Introduction

In this paper, we give some results concerning the dynamics of double Boolean automata circuits (DBAC’s for short), namely, networks associated to interaction graphs composed of two side-circuits that share a node. More precisely, a double circuit of left-size $\ell \in \mathbb{N}$ and of right-size $r \in \mathbb{N}$ is a graph that we denote by $D_{\ell,r}$. It has $n = \ell + r - 1$ nodes. Nodes that are numbered from 0 to $\ell - 1$ belong to the left-circuit and the others plus the node 0 (that belongs to both side-circuits) belong to the right-circuit. Node 0 is the only node with in- and out-degree 2. All other nodes have in- and out-degree 1.

![Double circuit $D_{\ell,r}$.](image)

Figure 1: Double circuit $D_{\ell,r}$. 

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A DBAC $D_{\ell,r} = (\mathbb{D}_{\ell,r}, F)$ is a network defined by (i) its interaction graph, a double-circuit $\mathbb{D}_{\ell,r}$, and (ii) a global transition function $F$ that updates the states of all nodes in parallel and that is defined as follows by the local transition functions $f_i$ of nodes $i < n$:

\[
\begin{align*}
\forall x \in \{0, 1\}^n,
F(x)_i &= f_i(x_{i-1}), \forall i \notin \{0, \ell\}, \\
F(x)_\ell &= f_\ell(x_0), \\
F(x)_0 &= f_0(x_{\ell-1}, x_{\ell+r-2}) = f^L_0(x_{\ell-1}) \star f^R_0(x_{\ell+r-2}) \text{ where } \star \in \{\wedge, \vee\}.
\end{align*}
\]

All local transition functions are supposed to be non-constant. Thus, $\forall i < n, f_i, f^L_0, f^R_0 \in \{\text{id, neg}\}$ where $\text{id} : a \mapsto a$ and $\text{neg} : a \mapsto \neg a$, $\forall a \in \{0, 1\}$. As a consequence all local transition functions are locally monotone. All arcs $(i, j)$ entering a node $j$ whose local transition function is $\text{id}$ (resp. $\text{neg}$), with respect to node $i$, are labeled with a + sign (resp. a − sign) and called positive arcs (resp. negative arcs). A side-circuit with an even number of negative arcs (resp. odd number of negative arcs) is called a positive (side-) circuit (resp. a negative (side-) circuit).

Given a configuration $x = (x_0, \ldots, x_{n-1}) \in \{0, 1\}^n$ of a DBAC $D_{\ell,r}$, we use the following notation:

\[x^L = (x_0, \ldots, x_{\ell-1}) \quad \text{and} \quad x^R = (x_0, x_{\ell}, \ldots, x_{n-1}).\]

A configuration $x(t) = (x_0(t), \ldots, x_{n-1}(t)) \in \{0, 1\}^n$ such that $\forall k \in \mathbb{N}, F^k p(x(t)) = x(t+k \cdot p) = x(t)$ is said to have period $p$. If $x(t)$ has period $p$ and does not also have period $d < p$, then $x(t)$ is said to have exact period $p$. An attractor of period $p \in \mathbb{N}$, or $p$-attractor, is the set of configurations belonging to the orbit of a configuration that has $p$ as exact period. Attractors of period 1 are called fixed points. The graph whose nodes are the configurations $x \in \{0, 1\}^n$ of a network and whose arcs represent the transitions $(x(t), x(t+1) = F(x(t)))$ is called the transition graph of the network.

In [1], the authors showed the following results:

**Proposition 1**

1. The transition graphs of two DBACs with same side-signs and side-sizes are isomorphic, whatever the definition of $f_0$ (i.e., whether $\star = \lor$ or $\star = \land$ in the definition [7] of $F$ above).

2. Attractor periods of a DBAC divide the sizes of the positive side-circuits if there are some and do not divide the sizes of the negative side-circuits if there are some.
3. If both side-circuits of a DBAC $D_{\ell,r}$ have the same sign, then, attractor periods divide the sum $N = \ell + r$.

4. If both side-circuits of a DBAC $D_{\ell,\ell}$ have the same sign and size, then $D_{\ell,\ell}$ behaves as an isolated circuit of that size and sign (i.e., the sub-transition graph generated by the periodic configurations of $D_{\ell,\ell}$ is isomorphic to the transition graph of an isolated circuit with the same sign and size).

