Self-consistent field theory of polarized BEC: dispersion of collective excitation

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We suggest the construction of a set of the quantum hydrodynamics equations for the Bose-Einstein condensate (BEC), where atoms have the electric dipole moment. The contribution of the dipole-dipole interactions (DDI) to the Euler equation is obtained. Quantum equations for the evolution of medium polarization are derived. Developing mathematical method allows to study effect of interactions on the evolution of polarization. The developing method can be applied to various physical systems in which dynamics is affected by the DDI. Derivation of Gross-Pitaevskii equation for polarized particles from the quantum hydrodynamics is described. We showed that the Gross-Pitaevskii equation appears at condition when all dipoles have the same direction which does not change in time. Comparison of the equation of the electric dipole evolution with the equation of the magnetization evolution is described. Dispersion of the collective excitations in the dipolar BEC, either affected or not affected by the uniform external electric field, is considered using our method. We show that the evolution of polarization in the BEC leads to the formation of a novel type of the collective excitations. Detailed description of the dispersion of collective excitations is presented. We also consider the process of wave generation in the polarized BEC by means of a monoenergetic beam of neutral polarized particles. We compute the possibilities of the generation of Bogoliubov and polarization modes by the dipole beam.

I. INTRODUCTION

After obtaining the Bose-Einstein condensate (BEC) in experiments with vapors of alkaline metal atoms, theoretical and experimental investigation of linear waves and nonlinear structures in the BEC have been performed. In recent years the interest to the polarized BEC has been increasing. It is connected with recent experimental progress in cooling of polarized atoms and molecules. At present day the BEC with magnetic polarization is realized on atoms $^{52}$Cr. There are a lot of attempt of experimental obtaining of the electrically polarized BEC (see reviews [1]-[4]). For this aim, Bose molecules having the electric dipole moment have been cooled. Particular interest to the electrically polarized BEC brought because of large dipole-dipole scattering length, and thereby because of both the big magnitude and large distance of interaction in compare with analogous quantities for the magnetized BEC.

Many processes in quantum systems are determined by the dynamics and the dispersion of collective excitations (CE) [5]. The law dispersion of the CE in the degenerate dilute Bose gas was obtained by Bogoliubov in 1947 [6,7]. Many authors studied the change of the Bogoliubov spectrum which arises when the short-range interaction accounted more carefully [8]-[14], or geometry of the system is complex [15], [16], [17]. In papers [18]-[22] authors studied the influence of the electric dipole moment (EDM) dynamics on the dispersion of the CE in the BEC. The contribution of polarization in the dispersion law of the Bogoliubov mode was obtained in Refs. [18]-[22]. Instability of the Bogoliubov spectrum in the 3D dipolar BEC (DBEC) with the repulsive short-range interaction (SRI) was shown in Refs. [18]-[20]. U. R. Fisher [21] obtained that the Bogoliubov mode in the 2D DBEC is stable for a wide range of system parameters. There are also review papers [2]-[4], where various aspects of physics of the polarized BEC were considered. In this paper we are interested in possibility of the polarization wave existence in the DBEC, i.e. in existence of new type of the CE. Dynamics of polarization is interesting not for the quantum gases only, but in the solid state physics and the physics of low-dimensional systems too. The polarization waves in the low dimensional and the multy-layer systems of conductors, dielectrics, and semiconductors are considered in the papers [23][24]. Analogously, in the BEC of molecules having the EDM we expect the existence of a polarization wave along with the Bogoliubov mode.

For investigation of the BEC, in system of particles with the EDM or the magnetic moments, various theoretical methods are used. One of the ways of theoretical description of the polarized BEC is the generalization of well-known Gross-Pitaevskii (GP) equation [25]-[29]. In this way next papers were made: for the spinor BEC [30]-[32], for the effect of magnetic moment on the BEC evolution [33]-[41] and for the influence of electrical polarization of atoms on the dynamic processes in the BEC [32]-[42]. Dipolar fermions also attract a lot of attention [43], [44].

The same generalization of the GP equation is usually used for the magnetic moments and the electric dipoles:

$$i\hbar \partial_t \Phi(r,t) = \left( -\frac{\hbar^2 \nabla^2}{2m} + \mu(r,t) + V_{\text{ext}}(r,t) + g \left| \Phi(r,t) \right|^2 \right) \Phi(r,t) + d^2 \int d\mathbf{r}' \frac{1 - 3 \cos^2 \theta'}{|\mathbf{r} - \mathbf{r}'|^3} \left| \Phi(\mathbf{r}',t) \right|^2 \Phi(\mathbf{r},t).$$

In equation (1) following designation are used: $\Phi(\mathbf{r},t)$ is the macroscopic wave function, $\mu$ is the chemical potential, $V_{\text{ext}}$ is the potential of external field, $g$ is the constant of short-range interaction, $d$ is the dipole electric moment of...
Along with the GP equation the method of quantum hydrodynamics (QHD) is also used. The method of QHD was used for investigations of various physical systems, particularly for the BEC and the polarization waves in conductors and semiconductors. Therefore, the method of QHD could be used to consider the possibility of arising polarization waves in the BEC and calculate of the dispersion law of these waves. 

At interpretation of the DDI in the quantum gases (QG) used the notion of the scattering length (SL) and the first Born approximation (FBA), analogously to the SRI described by the fourth term in the right-hand side of equation (1). Therefore, the problem reduces to the process of scattering. Different approximations based on scattering process was analyzed in Ref. [48]. Particularly, where were considered condition of the FBA using and presented generalization of the GP equation for the scattering of polarized atoms beyond the FBA.

Considering the QHD we do not formulate the scattering problem. We do not need to limit ourselves to a scattering process, because we are interested in interaction between atoms including interaction between several atoms at the same time. This statement is especially important for the polarized atoms because of the long-range interaction between dipoles. Each act of interaction between dipoles could be considered as the scattering process for system of particles with a low concentration. This approximation is possible as for the neutral polarized particles as for the charged particles. The kinetic theory for the last case was developed by L. D. Landau in 1936 [49]. Landau constructed collision term for the Coulomb particles, using analogy with the Boltzmann equation. The Landau collision integral depends on a scattering cross section. In this connection he considered the problem of the scattering of charged particles on small angles, that is suitable for rare systems. Strictly speaking, for particles with the long-range interaction the conception of the self-consistent field is more suitable. This conception was suggested by A. A. Vlasov in 1938, for system of the charged particles [50]. In fact, group of particles moves in the field of their neighbors, but state of motion of the neighbors depends on motion of this group of particles as well (This picture of interaction is described on Fig. 1). According to this idea Vlasov proposed a kinetic equation for the charged particles called the Vlasov equation. Actually the Landau collision integral and the Vlasov equation are the two opposite branches of a hypothetical general theory. In this paper we follow to the Vlasov approximation and derive the self-consistent field theory for the dipole-dipole interaction in the quantum gases. From this point of view, there is no need to consider the first Born approximation or scattering process. In Appendix B to this paper we present derivation of the GP equation for polarized particles from the QHD equations. We did it at the condition that dipoles are parallel each other and to an external field, they directions do not oscillate or vibrate around an equilibrium direction.

Other methods were also applied for investigation of the polarized BEC and other systems of particles having the EDM, along with the GP equation. A hydrodynamic formulation of the Hartree-Fock theory for particles with the significant EDM is considered in Ref. [51]. The Euler-type equation was derived in paper [51]., from the evolution of the density matrix. The EDM dynamics in dimer Mott insulators causes the rise of the low-frequency mode [52]. The spectrum of single-particle excitations and long-wave-length collective modes (zero sound) in the normal phase were obtained [53] in the Hartree-Fock approximation, which treats direct and exchange interactions on an equal footing. Minimal coupling model for description of spatial polarization changing on the BEC properties is constructed in Ref. [54]. The density functional method was used for the polarized quantum gas studying in Ref. [55]. The Hubbard model for the polarized BEC is used too [56]. A two-body quantum problem for the polarized molecules is analyzed in [57]. The existence of states with spontaneous interlayer coherence has been predicted in [58] in systems of the polar molecules. Calculations, in [58], were based upon the secondary quantization approach, where the Hamiltonian accounts for the molecular rotation and the dipole-dipole interactions. Superfluidity
FIG. 2. (Color online) The figure shows motion of dipoles without changing of direction of dipoles during motion. Dipoles stand at temperature equal to zero $T = 0$ and in external uniform static electric field $E$. This picture describes particles in two different moments of time. White dipoles presents system in the first moment of time, and violet dipoles are present system in the next moment.

