On universal central extensions of Hom-Leibniz algebras

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Abstract: In the category of Hom-Leibniz algebras we introduce the notion of representation as adequate coefficients to construct the chain complex to compute the Leibniz homology of Hom-Leibniz algebras. We study universal central extensions of Hom-Leibniz algebras and generalize some classical results, nevertheless it is necessary to introduce new notions of $\alpha$-central extension, universal $\alpha$-central extension and $\alpha$-perfect Hom-Leibniz algebra. We prove that an $\alpha$-perfect Hom-Lie algebra admits a universal $\alpha$-central extension in the categories of Hom-Lie and Hom-Leibniz algebras and we obtain the relationships between both. In case $\alpha = Id$ we recover the corresponding results on universal central extensions of Leibniz algebras.

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1 Introduction

The Hom-Lie algebra structure was initially introduced in [9] motivated by examples of deformed Lie algebras coming from twisted discretizations of vector fields. Hom-Lie algebras are $\mathbb{K}$-vector spaces endowed with a bilinear skew-symmetric bracket satisfying a Jacobi identity twisted by a map. When this map is the identity map, then the definition of Lie algebra is recovered.

From the introductory paper [9], this algebraic structure and other related ones as Hom-associative, Hom-Leibniz and Hom-Nambu algebras were studied in several papers [10, 13, 14, 15, 16, 17, 18, 19] and references given therein.

Following the generalization in [11] from Lie to Leibniz algebras, it is natural to describe same generalization in the framework of Hom-Lie algebras. In this way, the notion of Hom-Leibniz algebra was firstly introduced in [13] as $\mathbb{K}$-vector spaces $L$ together with a linear map $\alpha : L \to L$, endowed with a bilinear
bracket operation $-[,-]: L \times L \to L$ which satisfies the Hom-Leibniz identity
$[[\alpha(x), y], z]] = [[x, y], \alpha(z)] - [[x, z], \alpha(y)]$, for all $x, y, z \in L$, and it was the subject of the recent papers \cite{10, 14, 15, 18}. A (co)homology theory and an initial study of universal central extensions was given in \cite{6}.

Our goal in the present paper is to generalize properties and characterizations of universal central extensions of Leibniz algebras in \cite{3, 5, 8} to Hom-Leibniz algebras setting. But an important fact, which is the composition of two central extension is not central as the counterexample \cite{4, 9} below shows, doesn’t allow us to obtain a complete generalization of the classical results. Nevertheless it leads us to introduce new notions as $\alpha$-centrality or $\alpha$-perfection in order to generalize classical results. On the other hand, we prove that an $\alpha$-perfect Hom-Lie algebra admits a universal $\alpha$-central extension in the categories of Hom-Lie and Hom-Leibniz algebras, then one of our main results establishes the relationships between both universal $\alpha$-central extensions. When we rewrite these relationships for Lie and Leibniz algebras, i.e. the twisting endomorphism is the identity, we recover the corresponding results given in \cite{8}.

In order to achieve our goal we organize the paper as follows: section 2 is dedicated to introduce the background material on Hom-Leibniz algebras. In section 3 we introduce co-representations, which are the adequate coefficients to define the chain complex from which we compute the homology of a Hom-Leibniz algebra with coefficients. Low-dimensional homology $\mathbb{K}$-vector spaces are obtained. In case $\alpha = Id$ we recover the homology of Leibniz algebras \cite{11, 12}. In section 4 we present our main results on universal central extensions, namely we extend classical results and present a counterexample showing that the composition of two central extension is not a central extension. This fact lead us to define $\alpha$-central extensions: an extension $\pi: (K, \alpha_K) \to (L, \alpha_L)$ is said to be $\alpha$-central if $[\alpha(\text{Ker } (\pi)), K] = 0 = [K, \alpha(\text{Ker } (\pi))]$, and is said to be central if $[\text{Ker } (\pi), K] = 0 = [K, \text{Ker } (\pi)]$. Clearly central extension implies $\alpha$-central extension and both notions coincide in case $\alpha = Id$.

We can extend classical results on universal central extensions of Leibniz algebras in \cite{3, 5, 8} to the Hom-Leibniz algebras setting as: a Hom-Lie algebra is perfect if and only if admits a universal central extension and the kernel of the universal central extension is the second homology with trivial coefficients of the Hom-Leibniz algebra. Nevertheless, other result as: if a central extension $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ is universal, then $(K, \alpha_K)$ is perfect and every central extension of $(K, \alpha_K)$ is split only holds for universal $\alpha$-central extensions, which means that only lifts on $\alpha$-central extensions. Other relevant result, which cannot be extended in the usual way, is: if $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ is a universal $\alpha$-central extension, then $H_1^\alpha(K) = H_2^\alpha(K) = 0$. Of course, when the twisting endomorphism is the identity morphism, then all the new notions and all the new results coincide with the classical ones.
In section 5 we prove that an $\alpha$-perfect Hom-Lie algebra $(L, [-, -], \alpha_L)$, that is $L = [\alpha_L(L), \alpha_L(L)]$, admits a universal $\alpha$-central extension in the categories of Hom-Lie and Hom-Leibniz algebras, then we obtain the relationships between both. Our main results in this section generalize the relationships between the universal central extensions of a Lie algebra in the categories of Lie and Leibniz algebras given in [7] when we consider Leibniz algebras as Hom-Leibniz algebras, i.e. when the twisting endomorphism is the identity.

2 Hom-Leibniz algebras

In this section we introduce necessary material on Hom-Leibniz algebras which will be used in subsequent sections.

Definition 2.1 [14] A Hom-Leibniz algebra is a triple $(L, [-, -], \alpha_L)$ consisting of a $K$-vector space $L$, a bilinear map $[-, -] : L \times L \rightarrow L$ and a $K$-linear map $\alpha_L : L \rightarrow L$ satisfying:

$$[\alpha_L(x), [y, z]] = [[x, y], \alpha_L(z)] - [[x, z], \alpha_L(y)] \quad \text{(Hom-Leibniz identity)} \quad (1)$$

for all $x, y, z \in L$.

In terms of the adjoint representation $ad_x : L \rightarrow L, ad_x(y) = [y, x]$, the Hom-Leibniz identity can be written as follows:

$$ad_{\alpha_L(z)} \cdot ad_y = ad_{\alpha_L(y)} \cdot ad_z + ad_{[y, z]} \cdot \alpha$$

Definition 2.2 [17] A Hom-Leibniz algebra $(L, [-, -], \alpha_L)$ is said to be multiplicative if the $K$-linear map $\alpha_L$ preserves the bracket, that is, if $\alpha_L[x, y] = [\alpha_L(x), \alpha_L(y)]$, for all $x, y \in L$.

Example 2.3

a) Taking $\alpha = Id$ in Definition 2.1 we obtain the definition of Leibniz algebra [11]. Hence Hom-Leibniz algebras include Leibniz algebras as a full subcategory, thereby motivating the name "Hom-Leibniz algebras" as a deformation of Leibniz algebras twisted by a homomorphism. Moreover it is a multiplicative Hom-Leibniz algebra.

b) Hom-Lie algebras [9] are Hom-Leibniz algebras whose bracket satisfies the condition $[x, x] = 0$, for all $x$. So Hom-Lie algebras can be considered as a full subcategory of Hom-Leibniz algebras category. For any multiplicative Hom-Leibniz algebra $(L, [-, -], \alpha_L)$ there is associated the Hom-Lie algebra $(L_{Lie}, [-, -], \tilde{\alpha})$, where $L_{Lie} = L/L^{\text{ann}}$, the bracket is the canonical bracket induced on the quotient and $\tilde{\alpha}$ is the homomorphism naturally induced by $\alpha$. Here $L^{\text{ann}} = \{[x, x] : x \in L\}$. 

3
b) Let \((D, \sqsubset, \sqsupset, \alpha_D)\) be a Hom-dialgebra. Then \((D, \sqsubset, \sqsupset, \alpha_D)\) is a Hom-Leibniz algebra with respect to the bracket \([x, y] = x \sqsubset y - y \sqsupset x\), for all \(x, y \in A\) [18].

c) Let \((L, [-,-])\) be a Leibniz algebra and \(\alpha_L : L \to L\) a Leibniz algebra endomorphism. Define \([-,-]_\alpha : L \otimes L \to L\) by \([x, y]_\alpha = [\alpha(x), \alpha(y)]\), for all \(x, y \in L\). Then \((L, [-,-]_\alpha, \alpha_L)\) is a multiplicative Hom-Leibniz algebra.

d) Abelian or commutative Hom-Leibniz algebras are \(K\)-vector spaces \(L\) with trivial bracket and any linear map \(\alpha_L : L \to L\).

**Definition 2.4** A homomorphism of Hom-Leibniz algebras \(f : (L, [-,-], \alpha_L) \to (L', [-,-]', \alpha_L')\) is a \(K\)-linear map \(f : L \to L'\) such that

a) \(f([x, y]) = [f(x), f(y)]'\)

b) \(f \cdot \alpha_L(x) = \alpha_L' \cdot f(x)\)

for all \(x, y \in L\).

A homomorphism of multiplicative Hom-Leibniz algebras is a homomorphism of the underlying Hom-Leibniz algebras.

