More on a Market Share Theorem

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INTRODUCTION

Bell, Keeney, and Little [1] recently proved a market share theorem. They assumed that the sales potential of each product in a market can be summarized by one "attractiveness" number, and postulated several relationships between attractiveness ratings and market share. Then they proceeded to show that, in their model, a seller's proportion of total sales is simply the attractiveness of his product divided by the total attractiveness of all products in the market.

The authors noted that linear-normalized "us/(us + them)" expressions for market share often are hypothesized in the literature, and that one value of their result is its specification of conditions when such formulas are appropriate. But in addition to ratifying expressions that already have been assumed, the theorem has a prescriptive value. Suppose one has a set of mathematical functions of product attributes—one for each competitor in a market—that one believes related to the market shares of individual firms by the Bell-Keeney-Little (hereafter BKL) postulates. Then use of linear-normalized estimates for percentage distribution of sales is not only permissible under the theorem but is mandatory because of it. Certain broad assumptions about market behavior, when taken together, are surprisingly precise in their implications.

Because of the stringent conditions in the BKL axioms, the result is directly applicable only in limited circumstances. This paper describes an attempt to extend the market share theorem by dropping its assumptions of "symmetry" and "linearity" in favor of more general axioms. By use of arguments that rest ultimately on the BKL theorem itself, unique expressions for market shares are obtained in the wider framework considered. These results may make the market share theorem more accessible to those wishing to use it.

BKL MODEL

Associated with the n products in a particular market is an attractiveness vector \( a = (a_1, ..., a_n) \), where \( a_i \) is the attractiveness of product \( i \). BKL assume that the components of the attractiveness vector are governed and related to those of the market share vector \((m_1, ..., m_n)\) by the following rules.

1. \( a_i \geq 0 \) for all \( i \); \( \sum_{i=1}^{n} a_i > 0 \).
2. If \( a_i = 0 \), then \( m_i = 0 \).
3. If \( a_i = a_j \), then \( m_i = m_j \).
4. Let \( \Delta m_i(\delta)a = \Delta m_i(\delta)a \) the change in market share \( m_i \) of product \( i \) associated with a change in attractiveness vector from \( a \) to \( (a_1, a_2, ..., a_j + \delta, a_n) \).

Then for all \( j, k \neq i \) and all \( \Delta, \Delta m_i(\delta)a = \Delta m_k(\delta)a \).

Under these conditions, BKL proved that

\[ m = (m_1, m_2, ..., m_n) \quad \text{where} \quad m_i = \frac{a_i}{\sum_{j=1}^{n} a_j} (i = 1, ..., n) \]

is the only possible market share vector.

To use this theorem in an actual situation the model builder must construct—from vectors about price, advertising, quality, etc.—an attractiveness vector that satisfies the four postulates. In general this is a nontrivial task. Postulate 4 is in some sense the most "demanding" of the axioms; it specifies that a change \( \delta \) in another product's attractiveness affects a firm's market share the same way regardless of which competitor sustained the change (BKL called this "symmetry"). This symmetry requirement is sometimes difficult to satisfy. Clarke [2], for instance, found that sales-advertising cross-elasticities in a marketing problem were asymmetric. Furthermore, attractiveness changes enter this axiom only on an absolute scale ("linearity"). Thus, although an increment of \( \delta \) can mean much greater percentage growth in attractiveness for some products than for others, this is treated as irrelevant. The market analyst may find that many contemplated attractiveness vectors are tainted in practice by nonlinearity and asymmetry.

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He also may have problems with the postulate 3 requirement that equal attractiveness for two different products must mean equal market shares.

