SUFFICIENT CONDITIONS FOR SAMPLING AND INTERPOLATION ON THE SPHERE

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ABSTRACT. We obtain sufficient conditions for arrays of points, \( \mathcal{Z} = \{ \mathcal{Z}(L) \}_{L \geq 1} \), on the unit sphere \( \mathcal{Z}(L) \subset \mathbb{S}^d \), to be Marcinkiewicz-Zygmund and interpolating arrays for spaces of spherical harmonics. The conditions are in terms of the mesh norm and the separation radius of \( \mathcal{Z}(L) \).

1. INTRODUCTION

Let \( L^p(\mathbb{S}^d) \) the Banach space of measurable functions defined in the unit sphere \( \mathbb{S}^d \) with
\[
\|f\|_p^p = \int_{\mathbb{S}^d} |f(z)|^p d\sigma(z) < \infty,
\]
if \( 1 \leq p < \infty \), and
\[
\|f\|_\infty = \sup_{z \in \mathbb{S}^d} |f(z)| < \infty,
\]
when \( p = \infty \). Here \( \sigma \) stands for the Lebesgue surface measure in \( \mathbb{S}^d \).

For \( l \geq 0 \) an integer, let \( \mathcal{H}_l \) be the space of spherical harmonics of degree \( l \) in \( \mathbb{S}^d \) i.e. the space of eigenfunctions of the Laplace-Beltrami operator on \( \mathbb{S}^d \)
\[
\Delta_{\mathbb{S}^d} Y + l(l + d - 1)Y = 0, \quad Y \in \mathcal{H}_l.
\]
We denote by \( \Pi_L \), for \( L \) a nonnegative integer, the space of spherical harmonics of degree not exceeding \( L \)
\[
\Pi_L = \text{span} \bigcup_{l=0}^{L} \mathcal{H}_l.
\]
With respect to the inner product in \( L^2(\mathbb{S}^d) \) the spaces \( \mathcal{H}_l \) are orthogonal. We denote by \( h_l \) and \( \pi_L \) the dimension of \( \mathcal{H}_l \) and \( \Pi_L \), respectively. By Stirling’s formula, \( \pi_L \approx L^d \), when \( L \to \infty \). Let \( Y^1_l, \ldots, Y^{h_l}_l \) be an orthonormal basis of \( \mathcal{H}_l \). The reproducing kernel in \( \Pi_L \) is given by
\[
K_L(u, v) = \sum_{l=0}^{L} \sum_{j=1}^{h_l} Y^j_l(u) \overline{Y^j_l(v)} = C_{d,L} P_L^{(1+\lambda \lambda)}(\langle u, v \rangle), \quad u, v \in \mathbb{S}^d,
\]

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where \((u, v)\) stands for the scalar product in \(\mathbb{R}^{d+1}\), \(d = 2\lambda + 2\), \(C_{d,L}L^{-d/2}\) goes to a positive constant when \(L \to +\infty\) and \(P_L^{(\alpha, \beta)}\) are the Jacobi polynomials of degree \(L\) and index \((\alpha, \beta)\), normalized so that

\[
P_L^{(\alpha, \beta)}(1) = \begin{pmatrix} L + \alpha \\ L \end{pmatrix}.
\]

To discretize the \(L^p\)-norms in the space of spherical harmonics, we consider arrays of points on the sphere. More precisely, for any degree \(L\) we take \(m_L\) points in \(\mathbb{S}^d\)

\[
Z(L) = \{z_{L,j} \in \mathbb{S}^d : 1 \leq j \leq m_L\}, \quad L \geq 0.
\]

This yields an array of points \(Z = \{Z(L)\}_{L \geq 0}\) in \(\mathbb{S}^d\). Denote \(d(u, v) = \arccos(u, v)\) the geodesic distance between \(u, v \in \mathbb{S}^d\).

**Definition 1.1.** Let \(Z = \{Z(L)\}_{L \geq 0}\) be an array with \(m_L \geq \pi_L\) for all \(L\). We call \(Z\) an \(L^p\)-Marcinkiewicz-Zygmund array, denoted by \(L^p\)-MZ, if there exists a constant \(C_p > 0\) such that for all \(L \geq 0\) and \(Q \in \Pi_L\),

\[
\frac{C_p^{-1}}{\dim \Pi_L} \sum_{j=1}^{m_L} |Q(z_{L,j})|^p \leq \int_{\mathbb{S}^d} |Q(\omega)|^p d\sigma(\omega) \leq \frac{C_p}{\dim \Pi_L} \sum_{j=1}^{m_L} |Q(z_{L,j})|^p,
\]

if \(1 \leq p < \infty\), and

\[
\sup_{\omega \in \mathbb{S}^d} |Q(\omega)| \leq C \sup_{j=1, \ldots, m_L} |Q(z_{L,j})|,
\]

when \(p = \infty\).

