THE LARGEST \((k, \ell)\)-SUM FREE SUBSETS

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Abstract. Let \(\mathcal{M}_{(2,1)}(N)\) denotes the infimum of the size of the largest sum-free subset of any set of \(N\) positive integers. An old conjecture in additive combinatorics asserts that there are a constant \(c = c(2, 1)\) and a function \(\omega(N) \to \infty\) as \(N \to \infty\), such that \(cN + \omega(N) < \mathcal{M}_{(2,1)}(N) < (c + \varepsilon)N\) for any \(\varepsilon > 0\). The constant \(c(2, 1)\) is determined by Eberhard, Green, and Manners, while the existence of \(\omega(N)\) is still widely open.

In this paper, we study the analogue conjecture on \((k, \ell)\)-sum free sets and restricted \((k, \ell)\)-sum free sets. We determine the constant \(c(k, \ell)\) for every \((k, \ell)\), and confirm the conjecture for infinitely many \((k, \ell)\).

1. Introduction

In 1965, Erdős asked the following question \cite{9}. Given an arbitrary sequence \(A\) of \(N\) different positive integers, what is the size of the largest sum-free subsequence of \(A\)? By sum-free we mean that if \(x, y, z \in A\), then \(x + y \neq z\). Let

\[
\mathcal{M}_{(2,1)}(N) = \inf_{A \subseteq \mathbb{N}^{>0}} \max_{S \subseteq A \colon |S| = N} |S|.
\]

Using a beautiful probabilistic argument, Erdős showed that every \(N\)-element set \(A \subseteq \mathbb{N}^{>0}\) contains a sum-free subset of size at least \(N/3\), in other words, \(\mathcal{M}_{(2,1)}(N) \geq N/3\).

It turns out that it is surprisingly hard to improve upon this bound. The result was later improved by Alon and Kleitman \cite{2}, they showed that \(\mathcal{M}_{(2,1)}(N) \geq (N + 1)/3\). Bourgain \cite{5}, using an entirely different Fourier analytic argument, showed that \(\mathcal{M}_{(2,1)}(N) \geq (N + 2)/3\), which is the best lower bound on \(\mathcal{M}_{(2,1)}(N)\) to date. In particular, the following conjecture has been made in a series of papers, see \cite{9, 5, 8, 25} for example.

Conjecture 1. There is a function \(\omega(N) \to \infty\) as \(N \to \infty\), such that

\[
\mathcal{M}_{(2,1)}(N) > \frac{N}{3} + \omega(N).
\]

On the other hand, a recent breakthrough by Eberhard, Green, and Manners \cite{8} proved that \(\mathcal{M}_{(2,1)}(N) = (1/3 + o(1))N\). More precisely, they showed that for every

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\( \varepsilon > 0 \), when \( N \) is large enough, there is a set \( A \subseteq \mathbb{N}^{>0} \) of size \( N \), such that every subset of \( A \) of size at least \( (\varepsilon + 1/3)N \) contains \( x, y, z \) with \( x + y = z \). The result was later generalized by Eberhard \[7\] to \( k \)-sum-free. A set \( A \) is \( k \)-sum-free if for every \( y, x_1, \ldots, x_k \in A \), \( y \neq \sum_{i=1}^{k} x_i \). Eberhard proved that for every \( \varepsilon > 0 \), there is a set \( A \subseteq \mathbb{N}^{>0} \) of size \( N \), such that every subset of \( A \) of size at least \( (\varepsilon + 1/(k + 1))N \) contains a \( k \)-sum. For more background we refer to the survey \[25\].

In this paper, we study the analogue of the Erdős sum-free set problem on the \((k, \ell)\)-sum free sets. Given two positive integers \( k, \ell \) with \( k > \ell \), a set \( A \) is \((k, \ell)\)-sum free if for every \( x_1, \ldots, x_k, y_1, \ldots, y_\ell \in A \), \( \sum_{i=1}^{k} x_i \neq \sum_{j=1}^{\ell} y_j \). For example, using the notation of \((k, \ell)\)-sum free, sum-free is \((2, 1)\)-sum free; \( k \)-sum-free is \((k, 1)\)-sum free.

Let 
\[
\mathcal{M}_{(k, \ell)}(N) = \inf_{A \subseteq \mathbb{N}^{>0}, |A|=N} \max_{S \subseteq A} |S|.
\]

Thus, to determine \( \mathcal{M}_{(k, \ell)}(N) \) is a natural task. We can also make the following conjecture for \((k, \ell)\)-sum free set, which is an analogue of Conjecture \[\[\]

**Conjecture 2.** Let \( k > \ell > 0 \). There is a constant \( c = c(k, \ell) > 0 \), and a function \( \omega(N) \to \infty \) as \( N \to \infty \), such that
\[
cN + \omega(N) < \mathcal{M}_{(k, \ell)}(N) < (c + \varepsilon)N,
\]
for every \( \varepsilon > 0 \).

As we mentioned above, the constant \( c(k, \ell) \) in Conjecture \[\[\] for \((k, \ell) = (2, 1)\) is determined by Eberhard, Green, and Manners \[8\], and for \((k, \ell) = (k, 1)\) is determined by Eberhard \[7\]. The conjecture for \((k, \ell) = (3, 1)\) is confirmed by Bourgain \[5\].

Our first result determines the constant \( c(k, \ell) \) in Conjecture \[\[\] for every \((k, \ell)\), and confirms Conjecture \[\[\] for infinitely many \((k, \ell)\).

**Theorem 1.1.** Let \( k, \ell \) be positive integers and \( k > \ell \). Then the followings hold:

(i) for every \( k, \ell \), we have \( \mathcal{M}_{(k, \ell)}(N) \geq \frac{N}{k+\ell} \).

(ii) when \( k \geq \ell + 2 \), and either \( k + \ell \) is a prime, or \( p = \frac{k+\ell}{\gcd(k+\ell,k-\ell)} \) is a prime such that \( p \mid k-\ell \). Then
\[
\mathcal{M}_{(k, \ell)}(N) \geq \frac{N}{k+\ell} + c \frac{\log N}{\log \log N},
\]
where \( c > 0 \) is an absolute constant.

(iii) for every \( k, \ell \), we have \( \mathcal{M}_{(k, \ell)}(N) = \left(\frac{1}{k+\ell} + o(1)\right)N. \)
The upper bound construction given by Eberhard, Green, and Manners \cite{Eberhard2016} for the 
\((2,1)\)-sum free set actually works in a more general setting: the restricted \((2,1)\)-sum-
free set. A set \(A\) is restricted \((k,\ell)\)-sum free if for every \(k\) distinct elements \(a_1, \ldots, a_k\) 
in \(A\), and \(\ell\) distinct elements \(b_1, \ldots, b_\ell\) in \(A\), we have \(\sum_{i=1}^k a_i \neq \sum_{j=1}^\ell b_j\). Let

\[
\hat{M}(k,\ell)(N) = \inf_{A \subseteq \mathbb{N}^{\geq 0}} \max_{S \subseteq A \atop |S| = N} |S|.
\]

Clearly, we have that \(M(k,\ell)(N) \leq \hat{M}(k,\ell)(N)\). Our next theorem gives us an upper
bound on \(\hat{M}(k,\ell)(N)\) when \(k \leq 2\ell + 1\).

**Theorem 1.2.** Let \(k, \ell\) be positive integers, and \(k \leq 2\ell + 1\). Then

\[
\hat{M}(k,\ell)(N) = \left(\frac{1}{k + \ell} + o(1)\right)N.
\]

**Overview.** The paper is organized as follows. In the next section, we provide some
basic definitions and properties in additive combinatorics, harmonic analysis, and
model theory (or more precisely, nonstandard analysis) used later in the proof. Theorem \ref{thm:main}(i) is proved by using the probabilistic argument introduced by Erdős, 
and some structural results for the \((k,\ell)\)-sum free open set on the torus. We will prove
it in Section 3. Next, we consider Theorem \ref{thm:main}(ii). The special case for \((3,1)\)-sum
free set is proved by Bourgain \cite{Bourgain2008}, but his argument relies heavily on the fact that
a certain term of the Fourier coefficient of the characteristic function is multiplica-
tive, which is not true for the other \((k,\ell)\). Here we introduce a different and more
involved sieve function, which can sieve out integers containing only small prime
factors. We will discuss it in details in Section 4. In Sections 5 and 6, we prove
Theorem \ref{thm:main}(iii). The proof goes by showing that the constructions given by Eber-
hard \cite{Eberhard2016} for \((k,1)\)-sum free sets, the Følner sequence, is still the correct construction
for the other \((k,\ell)\)-sum free sets. The new ingredients contain structural results for
the large infinite \((k,\ell)\)-sum free sets, which can be viewed as a generalization of the
Luczak–Schoen Theorem \ref{thm:luczak}. We will prove Theorem \ref{thm:main}(ii) in Section 7. In Section 8,
we make some concluding remarks, and pose some open problems.

2. Preliminaries

We use standard definitions and notation in additive combinatorics as given in
\cite{Green2014}. Throughout the paper, let \(p\) be a prime, and let \(m, n, N\) ranging over positive
integers. Given \(a, b, N \in \mathbb{N}\) and \(a < b\), let \([a, b] := [a, b] \cap \mathbb{N}\), and let \([N] := [1, N]\).
We use the standard Vinogradov notation. That is, \(f \ll g\) means \(f = O(g)\), and
\(f \asymp g\) if \(f \ll g\) and \(f \gg g\).

