Cohomology of Lie semidirect products
and
poset algebras

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We study the cohomology of a Lie algebra acted upon by a toral algebra, and using Viviani’s lemma examine the cohomology of toral semidirect products. This is applied to compute the cohomology of Lie poset algebras when the coefficient module is either the trivial module or the algebra itself. In the latter case the cohomology is determined by that of a simplicial complex, as in the associative case, but the route to the result is quite different.

1 Introduction

This paper is influenced by the classical result of [4], summarized in its title Simplicial cohomology is Hochschild cohomology; it asserts that for every simplicial complex, finite or not, there is an associative algebra whose Hochschild cohomology with coefficients in itself is naturally isomorphic to that of the complex. In the finite case, from which the result ultimately follows, this algebra is a poset algebra, which can be represented as an an algebra of upper triangular matrices. The proof then uses the fact that the Hochschild cohomology of an algebra can be computed relative to any separable subalgebra containing its unit [5] and that for a poset algebra one has a convenient separable subalgebra, namely the diagonal matrices.

Here we consider Lie algebras. While the preceding holds over an arbitrary commutative unital coefficient ring, in the present paper the coefficients will generally be in a field \( \mathbf{k} \) of arbitrary characteristic (although there will be some restrictions later) and all algebras and vector spaces are assumed to be finite dimensional over \( \mathbf{k} \). We extend the notion of poset algebras to the Lie case and in certain cases compute their cohomology. A Lie algebra \( T \) acts torally on a vector space \( V \) if all of its elements act separably, i.e., become diagonalizable over the algebraic closure of \( \mathbf{k} \). This implies that the actions of any two elements of \( T \) commute, see, e.g., [9] p. 34], so the operations are simultaneously
diagonalizable over the algebraic closure of \( k \). We may generally assume that \( T \) itself is commutative. Since a Lie poset algebra \( g \) will here always have the form of a semidirect product \( g = h \ltimes \mathfrak{t} \) where \( h \) is Abelian and acting torally on \( \mathfrak{t} \), an important step will be to compute the Chevalley-Eilenberg cohomology of such algebras. Every associative poset algebra can be given a Lie poset algebra structure by taking the commutator product, so it is not surprising to find that their cohomology theories are closely related, but in general the Lie theory is richer.

Suppose that \( M \) is a module over a Lie algebra \( \mathfrak{t} \) and that \( h \) is an Abelian Lie algebra acting torally on both \( \mathfrak{t} \) and \( M \). If it also acts compatibly, i.e.,

\[
[h,\{\kappa,m\}] = \{[h,\kappa],m\} + \{\kappa,[h,m]\}
\]

for all \( h \in h, \kappa \in \mathfrak{t}, m \in M \), then \( M \) is naturally a module over the semidirect product \( g = h \ltimes \mathfrak{t} \) and the Chevalley-Eilenberg cohomology \( H^*(g, M) \) of \( g \) with coefficients in \( M \) also becomes such a module. The operation of \( h \) then induces a decomposition of \( H^*(\mathfrak{t}, M) \) into a direct sum of weight spaces indexed by elements of the dual vector space \( h^\vee \) of \( h \). The weight 0 component \( H^*(\mathfrak{t}, M)_0 \) consists of all elements annihilated by every element of \( h \). With this notation, the Hochschild-Serre spectral sequence \( \mathbb{S} \) collapses, yielding

\[
H^*(g, M) \cong \bigwedge h^\vee \otimes H^*(\mathfrak{t}, M)_0.
\]

Lie semidirect products occur quite naturally as instances of Lie poset algebras whose description follows. Let \( \mathfrak{sl}(N,k) \) denote the Lie algebra of \( N \times N \) matrices of trace zero over \( k \), \( \mathfrak{b} \) be its Borel subalgebra of upper triangular matrices, \( \mathfrak{n} \) be the strictly upper triangular matrices and \( \mathfrak{h} \) its Cartan subalgebra of diagonal matrices. A subalgebra \( g \) with \( \mathfrak{b} \supset g \supset \mathfrak{h} \) is then spanned by \( \mathfrak{h} \) and those \( e_{ij} \in \mathfrak{n} \) which it contains. Letting \( \mathfrak{t} \) denote the ideal spanned by these \( e_{ij} \), we have \( g = \mathfrak{h} \ltimes \mathfrak{t} \). One now has a a partial order on \( \{1, \ldots, N\} \) compatible with the linear order defined by setting \( i \leq j \) if \( e_{ij} \in g \); this defines a poset \( \mathcal{P} \). Conversely, suppose that we have a partial order on a finite set of order \( N \), which without loss of generality we may take the set to be \( \{1, \ldots, N\} \), and that the partial order is compatible with the linear order. We can then define \( g \) to be the span of \( \mathfrak{h} \) and those \( e_{ij} \) with \( i < j \) in the given partial order. These equivalent characterizations define the Lie poset subalgebras of \( \mathfrak{sl}(N,k) \) and we will write \( \mathfrak{g}(\mathcal{P}) \) to denote such a Lie poset algebra.

While the only simple associative algebras over an algebraically closed field of any characteristic are matrix algebras, for Lie algebras, even over \( \mathbb{C} \), there are more possibilities. Here we are considering only the \( A_n \) series but the characterization of “Lie poset algebras” as those contained in a Borel subalgebra and containing its Cartan subalgebra can be extended, in arbitrary characteristic, to those simple Lie algebras sometimes called Chevalley algebras: As Chevalley showed, the simple Lie algebras over \( \mathbb{C} \) all have bases for which the multiplication is defined over the ordinary integers and which, by extension of the coefficients, can therefore be defined over an arbitrary field. If the rank of the original algebra is \( \ell \) then the resulting algebra is simple when the characteristic of the field does not divide \( \ell + 1 \), and so, in particular, whenever it is greater
than \( \ell + 1 \). We conjecture that with the latter restriction all that follows can, in some form, be carried over to these Chevalley algebras.

Suppose now that \( g(\mathcal{P}) \) is a Lie poset subalgebra of \( \mathfrak{sl}(N, k) \) and that \( k \) is either of characteristic zero or \( p > N \). We compute the cohomology \( H^*(g(\mathcal{P}), M) \) of \( g(\mathcal{P}) \) in two basic cases, those where the module \( M \) is \( k \) and where it is \( g(\mathcal{P}) \) itself. In the former case we obtain

\[
H^*(g(\mathcal{P}), k) \cong \bigwedge h^\vee.
\]

To describe \( H^*(g(\mathcal{P}), g(\mathcal{P})) \), which governs the deformations of \( g(\mathcal{P}) \), observe first that the poset \( \mathcal{P} \) on \{1, \ldots, N\} may be viewed as a small category. Let the nerve of that category be the simplicial complex \( \Sigma \). Its 0-simplices are the elements of \( \mathcal{P} \); in this context denote these by \((i)\). We can augment \( \Sigma \) by the standard procedure of adjoining a unique formal simplex \( \sigma_{-1} \) of dimension -1 with the property that \( \partial(i) = \sigma_{-1} \) for all \( i \in \mathcal{P} \). The augmented nerve will be denoted \( \Sigma^+ \). Its homology and cohomology groups are frequently called the “reduced” cohomology, but differ from those of the original \( \Sigma \) only in dimension 0. Assuming that the characteristic of \( k \) is zero or greater than \( N \), we prove finally that

\[
H^*(g(\mathcal{P}), g(\mathcal{P})) = \bigwedge h^\vee \otimes H^*(\Sigma^+, k). \quad (1)
\]

This result in the Lie case is close to that of the associative case. There, if we let \( A(\mathcal{P}) \) denote the associative algebra spanned over \( k \) (which in this case, may be an arbitrary commutative unital ring) by \( g \) and the unit matrix, and if \( \Sigma \) is the same as before, then the classic result of [4] asserts that

\[
H^*(A(\mathcal{P}), A(\mathcal{P})) \cong H^*(\Sigma, k). \quad (2)
\]

Any deformation of an associative algebra induces one of the Lie algebra obtained from it by taking the commutator product. The generally larger cohomology we find here in the Lie case suggests that the Lie poset algebra \( g(\mathcal{P}) \) may have deformations not induced from those of its related associative algebra \( A(\mathcal{P}) \); the last section explicitly computes a simple example where the associative algebra is in fact “absolutely rigid”, i.e., has no infinitesimal deformations.

