$n$-LEVEL DENSITY OF THE LOW-LYING ZEROS OF PRIMITIVE
DIRICHLET $L$-FUNCTIONS

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Abstract. Katz and Sarnak conjectured that the statistics of low-lying zeros of various
family of $L$-functions matched with the scaling limit of eigenvalues from the random matrix
theory. In this paper we confirm this statistic for a family of primitive Dirichlet $L$-functions
matches up with corresponding statistic in the random unitary ensemble, in a range that
includes the off-diagonal contribution. To estimate the $n$-level density of zeros of the
$L$-functions, we use the asymptotic large sieve method developed by Conrey, Iwaniec and
Soundararajan. For the random matrix side, a formula from Conrey and Snaith allows us
to solve the matchup problem.

1. Introduction

Efforts to understand the location of zeros of the Riemann zeta function have played an
important role in the development of analytic number theory. Classically, information about
the horizontal distribution of these zeros yielded better understanding about the distribution
of prime numbers. Moreover, Montgomery [16] calculated statistics of the spacings of zeros
along the vertical line; more specifically, he examined the so called pair-correlation function,
which is a quantity roughly of the form

$$\frac{1}{N(T)} \sum_{0<\gamma, \gamma' \leq T} f \left( (\gamma - \gamma') \left( \frac{\log T}{2\pi} \right) \right),$$

where under the Riemann hypothesis (RH), $1/2 + i\gamma$ are non-trivial zeros of the Riemann
zeta function, $N(T)$ is the number of zeros such that $0 < \gamma \leq T$, and $f$ is a Schwartz
function on $\mathbb{R}$ such that its Fourier transform $\hat{f}$ is supported in $(-1,1)$. Then he showed
that as $T \to \infty$

$$\frac{1}{N(T)} \sum_{0<\gamma, \gamma' \leq T \atop \gamma \neq \gamma'} f \left( (\gamma - \gamma') \left( \frac{\log T}{2\pi} \right) \right) \to \int_{-\infty}^{\infty} f(x) W^{(2)}(x) \, dx,$$  \hspace{1cm} (1.1)

where $W^{(2)}(x) = 1 - \left( \frac{\sin \pi x}{\pi x} \right)^2$. (1.1) is expected to be true for any Schwartz functions, and
this is the Pair Correlation conjecture. Dyson later pointed out to Montgomery that the
factor $W^{(2)}(x)$ is the same as the distribution of the spacings of eigenvalues of the Gaussian
unitary ensemble (GUE) distribution from random matrix theory, which forshadowed a great
deal of work later. Indeed, the link between the Riemann zeta function and random matrix
theory has led to a better understanding of both moments and zeros of $L$-functions (see for
example [13], [14] and [20]).

Özlük [18] studied a $q$-analogue of Montgomery’s pair correlation result under the Generalized
Riemann hypothesis (GRH) for Dirichlet $L$-functions. In particular, he considered

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the pair correlation function of a family of Dirichlet $L$-functions averaging over character $\chi$ modulo $q$, where $q \in [Q, 2Q]$. The large size of the family ($\sim Q^2$) compared to the conductor ($\sim Q$) allows for an extension of the support of the Fourier transform of the test function beyond what is readily available. In this undertaking, Özlük dealt with the contribution of certain off diagonal terms, and he was able to succeed with the extra average over the modulus. Recently, the authors in joint work with Liu and Radziwill revisited Özlük’s pair correlation function but averaging over primitive characters instead, using an asymptotic large sieve introduced by Conrey, Iwaniec and Soundararajan. As a result, we improved the proportion of simple zeros of primitive Dirichlet $L$-functions.

The pair correlation conjecture has been extended to $n$-level correlation of the zeros of the Riemann zeta function through random matrix theory, which studies statistics involving $n$-tuples of zeros. In support of the conjecture, Rudnick and Sarnak proved the result for some special test functions $f$. To describe their results more precisely, assuming RH, let $1/2 + i\gamma_j$ be nontrivial zeros of the Riemann zeta function. Rudnick and Sarnak studied the sum of the form

$$ R(T; f, h) = \sum_{A_1 \ldots A_n} \sum_{j_1 \ldots j_n} h\left(\frac{\gamma_{j_1}}{T}\right) \ldots h\left(\frac{\gamma_{j_n}}{T}\right) f(\gamma_{j_1}, L, \ldots, \gamma_{j_n}, L), $$

where $L = \frac{\log T}{2\pi}$, $h$ is a rapidly decaying cut-off function, and the Fourier transform of the test function $f$ is compactly supported in the domain $|\xi_1| + \ldots + |\xi_n| < 2$. In addition, we demand that $f$ satisfies a couple of other technical conditions omitted for now. We define the $n$-level correlation density for the GUE model as

$$ W^{(n)}(x) := W^{(n)}(x_1, \ldots, x_n) := \det(K_0(x_j, x_k))_{j,k}, \tag{1.2} $$

where

$$ K_0(x, y) = \frac{\sin(\pi(x - y))}{\pi(x - y)}. $$

Then Rudnick and Sarnak showed that

$$ R(T; f, h) \sim N(T) \left(\int_{-\infty}^{\infty} h(r)^n \, dr\right) \int_{\mathbb{R}^n} f(x)W^{(n)}(x)\delta\left(\frac{1}{n}(x_1 + \cdots + x_n)\right) \, dx, $$

where $\delta$ is the Dirac-delta function. This result essentially reduces to (1.1) when $n = 2$. To deal with the sum over non-trivial zeros appearing in $R(T; f, h)$, they applied the explicit formula, which connects this sum over zeros to a sum over prime powers, basically of the form

$$ \sum_{n_1 \ldots n_r, m_1 \ldots m_s} c(n_1) \cdots c(n_r)c(m_1) \cdots c(m_s) \frac{A(n, m, T)}{\sqrt{n_1 \ldots n_r m_1 \ldots m_s}}. $$

where the factor $A(n, m, T)$ contains terms involving the Fourier transform of $f$. The restriction of the support of the Fourier transform of $f$ is required so that the contribution from the off diagonal terms $n_1 \ldots n_r \neq m_1 \ldots m_s$ can be ignored. Although it is not hard to evaluate the diagonal terms $n_1 \ldots n_r = m_1 \ldots m_s$, it was still a challenge to verify that their answers agree with the conjecture arising from the random matrix theory. Rudnick and Sarnak went through complicated combinatorial arguments involving random walks. Later, Conrey and Snaith presented a new formula for $n$-correlation from the random matrix theory side in [5] and applied it in [6] to straightforwardly match results from both sides. Although
this formula looks more intricate than the determinant form in \((1.2)\), it expresses the answer in terms of a test function, where the Fourier transform is supported in any range, and this allows one to naturally match answers from the number theory side.

In analogy with the Pair Correlation conjecture, we expect Rudnick and Sarnak’s result above to hold without any condition on the support of the Fourier transform of \(f\), where the off-diagonal terms also contribute. It is worth noting that this type of conjecture is quite powerful and appears currently intractable. In particular, Montgomery’s original Pair Correlation conjecture easily implies that there are infinitely many pairs of zeros of \(\zeta(s)\) which are far less than the average spacing apart, and this has deep consequences towards Siegel zeros. Typically, even extending the support of the Fourier transform beyond what is currently available is a challenging problem.

Katz and Sarnak \([13, \text{ appendix}]\) computed the \(n\)-level density of eigenvalues of various random matrices and conjectured that the statistics of low-lying zeros of various family of \(L\)-functions is the same as the corresponding one from the random matrix theory. Rubinstein \([19]\) studied a family of quadratic Dirichlet \(L\)-functions and proved that the \(n\)-level density for the family matched with the one for symplectic unitary ensemble in a certain range. Later Gao \([10]\) doubled the allowable range of the support of the Fourier transform of the test function, but he was not able to prove that his answer matched the conjecture from random matrix theory. This was then resolved by Entin, Roditty-Gershon and Rudnick through zeta functions over function fields \([8]\). Recently, Mason and Snaith \([15]\) presented an alternative proof of this result using a new formula for \(n\)-level densities of the random symplectic ensemble, analogous to the work of Conrey and Snaith in \([5]\) and \([6]\).

While only a symplectic family is considered in \([8, 10\] and \([19]\), we consider a family of primitive Dirichlet \(L\)-functions, which is a unitary case. To be more precise, let \(\chi\) be a primitive Dirichlet character modulo \(q > 1\), and a Dirichlet \(L\)-functions associated to it is defined to be

\[
L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}
\]

for \(\text{Re}(s) > 1\). Throughout this paper, we assume GRH for the Dirichlet \(L\)-function \(L(s, \chi)\) and write its nontrivial zeros as \(\frac{1}{2} + i \gamma_{\chi}^j, j = \pm 1, \pm 2, \ldots\), where

\[
\cdots \leq \gamma_{\chi}^{-3} \leq \gamma_{\chi}^{-2} \leq \gamma_{\chi}^{-1} < 0 \leq \gamma_{\chi}^1 \leq \gamma_{\chi}^2 \leq \cdots.
\]

We say that a function \(f : \mathbb{R}^n \to \mathbb{R}\) has the C4-Property provided that

**P1:** Each even function \(f_i : \mathbb{R} \to \mathbb{R}\) has a Fourier transform \(\hat{f_i}(u) := \int_{\mathbb{R}} f_i(x) e^{2\pi i x u} dx\) with a support contained in an interval \([-\eta_i, \eta_i]\) such that

\[
f(x) = f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_i(x_i).
\]

**P2:** \(\eta := \sum_{i=1}^{n} \eta_i < 4\) and \(\epsilon := 4 - \eta > 0\).

We define the \(n\)-level density function by

\[
\mathcal{L}_0(f, W, Q) = \sum_{q} \frac{W(q/Q)}{\varphi(q)} \sum_{\chi \pmod{q}} \sum_{j_1, \ldots, j_n} \sum_{U \gamma_{j_1}^\chi, \ldots, U \gamma_{j_n}^\chi} f(U \gamma_{j_1}^\chi, \ldots, U \gamma_{j_n}^\chi),
\]
where $W$ is a smooth function with a compact support in $[1, 2]$, the $\ast$-sum is over primitive Dirichlet characters modulo $q$, the $\sharp$-sum is over distinct indices $j_k$ and throughout this paper

$$U = \frac{\log Q}{2\pi}. \quad (1.3)$$

If $\eta < 2$, the off-diagonal terms in $L_0$ do not contribute to the main term, and the same method as for proving $n$-correlation of the Riemann zeta function can be applied here, and we do not even need extra average over $q$. For example, previously, Hughes and Rudnick [11] derived the same result as in Theorem 1.1 when $n = 1$ and averaging only over primitive characters of a fixed prime modulus. Otherwise, the off-diagonal terms also contribute to the main term in $L_0$. In this paper, we use the asymptotic large sieve technique to deal with the off-diagonal terms and evaluate

$$L_1(f, W, Q) := \int_\mathbb{R} \sum_q \frac{W(q/Q)}{\varphi(q)} \sum_{\chi \pmod{q}}^\ast \sum_{j_1, \ldots, j_n} f(U(\gamma_{j_1} - t), \ldots, U(\gamma_{j_n} - t)) e^{-t^2} dt.$$

The $t$-average is fairly short due to the rapid decay of $e^{-t^2}$ along the vertical line, and its appearance is to deal with certain unbalanced sums of the prime powers. Thus this average involves points very close to the real axis, and it is expected to have the same asymptotic formula as $L_0$ up to a constant factor. It would be very interesting to develop techniques to evaluate $L_0$ without the additional short average over $t$. The computation of the sixth [3] and eighth moment [1] of Dirichlet $L$-functions, averaging over the same family of primitive characters, also contains a similar $t$-average for the same reason.

Our goal is to prove the following theorem.

**Theorem 1.1.** Assume GRH for all primitive Dirichlet $L$-functions. Let $f$ have $C^4$-Property as described above. Then

$$\lim_{Q \to \infty} \frac{L_1(f, W, Q)}{D(W, Q)} = \int_{\mathbb{R}^n} f(x) W^{(n)}(x) dx,$$

where

$$D(W, Q) := \sum_q \frac{W(q/Q)}{\varphi(q)} \varphi^\ast(q) \int_{-\infty}^\infty e^{-t^2} dt. \quad (1.5)$$

$\varphi^\ast(q)$ is the number of primitive characters mod $q$ and $W^{(n)}(x)$ is defined in (1.2).

This is consistent with the $n$-correlation conjecture arising from the GUE model in random matrix theory where we are able to use a test function whose Fourier transform has double the support of the ones appearing in Rudnick and Sarnak’s work. This is the first time for unitary ensemble that the conjecture is verified for a wider range.

We note that stronger estimations for $n = 1$ without $t$-average were studied and conjectured. For details, see [9] and [11].

The proof contains a number of technical details, so we outline it here. In Section 2, we will apply a combinatorial sieving, which transforms the sum over distinct ordered zeros in $L_1$ to the unrestricted sums. By the explicit formula for a primitive Dirichlet $L$-function, we can express the sum over zeros as a sum over primes. Then, essentially we need to understand the sum $S$ in Proposition 5.1. The diagonal term is easy to be evaluated, but in our case there is an off-diagonal contribution. To deal with these, we apply the asymptotic large sieve technique developed in [4]. Certain delicate combinatorial arrangements appear...
in these terms along this process. This phenomena does not occur in the pair correlation work of [2] because it can be easily reduced to cases when \( m \) and \( n \) are prime numbers. The details will be covered in Section 5. As a result, the asymptotic formula for (1.4) is given in (5.19).

Finally, we verify that the result agrees with the random matrix conjecture through the new \( n \)-correlation formula of Conrey and Snaith [5], [6]. The detailed proof will appear in Section 6.

2. Initial Setup for the Proof of Theorem 1.1

In this section, we will explain how the sum over distinct ordered zeros in \( L_1(f, W, Q) \) can be deduced from the unrestricted sum by the combinatorial sieving. This sieving is also appeared in [20], but we describe it here for the sake of completeness.

A set partition \( G = \{G_1, \ldots, G_\nu\} \) of \( [n] = \{1, 2, \ldots, n\} \) is a decomposition of \( [n] \) into disjoint nonempty subsets \( G_1, \ldots, G_\nu \), where \( \nu = \nu(G) \). The collection \( \Pi_n \) of all set partitions of \( [n] \) forms a lattice with the partial ordering given by \( H \preceq G \) if every set \( G_i \) in \( G \) is a union of sets in \( H \). For example, \( \{\{1, 4\}, \{2\}, \{3\}\} \preceq \{\{1, 4\}, \{2, 3\}\} \) in \( \Pi_4 \). Hence the minimal element of \( \Pi_n \) is \( O = \{\{1\}, \{2\}, \ldots, \{n\}\} \) and the maximal element is \( [n] \).

**Lemma 2.1.** There exists the unique M"obius function \( \mu_n(H, G) \) of the poset \( \Pi_n \) such that for any function \( C, R: \Pi_n \to \mathbb{R} \), satisfying

\[
C_H = \sum_{H \preceq G} R_G,
\]

we have

\[
R_H = \sum_{H \preceq G} \mu_n(H, G) C_G.
\]

In particular,

\[
\mu_n(O, G) = \prod_{j=1}^\nu (-1)^{|G_j| - 1} (|G_j| - 1)!.
\]

Given a set partition \( G = \{G_1, \ldots, G_\nu\} \in \Pi_n \), define an embedding \( \iota_G: \mathbb{R}^\nu \to \mathbb{R}^n \) by \( \iota_G(x_1, \ldots, x_\nu) = (y_1, \ldots, y_n) \), where \( y_\ell = x_j \) if \( \ell \in G_j \). For example, when \( G = \{\{1, 4\}, \{2\}, \{3\}\} \), \( \iota_G(x_1, x_2, x_3) = (x_1, x_2, x_3, x_1) \). We also define

\[
R_{1,G} := \sum_q \frac{W(q/Q)}{\varphi(q)} \sum_{\chi \pmod q} \sum_{\gamma_1, \ldots, \gamma_\nu} g(\iota_G(\gamma_1^x, \ldots, \gamma_\nu^x))
\]

and

\[
C_{1,G} := \sum_q \frac{W(q/Q)}{\varphi(q)} \sum_{\chi \pmod q} \sum_{\gamma_1, \ldots, \gamma_\nu} g(\iota_G(\gamma_1^x, \ldots, \gamma_\nu^x))
\]

where

\[
g(u_1, \ldots, u_n) = \int_\mathbb{R} f(U(u_1 - t), \ldots, U(u_n - t)) e^{-t^2} dt.
\]

Then

\[
C_{1,H} = \sum_{H \preceq G} R_{1,G}.
\]
and by Lemma 2.1
\[ \mathcal{L}_1(f, \mathcal{W}, Q) = R_{1,\mathcal{G}} = \sum_{\mathcal{G} \in \Pi_n} \mu_n(Q,G)C_{1,\mathcal{G}}. \]

We focus on computing \( C_{1,\mathcal{G}} \). Let
\[ F_\ell(x) = \prod_{i \in \mathcal{G}_\ell} f_i(x) \]
for \( \mathcal{G} = \{G_1, \ldots, G_\nu\} \in \Pi_n \). Then by Claim 1 of [19] the Fourier transform \( \hat{F}_\ell(u) \) is supported in \([-\kappa_\ell, \kappa_\ell]\) with \( \kappa_\ell := \sum_{i \in \mathcal{G}_\ell} n_i \) and the function \( \prod_{\ell \leq \nu} F_\ell(x_\ell) \) has the C4-Property defined in Section 4. Thus, we see that
\[ C_{1,\mathcal{G}} = \int_\mathbb{R} \sum_q \mathcal{W}(q/Q) \sum_{\chi(\text{mod } q)} \sum_{\gamma_1, \ldots, \gamma_\nu} \prod_{\ell \leq \nu} F_\ell(U(\gamma_\ell^x - t)) e^{-t^2} dt \]
for \( \gamma \). Applying the explicit formula in Lemma 3.1 we find that
\[ C_{1,\mathcal{G}} = \sum_{s_1 + \cdots + s_4 = |\nu|} \int_\mathbb{R} \sum_q \mathcal{W}(q/Q) \sum_{\chi(\text{mod } q)} \sum_{\ell \in s_1} D_\ell(t) \prod_{\ell \in s_2} D_\ell(t) \prod_{\ell \in s_3} \hat{F}_\ell(0) \prod_{\ell \in s_4} E_\ell(t) e^{-t^2} dt, \]
where
\[ D_\ell(t) = -\frac{1}{\log Q} \sum_{m=1}^{\infty} \frac{\Lambda(m) \chi(m)}{m^{1/2+it}} \hat{F}_\ell \left( -\frac{\log m}{\log Q} \right) \]
and
\[ E_\ell(t) := E_{\hat{F}_\ell}(t) = O \left( \frac{\log(2 + |t|)}{\log Q} \right). \]

Here if \( A \) and \( B \) are sets of integers, \( A + B \) means a disjoint union of \( A \) and \( B \). Next we write
\[ \prod_{\ell \in s_1} D_\ell(t) = \frac{(-1)^{|s_1|}}{(\log Q)^{|s_1|}} \sum_{m=1}^{\infty} a_m(s_1) \chi(m) \frac{1}{m^{1/2+it}} \]
and
\[ \prod_{\ell \in s_2} D_\ell(t) = \frac{(-1)^{|s_2|}}{(\log Q)^{|s_2|}} \sum_{n=1}^{\infty} b_n(s_2) \chi(n) \frac{1}{n^{1/2-it}}, \]
where \(|S_i|\) is the number of elements in \( S_i \),
\[ a_m(s_1) = \sum_{\Pi_{\ell \in s_1} m_\ell = m} \left( \prod_{\ell \in s_1} \Lambda(m_\ell) \hat{F}_\ell \left( -\frac{\log m_\ell}{\log Q} \right) \right), \]
and
\[ b_n(s_2) = \sum_{\Pi_{\ell \in s_2} n_\ell = n} \left( \prod_{\ell \in s_2} \Lambda(n_\ell) \hat{F}_\ell \left( \frac{\log n_\ell}{\log Q} \right) \right). \]
Then
\[
C_{1G} = \sum_{S_1+\cdots+S_4=\nu} \left( \prod_{\ell \in S_3} \hat{F}_\ell(0) \right) \frac{(-1)^{|S_1|+|S_2|}}{(\log Q)^{|S_1|+|S_2|}} \times \int \sum_q \frac{W(q/Q)}{\varphi(q)} \sum_{\chi \pmod{q}} \left( \sum_{m=1}^{\infty} a_m(S_1) \chi(m) \right) \left( \sum_{n=1}^{\infty} b_n(S_2) \bar{\chi}(n) \right) \prod_{\ell \in S_4} E_\ell(t)e^{-t^2 dt}.
\]

(2.4)

We estimate $C_{1G}$ in Sections 4 and 5. In Section 4 we first prove that the main contribution to $C_{1G}$ comes from the cases $S_4 = \emptyset$ and squarefree $m, n$. As mentioned in the introduction, the main contribution is categorized into two types – diagonal terms ($m = n$), calculated in Section 4 and off-diagonal terms ($m \neq n$), estimated in Section 5.

3. Preliminary lemmas

In this section, we present lemmas required in the proof of Theorem 1.1.

