INNER RANK AND LOWER BOUNDS FOR MATRIX MULTIPLICATION

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Abstract. We develop a notion of inner rank as a tool for obtaining lower bounds on the rank of matrix multiplication tensors. We use it to give a short proof that the border rank (and therefore rank) of the tensor associated with $n \times n$ matrix multiplication over an arbitrary field is at least $2n^2 - n + 1$. While inner rank does not provide improvements to currently known lower bounds, we argue that this notion merits further study.

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1. Introduction

In this paper we introduce a new technique, that we call inner rank, to obtain lower bounds on the complexity of multiplying two matrices, where complexity is measured as the rank or border rank of the associated tensor. Inner rank is a simple idea that appears to have been overlooked in the literature until now. This idea allows us to give short proofs of some (mostly) known lower bounds on rank and border rank, and describes some interesting properties of any low rank formula for certain tensors including the matrix multiplication tensor.

Beyond this, we introduce some higher order tensors related to matrix multiplication\(^1\), and make some general remarks regarding other approaches to studying tensor rank.

The study of the rank and border rank of matrices has received a lot of attention since Strassen introduced his classical algorithm [Str69]; see Blaser’s excellent survey [Blä13] for a discussion and list of references. The development of an impressive set of tools to obtain lower bounds on rank and border rank is in progress (see [Blä03, Blä99, MR14, LO15, Lan14, LM17] and the references there). Traditionally one is most interested in the complexity of multiplying (1) two very small matrices, of interest in constructing practical algorithms, and (2) of two large matrices, a classical problem in complexity theory.

\(^1\) As this manuscript was completed, we found the work of [CZ16] that overlaps with this material.
Perhaps the most impressive application of inner rank in this article regards small matrices, namely to give a short proof of a lower bound of the border rank of seven for $2 \times 2$ matrix multiplication, valid over any field, matching Strassen’s algorithm; the lower bound for rank is a classical result, due independently to Winograd [Win71] and Hopcroft and Kerr [HK71]; the lower bound for border rank was proven by Landsberg [Lan06] over $\mathbb{C}$, but our result, valid over arbitrary fields, appears to be new.

We will argue that there is room for improvement in our methods per se, and that there is reasonable hope that our methods may be combined existing methods to give new results. Another reason why inner rank merits further study is that it can be relatively easy to apply, and it has not yet been fully explored.

The rest of this paper is organized as follows. In Section 2 we summarize our main results, saving precise definition and notation for Section 3. In Section 4 we introduce the idea behind inner rank and give a short proof that the border rank of $n \times n$ matrix over an arbitrary field is at least $2n^2 - n + 1$. In Section 5 we describe some theorems that might be used to improve the $2n^2 - n + 1$ bound via inner rank, and compare one of these theorems to bounds that do not use inner rank. In Section 6, we take the bound in Section 5 that involves inner rank and make some (rather speculative) conjectures and discuss their implications—and the questions that they suggest—in an attempt to get interesting lower bounds for rank for small $n$, focusing on $n = 3$. In Section 7 we give a more general form of the method of inner rank, and give some interesting special cases. We make some remarks about a class of special cases in Section 8. In Section 9 we introduce some techniques that one could use to study the tensor related to matrix multiplication. In Section 10 we make some closing remarks about inner rank. In Appendix A we make some closing remarks about inner rank. In Appendix A we make some closing remarks about inner rank. In Appendix B we list the matrices in Strassen’s algorithm [Str69] in our terminology, for ease of checking the various claims we make regarding this algorithm and the methods in this article.

2. Main Results and Context

In this section we summarize the notion of inner rank and state our main results in the context of known results; precise definitions will be given in the next section. In brief, we introduce a notion of inner rank that yields short proofs of known rank and border rank inequalities, although our results may be new for arbitrary fields. At present we can only give an $2n^2 - n + 1$ lower bound on the (border rank and)
rank of the tensor for $n \times n$ matrix multiplication; better lower bounds are known for border rank, and $3n^2 + o(n^2)$ lower bounds for rank are known. However, we argue that there is room for improvement both (1) in the methods per se, and (2) possibly in combination with known tools in the literature.

The best lower bounds known to us are in [Blä99, Lan14, LO15, LM17], and amount to: [Blä03] and Example 5.10 of [Blä13], for the rank of very small values of $n$ (useful in algorithm design); for rank for large $n$ and $n > 84$ in [Lan14]; and for border rank for all $n$ in [LO15, LM17]. We remark that best results for large $n$, given there, are $2n^2 + o(n^2)$ for border rank and $3n^2 + o(n^2)$ for rank; the pursuit of getting better error terms—especially interesting for applications to small $n$—has lead to an impressive set of tools.

Our inner rank inequality (in Section 4) states that if the tensor for multiplying two $n \times n$ matrices can be written as a sum of rank-1 tensors

$$\sum_{\rho=1}^{r} \alpha_{\rho} \otimes \beta_{\rho} \otimes \gamma_{\rho},$$

where $\alpha_{\rho}, \beta_{\rho}, \gamma_{\rho} \in \mathbb{F}^{n \times n}$ are $n \times n$ matrices with entries in a field, $\mathbb{F}$, then

$$n^3 \leq \sum_{\rho=1}^{r} \text{Rank}_{n \times n}(\gamma_{\rho}),$$

where $\text{Rank}_{n \times n}$ is the rank of $\gamma_{\rho}$ viewed as an $n \times n$ matrix, which we call the inner rank (to distinguish this notion of rank, interior to a factor of the tensor product space). The proof is simple, based on identifying the last of the three factors—each of which is a copy of $\mathbb{F}^{n \times n}$—with $\mathbb{F}^n \otimes \mathbb{F}^n$. Much of the rest of this paper consists of exploring variants of (2). Curiously, Strassen’s classical algorithm [Str69] satisfies (2) with equality.

Since tensor rank is invariant upon applying isomorphisms to any factors in the tensor product space, we conclude (2) with $\gamma_{\rho}$ replaced with $\mathcal{L}^2 \gamma_{\rho}$ for any invertible linear operator $\mathcal{L}$ on $\mathbb{F}^{n \times n}$. This gives an easy $2n^2 - n + 1$ lower bound on rank, which—by the same technique—can be easily adapted to give a border rank inequality. Curiously this precisely matches the bound in [LM17], which has been improved to $2n^2 - \log_2(n) - 1$ in [LM16]. The only possible new result in this article is that this bound holds for an arbitrary field.

Note (2) implies that the average rank of $\gamma_{\rho}$ there, or, more generally, $\mathcal{L} \gamma_{\rho}$ for any invertible linear operator $\mathcal{L}$ of $\mathbb{F}^{n \times n}$, is at least $n^3 / r$. Hence
inner rank

if the exponent of matrix multiplication is 2, the (inner) rank of the matrices involved are close to full rank. This may not be surprising, given that many algorithms embed many independent matrix multiplications into a single multiplication of larger matrices. However, this observation about average rank may be useful in future research.

The same argument shows that (2) holds for \(\langle n_1, n_2, n_3 \rangle \mathbb{F} \), i.e., the tensor over \(\mathbb{F} \) for multiplying an \(n_1 \times n_2\) with an \(n_2 \times n_3\) matrix, with \(n_3\) replaced with \(n_1 n_2 n_3\); of course, one can replace \(\gamma_\rho\) with either \(\alpha_\rho\) or \(\beta_\rho\) in (2). We give some broad generalizations of (2) in Sections 5—8, where we argue that our methods could be used in conjunction with previous methods.

One interesting generalization of the inner rank bound regarding \(L \gamma_\rho\) with \(L\) invertible is the following generalization (of Subsection 7.2): if \(\langle n_1, n_2, n_3 \rangle \mathbb{F}\) can be written as (1), and \(L\) is any linear operator on \(\mathbb{F}^{n_3 \times n_1}\), then

\[ n_2 \text{Rank}_{n_3 n_1 \times n_3 n_1}(L) \leq \sum_{\rho=1}^{r} \text{Rank}_{n_3 \times n_1}(L \gamma_\rho); \]

for \(L\) invertible and \(n_1 = n_2 = n_3\), reduces to (2) with \(L \gamma_\rho\) replacing \(\gamma_\rho\), and the case where \(L\) is the identity matrix is (2). A generalization to \(L\) not necessarily invertible may be useful in proving lower bounds in the case of high overlap in an algorithm, where overlap is a term that we define in Section 6.

In Section 6 we make some conjectures and describe an approach for getting improved lower bounds on the rank of multiplying two \(n \times n\) matrices for very small \(n\), such as \(n = 3\). In Section 9 we describe some other approaches to studying the rank matrix multiplication.

3. Preliminary Notation and Remarks

In this section review some notation that we will use, most of which is common in this field, and review some standard background. We refer the reader to [Blä13] and Chapters 14 and 15 of [BCS97], although some of our notation differs from these references.

Throughout this article \(\mathbb{F}\) will denote a field, arbitrary unless otherwise indicated. For an integer \(n \geq 1\), we use \([n]\) to denote \(\{1, \ldots, n\}\), and we let \(e_1, \ldots, e_n\) denote the standard basis of \(\mathbb{F}^n\). For integers \(n_1, n_2 \geq 1\), we use \([n_1 \times n_2]\) to denote \([n_1] \times [n_2]\), and similarly for \([n_1 \times n_2 \times n_3]\), etc.; we use \(\mathbb{F}^{n_1 \times n_2}\) to denote the field of \(n \times n\) matrices, with \(\{e_{ij}\}\) being the standard basis, i.e., \(e_{ij}\) has zero entries except for a 1 in position \((i, j)\). We use \(A^*\) to denote the dual \(\mathbb{F}\)-vector space of an \(\mathbb{F}\)-vector space \(A\); we use \(e_i^*\) to denote the standard basis vector for
\((\mathbb{F}^n)^*\) (taking a vector to its \(i\)-th coordinate); similarly for \(e^*_ij\). We often use the isomorphism \(\mathbb{F}^{n_1 \times n_2}\) with \(\mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2}\), via the map taking \(e_{ij}\) to \(e_i \otimes e_j\); similarly for \(\mathbb{F}^n\) and \((\mathbb{F}^n)^*\) via \(e_i \mapsto e^*_i\).

For any set \(S\) (such as \([n]\) or \([n_1 \times n_2]\) or etc.), and two elements \(u, v \in \mathbb{F}^S\) we set the dot product to be
\[
u \cdot v = \sum_{s \in S} u_sv_s;\]
we alert the reader that the function \(v \mapsto u \cdot v\) merely a convenient way of expressing an element of the dual space. For \(A \in \mathbb{F}^S\), the complement of \(A\) (understanding the ambient space \(\mathbb{F}^S\)) is the set
\[
A^\perp = \{ \zeta \in \mathbb{F}^S \mid \forall \alpha \in A, \; \zeta \cdot \alpha = 0 \},
\]
which is canonically isomorphic to \((\mathbb{F}^n/A)^*\). [One could, more generally, define \(A^\perp\) with respect to any perfect pairing \(\mathbb{F}^S \times \mathbb{F}^S \to \mathbb{F}\), but using the above dot product has notational advantages regarding matrix multiplication tensors.]

For brevity, an integer \(n \geq 1\) we use the notation
\[
F^n_1 = \mathbb{F}^n, \; F^n_2 = \mathbb{F}^{n \times n}, \; \ldots, \; F^n_{i,j} = F_i \otimes F_j, \; F^n_{i,j,k} = F_i \otimes F_j \otimes F_k, \; \ldots
\]
and we often write \(F_1^n\) for \(F^n_1\), etc., dropping the superscript \(n\) when confusion is unlikely to occur. Also, for brevity, we often write \(e_{i_1i_2i_3i_4}\) for the vector \(e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \otimes e_{i_4}\) in \(\mathbb{F}_{2,2}\), etc., when confusion is unlikely to occur.

One sets (as in [Blä13])
\[
\langle n_1, n_2, n_3 \rangle_\mathbb{F} \overset{\text{def}}{=} \sum_{i \in [n_1], \; j \in [n_2], \; k \in [n_2]} e_{ij} \otimes e_{jk} \otimes e_{ki},
\]
an element of \(\mathbb{F}^{n_1 \times n_2} \otimes \mathbb{F}^{n_2 \times n_3} \otimes \mathbb{F}^{n_3 \times n_1}\) (the last factor, \(e_{ki}\), in the above is sometimes taken to be \(e_{ik}\) in the literature), and we drop the subscript \(\mathbb{F}\) when confusion is unlikely to occur. It is often less cumbersome to illustrate our techniques in the case \(n_1 = n_2 = n_3\); we set \(\kappa_3(n) = \kappa_3(n)_\mathbb{F} = \langle n, n, n \rangle\), and more generally set
\[
\kappa_m(n)_\mathbb{F} = \sum_{i_1, \ldots, i_m \in [n]} e_{i_1i_2} \otimes \cdots \otimes e_{i_{m-1}i_m} \otimes e_{i_mi_1}.
\]
We will be interested in \(\kappa_m(n)\) for \(m > 3\) in Section 9.

If \(A, B\) are \(\mathbb{F}\)-vector spaces, we use “map” and “morphism” to mean a linear map \(A \to B\); hence the term automorphism implies \(A = B\), and an isomorphism means an invertible linear map. We use \(\text{Hom}(A, B)\) to denote the set of all linear maps from \(A\) to \(B\).
Given $\mathbb{F}$-vector spaces $A_1, \ldots, A_m$ the \textit{rank} of $\tau \in A_1 \otimes \cdots \otimes A_m$, denoted $R(\tau)$ is the smallest $r$ for which we may write $\tau$ as \textit{r rank-1} tensors, i.e., as
\[
\tau = \sum_{\rho=1}^{r} \alpha_{\rho}^1 \otimes \cdots \otimes \alpha_{\rho}^m
\]
with $\alpha^i_{\rho} \in A_i$.

If $A_i : A_i \to A'_i$ are linear maps, there is a map
\begin{equation}
A_1 \otimes \cdots \otimes A_m : A_1 \otimes \cdots \otimes A_m \to A'_1 \otimes \cdots \otimes A'_m
\end{equation}
which sends
\[
\alpha^1 \otimes \cdots \otimes \alpha^m \mapsto A_1(\alpha^1) \otimes \cdots \otimes A_m(\alpha^m).
\]
Hence $A_1 \otimes \cdots \otimes A_m$ does not increase tensor rank, and preserves tensor rank if each $A_i$ has a left inverse, i.e., is injective.

It will be useful to write $\tau \preceq \tau'$ whenever one can write
\[
\tau' = (A_1 \otimes \cdots \otimes A_m)\tau,
\]
in which case we say $\tau$ can be \textit{reduced} to $\tau'$. It follows that $R(\tau) \leq R(\tau')$ in this case, and similarly for $R$ replaced with $R$ defined below; also, $\preceq$ is clearly a preorder. Similar notation is used in \cite{BCS97} (Definition 14.27, replacing $W, W'$ there with their duals).

