Free Resolutions and Generalized Hamming Weights of binary linear codes

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Abstract. In this work, we explore the relationship between free resolution of some monomial ideals and Generalized Hamming Weights (GHWs) of binary codes. More precisely, we look for a structure smaller than the set of codewords of minimal support that provides us some information about the GHWs. We prove that the first and second generalized Hamming weight of a binary linear code can be computed (by means of a graded free resolution) from a set of monomials associated to a binomial ideal related with the code. Moreover, the remaining weights are bounded by the Betti numbers for that set.

Keywords: Generalized Hamming Weight · Graded free resolution · Second distance · Binary code.

1 Introduction

The study of the Generalized Hamming Weights (GHWs) has been motivated by several applications in cryptography [32] and they characterize the performance of a linear code when it is used for a given channel. There are few families of codes for which it is known the complete generalized weight hierarchy as for example: first-order Reed-Muller codes, binary Reed-Muller codes, Hamming code and its dual, Extended Hamming codes, Golay code, see [32]. On the other side, there has been an extensive research on GHWs and the second distance in particular for some classes of codes, see for example [S1012414161625283033] and the references therein. However, for the general case of a linear code, few properties are known.

In their seminal paper [17], Johnsen and Verdú showed how the GHWs of a linear code could be computed from a free resolution of a monomial ideal
associated to the set of codewords of minimal support of the code provided we
know this last set. That paper has produced a great avenue of research, see for example [9,11,18,20,21,22,23,24].

In the present work, we explore if one can find a set (smaller than the set
of all codewords of minimal support) that provides us some information on the
GHWs in the case of binary codes. The selected set of codewords is the so called
Gröbner test set related to the binomial ideal associated to a code defined in [5]. In
that paper it was proven that one can decode using that set and that the
minimal distance of the code (thus the first GHW) was reflected on it. Thus,
somehow some of the relevant information of the code lies in it. Moreover, in
[5] it was also shown how to compute the Gröbner test set avoiding some of the
most common disadvantages when one uses Gröbner basis. In this paper we will
show how one can compute also the second GHW of a binary linear code from
that binomial ideal associated with the code without the need of computing the
complete set of codewords of minimal support of the code as in [17]. Moreover,
in Theorem 3 we bound the remaining GHWs with the resolution of a monomial
ideal associated to this new set.

The structure of the contribution is simple, in Section 2 we revise some
results on GHW, free resolutions and the ideal associated with a code. Section 3
shows briefly the false conjecture and experiments that drive us to this study. In
Section 4 we show our main result and finally in Section 5 we show some future
lines of research and some conjectures related to the topic that we hope will be
helpful for future research.

2 GHW and minimal supports

Let \( \mathbb{F}_q \) be a finite field with \( q \) elements. Given two vectors \( \mathbf{x} = (x_1, \ldots, x_n) \) and
\( \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{F}_q^n \), the Hamming distance between \( \mathbf{x} \) and \( \mathbf{y} \) is defined as
\[
d_H(\mathbf{x}, \mathbf{y}) = |\{i \mid x_i \neq y_i\}|,
\]
where \( | \cdot | \) denotes the cardinality of the set. The Hamming weight of \( \mathbf{x} \) is given by \( w_H(\mathbf{x}) = d_H(\mathbf{x}, \mathbf{0}) \), where \( \mathbf{0} \) denotes the zero vector in \( \mathbb{F}_q^n \). The support of \( \mathbf{x} \) is
the set \( \text{supp}(\mathbf{x}) = \{i \mid x_i \neq 0\} \). A linear subspace \( \mathcal{C} \) in \( \mathbb{F}_q^n \) is called a linear code. The
elements of \( \mathcal{C} \) are called codewords. The basic parameters of \( \mathcal{C} \) are its length, its dimension and its minimum distance, which are denoted by \( n(\mathcal{C}) \), \( k(\mathcal{C}) \) and \( \delta(\mathcal{C}) \), respectively. In this case, we call \( \mathcal{C} \) an \( [n(\mathcal{C}), k(\mathcal{C}), \delta(\mathcal{C})]_q \) linear code. We
define a generator matrix of \( \mathcal{C} \) to be a matrix \( G \) over \( \mathbb{F}_q \) of size \( k(\mathcal{C}) \times n(\mathcal{C}) \) whose
row vectors span \( \mathcal{C} \), while a parity check matrix of \( \mathcal{C} \) is a matrix \( H \) over \( \mathbb{F}_q \) of size \( (n(\mathcal{C})- k(\mathcal{C})) \times n(\mathcal{C}) \) whose
null space is \( \mathcal{C} \).

**Definition 1.** Let \( \mathcal{C} \) be a linear code, we will say that a codeword \( \mathbf{m} \) has minimal
support if it is non-zero and \( \text{supp}(\mathbf{m}) \) is not contained in the support of any other
codewords. We will denote by \( \mathcal{M}_C \) the set of codewords of minimal support of \( \mathcal{C} \).

Note that computing a set of codewords of minimal support is a hard problem
for a general linear code, in fact it implies finding the minimum distance of the
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It is clear that if \( D \) is a one-dimensional subcode then, the support of \( D \) is equal to the Hamming weight of any of its nonzero codewords, i.e. \( d_1(C) = \delta(C) \).

Based on this idea, the \( h \)-th generalized Hamming weight of \( C \), denoted \( d_h(C) \), is the size of the smallest support of an \( h \)-dimensional subcode of \( C \) with \( h = 1, 2, \ldots, k(C) \). That is, if \( D_h \) is the set of all linear subspaces of the linear code \( C \) of dimension \( h \), then

\[
d_h(C) = \min \{ |\text{supp}(E)| \mid E \in D_h \}.
\]

Some basic facts on the Generalized Hamming Weights (GHW) are provided in the following proposition.

