi = \text{ran } (I - Q).
\]
Since ker \( \Psi \) is a submodule of \( F^2_m \otimes \mathcal{E}_* \), by the noncommutative version of the BLHT (see Theorem 2.2 in [21] or Theorem 1.2 in [22]), there exists an inner multi-analytic operator \( \Phi : F^2_m \otimes \mathcal{F} \to F^2_m \otimes \mathcal{E}_* \) for some Hilbert space \( \mathcal{F} \) such that
\[
\ker \Psi = \text{ran } (I - Q) = \Phi(F^2_m \otimes \mathcal{F}),
\]
Consequently,
\[ F^2_m \otimes \mathcal{E}_* = \text{ran} \Phi + \text{ran} \Theta. \]
Then one can define the invertible module map \( Z \) as in the necessary part and setting \( X = P_{\mathcal{H}_m} \Phi \) defines the required similarity, which completes the proof.

As mentioned in the introduction, specializing the preceding proof to the (commutative) \( m = 1 \) case yields a new proof of the old result on the similarity of contraction operators to unilateral shifts.

The main difference in the above proof and that of Theorem 3.2 for the commutative case is that here we can assume that \( \Theta \) has no kernel and one of the complemented submodule is isomorphic to a DA-module.

In the proof of Theorem 3.2, we did not use the fact that the characteristic function is an isometry. Hence we can state a more general result in terms of a module resolution.

**Theorem 5.4.** Let \( \mathcal{E} \) and \( \mathcal{E}_* \) be Hilbert spaces and \( \Theta : F^2_m \otimes \mathcal{E} \to F^2_m \otimes \mathcal{E}_* \) be a multi-analytic operator. Then the quotient space \( \mathcal{H}_\Theta \), given by \( (F^2_m \otimes \mathcal{E}_*)/\overline{\text{clos} \text{ran} \Theta} \) is similar to \( F^2_m \otimes \mathcal{F} \) for some Hilbert space \( \mathcal{F} \) if and only if \( \Theta \Psi \Theta = \Theta \) for some multi-analytic operator \( \Psi : F^2_m \otimes \mathcal{E}_* \to F^2_m \otimes \mathcal{E} \).

6. Concluding remarks

Observe that if \( \mathcal{H} \) is a Hilbert module over \( \mathbb{C}[z_1, \ldots, z_m] \) (or \( A(\Omega) \), where \( \Omega \) is a bounded connected open subset of \( \mathbb{C}^m \)), Theorem 3.2 remains true under the hypotheses that the analogue of the CLT holds for the class of Hilbert modules. In particular, Theorem 3.2 and the results in Section 4 can be generalized for any reproducing kernel Hilbert module where the kernel is given by a complete Nevanlinna-Pick kernel.

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