If $\mathbb{F}$ is algebraically closed, we define the \textit{border rank} \footnote{There are one or two other possible definitions of border rank, depending on $\mathbb{F}$; the Zariski rank is easily seen to be no larger than the others. See Appendix A.} of $\tau$, denoted $\overline{R}(\tau)$, to be the smallest $r$ such that $\tau$ lies in the Zariski closure of the set of tensors of rank at most $r$, where this set is identified with an affine space over $\mathbb{F}$ by choosing bases for the $A_i$ to coordinatize $A_1 \otimes \cdots \otimes A_m$ (the Zariski closure is clearly independent of this choice of bases). If $\mathbb{F}$ is not algebraically closed, we define the border rank of $\tau$ by base extension to the algebraic closure, i.e., we take an algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$, and define the border rank of $\tau$ to be that of $\tau'$ in $A'_1 \times \cdots \otimes A'_m$, where $A'_i = A_i \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ are $\overline{\mathbb{F}}$-vector spaces, and $\tau'$ is $\tau$ as an element of the aforementioned tensor product obtained by base extension.

For $\mathbb{F}$-vector spaces $M, N$, there is a canonical identification of $M \otimes N$ with $\text{Hom}(M^*, N)$ (and $\text{Hom}(N^*, M)$), and the rank of an element of $M \otimes N$ is the same as its rank as a map $M^* \to N$, which may be computed with respect to any bases of $M^*$ and $N$; therefore the set of elements of $M \otimes N$ of rank less than a given integer is Zariski closed.

By contrast, for tensor products of three or more spaces, rank is not Zariski semicontinuous (e.g., \cite{Blü13}, Section 6), and determining the rank of a three tensor is NP-complete \cite{Hs90}.
We alert the reader to the following peripheral facts. First, in this article, as in matrix multiplication in general, sets such as \([n]\) (or \([n_1 \times n_2 \times n_3]\)) are generally used for indices of matrices and tensors, without making use of any ordering on such sets. Second, when we identity \(F^S\) with \((F^S)^*\) by taking \(\zeta \in F^S\) to the map \(\alpha \mapsto \alpha \cdot \zeta\), we tend to destroy the functoriality; similarly when identify \(F_{n_1 \times n_2}\) with \((F_{n_1})^* \otimes F_{n_2}\) rather than \((F_{n_1})^* \otimes F_{n_2}\); a related matter is that—using the summation convention—we are working with the awkward tensor \(e_{i,j}^i e_{j,k}^j e_{k,i}^k\) rather than \(e_j^i e_k^j e_i^k\). Third, generally we use lower case Roman letters for integers and/or indices; generally upper case Roman letters are vector spaces, and lower case Greek letters tensors or morphisms of tensor spaces. We make the usual exceptions for \(\epsilon\), \(e_i\) in standard bases; we make a few exceptions, especially \(r, R, \rho\) to confirm to common notation in the matrix multiplication literature—and regarding the letters \(a, b, c\) and \(M, N, L\), which are a bit overloaded given our interest in 3-tensors.

4. The Inner Rank Bound

In this section we give the idea behind the inner rank bound, and use it to give a short proof that \(\kappa_3(n) = \langle n, n, n \rangle\) has (border rank and) rank at least \(2n^2 - n + 1\). We begin with the proof for \(n = 2\), and later extend it for general \(n\).

4.1. The Inner Rank Inequality.

**Lemma 4.1.** For any field, \(F\), and an integer, \(n\), let \(\pi: F_{2,2,2} \rightarrow F_{2,1} \otimes F_{2,1}\) be given by

\[
\pi(e_{ab} \otimes e_{cd} \otimes e_{fg}) = e_{abf} \otimes e_{cdg}
\]

(i.e., \(\pi\) is obtained by composing identifications of \(F_{2,2,2}\) with \(F_{2,2,1,1}\), then with \(F_{2,1,2,1}\), then with \(F_{2,1} \otimes F_{2,1}\)). Then \(\lambda \overset{\text{def}}{=} \pi(\kappa_3(n))\) is of full rank (i.e., rank \(n^3\)).

**Proof.** We have

\[
\lambda = \sum_{ijk} (e_{ij} \otimes e_k) \otimes (e_{jk} \otimes e_i);
\]

so viewing \(\lambda\) as a map \(F_{2,1}^* \rightarrow F_{2,1}\), for all \(a, b, c \in [n]\) we have

\[
\lambda(e_{ab}^* \otimes e_c^*) = \sum_{ijk} \left( (e_{ab}^* \otimes e_c^*) (e_{ij} \otimes e_k) \right) \otimes (e_{jk} \otimes e_i) = e_{bc} \otimes e_a.
\]
Hence the image of $\lambda$, as a morphism $F_{2,1} \to F_{2,1}$ is of all of $F_{2,1}$, and hence $\lambda$ is of rank $n^3$. \hfill \Box

**Lemma 4.2.** Let $\pi$ be as in (4), and let

$$\tau = \sum_{\rho=1}^{r} \alpha_{\rho} \otimes \beta_{\rho} \otimes \gamma_{\rho}$$

for vectors $\alpha_{\rho}, \beta_{\rho}, \gamma_{\rho}$ in $F_2$. Then the rank of $\pi \tau$ as a morphism of $F_{2,1}$ to itself is at most

$$\sum_{\rho=1}^{r} \text{Rank}_{n \times n}(\gamma_{\rho}).$$

**Proof.** If each $\gamma_{\rho}$ is of rank $r_{\rho}$, then we have

$$\tau = \sum_{\rho=1}^{r} \alpha_{\rho} \otimes \beta_{\rho} \otimes \sum_{s=1}^{r_{\rho}} \gamma_{\rho}^{1,s} \otimes \gamma_{\rho}^{2,s},$$

for some $\gamma_{\rho}^{1,s}, \gamma_{\rho}^{2,s}$, and hence

$$\pi \tau = \sum_{\rho,s} (\alpha_{\rho} \otimes \gamma_{\rho}^{1,s}) \otimes (\beta_{\rho} \otimes \gamma_{\rho}^{2,s}),$$

whose rank is therefore bounded by

$$\sum_{\rho} r_{\rho} = \sum_{\rho} \text{Rank}_{n \times n}(\gamma_{\rho}).$$

\hfill \Box

**Corollary 4.3.** If $\kappa_3(n) = \sum_{\rho=1}^{r} \alpha_{\rho} \otimes \beta_{\rho} \otimes \gamma_{\rho}$, then

$$n^3 \leq \sum_{\rho=1}^{r} \text{Rank}_{n \times n}(\gamma_{\rho}).$$

Curiously, Strassen’s classical algorithm for $\kappa_3(2)$ satisfies this corollary with equality.

Since tensor rank is left invariant by applying any isomorphism(s) to some of its factors (see the discussion regarding (3)), we conclude the following corollary.

**Corollary 4.4.** Let $\mathcal{L}$ be an isomorphism of $\mathbb{F}^{n \times n}$. Under the hypotheses of Corollary 4.3, we have

$$n^3 \leq \sum_{i=1}^{r} \text{Rank}_{n \times n}(\mathcal{L} \gamma_{\rho}).$$
4.2. A Short Proof of the Rank of $\kappa_3(2)$. We use Corollary 4.4 to give a short proof of the classical result $R((2,2,2))_F \geq 7$ (independently due to Hopcroft-Kerr and Winograd [Win71, HK71]), matching Strassen’s classical algorithm.

**Lemma 4.5.** Let $F$ be a field, and consider a set of four or five elements in $F^{2 \times 2}$ that (1) are all nonzero, and (2) span all of $F^{2 \times 2}$. Then there is an isomorphism $L$ of $F^{2 \times 2}$ taking each of these vectors to a matrix of rank one.

**Proof.** First consider the case of five vectors, $v_1, \ldots, v_5$, in the special case where $v_1, \ldots, v_4$ are linearly independent and $v_5$ is one of $v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_2 + v_3 + v_4$.

In the first, second, and fourth case, we define $L$ uniquely via

$$L(v_1) = e_{11}, \quad L(v_2) = e_{12}, \quad L(v_3) = e_{21}, \quad L(v_4) = e_{22},$$

each of rank one, and note that $L(v_5)$ is of rank one in these cases. In the third case we similarly define $L$ via

$$L(v_1) = e_{11}, \quad L(v_2) = e_{12}, \quad L(v_3) = e_{21} + e_{22}, \quad L(v_4) = e_{21};$$

again, each is of rank one, as is $L(v_5)$.

The general case involving five vectors is easily reduced to one of the above situations, since we may assume that $v_1, \ldots, v_4$ are linearly independent, and hence $v_5 = c_1 v_1 + \cdots + c_4 v_4 \neq 0$ for some $c_i \in F$; if $c_i \neq 0$ then we may replace $c_i v_i$ by $v_i$ by rescaling $v_i$, since rescaling leaves the rank (of any $2 \times 2$ matrix) invariant. The general case involving four vectors is handled via (5).

**Theorem 4.6.** For any field, $F$, $\kappa_3(2)_F = \langle 2,2,2 \rangle_F$ has rank greater than six.

**Proof.** Otherwise we have $\langle 2,2,2 \rangle = \sum_{i=1}^{r} \alpha_i \otimes \beta_i \otimes \gamma_i$ for some $r \leq 6$, with $\gamma_i \neq 0$ for all $r$. Clearly the $\gamma_i$ must span all of $F^{2 \times 2}$, so we may assume that $\gamma_1, \ldots, \gamma_4$ span $F^{2 \times 2}$. Identifying $F^{2 \times 2}$ with $F^{2 \times 2}$, define $L$ as in the lemma above, taking $v_i = \gamma_i$ for $i = 1, \ldots, \min(r,5)$. Then the sum of the ranks of $L(\gamma_i)$ is at most five plus the rank of $L(\gamma_6)$ (in the case $r = 6$), hence at most seven. But this would contradict Corollary 4.4.

4.3. The Border Rank of $\kappa_3(2)$. The method of our proof of Theorem 4.6 can be adapted to give the same bound for border rank (Landsberg [Lan06] proves this for subfields of $\mathbb{C}$; we don’t know if his proof works over arbitrary fields).
Theorem 4.7. Theorem 4.6 holds with “rank” replaced by “border rank.”

Proof. Let $\pi$ be as in (4). Let $C_4, C_{\leq 3}, B_{\leq 3}$, respectively, denote the sets of tensors in $F_{2,2}^2$ that can be written in the form

$$
\sum_{\rho=1}^{6} \alpha_{\rho} \otimes \beta_{\rho} \otimes \gamma_{\rho}
$$

for some $\alpha_{\rho}, \beta_{\rho}, \gamma_{\rho}$ where, respectively, (1) $C = \text{Span}_{\rho}(\gamma_{\rho}) \subset F_2$ is of dimension four, (2) $\mathcal{C}$ is of dimension less than four, and (3) $B = \text{Span}_{\rho}(\beta_{\rho}) \subset F_2$ is of dimension less than four. Since any $\tau \in F_{2,2}^2$ with $R(\tau) \leq 6$ lies in $C_4 \cup C_{\leq 3}$, it suffices to show that $\kappa_3(2)$ lies in the complement of the Zariski closures of both $C_4$ and $C_{\leq 3}$.

The proof of Theorem 4.6 implies that any if $\tau \in C_4$ then there is an isomorphism $L$ on $F_2$ for which

$$
\pi(I \otimes I \otimes L) \tau = M,
$$

where $M \in F_{2,1} \otimes F_{2,1}$ may be viewed as a linear map $F_{2,1}^* \rightarrow F_{2,1}$ whose rank is at most seven. Hence, viewing $\pi \tau$ as a map $F_{2,1}^* \rightarrow F_{2,1}$ is given by

$$
\pi \tau = M' \circ M, \quad \text{where } M' = \pi(I \otimes I \otimes L)^{-1} \pi^{-1}.
$$

Since $\pi \tau$ factors through a map of rank at most seven, it follows that $\pi \tau$ lies in the (Zariski closed) set, $Z$, of tensors $F_{2,1} \otimes F_{2,1}$ of rank at most seven. Since $\pi$ merely exchanges coordinates, it follows that $\pi^{-1}Z$ is Zariski closed, and hence $C_4 \subset \pi^{-1}Z$, and hence the closure of $C_4$ lies in $\pi^{-1}Z$. The proof of Theorem 4.6 (really Lemma 4.1) shows that $\pi \kappa_3(2) \notin Z$, and hence $\kappa_3(2) \notin \pi^{-1}Z$, and hence does not lie in the closure of $C_4$.

If $\tau \in B_{\geq 3}$, then the image of $\pi \tau$, viewed as a map $F_{2,1}^* \rightarrow F_{2,1}$, lies in $B \times F_1$, a subspace of dimension six. So again $\pi(\tau) \in Z$. Hence $\kappa_2(3)$ is not in the closure of $B_{\leq 3}$.

The fact that $\kappa_3(2)$ is not in the closure of $B_{\leq 3}$ immediately implies the same for that of $C_{\leq 3}$, since there is a simple permutation of co-ordinates, $\pi'$, on $F_{2,2,2}$ leaving $\kappa_3(2)$ invariant and taking $B_{\leq 3}$ to $C_{\leq 3}$, namely the one taking $e_{i_1i_2i_3i_4i_5i_6}$ to $e_{i_2i_1i_5i_6i_4i_3}$. [More conceptually, this argument amounts to applying inner rank technique to the second factor of $F_2$ in $F_{2,2,2}$, rather than to the third.]

4.4. The Immediate Extension to $\kappa_3(n)$. The $2 \times 2$ method has the immediate generalization.

Lemma 4.8. Let $\mathbb{F}$ be a field, $n \geq 2$ an integer, and consider a set of $n^2 + 1$ or fewer vectors in $F_2^n$ that (1) are all nonzero, and (2) span
all of $F_n^2$. Then there is an isomorphism $\mathcal{L}$ on $F_n^2$ taking each of these vectors to a matrix of rank one.

Proof. Like the proof of Lemma 4.5, it suffices to consider the case where

$$v_{n^2+1} = v_1 + \cdots + v_t;$$

if $m$ is the smallest integer with $mn \leq t$, then we set $\mathcal{L}(v_{n^2+1})$ to be the matrix with ones in the first $m$ rows and 0’s elsewhere; then set $\mathcal{L}(v_{1+a+nb}) = e_{a+1,b+1}$ for all $b$ and $0 \leq a \leq n$, with one exception when $mn \neq t$, namely we take $\mathcal{L}(v_{mn})$ to be the matrix supported on row $m$ in a way to have the $m$ rows of $v_{(m-1)n+1}, \ldots, v_{mn}$ add to $[1, \ldots, 1]$.

Theorem 4.9. For any field, $F$, and integer $n \geq 2$, $(R(\kappa_3(n))$ and) $\overline{R}(\kappa_3(n))$ are at least $2n^2 - n + 1$.

