DICHOTOMY SPECTRA OF TRIANGULAR EQUATIONS

CHRISTIAN PÖTzsche

Institut für Mathematik, Alpen-Adria Universität Klagenfurt
Universitätstraße 65–67
9020 Klagenfurt, Austria

(Communicated by Chongchun Zeng)

Abstract. Without question, the dichotomy spectrum is a central tool in the stability, qualitative and geometric theory of nonautonomous dynamical systems. In this context, when dealing with time-variant linear equations having triangular coefficient matrices, their dichotomy spectrum associated to the whole time axis is not fully determined by the diagonal entries. This is surprising because such a behavior differs from both the half line situation, as well as the classical autonomous and periodic cases. At the same time triangular problems occur in various applications and particularly numerical techniques.

Based on operator-theoretical tools, this paper provides various sufficient and verifiable criteria to obtain a corresponding diagonal significance for finite-dimensional difference equations in the following sense: Spectral and continuity properties of the diagonal elements extend to the whole triangular system.

1. Introduction. At least in finite dimensions, the local behavior of dynamical systems near constant or periodic solutions is generically determined by the spectrum of its linearization, i.e. by eigenvalues or Floquet multipliers. Provided the (Floquet) spectrum is disjoint from the stability boundary (the unit circle in discrete time resp. the imaginary axis in continuous time) one speaks of hyperbolicity. When extending this setting and dealing with general nonautonomous systems or aperiodic solutions, hyperbolicity is not a generic property anymore and cannot be characterized in terms of eigenvalues. On the one hand, this led to the theory of Lyapunov exponents, where the Multiplicative Ergodic Theorem (MET for short, cf., e.g. [27]) yields an ambient “nonautonomous linear algebra.” On the other hand, especially for systems being irregular in the sense of Lyapunov (see [8]), a further appropriate spectral notion is given in terms of the coarser dichotomy (dynamical, or Sacker-Sell) spectrum $\Sigma \subseteq \mathbb{R}$ (cf. [41, 27, 6]). This concept is particularly suitable to obtain stability information, and far beyond that to develop a geometric theory for time-dependent equations involving invariant manifolds and topological linearizations [28], as well as normal forms [43, 44]. In addition, it turned out to be beneficial to investigate several dynamically relevant subsets of the dichotomy spectrum for the following reasons: (1) These sets allow to classify nonautonomous...
bifurcations on a linear level [37]. (2) While $\Sigma$ is only upper-semicontinuous under general bounded perturbations, appropriate relations between its dichotomy subspectra yield even continuity for $\Sigma$ (see [39]).

Typically, the dichotomy spectrum is only accessible on a numerical basis. As a result, both the approximate computation (cf. [13, 24]), and also further properties (see [37]) of $\Sigma$ received attention over the recent years. Indeed, many of the computational methods are based on the strategy to transform a linear difference or differential equation to triangular form without affecting its spectrum (or stability properties), and then to extract the spectrum from their resulting diagonal: Since the diagonal elements are scalar functions, their dichotomy spectra are single intervals whose boundary consists of lower and upper Bohl exponents. The spectrum of the whole system then results as the union of the diagonal spectra. This is a valid technique, as long as the equations are merely dichotomic on a half line. Nonetheless, when dealing with exponential dichotomies and related spectra on the whole time axis, also information on elements off the diagonal is needed, or specific assumptions on the diagonal are necessary. Besides numerical techniques, another source for (block-) triangular linear problems are variational equations related to extinction equilibria of nonlinear models in e.g. population dynamics (we refer to [25]). From a theoretical perspective, also the proof of the MET in [27] rests upon a triangulation technique.

Indeed, the full axis dichotomy spectrum has more subtle (and weaker) perturbation properties than the related half line concept. This is due to the fact that a dichotomy on the entire axis requires dichotomies on both semiaxes and beyond that a stable subspace $X^+$ (in forward time) being complementary to the unstable subspace $X^-$ (in backward time). Thus, for instance the approximation algorithm in [14] has to verify the latter geometrical condition. Roughly speaking, this is achieved “Evans function-like” by choosing bases of $X^+$ and $X^-$, and showing the linear independence of their union. An alternative approach to obtain the subspaces $X^-, X^+$ in discrete time is given in [23].

These observations motivate our deeper analysis of spectral properties for w.l.o.g. (block) upper-triangular linear nonautonomous dynamical systems. For this purpose, the paper restricts to time-dependent finite-dimensional difference equations, since they provide a setting tailor-made to apply convenient operator-theoretical tools as previously exemplified in [10, 5] or [36, 37, 39]. This approach allows rather short proofs and can be seen as the main conceptional contribution of the present paper. Yet, we do not conceal that it somehow underplays the importance of $X^-, X^+$ being complementary. Our presentation begins in the subsequent Sect. 2 with preparations on characteristic, Lyapunov and Bohl exponents, as well as exponential dichotomies in discrete time. These concepts are illustrated by several examples to which we will return throughout the text. We also emphasize the close and central relationship between the dichotomy spectrum and weighted shift operators on the Hilbert space of square-summable sequences. Actually, this observation is crucial in order to apply corresponding results from operator theory and therefore for the entire paper. The following Sects. 3–4 illuminate how results on the spectra of abstract triangular operators provide sufficient conditions on the diagonal sequences, as well as on the off-diagonal elements, such that our desired diagonal significance holds: This means

- the dichotomy spectrum and its dynamically relevant subspectra are determined by the union of the corresponding diagonal spectra (cf. Sect. 4.2),
• continuity of the diagonal spectra w.r.t. the Hausdorff distance yields continuity of the full spectrum (cf. Sect. 4.4);

a precise definition follows below. Among others, these conditions are based on ambient compatibility conditions comparing a system’s growth in forward and backward time by means of their Lyapunov filtrations (cf. [8]). For instance, Sect. 3 tackles the basic situation of diagonal systems, whereas Sect. 4 studies upper block-triangular equations. Sufficient conditions for diagonal significance depending on the diagonal systems, or the off-diagonal entries are provided. The obtained prototype results extend to triangular equations by means of inductive arguments, which can be found in Sect. 5. An explicit example from population dynamics is presented in Sect. 6 in order to illustrate our results. Finally, for the reader’s convenience the paper closes with appendices covering the required basics of operator theory and matrix-weighted shifts.

Although the present paper sticks to a discrete time situation, we are about to explain in Sect. 7 to what extent the results are useful in an ODE context as well. In addition, we recently became aware of Flaviano Battelli’s and Ken Palmer’s preprint [9] dealing with dichotomies and the related spectrum of block-triangular equations in continuous time. They allow unbounded coefficients and obtain also necessary conditions for diagonal significance in the dichotomy spectrum. Moreover, a procedure to determine the full-axis spectrum from the half line spectra is given. The methods in [9] are different from ours though.

We start with the necessary terminology: Given a real interval $I \subseteq \mathbb{R}$, one denotes an intersection $I_\mathbb{Z} := I \cap \mathbb{Z}$ with the integers $\mathbb{Z}$ as discrete interval; for such a discrete interval $I$, set $I' := \{ k \in \mathbb{Z} : k + 1 \in I \}$. Here, $I$ will typically be unbounded, and e.g. of the form $I^+ := [\kappa, \infty)_\mathbb{Z}, I^- := (-\infty, \kappa]_\mathbb{Z}, \kappa \in \mathbb{Z}$, or $\mathbb{Z}$. Let us write $K$ for one of the fields $\mathbb{R}$ or $\mathbb{C}$. On $K^d$ we denote the Euclidean resp. unitary norm by $|\cdot|$, write $L(K^d)$ for the $d \times d$-matrices and $GL(K^d)$ for the invertible matrices. The space of square-summable sequences in $K^d$ is abbreviated as $\ell^2 = \ell^2(K^d)$. Throughout.

Let $K(K)$ denote the family of nonempty compact subsets of $K$ and

$$h : K(K) \times K(K) \to \mathbb{R}, \quad h(M_1, M_2) := \max \left\{ \sup_{x \in M_1} \text{dist}(x, M_2), \sup_{x \in M_2} \text{dist}(x, M_1) \right\}$$

be the Hausdorff distance. Then the pair $(K(K), h)$ becomes a metric space. Finally, the closure of a subset $M \subseteq K^d$ is denoted by $\overline{M}$, and $M^0$ is its interior.

2. Preliminaries. Consider a linear nonautonomous difference equation

$$x_{k+1} = A_k x_k \quad (\Delta_A)$$

with coefficient matrices $A_k \in GL(K^d), k \in I'$, fulfilling the assumption

$$\sup_{k \in I'} |A_k| < \infty.$$

We often identify $(\Delta_A)$ with the matrix sequence $A = (A_k)_{k \in I'}$ in the Banach space

$$\mathcal{L}^\infty(K^d) := \ell^\infty(\mathbb{R}^d \times d), \quad \|A\| := \sup_{k \in I'} |A_k|$$

of bounded matrix sequences. The solutions to \((\Delta_A)\) can be expressed in terms of the transition matrix

\[
\Phi : \mathbb{I} \times \mathbb{I} \to GL(\mathbb{K}^d), \quad \Phi(k, l) := \begin{cases} A_k \cdots A_l, & l < k, \\
\text{id}_{\mathbb{K}^d}, & k = l, \\
A_k^{-1} \cdots A_l^{-1}, & k < l. \end{cases}
\]

Along with \((\Delta_A)\) let us introduce the adjoint difference equation

\[
\dot{x}_k = A_k^{-1} x_{k+1}, \tag{\Delta_A^*}
\]

whose (adjoint) transition matrix is given by \(\Phi^*(k, \kappa) = \Phi(k + 1, k + 1)^*\).

The following subsections collect some basic notions, establish our terminology and provide elementary examples necessary for the remaining paper:

2.1. Characteristic exponents and Lyapunov filtration. Assume that the discrete interval \(\mathbb{I}\) is unbounded above. In order to capture the long-term behavior of \((\Delta_A)\) consider the (upper) characteristic exponent

\[
\chi_A(x) := \limsup_{k \to \infty} \sqrt[k]{|\Phi(k, \kappa)x|}
\]

of its solution starting in \(x \in \mathbb{K}^d\); this exponent is independent of the initial time \(\kappa \in \mathbb{I}\) and clearly fulfills \(\chi_A(0) = 0\). A difference eqn. \((\Delta_A)\) possesses up to \(d\) characteristic exponents which form its (upper) Lyapunov spectrum

\[
\{\chi_A(x) > 0 : x \in \mathbb{K}^d \setminus \{0\}\} = \{\lambda_1, \ldots, \lambda_n\}
\]

with \(n \leq d\). We suppose that the positive reals \(\lambda_j\) are ordered according to

\(0 < \lambda_1 < \ldots < \lambda_n\).

The sublevel sets \(W_j := \{x \in \mathbb{K}^d : \chi_A(x) \leq \lambda_j\}\) are linear subspaces of \(\mathbb{K}^d\) yielding the Lyapunov filtration of strict inclusions

\[
\{0\} = W_0 \subset W_1 \subset \ldots \subset W_n = \mathbb{K}^d.
\]

Concerning this, and more details on Lyapunov spectra we refer to [8, pp. 56ff].

For the adjoint difference eqn. \((\Delta_A^*)\) the characteristic exponent is defined by

\[
\chi^*_A(x) := \limsup_{k \to \infty} \sqrt[k]{|\Phi^*(k, \kappa)x|} = \limsup_{k \to \infty} \sqrt[k]{|\Phi(k + 1, k + 1)^*x|} \quad \text{for all } x \in \mathbb{K}^d.
\]

As above one obtains a finite Lyapunov spectrum

\[
\{\chi^*_A(x) > 0 : x \in \mathbb{K}^d \setminus \{0\}\} = \{\mu_1, \ldots, \mu_n\}
\]

and a Lyapunov filtration \(\{0\} =: V_0 \subset V_1 \subset \ldots \subset V_n = \mathbb{K}^d\) with \(n^* \leq d\) for \((\Delta_A^*)\).

2.2. Exponential dichotomies. Besides characteristic exponents and Lyapunov filtrations, a further and arguably more appropriate tool to capture the asymptotics of nonautonomous equations are exponential dichotomies.

Given an unbounded discrete interval \(\mathbb{I}\), a linear difference eqn. \((\Delta_A)\) has an exponential dichotomy on \(\mathbb{I}\) (ED for short, cf. [19, 6]), if there exists a sequence of projections \(P_k \in L(\mathbb{K}^d), k \in \mathbb{I}\), with \(P_{k+1}A_k = A_kP_k\) for all \(k \in \mathbb{I}\), growth rates \(\alpha \in (0, 1)\) and a constant \(K \geq 1\) such that the estimates

\[
|\Phi(k, l)P_l| \leq K \alpha^{-l}, \quad |\Phi(l, k)[\text{id}_{\mathbb{K}^d} - P_k]| \leq K \alpha^{-l} \quad \text{for all } l \leq k
\]

and \(k, l \in \mathbb{I}\) hold. Then dichotomy spectrum of \((\Delta_A)\) is defined as

\[
\Sigma(A) = \{\gamma > 0 : x_{k+1} = \gamma^{-1}A_k x_k \text{ does not have an ED on } \mathbb{I}\};
\]
it is empty or consists of up to \( d \) disjoint spectral intervals (cf. [6, Thm. 3.4])

\[
\Sigma(A) = \bigcup_{i=1}^{m-1} [\alpha_i, \beta_i] \cup \{(0, \beta_m) \cup [\alpha_m, \beta_m] \}
\]

with real numbers \( 0 < \alpha_m \leq \beta_m < \alpha_{m-1} \leq \cdots \leq \beta_1, \ m \leq d \). The invertibility assumption on \( A_k \) ensures that an empty spectrum or a spectral interval \((0, \beta_m)\) can be avoided precisely in case

\[
\sup_{k \in \mathbb{N}} |A_k^{-1}| < \infty. \tag{2.1}
\]

Due to its role for stability properties, \( \max \Sigma(A) \) is called stability radius of \((\Delta_A)\).

We speak of a discrete spectrum, if \( \Sigma(A) \) is finite. Discrete spectra on the half line \( \mathbb{I} = \mathbb{Z}_n^+ \) typically occur for asymptotically periodic equations, whereas on the whole line \( \mathbb{I} = \mathbb{Z} \), periodic (or autonomous) equations possess a discrete spectrum.

If we denote the dichotomy spectra associated with the discrete intervals \( \mathbb{Z}_k^+ \), \( \mathbb{Z}_k^- \) or \( \mathbb{Z} \) by \( \Sigma^+(A) \), \( \Sigma^{-}(A) \) resp. \( \Sigma(A) \), then the inclusions

\[
\{\lambda_1, \ldots, \lambda_n \} \subseteq \Sigma^+(A) \subseteq \Sigma(A), \quad \Sigma^{-}(A) \subseteq \Sigma(A)
\]

hold (see [28, p. 88, Thm. 5.13]).