Lemma 3.1 (Explicit Formula). Let $\chi$ be a primitive Dirichlet character modulo $q > 1$ and $F : \mathbb{R} \to \mathbb{R}$ be a smooth and rapidly decreasing function with a compact support. Define
\[
\kappa = \kappa_\chi = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1. \end{cases}
\]

Then we have
\[
\sum_{\gamma} F(\mathcal{U}(\gamma - t)) = -\frac{1}{\log Q} \sum_{m=1}^{\infty} \frac{\Lambda(m) \chi(m)}{m^{1/2+it}} \hat{F}\left(-\frac{\log m}{\log Q}\right) - \frac{1}{\log Q} \sum_{m=1}^{\infty} \frac{\Lambda(m) \bar{\chi}(m)}{m^{1/2-it}} \hat{F}\left(\frac{\log m}{\log Q}\right) + \hat{F}(0) + E_F(t),
\]
where $\mathcal{U} = (\log Q)/(2\pi)$ and
\[
E_F(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\mathcal{U}(u - t)) \Re \left[ \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} + iu + \kappa \right) \right] du = O\left( \frac{\log(2 + |t|)}{\log Q} \right).
\]

Proof. Define
\[
\xi(s, \chi) = L(s, \chi) \Gamma\left( \frac{s + \kappa}{2} \right) \left( \frac{q}{\pi} \right)^{(s+\kappa)/2}.
\]
Then $\xi(s, \chi)$ is an entire function and its zeros are exactly the nontrivial zeros of $L(s, \chi)$. By Cauchy’s integral formula
\[
\sum_{\gamma} F(\mathcal{U}(\gamma - t)) = \frac{1}{2\pi i} \int_{(1)} F(\mathcal{U}(-iw - t)) \frac{\xi'}{\xi} \left( \frac{1}{2} + w, \chi \right) dw
\]
\[
- \frac{1}{2\pi i} \int_{(-1)} F(\mathcal{U}(-iw - t)) \frac{\xi'}{\xi} \left( \frac{1}{2} + w, \chi \right) dw
\]
\[
:= I_1 + I_2.
\]
We shall estimate $I_1$ first.
\[
I_1 = \frac{1}{2\pi i} \int_{(1)} F(\mathcal{U}(-iw - t)) \left( \frac{L'}{L} \left( \frac{1}{2} + w, \chi \right) + \frac{1}{2} \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} + w + \kappa \right) + \frac{1}{2} \log \frac{q}{\pi} \right) dw
\]
Writing out $L'/L(s)$ in term of Dirichlet series and shifting the contour integration to $\text{Re}(w) = 0$, we have

\[ I_{11} = - \sum_{m=1}^{\infty} \frac{\Lambda(m)\chi(m)}{m^{1/2+it}} \int_{-\infty}^{\infty} F(u)m^{-iu/\mathcal{U}} du = - \frac{1}{\log Q} \sum_{m=1}^{\infty} \frac{\Lambda(m)\chi(m)}{m^{1/2+it}} \hat{F}\left(- \frac{\log m}{\log Q}\right); \]

\[ I_{12} = \frac{1}{4\pi} \int_{-\infty}^{\infty} F(U(u-t)) \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + w + \kappa\right) du; \]

\[ I_{13} = \frac{\log q/\pi}{2} \frac{\hat{F}(0)}{\log Q}. \]

Next we consider $I_2$. By the functional equation of $\xi(s, \chi)$,

\[ \frac{\xi'}{\xi}(s, \chi) = - \frac{\xi'}{\xi}(1-s, \bar{\chi}). \]

(See Section 10.1 of [17] for the detail.) Thus,

\[ I_2 = \frac{1}{2\pi i} \int_{(1)} F(U(iw-t)) \frac{\xi'}{\xi}\left(\frac{1}{2} + w, \bar{\chi}\right) dw \]

\[ = \frac{1}{2\pi i} \int_{(1)} F(U(iw-t)) \left(\frac{L'}{L}\left(\frac{1}{2} + w, \bar{\chi}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + w + \kappa\right)\right) + \frac{1}{2} \log q/\pi) dw \]

\[ =: I_{21} + I_{22} + I_{23}. \]

By the same argument as $I_1$, we obtain that

\[ I_{21} = - \frac{1}{\log Q} \sum_{m=1}^{\infty} \frac{\Lambda(m)\bar{\chi}(m)}{m^{1/2-it}} \hat{F}\left(\frac{\log m}{\log Q}\right); \]

\[ I_{22} = \frac{1}{4\pi} \int_{-\infty}^{\infty} F(U(u-t)) \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + iu + \kappa\right) du, \]

and

\[ I_{23} = \frac{\log q/\pi}{2} \frac{\hat{F}(0)}{\log Q} = I_{13}. \]

Hence,

\[ \sum_{\gamma} F(U(\gamma-t)) = - \frac{1}{\log Q} \sum_{m=1}^{\infty} \frac{\Lambda(m)\chi(m)}{m^{1/2+it}} \hat{F}\left(- \frac{\log m}{\log Q}\right) - \frac{1}{\log Q} \sum_{m=1}^{\infty} \frac{\Lambda(m)\bar{\chi}(m)}{m^{1/2-it}} \hat{F}\left(\frac{\log m}{\log Q}\right) \]

\[ + \frac{\log q/\pi}{\log Q} \hat{F}(0) + \frac{1}{2\pi} \int_{-\infty}^{\infty} F(U(u-t)) \Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + iu + \kappa\right)\right] du. \]

By Stirling’s formula, the integration above is bounded by

\[ \ll \int_{-10}^{10} |F(U(u-t))| du + \int_{|u|>10} |F(U(u-t))| |\log|u|| du \ll \frac{\log(2+|t|)}{\log Q}, \]

and we then obtain (3.1).
Lemma 3.2 (Large sieve inequality). For any complex numbers $a_n$ with $M < m \leq M + N$, where $N$ is a positive integer, we have

$$\sum_{Q < q \leq 2Q} \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}}^* \left| \sum_{M < m \leq M + N} a_m \chi(m) \right|^2 \ll \left( \frac{Q + N}{Q} \right) \sum_{M < m \leq M + N} |a_m|^2.$$ 

This is a consequence of Theorem 7.13 in [12].

Lemma 3.3. Let $D(W, Q)$ be defined as in (1.5). Then

$$D(W, Q) = \sqrt{\pi} \tilde{W}(1) Q \prod_p \left( 1 - \frac{1}{p^2} - \frac{1}{p^3} \right) + O(\sqrt{Q}),$$

where the product is over the prime numbers.

Proof. By Mellin inversion formula and

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi},$$

we have

$$D(W, Q) = \sqrt{\pi} \sum_{q} \frac{1}{2\pi i} \int_{(2)} \frac{Q^s}{\varphi(q) q^s} \tilde{W}(s) \varphi^*(q) ds.$$ 

Since $\varphi^*(q) = \sum_{cd=q} \varphi(c) \mu(d)$ and $\varphi(q) = \sum_{cd=q} c \mu(d)$, we obtain that

$$\sum_{q=1}^\infty \frac{\varphi^*(q)}{\varphi(q) q^s} = \frac{\zeta(s)}{\zeta(s+1)} G(s),$$

where

$$G(s) = \left( 1 - \frac{1}{p^{s+1}} \right)^{-1} \left( 1 - \frac{1}{(p-1)p^s} + \frac{1}{(p-1)p^{2s}} - \frac{1}{p^{2s+1}} \right),$$

and it is absolutely convergent when Re$(s) > 0$. Therefore

$$D(W, Q) = \sqrt{\pi} \frac{1}{2\pi i} \int_{(2)} Q^s \tilde{W}(s) \frac{\zeta(s)}{\zeta(s+1)} G(s) ds.$$ 

Moving the contour integration to the line Re$(s) = 1/2$, we pick up a simple pole at $s = 1$, bound the rest of integration trivially, and then derive the lemma. 

□

Lemma 3.4. Let $m$ be a positive integer. Then

$$\sum_{d \atop (d,m) = 1} \frac{1}{\varphi(cd)} d^s = \frac{1}{\varphi(c)} \zeta(1 + s) B(s) B_1(s, m) B_2(s, c),$$
where

\[ B(s) = \prod_p \left( 1 + \frac{1}{(p-1)p^{s+1}} \right) \]

\[ B_1(s, m) = \prod_{p|m} \left( 1 - \frac{1}{p^{s+1}} \right) \left( 1 + \frac{1}{(p-1)p^{s+1}} \right)^{-1} \]

\[ B_2(s, c) = \prod_{p|c} \left( 1 + \frac{1}{(p-1)p^{s+1}} \right)^{-1}. \]

This result is from Lemma 6 of [2], and the proof can be found there.

**Lemma 3.5.** Let \( \Psi \) be a nonprincipal Dirichlet character modulo \( d > 1 \). Suppose that \( c, d \leq Q^4 \). Assume GRH for \( L(s, \Psi) \). Define

\[ a_m = \mu^2(m) \sum_{p_1 \cdots p_k = m} \left( \prod_{j=1}^k \log p_j \hat{F}_j \left( -\frac{\log p_j}{\log Q} \right) \right) \]

and

\[ b_n = \mu^2(n) \sum_{p_{k+1} \cdots p_{k+r} = n} \left( \prod_{j=k+1}^{k+r} \log p_j \hat{F}_j \left( \frac{\log p_j}{\log Q} \right) \right), \]

where \( F_j \) is defined in (2.2). Let \( \alpha, \beta \in \mathbb{C} \) with \( \text{Re}(\alpha), \text{Re}(\beta) \in \left( \frac{1}{2} - \frac{10}{\log Q}, \frac{1}{2} + \frac{10}{\log Q} \right) \). Then

\[ \sum_{m,n} \frac{a_m \Psi(m) b_n \overline{\Psi}(n)}{m^\alpha n^\beta} \ll \left( \log \left( Q(2 + |\text{Im}(\alpha)|) \right) \right)^{2k} \left( \log \left( Q(2 + |\text{Im}(\beta)|) \right) \right)^{2r}, \]

where the implied constant depends on \( k \) and \( r \).

**Proof.** Define

\[ g(p_1, \ldots, p_{k+r}) := \prod_{j=1}^k \left( \frac{\Psi(p_j) \log p_j \hat{F}_j \left( -\frac{\log p_j}{\log Q} \right)}{p_j^\alpha} \right) \prod_{j=k+1}^{k+r} \left( \frac{\Psi(p_j) \log p_j \hat{F}_j \left( \frac{\log p_j}{\log Q} \right)}{p_j^\beta} \right) \]

for \( (p_1 \cdots p_{k+r}, c) = 1 \), and otherwise, \( g(p_1, \ldots, p_{k+r}) = 0 \). Furthermore, we define

\[ R_{0, \mathcal{G}} := \sum_{p_1, \ldots, p_\nu} g(\iota_{\mathcal{G}}(p_1, \ldots, p_\nu)) \]

and

\[ C_{0, \mathcal{G}} := \sum_{p_1, \ldots, p_\nu} g(\iota_{\mathcal{G}}(p_1, \ldots, p_\nu)), \]

where \( \mathcal{G} = \{ G_1, \ldots, G_\nu \} \) \( \in \Pi_{k+r} \) and \( \iota_{\mathcal{G}} \) are defined in Section 2. It is clear that

\[ C_{0, \mathcal{H}} = \sum_{\mathcal{H} \prec \mathcal{G}} R_{0, \mathcal{G}}. \]
and by Lemma 2.1 we have

\[
\sum_{\substack{m,n \\ (m,n) = 1}} \frac{a_m \Psi(m) b_n \overline{\Psi(n)}}{m^\alpha n^\beta} = R_{0,G} = \sum_{G \in \Pi_{k+r}} \mu_{k+r}(Q, G) C_{0,G}.
\]

For each \( G = \{G_1, \ldots, G_\nu \} \in \Pi_{k+r} \), we have

\[
C_{0,G} = \prod_{j=1}^\nu \left[ \sum_{(p,c) = 1} \left( \prod_{\ell \leq k} \Psi(p) \log p \, \hat{F}_\ell \left( - \frac{\log p}{\log Q} \right) \right) \left( \prod_{k < \ell \leq k+r} \Psi(p) \log p \, \hat{F}_\ell \left( \frac{\log p}{\log Q} \right) \right) \right].
\]

If \( |G_j| \geq 3 \), then

\[
\sum_{(p,c) = 1} \left( \prod_{\ell \leq k} \Psi(p) \log p \, \hat{F}_\ell \left( - \frac{\log p}{\log Q} \right) \right) \left( \prod_{k < \ell \leq k+r} \Psi(p) \log p \, \hat{F}_\ell \left( \frac{\log p}{\log Q} \right) \right) = O(1).
\]

When \( |G_j| = 2 \), \( \hat{F}_\ell \) is compactly supported in \([-\kappa_\ell, \kappa_\ell]\), where \( \sum_\ell \kappa_\ell < 4 \), and it follows that

\[
\sum_{(p,c) = 1} \left( \prod_{\ell \leq k} \Psi(p) \log p \, \hat{F}_\ell \left( - \frac{\log p}{\log Q} \right) \right) \left( \prod_{k < \ell \leq k+r} \Psi(p) \log p \, \hat{F}_\ell \left( \frac{\log p}{\log Q} \right) \right) = O \left( (\log Q)^2 \right).
\]

Hence

\[
C_{0,G} \ll \prod_{G_j = \{ \ell \}} \left| \sum_{(p,c) = 1} \frac{\Psi(p) \log p}{p^\alpha} \hat{F}_\ell \left( - \frac{\log p}{\log Q} \right) \right| \prod_{G_j = \{ \ell \}} \left| \sum_{k < \ell \leq k+r} \frac{\Psi(p) \log p}{p^\beta} \hat{F}_\ell \left( \frac{\log p}{\log Q} \right) \right| (\log Q)^{2g},
\]

where \( g \) is the number of \( j \) such that \( |G_j| = 2 \). When \( \text{Re}(s) \geq \frac{1}{2} + \frac{1}{\log Q} \), it is known that under GRH,

\[
\frac{\mathcal{L}'(s, \Psi)}{\mathcal{L}(s, \Psi)} = O \left( \log^2 (Q(2 + |\text{Im}(s)|)) \right)
\]

(e.g. Chapter 19 in [7]). By Fourier inversion formula, the fact that \( \hat{F}_\ell \) is supported in \([-\kappa_\ell, \kappa_\ell]\), and the integration by parts, we have for \( |y| \leq 1,000 \),

\[
F_\ell(v + iy) = \int_{\mathbb{R}} \hat{F}_\ell(w) e^{2\pi iyv} e^{2\pi iwy} dw \ll \frac{1}{1 + |v|^4}.
\]
for any nonnegative integer \( A \). Because \( \Psi \) is a non-principal character and using the bound above, we have

\[
\sum_{(p,c)=1} \frac{\Psi(p) \log p}{p^\alpha} \hat{F}_\ell \left( - \frac{\log p}{\log Q} \right) = \sum_n \frac{\Psi(n) \Lambda(n)}{n^\alpha} \hat{F}_\ell \left( - \frac{\log n}{\log Q} \right) + O(\log Q) \tag{3.4}
\]

\[
= -U \int_R \hat{F}_\ell \left( U \left( v - \frac{20i}{\log Q} \right) \right) \frac{L'}{L} \left( \frac{20}{\log Q} + \alpha + iv \right) dv + O(\log Q)
\]

\[
\ll U \int_R \frac{1}{1 + U^A |v|^A} \log^2 \left( Q(2 + |\text{Im}(\alpha)| + |v|) \right) dv
\]

\[
\ll \log^2 \left( Q(2 + |\text{Im}\alpha|) \right).
\]

Therefore,

\[
C_{0,Q} \ll \left( \log \left( Q(2 + |\text{Im}(\alpha)|) \right) \right)^{2k} \left( \log \left( Q(2 + |\text{Im}(\beta)|) \right) \right)^{2r},
\]

and the lemma follows from the above and Equation (3.3).

\[\square\]

**Lemma 3.6.** Assume RH and that \( F : \mathbb{R} \to \mathbb{R} \) is smooth and rapidly decreasing, and \( \hat{F} \) is supported in \([-\kappa, \kappa]\). Define

\[
R_\pm(\alpha, F) = \sum_p \frac{\log p}{p^\alpha} \hat{F} \left( \pm \frac{\log p}{\log Q} \right) - F(\pm iU(1 - \alpha)) \log Q.
\]

Then

\[
R_\pm(\alpha, F) = -\log Q \int_{-\infty}^0 \hat{F}(\pm w) Q^{1-\alpha} w dw + O(1) + O \left( \left( \frac{1}{\log Q} + |\alpha| \right) (\Re(\alpha) - 1/2)^{-3} \right)
\]

for \( \Re(\alpha) \geq 1/2 + 10/\log Q \), and

\[
R_\pm(\alpha, F) = O((\log Q)^2)
\]

for \(|\Re(\alpha) - 1/2| \leq 10/\log Q\).

**Proof.** Since \( F : \mathbb{R} \to \mathbb{R} \), we have \( F(w) = \hat{F}(-w) \) for \( w \in \mathbb{R} \). Therefore it is enough to consider only the positive case.

When \(|\Re(\alpha) - 1/2| \leq 10/\log Q\), by similar arguments to (3.4), we obtain that

\[
\sum_p \frac{\log p}{p^\alpha} \hat{F} \left( - \frac{\log p}{\log Q} \right) = F(iU(\alpha - 1)) \log Q + O((\log Q)^2).
\]

Now we prove the first assertion. Assume that \( \Re(\alpha) \geq 1/2 + 10/\log Q \). By the prime number theorem of the form

\[
\psi(x) := \sum_{p \leq x} \log p = x + O(\sqrt{x}(\log x)^2)
\]
under RH, we have
\[
\sum_p \frac{\log p}{p^\alpha} \hat{F} \left( -\frac{\log p}{\log Q} \right) = \int_1^\infty \frac{1}{v^\alpha} \hat{F} \left( -\frac{\log v}{\log Q} \right) dv + \int_1^\infty \frac{1}{v^\alpha} \hat{F} \left( -\frac{\log v}{\log Q} \right) d(\partial(v) - v)
\]
\[
= \log Q \int_{-\infty}^0 \hat{F}(w)Q^{(\alpha-1)w}dw + O(1) + O\left(\left(\frac{1}{\log Q} + |\alpha|\right) \int_1^\infty v^{-\alpha-1/2}(\log v)^2dv\right)
\]
\[
= F(i\mathcal{U}(\alpha - 1)) \log Q - \log Q \int_{-\infty}^0 \hat{F}(-w)Q^{(1-\alpha)w}dw + O(1) + O\left(\left(\frac{1}{\log Q} + |\alpha|\right)(\Re(\alpha) - 1/2)^{-3}\right).
\]

\[\square\]

**Lemma 3.7.** Let \( w_1, w_2 \) be complex numbers with \( \Re(w_1) = \delta_1 < \Re(w_2) = \delta_2 \). Let \( F : \mathbb{R} \to \mathbb{R} \) be a smooth and rapidly decreasing function with compactly supported \( \hat{F} \). Then
\[
\frac{1}{2\pi i} \int (\delta) F(iz) \left( \frac{1}{z - w_1} - \frac{1}{z - w_2} \right) dz = \begin{cases} 
\int_0^\infty \hat{F}(-u)e^{-2\pi i w_1 u}du - \int_0^\infty \hat{F}(-u)e^{-2\pi i w_2 u}du & \text{if } \delta < \delta_1 \\
\int_{-\infty}^0 \hat{F}(-u)e^{-2\pi i w_1 u}du + \int_0^\infty \hat{F}(-u)e^{-2\pi i w_2 u}du & \text{if } \delta_1 < \delta < \delta_2 \\
\int_{-\infty}^0 \hat{F}(-u)e^{-2\pi i w_1 u}du - \int_{-\infty}^0 \hat{F}(-u)e^{-2\pi i w_2 u}du & \text{if } \delta_2 < \delta.
\end{cases}
\]

**Proof.** Applying the inversion formula
\[
F(iz) = \int_{\mathbb{R}} \hat{F}(-u)e^{-2\pi i uz}du
\]
and then changing the order of integrals, we see that
\[
\frac{1}{2\pi i} \int (\delta) F(iz) \left( \frac{1}{z - w_1} - \frac{1}{z - w_2} \right) dz = \int_{-\infty}^0 \hat{F}(-u)\frac{1}{2\pi i} \int (\delta) \left( \frac{e^{-2\pi i uz}}{z - w_1} - \frac{e^{-2\pi i uz}}{z - w_2} \right) dz du
\]
\[
+ \int_0^\infty \hat{F}(-u)\frac{1}{2\pi i} \int (\delta) \left( \frac{e^{-2\pi i uz}}{z - w_1} - \frac{e^{-2\pi i uz}}{z - w_2} \right) dz du.
\]
For \( u \leq 0 \) we shift the \( z \)-integral to \(-\infty\); otherwise, we shift the \( z \)-integral to \( \infty \). By picking up residues properly, we can conclude the proof of the lemma. \[\square\]

4. **Extracting the main contribution of \( C_{1, L} \)**

We recall from Equation (2.4) that
\[
C_{1, L} = \sum_{s_1 + \cdots + s_4 = \nu} \left( \prod_{\ell \in S_3} \hat{F}_\ell(0) \right) \frac{(-1)^{|S_1| + |S_2|}}{(\log Q)^{|S_1| + |S_2|}}
\]
Lemma 4.1. Let all notations be defined as in Section 2. Then
\[
C_{1,\varnothing} = \sum_{S_1 + \ldots + S_4 = [\varnothing]} \left( \prod_{\ell \in S_3} \hat{F}_\ell(0) \right) \frac{(-1)^{|S_1| + |S_2|}}{(\log Q)^{|S_1| + |S_2|}} \times \sum_q \frac{\mathcal{W}(q/Q)}{\varphi(q)} \sum_{\chi (\text{mod } q)} \sum_{m,n} \frac{a_m(S_1)\chi(m)}{\sqrt{m}} \frac{b_n(S_2)\bar{\chi}(n)}{\sqrt{n}} \int_{\mathbb{R}} \left( \frac{n}{m} \right)^{it} \prod_{\ell \in S_4} E_\ell(t) e^{-t^2} dt.
\]

We first want to restrict the sum over \(m,n\) to the length at most \(Q^2\) with a small error, which will allow us to apply the large sieve inequality in Lemma 3.2.