Given \(A, B \subseteq \mathbb{Z}\), we write

\[
A + B := \{a + b \mid a \in A, b \in B\}, \quad \text{and} \quad AB := \{ab \mid a \in A, b \in B\}.
\]
When $A = \{x\}$, we simply write $x + B := \{x\} + B$ and $x \cdot B := \{x\} B$. Given $A \subseteq \mathbb{Z}$, let

$$kA := \{a_1 + \cdots + a_k \mid a_1, \ldots, a_k \in A\},$$

for integer $k \geq 2$. For example, $2 \cdot \mathbb{N}$ denotes the set of even natural numbers, while $2\mathbb{N}$ denotes $\mathbb{N} + \mathbb{N}$ which is still $\mathbb{N}$. Using this notation, a set $A$ is $(k, \ell)$-sum free if $kA \cap \ell A = \emptyset$.

We also define the restricted sums. Let

$$\hat{A} + B := \{a + b \mid a \in A, b \in B, a \neq b\},$$

$$\hat{k}A := \{a_1 + \cdots + a_k \mid a_1, \ldots, a_k \in A, \text{ all of them are distinct}\}.$$ 

Thus a set $A$ is restricted $(k, \ell)$-sum free if $\hat{k}A \cap \hat{\ell}A = \emptyset$.

Let $f : \mathbb{Z} \to \mathbb{C}$ be a function. Define $\hat{f} : \mathbb{T} \to \mathbb{C}$, where $\mathbb{T} = \mathbb{R} / \mathbb{Z}$ is a torus, and for every $r \in \mathbb{T}$,

$$\hat{f}(r) = \sum_x f(x)e(-rx),$$

where $e(\theta) = e^{2\pi i \theta}$. By Fourier Inversion, for every $x \in \mathbb{Z}$,

$$f(x) = \int_{\mathbb{T}} \hat{f}(r)e(rx)dr.$$ 

Let $\mu : \mathbb{N}^>0 \to \mathbb{C}$ be the Möbius function. Recall that $\mu$ is supported on the square-free integers, and $\mu(n) = (-1)^{\omega(n)}$ when $n$ is square-free, where $\omega(n)$ counts the number of distinct prime factors of $n$. By Inclusive-Exclusive Principle,

$$\sum_{d|n} \mu(d) = \begin{cases} 0 & \text{if } n > 1, \\ 1 & \text{if } n = 1. \end{cases}$$

We also make use of the weak Littlewood conjecture. The Littlewood problem [13] is to ask that, what is

$$I(N) := \min_{A \subseteq \mathbb{Z}, |A| = N} \int_{\mathbb{T}} \left| \sum_{n \in A} e^{inx} \right| d\mu(x)?$$

The strong Littlewood conjecture asserts that the minimum occurs when $A$ is an arithmetic progression. The conjecture is still widely open. However, the weak Littlewood conjecture, $I(N) \gg \log N$, is resolved by McGehee, Pigno, and Smith [19], and independently by Konyagin [15]. The analogous question in discrete setting is also well studied, we refer to [11, 21, 22] for the interested readers. In this paper, we use the following variant of the weak Littlewood conjecture proved by Bourgain [5].
\textbf{Theorem 2.1 (3)}. Let $\Lambda \subseteq \mathbb{N}^0$ be a finite set, and let $P \geq (\log |\Lambda|)^{100}$. Let $|a_n|, |b_n| \leq 1$ for all $n \in \mathbb{N}^0$. Then there exists $c \in \mathbb{R}^+$ such that
\[
\left\| \sum_{m \in \Lambda} e^{imkx} + \sum_{m \in \Lambda} \sum_{n \in \mathbb{N}^0} \frac{1}{n} (a_n e^{imnx} + b_n e^{-imnx}) \right\|_{L^1(T)} \geq c \log |\Lambda|,
\]
where $\mathcal{N}$ denotes set of positive integers $m$ such that every prime factor of $m$ is at least $P$.

Next, we give some basic definitions in nonstandard analysis which will be used later in the proof. For a more systematic accounts we refer to [3, 6]. Let $S$ denote an infinite set. An ultrafilter $\mathcal{U}$ on $S$ is a collection of subsets of $S$, such that the characteristic function $1_{\mathcal{U}} : 2^S \to \{0, 1\}$ is a finitely additive $\{0, 1\}$-valued probability measure on $S$. An ultrafilter is \textit{principal} if it consists of all sets containing some element $s \in S$. Let $\beta S$ denotes the collection of all ultrafilters. One can embed $S$ into $\beta S$, by mapping $x \in S$ to the principal ultrafilter generated by $x$. By Zorn’s Lemma, $\beta S \setminus S$ is non-empty.

Fix $\mathcal{U} \in \beta \mathbb{N} \setminus \mathbb{N}$, and let $M_n$ be a structure for each $n \in \mathbb{N}$. The ultraproduct $\prod_{n \to \mathcal{U}} M_n$ is a space consists of all ultralimits $\lim_{n \to \mathcal{U}} x_n$ of sequences $x_n$ defined in $M_n$, with $\lim_{n \to \mathcal{U}} x_n = \lim_{n \to \mathcal{U}} y_n$ if two sequences $\{x_n\}$ and $\{y_n\}$ agree on a set in $\mathcal{U}$. Let $\mathbb{R} := \prod_{n \to \mathcal{U}} \mathbb{R}$ be the hyperreal field. Every finite hyperreal number $\xi \in \mathbb{R}$ is infinitely close to a unique real number $r \in \mathbb{R}$, called the standard part of $\xi$. In this case, we use the notation $r = \text{st}(\xi)$.

Given a sequence of finite non-empty sets $F_n$, let $\mu_n(X) = |X \cap F_n|/|F_n|$ be the uniform probability measure. Let $F = \prod_{n \to \mathcal{U}} F_n$ be an ultraproduct. We define the \textit{Loeb measure} [16] $\mu_L$ on $F$ to be the unique probability measure on the $\sigma$-algebra generated by the Boolean algebra of internal subsets of $F$, such that when $X = \prod_{n \to \mathcal{U}} X_n$ is an internal subset of $F$, we have
\[
\mu_L(X) = \text{st} \left( \lim_{n \to \mathcal{U}} \mu_n(X_n) \right).
\]

3. \textit{\textbf{(k, $\ell$)-SUM FREE OPEN SETS IN THE TORUS}}

Let $T = \mathbb{R}/\mathbb{Z}$ be the 1-dimensional torus. In this section, we use $\mu_H$ as the Haar probability measure on $T$.

\textbf{Proposition 3.1.} Let $A \subseteq T$ be a $(k, \ell)$-sum free open set. Then $\mu_H(A) \leq \frac{1}{k+\ell}$.

\textit{Proof.} Since $A$ is $(k, \ell)$-sum free, we have $kA \cap \ell A = \emptyset$. In particular, $\mu_H(kA) + \mu_H(\ell A) \leq 1$. By Kneser’s inequality [14],
\[
(k + \ell) \mu_H(A) \leq \mu_H(kA) + \mu_H(\ell A) \leq 1,
\]
which implies that $\mu_H(A) \leq 1/(k + \ell)$. \hfill $\square$
Next, we construct the largest \((k, \ell)\)-sum free open sets in \(\mathbb{T}\). When \(k - \ell \geq 2\), our construction is asymmetric: this will help us to get a better lower bound on \(\mathcal{M}_{(k, \ell)}(N)\). We will discuss it in details in the next section.

**Lemma 3.2.** Let \(k, \ell\) be positive integers and \(k > \ell\).

(i) when \(k - \ell = 1\), set \(\Omega_1 = \left(\frac{1}{2} - \frac{1}{2(k+\ell)}, \frac{1}{2} + \frac{1}{2(k+\ell)}\right)\).

(ii) when \(k - \ell \geq 2\), set \(\Omega_t = \left(\frac{t-1}{k - \ell} + \frac{\ell}{k^2 - \ell^2}, \frac{t-1}{k - \ell} + \frac{k}{k^2 - \ell^2}\right)\) for every integer \(t \in [k - \ell]\).

Then \(\Omega_t\) is \((k, \ell)\)-sum free for every \(t\).

Lemma 3.2 is easy to verify, and we omit the details here. When \(k = \ell + 1\), the following observation shows that all the possible \((k, \ell)\)-sum free open sets with maximum measure are symmetric. Thus one cannot apply the method used in the next section to improve the lower bound for the case \(k = \ell + 1\).

**Lemma 3.3.** Let \(k = \ell + 1\). Suppose \(A \subseteq \mathbb{T}\) is a maximum \((k, \ell)\)-sum free open set. Then \(A\) is symmetric.

**Proof.** Since \(k = \ell + 1\), \(A\) is \((k, \ell)\)-sum free implies that \((\ell A - \ell A) \cap A = \emptyset\). Hence \(A \subseteq \mathbb{T} \setminus (\ell A - \ell A)\). By Kneser’s inequality,

\[
\mu_H(\mathbb{T} \setminus (\ell A - \ell A)) \leq 1 - 2\mu_H(A).
\]

By Proposition 3.1, \(\mu_H(A) = \frac{1}{2\ell + 1}\). Thus \(A = \mathbb{T} \setminus (\ell A - \ell A)\), and this implies that \(A\) is symmetric. \(\square\)

Using the argument by Erdős [9], Lemma 3.2 is able to give us the following lower bound on the maximum \((k, \ell)\)-sum free subsets of any set of \(N\) integers, which proves Theorem 1.1 (i).

**Proposition 3.4.** Let \(k, \ell\) be positive integers and \(k > \ell\). Then for every \(A \subseteq \mathbb{N}^{>0}\) of size \(N\), \(A\) contains a \((k, \ell)\)-sum free subsets of size at least \(\frac{1}{k + \ell}N\).

