Abstract

Starting from the symplectic construction of the Lie algebra $e_{7(7)}$ due to Adams, we consider an Iwasawa parametrization of the coset $E_{7(7)} / SU(8)$, which is the scalar manifold of $\mathcal{N} = 8$, $d = 4$ supergravity. Our approach, and the manifest off-shell symmetry of the resulting symplectic frame, is determined by a non-compact Cartan subalgebra of the maximal subgroup $SL(8, \mathbb{R})$ of $E_{7(7)}$.

In absence of gauging, we utilize the explicit expression of the Lie algebra to study the origin of $E_{7(7)} / SU(8)$ as scalar configuration of a $\frac{1}{8}$-BPS extremal black hole attractor. In such a framework, we highlight the action of a $U(1)$ symmetry spanning the dyonic $\frac{1}{8}$-BPS attractors. Within a suitable supersymmetry truncation allowing for the embedding of the Reissner-Nordstrom black hole, this $U(1)$ is interpreted as nothing but the global $R$-symmetry of pure $\mathcal{N} = 2$ supergravity.

Moreover, we find that the above mentioned $U(1)$ symmetry is broken down to a discrete subgroup $\mathbb{Z}_4$, implying that all $\frac{1}{8}$-BPS Iwasawa attractors are non-dyonic near the origin of the scalar manifold. We can trace this phenomenon back to the fact that the Cartan subalgebra of $SL(8, \mathbb{R})$ used in our construction endows the symplectic frame with a manifest off-shell covariance which is smaller than $SL(8, \mathbb{R})$ itself. Thus, the consistence of the Adams-Iwasawa symplectic basis with the action of the $U(1)$ symmetry gives rise to the observed $\mathbb{Z}_4$ residual non-dyonic symmetry.
1 Introduction

Local supersymmetry with \( \mathcal{N} = 8 \) supercharge spinor generators is the maximal one realized by a Lagrangian field theory with spin \( s \leq 2 \) in \( d = 4 \) space-time dimensions \([1, 2]\). No matter coupling is allowed, and the bosonic content of the unique gravity supermultiplet is given, besides the \emph{Vielbein}, by 28 Abelian vector fields and 70 real scalar fields. These latter coordinatize the symmetric coset

\[
M_{\mathcal{N}=8,d=4} = \frac{E_{7(7)}}{SU(8)},
\]

where \( E_{7(7)} \) is the U-duality group \([3]\) and \( SU(8) \) is its maximal compact subgroup. The 70 real scalar fields \( \phi_{[ijkl]} \) sit in the rank-4 completely antisymmetric irrepr. 70 of \( SU(8) \) (\( i,j = 1,..,8 \), in the fund. irrepr. 8 of \( SU(8) \)). On the other hand, the two-form Maxwell field strengths and their duals carry a symplectic index \( \hat{A} \) sitting in the fundamental irrepr. 56 of \( E_{7(7)} \), which define the symplectic embedding of the U-duality through the Gaillard-Zumino procedure \([4]\) (see also \( e.g. \) \([5]\)). Thus, the fluxes of the two-form Maxwell field strengths define the dyonic charge vector \( Q^{\hat{A}} \), which then splits into electric and magnetic charges in a manifestly \( SU(8) \)-covariant fashion as follows:

\[
Q^{\hat{A}} = (q_{ij}, p^{ij}),
\]

where antisymmetrization is understood in the pairs of \( SU(8) \)-indices.

\( \mathcal{N} = 8 \) supersymmetry constrains the theory in a remarkably peculiar way, which recently turned out to exhibit exceptional features. Indeed, apart from being studied as a candidate for the simplest quantum field theory \([6]\), \( \mathcal{N} = 8, d = 4 \) supergravity has been shown to have unexpected convergent ultraviolet properties, explicitly computed until four loops in perturbation theory \([7]\).

In absence of gauging, asymptotically flat, static, spherically symmetric, dyonic, extremal (i.e. zero temperature) black holes (BHs), with various degrees of BPS-saturation, emerge as classical solutions of the non-linear Einstein equations. According to \([8]\), these BHs can be seen as smooth solitonic \( p = 0 \)-branes, interpolating between two maximally supersymmetric \( d = 4 \) geometries, namely Minkowski space at spatial infinity and conformally flat \( AdS_2 \times S^2 \) Bertotti-Robinson \([9]\) near-horizon geometry.

At spatial infinity, such BHs are characterized by their ADM mass \([10]\), depending both on \( Q^{\hat{A}} \) and on the asymptotical unconstrained values \( \phi_{[ijkl]} \infty \) of the scalar fields. The area \( A_H \) of the BH
event horizon, and thus, through the Bekenstein-Hawking formula \([11]\), the BH entropy \(S_{BH}\), is given purely in terms \(Q^A\), thanks to the Attractor Mechanism \([12]-[16]\) \([17]\):

\[
\frac{S_{BH}}{\pi} = \frac{A_H}{4\pi} = \sqrt{|I_4|},
\]

where \(I_4\) is the unique quartic Cartan invariant \([18]\) of \(E_{7(7)}\), defined in terms of the rank-4 completely symmetric invariant tensor \(K_{(ABCD)}\) of the 56 of \(E_{7(7)}\) as follows:

\[
I_4 \equiv K_{ABCD} Q_A Q_B Q_C Q_D.
\]

Following the general analysis \([19, 20, 21, 22]\) of \(E_{7(7)}\)-invariant BPS conditions for the various classes of BH states, as well as of the corresponding charge orbits of the 56 of \(E_{7(7)}\), extremal BH attractors in \(\mathcal{N} = 8, d = 4\) supergravity were studied in \([23, 24]\) (see also \([25, 26]\), as well as the recent treatment in \([27]\)), by solving the criticality conditions \([16]\) for the effective BH potential

\[
V_{BH} \equiv \frac{1}{2} Z_{ij} \overline{Z}^{ij},
\]

where \(Z_{ij}\) is the \(\mathcal{N} = 8, d = 4\) complex antisymmetric central charge matrix (see e.g. \([19, 5]\), and Refs. therein). Then, in \([28]\) some simple configurations were considered, corresponding to the well-known typologies of Reissner-Nördstrom, Kaluza-Klein and axion-dilaton BHs. Through suitable branching decompositions of the relevant irreps. of the \(U\)-duality group \(E_{7(7)}\), these well-known solutions were shown to be embedded in maximal \(d = 4\) supergravity. Such an analysis has been further developed in \([29]\), where the relations between extremal \(d = 4\) BHs and extremal \(d = 5\) BHs and black strings have been studied, by exploiting the connection between \(E_{7(7)}\) and the \(d = 5\) \(U\)-duality group \(E_{6(6)}\). To this aim, \(\mathcal{N} = 8, d = 4\) supergravity has been formulated in a manifestly \(E_{6(6)}\)-covariant basis \([30]\), namely the one related to the Sezgin-Van Nieuwenhuizen \(d = 5 \rightarrow d = 4\) dimensional reduction \([31]\). This is not the same as the Cremmer-Julia \([1]\) or de Wit-Nicolai \([2]\) symplectic frame, whose maximal non-compact off-shell symmetry is \(SL(8, \mathbb{R})\). The relation between these two formulations, usually adopted to study \(d = 4\) maximal supergravity in absence of gauging, has been precisely discussed in \([28]\), and it amounts to dualizing several vector fields and therefore to interchanging the electric and magnetic charges of some of the 28 Abelian vector fields of the theory.

Furthermore, extremal BH attractors provide an interesting arena, in which the above mentioned issues of ultraviolet convergence of perturbative quantum field theory computations (leading to the conjecture of ultra-violet finiteness of \(\mathcal{N} = 8, d = 4\) supergravity) have been recently investigated (see \([32, 33]\), and Refs. therein; see also \([34, 35]\)).

\(Ca\ va\ sans\ dire,\ the\ Sezgin-Van\ Nieuwenhuizen\ \([31]\)\ and\ Cremmer-Julia\ \([1]\)\ or\ de\ Wit-Nicolai\ \([2]\)\ symplectic\ frames\ are\ not\ the\ only\ ones\ in\ which\ \(\mathcal{N} = 8, d = 4\)\ supergravity\ can\ be\ formulated.\ Apart\ from\ the\ bases\ related\ to\ the\ various\ possible\ gaugings\ of\ the\ theory\ (see\ e.g.\ \([36, 37, 38]\),\ and\ Refs.\ therein),\ other\ ungauged\ formulations\ can\ be\ considered,\ and\ they\ can\ be\ useful\ to\ unveil\ some\ interesting\ facets\ of\ the\ theory\ itself.\)

Hinted by Adam’s approach to the Lie algebra \(e_{7(7)}\) \([39]\), in this paper we explicitly perform an Iwasawa parametrization of the coset representative of the symmetric manifold \(\frac{E_{7(7)}}{SU(8)}\). The main feature of such a construction is the use of a completely non-compact 7-dimensional Cartan subalgebra of \(SL(8, \mathbb{R})\), which leads to the nilpotency of the matrix realization of the relevant coset generators, determining the maximal manifest covariance of the whole framework to be \(SL(7, \mathbb{R})\). Considering the expression of the coset \([1.1]\) to the first order, i.e. at the Lie algebra level, we then study the \(SU(8)\)-invariant origin of such a manifold as a \(\frac{1}{8}\)-BPS scalar configuration corresponding to an extremal BH attractor \([23, 28]\).

Within such an approach to \(\frac{1}{8}\)-BPS attractors, we remark the existence of a residual “degeneracy symmetry” \(U(1)\). This symmetry is residual, because it characterizes the (particular representative of
the) orbit of charge configurations which support $\frac{1}{8}$-BPS attractors. Furthermore, it is a “degeneracy symmetry” because it spans the dyonic nature of the $\frac{1}{8}$-BPS solutions exhibiting an Attractor Mechanism. As also pointed in the analysis of [28], this symmetry is decompactified to $SO(1, 1)$ in non-BPS attractors, thus not allowing for the origin of $E_{7(7)}^{\text{SU}(8)}$ to constitute a representative of non-BPS $\mathcal{N} = 8$ attractor scalar configurations.

The main result of the present investigation is the discovery that such a $U(1)$ symmetry, characterizing the $\frac{1}{8}$-BPS attractors in $E_{7(7)}^{\text{SU}(8)}$, actually gets spoiled within the coset construction à la Adams-Iwasawa performed in the paper. Indeed, $U(1)$ is broken down to a discrete $\mathbb{Z}_4$ subgroup, as it appears from the purely electric or purely magnetic nature of the solutions to the set of Attractor Equations governing the near-horizon dynamics of the scalar fields. By analyzing such a $U(1) \rightarrow \mathbb{Z}_4$ breaking in detail, we are able to trace its origin back to the maximal manifest off-shell covariance properties of the construction, i.e., to the choice of a 7-dimensional completely non-compact Cartan subalgebra of $SL(8, \mathbb{R})$, which breaks the maximal manifest off-shell covariance down to $SL(7, \mathbb{R})$, or, through a suitable Cayley rotation, to $SU(7)$.

