A Dynamic Solution to the Puzzle of Sea Battle

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Abstract

The puzzle of sea battle involves an argument that is an instantiation of reasoning by cases. Its premises include the conditionals “if there is a/no sea battle tomorrow, it is necessarily so”. It has a fatalistic conclusion. Two readings of necessity can be distinguished: absolute and relative necessity. The conditionals are valid for the latter reading. By the restrictor view of “if” in linguistics, the conditionals are not material implication. Instead, the if-clauses in them are devices for restricting the discourse domain that consists of possible futures. As a consequence, the argument is not sound. We present a dynamic temporal logic to formalize this idea. The base of this logic is CTL* without the operator until. The logic has a dynamic operator that shrinks models. The completeness of the logic is shown by reducing the dynamic operator.

1 The Puzzle of Sea Battle

The puzzle of sea battle is from Aristotle. Either there will be a sea battle tomorrow or not. If there is a sea battle tomorrow, it is necessarily so. If there is no sea battle tomorrow, it is necessarily so. So either necessarily there will be a sea battle tomorrow or necessarily there will be no sea battle tomorrow. Here “necessary” is understood as “inevitable”: something is necessary if it is the case no matter what we will do. The conclusion seems fatalistic and unacceptable.

There are two ways out: either arguing that the argument is not sound or arguing that its premises are not all true. The argument is a special case of reasoning by cases: from “φ₁ or φ₂”, “if φ₁ then ψ₁” and “if φ₂ then ψ₂”, we get “ψ₁ or ψ₂”. The argument has three premises. The first one may be called the principle of excluded future middle. The second and third may be called the principle of necessity of truth: true propositions are necessary.

The previous solutions to this puzzle presuppose the validity of the argument and adopt the latter strategy. They focus on the following issue: how do we ascribe truth values to the statements such as “there will be a sea battle tomorrow”? These statements are called future contingents in the literature: they are about the future but do not have an absolute sense.

These solutions include Łukasiewicz’s three-valued logic [5], Prior’s Peircean temporal logic [8], Prior’s Ockhamist temporal logic [8], the true futurist theory [11], the supervaluationist theory [11] and the relativist theory [4]. In the
first two, the principle of excluded future middle fails. In others, the principle of necessity of truth does not hold. Except Łukasiewicz’s three-valued logic, other solutions use branching time models. In a branching time model, time is represented as a tree. At any state, there is only one history but might be many possible futures. We refer to [3] for detailed comparison between these solutions.

There are two senses of necessity, absolute and relative necessity, determined by how we understand whatever we will do. Decision makes the difference. In reality we make decisions to do something or not to do something. So we will not do whatever we are able to do. We may read whatever we will do relative to the domain consisting of the things that we are able to do. We may also read it relative to the domain consisting of the things that we are able to do but have not decided to avoid. Note that doing a thing in the first domain but not in the second involves changing mind. A proposition is absolutely necessary if it is the case no matter which we choose to do from the big domain. A proposition is relatively necessary if it is the case no matter which we choose to do from the small domain. Accordingly, we have two principles of necessity of truth.

The principle of absolute necessity of truth does not hold for future contingents. Assume that the admiral is able to do two things: a and b. Doing a will cause a sea battle tomorrow but doing b will not. He decides to do a. In this case, it sounds plausible to say that there will be a sea battle tomorrow. However, it is strange of saying that there will be a sea battle tomorrow no matter what the admiral will do in the absolute sense.

We think that the principle of relative necessity of truth holds for future contingents. Assume that it is not necessary that \( \phi \) will be the case. Then someone has a way to act to make \( \phi \) not the case in the future without changing his decision. In this situation, it seems hard to say that \( \phi \) will be the case. Therefore, if \( \phi \) will be the case, it is necessary that \( \phi \) will be the case.

The puzzle of sea battle disappears under the absolute sense of necessity because the principle of necessity of truth fails for future contingents. However, it is still puzzling under the relative sense of necessity. The principle of excluded future middle is quite intuitive. So all the three premises of the puzzle seem valid. But the conclusion is still problematic. Suppose that doing a will cause a sea battle tomorrow but doing b will not. Assume that the admiral has not made the decision to do a or b. Then it is wrong to say that there will necessarily be a sea battle tomorrow. It is also wrong to say that there will necessarily not be a sea battle tomorrow. So the conclusion of the puzzle is false. How come we get a false conclusion from three valid premises by a sound inference?

The soundness of reasoning by cases presupposes that the two conditionals in it are material implications. However, if the two conditionals are something else, then reasoning by cases might not be a sound argument.

