TO MULTIDIMENSIONAL MELLIN ANALYSIS: BESOV SPACES, K-FUNCTOR, APPROXIMATIONS, FRAMES.

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Dedicated to 95th birthday of my teacher Paul L. Butzer

Abstract. In the setting of the multidimensional Mellin analysis we introduce moduli of continuity and use them to define Besov-Mellin spaces. We prove that Besov-Mellin spaces are the interpolation spaces (in the sense of J. Peetre) between two Sobolev-Mellin spaces. We also introduce Bernstein-Mellin spaces and prove corresponding direct and inverse approximation theorems. In the Hilbert case we discuss Laplace-Mellin operator and define relevant Paley-Wiener-Mellin spaces. Also in the Hilbert case we describe Besov-Mellin spaces in terms of Hilbert frames.

1. Introduction

By one-dimensional Mellin analysis we understand the following setup. For $p \in [1, \infty]$, denote by $\| \cdot \|_p$ the norm of the Lebesgue space $L^p(\mathbb{R}_+)$. In Mellin analysis, the analogue of $L^p(\mathbb{R}_+)$ are the spaces $X^p(\mathbb{R}_+)$ comprising all functions $f : \mathbb{R}_+ \mapsto \mathbb{C}$ such that $f(\cdot)^{-1/p} \in L^p(\mathbb{R}_+)$ with the norm $\|f\|_{X^p(\mathbb{R}_+)} := \|f(\cdot)^{-1/p}\|_p$. Furthermore, for $p = \infty$, we define $X^\infty(\mathbb{R}_+)$ as the space of all measurable functions $f : \mathbb{R}_+ \mapsto \mathbb{C}$ such that $\|f\|_{X^\infty(\mathbb{R}_+)} := \sup_{x > 0} |f(x)| < \infty$.

In spaces $X^p(\mathbb{R}_+)$ we consider the one-parameter $C_0$-group of operators $U(t), t \in \mathbb{R}$, where

$$U(t)f(x) = f(e^t x), \quad U(t + \tau) = U(t)U(\tau),$$

whose infinitesimal generator is

$$\frac{d}{dt}U(t)f(x)|_{t=0} = x \frac{d}{dx} f(x) = \Theta f(x).$$

The described setting closely related to the Mellin transform

$$\mathcal{M}f(s) = \int_0^\infty f(x)x^{s-1}dx,$$

and its inverse

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}f(s)x^{-s}ds, \quad s = c + it.$$
its history, and connections to other mathematical ideas see the beautiful paper [15]. A lot of relevant information can also be found in interesting articles [2], [17]. In particular, in a series of papers [3]-[11], [16], [40] authors developed in the framework of one-dimensional Mellin analysis analogs of such important topics as Bernstein spaces, Bernstein inequality, Paley-Wiener theorem, Riesz-Boas interpolation formulas, different sampling results including exponential sampling.

The present paper has two main objectives: 1) to go from one-dimensional setting to multidimensional, and 2) to concentrate on such important topics as Besov spaces, $K$-functor, approximations, frames. As well as we know, such topics were never discussed in the setting of the multidimensional Mellin analysis.

The paper is organized as follows. In section 2 we describe the basic assumptions and notations of multidimensional Mellin analysis. In section 3 in the setting of the multidimensional Mellin analysis we introduce moduli of continuity and use them to define Besov-Mellin spaces. In section 4 we prove that Besov-Mellin spaces are the interpolation spaces (in the sense of J.Peetre) between two Sobolev-Mellin spaces. In section 5 we introduce Bernstein-Mellin spaces and in section 6 prove corresponding direct and inverse approximation theorems including a Jackson-type inequality. Sections 7-9 are devoted to the Hilbert case. We introduce and discuss Laplace-Mellin operator in section 7. In section 8 we define relevant Paley-Wiener spaces and in section 9 consider a partition of unity on the spectrum of the Laplace-Mellin operator. In section 10 we describe Besov-Mellin spaces in terms of approximations by Paley-Wiener functions and in terms of some Hilbert frames. The article also contains extensions of the Landau-Kolmogorov-Stein inequality to the Mellin analysis setting. Throughout the paper we extensively use the Peetre’s theory of interpolation and approximation spaces. To make the paper self contained some related definitions, facts and proofs are collected in Appendices 1-3.

All our statements concerning the multidimensional Mellin analysis seems to be new. However, at least some of them are just adaptations and particular realizations of previously obtained (but maybe not very well known) more general results (see [14], [28]-[40], [42]). In this sense the article can be treated as a review paper. Our main goal was to attract attention to multidimensional Mellin analysis, which is currently underdeveloped.

2. Multidimensional Mellin Analysis

2.1. Space $X^p = X^p(\mathbb{R}_+^n, d\mu)$. Mellin translations and their infinitesimal operators. We set $\mathbb{R}_+ = \{x \in \mathbb{R}, x > 0\}$ and $\mathbb{R}_+^n = \{(x_1, \ldots, x_n), x_j \in \mathbb{R}_+, j = 1, \ldots, n\}$, and introduce the measure $d\mu = \frac{dx_1 \ldots dx_n}{x_1 \ldots x_n}$. The corresponding space $L^p$, $1 \leq p < \infty$ of complex valued functions defined on $\mathbb{R}_+^n$ is denoted as $X^p = X^p(\mathbb{R}_+^n, d\mu)$ and its norm is

$$\|f\|_{X^p} = \left(\int_{\mathbb{R}_+^n} |f|^p d\mu\right)^{1/p},$$

for $1 \leq p < \infty$. We define $X^\infty$ as the space of all measurable functions $f : \mathbb{R}_+^n \mapsto \mathbb{C}$ such that

$$\|f\|_{X^\infty} := \text{ess sup}_{x=(x_1,\ldots,x_n)\in\mathbb{R}_+^n} |f(x)| < \infty.$$
In spaces $X^p$, $1 \leq p < \infty$, we consider the one-dimensional $C_0$-groups of operators $U_j(t)$, $t \in \mathbb{R}$, $j = 1, \ldots, n$, where

$$(2.1) \quad U_j(t)f(x_1, \ldots, x_n) = f(x_1, \ldots, e^t x_j, \ldots, x_n),$$

whose infinitesimal generator is $\Theta_j = x_j \partial_j$, $j = 1, \ldots, n$, since

$$\lim_{t \to 0} \frac{1}{t} (U_j(t)f(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)) = \frac{d}{dt} U_j(t)f(x)|_{t=0} = x_j \partial_j f(x).$$

The general theory (see [14, 21]) implies that such an operator is closed in $X^p$, $1 \leq p < \infty$. Its domain $D(\Theta_j)$ is dense in $X^p$, $1 \leq p < \infty$, and it consists of all functions in $X^p$ for which the limit (2.2) exists in the norm of $X^p$, $1 \leq p < \infty$. Domains of the powers of $\Theta_j$ are $D(\Theta_j^k)$, $k \in \mathbb{N}$, and $D^\infty(\Theta_j) = \bigcap_{k=1}^\infty D(\Theta_j^k)$. We will also use the notation

$$D^\infty = D^\infty(\Theta_1, \ldots, \Theta_n) = \cap_{k \in \mathbb{N}} \cap_{|\alpha| = k} D(\Theta_1^{\alpha_1} \ldots \Theta_n^{\alpha_n}),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \in \mathbb{N}$, $1 \leq j \leq n$.

We note that the groups $U_j$ commute with each other

$$U_i(t_i)U_j(t_j)f = U_j(t_j)U_i(t_i)f, \quad f \in X^p, \quad 1 \leq p < \infty, \quad 1 \leq i, j \leq n, \quad t_i, t_j \geq 0.$$ 

Also, one can easily verify that $D^\infty$ is invariant under all $U_j$ and all $\Theta_j$ and

$$(2.3) \quad \Theta_j U_1(t_1)U_2(t_2)\ldots U_n(t_n)f = U_1(t_1)U_2(t_2)\ldots \Theta_j U_j(t_j)\ldots U_n(t_n)f, \quad f \in D^\infty.$$ 

One can also check that for the

$$Uf(t_1, t_2, \ldots, t_n) = U_1(t_1)U_2(t_2)\ldots U_n(t_n)f, \quad (t_1, \ldots, t_n) \in \mathbb{R}^n,$$

the following formula holds

$$(2.4) \quad \Theta_j Uf(t_1, \ldots, t_n) = \partial_j [Uf(t_1, \ldots, t_n)], \quad f \in D^\infty.$$

3. Sobolev-Mellin and Besov-Mellin spaces

**Definition 1.** The Banach space $W_p^k = W_p^k(\mathbb{R}_+^n, d\mu)$, $k \in \mathbb{N}$, $1 \leq p < \infty$, which we call the Sobolev-Mellin space, is the set of functions $f$ in $X^p = X^p(\mathbb{R}_+^n, d\mu)$ for which the following norm is finite

$$\|f\|_{W_p^k} = \|f\|_{X^p} + \sum_{|\alpha| = k} \|\Theta_1^{\alpha_1} \ldots \Theta_n^{\alpha_n} f\|_{X^p},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$.

By using closeness of the operators $\Theta_j$ one can prove the following fact.

**Lemma 3.1.** The norm $\|f\|_{W_p^k}$ is equivalent to the norm

$$\|\|f\|\|_{W_p^k} = \|f\|_{X^p} + \sum_{r=1}^{k} \sum_{1 \leq j_1, \ldots, j_r \leq n} \|\Theta_{j_1} \ldots \Theta_{j_r} f\|_{X^p},$$

where $r \in \mathbb{N}$, $f \in X^p$, $1 < p < \infty$.

One has the following version of the Landau-Kolmogorov-Stein inequality which follows from a general fact Theorem [12, 3] in Appendix 2.
Theorem 3.2. The following holds for $f \in D(\Theta_j^m)$, $1 \leq p < \infty$,
\[
\|\Theta_j^m f\|_{X^p}^m \leq C_{k,m}\|\Theta_j^m f\|_{X^p}^k \|f\|_{X^p}^{m-k}, \quad 0 \leq k \leq m,
\]
where $C_{k,m} = (K_{m-k})^m/(K_m)^{m-k}$, and
\[
K_j = \frac{4}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^{r+1}}{(2r+1)^{j+1}}, \quad j, r \in \mathbb{N},
\]
is a Krein-Favard constant.

