Fomenko–Mischenko Theory, Hessenberg Varieties, and Polarizations

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Abstract. The symmetric algebra $S(g)$ over a Lie algebra $g$ has the structure of a Poisson algebra. Assume $g$ is complex semisimple. Then results of Fomenko–Mischenko (translation of invariants) and A. Tarasev construct a polynomial subalgebra $H = \mathbb{C}[q_1, \ldots, q_b]$ of $S(g)$ which is maximally Poisson commutative. Here $b$ is the dimension of a Borel subalgebra of $g$. Let $G$ be the adjoint group of $g$ and let $\ell = \text{rank } g$. Using the Killing form, identify $g$ with its dual so that any $G$-orbit $O$ in $g$ has the structure (KKS) of a symplectic manifold and $S(g)$ can be identified with the affine algebra of $g$.

An element $x \in g$ will be called strongly regular if $\{(dq_i)_x \}, i = 1, \ldots, b$, are linearly independent. Then the set $g^{\text{reg}}$ of all strongly regular elements is Zariski open and dense in $g$ and also $g^{\text{reg}} \subset g^{\text{reg}}$ where $g^{\text{reg}}$ is the set of all regular elements in $g$. A Hessenberg variety is the $b$-dimensional affine plane in $g$, obtained by translating a Borel subalgebra by a suitable principal nilpotent element. Such a variety was introduced in [K2]. Defining Hess to be a particular Hessenberg variety, Tarasev has shown that Hess $\subset g^{\text{reg}}$.

Let $R$ be the set of all regular $G$-orbits in $g$. Thus if $O \in R$, then $O$ is a symplectic manifold of dimension $2n$ where $n = b - \ell$. For any $O \in R$ let $O^{\text{reg}} = g^{\text{reg}} \cap O$. One shows that $O^{\text{reg}}$ is Zariski open and dense in $O$ so that $O^{\text{reg}}$ is again a symplectic manifold of dimension $2n$. For any $O \in R$ let Hess$(O) = \text{Hess } \cap O$. One proves that Hess$(O)$ is a Lagrangian submanifold of $O^{\text{reg}}$ and that

$$\text{Hess} = \sqcup_{O \in R}\text{Hess}(O).$$

The main result of this paper is to show that there exists simultaneously over all $O \in R$, an explicit polarization (i.e., a “fibration” by Lagrangian submanifolds) of $O^{\text{reg}}$ which makes $O^{\text{reg}}$ simulate, in some sense, the cotangent bundle of Hess$(O)$.

Keywords: symplectic geometry, geometric quantization, Poisson manifolds, symplectic manifolds, Lagrangian submanifolds, Poisson algebras, group actions, invariant theory, group actions on affine varieties, rings and algebras

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0. Introduction

0.1. Let $g$ be a complex semisimple Lie algebra and put \text{rank } g = \ell. Let $G$ be the
adjoint group of \( g \). As one knows the symmetric algebra \( S(g) \) has the structure of a Poisson algebra. Identify \( g \) with its dual using the Killing form \((x, y)\) so that we can also regard \( S(g) \) as the algebra of polynomial functions on \( g \). But then \( g \) inherits the structure of a Poisson manifold. The corresponding symplectic leaves are the adjoint orbits \( O \) of \( G \). For any \( \varphi \in S(g) \) let \( \xi_\varphi \) be the “Hamiltonian” vector field on \( g \). If \( x \in g \), then one knows

\[
(\xi_\varphi) \in T_x(O),
\]

where \( O \) is the adjoint orbit of \( x \).

Let \( g = n_- + h + n \) be a standard triangular decomposition of \( g \). Let \( b = h + n \) (resp. \( b_- = h + n_- \)). Let \( b = \dim b \) (resp. \( n = \dim n \)) so that \( \dim g = b + n \). Let \( I_j \in S(g)^G, j = 1, \ldots, \ell \), be homogeneous generators of the algebra of \( G \)-invariants in \( S(g) \). Let \( d_j \) be the degree of \( I_j \). One knows that

\[
\sum_{j=1}^\ell d_j = b. \tag{0.1}
\]

For any \( u \in g \) let \( \partial_y \) be the directional partial derivative on \( g \) defined by \( y \). Thus by (0.1) one obtains \( b \) polynomials \( q_1, \ldots, q_b \) on \( g \) by considering all \( (\partial_y)^k I_j \) where \( j = 1, \ldots, \ell \), and \( k = 0, \ldots, d_j - 1 \). Let \( \mathcal{H}_y \) be the subalgebra of \( S(g) \) generated by the \( q_i, i = 1, \ldots, b \). Then

**Theorem 0.1.** (Fomenko–Mischenko) \( \mathcal{H}_y \) is Poisson commutative.

**0.2.** Let \( \Pi = \{\alpha_1, \ldots, \alpha_\ell\} \) be the set of simple positive (i.e., with respect to \( b \)) roots and let \( \{e_{\alpha_i}, i = 1, \ldots, \ell\} \) be corresponding roots vectors. Let \( \{w, e, f\} \) be the S-triple whose span \( u \) is the principal TDS where \( e = \sum_{i} e_{\alpha_i} \) and \( w \in h \) is defined so that \( \alpha_i(w) = 2 \) for \( i = 1, \ldots, \ell \).

Let \( g^{reg} \) be the dense Zariski open set of all regular elements \( x \in g \) (\( x \) is regular if \( \dim g^x = \ell \)). Fix \( y \in h \). We introduce the following terminology. An element \( z \in g \) will be said to be strongly regular if \( \{(dq_i)_z, i = 1, \ldots, b\} \) are linearly independent. An old criterion of ours for regularity implies

\[
g^{sreg} \subset g^{reg}. \tag{0.2}
\]

It is immediately obvious that if \( g^{sreg} \) is Zariski open in \( g \) and is Zariski dense if \( g^{sreg} \) is not empty. However it is true but not obvious that \( g^{sreg} \) is not empty.
**Theorem 0.2.** (Fomenko–Mischenko) $\mathfrak{g}^{\text{reg}}$ is not empty. In fact if $e_1 \in \mathbb{C}^\times$, then $e_1 \in \mathfrak{g}^{\text{reg}}$.

The Fomenko–Mischenko proof of Theorem 0.2 involves a case-by-case argument over all complex simple Lie algebras. In our paper here we give a case-independent proof (see Theorem 2.11) using the representation theory of the principal TDS $\mathfrak{u}$.

One immediate consequence of Theorem 0.2 is that the polynomials $q_i$ are algebraically independent so that

$$\mathcal{H}_y = \mathbb{C}[q_1, \ldots, q_b]$$

is a polynomial ring in $b$-variables.

We will refer to a translate of a Borel subalgebra of the form $e_1 + \mathfrak{b}_-$ as a generalized Hessenberg variety. In the case at hand if $N_- \subset G$ is the subgroup corresponding to $\mathfrak{n}_-$, then $e_1 + \mathfrak{b}_-$ is stable under the adjoint action of $N_-$. With positive and negative roots reversed we studied this action in [K2]. One outcome was the existence of a section of the action of $G$ on the set of regular $G$-orbits. To fix matters here let $e_1 \in \mathbb{C}^\times e$ be normalized so that $(e_1, f) = 1$ and put

$$\text{Hess} = e_1 + \mathfrak{b}_-.$$

In an all too brief note [T], A.A. Tarasev proved that $\mathcal{H}_y$ was a maximal Poisson commutative subalgebra of $S(\mathfrak{g})$. This solved the question of maximality raised by E. Vinberg. In addition Tarasev generalized Theorem 0.2 by proving that

$$\text{Hess} \subset \mathfrak{g}^{\text{reg}}. \quad (0.3)$$

Let

$$\Phi : \mathfrak{g} \to \mathbb{C}^b$$

be the morphism defined by putting $\Phi(x) = (q_1(x), \ldots, q_b(x))$. In [T] Tarasev implicitly proves that the restriction

$$\Phi : \text{Hess} \to \mathbb{C}^b \quad (0.4)$$

is an algebraic isomorphism.

**0.4.** In the present paper we apply the machinery above to simultaneously polarize a Zariski dense open set in all maximal $G$-orbits $O$, and in doing so simulate a cotangent bundle structure on these Zariski open sets.
Any $G$-adjoint orbit $O$ is a symplectic manifold, specifically with respect to KKS structure in the complex category. One has $\dim O \leq 2n$ and $O$ is an orbit of regular elements if and only if $\dim O = 2n$. Let $R$ be the set of all $G$-orbits of regular elements. For any $O \in R$ let

$$\text{Hess}(O) = O \cap \text{Hess}.\$$

In this paper we prove

**Theorem 0.3.** Let $O \in R$. Then $\text{Hess}(O)$ is a Lagrangian submanifold of $O$. In particular $\dim \text{Hess}(O) = n$. Furthermore one has the disjoint union

$$\text{Hess} = \sqcup_{O \in R} \text{Hess}(O).$$

In addition (0.5) is the decomposition of Hess into $N_-$ orbits.

If $O$ is any $G$-orbit, let $O^{\text{reg}} = O \cap \mathfrak{g}^{\text{reg}}$. It is immediate from (0.2) that $O^{\text{reg}}$ is empty if $O$ is not regular. On the other hand from Theorem 0.3 and (0.3) it follows that $O^{\text{reg}}$ is not empty if $O \in R$. In such a case $O^{\text{reg}}$ is necessarily open and Zariski dense in $O$. In particular $O^{\text{reg}}$ is then a symplectic manifold of dimension $2n$ and $\text{Hess}(O)$, by (0.3), is a Lagrangian submanifold of $O^{\text{reg}}$.

Now for any $x \in \mathfrak{g}^{\text{reg}}$ let $Z_x \subset T_x(\mathfrak{g})$ be the span of $(\xi_{q_i})_x$, $i = 1, \ldots, b$.

**Theorem 0.4.** The correspondence $x \mapsto Z_x$ defines an $n$-dimensional involutive distribution $Z$ on $\mathfrak{g}^{\text{reg}}$, so that by Frobenius (in the holomorphic category), one has a foliation

$$\mathfrak{g}^{\text{reg}} = \sqcup_{\lambda \in \Lambda} L_{\lambda}$$

for some parameter set $\Lambda$, where the $n$-dimensional leaves $L_{\lambda}$ are maximal connected integral submanifolds of $Z$. On the other hand $\Phi$ defines a fibration $\mathfrak{g}^{\text{reg}}$. Let $\Phi^{\text{reg}} = \Phi|_{\mathfrak{g}^{\text{reg}}}$ so that

$$\Phi^{\text{reg}} : \mathfrak{g}^{\text{reg}} \to \mathbb{C}^b.$$  

For any $x \in \text{Hess}$ let $F_x$ be the fiber of $\Phi^{\text{reg}}$ over $\Phi(x)$. That is

$$F_x = (\Phi^{\text{reg}})^{-1}(\Phi(x))$$

so that

$$\mathfrak{g}^{\text{reg}} = \sqcup_{x \in \text{Hess}} F_x.$$
It is of course clear that \( F_x \) is a Zariski closed (not necessarily irreducible) subvariety of \( \mathfrak{g}^{\text{reg}} \).