Proof. This is the evident generalization, using Lemma 4.8. For the rank bound, if we set $n^2 + 1$ of the $L_\gamma r$ to 1, to exceed a rank bound of $n^3$ we must have $r = (n^2 + 1) + m$ where $m$ is an integer such that

$$n^2 + 1 + mn \geq n^3,$$

i.e., $m \geq n^2 - n + 1/n$. So the smallest possible $m$ is $m = n^2 - n$, and we obtain a lower bound of

$$R(\kappa_3(n)) \geq n^2 + 1 + (n^2 - n) = 2n^2 - n + 1.$$

The proof of Theorem 4.7 shows that if we define the subset $C_n = C_n(r)$ of tensors of rank $r$ analogously, then $C_n$ is mapped under $\pi$ to the set of $n^3 \times n^3$ tensors of rank $\leq n^3 - 1$ if $r < 2n^2 - n + 1$. Similarly if we define $C_{n-1} = C_{n-1}(r)$ analogously, then $\pi C_{n-1}$ lies in the set of $n^3 \times n^3$ tensors of rank at most $(n^2 - 1)n$ (regardless of $r$), which is less than $n^3$.

Curiously, this bound equals that in [LM17], which has been improved to $2n^2 - \log_2(n) - 1$ in [LM16]. We will explain in Section 3 that the above theorem for rank is not particularly competitive concerning rank, where $3n^2 + o(n^2)$ has been achieved [LO15, MR14, Lan14].

Clearly one can improve the above results for rank provided that one can improve on the $\sum_\rho \text{Rank}(\mathcal{L}_\gamma r)$ bound obtained above by Lemma 4.8; similarly for border rank, if one can cover the set of all tensors of rank $r$ with a finite union of sets $^3$, each of whose closure excludes $\kappa_3(n)$.

---

$^3$ The finite number is generally necessary, for if $\tau$ is a tensor whose border rank is less than its rank (see [Bläi13], Section 6, for an example) in the sense that there is a family $\tau(\epsilon) = \tau + O(\epsilon)$ of rank less than $R(\tau)$, then $\{\tau(1/n)\}$ ranging over the positive integers $n$ is a countable union of singleton sets, each of which is equal to its closure (which excludes $\tau$). Also, one would be free to assume some property
4.5. **Parameter Counting.** We now speculate on the best possible improvements to Lemma 4.8 to obtain a better Theorem 4.9 via a crude “parameter count.” We shall see that this crude count is overly optimistic, even for $n = 2$.

Most optimistically, $\mathcal{L}$ is built from $n^4$ parameters, and the invertibility of $\mathcal{L}$ describes an open subset of $\mathbb{F}^{n^4}$; for simplicity we will not projectivize, since this will not affect the first order term in our rough computation. Imagine that for fixed $\gamma_1, \ldots, \gamma_r$ we want to have $\mathcal{L}(\gamma_i)$ of rank $q$. This places roughly $(n - q) \times (n - q)$ conditions per each of $r$ of the $\gamma_\rho$. To contradict $n^3 \leq rq$ one needs to take $q = n^3/r$, so most optimistically, from this crude count, one cannot expect better than $r$ with

$$n^4 = (n - n^3/r)^2r, \quad \text{i.e.,}$$

which for $r = Cn^2$ gives $1 = (1 - 1/C)^2C$, and hence gives $C$ equal to the square root of the golden ratio.

The strategy we used in Lemma 4.5 of taking $\gamma_1, \ldots, \gamma_{n^2}$ to $e_{ij}$ (in the generic case) may be simpler to analyze; one could, more generally, take $\gamma_1, \ldots, \gamma_{n^2}$ to be generic rank-1 tensors, yielding a $\mathcal{L}$ with roughly $2n^3$, would allow us to reduce the rank on some additional $Cn^2$ elements of $F_2$ to $q$, where $n^3 = (n - n^3/(Cn^2))^2Cn^2$, so $1/n = (1 - 1/C^2)C$. This makes $C$ now a function of $n$, roughly $n^{-1/2}$, optimistically a bound of $n^2 + n^{3/2}$, not impressive for large $n$; this may or not be interesting for $n$ small.

Now consider $n = 2$. Then $\mathcal{L}$, as an isomorphism of a four dimension space has 16 parameters (or 15 if we projectivize). But Strassen’s algorithm shows that it is impossible to take certain sets of seven vectors in $\mathbb{F}^{2 \times 2}$ all to rank one matrices via an isomorphism; since a two by two matrix is of rank one if it satisfies a single equation, this shows that parameter counting here is overly optimistic.

5. **Remarks on Annihilation Methods**

In this section we give an approach to inner rank where we try to annihilate some of the $\alpha_\rho$ and/or $\beta_\rho$ in Corollary 4.3 by applying linear maps (therefore with nontrivial kernels) to the first two factors, namely Theorem 5.4 below. We compare Theorem 5.4 to annihilation methods—presumably subsumed in the literature—that do not involve inner rank, because of certain similarities. In Section 6 we will use of the set of rank $r$ tensors that is known to hold in a Zariski open neighbourhood of $\kappa_3(n)$, but this assumption adds nothing: to show that this property holds as such is equivalent to showing that the complement of this property holds in its complement (which is Zariski closed and would exclude $\kappa_3(n)$).
Theorem 5.4 to speculate on certain conjectures that would lead to interesting improvements to rank lower bounds, especially for $R(\kappa_3(n))$ for small $n$. We finish this section by noting a curious fact related to inner rank, which can be derived from a standard technique used in previous lower bound methods; this fact seems related to the theorems in this section, but we do not have any particular application of it in this article.

5.1. Annihilation Lower Bounds. In this subsection we will state an inner rank lower bound that generalizes Corollary 4.4, in reference to $R(\langle n_1, n_2, n_3 \rangle)$. The idea is to incorporate $\alpha_\rho, \beta_\rho$ into the inner rank bound (Corollary 4.4), where they are conspicuously absent. We will compare this lower bound to analogous bounds that don’t use inner rank. Let us begin with some general remarks.

The idea is that if

$$\tau \in M \otimes N \otimes L_1 \otimes L_2,$$

then $R(\tau)$ gives an upper bound for the rank of the canonically associated linear map to $\tau$ in

$$\text{Hom}(M^* \otimes L_1^*, N \otimes L_2), \quad \text{Hom}(L_1^* \otimes L_2^*, M \otimes N), \quad \text{etc.}$$

Since an element of $\text{Hom}(U^*, V)$ has the same rank as that in $\text{Hom}(V^*, U)$, we have three possibly interesting ways of partitioning the four factors $M, N, L_1, L_2$ into two groups of size two each, and four possibly interesting ways into groups of sizes one and three. The best lower bound for $R(\tau)$ will depend on $\tau$, but will always be limited by the dimension of the source and target.

In order to improve the inner rank inequality, we will want to choose $A \subset M$ and $B \subset N$, and consider the image of $\tau$ under the natural map

$$M \otimes N \otimes L_1 \otimes L_2 \xrightarrow{(\cdot)^*/(\cdot)^* \otimes \text{id}_{L_1} \otimes \text{id}_{L_2}} (M/A) \otimes (N/B) \otimes L_1 \otimes L_2,$$

where $A$ denotes the canonical map $M \rightarrow M/A$ and similarly for $B$. We can then consider $\tau$ as a morphism

$$M/A \otimes (N/B) \rightarrow L_1 \otimes L_2 \quad \text{(7)}$$

or as one

$$M/A \otimes L_1^* \rightarrow (N/B) \otimes L_2^* \quad \text{(8)}$$

The latter was used for the inner rank bound (see Lemma 4.2), and as equivalent to that with $L_1$ and $L_2$ interchanged, since we are free to apply an isomorphism $L$ on $L_1 \otimes L_2$ as we did in Corollary 4.4; the former morphism does not use the inner rank at all, because $L_1$
and $L_2$ are together, and hence there is no need to factor the $\gamma_{\rho}$ in $L_1 \otimes L_2$. Hence the former is equivalent to previous approaches that do not involve inner rank.

Let us state bounds based on (7) and (8); in Section 7 we will study bounds based on the latter in more detail; they arise from the more general Lemma 7.1.

We will state two equivalent forms of the bound based on (7), in order to give intuition for the bound based on (8) (at the risk of trying the reader’s patience). The forms of the bound based on (7), which does not use inner rank at all, is undoubtedly subsumed in the previous literature; however these bounds are interesting to us because they have much commonality with the bound we state based on (8). We will also state a somewhat more general form of the theorem(s) based on (7).

The four theorems stated below are easily proven; their proofs will be given in Subsubsection 5.1.1.

**Theorem 5.1.** Let $\langle n_1, n_2, n_3 \rangle_\mathbb{F} = \sum_{\rho=1}^r \alpha_{\rho} \otimes \beta_{\rho} \otimes \gamma_{\rho}$ for some $n_i, r, \alpha_{\rho}, \beta_{\rho}, \gamma_{\rho}, \mathbb{F}$. Let $\mathcal{A} \subset \mathbb{F}^{n_1 \times n_2}$, and $\mathcal{B} \subset \mathbb{F}^{n_2 \times n_3}$. Let

$$S(\mathcal{A}, \mathcal{B}) \overset{\text{def}}{=} \text{Span}_{ik} \left( \sum_j [e_{ij}]_\mathcal{A} \otimes [e_{jk}]_\mathcal{B} \right) \subset \left( \mathbb{F}^{n_1 \times n_2} / \mathcal{A} \right) \otimes \left( \mathbb{F}^{n_2 \times n_3} / \mathcal{B} \right),$$

where $[e_{ij}]_\mathcal{A}$ denote the equivalence class of $e_{ij}$ in $\mathbb{F}^{n_1 \times n_2} / \mathcal{A}$, and similarly for $[e_{jk}]_\mathcal{B}$. Then for any isomorphism $L$ of $\mathbb{F}^{n_3 \times n_1}$ we have

$$\dim(S(\mathcal{A}, \mathcal{B})) \leq \sum_{\rho \in \text{supp}(/\mathcal{A}, /\mathcal{B})} 1,$$

where $\text{supp}(/\mathcal{A}, /\mathcal{B})$ is the set of $\rho \in [r]$ such that both $\alpha_{\rho} \notin \mathcal{A}$ and $\beta_{\rho} \notin \mathcal{B}$.

One point in the above lemma is that $\text{Rank}_{n_3 \times n_1}(L_{\gamma_{\rho}})$ is replaced with 1, since—as explained above—there is no need to factor $L_{\gamma_{\rho}}$.

Here is an essentially equivalent bound.

**Theorem 5.2.** Let $\langle n_1, n_2, n_3 \rangle_\mathbb{F} = \sum_{\rho=1}^r \alpha_{\rho} \otimes \beta_{\rho} \otimes \gamma_{\rho}$. Let $Z \subset \mathbb{F}^{n_1 \times n_2}$ and $H \subset \mathbb{F}^{n_2 \times n_3}$, and $L$ an isomorphism of $\mathbb{F}^{n_3 \times n_1}$. Then

$$\dim(\text{Span}(ZH)) \leq \sum_{\rho \in \text{supp}(ZH)} 1,$$

where

$$ZH \overset{\text{def}}{=} \{ \zeta \eta \in \mathbb{F}^{n_1 \times n_3} \mid \zeta \in Z, \eta \in H \},$$

(1)
(and where \( \zeta \eta \) is ordinary matrix multiplication, \( \text{Span} \) denotes their span in \( \mathbb{F}^{n_1 \times n_3} \), and \( \dim \) refers to the dimension of this subspace of \( n_1 \times n_3 \)), and

(2) \( \text{supp}(ZH) \subset [r] \) denotes the set of \( \rho \) for which there exists (both) some \( \zeta \in Z \) with \( \zeta \cdot \alpha_\rho \neq 0 \) and some \( \eta \in H \) with \( \eta \cdot \beta_\rho \neq 0 \).

In this theorem, \( \dim(\text{Span}(ZH)) \) is at most \( n_1 n_3 \). Our application to \( \kappa_3(n) \) will be when \( Z \) and \( H \) are of dimension \( n \), so we get a lower bound of \( n^2 \) in (10).

Let us mention a minor generalization of the above theorems.

**Theorem 5.3.** Consider the hypotheses of Theorem 5.2, with \( \langle n_1, n_2, n_3 \rangle \) replaced with

\[
\sum_{ijk \in [n_1 \times n_2 \times n_3]} f_{ij} \otimes g_{jk} \otimes e_{ki}
\]

for arbitrary matrices \( f_{ij} \in \mathbb{F}^{n_1 \times n_2} \) and \( g_{jk} \in \mathbb{F}^{n_2 \times n_3} \). Then (10) holds with \( \text{Span}(ZH) \) replaced with the dimension of the span of

\[
\{ \sum_j (\zeta \cdot f_{ij})(\eta \cdot g_{jk}) e_i \otimes e_k \mid \zeta \in Z, \eta \in H \} \subset \mathbb{F}^{n_1 \times n_3}.
\]

One nice aspect of the above theorem is that it shows although we can “preprocess” \( \mathbb{F}^{n_1 \times n_2} \) and \( \mathbb{F}^{n_2 \times n_3} \) by applying an isomorphism or an automorphism to each space (as we did in Corollary 4.3 to obtain Corollary 4.4), there is nonetheless a cost to such automorphisms, which is the complication of the formula for “matrix multiplication” \( \zeta \alpha \eta \) (which is morally a “co-multiplication” of linear functionals on the dual spaces of these matrices).

Now let us give a result based on inner rank and (8); we note its resemblance to the above theorems.

**Theorem 5.4.** Consider the hypotheses of Theorem 5.2, and let \( \mathcal{L} \) be any isomorphism of \( \mathbb{F}^{n_3 \times n_1} \). Then

\[
\text{rank}(\Psi_{Z,H}) \leq \sum_{\rho \in \text{supp}(ZH)} \text{Rank}_{n_3 \times n_1}(\mathcal{L}\gamma_\rho),
\]

where

(1) \( \text{supp}(ZH) \subset [r] \) is as in Theorem 5.2,

(2) \( \Psi_{Z,H} \) is the map

\[
\Psi_{Z,H} : Z \otimes (\mathbb{F}^{n_3})^* \to H^* \otimes \mathbb{F}^{n_1}
\]

which, under the identification \( H^* \otimes \mathbb{F}^{n_1} = \text{Hom}(H, \mathbb{F}^{n_1}) \) is the map such that for any \( \zeta \in Z \) and \( c \in [n_3] \) has

\[
\Psi_{Z,H}(\zeta \otimes e^*_c) = \left( \nu \mapsto \text{Col}_c(\zeta \nu) \right) \in \text{Hom}(H, \mathbb{F}^{n_1})
\]
where \( \text{Col}_c \) denotes the \( c \)-th column, i.e., the \( c \)-th column of the matrix of \( \zeta \in \mathbb{F}^{n_1 \times n_3} \);

(3) \( \Psi_{Z,H} \) is given (in coordinates) as the map

\[
\Psi_{Z,H} : \mathbb{F}^{m'} \otimes (\mathbb{F}^{n_3})^* \to (\mathbb{F}^{n'})^* \otimes \mathbb{F}^{n_1},
\]

where \( m' = \dim(Z) \) and \( n' = \dim(H) \), and \( \Psi_{Z,H} \) is given by

\[
(12) \quad \Psi_{Z,H}(e_a \otimes e_k^*) = \sum_{ib} e_k^* \otimes e_i (\zeta_a \eta_b)_{ik}
\]

where \( \zeta = \{\zeta_1, \ldots, \zeta_{m'}\} \) is a basis for \( Z \) and \( \eta = \{\eta_1, \ldots, \eta_{n'}\} \) is one for \( H \), (so \( m', n' \) are the dimensions of \( Z, H \) respectively) where \( \zeta_a \eta_b \in \mathbb{F}^{n_1 \otimes n_3} \) is ordinary matrix multiplication, and \( (\zeta_a \eta_b)_{ik} = (\zeta_a \eta_b) \cdot e_{ik} \) is the \( i,k \)-entry of the matrix \( \zeta_a \eta_b \).

