The geometry of spheres in free abelian groups

Moon Duchin
University of Michigan

joint work with Samuel Lelièvre and Christopher Mooney

Group Theory Webinar, 25 March 2010
A familiar question: given a property whose density you want to measure in a metric space, find the proportion of points in the ball $B_n$ having this property, and let $n \to \infty$.
A familiar question: given a property whose density you want to measure in a metric space, find the proportion of points in the ball $B_n$ having this property, and let $n \to \infty$.

**Example:** $\text{Prob(two random integers are relatively prime)} = \frac{6}{\pi^2}$.
A familiar question: given a property whose density you want to measure in a metric space, find the proportion of points in the ball $B_n$ having this property, and let $n \rightarrow \infty$.

**Example:** Prob(two random integers are relatively prime) = $6/\pi^2$.

Geometric interpretation is a ball-average:

$$\lim_{n \rightarrow \infty} \frac{1}{|B_n|} \sum_{x \in B_n} \frac{1}{n} f(x).$$
A familiar question: given a property whose density you want to measure in a metric space, find the proportion of points in the ball $B_n$ having this property, and let $n \to \infty$.

Example: $\text{Prob(two random integers are relatively prime)} = \frac{6}{\pi^2}$.

Geometric interpretation is a ball-average:

$$\lim_{n \to \infty} \frac{1}{|B_n|} \sum_{x \in B_n} \frac{1}{n} f(x).$$

Strictly harder problem: averaging over spheres.
A familiar question: given a property whose density you want to measure in a metric space, find the proportion of points in the ball $B_n$ having this property, and let $n \to \infty$.

Example: $\text{Prob(two random integers are relatively prime)} = \frac{6}{\pi^2}$.

Geometric interpretation is a ball-average:

$$\lim_{n \to \infty} \frac{1}{|B_n|} \sum_{x \in B_n} \frac{1}{n} f(x).$$

Strictly harder problem: averaging over spheres. We will study sphere-averages:

$$\lim_{n \to \infty} \frac{1}{|S_n|} \sum_{x \in S_n} \frac{1}{n} f(x).$$
Cone measure

Given finite genset \((\mathbb{Z}^d, S)\),

Figure: Some gensets: \(S_{\text{std}} = \pm\{e_1, e_2\}\), \(S_{\text{hex}} = \pm\{e_1, e_2, e_1 + e_2\}\), \(S_{\text{chess}} = \{((\pm 2, \pm 1), (\pm 1, \pm 2))\}\) (with irrelevant generator thrown in).
Cone measure

Given finite genset \((\mathbb{Z}^d, S)\), let \(Q\) be the convex hull of \(S\) in \(\mathbb{R}^d\),

**Figure:** Convex hulls \(Q\).
Cone measure

Given finite genset $(\mathbb{Z}^d, S)$, let $Q$ be the convex hull of $S$ in $\mathbb{R}^d$, let $L = \partial Q$.

Figure: Boundary polyhedra $L$.

As we will discuss below, $\mathbb{S}^n \to L$ as a Gromov-Hausdorff limit. We show counting measure on spheres converges to cone measure on $L$. 

Duchin Lelièvre Mooney (2010) The geometry of spheres in $\mathbb{Z}^d$ Group Theory Webinar 3-25
Cone measure

Given finite genset \((\mathbb{Z}^d, S)\), let \(Q\) be the convex hull of \(S\) in \(\mathbb{R}^d\), let \(L = \partial Q\), and let \(\hat{A}\) be the cone from \(A \subseteq L\) to \(0\), so that \(Q = \hat{L}\).

Figure: A cone.
Cone measure

Given finite genset $(\mathbb{Z}^d, S)$, let $Q$ be the convex hull of $S$ in $\mathbb{R}^d$, let $L = \partial Q$, and let $\hat{A}$ be the cone from $A \subseteq L$ to 0, so that $Q = \hat{L}$.

Define the cone measure by $\mu(A) = \mu_L(A) = \frac{\text{Vol}(\hat{A})}{\text{Vol}(Q)}$.