Anisotropy of polarized fermion systems was shown in [59] and their thermodynamic and correlation properties were investigated [60]. The effect of the EDM on a system of cooled neutral atoms which are used for the quantum computing and quantum memory devices was analyzed in Ref. [61].

The characteristic property of the BEC in a system of excitons inducing in semiconductors [62] is a significant value of the exciton EDM. This leads to strong interaction of excitons with an external electric field and emergence of the collective DDI in exciton systems. The QHD method may be applied to such systems along with quantum kinetics based on the nonequilibrium Green functions [63] or the density matrix equations [64].

Notable success has been reached in the Bose condensation of dense gases [65, 66]. Consequently, we need the detailed account of the short range interaction for investigation of the EDM dynamics. In this work we account the SRI up to the third order on interaction radius (TOIR) [14]. This approximation leads to nonlocality of the SRI [14], [67], [70].

Electrically polarized BEC can interact with the beam of charged and polarized particles by means of the charge-dipole and the dipole-dipole interaction. These interactions lead to transfer of energy from the beam to medium and, consequently, to generation of waves. In the plasma physics, the effects of generation of waves by electron [71] or magnetized neutron [72, 73] beam are well-known. When we use term beam we also mean stream of particles excited in the considered system, along with an external beam of particles passing through the system. In presented paper we consider similar effect in the polarized BEC.

At our treatment we derive and use the QHD equations for polarized particles. Set of the QHD equations consists of the continuity equation, the momentum balance equation (the Euler equation), the equation of polarization evolution and equation of polarization current evolution. We present first principles derivation of this equation. For this purpose we use the many-particle Schrödinger equation. We analytically calculate the dispersion properties of the DBEC. We show that the dynamics of the EDM leads to existence of new branch in the dispersion law. Consequently, there are the waves of polarization in the DBEC, along with the Bogoliubov mode. We obtain contribution of the DDI to the dispersion of the Bogoliubov mode. Then, we consider the process of wave generation in the DBEC by means of a monoenergetic beam of neutral polarized particles. We suggest new method of generation of both the Bogoliubov mode and the polarization wave. This paper is an extension and result of processing of our previous paper [74], where a brief description of the same results and ideas were presented. Some ideas were also described in Ref. [75].

This paper is organized as follows. We introduce the model Hamiltonian in Sec. II, and present the momentum balance equation. Further, in Sec. II, we derive equations of the polarization evolution and obtain the influence of the interactions on the evolution of polarization. In Sec. III we calculate the dispersion dependence of the CE in the DBEC. The polarization evolution is taken into account. We obtain the contribution of polarization in dispersion of the Bogoliubov mode and show the existence of new wave solution. We consider high frequency excitation appearing at large equilibrium polarizations. We also present calculations for the wave of polarization at constant concentration. In Sec. IV we study the wave generation in the polarized BEC by means of the neutral polarized particle beam. In Sec. V we present the brief summary of our results. App. A contains detail of derivation of equations of polarization evolution. In App. B derivation of the GP equation for polarized particles is presented. In App. C we consider distinction of evolution of spinning particle from the particles having EDM.
FIG. 4. (Color online) The figure describes how the polarization direction perturbation propagates through the system of particles. We consider system of dipoles at \( T = 0 \) and being in an external uniform electric field. Upper picture shows that if at the first moment there is a perturbation of the EDM, in next moment, this perturbation passes to several neighbors. At lower picture, particles in two different moments of time are described. White dipoles presents system in the first moment of time, when one dipole is perturbed only, but violet dipoles present system in the next moment.

THE MODEL

The term describing DDI in the GP equation, for both the EDM and the magnetic moment, is interpreted via the dipole scattering length (DSL) \( a_{dd} \). The GP equation does not contain information about evolution of the dipole direction. Changing of the dipole direction appears due to the interaction. If we have an external static uniform electric field, at close to zero temperatures, all dipoles will be directed parallel to the external field. At low temperatures, there is no temperature disorientation of dipoles and we have system of parallel dipoles, as it is shown on Fig. 2. In such systems, there are two type of dipoles motion. The first one is a motion, where the direction of dipoles preserves (see Fig.s 2, 3). The DDI leads to anisotropy of interaction in the BEC. At such form of motion the polarization is changed accordingly to the particle concentration

\[
P(r, t) = d_n(r, t),\]

where \( d \) is the EDM of single particle, this motion is depicted on the Fig. 3. The second type of motion is a motion of dipoles including change of dipole direction described on the Fig.s 4, 5. Perturbation of dipole direction can be caused by both external influence and local fluctuation of direction of other dipoles produced by the DDI. In consequence of the DDI, a perturbation will propagate through the system. Particularly, this propagation might have wave nature. We need to know how polarization of system evolves in different point of space during the time for description of such motion. Therefore we need to derive equations describing the evolution of polarization field and accounting influence of interaction on this evolution.

In this case dipoles in different areas of space have different directions Fig. 6 and it reflects on interaction via angle dependence under integral in the GP equation (1). Still, this equation does not contain information about propagation of dipoles direction perturbation.

At studying we pay attention for the SRI and DDI. We also consider the action of external fields on particles. We consider the action of external electric field on dipoles in explicit form and the external field describing trap presented in general form by trapping potential \( V_{\text{trap}}(r, t) \).

We used quasi-static approximation for the DDI. The quasi-static approximation means that we consider inter-
action between dipoles at fixed moment of time as they are motionless. The DDI leads to change of state of motion, and, in particularly, to variation of dipole direction. Then, we talk about change of direction we mean both the variation during precession around of rotation axis and deviation of rotation axis direction. Described approximation is an analog of the Coulomb interaction, where interaction between moving charges approximately described by means of interaction potential of motionless charges, this approximation usually used for description of non-relativistic plasmas [78].

Explicit form of the Hamiltonian of considering system in the quasi-static approximation is

\[
\hat{H} = \sum_i \left( \frac{1}{2m_i} \mathbf{\vec{p}}_i^2 - d_i \mathbf{\vec{E}}_{i,\text{ext}} + V_{\text{trap}}(\mathbf{r}_i, t) \right) + \frac{1}{2} \sum_{i,j \neq i} \left( U_{ij} - d_i^\alpha d_j^\beta G_{\alpha\beta}^{ij} \right). \tag{2}
\]

The first term in the Hamiltonian is the operator of the kinetic energy. The second term presents the interaction between dipoles at fixed moment of time as they are motionless. The DDI leads to change of state of motion, which are the Maxwell equation. The subsequent terms present the short-range stress tensor. They occur because of taking into account of the thermal velocities of particles and does not contribute into the BEC dynamics at temperatures near zero.

For investigation of the CE dynamics in the polarized BEC we derive set of the QHD equations. This system of equations consists of the continuity equation, Euler equation and, for the case of polarized particles, the equations of polarization evolution and equations of field (the Maxwell equations). The system of equations is derived by methods described in Ref. [14].