In particular, the rank of \( \Psi_{Z,H} \) is \( m'n_3 \) provided that its kernel is zero or, equivalently, with coordinates as above, for any \( x_1, \ldots, x_{m'} \in \mathbb{F}^{n_3} \) such that

\[
(13) \quad \sum_{a \in [m']} \zeta_a \eta x_a = 0 \in \mathbb{F}^{n_1}
\]

for all \( \eta \in H \) (or just all \( \eta = \eta_b \)) implies that \( x_a = 0 \) for all \( a \in [n_3] \).

Similarly, the rank of \( \Psi_{Z,H} \) is \( n'n_1 \), i.e., \( \Psi_{Z,H} \) is surjective, provided that for any some basis \( \eta_1, \ldots, \eta_{n'} \), for each \( b \in [n'] \), the sum of \( \text{Image}(\zeta \eta_b) \) over all \( \zeta \in Z \) such that \( \zeta \eta_{b'} = 0 \) for \( b' \neq b \) is all of \( \mathbb{F}^{n_1} \).

5.1.1. Proofs of Theorems 5.1–5.4. In this Subsubsection we prove the above four theorems by the following statements, each of which easily follows from the previous ones.

**Proofs of Theorems 5.1–5.4.** Let \( M, N \) be \( \mathbb{F} \)-vector spaces of the form \( \mathbb{F}^S \) for some \( S \) (or, more generally, any \( \mathbb{F} \)-vector spaces endowed with non-degenerate bilinear pairings), and let \( L : L \to L_1 \otimes L_2 \) be a map of \( \mathbb{F} \)-vector spaces. Fix subsets \( \mathcal{A} \subset M \) and \( \mathcal{B} \subset N \). Let \( (/\mathcal{A}) \) be the canonical map \( M \to M/\mathcal{A} \) and similarly for \( (/\mathcal{B}) \), and let \( \Omega : M \otimes N \otimes L \xrightarrow{(/\mathcal{A}) \otimes (/\mathcal{B}) \otimes L} (M/\mathcal{A}) \otimes (N/\mathcal{B}) \otimes L_1 \otimes L_2 \); clearly \( \Omega(\alpha \otimes \beta \otimes \gamma) = 0 \) if \( \alpha \in \mathcal{A} \) or \( \beta \in \mathcal{B} \). Hence if for some \( \tau \in M \otimes N \otimes L \) we have

\[
(14) \quad \tau = \sum_{\rho=1}^r \alpha_\rho \otimes \beta_\rho \otimes \gamma_\rho, \quad \mathcal{L} \gamma_\rho = \sum_{s=1}^{r_\rho} \ell_{1,\rho,s} \otimes \ell_{2,\rho,s},
\]

then

\[
\Omega(\tau) = \sum_{\rho,s} [\alpha_\rho]_\mathcal{A} \otimes [\beta_\rho]_\mathcal{B} \otimes \ell_{1,\rho,s} \otimes \ell_{2,\rho,s},
\]
where $[\alpha_{\rho}]_A$ denotes the $A$-class of $\alpha_{\rho}$ in $M/A$, and similarly for $[\beta_{\rho}]_B$, and hence
\begin{equation}
R(\Omega\tau) \leq \sum_{\rho \in I} \text{Rank}_{L_1 \otimes L_2}(L\gamma_{\rho}),
\end{equation}
where $I$ is the set of $\rho$ where $\alpha_{\rho} \notin A$ and $\beta_{\rho} \notin B$.

Now let $Z = A^\perp$. For each $\zeta \in Z \subset M$, we have that the morphism $\alpha \mapsto \zeta \cdot \alpha$ annihilates $A$. Hence viewing each $\zeta$ as inducing an element of $M^*$ that annihilates $A$, we get a natural map $\mu: Z \to (M/A)^*$; counting dimensions shows that $\mu$ is an isomorphism; the dual of $\mu$ is the map $\mu^*: (M/A) \to Z^*$, therefore also an isomorphism. Hence we get an isomorphism
\begin{equation}
\Phi = \mu^* \otimes \nu^* \otimes \text{id}_{L_1} \otimes \text{id}_{L_2} : (M/A) \otimes (N/B) \otimes L_1 \otimes L_2
\end{equation}
\begin{equation}
\longrightarrow Z^* \otimes H^* \otimes L_1 \otimes L_2.
\end{equation}

One way to bound $\Phi\Omega\tau$ is to consider it as a morphism
\begin{equation}
Z \otimes H \to L_1 \otimes L_2
\end{equation}
which takes $\zeta \otimes \eta \in Z \otimes H$ to

\begin{equation}
\sum_{\rho,s} (\alpha_{\rho} \cdot \zeta)(\beta_{\rho} \cdot \eta)\ell_1^{\rho,s} \otimes \ell_2^{\rho,s} = \sum_{\rho} (\alpha_{\rho} \cdot \zeta)(\beta_{\rho} \cdot \eta)\gamma_{\rho}.
\end{equation}

If $\zeta_* = \{\zeta_1, \ldots, \zeta_m\}$ is a basis for $Z$, and $\eta_* = \{\eta_1, \ldots, \eta_n\}$ one for $H$, then in coordinates $\Phi\Omega\tau$ is viewed as a morphism
\begin{equation}
\mathbb{F}^{m'} \otimes \mathbb{F}^{n'} \to L_1 \otimes L_2
\end{equation}

taking $e_a \otimes e_b \in \mathbb{F}^{m'} \otimes \mathbb{F}^{n'}$ to
\begin{equation}
\sum_{\rho} (\alpha_{\rho} \cdot \zeta_a)(\beta_{\rho} \cdot \eta_b)\gamma_{\rho}.
\end{equation}

In particular, for
\begin{equation}
\tau = \sum_{ijk \in [m_1 \times n_2 \times n_3]} f_{ij} \otimes g_{jk} \otimes e_{ki}
\end{equation}
and $L: \mathbb{F}^{n_3 \times n_1} \to \mathbb{F}^{n_3} \otimes \mathbb{F}^{n_1}$ the canonical identification, $\Phi\Omega\tau$ is viewed as a morphism taking $e_a \otimes e_b$ to
\begin{equation}
\sum_{ijk} (\zeta_a \cdot f_{ij})(\eta_b \cdot g_{ij})e_k \otimes e_i
\end{equation}
and Theorems 5.1, 5.2, and 5.3 follow. Since the $\gamma_\rho$ are not factored, this is part of the landscape of results that do not involve inner rank.

What is new in this article is what happens when we view $\Phi \Omega \tau$ as a morphism

$$Z \otimes L_1^* \rightarrow H^* \otimes L_2,$$

which takes $\zeta \otimes \lambda$ (so $\lambda \in L_1^*$) to

$$\sum_{\rho,s} (\alpha_\rho \cdot \zeta) \lambda(\ell^1_{\rho,s}) \nu^*([eta_\rho]_B) \otimes \ell^2_{\rho,s},$$

which, viewing this element of $H^* \otimes L_2$ as an element of $\text{Hom}(H, L_2)$, is the morphism

$$\eta \mapsto \sum_{\rho,s} (\alpha_\rho \cdot \zeta) \lambda(\ell^1_{\rho,s}) (\eta \cdot \beta) \ell^2_{\rho,s}.$$

Hence, in the special case of $\tau = (n_1, n_2, n_3)$ and $L: \mathbb{F}^{n_3 \times n_1} \rightarrow \mathbb{F}^{n_3} \otimes \mathbb{F}^{n_1}$ being the standard identification, if $\zeta_\bullet = \{\zeta_1, \ldots, \zeta_m\}$ is a basis for $Z$, then $\Phi \Omega \tau$ takes $\zeta_a \otimes e^*_c$ to the element of $\text{Hom}(H, \mathbb{F}^{n_1})$ given by

$$\eta \mapsto \sum_{ij} (\zeta_a \cdot e_{ij})(e^*_c(e_k))(\eta \cdot e_{jk})e_i,$$

i.e., the morphism

$$\left( \eta \mapsto \sum_i (\zeta_a \eta)_{ic} e_i = \text{Col}_c(\zeta_a \eta) \right) \in \text{Hom}(H, \mathbb{F}^{n_1}),$$

where $\text{Col}_c$ means the $c$-th column of the matrix.

To write this in coordinates (18) with $f_{ij}, g_{ij}$ both set to $e_{ij}$ imply that $\Phi \Omega \tau$ is the tensor

$$\sum_{abik} ((\zeta_a \eta_b) \cdot e_{ik}) e^*_a \otimes e^*_b \otimes (\zeta_a \eta_b) e_k \otimes e_i$$

and therefore, as a morphism $Z \otimes (\mathbb{F}^{n_3})^* \rightarrow H^* \otimes \mathbb{F}^{n_1}$, is given by (12).

The case of general $L$ that is an isomorphism $\mathbb{F}^{n_3 \times n_1} \rightarrow \mathbb{F}^{n_3} \otimes \mathbb{F}^{n_1}$ follows because such an $L$ does not change the ranks of $(n_1, n_2, n_3)$ and $\Psi_{Z,H}$, and does not change the formulas for $\Psi_{Z,H}$. □

5.2. A Rank Lemma. Landsberg [Lan14] (Lemma 2.2) eloquently states a standard lemma of use in [Blä03, Lan14]; we want to point out an interesting fact that one can prove based on this standard lemma, and its connection to inner rank. However, we do not know if this lemma can be applied to our methods.
Lemma 5.5. Let \( \mathbb{F} \) be a field, and \( \alpha_1, \ldots, \alpha_r \in \mathbb{F}^{n \times n} \) for integers \( r, n \) such that some linear combination of the \( \alpha_i \) is invertible, and \( \text{Rank}(\alpha_1) = s \). Then there exists an \( \mathbb{F} \)-linear combination of \( \alpha_1 \) and at most \( n - s \) of the \( \alpha_\rho \) with \( \rho \geq 2 \) that is invertible, provided that \( s \geq 2 \). If \( s = 1 \) then the same is true if \( \mathbb{F} \) is infinite or sufficiently large (as a function of \( n \)).

Proof. Let \( \mathbf{x} = (x_1, \ldots, x_r) \) be a vector of indeterminates, and let

\[
P(X) \overset{\text{def}}{=} \det(\sum_{\rho} x_\rho \alpha_\rho),
\]

Clearly \( P(\mathbf{x}) \) is of degree \( n \), and we now easily prove that that \( P(X) \) has a monomial involving \( x_1^s \): indeed, after applying appropriate isomorphisms of \( \mathbb{F}^n \) (one on the left, the other on the right) it suffices to consider the case where \( \alpha_1 \) is the diagonal matrix \( e_{11} + \cdots + e_{ss} \). The variable \( x_1 \) term appears in the minor determinant only in the upper left \( s \times s \) block, and is therefore multiplied by the lower determinant \( (n - s) \times (n - s) \) block.

Taking the \( n - s \) or fewer other variables of this monomial and restricting the other variables to 0 yields a nonzero polynomial in at most \( 1 + n - s \) variables of degree \( n \). Hence this nonzero polynomial (of degree \( n \)) attains a nonzero value in \( \mathbb{F}^n \) for \( \mathbb{F} \) sufficiently large\(^4\) as a function of \( n \), and for \( s \geq 2 \) and \( \mathbb{F} \) finite we apply the Chevalley-Warning theorem\(^5\). \(\Box\)

Inner rank implies that the average rank of matrices involved in a low rank expression for \( \kappa_3(n) \) is large. Since the matrices involved must span all of \( \mathbb{F}^{n \times n} \), we are wondering if facts like Lemma 5.5 may be interesting.

Corollary 5.6. Let \( \mathbb{F} \) be a field, and \( r, n, s, s' > 0 \) integers, and \( \alpha_1, \ldots, \alpha_r \in \mathbb{F}^{n \times n} \) such that some linear combination of the \( \alpha_i \) is invertible, and \( M = \alpha_1 + \cdots + \alpha_s' \) has rank \( s > s' \). Then there exists an \( \mathbb{F} \)-linear combination of \( M \) and at most \( n - s \) of the \( \alpha_\rho \) with \( \rho \geq s' + 1 \) that is invertible.

\(^4\) A quantified version of the fact that a nonzero polynomial in \( m \) variables has a nonzero value on “most” of its restrictions to a sufficiently large set is of use in randomized algorithms and appears in the works of DeMillo-Lipton, Schwartz, and Zippel (and perhaps others).

\(^5\) This theorem states that any finite field is pseudo algebraically closed, meaning that any polynomial of degree higher than its number of variables has a solution. This condition is motivated by the fact that it implies that the Brauer group of the field is trivial.
6. Some Remarks on $R(\kappa_3(n))$ for Small $n$

In this section we gather some definitions and make some optimistic conjectures to get an idea of how inner rank—especially Theorem 5.4—might have implications for lower bounds on $R(\kappa_3(n))$. To explore conjectures that would get new results for small $n$ (e.g., $n = 3$ or $n = 4$), we discuss what we call overlap and consider some ad hoc arguments that lead to interesting questions regarding reducing rank sums via an isomorphism $\mathcal{L}$ in $\mathbb{F}^{n \times n}$ akin to Lemma 4.8.

Let us motivate the notion of overlap. Consider the equation

$$\kappa_3(n) = \sum_{\rho=1}^{r} \alpha_{\rho} \otimes \beta_{\rho} \otimes \gamma_{\rho}$$

in view of the following conjecture.

**Conjecture 6.1.** Let $\mathbb{F}$ be a field and $n \geq 1$ an integer. Say that $\{\zeta_1, \ldots, \zeta_{n^2}\}$ and $\{\eta_1, \ldots, \eta_{n^2}\}$ be two bases for $\mathbb{F}^{n \times n}$. Then there exists a subspace $Z \subset \mathbb{F}^{n \times n}$ spanned by $n$ of the vectors $\zeta_{\rho}$, and a subspace $H$ by $n$ of the $\eta_{\rho}$, such that $\Psi_{Z,H}$ has (full) rank $n^2$.

We do not know if this conjecture has been studied. We do not necessarily believe or require the above conjecture; some version of it—in relation to $\zeta_{\rho}$ and $\eta_{\rho}$ related to (19)—would suffice. Less optimistically there are many pairs $(Z, H)$ with $\Psi_{Z,H}$ having rank at least $n^2$, but either the dimension of $Z$ or that of $H$ has to be larger than $n$.

