Semialgebraic and Continuous Solution of Linear Equation with Semialgebraic Coefficients

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Abstract

Starting from the results of Charles Fefferman and Janos Kollár in Continuous Solutions of Linear Equations [1], we adopt a new approach based on Fefferman’s techniques of Glaeser refinement to show a more general result than the one proved by Kollár by using techniques from algebraic geometry. Considering a system of linear equations with semialgebraic (not only polynomial as in [1]) coefficients on $\mathbb{R}^n$, we get a necessary and sufficient condition for the existence of a continuous and semialgebraic solution on $\mathbb{R}^n$. This is different from what Fefferman and Luli obtained in Semialgebraic Sections Over the Plane [3] since they stated their result for solutions of regularity $C^m$ on the plane $\mathbb{R}^2$. More in depth, we prove that a continuous and semialgebraic solution on $\mathbb{R}^n$ exists if and only if there is a continuous solution i.e., if the Glaeser-stable bundle associated to the system has no empty fiber.

1 Introduction

This work deal with the open problem of obtaining by analytical techniques necessary and sufficient conditions for the existence of a $C^m$ and semialgebraic solution of a system of linear equation with semialgebraic coefficients. In case $m = 0$ of a system with polynomial coefficients, the problem was solved by
Fefferman-Kollár [11] and Kollár [6] using algebraic techniques for systems with polynomial coefficients. In this work, by a new approach based on Fefferman’s analytic techniques of Glaser’s refinements, we solve the problem for the case of a $C^0$ and semialgebraic solution on a general $n$-dimensional space $\mathbb{R}^n$, extending Fefferman-Kollár’s result to the case of a system with semialgebraic (not only polynomial as in [1]) coefficients.

Let us go through a deeper explanation of our work and the context in which it is developed.

C. Fefferman proved in [1], by means of analysis techniques, a necessary and sufficient condition for the existence of a continuous solution $(\phi_1, \ldots, \phi_s)$ of the system

$$\phi = \sum_{i=1}^{s} \phi_i f_i$$

(1)

given the continuous functions $\phi$ and $f_i$. More precisely, by applying the theory of the Glaser refinements for bundles, he showed that system (1) has a continuous solution if and only if the affine Glaser-stable bundle associated with system (1) has no empty fiber.

Moreover J. Kollár, in the same (joint) paper [1], starting from the above result and making use of algebraic geometry techniques as blowing up at singular points, proved that fixed the polynomials $f_1, \ldots, f_s$ and assuming system (1) has a solution, then:

1) if $\phi$ is semialgebraic then there is a solution $(\psi_1, \ldots, \psi_s)$ of $\phi = \sum_{i} \psi_i f_i$ such that the $\psi_i$ are also semialgebraic;

2) let $U \subset \mathbb{R}^n \setminus Z$ (where $Z := (f_1 = \ldots = f_r = 0)$) be an open set such that $\phi$ is $C^m$ on $U$ for some $1 \leq m \leq \infty$ or $m = \omega$. Then there is a solution $\psi = (\psi_1, \ldots, \psi_s)$ of $\phi = \sum_{i=1}^{s} \psi_i f_i$ such that the $\psi_i$ are also $C^m$ on $U$.

Next, in [2] C. Fefferman and G.K. Luli exhibited generators of the module $\mathcal{M}$ (over the ring of polynomials on $\mathbb{R}^n$) of the vectors $f := (f_1, \ldots, f_s)$ of polynomials $f_1, \ldots, f_s$ such that

$$\sum_{j=1}^{M} A_{ij} F_j = f_i \quad (i = 1, \ldots, N),$$

(2)
(for unknown functions $F_1, \ldots, F_N \in C^m(\mathbb{R}^n)$, $m$ fixed) admits a $C^m$ solution.

Finally, in [3] C. Fefferman and G.K. Luli showed that if $\mathcal{H}$ is a semi-algebraic bundle with respect to the space of $\mathbb{R}^D$-valued functions on the plane $\mathbb{R}^2$ with continuous derivatives up to order $m$ (that space is called $C^m_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^D)$) and it has a $C^m_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^D)$ section, then $\mathcal{H}$ has a semi-algebraic and $C^m_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^D)$ section. Actually, the authors do not give an explicit method to compute that semi-algebraic $C^m_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^D)$ section: the $C^m_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^D)$ semialgebraic section is defined as the one satisfying equations (97), (98), and (99) at p.44 of [3].

