Behaviour of the order of Tate-Shafarevich groups for the quadratic twists of elliptic curves

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Abstract. We present the results of our search for the orders of Tate-Shafarevich groups for the quadratic twists of elliptic curves. We formulate a general conjecture, giving for a fixed elliptic curve $E$ over $\mathbb{Q}$ and positive integer $k$, an asymptotic formula for the number of quadratic twists $E_d$, $d$ positive square-free integers less than $X$, with finite group $E_d(\mathbb{Q})$ and $|\mathcal{M}(E_d(\mathbb{Q}))| = k^2$. This paper continues the authors previous investigations concerning orders of Tate-Shafarevich groups in quadratic twists of the curve $X_0(49)$. In section 8 we exhibit 88 examples of rank zero elliptic curves with $|\mathcal{M}(E)| > 63408^2$, which was the largest previously known value for any explicit curve. Our record is an elliptic curve $E$ with $|\mathcal{M}(E)| = 1029212^2$.

Keywords: elliptic curves, Tate-Shafarevich group, Cohen-Lenstra heuristics, distribution of central $L$-values

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1 Introduction

Let $E$ be an elliptic curve defined over $\mathbb{Q}$ of conductor $N_E$, and let $L(E, s)$ denote its $L$-series. Let $\mathcal{M}(E)$ be the Tate-Shafarevich group of $E$, $E(\mathbb{Q})$ the group of rational points, and $R(E)$ the regulator, with respect to the Néron-Tate height pairing. Finally, let $\Omega_E$ be the least positive real period of the Néron differential of a global minimal Weierstrass equation for $E$, and define $C_\infty(E) = \Omega_E$ or $2\Omega_E$ according as $E(\mathbb{R})$ is connected or not, and let $C_{\text{fin}}(E)$ denote the product of the Tamagawa factors of $E$ at the bad primes. The Euler product defining $L(E, s)$ converges for $\Re s > 3/2$. The modularity conjecture, proven by Wiles-Taylor-Diamond-Breuil-Conrad, implies that $L(E, s)$ has an analytic continuation to an entire function. The
Birch and Swinnerton-Dyer conjecture relates the arithmetic data of $E$ to the behaviour of $L(E, s)$ at $s = 1$.

**Conjecture 1** *(Birch and Swinnerton-Dyer)*

(i) $L$-function $L(E, s)$ has a zero of order $r = \text{rank } E(\mathbb{Q})$ at $s = 1$,

(ii) $\mathfrak{m}(E)$ is finite, and

$$\lim_{s \to 1} \frac{L(E, s)}{(s - 1)^r} = \frac{C_\infty(E)C_{\text{fin}}(E)R(E)|\mathfrak{m}(E)|}{|E(\mathbb{Q})_{\text{tors}}|^2}.$$

If $\mathfrak{m}(E)$ is finite, the work of Cassels and Tate shows that its order must be a square.

The first general result in the direction of this conjecture was proven for elliptic curves $E$ with complex multiplication by Coates and Wiles in 1976 [6], who showed that if $L(E, 1) \neq 0$, then the group $E(\mathbb{Q})$ is finite. Gross and Zagier [18] showed that if $L(E, s)$ has a first-order zero at $s = 1$, then $E$ has a rational point of infinite order. Rubin [26] proves that if $E$ has complex multiplication and $L(E, 1) \neq 0$, then $\mathfrak{m}(E)$ is finite. Let $g_E$ be the rank of $E(\mathbb{Q})$ and let $r_E$ the order of the zero of $L(E, s)$ at $s = 1$. Then Kolyvagin [20] proved that, if $r_E \leq 1$, then $r_E = g_E$ and $\mathfrak{m}(E)$ is finite. Very recently, Bhargava, Skinner and Zhang [1] proved that at least 66.48% of all elliptic curves over $\mathbb{Q}$, when ordered by height, satisfy the weak form of the Birch and Swinnerton-Dyer conjecture, and have finite Tate-Shafarevich group.

When $E$ has complex multiplication by the ring of integers of an imaginary quadratic field $K$ and $L(E, 1)$ is non-zero, the $p$-part of the Birch and Swinnerton-Dyer conjecture has been established by Rubin [27] for all primes $p$ which do not divide the order of the group of roots of unity of $K$. Coates et al. [5] [3], and Gonzalez-Avilés [17] showed that there is a large class of explicit quadratic twists of $X_0(49)$ whose complex $L$-series does not vanish at $s = 1$, and for which the full Birch and Swinnerton-Dyer conjecture is valid (covering the case $p = 2$ when $K = \mathbb{Q}(\sqrt{-7})$). The deep results by Skinner-Urban (29, Theorem 2) (see also Theorem 7 in section 8.4 below) allow, in specific cases (still assuming $L(E, 1)$ is non-zero), to establish $p$-part of the Birch and Swinnerton-Dyer conjecture for elliptic curves without complex multiplication for all odd primes $p$ (see examples in section 8.4 below, and section 3 in [10]).

The numerical studies and conjectures by Conrey-Keating-Rubinstein-Snaith [7], Delaunay [12][13], Watkins [31], Radziwill-Soundararajan [25] (see also the papers [11][10][9], and references therein) substantially extend the systematic tables given by Cremona.

This paper continues the authors previous investigations concerning orders of Tate-Shafarevich groups in quadratic twists of the curve $X_0(49)$. We
present the results of our search for the orders of Tate-Shafarevich groups for additional four elliptic curves (two with CM, and two without CM), for the same large ranges of the index (namely, $32 \cdot 10^9$). Our results support a general conjecture (Conjecture 2), giving for a fixed elliptic curve $E$ over $\mathbb{Q}$ and positive integer $k$, an asymptotic formula for the number of quadratic twists $E_d$, $d$ positive square-free integers less than $X$, with finite group $E_d(\mathbb{Q})$ and $|\mathfrak{w}(E_d(\mathbb{Q}))| = k^2$. Let us formulate explicitly the conjecture. Let $f_E(k, X)$ denote the number of positive square-free integers $d \leq X$, such that $(d, N_E) = 1$, $L(E_d, 1) \neq 0$, and $|\mathfrak{w}(E_d)| = k^2$.

**Conjecture** Let $E$ be an elliptic curve defined over $\mathbb{Q}$. For any positive integer $k$ there are constants $c_k(E) \geq 0$ and $d_k(E)$ such that

$$f_E(k, X) \sim c_k(E)X^{3/4}(\log X)^{d_k(E)}, \quad \text{as } X \to \infty.$$ 

The main results supporting the conjecture are reported in section 3; additional support is also given in subsection 7.1. In section 9 we give examples of elliptic curves $E$ with $c_r(E) = 0$, for some values $r$.

All the experiments concerning statistics of the $L$-values of quadratic twists of $X_0(49)$ (and related orders of Tate-Shafarevich groups) done in [9], are also confirmed for these four elliptic curves (see sections 4 - 6).

