LOCAL WELLPOSEDNESS FOR THE NON-RESISTIVE MHD EQUATIONS IN OPTIMAL SOBOLEV SPACES

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Abstract. In this paper, we consider the Cauchy problem of the non-resistive magnetohydrodynamics equations in $\mathbb{R}^d$ for $d = 2, 3$. We show that the system is locally well-posed in $H^{s-1} \times H^s$ by establishing a new commutator estimate and utilizing the heat smooth effect in Chemin-Lerner frame. The space $H^{s-1} \times H^s$ is optimal in Sobolev spaces for the local well-posedness of the system in the scaling sense. Therefore, we improve the results in [8].

1. Introduction

In this paper, we consider the Cauchy problem of the following incompressible non-resistive magnetohydrodynamics equations (NMHD) for $d = 2, 3$:

\[
\begin{aligned}
\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla \pi &= b \cdot \nabla b, \\
\partial_t b + u \cdot \nabla b &= b \cdot \nabla u, \\
\text{div} u &= \text{div} b = 0,
\end{aligned}
\]

where vector fields $u = (u^1, u^2, \ldots, u^d)$, $b = (b^1, b^2, \ldots, b^d)$ are the fluid velocity and the magnetic field respectively, the scalar function $\pi$ is the fluid pressure, and $\nu > 0$ is the viscosity coefficient. System (1.1) describes the dynamics of magnetic field in electrically conducting fluid. It has been extensively investigated by mathematicians in the last few decades. We can refer to [3, 4, 5, 7, 10, 11, 13].

In this paper, we concern the problem of local well-posedness in optimal Sobole spaces. Fefferman et al. obtained local-in-time existence of strong solutions to (1.1) in $\mathbb{R}^d$, $d = 2, 3$ with $(u_0, b_0) \in H^s \times H^s$ in [9] and $(u_0, b_0) \in H^{s-1+\varepsilon} \times H^s$ in [8]. The aim of this paper is to remove $\varepsilon$ in [8] and thus obtain local well-posedness in the optimal Sobolev space based on the natural scaling of system (1.1). The main difficulty comes from nonlinear terms in the transport equation due to the lack of the diffusion of space variable. However, by applying the frequency localization method and some harmonic analysis techniques, establishing a new commutator estimate and utilizing sufficiently the heat smooth effect in Chemin-Lerner Besov spaces, we overcome the disadvantage. Now, let’s state our main result as follows:

**Theorem 1.1.** Assume that the initial data that $u_0 \in H^{s-1}$, $b_0 \in H^s$, $s > d/2$, $d = 2, 3$. Then there exists a strictly positive maximum time $T_*$ such that a unique solution $(u, b)$
of the system (1.1) exists in the space \( C([0, T_*]; H^{s-1} \times H^s) \). Moreover, the solution \( u \in L^2([0, T_*); B_{2,2}^s) \cap L^1([0, T_*); B_{2,2}^{s+1}) \).

The paper is organized as follows. In Section 2, we recall Littlewood-Paley theory and give some properties of Besov space. In Section 3, we prove the local existence and uniqueness of the solution of system (1.1).

**Notation.** We denote by \( \langle \cdot, \cdot \rangle \) the inner product on \( \mathbb{R}^d \). Given a Banach space \( X \), we denote its norm by \( \| \cdot \|_X \). The uniform constant \( C \) may be different on different lines in this paper.

## 2. Preliminaries

In this section, we firstly recall some Littlewood-Paley theory. One can refer to [1] [12] for more details.

Let \( \varphi, \chi \in S(\mathbb{R}^d) \) be two smooth radial functions with values in \([0, 1]\), \( \varphi \) is supported in the annulus \( \{ \xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{3}{2} \} \), \( \chi \) is supported in the unit ball \( B(0,1) \) of \( \mathbb{R}^d \). They satisfy with

\[
\chi(\xi) + \sum_{j \geq 0} \varphi_j(\xi) = 1 \quad \text{for} \quad \xi \in \mathbb{R}^d,
\]

\[
\sum_{j \in \mathbb{Z}} \varphi_j(\xi) = 1 \quad \text{for} \quad \xi \in \mathbb{R}^d \setminus \{0\},
\]

where we denote \( \varphi_j(\xi) = \varphi(2^{-j}\xi) \).

Let us denote the Fourier transform on \( \mathbb{R}^d \) by \( \mathcal{F} \) and write \( h = \mathcal{F}^{-1} \varphi \) and \( \tilde{h} = \mathcal{F}^{-1} \chi \).

The homogeneous localization operator \( \dot{\Delta}_j \) and the homogeneous low-frequency cut-off operators \( \dot{S}_j \) are defined for all \( j \in \mathbb{Z} \) by

\[
\dot{\Delta}_j u = \varphi_j(D) u = 2^d \int_{\mathbb{R}^d} h(2^j y) u(x - y) \, dy,
\]

\[
\dot{S}_j u = \sum_{k \leq j-1} \dot{\Delta}_k u = 2^d \int_{\mathbb{R}^d} \tilde{h}(2^j y) u(x - y) \, dy,
\]

and the inhomogeneous localization operator:

\[
\Delta_j u = \varphi_j(D) u = \mathcal{F}^{-1}(\varphi_j(\xi) \hat{u}), \quad \text{if} \quad j \geq 0;
\]

\[
\Delta_{-1} u = \mathcal{F}^{-1}(\chi(\xi) \hat{u}); \quad \Delta_j u = 0, \quad \text{if} \quad j \leq -2.
\]

The inhomogeneous low-frequency cut-off operators \( S_j \) are defined for all \( j \in \mathbb{Z} \) by

\[
S_j u = \sum_{k \leq j-1} \Delta_k u = \mathcal{F}^{-1}(\chi(2^{-j} \xi) \hat{u}).
\]

One can easily verify that

\[
\dot{\Delta}_j \dot{\Delta}_{j'} u = 0 \quad \text{if} \quad |j - j'| \geq 2,
\]

\[
\dot{\Delta}_j (\dot{S}_{j'-1} u \dot{\Delta}_{j'} u) = 0 \quad \text{if} \quad |j - j'| \geq 5.
\]

Next, we recall Bony’s decomposition from [1]:

**Definition 2.1.** For any \( u, v \in S'/\mathcal{P}(\mathbb{R}^d) \), \( uv \) has the homogeneous Bony paraproduct decomposition:

\[
uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v),
\]
where
\[ \tilde{T}_u v := \sum_{j \leq k-2} \Delta_j u \Delta_k v = \sum_{j} \tilde{S}_{j-1} u \Delta_j v \quad \text{and} \quad \hat{R}(u, v) =: \sum_{|j-j'| \leq 1} \Delta_j u \Delta_j v. \]

For any \( u, v \in S' \), \( uv \) has the inhomogeneous Bony paraproduct decomposition:
\[ uv = T_u v + T_v u + R(u, v), \]
where
\[ T_u v := \sum_{j \leq k-2} \Delta_j u \Delta_k v = \sum_{j} S_{j-1} u \Lambda_j v \quad \text{and} \quad R(u, v) := \sum_{|j-j'| \leq 1} \Delta_j u \Delta_j v. \]

