On Diers’s theory of Spectrum I
Stable functors and right multi-adjoints

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Abstract

Diers developed a general theory of right multiadjoint functors leading to a purely categorical, point-set construction of spectra. Situations of “multiversal” properties return sets of canonical solutions rather than a unique one. In the case of a right multiadjoint, each object deploys a canonical cone of local units jointly assuming the role of the unit of an adjunction. This first part revolves around the theory of multiadjoint and recalls or precises results that will be used later on for geometric purpose. We also study the weaker notion of local adjoint, proving Beck-Chevalley conditions relating local adjunctions and the equivalence with the notion of stable functor. We also recall the link with the free-product completion, and describe factorization aspects involved in a situation of multi-adjunction.

Introduction

This paper, together with [15], is the first part of a twofold work on Diers construction of spectra through the notion of right multi-adjoint, and more generally, to a series of papers devoted to synthetise current approaches about the notion of spectrum and how they are related. Spectra have played a prominent role in several regions of mathematics: for instance, algebraic geometry could resume in some sense as the study of the different flavors of spectra of commutative rings, while Stone duality is about spectra of distributive lattices and ordered structure; more loosely, categorical model theory is in some sense the study of “spectra of theories”, for a still-to-define notion of 2-dimensional spectra. We could sum up the central philosophy behind this notion though the following claim: *spectra arise when free construction fails*. A very broad overview of the situation is the following: one starts with a category of algebraic “ambient” objects, and a class of objects and maps between them one wan to see as “local data”, but fails to associate canonically one local object under an ambient object: one ends up rather with a family of canonical local objects under an ambient object, which is universal in some sense. The spectrum of an ambient object is a space whose points index this canonical family under it, equipped with a structural sheaf whose purpose is to gather those local objects, and it defines a left adjoint to a comparison functor between categories of “structured spaces”.

Until now, several and rather independent proposals to construct spectra in a general way have been done. Contrasting to the topos-theoretic approaches of [4], [5], [12], or [13], of which we will also provide a synthesis in [16], Diers approach is more purely categorical in its premises, and strictly point-set in the way it processes to the construction. However, both notion of spectra follows a similar “scenario”: a first step identifying algebraic situations with a hidden geometric content, and a second step where this geometric content is used to construct a corresponding notion of spectrum. In the topos theoretic approach, the first step could be synthesized as revolving around what was variously called *admissibility* in [4], *geometry* in [13], *triples* in [5], which was stated in terms suited for topos constructions; in Diers approach, the starting point for constructing spectra was the notion of *right multi-adjoint*, which is at first sight far more abstract and algebraic than admissibility, and whose geometric meaning is more subtle. But modulo a slight additional assumption - we will refer as *Diers condition* in the second part - which is easily encountered in practice, this situation leads in a very natural and concrete way to a notion of spectrum. The topos theoretic approach is more abstract but also more “universal” as it is based on syntactic data, and start from a complicated situation (admissibility) to process to a natural construction; Diers way is more concrete and based on semantical data, and while the construction of the spectrum, being point-set, could seem a bit “handmade”, the algebraic situation it starts with is far more natural. Moreover this method subsumes not only most of the usual examples of algebraic
geometry or the structured versions of Stone dualities, but also a vast list of exotic examples which were unsuspected before Diers investigation, and are not necessarily suited for the general topos theoretic approach. Indeed, the later requires the categories it use as “local data” to construct the spectrum to be axiomatisable by geometric theories, as well as the factorization system need to be left generated - in some sense, also axiomatisable. In particular local objects must form a (non full) subcategory of ambient objects, and are models of a geometric extension of the theory behind the ambient objects. In Diers this condition is largely relaxed as local objects are related to ambient objects through a functor that is not required to be faithful nor injective on objects.

Those two papers will hence deals with Diers approach. While Diers work on multiversal constructions in category theory has been rather well known amongst category theory community, his presentation of the spectral construction seems to have been poorly acknowledged, perhaps due to a restricted circulation of the main paper [9]: this contrasts with the quality of the paper itself and the highly practical and comprehensive construction he proposed. We hope those two new papers will help to make more people aware of Diers work on spectra.

This first paper will revolves essentially about the first step in Diers approach. More precisely, we will focus on its notion of interest and its variations, providing different presentations of this situation and a purely categorical analysis of it, postponing the actual construction of the spectrum and the geometric analysis in the second paper. Most of the content of this paper is expository, and aimed at synthesizig as much as possible of the “algebraic” aspects of the construction, gathering and relating different notions dispatched in several papers as [7], [6], [17], [1]... However we try to present them as explicitly and originally as possible, providing alternative presentation and proofs of some already known results, and also providing new observations at some points. The last section will also provide a totally new method, whose relevance will however appear later with a 2-categorical version.

Right multi-adjoint were introduced by Diers and extensively studied in [7], [9] or [6], amongst “multi” versions of universal properties and usual categorical constructions. Multiversal properties are analogous of universal properties where, rather than having a unique solution representing a construction, one has a canonical small set of solution jointly assuming the universal property. The prototypical situation is the notion of multirepresentable functor into Set, that is, a functor that decompose as a coproduct of representable functors: the other situations are constructed from this as well as universal construction are done by representing a functor into Set. For instance, as well as a (co)limit over a diagram is an object representing the functor assigning to any object the set of (co)cones over the diagram with this object as tip, a multi (co)limit exist when this functor is multi-representable, and the “local representing objects” form a small family of (co)cone such that any other (co)cone factorizes uniquely through exactly one of them. The other main example of multi-constuction is the notion of right multi-adjoint, to which revolves the present paper. In an ordinary adjunction, any object in the category where the right adjoint lands admits one unit uniquely factorizing any map from this object toward an object in the range of the right adjoint: and the left adjoint is used to provide the codomain of this unit. In a multi-adjunction, there is no global left adjoint, hence no uniquely defined unit under a fixed object: one rather has a small cone of local units under a fixed object, which jointly play the role of the unit in the sense that any arrow from this object toward the right adjoint uniquely factorizes through exactly one of those units, followed by a morphism in the range of the right adjoint. This is a special situation of the more general notion of local right adjoint, were again one lacks a global left adjoint, but is able to construct local left adjoints to restrictions at slices: however in this case, while any object still posses a cone of local units, one cannot in general enforce the smallness of this cone, and right multi-adjointness amounts in fact exactly to “local adjointness plus small solution set condition”.

The second part of this work concerns a characterization of right multi-adjoint through free coproduct completion, and we reprove as explicitly as possible a result, already known in [7] but proved in a different way, stating that a functor is right multiadjoint if and only if its free coproduct extension is right adjoint. We will see in the second part that this result is the “discrete version” of the spectral adjunction.

The third section of this work is about the orthogonality aspects of local adjunctions, and gives some characterizations and properties of “diagonally universal morphism” as defined in [9], which will play a central role in the second part when defining the topology of the spectrum.
The last part contains new results. We examine a situation, in the context of a factorization system in the presence of a terminal object, producing a case of right multi-adjunction. However in this last section, to emphasize the geometric interpretation, we will work with a convention that produces actually left multi-adjunction. The interest of this construction will be revealed in a future work applying its bicategorical analog to the bicategory of Grothendieck toposes in order to construct notions of “2-geometries” and spectra for toposes.

1 Local right adjoints and stable functors

In this first section, we recall our three notions of interest, namely local right adjoints, right multi-adjoints and stable functors. We first give some technical points about the behavior of the local units of the local adjunctions. We also prove that for a local right adjoint, the local adjunctions enjoy automatically a Beck-Chevalley condition, which was seemingly unnoticed until now. Then we turn to different characterization of local adjointness in term of nerves and initial family, and introduce the stronger notion of right multi-adjoint and recall a variant of Freyd adjoint functor theorem for multi-adjoint. Finally we turn to the notion of stable functor, as studied by Taylor in [17], and also in [20], and we prove equivalence with the notion of local right adjoint.

**Definition 1.1.** A functor $U : A \to B$ is said to be a local right adjoint if for each object $A$ of $A$ the restriction of $U$ to the slice $A/A$ has a left adjoint

$$\begin{array}{ccc}
A/A & \overset{\perp}{\leftrightarrow} & B/U(A) \\
\downarrow & & \downarrow \\
U/A & & \\
\end{array}$$

where we denote $A_f$ the domain of the arrow $L_A(f)$ in $A/A$. In the following we will also denote $U/A$ as $U_A$ for concision. The maps $\eta_A^f$ for $f : B \to U(A)$ are called local units under $B$.

The definition of a local right adjoint means that for any arrow $f : B \to U(A)$ in $B \downarrow U$ and $u : A' \to A$ in $A/A$ we have triangles in $B$ and $A$ respectively

$$\begin{array}{ccc}
B & \xrightarrow{f} & U(A) \\
\downarrow & & \downarrow \\
U(A_f) & \xrightarrow{\eta^f_A} & U(A_f) \\
\end{array} \quad \begin{array}{ccc}
A' & \xrightarrow{u} & A \\
\downarrow & \nearrow \downarrow & \nearrow \\
L_A(U(u)) & \xrightarrow{\epsilon^u_A} & L_A(U(u)) \\
\end{array}$$

satisfying the triangle identities

$$\begin{array}{ccc}
U_A & \xrightarrow{\eta^u_A} & U_AL_AL_AU_A \\
\downarrow & \nearrow \downarrow & \nearrow \\
U_A & \xrightarrow{\eta^u_A} & U_A \\
\end{array} \quad \begin{array}{ccc}
L_A & \xrightarrow{\eta^u_A} & L_AL_AL_A \\
\downarrow & \nearrow \downarrow & \nearrow \\
L_A & \xrightarrow{\eta^u_A} & L_A \\
\end{array}$$

In other words we have the following rejections

$$\begin{array}{ccc}
U(A') & \xrightarrow{U_A(u)} & U(A) \\
\downarrow & \nearrow \downarrow & \nearrow \\
U(A') & \xrightarrow{U_A(u)} & U(A) \\
\end{array} \quad \begin{array}{ccc}
A_f & \xrightarrow{A_{U(u)}} & A_f \\
\downarrow & \nearrow \downarrow & \nearrow \\
A_f & \xrightarrow{A_{U(u)}} & A_f \\
\end{array}$$

defining an isomorphism

$A/A[L_A(f), u] \simeq B/U(A)[f, U(u)]$

sending an arrow $v : L_A(f) \to u$, resp. an arrow $g : f \to U(u)$, to the composite triangle on the left, resp. on the right

$$\begin{array}{ccc}
B & \xrightarrow{\eta^f_A} & U(A_f) \\
\downarrow & \nearrow \downarrow & \nearrow \\
U(A) & \xrightarrow{U_A(u)} & U(A) \\
\end{array} \quad \begin{array}{ccc}
A_f & \xrightarrow{A_{U(u)}} & A_f \\
\downarrow & \nearrow \downarrow & \nearrow \\
A & \xrightarrow{u} & A \\
\end{array}$$
Remark. Beware that in general we cannot enforce the counits to be pointwise iso, that is, to require each $U_A$ to be full and faithful. Hence the factorization of a morphism in the range of $U$ may not be trivial. Morally, the factorization through the unit only takes in account the object of $\mathcal{A}$ whose strict image is the codomain, while, even when the domain is in the image of $U$, the factorization may not remember from which precise object in $\mathcal{A}$ it comes from.

