GRADED PARABOLIC INDUCTION AND STRATIFIED MIXED
TATE MOTIVES

JENS NIKLAS EBERHARDT

Mathematisches Institut
Albert-Ludwigs-Universität Freiburg
jens.eberhardt@math.uni-freiburg.de

Abstract. We give a geometric and graded version of parabolic induction for
modules of semi-simple complex Lie algebras in the BGG-category \( \mathcal{O} \) in terms
of stratified mixed Tate motives on the corresponding flag varieties. We also
describe the effect of parabolic induction on the level of Soergel modules.

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1. Introduction

1.1. Parabolic Induction. Let \( \mathfrak{g} \) be a semi-simple complex Lie algebra, for
example \( \mathfrak{sl}_n(\mathbb{C}) \). Choose some parabolic subalgebra \( \mathfrak{p} \subset \mathfrak{g} \) and denote its reductive
Levi factor by \( \mathfrak{l} \). From any \( \mathfrak{l} \)-module \( M \) we can construct a \( \mathfrak{g} \)-module \( \text{Ind}^\mathfrak{g}_\mathfrak{p} M \)
by extending \( M \) trivially on the nilradical of \( \mathfrak{p} \) and then inducing it up. This process
is called parabolic induction and is of general interest in representation theory.

This article is concerned with parabolic induction of modules in regular integral
blocks of category \( \mathcal{O} \) (see [BGG71], [Hum08]). Translation functors reduce the
situation to the study of a family of functors \( \{ \text{Ind}_w \} \) between the principal
blocks

\[
\text{Ind}_w := \text{Ind}_\mathfrak{g}^\mathfrak{p} T_w^0 : \mathcal{O}_0(\mathfrak{l}) \to \mathcal{O}_0(\mathfrak{g})
\]

parametrised by shortest right coset representatives \( w \) of the quotient of the Weyl
groups of \( \mathfrak{g} \) and \( \mathfrak{l} \).

Our main goal is to provide a geometric and graded version of the functors \( \text{Ind}_w \)
in the following sense:

Denote by \( X \) and \( Y \) the full flag varieties of the Langlands dual algebraic groups
corresponding to \( \mathfrak{g} \) and \( \mathfrak{l} \). The mixed geometry of these varieties is – via the homotopy
category of Soergel modules – intimately related to the derived category \( \mathcal{O}'s \)
of \( \mathfrak{g} \) and \( \mathfrak{l} \) (see [BS94], [Soe90] and [BGS96]). This is made most precise in [SW14].

Date: February, 2016.
2010 Mathematics Subject Classification. 17B10, 22E46.
Key words and phrases. Parabolic induction, Lie algebras, category \( \mathcal{O} \), Soergel modules, Koszul
 duality, mixed Tate motives.
which provides us with categories $\text{Der}^b_{(B)}(X)$ and $\text{Der}^b_{(A)}(Y)$ of so called stratified mixed Tate motives on $X$ and $Y$. They are equivalent to the categories $\text{Der}^b(\mathcal{O}_0^Z(g))$ and $\text{Der}^b(\mathcal{O}_0^Z(l))$ of $[\text{BGS96}]$. We denote the functor forgetting the grading by $v$.

The big advantage of mixed Tate motives is that they are equipped with a six-functor formalism. We can hence construct a family of functors $\{h_{w,*}, p^!_{w}\}$ by defining maps

$$Y \leftarrow p_w Y \times A^Z_\mathbb{C}^{(w)} \rightarrow h_{w,*} X.$$

Our main result is:

**Theorem 1.** The following diagram commutes (up to natural isomorphism)

$$\begin{align*}
\text{Der}^b_{(A)}(Y) & \xrightarrow{h_{w,*}} \text{Der}^b_{(B)}(X) \\
\text{Der}^b(\mathcal{O}_0(l)) & \xrightarrow{\text{Ind}_w} \text{Der}^b(\mathcal{O}_0(g)).
\end{align*}$$

The functors $h_{w,*}, p^!_{w}$ are hence the sought-after graded and geometric versions of the Ind$_w$.

1.2. Soergel Modules and Plan of Proof. In order to proof Theorem 1 we need to work through the definition of the degrading maps $v$ which involve as an intermediate step the homotopy category of (graded) Soergel modules. So let $C_I$ and $C$ be the cohomology rings of $Y$ and $X$. Denote by the prefix Proj the full subcategory of projective modules and by the index $w = 0$ the full subcategory and weight 0 complexes, see $[\text{SW14}]$. Let furthermore $V_I$ and $V$ be the Kombinatorik-functors of $\mathcal{O}_0(l)$ and $\mathcal{O}_0(g)$ (see $[\text{Soe90}]$) as well as $H_I$ and $H$ be the hypercohomology functors for $Y$ and $X$. If we expand the definition of the degrading functors $v$ we obtain the following diagram.

$$\begin{align*}
\text{Der}^b_{(A)}(Y) & \xrightarrow{h_{w,*}} \text{Der}^b_{(B)}(X) \\
\text{Hot}^b(\text{Der}^b_{(A)}(Y)_{w=0}) & \xrightarrow{\text{Ind}_w} \text{Hot}^b(\text{Der}^b_{(B)}(X)_{w=0}) \\
\text{Hot}^b(C_I - \text{SMod}^Z_{ev}) & \xrightarrow{V_I} \text{Hot}^b(C - \text{SMod}^Z_{ev}) \\
\text{Hot}^b(\text{Proj} \mathcal{O}_0(l)) & \xrightarrow{\text{Ind}_w} \text{Hot}^b(\text{Proj} \mathcal{O}_0(g)).
\end{align*}$$

Here SMod (and SMod$^Z_{ev}$) denotes the category of (evenly graded) Soergel modules and $v'$ the functor forgetting the grading.

To prove the commutativity of the diagram we will study the effect of $h_{w,*}, p^!_{w}$ and Ind$_w$ on the level of Soergel modules. Let $\text{Res}: C_I - \text{Mod}^Z \rightarrow C - \text{Mod}^Z$ be functor induced by the canonical embedding $Y \hookrightarrow X$ and the pullback map of cohomology. We will show
Theorem 2. Let \( w = s_1 \cdots s_n \) be a reduced expression of \( w \) such that also \( s_1 \cdots s_i \) is a shortest right coset representative for all \( i \). Then on the homotopy category of (evenly graded) Soergel modules both \( \text{Ind}_w \) and \( h_{w,*}p^i_w \) induce (up to natural isomorphism) the same functor

\[
R_{s_n} \otimes_C \cdots \otimes_C R_{s_1} \otimes_C \text{Res}(-),
\]

where \( R_{s_i} \) denotes the Rouquier complex of \( s_i \).

Theorem 2 immediately implies Theorem 1. We will prove it using induction on the length of \( w \).

1.3. Structure of the paper. Section 2 will be concerned with category \( \mathcal{O} \). We mainly analyse the interaction of parabolic induction and translation and wall-crossing functors.