We will use repeatedly the following classical Bernstein-Type lemma

**Lemma 2.1.** [1] Let \( \mathcal{C} \) be an annulus, \( \mathcal{B} \) a ball, and \((p, q) \in [1, \infty]^2 \) with \( 1 \leq p \leq q \). Then for any vector filed \( f \in L^p(\mathbb{R}^d) \), there exist a constant \( C > 0 \), independent of \( f \) and \( \lambda \), such that for any \( k \in \mathbb{Z} \),
\[ \| D^k f \|_{L^q} \leq C^{k+1} \lambda^{k+\frac{d}{4}+\frac{1}{q}} \| f \|_{L^p} \quad \text{if} \quad \text{supp} \hat{f} \subset \lambda \mathcal{B}, \]
\[ C^{-k-1} \lambda^k \| f \|_{L^p} \leq \| D^k u \|_{L^p} \leq C^{k+1} \lambda^k \| f \|_{L^p} \quad \text{if} \quad \text{supp} \hat{f} \subset \lambda \mathcal{C}. \]

The definition of Besov spaces is as follows:

**Definition 2.2.** Let \( s \in \mathbb{R}, 1 \leq p, r \leq \infty \). \( S' \) be the space of tempered distributions and \( \mathcal{P} \) is the set of all polynomials. The homogeneous Besov space are defined as follows
\[ \dot{B}^s_{p, r}(\mathbb{R}^d) := \{ u \in S'(\mathbb{R}^d)/\mathcal{P} : \| u \|_{\dot{B}^s_{p, r}(\mathbb{R}^d)} < \infty \}, \]
where
\[ \| u \|_{\dot{B}^s_{p, r}(\mathbb{R}^d)} = (\sum_{j \in \mathbb{Z}} 2^{js} \| \Delta_j u \|_{L^r(\mathbb{R}^d)})^{\frac{1}{r}}. \]

The inhomogeneous Besov space are defined as follows
\[ \dot{B}^s_{p, r}(\mathbb{R}^d) := \{ u \in S'(\mathbb{R}^d) : \| u \|_{\dot{B}^s_{p, r}(\mathbb{R}^d)} < \infty \}, \]
where
\[ \| u \|_{\dot{B}^s_{p, r}(\mathbb{R}^d)} = (\sum_{j \geq -1} 2^{js} \| \Delta_j u \|_{L^r(\mathbb{R}^d)})^{\frac{1}{r}}. \]

**Remarks.** When \( p = r = 2 \), let us point out that for any \( s \in \mathbb{R} \), \( \dot{B}^s_{2, 2} \) and \( B^s_{2, 2} \) are the usual Sobolev space \( \dot{H}^s \) and \( H^s \), respectively. In addition, \( B^s_{2, 2} = \dot{B}^s_{2, 2} \cap L^2 \) with \( \delta \geq 0 \).

We also need use the Chemin-Lerner type homogeneous Besov space (see [1][2]):

**Definition 2.3.** Let \( s \in \mathbb{R}, 1 \leq p, q, r \leq \infty \), and \( T \in (0, \infty] \). the time homogeneous Besov space \( \dot{L}^s_T \dot{B}^s_{p, r}(\mathbb{R}^d) \) are defined as follows
\[ \| u \|_{\dot{L}^s_T \dot{B}^s_{p, r}(\mathbb{R}^d)} := (\sum_{j \in \mathbb{Z}} 2^{jsr} \| \Delta_j u \|_{L^r_T(\mathbb{R}^d)})^{\frac{1}{r}}. \]
By Minkowski’s inequality, it is easy to get that:
\[
\|u\|_{L^q_t(B^s_{p,r}(\mathbb{R}^d))} \leq \|u\|_{L^q_t(B^s_{p,r}(\mathbb{R}^d))} \quad \text{if } q \leq r,
\]
\[
\|u\|_{L^q_t(B^s_{p,r}(\mathbb{R}^d))} \geq \|u\|_{L^q_t(B^s_{p,r}(\mathbb{R}^d))} \quad \text{if } q \geq r.
\]
The inhomogeneous case is similar (see [1, 12]).

Some useful properties of the Besov spaces or the Chemin-Lerner type Besov space from [1] are collected as follows:

Lemma 2.2. For all \( s, s_1, s_2 \in \mathbb{R}, 1 \leq p, r, r_1, r_2, q_1, q_2 \leq +\infty, \frac{1}{p} \leq \frac{1}{r_1} + \frac{1}{r_2} \leq 1 \) and \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \),

(i) if \( s_1, s_2 \leq \frac{d}{p} \) such that \( s_1 + s_2 > d \max\{0, \frac{2}{p} - 1\} \), \( u \in B^s_{p,r_1} \) and \( v \in B^s_{p,r_2} \). Then there hold that
\[
\|uv\|_{B^{s_1+s_2-rac{d}{p}}_{p,r}} \leq C\|u\|_{B^{s_1}_{p,r_1}}\|v\|_{B^{s_2}_{p,r_2}};
\]
\[
\|uv\|_{L^q_t(B^{s_1+s_2-rac{d}{p}}_{p,r})} \leq C\|u\|_{L^q_t(B^{s_1}_{p,r_1})}\|v\|_{L^q_t(B^{s_2}_{p,r_2})};
\]

(ii) if \( s > 0 \), \( \|uv\|_{B^{s}_{p,r}} \leq C\|u\|_{L^\infty}\|v\|_{B^{s}_{p,r}} + \|u\|_{B^{s}_{p,r}}\|v\|_{L^\infty};
\]

(iii) if \( p_1 \leq p_2, r_1 \leq r_2 \), then
\[
\dot{B}^s_{p_1,r_1} \hookrightarrow B^s_{p_2,r_2}, \quad B^s_{p_1,r_1} \hookrightarrow B^s_{p_2,r_2};
\]

(iv) if \( s_1 \neq s_2 \) and \( \theta \in (0, 1) \), then
\[
\|u\|_{B^{s_1+(1-\theta)s_2}_{p,r}} \leq \|u\|_{B^{s_1}_{p,r}}\|u\|_{B^{s_2}_{p,r}}^{1-\theta}.
\]

We will present some estimates for the heat equation
\[(2.1) \quad \partial_t f - \nu \Delta f = g, \quad f|_{t=0} = f_0,\]
in homogenous Besov spaces (see [1, 12]).

Lemma 2.3. Let \( \rho, p_1, p, \) and \( r \) satisfy that \( 1 \leq p, r \leq \infty \), and \( 1 \leq \rho_1 \leq p \leq \infty \). \( f_0 \in \dot{B}^s_{p,r}, \quad g \in \tilde{L}^{p_1}_{L_t}(\dot{B}^{s-2+\frac{d}{2p_1}}_{p,r}) \) and \( f \) is a solution of equation (2.1). Then there exists an absolute constant \( C \) such that
\[
\nu^\frac{1}{p_1}\|f\|_{\tilde{L}^{p_1}_{L_t}(\dot{B}^{s_{-2+\frac{d}{2p_1}}}_{p,r})} \leq C\left(\|f_0\|_{B^s_{p,r}} + \nu^\frac{1}{p_1-1}\|g\|_{\tilde{L}^{p_1}_{L_t}(\dot{B}^{s-2+\frac{d}{2p_1}}_{p,r})}\right).
\]

We also use the notation of the commutator:
\[
[\dot{\Delta}_j, v \cdot \nabla]u = \dot{\Delta}_j(v \cdot \nabla) - v \cdot \nabla \dot{\Delta}_j u, \quad [\Delta_j, v \cdot \nabla]u = \Delta_j(v \cdot \nabla) - v \cdot \nabla \Delta_j u.
\]
There are two commutator estimates to be used. One will be applied for the transport equation and is as following:

Lemma 2.4. [1] Let \( s \in \mathbb{R}, 1 \leq r \leq \infty, 1 \leq p \leq p_1 \leq \infty, \) and \( s < \frac{d}{p} \) or \( s = \frac{d}{p} \) if \( r = 1 \).