Thus, our investigation points out that the dyonic nature of $\frac{1}{8}$-BPS extremal BH attractor in ungauged $\mathcal{N} = 8$, $d = 4$ supergravity essentially relies on the covariance properties exhibited by the parametrization chosen for the scalar manifold (1.1) itself. As explained in the concluding Sect. 6, each of the symplectic frames mentioned above is “natural” in order to explicit the maximal symmetry of a class of attractors. In this perspective, the Lie-algebra approach to the Adams-Iwasawa construction of $E_{7(7)}^{\text{SU}(8)}$, studied in the present paper highlights the action of a $U(1)$ symmetry in the dyonic attractor solutions pertaining to $\frac{1}{8}$-BPS BH states, and its breaking to a discrete subgroup.

It should be remarked that, in light of the embedding analysis performed in [28], the Lie algebra limit of $\frac{1}{8}$-BPS attractors is related to the embedding of the Reissner-Nördstrom extremal BH solution of pure $\mathcal{N} = 2$, $d = 4$ supergravity into a $\mathcal{N} = 8$ maximal theory. In this framework, the residual $U(1)$ “degeneracy symmetry” mentioned above is nothing but the $U(1)$ global $\mathcal{R}$-symmetry of pure $\mathcal{N} = 2$, $d = 4$ theory[1]11. In this context, the $U(1) \rightarrow \mathbb{Z}_4$ breaking due to the constraints on manifest covariance imposed by the Adams-Iwasawa construction, can be interpreted as a breaking of the $\mathcal{R}$-symmetry of the pure $\mathcal{N} = 2$, $d = 4$ theory, to which $\mathcal{N} = 8$, $d = 4$ supergravity gets effectively truncated in the sector of $\frac{1}{8}$-BPS attractors near the origin of (1.1).

The plan of the paper is as follows.

Sect. 2 is devoted to a detailed construction of the coset representative of the $\mathcal{N} = 8$, $d = 4$ scalar manifold (1.1), by exploiting Adams’ realization [39] of the Lie algebra of the $U$-duality group $E_{7(7)}$ (Subsect. 2.1). Then, using a completely non-compact 7-dimensional Cartan subalgebra of $SL(8, \mathbb{R})$ as a pivot, in Subsect. 2.2 a parametrization à la Iwasawa of the coset representative is worked out.

Then, Sect. 3 deals with the formulation of $\mathcal{N} = 8$, $d = 4$ ungauged supergravity theory within such a symplectic frame, computing the central charge matrix $Z_{ij}$ and the effective BH potential $V_{BH}$ in terms of the symplectic electric and magnetic sections.

At the Lie algebra level, exploring the Attractor Mechanism in the neighbourhood of the origin of the scalar manifold (1.1) itself, the $\frac{1}{8}$-BPS attractor solutions are studied in Sect. 4. From the analysis performed in Subsect. 4.1 only purely electric or purely magnetic Iwasawa solutions are obtained. The non-dyonic nature of such attractors is then investigated in Subsect. 4.2 in which it is found that such a phenomenon is due to the breaking of the residual “degeneracy symmetry” $U(1)$ down to a subgroup $\mathbb{Z}_4$.

Concerning the $d = 5$ uplift properties of maximal $d = 4$ supergravity, Sect. 5 reports some relations between scalar manifolds and moduli spaces of attractors, with some new observations related to the $c$-map [42] and thus to $d = 3$ (non-maximal) theories, hinting to further developments, also in view of

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[1] Indeed, in absence of scalars the $\mathcal{R}$-symmetry, usually contained in the stabilizer of the scalar manifold, gets promoted to a global ($U$-duality) symmetry [40].
2 Adams-Iwasawa Approach to \(e_7(7)/su(8)\)

2.1 Adams’ Symplectic Construction of \(e_7(7)\)

In order to obtain a direct construction of the maximally non-compact exceptional Lie algebra \(e_7(7)\), we follow Chapter 12 of [39].

Let \(V\) be an 8-dimensional real vector space and \(V^*\) its dual. The notation \(\Lambda^i V\) denotes the \(i\)-th external power of \(V\). By exploiting the isomorphism \(\Lambda^8 V \cong \mathbb{R}\), one can then define \(SL(V)\) as the group of automorphisms preserving such isomorphism. Then, the Lie algebra of \(SL(V)\) itself can be defined:

\[
L \equiv \mathfrak{sl}(V).
\]

\(L\) acts on the 56-dimensional real vector space

\[
W \equiv \Lambda^2 V \oplus \Lambda^2 V^*
\]

in the usual way, namely:

\[
L(W) = L(V) \wedge V \oplus L(V^*) \wedge V^* + V \wedge L(V) \oplus V^* \wedge L(V^*),
\]

where \(L(V^*)\) denotes the adjoint action.

If \(i + j = 8\), the pairing

\[
\Lambda^i V \otimes \Lambda^j V \longrightarrow \Lambda^8 V \cong \mathbb{R},
\]

given by the wedge product \(\wedge\), defines an isomorphism

\[
\Lambda^i V \cong \Lambda^j V^*.
\]

Such an isomorphism can then be used to define an action of

\[
\lambda^4 \equiv \Lambda^4 V
\]

(with \(dim_{\mathbb{R}} = \binom{8}{4} = 70\)) on \(W\) by means of the maps:

\[
\lambda^4 \otimes \Lambda^2 V \xrightarrow{\wedge} \Lambda^6 V \cong \Lambda^2 V^*;
\]

\[
\lambda^4 \otimes \Lambda^2 V^* \cong \Lambda^4 V^* \otimes \Lambda^2 V^* \xrightarrow{\wedge} \Lambda^6 V^* \cong \Lambda^2 V.
\]

Thus, it follows that

\[
A \equiv L \oplus \lambda^4
\]

is a 133-dimensional real vector space of operators acting on \(W\).

The following Theorem holds (cfr. Theorem 12.1, as well as the end of Chapter 12, of [39]):

**Theorem 1.** \(A\) is a Lie algebra of maps which acts on \(W\) in the same way as \(e_7(7)\) acts on its fundamental irrepr. \(56\), up to isomorphisms.

Then, \(A\) is a realization of the Lie algebra \(e_7(7)\) with irreducible representation \((A,W)\). Up to isomorphisms, this is indeed the smallest faithful representation of \(e_7(7)\).
2.1.1 Matrix Realization

After identifying $V$ with $\mathbb{R}^8$, the action of $L$ on $V$ is generated by the action of the traceless $8 \times 8$ matrices in $M(8, \mathbb{R})$. One can then choose a basis $\{e_i\}_{i=1, \ldots, 8}$ of $V$ and a basis $\{A_{kl}, S_{kl}, D_\alpha\}$ (with cardinality $8 \times 8 - 1 = 63$) for $M(8, \mathbb{R})$, defined as follows:\footnote{Throughout the whole treatment, as usual, the square brackets denote antisymmetrization of enclosed indices, according to the definition $A_{[kl]} \equiv \frac{1}{2}(A_{kl} - A_{lk})$, while the round brackets indicate symmetrization of enclosed indices: $S_{(kl)} \equiv \frac{1}{2}(S_{kl} + S_{lk})$. It is worth remarking that often the symmetry properties are used to introduce an ordering rule, and to restrict the range of the indices.}

\begin{align*}
A_{kl} e_i & \equiv \delta_{kl} e_k - \delta_{kl} e_l = A_{kl} e_i; \\
S_{kl} e_i & \equiv \delta_{kl} e_k + \delta_{kl} e_l = S_{(kl)} e_i; \\
D_\alpha & \equiv \text{diag}\{D^\alpha_1, \ldots, D^\alpha_8\}; \quad \text{Tr} (D_\alpha) = 0.
\end{align*}

Thus, $A_{kl}$’s and $S_{kl}$’s respectively are 28 antisymmetric and 28 symmetric\footnote{Notice that, despite their traceless symmetry, $S_{kl}$’s are only 28 (and not 35), because the index ordering $k < l$ has been enforced. The $8 - 1 = 7$ traceless diagonal (i.e. $k = l$) degrees of freedom of $S_{kl}$’s are implemented through the $D_\alpha$’s.} $8 \times 8$ matrices, whereas $D_\alpha$’s are 7 diagonal traceless $8 \times 8$ matrices, which can be identified with the Cartan subalgebra of $\mathfrak{e}_7(7)$ (see further below). Their normalization is chosen such that

\begin{align*}
\text{Tr} (A_{kl} A_{mn}) & \equiv -2 \delta_{kl} \delta_{mn}; \\
\text{Tr} (S_{kl} S_{mn}) & \equiv 2 \delta_{kl} \delta_{mn}; \\
\text{Tr} (D_\alpha D_\beta) & \equiv 2 \delta_{\alpha\beta}.
\end{align*}

It is now possible to extend the action of $L$ on $V$ (defined above) to $\Lambda^2 V$, and next to $W$. To this aim, let us introduce $\{e_{ij}\}_{i < j} \equiv e_i \wedge e_j$ as basis for $\Lambda^2 V$, and denote its dual by $\{\varepsilon_{ij}\}_{i < j}$. Thus, one reaches the following results:\footnote{In these sums we do not restrict $m < n$, rather we take into account that $\varepsilon_{mn} = -\varepsilon_{nm}$. It similarly holds for Eqs. (2.22)-(2.24).}

\begin{align*}
A_{kl} (e_{ij}) & = \sum_{m, n}(U^A_{klmn} D_{kljn} + D_{klmn} U^A_{kljn}) e_{mn}; \\
S_{kl} (e_{ij}) & = \sum_{m, n}(U^S_{klmn} D_{kljn} + D_{klmn} U^S_{kljn}) e_{mn}; \\
D_\alpha (e_{ij}) & = (D^i_\alpha + D^j_\alpha) e_{ij},
\end{align*}

where the quantities

\begin{align*}
U^A_{klmn} & \equiv \delta_{km} \delta_{li} - \delta_{ki} \delta_{lm}; \\
U^S_{klmn} & \equiv \delta_{km} \delta_{li} + \delta_{ki} \delta_{lm}; \\
D_{klmn} & \equiv \begin{cases} 
\delta_{lm} & \text{for } k \neq l \neq i; \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
Equations (2.22), (2.16), and (2.23), (2.17) define the $56 \times 56$ matrices representing the action on $W$ of the operators $A_{kl}$ and $S_{kl}$ respectively. We will keep the names $A_{kl}$ and $S_{kl}$ for such matrices. Similarly, (2.24) and (2.18) define the $56 \times 56$ matrices $h_{D_{\alpha}}$ corresponding to the diagonal matrices $D_{\alpha}$.

In order to determine the remaining 70 generators of $e_{7(7)}$ (which span $\lambda^4$ defined by (2.6)), we consider the action of

$$
\lambda_{i_1 i_2 i_3 i_4} \equiv e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge e_{i_4} \quad \text{(2.25)}
$$
on $W$. By exploiting the identifications (2.7) and (2.8), this yields to

$$
(e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge e_{i_4}) \otimes (e_{j_1 j_2}) \mapsto \frac{1}{2} \epsilon_{i_1 i_2 i_3 i_4 j_1 j_2 k_1 k_2} e_{k_1 k_2},
$$

(2.26)

$$
(e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge e_{i_4}) \otimes (\sigma_{j_1 j_2}) \mapsto \frac{1}{2} \delta_{i_1 i_2 i_3 i_4} e_{k_1 k_2},
$$

(2.27)

where $\epsilon$ is the standard 8-dimensional Levi-Civita tensor. Furthermore

$$
\delta_{i_1 i_2 i_3 i_4} \equiv \sum_{\sigma \in \mathcal{P}[1,2,3,4]} \epsilon_{\sigma(i_1)} \delta_{i_2}^{i_1} \delta_{i_3}^{i_2} \delta_{i_4}^{i_3} \delta_{i_4}^{i_4},
$$

(2.28)

where $\mathcal{P}[1,2,3,4]$ denotes the set of permutations of $[1,2,3,4]$, and $\epsilon_{\sigma}$ stands for the parity of permutation $\sigma$.