The mixed modulus of continuity is introduced as
\[
\Omega^p_r(s,f) =
\]
\[
\sum_{1 \leq j_1, \ldots, j_r \leq n} \sup_{0 \leq \tau_j \leq s} \sup_{0 \leq \tau_r \leq s} \|U_{j_1}(\tau_{j_1}) - I\) \ldots \ldots \ldots (U_{j_r}(\tau_{j_r}) - I) f\|_{X^p},
\]
where $f \in X^p$, $r \in \mathbb{N}$, and $I$ is the identity operator in $X^p$, $1 \leq p < \infty$.

Definition 2. For $\alpha \in \mathbb{R}_+$, $1 \leq p < \infty$, $1 \leq q \leq \infty$, and $\alpha$ is not integer the Besov-Mellin space $B^{\alpha}_{p,q}$ is defined as the subspace in $X^p$ of all the functions for which the following norm is finite
\[
\|f\|_{W^{\alpha}_{p,q}} + \sum_{1 \leq j_1, \ldots, j_\alpha \leq n} \left( \int_0^\infty \left( s^{\alpha-\alpha} \Omega_{p}^1(s,\Theta_{j_1} \ldots \Theta_{j_\alpha},f) \right) q \frac{ds}{s} \right)^{1/q},
\]
where $[\alpha]$ is the integer part of $\alpha$.

In the case when $\alpha = k \in \mathbb{N}$ is an integer the Besov-Mellin space $B^{\alpha}_{p,q}$, $1 \leq p < \infty$, $1 \leq q \leq \infty$, is defined as the subspace in $X^p$ of all the functions for which the following norm is finite (Zygmund condition)
\[
\|f\|_{W^{k}_{p,q}} + \sum_{1 \leq j_1, \ldots, j_{k-1} \leq n} \left( \int_0^\infty \left( s^{k-1} \Omega_{p}^2(s,\Theta_{j_1} \ldots \Theta_{j_{k-1}},f) \right) q \frac{ds}{s} \right)^{1/q}.
\]

4. The interpolation theorem for the Sobolev-Mellin and Besov-Mellin spaces

It is well known that the most important property of the classical Besov spaces is the fact that they are interpolation spaces (see section Appendix 1 below) between two Sobolev spaces [14], [22]. Here we formulate this result for Besov-Mellin spaces.

For the pair of Banach spaces $(X^p, W^1_p)$, $1 \leq p < \infty$, the $K$-functor is defined by the formula
\[
K(s^r, f, X^p, W^1_p) = \inf_{f=f_0+f_1, f_0 \in X^p, f_1 \in W^1_p} \left( \|f_0\|_{X^p} + s^r \|f_1\|_{W^1_p} \right).
\]

The results below are the particular case of the results in [29], [39] (see also [23], [28], [31], [32]) which correspond to a situation with commuting one-parameter semigroups.

We consider the so-called Hardy-Steklov-Mellin operator $P_r(s)$ which is defined on $X^p$, $1 \leq p < \infty$, by the formula
\[
P_r(s) = P_{1,r}(s)P_{2,r}(s) \ldots P_{n,r}(s)f, \quad f \in X^p, \quad 1 \leq p < \infty,
\]
where
\[ \mathcal{P}_{j,r}(s) = \frac{(s/r)^{-r}}{r!} \int_0^{s/r} \cdots \int_0^{s/r} \sum_{k=1}^{r} (-1)^k C_r^k U_j \left( k(\tau_j + \cdots + \tau_j) \right) f d\tau_j_1 \cdots d\tau_j_r, \]
and \( C_r^k \) are the binomial coefficients. As it is shown in [29], [39],
\[ \mathcal{P}_r(s) : X^p \mapsto \mathcal{W}_p^r, \quad 1 \leq p < \infty, \quad r \in \mathbb{N}, \quad s \in \mathbb{R}. \]
Moreover, there exist constants \( c > 0, \quad C > 0, \) such that for all \( f \in X^p, \quad s \geq 0, \quad 1 \leq p < \infty, \)
\begin{equation}
(4.1) \quad c \Omega_p^r(s, f) \leq K(s^r, f, X^p, \mathcal{W}_p^r) \leq C \left( \Omega_p^r(s, f) + \min(s^r, 1) \| f \|_{X^p} \right).
\end{equation}
This inequality implies density of \( \mathcal{W}_p^r, \quad 1 \leq p < \infty, \) in \( X^p. \)

**Definition 3.** The interpolation space \((X^p, \mathcal{W}_p^r)_{\alpha/r,q}^K, \quad 0 < \alpha < r \in \mathbb{N}, \quad 1 \leq p < \infty, \quad 1 \leq q \leq \infty, \) is defined as the space of all \( f \in X^p \) for which the following norm is finite
\begin{equation}
(4.2) \quad \| f \|_{\alpha/r,q}^K = \| f \|_{X^p} + \left( \int_0^\infty \left( s^\alpha K(s^r, f, X^p, \mathcal{W}_p^r) \right)^q \frac{ds}{s} \right)^{1/q}, \quad 1 \leq q < \infty,
\end{equation}
with the usual modifications for \( q = \infty. \)

**Theorem 4.1.** The following holds true.

1. The Besov-Mellin space \( \mathcal{B}_{pq}^\alpha \) coincides with the interpolation space \((X^p, \mathcal{W}_p^r)_{\alpha/r,q}^K, \quad 0 < \alpha < r \in \mathbb{N}, \quad 1 \leq p < \infty, \quad 1 \leq q \leq \infty, \) and the norms (3.3), (3.4), (4.2) are equivalent (for appropriate choice of indices) to the norm
\begin{equation}
(4.3) \quad \| f \|_{X^p} + \left( \int_0^\infty \left( s^{-\alpha} \Omega_p^r(s, f) \right)^q \frac{ds}{s} \right)^{1/q}, \quad 1 \leq q < \infty,
\end{equation}
with the usual modifications for \( q = \infty. \)

2. The following isomorphism holds true
\[ (X^p, \mathcal{W}_p^r)_{\alpha/r,q}^K = (\mathcal{W}_{p_{k_1}}^r \cap \mathcal{W}_{p_{k_2}}^r)_{(\alpha-k_1)/(k_2-k_1),q}^K, \]
where \( 0 \leq k_1 < \alpha < k_2 \leq r \in \mathbb{N}, \quad 1 \leq p < \infty, \quad 1 \leq q \leq \infty. \)

**Remark 4.2.** The results of the last two sections didn’t use the fact that the operators \( \Theta_j \) commute with each other. However, by using commutativity of \( \Theta_j \) one can show (see [42]) that the norm of \( \mathcal{W}_p^r \) is equivalent to the norm
\[ \| f \|_{X^p} + n \sum_{j=1} \| \Theta_j f \|_{X^p}, \]
and the norm (4.3) is equivalent to the norm
\[ \| f \|_{X^p} + n \sum_{j=1} \left( \int_0^\infty \left( s^{-\alpha} \omega_{j,p}^r(s, f) \right)^q \frac{ds}{s} \right)^{1/q}, \quad 1 \leq q < \infty,
\]
where
\[ \omega_{j,p}^r(s, f) = \sup_{0 \leq r \leq s} \| (U_j(r) - I)^m f \|_{X^p}. \]
5. Bernstein-Mellin spaces

Let’s remind that in the classical analysis a Bernstein class [1], [26], which is
denoted as $B^p_σ(\mathbb{R})$, $σ ≥ 0$, $1 ≤ p ≤ ∞$, is a linear space of all functions $f : \mathbb{R} → \mathbb{C}$
which belong to $L^p(\mathbb{R})$ and admit extension to $\mathbb{C}$ as entire functions of exponential
type $σ$. It is known, that for every such function there exists a constant $A > 0$ such
the following inequality holds

$$|f(x + iy)| ≤ Ae^{σ|y|}. $$

A function $f$ belongs to $B^p_σ(\mathbb{R})$ if and only if the following Bernstein inequality holds

$$\|f^{(k)}\|_{L^p(\mathbb{R})} ≤ σ^k \|f\|_{L^p(\mathbb{R})}$$

for all natural $k$. Using the distributional Fourier transform

$$\hat{f}(ξ) = \frac{1}{\sqrt{2π}} \int_{\mathbb{R}} f(x)e^{-ixξ} dx, \quad f ∈ L^p(\mathbb{R}), \quad 1 ≤ p ≤ ∞,$$

one can show (Paley-Wiener theorem) that $f ∈ B^p_σ(\mathbb{R})$, $1 ≤ p ≤ ∞$, if and only if
$f ∈ L^p(\mathbb{R})$, $1 ≤ p ≤ ∞$, and the support of $\hat{f}$ (in sens of distributions) is in $[-σ, σ]$.

**Definition 4.** The Bernstein-Mellin space $B^p_σ(Θ_1, ..., Θ_n)$, $σ > 0$, $1 ≤ p < ∞$, is
defined as a set of all functions $f$ in $X^p$ which belong to $D^∞(Θ_1, ..., Θ_n)$ and for which

$$\|\Theta_{Θ_1}^{j_1}...\Theta_{Θ_n}^{j_n}f\|_{X^p} ≤ σ^{|l|}\|f\|_{X^p},$$

where $l = (l_1, ..., l_n)$, $|l| = l_1 + ... + l_n$, $l_j ∈ \mathbb{N} ∪ \{0\}$, $k ∈ \mathbb{N}$.

The notation $B^p_σ(Θ_j)$, $1 ≤ p < ∞$, will be used for the set of all functions $f$ in
$X^p$ which belong to $D^∞(Θ_j)$ and for which

$$\|\Theta_{Θ_j}^{j}f\|_{X^p} ≤ σ^{|l|}\|f\|_{X^p}. $$

The next Lemma follows from a more general result Lemma[12.1] in Appendix 2.

**Lemma 5.1.** A function $f ∈ D^∞$ belongs to $B^p_σ(Θ_j)$, $1 ≤ j ≤ n$, $σ > 0$, $1 ≤ p < ∞$, if and only if the quantity

$$\sup_{k ∈ \mathbb{N}} σ^{-k}\|\Theta_{Θ_j}^{k}f\|_{X^p(\mathbb{R}^n)} = R(f, σ)$$

is finite.