The first equation of the QHD equation set is the continuity equation

\[
\partial_t n(\mathbf{r}, t) + \partial^\alpha (n(\mathbf{r}, t)v^\alpha(\mathbf{r}, t)) = 0. \tag{3}
\]

The momentum balance equation for the polarized BEC appears as

\[
mn(\mathbf{r}, t)(\partial_t + \mathbf{\nabla})v^\alpha(\mathbf{r}, t) + \partial_\beta P^{\alpha\beta}(\mathbf{r}, t)
\]

\[
- \frac{\hbar^2}{4m} \partial^\alpha \nabla n(\mathbf{r}, t) + \frac{\hbar^2}{4m} \partial^\beta \left( \frac{\partial^\alpha n(\mathbf{r}, t) \cdot \partial^\beta n(\mathbf{r}, t)}{n(\mathbf{r}, t)} \right)
\]

\[
= \Upsilon n(\mathbf{r}, t) \partial^\alpha n(\mathbf{r}, t) + \frac{1}{2} \Upsilon_2 \partial^\alpha n^2(\mathbf{r}, t)
\]

\[
+ P^{\beta}(\mathbf{r}, t) \partial^\alpha E^\beta(\mathbf{r}, t), \tag{4}
\]

where

\[
\Upsilon = \frac{4\pi}{3} \int dr(r) \frac{\partial U(r)}{\partial r}, \tag{5}
\]

and

\[
\Upsilon_2 = \frac{\pi}{30} \int dr(r) \frac{\partial U(r)}{\partial r}. \tag{6}
\]

In equation (4) we defined a parameter \( \Upsilon_2 \) as (6). This definition differs from the one in the paper [14]. Terms proportional to \( \hbar^2 \) appear as a result of using of quantum kinematics. The first two terms in the right-hand side of the equation (4) are the first terms of expansion of the quantum stress tensor. They occur because of taking into account of the SRI potential \( U_{ij} \). The interaction potential \( U_{ij} \) determines the macroscopic interaction constants \( \Upsilon \) and \( \Upsilon_2 \). The last two terms in the equation (4) describe force fields that affect the dipole moment in a unit of volume as the effect of the external electrical field and the field produced by other dipoles, respectively. The last term is written using the self-consistent field approximation [45]. [23]. \( p^{\alpha\beta}(\mathbf{r}, t) \) is the kinetic pressure tensor, which depends on the thermal velocities of particles and does not contribute into the BEC dynamics at temperatures near zero.

The first order of the interaction radius interaction constant for dilute gases has the form

\[
\Upsilon = -\frac{4\pi \hbar^2 a}{m}, \tag{7}
\]

where \( a \) is the scattering length (SL) [7,14]. The value \( \Upsilon_2 \) may be expressed approximately as

\[
\Upsilon_2 = -\frac{\theta a^2 \Upsilon}{8}, \tag{8}
\]

where \( \theta \) is a constant positive value about 1, which depends on the interatomic interaction potential. Finally, \( \Upsilon_2 \) takes the form

\[
\Upsilon_2 = -\frac{\pi \theta h^2 a^3}{2m}. \tag{9}
\]

We have also equations of field

\[
\nabla \mathbf{E}(\mathbf{r}, t) = -4\pi \nabla \mathbf{P}(\mathbf{r}, t), \tag{10}
\]

and

\[
\nabla \times \mathbf{E}(\mathbf{r}, t) = 0, \tag{11}
\]

which are the Maxwell equation.

In the case of particles having no dipole moment, the continuity equation and the momentum balance equation form a closed set of equations. When the dipole moment is taken into account in the momentum balance equation, a new physical value emerges, it is the polarization vector. For investigation of the CE dynamics in the polarized BEC we derive set of the QHD equations. This system of equations consists of the continuity equation, Euler equation and, for the case of polarized particles, the equations of polarization evolution and equations of field (the Maxwell equations). The system of equations is derived by methods described in Ref. [14].

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mn(\mathbf{r}, t)(\partial_t + \mathbf{\nabla})v^\alpha(\mathbf{r}, t) + \partial_\beta P^{\alpha\beta}(\mathbf{r}, t)
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- \frac{\hbar^2}{4m} \partial^\alpha \nabla n(\mathbf{r}, t) + \frac{\hbar^2}{4m} \partial^\beta \left( \frac{\partial^\alpha n(\mathbf{r}, t) \cdot \partial^\beta n(\mathbf{r}, t)}{n(\mathbf{r}, t)} \right)
\]

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= \Upsilon n(\mathbf{r}, t) \partial^\alpha n(\mathbf{r}, t) + \frac{1}{2} \Upsilon_2 \partial^\alpha n^2(\mathbf{r}, t)
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which are the Maxwell equation.

In the case of particles having no dipole moment, the continuity equation and the momentum balance equation form a closed set of equations. When the dipole moment is taken into account in the momentum balance equation, a new physical value emerges, it is the polarization vector field \( P^{\alpha}(\mathbf{r}, t) \). Thus, the set of equations is not closed.

Next equation we need for studying of the CE dispersion is the polarization evolution equation

\[
\partial_t P^{\alpha}(\mathbf{r}, t) + \partial^\beta R^{\alpha\beta}(\mathbf{r}, t) = 0, \tag{12}
\]

where \( R^{\alpha\beta}(\mathbf{r}, t) \) is the current of polarization.

Equation (12) does not contain information about the effect of the interaction on the polarization evolution. An
evolution equation of $R^{\alpha\beta}(r, t)$ can be obtained by analogy with the equations derived above. Method of the equations derivation is described in Appendix A. Using the self-consistent field approximation for the dipole-dipole interaction we obtain an equation for the polarization current $R^{\alpha\beta}(r, t)$ evolution

$$\partial_t R^{\alpha\beta}(r, t) + \partial^\gamma \left( R^{\alpha\beta}(r, t) v^\gamma(r, t) \right) + R^{\alpha\gamma}(r, t) v^\beta(r, t) - P^{\alpha}(r, t) v^\beta(r, t) v^\gamma(r, t)$$

$$+ \frac{1}{m} \partial^\gamma r^{\alpha\beta\gamma}(r, t) - \frac{\hbar^2}{4m^2} \partial_\beta \Delta P^{\alpha}(r, t)$$

$$+ \frac{\hbar^2}{8m^2} \partial^\gamma \left( \frac{\partial_\beta P^{\alpha}(r, t) \partial_\gamma n(r, t)}{n(r, t)} + \frac{\partial_\gamma P^{\alpha}(r, t) \partial_\beta n(r, t)}{n(r, t)} \right)$$

$$= \frac{1}{m} \Upsilon \partial^\beta \left( n(r, t) P^{\alpha}(r, t) \right)$$

$$+ \frac{\sigma}{m} \frac{P^{\alpha}(r, t) P^{\gamma}(r, t)}{n(r, t)} \partial^\beta E^{\gamma}(r, t),$$

(13)

where $r^{\alpha\beta\gamma}(r, t)$ represents the contribution of thermal movement of polarized particles into the dynamics of $R^{\alpha\beta}(r, t)$. As we deal with the BEC, the contribution of $r^{\alpha\beta\gamma}(r, t)$ may be neglected. The last term in the formula (13) includes both external electrical field and the self-consistent field created by particle dipoles. This term contains numerical constant $\sigma$. Approximations used in formula (13) are described in Appendix A. The first term in the right-hand side of equation (13) describes the short-range interaction.

We can see various interactions are included in the equations (4) and (13) additively. At short distances among particles the SRI and the dipole-dipole interaction act together. At large distances dipole-dipole interaction remains only.

We can see from equation (13) that change of the polarization arises from both the dipole-dipole interaction and the short range interaction. The SRI among particle leads to displacement of particles. Consequently, as particles have the EDM, there is motion of the EDM. Therefore, there is evolution of $R^{\alpha\beta}(r, t)$.

The terms proportional to $\hbar^2$ have the quantum origin. They are analogs of the Bohm quantum potential in the momentum balance equilibrium.

In the Appendix C we consider the difference between equations of electrical and magnetic polarization evolution. The QHD of spinning particles with spin-orbit interaction were developed in Ref. [73] for system of charged particles, however, obtained where force field can be used for neutral quantum gases too.