5. A DBAC has as many fixed points as it has positive side-circuits.

6. If both side-circuits of a DBAC $D_{\ell,r}$ are positive and $p$ divides $\ell$ and $r$, then number of attractors of period $p$ is given by $A_p$ (sequence A1037 of the OEIS [3]), namely, the number of attractors of period $p$ of an isolated positive circuit of size a multiple of $p$.

As a result of the first two points of Proposition 1, we may focus on canonical instances of DBACs. Thus, from now on, we will suppose that $\star = \lor$ and $\forall i \neq 0$, $f_i = id$. If the left-circuit is positive (resp. negative), we will suppose that the arc $(\ell - 1, 0)$ is positive (resp. negative) and $f_{\ell} = id$ (resp. $f_{\ell} = neg$) and similarly for the right-circuit. Thus, the only possible negative arcs on the DBACs we will study will be the arcs $(\ell - 1, 0)$ and $(n - 1, 0)$.

The last point of Proposition 1 yields a description of the dynamics of a doubly positive DBAC (in terms of combinatorics only but also gives a characterisation the configurations of period $p$, $\forall p \in \mathbb{N}$ for all types of DBACs). Thus, here, we will focus on the cases where the DBACs have at least one negative side-circuit.

For a negative-positive or a negative-negative DBAC $D_{\ell,r}$, we will use the following notations and results. The function $\mu$ is the Mobiüs function. It appears in the expressions below because of the Mobiüs inversion formula [2].

- The number of configurations of period $p$ is written:
  \[ C_p(\ell, r). \]

- The number of configurations of period exactly $p$ is written and given by:
  \[ C^*_p(\ell, r) = \sum_{q | p} \mu\left(\frac{p}{q}\right) \cdot C_p(\ell, r). \]
• The number of attractors of period $p$ (where $p$ is an attractor period) is written and given by:

$$A_p(\ell, r) = \frac{C_p(\ell, r)}{p} = \frac{1}{p} \sum_{q|p} \mu\left(\frac{p}{q}\right) \cdot C_p(\ell, r).$$

In particular, $C_1(\ell, r) = C_1^*(\ell, r) = 1$.

• The total number of attractors is written and given by:

$$T(\ell, r) = \sum_{p \text{ attractor period}} A_p(\ell, r).$$

For the sake of clarity, we will write $A_p(\ell, r) = A^\pm_p(\ell, r)$ (resp. $A_p(\ell, r) = A^\pm_p(\ell, r)$) and $T(\ell, r) = T^\pm(\ell, r)$ (resp. $T(\ell, r) = T^\pm(\ell, r)$) when $D_{\ell,r}$ will be a negative-positive DBAC (resp. a negative-negative DBAC).

1 Positive-Negative

We first concentrate on DBACs whose left-circuit is negative and whose right-circuit is positive. From Proposition 1 we know that all possible attractor periods of such DBACs divide $r$ and do not divide $\ell$. We also know that these networks have exactly one fixed point. In the sequel, we focus on attractor periods $p > 1$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{interaction_graph.png}
\caption{Interaction graph of a negative-positive DBAC. All arcs are positive (resp. all local transition functions are equal to $id$) except for the arc $((\ell - 1, 0)$ (resp. except for the local transition function $f_0^\ell = neg$).}
\end{figure}
Table 1: Total number of attractors of a negative-positive dbac $D_{\ell, r}$ (obtained by computer simulations). Each colour corresponds to a value of $gcd(\ell, r)$. The line $T^+_r$ (resp. the column $T^-_\ell$) gives the total number of attractors of an isolated positive (resp. negative) circuit of size $r$ (resp. $\ell$).
Characterisation of configurations of period $p$

Let $p$ be a divisor of $r = p \cdot q$ that does not divide $\ell = k \cdot p + d$, $d = \ell \mod p$. We write $\Delta_p = \gcd(p, \ell) = \gcd(p, d)$. Let $x = x(t)$ be a configuration of period $p$: $\forall m \in \mathbb{N}$, $x(t + m \cdot p) = x(t)$. Note that to describe the dynamics of $D_{\ell,r}$, it suffices to describe the behaviour of node 0 (see [1]). The configuration $x$ satisfies the following:

\[
x_0(t) = x_0(t + k \cdot p) = x_{t-1}(t + k \cdot p - 1) \lor x_{n-1}(t + k \cdot p - 1) = x_{t-k-p}(t) \lor x_0(t + (k - q) \cdot p) = -x_d(t) \lor x_0(t)
\]