**Proposition 1** ([32]). Let \( C \) be a linear code. Then:

1. \( 1 \leq d_1(C) < d_2(C) < \ldots < d_{k(C)}(C) \leq n(C) \)
2. (Generalized Singleton Bound) \( d_h(C) \leq n(C) - k(C) + h \).

From now on we assume that \( C \) is a nondegenerate code, that is, \( d_{k(C)}(C) = n(C) \).

The GHWs \( d_1(C), \ldots, d_{k(C)}(C) \) are completely determined by the underlying linear matroid structure of the code in a non-trivial manner, and the way to obtain them is not efficient as it was already pointed in [17]. Of course, as pointed before, calculating \( d_1(C) \) is equivalent to the problem of complete decoding linear codes [2], hence one can not expect a computational efficient approach. From now on, given a positive integer \( \ell \), we define \( [\ell] = \{1, \ldots, \ell\} \) and \( [\ell]_0 = \{0, \ldots, \ell\} \).

**Definition 3.** Let \( C \) be a \([n, k]_q\) code and let \( H \) be a parity check matrix of \( C \). Let \( H_i \) denote the \( i \)-th column of \( H \) and define the simplicial complex

\[
\Delta = \left\{ \sigma \in 2^{[n]} \mid \{H_i \mid i \in \sigma\} \text{ is linearly independent over } \mathbb{F}_q \right\}.
\]

Then, the pair \( \mathcal{M} = ([n], \Delta) \) is the linear matroid associated with the code \( C \).

The collection \( \Delta \) of subsets of \([n]\) are called independent sets of this matroid. A subset of \([n]\) that does not belong to \( \Delta \) is called dependent set. Minimal dependent subsets of \([n]\) are known as circuits of \( \mathcal{M} \). A set is said to be a cycle if it is a disjoint union of circuits.

We refer to [27] for a brief introduction on the theory of simplicial complexes, and to [29] for a thorough study of matroids.
Definition 4. Let $K$ be any field. We denote by $I_{\Delta}$ the ideal in the polynomial ring $R = K[X_1, \ldots, X_n]$ over a field $K$ generated by all square-free monomials supported on elements that are not in $\Delta$, i.e.

$$\prod_{i \in \tau} X_i \text{ with } \tau \in 2^n \setminus \Delta$$

That is, $I_{\Delta}$ is the ideal generated by monomials supported on the circuits of $\mathcal{M}$, or equivalently in a matroid associated with a code, supported on codewords of minimal support of $\mathcal{C}$.

$$I_{\Delta} = \left\langle \prod_{i \in \text{supp}(c)} X_i \mid c \in \mathcal{M}_C \right\rangle. \quad (1)$$

The quotient $R_{\Delta} = R/I_{\Delta}$ is called the Stanley-Reisner ring associated with $\Delta$. $R_{\Delta}$ is a finitely generated standard graded $K$-algebra of dimension $n(C) - k(C)$. Thus, it has a minimal graded free resolution. Moreover, since the generators of $I_{\Delta}$ are supported in the set of circuits of a matroid, by [4], one has that $\Delta$ is shellable and this implies that $R_{\Delta}$ is Cohen-Macaulay. So, by the Auslander-Buchsbaum formula, the projective dimension of $R_{\Delta}$ (i.e., the length of any minimal graded free resolution of $R_{\Delta}$) is $k(C)$ and it looks like

$$0 \rightarrow F_{k(C)} \rightarrow F_{k(C) - 1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow R_{\Delta} \rightarrow 0 \quad (2)$$

where $F_0 = R$ and each $F_i$ is a graded free $R$-module of the form

$$F_i = \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{i,j}} \text{ for } i \in [k(C)]_0.$$

We will refer to Equation (2) as a graded minimal free resolution of $\mathcal{C}$. The nonnegative integers $\beta_{i,j}$ are called Betti numbers of $\mathcal{C}$ and they depend only on $\mathcal{C}$ and not on the choice of the parity check matrix $H$, or the minimal free resolution of $R_{\Delta}$ or even the chosen field $K$ (this is because $I_{\Delta}$ comes from the set of circuits of a matroid, see [4]).

In [17], Johnsen and Verdure described the GHWs of a linear code $\mathcal{C}$ in terms of the shifts of the minimal graded free resolution of a Stanley-Reisner ideal $I_{\Delta}$ associated to $\mathcal{C}$. One of the main disadvantages of this method is that the generators of $I_{\Delta}$ correspond to the supports of all minimal support of $\mathcal{C}$. In general, the whole set of codewords of minimal support can be huge and computationally expensive to obtain and, in many cases, computing a minimal graded free resolution of an ideal with that many generators is unaffordable. More precisely, their result was:

Theorem 1. [17] Theorem 2] Let $\mathcal{C}$ be a $q$-ary linear code. Then,

$$d_i(\mathcal{C}) = \min\{ j \mid \beta_{i,j} \neq 0 \} \text{ for } j \in [k(C)]$$
Example 1 (Toy example). Let $C$ be the binary non-degenerate $[6, 3]$ code with generator matrix

$$ G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \in \mathbb{F}_2^{3 \times 6} $$

One can check that the set of codewords of minimal support is

$$ M_C = \{ w_1 = 100001, w_2 = 100110, w_3 = 011010, w_4 = 000111, w_5 = 111100, w_6 = 011101 \} $$

Its Stanley-Reisner ring is $R_{\Delta} = R/I_{\Delta}$ where $R = \mathbb{F}_2[x_1, \ldots, x_6]$ and the ideal $I_{\Delta}$ is generated by the monomials associated to $M_C$. If we compute a graded minimal free resolution of $R_{\Delta}$ we get the following Betti diagram:

|   | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 |
| 2 | 0 | 3 | 2 | 0 |
| 3 | 0 | 2 | 7 | 4 |

where the entry of the row indexed by $i$ and column indexed by $j$ indicates the value $\beta_{i,j}$ (for example $\beta_{3,5} = 7$). Hence, by Theorem 1, we have $d_1(C) = 2$, $d_2(C) = 4$ and $d_3(C) = 6$.

