Liouville field theory coupled to a critical Ising model: Non-perturbative analysis, duality and applications

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Abstract

Two different kinds of interactions between a $\mathbb{Z}_n$-parafermionic and a Liouville field theory are considered. For generic values of $n$, the effective central charges describing the UV behavior of both models are calculated in the Neveu-Schwarz sector. For $n = 2$ exact vacuum expectation values of primary fields of the Liouville field theory, as well as the first descendent fields are proposed. For $n = 1$, known results for Sinh-Gordon and Bullough-Dodd models are recovered whereas for $n = 2$, exact results for these two integrable coupled Ising-Liouville models are shown to exchange under a weak-strong coupling duality relation. In particular, exact relations between the parameters in the actions and the mass of the particles are obtained. At specific imaginary values of the coupling and $n = 2$, we use previous results to obtain exact information about: (a) Integrable coupled models like Ising-$\mathcal{M}_{p/p'}$, homogeneous sine-Gordon model $SU(3)_2$ or the Ising-XY model, (b) Neveu-Schwarz sector of the $\Phi_{13}$ integrable perturbation of $\mathcal{N} = 1$ supersymmetric minimal models. Several non-perturbative checks are done, which support the exact results.

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1 Introduction

Duality plays an important role in the analysis of quantum field theory (QFT) and statistical physics. This property allows to study the behavior of observable in the strong coupling region of one model in terms of the ones in the weak coupling region of the other (dual) model. In
different region of the coupling constant, it is then possible to use perturbative and semiclassical methods. For instance, in four dimensions the electro-magnetic duality conjectured in [1] and developed in [2] is the main ingredient in studying the spectrum and phase structure in $N = 2$ supersymmetric Yang-Mills theory.

Almost all two-dimensional relativistic theories can be understood as conformal field theories (CFTs) - describing a fixed point of the theory - perturbed by a relevant operator. If the perturbation preserves integrability, then the analysis remains exact beyond this point. In such cases, the non-perturbative analysis essentially simplifies. Besides its Lagrangian formulation, the QFT also possess an unambiguous definition in terms of factorized scattering theory. These data permit one to use non-perturbative methods for the analysis of integrable QFTs and make it possible, in some cases, to justify the existence of two (dual) Lagrangian representations of the theory.

Among two-dimensional dual models, one finds the well-known sine-Gordon/massive Thirring QFTs [3]. Another interesting example of duality is the weak-strong coupling duality flow from the (affine) Toda theories (A)TFTs based on the (affine) Lie algebra $G$ to the theory based on the dual (affine) Lie algebra $\tilde{G}$. More generally, integrable deformations of ATFTs also provide series of dual models [4, 5, 6] possessing many applications. For instance, the scattering data (S-matrix) for the ones based on Lie superalgebras with massive excitations only have been considered in detail [5, 6]. These models correspond to ATFTs (for the purely bosonic part) coupled with one or two Majorana fermions. This was done either for models based on $A^{(2)}(0, 2r-1)$ and $C^{(2)}(r+1)$ (whose bosonic root subsystem is simply laced) but also for $B^{(1)}(0, r)$ and $A^{(4)}(0, 2r)$ (non-simply laced subsystem corresponding respectively to $BC_r$ and $C_{r}^{(1)}$ roots).

In particular, these cases can be obtained as reductions of more general deformed ATFTs [7]. Furthermore, they can be seen as integrable perturbations of CFTs with extended symmetry based on $WB(0, r)$ (or also called fermionic $WB_r$) algebras.

Let us now consider some of the simplest cases above, i.e. corresponding to a rank $r = 1$. Then the models $C^{(2)}(2)$, $B^{(1)}(0, 1)$ and $A^{(4)}(0, 2)$ can be understood as integrable perturbations of $WB(0, 1)$ CFT, i.e. the $N = 1$ superLiouville (SL) field theory. Integrability can be shown either by using the so-called singular vector analysis [8] or by constructing explicitly local conserved quantities (see for instance [9] for the two latter cases). In the first model $C^{(2)}(2)$ the perturbation preserves supersymmetry and it corresponds to the well-known supersinh-Gordon model. In the two other cases, which will be the subjects of the present analysis, supersymmetry is broken.

If the theory contains only massive particles (like those presented above), the S-matrix data exhibit information about the long distance (IR) property of the theory. On the other hand, the knowledge of the CFT data and the relevant operator associated with the perturbation also define completely the theory. Once identified, the CFT contains all information about the UV behavior of the theory. It is consequently important to connect these two kinds of data in order to understand better the structure of integrable QFT. When the basic CFT admits a representation of the primary operators in terms of vertex operators the CFT data contain the so-called “reflection amplitudes”, which relate different vertex operators possessing the same quantum numbers. Considering the system on a circle of size $R$, it has been shown [10] that reflection amplitudes play a crucial role in the determination of the effective central charge $c_{eff}(R)$ in the UV domain ($R \rightarrow 0$) of the QFT. On the other hand, the effective central charge $c_{eff}(R)$ can be calculated independently from the S-matrix data using the TBA method. At small $R$, asymptotic can be compared with the one following from CFT data, if one knows explicitly the exact relation between the mass of the particles (IR data) and the (UV) parameter in the Lagrangian. The agreement of both approaches can be considered as a non trivial test for the S-matrix elements. On the other hand, reflection amplitudes are a key tool [11] in the calculations of exact vacuum expectation values (VEVs) in integrable QFTs. It has also been
shown that perturbative analysis and semiclassical methods support these conjectures. Both types of non-perturbative analysis have been performed in many cases \[10, 11, 12, 13, 14, 15, 16\]. Beside the confirmation of the $S$-matrix and calculation of exact quantities, they also provide a useful tool to check duality.

Perturbed CFTs have also attracted much attention since they can describe many physical systems in the vicinity of a critical point. Several models connected with either ATFTs\(^3\) or integrable deformations of ATFTs have received particular attention in the recent years, for a large class of strongly interacting solid state physics problems.

In a recent paper \[17\] we considered integrable coupled identical minimal models for which the interaction preserves integrability. These kind of models were introduced in \[18, 19\] for which the scattering properties were considered. In \[17\], exact results like VEVs and mass-parameter relations are obtained using exact results for $C^{(1)}_2$ ATFTs \[13\].

In this paper, we consider in detail the ATFTs based on super Lie algebras $B^{(1)}(0, 1)$ and $A^{(4)}(0, 2)$ which correspond to a critical Ising model coupled with a Liouville field theory. Using non-perturbative analysis based on reflection amplitudes, in the UV limit of both models the effective central charges are calculated in the Neveu-Schwarz (NS) sector (subsect. 2.1). Exact VEVs of primary fields which belong to the Liouville field theory and their first descendants are proposed (sect. 3). Observables of both models are shown to interchange with the flow of the coupling $b$. In particular they coincide at the self-dual point $b^2 = 1$. Duality relations between parameters in both Lagrangians are proposed as well as exact relations with the mass of the particles.

In section 4, taking analytic continuation of the coupling $b \to i\beta$ and fixing its value, several kind of coupled minimal models can be obtained. Namely, a critical Ising model and minimal model $\mathcal{M}_{p/p'}$ interacting through $\epsilon(x)\Phi_{12}(x)$ or $\epsilon(x)\Phi_{21}(x)$ where $\epsilon(x)$ denotes the energy operator of the critical Ising model and $\Phi_{12}(x)$ and $\Phi_{21}(x)$ are primary operators of the second model. For instance, together with the results of \[17\] this completes the analysis for Ising-Ising coupled models, as energy-spin interactions are considered here. Among other models, exact results are obtained for the Ising-3-state Potts models coupled by energy-energy which appear in the phase diagram of $Z_6$ spin models \[20\]. Furthermore, we will relate the critical and tricritical Ising models coupled through this type of interaction to a special case of the homogeneous sine-Gordon model (HSG) $SU(3)_2$ \[21\] (see \[24\] and references therein). This model which can be understood as an integrable perturbation of the WZNW-coset CFT $SU(3)_2/(U(1))^2$ have recently attracted attention \[21, 22, 23, 24\]. It possesses applications in one-dimensional quantum spin systems: such perturbations appear directly in the CFT study of a three-leg spin ladder which consists of three spin-1/2 Heisenberg chains weakly coupled by on-rung interaction \[23\].

However, there are also other coupled models which, although studied numerically, have not yet been studied analytically in great detail. For instance, the coupled Ising-XY model has attracted great attention as it is expected to describe the critical behavior of a large class of two dimensional classical XY models having the particularity to exhibit both discrete $\mathbb{Z}_2$ and continuous $U(1)$ degeneracy in their ground state. Among these models, one finds the fully frustrated XY model (FFXY) \[20, 27\], $J_1-J_2$ XY model \[28\], triangular lattice frustrated XY model \[23\], helical XY model \[30\], the Coulomb gas system of half-integer charges \[31\],... In particular, the FFXY model can be physically realized as a Josephson-junction array of large capacitance in a perpendicular magnetic field corresponding to a half-flux quantum per plaquette \[32\]. For imaginary values of the coupling, both models considered here possess such $\mathbb{Z}_2 \times U(1)$

\(^3\)For instance, the analytic continuation of $A_{n-1}$ ATFT provides exact results for leading thermal perturbation of $Z_n$-parafermionic CFTs \[13\]. Also, integrable perturbations of minimal models have been considered \[11\].

\(^2\)The model associated with $C^{(2)}(2)$, which corresponds to the $N = 1$ supersymmetric sine-Gordon model, has been suggested as a candidate for the coupled Ising-XY model by Foda \[34\]. Exact results for this case follow
symmetry, so that we apply previous results in section 5.

The models $B^{(1)}(0,1)$ and $A^{(4)}(0,2)$ describe integrable perturbations of $WB(0,1)$ superconformal minimal models, i.e. superLiouville theory. At specific values of the coupling, one can then obtain VEVs and exact mass-UV parameter relation in $N = 1$ superconformal minimal models perturbed by Neveu-Schwarz operators $\Phi_{13}^{(NS)}$, $\Phi_{31}^{(NS)}$ or $\Phi_{15}^{(NS)}$. For simplicity, only the first case is considered in detail in section 6. Some checks are done which support the results.

Finally, some remarks about the Ramond (R) sector of both models follow in the last section.

2 Liouville field theory coupled to $\mathbb{Z}_n$-parafermions

Instead of studying coupled Ising-Liouville models we rather prefer to consider two different kinds of interactions between a $\mathbb{Z}_n$-parafermionic [35] and a Liouville field theory. Although probably not integrable for generic values of $n \neq 1,2$, it provides a useful way to check the consistency of the expressions for $n = 1,2$. Consequently, we consider two QFTs which admit Lagrangian representations described in terms of a parafermionic field theory interacting with a Liouville-like term

\[
A^{(n)}_{t=1} = A^{(n)}_0 + \int d^2x \left[ \frac{1}{8\pi} (\partial_\nu \phi)^2 - \kappa \bar{\psi} \psi e^{-b \phi} + \mu e^{b \phi} \right], \\
A^{(n)}_{t=2} = A^{(n)}_0 + \int d^2x \left[ \frac{1}{8\pi} (\partial_\nu \phi)^2 - \tilde{\kappa} \bar{\psi} \psi e^{-b \phi} + \tilde{\mu} e^{2b \phi} \right].
\]

Here $A^{(n)}_0$ denotes the $\mathbb{Z}_n$ parafermionic CFT with central charge $c = 2 - \frac{6}{n+2}$ and the fields $\psi(\bar{\psi})$ are the holomorphic (antiholomorphic) parafermionic currents with spin $s = 1 - \frac{1}{n}$ ($\bar{s} = -s$). For $n = 1$, we have $\psi(\bar{\psi}) = \mathbb{I}$ in [1], [2]: the Lagrangians above coincide with the (integrable) well-known Sinh-Gordon (ShG) and Bullough-Dodd (BD) model, respectively. For $n = 2$, the parafermionic current $\psi$ is a Majorana fermion and to cancel fermion divergencies we have to add a counterterm $\tilde{\mu} e^{-2b \psi}$ in both cases [1], [2]. The resulting Toda-like part in the QFTs [1], [2] then becomes respectively a BD or a ShG model. It is known that these QFTs are integrable for $n = 2$ for which the factorized scattering theory has been studied in detail in [3].