Provided \((\Delta_A)\) can be described using a discrete-time skew-product flow over a compact base, it was established in [27] that the Lyapunov spectrum is contained between \( \Sigma \) and its boundary \( \partial \Sigma \). Generally speaking the Lyapunov spectrum is finer than the dichotomy spectra, and we refer to our concluding Ex. 2.6 for concrete examples illustrating these inclusions.

2.3. **Lyapunov and Bohl exponents.** While the often studied Lyapunov exponents measure exponential growth in a straight-forward manner, the related Bohl exponents (cf. [20, pp. 253ff, Sect. 3.3]) rather determine uniform growth of linear equations or individual solutions.

For the family of discrete intervals \( \mathbb{J} \subseteq \mathbb{I} \) with fixed length \( n \in \mathbb{N} \) one writes

\[
\mathbb{I}_n := \{ \mathbb{J} \subseteq \mathbb{I} : \mathbb{J} \text{ is a discrete interval with } \#\mathbb{J} = n \} \quad \text{for all } n \in \mathbb{N}.
\]

It has advantages to introduce Lyapunov and Bohl exponents abstractly: Suppose thereto that \( \mathcal{A} \) is a normed unital algebra over \( \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\} \) with norm \( |.| \). Let us define the lower resp. upper Bohl exponent of a sequence \( a = (a_k)_{k \in \mathbb{I}} \) in \( \mathcal{A} \) as

\[
\underline{\beta}(a) := \liminf_{n \to \infty} \sup_{J \in \mathbb{I}_n} \sqrt[n]{\prod_{j \in J} a_j}, \quad \overline{\beta}(a) := \limsup_{n \to \infty} \inf_{J \in \mathbb{I}_n} \sqrt[n]{\prod_{j \in J} a_j}. \tag{2.2}
\]

For fixed \( \mathbb{I} \) one writes \( \underline{\beta}(a) \) resp. \( \overline{\beta}(a) \). In comparison, Lyapunov exponents are

\[
\begin{align*}
\lambda_+(a) := \liminf_{n \to \infty} \sqrt[n]{\prod_{j=\kappa}^{n-1} a_j}, & \quad \lambda_-(a) := \liminf_{n \to \infty} \sqrt[n]{\prod_{j=-n}^{-1} a_j}, \\
\Lambda_+(a) := \limsup_{n \to \infty} \sqrt[n]{\prod_{j=\kappa}^{n-1} a_j}, & \quad \Lambda_-(a) := \limsup_{n \to \infty} \sqrt[n]{\prod_{j=-n}^{-1} a_j},
\end{align*}
\]

and independent of \( \kappa \in \mathbb{I} \), as long as all \( a_j \in \mathcal{A}, \ j \in \mathbb{I} \), are invertible. It goes without saying that \( \mathbb{I} \) has to be unbounded above in order to introduce \( \lambda_+(a), \lambda_-(a) \), while the definition of \( \Lambda_-(a), \Lambda_-(a) \) only makes sense for \( \mathbb{I} \) being unbounded below. When dealing with Bohl exponents \( \underline{\beta}_k(a), \overline{\beta}_k(a) \) it suffices that \( \mathbb{I} \) is solely unbounded.
Remark 2.1. In the Banach algebra $\mathcal{A} = \mathbb{K}$ the Lyapunov exponents of a sequence $a$ and the characteristic exponents of the corresponding scalar difference equation

$$x_{k+1} = a_k x_k$$

are related by

$$\bar{\lambda}_+(a) = x_{\alpha}(x), \quad \hat{\lambda}_+(a) = \chi_n(x)^{-1} \quad \text{for all } x \neq 0. \quad (2.3)$$

Properties of Lyapunov and particularly Bohl exponents, as well as the fact that the limits in (2.2) exist under natural assumptions, are given in

Proposition 2.2. On unbounded discrete subintervals $J \subseteq I$ one has

$$\beta_{\pm}(a) \leq \beta_{\pm}(a) \leq \beta_{\pm}(a) \leq \beta_{\pm}(a) \leq \beta_{\pm}(a) \quad \text{for all } \kappa \in \mathbb{Z} \quad (2.4)$$

and the positive homogeneity

$$\beta(\mu a) = |\mu| \beta(a), \quad \beta(\mu a) = |\mu| \beta(a), \quad (2.5)$$

Moreover, the left-hand limit in (2.2) exists and the characterizations

$$\begin{align*}
\beta_J(a) &:= \lim_{n \to \infty} \sup_{j \in J} \sqrt[n]{\prod_{j \in J} a_j} = \inf_{n \in \mathbb{N}} \sup_{j \in J} \sqrt[n]{\prod_{j \in J} a_j} \\
&= \inf \left\{ \rho > 0 : \exists K \geq 1 : \forall n \in \mathbb{N} : \sup_{j \in J} \prod_{j \in J} a_j \leq K \rho^n \right\} \quad (2.6)
\end{align*}$$

hold, where (2.7) necessitates the sequence $a$ to be bounded.

Proof. The inequalities relating Bohl exponents on different discrete intervals are evident from (2.2), as well as their homogeneity relations (2.5). Furthermore, define $\alpha := \sup_{j \in J} |a_j|$ and for the sake of a convenient notation abbreviate

$$\phi_n := \sup_{j \in \mathbb{N}} \sqrt[n]{\prod_{j \in J} a_j} = \sqrt[n]{\sup_{j \in \mathbb{N}} \prod_{j \in J} a_j} \quad \text{for all } n \in \mathbb{N}.$$ 

Therefore, because the norm $|\cdot|$ is submultiplicative, $|\prod_{j \in J} a_j| \leq \prod_{j \in J} |a_j| \leq \alpha^n$ for all $J \subseteq I$ implies $\sqrt[n]{\phi_n} \leq \alpha$ for every $n \in \mathbb{N}$ and consequently $\beta(a) \leq \alpha$.

(a) Let $m, n \in \mathbb{N}$ and suppose $J \subseteq I_{m+n}$ denotes an arbitrary discrete interval, e.g. of the form $J = [k, \kappa + m] \cup [\kappa + m, \kappa + m + n]$. Again the submultiplicativity of the norm allows us to obtain

$$\prod_{j \in J} a_j \leq \prod_{j = \kappa}^{\kappa + m - 1} a_j \prod_{j = \kappa + m}^{\kappa + m + n - 1} a_j \leq \phi_{m+n} \phi_n \quad \text{for all } m, n \in \mathbb{N}$$

and since $J \subseteq I_{m+n}$ was arbitrary, we can pass to the least upper bound over all such discrete intervals $J$ yielding $0 \leq \phi_{m+n} \leq \phi_m \phi_n$ for all $m, n \in \mathbb{N}$. Now it is well-known (see, e.g., [1, p. 246]) that the real sequence $(\sqrt[n]{\phi_n})_{n \in \mathbb{N}}$ converges to the value $\inf_{n \in \mathbb{N}} \sqrt[n]{\phi_n}$, which establishes (2.6).
In order to deduce the characterization (2.7), we abbreviate the right-hand side of the inequality required in (2.7) by $R$. Thus, for every $\varepsilon > 0$ there exists a $K \geq 0$ such that $\phi_n \leq K(R + \varepsilon)^n$ for all $n \in \mathbb{N}$ and $\overline{\beta}(a) \leq R$ follows from

$$
\overline{\beta}(a) = \limsup_{n \to \infty} \sqrt[n]{\phi_n} \leq \limsup_{n \to \infty} \sqrt[n]{K(R + \varepsilon)} = R + \varepsilon \quad \text{for all } \varepsilon > 0.
$$

Conversely, it remains to show $R \leq \overline{\beta}(a)$. From $\overline{\beta}(a) = \lim_{\nu \to \infty} \sup_{n \geq \nu} \sqrt[n]{\phi_n}$ we see that for every sufficiently small $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that the inequality $\sup_{n \geq \nu} \sqrt[n]{\phi_n} \leq \overline{\beta}(a) + \varepsilon$ holds for all $\nu \geq N$. Hence, it is

$$
\phi_n \leq (\overline{\beta}(a) + \varepsilon)^n \quad \text{for all } n \geq N \quad (2.8)
$$

and if we define $K := \sup_{1 \leq n < N} \left( \frac{\sup_{1 \leq j \leq n} |a_j|}{\overline{\beta}(a) + \varepsilon} \right)^n \geq 1$, then

$$
\phi_n \leq \left( \sup_{j \in I} |a_j| \right)^n \leq K (\overline{\beta}(a) + \varepsilon)^n \quad \text{for all } 1 \leq n < N.
$$

Combining this with (2.8), and since $\varepsilon > 0$ was arbitrary, we get $R \leq \overline{\beta}(a)$. \qed

**Corollary 2.3.** In algebras $A$ with multiplicative norm (i.e. $|ab| = |a||b|$, $a, b \in A$) one has $\overline{\beta}(|a|) = \overline{\beta}(a)$, $\underline{\beta}(|a|) = \underline{\beta}(a)$ and if every $a_k \in A$ is invertible, then

$$
\inf_{k \in I} |a_k| \leq \underline{\beta}(a) = \lim_{n \to \infty} \inf_{\ell \leq n} \sqrt[n]{\prod_{j \in J} a_j} = \sup_{n \in \mathbb{N}, k \in I} \sqrt[n]{\prod_{j \in J} a_j}. \quad (2.9)
$$

**Proof.** It is clear that both sequences $a$ and $|a|$ have the same Bohl exponents. When each $a_k \in A$, $k \in I$, is invertible, then (2.9) follows from the proof of Prop. 2.2 applied to $a_k := a_k^{-1}$. \qed

As temporary conclusion, we present the close connection between the dichotomy spectrum and Bohl exponents of scalar difference eqns. $(\Delta_a)$:

**Proposition 2.4** (see [25, Prop. B.4]). If $a = (a_k)_{k \in \ell}$ is a sequence in $\mathbb{K}$ with

$$
0 < \inf_{k \in \ell} |a_k| \leq \sup_{k \in \ell} |a_k| < \infty, \quad (10.20)
$$

then $(\Delta_a)$ has the dichotomy spectrum $\Sigma(a) = [\underline{\beta}(a), \overline{\beta}(a)]$.

2.4. **Weighted shift operators.** For one-sided time $I = \mathbb{Z}^+_\kappa$, $\kappa \in \mathbb{Z}$, the dichotomy spectrum $\Sigma^+(A)$ of difference eqns. $(\Delta^+)$ and the essential (Fredholm) spectrum $\sigma_F$ of the unilateral matrix-weighted shift

$$
T_A \phi := (0, A_k \phi_n, A_{k+1} \phi_{n+1}, \ldots) \quad \text{for all } \phi \in \ell^2
$$

are related by (cf. [10] or [37, Thm. 3.22])

$$
\Sigma^+(A) = \sigma_F(T_A) \cap \mathbb{R}^+ \quad (2.11)
$$

The set $\sigma_F(T_A)$ is rotationally invariant, i.e. consists of concentric rings and annuli in the complex plane. This observation has the striking advantage that information on the dichotomy spectrum can be obtained from results on shifts, like for instance

**Example 2.5** (asymptotically periodic scalar equations). Let $p \in \mathbb{N}$ and suppose $(a_k)_{\kappa \leq k}$ is a sequence in $\mathbb{K}$. If $(|a_k|)_{\kappa \leq k}$ is asymptotically $p$-periodic, i.e. there is some $p$-periodic positive real sequence $(p_k)_{k \in \ell}$ satisfying $\lim_{k \to \infty} (|a_k| - p_k) = 0$, then Ex. B.3 yields the dichotomy spectrum

$$
\Sigma^+(a) = \{ \epsilon \}
$$

with the asymptotic mean $c := \sqrt[p+k+p-1\cdots-p]{k}$. 

Difference eqns. \((\Delta_A)\) defined on the whole axis \(\mathbb{I} = \mathbb{Z}\) exhibit a richer spectral theory; it is based on bilateral matrix-weighted shifts

\[
(T_A \phi)_k := A_{k-1} \phi_{k-1} \quad \text{for all } k \in \mathbb{Z}, \phi \in \ell^2.
\]

As motivated in Sect. 1, it is advisable to distinguish different dichotomy spectra

\[
\Sigma_\alpha(A) = \sigma_\alpha(T_A) \cap \mathbb{R}^+ \quad \text{with } \alpha \in \{a, s, F, F_0, \pi\},
\]

where \(\Sigma_\alpha(A) := \Sigma(A)\) denotes the dichotomy spectrum of \((\Delta_A)\), while its subspectra \(\Sigma_\alpha(A), \Sigma_F(A), \Sigma_{F_0}(A)\) and \(\Sigma_\pi(A)\) are called surjectivity, Fredholm, Weyl resp. approximate point spectrum (see [37, 39]). They consist of all reals \(\gamma > 0\) such that \(L_\gamma \in L(\ell^2), (L_\gamma \phi)_k := \phi_{k+1} - \gamma^{-1} A_k \phi_k, k \in \mathbb{Z}\), fails to be onto, Fredholm, Weyl resp. bounded below. The corresponding spectra \(\sigma_\alpha\) are introduced in Sect. A.

2.5. Examples for \(\mathbb{I} = \mathbb{Z}\). The upcoming examples feature sequences whose Lyapunov and Bohl exponents can be obtained explicitly. This equips us with a number of difference equations sufficiently flexible to illustrate our results later on.

Example 2.6 (scalar equations). Choose \(\kappa \in \mathbb{Z}\) fixed and let \((a_k)_{k \in \mathbb{Z}}\) be a sequence in \(\mathbb{K}\) satisfying (2.10). The inclusions \(\partial \Sigma(a) \subseteq \Sigma_F(a) \subseteq \Sigma_{F_0}(a) \subseteq \Sigma(a)\), as well as \(\partial \Sigma(a) \subseteq \Sigma_s(a) \subseteq \Sigma(a)\) are fulfilled due to \([37, \text{Cors. } 4.26(d) \text{ and } 4.31]\). We can apply Prop. 2.4 in order to determine the dichotomy spectra of the eqns. \((\Delta_a)\). Moreover, based on the relations (2.3) one automatically has information concerning their characteristic exponents \(\chi_a, \chi_a^+\). Finally, combining (2.12) with the abstract results provided in Sect. B.2 implies the following concrete examples:

1. If \(|a_k| \equiv \bar{a} \) with \(\bar{a} > 0\), then all Bohl and Lyapunov exponents coincide, i.e.,

\[
\beta_{\bar{a}}^+ (a) = \beta_{\bar{a}}^\pm (a) = \beta_{\bar{a}}^- (a) = \beta_{\bar{a}}^\mp (a) = \lambda^+_a (a) = \lambda^-_a (a) = \lambda^+_a (a) = \bar{a}.
\]

One has discrete dichotomy spectra \(\Sigma_\alpha(a) = \{\bar{a}\}\) for all \(\alpha \in \{a, F_0, F, s, \pi\}\).