Section 3 is about the mixed geometry. We define the maps \( p_w \) and study the interaction of \( h_{w,*}p^i_w \) with the geometric analogues \( \pi_i^{-1} \pi_i \) of wall-crossing functors.

In Section 4 we study the effect of \( \text{Ind}_w \) and \( h_{w,*}p^i_w \) on the homotopy category of Soergel modules and prove Theorem 1 and 2.

1.4. Acknowledgments. I would like to thank Wolfgang Soergel for many instructive discussions and posing the questions treated in this article. I was financially supported by the DFG Graduiertenkolleg 1821 “Cohomological Methods in Geometry”.

2. Representation Theory

We need to first fix a lot of notation which is mostly taken from [Hum08, Chapter 9.1.]. Let \( \mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h} \) be a semi-simple complex Lie algebra with a fixed Borel and Cartan subalgebra. Denote by \( \mathfrak{h}^* \supset \Phi \supset \Phi^+ \supset \Delta \) the corresponding root system, positive and simple roots and by \( \mathcal{W} \supset \mathcal{S} \) the associated Weyl group and simple reflections. Each subset \( I \subset \Delta \) defines a root system \( \Phi_I \subset \Phi \) with positive and negative roots \( \Phi_I^+ \) and \( \Phi_I^- \) and Weyl group \( \mathcal{W}_I \subset \mathcal{W} \) generated by simple reflections \( S_I = \{ s_\alpha | \alpha \in I \} \). It also defines a standard parabolic subalgebra

\[
\mathfrak{p} = \mathfrak{p}_I = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_I \cap \Phi^+} \mathfrak{g}_\alpha,
\]

with nilradical \( \mathfrak{u} \) and reductive Levi factor \( \mathfrak{l} = \mathfrak{l}_I \). Denote by \( [l, l] =: \mathfrak{g} \supset \mathfrak{b}_I \supset \mathfrak{h}_I \) the maximal semi-simple subalgebra of \( \mathfrak{l} \) with its Borel and Cartan subalgebra and by \( \mathfrak{z} \subset \mathfrak{h} \) the center of \( \mathfrak{l} \).

In the following assume all weights to be integral. An integral weight \( \lambda \) will be called \( \Delta \)-dominant if \( \langle \lambda + \rho, \alpha^* \rangle \in \mathbb{Z}_{\geq 0} \) for all \( \alpha \in \Delta \) and \( I \)-dominant if \( \langle \lambda + \rho, \alpha^* \rangle \in \mathbb{Z}_{\geq 0} \) for all \( \alpha \in I \). Here \( \rho \) is the sum of all fundamental weights.

We parametrise integral blocks of \( \mathcal{O}(\mathfrak{g}) \) and \( \mathcal{O}(\mathfrak{l}) \) by \( \Delta \)-dominant and \( I \)-dominant weights, respectively. For \( \lambda \in \mathfrak{h}^* \) denote by \( M_I(\lambda), L_I(\lambda), P_I(\lambda) \) and \( M(\lambda), L(\lambda), P(\lambda) \) the Verma, simple and indecomposable projective module corresponding to \( \lambda \) in \( \mathcal{O}(\mathfrak{l}) \) and \( \mathcal{O}(\mathfrak{g}) \).

Denote by \( \mathcal{U}(\mathfrak{g}) \) the universal enveloping algebra. The functor \( \text{Ind}^\mathfrak{g}_\mathfrak{p} \) of parabolic induction is defined by

\[
\text{Ind}^\mathfrak{g}_\mathfrak{p} : \mathfrak{l} - \text{Mod} \to \mathfrak{g} - \text{Mod}, \ M \mapsto \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} M.
\]

The goal of this section is to analyse the functorial properties of parabolic induction when restricted to particular blocks of \( \mathcal{O}(\mathfrak{l}) \) in Section 2.1 and to define the functors \( \text{Ind}_w \) as well as to study their interaction with translation and wall-crossing functors in Section 2.2.
2.1. Functorial properties of parabolic induction and blocks in \( \mathcal{O} \). Let us recapitulate some standard facts about parabolic induction.

**Lemma 3.** 1.) \( \text{Ind}_p^\emptyset \) has a right adjoint given by \( M \mapsto M^\emptyset = \{ v \in M | uv = 0 \} \).
2.) \( \text{Ind}_p^\emptyset \) is exact and restricts to a functor

\[ \text{Ind}_p^\emptyset : \mathcal{O}(l) \to \mathcal{O}(g). \]
3.) For all \( \lambda \in \mathfrak{h}^* \) we have \( \text{Ind}_p^\emptyset(M_I(\lambda)) = M(\lambda) \).

Now let us study how \( \text{Ind}_p^\emptyset \) acts on the blocks of \( \mathcal{O} \). Denote by \( \mathcal{W}_f \) the set of shortest right coset representatives of \( \mathcal{W}_f \setminus \mathcal{W} \).

**Lemma 4.** Let \( \lambda \in \mathfrak{h}^* \) be \( \Delta \)-dominant and regular. Then:
1.) We have

\[ (\text{Ind}_p^\emptyset)^{-1}(\mathcal{O}_\lambda(g)) = \bigoplus_{w \in \mathcal{W}_f} \mathcal{O}_w(\lambda)(l). \]
2.) \( \text{Ind}_p^\emptyset|_{\mathcal{O}_\lambda(l)} \) maps projective to projective modules if and only if \( w = e \).
3.) Also \( \text{Ind}_p^\emptyset|_{\mathcal{O}_\lambda(l)} \) is fully faithful and \( \text{Ind}_p^\emptyset P_I(x \cdot \lambda) = P(x \cdot \lambda) \) for all \( x \in \mathcal{W}_f \).

**Proof.** 1.) The \( I \)-dominant weights in the orbit \( W \cdot \lambda \) are precisely the ones of the form \( w \cdot \lambda \) for \( w \in \mathcal{W}_f \).
2.) The only if part is clear since \( \text{Ind}_p^\emptyset M_I(w \cdot \lambda) = M(w \cdot \lambda) \) is only projective if \( w = e \). For the other direction we check that \( \text{Ind}_p^\emptyset|_{\mathcal{O}_\lambda(l)} \) has an exact right adjoint functor. Denote by \( \text{pr}_\lambda \) the projection onto the block \( \mathcal{O}_\lambda(l) \). Then \( \text{pr}_\lambda(-^\emptyset) \) is right adjoint to \( \text{Ind}_p^\emptyset|_{\mathcal{O}_\lambda(l)} \) and exact, see [SS15, Lemma 5.9, 5.11].
3.) By [SS15, Lemma 5.10] \( \text{pr}_\lambda(\text{Ind}_p^\emptyset|_{\mathcal{O}_\lambda(l)}(-)^\emptyset) \cong \text{id}_{\mathcal{O}_\lambda(l)} \). Hence \( \text{Ind}_p^\emptyset|_{\mathcal{O}_\lambda(l)} \) is fully faithful and maps indecomposable projectives to such. \( \square \)

2.2. \( \text{Ind}_w \) and translation functors. For an integral weight \( \nu \) denote by \( \tau \) and \( \tau_f \) the unique weight in the \( \mathcal{W} \) and \( \mathcal{W}_f \)-orbit (with respect to the regular action) of \( \nu \) with \( (\tau, \alpha') \in \mathbb{Z}_{>0} \) for all \( \alpha \in \Delta \) and \( (\tau_f, \alpha') \in \mathbb{Z}_{>0} \) for all \( \alpha \in I \).