In linguistics, a different view on conditionals is common, that is, the restrictor view. The conditional \( \text{if } \phi \text{ then } \psi \) is not a connective connecting two sentences. There is no such a connective in natural languages. Utterance of a sentence is always w.r.t. a specific discourse domain. The if-clause \( \text{if } \phi \text{ then } \psi \) is a device for restricting the domain. The conditional \( \text{if } \phi \text{ then } \psi \) is true w.r.t. a domain if \( \psi \) is true w.r.t. the resulted domain. Conditionals collapse to material implications when discourse domains are singletons. This view can be found in various works including [3], [2] and [1]. It can be tracked back to Ramsey Test in [9].
Reasoning by cases is not generally valid under the restrictor view on conditionals. Let \( \Delta \) be a discourse domain. Assume that if \( \phi_1 \) then \( \psi_1 \) and if \( \phi_2 \) then \( \psi_2 \) are true w.r.t. \( \Delta \). Let \( \Delta^{\phi_1} \) and \( \Delta^{\phi_2} \) be the respect results of restricting \( \Delta \) with if \( \phi_1 \) and \( \phi_2 \). Then what the two conditionals say is just that \( \psi_1 \) is true w.r.t. \( \Delta^{\phi_1} \) and \( \psi_2 \) true w.r.t. \( \Delta^{\phi_2} \). If neither the truth of \( \psi_1 \) nor the truth of \( \psi_2 \) is upward monotonic relative to discourse domains, then it is possible that neither \( \psi_1 \) nor \( \psi_2 \) is true w.r.t. \( \Delta \).

The discourse domain of the puzzle of sea battle consists of possible futures. The if-clauses “if there is a/no sea battle tomorrow” restrict the domain. However, relative necessity is not a upward monotonic notion w.r.t. the class of possible futures. Therefore, the argument concerning sea battle is not sound. In what follows we present a logic to formalize this idea.

### 2. Formal Settings

Let \( \Phi_0 \) be a countable set of atomic propositions and \( p \) range over it. Define a language \( \Phi_{XL} \) as follows:

\[
\phi ::= p \mid \top \mid \neg \phi \mid (\phi \land \phi) \mid X\phi \mid A\phi \mid [\psi]\phi
\]

where \( \psi \) is from \( \Phi_{XL} \), the sub-language of \( \Phi_{XL} \) generated from \( \Phi_0 \) under \( X \) and \([\cdot] \).

The featured formulas of \( \Phi_{XL} \) are read as follows:

1. \( X\phi \): \( \phi \) will be the case in the next moment.
2. \( A\phi \): no matter how the agent will act in the future, \( \phi \) is the case now, that is, \( \phi \) is necessary.
3. \([\psi]\phi \): given \( \psi \), \( \phi \) is the case.

The principle of excluded future middle is expressed as \( X\phi \lor X\neg \phi \) and the principle of necessity of truth as \([\phi]A\phi \).

\( X\phi \) is a temporal formula. \( p \), \( \top \) and \( A\phi \) are state formulas. \([\psi]\phi \) is a temporal formula if \( \phi \) is, or else a state formula. Later we will see that temporal formulas are evaluated at states into paths and state formulas evaluated just at states.

The other usual propositional connectives and the falsum \( \bot \) are defined in the usual way. \( E\phi \), defined as \( \neg A\neg \phi \), indicates that the agent has a way to act in the future s.t. \( \phi \) is the case now, that is, \( \phi \) is possible.

It seems strange to say that the agent has a way to act in the future s.t. \( \phi \) is the case now. Actually this is fine, as whether a sentence involving future is true or not now might be dependent on how the agent will act in the future. For example, whether a student will pass an exam is dependent on how he will study.

Let \( W \) be a nonempty set of states and \( R \) a binary relation on it. A sequence \( w_0 \ldots w_n \) of states is called a \( R \)-sequence if \( w_0 R \ldots Rw_n \). As a limit case, \( w \) is a \( R \)-sequence for any \( w \). \( (W, R) \) is a tree if there is a \( r \) s.t. for any \( w \), there is a unique \( R \)-sequence from \( r \) to \( w \). \( r \) is called the root. It can be seen that the root is unique and \( R \) is irreflexive. \( R \) is serial if for any \( w \), there is a \( u \) s.t. \( Rwu \).

We say that a tree \( (W, R) \) is serial if \( R \) is. A serial tree is understood as a time structure encoding an agent’s actions (the transitions) and states in time.
(the nodes). A branching in the tree is interpreted as a situation in which the agent can choose between different possible actions. The seriality corresponds to the fact that the agent can always perform an action at any given time.

Fix a serial tree \((W, R)\). Here are some auxiliary notations. A \(R\)-sequence \(w_0 \ldots w_n\) starting at the root is a history of \(w_n\). For any states \(w\) and \(u\), \(u\) is a historical state of \(w\) if there is a \(R\)-sequence \(u_0 \ldots u_n\) s.t. \(0 < n\), \(u_0 = u\) and \(u_n = w\). \(w\) is a future state of \(u\) if \(u\) is a historical state of \(w\). Note that a state can not be a historical or future state of itself.

An infinite \(R\)-sequence is a path. A path starting at the root is a timeline. A path \(w_0 \ldots\) passes through a state \(x\) if \(x = w_i\) for some \(i > 0\). Let \(\pi\) be a path. We use \(\pi(i)\) to denote the \(i + 1\)-th element of \(\pi\), \(\pi\) the prefix of \(\pi\) to the \(i + 1\)-th element, and \(\pi^i\) the suffix of \(\pi\) from the \(i + 1\)-th element. For example, if \(\pi = w_0 \ldots\), then \(\pi(2) = w_2\), \(\pi^i = w_0w_1w_2\) and \(\pi^2 = w_2\ldots\). For any history \(w_0 \ldots w_n\) and path \(u_0 \ldots\), if \(w_n = u_0\), let \(w_0 \ldots w_n \otimes u_0 \ldots\) denote the timeline \(w_0 \ldots w_nu_1 \ldots\).

