Fuzzy Topological Dimensions And Its Applications

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Abstract: 

The purpose of this paper is to present a formal definition of the computational fuzzy topology is shown which is based on the fuzzy $\omega$-interior operator and fuzzy $\omega$-closure operators. In spatial object modeling the fuzzy $\omega$-interior, fuzzy $\omega$-exterior, fuzzy $\omega$-closure and fuzzy $\omega$-boundary are computed based on computational fuzzy topology and we applied this studied and we determined the fuzzy $\omega$-interior, fuzzy $\omega$-exterior, fuzzy $\omega$-closure and fuzzy $\omega$-boundary of flood affected areas in Iraq and we obtained several properties.

Keywords: 3-Dimension fuzzy region, fuzzy $\omega$-interior, fuzzy $\omega$-exterior, fuzzy $\omega$-boundary

I. Introduction

The concept, which we will be considered in this paper, is the so called “fuzzy sets” which is totally different from the classical concept which is called “a crisp set”. The recent concept is introduced by Zadeh in 1965 [8], in which he defines fuzzy sets as a class of objects with a continuum of grades of membership and such a set is characterized by a membership function that assigns to each object a grade of membership ranging between zero and one, In (1968) Chang [2] introduced the definition of fuzzy topological spaces and extended in a straightforward manner some concepts of crisp topological spaces to fuzzy topological spaces. In geographic information system application fuzzy spatial objects have become very important but the classical set theory which was introduced by Gaal, 1964 [3] and Apostol 1974 [1] which is based on a crisp boundary, may not be fully suitable for handling problems of uncertainty, so the fuzzy sets provide a useful tool to describe uncertainty of single object in GIS. In this paper we a formal definition of the computational fuzzy topology is shown which is based on the fuzzy $\omega$-interior operator and fuzzy $\omega$-closure operators. In spatial object modeling the fuzzy $\omega$-interior, fuzzy $\omega$-exterior, fuzzy $\omega$-closure and fuzzy $\omega$-boundary are computed based on computational fuzzy topology. In this paper we presented the fuzzy topological relation for 3-Dimension fuzzy region also we presented the fuzzy definition in 3-Dimension fuzzy region such as (Support, $\alpha$-cut, $\omega$-interior and $\omega$-boundary). And we presented fuzzy topology induced by the $\omega$-interior and $\omega$-closure operators and we proved some propositions. In the last section we applied this studied and we determined the fuzzy $\omega$-interior, fuzzy $\omega$-exterior, fuzzy $\omega$-closure and fuzzy $\omega$-boundary of flood affected areas in Iraq which was studied for the first time, up to our knowledge.

II. Preliminaries

Definition 2.1 [8]:

Let $X$ be a non empty set, a fuzzy set $\tilde{A}$ in $X$ is characterized by a function $\mu_{\tilde{A}} : X \rightarrow [0,1]$, where

$I = [0,1]$ which is written as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X, 0 \leq \mu_{\tilde{A}}(x) \leq 1\}$, the collection of all fuzzy sets in $X$ will be denoted by $I^X$, that is $I^X = \{\tilde{A} : \tilde{A}$ is a fuzzy set in $X\}$ where $\mu_{\tilde{A}}$ is called the membership function.

Proposition 2.2 [8]:
Let $A$ and $B$ be two fuzzy sets in $X$ with membership functions $\mu_A$ and $\mu_B$ respectively, then for all $x \in X$: -

1. $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$.
2. $A = B$ if and only if $\mu_A(x) = \mu_B(x)$.
3. $C = A \cap B$ if and only if $\mu_C(x) = \min\{\mu_A(x), \mu_B(x)\}$.
4. $D = A \cup B$ if and only if $\mu_D(x) = \max\{\mu_A(x), \mu_B(x)\}$.

**Definition 2.3 [4]:**

The "support of a fuzzy set $A$, Supp ($A$), is the crisp set of all $x \in X$, such that $\mu_A(x) > 0$."

**Definition 2.4[2]:**

"A fuzzy topology is a family $T$ of fuzzy subsets in $X$, satisfying the following conditions:"

(a) $\emptyset, 1_X \in T$.
(b) If $A, B \in T$, then $A \cap B \in T$.
(c) If $A_i \in T$, $\forall i \in J$, where $J$ is any index set, then $\bigcup_{i \in J} A_i \in T$.

$T$ is called fuzzy topology for $X$, and the pair $(X,T)$ is a fuzzy topological space. Every member of $T$ is called open fuzzy set ($T$-open fuzzy set). A fuzzy set $C$ in $1_X$ is called closed fuzzy set ($T$-closed fuzzy set) if and only if its complement $C^c$ is $T$-open fuzzy set.