So we have defined the category \(\text{Hom-Leib}\) (respectively, \(\text{Hom-Leib}_{\text{mult}}\)) whose objects are Hom-Leibniz (respectively, multiplicative Hom-Leibniz) algebras and whose morphisms are the homomorphisms of Hom-Leibniz (respectively, multiplicative Hom-Leibniz) algebras. There is an obvious inclusion functor \(\text{inc} : \text{Hom-Leib}_{\text{mult}} \to \text{Hom-Leib}\). This functor has as left adjoint the multiplicative functor \((-)_{\text{mult}} : \text{Hom-Leib} \to \text{Hom-Leib}_{\text{mult}}\) which assigns to a Hom-Leibniz algebra \((L, [-,-], \alpha_L)\) the multiplicative Hom-Leibniz algebra \((L/I, [-,-], \tilde{\alpha})\), where \(I\) is the two-sided ideal of \(L\) spanned by the elements \(\alpha_L[x, y] - [\alpha_L(x), \alpha_L(y)]\), for all \(x, y \in L\).

In the sequel we refer Hom-Leibniz algebra to a multiplicative Hom-Leibniz algebra and we shall use the shortened notation \((L, \alpha_L)\) when there is not confusion with the bracket operation.

**Definition 2.5** Let \((L, [-,-], \alpha_L)\) be a Hom-Leibniz algebra. A Hom-Leibniz subalgebra \(H\) is a linear subspace of \(L\), which is closed for the bracket and invariant by \(\alpha_L\), that is,

a) \([x, y] \in H\), for all \(x, y \in H\)

b) \(\alpha_L(x) \in H\), for all \(x \in H\)
A Hom-Leibniz subalgebra $H$ of $L$ is said to be a two-sided Hom-ideal if $[x, y], [y, x] \in H$, for all $x \in H, y \in L$.

If $H$ is a two-sided Hom-ideal of $L$, then the quotient $L/H$ naturally inherits a structure of Hom-Leibniz algebra, which is said to be the quotient Hom-Leibniz algebra.

**Definition 2.6** Let $H$ and $K$ be two-sided Hom-ideals of a Hom-Leibniz algebra $(L, [-, -], \alpha_L)$. The commutator of $H$ and $K$, denoted by $[H, K]$, is the Hom-Leibniz subalgebra of $L$ spanned by the brackets $[h, k], h \in H, k \in K$.

Obviously, $[H, K] \subseteq H \cap K$ and $[K, H] \subseteq H \cap K$. When $H = K = L$, we obtain the definition of derived Hom-Leibniz subalgebra. Let us observe that, in general, $[H, K]$ is not a Hom-ideal, but if $H, K \subseteq \alpha_L(L)$, then $[H, K]$ is a two-sided ideal of $\alpha_L(L)$. When $\alpha = Id$, the classical notions are recovered.

**Definition 2.7** Let $(L, [-, -], \alpha_L)$ be a Hom-Leibniz algebra. The subspace $Z(L) = \{x \in L | [x, y] = 0 = [y, x], \text{ for all } y \in L\}$ is said to be the center of $(L, [-, -], \alpha_L)$.

When $\alpha_L : L \to L$ is a surjective homomorphism, then $Z(L)$ is a Hom-ideal of $L$.

## 3 Homology of Hom-Leibniz algebras

In this section, we introduce the notion of Hom-co-representation, construct a chain complex from which we define the homology $\mathbb{K}$-vector spaces of a Hom-Leibniz algebra with coefficients on a Hom-co-representation and we interpret low-dimensional homology $\mathbb{K}$-vector spaces as well.

**Definition 3.1** Let $(L, [-, -], \alpha_L)$ be a Hom-Leibniz algebra. A Hom-co-representation of $(L, [-, -], \alpha_L)$ is a $\mathbb{K}$-vector space $M$ together with two bilinear maps $\lambda : L \otimes M \to M, \lambda(l \otimes m) = l \cdot m$, and $\rho : M \otimes L \to M, \rho(m \otimes l) = m \cdot l$, and a $\mathbb{K}$-linear map $\alpha_M : M \to M$ satisfying the following identities:

1. $[x, y] \cdot \alpha_M(m) = \alpha_L(x) \cdot (y \cdot m) - \alpha_L(y) \cdot (x \cdot m)$.
2. $\alpha_L(y) \cdot (m \cdot x) = (y \cdot m) \cdot \alpha_L(x) - \alpha_M(m) \cdot [x, y]$.
3. $(m \cdot x) \cdot \alpha_L(y) = \alpha_M(m) \cdot [x, y] - (y \cdot m) \cdot \alpha_L(x)$.
4. $\alpha_M(x \cdot m) = \alpha_L(x) \cdot \alpha_M(m)$
5. $\alpha_M(m \cdot x) = \alpha_M(m) \cdot \alpha_L(x)$

for any $x, y \in L$ and $m \in M$
From the second and third identities above directly follows
\[ \alpha_L(y) \cdot (m \cdot x) + (m \cdot x) \cdot \alpha_L(y) = 0 \] (2)

**Example 3.2**

a) Let \( M \) be a co-representation of a Leibniz algebra \( L \) [12]. Then \( (M, Id_M) \) is a Hom-co-representation of the Hom-Leibniz algebra \( (L, Id_L) \).

b) Every Hom-Leibniz algebra \( (L, [-, -], \alpha_L) \) has a Hom-co-representation structure on itself given by the actions
\[ x \cdot m = -[m, x]; \quad m \cdot x = [m, x] \]
where \( x \in L \) and \( m \) is an element of the underlying \( \mathbb{K} \)-vector space to \( L \).

Let \( (L, [-, -], \alpha_L) \) be a Hom-Leibniz algebra and \( (M, \alpha_M) \) be a Hom-co-representation of \( (L, [-, -], \alpha_L) \). Denote \( CL_n^\alpha(L, M) := M \otimes L^\otimes n, n \geq 0 \). We define the \( \mathbb{K} \)-linear map
\[ d_n : CL_n^\alpha(L, M) \to CL_{n-1}^\alpha(L, M) \]
by
\[
d_n(m \otimes x_1 \otimes \cdots \otimes x_n) = m \cdot x_1 \otimes \alpha_L(x_2) \otimes \cdots \otimes \alpha_L(x_n) + \\
\sum_{i=2}^{n} (-1)^i x_i \cdot m \otimes \alpha_L(x_1) \otimes \cdots \otimes \alpha_L(x_i) \otimes \cdots \otimes \alpha_L(x_n) + \\
\sum_{1 \leq i < j \leq n} (-1)^{i+j} \alpha_M(m) \otimes \alpha_L(x_1) \cdots \otimes \alpha_L(x_{i-1}) \otimes [x_i, x_j] \otimes \cdots \otimes \alpha_L(x_{j-1}) \cdots \otimes \alpha_L(x_n)
\]

The chain complex \( (CL^\alpha_n(L, M), d_*) \) is well-defined, that is, \( d_n \cdot d_{n+1} = 0, n \geq 0 \). Indeed, if we define for any \( y \in L \) and \( n \in \mathbb{N} \) two \( \mathbb{K} \)-linear maps,
\[ \theta_n(y) : CL_n^\alpha(L, M) \to CL_n^\alpha(L, M) \]
given by
\[
\theta_n(y)(m \otimes x_1 \otimes \cdots \otimes x_n) = -y \cdot m \otimes \alpha_L(x_1) \otimes \cdots \otimes \alpha_L(x_n) + \\
\sum_{i=1}^{n} \alpha_M(m) \otimes \alpha_L(x_1) \otimes \cdots \otimes [x_i, y] \otimes \cdots \otimes \alpha_L(x_n)
\]
and
\[ i_n(\alpha_L(y)) : CL_n^\alpha(L, M) \to CL_{n+1}^\alpha(L, M) \]
given by
\[
i_n(\alpha_L(y))(m \otimes x_1 \otimes \cdots \otimes x_n) = (-1)^n m \otimes x_1 \otimes \cdots \otimes x_n \otimes y
\]
then the following formulas hold:
Proposition 3.3 (Generalized Cartan’s Formulas)
The following identities hold:

a) $d_{n+1} \cdot i_n (\alpha_L (y)) + i_{n-1} (\alpha^2_L (y)) \cdot d_n = \theta_n (y)$, for $n \geq 1$.

b) $\theta_n (\alpha_L (x)) \cdot \theta_n (y) - \theta_n (\alpha_L (y)) \cdot \theta_n (x) = -\theta_n ([x, y]) \cdot (\alpha_M \otimes \alpha_L^{\otimes n})$, for $n \geq 0$.

c) $\theta_n (x) \cdot i_{n-1} (\alpha_L (y)) - i_{n-1} (\alpha^2_L (y)) \cdot \theta_{n-1} (x) = i_{n-1} (\alpha_L ([y, x])) \cdot (\alpha_M \otimes \alpha_L^{\otimes n-1})$, for $n \geq 1$.

d) $\theta_{n-1} (\alpha_L (y)) \cdot d_n = d_n \cdot \theta_n (y)$, for $n \geq 1$.

e) $d_n \cdot d_{n+1} = 0$, for $n \geq 1$.

Proof. The proof is followed by a standard mathematical induction, so we omit it.

The homology of the chain complex $(CL_\alpha (L, M), d_\ast)$ is called the homology of the Hom-Leibniz algebra $(L, [\cdot, \cdot], \alpha_L)$ with coefficients in the Hom-co-representation $(M, \alpha_M)$ and we will denote it by:

$$HL_\ast^\alpha (L, M) := H_\ast (CL_\ast^\alpha (L, M), d_\ast)$$

Now we are going to compute low dimensional homologies. So, for $n = 0$, a direct checking shows that

$$HL_0^\alpha (L, M) = \frac{M}{M_L}$$

where $M_L = \{m \cdot l : m \in M, l \in L\}$.