Remark. Remark also that, contrary to the situation we are going to focus on, composing an arrow in the range of $U$ may change the unit. Indeed, while for any $u : A_1 \to A_2$ in $\mathcal{A}$, functoriality of $U$ makes the following square commute up to equality

$$
\begin{array}{ccc}
A/A_1 & \xrightarrow{U_{A_1}} & B/U(A_1) \\
\downarrow{A/A_2} & & \downarrow{B/U(u)} \\
A/A_2 & \xrightarrow{U_{A_2}} & B/U(A_2)
\end{array}
$$

its corresponding mate

$$
\begin{array}{ccc}
A/A_1 & \xleftarrow{L_{A_1}} & B/U(A_1) \\
\downarrow{A/A_2} & & \downarrow{B/U(u)} \\
A/A_2 & \xleftarrow{L_{A_2}} & B/U(A_2)
\end{array}
$$

defined as the composite

$$L_{A_2}B/U(u) \xrightarrow{L_{A_2}B/U(u)(\eta_{A_1})} L_{A_2}B/U(u)U_{A_1}L_{A_1} = L_{A_2}U_{A_2}A/uL_{A_1}.$$

This mates relates in a canonical way the unit of any $f : B \to U(A_1)$ and the unit of the composite $U(u)f : B \to U(A_2)$ as seen in the following diagram

But surprisingly, this mates is automatically an isomorphism because of the universal property of the units, as stated in the following proposition:

**Proposition 1.2.** Let be $U : \mathcal{A} \to \mathcal{B}$ a local right adjoint. Then for any $u : A_1 \to A_2$ in $\mathcal{A}$, we have the Beck-Chevalley condition at $u$, that is, the canonical transformation $\sigma^u$ is a point-wise isomorphism.

**Proof.** Remark that for each $u : A_1 \to A_2$ and $f : B \to U(A_1)$, the morphism $\sigma^u_f : L_{A_1}(B/U(u)(f)) \to L_{A_1}(f)$ is in $\mathcal{A}$, and we have a factorization

Observe that $\sigma^u_f$ is the unique arrow in $\mathcal{A}$ provided by the universal property of the unit $\eta_{U(u)f}$ at $\eta^A_{A_1}$ seen as an arrow $U(u)f \to U(u)U_{A_1}L_{A_1}(f)$ in $B/U(A_2)$. But on the other hand, by the
universal property of $\eta_A^f$ at $\eta_{U(w)f}^A$, seen as an arrow $f \to U_A, L_A(f)U(\eta_f^A)$ in $B/U(A_!)$, there exists a unique arrow $w : A_f \to A_U(w)_f$ in $A$ such that

$$
\begin{array}{c}
\begin{array}{c}
B \\
\eta_A^f
\end{array}
\xrightarrow{f} \\
\begin{array}{c}
U(A_f) \\
\eta_{U(w)f}^A
\end{array}
\xrightarrow{U_A, L_A(f)} \\
\begin{array}{c}
U(A_{U(w)}f) \\
\eta_{U(\sigma_f)^{-1}}^A
\end{array}
\xrightarrow{U(\sigma_f)^{-1}} \\
\begin{array}{c}
U(A_f)
\end{array}
\end{array}
$$

Now we prove that $w$ and $\sigma_f^w$ are mutual inverses in $A$. First, as

$$U(\sigma_f^w)\eta_{U(w)f}^A = \eta_A^f$$

and $1_{L_A(f)}$ is the unique map induced by $\eta_A^f$ seen as an arrow $f \to U_A, L_A(f)$, then necessarily we have a retraction in $A$

$$
\begin{array}{c}
A_f \\
w
\end{array}
\xrightarrow{w} \\
\begin{array}{c}
A_{U(w)f}
\end{array}
\xrightarrow{\sigma_f^w}
$$

but again, as now $\eta_{U(w)f}^A = U(w)\eta_A^f$ and $1_{L_A(U(w)f)}$ is the unique map induced by $\eta_{U(U(w))f}^A$ as an arrow $U(U(w)f) \to U_A, L_A(U(U(w)f))$, we have a retraction in $A$

$$
\begin{array}{c}
A_f \\
\sigma_f^w
\end{array}
\xrightarrow{w} \\
\begin{array}{c}
A_{U(w)f}
\end{array}
$$

and $\sigma_f^w$ defines both an iso $A_{U(w)f} \simeq A_f$ in $A$ and $L_{A_!} A/U(f) \simeq B/U(U(A_!))$ which can be shown to be natural.

**Corollary 1.3.** Let $f : B \to U(A)$: then we have $A_f \simeq A_{\eta_f^A}$ in $A$ and $\eta_{\sigma_f^w}^A \simeq \eta_f^A$ in $B \downarrow U_!$.

**Proof.** Consider the following diagram

$$
\begin{array}{c}
B \\
\eta_f^A
\end{array}
\xrightarrow{U_A, L_A(f)} \\
\begin{array}{c}
U(A_f) \\
\eta_{U(U(f))}^A
\end{array}
\xrightarrow{U_A, L_A(f)} \\
\begin{array}{c}
U(A_{U(U(f))}) \\
\eta_{U(U(f))}^A
\end{array}
\xrightarrow{U_A, L_A(f)} \\
\begin{array}{c}
U(A_f)
\end{array}
$$

Then by what precedes we have $\sigma_{\eta_f^A}^U$ is an iso as seen in the following diagram

$$
\begin{array}{c}
B \\
\eta_f^A
\end{array}
\xrightarrow{U_A, L_A(f)} \\
\begin{array}{c}
U(A_f) \\
\eta_{U(U(f))}^A
\end{array}
\xrightarrow{U_A, L_A(f)} \\
\begin{array}{c}
U(A_{U(U(f))}) \\
\eta_{U(U(f))}^A
\end{array}
\xrightarrow{U_A, L_A(f)} \\
\begin{array}{c}
U(A_f)
\end{array}
$$

But $U_A, L_A(f)\eta_f^A = f$, exhibiting an isomorphism

$$
\begin{array}{c}
B \\
\eta_f^A
\end{array}
\xrightarrow{U_A, L_A(f)} \\
\begin{array}{c}
U(A_f) \\
\eta_{U(U(f))}^A
\end{array}
\xrightarrow{U_A, L_A(f)} \\
\begin{array}{c}
U(A_f)
\end{array}
$$

□
A consequence is that local units that are related by an arrow in the range of $U$ must actually be isomorphic as objects under their domain:

**Corollary 1.4.** Let be $f_1 : B \to U(A_1)$ and $f_2 : B \to U(A_2)$, such that there exists a morphism $u$ in $\mathcal{A}$ and a triangle

$$
\begin{array}{ccc}
\eta_{f_1}^{A_1} & \xrightarrow{B} & \eta_{f_2}^{A_2} \\
\downarrow U(u) & & \downarrow U(u) \\
U(A_{f_1}) & \xrightarrow{u} & U(A_{f_2})
\end{array}
$$

then $u$ is an isomorphism.

**Proof.** By the previous proposition, we have that $\eta_{\eta_{f_1}^{A_1}}^{A_1} \simeq \eta_{\eta_{f_2}^{A_2}}^{A_2}$ and $\eta_{\eta_{f_2}^{A_2}}^{A_2} \simeq \eta_{f_2}^{A_2}$. But by Beck-Chevalley condition at $u$, we also have $\eta_{\eta_{f_1}^{A_1}}^{A_1} \simeq \eta_{U(u)\eta_{f_1}^{A_1}}^{A_2} = \eta_{\eta_{f_2}^{A_2}}^{A_2}$. □

**Corollary 1.5.** For any $f : B \to U(A)$, $\eta_f^U$ is initial in $(B \downarrow U) \downarrow f$.

**Definition 1.6.** Let be $C$ a category; *multi-initial family* in $C$ is a family of objects $(X_i)_{i \in I}$ such that for any $C$ in $C$ there is a unique $i \in I$ and a unique arrow $X_i \to C$. $X_i$ for $i \in I$ is a local initial object.

**Remark.** Observe that in this definition, if one has an arrow $f : C_1 \to C_2$, then $C_1$ and $C_2$ lie under the same local initial object. More generally, if two objects $C_1$, $C_2$ are in the same connected component and $X_i \to C_1$, $X_j \to C_2$ are the initial maps, then they lie under the same initial object: for there is a zigzag

$$
\begin{array}{cccccc}
& B_1 & \to & \cdots & \to & B_n \\
\leftarrow & \leftarrow & \leftarrow & \leftarrow & \leftarrow & \leftarrow
\end{array}
$$

and by uniqueness of the local initial object over a given object, necessarily the local object over $B_1$ is the same as the local initial object over $B_2$ because they both lies over $B_3$ and so on. Conversely any two objects under a same local initial object are in the same connected component. Hence there is exactly one local initial object by connected component.

**Proposition 1.7.** Let $U : \mathcal{A} \to \mathcal{B}$ be a local right adjoint and $B$ in $\mathcal{B}$: then the comma category $B \downarrow U$ has a multi-initial family.

**Proof.** We claim that the (large) class of local units under $B$ is a multi-initial family in $B \downarrow U$. First, for any $f : B \to U(A)$, we have by local adjunction in $\mathcal{A}$ an arrow $L_A(f)$ in $B \downarrow U$; but now suppose there is an other $g : B \to U(A')$ such that $\eta_{\eta_f^{A}}^A$ has a map $g \to f$ in $B \downarrow U$, that is there is a map $u$ in $\mathcal{A}$ such that $f = U(u)\eta_{\eta_f^{A}}^A$; but then by the universal property of the unit there is a unique factorization

$$
\begin{array}{ccc}
B & \xrightarrow{f} & U(A) \\
\downarrow \eta_f^A & & \downarrow U_A(f) \\
U(A_f) & \xrightarrow{U(u)} & U(A_{f'})
\end{array}
$$

But by corollary 1.4 this forces $\eta_f^A \simeq \eta_{f'}^A$. □

**Remark.** Observe that without further assumption, the multi-initial family of local units at a given $B$ may not be small. This is the point of the following notion.

**Definition 1.8.** A functor $U : \mathcal{A} \to \mathcal{B}$ is said to be a *right multi-adjoint* if for any $B$ in $\mathcal{B}$ there is a small multi-initial family in the comma $B \downarrow U$.

**Remark.** Observe that this definition is indexed by the domain, that is, by the object $B$ in $\mathcal{B}$, while the definition of local right adjoint was indexed by the objects $A$ of $\mathcal{A}$. However it is easy to see that any right multi-adjoint is in particular a local right adjoint: for any $f : B \to U(A)$ define $\eta_f^A$
to be the unique local initial object over \( f \), and \( L_A(f) \) to be the unique map \( \eta_f^A \rightarrow f \). Then \( \eta_f^A \) has the universal property of the unit as any triangle \( f \rightarrow U(u) \) in \( B/U(A) \)

\[
\begin{array}{ccc}
B & \xrightarrow{f} & U(A) \\
\downarrow{g} & & \downarrow{U(u)} \\
U(A') & & \\
\end{array}
\]

can be seen as a triangle \( g \rightarrow f \) in \( B \downarrow U \) forcing \( g \) and \( f \) to be in the same connected component. Therefore \( \eta_f^A \) is also the initial object over \( g \), inducing an unique arrow \( \eta_f^A \rightarrow g \) in \( B \downarrow U \), that is a unique arrow \( L_A(f) \rightarrow u \) in \( A/A \) as desired.

Existence of a multi-initial object in the comma is reminiscent of the so called solution set condition in Freyd Adjoint Functor Theorem. Let us precise this fact, in order to retrieve an analogous multi-adjoint theorem, in the vein of [14]. First we need a weakening of the notion of initial family

**Definition 1.9.** Let be \( C \) a category; a **weakly initial family** is a family \( (X_i)_{i \in I} \) such that for any object \( C \), there some \( i \in I \) and some arrow \( X_i \rightarrow C \).

**Remark.** Observe that in this definition, there is no requirement of uniqueness, nor for the index, nor for the arrow, so there may be several weakly initial objects over an arbitrary one. In the following we are interested in **small** weakly initial families:

**Definition 1.10.** A functor \( U : A \rightarrow B \) is said to satisfy the **Solution Set Condition** if each of the comma \( B \downarrow U \) admits a small weakly initial family: that is a family \( (n_i : B \rightarrow U(A_i))_{i \in I_B} \) such that any map from \( B \) to \( U \) factorizes through some (non necessarily unique) \( n_i \).

**Proposition 1.11.** A functor \( U : A \rightarrow B \) is a right multi-adjoint if and only if it is local right adjoint and satisfies the Solution Set Condition.