Thus, if $x_0(t) = 0$, then $x_d(t) = 1$ (and also, because $x_0(t) = -x_0(t - d) \lor x_0(t)$, if $x_0(t) = 0$, then $x_0(t - d) = 1$). From this, we may derive the following characterisation:

**Proposition 2** Let $p \in \mathbb{N}$ be divisor of $r = p \cdot q$ that does not divide $\ell$. A configuration $x = x(t)$ has period $p$ if and only if there exists a circular word $w \in \{0,1\}^p$ of size $p$ that does not contain the sub-sequence $0u0$, $u \in \{0,1\}^{d-1}$ and that satisfies:

\[
x^L = w^k w[0 \ldots d - 1] \quad \text{and} \quad x^R = w^q
\]

where $w[0 \ldots m] = w_0 \ldots w_m$. More precisely, the word $w$ satisfies:

\[
\forall i < p, w_i = x_0(t + i).
\]

Consequently, the property for a configuration to be of period $p$ depends only on $d = \ell \mod p$ and on $p$ (and not on $\ell$ nor on $r$).

As a result of the characterisation above, we may focus on negative-positive DBACs $D_{\ell,r}$ such that $\ell < r$ (i.e., $\ell = \ell \mod r$) and to count the number of attractors of period $p$ we may focus on DBACs $D_{\ell,r}$ such that $\ell < p$ (i.e., $\ell = d = \ell \mod p$). In other words, from Proposition 2

\[
A_p^\pm(\ell, r) = A_p^\pm(\ell \mod r, r) = A_p^\pm(\ell \mod p, r).
\]

**Combinatorics**

From Proposition 2, each configuration $x$ of period $p$ is associated with a circular word $w \in \{0,1\}^p$ that does not contain the sub-sequence $0u0$, $u \in \{0,1\}^{d-1}$, $d = \ell \mod p$. It is easy to check that this word $w$ can be written as an interlock of a certain number $N$ of circular words $w^{(1)}, w^{(2)}, \ldots, w^{(N)}$ of size $T = p/N$ that do not contain the sub-sequence 00 (see figure 3). More precisely, the size of a word $w^{(j)}$ satisfies $T \cdot d = K \cdot p$ for a certain $K$ such that $T$ and $K$ are minimal. In other words, $T \cdot d = \text{lcm}(d, p) = \frac{dp}{\gcd(d,p)}$ and thus $T = \frac{p}{\Delta_p}$. Consequently, $N = \Delta_p$ and we obtain the following lemma which explains why, in each column of Table 1, every cell of a same colour contains the same number.
Figure 3: The circular word \( w = w_0 \ldots w_{p-1} = x_0(t) \ldots x_0(t + p - 1) \) mentioned in Proposition 2 that characterises a configuration \( x(t) \) of period \( |w| = p \). In this example, \( p = 15 \), \( d = \ell \mod p = 6 \) so that \( w \) is made of an interlock of \( \Delta_p = \gcd(d, p) = 3 \) words \( w^{(j)} = w_0^{(j)} \ldots w_4^{(j)} \) of size \( p/\Delta_p = 5 \).

**Lemma 1** Let \( p \) be a divisor of \( r = p \cdot q \) that does not divide \( \ell \). The number of configurations of period \( p \), \( C_p(\ell, r) \), depends only on \( \Delta_p = \gcd(\ell, p) \) and on \( p \). Thus, we write:

\[
C_p(\ell, r) = C_p, \Delta_p.
\]

The number of circular words of size \( n \) that do not contain the sub-sequence \( 00 \) is counted by the Lucas sequence (sequence A204 of the OEIS [3]):

\[
\begin{align*}
L(1) &= 1 \\
L(2) &= 2 \\
L(n) &= L(n - 1) + L(n - 2) = \phi^n + \bar{\phi}^n = \phi^n + (-\frac{1}{\phi})^n,
\end{align*}
\]

where \( \phi = \frac{1 + \sqrt{5}}{2} \sim 1.61803399 \) is the golden ratio and \( \bar{\phi} = 1 - \phi = \frac{1 - \sqrt{5}}{2} \sim -0.61803399 \). Among the properties of \( \phi \) that will be useful to us in the sequel are the following:

\[
\phi^2 = 1 + \phi \quad \text{and} \quad \bar{\phi} = -\frac{1}{\phi}.
\]

