NEW QUANTUM CODES FROM CONSTACYCLIC CODES
OVER THE RING $R_{k,m}$

HABIBUL ISLAM, OM PRAKASH* AND RAM KRISHNA VERMA

Department of Mathematics
Indian Institute of Technology Patna
Patna- 801 106, India

(Communicated by Eimear Byrne)

Abstract. For any odd prime $p$, we study constacyclic codes of length $n$ over the finite commutative non-chain ring $R_{k,m} = \mathbb{F}_p^m[u_1, u_2, \ldots, u_k]/\langle u_i^2 - 1, u_i u_j - u_j u_i \rangle_{i \neq j = 1, 2, \ldots, k}$, where $m, k \geq 1$ are integers. We determine the necessary and sufficient condition for these codes to contain their Euclidean duals. As an application, from the dual containing constacyclic codes, several MDS, new and better quantum codes compare to the best known codes in the literature are obtained.

1. Introduction

Quantum computing is a fascinating topic for present research with a higher ability to solve severe problems faster than classical computers. The quantum error-correcting codes are used in the quantum computer to protect the quantum information from the noises that occurred during communication. After the pioneering work of Shor [35] in 1995, Calderbank et al. [5] proposed a prominent method to obtain quantum error-correcting codes from the classical error-correcting codes. The primary goal of this area is to construct better quantum codes employing state-of-art. In this connection, many significant works have been reported in the literature which provides better quantum codes over the finite fields, see [14, 15, 16, 26, 32]. It is also observed that the linear (e.g., cyclic, constacyclic) codes over finite non-chain rings produced a huge amount of good quantum codes [1, 2, 8, 11, 13, 19, 12, 27, 29, 30]. In 2015, Ashraf and Mohammad [1] studied quantum codes from cyclic codes over $\mathbb{F}_p + v\mathbb{F}_p$. Meantime, Dertli et al. [8] presented some new binary quantum codes obtained from the cyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$, and then Ashraf and Mohammad [2] generalized their work over the ring $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$ to derive new non-binary quantum codes. There are a lot of articles in which good quantum codes are obtained from the cyclic codes on different finite rings, see [11, 13, 19, 23, 31, 33, 32, 34]. On the other side, recently, Gao and Wang [12], Li et al. [27], Ma et al. [29, 30] considered the constacyclic codes over finite non-chain rings and obtained many new and better codes compare to the known codes. Based on the above studies, one can say that the constacyclic codes are a great resource to supply good quantum codes over finite rings. Hence, it is logical to study the

2020 Mathematics Subject Classification: 94B15, 94B05, 94B60.

Key words and phrases: Constacyclic code, gray map, self-dual code, quantum code.

The research is supported by the University Grants Commission (UGC) and the Council of Scientific & Industrial Research (CSIR), Govt. of India.

* Corresponding author: Om Prakash.
constacyclic codes over new and different non-chain rings to construct more new quantum codes.

Towards this, we study the constacyclic codes over the family of commutative non-chain rings \( R_{k,m} = \mathbb{F}_p[[u_1, u_2, \ldots, u_k]] / \langle u_1^2 - 1, u_1 u_j - u_j u_1 \rangle \) for all \( i, j = 1, 2, \ldots, k \), where \( p \) is an odd prime and \( m, k \) are positive integers. Note that for \( k = 2, m = 1 \) the constacyclic codes over \( R_{k,m} \) are studied in [21]. Further, authors constructed quantum codes based on cyclic codes over \( R_{2,1} \) in [22] and over \( R_{3,1} \) in [19], respectively. Therefore, the present article is a continuation and generalization of our previous studies in the context of new quantum codes construction. The main objective of the article is two-folded, first, we characterize the constacyclic codes over \( R_{k,m} \) (Section 4), and then by utilizing the structures we obtain new and better quantum codes (Section 5). To do so, we define a new Gray map \( \psi \) which is different from the usual canonical map and capable to produce many quantum MDS codes (Table 1) and better quantum codes (Table 3-5) compare to the best-known codes in the literature.

The presentation of the article is organized as follows: In Section 2, the results related to finite rings along with some basic definitions and properties have been discussed. Section 3 gives the structure of constacyclic codes, while Section 4 presents the construction of quantum codes and many examples of better codes. Section 5 concludes the article.

2. PRELIMINARY

Throughout the article, we use \( R_{k,m} := \mathbb{F}_p[[u_1, u_2, \ldots, u_k]] / \langle u_1^2 - 1, u_1 u_j - u_j u_1 \rangle \) where \( 1 \leq i, j \leq k \), \( p \) is an odd prime and \( k, m \) are positive integers. Thus \( R_{k,m} \) is a finite commutative ring (with unity) of characteristic \( p \) and order \( p^{2m} \). Note that any element \( r \in R_{k,m} \) has the expression \( r = \alpha_0 + \sum_{1 \leq i_1 \leq k} \alpha_{i_1} u_{i_1} + \sum_{1 \leq i_1 < i_2 \leq k} \alpha_{i_1,i_2} u_{i_1} u_{i_2} + \cdots + \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq k} \alpha_{i_1,i_2,\ldots,i_k} u_{i_1} u_{i_2} \cdots u_{i_k} \), where \( \alpha_0, \alpha_{i_1}, \alpha_{i_1,i_2}, \ldots, \alpha_{i_1,i_2,\ldots,i_k} \in \mathbb{F}_p^{2m} \), for all \( 1 \leq i_j \leq k \). Now, similar to Theorem 2.3 of [7], \( R_{k,m} \) has \( 2^k \) number of maximal ideals \( \langle v_1, v_2, \ldots, v_k \rangle \), where \( v_i \in \{1 - u_i, 1 + u_i\}, 1 \leq i \leq k \). Also, \( R_{k,m} \) is a principal ideal ring where any ideal \( I = \langle v_1, v_2, \ldots, v_l \rangle \) is principally generated by the element formed by the sum of all \( v_i \) and their products, see ([7], Theorem 2.6).

Therefore, by comparing above maximal ideals, we conclude that \( R_{k,m} \) is a non-chain semi-local Frobenius ring. For instance, if \( k = 1 \), then there are two maximal ideals \( I_1 = \langle 1 - u_1 \rangle, I_2 = \langle 1 + u_1 \rangle \) in \( R_{1,m} \) and \( I_1 \neq I_2 \). Clearly, \( R_{1,m} \) is a non-chain semi-local ring of order \( p^{2m} \). On the other hand, \( R_{k,m} \) contains \( (p^{2m} - 1)^2 \) units which is discussed in Lemma 3.2.

Recall that a nonempty subset \( C \) of \( R_{k,m}^n \) is a linear code of length \( n \) over \( R_{k,m} \) if it is an \( R_{k,m} \)-submodule of \( R_{k,m}^n \), and each element of \( C \) is called a codeword. The rank of a code \( C \) over \( R_{k,m} \) is the minimum number of elements which span \( C \). If \( K \) is the rank, then the code \( C \) is said to be an \([n, K]\) linear code. The Euclidean inner product of two elements \( a = (a_0, a_1, \ldots, a_{n-1}) \) and \( b = (b_0, b_1, \ldots, b_{n-1}) \) in \( R_{k,m}^n \) is defined as \( a \cdot b = \sum_{i=0}^{n-1} a_i b_i \). Let \( C \) be a linear code of length \( n \) over \( R_{k,m} \). Then the dual \( C^\perp := \{ a \in R_{k,m}^n \mid a \cdot b = 0 \ \forall \ b \in C \} \) is also a linear code. The code \( C \) is said to be self-orthogonal if \( C \subseteq C^\perp \) and self-dual if \( C^\perp = C \).
Let $A = \{i_1, i_2, \ldots, i_s\}$ be a subset of the set $S = \{1, 2, \ldots, k\}$ where $i_1 < i_2 < \cdots < i_s$ and $\zeta \in \mathbb{F}_p^m$ such that $2^k \zeta \equiv 1 \pmod{p}$. We define

$$u_A = \prod_{i \in A} u_i, \text{ and for } A = \phi, \ u_\phi = 1;$$

and $e_A^\Delta = \zeta \prod_{i_j \in A} (1-u_{i_j}) \prod_{i_k \not\in A} (1+u_{i_k}),$

where $|A| = \Delta$ ($1 \leq \Delta \leq k$), and $\Delta = 0$ for $A = \phi$. We use $e_A^0$ for $e_A^0 = \zeta \prod_{i=1}^k (1+u_{i_j})$, which can be obtained from above. From the definition of $e_A^\Delta$, it is clear that the superscript $\Delta$ is used to count the number of factors, like $(1-u_{i_j})$, present in $e_A^\Delta$. Let $B$ be a subset of $S$ different from $A$. Without loss of generality, let $i_j \in A$ and $i_j \not\in B$. Then, from the construction of $e_A^\Delta$, we must say $1-u_{i_j}$ divides $e_A^\Delta$, while $1+u_{i_j}$ divides $e_A^\Delta_B$. Since $u_{i_j}^2 = 1$, we have $e_A^\Delta e_B^\Delta = 0 \pmod{(u_{i_j}^2 - 1)}$. Further, since $(1 \pm u_{i_j})^2 = 2(1 \pm u_{i_j}) \pmod{(u_{i_j}^2 - 1)}$, for all $i_j$, therefore,

$$(e_A^\Delta)^2 = \left[ \zeta \prod_{i_j \in A} (1-u_{i_j}) \prod_{i_k \not\in A} (1+u_{i_k}) \right]^2$$

$$= \zeta^2 \prod_{i_j \in A} (1-u_{i_j})^2 \prod_{i_k \not\in A} (1+u_{i_k})^2$$

$$= 2^k \zeta^2 \prod_{i_j \in A} (1-u_{i_j}) \prod_{i_k \not\in A} (1+u_{i_k})$$

$$= \zeta \prod_{i_j \in A} (1-u_{i_j}) \prod_{i_k \not\in A} (1+u_{i_k})$$

$$= e_A^\Delta \pmod{(u_{i_j}^2 - 1)}.$$ 

Again, by induction on $k$ in $R_{k,m}$, we have $\sum_{A \subseteq S} e_A^\Delta = 1$. In the light of the above discussion, we conclude that

$$e_A^\Delta e_B^\Delta = \begin{cases} e_A^\Delta, & \text{if } A = B \\ 0, & \text{if } A \neq B. \end{cases}$$