2. If \(|a|\) is \(p\)-periodic, \(p \in \mathbb{N}\), then the Bohl and Lyapunov exponents become

\[
\beta_{\bar{a}}^+ (a) = \beta_{\bar{a}}^\pm (a) = \beta_{\bar{a}}^- (a) = \beta_{\bar{a}}^\mp (a) = \lambda^+_a (a) = \lambda^-_a (a) = \lambda^+_a (a) = \sqrt{|a_{k+p-1} \cdots a_k|}
\]

and we also arrive at the discrete spectra

\[
\Sigma_\alpha(a) = \left\{ \sqrt{|a_{p-1+k} \cdots a_k|} \right\} \quad \text{for all } \alpha \in \{a, F_0, F, s, \pi\} \text{ and } k \in \mathbb{Z}.
\]

3. The both-sided asymptotically constant situation \(\lim_{k \to \pm \infty} |a_k| = \alpha_\pm\) with reals \(\alpha_\pm > 0\) now illustrates a distinction between Bohl and Lyapunov exponents

\[
\beta_{\alpha_+} (a) = \beta_{\alpha_-} (a) = \lambda^+_a (a) = \lambda^-_a (a) = \alpha_+,
\]

\[
\beta_{\alpha_-} (a) = \min \{\alpha_-, \alpha_+\}, \quad \beta_{\alpha_+} (a) = \max \{\alpha_-, \alpha_+\},
\]

as well as their dependence on the discrete interval. Using [37, Ex. 5.3] one deduces

\[
\Sigma(a) = [\min \{\alpha_-, \alpha_+\}, \max \{\alpha_-, \alpha_+\}], \quad \Sigma_s(a) = \left\{ \{\alpha_-, \alpha_+\}, \alpha_- \leq \alpha_+ \right\},
\]

\[
\Sigma_{F_0}(a) = [\min \{\alpha_-, \alpha_+\}, \max \{\alpha_-, \alpha_+\}], \quad \Sigma_F(a) = \{\alpha_-, \alpha_+\}.
\]
Let \( p^+, p^- \in \mathbb{N} \). If \(|a|\) is asymptotically \( p^+ \)- resp. \( p^-\)-periodic to real positive sequences \((p^+_k)_{k \geq \kappa}, (p^-_k)_{k \leq \kappa}\) on \( \mathbb{Z}^+ \) or \( \mathbb{Z}^- \) by means of Ex. 2.5, then it follows

\[
\begin{align*}
\Sigma(a) &= \left[ \min \{c_+, c_-\}, \max \{c_-, c_+\} \right], \\
\Sigma_\alpha(a) &= \begin{cases} 
\{c_+, c_-\}, & c_- \leq c_+ , \\
\{c_+, c_-\}, & c_+ \leq c_-
\end{cases}
\end{align*}
\]

from Ex. B.4 with the asymptotic means \( c_k := \sqrt[m]{p^+_k p^-_{k-1} \cdots p^+_0} \).

(To clarify that the inequalities \((2.4)\) can be strict, for given reals \( \alpha, \beta > 0 \) consider a sequence \((a_k)_{k \geq 0}\) satisfying

\[
|a_k| = \begin{cases} 
\alpha, & k \in \left[ \frac{n(n+1)}{2}, \frac{(n+1)(n+2)}{2} \right) Z, n \text{ even,} \\
\beta, & k \in \left[ \frac{n(n+1)}{2}, \frac{(n+1)(n+2)}{2} \right) Z, n \text{ odd}
\end{cases}
\]

for all \( k \in \mathbb{Z}^+ \).

Hence, the modulus of \( a_k \) is alternately equal to the constant values \( \alpha \) resp. \( \beta \) on arithmetically increasing intervals (see Fig. 1). This yields the Bohl exponents

\[
\beta(a) = \min \{\alpha, \beta\}, \quad \beta(a) = \max \{\alpha, \beta\}.
\]

In order to obtain the Lyapunov exponents of \( a \), we observe that for every \( k \geq 1 \) there exist unique \( n \in \mathbb{N} \) and \( l \in [0, n)_Z \) with \( k = \frac{n(n+1)}{2} + l \). For even \( n \) this implies

\[
\prod_{j=0}^{k-1} a_j = \prod_{j=0}^{l} a_j = \alpha^{n/4} \beta^{n/4} - \frac{n}{2}
\]

and for odd \( n \) it is

\[
\prod_{j=0}^{k-1} a_j = \prod_{j=0}^{l} a_j = \alpha^{\frac{(n+1)^2}{4}} \beta^{\frac{n^2-1}{4}} - \frac{n}{2}.
\]

By means of these representations it is not difficult to deduce that the Lyapunov exponents are given as geometric mean \( \lambda_+ = \lambda_+ = \sqrt{\lambda_\beta} \) and fulfill the inequality \( \beta(a) \leq \lambda_+(a) = \lambda_+ \leq \beta(a) \); this corresponds with \((2.4)\).

![Figure 1. The sequence \((a_k)_{k \geq 0}\) from Ex. 2.6(5)](image)

Let us continue with a 2-dimensional problem from [39, Ex. 5.10] being useful in several situations: It illustrates both that the dichotomy spectrum is upper-semi-continuous, and that it might be smaller than the union of its diagonal spectra:

**Example 2.7.** Consider the planar real difference eqn. \((\Delta_A)\) with coefficients

\[
A_k := \begin{pmatrix}
a_k^1 & c_k \\
0 & a_k^2
\end{pmatrix} \in GL(\mathbb{R}^2)
\]

satisfying \( A \in L^\infty(\mathbb{R}^2) \) and involving the real sequences

\[
a_k^1 := \begin{cases} 
\alpha_+, & k \geq 0 , \\
\alpha_-, & k < 0 
\end{cases}, \quad a_k^2 := \begin{cases} 
\beta_+, & k \geq 0 , \\
\beta_-, & k < 0 
\end{cases}, \quad c_k := \begin{cases} 
\lambda, & k \geq 0 , \\
0, & k < 0
\end{cases}
\]

(2.13)
with reals $\alpha_\pm, \beta_\pm > 0$ and a parameter $\lambda \in \mathbb{R}$.

(1) Evidently, the background to obtain the Lyapunov spectra and filtrations are the transition matrices $\Phi(k, \kappa)$ of $(\Delta A)$ and $\Phi^*(k, \kappa)$ of $(\Delta^*_A)$:

- For $\alpha_+ \neq \beta_+$ they are given by
  \[
  \Phi(k, \kappa) = \begin{cases} 
  \begin{pmatrix} 
  \alpha_+^{k-\kappa} & \lambda(\alpha_+^{k-\kappa} - \beta_+^{k-\kappa}) \\
  0 & \beta_+^{k-\kappa} \\
  \alpha_-^{k-\kappa} & 0 \\
  0 & \beta_-^{k-\kappa} 
  \end{pmatrix} & \text{for all } k, \kappa \geq 0, \\
  \begin{pmatrix} 
  \alpha_+^{k-\kappa} & 0 \\
  \alpha_-^{k-\kappa} & 0 \\
  0 & \beta_-^{k-\kappa} \\
  0 & \beta_+^{k-\kappa} 
  \end{pmatrix} & \text{for all } k, \kappa \leq 0,
  \end{cases}
  \]

  $\Phi^*(k, \kappa)$ is defined similarly.

and consequently yield

- $n = 2$, $\{\lambda_1, \lambda_2\}$, $\{0\} = W_0 \subset W_1 \subset W_2 = \mathbb{R}^2$,
- $n^* = 2$, $\{\mu_1, \mu_2\}$, $\{0\} = V_0 \subset V_1 \subset V_2 = \mathbb{R}^2$.

The explicit values for these quantities can be found in Tab. 1.

| $\alpha_+ < \beta_+$ | $\lambda_1$ | $\lambda_2$ | $\mu_1$ | $\mu_2$ | $W_1$ | $V_1$ |
|----------------------|-------------|-------------|---------|---------|------|------|
| $\alpha_+ > \beta_+$ | $\beta_+$ | $\beta_+$ | $\frac{1}{\alpha_+}$ | $\frac{1}{\beta_+}$ | $\mathbb{R}e_1$ | $\mathbb{R}e_2$ |

$\mathbb{R}e_1(\beta_+ - \alpha_+)$

| $\alpha_+ = \beta_+$ | $\frac{1}{\alpha_+}$ | $\frac{1}{\beta_+}$ | $\mathbb{R}(\frac{\lambda}{\beta_+ - \alpha_+})$ | $\mathbb{R}(\frac{\lambda}{\beta_+ - \alpha_+})$ |

**Table 1.** Lyapunov spectra and filtrations for Ex. 2.7 with $\alpha_+ \neq \beta_+$

- For $\alpha_+ = \beta_+$ the transition matrices become

  $\Phi(k, \kappa) = \begin{cases} 
  \begin{pmatrix} 
  \alpha_+^{k-\kappa} & \lambda(\alpha_+^{k-\kappa} - \beta_+^{k-\kappa}) \\
  0 & \beta_+^{k-\kappa} \\
  \alpha_-^{k-\kappa} & 0 \\
  0 & \beta_-^{k-\kappa} 
  \end{pmatrix} & \text{for all } k, \kappa \geq 0, \\
  \begin{pmatrix} 
  \alpha_+^{k-\kappa} & 0 \\
  \alpha_-^{k-\kappa} & 0 \\
  0 & \beta_-^{k-\kappa} \\
  0 & \beta_+^{k-\kappa} 
  \end{pmatrix} & \text{for all } k, \kappa \leq 0,
  \end{cases}
  \]

  $\Phi^*(k, \kappa)$ is defined similarly.

and consequently yield

- $n = 1$, $\{\alpha_+\}$, $\{0\} = W_0 \subset W_1 = \mathbb{R}^2$,
- $n^* = 1$, $\{\frac{1}{\alpha_+}\}$, $\{0\} = V_0 \subset V_1 = \mathbb{R}^2$.

The related dichotomy spectra $\Sigma(A)$ for the various constellations of $\alpha_\pm \neq \beta_\pm$ were computed in [39, Ex. 5.10] already.

(2) We particularly focus on the situation

\[ \alpha_+ := \delta, \quad \alpha_- := \delta^{-1}, \quad \beta_- := \alpha_+, \quad \beta_+ := \alpha_- \quad (2.14) \]
for some real $\delta > 1$, where the diagonal sequences satisfy $\Sigma(a^1) = \Sigma(a^2) = [\alpha_-, \alpha_+]$ (cf. Prop. 2.4). One therefore obtains from [37, Ex. 5.5] that

$$\Sigma(A) = \begin{cases} \{\alpha_-, \alpha_+\}, & \lambda \neq 0, \\ [\alpha_-, \alpha_+], & \lambda = 0. \end{cases}$$

Hence, $\Sigma(A)$ suddenly shrinks and fulfills $\Sigma(A) \subset \Sigma(a^1) \cup \Sigma(a^2)$ for $\lambda \neq 0$.

3. Spectra of diagonal equations. As a foretaste of and distinction from the general triangular situation, let us initially tackle a simpler case: Results on scalar difference eqns. $(\Delta_\alpha)$ extend to $(\Delta_A)$ with diagonal coefficient matrices $A_k$, i.e.

$$x_{k+1} = A_k x_k, \quad A_k = \begin{pmatrix} a^1_k & \cdots & \cdots \\ \vdots & \ddots & \vdots \\ a^d_k & \cdots & \cdots \end{pmatrix}$$

and bounded diagonal sequences $(a^1_k)_{k \in \mathbb{Z}^+}, \ldots, (a^d_k)_{k \in \mathbb{Z}^+}$. Since the half line situation $\mathbb{I} = \mathbb{Z}^+$ was tackled in [37, Cor. 3.25] already, we restrict to the whole axis $\mathbb{I} = \mathbb{Z}$.

**Theorem 3.1.** Diagonal eqns. $(D)$ fulfill $\Sigma_\alpha(A) = \bigcup_{i=1}^d \Sigma_\alpha(a^i)$ for $\alpha \in \{a, s, F\}$. 

**Remark 3.2** (diagonal dominance). Combined with their $\ell^\infty$-roughness (see, for instance, [5, Cor. 3]), the above Thm. 3.1 can also be understood as diagonal dominance criterion for an ED on $\mathbb{Z}$. This means that the hyperbolicity assumption $1 \not\in \Sigma(a^i)$ for all $1 \leq i \leq d$, implies that also the perturbed equation

$$x_{k+1} = [A_k + B_k] x_k$$

has an ED, provided $\sup_{k \in \mathbb{I}} |B_k|$ is sufficiently small. In case the diagonal elements of each matrix $B_k$ vanish identically, this is actually related to the concept of diagonal dominance treated e.g. in [34] (for the lack of a reference in discrete time). Note that [34] states explicit conditions on the entries of $B_k$ guaranteeing an ED, whereas Thm. 3.1 only allows a qualitative statement.

**Proof.** For $1 \leq i \leq d$ we define the linear operators

$$L_\gamma = L(\ell^\infty(\mathbb{K}^d)), \quad (L_\gamma \phi)_k := \phi_{k+1} - \gamma^{-1} A_k \phi_k,$$

$$L^i_\gamma = L(\ell^\infty(\mathbb{K})), \quad (L^i_\gamma \phi)_k := \phi_{k+1} - \gamma^{-1} a^i_k \phi_k.$$

If $\alpha = s$, then due to [35, Prop. 1] we obtain the equivalences

$\gamma \not\in \Sigma_\alpha(A) \iff L_\gamma$ is onto

$$\iff \forall \psi \in \ell^\infty(\mathbb{K}^d) : \exists \phi \in \ell^\infty(\mathbb{K}^d) : L_\gamma \phi = \psi$$

$$\iff \forall \psi \in \ell^\infty(\mathbb{K}^d) : x_{k+1} = \gamma^{-1} A_k x_k + \psi_k \text{ has a bounded solution } \phi$$

$$\iff \forall 1 \leq i \leq d : \forall \psi^i \in \ell^\infty(\mathbb{K}) : x_{k+1} = \gamma^{-1} a^i_k x_k + \psi^i_k \text{ has a bounded solution } \phi^i$$

$$\iff \forall 1 \leq i \leq d : \forall \psi^i \in \ell^\infty(\mathbb{K}) : \exists \phi^i \in \ell^\infty(\mathbb{K}) : L^i_\gamma \phi^i = \psi^i$$

$$\iff L^i_\gamma$ is onto for all $1 \leq i \leq d$

$$\iff x_{k+1} = \gamma^{-1} a^i_k x_k \text{ has an ED for all } 1 \leq i \leq d \iff \gamma \not\in \bigcup_{i=1}^d \Sigma_s(a^i)$$. 

DICHOTOMY SPECTRA OF TRIANGULAR EQUATIONS 433


and the claim results in the logical contraposition. The situation $\alpha = a$ follows similarly by means of the characterization [19, p. 230, Thm. 7.6.5] applied to the operators $L_\gamma$ and $L'_\gamma$, while $\alpha = F$ was tackled in [37, Cor. 4.30].