If $M$ is a trivial Hom-co-representation, that is, $m \cdot l = l \cdot m = 0$, then

$$HL_1^\alpha (L, M) = \frac{M \otimes L}{\alpha_M (M) \otimes [L, L]}$$

Proposition 3.4 Let $(L, [\cdot, \cdot], \alpha_L)$ be a Hom-Leibniz algebra, which is considered as a Hom-co-representation of itself as in Example 3.2 b) and $\mathbb{K}$ as a trivial Hom-co-representation of $(L, [\cdot, \cdot], \alpha_L)$. Then

$$HL_n^\alpha (L, L) \cong HL_{n+1}^\alpha (L, \mathbb{K}), n \geq 0$$

Proof. Obviously $-Id : CL_n^\alpha (L, L) \to CL_{n+1}^\alpha (L, \mathbb{K}), n \geq 0$, defines a chain isomorphism, hence the isomorphism in the homologies.

Remark 3.5 Let $L$ be a Leibniz algebra. If we consider $L$ as a Hom-Leibniz algebra as in Example 2.3 a) and we rewrite Proposition 3.4 with this particular case, considering $(L, \alpha_L)$ with a Hom-co-representation structure as in Example 3.2 b), then we recover isomorphism 6.5 in [11] (see also application 3.1 in [7]).
4 Universal central extensions

Through this section we deal with universal central extensions of Hom-Leibniz algebras. We generalize classical results on universal central extensions of Leibniz algebras, but an inconvenient, which is the composition of central extensions is not central as Example 4.9 shows, doesn’t allow us to obtain the complete generalization of all classical results. Nevertheless, this fact lead us to introduce a new concept of centrality.

Definition 4.1 A short exact sequence of Hom-Leibniz algebras $(K) : 0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ is said to be central if $[M, K] = 0 = [K, M]$. Equivalently, $M \subseteq Z(K)$.

We say that $(K)$ is $\alpha$-central if $[\alpha_M(M), K] = 0 = [K, \alpha_M(M)]$. Equivalently, $\alpha_M(M) \subseteq Z(K)$.

Remark 4.2 Obviously, every central extension is an $\alpha$-central extension. Note that in the case $\alpha_M = Id_M$, both notions coincide.

Definition 4.3 A central extension $(K) : 0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ is said to be universal if for every central extension $(K') : 0 \to (M', \alpha_{M'}) \xrightarrow{i'} (K', \alpha_{K'}) \xrightarrow{\pi'} (L, \alpha_L) \to 0$ there exists a unique homomorphism of Hom-Leibniz algebras $h : (K, \alpha_K) \to (K', \alpha_{K'})$ such that $\pi' \cdot h = \pi$.

We say that the central extension $(K) : 0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ is universal $\alpha$-central if for every $\alpha$-central extension $(K) : 0 \to (M', \alpha_{M'}) \xrightarrow{i'} (K', \alpha_{K'}) \xrightarrow{\pi'} (L, \alpha_L) \to 0$ there exists a unique homomorphism of Hom-Leibniz algebras $h : (K, \alpha_K) \to (K', \alpha_{K'})$ such that $\pi' \cdot h = \pi$.

Remark 4.4 Obviously, every universal $\alpha$-central extension is a universal central extension. Note that in the case $\alpha_M = Id_M$, both notions coincide.

Definition 4.5 We say that a Hom-Leibniz algebra $(L, \alpha_L)$ is perfect if $L = [L, L]$.

Lemma 4.6 Let $\pi : (K, \alpha_K) \to (L, \alpha_L)$ be a surjective homomorphism of Hom-Leibniz algebras. If $(K, \alpha_K)$ is a perfect Hom-Leibniz algebra, then $(L, \alpha_L)$ is a perfect Hom-Leibniz algebra as well.

Lemma 4.7 Let $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ be an $\alpha$-central extension and $(K, \alpha_K)$ a perfect Hom-Leibniz algebra. If there exists a homomorphism of Hom-Leibniz algebras $f : (K, \alpha_K) \to (A, \alpha_A)$ such that $\tau \cdot f = \pi$, where $0 \to (N, \alpha_N) \xrightarrow{j} (A, \alpha_A) \xrightarrow{\tau} (L, \alpha_L) \to 0$ is a central extension, then $f$ is unique.

The proofs of these two last Lemmas use classical arguments, so we omit it.
Lemma 4.8 If $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ is a universal central extension, then $(K, \alpha_K)$ and $(L, \alpha_L)$ are perfect Hom-Leibniz algebras.

Proof. Assume that $(K, \alpha_K)$ is not a perfect Hom-Leibniz algebra, then $[K, K] \subsetneq K$. Consider $I$ the smallest Hom-ideal generated by $[K, K]$, then $(K/I, \tilde{\alpha})$, where $\tilde{\alpha}$ is the induced natural homomorphism, is an abelian Hom-Leibniz algebra and, consequently, is a trivial Hom-co-representation of $(L, \alpha_L)$. Consider the central extension $0 \to (K/I, \tilde{\alpha}) \to (K/I \times L, \tilde{\alpha} \times \alpha_L) \xrightarrow{pr} (L, \alpha_L) \to 0$, then the homomorphisms of Hom-Leibniz algebras $\varphi, \psi : (K, \alpha_K) \to (K/I \times L, \tilde{\alpha} \times \alpha_L)$ given by $\varphi(k) = (k + I, \pi(k))$ and $\psi(k) = (0, \pi(k)), k \in K$ verify that $pr \cdot \varphi = \pi = pr \cdot \psi$, so $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ cannot be a universal central extension. Lemma 4.6 ends the proof. 

Classical categories as groups, Lie algebras, Leibniz algebras and other similar ones share the following property: the composition of two central extensions is a central extension as well. This property is absolutely necessary in order to obtain characterizations of the universal central extensions. Unfortunately this property doesn’t hold for the category of Hom-Leibniz algebras as the following counterexample 4.9 shows. This problem lead us to introduce the notion of $\alpha$-central extensions in Definition 4.11 whose properties relative to the composition are given in Lemma 4.10.

Example 4.9 Consider the two-dimensional Hom-Leibniz algebra $(L, \alpha_L)$ with basis $\{b_1, b_2\}$, bracket given by $[b_2, b_1] = b_2, [b_2, b_2] = b_1$ (unwritten brackets are equal to zero) and endomorphism $\alpha_L = 0$.

Let $(K, \alpha_K)$ be the three-dimensional Hom-Leibniz algebra with basis $\{a_1, a_2, a_3\}$, bracket given by $[a_2, a_2] = a_1, [a_3, a_2] = a_3, [a_3, a_3] = a_2$ (unwritten brackets are equal to zero) and endomorphism $\alpha_K = 0$.

Obviously $(K, \alpha_K)$ is a perfect Hom-Leibniz algebra since $K = [K, K]$ and $Z(K) = \langle \{a_1\} \rangle$.

The linear map $\pi : (K, 0) \to (L, 0)$ given by $\pi(a_1) = 0, \pi(a_2) = b_1, \pi(a_3) = b_2$, is a central extension since $\pi$ is trivially surjective and is a homomorphism of Hom-Leibniz algebras:

$$\pi[a_2, a_2] = \pi(a_1) = 0; \quad \pi(a_2) = b_1; \quad [\pi(a_2), \pi(a_2)] = [b_1, b_1] = 0$$
$$\pi[a_3, a_2] = \pi(a_3) = b_2; \quad [\pi(a_3), \pi(a_2)] = [b_2, b_1] = b_2$$
$$\pi[a_3, a_3] = \pi(a_2) = b_1; \quad [\pi(a_3), \pi(a_3)] = [b_2, b_2] = b_1$$

Obviously, $0 \cdot \pi = \pi \cdot 0$ and $\ker(\pi) = \langle \{a_1\} \rangle$, hence $\ker(\pi) \subsetneq Z(K)$.

Now consider the four-dimensional Hom-Leibniz algebra $(F, \alpha_F)$ with basis $\{e_1, e_2, e_3, e_4\}$, bracket given by $[e_3, e_3] = e_2, [e_4, e_3] = e_4, [e_4, e_4] = e_3$ (unwritten brackets are equal to zero) and endomorphism $\alpha_F = 0$. 

9
Obviously, \(0 \cdot \rho = \rho \cdot 0, \text{Ker}(\rho) = \langle \{e_1\} \rangle \) and \(Z(F) = \langle \{e_1\} \rangle\), hence \(\text{Ker}(\rho) \subseteq Z(F)\).

The composition \(\pi \cdot \rho : (F, 0) \to (L, 0)\) is given by \(\pi \cdot \rho(e_1) = \pi(0) = 0, \pi \cdot \rho(e_2) = \pi(a_1) = 0, \pi \cdot \rho(e_3) = \pi(a_2) = b_1, \pi \cdot \rho(e_4) = \pi(a_3) = b_2\). Consequently, \(\pi \cdot \rho : (F, 0) \to (L, 0)\) is a surjective homomorphism, but is not a central extension, since \(Z(F) = \langle \{e_1\} \rangle\) and \(\text{Ker}(\pi \cdot \rho) = \langle \{e_1, e_2\} \rangle\), i.e. \(\text{Ker}(\pi \cdot \rho) \not\subseteq Z(F)\).

**Lemma 4.10** Let \(0 \to (M, \alpha_M) \overset{i}{\to} (K, \alpha_K) \overset{\pi}{\to} (L, \alpha_L) \to 0\) and \(0 \to (N, \alpha_N) \overset{j}{\to} (F, \alpha_F) \overset{\pi \cdot \rho}{\to} (K, \alpha_K) \to 0\) be central extensions with \(K, \alpha_K\) a perfect Hom-Leibniz algebra. Then the composition extension \(0 \to (P, \alpha_P) = \text{Ker}(\pi \cdot \rho) \to (F, \alpha_F) \overset{\pi \cdot \rho}{\to} (K, \alpha_K) \to 0\) is an \(\alpha\)-central extension.