It has long been known that the order of $\mathfrak{w}(E)[p]$ can be arbitrarily large for elliptic curves $E$ defined over $\mathbb{Q}$ and $p = 2, 3$ (for $p = 3$, the result is due to J. Cassels ([2]), and for $p = 2$ it is due to F. McGuinness ([21]), but no similar result is known for $p > 3$. We also stress that it has not yet been proven that there exist elliptic curves $E$ defined over $\mathbb{Q}$ for which $\mathfrak{w}(E)[p]$ is non-zero for arbitrarily large primes $p$. In our earlier papers, we have investigated (see [11], [9]) some numerical examples of $E$ defined over $\mathbb{Q}$ for which $L(E, 1)$ is non-zero and the order of $\mathfrak{w}(E)$ is large. We extend these numerical results here in section 8, with the largest proved examples of $\mathfrak{w}(E)$ having order $1029212^2 = 2^4 \cdot 79^2 \cdot 3257^2$.

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2 Formulae for the orders of $\omega(E_d)$ when $L(E_d, 1) \neq 0$

We can compute $L(E_d, 1)$ when it is non-zero for a huge range of positive square-free integers $d$ thanks to the remarkable ideas discovered by Waldspurger, and worked out explicitly in particular cases by many authors. These ideas show that $L(E_d, 1)$, when it is non-zero, is essentially equal to the $d$-th Fourier coefficient of an explicit modular form of weight $3/2$. We now recall some details for four elliptic curves (named $A$, $B$, $C$, and $D$ below).

**Notation.** Let $q := e^{2\pi iz}$, $\Theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2}$, $\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1-e^{2\pi inz})$.

**Example 1.** \[30\] Let $A : y^2 = x^3 - x$; for any square-free integer $d$ consider the quadratic twist $A_d : y^2 = x^3 - d^2 x$.

Let $f_i(z) := \eta(8z)\eta(16z)\Theta(2iz)z = \sum a_i(n)q^n$ $(i = 1, 2)$. Consequently, for odd $d$ we have $a_i(d) = |\{(x, y, z) \in \mathbb{Z}^3 : d = 2ix^2 + y^2 + 32z^2\}| - \frac{1}{2}|\{(x, y, z) \in \mathbb{Z}^3 : d = 2ix^2 + y^2 + 8z^2\}|$. If $d \geq 1$ is an odd square-free integer, then Tunnell proved that

$$L(A_{id}, 1) = \frac{2^{i-1}a_i(d)^2\Omega_A}{4\sqrt{2^{i-1}d}}.$$ 

Therefore assuming the Birch and Swinnerton-Dyer conjecture, $A_{id}$ has $\mathbb{Q}$-rank zero iff $a_i(d) \neq 0$. In addition, if $a_i(d) \neq 0$, then

$$|\omega(A_{id})| = \left(\frac{a_i(d)}{\tau(d)}\right)^2,$$

where $\tau(d)$ denotes the number of divisors of $d$. Let $a(d) := a_1(d)$.

**Example 2.** \[16, 15\] Let $B : y^2 = x^3 - 1$. For any square-free positive integer $d \equiv 1(\text{mod } 4)$, $(d, 6) = 1$, consider the quadratic twist $B_d : y^2 = x^3 - d^3$. Let $a(d)$ denote the $d$-th Fourier coefficient of $\eta^2(12z)\Theta(z)$. Frey (\[16\], page 232) proves the following result

$$L(B_{d}, 1) = \begin{cases} 0 & \text{if } d \equiv 3(\text{mod } 4) \\
 a(d)^2 L(B_{13}, 1)^{\frac{d}{26}} & \text{if } d \equiv 1(\text{mod } 24) \\
 \left(\frac{a(d)}{a(13)}\right)^2 \sqrt{\frac{15}{d}} L(B_{13}, 1) & \text{if } d \equiv 13(\text{mod } 24) \\
 \left(\frac{a(d)}{a(5)}\right)^2 \sqrt{\frac{5}{d}} L(B_5, 1) & \text{if } d \equiv 5(\text{mod } 24) \\
 \left(\frac{a(d)}{a(17)}\right)^2 \sqrt{\frac{17}{d}} L(B_{17}, 1) & \text{if } d \equiv 17(\text{mod } 24) \end{cases}$$

One can show \[15\] that $a(d) = \frac{1}{2} \sum (-1)^n$, where the sum is taken over all $m, n, k \in \mathbb{Z}$ satisfying $m^2 + n^2 + k^2 = d$, $3 \nmid m$, $3 \nmid n$ and $2 \nmid m + n$. 

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Let \( l = l_i + \left[ \frac{2i+1}{2} \right] \), where \( l_i = \left| \{ p|d : p \equiv i(\text{mod } 3) \} \right| \). Assuming the Birch and Swinnerton-Dyer conjecture, we obtain

\[
|\mathfrak{m}(B_d)| = \left( \frac{a(d)}{2^l} \right)^2,
\]

if \( a(d) \neq 0 \).

**Example 3.** Let \( C : y^2 = x^3 + 4x^2 - 16 \) be an elliptic curve of conductor 176. Consider the family \( C_d : y^2 = x^3 + 4dx^2 - 16d^3 \), where \( d \) runs over positive, odd, square-free integers, satisfying \( \left( \frac{d}{11} \right) = 1 \), i.e. such that \( d \equiv 1, 3, 4, 5, 9(\text{mod } 11) \). Let \( a(d) = \frac{n_d - m_d}{2} \), where \( n_d = \left| \{(x, y, z) \in \mathbb{Z}^3 : d = x^2 + 11y^2 + 11z^2 \} \right| \), and \( m_d = \left| \{(x, y, z) \in \mathbb{Z}^3 : d = 3x^2 + 2xy + 4y^2 + 11z^2 \} \right| \). All the groups \( C_d(\mathbb{Q})_{\text{tors}} \) are trivial, hence assuming the Birch and Swinnerton-Dyer conjecture, we obtain

\[
|\mathfrak{m}(C_d)| = \frac{a(d)^2}{C_{\text{fin}}(C_d)},
\]

if \( a(d) \neq 0 \). Moreover, in this case we have

\[
L(C_d, 1) = \frac{2^{s(d)}a(d)^2\Omega_C}{\sqrt{d}},
\]

where \( s(d) = 0 \) if \( d \equiv 1(\text{mod } 4) \) and \( s(d) = 1 \) if \( d \equiv 3(\text{mod } 4) \).