Let \( \lambda \) and \( \mu \) be some \( \Delta \)-dominant or \( I \)-dominant weights and \( \nu := \mu - \lambda \). Then the corresponding translation functors are given by

\[
T^-\lambda : \mathcal{O}_\lambda(l) \to \mathcal{O}_\mu(g), \quad M \mapsto \text{pr}_\mu(M \otimes L_I(\tau_f)),
\]
\[
T^\mu : \mathcal{O}_\mu(g) \to \mathcal{O}_\lambda(l), \quad M \mapsto \text{pr}_\mu(M \otimes L(\tau_f)),
\]
where \( \text{pr}_\mu \) denotes the projection to the corresponding block in \( \mathcal{O} \).

For \( w \in \mathcal{W}_f \) we define our main object of study:

\[
\text{Ind}_w := \text{Ind}_p^\emptyset T_w^0 : \mathcal{O}_0(l) \to \mathcal{O}_0(g).
\]

So \( \text{Ind}_w \) maps \( M_I(x_0) \) to \( M(wx_0) \). Remark will justify this definition by showing that the action of \( \text{Ind}_p^\emptyset \) on all regular integral blocks can be expressed in terms of the \( \text{Ind}_w \) which have the advantage of involving the principal blocks only.

The goal of the rest of the section is to understand the interaction of translation and wall-crossing functors and the \( \text{Ind}_w \). We start off our investigation with two technical lemmata.

**Lemma 5.** Let \( M \) be a \( \mathfrak{t} \)-module and \( E \) a finite dimensional \( \mathfrak{g} \)-module. Let \( \nu_1, \ldots, \nu_n \) be the weights of \( \mathfrak{g} \) on \( E \), ordered in a way that \( \nu_i \leq \nu_j \) (i.e. \( \nu_j - \nu_i, \alpha' \geq 0 \) for all \( \alpha \in \Delta \)) implies \( i \leq j \). Then \( \text{Ind}_p^\emptyset(M) \otimes E \) has a filtration

\[
\{ 0 \} = N_{n+1} \subset N_n \subset \cdots \subset N_1 = \text{Ind}_p^\emptyset(M) \otimes E
\]
with subquotients \( N_i/N_{i+1} \cong \text{Ind}_p^\emptyset(M \otimes E_{\nu_i}) \).
Lemma 6. Using the exactness of all involved functors we see that for general a filtration with subquotients of the form 

\[ w \]

furthermore the weights \( \nu \) contained in block \( x \).

Proof. The tensor identity yields 

\[ (\text{Ind}^g M) \otimes E = (U(g) \otimes_{U(p)} M) \otimes E \cong U(g) \otimes_{U(p)} (M \otimes E). \]

Now set \( M_i := \sum_{j=1}^n M \otimes E_{\nu_j}; \) this is clearly a \( t \)-submodule of \( M \otimes E \). Since 

furthermore the weights \( \nu_i \) are ordered in an ascending way, \( M_i \) is also a \( u \) and hence a \( p \)-submodule. The \( M_i \) define a filtration of \( M \otimes E \) as a \( p \)-module with subquotients \( M_i/M_{i+1} \cong M \otimes E_{\nu_{i+1}} \).

Let \( N_i := U(g) \otimes_{U(p)} M_i \). Using the exactness of parabolic induction and the tensor identity we see that the \( N_i \) define a filtration with the desired property. \( \square \)

Lemma 7. Let \( \nu \in \mathfrak{h}^* \) be some integral weight and \( \nu' := \nu|_1 \). Then, as \( l \)-module, \( L_I(\mathfrak{g}_I) \) appears with multiplicity one as direct summand of \( L(\mathfrak{g}_I) \).

Proof. Let \( w \in W \) with \( \mathfrak{g} = w(\nu) \). Write \( w = xy \) with \( x \) a shortest left coset representative of \( W/W_I \) and \( y \in W_I \). Then one easily proofs \( y(\nu) = \mathfrak{g}_I \) and \( xy(\nu) = \mathfrak{g} \).

Now choose some non-zero \( v^+ \in L(\mathfrak{g}_I) \). Then \( v^+ \) is a highest weight vector for \( l \) since for \( \alpha \in \Phi^+ \)

\[ \dim_{\mathbb{C}} L(\mathfrak{g}(\nu)+\alpha) = \dim_{\mathbb{C}} L(\mathfrak{g}(\nu)+\alpha) = 0 \]

because \( x(\alpha) \in \Phi^+ \) and all weights of \( L(\mathfrak{g}) \) are in \( \mathfrak{g} - \mathbb{Z}_{>0}\Phi^+ \). So indeed \( v^+ \) generates \( L_I(\mathfrak{g}_I) \) as \( l \)-module and \( q \) acts on it via \( \nu' = \nu|_1 = y(\nu)|_1 \). The multiplicity one statement follows from \( \dim_{\mathbb{C}} L(\mathfrak{g}_I) = 1 \). \( \square \)

We can now analyse what translating from the principal block to another integral (but not necessarily regular) block does to \( \text{Ind}^g_m \).

Lemma 7. Let \( M \in \mathcal{O}_0(1) \) and \( \mu \) some \( \Delta \)-dominant integral weight. Then there exists a natural isomorphism 

\[ T_0^m \text{Ind}^g M \cong \text{Ind}_0^g T_0^m M. \]

Proof. First observe that the statement is true for Verma modules and the exactness of all involved functors implies equality of characters of the left and right hand side of the equation. The exactness furthermore allows us to assume that \( M \) has a Verma flag since it can be resolved by modules with a Verma flag.

For abbreviation denote 

\[ N := \text{Ind}^g \text{Ind}^g T_0^m M \otimes L(\mathfrak{g}) = \text{Ind}_0^g T_0^m M \otimes L(\mathfrak{g}). \]

By applying Lemma 5 to \( \text{Ind}^g M \) and \( L(\mathfrak{g}) \) we see that, for suitable \( \nu \in \mathfrak{g}^* \), \( N \) has a filtration with subquotients of the form 

\[ \text{Ind}_0^g (T_0^m M \otimes L(\mathfrak{g})_{\nu}). \]

We will show that \( \text{pr}_m \) kills all subquotients of \( N \) except for the one with \( \nu_1 = w \cdot \mu|_1 \).