**Lemma 4.1.** Let all notations be defined as in Section 2. Then
\[
\mathcal{C} = \sum_{S_1 + \ldots + S_4 = [\varnothing]} \left( \prod_{\ell \in S_3} \hat{F}_\ell(0) \right) \frac{(-1)^{|S_1| + |S_2|}}{(\log Q)^{|S_1| + |S_2|}} \times \sum_q \frac{\mathcal{W}(q/Q)}{\varphi(q)} \sum_{\chi (\text{mod } q)} \sum_{m,n \leq Q^2} \frac{a_m(S_1)\chi(m)}{\sqrt{m}} \frac{b_n(S_2)\bar{\chi}(n)}{\sqrt{n}} \int_{\mathbb{R}} \left( \frac{n}{m} \right)^{it} \prod_{\ell \in S_4} E_\ell(t) e^{-t^2} dt
\]

\[+ O(e^{-\frac{Q^2}{6}(\log Q)^2}).\]

**Proof.** As previously mentioned, each \(\hat{F}_\ell(u)\) is supported in \(|u| \leq \kappa_\ell := \sum_{\ell \in F_\ell} \eta_\ell\). Thus, \(|m_\ell| \leq Q\kappa_\ell\) for \(\ell \in S_1\), \(|n_\ell| \leq Q\kappa_\ell\) for \(\ell \in S_2\) and
\[
|m| = \left| \prod_{\ell \in S_1} m_\ell \right| \leq Q^{\kappa(S_1)}, \quad |n| = \left| \prod_{\ell \in S_2} n_\ell \right| \leq Q^{\kappa(S_2)},
\]

where \(\kappa(S_1) := \sum_{\ell \in S_1} \kappa_\ell\) and \(\kappa(S_2) := \sum_{\ell \in S_2} \kappa_\ell\). Note that \(\sum_{\ell=1}^n \kappa_\ell = \sum_{i=1}^n \eta_i = \eta \leq 4 - \varepsilon\).

The Fourier transform of \(F_\ell(U(u - t))(1 + u^2)\) is
\[
\int_{\mathbb{R}} F_\ell(U(u - t))(1 + u^2)e^{2\pi i uv} du = e^{2\pi i t v} \int_{\mathbb{R}} F_\ell(Uu)(1 + t^2 + 2tu + u^2)e^{2\pi i uv} du
\]
\[= e^{2\pi i t v} \left( \frac{1 + t^2}{U} \hat{F}_\ell \left( \frac{v}{U} \right) + \frac{t}{\pi i U^2} \hat{F}_\ell \left( \frac{v}{U} \right) - \frac{1}{4\pi^2 U^3} \hat{F}_\ell'' \left( \frac{v}{U} \right) \right),
\]

so for each \(\ell \in S_4\), we have
\[
E_\ell(t) = \frac{1}{2\pi} \int_{\mathbb{R}} F_\ell(U(u - t))(1 + u^2) Re \left[ \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} \left( \frac{1}{2} + iu + \kappa \right) \right] \frac{du}{1 + u^2}
\]
\[= \frac{1}{2\pi} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{2\pi i tu} \left( \frac{1 + t^2}{U} \hat{F}_\ell \left( \frac{v}{U} \right) + \frac{t}{\pi i U^2} \hat{F}_\ell \left( \frac{v}{U} \right) - \frac{1}{4\pi^2 U^3} \hat{F}_\ell'' \left( \frac{v}{U} \right) \right) e^{-2\pi i u v} dv \right)
\]
\[\times Re \left[ \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} \left( \frac{1}{2} + iu + \kappa \right) \right] \frac{du}{1 + u^2}
\]
\[= \frac{1}{2\pi} \int_{\mathbb{R}} \left( \int_{-\kappa_\ell}^{\kappa_\ell} e^{2\pi i tu v} U \left( 1 + t^2 \hat{F}_\ell(v) + \frac{t}{\pi i U^2} \hat{F}_\ell(v) - \frac{1}{4\pi^2 U^3} \hat{F}_\ell''(v) \right) e^{-2\pi i u v} dv \right)
\]
\[\times Re \left[ \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} \left( \frac{1}{2} + iu + \kappa \right) \right] \frac{du}{1 + u^2}.
\]

Hence the \(t\)-integral
\[
\int_{\mathbb{R}} \left( \frac{n}{m} \right)^{it} \prod_{\ell \in S_4} E_\ell(t) e^{-t^2} dt
\]
in $C_{1,\mathcal{G}}$ is a combination of
\[ \int_{\mathbb{R}} \left( \frac{n}{m} \right)^{it} t^{A_1} e^{itv(S_4)\log Q} e^{-t^2} dt \]
with a nonnegative integer $A_1$, where
\[ v(S_4) := \sum_{\ell \in S_4} v_{\ell} \]
satisfying
\[ |v(S_4)| \leq \kappa(S_4). \]
It is known that
\[ \int_{-\infty}^{\infty} e^{-x^2} e^{i\xi x} dx = \sqrt{\pi} e^{-\xi^2/4}. \tag{4.2} \]
Taking $j^{th}$ derivative with respect to $\xi$ on both sides, we obtain that
\[ \int_{-\infty}^{\infty} (ix)^j e^{-x^2} e^{i\xi x} dx = e^{-\xi^2/4} P_j(\xi), \]
where $P_j(\xi)$ is an $j$-degree polynomial function. Therefore
\[ \int_{\mathbb{R}} \left( \frac{n}{m} \right)^{it} t^{A_1} e^{itv(S_4)\log Q} e^{-t^2} dt = i^{-A_1} e^{\frac{1}{4} \left(v(S_4)\log Q + \log \frac{n}{m}\right)^2} P_{A_1} \left( v(S_4)\log Q + \log \frac{n}{m} \right) . \]
If $\alpha_m(S_1) \neq 0$ for $m = \prod_{\ell \in S_1} m_\ell > Q^2$, then $\kappa(S_1) > 2$. Since $\kappa(S_1) + \kappa(S_2) + \kappa(S_4) \leq 4 - \varepsilon$, it follows that
\[ \kappa(S_2) + \kappa(S_4) < 2 - \varepsilon, \]
and so
\[ |v(S_4)\log Q + \log \frac{n}{m}| = \log m - \log(nQ^{\nu(S_4)}) \geq \log Q^2 - \log Q^{\kappa(S_2) + \kappa(S_4)} \geq \varepsilon \log Q. \]
Hence,
\[ \int_{\mathbb{R}} \left( \frac{n}{m} \right)^{it} t^{A_1} e^{itv(S_1)\log Q} e^{-t^2} dt \ll (\log Q)^2 e^{-\varepsilon^2/4 (\log Q)^2} \ll e^{-\frac{\varepsilon^2}{5} (\log Q)^2} . \]
Inserting the above bound in (1.1) and (2.4), we obtain that the contribution from the terms $m \geq Q^2$ is
\[ \ll Q^{A_2} e^{-\frac{\varepsilon^2}{5} (\log Q)^2} \ll e^{-\frac{\varepsilon^2}{6} (\log Q)^2} \]
for some constant $A_2 > 0$. The similar arguments can be applied to the terms $n \geq Q^2$, and this concludes the proof of the lemma.

Next, we will show that the main contribution of $C_{1,\mathcal{G}}$ comes from terms with $S_4 = \emptyset$.

**Lemma 4.2.** Let all notations be defined as in Section 3. Then
\[ C_{1,\mathcal{G}} = \sum_{S_1 + S_2 + S_3 = \nu} \left( \prod_{\ell \in S_3} \hat{F}_\ell(0) \right) \frac{(-1)^{|S_1| + |S_2|}}{(\log Q)^{|S_1| + |S_2|}} \sum_{q} \frac{\mathcal{W}(q/Q)}{\varphi(q)} \sum_{\chi \mod q} \sum_{m,n \leq Q^2} \frac{\alpha_m(S_1)\chi(m) b_n(S_2)\bar{\chi}(n)}{\sqrt{m} \sqrt{n}} \int_{\mathbb{R}} \left( \frac{n}{m} \right)^{it} e^{-t^2} dt + O \left( \frac{Q}{\log Q} \right) . \]
Proof. By the bound of $E_t(t)$ in Lemma 3.1 we obtain that the main term of $C_{1,2}$ in Lemma 4.1 is bounded above by

$$\int_{\mathbb{R}} \left( \frac{\log(2 + |t|)}{\log Q} \right)^{|S_4|} e^{-t^2} dt$$

Next, we apply the Cauchy-Schwarz inequality and the large sieve inequality (Lemma 3.2) and have that the above is bounded by

$$\ll \int_{\mathbb{R}} \left( \frac{\log(2 + |t|)}{\log Q} \right)^{|S_4|} e^{-t^2}$$

$$\times \left( \sum_{q} \frac{W(q/Q)}{\varphi(q)} \right)^{1/2} \left( \sum_{q} \frac{W(q/Q)}{\varphi(q)} \right)^{1/2} \left( \sum_{m \leq Q^2} a_m(S_1) \right)^2 \left( \sum_{n \leq Q^2} b_n(S_2) \right)^2$$

$$\ll Q \int_{\mathbb{R}} \left( \frac{\log(2 + |t|)}{\log Q} \right)^{|S_4|} e^{-t^2}$$

$$\times \left( \sum_{m \leq Q^2} \frac{|a_m(S_1)|^2}{m} \right)^{1/2} \left( \sum_{n \leq Q^2} \frac{|b_n(S_2)|^2}{n} \right)^{1/2}$$

Hence the contribution from $S_4 \neq \emptyset$ is at most $O(Q/\log Q)$.

Now we focus on the main term of Lemma 4.2. It is clear that the contribution of the case $S_1 = S_2 = \emptyset$ is

$$D(W, Q) \prod_{\ell = 1}^\nu \hat{F}_\ell(0).$$

If $S_2 = \emptyset$ but $S_1 \neq \emptyset$, then by (4.2) the contribution from these terms is bounded by

$$\ll \frac{Q}{\log Q} \sum_{m \leq Q^2} \frac{|a_m(S_1)|}{\sqrt{m}} \left| \int_{\mathbb{R}} \left( \frac{1}{m} \right)^{it} e^{-t^2} dt \right| \ll \frac{Q}{\log Q} \sum_{m \leq Q^2} \frac{|a_m(S_1)| e^{-(\log m)^2/4}}{\sqrt{m}} \ll \frac{Q}{\log Q}. \quad (4.3)$$

The same holds for the case $S_1 = \emptyset$ and $S_2 \neq \emptyset$. Thus we can now consider the case $S_1, S_2 \neq \emptyset$.

By repeated uses of the Cauchy-Schwarz inequality and Lemma 3.2 we can add the conditions such as $m, n$ are squarefree with an error $O(Q/\sqrt{\log Q})$. Then $m$ and $n$ can be written as products of distinct primes as the following:

$$m = \prod_{\ell \in S_1} p_{\ell}, \quad n = \prod_{\ell \in S_2} p_{\ell}.$$

However, $m, n$ might have a common prime divisor. Let $(m, n) = \prod_{\ell \in S_{11}} p_{\ell} = \prod_{\ell \in S_{21}} p_{\ell}$ for some $S_{11} \subseteq S_1$ and $S_{21} \subseteq S_2$. Then there is a unique bijection $\sigma : S_{11} \to S_{21}$ such that $p_{\ell} = p_{\sigma(\ell)}$ for all $\ell \in S_{11}$. Moreover, since $\hat{F}_j$ is compactly supported, by the similar arguments to the proof of Lemma 4.1 we can remove the conditions $m, n \leq Q^2$ with error term of size $O(e^{-\frac{1}{2} \log Q^2})$. Hence,

$$C_{1,2} = D(W, Q) \prod_{\ell = 1}^\nu \hat{F}_\ell(0) + \tilde{C}_{12} + O \left( \frac{Q}{\sqrt{\log Q}} \right), \quad (4.4)$$
where
\[ \widetilde{C}_G = \sum_{S_1 + S_2 + S_3 = \nu \atop S_1, S_2 \neq \emptyset} \left( \prod_{\ell \in S_3} \hat{F}_\ell(0) \right) \frac{(-1)^{|S_1| + |S_2|}}{(\log Q)^{|S_1| + |S_2|}} \int_{\mathbb{R}} e^{-t^2} \sum_{\sigma : S_1 \rightarrow S_3 \text{ bijection}} \sum_{S_1 + S_2 + S_3 = \nu \atop S_1, S_2 \neq \emptyset} \sum_{\ell} W(q/\ell) \sum_{q \neq \varphi(q)}^{*} \sum_{P \in \mathbb{R}} \mu^2(P) \left( \prod_{\ell \in S_1} \frac{|\chi(p_\ell)|^2 (\log p_\ell)^2}{p_\ell} \hat{F}_\ell \left( -\frac{\log p_\ell}{\log Q} \right) \hat{F}_\sigma(\ell) \left( \frac{\log p_\ell}{\log Q} \right) \right) \]
\times \sum_{m,n}^{m\neq n} \sum_{m_1, n_1} \mu^2(m)a_m(S_{12})\chi(m) \mu^2(n)b_n(S_{22})\bar{\chi}(n) \frac{dt}{m^{1/2 + it}} \frac{dt}{n^{1/2 - it}}.

and \( P = \prod_{\ell \in S_3} p_\ell \). Note that the sum over \( m \) is 1 if \( S_{12} = \emptyset \) and the sum over \( n \) is 1 if \( S_{22} = \emptyset \) and the sum over \( \sigma \) is 1 if \( S_{11} = S_{21} = \emptyset \). When \( S_{12} \neq \emptyset \) and \( S_{22} = \emptyset \), one can show that
\[ \int_{\mathbb{R}} e^{-t^2} \sum_{m} \mu^2(m)a_m(S_{12})\chi(m) \frac{dt}{m^{1/2 + it}} = O(1) \]
by the same method as in (4.3) and its contribution to \( \widetilde{C}_G \) is \( O(Q/\log Q) \). The same holds for the case \( S_{12} = \emptyset \) and \( S_{22} \neq \emptyset \). Let \( D_G \) be the above sum with the additional conditions \( S_{12}, S_{22} = \emptyset \) and \( N_G \) be the above sum with the additional conditions \( S_{12}, S_{22} \neq \emptyset \). Then we see that
\[ \widetilde{C}_G = D_G + N_G + O(Q/\log Q). \quad (4.5) \]
The term \( D_G \) is so-called “diagonal terms” and the term \( N_G \) is “off-diagonal terms”. \( D_G \) has a relatively simple representation as
\[ D_G = \sqrt{\pi} \sum_{S_1 + S_2 + S_3 = \nu \atop S_1, S_2 \neq \emptyset} \left( \prod_{\ell \in S_3} \hat{F}_\ell(0) \right) \frac{(-1)^{|S_1| + |S_2|}}{(\log Q)^{|S_1| + |S_2|}} \sum_{\sigma : S_1 \rightarrow S_3 \text{ bijection}} \sum_{S_1 + S_2 + S_3 = \nu \atop S_1, S_2 \neq \emptyset} \sum_{|S_1| = |S_2|} \sum_{P \in \mathbb{R}} W(q/\ell) \sum_{q \neq \varphi(q)}^{*} \sum_{P \in \mathbb{R}} \mu^2(P) \left( \prod_{\ell \in S_1} \frac{|\chi(p_\ell)|^2 (\log p_\ell)^2}{p_\ell} \hat{F}_\ell \left( -\frac{\log p_\ell}{\log Q} \right) \hat{F}_\sigma(\ell) \left( \frac{\log p_\ell}{\log Q} \right) \right), \]
where \( P = \prod_{\ell \in S_1} p_\ell \). Then we can obtain
\[ D_G = D(W, Q) \sum_{S_1 + S_2 + S_3 = \nu \atop S_1, S_2 \neq \emptyset} \left( \prod_{\ell \in S_3} \hat{F}_\ell(0) \right) \sum_{|S_1| = |S_2|} \prod_{\ell \in S_1} \int_{0}^{\infty} v \hat{F}_\ell(-v) \hat{F}_\sigma(\ell)(v)dv + O \left( \frac{Q}{\log Q} \right) \quad (4.6) \]
by the following lemma.

**Lemma 4.3.** Let \( \chi \) be a primitive Dirichlet character mod \( q \in [Q, 2Q] \) and \( P = \prod_{\ell \in S_1} p_\ell \). Then
\[ \sum_{P} \mu^2(P) \left( \prod_{\ell \in S_1} \frac{|\chi(p_\ell)|^2 (\log p_\ell)^2}{p_\ell} \hat{F}_\ell \left( -\frac{\log p_\ell}{\log Q} \right) \hat{F}_\sigma(\ell) \left( \frac{\log p_\ell}{\log Q} \right) \right) \]
\[ = (\log Q)^2 |S_1| \prod_{\ell \in S_1} \int_{0}^{\infty} v \hat{F}_\ell(-v) \hat{F}_\sigma(\ell)(v)dv + O((\log Q)^2 |S_1| - 1). \]
Proof. By the inclusion-exclusion principle, the prime number theorem and the fact that \( \sum_{p}(\log p)^{r}p^{-\alpha} \) is uniformly convergent and bounded for \( \alpha \geq 2 \) and \( r \leq 2|S_{1}| \), we have that
\[
\sum_{(P,q)=1} \mu^{2}(P) \left( \prod_{\ell \in S_{1}} \frac{|\chi(p_{\ell})|^{2}(\log p_{\ell})^{2}}{p_{\ell}} \hat{F}_{\ell}\left( -\frac{\log p_{\ell}}{\log Q} \right) \hat{F}_{\sigma(\ell)}\left( \frac{\log p_{\ell}}{\log Q} \right) \right)
= \prod_{\ell \in S_{1}} \left( \sum_{(p,q)=1} \frac{(\log p)^{2}}{p} \hat{F}_{\ell}\left( -\frac{\log p}{\log Q} \right) \hat{F}_{\sigma(\ell)}\left( \frac{\log p}{\log Q} \right) + O\left((\log Q)^{2|S_{1}|^{-2}}\right) \right).
\]
Since the number of primes diving \( q \) is \( O(\log q) \), the above is
\[
= \prod_{\ell \in S_{1}} \left( \sum_{p} \frac{(\log p)^{2}}{p} \hat{F}_{\ell}\left( -\frac{\log p}{\log Q} \right) \hat{F}_{\sigma(\ell)}\left( \frac{\log p}{\log Q} \right) + O(\log Q) + O((\log Q)^{2|S_{1}|^{-2}}) \right).
\]
By the prime number theorem and the partial summation, we obtain that
\[
\sum_{p} \frac{(\log p)^{2}}{p} \hat{F}_{\ell}\left( -\frac{\log p}{\log Q} \right) \hat{F}_{\sigma(\ell)}\left( \frac{\log p}{\log Q} \right) = (\log Q)^{2} \int_{0}^{\infty} v \hat{F}_{\ell}(-v) \hat{F}_{\sigma(\ell)}(v) dv + O(1).
\]
Thus the lemma holds.

Therefore, by (4.4), (4.5) and (4.6) we have
\[
C_{1,\underline{G}} = D(W, Q) \sum_{S_{1}+S_{2}+S_{3}=\nu} \left( \prod_{\ell \in S_{3}} \hat{F}_{\ell}(0) \right) \sum_{\sigma:S_{1}\rightarrow S_{2}} \left( \prod_{\ell \in S_{1}} \int_{0}^{\infty} v \hat{F}_{\ell}(-v) \hat{F}_{\sigma(\ell)}(v) dv \right)
+ N_{\underline{G}} + O\left(\frac{Q}{\sqrt{\log Q}}\right). \tag{4.7}
\]

5. Calculation of \( N_{\underline{G}} \)

In this section we will calculate \( N_{\underline{G}} \) defined in a line ahead of (4.5) using the asymptotic large sieve method. By the definition of \( N_{\underline{G}} \) and switching summations, it can be written as
\[
N_{\underline{G}} = \sum_{S_{1}+S_{2}+S_{3}=\nu} \left( \prod_{\ell \in S_{3}} \hat{F}_{\ell}(0) \right) \left( -1 \right)^{|S_{1}|+|S_{2}|} \frac{1}{(\log Q)^{|S_{1}|+|S_{2}|}} \sum_{S_{11}+S_{12}=S_{1}} \sum_{S_{21}+S_{22}=S_{2}} \sum_{S_{12},S_{22}\neq \emptyset} \sum_{\sigma:S_{1}\rightarrow S_{2}} \left( \prod_{\ell \in S_{11}} \left( \frac{(\log p_{\ell})^{2}}{p_{\ell}} \hat{F}_{\ell}\left( -\frac{\log p_{\ell}}{\log Q} \right) \hat{F}_{\sigma(\ell)}\left( \frac{\log p_{\ell}}{\log Q} \right) \right) \right) S(P; S_{12}, S_{22}), \tag{5.1}
\]
where \( P = \prod_{\ell \in S_{11}} p_{\ell} \) and \( S(P; S_{12}, S_{22}) \) denotes
\[
\int_{\mathbb{R}} e^{-\xi} \sum_{q_{1}=1}^{\varphi(q)} \frac{W(q_{1}/Q)}{\varphi(q)} \sum_{\chi(\mod q)}^{*} \sum_{n=1}^{\mu^{2}(n) a_{m}(S_{12}) \chi(n)}^{\mu^{2}(m) b_{n}(S_{22}) \chi(n)} \frac{m^{1/2+it}}{n^{1/2-it}} dt. \tag{5.2}
\]
Note that \( n \) is the positive integer introduced in Section 1 and let \( k \) and \( r \) be positive integers with \( k + r \leq n \). Due to the factor \( \mu^{2}(P) \), \( P \) is supported in squarefree positive integers and the number of prime divisors of \( P \) is less than or equal to \( n \). We start by
estimating $S(P; \{1, ... k\}, \{k+1, ..., k+r\})$, which is a special case of $S(P; S_{12}, S_{22})$. It will be apparent that our treatment of $S(P; \{1, ... k\}, \{k+1, ..., k+r\})$ can be generalized to deal with $S(P; S_{12}, S_{22})$.