\(\mathcal{M} = (W, R, r, V)\) is a model if \((W, R)\) is a serial tree with \(r\) as the root and \(V\) is a function from \(\Phi_0\) to \(2^W\). **Figure 1** illustrates a model.

![Figure 1: This figure indicates a model. \(w_0\) is the root. Arrows are transitions.](image)

**Definition 1** (Semantics). \(\mathcal{M}, \pi, i \models \phi\), the formula \(\phi\) being true at the state \(\pi(i)\) relative to the timeline \(\pi\) in the model \(\mathcal{M}\), meets the following conditions:

\[
\begin{align*}
\mathcal{M}, \pi, i \models p & \iff \pi(i) \in V(p) \\
\mathcal{M}, \pi, i \models \top & \iff \text{not } \mathcal{M}, \pi, i \models \phi \\
\mathcal{M}, \pi, i \models \phi \land \psi & \iff \mathcal{M}, \pi, i \models \phi \text{ and } \mathcal{M}, \pi, i \models \psi \\
\mathcal{M}, \pi, i \models X\phi & \iff \mathcal{M}, \pi, i + 1 \models \phi \\
\mathcal{M}, \pi, i \models A\phi & \iff \text{for any path } \rho \text{ starting at } \pi(i), \mathcal{M}, i \otimes \rho, i \models \phi \\
\mathcal{M}, \pi, i \models [\phi]\psi & \iff \mathcal{M}_{\pi(i)}^\phi, \pi, i \models \psi \text{ if } (\mathcal{M}_{\pi(i)}^\phi, \psi) \text{ is well given}
\end{align*}
\]

By \((\mathcal{M}_{\pi(i)}^\phi, \psi)\) is well given, we mean that \(\mathcal{M}_{\pi(i)}^\phi\) is defined and \(\pi\) is a path of it. Note that \(\mathcal{M}, \pi, i \models [\phi]\psi\) holds trivially if \((\mathcal{M}_{\pi(i)}^\phi, \psi)\) is not well given. \(\mathcal{M}_{w}^\phi\) is defined in parallel as follows. Fix a model \(\mathcal{M} = (W, R, r, V)\). We say that \(\phi\) is achievable at \(w\) in \(\mathcal{M}\) if \(\phi\) is true at \(w\) relative to a path from \(w\). Let \(W' \subseteq W\). A structure \((W', R', V')\) is called the restriction of \(\mathcal{M}\) to \(W'\) if \(R' = R \cap (W' \times W')\) and \(V'(p) = V(p) \cap W'\).
Definition 2 (Update of models). Let $\mathcal{M} = (W, R, r, V)$ be a model, $\phi$ a formula and $w$ a state. Assume that $\phi$ is achievable at $w$. Let $w_0 \ldots w_i$ be the history of $w$. Define a set $X^\phi_w$ of states as follows: for any $x \in W$, $x \in X^\phi_w$ if (i) $x$ is a future state of $w$ and (ii) there is no timeline $\rho$ passing through $w$ and $x$ such that $\mathcal{M}, \rho, w \models \phi$. Define $\mathcal{M}^\phi_w$ as $(W - X^\phi_w, R', r, V')$, the restriction of $\mathcal{M}$ to $W - X^\phi_w$. $\mathcal{M}^\phi_w$ is called the result of updating $\mathcal{M}$ at $w$ with $\phi$.

$\mathcal{M}^\phi_w$ is undefined if $\phi$ is not achievable at $w$.

The operator $A$ can be viewed as a universal quantifier over possible futures. The truth of a state formula at a state relative to a path is not dependent on the path, but this is not the case for temporal formulas. In some cases, if $\phi$ is a state formula, we write $\mathcal{M}, w \models \phi$ without specifying a path. Sometimes, if $\pi$ is a timeline of $\mathcal{M}$ containing $w$, we use $\mathcal{M}, \pi, w \models \phi$ instead of $\mathcal{M}, \pi(i) \models \phi$ where $\pi(i) = w$.

The meaning of $[\phi]$ lies in how $\phi$ updates models. Updating $\mathcal{M}$ with $\phi$ at $w$ is to shrink $\mathcal{M}$ by removing the states in $X^\phi_w$. $X^\phi_w$ can be understood as follows. Assume that the agent is at $w$ and decides to make $\phi$ true. After the decision is made, some future states are not possible anymore. A state becomes impossible if the agent travels to it, there would be no way to make $\phi$ true at $w$, no matter where he goes afterwards. $X^\phi_w$ is the collection of these states.

Figure 2 illustrates how a formula updates a model.

Figure 2: This figure illustrates how a model is shrunk by a formula. The right model is the result of updating the left one at $w_0$ with $X \neg r \wedge XX \neg r$.

