SUMMATIONS ASSOCIATED WITH GAMMA EXPONENTIATED EXPONENTIAL WEIBULL DISTRIBUTION

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Abstract. Considering the recently studied Gamma exponentiated exponential Weibull GEEW(θ)
probability distribution [7] surprising infinite summations are obtained for series which building blocks
are special functions like lower and upper incomplete Gamma, Fox–Wright Ψ, Meijer G or Whittaker
W functions.

1. Introduction and preliminaries

Adding parameters to an existing distribution enables one to obtain classes of more flexible distribu-
tions. Zografos and Balakrishnan [9] introduced an interesting method for adding a new parameter to
an existing distribution. The new distribution provides more flexibility to model various types of data.
The baseline distribution has the survivor function
\[ G(x) = 1 - G(x). \]
Then, the Gamma–exponentiated extended distribution has cumulative distribution function (CDF) \( F(x) \) given by
\[ F(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\log G(x)} t^{\alpha-1} e^{-t} dt, \quad \alpha > 0, \ x \in \mathbb{R}. \]
The related gamma–exponentiated extended probability density function (PDF) can be expressed in the
following form:
\[ f(x) = \frac{1}{\Gamma(\alpha)} (\log G(x))^{\alpha-1} g(x), \quad \alpha > 0, \ x \in \mathbb{R}, \]
where \( g = G' \). The so–called regularized Gamma function
\[ Q(a, z) = \frac{\Gamma(a, z)}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_z^{\infty} t^{a-1} e^{-t} dt, \quad \Re(a) > 0, \]
where \( \Gamma(a, x) \) denotes the familiar upper incomplete Gamma function. Both, regularized Gamma and incomplete Gamma, are in-built in Mathematica under GammaRegularized[a, z] and Gamma[a,z] respectively.

We reformulate their model by applying the gamma–exponentiated technique [9] in the following way.
Choosing the baseline distribution’s survivor function to be \( \overline{G}(x) = 1 - G(x) = \exp\{-h(x)\} \), where \( h: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) denotes a nonnegative Borel function. Accordingly, the rv \( X \), defined on a standard probability space \( (\Omega, \mathcal{F}, P) \), having CDF and PDF
\[ F(x) = (1 - Q(\alpha, h(x))) 1_{\mathbb{R}_+}(x) \]
\[ f(x) = \frac{h'(x)}{\Gamma(\alpha)} h^{\alpha-1}(x) e^{-h(x)} 1_{\mathbb{R}_+}(x), \]
we call Gamma exponentiated \( h \) distributed, noting this \( X \sim GE(\alpha,h) \). Here, and in what follows \( 1_A(x) \) denotes the characteristic function of the set \( A \), that is \( 1_A(x) = 1 \) when \( x \in A \) and equals 0 else. The well–known confluent hypergeometric form of the upper incomplete Gamma function enables to write
\[ F(x) = \frac{h^{\alpha}(x)}{\Gamma(\alpha + 1)} \, _1F_1(\alpha; \alpha + 1; -h(x)) 1_{\mathbb{R}_+}(x), \]

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where the confluent hypergeometric function’s (or in other words Kummer’s function of the first kind) series definition reads
\[ _1F_1(a; b; z) = \sum_{n \geq 0} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}. \]

The related Mathematica code is `Hypergeometric1F1[a, b, z].`

By choosing \( h(x) = \lambda x + \beta x^k \), that is when the baseline survivor function \( \bar{S}(x) = \exp\{-(\lambda x + \beta x^k)\} \), Pogány and Saboor introduced and discussed in detail the so-called Gamma–exponentiated exponential Weibull distribution \( \text{GEEW}(\theta), \theta = (\lambda, \beta, k, \alpha) > 0 \), see [7, Introduction]. The related CDF and PDF are
\[ F(x) = (1 - Q(\alpha, \lambda x + \beta x^k)) 1_{\mathbb{R}_+}(x) \]
\[ = \frac{(\lambda x + \beta x^k)^\alpha}{\Gamma(\alpha + 1)} _1F_1(\alpha; \alpha + 1; -(\lambda x + \beta x^k)) 1_{\mathbb{R}_+}(x), \]
\[ f(x) = \frac{1}{\Gamma(\alpha)} (\lambda + \beta k x^{k-1}) e^{-\lambda x-\beta x^k} (\lambda x + \beta x^k)^{\alpha-1} 1_{\mathbb{R}_+}(x), \]
respectively. Being our main goal to present summation formulae associated with \( \text{GEEW}(\theta) \), we need the following results regarding the moments of a rv \( X \sim \text{GEEW}(\theta) \) and to the a rv \( Y \sim \text{GE}(\alpha, h) \).