Therefore, $\{e_A \mid A \subseteq S\}$ is a set of pairwise orthogonal idempotent elements in $R_{k,m}$. Hence, by Chinese Remainder Theorem, we decompose the ring $R_{k,m}$ as

$$R_{k,m} = \bigoplus_{A \subseteq S} e_A^\Delta R_{k,m} \cong \bigoplus_{A \subseteq S} e_A^\Delta \mathbb{F}_p^m.$$ 

Thus, any element $r \in R_{k,m}$, where

$$r = \alpha_0 + \sum_{1 \leq i_1 \leq k} \alpha_{i_1} u_{i_1} + \sum_{1 \leq i_1 < i_2 \leq k} \alpha_{i_1, i_2} u_{i_1} u_{i_2} + \cdots$$

$$+ \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq k} \alpha_{i_1, i_2, \ldots, i_k} u_{i_1} u_{i_2} \cdots u_{i_k}$$

$$= \sum_{A \subseteq S} \alpha_A u_A$$
can be uniquely written as

\[ r = \beta_0 e_0^k + \sum_{1 \leq i_j \leq k} \beta_{i_j} e_{i_j}^1 + \sum_{1 \leq i_j < i_{j'} \leq k} \beta_{i_j,i_{j'}} e_{i_j,i_{j'}}^2 + \ldots + \sum_{1 \leq i_j < i_{j'} < \ldots < i_{k} \leq k} \beta_{i_j,i_{j'},\ldots,i_{k}} e_{i_j,i_{j'},\ldots,i_{k}}^k = \sum_{A \subseteq S} \beta_A e_A^S, \]

where, \( 1 \leq r \leq k \).

Let \( M \in GL_{2^k}(\mathbb{F}_{p^m}) \), where \( GL_{2^k}(\mathbb{F}_{p^m}) \) is the set of all \( 2^k \times 2^k \) invertible matrices over \( \mathbb{F}_{p^m} \). Now, we define a Gray map

\[ \psi : R_{k,m} \rightarrow \mathbb{F}_{p^m}^{2^k} \]

by

(1) \[ r \mapsto (\beta_0, \beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_k}, \beta_{i_1,i_2}, \beta_{i_1,i_3}, \ldots, \beta_{i_{k-1},i_k}, \ldots, \beta_{i_1,i_2\ldots,i_k}) \]

(2) \[ = (r_1, r_2, \ldots, r_{2^k}) \langle \mathbf{r} \rangle = r M \]

where, \( 1 \leq i_j \leq k \). Here, we enumerate the vector \((\beta_0, \beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_k}, \beta_{i_1,i_2}, \beta_{i_1,i_3}, \ldots, \beta_{i_{k-1},i_k}, \ldots, \beta_{i_1,i_2\ldots,i_k})\) as \((r_1, r_2, \ldots, r_{2^k}) = r\). Then the map \( \psi \) is linear and can be extended from \( R_{k,m}^n \) to \( \mathbb{F}_{p^m}^{2^k} \) componentwise. The Hamming weight of a codeword \( c = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C} \) is denoted by \( \text{wt}_H(c) \) and defined as the number of non-zero components in the codeword \( c \). The Hamming distance for the code \( \mathcal{C} \) is defined by \( d_H(\mathcal{C}) = \min\{d_H(c,c') \mid c \neq c', \text{for all } c, c' \in \mathcal{C} \} \), where \( d_H(c,c') \) is the Hamming distance between \( c, c' \in \mathcal{C} \) and \( d_H(c,c') = \text{wt}_H(c-c') \). Also, the Gray weight of any element \( r \in R_{k,m} \) is defined as \( \text{wt}_G(r) = \text{wt}_H(\psi(r)) \) and Gray weight for \( \bar{r} = (r_0, r_1, \ldots, r_{n-1}) \in R_{k,m}^n \) is \( \text{wt}_G(\bar{r}) = \sum_{i=0}^{n-1} \text{wt}_G(r_i) \). Further, the Gray distance between codewords \( c, c' \in \mathcal{C} \) is defined as \( d_G(c,c') = \text{wt}_G(c-c') \).

It is worth mentioning that in earlier works [1, 11, 17, 19], authors have used the canonical Gray maps which take every element into a vector consisting of its canonical components. But, we define the map \( \psi \) as the multiplication of a vector by an invertible matrix of order \( 2^k \). Such type of Gray maps one can also find in [13, 29, 30] with respect to their setup. One of the main advantages to choose such Gray maps, like \( \psi \), is to enhance the code parameters (particularly, dimension, and minimum distance, etc.) over the parameters obtained by the simple canonical Gray map. For example, using \( \psi \) we construct quantum code \([[22, 2, 7]]_5 \) in Example 4.6 whose minimum distance is larger than the quantum code \([[22, 2, 5]]_5 \) obtained in [17] under the usual canonical Gray map.

Now, we present an example for \( k = 2 \) to understand the ring structure based on the set of pairwise orthogonal idempotent elements and Gray map discussed above.

**Example 2.1.** Let \( k = 2 \) and \( R_{2,m} = \mathbb{F}_{p^m}[u_1, u_2]/(u_1^2 - 1, u_2^2 - 1, u_1 u_2 - u_2 u_1) \). Then \( R_{2,m} \) is a semi-local ring with four maximal ideals \( I_1 = \langle 1 - u_1, 1 - u_2 = 3 - 2u_1 - 2u_2 + u_1 u_2 \rangle, I_2 = \langle 1 - u_1, 1 + u_2 = 3 - 2u_1 + 2u_2 - u_1 u_2 \rangle, I_3 = \langle 1 + u_1, 1 - u_2 = 3 + 2u_1 - 2u_2 - u_1 u_2 \rangle \) and \( I_4 = \langle 1 + u_1, 1 + u_2 = 3 + 2u_1 + 2u_2 + u_1 u_2 \rangle \), respectively. Here, \( S = \{1, 2\} \), so the possible choices for \( A \subseteq S \) are \( \emptyset, \{1\}, \{2\}, \{1, 2\} \). Therefore,
an element \( r \in R_{2,m} \) has the expression

\[
 r = a_0 + a_1 u_1 + a_2 u_2 + a_{1,2} u_1 u_2 \\
= a_0 + \sum_{i=1}^{2} a_{i} u_i + \sum_{1 \leq i_1 < i_2 \leq 2} a_{i_1, i_2} u_{i_1} u_{i_2} \\
= \sum_{\lambda \subseteq S} \alpha_{\lambda} u_{\lambda},
\]

where \( a_0, a_i, a_{i_1, i_2}, \alpha_{i_1, i_2} \in \mathbb{F}_{p^m} \). Note that

\[
 e_0^0 = \zeta(1 + u_1)(1 + u_2), e_1^1 = \zeta(1 - u_1)(1 + u_2), \\
 e_2^1 = \zeta^2(1 + u_1)(1 - u_2),
\]

where \( 4k \equiv 1 \pmod{p} \). Then we can write \( r \) uniquely as

\[
 r = \beta_0 + \sum_{1 \leq i_1, i_2 \leq 2} \beta_{i_1, i_2} e_{i_1, i_2}^1 + \sum_{1 \leq i_1 < i_2 \leq 2} \beta_{i_1, i_2} e_{i_1, i_2}^2 = \sum_{\lambda \subseteq S} \beta_{\lambda} e_{\lambda}^2.
\]

In this case, the Gray map \( \psi : R_{2,m} \rightarrow \mathbb{F}_{p^m}^2 \) is defined by \( \psi(a_0 + a_1 u_1 + a_2 u_2 + a_{1,2} u_1 u_2) = (a_0 + a_1 + a_2 + a_{1,2}, a_0 - a_1 + a_2 - a_{1,2}, a_0 + a_1 - a_2 - a_{1,2}, a_0 - a_1 - a_2 + a_{1,2}) M \), where \( M \in GL_4(\mathbb{F}_{p^m}) \) is any \( 4 \times 4 \) invertible matrix.

Now, we review some important results on linear codes over \( R_{k,m} \). One can find the similar results in [7, 19, 30, 36].

**Theorem 2.2.** The Gray map \( \psi \) defined in equation (1) is linear and weight preserving from \( R_{k,m}^n \) (Gray weight) to \( \mathbb{F}_{p^m}^n \) (Hamming weight).

**Theorem 2.3.** The Gray image of an \([n, K, d_G]\) linear code over \( R_{k,m} \) is a \([2^k n, K, d_H]\) linear code over \( \mathbb{F}_{p^m} \) where \( d_G = d_H \).

**Proof.** Since the Gray map \( \psi \) is linear, \( \psi(C) \) is a linear code of length \( 2^k n \). Also, the map \( \psi \) is distance preserving, hence \( \psi(C) \) is a \([2^k n, K, d_H]\) linear code over the field \( \mathbb{F}_{p^m} \) where \( d_G = d_H \).

**Theorem 2.4.** Let \( C \) be a self-orthogonal (Euclidean) linear code over \( R_{k,m} \) and \( M \in GL_{2k}(\mathbb{F}_{p^m}) \) such that \( MM^t = cI_{2k} \), where \( M^t \) represents the transpose of \( M \), \( c \in \mathbb{F}_{p^m}^n = \mathbb{F}_{p^m} \setminus \{0\} \), and \( I_{2k} \) is the identity matrix of order \( 2k \). Then \( \psi(C) \) is a self-orthogonal (Euclidean) linear code of length \( 2^k n \) over \( \mathbb{F}_{p^m} \).