For any \( x \in \text{Hess} \) let

\[
\Lambda_x = \{ \lambda \in \Lambda \mid L_\lambda \subset F_x \}.
\]

**Theorem 0.5.** \( L_\lambda \), for any \( \lambda \in \Lambda \), is an irreducible, Zariski closed, nonsingular, \( n \)-dimensional subvariety of \( \mathfrak{g}^{\text{reg}} \). Furthermore

\[
\Lambda = \bigsqcup_{x \in \text{Hess}} \Lambda_x.
\]

In addition \( \Lambda_x \), for any \( x \in \text{Hess} \), is a finite set and one has

\[
F_x = \bigsqcup_{\lambda \in \Lambda_x} L_\lambda.
\] (0.9)

Moreover \( F_x \) is a nonsingular \( n \)-dimensional Zariski closed subvariety of \( \mathfrak{g}^{\text{reg}} \) and (0.9) is both the decomposition of \( F_x \) into the union of its irreducible components and simultaneously the decomposition of \( F_x \) into its connected (with respect to both its Zariski and ordinary Hausdorff topology) components.

Our final result is that the maximal Poisson commutative subalgebra \( \mathcal{H}_y \) of \( S(\mathfrak{g}) \) leads to a simultaneous polarization of \( O^{\text{reg}} \) for all regular \( G \)-orbits \( O \).

**Theorem 0.6.** Let \( O \in R \) and let \( x \in \text{Hess}(O) \). Then \( F_x \subset O^{\text{reg}} \). In fact \( F_x \) is a Lagrangian submanifold of \( O^{\text{reg}} \) and

\[
\mathfrak{g}^{\text{reg}} = \bigsqcup_{x \in \text{Hess}(O)} F_x, \quad (0.10)
\]

thereby defining a polarization of the \( 2n \)-dimensional symplectic manifold \( O^{\text{reg}} \). Even more the Lagrangian submanifold \( \text{Hess}(O) \) of \( O^{\text{reg}} \) is transversal to all the Lagrangian fibers \( F_x \), \( x \in \text{Hess}(O) \), so that (0.10) simulates on \( O^{\text{reg}} \) the structure of the cotangent bundle of \( \text{Hess}(O) \).

**Poisson structure and the generalized Hessenberg variety**

1. **Poisson bracket on \( \mathfrak{g} \) and the principal TDS**

   1.1. Let \( \mathfrak{g} \) be a complex semisimple Lie algebra and put \( \text{rank } \mathfrak{g} = \ell \). Let \( G \) be the adjoint group of \( \mathfrak{g} \). Identify \( \mathfrak{g} \) with its dual using the Killing form \( (x, y) \). The
symmetric algebra $S(\mathfrak{g})$ over $\mathfrak{g}$ then identifies with the algebra of polynomial functions on $\mathfrak{g}$, where if $x, y \in \mathfrak{g}$, then $x(y) = (x, y)$. If $q$ is any holomorphic function on $\mathfrak{g}$ (e.g., elements of $S(\mathfrak{g})$) and $x \in \mathfrak{g}$, let $dq(x) \in \mathfrak{g}$ be defined so that for any $z \in \mathfrak{g}$,

$$(dq(x), z) = \frac{d}{dt} (q(x + tz))_{t=0}.$$  \hfill (1.1)

Let $I_j$, $j = 1, \ldots, \ell$, be homogeneous generators of $S(\mathfrak{g})^G$. Let $d_j = m_j + 1$ be the degree of $I_j$. The following old result of ours is an immediate extension of Theorem 9, p. 382 in [K2].

**Theorem 1.1.** For any $x \in \mathfrak{g}$ and $I \in S(\mathfrak{g})^G$ one has

$$dI(x) \in \text{Cent } \mathfrak{g}^x.$$ \hfill (1.2)

Furthermore $dI_j(x)$, $j = 1, \ldots, \ell$, is a basis of $\mathfrak{g}^x$ if and only if $x \in \mathfrak{g}$ is regular. In particular if $x$ is regular semisimple, then $\mathfrak{g}^x$ is the unique Cartan subalgebra which contains $x$ so that $dI_j(x)$, $j = 1, \ldots, \ell$, is a basis of the Cartan subalgebra $\mathfrak{g}^x$.

**Proof.** Let $a \in G^x$ and $z \in \mathfrak{g}$. Then

$$(a \cdot dI(x), z) = (dI(x), a^{-1} \cdot z)$$

$$= \frac{d}{dt} (I(x + ta^{-1} \cdot z))_{t=0}$$

$$= \frac{d}{dt} (I(a^{-1} \cdot (x + tz))_{t=0}$$

$$= \frac{d}{dt} (I(x + tz))_{t=0} \text{ since } I \text{ is Ad } G\text{-invariant}$$

$$= (dI(x), z)$$

so that

$$dI(x) \in \mathfrak{g}^{G^x}.$$ \hfill (1.3)

But $x \in \mathfrak{g}^x$ so that

$$dI(x) \in \mathfrak{g}^x.$$ \hfill (1.4)

But (1.3) and (1.4) yield (1.2).

But now by Theorem 9, p. 382 in [K2], one has $dI_j(x)$, $j = 1, \ldots, \ell$, are linearly independent if and only if $x$ is regular. But, by definition, $x$ is regular if and only if $\text{dim } \mathfrak{g}^x = \ell$. But, clearly, this proves the theorem. QED
1.2. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\Delta$ be the set of roots for $(\mathfrak{h}, \mathfrak{g})$. For each $\varphi \in \Delta$ let $e_\varphi$ be a corresponding root vector. Let $\Delta_+ \subset \Delta$ be a choice of positive roots and let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be the set of simple positive roots. Let $w \in \mathfrak{h}$ be the unique element such that $\alpha(w) = 2$ for any $\alpha \in \Pi$. Let $f = \sum_{j=1}^\ell e_{-\alpha_j}$ so that

$$[w, f] = -2 f. \quad (1.5)$$

One has that $w$ is regular semisimple and $\mathfrak{g}^w = \mathfrak{h}$. Let $e \in \sum_{j=1}^\ell \mathbb{C} e_\alpha$ be such that $\{w, e, f\}$ is a principle $S$-triple spanning a principal TDS $\mathfrak{u}$. Then by [K1] one has a direct sum

$$\mathfrak{g} = \bigoplus_{j=1}^\ell \mathfrak{m}_j, \quad (1.6)$$

where

$$\dim \mathfrak{m}_j = 2m_j + 1 \quad (1.7)$$

is an $\text{ad} \mathfrak{u}$ irreducible module.

Let $\{z_j\}, j = 1, \ldots, \ell$, be a basis of $\mathfrak{h}$ such that

$$\mathfrak{m}_j \cap \mathfrak{h} = \mathbb{C} z_j. \quad (1.8)$$

Let $\mathfrak{b}_-$ (resp. $\mathfrak{b}$) be the Borel subalgebra spanned by $\mathfrak{h}$ and $\{e_{-\varphi}$(resp. $e_\varphi)$), $\varphi \in \Delta_+$. Let $\mathfrak{m}_- = [\mathfrak{b}_-, \mathfrak{b}_-]$ (resp. $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$). Put $\mathfrak{m}_j,- = \mathfrak{b}_- \cap \mathfrak{m}_j$. Let

$$z_{j,k} = (\text{ad} f/2)^k z_j, \quad k = 0, \ldots, m_j, \quad (1.9)$$

so that (since $\mathfrak{n}_-$ is the span of $\text{ad} w$ eigenvectors with negative eigenvalues) one has

**Proposition 1.2.** The set $\{z_{j,k}\}, k = 0, \ldots, m_j$, is a basis of $\mathfrak{m}_j,-$ and

$$\{z_{j,k}\}, \quad j = 1, \ldots, \ell, \quad k = 0, \ldots, m_j \quad (1.10)$$

is a basis of $\mathfrak{b}_-$.

For any $t \in \mathbb{C}$ let $u_t = \exp t/2 f$ so that

$$u_t \cdot w = w + t f. \quad (1.11)$$

Then one notes that, for $j = 1, \ldots, \ell$,

$$u_t \cdot z_j = z_j + t z_{j,1} + \cdots + t^{m_j} z_{j,m_j}, \quad (1.12)$$
But then by the usual Vandermonde argument one has

**Proposition 1.3.** Let $j = 1, \ldots, \ell$. If $c_i \in \mathbb{C}, i = 1, \ldots, d$, are distinct numbers where $d \geq d_j$, then

$$u_{c_i} \cdot z_j, \ i = 1, \ldots, d, \text{ spans } m_{j-}.$$  \hfill (1.13)

Obviously $w + tf$ is regular semisimple, by (1.11), and the Cartan subalgebra $\mathfrak{g}^{w+tf}$ equals $u_t \cdot \mathfrak{h}$ so that, by (1.12),

$$\{z_0 + t z_1 + \cdots + t^{m_j} z_j m_j, \ j = 1, \ldots, \ell\} \text{ is a basis of the Cartan subalgebra } \mathfrak{g}^{w+tf}.$$  \hfill (1.14)

Another basis of $\mathfrak{g}^{w+tf}$ is given by Theorem 1.1. That is,

$$dI_j(w + tj), \ j = 1, \ldots, \ell, \text{ is also a basis of } \mathfrak{g}^{w+tf}.$$  \hfill (1.15)

Now we recall that the Coxeter number $h$ of $\mathfrak{g}$ is the maximal value of $d_j$, $j = 1, \ldots, \ell$. We have proved

**Theorem 1.4.** For any $t \in \mathbb{C}$ and $j = 1, \ldots, \ell$, one has $dI_j(w + tf) \in \mathfrak{b}_-$. Furthermore $\mathfrak{b}_-$ is spanned by $dI_j(w + tf)$ for all $j = 1, \ldots, \ell$, and $t \in \mathbb{C}$. In fact it is already spanned by these elements where $t$ is restricted to take on a finite set of values whose cardinality is greater than or equal to the Coxeter number $h$.

