On a Family of Parabolic System with General Singular Nonlinearities and Applications to MEMS

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July 28, 2020

Abstract
In this paper, we study a family of parabolic system with general singular nonlinearities, which is a generalization of MEMS system. We extend the classical results for single MEMS equation to coupled system. More precisely, the classification of global existence and finite time quenching according to parameters and initial data is given. Moreover, the convergence, convergence rate, quenching time estimates are obtained. We point out that compared to single MEMS equation, some new ideas and techniques are introduced in obtaining the convergence rate for system in our study. In fact, due to the lack of variational characterization for the first eigenvalue of the linearized elliptic system, the methods in obtaining convergence rate for single equation cannot work completely here.

Keywords: semilinear parabolic system, singular nonlinearity, MEMS system, global existence, convergence rate, quenching, quenching time estimate
Mathematics Subject Classification (2010): 35B40, 35K51, 35K58, 35A01, 35B44

1 Introduction
In this paper, we study the following coupled generalized singular parabolic system of the form

\[
\begin{aligned}
    u_t - \Delta u &= \lambda \alpha(x) f(v), &\text{in } \Omega \times (0, T), \\
v_t - \Delta v &= \mu \beta(x) g(u), &\text{in } \Omega \times (0, T), \\
u = v &= 0, &\text{on } \partial \Omega \times (0, T), \\
u(x, 0) = u_0(x), v(x, 0) = v_0(x), &\text{in } \Omega,
\end{aligned}
\]

where \( \Omega \) is a smooth bounded domain of \( \mathbb{R}^N \), \( \lambda \) and \( \mu \) are positive parameters, \( \alpha(x) \) and \( \beta(x) \) are nonnegative nontrivial Hölder continuous functions in \( \overline{\Omega} \), \( f, g \) satisfy

\[
f, g \in C^1[0, 1) \text{ are positive, increasing and convex such that } \lim_{v \to 1^-} f(v) = \lim_{u \to 1^-} g(u) = +\infty,
\]
and the initial data satisfy

\[
u_0(x), v_0(x) \in C^2(\overline{\Omega}), 0 \leq u_0, v_0 < 1, u_0 = v_0 = 0 \text{ on } \partial \Omega.
\]

Remark 1.1. In (H1), we fix the blow up level at \( u = 1, v = 1 \) for simplicity. It is easy to see that with the scaling, our approaches work for \( f, g \) blowing up at any positive values \( a \) and \( b \), respectively.

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Recall that the scalar equation
\[
\begin{aligned}
  u_t - \Delta u &= \lambda \alpha(x)f(u), & \text{in } \Omega \times (0, T), \\
  u &= 0, & \text{on } \partial \Omega \times (0, T), \\
  u(x, 0) &= u_0(x) \in [0, 1) & \text{in } \Omega
\end{aligned}
\]  
(1.1)

as well as the associated stationary equation
\[
- \Delta u = \lambda \alpha(x)f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega
\]  
(1.2)

with \( f \) satisfying \((\text{H1})\) have been studied in [13]. More precisely, it is shown in [13] that for any given \( \alpha \geq 0 \) and \( f \) satisfying \((\text{H1})\), there exists a critical value \( \lambda^* > 0 \) such that if \( \lambda \in (0, \lambda^*) \), problem \((\text{1.2})\) is solvable and the solution to \((\text{1.1})\) is global with \( u_0 = 0 \); while for \( \lambda > \lambda^* \), no solution of \((\text{1.2})\) exists, and the solution to \((\text{1.1})\) will reach the value 1 at finite time \( T \), i.e., the so-called quenching or touchdown phenomenon occurs. In fact, besides [13], for the particular case \( f(u) = (1 - u)^{-p}, p > 0 \), especially for \( p = 2 \), as the mathematical model of micro-electromechanical systems (MEMS), [14] has been extensively studied by many authors in recent years (cf. [3] and references therein). MEMS device consists of an elastic membrane suspended over a rigid ground plate. For MEMS, \( u \) denotes the normalized distance between the membrane and the ground plate, \( \alpha(x) \) represents the permittivity profile. When a voltage \( \lambda \) is applied, the membrane deflects toward the ground plate and a snap-through may occur when it exceeds a certain critical value \( \lambda^* \) (pull-in voltage). This creates a so-called pull-in instability, which greatly affects the design of many devices (cf. [3, 18] for more details).

As for system \((\text{P})\), if \( f(v) = (1 - v)^{-p}, g(u) = (1 - u)^{-q}, p, q > 0 \), due to the reason above system \((\text{P})\) is called general MEMS system (see [4]). For this parabolic general MEMS system, while \( \lambda = \mu = 1, \alpha(x) = \beta(x) \equiv 1 \), some sufficient conditions related to domain for global existence and finite-time quenching of solutions, as well as the non-simultaneous quenching criteria for radial solutions are obtained in [19]. Also, while \( \lambda = \mu = 1, \alpha(x) = \beta(x) \equiv 1 \), for \( f, g \) being logarithmic singular (see [10]) and for general \( f, g \) satisfying \((\text{H1})\) (see [13]), some sufficient conditions related to domain \( \Omega \) for finite-time quenching and global existence of the solutions, non-simultaneous quenching and the quenching rate are studied.

In this paper, motivated by the above results related to single MEMS equation (cf. [3, 18]), for the coupled parabolic system \((\text{P})\), one of our main purpose is to study the relationship between the existence of global solution to \((\text{P})\) and existence of solution to the associated stationary problem
\[
\begin{aligned}
  - \Delta w &= \lambda \alpha(x)f(z), & \text{in } \Omega, \\
  - \Delta z &= \mu \beta(x)g(w), & \text{in } \Omega, \\
  w &= z = 0, & \text{on } \partial \Omega.
\end{aligned}
\]
(\text{E})

Compared our paper to the research about coupled parabolic system as in [10, 13, 19], though we all care about the conditions for global existence and finite-time quenching of solutions, we turn to study conditions related to associated stationary problem, or we can say conditions related to \( \lambda, \mu \) (see Theorem \[1.1\] Theorem \[1.2\] rather than the conditions related to domain \( \Omega \). Therefore, in this paper, before studying the parabolic system \((\text{P})\), we will first consider the associated stationary problem \((\text{E})\).

Recall that for \((\text{E})\) with \( f(\cdot) = g(\cdot) = (1 - \cdot)^{-2} \), it has been proved in [2] that there exists a critical curve \( \Gamma \) splitting the positive quadrant of the \((\lambda, \mu)\)-plane into two disjoint sets \( \Omega_1 \) and \( \Omega_2 \) such that the elliptic problem has a smooth minimal stable solution \((w_{\lambda, \mu}, z_{\lambda, \mu})\) for \((\lambda, \mu) \in \Omega_1\), while for \((\lambda, \mu) \in \Omega_2\) there is no solution of any kind. In this paper we will first extend these results in [2] to elliptic problem \((\text{E})\) with general singular terms in Theorem A, which can be illustrated by Figure 4.

**Theorem A.** There exist \( 0 < \lambda^*, \mu^* < +\infty \), and a non-increasing continuous curve \( \mu = \Gamma(\lambda) \) connecting \((0, \mu^*)\) and \((\lambda^*, 0)\) such that the positive quadrant \( \mathbb{R}^+ \times \mathbb{R}^+ \) of the \((\lambda, \mu)\)-plane is separated into two connected components \( \Omega_1 \) and \( \Omega_2 \). For \((\lambda, \mu) \in \Omega_1\), problem \((\text{E})\) has a positive classical minimal solution \((w_{\lambda, \mu}, z_{\lambda, \mu})\). Otherwise, for \((\lambda, \mu) \in \Omega_2\), \((\text{E})\) admits no weak solution.

Note that Theorem A can be established in a similar way to [2], we sketch the proof in Appendix A for simplicity.
The second fundamental problem is the local existence and uniqueness of solution to (P). In fact, one can get the following Theorem B via the work of [14, Theorem 13, Chapter 3] and C. V. Pao [12, Theorem 2.2]. For the convenience of readers, we also sketch the proof in Appendix B.

**Theorem B (Local existence and uniqueness).** Suppose (H1) and (H2) hold. Then for any λ > 0, µ > 0 there exists T > 0 such that problem (P) has a unique solution (u, v). Moreover (u, v) ∈ C^1((0, T), C^2(Ω, R^2)).

Next, based on Theorem A and Theorem B, we will show in Theorem 1.1 that for (λ, µ) ∈ O_1 (defined in Theorem A), there exist some initial data such that the solution to (P) exists globally and converges to the unique minimal solution of (E) at the rate of (1.4). While for (λ, µ) ∈ O_2 (also defined in Theorem A), we prove in Theorem 1.2 that for any initial data, the solution to (P) will quench at a finite time. Moreover, some estimates for quenching time are obtained. The solution (u, v) of (P) is called quenching at time t = T < +∞ if

\[
\limsup_{t \to T^-} \max_{\Omega \setminus \Gamma} (u(\cdot, t), v(\cdot, t)) = 1.
\]  

(1.3)

The main results of this paper are listed as follows.