Moreover, if \(0 \to (M, \alpha_M) \overset{i}{\to} (K, \alpha_K) \overset{\pi}{\to} (L, \alpha_L) \to 0\) is a universal \(\alpha\)-central extension, then \(0 \to (N, \alpha_N) \overset{j}{\to} (F, \alpha_F) \overset{\pi \cdot \rho}{\to} (K, \alpha_K) \to 0\) is split.

**Proof.** We must proof that \([\alpha_P(P), F] = 0 = [F, \alpha_P(P)]\).

Since \((K, \alpha_K)\) is a perfect Hom-Leibniz algebra, then every element \(f \in F\) can be written as \(f = \sum_i \lambda_i [f_{i_1}, f_{i_2}] + n, n \in N, f_{ij} \in F, j = 1, 2\). So, for all \(p \in P, f \in F\) we have

\[
[\alpha_P(p), f] = \sum_i \lambda_i ([p, f_{i_1}], \alpha_F(f_{i_2}) - [[p, f_{i_2}], \alpha_F(f_{i_1})]) + [\alpha_P(p), n] = 0
\]

since \([p, f_{ij}] \in \text{Ker}(\rho) \subset Z(F)\).

In a similar way we can check that \([f, \alpha_P(p)] = 0\).

For the second statement, if \(0 \to (M, \alpha_M) \overset{i}{\to} (K, \alpha_K) \overset{\pi}{\to} (L, \alpha_L) \to 0\) is a universal \(\alpha\)-central extension, then by the first statement, \(0 \to (P, \alpha_P) = \text{Ker}(\pi \cdot \rho) \to (F, \alpha_F) \overset{\pi \cdot \rho}{\to} (L, \alpha_L) \to 0\) is an \(\alpha\)-central extension, then there exists a unique homomorphism of Hom-Leibniz algebras \(\sigma : (K, \alpha_K) \to (F, \alpha_F)\) such that \(\pi \cdot \rho \cdot \sigma = \pi\). On the other hand, \(\pi \cdot \rho \cdot \sigma = \pi = \pi \cdot \text{Id}\) and \((K, \alpha_K)\) is perfect, then Lemma \(\square\) implies that \(\rho \cdot \sigma = \text{Id}\).
Theorem 4.11

a) If a central extension $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\tau} (L, \alpha_L) \to 0$ is a universal $\alpha$-central extension, then $(K, \alpha_K)$ is a perfect Hom-Leibniz algebra and every central extension of $(K, \alpha_K)$ splits.

b) Let $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ be a central extension.

If $(K, \alpha_K)$ is a perfect Hom-Leibniz algebra and every central extension of $(K, \alpha_K)$ splits, then $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ is a universal central extension.

c) A Hom-Leibniz algebra $(L, \alpha_L)$ admits a universal central extension if and only if $(L, \alpha_L)$ is perfect.

d) The kernel of the universal central extension is canonically isomorphic to $HL_2^\alpha(L)$.

Proof.

a) If $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ is a universal $\alpha$-central extension, then is a universal central extension by Remark 4.4, so $(K, \alpha_K)$ is a perfect Hom-Leibniz algebra by Lemma 4.8 and every central extension of $(K, \alpha_K)$ splits by Lemma 4.10.

b) Let us consider any central extension $0 \to (N, \alpha_N) \xrightarrow{j} (A, \alpha_A) \xrightarrow{\chi} (L, \alpha_L) \to 0$. Construct the pull-back extension $0 \to (N, \alpha_N) \xrightarrow{j} (P, \alpha_P) \xrightarrow{\pi} (K, \alpha_K) \to 0$, where $P = A \times_L K = \{(a, k) \in A \times K \mid \tau(a) = \pi(k)\}$ and $\alpha_P(a, k) = (\alpha_A(a), \alpha_K(k))$, which is central, consequently is split, that is, there exists a homomorphism $\sigma : (K, \alpha_K) \to (P, \alpha_P)$ such that $\pi \cdot \sigma = Id$.

Then $\overline{\pi} \cdot \sigma$, where $\overline{\pi} : (P, \alpha_P) \to (A, \alpha_A)$ is induced by the pull-back construction, satisfies $\tau \cdot \overline{\pi} \cdot \sigma = \pi$. Lemma 4.7 ends the proof.

c) and d) For a Hom-Leibniz algebra $(L, \alpha_L)$ consider the chain homology complex $CL^\alpha_2(L)$ which is $CL^\alpha_2(L, \mathbb{K})$ where $\mathbb{K}$ is endowed with the trivial Hom-corepresentation structure.

As $\mathbb{K}$-vector spaces, let $I_L$ be the subspace of $L \otimes L$ spanned by the elements of the form $- [x_1, x_2] \otimes \alpha_L(x_3) + [x_1, x_3] \otimes \alpha_L(x_2) + \alpha_L(x_1) \otimes [x_2, x_3], x_1, x_2, x_3 \in L.$ That is $I_L = \text{Im } (d_3 : CL^\alpha_3(L) \to CL^\alpha_2(L)).$

Now we denote the quotient $\mathbb{K}$-vector space $\frac{L \otimes L}{I_L}$ by $ucc(L)$. Every class $x_1 \otimes x_2 + I_L$ is denoted by $\{x_1, x_2\}$, for all $x_1, x_2 \in L$.

By construction, the following identity holds

$$\{\alpha_L(x_1), [x_2, x_3]\} = \{[x_1, x_2], \alpha_L(x_3)\} - \{[x_1, x_3], \alpha_L(x_2)\} \quad (3)$$

for all $x_1, x_2, x_3 \in L$. 

11
Corollary 4.12

Proof. 
\[
\text{Given by } \tilde{K}, \alpha \text{ then } (L, \alpha_L) \text{ is a homomorphism of Hom-Leibniz algebras. Actually, } \text{Im } u_L = [L, L], \text{ but } (L, \alpha_L) \text{ is a perfect Hom-Leibniz algebra, so } u_L \text{ is an epimorphism.}
\]

From the construction, it follows that Ker \( u_L = HL^0_2(L) \), so we have the extension
\[
0 \rightarrow (HL^0_2(L), \tilde{\alpha}_1) \rightarrow (ucc(L), \tilde{\alpha}) \xrightarrow{u_L} (L, \alpha_L) \rightarrow 0
\]
which is central, because \([\text{Ker } u_L, ucc(L)] = 0 = [ucc(L), \text{Ker } u_L] \), and universal because for any central extension \( 0 \rightarrow (M, \alpha_M) \rightarrow (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \rightarrow 0 \) there exists the homomorphism of Hom-Leibniz algebras \( \varphi : (ucc(L), \tilde{\alpha}) \rightarrow (K, \alpha_K) \) given by \( \varphi(\{x_1, x_2\}) = [k_1, k_2], \pi(k_i) = x_i, i = 1, 2, \) such that \( \pi \cdot \varphi = u_L \). Moreover, thanks to Lemma 4.8 and Lemma 4.7, \( \varphi \) is unique. 

\[\Box\]

Corollary 4.12

a) Let \( 0 \rightarrow (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \rightarrow 0 \) be a universal \( \alpha \)-central extension, then \( HL^0_1(K) = HL^0_2(K) = 0 \).

b) Let \( 0 \rightarrow (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \rightarrow 0 \) be a central extension such that \( HL^0_1(K) = HL^0_2(K) = 0 \), then \( 0 \rightarrow (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \rightarrow 0 \) is a universal central extension.

Proof. 

a) If \( 0 \rightarrow (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \rightarrow 0 \) is a universal \( \alpha \)-central extension, then \( (K, \alpha_K) \) is perfect by Remark 4.3 and Lemma 4.8, so \( HL^0_1(K) = 0 \). By Lemma 4.10 and Theorem 4.11 c), d) the universal central extension corresponding to \( (K, \alpha_K) \) is split, so \( HL^0_2(K) = 0 \).

b) \( HL^0_1(K) = 0 \) implies that \( (K, \alpha_K) \) is a perfect Hom-Leibniz algebra.

\( HL^0_2(K) = 0 \) implies that \( (ucc(K), \tilde{\alpha}) \xrightarrow{\sim} (K, \alpha_K) \). Theorem 4.11 b) ends the proof. 
\[\Box\]

Definition 4.13 An \( \alpha \)-central extension \( 0 \rightarrow (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \rightarrow 0 \) is said to be universal if for any central extension \( 0 \rightarrow (R, \alpha_R) \xrightarrow{j} (A, \alpha_A) \xrightarrow{\tau} (L, \alpha_L) \rightarrow 0 \) there exists a unique homomorphism \( \varphi : (K, \alpha_K) \rightarrow (A, \alpha_A) \) such that \( \tau \cdot \varphi = \pi \).

An \( \alpha \)-central extension \( 0 \rightarrow (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \rightarrow 0 \) is said to be \( \alpha \)-universal if for any \( \alpha \)-central extension \( 0 \rightarrow (R, \alpha_R) \xrightarrow{j} (A, \alpha_A) \xrightarrow{\tau} (L, \alpha_L) \rightarrow 0 \) there exists a unique homomorphism \( \psi : (K, \alpha_K) \rightarrow (A, \alpha_A) \) such that \( \tau \cdot \psi = \pi \).
Remark 4.14 Obviously, every \( \alpha \)-universal \( \alpha \)-central extension is an \( \alpha \)-central extension which is universal in the sense of Definition 4.13. In case \( \alpha_M = \text{Id} \) both notions coincide with the definition of universal central extension.