Let \( v \) be a divergence-free vector field over \( \mathbb{R}^d \). Assume that \( -1 - d\{\frac{1}{p_1}, \frac{1}{p}\} \leq s < 1 + \frac{d}{p_1} \), then
\[
\|\Delta_j, v \cdot \nabla\|_{B^s_{p,r}} \leq C\|\nabla v\|_{B^\frac{d}{p_1}r, \cap L^\infty}\|u\|_{B^s_{p,r}}.
\]
The other is established by us for the first time, which plays an important role in our proof. We give its detailed proof as follow.

**Proposition 2.5.** Let \( s > 0, 1 \leq p, q \leq \infty \), we have the following commutator estimate,

\[
\| [\hat{\Delta}_j, u \cdot \nabla]v \|_{L^p} \leq C_2 j^{-s(1)} \left( \|u\|_{L^\infty} \|v\|_{B^{s}_{p,q}} + \|v\|_{L^\infty} \|u\|_{B^{s}_{p,q}} \right),
\]

where \( \{c_j\}_{j \in \mathbb{Z}} \in \ell^q \).

**Proof.** Using Bony’s paraproduct decomposition by Definition 2.1, we have

\[
[\hat{\Delta}_j, u \cdot \nabla]v = [\hat{\Delta}_j, \hat{T}_u] \partial_v + \hat{\Delta}_j (\hat{T}_{\partial_v u^i}) + \hat{\Delta}_j (\hat{R}(u^i, \partial_v v)) - \hat{T}_{\partial \hat{\Delta}_j v} u^i - \hat{R}(u^i, \partial_v \hat{\Delta}_j v)
\]

For the term \([\hat{\Delta}_j, \hat{T}_u] \partial_v\). By the definition of \( \hat{\Delta}_j \) and Taylor’s formula, we have

\[
[\hat{\Delta}_j, \hat{T}_u] \partial_v = \sum_{|j-j'| \leq 4} \hat{\Delta}_j (\hat{S}_{j-1} u^i \hat{\Delta}_j \partial_v) - \hat{\Delta}_j (\hat{S}_{j-1} u^i \hat{\Delta}_j \partial_v)
\]

\[
= \sum_{|j-j'| \leq 4} 2^{j'} \int_{\mathbb{R}^d} \varphi(2^j y) \left( \hat{S}_{j-1} u^i (x-y) - \hat{S}_{j-1} u^i (x) \right) \hat{\Delta}_j \partial_v v(x-y) \, dy
\]

\[
= \sum_{|j-j'| \leq 4} 2^{j'} \int_{\mathbb{R}^d} \int_0^1 (-y) \cdot \nabla \hat{S}_{j-1} u^i (x-\tau y) \, d\tau \varphi(2^j y) \hat{\Delta}_j \partial_v v(x-y) \, dy.
\]

By Minkowski’s inequality and Lemma 2.1, we get that

\[
\| [\hat{\Delta}_j, \hat{T}_u] \partial_v \|_{L^p} \leq C \sum_{|j-j'| \leq 4} \| \hat{\Delta}_j (\hat{S}_{j-1} u^i \hat{\Delta}_j \partial_v) \|_{L^p} \| \hat{\Delta}_j v \|_{L^p} \leq C \sum_{|j-j'| \leq 4} 2^{j'} \| u \|_{L^\infty} \| \hat{\Delta}_j v \|_{L^p}
\]

For \( \hat{\Delta}_j (\hat{T}_{\partial_v u^i}) \) and \( \hat{T}_{\partial \hat{\Delta}_j v} u^i \), we have

\[
\| \hat{\Delta}_j (\hat{T}_{\partial_v u^i}) \|_{L^p} \leq C \sum_{|j-j'| \leq 4} \| \hat{\Delta}_j (\hat{S}_{j-1} \partial_v v) \|_{L^\infty} \| \hat{\Delta}_j u^i \|_{L^p} \leq C \sum_{|j-j'| \leq 4} 2^{j'} \| v \|_{L^\infty} \| \hat{\Delta}_j u^i \|_{L^p}
\]

and

\[
\| \hat{T}_{\partial \hat{\Delta}_j v} u^i \|_{L^p} \leq C \sum_{j' \geq j} \| \hat{\Delta}_j \hat{\Delta}_{j-1} \partial_v v \|_{L^\infty} \| \hat{\Delta}_j u^i \|_{L^p} \leq C \sum_{j' \geq j} 2^{j'} \| v \|_{L^\infty} \| \hat{\Delta}_j u^i \|_{L^p}.
\]

For the remainder terms \( \hat{\Delta}_j (\hat{R}(u^i, \partial_v v)) \) and \( \hat{R}(u^i, \partial_v \hat{\Delta}_j v) \), we have that

\[
\| \hat{\Delta}_j (\hat{R}(u^i, \partial_v v)) \|_{L^p} \leq C 2^j \sum_{j' \geq j-3} \| \hat{\Delta}_j u^i \|_{L^p} \| \hat{\Delta}_j v \|_{L^\infty} \leq C 2^j \sum_{j' \geq j-3} \| \hat{\Delta}_j u^i \|_{L^p} \| v \|_{L^\infty}
\]

and

\[
\| \hat{R}(u^i, \partial_v \hat{\Delta}_j v) \|_{L^p} \leq C 2^j \sum_{|j'-j| \leq 1} \| \hat{\Delta}_j u^i \|_{L^p} \| \hat{\Delta}_j v \|_{L^\infty} \leq C 2^j \sum_{|j'-j| \leq 1} \| \hat{\Delta}_j u^i \|_{L^p} \| v \|_{L^\infty}.
\]

Collecting all the above estimates, we obtain that

\[
2^j(1-s) \| [\hat{\Delta}_j, u \cdot \nabla]v \|_{L^p} \leq C \sum_{|j-j'| \leq 4} 2^{(j-j')(1-s)} \| \hat{\Delta}_j v \|_{L^p} \| u \|_{L^\infty} + C \sum_{|j-j'| \leq 4} 2^{(j-j')(1-s)} \| \hat{\Delta}_j u \|_{L^p} \| v \|_{L^\infty}.
\]
Lemma 2.6.
where

\[ p > 1 \]

Define the operator \( J_n \) and take full advantage of the heat smooth effect to obtain the uniform bound for approximate solutions in Chemin-Lerner type Besov space. Finally, we prove the strong convergence of the sequence and uniqueness in a weaker norm. The proof of Theorem 1.1 is transport, this leads to a loss of one derivative. Therefore, we establish Proposition 2.5 and need three steps.

(3.1)

\[ \begin{aligned}
& \partial_t u_n - \nu J_n \Delta u_n = -J_n \text{div}[(J_n u_n \cdot \nabla) J_n u_n] + J_n \text{div}[(J_n b_n \cdot \nabla) J_n b_n], \\
& \partial_t b_n + J_n [(J_n u_n \cdot \nabla) J_n b_n] = J_n [(J_n b_n \cdot \nabla) J_n u_n], \\
& (u_n, b_n)|_{t=0} = (J_n u_0, J_n b_0).
\end{aligned} \]

Define \( V_n := \{(u, b) \mid (u, b) \in L^2 \times L^2, \hat{u} \text{ and } \hat{b} \text{ are all supported in } B(0, n), \text{div } u = \text{div } b = 0 \} \) and endowed the norm with

\[ \|Z\|_2^2 \overset{\text{def}}{=} \|u\|^2_{L^2} + \|b\|^2_{L^2}, \text{ for any } Z = (u, b) \in V_n. \]

Finally, we state a nonlinear Gronwall’s inequality.

