BIFURCATION AND STABILITY OF A TWO-SPECIES DIFFUSIVE LOTKA-VOLTERRA MODEL

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ABSTRACT. This paper is devoted to a two-species Lotka-Volterra model with general functional response. The existence, local and global stability of boundary (including trivial and semi-trivial) steady-state solutions are analyzed by means of the signs of the associated principal eigenvalues. Moreover, the nonexistence and steady-state bifurcation of coexistence steady-state solutions at each of the boundary steady states are investigated. In particular, the coincidence of bifurcating coexistence steady-state solution branches is also described. It should be pointed out that the methods we applied here are mainly based on spectral analysis, perturbation theory, comparison principle, monotone theory, Lyapunov-Schmidt reduction, and bifurcation theory.

1. Introduction. Spatial characteristics of the environment play an important role in ecology and evolution. Using a prey-predation diffusion model, we shall illustrate the significant changes in dynamics caused by the changing of their maximal growth rates of the species, respectively. In this paper, we will study the following reaction-diffusion predator-prey model under the Neumann boundary condition

\[
\begin{cases}
\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + r_1(x)u(x,t) - Au^2(x,t) - B\phi(u(x,t))v(x,t), & x \in \Omega, \ t \geq 0, \\
\frac{\partial v(x,t)}{\partial t} = \Delta v(x,t) + r_2(x)v(x,t) - Dv^2(x,t) + C\phi(u(x,t))v(x,t), & x \in \Omega, \ t \geq 0, \\
\frac{\partial u(x,t)}{\partial \nu} = \frac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial \Omega, t \geq 0, \\
u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega,
\end{cases}
\]

(1)

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where \( u_0(x) \) and \( v_0(x) \) are both nonnegative continuous functions and not identically zero, \( A, B, C, \) and \( D \) are all strictly positive, \( \Delta \) denotes the Laplacian operator, \( \nu \) is the unit outward normal to \( \partial \Omega \), the functions \( r_1(x) \) and \( r_2(x) \) are strictly positive and continuous on \( \Omega \). In biological terms, \( u \) and \( v \) can be interpreted as the densities of prey and predator populations, respectively; \( A \) and \( D \) are self-limitation constants; \( r_1(x) \) and \( r_2(x) \) represent the (spatially inhomogeneous) intrinsic growth rates of the prey and predator populations of prey and predator populations, respectively; the habitat \( \Omega \) of the species is a bounded smooth domain in \( \mathbb{R}^N \). In the absence of predators and the diffusion effect, the prey species follows the logistic equation

\[
\frac{d}{dt} u(x,t) = \nu u(x,t) \left[ m_1(x) - A u(x,t) - B \phi(u(x,t)) v(x,t) \right],
\]

\( x \in \Omega, \ t \geq 0, \)

\[
\frac{d}{dt} v(x,t) = \Delta v(x,t) + b \left[ m_2(x) v(x,t) - D v^2(x,t) + C \phi(u(x,t)) v(x,t) \right],
\]

\( x \in \Omega, \ t \geq 0, \)

\[
\frac{\partial u(x,t)}{\partial \nu} = \frac{\partial v(x,t)}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t \geq 0,
\]

\[
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega,
\]

where the two continuous functions \( m_1(\cdot) \) and \( m_2(\cdot) \) are strictly positive on \( \Omega \) and satisfy that

\[
\max_{x \in \Omega} m_1(x) = 1 = \max_{x \in \Omega} m_2(x).
\]

The spatial component of ecological interactions has been identified as an important factor in how ecological communities are shaped, and understanding the role of space is challenging both theoretically and empirically \([7, 16, 17, 20, 21, 27, 34, 40, 41]\). Ecologically, positive solutions correspond to the existence of steady states of species. So the set of positive solutions may contain crucial clues for the stationary patterns. In this paper, we mainly focus on the following steady-state problem of system (1):

\[
0 = \Delta u(x) + a \left[ m_1(x) u(x) - A u^2(x) - B \phi(u(x)) v(x) \right], \quad x \in \Omega,
\]

\[
0 = \Delta v(x) + b \left[ m_2(x) v(x) - D v^2(x) + C \phi(u(x)) v(x) \right], \quad x \in \Omega,
\]

\[
0 = \frac{\partial u(x)}{\partial \nu} = \frac{\partial v(x)}{\partial \nu}, \quad x \in \partial \Omega.
\]

Based on the important role of the steady-state solutions, many researchers have been concentrating on the dynamics of spatially homogeneous steady-state solutions.
of diffusive systems, for example, Faria [6] considered a diffusive predator-prey system with one delay and a unique positive homogeneous steady-state solution $E^*$. Faria [6] studied the local stability of $E^*$ and described the Hopf bifurcation which occurs as the delay (taken as a parameter) crosses some critical values. Yi et al. [48] investigated a homogeneous reaction-diffusion model describing the control growth of mammalian hair, and found that when one of the dimensionless parameter is less than one, the unique positive homogeneous steady-state solution is globally asymptotically stable. Li and Li [24] studied a stage-structured predator-prey system with Holling type-III functional response and obtained the existence of a Hopf bifurcation at the homogeneous steady-state solution. Yi et al. [48] found that both spatially homogeneous and heterogeneous oscillatory solutions can be seen by using the composite form of some spatially independent parameters as a bifurcation parameter.

In the most of aforementioned literature, many researchers focused on the dynamics of spatially homogeneous steady-state solutions of the diffusive systems. However, the research of the spatially nonhomogeneous steady-state solutions is rare and complicated. There are many reasons why it is difficult to establish the dynamical behaviors near spatially nonhomogeneous steady-state solutions. The most difficulty is that even though the existence of spatially nonhomogeneous steady-state solutions can be investigated by means of topological and variational methods (see [39, 2, 5, 9, 32, 47]), to the best of our knowledge, the explicit algebraic form of these solutions cannot be derived. For example, Kuto and Tsujikawa [23] applied Leray-Schauder degree theory and just derived the existence of spatially nonhomogeneous solutions by regarding the diffusion coefficient $d$ as a bifurcation parameter. The stability of the spatially nonhomogeneous steady-state solutions highly depends on the characteristic equation of linearized system at spatially nonhomogeneous steady-state solutions. Hence, it is very important for us to obtain the explicit algebraic form of spatially nonhomogeneous steady-state solutions in order to investigate their stability. Recently, we [28] studied the dynamics of the delayed system (3) with the special case $m_1(x) \equiv m_2(x) \equiv 1$ and obtained the existence of spatially nonhomogeneous steady state solution by applying Lyapunov-Schmidt reduction. In [28], we also investigated the stability and nonexistence of Hopf bifurcation at the spatially nonhomogeneous steady-state solution. Unfortunately, this approach can not be applied to system (3) due to the effect of the spatially inhomogeneous intrinsic growth rate and the delay terms. Therefore, we have to resort to new methods.

So and Waltman [37] considered an unstirred competition chemostat model and obtained the local coexistence of positive solutions by the standard bifurcation theorems in one dimensional case. Later, Wu [42] obtained some relevant results in the $N$-dimensional case, and also described the local stability and global structure of the coexistence solutions. Nie and Wu [29] studied the uniqueness and stability of coexistence solutions by applying Lyapunov-Schmidt procedure and perturbation theory. A common feature of all these studies on the chemostat model with the Beddington-DeAngelis functional response is that coexistence solutions appear via codimension one bifurcation. A natural problem is what happens to the existence of coexistence solutions as the two bifurcation parameters simultaneously pass across some critical points. To the best of our knowledge, there are few works on this investigation. By means of bifurcation theory and perturbation method, Guo et
al. [8] and Li et al. [25] recently investigated the bifurcation from a double eigenvalue in the chemostat model with the Beddington-DeAngelis functional response. Motivated by Guo et al. [8], Li et al. [25] and Wu [42], in this paper we will consider the dynamical behaviors of (3). Of course, we cannot simply adopt the methods in [8, 25, 42] since the form of the chemostat models studied by Guo et al. [8] and Li et al. [25] are competitive and are much simpler than our model. Thus, we have to further develop the methods in [8, 25, 42] to overcome these difficulties.

The results of this paper is organized as follows. In section 2, we shall introduce some notations and preliminary results, which will be used in the subsequent sections. Section 3 is devoted to the existence, uniqueness, local and global stability of boundary steady states, including the trivial steady state \((0, 0)\) and two semi-trivial steady states \((\theta_a, 0)\) and \((0, \theta_b)\), by means of the sub-super solution, monotone method and generalized maximum principle, Krein-Rutman theorem [22, 35] and comparison principle. Finally, the nonexistence and steady-state bifurcation of coexistence steady-state solution of (2) at each of the boundary steady states are investigated in section 4 by using Lyapunov-Schmidt reduction and the bifurcation theory introduced by Crandall and Rabinowitz [3]. In particular, we also describe the coincidence of coexistence steady-state solution branches of (2).

2. Notations and preliminaries. In this section, we primarily introduce some basic notations and known results which will be used in this paper.

Throughout the paper, let \(L^2(\Omega)\) be the Lebesgue space of integrable nonnegative functions defined on \(\Omega\), \(C^k(\Omega)\) be the usual Banach space of \(k\)-times continuously differentiable functions defined on \(\Omega\) equipped with the supremum norm, \(X = X_0 = H^2(\Omega) \cap H_0^1(\Omega), Y = Y_0 = L^2(\Omega)\), \(X = X_0 \times X_0, Y = Y_0 \times Y_0\), where \(H^k(\Omega) (k \geq 0)\) are Sobolev spaces, and

\[
H_0^k(\Omega) = \left\{ u \in H^k(\Omega) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \right\}.
\]

We also introduce the following two spaces:

\[
C_0^k(\Omega) = \left\{ u \in C^k(\Omega) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \right\},
\]

\[
H_0^k(\Omega) = \left\{ u \in H^k(\Omega) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \right\}.
\]

Throughout the paper, the null space and range of a linear mapping \(A\) are denoted by \(\text{Ker}(A)\) and \(\text{Ran}(A)\). Let \(\text{dim}\) and \(\text{codim}\) denote dimension and codimension, respectively. For the complex-valued Hilbert space \(Y_{0C}\), we use the standard inner product \(\langle u, v \rangle = \int_{\Omega} \overline{u(x)} v(x) dx\).

In what follows, we collect some facts concerning the elliptic eigenvalue problem. Firstly, consider the following two eigenvalue problems

\[
\begin{cases}
\Delta \varphi + \lambda q(x) \varphi = 0, & \text{in } \Omega, \\
\frac{\partial \varphi}{\partial \nu} = 0, & \text{on } \partial \Omega,
\end{cases}
\]

and

\[
\begin{cases}
\Delta \varphi + q(x) \varphi = \mu \varphi, & \text{in } \Omega, \\
\frac{\partial \varphi}{\partial \nu} = 0, & \text{on } \partial \Omega.
\end{cases}
\]
Lemma 2.1 ([42, 46]). Assume that \(q(x) \in C(\overline{\Omega})\) and \(q(x) > 0\) on \(\overline{\Omega}\). Then all the eigenvalues of problem (4) satisfy

\[
0 < \lambda_1(q) < \lambda_2(q) \leq \cdots \rightarrow \infty
\]

with the corresponding eigenfunctions \(\varphi_1, \varphi_2, \cdots\), where the principal eigenvalue \(\lambda_1(q)\) is given by

\[
\lambda_1(q) = \inf_{\varphi \in H_1(\Omega) \backslash \{0\}} \frac{\int_{\Omega} |\nabla \varphi|^2 \, dx}{\int_{\Omega} q(x) \varphi^2 \, dx}
\]

and the principal eigenfunction \(\varphi_1\) is positive on \(\overline{\Omega}\). Moreover, \(\lambda_i(q_1) \leq \lambda_j(q_2)\) for \(i, j \geq 1\) if \(q_1(x) \geq q_2(x)\) on \(\overline{\Omega}\) and the strict inequality holds if \(q_1(x) \neq q_2(x)\).

Definition 2.2. Given a function \(q \in L^\infty(\Omega)\), we define \(\mu_k(q)\) to be the \(k\)-th eigenvalue (counting multiplicities) of (5). In particular, we call \(\mu_1(q)\) the first eigenvalue of (5) and have the following variational characterization

\[
\mu_1(q) = \sup_{\varphi \in H_1(\Omega) \backslash \{0\}} \frac{\int_{\Omega} q(x) \varphi^2 - |\nabla \varphi|^2 \, dx}{\int_{\Omega} \varphi^2 \, dx}.
\]

In view of (6) and page 95 of [1], we have

Lemma 2.3 ([15]). The first eigenvalue \(\mu_1(q)\) of (5) depends continuously on \(q \in L^\infty(\Omega)\). Moreover,

(i): \(\int_{\Omega} q \, dx \geq 0\) and \(q \not\equiv 0 \Rightarrow \mu_1(q) > 0\);

(ii): \(\int_{\Omega} q \, dx < 0\) and \(h\) changes sign in \(\Omega\), then \(\mu_1(q) > 0\) when \(\lambda_1(q) < 1\); \(\mu_1(q) = 0\) when \(\lambda_1(q) = 1\); \(\mu_1(q) < 0\) when \(\lambda_1(q) > 1\);

(iii): \(q_1(x) \leq q_2(x)\) in \(\Omega\), then \(\mu_1(q_1) \leq \mu_1(q_2)\) with equality holds if and only if \(q_1 = q_2\) a.e. in \(\Omega\).