**Proof.** Let \(\Omega_t\) be as in Lemma 3.2, and let \(\mathbb{1}_\Omega\) be the characteristic function of \(\Omega\) in \(\mathbb{T}\). Thus by Fubini’s Theorem,

\[
\int_{\mathbb{T}} \sum_{n \in A} \mathbb{1}_\Omega(nx) \, d\mu_H(x) = \sum_{n \in A} \int_{\mathbb{T}} \mathbb{1}_\Omega(nx) \, d\mu_H(x) = \frac{N}{k + \ell}.
\]

Therefore, by Pigeonhole principle, there exists \(x \in \mathbb{T}\) such that

\[
|\{n \in A \mid nx \in \Omega\}| \geq \frac{N}{k + \ell},
\]

finishes the proof. \(\square\)
4. Lower Bound when $k - \ell \geq 2$

Let $k, \ell$ be positive integers, and $k - \ell \geq 2$. Let $I = \{1, \ldots, k - \ell\}$ be the index set. Set

$$\Omega_t = \left( \frac{t - 1}{k - \ell} + \frac{\ell}{k^2 - \ell^2}, \frac{t - 1}{k - \ell} + \frac{k}{k^2 - \ell^2} \right),$$

for every $t \in I$. Let $\mathbb{1}_{\Omega_t}$ be the indicator function of $\Omega_t$. Given $A \subseteq \mathbb{N}^>0$ of size $N$. Let $\mathcal{M}(A)$ be the size of the maximum $(k, \ell)$-sum free subset of $A$. We have

$$\mathcal{M}(A) \geq \max_{x \in \mathbb{T}} \sum_{n \in A} \mathbb{1}_{\Omega_t}(nx),$$

since $\Omega_t$ is $(k, \ell)$-sum free for every $t$. Then

$$\max_{x \in \mathbb{T}} \sum_{n \in A} \mathbb{1}_{\Omega_t}(nx) = \frac{N}{k + \ell} + \max_{x \in \mathbb{T}} \sum_{n \in A} \left( \mathbb{1}_{\Omega_t} - \frac{1}{k + \ell} \right)(nx),$$

for every $t \in I$. We introduce the balanced function $f_t : \mathbb{T} \to \mathbb{C}$ defined by $f_t = \mathbb{1}_{\Omega_t} - \frac{1}{k + \ell}$. By orthogonality of characters we have

$$\hat{f}_t(x) = \begin{cases} 0 & \text{if } x = 0, \\ \mathbb{1}_{\Omega_t}(x) & \text{else.} \end{cases}$$

By Fourier inversion, when $n > 0$,

$$\hat{f}_t(n) = \int_{\mathbb{T}} \mathbb{1}_{\Omega_t}(x)e(-nx)d\mu(x) = \frac{1}{2\pi i n} \left( -e\left( -\frac{(t-1)n}{k - \ell} - \frac{nk}{k^2 - \ell^2} \right) + e\left( -\frac{(t-1)n}{k - \ell} - \frac{n\ell}{k^2 - \ell^2} \right) \right).$$

Therefore,

$$\hat{f}_t(n) = \frac{1}{2\pi n} e\left( \frac{(2t-1)n}{2(k-\ell)} \right) \left( \sin\left( \frac{2kn\pi}{k^2 - \ell^2} - \frac{\pi n}{k - \ell} \right) - \sin\left( \frac{2\ell n\pi}{k^2 - \ell^2} - \frac{\pi n}{k - \ell} \right) \right) = \frac{1}{\pi n} e\left( \frac{(2t-1)n}{2(k-\ell)} \right) \sin\left( \frac{n\pi}{k + \ell} \right).$$

Hence, for every $t \in I$ we have

$$f_t(x) = \sum_{n \neq 0} \hat{f}_t(n)e(nx) = \sum_{n \neq 0} \frac{1}{\pi n} e\left( \frac{(2t-1)n}{2(k-\ell)} \right) \sin\left( \frac{n\pi}{k + \ell} \right)e(nx).$$

Let $F := \sum_{t \in I} f_t$. Observe that

$$\sum_{t=1}^{k-\ell} \sin\left( \frac{(2t-1)n\pi}{k - \ell} \right) = 0.$$
Thus, let $M_1 = k - \ell$, we have
\[
F(x) = \frac{2M_1}{\pi} \sum_{n \geq 1} \frac{1}{n} \sin \left( \frac{n\pi}{k + \ell} \right) \alpha(n) \cos(nx),
\]
where
\[
\alpha(n) = \frac{1}{M_1} \sum_{t \in I} \cos \left( \frac{(2t - 1)n\pi}{k - \ell} \right) = \begin{cases} 
0 & \text{when } (k - \ell) \nmid n, \\
(-1)^s & \text{when } n = (k - \ell)s.
\end{cases}
\]
Therefore, we get
\[
F(x) = \frac{2M_1}{\pi} \sum_{n \geq 1} \frac{-1}{n} \sin \left( \frac{(k - \ell)n\pi}{k + \ell} \right) \cos((k - \ell)nx).
\]
Let $\Phi(n) = (-1)^{n-k+\ell} \sin\left(\frac{(k-\ell)n\pi}{k+\ell}\right)$, we have that $\|\Phi\|_\infty \leq 1$, and $\Phi$ is a periodic function with period $(k + \ell) / \gcd(k + \ell, k - \ell)$. Set $R_F = (k + \ell) / \gcd(k + \ell, k - \ell)$.

Let $I_1 = \{1, \ldots, \left[\frac{k-\ell}{2}\right]\}$ and $I_1 = \left\{\left[\frac{k-\ell}{2}\right] + 1, \ldots, k - \ell\right\}$. Now we consider two cases.

**Case 1. $k - \ell$ is prime.**

Define $G := \sum_{t \in I_1} f_t - \sum_{t \in I_2} f_t$. Similarly, observe that
\[
\sum_{t \in I_1} \cos \left( \frac{(2t - 1)n\pi}{k - \ell} \right) - \sum_{t \in I_2} \cos \left( \frac{(2t - 1)n\pi}{k - \ell} \right) = 0.
\]
Let $M_2 = 2\left[\frac{k-\ell}{2}\right]$, we have
\[
G(x) = \frac{2M_2}{\pi} \sum_{n \geq 1} \frac{1}{n} \sin \left( \frac{n\pi}{k + \ell} \right) \beta(n) \sin(nx),
\]
where
\[
\beta(n) = \frac{1}{M_2} \left( \sum_{t \in I_2} \sin \left( \frac{(2t - 1)n\pi}{k - \ell} \right) - \sum_{t \in I_1} \sin \left( \frac{(2t - 1)n\pi}{k - \ell} \right) \right).
\]
Note that $\beta(n)$ is a periodic function with period $2(k - \ell)$, $0 < \|\beta\|_\infty \leq 1$, $\beta(k - \ell) = 0$, and $\beta(x) = -\beta(x + k - \ell)$.

**Case 2. $k - \ell$ is composite.**

Let $p_1 < \cdots < p_s$ be all the prime factors of $k - \ell$. Note that for every factor $d$ of $k - \ell$, we always have $d \in I_1$. Observe that for every $t_1, t_2 \in I_1$, if $t_1 \neq t_2$, then
\[
\sin \left( \frac{(2t_1 - 1)n\pi}{k - \ell} \right) - \sin \left( \frac{(2t_2 - 1)n\pi}{k - \ell} \right) \neq 0
\]
for some \( n \in \mathbb{N}^{>0} \). Let
\[
M = \left\{ m \in \mathbb{N}^{>0} \mid m \leq \frac{k - \ell}{2}, \text{ and } m \text{ is a multiple of some } p_i \right\}.
\]
Thus \( |M| \leq \frac{(k - \ell)(1 - \Pi_{p \in \mathbb{P}}(1 - 1/p_i))}{2} < |I_1| \).

Now we consider a system of linear equations \( \mathcal{L} \) with \( |M| \) equations and \( |I_1| \) variables \( x_1, \ldots, x_{(k - \ell)/2} \), such that \( \mathcal{L} : \sum_{t \in I_1} x_t \sin\left(\frac{(2t - 1)n\pi}{k - \ell}\right) = 0 \) for every \( m \in M \). Since the rank of the system of linear equations is less than the number of variables, there are \( \lambda_1, \ldots, \lambda_{(k - \ell)/2} \in \mathbb{R} \), such that \( |\lambda_t| \leq 1 \) for every \( t \in I_2 \), not all \( \lambda_t \) are 0, and
\[
(3) \quad \sum_{t \in I_1} \lambda_t \sin\left(\frac{(2t - 1)n\pi}{k - \ell}\right) = 0
\]

For every \( j \in I_2 \), let \( \lambda_j = \lambda_{k - \ell + 1 - j} \). Define \( G := \sum_{t \in I_1} \lambda_t f_t - \sum_{j \in I_2} \lambda_j f_j \). Thus we obtain
\[
G(x) = \frac{2M_2}{\pi} \sum_{n \geq 1} \frac{1}{n} \sin\left(\frac{n\pi}{k + \ell}\right) \beta(n) \sin(nx),
\]

where
\[
\beta(n) = \frac{1}{M_2} \left( \sum_{t \in I_2} \lambda_t \sin\left(\frac{(2t - 1)n\pi}{k - \ell}\right) - \sum_{t \in I_1} \lambda_t \sin\left(\frac{(2t - 1)n\pi}{k - \ell}\right) \right).
\]

Note that \( 0 < \|\beta\|_\infty \leq 1 \), \( \beta(n) \) is a periodic function with period \( 2(k - \ell) \), and by (3), \( \beta(m) = 0 \) when \( m \in M \). By the symmetric property of the \( \sin \) function, this implies \( \beta(m) = 0 \) whenever \( m \) is a multiple of some \( p_i \) for \( i \in [s] \).

In either case, let \( \Psi(n) = \beta(n) \sin\left(\frac{n\pi}{k + \ell}\right) \), and set \( R_G := (k - \ell)(k + \ell) \). Thus \( \|\Psi\|_\infty \leq 1 \), and \( \Psi \) is a periodic function with period \( R_G \) since \( k + \ell \equiv k - \ell \pmod{2} \).

