Pedal Curves of Tangent Surfaces of Biharmonic \( B \)-General Helices according to Bishop Frame in Heisenberg Group Heis\(^ 3 \)

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ABSTRACT: In this paper, we study pedal curves of tangent surfaces of biharmonic \( B \)-general helices according to Bishop frame in the Heisenberg group Heis\(^ 3 \). We give necessary and sufficient conditions for \( B \)-general helices to be biharmonic according to Bishop frame. We characterize this pedal curves in the Heisenberg group Heis\(^ 3 \). Additionally, we illustrate our main theorem.

Key Words: Biharmonic curve, Bishop frame, Heisenberg group, Pedal curve.

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1. Introduction

Developable surfaces, which can be developed onto a plane without stretching and tearing, form a subset of ruled surfaces, which can be generated by sweeping a line through space. There are three types of developable surfaces: cones, cylinders (including planes) and tangent surfaces formed by the tangents of a space curve, which is called the cuspidal edge of this surface, [3].

In this paper, we study pedal curves of tangent surfaces of biharmonic \( B \)-general helices according to Bishop frame in the Heisenberg group Heis\(^ 3 \). We give necessary and sufficient conditions for \( B \)-general helices to be biharmonic according to Bishop frame. We characterize this pedal curves in the Heisenberg group Heis\(^ 3 \). Additionally, we illustrate our main theorem.

2. The Heisenberg Group Heis\(^ 3 \)

Heisenberg group Heis\(^ 3 \) can be seen as the space \( \mathbb{R}^3 \) endowed with the following multiplication:

\[
(\mathbf{\xi}, \mathbf{\eta}, \mathbf{\nu})(x, y, z) = (\mathbf{\xi} + x\mathbf{\eta} + y\mathbf{\nu} + z - \frac{1}{2}x\mathbf{\eta} + \frac{1}{2}y\mathbf{\nu})
\] (2.1)

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Heis$^3$ is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric $g$ is given by

$$g = dx^2 + dy^2 + (dz - xdy)^2.$$  

The Lie algebra of Heis$^3$ has an orthonormal basis

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z},$$  

for which we have the Lie products

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = [e_3, e_1] = 0$$

with

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$  

We obtain

$$\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = 0,$$

$$\nabla_{e_1} e_2 = -\nabla_{e_2} e_1 = \frac{1}{2} e_3,$$

$$\nabla_{e_1} e_3 = \nabla_{e_2} e_1 = -\frac{1}{2} e_2,$$

$$\nabla_{e_2} e_3 = \nabla_{e_3} e_2 = \frac{1}{2} e_1.$$  

3. Biharmonic $B$-General Helices with Bishop Frame In The Heisenberg Group Heis$^3$

Let $\gamma : I \rightarrow \text{Heis}^3$ be a non geodesic curve on the Heisenberg group Heis$^3$ parametrized by arc length. Let $\{T, N, B\}$ be the Frenet frame fields tangent to the Heisenberg group Heis$^3$ along $\gamma$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\nabla_T T = k_1 M_1 + k_2 M_2,$$

$$\nabla_T M_1 = -k_1 T,$$

$$\nabla_T M_2 = -k_2 T,$$  

where

$$g(T, T) = 1, \quad g(M_1, M_1) = 1, \quad g(M_2, M_2) = 1,$$

$$g(T, M_1) = g(T, M_2) = g(M_1, M_2) = 0.$$  

Here, we shall call the set $\{T, M_1, M_2\}$ as Bishop trihedra, $k_1$ and $k_2$ as Bishop curvatures.
With respect to the orthonormal basis \( \{ e_1, e_2, e_3 \} \) we can write
\[
T = T^1 e_1 + T^2 e_2 + T^3 e_3, \\
M_1 = M_1^1 e_1 + M_1^2 e_2 + M_1^3 e_3, \\
M_2 = M_2^1 e_1 + M_2^2 e_2 + M_2^3 e_3.
\]

To separate a general helix according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as \( \mathfrak{B} \)-general helix.

**Theorem 3.1.** Let \( \gamma_B : I \rightarrow \text{Heis}^3 \) be a unit speed biharmonic \( \mathfrak{B} \)-general helix with non-zero natural curvatures. Then the parametric equation of \( \gamma_B \) are
\[
x_B(s) = \frac{\sin \theta}{(k_1^2 + k_2^2) - \cos \theta} \sin \left[ \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right)^{\frac{1}{2}} s + \zeta_0 \right] + \zeta_2,
\]
\[
y_B(s) = -\frac{\sin \theta}{(k_1^2 + k_2^2) - \cos \theta} \cos \left[ \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right)^{\frac{1}{2}} s + \zeta_0 \right] + \zeta_3, \tag{3.2}
\]
\[
z_B(s) = (\cos \theta) s + \frac{\sin^2 \theta}{(k_1^2 + k_2^2) - \cos \theta} \frac{\zeta_1}{2} - \frac{\sin 2\left[ \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right)^{\frac{1}{2}} s + \zeta_0 \right]}{4\left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right)^{\frac{1}{2}}} - \frac{\zeta_1 \sin \theta}{(k_1^2 + k_2^2) - \cos \theta} \cos \left[ \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right)^{\frac{1}{2}} s + \zeta_0 \right] + \zeta_4,
\]
where \( \zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4 \) are constants of integration, \cite{[10]}.}

**4. Pedal Curves In The Heisenberg Group Heis^3**

The purpose of this section is to study pedal curves of tangent developable of biharmonic \( \mathfrak{B} \)-general helices with Bishop frame in the Heisenberg group Heis^3.