**Modifying the BKL Postulates**

Possible difficulties in using the BKL postulates as they stand lead to an obvious question: can the requirement that products must mean exactly as they stand lead to an obvious question: can the axiomatic structure be amended so that it is (1) flexible in the relationship between attractiveness and market shares yet (2) sufficiently rigid to ensure a unique market share vector for a given attractiveness vector? To investigate this question, let us first consider a class of asymmetric models. This particular system is very amenable to mathematical analysis and makes the crucial issues transparent. As will be shown, the introduction of some "bus or train" variable into the attractiveness functions might be sufficient to put things in the BKL framework. But there is something metaphysical about such variables; it may be preferable to deal with attractiveness vectors based on well-defined quantities even if they entail asymmetry.

**Trouble Develops**

Though the new postulates are not fettered by symmetry, there is another problem. Attempts to find market share functions that satisfy axioms 1–4 are doomed to failure. It is important to note that $m_1(a_1, a_2, a_3)$ can be written as $m_1(a_1, 2a_2 + a_3)$, i.e., product 1’s market share depends on $a_2$ and $a_3$ only through the weighted sum $2a_2 + a_3$. To see this, consider two consecutive changes in the attractiveness vector: (1) product 2’s attractiveness increases by $\Delta$, (2) product 2’s attractiveness decreases by $\Delta$. Obviously these two changes have no net effect on $m_1$. If the second alteration in the attractiveness vector were instead a decrease of $2\Delta$ in $a_3$, axiom 4A tells us that $m_1$ likewise would sustain no net change. Thus, whenever $a_2$ and $a_3$ change in such a way that $\Delta a_3 = -2\Delta a_2$ or, equivalently, that $2a_2 + a_3$ remains constant, $m_1$ remains constant unless $a_1$ varies. Therefore, $m_1$ can be considered a function of the two variables $a_1$ and $2a_2 + a_3$. Similarly, one can write $m_2(a_2, 2a_1 + a_3)$ and $m_3(a_3, a_1 + a_2)$.

We now examine five specific attractiveness vectors $(a_1, a_2, a_3)$ and reach an odd conclusion about market shares:

1. $a_1 = y$, $a_2 = 0$, $a_3 = y$ for some positive $y$.

From axiom 2, $m_2 = 0$ here; from 3 and the obvious requirement that $\Sigma_{i=1}^3 m_i = 1$, $m_1 = m_3 = .5$. Because $a_1$, $a_3$, $2a_2 + a_3$, and $a_1 + a_2$ all equal $y$, we see that $m_1(y, y) = m_3(y, y) = .5$.

2. $a_1 = y$, $a_2 = .5y$, $a_3 = 0$.

$A_1(a_1, 2a_2 + a_3) = m_1(y, y) = .5$ from the previous result. Because $m_3 = 0$ here, we can conclude that $m_2(a_2, 2a_1 + a_3) = m_2(.5y, 2y) = .5$.

In a wholly analogous way, examination of the attractiveness vectors 3:(0,y,y) and 4:(.5y,y,0) leads to the conclusion $m_1(.5y, 2y) = .5$.

3. $a_1 = .5y$, $a_2 = .5y$, $a_3 = y$.

Note that $2a_1 + a_3 = 2a_2 + a_3 = 2y$ here, thus the facts that $m_1(.5y, 2y) = m_1(.5y, 2y) = .5$ and $\Sigma_{i=1}^3 m_i = 1$ require that $m_1(a_1, a_2 + a_3) = m_1(y, y) = 0$. But heretofore we concluded that $m_1(y, y) = .5$. What a calamity has arisen—the axioms tell us that the third competitor has both none of the sales and half...
of them! At this point our attempts to generalize the BKL postulates hardly can be described as successful.