Next we are going to construct the sieve function, which is a crucial ingredient of our argument. Let \( R \) be either \( R_F \) or \( R_G \), and let \( \mathcal{S} = \{1, \ldots, R - 1\} \). For every \( k \in \mathcal{S} \), let \( a_1 = 1 \), and \( a_k \in \mathbb{N}^{>0} \) be the smallest prime \( p \) such that \( p \equiv k^{-1} \pmod{R} \) and \( \gcd(p, R) = 1 \). If such \( a_k \) does not exist (when \( \gcd(k, R) > 1 \)), we simply define \( a_k = \infty \). Let \( Q \) be the product of all finite \( a_k \), and let \( q \) be a prime such that \( \gcd(q, Q) = 1 \) and \( q \equiv Q^{-1} \pmod{R} \). Let \( \mathcal{M} \) be the set containing the square-free integers generated by all the finite \( a_k \) and \( q \). For every \( x \in \mathbb{N}^{>0} \), let \( \|x\|_R \in \mathcal{S} \cup \{0\} \) such that \( \|x\|_R \equiv x \pmod{R} \). Let \( P \) be a prime, and \( P \sim (\log N)^{100} \). Let \( \mu \) be the Möbius function.

We are now going to analyze the property of \( F \) and \( G \). The computation contains three steps. In the first step, we sieve out integers which do not contain \((Qq)^{\phi(R) - 2}\) as a factor, where \( \phi \) is the Euler totient function. In step two, we sieve out integers containing prime factors from \( \mathcal{M} \) with multiplicity at least \( \phi(R) - 1 \). Now, we may
still have some integers containing only small prime factors, all of which have the form 
\( u(qQ)^{\phi(R)-2} \), and \( \gcd(u, qQ) = 1 \). In the final step, we sieve out integers containing 
small prime factors which are coprime with \( qQR \). Note that for integers of the form 
\( u(qQ)^{\phi(R)-2} \), we may have \( u \not\equiv 1 \pmod{R} \), which will spoil the computation. In 
order to fix this, for each \( u \) with \( \gcd(u, R) = 1 \), we pair with an integer \( a_{\|u\|_R} \). An 
important fact is, all the remaining integers of the form \( u(qQ)^{\phi(R)-2} \) are divisible 
by some \( ua_{\|u\|_R} \) except for the first term \( (qQ)^{\phi(R)-2} \), and this is the reason why we 
cannot simply apply step three without the first two steps. Now, except for the 
term \( (qQ)^{\phi(R)-2} \), all the remaining integers either containing at least one large prime 
factor, or only containing prime factors which are also factors of \( R \). 

Note that in the computation, we actually require that \( \phi(R) - 2 \geq 1 \), which is not 
always true when \( R_F \) or \( R_G \) is small. In that case, we just simply let \( R = R_F^j \) for 
some positive integer \( j \) such that \( \phi(R) \) is large, and all the computations still work. 
Hence, we may assume \( \phi(R_F) \) and \( \phi(R_G) \) are not small for convenience. 

More precisely, let \( R = R_F \), for \( F(x) \) we have the following.

\[
\sum_{\substack{\ell | \| P \| \\ \gcd(t, RQq)=1}} \frac{\mu(t)}{ta_{\|t\|_R}} \sum_{k \in \mathbb{M}} \frac{\mu(k)}{k^{\phi(R)-1}} \frac{1}{(Qq)^{\phi(R)-2}} \sum_{m \in \mathbb{A}} F(ta_{\|t\|_R}k^{\phi(R)-1}(Qq)^{\phi(R)-2}m) \\
= 2M_1 \sum_{\substack{\ell | \| P \| \\ \gcd(t, RQq)=1}} \frac{\mu(t)}{ta_{\|t\|_R}} \sum_{k \in \mathbb{M}} \frac{\mu(k)}{k^{\phi(R)-1}} \sum_{m \in \mathbb{A}} (Qq)^{\phi(R)-2} m \frac{\Phi(n)}{n} \cos((k - \ell)ta_{\|t\|_R}k^{\phi(R)-1}mn) \\
= 2M_1 \sum_{\substack{\ell | \| P \| \\ \gcd(t, RQq)=1}} \frac{\mu(t)}{ta_{\|t\|_R}} \sum_{m \in \mathbb{A}} (Qq)^{\phi(R)-2} m \frac{\Phi(n)}{n} \cos((k - \ell)ta_{\|t\|_R}mn) \sum_{k \in \mathbb{M}} \frac{\mu(k)}{k^{\phi(R)-1}} \\
= 2M_1 \sum_{\substack{\ell | \| P \| \\ \gcd(t, RQq)=1}} \frac{\mu(t)}{ta_{\|t\|_R}} \sum_{m \in \mathbb{A}} \sum_{p \in \mathbb{P}} (Qq)^{\phi(R)-2} m \frac{\Phi(n)}{n} \cos((k - \ell)ta_{\|t\|_R}mn) \\
= 2M_1 \sum_{\substack{\ell | \| P \| \\ \gcd(t, RQq)=1}} \frac{\mu(t)}{ta_{\|t\|_R}} \sum_{m \in \mathbb{A}} \sum_{n \in \mathbb{N}} (Qq)^{\phi(R)-2} m \frac{\Phi(n)}{n} \cos((k - \ell)mn) \sum_{ta_{\|t\|_R} \mid n} \mu(t)
\]
the prime factors of $R$ and (7) still holds.

$M$ factors of $n = 2^\ell$ 
fore,
where (8) still holds.

Equality (4) follows from the fact that $R$ from the basic property of the M"obius function. The set $N$ denotes the set of integers $n > P$, or the only prime factors of $P$, or the only prime factors of $n$ that are at least $P$, or the only prime factors of $n$ that smaller than $P$ are the prime factors of $R$. Since $R = R_G = \frac{k + \ell}{\gcd(k + \ell, k - \ell)}$ is prime, and $\Phi(n) = 0$ when $R \mid n$. Thus we can let

$$\mathcal{N} = \{n : \forall p \mid n, \text{ if } p \neq 1, \text{ then } p > P\}.$$  

and (7) still holds.

We now use the similar argument on $G(x)$. Let $R = R_G$, and we define $Q'$, $q'$ and $\mathcal{M}_G$ in a similar way as we did for $F(x)$. We have

\begin{equation}
\sum_{\gcd(t, RQq') = 1} \frac{\mu(t)}{ta_{\mid t \mid}} \sum_{k \in \mathcal{M}_G} \mu(k) \frac{1}{k^{\phi(R) - 1}} (Q'q')^{\phi(R) - 2} \sum_{m \in A} G(ta_{\mid t \mid}, k^{\phi(R) - 1}(Q'q')^{\phi(R) - 2}mx)
\end{equation}

\begin{equation}
= \frac{2M_1}{\pi} \sum_{m \in A} \sum_{n \in \mathcal{N}_G} \frac{\Psi(n)}{n} \sin(mx) + \frac{2M_1}{\pi} \frac{(q'q')^{\phi(R) - 2}}{\Phi(q'q')} \sum_{m \in A} \sin((q'q')^{\phi(R) - 2}mx),
\end{equation}

where $\mathcal{N}_G$ denotes the set of integers $n > (q'q')^{\phi(R) - 2}$, such that either all the prime factors of $n$ are at least $P$, or the only prime factors of $n$ that smaller than $P$ are the prime factors of $R$. Since $R_G = (k + \ell)(k - \ell)$, and by the assumption, all the prime factors of $k + \ell$ are also the prime factors of $k - \ell$, and the fact that for every prime $t \mid k - \ell$, $\Psi(m) = 0$ whenever $t \mid m$, we can conclude that if we let $\mathcal{N}_G = \mathcal{N}$, (8) still holds.

Let

$$\eta = (Qq)^{\phi(R_G) - 2}, \quad \xi = (Q'q')^{\phi(R_G) - 2}.$$

Let $K = \frac{\eta \xi}{\gcd(\eta, \xi)}$. Now we apply Theorem 2.1 to bound the exponential sums. Therefore,

$$\log N \ll \sum_{m \in A} \left( \cos((k - \ell)Kmx) + i \sin((k - \ell)Kmx) \right)$$
By Merterns’ estimates we get

$$+ \sum_{m \in A} \sum_{n \in N} \frac{1}{n} \left( \frac{\Phi(n) \eta}{\Phi(1)} \cos((k - \ell)Kmnx) + i \frac{\Psi(n) \xi}{\Psi(1)} \sin((k - \ell)Kmnx) \right) \left\|_{L^1(\mathbb{T})} \right.$$ 

$$\ll \left\| \sum_{t \mid P!} \frac{\mu(t)}{t} \sum_{k \in M} \frac{\mu(k)}{k^{\phi(R_F)^{-1}}} \sum_{m \in A} F(ta\|t\|_R F k^{\phi(R_F)^{-1}} Kmx) \right\|_{L^1(\mathbb{T})}$$

$$+ \left\| \sum_{t \mid P!} \frac{\mu(t)}{t} \sum_{k \in M_G} \frac{\mu(k)}{k^{\phi(R_G)^{-1}}} \sum_{m \in A} G(ta\|t\|_R G k^{\phi(R_G)^{-1}} Kmx) \right\|_{L^1(\mathbb{T})}$$

$$\ll \prod_{p \leq P} \left( 1 + \frac{1}{p} \right) \left( \sum_{t = 1}^{\frac{k - \ell}{P}} \left\| \sum_{m \in A} f_t(mx) \right\|_{L^1(\mathbb{T})} + \sum_{t \in I_1 \cup I_2} \lambda_t \left\| \sum_{m \in A} f_t(mx) \right\|_{L^1(\mathbb{T})} \right).$$

By Merterns’ estimates we get

$$\prod_{p \leq P} \left( 1 + \frac{1}{p} \right) \ll \log P \approx \log \log N.$$

Hence there is $t \in I$ such that $\left\| \sum_{m \in A} f_t(mx) \right\|_{L^1(\mathbb{T})} \gg \frac{\log N}{\log \log N}$.