**Theorem 5.2.** A function $f ∈ D^∞$ belongs to $B^p_σ(Θ_1, ..., Θ_n)$, $σ > 0$, $1 ≤ p < ∞$, if and only if the quantity

$$\sup_{|l| ∈ \mathbb{N}} \sup_{l = (l_1, ..., l_n)} σ^{-|l|}\|\Theta_{Θ_1}^{l_1}...\Theta_{Θ_n}^{l_n}f\|_{X^p} = R(f, σ), \quad |l| = l_1 + ... + l_n,$$

is finite.

**Proof.** It is clear that if $f ∈ B^p_σ(Θ_1, ..., Θ_n)$ then $[5.3]$ holds. Now, we assume that
$[5.4]$ holds and we have to verify the inequality $[5.1]$ with $k = n$. The inequality
$[5.4]$ and the Lemma 5.1 imply that for every $1 ≤ j ≤ n$ the function $f$ belongs to
$B^p_σ(Θ_j)$, i.e.

$$\|\Theta_{Θ_j}^{α}f\|_{X^p} ≤ σ^α\|f\|_{X^p}, \quad α ∈ \mathbb{N}.$$
We proceed by induction assuming that for some \(1 \leq k < n\) the inequality (5.4) implies (5.2). Clearly, (5.3) implies that any \(\Theta_{j_1}^{a_1} \cdots \Theta_{j_k}^{a_k} f\) satisfies (5.2) for any \(\Theta_j\) and Lemma 6.1 implies the inequality
\[
\|\Theta_{j_{k+1}}^{a_{j_{k+1}}} \cdots \Theta_{j_k}^{a_k} f\|_{X^p} \leq \sigma^{\max\{\alpha_{j_{k+1}}, \ldots, \alpha_j\}} \|f\|_{X^p}.
\]
Therefore, (5.4) implies that any \(\Theta_\alpha X\) of Bochner for the abstract-valued function \(\mu\) implies (5.2). Clearly, (5.4) implies that any \(\Theta_\alpha X\) of Bochner for the abstract-valued function \(\mu\) and is finite, then (5.2) holds.

The next statement follows from the corresponding Theorem 12.4 in our Appendix. A J ACKSON-TYPE INEQUALITY.

The lemma below is a particular case of the Lemma 12.2 in Appendix 2.

**Lemma 6.1.** Suppose that \(\mu \in B^p_\sigma(\mathbb{R})\) is an entire function of exponential type \(\sigma\). Then for any \(f \in X^p\) the function
\[
P_{\sigma,f} = \int_{\mathbb{R}} \mu(t)U_j(t) f dt
\]
belongs to \(B^p_\sigma(\Theta_j)\). The integral here is understood in the sense of Bochner for the abstract-valued function \(\mu(t)U_j(t) f : \mathbb{R} \mapsto X^p\), \(1 \leq p < \infty\).

**Theorem 6.2.** Let \(\mu \in L_1(\mathbb{R})\), \(\|\mu\|_{L_1(\mathbb{R})} = 1\), be an entire function of exponential type \(\sigma\) then for any \(f \in X^p\), \(1 \leq p < \infty\), the function
\[
P_{\sigma,f} = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \mu(\tau_1) \cdots \mu(\tau_n) U_1(\tau_1) \cdots U_n(\tau_n) f d\tau_1 \cdots d\tau_n
\]
belongs to \(B^p_\sigma(\Theta_1, \ldots, \Theta_n)\), \(1 \leq p < \infty\). The integral here is understood in the sense of Bochner for the abstract-valued function \(\mu(\tau_1) \cdots \mu(\tau_n) U_1(\tau_1) \cdots U_n(\tau_n) f : \mathbb{R}^n \mapsto X^p\).

**Proof.** Indeed, due to commutativity and the previous Lemma one has
\[
\|\Theta_{j_1}^{a_1} \cdots \Theta_{j_k}^{a_k} P_{\sigma,f}\|_{X^p} = \left\|\Theta_{j_1}^{a_1} \int_{\mathbb{R}} \mu(\tau_1) U_{j_1}(\tau_1) [T_{j_1}(f)] d\tau_1\right\|_{X^p} \leq \sigma^{a_1} \|T_{j_1}(f)\|_{X^p},
\]
where
\[
T_{j_1}(f) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \mu(\tau_1) \cdots \mu(\tau_n) U_1(\tau_1) \cdots U_n(\tau_n) f d\tau_1 \cdots d\tau_n.
\]
where a wedge means that the corresponding term is missing. It is obvious that continuing this way we obtain
\[ \| \Theta_{j_1}^l \Theta_{j_2}^l \cdots \Theta_{j_k}^l P \sigma f \|_{\mathcal{X}_p} \leq \sigma^{[l]} \| f \|_{\mathcal{X}_p}. \]
Applying Theorem 5.2, we finish the proof. \( \square \)

Let’s consider the functionals
\[ \omega_{j,p}^m(s, f) = \sup_{0 \leq \tau \leq s} \| (U_j(\tau) - I)^m f \|_{\mathcal{X}_p}, \]
\[ \mathcal{E}_{j,p}(\sigma, f) = \inf_{g \in \mathcal{B}_{\sigma}(\Theta_j)} \| f - g \|_{\mathcal{X}_p}, \]
where \( 1 \leq p < \infty \). One has the following inequalities:
\[ (6.1) \quad \omega_{j,p}^m(s, f) \leq s \omega_{j,p}^m(s, \Theta_j^k), \quad 0 \leq k \leq m, \]
and
\[ (6.2) \quad \omega_{j,p}^m(as, f) \leq (1 + a)^m \omega_{j,p}^m(s, f), \quad a \in \mathbb{R}_+. \]
The first one follows from the identity
\[ (U_j(s) - I)^k f = \int_0^s \cdots \int_0^s U_j(\tau_1 + \cdots + \tau_k) \Theta_j^k f d\tau_1 \cdots d\tau_k, \]
where \( I \) is the identity operator and \( k \in \mathbb{N} \). The second one follows from the property
\[ \omega_{j,p}^1(s_1 + s_2, f) \leq \omega_{j,p}^1(s_1, f) + \omega_{j,p}^1(s_2, f) \]
which is easy to verify.

Our next objective is to prove an analog of the Jackson theorem. The proof of the lemma below is motivated by Theorem 5.2.1 in [26].

Let
\[ (6.4) \quad \rho(t) = a \left( \sin(t/N) \right)^N / t \]
where \( N = 2(m+3) \) and
\[ a = \left( \int_{-\infty}^{\infty} \left( \sin(t/N) \right)^N / t \right)^{-1}. \]

With such choice of \( a \) and \( N \) function \( \rho \) will have the following properties:
(1) \( \rho \) is an even nonnegative entire function of exponential type one;
(2) \( \rho \) belongs to \( L_1(\mathbb{R}) \) and its \( L_1(\mathbb{R}) \)-norm is 1;
(3) the integral
\[ (6.5) \quad \int_{-\infty}^{\infty} \rho(t) |t|^m dt \]
is finite.

Next, we observe the following formula for every \( 1 \leq j \leq n \)
\[ (-1)^{m+1} (U_j(s) - I)^m f = \]
\[ \sum_{k=0}^{m} (-1)^{m-k} C_m^k U_j(ks) f = \sum_{k=1}^{m} b_k U_j(ks) f - f, \]
\[ (-1)^{m+1} \sum_{k=0}^{m} (-1)^{m-k} C_m^k U_j(ks) f \]
where \( b_1 + b_2 + \ldots + b_m = 1 \). Consider the vector

(6.7) \[ Q_j(\sigma, m)(f) = \int_{-\infty}^{\infty} \rho(t) \left\{ (-1)^{m+1} \left( U_j \left( \frac{t}{\sigma} \right) - I \right) f \right\} dt. \]

According to (6.3) we have

\[ Q_j(\sigma, m)(f) = \int_{-\infty}^{\infty} \rho(t) \sum_{k=1}^{m} b_k U_j \left( \frac{k - t}{\sigma} \right) f dt. \]

Changing variables in each of integrals

\[ \int_{-\infty}^{\infty} \rho(t) U_j \left( \frac{k - t}{\sigma} \right) f dt, \quad 1 \leq k \leq m, \]

we obtain the formula

\[ Q_j(\sigma, m)(f) = \int_{-\infty}^{\infty} \Phi(t) U_j(t) f dt, \]

where

\[ \Phi(t) = \sum_{k=1}^{m} b_k \left( \frac{a}{k} \right) \rho \left( \frac{a}{k} \right), \quad b_1 + b_2 + \ldots + b_m = 1. \]

Since the function \( \rho(t) \) has exponential type one every function \( \rho \left( t \frac{\sigma}{k} \right) \) has the type \( \sigma / k, \quad 1 \leq k \leq m \), and because of this the function \( \Phi(t) \) has exponential type \( \sigma \). It also belongs to \( L^1(\mathbb{R}) \) and as it was just shown in the previous statement it implies that the function \( Q_j(\sigma, m)(f) \) belongs to \( B_{\tau}(\Theta_j) \).

