The instability spectra of near-extremal Reissner-Nordström-de Sitter black holes

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The linearized dynamics of charged massive scalar fields in the near-extremal charged Reissner-Nordström-de Sitter (RNdS) black-hole spacetime is studied analytically. Interestingly, it is proved that the non-asymptotically flat charged black-hole-field system is characterized by unstable (exponentially growing in time) complex resonant modes. Using a WKB analysis in the eikonal large-mass regime $M\mu \gg 1$, we provide a remarkably compact analytical formula for the quasinormal resonant spectrum \( \{\omega_n(M, Q, \Lambda, \mu, q)\}_{n=0}^\infty \) which characterizes the unstable modes of the composed RNdS-black-hole-charged-massive-scalar-field system [here \( \{M, Q, \Lambda\} \) are respectively the mass, electric charge, and cosmological constant of the black-hole spacetime, and \( \{\mu, q\} \) are the proper mass and charge coupling constant of the linearized field].

I. INTRODUCTION

The intriguing superradiant amplification phenomenon allows charged integer-spin fields to extract Coulomb energy and electric charge from various types of charged black holes [1–5]. As a consequence, a charged bosonic cloud surrounding a central charged black hole may grow exponentially over time if the extracted energy is not radiated fast enough to spatial infinity. Thus, the superradiant amplification mechanism imposes a non-trivial threat to the stability of charged black-hole spacetimes.

Interestingly, it has been proved in [6] that asymptotically flat charged Reissner-Nordström black-hole spacetimes are linearly stable to charged massive scalar field perturbations. On the other hand, it has recently been demonstrated numerically [7, 8] that non-asymptotically flat charged Reissner-Nordström-de Sitter (RNdS) black-hole spacetimes may become unstable to perturbations of charged scalar fields whose proper frequencies lie in the bounded superradiant regime [9]

\[
\frac{qQ}{r_c} < \omega < \frac{qQ}{r_+},
\]

where \( q \) is the charge coupling constant of the scalar field, \( \{Q, r_+\} \) are respectively the electric charge and outer horizon radius of the central black hole, and \( r_c \) is the radius of the cosmological horizon which characterizes the black-hole spacetime.

The main goal of the present paper is to use analytical techniques in order to explore the physical and mathematical properties of the intriguing superradiant instability phenomenon observed in the highly interesting numerical works [7, 8]. In particular, below we shall explicitly prove that the superradiant instability spectrum which characterizes the composed charged-RNdS-black-hole-charged-massive-scalar-field system can be determined analytically in the near-extremal \((r_c - r_+)/r_+ \ll 1\) regime.

II. DESCRIPTION OF THE SYSTEM

We shall analyze the quasinormal resonant modes that characterize the dynamics of a charged massive scalar field \( \Psi \) which is linearly coupled to a charged Reissner-Nordström-de Sitter black hole. The non-asymptotically flat curved black-hole spacetime is described by the line element [8–10]

\[
ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

where the radially-dependent metric function is given by the compact expression

\[
f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{r^2}{\Lambda^2}.
\]

Here \( \{M, Q, \Lambda \equiv 3/L^2\} \) are respectively the mass, electric charge [11], and the cosmological constant [12, 13] of the black-hole spacetime. The radii \( \{r_-, r_+, r_c\} \) of the inner (Cauchy) horizon, outer (event) horizon, and cosmological
horizon which characterize the RNdS black-hole spacetime are determined by the algebraic equation
\[ f(r_*) = 0 \quad \text{with} \quad * \in \{-, +, c\} . \]  

The linearized dynamics of a scalar field \( \Psi \) of proper mass \( \mu \) and charge coupling constant \( q \) in the RNdS black-hole spacetime is governed by the Klein-Gordon wave equation
\[ [(\nabla^\nu - iqA^\nu)(\nabla_\nu - iqA_\nu) - \mu^2] \Psi = 0 , \]  
where \( A_\nu = -\delta_0^\nu Q/r \) is the electromagnetic potential of the charged black-hole spacetime. It is convenient to expand the scalar field eigenfunction \( \Psi \) in the form
\[ \Psi(t, r, \theta, \phi) = \int \sum_{lm} \psi_{lm}(r; \omega) r Y_{lm}(\theta) e^{im\phi} e^{-i\omega t} d\omega , \]  
where the integer parameters \( \{l, m\} \) (which are characterized by the inequality \( l \geq |m| \)) denote the spherical and azimuthal harmonic indices of the charged massive scalar field eigen-modes. Substituting the scalar field decomposition into the Klein-Gordon wave equation, one obtains the ordinary differential equation
\[ \frac{d^2\psi}{dy^2} + V\psi = 0 \]  
for the radial part of the scalar eigenfunction, where the tortoise radial coordinate \( y \) is related to the areal coordinate \( r \) by the simple differential relation
\[ \frac{dy}{dr} = f^{-1}(r) . \]  