Note that for every $t \in I$,

$$\int_T \sum_{n \in A} f_t(nx) \, dx = 0.$$

Thus we have

$$\max_{x \in \mathbb{T}} \sum_{n \in A} f_t(nx) \geq \frac{1}{2} \left\| \sum_{n \in A} f_t(nx) \right\|_{L^1(\mathbb{T})}.$$

Together with (1) and (2), we get

$$\mathcal{M}(A) - \frac{N}{k + \ell} \gg \frac{\log N}{\log \log N},$$

and this proves Theorem 1.1 (ii).

5. Structures of the infinite $(k, \ell)$-sum free sets

Given $A \subseteq \mathbb{N}^{>0}$, the upper density of $A$ is

$$\overline{d}(A) = \limsup_{N \to \infty} \frac{|A \cap [N]|}{N}.$$
We also define the upper density on multiples of $A$ by
\[ \overline{d}(A) = \limsup_{N \to \infty} \limsup_{n \to \infty} \frac{|A \cap (N! \cdot [n])|}{n}. \]

In this section, we will prove the following theorem.

**Theorem 5.1.** Suppose $A \subseteq \mathbb{N}^{>0}$, and $A$ is $(k, \ell)$-sum free. Then $\overline{d}(A) \leq \frac{1}{k+\ell}$.

For this, we will need three lemmas. The first lemma says that if a $(k, \ell)$-sum free set $A$ contains a certain long arithmetic progression, then the upper density of $A$ is bounded.

**Lemma 5.2.** Let $A \subseteq \mathbb{N}^{>0}$ be a $(k, \ell)$-sum free set. Let $x, s, d, m$ be positive integers, such that $s \in \ell A - (k - 1)A$, $x + d \cdot [m] \subseteq A$, and $s$ is in the coset $x + d \cdot \mathbb{Z}$. Then
\[ \overline{d}(A) \leq \frac{m + k + \ell - 2}{(k + \ell)m + 2(k + \ell - 2)}. \]

**Proof.** Since $s \in \ell A - (k - 1)A$ and $A$ is $(k, \ell)$-sum free, we have $s \notin A$. We will only consider $s \leq x$, and the case when $s \geq x + m$ follows from the same proof. Since $x + d \cdot [m] \subseteq A$, then $(x + d \cdot [m]) \cap (\ell A - (k - 1)A) = \emptyset$. Thus, there is $s_0 \in x + d \cdot \mathbb{Z}$, such that $s_0 \in \ell A - (k - 1)A$, and
\[ (s_0 + d \cdot [m]) \cap (\ell A - (k - 1)A) = \emptyset. \]

Let $s_0 = \sum_{i=1}^{\ell} a_i - \sum_{j=1}^{k-1} b_j$, where $a_i, b_j \in A$ for every $1 \leq i \leq \ell$ and $1 \leq j \leq k - 1$. Let $B \subseteq A$ such that
\[ B := \{ b \in A \mid (b + d \cdot [m]) \cap A \neq \emptyset \}. \]

Set $a_0 = b_0 = 0$. Given integers $1 \leq u \leq k - 1$ and $2 \leq v \leq \ell$, let
\[ \mathcal{C}(u) = B + \sum_{j=1}^{k-u} b_j + (u - 1)a_1, \quad \mathcal{D}(v) = B + \sum_{j=0}^{\ell-v} a_j + \sum_{i=0}^{v-1} b_i, \]
and $\mathcal{C}(k) = A + (k - 1)a_1$, $\mathcal{D}(1) = A + \sum_{j=1}^{\ell-1} a_j$. Let $\mathcal{F} = \{ \mathcal{C}(u), \mathcal{D}(v) \mid u \in [k], v \in [\ell] \}$ be the collection of all $\mathcal{C}(u)$ and $\mathcal{D}(v)$.

**Claim 1.** Elements in $\mathcal{F}$ are pairwise disjoint.

**Proof of Claim 1.** Observe that for every $u \in [k]$ and $v \in [\ell]$, $\mathcal{C}(u) \cap \mathcal{D}(v) = \emptyset$. Otherwise, we will get $kA \cap \ell A \neq \emptyset$, contradicts that $A$ is $(k, \ell)$-sum free. Let $u_1, u_2 \in [k]$ and $u_1 < u_2$. Suppose that $\mathcal{C}(u_1) \cap \mathcal{C}(u_2) \neq \emptyset$. Then there exist $y_1 \in B$ and $y_2 \in A$, such that
\[ y_1 + \sum_{j=k-u_2+1}^{k-u_1} b_j = y_2 + (u_2 - u_1)a_1. \]
Then
\[ s_0 = \sum_{i=1}^{\ell} a_i - \sum_{j=1}^{k-1} b_j \]
\[ = y_1 + \sum_{i=2}^{\ell} a_i - y_2 - (u_2 - u_1 - 1)a_1 - \sum_{j \in [1,k-u_2] \cup [k-u_1+1,k-1]} b_j. \]

Since \( y_1 \in B \), thus there is \( r \in [m] \) such that \( y_1 + rd \in A \). This implies \( s_0 + rd \in \ell A - (k - 1)A \), contradicts \( (9) \).

Suppose \( D(v_1) \cap D(v_2) \neq \emptyset \) for some \( v_1, v_2 \in \ell \) and \( v_1 < v_2 \). Similarly, there exist \( y_1 \in A \) and \( y_2 \in B \), such that
\[ y_1 + \sum_{j=\ell-v_2+1}^{\ell-v_1} a_j = y_2 + \sum_{i=v_1}^{v_2-1} b_i. \]

Let \( c_0 = 0 \), and let \( c_1, \ldots, c_{v_2-v_1-1} \in A \) if \( v_2 > v_1 + 1 \). Therefore
\[ s_0 = y_2 + \sum_{j \in [0,\ell-v_2] \cup [\ell-v_1+1,\ell]} a_j + \sum_{t=0}^{v_2-v_1-1} c_t - y_1 - \sum_{i=0}^{v_2-v_1-1} b_i - \sum_{t=0}^{v_2-v_1-1} c_t. \]

Observe \( y_2 \in B \) implies that there is \( r \in [m] \), such that \( y_2 + rd \in A \). Hence \( s_0 + rd \in \ell A - (k - 1)A \), which contradicts \( (9) \).

By Claim 1, we obtain
\[ (k + \ell - 2)d(B) + 2d(A) \leq 1. \]

On the other hand, let \( \mathcal{N}(t) = A \setminus B + td \) for every \( t \in [m] \), and let
\[ \mathcal{G} = \left\{ A, A - (k - 1)x + \sum_{i=1}^{\ell-1} a_i \mathcal{N}(t) \mid t \in [m] \right\}. \]

Claim 2. Elements in \( \mathcal{G} \) are pairwise disjoint.

Proof of Claim 2. Suppose there are \( u, v \in [m], u < v \), such that \( \mathcal{N}(u) \cap \mathcal{N}(v) \neq \emptyset \). Thus we have \( c \in A \setminus B \) such that \( c_1 + (u - v)d \in A \), and this contradicts the assumption of \( B \). Same conclusion holds if \( A \cap \mathcal{N}(u) \neq \emptyset \). Observe that if \( A \cap (A - (k - 1)x + \sum_{i=1}^{\ell-1} a_i) \neq \emptyset \), it will contradict that \( A \) is \((k, \ell)\)-sum free. Finally, we assume that there are \( c_1, c_2 \in A, u \in [m] \) such that
\[ c_1 + ud = c_2 - (k - 1)x + \sum_{i=1}^{\ell-1} a_i. \]
Thus, $c_1 + x + ud + (k - 2)x = c_2 + \sum_{i=1}^{\ell-1} a_i$. Since $x + d \cdot [m] \subseteq A$, this contradicts $A$ is $(k, \ell)$-sum free.

Thus, by Claim 2 we get

$$(m + 2)d(A) - md(B) \leq 1.$$ 

Together with (10), this finishes the proof. \[\Box\]

The next lemma is a finite version of the Szemerédi Theorem [23], and we will use it to find the arithmetic progression in Lemma 5.2.

**Lemma 5.3** ([23]). For every $\varepsilon > 0$ and $m \in \mathbb{N}^>$, there is $L = L(\varepsilon, m) > 0$ such that every set $A \subseteq \mathbb{N}^>$ with $d(A) > \varepsilon$, there exist $x \in \mathbb{N}$, $d < L$, and $x + d \cdot [m] \subseteq A$.

Our final lemma says that a $(k, \ell)$-sum free set $A$ with large upper density should be periodic. This structural result can be viewed as a generalization of the Luczak–Schoen Theorem [18].

**Lemma 5.4.** Let $\varepsilon > 0$. Then there is $D > 0$ such that the following holds. Let $A \subseteq \mathbb{N}^>$ be a $(k, \ell)$-sum free set, and $\overline{d}(A) > \frac{1}{k+\ell} + \varepsilon$. Then $A$ is contained in a periodic $(k, \ell)$-sum free set with period $D$.