**Theorem 1.** Consider the rv \( X \sim \text{GE}(\alpha, h), \alpha > 0 \) defined by the CDF (1.1) and the related PDF (1.2). Then the rv \( Y = h(X) \sim \text{Gamma}(\alpha, 1) \) and the related CHF reads
\[ \phi_Y(t) = (1 - it)^{-\alpha}, \quad t \in \mathbb{R}, \]
where \( \text{Gamma}(\alpha, 1) \) stands for the two–parameter Gamma distribution and \( h: \mathbb{R}_+ \mapsto \mathbb{R}_+ \) is a non–decreasing Borel function. Moreover, we have \[ \mathbb{E}Y^s = (\alpha)_s, \quad \Re(s) > -\alpha. \]

**Proof.** By definition, the characteristic function (CHF) of an rv \( X \) is the Fourier transform of the related PDF \( f_X(x) \):
\[ \phi_Y(t) = \mathbb{E}e^{itY} = \int_{\mathbb{R}} e^{ith(x)} f(x) \, dx, \quad t \in \mathbb{R}. \]
Accordingly, substituting \( h(x) = y \) we confirm the statement (1.3). However, we immediately recognize \( \phi_Y(t) = (1 - it)^{-\alpha} \) as the CHF of a rv possessing \( \text{Gamma}(\alpha, 1) \) distribution. The rest is obvious. \( \square \)

The Lambert \( W \)-function is the inverse function of \( W \mapsto We^W \). Its principal branch \( W_0 \) is the solution of \( We^W = x \) for which \( W_0(x) \geq W_p(-e^{-1}) \). This function is implemented in Mathematica as `ProductLog[z]`. We are interested in \( W_0 \) exclusively for \( x \geq 0 \), where it is single–valued and monotone increasing, see [2].

Any nondecreasing function \( h \) possesses a so-called generalized inverse
\[ h^-(x) := \inf\{t \in \mathbb{R}_+: h(t) \geq x\}, \quad t \in \mathbb{R}_+, \]
with the convention that \( \inf \emptyset = \infty \). Moreover, if \( h \) is strong monotone increasing, that is \( h(x) < h(y) \) for all \( 0 < x < y \), then \( h^- \) coincides with the ‘ordinary’ inverse \( h^{-1} \).

Now, following the lines of the previous proof, we arrive at

**Theorem 2.** Consider random variables \( X \sim \text{GE}(\alpha, h) \) and \( Y = h(X) \exp\{\sigma h(X)\}, \sigma \geq 0 \). Then
\[ Y \sim \text{GE} \left( \alpha, h^-(\sigma^{-1} W_0(\sigma x)) \right). \]

Further, for all \( -\alpha < s < \sigma^{-1} \) it holds true
\[ \mathbb{E}h^+(X) \exp\{\sigma s h(X)\} = \frac{(\alpha)_s}{(1 - \sigma s)^{\alpha+s}}, \]
whenever \( h: \mathbb{R}_+ \mapsto \mathbb{R}_+ \) is a nondecreasing Borel function.
Proof. The rv $X \sim \text{GE}(\alpha, h)$ possesses CDF $F_X$ in the form (1.1). When $\sigma = 0$, then $Y \equiv X$. Letting $\sigma > 0$, the PDF $F_Y$ of the rv $Y = h(X) \exp\{\sigma h(X)\}$ becomes

$$F_Y(x) = P\{Y < x\} = P\{\sigma h(X) \exp\{\sigma h(X)\} < \sigma x\} = P\{h(X) < \sigma^{-1} W(x)\} = P\{X < h^{-1}(\sigma^{-1} W(x))\} = F_X[h^{-1}(\sigma^{-1} W(x))],$$

which is equivalent to the first assertion (1.4).