1.3. Let $\text{Hol}(\mathfrak{g})$ be the algebra of holomorphic functions on $\mathfrak{g}$. Now if $p, q \in \text{Hol}(\mathfrak{g})$ one defines the Poisson bracket $[p, q] \in \text{Hol}(\mathfrak{g})$ so that for any $x \in \mathfrak{g}$,

$$[p, q](x) = (x, [dp(x), dq(x)]).$$  \hfill (1.16)

One also defines the (holomorphic) Hamiltonian vector field $\xi_p$ on $\mathfrak{g}$ so that

$$(\xi_p(q))(x) = [p, q](x).$$  \hfill (1.17)

One notes that

$$(\xi_p(q))(x) = ([x, p(x)], dq(x))$$

$$= \frac{d}{dt}(q(x + t[x, dp(x)])|_{t=0}$$  \hfill (1.18)
so that one has

**Proposition 1.5.** Let $p \in \text{Hol}(\mathfrak{g})$ and $x \in \mathfrak{g}$. Then $(\xi_p)_x$ is tangent, at $x$, to the adjoint orbit of $x$, and in fact $(\xi_p)_x$ is the tangent vector, at $t = 0$, to the curve $(\text{Ad}\exp - t dp(x))(x)$. That is,

$$(\xi_p)_x = -[dp(x), x]. \quad (1.19)$$

**Remark 1.6.** Note that if $p = z \in \mathfrak{g}$, then for any $x \in \mathfrak{g}$,

$$dz(x) = z \quad (1.20)$$

so that (1.18) becomes

$$(\xi_z(q))(x) = \frac{d}{dt}(q(x + t[x, z]))_{t=0}. \quad (1.21)$$

In particular

$$(\xi_z)_x = -[z, x], \quad (1.22)$$

and hence of course

$$T_x(O) = \{((\xi_z)_x \mid z \in \mathfrak{g}\}. \quad (1.23)$$

2. Coadjoint orbits and Fomenko–Mischenko Theory

2.1. The main theorem (Theorem 2.11) of this section is due to Fomenko and Mischenko. Their proof is case-by-case verification over all simple Lie algebras. Here we give a general proof using results in [K1] on the adjoint action of a principal TDS on $\mathfrak{g}$ (see [K1]).

Let $x, z \in \mathfrak{g}$. If the context leads to no confusion, we may identify $z$ with the tangent vector $(\partial_z)_x$ at $x$ where, for $q \in \text{Hol}(\mathfrak{g})$ one has

$$(\partial_z)_x q = \frac{d}{dt}(q(x + tz))_{t=0}. $$

Let $O$ be the adjoint (= coadjoint) orbit containing $x$. We recall that $O$ has a symplectic structure (KKS) denoted by $(O, \omega_O)$, where if $\omega_x$ is the value of $\omega_O$ at $x$, then for $y, z \in \mathfrak{g}$,

$$\omega_x([-z, x], [-y, x]) = (x, [y, z]). \quad (2.1)$$
See §5.2, p. 180–183 in [K3] and Theorem 5.3.1, p. 184 in [K3]. Let $\text{Hol}(O)$ be the Poisson algebra of holomorphic functions on $O$. If $\varphi \in \text{Hol}(O)$, the corresponding Hamiltonian vector field $\xi_{\varphi}$ on $O$ is such that for any $v \in T_x(O)$,

$$\omega_x((\xi_{\varphi})_x, v) = v \varphi. \tag{2.2}$$

See (4.1.3), p. 166 in [K3]. Now by (1.22) we may choose $p \in \text{Hol}(g)$ so that $v = -[y, x]$ where $y = dp(x)$. Hence

$$\omega_x((\xi_{\varphi})_x, -[y, x]) = (-[y, x])\varphi(x). \tag{2.3}$$

Now assume that $\varphi = q|O$ where $q \in \text{Hol}(g)$. Then, by (1.18)

$$\omega_x((\xi_{\varphi})_x, -[y, x]) = \frac{d}{dt}(q(x + t[x, dp(x)])|_{t=0}
= (\xi_p(q))(x)
= [p, q](x)
= (x, [dp(x), dq(x)])
= \omega_x([-dq(x), x], v). \tag{2.4}$$

Hence by the nonsingularity of $\omega_x$ and (1.19) one has

$$(\xi_{\varphi})_x = [-dq(x), x]
= (\xi_q)_x. \tag{2.5}$$

As an immediate consequence of (2.5) one has

**Proposition 2.1.** Let $p, q \in \text{Hol}(g)$ and let $O$ be an adjoint orbit. Then

$$[p, q] \mid O = [p \mid O, q \mid O]. \tag{2.6}$$

Let $V \subset S(g)$ be a finite-dimensional space of polynomial functions. For any $x \in g$ let

$$\text{g}(V, x) = \{dp(x) \mid p \in V\}$$

so that $\text{g}(V, x)$ is a subspace of $g$. One notes that if $p_i$, $i = 1, \ldots, \dim V$, is a basis $V$ and $z_j$, $j = 1, \ldots, \dim g$, is a basis of $g$, then

$$\dim \text{g}(V, x) = \text{rank } M(V, x), \tag{2.7}$$
where $M(V, x)$ is the $\dim V \times \dim \mathfrak{g}$ matrix $M_{ij}(V, x)$ given by

$$M_{ij}(V, x) = (\partial_{z_j} p_i)(x). \quad (2.8)$$

Now let

$$m(V) = \max_{x \in \mathfrak{g}} \dim \mathfrak{g}(V, x), \quad (2.9)$$

and let $\mathfrak{g}(V) = \{ x \in \mathfrak{g} \mid \dim \mathfrak{g}(V, x) = m(V) \}$ so that clearly

**Proposition 2.2.** Let $V \subset S(\mathfrak{g})$ be any finite-dimensional space of polynomial functions where $V \neq 0$. Then $\mathfrak{g}(V)$ is a nonempty Zariski open subset of $\mathfrak{g}$.

2.2. Let $n = \text{card } \Delta_+$ and let $b = \ell + n$ so that

$$\dim \mathfrak{g} = \ell + 2n \quad (2.10)$$

$$\dim \mathfrak{b}_- = b,$$

and of course if $O$ is any adjoint orbit, then

$$\dim O \leq 2n \quad (2.11)$$

and

one has equality in (2.11) $\iff$ $O$ is an orbit of regular elements. \quad (2.12)

Let $\mathfrak{g}^{\text{reg}}$ be the set of regular elements in $\mathfrak{g}$ so that $\mathfrak{g}^{\text{reg}}$ is a nonempty Zariski open set in $\mathfrak{g}$.

**Proposition 2.3.** Assume that $V \subset S(\mathfrak{g})$ is a finite-dimensional space of Poisson commuting polynomial functions. Then

$$m(V) \leq b. \quad (2.13)$$

**Proof.** Since the intersection of two nonempty Zariski open sets in $\mathfrak{g}$ is again a nonempty Zariski open set it suffices to prove that

$$b \geq \dim \mathfrak{g}(V, x), \ \forall x \in \mathfrak{g}^{\text{reg}} \quad (2.14)$$

Let $x \in \mathfrak{g}^{\text{reg}}$ and let $O$ be the $(2n)$-dimensional adjoint orbit containing $x$. Now considering tangent and cotangent spaces for submanifolds. Let

$$\nu : T^*_x(\mathfrak{g}) \to T^*_x(O) \quad (2.15)$$
be the surjection defined by the embedding of $O$ into $\mathfrak{g}$. But the kernel of $\nu$ is clearly $\ell$-dimensional. Hence the kernel of the restriction of $\nu$ to $T^*_x[V] = \{dp_x \mid p \in V\}$ is at most $\ell$-dimensional. But, by (2.6), $\nu(T^*_x[V])$ corresponds to an $\omega_x$-isotropic subspace of $T^*_x(O)$ under the isomorphism $T^*_x(O) \rightarrow T^*_x(O)$ defined by $\omega_x$. Hence $\dim \nu(T^*_x[V]) \leq r$. Thus $b \geq \dim T^*_x[V]$. But of course $\dim T^*_x[V] = \dim \mathfrak{g}(V, x)$. QED

2.3. Let $j = 1, \ldots, \ell$, $t \in \mathbb{C}$, and $u \in \mathfrak{g}$. Consider the polynomial function on $\mathfrak{g}$ whose value at $x \in \mathfrak{g}$ is given by $I_j(tu + x)$. Note that, over all $t \in \mathbb{C}$, one obtains a finite-dimensional subspace $V_{j,u}$ of $S(\mathfrak{g})$. Indeed $V_{j,u}$ is spanned by the homogeneous polynomials $I_{j,u,k}(x)$, $k = 0, \ldots, m_j$, and constants, where we write

$$I_j(tu + x) = I_j(tu) + \sum_{k=0}^{m_j} I_{j,u,k}(x) t^k$$

where

$$\deg I_{j,u,k}(x) = d_j - k.$$

Now put

$$V_u = \sum_{j=1}^{\ell} V_{j,u}.$$  \hspace{1cm} (2.18)

**Theorem 2.4** (Fomenko–Mischenko) $V_u$ is Poisson commutative for any $u \in \mathfrak{g}$.

**Proof.** We must show that, for any $x \in \mathfrak{g}$ and any $p, q \in V_u$,

$$(x, [dp(x), dq(x)]) = 0.$$  \hspace{1cm} (2.19)

We first show that one has (2.19) if $p(x) = I_j(tu + x)$ and $q = I_k(tu + x)$ where $j, k = 1, \ldots, \ell$. Indeed, since exterior differentiation commutes with translation one has

$$dI_k(tu + x) \in \text{Cent} \mathfrak{g}^{t u + x}$$

for any $k = 1, \ldots, \ell$, by Theorem 1.1. Thus $[dp(x), dq(x)] = 0$, establishing (2.19) for this case. Now assume that $s, t \in \mathbb{C}$ are distinct. Let $p(x) = I_j(su + x)$. Then for any $z \in \mathfrak{g}$ one has

$$(su + x, [dp(x), z]) = ([su + x, dp(x)], z) = 0$$

since $dp(x) \in \mathfrak{g}^{su + x}$ by Theorem 1.1. But then using this argument twice one has that $[dp(x), dq(x)]$ is the Killing form, orthogonal to both $su + x$ and $tu + x$ if we put
\[ q(x) = I_k(tu + x). \] But, since \( s \neq t \), \( x \) is in the span of \( su + x \) and \( tu + x \). This proves (2.19). QED

2.4. If \( a \in G \) and \( u \in \mathfrak{g} \) it is clear that with respect to the adjoint action of \( a \) on \( S(\mathfrak{g}) \) one has \( a \cdot V_u = V_{a \cdot u} \). It follows therefore that the integer \( m(V_u) \) depends only on the conjugacy class of \( u \). We recall \( b = \ell + n \) so that \( b \) is the dimension of a Borel subalgebra of \( \mathfrak{g} \). In particular, recalling the notation in Theorem 1.4, one has

\[ b = \dim \mathfrak{b}_-. \quad (2.21) \]

By Proposition 2.3 one has

\[ m(V_u) \leq b \quad (2.22) \]

for any \( u \in \mathfrak{g} \). Let

\[ \mathcal{R} = \{ u \in \mathfrak{g} \mid m(V_u) = b \}. \quad (2.23) \]

**Theorem 2.5.** Any principal nilpotent element of \( \mathfrak{g} \) lies in \( \mathcal{R} \).

**Proof.** By conjugation it suffices to show that \( f \in \mathcal{R} \) where \( f \) is the principal nilpotent element given in Theorem 1.4. But, by Theorem 1.4, one has

\[ \mathfrak{g}(V_f, w) = \mathfrak{b}_- \quad (2.24) \]

proving the theorem. QED

**Remark 2.6.** Assume that \( u \in \mathcal{R} \). Then by definition there exists \( x \in \mathfrak{g} \) and \( q_{u,i} \in V_u, i = 1, \ldots, b \), such that the differentials \( (dq_{u,i})_x \) are linearly independent. But this implies that the polynomials \( q_{u,i} \) are algebraically independent. Thus if \( A_u \) is the subalgebra of \( S(\mathfrak{g}) \) generated by the \( q_{u,i} \), it follows that \( A_u \) is Poisson commutative and, as an algebra, is given as the polynomial algebra

\[ A_u = \mathbb{C}[q_{u,1}, \ldots, q_{u,b}] \quad (2.25) \]

in \( b \)-variables.