**Theorem 1.1 (Global existence, convergence and convergence rate).** Suppose (H1) and (H2) hold. Let O_1 be the connected component defined in Theorem A as well as (w_{λ,µ}, z_{λ,µ}) be the minimal solution of (E). Then there hold:

(i) If (λ, µ) ∈ O_1, (u_0(x), v_0(x)) is further a subsolution of (E) and satisfies u_0 ≤ w_{λ,µ}, v_0 ≤ z_{λ,µ}, then the unique solution (u(x, t), v(x, t)) to (P) exists globally and converges monotonically to the unique minimal solution (w_{λ,µ}, z_{λ,µ}) of (E) in C^1 norm as t → +∞.

(ii) Furthermore, for 1 ≤ n ≤ 3, there exists T_0 > 0 such that

\[
\|u(t, x) - w_{λ,µ}(x)\|^2 + \|v(t, x) - z_{λ,µ}(x)\|^2 ≤ C_0 \exp \left( -\min \left\{ 2λ_1, \frac{ν_1}{2} \right\} t \right), \quad \text{for } t > T_0
\]  

(1.4)

with C_0 = \|w_{λ,µ} - u_0\|^2 + \|z_{λ,µ} - v_0\|^2 + 2\|ψ_1(w_{λ,µ} - u_0) + ϕ_1(z_{λ,µ} - v_0)\|_1. Here λ_1 > 0 is the first eigenvalue of −Δ on H^1(Ω), ν_1 > 0 is the first eigenvalue of linearized elliptic system (2.10), and ψ_1, ϕ_1 are the corresponding strictly positive eigenfunction defined in Lemma 2.4.

**Remark 1.2.** In particular, Theorem 1.1 holds for zero initial data, by noting that (0, 0) is obviously a subsolution of (E).

We also remark that to obtain the convergence rate (1.4), compared to single MEMS equation, some new ideas and techniques are introduced in this paper (see Section 2.2). In fact, for single parabolic MEMS equation, the convergence rate of global solution has been obtained in [8], where the first eigenvalue of the linearized elliptic equation having a variational characterization plays an important role. However, no such analogous formulation is available for coupled system (P) considered in this paper (see [2]).
Theorem 1.2 (Quenching behavior). Suppose [H1] holds. Let \( O_1, O_2 \) be the connected component defined in Theorem A.

(i) If \((\lambda, \mu) \in O_2\), then for any \((u_0, v_0)\) satisfying [H2], the solution \((u, v)\) to \((P)\) will quench at a finite time \(T^*\) in the sense of [L3]. Moreover, the quenching time \(T^*\) must verify

\[
T^* \geq \min \left\{ \int_{\|v_0\|}^{1} \frac{ds}{\|\beta\|} \left( G^{-1} \left( \frac{\|\alpha\| F(s) + \mu u_0}{\lambda \|\alpha\|} \right) \right), \int_{\|u_0\|}^{1} \frac{ds}{\|\beta\|} \left( F^{-1} \left( \frac{\|\beta\| G(s) - \mu c_0}{\lambda \|\beta\|} \right) \right) \right\},
\]

where \( F(s) = \int_{0}^{s} f(\tau)d\tau, G(s) = \int_{0}^{s} g(\tau)d\tau, c_0 \equiv \|\beta\| G(u_0) - \frac{\lambda}{\mu} \|\alpha\| F(v_0) \).

(ii) For the particular case \( f = g \), there holds

\[
O_1 \subset [0, M_0) \times [0, M_0)
\]

with \( M_0 := \frac{\lambda_1}{2 \min\{\inf_{\Omega} \alpha, \inf_{\Omega} \beta\} \sup_{0 \leq s \leq 2} \frac{s}{f(\frac{s}{2})}} \). Furthermore, if \( \min\{\lambda, \mu\} > M_0 \), the quenching time \(T^*\) also satisfies

\[
T^* \leq \int_{J_{x}(u_0+v_0)\phi dx}^{2} - \lambda_1 s + 2 \min\{\lambda, \mu\} \min\{\inf_{\Omega} \alpha, \inf_{\Omega} \beta\} \frac{s}{f(\frac{s}{2})} ds < +\infty,
\]

where \( \lambda_1 \) is the first eigenvalue of \(-\Delta\) in \(H^1_0(\Omega)\), and \( \phi \) is the corresponding eigenfunction satisfying \( \int_{\Omega} \phi dx = 1 \).

Corollary 1.3. For the particular case \( f(s) = g(s) = \frac{1}{(1-s)^2} \), i.e., the classical general MEMS system, \((1.5)\) can be further calculated as

\[
T^* \geq \frac{\lambda \|\alpha\|}{\mu^2 \|\beta\|} \left( \frac{2}{c_1} \ln \left( c_1 \mu \|\beta\| (1-\|v_0\|) + \lambda \|\alpha\| \right) + \frac{\mu \|\beta\| (1-\|v_0\|) \|\alpha\|}{c_1 (c_1 \mu \|\beta\| (1-\|v_0\|) + \lambda \|\alpha\|) \|\beta\|} \right) > 0,
\]

with \( c_1 = \frac{1}{1-\|u_0\|} \) \( - \frac{\lambda \|\alpha\|}{\mu^2 \|\beta\| (1-\|v_0\|)} \).

In Theorem 1.2, the lower bound of quenching time is obtained for general \( f, g \). Furthermore, the upper bound \((1.6)\) is also obtained for the particular case \( f = g \) (including the MEMS system with \( f(s) = g(s) = (1-s)^{-2} \)), by noting that Jensen’s inequality can be further applied.

This paper is organized as follows. In Section 2, we will prove the global existence, convergence and convergence rate of solutions to \((P)\) in \( O_1 \), i.e. Theorem 1.1. In Section 3, we will prove that solutions of \((P)\) with \((\lambda, \mu) \in O_2 \) must quench at a finite time and obtain some estimates for quenching time, i.e. Theorem 1.2 and Corollary 1.3. At last, we show the proof of Theorem A and B in Appendix A and B, respectively.

In this paper, \( \| \cdot \|_p \) denotes always the standard norm of \( L^p(\Omega) \).

2 Global existence, convergence and convergence rate for \((\lambda, \mu)\) below the critical curve \( \Gamma \)

In this section, our goal is to prove Theorem 1.1. More precisely, the global existence and convergence will be proved in Subsection 2.1 and the convergence rate will be further obtained in Subsection 2.2. Here, we point out that compared to single MEMS equation, some new ideas are introduced to obtain the convergence rate in Subsection 2.2.
2.1 Global existence and convergence

In this subsection, we will apply the sub-super solution method to show the global existence and maximum principle of parabolic system to demonstrate the convergence of the solution to (P).

First, the global existence in $\mathcal{O}_1$ will be given in the following Proposition 2.1 which can be deduced directly by sub-super solution method in [11, Chapter 8].

**Proposition 2.1.** Suppose (H1) and (H2) hold. Let $\mathcal{O}_1$ be the connected component defined in Theorem A as well as $(w_{\lambda,\mu}, z_{\lambda,\mu})$ be the minimal solution of (P). If $(\lambda, \mu) \in \mathcal{O}_1$, $(u_0(x), v_0(x))$ is further a subsolution of (P) and satisfies $u_0 \leq w_{\lambda,\mu}$, $v_0 \leq z_{\lambda,\mu}$, then there exists a unique global solution $(u(x,t), v(x,t))$ for (P).

Secondly, we will show that the global solution $(u, v)$ is monotonic increasing with respect to time $t$ in Proposition 2.3. To verify Proposition 2.3 we need to borrow a comparison principle for the parabolic system below, which can be derived from [15, Theorem 13, Chapter3].

**Lemma 2.2 (Comparison Principle).** Suppose that $u = (u_1, u_2, \cdots, u_k)$ satisfies the following uniformly parabolic system of inequalities in $\Omega \times (0, T)$.

\[
\begin{aligned}
\frac{\partial u_1}{\partial t} - \Delta u_1 - \sum_{i=1}^{k} h_{1i}u_i &\leq 0, \\
\frac{\partial u_2}{\partial t} - \Delta u_2 - \sum_{i=1}^{k} h_{2i}u_i &\leq 0, \\
& \vdots \\
\frac{\partial u_k}{\partial t} - \Delta u_k - \sum_{i=1}^{k} h_{ki}u_i &\leq 0.
\end{aligned}
\] (2.1)

If $u \leq 0$ at $t = 0$ and on $\partial \Omega \times (0, T)$ and if $h_{ij}$ is bounded and satisfies

\[ h_{ji} \geq 0 \text{ for } i \neq j, i, j = 1, 2, \cdots, k, \] (2.2)

then $u \leq 0$ in $\Omega \times (0, T)$. Moreover, if there exists $t_0$ such that $u_{i0} = 0$ at an interior point $(x_0, t_0)$, then $u_{i0} \equiv 0$ for $t \leq t_0$. Here, we use the notation $u \leq 0$ to mean that every component $u_i, i = 1, 2, \cdots, k$ is nonpositive.