Suppose that $\phi$ is achievable at $w$. It can be verified that $r \notin X^\phi_w$ and $X^\phi_w$ is closed under $R$. So $(W - X^\phi_w, R')$ is a tree with $r$ as the root. It can also be verified that $R'$ is serial. It follows that $(W - X^\phi_w, R', r, V')$ is a model. Note that it is possible that $\pi$ is not a timeline of $\mathcal{M}^\phi_w$. So $(\mathcal{M}^\phi_{\pi(i)}, \pi)$ might not be well given. Later we will see that $\pi$ is a timeline of $\mathcal{M}^\phi_{\pi(i)}$ iff $\mathcal{M}, \pi, i \models \phi$.

A formula $\phi$ is valid if for any $\mathcal{M}, \pi$ and $i$, $\mathcal{M}, \pi, i \models \phi$. Let $\Gamma$ be a set of formulas and $\phi$ a formula. $\Gamma \models \phi$, $\Gamma$ entails $\phi$, if for any $\mathcal{M}, \pi$ and $i$, if $\mathcal{M}, \pi, i \models \Gamma$, then $\mathcal{M}, \pi, i \models \phi$. We in the sequel use $\mathcal{L}$ to denote the set of valid formulas.
3 The Puzzle Is Solved in a Way

The update with $\phi$ shrinks models. As a consequence, it restricts possible futures. The following theorem indicates that it restricts possible futures as we wish: it exactly excludes the possible futures which does not satisfy $\phi$.

Let $\mathbb{N}$ be the set of natural numbers. Define a function $\sigma : \Phi_X \rightarrow \mathbb{N}$ as follows:

\[
\begin{align*}
p^\sigma &= 0 \\
\tau^\sigma &= 0 \\
(\neg \phi)^\sigma &= \phi^\sigma \\
(\phi \land \psi)^\sigma &= \max\{\phi^\sigma, \psi^\sigma\} \\
(X\phi)^\sigma &= \phi^\sigma + 1 \\
([\phi]_\chi)^\sigma &= \max\{\phi^\sigma, \psi^\sigma\}
\end{align*}
\]

$\phi^\sigma$ intuitively indicates how far $\phi$ can see forward.

Lemma 1. Let $n$ be a natural number and $\phi$ a formula in $\Phi_X$ s.t. $\phi^\sigma \leq n$. Let $\mathcal{M}$ be a model and $w$ a state.

(a) For any timelines $\pi$ and $\tau$ of $\mathcal{M}$ passing through $w$, if they share the same $n$ elements after $w$, then $\mathcal{M}, \pi, w \models \phi$ iff $\mathcal{M}, \tau, w \models \phi$.

(b) For any timeline $\pi$ of $\mathcal{M}$ passing through $w$, $(\mathcal{M}_w^\pi, \pi)$ is well-given iff $\mathcal{M}, \pi, w \models \phi$.

Proof. We show the results (a) and (b) in parallel by inductions on $n$ and $\phi$. Note that if $\mathcal{M}, \pi, w \models \phi$, then $\mathcal{M}_w^\pi$ is defined and $\pi$ a path of $\mathcal{M}_w^\pi$ by Definition 2.

So the right-left direction of (b) holds. This direction will not be considered in the sequel.

Case $n = 0$. The subcase $\phi = \top$ is trivial and the subcase $\phi = X\psi$ impossible.

Subcase $\phi = p$.

(a) Clearly $\mathcal{M}, \pi, w \models p$ iff $\mathcal{M}, \tau, w \models p$.

(b) Assume $\mathcal{M}, \pi, w \not\models p$. Then $\mathcal{M}_w^\pi$ is undefined and $(\mathcal{M}_w^\pi, \pi)$ not well given.

Subcase $\phi = \neg \psi$.

(a) By the inductive hypothesis, $\mathcal{M}, \pi, w \models \neg \psi$ iff $\mathcal{M}, \tau, w \models \neg \psi$. Then $\mathcal{M}, \pi, w \models \neg \neg \psi$ iff $\mathcal{M}, \tau, w \models \neg \neg \neg \psi$.

(b) Assume $\mathcal{M}, \pi, w \not\models \neg \neg \psi$. Then $\mathcal{M}, \pi, w \not\models \neg \psi$. Let $\tau$ be an arbitrary timeline of $\mathcal{M}$ through $w$. By the inductive hypothesis, $\mathcal{M}, \tau, w \models \psi$. Then $\mathcal{M}, \tau, w \not\models \neg \psi$. Then $\mathcal{M}_w^\pi$ is undefined and $(\mathcal{M}_w^\pi, \pi)$ not well given.

Subcase $\phi = X \psi$.

(a) By the inductive hypothesis, $\mathcal{M}, \pi, w \models X \psi$ iff $\mathcal{M}, \tau, w \models X \psi$, and $\mathcal{M}, \pi, w \models \chi$ iff $\mathcal{M}, \tau, w \models \chi$. Then $\mathcal{M}, \pi, w \models X \psi \land \chi$ iff $\mathcal{M}, \tau, w \models X \psi \land \chi$.