**Proof.** It is obvious that a right multi-adjoint satisfies the Solution Set Condition as for any object \( B \) the small multi-initial family of \( B \downarrow U \) is in particular a small weakly initial family, and any right multi-adjoint is trivially local right adjoint. For the converse, if a local right adjoint satisfies the Solution Set Condition, consider the small weakly initial family \( (f_i : B \rightarrow U(A_i)) \subset U \) of \( B \downarrow U \) and then take the corresponding units \( (n_{f_i} : B \rightarrow U(A_{f_i}))_{i \in I} : \) we claim this is a small multi-initial family.

**Definition 1.12.** A **connected limit** is a limit indexed by a connected category; a **wide pullback** is a limit of a diagram over a set of arrows with common codomain. In particular any wide-pullback is a connected limit.

**Proposition 1.13.** A category has wide pullbacks if and only if each slice \( A/A \) has products. A functor preserves connected limits if and only if it preserves wide pullbacks.

In Freyd adjoint functor theorem, assuming completeness and local smallness of the domain category \( A \), one can prove that a functor preserving limits is right adjoint if it satisfies the solution set condition : this is achieved by constructing an initial object in any \( B \downarrow U \) as the limit \( \lim_{i \in I} U(A_i) \simeq U(\lim_{i \in I} A_i) \). Similarly :

**Proposition 1.14.** Let \( A \) be a category with wide pullbacks; then a functor \( U : A \rightarrow B \) is a right multi-adjoint if and only if it satisfies the solution set condition and preserves wide pullbacks.

**Proof.** Indeed, assuming that \( A \) has connected limit and that \( U \) preserves them, the solution set condition will enable us to prove that its restriction on any slice \( U_A : A/A \rightarrow B/U(A) \) is right adjoint by constructing an initial object in any of the comma \( f \downarrow U_A \) which \( f : B \rightarrow U(A) \in B/U(A) \) (this is nothing but the category of factorizations of \( f \) through \( U \)) as the following. If \( S_B = (n_i : B \rightarrow U(A_i))_{i \in I_B} \) is the small weakly initial family in \( B \downarrow U \) given by the solution set condition, define \( S_f \) the subfamily of \( S_B \) whose morphisms factorize \( f \); then we can do the wide pullback of the \( A_i \) over \( A \) and it is preserved by \( U \), so that we have

\[
\begin{array}{ccc}
U(A_i) & \xrightarrow{n_i} & U(A) \\
\downarrow{f} & & \downarrow{U(p_i)} \\
B & \xrightarrow{n_j} & U(A_j) \\
\end{array}
\]

\[
U(\times_{A}(A_i)_{i \in S_f}) \simeq \times_{U(A)}(U(A_i))_{i \in S_f}
\]
Now we claim that this unique arrow $B \to U(\times A_i)_{i \in S_f}$ is the desired initial object in $f \downarrow U_A$, as any factorization of $f$ factorizes itself through some of the $n_i$ in $S_f$ hence through the wide pullback. For the converse we will make use of stability in order to recover the Solution Set Condition, while the preservation of wide pullback just come from the fact that they are ordinary products in the slices, hence preserved by the restriction as it is right adjoint.

Now we turn to another facet of the local adjunction.

**Definition 1.15.** Let $U : A \to B$ be a functor; then the co-nerve of $U$ is the functor
\[
B \xrightarrow{\mathcal{N}_B^U} [A, \mathcal{S}et]
\]
sending each $B$ in $\mathcal{B}$ to the functor $\mathcal{N}_B^U = B[B, U(-)] : A \to \mathcal{S}et$.

For any $B$ in $\mathcal{B}$, we have a discrete opfibration
\[
\pi_B : \int \mathcal{N}_B^U \to A
\]
whose objects are pairs $(A, f)$ with $A$ in $\mathcal{A}$ and $f : B \to U(A)$, and morphisms $(A_1, f_1) \to (A_2, f_2)$ are $u : A_1 \to A_2$ with $U(u)f_1 = f_2$. There is a general result saying that a functor is a right adjoint if the projection from the category of elements of its co-nerve at each object has a limit; in fact this says that the co-nerve functor is representable, that is, there is an initial object in the category of elements, which is the unit. We give here the corresponding statement for a right multi-adjoint.

**Definition 1.16.** Let $F : I \to A$ be a functor; then a multi-limit of $F$ is a small family of cones
\[
(p^i_j : L_j \to F(i))_{i \in I, j \in J}
\]
such that for any cone $(f_i : X \to F(i))_{i \in I}$ there is a unique $j \in J$ and a unique factorization of the cone $(f_i)_{i \in I}$ through the cone $(p^i_j)_{i \in I}$. A functor $U : A \to \mathcal{B}$ preserves multi-limits (or also, is multi-continuous) if for any multi-limit $(p^i_j : L_j \to F(i))_{i \in I, j \in J}$ in $A$, there is a multi-limit $(q^{i,k}_j : M_k \to UF(i))_{i \in I, j \in J, k \in K}$ in $\mathcal{B}$ and for each $k \in K$ we have
\[
M_k \simeq \prod_{j \in J_k} U(L_j)
\]
where $J_k$ is the set of $j \in J$ such that the cone $(p^i_j : L_j \to X_i)_{i \in I}$ factorizes through $M_k$.

The following observation just gives the obvious analog of the characterization of right adjoint in terms of the existence of the limit of the projection of the category of elements of the nerve:

**Proposition 1.17.** A functor $U : A \to \mathcal{B}$ is a local right adjoint if and only if for any $B$ in $\mathcal{B}$, each connected component of $\int \mathcal{N}_B^U$ has an initial object. Moreover, $U$ is a right multi-adjoint if $\int \mathcal{N}_B^U$ has a set of connected components. Equivalently, $U$ is a right multi-adjoint if and only if the functor $\pi_B : \int \mathcal{N}_B^U \to A$ has a multi-limit in $A$ and $U$ preserves it.

The first half of this fact is tautological; for the second part, one can adapt [3][proposition 3.3.2].

Now from what was said, it appears that a right multi-adjoint is a functor such that the associated conerve in any object is “locally representable”. Indeed, any arrow from an object $B$ toward $U$ is determined first by the connected component it lies in, which corresponds to the local unit it factorizes through, and secondly by a choice of map in $A$. This amounts to the following:

**Proposition 1.18.** Let $U$ be a multi-right adjoint: then for each $B$ one has
\[
\mathcal{N}_B^U \simeq \prod_{i \in I_B} A[A_i, -]
\]
with $I_B$ the set of connected components of $B$ and $n_i : B \to U(A_i)$ the initial object of the connected component $i$.

Now we come to an alternative notion encapsulating the property of being a local right adjoint, but in a way that is more related to factorization systems. This was studied in [17] and [20], and we prove there that this notion coincides with local right adjointness. It relies on an alternative presentation of local unit in a more “orthogonality structure” spirit.
Definition 1.19. A candidate (diagonally universal toward \( U \) in the terminology of Diers), is a morphism \( n : B \to U(A) \) such that for any square of the following form there exist an unique morphism \( w : A \to A_1 \) such that \( U(w) \) diagonalizes uniquely the square and the left triangle already commutes in \( A \)

\[
\begin{array}{c}
B \xrightarrow{f} U(A_1) \\
\downarrow n \quad U(w) \xrightarrow{U(v)} \\
U(A) \quad \quad \quad \quad U(A_2) \\
\end{array}
\]

\[
\exists w \xrightarrow{U(w)} A_1
\]

Remark. Then the candidate for \( f \) is the initial object in the category of factorizations of \( f \) through the range of \( U \). Indeed, for any other factorization through \( U \)

\[
\begin{array}{c}
B \xrightarrow{f} U(A) \\
\downarrow g \quad \quad \quad \quad \quad U(u) \\
U(A') \\
\end{array}
\]

one gets a square as below, where \( n_f \) produces a unique \( w \) such that \( U(w) \) is a filler

\[
\begin{array}{c}
B \xrightarrow{g} U(A') \\
\downarrow n_f \quad U(v) \xrightarrow{U(u)} \\
U(A_f) \quad \quad \quad \quad U(A) \\
\end{array}
\]

Definition 1.20. A functor \( U : A \to B \) is stable when any morphism \( f : B \to U(A) \) factorizes uniquely through the range of \( U \) as

\[
\begin{array}{c}
B \xrightarrow{f} U(A) \\
\downarrow n_f \quad \quad \quad \quad \quad U(u_f) \\
U(A_f) \\
\end{array}
\]

where \( n_f : B \to U(A_f) \) is a candidate. We refer to this factorization as the stable factorization of \( f \) and to \( n_f \) as the candidate of \( f \).

Proposition 1.21. For a functor \( U : A \to B \) and \( B \) in \( B \) we have the following

- If a candidate \( n_1 : B \to U(A_1) \) admits an arrow \( n_2 \to n_1 \) from another candidate \( n_2 : B \to U(A_2) \) in \( B \downarrow U \), then we have \( n_1 \simeq n_2 \) in \( B \downarrow U \) and \( A_1 \simeq A_2 \) in \( A \).

- In particular, any two candidates in a same connected component of \( B \downarrow U \) are isomorphic.

- If \( f : B \to U(A) \) admits a stable factorization, then it is unique up to unique isomorphism.

- In particular, when \( U \) is stable, the stable factorization of any arrow is unique up to unique isomorphism.

Proof. The first item is easily shown to implies the other ones. Suppose we have \( n_1, n_2 \) and a triangle

\[
\begin{array}{c}
B \xrightarrow{n_1} U(A_1) \\
\downarrow n_2 \quad \quad U(u) \xrightarrow{U(u)} \\
U(A_2) \\
\end{array}
\]

Then we have a unique filler

\[
\begin{array}{c}
B \xrightarrow{n_2} U(A_2) \\
\downarrow n_1 \quad \quad U(v) \xrightarrow{U(u)} \\
U(A_1) \quad \quad \quad \quad U(A_2) \\
\end{array}
\]
But now there is a unique filler of the square

\[
\begin{array}{ccc}
B & \xrightarrow{n_1} & U(A_1) \\
\downarrow^{n_2} \ & \ & \downarrow^{U(u)} \\
U(A_2) & \xrightarrow{U(w)} & U(A_2)
\end{array}
\]

so that \( u : A_2 \to A_2 \) is both a retraction and a section in \( AA \), hence an isomorphism, so that \( n_1 \simeq n_2 \) in \( B \downarrow U \).

Proposition 1.22. For any square as below

\[
\begin{array}{ccc}
B_1 & \xrightarrow{f_1} & U(A_1) \\
\downarrow^{f} \ & \ & \downarrow^{U(u)} \\
B_2 & \xrightarrow{f_2} & U(A_2)
\end{array}
\]

the stable factorizations of \( f_1 \) and \( f_2 \) are related by a unique morphism in \( A \) such that

\[
\begin{array}{ccc}
B_1 & \xrightarrow{n_{f_1}} & U(A_{f_1}) \\
\downarrow^{f} \ & \ & \downarrow^{U(u)} \\
B_2 & \xrightarrow{n_{f_2}} & U(A_{f_2})
\end{array}
\]

Proof. The desired \( w_{g,u} \) is the filler of the square

\[
\begin{array}{ccc}
B_1 & \xrightarrow{n_{f_1}} & U(A_{f_1}) \\
\downarrow^{n_{f_2}f} \ & \ & \downarrow^{U(u_{f_{12}})} \\
U(A_{f_2}) & \xrightarrow{U(u_{f_{12}})} & U(A_2)
\end{array}
\]

Theorem 1.23. Stable functors and local right adjoints coincide.

Proof. Let \( U : A \to B \) be a stable functor. For each \( A \) defines the functor

\[
\begin{array}{ccc}
A/A & \xrightarrow{L_A} & B/U(A) \\
\downarrow^{U/A} & & \downarrow^{U(U(A))}
\end{array}
\]

where \( L_a \) returns the left part of the initial factorization of an arrow by its associated candidate:

\[
\begin{array}{ccc}
L_A : & B/U(A) & \xrightarrow{f} & U(A) \\
 & \xrightarrow{n_f} & U(A_f) & \xrightarrow{U(u_f)} & A \\
& U(A_f) & & U(u_f) & \xrightarrow{u_f} & A
\end{array}
\]

We can easily prove this functor is left adjoint to \( U/A \), but it is more direct to observe that the family of candidates under \( B \) is a multi-initial family in \( B \downarrow U \). Hence \( U \) is a local right adjoint.