The effective black-hole-field radial potential \( V = V(r; M, Q, \Lambda, \omega, q, \mu, l) \) in the Schrödinger-like differential equation is given by
\[ V(r) = \left( \omega - \frac{qQ}{r} \right)^2 - f(r)G(r) , \]  
where
\[ G(r) = \mu^2 + \frac{l(l + 1)}{r^2} + \frac{1}{r} \frac{df}{dr} . \]  

The quasinormal resonant modes, which characterize the composed RNdS-black-hole-charged-massive-scalar-field system, are determined by the Schrödinger-like ordinary differential equation with the physically motivated boundary conditions of purely ingoing waves at the outer (event) horizon of the black hole and purely outgoing waves at the cosmological horizon of the spacetime:
\[ \psi \sim \begin{cases} e^{-i(\omega - qQ/r_+)y} & \text{for} \quad r \to r_+ \quad (y \to -\infty) ; \\ e^{i(\omega - qQ/r_c)y} & \text{for} \quad r \to r_c \quad (y \to \infty) . \end{cases} \]  

Below we shall analyze the characteristic quasinormal resonant spectra \( \{\omega_n(M, Q, \Lambda, q, \mu, l)\}_{n=0}^{\infty} \) of the composed near-extremal-RNdS-black-hole-charged-massive-scalar-field system. In particular, we shall consider complex resonant frequencies of the form
\[ \omega = \omega_R - i\omega_1 . \]  
Taking cognizance of Eqs. and [12], one realizes that resonant black-hole-field eigen-frequencies with \( \omega_1 < 0 \) correspond to unstable (exponentially growing in time) charged field modes, whereas resonant eigen-frequencies with \( \omega_1 > 0 \) correspond to stable (exponentially decaying) field modes.
III. THE REGIME OF NEAR-EXTREMAL REISSNER-NORDSTRÖM-DE SITTER BLACK-HOLE SPACETIMES

Interestingly, as we shall explicitly show below, the characteristic quasinormal resonant frequencies \( \{\omega_n(M, Q, \Lambda, q, \mu, l)\}^n_{n=0} \) of the composed charged-RNdS-black-hole-charged-field system can be determined analytically in the regime

\[
\frac{r_c - r_+}{r_+ - r_-} \ll 1 \quad (13)
\]

of near-extremal black holes. In particular, in the regime \( (13) \) one finds the simple functional relations [see Eq. (8)]

\[
r(y) = \frac{r_c e^{2\kappa+y} + r_+}{1 + e^{2\kappa+y} + r_+} \quad (14)
\]

and

\[
f(y) = \frac{(r_c - r_+)\kappa_+}{2 \cosh^2(\kappa+y)}, \quad (15)
\]

where

\[
\kappa_+ = \frac{1}{2} \left( \frac{df}{dr} \right)_{r=r_+} \quad (16)
\]

is the characteristic surface gravity of the black-hole outer (event) horizon which, in the near-extremal regime \( (13) \), is given by the compact dimensionless relation \( \kappa_+ r_+ = (r_c - r_+)\left[1 - 2(Q/r_+)^2\right] \frac{1}{2r_+} \ll 1 \quad (17) \)

IV. THE QUASINORMAL RESONANCE SPECTRA OF THE CHARGED MASSIVE SCALAR FIELDS IN THE NEAR-EXTREMAL CHARGED RNdS BLACK-HOLE SPACETIMES

In the present section we shall explicitly show that the complex resonant spectra which characterize the composed near-extremal-RNdS-black-hole-charged-massive-scalar-field system can be determined analytically, using standard WKB techniques \( \[20\, 23\] \), in the dimensionless large-mass regime \( \[24\] \)

\[
\max\{\kappa_+ r_+, l(l+1)\} \ll \mu r_+ \quad , (18)
\]

in which case the effective black-hole-field radial potential \( \[9\] \) can be approximated by

\[
V[r(y)] = \left( \omega - \frac{qQ}{r} \right)^2 - \mu^2 f(r) \cdot \{1 + O((\mu r_+)^{-2})\} \quad (19)
\]