Let \(\lambda_0\) and \(\sigma_0\) be the principal eigenvalues of the following problems (7) and (8), respectively:

\[
\begin{cases}
\Delta \varphi + \lambda m_1(x) \varphi = 0, & \text{in } \Omega, \\
\frac{\partial \varphi}{\partial \nu} = 0, & \text{on } \partial \Omega,
\end{cases}
\]

and

\[
\begin{cases}
\Delta \psi + \sigma m_2(x) \psi = 0, & \text{in } \Omega, \\
\frac{\partial \psi}{\partial \nu} = 0, & \text{on } \partial \Omega
\end{cases}
\]

with the corresponding principal eigenfunctions \(\varphi_1, \psi_1 > 0\) on \(\overline{\Omega}\), which are uniquely determined by the normalization \(|\varphi_1|^2 = 1\) and \(|\psi_1|^2 = 1\).

3. Boundary steady-state solutions.

3.1. Existence. The boundary steady-state solutions of (2) take the form \((u, 0)\) or \((0, v)\) with \(u\) and \(v\) satisfying

\[
\begin{cases}
\Delta u + au \left(m_1(x) - A u\right) = 0, & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0, & \text{on } \partial \Omega,
\end{cases}
\]

and

\[
\begin{cases}
\Delta v + bv \left(m_2(x) - D v\right) = 0, & \text{in } \Omega, \\
\frac{\partial v}{\partial \nu} = 0, & \text{on } \partial \Omega,
\end{cases}
\]

respectively. For (9), we have the following results:
Theorem 3.1. If \( a \leq \lambda_0 \) then 0 is the unique nonnegative solution of (9). If \( a > \lambda_0 \) then (9) has a unique positive solution, denoted by \( \theta_a \), satisfying the following properties:

(i): \( 0 < \theta_a \leq \frac{1}{A} \);
(ii): \( \mu_k(L_1) < 0 \) for all \( k \in \mathbb{N} \), where \( L_1 = a(m_1(x) - 2A\theta_a) \);
(iii): \( \lim_{a \to \lambda_0} \theta_a = 0 \) uniformly for \( x \in \Omega \), and

\[
\lim_{a \to \lambda_0} \frac{\theta_a}{a - \lambda_0} = \frac{\varphi_1}{\lambda_0} \int_{\Omega} \varphi_1^2 dx \quad \text{for almost every } x \in \Omega; \tag{11}
\]

(iv): \( \theta_a \) is continuously differentiable with respect to \( a \in (\lambda_0, +\infty) \). In particular, if \( m_1(\cdot) \equiv 1 \), then \( \theta_a \) is pointwisely increasing with respect to \( a \).

Proof. We see that 0 is the unique nonnegative solution of (9) when \( a \leq \lambda_0 \) and that (9) with \( a > \lambda_0 \) has a unique positive solution, denoted by \( \theta_a \) provided \( a > \lambda_0 \) (see [30] for more details). In what follows, we shall prove the conclusions (i)-(iv).

First of all, conclusion (i) can be proved by means of the approach proposed in [42]. Here, we give detailed proof for the convenience of readers. Suppose that \( u \) is a nonnegative solution of (9) such that \( u \neq 0 \). If \( u(x_0) = 0 \) for some point \( x_0 \in \Omega \), we claim that \( x_0 \in \partial \Omega \). For otherwise, it follows from the minimum principle [31] that \( u \equiv 0 \), which is a contradiction. We apply the Hopf boundary condition to obtain \( \frac{\partial u}{\partial \nu} \bigg|_{x=x_0} < 0 \), which is also a contradiction. Thus, \( u > 0 \) on \( \Omega \). On the other hand, let \( u(y_0) = \max_{x \in \Omega} u(x) \), then it follows from the maximum principle that

\[
u(y_0) \leq \frac{1}{A} m_1(y_0) \leq \frac{1}{A} \max_{x \in \Omega} m_1(x) \leq \frac{1}{A}.
\]

This completes the proof of conclusion (i).

Let \( L_0 = a(m_1(x) - A\theta_a) \). Note that \( \theta_a > 0 \) is a solution to (9), then it follows from the Krein-Rutman theorem [36] that \( \mu_1(L_0) = 0 \) is the principal eigenvalue of the operator \( \Delta + L_0 \) in \( C^2_{\mathcal{B}} \) with an associated eigenfunction \( \theta_a \). Thus, the principal eigenvalue \( \mu_1(L_1) \) of \( \Delta + L_1 \) satisfies \( \mu_1(L_1) < \mu_1(L_0) = 0 \). This proves conclusion (ii).

In order to prove conclusion (iii), it suffices to show that (11) is satisfied because this also means that \( \lim_{a \to \lambda_0} \theta_a = 0 \). It follows from the fact that \( \theta_a \) is a solution to (9) that

\[
\int_{\Omega} |\nabla \theta_a|^2 dx = a \int_{\Omega} (m_1(x) - A\theta_a) \theta_a^2 dx.
\]

Let \( J(a) = \left[ \int_{\Omega} m_1(x) \theta_a^2 dx \right]^\frac{1}{2} \). Then we have

\[
\int_{\Omega} \left| \frac{\nabla \theta_a}{J(a)} \right|^2 dx \leq a.
\]

According to the above inequality and Theorem 3.1 (i), it is easy to see that \( \frac{\theta_a}{J(a)} \) is bounded in \( H^1(\Omega) \) as \( a \to \lambda_0 \). Note that \( \theta_a \) satisfies (9), then from Theorem 3.1(i) it follows that \( \frac{\theta_a}{J(a)} \) is bounded in \( H^2_{\mathcal{B}}(\Omega) \) as \( a \to \lambda_0 \). By the Sobolev embedding theorem and the standard elliptic regularity theory, there exists \( \phi \in C^1_{\mathcal{B}} \) such that \( \frac{\theta_a}{J(a)} \to \phi \) in \( C^1(\Omega) \) as \( a \to \lambda_0 \). Moreover,

\[
\int_{\Omega} |\nabla \phi|^2 dx \leq \lambda_0,
\]
satisfying the following properties:

\[ \theta_a \varphi_1(x) \Delta \varphi_a + a \varphi_1 \theta_a \frac{\partial \varphi}{\partial \nu} = 0. \]

Then using Green’s formula and noting that \( \Delta \varphi_1 + \lambda_0 \varphi_1 \varphi_1(x) = 0 \), we have

\[ \int \varphi_1 \left[ \Delta \frac{\theta_a}{J(a)} + a \frac{\theta_a}{J(a)} \varphi_1 \varphi_1(x) \right] dx = \int \frac{\theta_a}{J(a)} \left[ \Delta \varphi_1 + a \varphi_1 \varphi_1(x) \right] dx \]

where \( a > 0 \), it is well known that \( H(a, \theta_a) = 0 \) and the Fréchet derivative \( DH(a, \theta_a) = \lambda + L_1 \). In view of Theorem 3.1 (ii) and the implicit function theorem, there exists a \( C^1 \) function \( u_s : \mathbb{R} \rightarrow C^2_B(\Omega) \) defined on a neighbourhood of \( a \) such that \( u_s(x) = = \theta_a \) and \( H(s, u_s) = 0 \). Furthermore, \( s \rightarrow u_s \) is a \( C^1 \) mapping from \( \lambda_0 + \infty \) to \( C^2_B(\Omega) \). It follows from the uniqueness of the solution \( (s, u_s) \) close to \( (a, \theta_a) \) that \( \theta_a \) is continuously differentiable with respect to \( a \). In particular, if \( m_1(x) \equiv 1 \), then differentiating both sides of \( (9) \) with respect to \( a \) yields

\[ (\Delta + L_1) \xi_a = \Delta \xi_a + a \xi_a \varphi_1 \varphi_1(x) - 2A \theta_a = -a \theta_a (r - A \theta_a) < 0, \]

where \( \xi_a = \frac{\partial u_s}{\partial a} \). Since \( \mu_1(L_1) < 0 \), the associated principal eigenfunction \( \psi_a > 0 \) on \( \Omega \) satisfies that \( (\Delta + L_1) \psi = \mu_1(L_1) \psi < 0 \) in \( \Omega \) and that \( \frac{\partial \psi}{\partial \nu} = 0 \) on \( \partial \Omega \). It follows from the generalized maximum principle [31] that \( \frac{\partial \psi}{\partial \nu} \) does not attain a nonpositive minimum at \( x_0 \in \Omega \). In fact, if \( x_0 \in \Omega \) is the nonpositive minimum point of \( \xi_a \), then

\[ \frac{\partial \psi}{\partial \nu}(\xi_a)_{x=x_0} < 0, \]

which contradicts the boundary condition for \( \xi_a \). Therefore, \( \xi_a > 0 \), which implies the monotonicity of \( \theta_a \) with respect to \( a \). Thus, we complete the proof of Theorem 3.1.

For \( (10) \), we can obtain the following similar conclusions as Theorem 3.1.

**Theorem 3.2.** Suppose that \( b \leq \sigma_0 \). Then \( 0 \) is the unique nonnegative solution of \( (10) \). Suppose that \( b > \sigma_0 \). Then \( (10) \) has a unique positive solution, denoted by \( \theta_b \), satisfying the following properties:

(i): \( 0 < \theta_b \leq \frac{1}{b} \);

(ii): \( \mu_k(L_2) < 0 \) for all \( k \in \mathbb{N} \), where \( L_2 = b(m_2(x) - 2D \theta_b) \);

(iii): \( \lim_{b \rightarrow \sigma_0} \theta_b = 0 \) uniformly for \( x \in \Omega \), and

\[ \lim_{b \rightarrow \sigma_0} \frac{\theta_b}{b - \sigma_0} = \frac{\psi_1}{\sigma_0 \int_{\Omega} \psi_1^2 dx} \quad \text{for almost every} \quad x \in \Omega; \quad (12) \]
(iv): \( \theta_b \) is continuously differentiable with respect to \( b \in (\sigma_0, +\infty) \). In particular, if \( m_2 \equiv \text{const} > 0 \), then \( \theta_b \) is pointwisely increasing with respect to \( b \).

3.2. Local stability. In this subsection, we investigate the asymptotical stability of boundary steady states by examining the spectrum of the corresponding linearized operator. From the existence and uniqueness results of (9) and (10), it follows that system (3) has a trivial steady state \((0,0)\) and two semi-trivial steady states \((\theta_a,0)\) and \((0,\theta_b)\). Moreover, if a steady state \((u,v)\) satisfying \(u \geq 0\) and \(v \geq 0\) is neither a trivial nor a semi-trivial in \(\overline{\Omega}\), then we call \((u,v)\) a coexistence steady state. Now, let’s consider the local stability of the trivial steady state \((0,0)\) and two semi-trivial steady states \((\theta_a,0)\) and \((0,\theta_b)\).

Linearizing the steady-state problem of (2) at \((u^*,v^*)\), and letting \((u,v) = (\Phi e^{\mu t}, \Psi e^{\mu t})\) with \((\Phi, \Psi) \in X \setminus (0,0)\), we have
\[
\begin{align*}
\Delta \Phi + a (m_1(x) - 2Au^* - B\phi'(u^*)v^*) \Phi - aB\phi(u^*)\Psi &= \mu \Phi, & \text{in } \Omega, \\
\Delta \Psi + b (m_2(x) - 2Dv^* + C\phi(u^*)) \Psi + bC\phi'(u^*)v^* \Phi &= \mu \Psi, & \text{in } \Omega, \\
\frac{\partial \Phi}{\partial \nu} &= \frac{\partial \Psi}{\partial \nu} = 0, & \text{on } \partial \Omega.
\end{align*}
\]
(13)

If \((u^*,v^*)\) is a coexistence steady state, then the Krein-Rutman theorem [22, 35] means that (13) has a principal eigenvalue \(\mu_1 \in \mathbb{R}\), i.e., \(\mu_1\) is simple and has the least real part among all eigenvalues. If \((u^*,v^*)\) is a trivial or semi-trivial steady state, then the sign of the principal eigenvalue can be determined by means of Theorems 3.3, 3.5 and 3.7 stated below. In what follows, a steady state \((u^*,v^*)\) of (3) is said to be linearly stable (respectively, linearly unstable) if the principal eigenvalue \(\mu_1\) is negative (respectively, positive). As we known, a steady state of (3) is asymptotically stable (respectively, unstable) if it is linearly stable (respectively, linearly unstable). Here, the notions of stability and asymptotic stability are defined in the standard dynamical systems sense with the \(C(\overline{\Omega}) \times C(\overline{\Omega})\) topology.

To describe the asymptotical stability of \((0,0)\), it suffices to consider (13) with \((u^*,v^*) = (0,0)\), that is,
\[
\begin{align*}
\Delta \Phi + am_1(x) \Phi &= \mu \Phi, & \text{in } \Omega, \\
\Delta \Psi + bm_2(x) \Psi &= \mu \Psi, & \text{in } \Omega, \\
\frac{\partial \Phi}{\partial \nu} &= \frac{\partial \Psi}{\partial \nu} = 0, & \text{on } \partial \Omega.
\end{align*}
\]
(14)

It follows from the Riesz-Schauder theory that the spectrum \(\mu\) of the eigenvalue problem (14) consists of real eigenvalues. Moreover, we have the following result.