**Definition 2.5 [5]:**

If $B \in (X,T)$, the complement of $B$ referred to $1_X$ denoted by $B^c$, is defined by $B^c = 1_X - B$.

**Definitions 2.6 [8,6]:**

Let $B, C$ be a fuzzy set in a fuzzy topological space $(X,T)$ then:

- A fuzzy point $x_r$ is said to be quasi coincident with the fuzzy set $B$ if there exist $x \in X$ such that $\mu_{x_r}(x) + \mu_B(x) > \mu_A(x)$ and denote by $x_r q B$, $\forall x \in X$ then $x_r$ is not quasi coincident with a fuzzy set $B$ and is denoted by $x_r q B$.

- A fuzzy set $B$ is said to be quasi coincident (overlap) with a fuzzy set $C$ if there exist $x \in X$ such that $\mu_B(x) + \mu_C(x) > \mu_A(x)$ and denoted by $B q C$, if $\mu_B(x) + \mu_C(x) \leq \mu_A(x)$ then $B$ is not quasi coincident with a fuzzy set $C$ and is denoted by $B q C$.

**Definition 2.8 [2]:**

A fuzzy set $B$ in a fuzzy topological space $(X,T)$ is said to be a fuzzy neighborhood of a fuzzy point $x_r$ in $X$ if there is a fuzzy open set $G$ in $X$ such that $\mu_{x_r}(x) \leq \mu_G(x) \leq \mu_B(x)$, $\forall x \in X$.

**Definition 2.9 [7]:**

A fuzzy set $A$ in a fuzzy topological space $(X,T)$ is called a fuzzy uncountable if and only if supp($A$) is an uncountable subset of $X$.

**Definition 2.10:**
A fuzzy point $x_r$ of a fuzzy topological space $(X, \mathcal{F})$ is called a fuzzy condensation point of $\tilde{A}$ if $\tilde{B} \cap \tilde{A}$ is fuzzy uncountable for each fuzzy open set $\tilde{B}$ containing $x_r$. And the set of all fuzzy condensation point of $\tilde{A}$ is denoted by $\text{Cond}(\tilde{A})$

**Definition 2.11:**
A fuzzy subset $\tilde{A}$ in a fuzzy topological space $(X, \mathcal{F})$ is called a fuzzy $\omega$-closed set if it contains all its fuzzy condensation point. The complement fuzzy $\omega$-closed sets are called fuzzy $\omega$-open sets.

**Theorem 2.12:**
A fuzzy subset $\tilde{G}$ of a fuzzy topological space $(X, \mathcal{F})$ is fuzzy $\omega$-open set if and only if for each $x_r \in \tilde{G}$ there exist a fuzzy open set $\tilde{U}$ such that $x_r \in \tilde{U}$ and $\tilde{U} \setminus \tilde{G}$ is countable

**Proof:**
$\tilde{G}$ is fuzzy $\omega$-open set if and only if $1_X - \tilde{G}$ is fuzzy $\omega$-closed set

And $1_X - \tilde{G}$ is fuzzy $\omega$-closed set if and only if $\text{Cond}(1_X - \tilde{G}) \subseteq 1_X - \tilde{G}$

And $\text{Cond}(1_X - \tilde{G}) \subseteq 1_X - \tilde{G}$ if and only if each $x_r \in \tilde{G}$, $x_r \notin \text{Cond}(1_X - \tilde{G})$

Thus $x_r \notin \text{Cond}(1_X - \tilde{G})$ there exist a fuzzy open set $\tilde{U}$ such that $x_r \in \tilde{U}$ and $\tilde{U} \cap (1_X - \tilde{G}) = \tilde{U} - \tilde{G}$ is countable

**Theorem 2.13:**
A fuzzy subset $\tilde{G}$ of a fuzzy topological space $(X, \mathcal{F})$ is $\omega$-open set if and only if for each $x_r \in \tilde{G}$ there exist an fuzzy open set $\tilde{U}$ containing $x_r$ and countable fuzzy subset $\tilde{C}$ of $1_X$ such that $\tilde{U} \setminus \tilde{C} \subseteq \tilde{G}$

**Proof:**
$(\Rightarrow)$ suppose $\tilde{G}$ is fuzzy $\omega$-open set and let $x_r \in \tilde{G}$

Then there exist a fuzzy open set $\tilde{U}$ and $x_r \in \tilde{U}$ and $\tilde{U} - \tilde{G}$ is countable