As explicitly proved in \( \[20\, 21\] \), the WKB resonance condition which determines, in the large-frequency regime, the fundamental complex resonant frequencies of the Schrödinger-like radial equation \( \[7\] \) is given by the differential relation

\[
\frac{iV'(y_0)}{\sqrt{2V(2)(y_0)}} = n + \frac{1}{2} \quad ; \quad n = 0, 1, 2, ...
\]

(20)

where \( V^{(k)}(y_0) \equiv [d^k V/dy^k]_{y=y_0} \) and \( y_0(r_0) \) is the local extremum point [with \( V^{(1)}(y_0) = 0 \)] of the effective black-hole-field radial potential \( V(y) \). Taking cognizance of Eqs. \( \[8\] \) and \( \[19\] \), one finds that this extremum point is characterized by the functional relation

\[
\omega - \frac{qQ}{r_0} = \frac{\mu^2 r_0}{2qQ} \cdot \frac{f^{(1)}(y_0)}{f(y_0)} \quad , (21)
\]
where \( f^{(k)}(y_0) \equiv [d^k f/dy^k]_{y=y_0} \).

Substituting the effective black-hole-field radial potential \((19)\) into Eq. \((20)\), one finds

\[
\left( \omega - \frac{q Q}{r_0} \right)^2 - \mu^2 f(y_0) \frac{2}{\sqrt{4 \left( \frac{q Q}{r_0} \right)^2 - 8 \left( \omega - \frac{q Q}{r_0} \right) \frac{q Q}{r_0} f^2(y_0) + 4 \left( \omega - \frac{q Q}{r_0} \right) f^2(y_0) + 4 \left( \omega - \frac{q Q}{r_0} \right) f^{(1)}(y_0) - 2 \mu^2 f^{(2)}(y_0)}} = - i \left( n + \frac{1}{2} \right). \tag{22}\]

The WKB condition \((22)\), supplemented by the extremum relation \((21)\), determine the characteristic quasinormal resonant spectra of the composed near-extremal-RNdS-black-hole-charged-massive-scalar-field system. Interestingly, as we shall now show, this (rather cumbersome) equation can be solved analytically for the fundamental complex resonant frequencies of the system which, in the eikonal large-mass regime \((18)\), are characterized by the strong inequality [see Eqs. \((30)\) and \((32)\) below]

\[
\omega_R \gg |\omega_I| . \tag{23}\]

In particular, the real and imaginary parts of the WKB resonance condition \((22)\) can be decoupled in the large-frequency (large-mass) regime \((23)\). The real part of the resonance equation is given by

\[
\left( \omega_R - \frac{q Q}{r_0} \right)^2 - \mu^2 f(y_0) = 0 , \tag{24}\]

whereas, to leading order in the small dimensionless ratio \(|\omega_I|/\omega_R\) [see \((23)\)], the imaginary part of the resonance equation \((22)\) is given by

\[
2 \left( \omega_R - \frac{q Q}{r_0} \right) \omega_I = \sqrt{4 \left( \frac{q Q}{r_0} \right)^2 - 8 \left( \omega_R - \frac{q Q}{r_0} \right) \frac{q Q}{r_0} f^2(y_0) + 4 \left( \omega_R - \frac{q Q}{r_0} \right) f^2(y_0) + 4 \left( \omega_R - \frac{q Q}{r_0} \right) f^{(1)}(y_0) - 2 \mu^2 f^{(2)}(y_0)} \cdot \left( n + \frac{1}{2} \right) . \tag{25}\]

Substituting Eq. \((21)\) into Eq. \((24)\), one finds the relation \((26)\)

\[
\frac{f^{(1)}(y_0)}{f^{3/2}(y_0)} = \pm \frac{2q Q}{\mu r_+^2} \left[ 1 + O(\kappa_+ r_+) \right] \tag{26}\]

which, taking cognizance of the radial metric function \((15)\) that characterizes the near-extremal charged RNdS black-hole spacetimes, yields the compact functional expression

\[
\sinh(\kappa_+ y_0^\pm) = \pm \frac{q Q}{\mu r_+} \left[ 1 - 2(Q/r_+)^2 \right]^{-1/2} \cdot [1 + O(\kappa_+ r_+)] \tag{27}\]

for the locations of the extremum points \(y_0^\pm = y_0^\pm (r_+^\pm)\) which characterize the effective radial potential \((19)\) of the composed black-hole-field system \((26)\). From Eqs. \((13)\) and \((17)\) one finds the dimensionless ratio

\[
\frac{r(y)}{r_+} = 1 + \frac{2\kappa_+ r_+}{1 - 2(Q/r_+)^2} \left[ 1 + e^{-2\kappa_+ y} \right] \tag{28}\]

which, taking cognizance of Eq. \((27)\), yields the relation \((27)\)

\[
\frac{r_+^\pm}{r_+} = 1 + \frac{\kappa_+ r_+}{1 - 2(Q/r_+)^2} \cdot \left[ 1 \pm \frac{1}{\sqrt{1 + \left( \frac{\mu r_+}{q Q} \right)^2 \left[ 1 - 2(Q/r_+)^2 \right]}} \right] + O(\kappa_+^2 r_+) . \tag{29}\]