For this shortly assume that \( M = M_I(xw \cdot 0) \) for some \( x \in W_I \) and let \( \lambda \) be a weight of \( L(\mathfrak{g}) \). Then the only subquotient in the Verma flag of 

\[ N = \text{Ind}^g(M \otimes L(\mathfrak{g})) = M(xw \cdot 0) \otimes L(\mathfrak{g}) \]

contained in block \( \mathcal{O}_\nu(g) \) is of the form \( M(xw \cdot 0 + \lambda) = M(xw \cdot 0) \) by \[ \text{Jan79}, \text{Satz} \]

2.10] or \[ \text{Hum98}, \text{Lemma 7.5 and Theorem 7.6}]. To not be killed by \( \text{pr}_m \) hence \( \lambda \) has to be

\[ \lambda = xw \cdot \mu - xw \cdot 0 = xw(\mu) \] and therefore 

\[ \lambda|_3 = (xw \cdot \mu - xw \cdot 0)|_3 = xw(\mu)|_3 = w(\mu)|_3. \]

Using the exactness of all involved functors we see that for general \( M \) with a Verma flag 

\[ \text{pr}_m N = \text{pr}_m \left( \text{Ind}_0^g (T_0^m M \otimes L(\mathfrak{g})_{\nu_1}) \right). \]
Lemma 6 now ensures that, as l-module, \( L_I(w(\mu)) \) appears as a direct summand of \( L(\overline{\mu}) \). Hence
\[
\text{pr}_\mu N \supset \text{pr}_\mu \text{Ind}_p^O \left( T_0^{-w(\mu)} M \otimes L_I(w(\mu)) \right).
\]
By comparing characters we see that this is actually an equality, and that the right hand side equals
\[
\text{Ind}_p^O T_w^{-w(\mu)} M \cong \text{Ind}_p^O T_0^{-w(\mu)} M.
\]
The statement follows (the isomorphism is induced by the tensor identity and hence indeed natural).
\[\square\]

As a first application Lemma 7 can be used to express the effect of \( \text{Ind}_p^O \) on all regular integral blocks of \( \text{O}(l) \) in terms of the \( \text{Ind}_w \) and translation functors.

**Remark 1.** Each regular integral block of \( \text{O}(l) \) is of the form \( \text{O}_{w, \mu}(l) \) for a \( \Delta \)-dominant \( \mu \) and \( w \in \mathcal{W}^l \). Restricted to \( \text{O}_{w, \mu}(l) \) we have the following equivalence of functors
\[
\text{Ind}_p^O \cong \text{Ind}_p^O T_0^{-w(\mu)} M \cong \text{Ind}_w T_0^- M
\]
where we use \( T_0^{-w(\mu)} T_0^0 M \cong \text{Id} \) and Lemma 7.

Lemma 8 also serves as first step in analysing the interaction of wall-crossing functors with the \( \text{Ind}_w \). At first we study the case of «a wall outside of \( \mathfrak{b}_w^\circ \)» which will be used as induction step in the proof of the main theorem.

**Lemma 8.** Let \( w \in \mathcal{W}^l \) and \( s \in \mathcal{W} \) a simple reflection with \( ws > w \) and \( ws \in \mathcal{W}^l \). Denote by \( \theta_s \) a wall-crossing functor through the \( s \)-wall. Namely choose some \( \Delta \)-dominant weight \( \mu \) with stabilizer \( \mathcal{W}_\mu = \{1, s\} \) and put \( \theta_s = T_0^\mu T_0^- M \). Then for all \( M \in \text{O}_0(l) \) there is a short exact sequence, natural in \( M \),
\[
0 \longrightarrow \text{Ind}_w M \longrightarrow \theta_s \text{Ind}_w M \longrightarrow \text{Ind}_{ws} M \longrightarrow 0
\]
where the first morphism is the unit of the adjunction between \( T_0^\mu \) and \( T_0^- \).

**Proof.** This works analogously to the proof of Lemma 7 by carefully analysing \( T_0^\mu T_0^- \text{Ind}_w M \). The statement is again true for Verma modules and on the level of characters and we can assume that \( M \) has a Verma flag. For abbreviation denote
\[
N := T_0^\mu \text{Ind}_w M \otimes L(\overline{\mu}).
\]
By applying Lemma 5 to \( T_0^\mu \text{Ind}_w M \cong \text{Ind}_p^O T_0^{-w(\mu)} M \) and \( L(\overline{\mu}) \) we see that, for suitable \( \nu_i \in \mathfrak{z}^\circ \), \( N \) has a filtration with subquotients of the form
\[
\text{Ind}_p^O \left( T_0^{-w(\mu)} M \otimes L(\overline{\mu})_{\nu_i} \right).
\]
We will show that \( \text{pr}_0 \) kills all subquotients of \( N \) except for the ones with either
\[
\begin{align*}
(\text{a}) & \quad \nu_i = w(\overline{\mu})_1 \\
(\text{b}) & \quad \nu_i = w(s \cdot 0 - \mu)_1.
\end{align*}
\]
For this shortly assume that \( M = M_f(x \cdot 0) \) for some \( x \in \mathcal{W}_l \) and let \( \lambda \) be a weight of \( L(\overline{\mu}) \). Then the only subquotients in the Verma flag of
\[
N = T_0^\mu \text{Ind}_w M \otimes E(\overline{\mu}) = M(xw \cdot \mu) \otimes E(\overline{\mu})
\]
contained in \( \text{O}_0(\mathfrak{g}) \) are by [Hum08, Theorem 7.14] of the form
\[
\begin{align*}
(\text{a}) & \quad M(xw \cdot \mu + \lambda) = M(xw \cdot 0) \\
(\text{b}) & \quad M(xw \cdot \mu + \lambda) = M(xws \cdot 0).
\end{align*}
\]
Hence, to not be killed by $\text{pr}_0$, $\lambda$ has to be of the form

$$(a) \lambda = xw \cdot 0 - xw \cdot \mu = xw(-\mu)$$
and therefore

$$\lambda|_3 = (xw \cdot 0 - xw \cdot \mu)|_3 = xw(-\mu)|_3 = w(-\mu)|_3$$
or

$$(b) \lambda = xws \cdot 0 - xw \cdot \mu = xw(s \cdot 0 - \mu)$$
and therefore

$$\lambda|_3 = (xw \cdot 0 - xws \cdot \mu)|_3 = xw(s \cdot 0 - \mu)|_3 = w(s \cdot 0 - \mu)|_3.$$ 

Abbreviate $\nu_a := w(-\mu)|_3$ and $\nu_b := w(s \cdot 0 - \mu)|_3$ (Notice: they are independent of $x$). Then $\nu_a > \nu_b$ since $w(-\mu) - w(s \cdot 0 - \mu) = w(\alpha_0) \in \Phi^+ \setminus \Phi \mu$ which is easily proven using the assumptions on $w$ and $s$.