**COLLECTIVE EXCITATIONS IN THE POLARIZED BEC**

We can analyze the linear dynamics of the collective excitations in the polarized three dimensional BEC using the QHD equations (3), (4), (10), (12) and (13). Let’s assume that the system is placed in an external electrical field $E_0 = E_0 e_z$. The values of concentration $n_0$ and polarization $P_0 = \kappa E_0$ for the system in an equilibrium state are constant and uniform and corresponding values of the velocity field $v^\alpha(r, t)$ and tensor $R^{\alpha\beta}(r, t)$ equal to zero.

We consider small perturbations of the equilibrium state as

$$n = n_0 + \delta n, \quad v = 0 + v, \quad E = E_0 + \delta E,$$

$$P^{\alpha} = P_0^{\alpha} + \delta P^{\alpha}, \quad R^{\alpha\beta} = 0 + \delta R^{\alpha\beta}.$$  

Substituting these formulas in equations (3), (4), (12), (13) and (10) and neglecting nonlinear terms, we obtain a system of linear homogeneous equations in partial derivatives with constant coefficients. Passing to the following representation for small perturbations $\delta f$

$$\delta f = f(\omega, k) e^{i(\omega t + ikz)}$$

yields the homogeneous system of algebraic equations, where we consider wave propagation along the external electric field. In this case the equilibrium polarization gives maximum contribution. The electric field strength is assumed to have a nonzero value. Expressing all the quantities entering the system of equations in terms of the electric field, we come to the equation

$$\Lambda \cdot \delta E_z = 0,$$

where

$$\Lambda = \omega^2 - \frac{\hbar^2 k^4}{4m^2} + \frac{\Upsilon k^2 n_0}{2m} + 4\pi \sigma \frac{P_0^{2} k^2}{mn_0}$$

$$- \frac{2\pi \Upsilon k^4 P_0^2}{m^2 \omega^2 - \hbar^2 k^4 / \Lambda + \frac{\Upsilon k^2 n_0}{m} - \frac{\Upsilon k^4 n_0}{m}}.$$  

In this case, the dispersion equation is

$$\Lambda = 0.$$  

Solving this equation with respect to $\omega^2$ we obtain the following results.

The dispersion characteristic of the CE in the DBEC can be expressed in the form of

$$\omega^2 = \frac{1}{2m} \left( -\frac{3}{2} \Upsilon n_0 k^2 + \frac{\hbar^2 k^4}{2m} + \Upsilon_2 n_0 k^4 + 4\pi \sigma \frac{P_0^{2} k^2}{n_0} \right)$$

$$\pm \sqrt{\left( \frac{1}{2} \Upsilon n_0 k^2 - \Upsilon_2 n_0 k^4 + 4\pi \sigma \frac{P_0^{2} k^2}{n_0} \right)^2 - 8\pi \Upsilon k^4 P_0^2 \left( \frac{1}{2} \Upsilon n_0 k^2 - \Upsilon_2 n_0 k^4 + 4\pi \sigma \frac{P_0^{2} k^2}{n_0} \right)}}.$$  

(15)

In contrast to the non-polarized BEC, where the Bogoliubov mode exists only [6, 7, 14], a new wave solution
appears in the polarized system due to the polarization dynamics. The Bogoliubov mode corresponds to a solution of the equation (15) with minus sign before the square root. New wave solution is a wave of polarization. In general case presented by formula (15) the frequency of polarization dynamics in the BEC leads to a new solution. We can see wave propagation at an angle $\theta$ to the direction of the external field we find that we should put $P_0 \cos \theta$ instead of $P_0$ in equation (15).

Formula (15) demonstrates that taking into account of polarization dynamics in the BEC leads to a new solution. In general case presented by formula (15) the frequency of polarization wave in the BEC depends on $P_0$ and $\sigma$. To investigate the BEC polarization effect on the Bogoliubov mode and the dispersion characteristic of the new solution we analyze limit cases of the formula (15). If we consider wave propagation at an angle $\theta$ to the direction of the external field we find that we should put $P_0 \cos \theta$ instead of $P_0$ in equation (15).

Formula (15) demonstrates that taking into account of the DDI, although $a_{dd}$ does not connect with scattering.

Above we have discussed that we do not consider the scattering process, but analogy with the results of other papers we introduced an effective parameter $a_{dd}$ describing contribution of the DDI, although $a_{dd}$ does not connect with scattering.

In the following we describe a numerical analysis of formula (16). Reduced frequency $\xi$ for two waves presented by formula (16) is exhibited on Figs. 7 and 8. They
are obtained for the repulsive SRI \( a > 0, \ Y < 0 \) at \( a = 10^{-7} \text{cm} \) and \( n_0 = 10^{14} \text{cm}^{-3} \). These Fig.s show the reduced frequency \( \xi(k) \) for different values of equilibrium polarization and, consequently, for different DSL \( a_{dd} \).

Fig. 7 presents solution with sign "+" in front of the square root in formula (15), this solution corresponds to the Bogoliubov mode. For all \( a_{dd}/|a| \) the \( \xi(k) \) does not depend on \( a_{dd}/|a| \).

Fig. 8 shows strong dependence of the reduced frequency of the polarization mode ("+" in front of the square root in formula (15)) on equilibrium polarization. On Fig.8 as on other Fig.s this dependence is presented via dependence on the DSL divided on module of the SL of SRI \( a_{dd}/|a| \).

The form of \( \xi(k) \) for two solutions has similar form. We present both solutions on Fig. 9 at fixed \( a_{dd}/|a| = 1 \) to compare. From Fig. 9 we see that the reduced frequency of polarization mode is larger than one for Bogoliubov mode.

From the Fig.8 we obtain that polarization lead to growing of the reduced frequency \( \xi \) of the polarization mode. The reduced frequency \( \xi \) increases in 6 times at small wave vectors of order \( 10^4 \text{cm}^{-1} \) and for the \( a_{dd} = |a| \). For the \( a_{dd} = 100|a| \) increases in 60 times. With increasing of wave vector the influence of polarization becomes smaller. For example at \( k = 10^6 \text{cm}^{-1} \) and \( a_{dd} = 100|a| \) the reduced frequency \( \xi \) increase just in 1.6 times.

For the case of an attractive SRI in the absence of the polarization there is instability of the Bogoliubov mode at small wave vectors. At the presence of small polarization the Bogoliubov mode remains unstable, but the area of stability become some wider as it is shown on Fig. 10. There is fast widening of the region of stability at increasing of \( a_{dd}/|a| \). Fig. 11. Widening starts at approximately \( a_{dd} = 0.02|a| \).

For the case of the attractive SRI in the absence of polarization there is instability of polarization mode at small wave vectors (see Fig. 12). At \( a_{dd} = 0 \) we see instability of waves at wave vector \( k \) below \( 1.6 \cdot 10^4 \text{cm}^{-1} \). The EDM leads to increasing of \( \xi \) and fast growing of the region of stability. The wave is stable up to \( k = 10^6 \text{cm}^{-1} \) at \( a_{dd}/|a| = 1 \). Further increasing of \( a_{dd}/|a| \) leads to increasing of \( \xi \) up to 64. \( \xi \) become equal to 64 at \( a_{dd}/|a| \approx 100 \) and does not change at further increasing of \( a_{dd}/|a| \).

Comparing Fig.s 10 and 12 we can see that reduced frequency \( \xi \) of polarization mode much larger than the Bogoliubov mode, as it was for the repulsive SRI.

With increasing of strength of the SRI the role of \( \Upsilon_2 \) become more important. It’s influence reveals at \( |a| = 10^{-5} \text{cm} \) for the attractive SRI. In this case, there is instability region at relatively large wave vectors. For de-

FIG. 11. (Color online) The figure presents the dependence of reduced frequency \( \xi \) on the wave vector \( k \) for the Bogoliubov mode, \( a < 0, \ Y > 0 \) (attraction), the DSL \( a_{dd} = 0 \) (Red, the right-hand one), \( a_{dd} = 0.0005|a| \) (Blue), \( a_{dd} = 0.005|a| \) (Green), \( a_{dd} = 100|a| \) (Red, the left-hand one). We use \( \lg k \), where \( k \) is measured in \( \text{cm}^{-1} \). \( |a| = 10^{-7} \text{cm}, n_0 = 10^{14} \text{cm}^{-3}, \sigma = 1, \theta = 1 \).