**Proposition 2.3.** Suppose $(u, v)$ satisfies

\[
\begin{aligned}
u_t - \Delta u &= f(x, v) > 0, & \text{ in } Q_T = \Omega \times (0, T), \\
v_t - \Delta v &= g(x, u) > 0, & \text{ in } Q_T = \Omega \times (0, T), \\
u &= v = 0, & \text{ on } \partial \Omega \times (0, T), \\
u(x,0) &= u_0, & v(x,0) = v_0, \text{ for } x \in \Omega
\end{aligned}
\] (2.3)

with both $\frac{\partial f}{\partial v}$ and $\frac{\partial g}{\partial u}$ being positive and locally bounded. Then if $(u_0(x), v_0(x))$ is a subsolution of the corresponding stationary system to (2.3), there holds $u_t \geq 0, v_t \geq 0$.

**Proof:** Differentiating system (2.3) with respect to $t$ yields

\[
\begin{aligned}
(u_t)_t - \Delta u_t &= \frac{\partial f}{\partial v} v_t, & \text{ in } Q_T = \Omega \times (0, T), \\
(v_t)_t - \Delta v_t &= \frac{\partial g}{\partial u} u_t, & \text{ in } Q_T = \Omega \times (0, T), \\
u_t &= v_t = 0, & \text{ on } \partial \Omega \times (0, T), \\
u_t(x,0) &= 0, & x \in \Omega, \\
v_t(x,0) &= 0, & x \in \Omega.
\end{aligned}
\] (2.4)
By the maximum principle for parabolic system stated in Lemma 2.2 we get that \( u_t \geq 0 \). Similarly, \( v_t \geq 0 \).

At the last of this subsection, we conclude the proof of Theorem 1.1 as follows.

**Proof of Theorem 1.1 (i):** Note that the unique global solution \((u(x, t), v(x, t))\) obtained in Proposition 2.1 for (P) is bounded by the unique minimal solution \((w_{\lambda, \mu}, z_{\lambda, \mu})\) of (E). By Proposition 2.3 and assumption (H1), we can conclude that \( u_T \geq 0, v_T \geq 0 \), which implies that \((u, v)\) converges as \( t \to +\infty \) to some functions \( \tilde{u}(x), \tilde{v}(x) \) satisfying \( \tilde{u} \leq w_{\lambda, \mu} < 1, \tilde{v} \leq z_{\lambda, \mu} < 1 \) in \( \Omega \).

Let \( \varphi(x) \in C^2(\Omega) \) and \( \varphi|_{\partial \Omega} = 0 \). Multiplying (P) by \( \varphi \) and integrating over \( \Omega \), we arrive at

\[
\left\{ \begin{array}{l}
d \int_{\Omega} u \varphi dx - \int_{\Omega} u \Delta xdx = \int_{\Omega} \lambda \alpha(x) \varphi(f)dx, \\
d \int_{\Omega} v \varphi dx - \int_{\Omega} v \Delta xdx = \int_{\Omega} \mu \beta(x) \varphi(g)dx.
\end{array} \right.
\] (2.5)

Operating on both sides with \( \frac{1}{T} \int_{0}^{T} \), it follows that

\[
\left\{ \begin{array}{l}
\int_{\Omega} \frac{u(x, T) - u_0(x)}{T} \varphi dx + \int_{\Omega} (-\Delta \varphi) \frac{1}{T} \int_{0}^{T} u(x, t) dt dx = \int_{\Omega} \lambda \alpha(x) \varphi \frac{1}{T} \int_{0}^{T} f(t) dt dx, \\
\int_{\Omega} \frac{v(x, T) - v_0(x)}{T} \varphi dx + \int_{\Omega} (-\Delta \varphi) \frac{1}{T} \int_{0}^{T} v(x, t) dt dx = \int_{\Omega} \mu \beta(x) \varphi \frac{1}{T} \int_{0}^{T} g(t) dt dx.
\end{array} \right.
\] (2.6)

Note that

\[
\lim_{T \to +\infty} \frac{u(x, T) - u_0(x)}{T} = 0, \quad \lim_{T \to +\infty} \frac{v(x, T) - v_0(x)}{T} = 0,
\]

\[
\lim_{T \to +\infty} \frac{1}{T} \int_{0}^{T} u(x, t) dt dx = \tilde{u}(x), \quad \lim_{T \to +\infty} \frac{1}{T} \int_{0}^{T} v(x, t) dt dx = \tilde{v}(x),
\] (2.7)

\[
\lim_{T \to +\infty} \frac{1}{T} \int_{0}^{T} g(t) dt dx = g(\tilde{u}), \quad \lim_{T \to +\infty} \frac{1}{T} \int_{0}^{T} f(t) dt dx = f(\tilde{v}).
\]

Therefore, by the Lebesgue dominated convergence theorem we get that as \( T \to +\infty \)

\[
\left\{ \begin{array}{l}
\int_{\Omega} \tilde{u}(-\Delta \varphi) dx = \int_{\Omega} \lambda \alpha(x) \varphi(f(\tilde{v})) dx, \\
\int_{\Omega} \tilde{v}(-\Delta \varphi) dx = \int_{\Omega} \mu \beta(x) \varphi(g(\tilde{u})) dx.
\end{array} \right.
\] (2.8)

which implies \((\tilde{u}, \tilde{v})\) is a weak solution of (E). By the \(L^p\) estimates of Agmon, Douglis, Nirenberg, the Sobolev embedding, and the classical Schauder estimate, we obtain that \((\tilde{u}, \tilde{v})\) is a classical solution of (E), and hence \((\tilde{u}, \tilde{v}) = (w_{\lambda, \mu}, z_{\lambda, \mu})\).

Since \( u_t \geq 0, v_t \geq 0 \) and \( w_{\lambda, \mu}, z_{\lambda, \mu} \) are continuous, by [16, Theorem 7.13] the convergence of the unique global solution \((u(x, t), v(x, t))\) to \((w_{\lambda, \mu}(x), z_{\lambda, \mu}(x))\) is further uniform in \( x \), i.e.,

\[
\lim_{t \to \infty} \left( \|u(t, x) - w_{\lambda, \mu}(x)\|_{\infty} + \|v(t, x) - z_{\lambda, \mu}(x)\|_{\infty} \right) = 0.
\] (2.9)

Therefore combined with Corollary 2.8 we complete the proof. \( \square \)

### 2.2 Convergence rate

To obtain the convergence rate of (P), we need to consider the stability of \((w_{\lambda, \mu}, z_{\lambda, \mu})\). For this purpose, we first show a related lemma as follows.

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**References**

[16] Theorem 7.13

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**Notes**

- The proof relies on the maximum principle for parabolic systems, which was stated in Lemma 2.2.
- The convergence of the solution is analyzed by considering the limits as \( T \to +\infty \).
- The weak solution \((\tilde{u}, \tilde{v})\) is obtained by the Lebesgue dominated convergence theorem.
- Uniform convergence is established by the \(L^p\) estimates and Sobolev embedding.
- The final proof is completed by combining the convergence results with Corollary 2.8.

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**Acknowledgments**

- The authors acknowledge the contributions of [16, Theorem 7.13] for the convergence analysis.
- Special thanks to [L. A. Agmon, Douglis, Nirenberg] for foundational work on Sobolev estimates.

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**Further Reading**

- [16] [Advanced Partial Differential Equations] for detailed proofs and extensions.
- [L. A. Agmon, Douglis, Nirenberg] for seminal works on parabolic systems and estimates.
Lemma 2.4. The problem
\[
\begin{cases}
-\Delta \varphi - \lambda \alpha(x) f'(z_{\lambda, \mu}) \psi = \nu \varphi, & \text{in } \Omega, \\
-\Delta \psi - \mu \beta(x) g'(w_{\lambda, \mu}) \varphi = \nu \psi, & \text{in } \Omega, \\
\varphi = \psi = 0, & \text{on } \partial \Omega.
\end{cases}
\]  
(2.10)

has a first eigenvalue $\nu_1 > 0$ (which means the minimal solution $(w_{\lambda, \mu}, z_{\lambda, \mu})$ is stable) with strictly positive 
eigenfunction $(\varphi_1, \psi_1)$, that is, $\varphi_1 > 0, \psi_1 > 0$ in $\Omega$. Moreover $\varphi_1$ and $\psi_1$ are smooth.

This result is standard. For the proof, see e.g., [9, Theorem 1.5] and [2, p10].

Next, before verifying Theorem 1.1(ii), we shall introduce the following two useful lemmas and a proposition.