In other words, the \(L^p\)-norm in \(\mathbb{S}^d\) of a polynomial of degree \(L\) is comparable to the discrete version given by the weighted \(\ell^p\)-norm of its restriction to \(Z(L)\). For the unit circle, \(d = 1\), the spherical harmonics are trigonometric polynomials. In this case, for \(m_L = \pi_L = 2L + 1\), J. Marcinkiewicz and A. Zygmund proved that the array of roots of unity form an \(L^p\)-MZ array, [MZ37], observe that \(C_2 = 1\). In higher dimensions the situation is more delicate. For \(m_L = \pi_L\) there are no \(L^p\)-MZ arrays when \(p \neq 2\) and the case \(p = 2\) is open, see [Mar07]. For \(m_L\) big enough there are always \(L^p\)-MZ arrays, see for example [MNW00, FM11].

**Definition 1.2.** Let \(Z = \{Z(L)\}_{L \geq 0}\) be a triangular array with \(m_L \leq \pi_L\) for all \(L\). We say that \(Z\) is \(L^p\)-interpolating if, for arrays \(\{c_{L,j}\}_{L \geq 0, 1 \leq j \leq m_j}\) of complex values such that

\[
\sup_{L \geq 0} \frac{1}{\pi_L} \sum_{j=1}^{m_L} |c_{L,j}|^p < \infty,
\]

there exists a sequence of polynomials \(Q_L \in \Pi_L\) uniformly bounded in \(L^p\) such that

\[
Q(z_{L,j}) = c_{L,j}, \quad 1 \leq j \leq m_j.
\]

The concept of MZ and interpolating families is important in signal processing. Somehow these concepts are opposite in nature. MZ families are dense enough so that the \(L^p\)-norm in \(\mathbb{S}^d\) is comparable to a discrete version. On the other hand, interpolating families are sparse enough so that one can interpolate some given data.
Observe that \( Z \) is \( L^2 \)-MZ if and only if the normalized reproducing kernels of \( \Pi_L \) at the points \( Z(L) \) form a frame with frame bounds independent of \( L \). Therefore, \( Z \) is \( L^2 \)-MZ when \( Z(L) \) is a set of sampling for \( \Pi_L \) with constants independent of \( L \). Similarly, \( Z \) is \( L^2 \)-interpolating if and only if the normalized reproducing kernel of \( \Pi_L, k_L \), at the points \( Z(L) \) form a Riesz sequence i.e.

\[
C^{-1} \sum_{j=1}^{m_L} |a_{Lj}|^2 \leq \int_{S^d} \left| \sum_{j=1}^{m_L} a_{Lj} k_L(z, z_{L,j}) \right|^2 d\sigma(z) \leq C \sum_{j=1}^{m_L} |a_{Lj}|^2,
\]

for any \( \{a_{Lj}\}_{L,j} \) with \( C > 0 \) independent of \( L \). When \( Z \) is both \( L^2 \)-interpolating and \( L^2 \)-MZ the normalized reproducing kernels at the points \( Z(L) \) form a Riesz basis, for more about these concepts see [Sei95].

We denote by \( d(u, v) = \arccos \langle u, v \rangle \) the geodesic distance between \( u, v \in S^d \).

**Definition 1.3.** An array \( Z = \{Z(L)\}_{L \geq 0} \) is uniformly separated if there is a positive number \( \epsilon > 0 \) such that

\[
d(z_{Lj}, z_{Lk}) \geq \frac{\epsilon}{L+1}, \text{ if } j \neq k,
\]

for all \( L \geq 0 \).

The left hand side inequality in (1) holds if and only if \( Z \) is a finite union of uniformly separated arrays, also the \( L^p \) version of the right hand side inequality in (2) holds if and only if \( Z \) is uniformly separated, see [Mar07] or [OCP11] for the general case of a compact Riemannian manifold.

**Definition 1.4.** Let \( X \) be a subset of \( S^d \). The mesh norm of \( X \) is

\[
\rho(X) = \sup_{u \in S^d} d(u, X) = \sup_{u \in S^d} \inf_{z \in X} d(u, z).
\]

The separation radius of \( X \) is

\[
\delta(X) = \inf_{u \in X} \inf_{v \in X \setminus \{u\}} d(u, v).
\]

The mesh norm of \( X \subset S^d \) is therefore the maximal radius of a spherical cap which does not contain points from \( X \), and the separation radius is the minimal distance between points in \( X \).