**Theorem 3.3.** The trivial solution \((0,0)\) is asymptotically stable if \(a < \lambda_0\) and \(b < \sigma_0\) and is unstable if either \(a > \lambda_0\) or \(b > \sigma_0\).

To describe the asymptotical stability of \((\theta_a,0)\), it suffices to investigate the principal eigenvalue of (13) with \((u^*,v^*) = (\theta_a,0)\), that is,
\[
\begin{align*}
\Delta \Phi + L_1 \Phi - aB\phi(\theta_a) \Psi &= \mu \Phi, & \text{in } \Omega, \\
\Delta \Psi + \bar{L}_1 \Psi &= \mu \Psi, & \text{in } \Omega, \\
\frac{\partial \Phi}{\partial \nu} &= \frac{\partial \Psi}{\partial \nu} = 0, & \text{on } \partial \Omega.
\end{align*}
\]
(15)

where \(L_1\) is defined in Theorem 3.1 and \(\bar{L}_1 = b (m_2(x) + C\phi(\theta_a))\). Then, we have the following result.
Lemma 3.4. The principal eigenvalue \( \mu_1 \) of (15) satisfies \( \mu_1 = \max\{\mu_1(L_1), \mu_1(\tilde{L}_1)\} \).

Proof. Let \( \mu \) be an eigenvalue of (15) with an associated eigenfunction \((\Phi, \Psi)\). If \( \Psi \neq 0 \), then \( \mu \leq \mu_1(b(m_2(x) + C\phi(\theta_a))) \). Alternatively, if \( \Psi \equiv 0 \), then \( \Phi \neq 0 \), and \( \mu \leq \mu_1(L_1) \). This shows that

\[
\mu \leq \max\{\mu_1(\tilde{L}_1), \mu_1(L_1)\}.
\]

To show that the maximum is attainable, suppose first that \( \mu_1(L_1) \geq \mu_1(\tilde{L}_1) \). Let \( \nu_1 \) be an eigenfunction associated with the eigenvalue \( \mu_1(L_1) \), then \( \mu_1(\tilde{L}_1) \) is an eigenvalue of (15) with eigenfunction \((\nu_1, 0)\).

In what follows, we consider the case where \( \mu_1(L_1) < \mu_1(\tilde{L}_1) \). In this case, we have

\[
\mu_1(L_1 - \mu_1(\tilde{L}_1)) = \mu_1(L_1) - \mu_1(\tilde{L}_1) < 0
\]

and hence every eigenvalue of \( \Delta + L_1 - \mu_1(\tilde{L}_1) \) under zero Neumann boundary condition is negative, which implies that the operator \( \Delta + L_1 - \mu_1(\tilde{L}_1) \) under zero Neumann boundary condition is invertible. Let \( \nu_2 \) be an eigenfunction associated with \( \mu_1(L_1) \), then \( \mu_1(\tilde{L}_1) \) is an eigenvalue of (15) with an associated eigenfunction

\[
(\Phi, \Psi) = (aB(\Delta + L_1 - \mu_1(\tilde{L}_1))^{-1}[\phi(\theta_a)\nu_2], \nu_2).
\]

This completes the proof. \( \square \)

It follows from Theorem 3.1(ii), we have \( \mu_1(L_1) < 0 \). Thus, Lemma 3.4 implies that the sign of the principal eigenvalue \( \mu_1 \) of (15) is completely determined by \( \mu_1(\tilde{L}_2) \). Therefore, we have the following result.

Theorem 3.5. The linear stability of \((\theta_a, 0)\) is determined by \( \bar{b}(a) \), where \( \bar{b}(a) \) represents the principal eigenvalue of the following problem:

\[
\begin{cases}
\Delta \psi + b(m_2(x) + C\phi(\theta_a)) \psi = 0, & \text{in } \Omega, \\
\frac{\partial \psi}{\partial \nu} = 0, & \text{on } \partial \Omega,
\end{cases}
\tag{16}
\]

and \( \bar{\psi}(a) \) denotes the associated eigenfunction. More precisely, the semi-trivial solution \((\theta_a, 0)\) is asymptotically stable if \( b < \bar{b}(a) \) and is unstable \( b > \bar{b}(a) \).

In view of Lemma 2.1, we have

\[
\bar{b}(a) = \inf_{\psi} \frac{\int_{\Omega} |\nabla \psi|^2 dx}{\int_{\Omega} (m_2(x) + C\phi(\theta_a))\psi^2 dx}.
\tag{17}
\]

Next, we follow the ideas used in [18, 43] to investigate the properties of \( \bar{b}(a) \).

Lemma 3.6. The function \( \bar{b}(a) \) defined by (17) satisfies

(i): \( \bar{b}() \in C[\lambda_0, +\infty) \) and \( \bar{b}(\lambda_0) = \sigma_0 \);

(ii): \( \bar{b}() \in C^1(\lambda_0, +\infty) \), \( \text{sgn} \bar{b}'(a) = -\text{sgn}[\theta_a'(a)] \), and

\[
\lim_{a \to \lambda_0} \bar{b}'(a) = -\frac{C\sigma_0 \phi'(0)}{A\lambda_0} \frac{\int_{\Omega} \varphi_1 \psi_1^2 dx \int_{\Omega} m_1(x) \varphi_1^2 dx}{\int_{\Omega} \varphi_1^2 dx \int_{\Omega} m_2(x) \psi_1^2 dx}.
\tag{18}
\]

In particular, if \( m_1 \equiv 1 \), then \( \bar{b}'(a) < 0 \) for all \( a \in (\lambda_0, +\infty) \).
Proof. We adopt the methods in [8] to prove this lemma. Since $\bar{b}(a)$ is the principal eigenvalue of system (16), then we take the corresponding principal eigenfunction $\bar{\psi}(a)$ such that $\|\bar{\psi}(a)\|^2 = 1$ and $\bar{\psi}(a) > 0$ on $\Omega$. The infimum in (17) is attained by $\bar{\psi}(a)$, then
\[
\bar{b}(a) = \frac{\int_{\Omega} |\nabla \bar{\psi}(a)|^2 \, dx}{\int_{\Omega} [m_2(x) + C\phi(\theta_a)]\bar{\psi}^2(a) \, dx}
\leq \frac{\int_{\Omega} |\nabla \bar{\psi}(a + h)|^2 \, dx}{\int_{\Omega} [m_2(x) + C\phi(\theta_a)]\bar{\psi}^2(a + h) \, dx}
= \bar{b}(a + h) \frac{\int_{\Omega} [m_2(x) + C\phi(\theta_{a+h})]\bar{\psi}^2(a + h) \, dx}{\int_{\Omega} [m_2(x) + C\phi(\theta_a)]\bar{\psi}^2(a + h) \, dx}
\]
that is,
\[
\bar{b}(a) \frac{\int_{\Omega} [m_2(x) + C\phi(\theta_a)]\bar{\psi}^2(a + h) \, dx}{\int_{\Omega} [m_2(x) + C\phi(\theta_{a+h})]\bar{\psi}^2(a + h) \, dx} \leq \bar{b}(a + h).
\]
Similarly, exchanging $a$ and $a + h$, we have
\[
\bar{b}(a + h) = \frac{\int_{\Omega} |\nabla \bar{\psi}(a + h)|^2 \, dx}{\int_{\Omega} [m_2(x) + C\phi(\theta_{a+h})]\bar{\psi}^2(a + h) \, dx}
\leq \frac{\int_{\Omega} |\nabla \bar{\psi}(a)|^2 \, dx}{\int_{\Omega} [m_2(x) + C\phi(\theta_a)]\bar{\psi}^2(a) \, dx}
= \bar{b}(a) \frac{\int_{\Omega} [m_2(x) + C\phi(\theta_a)]\bar{\psi}^2(a) \, dx}{\int_{\Omega} [m_2(x) + C\phi(\theta_{a+h})]\bar{\psi}^2(a) \, dx}.
\]
Letting $h \to 0$, we obtain $\bar{b} \in C(\lambda_0, +\infty)$. Note that
\[
\bar{b}(a) = \frac{\int_{\Omega} |\nabla \bar{\psi}(a)|^2 \, dx}{\int_{\Omega} [m_2(x) + C\phi(\theta_a)]\bar{\psi}^2(a) \, dx}.
\]
By Theorem 3.1 (iii), we have $\bar{b}(a) \to \sigma_0$ as $a \to \lambda_0$. This completes the proof of conclusion (i).

By some simple computations, we obtain
\[
\bar{b}(a) \frac{\int_{\Omega} C [\phi(\theta_a) - \phi(\theta_{a+h})]\bar{\psi}^2(a + h) \, dx}{\int_{\Omega} [m_2(x) + C\phi(\theta_{a+h})]\bar{\psi}^2(a + h) \, dx} \leq \bar{b}(a + h) - \bar{b}(a)
\]
\[
\leq \bar{b}(a) \frac{\int_{\Omega} C [\phi(\theta_a) - \phi(\theta_{a+h})]\bar{\psi}^2(a) \, dx}{\int_{\Omega} [m_2(x) + C\phi(\theta_{a+h})]\bar{\psi}^2(a) \, dx}.
\]
Dividing both sides of the above inequality (19) by $h$ and letting $h \to 0$, we obtain
\[
\ddot{b}(a) = \frac{-C\bar{b}(a) \int_{\Omega} \phi(\theta_a)\theta'_a \bar{\psi}^2(a) \, dx}{\int_{\Omega} [m_2(x) + C\phi(\theta_a)]\bar{\psi}^2(a) \, dx}.
\]
that $\text{sgn}[\vec{b}'(a)] = \text{sgn}[\theta'_a(a)]$. In particular, if $m_1 \equiv 1$, then $\theta'_a(a) > 0$ and hence $\vec{b}'(a) < 0$ for all $a \in (\lambda_0, +\infty)$ by Theorem 3.1 (iv).

In what follows, we prove conclusion (18) using the identity of (20). It is obvious that
\[
\lim_{a \to \lambda_0} \phi'(\theta_a) = \phi'(0) \text{ in } C^1(\Omega)
\]
and
\[
\lim_{a \to \lambda_0} \phi(\theta_a) = \phi(0) = 0 \text{ in } C^1(\Omega).
\]
In view of Kato [19], we also have
\[
\lim_{a \to \lambda_0} \psi(a) = \psi_1 \text{ in } L^2(\Omega).
\]

It remains to show the dependence of $\theta'_a$ on $a$. By Theorem 3.1, we see that $\theta_a$ is the unique nontrivial solution of (9) for $a > \lambda_0$, we can apply the Crandall-Rabinowitz bifurcation theory [3] and obtain the expression of $\theta_a$ near $a = \lambda_0$. For this purpose, we define the operator $F : \mathbb{R} \times X \to Y$ by
\[
F(a; u) = \Delta u + au(m_1(x) - Au).
\]
It is obvious that $F(a; 0) = 0$, $F_u(\lambda_0; 0) = \Delta + \lambda_0 m_1(x)$, and $\text{Ker}(F_u(\lambda_0; 0)) = \text{span}\{\varphi_1\}$. Note that $F_u^*(\lambda_0; 0) = F_u(\lambda_0; 0)$, where $F_u^*(\lambda_0; 0)$ is the adjoint operator of $F_u(\lambda_0; 0)$, so we have
\[
\dim\text{Ker}(F_u(\lambda_0; 0)) = \text{codim}\text{Ran}(F_u(\lambda_0; 0)) = 1.
\]
Moreover, $F_u \varphi_1 = m_1(x)\varphi_1 \not\in \text{Ran}(F_u(\lambda_0; 0))$. Hence, the Crandall-Rabinowitz bifurcation theory [3] implies that there exists a function $(a(s), w(s)) \in C^1(\mathbb{Y} \times \mathbb{R}; [-s_0, s_0])$ for sufficiently small $s_0$ such that $w(0) = 0$, $a(0) = \lambda_0$ and $F(a(s), u(s)) = 0$, where $u(s) = s(\varphi_1 + w(s))$ and $w(s) \in \text{Ran}(F_u(\lambda_0; 0)) \cap X$. In view of the uniqueness of the nontrivial solution of (9), we have
\[
\theta_{a(s)} = s(\varphi_1 + w(s)).
\]
Substituting $(a(s), \theta_{a(s)})$ into (9), dividing $s$, differentiating with respect to $s$ and setting $s = 0$, we have
\[
\Delta w'(0) + a'(0)\varphi_1 m_1(x) + \lambda_0 w'(0)m_1(x) - A\lambda_0 \varphi_1^2 = 0.
\]
Now multiply the above equation by $\varphi_1$ and integrate it over $\Omega$ to get
\[
a'(0) \int_\Omega m_1(x)\varphi_1^2 dx = A\lambda_0 \int_\Omega \varphi_1^3 dx.
\]
Notice that
\[
\left. \frac{d\theta_a}{da} \right|_{a=\lambda_0} = \left. \frac{d\theta_a}{ds} \right|_{s=0} = \frac{\varphi_1}{a'(0)}.
\]
Therefore, the assertion (iii) follows from (20), (21), (22) and (23).

Finally, we consider the stability of the semi-trivial solution $(0, \theta_b)$. Using a similar argument as above, we can obtain the following result.