**Proposition 5.1.** Define

$$S := S(P; \{1, ... k\}, \{k+1, ..., k+r\})$$

$$= \int_{-\infty}^{\infty} e^{-t^2} \sum_{(q,P) = 1} \mathcal{W}(q/Q) \varphi(q) \sum_{\chi(\mod q)} \sum_{m,n \in (mn,P) = 1} a_m b_n \chi(m) \bar{\chi}(n) \left( \frac{n}{m} \right)^i dt,$$

where

$$a_m = \mu^2(m) \sum_{p_1 \cdots p_k = m} \left( \prod_{j=1}^k \log p_j \hat{F}_j \left( \frac{-\log p_j}{\log Q} \right) \right),$$

$$b_n = \mu^2(n) \sum_{p_{k+1} \cdots p_{k+r} = n} \left( \prod_{j=k+1}^{k+r} \log p_j \hat{F}_j \left( \frac{\log p_j}{\log Q} \right) \right).$$

Suppose that $\hat{F}_i(u)$ is supported in $|u| \leq \kappa_i$ for $1 \leq i \leq k+r$. Also for fixed $\varepsilon > 0$, we assume that $\kappa' + \kappa'' \leq 4 - \varepsilon$, where $\kappa' = \sum_{i=1}^k \kappa_i$ and $\kappa'' = \sum_{j=1}^r \kappa_{k+j}$. Then

$$S(P; \{1, ... k\}, \{k+1, ..., k+r\})$$

$$= Q(\log Q)^{k+r} \sqrt{\pi} \mathcal{W}(1) \prod_{p|P} \left( 1 - \frac{1}{p^2} - \frac{1}{p} \right) \prod_{p|P} \left( 1 - \frac{1}{p} \right) \mathcal{I}(k, r)$$

$$+ O \left( Q(\log Q)^{k+r-1} \right),$$

where

$$\mathcal{I}(k, r) := \sum_{1 \leq j_1 < k} \sum_{k+1 \leq j_2 < k+r} \sum_{T_1, W_1, T_2, W_2, T_3, W_3} \cdots \sum_{T_1+T_2+T_3 = \{j_1+1, ..., j_2+1\}} (-1)^{j_1+r+|W_2|+|W_3|} \int_{D_{k+r}(T, W)} \left( \prod_{j=1}^{k+r} \bar{F}_j(-u_j) \right) (1 - u_{j_1} - u(T)) \delta(u([k+r])) du,$$

$$D_{k+r}(T, W) := D_{k+r}(T_1, T_2, T_3, W_1, W_2, W_3)$$

$$:= \left\{ u \in \mathbb{R}^{k+r} : u_j < 0 \text{ for } j \in T_1 \cup T_3 \cup W_3, \text{ and } u_j > 0 \text{ for } j \in T_2 \cup W_1 \cup W_2 \right\},$$

$\delta(x)$ is the Dirac delta function, $u = (u_1, ..., u_{k+r})$, $du = du_1 \cdots du_{k+r}$, $u(S) := \sum_{j \in S} u_j$ for $S \subseteq [k+r] = \{1, ..., k+r\}$ and $u(T) := u(T_1) + u(T_2) + u(T_3)$.

We need new notations to extend Proposition [5.1] to general cases, so we will postpone it and complete the estimation of $N_{\infty}$ in Section [5.5].
Proof of Proposition 5.1: We start from applying the orthogonality relation of Dirichlet characters and obtain that
\[
\sum_q \frac{\mathcal{W}(q/Q)}{\varphi(q)} \sum_{\chi \mod q} \chi(m) \overline{\chi(n)} = \sum_q \frac{\mathcal{W}(q/Q)}{\varphi(q)} \sum_{d|q} \varphi(d) \mu \left( \frac{q}{d} \right) = \sum_{c,d} \mu(c) \frac{\varphi(d)}{\varphi(cd)} \mathcal{W} \left( \frac{cd}{Q} \right) .
\]

Since \(m\) is supported in products of \(k\) distinct primes, \(n\) is supported in products of \(r\) distinct primes and \((m, n) = 1, m \neq n\).

We have that
\[
S = \int_{-\infty}^{\infty} e^{-t^2} \sum_{m,n} \frac{a_mb_n}{\sqrt{mn}} \left( \frac{n}{m} \right)^{it} \sum_{c,d} \mu(c) \frac{\varphi(d)}{\varphi(cd)} \mathcal{W} \left( \frac{cd}{Q} \right) dt (5.3)
\]

say, where \(S_U\) is the sum over \(c > C\), and \(S_L\) is the sum over \(c \leq C\) with \(C = Q^{\varepsilon_1}\) for some \(\varepsilon_1 > 0\) to be determined later. The remaining part of the proof will be given in Section 5.4.

5.1. Evaluating \(S_U\). In this section we will prove the following lemma.

Lemma 5.2. Let all notations be as above. Recall that \(m, n, P\) are squarefree integers with the number of prime divisors less than or equal to \(n\) and pairwise relatively prime. Then for any \(\varepsilon > 0\)
\[
S_U = M_U + O \left( Q^{\varepsilon + \kappa''}/2 - 1 + \varepsilon + \frac{Q^{1+\varepsilon}}{C} \right),
\]

where
\[
M_U := -\mathcal{W}(0)B(0) \int_{-\infty}^{\infty} e^{-t^2} \sum_{m,n} \frac{a_mb_n}{\sqrt{mn}} \left( \frac{n}{m} \right)^{it} \sum_{c,d} \frac{\mu(c)B_2(0, c)}{\varphi(c)} B_1(0, mnP) dt (5.4)
\]

and \(B_2(0, c)\) and \(B_1(0, mnP)\) are defined as in Lemma 3.4.

Proof. Let
\[
S_U(m, n) := \sum_{c > C, d} \mu(c) \frac{\varphi(d)}{\varphi(cd)} \mathcal{W} \left( \frac{cd}{Q} \right),
\]

then
\[
S_U = \int_{-\infty}^{\infty} e^{-t^2} \sum_{m,n} \frac{a_mb_n}{\sqrt{mn}} \left( \frac{n}{m} \right)^{it} S_U(m, n) dt.
\]
Replacing the condition \( d\mid m - n \) by the orthogonality relation of a character sum, we have

\[
S_U(m, n) = \sum_{c > C, d} \frac{\mu(c)}{\varphi(cd)} W \left( \frac{cd}{Q} \right) \sum_{\Psi \pmod{d}} \Psi(m) \overline{\Psi(n)}
\]

\[
= \sum_{c > C, d} \frac{\mu(c)}{\varphi(cd)} W \left( \frac{cd}{Q} \right) + \sum_{(c, mnP) = 1} \sum_{(d, P) = 1} \frac{\mu(c)}{\varphi(cd)} W \left( \frac{cd}{Q} \right) \sum_{\Psi \neq \Psi_0 \pmod{d}} \Psi(m) \overline{\Psi(n)}
\]

\[
=: S_{U,0}(m, n) + S_{U,E}(m, n).
\]

We first evaluate \( S_{U,0}(m, n) \). By \( \sum_{c\mid q} \mu(c) = 0 \) for \( q > 1 \), writing \( W \) in terms of its Mellin transform \( \widetilde{W} \), we have

\[
S_{U,0}(m, n) = -\sum_{c \leq C, d} \frac{\mu(c)}{\varphi(cd)} W \left( \frac{cd}{Q} \right)
\]

\[
= -\sum_{c \leq C, d} \frac{\mu(c)}{\varphi(cd)} \frac{1}{2\pi i} \int \left( \frac{Q}{cd} \right)^s ds.
\]

Applying Lemma 5.1 to the sum over \( d \), we have

\[
S_{U,0}(m, n) = -\frac{1}{2\pi i} \int \left( \frac{Q}{s} \right)^s \sum_{c \leq C, d} \frac{\mu(c)}{\varphi(c)} \frac{B_2(s, c)}{\varphi(c)s} \zeta(1 + s) B(s) B_1(s, mnP) ds.
\]

We move the \( s \)-contour to \((-1 + \epsilon)\) and encounter a simple pole at \( s = 0 \). Then for any small \( \epsilon > 0 \),

\[
S_{U,0}(m, n) = -\widetilde{W}(0) B(0) \sum_{c \leq C, d} \frac{\mu(c)}{\varphi(c)} B_1(0, mnP) + O(CQ^{-1 + \epsilon}).
\]

Hence by the support of \( \widehat{F}_\epsilon \) in Proposition 5.1 we have

\[
S_{U,0} := \int_{-\infty}^{\infty} e^{-t^2} \sum_{m,n} a_{mn} b_n \sqrt{mn} \left( \frac{n}{m} \right)^it S_{U,0}(m, n) dt = M_U + O(CQ^{(\kappa' + \kappa'')/2 - 1 + \epsilon}).
\]

We next consider \( S_{U,E}(m, n) \). Define

\[
S_{U,E} := \int_{-\infty}^{\infty} e^{-t^2} \sum_{m,n} a_{mn} b_n \sqrt{mn} \left( \frac{n}{m} \right)^it S_{U,E}(m, n) dt
\]

\[
= \int_{-\infty}^{\infty} e^{-t^2} \sum_{c > C, d} \frac{\mu(c)}{\varphi(cd)} W \left( \frac{cd}{Q} \right) \sum_{\Psi \neq \Psi_0 \pmod{d}} \sum_{(m, n) = 1, \sqrt{mn}} a_{mn} \Psi(m) b_n \overline{\Psi(n)} \left( \frac{n}{m} \right)^it dt.
\]
By Lemma 3.5, we obtain that
\[ S_{U,E} \ll (\log Q)^{2(k+r)} \sum_{c \geq C, d} \frac{\varphi(d)}{\varphi(cd)} W \left( \frac{cd}{Q} \right) \ll \frac{Q^{1+\epsilon}}{C} \]
for any \( \epsilon > 0 \). We derive the lemma from the fact that \( S_U = S_{U,0} + S_{U,E} \).

5.2. Evaluating \( S_L \). In this section, we will treat the terms with \( c \leq C \). We write
\[ S_L = \int_{-\infty}^{\infty} e^{-t^2} \sum_{\substack{m, n \geq 1 \atop (m, n) = 1}} a_m b_n \frac{n^t}{\sqrt{mn}} S_L(m, n) dt, \]
where
\[ S_L(m, n) := \sum_{\substack{c \leq C, d \mid m-n \atop (mn, c) = 1}} \mu(c) \frac{\varphi(d)}{\varphi(cd)} W \left( \frac{cd}{Q} \right). \]

The conditions \( (m, n) = 1 \) and \( d \mid m-n \) imply \( (mn, d) = 1 \), so that we can remove the condition \( (mn, d) = 1 \) in the sum. By the identity
\[ \frac{\varphi(d)}{\varphi(cd)} = \frac{1}{\varphi(c)} \sum_{\substack{a \mid c \atop a \mid d}} \frac{\mu(a)}{a}, \]
we obtain that
\[ S_L(m, n) = \sum_{\substack{c \leq C \atop (mnP, c) = 1}} \sum_{a \mid c} \mu(a) \frac{\mu(c)}{a \varphi(c)} \sum_{\substack{d \mid m-n \atop (d, P) = 1}} W \left( \frac{acd}{Q} \right) \]
\[ = \sum_{\substack{c \leq C \atop (mnP, c) = 1}} \sum_{a \mid c} \mu(a) \frac{\mu(c)}{a \varphi(c)} \sum_{\substack{d \mid m-n \atop (d, P) = 1}} W \left( \frac{acd}{Q} \right) \]
\[ = \sum_{\substack{c \leq C \atop (mnP, c) = 1}} \sum_{a \mid c} \mu(a) \frac{\mu(c)}{a \varphi(c)} \sum_{\ell \mid P} \sum_{a \ell g \mid m-n} W \left( \frac{c|\ell}{gQ} \right) \frac{1}{\varphi(a \ell g)} \sum_{\Psi \equiv 1 (\mod a \ell g)} \Psi(m) \overline{\Psi(n)}. \]

We substitute the sum over \( d \) by the sum over \( g \) through the condition \( a\ell g = |m-n| \) and then write the condition \( a\ell g \mid m-n \) in term of Dirichlet characters. Hence
\[ S_L(m, n) = \sum_{\substack{c \leq C \atop (mnP, c) = 1}} \sum_{a \mid c} \mu(a) \frac{\mu(c)}{a \varphi(c)} \sum_{\ell \mid P} \sum_{a \ell g \mid m-n} W \left( \frac{c|\ell}{gQ} \right) \frac{1}{\varphi(a \ell g)} \sum_{\Psi \equiv 1 (\mod a \ell g)} \Psi(m) \overline{\Psi(n)} \]
\[ =: S_{L,0}(m, n) + S_{L,E}(m, n), \]
where
\[
S_{L,0}(m, n) = \sum_{c \leq C \atop \text{gcd}(mPc) = 1} \sum_{a | c} \sum_{g, \ell \mid P} \mu(a) \mu(c) \mu(\ell) \frac{a \varphi(c)}{a \varphi(c)} W\left(\frac{c|m - n|}{gQ}\right) \frac{1}{\varphi(a \ell g)}
\]
and
\[
S_{L,E}(m, n) = \sum_{c \leq C \atop \text{gcd}(mPc) = 1} \sum_{a | c} \mu(a) \mu(c) \frac{a \varphi(c)}{a \varphi(c)} \sum_{\ell \mid P} \sum_{g} W\left(\frac{c|m - n|}{gQ}\right) \frac{1}{\varphi(a \ell g)} \sum_{\Psi \pmod{a \ell g} \neq \Psi_0} \Psi(m) \overline{\Psi}(n).
\]
We remark that \((c, mnP) = 1\) and \(a | c\) imply that \((a, mnP) = 1\). Define
\[
S_{L,0} = \int_{-\infty}^{\infty} e^{-t^2} \sum_{m, n \atop (m, n) = 1} \frac{a_m b_n}{\sqrt{mn}} \left(\frac{n}{m}\right)^{it} S_{L,0}(m, n) \, dt
\]
and
\[
S_{L,E} = \int_{-\infty}^{\infty} e^{-t^2} \sum_{m, n \atop (m, n) = 1} \frac{a_m b_n}{\sqrt{mn}} \left(\frac{n}{m}\right)^{it} S_{L,E}(m, n) \, dt,
\]
so that
\[
S_L = S_{L,0} + S_{L,E}.
\]

We first estimate \(S_{L,E}\).

**Lemma 5.3.** Let \(S_{L,E}\) be defined as \((5.7)\). Then for any \(\epsilon > 0\),
\[
S_{L,E} \ll CQ^{-1+(\kappa' + \kappa'')/2+\epsilon},
\]
where \(\kappa'\) and \(\kappa''\) are defined as in Proposition \(5.7\).

**Proof.** We write out \(S_{L,E}\) as
\[
S_{L,E} = \int_{-\infty}^{\infty} e^{-t^2} \sum_{m, n \atop (m, n) = 1} \frac{a_m b_n}{\sqrt{mn}} \left(\frac{n}{m}\right)^{it} \sum_{c \leq C \atop \text{gcd}(mPc) = 1} \sum_{a | c} \mu(a) \mu(c) \frac{a \varphi(c)}{a \varphi(c)} \sum_{\ell \mid P} \sum_{g} W\left(\frac{c|m - n|}{gQ}\right) \frac{1}{\varphi(a \ell g)} \sum_{\Psi \pmod{a \ell g} \neq \Psi_0} \Psi(m) \overline{\Psi}(n) \, dt.
\]
If \(m\) or \(n\) is greater than \(Q^{(\kappa' + \kappa'')/2+\epsilon_1}\) for \(\epsilon_1 > 0\), then
\[
\min(m, n) \leq Q^{\min(\kappa', \kappa'')} < Q^{(\kappa' + \kappa'')/2+\epsilon_1} \leq \max(m, n),
\]
and
\[
\frac{\max(m, n)}{\min(m, n)} \geq Q^{(\kappa' + \kappa'')/2+\epsilon_1 - \min(\kappa', \kappa'')} \geq Q^{\epsilon_1}.
\]
It then follows that
\[
\int_{-\infty}^{\infty} e^{-t^2} \left(\frac{n}{m}\right)^{it} \, dt \ll e^{-(n/m)^2/4} \ll e^{-(\epsilon_1^2/4)(\log Q)^2}.
\]
Hence, we can restrict the range of $m, n$ up to $Q^{(\kappa + \kappa'')/2 + \epsilon_1}$ with an error of size $O(Q^{-A})$ for any positive integers $A$. For $m, n$ in this range, we have
\[ |m - n| \leq 2Q^{(\kappa + \kappa'')/2 + \epsilon_1}. \]
Since $W$ is supported in the interval $[1, 2]$, if
\[ W \left( \frac{c |m - n|}{gQ} \right) \neq 0, \]
then
\[ g \leq \frac{c |m - n|}{Q} \leq 2cQ^{-1 + (\kappa + \kappa'')/2 + \epsilon_1}. \]
Therefore we add the condition $g \leq \tilde{g} := 2cQ^{-1 + (\kappa + \kappa'')/2 + \epsilon_1}$ and then remove the restriction $m, n \leq Q^{(\kappa + \kappa'')/2 + \epsilon_1}$ from the sum over $m, n$ with an additional error $O(Q^{-A})$. Thus,
\[
S_{L,E} = \int_{-\infty}^{\infty} e^{-t^2} \sum_{m,n} \frac{a_mb_n}{\sqrt{mn}} \left( \frac{n}{m} \right)^{it} \sum_{c \leq C} \frac{\mu(a)\mu(c)}{ac} \psi_{(c,P) = 1} \int_{\Psi = 0}^{\Psi \neq 0} \frac{1}{a\ell g} \psi(a\ell g)g^{-s} \sum_{g \leq \tilde{g}} W \left( \frac{c |m - n|}{gQ} \right) dt ds + O(Q^{-100}),
\]
where $\tilde{W}$ is the Mellin transform of $W$. To separate $m$ and $n$, we apply the following identity
\[
|m - n|^{-s} = \frac{1}{2\pi i} \int_{(\delta_2)} \frac{\Gamma(1 - s)\Gamma(z)}{\Gamma(1 - s + z)} (m^{z-s}n^{-z} + n^{z-s}m^{-z}) \frac{dz}{dz} \quad (5.8)
\]
where $\delta_2 > 0$, $\text{Re}(s) < 0$ and $m \neq n$. The integral is absolutely convergent due to the product of gamma factors decaying like $|z|^{-1 + \text{Re}(s)}$. We write
\[
S_{L,E} = \frac{1}{2\pi i} \int_{(\delta_1)} \tilde{W}(s)Q^s \int_{-\infty}^{\infty} e^{-t^2} \sum_{c \leq C} \frac{\mu(a)\mu(c)}{ac} \psi_{(c,P) = 1} \frac{1}{a\ell g} \psi(a\ell g)g^{-s} \sum_{g \leq \tilde{g}} \left( S_{L,E,1} + S_{L,E,2} \right) dt ds + O(Q^{-100}),
\]
where
\[
S_{L,E,1} = \sum_{m,n} \frac{a_mb_n}{\sqrt{mn}} \left( \frac{n}{m} \right)^{it} \psi(m)\overline{\psi(n)}m^{z-s}n^{-z};
\]
\[ S_{L,E,2} = \sum_{\substack{m,n=1 \atop (m,n)\neq 1 \atop (mn,cP)\neq 1}} \frac{a_mb_n}{\sqrt{mn}} \left( \frac{n}{m} \right)^it \Psi(m)\overline{\Psi(n)}n^{z-s}m^{-z}. \]

We choose \( \delta_i = \frac{1}{\log Q} \). Applying Lemma 3.5 to \( S_{L,E,1} \) and \( S_{L,E,2} \) and using the fact that \( P \leq Q \) (due to support of \( \tilde{F} \)), we obtain that
\[ S_{L,E} \ll Q^{\epsilon} \sum_{c \leq C} \frac{1}{c} \sum_{a | c} \sum_{\ell | P} \sum_{g \leq \tilde{g}} \mu(c) \left( \frac{1}{c^{s+\varphi(c)}} \right) \ll CQ^{-1+(\kappa'+\kappa'')/2+\epsilon} \]
for any \( \epsilon > 0 \), concluding the proof of the lemma.