For the converse, suppose \( U \) is a local right adjoint. We claim that candidates are arrows \( n : B \to U(A) \) such that \( L_A(n) \) provides an iso \( A_f \simeq A \) in \( A \), hence \( n^\sim \simeq n \) in \( B \downarrow U \). Let be a square

\[
\begin{array}{ccc}
B & \xrightarrow{g} & U(A_1) \\
\downarrow^{n} \ & \ & \downarrow^{U(u)} \\
U(A) & \xrightarrow{U(u)} & U(A_2)
\end{array}
\]
Recall by proposition 1.2 we also have that composing with $U(v)$ does not modify the unit, as we have an isomorphism $\sigma^u_n : \eta^A_{U(v)n} \simeq \eta^A_f$. But the triangle

$$B \xrightarrow{g} U(A_1)$$

$$U(u)n \downarrow \quad \downarrow U(u)$$

$$U(A_2)$$

provides us with a unique arrow $w : A_{U(v)n} \to A_1$ such that

$$A_1 \xrightarrow{w} U(A_1)$$

$$\xrightarrow{g} U(A_1) \xrightarrow{U(u)} U(A_2)$$

Then by inserting the local inverses of $\eta^A_n$ and $\sigma^u_n$ in the square above and using the universal property of $\eta^A_n\eta^A_n$ at the triangle

$$B \xleftarrow{U(A)} U(A_1) \xrightarrow{U(u)} U(A_2)$$

provides us with a triangle as below and a diagonalization in $\mathcal{B}$

$$B \xleftarrow{U(A)} U(A_1) \xrightarrow{U(u)} U(A_2)$$

$$\xleftarrow{U(A_1)} \xrightarrow{U(u)} \xleftarrow{U(A_2)}$$

In particular, local units are candidates by corollary 1.3. Hence for any arrow $f : B \to U(A)$, the factorization through the unit as $f = U_A L_A(f) \eta^A_f$ provides a stable factorization through a candidate.

This achieve to prove that stable functors and local right adjoint can be used indifferently and are two ways of encapsulating the same property.

However in the following, and especially in the second paper, we will give more interest to right multi-adjoints for the smallness condition allows us to manipulate local units without size issue.

## 2 Right multi-adjoints through free product completion

In this section, we give the characterization of right multi-adjoints through the free product completion, following loosely [7] and [18]. In the second part of this work, we are going to show how the notion of spectrum is a way to turn a local right adjoint into a right adjoint, the spectrum functor being the desired left adjoint. But this construction, motivated by geometric and duality theoretic conceptions, process by extracting as much as possible topological and geometric information from a situation of local adjunction: in some way, it exploits the defect of universality on the algebraic side in order to produce richer structure on the geometric side. In this section, we
recall another way to turn a situation of multi-adjunction into a honest adjunction, which is purely algebraic and purer in some sense, but also devoid of any geometric content for this very reason. The relation between those two approach will be studied in more detail in the second part of this work, where this approach through free product completion will appear as the “discrete version” of the spectral adjunction.

The main intuition of this part is that, for a right multi-adjoint \( U : A \to B \), the cone of local units under a given object \( B \) defines a family of objects in \( A \) given by the codomains of those local units. Hence, at the level of families of objects, the multiversality of the construction can be fixed and \( U \) will induce an honest adjunction between categories of families of objects of \( A \) and \( B \). The good notion of “category of families” here is the one provided by the free product completion, the beginning of this part is devoted to.

**Definition 2.1.** For a category \( A \), the free product completion of \( A \) is the category \( \Pi A \) whose

- objects are functors \( A_{(-)} : I \to A \) (also denoted \( (A_i)_{i \in I} \)) with \( I \) a set,
- and arrows \( (A_i)_{i \in I} \to (B_j)_{j \in J} \) consist of the data of an application \( \alpha : J \to I \) and a natural transformation

\[
\begin{array}{ccc}
I & \xrightarrow{f} & A \\
\alpha \downarrow & & \downarrow \\
J & \xrightarrow{\beta} & B
\end{array}
\]

that is, a \( J \)-indexed family \( (f_j : A_{\alpha(j)} \to B_j)_{j \in J} \).

**Proposition 2.2.** We have the following properties of the free product completion, for a given category \( A \):

- \( \Pi A \) has small products
- There is a codense full embedding \( \iota_A : A \hookrightarrow \Pi A \) whose essential image is the subcategory \( (\Pi A)_{\text{co-conn}} \) of co-connected objects.
- Moreover, the embedding \( A \to \Pi A \) has a right adjoint if and only if \( A \) already had products
- We have a full embedding \( \Pi A \hookrightarrow [A, \text{Set}]^{op} \) whose essential image consists of all small products of representable
- For any category \( B \) with small products, we have an equivalence of categories

\[
[A, B] \simeq [\Pi A, B]_{\Pi}
\]

(where \( [\Pi A, B]_{\Pi} \) is the category of functors preserving small products) sending any \( F : A \to B \) on its right Kan extension \( \iota_A F \) and any \( G : \Pi A \to B \) on its restriction \( G \iota_A \).

**Proof.** For the first item: the product in \( \Pi A \) of a family of families \( ((A_i^j)_{i \in I_j})_{j \in J} \) has as indexing set the disjoint union \( \bigsqcup_{j \in J} I_j \) and whose member of index \( (j, i) \) is the object \( A_{(j, i)} = A_i^j \); the projections are given for each \( j \in J \) as the transformation

\[
\begin{array}{ccc}
\bigsqcup_{j \in J} I_j & \xrightarrow{(A_i^j)_{i \in I_j}} & \Pi A \\
q_j \downarrow & & \downarrow p_i \\
I_j & \xrightarrow{(A_i^j)_{i \in I_j}} & A
\end{array}
\]

where \( p_i \) is the pointwise equality \( A_{(j, i)} = A_i^j \).

For the second item, the embedding sends an object \( A \) of \( A \) to the one element family \( A : 1 \to A \) and a morphism \( f : A_1 \to A_2 \) to the natural transformation

\[
\begin{array}{ccc}
1 & \xrightarrow{A_1} & A \\
\downarrow f & & \downarrow \\
A_2 & \xrightarrow{A_2}
\end{array}
\]
Now we prove objects in the image of this embedding are coconnected, which says that for a family of families \((A_j)_{j \in J}\), we have

\[
\text{ILA} \left[ \prod_{j, i} (A_j), \iota_A(A) \right] \simeq \prod_{j, i} \text{ILA}[(A_j), \iota_A(A)]
\]

Indeed, any arrow \(\prod_{j, i} (A_j) \to \iota_A(A)\) defines an arrow \(1 \to \prod_{j \in J} I_j\) pointing at some pair \((j, i)\) with \(j \in J\) and \(i \in I_j\), and a natural transformation

\[
\begin{array}{ccc}
1 & \to & \prod_{j \in J} I_j \\
\downarrow & & \downarrow \\
& A & \to A
\end{array}
\]

which is nothing but an arrow \(f : A_{(j, i)} \to A\). But in such a case, as 1 is a connected object in \(\text{Set}\), this arrow \((j, i) : 1 \to \prod_{j \in J} I_j\) factorizes through \(I_j\) for some \(j \in J\), pointing the corresponding \(i \in I_j\), and the natural transformation \(f\) factorizes through the componentwise identity \(p_i : A_{(j, i)} = A_j^i\) so we have an arrow \(\prod_{j, i} (A_j) \to \iota_A(A)\) as desired. Conversely, any coconnected object is of the form \(\iota_A(A)\): indeed a family \((A_i)_{i \in I} : I \to \mathcal{A}\) is nothing but the product in \(\text{ILA}\) of the family \((\iota_A(A_i))_{i \in I}\) as the set \(I\) decomposes as the coproduct \(\prod_{i \in I} 1\) in \(\text{Set}\); and any family should be indexed by a connected set to be a coconnected object in \(\text{ILA}\), but 1 is the only connected set. This also suffice to prove that any object is a product of objects in the range of \(\iota_A\).

Now suppose that \(\mathcal{A}\) has products. Then for any family \((A_i)_{i \in I}\) in \(\text{ILA}\) one can compute the product in \(\mathcal{A}, \prod_{i \in I} A_i\). Now for an object \(A\) in \(\mathcal{A}\), we have

\[
\text{ILA}[(\iota_A(A), (A_i)_{i \in I})] \simeq \mathcal{A}[A, \prod_{i \in I} A_i]
\]

sending a family of arrows \((A \to A_i)_{i \in I}\) to the universal map \(A \to \prod_{i \in I} A_i\). The unit of this adjunction is is as \(\iota_A\) is full and faithful, while the counit is the transformation

\[
\begin{array}{ccc}
I & \to & \prod_{i \in I} A_i \\
\downarrow & & \downarrow \\
& A & \to A
\end{array}
\]

where \(\iota_{(A_i)_{i \in I}}\) has the projection \(p_i : \prod_{i \in I} A_i \to A_i\) has component in \(i\). For the converse, it is easy to see that any right adjoint of the embedding \(\iota_A\) sends a family on an object in \(\mathcal{A}\) with the universal property of the product.

The embedding \(\text{ILA} \hookrightarrow [\mathcal{A}, \text{Set}]^{op}\) just sends a family \((A_i)_{i \in I}\) to the coproduct \(\coprod_{i \in I} \mathcal{A}_i\) and an arrow \((\alpha, (f_j)_{j \in J}) : (A_i)_{i \in I} \to (B_j)_{j \in J}\) to the opposite of the induced map between the corresponding coproducts in \([\mathcal{A}, \text{Set}]\) as depicted below

\[
\begin{array}{ccc}
\mathcal{A}_i & \xrightarrow{\alpha_i} & \coprod_{i \in I} \mathcal{A}_i \\
\uparrow \quad & & \uparrow \\
\mathcal{B}_j & \xrightarrow{f_j} & \coprod_{j \in J} \mathcal{B}_j
\end{array}
\]

Finally, for a functor \(U : \mathcal{A} \to \mathcal{B}\) with \(\mathcal{B}\) having products; we claim that the right Kan extensions of \(U\) is pointwise and can be computed as

\[
\text{ran} \, \iota_A U(A_i)_{i \in I} = \prod_{i \in I} U(A_i)
\]

Indeed for any \((A_i)_{i \in I}\) the comma category \((A_i)_{i \in I} \downarrow \iota_A\) has a small initial \(I\)-indexed subcategory consisting of the objects \((i, 1_{A_i})\) for \(i \in I\), and this subcategory is discrete. Hence calculating the
poinwise right Kan extension resumes to calculating the product above. Moreover, as \( \iota_A \) is full and faithful, restricting back this Kan extension along \( \iota_A \) gives back \( U \), in fact up to equality in this context.

**Proposition 2.3.** The embedding \( A \hookrightarrow \Pi A \) creates connected limits in \( A \). Moreover, \( \Pi A \) is complete if and only if \( A \) has connected limits.

**Proof.** Let be \( D \) a connected category and \( F : D \to A \); we prove that the singleton \( \iota_A(\lim F) \) is the limit of \( \iota_A F \) in \( \Pi A \). Let be a cone \( (a_d, (f_d)_{d \in D} : (A_d)_{d \in D}) \) in \( \Pi A \) consisting for each \( d \in D \) of an arrow \( f_d : A_{a_d} \to F(d) \) where \( a_d : 1 \to I \) points to some index; but as \( D \) is connected and \( I \) is a set, necessarily the \( a_d \) are all equal to the same index \( \alpha \), so that we actually have a cone \( (f_d : A_\alpha \to F(d))_{d \in D} \) in \( A \), inducing a unique arrow \( f : A_\alpha \to \lim F \) in \( A \). This defines a unique arrow \( (\alpha, f) : (A_d)_{d \in I} \to \iota_A(\lim F) \). By what precedes, it is also clear that any connected cone that \( \iota_A \) sends to a limiting cone was already limiting.

