Reflection from black holes and space-time topology

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The quantum corrections make the black hole capable of reflection: any particle that approaches the event horizon can bounce back in the outside world. The albedo of the black hole depends on its temperature. The reflection shares physical origins with the phenomenon of Hawking radiation; both effects are explained as consequences of the singular nature that the event horizon exhibits on the quantum level.

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Intuitively black holes are defined as objects that do not emit anything and possess ideal absorption properties. A known limitation for this naive picture stems from thermodynamics of black holes. The first indication that gravitational fields could have entropy came when investigation of Christodoulou [3] of the Penrose process for extracting energy from a Kerr black hole showed that there is a quantity which could not go down. Hawking [2] found that it is proportional to the area of the horizon. Further research of Bardeen [3] demonstrated that black holes obey laws similar to the laws of thermodynamics. A known limitation for this naive picture stems from the fact that the wave function of the incoming particle acquires an unexpected admixture of the outgoing wave. This effect arises due to quantum corrections, becoming apparent in the semiclassical approximation. One can describe it in terms of the classical action of the probing particle that possesses a singularity on the horizon. The singularity, that persists in any coordinate frame, triggers the singularity of the wave function that describes the incoming particle. As a result the wave function of the incoming particle acquires an admixture of the outgoing wave. It is this admixture that explains the reflectivity of black holes. It is interesting that the Hawking radiation process can also be tracked down to the same origin. Having same basic roots, the two mentioned effects have different experimental manifestations because the flux and spectrum of incoming particles depend on the flux and spectrum of reflected particles, while the radiation depends entirely on properties of the black hole.

A consistent quantum treatment of the incoming particle agrees with the semiclassical results, revealing also an alternative interesting way to account for the quantum corrections. It is based on a topological charge that can be associated with events that happen with the incoming particle on the event horizon. This charge can be considered as a particular topological characteristic of the Schwarzschild geometry that becomes important for quantum description of the probing particle, being irrelevant at the classical level. The topological properties of the space-time of black holes within the frames of the classical approximation were discussed by Einstein and Rosen [11], Wheeler [12], and Misner and Wheeler [13].

Consider a black hole with the Schwarzschild geometry

\[ ds^2 = - \left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{1 - r_s/r} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \] (1)

Relativistic units \( \hbar = c = 1 \) supplemented by condition \( 2Gm = 1 \) imposed on the gravitational constant \( G \) and the black hole mass \( m \) are used. A sphere with the Schwarzschild radius \( r = r_g \equiv 1 \) represents the event horizon. Let us assume that the probing particle is a scalar with the mass \( \mu \) and describe its motion by the wave function \( \Phi \) that satisfies the Klein-Gordon equation

\[ g^{\lambda \mu} \nabla_\lambda \nabla_\mu \Phi = \mu^2 \Phi, \] where \( \nabla \) is the covariant derivative for a given metric. Separating the angular variables for the Schwarzschild metric [11] one finds that the radial wave function \( \phi(r) \) that describes the motion with the energy \( \varepsilon \) and momentum \( L \) satisfies

\[ \phi'' + \left( \frac{1}{r} + \frac{1}{r - 1} \right) \phi' + \frac{1}{1 - 1/r} \left( \frac{\varepsilon^2}{1 - 1/r} - \mu^2 - \frac{L(L + 1)}{r^2} \right) \phi = 0. \] (2)

Consider the proper incoming and outgoing waves \( \phi_{\text{in}}(r), \phi_{\text{out}}(r) \) that satisfy \( \phi_{\text{in}}(r) = [\phi_{\text{out}}(r)]^* \) and are distinguished by their asymptotic behavior at infinity \( \phi_{\text{in}}(r) \rightarrow B \exp[-i(pr + \cdots)], r \rightarrow \infty, \) where

\( B \) is a constant.
\( p = (\varepsilon^2 - \mu^2)^{1/2} \) is the momentum, the dots refer to the \( \mathcal{O}(\ln r/r) \) terms that appear due to long-range nature of the gravitational force, and \( B \) is a normalization factor. Eq. (2) has a regular singularity at the horizon, \( \phi(r) \to (r-1)^{\eta} \), \( r \to 1 \). Here the index \( \eta \) found from Eq. (3) equals \( \eta = \mp i \varepsilon \). For the incoming wave the sign minus here is to be chosen, which gives

\[
\phi^{(0)}_{in}(r) = \exp[-i \varepsilon \ln(r-1)], \quad r \to 1 . \tag{3}
\]