4. Block-triangular equations. Our subsequent analysis focuses on block-triangular systems ($\Delta _A$), capturing essential phenomena present in triangular equations. Thereto, let $d_1, d_2 \in \mathbb{N}$ be integers with $d_1 + d_2 = d$ and suppose w.l.o.g. that ($\Delta _A$) is in upper block-triangular form

$$x_{k+1} = A_k x_k,$$

$$A_k := \begin{pmatrix} A_k^1 & C_k \\ 0 & A_k^2 \end{pmatrix} \quad (B)$$

with blocks $A_k^1 \in \mathbb{K}^{d_1 \times d_1}$, $A_k^2 \in \mathbb{K}^{d_2 \times d_2}$, $C_k \in \mathbb{K}^{d_1 \times d_2}$, $k \in \mathbb{I}$, and $I$ being unbounded above. Due to $\det A_k = \det A_k^1 \det A_k^2$, one has $A_k \in GL(\mathbb{K}^d)$ if and only if both diagonal blocks $A_k^1, A_k^2$ are invertible.

This section provides several criteria guaranteeing that a block-triangular difference equation (B) is diagonally significant, i.e. its dichotomy spectrum is given as union of the block-diagonal spectra. While this is not an issue on the half line (see Thm. 4.1), the full axis situation requires more care. As main results, we show that $\Sigma (A)$ is always contained between the symmetric difference and the union of the block diagonal spectra $\Sigma (A^1)$ and $\Sigma (A^2)$ (for this, cf. Thm. 4.3). Besides the absence of interior points in $\Sigma (A^1) \cap \Sigma (A^2)$ we obtain further conditions guaranteeing $\Sigma (A) = \Sigma (A^1) \cup \Sigma (A^2)$. Similar criteria are also derived for the subspectra $\Sigma _\alpha$.

4.1. Equations on the half line $\mathbb{I} = \mathbb{Z}^+_1$. Here, diagonal significance holds:

**Theorem 4.1.** Block-triangular eqns. (B) satisfy $\Sigma ^+(A) = \Sigma ^+(A^1) \cup \Sigma ^+(A^2)$.

**Proof.** We represent points $x \in \mathbb{K}^d$ as pairs $(x^1, x^2)$ with the components $x^i \in \mathbb{K}^{d_i}$, $i = 1, 2$, and introduce the unilateral shift operator $T_A \in L(\ell^2)$ as

$$(T_A \phi)_k = \begin{cases} 0, & k = \kappa, \\
A_{k-1} \phi_{k-1}, & k > \kappa \end{cases} \quad (T_A^* \phi)_k = \begin{cases} 0, & k = \kappa, \\
A_{k-1}^* \phi_{k-1}, & k > \kappa \end{cases}$$

With the bounded projection $P \in L(\ell^2)$, $(P \phi)_k := (0_{d_1})$ and the closed subspaces $X := R(P), Y := N(P)$ of $\ell^2$ one obtains the direct sum $\ell^2 = X \oplus Y$ and

$$(T_A P \phi)_k = \begin{cases} 0, & k = \kappa, \\
\left( A_{k-1}^* \phi_{k-1} \right) & k > \kappa, \quad (T_A (\text{id}_{\ell^2} - P) \phi)_k = \begin{cases} 0, & k = \kappa, \\
\left( C_{k-1} \phi_{k-1} \right) & k > \kappa. \end{cases}$$

Furthermore, $T_A \in L(\ell^2)$ can be represented as upper-triangular matrix operator $T_A = \begin{pmatrix} T_{A^1} & T_C \\ 0 & T_{A^2} \end{pmatrix} \in L(X \oplus Y)$ with the unilateral shifts $T_{A^1} \in L(X), T_{A^2} \in L(Y)$,

$$(T_{A^1} \phi)_k := \begin{cases} 0, & k = \kappa, \\
A_{k-1} \phi_{k-1}, & k > \kappa \end{cases} \quad (T_{A^2} \phi)_k := \begin{cases} 0, & k = \kappa, \\
A_{k-1}^2 \phi_{k-1}, & k > \kappa \end{cases}$$

and $T_C \in L(Y, X), (T_C \phi)_k := \begin{cases} 0, & k = \kappa, \\
C_{k-1} \phi_{k-1}, & k > \kappa \end{cases}$ as blocks. Due to Prop. B.2(c) the operator $T_{A^2}$ has SVEP and [15, Thm. 2.3] implies $\sigma _F (T_A) = \sigma _F (T_{A^1}) \cup \sigma _F (T_{A^2})$. With (2.11) in mind this yields the claim. □
4.2. Equations on the whole line $I = \mathbb{Z}$. On the whole integer axis $I = \mathbb{Z}$ the statement of Thm. 4.1 is in general false and additional assumptions are required to obtain diagonal significance.

In the present context, diagonal significance of a block-triangular equation $(B)$ w.r.t. a spectrum $\Sigma_\alpha$, $\alpha \in \{a, F, F_0, s, \pi\}$, means that

$$\Sigma_\alpha(A) = \Sigma_\alpha(A^1) \cup \Sigma_\alpha(A^2)$$

(\textit{DS}_\alpha) holds. Indeed, the Ex. 2.7(2) shows that the dichotomy spectrum $\Sigma(A)$ of a block-triangular eqn. $(B)$ can be strictly smaller than the union $\Sigma(A^1) \cup \Sigma(A^2)$. However, this shrinking has no effect on the stability radius (see Cor. 4.4).

There are two approaches to determine subsets of $\Sigma(A)$. The first one is based on well-known relations between the half line spectra and the spectra on $\mathbb{Z}$:

**Proposition 4.2.** Keep $\alpha \in \{a, F, F_0\}$ fixed. Block-triangular eqns. $(B)$ satisfy

$$\Sigma_F(A) = \Sigma^+(A^1) \cup \Sigma^+(A^2) \cup \Sigma^-(A) \subseteq \Sigma_\alpha(A) \subseteq \Sigma(A)$$

and under (2.1) one can replace $\Sigma^-(A)$ by $\Sigma^-(A^1) \cup \Sigma^-(A^2)$.

**Proof.** Thanks to [37, Cor. 4.30] one has $\Sigma_F(A) = \Sigma^+(A) \cup \Sigma^-(A) \subseteq \Sigma_\alpha(A) \subseteq \Sigma(A)$, while the above Thm. 4.1 implies $\Sigma^+(A) = \Sigma^+(A^1) \cup \Sigma^+(A^2)$. Under the assumption (2.1), with the aid of [38, Prop. 2.1] one can also show $\Sigma^-(A) = \Sigma^-(A^1) \cup \Sigma^-(A^2)$ and the claim follows.

Concerning a second method to determine one set inclusion in $(\textit{DS}_\alpha)$ and a subset of $\Sigma_\alpha(A)$, we remind the reader that the symmetric difference of sets $M_1, M_2$ is

$$M_1 \triangle M_2 := (M_1 \cup M_2) \setminus (M_1 \cap M_2)$$

and contains all elements which are either in $M_1$ or in $M_2$. The intersection of sets distributes over the symmetric difference, i.e. for arbitrary sets $M$ one has

$$(M_1 \triangle M_2) \cap M = (M_1 \cap M) \triangle (M_2 \cap M).$$

(4.1)

**Theorem 4.3.** Keep $\alpha \in \{a, F, F_0\}$ fixed. Block-triangular eqns. $(B)$ satisfy:

\begin{itemize}
  \item[(a)] $\Sigma_\alpha(A^1) \triangle \Sigma_\alpha(A^2) \subseteq \Sigma_\alpha(A) \subseteq \Sigma_\alpha(A^1) \cup \Sigma_\alpha(A^2)$.
  \item[(b)] If $\Sigma_\alpha(A^1) \cap \Sigma_\alpha(A^2)$ has no interior points, then $(\textit{DS}_\alpha)$ holds.
\end{itemize}

The following construction has prototype character for our investigations and closely resembles the proof of Thm. 4.1.

**Proof.** Let us represent $x \in K^d$ as pairs $(x^1, x^2)$ with the components $x^i \in K^d$ for $i = 1, 2$. First, this allows us to introduce a bilateral shift $T_A \in L(\ell^2)$,

$$(T_A\phi)_k = A_{k-1}\phi_{k-1} = \left(\begin{array}{c}
A_{k-1}^1\phi_{k-1}^1 + C_{k-1}^1\phi_{k-1}^2 \\
A_{k-1}^2\phi_{k-1}^2
\end{array}\right)$$

for all $k \in \mathbb{Z}$, and second, $P \in L(\ell^2)$, $(P\phi)_k := (\phi^1_k)$ defines a projection. Therefore, $X := R(P)$, $Y := N(P)$ are closed subspaces of $\ell^2 = X \oplus Y$ and it holds

$$(T_A P\phi)_k = \left(\begin{array}{c}
A_{k-1}^1\phi_{k-1}^1 \\
0
\end{array}\right), \quad (TA_{k-1} - P\phi)_k = \left(\begin{array}{c}
C_{k-1}^1\phi_{k-1}^2 \\
0
\end{array}\right)$$

for all $k \in \mathbb{Z}$.

Moreover, $T_A$ can be written as $T_A = \left(\begin{array}{c@{,}c}
T_A^1 & T_C^1 \\
0 & T_A^2
\end{array}\right) \in L(X \oplus Y)$ with

$$(T_A^1\phi)_k := A_{k-1}^1\phi_{k-1}, \quad (T_A^2\phi)_k := A_{k-1}^2\phi_{k-1}, \quad (T_C\phi)_k := C_{k-1}\phi_{k-1}.$$
Theorem 4.6 (diagonal significance for $\Sigma$)

Proof. (a) It is $\sigma_\alpha(T_{A^1}) \setminus \sigma_\alpha(T_{A^2}) \subseteq \sigma_\alpha(T_A) \subseteq \sigma(T_{A^1}) \cup \sigma(T_{A^2})$ for $\alpha \in \{a, F, F_0\}$ due to [18, Cor. 4 for $a = a$, [46, proof of Thm. 3.1 for $a = F$] and [30, (6.1) for $a = F_0$]. The first claimed inclusion results from (4.1), if we set $M_i := \sigma(T_{A^i})$, $M := \mathbb{R}^+$ and use $\Sigma_\alpha(A^i) = \sigma_\alpha(T_{A^i}) \cap \mathbb{R}^+$, $i = 1, 2$, as well as $\Sigma_\alpha(A) = \sigma_\alpha(T_A) \cap \mathbb{R}^+$ (cf. (2.12)). The second claimed inclusion anw follows using (2.12).

(b) Thanks to $\sigma(T_{A^1}) \cap \sigma(T_{A^2}) = \{\lambda \in \mathbb{C} : |\lambda| \in \Sigma(A^1) \cap \Sigma(A^2)\}$ and our assumption on interior points the intersection $\sigma(T_{A^1}) \cap \sigma(T_{A^2}) \subseteq \mathbb{C}$ is a finite union of circles centered around $0$ (or $\emptyset$) and has thus no interior points. Then [18, Cor. 8] shows $\sigma(T_A) = \sigma(T_{A^1}) \cup \sigma(T_{A^2})$ and (2.12) yields the claim for $a = \alpha$. In case $a \in \{F, F_0\}$ one proceeds accordingly with [46, Cor. 3.2] resp. [30, Cor. 7].

Rather than its whole line dichotomy spectrum $\Sigma(A)$, the stability radius of (B) turns out to be fully determined by the diagonal blocks:

Corollary 4.4. $\max \Sigma(A) = \max \{\max \Sigma(A^1), \max \Sigma(A^2)\}$

Proof. The angular symmetry of $\sigma(T_A)$ and [22, Prop. 4] guarantee

$$\max \Sigma(A) (2.12) r(T_A) = \frac{2}{\max_i r(T_{A^i})} = \frac{2}{\max_i \max \Sigma(A^i)}$$

and this implies the claim.

Let us continue with general spectral inclusions:

Proposition 4.5. Block-triangular difference eqns. (B) satisfy:

(a) $\Sigma_\alpha(A) \subseteq \Sigma_\alpha(A^1) \cup \Sigma_\alpha(A^2)$ for all $a \in \{\pi, s\}$
(b) $\Sigma(A^2) \subseteq \Sigma_s(A^1) \cup \Sigma(A)$
(c) $\Sigma_\pi(A^1) \cup \Sigma_s(A^1) \subseteq \Sigma(A)$

Combined with Thm. 4.3(a) the inclusion (c) implies $(DS_a)$, provided the identity $\Sigma_\pi(A^1) \cup \Sigma_s(A^2) = \Sigma(A^1) \cup \Sigma(A^2)$ holds.

Proof. (a) follows as in the proof of Thm. 4.3 using [17, Prop. 1.1]. Concerning (b) let us apply [17, Cor. 2.2] and [22, Proof of Prop. 4] yields assertion (c).

For the following it is advisable to introduce the defect set

$$\mathcal{D}_\alpha(A) := (\Sigma_\alpha(A^1) \cup \Sigma_\alpha(A^2)) \setminus \Sigma_\alpha(A) \quad \text{for all } a \in \{a, F, F_0, \pi, s\}$$

of a block-triangular eqn. (B). By Thm. 4.3(a) and Prop. 4.5(a) one observes that diagonal significance $(DS_a)$ precisely holds for $\mathcal{D}_\alpha(A) = \emptyset$.

Theorem 4.6 (diagonal significance for $\Sigma$). One has

$$\Sigma(A) \cup (\Sigma(A^1) \setminus \Sigma_\pi(A^1) \cap \Sigma(A^2) \setminus \Sigma_s(A^2)) = \Sigma(A^1) \cup \Sigma(A^2).$$

We immediately locate the defect set as $\mathcal{D}_a(A) \subseteq \Sigma(A^1) \setminus \Sigma_\pi(A^1) \cap \Sigma(A^2) \setminus \Sigma_s(A^2)$.

Proof. With the shifts $T_A, T_{A^1}$ and $T_{A^2}$ defined in the proof of Thm. 4.3, we obtain

$$\sigma(T_A) \cup (\sigma(T_{A^1}) \setminus \sigma(T_{A^1}) \cap \sigma(T_{A^2})) = \sigma(T_{A^1}) \cup \sigma(T_{A^2})$$

from [45, (7)]. Because the intersection of both sides in this relation with $\mathbb{R}^+$ distributes over the set operations involved, the claim results with (2.12).

Corollary 4.7. If $(\Sigma_\pi(A^1) \cap \Sigma_\pi(A^2)) \setminus (\Sigma_\pi(A^1) \cap \Sigma_\pi(A^2))$ possesses no interior point, then $(DS_a)$ holds.