While the final result in the Lie case is close to that in the associative case, the steps needed to get there are different. The first is Viviani’s Lemma ([3]), which asserts, in particular, that when a toral subalgebra \( \mathfrak{h} \) of \( g \) acts torally and compatibly also on a \( g \)-module \( M \), then \( H^*(g, M) \) can be computed using only cochains of weight zero relative to \( \mathfrak{h} \). (This was observed independently by the second author [2], and possibly by others, but the first explicit statement we found seems to be in Viviani [12], so we have here attached his name to it.) The second ([4]) is the observation that the Hochschild-Serre filtration of the cochain complex \( C^*(g, M) \) associated with a subalgebra \( \mathfrak{h} \) of \( g \) is much better behaved when the subalgebra \( \mathfrak{h} \) is toral. The third step is to consider the case where \( g \) is a semidirect product \( \mathfrak{h} \ltimes \mathfrak{k} \), where \( \mathfrak{h} \) is an Abelian algebra acting torally on
the ideal $\mathfrak{t}$ and torally also on a $\mathfrak{g}$-module $M$. In this case, the Hochschild-Serre spectral sequence collapses and computation of $H^*(\mathfrak{g}, M)$ reduces to computing the weight zero part of the cohomology of $\mathfrak{t}$ with coefficients in $M$, where the weighting is that induced by the action of $\mathfrak{h}$. In the fourth and final step we show that when $\mathfrak{g}$ is a Lie poset algebra this last cohomology is essentially simplicial.

## 2 Toral Actions

Suppose that a Lie algebra $T$ acts torally on a vector space $V$. The dual vector space of $T$ will be denoted $T^\vee$ and similarly for other vector spaces. Then $V$ decomposes into a direct sum of irreducible weight spaces $V = \bigoplus_w V_w$, $w \in T^\vee$, where $V_w$ is the subspace consisting of all $v \in V$ such $\tau v = w(\tau)v$ for all $\tau \in T$. If $T$ acts torally on $V$ then it also does so on $V^\vee$ by setting $\langle \tau v^\vee, v \rangle = -\langle v^\vee, \tau v \rangle$, where $v \in V, v^\vee \in V^\vee$. In the weight space decomposition of $V^\vee$, the summand of weight zero consists of all $v^\vee$ which vanish on all homogeneous elements of $V$ having non-zero weight. Later we will need to consider only the common null space of the operators in $T$; its dimension does not change under any extension of coefficients.

If $T$ also acts torally on a space $W$ then it does so on $V \otimes W$ by setting $\tau (v \otimes w) = \tau v \otimes w + v \otimes \tau w$. It follows that there is a natural toral action of $T$ on $\bigwedge^r V$ and hence on $\text{Hom}_k(\bigwedge^r V, W)$ for all $r$. Suppose that $T$ operates torally and compatibly on both a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module $M$. Then it does so also on the Chevalley-Eilenberg cochains $F^n \in C^n(\mathfrak{g}, M)$, for the latter by definition is $\text{Hom}_k(\bigwedge^n \mathfrak{g}, M)$, and one has

$$(\tau F^n)(g_1, \ldots, g_n) = \tau(F^n(g_1, \ldots, g_n)) - \sum_{i=1}^n (-1)^iF^n(\tau g_i, g_1, \ldots, \hat{g}_i, \ldots, g_n)$$

where $\hat{g}_i$ denotes omission of $g_i$. This commutes with the coboundary operator, so $T$ acts torally on the cohomology $H^*(\mathfrak{g}, M)$. That cohomology then decomposes into a direct sum of weight spaces under the action of $T$. Viviani’s Lemma (34) then asserts that if $T$ is a subalgebra of $\mathfrak{g}$ (in which case it is necessarily Abelian) then any cocycle of weight different from 0 is a coboundary, so $H^*(\mathfrak{g}, M) = H^*(\mathfrak{g}, M)_0$.

Let $0 \to \mathfrak{t} \to \mathfrak{g} \to \mathfrak{h} \to 0$ be a split short exact sequence of Lie algebras. Then $\mathfrak{h}$ may be viewed as a subalgebra of $\mathfrak{g}$, hence acting on the ideal $\mathfrak{t}$, so we have a semidirect product $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{t}$. Suppose now that $\mathfrak{h}$ is Abelian and acts torally and compatibly on $\mathfrak{g}$ and $M$. As mentioned, the Hochschild-Serre spectral sequence (3) then collapses (5), yielding

$$H^*(\mathfrak{g}, M) \cong \bigwedge \mathfrak{h} \wedge \otimes H^*(\mathfrak{t}, M)_0,$$

where $H^*(\mathfrak{t}, M)_0$ is the weight 0 component of $H^*(\mathfrak{t}, M)$ relative to the decomposition induced by $\mathfrak{h}$. This need not be all of $H^*(\mathfrak{t}, M)$ since $\mathfrak{h}$ is not contained in $\mathfrak{t}$. For observe, in particular, that $H^0(\mathfrak{g}, M) = M^\mathfrak{h}$, the “invariants” of $M$ under the operation of $\mathfrak{g}$, i.e., those $m \in M$ such that $[g, m] = 0$ for
all \(g \in \mathfrak{g}, m \in M\), while \(H^0(\mathfrak{t}, M) = M^1\) is generally larger. However, equation \(3\) asserts that \(H^0(\mathfrak{g}, M) = (M^1)_0\), i.e., the subspace of \(M^1\) consisting of those elements which are invariant under \(\mathfrak{h}\). Since as a vector space \(\mathfrak{g}\) is the sum of \(\mathfrak{t}\) and \(\mathfrak{h}\), these are just the elements invariant under all of \(\mathfrak{g}\).

3 Viviani’s lemma

Let \(\mathfrak{g}\) be a Lie algebra (which in the following definitions may be over an arbitrary commutative unital coefficient ring \(k\)), and let \(M\) be a \(\mathfrak{g}\)-module. Denoting \(C^m(\mathfrak{g}, M)\) simply by \(C^m\), the Chevalley-Eilenberg coboundary operator \(\delta : C^n \to C^{n+1}\) is defined by setting

\[
(\delta F)(g_0, g_1, \ldots, g_n) = \sum_{0 \leq i \leq n} (-1)^i [g_i, F(g_0, \ldots, \hat{g}_i, \ldots, g_n)] + \sum_{0 \leq i < j \leq n} (-1)^{i+j} F([g_i, g_j], g_0, \ldots, \hat{g}_i, \ldots, \hat{g}_j, g_n),
\]

where \(\hat{\cdot}\) indicates omission of the argument \(g\). Following [8], it is useful to rearrange the terms on the right in the coboundary formula by taking first all those in which \(\text{ad} g_0\) appears as an operator on \(M\) or on \(\mathfrak{g}\). Denoting by \(\iota_{g_0} F\) the \((n-1)\)-cochain defined by setting \((\iota_{g_0} F)(g_1, \ldots, g_{n-1}) = F(g_0, g_1, \ldots, g_{n-1})\), the coboundary then takes the form

\[
(\delta F)(g_0, g_1, \ldots, g_n) = [g_0, F(g_1, \ldots, g_n)] + \sum_{1 \leq i \leq n} (-1)^i F([g_0, g_i], g_1, \ldots, \hat{g}_i, \ldots, g_n) - (\delta (\iota_{g_0} F))(g_1, \ldots, g_n).
\]

Since \((\delta F)(g_0, g_1, \ldots, g_n) = (\iota_{g_0}(\delta F))(g_1, \ldots, g_n)\), this can be rewritten as

\[
(\iota_{g_0}(\delta F))(g_1, \ldots, g_n) + (\delta (\iota_{g_0} F))(g_1, \ldots, g_n) = [g_0, F(g_1, \ldots, g_n)] + \sum_{1 \leq i \leq n} (-1)^i F([g_0, g_i], g_1, \ldots, \hat{g}_i, \ldots, g_n). \quad (4)
\]

Consider now the case where the toral algebra \(T\) is a subalgebra (necessarily Abelian) of \(\mathfrak{g}\), in which case it will be denoted by \(\mathfrak{h}\).

