NEW INVARIANTS AND ATTRACTING DOMAINS FOR
HOLOMORPHIC MAPS IN $\mathbb{C}^2$ TANGENT TO THE IDENTITY

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Abstract. We study holomorphic maps of $\mathbb{C}^2$ tangent to the identity at a
fixed point which have degenerate characteristic directions. With the help of
some new invariants, we give sufficient conditions for the existence of attracting
domains in these degenerate characteristic directions.

1. Introduction

A holomorphic map $f$ in $\mathbb{C}^n$ is said to be tangent to the identity at a
fixed point $p$, if the Jacobian $df_p$ is equal to the identity matrix. Choose local coordinates
$(z, w)$ with $p$ as the origin $O$. Let $f(z, w) = (z + \sum_{i=2}^{\infty} p_i(z, w), w + \sum_{i=2}^{\infty} q_i(z, w))$
be the homogeneous expansion of $f$, where $p_i(z, w)$ and $q_i(z, w)$ are homogeneous
of degree $i$. The order $k$ of $f$ at $O$ is by definition the minimum of $i$ such that either
$p_i(z, w) \neq 0$ or $q_i(z, w) \neq 0$. A direction $[z : w]$ is called a characteristic direction
of $f$, if there exists $\lambda \in \mathbb{C}$ such that $p_k(z, w) = \lambda z$ and $q_k(z, w) = \lambda w$. If $\lambda \neq 0$,
then $[z : w]$ is said to be non-degenerate, otherwise degenerate.

The study of local dynamics of holomorphic maps tangent to the identity has
been centered on generalizing the well-known Leau-Fatou Flower Theorem in the
one-dimensional case (see e.g. [4], [7]). In [11], Hakim proved such a generalization
for generic maps of $\mathbb{C}^n$, i.e. those with non-degenerate characteristic directions. In
[2], Abate obtained a full generalization in dimension two. In particular, Abate intro-
duced a “residual index” to each characteristic direction of a map tangent to the
identity and showed that there exist “parabolic curves” in directions with residual
index not in $\mathbb{Q}^+ \cup \{0\}$. In [13], Molino showed the existence of parabolic curves
in directions with non-zero residual index, under a mild extra assumption. In [12],
Hakim provided sufficient conditions for the existence of attracting domains in non-
degenerate characteristic directions. In [17], the author introduced a new invariant,
called the “non-dicritical order”, associated to each non-degenerate characteristic
direction and provided further sufficient conditions for the existence of attracting
domains in such directions. More recently, there have been some studies on the
local dynamics of so called “one-resonant” and “multi-resonant” biholomorphisms
(see e.g. [8, 9, 10, 14]).

The general theory for local dynamics in degenerate characteristic direc-
tions with zero residual index is yet to be developed. Characteristic directions of a
holomorphic map in $\mathbb{C}^2$ can be divided into three types: irregular, Fuchsian and

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apparent (cf. [5]). In [18, 19], Vivas gave sufficient conditions for the existence of attracting domains in the irregular and Fuchsian directions. However, no general sufficient conditions are known for the existence of attracting domains in the apparent directions.

In this paper, we define some new invariants associated to a degenerate characteristic direction of holomorphic maps in $\mathbb{C}^2$ and give sufficient conditions for the existence of attracting domains in such a direction. The currently known invariants such as non-degeneracy and residual index are defined using only the leading nonlinear terms of the homogeneous expansion of a map. However, examples show that sometimes the higher order nonlinear terms also play an important role in the local dynamics of a map. Our new invariants capture exactly this feature.

Our main result is the following

**Theorem 1.1.** Let $f$ be a holomorphic map in $\mathbb{C}^2$ tangent to the identity at a fixed point $p$. Assume that $[v]$ is a degenerate characteristic direction of $f$, which is essentially non-degenerate. If $f$ is transversally attracting in the direction $[v]$, then there exists an attracting domain in the direction $[v]$ for $f$ at $p$.

In section 2, we will define our new invariants and explain the terms essentially non-degenerate and transversally attracting. In section 3, we prove Theorem 1.1.

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2. New invariants

Let $f$ be a holomorphic map in $\mathbb{C}^2$, tangent to the identity at a fixed point $p$, with order $t + 1$, $t \geq 1$. Assume that $[v]$ is a degenerate characteristic direction of $f$. In suitable local coordinates $(z, w)$ around $p$, we can assume that $p$ is the origin $O$ and $[v] = [z : w] = [1 : 0]$. Write $f$ as

$$
\begin{align*}
(2.1) \quad z_1 &= z + wP_t(z, w) + O(t + 2), \\
w_1 &= w + wQ_t(z, w) + O(t + 2),
\end{align*}
$$

where $P_t(z, w)$ and $Q_t(z, w)$ are homogeneous of degree $t$. We say that $f$ is generically degenerate in the direction $[v]$ if $w \nmid P_t(z, w)$.