Proof. By means of (A.1) this follows as above using [45, Cor. 3.2] and (2.12).
A problem with the above criteria for diagonal significance is that certain dichotomy spectra, as well as their subspectra, have to be known in advance. In the following, we will thus obtain sufficient conditions on the basis of Lyapunov filtrations alone. As a further advantage, these criteria also provide diagonal significance of subspectra. To be precise, let us suppose that the diagonal systems \( (\Delta A) \) of subspectra. To be precise, let us suppose that the diagonal systems \( (\Delta A_i) \) and their adjoint eqns. \( (\Delta^* A_i) \) have Lyapunov spectra and filtrations

\[
\{\lambda_1^*, \ldots, \lambda_{n_i}^*\}, \quad W_0^i \subset \ldots \subset W_{n_i}^i = \mathbb{K}^{d_i} \quad \text{with} \quad n_i \leq d_i,
\]

\[
\{\mu_1^*, \ldots, \mu_{n_i}^*\}, \quad V_0^i \subset \ldots \subset V_{n_i}^i = \mathbb{K}^{d_i} \quad \text{with} \quad n_i^* \leq d_i
\]

for \( i \in \{1, 2\} \). Given this, one is able to formulate the following conditions on the Lyapunov spectra of the diagonal systems

\[
1 \leq \mu_1^* \limsup_{k \to \infty} \frac{1}{k} \left| \Phi_{A_i}^*(-k, \kappa)x \right| \quad \text{for all} \quad x \in V_j^i \setminus V_{j-1}^i, \quad 1 \leq j \leq n_1^*, \quad (S_{A_1}^*)
\]

\[
1 \leq \lambda_2^* \limsup_{k \to \infty} \frac{1}{k} \left| \Phi_{A_2}(-k, \kappa)x \right| \quad \text{for all} \quad x \in W_j^2 \setminus W_{j-1}^2, \quad 1 \leq j \leq n_2, \quad (S_{A_2}^*)
\]

Note that we will illustrate these conditions in the Exs. 4.10 and 4.14 below.

**Theorem 4.8** (diagonal significance for \( \Sigma_\pi \) and \( \Sigma_s \)). If a block-triangular difference eqn. \( (B) \) fulfills

(a) \( (S_{A_1}^*) \), then \( \Sigma_\pi(A) = \Sigma_\alpha(A^1) \cup \Sigma_\pi(A^2) \) for \( \alpha \in \{a, \pi\} \).

(b) \( (S_{A_2}^*) \), then \( \Sigma_s(A) = \Sigma_s(A^1) \cup \Sigma_s(A^2) \) for \( \alpha \in \{a, s\} \).

**Proof.** We borrow our notation from the proof of Thm. 4.3. Using (B.3) we know that the adjoint shift \( T_{A_i}^* \) has the SVEP if and only if

\[
1 \leq \chi_{A_i}^*(x) \limsup_{k \to \infty} \frac{1}{k} \left| \Phi_{A_i}^*(-k, \kappa)x \right| \quad \text{for all} \quad x \in \mathbb{K}^d \setminus \{0\}
\]

holds. Let us first establish that this inequality is equivalent to \( (S_{A_1}^*) \):

\( \Rightarrow \) The definition of the Lyapunov filtration for \( (\Delta A_i^*) \) guarantees \( \chi_{A_i}^*(x) = \mu_1^* \) for \( x \in V_j^1 \setminus V_{j-1}^1, \quad 1 \leq j \leq n_1^* \) and consequently \( (S_{A_1}^*) \) holds.

\( \Leftarrow \) Conversely, assume \( (S_{A_i}^*) \) is satisfied and choose \( x \in \mathbb{K}^d \setminus \{0\} \) arbitrarily. There exists a maximal \( 1 \leq j \leq n_1^* \) such that \( x \in V_j^1 \setminus V_{j-1}^1 \) and thus \( \chi_{A_i}^*(x) = \mu_1^* \). Hence,

\[
1 \leq \mu_1^* \limsup_{k \to \infty} \frac{1}{k} \left| \Phi_{A_i}^*(-k, \kappa)x \right| = \chi_{A_i}^*(x) \limsup_{k \to \infty} \frac{1}{k} \left| \Phi_{A_i}(-k, \kappa)x \right|
\]

and since \( x \neq 0 \) was arbitrary, \( T_{A_i}^* \) has the SVEP.

Analogously one uses (B.2) to show that \( (S_{A_2}^*) \) is equivalent to the SVEP of \( T_{A_2} \).

(a) Since \( T_{A_i}^* \) has the SVEP, by means of [45, Cor. 3.13] this implies \( (DS_s) \) and the claim results from (2.12) and Lemma A.2(b).

(b) Here, \( T_{A_2} \) has the SVEP. Then [45, Prop. 3.2] implies \( (DS_s) \) and the assertion follows with (2.12) and Lemma A.2(a). \( \square \)

**Theorem 4.9** (diagonal significance for \( \Sigma \) and \( \Sigma_F \)). Keep \( \alpha \in \{a, F\} \) fixed. Then \( (DS_\alpha) \) holds, if a block-triangular difference eqn. \( (B) \) fulfills one of the assumptions

(i) \( \Sigma_\alpha(A^1) \subseteq \Sigma_\alpha(A) \),
(ii) \( \Sigma_\alpha(A^2) \subseteq \Sigma_\alpha(A) \),
(iii) \( (S_{A_1}^*) \),
(iv) \( (S_{A_2}^*) \).
Proposition 4.12. Let \( i \in \{1, 2\} \). If \( \Sigma(A^i) \) is discrete, then \((S_{A^i})\) and \((S_{A^i}^*)\) hold.

Proof. Since \( \Sigma(A^i), i \in \{1, 2\} \), is discrete, due to (2.12) the spectrum \( \sigma(T_{A^i}) \) consists of finitely many concentric circles. Then the inclusion \( \partial \sigma(T_{A^i}) \subseteq \sigma_\alpha(T_{A^i}) \subset \sigma(T_{A^i}) \) implies \( \sigma_\alpha(T_{A^i}) = \partial \sigma(T_{A^i}) \) for \( \alpha \in \{\pi, s\} \) and Lemma A.3 guarantees that both weighted shifts \( T_{A^1} \) and \( T_{A^2}^* \) have the SVEP. Thanks to the characterization [11, Thm. 2.1 resp. Cor. 2.2] one shows as in the proof of Thm. 4.8 that this is equivalent to the estimates \((S_{A^i})\) resp. \((S_{A^i}^*)\).
Remark 4.13 (the classes $P_p(K^d)$ and $P_p^*(K^d)$). In [39] we consider linear difference eqns. $(\Delta_A)$ with coefficient sequences in the classes

\[
P_p(K^d) := \left\{ A \in L^\infty(K^d) : \Phi(k + 2p, k)^* \Phi(k + 2p, k) - 2r \Phi(k + p, k)^* \cdot \Phi(k + p, k)^* \text{ is positively-semidefinite for all } k \in \mathbb{Z}, r > 0 \right\},
\]

\[
P_p^*(K^d) := \left\{ A \in L^\infty(K^d) : \Phi(k + 2p, k)^* \Phi(k + 2p, k) - 2r \Phi(k + 2p, k)^* \cdot \Phi(k + 2p, k)^* \text{ is positively-semidefinite for all } k \in \mathbb{Z}, r > 0 \right\},
\]

which are related to the above assumptions. Indeed, by means of [39, Prop. A.3] one establishes the implications

\[A^i \in P_p(K^d) \Rightarrow (S_{A^i}), \quad A^i \in P_p^*(K^d) \Rightarrow (S_{A^i}^*) \text{ for all } i \in \{1, 2\}.
\]

Example 4.14. We revisit the planar upper-triangular difference eqn. $(\Delta_A)$ from Ex. 2.7. The dichotomy spectra for its diagonal sequences are given in Ex. 2.6(3) and moreover [37, Ex. 5.3] yields the surjectivity dichotomy spectra

\[
\Sigma_s(a) = \begin{cases} \{\alpha_-, \alpha_+\}, \quad \alpha_+ \leq \alpha_- & , \\ \{\alpha_+, \alpha_-\}, \quad \alpha_- \leq \alpha_+ & , \\ \{\beta_-, \beta_+\}, \quad \beta_+ \leq \beta_- & , \\ \{\beta_-, \beta_+\}, \quad \beta_- \leq \beta_+ & .
\end{cases}
\]

(1) Since the Fredholm spectra $\Sigma_F(a^i)$ are discrete, one obtains from Thm. 4.3(b) that $(DS_F)$ holds with $\Sigma_F(A) = \{\alpha_-, \alpha_+, \beta_-, \beta_+\}$ and that the estimates

\[\max \{\alpha_-, \alpha_+\} \leq \min \{\beta_-, \beta_+\} \quad \text{or} \quad \max \{\beta_-, \beta_+\} \leq \min \{\alpha_-, \alpha_+\}
\]

are sufficient for $(DS_A)$, $\alpha \in \{a, F_0\}$, to hold. The inequalities $\alpha_+ \leq \alpha_-$ and $\beta_- \leq \beta_+$ imply $\Sigma_s(a^1) \cup \Sigma_s(a^2) = [\alpha_+, \alpha_-] \cup [\beta_-, \beta_+] = \Sigma(a^1) \cup \Sigma(a^2)$ and therefore Prop. 4.5 guarantees $(DS_A)$. Note that this even holds under the weaker condition

\[\alpha_+ \leq \alpha_- \quad \text{or} \quad \beta_- \leq \beta_+,
\]

because Cor. 4.7 applies for discrete subspectra $\Sigma_s(a^1)$, $\Sigma_s(a^2)$, i.e. (4.2). (2) By means of Lyapunov exponent-like conditions we obtain the following criteria for diagonal significance. Analogously to the above Ex. 4.10 the condition

- $(S_{a^1}^*)$ is equivalent to $\alpha_+ \leq \alpha_-$ and so Thm. 4.8(a) leads to $(DS_{a^1})$
- $(S_{a^2}^*)$ is equivalent to $\beta_- \leq \beta_+$ and Thm. 4.8(b) guarantees $(DS_{a^2})$

Hence, under one of the conditions (4.2) our Thm. 4.9 implies $(DS_A)$, $\alpha \in \{a, F\}$. Our above Thm. 4.11 yields $(DS_F_{0})$, provided both (4.2) and the dual condition $\alpha_- \leq \alpha_+$ or $\beta_+ \leq \beta_-$ (cf. again Ex. 4.10) hold.

4.3. Conditions on $C$. In order to provide sufficient conditions for diagonal significance on basis of the sequence $C = (C_k)_{k \in \mathbb{Z}}$ alone, we define the linear spaces

\[
\mathcal{N}(A) := \left\{ X \in \ell^\infty(K^{d_1 \times d_2}) : X_{k+1}A_k^2 \equiv A_k^2 X_{k+1} \text{ on } \mathbb{Z} \right\},
\]

\[
\mathcal{R}(A) := \left\{ Y \in \ell^\infty(K^{d_1 \times d_2}) \mid \exists X \in \ell^\infty(K^{d_1 \times d_2}) : Y_k \equiv A_k^2 X_{k+1} - X_k A_k^2 \text{ on } \mathbb{Z} \right\}
\]

and immediately obtain

**Theorem 4.15.** If $C \in \mathcal{N}(A) + \mathcal{R}(A)$ is satisfied, then $(DS_A)$ holds.

In the block-diagonal situation $C = 0$ this ensures $\Sigma(A) = \Sigma(A^1) \cup \Sigma(A^2)$. 
Proof. We abbreviate $\ell^2_1 := \ell^2(\mathbb{K}^d_1)$ and for shifts $T_{A^i} \in L(\ell^2_i), i = 1, 2$, the
generalized derivation $\Delta : L(\ell^2_i, \ell^2_j) \to L(\ell^2_j, \ell^2_i), \Delta \Xi := T_{A^i} \Xi - \Xi T_{A^j}$ is bounded. If $C$
denotes a bounded sequence $(C_k)_{k \in \mathbb{Z}}$ of matrices $C_k \in \mathbb{K}^{d_1 \times d_2}$, then the operators

$$ (T_C\phi)_k := C_{k-1}\phi_{k-1}, \quad (M_C\phi)_k := C_k\phi_k \quad \text{for all } k \in \mathbb{Z} $$

fulfill $T_C, M_C \in L(\ell^2_1, \ell^2_1)$. First, provided $C \in \mathcal{N}(A)$ we obtain

$$ (\Delta T_C\phi)_k = (A^1_{k-1}C_{k-2} - C_{k-1}A^2_{k-2})\phi_{k-2} = 0 \quad \text{for all } k \in \mathbb{Z}, \phi \in \ell^2_1 $$

and thus $\Delta T_C = 0$, i.e. $T_C$ is in the kernel of $\Delta$. Second, in case $C \in \mathcal{R}(A)$ with $C_k = A^1_kX_k - X_{k+1}A^2_k$ for all $k \in \mathbb{Z}$ and some $X \in \ell^\infty(\mathbb{K}^{d_1 \times d_2})$ it is

$$ (\Delta M_X\phi)_k = (A^1_{k-1}X_{k-1} - X_kA^2_{k-1})\phi_{k-1} = C_{k-1}\phi_{k-1} \quad \text{for all } k \in \mathbb{Z}. $$

This yields $\Delta M_X = T_C$ and hence $T_C$ is in the range of $\Delta$. By linearity we conclude that for elements $C \in \mathcal{N}(A) + \mathcal{R}(A)$ the corresponding shifts $T_C$ are contained in the sum $N(\Delta) + R(\Delta)$. Consequently, [7, Thm. 1] implies $\sigma(T_A) = \sigma(T_{A^1}) \cup \sigma(T_{A^2})$ and the claim follows from (2.12).

Corollary 4.16. If $\Sigma(A^1) \cap \Sigma(A^2) = \emptyset$, then $\mathcal{N}(A) = \{0\}$. 

**Proof.** We use the terminology from the above proof. By [29, p. 256, Thm. 3.4.1] it is $\sigma(\Delta) = \sigma(T_{A^1}) - \sigma(T_{A^2})$. Thus, thanks to (2.12) we obtain that $\Delta \in L(\ell^2_1, \ell^2_1)$ is invertible and particularly $N(\Delta) = \{0\}$. The claim follows, because $C \in \mathcal{N}(A)$ implies $T_C \in N(\Delta) = \{0\}$ and therefore $C_k = 0, k \in \mathbb{Z}$. 

We finally illuminate the close relation between the assumption of Thm. 4.15 and exponential dichotomies resp. trichotomies as discussed in [35]:

**Remark 4.17.** Let $d_1 = d_2$ and suppose that all $A^2_k \in \mathbb{K}^{d_2 \times d_2}$ are invertible with

$$ \sup_{k \in \mathbb{Z}} |(A^2_k)^{-1}| < \infty. $$

(1) The linear space $\mathcal{R}(A)$ consists of all matrix sequences $Y \in \mathcal{L}^\infty(\mathbb{K}^{d_1})$ such that the matrix difference eqn. $X_{k+1} = (A^1_kX_k - Y_k)(A^2_k)^{-1}$ has a bounded solution. This, in turn, holds provided the linearly-homogenous equation

$$ X_{k+1} = A^1_kX_k(A^2_k)^{-1} \quad (4.3) $$

has an exponential trichotomy on $\mathbb{Z}$ (cf. [35, Prop. 1]).

(2) The stronger assumption that (4.3) is even exponentially dichotomic on $\mathbb{Z}$ corresponds precisely to the case $\mathcal{N}(A) = \{0\}$.