Within the basis $e \equiv e_{ij}$ ($i < j$), it is convenient to use a double-index notation for matrices, such that $e.g.$ the action of the matrix $M$ on $e$ reads:

$$
(M e)^{ij} = \sum_{k<l} M^{ij|kl} e_{kl}.
$$

(2.29)

Thus, the action of $\lambda_{i_1 i_2 i_3 i_4}$, written in block matrix form with respect to the decomposition (2.2), reads:

$$
\lambda_{i_1 i_2 i_3 i_4} = \left( \begin{array}{cc}
0 & \lambda_u(i_1, i_2, i_3, i_4)_{ijkl} \\
\lambda_d(i_1, i_2, i_3, i_4)_{ijkl} & 0
\end{array} \right) = \left( \begin{array}{cc}
0 & \epsilon_{i_1 i_2 i_3 i_4 j_1 j_2 kl} \\
\delta_{i_1 i_2 i_3 i_4} e_{j_1 j_2 kl} & 0
\end{array} \right),
$$

(2.30)

where it is worth pointing out that the matrices $\lambda_u$ and $\lambda_d$ are both symmetric.

It is now convenient to introduce the tetra-indices $I \equiv [i_1 i_2 i_3 i_4]$ (notice the complete antisymmetrization), endowed with the ordering rule $i_1 < i_2 < i_3 < i_4$. This in turn uniquely determines the complementary tetra-index $\bar{I}$, such that $\epsilon_{I\bar{I}} \neq 0$. As a consequence, it follows that

$$
(\lambda_{I})^T = \epsilon_{I\bar{I}} \lambda_{\bar{I}},
$$

(2.31)

This allows for a change of basis in $\lambda^4$ through the introduction of the symmetric matrices

$$
S_I \equiv \frac{1}{\sqrt{2}} (\lambda_I + \epsilon_{I\bar{I}} \lambda_{\bar{I}}),
$$

(2.32)

as well as of antisymmetric matrices

$$
A_I \equiv \frac{1}{\sqrt{2}} (\lambda_I - \epsilon_{I\bar{I}} \lambda_{\bar{I}}).
$$

(2.33)

Since each of the sets of tetra-indices $I \equiv \{I\}$ and $\bar{I} \equiv \{\bar{I}\}$ has cardinality 70, the definitions (2.32) and (2.33) exhibit a double over-counting, namely only half of the $S_I$’s and of the $A_I$’s is

\footnote{In (2.30) subscripts “u” and “d” simply stand for “up” respectively “down”, referring to the position of the block matrices within $\lambda_{i_1 i_2 i_3 i_4}$ (see also Eq. (2.58)).}
where $e^{2.2}$ Iwasawa Parametrization of subalgebra $C$

As the first step, one needs to choose a complete set of positive roots with respect to the Cartan underlying the 70-dim. real symmetric coset manifold consistent with (2.9), (2.10), (2.11), (2.12) and (2.34).

$$\{S_I, A_I\}_{I\in \mathcal{I}_8}, \quad (2.34)$$

with cardinality $35 + 35 = 70$.

Furthermore, through (2.9), a basis for $A$ is given by

$$\{A_{kl}, h_{Da}, S_{kl}, S_I\}, \quad (2.35)$$

$$\begin{cases} 1 \leq k < l \leq 8; \\ 1 \leq \alpha \leq 8; \\ I \in \mathcal{I}_8. \end{cases} \quad (2.36)$$

It is worth noting that, by construction, all matrices in (2.35) are orthogonal. The set of antisymmetric matrices $A_\mu \equiv \{A_{kl}, A_I\}$ is normalized as $Tr(A_\mu A_\nu) = -2\delta_{\mu\nu}$, and it has cardinality $28 + 35 = 63$ ($\mu = 1, \ldots, 63$), so that $A_\mu$ generates the maximal compact (symmetric) subgroup $SU(8)$ of $E_{7(7)}$ (see e.g. [44]). The remaining set of $7 + 28 + 35 = 70$ symmetric generators $S_\Lambda \equiv \{h_{Da}, S_{kl}, S_I\}$ ($\Lambda = 1, \ldots, 70$) is normalized as $Tr(S_\Lambda S_M) = 2\delta_{\Lambda M}$, so that it spans the non-compact part of $\varepsilon_{7(7)}$. In particular, as mentioned above, the 7 diagonal matrices $D_\alpha$ (or equivalently $h_{Da}$, see definition (2.55) below) generate a Cartan subalgebra

$$C \equiv \langle D_\alpha \rangle_\mathbb{R} \subseteq \varepsilon_{7(7)}, \quad (2.37)$$

containing no compact elements. Thus, the cardinality of the basis (2.35) of $A$ is $63 + 70 = 133$, consistent with (2.9), (2.10), (2.11), (2.12) and (2.34).

We can now proceed to perform an explicit Iwasawa parametrization of the algebra $\varepsilon_{7(7)}/su(8)$ underlying the 70-dim. real symmetric coset manifold $E_{7(7)}/SU(8)$ (namely, the scalar manifold of $N = 8, d = 4$ supergravity [1,1]).

### 2.2 Iwasawa Parametrization of $\varepsilon_{7(7)}/su(8)$

As the first step, one needs to choose a complete set of positive roots with respect to the Cartan subalgebra $C$ defined by (2.37). To this aim, it is convenient to introduce the following notation (recall definition (2.1)):

$$L = C \oplus \langle J^+ \rangle_\mathbb{R} \oplus \langle J^- \rangle_\mathbb{R}, \quad (2.38)$$

where

$$\langle J^+ \rangle_\mathbb{R} \equiv \left\{ J_{kl}^+ \equiv \frac{1}{\sqrt{2}}(S_{kl} + A_{kl}) \mid k < l \right\}_\mathbb{R}, \quad (2.39)$$

$$\langle J^- \rangle_\mathbb{R} \equiv \left\{ J_{kl}^- \equiv \frac{1}{\sqrt{2}}(S_{kl} - A_{kl}) \mid k < l \right\}_\mathbb{R}. \quad (2.40)$$

Furthermore, it holds that (recall (2.1) and (2.6), as well as (2.30))

$$\lambda^4 = \langle J \rangle_\mathbb{R}, \quad (2.41)$$

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Note that one might have instead fixed one index out of $\{i_1, i_2, i_3, i_4\}$ to one of the values $\{1, \ldots, 8\}$. All such choices are equivalent to fixing $i_4 = 8$, due to the complete antisymmetrization of the four indices $i_1i_2i_3i_4$.

Throughout our treatment, $\langle A \rangle_\mathbb{F}$ denotes the set of linear combinations of elements of $A$ with coefficients in the ground field $\mathbb{F}$.  

7 Throughout our treatment, $\langle A \rangle_\mathbb{F}$ denotes the set of linear combinations of elements of $A$ with coefficients in the ground field $\mathbb{F}$.  

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where
\[ \langle \mathcal{J} \rangle_R \equiv \{ \langle J_I \equiv \lambda_I | I \in \mathcal{I} \rangle \}_R \]. \tag{2.42} \]

It is then possible to prove the following

**Proposition 1.** The set \( J^+ \cup J^- \cup \mathcal{J} \) diagonalizes simultaneously the adjoint action of \( \mathcal{C} \).

**Proof.** The proof proceeds by direct computation. The adjoint action of \([ h_{D_{\alpha}} J_{kl}^\pm ] \) on \( e_{ij} \) and on \( \varepsilon_{ij} \) turns out to be:
\[
[h_{D_{\alpha}} J_{kl}^\pm] = \pm (D_k^\alpha - D_l^\alpha) J_{kl}^\pm, \tag{2.43}
\]
while for \( J_I = \lambda_{i_1 i_2 i_3 i_4} \) (recall definition (2.42)) one can compute that (recalling the tracelessness of \( D_{\alpha}'s \), see Eq. (2.12))
\[
[h_{D_{\alpha}} J_I] = (D_{i_1}^i + D_{i_2}^i + D_{i_3}^i + D_{i_4}^i) J_I. \tag{2.44}
\]

To proceed further, we split the set \( \mathcal{J} \) defined in (2.42) as follows:
\[
\mathcal{J} = J^+ \cup J^- \tag{2.45}
\]
where
\[
J^+ \equiv \{ \lambda_I \in \mathcal{J} : I \in \mathcal{I}_8 \}; \tag{2.46}
\]
\[
J^- \equiv \{ \lambda_I \in \mathcal{J} : I \notin \mathcal{I}_8 \}. \tag{2.47}
\]
Moreover, by extending the notation \( h_{D_{\alpha}} \) introduced above, in the treatment below we will denote by \( h_D \) the \( 56 \times 56 \) matrix representation of any \( 8 \times 8 \) traceless diagonal matrix \( D \) with real entries (which is then a “diagonal” element of \( \mathfrak{s}(8) \)). Then, one can prove the

**Proposition 2.** The set \( J^+ \cup J^+ \) defines a choice of positive roots of \( \mathfrak{e}_7(7) \). The corresponding roots are the operators
\[ \{ \beta_{kl} \}_{k<l} \cup \{ \beta_{i_1 i_2 i_3 i_4} \}_{i_1 i_2 i_3 i_4 \in \mathcal{I}_8}, \tag{2.48} \]
defined by
\[
\beta_{kl}(h_D) \equiv D^k - D^l, \quad k < l; \tag{2.49}
\]
\[
\beta_{i_1 i_2 i_3 i_4}(h_D) \equiv D^{i_1} + D^{i_2} + D^{i_3} + D^{i_4}, \quad i_1 i_2 i_3 i_4 \in \mathcal{I}_8. \tag{2.50}
\]

**Proof.** The fact that the set (2.49)-(2.50) is the subset of eigen-matrices for the adjoint action of \( \mathcal{C} \) corresponding to the set of given roots, follows immediately from Proposition 1. One only needs to check that such roots lie in a convex cone. To this end, it is sufficient to show that there exists at least one matrix \( c \in \mathcal{C} \) such that \( \beta_{k<l}(c) > 0 \) and \( \beta_{\mathcal{I}_8}(c) > 0 \). In fact, this is the case if \( e.g. \) the following \( c \) is chosen:
\[
c = \text{diag}\{-1, -2, -3, -4, -5, -6, -7, 28\}. \tag{2.51}
\]

Within the choices (2.49) and (2.50) for a complete set of positive roots of \( \mathfrak{e}_7(7) \), an Iwasawa parametrization for the representative of the irreducible, Riemannian, globally symmetric coset space \( E_7(7)/SU(8)/\mathbb{Z}_2 \). The extra factor \( \mathbb{Z}_2 \) is due to the fact that our construction is performed by starting from a completely non-compact Cartan subalgebra of \( SL(8, \mathbb{R}) \), and thus the associated maximal torus is the double cover of that of \( E_7(7) \). In other words, from the point of view of the corresponding supergravity theory, spinors transform according to the double cover of the stabilizer of the scalar manifold (see e.g. \[45, 46\], and Refs. therein).