1.1. **Main results.** Our results in this paper are the following sufficient conditions for an array \( Z \) to be \( L^p \)-MZ or \( L^p \)-interpolating.

**Theorem 1.5.** Let \( 1 \leq p \leq \infty \) and \( Z = \{Z(L)\}_{L \geq 0} \) be an array in \( S^d \) such that for all \( L \geq 0 \)

\[
\delta(Z(L)) > \frac{\theta}{L},
\]

where \( \theta > 2j_\lambda \), \( j_\lambda \) is the first zero of the Bessel function \( J_\lambda(t) \), and \( d = 2\lambda + 2 \). Then \( Z \) is \( L^p \)-interpolating family.
Theorem 1.6. Let $1 \leq p \leq \infty$ and $\mathcal{Z} = \{ \mathcal{Z}(L) \}_{L \geq 0}$ be a uniformly separated array in $\mathbb{S}^d$ such that for all $L \geq 0$

$$\rho(\mathcal{Z}(L)) < \frac{\pi}{2L},$$

then $\mathcal{Z}$ is an $L^p$-Marcinkiewicz-Zygmund array.

We want to point out that there are other results about sufficient conditions for $L^p$-MZ and interpolating families of points on compact manifolds, but such conditions do not provide precise constants, see [FM11, FM10, MNSW02, OCP12]. We observe that due to the result mentioned above about minimal $L^p$-MZ (or maximal $L^p$-interpolating) arrays, an array with $m_L = \pi_L$ cannot satisfy the conditions of Theorems 1.6, 1.5.

When $\mathbb{S}^2$ and for some particular arrays of points, there are some results about separation radius and mesh norm [Rei90, SW04, DM05]. The results we know are not very precise and we would just mention one to illustrate the use of our results.

The set $X = \{x_1, \ldots, x_N \} \subset \mathbb{S}^2$ is said to be in $s$-extremal configuration if $X$ maximizes the Riesz $s$-energy

$$E(X) = \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^s}$$

for subsets of $N$ points on the sphere. In [KS98] it is assumed in order to get an estimate for the separation radius that the Voronoi cells around points in $s$-extremal configuration are all hexagons. Then it is obtained that

$$\delta(X_N) \sim \left( \frac{8\pi}{\sqrt{3}} \right)^{1/2} N^{-1/2}.$$

The same way it seems reasonable to estimate the mesh norm by the value of the maximal radius of the hexagon getting

$$\rho(X_N) \sim \frac{1}{\sqrt{3}} \left( \frac{8\pi}{\sqrt{3}} \right)^{1/2} N^{-1/2}.$$

Therefore, if we take $(kL)^2$ points for degree $L$ in order to assure we get an $L^p$-interpolating array we need $k < 0.792$, and to get an $L^p$-MZ array we need $k > 1.4$.

1.2. Outline of the paper. In Section 2 we prove Theorem 1.5. We use the classical approach by Ingham to obtain sufficient conditions for interpolation, [Ing36]. Ingham idea has been used in different context, [OU10, KL05]. The main problem is the construction of appropriate pick functions, Lemma 2.1, which depend on an upper bound for the first eigenvalue of a spherical cap, [Pin81, BCG83]. Estimates for the first eigenvalue are known also in general Riemannian manifolds.

In Section 3 we prove Theorem 1.6. Our approach follow the classical ideas of Beurling to study sampling sequences in Bernstein space, [Beu89]. We define weak limits of an array, and relate uniqueness sets with the $L^\infty$-MZ property. Our result is consequence of a uniqueness result due also to Beurling.

In what follows, when we write $A \lesssim B$, $A \gtrsim B$ or $A \simeq B$, we mean that there are constants independent of $L$ such that $A \leq CB$, $A \geq CB$ or $C_1 B \leq A \leq C_2 B$, respectively. Also, the
value of the constants appearing during a proof may change but they will be still denoted with the same letter.

2. SUFFICIENT CONDITION FOR INTERPOLATION

In this section we prove Theorem [1.5]. We adapt a nice idea of Ingham for Dirichlet series, [Ing36]. The idea rely on the construction of some pick functions with appropriate spectral properties. The existence of such functions can be established studying the first eigenvalue/eigenfunction of the Laplace-Beltrami operator on the sphere from which a spherical cap has been removed, see [KL05, OU10]. Estimates on this first eigenvalue, [FH76, Pin81, BCG83], together with a result about perturbation of interpolating arrays, [Mar07, Lemma 4.11], provide the result. For the unit circle, \( d = 1 \), the condition in Theorem [1.5] is \( \theta = \pi \), and the proof is technically simpler.