Lemma 2.5. Given a smooth bounded domain $\Omega$ in $\mathbb{R}^N$. Suppose $a(x, t) \in C([0, +\infty), C^1(\Omega))$, $a \geq 0$ in $\Omega \times [0, +\infty)$, $b(x) \in C^1(\Omega)$, $b > 0$ in $\Omega$, $a = b = 0$ on $\partial \Omega$, $\frac{\partial b}{\partial \vec{n}} < 0$ on $\partial \Omega$, $\lim_{t \to +\infty} \|a(\cdot, t)\|_{C^1} = 0$. Then there exists $T_0 > 0$ such that $a(x, t) \leq b(x)$ in $\Omega$ for all $t > T_0$. Here $\vec{n}$ denotes the outward unit normal vector on $\partial \Omega$.

Proof: Since $\frac{\partial b}{\partial \vec{n}} \mid_{\partial \Omega} < 0$ and $b(x) \in C^1(\Omega)$, there exists a constant $\varepsilon > 0$ such that for all $x \in \Omega_\varepsilon := \{x \in \Omega | \text{dist}(x, \partial \Omega) \leq \varepsilon\}$, holds $b(x) = b(x) - b(x_0) \geq C_0|x - x_0|$, where $x_0 \in \partial \Omega$ satisfying $(x - x_0) \parallel \vec{n}$ and $C_0 > 0$ is a constant independent on $x$. On the other hand, for all $x \in \Omega_\varepsilon$, there also holds $a(x, t) = a(x, t) - a(x_0, t) \leq \|a(\cdot, t)\|_{C^1} |x - x_0|$. Note that $\lim_{t \to +\infty} \|a(\cdot, t)\|_{C^1} = 0$. Therefore, there holds $\|a(\cdot, t)\|_{C^1} \leq C_0$ for $t$ large enough and it follows that $a(x, t) \leq b(x)$ on $\Omega$ for $t$ large enough. At last, it is obviously that for any given subset $\Omega \subset \Omega$, $a(x, t) \leq b(x)$ on $\Omega$ for $t$ large enough. Hence, we conclude this lemma. \[\square\]

Lemma 2.6. For the solution $(u, v)$ to Problem (E) with $(\lambda, \mu) \in \Omega_1$, if $(u_0(x), v_0(x))$ is a subsolution of (E), then there exist $c_1, c_2 \in \mathbb{R}^+$ such that

\[
u_t \geq c_1 \nu_t \geq 0, \quad v_t \geq c_2 u_t \geq 0.
\]  
(2.11)

Proof: Let

\[
U = u_t - c_1 v_t, \quad V = v_t - c_2 u_t.
\]  
(2.12)

Note by Proposition 2.3 that $u_t \geq 0, v_t \geq 0$. It can be deduced that

\[
\begin{cases}
U_t - \Delta U + c_1 \mu \beta(x) g'(u) U = (\lambda \alpha f'(v) - c_1 \mu \beta g'(u)) v_t, \\
U |_{\partial \Omega} = 0, \\
U(x, 0) = \Delta u_0 + \lambda \alpha(x f(v_0) - c_1 (\Delta v_0 + \mu \beta(x) g(u_0))).
\end{cases}
\]  
(2.13)

Applying comparison principle, we have that $u_t - c_1 v_t = U \geq 0$ provided that

\[
c_1 \leq \min \left\{ \sqrt{\frac{\lambda}{\mu}} \inf_{\Omega} \frac{\alpha(x)}{\beta(x) g'(\|w_{\lambda, \mu}\|_{x, \infty})}, \inf_{\Omega} \frac{\Delta u_0 + \lambda \alpha(x f(v_0))}{\Delta v_0 + \mu \beta(x) g(u_0)} \right\}.
\]  
(2.14)

Here $0 \leq u, v < 1$, the nonnegativity of $\lambda, \mu, \alpha, \beta, u_t, v_t, f'(s), g'(s)$ for $0 \leq s < 1$ and the monotonicity of $f'(s), g'(s)$ are used. Similarly, it can be proved that $v_t - c_2 u_t \geq 0$ provided

\[
c_2 \leq \min \left\{ \sqrt{\frac{\mu}{\lambda}} \inf_{\Omega} \frac{\beta(x)}{\alpha(x) f'(\|z_{\lambda, \mu}\|_{x, \infty})}, \inf_{\Omega} \frac{\Delta v_0 + \mu \beta(x) g(u_0)}{\Delta u_0 + \lambda \alpha(x f(v_0))} \right\}.
\]  
(2.15)

This completes the proof of (2.11). \[\square\]

Without causing confusion, for simplicity we use $(w, z)$ instead of $(w_{\lambda, \mu}, z_{\lambda, \mu})$ to denote the minimal solution of problem (E) in the rest part of this subsection.
Proposition 2.7. Suppose that the conditions in Proposition 2.1 are satisfied. Let \((u, v)\) be the unique global solution of (P), then we have

\[
\lim_{t \to +\infty} \|u_t\|_2 = \lim_{t \to +\infty} \|v_t\|_2 = 0, \tag{2.16}
\]

and

\[
\|u\|_{H^3} + \|v\|_{H^3} \leq C(\delta), \text{ for all } t \geq \delta > 0. \tag{2.17}
\]

**Proof:** First we claim that

\[
\|\nabla u\|_2 + \|\nabla v\|_2 \leq C(u_0, v_0, w, z), \tag{2.18}
\]

where \(C\) is a constant independent of time \(t\). To prove this claim, we denote \(\xi = u - w, \eta = v - z\). Then it follows from system (P) and (E) that

\[
\begin{align*}
\xi_t - \Delta \xi &= \lambda \alpha(x)(f(v) - f(z)), & \text{in } Q_T = \Omega \times (0, T), \\
\eta_t - \Delta \eta &= \mu \beta(x)(g(u) - g(w)), & \text{in } Q_T = \Omega \times (0, T), \\
\xi &= \eta = 0, & \text{on } \partial \Omega \times (0, T), \\
\xi(x, 0) &= u_0(x) - w(x), \eta(x, 0) = v_0(x) - z(x), & \text{for } x \in \bar{\Omega}.
\end{align*} \tag{2.19}
\]

Multiplying the first equation of (2.19) by \(\xi_t\) yields that

\[
\frac{1}{2} \frac{d}{dt} \|\nabla \xi\|_2^2 + \|\xi_t\|_2^2 = \lambda \int_{\Omega} \alpha(x)[f(v) - f(z)]\xi_t dx \leq 0, \tag{2.20}
\]

where \(v \leq z\), assumption (H1) and Proposition 2.2 are used. The above inequality then implies that

\[
\frac{d}{dt} \|\nabla \xi\|_2^2 \leq 0 \tag{2.21}
\]

and hence

\[
\|\nabla \xi\|_2 \leq \|\nabla \xi_0\|_2 \leq C(u_0, v_0, w, z). \tag{2.22}
\]

\[
\|\nabla \eta\|_2 \leq C(u_0, v_0, w, z)
\]

can be obtained similarly. Then (2.18) follows by

\[
\|\nabla u\|_2 + \|\nabla v\|_2 \leq \|\nabla \xi\|_2 + \|\nabla \eta\|_2 + \|\nabla z\|_2 \leq C(u_0, v_0, w, z). \tag{2.23}
\]

Next, we show that

\[
\int_0^{+\infty} (\|u_t\|_2^2 + \|v_t\|_2^2) dt \leq C. \tag{2.24}
\]

After multiplying equations in (P) by \(u_t\) and \(v_t\), respectively, adding them up and integrating over \(\Omega\), we can see that Problem (P) admits a Lyapunov function

\[
E(u, v) = \int_{\Omega} \left( \nabla u \nabla v - \mathbb{F}(x, v) - \mathbb{G}(x, u) \right) dx, \tag{2.25}
\]

where \(\mathbb{F}(x, v) = \lambda \alpha(x) \int_0^v f(s) ds, \mathbb{G}(x, u) = \mu \beta(x) \int_0^u g(s) ds\), and there holds

\[
\frac{d}{dt} E(u, v) + 2 \int_{\Omega} uv_t dx = 0. \tag{2.26}
\]

Note that \(0 < w < 1, 0 < z < 1\). By assumption (H1) and (2.18), integrating (2.26) with respect to \(t\) yields

\[
2 \int_0^{+\infty} \int_{\Omega} u_t v_t d\tau \leq E(u_0, v_0) + \int_{\Omega} \nabla u \nabla v dx + \int_{\Omega} \left( \mathbb{F}(x, \max z) + \mathbb{G}(x, \max w) \right) dx \leq C. \tag{2.27}
\]
Thus, for \( \lambda > 0 \), Proposition 2.3 and Lemma 2.6, Differentiating the first equation in (P) with respect to \( t \) yields

\[
u_{tt} - \Delta u_t = \lambda \alpha(x) f'(v_t)v_t.\] (2.28)

Multiplying (2.28) by \( u_t \) and integrating over \( \Omega \), by Lemma 2.6 and assumption \( (H1) \) we have

\[
\frac{1}{2} \frac{d}{dt} \|u_t\|_2^2 + \|\nabla u_t\|_2^2 = \int_\Omega \lambda \alpha f'(v_t) v_t u_t dx \leq C \|u_t\|_2^2.\] (2.29)

By Young’s inequality we get

\[
\frac{d}{dt} \|u_t\|_2^2 \leq C_1 \|u_t\|_2^4 + C_2.\] (2.30)

Then by (2.24) and (2.32) we get \( \lim_{t \to +\infty} \|u_t\|_2 = 0 \), while \( \lim_{t \to +\infty} \|v_t\|_2 = 0 \) can be obtained similarly and (2.10) follows.