**Example 4.** Let \( D : y^2 = x^3 - x^2 - 8x - 16 \) be an elliptic curve of conductor 112. Consider the family \( D_d : y^2 = x^3 - dx^2 - 8d^2x - 16d^3 \), where \( d \) runs over positive, odd, square-free integers, satisfying \( \left( \frac{d}{7} \right) = 1 \), i.e. such that \( d \equiv 1, 2, 4(\text{mod } 7) \). Let \( a(d) = \frac{n_d - m_d}{2} \), where \( n_d = \left| \{(x, y, z) \in \mathbb{Z}^3 : d = x^2 + 14y^2 + 14z^2 \} \right| \), and \( m_d = \left| \{(x, y, z) \in \mathbb{Z}^3 : d = 2x^2 + 7y^2 + 14z^2 \} \right| \). Let us stress, that in case \( d \equiv 3(\text{mod } 8) \) we necessarily have \( a(d) = 0 \). Let \( s(d) = 2 \) if \( d \equiv 1(\text{mod } 4) \) and \( s(d) = 1 \) if \( d \equiv 3(\text{mod } 4) \). All the groups \( D_d(\mathbb{Q})_{\text{tors}} \) have order two, hence assuming the Birch and Swinnerton-Dyer conjecture, we obtain

\[
|\mathfrak{m}(D_d)| = \frac{2^{s(d)}a(d)^2}{C_{\text{fin}}(D_d)},
\]

if \( a(d) \neq 0 \). Moreover, in this case we have

\[
L(D_d, 1) = \frac{a(d)^2\Omega_D}{\sqrt{d}}.
\]

**Definitions.** We say, that a positive square-free odd integer \( d \) satisfies:

(i) condition \((*A)\), if \( d \equiv 1, 3(\text{mod } 8)\);
(ii) condition \((\ast_B)\), if \(d \equiv 1, 5 \pmod{12}\);
(iii) condition \((\ast_C)\), if \(d \equiv 1, 3, 4, 5, 9 \pmod{11}\);
(iv) condition \((\ast_D)\), if \(d \equiv 1, 2, 4 \pmod{7}\) and \(d \not\equiv 3 \pmod{8}\);
(v) condition \((\ast_E^*\)) if it satisfies condition \((\ast_E)\), and \(a(d) \neq 0\).

3 Frequency of orders of \(\varpi\)

Our data contains values of \(|\varpi(E_d)|\) for \(E \in \{A, B, C, D\}\), and \(d \leq 32 \cdot 10^9\) satisfying \((\ast_E)\). The data involves the proven odd orders of \(|\varpi(A_d)|\), and the proven (prime to 6) orders of \(|\varpi(B_d)|\). The non-trivial values of \(|\varpi(C_d)|\) and \(|\varpi(D_d)|\) are (mostly) the conjectural ones. Let \(k_0\) denote the largest positive integer such that for all \(k \leq k_0\) there is quadratic twist \(E_{d_k}\) (with \(d_k\) as above) with \(|\varpi(E_{d_k})| = k^2\). Let \(K_0\) denote the largest positive integer \(k\) such that for some \(d_k\) as above, we have \(|\varpi(E_{d_k})| = k^2\).

| \(E\) | \(k_0\) | \(K_0\) |
|------|------|------|
| \(A\) | 2277 | 2783 |
| \(B\) | 2037 | 3571 |
| \(C\) | 1727 | 4235 |
| \(D\) | 1914 | 2667 |

Note that 3571 is a prime. From our data it follows that \(|\varpi(C_{26650821201})| = 3917^2\), with 3917 the largest known (at the moment) prime dividing the order of \(\varpi(E_d)\) of an elliptic \(E \in \{A, B, C, D, X_0(49)\}\).

Our calculations strongly suggest that for any positive integer \(k\) there are infinitely many positive integers \(d\) satisfying condition \((\ast_E)\), such that \(E_d\) has rank zero and \(|\varpi(E_d)| = k^2\). Below we will state a more precise conjecture.

Let \(f_E(X)\) denote the number of integers \(d \leq X\), satisfying \((\ast_E)\) and such that \(|\varpi(E_d)| = 1\). Let \(g_E(X)\) denote the number of integers \(d \leq X\), satisfying \((\ast_E)\) and such that \(L(E_d, 1) = 0\).

We obtain the following graphs of the functions \(f_E(X)/g_E(X)\), for \(E \in \{A, B, C, D\}\).
We expect that $f_E(X)/g_E(X)$ tend to constants dependent on $E$ (the constant is 1 for $E = X_0(49)$, see section 11 in [9]).

We expect (Delaunay-Watkins [14], Heuristics 1.1):

$$g_E(X) \sim c_E X^{3/4} (\log X)^{b_E + \frac{3}{2}}, \quad \text{as} \quad X \to \infty,$$

where $c_E > 0$, and there are four different possibilities for $b_E$, largely dependent on the rational 2-torsion structure of $E$. Hence, we may expect similar asymptotic formula for $f_E(X)$ as well.

Now let $f_E(k, X)$ denote the number of integers $d \leq X$, satisfying (**$E$) and such that $|\omega(E_d)| = k^2$. Let $F_E(k, X) := \frac{f_E(X)}{f_E(k, X)}$. We obtain the following graphs of the functions $F_E(k, X)$ for $E \in \{A, B, C, D\}$ and $k = 2, 3, 4, 5, 6, 7$. 

Figure 1: Graphs of the functions $f_E(X)/g_E(X)$, for $E \in \{A, B, C, D\}$. 

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Figure 2: Graphs of the functions $F_A(k,X)$ for $k = 2, 3, 4, 5, 6, 7$.

Figure 3: Graphs of the functions $F_B(k,X)$ for $k = 2, 3, 4, 5, 6, 7$. 
The above calculations suggest the following general conjecture (compare Conjecture 8 in [9] for the case of quadratic twists of the curve $X_0(49)$). Let $f_E(k, X)$ denote the number of positive square-free integers $d \leq X$, such that $(d, N_E) = 1$, $L(E_d, 1) \neq 0$, and $|\mu(E_d)| = k^2$.

**Conjecture 2** Let $E$ be an elliptic curve defined over $\mathbb{Q}$. For any positive integer $k$ there are constants $c_k(E) \geq 0$ and $d_k(E)$ such that 
\[ f_E(k, X) \sim c_k(E) X^{3/4} (\log X)^{d_k(E)}, \quad \text{as} \quad X \to \infty. \]
In section 9 we give examples of elliptic curves $E$ with $c_{2m+1}(E) = 0$ and with $c_{4m+2}(E) = 0$.

Note that Park, Poonen, Voight, and Wood ([22], (11.2.2)) have formulated an analogous conjecture for the family of all elliptic curves over the rationals, ordered by height.

4 Cohen-Lenstra heuristics for the order of $\mathcal{U}$

Delaunay [13] has considered Cohen-Lenstra heuristics for the order of Tate-Shafarevich group. He predicts, among others, that in the rank zero case, the probability that $|\mathcal{U}(E)|$ of a given elliptic curve $E$ over $\mathbb{Q}$ is divisible by a prime $p$ should be $f_0(p) := 1 - \prod_{j=1}^{\infty} (1 - p^{-2j}) = \frac{1}{p} + \frac{1}{p^3} + \ldots$. Hence, $f_0(2) \approx 0.580577$, $f_0(3) \approx 0.360995$, $f_0(5) \approx 0.206660$, $f_0(7) \approx 0.145408$, $f_0(11) \approx 0.092$, and so on.

