AN ALTERNATIVE AND UNITED PROOF OF A DOUBLE INEQUALITY FOR BOUNDING THE ARITHMETIC-GEOMETRIC MEAN

FENG QI AND ANTHONY SOFO

Abstract. In the paper, we provide an alternative and united proof of a double inequality for bounding the arithmetic-geometric mean.

1. Introduction

The complete elliptic integral of the first kind was defined as

\[ K(t) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - t^2 \sin^2 \theta}} \] (1)

for \(0 < t < 1\), see [2, p. 132, Definition 3.2.1]. It can also be defined in the following way: For positive numbers \(a\) and \(b\),

\[ K(a, b) = \int_0^{\pi/2} \frac{1}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} \, d\theta. \] (2)

For positive numbers \(a = a_0\) and \(b = b_0\), let

\[ a_{k+1} = \frac{a_k + b_k}{2} \quad \text{and} \quad b_{k+1} = \sqrt{a_kb_k}. \] (3)

In [2, p. 134, Definition 3.2.2] and [4], the common limit \(M(a, b)\) of these two sequences \(\{a_k\}_{k=0}^\infty\) and \(\{b_k\}_{k=0}^\infty\) is called as the arithmetic-geometric mean. It was proved in [2, Theorem 3.2.3] and [4, Theorem 1] that

\[ \frac{1}{M(a, b)} = \frac{2}{\pi} K(a, b). \] (4)

For more information on the arithmetic-geometric mean and the complete elliptic integral of the first kind, please refer to [2, pp. 132–136], [4] and related references therein.

In [4, Theorem 4] and [6], it was proved that the inequality

\[ M(a, b) \geq L(a, b) \] (5)

holds true for positive numbers \(a\) and \(b\) and that the inequality (5) becomes equality if and only if \(a = b\), where

\[ L(a, b) = \frac{b - a}{\ln b - \ln a} \] (6)

stands for the logarithmic mean for positive numbers \(a\) and \(b\) with \(a \neq b\).

In [16, Theorem 1.3], it was turned out that

\[ M(a, b) < I(a, b) \] (7)

Key words and phrases. alternative and united proof, double inequality, arithmetic-geometric mean, complete elliptic integral of the first kind, generalized logarithmic mean.

The first author was partially supported by the China Scholarship Council.

This paper was typeset using \textsc{AMSLATEX}.
for positive numbers \( a \) and \( b \) with \( a \neq b \), where

\[
I(a, b) = \frac{1}{e} \left( \frac{b^a}{a^b} \right)^{1/(b-a)}
\]  

represents the exponential mean for for positive numbers \( a \) and \( b \) with \( a \neq b \).

It is known that a generalization of the logarithmic mean \( L(a, b) \) is the generalized logarithmic mean \( L(p; a, b) \) of order \( p \in \mathbb{R} \), which may be defined \([5, p. 385]\) by

\[
L(p; a, b) = \begin{cases} 
\left[ \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{1/p}, & p \neq -1, 0 \\
L(a, b), & p = -1 \\
I(a, b), & p = 0
\end{cases}
\]  

for positive numbers \( a \) and \( b \) with \( a \neq b \), and that \( L(p; a, b) \) is strictly increasing with respect to \( p \in \mathbb{R} \). Therefore, one may naturally pose the following problem.

**Problem 1.** What are the best constants \( 0 \geq \beta > \alpha \geq -1 \) such that the double inequality

\[
L(\alpha; a, b) < M(a, b) < L(\beta; a, b)
\]  

holds for all positive numbers \( a \) and \( b \) with \( a \neq b \)? In other words, are the constants \( \alpha = -1 \) and \( \beta = 0 \) the best possible in the inequality (10)?

It is easy to see that the complete elliptic integral \( K(a, b) \) of the first kind tends to infinity as the ratio \( \frac{b}{a} \) for \( a > b > 0 \) tends to zero, equivalently, the arithmetic-geometric mean \( M(a, b) \) tends to zero as \( \frac{b}{a} \to 0^+ \). As \( \frac{b}{a} \to 0^+ \), the logarithmic mean \( L(a, b) \) also tends to 0. However, the exponential mean \( I(a, b) \) does not tend to zero as \( \frac{b}{a} \to 0^+ \). These phenomena motivate us to put forward an alternative problem as follows.

**Problem 2.** What are the best constants \( \beta > \alpha \geq 1 \) such that the double inequality

\[
\alpha L(a, b) < M(a, b) < \beta L(a, b)
\]  

holds for all positive numbers \( a \) and \( b \) with \( a \neq b \)?