Set $\tilde{C} = \tilde{U} - \tilde{G}$, then $\tilde{C}$ is countable and $x_r \in \tilde{U} - \tilde{C} = \tilde{U} - (\tilde{U} - \tilde{G}) \subseteq \tilde{G}$

$(\Leftarrow)$ let $x_r \in \tilde{G}$ then by assumption there exist fuzzy open set $\tilde{U}$ containing $x_r$ and countable fuzzy subset $\tilde{C}$ of $1_X$ such that $\tilde{U} \setminus \tilde{C} \subseteq \tilde{G}$ since $\tilde{U} - \tilde{G} \subseteq \tilde{C}$ then $\tilde{U} - \tilde{G}$ is countable, hence $\tilde{G}$ is fuzzy $\omega$-open set

**Proposition 2.14:**
Every fuzzy open set is fuzzy $\omega$-open set

**Proof:**

Let $\tilde{G}$ be fuzzy open set and $x_r \in \tilde{G}$, Set $\tilde{U} = \tilde{G}$, $\tilde{C} = \emptyset$, then $\tilde{U}$ is fuzzy open set and $\tilde{C}$ countable set, Such that $x_r \in \tilde{U} - \tilde{C} \subseteq \tilde{G}$, thus $\tilde{G}$ is fuzzy $\omega$-open set

### III. Topological Relation For 3-Dimension fuzzy Region

**Definition 3.1:**
3-Dimension fuzzy region $\tilde{A}$ is characterized by its membership functions as $\tilde{A} = \{(x,y,z) \mid \mu_{\tilde{A}}(x,y,z)\}$ where $x,y,z \in \mathbb{R}^3$ and $\mu_{\tilde{A}} : \mathbb{R}^3 \to [0,1]$, here $\mu_{\tilde{A}}(x,y,z)$ is the membership functions of a set $\tilde{A}$ from the points in the continues 3-Dimension space to the real numbers between 0 and 1.
Definition 3.2:
Support of 3-Dimension fuzzy region \( \tilde{A} \), denoted by 3-Supp(\( \tilde{A} \)), can be defined as
\[
3\text{-}\text{Supp}(\tilde{A}) = \{(x, y, z) : \mu_{\tilde{A}}(x, y, z) > 0\}.
\]

Definition 3.3:
\( \alpha \)-cut of 3-Dimension fuzzy region \( \tilde{A} \), denoted by \( \tilde{A}_{\alpha} \) will be defined as
\[
\tilde{A}_{\alpha} = \{(x, y, z) : \mu_{\tilde{A}}(x, y, z) \geq \alpha\},
\] where \( \alpha \in (0, 1] \).

Definition 3.4:
For 3-Dimension fuzzy region \( \tilde{A} \), the membership functions of its \( \omega \)-interior is defined as
\[
\mu_{\omega-\text{Int}(\tilde{A})}(x, y, z) = \begin{cases} 
\mu_{\tilde{A}}(x, y, z) & \text{for } (x, y, z) \in \text{Supp}(\omega-\text{Int}(\tilde{A})) \\
0 & \text{otherwise}
\end{cases}
\]

Definition 3.5:
For 3-Dimension fuzzy region \( \tilde{A} \), the membership functions of its \( \omega \)-exterior is defined as
\[
\mu_{\omega-\text{Ext}(\tilde{A})}(x, y, z) = \begin{cases} 
\mu_{\tilde{A}}(x, y, z) & \text{for } (x, y, z) \in \text{Supp}(\omega-\text{Ext}(\tilde{A})) \\
0 & \text{otherwise}
\end{cases}
\]

Definition 3.6:
For 3-Dimension fuzzy region \( \tilde{A} \), the membership functions of its \( \omega \)-boundary is defined as
\[
\mu_{\omega-b(\tilde{A})}(x, y, z) = 2 \min \{\mu_{\tilde{A}}(x, y, z), 1-\mu_{\tilde{A}}(x, y, z)\}
\]

Remark 3.7:
In 3-Dimension fuzzy space, objects can be considered as having different dimension and therefore, can be divided into fuzzy points (\( \tilde{P} \)), fuzzy lines (\( \tilde{L} \)), fuzzy surfaces (\( \tilde{S} \)), and fuzzy bodies (\( \tilde{B} \)). Accordingly, different kinds of relations may be defined between two spatial objects. Such relations are related to fuzzy topological dimension of objects and can be denoted as \( R(\tilde{P}, \tilde{L}), R(\tilde{L}, \tilde{S}), \ldots \).

The fuzzy intersection between two objects depends on the fuzzy space rank, fuzzy topological dimension (0D, 1D, 2D, and 3D), and shape of the fuzzy boundary (continuous or not). For example, two regions cannot have intersection without considering their fuzzy boundary.