(b) Assume $\mathcal{M}, \pi, w \not\models X \psi \land \chi$. Then $\mathcal{M}, \pi, w \not\models \psi$ or $\mathcal{M}, \pi, w \not\models \chi$. Assume the former. Let $\tau$ be an arbitrary timeline of $\mathcal{M}$ through $w$. By the inductive hypothesis, $\mathcal{M}, \tau, w \not\models \psi$. Then $\mathcal{M}, \tau, w \not\models \psi \land \chi$. Then $\mathcal{M}_w^{\psi \land \chi}$ is undefined and $(\mathcal{M}_w^{\psi \land \chi}, \pi)$ not well given. In a similar way, we can get that if $\mathcal{M}, \pi, w \not\models \chi$, then $(\mathcal{M}_w^{\psi \land \chi}, \pi)$ is not well given either.

Subcase $\phi = [\psi]_\chi$.

(a) Assume $\mathcal{M}, \pi, w \models [\psi]_\chi$. Then if $(\mathcal{M}_w^{\psi}, \pi)$ is well given, $\mathcal{M}_w^{\psi}, \pi, w \models \chi$. By the inductive hypothesis, $(\mathcal{M}_w^{\psi}, \pi)$ is well given iff $\mathcal{M}, \pi, w \models \psi$, $\mathcal{M}, \pi, w \models \psi$ iff $\mathcal{M}, \tau, w \models \psi$. Then $\mathcal{M}, \pi, w \models \psi$. Then $\mathcal{M}_w^{[\psi]_\chi}$ is well given, and $\mathcal{M}_w^{\psi}, \pi, w \models \chi$ iff $\mathcal{M}_w^{[\psi]_\chi}, \pi$ is well given.
\[ M_w^x, \tau, w \models \chi. \] Then if \((M_w^x, \tau)\) is well given, \(M_w^x, \tau, w \models \chi. \) Then \(M, \tau, w \models [\psi]_X.\) Similarly, we can show that if \(M, \tau, w \models [\psi]_X\) then \(M, \pi, w \models [\psi]_X.\)

(b) Assume \(M, \pi, w \not\models [\psi]_X\). Then \((M_w^x, \tau)\) is well given but \(M_w^x, \pi, w \not\models \chi.\) Let \(\tau\) be a timeline of \(M\) through \(w.\) In a similar way as (a), we know \(M, \tau, w \not\models [\psi]_X.\) Then \(M_w^{[\psi]_X}\) is undefined and \((M_w^{[\psi]_X}, \pi)\) not well given.

Case \(n = k + 1.\) The subcases \(\phi = p\) and \(\phi = \tau\) are impossible.

**Subcase** \(\phi = \neg \psi.\)

(a) Similar arguments with the case \(n = 0.\)

(b) Assume \(M, \pi, w \not\models \psi.\) Then \(M, \pi, w \models \psi.\) Let \(\tau\) be an arbitrary timeline of \(M\) passing through \(w\) and sharing the same \(n\) elements after \(w\) with \(\pi.\) By the inductive hypothesis, \(M, \tau, w \models \psi.\) Then \(M, \tau, w \not\models \neg \psi.\) Let \(u\) be the \(n\)-th element of \(\pi\) after \(w.\) By **Definition 2**, \(u\) is not in \(M_w^{\psi}\) if \(M_w^{\psi}\) is defined. Then \((M_w^{[\psi]_X}, \pi)\) is not well given. In a similar way, we can get that if \(M, \pi, w \not\models \chi,\) then \((M_w^{[\psi]_X}, \pi)\) is not well given either.

**Subcase** \(\phi = \psi \land \chi.\)

(a) Similar arguments with the case \(n = 0.\)

(b) Assume \(M, \pi, w \not\models \psi \land \chi.\) Then \(M, \pi, w \not\models \psi\) or \(M, \pi, w \not\models \chi.\) Assume the former. Let \(\tau\) be an arbitrary timeline of \(M\) passing through \(w\) and sharing the same \(n\) elements after \(w\) with \(\pi.\) By the inductive hypothesis, \(M, \tau, w \not\models \psi,\) Then \(M, \tau, w \not\models \psi \land \chi.\) Let \(u\) be the \(n\)-th element of \(\pi\) after \(w.\) Then \(u\) is not in \(M_w^{[\psi \land \chi]}\) if \(M_w^{[\psi \land \chi]}\) is defined. Then \((M_w^{[\psi \land \chi]}, \pi)\) is not well given. In a similar way, we can get that if \(M, \pi, w \not\models \chi,\) then \((M_w^{[\psi \land \chi]}, \pi)\) is not well given either.

**Subcase** \(\phi = X \psi.\)

(a) Let \(\pi(i) = w.\) By the inductive hypothesis, \(M, \pi, i \models X \psi\) if \(M, \pi, i + 1 \not\models \psi\) iff \(M, \pi, i + 1 \not\models \psi\) iff \(M, \pi, i \not\models X \psi.\)

(b) Let \(\pi(i) = w.\) Assume \(M, \pi, i \not\models X \psi.\) Then \(M, \pi, i + 1 \not\models \psi.\) Let \(\tau\) be an arbitrary timeline of \(M\) passing through \(w\) and sharing the same \(n\) elements after \(w\) with \(\pi.\) Then \(\tau\) shares the same \(k\) elements after \(\pi(i + 1)\) with \(\pi.\) By inductive hypothesis, \(M, \tau, i + 1 \not\models \psi.\) Then \(M, \tau, i \not\models X \psi.\) Let \(u\) be the \(n\)-th element of \(\pi\) after \(w.\) Then \(u\) is not in \(M_w^{[\psi]}\) if \(M_w^{[\psi]}\) is defined. Then \((M_w^{[\psi]}, \pi)\) is not well given.