Now, recall that a category is complete if and only if it has connected limits and products. But \( \Pi A \) always has products, so we just have to show that \( \Pi A \) has connected limits if and only if \( A \) does. Let be \( D \) a connected category, and \( F : D \to \Pi A \) a functor, with \( F_d : I_d \to A \) its component in \( d \) and with transition morphism

\[
\begin{array}{ccc}
I_{d_1} & \xrightarrow{F_{d_1}} & A \\
\downarrow I_s & & \downarrow F_{s} \\
I_{d_2} & \xrightarrow{F_{d_2}} & A
\end{array}
\]

for each \( s : d_1 \to d_2 \) in \( D \). Then \( F \) defines an opplax cocone \( (F_d : I_d \to A)_{d \in D} \) in \( \text{Cat} \), defining uniquely a functor

\[
\int I F_{d \in D}
\]

where \( \int I \) is the category of elements of the functor \( I : D^{op} \to \text{Set} \) returning the indexing set \( I_d : I_{d_2} \to I_{d_1} \), and the transition map \( I_s \) for \( s : d_1 \to d_2 \); it is indeed well known that the category of elements is the opplax colimit in \( \text{Cat} \), and we see here the \( I_d \) as discrete categories. Now, as \( D \) was small and each \( I_d \) was a set, the category \( \int I \) has a small set \( \pi_0(\int I) \) of connected components. In this context, one can describe the connected components as follows. In set, the colimit of the diagram \( I \) is the quotient

\[
\colim_{d \in D} I_d \cong \prod_{d \in D} I_d / \sim_D
\]

where \( (d, i) \sim_D (d', i') \) if there is a zigzag in \( D \) relating \( i \) and \( i' \); this exactly amounts to say that \( (d, i) \) and \( (d', i') \) are in the same connected component of \( \int I \), so we also have that the connected components of \( \int I \) are exactly equivalence classes \( [(d, i)]_{\sim_D} \) and

\[
\prod_{d \in D} I_d / \sim_D \cong \pi_0(\int I)
\]

Now, if we restrict the induced functor \( (F_d)_{d \in D} \) along the inclusion of a connected component

\[
\begin{array}{ccc}
\int I & \xrightarrow{(F_d)_{d \in D}} & A \\
\uparrow \pi_0(\int I) & & \uparrow \pi_0(\int I) \\
[(d, i)]_{\sim_D} & \xrightarrow{F_{(d, i)}_{\sim_D}} & A
\end{array}
\]

we can compute the limit \( \lim F_{[(d, i)]_{\sim_D}} \) in \( A \), and this limit is preserved by the inclusion functor \( \iota_A \). So the desired limit of \( F \) in \( \Pi A \) is the family

\[
\pi_0(\int I) \to A
\]

sending the connected component \( [(d, i)]_{\sim_D} \) to the connected limit \( \lim F_{[(d, i)]_{\sim_D}} \), and this actually coincides with the product in \( \Pi A \) of the family \( (\lim F_{[(d, i)]_{\sim_D}} : 1 \to A)_{[(d, i)]_{\sim_D} \in \pi_0(\int I)} \).
Proposition 2.4. Any functor \( U : A \to B \) extends uniquely into a functor \( \Pi U \), called its free product extension, making the square below to commute up to equality

\[
\begin{array}{ccc}
A & \xrightarrow{U} & B \\
\downarrow{\iota_A} & & \downarrow{\iota_B} \\
\Pi A & \xrightarrow{\Pi U} & \Pi B
\end{array}
\]

Proof. The functor \( \Pi U \) just is the right Kan extension \( \text{ran} \, \iota_A \iota_B U \), and is defined by sending a family \((A_i)_{i \in I}\) to \((U(A_i))_{i \in I}\).

The following proposition is tautological:

Proposition 2.5. \( A \) has a multi-initial family if and only if \( \Pi A \) has an initial object.

The following proposition is the core idea of [7][part 4], though we present here a quite different proof.

Proposition 2.6. For a functor \( U : A \to B \), the following are equivalent:

1. \( U \) is a right multi-adjoint
2. \( U \) has a relative left adjoint along \( \iota_A \)
3. its free product extension \( \Pi U : \Pi A \to \Pi B \) is a right adjoint

Proof. Suppose that \( U \) is a right multi-adjoint, with \( I_B \) the set of local units \( \eta_B : B \to U(A_x) \) and \( \pi_B : I_B \to A \) its projection sending \( x \) to \( A_x \). Define a functor \( L : B \to \Pi A \) sending an object \( B \) to the family \( \pi_B : I_B \to A \), and an arrow \( f : B_1 \to B_2 \) to the transformation

\[
\begin{array}{ccc}
I_{B_1} & \xrightarrow{L_f} & A \\
\downarrow{\pi_{B_1}} & & \downarrow{\pi_B} \\
I_{B_2} & \xrightarrow{\eta_{B_1}} & \Pi A
\end{array}
\]

where \( I_f \) sends \( x \) to the index of the unit \( \eta_{B_1} : B \to U(A_{n_x}) = n_I f(x) \) and \( L_f \) has component \( L_{A_x}(n_x f) : A_{n_x f} \to A_x \) as provided in each \( x \in I_{B_2} \) by the factorization

\[
\begin{array}{ccc}
B_1 & \xrightarrow{f} & B_2 \\
\downarrow{\eta_{B_1}} & & \downarrow{n_x} \\
U(A_{n_x}) & \xrightarrow{U_{A_x}(L_{A_x}(n_x f))} & U(A_x)
\end{array}
\]

Observe that the local units of \( B \) define in particular a morphism of families

\[
\begin{array}{ccc}
1 & \xrightarrow{B} & B \\
\downarrow{\pi_B} & & \downarrow{U} \\
I_B & & A
\end{array}
\]

corresponding to the family \((n_x : B \to U(A_x))_{x \in I_B}\). Now it is easy to see that this functor is a relative left adjoint to \( U \) along \( \iota_A \), that is, that for any \( B \) in \( B \) and \( A \) in \( A \) we have

\[
\Pi A[L(B), \iota_A(A)] \simeq B[B, U(A)]
\]

Indeed, any arrow \( f : B \to U(A) \) factorizes through a unique \( n_x : B \to U(A_x) \), where \( x \) is the index of the unit \( \eta_f^x \), while \( L_A(f) : A_x \to A \) provides a morphism in \( \Pi A \)

\[
\begin{array}{ccc}
I_B & \xrightarrow{\pi_B} & A \\
\downarrow{L_A(f)} & & \downarrow{n_x} \\
A & & A
\end{array}
\]
Indeed a morphism of family $(\alpha, f) : (B_i)_{i \in I} \rightarrow (B'_j)_{j \in J}$, that is a family $(f_j : B_{\alpha(j)} \rightarrow B'_j)_{j \in J}$, defines an application $j \rightarrow \prod_{i \in I} I_{B_i}$ sending $i$ to the index of the local unit $n_{i,j} = \eta^A_{\alpha(j)} : B_{\alpha(j)} \rightarrow A_{\xi(i)}$, and a morphism of families

$$\prod_{i \in I} I_{B_i} \xrightarrow{\xi} A$$

Then for any $(A_j)_{j \in J}$ in $\mathsf{ILA}$ and $(B_i)_{i \in I}$ in $\mathsf{PIS}$ we have an isomorphism

$$\mathsf{ILA}[(B_i)_{i \in I}, (A_j)_{j \in J}] \cong \mathsf{PIS}[(B_i)_{i \in I}, (U(A_j))_{j \in J}]$$

Indeed a morphism of family $(\alpha, f) : (B_i)_{i \in I} \rightarrow (U(A_j))_{j \in J}$, that is a family $(f_j : B_{\alpha(j)} \rightarrow U(A_j))_{j \in J}$, defines an application $j \rightarrow \prod_{i \in I} I_{B_i}$ sending $i$ to the index of the local unit $n_{i,j} = \eta^A_{\alpha(j)} : B_{\alpha(j)} \rightarrow A_{\xi(i)}$, and a morphism of families

$$\prod_{i \in I} I_{B_i} \xrightarrow{\xi} A$$

while any arrow $(x, u) : L(B) \rightarrow \epsilon_A(A)$ can be pasted with the family of units

$$\begin{array}{ccc}
1 & \xrightarrow{\eta} & B \\
\downarrow & & \downarrow U \\
I_B & \xrightarrow{\pi_B} & A \\
\end{array}$$

This functor $L$ extends to $\mathsf{PIS}$ as follows: for a family $(B_i)_{i \in I}$, define $L(B_i)_{i \in I}$ as the family

$$\prod_{i \in I} I_{B_i} \xrightarrow{(\eta^A_{\alpha(i)})_{i \in I}} A$$

sending $(i, x)$ with $i \in I$ and $x \in I_{B_i}$ to the associated $A_x$: for an arrow $(\alpha, f = (f_i)_{i \in I}) : (B_i)_{i \in I} \rightarrow (B'_i)_{i \in I}$, that is a family $(f_i : B_{\alpha(i)} \rightarrow B'_i)_{i \in I}$, each pair $(i, x) \in \prod_{i \in I} I_{B_i}$ defines uniquely some $I_{f(x)} \in I_{B_{\alpha(i)}}$ indexing the unit through which factorizes the composite $x_{\alpha i} : B_{\alpha(i)} \rightarrow U(A_x)$, that is such that

$$\begin{array}{ccc}
B_{\alpha(i)} & \xrightarrow{f_i} & B'_i \\
\downarrow n_{(i, f(x))} & & \downarrow n_x \\
U(A_{f(x)}) & \xrightarrow{\eta^A_{\alpha(i)}} & U(A_x) \\
\end{array}$$

and define $I_{(\alpha, f)} : \prod_{i \in I} I_{B'_i} \rightarrow \prod_{i \in I} I_{B_i}$ as sending $(i, x)$ to this $I_{(\alpha, f)}(x)$, and define the desired morphism $L(\alpha, f)$ as

$$\begin{array}{ccc}
\prod_{i \in I} I_{B_i} & \xrightarrow{\prod_{i \in I} \pi_{I_{B_i}}_{|\epsilon\alpha(j)}} & \prod_{i \in I} I_{B_i} \\
\downarrow L_f & & \downarrow \prod_{i \in I} \pi_{I_{B_i}}_{|\epsilon\alpha(j)} \\
\prod_{i \in I} I_{B'_i} & \xrightarrow{\prod_{i \in I} \pi_{I_{B'_i}}_{|\epsilon\alpha(j)}} & \prod_{i \in I} I_{B'_i} \\
\end{array}$$

where $L_f$ denotes the family $(f_i : B_{\alpha(i)} \rightarrow B'_i)_{i \in I}$ and $\pi_I : \prod_{i \in I} I_{B_i} \rightarrow I$ is the projection sending $(i, x)$ on $i \in I$, $\pi_{I_{B_i}}_{|\epsilon\alpha(j)} : \prod_{i \in I} I_{B_i} \rightarrow A$ sends $(i, x)$ on $A_x$, and $n$ has as components

$$(n_{(i, x)} = n_x : B_i \rightarrow U(A_x))_{(i, x) \in \prod_{i \in I} I_{B_i}}$$
where \( L(A_j)_{j \in J} \) consists of all right part in \( \mathcal{A} \) of their factorization

\[
(L_{A_j}(f_j) : A_{\xi(j)} \to A_j)_{j \in J}
\]

For the converse, use the same argument as for the proof of the first implication by pasting.