**Theorem 1** Suppose that a toral subalgebra \(\mathfrak{h}\) of \(\mathfrak{g}\) acts torally also on a \(\mathfrak{g}\)-module \(M\). For \(F \in C^n(\mathfrak{g}, M)\) denote by \(F_\lambda\) the homogeneous component of \(F\) of weight \(\lambda \in \mathfrak{h}^\vee\). Then for all \(h \in \mathfrak{h}\) one has

\[
(\iota_h(\delta F_\lambda))(g_1, \ldots, g_n) + (\delta (\iota_h F_\lambda))(g_1, \ldots, g_n) = \lambda(h) F_\lambda (g_1, \ldots, g_n). \quad (5)
\]

If \(F\) is a homogeneous cochain of weight zero then

\[
(\iota_h(\delta F))(g_1, \ldots, g_n) = - (\delta (\iota_h F))(g_1, \ldots, g_n) \quad (6)
\]
for all $h \in \mathfrak{h}$, i.e., $\iota_h$ is (up to sign) a cochain mapping when restricted to the subcomplex $C^*(\mathfrak{g}, M)_0$ of cochains of weight 0.

**Proof.** When the arguments are all homogeneous elements of $\mathfrak{g}$ it is easy to see that the right side of $[H]$ collapses (after taking account of the signs) to the right side of $[G]$. This implies, however, that it is true in general, proving the first assertion. The second follows. $\square$

**Corollary 1 (Viviani’s Lemma)** Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field $k$, $M$ be a finite dimensional $\mathfrak{g}$-module, and $\mathfrak{h}$ be a toral subalgebra of $\mathfrak{g}$ acting torally also on $M$. Then any cocycle $F^n \in C^n(\mathfrak{g}, M)$ is cohomologous to its homogeneous part of weight zero, and so

$$H^n(\mathfrak{g}, M) \cong H^n(\mathfrak{g}, M)_0.$$

**Proof.** If $F$ is a cocycle then so are all its homogeneous summands, so $[H]$ implies that $\lambda(h)F\lambda$ is a coboundary. If, moreover, $\lambda \neq 0$ then there is some $h \in \mathfrak{h}$ such that $\lambda(h) \neq 0$, hence invertible, showing that $F$ is a coboundary. $\square$

## 4 Filtration of the cochain complex

Suppose that $\mathfrak{h}$ is a subalgebra, not necessarily toral, of a finite-dimensional Lie algebra $\mathfrak{g}$, and $M$ be a finite-dimensional $\mathfrak{g}$-module. If $F^n \in C^n(\mathfrak{g}, M)$ vanishes whenever $r$ or more of its arguments lie in $\mathfrak{h}$, then it is easy to check that $\delta F^n$ vanishes whenever $r + 1$ or more of its arguments lie in $\mathfrak{h}$. Let $F^jC^n(\mathfrak{g}, M)$ be the space of those cochains which vanish whenever $n + 1 - j$ or more arguments lie in $\mathfrak{h}$, frequently denoted by $C^{n-j}_\mathfrak{h}$. Then $F^jC^n(\mathfrak{g}, M)$ is a subcomplex of $C^*(\mathfrak{g}, M)$, giving the Hochschild-Serre descending filtration, $\mathfrak{H}$.

$$C^*(\mathfrak{g}, M) = F^0C^*(\mathfrak{g}, M) \supset F^1C^*(\mathfrak{g}, M) \supset \cdots \supset F^jC^*(\mathfrak{g}, M) \supset \cdots. \quad (7)$$

For any fixed dimension $n$ the filtration terminates since $F^{n+1}C^n(\mathfrak{g}, M) = 0$. The spectral sequence associated with the filtration will not be needed here, since in the case that concerns us it collapses, cf. $\mathfrak{K}$.

Now suppose, as before, that $\mathfrak{h}$ is a toral subalgebra of $\mathfrak{g}$ acting also torally and compatibly on the finite dimensional $\mathfrak{g}$-module $M$. Denoting the homogeneous part of weight 0 of $C^n(\mathfrak{g}, M)$ by $C^n(\mathfrak{g}, M)_0$, the cochain subcomplex $C^*(\mathfrak{g}, M)_0$ certainly inherits the filtration $\mathfrak{H}$, but now we can do better. By Theorem $\mathfrak{I}$ if $F^n \in C^n(\mathfrak{g}, M)_0$ vanishes whenever $r$ or more of its arguments lie in $\mathfrak{h}$, then $(\delta F^n)_0$ does also. Denoting the set of these by $F_rC^n(\mathfrak{g}, M)_0$ we have therefore $\delta F_rC^n(\mathfrak{g}, M)_0 \subset F_rC^{n+1}(\mathfrak{g}, M)_0$ and an ascending filtration

$$0 = F_0C^n(\mathfrak{g}, M)_0 \subset F_1C^n(\mathfrak{g}, M)_0 \subset \cdots \subset F_rC^n(\mathfrak{g}, M)_0 \subset \cdots,$$

where $F_rC^n(\mathfrak{g}, M)_0 = C^n(\mathfrak{g}, M)_0$ for all $r > n$. In addition, we can now identify the quotient $C^n(\mathfrak{g}, M)_0/F_rC^n(\mathfrak{g}, M)_0$. For Theorem $\mathfrak{I}$ implies that if
$F$ is a homogeneous $n$-cocycle of weight 0 and if we are given $h_1, \ldots, h_r \in \mathfrak{h}$ then $\iota_{h_r} \cdots \iota_{h_1} F$ is an $n - r$-cocycle whose cohomology class depends only on that of $F$ and the $h_i$. It is an alternating function of the latter, so we have a linear map

$$\bigwedge^r \mathfrak{h} \otimes C^n(\mathfrak{g}, M)_0 \rightarrow C^{n-r}(\mathfrak{g}, M)_0, \quad r \leq n,$$

whose kernel is precisely $\bigwedge^r \mathfrak{h} \otimes \mathcal{F}_r C^n(\mathfrak{g}, M)_0$. For all $r \leq n$ we therefore have an exact sequence

$$0 \rightarrow \bigwedge^r \mathfrak{h} \otimes \mathcal{F}_r C^n(\mathfrak{g}, M)_0 \rightarrow \bigwedge^r \mathfrak{h} \otimes C^n(\mathfrak{g}, M)_0 \rightarrow C^{n-r}(\mathfrak{g}, M)_0 \rightarrow 0.$$  \hfill (8)

Since by Viviani’s Lemma we only need to use cocycles of weight 0 to compute cohomology, \hfill (8) descends to cohomology, so we have also a linear map

$$\bigwedge^r \mathfrak{h} \otimes H^n(\mathfrak{g}, M) \rightarrow H^{n-r}(\mathfrak{g}, M), \quad r \leq n.$$  \hfill (9)

Recall that if $U, V,$ and $W$ are $k$-spaces with $\dim_k U < \infty$ then there is a canonical isomorphism

$$\text{Hom}(U \bigotimes V, W) \cong \text{Hom}(V, U^{\vee} \bigotimes W),$$

where $U^{\vee}$ is the dual vector space to $U$. Explicitly, suppose that $\dim U = q$, choose a basis $u_1, \ldots, u_q$ and let $u_1^{\vee}, \ldots, u_q^{\vee}$ be the dual basis. If a morphism $\phi : U \bigotimes V \rightarrow W$ is given, then the isomorphism (11) sends $\phi$ to the morphism $\psi : V \rightarrow U^{\vee} \bigotimes W$ sending $v \in V$ to $\sum_{i=1}^q u_i^{\vee} \otimes \phi(u_i \otimes v)$; note that this does not depend on the choice of basis. The inverse is given as follows. Suppose that $\psi : V \rightarrow U^{\vee} \bigotimes W$ is given. If $v \in V$ then $\psi(v)$ can be written in the form $\sum_{i=1}^q u_i^{\vee} \otimes w_i$ for some $w_i, i = 1, \ldots, q \in W$; set $\phi(u \otimes v) = \sum (u_i, v)w_i$.