**Subcase** \(\phi = [\psi]_X.\)

(a) Similar arguments with the case \(n = 0.\)

(b) Assume \(M, \pi, w \not\models [\psi]_X.\) Then \((M_w^\psi, \pi)\) is well given but \(M_w^\psi, \pi, w \not\models \chi.\) Let \(\tau\) be a timeline of \(M\) passing through \(w\) and sharing the same \(n\) elements after \(w\) with \(\pi.\) In a similar way as (a), we get \(M, \tau, w \not\models [\psi]_X.\) Let \(u\) be the \(n\)-th element of \(\pi\) after \(w.\) Then \(u\) is not in \(M_w^{[\psi]_X}\) if \(M_w^{[\psi]_X}\) is defined. Then \((M_w^{[\psi]_X}, \pi)\) is not well given.

**Theorem 1.** Let \(M\) be a model, \(w\) a state and \(\phi\) a formula of \(\Phi_{X[1]}\) achievable at \(w.\) For any timeline \(\pi\) of \(M\) passing through \(w,\) \(\pi\) is a timeline of \(M_w^\phi\) iff \(M, \pi, w \models \phi.\)

**Proof.** As \(\phi\) is achievable at \(w,\) \(M_w^\phi\) is defined. Let \(\pi\) be a timeline of \(M\) passing through \(w\). Assume \(M, \pi, w \models \phi.\) By the definition of \(M_w^\phi,\) all the elements of \(\pi\) are in \(M_w^\phi.\) So \(\pi\) is a timeline of \(M_w^\phi.\) Assume that \(\pi\) is a timeline of \(M_w^\phi.\) Let \(\pi = w_0, \ldots, w_i = w.\) By \(\Phi_{X[1]}\) \(M, \rho, w \models \phi.\) There exists \(\pi = w_0, \ldots, w_i = w.\) By **Lemma 1**, \(M, \pi, w \models \phi.\)

We now show that the principle of necessity of truth, \([\phi]_A \phi,\) is valid. Let \(\Phi_X\) denote the sub-language of \(\Phi_{X[1]}\) generated from \(\Phi_0\) under \(X.\)
Lemma 2. For any $\phi$ in $\Phi_{X[1]}$, there is a $\psi$ in $\Phi_X$ equivalent to $\phi$.

From Lemma 1 we can get a simple fact: for any $\phi$ and $\psi$ in $\Phi_X$, $[\phi]\psi$ is equivalent to $\phi \rightarrow \psi$. Based on this fact, we can show this lemma in an inductive way.

If a formula $\phi$ contains no $A$ and $[\ ]$, then its truth value at a state relative to a timeline is determined by the timeline itself. As $\Phi_{X[1]}$ can be reduced to $\Phi_X$, what follows is true.

Theorem 2. $[\phi]A\phi$ is valid.

Proof. Assume $M, i \not\models [\phi]A\phi$. Then $M^\phi_{(\pi(i), \pi, i \not\models A\phi}$. Then there is a path $\rho$ in $M^\phi_{(\pi(i)}}$ starting at $\pi(i)$ s.t. $M^\phi_{(\pi \otimes \rho, i \not\models \phi}$. By Lemma 3 $M, \pi \otimes \rho, i \not\models \phi$. By Lemma 1 $\rho$ is not a path of $M^\phi_{\pi(i)}$. We have a contradiction.

Let $s$ denote that there is a sea battle. The puzzle of sea battle can be formalized as the inference $Xs \land X\neg s, [Xs]AXs, [X\neg s]AX\neg s \equiv AXs \lor AX\neg s$. It is easy to see that $AXs \lor AX\neg s$ is not valid. It is also easy to get that the principle of excluded future middle, $X\phi \lor X\neg \phi$, holds. Therefore, the puzzle of sea battle is not a sound argument.

It can be verified that $\neg \phi \not\models [\phi]\psi$. For example, $\neg Xp \not\models [Xp]E\neg Xp$. So the notorious problem with material implication is not a problem here. $[\phi]\psi$ collapses to the material implication $\phi \rightarrow \psi$ if $\phi$ is a state formula. The reason is as follows. Assume that $\phi$ is a state formula. Fix a model $M$ and a state $w$. Then $\phi$ is true or false at $w$. Suppose that $\phi$ is true at $w$. Then the update with $\phi$ at $w$ does not change $M$. Then the truth conditions of $[\phi]\psi$ and $\phi \rightarrow \psi$ at $w$ relative to any timeline are the same. Suppose that $\phi$ is false at $w$. Then the update with $\phi$ at $w$ fails. Then both $[\phi]\psi$ and $\phi \rightarrow \psi$ are trivially true at $w$ relative to any timeline.

