The Deformed Matrix Model at Finite Radius and a New Duality Symmetry

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Abstract

The $1/x^2$ deformed $c = 1$ matrix model is studied at finite radius and non-zero cosmological constant. Calculational techniques are presented and illustrated in some examples. Furthermore, a new kind of $R \rightarrow 1/R$ duality is discovered, which mixes different genera.
1 INTRODUCTION

Recently the $1/x^2$ deformed $c = 1$ matrix model was introduced by Jevicki and Yoneya in [1]. The motivation was to find a matrix-model description of the eternal two-dimensional stringy black hole described in [2]. The new matrix model has been studied in several subsequent papers, [3–6]. It has been found that the model is clearly different from the standard $c = 1$. Furthermore, it has a few important properties, which makes it an interesting black-hole candidate, in particular the scaling properties and the position of poles in correlation functions.

In this paper the $1/x^2$ deformed $c = 1$ matrix model will be studied at finite radius $R$. This is important in view of the possible black-hole connection. In this case we would expect a compactified time coordinate in the Euclidean version. Unfortunately, the matrix model does not seem to prefer any particular value of the radius. Since the black-hole connection is just an educated guess, and we really have no good understanding of the physics of the deformed model, it is important to learn as much as possible. In fact, as we will see in Section 4, the model has some very interesting properties at finite radius.

In Section 2 I will describe some techniques to calculate arbitrary correlation functions in the presence of a world-sheet cosmological constant (i.e. time-independent tachyon background). To do this, the algebraic structure of the model is discussed. I will also recall the prescription introduced in [7] for finite radius correlation functions and explain why it works also in the deformed model.

In Section 3 some sample calculations are made to illustrate the methods of Section 2. A $\langle PTTT \rangle$ correlation function is calculated to genus 2 and an arbitrary tachyon $n$-point function is calculated to genus 1.

In Section 4 I will study the partition function and the issue of $R \to 1/R$ duality. A new kind of non-perturbative duality is discovered. This should prove useful for a deeper understanding of the theory.

2 HOW TO CALCULATE CORRELATION FUNCTIONS
2.1 Ingredient One: The Algebraic Structure

In [1] the matrix eigenvalue potential $-\frac{1}{2\alpha'}x^2$ of the standard $c = 1$ model is deformed into $-\frac{1}{2\alpha'}x^2 + \frac{\eta}{x^2}$. Using the notation in [3], this gives the following oscillator algebra for $\alpha' = 1$. The operators are

$$b = \frac{1}{2}(ip + x)^2 - \frac{\eta}{x^2};$$

its conjugate $b^\dagger$ and the Hamiltonian

$$H = \frac{1}{2}(p^2 + x^2) + \frac{\eta}{x^2}. \quad (2)$$

They obey

$$[b, b^\dagger] = 4H,$$

$$[H, b] = -2b \quad \text{and} \quad [H, b^\dagger] = 2b^\dagger. \quad (3)$$

I have, for convenience, continued $\alpha' \to -\alpha'$. A special tachyon of momentum $p = 2k$ is represented by $b^k$ (or $b^{\dagger k}$ depending on chirality). The use of the corresponding algebra in the undeformed case has been discussed in [8–10]. It is nothing but the matrix-model $W_\infty$ algebra. Through perturbation theory it is shown that

$$\langle PT_1...T_n \rangle \sim \langle P [T_1, [T_2, ... [T_{n-1}, T_n] ...]] \rangle$$

up to the factorized external leg factors; $T_1$ through $T_{n-1}$ has positive chirality while $T_n$ has negative. The right-hand side two-point function is given by

$$\langle PW \rangle = -\frac{1}{\pi^2} \sum_{n=0}^\infty \frac{\langle n\|W\|n \rangle}{E_n - \mu}; \quad (5)$$

$n$ labels the one-particle states in the potential. This also applies to the deformed model. To obtain an $n$-point function, we hence need the commutator

$$[b^{jk_1}, [b^{jk_2}, ... [b^{jk_{n-1}}, b^j] ...]] \quad (6)$$

with $\sum_{i=1}^{n-1} k_i = l$.

The following important relations can be obtained after some work

$$[b^l, b^{jk}] = 4^{k-r} \binom{k}{r} (l)_{r-k} b^{lr} b^{-k+r} (H - l + k)_{k-r} \quad (7)$$
and
\[ f(H)b^k = b^k f(H + 2k) \] (8)
for any function \( f(H) \). I use the notation \((x)_k = x(x+1)\ldots(x+k-1)\) and \((x)_{-k} = x(x-1)\ldots(x-k+1)\) for \( k \) positive.

These are all the commutation relations that we need.