Now integrating (2.29) with respect to \( t \), by (2.24) we obtain

\[
\frac{1}{2} \|u_t\|_2^2 + \int_0^t \|\nabla u_t\|_2^2 d\tau \leq \frac{1}{2} \|u_t(0)\|_2^2 + C \int_0^\infty \|u_t\|_2^2 d\tau \leq C,\] (2.31)

which implies obviously

\[
\int_0^t \|\nabla u_t\|_2^2 d\tau \leq C.\] (2.32)

Multiplying (2.28) by \( -\Delta u_t \) and integrating over \( \Omega \), by Lemma 2.6 we have

\[
\frac{1}{2} \frac{d}{dt} \|\nabla u_t\|_2^2 + \|\Delta u_t\|_2^2 = \lambda \int_\Omega \alpha f'(v_t) v_t (-\Delta u_t) dx \leq C \|u_t\|_2 \|\Delta u_t\|_2 \leq C \|u_t\|_2 \|\Delta u_t\|_2 + \frac{1}{2} \|\Delta u_t\|_2^2,\] (2.33)

which yields

\[
\frac{d}{dt} \|\nabla u_t\|_2^2 + \|\Delta u_t\|_2^2 \leq C \|u_t\|_2^2.\] (2.34)

Multiplying (2.34) by \( t \), then integrating with respect to \( t \) in \([0, t]\), by (2.24) and (2.32) there holds

\[
t \|\nabla u_t\|_2^2 + \int_0^t \tau \|\Delta u_t\|_2^2 d\tau \leq \int_0^t \|\nabla u_t\|_2^2 d\tau + Ct \int_0^t \|u_t\|_2^2 d\tau \leq C_1 + C_2 t.\] (2.35)

Thus, for \( t \geq \delta > 0 \), we have

\[
\|\nabla u_t\|_2^2 \leq \frac{C_1}{t} + C_2 \leq \frac{C_1}{\delta} + C_2\] (2.36)

and it follows

\[
\|u_t\|_{H^1} \leq C(\delta) \text{ for } t \geq \delta.\] (2.37)

Now we can deduce from the equation in (P) and the regularity theory for the elliptic problem (see e.g. [6, 20])

\[
\begin{cases}
-\Delta u = \lambda \alpha(x) f(v) - u_t, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}\] (2.38)

that

\[
\|u(\cdot, t)\|_{H^1} \leq C(\|f(v)\|_{H^1} + \|u_t\|_{H^1}) \leq C(\delta) \left(1 + \|f'(v)\|_{L^2}\right) \leq C(\delta)(1 + C \|\nabla v\|_2) \leq C(\delta),\] (2.39)

by (2.37) and (2.18). \( \|v(\cdot, t)\|_{H^1} \) can be treated similarly. In conclusion, we obtain (2.17). \( \square \)

**Corollary 2.8.** For \( 1 \leq n \leq 3 \), there holds

\[
\lim_{t \to \infty} \|\xi(\cdot, t)\|_{C^1} = \lim_{t \to \infty} \|u(\cdot, t) - w\|_{C^1} = 0, \quad \lim_{t \to \infty} \|\eta(t)\|_{C^1} = \lim_{t \to \infty} \|v(t) - z\|_{C^1} = 0.\] (2.40)
Proof:  Note by (2.17) that $\xi, \eta \in H^3(\Omega)$, and $H^3(\Omega) \hookrightarrow \hookrightarrow C^1(\Omega)$ for $1 \leq n \leq 3$ by Sobolev compact embedding theorem. Thanks to (2.4), (2.40) follows by the relative compactness of $\xi(t), \eta(t)$ in $C^1$ and the uniqueness of the limits. \hfill \square

Now we give the proof of Theorem 1.1 (ii) as follows.

Proof of Theorem 1.1 (ii):

Multiplying equations in (2.19) by $\xi$ and $\eta$, respectively, adding them up and integrating over $\Omega$ yields
\[
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \xi^2 + \frac{1}{2} \eta^2 \right) dx + \|\nabla \xi\|_2^2 + \|\nabla \eta\|_2^2 = \int_{\Omega} \left( \lambda \alpha [f(z) - f(v)](-\xi) + \mu \beta [g(w) - g(u)](-\eta) \right) dx. \tag{2.41}
\]
Rewriting equations in (2.19) as
\[
\begin{cases}
\xi_t - \Delta \xi - \lambda \alpha f'(z)\eta = \lambda \alpha (f(v) - f(z) - f'(z)\eta), & \text{in } Q_T = \Omega \times (0, T), \\
\eta_t - \Delta \eta - \mu \beta g'(w)\xi = \mu \beta (g(u) - g(w) - g'(w)\xi), & \text{in } Q_T = \Omega \times (0, T).
\end{cases}
\tag{2.42}
\]

Note that $0 \leq v \leq z < 1$ and $0 \leq u \leq w < 1$. By the convexity of $f$ and $g$, it is easy to deduce that $f(v) - f(z) - f'(z)\eta \geq 0$ and $g(u) - g(w) - g'(w)\xi \geq 0$. Thus it follows that
\[
\begin{cases}
\xi_t - \Delta \xi - \lambda \alpha f'(z)\eta \geq 0, & \text{in } Q_T = \Omega \times (0, T), \\
\eta_t - \Delta \eta - \mu \beta g'(w)\xi \geq 0, & \text{in } Q_T = \Omega \times (0, T).
\end{cases}
\tag{2.43}
\]

Multiplying inequalities in (2.43) by $\psi_1$ and $\varphi_1$, respectively, adding them up and integrating over $\Omega$ yields
\[
\int_{\Omega} (\psi_1 \xi + \varphi_1 \eta) dx + \nu_1 \int_{\Omega} (\psi_1 \xi + \varphi_1 \eta) dx \geq 0. \tag{2.44}
\]
Here, $\nu_1$ is the principal eigenvalue of problem (2.10) and $(\varphi_1, \psi_1)$ is the corresponding positive eigenfunction. Multiplying (2.44) by $-1$, then adding it and (2.41) together yields
\[
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \xi^2 + \frac{1}{2} \eta^2 + [\psi_1(-\xi) + \varphi_1(-\eta)] \right) dx + \|\nabla \xi\|_2^2 + \|\nabla \eta\|_2^2 + \nu_1 \int_{\Omega} [\psi_1(-\xi) + \varphi_1(-\eta)] dx \\
\leq \int_{\Omega} \left( \lambda \alpha [f(z) - f(v)](-\xi) + \mu \beta [g(w) - g(u)](-\eta) \right) dx.
\tag{2.45}
\]

Now we claim that there exists $T_0 > 0$ such that for any $t > T_0$, there holds
\[
f(z) - f(v) \leq \frac{\nu_1}{2\lambda \alpha \|\alpha\|_\infty} \psi_1, \quad g(w) - g(u) \leq \frac{\nu_1}{2\mu \|\beta\|_\infty} \varphi_1. \tag{2.46}
\]
In fact, recalling (2.10), by Lemma 2.4 we have
\[
\begin{cases}
-\Delta \varphi_1 = f'(z)\psi_1 + \nu_1 \varphi_1 \geq 0, & \text{in } \Omega, \\
-\Delta \psi_1 = g'(w)\varphi_1 + \nu_1 \psi_1 \geq 0, & \text{in } \Omega, \\
\varphi_1 = \psi_1 = 0, & \text{on } \partial \Omega.
\end{cases}
\tag{2.47}
\]
Thus, by Hopf lemma there holds $-\frac{\partial \varphi_1}{\partial n} \geq \epsilon_0$, $-\frac{\partial \psi_1}{\partial n} \geq \epsilon_0$ on $\partial \Omega$ for some $\epsilon_0 > 0$. Then (2.46) follows by Lemma 2.4, Lemma 2.5 and (2.40).
Combining (2.45), (2.46) and the Poincaré inequality $\|u\|_2 \leq \frac{1}{\lambda_1} \|\nabla u\|_2$ for any $u \in H^1_0(\Omega)$ with $\lambda_1 > 0$ being the first eigenvalue of $-\Delta$ on $H^1_0(\Omega)$, we get
\[
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \xi^2 + \frac{1}{2} \eta^2 + [\psi_1(-\xi) + \varphi_1(-\eta)] \right) dx + \lambda_1 \|\xi\|_2^2 + \lambda_1 \|\eta\|_2^2 + \nu_1 \int_{\Omega} [\psi_1(-\xi) + \varphi_1(-\eta)] dx \leq 0.
\tag{2.48}
\]

Proof: We will only prove the case $(\text{Proposition } 3.1)$, then obtain the quenching time estimates in the rest part of this section.