We remark that if the $\{\zeta_{\rho}\}$ and $\{\eta_{\rho}\}$ consist entirely of standard basis vectors $e_{ij}$, then the above conjecture holds, and there is a fair amount of freedom in choosing $Z$ and $H$. See Example 8.2 for a related remark.

**Theorem 6.2.** Assume that Conjecture 6.1 holds for some $\mathbb{F}$ and some value of $n \geq 2$. Assume that (19) holds. Then either:

1. $r \geq 3n^2 - 2n$, or
2. either there exists some subset of $n^2$ (or fewer if $r < n^2$) of the $\gamma_{\rho}$ that are linearly dependent, or similarly for the $\alpha_{\rho}$, or for the $\beta_{\rho}$.

**Proof.** Assume that Item (2) does not hold, and first consider the case $r \geq 2n^2$. Then $\alpha_1, \ldots, \alpha_{n^2}$ are linearly independent; let for $1 \leq a \leq n^2$, let $\zeta_a$ be any nonzero vector in $\mathbb{F}^{n \times n}$ (unique up to scalar) such that $\zeta_a \cdot \alpha_{a'} = 0$ for all $a' \neq a$ with $a' \leq n^2$; do similarly for $\beta_{n^2+1}, \ldots, \beta_{2n^2}$ and $\eta_b$ with $n^2+1 \leq b \leq 2n^2$. Now choose $Z$ and $H$ as in the conjecture, and let $I$ be $\lfloor r \rfloor$ where we discard $a \leq n^2$ if $\zeta_a$ was not chosen to span.

---

6 Of course, we already know that (19) implies that $r \geq 2n^2 - n + 1$, so $r \geq n^2$. 

Z, and similarly for $n^2 + 1 \leq b \leq n^2$. Hence $I$ is of size $2n$. Let $I'$ be $I$ and the integers $2n^2 + 1, \ldots, r$.

If $|I'| < n^2 - 1$, then the $\gamma_\rho$ with $\rho \in I'$ can be taken to elements of rank one by some isomorphism $\mathcal{L}$ of $\mathbb{F}^{n \times n}$. But the rank of $\Psi_{Z,H}$ as in (12) is $n^2$, while the left-hand-side (11) is at most $n^2 - 1$. Hence $|I'| \geq n^2$. Hence $r - 2(n^2 - n) \geq n^2$.

If $r < 2n^2$, then we can add $\alpha_\rho, \beta_\rho$ for $r < \rho \leq 2n^2$ to make $\{\alpha_1, \ldots, \alpha_{n^2}\}$ and $\{\beta_{n^2+1}, \ldots, \beta_{2n^2}\}$ be bases for $\mathbb{F}^{n \times n}$ and argue similarly. □

As an example, Strassen’s algorithm does not satisfy Item (2) above, as it has three $\alpha_\rho$ (and three $\beta_\rho$ and three $\gamma_\rho$) that are linearly dependent. Let us make some remarks about Item (2).

Definition 6.3. Let $v_1, \ldots, v_t \in V$ where $V$ is an $\mathbb{F}$-vector space of dimension $p$. By the overlap function of $v_1, \ldots, v_t$ we mean the function

$$\text{Overlap}(U) \overset{\text{def}}{=} |\{v_1, \ldots, v_t\} \cap U| - \dim(U)$$

ranging over all proper subspaces $U$ of $V$. Say that the maximum proper overlap of $v_1, \ldots, v_t$ is the maximum of Overlap($U$) over all proper subspaces, $U$, of $V$.

We remark that the function Overlap is clearly supermodular, i.e.,

$$\text{Overlap}(U_1) + \text{Overlap}(U_2) \leq \text{Overlap}(U_1 \cap U_2) + \text{Overlap}(U_1 + U_2),$$

where $U_1 + U_2$ denotes the span of $U_1$ and $U_2$. The supermodularity arises from the fact that

$$|\{v_1, \ldots, v_t\} \cap U_1| + |\{v_1, \ldots, v_t\} \cap U_2| \leq |\{v_1, \ldots, v_t\} \cap (U_1 + U_2)|$$

where inequality is possible. Just as with maximum excess ([Fri15]), this means that subspaces, $U$, at which Overlap($U$) takes its maximum values is a lattice under $\cap$ and $+$; this is also true if we restrict Overlap($U$) to an ambient space smaller than $V$ (i.e., for the function $U \mapsto \text{Overlap}(U \cap V')$ for any $V'$); this also true with $\dim(U)$ replaced with any multiple of $\dim(U)$, since $\dim(U)$ is modular.

Theorem 6.2 says that if Conjecture 6.1 holds for some $\mathbb{F}$, $n$ and (19) holds with $r < 3n^2 - 2n$, then at least one of the sets $\{\alpha_\rho\}$, $\{\beta_\rho\}$, and $\{\gamma_\rho\}$ has overlap at least one. For example, if $R(\kappa_3(3))_\mathbb{F} \leq 20$, then either Conjecture 6.1 fails for $\mathbb{F}$ or there must be overlap. We note that we do not need the full strength of Conjecture 6.1, but only some version of it applicable to the $\zeta_\alpha$ and $\eta_\beta$ that result from the method of Theorem 6.2, under the assumption that (19) holds (moreover, with (19) holding for $r$ below some upper bound).
It is instructive to see how the overlap of 1 in the families of matrices in Strassen’s algorithm \((n = 2)\) foils the method of the proof of Theorem 6.2. On the other hand, if there is a large amount of overlap in the \(\alpha_\rho, \beta_\rho, \gamma_\rho\) of (19), one might—yet more optimistically—conjecture that we could exploit this overlap to annihilate more of the \(\alpha_\rho\) and/or \(\beta_\rho\) than merely \(n^2 - n\) of each. Such conjectures lead to interesting questions: consider the following optimistic conjecture.

**Conjecture 6.4.** Assume that (19) holds for some \(F\) and \(n\). Then if the maximum overlap is \(p\), one can find \(Z, H\) — possibly by first exchanging the \(\alpha_\rho\) with the \(\gamma_\rho\), or the \(\beta_\rho\) with the \(\gamma_\rho\)—so that \(\Psi_{Z,H}\) has rank \(n^2\) and \(\text{supp}(Z, H)\) is at least \(n^2 - 2n + p\).

Notice that we do not assume that \(Z, H\) are necessarily each of dimension \(n\). Also, we are always free to exchange the order of the three factors of \(F^{n \times n}\) in the tensor product to assume that the overlap among the \(\{\alpha_\rho\}\) is no less than that among the \(\{\beta_\rho\}\), and which is no less than that among the \(\{\gamma_\rho\}\).

Again, we do not necessarily believe the above conjecture; however, consider its consequences for \(n = 3\), with an expression (19) with \(r\) as small as possible (i.e., equal to \(R(\kappa_3(3))\)): if the maximum overlap is \(p\), then:

1. for \(p = 0\), we have \(r \geq 21\);
2. for any \(p\), we have that \(r \geq 21\) provided that any set of \(8 - p\) vectors in \(F^{3 \times 3}\) that span a subspace of dimension at least \(8 - 2p\) can be taken via an isomorphism \(\mathcal{L}\) of \(F^{3 \times 3}\) to a set of elements whose sums of ranks is at most 8;
3. for any \(p\), we have that \(r \geq 20\) provided that any set of \(7 - p\) vectors in \(F^{3 \times 3}\) that span a subspace of dimension at least \(7 - 2p\) can be taken via an isomorphism \(\mathcal{L}\) of \(F^{3 \times 3}\) to a set of elements whose sums of ranks is at most 8;
4. etc.

If weaker conjectures holds, then one may get a set of similar questions. One can also consider variants of the above making \(\text{Rank}(\Psi_{Z,H})\) larger than \(n^2\). Even if one cannot improve upon known bounds with these methods, it would be interesting to consider such questions and conjectures and see what is their limit in establishing \(R(\kappa_3(n))\) lower bounds.

7. Generalizations of the Rank Lower Bound

In this section we give a broad generalization of the inner rank bound Corollary 4.4, and then discuss some special cases.
7.1. Abstract Generalization. Of course, the method of the previous section applies to matrix multiplication tensors of matrices that are not square, and we can embellish the method by introducing linear maps $\mathcal{M}, \mathcal{N}, \mathcal{L}$ on the three factors of the tensor product as follows.

**Lemma 7.1.** Let

\[
\mathcal{M}' : M \to M', \quad \mathcal{N}' : N \to N', \quad \mathcal{L}' : L \to L'_1 \otimes L'_2
\]

be morphisms of $\mathbb{F}$-vector spaces. Let $\pi$ be the natural isomorphism

\[
\pi : M' \otimes N' \otimes L' \to \text{Hom}((M' \otimes L'_1)^*, N' \otimes L'_2).
\]

Then if $\tau \in \sum_{\rho=1}^I \alpha_{\rho} \otimes \beta_{\rho} \otimes \gamma_{\rho} \in M \otimes N \otimes L$,

\[
(20) \quad \text{Rank}(\Psi) \leq \sum_{\rho \in I} \text{Rank}_{L'_1 \otimes L'_2}(\mathcal{L}\gamma_{\rho})
\]

where

\[
\Psi = \Psi_{\mathcal{M}', \mathcal{N}', \mathcal{L}'} \overset{\text{def}}{=} \pi(\mathcal{M}' \otimes \mathcal{N}' \otimes \mathcal{L}')\tau
\]

(so $\Psi : (M' \otimes L'_1)^* \to N' \otimes L'_2$, and

\[
I = \text{Supp}_{\rho \in [r]}(\mathcal{M}(\alpha_{\rho}) \otimes \mathcal{N}(\beta_{\rho}))
\]

is the set of $\rho$ for which both $\mathcal{M}(\alpha_{\rho})$ and $\mathcal{N}(\beta_{\rho})$ are nonzero.

In particular if for integers $n_1, n_2, n_3, m_1, m'_3, m'_1 \geq 1$, for the special case of $\tau = \langle n_1, n_2, n_3 \rangle$ and $L'_1 = \mathbb{F}^{m_3}$ and $L'_2 = \mathbb{F}^{m_1}$ in the above, we have

\[
\mathcal{M} : \mathbb{F}^{n_1 \times n_2} \to M', \quad \mathcal{N} : \mathbb{F}^{n_2 \times n_3} \to N', \quad \mathcal{L} : \mathbb{F}^{m_3 \times n_1} \to \mathbb{F}^{m_3} \otimes \mathbb{F}^{m_1},
\]

and then

\[
(21) \quad \Psi : (M' \otimes \mathbb{F}^{m_3})^* \to N' \otimes \mathbb{F}^{m_1}
\]

is the unique linear map such that for all $a \in [m']$ and $c \in [m_3]$ we have

\[
(22) \quad \Psi(e_a^* \otimes e_c^*) = \sum_{ijk \in [n_1] \times [n_2] \times [n_3]} (e_a \cdot \mathcal{M}(e_{ij})) \mathcal{N}(e_{jk}) \otimes (e_c^T \mathcal{L}(e_{ki}))
\]

where $e_c^T \mathcal{L}(e_{ki})$ denotes the contraction of $e_c$ into the first component of $\mathcal{L}(e_{ki})$, i.e., as multiplying a row vector by a matrix on the right. In coordinates, if

\[
(23) \quad \chi = \sum_{a \in [m'], c \in [m_3]} \chi_{ac} e_a^* \otimes e_c^*, \quad \chi_{ac} \in \mathbb{F},
\]
then
\[ \Psi(\chi) = \sum_{ac} \chi_{ac} \sum_{ijk} (e_s \cdot M(e_{ij})) N(e_{jk}) \otimes (e_c^T L(e_{kl})). \]

Let us make some remarks before giving the proof. We use (23) and (24), since sometimes it will be simpler to introduce coordinates. We warn the reader that in Lemma 4.1, the two indices \( a, b \) correspond to a single index \( a \) here; this is because \( M' \) here is general, as opposed to there where \( M = \text{id}_M \).

The proof of the above lemma is a simple adaptation of the proof of Corollary 4.4.

**Proof.** The proof is a simple generalization of the inner rank used before. The rank of \( \langle n_1, n_2, n_3 \rangle \) is no greater than that of its image under \( M \otimes N \otimes L \), which equals
\[
\sum_{\rho=1}^r \text{Rank}_{m_3 \times m_1}(L_{\gamma_{\rho}}) = \sum_{\rho} \sum_{s=1}^{\text{Rank}_{m_3 \times m_1}(L_{\gamma_{\rho}})} (M_{\alpha_{\rho}}) \otimes (N_{\beta_{\rho}}) \otimes \ell_{\rho}^{1,s} \otimes \ell_{\rho}^{2,s}
\]
for appropriate \( \ell_{\rho}^{i,s} \); of course, it suffices to sum over \( \rho \) for which both \( M_{\alpha_{\rho}} \) and \( N_{\beta_{\rho}} \) are nonzero. This shows that
\[
R(\Psi') \leq \sum_{\rho \in I} \text{Rank}_{m_3 \times m_1}(L_{\gamma_{\rho}}),
\]
where \( \Psi' \overset{\text{def}}{=} \pi(M \otimes N \otimes L) \tau \). Viewing \( \Psi' \) as a map \( \Psi \) as in (21), (22) is immediate, and (23) and (24) is just (22) in coordinates. \( \square \)

If \( m_3 = 1 \) or \( m_1 = 1 \) in Lemma 7.1, or \( L_1' \) or \( L_2' \) in the general case, then the inner rank is always one or zero. We call this **degenerate** inner rank; in this case Lemma 7.1 could be derived without using the idea of inner rank. Presumably any such inequality has appeared in the previous literature (at least implicitly).

To apply the above lemma we need to choose \( M, N, L \) judiciously and obtain reasonable bounds on the rank of the above map. In some of the subsections that follow, we consider some special cases of the lemma, which help to understand its scope and to yield concrete applications.

### 7.2. A Generalization of Inner Rank Bound

**Proposition 7.2.** Let \( L \) be any linear map on \( \mathbb{F}^{n_3 \times n_1} \). Then
\[
n_2 \text{Rank}_{n_3 n_1 \times n_3 n_1}(L) \leq \sum_{\rho=1}^r \text{Rank}_{n_3 \times n_1}(L_{\gamma_{\rho}}),
\]
Proof. Let $\mathcal{M}, \mathcal{N}$ be the identity maps. The image of $\Psi$

\[
\sum_{a,b,c \in [n_1 \times n_2 \times n_3]} \chi_{abc} \sum_{ijk} (e_{ab} \cdot e_{ij}) e_{jk} \otimes e^T_c \mathcal{L}(e_{ki})
\]

\[
= \sum_{a,b,c \in [n_1 \times n_2 \times n_3]} \chi_{abc} \sum_k e_{bk} \otimes e^T_c \mathcal{L}(e_{ka})
\]

taken over all choices of $\chi_{abc} \in \mathbb{F}$. Since the $b$ serves as a “place holder” as it appears in $e_{bk}$, the dimension of this space is $n_2$ times that of

\[
\sum_{a,c,k} \chi_{ac} e_k \otimes e^T_c \mathcal{L}(e_{ka})
\]

whose dimension is the same as that of

\[
\sum_{a,c,k} \chi_{ac} \mathcal{L}(e_{ka}) = \sum_{a,c} \chi_{ac} \mathcal{L}(e_{ca})
\]

which is just the rank of $\mathcal{L}$. □

7.3. The Very Degenerate Case.

Theorem 7.3. Let $\mathcal{M}, \mathcal{N}, \mathcal{L}$ be linear transformations as in Lemma 7.1 whose targets are $M' = N' = L' = \mathbb{F}$. Assume that

\[
\sum_{ijk} \mathcal{M}_{ij} \mathcal{N}_{jk} \mathcal{L}_{ki} \neq 0,
\]

where $\mathcal{M}_{ij} = \mathcal{M}(e_{ij})$, etc. Then for some $r \in [r]$ we have that $\mathcal{M}(\alpha_{r}), \mathcal{N}(\beta_{r}), \mathcal{L}(\gamma_{r})$ are all nonzero.