Proposition 4.15

a) Let \( 0 \to (M, \alpha_M) \overset{i}{\to} (K, \alpha_K) \overset{\pi}{\to} (L, \alpha_L) \to 0 \) and \( 0 \to (N, \alpha_N) \overset{j}{\to} (F, \alpha_F) \overset{\rho}{\to} (K, \alpha_K) \to 0 \) be central extensions. If \( 0 \to (N, \alpha_N) \overset{j}{\to} (F, \alpha_F) \overset{\rho}{\to} (K, \alpha_K) \to 0 \) is a universal central extension, then \( 0 \to (P, \alpha_P) = \ker(\pi \cdot \rho) \overset{\lambda}{\to} (F, \alpha_F) \overset{\pi \circ \rho}{\to} (L, \alpha_L) \to 0 \) is an \( \alpha \)-universal \( \alpha \)-central extension which is universal in the sense of Definition 4.13.

b) Let \( 0 \to (M, \alpha_M) \overset{i}{\to} (K, \alpha_K) \overset{\pi}{\to} (L, \alpha_L) \to 0 \) and \( 0 \to (N, \alpha_N) \overset{j}{\to} (F, \alpha_F) \overset{\rho}{\to} (K, \alpha_K) \to 0 \) be central extensions with \( (F, \alpha_F) \) a perfect Hom-Leibniz algebra. If \( 0 \to (P, \alpha_P) = \ker(\pi \cdot \rho) \overset{\lambda}{\to} (F, \alpha_F) \overset{\pi \circ \rho}{\to} (L, \alpha_L) \to 0 \) is an \( \alpha \)-universal \( \alpha \)-central extension, then \( 0 \to (N, \alpha_N) \overset{j}{\to} (F, \alpha_F) \overset{\rho}{\to} (K, \alpha_K) \to 0 \) is a universal central extension.

Proof.

a) If \( 0 \to (N, \alpha_N) \overset{j}{\to} (F, \alpha_F) \overset{\rho}{\to} (K, \alpha_K) \to 0 \) is a universal central extension, then \( (F, \alpha_F) \) and \( (K, \alpha_K) \) are perfect Hom-Leibniz algebras by Lemma 4.8.

On the other hand, \( 0 \to (P, \alpha_P) = \ker(\pi \cdot \rho) \overset{\lambda}{\to} (F, \alpha_F) \overset{\pi \circ \rho}{\to} (L, \alpha_L) \to 0 \) is an \( \alpha \)-universal \( \alpha \)-central extension by Lemma 4.10.

In order to obtain the universality, for any central extension \( 0 \to (R, \alpha_R) \overset{\tau}{\to} (L, \alpha_L) \to 0 \) construct the pull-back extension corresponding to \( \tau \) and \( \pi \), \( 0 \to (R, \alpha_R) \overset{\tau}{\to} (K \times_L A, \alpha_K \times A) \overset{\pi}{\to} (K, \alpha_K) \to 0 \). Since \( 0 \to (N, \alpha_N) \overset{j}{\to} (F, \alpha_F) \overset{\rho}{\to} (K, \alpha_K) \to 0 \) is a universal central extension, then there exists a unique homomorphism \( \varphi : (F, \alpha_F) \to (K \times_L A, \alpha_K \times A) \) such that \( \pi \cdot \varphi = \rho \). Then the homomorphism \( \pi \cdot \varphi \) satisfies that \( \tau \cdot \pi \cdot \varphi = \pi \cdot \rho \) and it is unique by Lemma 4.7.

b) \( (F, \alpha_F) \) perfect implies that \( (K, \alpha_K) \) is perfect by Lemma 4.6. In order to prove the universality of the central extension \( 0 \to (N, \alpha_N) \overset{j}{\to} (F, \alpha_F) \overset{\rho}{\to} (K, \alpha_K) \to 0 \), let us consider any central extension \( 0 \to (R, \alpha_R) \overset{\tau}{\to} (A, \alpha_A) \overset{\pi}{\to} (K, \alpha_K) \to 0 \), then \( 0 \to \ker(\pi \cdot \sigma) \overset{\lambda}{\to} (A, \alpha_A) \overset{\pi \circ \sigma}{\to} (L, \alpha_L) \to 0 \) is an \( \alpha \)-central extension by Lemma 4.10.

The \( \alpha \)-universality of \( 0 \to (P, \alpha_P) = \ker(\pi \cdot \rho) \overset{\lambda}{\to} (F, \alpha_F) \overset{\pi \circ \rho}{\to} (L, \alpha_L) \to 0 \) implies the existence of a unique homomorphism \( \omega : (F, \alpha_F) \to (A, \alpha_A) \) such that \( \pi \cdot \sigma \cdot \omega = \pi \cdot \rho \).

Since \( (F, \alpha_F) \) is a perfect Hom-Leibniz algebra and \( \text{Im}(\sigma \cdot \omega) \subseteq \text{Im}(\rho) + \ker(\pi) \), then easily is followed that \( \sigma \cdot \omega = \rho \). The uniqueness is followed from Lemma 4.7. \( \square \)
5 Relationships between universal $\alpha$-central extensions

Since the universal central extensions of a perfect Lie algebra in the categories of Lie and Leibniz algebras are related by means of the results given in \[8\], then it is natural to consider the generalization of these results to the framework of Hom-Leibniz algebras. But, once more, through the proofs it is necessary to deal with the composition of central extensions, that is, we need to apply Lemma 4.10. This fact causes no significant relationships between the universal central extensions of a perfect Hom-Lie algebra in the categories Hom-Lie and Hom-Leib, so it is not possible to obtain a complete generalization of the mentioned results in \[8\], but it is possible to obtain results in the more restricted context of universal $\alpha$-central extensions.

We start with the introduction of a new concept of perfection.

Definition 5.1 A Hom-Lie algebra (respectively, Hom-Leibniz) $(L,\alpha_L)$ is said to be $\alpha$-perfect if

$$L = [\alpha_L(L),\alpha_L(L)]$$

Example 5.2 The three-dimensional Hom-Lie algebra $(L,[,],\alpha_L)$ with basis $\{a_1,a_2,a_3\}$, bracket given by $[a_1,a_2] = -[a_2,a_1] = a_3; [a_2,a_3] = -[a_3,a_2] = a_1; [a_3,a_1] = -[a_1,a_3] = a_2$ (unwritten brackets are equal to zero) and endomorphism $\alpha_L$ represented by the matrix

$$
\begin{pmatrix}
\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\
0 & -1 & 0 \\
\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2}
\end{pmatrix}
$$

is an $\alpha$-perfect Hom-Lie algebra.

Remark 5.3

a) When $\alpha_L = Id$, the notion of $\alpha$-perfection coincides with the notion of perfection.

b) Obviously, if $(L,[,],\alpha_L)$ is an $\alpha$-perfect Hom-Lie algebra (respectively, Hom-Leibniz), then it is perfect. Nevertheless the converse is not true in general. For example, the three-dimensional Hom-Lie algebra $(L,[,],\alpha_L)$ with basis $\{a_1,a_2,a_3\}$, bracket given by $[a_1,a_2] = -[a_2,a_1] = a_3; [a_2,a_3] = -[a_3,a_2] = a_1; [a_3,a_1] = -[a_1,a_3] = a_2$ and endomorphism $\alpha_L = 0$ is perfect, but it is not $\alpha$-perfect.

c) If $(L,[,],\alpha_L)$ is $\alpha$-perfect, then $L = \alpha_L(L)$, i.e. $\alpha_L$ is surjective. Nevertheless the converse is not true. For instance, the two-dimensional Hom-Lie algebra with basis $\{a_1,a_2\}$, bracket given by $[a_1,a_2] = -[a_2,a_1] = a_2$ and endomorphism $\alpha_L$ represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Obviously the endomorphism $\alpha_L$ is surjective, but $[\alpha_L(L),\alpha_L(L)] = \langle\{a_2\}\rangle$. 

14
Lemma 5.4 Let $0 \to (M, \alpha_M) \xrightarrow{\iota} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ be a central extension and $(K, \alpha_K)$ an $\alpha$-perfect Hom-Lie (Hom-Leibniz) algebra. If there exists a homomorphism of Hom-Lie (Hom-Leibniz) algebras $f : (K, \alpha_K) \to (A, \alpha_A)$ such that $\tau \cdot f = \pi$, where $0 \to (N, \alpha_N) \xrightarrow{\iota} (A, \alpha_A) \xrightarrow{\pi} (L, \alpha_L) \to 0$ is an $\alpha$-central extension, then $f$ is unique.

Proof. Let us assume that there exist two homomorphisms $f_1$ and $f_2$ such that $\tau \cdot f_1 = \pi = \tau \cdot f_2$, then $f_1 - f_2 \in \text{Ker} \tau = N$, i.e. $f_1(k) = f_2(k) + n_k, n_k \in N$, and $\alpha_N(N) \subseteq Z(A)$.

Moreover, $f_1$ and $f_2$ coincide over $[\alpha_K(K), \alpha_K(K)]$. Indeed $f_1[\alpha_K(k_1), \alpha_K(k_2)] = [\alpha_A \cdot f_1(k_1), \alpha_A \cdot f_1(k_2)] = [\alpha_A \cdot f_2(k_1) + \alpha_A(n_{k_1}), \alpha_A \cdot f_2(k_2) + \alpha_A(n_{k_2})] = [\alpha_A \cdot f_2(k_1), \alpha_A \cdot f_2(k_2)] = f_2[\alpha_K(k_1), \alpha_K(k_2)]$.

Since $(K, \alpha_K)$ is $\alpha$-perfect, then $f_1$ coincides with $f_2$ over $K$. □

Theorem 5.5

a) An $\alpha$-perfect Hom-Lie algebra admits a universal $\alpha$-central extension.

b) An $\alpha$-perfect Hom-Leibniz algebra admits a universal $\alpha$-central extension.