Lemma 2.6. [2] Assume \( x \in W^{1,1}([0, T]) \cap C([0, T]) \) such that

\[ \dot{x} \leq c(t)x^p + e(t), \quad x(0) = x_0 \]

with \( p > 1, c, e \in L^1([0, T]). \) Then for each \( t \in [0, T], \) we have

\[ x(t) \leq (x_0 + \int_0^t e(\tau) \, d\tau)(1 - (p - 1)(x_0 + \int_0^t e(\tau) \, d\tau)^{p-1}) \int_0^t e(\tau) \, d\tau)^{-\frac{1}{p-1}}. \]

3. Proof of Theorem 1.1

In this section, we will prove the existence and uniqueness of the solution to the system (3.1). Firstly, we apply the very classical Friedrichs method to construct an approximate system of (1.1) in space \( \mathbb{R}^d. \) Next, because the second equation in the whole system is transport, this leads to a loss of one derivative. Therefore, we establish Proposition 2.5 and take full advantage of the heat smooth effect to obtain the uniform bound for approximate solutions in Chemin-Lerner type Besov space. Finally, we prove the strong convergence of the sequence and uniqueness in a weaker norm. The proof of Theorem 1.1 need three steps.

3.1. Construction of approximate solutions to the system (1.1).

We shall apply the classical Friedrichs method by cut-off in the frequency space. Define the operator \( J_n \) by

\[ J_n u(x) := \mathcal{F}^{-1}(1_{B(0, n)} \hat{u}(\xi)), \]

where \( \mathcal{F} \) denotes the Fourier transform in the space variables.

Let us construct the approximate system of (1.1) as follows

(3.1)

\[ \begin{aligned}
& \partial_t u_n - \nu J_n \Delta u_n = -J_n \text{div}[(J_n u_n \cdot \nabla) J_n u_n] + J_n \text{div}[(J_n b_n \cdot \nabla) J_n b_n], \\
& \partial_t b_n + J_n [(J_n u_n \cdot \nabla) J_n b_n] = J_n [(J_n b_n \cdot \nabla) J_n u_n], \\
& (u_n, b_n)|_{t=0} = (J_n u_0, J_n b_0).
\end{aligned} \]

Define

\[ V_n := \{(u, b) \mid (u, b) \in L^2 \times L^2, \hat{u} \text{ and } \hat{b} \text{ are all supported in } B(0, n), \text{div } u = \text{div } b = 0 \} \]
The system \((3.1)\) turns to be an ordinary differential system in the \(V_n\). Then, the Cauchy-Lipschitz theorem (see Theorem 3.1 in Majda and Bertozzi [1]) guarantees there exists a unique solution \((u_n, b_n)\) of the system \((3.1)\) on \([0, T_n)\) for every fixed \(n\) where \(T_n\) is strictly positive. Noting that \(J_n^2 = J_n\), we find that \((J_n u_n, J_n b_n)\) is also a solution to system \((3.1)\). By the uniqueness of the solution of ODE system \((3.1)\), we have \(J_n u_n = u_n, J_n b_n = b_n\). So \((u_n, b_n)\) is also the solution of the following system

\[
\begin{aligned}
\partial_t u_n - \nu \Delta u_n &= -J_n \mathbb{P}[(u_n \cdot \nabla)u_n] + J_n \mathbb{P}[(b_n \cdot \nabla)b_n], \\
\partial_t b_n + J_n[(u_n \cdot \nabla)b_n] &= J_n[(b_n \cdot \nabla)u_n] \\
(u_n, b_n)|_{t=0} &= (J_n u_0, J_n b_0).
\end{aligned}
\]

(3.2)

The solution will continue provided \(\|u_n\|_{H^{s-1}}\) and \(\|b_n\|_{H^s}\) remain finite.

3.2. A priori estimate.

In this section, we will establish the uniformly bound estimate for the smooth approximate solutions of system \((3.2)\), i.e. the following proposition:

**Proposition 3.1.** Under the initial condition of Theorem [1], there exists two positive constants \(C_* = C_*(\nu, u_0, \|b_0\|_{B^2_{0,2}})\) and \(T_* = T_*(\nu, u_0, \|b_0\|_{B^2_{0,2}})\) independent of \(n\) such that

\[
\|u_n\|^2_{L^\infty_t(B^{s-1}_{2,2})} + \|u_n\|^2_{L^2_t(B^{s+1}_{2,2})} + \|u_n\|^2_{L^2_t(B^s_{2,2})} + \|b_n\|^2_{L^\infty_t(B^s_{2,2})} \leq C_*.
\]

(3.3)

**Proof.** For convenience, we omit the indexes \(n\) and \(J_n\) in the system \((3.2)\). Therefore, we only need to do a priori estimate for the following system:

\[
\begin{aligned}
\partial_t u - \nu \Delta u + (u \cdot \nabla)u + \nabla \pi &= (b \cdot \nabla)b, \\
\partial_t b + (u \cdot \nabla)b &= (b \cdot \nabla)u, \\
\text{div } u &= \text{div } b = 0, \\
u|_{t=0} &= u_0, b|_{t=0} = b_0.
\end{aligned}
\]

(3.4)

Firstly, we do some \(L^2\) energy estimates. Multiplying the first equation of system \((3.4)\) by \(u\) and multiplying the second equation of system \((3.4)\) by \(b\), using the divergence free condition \(\text{div } u = 0\), integrating by parts, we get

\[
\frac{1}{2} \partial_t (\|u\|^2_{L^2} + \|b\|^2_{L^2}) + \nu \|\nabla u\|^2_{L^2} = 0.
\]

After integration in time on \([0, T]\), we have that

\[
\|u\|^2_{L^2} + \|b\|^2_{L^2} + \nu \int_0^T \|\nabla u\|^2_{L^2} dt = \|u\|^2_{L^2} + \|b\|^2_{L^2} \leq \|u_0\|^2_{B^{s-1}_{2,2}} + \|b_0\|^2_{B^s_{2,2}} =: M_0.
\]

(3.5)

Next, applying the frequency localization operator \(\hat{\Delta}_j\) to the first equation and \(\Delta_j\) to the second one of the system \((3.4)\), we have that

\[
\begin{aligned}
\partial_t \hat{\Delta}_j u - \nu \Delta \hat{\Delta}_j u + \hat{\Delta}_j (u \cdot \nabla u) + \nabla \hat{\Delta}_j \pi = \hat{\Delta}_j (b \cdot \nabla b), \\
\partial_t \Delta_j b + \Delta_j (u \cdot \nabla b) &= 0, \\
\text{div } \Delta_j u &= 0, \\
(\hat{\Delta}_j u, \Delta_j b)|_{t=0} &= (\hat{\Delta}_j u_0, \Delta_j b_0).
\end{aligned}
\]