In Fig. 1, fuzzy topological elements can be defined as follows:

| 0DC = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\} | 0DA = \{\tilde{x}_1', \tilde{x}_2', \tilde{x}_3', \tilde{x}_4'\} |
| 1DC = \{L_1, L_2, L_3, L_4, L_5, L_6\} | 1DA = \{L_1', L_2', L_3', L_4', L_5', L_6'\} |
| 2DC = \{\tilde{S}_1, \tilde{S}_2, \tilde{S}_3, \tilde{S}_4\} | 2DA = \{\tilde{S}_1', \tilde{S}_2', \tilde{S}_3', \tilde{S}_4'\} |
| 3DC = \{\tilde{B}\} | 3DA = \{\tilde{B}'\} |
Remark 3.8:
Each elements of $\tilde{A}$ can be related to any of the elements of $\tilde{C}$. Therefore we could create the fuzzy nine-intersection matrix and extract the number of possible relations among the 512 predicted relations. This is shown on Table 1.

|      | 0D$\tilde{A}$ | 1D$\tilde{A}$ | 2D$\tilde{A}$ | 3D$\tilde{A}$ |
|------|--------------|--------------|--------------|--------------|
| 0D$\tilde{C}$ | 16           | 24           | 16           | 4            |
| 1D$\tilde{C}$ | 24           | 36           | 24           | 6            |
| 2D$\tilde{C}$ | 16           | 24           | 16           | 4            |
| 3D$\tilde{C}$ | 4            | 4            | 4            | 1            |

Table 1 Number of possible fuzzy topological relations between two simple 3-Dimension objects

Remark 3.9:
Most of the relations cannot be found in the real world. But among all 512 states of the fuzzy nine-intersection matrix only eight are real. This is shown on fig 2.
Fig. 2. Topological relation between two simple 3-Dimension Fuzzy bodies

IV. Fuzzy Topological Induced By The $\omega$-Interior and $\omega$-Closure Operators

**Definition 4.1:**
Let $\mathcal{A}$ be a fuzzy set in $[0,1]^I = I^1$, for any fixed $\alpha \in [0,1]$, define the fuzzy $\omega$-interior and fuzzy $\omega$-closure operator on $[0,1]^I = I^1$ as

$A \xrightarrow{\omega} A_\alpha \in I^1$ and $A \xrightarrow{\omega} A^\alpha \in I^1$ respectively.

Where the membership functions of a fuzzy sets $A_\alpha$ and $A^\alpha$ in $I^1 \times \mathbb{R}$ are defined by

$\mu_{A_\alpha}(x) = \mu_{A}(x)$ if $\mu_{A}(x) > \alpha$ and $\mu_{A_\alpha}(x) = 0$ if $\mu_{A}(x) \leq \alpha$ and

$\mu_{A^\alpha}(x) = \mu_{A}(x)$ if $\mu_{A}(x) \leq \alpha$ and $\mu_{A^\alpha}(x) = 1$ if $\mu_{A}(x) > \alpha$.

**Remark 4.2:**
The geometric interpretation of the definition of the $\omega$-closure operator is that it rises up all fuzzy membership value greater than $\alpha$ to one. The geometric interpretation of the definition of the $\omega$-interior operator is that it cuts all fuzzy membership value that are less than or equal to $\alpha$.

**Proposition 4.3:**
Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{A}_i (i \in \Lambda)$ be fuzzy sets of $I^1$, then the following hold for all $\alpha \in [0,1]$

1. $0^\alpha = 0 = 0_\alpha$ and $1^\alpha = 1 = 1_\alpha$
2. $\mu_{A_\alpha}(x) \leq \mu_{A}(x) \leq \mu_{A^\alpha}(x)$
3. $\mu_{A_\alpha}(x) \leq \mu_{B_\alpha}(x) \Rightarrow \mu_{A_\alpha}(x) \leq \mu_{B_\alpha}(x)$ and $\mu_{A_\alpha}(x) \leq \mu_{B_\alpha}(x)$
4. $\mu_{(A \cup \alpha)}(x) = \mu_{A}(x)$ and $\mu_{(A \cup \alpha)}(x) = \mu_{A}(x)$
5. $\mu_{(A \cap \alpha)}(x) = \mu_{A}(x)$ and $\mu_{(A \cap \alpha)}(x) = \mu_{A}(x)$
6. $\alpha \leq \beta \Rightarrow \mu_{A_\beta}(x) \leq \mu_{A}(x)$ and $\mu_{(A_\beta \cup \alpha)}(x) = \mu_{A_\beta}(x)$
7. $\alpha \leq \beta \Rightarrow \mu_{A_\alpha}(x) \leq \mu_{A}(x)$ and $\mu_{(A_\alpha \cup \beta)}(x) = \mu_{A_\beta}(x)$
8. if $\Lambda$ is finite then $(\max_{\alpha \in \Lambda} \mu_{A}(x))^\alpha = \max_{\alpha \in \Lambda} \mu_{A}(x)$ and $(\min_{\alpha \in \Lambda} \mu_{A}(x))_\alpha = \min_{\alpha \in \Lambda} \mu_{A}(x)$
9. $(\max_{\alpha \in \Lambda} \mu_{A}(x))_\alpha = \max_{\alpha \in \Lambda} \mu_{A}(x)$ and $(\min_{\alpha \in \Lambda} \mu_{A}(x))^\alpha = \min_{\alpha \in \Lambda} \mu_{A}(x)$
10. $\mu_{A_\alpha}(x) \leq \mu_{A}(x) \leq \mu_{A^\alpha}(x)$