Thus, to build a circular word \( w \in \{0, 1\}^p \) without the sub-sequence \( 0u0 \), \( u \in \{0, 1\}^{\ell - 1} \), one needs to chose \( \Delta_p \) among \( L(\frac{p}{\Delta_p}) \) words \( w^{(j)} \) of size \( \frac{p}{\Delta_p} \) without the sub-sequence \( 00 \). As a result holds Proposition 3 below:
Proposition 3 The number of configurations of period \( p \) is given by:

\[ C_{p, \Delta_p} = L \left( \frac{p}{\Delta_p} \right)^{\Delta_p}. \]

Consequently, the number of attractors of period \( p \) is given by:

\[ A^+_{p, \Delta_p}(\ell, r) = A_{p, \Delta_p} = \frac{1}{p} \sum_{q \mid p} \mu \left( \frac{p}{q} \right) L \left( \frac{p}{\Delta_p} \right)^{\Delta_p}, \]

where \( \Delta_p = \gcd(\ell, p) \).

Number of configurations of period \( p \)

Let us develop the expression for \( C_{p}(\ell, r) = C_{p, \Delta_p} \):

\[
C_{p, \Delta_p} = L \left( \frac{p}{\Delta_p} \right)^{\Delta_p} \\
= L \left( \phi^{\frac{\ell}{\Delta_p}} + (-\phi)^{-\frac{\ell}{\Delta_p}} \right)^{\Delta_p} \\
= \sum_{k \leq \Delta_p} \left( \frac{\Delta_p}{k} \right) \cdot \phi^{\frac{\ell k}{\Delta_p}} \cdot (-\phi)^{-p \cdot \frac{\ell k}{\Delta_p}} \\
= (-\phi)^{-p} \cdot \sum_{k \leq \Delta_p} \left( \frac{\Delta_p}{k} \right) \cdot \phi^{2 \cdot \frac{\ell k}{\Delta_p}} \cdot (-1)^{\frac{\ell k}{\Delta_p}} \\
= \bar{\phi}^p \cdot \sum_{k \leq \Delta_p} \left( \frac{\Delta_p}{k} \right) \cdot (-\phi^2)^{\frac{\ell k}{\Delta_p}} \\
= \bar{\phi}^p \cdot ((-\phi^2)^{\frac{p}{\Delta_p}} + 1)^{\Delta_p} \\
= (-1)^p \cdot \bar{\phi}^p \cdot ((-1)^{\frac{\ell}{\Delta_p}} \cdot (\phi^2)^{\frac{\ell}{\Delta_p}} + 1)^{\Delta_p}.
\]

If \( p \) is odd, \( \frac{p}{\Delta_p} \) cannot be even. Thus, there are three cases only:

1. \( p \) and \( \frac{p}{\Delta_p} \) are odd. Thus, because \( \Delta_p \) is necessarily also odd:

\[
C_{p, \Delta_p} = \bar{\phi}^p \cdot (-1)^{\Delta_p} = \bar{\phi}^p \cdot ((\phi^2)^{\frac{\ell}{\Delta_p}} - 1)^{\Delta_p}. \tag{2}
\]

2. \( p \) is even and \( \frac{p}{\Delta_p} \) is odd. Thus, because \( \Delta_p \) is necessarily even:

\[
C_{p, \Delta_p} = \bar{\phi}^p \cdot (\phi^2)^{\frac{\ell}{\Delta_p}} + 1)^{\Delta_p} = \bar{\phi}^p \cdot ((\phi^2)^{\frac{\ell}{\Delta_p}} - 1)^{\Delta_p}. \tag{3}
\]