Like for most of two dimensional field theories, conformal perturbation theory (CPT) can be used to study these models. Indeed, QFT with action [1] (or [2] with the substitution $\kappa \rightarrow \tilde{\kappa}$) can be seen as two different deformations of the same following CFT

\[
A^{(n)}_{CFT} = A^{(n)}_0 + \int d^2x \left[ \frac{1}{8\pi} (\partial_\nu \phi)^2 - \kappa \bar{\psi} \psi e^{-b \phi} \right]
\]

which, for $n = 1,2$, coincides with the Liouville and $N = 1$ supersymmetric Liouville models (adding the counterterm), respectively. The stress-energy tensor

\[
T^{(n)}(z) = T^{(n)}_P(z) - \frac{1}{2} (\partial \phi)^2 - Q' \partial^2 \phi \quad \text{with} \quad Q' = \frac{b}{2} + \frac{1}{nb}
\]

and $T^{(n)}_P$ associated with the parafermionic CFT, ensures the local conformal invariance of [3]. Here, we denote $\partial = \frac{1}{2} (\partial_x - i \partial_y)$ and the fields are normalized such that

\[
\langle \phi(x) \phi(y) \rangle_{\text{Gaussian}} = -\ln |x-y|^2.
\]

from [16] so we will not discuss this model here.
For instance, we have the conformal dimensions
\[ \Delta(\psi\bar{\psi} e^{a\varphi}) = 1 - \frac{1}{n} - \frac{a(2Q' + a)}{2} . \] (6)

Besides the conformal symmetry, for general values of \( n \) the \( \mathbb{Z}_n \) parafermionic CFT possesses an additional symmetry generated by the parafermionic current \( \psi(\bar{\psi}) \) \[35\]. The basic fields in this CFT are the order parameters \( \sigma_j \) with \( j = 0, 1, \ldots, n-1 \) with conformal dimensions \( \delta_j = \frac{j(n-j)}{2n(n+2)} \). The operator algebra also contains \( \mathbb{Z}_n \) neutral fields \( \epsilon_j \) with conformal dimensions \( D_j = \frac{j(j+1)}{n+2} \). All fields in this CFT can be obtained from the field \( \sigma_j \) by application of the generators of the parafermionic symmetry \[35\]. It is thus natural to introduce as the basic operators the local fields \( \sigma_j e^{\alpha \varphi} \). However, here we will essentially focus on the Liouville part, i.e. we will take \( j = 0 \).

The exponential fields \( V_a = e^{\alpha \varphi} \) are spinless conformal primary fields of the CFT \[3\], with conformal dimensions \( -\frac{a(2Q' + a)}{2} \). It can be shown \[16\] that the fields \( V_a \) and \( V_{-2Q'-a} \) are reflection images of each other, i.e. they are related by the linear transformation
\[ e^{\alpha \varphi} = R_b^{(n)}(a) e^{-(2Q'+a)\varphi} . \] (7)

For \( n = 1 \), the reflection amplitude \( R_b^{(n)}(a) \) reduces to the so-called “Liouville reflection amplitude” proposed in \[10\]. For \( n = 2 \), it corresponds to the \( N = 1 \) superLiouville reflection amplitude in the NS sector proposed in \[36\]. For general values \( n \), the (NS for \( n = 2 \)) reflection amplitude \( R_b^{(n)}(a) \) associated with action \[3\] writes \[16\]
\[ R_b^{(n)}(a) = -\left[ \frac{\pi \kappa \gamma(Q'b)}{n(\frac{2b}{n})^{1/n}} \right]^{\frac{2(a+Q')}{b}} \frac{\Gamma(1 - (a + Q')b)\Gamma(1 - 2(a + Q')/nb)}{\Gamma(1 + (a + Q')b)\Gamma(1 + 2(a + Q')/nb)} . \] (8)

Here we denote \( \gamma(x) = \Gamma(x)/\Gamma(1-x) \) as usual.

### 2.1 Scaling functions for generic values of \( n \)

Consider now the CFT \[3\] on an infinite cylinder of circumference \( 2\pi \) with the cartesian coordinates \( x_1, x_2 \) where \( x_2 \) along the cylinder is defined as the imaginary time and \( x_1 \sim x_1 + 2\pi \) is the space coordinate. For simplicity, we will here focus on the sector in which only bosonic zero-modes appear. In particular, for \( n = 2 \) it corresponds to the NS sector. Then, the reflection amplitude \[3\] provides non-perturbative information in the study of quantum mechanical problem for bosonic zero modes

\[ \varphi_0 = \int_0^{2\pi} \varphi(x) \frac{dx_1}{2\pi} \] (9)

of the fields \( \varphi(x) \). In the semi-classical limit \( b \to 0 \), where one can neglect the oscillator modes of \( \varphi(x) \), the Schrödinger equation governing the zero-mode dynamics writes\[\footnote{For \( n = 2 \), one has for instance \[13\] \( \mathcal{H}_0^{(2)} = -\frac{1}{n} - (\frac{\partial}{\partial x_0})^2 + 2\pi \hat{\mu} e^{-2\pi \varphi_0} \)}
\[ \mathcal{H}_0^{(n)} \Psi_P(\varphi_0) = E_0^P \Psi_P(\varphi_0) \quad \text{with} \quad E_0^P = -\frac{n+1}{24} + P^2 \] (10)
the ground state energy where the momentum $P$ is a real vector. The full quantum effect can be implemented simply by introducing the exact reflection amplitudes which take into account also non zero-mode contributions [10].

The wave function in the asymptotic region can be obtained using the same arguments as the ones already applied for other models. Namely, the exponential term in the Hamiltonian is considered as a potential wall. An incident plane wave with momentum $P$ is then reflected to the plane wave with $-P$. The phase change corresponding to this process is associated with the reflection amplitude given above. Consequently, the wave function $\Psi_P(\varphi_0)$ is simply written as a superposition of two plane waves:

$$\Psi_P(\varphi_0) \simeq e^{iP\varphi_0} + S_b^{(n)}(P)e^{-iP\varphi_0} \quad \text{at} \quad \varphi_0 \to +\infty .$$  (11)

with $S_b^{(n)}(P) = R_b^{(n)}(-iP - Q')$. Using the approach proposed in [10], we can now obtain the scaling functions in the UV region of the QFTs (1) or (2) defined on a cylinder with circumference $R \to 0$.

Let us first consider the QFT (1). The additional term $\mu e^{b\varphi}$ in its action compared to the CFT introduces a new potential wall, which implies the quantization of the momentum $P$ in the wave function. It depends on the size of the enclosed region, which is proportional to $\ln(1/R)$. As we will see later, this quantized momentum $P(R)$ defines the scaling function $c_{\text{eff}}(R)$ in the UV region using eq. (10).

In action (1) the dimensions of the parameters are $\text{Dim}[\kappa] = \frac{2}{n} + b^2$ and $\text{Dim}[\mu] = 2 + b^2$. It is now convenient to rescale back the size of the system from $R$ to $2\pi$. The action (1) becomes

$$A^{(n)}_{\tau=1} = A^{(n)}_0 + \int d^2x \left[ \frac{1}{8\pi} (\partial_\nu \varphi)^2 - \kappa \left( \frac{R}{2\pi} \right)^{\frac{2+b^2}{2}} \overline{\psi} \psi e^{-b\varphi} + \mu \left( \frac{R}{2\pi} \right)^{2+b^2} e^{b\varphi} \right] .$$

Following previous analysis (see [10, 12] for instance) and due to the form of the perturbing term in (1) we obtain the following quantization condition for the momentum $P$:

$$\left( \frac{R}{2\pi} \right)^{-2iP(b+1/b)H_n} S_b^{(1)}(P)|_{\kappa \to -\mu} S_b^{(n)}(P) = 1 ,$$  (12)

where $S_b^{(1)}(P)$ is nothing but the so-called “Liouville reflection amplitude” proposed in [10]. For further convenience, here we introduced the “deformed” Coxeter number [37]:

$$H_n = \frac{2(n+1)}{n}(1 - B) + 2B \quad \text{with} \quad B = \frac{b^2}{1 + b^2} .$$  (13)

For the lowest energy state, in terms of the reflection phases, eq. (12) reduces to

$$LP = 2\pi - \delta_b^{(1)}(P) - \delta_b^{(n)}(P)$$  (14)

with

$$L = -\frac{2}{b} H_n(1 + b^2) \ln \left( \frac{R}{2\pi} \right) - \frac{1}{b} \ln \left[ \frac{\pi \kappa \gamma(1/n + b^2/2)}{n} \left( \frac{n b^2}{4} \right)^{1/n} \right]^2$$

and we used the convenient notation

$$\delta_b^{(n)}(P) = -i \ln \frac{\Gamma(1+Pb)\Gamma(1+i2P/nb)}{\Gamma(1-iPb)\Gamma(1-i2P/nb)} .$$
In the UV region $R \to 0$ we can solve eq. (14) perturbatively by expanding the reflection phases in powers of $P$. We obtain

$$\ell P = 2\pi - (\delta_{b,3}^{(1)} + \delta_{b,3}^{(n)}P^3 - (\delta_{b,5}^{(1)} + \delta_{b,5}^{(n)}P^5 + \ldots)$$

(15)

where we define

$$\delta_{b,1}^{(n)} = -2\gamma_E (b + \frac{2}{nb}), \quad \delta_{b,s}^{(n)} = (-)^{s-1} \frac{2}{s} \zeta(s)(b^s + (\frac{2}{nb})^s)$$

and introducing the Euler constant $\gamma_E$.

$$\ell \equiv L - L_0 = L - 2\gamma_E \frac{H_n(1 + b^2)}{b}.$$  

(16)

The ground state energy of the system on the circle of size $R$ is given by $E(R) = -\pi c_{eff}(R)/6R$ with the effective central charge $c_{eff}(R) = (n + 1)/2 - 12P^2$. Here, $P$ is the solution of the above quantization condition (14), and perturbatively (15) in powers of $1/\ell$. After some calculations, we find that the UV behavior of the QFT (1) - in the NS sector for $n = 2$ - is characterized by the scaling function

$$c_{eff}^{(n)}(R) = \frac{n + 1}{2} - 12 \left[ \frac{(2\pi/\ell)^2 - 2\zeta(3)(2b^3 + 8(n^3 + 1)/n^3b^3)}{3}\pi(\frac{2\pi}{\ell})^5 + \frac{2\zeta(5)(2b^5 + 32(n^5 + 1)/n^5b^5)}{5}\pi(\frac{2\pi}{\ell})^7 + O((\frac{2\pi}{\ell})^8) \right].$$ 

(17)

In particular, for $n = 1$ this result agrees perfectly with the sinh-Gordon one (1).

The same analysis can be performed along the same line for a different kind of perturbation, i.e. for instance in QFT (2). In this case, the dimension of the parameters in the action are $\text{Dim}[\tilde{\kappa}] = \frac{2}{n} + b^2$ and $\text{Dim}[\mu] = 2 + 4b^2$. The quantization condition is now

$$\left(\frac{R}{2\pi}\right)_{2b}^{2P(b+1/b)\tilde{H}_n S^{(1)}_{2b}(P)}|_{\kappa \to -\mu} S^{(n)}_b(P) = 1,$$

(18)

where we define the “deformed” Coxeter number $\tilde{H}_n = \frac{n+2}{n}(1 - B) + 3B$. As before, eq. (18) can be written in terms of the reflection phases as $\tilde{L}P = 2\pi - \delta_{2b}^{(1)}(P) - \delta_{2b}^{(n)}(P)$ with

$$\tilde{L} = -\frac{2}{b} \tilde{H}_n(1 + b^2) \ln \left(\frac{R}{2\pi}\right) - \frac{1}{b} \ln \left[ \frac{\pi \mu \gamma(2b^2)}{n} \frac{\gamma(1/n + b^2/2)}{\left(\frac{n^2b^4}{4}\right)^{1/n}} \right].$$

(19)

Consequently, it is straightforward to show that the scaling function for the UV behavior of the QFT (2) - in the NS sector for $n = 2$ - is given by the following expansion

$$\tilde{c}_{eff}^{(n)}(R) = \frac{n + 1}{2} - 12 \left[ \frac{(2\pi/\ell)^2 - 2\zeta(3)(9b^3 + (8 + n^3)/n^3b^3)}{3}\pi(\frac{2\pi}{\ell})^5 + \frac{2\zeta(5)(33b^5 + (32 + n^5)/n^5b^5)}{5}\pi(\frac{2\pi}{\ell})^7 + O((\frac{2\pi}{\ell})^8) \right]$$

(20)

with $\tilde{\ell}$ similar to $\ell$ in (18) but with the replacement $H_n \to \tilde{H}_n$. It can be checked that for $n = 1$, this result coincides exactly with the Bullough-Dodd (12) scaling function, as expected.

Notice that for the specific value of the coupling constant $b^2 = 1$, both scaling functions take the same form for any values of $n$. 

7
2.2 Thermodynamic Bethe ansatz for \( n = 2 \) and duality

For specific values of \( n \), the effective central charge calculated above from the CFT data (reflection amplitudes) can be compared with the same function determined from the numerical solution of the TBA equations for the QFT (1) and (2). For \( n = 1 \), this analysis has been done in [10] for the ShG and in [12] for the BD cases. Then, let us consider the case \( n = 2 \) in the QFTs (1) and (2). As was conjectured in [5] (even for higher rank \( r > 1 \) of the affine Toda part) these QFTs possess a weak-strong coupling duality: the analysis of the factorized scattering theory shows that there exists a QFT which possesses two (dual) perturbative regimes associated with action (1) and (2), respectively. An intermediate mass spectrum (consisting of two particles for the rank \( r = 1 \)) was proposed:

\[
M_{\psi} = \overline{m} \quad \text{and} \quad M_1 = 2\overline{m} \sin(\pi / H_2)
\]  

(21)

This mass spectrum flows from the one of the QFT (1) to the one associated with the QFT (2) while \( b \) increases.