**Proof.** Let \( x = (x_1, \cdots, x_{2k}), y = (y_1, \cdots, y_{2k}) \in \psi(C) \). Then there exist \( x' = (x'_1, \cdots, x'_n), y' = (y'_1, \cdots, y'_n) \in C \) and \( M \in GL_{2k}(\mathbb{F}_{p^m}) \) such that \( x = \psi(x') = (x'_1 M, \cdots, x'_n M), y = \psi(y') = (y'_1 M, \cdots, y'_n M) \).

Now, \( x \cdot y = x'y = \sum_{i=1}^{n} x'_i M y'_i = \sum_{i=1}^{n} x'_i c_{I_{2k}} y'_i = c \sum_{i=1}^{n} x'_i y'_i = 0 \). Hence, \( x \cdot y = 0 \), and consequently \( \psi(C) \) is a self-orthogonal linear code of length \( 2^k n \) over \( \mathbb{F}_{p^m} \).

For each \( 1 \leq s \leq 2^k \), define \( \bigoplus_{i=1}^{s} A_i := \{ a_1 + a_2 + \cdots + a_s \mid a_i \in A_i, 1 \leq i \leq s \} \).

Let \( C \) be a linear code of length \( n \) over \( R_{k,m} \), and for \( A \subseteq S \), \( C_A := \{ \beta_A \in \mathbb{F}_{p^m}^n \mid \text{there exist } \alpha_B \in \mathbb{F}_{p^m} \text{ for all } B \subseteq S \text{ distinct from } A \text{ such that } \beta_A = \sum_{B \subseteq S, A \neq B} \alpha_B c_{B} \in C \} \).

Then \( C_A \) is a linear code of length \( n \) over \( \mathbb{F}_{p^m} \) for all \( A \subseteq S \). Also, \( C \) can be expressed
Moreover, the generator matrix for the code $C$ is $M = (e_A M_A)_{A \subseteq S}$, where $M_A$ is a generator matrix of the code $C_A$ for $A \subseteq S$. Thus, $\psi(C)$ has a generator matrix $G = (\psi(e_A M_A))_{A \subseteq S}$ and its dimension $K = \sum_{A \subseteq S} k_A$, where $k_A$ is the dimension of $C_A$ for all $A \subseteq S$.

**Theorem 2.5.** Let $C = \bigoplus_{A \subseteq S} e_A C_A$ be a linear code of length $n$ over $R_{k,m}$. Then
\[
C^\perp = e_0^0 C_0^\perp \oplus \sum_{1 \leq i_1 \leq k} e_{i_1}^1 C_{i_1}^\perp \oplus \sum_{1 \leq i_1 < i_2 \leq k} e_{i_1,i_2}^2 C_{i_1,i_2}^\perp \oplus \cdots \oplus \sum_{1 \leq i_1 < i_2 \cdots < i_k \leq k} e_{i_1,i_2,\ldots,i_k}^k C_{i_1,i_2,\ldots,i_k}^\perp = \bigoplus_{A \subseteq S} e_A C_A^\perp.
\]
Moreover, $C$ is self-orthogonal over $R_{k,m}$ if and only if each $C_A$ is self-orthogonal over $\mathbb{F}_{p^m}$, for $A \subseteq S$.

**Proof.** It follows the similar argument of ([19], Theorem 5).

3. Constacyclic codes over $R_{k,m}$

In this section, we discuss the structural properties of constacyclic codes over $R_{k,m}$. These codes are used to obtain quantum codes in the subsequent section.

**Definition 3.1.** For $\gamma \in R_{k,m}^\ast$ (set of units in $R_{k,m}$), a linear code $C$ of length $n$ over $R_{k,m}$ is said to be a $\gamma$-constacyclic code if $\tau_\gamma(c) = (c_{n-1}, c_0, \ldots, c_{n-2}) \in C$ for $c = (c_0, c_1, \ldots, c_{n-1}) \in C$. The operator $\tau_\gamma$ is known as $\gamma$-constacyclic shift operator. The constacyclic code is cyclic if $\gamma = 1$ and negacyclic if $\gamma = -1$.

**Lemma 3.2.** An element $\gamma = \gamma_0 + \gamma_1 u_1 + \gamma_2 u_2 + \cdots + \gamma_{1,2} \ldots \ldots, k u_1 u_2 \ldots u_k \in R_{k,m}$ is a unit if and only if $\delta_0, \delta_1, \ldots, \delta_{1,2,\ldots,k}$ are units in $\mathbb{F}_{p^m}$ where
\[
\delta_0 e_0^0 + \sum_{1 \leq i_1 \leq k} \delta_{i_1} e_{i_1}^1 + \sum_{1 \leq i_1 < i_2 \leq k} \delta_{i_1,i_2} e_{i_1,i_2}^2 + \cdots + \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq k} \delta_{i_1,i_2,\ldots,i_k} e_{i_1,i_2,\ldots,i_k}^k = \sum_{A \subseteq S} \delta_A e_A^\Delta = \gamma.
\]

**Proof.** Let $\gamma = \gamma_0 + \gamma_1 u_1 + \gamma_2 u_2 + \cdots + \gamma_{1,2} \ldots \ldots, k u_1 u_2 \ldots u_k = \delta_0 e_0^0 + \sum_{1 \leq i_1 \leq k} \delta_{i_1} e_{i_1}^1 + \sum_{1 \leq i_1 < i_2 \leq k} \delta_{i_1,i_2} e_{i_1,i_2}^2 + \cdots + \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq k} \delta_{i_1,i_2,\ldots,i_k} e_{i_1,i_2,\ldots,i_k}^k$ be a unit in $R_{k,m}$.

Then there exists $\lambda \in R_{k,m}$ such that $\gamma \lambda = 1$, where $\lambda = \lambda_0 e_0^0 + \sum_{1 \leq i_1 \leq k} \lambda_{i_1} e_{i_1}^1 + \cdots + \sum_{1 \leq i_1 < i_2 \cdots < i_k \leq k} \lambda_{i_1,i_2,\ldots,i_k} e_{i_1,i_2,\ldots,i_k}^k$.
Proof. Let $\delta_A$ be a unit in $\mathbb{F}_{p^m}$, for all $A \subseteq S$. Therefore, $\delta_A$ is a unit in $\mathbb{F}_{p^m}$, for all $A \subseteq S$.

Conversely, let $\delta_A$ be a unit in $\mathbb{F}_{p^m}$, for all $A \subseteq S$. Let $C$ be a $\gamma$-constacyclic code of length $n$ on $R_{k,m}$. We represent each codeword $c = (c_0, c_1, \ldots, c_{n-1}) \in C$ as a polynomial $c(x) \in R_{k,m}[x]/(x^n - \gamma)$ under the correspondence

$$c = (c_0, c_1, \ldots, c_{n-1}) \mapsto c(x) = (c_0 + c_1 x + \cdots + c_{n-1} x^{n-1}) \mod (x^n - \gamma).$$

Under this identification, next result can easily be concluded.

Theorem 3.3. A linear code $C$ of length $n$ over $R_{k,m}$ is a $\gamma$-constacyclic code if and only if it is an ideal of the ring $R_{k,m}[x]/(x^n - \gamma)$.

Theorem 3.4. Let $C = \bigoplus_{A \subseteq S} e_A^C A$ be a linear code of length $n$ over $R_{k,m}$. Then $C$ is a $\gamma$-constacyclic code if and only if $C_A$ is the $\delta_A$-constacyclic code over $\mathbb{F}_{p^m}$, for every $A \subseteq S$ where $\delta_A$ is defined in Lemma 3.2.

Proof. Let $C = \bigoplus_{A \subseteq S} e_A^C A$ be a $\gamma$-constacyclic code. Let $a^A = (a_0^A, a_1^A, \ldots, a_{n-1}^A) \in C_A$, for $A \subseteq S$. Let $r_i = \sum_{A \subseteq S} e_A^C a_i^A$, for $0 \leq i \leq n-1$. Then $r = (r_0, r_1, \ldots, r_{n-1}) \in C$. Therefore,

$$\tau_r(r) = \sum_{A \subseteq S} e_A^C r_{\delta_A}(a^A) \in C = \bigoplus_{A \subseteq S} e_A^C A.$$

Hence, $\tau_{\delta_A}(a^A) \in C_A$, for $A \subseteq S$. Thus, $C_A$ is a $\delta_A$-constacyclic code of length $n$ over $\mathbb{F}_{p^m}$, for $A \subseteq S$.

Conversely, let $C_A$ be a $\delta_A$-constacyclic code of length $n$ over $\mathbb{F}_{p^m}$, for $A \subseteq S$. Let $r = (r_0, r_1, \ldots, r_{n-1}) \in C$ where $r_i = \sum_{A \subseteq S} e_A^C a_i^A$, for some $a_i^A \in \mathbb{F}_{p^m}$, $0 \leq i \leq n-1$. Now, $a^A = (a_0^A, a_1^A, \ldots, a_{n-1}^A) \in C_A$ for $A \subseteq S$. Therefore, $\tau_{\delta_A}(a^A) \in C_A$. Again, $\tau_r(r) = \sum_{A \subseteq S} e_A^C r_{\delta_A}(a^A) \in \bigoplus_{A \subseteq S} e_A^C A$. Hence, $C$ is a $\gamma$-constacyclic code of length $n$ over $R_{k,m}$.

Theorem 3.5. Let $C = \bigoplus_{A \subseteq S} e_A^C A$ be a $\gamma$-constacyclic code of length $n$ over $R_{k,m}$. Then there exists a polynomial $f(x) \in R_{k,m}[x]$ such that $C = (f(x))$ and $f(x) = \sum_{A \subseteq S} e_A^C f_A(x)$, is a factor of $(x^n - \gamma)$, where $f_A(x)$ is the generator polynomial of $C_A$, for $A \subseteq S$.