Let $F_E(X)$ denote the number of positive integers $d \leq X$, satisfying $(\ast E)$ and such that $L(E_d,1) \neq 0$. Let $F_E(p,X)$ denote the number of positive integers $d \leq X$, satisfying $(**E)$, such that $|\mathcal{U}(E_d)|$ is divisible by $p$. Let $f_E(p,X) := \frac{F_E(p,X)}{F_E(X)}$, and $f_E(p) := f_E(p,32 \cdot 10^9)$. We obtain the following table

| $E$ | $f_E(2)$ | $f_E(3)$ | $f_E(5)$ | $f_E(7)$ | $f_E(11)$ |
|-----|---------|---------|---------|---------|----------|
| A   | 0.565173| 0.348417| 0.192196| 0.130318| 0.076544 |
| B   | 0.500459| 0.427709| 0.197255| 0.135517| 0.081501 |
| C   | 0.387009| 0.355532| 0.233508| 0.139167| 0.085150 |
| D   | 0.607500| 0.424331| 0.193023| 0.131217| 0.077425 |

The papers of Quattrini [23][24] make a correction to Delaunay’s heuristics for $p$-divisibility of $|\mathcal{U}(E_d)|$ in the family of quadratic twists of a given elliptic curve $E$ of square-free conductor for odd primes dividing the order of $E(\mathbb{Q})_{\text{tors}}$. The author gives an explanation of why and when the original Cohen-Lenstra heuristics should be used for the prediction of the $p$-divisibility of the order of $\mathcal{U}(E_d)$. Roughly speaking, the proportion of values of $|\mathcal{U}(E_d)|$ divisible by a prime number $p$ among (imaginary) quadratic twists of $E$ is significantly bigger when $E$ has a $\mathbb{Q}$-rational point of order $p$, than in the general case where $|E(\mathbb{Q})_{\text{tors}}|$ is not divisible by $p$. In our situation, $|B_{-1}(\mathbb{Q})_{\text{tors}}| = |D_{-1}(\mathbb{Q})_{\text{tors}}| = 6$, $|C_{-1}(\mathbb{Q})_{\text{tors}}| = 5$, the original Cohen-Lenstra predictions for $p = 3$ (resp. for $p = 5$ are $\approx 0.439$ (resp. $\approx 0.239$), and it explains why the values $f_B(3)$, $f_D(3)$, and $f_C(5)$ deviate
from Delaunay’s predictions. We have no explanation why the values \( f_B(2) \), \( f_C(2) \), \( f_D(2) \) deviate from the expected one.

5 Numerical evidence for Delaunay asymptotic formulae

Let \( M_E(T) := \frac{1}{T^*} \sum |\mathcal{W}(E_d)| \), where the sum is over positive integers \( d \leq T \), satisfying \((**E)\), and \( T^* \) denotes the number of terms in the sum. Delaunay ([12], Conjecture 6.1) has conjectured that

\[
M_E(T) \sim c_E T^{1/2}(\log T)^{t_E}, \quad \text{as} \quad T \to \infty,
\]

where \(-1 < t_E < 1\) largely depend on the rational 2-structure of \( E \) (for instance, \( t_E = -5/8 \) for \( E = B \) or \( D \)). If we restrict to prime twists, then we obtain a similar conjecture, but without the log term (Conjecture 4.2 in [12]).

Let \( N_E(T) \) be a subsum of \( M_E(T) \), restricted to prime twists. Let \( f_E(T) := \frac{M_E(T)}{T^{1/2}} \), and \( g_E(T) := \frac{N_E(T)}{T^{1/2}} \). We obtain the following pictures confirming the conjectures 4.2 and 6.1 in [12] for the curves \( A, B, C, \) and \( D \) (compare numerical evidence for the curve \( X_0(49) \) done in [9]).

![Figure 6: Graphs of the functions \( f_E(T) \) and \( g_E(T) \), \( E \in \{A, B, C, D\} \), using the geometric sequence of arguments.](image)

Let us modify the functions \( f_E(T) \) for the curves \( B \) and \( D \), using the logarithmic factor predicted by Delaunay conjecture: \( f_E^*(T) := \left(\frac{\log T}{T^{1/2}}\right)^{5/8} M_E(T) \).
6 Distributions of $L(E_d, 1)$ and $|\omega(E_d)|$

6.1 Distribution of $L(E_d, 1)$

It is a classical result (due to Selberg) that the values of $\log |\zeta(\frac{1}{2} + it)|$ follow a normal distribution.

Let $E$ be any elliptic curve defined over $\mathbb{Q}$. Let $\mathcal{E}$ denote the set of all fundamental discriminants $d$ with $(d, 2N_E) = 1$ and $\epsilon_E(d) = \epsilon_E \chi_d(-N_E) = 1$, where $\epsilon_E$ is the root number of $E$ and $\chi_d = (d/\cdot)$. Keating and Snaith [19] have conjectured that, for $d \in \mathcal{E}$, the quantity $\log L(E_d, 1)$ has a normal distribution with mean $-\frac{1}{2} \log \log |d|$ and variance $\log \log |d|$; see [7] [9] for numerical data towards this conjecture.

Below we consider the families of quadratic twists $E_d$, where $E$ is one of the curve $A$, $B$, $C$, $D$, and $d$ runs over appropriate sets of positive integers dependent of $E$. Our data suggest that the values $\log L(E_d, 1)$ also follow an approximate normal distribution. Let $W_E = \{d \leq 32 \cdot 10^9 : d$ satisfies $(**E)\}$ and $I_x = [x, x+0.1]$ for $x \in \{-10, -9.9, -9.8, \ldots, 10\}$. We create histograms with bins $I_x$ from the data \{(\log L(E_d, 1) + \frac{1}{2} \log \log d) / \sqrt{\log \log d} : d \in W_E\}.

Below we picture these histograms.
Figure 8: Histogram of values $(\log L_{A,d}, 1) + \frac{1}{2} \log \log d) / \sqrt{\log \log d}$ for $d \in W_A$. 

Figure 9: Histogram of values $(\log L_{B,d}, 1) + \frac{1}{2} \log \log d) / \sqrt{\log \log d}$ for $d \in W_B$. 