In \([15]\) and \([16, \text{Theorem 1.3}]\), among others, the right-hand side inequality in (11) was verified to be valid for \( \beta = \frac{1}{2} \).

The aim of this paper is to confirm and sharpen the inequality (11) alternatively and unitedly.

Our main result may be recited as the following theorem.

**Theorem 1.** The double inequality (11) is valid for all positive numbers \( a \) and \( b \) with \( a \neq b \) if and only if \( \alpha \leq 1 \) and \( \beta \geq \frac{1}{2} \).

**Remark 1.** Some inequalities were established in \([3, 7, 8, 11, 12, 13, 14, 18]\) for bounding the complete elliptic integrals.

### 2. Lemmas

For proving our theorem alternatively and unitedly, we will employ the following lemmas.

**Lemma 1** \([10]\). Let \( a_k \) and \( b_k \) for \( k \in \mathbb{N} \) be real numbers and the power series

\[
A(x) = \sum_{k=1}^{\infty} a_k x^k \quad \text{and} \quad B(x) = \sum_{k=1}^{\infty} b_k x^k
\]

be convergent on \((-R, R)\) for some \( R > 0 \). If \( b_k > 0 \) and the ratio \( \frac{a_k}{b_k} \) is strictly increasing for \( k \in \mathbb{N} \), then the function \( \frac{A(x)}{B(x)} \) is also strictly increasing on \((0, R)\).
Lemma 2 ([17]). For \( n \in \mathbb{N} \),
\[
\int_0^{\pi/2} \sin^n x \, dx = \frac{\sqrt{\pi} \Gamma((n+1)/2)}{n \Gamma(n/2)} = \begin{cases} 
\frac{\pi}{2} \frac{(n-1)!!}{n!!} & \text{for } n \text{ even}, \\
\frac{(n-1)!!}{n!!} & \text{for } n \text{ odd},
\end{cases}
\]
where \( n!! \) denotes a double factorial.

Lemma 3. For \( k \in \mathbb{N} \),
\[
\sum_{i=1}^{k} \left( \frac{2i-2}{i-1} \right) \frac{1}{i^4} = \frac{1}{2} - \frac{2}{4^{k+1}} \binom{2k}{k},
\]
\[
\sum_{i=1}^{k} \left( \frac{2i-2}{i-1} \right) \frac{1}{(k-i+1)4^i} = \left[ 2 \ln 2 + \gamma + \psi \left( k + \frac{1}{2} \right) \right] \frac{\Gamma(k+1/2)}{4 \sqrt{\pi} \Gamma(k+1)}.
\]

Proof. For our own convenience, let us denote the two sequences in left-hand sides of (14) and (15) by \( h(k) \) and \( g(k) \) for \( k \in \mathbb{N} \) respectively.

When \( k = 1 \), the identity (14) is valid clearly. Suppose the identity (14) holds for some \( k > 1 \), then it follows that
\[
h(k+1) = h(k) + \left( \frac{2k}{k+1} \right) \frac{1}{(k+1)^{k+1}}
\]
\[
= \frac{1}{2} - \frac{2}{4^{k+1}} \binom{2k}{k} + \left( \frac{2k}{k+1} \right) \frac{1}{(k+1)^{k+1}}
\]
\[
= \frac{1}{2} - \frac{2}{4^{k+2}} \left( \frac{2k+2}{k+1} \right).
\]

Therefore, by induction, the identity (14) is valid for all \( k \in \mathbb{N} \).

Applying the Zeilberger algorithm (see [9, Chapter 6]) and (13) yields
\[
2(k+1)g(k+1) - (2k+1)g(k) = \frac{\Gamma(k+1/2)}{2 \sqrt{\pi} \Gamma(k+1)} = \frac{1}{2} \left( \frac{2k}{k} \right) \frac{1}{4^k}
\]
for \( k \in \mathbb{N} \), from which the identity (15) follows. \( \square \)

Remark 2. The identities (14) and (15) can also be verified easily by the famous softwares Maple and MATHEMATICA.

3. An alternative and united proof of Theorem 1

Now we are in a position to alternatively and unitedly verify Theorem 1.