Proof. a) For the Hom-Lie algebra $(L, [-,-], \alpha_L)$, consider the chain complex

$$C_n^\alpha(L, \mathbb{K}) : (C_n^\alpha(L, \mathbb{K}) = \Lambda^n L, d_n : C_n^\alpha(L, \mathbb{K}) \to C_{n-1}^\alpha(L, \mathbb{K}), d_n(x_1 \wedge \cdots \wedge x_n) = \sum_{1 \leq i < j \leq n} (-1)^{i+j}[x_i, x_j] \wedge \alpha_L(x_1) \wedge \cdots \wedge \alpha_L(x_i) \wedge \cdots \wedge \alpha_L(x_j) \wedge \cdots \wedge \alpha_L(x_n)),$$

where $\mathbb{K}$ is considered with a trivial Hom-L-module structure.

Now we consider the quotient $\mathbb{K}$-vector space $\text{ucc}_\alpha^\text{Lie}(L) = \frac{\alpha_L(L) \wedge \alpha_L(L)}{I_L}$, where $I_L$ is the vector subspace of $\alpha_L(L) \wedge \alpha_L(L)$ spanned by the elements of the form

$$-[x_1, x_2] \wedge \alpha_L(x_3) + [x_1, x_3] \wedge \alpha_L(x_2) - [x_2, x_3] \wedge \alpha_L(x_1)$$

for all $x_1, x_2, x_3 \in L$.

Observe that every summand of the form $[x_1, x_2] \wedge \alpha_L(x_3)$ is an element of $\alpha_L(L) \wedge \alpha_L(L)$, since $L$ is $\alpha$-perfect and then $[x_1, x_2] \in L = [\alpha_L(L), \alpha_L(L)] \subseteq \alpha_L(L)$.

We denote by $\{\alpha_L(x_1), \alpha_L(x_2)\}$ the equivalence class $\alpha_L(x_1) \wedge \alpha_L(x_2) + I_L$. $\text{ucc}_\alpha^\text{Lie}(L)$ is endowed with a structure of Hom-Lie algebra with respect to the bracket

$$\{\alpha_L(x_1), \alpha_L(x_2), \alpha_L(y_1), \alpha_L(y_2)\} = \{\alpha_L(x_1), \alpha_L(x_2), \alpha_L(y_1), \alpha_L(y_2)\}$$

and the endomorphism $\tilde{\alpha} : \text{ucc}_\alpha^\text{Lie}(L) \to \text{ucc}_\alpha^\text{Lie}(L)$ given by $\tilde{\alpha}(\{\alpha_L(x_1), \alpha_L(x_2)\}) = \{\alpha_L^2(x_1), \alpha_L^2(x_2)\}$. 

15
The restriction of $d_2 : C^2_{\alpha}(L) = L \wedge L \rightarrow C^1_{\alpha}(L) = L$ to $\alpha_L(L) \wedge \alpha_L(L)$ vanishes on $I_L$, then it induces a linear map $u_\alpha : \text{ucc}_{\alpha}^{\text{Lie}}(L) \rightarrow L$, which is defined by $u_\alpha([\alpha_L(x_1), \alpha_L(x_2)]) = [\alpha_L(x_1), \alpha_L(x_2)]$. Moreover, since $(L, [-, -], \alpha_L)$ is $\alpha$-perfect, then $u_\alpha$ is a surjective homomorphism, because $\text{Im}(u_\alpha) = [\alpha_L(L), \alpha_L(L)] = L$ and

\[
\cdot \quad u_\alpha([\alpha_L(x_1), \alpha_L(x_2)], [\alpha_L(y_1), \alpha_L(y_2)]) = u_\alpha([\alpha_L(x_1), [\alpha_L(x_2), \alpha_L(y_2)]]) = [u_\alpha(\alpha_L(x_1), \alpha_L(x_2), \alpha_L(y_2))].
\]

\[
\cdot \quad \alpha_L \cdot u_\alpha(\alpha_L(x_1), \alpha_L(x_2)) = \alpha_L[\alpha_L(x_1), \alpha_L(x_2)] = [\alpha_L^2(x_1), \alpha_L^2(x_2)] = u_\alpha(\alpha_L^2(x_1), \alpha_L^2(x_2)) = u_\alpha(\{\alpha_L(x_1), \alpha_L(x_2)\}).
\]

Then we have constructed the extension

\[
0 \rightarrow (\text{Ker}(u_\alpha), \tilde{\alpha}) \rightarrow (\text{ucc}_{\alpha}^{\text{Lie}}(L), \tilde{\alpha}) \xrightarrow{u_\alpha} (L, \alpha_L) \rightarrow 0
\]

which is a central extension. Indeed, for any $\{\alpha_L(x_1), \alpha_L(x_2)\} \in \text{Ker}(u_\alpha)$, $\{\alpha_L(y_1), \alpha_L(y_2)\} \in \text{ucc}_{\alpha}^{\text{Lie}}(L)$, one verifies

\[
[\{\alpha_L(x_1), \alpha_L(x_2)\}, \{\alpha_L(y_1), \alpha_L(y_2)\}] = [[\alpha_L(x_1), \alpha_L(x_2)], [\alpha_L(y_1), \alpha_L(y_2)]] = 0
\]

since $\{\alpha_L(x_1), \alpha_L(x_2)\} \in \text{Ker}(u_\alpha) \iff [\alpha_L(x_1), \alpha_L(x_2)] = 0$.

Finally, this central extension is universal $\alpha$-central. Indeed, consider an $\alpha$-central extension $0 \rightarrow (M, \alpha_M) \xrightarrow{j} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \rightarrow 0$. We define the map $\Phi : (\text{ucc}_{\alpha}^{\text{Lie}}(L), \tilde{\alpha}) \rightarrow (K, \alpha_K)$ by $\Phi(\{\alpha_L(x_1), \alpha_L(x_2)\}) = [\alpha_K(k_1), \alpha_K(k_2)]$, where $\pi(k_i) = x_i, x_i \in L, k_i \in K, i = 1, 2$. Now we are going to check that $\Phi$ is a homomorphism of Hom-Lie algebras such that $\pi \cdot \Phi = u_\alpha$. Indeed:

\[
\cdot \quad \text{Well-definition: if } \pi(k_i) = x_i = \pi(k_i'), i = 1, 2, \text{ then } k_i - k_i' \in \text{Ker}(\pi) = M, \text{ so } \alpha_K(k_i) - \alpha_K(k_i') \in \alpha_M(M). \text{ Hence, } \Phi(\{\alpha_L(x_1), \alpha_L(x_2)\}) = [\alpha_K(k_1), \alpha_K(k_2)] = [\alpha_K(k_1'), \alpha_M(m_1), \alpha_L(k_2') + \alpha_M(m_2)] = [\alpha_K(k_1'), \alpha_L(k_2')].
\]

Observe that the condition of $\alpha$-centrality is essential at this step in order to be guaranteed that $[\alpha_M(M), K] = 0$.

\[
\cdot \quad \Phi(\{\alpha_L(x_1), \alpha_L(x_2)\}, \{\alpha_L(y_1), \alpha_L(y_2)\}) = \Phi([\alpha_L(x_1), \alpha_L(x_2), \alpha_L(y_1), \alpha_L(y_2)]) = ([\alpha_L(k_1), \alpha_L(k_2)], [\alpha_L(l_1), \alpha_L(l_2)] = [\Phi(\alpha_L(x_1), \alpha_L(x_2)), \Phi(\alpha_L(y_1), \alpha_L(y_2))]
\]

where $\pi(k_i) = x_i, \pi(y_i) = l_i, i = 1, 2$. We have

\[
\cdot \quad \alpha_K \cdot \Phi(\{\alpha_L(x_1), \alpha_L(x_2)\}) = \alpha_K[\{\alpha_K(k_1), \alpha_K(k_2)\}] = [\alpha_K^2(k_1), \alpha_K^2(k_2)] = \Phi(\alpha_L^2(x_1), \alpha_L^2(x_2)) = \Phi(\tilde{\alpha}(\alpha_L(x_1), \alpha_L(x_2))).
\]

\[
\cdot \quad \pi \cdot \Phi(\{\alpha_L(x_1), \alpha_L(x_2)\}) = \pi[\alpha_K(k_1), \alpha_K(k_2)] = [\pi \cdot \alpha_K(k_1), \pi \cdot \alpha_K(k_2)] = [\alpha_L \cdot \pi(k_1), \alpha_L \cdot \pi(k_2)] = [\alpha_L(x_1), \alpha_L(x_2)] = u_\alpha(\{\alpha_L(x_1), \alpha_L(x_2)\}).
\]
To end the proof, we must verify the uniqueness of $\Phi$: first at all, we check that $\text{ucc}^\text{Lie}_\alpha(L)$ is $\alpha$-perfect. Indeed,

$$\{\alpha_L(x_1), \alpha_L(x_2)\} = \{\alpha_L^2(x_1), \alpha_L^2(x_2)\} = \{\alpha_L^2(y_1), \alpha_L^2(y_2)\}$$

i.e. $[\alpha(\text{ucc}^\text{Lie}_\alpha(L)), \alpha(\text{ucc}^\text{Lie}_\alpha(L))] \subseteq \text{ucc}^\text{Lie}_\alpha(L)$.

For the converse inclusion, having in mind that $\alpha_L(x_i) \in L = [\alpha_L(L), \alpha_L(L)]$, we have that:

$$\{\alpha_L(x_1), \alpha_L(x_2)\} = \{\alpha \left( \sum_i \lambda_i [\alpha_L(l_{i1}), \alpha_L(l_{i2})] \right), \alpha \left( \sum_j \mu_j [\alpha_L(l'_{j1}), \alpha_L(l'_{j2})] \right) \}$$

Now Lemma 5.4 ends the proof.

b) For the Hom-Leibniz algebra $(L, [-, -], \alpha_L)$, we consider the chain complex $CL^\alpha_*(L, K)$, where $K$ is considered with trivial Hom-co-representation structure.