In the case $m = 0$ the problems of

- determining necessary and sufficient conditions for the existence of a continuous solution of (2) where $A_{ij}$ and $f_j$ are given functions,
- exhibiting generators of the module $\mathcal{M}$ with $A_{ij}$ given polynomials,
- determining necessary and sufficient conditions for the existence of a continuous and semi-algebraic solution of (2) where $A_{ij}$ and $f_j$ are given polynomials and (2) admits a continuous solution,

were posed by Brenner [4], and Epstein-Hochster [5], and solved by Fefferman-Kollár [1] and Kollár [6].

In this paper by a new approach, based on Fefferman’s techniques, we generalize and solve the third of the above problems showed by Kollár through algebraic techniques for $m = 0$. More precisely, we prove that if a semi-algebraic bundle associated to a system with coefficients and right-hand side that are semi-algebraic (but not necessarily continuous) on $\mathbb{R}^n$ has a continuous section then it has also a continuous and semi-algebraic section. We show it without employing the algebraical blow-up theory but only by using the analytical Fefferman-Glaeser theory with the aim of determining an explicit method to construct a continuous and semi-algebraic section.

Let us provide a more detailed description of the problem we deal with. We consider a semi-algebraic compact metric space $Q \subseteq \mathbb{R}^n$ and a system of
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linear equations

\[ A(x) \phi(x) = \gamma(x), \quad x = (x_1, \ldots, x_n) \in Q \]  

where

\[ Q \ni x \mapsto A(x) = (a_{ij}(x)) \in M_{r,s}(\mathbb{R}) \]

is semialgebraic, with \( M_{r,s}(\mathbb{R}) \) denoting the set of real \( r \times s \) matrices and

\[ Q \ni x \mapsto \gamma(x) \in \mathbb{R}^r, \gamma(x) = \begin{bmatrix} \gamma_1(x) \\ \vdots \\ \gamma_r(x) \end{bmatrix} \in \mathbb{R}^r \]

being themselves semialgebraic functions on \( Q \subseteq \mathbb{R}^n \).

Our aim is to find a necessary and sufficient condition for the existence of a solution \( Q \ni x \mapsto \phi(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_s(x) \end{bmatrix} \in \mathbb{R}^s \) of system (3), with the \( \phi_i : Q \to \mathbb{R} \) continuous and semialgebraic.

We notice that the semialgebraicity of \( Q \) is a necessary condition for the existence of a semialgebraic solution of system (3) by the definition of semialgebraic function (i.e. a function with semialgebraic graph) and by the Tarski-Seidenberg theorem.

The plan of the paper is the following. In Section 2 we fix some notations and give some definitions that will be used in Section 3.

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\[ \text{Tarski-Seidenberg Theorem} \] Let \( A \) a semialgebraic subset of \( \mathbb{R}^{n+1} \) and \( \pi : \mathbb{R}^{n+1} \to \mathbb{R}^n \), the projection on the first \( n \) coordinates. Then \( \pi(A) \) is a semialgebraic subset of \( \mathbb{R}^n \).