3. \( p \) and \( \frac{p}{\Delta_p} \) are both even. Thus:

\[
C_{p, \Delta_p} = \bar{\phi}^p \cdot ((\phi^2)^{\frac{\ell}{\Delta_p}} - 1)^{\Delta_p}. \tag{4}
\]
To sum up, we give below Proposition 4 whose last part can be derived from the relation between the Euler totient $\psi(\cdot)$ and the Mobiüs function $\mu(\cdot)$, $\psi(n) = \sum_{m|n}(n/m) \cdot \mu(m)$, and from the following equations where $\Delta_q = \gcd(q, \ell)$:

$$T^\pm(\ell, r) = \sum_{p|r} \sum_{q|p} \frac{1}{p} \cdot \mu\left(\frac{p}{q}\right) \cdot C_{q, \Delta_q} = \frac{1}{r} \cdot \sum_{p|r} \sum_{q|p} C_{q, \Delta_q} \cdot \frac{r}{(p/q) \cdot q} \cdot \mu\left(\frac{p}{q}\right)$$

$$= \frac{1}{r} \cdot \sum_{q|r} C_{q, \Delta_q} \sum_{k|\frac{r}{q}} \frac{r}{k \cdot q} \cdot \mu(k) = \frac{1}{r} \cdot \sum_{q|r} \psi\left(\frac{r}{q}\right) \cdot C_{q, \Delta_q}.$$

**Proposition 4** Let $\Delta_p = \gcd(\ell, p)$ where $p$ is a divisor of $r$ that does not divide $\ell$. Then, the number of configurations of period $p$ is given by:

$$C_{p, \Delta_p} = \begin{cases} \left\lfloor \frac{\phi(p)}{p} \right\rfloor \cdot ((\phi^2)^{\frac{\ell}{p}} - 1)^{\Delta_p} & \text{if } \frac{p}{\Delta_p} \text{ is odd}, \\ \left\lfloor \frac{\phi(p)}{p} \right\rfloor \cdot ((\phi^2)^{\frac{\ell}{p}} + 1)^{\Delta_p} & \text{if } \frac{p}{\Delta_p} \text{ is even.} \end{cases}$$

In particular, $C_{r, \gcd(\ell, r)}$ counts the total number of periodic configurations of the network. The number of attractors of period $p$ and the total number of attractors are respectively given by:

$$A^\pm_p(\ell, r) = A^\pm_{p, \Delta_p} = \frac{1}{p} \sum_{q|p} \mu\left(\frac{p}{q}\right) \cdot C_{q, \Delta_q}, \text{ and } T^\pm(\ell, r) = \frac{1}{r} \sum_{p|r, \neg(p|\ell)} C_{p, \Delta_p}.$$

**Upper bounds**

From equations (2), (3) and (4), one can derive that $C_{p, \Delta_p}$ and thus $A^\pm_p(\ell, r) = A^\pm_{p, \Delta_p}$ are maximal when $\Delta_p$ is minimal (i.e., $\Delta_p = 1$), if $p$ is odd and if $p$ is even, on the contrary, $C_{p, \Delta_p}$ and $A^\pm_p(\ell, r)$ are maximal when $\Delta_p$ is maximal (i.e., $\Delta_p = \frac{p^2}{2}$). Thus, we have:

$$C_{p, \Delta_p} \leq \left\lfloor \frac{\phi(p)}{p} \right\rfloor \cdot ((\phi^2)^{\frac{\ell}{p}} + 1)^{\Delta_p}$$

$$\leq \left\lfloor \frac{\phi(p)}{p} \right\rfloor \cdot ((\phi^4) + 1)^{\frac{p}{2}} = C_{p, \frac{p^2}{2}}$$

$$= \frac{(3+3\phi)^{\frac{p}{2}}}{\phi^p}$$

$$= \left(\frac{3+3\phi}{1+\phi}\right)^{\frac{p}{2}}$$

$$= 3^{\frac{p}{2}},$$
In addition, $A_{p,\Delta}^\pm \leq \frac{1}{p} \cdot F_p(\sqrt{3})$ where $F_p(a) = \sum_{d|p} \mu(\frac{p}{d}) \cdot a^d$. Now, it can be shown that:

$$\forall a > \phi, \forall p \neq 2, \quad a^{p-1} < F_p(a) = \sum_{d|p} \mu(\frac{p}{d}) \cdot a^d < a^p.$$  

Consequently,

$$\forall p \neq 2, \quad A_{p,\Delta}^\pm \leq \frac{1}{p} \cdot F_p(\sqrt{3}) < 2 \cdot (\frac{\sqrt{3}}{2})^p \cdot \frac{1}{p} \cdot F_p(2) = 2 \cdot (\frac{\sqrt{3}}{2})^p \cdot A_p^+.$$  