(3.6)
Taking the $L^2$ inner product of the first equation of (3.6) with $\Delta_j u$, we get that, by integrations by parts,

$$\frac{1}{2} \partial_t \|\Delta_j u\|_{L^2}^2 + \nu \|\nabla \Delta_j u\|_{L^2}^2 \leq -\int_{\mathbb{R}^d} [\Delta_j, u \cdot \nabla] u \cdot \Delta_j u \, dx + \int_{\mathbb{R}^d} [\Delta_j, b \cdot \nabla] b \cdot \Delta_j u \, dx$$



$$\leq \|\Delta_j, u \cdot \nabla\|_{L^2} \|\Delta_j u\|_{L^2} + \|\Delta_j, b \cdot \nabla\|_{L^2} \|\Delta_j u\|_{L^2}.$$

The above inequality holds since $\text{div} \, u = 0$ which implies that

$$\int_{\mathbb{R}^d} (u \cdot \nabla) \Delta_j u \cdot \Delta_j u \, dx = 0, \quad \int_{\mathbb{R}^d} (b \cdot \nabla) \Delta_j b \cdot \Delta_j u \, dx = 0.$$

Integrating the above inequality over $[0, t]$, by Lemma 2.1, we have that

$$\|\Delta_j u(t)\|_{L^2}^2 + c2\nu 2^j \|\Delta_j u\|_{L^2([0,t] \times \mathbb{R}^d)}^2 \leq \|\Delta_j u_0\|_{L^2}^2 + \int_0^t \|\Delta_j, u \cdot \nabla\|_{L^2} \|\Delta_j u\|_{L^2} + \|\Delta_j, b \cdot \nabla\|_{L^2} \|\Delta_j u\|_{L^2} \, dt'.$$

Taking $L^\infty([0,t])$ of the above inequality on $t$, then using Lemma 2.2 and Proposition 2.5, we deduce that

$$\|\Delta_j u\|_{L^\infty([0,t]; L^2)}^2 + c2\nu 2^j \|\Delta_j u\|_{L^2([0,t] \times \mathbb{R}^d)}^2 \leq \|\Delta_j u_0\|_{L^2}^2 + \int_0^t \|u\|_{L^\infty} \|u\|_{B_{2,2}^j} \|\Delta_j u\|_{L^2} \, dt' + Cc_j 2^{-j(s-1)} \int_0^t \|b\|_{L^\infty} \|b\|_{B_{2,2}^j} \|\Delta_j u\|_{L^2} \, dt'.$$

Multiplying both sides of the above inequality by $2^{2j(s-1)}$ and then summing over $j$, we get that

$$\|u\|^2_{L^\infty([0,t]; B_{2,2}^{s-1})} + c2\nu \|u\|^2_{L^2([0,t]; B_{2,2}^j)} \leq \|u_0\|^2_{B_{2,2}^{s-1}} + C\int_0^t \|u\|_{L^\infty} \|u\|_{B_{2,2}^j} \|u\|_{B_{2,2}^{-1}} \, dt' + C\int_0^T \|b\|_{L^\infty} \|b\|_{B_{2,2}^j} \|u\|_{B_{2,2}^{s-1}} \, dt'.$$

This, together with (3.5), yields that

$$\|u\|^2_{L^\infty([0,t]; B_{2,2}^{s-1})} + c2\nu \|u\|^2_{L^2([0,t]; B_{2,2}^j)} \leq (1 + c2\nu t) M_0 + C\int_0^t \|u\|_{L^\infty} \|u\|_{B_{2,2}^j} \|u\|_{B_{2,2}^{-1}} \, dt' + C\int_0^t \|b\|_{B_{2,2}^j} \|u\|_{B_{2,2}^{s-1}} \, dt'.$$

Since $s > \frac{d}{2}$, we can choose a $r > 2$ such that $s - 1 + \frac{2}{r} \in \left(\frac{d}{2}, s\right)$. Then using the embedding:

$$\|f\|_{L^\infty} \leq C\|f\|_{B_{2,2}^\theta}, \quad \theta > d/2,$$

Young’s inequality and the interpolation inequality in Lemma 2.2 deduces that

$$\|u\|^2_{L^\infty([0,t]; B_{2,2}^{s-1})} + c2\nu \|u\|^2_{L^2([0,t]; B_{2,2}^j)} \leq 2(1 + c2\nu t) M_0 + C\int_0^t \|u\|_{B_{2,2}^{s-1}} \|u\|_{B_{2,2}^j} \|u\|_{B_{2,2}^{-1}} \, dt' + C\int_0^t \|b\|_{B_{2,2}^j} \|u\|_{B_{2,2}^{s-1}} \, dt'.$$
\[ \leq 2(1 + 2cvt)M_0 + C \int_0^t \|u\|^{\frac{2}{B_{2,2}^*}} \|u\|_{B_{2,2}^*} dt' + C \int_0^t \|b\|^{\frac{2}{B_{2,2}^*}} \|u\|_{B_{2,2}^*} dt' \]

\[ \leq 2(1 + 2cvt)M_0 + cv \|u\|_{L^2([0, t]; B_{2,2}^*)}^2 dt' + \frac{t}{2} \|u\|_{L^\infty([0, t]; B_{2,2}^{-1})}^2 \]

\[ + C(\nu) \int_0^t \|u\|_{L^\infty([0, t]; B_{2,2}^{-1})}^{\frac{4}{1 + \frac{1}{2}}} dt' + C \int_0^t \|b\|_{L^\infty([0, t]; B_{2,2}^*)}^{\frac{4}{3}} dt'. \]

Hence, we get that for any \( t \leq 1 \),

\[ \|u(t)\|_{L^\infty([0, t]; B_{2,2}^{-1})}^2 + \|u\|_{L^2([0, t]; B_{2,2}^*)}^2 \]

\[ \leq C_1(\nu)M_0 + C_1(\nu)\int_0^t \|u\|_{L^\infty([0, t]; B_{2,2}^{-1})}^{\frac{4}{1 + \frac{1}{2}}} dt' + C_2 \int_0^t \|b\|_{L^\infty([0, t]; B_{2,2}^*)}^{\frac{4}{3}} dt'. \tag{3.8} \]

On the other hand, by Lemma 2.23 and Proposition 2.5, we obtain that

\[ \|u\|_{L^1([0, T]; B_{2,2}^*]}^\sigma \leq C(\nu)\|e^{\nu\Delta}u_0\|_{L^1(B_{2,2}^*)} + C(\nu)\|u \cdot \nabla u - b \cdot \nabla b\|_{L^1(B_{2,2}^*)} \]

\[ \leq C(\nu)\|e^{\nu\Delta}u_0\|_{L^1(B_{2,2}^*)} + C(\nu)\|u \cdot \nabla u - b \cdot \nabla b\|_{L^1(B_{2,2}^*)} \]

\[ \leq C(\nu)\|e^{\nu\Delta}u_0\|_{L^1(B_{2,2}^*)} + C(\nu)T^{\frac{\sigma - 2}{\sigma}} \|u\|_{L^\infty([0, T]; L^\infty)} \|u\|_{L^2(B_{2,2}^*)} \]

\[ + C(\nu)\int_0^T \|b\|_{B_{2,2}^*}^2 dt. \]