4 Completeness by Reduction

The idea of showing the completeness of $XL$ is to reduce the dynamic operator $[\phi]$ in $[\phi]\psi$. This idea is from dynamic epistemic logic [12]. To reduce $[\phi]$, a strategy is to massage $[\phi]$ into $\psi$ deeper and deeper until it meets atomic propositions. A difficulty arises when $[\phi]$ meets the operator $X$. To handle this, some pretreatment of $\phi$ is needed.

Let $\Phi_{PC}$ be the set of formulas of Propositional Calculus.

Lemma 4. For any $\phi$ in $\Phi_X$, there are $\psi_1, \ldots, \psi_n$ in $\Phi_{PC}$ and $\chi_1, \ldots, \chi_n$ in $\Phi_X$ s.t. $\phi$ is equivalent to $(\psi_1 \lor X\chi_1) \land \cdots \land (\psi_n \lor X\chi_n)$.

The operator $X$ can freely go into and out of conjunctions and disjunctions: $X(\phi \land \psi) \leftrightarrow (X\phi \land X\psi)$ and $X(\phi \lor \psi) \leftrightarrow (X\phi \lor X\psi)$ are valid. By making a reflection we can see that this lemma holds.

Lemma 5. $[\phi \land \psi] X \leftrightarrow [\phi]([\psi] \chi)$ is valid.
Lemma 7. \[ \implies \text{what follows is the case:} \]

Proof. Firstly, we show that \( M^\phi_{\pi_\psi} \) is defined iff \( (M^\phi_{\pi_\psi})^w \) is defined. Assume that \( M^\phi_{\pi_\psi} \) is defined. Then \( M, \pi, w \vdash \phi \land \psi \) for some \( \pi \) through \( w \). By Lemma 1, \( M^\phi_{\pi_\psi} \) is defined and \( \pi \) is a path of \( M^\phi_{\pi_\psi} \). By Lemma 3, \( M^\phi_{\pi_\psi}, \pi, w \vdash \psi \). Then \( (M^\phi_{\pi_\psi})^w \) is defined. Now assume that \( (M^\phi_{\pi_\psi})^w \) is defined. Then \( M^\phi_{\pi_\psi}, \pi, w \vdash \psi \) for some \( \pi \) through \( w \). By Theorem 2, \( M^\phi_{\pi_\psi}, \pi, w \vdash \phi \) for any \( \pi \) through \( w \). Then \( M^\phi_{\pi_\psi}, \pi, w \vdash \phi \land \psi \). By Lemma 3, \( M, \pi, w \vdash \phi \land \psi \). Then \( M^\phi_{\pi_\psi} \) is defined.

Secondly, we show \( M^\phi_{\pi_\psi} \) and \( (M^\phi_{\pi_\psi})^w \) have the same timelines through \( w \). By Lemma 1 and Lemma 3, what follows is the case: \( \pi \) is a timeline of \( M^\phi_{\pi_\psi} \) through \( w \) \( \iff \) \( M, \pi, w \vdash \phi \land \psi \) \( \iff \) \( \pi \) is a timeline of \( M^\phi_{\pi_\psi} \) through \( w \) and \( \phi \land \psi \) \( \iff \) \( \pi \) is a timeline of \( (M^\phi_{\pi_\psi})^w \) through \( w \).

By the previous results, what follows is the case: \( M, \pi, w \nvdash [\phi \land \psi]_\chi \) \( \iff \) \( M^\phi_{\pi_\psi}, \pi, w \nvdash \chi \) \( \iff \) \( M^\phi_{\pi_\psi}, \pi, w \nvdash \chi \) \( \iff \) \( M^\phi_{\pi_\psi}, \pi, w \nvdash [\psi]_\chi \) \( \iff \) \( M, \pi, w \nvdash [\phi]_\psi \chi \).

\[ \square \]

From this result and Lemma 2 it follows that \( [\phi]_\psi \chi \) is equivalent to \( [\phi \land [\phi]_\psi] \chi \).

Lemma 6. Let \( \phi \) be in \( \Phi_{X\delta} \) and \( M, \pi, i \vdash X_\delta \phi \). Then the generated submodels of \( M^\phi_{\pi(i)} \) and \( M^\phi_{\pi(i+1)} \) at \( \pi(i+1) \) are identical.

Lemma 7.