Now return to the case of a general $M$ with Verma flag. The above arguments show that the only subquotients of $N$ not entirely killed by $\text{pr}_0$ are

$$(a) \quad N_a := \text{Ind}_\mathcal{H}^\mathcal{G} (T_0^{w \cdot \mu} M \otimes L(-\mu)_{\nu_a})$$
and

$$(b) \quad N_b := \text{Ind}_\mathcal{H}^\mathcal{G} (T_0^{w \cdot \mu} M \otimes L(-\mu)_{\nu_b})$$
and that $N_a$ appears below $N_b$ as subquotients in the filtration of $N$. Hence there is a unique direct summand $\tilde{N}$ of $N$ fitting a short exact sequence

$$0 \rightarrow N_a \rightarrow \tilde{N} \rightarrow N_b \rightarrow 0$$

such that $\text{pr}_0(N) = \text{pr}_0(\tilde{N})$. Therefore we now have to show $\text{pr}_0(N_a) = \text{Ind}_w M$ and $\text{pr}_0(N_b) = \text{Ind}_{w_3} M$. We only do this in the case (b) with $\nu_b = w(s \cdot 0 - \mu)|_3$.

Use [Jantzen Satz 2.9]{#fn1} to see that $w(s \cdot 0 - \mu)$ is a $\mathcal{W}$-conjugate of $-\mu$ and hence $w(s \cdot 0 - \mu) = -\mu$. Apply Lemma 6 to $w(s \cdot 0 - \mu)$ which implies that $-\mu$ as $\mathcal{L}$-module $L_I(\nu_{b I})$ appears as a direct summand of $L(-\mu)_{\nu_b}$. Hence

$$\text{pr}_\mu N_b \supset \text{pr}_\mu \text{Ind}_\mathcal{H}^\mathcal{G} (T_0^{w \cdot \mu} M \otimes L_I(\nu_{b I})).$$

By comparing characters we see that this is actually an equality and that the right hand side equals

$$\text{Ind}_\mathcal{H}^\mathcal{G} T_0^{w \cdot \mu} T_0^{w \cdot 0} M \cong \text{Ind}_\mathcal{H}^\mathcal{G} T_0^{w \cdot 0} M = \text{Ind}_{w_3} M.$$ 

The same proof works for $N_a$ and it is not hard to see that the embedding of $N_a$ in $\tilde{N}$ can be chosen to be the unit of the adjunction. The statement follows. \hfill $\square$

Now we come to the second important case of crossing «a wall inside of $\mathfrak{h}_T^*$». This will be used for the base case concerning $\text{Ind}_c$ of the induction proving the main theorem.

**Lemma 9.** Let $M \in \mathcal{O}_0(I)$ and $\mu \in \mathfrak{h}_T^*$ a $I$-dominant integral weight with $\mathcal{W}_{I, \mu} = \{1, s\}$ for some simple reflection $s \in \mathcal{W}_I$. Denote by abuse of notation by $\theta_s = T_0^s T_0^\mu$ the wall-crossing functor for both $\mathcal{O}_0(I)$ and $\mathcal{O}_0(g)$. Then there is a natural isomorphism

$$\theta_s \text{Ind}_c M \cong \text{Ind}_c \theta_{s} M.$$ 

**Proof.** Everything works the same as in the proof of Lemma 8 for $w = e$ up to and including the point where $\nu_a = w(-\mu)|_3 = -\mu|_3$ and $\nu_b = w(s \cdot 0 - \mu)|_3 = s \cdot 0 - \mu|_3$ are defined. But this time we do not have $\nu_a > \nu_b$ but $\nu_a = \nu_b = 0$ since both $\mu$ and $s \cdot 0$ are in $\mathfrak{h}_I$. Hence

$$T_0^\mu T_0^s \text{Ind}_c M = \text{pr}_0(\text{Ind}_\mathcal{H}^\mathcal{G} (T_0^{w \cdot \mu} M \otimes L(-\mu)_{\nu_0})).$$

Applying Lemma 6 to $-\mu$ shows that $L_I(-\mu_I)$ appears as a direct summand of $L(-\mu_I)$ and again a character argument yields the statement. \hfill $\square$
3. Mixed Geometry

Let $G \supset B \supset T$ the Langlands dual algebraic group, Borel subgroup and maximal torus corresponding to $g \supset b \supset h$ (we omit the $(-)^*$ since we only consider the Langlands dual). Denote by $P \subset G$ the standard parabolic corresponding to $p$, its unipotent radical by $U$ and its Levi factor by $L$. Let $A = B \cap L$ be the Borel subgroup of $L$. For $\alpha \in \Phi$ denote by $U_\alpha \subseteq G$ the unipotent one-parameter subgroup with Lie algebra $g_\alpha$.

Using the formalism of [CD09, SW14] defines categories of stratified mixed Tate motives (with respect to the Bruhat stratification) on $X := G/B$ and $Y := L/A$. Denote them by $\text{Der}^{b,\mathbb{Z}}_B(X)$ and $\text{Der}^{b,\mathbb{Z}}_A(Y)$, respectively. For a motive $\mathcal{F}$ let $\mathcal{F}((n)[m])$ be its $n$-th Tate twist and $m$-th shift. Denote by $w$ the weight structure on $\text{Der}^{b,\mathbb{Z}}_B(X)$ and $\text{Der}^{b,\mathbb{Z}}_A(Y)$, see [SW14, Section 5], and by the index $w = 0$ their full subcategories of weight 0 complexes. For a brief description of the six-functor formalism those categories are equipped with see [SW14, Remark 2.4]. Just think about them as categories of constructible (with respect to the Bruhat stratification) sheaves with an additional Tate twist and a weight structure.

The goal of this section is to construct a geometric analogue of the functors $\text{Ind}_{w}$. In Section 3.1 we start off by defining for all $w \in \mathcal{W}$ maps

$$Y \xleftarrow{p_w} Y \times \mathbb{A}^{l(w)} \xrightarrow{h_w} X$$

under which the Bruhat cell $AxA/A$ in $Y$ corresponds to the Bruhat cell $BwxB/B$ in $X$. We also analyse the functorial properties of the functors $h_{w,*}p_w^!$. In Section 3.2 we show how the $h_{w,*}p_w^!$ are related to one another.

3.1. Construction and functorial properties of $h_{w,*}p_w^!$. Denote by $U$ and $U^-$ the unipotent radical of $B$ and $B^-$, where by $B^-$ we denote the opposite Borel. For $x \in \mathcal{W}$ define

$$U_x := U \cap xU^-x^{-1} = \langle U_\alpha | \alpha \in x(\Phi^-) \cap \Phi^+ \rangle \subset B.$$

Let $w \in \mathcal{W}$. Then the following statements hold.

Lemma 10. 1.) As variety $U_w \cong \mathbb{A}^{l(w)}$.
2.) As algebraic groups $LU_w \cong L \times U_w$.
3.) As variety $LU_w/A \cong L/A \times U_w$.

Proof. Omitted. See [Hum98, Chapter 28] (There $U_w$ is denoted by $U'_w$). \qedsymbol

Define

$$h_w: LU_w/A \rightarrow X, \ gA/A \mapsto gwB/B \quad \text{and} \quad p_w: LU_w/A \cong Y \times U_w \rightarrow Y, \ (gB/B, u) \mapsto g/B.$$

Those maps fulfill the desired properties.