### 2.2 Ingredient Two: The Matrix Elements

To complete the calculation we also need some matrix elements. As shown in [3], we have
\[ \langle n | b^k b^\dagger | n \rangle = 2^{2k} \left( \frac{n}{2} + a + 1 \right) \left( \frac{n+1}{2} \right)_k \] (9)
and
\[ \langle n | b^\dagger b^k | n \rangle = 2^{2k} \left( \frac{n}{2} + a + 1 - k \right) \left( \frac{n+1}{2} - k \right)_k. \] (10)

Here \( a = -\frac{1}{2} + \sqrt{\frac{1}{4} + 2\eta} \). These matrix elements will appear as part of a two-point function, as in (5). As explained in [3, 4], the sum over states in (5) should only be over odd \( n \). This is to make sure that all wave functions are normalizable. The sum is obtained by using the formal equality
\[ \sum_{n=0}^{\infty} \frac{f(n)}{n+z} = f(-z) \sum_{n=0}^{\infty} \frac{1}{n+z}. \] (11)

The sum is defined by taking derivatives in \( z \). This gives \( n \rightarrow -y + i\mu \), where \( y = \sqrt{\frac{1}{4} + 2\eta} \), since, as explained in [3, 4], \( E_n = -\frac{1}{2\sqrt{\alpha'}}(2n+1+2\alpha) \) in the deformed model. The imaginary energies arise from the convenient continuation \( \alpha' \rightarrow -\alpha' \) introduced in [4]. Powers of \( H \) in the matrix elements are easily taken care of inside the sum according to
\[ \langle n | b^k b^\dagger H^l | n \rangle \rightarrow (i\mu)^l \langle n | b^k b^k | n \rangle. \] (12)

If we expand in large \( y \) (corresponding to large \( \eta \) and hence weak coupling) we find
\[ \sum_{n \text{ odd}} \frac{\langle n \| b^k b^k H^l \| n \rangle}{E_n - \mu} = (1) \int \frac{d^2x}{(2\pi)^2} e^{i \int \frac{dx}{2R} R \frac{\partial}{\partial \mu} - e^{i \int \frac{dx}{2R} R \frac{\partial}{\partial \mu}} \sim 1 + \frac{1}{24R^2} \frac{\partial^2}{\partial \mu^2} + \frac{7}{5760R^4} \frac{\partial^4}{\partial \mu^4} + \ldots \] (14)

The easiest way to see this, \([11]\), is to consider the puncture two point function \(< PP >\) given by

\[ \frac{1}{\pi R} R \sum_{n,m=0}^{\infty} \left( \frac{2n+1}{2\sqrt{\alpha}} + \frac{2m+1}{2R} + i\mu \right)^2. \] (15)

This can be obtained from the non-relativistic fermionic field theory description of the \(c = 1\) matrix model \([12]\), and an equivalent expression was first derived in \([13]\). The \(R \to \infty\) limit of this is clearly

\[ -\frac{1}{\pi} R \sum_{n=0}^{\infty} \frac{1}{2n+1 + i\mu}. \] (16)

Applying (14) on (16) gives (13). Other correlation functions are obtained through perturbation theory, a procedure that clearly commutes with (14). This was shown in the standard \(c = 1\) case. However, as should be clear from...
above, the same reasoning applies to the case of the deformed matrix model. Just recall that
\[
\langle PP \rangle = \frac{1}{\pi R} \mathcal{R} \sum_{n,m=0}^{\infty} \left( \frac{2(2n+1)+1+\alpha}{2\sqrt{\alpha^{2}}} + \frac{2m+1}{2R} + i \mu \right)^{2}\tag{17}
\]
and it is clear that everything goes through.

3 SOME EXAMPLES

3.1 Four-point Correlation Function to Genus 2

In this section I will calculate some correlation functions at finite radius. I will begin with \( \langle PTTT \rangle \) up to genus 2. That is, the tachyon four-point function with one of the momenta put to zero.

The needed commutator is
\[
\left[ b^{l_{k_1}}, \left[ b^{l_{k_2}}, b^{l} \right] \right] = b^{l_{k_1}+k_2}b^{l} - b^{l_{k_1}}b^{l}b^{l_{k_2}} - b^{l_{k_2}}b^{l}b^{l_{k_1}} + b^{l}b^{l_{k_1}+k_2}, \tag{18}\]
where \( l = k_1 + k_2 \). Using (7) and (8) we find
\[
\sum_{s=0}^{l-1} 4^{l-s-1} \binom{l}{s} (l-s)_{l+s} b^{s}(H)_{l-s} - \sum_{s=0}^{k_1-1} 4^{k_1-s} \binom{k_1}{s} (l_{k_1+s})_{l-k_1} b^{l_{k_1}+s}b^{l_{k_2}+s}(H-k_2)_{k_1-s} \]
\[
- \sum_{s=0}^{k_2-1} 4^{k_2-s} \binom{k_2}{s} (l_{k_2+s})_{l-k_2} b^{l_{k_1}+s}b^{l_{k_2}+s}(H-k_1)_{k_2-s}. \tag{19}\]