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6.2 Distribution of $|\mathfrak{w}(E_d)|$

It is an interesting question to find results (or at least a conjecture) on distribution of the order of the Tate-Shafarevich group for rank zero quadratic twists of an elliptic curve over $\mathbb{Q}$. It turns out that the values of $\log(|\mathfrak{w}(E_d)|/\sqrt{d})$ are the natural ones to consider (compare Conjecture 1 in [25], and numerical experiments in [9]). Below we create histograms from the data.
\[
\left\{ \left( \log(|W(E_d)|/\sqrt{d}) - \mu_E \log \log d \right) / \sqrt{\sigma^2_E \log \log d} : d \in W_E \right\},
\]
for the curves \( E \in \{A, B, C, D\} \), and where we take, according to Conjecture 1 in [25], \( \mu_A = -\frac{1}{2} - 2 \log 2 \), \( \mu_B = \mu_D = -\frac{1}{2} - \frac{3}{2} \log 2 \), \( \mu_C = -\frac{1}{2} - \frac{5}{6} \log 2 \), \( \sigma^2_A = 1 + 4(\log 2)^2 \), \( \sigma^2_B = \sigma^2_D = 1 + \frac{1}{2}(\log 2)^2 \), and \( \sigma^2_C = 1 + \frac{7}{6}(\log 2)^2 \).

Our data suggest that the values \( \log(|W(E_d)|/\sqrt{d}) \) also follow an approximate normal distribution. Below we picture these histograms.

**Figure 12:** Histogram of values \( \left( \log(|W(E_d)|/\sqrt{d}) - \mu_A \log \log d \right) / \sqrt{\sigma^2_A \log \log d} \) for \( d \in W_A \).

**Figure 13:** Histogram of values \( \left( \log(|W(B_d)|/\sqrt{d}) - \mu_B \log \log d \right) / \sqrt{\sigma^2_B \log \log d} \) for \( d \in W_B \).
7 More observations

7.1 Additional support towards the Conjecture 2

Let $E \in \{A, B, C, D, X_0(49)\}$. For any elliptic curve $F$ in the isogeny class of $E$, we were able to confirm numerically the Conjecture 2. In all these cases we obtain $c_k(F) > 0$, for $1 \leq k \leq 7$. 
For any integer \( l \geq 2 \), let \( \mathcal{d}_l(E) \) denote the minimal \( 1 \leq d \leq 32 \cdot 10^9 \), satisfying \((**_E)\), and such that \(|\mathcal{W}(E_d)| = l^2\). For each \( 2 \leq l \leq 12 \), we have considered quadratic twists of \( E_{\mathcal{d}_l(E)} \) by \( \frac{d}{\mathcal{d}_l(E)} \), with \( d \leq 32 \cdot 10^9 \) satisfying \((**_E)\). For any of these 55 elliptic curves \( E_{\mathcal{d}_l(E)} \), we were able to confirm numerically the Conjecture 2 as well. In all these cases we obtain \( c_k(E_{\mathcal{d}_l(E)}) > 0 \), for \( 1 \leq k \leq 12 \).

One extra observation is that, in some cases, the number of twists with even order of \( \mathcal{W} \) is much larger than the number of twists with odd order of \( \mathcal{W} \). Examples include: \( B_{\mathcal{d}_8(B)} \), \( D_{\mathcal{d}_8(D)} \), and \( E_{\mathcal{d}_l(E)} \), where \( E = X_0(49) \), \( l = 2, 4, 6, 12 \).

### 7.2 On a question of Coates-Li-Tian-Zhai

Coates et al. \[5\] have proved (among others) the following remarkable result. Let \( E = X_0(49) \).

**Theorem 3** \([5]\), Theorem 1.2) Let \( d = p_1 \cdots p_l \) be a product of \( \geq 0 \) distinct primes, which are \( \equiv 1 \mod 4 \) and inert in \( \mathbb{Q}(\sqrt{-7}) \). Then \( L(E_d,1) \neq 0 \), \( E_d(\mathbb{Q}) \) is finite, the Tate-Shafarevich group of \( E_d \) is finite of odd cardinality, and the full Birch-Swinnerton-Dyer conjecture is valid for \( E_d \).

At the end of his paper, the authors say: “for every elliptic curve \( E \) defined over \( \mathbb{Q} \), we believe there should be some analogues of Theorems 1.1 - 1.4 for the family of quadratic twists of \( E \), and it seems to us to be an important problem to first formulate precisely what such analogues should be, and then to prove them.”

Below we formulate analogues of Theorem 1.2 for the curves \( A \) and \( C \).

Let \( E = A \). Let \( d = p_1 \cdots p_l \). Define the directed graph \( G(d) \) by \( V(G(d)) := \{p_1, \ldots, p_l\} \) and \( E(G(d)) := \{\overrightarrow{p_ip_j} : (p_i, p_j) = -1, 1 \leq i \neq j \leq l\} \). \( G(d) \) is called odd provided that its only even partitions are trivial. Zhao \[32\] proved the following remarkable result.

**Theorem 4** Suppose \( d = p_1 \cdots p_l \), \( p_1 \equiv 3(\text{mod} 8) \), \( p_2, \ldots, p_l \equiv 1(\text{mod} 8) \). Then \( L(E_d,1) \neq 0 \), \( E_d(\mathbb{Q}) \) is finite, and the Tate-Shafarevich group of \( E_d \) is finite of odd cardinality if \( G(d) \) is odd. If \( G(d) \) is odd, then the full Birch-Swinnerton-Dyer conjecture is valid for \( E_d \).

Let \( E = C \), and \( K = \mathbb{Q}(E[2]) \). Then Corollary 3.11 in \[3\] allows to formulate the following
Theorem 5 Let $d = p_1 \cdots p_l$ be a product of $\geq 0$ distinct odd primes, which are prime to 11 and split into two primes over $K$. Then $L(E_d, 1) \neq 0$, $E_d(\mathbb{Q})$ is finite, and the Tate-Shafarevich group of $E_d$ is finite of odd cardinality.

7.3 Some folklore conjecture

It is known [8] that for any positive integer $m$, there are pairwise non-isogenous elliptic curves $E^1, \ldots, E^m$ defined over the rationals such that the rank of the Mordell-Weil group of the $p$-twist of $E^i$, $i = 1, \ldots, m$, has rank zero for a positive proportion of primes $p$. We propose the following (optimistic)

Conjecture 6 For any positive integers $m$ and $k$, there are pairwise non-isogenous elliptic curves $E^4, \ldots, E^m$ defined over the rationals such that the rank of the Mordell-Weil group of the $p$-twist of $E^i$ has rank zero for a positive proportion of primes $p$, and $|W(E^i)| = k^2$, for all $i = 1, \ldots, m$.

Our data for $E \in \{A, B, C, D\}$ (and for $E = X_0(49)$ in [9]) support the above conjecture for $m = 3$ and small values of $k$.