Proof. Let $C = \bigoplus_{A \subseteq S} e_A^C A$ be a $\gamma$-constacyclic code of length $n$ over $R_{k,m}$. Therefore, by Theorem 3.4, each $C_A$ is the $\delta_A$-constacyclic code of length $n$ over $\mathbb{F}_{p^m}$. Let $C_A = \langle f_A(x) \rangle$ where $f_A(x) \mid (x^n - \delta_A)$ for all $A \subseteq S$. Let $f(x) = \sum_{A \subseteq S} e_A^C f_A(x)$. Clearly, $\langle f(x) \rangle \subseteq C$. Also, $f_A(x)e^A_{\gamma} = f(x)e^A_{\gamma}$ for all $A \subseteq S$, which implies that $C \subseteq \langle f(x) \rangle$. Hence, $C = \langle f(x) \rangle$. 

Corollary 3.6. Every ideal of \( R \) is principally generated.

Corollary 3.7. Let \( C = \bigoplus_{A \subseteq S} e_A^\perp C_A \) be a \( \gamma \)-constacyclic code of length \( n \) over \( R \). Then, by Theorem 3.4, \( C_A \) is a \( \delta_A \)-constacyclic code of length \( n \) over \( \mathbb{F}_p \), for all \( A \subseteq S \). Therefore, \( C_A^\perp \) is a \( \delta_A^{-1} \)-constacyclic code over \( R \).

Recall that a \( q \)-ary quantum code of length \( n \) and size \( K \) is a \( K \)-dimensional subspace of \( q^n \)-dimensional Hilbert space \( \mathcal{C}^n \), where \( q = p^m \). Precisely, a quantum code is represented as \([[[n,k,d]]_q] \), where \( n \) is the length, \( d \) is the minimum distance and \( K = q^k \). The quantum code \([[[n,k,d]]_q] \) satisfies the singleton bound \( k + 2d \leq n + 2 \), and known as quantum MDS (maximum-distance-separable) if it attains the bound. In this section, we construct several new \( q \)-ary quantum codes by using the structure of \( \gamma \)-constacyclic codes over \( R_k \). Also, the necessary and sufficient conditions for these codes to contain their duals are obtained. We first recall the CSS construction (Lemma 4.1) which plays an important role to obtain the quantum codes.

Lemma 4.1 ([16], Theorem 3). Let \( C_1 = [n,k_1,d_1]_q \) and \( C_2 = [n,k_2,d_2]_q \) be two linear codes over \( GF(q) \) with \( C_2^\perp \subseteq C_1 \). Then there exists a quantum error-correcting code \( C = [[n,k_1 + k_2 - n,d,\min\{w(v) : v \in (C_1 \setminus C_2^\perp) \cup (C_2 \setminus C_1^\perp)\}] \geq \min\{d_1, d_2\} \}. Further, if \( C_1^\perp \subseteq C_1 \), then there exists a quantum error-correcting code \( C = [[n,2k_1 - n,d_1]] \), where \( d_1 = \min\{w(v) : v \in C_1 \setminus C_1^\perp\} \).

Lemma 4.2 ([5], Theorem 13). Let \( C \) be a \( \lambda \)-constacyclic code of length \( n \) over \( \mathbb{F}_p \) with the generator polynomial \( f(x) \). Then \( C^\perp \subseteq C \) if and only if \((x^n - \lambda) \equiv 0 \mod f(x)f^*(x))\), where \( f^*(x) \) is the reciprocal polynomial of \( f(x) \) and \( \lambda = \pm1 \).
In the light of Lemma 4.1, one must say that the dual containing linear code is the key to obtain quantum codes under the CSS construction. Therefore, by using Lemma 4.2, we present the necessary and sufficient conditions of the constacyclic codes to contain their duals in the next result.

**Theorem 4.3.** Let $\mathcal{C} = \bigoplus_{A \subseteq S} e_A^C \mathcal{C}_A$ be a $\gamma$-constacyclic code of length $n$ over $R_{k,m}$ and $\mathcal{C} = \langle f(x) \rangle = \langle \sum_{A \subseteq S} e_A^C f_A(x) \rangle$, where $f_A(x)$ is the generator of $\mathcal{C}_A$ and $\delta_A = \pm 1$, for $A \subseteq S$. Then $\mathcal{C}^\perp \subseteq \mathcal{C}$ if and only if

$$(x^n - \delta_A) \equiv 0 \pmod{f_A(x) f_A^*(x)},$$

where $f_A^*(x)$ is the reciprocal polynomial of $f_A(x)$, for $A \subseteq S$.

**Proof.** Let $(x^n - \delta_A) \equiv 0 \pmod{f_A(x) f_A^*(x)}$, for $A \subseteq S$. Then, by Lemma 4.2, we have $\mathcal{C}_A^\perp \subseteq \mathcal{C}_A$. Therefore, $e_A^C \mathcal{C}_A^\perp \subseteq e_A^C \mathcal{C}_A$, for $A \subseteq S$. Hence, $\mathcal{C}^\perp = \bigoplus_{A \subseteq S} e_A^C \mathcal{C}_A^\perp \subseteq \bigoplus_{A \subseteq S} e_A^C \mathcal{C}_A = \mathcal{C}$.

Conversely, let $\mathcal{C}^\perp \subseteq \mathcal{C}$. Then $\bigoplus_{A \subseteq S} e_A^C \mathcal{C}_A^\perp \subseteq \bigoplus_{A \subseteq S} e_A^C \mathcal{C}_A$. Since $\mathcal{C}_A$ is the $p^m$-ary code over $\mathbb{F}_{p^m}$ such that $e_A^C \mathcal{C}_A = \mathcal{C} \pmod{e_A^C}$ for $A \subseteq S$, $\mathcal{C}_A^\perp \subseteq \mathcal{C}_A$ for $A \subseteq S$. Again, by Lemma 4.2, we have

$$(x^n - \delta_A) \equiv 0 \pmod{f_A(x) f_A^*(x)},$$

where $f_A^*(x)$ is the reciprocal polynomial of $f_A(x)$, for $A \subseteq S$. \hfill $\square$

**Corollary 4.4.** Let $\mathcal{C} = \bigoplus_{A \subseteq S} e_A^C \mathcal{C}_A$ be a $\gamma$-constacyclic code of length $n$ over $R_{k,m}$. Then $\mathcal{C}^\perp \subseteq \mathcal{C}$ if and only if $\mathcal{C}_A^\perp \subseteq \mathcal{C}_A$, for $A \subseteq S$.

Now, Theorem 4.3 and Lemma 4.1 help us to obtain quantum codes as follows:

**Theorem 4.5.** Let $\mathcal{C} = \bigoplus_{A \subseteq S} e_A^C \mathcal{C}_A$ be a $\gamma$-constacyclic code of length $n$ over $R_{k,m}$ and $[2^k n, K, d_G]$, the parameters of $\psi(\mathcal{C})$, where $d_G$ is the minimum Gray distance of $\mathcal{C}$. If $\mathcal{C}^\perp \subseteq \mathcal{C}$, then there exists a quantum code with parameters $[[2^k n, 2K - 2^k n, d_G]]$ over $\mathbb{F}_{p^n}$.

**4.1. Examples.** With the help of Theorem 4.5, here we present some better quantum codes compare to the existing codes in the literature. It is remarkable that few recent articles in the reference list contain quantum codes inferior to the best-known codes. Hence, we compare our obtained codes with the best-known codes irrespective of their publication years. For example, the code $[[60, 48, 2]]_5$ presented by [3] in 2019 is inferior to $[[60, 48, 3]]_5$ appeared in the article [29] in 2018. Therefore, as seen in the Table 3, we compare our code $[[60, 50, 3]]_5$ to the best-known code $[[60, 48, 3]]_5$, rather with $[[60, 48, 2]]_5$. In this way, we show the superiority of our codes over the known codes to the best of our survey and knowledge. All computations involved in these examples are carried out by using the Magma computation system [4].

**Example 4.6.** Let $k = m = 1, p = 5, n = 11$ and $R_{1,1} = \mathbb{F}_5[u_1]/\langle u_1^2 - 1 \rangle$. Then $e_0^0 = 3(1 + u_1), e_1^1 = 3(1 - u_1)$. If $\gamma = u_1$, then $\delta_0 = 1, \delta_1 = -1$. Now, in $\mathbb{F}_5[x]$, we have

$$x^{11} - 1 = (x + 4)(x^5 + 2x^4 + 4x^3 + x^2 + 4)(x^5 + 4x^4 + 4x^3 + x^2 + 3x + 4)$$

$$x^{11} + 1 = (x + 1)(x^5 + x^4 + 4x^3 + 4x^2 + 3x + 1)(x^5 + 3x^4 + 4x^3 + 4x^2 + x + 1).$$
Let \( f_0(x) = x^5 + 2x^4 + 4x^3 + x^2 + x + 4, \) and \( f_1(x) = x^5 + x^4 + 4x^3 + 4x^2 + 3x + 1. \) Then \( C = \langle e_0^0 f_0(x) + e_1^1 f_1(x) \rangle \) is a \( u_1 \)-constacyclic code of length 11 over \( R_{1,1}. \) Let

\[
M = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \in GL_2(F_5),
\]

satisfying \( MM^t = 2I_2. \) Then the Gray image \( \psi(C) \) has the parameters \([22, 12, 7].\) Also, \( (x^{11} - \delta_i) \equiv 0 (mod \ f_i(x)f_i^*(x)), \) for \( i = 0, 1. \) Therefore, by Theorem 4.3, \( C^\perp \subseteq C. \) Hence, by Theorem 4.5, there exists a quantum code \([22, 2, 7])_5, \) which has the larger distance compare to the known code \([22, 2, 5])_5 \) given by \([10, 17].\)