\[ \text{Corollary} \] If \( A \) is a semialgebraic subset of \( \mathbb{R}^{n+k} \), its image by the projection on the space of the first \( n \) coordinates is a semialgebraic subset of \( \mathbb{R}^n \).
In Section 3 we prove that if a semialgebraic bundle associated to a system of semialgebraic (but not necessarily continuous) function on a semialgebraic compact set $Q$ has a continuous section then it has also a semialgebraic and continuous one. The main idea is to prove the result by an induction argument on the dimension $d$ of $Q$. (We recall that the dimension of a semialgebraic set $E \subset \mathbb{R}^n$ is the maximum of the dimensions of all the embedded, not necessarily compact, submanifolds of $\mathbb{R}^n$ that are contained in $E$..) In fact, for the case $d = 1$ we use the fact that a semialgebraic function on a subset of $\mathbb{R}$ has finitely many isolated discontinuity points (the set on which a semialgebraic function is not continuous is a semialgebraic subset of its domain of strictly lower dimension and a semialgebraic set of dimension 0 is finite i.e. it is made by finitely many isolated points). Hence, we construct a local semialgebraic and continuous section of the bundle on a neighbourhood of each point of $Q$ and we glue the semialgebraic and continuous sections by a semialgebraic and continuous partition of the unity. Next, in the case $d \geq 2$, by induction hypothesis there is a continuous and semialgebraic section on an appropriate compact subset of $Q$ of dimension $\leq d - 1$ (which will be defined in the proof of Theorem 3.1) and we extend it thanks to a semialgebraic version of Tietze-Uryshon Theorem. Finally, we need to compute the projection of the extension on the fibers of $\mathcal{H}^{Gl}$ (i.e. the Glaeser-stable bundle associated to the system (3)) to obtain a continuous and semialgebraic section of $\mathcal{H}^{Gl}$ i.e. a semialgebraic and continuous solution of (3).

The result of Section 3 is obtained without the use of algebraic geometrical tools, but only by the analysis techniques such as the Glaeser refinement and the theory of bundle sections developed by Fefferman. This paper gives an explicit method for the construction of a semialgebraic continuous solution of system (3) by finitely many induction steps.

2 The setting

Let us start by setting some notations and definitions that will be used to pursue our goal. We shall endow every $\mathbb{R}^s$ used here with euclidean norm.
**Notation:** Let \( V \subseteq \mathbb{R}^s \) be an affine space in \( \mathbb{R}^s \) and \( w \in \mathbb{R}^s \). We denote the projection of \( w \) on \( V \) (i.e. the point \( v \in V \) that makes the euclidean norm of \( v - w \) as small as possible) by \( \Pi_V w \).

Let us consider a singular affine bundle (or bundle for short) (see [1]), meaning a family \( H = (H_x)_{x \in Q} \) of affine subspaces \( H_x \subseteq \mathbb{R}^s \), parametrized by the points \( x \in Q \). The affine subspaces

\[
H_x = \{ \lambda \in \mathbb{R}^s : A(x) \lambda = \gamma(x) \}, \quad x \in Q
\]

are the fibers of the bundle \( H \). (Here, we allow the empty set \( \emptyset \) and the whole space \( \mathbb{R}^s \) as affine subspaces of \( \mathbb{R}^s \).)

Now we call \( H^{(k)} \) the \( k \)-th Glaeser refinement of \( H \) i.e. \( H^{(0)} := H \) and for all \( k \geq 1 \) the fibers of \( H^{(k)} \) are

\[
H^{(k)}_x := \{ \lambda \in H^{(k-1)}_x ; \ \text{dist}(\lambda, H^{(k-1)}_y) \to 0 \ \text{as} \ y \to x \ \ (y \in Q) \},
\]

for all \( x \in Q \) (see Chapter 2 of [1]). We notice that \( H^{(k)} \) is a subbundle of \( H^{(k-1)} \) for all \( k \geq 1 \). By Lemma 5 of [1] the procedure of refinement leads to a Glaeser-stable refinement of \( H \) i.e. there is a \( r \in \mathbb{N} \) such that \( H^{(2r+1)} = H^{(2r+2)} = \cdots \). We denote \( H^{(2r+1)} \) by \( H^{Gl} \) and we will call it the Glaeser-stable refinement of \( H \) (its fibers will respectively be denoted by \( H^{Gl}_x \) for all \( x \in Q \)). Notice that the projection on the fibers of \( H^{Gl} \) is not linear as the fibers are affine spaces and not vector spaces.

Given a continuous solution \( f \) of system (4) we define

\[
Q \ni y \mapsto \omega(y) := \Pi_{(H^{Gl}_y)^\perp} f(y),
\]

and we notice that \( \omega \) does not depend on the choice of the continuous solution \( f \). More precisely, for all \( y \in Q \) the value \( \omega(y) \) can be computed by projecting \( 0 \) on \( H^{Gl}_y \). Moreover, we define

\[
Q \ni y \mapsto \tilde{\Pi}_1(y)v := \Pi_{(H^{Gl}_y)^\perp} v,
\]
for all $v \in \mathbb{R}^s$. We say that $\tilde{\Pi}_1$ is continuous if $y \mapsto \tilde{\Pi}_1(y)e_j$ is continuous for all $j = 1, \ldots, s$ with $(e_1, \ldots, e_s)$ the canonical basis of $\mathbb{R}^s$.