8 Elliptic curves with exceptionally large analytic order of the Tate-Shafarevich groups

It has been known for a long time that $|W(E)|$ (provided is finite) can take arbitrarily large values (Cassels). The previously largest value for $|W(E)|$ was 63408$^2$, found by Dąbrowski and Wodzicki [11]. In ([9], section 5) we propose a candidate with $|W(E)| > 100000^2$. Below we present the results of our search for elliptic curves with exceptionally large analytic order of the Tate-Shafarevich groups. We exhibit 88 examples of rank zero elliptic curves with $|W(E)| > 63408^2$. Our record is an elliptic curve $E = E_2(23, -348)$ with $|W(E)| = 1029212^2$. It is proven that it is the true order of $W$ in this case (see 8.4). Also note that the prime 19861 divides the orders of $W(E_i(22, 304))$ - the largest (at the moment) prime dividing the order of $W(E)$ of an elliptic curve over $\mathbb{Q}$.

8.1 Preliminaries

In this section we compute the analytic order of $W(E)$, i.e., the quantity

$$|W(E)| = \frac{L(E, 1) \cdot |E(\mathbb{Q})_{\text{tors}}|^2}{C_{\infty}(E)C_{\text{fin}}(E)},$$
for certain special curves of rank zero. We use the following approximation of $L(E, 1)$

\[ S_m = 2 \sum_{n=1}^{m} \frac{a_n}{n} e^{-\frac{2\pi}{\sqrt{N}}}, \]

which, for

\[ m \geq \frac{\sqrt{N}}{2\pi} \left( 2 \log 2 + k \log 10 - \log(1 - e^{-2\pi/\sqrt{N}}) \right), \]

differs from $L(E, 1)$ by less than $10^{-k}$.

Consider (as in [11]) the family $E_1(n,p)$:

\[ y^2 = x(x + p)(x + p - 4 \cdot 3^{2n+1}), \]

with $(n, p) \in \mathbb{N} \times (\mathbb{Z} \setminus \{0\})$. Any member of the family admits three isogenous (over $\mathbb{Q}$) curves $E_i(n, p)$ ($i = 2, 3, 4$):

- $E_2(n, p)$: $y^2 = x^3 + 4(2 \cdot 3^{2n+1} - p)x^2 + 16 \cdot 3^{4n+2}x,$
- $E_3(n, p)$: $y^2 = x^3 + 2(4 \cdot 3^{2n+1} + p)x^2 + (4 \cdot 3^{2n+1} - p)^2x,$
- $E_4(n, p)$: $y^2 = x^3 + 2(p - 8 \cdot 3^{2n+1})x^2 + p^2x.$

In our calculations, we focused on the pairs of integers $(n, p)$ within the bounds $20 \leq n \leq 24$ and $0 < |p| \leq 5000$. Recall that the calculations in [11] were focused on the pairs $(n, p)$ within the bounds $3 \leq n \leq 19$ and $0 < |p| \leq 1000$.

The conductors, $L$-series and ranks of isogenous curves coincide, what may differ is the orders of $E(\mathbb{Q})_{\text{tors}}$ and $\mathcal{W}(E)$, the real period $\Omega_E$, and the Tamagawa number $C_{\text{fin}}(E)$. In our situation we are dealing with 2-isogenies, thus the analytic order of $\mathcal{W}(E)$ can only change by a power of 2.

**Notation.** Let $N(n, p)$ denote the conductor of the curve $E_i(n, p)$. We put $|\mathcal{W}_i| = |\mathcal{W}(E_i)|$.

### 8.2 Elliptic curves $E_i(n, p)$ with $50000^2 \leq \max(|\mathcal{W}_i|) < 250000^2$

| $(n, p)$     | $N(n, p)$     | $|\mathcal{W}_1|$ | $|\mathcal{W}_2|$ | $|\mathcal{W}_3|$ | $|\mathcal{W}_4|$ |
|--------------|---------------|-------------------|-------------------|-------------------|-------------------|
| $(20, -756)$ | 42551829106699251024 | 27993^2           | 55986^2           | 27993^2           | 27993^2           |
| $(20, -2000)$| 19029389411760627320 | 15081^2           | 60324^2           | 15081^2           | 60324^2           |
| $(20, 192)$  | 109418989131512359065 | 3780^2           | 60480^2           | 945^2            | 60480^2           |

Continued on next page
### 8.3 Elliptic curves $E_i(n, p)$ with $\max(|w_i|) \geq 25000^2$

| $(n, p)$ | $N(n, p)$ | $|w_1|$ | $|w_2|$ | $|w_3|$ | $|w_4|$ |
|----------|------------|--------|--------|--------|--------|
| $(22, -692)$ | 1197814802352835313168 | 151942 | 303882 | 75972 | 607762 |
| $(21, -128)$ | 1969541804367222465954 | 34232 | 68462 | 34232 | 68462 |
| $(20, -180)$ | 60788327295286424080 | 209702 | 419402 | 104852 | 838802 |
| $(20, -2448)$ | 1653442502431742344680 | 22062 | 88112 | 22062 | 88112 |
| $(20, -2704)$ | 1137957486977285146824 | 485382 | 970762 | 485382 |
| $(21, 12)$ | 28136311490603209392 | 127682 | 1021442 | 31922 | 1021442 |
| $(20, -608)$ | 16631686347989878669080 | 257872 | 1031482 | 515742 | 515742 |
| $(21, 192)$ | 984770902183611232737 | 54642 | 1092962 | 273242 | 1092962 |
| $(20, 4788)$ | 258715209687343639456 | 277452 | 1109802 | 277452 | 277452 |
| $(20, 2680)$ | 2393819517362147862720 | 144742 | 1157922 | 144742 | 578962 |
| $(20, -801)$ | 34625031227391434115352 | 293382 | 586762 | 293382 | 117352 |
| $(22, 1344)$ | 620405668375650764447 | 609302 | 1218602 | 304602 | 609302 |
| $(20, -1436)$ | 183236931070381048288 | 324552 | 1298202 | 324552 | 1298202 |
| $(20, 4768)$ | 10032879618827902147272 | 162542 | 1300322 | 81272 | 650162 |
| $(21, -24)$ | 31516686897555452768 | 340922 | 681842 | 170462 | 1363682 |
| $(20, -1376)$ | 376401322612402192904 | 700102 | 14000202 | 700102 | 700102 |
| $(22, 64)$ | 8862938119652501095881 | 723062 | 146612 | 361522 | 146612 |
| $(21, -1536)$ | 196954180436722468066 | 758972 | 1517922 | 517922 | 1517922 |
| $(20, -6)$ | 1400563068883581979328 | 192482 | 769922 | 48122 | 153982 |
| $(22, 304)$ | 2749319579973837074584 | 397222 | 1588882 | 198612 | 79442 |
| $(21, 1516)$ | 11663308372737145525968 | 258662 | 2069282 | 129332 | 103642 |
| $(21, 480)$ | 3939603608744449300840 | 541102 | 2164402 | 270552 | 1082202 |
| $(20, 1452)$ | 645169760180585648272 | 556982 | 2217022 | 278492 | 227792 |

\[8.3\] Elliptic curves $E_i(n, p)$ with $\max(|w_i|) \geq 25000^2$
8.4 Birch and Swinnerton-Dyer conjecture for elliptic curves with exceptionally large analytic order of Tate-Shafarevich groups

In this subsection, we will use the deep results by Skinner-Urban [29], to prove the full version of the Birch-Swinnerton-Dyer conjecture for some elliptic curves $E_i(n, p)$ with exceptionally large analytic order of Tate-Shafarevich groups.