Figure: Cone measure.
Cone measure

Given finite genset \((\mathbb{Z}^d, S)\), let \(Q\) be the convex hull of \(S\) in \(\mathbb{R}^d\), let \(L = \partial Q\), and let \(\hat{A}\) be the cone from \(A \subseteq L\) to \(0\), so that \(Q = \hat{L}\).

Define the cone measure by \(\mu(A) = \mu_L(A) = \frac{\text{Vol}(\hat{A})}{\text{Vol}(Q)}\).

As we will discuss below, \(\frac{1}{n}S_n \rightarrow L\) as a Gromov-Hausdorff limit. We show counting measure on spheres converges to cone measure on \(L\).
$g : \mathbb{R}^d \rightarrow \mathbb{R}$ is called *homogeneous* if $g(ax) = ag(x)$ for $a \geq 0$. 
Limit shape, limit measure

- \( g : \mathbb{R}^d \to \mathbb{R} \) is called *homogeneous* if \( g(ax) = ag(x) \) for \( a \geq 0 \).

- \( f : \mathbb{Z}^d \to \mathbb{R} \) is *coarsely homogeneous* if \( \exists \) homog \( g \) with \( f \asymp g \), meaning \( |g(x) - f(x)| \) is uniformly bounded over \( x \in \mathbb{Z}^d \).

Theorem (Limit shape and limit measure)

For any finite presentation \((\mathbb{Z}^d, S)\) and any function \( f : \mathbb{Z}^d \to \mathbb{R} \) asymptotic to a homogeneous \( g : \mathbb{R}^d \to \mathbb{R} \),

\[
\lim_{n \to \infty} \frac{1}{|S_n|} \sum_{x \in S_n} 1_{f(x)} = \int Lg(x) d\mu(x).
\]

So group averaging problem reduces to a problem in convex geometry.

Duchin Lelièvre Mooney (2010)
Limit shape, limit measure

- \( g : \mathbb{R}^d \rightarrow \mathbb{R} \) is called *homogeneous* if \( g(ax) = ag(x) \) for \( a \geq 0 \).

- \( f : \mathbb{Z}^d \rightarrow \mathbb{R} \) is *coarsely homogeneous* if \( \exists \) homog \( g \) with \( f \overset{\dagger}{\sim} g \), meaning \( |g(x) - f(x)| \) is uniformly bounded over \( x \in \mathbb{Z}^d \).

- \( f \) is *asymptotically homogeneous* if \( \exists \) homog \( g \) with \( f \sim g \), meaning \( f(x)/g(x) \rightarrow 1 \) as \( x \rightarrow \infty \).
Limit shape, limit measure

- $g : \mathbb{R}^d \to \mathbb{R}$ is called **homogeneous** if $g(ax) = ag(x)$ for $a \geq 0$.
- $f : \mathbb{Z}^d \to \mathbb{R}$ **coarsely homogeneous** if $\exists$ homog $g$ with $f \equiv g$, meaning $|g(x) - f(x)|$ is uniformly bounded over $x \in \mathbb{Z}^d$.
- $f$ is **asymptotically homogeneous** if $\exists$ homog $g$ with $f \sim g$, meaning $f(x)/g(x) \to 1$ as $x \to \infty$. (So $f \equiv g, g \neq 0 \implies f \sim g$.)
Limit shape, limit measure

- \( g : \mathbb{R}^d \to \mathbb{R} \) is called *homogeneous* if \( g(ax) = ag(x) \) for \( a \geq 0 \).
- \( f : \mathbb{Z}^d \to \mathbb{R} \) *coarsely homogeneous* if \( \exists \) homog \( g \) with \( f \preceq g \), meaning \( |g(x) - f(x)| \) is uniformly bounded over \( x \in \mathbb{Z}^d \).
- \( f \) is *asymptotically homogeneous* if \( \exists \) homog \( g \) with \( f \sim g \), meaning \( f(x)/g(x) \to 1 \) as \( x \to \infty \). (So \( f \preceq g, g \neq 0 \implies f \sim g \).)