To prove Theorem [1.5] we use the functions given by the following lemma.

Lemma 2.1. Given \( L \) and \( \theta \) as in Theorem [1.5] there exists functions \( F_L \) such that

1. \( F_L \in L^2([-1, 1]) \).
2. \( \text{supp} \, F_L \subset [\cos(\theta/L), 1] \).
3. \( [F_L(\langle u, \cdot \rangle)](\ell, j) \leq 0 \) for all \( \ell > L \) and \( [F_L(\langle u, \cdot \rangle)](\ell, j) \leq \theta d/2 \) for all \( \ell \leq L \).
4. \( F_L(1) \approx \pi L \).

where \( [F_L(\langle u, \cdot \rangle)](\ell, j) \) stands for the Fourier coefficient \( \int_{S^d} F_L(\langle z_Lj, u \rangle)\overline{Y^j_\ell(u)}d\sigma(u) \).

Before establishing the existence of such functions we prove our main result.

Proof. [Theorem [1.5]] Recall that the normalized reproducing kernel can be written as

\[
k_L(z, z_Lj) = \frac{1}{\sqrt{\pi_L}} \sum_{\ell=0}^{L} \sum_{k=1}^{h_L} Y_\ell^k(z)Y_\ell^k(z_Lj).
\]

Let \( F_L(x) \) be given by Lemma 2.1 i.e. \( F_L \) is a continuous function defined in \(-1 \leq x \leq 1\) such that

\[
\text{supp} \, F_L \subset [\cos(\theta/L), 1],
\]

and \( F_L(1) \approx \pi L \). Moreover, the Fourier coefficients of \( F_L \) are negative for \( \ell > L \) and uniformly bounded by \( C\theta d/2 \) for \( \ell \leq L \). Thus by using these estimates and Funk-Hecke formula we get

\[
\int_{S^d} \left| \sum_{j=1}^{m_L} c_{Lj} k_L(z, z_Lj) \right|^2 d\sigma(z) = \frac{1}{\pi L} \int_{S^d} \sum_{\ell=0}^{L} \sum_{k=1}^{h_L} \left( \sum_{j=1}^{m_L} c_{Lj} \overline{Y^j_\ell(z_Lj)} \right) \overline{Y^k_\ell(z)} \left| \sum_{j=1}^{m_L} c_{Lj} \overline{Y^j_\ell(z_Lj)} \right|^2 d\sigma(z)
\]

\[
\geq \frac{1}{\theta d/2} \frac{1}{\pi L} \sum_{\ell=0}^{\infty} \sum_{k=1}^{m_L} \sum_{i,j=1}^{m_L} [F_L(\langle z_Lj, \cdot \rangle)](\ell, k)c_{Lj}c_{Li} \overline{Y^j_\ell(z_Lj)} Y^k_\ell(z_Li)
\]
\[
= \frac{1}{\theta^{d/2}} \frac{1}{\pi L} \sum_{i,j=1}^{m_L} c_{Li} c_{Lj} \sum_{\ell=0}^{+\infty} \sum_{k=1}^{h_\ell} |F_L(\langle z_{Lj}, \cdot \rangle)](\ell, k) Y^k(\cos \theta L) Y^k(\cos \theta L_i)
\]

Thus, the perturbed array \( Z \) is therefore \( L^2 \)-interpolating, because the other inequality in (2) follows directly from the separation.

In order to prove the result for other \( p \neq 2 \) we define, for \( \delta > 0 \), the perturbed array \( Z_\delta = \{ Z(L_{1+\delta}) \}_{L} \), where \( L_{1+\delta} = [L(1+\delta)] \). It was proved in \([\text{Mar07}, \text{Lemma 4.11}]\) that if \( Z \) is \( L^2 \)-interpolating then \( Z_{-\delta} \) is \( L^p \)-interpolating for all \( p \in [1, \infty] \). Therefore, assume that \( Z \) satisfies the geometric separation condition

\[
\eta := L \min_{i \neq j} d(z_{Li}, z_{Lj}) > \theta.
\]

Let \( \delta > 0 \) be small enough so that \( \eta > \theta + \delta \theta \). We assume that \( L >> 1 \) so that \( L\delta > 1 \). Then

\[
L d(z_{L_{1+\delta}}, z_{L_{1+\delta}}) > \frac{L}{L_{1+\delta}} \eta > \theta \frac{L}{L_{1+\delta}} (1 + \delta) \geq \theta.
\]

Thus, the perturbed array \( Z_\delta \) satisfies the same separation condition as \( Z \), then \( Z_\delta \) is \( L^2 \)-interpolating and \( Z = (Z_\delta)_{-\delta} \) is \( L^p \)-interpolating for all \( 1 \leq p \leq \infty \).