This uses nothing about inner rank, and we presume this is known or at least strongly related to known methods. Its proof is immediate from Lemma 7.1, since then $\Psi$ is a map $\mathbb{F} \to \mathbb{F}$, and there is only one value for both $a$ and $c$.

Of course,

\[
\sum_{ijk} \mathcal{M}_{ij} \mathcal{N}_{jk} \mathcal{L}_{ki} = \sum_{ij} \mathcal{M}_{ij}(\mathcal{N}\mathcal{L})_{ji},
\]

an expression akin to the usual Frobenius inner product of real matrices.

\footnote{More formally, we say that $n_2$ subspaces $A_1, \ldots, A_{n_2}$ of some ambient vector space are linearly independent if $a_i \in A_i$, and $\sum_i a_i = 0$ implies that $a_i = 0$ for all $i$. This independence holds if $A_k$ are the span $e_{bk} \otimes v$ for arbitrary $k \in [n_3]$ and $v \in \mathbb{F}^{N_3}$, and furthermore each $A_k$ is isomorphic to the dimension of the sum where $b$ is dropped from $\chi_{abc}$, from $e_{bk}$, and from the summation.}
As mentioned in Subsection 5.1, the $M_{ij}$ are morally the coefficients of

$$M = \sum_{ij} M_{ij} e_{ij}^\ast,$$

and hence a product of matrices $N L$ morally is a “co-multiplication” in $(\mathbb{R}^{n_2 \times n_3})^\ast$, i.e. the rule

$$e_{kj}^\ast \text{ co-prod } e_{ji}^\ast = e_{ki}^\ast \delta_{jj'}$$

with $\delta_{jj'}$ being the Dirac delta function.

7.4. Degenerate and Small $M, N$. Another interesting case of Lemma 7.1 is the case where $M' = N' = \mathbb{F}$. This would be designed to make the set $I$ in the lemma—i.e., the set where $M \alpha_\rho$ and $N \beta_\rho$ are both nonzero—as small as possible, while retaining the inner rank method. To fix ideas, first take $L$ to be an invertible operator on $\mathbb{F}^{n_3 \times n_1}$. We get that the rank of the resulting

$$\Psi: \mathbb{F}^{n_3} \rightarrow \mathbb{F}^{n_1}$$

is bounded below by $\sum_{\rho \in I} \text{Rank}(L \gamma_\rho)$. The rank of this $\Psi$ is at most $\min(n_3, n_1)$.

If wants to make use that the flexibility in $L$ allows us to send any basis of $\mathbb{F}^{n_3n_1}$ to rank one matrices, it may be better to take $M', N'$ of small dimension—of size roughly $n_1$ or $n_3$—rather than of dimension one.

In the case of $\kappa_3(n)$, this means we might take $M', N'$ of dimension, say, $n + 1$ (perhaps a bit larger), so if $I$ is of size $n^2$ we choose $L$ to have $\sum_{\rho \in I} \text{Rank}(L \gamma_\rho) = n^2$, which would yield a contradiction if we could find $M, N$ so that the resulting $\Psi$ would have full rank $n(n + 1)$, or at least rank $n^2 + 1$.

The above approach is essentially what was considered in Sections 5 and 6.

7.5. Trivial Tensoring. The linear map $\Psi$ of Lemma 7.1 is naturally associated to the tensor $\Psi' = \pi(M \otimes N \otimes L) \tau$ in the proof, which is therefore built from a given $\tau$ and morphisms

$$M: M \rightarrow M', \ N: N \rightarrow N', \ L: L \rightarrow L',$$

where $M, N, L$ are determined from the tensor $\langle n_1, n_2, n_3 \rangle$. One might ask if for fixed $n_1, n_2, n_3$ the rank of $\Psi$—which produces lower bounds—is bounded as a function of $n_1, n_2, n_3$. We claim that for trivial reasons the answer is no, but this trivial reason doesn’t change the bounds we get.
For example, in the above situation, if \(d\) is any fixed integer, define \(L_d: L \rightarrow L' \otimes \mathbb{F}^{d^2}\) defined by \(L_d(\gamma) = \gamma \otimes I_d\), where \(I_d \in \mathbb{F}^{d \times d}\) is the identity matrix; since \(L' = \mathbb{F}^{m_3 \times m_1}\), may permute coordinates to obtain a map \(L'_d: L \rightarrow \mathbb{F}^{(m_3 d) \times (m_1 d)}\). We easily see that replacing \(L'_d\) for \(L_d\) in the original situation, with \(\pi_d\) being the evident modification of \(\pi\), yields the modified map

\[
(26) \quad \Psi_d: (M' \otimes \mathbb{F}^{m_3 d})^* \rightarrow N' \otimes \mathbb{F}^{m_1 d}
\]

given by \(\Psi_d = \Psi \otimes I_d\), which has \(d\) times the rank of \(\Psi\); so as \(d \rightarrow \infty\), this rank is arbitrarily large (unless \(\Psi = 0\)). Of course,

\[
\text{Rank}_{(m_3 d) \times (m_1 d)}(L'_d \gamma \rho) = d \text{Rank}_{m_3 \times m_1}(L \gamma \rho),
\]

so the resulting lower bound is simply multiplied by \(d\) on both sides and achieves nothing. We therefore call this construction \textit{trivial tensoring}.

This suggests the possibility that one might replace the spaces \(\mathbb{F}^{m_3}, \mathbb{F}^{m_1}\) by spaces \(\mathbb{F}^{m_3 d}, \mathbb{F}^{m_1 d}\), in a way that takes a \(\otimes d\) tensor power but “twists” these spaces—rather than mere tensoring by \(I_d\)—and consider if some twists can improve lower bounds; in Subsection 7.7 in will discuss something along these lines.

7.6. Inequality Tensoring. For \(i = 1, 2\), if \(\Psi_i\) results from maps

\[
M_i: M_i \rightarrow M'_i, \quad N_i: N_i \rightarrow N'_i, \quad L_i: L_i \rightarrow L'_i,
\]

for \(i = 1, 2\), then forming \(M_{12}(\alpha) \overset{\text{def}}{=} M_1(\alpha) \otimes M_2(\alpha)\), and \(N_{12}, L_{12}\) similarly, there results a \(\Psi_{12}\) that is just \(\Psi_1 \otimes \Psi_2\). This has the effect of multiplying two inequalities together, which is no stronger an inequality than the original two considered individually. However, it may be easier to study the rank of \(\Psi_{12}\) and than that of the individual \(\Psi_i\), and similarly for \(\sum_{\rho \in I_1 \times I_2} \text{Rank}(L_{12} \gamma \rho)\), with \(I_1, I_2\) the analogs of \(I\) in (20); similarly for tensoring together \(M_i, N_i, L_i\) for three or more values of \(i\).

In the case where \(M_i, N_i, L_i\) arise from tensors \(\langle n_1^1, n_1^2, n_1^3, n_2^1, n_2^2, n_2^3, n_3^1, n_3^2, n_3^3 \rangle\).

An example of this inequality tensoring is to take the \(n_1 = n_2 = n_3 = 1\) and \(M = N = L\) to be the identity maps; if \(\Psi\) is formed from the tensoring with \(I_d\) for some \(d \geq 1\) in the manner of trivial tensoring of Subsection 7.5, we get a trivial inequality \(d \leq d\). If we tensor this with any other instance of Lemma 7.1, we are performing the more general instance of trivial tensoring.

As in the previous sections, this suggests that one might want to tensor two inequalities in a way that twists them somehow, akin to Subsection 7.7 below.
7.7. Strassen’s Equations, Following Landsberg-Ottaviani. A possible way to strengthen Lemma 7.1 is evident from the elegant conceptual description of Strassen’s equations [Str83] by Landsberg and Ottaviani and their generalization [Lan14, LO15]. Here is their view.

If \( F \) is a field and \( A, B, C \) are \( F \)-vector spaces, then an element of \( A \otimes B \otimes C \) naturally gives rise to a map \( \phi: B^* \to A \otimes C \). For \( \kappa_3(n) \) this produces a linear map of rank at most \( n^2 \), as \( A, B, C \) are each of dimension \( n^2 \), which does not give an interesting rank lower bound as is. Landsberg and Ottaviani ([LO15], Section 2) take \( \phi \) and produce the map

\[
\pi: B^* \otimes \Lambda^p(A) \to (A \otimes C) \otimes \Lambda^p(A) \to \Lambda^{p+1}(A) \otimes C,
\]

and explain the equations of Strassen [Str83] as the case \( p = 1 \). Then they take a subspace \( A' \subset A \) and restrict, obtaining a map with \( A' \) replacing \( A \) (Theorem 3.1 of [LO15], or, more simply, equation (1) of [Lan14]).

It would be interesting to know if such methods can be combined with the method of inner rank. From our point of view, the tensor \( \tau = (n_1, n_2, n_3) \) of \( A \otimes B \otimes C \), is associated to \( \pi \tau \in B^* \to A \otimes C \) in a degenerate form Lemma 7.1, with \( m_3 = 1 \) and \( \pi \) as in the proof of Lemma 7.1. Tensoring gives

\[
\pi \tau \otimes \text{id}_L \colon B^* \otimes L \to (A \otimes C) \otimes L
\]

where \( L \) is any vector space and \( \text{id}_L \) the identity there.

One could view tensoring with \( L \) as producing a (trivial) “moduli space” or “family” of maps \( B^* \to A \otimes C \), parameterized by \( L^* \), since each element of \( L^* \) can be “applied” (or contracted into) to “both sides” of (28) to obtain a map \( B^* \to A \otimes C \) (this parameterization in invariant under scaling and hence also lives on the projectivization of \( L^* \)). From this point of view we are not producing \( M, N, L \) with larger targets through which \( \pi \tau \) acts, but fixing \( M, N, L \) and creating a moduli space of such maps; unfortunately merely applying \( \otimes L \) to both sides yields the moduli space of morphisms \( B^* \to A \otimes C \), each of which is the same morphism. The last step is to choose \( L = \Lambda^p A \), which gives us a type of “twist” that we were looking for earlier by linking \( A \) and \( L \) together in the map \( A \otimes L \to L' = \Lambda^{p+1} A \). After this, comes the serous matter of getting this to work, including introducing \( A' \subset A \) there and making some computation.

However, regardless of how one conceptually explains things (there might be different and more useful explanations), the bottom line is that a small rank for \( \tau \) implies one for the resulting linear map. Certainly the idea behind inner rank is in the same vein as these previous
works: since $A, B, C$ are of equal dimension, we split $C$ as $C_1 \otimes C_2$ to obtain maps $A^* \otimes C_1^* \rightarrow B \otimes C_2$ whose ranks can be as large as $n^3$, rather than $n^2$ when $A, B, C$ are divvied up between source and target without splitting. For inner rank the downside is that the (inner) rank of our $\gamma$ can be as large as $n$.

We therefore find that the idea of trying to combine inner rank with the above “twisting methods” merits further study.

8. The Quotient Case of Lemma 7.1

In this section we study what we call the “quotient question,” which we now formalize.

**Definition 8.1.** A *quotient question* consists of the following data, $Q$:

1. $\tau \in M \otimes N \otimes L$, a tensor product of $\mathbb{F}$-vector spaces,
2. a morphism $\mathcal{L}: L \rightarrow L_1' \otimes L_2'$,
3. subspaces $A \subset M$ and $B \subset N$.

Given this data, $Q = (\tau, A, M, B, N, L, \mathcal{L})$ (with $\mathbb{F}$ understood), we set

$$
\text{QuotQuest}(Q) = \text{QuotQuest}(\tau, A, M, B, N, L, \mathcal{L})
$$

to be the set of all possible values of $\Psi$ over all $\Psi = \Psi_{M,N}$ as in (21) in the special case

$$
M' = M/\mathcal{A}, \quad N' = N/\mathcal{B}.
$$

We define

$$
\text{MaxRank}(Q) \overset{\text{def}}{=} \max\{\text{Rank}(\Psi) \mid \Psi \in \text{QuotQuest}(Q)\}.
$$

Often we will write $Q = (L; A, B)$, or even omit the $\mathcal{L}$, when the other parameters are understood (which they are just from $\tau$).

The rank

$$
\Psi: (M' \otimes L_1')^* \rightarrow N' \otimes L_2',
$$

is bounded by both the dimension of its source and its target. Theorem 5.4 discusses a special case of a quotient question.

We will consider this question in the case $\tau = (n_1, n_2, n_3)$ and first study it in the case $\mathcal{L} = \text{id}$ being the identity (this means that $L = L_1' \otimes L_2'$). In this case we have

$$
\text{MaxRank}(A, B) \leq n_1n_2n_3 - \text{Optimistic}(A, B), \quad \text{(29)}
$$

where

$$
\text{Optimistic}(A, B) \overset{\text{def}}{=} \min\{n_3 \dim(A), n_2 \dim(B)\}.
$$

Our goal in studying this *quotient question* is to give ways to compute or estimate MaxRank($Q$), especially when (29) is satisfied.
8.1. **Motivation.** This subsection shows that, in principle, any rank lower bounds that can be produced with Lemma 7.1 can also be produced with a quotient question; hence, assuming we can solve and understand all quotient questions, there is no point considering any further case of Lemma 7.1. Of course, there might be an instance of Lemma 7.1 that might be more convenient to work with than a quotient question. To explain this principle, let us introduce some notation.

Following, for example, [BCS97] (Definition 14.27, replacing $W, W'$ there with their duals), we write $\tau \preceq \tau'$ whenever one can write $\tau' = (A_1 \otimes \ldots \otimes A_m) \tau$ and we will say $\tau$ can be reduced to $\tau'$. It follows that $R(\tau) \geq R(\tau')$ in this case (and similarly for $R$ replaced with $\overline{R}$ defined below), and $\preceq$ is a preorder; see [BCS97] for more on this preorder. Here is an example.

If in the general situation of Lemma 7.1 we have $M': M' \to M''$, $N': N' \to N''$, and $L' = \text{id}_{L'_1 \otimes L'_2}$ is the identity, then we easily see that $\Psi_{M',N',L'} \preceq \Psi_{M,M''},N',L'$, and hence

$$\text{Rank}(\Psi_{M,N,L}) \geq \text{Rank}(\Psi_{M',M''},N',L').$$

and hence if we can compute left-hand-side rank in some case, the right-hand-side produces no better a lower bound in (20). Furthermore, these two ranks are equal if $M', N', L'$ are injections.