3 Existence of a continuous semialgebraic solution

In this section we prove that system (3) on a semialgebraic compact space $Q \subseteq \mathbb{R}^n$ has a semialgebraic and continuous solution if and only if it has a continuous one. We do it by induction on the dimension of $Q$ on which the problem is defined.

**Theorem 3.1.** Consider a semialgebraic compact metric space $Q \subseteq \mathbb{R}^n$ and a system of linear equations

$$A(x)\phi(x) = \gamma(x), \quad x \in Q,$$

where the entries of

$$A(x) = (a_{ij}(x_1, \ldots, x_n)) \in M_{r,s}(\mathbb{R}) \quad \text{and} \quad \gamma(x) = (\gamma_i(x)) \in \mathbb{R}^r$$

are themselves semialgebraic functions on $\mathbb{R}^n$.

Then system (3) has a continuous semialgebraic solution $\phi : Q \to \mathbb{R}^s$ if and only if $\mathcal{H}^{Gl}$ has no empty fiber.

**Proof.** We start by proving the forward implication which is trivial. In fact, if system (3) has a continuous solution then $\mathcal{H}^{Gl}$ has no empty fiber (see [1]).

Now, we prove the reverse implication. To do it we proceed by induction on the dimension $d \in \{1, \ldots, n\}$ of $Q$. (We notice that if $d = 0$ then $Q$ is a finite set and, hence, any selection of $\mathcal{H}^{Gl}$ is a semialgebraic and continuous section of $\mathcal{H}^{Gl}$. A selection of $\mathcal{H}^{Gl}$ exists since $\mathcal{H}^{Gl}$ has no empty fiber.) Actually, before starting the proof by induction, we need to show that $\omega$ is semialgebraic. To do this we need to verify that the set

$$\mathcal{H}_Q^{Gl} := \{(x, v) \in \mathbb{R}_x^n \times \mathbb{R}_v^s; \ x \in Q, \ v \in H_x^{Gl}\}$$

is semialgebraic.
Hence, we prove by induction on $k \geq 0$ that

$$
\mathcal{H}_Q^{(k)} := \{(x, v) \in \mathbb{R}^n_x \times \mathbb{R}^s_v; x \in Q, v \in H^{(k)}_x\}
$$

is semialgebraic for all $k \geq 0$. In fact,

$$
\mathcal{H}_Q^{(0)} = \{(x, v) \in Q \times \mathbb{R}^s; A(x)v = \gamma(x)\}
$$

is semialgebraic and after supposing that $\mathcal{H}_Q^{(k-1)}$ is semialgebraic ($k \geq 1$), $\mathcal{H}_Q^{(k)}$ can be rewritten as

$$
\{(x, v) \in \mathcal{H}_Q^{(k-1)}; \forall \varepsilon > 0, \exists \delta > 0, \\
\forall (y, v') \in (B(x, \delta) \times \mathbb{R}^s) \cap \mathcal{H}_Q^{(k-1)} : \|v - v'\| < \varepsilon\},
$$

that is semialgebraic by elimination of quantifiers. Now, $\omega$ has graph given by

$$
\{(x, v) \in \mathcal{H}_Q^{Gl}; \exists (x', v') \in \mathcal{H}_Q^{Gl}, x' = x, \|v\| > \|v'\|\},
$$

with $\|\cdot\|$ the euclidean norm on $\mathbb{R}^s$ and, hence, it is semialgebraic by elimination of quantifiers.

In a similar way we have that if $(e_1, ..., e_s)$ is the canonical basis of $\mathbb{R}^s$ then $y \mapsto \tilde{P}_1(y)e_j$ is semialgebraic for all $j$ since its graph can be written as

$$
\{(y, v) \in Q \times \mathbb{R}^s; \exists (x', v') \in (\mathcal{H}_Q^{Gl} - \omega), x' = y, \langle v', v \rangle = 0, v + v' = e_j\},
$$

which is semialgebraic by elimination of quantifiers because

$$
(\mathcal{H}_Q^{Gl} - \omega)_Q := \{(x, v) \in Q \times \mathbb{R}^s; \exists (x', v') \in \mathcal{H}_Q^{Gl} - \omega, x' = x, v = v' - \omega(x)\}
$$

is semialgebraic again by elimination of quantifiers.