**Theorem 3.7.** The linear stability of $(0, \theta_b)$ is determined by $\pi(b)$, where $\pi(b)$ represents the principal eigenvalue of the following problem:
\[
\begin{aligned}

\Delta \chi + a(m_1(x) - B\phi'(0)\theta_b) \chi &= 0, & \text{in } \Omega,

\frac{\partial \chi}{\partial \nu} &= 0, & \text{on } \partial \Omega,
\end{aligned}
\]
and $\chi(b)$ denotes the associated eigenfunction. More precisely,

$$\bar{\lambda}(b) = \inf_{\chi} \frac{\int_{\Omega} |\nabla \chi|^2 dx}{\int_{\Omega} (m_1(x) - B\phi'(0)\theta_b)\chi^2 dx}. \quad (25)$$

Moreover, the semi-trivial solution $(0, \theta_b)$ is asymptotically stable if $a < \bar{\lambda}(b)$ and is unstable $a > \bar{\lambda}(b)$.

Similar to Lemma 3.6, we have the following results on the properties of $\bar{\lambda}(b)$.

**Lemma 3.8.** The function $\bar{\lambda}(b)$ defined by (25) satisfies

(i): $\bar{\lambda}(\cdot) \in C[\sigma_0, +\infty)$ and $\bar{\lambda}(\sigma_0) = \lambda_0$;

(ii): $\bar{\lambda}(\cdot) \in C^1(\sigma_0, +\infty)$, $\text{sgn}[\bar{\lambda}'(b)] = \text{sgn}[\theta'(b)],$ and

$$\lim_{b \to \sigma_0} \bar{\lambda}'(b) = -\frac{B\phi'(0)\lambda_0}{D\sigma_0} \cdot \frac{\int_{\Omega} \phi_1^2 \psi_1 \int_{\Omega} m_2(x) \psi_1^2 dx}{\int_{\Omega} \psi_1^2 dx \int_{\Omega} m_1(x) \phi_1^2 dx}. \quad (26)$$

In particular, if $m_2 \equiv 1$, then $\bar{\lambda}'(b) > 0$ for all $b \in (\sigma_0, +\infty)$.

### 3.3. Global stability

This subsection is devoted to the global stability of boundary steady states of (2). Note that

$$(f(u, v), g(u, v)) = (a(m_1(x)u - Au^2 - B\phi(u)v), b(m_2(x)v - Dv^2 + C\phi(u)v))$$
satisfies the Lipschitz condition in a bounded set $[u_1, u_2] \times [v_1, v_2]$. Therefore, by applying Theorem 12.5 in [30], we have

**Theorem 3.9.** Let $(u, v)$ be the positive solution of (2) with initial value $(u_0, v_0)$ satisfying $0 \leq u_0 \neq 0$ and $0 \leq v_0 \neq 0$, then

$$(u, v) \in (\pi, \pi^*), \quad (x, t) \in \Omega \times (0, \infty),$$

where $\pi$ and $\pi^*$ are the solutions to the following two equations, respectively:

$$\begin{align*}
\frac{\partial u(x, t)}{\partial t} &= \Delta u(x, t) + au(x, t) [m_1(x) - Au(x, t)], \quad x \in \Omega, \quad t \geq 0, \\
\frac{\partial u(x, t)}{\partial \nu} &= 0, \quad x \in \partial \Omega, \quad t \geq 0, \\
&u(x, 0) = u_0(x), \quad x \in \Omega,
\end{align*} \quad (27)$$

and

$$\begin{align*}
\frac{\partial v(x, t)}{\partial t} &= \Delta v(x, t) + bv(x, t) [m_2(x) - Dv(x, t) + C\phi(\pi)], \quad x \in \Omega, \quad t \geq 0, \\
\frac{\partial v(x, t)}{\partial \nu} &= 0, \quad x \in \partial \Omega, \quad t \geq 0, \\
v(x, 0) &= v_0(x), \quad x \in \Omega.
\end{align*} \quad (28)$$

**Theorem 3.10.** Let $(u, v)$ be a positive solution of (2), then

(i): If $a \leq \lambda_0, \ b \leq \sigma_0$, then $(u, v) \to (0, 0)$ as $t \to \infty$.

(ii): If $a > \lambda_0, \ b > \sigma_0$, then $(u, v) \to (\theta_a, 0)$ as $t \to \infty$.

(iii): If $a \leq \lambda_0, \ b > \sigma_0$, then $(u, v) \to (0, \theta_b)$ as $t \to \infty$.

**Proof.** (i) Assume that $(u, v)$ is the positive solution of (2) with $a \leq \lambda_0, \ b \leq \sigma_0$, and the initial condition $(u(x, 0), v(x, 0)) = (u_0(x), v_0(x))$, and that $\bar{\lambda}$ is the unique solution of (27). According to Theorem 3.9, we have $0 \leq u(x, t) \leq \bar{\lambda}(x, t)$. Thus, if $a \leq \lambda_0$ then $\bar{\lambda}(x, t) \to 0$ uniformly as $t \to \infty$ (see more details in [33]), and hence
$u(x, t) \to 0$ uniformly as $t \to \infty$. For every given $\varepsilon > 0$, there exists $T_{\varepsilon} \geq 0$ such that $u(x, t) \leq \varepsilon$ for all $t \geq T_{\varepsilon}$. Thus,

$$\frac{\partial v}{\partial t} - \Delta v = bv [m_2(x) - Dv + C\phi(u)] \leq bv [m_2(x) - Dv + C\phi(\varepsilon)] \text{ for all } t \geq T_{\varepsilon}.$$  

Let $v_{\varepsilon}$ be the solution of the following equation

$$\begin{cases}
\frac{\partial v_{\varepsilon}}{\partial t} = \Delta v_{\varepsilon} + bv_{\varepsilon} [m_2(x) - Dv_{\varepsilon} + C\phi(\varepsilon)], & x \in \Omega, t \geq 0, \\
\frac{\partial v_{\varepsilon}}{\partial \nu} = 0, & x \in \partial \Omega, t \geq 0, \\
v_{\varepsilon}(x, T_{\varepsilon}) = v(x, T_{\varepsilon}), & x \in \Omega.
\end{cases}$$

By applying the comparison principle, we have $0 \leq v(x, t) \leq v_{\varepsilon}(x, t)$ for $t \geq T_{\varepsilon}$. It follows from [33] again that $v_{\varepsilon}(x, t) \to 0$ as $\varepsilon \to 0$ and $t \to \infty$ when $b \leq \sigma_0$.

(ii) It follows from Theorem 3.9 that $u(x, t) \leq \bar{u}(x, t)$. If $a > \lambda_0$ then $\bar{u}(x, t) \to \theta_a$ uniformly as $t \to \infty$ by (see [15] for more details). For any $\varepsilon > 0$ small enough, there exists $T_{\varepsilon}' \geq 0$ such that

$$u(x, t) \leq \theta_a + \varepsilon \text{ for all } t \geq T_{\varepsilon}'.$$  

(29)

On the other hand,

$$\frac{\partial v}{\partial t} - \Delta v = bv [m_2(x) - Dv + C\phi(u)] \leq bv [m_2(x) - Dv + C\phi(\theta_a + \varepsilon)].$$

Let $\varepsilon \to 0$, then $v(x, t) \to 0$ as $t \to \infty$ when $b < \delta(a)$. Hence, there exists some $T_{\varepsilon}''$ such that $v(x, t) \leq \varepsilon$ for all $t > T_{\varepsilon}''$, and hence

$$\frac{\partial u}{\partial t} - \Delta u = a [m_1(x)u - Au^2 - B\phi(u)v] \geq a [m_1(x)u - Au^2 - B\phi(u)\varepsilon] \geq au [(m_1(x) - B\|\phi\|\varepsilon) - Au].$$

Note that

$$a > \lambda_0 = \inf_\varphi \frac{\int_\Omega ||\varphi||^2 dx}{\int_\Omega m_1(x)\varphi^2 dx},$$

then there exists sufficiently small $\varepsilon > 0$ such that

$$a > \inf_\varphi \frac{\int_\Omega \|\varphi\|^2 dx}{\int_\Omega (m_1(x) - B\|\phi\|\varepsilon)\varphi^2 dx}.$$  

Let $u_{\varepsilon}$ be the unique positive solution to the following equation

$$\begin{cases}
\frac{\partial u_{\varepsilon}}{\partial t} = \Delta u_{\varepsilon} + au_{\varepsilon} [(m_1(x) - B\|\phi\|\varepsilon) - Au_{\varepsilon}], & x \in \Omega, t \geq 0, \\
\frac{\partial u_{\varepsilon}}{\partial \nu} = 0, & x \in \partial \Omega, t \geq 0, \\
u_{\varepsilon}(x, T_{\varepsilon}'') = u(x, T_{\varepsilon}''), & x \in \Omega.
\end{cases}$$

By virtue of the comparison principle again, we obtain

$$u(x, t) \geq u_{\varepsilon}(x, t) \text{ for all } t \geq T_{\varepsilon}''.$$

Note that $u_{\varepsilon}(x, t) \to \bar{u}(x, t)$ as $\varepsilon \to 0$ and that $\bar{u}(x, t) \to \theta_a$ as $t \to \infty$, where $\bar{u}(x, t)$ is the unique solution of the initial value problem (27). Then we have $u(\cdot, t) \to \theta_a$ as $t \to \infty$, that is to say, $(u, v) \to (\theta_a, 0)$ as $t \to \infty$. 

(iii) It follows from Theorem 3.9 that \( u(x, t) \leq \overline{u}(x, t) \). If \( a \leq \lambda_0 \) then \( \overline{u}(x, t) \to 0 \) uniformly as \( t \to \infty \) by [15]. Thus, for any \( \varepsilon > 0 \) small enough, there exists \( T''_\varepsilon \geq 0 \) such that
\[
u(x, t) \leq \varepsilon \text{ for all } t \geq T''_\varepsilon.
\]
On the other hand,
\[
be(m_2(x) - Dv) \leq \frac{\partial v}{\partial t} - \Delta v \leq be(m_2(x) - Dv + C\phi(\varepsilon)).
\]
Using a similar argument as above, we see that \( v(x, t) \to \theta_b \) as \( t \to \infty \) for \( b > \sigma_0 \), and hence that \( (u, v) \to (0, \theta_b) \) as \( t \to \infty \). The proof is completed.

4. Coexistence steady-state solutions. In this section, we consider the nonexistence, existence, and stability of coexistence solutions to (3).

4.1. Nonexistence of coexistence solutions. First of all, we have the following result on the nonexistence of the coexistence steady-state solutions when the maximal growth rates are sufficiently small.

**Lemma 4.1.** If \( (u, v) \) is a nonnegative steady state solution of (3) satisfying \( u \neq 0 \) and \( v \neq 0 \), then both \( u \) and \( v \) are strictly positive on \( \Omega \) and \( a > \lambda_0, b > \tilde{\sigma}_0 \), where
\[
\tilde{\sigma}_0 = \inf_\varphi \frac{\int_\Omega |\nabla \varphi|^2 \, dx}{\int_\Omega [m_2(x) + C\phi(1)] \, dx}.
\]

**Proof.** Since \( u \geq 0, u \neq 0 \), it follows from (3) and the maximum principle that \( u > 0 \) on \( \Omega \). Multiplying the first equation of (3) by \( u \) and integrating over \( \Omega \), we obtain
\[
\int_\Omega |\nabla u|^2 \, dx = \int_\Omega au [m_1(x)u - Au^2 - B\phi(u)v] \, dx < a \int_\Omega m_1(x)u^2 \, dx.
\]
On the other hand, it follows from the variational property of the principle eigenvalue that
\[
\int_\Omega |\nabla u|^2 \, dx \geq \lambda_0 \int_\Omega m_1(x)u^2 \, dx.
\]
Hence \( a > \lambda_0 \). Similarly, we have \( b > \tilde{\sigma}_0 \). \( \square \)

**Lemma 4.2.** If \( (u, v) \) is a nonnegative steady state solution of (3) satisfying \( u \neq 0 \) and \( v \neq 0 \), then
\[
0 < u \leq \theta_a \quad \text{and} \quad \theta_b \leq v < \frac{1}{D} \left[ 1 + C\phi \left( \frac{1}{A} \right) \right] \quad \text{on} \quad \overline{\Omega}.
\]

**Proof.** From Lemma 4.1 it follows that \( u > 0, v > 0 \) on \( \overline{\Omega} \). Note that \( \theta_a \) is the unique positive solution of (9) and that \( \underline{u} = u \) is a subsolution of (9), then we have \( \theta_a \geq u \). Similarly, we observe that \( \overline{v} = v \) is a supersolution of (10), and hence \( \theta_b \leq v \). On the other hand, taking \( v(x_0) = \max_{x \in \overline{\Omega}} v(x) \) and then applying the maximum principle, we obtain
\[
v(x) \leq v(x_0) \leq \frac{1}{D} \left[ m_2(x_0) + C\phi(u(x_0)) \right]
\]
\[
\leq \frac{1}{D} \left[ \max_{\overline{\Omega}} m_2(x) + C\phi(\theta_a(x_0)) \right] < \frac{1}{D} \left[ 1 + C\phi \left( \frac{1}{A} \right) \right],
\]
and hence complete the proof of this lemma. \( \square \)
4.2. Bifurcation from the trivial solution. We investigate the existence and stability of positive solutions \((u, v)\) to (3) with \((a, b, u, v)\) lying in a neighbourhood of \((\lambda_0, \sigma_0, 0, 0)\). For this purpose, define a nonlinear mapping \(F: \mathbb{R} \times \mathbb{R} \times X \to Y\) by

\[
F(a, b; U) = \begin{pmatrix}
\Delta u + a(m_1(x)u - Au^2 - B\phi(u)v) \\
\Delta v + b(m_2(x)v - Dw^2 + C\phi(w)v)
\end{pmatrix}
\]

for \(U = (u, v)^T \in X\). Then \(F(a, b; 0) = 0\) for all \((a, b)\). Let \(T\) be the linearized operator of \(F(a, b; U)\) with respect to \(U\) evaluated at \((\lambda_0, \sigma_0; 0)\), i.e.,

\[
T = \begin{pmatrix}
\Delta + am_1(x) & 0 \\
0 & \Delta + bm_2(x)
\end{pmatrix}.
\]

It’s clear that \(T\) is a Fredholm operator and zero is a double eigenvalue of \(T\) with associated eigenfunctions \(\Phi = (\varphi_1, 0)^T\) and \(\Psi = (0, \psi_1)^T\), and hence \(\text{Ker}(T) = \text{span}\{\Phi, \Psi\}\) and \(\dim[\text{Ker}(T)] = \text{codim}[\text{Ran}(T)] = 2\). Moreover, we have \((k_1, k_2)^T \in \text{Ran}(T)\) if and only if \((k_1, \varphi_1) = (k_2, \psi_1) = 0\).