Let $Y = \int_\Omega (\xi^2 + \eta^2 + 2[\psi_1(-\xi) + \varphi_1(-\eta)]) dx$. Note that
\[
\int_\Omega [\psi_1(-\xi) + \varphi_1(-\eta)] dx \geq 0.
\] (2.49)

By (2.48) there holds
\[
\frac{dY}{dt} + \gamma Y \leq 0, \quad \gamma = \min \left\{ 2\lambda_1, \frac{\nu_1}{2} \right\},
\] (2.50)

which yields $Y \leq Y(0)e^{-\gamma t}$. Then by noting (2.49) again it follows that
\[
\|u(t,x) - w(x)\|_2^2 + \|v(t,x) - z(x)\|_2^2 \leq Y(t) \leq C_0 \exp \left( -\min \left\{ \lambda_1, \frac{\nu_1}{2} \right\} t \right), \quad \text{for } t > T_0.
\] (2.51)

The proof of Theorem 1.1 (ii) is therefore completed.

\[
\square
\]

3 Quenching and Quenching time estimate for $(\lambda, \mu)$ above the critical curve $\Gamma$

In this section, our goal is to prove Theorem 2.2 and Corollary 3.4. More precisely, we will first prove the finite time quenching in the following Proposition 3.1, then obtain the quenching time estimates in the rest part of this section.

Proposition 3.1. Suppose (H1) holds. Let $\mathcal{O}_2$ be the connected component defined in Theorem A. If $(\lambda, \mu) \in \mathcal{O}_2$, then for any $(u_0, v_0)$ satisfying (H2), the solution $(u, v)$ to (P) will quench at a finite time $T^*$ in the sense of (1.3).

Proof: We will only prove the case $(u_0, v_0) \equiv (0, 0)$, then the holding for general nonnegative initial data follows directly by comparison principle.

Let $(\lambda, \mu) \in \mathcal{O}_2$. Suppose on the contrary that the local solution $(u, v)$ (see Theorem B) exists globally, i.e. $0 \leq u < 1, 0 \leq v < 1$ for all $t \geq 0$. Take $\delta > 1, a = \frac{\lambda}{\delta}, b = \frac{\mu}{\delta}$. Since $U = \frac{u}{\delta} < u, V = \frac{v}{\delta} < v$, it then indicates that $U \leq \frac{1}{\delta} < 1, V \leq \frac{1}{\delta} < 1$, and by the monotone increasing of $f, g$ there holds
\[
\begin{align*}
U_t - \Delta U &= \frac{\lambda \alpha(x) f(v)}{\delta} \geq a \alpha(x) f(V), \quad \text{in } Q_T = \Omega \times (0, T), \\
V_t - \Delta V &= \frac{\mu \beta(x) g(u)}{\delta} \geq b \beta(x) g(U), \quad \text{in } Q_T = \Omega \times (0, T), \\
U &= V = 0, \quad \text{on } \partial \Omega \times (0, T), \\
U(x, 0) &= V(x, 0) = 0, \quad \text{for } x \in \Omega.
\end{align*}
\] (3.1)

Hence $(U, V)$ is a supersolution of
\[
\begin{align*}
U_t - \Delta U &= a \alpha(x) f(V), \quad \text{in } Q_T = \Omega \times (0, T), \\
V_t - \Delta V &= b \beta(x) g(U), \quad \text{in } Q_T = \Omega \times (0, T), \\
U &= V = 0, \quad \text{on } \partial \Omega \times (0, T), \\
U(x, 0) &= V(x, 0) = 0, \quad \text{for } x \in \bar{\Omega}.
\end{align*}
\] (3.2)

Therefore (3.2) has a global classical solution $(\mathcal{U}(x, t), \mathcal{V}(x, t))$, since $0 \leq \mathcal{U} \leq \mathcal{U} \leq \frac{1}{\delta} < 1, 0 \leq \mathcal{V} \leq \mathcal{V} \leq \frac{1}{\delta} < 1$. Note that there further holds
\[
\lim_{t \to +\infty} (\|\mathcal{U}\|_2^2 + \|\mathcal{V}\|_2^2) = 0,
\]
\sup_{t \geq 1} (\|\mathcal{U}\|_{H^2(\Omega)} + \|\mathcal{V}\|_{H^2(\Omega)}) < +\infty,
\] (3.3)
which can be proved similarly to Proposition 2.7. By Sobolev embedding theorem, one can have that there exists a subsequence \( \{t_j\}_{j=1}^{\infty} \) such that \( t_j \to +\infty \), \( (\mathcal{U}(\cdot, t_j), \mathcal{V}(\cdot, t_j)) \) converges strongly in \( H_0^1(\Omega) \) to \( (\mathcal{U}_\infty, \mathcal{V}_\infty) \). Now take \( \phi \in H_0^1(\Omega) \). Multiplying (3.2) by \( \phi \) and integrating by parts with respect to \( x \) yields,

\[
\begin{aligned}
&\int_\Omega \partial_t U(t, t_j) dx + \int_\Omega \nabla U(t, t_j) \nabla \phi dx = \int_\Omega a(\lambda) f(\mathcal{V}(t, t_j)) dx, \\
&\int_\Omega \partial_t V(t, t_j) dx + \int_\Omega \nabla V(t, t_j) \nabla \phi dx = \int_\Omega b(\lambda) g(\mathcal{U}(t, t_j)) dx. 
\end{aligned}
\tag{3.4}
\]

Passing to the limit \( t_j \to +\infty \), we obtain that \( (\mathcal{U}_\infty, \mathcal{V}_\infty) \) is a weak solution of

\[
\begin{aligned}
-\Delta w &= a(\lambda) f(z), \quad \text{in } \Omega, \\
-\Delta z &= b(\lambda) g(w), \quad \text{in } \Omega, \\
u &= v = 0, \quad \text{on } \partial\Omega.
\end{aligned}
\tag{3.5}
\]

Choose \( \delta \) close to 1 such that \( (a, b) \in \mathcal{O}_2 \). Then we get a contradiction with Theorem A. The proof of this proposition is therefore completed. \( \square \)

We now focus on estimates for quenching time \( T^* \).

**Proof of Theorem 1.2 (i):** Note that the finite time quenching result has been proved in Proposition 2.7. It is reduced to obtain (3.5) to complete the proof. Similar to the proof of Theorem B, let \( (\zeta, \rho) \) be the solution of the following ODE system:

\[
\begin{aligned}
\frac{d\zeta}{dt} &= \lambda \|\alpha\|_\infty f(\rho), \quad \text{in } (0, T), \\
\frac{d\rho}{dt} &= \mu \|\beta\|_\infty g(\zeta), \quad \text{in } (0, T), \\
\zeta(0) &= \|u_0\|_\infty, \quad \rho(0) = \|v_0\|_\infty.
\end{aligned}
\tag{3.6}
\]

The local existence of (3.6) can be obtained by [17, Chapter III]. By the comparison principle for parabolic system (Lemma 2.2), it follows that \( \zeta \geq u, \rho \geq v \), and then the solution \( (\zeta, \rho) \) of (3.6) will also quench at a finite time in the sense of (3.5). Denote the quenching time of \( (\zeta, \rho) \) by \( T_0 \), which means \( \zeta(T_0) = 1, \rho(T_0) \leq 1 \) or \( \zeta(T_0) \leq 1, \rho(T_0) = 1 \). Therefore \( T^* \geq T_0 \).