Let $\bar{\rho}_{E,p} : \text{Gal}({\overline{\mathbb{Q}}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F}_p)$ denote the Galois representation on the $p$-torsion of $E$. Assume $p \geq 3$.

**Theorem 7** ([29], Theorem 2) Let $E$ be an elliptic curve over $\mathbb{Q}$ with conductor $N_E$. Suppose: (i) $E$ has good ordinary reduction at $p$; (ii) $\bar{\rho}_{E,p}$ is irreducible; (iii) there exists a prime $q \neq p$ such that $q || N_E$ and $\bar{\rho}_{E,p}$ is ramified at $q$; (iv) $\bar{\rho}_{E,p}$ is surjective. If moreover $L(E, 1) \neq 0$, then the $p$-part of the Birch and Swinnerton-Dyer conjecture holds true, and we have

$$\text{ord}_p(|\mathcal{U}(E)|) = \text{ord}_p \left( \frac{|E(\mathbb{Q})_{\text{tors}}|^2 L(E, 1)}{C_\infty(E) C_{\text{fin}}(E)} \right).$$

The following curves from the tables in 8.2 and 8.3 satisfy the conditions $(N_{E_i}, \min(|\mathcal{U}_i|)) = 1$: $E_i(22, 692)$, $E_i(20, -608)$, $E_i(20, -1436)$, $E_i(23, -84)$, $E_i(20, 4788)$, $E_i(22, 304)$, $E_i(21, 1248)$, $E_i(22, 480)$, $E_i(23, 960)$, and $E_i(23, -348)$. In these cases, all the assumptions of the result by Skinner-Urban are satisfied.

Let us give some details for the curves $E_i = E_i(20, -1436)$. We can use Theorem 2 to show that $|\mathcal{U}_1| = 5^2 6491^2$ is the true order of $\mathcal{U}(E_1)$ (and, hence, all $|\mathcal{U}_i|$ are the true orders of $\mathcal{U}(E_i)$). (i) $E_1$ has good ordinary reduction at 5 and 6491: $(N_{E_1}, 5) = (N_{E_1}, 6491) = 1$, and $a_5(E_1) = 2$, $a_{6491}(E_1) = 108$. (ii) We have the following general result of B. Mazur. Let $E$ be an elliptic curve over $\mathbb{Q}$ with all its 2-division points defined over $\mathbb{Q}$. Then $\bar{\rho}_{E,p}$ is absolutely irreducible for any prime $p \geq 5$. (iii) Take $q = 7$. Then $7 || N_{E_1}$, and $\bar{\rho}_{E_1,p}$ is ramified at 7 for $p = 5$ and 6491, since these $p$’s do not divide $\text{ord}_7(\Delta_{E_1})$. (iv) We have $j_{E_1} = \frac{2^{28} 3^{43} 13^{20} 23^{23} 257^{2} 339^{2} 1062^{1} 888^{1} 304^{1} 343^{1} 1062^{1} 888^{1} 304^{1} 343^{1}}{3^{4} 7^{3} 3^{1} 257^{1} 339^{1} 1062^{1} 888^{1} 304^{1} 343^{1}}$. The representation $\bar{\rho}_{E_1,p}$ is surjective for any prime $p \geq 19$ by Prop. 1.8 in [33]. On the other hand, Prop. 6.1 in [33] gives a criterion to determine whether $\bar{\rho}_{E,p}$ is surjective or not for any non-CM elliptic curve $E$ and any prime $p \leq 11$. For instance, the representation $\bar{\rho}_{E,5}$ is not surjective if and only if $j_E = \frac{5^{3(t+1)(2t+1)^2(2t^2-3t+3)^3}}{(t^2+1)^3}$ or $j_E = \frac{5^{3(t+10t+5)^3}}{t^5}$ or $j_E = t^3(t^2 + 5t + 40)$ for some $t \in \mathbb{Q}$. Taking $t = \frac{a}{b}$, where $a$, $b$ are relatively prime integers with
In the first case note, that $a^2 + ab - b^2$ is relatively prime to 3, a contradiction. The second case is impossible, since necessarily $ab \equiv 1 \pmod{8}$, and consequently 8 divides $a^2 + 10ab + 5b^2$, a contradiction with $\text{ord}_2(j_{E_1}) = 6$. The last case is impossible, since a denominator of $j_{E_1}$ is not a fifth power of an integer.

9 Examples of elliptic curves with $c_{2m+1}(E) = 0$ and $c_{4m+2}(E) = 0$

9.1 Numerical data

Let $E^i = E_i(5, 2)$. For all positive, square-free integers $d \leq 2495$, prime to $N_{E^i}$, and such that the groups $E^i_d(\mathbb{Q})$ are finite, we calculated the orders of $\mathfrak{w}(E^i_d)$. A part of the data is collected below in the following table.

| $d$  | $[\mathfrak{w}(E^i_1)]$ | $[\mathfrak{w}(E^i_2)]$ | $[\mathfrak{w}(E^i_3)]$ | $[\mathfrak{w}(E^i_4)]$ |
|------|--------------------------|--------------------------|--------------------------|--------------------------|
| 1    | $3^2$                    | $6^2$                    | $3^2$                    | $12^2$                   |
| 5    | $6^2$                    | $24^2$                   | $3^2$                    | $24^2$                   |
| 11   | $6^2$                    | $24^2$                   | $3^2$                    | $24^2$                   |
| 23   | $6^2$                    | $24^2$                   | $6^2$                    | $24^2$                   |
| 55   | $3^2$                    | $12^2$                   | $3^2$                    | $24^2$                   |
| 59   | $12^2$                   | $48^2$                   | $6^2$                    | $48^2$                   |
| 61   | $6^2$                    | $12^2$                   | $3^2$                    | $24^2$                   |
| 71   | $6^2$                    | $24^2$                   | $3^2$                    | $24^2$                   |
| 73   | $2^2$                    | $4^2$                    | $1^2$                    | $8^1$                    |
| 79   | $4^2$                    | $8^2$                    | $2^2$                    | $16^2$                   |
| 83   | $2^2$                    | $8^2$                    | $2^2$                    | $8^2$                    |
| 97   | $2^2$                    | $4^2$                    | $1^2$                    | $8^1$                    |
| 101  | $20^2$                   | $80^2$                   | $10^2$                   | $80^2$                   |
| 109  | $4^2$                    | $8^2$                    | $2^2$                    | $16^2$                   |
| 113  | $8^2$                    | $32^2$                   | $4^2$                    | $32^2$                   |
| 115  | $2^2$                    | $8^2$                    | $1^2$                    | $8^2$                    |
| 119  | $8^2$                    | $32^2$                   | $8^2$                    | $64^2$                   |
| 127  | $8^2$                    | $16^2$                   | $4^2$                    | $32^2$                   |
| 143  | $3^2$                    | $12^2$                   | $3^2$                    | $24^2$                   |

Continued on next page
Our data strongly suggest that $c_{2m+1}(E_d^2) = c_{2m+1}(E_d^4) = c_{4m+2}(E_d^2) = 0$. We prove these statements in the next subsection.