5.3. Evaluating \( \mathcal{M}_U + S_{L,0} \). Next, we compute \( S_{L,0} \). Indeed, we will show that one of the main terms from \( S_{L,0} \) will cancel out with the main term of \( S_U \), which is \( \mathcal{M}_U \) defined in (5.4). Let \( \mathcal{I}(k, r) \) be defined as in Proposition 5.1. In this section, we will show the following:
\[ \mathcal{M}_U + S_{L,0} = Q(\log Q)^{k+r} \sqrt{\pi} \overline{\mathcal{W}}(1) \prod_{p|P} \left( 1 - \frac{1}{p^2} - \frac{1}{p^3} \right) \prod_{p|P} \left( 1 - \frac{1}{p} \right) \mathcal{I}(k, r) + O \left( Q(\log Q)^{k+r-1} \right). \]

First we write \( S_{L,0} \) in (5.6) in terms of the Mellin transform of \( \mathcal{W} \). For small \( \delta_1 > 0 \),
\[ S_{L,0} = \frac{1}{2\pi i} \int_{(-\delta_1)} \mathcal{W}(s)Q^s e^{-t^2} \sum_{(m,n)=1} \frac{a_mb_n}{\sqrt{mn}} \left( \frac{n}{m} \right)^it |m-n|^{-s} \sum_{(c, mnP)=1} \frac{\mu(c)}{c^{s+\varphi(c)}} \times \sum_{a | c} \frac{\mu(a)}{a} \sum_{\ell | P} \frac{\mu(\ell)}{\ell, \ell mn = 1} \frac{1}{\varphi(\ell)(a)f)^g-s} dt ds. \]

By Lemma 3.4, the sum over \( g \) is
\[ \sum_{(g, mn)=1} \frac{1}{\varphi(\ell)(a)f)^g-s} = \frac{1}{\varphi(\ell)(a)f)^g-s} \zeta(1-s)B(-s)B_{1}(-s, mn)B_{2}(-s, a\ell), \]
where the functions \( B, B_1 \) and \( B_2 \) are defined in the lemma. Since \( a|c, (c, P) = 1 \) and \( \ell|P \), it follows that \( (a, \ell) = 1, \varphi(\ell) = \varphi(a)\varphi(\ell) \) and
\[ B_2(-s, a\ell) = B_2(-s, a)B_2(-s, \ell). \]

Define
\[ B_3(s, c) := \sum_{a|c} \frac{\mu(a)B_2(s, a)}{a\varphi(a)} = \prod_{p|c} \left( 1 - \frac{1}{p(p-1)} \left( 1 + \frac{1}{(p-1)p+1} \right)^{-1} \right), \] (5.10)
and
\[ B_4(s, P) := \sum_{\ell|P} \frac{\mu(\ell)B_2(s, \ell)}{\varphi(\ell)} = \prod_{p|P} \left( 1 - \frac{1}{p-1} \left( 1 + \frac{1}{(p-1)p+1} \right)^{-1} \right). \] (5.11)
Applying (5.8) and the fact that \((m, n) = 1\), we obtain that
\[
S_{L,0} = \frac{1}{(2\pi i)^2} \int_{(\delta_1)} \tilde{W}(s) Q^s \zeta(1-s) B(-s) B_4(-s, P) \int_{-\infty}^{\infty} e^{-t^2} \sum_{c \leq C \atop (c, P) = 1} \mu(c) B_3(-s, c) e^{\varphi(c)}
\]
\[
\times \int_{(\delta_2)} \frac{\Gamma(1-s) \Gamma(z)}{\Gamma(1-s + z)} \left( \mathcal{H}_0(s, s - z + it, z - it) + \mathcal{H}_0(s, z + it, s - z - it) \right) \, dt \, dz \, ds
\]
where \(\delta_2\) is a small positive number and
\[
\mathcal{H}_0(s, \alpha, \beta) := \sum_{m, n} \sum_{(m, n) = 1 \atop (mn, cP) = 1} \frac{a_m B_1(-s, m) b_n B_1(-s, n)}{m^{1/2 + \alpha} n^{1/2 + \beta}}.
\]
We now want to estimate \(\mathcal{H}_0\) using Lemma 2.1. For \(K = \{K_1, \ldots, K_r\} \in \Pi_{k+r}\), let
\[
R_K(cP; s, \alpha, \beta) = \sum_{j=1}^{k} \mathcal{J}_{cP; s, \alpha, \beta}(t_K(p_1, \ldots, p_r))
\]
\[
C_K(cP; s, \alpha, \beta) = \sum_{j=1}^{k} \mathcal{J}_{cP; s, \alpha, \beta}(t_K(p_1, \ldots, p_r))
\]
where \(\sum_{j=1}^{k}\) is the sum over distinct primes, and
\[
\mathcal{J}_{cP; s, \alpha, \beta}(p_1, \ldots, p_{k+r}) = \prod_{j=1}^{k} \log p_j B_1(-s, p_j) \widetilde{F}_j \left( -\frac{\log p_j}{\log Q} \right) \prod_{j=k+1}^{k+r} \log p_j B_1(-s, p_j) \widetilde{F}_j \left( \frac{\log p_j}{\log Q} \right)
\]
if \((p_1 \cdots p_{k+r}, cP) = 1\), and equals to 0 otherwise. Then
\[
C_H(cP; s, \alpha, \beta) = \sum_{H \leq K} R_K(cP; s, \alpha, \beta)
\]
for any \(H \in \Pi_{k+r}\). By Lemma 2.1, we have
\[
\mathcal{H}_0(s, \alpha, \beta) = R_Q(cP; s, \alpha, \beta) = \sum_{K \in \Pi_{k+r}} \mu_{k+r}(Q, K) C_K(cP; s, \alpha, \beta).	ag{5.12}
\]
Thus, we have
\[
S_{L,0} = \sum_{K \in \Pi_{k+r}} \mu_{k+r}(Q, K) S_{L,0}(K),
\]
where
\[
S_{L,0}(K) := \frac{1}{(2\pi i)^2} \int_{(\delta_1)} \tilde{W}(s) Q^s \zeta(1-s) B(-s) B_4(-s, P) \int_{-\infty}^{\infty} e^{-t^2} \sum_{c \leq C \atop (c, P) = 1} \mu(c) B_3(-s, c) e^{\varphi(c)}
\]
\[
\times \int_{(\delta_2)} \frac{\Gamma(1-s) \Gamma(z)}{\Gamma(1-s + z)} \left( C_K(cP; s, \alpha_1, \beta_1) + C_K(cP; s, \alpha_2, \beta_2) \right) \, dz \, dt \, ds
\]
with \(\alpha_1 = s - z + it\), \(\beta_1 = z - it\), \(\alpha_2 = z + it\) and \(\beta_2 = s - z - it\).
Next, we find a similar representation for $\mathcal{M}_U$. By switching the order of summations in (5.4) and using the coprime conditions, we see that

$$
\mathcal{M}_U = -\tilde{\mathcal{W}}(0)B(0)B_1(0, P) \int_{-\infty}^{\infty} e^{-t^2} \sum_{c \leq C, (c,P)=1} \frac{\mu(c)B_2(0,c)}{\varphi(c)} \mathcal{H}_0(0, it, -it) \, dt.
$$

By (5.12) we have

$$
\mathcal{M}_U = \sum_{K \in \Pi_{k+r}} \mu_{k+r}(\bar{Q}, \bar{K}) \mathcal{M}_U(\bar{K}),
$$

where

$$
\mathcal{M}_U(\bar{K}) := -\tilde{\mathcal{W}}(0)B(0) \int_{-\infty}^{\infty} e^{-t^2} \sum_{c \leq C, (c,P)=1} \frac{\mu(c)B_2(0,c)}{\varphi(c)} C_K(cP; 0, it, -it) \, dt
$$

for $K \in \Pi_{k+r}$.

We now compute $C_K$, which will yield the estimation of $\mathcal{S}_{L,0}$ and $\mathcal{M}_U$. Define

$$
K_{K_j}(cP; s, \alpha, \beta) := \sum_{(p,cP)=1} \left( \prod_{\ell \leq k} \frac{\log p B_1(-s,p)}{p^{1/2+\alpha}} \right) \left( \prod_{k < \ell \leq k+r} \frac{\log p B_1(-s,p)}{p^{1/2+\beta}} \right) \tilde{F}_\ell \left( -\frac{\log p}{\log Q} \right)
$$

for each $j \leq \tau$, then we have

$$
C_K(cP; s, \alpha, \beta) = \prod_{j \leq \tau} K_{K_j}(cP; s, \alpha, \beta).
$$

Here $(\alpha, \beta)$ represents $(it, -it), (s - z + it, z - it)$ or $(z + it, s - z - it)$. If $K_j = \{\ell\}$ for some $\ell \leq k$, then

$$
K_{K_j}(cP; s, \alpha, \beta) = \sum_{(p,cP)=1} \frac{\log p}{p^{1/2+\alpha}} \left( 1 - \frac{1}{p^{s+1}} \right) \left( 1 + \frac{1}{(p-1)p^{s+1}} \right)^{-1} \tilde{F}_\ell \left( -\frac{\log p}{\log Q} \right)
$$

$$
= \sum_{0 \leq i \leq 3} K_{K_j,i}(cP; s, \alpha, \beta),
$$

where

$$
K_{K_j,0}(cP; s, \alpha, \beta) := F_\ell(-i\mathcal{U}(1/2 - \alpha)) \log Q,
$$

$$
K_{K_j,1}(cP; s, \alpha, \beta) := -\log Q \int_{-\infty}^{0} Q^u(1/2-\alpha) \tilde{F}(-v) \, dv,
$$

$$
K_{K_j,2}(cP; s, \alpha, \beta) := -\log Q \int_{0}^{\infty} Q^u(-1/2+s-\alpha) \tilde{F}(-v) \, dv,
$$

$$
K_{K_j,3}(cP; s, \alpha, \beta) := K_K(cP; s, \alpha, \beta) - \sum_{0 \leq i \leq 2} K_{K_j,i}(cP; s, \alpha, \beta).
$$
Similarly if $K_j = \{\ell\}$ for some $k < \ell \leq k + r$, then

$$\mathcal{K}_{K_j}(cP; s, \alpha, \beta) = \sum_{p \mid p \mid cP = 1} \frac{\log p}{p^{1/2 + \beta}} \left( 1 + \frac{1}{p^{1/2 + 1}} \right) \left( 1 + \frac{1}{(p - 1) p^{-s/2 + 1}} \right)^{-1} \hat{F}_\ell \left( \frac{\log p}{\log Q} \right)$$

$$= \sum_{0 \leq i \leq 3} \mathcal{K}_{K_{j,i}}(cP; s, \alpha, \beta),$$

where

$$\mathcal{K}_{K_{j,0}}(cP; s, \alpha, \beta) := F_\ell(iU(1/2 - \beta)) \log Q,$$

$$\mathcal{K}_{K_{j,1}}(cP; s, \alpha, \beta) := -\log Q \int_{-\infty}^{0} Q^{v(1/2 - \beta)} \hat{F}(v) dv,$$

$$\mathcal{K}_{K_{j,2}}(cP; s, \alpha, \beta) := -\log Q \int_{0}^{\infty} Q^{v(-1/2 + s - \beta)} \hat{F}(v) dv,$$

$$\mathcal{K}_{K_{j,3}}(cP; s, \alpha, \beta) := \mathcal{K}_{K_j}(cP; s, \alpha, \beta) - \sum_{0 \leq i \leq 2} \mathcal{K}_{K_{j,i}}(cP; s, \alpha, \beta)$$

Then by Lemma 3.6

$$\mathcal{K}_{K_{j,3}}(cP; s, \alpha, \beta) = O \left( \sum_{p \mid \mathfrak{c}} 1 \right)$$

holds uniformly for $\text{Re}(s) \leq 1 + \epsilon$ and $-\epsilon \leq \text{Re}(\alpha), \text{Re}(\beta)$. Note that the bound does not depend on $P$ because $\sum_{p \mid \mathfrak{c}} 1 \leq n$, where $n$ is defined in Section 1. Let $\delta_1 = \delta_2 = \frac{1}{\log Q}$. Thus $|\text{Re}(\alpha)|, |\text{Re}(\beta)| \leq 2/\log Q$. If $|K_j| > 1$, then for any $\epsilon > 0$

$$\mathcal{K}_{K_j}(cP; s, \alpha, \beta) = O(Q^\epsilon).$$

Moreover, if $|K_j| = 1$, then for $i = 1, 2, 3$ and for any $\epsilon > 0$,

$$\mathcal{K}_{K_{j,i}}(cP; s, \alpha, \beta) = O(Q^\epsilon).$$

Since

$$C_{\mathcal{K}}(cP; s, \alpha, \beta) = \left( \sum_{d_j=0,1,2,3 \text{ for } |K_j|=1} \prod_{|K_j|=1} \mathcal{K}_{K_{j,d_j}}(cP; s, \alpha, \beta) \right) \prod_{|K_j| \geq 2} \mathcal{K}_{K_j}(cP; s, \alpha, \beta),$$

we have

$$C_{\mathcal{K}}(cP; s, \alpha, \beta) = C'_{\mathcal{K}}(cP; s, \alpha, \beta) + O(Q^\epsilon),$$

where

$$\mathcal{K} \in \Pi' := \Pi'_{k+r} = \{ \mathcal{K} = \{K_1, \ldots, K_r\} \in \Pi_{k+r} : |K_j| = 1 \text{ for some } j \leq r \},$$

and

$$C'_{\mathcal{K}}(cP; s, \alpha, \beta) := \left( \sum_{\prod_{|K_j|=1} d_j = 0} \prod_{|K_j|=1} \mathcal{K}_{K_{j,d_j}}(cP; s, \alpha, \beta) \right) \prod_{|K_j| \geq 2} \mathcal{K}_{K_j}(cP; s, \alpha, \beta).$$

If $\mathcal{K} \in \Pi_{k+r} \setminus \Pi'$, then

$$C_{\mathcal{K}}(cP; s, \alpha, \beta) = O(Q^\epsilon).$$
Because the integral over $z$ is absolutely convergent when $\text{Re}(s) < 0$,

$$S_{L,0} = \sum_{K \in \Pi'} \mu_{k+r}(Q, K) S'_{L,0}(K) + O(Q^e),$$

$$\mathcal{M}_U = \sum_{K \in \Pi'} \mu_{k+r}(Q, K) \mathcal{M}'_U(K) + O(Q^e),$$

where

$$S'_{L,0}(K) := \frac{1}{(2\pi i)^2} \int_{(-\delta_1)} \tilde{W}(s) Q^s \zeta(1-s) B(-s) B_4(-s, P) \int_{-\infty}^\infty e^{-t^2} \sum_{c \leq C \atop (c, P) = 1} \frac{\mu(c) B_3(-s, c)}{c^s \varphi(c)}
\times \int_{(\delta_2)} \frac{\Gamma(1-s) \Gamma(z)}{\Gamma(1-s + z)} \left( C'_K(cP; s, s - z + it, z - it) + C'_K(cP; s, z + it, s - z - it) \right) dz \, dt \, ds$$

and

$$\mathcal{M}'_U(K) := -\tilde{W}(0) B(0) B_1(0, P) \int_{-\infty}^\infty e^{-t^2} \sum_{c \leq C \atop (c, P) = 1} \frac{\mu(c) B_3(0, c)}{\varphi(c)} C'_K(cP; 0, it, -it) dt$$

for $K \in \Pi'$.

Now each term has a factor $K_{K,j,0}$ for some $j$ which decays rapidly, so the integrand in $S'_{L,0}(K)$ is absolutely convergent even when $\text{Re}(s) > 0$. Thus, we can shift the $z$-contour to $\text{Re}(z) = 1/2$, change the order of $s$ and $z$ integrals and then shift the $s$-contour to $\text{Re}(s) = 1 - 1/U$. Since $\zeta(1-s)$ has a simple pole at $s = 0$ with the residue $-1$, we obtain

$$S'_{L,0}(K) = \int_{-\infty}^\infty e^{-t^2} \frac{1}{(2\pi i)^2} \int_{(1/2)} \int_{(1-1/U)} \tilde{W}(s) Q^s \zeta(1-s) B(-s) B_4(-s, P) \sum_{c \leq C \atop (c, P) = 1} \frac{\mu(c) B_3(-s, c)}{c^s \varphi(c)}
\times \frac{\Gamma(1-s) \Gamma(z)}{\Gamma(1-s + z)} \left( C'_K(cP; s, s - z + it, z - it) + C'_K(cP; s, z + it, s - z - it) \right) ds \, dz \, dt
\times \int_{-\infty}^\infty e^{-t^2} \frac{1}{2\pi i} \int_{(1/2)} \left( C'_K(cP; 0, -z + it, z - it) + C'_K(cP; 0, z + it, -z - it) \right) \frac{dz}{z} \, dt.$$

From the residue theorem

$$\frac{1}{2\pi i} \int_{(1/2)} C'_K(cP; 0, -z+it, z-it) \frac{dz}{z} = \frac{1}{2\pi i} \int_{(-1/2)} C'_K(cP; 0, -z+it, z-it) \frac{dz}{z} + C'_K(cP; 0, it, -it),$$

and by the change of variable

$$\frac{1}{2\pi i} \int_{(-1/2)} C'_K(cP; 0, -z+it, z-it) \frac{dz}{z} = -\frac{1}{2\pi i} \int_{(1/2)} C'_K(cP; 0, z+it, -z-it) \frac{dz}{z}.$$

Therefore

$$\frac{1}{2\pi i} \int_{(1/2)} \left( C'_K(cP; 0, -z+it, z-it) + C'_K(cP; 0, z+it, -z-it) \right) \frac{dz}{z} = C'_K(cP; 0, it, -it).$$
Since \( B_3(0, c) = B_2(0, c) \) and \( B_1(0, P) = B_1(0, P) \), it is not difficult to see that the main term of \( \mathcal{M}'(K) \) cancels out the residue at \( s = 0 \) of \( S_{L,0}'(K) \). Therefore, we derive at

\[
\mathcal{M}'(K) + S_{L,0}'(K) = \int_{-\infty}^{\infty} e^{-t^2} \frac{1}{2\pi i} \int_{(1/2)} \frac{1}{2\pi i} \int_{(1-1/\mathcal{U})} \tilde{W}(s)Q^s\zeta(1-s)B(-s)B_4(-s, P) \sum_{c \leq C, (c, P) = 1} \frac{\mu(c)B_3(-s, c)}{c^s\varphi(c)}
\]

\[
\times \frac{\Gamma(1-s)\Gamma(z)}{\Gamma(1-s+z)} (C'_K(cP; s, s - z + it, z - it) + C'_K(cP; s, s + it, s - z - it)) ds\, dz\, dt
\]

for each \( K \in \mathcal{K}' \). When \( \text{Re}(s) = 1 - 1/\mathcal{U} \) and \( \text{Re}(z) = 1/2 \),

\[
\mathcal{K}_{K_j}(cP; s, \alpha, \beta) = O(1)
\]

for \( |K_j| > 1 \), and

\[
\mathcal{K}_{K_j,i}(cP; s, \alpha, \beta) = O(\log Q)
\]

for \( |K_j| = 1 \) and \( i = 0, 1, 2, 3 \). We let

\[
B_5(s) = \frac{B(-s)}{\zeta(2-s)} = \prod_p \left( 1 + \frac{1}{(p-1)p^{1-s}} \right) \left( 1 - \frac{1}{p^{2-s}} \right). \tag{5.15}
\]

The function \( B_5(s) \) is absolutely convergent when \( \text{Re}(s) < 3/2 \) and \( B_5(1) = 1 \). When \( \text{Re}(s) = 1 - 1/\mathcal{U}, B(-s) \ll |s - 1|^{-1} \) and \( \Gamma(1-s) \ll |s - 1|^{-1} \). Hence,

\[
\mathcal{M}'(K) + S_{L,0}'(K) \ll Q(\log Q)^{k+r-2} \int_{(1-1/\mathcal{U})} \frac{1}{|s - 1|^2} ds \ll Q(\log Q)^{k+r-1}
\]

if \( K \in \mathcal{K}' \) contains a set \( K_j \) with \( |K_j| > 1 \). It is then enough to consider the case \( K = \mathcal{Q} = \{ \{1\}, \ldots, \{k + r\} \} \). In particular, we see that

\[
\mathcal{M}' + S_{L,0} = \mathcal{M}_1 + \mathcal{M}_2 + O(\log Q)^{k+r-1}, \tag{5.16}
\]

where

\[
\mathcal{M}_1 = \int_{-\infty}^{\infty} e^{-t^2} \frac{1}{(2\pi i)^2} \int_{(1/2)} \frac{1}{2\pi i} \int_{(1-1/\mathcal{U})} \tilde{W}(s)Q^s\zeta(1-s)\zeta(2-s)B_5(s)B_4(-s, P) \frac{\Gamma(1-s)\Gamma(z)}{\Gamma(1-s+z)}
\]

\[
\times \sum_{c \leq C, (c, P) = 1} \frac{\mu(c)B_3(-s, c)}{c^s\varphi(c)} \sum_{\Pi_{j \leq k+r, d_j = 0}} \mathcal{K}_d(cP; s, s - z + it, z - it) ds\, dz\, dt,
\]

\[
\mathcal{M}_2 = \int_{-\infty}^{\infty} e^{-t^2} \frac{1}{(2\pi i)^2} \int_{(1/2)} \frac{1}{2\pi i} \int_{(1-1/\mathcal{U})} \tilde{W}(s)Q^s\zeta(1-s)\zeta(2-s)B_5(s)B_4(-s, P) \frac{\Gamma(1-s)\Gamma(z)}{\Gamma(1-s+z)}
\]

\[
\times \sum_{c \leq C, (c, P) = 1} \frac{\mu(c)B_3(-s, c)}{c^s\varphi(c)} \sum_{\Pi_{j \leq k+r, d_j = 0}} \mathcal{K}_d(cP; s, s + it, s - z - it) ds\, dz\, dt,
\]

and

\[
\mathcal{K}_d(cP; s_1, s_2, s_3) := \prod_{j \leq k+r} \mathcal{K}_{(j)}(cP; s_1, s_2, s_3)
\]

for \( d = (d_1, \ldots, d_{k+r}) \).
Lemma 5.4. Let all notations be defined as above and Proposition 5.1. Then

\[
\mathcal{M}_i = \frac{1}{2} Q (\log Q)^{k+r} \sqrt{\pi} \widetilde{W}(1) \prod_{p \mid P} \left( 1 - \frac{1}{p^2} - \frac{1}{p^3} \right) \prod_{p \mid P} \left( 1 - \frac{1}{p} \right) \mathcal{I}(k, r) + O \left( (\log Q)^{k+r-1} \right)
\]

for \( i = 1, 2 \).

Note that Equation (5.9) follows from Equation (5.16) and the above lemma.