FIG. 12. (Color online) The figure presents the dependence of reduced frequency \( \xi \) on the wave vector \( k \) for the polarization mode, \( a < 0, \ Y > 0 \) (attraction), DSL \( a_{dd} = 0 \) (Red), \( a_{dd} = |a| \) (Blue), \( a_{dd} = 10|a| \) (Green), \( a_{dd} = 100|a| \) (Brown). We use \( \lg k \), where \( k \) is measured in \( \text{cm}^{-1} \). \( |a| = 10^{-7} \text{cm}, n_0 = 10^{14} \text{cm}^{-3}, \sigma = 1, \theta = 1 \).
where $k$ is a region, where we see a gap in curves. Figure obtained for the attractive SRI ($a < 0$, $\Upsilon > 0$), $a_{dd} = 0$ (Red), $a_{dd} = 0.1|a|$ (Blue), $a_{dd} = 0.2|a|$ (Green), $a_{dd} = |a|$ (Black). We use $\lg k$, where $k$ is measured in cm$^{-1}$. $|a| = 10^{-5}$ cm, $n_0 = 10^{14}$ cm$^{-3}$, $\sigma = 1$, $\theta = 1$.

Below, we analytically consider limit cases of general dispersion solution (15).

We now proceed to derive a dispersion dependence when the effect of polarization is small in comparison with the contribution of the short-range effects, i.e. we consider a limit when the terms proportional to $\Upsilon$ and $\Upsilon_2$ are compara-

rably large. Consequently, formula (15) takes the form

$$\omega^2_B = \frac{\hbar^2 k^4}{4m^2} - \frac{\Upsilon n_0 k^2}{m} + \frac{\Upsilon_2 n_0 k^4}{m},$$

(17)

and

$$\omega^2_P = \frac{\hbar^2 k^4}{4m^2} - \frac{1}{2} \frac{\Upsilon n_0 k^2}{m} + \frac{4\pi P_0^2 k^2 \Upsilon (\sigma - 1) - 2\sigma \Upsilon_2 k^2}{\Upsilon - 2\Upsilon_2 k^2},$$

(18)

where we use indexes "B" for the Bogoliubov mode and "P" for new polarization mode. At derivation of formulas (12) and (18) we expanded a sub-radical expression in (15) and took only first two terms of the expansion to estimate the influence of polarization on the wave dispersion.

As it follows from equations (12) and (13), changes of polarization can occur due to the dipole-dipole interactions, the SRI and the quantum Bohm potential (terms proportional to $\hbar^2$). This is the reason for existence of other ways of polarization changing when the contribution of equilibrium polarization $P_0$ is negligible. In described case, changes in the polarization do not affect the concentration evolution in linear approximation.

Formula (15) is valid even in absence of the external electrical field when equilibrium polarization equals to zero $P_0 = 0$. In this case, equations (15) appears as

$$\omega^2_B = \frac{1}{m} \left( \frac{\hbar^2 k^4}{4m} - \Upsilon n_0 k^2 + \Upsilon_2 n_0 k^4 \right),$$

(19)
The waves of polarization can exist at the absence of external electric field $E_0 = 0$. In this case the equilibrium state is not polarized and dipole direction of particles is distributed accidentally.

If the contribution of equilibrium polarization in the BEC dispersion (15) is comparable to the contribution of the SRI in the third order by the interaction radius $\Upsilon_2$ transforms into

$$\omega^2_B = \frac{\hbar^2 k^4}{4m^2} - \frac{\Upsilon n_0 k^4}{m} + 4\pi P_0^2 k^2 \frac{n_0}{mn_0},$$

and

$$\omega^2_p = \frac{\hbar^2 k^4}{4m^2} - \frac{1}{2} \frac{\Upsilon n_0 k^4}{m} + 4\pi P_0^2 k^2 \frac{\Upsilon}{mn_0}(\sigma - 1).$$

(21)

Using Feshbach resonance [76, 77] we can tune the SRI potential that $\Upsilon = 0$ while $\Upsilon_2 \neq 0$. In this situation formula (15) turns into

$$\omega^2_B = \frac{\hbar^2 k^4}{4m^2} + \frac{\Upsilon n_0 k^4}{m}.$$  

(23)

and

$$\omega^2_p = \frac{\hbar^2 k^4}{4m^2} + 4\pi \sigma P_0^2 k^2 \frac{n_0}{mn_0}.$$  

(24)

In this paper we primarily focus on the influence of the BEC polarization on its dispersion characteristics. So, let us consider the case when the contribution into the dispersion of the SRI at the first order of the interaction radius, i.e. terms proportional to $\Upsilon$, is comparable to the contribution of polarization, and their total effect is much larger than the contribution of terms proportional to $\Upsilon_2$. We find that formula (15) turns into

$$\omega^2_B = \frac{\hbar^2 k^4}{4m^2} - \frac{\Upsilon n_0 k^4}{m} \left( \frac{\sigma - 1}{\sigma} \right).$$

(25)

and

$$\omega^2_p = \frac{\hbar^2 k^4}{4m^2} + 4\pi \sigma \frac{P_0^2 k^2}{mn_0} \frac{\Upsilon n_0 k^4}{m} - \frac{1}{2} \frac{\Upsilon n_0 k^4}{m} \left( \frac{\sigma + 4}{2\sigma} \right).$$

(26)

polarization evolution at constant concentration

The DBEC is the rare gas and we cannot use condition incompressibility, but we consider the case where concentration of particles is not change in time. We are interested in dynamics of wave of polarization only. For this aim we need to use equations (12) and (13) at additional condition $n(r, t) = n_0 = const$. This is means the polarization changes due to evolution of dipoles direction. Equation of polarization evolution (13) has no change in this approximation. Nonlinear terms in equation (13) disappear and the term describing the SRI simplifies, so we have

$$\partial_t R^\alpha\beta(r, t) + \partial^\gamma \left( R^\alpha\beta(r, t) v^\gamma(r, t) \right) + R^\alpha\gamma(r, t) v^\beta(r, t) \right) - P^\alpha(r, t) v^\beta(r, t) v^\gamma(r, t)$$

$$- \frac{\hbar^2}{4m^2} \partial_\beta \Delta P^\alpha(r, t) = n_0 \Upsilon \partial^\beta \left( P^\alpha(r, t) \right) + \frac{\sigma}{mn_0} P^\alpha(r, t) P^\gamma(r, t) \partial^\beta E^\gamma(r, t).$$

(27)

in the set with equations of field (10) and (11).

Only one wave solution exists in this approximation, it is the wave of polarization. Its dispersion is

$$\omega^2 = \frac{\hbar^2 k^4}{4m^2} + \left( \frac{4\pi \sigma P_0^2}{mn_0} - \frac{n_0 \Upsilon}{m} \right) k^2.$$  

(28)

Formula (28) reduces to

$$\omega^2 = \frac{4\pi \sigma P_0^2}{mn_0} k^2$$

than polarization dominates. For the particles with the EDM equal to 1 Debye at concentration $10^{12}$ cm$^{-3}$ and wave vector $k \approx 10^7$ cm$^{-1}$ the frequency $\omega$ reach value of order $10^6 - 10^7$ s$^{-1}$.

Dispersion of the 1D and 2D wave of polarization is described in Ref. [23].

GENERATION OF WAVES IN POLARIZED BEC

In this section we consider the process of wave generation in the BEC by means of the beam of neutral polarized particles. The interaction between the beam and the BEC has the dipole-dipole origin.