**Proof:**
(1) And (2) are obvious

The geometric interpretation of \( \mu_{\tilde{A}_{\alpha}}(x) \leq \mu_{\tilde{A}}(x) \) is equal to the cutting of the tall of \( \tilde{A} \) and The geometric interpretation of \( \mu_{\tilde{A}}(x) \leq \mu_{\tilde{A}_{\alpha}}(x) \) is that \( \tilde{A} \) is exactly to \( \tilde{A}_{\alpha} \) except the raising part.

(3) For all \( x \in X \) \( \mu_{\tilde{A}}(x) \geq \alpha \Rightarrow \mu_{\tilde{A}_{\alpha}}(x) \geq \alpha \) so

\[
\{ x \in X : \mu_{\tilde{A}}(x) \geq \alpha \} \subseteq \{ x \in X : \mu_{\tilde{A}_{\alpha}}(x) \geq \alpha \}
\]

\( \mu_{\tilde{A}}(x) < \alpha \Rightarrow \mu_{\tilde{A}_{\alpha}}(x) = \min(\mu_{\tilde{A}}(x),1) \)

Combining these two, we have \( \mu_{\tilde{A}_{\alpha}}(x) \leq \mu_{\tilde{A}}(x) \) For all \( x \in X \)

Similarly we have \( \mu_{\tilde{A}_{\beta\alpha}}(x) \leq \mu_{\tilde{A}_{\beta}}(x) \)

(4) For all \( x \in X \) \( \mu_{\tilde{A}}(x) \geq \alpha \Rightarrow \mu_{\tilde{A}_{\alpha}}(x) = 1 \Rightarrow \mu_{\tilde{A}_{\alpha}}(x) \geq \alpha \)

\( \mu_{\tilde{A}}(x) < \alpha \Rightarrow \mu_{\tilde{A}_{\alpha}}(x) = \mu_{\tilde{A}}(x) \Rightarrow \mu_{\tilde{A}_{\alpha}}(x) = \mu_{\tilde{A}}(x) \) therefore \( \mu_{\tilde{A}_{\alpha}}(x) = \mu_{\tilde{A}}(x) \)

Similarly we have \( \mu_{\tilde{A}_{\beta\alpha}}(x) = \mu_{\tilde{A}_{\beta}}(x) \)

(5) \( \mu(\tilde{A}_{\alpha})^{\epsilon}(x) = 1 - \mu_{\tilde{A}_{\alpha}}(x) = 1 - \mu_{\tilde{A}}(x) \) if \( \mu_{\tilde{A}}(x) < \alpha \) and \( \mu(\tilde{A}_{\alpha})^{1-\alpha}(x) = 1 - \mu_{\tilde{A}}(x) \) if \( \mu_{\tilde{A}}(x) \geq \alpha \)

Similar to (6)

(7) Similar to (6)

(8) \( \mu_{\tilde{A}}(x) \geq \alpha \) for some \( i \in \Lambda \) \( \Rightarrow \max_{i \in \Lambda} \mu_{\tilde{A}_{\alpha}}(x) \geq \alpha \)

Hence \( \mu_{\tilde{A}_{\alpha}}(x) = 1 \) \( \Rightarrow (\max_{i \in \Lambda} \mu_{\tilde{A}_{\alpha}}(x))^\alpha = 1 \)  

Therefore \( (\max_{i \in \Lambda} \mu_{\tilde{A}_{\alpha}}(x))^\alpha = \max_{i \in \Lambda} \mu_{\tilde{A}_{\alpha}}(x) \)  