**Proof.** We pick $m \in \mathbb{N}^>$ such that

$$(11) \quad \frac{m + k + \ell - 2}{(k + \ell)m + 2(k + \ell - 2)} < \frac{1}{k + \ell} + \varepsilon.$$ 

Let $L = L(\varepsilon, m)$ be as in Lemma 5.3. Let $D = L!$. Suppose the lemma fails. Let $C \subseteq \mathbb{N}^>$ be a periodic set with period $D$, consists of all positive integers in every coset $a + D \cdot \mathbb{Z}$ for $a \in A$. Thus $C$ is not $(k, \ell)$-sum free. This means, there are $a_1, \ldots, a_\ell$ and $b_1, \ldots, b_k$ in $C$ such that $\sum_{i=1}^{\ell} a_i = \sum_{j=1}^{k} b_j$. Let $P$ be the “$(k, \ell)$-sum free” part of $C$. That is,

$$P = C \setminus (\ell C - (k - 1) C).$$

Set $a_0 = b_0 = 0$. For every $u \in [k]$ and $v \in [\ell]$, let

$$\mathcal{M}(u) = P + \sum_{j=0}^{k-u} b_j + (u - 1) a_1, \quad \mathcal{N}(v) = P + \sum_{i=0}^{\ell-v} a_i + (v - 1) b_1.$$ 

Let $\mathcal{F}$ be the collection of all $\mathcal{M}(u)$ and $\mathcal{N}(v)$.

**Claim 3.** Elements in $\mathcal{F}$ are pairwise disjoint.
Proof of Claim \(3\). Observe that for every \(u \in [k]\) and \(v \in [\ell]\), \(\mathcal{M}(u) \cap \mathcal{N}(v) = \emptyset\). Otherwise there are \(p_1, p_2 \in P\), such that

\[
p_1 = p_2 + \sum_{i=0}^{\ell-v} a_i + (v-1)b - \sum_{j=0}^{k-u} b_j - (u-1)a_1 \in \ell C - (k-1)C,
\]

contradicts the assumption of \(P\). Now, suppose \(u_1, u_2 \in [k]\), \(u_1 < u_2\), such that \(\mathcal{M}(u_1) \cap \mathcal{M}(u_2) \neq \emptyset\). The case that \(\mathcal{N}(v_1) \cap \mathcal{N}(v_2) \neq \emptyset\) can be proved in the same way. Thus, there exist \(p_1, p_2 \in P\), such that

\[
p_1 + \sum_{j=k-u_2+1}^{k-u_1} b_j = p_2 + (u_2 - u_1)a_1.
\]

This implies

\[
0 = \sum_{j=1}^{k} b_j - \sum_{i=1}^{\ell} a_i = p_2 + (u_2 - u_1 - 1)a_1 + \sum_{j \in [0,k-u_2] \cup [k-u_1+1,k]} b_j - \sum_{i=2}^{k-1} a_i - p_1,
\]

hence \(P \cap (\ell C - (k-1)C) \neq \emptyset\), contradiction. \(\Box\)

By Claim \(3\) we obtain that \(\overline{d}(P) \leq \frac{1}{k+\ell}\). This means, \(\overline{d}(A \setminus P) \geq \varepsilon\). By Lemma \(5.3\) \(A \setminus P\) contains a progression \(x + d \cdot [m]\), and \(d < L\). By the way we construct \(P\), there are \(s_1, \ldots, s_\ell\) and \(t_1, \ldots, t_{k-1}\) in \(C\) such that

\[
x + dm = \sum_{i=1}^{\ell} s_i - \sum_{j=1}^{k-1} t_j.
\]

Hence there are \(e_1, \ldots, e_\ell\) and \(f_1, \ldots, f_{k-1}\) in \(A\), such that for every \(i \in [\ell]\) and \(j \in [k-1]\), we have that \(e_i \in s_i + D \cdot \mathbb{Z}\), and \(f_j \in t_j + D \cdot \mathbb{Z}\). Let \(s = \sum_{i=1}^{\ell} e_i - \sum_{j=1}^{k-1} f_j\), thus \(s \in \ell A - (k-1)A\), and \(s \in x + D \cdot \mathbb{Z}\). Since \(d \mid D\), we have \(s \in x + d \cdot \mathbb{Z}\). By Lemma \(5.2\) we have that

\[
\overline{d}(A) \leq \frac{m + k + \ell - 2}{(k + \ell)m + 2(k + \ell - 2)},
\]

and this contradicts (II). \(\square\)

Now we can prove the main result of this section.

Proof of Theorem \(5.7\). Let \(A/N! := \{a \mid aN! \in A\}\). Thus \(\overline{d}(A) > 0\) implies that \(A/N!\) contains a multiple of every natural number. In particular, \(A/N!\) is not contained in a periodic \((k, \ell)\)-sum free set. By Lemma \(5.4\) \(\overline{d}(A/N!) \leq \frac{1}{k+\ell}\). Observe that \(\overline{d}(A) = \lim \sup_{N \to \infty} \overline{d}(A/N!)\), thus \(\overline{d}(A) \leq \frac{1}{k+\ell}\). \(\square\)
6. Upper bound constructions

Recall a Følner sequence in \((\mathbb{N}, \cdot)\) is any sequence \(\Phi : m \mapsto \Phi_m\) of finite non-empty subsets of \(\mathbb{N}\), such that for every \(a \in \mathbb{N}\),

\[
\lim_{m \to \infty} \frac{|\Phi_m \triangle (a \cdot \Phi_m)|}{|\Phi_m|} = 0.
\]

Følner sequence has been used as some good constructions in many additive combinatorics problems, see [20, 4] for example. In this section, we will show that the sets in Følner sequence will never have large \((k, \ell)\)-sum free subsets. In fact, we will prove the following theorem.

**Theorem 6.1.** Let \(\Phi = \{\Phi_m\}\) be a Følner sequence in \((\mathbb{N}, \cdot)\). Suppose there are infinitely many \(m\) such that \(\Phi_m\) has a \((k, \ell)\)-sum free set of size at least \(\delta|\Phi_m|\) for some positive real number \(\delta \leq 1\). Then there exists a \((k, \ell)\)-sum free set \(A \subseteq \mathbb{N}\) such that \(\tilde{d}(A) \geq \delta\).

Theorem 1.1 (iii) follows easily from Theorem 6.1 and Theorem 5.1.

**Proof of Theorem 6.1.** By passing to a subsequence, we may assume for every \(\Phi_m \in \Phi\), there is a \((k, \ell)\)-sum free set \(\phi_m \subseteq \Phi_m\), such that \(|\phi_m|/|\Phi_m| \geq \delta\). Let \(\beta \mathbb{N}\) be the collection of ultrafilters, and let \(\mathcal{U} \in \beta \mathbb{N} \setminus \mathbb{N}\) be a non-principal ultrafilter. Let \(*\mathbb{Z} = \prod_{m \to \mathcal{U}} \mathbb{Z}\) be the ultrapower of \(\mathbb{Z}\). Let \(\Sigma\) be the Loeb \(\sigma\)-algebra on \(*\mathbb{Z}\). Let \(\mu_m\) be the Loeb measure induced by \(\mu_m\), where \(\mu_m(X) = |X \cap \Phi_m|/|\Phi_m|\) for every \(X \subseteq \mathbb{Z}\). Let \(\phi = \prod_{m \to \mathcal{U}} \phi_m\) be the internal set. Then by Los’s Theorem, \(\phi\) is \((k, \ell)\)-sum free, and

\[
\mu_L(\phi) = \text{st} \left( \lim_{m \to \mathcal{U}} \mu_m(\phi_m) \right) \geq \delta.
\]

**Claim 4.** For every \(a \in \mathbb{N}\), the map \(x \mapsto ax\) is \(\Sigma\)-measurable and \(\mu_L\)-preserving.

**Proof of Claim 4.** Note that \(x \mapsto ax\) sends internal sets to internal sets, thus it is \(\Sigma\)-measurable. For every \(X \subseteq \mathbb{Z}\), since

\[
\mu_m(X) - \mu_m(a \cdot X) = \frac{|X \cap \Phi_m| - |(a \cdot X) \cap \Phi_m|}{|\Phi_m|} \leq \frac{|(a \cdot \Phi_m) \triangle \Phi_m|}{|\Phi_m|} \to 0
\]

as \(m \to \infty\), it preserves the Loeb measure \(\mu_L\).

Now we are able to apply the probabilistic argument used in the proof of Proposition 3.1 on the set \(\phi\). For every \(x \in *\mathbb{Z}\), let \(A_x := \{a \in \mathbb{N} \mid ax \in \phi\}\). Thus \(A_x\) is \((k, \ell)\)-sum free. By Claim 4 \(\tilde{d}(A_x)\) is \(\Sigma\)-measurable on \(x\). Suppose \(x\) is chosen
uniformly at random with respect to the measure $\mu_L$. By Fatou’s Lemma,

$$\mathbb{E}(\tilde{d}(A_x)) \geq \limsup_{N \to \infty} \limsup_{n \to \infty} \mathbb{E} \left( \frac{|A_x \cap (N! \cdot [n])|}{n} \right)$$

$$= \limsup_{N \to \infty} \limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \mathbb{P}(jN!x \in \phi).$$

By Claim 4, we have

$$\mathbb{E}(\tilde{d}(A_x)) \geq \limsup_{N \to \infty} \limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \mathbb{P}(x \in \phi) = \mu_L(\phi) \geq \delta.$$

Thus by Pigeonhole Principle, there exists a set $A_x \subseteq \mathbb{N}$ for some $x \in \ast \mathbb{Z}$ such that $\tilde{d}(A_x) \geq \delta$. □

7. Restricted $(k, \ell)$-sum free sets

In this section, we prove Theorem 1.2. Since restricted $(k, \ell)$-sum free can be expressed by first order formula, once we prove the conclusion in Theorem 5.1 also works for restricted $(k, \ell)$-sum free sets, Theorem 1.2 follows by using the same proof in Theorem 6.1. We first consider the analogue of Lemma 5.2 for restricted $(k, \ell)$-sum free sets. The similar argument also works here, with a different and more involved constructions of sets $C(u), D(v)$, and $N(t)$, and a more careful analysis. These new constructions will lead a different structure for the large infinite restricted $(k, \ell)$-sum free sets in Lemma 7.2, compared to the non-restricted setting.