According to the embedding that can be easily proved

\[ \|h\|_{L^\infty([0, T]; L^\infty)} \leq C\|h\|_{L^\nu([0, T]; B_{2,2}^*)}, \quad \forall \nu \in [1, \infty], \quad \sigma > d/2, \]

and the interpolation inequality in Lemma 2.22 we get that

\[ \|u\|_{L^1(B_{2,2}^*)} \]

\[ \leq C(\nu)\left(\|e^{\nu\Delta}u_0\|_{L^1(B_{2,2}^*)} + T^{\frac{\sigma - 2}{\sigma}} \|u\|_{L^\infty((0, T); B_{2,2}^*)} \|u\|_{L^2(B_{2,2}^*)} + \int_0^T \|b\|_{B_{2,2}^*}^2 dt \right) \]

\[ \leq C(\nu)\left(\|e^{\nu\Delta}u_0\|_{L^1(B_{2,2}^*)} + T^{\frac{\sigma - 2}{\sigma}} \|u\|_{L^\infty((0, T); B_{2,2}^*)} \|u\|_{L^2(B_{2,2}^*)} \right) + \int_0^T \|b\|_{B_{2,2}^*}^2 dt. \]

This, together with (3.5), gives that

\[ \|u\|_{L^1(B_{2,2}^*)} \leq C(\nu)\|e^{\nu\Delta}u_0\|_{L^1(B_{2,2}^*)} + TM_0^{\frac{\sigma}{\sigma}} + C_2(\nu)\int_0^T \|b\|_{B_{2,2}^*}^2 dt \]

\[ + C_2(\nu)T^{\frac{\sigma - 2}{\sigma}} \|u\|_{L^\infty([0, T]; L^\infty)} \|u\|_{L^2(B_{2,2}^*)} \]

(3.10)

Now, let’s deal with the term \( \|b\|_{B_{2,2}^*}^2 \). Taking the \( L^2 \) inner product of the second equation of (3.6) with \( \Delta_j b \) on \( \mathbb{R}^d \), by Hölder’s inequality and the fact that

\[ \int_{\mathbb{R}^d} (u \cdot \nabla) \Delta_j b \cdot \Delta_j b \, dx = 0, \]

we have that

\[ \frac{1}{2} \frac{d}{dt} \|\Delta_j b\|_{L^2}^2 \leq \|\Delta_j, u \cdot \nabla\|_{L^2} \|\Delta_j b\|_{L^2}^2 + \|\Delta_j (b \cdot \nabla u)\|_{L^2} \|\Delta_j b\|_{L^2} \|\Delta_j b\|_{L^2}. \]
By Lemma 2.4 and Proposition 2.2 and the embedding relationship $B_{2,2}^s \hookrightarrow B_{2,1}^{\frac{s}{2}}$, we obtain that
\begin{align}
\int_0^t \| \Delta_j, u \cdot \nabla b \|_{L^2} + \| \Delta_j, (b \cdot \nabla u) \|_{L^2} dt' \\
\leq Cc_j 2^{-js} \int_0^t \| \nabla u \|_{B_{2,\infty}^{\frac{s}{2}} \cap L^\infty} \| b \|_{B_{2,2}^s} dt \\
\leq Cc_j 2^{-js} \| \nabla u \|_{L^1([0,t]; B_{2,2}^s)} \| b \|_{L^\infty([0,t]; B_{2,2}^s)} \\
\leq Cc_j 2^{-js} \| u \|_{L^1([0,t]; B_{2,2}^{s+1})} \| b \|_{L^\infty([0,t]; B_{2,2}^s)}.
\end{align}

Therefore, dividing both sides of (3.11) by $\| \Delta_j b \|_{L^2}$, taking $L^1([0,t])$-norm, plugging (3.12) into (3.11), we obtain that
\[ \| \Delta_j b(t) \|_{L^\infty(B_{2,2}^s)} \leq \| \Delta_j b_0 \|_{L^2} + Cc_j 2^{-js} \| u \|_{L_t^1(B_{2,2}^{s+1})} \| b \|_{L_t^\infty(B_{2,2}^s)} . \]

Multiplying both sides of the above inequality by $2^j$ and taking $\ell^2(j \geq -1)$-norm deduces that
\[ \| b \|_{L_t^\infty(B_{2,2}^s)} \leq \| b_0 \|_{B_{2,2}^s} + C_3 \| u \|_{L_t^1(B_{2,2}^{s+1})} \| b \|_{L_t^\infty(B_{2,2}^s)} . \]

Next, we will show that there exists a $T_*$ such that $\| b \|_{L^\infty([0,t]; B_{2,2}^s)} \leq 2\| b_0 \|_{B_{2,2}^s}$ for any $t \in [0, T_*]$. Notice that
\[ \| e^{\tau \Delta} u_0 \|_{L_t^1(B_{2,2}^{s+1})} \leq \sum_j 2^{2j(s-1)} \| \Delta_j u_0 \|_{L^2(1-e^{-C2^j \tau})} \leq \| u_0 \|_{B_{2,2}^{s-1}}, \]

set
\[ T' \triangleq \sup \{ T \in [0, T_*] \mid \| b \|_{L_t^\infty(B_{2,2}^s)} \leq 2\| b_0 \|_{B_{2,2}^s} \} \]
where $T_*$ satisfies
\begin{align}
\left( 1 - \frac{r}{r - 2} C_1(\nu) T_*(C_1(\nu) M_0 + 16 C_2 \| b_0 \|_{B_{2,2}^s} \| b_0 \|_{B_{2,2}^s}) \right)^{-\frac{r-2}{r-4}} < 2.
\end{align}

and
\begin{align}
C_2(\nu) \| e^{\tau \Delta} u_0 \|_{L_t^1(B_{2,2}^{s+1})} + T'(M_0^\frac{1}{2} + C_2(\nu) M_0) \\
+ C_2(\nu)(T')^{-\frac{r-2}{r-4}}(2 C_1(\nu) M_0 + 32 C_2 M_0^\frac{1}{2}) < \frac{1}{2 C_3}.
\end{align}

Suppose that $T' < T_*$, we get that from (3.8)
\[ \| u \|_{L_t^\infty([0,T']; B_{2,2}^{s-1})} + c 2\nu \| u \|_{L_t^3([0,T']; B_{2,2}^s)} \]
\[ \leq C_1(\nu) M_0 + C_1(\nu) \int_0^{T'} \| u \|_{L_t^4([0,T']); B_{2,2}^{s-1}} dt + 16 C_2 T' \| b_0 \|_{B_{2,2}^s}. \]

Applying Lemma 2.6 to the above inequality, from (3.14), we have that
\begin{align}
\| u \|_{L_t^\infty(B_{2,2}^{s-1})} + c 2\nu \| u \|_{L_t^2(B_{2,2}^s)} \leq 2 C_1(\nu) M_0 + 32 C_2 T' \| b_0 \|_{B_{2,2}^s}.
\end{align}
Substituting (3.16) into (3.10), we get that
\[
\|u\|_{L^1_t(B^{2+1}_{2,2})} \leq C_2(\nu)\|e^{\nu t}u_0\|_{L^1_t(B^{2+1}_{2,2})} + T'\|M_0\|_2^2 + C_2(\nu)T'\|b_0\|_{B^2_{2,2}}^2
\]
\[
+ C_2(\nu)(T')^{-\frac{2}{p'}}(2C_1(\nu)M_0 + 32C_2T'\|b_0\|_{B^2_{2,2}}^2)
\]
\[
\leq C_2(\nu)\|e^{\nu t}u_0\|_{L^1_t(B^{2+1}_{2,2})} + T'\|M_0\|_2^2 + C_2(\nu)T'\|M_0\|_2^2
\]
\[
+ C_2(\nu)(T')^{-\frac{2}{p'}}(2C_1(\nu)M_0 + 32C_2M_0^4) < \frac{1}{2C_3}.
\]
This, together with (3.5), implies that
\[
\|b\|_{L^2_t(B^2_{2,2})} < 2\|b_0\|_{B^2_{2,2}},
\]
contradicting the maximality of T. Hence T = T_*. This fact, together with (3.8), (3.10) and (3.13), entails that the require results (3.11) \(\square\)