4.4. **Spectral continuity.** While the dichotomy spectra interpreted as mappings $\Sigma_\alpha : \mathcal{L}^\infty(\mathbb{K}^d) \to K(\mathbb{R}), \Sigma_\alpha(A) := \Sigma_\alpha(A) \cup \{0\}$ are only upper-semicontinuous in general (cf. [36, Cor. 4] for $I = \mathbb{Z}$ and [37, Cor. 3.24] on the half line $I = \mathbb{Z}^+_n$), let us next provide continuity criteria for triangular coefficient sequences.

In this context, diagonal significance means that points of continuity of the dichotomy spectra $\Sigma_\alpha(A^i)$ for the block subsystems guarantee that also the full system (B) has a corresponding spectrum being continuous at $A$.

Our analysis is fundamentally follows from the geometrically evident

**Proposition 4.18** (see [39, Prop. 5.3]). *Keep $\alpha \in \{a, F_0\}$ fixed. If $\sigma_\alpha : L(\ell^2) \to K(\mathbb{C})$ is continuous at $T_A$, then $\Sigma_\alpha : \mathcal{L}^\infty(\mathbb{K}^d) \to K(\mathbb{R})$ is continuous at $A$. *
The remaining section is based on the assumption that \( A, A^1 \) and \( A^2 \) fulfill (2.1). Then, in our preparatory paper [39] we have shown that

\[
\Sigma(A) = \Sigma_{F_0}(A) \setminus \Sigma_F(A)
\]

is a sufficient condition for \( \Sigma \) to be continuous at \((\Delta A)\), while

\[
\Sigma_{F_0}(A) = \Sigma_{F_0}(A) \setminus \Sigma_F(A)
\]

guarantees the corresponding continuity of the Weyl spectrum \( \Sigma_{F_0} \).

**Theorem 4.19.** Keep \( \alpha \in \{a, F_0\} \) fixed. If a block-triangular eqn. \((B)\) satisfies

(i) both the continuity conditions \((C^a_{A^1})\) and \((C^a_{A^2})\),

(ii) \(\Sigma_\alpha(A^1) \cap \Sigma_\alpha(A^2) = \emptyset\),

then \( \Sigma_\alpha \) is continuous at \( A \).

**Proof.** Let \( \alpha \in \{a, F_0\} \) be fixed. On the one hand, condition (i) ensures that the shift \( T_{A^i} \) is a point of continuity for \( \sigma \) (cf. [39, Proof of Thm. 5.4]) resp. \( \sigma_{F_0} \) (see [39, Proof of Cor. 5.5]) with \( i = 1, 2 \). On the other hand, assumption (ii) guarantees \( \sigma_\alpha(T_{A^1}) \cap \sigma(T_{A^2}) = \emptyset \). From [42, Thm. 7] we get that \( \sigma_\alpha \) is continuous at \( T_A \). Then Prop. 4.18 implies that also the dichotomy spectrum \( \Sigma_\alpha \) is continuous at \((\Delta A)\).

**Theorem 4.20.** If a block-triangular difference eqn. \((B)\) satisfies

(i) both the continuity conditions \((C^a_{A^1})\) and \((C^a_{A^2})\),

(ii) the estimates \((S^*_{A^1})\),

then \( \Sigma \) is continuous at \( A \).

**Proof.** In the previous proof of Thm. 4.19 we justified that \( \sigma \) is continuous at \( T_{A^1} \) and \( T_{A^2} \). Moreover, in the proof of Thm. 4.8 it was shown that the adjoint \( T_{A^1}^* \) has the SVEP. If we combine Prop. 4.18 with [42, Cor. 9], then the claim follows.

Let us eventually discuss the upper-triangular difference equation from Ex. 2.7(2) having a discontinuous dichotomy spectrum in the light of Thm. 4.19 and 4.20:

**Example 4.21.** Given some real \( \delta > 1 \) we consider \((\Delta A)\) as defined in Ex. 2.7 with parameters satisfying the conditions (2.14). Then Ex. 2.6(3) yields the spectra

\[
\Sigma(a^i) = \Sigma_{F_0}(a^i) = [\alpha_-, \alpha_+], \quad \Sigma_F(a^i) = \{\alpha_-, \alpha_+\}
\]

and therefore the continuity conditions \((C^a_{A^i})\) hold for \( \alpha \in \{a, F\} \) and \( i = 1, 2 \). However, the assumption (ii) in both Thms. 4.19 and 4.20 are violated.

5. **Triangular equations.** This section is concerned with linear eqns. \((\Delta A)\) whose coefficient matrices \( A_k \) are triangular, where w.l.o.g. we restrict to the upper-triangular situation. Hence, they are of the form

\[
x_{k+1} = B_k x_k, \quad B_k := \begin{pmatrix}
    b^1_k & b^{1,2}_k & b^{1,3}_k & \cdots & b^{1,d}_k \\
    b^2_k & b^{2,3}_k & b^{2,4}_k & \cdots & b^{2,d}_k \\
    & \ddots & \ddots & \ddots & \vdots \\
    & & \ddots & \ddots & b^{d-1,d}_k \\
    & & & b^d_k
\end{pmatrix} (T)
\]

with bounded diagonal sequences \((b^i_k)_{k \in \mathbb{N}}\) and bounded super-diagonal sequences \((b^{i,j}_k)_{k \in \mathbb{N}}\) for indices \( 1 \leq i < j \leq d \) in \( \mathbb{K} \).

As already pointed out, the importance of triangular equations \((T)\) is due to the close connection between the dichotomy spectrum and the Bohr exponents of
their diagonal sequences. For the reader's convenience we therefore illustrate how an arbitrary linear equation \((\Delta A)\) can be brought into triangular form:

**Algorithm 5.1.** Choose an initial time \(\kappa \in \mathbb{I}\), e.g. \(\kappa := \min \mathbb{I}\) when the discrete interval \(\mathbb{I}\) is bounded below. In case \(\mathbb{K} = \mathbb{R}\):

1. Set \(T_\kappa := \text{id}_{\mathbb{R}^d}\) and determine the QR-decomposition
   \[ A_\kappa = T_{\kappa+1}B_\kappa \]
   of \(A_\kappa\) with an orthogonal \(T_{\kappa+1} \in L(\mathbb{R}^d)\) and an upper-triangular \(B_\kappa \in L(\mathbb{R}^d)\)
   (see [21, p. 89, Thm. 2.1.14])

2. For \(k > \kappa\) recursively compute the corresponding QR-decompositions
   \[ A_kT_k = T_{k+1}B_k \]

3. For \(k \leq \kappa\) determine the RQ-factorizations (cf. [21, p. 91])
   \[ T_k^{-1}A_k^{-1} = B_k^{-1}T_k^{-1} \]

In case \(\mathbb{K} = \mathbb{C}\) the same procedure applies, where \(T_k \in L(\mathbb{C}^d)\) is a sequence of unitary matrices and the transposes \(T_k^T\) have to replaced by the conjugate transposes \(T_k^*\).

By construction, the linear equation \((\Delta A)\) is kinematically similar to the upper-triangular equation \((T)\) by means of the Lyapunov transformation \((T_k)_{k \in \mathbb{I}}\). Because of [37, Rem. 4.33] this has no effect on the dichotomy spectra \(\Sigma_\alpha\).

On the half line \(\mathbb{I} = \mathbb{Z}_+\) it is known that \(\Sigma^+(B)\) is simply the union of spectra for the corresponding diagonal eqns. \((\Delta b)\) (see [37, Cor. 3.25]). As again demonstrated in Ex. 2.7(2), the situation for \(\mathbb{I} = \mathbb{Z}\) is more complicated.

So let us consider the whole line case \(\mathbb{I} = \mathbb{Z}\) from now on. In Thm. 5.2 we first provide a constructive tool to embed the dichotomy spectrum \(\Sigma(B)\) between a sub- and a superset. The “empty interior” condition from Thm. 4.3(b) extends to triangular equations (see Cor. 5.6), but we are able to complement it as follows: Rather than the Bohl exponents of the diagonal sequences, it suffices to assume appropriate inequalities for their Lyapunov exponents, in order to obtain diagonal significance (see Thm. 5.7).

**Theorem 5.2.** The dichotomy spectrum \(\Sigma(B)\) of \((T)\) satisfies

\[ \Sigma(b^1) \triangle \Sigma_1 \subseteq \Sigma(B) \subseteq \Sigma(b^1) \cup \Sigma_1, \]

with a set \(\Sigma_1\) given by

\[ \Sigma_1 := \begin{cases} \emptyset, & d = 1, \\ \Sigma(b^2), & d = 2 \end{cases} \]

and for \(d > 2\) allowing the recursive construction

\[ \Sigma(b^{d-1}) \triangle \Sigma(b^d) \subseteq \Sigma_{d-2} \subseteq \Sigma(b^{d-1}) \cup \Sigma(b^d), \]

\[ \Sigma(b^j) \triangle \Sigma_j \subseteq \Sigma_{j-1} \subseteq \Sigma(b^j) \cup \Sigma_j \text{ for all } 2 \leq j < d - 1. \]

**Remark 5.3.** (1) The above procedure shows \(\Sigma(B) \subseteq \bigcup_{i=1}^d \Sigma(b^i)\).

(2) Under the assumption \(\inf_{k \in \mathbb{I}} \|b^i_k\| > 0\) for all \(1 \leq i \leq d\) one can substitute the sets \(\Sigma(b^i)\) by the closed intervals \([\beta(b^i), \bar{\beta}(b^i)]\) (cf. Prop. 2.4). Moreover, the symmetric differences in Thm. 5.2 can be replaced by their closures.
Proof. For $d = 1, 2$ the claim follows directly from Thm. 4.3(a). For $d > 2$ write

$$B_k := \begin{pmatrix} b_1^k & c_1^k \\ b_2^k & c_2^k \\ \vdots & \vdots \\ b_{d-1}^k & c_{d-1}^k \\ b_d^k & c_d^k \end{pmatrix}$$

for all $k \in \mathbb{Z}$

with rows $c_k^j \in \mathbb{K}^{1 \times (d-j)}$ and diagonal sequences $b^j$ yielding

$$B_k^{j-1} = \begin{pmatrix} b_k^j & c_k^j \\ b_k^d & B_k^d \end{pmatrix}$$

for all $1 \leq j < d$. (5.1)

Here, the square matrix $B_k^j \in \mathbb{K}^{(d-j) \times (d-j)}$ is defined by simultaneously discarding the first $j \in [0, d-2]$ columns and rows of $B_k$; it is $B_k = B_k^0$. Setting $\Sigma_j := \Sigma(A^j)$ and applying Thm. 4.3(a) to (5.1) we deduce

$$\Sigma(b^j) \cap \Sigma_j \subseteq \Sigma_{j-1} \subseteq \Sigma(b^j) \cup \Sigma_j$$

for all $1 \leq j < d$,

particularly $\Sigma(b^j) \cap \Sigma_1 \subseteq \Sigma(B) \subseteq \Sigma(b^j) \cup \Sigma_1$ and $B_k^{d-1} = b_k^d$ yields our claim. \qed

The above Thm. 5.2 allows to circumscribe the dichotomy spectrum using exclusively the diagonal spectral intervals. As a concrete example we consider

Example 5.4. For a difference eqn. (T) in $\mathbb{R}^d$ with diagonal sequences, the spectra

$$\Sigma(b^1) = [1, 2], \quad \Sigma(b^2) = [1, 3], \quad \Sigma(b^3) = [3, 5], \quad \Sigma(b^4) = [4, 5]$$

and bounded super-diagonal entries, the following holds: In the terminology of Thm. 5.2 with $d = 4$ we obtain the inclusions

$$[3, 4] = \Sigma(b^4) \triangle \Sigma_2 \subseteq \Sigma_2 \subseteq \Sigma(b^3) \cup \Sigma(b^4) = [3, 5],$$

$$\Sigma(b^3) \triangle \Sigma_2 \subseteq \Sigma_1 \subseteq \Sigma(b^2) \cup \Sigma_2 = [1, 3] \cup [1, 5],$$

which guarantee $\Sigma(b^3) \cap \Sigma_2 = [3]$, consequently $[1, 4] \subseteq \Sigma(b^2) \Delta \Sigma_2$ and thus the inclusions $[1, 4] \subseteq \Sigma_1 \subseteq [1, 5]$. Now $\Sigma(b^1) = [1, 2] \subseteq \Sigma_1$ implies

$$[2, 4] \subseteq \Sigma_1 \setminus \Sigma(b^1) = \Sigma(b^1) \setminus \Sigma_1 \subseteq \Sigma(B) \subseteq [1, 5].$$

Yet, one obtains the stability radius of (T) from the diagonal sequences:

Corollary 5.5. $\max \Sigma(B) = \max_{j=1}^d \overline{\sigma}(b^j)$.\qed

Proof. With Cor. 4.4 the formula for $\max \Sigma(B)$ follows by induction over $j$.

In the following, we provide several sufficient criteria for diagonal significance of triangular systems (T), i.e. the fact that $\Sigma(B)$ is the union of the dichotomy spectra of the diagonal sequences. The corresponding results yield from our preparations for block-triangular equations (B) in Subsect. 4.2 by means of mathematical induction. We exemplify this in case of Thm. 4.3 and leave it to the interested reader to deduce counterparts to e.g. Thms. 4.8, 4.9 or 4.11:

Corollary 5.6 (exponential separation). If all the intersections $\Sigma(b^i) \cap \Sigma(b^j)$ for $i \neq j$ have empty interior, then $\Sigma(B) = \bigcup_{i=1}^d \Sigma(b^i)$.\qed
According to Ex. 2.7(2) one knows that Cor. 5.6 is wrong without the additional assumption on interior points.

**Proof.** Retaining the notation from the proof of Thm. 5.2, set \( \sigma_j := \Sigma(b_i) \). By assumption, the intersection \( \bigcap_{i=1}^d \sigma_i \) has no interior points and Thm. 4.3(b) implies \( \Sigma_{d-2} = \bigcup_{i=d-1}^d \sigma_i \). Our induction is based on the hypothesis \( \Sigma_j = \bigcup_{i=j+1}^d \sigma_i \) and for the induction step \( j \to j-1 \) we proceed as follows: Thanks to (5.1) it is

\[
\sigma_j \triangle \Sigma_j \subseteq \Sigma_{j-1} \subseteq \sigma_j \cup \Sigma_j,
\]

the induction hypothesis guarantees \( \sigma_j \cap \Sigma_j = \sigma_j \cap \bigcup_{i=j+1}^d \sigma_i = \bigcup_{i=j+1}^d (\sigma_i \cap \sigma_j) \), and if we invest our assumption, \( \sigma_j \cap \Sigma_j \) has no interior points. Hence, Thm. 4.3(b) yields \( \Sigma_{j-1} = \sigma_j \cup \Sigma_j = \bigcup_{i=j}^d \sigma_i \).

The next criteria for diagonal significance involve only Lyapunov exponents of the first resp. last \( d-1 \) diagonal sequences. With Ex. 2.6(5) one easily constructs diagonally significant equations having overlapping diagonal spectral intervals.