Next, in turn

$$E h^\alpha(X) \exp\{\sigma s h(X)\} = \frac{1}{\Gamma(\alpha)} \int_0^\infty h^{\alpha+s-1}(x) e^{-(1-\sigma s)h(x)} dh(x),$$

where the convergence of the integral is controlled by the condition $\sigma s < 1$, being $h$ nondecreasing and positive at the infinity. □

The following straightforward consequence of previous results we will need in the summation derivations.

**Corollary 1.** Let $X \sim \text{GEEW}(\theta)$, $\theta = (\lambda, \beta, k, \alpha) > 0$. Then consider the rv $\lambda X + \beta X^k \sim \text{Gamma}(\alpha, 1)$ and for all $\Re(s) > -\alpha$ we have

$$E(\lambda X + \beta X^k)^s = \frac{\Gamma(\alpha + s)}{\Gamma(\alpha)} = (\alpha)_s.$$  \tag{1.5}

2. **By–product Summation Formulae for Hypergeometric Type Special Functions**

The following interesting facts turn out to be a consequences of the exponentiation procedure. Employing the derived expressions which concerns computation formulae of higher transcendental function terms of hypergeometric type, that is lower and upper incomplete Gamma functions $\gamma(a, z), \Gamma(a, z)$; confluent (unified confluent) Fox–Wright generalized hypergeometric function $_1\Psi_0(1, \Psi_0)$; Meijer $G_{31}^{13}$ function and Whittaker function of the second kind $W_{\nu, \mu}$.

Certain attractive special cases of the Theorem 3, which are evidently not so obvious corollaries of (2.8) below, are our main results. In turn, let us recall in short the definitions of the above mentioned special functions. Firstly,

$$p\Psi_q\left[\begin{array}{c} (a, A)_p \\ (b, B)_q \end{array} \right| z \right] = \sum_{n=0}^\infty \frac{\prod_{j=1}^p (a_j A_n)}{\prod_{j=1}^q (b_j B_n)} \frac{z^n}{n!}\)$$

is the unified variant of the Fox–Wright generalized hypergeometric function with $p$ upper and $q$ lower parameters; $(a, A)_p$ denotes the parameter $p$–tuple $(a_1, A_1), \ldots, (a_p, A_p)$ and $a_j \in \mathbb{C}$, $b_i \in \mathbb{C} \setminus \mathbb{Z}_0^+$, $A_j, B_i > 0$ for all $j = 1, p, i = 1, q$, while the series converges for suitably bounded values of $|z|$ when

$$\Delta_{p,q} := 1 - \sum_{j=1}^p A_j + \sum_{j=1}^q B_j > 0.$$  

In the case $\Delta_{p,q} = 0$, the convergence holds in the open disc $|z| < \beta = \prod_{j=1}^p B_j \cdot \prod_{j=1}^q A_j^{-A_j}$. The convergence condition $\Delta_{1,0} = 1 - A_1 > 0$ is of special interest for us.

We point out that another definition of the Fox–Wright function $p\Psi_q[z]$ (consult the monographs [3], [5]) contains Gamma functions instead of the here used generalized Pochhammer symbols. However, these two functions differ only up to constant multiplying factor, that is

$$p\Psi_q\left[\begin{array}{c} (a, A)_p \\ (b, B)_q \end{array} \right| z \right] = \frac{\prod_{j=1}^p \Gamma(a_j)}{\prod_{j=1}^q \Gamma(b_j)} \frac{\Gamma(a)}{\Gamma(b)} p\Psi_q\left[\begin{array}{c} (a, A)_p \\ (b, B)_q \end{array} \right| z \right].$$
The unification’s motivation is clear; for \( A_1 = \cdots = A_p = B_1 = \cdots = B_q = 1, \rho P_0^r[z] \) one reduces exactly to the generalized hypergeometric function \( \rho F_2^1[z] \).