**Theorem (Limit shape and limit measure)**

*For any finite presentation \((\mathbb{Z}^d, S)\) and any function \( f : \mathbb{Z}^d \to \mathbb{R} \) asymptotic to a homogeneous \( g : \mathbb{R}^d \to \mathbb{R} \),

\[
\lim_{n \to \infty} \frac{1}{|S_n|} \sum_{x \in S_n} \frac{1}{n} f(x) = \int_L g(x) \, d\mu(x).
\]

So group averaging problem reduces to a problem in convex geometry.
Word length is coarsely homogeneous

- $L$ induces a Minkowski norm $\| \cdot \|_L$
  (unique norm on $\mathbb{R}^d$ with $L$ as unit sphere) — **Ex:** std induces $\ell^1$

---

Duchin Lelièvre Mooney (2010)  The geometry of spheres in $\mathbb{Z}^d$  Group Theory Webinar 3-25
Word length is coarsely homogeneous

- $L$ induces a **Minkowski norm** $\|\cdot\|_L$ (unique norm on $\mathbb{R}^d$ with $L$ as unit sphere) — **Ex:** std induces $\ell^1$
- The annular region $\Delta_nL := nQ \setminus (n-1)Q$ is covered by $S_{n-1} + Q$.

**Figure:** The *chess-knight metric.*
Word length is coarsely homogeneous

- $L$ induces a *Minkowski norm* $\| \cdot \|_L$ (unique norm on $\mathbb{R}^d$ with $L$ as unit sphere) — **Ex:** std induces $\ell^1$
- The annular region $\Delta_n L := nQ \setminus (n - 1)Q$ is covered by $S_{n-1} + Q$.
- How much word-length is used to fill in $Q$? Let $K = \max |\mathbb{Z}^2 \cap Q|$. Then $|w|$ and $\|w\|_L$ differ by at most $K$. (Burago 1992)

*Figure: The chess-knight metric.*
$\mathbb{Z}^2$ with $S = \{6e_1, e_1, 6e_2, e_2\}$. Watch $\frac{1}{n} S_n$ converge to $L$. 

$n = 1$
$\mathbb{Z}^2$ with $S = \{6e_1, e_1, 6e_2, e_2\}$. Watch $\frac{1}{n} S_n$ converge to $L$. 

$n = 2$
Convergence to the limit shape

\( \mathbb{Z}^2 \) with \( S = \{ 6e_1, e_1, 6e_2, e_2 \} \). Watch \( \frac{1}{n} S_n \) converge to \( L \).
Convergence to the limit shape

$\mathbb{Z}^2$ with $S = \{ 6e_1, e_1, 6e_2, e_2 \}$. Watch $\frac{1}{n} S_n$ converge to $L$. 

$n = 4$
Convergence to the limit shape

\(\mathbb{Z}^2\) with \(S = \{ 6e_1, e_1, 6e_2, e_2 \}\). Watch \(\frac{1}{n} S_n\) converge to \(L\).
Convergence to the limit shape

\( \mathbb{Z}^2 \) with \( S = \{ 6e_1, e_1, 6e_2, e_2 \} \). Watch \( \frac{1}{n} S_n \) converge to \( L \).
Convergence to the limit shape

$\mathbb{Z}^2$ with $S = \{ 6e_1, e_1, 6e_2, e_2 \}$. Watch $\frac{1}{n} S_n$ converge to $L$. 

$n = 7$
$\mathbb{Z}^2$ with $S = \{6e_1, e_1, 6e_2, e_2\}$. Watch $\frac{1}{n} S_n$ converge to $L$. 

$n = 8$
Convergence to the limit shape

\( \mathbb{Z}^2 \) with \( S = \{ 6e_1, e_1, 6e_2, e_2 \} \). Watch \( \frac{1}{n} S_n \) converge to \( L \).

\[ n = 9 \]
Convergence to the limit shape

\( \mathbb{Z}^2 \) with \( S = \{ 6e_1, e_1, 6e_2, e_2 \} \). Watch \( \frac{1}{n} S_n \) converge to \( L \).
Convergence to the limit shape

$\mathbb{Z}^2$ with $S = \{6e_1, e_1, 6e_2, e_2\}$. Watch $\frac{1}{n} S_n$ converge to $L$. 