Lemma 11. The map $h_w$ is well-defined and injective. Furthermore

$$h_w(p_w^{-1}(AxA/A)) = BwxB/B \subset X$$

for all $x \in \mathcal{W}$.

Proof. Follows from standard $BN$-pair arguments. \qedsymbol

This Lemma also ensures that $h_{w,*}p_w^!$ really induces a functor

$$h_{w,*}p_w^!: \text{Der}^{b,\mathbb{Z}}_A(Y) \rightarrow \text{Der}^{b,\mathbb{Z}}_B(X),$$

i.e. maps mixed stratified Tate motives with respect to the Bruhat stratification as defined in [SW14, Section 4] to such. Now we can come to the properties of the $h_{w,*}p_w^!$. 
Lemma 12. 1.) Let $x \in W_I$. Then $h_w, p_w^! (\mathbb{D}_X A x A / A) = \mathbb{D}_X B x w B / B$, where $A x A / A$ and $B x w B / B$ denote the constant motives on the Bruhat cells extended by zero and $\mathbb{D}$ the Verdier duality functor.

2.) Let $w = e$, then $h_w, p_w^! = h_e, p_e^!$ is weight exact and restricts to a fully faithful functor $\text{Der}^b_{(A)} (Y)_{w=0} \rightarrow \text{Der}^b_{(B)} (X)_{w=0}$.

Proof. 1.) Follows from Lemma 11.

2.) In this case $h_e : Y \hookrightarrow X$ is a proper embedding. Hence $h_x, p_x^!$ is weight left and right exact (see [SW14, Theorem 2.13, Remark 2.4]). \[ \square \]

3.2. A Gysin Triangle. The geometric analogue of the wall-crossing functor $\theta_s$ is $\pi_s^! \pi_s^*$ for $\pi_s : G / B \rightarrow G / P_s$, where $P_s = B \cup B s B$ denotes the minimal parabolic for $s \in S$. In this section we will hence study the interaction of the $h_w, p_w^!$ and $\pi_s^! \pi_s^*$.

We start with a statement similar to Lemma 8 which will serve as an argument in the induction step of the proof of our main theorem.

Lemma 13. Let $w \in W_I$, $s \in S$ with $w s > w$ and $w s \in W_I$. Then for $F \in \text{Der}^b_{(A)} (Y)$ there exists a distinguished triangle

$$ h_w, p_w^! F \rightarrow \pi_s^! \pi_s^* h_w, p_w^! F \rightarrow h_w, p_w^! h_{w s}, p_{w s}^! F \rightarrow $$

where the first morphism is the unit of the adjunction.

Proof. Abbreviate $\pi := \pi_s$. Consider the following diagram where the square is cartesian and both triangles commute.

$$\begin{array}{ccc}
LU_w / A & \xrightarrow{id} & LU_w / A \\
\downarrow h_w & & \downarrow \pi h_w \\
X & \xrightarrow{k} & G / P_s \\
\downarrow k & & \\
Z & \xrightarrow{q} & LU_w / A
\end{array}$$

Here $Z$ is defined to be the pullback of the square. Since $w s > w$, $\pi h$ is injective we have

$$ Z \cong \pi^{-1} (\pi h_w (LU_w / A)) = h_{w s} (LU_{w s} / A) \oplus h_w (LU_w / A), $$

where $h_{w s} (LU_{w s} / A)$ is open and $h_w (LU_w / A)$ is closed in $Z$. The inclusions are denoted by $i : LU_w / A \hookrightarrow Z \hookleftarrow LU_{w s} / A : j$.

Now we apply $h_w, \pi$ to the unit of the adjunction $(\pi, \pi^!)$ and transform it.

$$\begin{array}{ccc}
h_{w, \pi} & \rightarrow \pi^! \pi h_w, \pi \\
\downarrow k_s i & & \downarrow k_s q^! \\
k_s i & \rightarrow k_s q^!
\end{array}$$

Here we used base change for the upper right equality, $i q = \text{id}$ and $i = i_s$ as well as $\pi = \pi_s$ since both are proper. We claim that the lower morphism is induced by the counit of the adjunction $(i_i, i^!)$.

By [SW14, Theorem 11.3] it suffices to show this in the category of constructible sheaves, where it follows from an explicit
computation. It can also be shown using the formalism of fibred categories. Using the Gysin/Localisation triangle we hence obtain the distinguished triangle

\[ h_{w, *} \longrightarrow \pi_! h_{w, *} \longrightarrow k_! j_! q_! \overset{+1}{\longrightarrow} \]

Now apply this triangle to \( p_w^! \mathcal{F} \) and use \( k_! j_! q_! \) to get

\[ h_{w, *}, \pi_! \]

\[ \cong \]

\[ h_{w, *}, \pi_! \]

which can be easily seen. □

The following statement concerns in parallel to Lemma 9 the case of \( s \in S_I \) and will too be used in the base case of the induction.

**Lemma 14.** Let \( s \in S_I \) and denote \( \kappa_s : Y \to L/Q_s \) for the minimal parabolic \( Q_s = A \cup AsA \). Then for \( F \in \text{Der}^{b}_{(A)}(Y) \) there is a natural isomorphism

\[ \pi_! \kappa_s, \pi_! h_{e, *} \cong \pi_! h_{e, *}, \kappa_s, \pi_! \]

\[ \text{Der}^{b}_{(A)}(Y) \to \text{Der}^{b}_{(B)}(X) \]

**Proof.** The following diagram is cartesian.

\[ \begin{array}{ccc}
Y & \xrightarrow{h_s} & X \\
\kappa_s \downarrow & & \downarrow \pi_s \\
L/Q_s & \xrightarrow{\kappa_s} & G/P_s
\end{array} \]

By the commutativity of the diagram we have

\[ \pi_! \pi_! h_{e, *} \cong \pi_! h_{e, *}, \kappa_s, \pi_! \]

and by base change

\[ \cong h_{e, *}, \kappa_s, \pi_! \]

Now use \( \pi_! \cong \pi_! \) and \( \kappa_s, \pi_! \cong \kappa_s, \pi_! \) since both \( \pi_! \) and \( \kappa_s \) are proper. □

4. **Soergel Modules and Proof**

We need to introduce some notation mostly taken from [Soe90] and [SW14]. Let \( C \) and \( C_I \) be the cohomology rings of \( X \) and \( Y \). Denote by \( P \) and \( P_I \) the antidominant projectives in \( \mathcal{O}_0(g) \) and \( \mathcal{O}_0(l) \), respectively, and let

\[ \mathbb{V} = \text{Hom}_g(P, -) : \mathcal{O}_0(g) \to C - \text{Mod} \]

\[ \mathbb{V}_I = \text{Hom}_l(P_I, -) : \mathcal{O}_0(l) \to C_I - \text{Mod} \]

be the \textit{Kombinatorik} functors. Denote by

\[ \mathbb{H} = \text{fin}, \mathbb{H}_g(X, -) : \text{Der}^{b}_{(B)}(X) \to C - \text{Mod} \]

\[ \mathbb{H}_I = \text{fin}, \mathbb{H}_l(Y, -) : \text{Der}^{b}_{(A)}(Y) \to C_I - \text{Mod} \]

the hypercohomology functors of \( X \) and \( Y \). Here \( \text{fin} \) denotes the projection to the point and \( \mathbb{H} \) is the internal \textit{Hom}-motive (see [SW14, Definition 8.2]). The category of (evenly graded) Soergel modules denoted by \( \text{SMod} \) and \( \text{S Mod}^{ev} \) is defined as the essential image of all projectives and all weight zero complexes under the \textit{Kombinatorik} and hypercohomology functors.