**Theorem 2.7.** \( \mathcal{R} \) is a nonempty Zariski open subset of \( \mathfrak{g} \). Furthermore \( \mathcal{R} \) is closed under multiplication by \( \mathbb{C}^\times \).
Proof. The last statement is obvious from the definition of $V_u$. Now $\mathcal{R}$ is nonempty by Theorem 2.5. Let $u \in \mathcal{R}$. Let $q_{u,i}$ and $x$ be as in Remark 2.6. Replacing $u$ by $v \in \mathfrak{g}$ in the definition of $q_{u,i}$ it is clear, from the matrix argument in (2.8), that the set

$$s = \{ v \in \mathfrak{g} \mid (dq_{v,i})_x, i = 1, \ldots, b, \text{ are linearly independent} \}$$

is a Zariski open neighborhood of $u$. But this proves the theorem. QED

But now one has

**Theorem 2.8.** $\mathcal{R}$ contains all regular semisimple elements.

**Proof.** Let $u$ be regular semisimple. By conjugacy we may assume that $u \in \mathfrak{h}$, using the notation of §1.2. Recall (see §1.2) that $\mathfrak{n}_- = [\mathfrak{b}_-, \mathfrak{b}_-]$. Since $\mathcal{R}$ is closed under scalar multiplication it suffices to show that $\lambda u \in \mathcal{R}$ for some nonzero scalar $\lambda$. Let $N_- \subset G$ be the subgroup corresponding to $\mathfrak{n}_-$. Consider the adjoint action of $N_-$ on $\mathfrak{b}_-$. Since $u$ centralizes no nonzero element in $\mathfrak{n}_-$ one knows (e.g., by the Kostant–Rosenlicht theorem, see e.g., bottom of p. 36 and 2.4.14 in [Sp]) that, for any $\lambda \in \mathbb{C}^\times$,

$$N_- \cdot \lambda u = \lambda u + \mathfrak{n}_-. \quad (2.26)$$

In particular (using notation in §1.5) $\lambda u + f$ is conjugate to $\lambda u$. But any Zariski open neighborhood of $f$ contains $\lambda u + f$ for some sufficiently “small” $\lambda$. Hence $u \in \mathcal{R}$ by Theorem 2.7. QED

2.5. Let $J_k \in S^k(\mathfrak{g})$. Let $x, y \in \mathfrak{g}$. Then using the inner product on $S(\mathfrak{g})$ which extends the Killing form one has

$$J_k(x + ty) = (J_k, (x + ty)^k/k!)$$

$$= \sum_{j=0}^{k} (J_k, x^{k-j}/(k-j)! y^j/j! t^j/j!)$$

$$= \sum_{j=0}^{k} \left( \frac{1/j!}{(1/j!)} (\partial^j) y^j J_k(x) t^j \right)$$

$$= \sum_{j=0}^{k} J_{k-j,y}(x) t^j \quad (2.27)$$

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where
\[ J_{k-j,y} = (1/j!) (\partial_y)^j J_k \in S^{k-j}(g). \] (2.28)

Now as a function of \( x \) one has \( dJ_k(x + ty) \in \mathfrak{g} \) where for any \( z \in \mathfrak{g} \),

\[
(dJ_k(x + ty), z) = d/ds(J_k(x + sz + ty)|_{s=0} t^j
= \sum_{j=0}^{k-1} (1/(k-j-1)!)((\partial_x)^{k-j-1} J_{k-j,y}) (z)
\] (2.29)

so that
\[
dJ_k(x + ty) = \sum_{j=0}^{k-1} (1/(k-j-1)!)((\partial_x)^{k-j-1} J_{k-j,y}) t^j
= \sum_{j=0}^{k-1} 1/(j! (k-j-1)!)(\partial_y)^j(\partial_x)^{k-j-1} J_k t^j.
\] (2.30)

Now for \( j = 0, \ldots k-1 \), write \( v_{k-j-1} = 1/(j!(k-j-1)!)(\partial_y)^j(\partial_x)^{k-j-1} J_n \) so that
\[
dJ_k(x + ty) = \sum_{j=0}^{k-1} v_{k-j-1} t^j.
\] (2.31)

Now assume that \( J_k \in (S^k(g))^G \) so that \([x + ty, dJ_k(x + ty)] = 0\) and hence equating coefficients of powers of \( t \), one has

**Proposition 2.9.** (Fomenko–Mischenko)
\[
[x, v_{k-1}] = 0
\]
\[
[v_0, y] = 0,
\] (2.32)

and for \( j = 0, \ldots, k-2 \),
\[
[x, v_{k-j-2}] = [v_{k-j-1}, y].
\] (2.33)

Now fix \( y \) to be a regular element of \( \mathfrak{h} \) and let \( x = e \) recalling §1.2 so that \( e \) is a principal nilpotent element in the TDS \( u \). For \( i = 1, \ldots, h-1 \), let \( \mathfrak{b}_j \subset \mathfrak{b} \) be the span of all \( e_\varphi \) where \((\varphi, w) = 2j\). Put \( \mathfrak{b}_0 = \mathfrak{h} \) so that
\[
\mathfrak{b} = \bigoplus_{j=0}^{h-1} \mathfrak{b}_j.
\]
One has, for \( j = 0, \ldots, h - 2 \),
\[
\text{ad} \, e : b_j \to b_{j+1}. \quad (2.34)
\]

Now since \( y \in h \) is regular it follows that \( b_j \) is stable under \( \text{ad} \, y \) and \( \text{ad} \, y \mid b_j \) is nonsingular for \( j > 0 \). In particular if \( \text{ad}_n y = \text{ad} \, y \mid n \), then \( \text{ad}_n y \) is invertible. Let \( \zeta : b \to n \) be given by putting \( \zeta = \frac{-1}{\text{ad} \, e \mid b} \) so that for \( i = 0, \ldots, h - 2 \),
\[
\zeta : b_i \to b_{i+1}, \quad (2.35)
\]

Now in the notation of (2.33) one notes that if \( i = k - j - 1, j = 0, \ldots, k - 1 \), then
\[
v_i \in m_i, \; i = 0, \ldots, k - 1, \quad (2.36)
\]
and (2.32) and (2.33) are the statements
\[
[y, v_0] = 0
\]
\[
[e, v_{k-1}] = 0
\]
\[
\zeta(v_i) = v_{i+1}, \; i = 0, \ldots, k - 2. \quad (2.37)
\]

**Remark 2.10.** It is important to note that \( \zeta \) is independent of \( k \) and \( J_k \).

For \( j = 1, \ldots, \ell, \) and \( i = 0, \ldots, h - 1 \), we define \( v_i(I_j) = 0 \) if \( i \geq d_j \) and \( v_i(I_j) = v_i \) using the notation of (2.37) where \( k = d_j \) and \( J_k = I_j \). One then has
\[
\{dI_j(e + ty) \mid t \in \mathbb{C}\} = \sum_{i=0}^{h-1} \mathbb{C} \, v_i(I_j) \quad (2.38)
\]
and
\[
\{dI_j(e + ty) \mid t \in \mathbb{C}\} \cap m_i = \mathbb{C} v_i(I_j); \quad (2.39)
\]
and where \( v_h(I_j) = 0, \)
\[
\zeta(v_i(I_j)) = v_{i+1}(I_j). \quad (2.40)
\]

Recalling §2.1 one has
\[
\mathfrak{g}(V_y, e) = \oplus_{i=0}^{h-1} (\mathfrak{g}(V_y, e))_i \quad (2.41)
\]
where
\[
(\mathfrak{g}(V_y, e))_i = \mathfrak{g}(V_y, e) \cap m_i \quad (2.42)
\]
and \((g(V_y, e))_i\) is given by

\[
(g(V_y, e))_i = \sum_{i=1}^{\ell} C_{i_j}.
\]  

(2.43)

We can now prove the following result of Fomenko–Mischenko (see, Lemma 4.3, p. 383 and Lemma 44, p.384 in [F-M]). The proof of this result in [F-M] depends on the fact that (2.34) (and hence (2.35)) is surjective. The authors assert that this can be proved by considering the question case-by-case. We will give a general proof using the representation theory of the TDS \(u\). Namely, one has that (2.34) is surjective since the spectrum of \(\text{ad} w\) on \(n\) is strictly positive.

**Theorem 2.11** (Fomenko–Mischenko). Let \(y \in \mathfrak{h}\) be regular. Then

\[
g(V_y, e) = \mathfrak{b}.
\]  

(2.44)

**Proof.** The proof will be by induction on \(i\) using (2.43). One has \((g(V_y, e))_0 = g^y\) by Theorem 1.1. But \(g^y = \mathfrak{h}\) and \(\mathfrak{h} = \mathfrak{b}_0\). Assume inductively that \(\sum_{m=0}^j b_m \subset g(V_y, e)\) for \(j \leq n - 2\). Let \(v_{j+1} \in \mathfrak{b}_{j+1}\) be arbitrary. By the surjectivity of (2.35) there exists \(v_j \in \mathfrak{b}_j\) such that \(\zeta(v_j) = v_{j+1}\). But by induction \(v_j \in g(V_y, e)_j\). Thus there exists constants \(c_k \in \mathbb{C}, k = 1, \ldots, \ell\), such that

\[
v_j = \sum_{k=1}^{\ell} c_k v_j(I_k),
\]

But then

\[
v_{j+1} = \sum_{k=1}^{\ell} c_k v_{j+1}(I_k)
\]

by (2.40). Thus \(v_{j+1} \in g(V_y, e)\). QED

**Remark 2.12.** Note that upon conjugating (2.44) by an element in \(\exp \mathfrak{h}\) we may replace \(e\) in (2.44) by any element \(e_1\) of the form

\[
e_1 = \sum_{i=1}^{\ell} b_i e_{\alpha_i},
\]

where all \(b_i\) are in \(\mathbb{C}^\times\).
3. The generalized Hessenberg variety

3.1 Let $q = Cf + b$. We refer to an affine plane of the form $f + b$ as a generalized Hessenberg variety and $q$ as its linearization. We will also consider the opposed linearized Hessenberg variety $q_- = Ce + b_-$. In essence the results in §3 are due to A.A. Tarasev. They are either implicit or explicit in the very brief note [T]. For what we believe is greater clarity we will reestablish Tarasev’s results and place them in a context which will lead to the results of §4. The proof here is along the lines leading to our result in [K2] that $f + g^e$ is a section of the adjoint action of $G$ on $g_{reg}$. The generalized Hessenberg variety was introduced in [K2]. See §4 in [K2].