Making use of the power series expansion
\[
\frac{1}{\sqrt{1-s}} = \sum_{i=0}^{\infty} \frac{(2i)!}{4^i (i!)^2} s^i, \quad 0 < s < 1,
\]
it is obtained that
\[
\frac{1}{\sqrt{1-s^2 \sin^2 \theta}} = \sum_{i=0}^{\infty} \frac{(2i)!}{4^i (i!)^2} s^{2i} \sin\theta, \quad 0 < s < 1.
\]
From the celebrated Wallis sine formula (13) in Lemma 2, it is obtained that
\[
\int_0^{\pi/2} \sin^{2i} \theta \, d\theta = \frac{1}{4^i} \binom{2i}{i} \frac{\pi}{2} = \sqrt{\pi} \frac{\Gamma(i+1/2)}{\Gamma(i+1)}, \quad i \in \mathbb{N}.
\]
Integrating on both sides of (17) with respect to $\theta$ from 0 to $\frac{\pi}{2}$ and using the identity (18) yield
\[
\frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-s^2 \sin^2 \theta}} = \sum_{i=0}^{\infty} \frac{1}{2^{2i}} \left( \frac{2i}{i} \right)^2 s^{2i} = \frac{1}{\pi} \sum_{i=0}^{\infty} \left[ \frac{\Gamma(i+1/2)}{\Gamma(i+1)} \right]^2 s^{2i} \quad (19)
\]
for $0 < s < 1$. Letting $s^2 = 1-t^2$ in (19) yields
\[
\frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-(1-t^2) \sin^2 \theta}} = \sum_{i=0}^{\infty} \frac{1}{2^{2i}} \left( \frac{2i}{i} \right)^2 (1-t^2)^i \overset{!}{=} \sum_{k=0}^{\infty} b_k (1-t^2)^k
\]
for $0 < t < 1$.

It is easy to see that
\[
\left( \frac{1}{t} \right)^{(k)} = \frac{(-1)^k k!}{t^{k+1}} \quad \text{and} \quad (\sqrt{t})^{(k)} = \frac{(-1)^{k+1} (2k-1)!!}{2k(2k-1) t^{k-1/2}} \quad (21)
\]
for $k \in \mathbb{N}$. Then, by Leibniz’s Theorem [1, p. 12, 3.38] for differentiation of a product, we gain that
\[
\left[ (1 + \sqrt{t}) \ln t \right]^{(k)} = (1 + \sqrt{t}) \left( \frac{1}{t} \right)^{(k-1)} + (1 + \sqrt{t})^{(k)} \ln t \quad + \sum_{i=1}^{k-1} \left( \frac{k}{i} \right) (1 + \sqrt{t})^{(i)} \left( \frac{1}{t} \right)^{(k-i-1)} = (1 + \sqrt{t}) \frac{(-1)^{k-1} (k-1)!}{t^k} \quad + (\sqrt{t})^{(k)} \ln t + \frac{(-1)^k k!}{t^{k-1/2}} \sum_{i=1}^{k-1} \frac{(2i-1)!!}{(2i)!!(2i-1)(k-i)} \quad \text{where, and elsewhere in this paper, an empty sum is understood to be nil. Thus,}
\]
\[
\left[ (1 + \sqrt{t}) \ln t \right]^{(k)} \bigg|_{t=1} = (-1)^k k! \left[ \sum_{i=1}^{k-1} \frac{(2i-1)!!}{(2i)!!(2i-1)(k-i)} - \frac{2}{k} \right] \quad (22)
\]
for $k \in \mathbb{N}$. Hence,
\[
(1 + \sqrt{t}) \ln t = \sum_{k=1}^{\infty} \left[ \frac{1}{2} \sum_{i=1}^{k-1} \frac{(2i-1)!!}{(2i)!!(2i-1)(k-i)} - \frac{2}{k} \right] (1-t)^k \quad (23)
\]
which can be reduced by replacing $\sqrt{t}$ by $t$ to
\[
(1 + t) \ln t = \sum_{k=1}^{\infty} \left[ \frac{1}{2} \sum_{i=1}^{k-1} \frac{(2i-1)!!}{(2i)!!(2i-1)(k-i)} - \frac{1}{k} \right] (1-t^2)^k, \quad (24)
\]
and so
\[
\frac{\ln t}{t-1} = \frac{(1+t) \ln t}{t^2-1} = \sum_{k=0}^{\infty} \left[ \frac{1}{k+1} - \frac{1}{2} \sum_{i=1}^{k} \frac{(2i-1)!!}{(2i)!!(2i-1)(k-i+1)} \right] (1-t^2)^k-1 \quad = \sum_{k=0}^{\infty} \left[ \frac{1}{k+1} - \frac{1}{2} \sum_{i=1}^{k} \frac{(2i-1)!!}{(2i)!!(2i-1)(k-i+1)} \right] (1-t^2)^k \overset{!}{=} \sum_{k=0}^{\infty} a_k (1-t^2)^k
\]
for $0 < t < 1$.