(5)

where $A_p^+$ refers to the number of attractors of period $p$ of an isolated positive circuit whose size is a multiple of $p$. Then, from (5), we may derive the following result:

**Proposition 5** For all $p \neq 2$, the number of attractors of period $p$ satisfies:

$$A_p^\pm(\ell, r) = A_{p,\Delta}^\pm < 2 \cdot (\frac{\sqrt{3}}{2})^p \cdot A_p^+$$

and for $p = 2$: $A_{2,\Delta}^\pm = 1 = A_{2}^\pm$.

Let us denote by $T_n^+$ the number of attractors of a positive circuit of size $n$. From Proposition 5 for all $r$, the total number of attractors of a negative-positive DBAC satisfies:

$$T_{\ell,r}^+ < 2 \cdot (\frac{\sqrt{3}}{2})^r \cdot T_r^+.$$  

However, computer simulations (cf Table 1 last line) show that this bound on $T_{\ell,r}^+$ is too large. We leave open the problem of finding a better bound.

## 2 Negative-Negative

We now concentrate on doubly negative DBACs. The canonical DBAC we will use in the discussion below is defined in Figure 4.

Let $p \in \mathbb{N}$ be a possible attractor period of $D_{\ell,r}$ ($p$ divides $N = \ell + r$ but divides neither $\ell$ nor $r$). Without loss of generality, suppose $\ell \mod p > r \mod p = d$. Because $p$ divides $\ell + r$, it holds that $\ell \mod p = p - d$. Then, for any configuration $x = x(t) \in \{0,1\}^n$ of period $p$, we have the following:

\[
\begin{align*}
x_0(t) &= -x_{\ell-1}(t - 1) \lor -x_{n-1}(t - 1) \\
&= -x_0(t - \ell) \lor -x_0(t - r) \\
&= -x_0(t + r) \lor -x_0(t + \ell) \\
&= -x_0(t + d) \lor -x_0(t - d)
\end{align*}
\]

As a consequence, if $x_0(t) = 0$, then $x_0(t + d) = x_0(t - d) = 1$ and if $x_0(t) = 1$, then either $x_0(t + d) = 0$, or $x_0(t - d) = 0$. Thus, the circular word $w =
$\ell - 1 = \ell + r - 2$

Figure 4: Interaction graph of a negative-negative DBAC. All arcs are positive (resp. all local transition functions are equal to id) except for the arcs $(\ell - 1, 0)$ and $(n - 1, 0)$ (resp. except for the local transition functions $f_0^L = \text{neg}$ and $f_0^R = \text{neg}$).

$x_0(t) \ldots x_0(t + p - 1)$ contains neither the sub-sequence $0u0$ nor the sub-sequence $1u1u'1$ ($u, u' \in \{0, 1\}^{d-1}$).

Let $\Delta = \text{gcd}(\ell, r)$. As in the previous section, $w$ can be written as an interlock of $\Delta_p = \text{gcd}(d, p) = \text{gcd}(\Delta, p)$ words $w^{(j)}$ of size $p/\Delta_p$ that do not contain the sub-sequences $00$ and $111$. As one may show by induction, the number of such words is counted by the Perrin sequence [4], sequence A1608 of the OEIS [3]:

$$
\begin{align*}
P(0) &= 3, \\
P(1) &= 0, \\
P(2) &= 2, \\
P(n) &= P(n - 2) + P(n - 3) = \alpha^n + \beta^n + \overline{\beta}^n,
\end{align*}
$$

where $\alpha, \beta = \frac{1}{2} \cdot (-\alpha + i \cdot \sqrt{\frac{5}{2} - 1})$ and $\overline{\beta}$ are the three roots of $x^3 - x - 1 = 0$, and $\alpha$, the only real root of this equation, is called the plastic number [5].

Using similar arguments to those used in the previous section, we derive Proposition 6 below. This proposition explains why, in Table 3, all cells of a same diagonal (i.e., when $N = \ell + r$ is kept constant) that have the same colour also contain the same number: the number of attractors depends only on $N = \ell + r$ and on $\Delta = \text{gcd}(\ell, r)$ and not on $\ell$ nor $r$. Equations in Proposition 6 exploit, in particular, the fact that if $p$ divides $\ell$ or $r$ then $C_{p, \Delta_p} = 0$ (because then $P(\frac{p}{\Delta_p}) = P(1) = 0$).