**Lemma 6.3.** [38] For a given natural \( m \) there exists a constant \( c = c(m) > 0 \) such that for all \( \sigma > 0 \) and all \( f \in X^p, 1 \leq p < \infty \),

(6.8) \[ \mathcal{E}_{j,p}(\sigma, f) \leq c \omega_{j,p}^{m-k} (1/\sigma, \Theta_j^k f), \quad 0 \leq k \leq m. \]

Moreover, for any \( 1 \leq k \leq m \) there exists a \( C = C(m, k) > 0 \) such that for any \( f \in \mathcal{D}(\Theta_j^k) \) one has

(6.9) \[ \mathcal{E}_{j,p}(\sigma, f) \leq \frac{C}{\sigma_k} \omega_{j,p}^{m-k} (1/\sigma, \Theta_j^k f), \quad 0 \leq k \leq m. \]

**Proof.** We estimate the error of approximation of \( Q_j(\sigma, m)(f) \) to \( f \). Since by (6.7)

\[ Q_j(\sigma, m)(f) - f = (-1)^{m+1} \int_{-\infty}^{\infty} \rho(t) \left( U_j \left( \frac{t}{\sigma} \right) - I \right)^m f dt \]

we obtain by using (6.2)

\[ \mathcal{E}_{j,p}(\sigma, f) \leq \| f - Q_j(\sigma, m)(f) \|_{X_p} \leq \int_{-\infty}^{\infty} \rho(t) \left\| \left( U_j \left( \frac{t}{\sigma} \right) - I \right)^m f \right\|_{X_p} dt \leq \]

(6.10) \[ \int_{-\infty}^{\infty} \rho(t) \omega_{j,p}^{m-k} (t/\sigma, f) dt \leq c \omega_{j,p}^{m-k} (1/\sigma, f), \quad c = \int_{-\infty}^{\infty} \rho(t)(1 + |t|)^m dt. \]

If \( f \in \mathcal{D}(\Theta_j^k) \) then by using (6.11) we have

\[ \mathcal{E}_{j,p}(\sigma, f) \leq \int_{-\infty}^{\infty} \rho(t) \omega_{j,p}^{m-k} (t/\sigma, f) dt \leq \]

(6.11) \[ \frac{\omega_{j,p}^{m-k} (1/\sigma, \Theta_j^k f)}{\sigma_k} \int_{-\infty}^{\infty} \rho(t)|t|^k (1 + |t|)^{m-k} dt \leq \frac{C}{\sigma_k} \omega_{j,p}^{m-k} (1/\omega, \Theta_j^k f), \]
where
\[ C = \int_{-\infty}^{\infty} \rho(t)|t|^k(1+|t|)^{m-k}dt, \]
is finite by the choice of \( \rho \). The inequalities (6.8) and (6.9) are proved. \( \square \)

We define the following functional
\[ (6.12) \quad E_p(\sigma, f) = \inf_{g \in B_p^\sigma(\Theta_1, \ldots, \Theta_n)} \|f - g\|_{X_p}, \quad 1 \leq p < \infty. \]
Since the operator \( Q_j(\sigma, m), \quad 1 \leq j \leq n, \) maps \( X_p, \quad 1 \leq p < \infty, \) into \( B_p^\sigma(\Theta_j), \quad 1 \leq p < \infty, \) and since all the operators \( \Theta_j, U_j, \quad 1 \leq j \leq n, \) commute with each other one concludes that for any \( f \in X_p \) the function
\[ Q_1(\sigma, m)\ldots Q_n(\sigma, m)(f) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \Phi(\tau_1)\ldots\Phi(\tau_n)U_j(\tau_1 + \ldots + \tau_n)f d\tau_1 \ldots d\tau_n, \]
belongs to \( B_p^\sigma(\Theta_1, \ldots, \Theta_n). \)

Now we can formulate the following analog of the Jackson inequality.

**Theorem 6.4.** There exists a constant \( C \) such that for the same notations as above the next inequality holds
\[ (6.13) \quad E_p(\sigma, f) \leq C \sum_{j=1}^{n} \omega_{j,p}^m(1/\sigma, f) \leq C \Omega_p^m(1/\sigma, f). \]

**Proof.** By using (6), the formal identity
\[ 1 - a_1a_2\ldots a_n = (1 - a_1) + a_1(1 - a_2) + \ldots + a_1a_2\ldots a_{n-1}(1 - a_n), \]
and boundness of every operator \( Q_j(\sigma, m) \), we obtain the following
\[ E_p(\sigma, f) \leq \|f - Q_1(\sigma, m)Q_2(\sigma, m)\ldots Q_n(\sigma, m)f\|_{X_p} \leq \]
\[ (6.14) \quad C \sum_{j=1}^{n} \|f - Q_j(\sigma, m)f\|_{X_p} \leq C \sum_{j=1}^{n} \omega_{j,p}^m(1/\sigma, f) \leq C \Omega_p^m(1/\sigma, f). \]
The inequality (6.13) is proved. \( \square \)

Set
\[ B = \bigcup_{\sigma > 0} B_p^\sigma(\Theta_1, \ldots, \Theta_n), \quad 1 \leq p < \infty. \]
According to the general theory (see Appendix 1) the space of all \( f \in X_p \) for which the integral
\[ \left( \int_{0}^{\infty} (\tau^n E_p(\tau, f))^q \frac{d\tau}{\tau} \right)^{1/q}, \quad 1 \leq p < \infty, \quad 1 \leq q \leq \infty, \]
is finite is known as the approximation space and denoted by \( E_{\alpha, q}(X_p, B). \) We have the following Direct Approximation Theorem.
Theorem 6.5. The following continuous embedding holds, \( 1 \leq p < \infty, 1 \leq q \leq \infty, \)
\[
\mathcal{B}_{p,q}^\alpha \subset \mathcal{E}_{\alpha,q}(X^p, B),
\]
and for some \( C > 0 \) the inequalities hold
\[
\int_0^\infty (\tau^\alpha \mathcal{E}_p(\tau, f))^q \frac{d\tau}{\tau} \leq C \sum_{j=1}^n \int_0^\infty (s^{-\alpha}\omega_{j,p}(s,f))^q \frac{ds}{s} \leq C \int_0^\infty (s^{-\alpha}\Omega_p^r(s,f))^q \frac{ds}{s}.
\]

Proof. According to (6.13) we have
\[
s^{-\alpha}\mathcal{E}_p(s^{-1}, f) \leq Cs^{-\alpha}\Omega_p^r(s,f).
\]
Since
\[
\int_0^\infty (s^{-\alpha}\mathcal{E}_p(s^{-1}, f))^q \frac{ds}{s} = \int_0^\infty (\tau^\alpha \mathcal{E}_p(\tau, f))^q \frac{d\tau}{\tau},
\]
and since the norm of the Besov space \( \mathcal{B}_{p,q}^\alpha \) is equivalent to the norm
\[
\|f\|_{X^p} + \left( \int_0^\infty (s^{-\alpha}\Omega_p^r(s,f))^q \frac{ds}{s} \right)^{1/q}, \quad \alpha < r,
\]
our claim is proven. \( \square \)

In order to obtain an Inverse Approximation theorem, we are using Theorem 11.2 with \( E = X^p, F = W_p^m, T = B. \) On the linear space \( B = \cup_{\sigma>0} \mathcal{B}_p^\sigma(\Theta_1, ..., \Theta_n) \) we consider the quasi-norm defined as
\[
|f|_B = \inf \{ \sigma > 0 : f \in \mathcal{B}_p^\sigma(\Theta_1, ..., \Theta_n) \}.
\]
Thus the assumption of Theorem 11.2 simply means that the Bernstein-Mellin inequality for functions in \( B \) holds true, i.e. if \( f \in B \) belongs to \( \mathcal{B}_p^\sigma(\Theta_1, ..., \Theta_n) \) then
\[
\|f\|_{W_p^m} \leq C \|f\|_{X^p}^\sigma.
\]
It implies the Inverse Approximation Theorem meaning that the following continuous imbedding holds true
\[
\mathcal{E}_{\alpha,q}(X^p, B) \subset \mathcal{B}_{p,q}^\alpha, \quad 1 \leq p < \infty, 1 \leq q \leq \infty, \quad \alpha \in \mathbb{R}^+.
\]

Remark 6.6. We obtain the Direct Approximation Theorem 6.5 directly as a consequence of the Jackson-type inequality (6.13). However, one could also use the abstract Direct Theorem 11.1 to obtain Theorem 6.5. Indeed, first we note that by using (6.1), (6.13) and (6.1), (6.2) one has
\[
\mathcal{E}_p(\sigma, f) \leq C \sum_{j=1}^n \|f - Q_j(\sigma, m)f\|_{X^p} \leq C_1 \sum_{j=1}^n \int_{-\infty}^{\infty} \rho(t)\omega_{j,p}^m(t/\sigma, f) \ dt \leq C_1 \sum_{j=1}^n \frac{\omega_{j,p}^{m-k}(1/\sigma, \Theta_j)}{\sigma^k} \int_{-\infty}^{\infty} \rho(t)|t|^k(1+|t|)^{m-k} \ dt \leq C_2 \sum_{j=1}^n \frac{1}{\sigma^k} \omega_{j,p}^{m-k}(1/\omega, \Theta_j) f,
\]
and for \( m = k \) the last inequality implies that
\[
\mathcal{E}_p(\sigma, f) \leq C_2 \sigma^{-m} \|f\|_{W_p^m}, \quad f \in W_p^m.
\]
Comparing our situation with Theorem 11.1 we see that \( E = X^p, F = W_p^m, T = B, \) and the assumption of this theorem is exactly (6.17) with the \( \beta = m. \) All together it gives the imbedding (6.15).
7. The Laplace-Mellin operator in the Hilbert space \( X^2 \)

Since \( \Theta_j = x_j \partial_j \) is a skew-symmetric operator in \( X^2 \) its negative square

\[
-\Theta_j^2 = -(x_j \partial_j)^2 = -x_j^2 \partial_j^2 - x_j \partial_j
\]

is a self-adjoint non-negative operator. We will use notation \( e^{-it\Theta_j^2} \) for the corresponding group of unitary operators.

**Remark 7.1.** We note that the operator \( -\Theta_j^2 = -(x_j \partial_j)^2 = -x_j^2 \partial_j^2 - x_j \partial_j \) is a "multiple" of the Bessel operator

\[
B_j = -\partial_j^2 - \frac{1}{x_j} \partial_j
\]

in the sense that \( \Theta_j^2 = x_j^2 B_j \).

In the Hilbert space \( \mathcal{W}_2^k \) we introduce the Laplace-Mellin operator as

\[
(7.1) \quad L_{\mathcal{M}} = -\sum_{j=1}^{n} \Theta_j^2 = -\sum_{j=1}^{n} (x_j^2 \partial_j^2 + x_j \partial_j).
\]

The self-adjoint operator \( L_{\mathcal{M}} \) generates a group of unitary operators \( e^{-itL_{\mathcal{M}}} \) and since the operators \( \Theta_j^2, j = 1, 2, \ldots, n \), commute with each other, the following formula holds

\[
e^{-itL_{\mathcal{M}}} = e^{-it\Theta_1^2} e^{-it\Theta_2^2} \ldots e^{-it\Theta_n^2}.
\]

The operator \( L_{\mathcal{M}} \) is not negative and it has a unique self-adjoint non-negative square root \( L_{\mathcal{M}}^{1/2} \). One can introduce another scale of Sobolev-Mellin spaces which are domains \( \mathcal{D}(L_{\mathcal{M}}^{k/2}) \), \( k \in \mathbb{N} \), of powers of \( L_{\mathcal{M}}^{1/2} \) with the graph norm:

\[
\|f\|_{\mathcal{D}(L_{\mathcal{M}}^{k/2})} = \|f\|_{X^2} + \|L_{\mathcal{M}}^{k/2} f\|_{X^2}.
\]

**Theorem 7.2.** The spaces \( \mathcal{D}(L_{\mathcal{M}}^{k/2}) \) and \( \mathcal{W}_2^k \), \( k \in \mathbb{N} \), coincide and their norms are equivalent.