Substituting Eqs. \((15)\), \((27)\), and \((29)\) into Eq. \((21)\), one obtains the expression

\[
\omega_R^\pm = \frac{q Q}{r_+} - \frac{q Q}{1 - 2(Q/r_+)^2} \left[ 1 \pm \frac{1}{\sqrt{1 + \left( \frac{\mu r_+}{q Q} \right)^2 \left[ 1 - 2(Q/r_+)^2 \right]}} \right] \cdot \kappa_+ + O(\kappa_+^2 r_+) \tag{30}\]

for the real parts of the resonant frequencies which characterize the composed near-extremal-RNdS-black-hole-charged-massive-scalar-field system.
Before proceeding, it is worth stressing the fact that the resonant frequencies \( \omega_{\pm} \) are characterized by the inequalities
\[
\omega_+^R < \frac{qQ}{r_+} \quad \text{and} \quad \omega_-^R > \frac{qQ}{r_+}.
\] (31)

Taking cognizance of the relations (1) and (31), one deduces that the resonant frequency \( \omega_+^R \) lies within the superradiant regime of the composed charged-black-hole-charged-field system, whereas the resonant frequency \( \omega_-^R \) lies outside the superradiant regime of the system. Thus, the resonant frequency \( \omega_+^R \) has the potential of triggering exponentially growing superradiant instabilities [with \( \omega_I < 0 \), see Eqs. (6) and (12)] in the composed black-hole-field system. We shall now explicitly confirm this physical expectation.

Substituting Eqs. (8), (15), (27), and (30) into Eq. (25), one obtains the remarkably compact functional relation
\[
\omega_\pm^I = \pm \kappa_+ \cdot \left( n + \frac{1}{2} \right) \cdot \left[ 1 + O(\kappa^r) \right] (32)
\]
for the imaginary parts of the quasinormal resonant frequencies which, in the eikonal large-mass regime (18), characterize the linearized dynamics of the charged massive scalar fields in the near-extremal \( (\kappa r_+ \ll 1) \) charged RNdS black-hole spacetimes. Taking cognizance of Eqs. (23), (30), and (32), one learns that our analysis is valid in the dimensionless regime \( \kappa^r \cdot (n + \frac{1}{2}) \ll qQ \). (33)

V. THE REGIME OF VALIDITY OF THE WKB APPROXIMATION

It is important to stress the fact that the WKB functional expressions (30) and (32) for the real and imaginary parts of the complex resonant frequencies which characterize the composed near-extremal-RNdS-black-hole-charged-massive-scalar-field system are valid under the assumption that higher-order correction terms that appear in the WKB resonance condition (20) can be neglected. In particular, it was proved in \( [20-23] \) that, taking into account higher-order spatial derivatives of the effective radial potential \( V(y) \) [see Eq. (19)], one obtains the higher-order correction term
\[
\Lambda(n) = \frac{1}{\sqrt{2V_0^{(2)}}} \left[ -\frac{7 + 15(2n + 1)^2}{288} \cdot \frac{V_0^{(3)}}{V_0^{(2)}} + \frac{1 + (2n + 1)^2}{32} \cdot \frac{V_0^{(4)}}{V_0^{(2)}} \right] (34)
\]
on the r.h.s of the black-hole-field resonance condition (20). Thus, our analytically derived results (30) and (32) are valid provided [see Eqs. (20) and (34)]
\[
\frac{\Lambda(n)}{n + \frac{1}{2}} \ll 1. \] (35)

Using Eqs. (8), (15), (19), and (27), one finds the expression
\[
\Lambda(n) = -\frac{4(2n + 1)^2 \left( \frac{qQ}{\mu r_+} \right)^2 + \left[ 1 + (2n + 1)^2 \right][1 - 2(Q/r_+)^2]}{8\sqrt{(qQ)^2 + (\mu r_+)^2}[1 - 2(Q/r_+)^2]} (36)
\]
for the higher-order correction term (34) in the WKB approximation. Taking cognizance of (35) and (36), one deduces that the analytical expressions (30) and (32) for the real and imaginary parts of the composed black-hole-field quasinormal resonant frequencies, which have been derived using the WKB resonance condition (20), are valid in the dimensionless physical regime
\[
\max \left\{ \left( \frac{qQ}{\mu r_+} \right)^2, 1 - 2(Q/r_+)^2 \right\} \ll (n + \frac{1}{2})^{-1}. \] (37)