This result shows that the determination of the Betti numbers of the monomial ideal $I_{\Delta}$ related to a code completely determine the weight hierarchy. However, as mentioned before, this is usually a hard problem except in some special cases. For example, Johnsen and Verdure in [18] explicitly determine the Betti Numbers for MDS codes, since the minimal free resolution of these codes is linear. Moreover, in [18] the authors prove that the resolution of the first order Reed-Muller code is pure. And a similar result can be deduced for constant weight codes [11]. Thus, simplex codes or dual Hamming codes, which are constant weight codes also have pure resolution, although not necessarily linear.

Remark 1. The resolution (2) is said to be pure of type $(d_0, \ldots, d_k(C))$ if for each $i \in [k]_D$, the Betti number $\beta_{i,j}$ is nonzero if and only if $j = d_i$. If, in addition $d_1, \ldots, d_k$ are consecutive, then the resolution is said to be linear.

The rest of this work is devoted to compute a simpler and smaller structure than the whole set of codewords of minimal support that allows, at least partially, know something about the GHW of the code.

2.1 Test-Sets of a binary code

From now on, we will restrict our study to binary codes. Let $C$ be a binary linear code. Let $X$ be a vector with $n = n(C)$ variables $x_1, \ldots, x_n$. A monomial in $X$ is a product of the form $X^a := \prod_{i=1}^n x_i^{a_i}$, where $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$. The
total degree of $X^a$ is $\deg(X^a) = \sum_{i=1}^n a_i$. When $a = \mathbf{0}$, note that $X^a = 1$. The polynomial ring $K[X]$ is the set of all polynomials in $X$ with coefficients in $K$, where $K$ denotes an arbitrary finite field.

**Remark 2.** For abuse of notation we will write $X^a$ with $a \in F_2^n$. In this case, we understand that the classes of 0, 1 are replaced by the same symbols regarded as integers. Moreover, we will use the notation $X^I$ for the square free monomial with support $I \subseteq [n]$, that is,

$$X^I = \prod_{i \in I} x_i \text{ with } I \subseteq [n].$$

Let $g = X^A - X^B$ be a binomial with $A, B \subseteq [n]$, we define supp$(g) = A \cup B$. We say that $g$ is in *standard form* if $A \cap B = \emptyset$ or, equivalently, if supp$(g) = A \cup B$,

where $\cup$ denotes the disjoint union of $A$ and $B$.

**Definition 5.** Let $K$ be any field, we define the ideal associated with $C$ over $K$ as the binomial ideal:

$$I(C) = \langle X^a - X^b \mid a, b \in F_2^n, a + b \in C \rangle + \langle x_i^2 - 1 \mid i \in [n] \rangle \subseteq K[X].$$

(3)

Note that $I(C)$ is a zero-dimensional ideal since the quotient ring $R = K[X]/I(C)$ is a finite-dimensional vector space (i.e. $\dim(R) < \infty$). Moreover, its dimension is equal to the number of cosets in $F_2^n/C$. For $a, b \in F_2^n$ one has that $X^a - X^b \in I(C)$ if and only if $a - b \in C$. The following result shows how to obtain a set of generators of the ideal $I(C)$ from a generator matrix of $C$.

**Proposition 2.** [3, Theorem 1] Let $\{w_1, \ldots, w_k\}$ be the row vectors of a generator matrix for $C$. Then

$$I(C) = \left\langle \{X^{w_i} - 1\}_{i \in [k]} \cup \{x_i^2 - 1\}_{i \in [n]} \right\rangle$$

If we fix a term order $\prec$, then the leading term of a polynomial $f$ with respect to $\prec$, denoted by $\text{LT}_\prec(f)$, is the largest monomial among all monomials which occur with non-zero coefficient in the expansion of $f$. Let $I$ be an ideal in $K[X]$, then the initial ideal $\text{in}_\prec(I)$ is the monomial ideal generated by the leading term of all the polynomials in $I$, i.e., $\text{in}_\prec(I) = \langle \{\text{LT}_\prec(f) \mid f \in I\} \rangle$. By definition, $\text{in}_\prec(I)$ is a monomial ideal and, thus, it has a unique minimal generating set formed by monomials. These monomials will be called *minimal generators of $\text{in}_\prec(I)$*.

**Definition 6.** A finite set of nonzero polynomials $\mathcal{G} = \{g_1, \ldots, g_m\}$ of the ideal $I$ is a Gröbner basis of $I$ with respect to the term order $\prec$ if the leading terms of the elements of $\mathcal{G}$ generate the initial ideal $\text{in}_\prec(I)$. Moreover $\mathcal{G}$ is reduced if

1. $g_i$ is monic (i.e., its leading coefficient is 1) for all $i \in [m]$, and
2. none of the monomials appearing in the expansion of $g_j$ is divisible by $\text{LT}_\prec(g_i)$ for all $i \neq j$. 

For a given monomial order \( \prec \), every ideal has a unique reduced Gröbner basis (see, e.g., [1]). Since \( I(C) \) is generated by binomials (differences of monomials), then all its reduced Gröbner bases consist of binomials (see [7]). In [5] it is shown that, if \( C \) is a binary code and we fix a degree compatible term order \( \prec \) on \( K[X] \), then the reduced Gröbner basis \( G_\prec \) for the code ideal \( I(C) \) can be computed by a linear algebra (an FGLM-like) algorithm. Moreover, the reduction provided by \( G_\prec \) gives a decoding procedure. Along the way, they also prove that the support of every binomial in \( G_\prec \) different from \( x_{2i}^2 - 1 \) for \( i = 1, \ldots, n \) provides a codeword of minimal support of \( C \), and that there is a word of Hamming weight \( d_1(C) \) that can be obtained in this way. More precisely:

**Proposition 3.** [5] Let \( G_\prec \) be the reduced Gröbner basis of \( I(C) \) with respect to a degree compatible term order \( \prec \). For every binomial \( X^a - X^b \in G_\prec \setminus \{x_{2i}^2 - 1 \mid i \in [n] \} \), then \( a + b \in F_n^2 \) is a codeword of minimal support of \( C \). Moreover, there exists \( X^a - X^b \in G_\prec \) such that \( w_H(a + b) = d_1(C) \).