**Remark 2.1.** A holomorphic map $f$ of order $t + 1$ at the origin, with a characteristic direction $[v]$ in suitable local coordinates $(z, w)$, can be written as

$$
\begin{align*}
(2.1) \quad z_1 &= z + p_{t+1}(z, w) + O(t + 2), \\
w_1 &= w + q_{t+1}(z, w) + O(t + 2),
\end{align*}
$$

where $p_{t+1}(z, w) = \sum_{i=0}^{t+1} a_i z^{t+1-i} w^i$ and $q_{t+1}(z, w) = \sum_{j=1}^{t+1} b_j z^{t+1-j} w^j$. Then $[1 : 0]$ is a non-degenerate characteristic direction if and only if $a_0 \neq 0$, a generic condition. And $[1 : 0]$ is a generically degenerate characteristic direction if and only if $a_0 = 0$ and $b_1 \neq 0$, a generic condition among degenerate characteristic directions.

**Remark 2.2.** A generically degenerate characteristic direction is an apparent characteristic direction.
Let $G$ be the group of local changes of coordinates which preserves the degenerate characteristic direction as $[1 : 0]$. Then each element $(Z, W) = \Phi(z, w)$ of $G$ takes the form
\[
(2.2) \begin{cases}
Z = lz + mw + O(2), \\
W = nw + O(2),
\end{cases}
\]
with $l, n \neq 0$. Under such a change of coordinates, we write $f$ as
\[
(2.3) \begin{cases}
Z_1 = Z + P_\Phi(Z) + WS_\Phi(Z, W), \\
W_1 = W + Q_\Phi(Z) + WR_\Phi(Z) + W^2T_\Phi(Z, W).
\end{cases}
\]
It is easy to check that $\text{ord}_R \Phi(Z) = t$ for every $\Phi \in G$, thus being generically degenerate is well defined.

We define the virtual order to be
\[
s := \max_{\Phi \in G} \text{ord}_Q \Phi(Z),
\]
and set $\nu := s - t$, which we call the weight.

**Remark 2.3.** We allow $s = \infty$, which is the case if and only if there exists an $f$-invariant curve passing through $p$ tangent to $[v]$.

Let $H$ be the subgroup of $G$ consisting of $\Phi$’s with $\text{ord}_Q \Phi(Z) = s$. Then we define the essential order to be
\[
\mu := \max_{\Phi \in H} \text{ord}_P \Phi(Z).
\]
Obviously both the virtual order and the essential order are invariants associated to $f$ in the degenerate characteristic direction $[v]$. Finally, we say that $[v]$ is an essentially non-degenerate characteristic direction of $f$ if
\[
\mu < s.
\]
Note that this implies $\nu \geq 3$.

**Remark 2.4.** For an essentially non-degenerate characteristic direction, we certainly have $\mu < \infty$. However, the case $\mu = \infty$ is also interesting. For instance, if $\mu = s = \infty$ then there exists a line of fixed points through $p$. For the dynamics in this case, see e.g. [6].

For later convenience, set $r := \mu - 1$. Rewrite (2.1) as
\[
(2.4) \begin{cases}
z_1 = z + az^{r+1} + P(z) + wS(z, w), \\
w_1 = w + bw^t w + dz^s + Q(z) + wR(z) + w^2T(z, w),
\end{cases}
\]
with $r \geq t + 1$, $a, b \neq 0$, $d \neq 0$ if $s < \infty$, $P(z) = O(z^{r+2})$, $Q(z) = O(z^{s+1})$, $R(z) = O(z^{t+1})$, $S(z, w) = O(t)$ and $T(z, w) = O(t - 1)$.

Define the director in the characteristic direction $[v]$ to be
\[
\alpha := -b(-a)^{-t/r}.
\]
We say that $f$ is transversally attracting in $[v]$ if
\[
(2.5) \quad \text{Re} \alpha > 0.
\]
Remark 2.6. The generalization of Hakim’s results in [11] to the degenerate case.

characteristic direction defined in [11]. In this sense, our main theorem is a natural
r
a non-degenerate characteristic direction is essentially non-degenerate and we have

can be made replacing generically degenerate by non-degenerate. In the latter case,

a degenerate characteristic direction for \( f \)

weak. In fact, one readily checks that the only case when this condition fails is when

\( t/r = 1/2 \) and \( \alpha \) is purely imaginary.