Now we prove that if \( U \) is such that \( \Pi U \) is right adjoint to a functor \( L \), then \( U \) is multi-adjoint. Observe that with this hypothesis, we have in particular for any \( B \in \mathcal{B} \) a unit \( \eta_{B, B} : (\ast, B) \to \Pi UL(\ast, B) \). So if \( I_B \) denote the indexing set of \( \Pi UL(\ast, B) \) and \( A_i \) is the object in \( \mathcal{A} \) corresponding to the \( i \)-th index of \( I_B \) in this family, then we have a cone \( (\eta_i : B \to U(A_i))_{i \in I_B} \). Now we prove this is a cone of local units. For any \( A \in \mathcal{A} \), the unit property of \( \Pi UL(\ast, B) \) provides a factorization

\[
\ast \xrightarrow{i_f} \ast \rightarrow I_B \rightarrow U(B) \rightarrow U(\ast) \rightarrow \mathcal{B}
\]

for \( i_f : \ast \to I_B \) pointing at the index of the local unit \( \eta_f^A \) and \( L_A(f) \) returning the image of \( f \) along the local left adjoint at \( A \).

\[
\ast \xrightarrow{g} C \xrightarrow{d} U(A_1) \xrightarrow{U(u)} U(A_2) \xrightarrow{\phi} A
\]

### 3 Factorization aspects

As suggested by the definition of candidates in the notion of stability, orthogonality structures are hidden in the notion of local adjunction. In the same vein, one could ask whether the stable factorization of arrows toward \( U \) can be generalized to any arrow, that is, if the orthogonality structure provided by the candidates on the left and the morphisms in the range of \( U \) on the right can be completed into a factorization system. In the context studied in [9], this is possible through a small object argument in the context of locally finitely presentable category. This step is essential in general in the construction of spectra, and also takes place in the topos-theoretic approach of [5], though it is mostly left implicit. The reference for this is [2], we mostly follow there modulo some adaptations, and in combination with elements from [9].

**Definition 3.1.** Let \( U : \mathcal{A} \to \mathcal{B} \) a local right adjoint. A morphism \( n : B \to C \) is said to be **diagonally universal** if it is left orthogonal to morphisms in the range of \( U \), that is, if for any morphism \( u : A_1 \to A_2 \) in \( \mathcal{A} \) and any square as below, there exists a unique filler \( d : C \to U(A_1) \) making both the upper and lower triangles to commute

\[
B \xrightarrow{f} U(A_1) \\
\downarrow{n} \quad \downarrow{U(u)} \\
C \xrightarrow{g} U(A_2)
\]

**Remark.** Beware that without further assumption, a diagonally universal morphism with codomain in the range of \( U \), that is of the form \( n : B \to U(A) \), needs not to be a candidate, as the filler needs not to arise from a morphism in \( \mathcal{A} \).

As a left class in an orthogonality structure, diagonally universal morphisms enjoy the following properties, which are standards and then do not need proofs here.

**Proposition 3.2.** We have the following:

- diagonally universal morphisms are stable under composition and contain isomorphisms
- if \( n : B \to C \) is diagonally universal and \( m : C \to D \) is such that \( mn \) is diagonally universal, then \( m \) is also diagonally universal. In particular, diagonally universal morphisms are stable under retracts.
- diagonally universal morphisms are stable under pushout along arbitrary morphisms
- diagonally universal morphisms are stable under colimits in the arrow category \( \mathcal{B}^2 \)
**Proposition 3.3.** For any \( u : A_1 \to A_2 \), \( U(u) \) is diagonally universal if and only if \( U(u) \) is an isomorphism. If moreover \( U \) is relatively full and faithful, then \( U(u) \) is diagonally universal if and only if \( u \) is an isomorphism.

**Proof.** The unique filler of the square

\[
\begin{array}{ccc}
U(A_1) & \xrightarrow{U(u)} & U(A_2) \\
\downarrow & \searrow d & \downarrow \searrow \\
U(A_2) & \xrightarrow{U(u)} & U(A_2)
\end{array}
\]

is both a right and left inverse to \( U(u) \); and if moreover \( U \) is relatively full and faithful, it comes from a unique morphism \( d = U(v) \) which is both a section of \( u \) from the lower triangle, but it is also a retraction because there is a unique morphism in \( \mathcal{A} \) lifting \( U(u) \) in the upper triangle, and this must be \( u \).

**Proposition 3.4.** A morphism \( n : B \to U(A) \) is diagonally universal if and only if \( U_AL_A(n) \) is an isomorphism.

**Proof.** The unique lifter \( d \) in the following square

\[
\begin{array}{ccc}
B & \xrightarrow{\eta_n^A} & U(A_n) \\
\downarrow n & \nearrow d & \downarrow U_AL_A(n) \\
U(A) & \xrightarrow{n} & U(A)
\end{array}
\]

is a section of \( U_AL_A(n) \). This provide also a filler of the square

\[
\begin{array}{ccc}
B & \xrightarrow{\eta_n^A} & U(A_n) \\
\downarrow \eta_n^A & \nearrow dU_AL_A(n) & \downarrow \Upsilon_U_AL_A(n) \\
U(A_n) & \xrightarrow{U_AL_A(n)} & U(A)
\end{array}
\]

But \( 1_{U(A)} = U(1_A) \) is the only filler of the square because \( \eta_n^A \) is a candidate. So \( d \) is also a retraction of \( U_AL_A \), which is hence an isomorphism. Conversely, if \( U_AL_A(n) \) has an inverse, then one can use the candidate property at \( \eta_n^A \) to get a filler in any square with a morphism in the range of \( U \) on the right.

**Remark.** Beware that \( U \) needs not be conservative, so that the inverse of \( U_AL_A(n) \) needs not comes from a morphism in \( \mathcal{A} \) making \( L_A(f) \) an isomorphism itself, so that the remark above does not says that \( n \simeq \eta_n^A \) in \( B \downarrow U \); in particular \( n : B \to U(A) \) may be diagonally universal without being a candidate. However, in case where \( U \) is relatively full and faithful, the filler we had above must come from a unique morphism \( d = U(v) \) which must satisfies the same commutations, hence provides an inverse of \( L_A(n) \): hence the following corollary.

**Corollary 3.5.** Suppose that \( U \) is relatively full and faithful; then for a morphism \( n : B \to U(A) \), the following are equivalent:

- \( n \) is diagonally universal
- \( n \) is a candidate
- \( L_A(n) \) is an isomorphism

We defined \( \text{Diag} \) as the left orthogonal \( \perp^U(A^2) \). Hence we end with an orthogonality structurutre \( (\text{Diag}, \text{Diag}^\perp) \) where \( \text{Diag}^\perp \) is the double-orthogonal \( (\perp^U(A^2))^\perp \). Arrows in \( \text{Diag}^\perp \) lies now out of the essential image of \( U \) and may have arbitrary domain and codomain. However, we have the following fullness property of the essential image of \( U \) relatively to \( \text{Diag}^\perp \):

**Proposition 3.6.** Let be \( u : U(A_1) \to U(A_2) \) be an arrow in \( \text{Diag}^\perp \). Then \( u \simeq U(L_{A_2}(u)) \) in \( B/U(A_2) \) and \( \eta_n^A \) is an isomorphism. In particular \( u \) is an arrow in the essential image of \( U \).
Proof. Indeed, \( u \) is right orthogonal to the local unit in its stable factorization, so there exists a unique \( w \) as below

\[
\begin{align*}
U(A_1) & \xrightarrow{\eta_{A_2}^A} U(A_1) \\
\eta_{A_2}^A & \downarrow \quad w \quad \downarrow u \\
U(A_u) & \xrightarrow{U(L_{A_2}(u))} U(A_2)
\end{align*}
\]

which is both diagonally universal by left cancellation, and in \( \text{Diag}^\perp \) by right cancellation, and is hence an isomorphism. In particular \( \eta_{A_2}^A \) is an iso, being section of an iso. \( \square \)

Now, we explain how, in a suitable context, the stable factorization of morphisms towards \( U \) extends to a factorization system in \( \mathcal{B} \), where the diagonally universal morphisms form the left class. To do so, we are going to adapt [2] version of the small object argument in the context of locally presentable categories.

In the following we suppose that \( \mathcal{B} \) is a locally finitely presentable category and \( U : \mathcal{A} \to \mathcal{B} \) is a local right adjoint. Then denote \( \mathcal{D} \) the class of diagonally universal morphisms between finitely presented objects. This coincides with the intersection of the class of diagonally universal morphisms and the class of finitely presented morphisms, that is, \( \mathcal{D} = \text{Diag} \cap \mathcal{B}_2^\omega \). We are going to use \( \mathcal{D} \) to left-generate a factorization system, which will enjoy some degree of accessibility.

**Proposition 3.7.** The class \( \mathcal{D} \) has the following properties:

- \( \mathcal{D} \) is closed under composition and contains isomorphisms between finitely presented objects
- \( \mathcal{D} \) is left cancellative
- \( \mathcal{D} \) is closed under pushouts along arbitrary finitely presented morphisms
- \( \mathcal{D} \) is closed under finite colimit in the arrow category \( \mathcal{B}^2 \)
- Any filtered colimit in \( \mathcal{B}_2 \) of morphisms in \( \mathcal{D} \) is diagonally universal.

**Proof.** The two first propositions are obvious. The third is an easy consequence of the universal property of the pushout. The fourth comes from the fact that \( \mathcal{B}_2^\omega \) is itself closed under finite colimits in \( \mathcal{B}_2 \) because \( \mathcal{B}_2 \) is so in \( \mathcal{B} \) and colimits in the arrow category are sent to colimit by domain and codomain functors, while \( \text{Diag} \) is also closed under colimit as a left class in an orthogonality structure. This last argument also proves the last item. \( \square \)

Recall that \( \mathcal{B}_2 \) also is locally finitely presentable, with \( \mathcal{B}_2^\omega \) as generator of finitely presented objects. We have now a functor preserving finite colimits

\[
\mathcal{D} \xrightarrow{\text{Ind}(\mathcal{D})} \mathcal{B}_2^\omega
\]

which extends into a locally finitely presentable functor

\[
\begin{array}{ccc}
\mathcal{B}^2 & \xrightarrow{\text{Ind}(\mathcal{D})} & \text{Ind}(\mathcal{D}) \\
\parallel & \downarrow & \\
\mathcal{D} & \xrightarrow{\text{Ind}(\mathcal{D})} & \text{Ind}(\mathcal{D})
\end{array}
\]

where \( \text{Ind}(\mathcal{D}) \) itself is locally finitely presentable. This gives rise to an idempotent comonad, one could see as returning the left part of a factorization system. Moreover, objects in \( \text{Ind}(\mathcal{D}) \) are then models of a finite limit theory, which motivates the following definition:

**Definition 3.8.** A diagonally universal morphism is said to be *axiomatisable* if it lies in \( \text{Ind}(\mathcal{D}) \).

**Remark.** Observe however that in general one cannot force arbitrary diagonally universal morphisms to be decomposable as a filtered colimit of morphisms in \( \mathcal{D} \). That is, we only have \( \text{Ind}(\mathcal{D}) \subseteq \text{Diag} \) (and hence \( \text{Diag}^\perp \subseteq \text{Ind}(\mathcal{D})^\perp \)) in the general case. For this reason, with the inclusion of right class a morphism between orthogonality structures, the factorization system \( (\text{Ind}(\mathcal{D}), \text{Ind}(\mathcal{D})^\perp) \) is the free left generated factorization system associated to \( (\text{Diag}, \text{Diag}^\perp) \).

Now we invoke results of [2] to construct a factorization system \((\text{Ind}(\mathcal{D}), \mathcal{D}^\perp)\). In our context, we can use the small class of finitely presented diagonally universal morphism to left-generate a factorization system. We recall here the process:
Definition 3.9. A **saturated class** is a set $\mathcal{V} \subseteq \mathcal{B}_\omega$ of finitely-presented maps such that:
- $\mathcal{V}$ contains isomorphisms and is stable by composition,
- $\mathcal{V}$ is closed under finite colimits in $\mathcal{B}_\omega$,
- $\mathcal{V}$ is closed under pushouts along arbitrary maps between finitely presented objects.

A saturated class is always small as long as being in the essentially small generator $\mathcal{B}_\omega$. In our case, the class $\mathcal{D}$ of finitely presented diagonally universal morphisms is a saturated class.