Take the quotient vector space $\text{ucc}^\text{Leib}_\alpha(L) = \alpha_L(L) \otimes_{I_L} I_L^{-1}$, where $I_L$ is the vector subspace of $\alpha_L(L) \otimes \alpha_L(L)$ generated by the elements of the form

$$-[x_1, x_2] \otimes \alpha_L(x_3) + [x_1, x_3] \otimes \alpha_L(x_2) + \alpha_L(x_1) \otimes [x_2, x_3]$$

for all $x_1, x_2, x_3 \in L$.

Observe that every summand of the form $[x_1, x_2] \otimes \alpha_L(x_3)$ or $\alpha_L(x_1) \otimes [x_2, x_3]$ belongs to $\alpha_L(L) \otimes \alpha_L(L)$, since $L$ is $\alpha$-perfect, then $[x_1, x_2] \in L = [\alpha_L(L), \alpha_L(L)] \subseteq \alpha_L(L)$.

We denote by $\{\alpha_L(x_1), \alpha_L(x_2)\}$ the equivalence class $\alpha_L(x_1) \otimes \alpha_L(x_2) + I_L$.

$\text{ucc}^\text{Leib}_\alpha(L)$ is endowed with a structure of Hom-Leibniz algebra with respect to the bracket

$$\{[\alpha_L(x_1), \alpha_L(x_2)], [\alpha_L(y_1), \alpha_L(y_2)]\} = \{[\alpha_L(x_1), \alpha_L(x_2)], [\alpha_L(y_1), \alpha_L(y_2)]\}$$

and the endomorphism $\pi : \text{ucc}^\text{Leib}_\alpha(L) \rightarrow \text{ucc}^\text{Leib}_\alpha(L)$ defined by $\pi(\{\alpha_L(x_1), \alpha_L(x_2)\}) = \{\alpha_L^2(x_1), \alpha_L^2(x_2)\}$.

The restriction of the differential $d_2 : CL^2_*(L) = L \otimes L \rightarrow CL^1_*(L) = L$ to $\alpha_L(L) \otimes \alpha_L(L)$ vanishes on $I_L$, so it induces a linear map $U_\alpha : \text{ucc}^\text{Leib}_\alpha(L) \rightarrow L$, that is given by $U_\alpha(\{\alpha_L(x_1), \alpha_L(x_2)\}) = [\alpha_L(x_1), \alpha_L(x_2)]$. Moreover, thanks to be $(L, [-, -], \alpha_L)$ $\alpha$-perfect, then $U_\alpha$ is a surjective homomorphism, since $\text{Im}(U_\alpha) = [\alpha_L(L), \alpha_L(L)] = L$ and
• \( U_\alpha[[\alpha_L(x_1), \alpha_L(x_2)], \{\alpha_L(y_1), \alpha_L(y_2)\}] = U_\alpha[[\alpha_L(x_1), \alpha_L(x_2)], [\alpha_L(y_1), \alpha_L(y_2)]] = [[\alpha_L(x_1), \alpha_L(x_2)], [\alpha_L(y_1), \alpha_L(y_2)]] = [U_\alpha\{\alpha_L(x_1), \alpha_L(x_2)\}, U_\alpha\{\alpha_L(y_1), \alpha_L(y_2)\}].\)

• \( \alpha_L \cdot U_\alpha\{\alpha_L(x_1), \alpha_L(x_2)\} = \alpha_L[\alpha_L(x_1), \alpha_L(x_2)] = [\alpha_L^2(x_1), \alpha_L^2(x_2)] = U_\alpha[\alpha_L^2(x_1), \alpha_L^2(x_2)] = U_\alpha[\alpha_L(x_1), \alpha_L(x_2)].\)

so we have constructed the extension
\[
0 \to (Ker(U_\alpha), \overline{\alpha}) \to (ucc_{\alpha}^{Leib}(L), \overline{\alpha}) \xrightarrow{U_\alpha} (L, \alpha_L) \to 0
\]
which is central. Indeed, for any \( \{\alpha_L(x_1), \alpha_L(x_2)\} \in Ker(U_\alpha), \{\alpha_L(y_1), \alpha_L(y_2)\} \in ucc_{\alpha}^{Leib}(L), \) we have that
\[
[[\alpha_L(x_1), \alpha_L(x_2)], \{\alpha_L(y_1), \alpha_L(y_2)\}] = [[\alpha_L(x_1), \alpha_L(x_2)], [\alpha_L(y_1), \alpha_L(y_2)]] = 0
\]
since \( \{\alpha_L(x_1), \alpha_L(x_2)\} \in Ker(U_\alpha) \iff [\alpha_L(x_1), \alpha_L(x_2)] = 0.\)

In a similar way it is verified that \( [ucc_{\alpha}^{Leib}(L), Ker(U_\alpha)] = 0.\)

The checking that this central extension is universal \( \alpha \)-central follows parallel arguments as in Hom-Lie case, so we omit it. \( \Box \)

Let \((L, \alpha_L)\) be an \( \alpha \)-perfect Hom-Lie algebra. By Theorem 5.5 \((L, \alpha_L)\) admits a universal \( \alpha \)-central extension \((ucc_{\alpha}^{Lie}(L), \overline{\alpha})\) in the category Hom-Lie of Hom-Lie algebras and a universal \( \alpha \)-central extension \((ucc_{\alpha}^{Leib}(L), \overline{\alpha})\) in the category Hom-Leibniz. The following result provides a relationship between both extensions.

**Proposition 5.6** \((ucc_{\alpha}^{Leib}(L), \overline{\alpha})\) is the universal central extension of the \( \alpha \)-perfect Hom-Lie algebra \((ucc_{\alpha}^{Lie}(L), \overline{\alpha})\) in the category Hom-Leib.

Moreover there is an isomorphism of Hom-Lie algebras
\[
(ucc_{\alpha}^{Lie}(L), \overline{\alpha}) \cong \left(\left(ucc_{\alpha}^{Leib}(L)\right)_{Lie}, \overline{\alpha}_{Lie}\right).
\]

**Proof.** Since \((L, \alpha_L)\) is an \( \alpha \)-perfect Hom-Lie algebra, then by Theorem 5.5 it admits a universal \( \alpha \)-central extension in the categories Hom-Lie and Hom-Leib. This situation is described in the following diagram:

\[
\begin{array}{cccc}
0 & \to & (Ker(U_\alpha), \overline{\alpha}) & \xrightarrow{U_\alpha} (ucc_{\alpha}^{Leib}(L), \overline{\alpha}) \\
& & \downarrow \exists \Phi & \\
0 & \to & (Ker(u_\alpha), \overline{\alpha}) & \xrightarrow{u_\alpha} (ucc_{\alpha}^{Lie}(L), \overline{\alpha}) \\
\end{array}
\]

Since \( U_\alpha : (ucc_{\alpha}^{Leib}(L), \overline{\alpha}) \to (L, \alpha_L) \) is a universal \( \alpha \)-central extension, by Remark 4.4 it is a universal central extension; since \( u_\alpha : (ucc_{\alpha}^{Lie}(L), \overline{\alpha}) \to (L, \alpha_L) \) is a central extension, then there exists a unique homomorphism of Hom-Leibniz algebras \( \Phi : (ucc_{\alpha}^{Leib}(L), \overline{\alpha}) \to (ucc_{\alpha}^{Lie}(L), \overline{\alpha}) \) such that \( u_\alpha \Phi = U_\alpha. \)
We only need to prove that the extension
\[
0 \to (\ker(\Phi), \bar{\alpha}) \to (uce_{\alpha}^{Lie}(L), \bar{\alpha}) \xrightarrow{\Phi} (uce_{\alpha}^{Lie}(L), \bar{\alpha}) \to 0
\]
is a universal central extension in the category Hom-Leib.

First at all, by construction, we have that \( \ker(\Phi) \subseteq \ker(U_\alpha) \subseteq \text{Z}(uce_{\alpha}^{Lie}(L)) \).

On the other hand, \( \Phi \) is a surjective homomorphism, since \( uce_{\alpha}^{Lie}(L) \subseteq \im(\Phi) + \ker(u_\alpha) \). Indeed, for \( z' \in uce_{\alpha}^{Lie}(L), u_\alpha(z') \in L \), then there exists any \( z \in uce_{\alpha}^{Lie}(L) \) such that \( U_\alpha(z) = u_\alpha(z') \), but \( U_\alpha = u_\alpha \cdot \Phi \), then \( z' - \Phi(z) \in \ker(u_\alpha) \) and, consequently, \( z' \in \im(\Phi) + \ker(u_\alpha) \).

Since \( u_\alpha : (uce_{\alpha}^{Lie}(L), \bar{\alpha}) \to (L, \alpha_L) \) is a universal \( \alpha \)-central extension, then by Remark 4.4 in [4] is a universal central extensions and, by Lemma 4.8 in [4], we know that \( (uce_{\alpha}^{Lie}(L), \bar{\alpha}) \) is perfect. Hence:
\[
uce_{\alpha}^{Lie}(L) = [uce_{\alpha}^{Lie}(L), uce_{\alpha}^{Lie}(L)] \subseteq \[\im(\Phi) + \ker(u_\alpha), \im(\Phi) + \ker(u_\alpha)\] \subseteq \im(\Phi)
\]
since \( \im(\Phi) = \ker(u_\alpha) \).