This result motivates the definition of a \( G_\prec \)-test, which by the above proposition is a subset of the set of codewords of minimal support and contains a word of minimum weight.

**Definition 7.** Given a binary code \( C \) and a degree compatible term order \( \prec \), we will call \( G_\prec \)-test set of \( C \) to the subset of codewords of minimal support of \( C \) whose supports are given by those binomials in the reduced Gröbner basis of \( I(C) \) different from \( x_{2i}^2 - 1 \) for all \( i \in [n] \).

**Example 2.** We continue the toy Example [1]. Then, the ideal associated with \( C \) over \( R \) is defined as

\[
I(C) = \left\{ x_1x_6 - 1, x_2x_3x_5 - 1, x_4x_5x_6 - 1 \right\} \cup \{x_{2i}^2 - 1 \mid i \in [6]\} \subseteq R
\]

Now we consider the degree reverse lexicographic order \( \prec \) with \( x_6 \prec \ldots \prec x_1 \). Then, the reduced Gröbner basis \( G_\prec \) of \( I(C) \) with respect to \( \prec \) has 14 elements. An \( G_\prec \)-test-set of \( C \) is given by codewords of \( M_C \) whose supports are given by those binomials of \( G_\prec \) different from \( x_{2i}^2 - 1 \), i.e. the set

\[
\{ w_1 = 100001, w_3 = 011010, w_4 = 000111, w_6 = 011101 \} \subseteq M_C.
\]

### 3 On a false conjecture

Note that the second and the third author of this work have conjectured for a long time the following:

**Conjecture (false)** If one considers the monomial ideal \( M \) associated with the supports of the binomials in the \( G_\prec \)-test set, then \( d_i = \min \{j \mid \beta_{i,j}(R/M) \neq 0 \} \) for \( i \in \{1, \ldots, k(C)\} \), that is, the \( G_\prec \)-test set determines the GHWs of the codes.
This conjecture was supported by computational evidence (see Examples 3 and 4 for some example satisfying this conjecture), but unfortunately this is not true as shown in the counterexample 5. Theorem 3 in Section 4 in this paper will prove this fact for $i = 2$ (and it was known for $i = 1$ [5]). Note also that in [26], the authors show that from the Graver basis associated to $I(C)$ one can purge the set of codewords of minimal support of $C$, i.e. $M_C$.

**Example 3.** We continue the Example 1 where we have computed the Betti diagram associated to $R_{\Delta} = R/I_{\Delta}$ where $I_{\Delta}$ is generated by the monomials associated to $M_C$. Then, in Example 2 we compute a $G_{\prec}$-test-set $T_C$ of $C$ with respect to the degree reverse lexicographic order $\prec$. Note that $T_C \subset M_C$ with just 4 elements. If we consider $M$ the corresponding monomial ideal related to $T_C$ and compute a graded minimal free resolution of $R/M$ we get the following Betti diagram:

| $i$ | $j$ | $\beta_{i,j}$ |
|-----|-----|--------------|
| 0   | 1   | 2            |
| 0   | 0   | 0            |
| 1   | 0   | 0            |
| 2   | 2   | 1            |
| 3   | 4   | 4            |
| 4   | 4   | 2            |

The Betti numbers of $R/M$ are smaller than those of $R_{\Delta}$ and in this example the sequence

$$\left(\min\{j \mid \beta_{i,j} \neq 0\}; 1 \leq i \leq 3\right) = (2, 4, 6),$$

coincides with the GHW of $C$.

The following example is a less trivial case where the difference between our structure $T_C$ and the set of codewords of minimal support of $C$, $M_C$, is larger.

**Example 4.** Let $C$ be the binary non-degenerate $[14, 9]$-code with generator matrix

$$G = \begin{pmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \in \mathbb{F}_2^{9 \times 14}$$

Its Stanley-Reisner ring is $R_{\Delta} = R/I_{\Delta}$ where $R = \mathbb{F}_2[x_1, \ldots, x_{14}]$ and $I_{\Delta}$ is minimally generated by 147 monomials. If we compute a graded minimal free resolution of $R_{\Delta}$ we get the following Betti diagram:
Hence, by Theorem 1, we have that the GHW are 
$$(d_1(\mathcal{C}), \ldots, d_9(\mathcal{C})) = (2, 4, 6, 7, 9, 10, 12, 13, 14).$$

Moreover, if we consider $\prec$ the graded degree lexicographic order with $x_1 \succ \cdots \succ x_{14}$, we get that the $G_{\prec}$-test set has 24 elements. If we consider $M$ the corresponding monomial ideal and compute a graded minimal free resolution of $R/M$ we get the following Betti diagram:

| 0 1 2 3 4 5 6 7 8 9 |
|----------------------|
| 0 1 0 0 0 0 0 0 0 0 |
| 1 0 2 0 0 0 0 0 0 0 |
| 2 0 8 5 0 0 0 0 0 0 |
| 3 0 34 82 8 0 0 0 0 0 |
| 4 2 52 441 897 753 289 42 0 0 0 |
| 5 0 51 1345 7410 18309 25248 21008 10579 2990 366 |

As one can observe, the Betti numbers of $R/M$ are smaller than those of $R_{\Delta}$. Moreover, in this example the sequence $(\min\{j \mid \beta_{i,j} \neq 0\}; 1 \leq i \leq 9)$ is given by $(2, 4, 6, 7, 9, 10, 12, 13, 14)$ and it coincides with the GHW of $\mathcal{C}$.