Using the corresponding (diagonal) FST proposed in [5] - we report the reader to this work for details - one writes the TBA equations. Namely,

\[
c_{\text{eff}}^{(\text{TBA})}(R) = \frac{3R}{\pi^2} \int \cosh \theta \left[ M_{\psi} \ln(1 + e^{-\epsilon_{\psi}(\theta,R)}) + M_1 \ln(1 + e^{-\epsilon_1(\theta,R)}) \right] d\theta
\]

(22)

where the functions \( \epsilon_{\psi}(\theta,R), \epsilon_1(\theta,R) \) satisfy the system of 2 coupled integral equations

\[
M_i R \cosh \theta = \epsilon_i(\theta,R) + \sum_{j \in \{1,\psi\}} \int \varphi_{ij}(\theta - \theta') \ln(1 + e^{-\epsilon_j(\theta',R)}) \frac{d\theta'}{2\pi} \quad \text{for} \quad i \in \{1,\psi\}
\]

with the kernels \( \varphi_{ij} \) defined as the logarithmic derivatives of the \( S \)-matrix elements obtained in [5]. However, the function \( E^{(\text{TBA})}(R) \) defined from the TBA equations differs from the ground state energy \( E(R) \) of the system on the circle of size \( R \) by the bulk term \( E^{(\text{TBA})}(R) = E(R) - f^{(2)}_\tau R \), where \( f^{(2)}_\tau \) is a specific bulk free energy of the QFT. To compare the same functions we should then substract this term from the function \( E(R) \) defined by \( c_{\text{eff}}^{(2)}(R) \), i.e. for instance in case (1) we have

\[
c_{\text{eff}}^{(\text{TBA})}(R) = c_{\text{eff}}^{(2)}(R) + \frac{6R^2}{\pi} f^{(2)}_{\tau=1}
\]

(23)

and similarly for case (2). Notice that the contribution of bulk term \( f^{(2)}_\tau \) becomes quite essential at \( R \sim O(1) \). In many examples of known QFTs, this quantity can be calculated explicitly using Bethe Ansatz (BA) method (see for instance [38, 39]). Here, we propose the following expression for the specific bulk free energy in the QFT (1) (the same quantity for the QFT (2) follows using the duality \( B \rightarrow 1 - B \) and \( H_2 \rightarrow \tilde{H}_2 \)):

\[
f^{(2)}_{\tau=1} = \frac{\overline{m}^2}{8} \frac{\sin(\pi / H_2)}{\sin(\pi B / H_2) \sin(\pi (1 - B) / H_2)}.
\]

(24)

Finally, to compare the expansion coming from the CFT data and the one from the numerical analysis of TBA equations, we need the exact relations between the UV parameters in the actions (1) and (2) and the IR mass scale for the particles \( \overline{m} \). Here, we propose

\[
[- \pi \mu \gamma(1 + b^2/2)]^2 \left[ \frac{\pi \kappa}{2} \gamma \left( \frac{1 + b^2}{2} \right) \right]^2 = \left[ \frac{\overline{m} \Gamma(1 + \frac{B}{R}) \Gamma(1 - B)}{2^{1+2/H_2} \Gamma(1/H_2)} \right]^{2H_2(1 + b^2)}
\]

(25)
for the QFT (1) and

\[ [ - \pi \tilde{\mu} 2^{-4b^2-2} \gamma(1 + 2b^2)] \left[ \frac{\pi \tilde{\kappa}}{2} \gamma \left( \frac{1 + b^2}{2} \right) \right]^2 = \left[ \frac{m \Gamma(1 + \frac{B}{\tilde{H}}) \Gamma \left( \frac{1-B}{\tilde{H}} \right)}{2^{1+2/\tilde{H}} \Gamma(1/\tilde{H})} \right]^{2\tilde{H}^2(1+b^2)} \]

(26)

for the QFT (4). One should mention that under the weak-strong coupling duality transformation \( b \leftrightarrow 1/b \), these relations exchange perfectly if the parameters in action (1) and (2) satisfy the duality relations

\[ \pi \mu \gamma \left( \frac{b^2}{2} \right) = \left[ \pi \tilde{\mu} \gamma \left( \frac{2}{b^2} \right) \right]^{b^2/2} \]

\[ \left[ \frac{\pi \kappa b^2}{2} \gamma \left( \frac{1 + b^2}{2b^2} \right) \right]^b = \left[ \frac{\pi \tilde{\kappa}}{2b^2} \gamma \left( \frac{1 + b^2}{2} \right) \right]^{1/b} \]

(27)

Using the first relation (23) in (14) for \( n = 2 \), it is now possible to compare the expansion (17) for \( n = 2 \) to (22) obtained numerically using (21), (23) and (24). The good agreement supports the approach based on the reflection amplitudes, the exact \( \mu - \kappa - \mu \) (and their dual) relations given above, the bulk free energies (24) of (1) (and its dual for (4)) as well as the \( S \)-matrix elements conjectured in [5]. Due to the following relations

\[ c_{eff}^{(2)}(R)|_b = c_{eff}^{(2)}(R)|_{1/b} \quad \text{as} \quad \ell|_b = \tilde{\ell}|_{1/b} \]

(28)

the weak-strong coupling duality property between the models with action (1) and (4) for \( n = 2 \) proposed in [5] is indeed confirmed at the on-shell level.

3 Vacuum expectation values of local fields

For generic values of the parameter \( n \neq 1, 2 \), there is no reason to expect the QFTs (1) and (2) to be integrable. However, for \( n = 1 \) the ShG and BD integrable models are recovered. For \( n = 2 \) the parafermionic current \( \psi \) is a Majorana fermion. As the scaling limit of the Ising model with zero external magnetic field is described by a free Majorana fermion field theory, both models describe different interactions between a Liouville field theory and a critical Ising model (6). In this case, it can be shown that both models are also integrable [8], and that respective conserved quantities exchange under weak-strong coupling duality [9].

In the previous subsection we considered the UV asymptotic of the effective central charges in these QFTs. The calculations were based on the reflection amplitudes, i.e. using CFT data. However, these functions play a crucial role in the calculation of vacuum expectation values in perturbed CFT [11]. Here, at the off-shell level, we will see that VEVs also satisfy such duality property.

3.1 Expectation values of primary fields and the Ising-Liouville dual models

Here, we consider the expectation values of the simplest local fields which belong to the Liouville field theory, i.e.

\[ G^{(n)}(\tau_a) = \langle e^{a\phi}(x) \rangle_{\tau,n} \quad \text{with} \quad \tau = 1, 2 \]

(29)

\(^5\)I am very grateful to P. Dorey and R. Tateo for these numerical checks.

\(^6\)Notice that the form of the interaction remains the same but the definition of the Liouville coupling constant changes.
From the discussion in the previous subsection and using once again CPT framework with (4), the last term in action (1) or (2) can be considered as a perturbation. It is then expected that for any $\tau$ and $n$ the VEV (24) satisfy the reflection relation

$$G^{(n)}_{\tau}(-a) = R_{b}^{(n)}(-a) G^{(n)}_{\tau}(-2Q' + a) .$$

Let us first study the QFT (1). Instead of the previous picture, it is also possible to consider this QFT as a perturbed Liouville field theory with parameter and coupling $(\mu, b)$. The stress-energy tensor for the conformal invariant part is then

$$T_{L}(z) = -\frac{1}{2} (\partial \varphi)^{2} + Q \partial^{2} \varphi \quad \text{with} \quad Q = \frac{b}{2} + \frac{1}{b} .$$

Using the CPT framework, the VEV is then expected to satisfy the reflection relation

$$G^{(n)}_{\tau=1}(a) = R_{b}^{(1)}(-a)|_{\kappa \to -\mu} G^{(n)}_{\tau=1}(2Q - a)$$

Obviously, it is not possible to find a solution to these reflection equations without any strong analyticity assumptions. Assuming that the VEV $G^{(n)}_{\tau=1}(a)$ is a meromorphic function in $a$, the “minimal” solution of the reflection relations (30) and (32) can be explicitly obtained with the result

$$G^{(n)}_{\tau=1}(a) = \left[ - \pi \mu \gamma(1 + b^{2}/2)^{2} \frac{\pi \kappa}{n} \gamma(1/n + b^{2}/2)^{2} \right] \frac{a^{2} + 2Qa}{\mu \kappa a^{2} + 2Qa} \left[ - \pi \mu \gamma(1 + b^{2}/2)^{2} \right]^{\frac{n}{b}} \times \exp \int_{0}^{\infty} \frac{dt}{t} \left[ a^{2} e^{-2t} - \frac{\sinh(abt) \Psi_{n}(t, b, a)}{\sinh(t) \sinh(tb^{2}) \sinh(ntb^{2}/2) \sinh(H_{n}(1 + b^{2})t)} \right]$$

where we define for $-(b^{2}/2 + \frac{1}{n}) < \text{Re}(ab) < b^{2}/2 + 1$

$$\Psi_{n}(t, b, a) = 2 \sinh((ab + 2Q'bt) \sinh(Qbt) \sinh(ntb^{2}/2) \cosh(tb^{2}/2) + \sinh((ab - 2Qbt) \sinh(nQbt) \sinh(tb^{2}) .$$

For several other QFTs (Toda, deformed Toda, ...) for which exact VEVs were proposed explicitly, it has been shown that the expectation values of the fundamental field $\langle \varphi \rangle$ for $b \to 0$ agrees with the same quantity calculated within perturbation theory. It has also been shown that semiclassical analysis supports the conjectures for the VEVs. Consequently, here we expect the same feature to be satisfied.

We can proceed similarly for the QFT (2). This QFT can be understood as a perturbed Liouville field theory with parameter and coupling $(\tilde{\mu}, 2b)$. One has to consider now the stress-energy tensor with a form similar to (31) but with the substitution

$$Q \quad \longrightarrow \quad \tilde{Q} = b + \frac{1}{2b} .$$

The reflection relation satisfied by the VEV immediately follows

$$G^{(n)}_{\tau=2}(a) = R_{2b}^{(1)}(-a)|_{\kappa \to \tilde{\mu}} G^{(n)}_{\tau=2}(2\tilde{Q} - a) .$$

As before, using the reflection relations (30) and (33) simultaneously and assuming similarly strong analytical assumptions we obtain the following conjecture

$$G^{(n)}_{\tau=2}(a) = \left[ - \pi 2^{-4b^{2} - \tilde{\mu} \gamma(1 + b^{2})^{2}} \frac{\pi \kappa}{n} \gamma(1/n + b^{2}/2)^{2} \right] \frac{a^{2} + 2Qa}{\mu \kappa a^{2} + 2Qa} \left[ - \pi 2^{-4b^{2} - \tilde{\mu} \gamma(1 + b^{2})^{2}} \right]^{\frac{n}{b}} \times \exp \int_{0}^{\infty} \frac{dt}{t} \left[ a^{2} e^{-2t} - \frac{\sinh(abt) \Psi_{n}(t, b, a)}{\sinh(t) \sinh(tb^{2}) \sinh(ntb^{2}/2) \sinh(H_{n}(1 + b^{2})t)} \right]$$
with
\[ \tilde{\Psi}_n(t, b, a) = 2 \sinh((ab + 2Q'b)t) \sinh(\tilde{Q}bt) \sinh(ntb^2/2) \cosh(t/2) \\
+ \sinh((ab - 2\tilde{Q}b)t) \sinh(nQ'bt) \sinh(tb^2) . \tag{36} \]

Here, the integral is convergent if \(-b^2 + \frac{1}{n} < Re(ab) < b^2 + \frac{1}{n}\).

For \(n = 1\), it is straightforward to check that (33) and (36) agree respectively with BD and ShG results. For \(n = 2\), using the duality transformations of the parameters (27) it is also easy to check that
\[ G^{(2)}_{\tau=1}(a)|_b = G^{(2)}_{\tau=2}(a)|_{1/b} . \tag{37} \]

Finally, using the VEVs (33) (or similarly (36)) for \(n = 2\), the bulk free energy of (1) can be calculated as we have the relation
\[ \mu G^{(2)}_{\tau=1}(b) = \frac{2(1 - B)}{H_2} f^{(2)}_{\tau=1} . \tag{38} \]

Using (23), it is easy to show that the result for \(f^{(2)}_{\tau=1}\) which follows is in perfect agreement with the expression proposed in (24) (and similarly for the QFT (2)).