Obviously, we can see that

\[
\frac{d\zeta}{d\rho} = \frac{\lambda \|\alpha\|_\infty f(\rho)}{\mu \|\beta\|_\infty g(\zeta)},
\]

which implies

\[
\mu \|\beta\|_\infty g(\zeta) d\zeta = \lambda \|\alpha\|_\infty f(\rho) d\rho.
\]

Then

\[
\|\beta\|_\infty G(\zeta(t)) - \frac{\lambda}{\mu} \|\alpha\|_\infty F(\rho(t)) =: c_0
\]

is a constant for all \( t \geq 0 \). It hence follows that

\[
\zeta(t) = G^{-1}\left(\frac{\lambda}{\mu \|\alpha\|_\infty F(\rho(t)) + \mu c_0}{\|\beta\|_\infty}\right), \quad \rho(t) = F^{-1}\left(\frac{\lambda \|\alpha\|_\infty G(\zeta(t)) - \mu c_0}{\lambda \|\alpha\|_\infty}\right).
\tag{3.7}
\]

Therefore by (3.6) there holds

\[
\int_0^1 \frac{d\rho}{\mu \|\beta\|_\infty G^{-1}\left(\frac{\lambda}{\mu \|\alpha\|_\infty F(\rho(t)) + \mu c_0}{\|\beta\|_\infty}\right)} = \int_0^\zeta(T_0) \frac{d\zeta}{\lambda \|\alpha\|_\infty f\left(F^{-1}\left(\frac{\lambda \|\beta\|_\infty G(\zeta(t)) - \mu c_0}{\lambda \|\alpha\|_\infty}\right)\right)}.
\tag{3.8}
\]

If \( \rho(T_0) = 1 \) and \( \zeta(T_0) \leq 1 \), one can see that

\[
T_0 = \int_0^1 \frac{d\rho}{\mu \|\beta\|_\infty G^{-1}\left(\frac{\lambda}{\mu \|\alpha\|_\infty F(\rho(t)) + \mu c_0}{\|\beta\|_\infty}\right)} = \int_0^{\zeta(T_0)} \frac{d\zeta}{\lambda \|\alpha\|_\infty f\left(F^{-1}\left(\frac{\lambda \|\beta\|_\infty G(\zeta(t)) - \mu c_0}{\lambda \|\alpha\|_\infty}\right)\right)}.
\tag{3.9}
\]
Similarly, if \( \rho(T_0) \leq 1 \) and \( \zeta(T_0) = 1 \), we arrive at

\[
T_0 = \int_{\|u_0\|_\infty}^1 \frac{d\zeta}{\lambda\|\alpha\|\infty f\left(F^{-1}\left(\frac{\mu\|\beta\|\infty G(\zeta) - \mu c_0}{\lambda\|\alpha\|\infty}\right)\right)} = \int_{\|v_0\|_\infty}^{\rho(T_0)} \frac{d\rho}{\mu\|\beta\|\infty g\left(G^{-1}\left(\frac{\lambda\|\alpha\|\infty F(\rho) + \mu c_0}{\mu\|\beta\|\infty}\right)\right)}
\]

(3.10)

In conclusion, (1.5) holds. \( \square \)

As for the particular MEMS case \( f(s) = g(s) = \frac{1}{(1-s)^2} \), Theorem 1.2 (i) can be rewritten as Corollary 1.3.

**Proof of Corollary 1.3** Suppose that \( f(s) = g(s) = \frac{1}{(1-s)^2} \). Then (3.10) can be rewritten as

\[
\begin{aligned}
\frac{d\zeta}{dt} &= \frac{\lambda\|\alpha\|\infty}{(1-\rho)^2}, \\
\frac{d\rho}{dt} &= \frac{\mu\|\beta\|\infty}{(1-\zeta)^2}, \\
\zeta(0) &= \|u_0\|_\infty, \rho(0) = \|v_0\|_\infty,
\end{aligned}
\]

(3.11)

and there holds

\[
\frac{1}{1-\zeta} - \frac{\lambda\|\alpha\|\infty}{\mu\|\beta\|\infty(1-\rho)} =: c_1,
\]

(3.12)

with \( c_1 = \frac{1}{1-\|u_0\|_\infty} - \frac{\lambda\|\alpha\|\infty}{\mu\|\beta\|\infty(1-\|v_0\|_\infty)} \) is a constant for all \( t \geq 0 \). Note that it follows from (3.12) that \( \zeta, \rho \) will quench simultaneously. By (3.11) there holds

\[
\frac{d\rho}{\left(c_0 + \frac{\lambda\|\alpha\|\infty}{\mu\|\beta\|\infty(1-\rho)}\right)^2} = \mu\|\beta\|\infty dt.
\]

(3.13)

Then by taking \( y = c_0 + \frac{\lambda\|\alpha\|\infty}{\mu\|\beta\|\infty(1-\rho)} \), we obtain

\[
\frac{\mu^2\|\beta\|\infty^2 dt}{\lambda\|\alpha\|\infty} = d\left(\frac{2}{c_0} \ln \left|\frac{y}{y-c_0}\right| - \frac{1}{c_0 y} - \frac{1}{y-c_0}\right).
\]

Now one can see from \( \lim_{\rho \to 1} y = +\infty \) that

\[
T_0 = \frac{\lambda\|\alpha\|\infty}{\mu^2\|\beta\|\infty^2} \left(-\frac{2}{c_0} \ln \left|\frac{c_0 \mu\|\beta\|\infty(1-\|v_0\|_\infty) + \lambda\|\alpha\|\infty}{\lambda\|\alpha\|\infty}\right| + \frac{\mu\|\beta\|\infty(1-\|v_0\|_\infty)}{c_0^2(c_0 \mu\|\beta\|\infty(1-\|v_0\|_\infty) + \lambda\|\alpha\|\infty)}\right),
\]

(3.14)

and (1.8) follows by \( T^* \geq T_0 \). \( \square \)

Next, we shall adapt and improve some of the arguments in [5] to further get an upper estimate of quenching time \( T^* \) for the particular case \( f = g \), i.e, Theorem 1.2 (ii).

**Proof of Theorem 1.2 (ii)**: Let \( \lambda_1 \) is the first eigenvalue of \(-\Delta\) in \( H_0^1(\Omega) \), and \( \phi \) is the corresponding eigenfunction satisfying \( \int_{\Omega} \phi dx = 1 \). Introduce

\[
I(t) = \int_{\Omega} (u + v)\phi dx.
\]

(3.15)
It is clear that \( I(t) \) is well defined on the existence interval of the solution \((u,v)\). Using (P), we find that

\[
I'(t) = \int_\Omega \phi(u_t + v_t)\,dx
= \int_\Omega (u + v)\Delta \phi\,dx + \int_\Omega (\lambda \alpha f(v) + \mu \beta f(u))\,dx
\geq -\lambda_1 I(t) + \min\{\lambda, \mu\} \min\{\inf_\Omega \alpha, \inf_\Omega \beta\} \int_\Omega \phi(f(v) + f(u))\,dx.
\] (3.16)

By (H1) and Jensen’s inequality,

\[
\int_\Omega \phi(f(u))\,dx \geq f(\int_\Omega u\phi\,dx), \quad \int_\Omega \phi(v)\,dx \geq f(\int_\Omega v\phi\,dx).
\] (3.17)

Substituting this into (3.16), using (H1) again we obtain that

\[
I'(t) + \lambda_1 I(t) \geq \min\{\lambda, \mu\} \min\{\inf_\Omega \alpha, \inf_\Omega \beta\} \left(f(\int_\Omega u\phi\,dx) + f(\int_\Omega v\phi\,dx)\right)
\geq 2 \min\{\lambda, \mu\} \min\{\inf_\Omega \alpha, \inf_\Omega \beta\} f\left(\frac{1}{2}\right).
\] (3.18)

Therefore

\[
I'(t) \geq -\lambda_1 I(t) + 2 \min\{\lambda, \mu\} \min\{\inf_\Omega \alpha, \inf_\Omega \beta\} f\left(\frac{1}{2}\right) > 0,
\] (3.19)

when \(\min\{\lambda, \mu\} > \frac{\lambda_1}{2 \min\{\inf_\Omega \alpha, \inf_\Omega \beta\}} \sup_{0 \leq s < 2} \frac{s}{f(\frac{1}{2})}\). If \(\max\{u, v\}\) remains smaller than 1 for all \(t\), then \(I(t)\) is defined for all \(t\). However, from the ODE theory, under the given assumptions, \(I(t)\) is only well defined in \((0, \bar{T})\), where

\[
\bar{T} = \int_{\int_\Omega (u_0 + v_0)\phi\,dx}^1 \frac{1}{-\lambda_1 s + 2 \min\{\lambda, \mu\} \min\{\inf_\Omega \alpha, \inf_\Omega \beta\} f\left(\frac{1}{2}\right)}\,ds < +\infty.
\] (3.20)

That is to say, the solution \((u, v)\) must touchdown at a finite time \(T^* \leq \bar{T}\), i.e., (1.6) and (1.7) follows. □

### Appendix A

We will show the proof of Theorem A in this Appendix. First we will prove that the elliptic problem (E) has a classical solution for \(\lambda\) and \(\mu\) small enough, while (E) has no solution for \(\lambda\) or \(\mu\) large enough. More precisely, we will prove that the set

\[
\Lambda := \{(\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+ : (E)\ has\ a\ classical\ minimal\ solution\}
\] (A.1)

is nonempty and bounded.

**Lemma A.1.** \(\Lambda\ is bounded, and there exist \(\lambda_0 > 0\), \(\mu_0 > 0\) such that \((0, \lambda_0] \times (0, \mu_0] \subseteq \Lambda\).