**Remark 1.** In the above example, we have calculated that the Gray image \( \psi(C) \) is a \([22, 12, 7]) \) linear code over \( F_5. \) Note that \( \psi(C) \) has length \( = 2^k n = 2^1 \cdot 11 = 22 \) and dimension is equal to the sum of the dimensions of linear codes generated by polynomials \( f_0(x) \) and \( f_1(x), \) respectively. Now, putting the generator matrix \( G \) of linear code \( \psi(C), \) we have computed the minimum distance 7 by the Magma computation system \([4].\)

**Example 4.7.** Let \( k = 1, m = 2, p = 3, n = 8 \) and \( R_{1,2} = F_9[u_1]/\langle u_1^2 - 1 \rangle. \) Then \( e_0^0 = 2(1 + u_1), \) \( e_1^1 = 2(1 - u_1). \) If \( \gamma = 1, \) then \( \delta_0 = \delta_1 = 1. \) Now, in \( F_9[x], \) we have

\[
x^8 - 1 = (x + 1)(x + w)(x + w^2)(x + w^3)(x + 2)(x + w^5)(x + w^9)(x + w^7).
\]

Let \( f_0(x) = (x + w)(x + w^2) = x^2 + w^3 x + w^3, \) \( f_1(x) = x + w^2. \) Then \( C = \langle e_0^0 f_0(x) + e_1^1 f_1(x) \rangle \) is a cyclic code of length 8 over \( R_{1,2}. \) Let

\[
M = \begin{bmatrix} w & -1 \\ 1 & w \end{bmatrix} \in GL_2(F_9),
\]

satisfying \( MM^t = (1 + w^2)I_2. \) Then the Gray image \( \psi(C) \) has the parameters \([16, 13, 3].\) Also, \( (x^8 - \delta_i) \equiv 0 (mod \ f_i(x)f_i^*(x)), \) for \( i = 0, 1. \) Therefore, by Theorem 4.3, \( C^\perp \subseteq C. \) Hence, by Theorem 4.5, there exists a quantum code \([16, 10, 3])_9, \) which has the larger code rate than the known code \([16, 8, 3])_9 \) given by \([30].\)

**Example 4.8.** Let \( k = m = 1, p = 13, n = 6 \) and \( R_{1,1} = F_{13}[u_1]/\langle u_1^2 - 1 \rangle. \) Then \( e_0^0 = 7(1 + u_1), \) \( e_1^1 = 7(1 - u_2). \) Also, let \( \gamma = u_1, \) then \( \delta_0 = 0, \delta_1 = -1. \) Now, in \( F_{13}[x], \) we have

\[
x^6 - 1 = (x + 1)(x + 3)(x + 4)(x + 9)(x + 10)(x + 12) \\
x^6 + 1 = (x + 2)(x + 5)(x + 6)(x + 7)(x + 8)(x + 11).
\]

Let \( f_0(x) = x + 3, \) \( f_1(x) = x + 5. \) Then \( C = \langle e_0^0 f_0(x) + e_1^1 f_1(x) \rangle \) is a \( u_1 \)-constacyclic code of length 6 over \( R_{1,1}. \) Let

\[
M = \begin{bmatrix} 3 & 3 \\ 3 & 10 \end{bmatrix} \in GL_2(F_{13}),
\]

satisfying \( MM^t = 5I_2. \) Then the Gray image \( \psi(C) \) has the parameters \([12, 10, 3].\) Also, \( (x^6 - \delta_i) \equiv 0 (mod \ f_i(x)f_i^*(x)), \) for \( i = 0, 1. \) Therefore, by Theorem 4.3, \( C^\perp \subseteq C. \) Hence, by Theorem 4.5, there exists a quantum code \([12, 8, 3])_{13} \) which satisfies \( n - k + 2 - 2d = 0, \) i.e., a quantum MDS code.
Example 4.9. Let $k = m = 1, p = 11, n = 15$ and $R_{1,1} = \mathbb{F}_{11}[u_1]/(u_1^2 - 1)$. Then $e_0 = 6(1 + u_1), e_1 = 6(1 - u_2)$. If $\gamma = u_1$, then $\delta_0 = 1, \delta_1 = -1$. Now, in $\mathbb{F}_{11}[x]$, we have

$$x^{15} - 1 = (x + 2)(x + 6)(x + 7)(x + 8)(x + 10)(x^2 + x + 1)(x^2 + 3x + 9)$$

$$(x^2 + 4x + 5)(x^2 + 5x + 3)(x^2 + 9x + 4)$$

$$x^{15} + 1 = (x + 1)(x + 3)(x + 4)(x + 5)(x + 9)(x^2 + 2x + 4)(x^2 + 6x + 3)$$

$$(x^2 + 7x + 5)(x^2 + 8x + 9)(x^2 + 10x + 1).$$

Let $f_0(x) = (x^2 + 3x + 9)(x^2 + 9x + 4)(x + 2)(x + 7) = x^6 + 10x^5 + 3x^4 + 8x^3 + 2x^2 + 10x + 9, f_1(x) = (x^2 + 2x + 4)(x^2 + 7x + 5) = x^4 + 9x^3 + 12x^2 + 3x + 9$. Then $C = \langle e_0^0 f_0(x) + e_1^1 f_1(x) \rangle$ is a $u_1$-constacyclic code of length 15 over $R_{1,1}$. Let

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 10 \end{bmatrix} \in GL_2(\mathbb{F}_{11}),$$

satisfying $MM^t = 2I_2$. Then the Gray image $\psi(C)$ has the parameters $[30, 20, 6]$. Also, $(x^{15} - \delta_1) \equiv 0 \ (mod \ f_i(x)f_i^*(x))$, for $i = 0, 1$. Therefore, by Theorem 4.3, $C^\perp \subseteq C$. Hence, by Theorem 4.5, there exists a quantum code $[[30, 10, 6]]_{11}$. The obtained quantum code has the larger distance compare to the known code $[[30, 10, 5]]_{11}$ given by [25].

Example 4.10. Let $k = 2, m = 1, p = 17, n = 8$ and $R_{2,1} = \mathbb{F}_{17}[u_1, u_2]/(u_1^2 - 1, u_2 - 1, u_1 u_2 - u_2 u_1)$. Then $e_0 = 13(1 + u_1)(1 + u_2), e_1 = 13(1 - u_1)(1 + u_2), e_2 = 13(1 + u_1)(1 - u_2), e_2^2 = 13(1 - u_1)(1 - u_2)$. If $\gamma = u_1 u_2$, then $\delta_0 = \delta_1 = 1, \delta_2 = -1$. Now, in $\mathbb{F}_{17}[x]$, we have

$$x^8 - 1 = (x + 1)(x + 2)(x + 4)(x + 9)(x + 13)(x + 15)(x + 16)$$

$$x^8 + 1 = (x + 3)(x + 5)(x + 6)(x + 7)(x + 10)(x + 11)(x + 12)(x + 14).$$

Let $f_0(x) = (x + 2)(x + 4)(x + 15), f_1(x) = (x + 3)(x + 5)(x + 10)(x + 11), f_2(x) = x + 5, f_{1,2}(x) = x + 2$. Then $C = \langle e_0^0 f_0(x) + e_1^1 f_1(x) + e_2^2 f_2(x) + e_2^2 f_{1,2}(x) \rangle$ is a $u_1 u_2$-constacyclic code of length 8 over $R_{2,1}$. Let

$$M = \begin{bmatrix} 12 & 9 & 14 & 16 \\ 9 & 5 & 16 & 3 \\ 14 & 1 & 5 & 9 \\ 16 & 14 & 8 & 5 \end{bmatrix} \in GL_4(\mathbb{F}_{17}),$$

satisfying $MM^t = 14I_4$. Then the Gray image $\psi(C)$ has the parameters $[32, 23, 6]$. Also, $(x^8 - \delta_A) \equiv 0 \ (mod \ f_A(x)f_A^*(x))$, for $A \subseteq \{1, 2\}$. Therefore, by Theorem 4.3, $C^\perp \subseteq C$. Hence, by Theorem 4.5, there exists a quantum code $[[32, 14, 6]]_{17}$, which has larger dimension compare to the known code $[[32, 12, 6]]_{17}$ given by [13]. Therefore, our code has larger code rate than the known.

Example 4.11. Let $k = 2, m = 1, p = 17, n = 4$ and $R_{2,1} = \mathbb{F}_{17}[u_1, u_2]/(u_1^2 - 1, u_2 - 1, u_1 u_2 - u_2 u_1)$. Then $e_0 = 13(1 + u_1)(1 + u_2), e_1 = 13(1 - u_1)(1 + u_2), e_2 = 13(1 + u_1)(1 - u_2), e_2^2 = 13(1 - u_1)(1 - u_2)$. If $\gamma = u_1 u_2$, then $\delta_0 = \delta_1 = 1, \delta_2 = -1$. Now, in $\mathbb{F}_{17}[x]$, we have

$$x^4 - 1 = (x + 1)(x + 4)(x + 13)(x + 16)$$

$$x^4 + 1 = (x + 2)(x + 8)(x + 9)(x + 15).$$
Let $f_0(x) = x + 4, f_1(x) = x + 2, f_2(x) = x + 8, f_{1,2}(x) = x + 13$. Then $C = \langle e_0^0 f_0(x) + e_1^1 f_1(x) + e_2^2 f_2(x) + e_{1,2}^7 f_{1,2}(x) \rangle$ is a $u_1 u_2$-constacyclic code of length 4 over $R_{2,1}$. Let

$$ M = \begin{bmatrix} 9 & 10 & 12 & 15 \\ 10 & 8 & 15 & 5 \\ 12 & 2 & 8 & 10 \\ 15 & 12 & 7 & 8 \end{bmatrix} \in GL_4(\mathbb{F}_{17}). $$

Then $M$ satisfies $M M^t = 6 I_4$ and the Gray image $\psi(C)$ has the parameters $[16, 12, 4]$. Since $(x^4 - \delta_A) \equiv 0 \pmod{f_A(x) f_A^*(x)}$, for $A \subseteq \{1, 2\}$, by Theorem 4.3, we have $C^\perp \subseteq C$. Hence, by Theorem 4.5, there exists a quantum code $[[16, 8, 4]]_{17}$. Now, by comparing with the known code $[[16, 8, 3]]_{17}$, we conclude that our code has larger distance than that code.