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\[8\] To be more precise, it is worth mentioning that the coset manifold parametrized à la Iwasawa by (2.53) is actually \( E_7(7)/SU(8)/\mathbb{Z}_2 \). The extra factor \( \mathbb{Z}_2 \) is due to the fact that our construction is performed by starting from a completely non-compact Cartan subalgebra of \( SL(8, \mathbb{R}) \), and thus the associated maximal torus is the double cover of that of \( E_7(7) \). In other words, from the point of view of the corresponding supergravity theory, spinors transform according to the double cover of the stabilizer of the scalar manifold (see e.g. \[45, 46\], and Refs. therein).
can be written down as follows\footnote{In the parametrization \eqref{eq:2.53}, the Abelianity of the completely non-compact Cartan subalgebra $C$ of $\mathfrak{e}_7(7)$ defined by \eqref{eq:2.37} has been implemented:}
\begin{equation}
\prod_{\alpha=1}^{7} \exp(x^{\alpha} h_{D_{\alpha}}) = \exp \left( \sum_{\alpha=1}^{7} x^{\alpha} h_{D_{\alpha}} \right).
\end{equation}

The explicit expression of the three typologies of matrices $h_{D_{\alpha}}$, $J^+_{ij}$ and $J^+_{I}$ appearing in \eqref{eq:2.53} can be obtained as follows.

Let us start by observing that all such $56 \times 56$ matrices can be rewritten in terms of $28 \times 28$ blocks $M^m_{ij}$ (with $1 \leq m < n \leq 8$ selecting the columns, and $1 \leq i < j \leq 8$ respectively the rows). In particular
\begin{equation}
\delta^m_{ij} \equiv \frac{1}{2}(\delta^m_j \delta^n_i - \delta^n_j \delta^m_i)
\end{equation}
is the antisymmetric identity. Thus, the diagonal generators of completely non-compact Cartan subalgebra $C$ of $\mathfrak{e}_7(7)$ read (recall Eqs. \eqref{eq:2.18} and \eqref{eq:2.24})
\begin{equation}
h_{D_{\alpha}} \equiv \left(
\begin{array}{cc}
(D^i_{\alpha} + D^j_{\alpha})\delta^m_{ij} & 0 \\
0 & -(D^i_{\alpha} + D^j_{\alpha})\delta^m_{ij}
\end{array}
\right).
\end{equation}

On the other hand, the $J^+_{kl}$ matrices read
\begin{equation}
J^+_{kl} = \left(
\begin{array}{cc}
M^m_{ij} & 0 \\
0 & N^i_{mn}
\end{array}
\right) \equiv \sqrt{2} \left(
\begin{array}{cc}
\delta_{kl} \delta^m_{ij} - \delta_{ij} \delta^m_{kl} & 0 \\
0 & \delta_{kn} \delta^m_{ij} - \delta_{jm} \delta^m_{kn}
\end{array}
\right),
\end{equation}
thus implying
\begin{equation}
(J^+_{kl})^2 = 0.
\end{equation}
Concerning the remaining matrices, definition \eqref{eq:2.46} implies $J^+_{I} = \lambda_I$. Thus, by recalling \eqref{eq:2.30} one obtains
\begin{equation}
J^+_{1i_1 i_2 i_3 i_4} = \left(
\begin{array}{cc}
0 & \lambda_a(i_1, i_2, i_3, i_4)\delta^m_{ij} \\
0 & \delta^m_{i_1 i_2 i_3 i_4}
\end{array}
\right) \equiv \left(
\begin{array}{cc}
0 & \epsilon_{i_1 i_2 i_3 i_4 j} \delta^m_{ij} \\
0 & \delta^m_{i_1 i_2 i_3 i_4}
\end{array}
\right),
\end{equation}
again implying
\begin{equation}
(J^+_{1i_1 i_2 i_3 i_4})^2 = 0.
\end{equation}

Attention should be paid to the fact that, consistently with the very definition \eqref{eq:2.52} of coset representative $C$ [2]; also cfr. Eq. (3.1) of [28], in all expressions \eqref{eq:2.54}, \eqref{eq:2.55}, \eqref{eq:2.56} and \eqref{eq:2.58}, $i j$ are $SU(8)$-indices, whereas $mn$ are $E_7(7)$-indices. However, note that, within the explicit coset construction à la Iwasawa performed above, the indices $i j$ actually belong to a representation of $SL(8, \mathbb{R})$, so they are not fully $SU(8)$-covariant, but rather covariant only under $SO(8) = SU(8) \cap SL(8, \mathbb{R})$. This covariance is manifest by construction. The physical implications within the theory of extremal black hole attractors in $\mathcal{N} = 8, d = 4$ supergravity will be discussed in the next Sections.
Furthermore, it should be remarked that the difference among $E_{7(7)}$-covariant and $SU(8)$-covariant indices can be removed by suitably performing an $SU(8)$-gauge-fixing and then retaining only manifest invariance with respect to the rigid diagonal subgroup of $E_{7(7)} \times SU(8)$, without distinction among the two types of indices (see e.g. [2], as well as Sect. 3 of [28]; see also [49]).

Let us also point out that all matrices constructed so far are infinitesimally symplectic, i.e. they belong to $\mathfrak{sp}(56, \mathbb{R})$. In fact, their structure reads
\[
M = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix},
\]
where $B$ and $C$ are $28 \times 28$ symmetric matrices, and thus they do satisfy the infinitesimal symplectic condition:
\[
M^T \Omega + \Omega M = 0,
\]
where $\Omega$ is the symplectic metric ($I$ denoting the $28 \times 28$ identity):
\[
\Omega \equiv \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.
\]

In particular, the embedding of $\mathfrak{su}(8)$ into $\mathfrak{sp}(56, \mathbb{R})$ symplectic algebra is provided by the $56 \times 56$ matrices of the form
\[
U = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix},
\]
where $X$ is a linear combination of the $28$ antisymmetric $28 \times 28$ matrices $A_{kl}$’s (defined by (2.10), and then extended by (2.16) and (2.19)-(2.21)), while $Y$ is a linear combination of the $28$ symmetric $28 \times 28$ matrices $S_{kl}$ (defined by (2.11), and then extended by (2.17) and (2.19)-(2.21)) and of the $7$ diagonal traceless matrices $D_\alpha$ (defined by (2.12), and then extended by (2.18) and (2.24)).

### 3 $N = 8, d = 4$ Supergravity à la Iwasawa

Consistent with the notation of [28], one can rewrite the coset\(^\text{10}\) (2.53) as:
\[
C(x^\alpha, x^{ij}, x^I) = \exp \left( \sum_{\alpha=1}^{7} x^\alpha h_{D_\alpha} \right) \prod_{i<j} \exp \left( x^{ij} J_{ij}^+ \right) \prod_{I \in I_8} \exp \left( x^I J_I^+ \right)
\]
\[
= \frac{1}{\sqrt{2}} \begin{pmatrix} (W_1)^{mn}_{ij} & (V_1)^{ij[mn]} \\ (V_2)^{ij[mn]} & (W_2)^{ij}_{mn} \end{pmatrix},
\]
where $i, j = 1, ..., 8$ are in the fundamental irrepr. $8$ of $SL(8, \mathbb{R})$, and all the indices are antisymmetrized, i.e. $ij = [ij], mn = [mn]$ is understood throughout.

As pointed out in Eq. (2.1) of [28], the block-writing (3.1) of the $E_7(7) \times SU(8)$-coset representative $C$ corresponds to the following branching of the (fundamental irrepr. $56$ of the) $N = 8, d = 4$ $U$-duality group:
\[
E_7(7) \supset_{\text{symm}} \times SL(8, \mathbb{R})
\]
\[
56 \rightarrow 28 + 28',
\]
\(^\text{10}\)An equivalent Iwasawa parametrization might be
\[
C(x^\alpha, x^{kl}, x^f) = \exp \left( \sum_{\alpha=1}^{7} x^\alpha h_{D_\alpha} \right) \exp \left( \sum_{k<l=1}^{8} x^{kl} J_{kl}^+ \right) \exp \left( \sum_{I \in I_8} x^I J_I^+ \right).
\]
However, for computational purposes, it is more convenient to choose the product of the exponentials of the generators, rather than the exponential of their linear combination.
where the prime denotes the contragradient irrepr(s.) throughout. Thus, in the symplectic basis under consideration, the maximal symmetry of the Lagrangian density of $\mathcal{N} = 8$, $d = 4$ supergravity is $SL(8,\mathbb{R})$, whose maximal compact subgroup (mcs) - with symmetric embedding - reads:

$$SO(8) = \text{mcs} (SL(8,\mathbb{R})) = SU(8) \cap SL(8,\mathbb{R}),$$  \hspace{1cm} (3.3)

as also noticed in [28] (see Sects. 2 and 3 therein), and mentioned above. The $SU(8)$ symmetry is recovered only on-shell, and it is clearly the maximal compact (local) symmetry of the non-linear sigma model of scalars (whose representative is $C$). This is the very same situation as in the de Wit-Nicolai’s framework (see [2], and also [28] and [29] for recent treatments).

It is possible to switch to a complex manifestly $SU(8)$-covariant basis through a Cayley transformation\(^{11}\). Such a rotation can be described by the unitary matrix:

$$R \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} I & iI \\ iI & I \end{pmatrix} \leftrightarrow R^{-1} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} I & -iI \\ -iI & I \end{pmatrix} = \bar{R}. \hspace{1cm} (3.4)$$

Therefore, the manifestly $SU(8)$-covariant $E_{7(7)}/SU(8)$-coset representative is (cfr. Eq. (3.1) of [28])

$$\mathcal{V} \equiv R \, C \, (x^\alpha, x^{ij}, x^i) \, R^{-1} =$$

$$= \frac{1}{2\sqrt{2}} \begin{pmatrix} (W_1 + W_2 + i(V_2 - V_1))_{ij \, mn} & (V_1 + V_2 - i(W_1 - W_2))_{ij \, mn} \\ (V_1 + V_2 + i(W_1 - W_2))_{ij \, mn} & (W_1 + W_2 + i(V_1 - V_2))_{ij \, mn} \end{pmatrix} =$$

$$\equiv \begin{pmatrix} u_{ij \, mn} & v_{ij \, mn} \\ v_{ij \, mn} & u_{ij \, mn} \end{pmatrix}. \hspace{1cm} (3.5)$$