Definition 3.10. Let be $B$ in $\mathcal{B}$: define $\mathcal{D}_{\text{diag}}^B$ the category of diagonally universal morphisms with domain $B$, and $\mathcal{D}_B \hookrightarrow \mathcal{D}_{\text{diag}}^B$ the full subcategory whose objects are arrows $n : B \to C$ such that there exists some finitely presented diagonally universal morphism $k : K \to K'$ in $\mathcal{D}$ and $a : K \to B$ such that we can exhibit $n$ as a pushout $K \xrightarrow{k} K' \xleftarrow{a} B \xrightarrow{r} C$.

Remark. By left cancellation, $\mathcal{D}_{\text{diag}}^B$ is itself a full subcategory of the coslice $B \downarrow \mathcal{B}$.

Proposition 3.11. The category $\mathcal{D}_B$ is closed under finite colimits in $B \downarrow \mathcal{B}$.

Proof. Let be $(n_i, a_i)_{i \in I}$ with $K_i \xrightarrow{t_i} K'_i \xleftarrow{a_i} B$ a finite diagram in $\mathcal{D}_B$. We can use the fact that $\mathcal{D}$ is closed under finite colimit to compute the finite colimit of $F$ as $K_i \xrightarrow{t_i} K'_i \xleftarrow{a_i} B$ where $(q_i, q'_i) : n_i \to \text{colim}_{i \in I} l_{t_i}$ is the inclusion in the colimit computed in the category of arrows and $\text{colim}_{i \in I} l_{t_i}$ is still in $\mathcal{D}$. Then by commutation of pushouts with colimits we have

$$\langle a_i \rangle_{i \in I} \text{colim}_{i \in I} l_{t_i} = \text{colim}_{i \in I} a_i * l_{t_i}$$

and this map is still in $\mathcal{D}_B$.

As a consequence, for any $f : B \to C$, the category $\mathcal{D}_B \downarrow f$ of diagonally universal morphisms under $B$ above $f$ is filtered. Moreover, recall that the codomain functor $B \downarrow \mathcal{B}$ preserves filtered colimits. Now we can construct the factorization of any arrow $f$ in $\mathcal{B}$.

Proposition 3.12. For any $f : B \to C$ in $\mathcal{B}$ we have a factorization

$$B \xrightarrow{\text{colim} \mathcal{D}_B \downarrow f} C \xrightarrow{\text{colim} \mathcal{D}_B \downarrow f} r_f$$

with $l_f$ an axiomatizable diagonally universal morphism and $r_f$ is in $\mathcal{D}^\perp$.

For a complete proof of the statement, see [2][section 2.3] and in particular [theorem 14].

Proposition 3.13. The factorization above is orthogonal, that is, $\text{Ind}(\mathcal{D}) = \perp (\mathcal{D}^\perp)$. 
Proof. First recall that diagonally universal are closed under (filtered) colimits. For \( l \in \mathcal{D}^+ \) with factorization \( f = r_l f_l \), \( f \) is left orthogonal to its own right part, therefore there is a unique filler in the diagram below

\[
\begin{array}{c}
\xymatrix{ B & C_f \\
& C_f \ar[ur]^f \\
C & C \ar[ul]^{r_f} \ar[uu]_f \\
& C \ar[ul]^{r_f} } \\
\end{array}
\]

But now left cancellation of left maps, together with cancellation of right maps, enforces that this filler is an isomorphism, which forces \( r_f \) to be iso, so that \( f \) is in \( \text{Ind}(\mathcal{D}) \).

Hence we have for any local right adjoint \( U \) a factorization system \( (\text{Ind}(\mathcal{D}), \mathcal{D}^+) \). Observe that in the general case, the local units under a given object may not be obtained as filtered colimit of finitely presented diagonal universal morphisms under them. If one successively take the stable factorization and the \( (\text{Ind}(\mathcal{D}), \text{Ind}(\mathcal{D})^+) \)-factorization

\[
\begin{array}{c}
\xymatrix{ \text{colim} \ar[r]^{f} & U(A) \\
\text{colim} \ar[r]^{\eta_A^f} & D_{B \downarrow U(A)}(U(A)) } \\
\end{array}
\]

then \( u_f = U(L_A(f))u_{\eta_A^f} \) is in \( \text{Ind}(\mathcal{D})^+ \) by uniqueness of the factorization because \( \text{Diag}^+ \subseteq \text{Ind}(\mathcal{D})^+ \), so that the right part of \( \eta_f^A \) is also the right part of \( f \), so that we know that the functor

\[
\mathcal{D}_B \downarrow \eta_f^A \rightarrow \mathcal{D}_B \downarrow f
\]

is cofinal since it induces the same colimit. Remark also that \( u_{\eta_A^f} \) is in \( \text{Diag} \cap \text{Ind}(\mathcal{D})^+ \), which however does not forces it however to be an isomorphism.

Moreover, this situation cannot even be improved in the case of a right multi-adjoint, where the local units under a given object form a small set. This is why, as we shall see in the second part, we have to impose explicitly the condition that local units are filtered colimits of finitely presented diagonal universal morphisms above them amongst conditions isolated by Diers to enable the construction of a spectra from a right multi-adjoint.

**Definition 3.14.** A local right adjoint functor \( U \) is said to be diagonally axiomatisable if we have \( \text{Diag} = \text{Ind}(\mathcal{D}) \). It is said to satisfy Diers condition if for any \( B \) and any \( f : B \rightarrow U(A) \), the local unit \( \eta^A_f \) is in \( \text{Ind}(\text{Diag}_B) \).

**Remark.** Diers condition says that local units coincide with the diagonally axiomatisable morphism in the induced factorization. This is a strictly weaker condition than being diagonally axiomatisable as it does not requires any diagonally universal morphism to be axiomatisable.

However in some situation we can produce local right adjoints with the desired property if we start from a left generated factorization system and a class of objects enjoying an adequate “gliding” condition:

**Definition 3.15.** Let be a functor \( U : A \rightarrow B \) and \( R \) a class of maps in \( B \). We say that \( U \) lifts \( R \) maps if for any \( A \) in \( A \) and any \( r : B \rightarrow U(A) \), there exists \( u : A_0 \rightarrow A \) and an isomorphism \( \alpha : U(A_0) \simeq B \) such that \( ra = U(u) \).

Let be \( V \) a saturated class in a locally finitely presentable category \( B \) and \( (\mathcal{L}, \mathcal{R}) \) the associated left generated system, with \( \mathcal{L} = \text{Ind}(\mathcal{V}) \) and \( \mathcal{R} = \mathcal{V}^\perp \). Now suppose that \( U_0 : A_0 \rightarrow B \) is a functor lifting \( \mathcal{R} \)-maps. Then define \( \iota_0 : A \rightarrow A_0 \) as the wide subcategory whose arrows are those whose image under \( U_0 \) are in \( \mathcal{R} \) and \( U \) as the restriction \( U = U_0 \iota_0 \).

**Proposition 3.16.** Suppose that \( U_0 \) is relatively full and faithful; then the induced functor \( U : A \rightarrow B \) is stable and diagonally axiomatisable.
Proof. For any \( f : B \to U(A) \), consider the axiomatisable factorization

\[
\begin{array}{ccc}
B & \xrightarrow{f} & U(A) \\
\downarrow{l_f} & & \downarrow{r_f} \\
C_f & & \\
\end{array}
\]

where \( l_f \) is obtained as the filtered colimit \( l_f = \text{colim}_{D \in B} f \) in \( B \downarrow B \). For \( U_0 \) lifts along \( R \)-maps, there exists \( u_f : A_f \to A \) in \( A \) sent by \( U \) to \( r_f \), and moreover, this morphism is essentially unique in \( A \) as \( U_0 \) is relatively full and faithful. But then \( l_f \) is the local unit of \( U \), or equivalently, is a candidate for \( U \) for it is diagonally universal with its image in the range of \( U \) by corollary 3.5.

4 Co-stable functors from factorization systems

We saw in the previous section how stable functors towards a locally presentable category induced a factorization system through a small object argument. In this part we also make use of a factorization system, but for a different purpose: in the context of a category with a terminal object with suitable property, we show how to construct a co-stable inclusion from a class of “left objects” (aka, objects with a left map as their terminal map). The motivation for this construction will be made clearer in a future work, where the bicategorical version of this process will be developed and applied in the bicategory of Grothendieck topoi relatively to several factorization systems, and provide a notion of “2-geometry” for Grothendieck topoi. However, as we chose to remain purely 1-categorical in this paper, we do not go into the 2-dimensional version of this construction and give it as an autonomous construction to prepare for its future involvement.

Throughout this section, we want to emphasize the “geometric” intuition leading our work. Hence, while the remaining of the paper followed “algebraic convention”, where we studied stables functor in to categories of objects to be seen as algebraic (for instance, living in a locally presentable category), this section will produce co-stable functors rather than stable ones, and the object of the ambient category should be seen as spaces.

Before anything, let us precise that we call a functor \( F : A \to C \) co-stable if the corresponding functor between the opposite categories \( F^{op} : A^{op} \to C^{op} \) is stable.

We fix a category \( C \) with a terminal object. This object should be seen as the generic point, as is the point \(*\) in the category of topological spaces, or the topos of Sets in the bicategory of Grothendieck toposes, or the 2-elements lattice 2 seen as the trivial locale. Now we equip \( C \) with a factorization system \((L, R)\). We recall first some general properties of the left and right classes of a factorization system:

**Proposition 4.1.** For a factorization system \((L, R)\):

- \( L \) contains all isomorphisms and is closed under composition,
- \( L \) is left-cancellative: if one has \( C_1 \xrightarrow{l'} C_2 \xrightarrow{l} C_3 \) then \( f \) is also in \( L \)
- \( R \) contains all isomorphisms and is closed under composition,
- \( R \) is right-cancellative: if one has \( C_1 \xrightarrow{r'} C_2 \xrightarrow{r} C_3 \) then \( f \) is also in \( R \)
- \( L \) is closed under colimits in \( C \)
- \( R \) is closed under limits in \( C \)

Now the core idea of this section is that one can classify objects of \( C \) as left or right depending whether their terminal map is left or right; though arbitrary objects may not lie in either of those two classes, it is well known that right classes always define reflective subcategories by this process, which means that any object admits a “right replacement”.

**Definition 4.2.** Let \( C \) be a category equipped with a factorization system \((L, R)\) and a terminal object. Let define
the class of $\mathcal{L}$-objects as those $L$ such that $L \xrightarrow{1} 1$ is in $\mathcal{L}$; together with the $L$ maps they form a subcategory $\mathcal{L}Ob_j$.

the class of $\mathcal{R}$-objects as those $R$ such that $R \xrightarrow{1} 1$ is in $\mathcal{R}$; together with the $R$ maps they form a subcategory $\mathcal{R}Ob_j$.

Remark. Now observe the following:

1 is, up to iso, the only object up to iso to be both left and right.

By right cancellation, any point $1 \to R$ of a right object is a right map.

Any arrow toward a right object factorizes through a right object by right cancellability of the right class.

The category $\mathcal{R}Ob_j$ of right objects and right maps is reflexive in $\mathcal{C}$ because of the factorization of terminal maps

\[
\begin{array}{ccc}
A & \xrightarrow{l_A} & 1 \\
\downarrow{l_A} & & \downarrow{r_A} \\
\mathcal{R}(A) & & \\
\end{array}
\]

Indeed the reflector $A \xrightarrow{l_A} \mathcal{R}(A)$ (which is a left map) is initial amongst those arrows toward a right object: for any $A \xrightarrow{f} R$, the statute of $(l_A, r_A)$ as the terminal factorization of $!_A$ with a left map on the left induces the dashed arrow in the following:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & R \\
\downarrow{l_A} & & \downarrow{r_f} \\
\mathcal{R}(A) & \xrightarrow{\exists \in \mathcal{L}} & R_f \\
\uparrow{r_A} & & \downarrow{!_R} \\
1 & & \\
\end{array}
\]

In particular the right reflection of a left object is necessarily 1 by applying cancellation property of both classes to its terminal map.