**Proposition 6** Let $N = \ell + r$ and let $p \in \mathbb{N}$ be a possible attractor period of $D_{\ell,r}$ ($p$ divides $N$ but divides neither $\ell$ nor $r$). Let also $\Delta = \text{gcd}(\ell, r)$ and $\Delta_p = \text{gcd}(\Delta, p)$. 
Then, the number of configurations of period $p$ of the doubly negative DBAC $D_{\ell,r}$ depends only on $p$ and $\Delta_p$. It is given by:

$$C_p(\ell,r) = C_{p,\Delta_p} = P\left(\frac{p}{\Delta_p}\right)^{\Delta_p}.$$ 

The number of $p$-attractors and the total number of attractors of a doubly negative DBAC $D_{\ell,r}$ are respectively given by:

$$A_p(\ell,r) = A_{p,\Delta_p} = 1 \cdot \sum_{q|p} \mu\left(\frac{p}{q}\right) \cdot P\left(\frac{q}{\Delta_q}\right)^{\Delta_q},$$

$$T^\pm(\ell,r) = T_{N,\Delta}^\pm = \frac{1}{N} \cdot \sum_{p|N} \psi\left(\frac{N}{p}\right) \cdot P\left(\frac{p}{\Delta_p}\right)^{\Delta_p}.$$ 

The expression for $T_{N,\Delta}^\pm$ in Proposition 6 above simplifies into the following if $K = \frac{N}{\Delta}$ is a prime:

$$T_{N,\Delta}^\pm = \frac{1}{N} \cdot \sum_{q|\Delta, \gcd(q,K)=1} \psi(q) \cdot P(K)^\frac{q}{\Delta_q}.$$ 

In particular, if $K = \frac{N}{\Delta} = 2$ or $K = \frac{N}{\Delta} = 3$, then because $P(2) = 2$ and $P(3) = 3$,

$$T^\pm(\frac{N}{2},\frac{N}{2}) = T_{\frac{N}{2},\frac{N}{2}}^\pm = \frac{1}{N} \cdot \sum_{q|\frac{N}{2}, \gcd(q,2)=1} \psi(q) \cdot 2^{\frac{N}{2}},$$

$$T^\pm(\frac{N}{3},\frac{2N}{3}) = T_{\frac{N}{3},\frac{2N}{3}}^\pm = \frac{1}{N} \cdot \sum_{q|\frac{N}{3}, \gcd(q,3)=1} \psi(q) \cdot 3^{\frac{N}{3}}.$$ 

From the computer simulations we performed (see Tables 2 and 3), we observe the following:

1. Given $\ell$, $T^\pm(\ell,r)$ is maximal when $r = \ell$.

2. Given an integer $N$ which is not a multiple of 3, $T_{N,\Delta}^\pm$ is maximal when $\Delta$ is maximal (in particular, if $N$ is even without being a multiple of 3, then $T_{N,\Delta}^\pm \leq T_{\frac{N}{2},\frac{N}{2}}^\pm$).

3. Given an integer $N$ which is a multiple of 3, $T_{N,\Delta}^\pm$ is maximal when $\Delta = \frac{N}{3}$.

We leave the proofs of these three points as an open problem.
\[ \ell = \rho \]

\[ \ell = 2 \times r \]

\[ \ell + r \text{ prime} \]

\[ \ell + r \text{ even} \]

\[ \ell + r \text{ odd} \]

Table 2: Total number of attractors of a negative-negative DBAC $D_{\ell,r}$ (obtained by computer simulations).
\[
\ell = r
\]
\[
\ell = 2 \times r
\]
\[
\ell = 4 \times r
\]
\[
\gcd(\ell, r) = \]

| \( \ell \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 | 8192 | 16384 |

Table 3: Total number of attractors of a negative-negative DBAC \( D_{\ell, r} \) (obtained by computer simulations). Each colour corresponds to a value of \( \gcd(\ell, r) \). The last column gives the total number \( T_{\ell}^- \) of attractors of an isolated negative circuit.

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