One must then use Section 2.2. The result must also be reexpanded in terms of \( \eta \) instead of \( y \). Furthermore, we should recall that \( p = 2k \) at \( \alpha' = 1 \). When all this is done, and the normalization fixed to that of collective field theory, we find
\[
\langle PTTT \rangle \sim
pp_1p_2 \left( \eta^{p/2-1} - \frac{1}{48}(p-2)(p^2 + 3(p_1^2 + p_2^2) - 4p - 15)\eta^{p/2-2} + \frac{3}{4}(p-2)\mu^2\eta^{p/2-2}
+ \frac{(p-2)(p-4)}{23040}(80(p_1^4 + p_2^4) - 288(p_1^2 + p_2^2) - 728(p_1^2 + p_2^2) + 1176p + 2085
+ 240p_1p_2 + 80(p_1p_2^2 + p_2p_1^2) + 384(p_1p_2^2 + p_2p_1^2) + 44p_1p_2)\eta^{p/2-3}
- \frac{(p-2)(p-4)}{64}(l^2 + 5(p_1^2 + p_2^2) - 8p - 23)\eta^{p/2-3}\mu^2
+ \frac{5}{32}(p-2)(p-4)\eta^{p/2-3}\mu^4 + ... \right). \tag{20}\]
Here a zero due to the single puncture has been extracted. A comment about the genus expansion is needed. It is not only \( \eta \sim 1/g^2 \) that carries powers of the string coupling constant, we also have \( \mu \sim 1/g \). This means, for instance, that the \( \mu^2 \eta^{p/2-2} \) term above is at genus 0. Indeed, taking two \( \mu \) derivatives gives \( \langle PPPTT \rangle \) at genus 0. Hence, from this point of view, the above expression, which only keeps the first few terms, is not complete at a fixed genus. However, it includes everything that is needed when we turn to finite radius at zero \( \mu \).

As we saw in Section 2.3, the finite radius expression at \( \mu = 0 \) is obtained immediately by substituting

\[
\mu^2 \rightarrow \frac{1}{12R^2} \tag{21}
\]

and

\[
\mu^4 \rightarrow \frac{7}{240R^4}. \tag{22}
\]

This completes the calculation of \( \langle PTTT \rangle \) to genus 2 at finite radius.

### 3.2 n-point Correlation Function to Genus 1

Next let me consider a general \( n \)-point function. This time to genus 1. However, I will begin with \( \langle PT_1...T_n \rangle \) where \( n \) is even. This will allow us to verify, as noted in [1, 4], that odd-point functions vanish when \( \mu = 0 \).

Generalizing the above calculation, one finds

\[
\langle PT_1...T_n \rangle \sim \sum_{i=1}^{n-1} A(k_i) - \sum_{i_1>i_2} A(k_{i_1}+k_{i_2}) + \sum_{i_1>i_2>i_3} A(k_{i_1}+k_{i_2}+k_{i_3}) - ... \tag{23}
\]

where

\[
A(k) = \sum_{s=0}^{k-1} 4^{k-s} \binom{k}{s} (l)_{-k+s} b^{l-k+s} y^{l-k+s} (H - l + k)_{k-s}. \tag{24}
\]

As before \( \sum_{i=1}^{n-1} k_i = l \). If we are interested only in lower genera, we should keep only the terms with the highest values of \( s \). We can then write

\[
\langle PT_1...T_n \rangle \sim \frac{4^n}{(2^n)!} (l)_{-\frac{n}{2}} B_{n,0} b^{l-\frac{n}{2}} y^{l-\frac{n}{2}} + \frac{4^{n+2}}{(2^{n+1})!} (l)_{-\frac{n+2}{2}} B_{n,1} b^{l-\frac{n+2}{2}} y^{l-\frac{n+2}{2}} + ... \tag{25}
\]
where

\[ B_{n,m} = \sum_{i=1}^{n-1} (k_i - \frac{n-1}{2} - m(i\mu - l + k_i)) \frac{n-1}{2} + m \]

\[ - \sum_{i_1 > i_2} (k_{i_1} + k_{i_2}) - \frac{n-1}{2} - m(i\mu - l + k_{i_1} + k_{i_2}) \frac{n-1}{2} + m + \ldots \] (26)