**Example 5.** Let $\mathcal{C}$ be the binary non-degenerate $[10, 7]$-code with generator matrix

$$G = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix} \in \mathbb{F}_2^{7 \times 10}$$

Its Stanley-Reisner ring is $R_{\Delta} = R/I_{\Delta}$ where $R = \mathbb{F}_2[x_1, \ldots, x_{10}]$ and $I_{\Delta}$ is minimally generated by 42 monomials. If we compute a graded minimal free resolution of $R_{\Delta}$ we get the following Betti diagram:

| 0 1 2 3 4 5 6 7 |
|-------------------|
| 0 1 0 0 0 0 0 0 |
| 1 0 4 0 0 0 0 0 |
| 2 0 18 48 32 7 0 0 0 |
| 3 0 20 214 637 874 637 242 38 |
Hence, by Theorem 1, we have $d_1(C) = 2$, $d_2(C) = 4$, $d_3(C) = 5$, $d_4(C) = 6$, $d_5(C) = 8$, $d_6(C) = 9$, $d_7(C) = 10$.

If we compute a $G_\prec$-test set $T$ with respect to the degree reverse lexicographical order, we get that $T$ has only 10 elements. Computing a minimal graded free resolution of $R/M$ being the ideal $M := \langle \{X^c \mid c \in T\} \rangle$, we get the following Betti diagram.

$$
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 4 & 0 & 0 & 0 & 0 \\
2 & 0 & 4 & 14 & 5 & 0 & 0 & 0 \\
3 & 0 & 2 & 23 & 56 & 48 & 17 & 2
\end{array}
$$

Thus, we can recover the correct values of $d_1(C)$, $d_2(C)$, $d_3(C)$ from this resolution but not $d_i(C)$ for $i = 4, 5, 6, 7$.

Anyway, the experiments we dealt trying to prove that conjecture drove us the results in Section 4 and to propose some further conjectures and future work based on experimental evidences showed in Section 5.

### 4 Second GHW obtained from a $G_\prec$-test set

In this section we will explain how to compute the second generalized Hamming weight from a $G_\prec$-test set. Throughout this section, $C$ will denote a binary linear code and let $G_\prec$ be the reduced Gröbner basis of the ideal $I(C)$ with respect to $\prec$, where we take $\prec$ to be any degree compatible ordering on $K[X]$. The following result is a technical lemma whose proof is just an easy exercise in set theory. We will denote by $A \triangle B$ the symmetric difference of the subsets $A, B \subseteq [n]$, which is the set of elements which are in the union of the two sets $A \cup B$, minus their intersection $A \cap B$. Moreover, for a set $A$, its cardinality is denoted by $|A|$.

**Lemma 1.** Let $A, B \subseteq X$ with $|A \cap B| > \frac{|A|}{2}$. Then $C = A \triangle B$ satisfies the following statements:

1. $A \cup B = A \cup C$
2. $|A \cap C| < \frac{|A|}{2}$
3. $|C| < |B|$.

Consider the set $M$ of codewords in $C$ belonging to a linear subspace of dimension two of minimal support, more precisely,

$$
M = \{m \in C \mid \exists m' \in C \text{ such that } d_2(C) = \text{supp}(\langle m, m' \rangle)\},
$$

and define $m_1, m_2 \in M$ as follows:

(a) $m_1 := \min_\prec(M)$, i.e., $m_1$ is the smallest codeword with respect to $\prec$ in $M$,
(b) $m_2 := \min_\prec\{m \in M \mid d_2(C) = \text{supp}(\langle m_1, m \rangle)\}$, i.e., $m_2$ is the smallest codeword with respect to $\prec$ such that $d_2(C) = \text{supp}(\langle m_1, m_2 \rangle)$.

With these conditions we define:

$I = \text{supp}(m_1)$ and $J = \text{supp}(m_2)$. 

Remark 3. Since \( < \) is degree compatible and, by Lemma \( \mathbf{1} \) we have that
\[
|I \cap J| \leq \frac{|I|}{2} \leq \frac{|J|}{2}.
\]

Proposition 4. There exists a binomial \( f \in G_{<} \subseteq K[X] \) such that \( \text{supp}(f) = I \).

Proof. Consider \( f = X^{I_1} - X^{I_2} \in I(C) \) any binomial with \( I = I_1 \cup I_2 \), \( |I_1| - 1 \leq |I_2| \leq |I_1| \) and \( X^{I_1} \succ X^{I_2} \), we will show that \( f \in G_{<} \). For proving this, it suffices to check that:

(a) \( X^{I_1} = \text{LT}_{<}(f) \) is a minimal generator of \( \text{in}_{<}(I(C)) \), and
(b) \( X^{I_2} \notin \text{in}_{<}(I(C)) \).

Proof of (a). By contradiction suppose that there exists a binomial \( h = X^{K_1} - X^{K_2} \in G_{<} \) with \( K = K_1 \cup K_2 \), \( \text{LT}_{<}(h) = X^{K_1} \) (and, in particular, \( |K_1| \geq |K_2| \)) such that \( K_1 \nsubseteq I_1 \) and \( X^{K_1} \) is divisible by \( X^{K_2} \), i.e. \( K_1 \nsubseteq I_1 \).

Claim: \( d_2(C) \leq |I \cup K| - 1 \).

Proof of the claim: We have that \( |K_2| \leq |K_1| \leq |I_1| - 1 \leq |I_2| \). Then \( |K| < |I| \) and, in particular, \( X^K \prec X^I \prec X^J \). Hence, by the choice of \( m_2 \), we have that \( |I \cup J| < |I \cup K| \) and the Claim follows.