**Proof.** [Lemma 2.1] Let \( C_{\theta/2L} \) be the spherical cap of those \( x = (x_1, \ldots, x_{d+1}) \in S^d \) such that \( \cos \frac{\theta}{2L} < x_{d+1} \leq 1 \). We denote by \( f_0 \) the eigenfunction of the problem

\[
\Delta_{S^d} f_0 + \lambda_{0,L} f_0 = 0
\]

in \( C_{\theta/2L} \) corresponding to the first eigenvalue \( \lambda_{0,L} \) of the Laplace-Beltrami operator \( \Delta_{S^d} \). It is known, see \([\text{FH76}]\), that \( f_0 \) belongs to the class of zonal functions, Lipschitzian, nonnegative, non identically zero and with support in \([0, \frac{\theta}{2L}]\).

We normalize \( \|f_0\|_2^2 \simeq \pi L \) and define the zonal function

\[
F_L = \left( 1 + \frac{\Delta_{S^d}}{L(L + d - 1)} \right) (f_0 * f_0).
\]

with support in \([\cos \frac{\theta}{2L}, 1]\).

Let \( \{Y^j_\ell\}_{j,\ell} \) be the orthonormal basis in \( L^2(S^d) \) given by the spherical harmonics, then

\[
F_L(\langle u, v \rangle) = \sum_{\ell \geq 0} \sum_{j=1}^{h_\ell} [F_L(\langle \cdot, \cdot \rangle)](\ell, j) Y^j_\ell(v),
\]

where by Funk-Hecke

\[
[F_L(\langle u, \cdot \rangle)](\ell, j) = \int_{S^d} F_L(\langle u, v \rangle) Y^j_\ell(v) d\sigma(v) = \hat{F}_L(\ell) Y^j_\ell(u),
\]
and
\[
\hat{F}_L(\ell) = \frac{\sigma(S^d)}{C^{(d-1)/2}_\ell(1)} \frac{\int_{-1}^1 F_L(t) C^{(d-1)/2}_\ell(t)(1 - t^2)^{(d-2)/2}dt}{\int_{-1}^1 (1 - t^2)^{(d-2)/2}dt},
\]
here \( C^{\alpha}_\ell \) is the Gegenbauer polynomial of order \( \alpha \) and degree \( \ell \).

On the other hand
\[
[F_L(\langle u, \cdot \rangle)](\ell, j) = \int_{S^d} \left( 1 + \frac{\Delta_{S^d}}{L(L + d - 1)} \right) (f_0 * f_0)(\langle u, v \rangle) Y^2_\ell(v) d\sigma(v) \\
= \int_{S^d} (f_0 * f_0)(\langle u, v \rangle) Y^2_\ell(v) d\sigma(v) + \int_{S^d} \Delta_{S^d}(f_0 * f_0)(\langle u, v \rangle) Y^2_\ell(v) d\sigma(v) \\
= \hat{f}_0(\ell)^2 Y^2_\ell(u) + \frac{1}{L(L + d - 1)} \int_{S^d} (f_0 * f_0)(\langle u, v \rangle) \Delta_{S^d} Y^2_\ell(v) d\sigma(v) \\
= \hat{f}_0(\ell)^2 Y^2_\ell(u) - \frac{\ell(\ell + d - 1)}{L(L + d - 1)} \int_{S^d} (f_0 * f_0)(\langle u, v \rangle) Y^2_\ell(v) d\sigma(v) \\
= \left( 1 - \frac{\ell(\ell + d - 1)}{L(L + d - 1)} \right) \hat{f}_0(\ell)^2 Y^2_\ell(u).
\]

So we have proved that
\[
\hat{F}_L(\ell) = \left( 1 - \frac{\ell(\ell + d - 1)}{L(L + d - 1)} \right) \hat{f}_0(\ell)^2
\]

Note that the coefficients \( \hat{F}_L(\ell) \leq 0 \) for all \( \ell > L \). Now we are going to prove that the coefficients \( \hat{f}_0(\ell) \) are bounded for \( \ell \leq L \).