Now, by following the same procedure as in Eq. (3.2) of [28] and exploiting the symplectic formalism for extended supergravities recently reviewed in [50] (see therein for further Refs.), the electric and magnetic symplectic sections of $\mathcal{N} = 8$, $d = 4$ supergravity can be defined respectively as:

$$f_{ij \, mn} \equiv \frac{1}{\sqrt{2}} (u + v)_{ij \, mn} = \frac{1}{4} [(W_1 + W_2 + V_1 + V_2) + i(-W_1 + W_2 - V_1 + V_2)]_{ij \, mn}; \hspace{1cm} (3.6)$$

$$h_{ij \, mn} \equiv -\frac{i}{\sqrt{2}} (u - v)_{ij \, mn} = \frac{1}{4} [(W_1 - W_2 - V_1 + V_2) + i(W_1 - W_2 + V_1 + V_2)]_{ij \, mn}. \hspace{1cm} (3.7)$$

Then, it can easily be checked by direct computation that the usual relations among symplectic sections hold, namely (e.g. cfr. Eqs. (3.10) of [28]):

$$i \left( f^\dagger \mathbf{h} - \mathbf{h}^\dagger f \right) = \mathbf{I}; \hspace{1cm} (3.8)$$

$$\mathbf{h}^T f - f^T \mathbf{h} = 0. \hspace{1cm} (3.9)$$

\(^{11}\)Notice that usually a Cayley rotation is represented by a matrix of the form

$$\tilde{R} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} I & iI \\ iI & -i \end{pmatrix}.$$  

However, the map (3.4) that we use is a transformation from the Siegel upper half-plane to the unit disk, as well. Thus, for our purposes (3.4) is equivalent to a standard Cayley rotation, and it yields the consistent expressions (3.6)-(3.7) for the symplectic sections $f_{ij \, mn}$ and $h_{ij \, mn}$.
which also directly follows from the finite symplecticity \((Sp(56, \mathbb{R}))\) condition satisfied by \(C\) itself (recall definition (2.62)):

\[
C^T \Omega C = \Omega. \tag{3.10}
\]

Consequently, by recalling the definition of the \(8 \times 8\) antisymmetric central charge matrix \(Z_{ij}\) (see e.g. Eq. (3.14) of \([28]\), as well as the treatment of \([23]\) and \([50]\)), one can compute:

\[
Z_{ij} \left( x^\alpha, x^{kl}, x^l; q_{mn}, p^{mn} \right) = f_{ij}^{\ mn} q_{mn} - h_{ijmn} p^{mn} = \\
\frac{1}{4} \left[ (W_1 + W_2 + V_1 + V_2) + i (-W_1 + W_2 - V_1 + V_2) \right]_{ij}^{\ mn} \left( x^\alpha, x^{kl}, x^l \right) q_{mn} + \\
\frac{1}{4} \left[ (W_1 - W_2 - V_1 + V_2) + i (-W_1 - W_2 + V_1 + V_2) \right]_{ijmn} \left( x^\alpha, x^{kl}, x^l \right) p^{mn}. \tag{3.11}
\]

It follows that the positive definite effective black hole potential \([14]\) can be written as follows (\([23, 50]\); recall definition \([1, 5]\), and cfr. Eq. (3.17) of \([28]\), as well):

\[
V_{BH} = \frac{1}{2} Tr \left( ZZ^t \right) = \frac{1}{2} Z_{ij} Z^{ij} = \\
\frac{1}{25} [(W_1 + W_2 + V_1 + V_2) + i (-W_1 + W_2 - V_1 + V_2)]_{ij}^{\ mn} \cdot \\
\left[ (W_1 + W_2 + V_1 + V_2) - i (-W_1 + W_2 - V_1 + V_2) \right]_{ij}^{\ rs} q_{mn} q_{rs} + \\
\frac{1}{25} [(W_1 - W_2 - V_1 + V_2) + i (-W_1 - W_2 + V_1 + V_2)]_{ij}^{\ mn} \cdot \\
\left[ (W_1 - W_2 - V_1 + V_2) - i (-W_1 - W_2 + V_1 + V_2) \right]_{ij}^{\ rs} q_{mn} p^{rs} + \\
\frac{1}{25} [(W_1 - W_2 - V_1 + V_2) + i (-W_1 + W_2 + V_1 + V_2)]_{ij}^{\ rs} q_{mn} p^{rs} + \\
\frac{1}{25} [(W_1 + W_2 - V_1 + V_2) - i (-W_1 - W_2 + V_1 + V_2)]_{ij}^{\ mn} \cdot \\
\left[ (W_1 - W_2 - V_1 + V_2) - i (-W_1 - W_2 + V_1 + V_2) \right]_{ij}^{\ rs} p^{mn} p^{rs}. \tag{3.12}
\]

### 3.1 Scalar Covariance

Before proceeding further with the computations, a comment on the manifest covariance properties of the 70 real scalars \(\{x^\alpha, x^{kl}, x^l\}\) in our parametrization \(\text{à la Iwasawa}\) given by \(C\) \([3.1]\) and \(V\) \([3.5]\) is needed.

It is worth recalling that in the de Wit-Nicolai parametrization (\([2]\); see also Sects. 2 and 3 of \([28]\), and Refs. therein), the 70 real scalars \(\phi_{ijkl} = \phi_{ijkl}\) coordinatizing \(E_{(7)}^{(15)}\) sit in the rank-4 antisymmetric irrepr. \(70\) of \(SU(8)\), which is constrained by a self-reality condition (see e.g. Eq. (3.4) of \([23]\)):

\[
\phi_{ijkl} = \frac{1}{4!} e^{ijklmnpq} \phi_{mnpq}. \tag{3.13}
\]

On the other hand, in the explicit construction and subsequent Iwasawa parametrization performed above, the scalars have different types of indices, and thus different covariance properties, namely:

\[
\left\{ x^\alpha, x^{kl}, x^l \right\}, \tag{3.14}
\]
where:

- $I \in I_8$ is in the rank-3 antisymmetric irreducible repr. $35$ of $SL(7, \mathbb{R})$;
- $kl = [kl]$ is in the rank-2 antisymmetric (contra-gradient) $28'$ irrep. of $SL(8, \mathbb{R})$;
- $\alpha$ is in the fundamental irrep. $7$ of $SL(7, \mathbb{R})$.

Thus, the maximal common covariance of the scalars (3.14) is $SL(7, \mathbb{R})$, yielding to the following split of indices $kl$:

$$SL(8, \mathbb{R}) \rightarrow SL(7, \mathbb{R}) \times SO(1, 1)$$

$$28' \rightarrow 21'_1 + 7'_3,$$

where $21$ is the rank-2 antisymmetric (contra-gradient) irrep. of $SL(7, \mathbb{R})$, and the subscripts denote the weights with respect to $SO(1, 1)$.

### 4 The Origin of $E_{7(7)}/SU(8)$ as $\frac{1}{8}$-BPS Attractor

From Eq. (3.15) of [28], it follows that at the $\frac{1}{8}$-BPS attractor solution (recall $i = 1, ..., 8$ throughout)

$$\phi_{ijkl} = 0$$

the $N = 8, d = 4$ central charge matrix $Z_{ij}$ simply reads (cfr. Eq. (3.16) of [28])

$$Z_{ij} |_{\phi=0} = \frac{1}{\sqrt{2}} Q_{ij} \equiv \frac{1}{2} (q_{ij} + i p^{ij})$$

The origin (4.1) of the the scalar manifold $E_{7(7)}/SU(8)$, as a $\frac{1}{8}$-BPS attractor solution, is supported by the skew-diagonal charge configuration [20, 23, 28]

$$Q_{ij} = \left( \begin{array}{cc} 0_{1 \times 3} & 0_{1 \times 3} \\ 0_{3 \times 1} & Q_{03} \end{array} \right) \otimes \epsilon,$$

where

$$Q \equiv Q_{12} \equiv \frac{1}{\sqrt{2}} \left( q_{12} + i p^{12} \right) \equiv \frac{1}{\sqrt{2}} (q + ip) \in \mathbb{C},$$

and $\epsilon$ is the $2 \times 2$ symplectic metric:

$$\epsilon \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus, Eq. (4.2) implies that at the $\frac{1}{8}$-BPS attractor solution (4.1) supported by the charge configuration (4.3)-(4.4), $Z_{ij}$ reads as follows:

$$Z_{ij,\frac{1}{8}-BPS,\text{large}} = \begin{pmatrix} \frac{1}{\sqrt{2}} (q + ip) & 0 \\ 0 & 0 \end{pmatrix} \otimes \epsilon.$$

By comparing Eq. (4.6) with the usual parametrization of $Z_{ij,\frac{1}{8}-BPS,\text{large}}$ (see e.g. Eq. (2.20) of [52]), it follows that the absolute value and the phase of the unique non-vanishing skew-eigenvalue of $Z_{ij,\frac{1}{8}-BPS,\text{large}}$ reads

$$\rho_{BPS} = \frac{1}{\sqrt{2}} |q + ip| = \frac{\sqrt{q^2 + p^2}}{\sqrt{2}};$$

$$\psi = \frac{\sqrt{q^2 + p^2}}{q}.$$

$$\phi_{Q_{12}} = \psi_{Q_{12}} : \tan (\psi_{Q_{12}}) = \frac{p}{q}.$$
Concerning (4.8), the stabilization of the phase $\varphi$ in terms of the ratio of the charges $p \, q$ might seem to be inconsistent with the well known fact that the overall phase of $\mathcal{N} = 8$, $d = 4$ central charge matrix $Z_{ij}$ is undetermined at $\frac{1}{8}$-BPS attractors \cite{19, 20, 23}. But it should be recalled that when taking the Lie algebra limit, the scalar configuration (4.1) is picked as the starting point, which fixes the $\frac{1}{8}$-BPS attractor to be the origin of the scalar manifold $M_{\mathcal{N}=8,d=4} = \frac{E_{7(7)}}{SU(8)}$ (recall Eq. (1.1)). As it is well known, not all scalar fields are stabilized in terms of the electric and magnetic charges at the event horizon of $\frac{1}{8}$-BPS attractors, but rather a certain subset of them spans a related moduli space (see also Sect. 5). According to \cite{53, 54, 28, 35}, the moduli space of $\frac{1}{8}$-BPS attractors $M_{\mathcal{N}=8,d=4}$ is a symmetric quaternionic space of real dimension 40, given by Eqs. (5.5)-(5.6) below. In the maximal symmetric embedding \cite{53, 55, 28}

$$SU(8) \supseteq SU(6) \times SU(2) \times U(1) \tag{4.9}$$

$$70 \rightarrow (15, 1)_{-2} + (\overline{15}, 1)_{2} + (20, 2)_{0},$$

the 40 real scalar degrees of freedom pertaining to $M_{\mathcal{N}=8,d=4}$ fit into the $(20, 2)_{0}$ irrepr. of $SU(6) \times SU(2) \times U(1)$, where $15$, $\overline{15}$ and $20$ respectively are the rank-2 antisymmetric, complex conjugate rank-2 antisymmetric, and rank-3 antisymmetric irreprs. of $SU(6)$, and the subscripts denote the $U(1)$ charges. Thus, by specifying (4.1), both the 30 stabilized scalar degrees of freedom (coordinatizing the submanifold $M_{\mathcal{N}=8,d=4}$) are set to zero. Such a “polarization” of the moduli space $M_{\mathcal{N}=8,d=4}$ implies the phase $\varphi$ to be actually determined in terms of relevant charges, as given by Eq. (4.8).