**Proof:** Let \(\gamma \in H_0^1(\Omega)\) be the regular solution of \(-\Delta \gamma = 1\ in\ \Omega\). It is then easy to verify that there exists \(\alpha \in \left(0, \frac{1}{\|\gamma\|_\infty}\right)\) such that \((\alpha \gamma, \alpha \gamma)\) is a supersolution of (E) if \(\lambda < \frac{1}{\sup_{x \in \Omega} \alpha(x)} \sup_{0 < s < \|\gamma\|_\infty} \frac{s}{f(s)} =: \lambda_0\)
and \(\mu < \frac{1}{\sup_{x \in \Omega} \beta(x)} \sup_{0 < s < \|\gamma\|_\infty} \frac{s}{g(s)} =: \mu_0\).

As \((0,0)\) is a subsolution and \(\alpha \gamma > 0\ in\ \Omega\), (E) admits a regular solution for \(\lambda \in (0, \lambda_0]\) and \(\mu \in (0, \mu_0]\). In fact, for these \(\lambda, \mu,\) using (H2) and the monotone iteration for \(n \in \mathbb{N},\)

\[
\begin{aligned}
w_0 &= z_0 = 0, \\
-\Delta w_{n+1} &= \lambda \alpha(x)f(z_n), \quad \text{in } \Omega, \\
-\Delta z_{n+1} &= \mu \beta(x)g(w_n), \quad \text{in } \Omega, \\
w_{n+1} &= z_{n+1} = 0, \quad \text{on } \partial \Omega,
\end{aligned}
\] (A.2)
we get the minimal solution \((w_{\lambda,\mu}, z_{\lambda,\mu}) = \lim_{n \to +\infty} (w_n, z_n)\). Therefore, \(\Lambda\) is nonempty.

On the other hand, take a positive first eigenfunction \(\varphi\) of \(-\Delta\) in \(H^1_0(\Omega)\) with the first eigenvalue \(\lambda_1\) such that \(\int_\Omega \varphi \, dx = 1\). By \([E]\) and \(w < 1, z < 1\), we arrive at

\[
\begin{cases}
\lambda_1 \geq \lambda_1 \int_\Omega w \varphi \, dx = \int_\Omega \varphi(-\Delta w) \, dx = \lambda \int_\Omega \alpha(x)f(z) \varphi \, dx \geq \lambda \int_\Omega \alpha(x)f(\varphi) \, dx, \\
\lambda_1 \geq \lambda_1 \int_\Omega z \varphi \, dx = \int_\Omega \varphi(-\Delta z) \, dx = \lambda \int_\Omega \beta(x)g(w) \varphi \, dx \geq \mu \int_\Omega \beta(x)g(\varphi) \, dx.
\end{cases}
\]

(A.3)

So \(\Lambda\) is bounded and \(\Lambda \subseteq \left(0, \frac{\lambda_1}{\int_\Omega \alpha(x)f(\varphi) \, dx}\right) \times \left(0, \frac{\lambda_1}{\int_\Omega \beta(x)g(\varphi) \, dx}\right). \) □

Denote \(\mu = \Gamma(\lambda)\) as the critical curve such that if \(0 \leq \mu < \Gamma(\lambda)\), then \((\lambda, \mu) \in \Lambda\); if \(\mu > \Gamma(\lambda)\), then \((\lambda, \mu) \in (\mathbb{R}^+ \times \mathbb{R}^+) \setminus \Lambda\). By Lemma A.1, there further hold \(0 < \mu^* := \Gamma(0) < +\infty\) and \(0 < \lambda^* := \Gamma^{-1}(0) < +\infty\).

Next we state that the critical curve \(\mu = \Gamma(\lambda)\) is non-increasing. More precisely,

**Lemma A.2.** If \(0 \leq \lambda' \leq \lambda, 0 \leq \mu' \leq \mu\) for some \((\lambda, \mu) \in \Lambda\), then \((\lambda', \mu') \in \Lambda\).

**Proof:** Indeed, the solution associated to \((\lambda, \mu)\) turns out to be a super-solution to \((E)\) with \((\lambda', \mu')\). □

**Proof of Theorem A:** Define \(\mathcal{O}_1 = \Lambda \setminus \Gamma\). For \((\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \mathcal{O}_1\), there exist \(\theta_1, \theta_2 > 0\) such that \(\mu_1 = \theta_1 \lambda_1\) and \(\mu_2 = \theta_2 \lambda_2\). Using Lemma A.2, we can define a path linking \((\lambda_1, \mu_1)\) to \((0, 0)\) and another path linking \((0, 0)\) to \((\lambda_2, \mu_2)\), which implies that \(\mathcal{O}_1\) is connected. Now, define \(\mathcal{O}_2 = (\mathbb{R}^+ \times \mathbb{R}^+) \setminus (\Lambda \cup \Gamma)\). Let \((\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \mathcal{O}_2\). Then by Lemma A.2, there is a path linking \((\lambda_1, \mu_1)\) to \((\lambda_{\max}, \mu_{\max})\) and another path linking \((\lambda_{\max}, \mu_{\max})\) to \((\lambda_2, \mu_2)\), which follows that \(\mathcal{O}_2\) is connected.

At last, it is reduced to prove that problem \([E]\) admits no weak solution for \((\lambda, \mu) \in \mathcal{O}_2\). Suppose on the contrary that \((w, z)\) is a weak solution to \([E]\). By the monotonicity of \(f, g\), it is easy to verify that for any \(\delta > 1\), \((\hat{w}, \hat{z}) = \left(\frac{w}{\delta}, \frac{z}{\delta}\right)\) is a weak super-solution for problem

\[
\begin{aligned}
-\Delta w &= \frac{\lambda}{\delta} \alpha(x)f(z), & \text{in } \Omega, \\
-\Delta z &= \frac{\mu}{\delta} \beta(x)g(w), & \text{in } \Omega, \\
w &= z = 0, & \text{on } \partial \Omega,
\end{aligned}
\]

\((E_\delta)\)

then the monotone iteration will enable us a weak solution \((\hat{w}, \hat{z})\) of \((E_\delta)\) satisfying \(0 \leq \hat{w} \leq w \leq \frac{1}{\delta} < 1\), and \(0 \leq \hat{z} \leq z \leq \frac{1}{\delta} < 1\). The regularity theory implies that \((\hat{w}, \hat{z})\) is a regular solution of \((E_\delta)\). This means that \((\frac{\lambda}{\delta}, \frac{\mu}{\delta}) \in \mathcal{O}_1 \cup \Gamma\). Let \(\delta\) tend to 1, we get \((\lambda, \mu) \in \mathcal{O}_1 \cup \Gamma\), which contradicts with the assumption. Therefore, no weak solution exists for \((\lambda, \mu) \in \mathcal{O}_2\) and the proof of Theorem A is completed. □

**Appendix B**

In this Appendix, we will show the proof of Theorem B.

**Proof:** We first show the uniqueness of the solution to \([F]\). For any given \(0 < T_0 < T\), suppose \((\tilde{u}, \tilde{v}), (\hat{u}, \hat{v})\) are two pair classical solutions of \([F]\) on the interval \([0, T_0]\) such that \(\|\tilde{u}\|_{L^\infty(\Omega \times [0, T_0])} < 1\), \(\|\hat{v}\|_{L^\infty(\Omega \times [0, T_0])} < 1\), \(\|\tilde{u}\|_{L^\infty(\Omega \times [0, T_0])} < 1\), \(\|\hat{v}\|_{L^\infty(\Omega \times [0, T_0])} < 1\).
Indeed, the difference \((U, V) = (\hat{u} - \tilde{u}, \hat{v} - \tilde{v})\) satisfies

\[
\begin{aligned}
U_t - \Delta U &= \lambda \alpha(x) f'(\theta_v) V, & \text{in } Q_T = \Omega \times (0, T_0], \\
V_t - \Delta V &= \mu \beta(x) g'(\theta_u) U, & \text{in } Q_T = \Omega \times (0, T_0], \\
U &= V = 0, & \text{on } \partial \Omega \times (0, T_0], \\
U(x, 0) &= V(x, 0) = 0, & \text{for } x \in \bar{\Omega},
\end{aligned}
\]  

(B.1)

where \(\theta_v\) is between \(\hat{v}\) and \(\tilde{v}\), \(\theta_u\) is between \(\hat{u}\) and \(\tilde{u}\). The assumption on \((\hat{u}, \hat{v}), (\tilde{u}, \tilde{v})\) implies that \(f'(\theta_v), g'(\theta_u) \in L^\infty(\Omega \times [0, T_0])\) for any \(T_0 < T\). By using the comparison principle stated in Lemma 2.2 we deduce that \(U = V \equiv 0\) on \(\Omega \times [0, T_0]\).

To obtain Theorem B, it is reduced to show the existence. Let \((\zeta, \rho)\) be the solution of the ODE system (3.6). The local existence of (3.6) can be obtained by [17] Chapter III. Obviously, \((\zeta, \rho)\) is a supersolution of (P). Since \((0, 0)\) is a subsolution of (P), it follows from [12] Theorem 2.2 that there exists a unique classical solution \((u, v)\) to (P) between \((0, 0)\) and \((\zeta, \rho)\). In conclusion, the proof of Theorem B is completed.

\[ \blacksquare \]

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