The two identities in Lemma 3 and the equality in the right-hand side of the inequality (18) give
\[
\frac{1}{k+1} - a_k = \frac{1}{2} \sum_{i=1}^{k} \frac{(2i-1)!!}{2^{i-1}(i-1)!!2^i(2i-1)(k-i+1)}
\]
It is clear that
\[ \frac{\psi k}{a k} \leq \frac{a k + 1}{b k + 1} \iff \frac{b k + 1}{b k} \leq \frac{a k + 1}{a k} \iff \left( \frac{2 k + 1}{2 k + 2} \right)^2 \leq \frac{a k + 1}{a k} \]
which is equivalent to

\[ \sum_{i=2}^{k+1} \frac{1}{2i-1} - \frac{(k + 1/2)(k + 2)}{(k + 1)^2} \sum_{i=2}^{k} \frac{1}{2i-1} \frac{\Gamma(k + 3/2)}{\sqrt{\pi} \Gamma(k + 2)} \leq 1 - \frac{(k + 1/2)^2(k + 2)}{(k + 1)^3} \]

for \( k \geq 2 \). Furthermore, easy calculation gives

\[
\begin{align*}
\sum_{i=2}^{k+1} \frac{1}{2i-1} &- \frac{(k + 1/2)(k + 2)}{(k + 1)^2} \sum_{i=2}^{k} \frac{1}{2i-1} \\
&= \frac{1}{2k + 1} + \left[ 1 - \frac{(k + 1/2)(k + 2)}{(k + 1)^2} \right] \sum_{i=2}^{k} \frac{1}{2i-1} \\
&= \frac{1}{2k + 1} - \frac{k}{2(k + 1)^2} \sum_{i=2}^{k} \frac{1}{2i-1} \\
&= \frac{k}{2(k + 1)^2} \left[ \frac{2(k + 1)^2}{k(2k + 1)} - \sum_{i=2}^{k} \frac{1}{2i-1} \right].
\end{align*}
\]
Since the sequence $\frac{2(k+1)^2}{k(2k+1)}$ for $k \geq 2$ is strictly decreasing and tends to 1 as $k \to \infty$ and the sequence $\sum_{i=2}^{k} \frac{1}{2i-1}$ for $k \geq 2$ is strictly increasing and diverges to $\infty$, the sequence

$$S_k \triangleq \frac{2(k+1)^2}{k(2k+1)} - \sum_{i=2}^{k} \frac{1}{2i-1} \quad (29)$$

for $k \geq 2$ is strictly decreasing and diverges to $-\infty$. As a result, there exists an integer $k_0 \geq 2$ such that the sequence $S_k$ is negative for all $k \geq k_0$. From the fact that $S_9 = 0.01\cdots$ and $S_{10} = -0.04\cdots$, it follows that $k_0 = 10$. Therefore, considering the facts that

$$\frac{(k+1/2)^2(k+1)}{(k+1)^3} \leq 1$$

and

$$\frac{\Gamma(k+3/2)}{\sqrt\pi \Gamma(k+2)} > 0$$

for $k \geq 2$, it readily follows that the inequality (28) holds for all $k \geq 10$.

Straightforward computations reveal that

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|----|
| $\frac{n_k}{a_k}$ | $\frac{1}{12}$ | $\frac{71}{420}$ | $\frac{71}{840}$ | $\frac{182}{852}$ | $\frac{25}{150}$ | $\frac{196}{840}$ | $\frac{224}{872}$ | $\frac{301}{1310}$ | $\frac{361}{1532}$ | $\frac{400}{1728}$ |

and that

$$\frac{2k+1}{2k+3} \frac{2}{7} = 0.583\cdots \quad \frac{2}{7} = 0.714\cdots \quad \frac{3}{7} = 0.782\cdots$$

$$\frac{2k+1}{2k+3} \frac{2}{9} = 0.562\cdots \quad \frac{2}{9} = 0.694\cdots \quad \frac{3}{9} = 0.765\cdots$$