**Proof.** One has

\[
\sum_{j=1}^{n} \|Q_j f\|^2_{X^2} = \sum_{j=1}^{n} \langle \Theta_j f, \Theta_j f \rangle_{X^2} = \left( -\sum_{j=1}^{n} \Theta_j^2 f, f \right)_{X^2} = \langle L_{\mathcal{M}} f, f \rangle_{X^2} = \|L_{\mathcal{M}}^{1/2} f\|_{X^2}.
\]

Thus the spaces \( \mathcal{D}(L_{\mathcal{M}}^{1/2}) \) and \( \mathcal{W}_2^1 \) coincide and their norms equivalent. Now, since \( \Theta_j \) are commuting with each other, we have

\[
\sum_{1 \leq k, j \leq n} \|\Theta_k \Theta_j f\|^2_{X^2} = \sum_{j=1}^{n} \|L_{\mathcal{M}}^{1/2} \Theta_j f\|^2_{X^2} = \sum_{j=1}^{n} \|\Theta_j L_{\mathcal{M}}^{1/2} f\|^2_{X^2} = \|L_{\mathcal{M}} f\|^2_{X^2},
\]

which proves the statement for \( k = 2 \). Continue this way we can finish the proof. \( \Box \)

We set

\[
w_2^s(s, f; L_{\mathcal{M}}) = \sup_{0 \leq \tau \leq s} \| (e^{-i\tau L_{\mathcal{M}}} - I)^\tau f \|_{X^2}.
\]

The previous Theorem implies the following Corollary.
Corollary 7.1. The norm of the Besov-Mellin space $B^q_{2,q}$, $1 \leq q \leq \infty$, is equivalent to

$$\|f\|_{X^2} + \left( \int_0^\infty (s^{-\alpha} w_2^q(s, f; L_M))^q \frac{ds}{s} \right)^{1/q}.$$  \(7.2\)

Moreover, for $\alpha$ not-integer it is equivalent to

$$\|f\|_{D(L_{M}^{\alpha+1})} + \left( \int_0^\infty (s^{-\alpha} w_2^q(s, L_M^{[\alpha]} f; L_M))^q \frac{ds}{s} \right)^{1/q} \quad \text{(7.3)}$$

where $[\alpha]$ is the integer part of $\alpha$, and for $\alpha = k \in \mathbb{N}$ integer its norm is equivalent to the norm (Zygmund condition)

$$\|f\|_{D(L_{M}^{k-1})} + \left( \int_0^\infty (s^{-1} w_2^q(s, L_{M}^{k-1} f; L_M))^q \frac{ds}{s} \right)^{1/q} \quad \text{(7.4)}$$

Note that according to the general theory (see for example [22], [42]) one has the following isomorphisms for the self-adjoint operator $L_M^{1/2}$ for $0 < \alpha < r$:

$$B^q_{2,q} = (X^2, W_2^q)_{\alpha/r, 2} = (X^2, L_M^{r/2})_{\alpha/r, 2} = D(L_M^{1/2}).$$

Thus when $q = 2$ the norms (7.2) and (7.4) are equivalent to the graph norm of a corresponding space $D(L_M^{1/2})$.

Let $D^\infty(L_{M}^{1/2}) = \cap_{m \in \mathbb{N}} D(L_{M}^{m/2})$, where $D(L_{M}^{m/2})$ is the domain of $L_{M}^{m/2}$. One has the following version of the Landau-Kolmogorov-Stein inequality which follows from Theorem 12.3:

Theorem 7.3. The following holds for $f \in D(L_{M}^{m/2})$

$$\|L_{M}^{k/2} f\|_{X^2}^{m} \leq C_{k,m} \|L_{M}^{m/2} f\|_{X^2}^k \|f\|_{X^2}^{m-k}, \quad 0 \leq k \leq m,$$  \(7.5\)

where $C_{k,m} = (K_{m-k})^m / (K_m)^{m-k}$, and $K_j$ is the Favard constant.

Definition 5. Let $B^2_{\sigma}(L_{M}^{1/2})$, be the space of all $f \in X^2$ for which the following inequality holds

$$\|L_{M}^{k/2} f\|_{X^2} \leq \sigma^k \|f\|_{X^2}, \quad \text{for all natural } k.$$  \(7.6\)

for all natural $k$. For a function $f \in \cup_{\sigma > 0} B^2_{\sigma}(L_{M}^{1/2})$, let the $\sigma_f$ be the smallest $\sigma$ for which the inequality (7.6) holds.

The next Theorem follows from Theorem 12.4 in Appendix 2.

Theorem 7.4. Let $f \in X^2$ belong to a space $B^2_{\sigma}(L_{M}^{1/2})$, for some $0 < \sigma < \infty$. Then the following limit exists

$$d_f = \lim_{k \to \infty} \|L_{M}^{k/2} f\|_{X^2}^{1/k}, \quad \text{and } d_f = \sigma_f.$$  \(7.7\)

and $d_f = \sigma_f$. Conversely, if $f \in D^\infty(L_{M}^{1/2})$ and $d_f = \lim_{k \to \infty} \|L_{M}^{k/2} f\|_{X^2}^{1/k}$, exists and is finite, then $f \in B^2_{d_f}(L_{M}^{1/2})$ and $d_f = \sigma_f$. 

8. Paley-Wiener-Mellin spaces $PW_\sigma(L^{1/2}_M)$

8.1. The spectral theorem approach. Consider a self-adjoint non-negative definite operator $L_M$ in the Hilbert space $X^2(\mathbb{R}_+^n, d\mu)$ and let $\sqrt{L_M}$ be the non-negative square root of $L_M$.

According to the spectral theory for self-adjoint operators there exists a direct integral of Hilbert spaces $H = \int H(\lambda) d\mu(\lambda)$ and a unitary operator $F$ from $X^2$ onto $H$, which transforms the domains of $L_M^{k/2}$, $k \in \mathbb{N}$, onto the sets $H_k = \{ x \in H | \lambda^k x \in H \}$ with the norm

$$\| x \|_{H_k} = (\int_0^\infty \lambda^{2k} \| x(\lambda) \|^2_{H(\lambda)} d\mu(\lambda))^{1/2},$$

and satisfies the identity $F(L_M^{k/2} f)(\lambda) = \lambda^k (F f)(\lambda)$, if $f$ belongs to the domain of $L_M^{k/2}$. We call the operator $F$ the Spectral Fourier Transform. As known, $H$ is the set of all $m$-measurable functions $\lambda \mapsto x(\lambda) \in H(\lambda)$, for which the following norm is finite:

$$\| x \|_H = \left( \int_0^\infty \| x(\lambda) \|^2_{H(\lambda)} d\mu(\lambda) \right)^{1/2}.$$

For a function $F$ on $[0, \infty)$ which is bounded and measurable with respect to $d\mu$ one can introduce the operator $F(L_M)$ as a multiplication operator by using the formula

$$F(L_M) f = F^{-1} F F f, \quad f \in X^2.$$

If $F$ is real-valued the operator $F(L_M)$ is self-adjoint.

Definition 6. We say that a function $f \in X^2(\mathbb{R}_+^n, d\mu)$ belongs to the Paley-Wiener-Mellin space $PW_\sigma\left(L^{1/2}_M\right)$ if the support of the Spectral Fourier Transform $F f$ is contained in $[0, \sigma]$.

The next two facts are obvious.

Theorem 8.1. The spaces $PW_\sigma\left(L^{1/2}_M\right)$ have the following properties:

1. the space $PW_\sigma\left(L^{1/2}_M\right)$ is a linear closed subspace in $X^2$,
2. the space $\bigcup_{\sigma > 0} PW_\sigma\left(L^{1/2}_M\right)$ is dense in $X^2$.

The next theorem contains generalizations of several results from classical harmonic analysis (in particular the Paley-Wiener theorem). It follows from our results in (see also Appendix 3).

Theorem 8.2. The following statements hold:

1. (Bernstein-Mellin inequality) $f \in PW_\sigma\left(L^{1/2}_M\right)$ if and only if $f \in \mathcal{D}^\infty(L_M) = \bigcap_{k=1}^\infty \mathcal{D}(L_M^k)$, and the following Bernstein-type inequalities holds true

$$\| L_M^{s/2} f \|_{X^2} \leq \sigma^s \| f \|_{X^2} \quad \text{for all } s \in \mathbb{R}_+;$$

2. (Paley-Wiener-Mellin theorem) if and only if for every $g \in X^2$ the scalar-valued function of the real variable $t \mapsto (e^{-it\sqrt{L_M}} f, g)$ is bounded on the real line and has an extension to the complex plane as an entire function of the exponential type $\sigma$. 
(3) (Riesz-Boas-Mellin interpolation formula) \( f \in PW_\sigma \left( L_{\mathcal{M}}^{1/2} \right) \) if and only if \( f \in D^\infty (L_{\mathcal{M}}) \) and the following Riesz-Boas-Mellin interpolation formula holds for all \( \sigma > 0 \):

\[
i \sqrt{L_{\mathcal{M}} f} = \frac{\sigma}{\pi^2} \sum_{k \in \mathbb{Z}} (-1)^{k-1} \frac{1}{(k-1/2)^2} e^{-i(k-1/2)\sqrt{L_{\mathcal{M}}} f}.
\]

Proof. (1) follows immediately from the definition and representation \( (8.1) \). To prove (2) it is sufficient to apply the classical Bernstein inequality \( [26] \) in the uniform norm on \( \mathbb{R} \) to every function \( \langle e^{-it\sqrt{L_{\mathcal{M}}}} f, g \rangle, \ g \in X^2 \). To prove (3) one has to apply the classical Riesz-Boas interpolation formula on \( \mathbb{R} \) \( [26], [38] \), to the function \( \langle e^{-it\sqrt{L_{\mathcal{M}}}} f, g \rangle \).