Proof. We will first show the calculation of \( \mathcal{M}_1 \) in details and briefly mention how to modify it to obtain the result for \( \mathcal{M}_2 \) at the end of this proof. Define

\[
\mathcal{M}_{1,d} = \int_{-\infty}^{\infty} e^{-it} \frac{1}{(2\pi i)^2} \int_{(1/2)} \int_{(1-\mu i)} \widetilde{W}(s) Q^s \zeta(1-s) \zeta(2-s) B_5(s) B_4(-s, P) \frac{\Gamma(1-s) \Gamma(z)}{\Gamma(1-s+z)}
\]

\[
\times \sum_{c \leq C \atop (c,P)=1} \frac{\mu(c) B_5(-s,c)}{c^s \varphi(c)} K_d(cP; s, s - z + it, z - it) \, ds \, dz \, dt.
\]

Then

\[
\mathcal{M}_1 = \sum_{d} \mathcal{M}_{1,d}.
\]

If \( d \) satisfies the property that \( d_j \neq 0 \) for all \( j \leq k \), then we move the \( s \)-contour to \( \text{Re}(s) = 1 - \epsilon \) for \( \epsilon > 0 \) and the \( z \)-contour remains on the line \( \text{Re}(z) = 1/2 \). Then

\[
K_d(cP; s, s - z + it, z - it) = O((\log Q)^{k+r})
\]

and

\[
\mathcal{M}_{1,d} = O(\log Q)^{k+r}).
\]

If \( d \) satisfies the property that \( d_j \neq 0 \) for all \( k < j \leq k + r \), then we move the \( s \)-contour to \( \text{Re}(s) = 1 - \epsilon - 1/\log Q \) for \( \epsilon > 0 \) and the \( z \)-contour to \( \text{Re}(z) = 1/2 - \epsilon \). Then

\[
K_d(cP; s, s - z + it, z - it) = O((\log Q)^{k+r})
\]

and

\[
\mathcal{M}_{1,d} = O(\log Q)^{k+r}).
\]

The other \( d' \)’s satisfy that \( d_{j_1} = d_{j_2} = 0 \) for some \( j_1 \) and \( j_2 \) such that \( j_1 \leq k < j_2 \). Hence,

\[
\mathcal{M}_1 = \sum_{d} \mathcal{M}_{1,d} + O(\log Q)^{k+r}).
\]

Now we consider the sum

\[
\sum_{d} K_d(cP; s, s - z + it, z - it) = \sum_{1 \leq j_1 \leq k \atop k+1 \leq j_2 \leq k+r} \sum_{d} \mathcal{M}_{1,d} + O(\log Q)^{k+r})
\]

\[
= : \sum_{1 \leq j_1 \leq k \atop k+1 \leq j_2 \leq k+r} \mathcal{P}_{j_1,j_2}(cP; s, z, t).
\]
We can write $P_{j_1, j_2}(cP; s, z, t)$ as the following product

$$K_{(j_1),0}K_{(j_2),0} \prod_{1 \leq j < j_1 \text{ or } k < j < j_2} (K_{(j),1} + K_{(j),2} + K_{(j),3}) \prod_{j_1 < j \leq k \text{ or } j_2 < j \leq k+r} (K_{(j),0} + K_{(j),1} + K_{(j),2} + K_{(j),3}).$$

Let $\alpha = s - z + it$ and $\beta = z - it$. For $j \leq k$, we obtain that

$$K_{(j),0} + K_{(j),1} = \log Q \int_{-\infty}^{\infty} Q^{v(1/2-\alpha)} \hat{F}_j(-v) dv,$$

and for $j > k$,

$$K_{(j),0} + K_{(j),1} = \log Q \int_{-\infty}^{\infty} Q^{v(1/2-\beta)} \hat{F}_j(v) dv = \log Q \int_{-\infty}^{0} Q^{-v(1/2-\beta)} \hat{F}_j(-v) dv.$$

Thus

$$P_{j_1, j_2}(cP; s, z, t)$$

$$= (\log Q)^{k+r}(-1)^{j_1+r} \int_{-\infty}^{\infty} Q^{u_{j_1}(1/2-\alpha)} \hat{F}_{j_1}(-u_{j_1}) du_{j_1} \int_{-\infty}^{\infty} Q^{-u_{j_2}(1/2-\beta)} \hat{F}_{j_2}(-u_{j_2}) du_{j_2}$$

$$\times \prod_{1 \leq j < j_1 \text{ or } k+1 \leq j < j_2} \left( \int_{-\infty}^{0} Q^{u_j(1/2-s+z-it)} \hat{F}_j(-u_j) du_j + \int_{0}^{\infty} Q^{u_j(-1/2+z-it)} \hat{F}_j(-u_j) du_j - \frac{K_{(j),3}}{\log Q} \right)$$

$$\times \prod_{j_1 < j \leq k} \left( \int_{0}^{\infty} Q^{u_j(1/2-s+z-it)} \hat{F}_j(-u_j) du_j - \int_{-\infty}^{0} Q^{u_j(-1/2+z-it)} \hat{F}_j(-u_j) du_j + \frac{K_{(j),3}}{\log Q} \right).$$

Define

$$P'_{j_1, j_2}(s, z, t)$$

$$:= (\log Q)^{k+r}(-1)^{j_1+r} \int_{-\infty}^{\infty} Q^{u_{j_1}(1/2-s+z-it)} \hat{F}_{j_1}(-u_{j_1}) du_{j_1} \int_{-\infty}^{\infty} Q^{u_{j_2}(-1/2+s-it)} \hat{F}_{j_2}(-u_{j_2}) du_{j_2}$$

$$\times \prod_{1 \leq j < j_1 \text{ or } k+1 \leq j < j_2} \left( \int_{-\infty}^{0} Q^{u_j(1/2-s+z-it)} \hat{F}_j(-u_j) du_j + \int_{0}^{\infty} Q^{u_j(-1/2+s-it)} \hat{F}_j(-u_j) du_j \right)$$

$$\times \prod_{j_1 < j \leq k} \left( \int_{0}^{\infty} Q^{u_j(1/2-s+z-it)} \hat{F}_j(-u_j) du_j - \int_{-\infty}^{0} Q^{u_j(-1/2+s-it)} \hat{F}_j(-u_j) du_j \right).$$

Since $K_{(j),3} = O(\log c)$ uniformly for $|s - 1| \leq \epsilon$, $|z - 1/2| \leq \epsilon$ and $t \in \mathbb{R}$, it implies that the contribution of $P_{j_1, j_2} - P'_{j_1, j_2}$ to $M_1$ is $O(Q(\log Q)^{k+r-1})$. This can be done by the same
method as in the estimation of $\mathcal{P}'_{j_1,j_2}$, so we omit the proof. Thus, we have

$$
\mathcal{M}_1 = \sum_{1 \leq j_1 \leq k} \sum_{k+1 \leq j_2 \leq k+r} \int_{-\infty}^{\infty} e^{-t^2} \frac{1}{(2\pi i)^2} \int_{(1/2)} \int_{(1-\epsilon/t)} \tilde{W}(s)Q^s \zeta(1-s)\zeta(2-s)B_5(s)B_4(-s, P) \\
\times \frac{\Gamma(1-s)\Gamma(z)}{\Gamma(1-s+z)} \sum_{c \leq C' \atop (c,P)=1} \frac{\mu(c) B_3(-s,c)}{c^s \varphi(c)} \mathcal{P}'_{j_1,j_2}(s, z, t) \, ds \, dz \, dt + O(Q(\log Q)^{k+r-1}).
$$

Notice that $\mathcal{P}'_{j_1,j_2}$ is independent to $c$ and $P$. The sum over $c$ is asymptotic to

$$
B_6(s, P) := \sum_{c=1}^{\infty} \frac{\mu(c) B_3(-s,c)}{c^s \varphi(c)}
$$

with an error $O(C^{-1+\varepsilon})$ for $|s-1| \leq \varepsilon/2$ and any $\varepsilon > 0$. Hence,

$$
\mathcal{M}_1 = \sum_{1 \leq j_1 \leq k} \sum_{k+1 \leq j_2 \leq k+r} \int_{-\infty}^{\infty} e^{-t^2} \frac{1}{(2\pi i)^2} \int_{(1/2)} \int_{(1-\epsilon/t)} \tilde{W}(s)Q^s \zeta(1-s)\zeta(2-s)B_5(s)B_4(-s, P) \\
\times \frac{\Gamma(1-s)\Gamma(z)}{\Gamma(1-s+z)} B_6(s, P) \mathcal{P}'_{j_1,j_2}(s, z, t) \, ds \, dz \, dt + O(Q(\log Q)^{k+r-1}).
$$

By expanding the products in $\mathcal{P}'_{j_1,j_2}$ and changing the order of integrals, we have

$$
\mathcal{P}'_{j_1,j_2}(s, z, t) = (\log Q)^{k+r} (-1)^{j_1+r} \sum_{T_1,W_1,T_2,W_2,T_3,W_3} \sum_{T_1+W_1=\{1,...,j_1-1\} \cup \{k+1,...,j_2-1\}} \sum_{T_2+W_2=\{j_1+1,...,k\}} \sum_{T_3+W_3=\{j_2+1,...,k+r\}} (-1)^{|W_2|+|W_3|} \\
\times \int_{D_{k+r}(\vec{T}, \vec{W})} \left( \prod_{j=1}^{k+r} \tilde{F}_j(-u_j) \right) Q^{1-s(u_{j_1}+u(\vec{T})+(-1/2+z-it)u([k+r]))} \, du,
$$

where $D_{k+r}(\vec{T}, \vec{W})$, $u(\vec{T})$ and $u([k+r])$ are defined in Proposition 5.1. Hence,

$$
\mathcal{M}_1 = (\log Q)^{k+r} \sum_{1 \leq j_1 \leq k} \sum_{k+1 \leq j_2 \leq k+r} \sum_{T_1,W_1,T_2,W_2,T_3,W_3} \sum_{T_1+W_1=\{1,...,j_1-1\} \cup \{k+1,...,j_2-1\}} \sum_{T_2+W_2=\{j_1+1,...,k\}} \sum_{T_3+W_3=\{j_2+1,...,k+r\}} (-1)^{j_1+r+|W_2|+|W_3|} \mathcal{M}_1(j_1, j_2, \vec{T}, \vec{W}) \\
+ O(Q(\log Q)^{k+r-1}),
$$

where

$$
\mathcal{M}_1(j_1, j_2, \vec{T}, \vec{W}) := \int_{-\infty}^{\infty} e^{-t^2} \frac{1}{(2\pi i)^2} \int_{(1/2+\epsilon_1)} \int_{(1-\epsilon_2)} \tilde{W}(s)\zeta(1-s)\zeta(2-s)B_5(s)B_4(-s, P)B_6(s, P) \\
\times \frac{\Gamma(1-s)\Gamma(z)}{\Gamma(1-s+z)} \sum_{c \leq C' \atop (c,P)=1} \frac{\mu(c) B_3(-s,c)}{c^s \varphi(c)} \prod_{j=1}^{k+r} \tilde{F}_j(-u_j) \, Q^{1-s(u_{j_1}+u(\vec{T})+(-1/2+z-it)u([k+r]))} \, du \, ds \, dt.
$$
for \(0 < \epsilon_1 < \epsilon_2 < 1/100\). To make the \(z\)-integral absolutely convergent for \(\text{Re}(s)\) near 1, we integrate the \(u_{j_2}\)-integral by parts twice.

\[
\mathcal{M}_1(j_1, j_2, \vec{T}, \vec{W}) = \frac{1}{(\log Q)^2} \int_{-\infty}^{\infty} \frac{1}{(2\pi i)^2} \int_{(1/2-\epsilon_1)}^{1} \int_{(1-\epsilon_2)} \vec{W}(s) \zeta(1-s) \zeta(2-s) B_5(s) B_4(-s, P) B_6(s, P) \frac{\Gamma(1-s) \Gamma(z)}{\Gamma(1-s+z)} \\
	imes \int_{D_{k+r}(\vec{T}, \vec{W})} \left( \prod_{j \neq j_2} \hat{F}_j(-u_j) \right) \hat{F}_{j_2}''(-u_{j_2}) Q^{1+(1-s)(u_{j_1}+u(\vec{T})-1)+(1/2+z-it)u([k+r])} e^{-t^2} du \, ds \, dz \, dt \left(1/2 - z + it\right)^2
\]

Based on the exponent of \(Q\) in the integrand, we split the domain \(D_{k+r}(\vec{T}, \vec{W})\) into the following four subsets:

\[
\begin{align*}
D_1 &= \{ u \in D_{k+r}(\vec{T}, \vec{W}) : u_{j_1} + u(\vec{T}) - 1 > 0, \quad u([k+r]) < 0 \}, \\
D_2 &= \{ u \in D_{k+r}(\vec{T}, \vec{W}) : u_{j_1} + u(\vec{T}) - 1 > 0, \quad u([k+r]) \geq 0 \}, \\
D_3 &= \{ u \in D_{k+r}(\vec{T}, \vec{W}) : u_{j_1} + u(\vec{T}) - 1 \leq 0, \quad u([k+r]) < 0 \}, \\
D_4 &= \{ u \in D_{k+r}(\vec{T}, \vec{W}) : u_{j_1} + u(\vec{T}) - 1 \leq 0, \quad u([k+r]) \geq 0 \}.
\end{align*}
\]

Clearly,

\[
\mathcal{M}_1(j_1, j_2, \vec{T}, \vec{W}) = \sum_{i=1}^{4} \mathcal{M}_1(j_1, j_2, \vec{T}, \vec{W}; D_i),
\]

where each \(\mathcal{M}_1(j_1, j_2, \vec{T}, \vec{W}; D_i)\) is defined analogously to \(\mathcal{M}_1(j_1, j_2, \vec{T}, \vec{W})\) with \(D_i\) in place of \(D_{k+r}(\vec{T}, \vec{W})\). We now compute each \(\mathcal{M}_1(j_1, j_2, \vec{T}, \vec{W}; D_i)\) as follows, expecting that the main contribution comes from the region \(D_1\) (Case 1).

**Case 1:** \(\mathcal{M}_1(j_1, j_2, \vec{T}, \vec{W}; D_1)\). The integrand has a double pole at \(s = 1\). By shifting the \(s\)-integral to \(1 + \epsilon\), we pick up the residue at \(s = 1\).

\[
\begin{align*}
\mathcal{M}_1(j_1, j_2, \vec{T}, \vec{W}; D_1) &= \frac{1}{(\log Q)^2} \int_{-\infty}^{\infty} \frac{1}{(2\pi i)^2} \int_{(1/2-\epsilon_1)}^{1} \int_{(1+\epsilon)} \vec{W}(s) \zeta(1-s) \zeta(2-s) B_5(s) B_4(-s, P) B_6(s, P) \frac{\Gamma(1-s) \Gamma(z)}{\Gamma(1-s+z)} \\
&\quad \times \int_{D_1} \left( \prod_{j \neq j_2} \hat{F}_j(-u_j) \right) \hat{F}_{j_2}''(-u_{j_2}) Q^{1+(1-s)(u_{j_1}+u(\vec{T})-1)+(1/2+z-it)u([k+r])} e^{-t^2} du \, ds \, dz \, dt \left(1/2 - z + it\right)^2
\end{align*}
\]

By shifting the \(z\)-integral to \(1/2 + \epsilon\), we encounter a pole at \(z = 1/2\) and obtain that the above integral is \(\ll Q/\log Q\). Next we compute the residue at \(s = 1\). Since

\[
\zeta(2-s) \Gamma(1-s) = \frac{1}{(s-1)^2} + O(1)
\]
as \( s \to 1 \), we have

\[
\text{Res}_{s=1} = - \frac{1}{(\log Q)^2} \int_{-\infty}^{\infty} \frac{1}{2\pi i} \int_{(1/2-\epsilon)} \frac{\partial}{\partial s} \left[ \tilde{W}(s) \zeta(1-s) B_5(s) B_4(-s, P) B_6(s, P) \frac{\Gamma(z)}{\Gamma(1-s+z)} \right]_{s=1} d u \ d z
\]

\[
\times \int_{D_1} \left( \prod_{j \neq j_2} \tilde{F}_{j_2}(-u_j) \right) \tilde{F}_{j_2}''(-u_{j_2}) Q^{1+(-1/2+z-it)u([k+r])} \frac{e^{-t^2}}{(1/2 - z + it)^2} \ d u \ d z \ d t
\]

\[
+ \frac{1}{(\log Q)^2} \int_{-\infty}^{\infty} e^{-t^2} \frac{1}{2\pi i} \int_{(1/2-\epsilon)} \tilde{W}(1) \zeta(0) B_5(1) B_4(-1, P) B_6(1, P)
\]

\[
\times \int_{D_1} \left( \prod_{j \neq j_2} \tilde{F}_{j_2}(-u_j) \right) \tilde{F}_{j_2}''(-u_{j_2}) Q^{1+(-1/2+z-it)u([k+r])} (u_{j_1} + u(T) - 1) \log Q \frac{e^{-t^2}}{(1/2 - z + it)^2}.
\]

For both terms, we shift the \( z \)-integral to \( 1/2 + \epsilon \) and picking up a residue at \( z = 1/2 + it \). We bound both shifted \( z \)-integral and obtain that it is bounded by \( \ll Q/\log Q \). For the first integral, the residue also contributes \( O(Q/\log Q) \). Therefore

\[
\mathcal{M}_1(j_1, j_2, \tilde{T}, \tilde{W}; D_1)
\]

\[
= -Q \sqrt{\pi} \tilde{W}(1) \zeta(0) B_4(-1, P) B_6(1, P) \int_{D_1} \left( \prod_{j \neq j_2} \tilde{F}_{j_2}(-u_j) \right) \tilde{F}_{j_2}''(-u_{j_2}) u([k+r])(u_{j_1} + u(T) - 1) \ d u
\]

\[
+ O(Q/\log Q).
\]

We observe that for \( \tilde{u}_{j_2} := u([k+r]) - u_{j_2} \), the \( u_{j_2} \)-integral is

\[
\int_{-\infty}^{-\tilde{u}_{j_2}} \tilde{F}_{j_2}''(-u_{j_2}) u([k+r]) \ d u_{j_2} = \int_{-\infty}^{-\tilde{u}_{j_2}} \tilde{F}_{j_2}'(-u_{j_2}) \ d u_{j_2} = -\tilde{F}_{j_2}(\tilde{u}_{j_2})
\]

\[
= - \int_{\mathbb{R}} \tilde{F}_{j_2}(-u_{j_2}) \delta(u([k+r])) \ d u_{j_2},
\]

where \( \delta \) is the Dirac delat function. Hence,

\[
\mathcal{M}_1(j_1, j_2, \tilde{T}, \tilde{W}; D_1)
\]

\[
= Q \sqrt{\pi} \tilde{W}(1) \zeta(0) B_4(-1, P) B_6(1, P) \int_{D_{k+r}(\tilde{T}, \tilde{W})} \left( \prod_{u_{j_1} + u(T) > 1} \tilde{F}_{j_1}(-u_{j_1}) \right) (u_{j_1} + u(T) - 1) \delta(u([k+r])) \ d u
\]

\[
+ O(Q/\log Q).
\]

**Case 2**: \( \mathcal{M}_1(j_1, j_2, \tilde{T}, \tilde{W}; D_2) \). We shift the \( s \)-contour to the line \( \text{Re}(s) = 1 + \epsilon \) as in the first case and pick up the residue at \( s = 1 \). After that we bound the shifted integral and the residue trivially and obtain that

\[
\mathcal{M}_1(j_1, j_2, \tilde{T}, \tilde{W}; D_2) \ll \frac{Q}{\log Q}.
\]
Case 3: \( \mathcal{M}_1(j_1, j_2, \bar{T}, \bar{W}; D_3) \). For this case, we do not shift the contour integration and just bound it trivially. Therefore
\[
\mathcal{M}_1(j_1, j_2, \bar{T}, \bar{W}; D_3) \ll \frac{Q}{(\log Q)^2}.
\]

Case 4: \( \mathcal{M}_1(j_1, j_2, \bar{T}, \bar{W}; D_4) \). We shift the \( z \)-contour integral to the line \( \text{Re}(z) = 1/2 + \epsilon \) with a residue at \( z = 1/2 + it \). After that we bound the shifted integral and the residue trivially and obtain that
\[
\mathcal{M}_1(j_1, j_2, \bar{T}, \bar{W}; D_4) \ll \frac{Q}{\log Q}.
\]

Combining Cases 1-4 and the facts that \( \zeta(0) = -1/2 \), \( B_4(-1, P) = \prod_{p \mid p}(1 - p^{-2} - p^{-3}) \) and \( B_6(1, P) = \prod_{p \mid p}(1 - p^{-4}) \), we derive that
\[
\mathcal{M}_1(j_1, j_2, \bar{T}, \bar{W}) = \frac{Q \sqrt{\pi}}{2} \tilde{W}(1) \prod_{p \mid P} \left( 1 - \frac{1}{p^2} - \frac{1}{p^3} \right) \prod_{p \mid P} \left( 1 - \frac{1}{p} \right)
\times \int_{D_{k+r}(\bar{T}, \bar{W})} \left( \prod_{j=1}^{k+r} \hat{F}_j(-u_j) \right) (1 - u_{j_1} - u(\bar{T}))\delta(u([k + r])) \, du
+ O(Q/\log Q).
\]

Therefore,
\[
\mathcal{M}_1 = Q(\log Q)^{k+r} \frac{\sqrt{\pi}}{2} \tilde{W}(1) \prod_{p \mid P} \left( 1 - \frac{1}{p^2} - \frac{1}{p^3} \right) \prod_{p \mid P} \left( 1 - \frac{1}{p} \right)
\times \sum_{1 \leq j_1 \leq k} \sum_{k+1 \leq j_2 \leq k+r} \sum_{T_1, W_1, T_2, W_2, T_3, W_3}
\sum_{j_1 + u(\bar{T}) > 1} \left( \prod_{j=1}^{k+r} \hat{F}_j(-u_j) \right) (1 - u_{j_1} - u(\bar{T}))\delta(u([k + r])) \, du
+ O(Q(\log Q)^{k+r-1}),
\]

We next consider \( \mathcal{M}_2 \). By the change of variable \( z \) to \( s - z \), we can write
\[
\mathcal{M}_2 = \int_{-\infty}^{\infty} e^{-t^2} \frac{1}{(2\pi i)^2} \int_{(1/2)} \int_{(1-1/\epsilon)} \tilde{W}(s)Q^s\zeta(1-s)\zeta(2-s)B_5(s)B_4(-s, P)\Gamma(1-s)\Gamma(s-z) \Gamma(1-z)
\times \sum_{c \leq C} \mu(c)B_3(-s, c) e^{s\varphi(c)} \sum_{\Pi_{j \leq k+r, d_j=0}} K_d(cP; s, s-z + it, z-it) \, ds \, dz \, dt
\]