To conform to later notation, suppose that $\dim \mathfrak{h} = N - 1$. Choose a basis $\eta_1, \ldots, \eta_{N-1},$ let $\eta_i^{\vee}, i = 1, \ldots, N - 1$ be the dual basis, and fix an integer $r$ with $0 \leq r \leq N - 1$. Then the $\eta_1 \wedge \cdots \wedge \eta_r$, with $1 \leq i_1 \prec \cdots \prec i_r \leq N - 1$ form a basis for $\bigwedge^r \mathfrak{h}$, where for $r = 0$ this is $1 \in k$. For simplicity, denote $\eta_{i_1} \wedge \cdots \wedge \eta_{i_r}$ by $\eta_I$, where $I$ denotes the linearly ordered set of indices $i_1, \ldots, i_r$. Then the $\eta_I = \eta_{i_1}^{\vee} \wedge \cdots \wedge \eta_{i_r}^{\vee}$ form the dual basis for $\bigwedge^r \mathfrak{h}^{\vee}$. If $F^n \in C^n(\mathfrak{g}, M)_0$ then we will write also $\iota_I F^n$ for $\iota_{\eta_{i_1}} \iota_{\eta_{i_2}} \cdots \iota_{\eta_{i_r}} F^n$. As $U$ of (11) now take $\bigwedge^r \mathfrak{h}$ and view $\bigwedge^r \mathfrak{h}^{\vee} \otimes C^r(\mathfrak{g}, M)$ as a cochain complex by letting $\delta$ operate only on the factor $C^r(\mathfrak{g}, M)$. For every $r \leq n$ there is then a cochain morphism

$$\sigma : C^n(\mathfrak{g}, M)_0 \rightarrow \bigwedge^r \mathfrak{h}^{\vee} \otimes C^{n-r}(\mathfrak{g}, M)_0$$

sending $F^n \in C^n(\mathfrak{g}, M)_0$ to $\sum_I \eta_I^{\vee} \otimes \iota_I F^n$. The kernel of this map is again just $\mathcal{F}_r C^n(\mathfrak{g}, M)_0$, for it consists of those $F$ such that $\iota_I F = 0$ for all $I$ with $\#I = r$, i.e., those $F$ which vanish when $r$ or more of its arguments lie in $\mathfrak{h}$. This gives a more useful form of (11): For all $r \leq n$ one has

$$0 \rightarrow \mathcal{F}_r C^n(\mathfrak{g}, M)_0 \rightarrow C^n(\mathfrak{g}, M)_0 \rightarrow \bigwedge^r \mathfrak{h}^{\vee} \otimes C^{n-r}(\mathfrak{g}, M)_0 \rightarrow 0.$$  \hfill (13)
By Viviani’s Lemma, there is also a linear map
\[ H^n(g, M) \to \bigwedge^r h^n \otimes H^{n-r}(g, M). \] (14)

Since, as one can readily check, \( \bigwedge^r h^n \cong H^r(h, k) \) one has

**Theorem 2** For all \( r \) with \( 0 \leq r \leq n \) there is a linear map
\[ H^n(g, M) \to H^r(h, k) \otimes H^{n-r}(g, M). \] \( \square \)

Following [8], two \( g \)-modules \( M \) and \( M' \) are said to be paired to a third, \( P \), if there is a bilinear map \((m, m') \mapsto m \wedge m' \) of \( M \times M' \) to \( P \) such that for all \( g \in g \) one has \([g, m \wedge m'] = [g, m] \wedge m + m \wedge [g, m']\). If \( F^r \in C^r(g, M), G^s \in C^s(g, M') \) then we can define \( F^r \wedge G^s \in C^{r+s}(g, P) \) as follows. Let \( I = (i_1, \ldots, i_r) \) be an ordered subset of \( \{1, \ldots, r + s\} \), for simplicity set \( g_I = g_{i_1} \wedge \cdots \wedge g_{i_r} \) and if \( J \) is its complement define \( g_J \) similarly. Then \( I \cup J \) is a permutation of \( \{1, \ldots, r + s\} \). Letting \( \nu(I) \) denote its signum, one has

\[ F^r \wedge G^s(g_1, \ldots, g_{r+s}) = \sum \nu(I) F^r(g_I) \wedge G^s(g_J), \]

where the sum is over all partitions of \( \{1, \ldots, r + s\} \) into a disjoint union \( I \cup J \) with \#\( I \) = \( r \), \#\( J \) = \( s \). Moreover,

\[ \delta(F^r \wedge G^s) = \delta F^r \wedge G^s + (-1)^r F^r \wedge \delta G^s, \]

so it follows that the cup product descends to cohomology.

Since \( M \) and \( M' \) are always paired to \( M \otimes M' \) and \( k \otimes M = M \), the coefficient ring \( k \), considered as a trivial \( g \)-module, is always paired with any \( g \)-module \( M \) to the same \( M \). Hence, there are morphisms

\[ C^r(g, k) \otimes C^s(g, M) \to C^{r+s}(g, M) \]

and clearly also

\[ C^r(g, k)_0 \otimes C^s(g, M)_0 \to C^{r+s}(g, M)_0, \] (15)

from either of which one has

\[ H^*(g, k) \otimes H^*(g, M) \to H^{r+s}(g, M). \] (16)

## 5 Toral semidirect products

When \( h \) is a Lie algebra acting as derivations on another Lie algebra \( \mathfrak{t} \) then we can form the semidirect product \( h \ltimes \mathfrak{t} \). If, moreover, \( h \) is an Abelian algebra acting torally on \( \mathfrak{t} \) then \( h \) operates torally on all of \( g \). This we now assume and call the entire semidirect product \( \mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{t} \) toral. If \( h \) also acts torally and compatibly on the \( g \)-module \( M \), then the subcomplex of \( C^n(g, M)_0 \) of \( C^*(g, M) \)
consisting of cochains of weight 0 relative to the weighting induced by \( h \) inherits the filtration \([I]\). However, we will see that the filtration now actually arises from a gradation on \( C^n(g, M)_0 \), causing the associated spectral sequence to collapse.

As a vector space \( g \) is just the direct sum of \( h \) and \( t \), so we may obtain an ordered basis of \( g \) by first taking the ordered basis \( \eta_1, \ldots, \eta_{N-1} \) of \( h \), followed by an ordered basis \( \kappa_1, \ldots, \kappa_K \) of \( t \). A cochain \( F \in C^n(g, M) \) will be completely determined by its values when its arguments \( g_1, \ldots, g_n \) are taken from this basis of \( g \) and are in the prescribed order, something which we will henceforth always assume.

There is a morphism \( g \to h \), namely reduction modulo \( t \), and consequently there is also an induced morphism \( H^*(h, k) \to H^*(g, k) \). When \( h \) is Abelian one has \( H^*(h, k) \cong \wedge(h^\vee, k) \), which together with \([E]\), produces morphisms

\[
\tau : \wedge^r h^\vee \otimes C^{n-r}(g, M)_0 \to C^n(g, M)_0, \quad \text{all } r
\]

and

\[
\wedge^r h^\vee \otimes H^{n-r}(g, M) \to H^n(g, M), \quad \text{all } r.
\]

Recalling the definition of the morphism \( \sigma \) from \([12]\), the following fundamental result gives the collapse of the spectral sequence.

**Theorem 3** The composite morphism

\[
\wedge^r h^\vee \otimes C^{n-r}(g, M)_0 \xrightarrow{\tau} C^n(g, M)_0 \xrightarrow{\sigma} \wedge^r h^\vee \otimes C^{n-r}(g, M)_0
\]

is the identity; the sequence \([E]\) splits.

**Proof.** Since \( \wedge^r h^\vee \otimes C^{n-r}(g, M)_0 \) is spanned by elements of the form \( \eta_I \otimes G \) with \( G \in C^{n-r}(g, M)_0 \) and \( \eta_I \) of the form \( \eta_{i_1} \wedge \cdots \wedge \eta_{i_r} \), it is sufficient to prove that \( \sigma \tau = \text{identity} \) on such an element. Set \( \tau(\eta_I \otimes G) = F \). With the preceding convention, \( F(g_1, \ldots, g_n) \) vanishes unless \( g_i = \eta_{i_i}, \ldots, g_r = \eta_{i_r}, \) in which case its value is \( G(g_{r+1}, \ldots, g_n) \). Now recall that \( \sigma F = \sum \eta_{J} \otimes \iota_{\eta_{J}} F \). By the definition of \( F \) we have \( \iota_{\eta_{J}} F = 0 \) unless \( J = I \) and \( \iota_{\eta_{J}} F = G \). \( \square \)

Viewing \( \wedge^r h^\vee \otimes H^{n-r}(g, M) \) now as a submodule of \( H^n(g, M) \), the morphism \( H^n(g, M) \to \wedge^r h^\vee \otimes H^{n-r}(g, M) \) is just the projection of \( H^n(g, M) \) onto the submodule spanned by the classes of those cocycles which vanish when \( n + 1 - r \) or more of the arguments lie in \( t \).