### 3.2 Expectation values of the first descendant fields in the Ising-Liouville dual models

By adding the correct counterterms, for \(n = 2\) both models (1) and (2) can now be understood as two different perturbations of the \(N = 1\) superLiouville field theory. For instance, for \(\tau = 1\) it is given by the action
\[ A_{SL} = \int d^2x \left[ \frac{1}{2\pi} (\bar{\psi} \partial \psi + \psi \partial \bar{\psi}) + \frac{1}{8\pi} (\partial \varphi)^2 - \kappa \psi \psi e^{-b\varphi} + \bar{\mu} e^{-2b\varphi} \right] \tag{39} \]

and similarly with the change \(\kappa \rightarrow \tilde{\kappa}\) for \(\tau = 2\). Superconformal transformations in the SL theory are generated by the super stress-energy tensor \(\bar{T}(z) = -S(z)/2 + \theta \bar{T}(z)\) and similarly for the antiholomorphic part with \((\theta, \bar{\theta})\) the corresponding Grassmann coordinates. Here \(T(z)\) denotes the usual (bosonic) holomorphic stress-energy tensor given in eq. (4) for \(n = 2\) and \(S(z)\) is a spin 3/2 conserved current:
\[ S(z) = \psi \partial \varphi + 2Q' \partial \psi \tag{40} \]
and similarly for the antiholomorphic part. With the background charge given in eq. (4) for \(n = 2\), the central charge of the superLiouville model is then \(c_{SL} = \frac{3}{2}(1 + 8Q'^2)\).

The basics fields in the SL theory are the operators \(\sigma_j \exp(a \varphi)\) which belong either to the Neveu-Schwarz sector (NS) (for \(j = 0\)) or the Ramond sector (R) (for \(j = 1\), also called “twisted” fields). These primary fields have conformal dimension
\[ \Delta_j(a) = \frac{j(2-j)}{16} - \frac{a(2Q' + a)}{2} \quad \text{for} \quad j = 0, 1 . \]

Then, using the Laurent expansion
\[ T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \quad \text{and} \quad S(z) = \sum_{r \in \mathbb{Z}/2} S_r z^{-r-3/2} \tag{41} \]
and similarly for the antiholomorphic part, all the other fields can be obtained via the action of the Neveu-Schwarz (Ramond) algebra generators $L_n$, $S_r$ for $n \in \mathbb{Z}$, $r \in \mathbb{Z} + 1/2$ ($n \in \mathbb{Z}$, $r \in \mathbb{Z}$). In general, we denote these descendent fields

$$L_{[n]} \bar{L}_{[m]} S_{[r]} \bar{S}_{[s]} \sigma_j e^{a \varphi} = L_{-n_1} \ldots L_{-n_N} \bar{L}_{-m_1} \ldots \bar{L}_{-m_K} S_{-r_1} \ldots S_{-r_{N'}} \bar{S}_{-s_1} \ldots \bar{S}_{-s_K} \sigma_j e^{o \varphi}$$

(42)

where $[u] = [-u_1, \ldots, -u_p]$ are arbitrary strings. The descendent fields (42) and the ones obtained after the reflection $a \rightarrow -(2Q' + a)$ possess the same quantum numbers. It is then possible to show that the reflection property extends to all these descendents and arguments based on CPT approach suggest the following reflection relation for $j = 0$:

$$\langle L_{[n]} \bar{L}_{[m]} S_{[r]} \bar{S}_{[s]} e^{a \varphi} \rangle_{\tau} = R_b^{(2)}(a) \langle L_{[n]} \bar{L}_{[m]} S_{[r]} \bar{S}_{[s]} e^{-(2Q' + a) \varphi} \rangle_{\tau}$$

and $\tau = 1, 2$,

where $R_b^{(2)}(a)$ is the SL reflection amplitude in the NS sector calculated in [36]. For simplicity, here we only consider the VEV of the simplest descendent fields. Using the relations (40) and (41) above, we have

$$S_{-1/2} \bar{S}_{-1/2} e^{a \varphi} = -a^2 \bar{\psi} \psi e^{a \varphi}.$$  

(43)

Consequently, expectation values of these operators in the perturbed theories (1) or (2) for $n = 2$ are expected to satisfy similar reflection equations. Let us consider the ratio

$$H_{\tau}(a) = \frac{\langle \bar{\psi} \psi e^{a \varphi} \rangle_{\tau}}{\langle e^{a \varphi} \rangle_{\tau}}.$$  

(44)

The model with action (1) and $n = 2$ can either be considered as a perturbed SL theory or a perturbed Liouville theory. As before, approach based on CPT then suggests the following reflection equations

$$H_{\tau=1}(a) = \frac{(2Q' + a)^2}{a^2} H_{\tau=1}(-2Q' - a) \text{ for } a \neq 0,$$

$$H_{\tau=1}(a) = H_{\tau=1}(2Q - a).$$

The “minimal” solution of these reflection equations is defined up to an overall constant. To fix it, it is sufficient to notice that $H_{\tau=1}(-b)$ can be related with the bulk free energy of the system as follow

$$H_{\tau=1}(-b) \langle e^{-b \varphi} \rangle_{\tau=1} = \frac{\partial f^{(2)}_{\tau=1}}{\partial \kappa},$$

(45)

which leads to the result

$$\langle \bar{\psi} \psi e^{a \varphi} \rangle_{\tau=1} = -\frac{m b^2}{2(3 + 2b^2)^2} \frac{\Gamma(\frac{1}{3 + 2b^2}) \Gamma(\frac{b^2}{3 + 2b^2})}{\Gamma(\frac{2 + b^2}{3 + 2b^2})} \gamma\left(\frac{b^2 + 2 - ab}{3 + 2b^2}\right) \gamma\left(\frac{ab}{3 + 2b^2}\right) G^{(2)}_{\tau=1}(a)$$

(46)

for $a \neq 0$. It is straightforward to do the same analysis for the model with action (2) for $n = 2$. It can be shown that the result for $\langle \bar{\psi} \psi e^{a \varphi} \rangle_{\tau=2}$ follows from (46) with the substitution $b \rightarrow 1/b$. In Appendix A, we give a further support to (46). In conclusion, all previous results for $n = 2$ confirm at the off-shell level the weak-strong coupling duality.
4 Application to integrable coupled minimal models and the homogeneous sine-Gordon $SU(3)_2$ model

For $n = 2$, the QFT with action (2) possesses interesting applications. As is well known, the “minimal model” $\mathcal{M}_{p/p'}$ with central charge $c = 1 - 6\frac{(p-p')^2}{pp'}$ can be obtained from the Liouville model. Consequently, adding the counterterm $\tilde{\mu}e^{-2b\phi}$ in action (2), the resulting QFT can be identified with a minimal model interacting with a critical Ising model if we substitute

$$b \rightarrow i\beta, \quad \tilde{\mu} \rightarrow -\tilde{\mu}, \quad \tilde{\kappa} \rightarrow i\tilde{\kappa},$$

and make the choice either

$$\beta^2 = \beta_+^2 = p/2p' \quad \text{or} \quad \beta^2 = \beta_-^2 = p'/2p \quad \text{with} \quad p < p'.$$

For both values of the coupling and using the background charge (34) which ensures the local conformal invariance of the Liouville field theory, the conformal dimension of the perturbing term becomes $\Delta(\psi\bar{\psi}e^{-i\beta\phi}) = 3\beta^2/2$, which is relevant for $\beta^2 < 2/3$.

In the following, we define $\{\Phi_{rs}\}$ as the set of primary fields with conformal dimensions

$$\Delta_{rs} = \frac{(p'r - ps)^2 - (p - p')^2}{4pp'} \quad \text{for} \quad 1 \leq r < p, \quad 1 \leq s < p', \quad \text{and} \quad p < p'.$$

Using the Coulomb gas representation, they are simply related with the vertex operators of the Liouville field theory through the relation

$$\Phi_{rs}(x) = N_{rs}^{-1} \exp(i\eta^r_s \varphi(x)) \quad \text{with} \quad \eta^r_s = -\frac{(1 - r)}{2\beta} + (1 - s)\beta,$$

where we have introduced the normalization factors $N_{rs}$. These numerical factors depend on the normalization of the primary fields. Here, they are chosen in such a way that the primary fields satisfy the conformal normalization condition:

$$\langle \Phi_{rs}(x)\Phi_{rs}(y) \rangle_{\text{CFT}} = \frac{1}{|x - y|^{4\Delta_{rs}}}.$$

For further convenience, we write these coefficients $N_{rs} = N(\eta^r_s)$ where:

$$N(\eta) = [\pi\tilde{\mu}\gamma(-2\beta^2)]^{-\beta_+^2} \left[\frac{\Gamma(2\beta^2 - 2\eta\beta)\Gamma(1/2\beta^2 + \eta/\beta)\Gamma(2 - 2\beta^2)\Gamma(2 - 1/2\beta^2)}{\Gamma(2 - 2\beta^2 + 2\eta\beta)\Gamma(2 - 1/2\beta^2 - \eta/\beta)\Gamma(2\beta^2)\Gamma(1/2\beta^2)}\right]^{\frac{1}{2}}. (52)$$

Taking specific values of the coupling constant in action (2), it is now possible to obtain non-perturbative information for two planar systems which interact through a relevant operator preserving integrability. The first system - described by the free Majorana fermion part - is identified with a critical Ising model denoted $\mathcal{M}_{3/4}$ using the standard conventions. The second one - obtained from the QG restriction of the Liouville field theory - is identified with a minimal model denoted $\mathcal{M}_{p/p'}$. Then, the action can be written either

$$A = \mathcal{M}_{3/4} + \mathcal{M}_{p/p'} + \lambda \int d^2x \epsilon(x)\Phi_{12}(x) \quad \text{for} \quad \beta^2 = \beta_+^2, (53)$$

or

$$\dot{A} = \mathcal{M}_{3/4} + \dot{\mathcal{M}}_{p/p'} + \dot{\lambda} \int d^2x \epsilon(x)\Phi_{21}(x) \quad \text{for} \quad \beta^2 = \beta_-^2, (54)$$
where the parameters $\lambda$ and $\hat{\lambda}$ characterize the strength of the interaction and the energy operator of the Ising model is defined by $\epsilon \equiv i\bar{\psi}\psi$. Also, both QFTs make sense if $3p < 4p'$ in (53) and $3p' < 4p$ in (54), which ensure the perturbation to be relevant.

For imaginary values of the coupling $b = i\beta$, it is expected that the QFT (2), possesses complex soliton solutions which interpolate between the degenerate vacua. It can be shown that this model possesses the QG symmetry associated with $U_q(B(0,1)^{(1)})$ where $q$ is the deformation parameter. Naturally, there are good reasons to believe that the $S$-matrix of this model can be expressed in terms of the $R$-matrix associated with this deformed affine Lie superalgebra. Then, in the following we assume that a breather-particle identification holds by comparing the resulting $S$-matrix elements of the lowest-breathers (breathers with lowest mass) with the $S$-matrix elements for the quantum particles in the real coupling case. Denoting $M$ as the mass of the lowest kink, we suggest the identification

$$\overline{m} = 2M \sin \left(\frac{\pi\xi}{2 - \xi}\right) \quad \text{with} \quad \xi = \frac{\beta^2}{1 - \beta^2} . \quad (55)$$

In the following, we will study successively the models with action (53) and (54). However, the vacuum structure in both cases is not clearly understood so the prefactor associated with the vacuum structure in both cases is not clearly understood so the prefactor associated with the QG restriction is omitted for simplicity. Consequently, we denote all expectation values $\langle \ldots \rangle \equiv \langle 0 \vert \ldots \vert 0 \rangle$ where $\vert 0 \rangle$ is one of the many ground states.

Let us first consider the coupled minimal models with action (53). With the previous identification and using the exact $\overline{m}$-$\bar{\kappa}$-$\bar{\mu}$ relations (24) proposed in the previous section, it is now straightforward to obtain the following exact $M$-$\lambda$ relations

$$\lambda^2 = \frac{1}{\pi^2} \frac{\gamma(3\xi - 1)}{\gamma^2(2(1 + \xi))} \left[ \frac{\pi M \Gamma(\frac{1 + \xi}{2 + \xi})}{\Gamma'(\frac{\xi}{2 + \xi})\Gamma(\frac{1}{2 + \xi})} \right]^{2(2 - \xi)} \quad \text{for} \quad \xi = \frac{p}{2p' - p} . \quad (56)$$

Consequently, according to (56) and $\beta^2 < 2/3$, the coupled minimal models (53) develop a massive spectrum for:

$$\begin{align*}
(i) \quad & 1/3 < \xi < 1, \quad \Im m(\lambda) = 0 \quad \text{{i.e.}} \quad \frac{1}{2} < \frac{p}{p'} < 1 , \\
(ii) \quad & 0 < \xi < 1/3, \quad \Re e(\lambda) = 0 \quad \text{{i.e.}} \quad 0 < \frac{p}{p'} < \frac{1}{2} .
\end{align*} \quad (57)$$

In particular, the condition (i) is always satisfied for the coupled unitary minimal models defined by (53) with $p' = p + 1$. Notice that for $p = 2, p' = 3$ the model (53) corresponds to an off-critical Ising model as we have $\Phi_{12} = \mathbb{I}$ in this case. If we make the identification $\overline{m} = 2M \sin(\pi/3) \equiv M_{SG}$ where $M_{SG}$ is the SG soliton mass, then the $M$-$\lambda$ relation given above coincides with the one associated with the off-critical Ising model, as expected.