**Example 4.12.** Let $k = 2, m = 1, p = 13, n = 6$ and $R_{2,1} = F_{13}[u_1, u_2]/(u_1^2 - 1, u_2^2 - 1, u_1 u_2 - u_2 u_1)$. Then $e_0 = 10 (1 + u_1)(1 + u_2), e_1 = 10 (1 - u_1)(1 + u_2), e_2 = 10(u_1 + u_2)(1 - u_1)(1 - u_2), e_{1,2} = 10(1 - u_1)(1 - u_2)$. If $\gamma = u_1 u_2$, then $\delta_0 = \delta = \delta_1 = \delta_2 = -1$. Now, in $F_{13}[x]$, we have

$$ x^6 - 1 = (x + 1)(x + 3)(x + 4)(x + 9)(x + 10)(x + 12) $$

$$ x^6 + 1 = (x + 2)(x + 5)(x + 6)(x + 7)(x + 8)(x + 11). $$

Let $f_0(x) = (x + 3)(x + 4), f_1(x) = (x + 2)(x + 5), f_2(x) = (x + 6)(x + 7), f_{1,2}(x) = (x + 4)(x + 9)$. Then $C = \langle e_0^0 f_0(x) + e_1^1 f_1(x) + e_2^2 f_2(x) + e_{1,2}^7 f_{1,2}(x) \rangle$ is a $u_1 u_2$-constacyclic code of length 6 over $R_{2,1}$. Let

$$ M = \begin{bmatrix} 10 & 7 & 9 & 8 \\ 7 & 3 & 8 & 4 \\ 9 & 5 & 3 & 7 \\ 8 & 9 & 6 & 3 \end{bmatrix} \in GL_4(\mathbb{F}_{13}). $$

Then $M$ satisfies $M M^t = 8 I_4$ and the Gray image $\psi(C)$ has the parameters $[24, 16, 6]$. Since $(x^6 - \delta_A) \equiv 0 \pmod{f_A(x) f_A^*(x)}$, for $A \subseteq \{1, 2\}$, by Theorem 4.3, we have $C^\perp \subseteq C$. Hence, by Theorem 4.5, there exists a quantum code $[[24, 8, 6]]_{13}$, which has larger distance than the known code $[[24, 8, 4]]_{13}$ appeared in [13].

### 4.2. Computation tables

The Table 2 gives the set of matrices over the finite field $\mathbb{F}_{p^m}$, which are used to compute the Gray images of constacyclic codes in Table 1 and Table 3-5, respectively. Also, Table 1 presents some quantum MDS codes while Table 3-5 include new and better quantum codes than previously known codes from the constacyclic codes over $R_{1,m} = F_{p^m}[u_1]/(u_1^2 - 1)$. In Table 1 and Table 3-5, we used different columns as below:

1. **1st column-values of $p^m$,**
2. **2nd column-lengths of the codes,**
3. **3rd column-values of the unit $\gamma$,**
4. **4th column-corresponding values of the units $\delta_0, \delta_1$,**
5. **5th & 6th column-generator polynomials for the constacyclic codes,**
6. **7th column-used matrices to compute parameters of Gray images,**
7. **8th column-parameters of Gray images $\psi(C)$ of the constacyclic codes,**
8. **9th column-parameters $[[n, k, d]]_{p^m}$ of the obtained quantum codes,**
9. **10th column-parameters $[[n', k', d']]_{p^m}$ of the best-known quantum codes.**

In order to compare our obtained quantum codes with best-known codes, we include the 10th column from different references as mentioned in the column. We
New quantum codes from constacyclic codes over the ring $R_{k,m}$

Table 1. Quantum MDS codes $[[n, k, d]]_{p^m}$ from constacyclic codes over $R_{1,m} = \mathbb{F}_{p^m}[u]/(u^2 - 1)$

| $p^m$ | $n$ | $\gamma$ | $(\delta_0, \delta_1)$ | $f_0(x)$ | $f_1(x)$ | $M$ | $\psi(C)$ | $[[n, k, d]]_{p^m}$ |
|-------|-----|-----------|----------------------|----------|----------|-----|---------|---------------------|
| 5     | 2   | $-1$      | $(-1, -1)$           | 13       | 1        | $M_4$| [4,3,2] | $[[4,2,2]]_5$      |
| 13    | 6   | $u_1$     | $(1, -1)$            | 13       | 15       | $M_7$|[12,10,3] | $[[12,8,3]]_{13}$  |
| 11    | 5   | $u_1$     | $(1, -1)$            | 12       | 14       | $M_8$|[10,8,3]  | $[[10,6,3]]_{11}$  |
| 11    | 5   | $u_1$     | $(1, -1)$            | 12       | 184      | $M_8$|[10,7,4]  | $[[10,4,4]]_{11}$  |
| 17    | 8   | $u_1$     | $(1, -1)$            | 12       | 17       | $M_9$|[16,14,3] | $[[16,12,3]]_{17}$ |
| 23    | 11  | $u_1$     | $(1, -1)$            | 17       | 13       | $M_{10}$|[22,20,3] | $[[22,18,3]]_{23}$ |
| 19    | 9   | $u_1$     | $(1, -1)$            | 13       | 14       | $M_{11}$|[18,16,3] | $[[18,14,3]]_{19}$ |
| 30    | 14  | $u_1$     | $(1, -1)$            | 1((3)    | 13       | $M_{12}$|[28,26,3] | $[[28,24,3]]_{29}$ |

have seen that our obtained codes given in the 9th column are better than the codes shown in the 10th column by means of larger code rates and larger minimum distances. To represent the generator polynomials $f_0(x), f_1(x)$, we write their coefficients in decreasing order, e.g., we use 124114 to represent the polynomial $x^5 + 2x^4 + 4x^3 + x^2 + x + 4$.

**Remark 2.** Recall that a code $[[n, k, d]]_q$ satisfying $n - k + 2 - 2d = t$ is known as a quantum code with singleton defect $t$. Obviously $t \geq 0$ and when $t = 0$, it is a quantum MDS code. Also, $t$ is the judgmental parameter to determine a code how much close to MDS. In fact, smaller $t$ implies code is close to MDS. Hence, the main objective should be to obtain the code whose $t$ is closer to zero, as much as possible. In the above tables, we have seen that the quantum codes with singleton defect $t = 2$ are

$$[[24,16,4]]_3, [[60,56,2]]_3, [[72,68,2]]_3, [[20,14,3]]_5, [[120,116,2]]_5, [[168,164,2]]_7, [[16,10,3]]_9, [[12,4,4]]_{13}, [[24,18,3]]_{13}, [[36,32,2]]_{13}, [[16,10,3]]_{17}, [[24,18,3]]_{17}.$$  

On the other hand, quantum codes with singleton defect $t = 4$ are

$$[[40,32,5]]_5, [[14,6,3]]_7, [[28,20,3]]_7, [[28,18,4]]_7, [[16,8,3]]_{13}, [[18,10,3]]_{13}, [[26,18,3]]_{13}, [[26,14,5]]_{13}, [[26,12,6]]_{13}, [[48,40,3]]_{17}, [[48,38,4]]_{17}.$$
Table 2. Matrix Encoding

| $M_1$ | $GL_2(\mathbb{F}_{p^m})$ | $M_1M_1^t = 2I_2$ | $M_7$ | $GL_2(\mathbb{F}_{p^m})$ | $M_7M_7^t = 5I_2$ |
| --- | --- | --- | --- | --- | --- |
| $\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$ | $GL_2(\mathbb{F}_3)$ | | $\begin{bmatrix} 3 & 3 \\ 3 & 10 \end{bmatrix}$ | $GL_2(\mathbb{F}_{13})$ | |
| $M_2$ | $GL_2(\mathbb{F}_3)$ | $M_2M_2^t = 2I_2$ | $M_8$ | $GL_2(\mathbb{F}_{11})$ | $M_8M_8^t = 2I_2$ |
| $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ | | | $\begin{bmatrix} 1 & 1 \\ 1 & 10 \end{bmatrix}$ | | |
| $M_3$ | $GL_2(\mathbb{F}_9)$ | $M_3M_3^t = (1 + w^2)I_2$ | $M_9$ | $GL_2(\mathbb{F}_{17})$ | $M_9M_9^t = 8I_2$ |
| $\begin{bmatrix} w & -1 \\ 1 & w \end{bmatrix}$ | | | $\begin{bmatrix} 2 & 2 \\ 2 & 15 \end{bmatrix}$ | | |
| $M_4$ | $GL_2(\mathbb{F}_5)$ | $M_4M_4^t = 2I_2$ | $M_{10}$ | $GL_2(\mathbb{F}_{23})$ | $M_{10}M_{10}^t = 2I_2$ |
| $\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$ | | | $\begin{bmatrix} 1 & 1 \\ 1 & 22 \end{bmatrix}$ | | |
| $M_5$ | $GL_2(\mathbb{F}_7)$ | $M_5M_5^t = 4I_2$ | $M_{11}$ | $GL_2(\mathbb{F}_{19})$ | $M_{11}M_{11}^t = 8I_2$ |
| $\begin{bmatrix} 3 & 4 \\ 3 & 3 \end{bmatrix}$ | | | $\begin{bmatrix} 2 & 2 \\ 2 & 17 \end{bmatrix}$ | | |
| $M_6$ | $GL_2(\mathbb{F}_7)$ | $M_6M_6^t = 2I_2$ | $M_{12}$ | $GL_2(\mathbb{F}_{29})$ | $M_{12}M_{12}^t = 8I_2$ |
| $\begin{bmatrix} 1 & 1 \\ 1 & 6 \end{bmatrix}$ | | | $\begin{bmatrix} 2 & 2 \\ 2 & 27 \end{bmatrix}$ | |
Table 3. New Quantum codes $[[n, k, d]]_{p^m}$ from constacyclic codes over $R_{1,m} = \mathbb{F}_{p^m}[u_1]/\langle u_1^2 - 1 \rangle$