By the previous Claim we have that
\[
|I \cup J| = d_2(C) \leq |I \cup K| - 1
\]
\[
= |I \cup K_2| - 1 \quad \text{(since } K_1 \subseteq I \text{)}
\]
\[
= |I| + |K_2| - |I \cap K_2| - 1
\]
\[
\leq |I| + |K_1| - |I \cap K_2| - 1 \quad \text{(since } X^{K_1} \succ X^{K_2} \text{)}.
\]

Thus,
\[
|I| + |J| - |I \cap J| = |I \cup J| \leq |I| + |K_1| - |I \cap K_2| - 1
\]
or equivalently \( |J| - |I \cap J| \leq |K_1| - |I \cap K_2| - 1 \). Thus,
\[
|I_1| - \frac{1}{2} \leq \frac{|I|}{2} \leq \frac{|J|}{2} \leq |J| - |I \cap J| \quad \text{(by Remark } \mathbf{3} \text{)}
\]
\[
\leq |K_1| - |I \cap K_2| - 1 \leq |K_1| - 1.
\]

Therefore, \( |I_1| \leq |K_1| - \frac{1}{2} \) and \( K_1 \subseteq I_1 \), a contradiction.

Proof of (b). By contradiction, we assume that there exists \( h = X^{K_1} - X^{K_2} \in G_{<} \) such that \( \text{LT}_{<}(h) = X^{K_1} \) divides \( X^{I_2} \) or, equivalently, \( K_1 \subseteq I_2 \). Then \( K_2 \prec K_1 \prec I_2 \prec I_1 \) and, in particular, \( X^K \prec X^I \prec X^J \). Hence, by the choice of \( m_2 \), we have that \( |I \cup J| < |I \cup K| \).

\[
d_2(C) = |I \cup J| \leq |I \cup K| - 1 = |I \cup K_2| - 1 = |I| + |K_2| - |I \cap K_2| - 1
\]
\[
\leq |I| + |K_2| - 1 \leq |I| + |K_1| - 1.
\]

Therefore, \( |J| - |I \cap J| \leq |K_1| - 1 \). And, by Remark \( \mathbf{3} \) we deduce that
\[
|I_2| \leq \frac{|I|}{2} \leq \frac{|J|}{2} \leq |J| - |I \cap J| \leq |K_1| - 1.
\]

Therefore, \( |I_2| < |K_1| \) and \( K_1 \subseteq I_2 \), which is a contradiction. \( \square \)
One can check that the same result (and the same proof) holds for all $I'$ such that $I' = \text{supp}(m)$ with $m \in M$ and $|I'| = |I|$. As a consequence of this observation we have that:

**Corollary 1.** If $|I| = |J|$. Then, we can always find binomials $f, g \in G_\prec \subseteq K[X]$ such that $\text{supp}(f) = I$ and $\text{supp}(g) = J$.

We just found that $I$ is always involved in the supports associated with $G_\prec$ and that if $J$ has the same cardinal as them, also the second GHW can be derived from the Gröbner basis. Let us prove now the general case.

**Proposition 5.** There exists a binomial $g \in G_\prec \subseteq K[X]$ such that $\text{supp}(g) = J$.

**Proof.** By Remark 3, we know that $|I \cap J| \leq |J|/2$, then one may consider

$$g = X^{\lfloor n/2 \rfloor} X^{J_1} - X^{J_2} \in I(C),$$

such that $J_1 \cup J_2 = J - I$, $J_1 \cap J_2 = \emptyset$ and $|J_2| + 1 \geq |I \cap J| + |J_1| \geq |J_2|$. Our goal is to prove that $g$ (or $-g$) is in $G_\prec$. We split the proof in two:

**Case LT$_\prec(g) = X^{\lfloor n/2 \rfloor} X^{J_1}$.** We are going to see that $g \in G_\prec$. First, we show that $X^{\lfloor n/2 \rfloor} X^{J_1}$ is a minimal generator of $\text{in}_\prec(I(C))$. By contradiction, suppose that there is a binomial $h = X^{K_1} - X^{K_2} \in G_\prec$ with $\text{LT}_\prec(h) = X^{K_1}$ such that $K_1 \subseteq (I \cap J) \cup J_1$, and denote $K = K_1 \cup K_2$. We have that $|J_2| \geq |I \cap J| + |J_1| - 1 \geq |K_1| \geq |K_2|$ and, in particular, $|J| > |K|$.

However,

$$d_2(C) = |I \cup J| = |I| + |J| - |I \cap J|$$

$$= |I| + |J_1| + |J_2|$$

$$\geq |I| + |J_1| + |K_2|$$

$$\geq |I \cup K|,$$

which cannot happen by the choice of $J = \text{supp}(m)$. Therefore, $X^{\lfloor n/2 \rfloor} X^{J_1}$ is a minimal generator of $\text{in}_\prec(I(C))$.

Now, we will show that $X^{J_2} \notin \text{in}_\prec(I(C))$. Since $X^{\lfloor n/2 \rfloor} X^{J_1}$ is a minimal generator of $\text{in}_\prec(I(C))$, then there exists $h = X^{\lfloor n/2 \rfloor} X^{J_1} - X^L \in G_\prec$. We have that $X^L \notin \text{in}_\prec(I(C))$ and let us see that $L = J_2$. Suppose that $L \neq J_2$, then $0 \neq h - g = x^{J_2} - x^L \in I(C)$ and $X^{J_2} \not\succ X^L$, because $X^L \notin \text{in}_\prec(I(C))$.

However,

$$|I \cup \text{supp}(h)| \leq |I| + |J_1| + |L| \leq |I| + |J_1| + |J_2| = |I \cup J| = d_2(C),$$

which is a contradiction.