Clearly, the Crandall-Rabinowitz bifurcation theorem [3, 4] does not work here. Hence, we shall resort to Lyapunov-Schmidt reduction method [12, 13, 26, 14, 28, 44, 50, 51].

Firstly, we define the operator \(P\) by \(PU = \langle u, \varphi_1 \rangle \Phi + \langle v, \psi_1 \rangle \Psi\) for \(U = (u, v)^T \in X\) and decompose \(X\) as \(X = X_1 \oplus X_2\) with \(X_1 = PX\) and \(X_2 = (I-P)X\). Similarly, \(Y\) can be decomposed as \(Y = Y_1 \oplus Y_2\) with \(Y_2 = \text{Ran}(T)\). Next, we apply the implicit function theorem to look for the following kind of solutions to \(F(a, b; U) = 0\):

\[
U = s(\cos \omega \Phi + \sin \omega \Psi + W(s)), \quad W(s) = (w_1(s), w_2(s))^T \in X_2,
\]

with \(s, \omega \in \mathbb{R}\). We shall restrict \(\omega\) to \((0, \frac{\pi}{2})\) since in population models, negative steady states are meaningless and what we are only interested in is the existence and stability of positive solutions.

For each fixed \(\omega \in (0, \frac{\pi}{2})\), we define a new nonlinear mapping \(H(\alpha, \beta, W; s): \mathbb{R} \times \mathbb{R} \times X_2 \times \mathbb{R} \to Y\) by

\[
H(\alpha, \beta, W; s) = s^{-1}F(\lambda_0 + \alpha(s), \sigma_0 + \beta(s), s(\cos \omega \Phi + \sin \omega \Psi + W(s))).
\]

It is obvious that \(H: \mathbb{R} \times \mathbb{R} \times X_2 \times \mathbb{R} \to Y\) is continuously differentiable and satisfies \(H(0, 0, 0, 0) = 0\). The Fréchet derivative of \(H\) with respect to \((\alpha, \beta, W)\) at \((0, 0, 0, 0)\) takes the form

\[
H_{(\alpha, \beta, W)}(0, 0, 0; 0)(\hat{\alpha}, \hat{\beta}, \hat{W}) = T\hat{W} + \hat{\alpha}m_1 \cos \omega \Phi + \hat{\beta}m_2 \sin \omega \Psi
\]

for \((\hat{\alpha}, \hat{\beta}, \hat{W}) \in \mathbb{R} \times \mathbb{R} \times X_2\). We first claim that \(H(\alpha, \beta, W; s): \mathbb{R} \times \mathbb{R} \times X_2 \times \mathbb{R} \to Y\) is an isomorphism, i.e., \(H_{(\alpha, \beta, W)}(0, 0, 0; 0)(\hat{\alpha}, \hat{\beta}, \hat{W})\) is not only a injection but also a surjection. In fact, let

\[
H_{(\alpha, \beta, W)}(0, 0, 0; 0)(\hat{\alpha}, \hat{\beta}, \hat{W}) = 0,
\]

that is,

\[
\begin{align*}
\Delta \hat{w}_1 + \lambda_0 m_1(x) \hat{w}_1 + \hat{\alpha}m_1(x) \cos \omega \varphi_1 &= 0, \\
\Delta \hat{w}_2 + \sigma_0 m_2(x) \hat{w}_2 + \hat{\beta}m_2(x) \sin \omega \psi_1 &= 0.
\end{align*}
\]

In view of the decomposition of \(Y\), we have

\[
T\hat{W} = 0 \quad \text{and} \quad \hat{\alpha}m_1(x) \cos \omega \varphi_1 = \hat{\beta}m_2(x) \sin \omega \psi_1 = 0.
\]
Notice that $T$ is an isomorphism from $X_2$ to $Y_2$ and hence we have $\widehat{W} = 0$. Further, $\varphi_1, \psi_1 > 0$ in $\Omega$ and $\omega \in (0, \frac{\pi}{2})$, then we obtain $\hat{\alpha} = \hat{\beta} = 0$. This implies that $H_{(\alpha, \beta, W)}(0, 0, 0; 0)$ is injective.

Next, we shall show that $H_{(\alpha, \beta, W)}(0, 0, 0; 0)$ is surjective. It suffices to show that for every given $(h, l)^T \in Y$ there exists $(\hat{\alpha}, \hat{\beta}, \hat{W}) \in \mathbb{R} \times \mathbb{R} \times X_2$ such that $H_{(\alpha, \beta, W)}(0, 0, 0; 0)(\hat{\alpha}, \hat{\beta}, \hat{W}) = (h, l)^T$. By applying the decomposition of $Y$ again, we see that

$$(h, l)^T = (h_1, l_1)^T + (h_2, l_2)^T,$$

with $(h_1, l_1)^T \in \mathbb{Y}_1$ and $(h_2, l_2)^T \in \mathbb{Y}_2$, and hence

$$
\begin{align*}
T \hat{W} &= (h_2, l_2)^T, \\
\hat{\alpha} m_1(x) \cos \omega \Phi + \hat{\beta} m_2(x) \sin \omega \Psi = (h_1, l_1)^T.
\end{align*}
$$

Since $T$ is an isomorphism from $X_2$ to $Y_2$, and hence we have $\hat{W} = T^{-1}(h_2, l_2)^T$. On the other hand, we have

$$
\hat{\alpha} = (m_1(x) \cos \omega \varphi_2)^{-1} h_1 \quad \text{and} \quad \hat{\beta} = (m_2(x) \sin \omega \psi_1)^{-1} l_1.
$$

Therefore, we can find out $(\hat{\alpha}, \hat{\beta}, \hat{W}) \in \mathbb{R} \times \mathbb{R} \times X_2$ such that

$$H_{(\alpha, \beta, W)}(0, 0, 0; 0)(\hat{\alpha}, \hat{\beta}, \hat{W}) = (h, l)^T.$$ 

This means that $H_{(\alpha, \beta, W)}(0, 0, 0; 0)$ is surjective. Hence, $H_{(\alpha, \beta, W)}(0, 0, 0; 0) : \mathbb{R} \times \mathbb{R} \times X_2 \times \mathbb{R} \rightarrow \mathbb{Y}$ is an isomorphism.

From the implicit function theorem it follows that there exist a continuously differentiable mappings $(\hat{\alpha}(s), \hat{\beta}(s), \hat{W}(s))$ for sufficiently small $s > 0$ such that $\hat{\alpha}(0) = \hat{\beta}(0) = 0$, $\hat{W}(0) = (0, 0)^T$ and $H(\hat{\alpha}(s), \hat{\beta}(s), \hat{W}(s)) = 0$, where $\hat{W}(s) = (\hat{w}_1(s), \hat{w}_2(s))^T$ satisfies $\langle \hat{w}_1(s), \varphi_1 \rangle = 0$ and $\langle \hat{w}_2(s), \psi_1 \rangle = 0$. Set

$$
\begin{align*}
\hat{\alpha}(s) &= \lambda_0 + \hat{\alpha}(s), \\
\hat{\beta}(s) &= \sigma_0 + \hat{\beta}(s), \\
\hat{u}(s) &= s(\cos \varphi_1 + \hat{\omega}_1(s)), \\
\hat{v}(s) &= s(\sin \psi_1 + \hat{\omega}_2(s)),
\end{align*}
$$

then a positive solution $(\hat{\alpha}(s), \hat{\beta}(s), \hat{W}(s))$ of $F(a, b; U) = 0$ emerges from $(\lambda_0, \sigma_0; (0, 0)^T)$, where $\hat{U}(s) = (\hat{u}(s), \hat{v}(s))^T$. Thus, we obtain the following result.

**Theorem 4.3.** $(\lambda_0, \sigma_0; (0, 0)^T)$ is a bifurcation point of $F(a, b; U) = 0$. More precisely, there exists a curve of non-constant positive solutions $(\hat{\alpha}(s), \hat{\beta}(s), \hat{W}(s))$ of $F(a, b; U) = 0$ for sufficiently small $s > 0$, where $\hat{\alpha}(s), \hat{\beta}(s), \hat{U}(s)$ are defined by (31) and (32).

In the following, we shall investigate the asymptotical stability of $(\hat{\alpha}(s), \hat{\beta}(s))$. It suffices to consider the following eigenvalue problem

$$H_0(\hat{\alpha}(s), \hat{\beta}(s); \hat{U}(s)) \chi = \gamma(s) \chi,$$

with $\gamma(0) = 0$. And we want to find out the eigenfunctions $\chi$ of the form

$$\chi = \Phi + p\Psi + \overline{W}(s), \quad \overline{W} = (\overline{w}_1, \overline{w}_2)^T \in X_2,$$

with $\overline{W}(s)$ satisfying $\overline{W}(0) = (0, 0)^T$. Substituting (34) into (33) yields

$$
\gamma(s) (\varphi_1 + \overline{w}_1) + \hat{\alpha}(m_1(x) - 2A\hat{u} - B\phi'(\hat{u})\hat{v}) (\varphi_1 + \overline{w}_1) - \hat{\alpha}B\phi(\hat{u})(p\psi_1 + \overline{w}_2),
$$

(35)
and
\[ \gamma(s)(p\psi_1 + w_2) = \Delta(p\psi_1 + w_2) + \hat{b}(m_2(x) - 2D\hat{v} + C\phi(\hat{u}))(p\psi_1 + w_2) + \hat{b}C\phi'(\hat{u})(\varphi_1 + w_1). \] (36)

Multiplying (35) by \( \varphi_1 \), integrating by parts and using Green’s formula, we have
\[ \gamma(s) = \int_\Omega \left[ \Delta(\varphi_1 + w_1)\varphi_1 + (\lambda_0 + \hat{\alpha}(s))(m_1(x) - 2A\hat{u} - B\phi'(\hat{u})\hat{v})(\varphi_1 + w_1)\varphi_1 \right. \]
\[ \left. \quad - \hat{a}B\phi(\hat{u})(p\psi_1 + w_2)\varphi_1 \right] dx \]
\[ = \hat{\alpha}(s) \int_\Omega m_1(\varphi_1 + w_1)\varphi_1 dx - (\lambda_0 + \hat{\alpha}(s))(\int_\Omega (\Phi + p\Psi + \mathbb{W}), \Phi) \] (37)

Similarly, multiplying (36) by \( \psi_1 \), integrating by parts and using Green’s formula, we obtain
\[ p\gamma(s) = \int_\Omega \left[ \Delta(p\psi_1 + w_2)\psi_1 + (\sigma_0 + \hat{\beta})(m_2(x) - 2D\hat{v} + C\phi(\hat{u}))(p\psi_1 + w_2)\psi_1 \right. \]
\[ \left. \quad + \hat{b}C\phi'(\hat{u})(\varphi_1 + w_1)\psi_1 \right] dx \]
\[ = \hat{\beta}(s) \int_\Omega m_2(p\psi_1 + w_2)\psi_1 dx + (\sigma_0 + \hat{\beta}(s)) \int_\Omega (\Phi - 2D\hat{v})(p\psi_1 + w_2)\psi_1 \]
\[ \left. \quad + C\phi'(\hat{u})\hat{v}(\varphi_1 + w_1)\psi_1 \right] dx \]
\[ = \hat{\beta}(s) \int_\Omega m_2(p\psi_1 + w_2)\psi_1 dx + (\sigma_0 + \hat{\beta}(s))\int_\Omega (\Phi + p\Psi + \mathbb{W}), \Psi), \] (38)

where
\[ N(s) = \begin{pmatrix} 2A\hat{u} + B\phi'(\hat{u})\hat{v} & B\phi(\hat{u}) \\ C\phi'(\hat{u})\hat{v} & C\phi(\hat{u}) - 2D\hat{v} \end{pmatrix}. \]