Let us describe the incoming particle in the outside region \( r > 1 \) by the wave function

\[
\phi_{in}(r) = \phi^{(0)}_{in}(r) + \mathcal{R} \phi^{(0)}_{out}(r) . \tag{4}
\]

The first term here, the proper incoming wave, is definitely present because we do consider the incoming particle. The second term, the proper outgoing wave, is introduced in order to allow for the possibility of reflection of the incoming particle off the black hole. The magnitude of this, hypothetical at the moment, reflection is measured by the coefficient \( \mathcal{R} \). The unitarity condition requires that \( |\mathcal{R}| \leq 1 \). Intuitive arguments based on an experience with classical trajectories would indicate that the black hole is incapable of reflection, i.e. the reflection coefficient is 0. However, the quantum corrections do produce the phenomenon of reflection, making \( |\mathcal{R}| > 0 \), as verified below.

Let us consider what happens with the incoming wave in Eq. (4) in the inside region \( r < 1 \). The horizon is a classically forbidden area for the outside observer who needs to wait an infinite interval of time to see the incoming particle to cross the horizon. However, this fact does not prevent the incoming wave from penetrating into the inside region simply because this is a stationary wave that exists during the infinite interval of time. This claim is supported by the asymptotic behavior of the wave function on the horizon. Continuing this wave function analytically from the outside region \( r > 1 \) into the inside region \( r < 1 \) over the lower semiplane of the complex \( r \)-plane, see the counter \( C_1 \) in Fig. 1 one finds that this wave exists in the region \( r < 1 \), being suppressed compared with the outside region \( r > 1 \) by the factor \( k = \exp(-\pi \varepsilon) \). We will return to this point below. Consider now the singularity of Eq. (2) at the origin \( r = 0 \). There are two solutions here, the regular \( \phi(r) \propto 1 \), and the singular \( \phi(r) \propto \ln r \). Conventional quantum mechanical arguments favor the regular boundary condition \( \phi(0) = \text{const} \), which cannot be satisfied by the incoming wave by itself. One needs to introduce its complex conjugate, i.e. the proper outgoing wave, and take a linear combination of these two waves. These arguments show that the outgoing wave certainly exists in the inside region \( r < 1 \) prompting one to consider what happens with this wave on the horizon. The problem is similar to the one discussed above for the incoming wave. The outgoing wave is stationary, hence it is able to penetrate into the outside region \( r > 1 \). Its analytical continuation along the contour \( C_2 \) in Fig. 1 gives the necessary suppression factor, which again proves be equal to \( k = \exp(-\pi \varepsilon) \).

We find that the wave function for the incoming particle has the form of Eq. (4) that incorporates the expected proper incoming wave, represented by the first term, and the surprising admixture of the proper outgoing wave, the second term. The above discussion shows that the reflection arises after the horizon is traversed twice, along \( C_1 \) and \( C_2 \), each crossing introducing a suppression factor \( k = \exp(-\pi \varepsilon) \). Hence the reflection coefficient in Eq. (4) satisfies

\[
|R| = k^2 = \exp(-2\pi \varepsilon) . \tag{5}
\]

The admixture of the outgoing wave in Eq. (4) shows that black holes are capable of reflection, see Ref. [14] that derives this fact from symmetry conditions. In the presented approach the effect can be tracked down to the singularity that exists in the wave equation (2) at the horizon. The above arguments can be cast into conventional mathematical terms. The singularity \( r = 1 \) is associated with the \( 2 \times 2 \) monodromy matrix \( \mathcal{M} \). The eigenvalues \( \lambda_{1,2} \) of this matrix are related to the indexes \( \eta = \pm i \varepsilon, \lambda_{1,2} = \exp(2 \pi i \eta) = \exp(\mp 2 \pi i \varepsilon) \). The absolute value for the reflection coefficient in Eq. (5) coincides with the smaller eigenvalue. This construction has a topological bearing. A closed contour on the complex \( r \)-plane around the singularity on the horizon \( r = 1 \) can be associated with the topological charge \( n \) that counts its winding around the point \( r = 1 \) and gives the eigenvalues \( \exp(\mp 2 \pi i \varepsilon n) \) for the \( n \)-th power of the matrix \( \mathcal{M} \). An interesting part of this construction is that the reflective ability of black holes can be associated with those events that are distinguished by the topological number \( n = 1 \). Continuing the incoming wave along the counter \( C' \) with \( n = 1 \) in Fig. 1 one reproduces the reflection coefficient Eq. (5) as an eigenvalue of this transformation.