**PICKING UP THE PIECES**

A crude, qualitative explanation of what went wrong is in order. The contradiction found is similar to that which would arise if one tried calculating the slope of a straight line through three points on a circle. In both problems, the conditions imposed on the solution include some that cannot be satisfied simultaneously. We have noted at length the asymmetric form of postulate 4A, but paid too little attention to the fact that axiom 3 (equal attractiveness implies equal market share) not only is symmetric in structure but also is a very strong requirement on all contemplated market share functions. The underlying cause of trouble is that axioms 3 and 4A "clash" in their implications in a way that cannot be reconciled by single-valued expressions for \( m, \) consistent with the rules

\[
\begin{align*}
(1) & \quad a_i \geq 0, \sum_{i=1}^{3} a_i > 0. \\
(2) & \quad a_i = 0 \Rightarrow m_i = 0. \\
(4A) & \quad (i) \quad \Delta_{12a}(b) = \Delta_{13a}(2a) \\
& \quad (ii) \quad \Delta_{21a}(b) = \Delta_{23a}(2a) \\
& \quad (iii) \quad \Delta_{31a}(b) = \Delta_{32a}(2a)
\end{align*}
\]

It is no longer the case that there are no \( m, \) \((a_1, a_2, a_3)\) \((i = 1, 2, 3)\) satisfying the axioms and the requirement that \( \sum_{i=1}^{3} m_i = 1. \) Indeed, there is now a linear-normalized solution:

\[
\begin{align*}
m_1 &= \frac{2a_1}{2a_1 + 2a_2 + a_3}, \\
m_2 &= \frac{2a_2}{2a_1 + 2a_2 + a_3}, \\
m_3 &= \frac{a_3}{2a_1 + 2a_2 + a_3}.
\end{align*}
\]

There is a danger, however, that the axiomatic structure is so "weakened" that there are many, possibly even an infinite number of, well-defined sets of \( m, \)'s consistent with it.

At this point we make a linear change of variables in the components of the attractiveness vector. The new vector \( b = (b_1, b_2, b_3) \) is related to the original \( a = (a_1, a_2, a_3) \) by the relationships: \( b_1 = 2a_1, b_2 = 2a_2, b_3 = a_3. \) In the new system, axioms 1, 2, and 4A appear as:

\[
\begin{align*}
(1) & \quad b_i \geq 0, \sum_{i=1}^{3} b_i > 0. \\
(2) & \quad b_i = 0 \Rightarrow m_i = 0. \\
(4B) & \quad \Delta_{12b}(b) = \Delta_{13b}(2b) \\
& \quad \Delta_{21b}(b) = \Delta_{23b}(2b) \\
& \quad \Delta_{31b}(b) = \Delta_{32b}(2b).
\end{align*}
\]

Postulates 1 and 2 are unchanged but 4B, which is equivalent to the former 4A, is no longer asymmetric; 4B is the same as postulate 4 in the original BKL model. With the new attractiveness variables, therefore, three of the four specifications made by BKL are satisfied. This is only a curiosity in the absence of the other axiom (equal attractiveness implies equal market share) unless it could be shown somehow that the three we already have imply the fourth; in that case, the market share theorem would be directly relevant to the current problem. Strangely enough, postulates 1, 2, and 4B do imply the remaining axiom.

**Theorem:** Suppose one has an attractiveness vector \( b = (b_1, b_2, b_3) \) and market share functions \( m, \) \((b_1, b_2, b_3)\) satisfying rules 1, 2, and 4B. Then if \( b_1 = b_2, m_1 = m_2. \)

**Proof:** Let \( b_1 = x, b_2 = x, \) and \( b_3 = z \) for any \( x \) and/or \( z > 0 \); we will show that \( m_1(x, x, z) = m_2(x, x, z) \). The argument is wholly parallel when \( b_1 = b_3 \) or \( b_2 = b_3. \)

With the same reasoning used heretofore, we can establish that under postulate 4B, \( m_1 \) depends on \( b_2 \) and \( b_3 \) only through their sum \( b_2 + b_3, m_2 \) depends on \( b_1 \) and \( b_3 \) only through \( b_1 + b_3, \) and \( m_1 \) depends on \( b_1 \) and \( b_2 \) only through \( b_1 + b_2. \) (In this argument we write \( m_1(b_1, b_2, b_3) \) rather than \( m_1(b_1, b_2, b_2 + b_3) \) etc. but utilize the property just described.) We demonstrate that \( m_1 = m_2 \) by considering some other attractiveness vectors \( (b_1, b_2, b_3). \)