Now, we are ready to begin the proof by induction.

We start with the case $d = 1$. For every given $x \in Q$ there exists $v_x \in$
$H^\text{Gl}_x \neq \emptyset$ and a ball $B(x, r_{v_x}) \subseteq \mathbb{R}^n$ such that

$$Q \cap B(x, r_{v_x}) \ni y \mapsto \tilde{\gamma}_{v_x}(y) := \Pi_{H_x^\text{Gl}} y_{v_x} = \omega(y) + v_x - \tilde{\Pi}_1 y_{v_x}$$

is semialgebraic since $\omega$ and $\tilde{\Pi}_1$ are semialgebraic. We show that $\tilde{\gamma}_{v_x}$ is continuous for $r_{v_x}$ small enough. In fact, if we suppose by contradiction that there is no $r_{v_x}$ such that $\tilde{\gamma}_{v_x}$ is continuous then for all $n \in \mathbb{N}$ there is $y_n \in B(x, r_{v_x})$ such that $\tilde{\gamma}_{v_x}$ is discontinuous at $y_n$. Hence, there are two possibilities:

1. $\forall n \in \mathbb{N}, y_n \neq x$. A semialgebraic function is real analytic on the complementary of a semialgebraic set of dimension strictly less than the one of its domain. (In fact, the domain of a semialgebraic function is semialgebraic by the Tarski-Seidenberg theorem.) Thus, since $\gamma_{v_x}$ is semialgebraic on $Q$ the discontinuity points of $\gamma_{v_x}$ are finitely many. Hence, we come to a contradiction;

2. $\exists \pi \in \mathbb{N}$ such that $y_{\pi} = x$. Now, since $v_x \in H^\text{Gl}_x$ on the one hand

$$\text{dist}(v_x, H^\text{Gl}_y) \underset{y \to x}{\to} 0,$$

and, on the other,

$$\left\| \Pi_{H_x^\text{Gl}} y_{v_x} - v_x \right\| = \text{dist}(v_x, H^\text{Gl}_y).$$

Therefore

$$\Pi_{H_x^\text{Gl}} y_{v_x} \underset{y \to x}{\to} v_x = \Pi_{H_x^\text{Gl}} v_x.$$

This is impossible since $\gamma_{v_x}$ would be continuous at $x$, contrary to the assumption. Thus $\tilde{\gamma}_{v_x}$ is continuous upon possibly reducing the ball radius $r_{v_x}$.

Now we glue these local solutions thanks to a semialgebraic continuous partition of the unity. In fact, we notice that the set of balls $\{B(x, r_{v_x})\}_{x \in Q}$, where $v_x$ is chosen in $H^\text{Gl}_x$, is an open cover of the compact space $Q$. Then
there is \( N \) such that \( \{ B(x_i, \overline{v}_{x_i}) \}_{i=1,...,N} \) is an open cover of \( Q \). Consider

\[
\mu(x,r)(y) := \begin{cases} \sqrt{r^2 - \| y - x \|^2} & \text{for } y \in B(x,r), \\ 0 & \text{for } y \notin B(x,r). \end{cases}
\]  

Notice that \( \mu(x,r)(y) \) is semialgebraic and continuous on \( Q \), \( \forall x \in Q, \forall r \in \mathbb{R}^+ \) and that \( \sum_{i=1}^{N} \mu(x_i, \overline{v}_{x_i})(y) > 0 \) for each \( y \in Q \) as \( \mu(x,r)(y) \geq 0 \) for every \( y \in Q \) and \( \mu(x,r)(y) > 0 \) for all \( y \in B(x,r) \). Moreover, for all \( y \in Q \) there is \( B(x_i, \overline{v}_{x_i}) \) as above such that \( y \in B(x_i, \overline{v}_{x_i}) \) since \( \{ B(x_i, \overline{v}_{x_i}) \}_{i=1,...,N} \) is an open covering of \( Q \). Hence the function

\[
Q \ni y \mapsto \phi(y) := \frac{1}{\sum_{i=1}^{N} \mu(x_i, \overline{v}_{x_i})(y)} \prod_{j=1}^{N} \mu(x_j, \overline{v}_{x_j})(y) \prod \Pi_{H^G_{\overline{v}_{x_j}}}
\]

is a semialgebraic and continuous solution of the system on \( Q \). (We also notice that \( \phi(y) \in H^G_y \) for all \( y \in Q \).)