Consequently, the inequality (28) holds for all $k \in \mathbb{N}$. This demonstrates that the ratio $\frac{n_k}{a_k}$ for $k \in \mathbb{N}$ is strictly increasing. By virtue of Lemma 1, it is obtained that the function

$$\frac{(\ln t)/(t-1)}{(2/\pi) \int_{0}^{\pi/2} 1/\sqrt{1-(1-t^2)\sin^2 \theta} \, d\theta} \quad (30)$$

is strictly decreasing in $t \in (0, 1)$. The well-known Hôpital's rule yields that the limits of the function (30) are $\frac{2}{7}$ and $1$ as $t$ tends to $0^+$ and $1^-$ respectively. Hence, the double inequality

$$\frac{2}{\pi} \cdot \frac{\ln t}{t-1} < \frac{2}{\pi} \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1-(1-t^2)\sin^2 \theta}} < \frac{\ln t}{t-1} \quad (31)$$

for $t \in (0, 1)$ is valid and sharp. Letting $t = \frac{a}{b}$ for $b > a > 0$ in (31) leads to

$$\frac{2}{\pi} \cdot \frac{\ln a - \ln b}{a-b} < \frac{2}{\pi} \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}} < \frac{\ln a - \ln b}{a-b} \quad (32)$$

It is easy to see that the inequality (32) is valid for all positive numbers $a$ and $b$ with $a \neq b$. This implies that the double inequality (11) is valid for all positive numbers $a$ and $b$ with $a \neq b$ if and only if $\alpha \leq \frac{2}{7}$ and $\beta \geq 1$. The proof of Theorem 1 is complete.
Acknowledgements. The authors appreciate Associate Professor Chao-Ping Chen in China for his recommending the reference [4] and some valuable comments in an e-mail to the first author on 8 June 2008. This paper was completed while the first author was visiting the Research Group in Mathematical Inequalities and Applications, Victoria University, Melbourne, Australia between March 2008 and February 2009 by the grant from the China Scholarship Council. So the first author expresses many thanks to the local colleagues such as Professors Pietro Cerone and Sever S. Dragomir for their invitation and hospitality throughout this period.

References

[1] M. Abramowitz and I. A. Stegun (Eds), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards, Applied Mathematics Series 55, Washington, 1972.
[2] G. E. Andrews, R. A. Askey, and R. Roy, Special Functions, Cambridge University Press, Cambridge, 1999.
[3] Á. Baricz, Turán type inequalities for generalized complete elliptic integrals, Math. Z. 256 (2007), no. 4, 895–911.
[4] P. Bracken, An arithmetic-geometric mean inequality, Expo. Math. 19 (2001), no. 3, 273–279.
[5] P. S. Bullen, Handbook of Means and Their Inequalities, Mathematics and its Applications, Volume 560, Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.
[6] B. C. Carlson and M. Vuorinen, An inequality of the AGM and the logarithmic mean, SIAM Rev. 33 (1991), Problem 91-17, 655.
[7] B.-N. Guo and F. Qi, Some bounds for the complete elliptic integrals of the first and second kinds, submitted.
[8] B.-N. Guo, F. Qi and Sh.-J. Jing, Improvement for the upper bound of a class of elliptic integral, Jiaozuo Kuangyu Xueyu University College (Journal of Jiaozuo Mining Institute) 14 (1995), no. 6, 125–128. (Chinese)
[9] M. Petkovšek, H. S. Wilf, and D. Zeilberger, A = B, Wellesley, Massachusetts, 1996.
[10] S. Ponnusamy and M. Vuorinen, Asymptotic expansions and inequalities for hypergeometric functions, Mathematika 44 (1997), 278–301.
[11] F. Qi, L.-H. Cui, and S.-L. Xu, Some inequalities constructed by Tchebysheff’s integral inequality, Math. Inequal. Appl. 2 (1999), no. 4, 517–528.
[12] F. Qi and B.-N. Guo, Estimate for upper bound of an elliptic integral, Shuxue de Shijian yu Renshi (Math. Practice Theory) 26 (1996), no. 3, 285–288. (Chinese)
[13] F. Qi and B.-N. Guo, The estimation of inferior bound for an ellipse integral, Gongke Shuxue (Journal of Mathematics for Technology) 10 (1994), no. 1, 87–90. (Chinese)
[14] F. Qi and Zh. Huang, Inequalities of the complete elliptic integrals, Tammang J. Math. 29 (1998), no. 3, 165–168.
[15] J. Sándor, On certain inequalities for means, II, J. Math. Anal. Appl. 199 (1996), 629–635.
[16] M. K. Vamanamurthy and M. Vuorinen, Inequalities for means, J. Math. Anal. Appl. 183 (1994), 155–166.
[17] E. W. Weisstein, Wallis Cosine Formula, From MathWorld—A Wolfram Web Resource; Available online at http://mathworld.wolfram.com/WallisFormula.html.
[18] L.-Q. Yu, F. Qi and B.-N. Guo, Estimates for upper and lower bounds of a complete elliptic integral, Kuant Yè (Mining) (1995), no. 1, 35–38. (Chinese)