Let \( C^{n-r}(g, M)_0 \) now denote the subspace of \( C^n(g, M)_0 \) spanned by those \( n \)-cochains which, when its arguments are chosen as above, vanish unless \( r \) of its arguments are amongst the \( \eta_i \). Then \( \delta C^{n-r}(g, M)_0 \subset C^{n-r+1}(g, M)_0 \). From \([10]\) we have \( \delta F = (-1)^{r+1} \delta(\iota_{h_1} \cdots \iota_{h_{r+1}} F) \) from \([10]\), but the right side vanishes by hypothesis. Since every \( F \in C^n(g, M)_0 \) can be written uniquely as a sum of components in the various \( C^{n-r}(g, M)_0 \) we have the following decomposition into a direct sum of subcomplexes.

9
Theorem 4

\[ C^*(\mathfrak{g}, M)_0 = \bigoplus_{r=0}^{N-1} C^{*+r}(\mathfrak{g}, M)_0 \]

With \( r \) fixed, let \( I \) again denote an \( r \)-tuple of integers \( 1 \leq i_1 < i_2 \cdots < i_r \leq N-1 \) and set \( \eta_I = \eta_{i_1} \wedge \cdots \wedge \eta_{i_r} \). The set of these is a basis for \( \bigwedge^r \mathfrak{h} \). Note that \( r \) may be zero, in which case \( \eta_I = 1 \). With \( n \) also fixed, similarly define \( \kappa_J \) for a set of \( n-r \) distinct integers between 1 and \( K \). When \( F \in C^{n-r} \) has arguments restricted to the chosen basis elements, one can express its value simply as \( F(\eta_I, \kappa_J) \) for suitable \( I \) and \( J \). If also \( F \in C^0 \) then \( \iota_I F = i_{i_1}i_{i_2-1} \cdots i_1 F \) is an \( n-r \) cochain which also has weight 0 since every \( \eta \) has weight 0. It vanishes when any argument is in \( \mathfrak{h} \) and so may be viewed as an element of \( C^{n-r}(\mathfrak{k}, M)_0 \). By repeated use of (5), we therefore have a cochain morphism (up to sign)

\[ \bigwedge^r \mathfrak{h} \bigotimes C^{*+r}(\mathfrak{g}, M)_0 \rightarrow C^{*+r}(\mathfrak{k}, M)_0, \]

where on the left the coboundary operator operates only on \( C^{*+r}(\mathfrak{g}, M)_0 \). From the preceding section, this is identical with a cochain morphism

\[ \phi : C^{*+r}(\mathfrak{g}, M)_0 \rightarrow \bigwedge^r \mathfrak{h} \bigotimes C^{*+r}(\mathfrak{k}, M)_0, \]

where on the right the coboundary operator operates only on \( C^{*+r}(\mathfrak{k}, M)_0 \).

Now, however, we also have a cochain morphism

\[ \psi : \bigwedge^r \mathfrak{h} \bigotimes C^{*+r}(\mathfrak{k}, M)_0 \rightarrow C^{*+r}(\mathfrak{g}, M)_0 \]

defined as follows. If \( \xi \in \mathfrak{h}^r, f \in \bigwedge^r \mathfrak{g} \), then to define \( F = \psi(\xi \otimes f) \in C^{n-r}(\mathfrak{g}, M)_0 \) we only have to give its values when its arguments are amongst the chosen basis element of \( \mathfrak{h} \) and \( \mathfrak{k} \). Let the set of arguments be written, as above, in the form \( \eta_I, \kappa_J \), where \( I \) is an \( r' \)-tuple of integers and \( J \) an \( s' \)-tuple, with \( r' + s' = \dim \mathfrak{g} \). Then set \( F(\eta_I, \kappa_J) = 0 \) unless \( r' = r \), in which case set \( F(\eta_I, \kappa_J) = \langle \xi, \eta_I \rangle f(\kappa_J) \). It is clear then that \( \psi \) is the inverse of \( \phi \). With \( n \) fixed, summing over \( r \) gives

\[ C^n(\mathfrak{g}, M)_0 = \bigoplus_{r=0}^n \bigwedge^r \mathfrak{h} \bigotimes C^{n-r}(\mathfrak{k}, M)_0, \]

so there is an isomorphism of complexes

\[ \Phi : C^*(\mathfrak{g}, M)_0 \cong \bigwedge^r \mathfrak{h} \bigotimes C^*(\mathfrak{k}, M)_0. \]

Taking cohomology, we have the following basic isomorphism.

Theorem 5 Let \( \mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{k} \) be a semidirect product where \( \mathfrak{h} \) is Abelian and acts torally both on \( \mathfrak{g} \) and on a \( \mathfrak{g} \)-module \( M \). Then with respect to the weighting induced by the action of \( \mathfrak{h} \) we have

\[ H^*(\mathfrak{g}, M) \cong \bigwedge^r \mathfrak{h} \bigotimes H^*(\mathfrak{k}, M)_0. \]
6 Cohomology of poset algebras

To every poset $\mathcal{P}$ we can associate a simplicial complex $\Sigma(\mathcal{P})$, which is a special case of that which can be associated to every small category.

The nerve of a small category $\mathcal{C}$ is the simplicial complex $\Sigma(\mathcal{C})$ whose 0-simplices $\sigma^0$ are its objects $(i, (j), \ldots)$, its 1-simplices $\sigma^1$ are its morphisms $\sigma^1 = (i_0 \xrightarrow{\mu} i_1)$, and its $n$-simplices for $n > 1$ are its $n$-tuples of composable morphisms

$$\sigma^n = (i_0 \xrightarrow{\mu_1} i_1 \xrightarrow{\mu_2} i_2 \xrightarrow{\mu_3} \cdots \xrightarrow{\mu_n} i_n).$$

Usually the 0-simplices are considered to have no faces and therefore to have zero boundary. This will be the assumption here when dealing with associative poset algebras, but in the Lie case it will be convenient to augment $\Sigma(\mathcal{C})$ by adjoining a unique simplex $\sigma_{-1}$ of dimension $-1$ which will act as the boundary of every 0-simplex. For $n \geq 1$, an $n$-simplex $\sigma^n$ has $n + 1$ faces $\partial_r \sigma^n$, $r = 0, \ldots, n$ of dimension $n - 1$ defined as follows. If $\sigma^1 = (i \to j)$ then $\partial_0 \sigma^1 = (i)$, $\partial_1 \sigma^1 = (j)$. For $\sigma^n$ with $n > 1$, the $(n - 1)$-dimensional faces $\partial_0 \sigma^n$ and $\partial_n \sigma^n$ are obtained by deleting the morphisms $\mu_0$ and $\mu_n$, respectively, while for $1 \leq r \leq n$, one defines $\partial_r \sigma^n$ by omitting $i_r$ and replacing the successive morphism $\mu_r$ and $\mu_{r+1}$ by their composite. On the free Abelian group generated by the simplices one can define a boundary operator $\partial$ by setting $\partial \sigma^n = \sum_{r=0}^{n} (-1)^r \partial_r \sigma^n$ and extending this linearly to sums of simplices; one then has $\partial \partial = 0$. Simplices where some component morphism is an identity map from an object of $\mathcal{C}$ to itself are called degenerate. The reduced nerve of $\mathcal{C}$ is the quotient of the module generated by all simplices by that generated by the degenerate ones. As the boundary of a degenerate simplex always vanishes, the boundary operator remains well-defined on this reduced nerve. One can then define its homology and cohomology groups with coefficients in $\mathbf{k}$, which are identical with those of the unreduced nerve. While the boundary of a non-degenerate simplex may contain degenerate simplices these are now simply discarded. By “nerve” here we will always mean the reduced nerve.

A poset $\mathcal{P}$ with partial order $\preceq$ may be viewed as a category in which there is a unique morphism $i \to j$ whenever $i \preceq j$ and none otherwise. Since a morphism is then uniquely defined by its domain and range, a non-degenerate $n$-simplex $\sigma^n = (i_0 \to i_1 \to \cdots \to i_n)$ can be identified with the ordered $(n+1)$-tuple of $(i_0, \ldots, i_n)$ of elements of $\mathcal{P}$, where $i_0 \preceq \cdots \preceq i_n$. With this notation, the $r$th face is then obtained simply by removing $i_r$. The resulting simplicial complex will be denoted $\Sigma(\mathcal{P})$, or just $\Sigma$ when $\mathcal{P}$ is understood. The associative poset algebra $A(\mathcal{P})$ is defined to be the free module spanned over $\mathbf{k}$ by all formal elements $e_{ij}$ with $i \preceq j$, with multiplication given by $e_{ij}e_{j'k} = e_{ik}$ if $j = j'$ and 0 otherwise.