On the other hand, \( \mu_{\tilde{A}_{\alpha}}(x) < \alpha \) for all \( i \in \Lambda \) \( \Rightarrow \max_{i \in \Lambda} \mu_{\tilde{A}_{\alpha}}(x) < \alpha \)

Therefore \( \mu_{\tilde{A}_{\alpha}}(x) < \alpha \) for all \( i \in \Lambda \) \( \Rightarrow (\max_{i \in \Lambda} \mu_{\tilde{A}_{\alpha}}(x))^\alpha = \max_{i \in \Lambda} \mu_{\tilde{A}_{\alpha}}(x) \)  

Hence \( (\max_{i \in \Lambda} \mu_{\tilde{A}_{\alpha}}(x))^\alpha = \max_{i \in \Lambda} \mu_{\tilde{A}_{\alpha}}(x) \)

For the second statement

\( (\max_{i \in \Lambda} \mu_{\tilde{A}_{\alpha}}(x))^{\beta} = \min_{i \in \Lambda} \mu_{\tilde{A}_{\alpha}}^{\beta\alpha}(x) = \min_{i \in \Lambda} \mu_{\tilde{A}_{\alpha}}^{\beta}(x) \)

And \( ((\max_{i \in \Lambda} \mu_{\tilde{A}_{\alpha}}(x))^{\beta})^{\alpha} = ((\max_{i \in \Lambda} \mu_{\tilde{A}_{\alpha}}(x))^{\beta})^{1-\alpha} = \min_{i \in \Lambda} \mu_{\tilde{A}_{\alpha}}^{\beta}(x) = s \min_{i \in \Lambda} \mu_{\tilde{A}_{\alpha}}^{\beta}(x) \)
(min_{i \in A} μ_{\tilde{A}_i}(x))_{1-α} and if \( \tilde{A}_i^c \) and 1 - α are replaced by \( \tilde{A}_i \) and α respectively we get the definition result.

(9) For \( \min_{i \in A} μ_{\tilde{A}_i}(x) < α \Rightarrow μ_{\tilde{A}_i}(x) < α \) for some \( i \in A \Rightarrow μ_{\tilde{A}_i}(x) = (\min_{i \in A} μ_{\tilde{A}_i}(x))^α. \)

For \( \min_{i \in A} μ_{\tilde{A}_i}(x) ≥ α \Rightarrow μ_{\tilde{A}_i}(x) ≥ α \) for all \( i \in A \Rightarrow μ_{\tilde{A}_i}(x) = 1 \) for all \( i \in A \Rightarrow \min_{i \in A} μ_{\tilde{A}_i}(x) = 1. \)

On the other hand \( \min_{i \in A} μ_{\tilde{A}_i}(x) ≥ α \Rightarrow (\min_{i \in A} μ_{\tilde{A}_i}(x))^α = 1 \)
This proved \( (\min_{i \in A} μ_{\tilde{A}_i}(x))^α = \min_{i \in A} μ_{\tilde{A}_i}(x) \)

For \( \max_{i \in A} μ_{\tilde{A}_i}(x) > α \Rightarrow μ_{\tilde{A}_i}(x) > α \) for some \( i \in A \Rightarrow μ_{\tilde{A}_i}(x) = μ_{\tilde{A}_i}(x) > α \) for some \( i \in A \Rightarrow μ_{\tilde{A}_i}(x) = (\max_{i \in A} μ_{\tilde{A}_i}(x))^α. \)

For \( \max_{i \in A} μ_{\tilde{A}_i}(x) ≤ α \Rightarrow μ_{\tilde{A}_i}(x) ≤ α \) for all \( i \in A \Rightarrow μ_{\tilde{A}_i}(x) = 1 \) for all \( i \in A \Rightarrow \max_{i \in A} μ_{\tilde{A}_i}(x) = 0. \)
On the other hand \( \max_{i \in A} μ_{\tilde{A}_i}(x) ≤ α \Rightarrow (\max_{i \in A} μ_{\tilde{A}_i}(x))^α = 0 \)
This proved \( (\max_{i \in A} μ_{\tilde{A}_i}(x))^α = \max_{i \in A} μ_{\tilde{A}_i}(x) \)

(10) For all \( x \in X \) \( μ_{\tilde{A}_a}(x) ≤ μ_{\tilde{A}}(x) \) and \( μ_{\tilde{A}}(x) ≤ μ_{\tilde{A}^{1-α}}(x) \)

Definition 4.4 :

For \( 0 < α < 1 \) define \( \tilde{T}_a = \{ A^a : \tilde{A} ∈ I^a \} \) and \( \tilde{T}α = \{ A^a : \tilde{A} ∈ I^a \} \)