Alternatively, the effect of reflection can be described in the semiclassical approximation, with the help of the classical action \( S(r, t) \). For the metric Fig. 1 the variables are separated \( S(r, t) = -\varepsilon t + L\varphi + S(r) \), where \( \varphi \) is the

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**FIG. 1:** The complex \( r \)-plane. To verify Eqs. (4, 5) it is sufficient to continue analytically the incoming wave along \( C_1 \) and the outgoing wave along \( C_2 \). In the semiclassical approximation the shift along the contour \( C \) transforms the incoming wave into the outgoing wave. The reflection coefficient \( \mathcal{R} \) can be reproduced as an eigenvalue of a along shift the contour \( C' \), which has the topological number \( n = 1 \).
azimuthal angle. The radial part of the action $S(r)$ found from the Hamilton-Jacobi classical equations of motion $g^{\alpha\beta}\partial_\alpha S\partial_\beta S = -\mu^2$ reads $S(r) = \mp S_0(r)$ (see e.g. [10]), where

$$S_0(r) = \int_0^r \left[ \epsilon^2 - \left( \mu^2 + L^2/r^2 \right) (1 - 1/r) \right]^{1/2} \frac{dr}{1 - 1/r}. \quad (6)$$

On the horizon this action possesses the logarithmic singularity

$$S_0(\rho) \simeq \epsilon \ln(r - 1), \quad r \to 1. \quad (7)$$

Eqs. (6, 7) shows that the wave function on the horizon exhibits a semiclassical behavior $\phi_{in}^{(0)}(r) = \exp[-i S_0(r)]$. Therefore the found above suppressing factor $k = \exp(-\pi \epsilon)$ that describes the crossing of the horizon can be attributed to the semiclassical approximation. Moreover, all arguments regarding the behavior of the incoming and outgoing waves in Eq. (4) can be reproduced semiclassically. One takes the proper incoming wave for $r > 1$, then continue it along the contour $C$ in Fig. 1 (which can be transformed into a series of contours $C_1 - C_2$) and end up with the proper outgoing wave. These purely semiclassical arguments supports validity of Eqs. (3, 5). The derivation given holds for arbitrary $\mu, \epsilon, L$. However, to allow the particle a classically allowed propagation from the horizon towards the infinity the classical momentum must be real for $1 < r < \infty$. To satisfy this condition it suffices to have either $L/2 \leq \mu$ for arbitrary energy $\epsilon \leq \mu$, or $L \leq (3\sqrt{3}/2) \epsilon$ for the ultrarelativistic case $\epsilon \gg \mu$.

Thus the reflection ability of black holes arises from the logarithmic singularity of the classical action [4]. It is important that this singularity is an invariant property of the action that persists even in those coordinates that eliminate the singularity of the metric and classical equations of motion. Consider for example convenient Kruskal coordinates $U, V$ that are defined by equations

$$U/V = -\exp(-t), \quad UV = (1 - r) \exp(r). \quad (8)$$

Fig. 2 shows that there are four regions of the spacetime, see Ref. [11] for detailed discussion. Areas I and III represent two identical copies of the outside region; II, IV show two inside regions. They all are divided by the horizon located at $U = 0$, or $V = 0$. The incoming particle follows the trajectory $AB$ and, crossing the horizon, resides in II. There are also the outgoing trajectories; following the trajectory $CD$ the particle escapes from the inside region IV into the outside world I. Areas II and IV are not connected, which ensures (classical) confinement of the incoming particle trapped in II. In Kruskal coordinates the metric is regular on the horizon, also regular are the classical equations of motion that on the horizon read simply $V = const$ for $|U| \ll |V| \ll 1$ and $U = const$ for $|V| \ll |U| \ll 1$. However, the singularity of the action remains intact. It can be conveniently presented in terms of the action $S = -\epsilon t \mp S_0(r)$, where the minus and plus signs are for the incoming and outgoing trajectories respectively,

$$S \simeq \begin{cases} -\epsilon \ln(V^2), & |U| \ll |V| \ll 1, \\ \epsilon \ln(U^2), & |V| \ll |U| \ll 1. \end{cases} \quad (9)$$