$n = 11$
Convergence to the limit shape

$\mathbb{Z}^2$ with $S = \{6e_1, e_1, 6e_2, e_2\}$. Watch $\frac{1}{n} S_n$ converge to $L$. 

$n = 12$
Convergence to the limit shape

\( \mathbb{Z}^2 \) with \( S = \{ 6e_1, e_1, 6e_2, e_2 \} \). Watch \( \frac{1}{n} S_n \) converge to \( L \).
Convergence to the limit shape

\( \mathbb{Z}^2 \) with \( S = \{ 6e_1, e_1, 6e_2, e_2 \} \). Watch \( \frac{1}{n} S_n \) converge to \( L \).
Convergence to the limit shape

\( \mathbb{Z}^2 \) with \( S = \{ 6e_1, e_1, 6e_2, e_2 \} \). Watch \( \frac{1}{n} S_n \) converge to \( L \).
Convergence to the limit shape

\( \mathbb{Z}^2 \) with \( S = \{6e_1, e_1, 6e_2, e_2\} \). Watch \( \frac{1}{n} S_n \) converge to \( L \).
Convergence to the limit shape

\( \mathbb{Z}^2 \) with \( S = \{ 6e_1, e_1, 6e_2, e_2 \} \). Watch \( \frac{1}{n} S_n \) converge to \( L \).
Convergence to the limit shape

$\mathbb{Z}^2$ with $S = \{6e_1, e_1, 6e_2, e_2\}$. Watch $\frac{1}{n} S_n$ converge to $L$. 

$n = 18$
Convergence to the limit shape

$\mathbb{Z}^2$ with $S = \{6e_1, e_1, 6e_2, e_2\}$. Watch $\frac{1}{n} S_n$ converge to $L$. 

$n = 19$
Convergence to the limit shape

$\mathbb{Z}^2$ with $S = \{6e_1, e_1, 6e_2, e_2\}$. Watch $\frac{1}{n} S_n$ converge to $L$. 

$n = 20$
Theorem (Density)

For any asymptotically homogeneous function $f : \mathbb{Z}^d \to \mathbb{R}$,

$$
\lim_{n \to \infty} \frac{1}{|B_n|} \sum_{x \in B_n} \frac{1}{n} f(x) = \left( \frac{d}{d + 1} \right) \lim_{n \to \infty} \frac{1}{|S_n|} \sum_{x \in S_n} \frac{1}{n} f(x).
$$

Example: for any genset on $\mathbb{Z}^2$, the expected position of a point in $B_n$ is on $S_n/3$.

Notably different from scale-invariant functions on $\mathbb{Z}^d$, or from any functions on hyperbolic groups, where ball-average equals sphere-average.
Application: Density

Theorem (Density)

For any asymptotically homogeneous function $f : \mathbb{Z}^d \to \mathbb{R}$,

$$
\lim_{n \to \infty} \frac{1}{|B_n|} \sum_{x \in B_n} \frac{1}{n} f(x) = \left( \frac{d}{d+1} \right) \lim_{n \to \infty} \frac{1}{|S_n|} \sum_{x \in S_n} \frac{1}{n} f(x).
$$

Example: for any genset on $\mathbb{Z}^2$, the expected position of a point in $B_n$ is on $S_{2n/3}$. Notably different from scale-invariant functions on $\mathbb{Z}^d$, or from any functions on hyperbolic groups, where ball-average equals sphere-average.
Theorem (Density)

For any asymptotically homogeneous function \( f : \mathbb{Z}^d \to \mathbb{R} \),

\[
\lim_{n \to \infty} \frac{1}{|B_n|} \sum_{x \in B_n} \frac{1}{n} f(x) = \left( \frac{d}{d + 1} \right) \lim_{n \to \infty} \frac{1}{|S_n|} \sum_{x \in S_n} \frac{1}{n} f(x).
\]

Example: for any genset on \( \mathbb{Z}^2 \), the expected position of a point in \( B_n \) is on \( S_{2n/3} \).