3.3. Convergence of the solution sequences.

**Proposition 3.2.** The solutions \((u_n, b_n)\) of approximate system (3.2) is Cauchy with respect to n in \(C([0, T_*]; L^2(\mathbb{R}^d)) \times C([0, T_*]; L^2(\mathbb{R}^d))\). Moreover, the limits satisfy with that
\[
\begin{align*}
  u &\in C([0, T_*]; B^{2+1}_{2,2}(\mathbb{R}^d)) \cap L^2([0, T_*]; B^2_{2,2}(\mathbb{R}^d)) \cap L^1([0, T_*]; B^{2+1}_{2,2}(\mathbb{R}^d)), \\
  b &\in C([0, T_*]; B^2_{2,2}(\mathbb{R}^d)).
\end{align*}
\]

**Proof.** We firstly prove the solutions \((u_n, b_n)\) of the approximate system (3.2) is Cauchy with respect to n. Assume \((u_n, b_n), (u_{p,p})\) are any two solutions of the approximate system (3.2). They all satisfy with Proposition 3.3
\[
\begin{align*}
  \|u_n\|_{C([0,T_*]; B^{2+1}_{2,2})} + \|u_n\|_{L^1_t(B^{2+1}_{2,2})} + \|u_n\|_{L^2_t(B^{2}_{2,2})} + \|b_n\|_{C([0,T_*]; B^2_{2,2})} \leq C_*. 
\end{align*}
\]

The differences \(u_n - u_p, b_n - b_p\) (suppose \(p > n\)) satisfy the following system:
\[
\begin{align*}
  \partial_t (u_n - u_p) - \nu \Delta (u_n - u_p) &= J_n\mathbb{P}(b_n \cdot \nabla b_n) - J_p\mathbb{P}(b_p \cdot \nabla b_p) \\
  - J_n\mathbb{P}[(u_n \cdot \nabla)u_n] + J_p\mathbb{P}[(u_p \cdot \nabla)u_p], \\
  \partial_t (b_n - b_p) &= - J_n(u_n \cdot \nabla b_n) + J_p(b_p \cdot \nabla u_p) \\
\end{align*}
\]

Applying localization operator \(\hat{\Delta}_j\) to both sides of the above system, by the \(L^2\) energy estimate, divergence free condition and integrating on \([0, t]\), we obtain that
\[
\begin{align*}
  \|\hat{\Delta}_j(u_n - u_p)\|_{L^2_t}^2 + \|\hat{\Delta}_j(b_n - b_p)\|_{L^2_t}^2 + \nu \int_0^t \|\hat{\Delta}_j \nabla (u_n - u_p)\|_{L^2_t}^2 \\
  = \|\hat{\Delta}_j(u_0 - u_{0p})\|_{L^2_t}^2 + \|\hat{\Delta}_j(b_0 - b_{0p})\|_{L^2_t}^2 \\
  + \int_0^t \langle \hat{\Delta}_j(J_n\mathbb{P}(b_n \cdot \nabla b_n) - J_p\mathbb{P}(b_p \cdot \nabla b_p)), \hat{\Delta}_j(u_n - u_p) \rangle \\
  - \langle (J_n\mathbb{P}((u_n \cdot \nabla)u_n) - J_p\mathbb{P}([u_p \cdot \nabla]u_p, \hat{\Delta}_j(u_n - u_p)) \\
  + J_n\hat{\Delta}_j((b_n \cdot \nabla)u_n) - J_p\hat{\Delta}_j((b_p \cdot \nabla)u_p), \hat{\Delta}_j(b_n - b_p) \rangle \\
\end{align*}
\]
where \( E_0 = \| \hat{\Delta}_j(u_{0n} - u_{0p}) \|_{L^2}^2 + \| \hat{\Delta}_j(b_{0n} - b_{0p}) \|_{L^2}^2 \).

Split each \( E_i \) \( (i = 1, 2, 3) \) into three parts. We only deal with the two more difficult terms: \( E_3 \) and \( E_4 \).

\[
E_3 = \int_0^t \langle J_n \hat{\Delta}_j(b_n \cdot \nabla u_n) - J_p \hat{\Delta}_j(b_p \cdot \nabla u_p), \hat{\Delta}_j(b_n - b_p) \rangle \, dt' =: \sum_{i=0}^4 E_i.
\]

For \( E_{31} \), summing up over \( j \in \mathbb{Z} \), we obtain that by Hölder’s inequality

\[
\sum_j |E_{31}| \leq \sum_j \int_0^t \|(J_n - J_p) \hat{\Delta}_j(b_n \cdot \nabla u_n)\|_{L^2} \|\hat{\Delta}_j(b_n - b_p)\|_{L^2} \leq \bigg\{ \int_0^t \|(J_n - J_p) \hat{\Delta}_j(b_n \cdot \nabla u_n)\|_{L^2} dt \bigg\} \| b_n - b_p \|_{L^\infty(B_{2,2}^1)}^{(\epsilon)} \leq \frac{C}{\epsilon} \| b_n \cdot \nabla u_n \|_{L^1(\mathbb{R}^2)} \| b_n - b_p \|_{L^\infty(B_{2,2}^1)}
\]

(3.17)

where we choose \( \epsilon \) satisfying \( 0 < \epsilon < s - \frac{d}{2} \) and use the embedding that \( B_{2,2}^s \hookrightarrow B_{2,2}^r \hookrightarrow \dot{B}_{2,2}^s \) together with Lemma 222.

\( E_{32} \) and \( E_{33} \) can be estimated similarly as follows

\[
\sum_j \sum_{k=2}^3 |E_{3k}| \leq \frac{1}{2} \| \nabla u_n - \nabla u_p \|_{L^2(B_{2,2}^1)}^2 + C \bigg( \| u_p \|_{L^1(B_{2,2}^{s+1})} + t \| b_n \|_{L^\infty(B_{2,2}^1)}^2 \bigg) \| b_n - b_p \|_{L^\infty(B_{2,2}^1)}^2
\]

where we use the Young’s inequality. Then, we have that

\[
\sum_j \sum_{k=1}^3 |E_{3k}| \leq C \int_0^t \| \nabla u_n - \nabla u_p \|_{B_{2,2}^s} \| b_n - b_p \|_{L^\infty([0,t];B_{2,2}^s)} dt' + \big( \| \nabla u_p \|_{L^1([0,t];B_{2,2}^s)} + t \| b_n \|_{L^\infty([0,t];B_{2,2}^s)} \big) \| b_n - b_p \|_{L^\infty([0,t];B_{2,2}^s)}^2 \leq \frac{C}{\epsilon} \big( \| u_n \|_{L^1([0,t];B_{2,2}^{s+1})} + t \| b_n \|_{L^\infty([0,t];B_{2,2}^s)} \big) \| b_n - b_p \|_{L^\infty([0,t];B_{2,2}^s)}^2 + C \big( \| u_p \|_{L^1([0,t];B_{2,2}^{s+1})} + t \| b_n \|_{L^\infty([0,t];B_{2,2}^s)} \big) \| b_n - b_p \|_{L^\infty([0,t];B_{2,2}^s)}^2
\]