\[
|\hat{f}_0(\ell)| \lesssim \frac{1}{C^{(d-1)/2}_\ell(1)} \left| \int_{-1}^1 f_0(t) C^{(d-1)/2}_\ell(t)(1 - t^2)^{(d-2)/2}dt \right| \\
\lesssim \frac{1}{C^{(d-1)/2}_\ell(1)} \|f_0\|_2 \left( \int_0^{\theta/2L} C^{(d-1)/2}_\ell(\cos \theta)^2 \sin^{d-1} \theta d\theta \right)^{1/2} \\
\lesssim \frac{\sqrt{\pi_L}}{C^{(d-1)/2}_\ell(1)} \left( \int_0^{\theta/2L} C^{(d-1)/2}_\ell(\cos \theta)^2 \sin^{d-1} \theta d\theta \right)^{1/2} \lesssim \theta^{d/2},
\]
where we have used that \( C^{(d-1)/2}_\ell(1) \simeq \ell^{(d-2)/2} \) and \( C^{(d-1)/2}_\ell(t) \leq C \ell^{\max\left(\frac{d-1}{2}, -\frac{1}{2}\right)} \) (see [Sze39, Section 7.32]).

On the other hand, note that
\[
(f_0 * f_0)(1) = (f_0 * f_0)(\langle N, N \rangle) = \int_0^\pi f_0^2(\cos \theta) \sin^{d-1} \theta d\theta \simeq \pi_L.
\]
Thus,
\[
F_L(1) = (f_0 * f_0)(1) + \frac{1}{L(L + d - 1)} \Delta_{S^d}(f_0 * f_0)(1) \\
= (f_0 * f_0)(1) + \frac{1}{L(L + d - 1)}(f_0 * \Delta_{S^d}f_0)(1) \\
= \left(1 - \frac{\lambda_{0,L}}{L(L + d - 1)}\right)(f_0 * f_0)(1) \approx \pi_L \left(1 - \frac{\lambda_{0,L}}{L(L + d - 1)}\right).
\]

So we need to find the smallest \(\theta\) so that the quantity
\[
1 - \frac{\lambda_{0,L}}{L(L + d - 1)} > 0.
\]

Equivalently, we need the smallest \(\theta\) so that \(\lambda_{0,L} < L(L + d - 1)\). Using the upper bound from [BCG83], we get that
\[
\lambda_{0,L} \frac{\theta^2}{4L^2} < j_{d-2}^2,
\]
where \(j_{(d-2)/2}\) is the first zero of the Bessel function \(J_{(d-2)/2}\). So taking \(\theta\) as in the hypothesis we have the result. 

\[\square\]

3. SUFFICIENT CONDITION FOR SAMPLING

In this section we follow the classical approach used by Beurling to study sampling sequences in the space of bounded bandlimited functions, [Beu89]. First we identify the space of spherical harmonics composed with the exponential map as a subspace of the space of bounded bandlimited functions. Then we define the concept of weak limit of an array, and relate uniqueness sets with \(L_\infty\)-MZ arrays. Finally, we get Theorem [1.6] by using a result of Beurling about uniqueness sets and a result about perturbation of MZ arrays, [Mar07, Lemma 4.9.].

Let \(x = (x_1, \ldots, x_d) \in \mathbb{R}^d\) and denote \(\phi(x) = (x_1^2 + \cdots + x_d^2)^{1/2}\). The exponential map in \(S^d\) is defined by
\[
\exp(x) = \left(x_1 \frac{\sin \phi(x)}{\phi(x)}, \ldots, x_d \frac{\sin \phi(x)}{\phi(x)}, \cos \phi(x)\right) \in S^d.
\]

Observe that \(\exp(z)\) is defined also for \(z \in \mathbb{C}^d\) and is an entire function.

Given \(Q_L \in \Pi_L\) we define the function
\[
\tilde{Q}_L(z) = Q_L(\exp(z/L)), \quad z \in \mathbb{C}^d,
\]
and the corresponding space \(\tilde{\Pi}_L\). Observe that
\[
\sup_{u \in S^d} |Q_L(u)| = \sup_{x \in \mathbb{R}^d} |\tilde{Q}_L(x)|.
\]

The following result shows that functions in \(\tilde{\Pi}_L\) are entire in \(\mathbb{C}^d\) with Fourier-Laplace transform supported in the unit ball.

**Proposition 3.1.** If \(Q \in \Pi_L\) then the Fourier transform of \(\tilde{Q}\) has support in the unit ball of \(\mathbb{R}^d\).
Proof. The reproducing kernel of $\Pi_L$ centered at $v \in S^d$ is, up to constants, the Jacobi polynomial $u \mapsto P_L^{(1+\lambda,\lambda)}(\langle u, v \rangle)$. Let $y \in \mathbb{R}^d$ be such that $\exp(y/L) = v$. Consider the entire function $C^d \ni z \mapsto P_L^{(1+\lambda,\lambda)}(\langle \exp(z/L), \exp(y/L) \rangle)$. It is enough to see that for $\ell \leq L$ and for some constants $C, N \geq 0$ (that may depend on $L$)

$$|\langle \exp(z/L), \exp(y/L) \rangle|^\ell \leq C(1 + |z|)^N e^{|z^2|}, \quad z \in \mathbb{C}.$$ 