Notably different from scale-invariant functions on \( \mathbb{Z}^d \), or from any functions on hyperbolic groups, where ball-average equals sphere-average.
Introducing **sprawl**

**Question:** what is the average distance between two points in a large sphere?

\[ E(G, S) := \lim_{n \to \infty} \frac{1}{|S_n|^2} \sum_{x, y \in S_n} \frac{1}{n} d(x, y). \]
Question: what is the average distance between two points in a large sphere?

\[
E(G, S) := \lim_{n \to \infty} \frac{1}{|S_n|^2} \sum_{x, y \in S_n} \frac{1}{n} d(x, y).
\]

Note that since \(0 \leq d(x, y) \leq 2n\), the value is always between 0 and 2. \(E = 2\) means that one can almost always pass through the origin without taking a significant detour.
**Introducing sprawl**

**Question:** what is the average distance between two points in a large sphere?

\[
E(G, S) := \lim_{n \to \infty} \frac{1}{|S_n|^2} \sum_{x, y \in S_n} \frac{1}{n} d(x, y).
\]

Note that since \(0 \leq d(x, y) \leq 2n\), the value is always between 0 and 2. \(E = 2\) means that one can almost always pass through the origin without taking a significant detour. (Sounds like hyperbolicity.)
Introducing *sprawl*

**Question:** what is the average distance between two points in a large sphere?

\[
E(G, S) := \lim_{n \to \infty} \frac{1}{|S_n|^2} \sum_{x, y \in S_n} \frac{1}{n} d(x, y).
\]

Note that since \(0 \leq d(x, y) \leq 2n\), the value is always between 0 and 2. \(E = 2\) means that one can almost always pass through the origin without taking a significant detour. (Sounds like hyperbolicity.)

**Theorem**

- *Trees can have any* \(0 \leq E \leq 2\), *or E may not exist.*
- *If G is a non-elem. hyperbolic group, then* \(E(G, S) = 2\) *for all S.*
- \(E(\mathbb{Z}^d, S)\) *depends on S.*
- *For* \((\mathbb{Z}^d, S),\) *we have* \(E(G, S) = \int_{L^2} \|x - y\|_L d\mu^2\)
Introducing sprawl

**Question:** what is the average distance between two points in a large sphere?

\[ E(G, S) := \lim_{n \to \infty} \frac{1}{|S_n|^2} \sum_{x, y \in S_n} \frac{1}{n} d(x, y). \]

Note that since \(0 \leq d(x, y) \leq 2n\), the value is always between 0 and 2. \(E = 2\) means that one can almost always pass through the origin without taking a significant detour. (Sounds like hyperbolicity.)

**Theorem**

- Trees can have any \(0 \leq E \leq 2\), or \(E\) may not exist.
- If \(G\) is a non-elem. hyperbolic group, then \(E(G, S) = 2\) for all \(S\).
- \(E(\mathbb{Z}^d, S)\) depends on \(S\).
- For \((\mathbb{Z}^d, S)\), we have \(E(G, S) = \int_{L^2} \|x - y\| \, d\mu^2 =: E(L)\).

Note \(E(L) = E(TL)\) for \(L \in GL(d, \mathbb{R})\).
Sprawl in $\mathbb{Z}^2$

We developed the *cutline* algorithm for computing sprawls of polygons and used it to make calculations, showing the extent of dependence on $S$.

\[
E(P_4) = \frac{4}{3} \quad \text{and} \quad E(P_6) = \frac{23}{18}
\]
Sprawl in $\mathbb{Z}^2$

We developed the *cutline* algorithm for computing sprawls of polygons and used it to make calculations, showing the extent of dependence on $S$.

\[ E(P_4) = \frac{4}{3} \]
\[ E(P_6) = \frac{23}{18} \]
\[ E(P_8) = \frac{1+2\sqrt{2}}{3} \]
Sprawl in $\mathbb{Z}^2$

We developed the cutline algorithm for computing sprawls of polygons and used it to make calculations, showing the extent of dependence on $S$.

\[
E(P_4) = \frac{4}{3} \quad E(P_6) = \frac{23}{18}
\]

\[
E(P_8) = \frac{1+2\sqrt{2}}{3} \quad E(P_n) \rightarrow E(S^1) = \frac{4}{\pi}
\]
Sprawl in $\mathbb{Z}^2$