**Lemma 4.4.**

\[ \lim_{s \to 0} \frac{\gamma(s)}{s} = \gamma'(0) = -\lambda_0 \cos \omega \left[ A \int_\Omega \varphi_1^2 dx + pB\phi'(0) \int_\Omega \varphi_1^2 \psi_1 dx \right] \]
\[ = \sigma_0 \sin \omega \left[ -D \int_\Omega \psi_1^2 dx + \frac{1}{p} C\phi'(0) \int_\Omega \varphi_1^2 \psi_1^2 dx \right]. \] (39)

**Proof.** In view of (54) and (55), for sufficiently small \( s > 0 \), we have
\[ \hat{\alpha}(s) = s^{-\lambda_0} \int_\Omega \left[ A \cos \omega \varphi_1 + B\phi'(0) \sin \omega \varphi_1 \right] \varphi_1^2 dx + o(s) \triangleq \hat{\alpha}'(0)s + o(s) \] (40)

and
\[ \hat{\beta}(s) = s \sigma_0 \int_\Omega \left[ D \sin \omega \psi_1 - C\phi'(0) \cos \omega \varphi_1 \right] \psi_1^2 dx + o(s) \triangleq \hat{\beta}'(0)s + o(s). \] (41)

By (32), we have
\[ \hat{u}(s) = s(\cos \omega \varphi_1 + \hat{\omega}_1(s)), \quad \hat{v}(s) = s(\sin \omega \psi_1 + \hat{\omega}_2(s)). \]
Substituting (40) and (41) and the above equation into (37) and (38) and dividing by \(s\) both sides of (37) and (38), respectively, we obtain

\[
\frac{\gamma(s)}{s} = \bar{\alpha}'(0) \int_{\Omega} m_1 \varphi_1^2 dx - \lambda_0 \int_{\Omega} [2A \cos \omega \varphi_1 + B \phi'(0) \sin \omega \psi_1] \varphi_1^2 dx \\
- \lambda_0 p B \phi'(0) \int_{\Omega} \cos \omega \varphi_1^2 \psi_1 dx + \frac{o(s)}{s} = -\lambda_0 \left[ A \cos \omega \int_{\Omega} \varphi_1^2 dx + p B \phi'(0) \int_{\Omega} \cos \omega \varphi_1^2 \psi_1 dx \right] + \frac{o(s)}{s},
\]

and

\[
p \frac{\gamma(s)}{s} = p \bar{\beta}'(0) \int_{\Omega} m_2 \psi_1^2 dx + p \sigma_0 \int_{\Omega} [C \phi'(0) \cos \omega \varphi_1 - 2D \sin \omega \psi_1] \psi_1^2 dx \\
+ \sigma_0 C \phi'(0) \int_{\Omega} \sin \omega \varphi_1 \psi_1^3 dx + \frac{o(s)}{s} = \sigma_0 \left[ -D \sin \omega \int_{\Omega} \psi_1^3 dx + C \phi'(0) \int_{\Omega} \sin \omega \varphi_1 \psi_1^2 dx \right] + \frac{o(s)}{s},
\]

where we used the Taylor expression of \(\phi(x) = \phi'(0)x + o(x)\) as \(x\) is sufficiently close to zero since \(\phi(0) = 0\). Taking the limits in (42) and (43) yields the conclusion of this lemma.

In order to determine the sign of \(\gamma(s)\) for sufficiently small \(s\), it follows from Lemma 4.4 that we still need to compute the expression of \(p\). A series of straightforward but tedious computations show that

\[
\langle N(s) \Phi, \Phi \rangle = s \int_{\Omega} [2A \cos \omega \varphi_1 + B \phi'(0) \sin \omega \psi_1] \varphi_1^2 dx + o(s) \triangleq sa_{11} + o(s),
\]

\[
\langle N(s) \Psi, \Phi \rangle = sB \phi'(0) \int_{\Omega} \cos \omega \varphi_1 \psi_1^2 dx + o(s) \triangleq sa_{21} + o(s),
\]

\[
\langle N(s) \Psi, \Psi \rangle = s \int_{\Omega} [C \phi'(0) \cos \omega \varphi_1 - 2D \sin \omega \psi_1] \psi_1^2 dx + o(s) \triangleq sa_{22} + o(s).
\]

In view of Lemma 4.4, we have \(\gamma(s) = \gamma'(0)s + o(s)\) for \(s\) sufficiently small and hence

\[
\langle N(s) \Phi, \Psi \rangle = s C \phi'(0) \int_{\Omega} \sin \omega \varphi_1 \psi_1^2 dx + o(s) \triangleq sa_{12} + o(s).
\]

Thus, we have

\[
\gamma(s) = s \lambda_0 \left[ \int_{\Omega} (A \cos \omega \varphi_1 + B \phi'(0) \sin \omega \psi_1) \varphi_1^2 dx - a_{11} - pa_{21} \right] + o(s).
\]

Recall (1), then we have

\[
\lambda_0 a_{21} p^2 + \varepsilon p + a_{12} \sigma_0 + o(1) = 0,
\]

where \(\varepsilon = A \lambda_0 \int_{\Omega} \cos \omega \varphi_1^2 dx - \sigma_0 D \int_{\Omega} \sin \omega \psi_1^2 dx\). It follows from (45) that

\[
p = p_{\pm} \triangleq \frac{-\varepsilon \pm \sqrt{\varepsilon}}{2 \lambda_0 a_{21}}.
\]
where $\zeta = \varepsilon^2 - 4\lambda_0\sigma_0a_2a_{12}$, that is,
\[
\zeta = \left[ A\lambda_0 \int_{\Omega} \cos \omega \varphi_1^3 dx - \sigma_0 D \int_{\Omega} \sin \omega \psi_1^3 dx \right]^2
- 4BC[\phi'(0)]^2 \lambda_0\sigma_0 \int_{\Omega} \cos \omega \varphi_1^3 dx \int_{\Omega} \sin \omega \psi_1^3 dx.
\]
Substituting (46) into (44) yields
\[
\gamma_{\pm}(s) = -\frac{s}{2} \left[ \sigma_0 D \int_{\Omega} \sin \omega \psi_1^3 dx \pm \sqrt{\zeta} \right] + o(s).
\]

In what follows, we consider the sign of $\gamma_{\pm}(s)$. If $\zeta \leq 0$, it is obvious that $\Re(\gamma_{\pm}(s)) < 0$ for sufficiently small $s > 0$; if $\zeta > 0$, then $\gamma_{+}(s) < 0$. Hence, it suffices to determine the sign of $\gamma_{-}(s)$ in the case of $\zeta > 0$. Set
\[
\hat{I} = A^2\lambda_0 \int_{\Omega} \cos \omega \varphi_1^3 dx - (2AD + 4BC[\phi'(0)]^2) \sigma_0 \int_{\Omega} \sin \omega \psi_1^3 dx,
\]
then we have
(i) If $\hat{I} < 0$, then $\sqrt{\zeta} < \sigma_0 D \int_{\Omega} \sin \omega \psi_1^3 dx$, that is, $\gamma_{-}(s) < 0$ for sufficiently small $s$;
(ii) If $\hat{I} > 0$, then $\sqrt{\zeta} > \sigma_0 D \int_{\Omega} \sin \omega \psi_1^3 dx$, that is, $\gamma_{-}(s) > 0$ for sufficiently small $s$.

Therefore, we have the following result.

**Theorem 4.5.** For each $(a, b)$ lying in a small enough neighbourhood of $(\lambda_0, \sigma_0)$, (3) has a coexistence solution, which is asymptotically stable (respectively, unstable) if either $\zeta \leq 0$ or $\zeta > 0$ and $\hat{I} < 0$ (respectively, $\zeta > 0$ and $\hat{I} > 0$).

**Remark 1.** If $m_1 = m_2$ on $\overline{\Omega}$, then $\lambda_0 = \sigma_0$ and $\varphi_1 = \psi_1$. Assume further that $A^2 < 2AD + 4BC[\phi'(0)]^2$ and $\omega \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, then $\hat{I} < 0$ and hence the coexistence solution established in Theorem 4.5 is asymptotically stable. However, if $A^2 > 2AD + 4BC[\phi'(0)]^2$ and $\omega \in \left[0, \frac{\pi}{4}\right]$, then $\hat{I} > 0$ and hence the coexistence solution is unstable.

4.3. **Bifurcation from the semi-trivial solution $(\theta_a, 0)$.** First of all, we introduce the following useful bifurcation result (see Theorem 1.7 of [3] for the detailed proof).

**Lemma 4.6.** Suppose that $\lambda_0 \in \mathbb{R}$ and $F : \mathbb{R} \times X \to Y$ is a twice continuously differentiable mapping and also that
(i): $F(\lambda, 0) = 0$ for $\lambda \in I$,
(ii): $\dim \ker(F_x(\lambda_0, 0)) = \text{codim} \text{Ran}(F_x(\lambda_0, 0)) = 1$,
(iii): $F_{xx}(\lambda_0, 0)x_0 \notin \text{Ran}(F_x(\lambda_0, 0))$ where $x_0 \in X$ spans $\ker(F_x(\lambda_0, 0))$.

Let $X_1$ be any complement of $\text{span}\{x_0\}$ in $X$. Then there exist an open interval $\hat{I}$ containing 0 and a continuously differentiable function $\lambda : \hat{I} \to X_1$ such that $\lambda(0) = \lambda_0$, $\psi(0) = 0$, and $x(s) = sx_0 + s\psi(s)$ satisfies $F(\lambda(s), x(s)) = 0$. Moreover, $F^{-1}(\{0\})$ near $(\lambda_0, 0)$ consists precisely of $x = 0$ and the curves $(\lambda(s), x(s)), s \in \hat{I}$.

In this subsection, we discuss the existence and stability of coexistence solutions to (3) bifurcating from $(\theta_a, 0)$ near $b = \overline{b}(a)$. 
Theorem 4.7. Assume that \( a > \lambda_0 \). Then near \( b = \tilde{b}(a) \) a steady-state bifurcation occurs at the semi-trivial steady-state \((\theta_a, 0)\) of system (3). More precisely, there exists a curve of non-constant coexistence mappings \((b(s); u(s), v(s))\) for sufficiently small \( s > 0 \), satisfying \( b(s) = \tilde{b}(a) + o(s), \ u(s) = \theta_a + s(z_1 + \rho(s)), \) and \( v(s) = s(\tilde{\psi}(a) + \kappa(s)) \), such that system (3) with \( b = b(s) \) has a coexistence solution \((u(s), v(s))\), where \( \tilde{\psi}(a) \) is given in Theorem 3.5, \(((\rho(s), \kappa(s))^T, (z_1, \tilde{\psi}(a))^T) = 0, \) and \( z_1 \) satisfies
\[
(\Delta + L_1)z_1 = aB\phi(\theta_a)\tilde{\psi}(a).
\]

Proof. Let
\[
G(b, z, \chi) = \left( (\Delta(\theta_a + z) + a(m_1(\theta_a + z) - A(\theta_a + z)^2 - B\phi(\theta_a + z)\chi)) + b(m_2\chi - D\chi^2 + C\phi(\theta_a + z)\chi) \right),
\]
then \((u, v) = (\theta_a + z, \chi)\) is a solution to (3) if and only if \((z, \chi)\) is a solution to \(G(b, z, \chi) = 0\). Clearly, \(G(b, 0, 0) = 0\). Our purpose is to find out nontrivial coexistence solutions to \(G(b, z, \chi) = 0\). Let \(L\) be the Fréchet derivative of \(G\) with respect to \((z, \chi)\) at \((\tilde{b}(a), 0, 0)\), then we have
\[
L(z, \chi) = \begin{pmatrix}
\Delta z + L_1 z - aB\phi(\theta_a)\chi \\
\Delta \chi + b(\theta_a)(m_2 + C\phi(\theta_a))\chi
\end{pmatrix}.
\]
Thus, \(\text{Ker}(L) = \text{span}\{U_0\}\) with \(U_0 = (z_1, \tilde{\psi}(a))\), where \(\tilde{\psi}(a)\) is the principal eigenfunction of \(\tilde{b}(a)\) established in Theorem 3.5, and \(z = z_1 \triangleq aB(\Delta + L_1)^{-1}(\phi(\theta_a)\tilde{\psi}(a))\), and \((\Delta + L_1)^{-1}\) is the inverse of the operator \(\Delta + L_1\) given in Theorem 3.1.