Lemma 7.1. Let $k, \ell$ be positive integers, and $\ell < k \leq 2\ell + 1$. Suppose $A \subseteq \mathbb{N}^+ > 0$ be a restricted $(k, \ell)$-sum free set. Define $W \subseteq \mathbb{N}^+ > 0$, satisfies that for every $w \in W$, there are $\ell$ distinct elements $y_1, \ldots, y_{\ell} \in A$, and $k - 1$ distinct elements $z_1, \ldots, z_{k-1} \in A$, such that $w \neq z_i$ for $i \in [k-1]$, and $w = \sum_{j=1}^{\ell} y_j - \sum_{i=1}^{k-1} z_i$. Let $x, s, d, m$ be integers, such that $s \in W$, $m > k + \ell$, $x + d \cdot [m] \subseteq A$, and $s$ is in the coset $x + d \cdot \mathbb{Z}$. Then

$$\bar{d}(A) \leq \frac{m - 2}{(k + \ell)(m - k - \ell) + 2(k + \ell - 2)}.$$

Proof. $s \in W$ implies that $s \notin A$ since $A$ is restricted $(k, \ell)$-sum free. We only consider the case when $s < x$. Since $A \cap W = \emptyset$, there is $s_0 \in x + d \cdot \mathbb{Z}$ such that $s_0 \in W$ and $(s_0 + d \cdot [m]) \cap W = \emptyset$. Thus there are $\ell$ distinct elements $a_1, \ldots, a_{\ell} \in A$, and $k - 1$ distinct elements $b_1, \ldots, b_{k-1} \in A$, $s_0 \neq b_j$ for every $j \in [k - 1]$, and $s_0 = \sum_{i=1}^{\ell} a_i - \sum_{j=1}^{k-1} b_j$. Let $E$ consists of $k - 1$ distinct elements $e_1, \ldots, e_{k-1} \in A$,
and all of them are disjoint from \( \{a_i\}_{i=1}^{\ell}, \{b_j\}_{j=1}^{k-1}, \{s_0\} \) and \( s_0 + d \cdot [m] \). Let

\[
A' = A \setminus \left( \bigcup_{i=1}^{\ell} \{a_i\} \cup \bigcup_{j=1}^{k-1} \{b_j\} \cup E \cup \{s_0\} \cup (s_0 + d \cdot [m]) \right).
\]

Observe that

\[
(s_0 + d \cdot [m]) \cap \{b_j\}_{j=1}^{k-1} = \emptyset,
\]

since \( b_j \in W \) for every \( j \in [k-1] \). Let \( m' = m - k - \ell \), we claim that

\[
(s_0 + d \cdot [m']) \cap \{a_i\}_{i=1}^{\ell} = \emptyset.
\]

Otherwise, suppose there is \( r \in [m'] \) such that \( s_0 + rd = a_t \) for some \( t \in [\ell] \). Then

\[
x' + \sum_{j=1}^{k-1} b_j = x' + rd + \sum_{j=1, j \neq t}^{\ell} a_j.
\]

By taking \( x' \in x + d \cdot [0, m-r] \), then both \( x' \) and \( x' + rd \) are in \( A \). Since \( m-r \geq k+\ell \), there is \( \alpha \in [0, m-r] \) such that \( x + \alpha d \notin \{b_j\}_{j=1}^{k-1} \), and \( x + (\alpha + r)d \notin \{a_i\}_{i=1}^{\ell} \). This contradicts that \( A \) is restricted \((k, \ell)\)-sum free.

Let \( B = \{b \in A' \mid (b + d \cdot [m']) \cap A \neq \emptyset\} \), and let

\[
B' = B \setminus \left( \bigcup_{i=1}^{\ell} \{a_i\} \cup E \right) - d \cdot [m']
\]

Let \( c_0 = 0, c_i = a_i \) when \( i \in [\ell] \), and \( c_j = a_{j-\ell} \) when \( j \in [\ell+1, k-1] \). For \( u \in [k-1] \) and \( v \in [2, \ell] \), let

\[
C(u) = B' + \sum_{j=1}^{k-u} b_j + \sum_{i=0}^{u-1} c_i, \quad D(v) = B' + \sum_{i=0}^{v-1} a_i + \sum_{j=0}^{u-1} b_j,
\]

and \( C(k) = A' + \sum_{i=0}^{k-1} c_i, D(1) = A' + \sum_{i=0}^{\ell-1} a_i \). Let \( \mathcal{F} \) consists of all \( C(u) \) and \( D(v) \), then Claim \( \square \) still holds. In fact, suppose there are \( u_1, u_2 \in [k], u_1 < u_2 \) such that \( C(u_1) \cap C(u_2) \neq \emptyset \) (the case when \( D(v_1) \cap D(v_2) \neq \emptyset \) is simpler). Then there exist \( y_1 \in B', y_2 \in A' \) such that

\[
y_1 + \sum_{j=k-u_2+1}^{k-u_1} b_j = y_2 + \sum_{i=u_1}^{u_2-1} c_i.
\]

Let \( e_0 = 0, e_1, \ldots, e_{u_2-u_1-1} \in E \) if \( u_2 > u_1 + 1 \). If \( u_2 \leq \ell \), we have

\[
s_0 = y_1 + \sum_{i \in [0, u_1-1] \cup [u_2, \ell]} a_i + \sum_{t=0}^{u_2-u_1-1} e_t - y_2 - \sum_{j \in [0, k-u_2] \cup [k-u_1+1, k-1]} b_j - \sum_{t=0}^{u_2-u_1-1} e_t.
\]
If \( u_1 \geq \ell + 1 \), we get
\[
s_0 = y_1 + \sum_{i \in [0,u_1-\ell-1]} a_i + \sum_{t=0}^{u_2-u_1-1} e_t - y_2 - \sum_{j \in [0,k-u_2] \cup [k-u_1+1,k-1]} b_j - \sum_{t=0}^{u_2-u_1-1} e_t.
\]

If \( u_1 \leq \ell \), \( u_2 \geq \ell + 1 \), and \( u_2 - u_1 + 1 \leq \ell \),
\[
s_0 = y_1 + \sum_{i \in [u_2-\ell,u_1-1]} a_i + \sum_{t=0}^{u_2-u_1-1} e_t - y_2 - \sum_{j \in [0,k-u_2] \cup [k-u_1+1,k-1]} b_j - \sum_{t=0}^{u_2-u_1-1} e_t.
\]

If \( u_1 \leq \ell \), \( u_2 \geq \ell + 1 \), and \( u_2 - u_1 \geq \ell \). Let \( e_0, e_1, \ldots, e_{\ell-1} \in \mathcal{E} \) if \( \ell > 1 \). Thus
\[
s_0 = y_1 + \sum_{t=0}^{\ell-1} e_t - y_2 - \sum_{j \in [0,k-u_2] \cup [k-u_1+1,k-1]} b_j - \sum_{t=0}^{\ell-1} e_t - \sum_{i=u_1}^{u_2-1-\ell} a_i.
\]

Note that \( k \leq 2\ell + 1 \) implies \( u_2 - 1 - \ell \leq \ell \).

In any case, since \( y_1 \in B \), by (12), (13), and (14), there is \( r \in [m'] \) such that \( s_0 + rd \in W \), which contradicts the assumption of \( s_0 \). Therefore,
\[
(k + \ell - 2)d(B) + 2d(A) \leq 1,
\]
since \( d(A') = d(A) \) and \( d(B') = d(B) \).

We also modify the construction of \( \mathcal{N}(t) \) in a similar way. For every \( t \in [m'] \), let \( \mathcal{N}(t) = A' \setminus B + td \). Let \( e_0 = 0 \), and \( e_1, \ldots, e_{k-2} \in \mathcal{E} \) if \( k \geq 3 \). Let \( A'' = A' \setminus (x + d \cdot [m']) \).

Define
\[
\mathcal{G} = \left\{ \mathcal{N}(t), A', A'' + \sum_{i=1}^{\ell-1} a_i - x - \sum_{j=0}^{k-2} e_j \mid t \in [m'] \right\}
\]

Then by using the similar argument, it is easy to see that Claim 2 still holds. We omit the details here. We have
\[
(m - k - \ell + 2)d(A) - (m - k - \ell)d(B) \leq 1,
\]
since \( d(A'') = d(A) \). Together with (15), finishes the proof. \( \square \)

Next, we consider the analogue of Lemma 5.4 for restricted \((k, \ell)\)-sum free sets. The structure here is slightly different from the \((k, \ell)\)-sum free sets.

**Lemma 7.2.** Let \( \varepsilon > 0 \) and let \( k, \ell \) be positive integers with \( \ell < k \leq 2\ell + 1 \). Then there is \( D > 0 \) such that the following holds. Let \( A \subseteq \mathbb{N}^+ \) be a restricted \((k, \ell)\)-sum free set, and \( d(A) > \frac{1}{k + \ell} + \varepsilon \). Then after removing at most \( D(2k + \ell) \) elements from \( A \), it is contained in a periodic restricted \((k, \ell)\)-sum free set with period \( D \).
We pick \( m > k + \ell \) such that
\[
\frac{m - 2}{(k + \ell)(m - k - \ell) + 2(k + \ell - 2)} < \frac{1}{k + \ell} + \varepsilon.
\]

Let \( L = L(\varepsilon, m) \) be as in Lemma \[5.3\] and let \( D = L! \). We consider the partition of \( \mathbb{N} \) into cosets:
\[
\mathbb{N} = \bigcup_{x \in [D]} x + D \cdot \mathbb{N}.
\]

For every \( x \in [D] \), let \( \mathbb{N}_x = x + D \cdot \mathbb{N} \), and \( A_x = A \cap \mathbb{N}_x \). Let \( A' \) be a subset of \( A \), obtained by removing \( A_x \) from \( A \) when \( |A_x| < 2k + \ell \). Hence \( \overline{d}(A') = \overline{d}(A) \). Next, we are going to show that \( A' \) is contained in a periodic restricted \((k, \ell)\)-sum free set with period \( D \). Suppose this is not the case. Let
\[
C = \left( \bigcup_{a \in A'} a + D \cdot \mathbb{Z} \right) \cap \mathbb{N}^0.
\]

Thus \( C \) is not restricted \((k, \ell)\)-sum free. This means, there are \( \ell \) distinct elements \( a_1, \ldots, a_\ell \in C \) and \( k \) distinct elements \( b_1, \ldots, b_k \in C \), such that \( \sum_{i=1}^\ell a_i = \sum_{j=1}^k b_j \).