**Theorem 5.7.** If a triangular difference eqn. (\( T \)) fulfills one of the assumptions

(i) \( \underline{\lambda}_-(b_i) \leq \overline{\lambda}_+(b_i) \) for \( 1 < i \leq d \),  
(ii) \( \underline{\lambda}_+(b_i) \leq \overline{\lambda}_-(b_i) \) for \( 1 \leq i < d \),  

then

\[
\Sigma_\alpha(B) = \bigcup_{i=1}^d \Sigma_\alpha(b_i) \quad \text{for all } \alpha \in \{a, F\}.
\]

**Proof.** First of all, using Lemma B.5 we obtain from assumption (i) that every \( T_{b_i} \) has the SVEP for \( 1 < i \leq d \), while assumption (ii) guarantees the SVEP of \( T_{b_i} \) for all \( 1 \leq i < d \). Because mathematical induction on basis of [15, Thm. 2.3] implies the relation \( \sigma_\alpha(T_A) = \bigcup_{i=1}^d \sigma_\alpha(T_{b_i}) \), the claim results from (2.12).

**Corollary 5.8.** (a) If \( \underline{\lambda}_-(b_i) \leq \overline{\lambda}_+(b_i) \) holds for \( 1 \leq i \leq d \), then

\[
\Sigma(b_i) = \Sigma_\sigma(b_i) \quad \text{for all } 1 \leq i \leq d,
\]

\[
\Sigma(B) = \Sigma_\sigma(B) = \bigcup_{i=1}^d \Sigma_\sigma(b_i).
\]

(b) If \( \underline{\lambda}_+(b_i) \leq \overline{\lambda}_-(b_i) \) holds for \( 1 \leq i \leq d \), then

\[
\Sigma(b_i) = \Sigma_\sigma(b_i) \quad \text{for all } 1 \leq i \leq d,
\]

\[
\Sigma(B) = \Sigma_\sigma(B) = \bigcup_{i=1}^d \Sigma_\sigma(b_i).
\]

In both cases one has \( \Sigma_\alpha(B) = \bigcup_{i=1}^d \Sigma_\alpha(b_i) \) for \( \alpha \in \{F, F_0\} \).

**Proof.** (a) As in the proof of Thm. 5.7(a) one argues that every \( T_{b_i} \) has the SVEP and thus Lemma A.2(a) implies \( \Sigma_\sigma(b_i) = \sigma_\sigma(T_{b_i}) \cap \mathbb{R}^+ = \sigma(T_{b_i}) \cap \mathbb{R}^+ = \Sigma(b_i) \) for all \( 1 \leq i \leq d \) (cf. (2.12)). Moreover, an inductive argument based on [2, p. 62, Thm. 2.9] establishes the SVEP for \( T_A \) and once more Lemma A.2(a) with (2.12) implies \( \Sigma(B) = \Sigma_\sigma(B) \). Then (5.3) follows from (5.2).

(b) The assertions are “dual” to (a) and based on Lemma A.2(b).

Finally, given \( \alpha \in \{F, F_0\} \) the representation \( \Sigma_\alpha(B) = \bigcup_{i=1}^d \Sigma_\alpha(b_i) \) follows from [16, Prop. 3.5(ii) and (iii)] for (a) and from [16, Prop. 3.6(ii) and (iii)] for (b).
6. Application: A nonautonomous Leslie-Gower equation. In order to release this paper from its somehow theoretical flavor, let us consider the nonlinear planar difference equation

\[
\begin{align*}
x_{k+1} &= \frac{a_k x_k}{1 + x_k + b_k y_k}, \\
y_{k+1} &= \frac{c_k x_k}{1 + d_k x_k + y_k}.
\end{align*}
\] (6.1)

Concerning the relevance of (6.1) in population dynamics we refer to [12], where it is known as Leslie-Gower equation. The real nonnegative sequences \((a_k)_{k \in \mathbb{Z}}, (b_k)_{k \in \mathbb{Z}}, (c_k)_{k \in \mathbb{Z}}\) and \((d_k)_{k \in \mathbb{Z}}\) are assumed to be bounded.

In the extinction equilibrium \((0, 0)\) the variational equation of (6.1) is of diagonal form \((D)\) with

\[ A_k = \begin{pmatrix} a_k & 0 \\ 0 & c_k \end{pmatrix} \text{ for all } k \in \mathbb{Z} \]

and Thm. 3.1 yields the dichotomy spectra \(\Sigma_\alpha(A) = \Sigma_\alpha(a) \cup \Sigma_\alpha(c)\) for \(\alpha \in \{a, s, F\}\).

In particular, for sequences \(a, c\) satisfying (2.10) it is \(\Sigma(A) = [\beta(a), \beta(a)] \cup [\beta(c), \beta(c)]\) (see Prop. 2.4) and this set determines whether the origin is a (nonautonomous) sink, saddle of source.

More interesting is the nontrivial solution

\[ \left( \begin{array}{c} x_k^* \\ y_k^* \end{array} \right) = \left( \begin{array}{c} \sum_{i=-\infty}^{i=k} \Phi_{a, i, k} \\ 0 \end{array} \right) \]

of (6.1), which remains on the \(x\)-axis. Along this solution, the variational equation has the coefficients

\[ A_k = \begin{pmatrix} a_k & b_k x_k^* \\ 0 & c_k \end{pmatrix} \frac{(1 + x_k^*)}{1 + d_k x_k^*} \]

and we distinguish two cases:

(a) For a sequence \(a_k \equiv \lambda > 1\) one obtains \(x_k^* \equiv \lambda - 1\) and \(A_k = \begin{pmatrix} \frac{1}{\lambda - 1} & \frac{-b_k}{\lambda - 1} \\ 0 & \frac{c_k}{\lambda - 1} \end{pmatrix} \); hence one spectral interval is a singleton \(\{\frac{1}{\lambda - 1}\}\). Then Thm. 4.3(b) implies diagonal significance \(\Sigma(A) = \{\frac{1}{\lambda - 1}\} \cup \Sigma_\alpha(1 + \lambda)\) and the dependence of \(\Sigma(A)\) on \(\lambda\) is shown in Fig. 2. It indicates uniform asymptotic stability of the constant extinction solution \((\lambda - 1, 0)\) (at least) for \(\lambda \geq 2\), while it is a nonautonomous saddle for \(\lambda > 1\) near 1.

(b) Allowing the sequence \(a\) to vary and considering \(c_k \equiv \lambda > 0\) as parameter, the solution \(x^*\) is not constant anymore. Thus, the spectra of the diagonal sequences can be closed intervals (of positive length, see Fig. 3). As long as
Figure 3. Case (b): Dependence of $\Sigma(A)$ on $\lambda \in (1, 5]$ with given sequences $a_k = 2 + \sin k$ and constant $d_k \equiv 4$.

these two intervals are disjoint, Thm. 4.3(b) ensures that their union forms $\Sigma(A)$. Otherwise Thm. 5.7 might apply and one of the conditions

$$\overline{\lambda}_+\left(\frac{a}{1 + x^2}\right) \leq \overline{\lambda}_-\left(\frac{a}{1 + x^2}\right) \quad \text{or} \quad \overline{\lambda}_+\left(\frac{\lambda}{1 + dx^2}\right) \leq \overline{\lambda}_-\left(\frac{\lambda}{1 + dx^2}\right)$$

guarantees diagonal significance.

7. **Outlook and remarks on the continuous time situation.** As both a conclusion and outlook let us point out that our results are useful in an ODE context as well: Thereto, consider a linear ODE

$$\dot{x} = A(t)x$$

(7.1)

with e.g. a locally integrable coefficient matrix $A : \mathbb{R} \to L(K^d)$ and the transition matrix $U(t, s) \in L(K^d)$.

On the one hand, the discrete time results of this paper might be applied to the time-1-difference equation $(\Delta A)$ with the invertible coefficients

$$A_k := U(k + 1, k) \quad \text{for all } k \in \mathbb{Z};$$

they are bounded under the assumption $\sup_{k \in \mathbb{Z}} |U(k + 1, k)| < \infty$. It is not difficult to see that a (block-)triangular structure of (7.1) extends to the matrices $A_k$. Furthermore, we restricted ourselves to linear difference equations with invertible coefficient matrices. This excludes various features present in discrete but not in continuous time. Thus, as long as the dichotomy spectrum is concerned, similar results as in Sects. 3–5 are expected to hold for (block-)triangular ODEs (7.1). Yet, [9] specifically treats ODEs (7.1) and provides not only necessary and sufficient conditions for diagonal significance in the dichotomy spectrum, but also a procedure to obtain $\Sigma(A)$ from the half line spectra.

On the other hand, the subspectra $\Sigma_s, \Sigma_F$ and $\Sigma_{F_0}$ were introduced in order to classify nonautonomous bifurcations (cf. [37]). They appear equally important for (7.1) and nonautonomous nonlinear ODEs. Yet, their corresponding continuous time theory has not been established so far.

**Appendix A. Operators on Hilbert spaces.** Let $X$ be an infinite-dimensional separable and complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. The set of linear bounded operators between $X$ and a normed space $Y$ is abbreviated as $L(X, Y)$; we write $N(S) \subseteq X$ for the kernel and $R(S) \subseteq Y$ for the range of some $S \in L(X, Y)$. Furthermore, $L(X) := L(X, X)$ is the Banach algebra of bounded linear operators on $X$ with identity $\text{id}_X$. 

The Bohl exponents in Figs. 2 and 3 were computed using the scheme [25, Rem. B.3].
Given an operator $T \in L(X)$, let $\sigma_\alpha(T) := \sigma(T)$, $\sigma_\pi(T)$, $\sigma_s(T)$, $\sigma_F(T)$ and $\sigma_{\emptyset}(T)$ be its spectrum, approximate point spectrum, surjectivity, essential and Weyl spectrum, respectively (see [1, 2, 3, 9]). Since $X$ is a Hilbert space, the left spectrum $\sigma_l(T)$ resp. the right spectrum $\sigma_r(T)$ satisfy
\[ \sigma_l(T) = \sigma_\pi(T), \quad \sigma_r(T) = \sigma_s(T). \] (A.1)

Let us write $r(T)$ for the spectral radius and $r_F(T) := \sup_{\lambda \in \sigma_F(T)} |\lambda|$ for the essential spectral radius of $T$. When $T^* \in L(X)$ denotes the (Hilbert space) adjoint operator of $T$, then the spectra of $T$ and $T^*$ are related by (cf. [1, p. 244, Thm. 6.14 and p. 300, Lemma 7.41], [2, p. 79, Thm. 2.42])
\[ \sigma_\alpha(T^*) = \sigma_\alpha(T)^* \quad \text{for all } \alpha \in \{a,F,\emptyset\}, \] (A.2)
\[ \sigma_s(T) = \sigma_\pi(T^*), \quad \sigma_s(T^*) = \sigma_s(T), \]
where $\Omega^* := \{\lambda \in \mathbb{C} : \lambda \in \Omega\}$ for every $\Omega \subseteq \mathbb{C}$.

**Lemma A.1** ([2, p. 79, Thm. 2.42]). $\partial \sigma(T) \subseteq \sigma_\pi(T) \cap \sigma_s(T)$.

An operator $T \in L(X)$ possesses the single-valued extension property (SVEP for short) at a point $\lambda_0$, provided for every neighborhood $U \subseteq \mathbb{C}$ of $\lambda_0$ the only analytic function $f : U \to X$ satisfying $(\lambda_0 I - T) f(\lambda) = 0$ on $U$ is identically vanishing. If the SVEP holds for every $\lambda_0 \in \mathbb{C}$, then the operator $T$ is said to have the SVEP. The associate set (cf. [3, p. 65ff])
\[ \mathcal{S}(T) := \{\lambda \in \mathbb{C} : T \text{ do not have the SVEP at } \lambda\} \]
is open and fulfills $\mathcal{S}(T) \subseteq \sigma(T)^c$. Clearly, $T$ has the SVEP, if and only if $\mathcal{S}(T) = \emptyset$.

**Lemma A.2** ([2, p. 80, Cor. 2.45]).
(a) If $T$ has the SVEP, then $\sigma(T) = \sigma_s(T)$.
(b) If $T^*$ has the SVEP, then $\sigma(T) = \sigma_\pi(T)$.

**Lemma A.3** ([2, p. 85, Thm. 2.52]).
(a) If $\partial \sigma(T) = \sigma_s(T)$, then $T$ has SVEP.
(b) If $\partial \sigma(T) = \sigma_\pi(T)$, then $T^*$ has the SVEP.

**Appendix B. Weighted shift operators.** Let $I$ be a discrete interval unbounded above. We denote by $\ell^2$ the linear space of square-summable sequences $\phi = (\phi_k)_{k \in I}$ in $\mathbb{K}^I$ equipped with the inner product
\[ \langle \phi, \psi \rangle := \sum_{k \in I} \langle \phi_k, \psi_k \rangle \quad \text{for all } \phi, \psi \in \ell^2 \]
and norm $\|\phi\| = \sqrt{\langle \phi, \phi \rangle}$; $\ell^2$ is the prototype of a separable Hilbert space.

For a bounded weight sequence $A = (A_k)_{k \in I}$ in $L(\mathbb{K}^I)$, we define the left shift
\[ T_A : \ell^2 \to \ell^2, \quad (T_A \phi)_k := A_{k-1} \phi_{k-1} \quad \text{if } I = \mathbb{Z}, \]
\[ T_A \phi := (0, A_0 \phi, A_k \phi, A_{k+1} \phi, \ldots) \quad \text{if } I = \mathbb{Z}_+^i; \] (B.1)

$T_A$ is bounded with $\|T_A\| = \sup_{k \in I} |A_k|$ and $\beta(A) = r(T_A)$ (cf. [5, Thm. 1(i)]). Since the SVEP is invariant under similarity, $\mathcal{S}(T_A)$ is rotationally symmetric w.r.t. 0.

**Lemma B.1.** The adjoint of $T_A$ is given by $T_A^* \in L(\ell^2)$, $(T_A^* \phi)_k = A_{-k}^* \phi_{k+1}$, $k \in I$.