Any object in $\mathcal{C}$ admits exactly one $\mathcal{L}$-map into an $\mathcal{R}$-object: its own reflection map, because post composing it with the terminal map of this $\mathcal{R}$-object returns the factorization of the terminal map of $\mathcal{C}$ which is unique.

Both right and left objects possess a gliding condition along their associated map by stability by composition: in the following

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{l_A} & & \downarrow{l_B} \\
1 & & \downarrow{!_B} \\
\end{array}
\]

Then $!_A$ is either in $\mathcal{R}$ or $\mathcal{L}$ as soon as both $f$ and $!_B$ are. However, as we are in the geometric side, this is not the condition from which we want to deduce stability.

Left objects have moreover the co-gliding condition relatively to the $\mathcal{L}$-maps: in the following

\[
\begin{array}{ccc}
L & \xrightarrow{t \in \mathcal{L}} & A \\
\downarrow{l_L} & & \downarrow{l_A} \\
1 & & \\
\end{array}
\]

if $L$ is a left object sending a left map toward an object $A$, then $A$ has to be a left object by left cancellation. As a consequence, in the opposite side we have a stable inclusion:

$\mathcal{L}Ob_j^{op} \hookrightarrow \mathcal{C}^{op}$

However notify we cannot infer any smallness condition, so this has to be token as a local reflexivity. But this just forces us to admits the use of large spaces with a proper (at least a moderate) class of points.
Most of those properties have a relativised version, that is, a corresponding statement where 1 has been replaced by an arbitrary base object \( B \) of \( \mathcal{C} \).

**Definition 4.3.** For \( B \) in \( \mathcal{C} \), we define \( \mathcal{L} \text{Obj}[B] \) (resp. \( \mathcal{R} \text{Obj}[B] \)) as the categories whose objects are respectively left maps \( l : C \to B \in \mathcal{L} \) (resp. \( r : C \to B \in \mathcal{R} \)) and \( \mathcal{L} \)-maps (resp \( \mathcal{R} \)-maps) between them.

**Proposition 4.4.** We have the same properties as over 1:

- \( \mathcal{R} \text{Obj}[B] \) is a reflective subcategory of \( \mathcal{C}/B \) where the free right object of a map \( f : C \to B \) is \( r_f : C_f \to B \) and its unit is given by the left map \( l_f : C \to C_f \);
- isomorphism \( B' \simeq B \) are the unique object to be both right and left;
- the category \( \mathcal{L} \text{Obj}[B] \) is co-stable in \( \mathcal{C}/B \) because of the glidding condition induced by the left cancellation property of the left maps.

However it may happen that we want to restrict our notion of local objects and not to take all the \( \mathcal{L} \)-objects. In the context above, \( \mathcal{L} \)-objects can be seen as “connected objects”, while \( \mathcal{R} \)-objects behaves as discrete objects, and similarly in the relative case, any \( \mathcal{L} \)-maps \( l : C \to B \in \mathcal{L} \) (resp. any \( \mathcal{R} \)-maps \( r : C \to B \in \mathcal{R} \)) exhibits \( C \) as a connected (resp. discrete) object over \( B \). In particular we want to see \( \mathcal{R} \)-maps \( r : C \to B \) as playing the same role as étale maps in the case of spectra, that is, as étale domains of a certain geometry. In particular the \( \mathcal{R} \)-map from a \( \mathcal{L} \)-object \( r : L \to B \) with \( L \) a left object can be seen as a connected component of \( B \), while the reflection \( \mathcal{R}(C) \) of a given object \( C \) can be seen as the object of connected components of \( C \).

Let \( (\mathcal{L}, \mathcal{R}) \) be a factorization system on \( \mathcal{C} \) as above. Now we are going to chose as left objects a subclass \( \mathcal{L}' \subseteq \mathcal{L} \) behaving at least as a left class in an orthogonality structure, while not constituting necessarily the whole left class of a factorization system, plus some additional condition of closure by pullback and cancellability along \( \mathcal{L} \)-maps.

**Definition 4.5.** Let be \( \mathcal{C} \) and \( (\mathcal{L}, \mathcal{R}) \) as above with a distinguished class \( \mathcal{L}' \subseteq \mathcal{L} \). We say that \( \mathcal{L}' \) has left \( \mathcal{L} \)-cancellation property when for each triangle as below

\[
\begin{array}{ccc}
A & \xrightarrow{t \in \mathcal{L}} & B \\
\phantom{t} & \searrow & \nearrow f \\
\phantom{t} & l' \in \mathcal{L}' & C \\
\end{array}
\]

then \( l \) being in \( \mathcal{L} \) and \( l' \) in \( \mathcal{L}' \) forces \( f \) to be in \( \mathcal{L}' \).

In the following we fix a class of map \( \mathcal{L}' \subseteq \mathcal{L} \) satisfying the left \( \mathcal{L} \)-cancellation property which moreover we suppose stable by pullback and composition and containing all iso. In particular \( \mathcal{L}' \)-maps are left orthogonal to \( \mathcal{R} \)-maps.

**Definition 4.6.** We define the category \( \mathcal{L}' \text{Obj}\mathcal{L}' \) as having \( \mathcal{L}' \)-objects as objects and \( \mathcal{L} \)-maps between them. Similarly, for any \( B \) in \( \mathcal{C} \), we define \( \mathcal{L}' \text{Obj}\mathcal{L}'[B] \) as having \( \mathcal{L}' \)-maps \( l : C \to B \) for objects and triangles with \( \mathcal{L} \)-maps between them as morphisms.

**Proposition 4.7.** In a context above with \( \mathcal{L}' \) left \( \mathcal{L} \)-absorbing, we have a stable inclusion

\[
(\mathcal{L}' \text{Obj}\mathcal{L}')^{\text{op}} \hookrightarrow \mathcal{C}^{\text{op}}
\]

Indeed it is easy to see that we then have the op-gliding condition above any object by absorption of the terminal map of the intermediate object of any \( (\mathcal{L}, \mathcal{R}) \)-factorization: indeed in the situation below

\[
\begin{array}{ccc}
L' & \xrightarrow{f} & A \\
\phantom{f} & l_f & \searrow r_f \\
\phantom{f} & l' \in \mathcal{L}' & C \\
\end{array}
\]

\( l_f \) is in \( \mathcal{L}' \) as soon as \( l' \) is.

**Definition 4.8.** In the spirit of \([2]\), we define as \( \mathcal{L}' \)-form of an object \( C \) all \( \mathcal{R} \)-maps \( r : L \to C \) with a \( \mathcal{L}' \)-object as domain. In particular any arrow between \( \mathcal{L}' \)-forms is in \( \mathcal{R} \) and should be seen as an inclusion of fundamental neighborhood, or equivalently, as generalized specialization order.
This means that any left form of $A$ induces a point of $\mathcal{R}(A)$: to any $r_x : L_x \to A$ we can associate $p_x = R(r_x) : 1 \to \mathcal{R}(A)$. Moreover, we could think this assignment as “continuous” in the sense that any $\mathcal{R}$-neighborhood $R' \to R$ of $p_x$ induces a $\mathcal{R}$-neighborhood of $r_x$ by pullback

![Diagram](image)

However in general, it is clear by functoriality of $\mathcal{R}(-)$ that the reflection of an object $A$ does not correctly distinguish local forms of $A$ as it collapses the specialization order when this one is non trivial: for a triangle of $\mathcal{R}$-maps

![Diagram](image)

then $\mathcal{R}(r_1) = \mathcal{R}(r_2) = \mathcal{R}!1$ so that both define the same point of $\mathcal{R}(A)$. Intuitively, $\mathcal{R}$ collapses the local components and produces a disconnected set of points with trivial specialization order.

In this context, $\mathcal{R}$-objects still play the role of “discrete” objects and $\mathcal{L}$-objects the role of connected object, while $\mathcal{L}'$-objects as local objects are to be seen as the “fundamental neighborhood” of the geometry, while the $\mathcal{R}$-maps codes its etale domain.

The $\mathcal{L}'$-maps from an arbitrary object toward an object will then be examples of “bundles of $\mathcal{L}'$-objects” over a space, that is, bundles that are locally $\mathcal{L}'$-objects in the sense that their fibers are $\mathcal{L}'$-objects and $\mathcal{L}'$-form of the domain of the bundle, from our key assumption that $\mathcal{L}$-maps are closed by pullback:

![Diagram](image)

**Definition 4.9.** Define $\mathcal{C}_{\text{Spaces}}$ as the category of “$\mathcal{C}$-modelled spaces” whose

- objects are arbitrary maps $f : C \to B$,
- and morphisms $f_1 \to f_2$ are triples $(g, a^f, a_g)$ of the form

![Diagram](image)

**Remark.** Actually, in the diagram above, $a^f$ and $a_g$ are obviously mutually determined, but this emphasizes the analogy with modelled toposes as they correspond to the dual maps of the inverse and direct comorphisms. Indeed, one would expect $f_1, f_2$ to code for sheaves of $\mathcal{C}^{op}$ objects over $\mathcal{R}$-spaces with $g$ a continuous map between them, along which one can either pull or push those sheaves. In this point of view, $g^*f_2 : g^*C_2 \to B_1$ is the inverse image of $f_2$ over $B_1$ and $g \circ f_1$ really is the direct image $r, f_1$ of $f_1$ over $B_2$.

Now we turn to the objects we want to see as locally modelled objects:
Definition 4.10. For each object $B$, we define $\mathcal{L}oc_L^L[B]$ as the category whose:

- objects are $C \to B$ that are stalkwise in $\mathcal{L}'$
- maps are triangles that are stalkwise in $\mathcal{L}$

Then we define the category $\mathcal{L}oc_L^L$ as the wide subcategory of $\int \mathcal{L}oc_L^L[\_\_]$ whose arrows are restricted to the $(g, a^\sharp)$ with $a^\sharp$ stalkwise in $\mathcal{L}$. Observe that in particular this is a non full subcategory of $C \to \text{spaces}$.

Remark. Observe we have to require the comorphisms to be locally in $\mathcal{L}$ as requiring it globally would not be sufficient from the non stability along pullback; actually, they do not even need to be so globally. This mimics the idea of local morphism, which returns local maps at stalks, in the usual spectral construction.

In the following diagram

$$
\begin{array}{ccc}
C_1 & \xrightarrow{l \in \mathcal{L}} & C_2 \\
\downarrow{f_1} & & \downarrow{f_2} \\
C & & C
\end{array}
$$

Then, if $f_1$ is in $\mathcal{L}$-cancellation. If $f_1$ locally is in $\mathcal{L}$, then by right cancellation of pullback square we know that for any point $p$, $p^*a^\sharp$ is in $\mathcal{L}$ so that one has over $1$

$$
\begin{array}{ccc}
p^*C_1 & \xrightarrow{p^*l \in \mathcal{L}} & p^*C_2 \\
p^*f_1 \in \mathcal{L} & & p^*f_2 \\
1 & & 1
\end{array}
$$

which forces each $p^*f_2$ to be in $\mathcal{L}$.

Proposition 4.11. One has a relatively full and faithful costable inclusion in each $C$

$$\mathcal{L}oc_L^L[C] \hookrightarrow C/C$$

Proof. This just comes from the $\mathcal{L}$-absorption property of $\mathcal{L}'$ applied over $1$ in each point of $\mathcal{R}$. The inclusion functor $\mathcal{L}oc_L^L[C]$ always is relatively full and faithful as indeed in the following situation

$$
\begin{array}{ccc}
A_2 & \xrightarrow{f} & A_3 \\
\downarrow{l' \in \mathcal{L}'} & & \downarrow{l_3} \\
A_1 & & C
\end{array}
$$

$f$ is forced to be in $\mathcal{L}'$ by the absorption property of $\mathcal{L}'$ in $\mathcal{L}$ (this actually does not depends of $l_1, l_2, l_3$ being in $\mathcal{L}'$).

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