To get the dispersion solution we use the system of the QHD equations for each sort of particles (2), (3), (12), (13) and the equations of field (10) and (11). The equilibrium state of system is characterized by following values of the BEC parameters:

$$n_B = n_{B0} + \delta n_B, \quad v_B^a = 0 + v_B^a.$$
\[ P_B^\alpha = P_{B0}^\alpha + \delta P_B^\alpha, \quad R_B^{\alpha\beta} = 0 + \delta R_B^{\alpha\beta}, \]
and values of the beam parameters:
\[ n_b = n_{b0} + \delta n_b, \quad v_b^\alpha = U\delta z^\alpha + \delta v_b^\alpha, \]
\[ P_b^\alpha = P_{b0}^\alpha + \delta P_b^\alpha, \quad R_b^{\alpha\beta} = R_{b0}^{\alpha\beta} + \delta R_b^{\alpha\beta}. \]

The polarization \( P_{0b}^\alpha \) is proportional to the external electric field \( E_0^\alpha \). We consider the case then \( E_0^\alpha = [E_0\sin \varphi, 0, E_0\cos \varphi] \). In this case the tensor \( R_{0b}^{\alpha\beta} \) has only two nonzero elements: \( R_{0b}^{\alpha\beta} = R_{0b0}^\alpha \sin \varphi \) and \( R_{0b}^{\alpha\beta} = R_{0b0}^\alpha \cos \varphi \). For the process, under consideration the dispersion relation is:
\[ 1 + 4\pi k^2 \left( \frac{P_0^2}{\omega^2 - \frac{h^2 k^4}{4m^2} + \frac{\Upsilon n_0 k^2}{2m}} \times \right. \]
\[ \left( \frac{\Upsilon k^2/(2m^2)}{\omega^2 - \frac{h^2 k^4}{4m^2} + \frac{\Upsilon n_0 k^2}{2m} - \frac{\Upsilon n_0 k^2}{2m}} - \frac{\sigma}{m n_0} \right) \]
\[ + \frac{1}{(\omega - k_z U)^2 - \frac{h^2 k^4}{4m^2}} \times \]
\[ \left( \frac{2(\omega - k_z U) P_{0b} k_z (P_{0b} U - R_{0b})}{m_b n_{b0} (\omega - k_z U)^2 - \frac{h^2 k^4}{4m^2}} - \frac{\sigma_0 P_{0b}^2}{m_b n_{b0}} \right) \] = 0. \quad (31)

Using relation \( P_{0b} U = R_{0b} = 0 \), we can simplify the equation (31) and obtain
\[ 1 + \frac{\omega_2^2}{\omega^2 - \omega_1^2} \left( \frac{\Upsilon n_0 k^2/(2m^2)}{\omega^2 - \omega_1^2} - \frac{\sigma_0 \omega_{Db}^2}{(\omega - k_z U)^2 - \frac{h^2 k^4}{4m^2}} \right) \]
\[ - \frac{\sigma_0 \omega_{Db}^2}{(\omega - k_z U)^2 - \frac{h^2 k^4}{4m^2}} = 0. \quad (32)\]

In this formula the following designations are used
\[ \omega_1^2 = \frac{h^2 k^4}{4m^2} - \frac{\Upsilon n_0 k^2}{2m}, \quad (33) \]
\[ \omega_2^2 = \frac{h^2 k^4}{4m^2} - \frac{\Upsilon n_0 k^2}{m} + \frac{\Upsilon n_0 k^2}{m}, \quad (34) \]
\[ \omega_{Db}^2 = \frac{4\pi P_{0b}^2 k^2}{m_b n_{b0}}, \quad (35) \]

where \( i \) is the index of sorts of particles, the BEC \((i = B)\) or the beam \((i = b)\).

The equation (32) has two beam related solutions, in the absence of BEC medium:
\[ \omega = k_z U \pm \sqrt{\frac{h^2 k^4}{4m_b^2} + \sigma_0 \omega_{Db}^2}. \quad (36) \]

The sign of \( \omega \) will consider the possibilities of instability for the case of low-density beam \( \omega_{Db} \sim n_{b0} \rightarrow 0 \). In this case we can neglect the last term under the square root in (36). The resonance interaction of beam with the BEC realizing at
\[ k_z U \pm \frac{\hbar k^2}{2m_b} = \omega(k), \quad (37) \]
and could lead to instabilities. The quantity \( \omega(k) \) is the dispersion of BEC modes (15). The frequency in this case can be presented in the form
\[ \omega = k_z U \pm \frac{\hbar k^2}{2m_b} + \delta \omega. \quad (38) \]

Let us to consider two limit cases.

**Small frequency shift limit**

Under condition
\[ \delta \omega \ll \hbar k^2 / m \]
the frequency shift emerges as
\[ \delta \omega^2 = \pm \frac{2\sigma_0 m_b m^2 \omega_{Db}^2 (\omega^2 - \omega_1^2)^2 (\omega^2 - \omega_2^2)^2}{\omega_{Db}^2 // \Upsilon n_0 k^2 (\omega^2 - \omega_1^2)^2}, \quad (40) \]
and the frequency \( \omega \) determined with formula (15). The instabilities take place when \( \delta \omega^2 < 0 \). The sign of \( \delta \omega^2 \) depends on the sign of \( W \).

For the case resonance interaction of beam with the waves in BEC there are instabilities, for the first beam mode in (36) at \( W < 0 \) and for the second beam mode in formula (36) at \( W > 0 \). For the polarization mode \( W \) is positive. It means that interaction of polarization mode with the second beam related mode results in the instability. For the Bogoliubov mode the sign of \( W \) depends on \( \sigma \).

We can consider the following cases:
(i) the contribution of equilibrium polarization to \( \omega(k) \) is dominant; then \( W > 0 \);
(ii) main contribution in \( \omega(k) \) gives term proportional to \( \Upsilon, \) and, equilibrium polarization and SRI in the TOIR approximation are comparable to each other. In this case, there is a value of \( \sigma \), designated as \( \sigma_0 \), which is
\[ \sigma_0 = 1 + \frac{\Upsilon n_0 k^4}{m} + \Upsilon n_0 k^2/(2m) + 2\Upsilon n_0 k^4/m - 4\omega_{Db}^2. \]

The sign of \( W \) varies at \( \sigma = \sigma_0 \). The dependence of sign of \( W \) on \( \sigma \) is presented in the Table 1.
large frequency shift limit

In opposite limit
\[ \delta \omega \gg \hbar k^2 / m \] 
we have
\[ \delta \omega = \xi \sqrt{\frac{\sigma m^2 \omega^2_{pe} (\omega^2 - \omega_1^2)^2 (\omega^2 - \omega_2^2)^2}{\omega^2 \nu^2 + \nu^4 | W |}}, \] 
where \( \xi \) equal to \( \xi_1 = \sqrt{\nu} \) for \( W > 0 \) or \( \xi_1 = \sqrt{\nu} \) for \( W < 0 \). Explicit form of quantities \( \xi_1 \) and \( \xi_1 \) are
\[ \xi_1 = [-1, \frac{1 + \sqrt{3}}{2}, \frac{1 - \sqrt{3}}{2}], \] 
and
\[ \xi_1 = [1, \frac{1 + \sqrt{3}}{2}, \frac{1 - \sqrt{3}}{2}]. \]

The consideration concerning the sign of \( W \), reflected in the table 1, are valid also for the limit condition \( \frac{\omega^2}{\omega^2_{pe}} \).

From the formulas (40) and (43) we can see that neutral particles beam leads to instabilities for both the Bogoliubov and polarization waves.

CONCLUSION

We developed the self-consistent method for description of the DBEC dynamics. This method accounts effect of polarization on changes of the particle concentration and velocity field, which are determined in general by the continuity equation and Euler equation. We derived the evolution equations of the polarization and the polarization current. The derived equations contain information about the influence of the interactions on the polarization evolution. We studied the effect of polarization on the BEC dynamics and influence of the SRI on the polarization evolution. An expression of the SRI contribution in the equation of polarization current evolution via concentration, polarization and the SRI potential \( \Upsilon = \Upsilon(U_{ij}) \) was derived. With the assumption that the state of polarized Bose particles in the form of BEC can be described with some single-particle wave function. Changes in polarization due to SRI are shown to be determined at the first order of the interaction radius by the same interaction constant that occurs in Euler’s equation and Gross-Pitaevskii equation.