(a) Let \( b_1 = x, b_2 = 0, b_3 = x + z. \) By 2, \( m_2 = 0; \) let \( m_1(x, 0, x + z) = \gamma. \) Because \( b_1 \) and \( b_2 + b_3 \) take the same values as at \((x, x, z), m_1(x, 0, x + z) = m_1(x, x, z) = 1 - \gamma. \)

(b) Let \( b_1 = 0, b_2 = x, b_3 = x + z. \) Here \( m_1 = 0; \) because \( b_1 + b_3 \) and \( b_2 \) retain their values in (a) \( m_1(0, 0, x + z) = \gamma. \) Thus \( m_2(0, 0, x + z) = 1 - \gamma \) and therefore, because \( m_2(0, 0, x + z) = m_2(x, x, z) \), it follows that \( m_1(x, x, z) = m_2(x, x, z). \) Q.E.D.

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1Original numbering of postulates is retained to avoid confusion.
The theorem just proved means that 1, 2, and 4A imply all four BKL conditions in the new variable system. But then the market share theorem requires that

\[ m_i = \frac{b_i}{b_1 + b_2 + b_3} \quad \text{for} \quad i = 1, 2, 3 \]

or, transforming back to the original variables,

\[ m_1 = \frac{2a_1}{2a_1 + 2a_2 + a_3}, \]
\[ m_2 = \frac{2a_2}{2a_1 + 2a_2 + a_3}, \]
\[ m_3 = \frac{a_3}{2a_1 + 2a_2 + a_3}, \]

is the only set of market share functions consistent with 1, 2, and 4A. Thus the elimination of axiom 3 created an asymmetric set of postulates that did in fact prescribe a unique distribution of percentage sales for a given attractiveness vector. The distribution itself was found through an appropriate application of the market share theorem. In the next section, we use the insights obtained to handle asymmetry and nonlinearity in a more general setting.

MORE NONLINEARITY AND ASYMMETRY

Consider the following model of market behavior. There are \( N \) competing products \((N \geq 3)\) and an attractiveness vector \((a_1, a_2, \ldots, a_N)\). Associated with each product \( i (i = 1, \ldots, N) \) is a positive mathematical function \( h_i(x) \) defined for all \( x \geq 0 \). The relationships governing market share \( m_i(a_1, \ldots, a_N) \) \((i = 1, \ldots, N)\) and attractiveness are

\[ a_i \geq 0, \quad \sum_{i=1}^{N} a_i > 0. \]
\[ a_i = 0 \Rightarrow m_i = 0. \]

(3A) \( m_i(a_1, \ldots, a_N) \) is a differentiable function of each \( a_j (j = 1, \ldots, n) \) and for all \( K, L \neq i \), the following equation is satisfied:

\[ \frac{\partial m_i}{\partial a_K} = \frac{h_K(a_K)}{h_L(a_L)} \frac{\partial m_i}{\partial a_L}. \]

The key features of this model are contained in axiom 3A. Unlike the BKL problem and the example we used heretofore, this formulation explicitly assumes each \( m_i \) differentiable in the attractiveness variables. This is probably not a great practical restriction, especially because the market share functions obtained in the previous problems turned out to be differentiable. (There is a way to avoid a differentiability assumption in 3A but it is so cumbersome as to seem not worthwhile.) The equation in 3A makes clear that the model is asymmetric because the impact of a change in attractiveness depends in general on which product sustains the change. And the factor

\[ \frac{h_K(a_K)}{h_L(a_L)} \]

relates the significance of changing a product’s attractiveness directly to its current level, introducing—for the first time—the element of nonlinearity. The functions \( h_i(x) \) indicate the “elasticity” of a competitor’s market share in product \( i \)’s attractiveness level, and allow different assessments of market behavior for different products.