Then by the same method as in the estimation of \( \mathcal{M}_1 \), we can show that \( \mathcal{M}_2 \) satisfies the same asymptotic formula as \( \mathcal{M}_1 \). This completes the proof of the lemma.

\[\square\]
5.4. Conclusion of the proof of Proposition 5.1. By Equation (5.3), and Lemmas 5.2 and 5.3 we have

\[ S = M_U + S_{L,0} + O \left( CQ^{(\kappa' + \kappa'')/2 - 1 - \epsilon} + \frac{Q^{1+\epsilon}}{C} \right) \]

for any \( \epsilon > 0 \). Since \( \kappa' + \kappa'' \leq 4 - \epsilon \), by letting \( C = Q^{\epsilon/3} \), the \( O \)-terms above are \( o(Q) \). Therefore, by (5.9) we finally have

\[ S = Q(\log Q)^{k+r} \sqrt{\pi \tilde{W}(1)} \prod_{p \mid P} \left( 1 - \frac{1}{p^2} - \frac{1}{p^3} \right) \prod_{p \mid P} \left( 1 - \frac{1}{p} \right) I(k, r) \]

\[ + O(Q(\log Q)^{k+r-1}). \]

5.5. The estimation of \( N_{\tilde{G}} \). To complete the calculation of \( N_{\tilde{G}} \), we first evaluate \( S(P; S_{12}, S_{22}) \) defined in (5.2). We list sets \( S_{12} \) and \( S_{22} \) in increasing order as \( S_{12} = \{ \alpha_1, \ldots, \alpha_{|S_{12}|} \} \) and \( S_{22} = \{ \beta_1, \ldots, \beta_{|S_{22}|} \} \). By modifying arguments of Proposition 5.1 we find that

\[ S(P; S_{12}, S_{22}) = Q(\log Q)^{|S_{12}| + |S_{22}|} \sqrt{\pi \tilde{W}(1)} \prod_{p \mid P} \left( 1 - \frac{1}{p^2} - \frac{1}{p^3} \right) \prod_{p \mid P} \left( 1 - \frac{1}{p} \right) I(S_{12}, S_{22}) \]

\[ + O(Q(\log Q)^{|S_{12}| + |S_{22}| - 1}), \]

where

\[ I(S_{12}, S_{22}) := \sum_{1 \leq j_1 \leq |S_{12}|} \sum_{1 \leq j_2 \leq |S_{22}|} \cdots (-1)^{j_1 + |S_{22}| + |W_2| + |W_3|} \]

\[ \times \int_{D_{|S_{12}| + |S_{22}|}(\tilde{T}, \tilde{W})} \left( \prod_{j \in S_{12} \cup S_{22}} \tilde{F}_j(-u_j) \right) (1 - u_{\alpha j_1} - u(\tilde{T})) \delta(u(S_{12}) + u(S_{22})) \, du \]

and

\[ D_{|S_{12}| + |S_{22}|}(\tilde{T}, \tilde{W}) = \left\{ u = (u_{\alpha_1}, \ldots, u_{\alpha_{|S_{12}|}}, u_{\beta_1}, \ldots, u_{\beta_{|S_{22}|}}) \in \mathbb{R}^{|S_{12}| + |S_{22}|} \right\} \]

\[ : u_j < 0 \text{ for } j \in T_1 \cup T_3 \cup W_3, \text{ and } u_j > 0 \text{ for } j \in T_2 \cup W_1 \cup W_2 \}

with \( du = du_{\alpha_1} \cdots du_{\alpha_{|S_{12}|}} du_{\beta_1} \cdots du_{\beta_{|S_{22}|}} \). By Equation (5.1) and Lemma 3.3 we have

\[ N_{\tilde{G}} = D(W, Q) \sum_{S_1 + S_2 + S_3 = \nu} \left( \prod_{\ell \in S_3} \tilde{F}_\ell(0) \right) (-1)^{|S_1| + |S_2|} \sum_{S_{11} + S_{12} = S_1 \atop S_{21} + S_{22} = S_2 \atop \nu = |S_{11}| + |S_{21}|} \sum_{\sigma: S_{11} \to S_{21} \text{ bijection}} \frac{1}{(\log Q)^{2|S_{11}|}} \]

\[ \times \sum_P \mu^2(P) \left( \prod_{\ell \in S_{11}} \left( \frac{\log p_\ell}{p_\ell} \right)^2 \tilde{F}_\ell \left( - \frac{\log p_\ell}{\log Q} \right) \tilde{F}_\sigma(\ell) \left( \frac{\log p_\ell}{\log Q} \right) \right) \prod_{p \mid P} \left( 1 - \frac{1}{p^2} - \frac{1}{p^3} \right)^{-1} \left( 1 - \frac{1}{p} \right) \]

\[ + O(Q / \log Q). \]
Modifying the proof of Lemma 4.3, we can show that

\[ N_{Q} = D(W, Q) \sum_{S_1 + S_2 + S_3 = [\nu]} \left( \prod_{\ell \in S_3} \hat{F}_{\ell}(0) \right) (-1)^{|S_1|+|S_2|} \times \sum_{\sigma : S_1 \rightarrow S_2} I(S_{12}, S_{22}) \sum_{\text{bijections}} \left( \prod_{\ell \in S_{11}} \int_{0}^{\infty} v \hat{F}_{\ell}(-v) \hat{F}_{\sigma(\ell)}(v) dv \right) \]  

(5.18)

where

\[ \mu_{n}(Q, G) = \prod_{j=1}^{\nu} (-1)^{|G_j|-1}(|G_j| - 1)! \]

and \(I(S_{12}, S_{22})\) is defined in (5.17).

6. Comparison with Random Matrix Theory

In this section, we will complete the proof of Theorem 1.1 by comparing (5.19) with the integral in (1.3). First we need the following lemma, which expresses the integral as the limit of \(n\)-correlation of eigenvalues of random unitary matrices of size \(N \rightarrow \infty\).

Lemma 6.1. Let \(f : \mathbb{R}^{n} \rightarrow \mathbb{R}\) be smooth and rapidly decreasing. For an \(N \times N\) unitary matrix \(X_N\), write its eigenvalues as \(e^{i\theta_j}\) with \(-\pi \leq \theta_1 \leq \cdots \leq \theta_N < \pi\). Then

\[ \lim_{N \rightarrow \infty} \int_{U(N)} \sum_{1 \leq j_1, \ldots, j_n \leq N} f(\frac{N\theta_{j_1}}{2\pi}, \ldots, \frac{N\theta_{j_n}}{2\pi}) dX_N = \int_{\mathbb{R}^{n}} f(x) W^{(n)}(x) dx, \]

where \(dX_N\) is the Haar measure on the group of \(N \times N\) unitary matrices \(U(N)\) and \(W^{(n)}(x)\) is defined in (1.3).

Note that the condition \(-\pi \leq \theta_1 \leq \cdots \leq \theta_N < \pi\) in the above lemma is also required for Theorem 3.4 of [3], which will be used in the proof of Proposition 5.2.
Proof. By Theorem 3.1 of [5], we have
\[
\int_{U(N)} \sum_{1 \leq j_1, \ldots, j_n \leq N}^* f \left( \frac{N \theta_{j_1}}{2\pi}, \ldots, \frac{N \theta_{j_n}}{2\pi} \right) dX_N
\]
\[
= \frac{1}{(2\pi)^n} \int_{[-\pi, \pi]^n} f \left( \frac{N x_1}{2\pi}, \ldots, \frac{N x_n}{2\pi} \right) \det S_N(x_k - x_j) dx
\]
\[
= \int_{[-N/2, N/2]^n} f(x_1, \ldots, x_n) \frac{1}{N^{n \times n}} \det S_N \left( \frac{2\pi}{N} (x_k - x_j) \right) dx,
\]
where
\[
S_N(x) = \frac{\sin(Nx/2)}{\sin(x/2)}.
\]
It is easy to see that
\[
\lim_{N \to \infty} \frac{1}{N^{n \times n}} \det S_N \left( \frac{2\pi}{N} (x_k - x_j) \right) = W^{(n)}(x).
\]
Since \( f \) has a rapid decay, we have
\[
\lim_{N \to \infty} \int_{U(N)} \sum_{1 \leq j_1, \ldots, j_n \leq N}^* f \left( \frac{N \theta_{j_1}}{2\pi}, \ldots, \frac{N \theta_{j_n}}{2\pi} \right) dX_N
\]
\[
= \int_{\mathbb{R}^n} f(x_1, \ldots, x_n) \lim_{N \to \infty} \frac{1}{N^{n \times n}} \det S_N \left( \frac{2\pi}{N} (x_k - x_j) \right) dx = \int_{\mathbb{R}^n} f(x) W^{(n)}(x) dx.
\]
By the above lemma, Theorem 1.1 is equivalent to
\[
\lim_{Q \to \infty} \frac{L_1(f, \mathcal{W}, Q)}{D(W, Q)} = \lim_{N \to \infty} \int_{U(N)} \sum_{1 \leq j_1, \ldots, j_n \leq N}^* f \left( \frac{N \theta_{j_1}}{2\pi}, \ldots, \frac{N \theta_{j_n}}{2\pi} \right) dX_N.
\]
Let \( f(x_1, \ldots, x_n) = \prod_{i=1}^n f_i(x_i) \) have C4-property and let \( \mathcal{G} = \{G_1, \ldots, G_\nu\} \in \Pi_n \) be a partition of \( [n] = \{1, 2, \ldots, n\} \). Define
\[
F_\ell(x) = \prod_{i \in G_\ell} f_i(x).
\]
By combinatorial sieving in Lemma 2.1 we have
\[
\lim_{N \to \infty} \int_{U(N)} \sum_{1 \leq j_1, \ldots, j_n \leq N}^* f \left( \frac{N \theta_{j_1}}{2\pi}, \ldots, \frac{N \theta_{j_n}}{2\pi} \right) dX_N
\]
\[
= \lim_{N \to \infty} \int_{U(N)} \sum_{\mathcal{G} \in \Pi_n} \mu_{\mathcal{G}}(Q, \mathcal{G}) \sum_{1 \leq j_1, \ldots, j_\nu \leq N} \prod_{\ell=1}^\nu F_\ell \left( \frac{N \theta_{j_\ell}}{2\pi} \right) dX_N. \tag{6.1}
\]
Then Theorem 1.1 can be deduced from Equations 2.1, (6.1) and the following proposition.

**Proposition 6.2.** Let \( D(W, Q) \) and \( C_{1, \mathcal{G}} \) be defined in Equations 1.5 and 2.3, respectively. Then
\[
\lim_{Q \to \infty} \frac{C_{1, \mathcal{G}}}{D(W, Q)} = \lim_{N \to \infty} \int_{U(N)} \sum_{1 \leq j_1, \ldots, j_\nu \leq N} \prod_{\ell=1}^\nu F_\ell \left( \frac{N \theta_{j_\ell}}{2\pi} \right) dX_N.
\]
Proof of Proposition 6.2. Applying Theorems 3.3 and 3.4 in [5], we have

\[ R := \lim_{N \to \infty} \int_{U(N)} \prod_{1 \leq j_1, \ldots, j_\nu \leq N} F_\ell \left( \frac{N \theta_j}{2\pi} \right) dX_N \]

\[ = \lim_{N \to \infty} \frac{1}{(2\pi i)^\nu} \sum_{S_1 + S_2 + S_3 = [\nu]} (-1)^{|S_1| + |S_3|} N^{|S_3|} \]

\[ \times \int_{c_+^{|S_2|}} \int_{c_-^{|S_1| + |S_3|}} J^*(z_{S_2}; -z_{S_1}) \prod_{\ell=1}^\nu F_\ell \left( \frac{iN}{2\pi} z_\ell \right) dz_{S_3} dz_{S_1} dz_{S_2}, \]

where \( c_+ \) denotes the path from \( \delta_1 - \pi i \) up to \( \delta_1 + \pi i \), \( c_- \) denotes the path from \( -\delta_1 + \pi i \) down to \( -\delta_1 - \pi i \) for some \( \delta_1 > 0 \), \( z_{S_1} = \{ z_\ell : \ell \in S_1 \} \), \( -z_{S_1} = \{ -z_\ell : \ell \in S_1 \} \) and \( dz_{S_i} = \prod_{\ell \in S_i} dz_\ell \).

\[ J^*(A; B) := \sum_{S,A,T \subseteq B \atop |S| = |T|} e^{-N(\sum_{\alpha \in S} \hat{\alpha} + \sum_{\beta \in T} \hat{\beta})} \frac{Z(S, T)Z(S^-, T^-)}{Z(S, S^-)Z(T, T^-)} \sum_{y = (A-S)+(B-T)} \prod_{y = 1}^y H_{S,T}(U_y), \]

where \( S^- = \{ \hat{\alpha} : \hat{\alpha} \in S \} \),

\[ Z(A, B) = \prod_{\alpha \in A} z(\alpha + \beta), \quad Z^\dagger(A, B) = \prod_{\alpha \in A, \beta \in B \atop \alpha + \beta \neq 0} z(\alpha + \beta) \]

with \( z(x) = (1 - e^{-x})^{-1} \), and

\[ H_{S,T}(W) = \begin{cases} \sum_{\tilde{\alpha} \in S} \frac{z'}{z} (\alpha - \tilde{\alpha}) - \sum_{\tilde{\beta} \in T} \frac{z'}{z} (\alpha + \tilde{\beta}) & \text{if } W = \{ \alpha \} \subset A - S, \\ \sum_{\tilde{\beta} \in T} \frac{z'}{z} (\beta - \tilde{\beta}) - \sum_{\tilde{\alpha} \in S} \frac{z'}{z} (\tilde{\beta} + \tilde{\alpha}) & \text{if } W = \{ \beta \} \subset B - T, \\ \left( \frac{z'}{z} \right) (\alpha + \beta) & \text{if } W = \{ \alpha, \beta \} \text{ with } \alpha \in A - S, \beta \in B - T, \text{ otherwise.} \end{cases} \]

The innermost sum of \( J^*(A; B) \) is the sum over all partitions of \( (A - S) + (B - T) \) into singletons or doubletons \( U_1, \ldots, U_y \).

We change the orientation of the \( z_\ell \)-integral in (6.2) for each \( \ell \in S_1 \cup S_3 \) and this removes the factor \( (-1)^{|S_1| + |S_3|} \). Since \( F_\ell \) is rapidly decreasing, we can extend each vertical integrals. Thus

\[ R = \lim_{N \to \infty} \frac{1}{(2\pi i)^\nu} \sum_{S_1 + S_2 + S_3 = [\nu]} N^{|S_3|} \int_{(\delta_1)^{|S_2|}} \int_{(\delta_1)^{|S_1| + |S_3|}} J^*(z_{S_2}; -z_{S_1}) \prod_{\ell=1}^\nu F_\ell \left( \frac{iN}{2\pi} z_\ell \right) dz_{S_3} dz_{S_1} dz_{S_2}. \]

Since

\[ \frac{N}{2\pi i} \int_{(\delta_1)} F_\ell \left( \frac{iN}{2\pi} z_\ell \right) = \frac{1}{i} \int_{(0)} F_\ell(iz) dz = \hat{F}_\ell(0) \]

for each \( \ell \in S_3 \),

\[ R = \sum_{S_1 + S_2 + S_3 = [\nu]} \left( \prod_{\ell \in S_3} \hat{F}_\ell(0) \right) R(S_1, S_2), \]
where
\[ \mathcal{R}(S_1, S_2) := \lim_{N \to \infty} \frac{1}{(2 \pi i)^{|S_1|+|S_2|}} \int_{(\delta_1)|S_1|} \int_{(-\delta_1)|S_1|} J^*(z_{S_2}; -z_{S_1}) \prod_{\ell \in S_1 \cup S_2} F_\ell \left( \frac{iN}{2\pi} z_\ell \right) \ dz_{S_1} dz_{S_2}. \]

We now consider \( J^*(z_{S_2}; -z_{S_1}) \). When \( |S| = |T| \geq 2 \),
\[ |e^{-N(\sum_{\alpha \in S} \alpha + \sum_{\beta \in T} \beta)}| \leq e^{-4N\delta_1}. \]
Combining above with the support condition of \( J \), we have
\[ \prod_{\ell=1}^{\nu} F_\ell \left( \frac{iN}{2\pi} z_\ell \right) \ll \prod_{\ell=1}^{\nu} \int_{-\infty}^{\infty} |\hat{F}_\ell(v)| e^{Nz_\ell v} dv \ll e^{N\delta_1(k' + \kappa'')}. \]
Since \( k' + \kappa'' \leq 4 - \varepsilon \), we obtain that the contribution to \( \mathcal{R} \) is
\[ \ll N^{\nu} e^{N\delta_1(k' + \kappa'' - 4)} \leq N^{\nu} e^{-\varepsilon N\delta_1} \to 0 \]
as \( N \to \infty \). Hence, the main contribution of \( \mathcal{R} \) comes solely from the cases \( |S| = |T| = 0, 1 \).
Let \( \mathcal{J}_i \) be the contribution from the case \( |S| = |T| = i \) for \( i = 0, 1 \). Then
\[ \mathcal{R}(S_1, S_2) =: \mathcal{J}_0(S_1, S_2) + \mathcal{J}_1(S_1, S_2). \]
Define
\[ \mathcal{R}_i = \sum_{S_1 + S_2 + S_3 = \nu} \left( \prod_{\ell \in S_3} \hat{F}_\ell(0) \right) \mathcal{J}_i(S_1, S_2) \]
for each \( i = 0, 1 \), so that
\[ \mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1. \]

6.1. **Calculation of \( \mathcal{J}_0(S_1, S_2) \).** In this section, we will show the following lemma.

**Lemma 6.3.** Let notations be defined above. Then
\[ \mathcal{J}_0(S_1, S_2) = \sum_{\sigma: S_1 \to S_2} \prod_{\ell \in S_1} \left( \int_{0}^{\infty} v \hat{F}_\ell(-v) \hat{F}_\sigma(\ell)(v) \ dv \right), \]
and so it can be easily deduced that
\[ \mathcal{R}_0 = \sum_{S_1 + S_2 + S_3 = \nu} \left( \prod_{\ell \in S_3} \hat{F}_\ell(0) \right) \sum_{\sigma: S_1 \to S_2} \prod_{\ell \in S_1} \left( \int_{0}^{\infty} v \hat{F}_\ell(-v) \hat{F}_\sigma(\ell)(v) \ dv \right). \]

**Proof.** For this case, \( S = T = \emptyset \), and so \( \mathcal{J}_0(S_1, S_2) \) equals
\[ \lim_{N \to \infty} \frac{1}{(2 \pi i)^{|S_1|+|S_2|}} \int_{(\delta_1)|S_1|} \int_{(-\delta_1)|S_1|} \prod_{y=1}^{\nu} H_{\emptyset, \emptyset}(U_y) \prod_{\ell \in S_1 \cup S_2} F_\ell \left( \frac{iN}{2\pi} z_\ell \right) \ dz_{S_1} dz_{S_2} \]
for \( \delta_1 > 0 \), where
\[ H_{\emptyset, \emptyset}(W) = \begin{cases} (\frac{i}{z})' (\alpha + \beta) & \text{if } W = \{\alpha, \beta\} \text{ with } \alpha \in z_{S_2}, \beta \in -z_{S_1}, \\ 0 & \text{otherwise}. \end{cases} \]
Notice that $\prod_{y=1}^{Y} H_{0,\emptyset}(U_y)$ is non-zero if and only if every $U_j$ contains one element from $-z_{S_1}$ and the other element from $z_{S_2}$. Thus, for each partition $U_1 + \cdots + U_y$, there is a natural bijection $\sigma : S_1 \to S_2$, defined by

$$\sigma(\ell) = \ell'$$

when $-z\ell, z\ell' \in U_j$ for some $j$. Hence $J_0(S_1, S_2)$ equals

$$\sum_{\sigma : S_1 \to S_2 \text{ bijection}} \lim_{N \to \infty} \prod_{\ell \in S_1} \left( \frac{1}{(2\pi i)^2} \int_{(\delta_1)} \int_{(-\delta_1)} \left( \frac{z'}{z} \right)' \left( z_{\sigma(\ell)} - z\ell \right) F_\ell \left( \frac{iN}{2\pi} z_{\ell} \right) F_\ell \left( \frac{iN}{2\pi} z\ell_{\sigma(\ell)} \right) dz\ell \ d\sigma(\ell) \right)$$

$$= \sum_{\sigma : S_1 \to S_2 \text{ bijection}} \lim_{N \to \infty} \prod_{\ell \in S_1} \left( \frac{1}{(iN)^2} \int_{(\delta)} \int_{(-\delta)} \left( \frac{2\pi}{N} \right)^{\ell} \left( z_{\sigma(\ell)} - z\ell \right) F_\ell(i z\ell) F_\ell(iz_{\sigma(\ell)}) dz\ell \ d\sigma(\ell) \right).$$

Since

$$\lim_{N \to \infty} \frac{1}{(iN)^2} \left( \frac{2\pi}{N} \right)^{\ell} \left( z_{\sigma(\ell)} - z\ell \right) = \frac{1}{(2\pi i)^2 (z_{\sigma(\ell)} - z\ell)^2},$$

we have

$$J_0(S_1, S_2) = \sum_{\sigma : S_1 \to S_2 \text{ bijection}} \prod_{\ell \in S_1} \left( \frac{1}{(2\pi i)^2} \int_{(\delta)} \int_{(-\delta)} F_\ell(i z\ell) F_\ell(iz_{\sigma(\ell)}) \frac{1}{(z\ell - z_{\sigma(\ell)})^2} dz\ell \ d\sigma(\ell) \right).$$