| $p^m$ | $n$  | $\gamma$ | $(\delta_0, \delta_1)$ | $f_0(x)$ | $f_1(x)$ | $M$     | $\psi(C)$ | $[[n, k, d]]_{p^m}$ | $[[n', k', d']]_{p^m}$ |
|-------|------|----------|------------------------|----------|----------|---------|-----------|---------------------|---------------------|
| 3     | 12   | $-1$     | $(-1, -1)$             | 112      | 122      | $M_1$   | [24, 20, 4] | $[[24, 16, 4]]_3$ | $[[26, 16, 4]]_3 [9]$ |
| 3     | 15   | $u_1$    | $(1, -1)$              | 11111    | 12121    | $M_1$   | [30, 22, 6] | $[[30, 14, 6]]_3$ | $[[31, 13, 6]]_3 [9]$ |
| 3     | 18   | $u_1$    | $(1, -1)$              | 12021    | 10201    | $M_2$   | [36, 28, 3] | $[[36, 20, 3]]_3$ | -                   |
| 3     | 30   | 1        | $(1, 1)$               | 11       | 12       | $M_2$   | [60, 58, 2] | $[[60, 56, 2]]_3$ | $[[60, 54, 2]]_3 [12]$ |
| 3     | 36   | 1        | $(1, 1)$               | 12       | 12       | $M_2$   | [72, 70, 2] | $[[72, 68, 2]]_3$ | $[[72, 66, 2]]_3 [12, 13]$ |
| 5     | 10   | $u_1$    | $(1, -1)$              | 131      | 12       | $M_4$   | [20, 17, 3] | $[[20, 14, 3]]_5$ | $[[22, 14, 3]]_5 [10]$ |
| 5     | 10   | $u_1$    | $(1, -1)$              | 1441     | 143122   | $M_4$   | [20, 12, 5] | $[[20, 4, 5]]_5$  | $[[19, 1, 5]]_5 [10]$ |
| 5     | 11   | $u_1$    | $(1, -1)$              | 124114   | 114431   | $M_4$   | [22, 12, 7] | $[[22, 2, 7]]_5$  | $[[22, 2, 5]]_5 [10, 17]$ |
| 5     | 12   | $u_1$    | $(1, -1)$              | 10224    | 12041    | $M_4$   | [24, 16, 5] | $[[24, 8, 5]]_5$  | $[[23, 6, 5]]_5 [10]$ |
| 5     | 15   | $u_1$    | $(1, -1)$              | 1003001  | 1003421  | $M_4$   | [30, 18, 6] | $[[30, 6, 6]]_5$  | $[[60, 8, 6]]_5 [6]$ |
| 5     | 15   | $u_1$    | $(1, -1)$              | 1003001  | 11021    | $M_4$   | [30, 20, 4] | $[[30, 10, 4]]_5$ | $[[30, 10, 2]]_5 [6]$ |
| 5     | 20   | 1        | $(1, 1)$               | 1034     | 12       | $M_4$   | [40, 36, 3] | $[[40, 32, 3]]_5$ | $[[40, 24, 3]]_5 [30]$ |
| 5     | 22   | $u_1$    | $(1, -1)$              | 13024212034 | 111212 | $M_4$   | [44, 29, 8] | $[[44, 14, 8]]_5$ | $[[44, 8, 8]]_5 [30]$ |
| 5     | 30   | $u_1$    | $(1, -1)$              | 13431    | 13       | $M_4$   | [60, 55, 3] | $[[60, 50, 3]]_5$ | $[[60, 48, 3]]_5 [29]$ |
Table 4. New Quantum codes $[[n, k, d]]_{p^m}$ from constacyclic codes over $R_{1,m} = F_{p^m}[u_1]/(u_1^2 - 1)$

| $p^m$ | $n$  | $\gamma$ | $(\delta_0, \delta_1)$ | $f_0(x)$ | $f_1(x)$ | $M$   | $\psi(C)$ | $[[n, k, d]]_{p^m}$       | $[[n', k', d']]_{p^m}$ |
|-------|------|-----------|------------------------|----------|----------|-------|-----------|--------------------------|-------------------------|
| 5     | 60   | 1         | $(1, 1)$               | 13       | 12       | $M_4$ |           | $[120, 118, 2]$           | $[120, 116, 2]_{5}$     |
| 5     | 70   | $u_1$     | $(1, -1)$              | 134444431| 13       | $M_4$ |           | $[140, 131, 3]$           | $[140, 122, 3]_{5}$     |
| 7     | 7    | $u_1$     | $(1, -1)$              | 151      | 121      | $M_5$ |           | $[14, 10, 3]$             | $[14, 6, 3]_{7}$        |
| 7     | 7    | $u_1$     | $(1, -1)$              | 1436     | 1331     | $M_5$ |           | $[14, 8, 4]$              | $[14, 2, 4]_{7}$        |
| 7     | 14   | 1         | $(1, 1)$               | 1661     | 16       | $M_5$ |           | $[28, 24, 3]$             | $[28, 20, 3]_{7}$       |
| 7     | 14   | 1         | $(1, 1)$               | 15026    | 11       | $M_5$ |           | $[28, 23, 4]$             | $[28, 18, 4]_{7}$       |
| 7     | 21   | $u_1$     | $(1, -1)$              | 1054214515| 1515511  | $M_6$ |           | $[42, 27, 7]$             | $[42, 12, 7]_{7}$       |
| 7     | 84   | 1         | $(1, 1)$               | 12       | 13       | $M_6$ |           | $[168, 166, 2]$           | $[168, 164, 2]_{7}$     |
| 9     | 8    | 1         | $(1, 1)$               | $1w^3w^3$| $1w^2$   | $M_3$ |           | $[16, 13, 3]$             | $[16, 10, 3]_{9}$       |
| 9     | 8    | $u_1$     | $(1, -1)$              | $1w^7w_5$| $10w^201$| $M_3$ |           | $[16, 9, 5]$              | $[16, 2, 5]_{9}$        |
| 9     | 12   | $u_1$     | $(1, -1)$              | $102w^60w^2$| $1w^3$  | $M_3$ |           | $[24, 18, 4]$             | $[24, 10, 4]_{9}$       |
| 11    | 15   | $u_1$     | $(1, -1)$              | $(1(10)382(10)9)$| $19(12)39$| $M_8$ |           | $[30, 20, 6]$             | $[30, 10, 6]_{11}$      |
| 11    | 26   | $-1$      | $(-1, -1)$             | $1342443749481$| $184943442431$| $M_8$ |           | $[52, 28, 10]$            | $[52, 4, 10]_{11}$      |
| 11    | 33   | $u_1$     | $(1, 1)$               | $191(10)2(10)$| 11      | $M_8$ |           | $[66, 60, 4]$             | $[66, 54, 4]_{11}$      |
| 11    | 33   | $u_1$     | $(1, 1)$               | $191949191$| 11      | $M_8$ |           | $[66, 57, 5]$             | $[66, 48, 5]_{11}$      |
Table 5. New Quantum codes $[[n, k, d]]_{p^m}$ from constacyclic codes over $R_{1,m} = \mathbb{F}_{p^m}[u_1]/(u_1^2 - 1)$

| $p^m$ | $n$ | $\gamma$ | $(\delta_0, \delta_1)$ | $f_0(x)$ | $f_1(x)$ | $M$ | $\psi(C)$ | $[[n, k, d]]_{p^m}$ | $[[n', k', d']]_{p^m}$ |
|-------|-----|-----------|------------------------|--------|--------|-----|----------|----------------|----------------|
| 13    | 6   | $u_1$     | $(1, -1)$              | 17(12) | 17(10) | $M_7$ | 12, 8, 4 | $[[12, 4, 4]]_{13}$ | $[[12, 4, 3]]_{13}$ |
| 13    | 8   | $u_1$     | $(1, -1)$              | 15     | 155(12)| $M_7$ | 16, 12, 3| $[[16, 8, 3]]_{13}$ | $[[16, 8, 2]]_{13}$ |
| 13    | 9   | $u_1$     | $(1, -1)$              | 1(10)  | 1003   | $M_7$ | 18, 14, 3| $[[18, 10, 3]]_{13}$ | $[[12, 4, 3]]_{13}$ |
| 13    | 12  | $u_1$     | $(1, -1)$              | 12     | 102    | $M_7$ | 24, 21, 3| $[[24, 18, 3]]_{13}$ | $[[24, 16, 3]]_{13}$ |
| 13    | 13  | $u_1$     | $(1, -1)$              | 1(11)  | 121    | $M_7$ | 26, 22, 3| $[[26, 18, 3]]_{13}$ | $[[36, 20, 3]]_{13}$ |
| 13    | 13  | $u_1$     | $(1, -1)$              | 1(11)  | 14641  | $M_7$ | 26, 20, 5| $[[26, 14, 5]]_{13}$ | $[[24, 8, 5]]_{13}$ |
| 13    | 13  | $u_1$     | $(1, -1)$              | 1(11)  | 15(10)(10) | $M_7$ | 26, 19, 6| $[[26, 12, 6]]_{13}$ | $[[24, 4, 6]]_{13}$ |
| 13    | 13  | $u_1$     | $(1, -1)$              | 13     | 12     | $M_7$ | 36, 34, 2| $[[36, 32, 2]]_{13}$ | $[[36, 30, 2]]_{13}$ |
| 13    | 13  | $u_1$     | $(1, -1)$              | 130780(12)(10) | 120830(12)(11) | $M_7$ | 36, 22, 6| $[[36, 8, 4]]_{13}$ | $[[36, 8, 4]]_{13}$ |
| 17    | 8   | $u_1$     | $(1, -1)$              | 168    | 15     | $M_9$ | 16, 13, 3| $[[16, 10, 3]]_{17}$ | $[[16, 8, 3]]_{17}$ |
| 17    | 12  | $u_1$     | $(1, -1)$              | 14     | 124    | $M_9$ | 24, 21, 3| $[[24, 18, 3]]_{17}$ | $[[24, 18, 2]]_{17}$ |
| 17    | 16  | $u_1$     | $(1, -1)$              | 1(14)(3) | 1010(10)(0|14) | $M_9$ | 32, 22, 7| $[[32, 12, 7]]_{17}$ | $[[32, 12, 6]]_{17}$ |
| 17    | 16  | $u_1$     | $(1, -1)$              | 1(15)(12)(24)(13)(12) | 10(12)040501 | $M_9$ | 32, 18, 10| $[[32, 4, 10]]_{17}$ | $[[32, 4, 8]]_{17}$ |
| 17    | 24  | $u_1$     | $(1, -1)$              | 16(12) | 17     | $M_9$ | 48, 44, 3| $[[48, 40, 3]]_{17}$ | $[[48, 36, 3]]_{17}$ |
| 17    | 24  | $u_1$     | $(1, -1)$              | 1(14)(9)(10) | 1(11) | $M_9$ | 48, 43, 3| $[[48, 38, 4]]_{17}$ | $[[48, 30, 4]]_{17}$ |
5. Conclusion