Furthermore, the maximal compact symmetry exhibited by $Z_{ij,\frac{1}{8}}$-BPS, large given by Eq. (4.6) (or equivalently, through Eq. (4.2), by $Q_{ij}$ given by Eqs. (4.3) and (4.4)) is

$$SU(6) \times SU(2) = mcs \left( E_{6(2)} \right), \tag{4.10}$$

(see e.g. \cite{44}). In other words, $Q_{ij}$ given by Eqs. (4.3) and (4.4) is a representative of the $\frac{1}{8}$-BPS “large” (i.e. attractive) orbit of the 56 fundamental representation space of $E_{7(7)}$, exhibiting the maximal (compact) symmetry (4.10) \cite{20, 21}:

$$Q_{ij} \in O_{\frac{1}{8}} \supseteq \frac{E_{7(-7)}}{E_{6(2)}}, \tag{4.11}$$

Notice that $Q$ and $\overline{Q}$ are charged with respect to the $U(1)$ in the extreme right-hand side of the following chain of maximal symmetric group embeddings (see e.g. \cite{44})

$$E_{7(7)} \overset{mcs}{\supseteq} SU(8) \supseteq SU(6) \times SU(2) \times U(1). \tag{4.12}$$

Indeed, $Q$ and $\overline{Q}$ respectively are the $(SU(6) \times SU(2))$-singlets $(1, 1)_{-3}$ and $(1, 1)_{+3}$ in the subsequent decomposition of the fundamental irrepr. 56 of $E_{7(7)}$ \cite{28}:

$$E_{7(7)} \rightarrow SU(8) \rightarrow SU(6) \times SU(2) \times U(1);$$

$$56 \rightarrow 28 + 28 \rightarrow (15, 1)_{+1} + (6, 2)_{-1} + (1, 1)_{-3} + (1, 1)_{+3},$$

where the subscripts denote the charge with respect to $U(1)$. Thus, by suitably renormalizing the $U(1)$-phase $\Phi$, $Q$ and $\overline{Q}$ transform under considered $U(1)$ respectively as follows:

$$U(1) : \begin{cases} Q \rightarrow Q e^{-i\Phi} ; \\ \overline{Q} \rightarrow \overline{Q} e^{i\Phi} , \end{cases} \tag{4.14}$$

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or, equivalently, in the 2 of $SO(2)$:

$$SO(2) : \left( \begin{array}{c} q \\ p \end{array} \right) \rightarrow \left( \begin{array}{cc} \cos \Phi & \sin \Phi \\ -\sin \Phi & \cos \Phi \end{array} \right) \left( \begin{array}{c} q \\ p \end{array} \right).$$

(4.15)

Thus, the $U(1)$-transformation properties of $\rho_{BPS}$ and $\psi_{Q_{12}}$, given by Eqs. (4.7) and (4.8), respectively are

$$\rho_{BPS} = \sqrt{Q_\alpha Q_\gamma} \rightarrow \rho_{BPS};$$

(4.16)

$$\psi_{Q_{12}} \rightarrow \arctan \left( \frac{p}{q} \right) - \Phi.$$  

(4.17)

As a consequence, $\rho_{BPS}$ is invariant under $SU(6) \times SU(2) \times U(1)$, whereas $\psi_{Q_{12}}$ only under $SU(6) \times SU(2)$. Notice that the fact that $\psi_{Q_{12}}$ is not invariant under the considered $U(1)$ (as given by Eq. 4.17) does not affect the attractor nature of the origin (4.1) of $E_7(7)$ as $SU(8)$-invariant (see Footnote 12). As mentioned in the Introduction (see also final Sect. 6), the $U(1)$ under consideration should be seen as a residual “degeneracy symmetry” at 1/8-BPS attractor solutions in the Lie algebra limit (namely, at the origin (4.1) of the scalar manifold $E_7(7)$).

4.1 Iwasawa Solutions

We are now going to analyze the consequences of the above reasoning for the explicit construction à la Adams-Iwasawa performed in Subsect. 2.1.

Thus, we start and study how the 1/8-BPS Attractor Eqs. can be implemented in the Iwasawa construction performed above, confining ourselves to the origin (4.1) of $E_7(7)/SU(8)$, as attractor solution. Therefore, we have to set all scalar fields to zero in the parametrization (3.1) of the coset representative $C$. Since the Attractor Eqs. are nothing but criticality conditions for the effective black hole potential $V_{BH}$, namely

$$\partial_\phi V_{BH} = 0 : V_{BH}|_{\partial_\phi V_{BH}=0} \neq 0,$$

(4.18)

this implies that one only needs to compute the terms of $V_{BH}$ which are linear in the scalar fields. To the first order in scalar fields, the coset representative $C$ (3.1) reads ($I_{56}$ denoting the $56 \times 56$ identity)

$$C(x^\alpha, x^{ij}, x^I) = I_{56} + \sum_{\alpha=1}^{7} x^\alpha h_{D_\alpha} + \sum_{i<j=2}^{8} x^{ij} J_+^{ij} + \sum_{I \in I_8} x^I J_+^{I} + \mathcal{O} \left( \{x^\alpha, x^{ij}, x^I\}^2 \right).$$

(4.19)

Thus, by neglecting $\mathcal{O} \left( \{x^\alpha, x^{ij}, x^I\}^2 \right)$, Eq. (3.1) yields that

$$(W_1)_{ij}^{mn} = I + \sum_{\alpha=1}^{7} x^\alpha (D^i_\alpha + D^j_\alpha) \delta_{ij}^{mn} + \sum_{k<l=2}^{8} x^{kl} (J_+^{kl})_{ij}^{mn};$$

(4.20)

$$(W_2)^{ij}_{mn} = I - \sum_{\alpha=1}^{7} x^\alpha (D^i_\alpha + D^j_\alpha) \delta_{ij}^{mn} - \sum_{k<l=2}^{8} x^{kl} (J_+^{kl})^T_{ij}^{mn};$$

(4.21)

$$(V_1)_{ij}^{mn} = \sum_{I \in I_8} x^I \epsilon_{Iijmn};$$

(4.22)

$$(V_2)^{ij}_{mn} = \sum_{I \in I_8} x^I \epsilon^I_{ij}^{mn}.$$  

(4.23)
By plugging results (4.20)-(4.23) into Eq. (3.12) and evaluating the resulting Attractor Eqs. (4.18), the following system is obtained:

\[
\begin{align*}
(h_{D_\alpha})^{mn}_{rs}q_{mn}p^{rs} &= 0; \\
(J^+_{kl} + J^+_{kl}^T)^{mn}_{rs}q_{mn}p^{rs} &= 0; \\
(\epsilon + \delta)^{mnrs}q_{mn}q_{rs} - (\epsilon + \delta)^{lmnrsp}_{mn}p^{rs} &= 0.
\end{align*}
\]

(4.24)

By introducing complex charges \(Q^{ij}\) (recall Eqs. (4.3), (4.4)) such that

\[
q_{ij} = \frac{1}{\sqrt{2}}(Q^{ij} + \bar{Q}^{ij}); \quad p_{ij} = \frac{1}{i\sqrt{2}}(Q^{ij} - \bar{Q}^{ij}),
\]

(4.25)

the system (4.24) can be recast in the following form:

\[
\begin{align*}
I. \quad (h_{D_\alpha})^{mn|rs}(Q^{mn}Q^{rs} - \bar{Q}^{mn}\bar{Q}^{rs}) &= 0; \\
II. \quad (J^+_{kl} + J^+_{kl}^T)^{mn|rs}(Q^{mn}Q^{rs} - \bar{Q}^{mn}\bar{Q}^{rs}) &= 0; \\
III. \quad (\epsilon + \delta)_{lmnrsp}(Q^{mn}Q^{rs} + \bar{Q}^{mn}\bar{Q}^{rs}) &= 0.
\end{align*}
\]

(4.26)

where the symmetry of the matrices involved was used. The system (4.26) is nothing but the set of (necessarily \(1/8\)-BPS) Attractor Eqs. consistent with the origin (4.1) of \(E_7(7)\) \(SU(8)\). Thus, Eqs. (4.26) express conditions on the complex dyonic charges \(Q^{ij}\) such that

\[
\phi_{ijkl} = 0 \Leftrightarrow \begin{cases} 
  x^\alpha = 0; \\
  x^{ij} = 0; \\
  x^I = 0 
\end{cases}
\]

(4.27)

is a (necessarily \(1/8\)-BPS) attractor scalar configuration in the background of a static, spherically symmetric, asymptotically flat extremal black hole of \(N = 8, d = 4\) (ungauged) supergravity.

Consistent with Eq. (4.3), we look for solutions of (4.26) within the following structural Ansatz:

\[
Q = e^{i\varphi/4}\text{diag}(r,0,0,0) \otimes \epsilon,
\]

(4.28)

with \(r \in \mathbb{R}_0^+\) and \(\varphi \in [0,8\pi]\). By recalling Eq. (4.11), \(Q\) is a singlet of \(SU(2) \times SU(6)\), stabilizer of the \(1/8\)-BPS “large” orbit \(O_{1/8-BPS,large}^{20,21}\). By inserting (4.28) into the system (4.26), Eqs. II and III are automatically satisfied, whereas Eqs. I yield the conditions

\[
(D^1_\alpha + D^2_\alpha)\sin\frac{\varphi}{2} = 0, \quad \forall \alpha = 1,\ldots,7.
\]

(4.29)

Note that \(D^1_\alpha + D^2_\alpha\) cannot vanish for all \(\alpha\)’s, because \(D_\alpha\)’s defined by (2.12) are a basis for the 8 \(\times\) 8 diagonal traceless matrices with real entries: if \(D^1_\alpha + D^2_\alpha = 0 \quad \forall \alpha = 1,\ldots,7\), then the \(D_\alpha\)’s would not generate, e.g., the diagonal traceless matrix \(h = \text{diag}\{1,1,-2,0,0,0,0,0\}\).

Thus, within the Ansatz (4.28) exhibiting the maximal (compact) symmetry \(SU(2) \times SU(6)\), the solution of conditions (4.29), and thus of Attractor Eqs. (4.26), reads

\[
\begin{cases}
  r \in \mathbb{R}_0; \\
  \varphi = 2n\pi, \quad n \in \mathbb{Z}.
\end{cases}
\]

(4.30)

Consequently, two kinds of solutions are obtained:
1. the purely electric solution:
\[
\begin{align*}
  r &= q; \\
  \varphi &= 4k\pi, \ k \in \mathbb{Z}.
\end{align*}
\] (4.31)

2. the purely magnetic solution:
\[
\begin{align*}
  r &= p; \\
  \varphi &= 2(2k + 1)\pi, \ k \in \mathbb{Z}.
\end{align*}
\] (4.32)

4.2 Analysis of Solutions and Breaking $U(1) \rightarrow \mathbb{Z}_4$

In order to analyze the obtained solutions (4.31) and (4.32), which support the origin (4.27) as an $\frac{1}{8}$-BPS attractor, let us consider the following charge configuration:
\[
\tilde{Z}_{ij, \frac{1}{8}}^{BPS, large} = \left( \begin{array}{ccc}
  \frac{1}{\sqrt{2}}q & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0 \\
\end{array} \right) \otimes \epsilon. \quad (4.33)
\]

By recalling Eqs. (4.6), (4.7), (4.8), this is an $\frac{1}{8}$-BPS attractor solution at the origin (4.1) of $E_7(7)$ $SU(8)$, with
\[
\tilde{\rho}_{BPS} = \frac{|q|}{\sqrt{2}}; \quad (4.34)
\]
\[
\frac{\tilde{\psi}}{4} = \tilde{\psi}_{Q_{12}} = \left\{ \begin{array}{l}
  0, \ q > 0; \\
  \pi, \ q < 0.
\end{array} \right. \quad (4.35)
\]

Thus, it is of purely electric nature, namely of the kind (4.31) obtained above.