Remark 2.6. Although above definitions were given under the assumption that \([v]\) is
degenerate characteristic direction for \( f \), it is easy to see that the same definitions

be made replacing generically degenerate by non-degenerate. In the latter case,

a non-degenerate characteristic direction is essentially non-degenerate and we have

Then our director minus one is exactly the director of

Not depend on the choice of \( \Phi \).

It is then easy to see that the director is well-defined and the condition (2.5) does

not depend on the choice of \( \Phi \).

We now prove Theorem 1.1.

First assume that \( (\mathbf{v}, \mathbf{w}) \) is of the form (2.2), (2.4) becomes

\[
\begin{align*}
Z_1 &= Z + at^{-r}Z^{r+1} + \tilde{P}(Z) + W\tilde{S}(Z, W), \\
W_1 &= W + bl^{-s}Z^sW + dnl^{-s}Z^s + \tilde{Q}(Z) + W\tilde{R}(Z) + W^2\tilde{T}(Z, W).
\end{align*}
\]

It is then easy to see that the director is well-defined and the condition (2.5) does

\( \alpha \) different values. Hence

\( (3.1) \)

\( (3.2) \)

Denote by \( D \) the open set

\[
\{ (z, w) \in \mathbb{C}^2 : z \in V_{r, \delta}, \ |w| < |z|^{\nu - \tau} \},
\]

for \( 0 < \tau \ll 1 \). Write \( (z_n, w_n) = f^n(z, w) \). We want to show that \( f(D) \subset D \) and

\( (z_n, w_n) \to (0, 0) \) as \( n \to \infty \).

Set \( l := \min \{i + j\nu : z^i w^j \in S(z, w) \} \). By assumption, we have \( \mu < s \), which

easily implies that \( \mu < l + \nu \). Thus for \( (z, w) \in D \), from (3.1), we have

\( (3.3) \)

\( (3.4) \)

Write \( |w| = |z|^\gamma \) for some \( \gamma = \gamma(z, w) > 1 \). If \( \gamma < \nu \), then

\[
\frac{|w_1|}{|z_1|^{\nu - \tau}} \leq \frac{|w|}{|z|^{\nu - \tau}} |1 - cz^t + \lambda z^{\nu - \gamma}z^t + o(z^t)| < \frac{|w|}{|z|^{\nu - \tau}} < 1.
\]
If $\gamma \geq \nu$, then
\[
\frac{|w_1|}{|z_1|^{|\nu-\gamma|}} \leq \frac{|w_1|}{|z|^\tau} |z|^7 |1 + o(1)| + \lambda |z|^4 |1 + o(1)| < 1.
\]
Thus, we have $|w_1| < |z|^{|\nu-\gamma|}$.

Write $z = e(z) e^{i \delta(z)}$ with $0 < \epsilon(z) < \epsilon$ and $|\delta(z)| < \delta$. Denote $x = 1 - z^*(1/r + o(1))$. Then it is easy to see that $|x| < 1$ and $\arg x$ is of different sign as $\delta(z)$ with $|\arg x| < |\delta(z)|$. From (3.3) we have
\[
|z_1| = \epsilon(z) |x| < \epsilon, \quad |\arg z_1| = |\delta(z) + \arg x| < |\delta(z)| < \delta.
\]
Therefore, we have shown that $f(D) \subset D$.

From (3.3), we have
\[
\frac{1}{|z_1|} = \frac{1}{z^r} + 1 + o(1),
\]
from which we get the estimate
\[
(3.6) \quad z_n \sim \frac{1}{n^{1/r}}.
\]
Set $b_k = 1 - cz_k^4 + o(z_k^4)$. From (3.4), we have
\[
(3.7) \quad w_n = w \prod_{k=0}^{n-1} b_k + \lambda \sum_{l=0}^{n-1} z_l^s \prod_{m=l+1}^{n-1} b_m + h.o.t..
\]
For $z$ small enough, we have
\[
(3.8) \quad \prod_{m=l+1}^{n-1} b_m \sim e^{\sum_{m=l+1}^{n-1} \log b_m} \sim e^{-c \sum_{m=l+1}^{n-1} z_m^4}.
\]
From (3.6) and (3.8), we get
\[
(3.9) \quad \prod_{m=l+1}^{n-1} |b_m| \sim e^{-\beta \sum_{m=l+1}^{n-1} m^{-\frac{\nu}{r}}} \sim e^{-\beta(n^{1-r}-l^{1-r})}.
\]
Therefore
\[
(3.10) \quad \sum_{l=0}^{n-1} |z_l|^s \prod_{m=l+1}^{n-1} |b_m| \sim \sum_{l=1}^{n-1} l^{-\frac{\nu}{r}} e^{-\beta(n^{1-r}-l^{1-r})} = e^{-\beta n^{1-r}} \sum_{l=1}^{n-1} l^{-\frac{\nu}{r}} e^{\beta l^{1-r}}.
\]
For $n$ large, we have
\[
(3.11) \quad \sum_{l=1}^{n-1} l^{-\frac{\nu}{r}} e^{\beta l^{1-r}} \sim \int_1^n x^{-\frac{\nu}{r}} e^{\beta x^{1-r}} dx \sim n^{-\frac{\nu}{r} + \frac{\nu}{r} e^{\beta n^{1-r}}}.
\]
From (3.7), (3.9), (3.10) and (3.11), we finally have the estimate
\[
(3.12) \quad |w_n| \sim \frac{1}{n^{1/r}}, \quad (s < \infty).
\]
For $s = \infty$, denote by $D$ the open set
\[
\{(z,w) \in C^2 : z \in V_{\epsilon, \delta}, |w| < |z|^\kappa\},
\]
for $\kappa \gg 1$. Then a similar (and simpler) argument as above shows that $f(D) \subset D$ and
\[
(3.13) \quad z_n \sim \frac{1}{n^{1/r}}; \quad |w_n| \sim e^{-\beta n^{(r-1)/r}}, \quad (s = \infty).
\]
This completes the proof of Theorem 1.1.