Conversely, to every simplicial complex $\Sigma$ one can also associate a poset $\mathcal{P}(\Sigma)$ whose elements are the simplices of the complex with the partial order given by $\sigma \preceq \tau$ when $\sigma$ is a face of $\tau$. The simplicial complex $\Sigma(\mathcal{P}(\Sigma))$ associated to this poset is just the barycentric subdivision of the original $\Sigma$ and therefore
has the same cohomology as $\Sigma$ relative to any given ground ring $k$. One then has the finite case of the basic result of [4]:

**Theorem 6** With the foregoing notation, there is a canonical isomorphism

$$H^*(A(\mathcal{P}), A(\mathcal{P})) \cong H^*(\Sigma, k),$$

where on the left one has Hochschild cohomology and on the right simplicial.

The isomorphism respects the ring structures and carries the squaring operation in the former to the the Steenrod square in the latter. (The more difficult part of [4] is showing that the finiteness condition can be removed.) When $\mathcal{P}$ is finite of order $N$ then we may assume that its underlying set is $\{1, \ldots, N\}$ and that the partial order is compatible with the usual linear order. The $e_{ij}$ are then the usual matrix units and $A(\mathcal{P})$ becomes an algebra of upper triangular matrices.

**Example.** Consider the poset $\mathcal{P}$ on the four element set $\{1, 2, 3, 4\}$ defined by the order relations: $1 \preceq 1, 3, 4$; $2 \preceq 2, 3, 4$; $3 \preceq 3$, and $4 \preceq 4$. These relationships may be organized into a Hasse diagram below. This diagram can be untwisted to visualize the simplicial 1-complex, $\Sigma(\mathcal{P})$. The associated poset algebra $A = A(\mathcal{P})$ is an eight dimensional subalgebra of $\mathfrak{gl}(4, k)$, see Figure 1, in which in the positions labeled by $*$ contain arbitrary element of $k$. Theorem [4] implies that $\dim H^n(A, A) = 1$ for $n = 0, 1$ and is 0 otherwise, something evident from the simplicial complex but not directly from the algebra. Since $H^2(A, A) = 0$ this associative poset algebra is absolutely rigid, but we will see that the related Lie poset algebra admits a non-trivial deformation.

![Figure 1: Hasse diagram of $\mathcal{P}$, simplicial complex $\Sigma(\mathcal{P})$, and poset algebra $A(\mathcal{P})$](image)

**7 Lie poset subalgebras of $\mathfrak{sl}(N, k)$**

From this point on we will be concerned with subalgebras of $\mathfrak{sl}(N, k)$, the Lie algebra of all $N \times N$ matrices over $k$ of trace zero; $e_{ij}$ will be the basic matrix whose $(ij)$ component is 1 and all others are 0. If we have a partial order $\mathcal{P}$ on $\{1, \ldots, N\}$ compatible with the linear order then in analogy with the foregoing we define the **Lie poset algebra** $\mathfrak{g}(\mathcal{P})$ to be the set of matrices of trace zero in $A(\mathcal{P})$, using the commutator multiplication. This contains the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices of trace zero, whose dimension is $N - 1$, and is contained in the Borel subalgebra $\mathfrak{b}$ of all upper triangular matrices. Let $\mathfrak{n}$ be the
ideal of \( \mathfrak{b} \) consisting of all strictly upper triangular matrices. For all \( h \in \mathfrak{h} \) we will denote by \( h(i) \) its \( i \)th diagonal component. Where previously \( \eta_1, \ldots, \eta_{N-1} \) denoted any basis of \( \mathfrak{h} \), henceforth \( \eta_i \), \( i = 1, \ldots, N - 1 \) will specifically denote that diagonal matrix with \( \eta_i(i) = 1, \eta_i(i + 1) = -1 \), and all other components equal to zero. Suppose now that \( \mathfrak{g} \) is a subalgebra of \( \mathfrak{sl}(N, \mathbb{K}) \) with \( \mathfrak{b} \supset \mathfrak{g} \supset \mathfrak{h} \) and set \( \mathfrak{t} = \mathfrak{g} \cap \mathfrak{n} \), an ideal of \( \mathfrak{g} \).

**Theorem 7** With the foregoing notation we have \( \mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{t} \), and \( \mathfrak{t} \) is spanned by those \( e_{ij} \) which it contains. Define a partial order \( \mathcal{P} \) on \( \{1, \ldots, N\} \) by setting \( i \leq j \) if either \( i = j \) or \( e_{ij} \in \mathfrak{t} \); then \( \mathfrak{g} = g(\mathcal{P}) \).

**Proof.** The first assertion is immediate from the fact that an upper triangular matrix is uniquely a sum of an element of \( \mathfrak{h} \) and an element of \( \mathfrak{n} \). For the second it is sufficient to show that if some linear combination \( a = \sum_{k=1}^{p} c_k e_{i_k,j_k}, \ i_k < j_k, c_k \neq 0 \) is in \( \mathfrak{t} \) then at least one of the \( e_{i_k,j_k} \) is already in \( \mathfrak{t} \); this will imply that all are in \( \mathfrak{t} \). If not, suppose that the given \( a \in \mathfrak{t} \) is one with minimal \( r \) having no summand \( e_{i_k,j_k} \) in \( \mathfrak{t} \); surely \( r \geq 2 \). Then \( [\eta_i, a] \) is not a multiple of \( a \) but is a linear combination of the same summands \( e_{i_k,j_k}, k = 1, \ldots, r \), so there is a linear combination of \( a \) and \( [\eta_i, a] \) which is not zero and contains no more than \( r - 1 \) of these summands. One of them is consequently already in \( \mathfrak{t} \), a contradiction. The last assertion follows. \( \square \)

Theorem 7 gives an alternate description of Lie poset algebras which is meaningful for all Lie algebras of Chevalley type. As mentioned in the Introduction, we believe that the results that follow carry over in some way to these, with the restriction that if the rank of the algebra is \( \ell \) then the characteristic \( p \) must exceed \( \ell + 1 \). Here the requirement is that the characteristic of \( \mathfrak{t} \) be greater than \( N \), which we henceforth suppose.

Since the \( e_{ij} \) in \( \mathfrak{sl}(N, \mathbb{K}) \) are all simultaneous eigenvectors for the operations of \( \mathfrak{h} \) they determine elements of \( \mathfrak{h}^\vee \). The weight defined by \( e_{ij} \) will be denoted \( w_{ij} \), so \([h, e_{ij}] = w_{ij}(h)c_{ij}\) for \( h \in \mathfrak{h} \). Note that every \( w_{ij} \) with \( i < j \) is a sum of weights of simple positive roots: \( w_{ij} = w_{i,i+1} + w_{i+1,i+2} + \cdots + w_{j-1,j} \).

**Theorem 8** Suppose that \( e_{i_1,j_1}, \ldots, e_{i_k,j_k}, i_r < j_r, r = 1, \ldots, k \) are distinct elements of \( \mathfrak{sl}(N, \mathbb{K}) \) and that the characteristic of \( \mathbb{K} \) is greater than \( N \). Then

1. \( \sum_{r=1}^{k} w_{i_r,j_r} \neq 0 \)
2. \( \sum_{r=1}^{k} w_{i_r,j_r} \neq w_{ij} \) for any \( e_{ij}, i < j \) unless the \( e_{i_r,j_r} \) can be so ordered that \( i = i_1, j_r = i_{r+1}, r = 1, \ldots, k - 1 \) and \( j_k = j \).

**Proof.** Suppose the first assertion were false. Reordering if necessary, we may assume that \( i_1 \) is minimal amongst the \( i_r \). Then we may assume that \( i_1 = 1 \), else we could reduce the value of \( N \). The vanishing of the sum in 1. is equivalent to having \( [\eta_i, \sum_{r=1}^{k} e_{i_r,j_r}] = 0 \) for all \( i \). Consider the summands on the right of

---

1. The cohomology of \( \mathfrak{n} \) with coefficients in an arbitrary module has been computed by Kostant, [10].
the form \( w_{1j} \). Since \( [\eta_1, e_{12}] = 2 \) while \( [\eta_1, e_{1j}] = 1 \) for \( j > 1 \), writing all \( w_{i_r,j_r} \) as sums of weights of simple positive roots, one sees that the sum of these can not vanish. For \( w_{12} \) appears, but can not appear with coefficient greater than \( N \) since there are no more than \( N - 1 \) distinct possible summands of the form \( w_{1j} \); since we have assumed that \( p > N \) it can not be zero modulo \( p \).