Clearly the $b + 1$-dimensional subspaces $q$ and $q_-$ are nonsingularly paired by the Killing form. Let $q_-^\perp$ be the Killing form orthocomplement of $q_-$ in $g$ so that $q_-^\perp \subset n$ and

$$g = q \oplus q_-^\perp.$$  \hspace{1cm} (3.1)

If $X$ is an affine variety, then $A(X)$ will denote the affine algebra of $X$. Consider $A(q_-)$. By restricting the polynomial functions on $g$ (namely $S(g)$) to $q_-$ one has an exact sequence

$$0 \rightarrow (q_-^\perp) \rightarrow S(g) \rightarrow A(q_-) \rightarrow 0$$  \hspace{1cm} (3.2)

where $(q_-^\perp)$ is the ideal in $S(g)$ defined by $q_-^\perp$. On the other hand one has the direct sum

$$S(g) = (q_-^\perp) \oplus S(q)$$  \hspace{1cm} (3.3)

so that the restriction of the third map in (3.2) to $S(q)$ defines an algebra isomorphism

$$S(q) \rightarrow A(q_-).$$  \hspace{1cm} (3.4)

Let

$$Q : S(g) \rightarrow S(q)$$  \hspace{1cm} (3.5)

be the projection defined by (3.3) so that for any $p \in S(g)$ the image of both $p$ and $Q(p)$ in $A(q_-)$ are the same. \hspace{1cm} (3.6)

One notes then that

$$Q(e_{-\alpha_i}) = c_i f, \ i = 1, \ldots, \ell, \ \text{for some } c_i \in \mathbb{C}^\times$$

$$Q([n_-, n_-]) = 0$$

$$Q = \text{Id on } S(b).$$
The decomposition (3.3) is clearly stable under $ad w$ so that $Q$ commutes with $ad w$. In particular $Q$ maps $S(g)^w$ into $S(q)^w$. But of course $Q$ is the identity on $S(q)^w$ so that

$$Q(S(g)^w) = S(q)^w. \quad (3.7)$$

**Remark 3.1.** Since (3.3) is clearly a decomposition of graded vector spaces note that (3.7) on homogeneous components may be written

$$Q(S^m(g)^w) = (S^m(q)^w).$$

Let $e_1 \subset C e$ be the normalization so that

$$(e_1, f) = 1, \quad (3.8)$$

and let $\text{Hess} \subset q_-$ be the fixed affine variety defined by putting

$$\text{Hess} = e_1 + b_- \cdot (3.9)$$

In particular $\text{Hess}$ is a $b$-dimensional affine plane and $A(\text{Hess})$ is the affine ring of $\text{Hess}$. Again restriction of functions defines a surjection

$$\sigma_{\text{Hess}} : S(g) \rightarrow A(\text{Hess}). \quad (3.10)$$

If $\sigma : S(g) \rightarrow A(q_-)$ is defined by restriction of functions (so that $\sigma$ is the composite of $Q$ and the isomorphism (3.4)), then of course

$$\sigma_{\text{Hess}} = \tau_{\text{Hess}} \circ \sigma,$$

where

$$\tau_{\text{Hess}} : A(q_-) \rightarrow A(\text{Hess}) (3.11)$$

is defined by restriction of functions. We note that

$$\sigma_{\text{Hess}} : S(b) \rightarrow A(\text{Hess}) \quad (3.12)$$

is an algebra isomorphism of algebras

$$\sigma_{\text{Hess}}(f) = 1. \quad (3.13)$$
Now for any \( k \in \mathbb{Z}_+ \) let \( S(\mathfrak{g})_{[k]} \) be the graded subspace of \( S(\mathfrak{g}) \) (with homogeneous components \( S^m(\mathfrak{g})_{[k]} \) ) defined by putting
\[
S(\mathfrak{g})_{[k]} = \{ p \in S(\mathfrak{g}) \mid \text{ad } w(p) = kp \}.
\]
But now if \( q \in S(\mathfrak{q})^w \), clearly we may uniquely write, as a finite sum,
\[
q = \sum_{k=0}^{m-1} f^k q_{[k]}, \quad (3.14)
\]
where
\[
q_{[k]} \in S(\mathfrak{b})_{[k]} . \quad (3.15)
\]
One notes that if \( q \in S^m(\mathfrak{q})^w \), then the sum in (3.14) can be taken for \( k \leq m \), and one has
\[
q_{[k]} \in S^{m-k}(\mathfrak{b})_{[k]} . \quad (3.16)
\]
But now if \( m > 0 \), one has \( q_{[m]} = 0 \) since of course \( S^0(\mathfrak{b})_{[m]} = 0 \) so that for \( m > 0 \) and \( q \in S^m(\mathfrak{q})^w \), one has
\[
q = \sum_{k=0}^{m-1} f^k q_{[k]} \quad \text{with } q_{[k]} \in S^{m-k}(\mathfrak{b})_{[k]} . \quad (3.17)
\]
Since the affine space Hess is a translation of \( \mathfrak{b}_+ \) the tangent space to Hess at \( e_1 \) identifies with \( \mathfrak{b}_- \). Consequently, using the Killing form nonsingular pairing of \( \mathfrak{b} \) and \( \mathfrak{b}_- \), one has an identification
\[
T^*\text{(Hess)} = \mathfrak{b} . \quad (3.18)
\]

**Proposition 3.2.** Assume \( m > 0 \). Let \( p \in S^m(\mathfrak{q})^w \) and let \( q = Q(p) \) so that \( q \in S^m(\mathfrak{q})^w \). See Remark 3.1. Then
\[
\sigma_{\text{Hess}}(p) = \sigma_{\text{Hess}}(q) = \sigma_{\text{Hess}}(\sum_{k=0}^{m-1} q_{[k]}), \quad (3.19)
\]
where \( q_{[k]} \in S^{m-k}(\mathfrak{b})_{[k]} \). In particular
\[
q_{[m-1]} \in \mathfrak{b}_{m-1} \quad (3.20)
\]
and in fact, using (3.18),
\[
q_{[m-1]} = d(\sigma_{\text{Hess}}q)_{e_1} . \quad (3.21)
\]
In addition if \( m > 1 \), then one has
\[
\sum_{k=0}^{m-2} q[k] \in S\left( \sum_{k=0}^{m-2} b_k \right). \tag{3.22}
\]

**Proof.** The equality (3.19) follows from (3.6) and (3.13). Of course (3.20) is given by (3.17). If \( z \in b_- \) then, since \( \sigma_{\text{Hess}}(S^k(b)) \) vanishes at \( e_1 \) for all positive integers \( k \), clearly (3.18) implies
\[
d/dt(q(e_1 + t z))_{t=0} = (q[m-1], z). \tag{3.23}
\]
But this establishes (3.21). Finally write \( q[k], k \leq m - 2 \), as a linear combination of the basis of \( S(b) \) formed by all product monomials of the root vectors \( e_\varphi, \varphi \in \Delta_+ \), and a basis of \( \mathfrak{h} \). It follows from the condition that \( q[k] \in S(b)[k] \) that the coefficient of every monomial containing any \( e_\varphi \in \sum_{j \geq m-1} b_j \) is zero. But this implies (3.22).

QED

3.2. Let \( y \in \mathfrak{h} \) be regular. By Theorem 2.11 and Remark 2.12 one has
\[
g(V_y, e_1) = b. \tag{3.24}
\]
In particular
\[
dim V_y \geq b. \tag{3.25}
\]
However in the notation of §1.1, Proposition 1.2, one has
\[
b = \sum_{j=1}^\ell d_j. \tag{3.26}
\]
But by (2.18) one has
\[
V_y = \sum_{j=1}^\ell V_{j,y}, \tag{3.27}
\]
where
\[
dim V_{j,y} \leq d_j \tag{3.28}
\]
by definition of \( V_{j,y} \) in §2.3, recalling that \( V_{j,y} \) is spanned by the homogeneous polynomials \( I_{j,y,k}, k = 0, \ldots, m_j \), and
\[
I_{j,y,k} \in S^{d_j-k}. \tag{3.29}
\]
Thus one has

**Theorem 3.3.**
\[
\dim V_y = b \\
\dim V_{j,y} = d_j.
\]

(3.30)

Also (3.27) is a direct sum and the homogeneous polynomials \(I_{j,y,k}, \ j = 1, \ldots, \ell, \ k = 0, \ldots, m_j, \) are a basis of \(V_y.\)

We assume the \(d_j\) are nondecreasing in \(j\) so that \(d_\ell = h\) where we recall \(h\) is the Coxeter number. Let the partition of \(b,\) dual to (3.26), be given as
\[
b = \sum_{m=1}^{h} r_m,
\]
where the \(r_m\) are nonincreasing. It follows easily from (3.29) that
\[
r_m = \dim V^m_y,
\]
where
\[
V^m_y = V_y \cap S^m(g).
\]
(3.33)

On the other hand from the representation theory of the TDS \(u\) (yielding the surjectivity of (2.34)), one readily has that
\[
r_m = \dim b_{m-1},
\]
and hence proving

**Theorem 3.4.** One has
\[
\dim V^m_y = \dim b_{m-1}, \ m = 1, \ldots, h.
\]
(3.35)

**Remark 3.5.** For \(m = 1, \ldots, h,\) consider the set of all pairs \(\{d_j, k\}\) in (3.29) such that \(d_j - k = m.\) For any such pair let \(i = \ell + 1 - j\) and put
\[
J_{m,y,i} = I_{j,y,k}.
\]
(3.36)

Then one notes that
\[
\{J_{m,y,i}\}, \ i = 1, \ldots, r_m, \ is a basis of \ V^m_y.
\]
(3.37)
3.3. As above let $y \in \mathfrak{h}$ be regular. Let $\mathcal{H}_y \subset S(\mathfrak{g})$ be the (Poisson commutative—see Theorem 2.4) subalgebra generated by the $b$-dimensional subspace $V_y$. In dealing with (3.36) it is convenient to simply order the pairs $(m, i)$. Let $\mathcal{B} = \{1, \ldots, b\}$ and let $\mathcal{P} = \{(m, i) \mid m = 1, \ldots, h-1, \ i = 1, \ldots, r_m\}$. Recalling (3.31) let

$$\mathcal{B} \to \mathcal{P}, \ \beta \mapsto (m(\beta), i(\beta))$$

be a bijection where if $\beta < \beta'$, then $m(\beta) \leq m(\beta')$ so that if $J_{y;\beta} = J_{m(\beta), y, i(\beta)}$, then (see Theorem 3.3)

$$\{J_{y;\beta} \mid \beta \in \mathcal{B}\}$$

is a basis of $V_y$ and also generates $\mathcal{H}_y$. (3.39)

**Theorem 3.6.** The function restriction map (see (3.9))

$$\mathcal{H}_y \to A(\text{Hess})$$

is an algebra isomorphism. Furthermore $\mathcal{H}_y$ is a polynomial algebra. In fact

$$\mathcal{H}_y = \mathbb{C}[J_{y;1}, \ldots, J_{y;b}].$$

Moreover not only are the $J_{y;\beta}$ algebraically independent but in fact the differentials

$$\{(dJ_{y;\beta})_v, \ \beta \in \mathcal{B}\}$$

is a basis of $T^*_v(\text{Hess})$ at any point $v$ of the Hessenberg Hess—(not just at $e_1$). (3.42)

In fact, even stronger, (3.42) remains true if the $J_{y;\beta}$ are replaced by the restrictions $J_{y;\beta} \mid \text{Hess}$. Indeed the restrictions $J_{y;\beta} \mid \text{Hess}$, $\beta \in \mathcal{B}$ define a “coordinate system” on Hess. In fact the map

$$\text{Hess} \to \mathbb{C}^b, \ v \mapsto (J_{y;1}(v), \ldots, J_{y;b}(v))$$

is an algebraic isomorphism.