The writer’s major finding follows.

**Theorem:** Under axioms 1-3A,

\[ m_i(a_1, \ldots, a_N) = \frac{\int_{0}^{a_i} h_i(x)dx}{\sum_{j=1}^{N} \int_{0}^{a_j} h_j(x)dx} \quad \text{uniquely}. \]

**Proof:** As before, we introduce a change of attractiveness variables. Let \( b_i = \int_{0}^{a_i} h_i(x)dx \). Then equivalent to postulates 1, 2, and 3A are the rules:

\[ b_i \geq 0, \quad \sum_{i=1}^{N} b_i > 0. \]
\[ b_i = 0 \Rightarrow m_i = 0. \]

(3B) \( \frac{\partial m_i}{\partial b_K} = \frac{\partial m_i}{\partial b_l} \quad \text{for all} \quad K, l \neq i. \)

Rule 2 follows at once from the previous rule 2 and the definition of \( b_i \); rule 1 is an immediate consequence of its antecedent in \((a_1, \ldots, a_N)\) and the assumption that all \( h_i(x) \)'s are positive functions. 3B results from 3A and the observation that

\[ \frac{\partial m_i}{\partial b_i} h_i(a_i) = \frac{\partial m_i}{\partial a_i} h_i(a_i). \]

A moment’s reflection makes apparent that 3B is the “differentiable” equivalent to BKL postulate 4. Thus BKL axioms 1, 2, and 4 are satisfied under the new attractiveness variables; because, as we have seen, 1, 2, and 4 imply BKL axiom 3, we can use the market share theorem to state that

\[ m_i = \frac{b_i}{\sum_{j=1}^{N} b_j} \]

Rewriting \( m_i \) in terms of \((a_1, \ldots, a_N)\) yields the theorem.

Note that 3 would not have become \( b_i = b_i \Rightarrow m_i = m_i \) in the new system.
What we have done is consider a special set of nonlinear and asymmetric situations that can be transformed into linear symmetric problems with relatively simple changes of variables. However, the generality in the functional forms of the \( h_i(x) \)'s allows a number of potentially practical cases to fall into this set.

**Example**

Consider a three-product market with a non-negative attractiveness measure for each competitor. Suppose one believes that zero attractiveness implies no sales, attractiveness numbers. On the market a somewhat shorter time than their competitor, the model-builder might believe a weaker

\[
\frac{\partial m_1}{\partial a_2} = \frac{\sqrt{1 + a_2}}{1 + a_2} \quad \frac{\partial m_2}{\partial a_2} = \frac{\sqrt{1 + a_2}}{1 + a_2} \quad \frac{\partial m_3}{\partial a_2} = \frac{\sqrt{1 + a_2}}{1 + a_2}
\]

Because \( h_1(x) = h_2(x) = \sqrt{1 + x} \) and \( h_3(x) = 1 + x \), the result

\[
m_i = \frac{\int_0^x h_i(x) dx}{\sum_{i=1}^3 \int_0^x h_i(x) dx} \quad \text{just proved}
\]

becomes:

\[
m_1(a_1, a_2, a_3) = \frac{2/3((1 + a_1)^{3/2} - 1)}{2/3((1 + a_1)^{3/2} + (1 + a_2)^{3/2}/2 + a_3 + a_3^2/2}
\]

\[
m_2(a_1, a_2, a_3) = \frac{2/3((1 + a_2)^{3/2} - 1)}{2/3((1 + a_2)^{3/2} + (1 + a_3)^{3/2}/2 + a_3 + a_3^2/2}
\]

\[
m_3(a_1, a_2, a_3) = \frac{a_3 + a_3^2/2}{2/3((1 + a_3)^{3/2} + (1 + a_2)^{3/2}/2 + a_3 + a_3^2/2}
\]