It should be stressed that (4.33) does not exhibit the symmetry enhancement
\[
SU(6) \times SU(2) \rightarrow SU(6) \times SU(2) \times U(1), \quad (4.36)
\]
despite the fact that, consistent with Eqs. (4.3)-(4.4), it is supported by
\[
\tilde{Q}_{ij} = \left( \begin{array}{ccc}
  \tilde{Q} & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0 \\
\end{array} \right) \otimes \epsilon; \quad (4.37)
\]
\[
\tilde{Q} \equiv \frac{1}{\sqrt{2}}q \in \mathbb{R}. \quad (4.38)
\]

Indeed, under the commuting $U(1)$ in the r.h.s. of (4.36), the explicit expression of $\tilde{\rho}_{BPS}$ changes as follows:
\[
\tilde{\rho}_{BPS} \rightarrow \frac{|q\cos \Phi - iq\sin \Phi|}{\sqrt{2}}, \quad (4.39)
\]
although its value remains the same because
\[
|q\cos \Phi - iq\sin \Phi| = |q|. \quad (4.40)
\]

Eq. (4.39) is a trivial consequence of the transformation described by Eq. (4.15):
\[
SO(2) : \left( \begin{array}{c}
  q \\
  0
\end{array} \right) \rightarrow \left( \begin{array}{cc}
  \cos \Phi & \sin \Phi \\
  -\sin \Phi & \cos \Phi
\end{array} \right) \left( \begin{array}{c}
  q \\
  0
\end{array} \right) = \left( \begin{array}{c}
  q \cos \Phi \\
  -q\sin \Phi
\end{array} \right) \equiv \left( \begin{array}{c}
  q' \\
  p'
\end{array} \right). \quad (4.41)
\]
On the other hand, $\tilde{\psi}_{Q_{12}}$ transforms as follows (recall Eq. (4.17)):

$$\tilde{\psi}_{Q_{12}} \rightarrow \tilde{\psi}'_{Q_{12}} \equiv \text{arctan} \left( \frac{p'}{q'} \right) = -\Phi \ (\pm k\pi, \ k \in \mathbb{Z}).$$ \hspace{1cm} (4.42)

Through this reasoning, we now address the following question: why are the Iwasawa $\frac{1}{8}$-BPS attractor solutions (4.31) and (4.32) respectively purely electric and purely magnetic, and thus non-dyonic?

As we will see below, the non-dyonicity of Iwasawa solutions (4.31) and (4.32) is intrinsic to the Adams-Iwasawa construction approach to the $E_{7(7)}/SU(8)$ coset representative, exploited in Sect. 2. Indeed, such a construction explicitly breaks the residual $U(1)$-symmetry (parametrizing the dyonic nature of $\frac{1}{8}$-BPS attractor solutions at the coset origin (4.27); recall Eqs. (4.14)-(4.17)) into a discrete subgroup $Z_4$. Thus, as an intrinsic feature of the Iwasawa parametrization we performed, the residual “degeneracy symmetry” $U(1)$ of the charge orbit $O_{\frac{1}{8}\text{-BPS}}$ large (4.11) supporting the origin (4.27) of the coset $E_{7(7)}/SU(8)$ as a $\frac{1}{8}$-BPS attractor gets broken, and discretized, to a finite subgroup $Z_4$.

In order to show this, let us consider Eqs. (4.14)-(4.17). These Eqs. express the action of the $U(1)_A$ factor in the second group embedding of (4.13):

$$SU(2) \times SU(6) \times U(1)_A \subset_{\text{symm}} SU(8).$$ \hspace{1cm} (4.43)

which is maximal (“max”) and symmetric (“symm”). We denote such an extra commuting $U(1)$-factor with a subscript “$A$”, in order to discriminate it from other $U(1)$’s we will consider below. (4.43) means that the maximal compact symmetry of $O_{\frac{1}{8}\text{-BPS}}$ large (4.11) embeds into the maximal compact (local) symmetry of the non-linear sigma model of scalars (and maximal compact on-shell symmetry of the whole $\mathcal{N} = 8$, $d = 4$ ungauged Lagrangian density, as well; see the end of Subsect. 2.2) through an extra commuting $U(1)_A$. Such a $U(1)_A$ is a residual “degeneracy symmetry”, in the sense that it spans all possible charge configurations supporting $\frac{1}{8}$-BPS attractor points in $E_{7(7)}/SU(8)$ (these latter are exemplified by the manifestly $SU(8)$-symmetric\textsuperscript{12} point of the scalar manifold (1.1), namely by the origin (4.27)).

However, as discussed at the start of Sect. 3, as well as in Subsect. 3.1 (recall Eq. (3.15)), the performed Adams-Iwasawa construction of the $E_{7(7)}/SU(8)$ coset representative $\mathcal{C}$ (3.1) explicitly breaks the maximal covariance from $SL(8, \mathbb{R})$ down to $SL(7, \mathbb{R})$. Through the Cayley-like transformation (3.4), yielding to the manifestly $SU(8)$-covariant $E_{7(7)}/SU(8)$ coset representative $\mathcal{V}$ (3.5), the breaking (3.15) becomes

$$SU(7) \times U(1)_{\varepsilon} \subset_{\text{symm}} SU(8).$$ \hspace{1cm} (4.44)

The $\frac{1}{8}$-BPS “large” charge orbit $O_{\frac{1}{8}\text{-BPS}}$ large (4.11) gives rise to a further decomposition into $SU(6)$, and thus the embedding (3.15) can be completed as follows:

$$SU(6) \times U(1)_{\varepsilon} \subset_{\text{symm}} SU(7) \times U(1)_{\varepsilon} \subset_{\text{symm}} SU(8).$$ \hspace{1cm} (4.45)

The symmetry breaking (3.15), intrinsic to the performed Iwasawa parametrization (exploited by singling the Cartan subalgebra $\mathcal{C}$ (2.37) out), is crucial. Indeed, by recalling Eqs. (4.3)-(4.4), the embeddings (4.43) and (4.45) necessarily lead to conclude that (if any) the residual symmetry group

\textsuperscript{12}More precisely, the origin $\phi_{ijkl} = 0$ (or equivalently (4.27)) is the \textit{unique} $SU(8)$-symmetric point of $E_{7(7)}/SU(8)$, such that its symmetry $SU(8)$ coincides with the stabilizer of the coset itself. In other words, consistent with the geometric construction of the non-compact, irreducible, Riemannian, globally symmetric coset $E_{7(7)}/SU(8)$, the action of the stabilizer $SU(8)$ has a \textit{unique} fixed point, \textit{i.e.} the origin $\phi_{ijkl} = 0$ of the coset itself.
Γ of $\frac{1}{8}$-BPS attractors in the Adams-Iwasawa construction performed above must be given by the intersection:

$$\Gamma \equiv U(1)_A \cap U(1)_E.$$  \hfill (4.46)

We now show that $\Gamma$ actually is a $\mathbb{Z}_4$ discrete finite group.

In order to do this, it should be remarked that the original $SL(8, \mathbb{R})$ group on which Adams’ construction is based (see Subsect. 2.1), is broken, by the considered Iwasawa parametrization (see Subsect. 2.2), down to $SL(7, \mathbb{R}) \times SO(1, 1)$ (as given by (3.15)), where the generator of $SO(1, 1)$ can be written as $(I_2, I_6$ and $I_7$ respectively denoting the $2 \times 2$, $6 \times 6$ and $7 \times 7$ identity)

$$\tau \equiv \begin{pmatrix} I_7 & 0 \\ 0 & -7 \end{pmatrix}. \hfill (4.47)$$

After a suitable Cayley transformation $^{13}R$, it takes the form:

$$\tilde{\tau} \equiv R\tau R^{-1} = \begin{pmatrix} \tau E & 0 \\ 0 & I_6 \end{pmatrix}; \hfill (4.48)$$

$$R \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -i & 0 \\ -1 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 1 & 0 \\ i & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}; \hfill (4.49)$$

$$\tau E \equiv \begin{pmatrix} -3 & 4i \\ -4i & -3 \end{pmatrix}. \hfill (4.50)$$

On the other hand, the decomposition of the fundamental irrepr. 8 of $SU(8)$ under (4.43) reads (subscripts denote charges with respect to $U(1)_A$):

$$SU(8) \rightarrow SU(2) \times SU(6) \times U(1)_A$$

$$8 \rightarrow (1, 6)_1 + (2, 1)_{-3}, \hfill (4.51)$$

so that the generator of $U(1)_A$ can be written as

$$\tau_A \equiv \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}. \hfill (4.52)$$

Thus, the group $\Gamma$ defined by (4.46) is explicitly determined by the solution of the following $2 \times 2$ matrix Eq.:

$$\exp(is\tau_A) = \exp(is\tau_E). \hfill (4.53)$$

By using definitions (4.50) and (4.52), Eq. (4.53) can be explicit as follows:

$$I_2 = \exp(-4s\sigma_2) = \cos(4s)I_{2\times2} - i \sin(4s)\sigma_2, \hfill (4.54)$$

$^{13}$To be more precise, $R \equiv R_{1-4}R_{2-8}R_0$, where $R_{i-j}$ is the rotation of $\pi/2$ of the $i - j$ plane and $R_0$ is the $8 \times 8$ Cayley transformation (also recall Footnote 11)

$$R_0 \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} I_4 & -iI_4 \\ -iI_4 & I_4 \end{pmatrix}.$$
where $\sigma_2$ is the second Pauli matrix
\[
\sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\] (4.55)

The solution of Eq. (4.54) reads
\[
s = \frac{\kappa \pi}{2}, \ \kappa \in \mathbb{Z},
\] (4.56)

implying that
\[
\Gamma = \langle \exp(-3i\kappa \frac{\pi}{2} I_2) \rangle_{\kappa \in \mathbb{Z}} \equiv \mathbb{Z}_4,
\] (4.57)
as claimed.

The residual symmetry $\Gamma$ (4.57) of $\frac{1}{8}$-BPS attractors within Adams-Iwasawa approach to $\mathcal{N} = 8$, $d = 4$ ungauged supergravity performed above is consistent with the fact that the charge configurations found to support the manifestly $SU(8)$-invariant representative point (4.27) as a $\frac{1}{8}$-BPS attractor are not dyonic, but rather purely electric (4.31) or purely magnetic (4.32).

This has the following physical interpretation: the Iwasawa parametrization (2.53) is not as symmetric as the Cartan construction [2], but it nevertheless remarkably provides the purely electric / purely magnetic disentangling of generically dyonic $\frac{1}{8}$-BPS "large" $d = 4$ extremal black holes. In this sense, by recalling Eqs. (4.3)-(4.4) and solutions (4.31)-(4.32), the performed intersection (4.46) can be interpreted as a constraint selecting the real (or, equivalently, imaginary) part of the $\frac{1}{8}$-BPS "large" charge orbit $O^{1/8}_{\text{BPS,large}}$ (4.11).