Now we check that the conditions of Theorem 4.11 hold:

* Since \( U_\alpha : (uce_{\alpha}^{Lie}(L), \bar{\alpha}) \to (L, \alpha_L) \) is a universal \( \alpha \)-central extension, then by Remark 4.4 is a universal central extension and, by Lemma 4.8, we know that \( (uce_{\alpha}^{Lie}(L), \bar{\alpha}) \) is perfect.

** Now we must check that any central extension of the form \( 0 \to (P, \alpha_P) \to (K, \alpha_K) \xrightarrow{\rho} (uce_{\alpha}^{Lie}(L), \bar{\alpha}) \) is split.

For that, we consider the composition extension \( U_\alpha \cdot \rho : (K, \alpha_K) \to (L, \alpha_L) \), which is an \( \alpha \)-central extension thanks to Lemma 4.10. Now Definition 4.3 guaranties the existence of a unique homomorphism of Hom-Leibniz algebras \( \sigma : (uce_{\alpha}^{Lie}(L), \bar{\alpha}) \to (K, \alpha_K) \) such that \( U_\alpha \cdot \rho \cdot \sigma = U_\alpha \).

Since \( U_\alpha \cdot \text{Id} = U_\alpha \), then Lemma 4.7 implies that \( \rho \cdot \sigma = \text{Id} \).

In order to prove the isomorphism established in the second statement, we consider the following diagram where \( \Phi \) is induced by \( \Phi \) and \( \gamma \) by \( U_\alpha \):

\[
\begin{array}{ccc}
(uce_{\alpha}^{Lie}(L), \bar{\alpha}) & \xrightarrow{U_\alpha} & (L, \alpha_L) \\
\downarrow \Phi & & \downarrow \text{Id} \\
(uce_{\alpha}^{Lie}(L), \bar{\alpha}) & \xrightarrow{u_\alpha} & (L, \alpha_L)
\end{array}
\]

\[
\begin{array}{ccc}
(uce_{\alpha}^{Lie}(L), \bar{\alpha}) & \xrightarrow{\gamma} & (uce_{\alpha}^{Lie}(L), \bar{\alpha}) \\
\downarrow \text{ann} & & \downarrow \text{ann} \\
(uce_{\alpha}^{Lie}(L), \bar{\alpha}) & \xrightarrow{U_\alpha} & (L, \alpha_L) \\
\end{array}
\]
\( \gamma : \left( \text{ucc}^\text{Lie}_\alpha (L) \right)_{\text{Lie}, \overline{\alpha}_{\text{Lie}}} \rightarrow (L, \alpha_L) \) is a central extension since \( \gamma \) is a surjective homomorphism. \( \text{Ker} (\gamma) \subseteq Z \left( \left( \text{ucc}^\text{Lie}_\alpha (L) \right)_{\text{Lie}} \right) \) since for \( \varpi \in \text{Ker} (\gamma) \) and \( y \in \left( \text{ucc}^\text{Lie}_\alpha (L) \right)_{\text{Lie}}, \left[ \varpi, y \right] = [x, y] = 0 \), because \( 0 = \gamma (\varpi) = U_\alpha (x) \), which implies that \( x \in \text{Ker} (U_\alpha) \subseteq Z (\text{ucc}^\text{Lie}_\alpha (L)) \).

As \( u_\alpha : \left( \text{ucc}^\text{Lie}_\alpha (L), \overline{\alpha} \right) \rightarrow (L, \alpha_L) \) is a universal \( \alpha \)-central extension, then is a universal central extension by Remark 4.4 in [4], and since \( \gamma : \left( \left( \text{ucc}^\text{Lie}_\alpha (L) \right)_{\text{Lie}}, \overline{\alpha}_{\text{Lie}} \right) \rightarrow (L, \alpha_L) \) is a central extension, then there exists a unique homomorphism \( \Psi : \left( \text{ucc}^\text{Lie}_\alpha (L), \overline{\alpha} \right) \rightarrow \left( \left( \text{ucc}^\text{Lie}_\alpha (L) \right)_{\text{Lie}}, \overline{\alpha}_{\text{Lie}} \right) \) such that \( \gamma \cdot \Psi = u_\alpha \).

To end the proof we must check that \( \overline{\varpi} \) and \( \Psi \) are inverse to each other. Indeed,

\[
\begin{align*}
\gamma \cdot (\Psi \cdot \overline{\varpi}) &= (\gamma \cdot \Psi) \cdot \overline{\varpi} = u_\alpha \cdot \overline{\varpi} = \gamma \\
\gamma \cdot (\Psi \cdot \overline{\varpi}) &= (\gamma \cdot \Psi) \cdot \overline{\varpi} = u_\alpha \cdot \overline{\varpi} = \gamma
\end{align*}
\]

Since \( \left( \text{ucc}^\text{Lie}_\alpha (L), \overline{\alpha} \right) \) and \( \left( \left( \text{ucc}^\text{Lie}_\alpha (L) \right)_{\text{Lie}}, \overline{\alpha}_{\text{Lie}} \right) \) are perfects, then by Lemma 4.7 in [4] we conclude that \( \Psi \cdot \overline{\varpi} = \Psi = \Psi \cdot \overline{\varpi} = Id.\)

Let \( (L, [-, -], \alpha_L) \) be a perfect Hom-Lie algebra. By Theorem 4.11 c) and d) in [4], \( (L, [-, -], \alpha_L) \) admits a universal central extension in the category Hom-Lie of the form:

\[
0 \rightarrow (H^\alpha_2 (L), \overline{\alpha}) \rightarrow (\text{ucc}_{\text{Lie}} (L), \overline{\alpha}) \xrightarrow{u_L} (L, \alpha_L) \rightarrow 0
\]

Here \( \text{ucc}_{\text{Lie}} (L) \) denotes the quotient \( \mathbb{K} \)-vector space \( \frac{L \wedge L}{I_L} \), where \( I_L \) is the subspace of \( L \wedge L \) spanned by the elements of the form \(-x_1, x_2 \wedge \alpha_L (x_3) + [x_1, x_3] \wedge \alpha_L (x_2) - [x_2, x_3] \wedge \alpha_L (x_1), x_1, x_2, x_3 \in L \), that is, \( I_L = \text{Im} (d_3 : C_3^\alpha (L) \rightarrow C_2^\alpha (L)) \) and \( (H^\alpha_2 (L), \overline{\alpha}) \) denotes the second homology with trivial coefficients of the Hom-Lie algebra \( (L, \alpha_L) \) (see [4] for details).

On the other hand, since a perfect Hom-Lie algebra is a perfect Hom-Leibniz algebra as well, then by Theorem 4.11 c) and d), \( (L, [-, -], \alpha_L) \) also admits a universal central extension in the category Hom-Leib of the form:

\[
0 \rightarrow (HL^\alpha_2 (L), \overline{\alpha}) \rightarrow (\text{ucc}_{\text{Leib}} (L), \overline{\alpha}) \xrightarrow{u_L} (L, \alpha_L) \rightarrow 0
\]

**Proposition 5.7** Let \( (L, \alpha_L) \) be a perfect Hom-Lie algebra and let \( (\text{ucc}_{\text{Lie}} (L), \overline{\alpha}) \) and \( (\text{ucc}_{\text{Leib}} (L), \overline{\alpha}) \) be its universal central extensions in the categories Hom-Lie and Hom-Leib, respectively.

\[
(\text{ucc}_{\text{Lie}} (L), \overline{\alpha}) \simeq (\text{ucc}_{\text{Leib}} (L), \overline{\alpha}) \iff H^\alpha_2 (L) \simeq HL^\alpha_2 (L).
\]

**Proof.** If \( (L, \alpha_L) \) is perfect in the category Hom-Lie, then it is perfect in the category Hom-Leib as well. Consequently, by Theorem 4.11 c) in [4] and Theorem 4.11 c), \( (L, \alpha_L) \) admits universal central extension in both categories, hence we
can construct the following commutative diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & H_{L_2}(L) \\
\downarrow \sigma & & \downarrow \exists \sigma \\
0 & \rightarrow & H^\alpha_2(L) \\
\end{array}
\]

\[
\begin{array}{ccc}
(\text{uce}_{\text{Leib}}(L), \alpha) & \xrightarrow{U} & (L, \alpha_L) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \\
(\text{uce}_{\text{Lie}}(L), \tilde{\alpha}) & \xrightarrow{u} & (L, \alpha_L) & \rightarrow & 0
\end{array}
\]

\[U : (uce_{\text{Leib}}(L), \alpha) \rightarrow (L, \alpha_L)\] is a universal central extension in the category Hom-Leib and \[u : (uce_{\text{Lie}}(L), \tilde{\alpha}) \rightarrow (L, \alpha_L)\] is a central extension in the same category, then there exists a unique homomorphism \(\sigma : (uce_{\text{Leib}}(L), \alpha) \rightarrow (uce_{\text{Lie}}(L), \tilde{\alpha})\) such that \(u \cdot \sigma = U\).

If \(\sigma\) is an isomorphism, then its restriction \(\sigma|\) is an isomorphism as well, hence \(H^\alpha_2(L) \simeq H_{L_2}(L)\).

Conversely, if \(\sigma| : H_{L_2}(L) \rightarrow H^\alpha_2(L)\) is an isomorphism, then \(\sigma\) is an isomorphism as well, by the the Short Five Lemma \([2]\) and having in mind that Hom-Leib is a semi-abelian category.

\section{Remark 5.8}

When Leibniz (Lie) algebras are considered as Hom-Leibniz (Hom-Lie) algebras, i.e. when \(\alpha = \text{Id}\), then the above results recover the relationship between universal central extensions of a perfect Lie algebra in the categories of Lie algebras and Leibniz algebras given in \([3]\).

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