**Case LT$_\prec(g) = X^{J_2}$.** We are going to see that $-g \in G_\prec$. First, we show that $X^{J_2}$ is a minimal generator of $\text{in}_\prec(I(C))$. Suppose that there exists a binomial $h = X^{K_1} - X^{K_2} \in G_\prec$ with $X^{K_1} \succ X^{K_2}$ and $K_1 \subseteq J_2$. Consider $L := J \triangle K$, we have that $L \subseteq (J - K_1) \cup K_2$ and, hence, $X^L \prec X^{J - K_1} X^{K_2} \prec X^J$.

However,

$$|I \cup L| \leq |I \cup (J - K_1) \cup K_2| \leq |I \cup J| - |K_1| + |K_2| \leq |I \cup J| = d_2(C),$$

and again it is a contradiction.
We will show now that $X^{I \cap J} X^J \notin \text{in}_\prec (I(C))$, by contradiction. Suppose that $X^{I \cap J} X^J \in \text{in}_\prec (I(C))$, then there exists a binomial $X^{I \cap J} X^J - X^L$ in $I(C)$. Consider now $K = (I \cap J) \cup J_1 \cup L$, again one can compute that $|I \cup J| \geq |I \cup K|$ and that $K \prec J$, which again contradicts the choice of $m_2$.

From the above results the main theorems of this paper will follow.

**Theorem 2.** Let $\prec$ a degree compatible order in $\mathbb{K}[X]$, then there are $f, g$ in the reduced Gröbner basis $\mathcal{G}_\prec$ of $I(C)$ with respect to $\prec$, such that $d_2 = |\text{supp}(f) \cup \text{supp}(g)|$.

**Example 6.** Let $C$ be the binary non-degenerate $[6, 3]$-code with generator matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \in \mathbb{F}_2^{3 \times 6}$$

One can check that every codeword different from 0 is a codeword of minimal support and, hence, there are 7 codewords of minimal support, namely $w_1 = 111000, w_2 = 110011, w_3 = 101101, w_4 = 101110, w_5 = 010101, w_7 = 001011$. Moreover, we have that $d_1(C) = 3$ and $d_2(C) = 5$ and $d_3(C) = 6$ and the set $M$ described in (3) coincides with $C - \{0\}$.

Consider the degree reverse lexicographic order $\prec_1$ with $x_6 \prec_1 \cdots \prec_1 x_1$, then

- $m_1 := \min_{\prec_1}(M) = 001011 = w_7$, and
- $m_2 := \min_{\prec_1}\{m \in M \mid d_2(C) = \text{supp}(\langle m, m \rangle)\} = 010101 = w_6$.

The reduced Gröbner basis $\mathcal{G}_1$ of $I(C)$ with respect to $\prec_1$ has 20 elements. As proved in Proposition 4 the binomial $f = x_3 x_6 - x_3$ belongs to $G_1$ and has $\text{supp}(f) = \text{supp}(w_7) = \{3, 5, 6\}$ and, by Proposition 5 the binomial $g = x_4 x_6 - x_2$ belongs to $G_1$ and has $\text{supp}(g) = \text{supp}(w_6) = \{2, 4, 6\}$. Moreover, $d_2 = |\text{supp}(f) \cup \text{supp}(g)| = |\text{supp}(\langle w_6, w_7 \rangle)| = |\{2, 3, 4, 5, 6\}| = 5$.

If we consider the degree reverse lexicographic order $\prec_2$ with $x_1 \prec_2 \cdots \prec_2 x_6$, then

- $m_1' := \min_{\prec_2}(M) = 111000 = w_1$, and
- $m_2' := \min_{\prec_2}\{m \in M \mid d_2(C) = \text{supp}(\langle m', m \rangle)\} = 100110 = w_4$.

The reduced Gröbner basis $\mathcal{G}_2$ of $I(C)$ with respect to $\prec_2$ has 20 elements. As proved in Proposition 4 the binomial $f' = x_1 x_2 - x_3$ belongs to $\mathcal{G}_2$ and has $\text{supp}(f) = \text{supp}(w_1) = \{1, 2, 3\}$ and, as proved in Proposition 5 the binomial $g' = x_1 x_4 - x_5$ belongs to $\mathcal{G}_2$ and has $\text{supp}(g) = \text{supp}(w_4) = \{1, 4, 5\}$. Moreover, $d_2 = |\text{supp}(f') \cup \text{supp}(g')| = |\text{supp}(\langle w_1, w_4 \rangle)| = |\{1, 2, 3, 4, 5\}| = 5$.

As a consequence, we have that the two first GHWs $d_1(C)$ and $d_2(C)$ can be obtained from the minimal graded free resolution associated with the supports in the $\mathcal{G}_\prec$-test set. Moreover, from this resolution one can also obtain upper bounds for all the $d_i(C)$ where $i \in [k(C)]$. More precisely, we have the following.
Theorem 3. Let $C \subseteq \mathbb{F}_2^n$ be a binary code, $< \, a$ degree compatible monomial order in $R$. Let $T$ denote the $\mathcal{G}_\alpha$-test, define the square-free monomial ideal

$$M := \langle \{X^c \mid c \in T\} \rangle \subseteq R,$$

and consider a minimal graded free resolution of $R/M$:

$$0 \rightarrow F_p \rightarrow \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow R \rightarrow R/M \rightarrow 0,$$

where each $F_i$ is a graded free $R$-module of the form

$$F_i = \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{i,j}(R/M)} \text{ for } i \in [p]_0.$$

Then,

(a) $p \leq k(C)$
(b) $d_i(C) \leq \min\{j \mid \beta_{i,j}(R/M) \neq 0\}$ for all $j \in \{3, \ldots, p\}$, and
(c) $d_i(C) = \min\{j \mid \beta_{i,j}(R/M) \neq 0\}$, for $i = 1, 2$.

Proof. By Proposition 3 every $c \in T$ is a codeword of minimal support and, hence, $\{X^c \mid c \in T\}$ is a subset of the generators of the monomial ideal supported on all the codewords of minimal support. Thus, (a) and (b) follow from Theorem 1.