The wave function (11) can be presented in a more general form

$$|\phi_{in}\rangle = |\text{in}\rangle + R |\text{out}\rangle. \quad (10)$$

Here the variable $t$ is included in the wave function $|\phi_{in}\rangle \equiv \exp(-i\epsilon t)\phi_{in}(r)$, permitting convenient presentation for the proper incoming $|\text{in}\rangle$ and outgoing $|\text{out}\rangle$ waves, which in the vicinity of the horizon read

$$|\text{in}\rangle = \exp[-i\epsilon \ln(V^2)], \quad |\text{out}\rangle = \exp[i\epsilon \ln(U^2)]. \quad (11)$$

Since the semiclassical picture is valid on the horizon, one can associate the proper incoming wave, i.e. the first term in (11), with the incoming trajectories, see $AB$ in Fig. 2. Correspondingly, the second term, the proper outgoing wave, is associated with the outgoing trajectories, see $CD$. Eq. (10) shows that effects associated with these different classical trajectories are mixed in the wave function of the incoming particle.

Notice that the effects that look completely isolated in the classical description interfere in the wave function

\[\]
We proved this fact above for the outside region $I$, where $U < 0$, $V > 0$. However, it can be argued that this form of the wave function remains valid in a more general case because the wave functions in (10) are symmetrical under the transformations $U \rightarrow -U$, $V \rightarrow -V$. These transformations convert an event in the outside region $I$ into an event in one of the inside regions $II$, or $IV$. Following this line of arguments one concludes that quantum description of any event in any region, inside or outside, is based on the wave function (10) that includes two terms: the proper incoming and outgoing waves.

Consider implications of this fact for a particle that in the classical approximation is confined in the inside region. Describing its behavior with the help of the wave function (10) one concludes that there exists a probability $P_{\text{esc}} \propto |R|^2 = \exp(-\varepsilon/T)$ for this particle to populate the outgoing wave and escape into the outside world. We see that on the quantum level confinement cannot be absolute, the locked in particle can escape into the outside world. Thus the black hole creates a flux of outgoing particles escaping from its inside region. It is instructive to consider the black hole behavior inside the temperature bath with the temperature equal to the Hawking temperature $T = 1/(4\pi)$. In this imaginary experiment the black hole absorbs the particles produced by the temperature bath and emits particles that escape from its inside region. Considering the ratio of the probability to escape to the probability to be absorbed $P_{\text{esc}}/P_{\text{abs}}$, one finds

$$P_{\text{esc}}/P_{\text{abs}} = \exp(-\varepsilon/T).$$

This shows that the black hole remains in equilibrium with the temperature bath and hence has the temperature $T = 1/(4\pi)$, as was discovered in Ref. [7], see also discussion in Ref. [13]. The phenomenon of the Hawking radiation is usually explained by creation of pairs in the gravitational field. As we see, Eq. (11) proposes an alternative simple explanation: the radiation happens because the wave function of a particle that is confined inside the horizon incorporates an admixture of the outgoing wave that gives the particle a chance to escape into the outside world.

Quantum corrections alter the fundamental properties of the event horizon. In the classical approximation this is an area that the incoming particle penetrates through smoothly going inside, but which becomes impassable in the opposite direction. Quantum effects change both these properties. They make possible the reflection from the horizon that shows that the horizon presents some obstacle for the incoming particle. The reflection is the more prominent the lower the energy, for $\varepsilon < T$ the black hole behaves as a mirror, which is surprising. Quantum corrections also make possible the Hawking radiation that can be described as an escape of the particle from the inside region of a black hole, which shows that the horizon is transparent for the inside-out transition.

The discussed phenomena follow from the fact that the classical action is singular on the horizon [13], making singular the semiclassical wave function and ultimately resulting in Eq. (11). There is a topological aspect of the quantum problem. It arises because the singularity of the wave equation on the horizon can be associated with the particular topological charge $n$ that counts winding of the trajectory of the probing particle around the horizon on the complex $r$-plane. The reflective and radiative abilities of black holes are associated with topologically nontrivial trajectories with $n = 1$. This result makes the topological charge an interesting characteristic of the Schwarzschild geometry of the space-time. Generalization of the discussed results for the Kerr–Newmann charged rotating black holes is given in Ref. [10]. It verifies validity of Eq. (11) for this case as well, deriving from it the phenomena of reflection and Hawking radiation.

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