**Proof.** Let $\beta \in \mathcal{B}$. For notational convenience put $p = J_{y;\beta}$ so that $p = J_{m,y,i}$ where $m = m(\beta)$ and $i = i(\beta)$. Now recall the notation of Proposition 3.2. Of course $p \in S^m(\mathfrak{g})^w$. As in Proposition 3.2 let $q = Q(p)$ so that $q \in S^m(\mathfrak{q})^w$. Write $q = q_\beta$ and $q[k] = q_{\beta,[k]}$ so that by (3.18),

$$\sigma_\text{Hess}(J_{y;\beta}) = \sigma_\text{Hess}(\sum_{k=0}^{m(\beta)-1} q_{\beta,[k]}).$$

(3.44)
Also, by Proposition 3.2, one has
\[ q_{\beta, [m(\beta) - 1]} \in \mathfrak{b}_{m(\beta) - 1}, \] (3.45)
and if \( m(\beta) > 1 \),
\[ \sum_{k=0}^{m(\beta) - 2} q_{\beta, [k]} \in S\left( \sum_{k=0}^{m(\beta) - 2} b_k \right). \] (3.46)
In addition, by (3.18) and (3.21),
\[ q_{\beta, [m(\beta) - 1]} = d\left( \sigma_{\text{Hess}}(J_{y;\beta}) \right) e_1. \] (3.47)

For notational convenience put \( z_\beta = q_{\beta, [m(\beta) - 1]} \). We now assert that
\[ \{ z_\beta \mid \beta \in \mathcal{B} \} \text{ is a basis of } \mathfrak{b} \] (3.48)
and hence in particular, by (3.45), for \( m = 1 \ldots, h \),
\[ \{ z_\beta \mid m(\beta) = m \} \text{ is a basis of } \mathfrak{b}_{m-1}. \] (3.49)
Indeed, by dimension, Theorem 2.11 and (3.39),
\[ \{ dJ_{y;\beta}(e_1) \mid \beta \in \mathcal{B} \} \text{ is a basis of } \mathfrak{b}. \] (3.50)
However since \( dJ_{y;\beta}(e_1) \) is an element of \( \mathfrak{b} \) it is immediate from definitions that
\[ dJ_{y;\beta}(e_1) = (d\sigma_{\text{Hess}}(J_{y;\beta})) e_1. \] (3.51)
But this proves (3.48) (and (3.49)). Note also that (3.50) and (3.51) establish that
\( \{ \sigma_{\text{Hess}}(J_{y;\beta}) \mid \beta \in \mathcal{B} \} \) (and a fortiori \( \{ J_{y;\beta} \mid \beta \in \mathcal{B} \} \)) are algebraically independent. But now (3.48) implies that \( S(\mathfrak{b}) \) as a polynomial algebra can be given as
\[ S(\mathfrak{b}) = \mathbb{C}[z_1, \ldots, z_h]. \] (3.52)
On the other hand since \( \text{Hess} \) is the \( e_1 \) translate of \( \mathfrak{b}_- \) one has an algebra isomorphism
\[ \sigma_{\text{Hess}} : S(\mathfrak{b}) \to A(\text{Hess}) \] (3.53)
and hence the map
\[ \text{Hess} \to \mathbb{C}^b, \; v \mapsto (z_1(v), \ldots, z_h(v)) \] (3.54)
is an algebraic isomorphism. But we wish to show that the $z_\beta$ in (3.54) can be replaced by the $J_{y;\beta}$. To do this we first show, inductively, that for all $\beta \in B$ one has

$$\sigma_{\text{Hess}}(\mathbb{C}[z_1, \ldots, z_\beta]) \subset \sigma_{\text{Hess}}(\mathbb{C}[J_{y;1}, \ldots, J_{y;\beta}]).$$

But now if $m(\beta) = 1$, then (3.55) is true since if $i \leq \beta$, then $m(i) = 1$ and hence by (3.44) one has

$$\sigma_{\text{Hess}}(J_{y;i}) = \sigma_{\text{Hess}}(z_i).$$

Now assume $m(\beta) > 1$ (which implies that $\beta > 1$), and assume that (3.55) is true for $\beta - 1$. But then by the induction assumption and (3.46), one has

$$\sigma_{\text{Hess}}(m(\beta) - 2 \sum_{n=0}^{m(\beta)-2} q_{\beta,[n]} \in \sigma_{\text{Hess}}(\mathbb{C}[J_{y;1}, \ldots, J_{y;m(\beta)-1}]).$$

Here we are implicitly using the obvious fact that if $k \leq m(\beta) - 1$ and $m(\beta') = k$, then $\beta' \leq \beta - 1$. But now, by (3.44), (3.46) and (3.57) we may write

$$\sigma_{\text{Hess}}(J_{y;\beta}) = \sigma_{\text{Hess}}(z_\beta) + u,$$

where $u \in \sigma_{\text{Hess}}(\mathbb{C}[J_{y;1}, \ldots, J_{y;m(\beta)-1}])$. Thus we may solve for $\sigma_{\text{Hess}}(z_\beta)$, establishing the inductive step (3.55). But then the remaining statements of Theorem 3.6 are immediate consequences of (3.52), (3.53) and (3.54). QED

By (3.29), (3.36), (3.39), (3.41) and also (3.31), (3.32) we can write down the Poincaré series $P_y(t)$ of $\mathcal{H}_y$. Namely one has (clearly independent of the regular element $y \in \mathfrak{b}$)

$$P_y(t) = \prod_{j=1}^{d_j} \frac{1}{1-t} \cdots \frac{1}{1-t^{d_j}} = \prod_{m=1}^{h} \frac{1}{(1-t^m)^{r_m}}.$$ (3.59)

4. Strong regularity and Zariski dense cotangent structure on regular orbits

4.1. Recall $n = b - \ell$ so that

$$b + n = \dim \mathfrak{g}.$$ (4.1)
Let $\mathcal{N} = \{1, \ldots, n\}$. Fix a regular semisimple element $y \in \mathfrak{h}$. By (2.16) one has

$$I_{j, y, 0} = I_j, \; j = 1, \ldots, \ell,$$  \hspace{1cm} (4.2)

and hence by (3.36)

$$J_{d_j, y, \ell+1-j} = I_j.$$  \hspace{1cm} (4.3)

Thus, recalling the notation of Theorem 3.6, let $\mathcal{I}$ be the subset of cardinality $\ell$ given by

$$\mathcal{I} = \{ \beta \in \mathcal{B} \mid m(\beta) = d_j, \; i(\beta) = \ell + i - j, \; j = 1, \ldots, \ell \}$$  \hspace{1cm} (4.4)

so that

$$\mathcal{B} = \mathcal{N} \sqcup \mathcal{I}$$

and

$$\{ J_{y, \beta}, \; \beta \in \mathcal{I} \} = \{ I_j, \; j = 1, \ldots, \ell \}.$$  \hspace{1cm} (4.5)

For notational convenience let $\{ q_i, i \in \mathcal{B} \}$ be a reordering of the $J_{y, \beta}$ so that one retains

$$\{ q_{\beta}, \; \beta \in \mathcal{I} \} = \{ I_j, \; j = 1, \ldots, \ell \}.$$  \hspace{1cm} (4.6)

Then, first of all let $\Phi$ be the morphism of $\mathfrak{g}$ to $\mathbb{C}^b$ given by

$$\Phi(z) = (q_1(z), \ldots, q_b(z)).$$  \hspace{1cm} (4.7)

Then Theorem 3.6 asserts

\textbf{Theorem 4.1.} \textit{The morphism $\Phi$ is surjective and in fact the restriction}

$$\Phi : \text{Hess} \to \mathbb{C}^b$$  \hspace{1cm} (4.8)

\textit{is an algebraic isomorphism. In particular the isomorphism}

$$(\Phi | \text{Hess})^{-1} : \mathbb{C}^b \to \text{Hess}$$  \hspace{1cm} (4.9)

\textit{is a cross-section of $\Phi$.}

Let

$$\mathfrak{g}^{\text{reg}} = \{ z \in \mathfrak{g} \mid (dq_j)_j, \; j \in \mathcal{B} \text{ be linearly independent} \}.$$  \hspace{1cm} (4.10)
Then $\mathfrak{g}^{\text{reg}}$ is not empty by Theorem 3.6, and in fact Theorem 3.6 asserts that

$$\text{Hess} \subset \mathfrak{g}^{\text{reg}}. \quad (4.11)$$

The elements in $\mathfrak{g}^{\text{reg}}$ are regular by (4.6) and our criterion for regularity (see reference at the end of §1.1). Thus

$$\mathfrak{g}^{\text{reg}} \subset \mathfrak{g}^{\text{reg}}. \quad (4.12)$$

We will refer to the elements in $\mathfrak{g}^{\text{reg}}$ as strongly regular.