Finally, set (\( \Phi f \)) so that \( \Phi \) is an alternating multilinear function of these arguments. For the second assertion suppose that we have an equality of the kind given. If \( i_1 \) is minimal amongst the \( i_r \) that appear on the left then \( i_1 = i \) for it clearly can not be greater, while it can not be less by the same argument as before; the summands of the form \( w_{i_1,j_r} \) could not cancel. Similarly there can not be more than one summand of the form \( w_{i_j,j_r} \), so \( i_1 = i \) and with this, \( r \) can only be 1. If now \( i_2 > i_1 \) is minimal amongst the remaining \( i_r \) then we must have \( i_2 \geq j_1 \). Otherwise, writing every weight on the left as a sum of weights of simple positive roots, observe that \( w_{i_2,j_2+1} \) would occur at least twice, but its multiplicity can not then be 1 modulo \( p \) since there can be no more than \( p - 1 \) elements of the form \( w_{i_2,j_r} \) on the left. Continuing, we see that after possible reordering we must have \( i = i_1, j_1 \leq i_2, j_2 \leq i_3, \ldots \), from which it is clear that the assertion must hold. \( \square \)

It follows that \( H^n(\mathfrak{g}, \mathfrak{k})_0 = 0 \) for \( n > 0 \), for a cochain \( F \) of weight 0 evaluated on elements of \( \mathfrak{g} \) must have non-zero weight, but all elements of the coefficient module have weight 0. On the other hand, \( H^0(\mathfrak{g}, \mathfrak{k})_0 = \mathfrak{k} \), so from Theorem 9 we have the result of 8.

**Theorem 9** \( H^*(\mathfrak{g}(\mathcal{P}), \mathfrak{k}) \cong \bigwedge \mathfrak{h}^* \) \( \square \)

Theorem 8 will permit us to identify \( H^n(\mathfrak{f}, \mathfrak{k})_0 \) with \( H^n(\Sigma, \mathfrak{f}) \) for \( n > 0 \), in turn allowing the application of Theorem 9. For recall (10) that a non-degenerate \( n \)-simplex of the partially ordered set \( \{1, \ldots, N\} \), can be identified with an ordered \( n+1 \) tuple \((i_0, i_1, \ldots, i_n)\) of integers with \( i_0 < i_1 < \cdots < i_n \) in the partial order induced by \( \mathfrak{g} \), and that an \( n \)-cochain in \( C^n(\Sigma, \mathfrak{k}) \) can then be considered as a function \( f^n(i_0, i_1, \ldots, i_n) \) from such \( n \)-tuples to \( \mathfrak{k} \), where \( \Sigma \) is the associated simplicial complex. We now define a mapping \( \Phi^n : C^n(\Sigma, \mathfrak{k}) \to C^n(\mathfrak{f}, \mathfrak{g}(\mathcal{P}))_0 \) for all \( n > 0 \). A cochain is uniquely determined by its values when all its arguments are basis elements. If \( f^n \in C^n(\Sigma, \mathfrak{k}) \) then set the value of \( \Phi^n f^n \) equal to zero if any argument is \( \eta_i \); it then vanishes if any argument is in \( \mathfrak{h} \). If \( i_0, \ldots, i_n \in \{1, \ldots, N\} \) with \( i_0 < i_1 < \cdots < i_n \) (so by hypothesis \( e_{i_0,i_1}, e_{i_1,i_2}, \ldots, e_{i_{n-1},i_n} \in \mathfrak{t} \)), then set \((\Phi f^n)(e_{i_0,i_1}, e_{i_1,i_2}, \ldots, e_{i_{n-1},i_n}) = f^n(i_0, \ldots, i_n) \) and extend the definition so that \( \Phi f^n \) is an alternating multilinear function of these arguments. Finally, set \((\Phi f^n)(e_{i_0,i_0}, e_{i_1,i_1}, \ldots, e_{i_{n-1},i_{n-1}}) = 0 \) if the arguments can not be so reordered that \( j_0 = i_1, j_1 = i_2, \ldots, j_{n-2} = i_{n-1} \). Then \( \Phi f^n \) is homogeneous of weight 0 relative to the weighting induced by the toral subalgebra \( \mathfrak{h} \) of \( \mathfrak{g}(\mathcal{P}) \), hence an element of \( C^n(\mathfrak{f}, \mathfrak{g}(\mathcal{P}))_0 \). Theorem 8 asserts that \( \Phi^n \) is onto.

When \( n = 0 \) we must make a slight modification. Note that \( C^0(\Sigma, \mathfrak{k}) \) consists of all maps \( f : \{1, \ldots, N\} \to \mathfrak{k} \). Define \( C^0(\Sigma, \mathfrak{k}) \) to consist only of those maps \( f^0 \) such that \( \sum_{i=0}^N f^0(i) = 0 \). There is a bijection \( \Phi : C^0(\Sigma, \mathfrak{k}) \to \mathfrak{h} \) sending \( f^0 \) to \( \sum f^0(i)e_{ii} \). Setting \( C^n(\Sigma, \mathfrak{k}) = C^n(\Sigma, \mathfrak{k}) \) for \( n > 0 \) we then have a bijection.
Φ : C^n(Σ, t) → C^n(t, g(P)) = 0 for all n. Obviously C^n(Σ, k) is a subcomplex of C^*(Σ, k) which differs from it only at dimension 0.

**Theorem 10** The mapping Φ : C^n(Σ, k) → C^n(t, g(P)) = 0 is a cochain isomorphism.

**Proof.** We must show that δΦf^n = Φδf^n for all f^n ∈ C^n(Σ, k). It is sufficient to show that the two sides coincide whenever the arguments g_0,...,g_n are taken from the basis we have chosen for g(P), namely η_1,...,η_{N-1} and the e_ij, i < j. Consider first the case where some g_i, say g_0, is in h. In that case the right side vanishes by the definition of Φ, so we must show that the left, which can now be written as (η_0δΦf^n)(g_1,...,g_n) also vanishes. However, Φf^n has weight 0, so this vanishes in view of (6). Suppose now no argument is in h, so the ordered arguments (g_0,...,g_n) of δΦf^n are amongst the e_ij, i < j. Since δΦf^n is a homogeneous cocycle of weight 0 it must vanish, by Theorem 9, unless the arguments can be reordered to be of the form (e_{i_0,i_1},...,e_{i_{n-1},i_n}) with i_0 < i_1 < ••• < i_n, in which case the value must be a multiple of e_{i_0,i_n}. It follows, cf. Theorem 9 that the only non-zero terms in (δΦf^n)(e_{i_0,i_1},e_{i_1,i_2},...,e_{i_n-1,i_n}) can be ones of the form [e_{i_0,i_1},(Φf^n)(e_{i_1,i_2},...,e_{i_n-1,i_n})], [e_{i_1,i_2},...,e_{i_n-1,i_n}], [Φf^n](e_{i_0,i_1},...,e_{i_n-2,i_n-1}), and (Φf^n)[e_{i_r,i_{r+1}},e_{i_{r+1},i_{r+2}},...,e_{i_{n-1},i_n}], r = 1,...,n - 1. Examining the signs that appear shows that δΦf^n(e_{i_0,i_1},e_{i_1,i_2},...,e_{i_n-1,i_n}) = (δf^n)(i_0,...,i_n)e_{i_0,i_n} = Φ(δf^n)(e_{i_0,i_1},e_{i_1,i_2},...,e_{i_n-1,i_n}). □

**Theorem 11** Let H^n(Σ, k) denote the cohomology of C^n(Σ, k). Then

H^n(Σ, k) = H^n(Σ, k), n ≥ 1.

**Proof.** Since Φ : C^n(Σ, k) and Φ : C^n(Σ, k) differ only at dimension 0, it follows that we certainly have H^n(Σ, k) = H^n(Σ, k) for n > 1. However, we claim that also H^0(Σ, k) = H^1(Σ, k). What must be shown is that δC^0(Σ, k) = δC^0(Σ, k). However, every 0-cochain f : {1,...,N} → k in C^0(Σ, k) can be written uniquely as a sum f = f' + f'' with f' ∈ C^0(Σ, k) and f'' having the same constant value on all i ∈ {1,...,N}: say f''(i) = (1/N) ∑_{j=1}^{N} f(j) for all i and let f''(i) = (1/N) ∑_{j=1}^{N} f(j) all i. This is well-defined since the characteristic is greater than N. Then δf'' = 0, which proves the assertion. □

Recalling that H^0(t, g(P)) = 0 is just the center c(g(P)) of g(P), we have the following.