The Bernstein inequality \( [38] \) and Theorem 7.2 imply the following.

**Theorem 8.3.** The Bernstein-Mellin space \( \cup_{\sigma > 0} \mathcal{B}^2_\sigma(\Theta_1, \ldots, \Theta_n) \) and the Paley-Wiener-Mellin space \( \cup_{\sigma > 0} PW_\sigma (L_{\mathcal{M}}^{1/2}) \) coincide.

Thus the best approximation functional \( \mathcal{E}_2 (s, f) \) defined in \( (6.12) \) can be redefined as

\[
\mathcal{E}_2 (\sigma, f) = \mathcal{E}_2 (\sigma, f; L_{\mathcal{M}}) = \inf_{g \in PW_\sigma (L_{\mathcal{M}}^{1/2})} \| f - g \|_{X^2}.
\]

**8.2. The Mellin transform approach.** For an approach to the Paley-Wiener-Mellin spaces in one-dimensional case in terms of the Mellin transform we refer to the very interesting papers \( [7], [8], [10] \). Note that in one-dimensional case this approach is simply identical to the approach described in the previous section.

Namely, it is known \( [15], [16] \) that the Mellin transform

\[
\mathcal{M} f(t) = \int_0^\infty f(x)x^{it} \frac{dx}{x},
\]

which is originally defined for functions in \( X^1 (\mathbb{R}^+, d\mu) \cap X^2 (\mathbb{R}^+, d\mu) \) can be extended to an isomorphism of \( X^2 (\mathbb{R}^+, d\mu) \) onto \( L^2 (i\mathbb{R}, dt) \) and the following relation holds

\[
\mathcal{M} [\Theta^r f] (it) = (-it)^r \mathcal{M} f(t), \ f \in D(\Theta^r),
\]

where \( \Theta = x^{1/t} \) is the Mellin differentiation operator in the Hilbert space \( X^2 (\mathbb{R}^+, d\mu) \) and \( D(\Theta^r) \) is the domain of \( \Theta^r, \ r \in \mathbb{N} \).

It implies, that the Mellin transform in \( X^2 (\mathbb{R}^+, d\mu) \) provides a diagonalization of the self-adjoint, non-negative Laplace operator \( L_{\mathcal{M}} = -\Theta^2 \) in \( X^2 (\mathbb{R}^+, d\mu) \) in the sense that the following formula holds

\[
\mathcal{M} [L_{\mathcal{M}} f] (t) = \mathcal{M} [\Theta^2 f] (t) = -t^2 \mathcal{M} f(t), \ f \in D(\Theta^2).
\]

In the \( n \)-dimensional case a similar approach can be implemented by using the \( n \)-dimensional version of the Mellin transform (see \( [43] \)). However, this approach is slightly different from the approach of the previous subsection since it is relying on a different form of the Spectral Theorem.

One can show that for the transform

\[
\mathcal{M}^n f = \int_0^\infty \ldots \int_0^\infty f(x_1, \ldots, x_n)x_1^{it_1} \ldots x_n^{it_n} \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n},
\]

the following equality holds

\[
\mathcal{M}^n [L_{\mathcal{M}} f] (t_1, \ldots, t_n) = -(t_1^2 + \ldots + t_n^2) \mathcal{M}^n f(t_1, \ldots, t_n), \ f \in X^2 (\mathbb{R}^n_+, d\mu),
\]
where \(-L_M = \Theta_1^2 + \ldots + \Theta_n^2\).

Next, one can define \(\widehat{PW}_\sigma \left( L_M^{1/2} \right)\) as a subspace of functions \(f \in X^2(\mathbb{R}_+^n, d\mu)\) for which the support of \(M^n f\) is in the ball \(\sqrt{t_1^2 + \ldots + t_n^2} \leq \sigma\). This definition is essentially equivalent to the our Definition 6.

9. Paley-Wiener-Mellin frames in \(X^2(\mathbb{R}_+^n, d\mu)\).

The construction of frequency-localized frames is achieved via spectral calculus. The idea is to start from a partition of unity on the positive real axis. In the following, we will be considering two different types of such partitions, whose construction we now describe in some detail. The construction below was described in [19].

Let \(g \in C^\infty(\mathbb{R}_+)\) be a non-increasing function such that \(\text{supp}(g) \subset [0, 2]\), and \(g(\lambda) = 1\) for \(\lambda \in [0, 1]\), \(0 \leq g(\lambda) \leq 1\), \(\lambda > 0\). We now let

\[
h(\lambda) = g(\lambda) - g(2\lambda).
\]

The function \(h\) is supported in \([2^{-1}, 2]\) and use it to define

\[
F_0(\lambda) = \sqrt{g(\lambda)}, F_j(\lambda) = \sqrt{h(2^{-j}\lambda)}, j \geq 1,
\]

as well as \(G_j(\lambda) = [F_j(\lambda)]^2 = F_j^2(\lambda), j \geq 0\). As a result of the definitions, we get for all \(\lambda \geq 0\) the equations

\[
\sum_{j=0}^{k} G_j(\lambda) = \sum_{j=0}^{k} F_j^2(\lambda) = g(2^{-k}\lambda),
\]

and as a consequence

\[
\sum_{j \geq 0} G_j(\lambda) = \sum_{j \geq 0} F_j^2(\lambda) = 1, \quad \lambda \geq 0,
\]

with finitely many nonzero terms occurring in the sums for each fixed \(\lambda\). We call the sequence \((G_j)_{j \geq 0}\) a (dyadic) partition of unity, and \((F_j)_{j \geq 0}\) a quadratic (dyadic) partition of unity. As will become soon apparent, quadratic partitions are useful for the construction of frames. Using the spectral theorem one has

\[
F_j^2(L_M)f = F^{-1} F_j^2 \mathcal{F} f, \quad j \geq 1,
\]

and thus

\[
(9.1) \quad f = F^{-1} \mathcal{F} f = F^{-1} \left( \sum_{j \geq 0} F_j^2 \right) \mathcal{F} f = \sum_{j \geq 0} (F^{-1} F_j^2 \mathcal{F} f) = \sum_{j \geq 0} F_j^2(L_M)f
\]

Taking inner product with \(f\) gives

\[
\|F_j(L_M)f\|^2_{X^2} = \langle F_j^2(L_M)f, f \rangle_{X^2},
\]

and

\[
(9.2) \quad \|f\|^2_{X^2} = \sum_{j \geq 0} \langle F_j^2(L_M)f, f \rangle_{X^2} = \sum_{j \geq 0} \|F_j(L_M)f\|^2_{X^2}.
\]

Similarly, we get the identity

\[
\sum_{j \geq 0} G_j(L_M)f = f.
\]
Moreover, since the functions $G_j, F_j$, have their supports in $[2^{j-1}, 2^{j+1}]$, the elements $F_j(L_M)f$ and $G_j(L_M)f$ are bandlimited to $[2^{j-1}, 2^{j+1}]$, whenever $j \geq 1$, and to $[0, 2]$ for $j = 0$.

10. More about Besov-Mellin spaces $B^\sigma_{2,q}$

10.1. **Besov-Mellin spaces $B^\sigma_{2,q}$ in terms of approximations.** In this section we apply Theorems 11.1 and 11.2 in Appendix 1 to a situation where $E = X^2, F = W^p, B^0_{2,q} = (X^2, W^p)_{\alpha/r, q}$, and $T = \cup_{\omega > 0} PW_\omega (L^{1/2}_M)$ is the abelian additive group of the underlying vector space, with the quasi-norm

$$||f||_T = \inf \{ \omega' > 0 : f \in PW_\omega (L^{1/2}_M) \} .$$

In previous sections we already proved that in our case the assumptions of Theorems 11.1 and 11.2 are satisfied. It allows us to formulate the following result.

**Theorem 10.1.** For $\alpha > 0, 1 \leq q \leq \infty$, the norm of $B^\sigma_{2,q}$, is equivalent to

\[
\| f \|_{X^2} + \left( \sum_{j=0}^{\infty} (2^{j\alpha} E_2(2^j, f; L_M))^q \right)^{1/q}.
\]

Let the functions $F_j$ be as in section 9.

**Theorem 10.2.** For $\alpha > 0, 1 \leq q \leq \infty$, the norm of $B^\sigma_{2,q}$, is equivalent to

\[
f \mapsto \left( \sum_{j=0}^{\infty} (2^{j\alpha} \| F_j(L_M)f \|_{X^2})^q \right)^{1/q},
\]

with the standard modifications for $q = \infty$.

**Proof.** We obviously have

$$E_2(2^j, f; L_M) \leq \sum_{j \geq 1} \| F_j(L_M)f \|_{X^2} .$$

By using a discrete version of Hardy’s inequality [11] we obtain the estimate

\[
\| f \|_{X^2} + \left( \sum_{l=0}^{\infty} (2^{l\alpha} E_2(2^l, f; L_M))^q \right)^{1/q} \leq C \left( \sum_{j=0}^{\infty} (2^{j\alpha} \| F_j(L_M)f \|_{X^2})^q \right)^{1/q}.
\]

Conversely, for any $g \in PW_{2^{j-1}} (L^{1/2}_M)$ we have

$$\| F_j(L_M)f \|_{X^2} = \| F_j(L_M)(f - g) \|_{X^2} \leq \| f - g \|_{X^2} .$$

This implies the estimate

$$\| F_j(L_M)f \|_{X^2} \leq E_2(2^{j-1}, f; L_M) ,$$

which shows that the inequality opposite to (10.1) holds. The proof is complete. □
10.2. Paley-Wiener-Mellin frames in $X^2(\mathbb{R}_+^n, d\mu)$. We consider the Laplace-Mellin operator $L_M$ defined in \((\ref{7.1})\) in the Hilbert spaces $X^2(\mathbb{R}_+^n, d\mu)$.

**Definition 7.** The notation $PW_{[2^{j-1}, 2^{j+1})} \left( L^{1/2}_M \right)$, $j \in \mathbb{N}$ will be used for the space of functions $f$ in $X^2$ whose Spectral Fourier Transform $\mathcal{F}f$ has support contained in $[2^{j-1}, 2^{j+1})$.