The derivation of the GP equation for polarized particles from the QHD equations was obtained. The conditions of validity of the GP equation was presented. Comparison of evolution equation of the electrically polarized BEC and the magnetized BEC was discussed and differences were described.

Physical meaning of the self-consistent approximation was described. Suitability of the self-consistent approximation for electrically polarized BEC was shown. Distinctions of developed self-consistent approximation for dipole-dipole interaction from the scattering process are discussed.

Correct form of the dipole-dipole interaction Hamiltonian is discussed. The arguments for choosing of correct form of Hamiltonian for electric and magnetic dipoles are presented.

The dispersion of the CE in the polarized BEC was analyzed. We shown that polarization evolution in the BEC causes a novel type of waves. The effect of polarization on the dispersion of the Bogoliubov mode and the dispersion of a new wave mode were studied.

We shown the possibility of wave generation in the polarized BEC by means of the monoenergetic beam of neutral polarized particles.

APPENDIX A

Correct form of Hamiltonian for dipole-dipole interaction

At derivation of the QHD equations we need to write explicit form of the Hamiltonian of dipole-dipole interaction. In some works the dynamics of the magnetic dipole moment and of the EDM [32] [79] are analyzed in similar ways. Usual expression of the Hamiltonian for the dipole-dipole interaction for the electric and the magnetic dipoles equals
\[ H_{dd} = \frac{\delta^{a \beta} - 3 \sigma^a \sigma^\beta / r^2}{r^3} d_1^a d_2^\beta. \] 
Generally speaking the Hamiltonian for the spin-spin and the electric dipole-dipole interaction are different, and they also differ from (44). All Hamiltonians contain a term proportional to the Dirac delta function. These terms have different numerical coefficient for two kind of BECs. In this paper for interaction of electric dipoles we used following Hamiltonian
\[ H_{dd} = -\bar{\sigma}^a \bar{\sigma}^\beta \frac{1}{r} d_1^a d_2^\beta, \] 
which follows from potential of the electric field caused by electric dipole [84]. There is well-known identity
\[ -\bar{\sigma}^a \bar{\sigma}^\beta \frac{1}{r} = \frac{\delta^{a \beta} - 3 \sigma^a \sigma^\beta / r^2}{r^3} + \frac{4 \pi}{3} \delta^{a \beta} \delta(r), \]
so we can see the difference between usually using Hamiltonian and one’s used in this paper. The corrections of our selection followed from the fact that the equations obtained in the paper coincide to the Maxwell equations. Derivation of the QHD equations in the self-consistent approximation leads to field equation (10) and (11), we rewrite them here
\[
\nabla E(r, t) = -4\pi \nabla P(r, t),
\]
and
\[
\nabla \times E(r, t) = 0,
\]
but if we used Hamiltonian (44) we would obtain
\[
\nabla E(r, t) = \frac{8\pi}{3} \nabla P(r, t),
\]
instead of (46). Using formula (44) leads to non-fulfilment of equation (47).

It has been shown by Breit [81] that the Hamiltonian for the spin-spin interaction contains a term proportional to the Dirac \(\delta\)-function \(\delta(r_1 - r_2)d_1^\alpha d_2^\beta\) along with (44). The coefficient at the \(\delta\) function was refined later [82], so it was shown that the Hamiltonian is in according with the Maxwell equations (46), (47). The conclusive expression for the spin-spin interaction Hamiltonian appears as
\[
H_{\mu\mu} = -\left(4\pi \delta^{\alpha\beta} \delta(r_{12}) + \nabla_1 \nabla_1^{\beta}(1/r_{12})\right)\mu_1^\alpha \mu_2^\beta, \tag{49}
\]
or
\[
H_{\mu\mu} = \left(\frac{\delta^{\alpha\beta} - 3r^\alpha r^\beta/r^2 - \frac{8\pi}{3} \delta^{\alpha\beta}(r)}{r^3}\right)\mu_1^\alpha \mu_2^\beta. \tag{50}
\]

Thus, the differences in the dipole-dipole interactions of the electric dipoles, and the magnetic dipoles, have to be taken into account at development of theoretical field apparatus.

**Method of equations derivation**

The Schrödinger equation defines wave function in the 3N-dimensional configuration space while physical processes in systems that involve large number of particles occur in the three-dimensional physical space [83]. This is why a problem evolves obtaining of the quantum-mechanical description of a system of particles in terms of material fields, e.g. the particle concentration, the momentum density, the energy density and other fields of various tensor dimension that are defined in the three-dimensional space.

The first step in the many-particle QHD equation derivation is the definition of particles concentration. We define the quantum particles concentration as the quantum average of the classic microscopic concentration in the coordinate representation on many-particle wave function \(\psi(R, t)\). Thus, the particles concentration explicit form to be
\[
n(r, t) = \int dR \sum_i \delta(r - r_i)\psi^*(R, t)\psi(R, t),
\]
where \(dR = \prod_{i=1}^N dr_i\). Differentiating this function over time and using the Schrödinger equation with the Hamiltonian (2), we derive the continuity equation (3). In the continuity equation the new quantity appears, it is the particle current or momentum density. We differentiate this quantity over time and use the Schrödinger equation with the Hamiltonian (2). In the results we obtain the momentum balance equation. In this way we can find an evolution equation for any additive physical quantity, but we must know its definition. The definition of new physical quantities appears at derivation of the QHD equation. Several quantities appears at derivation of the momentum balance equation (4), one of them is the polarization presented below (51). Knowledge of the polarization definition gives possibility to derive the equation of polarization evolution (13) and so on.

In equations (4), (10) polarization occurs in the form of
\[
P^\alpha(r, t) = \int dR \sum_i \delta(r - r_i)\psi^*(R, t)\partial_\alpha \psi(R, t), \tag{51}
\]
where \(r_i\) is the coordinate of i-th particle, \(dR = \prod_{i=1}^N dr_i\).

In the right-hand side of equation (13), where is the force-like field \(F^{\alpha\beta}(r, t)\) which gives rise to evolution of the polarization current \(R^{\alpha\beta}(r, t)\). In general case, for \(F^{\alpha\beta}(r, t)\), we can write:
\[
F^{\alpha\beta}(r, t) = -\frac{1}{m} \partial_\gamma \Sigma^{\alpha\beta\gamma}(r, t) + \frac{1}{m} D_{\alpha\beta}(r, t)\partial^\beta E^\gamma(r, t). \tag{52}
\]

The tensor
\[
D^{\alpha\beta}(r, t) = \int dR \sum_i \delta(r - r_i)d_i^\alpha d_i^\beta \psi^*(R, t)\psi(R, t), \tag{53}
\]
occurs in the term that presents the dipole-dipole interaction and an interaction of the dipole with an external electrical field. This value can be approximately presented as
\[
D^{\alpha\beta}(r, t) = \sigma \frac{P^{\alpha}(r, t)P^\beta(r, t)}{n(r, t)}, \tag{54}
\]
that based on the reasons of dimensions.

The SRI causes the tensor \(\Sigma^{\alpha\beta\gamma}(r, t)\) to occur in the equation (13). Taken at the first order by the interaction radius it has the form
\[
\Sigma^{\alpha\beta\gamma}(r, t) = -\frac{1}{2} \int dR \sum_{i,j} \delta(r - R_{ij})
\times \frac{r_i^\beta r_j^\gamma}{r_{ij}} \partial t_{ij} \psi^*(R, t)\partial_\alpha \psi(R, t), \tag{55}
\]
Tensor $\Sigma^{\alpha\beta\gamma}(r, t)$ describes the influence of the SRI on evolution of polarization. Formula (55) describes the SRI.