Further, the symbol \( G_{\rho,p,q}^{m,n}(\cdot) \) denotes Meijer’s \( G \)-function [6] defined in terms of the Mellin–Barnes integral reads

\[
G_{\rho,p,q}^{m,n}(z \mid a_1, \ldots, a_p; b_1, \ldots, b_q) = \frac{1}{2\pi i} \int_{C} \prod_{j=m+1}^{p} \Gamma(b_j - s) \prod_{j=1}^{r} \Gamma(1 - b_j + s) z^s ds,
\]

where \( 0 \leq m \leq q, 0 \leq n \leq p \) and the poles \( a_j, b_j \) are such that no pole of \( \Gamma(b_j - s), j = 1, \infty \) coincides with any pole of \( \Gamma(1 - a_j + s), j = 1, n \); i.e. \( a_k - b_j \notin \mathbb{N} \), while \( z \neq 0 \). \( C \) is a suitable integration contour which starts at \( -\infty \) and goes to \( \infty \) separating the poles of \( \Gamma(b_j - s), j = 1, m \) which lie to the right of the contour, from all poles of \( \Gamma(1 - a_j + s), j = 1, n \), which lie to the left of \( C \). The integral converges if \( \delta = m + n - \frac{1}{2}(p + q) > 0 \) and \( |\arg(z)| < \delta \pi \), see [4, p. 143] and [6]. Let us mention that the \( G \) function’s Mathematica code reads

\[
\text{MeijerG}[\{(a_1, \ldots, a_p), \{a_{n+1}, \ldots, a_p\}\}, \{(b_1, \ldots, b_q), \{b_{m+1}, \ldots, b_q\}\}, z].
\]

Finally, we formulate a further special summation, where the moment \( E X^r \) we express in terms of the Whittaker function of the second kind \( W_{a,b}(z) \) (see [2, §13.14]). One of the numerous connecting formulæ close to our recent considerations reads as follows:

\[
W_{a,b}(z) = e^{-\frac{1}{2}z^2} z^{1/2 + b} \left( -\frac{z^2}{2} \right)^{-a} \Gamma(b - a + \frac{1}{2}) \left( -b - a + \frac{1}{2} \right) \text{I}_F(b - a + \frac{1}{2} + 2b; z)
\]

\[
+ \frac{\Gamma(-2b)}{\Gamma(-b - a + \frac{1}{2})} \text{I}_F\left(b - a + \frac{1}{2} + 2b; z\right), \quad 2b \notin \mathbb{Z}.
\]

The associated Mathematica code is \texttt{MeijerG[a, b, z]}.

\textbf{Lemma 1.} [7, Lemma 3.1] For all positive \( (\mu, a, \nu, \rho) \), for which \( \rho + \nu^{-1} \ell \notin \mathbb{N} \) when \( \ell \in \mathbb{N} \), we have

\[
I_{\mu}(a, \nu, \rho) = \int_{0}^{\infty} x^{\mu-1}(1 + ax^\nu)^\rho e^{-x} dx
\]

\[
= \sum_{n \geq 0} \frac{(-1)^n (\rho)_n}{n!} \Bigg\{ a^n \gamma(\mu + \nu n, a^{-1/\nu}) + a^{\rho-n} \Gamma(\mu + \nu(\rho - n), a^{-1/\nu}) \Bigg\}, \quad \text{for } (2.6)
\]

where \( \gamma(a, z) = \Gamma(a) - \Gamma(a, z), \Re(a) > 0 \) signifies the lower incomplete Gamma function.

Moreover, specifying \( \rho \in \mathbb{N}_0 \) in (2.6) \( \mu, a, \nu \) remain positive, \( I_{\mu}(a, \nu, \rho) \) one reduces to a polynomial in \( a \) of \( \deg(I_{\mu}) = \rho \):

\[
I_{\mu}(a, \nu, \rho) = \Gamma(\mu) \text{MeijerG}[\{(a, \mu), \{a, a\}\}, \{(\mu, \nu), \{\mu, \mu\}\}, z].
\]

Now, we formulate a set of special summation results.

\textbf{Theorem 3. a.} For all \( \alpha \in \mathbb{R}_+ \setminus \mathbb{N} \), denoting \( a_0 = (\beta \lambda^{-k})^{-1/(k-1)} \), we have

\[
\frac{\sin(\pi \alpha)}{\pi} \sum_{m,n \geq 0} \frac{(-\beta \lambda^{-k})^m (\rho - 1)^n \Gamma(1 - \alpha + n)}{m! n!} \left\{ \frac{\beta}{\lambda^k} \right\} \left( \Gamma(\alpha + 1 + km + (k - 1)n, a_0) \right.
\]