Remark 3.1. For $r = 2$ and $t = 1$, a special case of maps as in (2.4) was studied in [15]. (As noted there, we do not get a “flower” of attracting domains, but only some “petals”.) Such maps also show up in the study of the local dynamics of holomorphic maps of $\mathbb{C}^2$ with a Jordan fixed point (cf. [16], see also [1, 3]).

REFERENCES

[1] M. Abate; Diagonalization of non-diagonalizable discrete holomorphic dynamical systems, Amer. J. Math. 122 (2000), 757-781.
[2] M. Abate; The residual index and the dynamics of holomorphic maps tangent to the identity, Duke Math. J. 107 (2001), 173-207.
[3] M. Abate; Basin of attraction in quadratic dynamical systems with a Jordan fixed point, Nonlinear Anal. 51 (2002), 271-282.
[4] M. Abate; Discrete holomorphic local dynamical systems, In "Holomorphic Dynamical Systems", Eds. G. Gentili, J. Guenot, G. Patrizio, Lect. Notes in Math. 1998, Springer, Berlin, 2010, 1-55.
[5] M. Abate, F. Tovena; Poincaré-Bendixson theorems for meromorphic connections and homogeneous vector fields, J. Differential Equations 251 (2011), 2612-2684.
[6] F. Bracci; The dynamics of holomorphic maps near curves of fixed points, Annali Scuola Norm. Sup. Pisa, Cl. Sci. (5) Vol. II (2003), 493-520.
[7] F. Bracci; Local dynamics of holomorphic diffeomorphisms, Boll. UMI (8) 7-B (2004), 609-636.
[8] F. Bracci, J. Raissy, D. Zaitsev; Dynamics of multi-resonant biholomorphisms, Int. Math. Res. Not. 20 (2013), 4772-4797.
[9] F. Bracci, F. Rong; Dynamics of quasi-parabolic one-resonant biholomorphisms, J. Geom. Anal., to appear (doi: 10.1007/s12220-012-9382-5).
[10] F. Bracci, D. Zaitsev; Dynamics of one-resonant biholomorphisms., J. Eur. Math. Soc. 15 (2013), 179-200.
[11] M. Hakim; Analytic transformations of $(\mathbb{C}^2,0)$ tangent to the identity, Duke Math. J. 92 (1998), 403-428.
[12] M. Hakim; Transformations tangent to the identity. Stable pieces of manifolds, Preprint, 1998.
[13] L. Molino; The dynamics of maps tangent to the identity and with nonvanishing index, Trans. Amer. Math. Soc. 361 (2009), 1597-1623.
[14] J. Raissy, L. Vivas; Dynamics of two-resonant biholomorphisms, Preprint, 2012, arXiv: 1211.3103.
[15] F. Rong; Quasi-parabolic analytic transformations of $\mathbb{C}^n$, J. Math. Anal. Appl. 343 (2008), 99-109.
[16] F. Rong; Local dynamics of holomorphic maps in $\mathbb{C}^2$ with a Jordan fixed point, Michigan Math. J. 62 (2013), 843-856.
[17] F. Rong; The non-dicritical order and attracting domains of holomorphic maps tangent to the identity, Internat. J. Math. 25 (2014), 1450003, 10 pp.
[18] L. Vivas; Basins of attraction along degenerate characteristic directions, J. Geom. Anal. 22 (2012), 352-382.
[19] L. Vivas; Degenerate characteristic directions for maps tangent to the identity, Indiana U. Math. J. 61 (2012), 2019-2040.