But any map $\mathcal{M}: M \to M'$ factors uniquely as a surjection followed by an injection, namely as $M \to M A \to M'$ where $A = \ker(\mathcal{M})$. Hence the discussion from the previous paragraph implies that for any instance of Lemma 7.1 we have

$$\text{Rank}(\Psi_{M,N,L}) \leq \text{MaxRank}(L, A, \mathcal{B}).$$

Hence knowing the “solution” to all the quotient questions gives the “best possible lower bounds” in (20).

8.2. **An Illustrative Example.** Ideally, given $\mathcal{M}, \mathcal{N}, \mathcal{L}$ we would be able to determine $\text{Rank}(\Psi_{M,N,L})$. Of course, one could compute this rank in polynomial by doing so modulo a sufficiently large set of primes. However, we do not have good theorems that allow us to compute this rank in many cases. So in this subsection we one interesting example where this rank is easy to determine.

**Example 8.2.** Let $n_1, n_2, n_3 \geq 1$ be integers, and $\mathbb{F}$ a field. Let $J_1 \subset [n_1] \times [n_2]$ and let $\neg J_1$ be its complement in $[n_1] \times [n_2]$: let $\mathcal{A}$ be the
span of $e_{ij}$ over those $(i, j) \in J_1$; and similarly for $J_2 \subset [n_2] \times [n_3]$, $\neg J_2$ and $\mathcal{B}$. The fact that $\langle n_1, n_2, n_3 \rangle$ induces an isomorphism

$$e_{a_1, a_2}^* \otimes e_c^* \mapsto e_{a_2} \otimes e_{a_1}$$

(via our usual $\pi$) makes it easy to see that

$$\text{Rank}(\Psi/\mathcal{A},/\mathcal{B},\text{id}) = |\{(i, j, k) \ | \ (i, j) \in \neg J_1, (j, k) \in \neg J_2\}|.$$ Such an example is easy because of the “compatibility” of the spaces $\mathcal{A}$ and $\mathcal{B}$.

9. Other Remarks

In this section we discuss some briefly discuss some other approaches to studying $\kappa_3(n)$ which may prove useful in future work. We introduce some higher order tensors related to matrix multiplication, valid for any field $\mathbb{F}$. We discuss a number of ways that one might exploit standard linear algebra involving inner products and orthogonality, in the case where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$ (or a subfield thereof). Some of our remarks involve standard facts regarding tensor products; we explain why they may be of interest in the study of $\kappa_3(n)$.

We assume that $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, unless otherwise indicated. We note that common applications often only concern this assumption on $\mathbb{F}$, and much of the literature deals only with these cases.

9.1. Some Related Tensors. In this subsection, $\mathbb{F}$ is an arbitrary field.

**Definition 9.1.** For a field, $\mathbb{F}$, an integer $m \geq 1$, and integers $n_1, \ldots, n_m$, define the **cyclic tensor on** $n_1, \ldots, n_m$ **over** $\mathbb{F}$ to be

$$\langle n_1, \ldots, n_m \rangle_{\mathbb{F}} \overset{\text{def}}{=} \sum_{i_j \in [n_j]} e_{i_1, i_2} \otimes e_{i_2, i_3} \otimes \cdots \otimes e_{i_{m-1}, i_m} \otimes e_{i_m, i_1}$$

which is therefore lies in a tensor product of $m$ matrix spaces; we similarly define the **open cyclic tensors on** $n_1, \ldots, n_m$ **over** $\mathbb{F}$ to be the collection of tensors indexed on $i_1 \in [n_1]$ and $i_m \in [n_m]$

$$\omega_m(i_1, i_m)(n_1, \ldots, n_m)_{\mathbb{F}} \overset{\text{def}}{=} \sum_{i_j \in [n_j]} \text{ for } 1 < j < m \quad e_{i_1, i_2} \otimes e_{i_2, i_3} \otimes \cdots \otimes e_{i_{m-1}, i_m},$$

which is therefore lies in a tensor product of $m - 1$ matrix spaces. For a positive integer, $n$, we use $\kappa_m(n)_{\mathcal{F}}$ to denote the case $n_1 = \cdots = n_m = [n]$, and write $\kappa_m(n)$ if $\mathbb{F}$ is understood.
Notice that if
\[ E_{i,i'} \overset{\text{def}}{=} \text{Span}_{i_2,\ldots,i_{m-1}} \left( e_{i,i_2} \otimes \cdots \otimes e_{i_{m-1},i'} \right), \]
then \( \omega_m(i,i')(n) \) is a projection of \( \kappa_m(n) \) onto \( E_{i,i'} \) (induced by writing elements of this \( k-1 \) tensor product using the standard basis, where the projection restricts to the basis subset of elements \( e_{i,i_2} \otimes \cdots \otimes e_{i_{k-1},i'} \) and, conversely,
\[
\kappa_m(n) = \sum_{i=i'} \omega_m(i,i')(n).
\]

We shall make some remarks on the rank of the above tensors; for brevity we restrict our attention to \( \kappa_m(n) \) and \( \omega_m(i,i')(n) \); these remarks generalize, and we will indicate any such generalizations or lack thereof when the generalized remarks are not straightforward.

First let us organize some remarks about these tensors, most of which easily from general remarks regarding tensors (such as in, for example, [Blä13] or [BCS97]).

**Proposition 9.2.** Fix a field, \( \mathbb{F} \), and integers \( m,m',n,n' \geq 1 \) we have:

1. \( R(\kappa_m(n)) \) is submultiplicative in \( n \).
2. \( R(\kappa_m(n)) = n^k \) for \( k \) even.
3. \( R(\kappa_m(n)) \) equals the smallest number of rank-1 tensors whose span includes \( \omega_m(i,i') \) for all \( i,i' \in [n] \).
4. \( R(\kappa_{m+m'-1}(n)) \leq R(\kappa_{m-1}(n))R(\kappa_{m'-1}(n)) \).
5. For any \( m \) and \( n \) we have
\[
R(\kappa_m(n)) \leq R(\kappa_{m+1}(n)) \leq n^2 R(\kappa_m(n)).
\]

Furthermore, claims (1),(2),(4),(5) hold with \( R \) replaced with \( \overline{R} \), or, more generally, the border rank defined by closure under any topology no coarser than the Zariski topology.

**Proof.** Let us begin with the statements for \( R \) (i.e., rank, not one of its closures). The first claim follows just as in the case \( m = 3 \) in matrix rank, namely by tensoring. The second claim follows by the permutation of factors, \( \pi \), for which
\[
\pi(e_{i_1,i_2} \otimes \cdots \otimes e_{i_m,i_1}) = (e_{i_1,i_2} \otimes e_{i_3,i_4} \otimes \cdots e_{m-1,m}) \otimes (e_{i_2,i_3} \cdots e_{i_m,i_1}),
\]
which, viewed as a linear operator on 
\[
(\mathbb{F}^{n \times n})^{\otimes (m/2)},
\]
is clearly an isomorphism; this gives the lower bound on the rank \( \kappa_m(n) \), and the upper bound is immediate from the definition of \( \kappa_m(n) \).
third claim is immediate, and the fourth claim follows since

\( \omega_{m+m'-2}(i, i')(n) = \sum_{i''} \omega_{m-2}(i, i'')(n) \otimes \omega_{m'-2}(i'', i')(n), \)

so if the former factors are spanned by a set of \( r \) rank-1 tensors, and
the latter by a set of \( r' \), then the LHS tensors are spanned by \( rr' \) rank-1
tensors. The fifth claim follows from the third claim and the fact that

\( \omega_{m+1}(i, i')(n) = \sum_{i''} \omega_m(i, i'')(n) \otimes e_{i'', i'} \)

and

\( \omega_m(i, i')(n) = \sum_{i''} \text{Contr}(\omega_{m+1}(i, i'')(n), e_{i'', i'}) \)

where \( \text{Contr} \) is the contraction of \( e_{i'', i'} \) into the last factor of \( \omega_{m+1}(n, i, i'') \).

For border rank, Claims (1), (4), and (5) are easily obtained by tak-
ing closures under any topology at least as fine as the Zariski topology;
for example, (1) really says that \( \kappa n^1 \) lies in the tensor product of
the subspaces of rank \( R(\kappa_n) \), and hence in the closure of the product
of these spaces, which is closed in any topology as fine as the Zariski
 topology.

Similarly, Claim (2) also follows with the same proof of Theorem 4.7,
since we are really showing that for an appropriate rearrangement of
coordinates map \( \pi \), we have that \( \pi \kappa_m(n) \) is in the Zariski open subset
of of full rank matrices.

In particular, if the exponent of matrix multiplication is \( \omega \), then for
integer \( m \geq 1 \) we have

\( R(\kappa_{2m+1}(n)) \leq n^{m \omega}. \)

Hence if the exponent of matrix multiplication is \( \omega = 2 \), then (31) holds
with “rough equality” on the left for \( k \) even (fixed, and large \( n \)), and
on the right for \( k \) odd.

9.2. **Inner Rank and \( \kappa_m(n) \) for \( m \) Odd.** In this subsection, \( \mathbb{F} \) is an
arbitrary field.

We can easily formulate inner rank inequalities for \( \kappa_m(n) \) for \( m \) odd.
For example, if

\( \kappa_m(n) = \sum_{\rho=1}^r \alpha_1^1 \otimes \cdots \otimes \alpha_m^m, \)

then for any \( i = 1, \ldots, m \) and any invertible \( L : F_2[n] \), we have

\( n^m \leq \sum_{\rho=1}^r \text{Rank}_{n \times n}(L \alpha_\rho^i). \)
However, by taking $n^2 + 1$ vectors in $\mathbb{F}^{n \times n}$ to rank one matrices, we can only conclude that for $m$ odd, we have a lower bound of $n^{m-1} + n^2 - n + 1$, which does not have interesting direct implications regarding $\kappa_3(n)$.

9.3. Cosines and Relaxation Hierarchies. Once a vector space over $\mathbb{R}$ or $\mathbb{C}$ is endowed with an inner product, one can speak of the cosine between two vectors. A similar remark regards tensor products, and contractible tensors.

So consider inner product spaces $A, B, C$ over $\mathbb{R}$ or $\mathbb{C}$. It is a standard fact that any tensor product of inner product spaces has a canonical inner product inherited from its factors, and if $\{a_i\}$ and $\{b_j\}$ are orthonormal bases for $A$ and $B$, then $\{a_i \otimes b_j\}$ is an orthonormal basis for $A \otimes B$, and similarly for any finite tensor product.

There is a natural contraction

(33) \[ \text{Contr} : (A \otimes B) \times (B \otimes C) \to A \otimes C \]

which allows us to define for $\tau \in A \otimes B$ and $\sigma \in B \otimes C$

\[
\cos(\tau, \sigma) \overset{\text{def}}{=} \|\text{Contr}(\tau, \sigma)\|_{A \otimes C} / (\|\tau\|_{A \otimes B}, \|\sigma\|_{B \otimes C})^{1/2}.
\]

The case $A = C = \mathbb{F}$ is the standard case. This cosine has Pythagorean laws, that for inner product spaces that says that if $v_1, \ldots, v_n$ is an orthonormal basis for an inner product space, $V$, and $w \in V$ is a unit vector, then $\sum_i \cos(w, u_i)^2 = 1$. For example, if $A = \mathbb{F}$, so that (33) refers to the contraction $B \times (B \times C) \to C$, then if $\{b_i\}$ is an orthonormal basis for $B$ and $w \in B \otimes C$ is a unit vector, then

\[
\sum_i \cos^2(b_i, c) = 1.
\]

We presume that if $\tau$ is a tensor with $\tau = R(\tau) < R(\tau)$, then when writing $\tau + O(\epsilon)$ as the sum of $\tau$ rank-1 tensors, the rank-1 tensors must degenerate as $\epsilon \to 0$, and hence one might measure this degeneration using cosines.

A relaxation that has appeared in the matrix multiplication literature is to study the minimum of $R(\tau)$ over all tensors

\[
\tau = \sum_{ijk} t_{ijk} e_{ij} \otimes e_{jk} \otimes e_{ki}
\]

[This is used constantly in Hodge theory.] One defines an inner product by choosing orthonormal bases $\{a_i\}$ and $\{b_j\}$ for, respectively, $A$ and $B$, and declaring $\{a_i \otimes b_j\}$ to be an orthonormal basis for $A \otimes B$. To check that the inner product on $A \otimes B$ is independent of the choice of basis, it suffices to note that if $Q_1, Q_2$ are orthogonal matrices for $\mathbb{F} = \mathbb{R}$ (i.e., $Q_i^{-1} = Q_i^T$), or unitary matrices for $\mathbb{F} = \mathbb{C}$ (i.e., $Q_i^{-1} = Q_i^H$), then the same is true for $Q_1 \otimes Q_2$.}
where \( t_{ijk} \) are not necessarily one, but must satisfy \( |t_{ijk}| = 1 \) (or some other constraint). The constraint \( |t_{ijk}| = 1 \) certainly implies

\[
\cos(\sigma, \tau) = \cos(\sigma, \kappa_3(n)),
\]

and a Pythagorean law shows that the converse is true.

A similar relaxation can be made for any tensor \( \kappa \in A_1 \otimes A_m \), if \( I \subset [m] \): say that \( \tau \) is \( I \)-equivalent to \( \tau' \) if

\[
\cos_I(\sigma, \tau) = \cos_I(\sigma, \tau')
\]

for all

\[
\sigma \in A_I \overset{\text{def}}{=} \bigotimes_{i \in I} A_i,
\]

and where \( \cos_I \) refers to the contraction of \( A_I \) with \( A_I[m] \) in the natural sense (i.e., for \( i \in I \), the \( A_i \) factor in \( A_I \) is contracted with the one in \( A_I[m] \)). \( I \)-equivalence is clearly an equivalence relation, and if \( I \subset I' \) then a Pythagorean law shows that \( I \)-equivalence is implied by \( I' \)-equivalence. Hence one gets a hierarchy of studying \( R_I(\tau) \), defined to be the minimum of \( R_I(\tau') \) equivalent to \( \tau \), since \( I \subset I' \) implies that \( R_I(\tau) \leq R_{I'}(\tau) \). However, in this hierarchy it is not clear to us, when \( I \subset I' \), which is easier to determine; it is only clear that a lower bound on \( R_I \) implies one for \( R_{I'} \), and vice versa for upper bounds.

These remarks lead to a “double” hierarchy of determining \( R_I(\kappa_m(n)) \) as a function of \( n \), with \( I \) and \( m \) as varying; there is not only a hierarchy with \( I \) varying, but also with \( m \), in view of the relation between them given in Proposition 9.2, its proof, and other remarks in Subsection 9.1.