\[
+ \frac{\beta (k + 1)}{\lambda^k} \gamma(\alpha + k(m + 1) + (k - 1)n, a_0) + \frac{\beta^2 k}{\lambda^{2k}} \gamma(\alpha + k(m + 2) - 1 + (k - 1)n, a_0)
\]

\[
+ \left( \frac{\beta}{\lambda^k} \right)^{\alpha - n} \left( \lambda^k \Gamma(\alpha - 1 + m) + 2 - (k - 1)n, a_0) \right) \right\} = 1.
\]

\[
= (k + 1) \Gamma(k(a + m) + 1 - (k - 1)n, a_0) + \frac{\beta k}{\lambda^k} \Gamma(k(a + m + 1) - (k - 1)n, a_0)
\]
b. For all $\alpha \in \mathbb{N}; (\lambda, \beta, k) > 0$ we have

$$
\sum_{n=0}^{\alpha-1} \frac{(1-\alpha)_n}{n!} \left( \frac{-\beta}{\lambda^k} \right)^n \left\{ \Gamma(\alpha + 1 + kn) \psi_0 \left[ (\alpha + 1 + (k-1)n, k) \mid -\frac{\beta}{\lambda^k} \right] - \frac{1}{\lambda^k} \Gamma(k) \psi_0 \left[ (\alpha + k(1+n), k) \mid -\frac{\beta}{\lambda^k} \right] \right\} = a.
$$

$$
+ \frac{1}{\lambda^k} \Gamma(k) \psi_0 \left[ (\alpha + k(1+n), k) \mid -\frac{\beta}{\lambda^k} \right] \frac{\beta}{\lambda^k} \Gamma(2k-1) \psi_0 \left[ (\alpha + k(2+n) - 1, k) \mid -\frac{\beta}{\lambda^k} \right] = a.
$$

For all $c.\text{ For all } \lambda > 0, a > 0, \alpha > 0$ we have

$$
\sin(\pi \alpha) \sum_{n=0}^{\infty} \frac{\left( \frac{n}{\lambda^2} \right)}{n!} \left\{ G_{13} \left( \frac{(u\lambda)^2}{4} \mid \frac{1-\alpha-3\alpha}{2}, 0, \frac{1}{2} \right) + \frac{3}{\lambda^2} G_{13} \left( \frac{(u\lambda)^2}{4} \mid \frac{-\alpha+3\alpha}{2}, 0, \frac{1}{2} \right) \right\} = \frac{2\sqrt{\pi}}{u^{\alpha+1} \lambda^2}.
$$

d. For all $\lambda > 0, b > 0, \alpha > 0$, we have

$$
\lambda \sum_{n=0}^{\infty} \frac{(b\lambda)^n}{n! \Gamma(\alpha + 1 + 2n)} \left\{ W_{\frac{\alpha-1}{2}, \frac{\alpha+1}{2} + 1} (b\lambda) + 3 \frac{W_{\frac{\alpha-1}{2}, \frac{\alpha+1}{2} + 1} (b\lambda)}{\lambda^2 b\lambda (\alpha + 1 + 2n)} \right\} = \alpha \left( \frac{\lambda}{b} \right)^{\alpha+1} e^{-\frac{1}{2}b\lambda}.
$$

Proof. a. Let us consider the rv $X \sim \text{GEEW}(\theta, \lambda, \beta, k, \alpha)$ having PDF

$$
f(x) = \frac{\lambda^n}{\Gamma(\alpha)} x^{\alpha-1} \left( 1 + \frac{\beta}{\lambda} x^{k-1} \right)^{\alpha-1} e^{-x(\lambda+\beta x^k)} 1_{\mathbb{R}_+}(x)
$$

$$
= \frac{\lambda^n}{\Gamma(\alpha)} x^{\alpha-1} \left( 1 + \frac{\beta}{\lambda} k x^{k-1} \right)^{\alpha-1} e^{-x(\lambda+\beta x^k)} 1_{\mathbb{R}_+}(x).
$$

By expanding the exponential term $\exp\{-\beta x^k\}$ into the Maclaurin series and using (2.6) we get $\mathbb{E}X^r$ as the linear combination of two $f_{\mu}(\alpha, \nu, \rho)$ integrals:

$$
\mathbb{E}X^r = \frac{\lambda^n}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \left( \frac{-\beta}{\lambda^m} \right)^m \left\{ I_{\alpha+k+m} \left( \frac{\beta}{\lambda^k} x + 1, \alpha - 1 \right) + \frac{\beta k}{\lambda^k} I_{\alpha+k+m+1} \left( \frac{\beta}{\lambda^k} x + 1, \alpha - 1 \right) \right\}.
$$

Now, highlighting the case $s = 1$ of (1.5) which leads to $\lambda \mathbb{E}X + \beta \mathbb{E}X^k = \alpha$ and taking $r = 1, k$ above in (2.8) respectively, we deduce the formula (2.7).