On the other hand, the adjoint operator \(L^*\) of \(L\) is defined as
\[
L^*(z, \chi) = \begin{pmatrix}
\Delta z + L_1 z \\
\Delta \chi + b(\theta_a)(m_2 + C\phi(\theta_a))\chi - aB\phi(\theta_a)z
\end{pmatrix}.
\]
Note that all of the eigenvalues of \(\Delta + L_1\) are negative by Theorem 3.1 (ii), then
\[
\text{Ker}(L^*) = \text{span}\{U^*\}, \quad U^* = (0, \tilde{\psi}(a))^T.
\]
By applying the Fredholm alternative, the range of \(L\) can be figured out as follows:
\[
\text{Ran}(L) = \left\{ (z, \chi) \in X : \int_{\Omega} \chi \tilde{\psi}(a) dx = 0 \right\},
\]
which means that \(\text{codimRan}(L) = 1\). Let \(L_1\) be the mixed second-order partial derivative of \(G\) with respect to \((z, \chi)\) and \(b\) at \((\tilde{b}(a), 0, 0)\), then we have \(L_1(z_1, \tilde{\psi}(a)) = (0, (m_2 + C\phi(\theta_a))\tilde{\psi}(a)) \notin \text{Ran}(L)\). It follows from Lemma 4.6 that there exist \(\delta > 0\) and a continuously differentiable mapping \((b(s), \rho(s), \kappa(s)) : (-\delta, \delta) \rightarrow \mathbb{R} \times X\) such that \(b(0) = \tilde{b}(a), \ q(0) = 0, \ \rho(0) = 0, \ \kappa(0) = 0, \ (\rho(s), \kappa(s)) \in X_1, \) and
\[
(b(s), z(s), \chi(s)) = (\tilde{b}(a) + q(s), s(z_1 + \rho(s)), s(\tilde{\psi}(a) + \kappa(s)))
\]
satisfies \(G(b(s), z(s), \chi(s)) = 0\), where \(X = X_1 \oplus \text{Ker}(L)\). Thus, we conclude that \((b(s), u(s), v(s))\) with \(|s| < \delta\) is a solution branch of (3), where
\[
\begin{align*}
b(s) &= \tilde{b}(a) + q(s), \\
u(s) &= \theta_a + z(s) = \theta_a + s(z_1 + \rho(s)), \\
v(s) &= \chi(s) = s(\tilde{\psi}(a) + \kappa(s)).
\end{align*}
\]
The proof is completed. \(\square\)
In what follows, we shall determine the stability of the coexistence solution close to the bifurcation point \((b(a), \theta_a, 0)\). Let \(K : X \rightarrow Y\) be the inclusion mapping, and \(B(X, Y)\) denote the set of bounded linear maps of \(X\) into \(Y\).

**Definition 4.8** ([4]). Let \(T, K \in B(X, Y)\). Then \(\mu \in \mathbb{R}\) is a \(K\)-simple eigenvalue of \(T\) if

(i): \(\dim \ker(T - \mu K) = \operatorname{codim} \operatorname{ran}(T - \mu K) = 1\) and \(\ker(T - \mu K) = \text{span}\{x_0\}\),

(ii): \(Kx_0 \not\in \operatorname{ran}(T - \mu K)\).

We have the following result.

**Lemma 4.9.** \(0\) is a \(K\)-simple eigenvalue of \(L\) given in (47).

**Proof.** Suppose \(L(z, \chi) = 0\), that is,

\[
\begin{align*}
\Delta z + a(m_1(x) - 2A \theta_a)z - aB \phi(\theta_a) \chi &= 0, & & \text{in } \Omega, \\
\Delta \chi + b(m_2(x) + C \phi(\theta_a)) \chi &= 0, & & \text{in } \Omega, \\
\frac{\partial z}{\partial \nu} &= \frac{\partial \chi}{\partial \nu} = 0, & & \text{on } \partial \Omega.
\end{align*}
\]

This implies that \(\ker(L) = \text{span}\{(z_1, \chi_1)\}\), where

\[
\chi_1 \triangleq \psi(a), \quad z_1 = -aB(\Delta + L_1)^{-1}(\phi(\theta_a)\chi_1).
\]

On the other hand, we denote by \(L^*\) the adjoint operator of \(L\). \(L^*(z, \chi) = 0\) means that

\[
\begin{align*}
\Delta z + a(m_1(x) - 2A \theta_a)z &= 0, & & \text{in } \Omega, \\
\Delta \chi + b(m_2(x) + C \phi(\theta_a)) \chi &= aB \phi(\theta_a)z, & & \text{in } \Omega, \\
\frac{\partial z}{\partial \nu} &= \frac{\partial \chi}{\partial \nu} = 0, & & \text{on } \partial \Omega.
\end{align*}
\]

Recall the fact that all the eigenvalues of \(\Delta + L_1\) are negative, then we have \(z = 0\), and hence \(\ker(L^*) = \text{span}\{(0, \chi_1)\}\). Therefore, the range of \(L\) is given by

\[
\operatorname{ran}(L) = \left\{(u, v) \in Y : \int_\Omega v \cdot \chi_1 dx = 0\right\},
\]

and \(K(z_1, \chi_1) \not\in \operatorname{ran}(L)\), which completes the proof.

**Lemma 4.10.** \(0\) is the eigenvalue with the largest real part among all the eigenvalues of \(L\).

**Proof.** In view of Lemma 4.9, 0 is a simple eigenvalue of \(L\). Now, we will prove that 0 is the largest eigenvalue of \(L\). Let \(J = \Delta + b(a)(m_2(x) + C \phi(\theta_a))\). It is obvious that 0 is the principal eigenvalue of \(J\) with an eigenfunction \(\psi(a)\). If the conclusion is not true, then we can find an eigenvalue \(\lambda_*\) of \(L\) such that \(\Re\lambda_* > 0\), where \((\varphi, \chi)\) is the corresponding eigenfunction, then \(L(\varphi, \chi) = \lambda_* (\varphi, \chi)\), i.e.,

\[
\begin{align*}
\Delta \varphi + a(m_1(x) - 2A \theta_a) \varphi - aB \phi(\theta_a) \chi &= \lambda_* \varphi, & & \text{in } \Omega, \\
\Delta \chi + b(a)(m_2(x) + C \phi(\theta_a)) \chi &= \lambda_* \chi, & & \text{in } \Omega, \\
\frac{\partial \varphi}{\partial \nu} &= \frac{\partial \chi}{\partial \nu} = 0, & & \text{on } \partial \Omega.
\end{align*}
\]

If \(\chi = 0\), it follows from the fact that the operator \((L_1 - \lambda_*)\) is invertible that \(\varphi \equiv 0\), which is a contradiction. Thus, \(\chi \neq 0\), which means that \(\lambda_*\) is an eigenvalue of \(J\), and hence \(\lambda_* \in \mathbb{R}\). Therefore, we have \(\lambda_* > 0\), which contradicts the fact that 0
is the principal (maximal) eigenvalue of $J$ with an eigenfunction $\overline{\psi}(a)$. Thus we complete the proof of Lemma 4.10. \qed

In view of Corollary 1.13 and Theorem 1.16 of [4], we have the following result.

**Lemma 4.11.** Let $0$ be a $K$-simple eigenvalue of $L$, then there exists a $C^1$ mapping $s \to (\mu(s), U(s))$ from $\mathbb{R}$ into a neighbourhood of $(\overline{b}(a), 0)$ in $\mathbb{R} \times \mathbb{R}$ such that $(\mu(0), U(0)) = (0, (z_1, \overline{w}(a)))$ and that $U(s) = (\Phi(s), \Psi(s))$, $u(s) = \theta_a + s(z_1 + \rho(s))$, and $v(s) = s(\chi_1 + \kappa(s))$ with $\chi_1 \equiv \overline{w}(a)$ satisfy $L(b(s), u(s), v(s))U(s) = \mu(s)U(s)$ for $|s| \ll 1$, where

$$L(b, u, v) = \begin{pmatrix} \Delta + a(m_1(x) - 2Au - B\phi'(u)v) & -aB\phi(u) \\ bC\phi'(u)v & \Delta + b(m_2(x) - 2Dv + C\phi(u)) \end{pmatrix}.$$

In order to obtain the sign of $\mu(s)$ for $|s| \ll 1$ and $s > 0$, it suffices to investigate the sign of $\mu'(0)$ since $\text{sign}(\mu(s)) = \text{sign}(\mu'(0))$ for $|s| \ll 1$ and $s > 0$. In fact, we have the following result.

**Lemma 4.12.**

$$b'(0) = \frac{\overline{b}(a) \int_{\Omega} \chi_1^2(D\chi_1 + C\phi'(\theta_a)z_1)dx}{\int_{\Omega} \chi_1^2(m_2 + C\phi(\theta_a))dx}.$$ 

**Proof.** Substituting $(b(s), u(s), v(s))$ into the second equation of (3), and then dividing it by $s$, we obtain

$$\Delta(\chi_1 + \kappa(s) + b(s)[m_2(x) - Ds(\chi_1 + \kappa(s)) + C\phi(\theta_a + s(z_1 + \rho(s))]](\chi_1 + \kappa(s)) = 0.$$ 

Differentiating the both sides of the above equation with respect to $s$, and then setting $s = 0$, we obtain

$$\Delta\kappa'(0) + b'(0)(m_2(x) + C\phi(\theta_a))\chi_1 + \overline{b}(a)\kappa'(0)(m_2(x) + C\phi(\theta_a)) = \frac{b'(0)\chi_1^2(D\chi_1 + C\phi'(\theta_a)z_1)}{\int_{\Omega} \chi_1^2(m_2 + C\phi(\theta_a))dx},$$

where $\kappa'(0)$ is the differential of $\kappa$ with respect to $s$ at $s = 0$. Multiplying the above equation by $\chi_1$, integrating it over $\Omega$, and applying Green’s formula, we can easily obtain the conclusion. \qed

We now give another useful result which determines the stability of the solution established by Theorem 4.7.

**Lemma 4.13.** $\mu'(0) = \overline{b}(a) \int_{\Omega} \chi_1^2(C\phi'(\theta_a)z_1 - D\chi_1)dx$.

**Proof.** Assume that $\mu(s)$ is the eigenvalue of $L(b(s), u(s), v(s))$ at $u(s) = \theta_a + s(z_1 + \rho(s))$, $v(s) = s(\chi_1 + \kappa(s))$, and $(\Phi(s), \Psi(s))$ is the corresponding eigenfunction, that is,

$$\begin{align*}
\begin{cases}
\Delta \Phi(s) + a(m_1(x) - 2Au(s) - B\phi'(u(s))v(s))\Phi(s) - aB\phi(u(s))\Psi(s) \\
\Delta \Psi(s) + b(s)(m_2(x) - 2Dv(s) + C\phi(u(s)))\Psi(s) + b(s)C\phi'(u(s))v(s)\Phi(s)
\end{cases}
= \mu(s)\Phi(s) \text{ in } \Omega, \\
\begin{cases}
\frac{\partial \Phi(s)}{\partial \nu} = \frac{\partial \Psi(s)}{\partial \nu} = 0 \text{ on } \partial \Omega.
\end{cases}
\end{align*}$$

(49)

Note that $L(b(0), u(0), v(0)) = \mathcal{L}$ and $\text{Ker}(\mathcal{L}) = \{(z_1, \chi_1)\}$, $\text{Ker}(\mathcal{L}^*) = \{(0, \chi_1)\}$, then $(\Phi(s), \Psi(s))$ can be represented as

$$\Phi(s) = z_1 + sh_1(s, x), \quad \Psi(s) = \chi_1 + sh_2(s, x),$$

(50)
where \((h_1(s, x), h_2(s, x)) \in x_1\). If (49) has a solution \((\mu, (\Phi, \Psi)^T)\) with \(\Phi(s) = z_1 + s h_1(s, x)\) and \(\Psi(s) = \chi_1 + s h_2(s, x)\) for some \((h_1, h_2) \in X_1\), and then multiplying (49) by \((0, \chi_1)\) and integrating it on \(\Omega\) yield

\[
\mu(s) \int_{\Omega} \Psi(s) \chi_1 \, dx \\
= \int_{\Omega} \chi_1 [\Delta \Psi(s) + b(s) (m_2(x) - 2 D s (\chi_1 + \kappa(s))) + C \phi(\theta_a + s (z_1 + \rho(s))) + b(s) C \phi'(\theta_a + s (z_1 + \rho(s))) s (\chi_1 + \kappa(s)) (z_1 + s h_1(s, x))] \, dx
\]

It follows from (4.17), (49) and (50) that the above equation can be reduced to

\[
I \equiv \mu(s) \int_{\Omega} (\chi_1 + s h_2(s, x)) \chi_1 \, dx \\
= \int_{\Omega} \chi_1 [\Delta (\chi_1 + s h_2(s, x)) + b(s) (m_2(x) - 2 D s (\chi_1 + \kappa(s))) + C \phi(\theta_a + s (z_1 + \rho(s))) + b(s) C \phi'(\theta_a + s (z_1 + \rho(s))) s (\chi_1 + \kappa(s)) (z_1 + s h_1(s, x))] \, dx
\]

Dividing it by \(s\), taking the limit \(s \to 0\), and noticing the fact that \(\mu(0) = 0\), \(b(s) = \overline{b}(a) + q(s)\), and

\[
b'(0) = \overline{b}(a) \frac{\int_{\Omega} \chi_1^2 (D \chi_1 + C \phi'(\theta_a) z_1) \, dx}{\int_{\Omega} \chi_1 (m_2(x) + C \phi(\theta_a)) \, dx},
\]

we have

\[
\lim_{s \to 0} \frac{I}{s} = \mu'(0) \int_{\Omega} \chi_1^3 \, dx = \mu'(0),
\]

\[
\lim_{s \to 0} \frac{I_1}{s} = - \overline{b}(a) C \int_{\Omega} \phi'(\theta_a) z_1 \chi_1^2 \, dx + b'(0) \int_{\Omega} \chi_1^2 (m_2(x) + C \phi(\theta_a)) \, dx
\]

\[
= \overline{b}(a) D \int_{\Omega} \chi_1^3 \, dx,
\]

\[
\lim_{s \to 0} \frac{I_2}{s} = 0,
\]

\[
\lim_{s \to 0} \frac{I_3}{s} = - 2 \overline{b}(a) D \int_{\Omega} \chi_1^3 \, dx,
\]

\[
\lim_{s \to 0} \frac{I_4}{s} = \overline{b}(a) C \int_{\Omega} \phi'(\theta_a) z_1 \chi_1^2 \, dx.
\]

This completes the proof of Lemma 4.12. \(\square\)

In view of Lemmas 4.11 and 4.12, we have

**Theorem 4.14.** If \(D \equiv \int_{\Omega} \chi_1^2 (C \phi'(\theta_a) z_1 - D \chi_1) \, dx < 0\) (respectively, \(D > 0\)), then \(\mu(s) < 0\) (respectively, \(\mu(s) > 0\)) for \(0 < s \ll 1\) and the coexistence solutions \((u(s), v(s))\) defined by Theorem 4.7 are stable (respectively, unstable).
4.4. Bifurcating from the semi-trivial solution \((0, \theta_b)\). Similarly to the discussion about the case of \((\tilde{b}(a); \theta_a, 0)\), we have the following results on the case of \((\overline{\pi}(b); 0, \theta_b)\).