Interestingly, one learns from (37) that, for \( \mu r_+ \gg n + 1/2 \), the superradiant instability of the near-extremal charged RNdS black-hole spacetimes can occur for arbitrarily small [with the constraint (33)] values of the dimensionless charge coupling constant \( qQ \). In addition, one deduces from (37) that the physical parameter \( qQ \) is bounded from above by the simple dimensionless relation
\[
qQ \ll (\mu r_+)^2 \cdot (n + \frac{1}{2})^{-1}. \] (38)
VI. SUMMARY

The Reissner-Nordström-de Sitter black-hole spacetimes describe a family of charged solutions to the coupled Einstein-Maxwell field equations in non-asymptotically flat de Sitter spacetimes. It has recently been demonstrated numerically in the very interesting works [6, 8] that, as opposed to the asymptotically flat Reissner-Nordström black-hole spacetimes, the non-asymptotically flat Reissner-Nordström-de Sitter black-hole spacetimes may become unstable to perturbations of charged scalar fields in the bounded superradiant regime [1].

In the present paper we have studied analytically the superradiant instability spectrum which characterizes the linearized dynamics of charged massive scalar fields in the charged near-extremal (with $\kappa_+ r_+ \ll 1$) Reissner-Nordström-de Sitter black-hole spacetimes. In particular, using WKB techniques in the dimensionless physical regime [see Eqs. (16), (17)]

$$n + \frac{1}{2} \ll \mu r_+ \quad \text{with} \quad \kappa_+ r_+ \cdot (n + \frac{1}{2}) \ll q Q \ll (\mu r_+)^2 \cdot (n + \frac{1}{2})^{-1},$$

we have derived the remarkably compact analytical formula [see Eqs. (30) and (32)]

$$\omega_R = \frac{q Q}{r_+} - \frac{q Q}{1 - 2 (Q/r_+)^2} \left\{ 1 + \sqrt{1 + \left( \frac{\mu r_+}{q Q} \right)^2 \left[ 1 - 2 (Q/r_+)^2 \right]^2} \right\} \cdot \kappa_+ - i \kappa_+ \cdot (n + \frac{1}{2}) + O(\kappa_+^2 r_+)$$

for the complex resonant frequencies which characterize the superradiant instability spectrum of the composed charged-near-extremal-RNdS-black-hole-charged-massive-scalar-field system.

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Here we have used the relation \( r_0 = r_+ [1 + O(\kappa_+ r_+)] \) for the near-extremal black holes in the dimensionless regime (13).

It is important to emphasize the fact that the effective black-hole-field radial potential (19) is frequency-dependent. Thus, for a given value of the resonant frequency \( \omega_R \) [see Eq. (30) below], the frequency-dependent radial potential (19) is characterized by only one local extremum point which, as we shall explicitly prove below, is characterized by the relation \( V^{(2)}(y_0) > 0 \).

Here we have used the identities \( 2/(1 + e^{-2\alpha}) = 1 + \tanh(\alpha) \) and \( \cosh(\alpha) = \sqrt{1 + \sinh^2(\alpha)} \).

It is worth emphasizing again that in the present paper we consider the physical regime of near-extremal RNdS black-hole spacetimes with \( \kappa_+ r_+ \ll 1 \). Thus, the inequality (33) does not necessarily imply that the dimensionless charge coupling constant \( qQ \) is large [see Eq. (38) below].

It is important to point out that the strong inequality (37) implies the dimensionless relation \( \mu r_+ \gg n + 1/2 \). To see this, we note that the relation \( \sqrt{(qQ)^2 + (\mu r_+)^2 [1 - 2(Q/r_+)^2]} \gg n + 1/2 \) provides a necessary condition for the validity of (37). This implies that \( \mu r_+ \gg n + 1/2 \) and/or \( qQ \gg n + 1/2 \). Let us assume that \( \mu r_+ \) is not large [and therefore \( qQ \) must be large with \( qQ \gg \max\{n + 1/2, \mu r_+\} \)]. In this case, one deduces from (37) the strong inequalities \( \max\{n + 1/2, \mu r_+\} \ll qQ \ll (\mu r_+)^2 \cdot (n + \frac{1}{2})^{-1} \), which immediately imply the dimensionless relation \( \mu r_+ \gg n + 1/2 \).