**Proof.** For arbitrary $\phi, \psi \in \ell^2$ we obtain
\[ \langle T_A \phi, \psi \rangle = \sum_{k \in I} \langle A_k \phi_k, \psi_{k+1} \rangle = \sum_{k \in I} \langle \phi_k, A_k^* \psi_{k+1} \rangle = \langle \phi, T_A^* \psi \rangle \]
with $(T_A^* \psi)_k := A_k^* \psi_{k+1}$ for all $k \in I$. \qed
B.1. Unilateral shifts. On a discrete interval \( I = \mathbb{Z}_k^+ \), \( k \in \mathbb{Z} \), one denotes (B.1) as unilateral shift. The essential properties of unilateral shifts are summarized in

Proposition B.2. If each \( A_k, k \in \mathbb{Z}_k^+ \), is invertible, then the following holds:

(a) \( \sigma_F(T_A) = \sigma_\pi(T_A) \subseteq \{ \lambda \in \mathbb{C} : \beta(A) \leq |\lambda| \leq \overline{\beta}(A) \} \)

(b) \( \beta(A) > 0 \) if and only if \( \sup_{k \in \mathbb{Z}_k^+} |A_k^{-1}| < \infty \)

(c) \( \mathcal{G}(T_A) = \emptyset \) and \( \mathcal{G}(T_A^*) = \{ \lambda \in \mathbb{C} : |\lambda| < \lambda_+(A) \} \)

Proof. (a) is from [32], (b) by [26, p. 134, Thm. 4.6.10] and (c) by [31, Thm. 2.1]. \( \square \)

Assume now that \( a = (a_k)_{k \leq \kappa} \) is a bounded sequence in \( \mathbb{K} \). Then the results [40, Thm. 1] combined with Prop. B.2(a) yield the relations

\[
\sigma_\alpha(T_a) = \delta(T_a) = \begin{cases} 
\{ \lambda \in \mathbb{C} : \beta(a) \leq |\lambda| \leq \overline{\beta}(a) \}, & a_k \neq 0 \text{ for all } k \in \mathbb{Z}_k^+, \\
\{0\} \cup \{ \lambda \in \mathbb{C} : \beta(\tilde{a}) \leq |\lambda| \leq \overline{\beta}(\tilde{a}) \}, & a_k = 0 \text{ for finitely many } k \in \mathbb{Z}_k^+, \\
\{ \lambda \in \mathbb{C} : |\lambda| \leq \beta(a) \}, & a_k = 0 \text{ for infinitely many } k \in \mathbb{Z}_k^+ 
\end{cases}
\]

with \( \alpha \in \{ \mathbb{F}, \pi \} \), where the K-valued sequence \( \tilde{a} \) is defined as \( \tilde{a}_k := a_{k-\kappa+K} \) for every \( \kappa \leq k \) with \( K \) being the minimal integer such that \( a_k \neq 0 \) for all \( k \geq K \).

Example B.3 (asymptotically periodic case). Let \( p \in \mathbb{N} \). A sequence \( (a_k)_{k \leq \kappa} \) in \( \mathbb{K} \) is asymptotically \( p \)-periodic, if there exists a \( p \)-periodic sequence \( (p_k)_{k \leq \kappa} \) with

\[
\lim_{k \to \infty} |a_k - p_k| = 0
\]

and \( c := \sqrt[p_k+p-1]{p_k} \) is denoted as its asymptotic mean. Provided \( (|a_k|)_{k \leq \kappa} \) is asymptotically \( p \)-periodic, then [40, Thm. 2] and Prop. B.2(a) yield

\[
\sigma_F(T_a) = \sigma_\pi(T_a) = \begin{cases} 
\{ \lambda \in \mathbb{C} : |\lambda| = c \}, & a_k \neq 0 \text{ for all } k \in \mathbb{Z}_k^+, \\
\{0\} \cup \{ \lambda \in \mathbb{C} : |\lambda| = c \}, & a_k = 0 \text{ for finitely many } k \in \mathbb{Z}_k^+, \\
\{ \lambda \in \mathbb{C} : |\lambda| \leq \beta(a) \}, & a_k = 0 \text{ for infinitely many } k \in \mathbb{Z}_k^+ 
\end{cases}
\]

B.2. Bilateral shifts. For \( I = \mathbb{Z} \) one speaks of bilateral shifts \( T_A \in L(\mathbb{K}^d) \). In case of invertible weights \( A_k \in \mathbb{K}^{d \times d} \) the condition \( \sup_{k \in \mathbb{Z}} |A_k^{-1}| < \infty \) implies \( \sigma(T_A) \in \mathcal{G}(T_A) \) is not a singleton.

As opposed to unilateral shifts, a characterization of the SVEP is more involved:

- \( T_A \) has the SVEP if and only if (cf. [11, Thm. 2.1])

\[
\liminf_{k \to \infty} \frac{\sqrt[k]{|\Phi(-k, \kappa)x|}}{|x|} \leq \limsup_{k \to \infty} \frac{\sqrt[k]{|\Phi(k, \kappa)x|}}{|x|} \quad \text{for all } x \in \mathbb{K}^d \setminus \{0\}. \quad \text{(B.2)}
\]

- \( T_A^* \) has the SVEP if and only if (cf. [11, Cor. 2.2])

\[
\liminf_{k \to \infty} \frac{\sqrt[k]{|\Phi(k, \kappa)x|}}{|x|} \leq \limsup_{k \to \infty} \frac{\sqrt[k]{|\Phi(k, -k)^*x|}}{|x|} \quad \text{for all } x \in \mathbb{K}^d \setminus \{0\} \quad \text{(B.3)}
\]

for some \( \kappa \in \mathbb{Z} \). Different from the scalar situation (cf. Lemma B.5 below), both \( T_A \) and \( T_A^* \) might fail to possess the SVEP (see [11, Ex. 2.3]).

Particularly for scalar shifts with weights \( a \in \ell^\infty(\mathbb{K}) \), thanks to [40, Thm. 3] it is

\[
\sigma(T_a) = \begin{cases} 
\sigma(T_a^*) = \{ \lambda \in \mathbb{C} : \beta(a) \leq |\lambda| \leq \overline{\beta}(a) \}, \\
\sigma_\pi(T_a) = \begin{cases} 
\{ \lambda \in \mathbb{C} : \beta_{z_-}(a) \leq |\lambda| \leq \overline{\beta}_{z_-}(a) \} \\
\cup \{ \lambda \in \mathbb{C} : \beta_{z_+}(a) \leq |\lambda| \leq \overline{\beta}_{z_+}(a) \} \\
\sigma(T_a)
\end{cases}, & \overline{\beta}_{z_-}(a) < \beta_{z_+}(a),
\end{cases}
\]

otherwise.
Example B.4 (asymptotically periodic case). Let $\kappa \in \mathbb{Z}$, $p^+, p^- \in \mathbb{N}$, and $(a_k)_{k \in \mathbb{Z}}$ be a sequence in $\mathbb{K}$. If $(|a_k|)_{k \geq \kappa}$ is asymptotically $p_+-$periodic and $(|a_k|)_{k \leq \kappa}$ is asymptotically $p_-$periodic with asymptotic means $c_+, c_-$, then [40, Thm. 5] showed

$$
\sigma_+(T_a) = \left\{ \lambda \in \mathbb{C} : c_+ \leq |\lambda| \leq c_- \right\}, \quad c_+ \leq c_-,
$$

$$
\sigma_-(T_a) = \left\{ \lambda \in \mathbb{C} : |\lambda| \in \{c_+, c_-\} \right\}, \quad c_- < c_+,
$$

where

$$
\sigma(T_a) = \left\{ \lambda \in \mathbb{C} : \min \{c_+, c_-\} \leq |\lambda| \leq \max \{c_-, c_+\} \right\}.
$$

Lemma B.5 ([33, Prop. 2.5] and [4, Thm. 18 and Cor. 19]). Either $T_a$ or $T_a^*$ has the SVEP and

$$
\mathcal{G}(T_a) = \left\{ \lambda \in \mathbb{C} : \overline{\Lambda}_+(a) < |\lambda| < \underline{\Lambda}_-(a) \right\},
$$

$$
\mathcal{G}(T_a^*) = \left\{ \lambda \in \mathbb{C} : \overline{\Lambda}_-(a) < |\lambda| < \underline{\Lambda}_+(a) \right\}.
$$

Acknowledgments. The author thanks Ken Palmer for the preprint [9]. Moreover, he is grateful to the referee for her/his constructive and insightful remarks improving the presentation and scope of the paper.

REFERENCES

[1] Y. Abramovich and C. Aliprantis, An Invitation to Operator Theory, Graduate Studies in Mathematics, 50. American Mathematical Society, Providence, RI, 2002.

[2] P. Aiena, Fredholm and Local Spectral Theory, with Applications to Multipliers, Kluwer, Dordrecht, 2004.

[3] P. Aiena, Semi-Fredholm Operators, Perturbation Theory and Localized SVEP, XX Escuela Venezolana de Matemáticas, Caracas, Venezuela, 2007.

[4] P. Aiena, T. Miller and M. Neumann, On a localized single-valued extension property, Math. Proc. R. Ir. Acad., 104A (2004), 17–34.

[5] B. Aulbach, N. Van Minh and P. Zabreiko, The concept of spectral dichotomy for linear difference equations, J. Math. Anal. Appl., 185 (1994), 275–287.

[6] B. Aulbach and S. Siegmund, A spectral theory for nonautonomous difference equations, In: Lázaro-Fernández J., e.a. (ed.) Proceedings of the 5th Intern. Conference of Difference Eqns. and Applications (Temuco, Chile, 2000), 45–55. Taylor & Francis, London, 2002.

[7] M. Barraa and M. Boumazgour, A note on the spectrum of an upper triangular operator matrix, Proc. Am. Math. Soc., 131 (2006), 3083–3088.

[8] L. Barreira and Y. Pesin, Introduction to Smooth Ergodic Theory, Graduate Studies in Mathematics 148, AMS, Providence, RI, 2013.

[9] F. Battelli and K. Palmer, Criteria for exponential dichotomy for triangular systems, Journal of Mathematical Analysis and Applications, 428 (2015), 525–543.

[10] A. Ben-Artzi and I. Gohberg, Dichotomies of perturbed time varying systems and the power method, Indiana Univ. Math. J., 42 (1993), 699–720.

[11] A. Bourhim and C. Chidume, The single-valued extension property for bilateral operator weighted shifts, Proc. Am. Math. Soc., 133 (2005), 485–491.

[12] J. Cushing, S. LeVarge, N. Chitnis and S. Henson, Some discrete competition models and the competitive exclusion principle, J. Difference Equ. Appl., 10 (2004), 1139–1151.

[13] L. Dieci and E. Van Vleck, Lyapunov and other spectra: A survey, In Collected Lectures on the Preservation of Stability under Discretization, SIAM, (2002), 197–218.

[14] L. Dieci, C. Elia and E. Van Vleck, Exponential dichotomy on the real line: SVD and QR methods, J. Differ. Equations, 248 (2010), 287–308.

[15] S. Djordjević and Y. Han, A note on Weyl’s theorem for operator matrices, Proc. Am. Math. Soc., 131 (2003), 2543–2547.

[16] B. Duggal, Upper triangular operators with SVEP: Spectral properties, Filomat, 21 (2007), 25–37.

[17] H. Elbjaoui and E. Zerouali, Local spectral theory for $2 \times 2$ operator matrices, Int. J. Math. Math. Sci., 42 (2003), 2667–2672.

[18] J. Han, H. Lee and W. Lee, Invertible completions of $2 \times 2$ upper triangular operator matrices, Proc. Am. Math. Soc., 128 (2000), 119–123.
[19] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lect. Notes Math., 840. Springer, Berlin etc., 1981.
[20] D. Hinrichsen and A. Pritchard, *Mathematical Systems Theory I – Modelling, State Space Analysis, Stability and Robustness*, Texts in Applied Mathematics, 48. Springer, Heidelberg etc., 2005.
[21] R. Horn and C. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 2013.
[22] D. Hong-Ke and P. Jin, Perturbations of spectrums of $2 \times 2$ operator matrices, *Proc. Am. Math. Soc.*, **121** (1994), 761–766.
[23] T. Hüls, Numerical computation of dichotomy rates and projectors in discrete time, *Discrete Contin. Dyn. Syst. (Series B)*, **12** (2009), 109–131.
[24] T. Hüls, Computing Sacker-Sell spectra in discrete time dynamical systems, *SIAM Journal on Numerical Analysis*, **48** (2010), 2043–2064.
[25] T. Hüls and C. Pötzche, Qualitative analysis of a nonautonomous Beverton-Holt Ricker model, *SIAM Journal of Applied Dynamical Systems*, **13** (2014), 1442–1488.
[26] C. Jiang and Z. Wang, *Structure of Hilbert Space Operators*, World Scientific, New Jersey, 2006.
[27] R. Johnson, K. Palmer and G. Sell, Ergodic properties of linear dynamical systems, *SIAM J. Math. Anal.*, **18** (1987), 1–33.
[28] K. Kloeden and M. Rasmussen, *Nonautonomous Dynamical Systems*, Mathematical Surveys and Monographs, 176. AMS, Providence, RI, 2011.
[29] K. Laursen and M. Neumann, *An Introduction to Local Spectral Theory*, Oxford Science Publications, Oxford, 2000.
[30] W. Lee, Weyl spectra of operator matrices, *Proc. Am. Math. Soc.*, **129** (2000), 131–138.
[31] J. Li, The single valued extension property for operator weighted shifts, *Northeast. Math. J.*, **10** (1994), 99–103.
[32] J. Li, Y. Ji and S. Sun, The essential spectrum and Banach reducibility of operator weighted shifts, *Acta Mathematica Sinica*, **17** (2001), 413–424.
[33] T. Miller, V. Miller and M. Neumann, Local spectral properties of weighted shifts, *J. Operator Theory*, **51** (2004), 71–88.
[34] K. Palmer, A diagonal dominance criterion for exponential dichotomy, *Bull. Austral. Math. Soc.*, **17** (1977), 363–374.
[35] G. Papaschinopoulos, On exponential trichotomy of linear difference equations, *Appl. Anal.*, **40** (1991), 89–109.
[36] C. Pötzche, A note on the dichotomy spectrum, *J. Difference Equ. Appl.*, **15** (2009), 1021–1025, (see also the corrigendum in *J. Difference Equ. Appl.*, **18** (2009), 1257–1261 (2012)).
[37] C. Pötzche, Fine structure of the dichotomy spectrum, *Integral Equations Oper. Theory*, **73** (2012), 107–151.
[38] C. Pötzche, Continuity of the dichotomy spectrum on the half line, Submitted, 2014.
[39] C. Pötzche and E. Russ, Continuity and invariance of the dichotomy spectrum, Submitted, 2014.
[40] W. Ridge, Approximate point spectrum of a weighted shift, *Trans. Am. Math. Soc.*, **147** (1970), 349–356.
[41] R. Sacker and G. Sell, A spectral theory for linear differential systems, *J. Differ. Equations*, **27** (1978), 329–358.
[42] S. Sánchez-Perales and S. Djordjević, Continuity of spectrum and approximate point spectrum on operator matrices, *J. Math. Anal. Appl.*, **378** (2011), 289–294.
[43] S. Siegmund, Normal forms for nonautonomous differential equations, *J. Differ. Equations*, **178** (2002), 541–575.
[44] S. Siegmund, Normal forms for nonautonomous difference equations, *Comput. Math. Appl.*, **45** (2003), 1059–1073.
[45] E. Zerouali and H. Zguitti, Perturbation of spectra of operator matrices and local spectral theory, *J. Math. Anal. Appl.*, **324** (2006), 992–1005.
[46] Y. Zhang, H. Zhong and L. Lin, Browder spectra and essential spectra of operator matrices, *Acta Mathematica Sinica*, **24** (2008), 947–955.

Received August 2014; revised March 2015.

E-mail address: christian.poetzche@aau.at