Now since any $I \in S(\mathfrak{g})^G$ Poisson commutes with any $p \in S(\mathfrak{g})$ one has $\xi_{q_i} = 0$ for $i \in \mathcal{I}$. For $i \in \mathcal{N}$ let $\xi_i = \xi_{q_i}$. By Poisson commutativity

$$[\xi_i, \xi_j] = 0, \quad \forall i, j \in \mathcal{N}. \quad (4.13)$$

On the other hand if $x \in \mathfrak{g}$ and $O$ is the $G$-adjoint orbit containing $x$ then, for $i \in \mathcal{N}$,

$$(\xi_i)_x \in T_x(O) \quad (4.14)$$

by (2.5). If $x \in \mathfrak{g}^{\text{reg}}$, then as noted in (2.12), one has

$$\dim O = 2n. \quad (4.15)$$

Let $R$ be the set of all $G$-orbits in $\mathfrak{g}^{\text{reg}}$. If $x$ is strongly regular, let $Z_x$ be the span of $(\xi_i)_x$, $i \in \mathcal{N}$ so that $Z_x \subset T_x(O)$.

**Theorem 4.2.** Let $x \in \mathfrak{g}^{\text{reg}}$ and let $O \in R$ be the regular orbit containing $x$. Then $Z_x$ is a Lagrangian subspace of $T_x(O)$ and

$$\{(\xi_i)_x, i \in \mathcal{N}, \text{ is a basis of } Z_x. \quad (4.16)$$

**Proof.** Since $\{(dq_j)_x\}, j \in \mathcal{B}$ are linearly independent and since $\{q_k\}, k \in \mathcal{I}$, are constant on $O$ it is immediate that $\{(dq_k)_x\}, k \in \mathcal{I}$, are a basis of the orthocomplement of $T_x(O)$ in $T^*_x(\mathfrak{g})$. But then necessarily the differentials $\{d(q_i|O)_x\}, i \in \mathcal{N}$, are a basis of $T^*_x(O)$. But this implies (4.16). But then $Z_x$ is Lagrangian by (4.13). QED

4.2. For any $O \in R$ and $j \in \{1, \ldots, \ell\}$ let $I_j(O)$ be the constant value that $I_j$ takes on $O$. We recall (see Theorem 2, p. 360 in [K2]) that if $\eta : R \to \mathbb{C}^\ell$ is the map given by putting $\eta(O) = (I_1(O), \ldots, I_\ell(O))$, then

$$\eta : R \to \mathbb{C}^\ell \text{ is a bijection.} \quad (4.17)$$
Now for any $O \in R$ let

$$\text{Hess}(O) = O \cap \text{Hess}.$$  

Elsewhere we have proved (reversing the roles of $n$ and $n_-$)

**Theorem 4.3.** Let $O \in R$. Then

(a) $\text{Hess}(O)$ is the subvariety of $\text{Hess}$ given by

$$\text{Hess}(O) = \{v \in \text{Hess} \mid I_j(v) = I_j(O), \ j = 1, \ldots, \ell\};$$

(b) $\text{Hess}(O)$ is a principal (i.e., with trivial isotropy subgroup) $N_-$ orbit in $\text{Hess}$ so that in particular $\text{Hess}(O)$ is a nonsingular $n$-dimensional subvariety of $\text{Hess}$.

(c) One has

$$\text{Hess} = \bigsqcup_{O \in R} \text{Hess}(O) \quad (4.18)$$

so that (4.18) is the $N_-$-orbit decomposition of $\text{Hess}$.

**Proof.** Theorem 4.3 is an immediate consequence of Theorem 7, p. 381, and Theorem 8, p. 382 in [K2] and Theorem 1.2. p. 109 in [K4]. QED

Let $O \in R$. As an adjoint orbit we recall that $O$ is a $2n$-dimensional symplectic manifold. On the other hand $\text{Hess}(O)$ is an $n$-dimnesional submanifold of $O$ by (b) in Theorem 4.3. In fact

**Theorem. 4.4.** Let $O \in R$. Then $\text{Hess}(O)$ is a Lagrangian submanifold of $O$.

**Proof.** Let $v \in \text{Hess}(O)$. Let $\tau_1, \tau_2 \in T_v(\text{Hess}(O))$. Then if, as in (2.2), $\omega_v$ is the symplectic form $\omega_O$ at $v$, we are to show that $\omega_v(\tau_1, \tau_2) = 0$. But by Theorem 4.3 there exists $z_i \in n_-$, for $i = 1, 2$, such that $-[z_i, v] = \tau_i$. But then, by (2.1),

$$\omega_v(\tau_1, \tau_2) = (v, [z_2, z_1]). \quad (4.19)$$

But $[z_2, z_1] \in [n_-, n_-]$ and clearly

$$[n_-, n_-] \subset \sum_{k=2}^{h-1} g[-k], \quad (4.20)$$

where $g[k] = \{x \in g \mid [w, x] = 2kx\}$. On the other hand $v \in g[1] + h + \sum_{k=1}^{h-1} g[-k]$. Hence $(v, [z_2, z_1]) = 0$. Thus $\text{Hess}(O)$ is Lagrangian. QED
4.2. Now clearly $g^{\text{reg}}$ is a nonempty (by e.g., (4.11)) Zariski open subset of $g$. In particular $g^{\text{reg}}$ is a quasi-affine nonsingular irreducible algebraic variety. Let $\Phi^{\text{reg}} = \Phi|_{g^{\text{reg}}}$ (see (4.7)), so that by (4.8) one has the surjective morphism

$$\Phi^{\text{reg}} : g^{\text{reg}} \to C^b$$

and

$$\Phi^{\text{reg}} : \text{Hess} \to C^b$$

is an algebraic isomorphism. For any $c \in C^b$ let

$$F_c = (\Phi^{\text{reg}})^{-1}(c)$$

so that $F_c$ is a closed subvariety of $g^{\text{reg}}$, (noting that variety in our notation here does not require irreducibility) and

$$g^{\text{reg}} = \bigsqcup_{c \in C^b} F_c. \tag{4.21}$$

Of course if $c = (c_1, \ldots, c_b) \in C^b$, then

$$F_c = \{z \in g^{\text{reg}} \mid q_i(z) = c_i, \ i = 1, \ldots, b\}. \tag{4.22}$$

On the other hand one knows that for any $z \in g^{\text{reg}}$ the differentials $(dq_i)_z, i = 1, \ldots, b,$ are linearly independent.

**Theorem 4.5.** $F_c$ is a nonsingular variety of dimension $n$ for any $c \in C^b$.

**Proof.** Let $z \in F_c$. Then Theorem 4.5 is an immediate consequence of Theorem 4, §4 in Chapter III, p. 172 in [M] where $U \subset g^{\text{reg}}$ is an affine neighborhood of $z$ and $f_1, \ldots, f_b$ are the images of $q_1, \ldots, q_b$ in $A(U)$. QED

**Theorem 4.6.** Let $c \in C^b$. Then the analytic space $F_c$ is a nonsingular analytic manifold of dimension $n$.

**Proof.** This is immediate from Corollary 2 , §4 in Chapter III, p. 168 in [M]. QED

For any $G$-orbit $O$ in $g$ let

$$O^{\text{reg}} = g^{\text{reg}} \cap O.$$
Proposition 4.7. Let $O$ be a $G$-orbit in $\mathfrak{g}$. Then $O^{\text{reg}}$ is nonempty if and only if $O \in R$. In fact if $O \in R$, then $O^{\text{reg}}$ is an open Zariski dense subvariety of $O$. In particular $O^{\text{reg}}$ is a $2n$-dimensional symplectic submanifold of $O$, and one has

$$\mathfrak{g}^{\text{reg}} = \bigsqcup_{O \in R} O^{\text{reg}}.$$ (4.23)

Proof. The proposition is immediate from (4.11), (4.12), Theorem 4.3 and of course from the fact that $\mathfrak{g}^{\text{reg}}$ is Zariski open in $\mathfrak{g}$. QED

Recalling (4.6), let $j(\beta) \in \{1, \ldots, \ell\}$ be defined for $\beta \in I$ so that

$$q_\beta = I_{j(\beta)}.$$ (4.24)

Then for $O \in R$, let

$$\mathbb{C}^b(O) = \{c = (c_1, \ldots, c_b) \in \mathbb{C}^b \mid c_\beta = I_{j(\beta)}(O) \quad \forall \beta \in \mathcal{I}\}.$$ Of course

$$\mathbb{C}^b = \bigsqcup_{O \in R} \mathbb{C}^b(O).$$ (4.25)

Then one has the following fibration (with $n$-dimensional fibers) of $O^{\text{reg}}$ for any $O \in R$.

Theorem 4.8. Let $O \in R$. Then

$$O^{\text{reg}} = \bigsqcup_{c \in \mathbb{C}^b(O)} F_c.$$ (4.26)

Proof. This is immediate from Theorem 2, p. 360 in [K2]. This result asserts that any element $x \in \mathfrak{g}^{\text{reg}}$ is uniquely determined, up to $G$-conjugacy, by the vector $(I_1(x), \ldots, I_\ell(x)) \in \mathbb{C}^\ell$, and any such vector can be achieved by some $x \in \mathfrak{g}^{\text{reg}}$. QED

Theorem 4.2 asserts that $x \mapsto Z_x$ for $x \in \mathfrak{g}^{\text{reg}}$ is an $n$-dimensional distribution (in the sense of differential geometry) $Z$ on the analytic manifold $\mathfrak{g}^{\text{reg}}$. But then (4.13) asserts that $Z$ is involutory. Thus by the Frobenius theorem, in the complex analytic category, one has a foliation of $\mathfrak{g}^{\text{reg}}$ by a family $\mathcal{L}$ of maximal integral connected ($n$-dimensional) manifolds of $Z$. For the validity of the use of the Frobenius theorem in the complex analytic category, see Theorem 1.3.6, p. 30 in [V] and the comment at the end of §1.3 in Chapter 1, p. 31, in [V]. We refer to the elements $L$ of $\mathcal{L}$ as leaves of $Z$. Let $\Lambda$ be an index set for $\mathcal{L}$ so that $\mathcal{L} = \{L_\lambda \mid \lambda \in \Lambda\}$, and one has

$$\mathfrak{g}^{\text{reg}} = \bigsqcup_{\lambda \in \Lambda} L_\lambda.$$ (4.27)
Recalling the notation of Theorem 4.2 note that, by definition of integral manifold, for any $\lambda \in \Lambda$ and $x \in L_\lambda$, one has

$$T_x(L_\lambda) = Z_x. \quad (4.28)$$

A complex algebraic variety has, besides the Zariski topology, the ordinary Hausdorff topology, which following Chapter 7 in [S], we refer to as the complex topology.

**Theorem 4.9.** Let $\lambda \in \Lambda$. Then there exists a unique $c \in C^b$ such that $L_\lambda$ is an open, in the complex topology, submanifold of the fiber $F_c$. In particular if $x \in L_\lambda$, then

$$T_x(F_c) = Z_x. \quad (4.29)$$

**Proof.** If $i, j = 1, \ldots, b$, then by Theorem 2.4 one has $[q_i, q_j] = 0$. Thus $\xi_i q_j = 0$. But then $q_j$ is constant on $L_\lambda$ for $j = 1, \ldots, b$. But this implies that there exists $c \in C^b$ such that $L_\lambda \subset F_c$. Since both $L_\lambda$ and $F_c$ are (nonsingular) analytic manifolds of dimension $n$, this implies that $L_\lambda$ is open in $F_c$ in the complex topology and also that (4.28) implies (4.29). QED

Now for any $c \in C^b$ let

$$\Lambda_c = \{ \lambda \in \Lambda \mid L_\lambda \subset F_c \}.$$

Recalling (4.27) the following statement is an immediate corollary of Theorem 4.9.