For every $j \in \mathbb{N}$ let

\[
\{ \Phi^j_k \}_{k=1}^{K_j}, \quad \Phi^j_k \in PW_{[2^{j-1}, 2^{j+1})} \left( L^{1/2}_M \right),
\]

\[K_j \in \mathbb{N} \cup \{\infty\},\]

be a frame in $PW_{[2^{j-1}, 2^{j+1})} \left( L^{1/2}_M \right)$ with the fixed constants $a, b$, i.e.

\[
a \|f\|_{X^2}^2 \leq \sum_{k=1}^{K_j} \left| \left< f, \Phi^j_k \right>_{X^2} \right|^2 \leq b \|f\|_{X^2}^2, \quad f \in PW_{[2^{j-1}, 2^{j+1})} \left( L^{1/2}_M \right).
\]

The formula \((\ref{10.2})\) and the general theory of frames imply the following statement.

**Theorem 10.3.** (1) The set of functions $\{\Phi^j_k\}$ will be a frame in the entire space $X^2$ with the same frame constants $a$ and $b$, i.e.

\[
a \|f\|_{X^2}^2 \leq \sum_j \sum_k \left| \left< f, \Phi^j_k \right>_{X^2} \right|^2 \leq b \|f\|_{X^2}^2, \quad f \in X^2.
\]

(2) The canonical dual frame $\{\Psi^j_k\}$ also consists of bandlimited vectors $\Psi^j_k \in PW_{[2^{j-1}, 2^{j+1})}(L^{1/2}_M)$, $j \in \mathbb{N}$, $k = 1, \ldots, K_j$, and has the frame bounds $b^{-1}$, $a^{-2}$.

(3) The reconstruction formulas hold for every $f \in X^2$

\[
f = \sum_j \sum_k \left< f, \Phi^j_k \right>_{X^2} \Psi^j_k = \sum_j \sum_k \left< f, \Psi^j_k \right>_{X^2} \Phi^j_k.
\]

The formula \((\ref{10.2})\) implies that in this case the set of functions $\{\Phi^j_k\}$ will be a frame in the entire $X^2$ with the same frame constants $a$ and $b$, i.e.

\[
a \|f\|_{X^2}^2 \leq \sum_j \sum_k \left| \left< f, \Phi^j_k \right>_{X^2} \right|^2 \leq b \|f\|_{X^2}^2, \quad f \in X^2.
\]

**Theorem 10.4.** For $\alpha > 0$, $1 \leq q \leq \infty$, the norm of $\mathcal{B}^\alpha_{2,q}$ is equivalent to

\[
\left( \sum_{j=0}^{\infty} 2^{j \alpha q} \left( \sum_k \left| \left< F_j(L_M)f, \Phi^j_k \right>_{X^2} \right|^2 \right)^{q/2} \right)^{1/q} \asymp \|f\|_{\mathcal{B}^\alpha_{2,q}},
\]

with the standard modifications for $q = \infty$.

**Proof.** For $f \in X^2$ and operator $F_j(L_M)$ we have according to \((\ref{10.4})\)

\[
a \|F_j(L_M)f\|_{X^2}^2 \leq \sum_k \left| \left< F_j(L_M)f, \Phi^j_k \right>_{X^2} \right|^2 \leq b \|F_j(L_M)f\|_{X^2}^2,
\]

and then by means of \((\ref{10.2})\) we obtain the statement. Theorem is proven. $\square$
11. Appendix 1.

11.1. \textit{K-functional and interpolation spaces.} The goal of the section is to introduce basic notions of the theory of interpolation spaces\cite{21},\cite{22},\cite{23},\cite{27}, and approximation spaces\cite{12},\cite{13},\cite{23},\cite{27}. It is important to realize that the relations between interpolation and approximation spaces cannot be described in the language of normed spaces. We have to make use of quasi-normed linear spaces in order to treat them simultaneously.

A quasi-norm $\| \cdot \|_E$ on linear space $E$ is a real-valued function on $E$ such that for any $f, f_1, f_2 \in E$ the following holds true:

1. $\|f\|_E \geq 0$;
2. $\|f\|_E = 0 \iff f = 0$;
3. $\| - f \|_E = \| f \|_E$;
4. there exists some $C_E \geq 1$ such that $\|f_1 + f_2\|_E \leq C_E(\|f_1\|_E + \|f_2\|_E)$.

Two quasi-normed linear spaces $E$ and $F$ form a pair if they are linear subspaces of a common linear space $A$ and the conditions $\|f_k - g\|_E \to 0$, and $\|f_k - h\|_F \to 0$ imply equality $g = h$ (in $A$). For any such pair $E, F$ one can construct the space $E \cap F$ with quasi-norm

$$\|f\|_{E \cap F} = \max \{ \|f\|_E, \|f\|_F \}$$

and the sum of the spaces, $E + F$ consisting of all sums $f_0 + f_1$ with $f_0 \in E, f_1 \in F,$ and endowed with the quasi-norm

$$\|f\|_{E + F} = \inf_{f = f_0 + f_1, f_0 \in E, f_1 \in F} (\|f_0\|_E + \|f_1\|_F).$$

Quasi-normed spaces $H$ with $E \cap F \subset H \subset E + F$ are called intermediate between $E$ and $F$. If both $E$ and $F$ are complete the inclusion mappings are automatically continuous. An additive homomorphism $T : E \to F$ is called bounded if

$$\|T\| = \sup_{f \in E, f \neq 0} \|Tf\|_F/\|f\|_E < \infty.$$ 

An intermediate quasi-normed linear space $H$ interpolates between $E$ and $F$ if every bounded homomorphism $T : E + F \to E + F$ which is a bounded homomorphism of $E$ into $E$ and a bounded homomorphism of $F$ into $F$ is also a bounded homomorphism of $H$ into $H$. On $E + F$ one considers the so-called Peetre’s $K$-functional

$$K(f, t) = K(f, t, E, F) = \inf_{f = f_0 + f_1, f_0 \in E, f_1 \in F} (\|f_0\|_E + t \|f_1\|_F).$$

The quasi-normed linear space $(E, F)_{\theta, q}^K$ with parameters $0 < \theta < 1$, $0 < q \leq \infty$, or $0 \leq \theta \leq 1$, $q = \infty$, is introduced as the set of elements $f$ in $E + F$ for which

$$\|f\|_{\theta, q} = \left( \int_0^\infty \left( t^{-\theta} K(f, t) \right)^q \frac{dt}{t} \right)^{1/q} < \infty.$$  

(11.1)

It turns out that $(E, F)_{\theta, q}^K$ with the quasi-norm $\text{(11.1)}$ interpolates between $E$ and $F$.

11.2. \textit{Approximation spaces.} Let us introduce another functional on $E + F$, where $E$ and $F$ form a pair of quasi-normed linear spaces

$$\mathcal{E}(f, t) = \mathcal{E}(f, t, E, F) = \inf_{g \in F, \|g\|_F \leq t} \|f - g\|_E.$$
**Definition 8.** The approximation space $\mathcal{E}_{\alpha,q}(E,F)$, $0 < \alpha < \infty$, $0 < q \leq \infty$ is the quasi-normed linear spaces of all $f \in E + F$ for which the quasi-norm

$$\|f\|_{\mathcal{E}_{\alpha,q}(E,F)} = \left( \int_0^\infty \left( t^\alpha \mathcal{E}(f,t) \right)^q \frac{dt}{t} \right)^{1/q}$$

is finite.

The next two theorems represent a very abstract version of what is known as a Direct and an Inverse Approximation Theorems [27, 12]. In the form it is stated below they were proved in [23].

**Theorem 11.1.** Suppose that $\mathcal{T} \subset F \subset E$ are quasi-normed linear spaces and $E$ and $F$ are complete. If there exist $C > 0$ and $\beta > 0$ such that the following Jackson-type inequality is satisfied

$$t^\beta \mathcal{E}(t,f,\mathcal{T},E) \leq C \|f\|_F, \ t > 0, \ f \in F,$n

then the following embedding holds true

$$\mathcal{E}_{\theta \beta,q}(E,\mathcal{T}) \subset (E,F)_k^\kappa, \ 0 < \theta < 1, \ 0 < q \leq \infty.$$ 

**Theorem 11.2.** If there exist $C > 0$ and $\beta > 0$ such that the following Bernstein-type inequality holds

$$\|f\|_F \leq C \|f\|_E^\beta \|f\|_E, \ f \in \mathcal{T},$$

then the following embedding holds true

$$\mathcal{E}_{\theta \beta,q}(E,\mathcal{T}) \subset (E,F)_k^\kappa, \ 0 < \theta < 1, \ 0 < q \leq \infty.$$ 

**11.3. Frames in Hilbert spaces.** A family of vectors $\{\Phi_j\}$ in a Hilbert space $H$ is called a frame if there exist constants $A, B > 0$ such that

$$A \|f\|_H^2 \leq \sum_j |\langle f, \Phi_j \rangle|^2 \leq B \|f\|_H^2 \quad \text{for all} \quad f \in H.$$ 

The largest $A$ and smallest $B$ are called lower and upper frame bounds.

The family of scalars $\{\langle f, \Phi_j \rangle\}$ represents a set of measurements of a vector $f$. In order to reconstruct the vector $f$ from this collection of measurements in a linear way one has to find another (dual) frame $\{\Psi_j\}$. Then a reconstruction formula is

$$f = \sum_j \langle f, \Phi_j \rangle \Psi_j.$$ 

Dual frames are not unique in general. Moreover it may be difficult to find a dual frame in concrete situations. If $A = B = 1$ the frame is said to be tight or Parseval. Parseval frames are similar in many respects to orthonormal wavelet bases. For example, if in addition all vectors $\Phi_j$ are unit vectors, then the frame is an orthonormal basis. The main feature of Parseval frames is that decomposing and synthesizing a vector from known data are tasks carried out with the same family of functions, i.e., the Parseval frame is its own dual frame.

For more details on Interpolation and Approximation spaces see [14], [12], [18], [22], [23], [25], [27], [12].

12. Appendix 2.

Let $D$ be a generator of a strongly continuous one parameter group $T(t), t \in \mathbb{R}$, of isometries of a Banach space $B$ with the norm $\| \cdot \|$. Let $D^\infty = \cap_{m \in \mathbb{N}} D(D^m)$, where $D(D^m)$ is the domain of $D^m$. Let $B_\sigma(D)$ be the space of all $f \in B$ for which the Bernstein-type inequality holds, i.e.

$$\|D^k f\| \leq \sigma^k \|f\|, \ k \in \mathbb{N}.$$
The following statements [35, 38], imply Lemma 5.1 and Lemma 6.1 respectively.