Let \( P \) be the \("(k, \ell)\)-sum free" part of \( C \), that is, for every \( w \in P \), every \( k - 1 \) distinct elements \( y_1, \ldots, y_{k-1} \in C \setminus \{w\} \), and every \( \ell \) distinct elements \( z_1, \ldots, z_\ell \in C \), we have \( w + \sum_{i=1}^{k-1} y_i \neq \sum_{j=1}^{\ell} z_j \). Let \( e_0 = 0 \), and let \( E \) consists of \( k - 1 \) distinct elements \( e_1, \ldots, e_{k-1} \in C \), such that \( E \) is disjoint from \( \{a_i\}_{i=1}^\ell \) and \( \{b_j\}_{j=1}^k \).

\[
P' = P \setminus \left( \bigcup_{i=1}^\ell \{a_i\} \cup \bigcup_{j=1}^k \{b_j\} \cup E \right).
\]

Set \( a_0 = b_0 = c_0 = 0 \). Let \( c_t = a_t \) when \( t \in [k] \), and \( c_t = a_{t-\ell} \) when \( t \in [\ell + 1, k-1] \). For every \( u \in [k] \) and \( v \in [\ell] \), let
\[
\mathcal{M}(u) = P' + \sum_{j=0}^{k-u} b_j + \sum_{t=0}^{u-1} c_t, \quad \mathcal{N}(v) = P' + \sum_{i=0}^{\ell-v} a_i + \sum_{t=0}^{v-1} b_t.
\]

Let \( \mathcal{F} \) be the collection of all \( \mathcal{M}(u) \) and \( \mathcal{N}(v) \). Then elements in \( \mathcal{F} \) are pairwise disjoint. Otherwise, suppose there are \( u_1, u_2 \in [k], u_1 < u_2 \) such that \( \mathcal{M}(u_1) \cap \mathcal{M}(u_2) \neq \emptyset \) (the case when \( \mathcal{N}(v_1) \cap \mathcal{N}(v_2) \neq \emptyset \) is simpler). Thus, there are \( y_1, y_2 \in P' \), such that
\[
y_1 + \sum_{k-u_2+1}^{k-u_1} b_j = y_2 + \sum_{t=u_1}^{u_2-1} c_t.
\]
Let $e_1, \ldots, e_{u_2-u_1-1} \in \mathcal{E}$ if $u_2 > u_1 + 1$. If $u_2 \leq \ell$, we have

$$0 = \sum_{i=1}^{\ell} a_i - \sum_{j=1}^{k} b_j$$

$$= y_1 + \sum_{i \in [0,u_1-1] \cup [u_2,\ell]} a_i + \sum_{t=0}^{u_2-u_1-1} e_t - y_2 - \sum_{j \in [0,k-u_2] \cup [k-u_1+1,k]} b_j - \sum_{t=0}^{u_2-u_1-1} e_t.$$  

If $u_1 \geq \ell + 1$, we have

$$0 = y_1 + \sum_{i \in [0,u_1-1-\ell] \cup [u_2-\ell,\ell]} a_i + \sum_{t=0}^{u_2-u_1-1} e_t - y_2 - \sum_{j \in [0,k-u_2] \cup [k-u_1+1,k]} b_j - \sum_{t=0}^{u_2-u_1-1} e_t.$$  

If $u_1 \leq \ell$, $u_2 \geq \ell + 1$, and $\ell \geq u_2 - u_1$, we get

$$0 = y_1 + \sum_{i=0}^{u_1-1} a_i + \sum_{t=0}^{u_2-u_1-1} e_t - y_2 - \sum_{j \in [0,k-u_2] \cup [k-u_1+1,k]} b_j - \sum_{t=0}^{u_2-u_1-1} e_t.$$  

If $u_1 \leq \ell$, $u_2 \geq \ell + 1$, and $\ell < u_2 - u_1$. Let $e_1, \ldots, e_{\ell-1} \in \mathcal{E}$ if $\ell > 1$, we get

$$0 = y_1 + \sum_{t=0}^{\ell-1} e_t - y_2 - \sum_{j \in [0,k-u_2] \cup [k-u_1+1,k]} b_j - \sum_{i=1}^{u_1-1} a_i - \sum_{t=0}^{\ell-1} e_t.$$  

In any case, we get a contradiction with the assumption of $P'$ and the fact that $y_2 \in P'$. Therefore,

$$\overline{d}(P) \leq \frac{1}{k+\ell},$$

since $\overline{d}(P') = \overline{d}(P)$. This means, $\overline{d}(A' \setminus P) \geq \varepsilon$. By Lemma 5.3, $A' \setminus P$ contains a progression $x + d \cdot [m]$, and $d < L$. By the way we construct $P$, there are $\ell$ distinct elements $s_1, \ldots, s_\ell \in C$ and $k-1$ distinct elements $t_1, \ldots, t_{k-1}$ in $C \setminus \{x+m\}$ such that

$$x + m = \sum_{i=1}^{\ell} s_i - \sum_{j=1}^{k-1} t_j.$$  

By the way we construct $A'$, for every $r \in [D]$, if $|A' \cap N_r| > 0$, then $|A' \cap N_r| \geq 2k+\ell$. Thus, there are $\ell$ distinct elements $\alpha_1, \ldots, \alpha_\ell \in A'$ and $k-1$ distinct elements $\beta_1, \ldots, \beta_{k-1} \in A'$, such that for every $i \in [\ell]$ and $j \in [k-1]$, we have that $\alpha_i \in s_i + D \cdot \mathbb{Z}$, and $\beta_j \in t_j + D \cdot \mathbb{Z}$. Let $s = \sum_{i=1}^{\ell} \alpha_i - \sum_{j=1}^{k-1} \beta_j$. Note that $|A' \cap N_r| \geq 2k+\ell$ also implies that there is $r' \in [\ell]$, and $M \subseteq N^{>0}$, $|M| \geq k$, such
that
\[
\alpha_{r'} + D \cdot M \subseteq A', \quad (\alpha_{r'} + D \cdot M) \cap \bigcup_{i=1}^{\ell} \{\alpha_i\} = \emptyset.
\]
Thus if \( s \cap \{\beta_j\}_{j=1}^{k-1} \neq \emptyset \), then by changing \( \alpha_{r'} \) by \( \alpha_{r'} + nD \) for some \( n \in M \), one can make \( s + nD \cap \{\beta_j\}_{j=1}^{k-1} = \emptyset \). Since \( d \mid D \), we have \( s \in x + d \cdot \mathbb{Z} \). By Lemma 5.2, we have that
\[
\overline{d}(A) \leq \frac{m-2}{(k+\ell)(m-k-\ell) + 2(k+\ell - 2)},
\]
and this contradicts (10). \( \square \)

Let \( A \) be a restricted \((k, \ell)\)-sum free set, and let \( A' \) be a subset of \( A \) obtained by removing finitely many elements from \( A \). Observe that, if \( A' \) is contained in a periodic restricted \((k, \ell)\)-sum free set, then \( A \) cannot contain a multiple of every natural number. Thus, using the same proof in Theorem 5.1, we conclude that \( \overline{d}(A) \leq \frac{1}{k+\ell} \) if \( A \) is restricted \((k, \ell)\)-sum free.

8. Concluding Remarks

In this paper, we study \( \mathcal{M}_{(k, \ell)}(N) \) and \( \hat{\mathcal{M}}_{(k, \ell)}(N) \). In particular, we prove that Conjecture 2 is true for infinitely many \((k, \ell)\). While solving Conjecture 2 might not be a realistic target at the moment, the following conjecture for the case when \( k - \ell \geq 2 \) might be feasible.

Conjecture 3. Let \( k, \ell \) be positive integers and \( k \geq \ell + 2 \). Then there is a function \( \omega(N) \rightarrow \infty \) as \( N \rightarrow \infty \), such that
\[
\mathcal{M}_{(k, \ell)}(N) \geq \frac{N}{k+\ell} + \omega(N).
\]

A \((k, \ell)\)-sum free set is a set forbidding a linear equation \( \sum_{i=1}^{\ell} x_i = \sum_{j=1}^{k} y_j \). Another interesting direction is to consider the analogue problem on sets forbidding a system of linear equations. One of the most interesting problems along this line might be forbidding the projective cubes. Given a multiset \( S = \{s_1, \ldots, s_d\} \), a \( d \)-dimensional projective cube generated by \( S \) is
\[
\square^d(S) := \left\{ \sum_{i \in I} s_i \mid \emptyset \neq I \subseteq [d] \right\}.
\]
A set is \( \square^d \)-free if it does not contain any \( d \)-dimensional projective cubes as its subsets. Extremal properties of projective cubes have a vast literature, see e.g. [11, 10, 12, 17]. The problem on forbidding \( d \)-dimensional projective cubes can be viewed as a generalization of sum-free sets in another direction, since a sum-free set is also a \( \square^2 \)-free set. Thus, the following problem is worthwhile to pursue.
The largest \((k, \ell)\)-sum free subsets

**Question 4.** Let \(d \geq 3\) be an integer. Define

\[
\mathcal{M}_{\square d}(N) := \inf_{A \subseteq \mathbb{N}, |A| = N} \max_{B \subseteq A} |B|. 
\]

Determine \(\mathcal{M}_{\square d}(N)\).

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