For \( E_4 \), we decompose into

\[
E_4 = - \int_0^t \langle J_n \hat{\Delta}_j(u_n \cdot \nabla b_n) - J_p (u_p \cdot \nabla b_p), \hat{\Delta}_j(b_n - b_p) \rangle \, dt'
\]
$$= - \int_0^t \langle (J_n - J_p) \Delta_j (u_n \cdot \nabla b_n), \Delta_j (b_n - b_p) \rangle$$
$$- \langle J_p \Delta_j [(u_n - u_p) \cdot \nabla b_n], \Delta_j (b_n - b_p) \rangle$$
$$- \langle J_p \Delta_j [u_p \cdot \nabla (b_n - b_p)], \Delta_j (b_n - b_p) \rangle dt' =: \sum_{i=1}^3 E_{4i}$$

Just as $E_{31}$, we have

$$\sum_j |E_{41}| \leq \frac{1}{\eta^\epsilon} \|u_n\|_{L^\infty_t(B^{s+1}_{2,2})} \|b_n\|_{L^\infty_t(B^{s}_{2,2})} \|b_n - b_p\|_{L^\infty_t(B^{s}_{2,2})}.$$

Before dealing with $E_{42}$, we need the following estimates:
when $d = 2$, $0 < \epsilon < s - \frac{d}{2}$, we have

$$\|fg\|_{L^2} \leq C \|f\|_{L^4} \|g\|_{L^\infty} \leq C \|f\|_{L^{4s+1}_2} \|g\|_{L^{2s+1}_2} \leq C \|f\|_{L^{2s+1}_2} \|g\|_{L^{2s+1}_2};$$
when $d = 3$, we have

$$\|fg\|_{L^2} \leq C \|f\|_{L^2} \|g\|_{L^2} \leq C \|f\|_{L^{2s+1}_2} \|g\|_{L^{2s+1}_2} \leq C \|f\|_{L^{2s+1}_2} \|g\|_{L^{2s+1}_2}$$

by Lemma 17.2 as $0 < \delta < s - \frac{d}{2}$. This implies

$$\sum_j |E_{4j}| \leq C \int_0^t \|u_n - u_p\|_{B^{s+1}_{2,2}} \|\nabla b_n\|_{B^{s+1}_{2,2}} \|b_n - b_p\|_{L^\infty_t(B^{s}_{2,2})} dt'$$
$$\leq \frac{1}{\eta^\epsilon} \|u_n - u_p\|_{L^\infty_t(B^{s}_{2,2})}^2 + C \|b_n\|_{L^\infty_t(B^{s}_{2,2})} \|b_n - b_p\|_{L^\infty_t(B^{s}_{2,2})}^2.$$

Noting that fact $J_p J_n = J_n$, when $p > n$, we get $E_{43} = 0$ by integrating by parts and using the divergence free condition.

In addition, $E_1, E_2$ can be easily estimated similarly.

Plugging the above estimates together with estimates for $E_1, E_2$ and using Young's inequality, we obtain that

$$\|u_n - u_p\|_{L^\infty_t(B^{s}_{2,2})}^2 + \|b_n - b_p\|_{L^\infty_t(B^{s}_{2,2})}^2 + \|\nabla (u_n - u_p)\|_{L^\infty_t(B^{s}_{2,2})}^2$$
$$\leq \frac{C}{\eta^\epsilon} \left( \|u_n\|_{B^{s+1}_{2,2}}^2 + \|b_n\|_{B^{s+1}_{2,2}}^2 + \|u_n\|_{L^\infty_tB^{s+1}_{2,2}}^2 + \|b_n\|_{L^\infty_tB^{s+1}_{2,2}}^2 \right)$$
$$\times \left( \|u_n - u_p\|_{L^\infty_t(B^{s}_{2,2})}^2 + \|b_n - b_p\|_{L^\infty_t(B^{s}_{2,2})}^2 \right)$$
$$+ \tilde{C} (t + \|u_p\|_{L^1_tB^{s}_{2,2}} + \|b_n\|_{L^\infty_tB^{s}_{2,2}}^2) (\|u_n - u_p\|_{L^\infty_t(B^{s}_{2,2})}^2 + \|b_n - b_p\|_{L^\infty_t(B^{s}_{2,2})}^2).$$

Denote

$$Y(t) = \|u_n - u_p\|_{L^\infty_t(B^{s}_{2,2})}^2 + \|b_n - b_p\|_{L^\infty_t(B^{s}_{2,2})}^2.$$

By Proposition 3.3 and let $t$ small enough such that $\tilde{C} (t + \|u_p\|_{L^1_tB^{s}_{2,2}} + \|b_n\|_{L^\infty_tB^{s}_{2,2}}^2) \leq \frac{1}{2}$, we get that

$$Y(t) \leq \frac{CC^*}{\eta^\epsilon} \rightarrow 0, n \rightarrow +\infty.$$

Hence, $\{(u_n, b_n)\}_n$ is Cauchy for $n$ in $C([0, T_*); L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$, by interpolation, in $C([0, T_*); B^{s}_{2,2} \times C([0, T_*); B^{s'}_{2,2})$ for any $s' < s$. The limit $(u, b)$ is in $L^\infty([0, T_*); B^{s-1}_{2,2}) \times$
difference satisfies the following system:

$$u \in C([0, T_*); B_{2,2}^{s-1}(\mathbb{R}^d)) \cap L^2([0, T_*); B_{2,2}^s(\mathbb{R}^d)) \cap \widetilde{L}_1([0, T_*); B_{2,2}^{s+1}(\mathbb{R}^d)),$$

and

$$b \in C([0, T_*); B_{2,2}^s(\mathbb{R}^d)).$$

Then we have proved Proposition 3.2. \hfill \Box

3.4. Uniqueness.

**Proposition 3.3.** The solution \((u, b)\) of system (1.1) in the previous step is unique.

**Proof.** Assume \((u_1, b_1), (u_2, b_2)\) are two any solutions of the system (1.1). Certainly, they all satisfy with Proposition 3.1. Denote \(u = u_1 - u_2, b = b_1 - b_2, \pi = \pi_1 - \pi_2\), then the difference satisfies the following system:

\[
\begin{aligned}
\partial_t u - \nu \Delta u + u \cdot \nabla u + u \cdot \nabla u_2 + \nabla \pi &= b_1 \cdot \nabla b + b \cdot \nabla b_2, \\
\partial_t b + u_1 \cdot \nabla b + u \cdot \nabla b_2 &= b_1 \cdot \nabla u + b \cdot \nabla u_2, \\
\text{div } u = \text{div } b &= 0, \\
(u, b)|_{t=0} &= (0, 0).
\end{aligned}
\]  

(3.18)

The proof sketch is very similar to prove Proposition 3.2, we omit it and finally have small \(t_1 > 0\) enough such that for any \(0 < t \leq t_1\)

$$
\|u\|_{L_x^\infty(B_{2,2}^s)}^2 + \|b\|_{L_x^\infty(B_{2,2}^s)}^2 \leq \frac{1}{2}(\|u\|_{L_x^\infty(B_{2,2}^s)}^2 + \|b\|_{L_x^\infty(B_{2,2}^s)}^2).
$$

Therefore, we have \((u, b) = 0\) on \([0, t_1]\). Using a continuity argument ensures that \((u_1, b_1) = (u_2, b_2)\) on \([0, T]\). This concludes the proof. \hfill \Box

Combining Proposition 3.1, Proposition 3.2 and Proposition 3.3 together, we complete the proof of Theorem 1.

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