Proof. 5. Assume \( M, \pi, i \nvdash [\psi \lor \chi \psi]_X \chi \). Then \( M, \pi, i \nvdash \phi \lor \chi \psi \) and \( M^\phi_{\pi(i)} \), \( \pi, i \nvdash \chi \psi \). Assume \( M, \pi, i \nvdash \phi \). As \( \phi \) is in \( \Phi_{X\delta} \), \( M^\phi_{\pi(i)} = M \). Then \( M, \pi, i \nvdash X \chi \). Then \( M, \pi, i \nvdash \phi \rightarrow X \chi \). Assume \( M, \pi, i \nvdash \chi \). Then \( M, \pi, i \nvdash \phi \lor \chi \). As \( \phi \) is in \( \Phi_{X\delta} \), \( M^\phi_{\pi(i+1)} = M^\phi_{\pi(i)} \). Then \( M^\phi_{\pi(i)} = M^\phi_{\pi(i)} \). Then \( M^\phi_{\pi(i)} = M^\phi_{\pi(i)} \). Then \( M^\phi_{\pi(i)} \), \( \pi, i + 1 \nvdash \chi \). By Lemma 6, \( M^\phi_{\pi(i+1)} \), \( \pi, i + 1 \nvdash \chi \). Then \( M, \pi, i \nvdash \chi \psi \). Then \( M, \pi, i \nvdash [\psi \lor \chi \psi]_X \chi \). Assume \( M, \pi, i \nvdash \phi \rightarrow X \chi \). Then \( M, \pi, i \nvdash [\phi \lor \chi \psi]_X \chi \).

6. Assume \( M, \pi, i \nvdash \phi \rightarrow A_\psi[\phi] \psi \). Then \( M, \pi, i \nvdash \phi \) but \( M, \pi, i \nvdash A_\psi[\phi] \psi \).

By Lemma 1, \( \pi \) is in \( M^\phi_{\pi(i)} \). Then there is a \( \rho \) starting at \( \pi(i) \) s.t. \( M, \pi, i \bullet \rho \), \( i \nvdash \phi \psi \). Then \( M^\phi_{\pi(i)} \), \( i \bullet \rho \), \( i \nvdash [\phi] \psi \). Then \( M^\phi_{\pi(i)} \), \( i \bullet \rho \), \( i \nvdash [\phi] \psi \).

Assume \( M, \pi, i \nvdash [\phi] A_\psi \). Then \( (M^\phi_{\pi(i)}) \) is well given but \( M^\phi_{\pi(i)} \), \( i \nvdash A_\psi \).

By Lemma 1, \( M, \pi, i \nvdash \phi \). Then there is a \( \rho \) starting at \( \pi(i) \) in \( M^\phi_{\pi(i)} \) s.t.
Let $\Phi_{X_A}$ denote the language generated from $\Phi_0$ under $X$ and $A$.

**Theorem 3.** The language $\Phi_{X_L}$ can be reduced to $\Phi_{X_A}$.

**Proof.** Let $\theta$ be a formula in $\Phi_{X_L}$. We pick a subformula of $\theta$ which is in the form of $[\phi]\psi$ where $\psi$ contains no $[\cdot]$. Note that if $\theta$ has no such a subformula, then $\theta$ contains no $[\cdot]$ and is already in $\Phi_{X_A}$. Then we do the following things:

1. By Lemma 2 we can find $\phi'$ in $\Phi_X$ equivalent to $\phi$. We transform $[\phi]$ to $[\phi']$.

2. By Lemma 4 we can find $\psi_1, \ldots, \psi_n$ in $\Phi_{PC}$ and $\chi_1, \ldots, \chi_n$ in $\Phi_X$ s.t. $(\psi_1 \lor X\chi_1) \land \cdots \land (\psi_n \lor X\chi_n)$ is equivalent to $\phi'$. We transform $[\phi']$ to $[(\psi_1 \lor X\chi_1) \land \cdots \land (\psi_n \lor X\chi_n)]$.

3. We transform $[(\psi_1 \lor X\chi_1) \land \cdots \land (\psi_n \lor X\chi_n)]$ to $[(\psi_1 \lor X\chi_1)] \land \cdots \land [(\psi_n \lor X\chi_n)]$. By Lemma 5 they have the same effects.

4. In the way specified by the last four items of Lemma 7 we let $[(\psi_n \lor X\chi_n)]$ into $\psi$ deeper and deeper until it meets atomic propositions. Note when $[(\psi_n \lor X\chi_n)]$ meets $X$, it becomes $[\chi_n]$. Then we reduce $[(\psi_n \lor X\chi_n)]$ in the way specified by the first two items of Lemma 7. We repeat until $[(\psi_1 \lor X\chi_1)]$ is reduced.

We repeat until $\theta$ contains no $[\cdot]$.

The completeness of the logic for the language $\Phi_{X_A}$ is already shown in the literature. Then we can get the completeness of the logic $X_L$.

## 5 Conclusive Remarks

The argument in the puzzle of sea battle is a special case of reasoning by cases. Its premises include the conditionals “if there is a/no sea battle tomorrow, it is necessarily so”. It has a fatalistic conclusion: we can not interfere with whether there will be a sea battle tomorrow. The principle expressed by the conditionals are plausible if we read necessity in the relative sense. The conditionals are not material implication. Instead, the if-clauses in them are devices for restricting possible futures. Necessity is not a upward monotonic notion. So the argument is not valid. It is fine that an invalid argument has a fatalistic conclusion. We present a formal way to make this precise.

Reading conditionals as material implication makes reasoning by cases valid. This causes others puzzles. Among them is the Puzzle of Miners presented by [10]. [3] proposes a deontic logic based on an extension of $\text{CTL}^*$. By applying the approach in this work to the deontic logic, we might get a solution to the Puzzle of Miners. This is our future work.

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