In this particular model, the "elasticity" \( \partial m_i/\partial a_i \) market shares for products 1 and 2 goes up, roughly speaking, with \( a_i^{-3/2} \), whereas product 3's "elasticity" varies with \( (a_3)^{-2} \). When might such a situation actually arise? Suppose current advertising expenditures seem appropriate as major components of the attractiveness numbers. If products 1 and 2 have been on the market a somewhat shorter time than their competitor, the model-builder might believe a weaker

"law of diminishing returns" applies to these products than to 3. In some cases the foregoing equations might quantify that notion adequately. Note that when \( a_2 = a_3 \)

\[
\frac{\partial m_1}{\partial a_3} > \frac{\partial m_1}{\partial a_2} \quad \text{(except when } a_2 = a_3 = 0); \quad \text{similarly}
\]

\[
\frac{\partial m_2}{\partial a_3} > \frac{\partial m_2}{\partial a_2} \quad \text{when } a_1 = a_3 > 0.
\]

This indicates that, other factors being equal, products 1 and 2 compete more directly with product 3 than with each other. We have discussed a circumstance in which such asymmetric competition might occur (2 bus lines, 1 rail line).

The foregoing analysis assumes that an attractiveness vector consistent with the postulates already has been found. As the reader might suspect, finding such a vector is a difficult task. Constructing attractiveness functions for products from raw data might well be a fruitful area for empirical research.

**Relationship to Current Research**

It is natural to wonder how the results described relate to methods currently used to estimate market shares. In a market with \( N \) products and \( K \) attribute-related variables for each, the market share of any competitor is a function of \( NK \) variables. Specifying such functions for all products is extremely difficult, so much current work is based on one of two simplifying procedures. The first set of models creates for each product a function of its \( K \) attribute variables; the same function is used for all \( N \) products. The market share of any product is estimated by the ratio of its function's value to the sum of those for all products. Symmetric models of this kind are the subject of the BKL paper [1]; many are cited therein. The second major approach assumes that levels of all but one particular attribute (e.g. advertising, outlet share; see [2-4]) are effectively held constant for all products, and that the market share of a given product is a function of the levels for all products of the attribute studied (i.e. a function of \( N \) variables). Asymmetric cross-elasticities usually are allowed in models of this kind.

The results presented herein might be useful in an approach that combines features of the two just described to reduce the potential deficiencies of each. One could continue to present a product's attractiveness as an explicit function of all—not just one—of its attribute variables; to do so is desirable because several attributes (e.g. price, advertising, distribution pattern) often change simultaneously, making the identification of the effects of any one somewhat perilous. However, one could allow for market-share elasticities that are asymmetric with respect to different competitors and sensitive to the current at-
tractiveness level of a given competitor. Thus, for instance, the effect on product X of a $10,000 rise in advertising for another product need not be assumed independent of (i) the identity of the competing product and (ii) its previous advertising budget. Such independence is implied in “us/(us + them)” models with attractiveness linear in advertising. It would be presumptuous to say now that the results herein can improve market-share estimates, but investigation of this possibility might be appropriate.

CONCLUSIONS

The results should be viewed not as generalizations of the market share theorem but rather as elaborations of it. Direct use of the Bell-Keeney-Little theorem requires an attractiveness vector related linearly and symmetrically to the vector of market shares. Results were obtained in some asymmetric, nonlinear situations by making variable changes that led to the BKL framework. This work was facilitated greatly by the finding that the axiom “equal attractiveness implies equal market share” was not essential to reaching the BKL result; thus no corresponding postulate was needed in the systems considered. Direct imposition of that axiom upon an asymmetric system led to contradictory and meaningless results.

The findings mean that the market share theorem is of even broader relevance than one might at first realize. Persons trying to understand market behavior may find this information useful.

REFERENCE

1. Bell, D., R. Keeney and J. Little. “A Market Share Theorem,” Journal of Marketing Research, 12 (May 1975).