Definition 4.5 :

For \( 0 < α ≤ 1 \) define the fuzzy \( ω \)-boundary of a fuzzy set \( \tilde{A} \) in a fuzzy topological space \( (X, \tilde{T}_a) \) as \( μ_{\tilde{A}(\tilde{A})}(x) = \min \{ μ_{\tilde{A}^{1-α}}(x), μ_{\tilde{A}^{1-α}}(x) \} \)

Proposition 4.6 :

\( μ_{\tilde{A}(\tilde{A})}(x) = \min \{ μ_{\tilde{A}^{1-α}}(x), μ_{\tilde{A}^{1-α}}(x) \} \)

Proof :

By proposition 4.3 (5)

Proposition 4.7 :

For all \( x \in X \) \( μ_{\tilde{A}(\tilde{A})}(x) = 0 \) if and only if \( μ_{\tilde{A}}(x) = 1 \) or \( μ_{\tilde{A}}(x) = 0 \)

Proof :

For \( 0 < α ≤ 1 \) and \( x \in X \)

\( μ_{\tilde{A}(\tilde{A})}(x) = 0 \Leftrightarrow μ_{\tilde{A}^{1-α}}(x) = 0 \) or \( μ_{\tilde{A}^{1-α}}(x) = 0 \)

\( μ_{\tilde{A}}(x) = 0 \) or \( μ_{\tilde{A}^{1-α}}(x) = 0 \)

\( μ_{\tilde{A}}(x) = 0 \) or \( μ_{\tilde{A}}(x) = 1 \)

Proposition 4.8 :

If \( α < \frac{1}{2} \) then \( μ_{\tilde{A}(\tilde{A})}(x) < 1 - α \) for all \( x \in X \)

Proof :

For \( α < \frac{1}{2} \Rightarrow α < 1 - α \), since by definition \( μ_{\tilde{A}(\tilde{A})}(x) = \min \{ μ_{\tilde{A}^{1-α}}(x), μ_{\tilde{A}^{1-α}}(x) \} \). If \( μ_{\tilde{A}^{1-α}}(x) ≥ 1 - α \), then \( μ_{\tilde{A}^{1-α}}(x) ≥ 1 - α \) and \( 1 - μ_{\tilde{A}}(x) ≤ α < 1 - α \Rightarrow μ_{\tilde{A}^{c}}(x) < 1 - α \Rightarrow μ_{\tilde{A}^{c}}(x) < 1 - α \). If \( μ_{\tilde{A}^{c}}(x) ≥ 1 - α \Rightarrow μ_{\tilde{A}^{c}}(x) ≥ 1 - α \Rightarrow 1 - μ_{\tilde{A}}(x) ≥ 1 - α \Rightarrow μ_{\tilde{A}}(x) ≤ α < 1 - α \Rightarrow μ_{\tilde{A}^{1-α}}(x) < 1 - α \)
Remark 4.9:
In proposition above if $\alpha < 1 - \alpha$ then the fuzzy value of the $\omega$-boundary is less than one. But if $\alpha > 1 - \alpha$ then he fuzzy value of the $\omega$-boundary may be equal one for some $x \in X$, this means by definition of fuzzy $\omega$-boundary $\mu_{\omega}(x, a) = \min\{\mu_{\omega}(x), \mu_{\omega}(x')\}$ in case $\alpha > 1 - \alpha$, for some $x \in X$ $\mu_{\omega}(x)=1$ and $\mu_{\omega}(x')=1$ figure 3 is shown.

Example 4.10:
Define a fuzzy set $A : R^2 \rightarrow [0,1]$ by

$$\mu_A(x, y) = \begin{cases} 
1 & \text{if } x^2 + y^2 < 0.5 \\
\exp(-x^2 - y^2) & \text{if } 0.5 \leq x^2 + y^2 \leq 2 , \text{Then} \\
0 & \text{otherwise} 
\end{cases}$$

$$\mu_{\omega}(x, y) = \begin{cases} 
0 & \text{if } x^2 + y^2 < 0.5 \\
1 - \exp(-x^2 - y^2) & \text{if } 0.5 \leq x^2 + y^2 \leq 2 , \text{For } \alpha = 0.3 \text{ then} \\
1 & \text{otherwise} 
\end{cases}$$

**Fuzzy $\omega$-interior** $\mu_{A, \omega}(x, y) = \begin{cases} 
1 & \text{if } x^2 + y^2 < 0.5 \\
\exp(-x^2 - y^2) & \text{if } 0.5 \leq x^2 + y^2 \leq -\ln(0.3) \\
0 & \text{otherwise} 
\end{cases}$