5 Comments on $d = 4 / d = 5$ Relations

The present Section is devoted to consider some relations between maximal supergravities in $d = 4$ and $d = 5$, at the level of scalar manifolds ($\mathcal{M}$), "large" orbits ($\mathcal{O}$) and moduli spaces of attractors ($\mathcal{M}$).

While many relations have been derived in [54, 51, 35], and detailed treatments of related issues are given in [20, 35], we believe that the relations to $d = 3$ theories given below can hint some further interesting developments, especially in light of very recent and intriguing advances (see e.g. [43]).

- The scalar manifold and the unique "large" charge orbit (with related moduli space) of $\mathcal{N} = 8$, 14

\footnote{However, it should be pointed out that the non-dyonic (namely, purely electric or purely magnetic) $\frac{1}{8}$-BPS attractor solutions with $\phi_{ijkl} = 0$ obtained in $d = 4$ (see Eqs. (4.31) and (4.32), respectively) do not necessarily uplift to $d = 5$ extremal (electric) black holes or extremal (magnetic) black strings, respectively. Indeed, the Iwasawa symplectic basis considered in the present paper does not coincide with the Sezgin-Van Nieuwenhuizen [31] symplectic frame, in which the maximal non-compact covariance is nothing but the $d = 5 U$-duality group $E_{6(6)}$.

Clearly, these two symplectic bases are related through a finite symplectic transformation ($\in Sp(56, \mathbb{R})$, in the semi-classical regime of large, continuous charges). We leave the determination of such a transformation, along with the study of the $d = 5$ uplifts of the obtained Iwasawa non-dyonic $d = 4 \frac{1}{8}$-BPS "large" solutions, as an issue for future investigations.}
\[ d = 5 \] ungauged supergravity respectively read

\[
M_{\mathcal{N}=8,d=5} = \frac{E_{6(6)}}{USp(8)}; \quad (5.1)
\]

\[
\mathcal{O}_{\frac{8}{5}} - \text{BPS} = \frac{E_{6(6)}}{F_{4(4)}}; \quad (5.2)
\]

\[
\mathcal{M}_{\frac{8}{5}} - \text{BPS} = \frac{F_{4(4)}}{USp(6) \times USp(2)} = M_{\mathcal{N}=4,J_{3}^{R\mathbb{C}},d=3} = \frac{c \left( Sp(6,\mathbb{R}) \right)}{SU(3) \times U(1)}; \quad (5.3)
\]

\[
\text{with } \text{“}c\text{” denotes the } c\text{-map } \left[42\right].
\]

• The scalar manifold and the “large” charge orbits of \( \mathcal{N} = 8, d = 4 \) ungauged supergravity are respectively given by Eqs. (1.1), (4.11) and (6.1). The related moduli spaces of attractors respectively read

\[
\mathcal{M}_{\frac{8}{5}} - \text{BPS,large} = \frac{E_{6(2)}}{SU(6) \times SU(2)} = M_{\mathcal{N}=4,J_{3}^{C},d=3} = \frac{c \left( SU(3,3) \right)}{SU(3) \times SU(3) \times U(1)}; \quad (5.6)
\]

\[
\mathcal{M}_{\text{nBPS}} = \frac{E_{6(6)}}{USp(8)} = M_{\mathcal{N}=8,d=5}. \quad (5.7)
\]

Both \( \frac{1}{8} \)-BPS and non-BPS \( d = 4 \) extremal black hole attractors are descendants of the \( \frac{1}{8} \)-BPS extremal \( d = 5 \) (black hole or black string) attractors. Thus, it holds that

\[
\mathcal{M}_{\frac{1}{8} - \text{BPS},d=5} \subseteq \left( \mathcal{M}_{\text{nBPS},d=4} \cap \mathcal{M}_{\frac{1}{8} - \text{BPS,large},d=4} \right); \quad (5.8)
\]

\[
M_{\mathcal{N}=4,J_{3}^{R\mathbb{C}},d=3} \subseteq \left( M_{\mathcal{N}=8,d=5} \cap M_{\mathcal{N}=4,J_{3}^{C},d=3} \right); \quad (5.9)
\]

\[
\frac{F_{4(4)}}{USp(6) \times USp(2)} \subseteq \left( \frac{E_{6(6)}}{USp(8)} \cap \frac{E_{6(2)}}{SU(6) \times SU(2)} \right). \quad (5.10)
\]

While (5.8) (or explicitly (5.10)) has been obtained in [35], its re-interpretation (5.9) is new. It involves the quaternionic scalar manifolds of \( \mathcal{N} = 4, d = 3 \) “magic” supergravity theories based on the rank-3 Euclidean Jordan algebras \( J_{3}^{R\mathbb{C}} \) and \( J_{3}^{C\mathbb{C}} \), or equivalently in terms of the hypermultiplets’ scalar manifolds of the same theories in \( d = 4 \). In light of very recent developments on the timelike \( d = 4 \rightarrow d = 3 \) reduction of \( \text{stu} \) model (see e.g. [43]), the relations (5.4), (5.6) and (5.9) are interesting, because \( \text{stu} \) is a common sector of all symmetric rank-3 special Kähler geometries in \( d = 4 \), as its \( d = 3 \) spacelike (timelike) reduction is a common sector of the \( c\)-map (\( c^\ast\)-map) images of all symmetric rank-3 special Kähler geometries in \( d = 4 \), namely of all symmetric rank-4 (pseudo-)quaternionic Kähler geometries in \( d = 3 \). In particular:

\[
M_{\mathcal{N}=2,\text{stu},d=4} = \left( \frac{SL(2,\mathbb{R})}{U(1)} \right)^3 \downarrow c
\]

\[
M_{\mathcal{N}=4,\text{stu},d=3} = \frac{SO(4,4)}{SO(4) \times SO(4)} \subseteq \left\{ \begin{array}{c}
M_{\mathcal{N}=4,J_{3}^{R\mathbb{C}},d=3} = M_{\frac{1}{8} - \text{BPS,}\mathcal{N}=8,d=5}; \\
M_{\mathcal{N}=4,J_{3}^{C\mathbb{C}},d=3} = M_{\frac{1}{8} - \text{BPS,large,}\mathcal{N}=8,d=4}.
\end{array} \right\} \quad (5.11)
\]
6 Conclusion

Through a comparison with some other approaches to $\mathcal{N} = 8, d = 4$ ungauged supergravity, namely with the ones by Sezgin-van Nieuwenhuizen [31,28,29] and Cremmer-Julia or de Wit-Nicolai [1,2,28], the main properties of the various symplectic frames can be summarized as follows.

1. The Sezgin-van Nieuwenhuizen [31,28,29] construction has $USp(8) \subset_{\text{symm}}^{\text{max}} SU(8)$ as maximal compact subgroup, which is nothing but the maximal compact subgroup of the $\mathcal{N} = 8, d = 5$ U-duality group $E_{6(6)}$: $USp(8) = \text{mcs} (E_{6(6)})$. By recalling the explicit form of “large” non-BPS charge orbit in $\mathcal{N} = 8, d = 4$ supergravity [20,21]:

\[
\mathcal{O}_{nBPS} = \frac{E_{7(7)}}{E_{6(6)}},
\]

it is easy to realize that this is the natural context in which “large” dyonic non-BPS $d = 4$ extremal black holes can be treated [28,29].

2. The Cremmer-Julia or de Wit-Nicolai [1,2,28] parametrization privileges the subgroup $SO(8) = \text{mcs} (SL(8,\mathbb{R})) \subset_{\text{symm}}^{\text{max}} E_{7(7)}$, providing a natural context in which $\frac{1}{8} - \text{BPS} “\text{large}”$ extremal $d = 4$ black holes can be treated (in particular, in the Lie algebra approach to the scalar manifold, with all scalars vanishing at the horizon [28]).

3. In the Iwasawa construction performed in the present paper, we started from a realization of $E_{7(7)}$ (Adams’ approach, Subsect. 2.1) with a natural underlying $SL(8,\mathbb{R})$ symmetry. However, this symmetry is soon broken explicitly down to $SL(7,\mathbb{R}) \times SO(1,1)$ by the selection of a Cartan subalgebra (see Eq. (3.15)). This is essentially due to the maximally non-compact nature of the Cartan subalgebra $C (2.37)$ of $\varepsilon_{7(7)}$, used in our construction. Therefore, the maximal off-shell symmetry of the whole $\mathcal{N} = 8, d = 4$ Lagrangian density is actually $SL(7,\mathbb{R}) \times SO(1,1)$, with $SO(7)$ as maximal compact subgroup (with symmetric embedding). Since

\[
\text{mcs} (SL(7,\mathbb{R}) \times SO(1,1)) = SO(7) \subset_{\text{symm}}^{\text{max}} SO(8) = \text{mcs} (SL(8,\mathbb{R})),
\]

it is natural to expect the residual “degeneracy” symmetry of $\frac{1}{8} - \text{BPS} “\text{large}”$ non-dyonic extremal $d = 4$ black holes, represented by solutions (4.31) and (4.32).

In light of previous reasonings, this breaking can also be traced back to the “real selection rule” inherited from the Cayley transformation $SL(8,\mathbb{R}) \mapsto SU(8)$ (recall (3.4) and (3.5)), thus giving rise only to $\frac{1}{8} - \text{BPS} “\text{large}”$ non-dyonic extremal $d = 4$ black holes, represented by solutions (4.31) and (4.32).

The origin of $Z_4$ of (6.3) can also be explained as follows. The breaking (3.15) is clearly incompatible with the branching

\[
SL(8,\mathbb{R}) \rightarrow SL(6,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(1,1),
\]

where $\mu$ is a primitive generator of $Z_4$.\footnote{It is worth mentioning that there are essentially two distinct embeddings of $Z_4 \hookrightarrow U(1)$ given by $\mu \mapsto \pm i$.}
unless the $SL(7,\mathbb{R})$ in (3.15) breaks down as

$$SL(7,\mathbb{R}) \rightarrow SL(6,\mathbb{R}) \times \Gamma,$$

(6.5)

where $\Gamma$ is some subgroup commuting with $SL(2,\mathbb{R}) \times SO(1,1)$ in (6.4). Indeed, by performing the Cayley transformation (4.48), the result (4.57) has been achieved.

As mentioned in the Introduction, the embedding analysis of [28] has pointed out that the Lie algebra approach to $\frac{1}{2}$-BPS attractors is related to the embedding of the Reissner-Nördstrom extremal BH solution of pure $\mathcal{N} = 2$, $d = 4$ supergravity into $\mathcal{N} = 8$ theory itself. Consequently, the $U(1) \rightarrow \mathbb{Z}_4$ breaking (6.3), due to the constraints on manifest off-shell covariance originated from the explicit Adams-Iwasawa construction of the coset representative of $\frac{E_7}{SU(8)}$ performed in the present paper, can be nicely interpreted as a breaking of the global $U(1)\mathcal{R}$-symmetry [40] of the pure $\mathcal{N} = 2$, $d = 4$ theory.

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