Although the model (2) with imaginary coupling is very different from the real coupling case in its physical content (the model contains solitons and excited solitons), there are good reasons to believe that the expectation values obtained in the real coupling case provide also the expectation values for imaginary coupling. Similarly to the analysis done for the ShG and BD models (11), we obtain the VEV of primary operators which belong to the second minimal model using eqs. (36) for $b = i\beta_+$, (54) and (52)

$$\langle \Phi_{rs}(x) \rangle = \left[ \frac{\pi^2 \lambda^2 \gamma^2(2(1 + s))}{2\pi^2 \gamma(3\xi - 1)} \right]^{(1 + \xi)} \frac{\Gamma(\frac{1 + \xi}{2 + \xi})}{\Gamma'(\frac{\xi}{2 + \xi})\Gamma(\frac{1}{2 + \xi})} \Delta_{rs}^{(1 + \xi)} \exp \mathcal{Q}_{12}((1 + \xi)r - 2\xi s) . \quad (58)$$

7The parameter $\lambda$ is real and we obtain $M_{SG} = 2\pi|\lambda|$.
The function \( Q_{e12}(\theta) \) for \(|\theta| < 2\xi\) and \( \frac{1}{3} < \xi < 1\) is given by the integral

\[
Q_{e12}(\theta) = \int_0^\infty \frac{dt}{t} \frac{\Psi_{e12}(\theta, t)}{\sinh((1 + \xi)t) \sinh(t \xi) \sinh((2 - \xi)t)} - 2\Delta_{rs} e^{-2t}
\]

with

\[
\Psi_{e12}(\theta, t) = \frac{\sinh(t)}{2} \left[ \cosh(\theta t) - \cosh((1 - \xi)t) \right] + \left[ \cosh(\theta t) \cosh((2 - \xi)t) - \sinh((2 - \xi)t) \sinh((1 - \xi)t) - \cosh(t) \right] \sinh\left(\frac{(1 - \xi)t}{2}\right) \cosh\left(\frac{(1 + \xi)t}{2}\right)
\]

and defined by analytic continuation outside this domain. Notice that eq. (58) satisfies

\[
\langle \Phi_{rs}(x) \rangle = \langle \Phi_{p-r} p' - s(x) \rangle .
\]  

Finally, the expectation value (58) can be used to derive the bulk free energy \( f_{e12} = -\lim_{V \to \infty} \frac{1}{V} \ln Z \) where \( V \) is the volume of the 2D space and \( Z \) is the singular part of the partition function associated with action (53). The result for the bulk free energy follows from the analytic continuation of (24) and eq. (53), i.e.

\[
f_{e12} = -\frac{M^2}{2} \sin\left(\frac{\pi \xi}{2 - \xi}\right) \sin\left(\frac{\pi (1 + \xi)}{2 - \xi}\right).
\]

If we look at the action (53), we can now use the relation \( \partial_\lambda f_{e12} = \langle \epsilon \Phi_{12} \rangle \) to deduce the expectation value

\[
\langle \epsilon(x) \Phi_{12}(x) \rangle = -\frac{1}{\lambda} \left[ \frac{\lambda^2 \pi^2 \gamma^2 (\frac{1}{2(1 + \xi)})}{\gamma(\frac{3\xi - 1}{1 + \xi}) \gamma(\frac{1 - \xi}{1 + \xi})} \right]^{\frac{1 + \xi}{2 - \xi}} \frac{1 + \xi}{2 - \xi} \frac{\Gamma^2(\frac{1}{2 - \xi})}{\Gamma(\frac{1 - \xi}{2 - \xi}) \Gamma(\frac{1 + \xi}{2 - \xi})} \sin\left(\frac{\pi \xi}{2 - \xi}\right) \sin\left(\frac{\pi (1 + \xi)}{2 - \xi}\right).
\]

Instead of taking \( \Phi_{12} \) inside the expectation value written above, one can now consider the one associated with more general primary operator of the second minimal model. The corresponding VEV follows from the dual result of eq. (46), i.e. we obtain

\[
\langle \epsilon(x) \Phi_{rs}(x) \rangle = -\left[ \frac{\lambda^2 \pi^2 \gamma^2 (\frac{1}{2(1 + \xi)})}{\gamma(\frac{3\xi - 1}{1 + \xi}) \gamma(\frac{1 - \xi}{1 + \xi})} \right]^{\frac{1 + \xi}{2 - \xi}} \frac{1 + \xi}{2 - \xi} \frac{\Gamma^2(\frac{1}{2 - \xi})}{\Gamma(\frac{1 - \xi}{2 - \xi}) \Gamma(\frac{1 + \xi}{2 - \xi})} \frac{1}{\gamma(\frac{3 - \xi + (1 + \xi)r - 2\xi s}{4 - 2\xi})} \langle \Phi_{rs}(x) \rangle .
\]

Let us now turn to the coupled minimal models with action (54). In this case, the condition \( 4p > 3p' \) guarantees that the perturbing operator is relevant. Then, the vacuum structure is expected to be similar to that of (53). Using the substitutions

\[
p \leftrightarrow p', \quad r \leftrightarrow s, \quad \xi \rightarrow \frac{1 + \xi}{3\xi - 1}
\]

in the previous expressions, the breather-particle relation is now given by

\[
\overline{m} = 2M \sin\left(\frac{\pi (1 + \xi)}{5\xi - 3}\right)
\]
and the exact relation between the mass of the lightest kink $M$ and $\hat{\lambda}$ follows

$$
\hat{\lambda}^2 = \frac{1}{\pi^2} \gamma(\frac{1}{2})\gamma(\frac{3\xi-1}{2\xi}) \left[ \frac{\pi M\Gamma(\frac{4\xi}{5\xi-3})}{\Gamma(\frac{3\xi-1}{5\xi-3})\Gamma(\frac{1+\xi}{5\xi-3})} \right]^{\frac{5\xi-3}{4\xi}} .
$$

(65)

Then, for the coupled minimal models defined by (54), the massive phase corresponds to the domain:

$$(iii) \quad \frac{3}{5} < \xi < 1, \quad \Im(\hat{\lambda}) = 0 \quad \text{i.e.} \quad \frac{3}{4} < \frac{p}{p'} < 1 .
$$

(66)

From eq. (58) and (53), we obtain the following expression for the VEV

$$
\langle \Phi_{rs}(x) \rangle = \left[ \frac{\pi^2\hat{\lambda}^2\gamma^2(\frac{3\xi-1}{8\xi})}{2^{\frac{11}{2}}\gamma(\frac{1}{2})\gamma(\frac{3\xi-1}{2\xi})} \right]^{\frac{4\xi}{5\xi-3} \Delta_{rs}} \exp Q_{\epsilon 21}((1 + \xi)r - 2\xi s) .
$$

(67)

The function $Q_{\epsilon 21}(\theta)$ for $|\theta| < 2\xi$ is given by the integral

$$
Q_{\epsilon 21}(\theta) = \int_0^\infty \frac{dt}{t} \left( \frac{\Psi_{\epsilon 21}(\theta, t)}{\sinh((1 + \xi)t) \sinh(4t\xi) \sinh((5\xi - 3)t)} - 2\Delta_{rs}e^{-2t} \right)
$$

with

$$
\Psi_{\epsilon 21}(\theta, t) = \frac{\sinh((3\xi - 1)t)}{2} \left[ \cosh(2\theta t) - \cosh((2 - 2\xi)t) \right] + \left[ \cosh(2\theta t) \cosh((5\xi - 3)t) \right. \\
- \left. \sinh((2 - 2\xi)t) \sinh((5\xi - 3)t) - \cosh((3\xi - 1)t) \right] \sinh((\xi - 1)t) \cosh((2\xi)t)
$$

and is defined by analytic continuation outside this domain. As was considered above, the exact bulk free energy can be also calculated from the results of the previous section,

$$
f_{\epsilon 21} = -\frac{M^2 \sin(\frac{\pi(3\xi-1)}{5\xi-3}) \sin(\frac{\pi(1+\xi)}{5\xi-3})}{2 \sin(\frac{\pi(4\xi)}{5\xi-3})} .
$$

(68)

Finally, this latter expression provides us the VEV

$$
\langle \epsilon(x)\Phi_{21}(x) \rangle = -\frac{1}{\lambda} \left[ \hat{\lambda}^2 \pi^2\gamma^2(\frac{3\xi-1}{8\xi}) \right]^{\frac{4\xi}{5\xi-3}} \frac{4\xi}{5\xi - 3} \frac{\Gamma^2(\frac{3\xi-1}{5\xi-3})}{\Gamma(\frac{3\xi-1}{5\xi-3})\Gamma(\frac{4\xi}{5\xi-3})} \sin(\frac{\pi(3\xi-1)}{5\xi-3}) ,
$$

(69)

whereas for a more general primary operator of the second minimal model, we now obtain

$$
\langle \epsilon(x)\Phi_{rs}(x) \rangle = -\left[ \hat{\lambda}^2 \pi^2\gamma^2(\frac{3\xi-1}{8\xi}) \right]^{\frac{2\xi}{5\xi-3}} \frac{4\xi}{5\xi - 3} \frac{\Gamma^2(\frac{3\xi-1}{5\xi-3})}{\Gamma(\frac{3\xi-1}{5\xi-3})\Gamma(\frac{4\xi}{5\xi-3})} \sin(\frac{\pi(3\xi-1)}{5\xi-3}) \langle \Phi_{rs}(x) \rangle .
$$

(70)

Some checks of these expressions are desirable. For instance, the case $p = 5$ and $p' = 6$ in action (54) corresponds to a critical Ising model coupled to a critical $\mathbb{Z}_3$-Potts model through
the interaction $\epsilon \mathcal{E}$ where $\mathcal{E} \equiv \Phi_{21}$ is the leading energy operator of the $\mathbb{Z}_3$-Potts model. It is rather interesting to recall that the decoupled critical Ising-3-state Potts models appear in the phase diagram of the $\mathbb{Z}_6$ spin model in the vicinity of a renormalization group fixed point. Using continuum field theory approach, which is valid in the scaling region around this point, Zamolodchikov’s counting argument \cite{20} can be used to show that the $\epsilon \mathcal{E}$ perturbation preserves conserved charges of spin $\pm 3$, $\pm 5$. As was suggested in \cite{21} and confirmed here by explicitly constructing the QFT, this perturbation of the Ising-3-state Potts model is therefore integrable. Although the exact $S$-matrix for these coupled models is not known explicitly, a TBA system based on $E_7$ Dynkin diagram has already been considered as a good candidate \cite{22}. The exact relation \cite{53} as well as the breather-particle relation can been checked using this TBA system. The good agreement \cite{54} supports the exact results above.

Let us now explain how the above results can be relevant in the study of the $SU(3)_2$-HSG model. The simplest HSG model is the complex sine-Gordon model associated with an integrable perturbation of the WZNW-coset model $SU(2)_k/U(1)$, whereas more complicated HSG theories can be viewed as interacting copies of complex sine-Gordon theories. At classical level, the corresponding equations of motion correspond to non-abelian Toda theories which are known to be integrable. At quantum level it has also been shown that integrability is preserved \cite{21}, and assuming factorization of scattering process $S$-matrices have been proposed \cite{22}. Among the generalizations of the complex sine-Gordon model, the $SU(3)_2$-HSG can describe a WZNW-coset model $SU(3)_2/(U(1))^2$ with central charge $c = 6/5$ perturbed by an operator with conformal dimension $\Delta_{\text{pert}} = 3/5$, as was shown in \cite{21}. On the other hand, the case $p = 4$ and $p' = 5$ in action \cite{53} leads to a critical Ising model coupled to a tricritical Ising model (with a total central charge $c = 1/2 + 7/10 = 6/5$ in the UV limit). It is then interesting to notice that conformal dimensions $\Delta(\epsilon(x)\Phi_{12}(x)) = \Delta(\Phi_{13}(x)) = 3/5$. Obviously, in the deep UV the operator content of the coupled models \cite{53} is bigger than the operator content of the $SU(3)_2$-HSG, which only consists of operators of conformal dimension $0, 1/2, 1/10$ and $3/5$ \cite{24}. However, the operator content of \cite{53} possesses a closed subset of operators \{1, $\epsilon$, $\Phi_{12}$, $\epsilon \Phi_{12}$, $\Phi_{13}$\} with conformal dimensions given above. Consequently, using the notations of \cite{24} for the $SU(3)_2$-HSG, we propose the identification $\mathcal{O}_{0,2}^{1,1} \equiv \Phi_{12}$, the trace of the energy momentum tensor $\Theta \equiv \{\epsilon \Phi_{12} \cup \Phi_{13}\}$ whereas the remaining operator is the Ising energy density $\epsilon$. Then, the action associated with the $SU(3)_2$-HSG model can be seen as a subsector of the following (integrable \cite{24}) action:

$$S_{HSG[SU(3)_2]} \sim \mathcal{M}_{3/4} + \mathcal{M}_{4/5} + \lambda \int d^2 x \Phi_{\text{pert}}(x)$$

(71)

with $\Phi_{\text{pert}}(x) = \epsilon(x)\Phi_{12}(x) + \rho \Phi_{13}(x)$ and $\rho$ is a $c-$number. At the special value $\rho = 0$, one recovers \cite{53}. From the above analysis \cite{57}, it follows that the $SU(3)_2$-HSG is a massive theory in agreement with the results of \cite{21}. Also, taking the value $\xi = 2/3$ in \cite{55} gives $m = 2M$, i.e., the formation of stable particles via fusing of the soliton mass $M$ is not possible in agreement with \cite{23}. At $\rho = 0$, its bulk free energy follows from \cite{60} whereas the exact relation between the soliton mass $m$ and the parameter in the Lagrangian $\lambda$ is given in \cite{56} for $\xi = 2/3$.