We developed the *cutline* algorithm for computing sprawls of polygons and used it to make calculations, showing the extent of dependence on $S$.

$$E(P_4) = \frac{4}{3} \quad \quad E(P_6) = \frac{23}{18}$$

$$E(P_8) = \frac{1+2\sqrt{2}}{3} \quad \quad E(P_n) \to E(S^1) = \frac{4}{\pi}$$

*Figure*: Ranges of sprawls.
Sprawls of hexagons and three-generator presentations

We know that $E(\Omega)$ can take all values in $[4/\pi, 4/3]$, which implies that $E(\mathbb{Z}^2, S)$ can take a dense set of values in that range. (Rational approximation.)

Conjecture

*That’s it.*
Sprawls of hexagons and three-generator presentations

We know that $E(\Omega)$ can take all values in $[4/\pi, 4/3]$, which implies that $E(\mathbb{Z}^2, S)$ can take a dense set of values in that range. (Rational approximation.)

Conjecture

That’s it. \( \{E(\Omega) : \Omega \subset \mathbb{R}^2 \text{ convex, cent.sym.}\} = [4/\pi, 4/3]. \)
Sprawls of hexagons and three-generator presentations

We know that $E(\Omega)$ can take all values in $[4/\pi, 4/3]$, which implies that $E(\mathbb{Z}^2, S)$ can take a dense set of values in that range. (Rational approximation.)

Conjecture

That’s it. \[ \{ E(\Omega) : \Omega \subset \mathbb{R}^2 \text{ convex, cent.sym.} \} = [4/\pi, 4/3]. \]

Besides lots of empirical evidence, here is some good rigorous evidence.

Theorem

\[ \{ E(H) : \text{hexagons } H \} = [23/18, 4/3]. \]

Thus, $23/18 \leq E(\mathbb{Z}^2, S) \leq 4/3$ whenever $|S| \leq 6$. 
As $d \to \infty$, we find $E(\text{Sphere}_d) \to \sqrt{2}$, $E(\text{Cube}_d) \to 2$. 
Sprawl in $\mathbb{Z}^d$

As $d \to \infty$, we find $E(\text{Sphere}_d) \to \sqrt{2}$, $E(\text{Cube}_d) \to 2$.

Figure: Ranges of sprawls: $d = 2, 3, 4, 5, 100, \infty$. 

Theorem: $E(\mathbb{Z}^d, \text{std}) = E(\text{Orth}_d) = \frac{3d}{2}$. 

Conclusion: to make a free abelian group look as hyperbolic as possible, use the nonstandard generators $S_{\text{cube}} = \{\pm e_1 \pm e_2 \cdots \pm e_d\}$. 

Duchin Lelièvre Mooney (2010)
Sprawl in $\mathbb{Z}^d$

As $d \to \infty$, we find $E(\text{Sphere}_d) \to \sqrt{2}$, $E(\text{Cube}_d) \to 2$.

**Theorem**

$$E(\mathbb{Z}^d, \text{std}) = E(\text{Orth}_d) = \frac{3d-2}{2d-1} \to \frac{3}{2}.$$
Sprawl in $\mathbb{Z}^d$

As $d \to \infty$, we find $E(\text{Sphere}_d) \to \sqrt{2}$, $E(\text{Cube}_d) \to 2$.

Figure: Ranges of sprawls: $d = 2, 3, 4, 5, 100, \infty$.

Theorem

$$E(\mathbb{Z}^d, \text{std}) = E(\text{Orth}_d) = \frac{3d-2}{2d-1} \to \frac{3}{2}.$$
Sprawl in $\mathbb{Z}^d$

As $d \to \infty$, we find $E(\text{Sphere}_d) \to \sqrt{2}$, $E(\text{Cube}_d) \to 2$.

**Theorem**

$$E(\mathbb{Z}^d, \text{std}) = E(\text{Orth}_d) = \frac{3d-2}{2d-1} \to \frac{3}{2}.$$  

**Conclusion:** to make a free abelian group look as hyperbolic as possible, use the nonstandard generators $S_{\text{cube}} = \{\pm e_1 \pm e_2 \cdots \pm e_d\}$.