**Theorem 4.10.** Let $c \in C^b$. Then

$$F_c = \sqcup_{\lambda \in \Lambda} L_\lambda. \quad (4.30)$$

Moreover (4.30) is the decomposition of the fiber $F_c$ into its connected components with respect to its complex topology.

But now recall (see Theorem 4.5) that the Fiber $F_c$ is a nonsingular algebraic variety. Using the Zariski topology this leads to a much more interesting statement than Theorem 4.10.

**Theorem 4.11.** The leaf $L_\lambda$ is a (Zariski) closed nonsingular algebraic irreducible subvariety of $g^{\text{reg}}$ for any $\lambda \in \Lambda$. Furthermore if $c \in C^b$, then $\Lambda_c$ is finite and
the decomposition (4.30) is both the decomposition of the fiber $F_c$ into the union of its (algebraic) Zariski irreducible components and also the decomposition of $F_c$ into the union of its Zariski connected components.

**Proof.** Let $c \in C^b$ and for some index set $\Gamma$ let \( \{ F_c^\gamma, \gamma \in \Gamma \} \) be the set of all Zariski connected components of $F_c$. Thus

\[ F_c = \bigsqcup_{\gamma \in \Gamma} F_c^\gamma. \quad (4.31) \]

But since $F_c$ is nonsingular the set of Zariski connected components of $F_c$ is the same as the set of (Zariski) irreducible components of $F_c$. See Corollary 17.2, p. 74 in [B]. Thus $\Gamma$ is finite and hence all the $F_c^\gamma$ are Zariski open and closed in $F_c$. But Zariski open implies complex open. Thus if $\lambda \in \Lambda_c$ and $\gamma \in \Gamma$ one has, since $L_\lambda$ is complex connected, either $L_\lambda \subset F_c^\gamma$ or $L_\lambda \cap F_c^\gamma$ is empty, Thus there exists a subset $\Lambda_c^\gamma \subset \Lambda_c$ such that

\[ F_c^\gamma = \bigsqcup_{\lambda \in \Lambda_c^\gamma} L_\lambda. \quad (4.32) \]

But Theorem 4.30 implies that (4.32) is the decomposition of $F_c^\lambda$ into its complex connected components. But $F_c^\gamma$ is complex connected by Theorem, §2 in Chapter 7, p. 321 in [S]. Thus $\Lambda_c^\gamma$ must only have one element. But this clearly proves the theorem since $c \in C^b$ is arbitrary. QED

We note in passing that we recover the theorem of A.A. Tarasev in [T] to the effect that $\mathcal{H}_y$ (see §3.3) is maximally Poisson commutative in $S(\frak{g})$. Tarasev’s result was in response to the question of maximality posed by E. Vinberg.

**Theorem 4.12.** The subalgebra $\mathcal{H}_y$ of §3.3 is maximally Poisson commutative in $S(\frak{g})$.

**Proof.** Let $x \in \text{Hess}$. Then by the local Frobenius theorem (see Theorem 1.3.3, §1.3, Chapter 1, p.28 in [V] and the statement of its applicability in the complex analytic case at the end of §1.3 on p.31) there exists an complex open neighborhood $U'$ of $x$ in $\frak{g}^{\text{reg}}$ which admits a foliation

\[ U' = \bigsqcup_{\delta \in \Delta'} E_\delta \quad (4.33) \]

where, for each $\delta$ in the parameter set $\Delta'$, $E_\delta$ is a connected integral manifold for the distribution $Z$ (see above in §4.2). Let $\Phi_U = \Phi|U'$ (see Theorem 4.1) and for any
z ∈ U' let δ(z) ∈ Δ' be such that z ∈ E_δ(z). Clearly

\[ E_\delta(z) \subset F_c, \quad (4.34) \]

where \( c = \Phi_U(z) \). Now let

\[ U = \{ z \in U' \mid \text{Hess} \cap E_\delta(z) \neq \emptyset \}. \]

Note that \( U \) is not empty since \( x \in U \). We assert that \( U \) is complex open in \( U' \). Indeed let \( z \in U \) and let \( c = \Phi_U(z) \). Then, recalling (4.9), one must have \( v \in \text{Hess} \cap E_\delta(z) \) where

\[ v = (\Phi|\text{Hess})^{-1}(c). \quad (4.35) \]

But now \( U' \cap \text{Hess} \) is a complex open neighborhood of \( v \) in \( \text{Hess} \). By continuity there exists a complex open neighborhood \( D_c \) of \( c \) in \( \mathbb{C}^b \) such that

\[ (\Phi|\text{Hess})^{-1}(D_c) \subset U' \cap \text{Hess}. \]

But by the continuity of \( \Phi_U \) there exists a complex open neighborhood \( W \) of \( z \) in \( U' \) such that \( \Phi_U(W) \subset D_c \). But then it is immediate that \( W \subset U \). Hence \( U \) is open and clearly there exists a subset \( \Delta \subset \Delta' \) such that

\[ U = \sqcup_{\delta \in \Delta} E_\delta. \quad (4.36) \]

Now assume that \( f \in S(\mathfrak{g}) \) Poisson commutes with all \( q \in \mathcal{H}_y \). But then by Theorem 3.6 there exists \( p \in \mathcal{H}_y \) such that

\[ f|\text{Hess} = p|\text{Hess}. \]

But both \( p \) and \( f \) are constant on any connected integral manifold of \( \mathcal{Z} \). But then \( f - p \) vanishes on \( U \) by (4.36). Since \( u \) is complex open in \( \mathfrak{g} \) this implies \( f = p \). QED

4.3. To state our final results it will be convenient to replace \( \mathbb{C}^b \) as parameters for the fibers \( F_c \) of \( \Phi|\mathfrak{g}_{\text{reg}} \) by Hess (see (4.8) and (4.21)). For any \( x \in \text{Hess} \) put

\[ F[x] = F_{\Phi(x)}. \]

We recall that

\[ T_x(F[x]) = Z_x; \quad (4.37) \]
(see (4.28) and Theorems 4.2 and 4.9).

We also recall that Hess($O$) is a Lagrangian submanifold of $O$ for any $O \in R$ (see Theorems 4.3 and 4.4). On the other other hand one has

**Theorem 4.13.** Let $O \in R$. and let $x \in \text{Hess}(O)$. Then $F_{[x]}$ is a Lagrangian submanifold (not necessarily connected) of $O^{\text{sreg}}$. Furthermore

$$T_x(O) = T = T_x(F_{[x]}) \oplus T_x(\text{Hess}(O))$$

(4.38)

so that the two Lagrangian subspaces $T_x(F_{[x]})$ and $T_x(\text{Hess}(O))$ of $T_x(O)$ are nonsingularly paired by $\omega_x$ (see (2.1)).

**Proof.** The first conclusion of Theorem 4.13 is immediate from Theorems 4.2 and 4.9. To prove the final statement of Theorem 4.13 it is, by dimension, enough to prove that

$$T_x(F_{[x]}) \cap T_x(\text{Hess}(O)) = 0.$$

(4.39)

But if $\Phi_*$ is the differential of $\Phi$ (operating on $T(g)$), one has $T_x(F_{[x]}) \subset \text{Ker}\Phi_*$ by definition of the fiber $F_{[x]}$. On the other hand $\Phi_*|T_x(\text{Hess}(O))$ is injective by Theorem 4.1. This proves (4.39). QED

For convenience, before our final statement we will recount some of the definitions and previous results. We have chosen and fixed a regular semisimple element $y \in \mathfrak{h}$. Using $y$ and the translation of invariant procedure of Fomenko–Mischenko one constructs a maximal Poisson commutative subalgebra $\mathcal{H}_y$ of $S(\mathfrak{g})$. We use this Poisson commutative subalgebra to introduce the definition of strong regularity in $\mathfrak{g}$ and the corresponding open Zariski dense variety $\mathfrak{g}^{\text{sreg}}$ in $\mathfrak{g}$. Intersecting with an adjoint orbit $O$ of regular elements one has an open Zariski dense subvariety $O^{\text{sreg}}$ of $O$ which then is also a symplectic submanifold of $O$. One has that Hess $\subset \mathfrak{g}^{\text{sreg}}$ where Hess $= e_1 + \mathfrak{b}_-$. Intersecting Hess with $O$ defines a Lagrangian submanifold Hess($O$) of $O^{\text{sreg}}$. The Hamiltonian vector fields which arise from $\mathcal{H}_y$, restricted to $\mathfrak{g}^{\text{sreg}}$, define an involutive distribution $\mathcal{Z}$ on $\mathfrak{g}^{\text{sreg}}$. The leaves of $\mathcal{Z}$ define a foliation of $\mathfrak{g}^{\text{sreg}}$. A choice of generators of $\mathcal{H}_y$ defines a surjective morphism $\Phi^{\text{sreg}} : \mathfrak{g}^{\text{sreg}} \to \mathbb{C}^b$ whose restriction to Hess is an algebraic isomorphism Hess $\to \mathbb{C}^b$. Here $b$ is the dimension of a Borel subalgebra of $\mathfrak{g}$. The irreducible components of any fiber $F_{[x]}$, $x \in \text{Hess}$, of $\Phi^{\text{sreg}}$ are maximal complex connected integral submanifolds of $\mathcal{Z}$, establishing therefore that
there are only a finite number of maximal complex connected integral submanifolds of $Z$ in $F_x$ for any $x \in Hess$.

The following result says in effect that then $H_y$ simultaneously polarizes $O^{\text{reg}}$ for all regular orbits of $G$. Even more $O^{\text{reg}}$ simulates a cotangent bundle structure over a base manifold $rm Hess(O)$.

**Theorem 4.14.** Let $O \in R$ so that $O$ is an arbitrary regular $G$-orbit in $\mathfrak{g}$. Then

$$O^{\text{reg}} = \sqcup_{x \in Hess(O)} F_x$$

(4.40)

defines a polarization of the symplectic manifold $O^{\text{reg}}$, noting that $Hess(O)$ is a Lagrangian submanifold of $O^{\text{reg}}$ and that $Hess(O)$ is transversal to all the Lagrangian fibers $F_{x}$, $x \in Hess(O)$ of the polarization.

**Proof.** Recalling the notation of (4.25) note that $\Phi$ induces an isomorphism $Hess(O) \to C^b(O)$ by (a) of Theorem 4.3. But then (4.40) follows from Theorem 4.8. The remaining statements follow from Theorem 4.4 and Theorem 4.13. QED

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