In this article, we studied the constacyclic codes over the family of commutative non-chain rings $R_{k,m}$ to obtain new non-binary quantum codes over finite fields. In the above tables, we have determined many new quantum codes which are superior to the best-known codes in the literature. Therefore, we believe that our work will motivate researchers to unfold the existence of many new quantum codes which can be obtained from this class of constacyclic codes.

Acknowledgement

The authors are thankful to the University Grants Commission (UGC) and the Council of Scientific & Industrial Research (CSIR), Govt. of India for their financial support. The authors would also like to thank the Editor and anonymous referee(s) for careful reading and constructive suggestions to improve the presentation of the manuscript.

References

[1] M. Ashraf and G. Mohammad, Construction of quantum codes from cyclic codes over $F_p + vF_p$, Int. J. Inf. Coding Theory, 3 (2015), 137–144.
[2] M. Ashraf and G. Mohammad, Quantum codes from cyclic codes over $F_q + uF_q + vF_q + uvF_q$, Quantum Inf. Process., 15 (2016), 4089–4098.
[3] M. Ashraf and G. Mohammad, Quantum codes over $F_p$ from cyclic codes over $F_p[u,v]/(u^2 - 1, v^3 - v, uv - vu)$, Cryptogr. Commun., 11 (2019), 325–335.
[4] W. Bosma and J. Cannon, Handbook of Magma Functions, University of Sydney, 1995.
[5] A. R. Calderbank, E. M. Rains, P. M. Shor and N. J. A. Sloane, Quantum error correction via codes over $GF(4)$, IEEE Trans. Inform. Theory, 44 (1998), 1369–1387.
[6] Y. Cengellenmis and A. Dertli, The Quantum Codes over $F_4$ and quantum quasi-cyclic codes over $F_4$, Math. Sci. Appl. E-Notes, 7 (2019), 87–93.
[7] Y. Cengellenmis, A. Dertli and S. T. Dougherty, Codes over an infinite family of rings with a Gray map, Des. Codes Cryptogr., 72 (2014), 559–580.
[8] A. Dertli, Y. Cengellenmis and S. Eren, On quantum codes obtained from cyclic codes over $A_2$, Int. J. Quantum Inf., 13 (2015), 1550031, 9 pp.
[9] M. F. Ezerman, S. Ling, B. Qzkaya and P. Sole, Good stabilizer codes from quasi-cyclic codes over $F_3$ and $F_9$, IEEE International Symposium on Information Theory (ISIT), Paris, France, 2019, 2898–2902.
[10] Y. Edel, Some good quantum twisted codes, https://www.mathi.uni-heidelberg.de/~yves/\textifo\textit{Matritzen/QTBCH/QTBCHIndex.html}.
[11] J. Gao, Quantum codes from cyclic codes over $F_q + vF_q + v^2F_q + v^3F_q$, Int. J. Quantum Inf., 13 (2015), 1550063(1-8).
[12] J. Gao and Y. Wang, $u$-Constacyclic codes over $F_p + uF_p$ and their applications of constructing new non-binary quantum codes, Quantum Inf. Process., 17 (2018), Art. 4, 9 pp.
[13] Y. Gao, J. Gao and F. W. Fu, On Quantum codes from cyclic codes over the ring $F_q + v_1F_q + \cdots + v_nF_q$, Appl. Algebra Engrg. Comm. Comput., 30 (2019), 161–174.
[14] G. Gaurdia and R. Palazzo Jr., Constructions of new families of nonbinary CSS codes, Discrete Math., 310 (2010), 2935–2945.
[15] D. Gottesman, An introduction to quantum error-correction, Proc. Symp. Appl. Math., 68 (2010), 13–27.
[16] M. Grassl and T. Beth, On optimal quantum codes, Int. J. Quantum Inf., 2 (2004), 55–64.
[17] M. Guzeltepe and M. Sari, Quantum codes from cyclic codes over the ring $F_q + \alpha F_q$, Quantum Inf. Process., 18 (2019), Art. 365, 21 pp.
[18] X. He, L. Xu and H. Chen, New q-ary quantum MDS codes with distances bigger than $\frac{q}{2}$, Quantum Inf. Process., 15 (2016), 2745–2758.
[19] H. Islam and O. Prakash, Quantum codes from the cyclic codes over $F_{p}[u,v,w]/(u^2 - 1, v^2 - 1, w^2 - 1, uv - vu, vw - vw, uu - uu)$, J. Appl. Math. Comput., 60 (2019), 625–635.
New quantum codes from constacyclic codes over the ring $R_{k,m}$

[20] H. Islam, O. Prakash and D. K. Bhunia, Quantum codes obtained from constacyclic codes, Int J Theor Phys., 58 (2019), 3945–3951.

[21] H. Islam, R. K. Verma and O. Prakash, A family of constacyclic codes over $F_p^m[u,v]/(u^2 - 1, u^2 - 1, u^2 - 1, uv - uv)$, Int. J. Inf. Coding Theory, (2020).

[22] H. Islam, O. Prakash and R. K. Verma, Quantum codes from the cyclic codes over $F_p^m[u,v]/(u^2 - 1, v^2 - 1, uv - uv)$, Springer Proceedings in Mathematics & Statistics, 307 (2020).

[23] X. Kai and S. Zhu, Quaternary construction of quantum codes from cyclic codes over $F_4 + uF_4$, Int. J. Quantum Inf., 9 (2011), 689–700.

[24] X. Kai, S. Zhu and P. Li, Constacyclic codes and some new quantum MDS codes, IEEE Trans. Inform. Theory, 60 (2014), 2080–2086.

[25] M. E. Koroglu and I. Siap, Quantum codes from a class of constacyclic codes over group algebras, Malays. J. Math. Sci., 11 (2017), 289–301.

[26] R. Li, Z. Xu and X. Li, Binary construction of quantum codes of minimum distance three and four, IEEE Trans. Inform. Theory, 50 (2004), 1331–1336.

[27] R. Li and Z. Xu, Construction of $[[n, n - 4, 3]]_q$ quantum codes for odd prime power $q$, Phys. Rev. A, 82 (2010), 052316, 1–4.

[28] J. Li, J. Gao and Y. Wang, Quantum codes from $(1 - 2v)$-constacyclic codes over the ring $F_q[u,v]/(u^2 - 1, v^2 - 1, uv - vu)$, Discrete Math. Algorithms Appl., 10 (2018), 1850046, 8 pp.

[29] F. Ma, J. Gao and F. W. Fu, Constacyclic codes over the ring $F_p + vF_p + v^2F_p$ and their applications of constructing new non-binary quantum codes, Quantum Inf. Process., 17 (2018), Art. 122, 19 pp.

[30] F. Ma, J. Gao and F. W. Fu, New non-binary quantum codes from constacyclic codes over $F_p[u,v]/(u^2 - 1, v^2 - 1, uv - vu)$, Adv. Math. Commun., 13 (2019), 421–434.

[31] M. Ozen, N. T. Ozzaim and H. Ince, Quantum codes from cyclic codes over $F_3 + uF_3 + vF_3 + uvF_3$, Int. Conf. Quantum Sci. Appl. J. Phys. Conf. Ser., 766 (2016).

[32] J. Qian, W. Ma and W. Gou, Quantum codes from cyclic codes over finite ring, Int. J. Quantum Inf., 7 (2009), 1277–1283.

[33] M. Sari and I. Siap, On quantum codes from cyclic codes over a class of nonchain rings, Bull. Korean Math. Soc., 53 (2016), 1617–1628.

[34] A. K. Singh, S. Pattanayek, P. Kumar and K. P. Shum, On Quantum codes from cyclic codes over $F_2 + uF_2 + u^2F_2$, Asian-Eur. J. Math., 11 (2018), 1850009, 11 pp.

[35] P. W. Shor, Scheme for reducing decoherence in quantum memory, Phys. Rev. A, 52 (1995), 2493–2496.

[36] X. Zheng and B. Kong, Constacyclic codes over $F_{p^m} = [u_1, u_2, \ldots, u_k]/(u_i^2 = u_i, u_iu_j = u_ju_i)$, Open Math., 16 (2018), 490–497.

Received December 2019; 1st revision May 2020.

E-mail address: habibul.pma17@iitp.ac.in
E-mail address: om@iitp.ac.in
E-mail address: ram.pma15@iitp.ac.in