As to the assertions b. it is enough the point out the result [7, Theorem 2]

$$
\mathbb{E}X^r = \frac{(r)_a}{\lambda^r} \sum_{n=0}^{\alpha-1} \frac{(1-\alpha)_n (r + \alpha)_n}{n!} \psi_0 \left[ (r + \alpha + (k-1)n, k) \mid -\frac{\beta}{\lambda^k} \right] \left( \frac{-\beta}{\lambda^k} \right)^n
$$

$$
+ \frac{(r + k - 1)_a}{\lambda^r} \sum_{n=0}^{\alpha-1} \frac{(1-\alpha)_n (r + \alpha + k - 1)_n}{n!} \psi_0 \left[ (r + \alpha - 1 + (k+1), k) \mid -\frac{\beta}{\lambda^k} \right] \left( \frac{-\beta}{\lambda^k} \right)^n,
$$

valid for a rv $X \sim \text{GEEW}(\theta)$ where $\alpha \in \mathbb{N}; (\lambda, \beta, k) > 0$ and for all $r > \max\{-\alpha, 1 - \alpha - k\}$. Repeating the proving procedure of a. by setting $r = 1, k$ in (2.9), we arrive at the stated formula b.
c. Similarly, having in mind [7, Theorem 3] constituted for a rv $X \sim \text{GEEW}(\lambda, (u\lambda)^{-2}, 3, \alpha)$, for which, under constraints $\lambda > 0, u > 0, \alpha > -1$ and for all $r > -1$, there holds

$$E X^r = \frac{u^{\alpha+\lambda^r} \sin(\pi\alpha)}{2 \pi} \sum_{n \geq 0} C_{13}^{21} \left( \frac{(u\lambda)^2}{4} \left| \begin{array}{c} 1 - \frac{r+\alpha+3n}{2} \\ 0, \frac{1}{2} \end{array} \right. \right) + \frac{3}{\lambda^3} C_{13}^{21} \left( \frac{(u\lambda)^2}{4} \left| \begin{array}{c} \frac{-r+\alpha+3n}{2} \\ 0, \frac{1}{2} \end{array} \right. \right) \left( \frac{-u}{\lambda^2} \right)^n,$$

we infer the stated summation which involves Meijer $G$ function terms.

Finally, statement d. we achieve employing [7, Theorem 3]. When $X \sim \text{GEEW}(\lambda, (b\lambda)^{-1}, 2, \alpha)$, then for all $\lambda > 0, b > 0, \alpha > -1$ and for all $r > -1$, we have

$$E X^r = \left( \frac{b}{\lambda} \right)^{\frac{\alpha+1}{2}} e^{\frac{b\lambda}{2}} \sum_{n \geq 0} \frac{(b\lambda)^n}{n! (r + \alpha + 2n)} \left\{ \begin{array}{c} W_{\alpha - r - 1} - n, \frac{3\alpha + r - 2}{2} + n (b\lambda) \\ W_{\alpha - r - 1} - n, \frac{3\alpha + r - 2}{2} + n (b\lambda) \end{array} \right\}.$$

Now, the same treatment of the restricted formula (1.5) under $s = 1$ finishes the proof. \qed

**Remark 1.** Similar summations neither contain the celebrated formula collection by Hansen [1] nor the comprehensive and exhaustive classical monograph series Prudnikov et. al. [8].

**Remark 2.** Considering the result of Corollary 1 for $s \in \mathbb{N}_2 = \{2, 3, \ldots\}$, expanding the left–hand–side expression in (1.5) into a binomial sum, we can conclude a set of further summation results following the traces of the proof of Theorem 2. However, using into account this simple derivation procedure we lose the elegance in the resulting very complicated, hardly reducible resulting sum–product expressions.

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