As in the introduction let \( \text{Res} : C_I - \text{Mod} \to C - \text{Mod} \) be functor induced by \( h_* : X_I \to X \) and the pullback map of cohomology. For \( s \in S \) there is a complex of Soergel bimodules

\[ R_s : \cdots \longrightarrow C \longrightarrow C \otimes_{C^*} C(2) \longrightarrow \cdots \]
called the Rouquier complex. Here \((-\cdot)\) denotes the degree shift in \(C - \text{SMod}^Z\) and \(C\) sits in cohomological degree \(-1\). The map is induced by the unit of the adjunction coming from the Frobenius extension \(C^e \subset C\). The complex acts by tensoring on the homotopy category of (evenly graded) Soergel modules. The goal of this section is the proof of

**Theorem 2.** Let \(w \in \mathcal{W}_I\) and \(w = s_1 \cdots s_n\) a reduced expression such that \(s_1 \cdots s_i \in \mathcal{W}_I\) for all \(i\). Then on the homotopy category of (evenly graded) Soergel modules both \(\text{Ind}_w\) and \(h_{w_*,p_w}^\land\) induce (up to natural isomorphism) the same functor

\[
R_{s_n} \otimes_C \cdots \otimes_C R_{s_1} \otimes_C \text{Res}(-).
\]

which finally amounts to our Theorem 1. The proof will work by induction on the length of \(w\). We deal with the base case in Section 4.1 and the induction step in Section 4.2.

### 4.1. The case \(w = e\)

The adjunction \((h_e^*, h_{e,*})\) immediately yields

**Lemma 15.** There is an equivalence of functors

\[
\mathbb{H} h_{e,*} \cong \text{Res} H_I : \text{Der}^b_{(A)}(Y)_{w=0} \to C - \text{SMod}^Z_{ev}.
\]

**Lemma 16.** Let \(s \in S_I\). Then we have an equivalence of functors

\[
\text{Res}(C_I \otimes C_I -) \cong C \otimes C : \text{Res}(-) : C_I - \text{SMod}^Z_{ev} \to C - \text{SMod}^Z_{ev}.
\]

**Proof.** We have the following chain of equivalences of functors \(\text{Der}^b_{(A)}(Y)_{w=0} \to C - \text{SMod}^Z_{ev}:

\[
C \otimes C : \text{Res} H_I \cong C \otimes C : \mathbb{H} h_{e,*} \quad (\text{Lemma 15})
\cong \mathbb{H} \pi_s^* \pi_s h_{e,*} \quad (\text{Soe90, Korollar 2})
\cong \mathbb{H} h_{e,*} \nu_s^* \nu_s^* \quad (\text{Lemma 14})
\cong \text{Res} H_I \nu_s^* \nu_s^* \quad (\text{Lemma 15})
\cong \text{Res} C_I \otimes C_I H_I \quad (\text{Soe90, Korollar 2}).
\]

Now use that \(H_I\) induces an equivalence of categories. \(\square\)

We want to analyse how \(\text{Ind}_e\) interacts with the functors \(\mathbb{V}_I\) and \(\mathbb{V}\).

**Lemma 17.** There is an equivalence of functors

\[
\mathbb{V} \text{Ind}_e \cong \text{Res} \mathbb{V}_I : \text{Proj} O_0(l) \to C - \text{SMod}^Z_{ev}.
\]

**Proof.** One easily sees \(\mathbb{V} \text{Ind}_e M_I(0) = C = \text{Res} \mathbb{V}_I M_I(0)\). Now \(\text{Proj} O_0(l)\) is the Karoubian envelope of its full additive subcategory of projectives of the form \(\theta_{s_n} \cdots \theta_{s_1} M_I(0)\) where \(s_1, \ldots, s_n \in S_I\) and \(s_1 \cdots s_n \in \mathcal{W}_I\) is a reduced expression. For modules of this form we have

\[
\mathbb{V} \text{Ind}_e(\theta_{s_n} \cdots \theta_{s_1} M_I(0)) \cong \mathbb{V} \theta_{s_n} \cdots \theta_{s_1} M(0) \quad (\text{Lemma 9})
\cong C \otimes C^* \cdots \otimes C^* : C \quad (\text{Soe90, Korollar 1})
\cong \text{Res}(C_I \otimes C_I^* \cdots \otimes C_I^*) : C \quad (\text{Lemma 16})
\cong \text{Res} \mathbb{V}_I(\theta_{s_n} \cdots \theta_{s_1} M_I(0)) \quad (\text{Soe90, Korollar 1})
\]

where all isomorphisms are natural. The statement follows for general projectives by the properties of Karoubian envelopes. \(\square\)

Since \(\text{Ind}_e\) maps projectives to projectives and \(h_{e,*}\) is weight exact, both act on the homotopy category of projectives and weight 0 complexes by pointwise application. Hence Lemma 15 and 17 imply Theorem 1 and 2 in the case \(w = e\).
4.2. Induction Step. Assume that we already constructed the equivalence for a fixed \( w \in \mathcal{W} \) with reduced expression \( w = s_1 \cdots s_n \) such that \( s_1, \ldots, s_i \in \mathcal{W} \) for all \( i \). Denote \( R_w := R_{s_n} \otimes_C \cdots \otimes_C R_{s_1} \). Now let \( s \in S \) with \( ws > w \) and \( ws \in \mathcal{W} \). Denote \( R_{ws} := R_w \otimes_C R_s \).

In the following Lemmata we will need functorial mapping cones. In general this is not possible and is equivalent to the uniqueness of morphisms completing a certain diagram of distinguished triangles, c.f. [GM03, IV.7. A Cone]. The following Lemma gives a sufficient condition for this uniqueness.

**Lemma 18.** Let \( C \) be a triangulated category. Then for any commutative diagram of distinguished triangles of the form

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow & & \downarrow \\
X' & \rightarrow & Y'
\end{array}
\]

\[
\begin{array}{ccc}
& & +1 \\
& \uparrow f & \\
& & +1
\end{array}
\]

there is a \( f \in \text{Hom}_C(Z, Z') \) completing the diagram to a morphism of triangles. If \( \text{Hom}_C(X[1], Z') = 0 \) then \( f \) is unique.