**Fuzzy $\omega$-boundary** $\mu_{\partial(A, \omega)}(x, y) = \begin{cases} 
\exp(-x^2 - y^2) & \text{if } 0.5 \leq x^2 + y^2 \leq -\ln(0.5) \\
1 - \exp(-x^2 - y^2) & \text{if } -\ln(0.5) \leq x^2 + y^2 \leq -\ln(0.3) \\
0 & \text{otherwise} 
\end{cases}$

For $\alpha = 0.6$ then

**Fuzzy $\omega$-interior** $\mu_{A, \omega}(x, y) = \begin{cases} 
1 & \text{if } x^2 + y^2 < 0.5 \\
0 & \text{otherwise} 
\end{cases}$

**Fuzzy $\omega$-boundary** $\mu_{\partial(A, \omega)}(x, y) = \begin{cases} 
\exp(-x^2 - y^2) & \text{if } 0.5 \leq x^2 + y^2 \leq -\ln(0.5) \\
1 - \exp(-x^2 - y^2) & \text{if } -\ln(0.5) \leq x^2 + y^2 \leq -\ln(0.3) \\
0 & \text{otherwise} 
\end{cases}$

Remark 4.11:
If the value of $\alpha$ is small then the value of intersection of fuzzy $\omega$-interior and fuzzy $\omega$-boundary is large. But if the value of $\alpha$ is large then the value of intersection of fuzzy $\omega$-interior and fuzzy $\omega$-boundary is small. The figure 4 is shown.
V. Application to Computing the fuzzy Topological relations of spatial objects For Geographic Information System

Definitions 5.1:

1- The photo is viewed as a fuzzy space
2- Each area affected in this photo is viewed as a fuzzy set
3- The size of each area affected in this photo used to calculate the fuzzy value of the fuzzy sets
4- The fuzzy value of each area affected in this photo Calculated as follows
   \[
   \frac{\log(\text{Area of certain affected area})}{\log(\text{Total area of affected area})} \quad \text{if } \frac{\log(A)}{\log(7)} > 0
   
   0 \quad \text{otherwise}
   \]
   Which is well-defined function from the interval \([1, \infty]\) to the interval \([0,1]\)
5- The fuzzy \(\omega\)-Interior and fuzzy \(\omega\)-Closure and \(\omega\)-Exterior and fuzzy \(\omega\)-Boundary Calculated for \(\alpha\) equal 0.3 , 0.45 and 0.6 respectively

Remark 5.2:
For the $\omega$-Interior, if the value is closer to one this means that the effect is high in relation to the overall affected area. If the value is closer to zero this means that the effect is low in relation to the overall affected area in the below table.
| ID | Area m² | Fuzzy value | a=0.3  | a=0.45 | a=0.6  |
|----|---------|-------------|--------|--------|--------|
|    |         |             | ω-interior | ω-Exterior | ω-Boundary | ω-interior | ω-Exterior | ω-Boundary | ω-interior | ω-Exterior | ω-Boundary |
| 1  | 7.99    | 0.22        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 2  | 8.20    | 0.22        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 3  | 9.44    | 0.24        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 4  | 10.12   | 0.25        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 5  | 10.55   | 0.25        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 6  | 15.76   | 0.29        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 7  | 17.87   | 0.31        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 8  | 19.53   | 0.32        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 9  | 19.74   | 0.32        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 10 | 28.76   | 0.36        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 11 | 29.23   | 0.36        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 12 | 31.65   | 0.37        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 13 | 38.12   | 0.39        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 14 | 39.11   | 0.40        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 15 | 45.76   | 0.41        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 16 | 53.76   | 0.43        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 17 | 55.90   | 0.43        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 18 | 57.23   | 0.44        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 19 | 60.10   | 0.44        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 20 | 61.21   | 0.44        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 21 | 73.29   | 0.46        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 22 | 74.89   | 0.47        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 23 | 75.61   | 0.47        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 24 | 80.34   | 0.47        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 25 | 82.41   | 0.48        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 26 | 85.66   | 0.48        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 27 | 88.80   | 0.48        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 28 | 90.11   | 0.49        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 29 | 91.13   | 0.49        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 30 | 95.90   | 0.49        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 31 | 97.50   | 0.49        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 32 | 99.90   | 0.50        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 33 | 101.12  | 0.50        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
| 34 | 104.67  | 0.50        | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00      |
VI. CONCLUSION

It is found that not only we get information on fuzzy topological relations between two objects (Such as the value of the interior, boundary, and exterior) but a quantitative level of these topological relations between simple fuzzy regions can be obtained.

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