Next, we suppose that \( Q \) is a semialgebraic subset of dimension \( \tilde{d} \leq n \) and that we can write a semialgebraic and continuous section of \( H^G \) on any compact semialgebraic subset of \( Q \) of dimension \( d \leq \tilde{d} - 1 \) of \( Q \). We want to construct a semialgebraic and continuous section of \( H^G \) on \( Q \). We will call \( U \) the subset of \( Q \) where \( \omega \) or \( \tilde{H}_1 \) is not continuous. Since we proved that \( \omega \) and \( y \mapsto \tilde{H}_1(y)e_j \) are semialgebraic (for all \( j \)), \( U \) is a semialgebraic set of dimension \( \leq \tilde{d} - 1 \). (A zero-dimensional semialgebraic subset of \( \mathbb{R}^n \) is finite. A one-dimensional semialgebraic subset of \( \mathbb{R}^n \) is a union of finitely many real-analytic arcs and finitely many points. See Chapter 2 of [7].)

Thus, by inductive hypothesis there is a semialgebraic and continuous section \( S \) of \( H^G \) on \( \overline{U} \). In fact, \( \overline{U} \) is a compact semialgebraic subset of \( Q \) of dimension \( \tilde{d} - 1 \). Now, \( S \) can be extended to a semialgebraic and continuous function on \( Q \) by Proposition 2.6.9 at p. 45 of [7] which is a semialgebraic version of Tietze-Uryshon Theorem and we will call \( S \) that extension again. Actually, \( S \) is defined on \( Q \), but it is a section of \( H^G \) only on \( \overline{U} \). Hence, we
compute the projection of $S$ on the fibers of $\mathcal{H}^{Gl}$ i.e.

$$Q \ni y \mapsto \sigma(y) := \Pi_{H^y} S(y) = \omega(y) + S(y) - \tilde{\Pi}_1(y) S(y).$$

We notice that $\sigma$ is semialgebraic and also that $\sigma$ is continuous on $Q \setminus \overline{U}$ since $\omega$ and $y \mapsto \tilde{\Pi}_1(y) S(y)$ is continuous on $Q \setminus \overline{U}$. Moreover, $\sigma$ is continuous on $\overline{U}$ since $S(x) \in H^x_{Gl}$ for all $x \in \overline{U}$ and, hence, we can proceed as done to prove that $x$ is not a discontinuity point for $\tilde{\gamma}_{ux}$ in the case $d = 1$. In fact, for all $x \in \overline{U}$ and all $y \in Q$

$$\left\| \Pi_{H^y} S(y) - \Pi_{H^x} S(x) \right\| = S(x) \in H^x_{Gl} \leq \left\| \Pi_{H^y} S(y) - S(y) \right\| + \left\| S(y) - S(x) \right\|$$

where the second inequality follows from the minimal distance property of the projection (we notice that $\Pi_{H^y} S(x) \in H^y_{Gl}$). Now, $\left\| S(y) - S(x) \right\| \xrightarrow{y \rightarrow x} 0$ by the continuity of $S$ and $\left\| \Pi_{H^y} S(x) - S(x) \right\| = \text{dist}(S(x), H^y_{Gl}) \xrightarrow{y \rightarrow x} 0$ since $S(x) \in H^x_{Gl}$.

The proof is complete.

**Remark.** Since the absence of empty fiber of $\mathcal{H}^{Gl}$ is equivalent to the existence of a continuous section of $\mathcal{H}^{Gl}$ (see [1]) and, hence, of $\mathcal{H}$, we have just proved that system [4] on a semialgebraic compact space $Q \subseteq \mathbb{R}^n$ has a semialgebraic and continuous solution if and only if it has a continuous one.

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