The double integrals above is

$$= \frac{1}{(2\pi i)^2} \int_{(\delta)} \int_{(-\delta)} \frac{1}{(z_{\sigma(\ell)} - z\ell)^2} F_\ell(iz_{\sigma(\ell)}) \int_{\mathbb{R}} e^{-2\pi v z\ell} \widehat{F}_\ell(-v) \ dv \ dz\ell \ d\sigma(\ell)$$

$$= \frac{1}{(2\pi i)^2} \int_{(\delta)} \int_{\mathbb{R}} \int_{(-\delta)} \frac{1}{(z_{\sigma(\ell)} - z\ell)^2} \ dv \ F_\ell(iz_{\sigma(\ell)}) \widehat{F}_\ell(-v) \ dv \ d\sigma(\ell)$$

$$= \frac{1}{2\pi i} \int_{(\delta)} \int_{0}^{\infty} 2\pi v e^{-2\pi v z\sigma(\ell)} F_\ell(iz_{\sigma(\ell)}) \widehat{F}_\ell(-v) \ dv \ dz\sigma(\ell)$$

$$= \int_{0}^{\infty} v \widehat{F}_\ell(-v) \widehat{F}_\ell(\ell)(v) \ dv,$$

and this completes the proof of the lemma.

\[ \square \]

6.2. Calculation of $J_1(S_1, S_2)$. This is the case $|S| = |T| = 1$ in $\mathcal{R}(S_1, S_2)$. There exist $\alpha \in S_1$ and $\beta \in S_2$ such that $S = \{z_\beta\}$ and $T = \{-z_\alpha\}$. For $\delta_1 > 0$, we then have

$$J_1(S_1, S_2) := \lim_{N \to \infty} \frac{1}{(2\pi i)^{|S_1|+|S_2|}} \int_{(\delta_1)}^{\delta_1} \int_{(-\delta_1)}^{\delta_1} \prod_{\ell \in S_1 \cup S_2} F_\ell \left( \frac{iN}{2\pi} z\ell \right)$$

$$\times \sum_{\substack{\alpha \in S_1 \\beta \in S_2}} \frac{e^{N(z_\alpha - z_\beta)}}{(1 - e^{z_\alpha - z_\beta})(1 - e^{-z_\alpha + z_\beta})} \sum_{\substack{y = U_1 + \cdots + U_y \\|U_y\| \leq 2}} \prod_{\ell \in S_1 \cup S_2} H_{\ell, \{z_\beta\}, \{-z_\alpha\}}(U_y) dz_\alpha dz_\beta.$$
where for \( \ell_1 \in S_1 \setminus \{\alpha\} \) and \( \ell_2 \in S_2 \setminus \{\beta\} \),

\[
H_{\{z_\beta\},\{-z_\alpha\}}(\{z_\ell\}) = \frac{1}{1 - e^{z_\beta - z_\alpha}} - \frac{1}{1 - e^{z_\beta - z_\alpha}},
\]

\[
H_{\{z_\beta\},\{-z_\alpha\}}(\{-z_\ell\}) = \frac{1}{1 - e^{z_\alpha - z_\ell}} - \frac{1}{1 - e^{z_\alpha - z_\ell}},
\]

\[
H_{\{z_\beta\},\{-z_\alpha\}}(\{z_\ell_2, -z_\ell_1\}) = \frac{e^{z_\ell_2 - z_\ell_1}}{(1 - e^{z_\ell_2 - z_\ell_1})^2}.
\]

and otherwise, \( H_{\{z_\beta\},\{-z_\alpha\}}(W) = 0 \). As indicated in Remark 3.2 of [5], even though each term \( H_{\{z_\beta\},\{-z_\alpha\}}(\{z_\ell_2, -z_\ell_1\}) \) has a singularity on the contour, the integrand has no poles because they cancel. We would like to shift contours in such a way that we avoid singularities of the integrand. If \( S_1 = \{\alpha_1, \ldots, \alpha_{|S_1|}\} \) with \( \alpha_1 < \cdots < \alpha_{|S_1|} \) and \( S_2 = \{\beta_1, \ldots, \beta_{|S_2|}\} \) with \( \beta_1 < \cdots < \beta_{|S_2|} \), then define

\[
\int_{S_1^-} := \int_{-\delta_1} \cdots \int_{-\delta_{|S_1|}}, \quad \int_{S_2^+} := \int_{(\delta_1)} \cdots \int_{(\delta_{|S_2|})}
\]

for some \( 0 < \delta_1 < \cdots < \delta_{\max(|S_1|,|S_2|)} \). After we replace \( \int_{(-\delta_1)^{|S_1|}} \) and \( \int_{(\delta_1)^{|S_2|}} \) by \( \int_{S_1^-} \) and \( \int_{S_2^+} \), respectively, changing the order of integrals and summations is legitimate.

We next estimate the integrand of \( \mathcal{J}_1(S_1, S_2) \). Define

\[
S_{11} = S_1 \setminus S_{12}, \quad S_{12} = \{\alpha\} \cup \{\ell \in S_1 : \{-z_\ell\} = U_y \text{ for some } y\},
\]

\[
S_{21} = S_2 \setminus S_{22}, \quad S_{22} = \{\beta\} \cup \{\ell \in S_2 : \{z_\ell\} = U_y \text{ for some } y\}.
\]

Further let a bijection \( \sigma : S_{11} \rightarrow S_{21} \) be defined such that for any \( \ell \in S_{11}, \{-z_\ell, z_\sigma(\ell)\} = U_y \) for some \( y \). Hence,

\[
\sum_{\alpha \in S_1, \beta \in S_2} \frac{e^{N(z_\alpha - z_\beta)}}{(1 - e^{z_\alpha - z_\beta})(1 - e^{-z_\alpha + z_\beta})} \sum_{y=1}^{\gamma} \prod_{\ell \in S_{12} \setminus \{\alpha\}, |U_y| \leq 2} H_{\{z_\beta\},\{-z_\alpha\}}(U_y)
\]

\[
= \sum_{S_1 = S_{11} + S_{12}} \sum_{S_2 = S_{21} + S_{22}, S_{12}, S_{22} \neq \emptyset} \frac{e^{N(z_\alpha - z_\beta)}}{(1 - e^{-z_\alpha - z_\beta})(1 - e^{-z_\alpha + z_\beta})} \prod_{\ell \in S_{12} \setminus \{\alpha\}} H_{\{z_\beta\},\{-z_\alpha\}}(\{-z_\ell\}) \prod_{\ell \in S_{22} \setminus \{\beta\}} H_{\{z_\beta\},\{-z_\alpha\}}(\{z_\ell\})
\]

\[
\times \sum_{\sigma : S_{11} \rightarrow S_{21}} \prod_{\ell \in S_{11}} H_{\{z_\beta\},\{-z_\alpha\}}(\{z_\sigma(\ell), -z_\ell\}).
\]

We then apply the above identity to \( \mathcal{J}_1(S_1, S_2) \), substitute \( z_\ell \) by \( 2\pi z_\ell/N \) for all \( \ell \in S_1 \cup S_2 \) and take the limit \( N \to \infty \). Since

\[
\lim_{N \to \infty} iN(1 - e^{2\pi x/N}) = -2\pi ix,
\]
we find that

\[
J_1(S_1, S_2) = \sum_{\substack{S_1 = S_{11} + S_{12} \\ S_2 = S_{21} + S_{22} \\ S_{11,22} \neq \emptyset}} \frac{1}{(2\pi i)^{|S_1| + |S_2|}} \int_{S_2^+} \int_{S_2^-} \prod_{\ell \in S_1 \cup S_2} F_\ell(i\zeta_\ell)
\]

\[
\times \sum_{\alpha \in S_{12} \atop \beta \in S_{22}} -\frac{e^{2\pi(z_\alpha - z_\beta)}}{(z_\alpha - z_\beta)^2} \prod_{\ell \in S_1 \setminus \{\alpha\}} \left( \frac{1}{-z_\alpha + z_\ell} - \frac{1}{-z_\beta + z_\ell} \right) \prod_{\ell \in S_2 \setminus \{\beta\}} \left( \frac{1}{-z_\ell + z_\beta} - \frac{1}{-z_\ell + z_\alpha} \right)
\]

\[
\times \sum_{\sigma:S_{11} \to S_{21} \atop \ell \in S_{11}} \frac{1}{(z_\sigma(\ell) - z_\ell)^2} dz_{S_{11}}, dz_{S_{22}}.
\]

By Equation (6.5) and combining the products on \(S_{12} \setminus \{\alpha\}\) and \(S_{22} \setminus \{\beta\}\), we have

\[
J_1(S_1, S_2) = \sum_{\substack{S_1 = S_{11} + S_{12} \\ S_2 = S_{21} + S_{22} \\ S_{11,22} \neq \emptyset}} \left( \sum_{\sigma:S_{11} \to S_{21} \atop \ell \in S_{11}} \prod_{\ell \in S_{11}} \int_0^\infty \tilde{v} F_\ell(-v) \tilde{F}_\sigma(\ell)(v) dv \right) J(S_{12}, S_{22}),
\]

where

\[
J(S_{12}, S_{22}) := \sum_{\alpha \in S_{12} \atop \beta \in S_{22}} \frac{-1}{(2\pi i)^{|S_{12}| + |S_{22}|}} \int_{S_{22}^+} \int_{S_{12}^-} \left( \prod_{\ell \in S_{12} \cup S_{22}} F_\ell(i\zeta_\ell) \right) e^{2\pi(z_\alpha - z_\beta)}
\]

\[
\times \prod_{\ell \in S_{12} \setminus S_{22} \setminus \{\alpha, \beta\}} \left( \frac{1}{z_\ell - z_\alpha} - \frac{1}{z_\ell - z_\beta} \right) dz_{S_{12}} dz_{S_{22}}.
\]

The evaluation of \(J(S_1, S_2)\) follows from the calculation of \(J(S_{12}, S_{22})\) below.

**Lemma 6.4.** Let \(I(S_{12}, S_{22})\) be defined as Equation (5.17). Then

\[
J(S_{12}, S_{22}) = (-1)^{|S_{12}| + |S_{22}|} I(S_{12}, S_{22}).
\]

**Proof.** Write \(S_{12} = \{\alpha_1, \ldots, \alpha_{|S_{12}|}\}\) with \(\alpha_1 < \cdots < \alpha_{|S_{12}|}\) and \(S_{22} = \{\beta_1, \ldots, \beta_{|S_{22}|}\}\) with \(\beta_1 < \cdots < \beta_{|S_{22}|}\). Let \(\alpha = \alpha_j_1\) and \(\beta = \beta_j_2\) in the above sum. Then

\[
J(S_{12}, S_{22}) = \sum_{\substack{j_1 \leq |S_{12}| \\ j_2 \leq |S_{22}|}} \frac{1}{(2\pi i)^2} \int_{(-\delta_1)} \int_{(\delta_2)} F_{\alpha_j_1}(i\zeta_{\alpha j_1}) F_{\beta_j_2}(i\zeta_{\beta j_2})
\]

\[
eq \frac{e^{2\pi(z_{\alpha j_1} - z_{\beta j_2})}}{(z_{\alpha j_1} - z_{\beta j_2})^2}
\]

\[
\times \prod_{\alpha \in S_{12} \setminus \{\alpha_{j_1}\}} \frac{1}{2\pi i} \int_{(-\delta_1)} F_{\alpha}(i\zeta_\alpha) \left( \frac{1}{z_\alpha - z_{\alpha j_1}} - \frac{1}{z_\alpha - z_{\beta j_2}} \right) dz_\alpha
\]

\[
\times \prod_{\beta \in S_{22} \setminus \{\beta_{j_2}\}} \frac{1}{2\pi i} \int_{(\delta_2)} F_{\beta}(i\zeta_\beta) \left( \frac{1}{z_\beta - z_{\beta j_1}} - \frac{1}{z_\beta - z_{\beta j_2}} \right) dz_\beta.
\]

By Lemma 3.7, \(J(S_{12}, S_{22})\) becomes

\[
\sum_{\substack{j_1 \leq |S_{12}| \\ j_2 \leq |S_{22}|}} \frac{1}{(2\pi i)^2} \int_{(-\delta_1)} \int_{(\delta_2)} \left( z_{\alpha j_1} - z_{\beta j_2} \right)^2 (-1)^{|S_{12}| - j_1} F_{\beta_j_2}(i\zeta_{\beta j_2}) \int_{\mathbb{R}} \tilde{F}_{\alpha_j_1}(-u_{\alpha j_1}) e^{-2\pi u_{\alpha j_1} z_{\alpha j_1}} du_{\alpha j_1}
\]
\[
\times \prod_{\ell<j_1} \left( \int_{-\infty}^{0} \hat{F}_{\alpha\ell}(-u_{\alpha\ell})e^{-2\pi u_{\alpha\ell}z_{\alpha j_1}} du_{\alpha\ell} + \int_{0}^{\infty} \hat{F}_{\alpha\ell}(-u_{\alpha\ell})e^{-2\pi u_{\alpha\ell}z_{\beta j_2}} du_{\alpha\ell} \right)
\times \prod_{\ell<j_2} \left( \int_{-\infty}^{0} \hat{F}_{\beta\ell}(-u_{\beta\ell})e^{-2\pi u_{\beta\ell}z_{\alpha j_1}} du_{\beta\ell} + \int_{0}^{\infty} \hat{F}_{\beta\ell}(-u_{\beta\ell})e^{-2\pi u_{\beta\ell}z_{\beta j_2}} du_{\beta\ell} \right)
\times \prod_{j_1<\ell \leq |S_{12}|} \left( \int_{0}^{\infty} \hat{F}_{\alpha\ell}(-u_{\alpha\ell})e^{-2\pi u_{\alpha\ell}z_{\alpha j_1}} du_{\alpha\ell} - \int_{0}^{\infty} \hat{F}_{\alpha\ell}(-u_{\alpha\ell})e^{-2\pi u_{\alpha\ell}z_{\beta j_2}} du_{\alpha\ell} \right)
\times \prod_{j_2<\ell \leq |S_{22}|} \left( \int_{-\infty}^{0} \hat{F}_{\beta\ell}(-u_{\beta\ell})e^{-2\pi u_{\beta\ell}z_{\alpha j_1}} du_{\beta\ell} - \int_{-\infty}^{0} \hat{F}_{\beta\ell}(-u_{\beta\ell})e^{-2\pi u_{\beta\ell}z_{\beta j_2}} du_{\beta\ell} \right) dz_{\beta j_2} dz_{\alpha j_1} .
\]

Note that there is no \(u_{\beta j_2}\)-integral above. After we expand the products and combine the \(u_j\)-integrals together, the above equals

\[
\sum_{j_1 \leq |S_{12}|} \sum_{j_2 \leq |S_{22}|} \frac{1}{(2\pi)^2} \int_{(-\delta_{j_1})} \int_{(-\delta_{j_2})} e^{2\pi (z_{\alpha j_1} - z_{\beta j_2})} F_{\beta j_2}(i z_{\beta j_2}) \sum_{T_1+T_2+T_3+T_4+T_5+T_6+T_7+T_8+T_9+T_{10} \in \{\alpha_j, \ldots, \alpha_{j-1}\}, \{\beta_{j_1}, \ldots, \beta_{j_2-1}\}} \sum_{T_{10} \subseteq \{\alpha_j, \ldots, \alpha_{j-1}\}, \{\beta_{j_1}, \ldots, \beta_{j_2-1}\}} (-1)^{|T_{10}|+|\{\alpha_j, \ldots, \alpha_{j-1}\}|+|\{\beta_{j_1}, \ldots, \beta_{j_2-1}\}|} \int_{D_{|S_{12}|+|S_{22}|}(T, W: \beta_{j_2})} \left( \prod_{j \in S_{12} \cup S_{22} \setminus \{\beta_{j_2}\}} \hat{F}_{j}(-u_j) \right) e^{-2\pi u_{\alpha_j}z_{\alpha j_1} + 2\pi u_{(\hat{T})}z_{\beta j_2}} du dz_{\beta j_2} dz_{\alpha j_1},
\]

where \(D_{|S_{12}|+|S_{22}|}(T, W: \beta_{j_2})\) is defined analogously to \(D_{|S_{12}|+|S_{22}|}(T, W)\) but without the \(u_{\beta j_2}\)-coordinate. By changing the order of integrals, we see that

\[
\mathcal{J}(S_{12}, S_{22}) = \sum_{j_1 \leq |S_{12}|} \sum_{j_2 \leq |S_{22}|} \frac{1}{2\pi i} \int_{(-\delta_{j_1})} \int_{(-\delta_{j_2})} F_{\beta j_2}(i z_{\beta j_2}) \sum_{T_1+T_2+T_3+T_4+T_5+T_6+T_7+T_8+T_9+T_{10} \in \{\alpha_j, \ldots, \alpha_{j-1}\}, \{\beta_{j_1}, \ldots, \beta_{j_2-1}\}} \sum_{T_{10} \subseteq \{\alpha_j, \ldots, \alpha_{j-1}\}, \{\beta_{j_1}, \ldots, \beta_{j_2-1}\}} (-1)^{|T_{10}|+|\{\alpha_j, \ldots, \alpha_{j-1}\}|+|\{\beta_{j_1}, \ldots, \beta_{j_2-1}\}|} \int_{D_{|S_{12}|+|S_{22}|}(T, W: \beta_{j_2})} \left( \prod_{j \in S_{12} \cup S_{22} \setminus \{\beta_{j_2}\}} \hat{F}_{j}(-u_j) \right) \times \frac{-1}{2\pi i} \int_{(-\delta_{j_1})} e^{-2\pi u_{\alpha_j}z_{\alpha j_1} + 2\pi u_{(\hat{T})}z_{\beta j_2} - 2\pi u_{(\hat{T})}z_{\beta j_2} + |\{\alpha_j, \ldots, \alpha_{j-1}\}|+|\{\beta_{j_1}, \ldots, \beta_{j_2-1}\}|} du dz_{\beta j_2} dz_{\alpha j_1}.
\]

The last integral is nonzero only when \(u_{\alpha j_1} + u_{(\hat{T})} > 1\). In such a case, we shift the \(z_{\alpha j_1}\)-integral to \(\infty\) and obtain that the above equals

\[
\sum_{j_1 \leq |S_{12}|} \sum_{j_2 \leq |S_{22}|} \frac{1}{i} \int_{(-\delta_{j_2})} F_{\beta j_2}(i z_{\beta j_2}) \sum_{T_1+T_2+T_3+T_4+T_5+T_6+T_7+T_8+T_9+T_{10} \in \{\alpha_j, \ldots, \alpha_{j-1}\}, \{\beta_{j_1}, \ldots, \beta_{j_2-1}\}} \sum_{T_{10} \subseteq \{\alpha_j, \ldots, \alpha_{j-1}\}, \{\beta_{j_1}, \ldots, \beta_{j_2-1}\}} (-1)^{|T_{10}|+|\{\alpha_j, \ldots, \alpha_{j-1}\}|+|\{\beta_{j_1}, \ldots, \beta_{j_2-1}\}|} \int_{\mathbb{R}} e^{-2\pi u_{\alpha j_1} + 2\pi u_{(\hat{T})}z_{\beta j_2} + 2\pi u_{(\hat{T})}z_{\beta j_2} + |\{\alpha_j, \ldots, \alpha_{j-1}\}|+|\{\beta_{j_1}, \ldots, \beta_{j_2-1}\}|} du dz_{\beta j_2} dz_{\alpha j_1}.
\]
We then interchange the order of integrals again and replace the $\zeta_{\beta_j}$-integral by $\hat{F}_{\beta_j}$. Thus from Equations (5.18), (6.6) and Lemma 6.4, we obtain that $\mathcal{J}(S_{12}, S_{22})$ equals

$$\sum_{j_1 \leq |S_{12}|} \sum_{j_2 \leq |S_{22}|} \sum_{T_1, T_2, W_1, W_2, T_3, W_3} (-1)^{j_1 + |S_{12}| + |W_2| + |W_3|} \times \int_{D [S_{12} + |S_{22}|]} \left( \prod_{j \in S_{12} \cup S_{22} \setminus \{\beta_j\}} \hat{F}_j(-u_j) \right) (1 - u_{\alpha_j} - u(\hat{T})) e^{-2\pi (u_{\alpha_j} + u(\hat{T}) + u(W))} d\hat{u}$$

$$\times \int_{D [S_{12} + |S_{22}|]} \left( \prod_{j \in S_{12} \cup S_{22} \setminus \{\beta_j\}} \hat{F}_j(-u_j) \right) (1 - u_{\alpha_j} - u(\hat{T})) \hat{F}_{\beta_j} (u_{\alpha_j} + u(\hat{T}) + u(W)) d\hat{u}$$

$$= \sum_{j_1 \leq |S_{12}|} \sum_{j_2 \leq |S_{22}|} \sum_{T_1, T_2, W_1, W_2, T_3, W_3} (-1)^{j_1 + |S_{12}| + |W_2| + |W_3|} \times \int_{D [S_{12} + |S_{22}|]} \left( \prod_{j \in S_{12} \cup S_{22}} \hat{F}_j(-u_j) \right) (1 - u_{\alpha_j} - u(\hat{T})) \delta(u(S_{12}) + u(S_{22})) du,$$

which is the same as $(-1)^{|S_{12}| + |S_{22}|} \mathcal{I}(S_{12}, S_{22})$, finishing the lemma.

6.3. Conclusion of the proof of Proposition 6.2 From Equation (1.7) and Lemma 6.3

$$\lim_{Q \to \infty} \frac{C_1 G}{D(W, Q)} = \mathcal{R}_0 + \lim_{Q \to \infty} \frac{N_G}{D(W, Q)}.$$

Then from Equations (5.18), (6.6) and Lemma 6.4, we obtain that

$$\lim_{Q \to \infty} \frac{N_G}{D(W, Q)} = \mathcal{R}_1.$$

Thus by Equation (5), we derive at

$$\lim_{Q \to \infty} \frac{C_1 G}{D(W, Q)} = \mathcal{R}_0 + \mathcal{R}_1 = \mathcal{R}$$

as desired.

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