Accepting the conjectures \cite{58} and \cite{57}, one can then make interesting predictions for numerical values of VEVs. We report the reader to the Appendix B where various examples are considered.

\footnote{P. Dorey and R. Tateo, private communication}
5 Relation to the coupled Ising-XY model

Over the years, the critical behavior of the two dimensional Ising-XY model, consisting of Ising and XY models coupled through their energy densities, has been studied numerically in some detail. A Hamiltonian for this model has been proposed which writes

\[ \mathcal{H}/kT = -\sum_{i,j} [(A + B\sigma_i\sigma_j) \cos(\theta_i - \theta_j) + C\sigma_i\sigma_j] \quad (72) \]

where \( A, B \) and \( C \) are effective couplings. The model with \( A \neq B \) is relevant for the anisotropic frustrated XY model [27] and anti-ferromagnetic RSOS model [42] whereas the subspace \( A = B \) is relevant for the isotropic FFXY model or its one dimensional quantum version [13]. In this latter case, a phase diagram has been proposed (see figure 1). It consists of three branches joining at a point \( P \) in the ferromagnetic region \( A > 0, A + C > 0 \). One of these branches corresponds to a single transition with simultaneous loss of Ising and XY order and the other two to separate Kosterlitz-Thouless (KT) and Ising transitions. Monte Carlo transfer-matrix methods [33] yields that the central charge seems to vary continuously from \( c \approx 1.5 \) near \( P \) to \( c \approx 2 \) at \( T \), which contradicts the hypothesis that the line \( PT \) can be simply described in terms of a superposition of critical Ising and XY models with \( c = 1.5 \) as was suggested by Foda in [34].

The only possibility would be the existence of a parameter changing along the line \( PT \) that does not affect the symmetry. As was argued in [44] there are three possible explanations: (a) The system is not conformally invariant; (b) The result is an artifact of limited strip widths; (c) It is a new effect. Consequently, due to the limited strip widths of the numerical analysis, there are some reasons to believe that the phase diagram analysis is not yet complete, in particular, first order transitions may appear along the line denoted \( PT \). More recent numerical analysis of the Ising-XY model based on the coupled Ising-RSOS model [45] also supports this hypothesis. Then, if we opt for scenario (a), it is natural to consider all possible integrable perturbations of a superposition of critical Ising and XY model as a starting point to study the vicinity of the point \( P \). It can be shown that there exists only three kinds of integrable perturbations which can provide interesting candidates: the supersymmetric sine-Gordon model [46] and the models with action (1) and (2) for \( n = 2 \) and imaginary coupling.

![Figure 1: The phase diagram for the coupled Ising-XY models](image)

\( ^9 \)Notice that the continuum limit of the generalized Coulomb-gas representation of the FFXY model containing fractional charges is nothing but the (conformally invariant) action of a free Majorana fermion and a free boson.
Consequently, it would be rather interesting to obtain non-perturbative results for these cases. As some exact off-shell results (VEVs, bulk free energy,...) for the supersymmetric SG model can be found in [14], here we naturally focus on action (2) for $n = 2$ \(\square\). Using the analytic continuation \(b \rightarrow i\beta\) and \(\bar{\mu} \rightarrow -\bar{\mu}\), we will consider in the vicinity of the point \(P\) of the phase diagram the following action:

\[
A_\beta = \int d^2 x \left[ \frac{1}{2\pi} \left( \psi \partial \bar{\psi} + \bar{\psi} \partial \psi \right) + \frac{1}{8\pi} (\partial \psi e^{-i\beta \varphi} - 2\bar{\mu} \cos(2\beta \varphi)) \right]. \tag{73}
\]

Before going further, let us recall known results for the XY model - which also corresponds to the nonlinear $O(2)$ $\sigma$-model. Kosterlitz and Thouless showed that spin configurations are mixture of topologically trivial configurations (called spin waves) and a gas of vortices with integer topological charge. Both are decoupled and the vortices interact through a logarithmic potential which is therefore identical to a two dimensional Coulomb gas. At a specific finite critical coupling, it has been shown rigorously [17] that this Coulomb gas possesses a phase transition. In the vicinity of that point, only vortices of topological charges $\pm 1$ are important and higher charge vortices can be neglected. In the Coulomb gas formalism, various configurations are then associated with the operators $O_{e,m}$ with dimension $d_{e,m} = e^2/R^2 + mR^2/4$ and spin $s_{e,m} = em$. In particular, the “electric” $e$ and “magnetic” $m$ charges exchange each other under the weak-strong coupling duality $R \leftrightarrow 2/R$. Also, the “electric” operators can be written in terms of vertex operators in the following way

\[
O_{e,0}(x) = N^{-1}(e\beta) \exp(i e \varphi / R)(x) \quad \text{with} \quad e \in \mathbb{Z} \tag{74}
\]

where we choose the normalization factor (the mass scale $M$ has been introduce in (55))

\[
N(\eta) = \left[ \frac{\pi M \Gamma(\frac{1}{2-3\beta^2})}{2 \frac{2^{1-3 \beta^2}}{2-3 \beta^2} \Gamma(\frac{1-\beta^2}{2-3\beta^2}) \Gamma(\frac{\beta^2}{2-3\beta^2})} \right]^{2\beta^2-1} \left( \pi \bar{\mu} 2^{4\beta^2 - 2\gamma(1-2\beta^2)} \right)^{1/2}. \tag{75}
\]

Returning to action (73) it is now quite natural to take $\beta = 1/\sqrt{2}$ in order to consider the Ising-XY model. For this value and $R = \sqrt{2}$, the operators $O_{\pm 2,0}$ become marginal as well as the last part of the action (cosine term) (73). For $e > 2$, operators are irrelevant. Then, we obtain the following action

\[
A_{1/\sqrt{2}} = \mathcal{M}_{3/4} + \frac{1}{8\pi} \int d^2 x (\partial \varphi)^2 + \Lambda_{\beta = 1/\sqrt{2}} \int d^2 x \epsilon(x) O_{-1,0}(x). \tag{76}
\]

For general values of $\beta$, the exact relation between the mass scale $M$ and the parameter $\Lambda_\beta$ follows from section 2 and eq. (55):

\[
\Lambda_\beta = \frac{2}{\pi \gamma(1-\beta^2)} \left[ \frac{\pi M \Gamma(\frac{1}{1-\beta^2})}{2 \frac{2^{1-3 \beta^2}}{2-3 \beta^2} \Gamma(\frac{1-\beta^2}{2-3\beta^2}) \Gamma(\frac{\beta^2}{2-3\beta^2})} \right]^{1-\beta^2}. \tag{77}
\]

\[\text{We have seen in the previous section that the model (3) for } n = 2 \text{ can describe a critical RSOS model coupled to a critical Ising model through their energy densities. Furthermore, since for } n = 2 \text{ both models with actions (1) and (3) possess the same limit at } \beta^2 = 1/2, \text{ it is sufficient to focus on one only.}\]
Also, using the conventions defined above, an exact expression for the VEV of the “electric” operators can be obtained:

\[
\langle O_{e,0}(x) \rangle = \left[ \frac{\pi M \Gamma \left( \frac{1}{2-3\beta^2} \right)}{2^{2(1-\beta^2)} \Gamma \left( \frac{\beta^2}{2-3\beta^2} \right) \Gamma \left( \frac{1-\beta^2}{2-3\beta^2} \right)} \right]^{2\beta^2} e^{2\beta^2} \\
\times \exp \int_0^\infty \frac{dt}{t} \left[ -e^{2\beta^2} e^{-2t} - \frac{\sinh(e^{\beta^2} t) \Psi_\beta(t)}{\sinh(t) \sinh(t^2) \sinh((2-3\beta^2)t)} \right]
\]

with

\[
\Psi_\beta(t) = 2 \sinh((1 - (e + 1)\beta^2) t) \sinh((1/2 - \beta^2) t) \cosh(t/2) \\
- \sinh((1 + (e - 2)\beta^2) t) \sinh((1 - \beta^2) t).
\]

For the choice \( \beta = 1/\sqrt{2} \), the model (76) becomes massive if \( \Im \Lambda_{\beta=1/\sqrt{2}} = 0 \), corresponding to a first order phase transition. One can then check that \( \langle O_{\pm,0} \rangle = \langle O_{0,\pm} \rangle \) and \( \langle O_{\pm,0} \rangle = 0 \) as expected. As the cosine term in the action is marginal, one can show that exactly the same results can be obtained if we had started from action (1) instead. This is not surprising as both models possess the same limit at \( \beta^2 = 1/2 \).

6 Neveu-Schwarz sector of perturbed \( N = 1 \) supersymmetric unitary minimal models

Instead of considering a restriction of action (2), it is also interesting to study action (1) for \( n = 2 \) at specific imaginary values of the coupling \( b \). As it is known, the minimal series of superconformal unitary models can be described from the superLiouville field theory using the analytic continuation \( b \to i\beta \) in (39) and taking the specific value

\[
\beta^2 = \frac{K}{K+2} \quad \text{with} \quad K \geq 2.
\]

Their corresponding central charge is \( c_{SUSY} = \frac{3}{2}(1 - \frac{8}{K(K+2)}) \) and the finite number of primary operators belonging to the NS sector are labelled by the conformal dimensions (for \( r - s \in 2\mathbb{Z} \))

\[
\Delta_{rs}^{(NS)} = \frac{(r(K+2) - sK)^2 - 4}{8K(K+2)} \quad \text{with} \quad 1 \leq r < K \ , \ 1 \leq s < K+2.
\]

Using the vertex operator representation, they can be written in terms of the exponential fields of the superLiouville theory as follows:

\[
\Phi_{rs}^{(NS)}(x) = \mathcal{N}_{rs}^{-1} \exp(i\eta^{rs} \varphi(x)) \quad \text{with} \quad \eta^{rs} = \frac{(1-r)}{2\beta} - \frac{(1-s)}{2} \beta \quad \text{and} \quad r - s \in 2\mathbb{Z}
\]

where we have introduced the normalization factors \( \mathcal{N}_{rs} \). Choosing a condition similar to (31), they can be expressed in terms of the reflection amplitude in the NS sector of the superLiouville field theory (36) as \( \mathcal{N}_{rs} \equiv \mathcal{N}^{(NS)}(\eta_{rs}) \) where

\[
\mathcal{N}^{(NS)}(\eta) = \left[ \frac{\pi K}{2\beta^2} \right]^{\frac{n}{2}} \left[ \frac{\Gamma(1/2 + \beta^2/2 + \eta\beta) \Gamma(1/2 + 1/2\beta^2 - \eta/\beta) \Gamma(3/2 - 3\beta^2/2) \Gamma(3/2 - 1/2\beta^2)}{\Gamma(3/2 - 2\beta^2/2 - \eta\beta) \Gamma(3/2 - 1/2\beta^2 + \eta/\beta) \Gamma(1/2 + \beta^2/2) \Gamma(1/2 + 1/2\beta^2)} \right].
\]
Adding the perturbing term with conformal dimension \( \Delta_{\text{pert}} = \Delta_{SL}(e^{b\phi}) = -b^2 - 1/2 \) in (39), the analytic continuation \( b \to i\beta \) then gives perturbed \( N = 1 \) supersymmetric minimal model \([3]\). Notice that for any values of \( K \), the perturbation is relevant and is identified to \( \Phi^{(NS)}_{13} \) with conformal dimension \( \Delta_{13} = \frac{K-2}{2(K+2)} \). The resulting action writes

\[
\tilde{A} = \mathcal{M}^{N=1}_K + \tilde{\lambda} \int d^2x \Phi^{(NS)}_{13}(x).
\]