**Theorem 4.15.** Assume that \(b > \sigma_0\). Then near \(b = \overline{\pi}(b)\) a steady-state bifurcation occurs at the semi-trivial steady-state \((0, \theta_b)\) of system (3). More precisely, there exists a curve of non-constant coexistence mappings \((a(s); \overline{\pi}(s), \overline{\nu}(s))\) for sufficiently small \(s > 0\), satisfying \(a(s) = \overline{\pi}(b) + o(s)\), \(\overline{\pi}(s) = s(1(b) + \overline{\nu}(s))\), \(\nu(s) = \theta_b + s(1 + \overline{\nu}(s))\), such that system (3) with \(a = a(s)\) has a coexistence solution \((\overline{\pi}(s), \overline{\nu}(s))\), where \((\overline{\pi}(s), \overline{\nu}(s))^\top, (\overline{1}(b), \overline{1})^\top) = 0\), and \(\overline{1}\) satisfies

\[
(\Delta + L_2)\overline{1} = bC\phi'(0)\phi(\theta_b)\overline{1}(b).
\]

Let

\[
\overline{L}(a, u, v) = \begin{pmatrix}
\Delta + a(m_1(x) - 2Au - B\phi'(u)v) & -aB\phi(u) \\
bC\phi'(u)v & \Delta + b(m_2(x) - 2Dv + C\phi(u))
\end{pmatrix}.
\]

Then we have the following result.

**Lemma 4.16.** \(0\) is a K-simple eigenvalue of \(\overline{L}(\overline{\pi}(b), 0, \theta_b)\). Moreover, \(0\) is the eigenvalue with the largest real part among all the eigenvalues of \(\overline{L}(\overline{\pi}(b), 0, \theta_b)\).

In view of the above analysis and Corollary 1.13 and Theorem 1.16 of [4], we have

**Lemma 4.17.** There exists a \(C^1\) mapping \(s \rightarrow (\overline{\pi}(s), \overline{\nu}(s))\) from \(\mathbb{R}\) into the neighbourhood of \((\overline{\pi}(b), 0, 0)\) in \(\mathbb{R} \times \mathbb{R}\), such that \((\overline{\pi}(0), \overline{\nu}(0)) = (0, (\overline{1}(b), \overline{1}))\) and that \(\overline{\nu}(s) = (\overline{\pi}(s), \overline{\nu}(s))\) with \(\overline{\pi}(s) = s(\overline{1}(b) + \overline{\nu}(s))\) and \(\nu(s) = \theta_b + s(1 + \overline{\nu}(s))\) satisfies

\[
\overline{L}(a(s), \overline{\pi}(s), \overline{\nu}(s))\overline{U}(s) = \overline{\pi}(s)\overline{U}(s) \quad \text{for} \quad |s| \ll 1.
\]

In order to obtain the sign of \(\mu(s)\) for \(0 < s \ll 1\), it suffices to investigate the sign of \(\mu'(0)\), since \(\text{sign}(\overline{\mu}(s)) = \text{sign}(\overline{\mu}'(0))\) for \(|s| \ll 1\) and \(s > 0\). Using a similar argument as the previous subsection, we have

\[
a'(0) = \overline{\pi}(b) \int_\Omega \overline{1}^2(b)(A\overline{1}(b) - B\phi'(0)\overline{1})d\chi,
\]

and

\[
\overline{\mu}'(0) = \overline{\pi}(b) \int_\Omega \overline{1}^2(b)(B\phi'(0)\overline{1} - A\overline{1}(b))d\chi.
\]

In view of Lemmas 4.17, we have

**Theorem 4.18.** If \(\overline{D} \triangleq \overline{\pi}(b) \int_\Omega \overline{1}^2(b)(B\phi'(0)\overline{1} - A\overline{1}(b))d\chi < 0\) (respectively, \(\overline{D} > 0\)), then \(\overline{\pi}(s) < 0\) (respectively, \(\overline{\pi}(s) > 0\)) for \(0 < s \ll 1\) and the coexistence solutions \((\overline{\pi}(s), \overline{\nu}(s))\) defined by Theorem 4.15 are stable (respectively, unstable).

4.5. Coincidence of coexistence steady-state solution branches. Substitute \((\tilde{a}(s), \tilde{b}(s); \tilde{U}(s))\) to (3) and divide by \(s\), then we have

\[
\Delta[\cos \omega_1 + \tilde{\omega}_1(s)] + (\lambda_0 + \tilde{\alpha}(s)) m_1(\cos \omega_1 + \tilde{\omega}_1(s)) - sA(\cos \omega_1 + \tilde{\omega}_1(s))^2 - B\phi(\tilde{\alpha}(s))(\sin \omega_1 + \tilde{\omega}_2(s)) = 0,
\]

and

\[
\Delta[\sin \omega_1 + \tilde{\omega}_2(s)] + (\sigma_0 + \beta(s)) m_2(\sin \omega_1 + \tilde{\omega}_2(s)) - sD(\sin \omega_1 + \tilde{\omega}_2(s))^2 + C\phi(\tilde{\alpha}(s))(\sin \omega_1 + \tilde{\omega}_2(s)) = 0.
\]
Differentiating (52) with respect to $s$ and then evaluating it at $s = 0$ yield
\[
\Delta \hat{w}_1'(0) + \alpha(0)m_1 \varphi_1 + \lambda_0 m_1 \hat{w}_1(0) - \lambda_0 (A \cos^2 \omega \varphi_1^2 + B \phi'(0) \cos \omega \varphi_1 \sin \omega \psi_1) = 0.
\]
Multiplying the above equation by $\varphi_1$ and integrating it over $\Omega$, we get
\[
\hat{\alpha}'(0) \int_{\Omega} m_1(x) \varphi_1^2 \mathrm{d}x = \lambda_0 \int_{\Omega} [A \cos \omega \varphi_1 + B \phi'(0) \sin \omega \psi_1] \varphi_1^2 \mathrm{d}x.
\] (54)

Similarly, differentiating (53) with respect to $s$ and setting $s = 0$ yield
\[
\Delta \hat{w}_2'(0) + \beta'(0)m_2 \psi_1 + \sigma_0 m_2 \hat{w}_2(0) - \sigma_0 (D \sin^2 \omega \psi_1^2 - C \phi'(0) \cos \omega \varphi_1 \sin \omega \psi_1) = 0.
\]
Multiplying the above equation by $\psi_1$ and integrating it over $\Omega$, we get
\[
\beta'(0) \int_{\Omega} m_2(x) \psi_1^2 \mathrm{d}x = \sigma_0 \int_{\Omega} [D \sin \omega \psi_1 - C \phi'(0) \cos \omega \varphi_1] \psi_1^2 \mathrm{d}x.
\] (55)

Therefore, it follows from (54) and (55) that
\[
\lim_{s \to 0} \frac{\hat{b}(s) - \sigma_0}{\alpha(s) - \lambda_0} = \lim_{s \to 0} \frac{\hat{\beta}(s)}{\alpha(s)} = \lim_{s \to 0} \frac{\hat{\beta}'(s)}{\alpha(s)} = \hat{\beta}'(0) = l_1(\omega),
\] (56)

where
\[
l_1(\omega) = \frac{\sigma_0}{\lambda_0} \int_\Omega (D \sin \omega \psi_1 - C \phi'(0) \cos \omega \varphi_1) \psi_1^2 \mathrm{d}x \int_\Omega m_1(x) \varphi_1^2 \mathrm{d}x.
\]

Set $l_2(\omega) = \frac{1}{l_1(\omega)}$, then we have
\[
\lim_{\omega \to -2} l_1(\omega) = -\frac{\sigma_0}{\lambda_0} \int_\Omega (D \sin \omega \psi_1 - C \phi'(0) \cos \omega \varphi_1) \psi_1^2 \mathrm{d}x \int_\Omega m_1(x) \varphi_1^2 \mathrm{d}x,
\]

and
\[
\lim_{\omega \to -2} l_2(\omega) = \frac{\lambda_0}{\sigma_0} \frac{B \phi'(0) \int_\Omega \varphi_1^2 \psi_1 \mathrm{d}x \int_\Omega m_2(x) \psi_1^2 \mathrm{d}x}{D \int_\Omega \psi_1^2 \mathrm{d}x \int_\Omega m_1(x) \varphi_1^2 \mathrm{d}x}.
\]

In view of Lemma 3.6 and Theorem 4.7, let
\[
\Gamma_1 = \{(b(s), \theta_a + s(z_1 + \rho(s)), s(\bar{\psi}(a) + \kappa(s)) : 0 < s < \delta\}
\]
be the positive solution branch bifurcating from $(\bar{b}(a); \theta_a, 0)$ and satisfying
\[
\lim_{a \to \lambda_0} \hat{b}'(a) = \lim_{\omega \to 0} l_1(\omega).
\] (57)

In view of Lemma 3.8 and Theorem 4.15, let
\[
\Gamma_2 = \{(a(s), s(\bar{\chi}_1(b) + \bar{\kappa}(s)), \theta_b + s(\bar{\psi}_1(b) + \bar{\kappa}(s)) : 0 < s < \delta\}
\]
be the positive solution branch bifurcating from $(\bar{a}(b); 0, \theta_b)$ and satisfying
\[
\lim_{b \to \sigma_0} \bar{\psi}'(b) = \lim_{\omega \to -2} l_2(\omega).
\] (58)

In view of (57) and (58), we have the following results.

**Theorem 4.19.** (i): For any $(a, \omega) \in \mathbb{R} \times (0, \frac{\pi}{2})$ sufficiently close to $(\lambda_0, 0)$, 
$(\bar{u}(s), \bar{v}(s))$ obtained in Theorem 4.3 coincides with the coexistence positive solutions bifurcating from $(\theta_a, 0)$ at the curve $\Gamma_1$;

(ii): For any $(b, \omega) \in \mathbb{R} \times (0, \frac{\pi}{2})$ sufficiently close to $(\sigma_0, \frac{\pi}{2})$, 
$(\bar{u}(s), \bar{v}(s))$ obtained in Theorem 4.3 coincides with the coexistence positive solutions bifurcating from $(0, \theta_b)$ at the curve $\Gamma_2$. 
Both \((\theta_a,0)\) and \((0,\theta_b)\) can be regarded as the first steady-state bifurcation branches of system (3) at the trivial solution because they appear as \(a\) and \(b\) increase and passes through the critical values \(\lambda_0\) and \(\sigma_0\), respectively. For the solution branch \((\theta_a,0)\) (respectively, \((0,\theta_a)\)), system (3) undergoes the second steady-state bifurcation as \(b\) (respectively, \(a\)) increases and passes through the critical value \(\pi(b)\) (respectively, \(\pi(a)\)), and a coexistence steady-state solution branches \(\Gamma_1\) (respectively, \(\Gamma_2\)) appears. Theorem 4.19 implies that \((\tilde{a}(s),\tilde{b}(s);\tilde{u}(s),\tilde{v}(s))\) obtained in Theorem 4.3 coincides the coexistence steady-state solution branch \(\Gamma_1\) (respectively, \(\Gamma_2\)) as \((s,\omega)\) tends to \((0,0)\) (respectively, \((0,\frac{\pi}{2})\)). In this sense, \((\tilde{a}(s),\tilde{b}(s);\tilde{u}(s),\tilde{v}(s))\) obtained in Theorem 4.3 can be regarded as the connection of the two coexistence steady-state solution branches \(\Gamma_1\) and \(\Gamma_2\).

5. Conclusion. In this paper, the dynamics of a two-species diffusive Lotka-Volterra model is investigated. By employing the signs of the associated principal eigenvalues, bifurcation theorems, perturbation theory, comparison principle, monotone theory and Lyapunov-Schmidt reduction method, we obtain the existence, local and global stability of boundary (including trivial and semi-trivial) steady-state solutions. Moreover, we also study the nonexistence and steady-state bifurcation of coexistence steady-state solutions at each of the boundary steady states. At the same time, the coincidence of bifurcating coexistence steady-state solution branches is also described. The functional \(\phi\) of our model represents a general functional response of the predator function, which contains many types of prey-predator model and hence the results of our paper can be applicable to many models and possess strong theoretical guidance.

Finally, the study of the existence, nonexistence, stability and bifurcation of the steady states and semi-trivial steady state solutions is also an interesting and important problem in the investigation of delayed diffusive prey-predator models and stochastic population models; See, for example, [10, 11, 45, 49]. Can the relevant results obtained in our paper still hold for the models with delay or stochastic term? This topic is left for our future study.

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