For any $\zeta \in \mathbb{C}$ one has $2(|\zeta|^2 - \Re \zeta^2)$. Therefore for any $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$

$$\sum_{i=1}^d \Re z_i^2 + 2 \sum_{i=1}^d (\Im z_i)^2 = \sum_{i=1}^d |z_i|^2,$$

and we get by the triangle inequality that

$$|\Im(z_1^2 + \cdots + z_d^2)|^{1/2} \leq \left(\sum_{i=1}^d (\Im z_i)^2\right)^{1/2}.$$ 

Finally

$$|\langle \exp(z/L), \exp(y/L) \rangle| = \left|\frac{\langle z, y \rangle \sin \phi(z)/L}{L|y|} \frac{|y|}{L} + \cos \phi(z)/L \cos |y|/L\right| \leq \left(1 + \left|\frac{\langle z, y \rangle}{L|y|} \sin |y|/L\right|\right) e^{\Im \phi(z)/L} \leq C_yL(1 + |z|) e^{\phi(\Im z)/L},$$

and $\tilde{Q}$ is the Fourier-Laplace transform of a distribution supported in the unit ball of $\mathbb{R}^d$. \hfill\blacksquare

Let $B$ be the Bernstein space of entire functions in $C^d$, bounded in $\mathbb{R}^d$ with Fourier transform supported in the unit ball of $\mathbb{R}^d$, endowed with the uniform norm.

Given an array $Z$ we send the points in $Z(L)$ to $\mathbb{R}^d$ via the exponential map and define the corresponding family of weak limits. The Fréchet distance between the closed sets $A, B \subset \mathbb{R}^d$ is given by

$$[A, B] = \inf_{t > 0}\{A \subset B + B(0, t), B \subset A + B(0, t)\}.$$ 

Definition 3.2. Let $Z = \{Z(L)\}_{L \geq 0}$ be an array in $\mathbb{S}^d$. We say that $\Lambda \subset \mathbb{R}^d$ is a weak limit of $Z$, denoted as $\Lambda \in W(Z)$, if there exist rotations $\rho_L \in SO(d+1)$, such that

$$L \exp^{-1}(\rho_L Z(L)) \rightarrow \Lambda,$$

where the above expression means that for any $K \subset \mathbb{R}^d$ compact

$$[(L \exp^{-1}(\rho_L Z(L))) \cap K] \cup \partial K, (\Lambda \cap K) \cup \partial K \rightarrow 0, \quad L \rightarrow \infty.$$ 

Proposition 3.3. If any $\Lambda \in W(Z)$ is a uniqueness set for $B$ then $Z$ is a $L^\infty$-MZ array.

Proof. We argue by contradiction. Suppose that $\Lambda \in W(Z)$ is a uniqueness set for $B$ but $Z$ is not $L^\infty$-MZ. For any $n \in \mathbb{N}$ there exists $Q_n \in \Pi_{L_n}$ such that $Q_n(N) = \|Q_n\|_\infty = 1$ and

$$\frac{1}{n} > \sup_{j=1,\ldots,m_{L_n}} |Q_n(z_{L_nj})|.$$ 

(5)
From the sequence \((\widetilde{Q}_n)_n\) defined as in (4) it is possible to select a subsequence (see [Nik75, 3.3.6.]) converging uniformly on compact sets of \(C^d\) to some function \(f \in B\) with \(f(N) = 1\). We denote this subsequence as before. For any \(\lambda \in \Lambda\) there exists a sequence \(z_{L_n k} \in Z(L_n)\) such that

\[
\mathbb{R}^d \ni w_{L_n k} = L_n \exp^{-1}(z_{L_n k}) \to \lambda, \quad k \to \infty.
\]

We denote this subsequence as before and for any \(\lambda \in \Lambda\) there exists a sequence \(z_{L_n k} \in Z(L_n)\) such that

\[
R^d \ni w_{L_n k} = L_n \exp^{-1}(z_{L_n k}) \to \lambda, \quad k \to \infty.
\]

We denote this subsequence as before and for any \(\lambda \in \Lambda\) there exists a sequence \(z_{L_n k} \in Z(L_n)\) such that

\[
R^d \ni w_{L_n k} = L_n \exp^{-1}(z_{L_n k}) \to \lambda, \quad k \to \infty.
\]