**Proof.** For example [GM03, IV.1 Corollary 5]. \( \square \)

**Lemma 19.** The following diagram commutes (up to a natural transformation).

\[
\begin{array}{ccc}
\text{Hot}^b(C_I \rightarrow \text{SMod}) & \rightarrow & \text{Hot}^b(C \rightarrow \text{SMod}) \\
\downarrow V_I & & \downarrow V \\
\text{Hot}^b(\text{Proj} O_0(0)) & \rightarrow & \text{Hot}^b(\text{Proj} O_0(\mathfrak{g}))
\end{array}
\]

\[
\begin{array}{ccc}
\bigtriangleup \text{Ind}^w M & \rightarrow & \bigtriangleup \theta_s \text{Ind}^w M \\
\text{Der}^b(O_0(0)) & \rightarrow & \text{Der}^b(O_0(\mathfrak{g}))
\end{array}
\]

**Proof.** Let \( M \in \text{Der}^b(O_0(0)) \) and denote by \( \Delta \) the equivalence between the derived category and the homotopy category of projectives. Then by applying \( \bigtriangleup \) to the distinguished triangle induced by Lemma 8 we obtain a distinguished triangle in \( \text{Hot}^b(C \rightarrow \text{SMod}) \)

\[
\bigtriangleup \text{Ind}^w M \rightarrow \bigtriangleup \theta_s \text{Ind}^w M \rightarrow \bigtriangleup \text{Ind}^w M +1
\]

Using the induction hypothesis, the fact that \( \theta_s \) maps projectives to projectives and [Soe01, Korollar 1] this is isomorphic to

\[
\text{Res}_w \otimes_C V_I \Delta M \rightarrow C \otimes_C \text{Res}_w \otimes_C V_I \Delta M \rightarrow \bigtriangleup \text{Ind}^w M +1
\]

and it follows that \( \bigtriangleup \text{Ind}^w M \) is isomorphic to the mapping cone of the first morphism which is precisely \( \text{Res}_w \otimes_C V_I \Delta M \). Here we use that the morphism \( \text{Ind}^w M \rightarrow \theta_s \text{Ind}^w M \) is induced by the adjunction morphism \( 1 \rightarrow \theta_s \).

This indeed yields a *natural* isomorphism between the mapping cone and \( \bigtriangleup \text{Ind}^w M \) since it is unique: By Lemma 18 it suffices to show that for all \( M, N \in \text{Der}^b(O_0(0)) \) we have

\[
\text{Hom}_{\text{Der}^b(O_0(0))}(\text{Ind}^w M, \text{Ind}^w N) = 0.
\]

This is true since both \( \text{Ind}^w M \) and \( \text{Ind}^w N \) are quasi-isomorphic to a complexes of modules with Verma flags consisting of \( M(xw \cdot 0) \) and \( M(yws \cdot 0) \), respectively, for \( x \in \mathcal{W}_I \), Now use that \( \text{Ext}^n(M(xw \cdot 0), M(yws \cdot 0)) = 0 \) for all \( n \) and \( x, y \in \mathcal{W}_I \), see [Hum08, Theorem 6.11]. \( \square \)

**Lemma 20.** The following diagram commutes (up to a natural transformation).
\[
\begin{array}{c}
\text{Det}^b (Y) \xrightarrow{h_{ws},, p_{ws}} \text{Det}^b (X) \\
\text{Hot}^b (\text{Det}^b (Y)_{w=0}) \xrightarrow{i} \text{Hot}^b (\text{Det}^b (X)_{w=0}) \\
\text{Hot}^b (C_t - \text{SMod}_{ev}^Z) \xrightarrow{R_{w, C \otimes \text{Res} (-)}} \text{Hot}^b (C - \text{SMod}_{ev}^Z)
\end{array}
\]

Proof. Let \( F \in \text{Der}^b (Y) \). Abbreviate the tilting equivalence functor by \( \Delta \). We apply \( \mathbb{H} \Delta \) to the distinguished triangle in Lemma 13 to obtain a distinguished triangle in \( \text{Hot}^b (C - \text{SMod}_{ev}^Z) \):

\[
\mathbb{H} \Delta h_{w,*}p_{w}^*F \longrightarrow \mathbb{H} \Delta \pi^*_s \pi^*_s h_{w,*}p_{w}^*F \longrightarrow \mathbb{H} \Delta h_{w,*}p_{w}^*F \xrightarrow{+1}
\]

by the induction hypothesis, the weight exactness of \( \pi^*_s \pi^*_s \) and \([\text{Soe90, Korollar 2}]\) this is naturally isomorphic to

\[
R_{w} \otimes_C \text{Res} \mathbb{H}_I \Delta F \longrightarrow C \otimes_C R_{w} \otimes_C \text{Res} \mathbb{H}_I \Delta F (2) \longrightarrow \mathbb{H} \Delta h_{w,*}p_{w}^*F \xrightarrow{+1}
\]

Hence \( \mathbb{H} \Delta h_{w,*}p_{w}^*F \) is isomorphic to the mapping cone of the first morphism which is precisely \( R_{w} \otimes_C \text{Res} \mathbb{H}_I \Delta F \). This indeed yields a natural isomorphism since it is unique: By Lemma 13 it suffices to show that for all \( F, G \in \text{Der}^b (Y) \) we have

\[
\text{Hom}_{\text{Der}^b (X)} (h_{w,*}p_{w}^*F, h_{w,*}p_{w}^*G) = 0.
\]

This is simply a matter of their support. Let

\[
U := \bigcup_{x \in W_{I}} B_{w,x}B \xrightarrow{j} Z := U \cup W \xleftarrow{i} W := \bigcup_{x \in W_{I}} B_{w,x}B
\]

and denote by \( k \) the inclusion of \( Z \) in \( X \). Notice that \( U \) is open in \( Z \). We have

\[
\text{Hom}_{\text{Der}^b (X)} (h_{w,*}p_{w}^*F, h_{w,*}p_{w}^*G) \\
= \text{Hom}_{\text{Der}^b (X)} (h_{w,*}p_{w}^*F, h_{w,*}p_{w}^*G) \quad \text{(duality)} \\
= \text{Hom}_{\text{Der}^b (Y)} (k^*h_{w,*}p_{w}^*F, k^*h_{w,*}p_{w}^*G) \quad \text{(support \subseteq Z)} \\
= \text{Hom}_{\text{Der}^b (Y)} (j^*k^*h_{w,*}p_{w}^*F, j^*k^*h_{w,*}p_{w}^*G) \quad \text{(support \subseteq W, resp. U)} \\
= \text{Hom}_{\text{Der}^b (Y)} (j^*k^*h_{w,*}p_{w}^*F, j^*k^*h_{w,*}p_{w}^*G) \quad \text{(adjunction and } j^* = j^!) \\
= 0 \quad \text{(since } j^*h = 0)\]

and the claim follows. \( \square \)

Both Theorem 2 and our main theorem follow.

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