The tangent surface of \( \gamma_B \) is a ruled surface
\[
\mathcal{R}(s, u) = \gamma_B(s) + uT(s). \tag{4.1}
\]

Let \( \mathcal{R} \) be a developable ruled surface given by equation (4.1) in Heis^3. Since the tangent plane is constant along rulings of \( \mathcal{R} \), it is clear that the pedal of \( \mathcal{R} \) is a curve. Thus, for the pedal of \( \mathcal{R} \), we can write
\[
\bar{\gamma}(s) = \gamma_B(s) + \Pi(s) T(s),
\]
where \( \Pi(s) \) is the distance between the points \( \gamma(s) \) and \( \bar{\gamma}(s) \).
Theorem 4.1. Let $\gamma_0 : I \rightarrow \text{Heis}^3$ be a unit speed biharmonic $B$-general helix and $\hat{\gamma}$ its pedal curve. Then, the parametric equations of pedal curve are

\[
\begin{align*}
x_\gamma (s) & = \frac{\sin \theta}{(k_1 + k_2 \sin^2 \theta - \cos \theta)^{\frac{1}{2}} \sin \frac{1}{2} \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2}} + \Pi(s) \sin \theta \cos \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0] + \zeta_2, \\
y_\gamma (s) & = -\frac{\sin \theta}{(k_1 + k_2 \sin^2 \theta - \cos \theta)^{\frac{1}{2}} \cos \frac{1}{2} \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2}} + \Pi(s) \sin \theta \sin \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0] + \zeta_3, \\
z_\gamma (s) & = (\cos \theta) s + \frac{\sin^2 \theta}{(k_1 + k_2 \sin^2 \theta - \cos \theta)^{\frac{1}{2}}} - \frac{\zeta_1 \sin \theta}{(k_1 + k_2 \sin^2 \theta - \cos \theta)^{\frac{1}{2}}} \cos \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0] + \Pi(s) \cos \theta \\
& \quad + \Pi(s) \sin^2 \theta \sin \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0] + \Pi(s) \zeta_4 \sin \theta \sin \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0] + \zeta_4,
\end{align*}
\]

where $\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4$ are constants of integration and $\Pi(s)$ is the distance between the points $\gamma(s)$ and $\hat{\gamma}(s)$.

Proof: From orthonormal basis (2.2) and (3.8), we obtain

\[
\begin{align*}
T & = (\sin \theta \cos \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0] \sin \theta \sin \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0], \\
& = \cos \theta + \frac{\sin^2 \theta}{(k_1 + k_2 \sin^2 \theta - \cos \theta)^{\frac{1}{2}}} \sin \frac{1}{2} \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0] (4.3)
\end{align*}
\]

where $\zeta_1$ is constant of integration.

Using above equation, we have (4.2), the theorem is proved.

\[\square\]

Theorem 4.2. Let $\gamma_0 : I \rightarrow \text{Heis}^3$ be a unit speed biharmonic $B$-general helix.
and \( \bar{\gamma} \) its pedal curve. Then the equation of pedal curve is

\[
\bar{\gamma}(s) = \left[ \frac{\sin \theta}{(k_1^2 + k_2^2) \sin^2 \theta - \cos \theta} \right] \sin \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right)^{\frac{1}{2}} s + \zeta_0
\]

\[
+ \Pi(s) \sin \theta \cos \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right)^{\frac{1}{2}} s + \zeta_0 \right] + \zeta_1 \mathbf{e}_1
\]

\[
+ \left[ -\frac{\sin \theta}{(k_1^2 + k_2^2) \sin^2 \theta - \cos \theta} \right] \sin \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right)^{\frac{1}{2}} s + \zeta_0 \right] + \zeta_2 \mathbf{e}_2
\]

\[
+ \Pi(s) \sin \theta \sin \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right)^{\frac{1}{2}} s + \zeta_0 \right] + \zeta_3 \mathbf{e}_2
\]

\[
+ \left[ -\frac{\sin \theta}{(k_1^2 + k_2^2) \sin^2 \theta - \cos \theta} \right] \sin \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right)^{\frac{1}{2}} s + \zeta_0 \right] + \zeta_4 \mathbf{e}_3
\]

\[
+ (\cos \theta) s + \frac{\sin^2 \theta}{(k_1^2 + k_2^2) \sin^2 \theta - \cos \theta} \left( \frac{s}{2} - \frac{\sin 2\left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right)^{\frac{1}{2}} s + \zeta_0 }{4(k_1^2 + k_2^2) \sin^2 \theta - \cos \theta} \right)
\]

\[
- \zeta_1 \sin \theta \cos \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right)^{\frac{1}{2}} s + \zeta_0 \right] + \Pi(s) \cos \theta + \zeta_4 \mathbf{e}_3,
\]

where \( \zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4 \) are constants of integration and \( \Pi(s) \) is the distance between the points \( \gamma(s) \) and \( \bar{\gamma}(s) \).

**Proof:** From section 3, we immediately arrive at

\[
T = \sin \theta \cos \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right)^{\frac{1}{2}} s + \zeta_0 \right] + \sin \theta \sin \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right)^{\frac{1}{2}} s + \zeta_0 \right] + \mathbf{e}_2
\]

\[
+ \cos \theta \mathbf{e}_3.
\]

(4.5)

Using above equation and theorem we easily have (4.4). \( \square \)
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