The exact relation between the parameter \( \tilde{\lambda} \) and the mass scale \( m \) introduced in eq. (21) can be obtained using \( \tilde{\lambda} = -\mu N^{(NS)}(\beta) \) with the result

\[
\tilde{\lambda} = -\frac{(1 + \tilde{\xi})}{\pi \gamma(\frac{2 + \tilde{\xi}}{2 + 2\tilde{\xi}})} \left[ \frac{\Gamma\left(\frac{1 + \tilde{\xi}}{2 + 2\tilde{\xi}}\right)\Gamma\left(\frac{3 + 2\tilde{\xi}}{2 + 2\tilde{\xi}}\right)}{(2\tilde{\xi} - 1)\Gamma\left(\frac{3}{2 + 2\tilde{\xi}}\right)\Gamma\left(\frac{1 + 2\tilde{\xi}}{2 + 2\tilde{\xi}}\right)} \right] \frac{1 + \tilde{\xi}}{\tilde{\xi}} \frac{1 + \tilde{\xi}}{3 + \tilde{\xi}} \frac{1 + \tilde{\xi}}{2 + 2\tilde{\xi}} \Gamma\left(\frac{1 + 4\tilde{\xi}}{2 + 2\tilde{\xi}}\right) \Gamma\left(\frac{3 + 2\tilde{\xi}}{2 + 2\tilde{\xi}}\right) 2\tilde{\xi} - 1 4(1 + \tilde{\xi})^2 \frac{1 + \tilde{\xi}}{3 + \tilde{\xi}} \frac{1 + \tilde{\xi}}{3 + \tilde{\xi}} \frac{1 + \tilde{\xi}}{2 + 2\tilde{\xi}} \sin\left(\frac{\pi}{3 + \tilde{\xi}}\right) \sin\left(\frac{\pi}{3 + \tilde{\xi}}\right) \sin\left(\frac{\pi}{3 + \tilde{\xi}}\right). \]

For any values of \( K \) we have \( 1/2 < \beta^2 < 1 \). Consequently, the model (80) develops a massive phase for \( \mathcal{M}(\tilde{\lambda}) = 0 \). The calculation of the bulk free energy leads to the result

\[
f_{13}^{SUSY} = -\frac{m^2}{8} \frac{\sin\left(\frac{\pi}{3 + \tilde{\xi}}\right)}{\sin\left(\frac{\pi(1 + \tilde{\xi})}{3 + \tilde{\xi}}\right) \sin\left(\frac{\pi(1 + \tilde{\xi})}{3 + \tilde{\xi}}\right) \sin\left(\frac{\pi(1 + \tilde{\xi})}{3 + \tilde{\xi}}\right)} \]

where we have \( \tilde{\xi} = K/2 \), which gives the following expression for the exact vacuum expectation value of the perturbing operator

\[
\langle \Phi^{(NS)}_{13} \rangle = -\frac{1}{\lambda} \left[ \frac{\pi^2 \lambda^2}{\pi^2 \lambda^2} \frac{\Gamma\left(\frac{3 + 2\tilde{\xi}}{2 + 2\tilde{\xi}}\right)\Gamma\left(\frac{1 + 2\tilde{\xi}}{2 + 2\tilde{\xi}}\right)}{(2\tilde{\xi} - 1)\Gamma\left(\frac{3}{2 + 2\tilde{\xi}}\right)\Gamma\left(\frac{1 + 2\tilde{\xi}}{2 + 2\tilde{\xi}}\right)} \right] \frac{1 + \tilde{\xi}}{3 + \tilde{\xi}} \frac{1 + \tilde{\xi}}{2 + 2\tilde{\xi}} \Gamma\left(\frac{1 + 4\tilde{\xi}}{2 + 2\tilde{\xi}}\right) \Gamma\left(\frac{3 + 2\tilde{\xi}}{2 + 2\tilde{\xi}}\right) 2\tilde{\xi} - 1 4(1 + \tilde{\xi})^2 \frac{1 + \tilde{\xi}}{3 + \tilde{\xi}} \frac{1 + \tilde{\xi}}{3 + \tilde{\xi}} \frac{1 + \tilde{\xi}}{2 + 2\tilde{\xi}} \sin\left(\frac{\pi}{3 + \tilde{\xi}}\right) \sin\left(\frac{\pi}{3 + \tilde{\xi}}\right) \sin\left(\frac{\pi}{3 + \tilde{\xi}}\right). \]

From the exact expression of the vacuum expectation value (33) and the definition of the normalization factor, for more general VEVs of primary fields one obtains:

\[
\langle \Phi^{(NS)}_{rs} \rangle = \left[ \frac{\Gamma\left(\frac{3 + 2\tilde{\xi}}{2 + 2\tilde{\xi}}\right)\Gamma\left(\frac{1 + 2\tilde{\xi}}{2 + 2\tilde{\xi}}\right)}{\Gamma\left(\frac{1}{3 + \tilde{\xi}}\right)} \right]^{2\Delta_{rs}} \frac{1}{2^{1 + \frac{\tilde{\xi}}{3 + \tilde{\xi}}} \Gamma\left(\frac{1}{3 + \tilde{\xi}}\right)} \times \exp \int_0^\infty \frac{dt}{t} \left[ -2 \Delta_{rs} e^{-4t} = \frac{\mathcal{F}(1 + \tilde{\xi})r - \tilde{\xi} s, \tilde{\xi}, t}{2 \sinh(2(1 + \tilde{\xi})t) \sinh(2\tilde{\xi}t) \sinh((6 + 2\tilde{\xi})t)} \right],
\]

\(^1\)To describe other perturbations of supersymmetric minimal models, one may have also considered the analytic continuation of the model with action (3). However, for \( \beta^2 = K/(K + 2) \) the perturbation corresponds to \( \Phi^{(NS)}_{13} \), which is relevant only for \( K = 4 \). On the other hand, the analytic continuation of action (1) at the (dual) value \( \beta^2 = (K + 2)/2 \) with \( K \leq 4 \) leads to a relevant perturbation, namely \( \Phi^{(NS)}_{31} \). For simplicity we will not consider this case here.
where we introduce the function

\[
\mathcal{F}(\theta, \tilde{\xi}, t) = \sinh((1 - \theta)t)(\cosh((\theta + 3 + 2\tilde{\xi})t) + \cosh((\theta + 3)t) - \cosh((\theta - 1)t) - \cosh((\theta - 1 - 2\tilde{\xi})t) + (\cosh((\theta - 3 - 2\tilde{\xi})t) - \cosh((\theta - 7 - 2\tilde{\xi})t) + \cosh((\theta + 9 + 2\tilde{\xi})t) - \cosh((\theta + 5 + 2\tilde{\xi})t)/2).
\]

(83)

For \( K = 2 \), i.e. \( \tilde{\xi} = 1 \) the central charge of the supersymmetric minimal model is \( c_{SUSY} = 0 \). Then the perturbing operator reduces to \( \Phi^{(NS)}_{13} \equiv \mathbb{I} \), i.e. the identity operator. In this case, one can check that the above relation \( \tilde{\lambda} = -m^2/8 \) which implies \( f_{13}^{SUSY} = \tilde{\lambda} \), as expected.

For \( K = 3 \), the model with action (80) describes an integrable perturbation of the tricritical Ising model. On the other hand, the same model can be obtained starting from an integrable perturbation of a nonsupersymmetric minimal models, as \( \mathcal{M}_{4/5} \equiv \mathcal{M}_{3}^{N=1} \). Using notations of the previous section, we have the correspondence \( \Phi_{12} \equiv \Phi^{(NS)}_{13} \). For this case, the VEVs, exact relations between the mass scale of the particles and the parameter in the action have been proposed in [11]. It is straightforward to check that the above results perfectly agree with these ones as long as the mass scale \( m \equiv M \) of [11].

For \( K = 6 \), the central charge of the conformal part becomes \( c = 5/4 \), i.e. the resulting model \( \mathcal{M}_{6}^{N=1} \) is identified with a critical \( \mathbb{Z}_6 \) spin model perturbed by the thermal operator with conformal dimension \( D_1 = 1/4 \) denoted \( \epsilon^{(1)} \) in [35]. In general, \( \mathbb{Z}_n \) spin models are the natural generalizations of the Ising model, when the spin variable takes its values in group \( \mathbb{Z}_n \) [35]. These self-dual models possess critical points which are associated with \( \mathbb{Z}_n \) parafermionic CFTs with central charge \( c = \frac{2(n-1)}{n+2} \) briefly described in beginning of section 2. Besides the parafermionic symmetry, these CFTs possess \( W(A_{n-1}) \) symmetry and can be described by the \( \mathcal{M}_{n+1}(A_{n-1}) \) minimal model at the specific coupling \( \beta^2 = \frac{n+1}{n+2} \). It has been shown that integrability is preserved by adding the perturbing first thermal operator \( \epsilon^{(1)} \) with conformal dimension \( 2/(n+2) \). This operator is anti self-dual, i.e. his sign changes under duality transformation. Depending of the sign of the parameter characterizing the strength of the perturbation, the perturbed theory will be either in ordered or disordered phase. For general values of \( n \), exact relation between this parameter and the mass of the particles have been proposed in [13]. However, to compare our results to the ones associated with the perturbed model \( \mathcal{M}_7(A_5) \), we need to identify the lowest kink in both models. Similarly to the case of the coupled models in section 4, for general values of \( \tilde{\xi} \) here we expect the identification

\[
\overline{m} = 2M \sin \left( \frac{\pi \tilde{\xi}}{3 + \tilde{\xi}} \right)
\]

(84)

where \( M \) denotes the mass of the lowest kink. Taking \( \tilde{\xi} = 3 \) and using the substitution above in (81) and (82), it is straightforward to check that previous results coincide perfectly with the ones in [13].

7 Concluding remarks

In a first part, we have studied in detail two of the simplest dual representations of integrable deformations of affine Toda theories, which corresponds to a critical Ising model coupled with a Liouville field theory depending on the coupling \( b \). In the deep UV, the effective central charges which characterize the behavior of both models are obtained explicitly. Also, we propose the exact relations between the parameters in the Lagrangians (UV data) and the mass scale of
the particles (IR data), which are necessary to perform properly the TBA analysis. Various vacuum expectation values, namely the one associated with the primary field $\langle e^{i\varphi}(x)\rangle$ and the first descendant field $\langle \bar{\psi}\psi e^{i\varphi}(x)\rangle$, as well as the bulk free energy are obtained explicitly. All previous results are shown to exchange under a weak-strong coupling duality $b \leftrightarrow 1/b$, confirming the duality relation between both Lagrangian representations conjectured in \[3,9\].

In the second part, we consider various applications of these results. From the model \(2\) for $n = 2$, we were able to describe several kind of coupled models. First, we studied in detail the case $b^2 = -p/2p'$ or $b^2 = -p'/2p$ from which we obtain the coupled minimal models \(33\) or \(54\), respectively. The main difference between both actions is the presence of an extra term in \(33\), associated with the $\Phi_{13}$ primary operator. It comes from the counterterm in the real coupling QFT. In both cases, we propose exact relation between the mass of the lowest breather $m$ and the mass of the lowest kink $M$, as well as the exact relation between the parameters $\lambda$ or $\tilde{\lambda}$ and $M$. For a critical Ising model coupled with unitary minimal models, the QFTs \(33\) and \(54\) are found to be massive. Exact VEVs and bulk free energies are obtained. Some special cases have been independently checked and shown to agree using different methods\(^5\). Among these coupled models, we propose an identification between a special case ($\rho = 0$) of the $SU(3)_2$-HSG model for a finite resonance parameter and a subsector of the Ising-Tricritical Ising models coupled through energy-energy. In particular, we find that the lowest breather disappears from the spectrum, unstable as expected \[23\].

Secondly, we propose to study the vicinity of the critical point $P$ in the phase diagram of the coupled Ising-XY model depicted in figure 1. Three candidates, integrable perturbations of a $c = 3/2$ CFT, can be considered. The supersymmetric sine-Gordon suggested in \[34\] or the models \(1\) or \(2\) for imaginary coupling and $n = 2$. Here we focused on the QFT \(76\) obtained from \(3\) for $n = 2$ as the other results can be found in \[16\]. The model is a massive QFT, corresponding to a first order phase transition along the line $PT$ in figure 1. Exact VEV of the electric/magnetic operator $\langle O_{\pm1,0} \rangle = \langle O_{0,\pm1} \rangle$ is calculated.

From the model \(1\) for $n = 2$, we study in detail the NS sector of the QG restriction associated with $b^2 = -K/K + 2$, which leads to the $\Phi_{13}^{SUSY}$ integrable perturbation of $N = 1$ unitary minimal models. We show that results agree with the known ones obtained for the perturbed Tricritical Ising model \(1\) for $K = 3$ and the perturbed $Z_6$ parafermionic CFTs \[13\] for $K = 6$. Finally we would like to add a few remarks:

- **Scattering theory of solitons for imaginary coupling.**
  For imaginary coupling both models studied here admit a quantum group symmetry based on $U_q(A^{(1)}(0, 2))$ and $U_q(B^{(1)}(0, 1))$, respectively. Then, it would be interesting to construct the corresponding $R$-matrices in such a way to obtain the exact $S$-matrices for the scattering of solitons. The identification of the scattering amplitudes for the lowest breathers with the $S$-matrix for quantum particles in the real coupling case should confirm eq. \(53\). Also, the restricted $R$-matrix with respect to the different subalgebras should provide the scattering amplitudes in the coupled models described here. Understanding the vacuum structure of all the models described above would also allow one to fix the form of the prefactor associated with the QG restriction.