If we apply the procedure described in Ref. [14] to calculate the quantum stress tensor, and neglect the contribution of the thermal effects, when $\Sigma^{\alpha\beta\gamma}(r, t)$, for the BEC, takes the following form
\[ \Sigma_{BEC}^{\alpha\beta\gamma}(r, t) = -\frac{1}{2} \delta^{\beta\gamma} n(r, t) P^\alpha(r, t). \] (56)

Formula (56) is obtained for particles located in state with the lowest energy, which can be described by one particle wave function. However, this state may be the product of the lowest energy, which can be described by one particle wave function $\theta(r, t)$, depending on $\Sigma$ at the first order by the interaction radius $|\Sigma|_{14}$.

### APPENDIX B: DERIVATION OF THE NON-LINEAR SCHRODINGER EQUATION

A derivation of GP equation was performed for non-polarized particles from the QHD equations [14], here we present one for particles having electric dipole moment. The NLSE comes from the continuity equation [5] and the Cauchy integral of the momentum balance equation [3]. The Cauchy integral exists if the velocity field can be expressed as
\[ v^\alpha(r, t) = \frac{1}{m} \partial^\alpha \theta(r, t) \] (57)
where $\theta(r, t)$ is the velocity field potential.

Starting from the QHD equations we can derive an equation for evolution of a model function defined in terms of hydrodynamic variables. Thus, the macroscopic single-particle wave function may be defined as
\[ \Phi(r, t) = \sqrt{n(r, t)} \exp\left(\frac{i}{\hbar} m \theta(r, t)\right). \] (58)

If we differentiate this function with respect to time and apply the QHD equations when we obtain
\[
\hbar \partial_t \Phi(r, t) = \left( -\frac{\hbar^2}{2m} \nabla^2 + \Gamma(r, t) + g |\Phi(r, t)|^2 \right.
\]
\[
- \int_{r_0}^{r} dr' \frac{1}{n(r', t)} P^\beta(r', t) \times
\]
\[
\nabla' \int dr'' G^{\beta\gamma}(r', r'') P^\gamma(r'', t) \right) \Phi(r, t),
\] (59)

here we use designation $\Gamma(r, t)$ described by formula
\[ \Gamma(r, t) = \int_{r_0}^{r} dr' \frac{\nabla P(r', t)}{n(r', t)}. \] (60)

Quantity $\Gamma(r, t)$ can be considered as chemical potential $\mu(r, t)$ and we use this below. The equation (59) has the form of a NLSE. A NLSE with the DDI was obtained in Ref. [23], where the sign "-" was lost in front of integral in formulas (A6) and (A8).

The NLSE (59) describes collective properties of the many particles system. This follows from the derivation of the NLSE and from the definition of the many-particle wave function $\Phi(r, t)$ (58). Function $\Gamma(r, t)$ (60) is the contribution of the kinetic pressure and does not contain any interaction.

If we introduce a function $d^\alpha(r, t)$ as $d^\alpha(r, t) = P^\alpha(r, t)/n(r, t)$ and if also we suppose that $d^\alpha(r, t)$ is a constant $d^\alpha$, when we obtain
\[
\hbar \partial_t \Phi(r, t) = \left( -\frac{\hbar^2}{2m} \nabla^2 + \mu(r, t) + g |\Phi(r, t)|^2 \right.
\]
\[
- d^\beta d^\gamma \int dr' G^{\beta\gamma}(r', r') |\Phi(r', t)|^2 \right) \Phi(r, t) \] (61)

from equation (59), we have used that $n(r, t) = |\Phi(r, t)|^2$.

At the temperature $T$ equal to zero in external electric field $E_0$, all dipoles directed parallel to external field. If $E_0 = E_0 e_z$, we have $d = d_0 e_z$. In this case in equation (61) only one component of Green function $G^{zz}$ is nonzero, it is $G^{zz} = (1 - 3z^2/r^2)/r^3 + 4\pi/3 \delta(r)$, we obtain the GP equation [11] presented in introduction at the absence of $\delta$ function from (61) and using relation $z^2/r^2 = \cos^2 \theta$.

The GP equation was obtained for the case of system of parallel dipoles. We suppose it can be approximately used for analyze of dipoles whose direction slowly changed in space.

### APPENDIX C

We consider the QHD equations for the spinning particles to present it’s difference from the QHD equations for particles having electric dipole moment.

Contribution of magnetization $M$ appears in the Euler equation [4] instead of polarization $P$. The method of QHD also allows to obtain equation of the magnetization evolution
\[
\partial_t M^\alpha(r, t) + \nabla^\beta J_M^{\alpha\beta}(r, t) = \frac{\gamma}{\hbar} e^\alpha e^\beta M^\beta(r, t) B^\alpha(r, t),
\] (62)

where $J_M^{\alpha\beta}$ arises. Vanishing by thermal motion we have $J_M^{\alpha\beta} = M^\alpha v^\beta$.

For obtaining of the QHD equations we started from the many-particle Schrodinger equation
\[
\hbar \partial_t \psi(R, t) = \left( \sum_i \left( \frac{\vec{p}_i^2}{2m_i} - \gamma_i s_i B_i^{\alpha}(ext) \right) \right)
\]
\[ + \frac{1}{2} \sum_{i,j \neq i} \left( U_{ij} - \gamma_i \gamma_j \gamma_{ij} \gamma_{ij} \right) \left( R, t \right) \psi \right) \] (R, t). \quad (63)

where we include the short-range and spin-spin interactions, and action of an external magnetic field on spin. In the Schrodinger equation (63) we use following designations: \( \gamma_i \) is the gyromagnetic ratio, \( \mathbf{p}_i = -i \hbar \nabla_i \) is the operator of momentum, \( U_{ij} \) presents the short-range interaction, the Green function of spin-spin interaction has form

\[ G_{ij}^{\alpha \beta} = 4\pi \delta_{\alpha \beta} \delta(\mathbf{r}_{12}) + \nabla_1 \nabla_2 (1/|\mathbf{r}_{12}|). \]

For spin matrices \( \hat{s}_i^\alpha \) the commutation relations are

\[ [\hat{s}_i^\alpha, \hat{s}_j^\beta] = i\delta_{ij}\varepsilon^{\alpha \beta \gamma} \hat{s}_i^\gamma. \]

Therefore we consider Bose particles we present here the explicit form of the spin matrices \( \hat{s}_i^\alpha \) for particles with spin equal to 1:

\[ \hat{s}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{s}_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{s}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]

Derivation of the QHD equation from the Schrodinger equation we start as usual from definition of the concentration of particles in vicinity of a point \( \mathbf{r} \) of the physical space:

\[ n(r, t) = \sum_s \int dR \sum_i \delta(\mathbf{r} - \mathbf{r}_i) \psi_i^+(R, t) \psi_i(R, t), \]

where \( dR = \prod_{i=1}^N d\mathbf{r}_i \). We obtain equations analogous to (4) and (5), where magnetization \( \mathbf{M}(\mathbf{r}, t) \) appears instead of polarization \( \mathbf{P}(\mathbf{r}, t) \).

The magnetization arise in the form

\[ M^\alpha(r, t) = \sum_s \int dR \sum_i \delta(\mathbf{r} - \mathbf{r}_i) \times \gamma_i \psi_i^+(R, t) \times \hat{s}_i^\alpha \psi_i(R, t) \right)_s. \quad (64) \]

Differentiating magnetization (64) with respect to time and using Schrodinger equation (63) we came to equation (62).

More details of obtaining of the QHD equations for spinning particles are presented in Refs. [23], [73].

We can perform derivation of a NLSE for spinning particles, in the result we obtain an equation analogous to (59). In the case parallel spins we have following equation, where the difference between Green functions of the spin-spin and the EDM interactions is accounted

\[ +\mu_0^2 \int d\mathbf{r} \frac{1 - 3 \cos^2 \theta'}{3 - |\mathbf{r}|^2} |\Phi(\mathbf{r}', t)|^2 - \frac{8\pi}{3} \mu_0^2 |\Phi(\mathbf{r}, t)|^2 \Phi(\mathbf{r}, t). \quad (65) \]

Therefore, features of the spin-spin interaction give the additional term in the GP equation for spinning particles, which is the last term in equation (65). This term caused by the long-range interaction, but it has form analogous to the SRI.

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