The first term on the right side clearly goes to zero. Also, the last term goes to zero because of (5). For the second term, using Bernstein’s inequality we get

\[
|\widetilde{Q}_n(\lambda) - \widetilde{Q}_n(z_{L_n j n})| = |Q_n(\exp(\lambda/L_n)) - Q_n(z_{L_n j n})| = L_n d(\exp(\lambda/L_n), z_{L_n j n}) \|Q_n\|_\infty \to 0, \quad n \to \infty,
\]

because, see [BC73, p. 229],

\[
d(\exp x, \exp y) = |x - y| + o(L^{-1}), \quad x, y \in \mathbb{R}^d.
\]

We get that \(f\) vanish in \(\Lambda\), but \(f \in B\) and therefore \(f = 0\).

To prove our main result we use the following result about uniqueness due to Beurling, [Beu89, p. 310].

**Theorem 3.4 (Beurling).** Let \(f\) be an entire function in \(C^d\). Assume that

\[
\limsup_{|\xi| \to \infty} \frac{\log |f(\xi)|}{|\xi|} = r < \infty, \quad \int_{1}^{\infty} \max_{|\xi| \leq t, \xi \in \mathbb{R}^d} \log |f(\xi)| \frac{dt}{t^2} < \infty.
\]

Assume \(f = 0\) on a discrete set \(\Lambda \subset \mathbb{R}^d\) such that

\[
r \limsup_{\mathbb{R}^d \ni x \to \infty} \inf_{\lambda \in \Lambda} |x - \lambda| < \frac{\pi}{2}.
\]

Then \(f = 0\).

**Proof.** [Theorem 1.6] Let \(Z\) be such that for all \(L\) big enough

\[
\rho(Z(L)) < \frac{\pi}{2L}.
\]

Let \(\Lambda \in W(Z)\) and \(X_L = L \exp^{-1}(\rho_L Z(L)) \subset \mathbb{R}^d\) with \(X_L \to \Lambda\). We want to see that \(\Lambda\) is a uniqueness set for \(B\).

For \(f \in B\) and \(\epsilon > 0\) there exists \(A_\epsilon > 0\) such that

\[
|f(\xi)| \leq A_\epsilon e^{(1+\epsilon)|\xi|}, \quad \xi \in \mathbb{C}^d.
\]

Also \(|f(x)| \leq M < \infty\) for \(x \in \mathbb{R}^d\), so we can apply Beurling’s result (with \(r = 1\)) getting that any \(\Lambda \subset \mathbb{R}^d\) such that

\[
\limsup_{\mathbb{R}^d \ni x \to \infty} \inf_{\lambda \in \Lambda} |x - \lambda| < \frac{\pi}{2},
\]

is a uniqueness set for \(B\).
If
\[
\sup_{|x| < \pi L} \inf_{z \in X_L} |x - z| < \frac{\pi}{2},
\]
for any \( L \) big enough, then we can deduce (6) which is equivalent to
\[
\sup_{\omega \in S^d} \min_{Z \in L \exp^{-1}(\rho_L Z(L))} |L \exp^{-1}(\omega) - z| < \frac{\pi}{2}.
\]
But this follows from the condition on the mesh norm the property
\[
d(\exp \frac{x}{L}, \exp \frac{y}{L}) = \frac{|x - y|}{L} + o(L^{-1}), \quad x, y \in \mathbb{R}^d.
\]

Therefore the condition on \( Z \) implies that it is an \( L^\infty \)-MZ array. In order to deduce the result for \( 1 \leq p < \infty \) we define, for \( \delta > 0 \), the associated arrays \( Z_\delta, Z_{-\delta} \) by \( Z_\delta(L) = Z([1 + \delta L]) \), \( Z_{-\delta}(L) = Z([(1 - \delta) L]) \).

It was proved in [Mar07, Lemma 4.9.] that if \( Z \) is an \( L^\infty \)-MZ array then \( Z_\delta \) is an \( L^p \)-MZ array for all \( 1 \leq p < \infty \).

Suppose that
\[
\eta = \sup_{L \geq 0} L \rho(Z(L)) < \frac{\pi}{2},
\]
and \( \delta > 0 \) be such that
\[
\eta < \frac{\pi}{2} - \delta \pi.
\]

For \( L \) big enough with \( L \delta > 1 \) and any \( u \in S^d \) we have
\[
Ld(u, Z_{L_{1-\delta}}) \leq \frac{L}{L_{1-\delta}} \eta < \frac{\pi}{2},
\]
therefore \( Z_{-\delta} \) is \( L^\infty \)-MZ and \( (Z_{-\delta})_\delta = Z \) is \( L^p \)-MZ.

\[\Box\]

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