- **Homogeneous sine-Gordon $SU(3)_2$ model.**
  As we mentioned above, the perturbing field $\Phi_{pert}\propto \Theta(x)$ with conformal dimension $3/5$ can be written as $\Phi_{pert}(x) = \epsilon(x)\Phi_{12}(x) + \rho\Phi_{13}(x)$ where $\rho$ is a $c$–number. For a finite resonance parameter \(21\) \(24\) the $SU(3)_2$-HSG model contains only two self-conjugate solitons with masses $M$ and $M'$. Then, it would be rather interesting to obtain the exact relation between the ratio $M/M'$ and the parameter $\rho$. With the $\lambda$-$M$ relation ($M$ is the lightest mass of the two self-conjugate solitons) obtained from eq. \(58\) for $\xi = 2/3$, i.e. $|\lambda| = 0.2790872531...M^{4/5}$ and using \(58\) and
it would be straightforward to obtain the expectation values \( \langle O_{0,1}^\text{pert}(x) \rangle \) and \( \langle \Phi_{\text{pert}}(x) \rangle \) in the HSG-SU(3)_2 model for a finite resonance parameter.

Furthermore, it is well-known that any correlation function of local fields \( \langle O(r)O(0) \rangle \) in the short-distance limit can be reduced down to one-point functions \( \langle O'(r) \rangle \) by successive application of the operator product expansion \[25, 49\]. Along the line of \[25, 50\], using the exact results obtained here as well as the three-point functions calculated in \[30\] and the relation between the HSG-SU(3)_2 and the coupled Ising-tricritical Ising models proposed in section 4, the UV behavior of the two-point correlation functions can be studied. On the other hand, the long-distance behavior (IR) of the two-point correlation functions in the HSG-SU(3)_2 model has been studied in detail in \[24\] using the so-called form-factors approach. Then, it would be interesting to compare both results which should agree in a good approximation in the intermediate region \( M_r \sim 1 \).

- Ramond sector of \( \mathcal{N} = 1 \) superconformal unitary minimal models.

In this work we mainly focused on the NS sector of both models. However one may be interested in the R sector, for instance, the VEVs of the primary fields \( \langle \sigma e^{a\varphi} \rangle \). In this case we have to consider the reflection amplitudes in the R sector of the SL field theory instead of the NS one. This reflection amplitude has been calculated in \[30\] with the result

\[
R_b^{(R)}(a) = \left[ \frac{\pi \kappa}{2b^2} \right]^{2(a+Q')} \frac{\Gamma(\frac{1}{2} - (a + Q')b)\Gamma(\frac{1}{2} - (a + Q')/b)}{\Gamma(\frac{1}{2} + (a + Q')b)\Gamma(\frac{1}{2} + (a + Q')/b)}
\]

where \( 2Q' = b + 1/b \). If we define \( G^{(R)}(a) \equiv \langle \sigma e^{a\varphi} \rangle \) for the VEVs of the R primary fields in both models, instead of \[30\] we have

\[
G^{(R)}(a) = R_b^{(R)}(-a) G^{(R)}(-2Q' + a) .
\]

For the QFT \( \mathcal{I} \), the minimal solution of this reflection equation together with \[32\] can be obtained as before. The result can be written in terms of the NS one, i.e.

\[
\langle \sigma e^{a\varphi} \rangle = \langle \sigma \rangle G^{(2)}_{\sigma} (a) \exp \int_0^\infty \frac{dt}{t} \frac{\sinh(abt)\sinh((ab - 2Qb)t)f(t,b)}{\sinh(t)\sinh(tb^2)\sinh(H_2(1+b^2)t)}
\]

with

\[
f(t,b) = \sinh((b^2 + 1)t) - \sinh(b^2t) - \sinh(t)
\]

and \( \langle \sigma \rangle \) is the VEVs of the Ising spin field in the QFT \( \mathcal{I} \). The result for the QFT \( \mathcal{I} \) is straightforward.

Considering specific values of the coupling constant in the QFT \( \mathcal{I} \) it is then possible to obtain the VEVs \( \langle \sigma \Phi_{rs} \rangle \) in the coupled minimal models studied in sect. 4, which provides the leading term of the two-point function between operators which belong to the critical Ising model and minimal model, respectively. Also, from the QFT \( \mathcal{I} \) one obtains the primary operators in the R sector of the perturbed \( \mathcal{N} = 1 \) superconformal minimal models.

To conclude, as we mentioned briefly in the Introduction the models studied here belong to a more general family of deformed Toda models based on Lie superalgebras. In these cases, previous analysis can be performed along the same line. At specific values of the coupling, beyond describing different kind of coupled models, these series are identified with integrable perturbation of CFTs associated with \( WB(0, r) \) algebras with central charge

\[
c_{WB(0,r)} = (r + 1/2)(1 - 2r(2r - 1)\frac{(p - p')^2}{pp'})
\]
for \( p, p' \geq 2r - 1 \). We intend to return to these models in the future.

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**Appendix A: Normalization of the VEV of the first descendant field**

To calculate the VEV of the first descendant field \( \langle \psi \overline{\psi} \exp(a\varphi) \rangle_\tau \), we considered \( H_\tau(a) \) defined in eq. (44) as the minimal solution of certain reflection equations. However, we used the exact result for the bulk free energy to fix its overall coefficient. It is then rather important to possess an independent derivation for this coefficient, based on the “resonance condition” [12].

For instance, let us consider the analytic continuation \( b \to i\beta \) and \( \mu \to -\mu \) of model (1) with \( n = 2 \). In the free field theory, the composite field \( \psi \overline{\psi} \exp(i\alpha \varphi) \) is spinless with scale dimension

\[
D \equiv \Delta + \overline{\Delta} = 1 + \alpha^2. \tag{85}
\]

For generic values of the coupling \( \beta \) some divergences arise in the VEVs of the descendant fields due to the perturbation in (1) with imaginary coupling. They are generally cancelled if we add specific counterterms which contain spinless local fields with cut-off dependent coefficients. If the perturbation is relevant, a finite number of lower scale dimension counterterms are then sufficient. However, this procedure is regularization scheme dependent, i.e. one can always add finite counterterms. For generic values of \( \alpha \) this ambiguity in the definition of the renormalized expression for the descendant fields can be eliminated by fixing their scale dimensions to be (85). Here, this situation arises if two fields, say \( \mathcal{O}_\alpha \) and \( \mathcal{O}_{\alpha'} \), satisfy the “resonance condition” [51].

\[
D_\alpha = D_{\alpha'} + k(1 - \beta^2) + l(2 - \beta^2) \quad \text{with} \quad (k, l) \in \mathbb{N} \tag{86}
\]

associated with the ambiguity

\[
\mathcal{O}_\alpha \longrightarrow \mathcal{O}_\alpha + \kappa^k \mu^l \mathcal{O}_\alpha. \tag{87}
\]

In this specific case one says that the renormalized field \( \overline{\mathcal{O}}_\alpha \) has an \( (k|l) \)-th resonance [51] with the field \( \mathcal{O}_{\alpha'} \). In particular, the first descendant field \( \psi \overline{\psi} \exp(i\alpha \varphi) \) has a \( (1|0) \) resonance with the field \( \exp(i(\alpha - \beta)\varphi) \) at \( \alpha = 0 \). If one looks at the short distance behavior in \( r = |x| \to 0 \) of the two-point function

\[
\langle \psi \overline{\psi}'(x) \exp(i \alpha_1 \varphi(0)) \rangle_{\tau = 1} \quad \text{with} \quad \alpha_1 + \alpha_2 = \alpha, \tag{88}
\]

the contributions brought by \( H_{\tau = 1}(i\alpha) \) and \( \langle \exp(-i\beta \varphi) \rangle_{\tau = 1} \) have the same power behavior in \( r \). Furthermore, \( H_{\tau = 1}(i\alpha) \) and the coefficient in front of the second VEV in the short distance

\[\text{[12] I am very grateful to Al. Zamolodchikov for suggesting this check.}\]
expansion of (88) both exhibit a pole at $\alpha = 0$. By analogy with the method used in [51], we require that the divergent contributions compensate each other. This leads to the relation

$$\text{Res}_{\alpha=0} H_{\tau=1}(i\alpha) = \frac{\pi \kappa}{\beta} \langle \exp(-i\beta \varphi) \rangle_{\tau=1}. \quad (89)$$

It is straightforward to check that (44) with $\tau = 1$ and $b \to i\beta$ indeed satisfies this requirement. In particular, for $\tau = 1, 2$ this gives a further support to $\langle \psi \psi e^{-b\varphi} \rangle_\tau$ and the exact bulk free energy proposed for both models.

Appendix B: Numerical values for coupled minimal models

- **Two energy-spin coupled Ising models.**
  It corresponds to $p = 3, p' = 4$ in (53), i.e. $\xi = 3/5$ in (58). In this case $\Phi_{12}$ is the spin operator of the second model in (53) with conformal dimension $\Delta_{12} = 1/16$ whereas $\Phi_{13}$ with conformal dimension $\Delta = 1/2$ is the energy operator. We obtain

$$\langle \Phi_{12}(x) \rangle = 1.281110557...M^{1/8};$$
$$\langle \epsilon(x)\Phi_{12}(x) \rangle = 7.253910604...M^{9/8};$$

where the parameter $\lambda$ is related to the mass of the lowest kink by $\lambda^2 = 0.0766062552...M^{7/4}$.

- **Two energy-energy coupled Ising-tricritical Ising models.**
  The case $p = 4, p' = 5$ in (53) describes a critical Ising model which interacts with a tricritical Ising model through their leading energy density operators. For the second model $\Phi_{12}$ has conformal dimension $\Delta_{12} = 1/10$. It corresponds to $\beta^2 = 2/5$ i.e. $\xi = 2/3$ in (58). This model also contains the sub-leading energy density operator $\Phi_{13}$ with $\Delta_{13} = 3/5$ (“vacancy operator”), two magnetic operators $\Phi_{22}$ with $\Delta_{22} = 3/80$, $\Phi_{21}$ with $\Delta_{21} = 7/16$ and $\Phi_{14}$. We have for instance

$$\langle \Phi_{12}(x) \rangle = 1.495279412...M^{1/5};$$
$$\langle \Phi_{22}(x) \rangle = 1.133076821...M^{3/40};$$
$$\langle \epsilon(x)\Phi_{12}(x) \rangle = 4.478886063...M^{6/5};$$

where the parameter $\lambda$ is related to the mass of the lowest kink by $\lambda^2 = 0.07788969384...M^{8/5}$.

- **Two Ising-$A_5$ RSOS coupled models.**
  It corresponds to the choice $p = 5, p' = 6$ in (53). The $A_5$ RSOS model possesses a primary operator $\Phi_{12}$ with conformal dimension $\Delta_{12} = 1/8$ and $\Phi_{22}$ with $\Delta_{22} = 1/40$. Taking $\xi = 5/7$ in (58), we obtain

$$\langle \Phi_{22}(x) \rangle = 1.081610664...M^{1/20};$$
$$\langle \Phi_{12}(x) \rangle = 1.673156742...M^{1/4};$$
$$\langle \epsilon(x)\Phi_{12}(x) \rangle = 3.478873566...M^{5/4};$$

where the parameter $\lambda$ is related to the mass of the lowest kink by $\lambda^2 = 0.07848322325...M^{3/2}$. 
• Two energy-spin coupled Ising-tricritical Ising models.
For \( p = 4, p' = 5 \) in (54), the Ising energy operator is coupled to the subleading spin operator of the tricritical Ising model. For \( \xi = 2/3 \) in (67) we have for instance \( \langle \Phi_{22}(x) \rangle = 1.2148...M^{3/40} \) where the parameter \( \lambda \) is related to the mass of the lowest kink by \( \lambda^2 = 0.0319842...M^{1/4} \).

• Two energy-energy coupled Ising-3-state Potts models.
The case \( p = 5, p' = 6 \) in (54) describes a critical Ising model coupled to a critical 3-state Potts models through their energy density operator \( \epsilon \) and \( \Phi_{21} \) with conformal dimension \( \Delta_{21} = 2/5 \), respectively. It corresponds to \( \xi = 5/7 \) in (67). The 3-state Potts model also contains the primary operator \( \Phi_{23} \) - the spin operator - with \( \Delta_{23} = 1/15 \). We obtain for instance \( \langle \Phi_{23}(x) \rangle = 1.3378...M^{2/15} \) where the parameter \( \lambda \) is related to the mass of the lowest kink by \( \lambda^2 = 0.0420507...M^{2/5} \).

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