9.4. Orthogonality. The tensor \( \kappa_3(n) \) can be symmetrized as

\[
\sum_{ijk}(e_{ij} + e_{ji}) \otimes (e_{jk} + e_{kj}) \otimes (e_{ki} + e_{ik}),
\]

which allows us to work with symmetric matrices; in this case when we write \( \mathcal{L}\gamma'_{\rho} \) as a sum of rank-1 tensors, where \( \gamma'_{\rho} \) is the symmetrized version of the \( \gamma_{\rho} \) involved in the \( \kappa_3(n) \) sum; this yields an an orthonormal eigenbasis for each \( \mathcal{L}\gamma_{\rho} \). It would be interesting to know if such orthonormal bases can give information about rank and border rank; for example, in the case where border rank is less than rank as mentioned above, so that for \( \tau = \kappa_3(n) \) there exists a family \( \tau_\epsilon = \tau + O(\epsilon) \) with smaller rank than \( \tau \), we are curious to know if the degeneracy of the sum of rank-1 tensors in \( \tau_\epsilon \) manifests itself in some type of spectral information.

Of course, the map \( e_{ij} \mapsto e_{ij} + e_{ji} \), which needs to be applied to each factor, results in a loss of information, since the map is of rank \( n(n + 1)/2 \); hence this map, which needs to be applied to each factor, could
strictly reduce the rank (for example, if the $\alpha_\rho, \beta_\rho, \gamma_\rho$ in an optimal rank-1 decomposition are sent to zero under $e_{ij} \mapsto e_{ij} + e_{ji}$).

9.5. **Biasing Dimension Counts.** To study an equation

$$\kappa_3(n) = \sum_{\rho=1}^{r} \alpha_\rho \otimes \beta_\rho \otimes \gamma_\rho,$$

one may try to count dimensions in a biased fashion if the the $\gamma_\rho$ are in $\mathbb{F}^{n \times n}$ are not “evenly distributed” among the $3n^2 + o(n^2)$ or more such matrices. If the $\gamma_\rho$ are distributed more in some directions in $\mathbb{F}^{n \times n}$, one may be able to “bias” the way we view directions and/or dimension counts in $\mathbb{F}^{n \times n}$ to get a dimension count in $\mathbb{F}^{n \times n}$ (or inner rank counts, etc.) to get a view of dimensions in $\mathbb{F}^{n \times n}$ that is more appropriate to analyzing expressions for $\kappa_3(n)$ as above. Here we give one such way of “dimension biasing,” and discuss its benefits and shortcomings. We will assume $\mathbb{F} = \mathbb{R}$; the same holds for $\mathbb{F} = \mathbb{C}$, where we substitute “Hermitian” for “symmetric,” etc. Such dimension biasing may have other applications, such as defining a dimension biased notion of the maximum excess of sheaves on graphs [Fri15].

**Definition 9.3.** To a symmetric matrix, $D \in \mathbb{R}^n$ and nonzero $v \in \mathbb{R}$ we associate the *Rayleigh quotient*

$$\mathcal{R}_D(v) \overset{\text{def}}{=} \frac{v^T D v}{v^T v}.$$  

For a subspace $S \subset \mathbb{R}^n$, let the *orthogonal restriction of $D$ to $S$*, denoted $D|_S$, be the morphism $S \to S$ given by $\pi_S D$ where $\pi_S$ is the orthogonal projection $\mathbb{R}^n \to S$. Define the *$D$-biased dimension of $S$*, denoted $\dim_D(S)$, to be the trace of $D|_S$.

Notice that we don’t assume that $D$ is positive semidefinite; however, the facts below show that if $D$ is not, then $\dim_D(S)$ is not a non-decreasing function of $S$ (i.e., $S_1 \subset S_2$ implies $\dim_D(S_1) \leq \dim_D(S_2)$).

We now give a number of easy facts related to the setting in the above definition. If $s_1, \ldots, s_d$ is an orthonormal basis of $S$, then

$$\dim_D(S) = \sum_{i=1}^{d} \mathcal{R}_D(s_i).$$

The same is true of $s_1, \ldots, s_d$ is a sequence of nonzero, mutually orthogonal vectors. Second, if $S \subset \mathbb{R}^n$, and $S^\perp$ is its orthogonal complement, then

$$\dim_D(S) + \dim_D(S^\perp) = \dim_D(\mathbb{R}^n) = \text{Trace}(D).$$
If \( D \) maps \( S \) to itself, then \( \pi_S D \) is just the restriction of \( D \) to \( S \). If \( s_1, \ldots, s_d \) is any basis for \( S \), then \( \dim_D(S) \) is just the trace of the matrix expressing the restriction to \( S \) of \( D \) in the basis \( s_1, \ldots, s_d \). Whether or not \( D \) maps \( S \) to itself, \( \dim_D(S) \) is the sum of the eigenvalues of \( D|_S \).

Here is the problem with biasing dimension with matrix \( D \) that is not proportional to the identity matrix: the function \( \dim_D \) is modular on right-angled subsets, in the sense that if \( S_1, S_2 \subset \mathbb{R}^n \) such that \( (S_1 \cap S_2)^\perp \cap S_1 \) and \( (S_1 \cap S_2)^\perp \cap S_2 \) are orthogonal, then

\[
\dim_D(S_1 \cap S_2) + \dim(S_1 + S_2) = \dim_D(S_1) + \dim_D(S_2);
\]

however, this does not generally hold unless \( D \) has multiple eigenvalues and, for example, \( S_1, S_2 \) both lie in a single eigenspace. This modularity holds for all \( S_1, S_2 \) if \( D \) has a single eigenspace, i.e., since \( D \) is symmetric, for \( D \) proportional to the identity matrix.

The above shortcoming is illustrated in the example,

\[
D = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}
\]

for some \( \lambda \in \mathbb{R} \); if \( A_1 \) spanned by \([1 0]^T \) and \( A_2 \) spanned by \([1 1]^T \), we have

\[
\dim_D(A_1 + A_2) = \dim_D(\mathbb{R}^2) = \text{Trace}(D) = 1 + \lambda
\]

and

\[
\dim_D(A_1) + \dim_D(A_2) = 1 + \frac{1 + \lambda}{2}
\]

and the difference of these two expressions is \((1 - \lambda)/2\). Hence the more one biases dimensions—say in trying to improve rank bounds—the farther \( \lambda \) is from 1, the greater the violation of the modularity in \( \dim_D \), at least for certain spaces (that are not orthogonal).

10. Closing Remarks

We finish with a few remarks regarding inner rank.

The existence of matrix multiplication algorithms also shows us the limits of our methods. For example, Strassen’s algorithm for \( \kappa_3(2) \), written as the sum of seven rank-1 tensors, shows us that there exist seven vectors that span \( \mathbb{R}^{2 \times 2} \) such that no invertible linear operator on this space can take all seven vectors to rank-1 matrices.

Consider the form of Lemma 7.1 where \(\mathcal{M}, \mathcal{N}, \mathcal{L} \) are each linear operators, then these operators are parameterized by \((n_1n_2)^2, (n_2n_3)^2, (n_3n_1)^2 \) parameters. So for \( n = n_1 = n_2 = n_3 \) this gives \( n^4 \) parameters for each matrix. In particular, if \( \mathcal{M} = \mathcal{N} \) is the identity matrix, then our choice of \( \mathcal{L} \) involves only \( n^4 \) parameters. Hence we are not necessarily optimistic that we can use these \( n^4 \) parameters alone to lower
It may be interesting to ask if one can prove that

\[ f(n, r)n^3 \leq \sum_{\rho=1}^{\rho} \text{Rank}_{n \times n} \gamma_{\rho} \]

for a function \( f(n, r) \) that for some values of large \( n \) and \( r \) (i.e., \( r = (3.01)n^2 \) and \( n \) large) is strictly greater than one. In other words, when \( r \) is optimal or close to it, so at present is conceivably as small as \( r = 3n^2 + o(n) \), does the inner rank sum have to be significantly larger than \( n^3 \). Of course, for \( r \) as in Strassen’s algorithm \[Str69\] and \( n \) a power of 2, the inner rank inequality with \( n^3 \) is tight.

The methods of \[Blä99, Lan14, LO15, LM17\] have a way of taking a border rank lower bound and increasing this bound on the rank of the same tensor (e.g., from \( 2n^2 + o(n) \) to \( 3n^2 + o(n^2) \)). We would be interested to know if this is possible here; however, since our inner rank bound involve the fact that \( \pi_{K5}(n) \) is of high rank (i.e., \( n^3 \)), we are not necessarily optimistic.

Nick Harvey has asked us whether one can use sheaf theoretic methods (e.g., \[Fri15\]) regarding our methods (or others) to study this tensor. Although rank can be expressed in terms of finite diagrams, we have yet to find such an expression where short/long exact sequences or other ideas in sheaf theory might be applicable.

**Appendix A. Remarks on Border Rank**

In the literature there are two different definitions of border rank of an element, \( \tau \), of a tensor product of \( \mathbb{F} \)-vector spaces the infinitesimal border, \( R_{inf}(\tau) \), and the Zariski border rank, \( R_{Zar}(\tau) \); if \( \mathbb{F} = \mathbb{R}, \mathbb{C} \) there is a third definition, the norm border rank, \( R_{norm}(\tau) \) (which would work over any local field, \( \mathbb{F} \), or algebra thereof). In this section we give the definitions of these border ranks and make some remarks regarding them. First we wish to give the short argument that

\[ R_{inf}(\tau) \geq R_{Zar}(\tau), \]

and then that \( R_{norm}(\tau) \) lies between these two. Then we explain the importance of \( R_{inf}(\tau) \).

As we understand it, \[Adl83\] contains a proof that \( R_{inf}(\tau) = R_{Zar}(\tau) \) if \( \mathbb{F} \) is algebraically closed (see Strassen \[Str83\] (page 647, just below (1.5) there). [At present we do not know of a more readily available reference for a proof. Also, we do not know if the same is true.
for arbitrary fields, as the Zariski border rank by fiat passes to an algebraic closure, $\mathbb{F}$, of the field $\mathbb{F}$, whereas the infinitesimal border rank need not and it is not clear to us that $\alpha^i_\rho(\epsilon) \in A_i \otimes_\mathbb{F} [\epsilon]$ in (34) below necessarily can be modified appropriately to lie in $A_i \otimes_\mathbb{F} [\epsilon]$; perhaps this is known or can be answered with Galois cohomology.

Provided that $\mathbb{F}$ is algebraically closed, we define the Zariski border rank of $\tau$, denoted $R_{\text{Zar}}(\tau)$, to be the smallest integer $r$ such that $\tau$ is in the Zariski closure of set of all tensors of rank at most $r$, where this set is identified with an affine space over $\mathbb{F}$ by choosing bases for the $A_i$ to coordinatize $A_1 \otimes \cdots \otimes A_m$ (the Zariski closure is clearly independent of this choice of bases). If $\mathbb{F}$ is not algebraically closed, we embed $\mathbb{F}$ in one of its algebraic closures, $\bar{\mathbb{F}}$, and define $R_{\text{Zar}}(\tau)$ to be its rank there (i.e., via the functor $V \mapsto V \otimes_\mathbb{F} \bar{\mathbb{F}}$); clearly this is independent of the choice of $\mathbb{F}$.

We can similarly define the border rank with respect to any topology on $\mathbb{F}$-vector spaces (or $\mathbb{F}$-modules if $\mathbb{F}$ is ring, which we will not consider in this article). The common topology, aside from the Zariski topology, is the standard (or norm) topology on subfield of $\mathbb{R}$ or $\mathbb{C}$ (induced from the norm on $\mathbb{R}$ or $\mathbb{C}$, or on, say, any local field); we call this the norm Border rank, denoted $R_{\text{norm}}(\tau)$. Since any polynomial is continuous in the norm topology, $R_{\text{norm}}(\tau) \geq R_{\text{Zar}}(\tau)$. Since the norm topology is given by a metric, one can equivalently define $R_{\text{norm}}(\tau)$ as the smallest $r$ such that $\tau$ is a limit tensors of rank at most $r$.

Say that $\tau$ is of $h$-infinitesimal rank $r$ if it can be written as

\begin{equation}
\sum_{i=1}^{r} \alpha^i_\rho(\epsilon) \otimes \cdots \otimes \alpha^m_\rho(\epsilon) = \epsilon^h \tau + O(\epsilon^{h+1})
\end{equation}

where we extend scalars to $\mathbb{F}[\epsilon]$, and $\alpha^i_\rho(\epsilon)$ are therefore elements of $A_i \otimes_\mathbb{F} [\epsilon]$; see [Blå13], Section 6 or [BCS97], degeneration of order $q$ involving equation (15.6). Define $R_{\text{inf}}(\tau)$ to be the smallest such $r$ possible over all integers $h$. It is easy to see that any such $\tau$ satisfying (34) has Zariski border rank at least $r$: indeed, since the rank of a tensor is invariant under scaling, if $P$ is a polynomial that vanishes on all tensors of rank at most $r$, then the same is true of the homogeneous part, $P_i$, of $P$ of terms of total degree $i$; but if $P_i$ is homogeneous of degree $i$ and vanishes on $\epsilon^h \tau + O(\epsilon^{h+1})$, then by homogeneity it vanishes on $\tau + O(\epsilon)$ and hence on $\tau$. Hence $P_i(\tau) = 0$; since $P = P_0 + \cdots + P_d$, where $d$ is a bound on the total degree of $P$, we have $P(\tau) = 0$; hence $\tau$ lies in the aforementioned Zariski closure and hence has Zariski border rank at most $r$. 

\[\text{(34)}\]
If $F$ is a local field (or even an algebra over a local field), then one has a norm topology; in case (34) implies that $\tau + O(\epsilon)$ has rank at most $r$, and letting $\|\epsilon\| \to 0$ we conclude that $R_{\text{norm}}(\tau) \leq r$. Hence $R_{\text{norm}}(\tau)$, when it makes sense, lies between $R_{\text{inf}}(\tau)$ and $R_{\text{Zar}}(\tau)$.

The algorithmic importance of $h$-infinitesimal border rank is that if $\langle N, N, N \rangle$ has $h$-infinitesimal border rank $r$ for some $N, r$, then $\langle n, n, n \rangle$ has rank $n^{\omega+\epsilon}$ for large $n$ for any $\epsilon > 0$, where $\omega = \log_N r$ (see [Blä13], Theorem 6.6). [Hence an $h$-infinitesimal border rank upper bound is essentially as good as a rank upper bound for the purpose of algorithms regarding the rank of large matrices.]

**Appendix B. Strassen’s Algorithm**

In a number of places in this article (especially Section 6), it is helpful to consider Strassen’s algorithm [Str69] that expresses $\kappa_3(2)$ as $\sum_{\rho=1}^7 \alpha_\rho \otimes \beta_\rho \otimes \gamma_\rho$, given by the following table; we give $\gamma_\rho^T$, the transpose of $\gamma_\rho$, to make this easier to reconcile our table below with standard textbooks on matrix multiplication.
\[ \begin{array}{cccccc}
\rho & \alpha_{\rho} & \beta_{\rho} & \gamma_{\rho}^T & (\gamma_{\rho})^T & \text{Decomp} \\
1 & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
2 & \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\
3 & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \\
4 & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\
5 & \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \\
6 & \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
7 & \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\end{array} \]

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