9.2 Lower bounds for the 2-rank of $\mathcal{U}(E_d')$

Consider an elliptic curve $E = E_1(5, 2)$ given by the equation $y^2 = x^3 - 708584x^2 - 1417172$, of conductor $N_E = 2^6.3.19.29.643$. Let $E' = E_2(5, 2)$ be the elliptic curve given by the equation $y^2 = x^3 + 1417172x^2 + 502096953744$. Let $E_d$ and $E'_d$ denote the quadratic twists by positive, square-free integers $d$, prime to $N_E$. Consider the two-isogeny $\phi : E_d \to E'_d$, defined by $\phi((x, y)) = (y^2/x^2, -y(1417172x^2 + x^2)/x^2)$; let $\hat{\phi}$ denote the dual isogeny. Consider the Selmer groups $S(\phi)(E_d/\mathbb{Q})$, and $S(\phi)(E'_d/\mathbb{Q})$. We use the notations and results from chapter X of Silverman’s book [28]. Let

$$C_r^{(d)} : ry^2 = r^2 + 2^7.23.3851rdx^2 + 2^4.3^{22}d^2x^4,$$

$$C_r'(d) : ry^2 = r^2 - 2^5.23.3851rdx^2 - 2^6.19.29.643d^2x^4,$$

be the principal homogeneous spaces under the actions of the elliptic curves previously defined. Let $\Sigma(M)$ and $\Delta(M)$ be the support of an integer $M$ in the set of prime numbers and the set of divisors of $M$ in $\mathbb{Z}$ respectively. Using [28, Proposition 4.9, p.302], we have the following identifications:

| $d$  | $|\mathcal{U}(E_d^2)|$ | $|\mathcal{U}(E_d^4)|$ | $|\mathcal{U}(E_d^2)|$ | $|\mathcal{U}(E_d^4)|$ |
|------|----------------|----------------|----------------|----------------|
| 157  | $2^2$          | $4^2$          | $1^2$          | $8^2$          |
| 163  | $6^2$          | $12^2$         | $3^2$          | $24^2$         |
| 173  | $2^2$          | $8^2$          | $1^2$          | $8^2$          |
| 179  | $28^2$         | $112^2$        | $14^2$         | $112^2$        |
| 197  | $10^2$         | $40^2$         | $5^2$          | $40^2$         |
| 199  | $1^2$          | $2^2$          | $1^2$          | $8^2$          |
| 205  | $4^2$          | $16^2$         | $2^2$          | $16^2$         |
| 217  | $2^2$          | $8^2$          | $2^2$          | $16^2$         |
| 227  | $4^2$          | $8^2$          | $1^2$          | $8^2$          |
| 237  | $2^2$          | $8^2$          | $1^2$          | $8^2$          |
| 263  | $6^2$          | $24^2$         | $3^2$          | $24^2$         |
| 271  | $3^2$          | $6^2$          | $3^2$          | $12^2$         |
| 277  | $9^2$          | $18^2$         | $9^2$          | $36^2$         |
| 281  | $27^2$         | $108^2$        | $27^2$         | $108^2$        |
| 283  | $2^2$          | $4^2$          | $1^2$          | $8^2$          |
| 295  | $3^2$          | $12^2$         | $3^2$          | $24^2$         |
| 299  | $2^2$          | $8^2$          | $1^2$          | $8^2$          |
| 301  | $2^2$          | $8^2$          | $1^2$          | $8^2$          |
| 305  | $1^2$          | $4^2$          | $1^2$          | $8^2$          |
$S^{(\phi)}(E_d/\mathbb{Q}) \simeq \{ r \in \Delta(2.3.d) : C_r^{(d)}(\mathbb{Q}) \neq \emptyset \ \forall l \in \Sigma(2.3.23.3851.d) \cup \{ \infty \} \},$

$S^{(\hat{\phi})}(E'_d/\mathbb{Q}) \simeq \{ r \in \Delta(2.19.29.643.d) : C_r^{(d)}(\mathbb{Q}) \neq \emptyset \ \forall l \in \Sigma(2.3.19.29.643.d) \cup \{ \infty \} \}.$

**Theorem 8** Let $d$ be a positive, square-free integer, prime to $N_E$, and such that the group $E(\mathbb{Q})$ is finite. Then the group $\mathcal{M}(E'_d)[2]$ is non-trivial.

**Proof.** We have $\dim_2 S^{(\phi)}(E_d/\mathbb{Q}) \geq \dim_2 \mathcal{M}(E_d)[\phi]$. If $E_d(\mathbb{Q})$ is finite, then using the fundamental formula (see, for instance, [28], p. 314), we obtain $\dim_2 \mathcal{M}(E'_d)[\hat{\phi}] \geq \dim_2 S^{(\hat{\phi})}(E'_d/\mathbb{Q}) - 2$. Here $\mathcal{M}(E_d)[\phi]$ is the kernel of the mapping $\mathcal{M}(E_d) \to \mathcal{M}(E'_d)$ induced by $\phi$, and $\mathcal{M}(E'_d)[\hat{\phi}]$ is defined similarly; also, we write $\dim_2$ for $\dim_{\mathbb{F}_2}$.

Now, it is sufficient to prove the following result.

**Lemma 1** We have $\dim_2 S^{(\hat{\phi})}(E'_d/\mathbb{Q}) \geq 3$.

**Proof of Lemma 1.** We will exhibit 8 elements in $S^{(\hat{\phi})}(E'_d/\mathbb{Q})$. More precisely, it is sufficient to check that $< 19.643, -19.29, r > \subset S^{(\hat{\phi})}(E'_d/\mathbb{Q})$, where $r$ denotes any prime divisor of $d$, and

$$r_d = \begin{cases} r, & \text{if } r \equiv 1(\mod 8), \\ 19r, & \text{if } r \equiv 3(\mod 8), \\ 29r, & \text{if } r \equiv 5(\mod 8), \\ -r, & \text{if } r \equiv 7(\mod 8). \end{cases}$$

We omit the standard calculations using the Hensel Lemma.

Similarly, one can prove that $c_{2m+1}(E'_d) = 0$.

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