Terms with higher powers of \( b^\dagger \) and \( b \) vanish. One can calculate that

\[ B_{n,0} = \frac{n!}{2} \mu k_1 \ldots k_{n-1} \] (27)

and

\[ B_{n,1} = \frac{n!(n + 2)}{24} i\mu k_1 \ldots k_{n-1} \times \left[ -\frac{n^2 + n}{2} + \frac{3n}{2} \sum_{i=1}^{n-1} k_i + \frac{n-2}{2} \sum_{i=1}^{n-1} k_i^2 \right. \]

\[ - 3 \sum_{i>j} k_i k_j + 3i\mu \left( \sum_{i=1}^{n-1} k_i - \frac{n}{2} \right) - \frac{n-2}{2} \mu^2 \]. \] (28)

It is seen that the correlation function will be proportional to \( \mu \) and vanish at \( \mu = 0 \) unless we take a \( \mu \) derivative, i.e. insert an extra puncture. Using the results obtained in this way for \( \langle PPT_1 \ldots T_n \rangle \) it is easy to generalize to \( \langle T_1 \ldots T_n \rangle \). The final answer is, after some work,

\[ \langle T_1 \ldots T_n \rangle \sim (n - 3)!! \mu (p - 2) \ldots (p - (n - 4)) \prod_{i=1}^{n-1} p_i \left[ \eta^{p/2 - n/2 + 1} \right. \]

\[ - (p - (n - 2)) \left( \frac{p^2 + (n-1) \sum_{i=1}^{n-1} p_i^2 - 2(n-2)p - 4n + 1}{48} - \frac{(n-1)}{48R^2} \right) \eta^{p/2 - n/2} + \ldots \] (29)

when normalized to collective field theory. So, for the 2-point function we find

\[ pp^{p/2 - \frac{1}{48} p^2 \left( 2p^2 - 7 - \frac{1}{R^2} \right) \eta^{p/2 - 1} + \ldots \] (30)

and for the 4-point one

\[ pp_1 p_2 p_3 \left( \eta^{p/2 - 1} - \frac{1}{48} (p - 2) \left( p^2 + 3(p_1^2 + p_2^2 + p_3^2) - 4p - 15 - \frac{3}{R^2} \right) \eta^{p/2 - 2} \right) \]

\[ + \ldots \] (31)
4 A NEW DUALITY SYMMETRY

Duality is a property of the string theory partition function (and puncture correlation functions). As observed in [13] the function (15) is invariant under

\[
\left\{ \begin{array}{l}
R \to \alpha'/R \\
\mu \to \frac{R}{\sqrt{\alpha'}} \mu
\end{array} \right.
\] (32)

(i.e. \( g \to \frac{\sqrt{\alpha'}}{R} g \), where \( g \) is the string coupling). Correlation functions with non-zero momentum tachyons are clearly not invariant, as discussed in [7]. Instead they are transformed into correlation functions with non-zero winding.

Does the deformed matrix model also have a duality symmetry? In [4] the answer was negative. Indeed, although the standard duality transformation above (with \( \mu^2 \) replaced by \( \eta \)) works to genus 1 (for the partition function), it breaks down for higher genera as shown in [4]. However, I will show below the existence of a new kind of non-perturbative duality. This is the most important result in the paper.

Consider (17). By the same reasoning as for the undeformed case, one finds a symmetry under

\[
\left\{ \begin{array}{l}
R \to \alpha'/4R \\
\frac{1+2\eta}{2\sqrt{\alpha'}} \to \frac{2\eta}{\sqrt{\alpha'}} \frac{1+2\eta}{2\sqrt{\alpha'}} + \frac{1}{8}
\end{array} \right.
\] (33)

for the partition function and all puncture correlation functions. Note that the self-dual radius is \( R = \frac{1}{2}\sqrt{\alpha'} \) for this transformation, rather than \( R = \sqrt{\alpha'} \) as for (32). This is due to the summation being done only over odd states. The peculiar nature of this duality becomes clear when we express it in terms of \( \eta \sim 1/g^2 \). We find that the second transformation is

\[
\eta \to \frac{4R^2}{\alpha'} \eta + \frac{R^2}{2\alpha'} - \frac{1}{8}.
\] (34)

Note the translational piece. Its presence implies that the duality symmetry mixes different genera! A partition function at a fixed genus is not invariant. Only the full sum over all genera is. It is a simple exercise to check the symmetry in the perturbative expansion. It is then seen how contributions from different genera cancel against each other.
5 CONCLUSIONS

The duality symmetry discovered in this paper is difficult to understand from a world sheet point of view. Duality is usually thought to work genus by genus. The deformed matrix model tells us that this might not always be the case. It will be very interesting to understand the physical explanation, and to find out what kind of string theory have this peculiar property, regardless of whether it is a black hole or not.

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