The rest of the proof concerns (c). Again by Proposition 3 $\{X^c \mid c \in T\}$ is the minimal monomial generating set of $M$. Since $\beta_{1,i}(R/M)$ equals the number of minimal generators of $M$ of degree $i$, then, $\beta_{1,i}(R/M) = |\{c \in T \mid w_H(c) = i\}|$.

By Proposition 3 there is a $c \in T$ such that $w_H(c) = d_1$ and, thus,

$$d_1(C) = \min\{w_H(c) \mid c \in T\} = \min\{i \mid \beta_{1,i}(R/M) \neq 0\}.$$

Assume now that $T = \{c_1, \ldots, c_r\}$, and consider $T$ the Taylor resolution of $M = \langle X^{c_1}, \ldots, X^{c_r} \rangle$. The first steps of this resolution are given by

$$\mathcal{T} : \cdots \rightarrow F'_2 \xrightarrow{\varphi_2} F'_1 \rightarrow R \rightarrow R/M \rightarrow 0,$$

where $F'_1 := \bigoplus_{|I| = 1} R(-|\text{supp}(c_j) \mid j \in I|)$ and $F'_1 := \bigoplus_{1 \leq i \leq r} R(-|\text{supp}(c_i)|)$. Hence, the shifts in the second step of $\mathcal{T}$ are given by $|\text{supp}(c_i, c_j)|$ for $1 \leq i < j \leq r$ and, as a consequence, $d_2(C) \leq \min\{|\text{supp}(c_i, c_j)| \mid 1 \leq i < j \leq r\}$. Moreover, by Theorem 2, this is indeed an inequality.

In general, the Taylor resolution is not minimal (it is usually very far from minimal). However, it can be pruned to get a minimal one. Consider now

$$\mathcal{F} : \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow R \rightarrow R/M \rightarrow 0,$$

a minimal graded free resolution of $R/M$ obtained after pruning the Taylor one. As we said before, $\{X^{c_1}, \ldots, X^{c_r}\}$ is the minimal monomial generating set of $M$ and, hence, $F'_1 = F_1$. As a consequence, $\min\{i \mid \beta_{2,i}(R/M) \neq 0\} = \min\{|\text{supp}(c_i, c_j)| \mid 1 \leq i < j \leq r\} = d_2(C).$ \qed
5 Final remarks, future work & new conjectures

In this work, we build on the results of Borges-Quintana et al. [5] and propose \( \mathcal{G}_\prec \)-test sets as a smaller structure from where one can obtain the values of \( d_1(C) \) and \( d_2(C) \) for binary codes. Several experiments with SageMath [31] suggest that Theorem 3 can also be extended for \( i = 3 \). More precisely:

**Question-Conjecture 1.** Let \( C \subseteq \mathbb{F}_2^n \) be a binary code, \( \prec \) a degree compatible monomial order in \( R \). Consider the \( \mathcal{G}_\prec \)-test set \( T \) and define the square-free monomial ideal

\[
M := \langle \{X^e \mid c \in T\} \rangle \subseteq R.
\]

Is \( d_3(C) = \min\{i \mid \beta_{3,i}(R/M) \neq 0\} \)?

In Example 5 one has that the projective dimension (i.e., the number of steps of the resolution) of \( R/M \) is \( \text{pd}(R/M) = 6 \), while the dimension of \( C \) is \( k(C) = 7 \). In all the counterexamples to the original conjecture that we have found it turns out that \( \text{pd}(R/M) < k(C) \). This motivates us to ask if the conjecture holds provided that \( \text{pd}(R/M) = k(C) \). More precisely:

**Question-Conjecture 2.** Whenever \( \text{pd}(R/M) = k(C) \), is it true that

\[
d_i(C) = \min\{j \mid \beta_{i,j}(R/M) \neq 0\}
\]

for all \( i \in \{1, \ldots, k(C)\} \)?

Also, the following natural questions arise.

**Question-Conjecture 3.** What is in between the test set and the complete set of codewords of minimal support? i.e. Can we characterize a mid-way structure that provides the complete set of GHWs?

A possible candidate for that intermediate set could be the union of all \( \mathcal{G}_\prec \)-test sets for all \( \prec \) degree compatible orderings. In general, this set can be smaller than the whole set of codewords of minimal support and can be computed by the algorithm proposed in [6]. For example, for the [7, 4] binary Hamming code, i.e., the code with generator matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix} \in \mathbb{F}_2^{4 \times 7},
\]

has 14 codewords of minimal support, half of them with weight 3, and the rest with weight 4. Moreover, the sequence of GHW is \( (d_1(C), \ldots, d_4(C)) = (3, 5, 6, 7) \). One has that all the \( \mathcal{G}_\prec \)-test sets when \( \prec \) ranges over all degree compatible orderings consists of the 7 codewords of Hamming weight 3. If one computes the Betti diagram of the monomial ideal \( M \subseteq \mathbb{F}_2[x_1, \ldots, x_7] \) corresponding to this set, one gets the following
Hence, the sequence \( \left( \min \{ j \mid \beta_{i,j} \neq 0 \}; 1 \leq i \leq 4 \right) = (3, 5, 6, 7) \) coincides with the sequence of GHWs of the code \( \mathcal{C} \).

**Question-Conjecture 4.** Can we say something in the non-binary case?

In order to answer this question, one could try to apply the techniques in [20], where a generalization of the ideal \( I(\mathcal{C}) \) for non-binary codes is studied.

**Question-Conjecture 5.** Recently, E. Gorla and A. Ravagnani in [13] extend and generalize the results of Johnsen and Verdure [17] to compute the generalized weights of a code with respect to a different notions of weight. It would be interesting to see if these generalized weights can be computed from a \( \mathcal{G}_\prec \)-test set of \( \mathcal{C} \).

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