**Lemma 12.1.** A function \( f \in D^\infty(D) \) belongs to \( B_\sigma(D) \), \( \sigma > 0 \), if and only if the quantity

\[
\sup_{k \in \mathbb{N}} \sigma^{-k} \|D^k f\| = R(f, \sigma)
\]

is finite.

**Theorem 12.2.** If \( T \) is a strongly continuous one parameter group \( T(t), t \in \mathbb{R} \), of isometries generated by \( D \) and \( \mu \in B^1_\sigma(\mathbb{R}) \), \( \|\mu\|_{L^1(\mathbb{R})} = 1 \), be an entire function of exponential type \( \sigma \) then for any \( f \in B \) the function

\[
P_\sigma f = \int_{-\infty}^{\infty} \mu(t)T(t)f \, dt
\]

belongs to \( B^n_\sigma(D) \). The integral here is understood in the sense of Bochner for the abstract-valued function \( \mu(t)T(t)f : \mathbb{R} \to B \).

**12.1. Landau-Kolmogorov-Stein inequality.** Let us introduce the Krein-Favard constants (see [1], Ch. V) which are defined as

\[
K_j = \frac{4}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^{r+1}}{(2r+1)^{j+1}}, \quad j, \quad r \in \mathbb{N}.
\]

It is known, that the sequence of all Krein-Favard constants with even indices is strictly increasing and belongs to the interval \([1, 4/\pi]\) and the sequence of all Krein-Favard constants with odd indices is strictly decreasing and belongs to the interval \((\pi/4, \pi/2]\), i.e.,

\[
K_{2j} \in [1, 4/\pi), \quad K_{2j+1} \in (4/\pi, \pi/2],
\]

or

\[
1 = K_0 \leq K_2 \leq ... < 4/\pi < ... \leq K_3 \leq K_1 = \pi/2.
\]

In what follows the constant

\[
C_{k,m} = (K_{m-k})^m / (K_m)^{m-k} \leq (\pi/2)^m
\]

is playing an important role. We are going to prove a generalization of the classical Landau-Kolmogorov-Stein inequality [24, 29, 41]. This inequality was first proved by Landau and Kolmogorov for \( L^\infty(\mathbb{R}) \) and later extended to \( L^p(\mathbb{R}) \) for \( 1 < p < \infty \) by Stein (see also [36]).

**Lemma 12.3.** If \( f \in D(D^m) \) then the following inequality holds

\[
\|D^k f\|_m^m \leq C_{k,m} \|D^m f\| \|f\|^{m-k}, \quad 0 \leq k \leq m.
\]

**Proof.** Indeed, for any \( h^* \in B^* \) the Kolmogorov inequality [41] applied to the entire function \( \langle T(t)f, h^* \rangle \) gives

\[
\left\| \frac{d}{dt} \right\|^k \langle T(t)f, h^* \rangle \right\|_{C(\mathbb{R}^1)}^m \leq C_{k,m} \left\| \frac{d}{dt} \right\|^n \langle T(t)f, h^* \rangle \right\|_{C(\mathbb{R}^1)}^k \times
\]

\[
\| \langle T(t)f, h^* \rangle \|_{C(\mathbb{R}^1)}^{m-k}, \quad 0 < k < m,
\]

or

\[
\| \langle T(t)f, h^* \rangle \|_{C(\mathbb{R}^1)}^m \leq C_{k,m} \| \langle T(t)f, h^* \rangle \|_{C(\mathbb{R}^1)}^k \| \langle T(t)f, h^* \rangle \|_{C(\mathbb{R}^1)}^{m-k}.
\]
We obtain

\[ \| \langle T(t)D^k f, h^* \rangle \|_{C^m(\mathbb{R}^1)}^m \leq C_{k,m}\|h^*\|^k \|D^m f\|^k \|f\|^m - k \]

which yields when \( t = 0 \)

\[ \| \langle D^k f, h^* \rangle \|_{C^m(\mathbb{R}^1)}^m \leq C_{k,m}\|h^*\|^m \|D^m f\|^k \|f\|^m - k. \]

By choosing \( h \) such that \( |\langle D^k f, h^* \rangle| = \|D^k f\| \) and \( \|h^*\| = 1 \) we obtain (12.4). \( \Box \)

12.2. A characterization of \( \mathbb{B}_\sigma(D) \). For a vector \( f \in \cup_{\sigma > 0} \mathbb{B}_\sigma(D) \), the \( \sigma_f \) will denote the smallest \( \sigma \) for which the inequality (12.1) holds.

**Theorem 12.4.** Let \( f \in \mathbb{B} \) belong to a space \( \mathbb{B}_\sigma(D) \), for some \( 0 < \sigma < \infty \). Then the following limit exists

\[ (12.5) \]

\[ d_f = \lim_{k \to \infty} \| D^k f \|^{1/k}, \]

and \( d_f = \sigma_f \). Conversely, if \( f \in D^\infty \) and \( d_f = \lim_{k \to \infty} \| D^k f \|^{1/k} \) exists and is finite, then \( f \in \mathbb{B}_d(D) \) and \( d_f = \sigma_f \).

**Proof.** According to Lemma 12.4 we have

\[ \| D^k f \|^m \leq C_{k,m} \| D^m f \|^k \|f\|^m - k, \quad 0 \leq k \leq m. \]

Without loss of generality, let us assume that \( \|f\| = 1 \). Thus,

\[ \| D^k f \|^{1/k} \leq (\pi/2)^{1/km} \| D^m f \|^{1/m}, \quad 0 \leq k \leq m. \]

Let \( k \) be arbitrary but fixed. It follows that

\[ \| D^k f \|^{1/k} \leq (\pi/2)^{1/km} \| D^m f \|^{1/m}, \quad \text{for all } m \geq k, \]

which implies that

\[ \lim_{k \to \infty} \| D^k f \|^{1/k} \leq \lim_{m \to \infty} \| D^m f \|^{1/m}. \]

But since this inequality is true for all \( k > 0 \), we obtain that

\[ \lim_{k \to \infty} \| D^k f \|^{1/k} \leq \lim_{m \to \infty} \| D^m f \|^{1/m}, \]

which proves that \( d_f = \lim_{k \to \infty} \| D^k f \|^{1/k} \) exists. Since \( f \in \mathbb{B}_\sigma(D) \) the constant \( \sigma_f \) is finite and we have

\[ \| D^k f \|^{1/k} \leq \sigma_f \|f\|^{1/k}, \]

and by taking the limit as \( k \to \infty \) we obtain \( d_f \leq \sigma_f \). To show that \( d_f = \sigma_f \), let us assume that \( d_f < \sigma_f \). Therefore, there exist \( M > 0 \) and \( \sigma \) such that \( 0 < d_f < \sigma < \sigma_f \) and

\[ \| D^k f \|^{1/k} \leq M\sigma^k, \quad \text{for all } k > 0. \]

Thus, by Lemma 12.4 we have \( f \in \mathbb{B}_\sigma(D) \), which is a contradiction to the definition of \( \sigma_f \).

Conversely, suppose that \( d_f = \lim_{k \to \infty} \| D^k f \|^{1/k} \) exists and is finite. Therefore, there exist \( M > 0 \) and \( \sigma > 0 \) such that \( d_f < \sigma \) and

\[ \| D^k f \|^{1/k} \leq M\sigma^k, \quad \text{for all } k > 0, \]

which, in view of Lemma 12.4 implies that \( f \in \mathbb{B}_\sigma(D) \). Now by repeating the argument in the first part of the proof we obtain \( d_f = \sigma_f \), where \( \sigma_f = \inf \{ \sigma : f \in \mathbb{B}_\sigma(D) \} \).

Theorem 12.4 is proven.

\[ \Box \]
13. Appendix 3

The Appendix contains some comments on the proof of Theorem 8.2. To prove the first item of it we proceed as follows. If \( f \in PW_{\sigma} (L^{1/2}_{M}) \), then one has for \( \mathcal{F} f \):

\[
\| L^{s/2}_{M} f \|_{X^2} = \left( \int_{0}^{\infty} \lambda^{2s} \| \mathcal{F} f (\lambda) \|_{H(\lambda)}^2 \, dm(\lambda) \right)^{1/2} = (13.1)
\]

which gives Bernstein inequality for \( f \). Conversely, if \( f \) satisfies Bernstein inequality then \( x = \mathcal{F} f \) satisfies \( \| x \|_{H_k}^2 \leq \sigma^{2k} \| x \|_H^2 \). Suppose that there exists a set \( \mu \subset [0, \infty] \setminus [0, \mu] \) whose m-measure is not zero and \( x|_{\mu} \) is not zero almost everywhere. We can assume that \( \mu \subset [\sigma + \epsilon, \infty) \) for some \( \epsilon > 0 \). Then for any \( k \in \mathbb{N} \) we have

\[
\int_{\mu} \| x(\lambda) \|_{H(\lambda)}^2 \, dm(\lambda) \leq \int_{\sigma + \epsilon}^{\infty} \lambda^{-2k} \| x(\lambda) \|_{H(\lambda)}^2 \, dm(\lambda) \leq \| x \|_H^2 \left( \frac{\sigma}{\sigma + \epsilon} \right)^{2k}.
\]

Since \( k \) here can be arbitrarily large, it implies that either \( x(\lambda) \) is zero on \( \mu \) or \( \mu \) has measure zero.

To prove the second item of Theorem 8.2 one has to show that the above Bernstein inequality \( 13.1 \) implies that every function

\[
t \mapsto \langle e^{it\sqrt{L_M}} f, g \rangle_{X^2}, \quad t \in \mathbb{R}, \ f, g \in X^2,
\]

is a regular Bernstein function in \( B_{\sigma}^\infty (\mathbb{R}) \) which is bounded on the real line. This fact implies its properties described in the second item.

To prove the third item of Theorem 8.2 we use the fact that the function \( 13.2 \) belongs to the regular space \( B_{\sigma}^\infty (\mathbb{R}) \) and apply to it the classical Riesz-Boas interpolation formula (see [37]).

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