**Theorem 12** Suppose that k is a field of characteristic 0 or of characteristic p > N. Let b be the Borel subalgebra of sl(N, k) consisting of all upper triangular matrices of trace zero, h be the Cartan subalgebra of all diagonal matrices of trace zero, g(P) be a subalgebra of b containing h, c(g) be the center of g, and Σ(P) be the associated simplicial complex, then for all n ≥ 0 one has

H^n(g(P), g(P)) = (C^n(g(P), g(P)) ⊕ (C^n ∆ γ C^n ∆ γ C^n))
The center of $g(P)$ is spanned by those diagonal matrices $h$ such that $h(i) = h(j)$ whenever $e_{ij} \in \mathfrak{t}$. Adjoining to $\Sigma$ a single simplex $\sigma_{-1}$ of dimension -1 to serve as boundary of every 0-simplex and denoting the enlarged complex by $\Sigma^+$, those 0-cochains $f''$ having the same constant value on all $i \in \{1, \ldots, N\}$ are now coboundaries. A 0-cocycle of $\Sigma^+$ is a cochain $f$ such that $f(i) = f(j)$ whenever $e_{ij} \in \mathfrak{t}$. For each there is now a unique cohomologous cochain with $\sum_{i=1}^{N} f(i) = 0$, so we have an isomorphism, $\alpha(g(P)) \cong H^0(\Sigma^+, k)$. We can finally put the preceding theorem in the more concise form in the Introduction, analogous to the corresponding result for the associative case.

**Corollary 2** With the foregoing hypotheses,

$$H^*(g(P), g(P)) = \bigwedge \mathfrak{h} \bigotimes H^*(\Sigma^+, k). \quad \square$$

### 8 Some special cases; deformations

Using the notation established previously, we note that the associative poset algebra $A(P)$ may be rigid while the associated Lie poset algebra $g(P)$ may not be. The simplest case is that where the partial order is vacuous, so $A(P)$ is the algebra of all diagonal $N \times N$ matrices while $g(P)$ consists of those of trace zero. The former, an associative algebra, is just the direct sum of $N$ copies of $k$. It is separable, therefore has trivial cohomology, and consequently is rigid. The latter, a Lie algebra, is Abelian and therefore deformable to any algebra of the same dimension. Its cohomology must therefore be quite large, as one sees from the theorems of the preceding section.

For more interesting examples, let us construct some small posets whose corresponding simplicial complexes have geometric realizations which are spheres, cf. [6], pp 138, 139] by starting with the zero sphere $S^0$ which consists of just two points, and repeatedly taking two-point suspensions. For $S^0$ we can take $\{1, 2\}$ with vacuous partial order. For $S^1$ take its two-point suspension $\{1, 2, 3, 4\}$ with $1 \prec 3, 4; 2 \prec 3, 4$ and no other relations, for $S^2$ one has $\{1, 2, 3, 4, 5, 6\}$ with $1 \prec 3, 4; 2 \prec 3, 4; 3 \prec 5, 6; 4 \prec 5, 6$ and no relations other than those following from these. With $k$ understood, denote the corresponding associative poset algebras as $A(S^n)$ and the Lie poset algebras by $g(S^n)$. Since $H^2(A(S^0), A(S^0)) = H^2(S^0, k) = 0$ except for $n = 2$, when it is $k$, it follows that $A(S^n)$ is rigid for $n \neq 2$, but for $n = 2$ there is no obstruction to any infinitesimal deformation since $H^3(A(S^2), A(S^2)) = 0$. When $k$ is a field there is therefore a one-parameter family of non-trivial deformations. Assuming that the characteristic of $k$ is at least 7, Theorem [12] asserts that $H^2(g(S^2), g(S^2))$ is likewise one-dimensional, but $H^3(g(S^2), g(S^2))$ does not vanish, since it is $\mathfrak{h} \bigotimes H^2(g(S^2), g(S^2))$. (Note that the center of $g(S^n)$ is 0 except for $n = 0$.) Nevertheless, the existing single infinitesimal deformation (up to non-zero constant multiple) of $g(S^2)$ is not obstructed and does give rise to a one parameter family of deformations. For observe that if we have a deformation of an associative algebra $A$ then the Lie algebra obtained by taking commutators is
simultaneously deformed, and that is what is happening here. (The unit element of \(A(S^2)\) is omitted from \(g(S^2)\) but is not involved in the deformation; in any deformation of a unital associative algebra we may assume, by taking an equivalent deformation, that the unit remains the unit.)

With suitable restriction on the characteristic of \(k\) one sees from Theorem \ref{thm:non-inner-derivation} that \(g(S^n)\) also has no infinitesimal deformations for \(n > 2\) and hence also is rigid, so let us consider the case \(n = 1\). Here \(A(S^1)\) has no infinitesimal deformations and therefore is rigid, while we will see that \(g(S^1)\) in fact allows non-trivial deformations. This is a special case of the following. Returning to the \(g(P)\) at the start of this section, note that if \(H^1(\Sigma(P), k) \neq 0\) then Theorem \ref{thm:non-inner-derivation} implies that \(H^2(g(P), g(P)) \neq 0\) since it contains \(h^\vee \otimes H^1(\Sigma(P), k)\). The elements of the latter give rise to non-trivial deformations of a special kind. Suppose that \(f^1\) is a non-trivial 1-cocycle in \(H^1(\Sigma(P), k)\). With the notation of the preceding section, \(\Phi f^1\) is then a non-trivial 1-cocycle in \(C^1(g(P), g(P))\), that is, a non-inner derivation \(D\) which annihilates \(h\) and is homogeneous of weight 0. Since it therefore carries every \(e_{ij}, i \prec j\) to a multiple of itself and multiplication by an element of \(h\) does the same, this \(D\) commutes with multiplication by any element of \(h\). Choosing any element \(h^\vee\) in the dual space \(h^\vee\) we can now deform \(g(P)\). The products of elements of the form \(e_{ij}, i \prec j\) are to be unchanged, but the operation of \(h\) is changed by setting \([h, e_{ij}]^* = [h, e_{ij}] + t\langle h^\vee, h \rangle D(e_{ij})\), where \([\cdot, \cdot]^*\) denotes the deformed multiplication and \(t\) is the deformation parameter. The power series one expects in a deformation is here simply a polynomial of degree 1 in \(t\). As an illustration, we compute explicitly the deformation associated with one particular non-trivial cocycle.

In \(g(S^1)\) the ideal \(\mathfrak{f}\) is spanned by \(e_{13}, e_{14}, e_{23}, e_{24}\), and is Abelian (cf. Figure 1). Equivalently \(\Sigma(P)\) has no 2-simplices, so any 1-cochain \(f\) there is already a 1-cocycle. One can check that the condition for \(f = f^1\) to be a coboundary is that \(f(1, 2) - f(2, 3) + f(2, 4) - f(1, 4) = 0\). (Viewing \(f\) as an alternating function of its arguments, this is just \(f(1, 2) + f(2, 3) + f(3, 4) + f(4, 1) = 0\).) So, for example, choosing \(f(1, 2) = 1, f(1, 3) = f(2, 3) = f(2, 4) = 0\) gives a 1-cocycle whose class generates \(H^1(\Sigma(P), k)\). The Cartan subalgebra \(\mathfrak{h}\) of \(g(S^1)\) is 3-dimensional with basis \(\{\eta_i, i = 1, 2, 3\}\) in the notation of the preceding section. Let \(\{\eta_i^\vee, i = 1, 2, 3\}\) be the dual basis. Then \(z = \eta_1 \otimes \Phi f\) is a non-trivial 2-cocycle of \(g(S^1)\). As basis for \(g(S^1)\) we have \(\eta_1, \eta_2, \eta_3, e_{13}, e_{14}, e_{23}, e_{24}\). The deformed product \([\cdot, \cdot]^*\) induced by \(z\) on these basis elements is exactly the same as the original except in one instance: one has \(\eta_1, e_{12}^* = (2 + t)e_{12}\). This is a non-trivial deformation of the Lie algebra \(g(S^1)\) not coming from any deformation of the associative algebra \(A(S^1)\).
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