Device-to-Device Aided Multicasting

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Abstract—We consider a device-to-device (D2D) aided multicast channel, where a transmitter wishes to convey a common message to many receivers and these receivers cooperate with each other. We propose a simple computationally efficient scheme requiring only statistical channel knowledge at transmitter. Our analysis in general topologies reveals that, when the number of receivers $K$ grows to infinity, the proposed scheme guarantees a multicast rate of $\frac{1}{2} \log_2(1 + \beta \ln K)$ with high probability for any $\beta < \beta^*$ where $\beta^*$ depends on the network topology. This scheme undergoes a phase transition at threshold $\beta^* \ln K$ where transmissions are successful/unsuccessful with high probability when the SNR is above/below this threshold. We also analyze the outage rate of the proposed scheme in the same setting.

Index Terms—Device-to-Device, Multicasting.

I. INTRODUCTION

We consider the multicast channel, where a single transmitter sends a common message to many receivers in the presence of fading [1–3]. Since the common message must be decoded by all receivers, the multicast capacity of the fading broadcast channel is limited by the worst user. For the case of the i.i.d. Rayleigh fading channel, it is known that the multicast capacity vanishes inversely to the number $K$ of users as $K$ grows [4]. Despite its vanishing rate, the multicast channel is relevant to two scenarios in wireless crowded networks. The first scenario is the wireless edge caching. It has been shown that the traffic during the peak hours can be significantly reduced by caching popular contents during off-peak hours, and delivering these popular contents using multicasting (see e.g. [5], [6] and references therein). A novel user selection scheme requiring only statistical channel knowledge has been proposed recently in [7] for the case of i.i.d. Rayleigh fading channel. The second scenario is so-called enhanced Multimedia Broadcast Multicast Service (eMBMS) aided by Proximity Service (ProSe), standardized by the 3rd Generation Partnership Project (3GPP). These services reduce the network load by multicasting common data to public safety devices [8] which can use Device-to-Device (D2D) communication [9]. In this work, motivated by these two scenarios, we study D2D-aided multicasting without Channel State Information at the Transmitter (CSIT) to overcome the vanishing effect of the multicast rate. We consider a general network topology where the transmitter has only statistical channel knowledge. In this setup, we address the fundamental question: can D2D without CSIT increase the achievable multicasting rate?

We propose a two-stage transmission scheme assuming no CSIT: in the first stage the transmitter sends a common message, while in the second stage, the subset of users whom have successfully decoded retransmit this information simultaneously by using the same codeword. We study this two-stage scheme for two metrics of interest:

- The average multicast rate, which is the expected number of successfully decoded bits per channel use achieved by a user chosen uniformly at random;
- The outage rate, which is the maximum rate at which all the users can decode the message with an error probability of $\epsilon$.

We show that by carefully choosing the transmission rate, we can answer positively to our main question. More specifically, our contributions is two-fold and summarized below:

1) We identify conditions on the network topology for which the average multicast rate grows with the number of users $K$ and the outage rate does not vanish with $K$.
2) We provide tractable, asymptotic expressions for both the multicast rate and the outage rate in the regime of a large number of users.

A similar two-phase scheme has been studied in [10]. However this work is different from ours in its assumption and concept. First, the channel statistic of all users is assumed to be symmetric in [10]. In fact, this is a special case of our model corresponding to a single class $C=1$ (see subsection II-D). Second, the metric of the work [10] is on the error probability decay by exploiting channel hardening via space-time code, while we aim at achieving a scalable multicast rate by exploiting multiuser diversity.

The rest of the paper is organized as follows. Section II presents the system model. Sections III and IV studies the multicast and outage rate metrics respectively. Numerical experiments are provided in Section V. Section VI concludes the paper. We denote by $\ln$ the natural logarithm and $\log_2$ the binary logarithm.

II. MODEL

A. Channel Model

We consider a multicast channel, where a station (indexed by $0$) wants to convey the same message to $K$ users indexed by $i = 1, \ldots, K$. Time is slotted, and in each time slot, $y_i$ the received signal by $i = 1, \ldots, K$ is given by:

$$y_i = \sum_{j=0}^{K} x_j h_{j,i} + n_i,$$
with \((x_i)_{i=0,...,K}\) the signal transmitted by \(i = 0,\ldots, K\), \(h = (h_{i,j})_{i,j=0,...,K}\) are the channel coefficients and \(n_i \sim \mathcal{N}(0, 1)\) is Additive White Gaussian Noise (AWGN). The channel coefficients \(h\) are assumed to be independent, Gaussian complex random variables with mean 0 and variance \(\gamma_{i,j} = \mathbb{E}|h_{i,j}|^2\).

The transmitted signals have unit power \(\mathbb{E}(|x_i|^2) \leq 1\). This is without loss of generality, otherwise simply replace \(x_i\) by \(\sqrt{2\mathbb{E}(|x_i|^2)}\) and \(h_{i,j}\) by \(\sqrt{\mathbb{E}(|x_i|^2)}h_{i,j}\).

Matrix \(\Gamma = (\gamma_{i,j})_{i,j=0,...,K}\) represents the channel statistics, and captures the topology of the network. Channel coefficients during successive time slots are assumed to be independent. No channel state information is available at the transmitter: the transmission strategy should only depend on the statistics of the channel (i.e. \(\Gamma\)), but not on the channel coefficients \(h\).

B. Proposed scheme

Throughout the article we will study the following transmission scheme, which utilizes two time slots. Let \(h, h'\), the channel coefficients during the two successive time slots.

- During the first time slot, the station broadcasts a message at rate \(\log_2(1 + s)\), where \(s\) is a parameter of the scheme and can be chosen depending on the channel statistics \(\Gamma\).
- At the end of the first time slot, users attempt to decode, and user \(i\) decodes successfully if and only if the received Signal to Noise Ratio (SNR) is greater than \(s\), i.e. \(|h_{0,i}|^2 \geq s\).
- All users that have successfully decoded the message in the first slot retransmit this message in the second slot. As before, user \(j\) decodes successfully at the end of the second time slot if the SNR is above \(s\), i.e. \(\sum_{j=1}^{K} |Z_j(s)h'_{j,i}|^2 \geq s\), where \(Z_j(s)\) are binary variables with \(Z_j(s) = 1\) if \(j\) has decoded successfully during the first time slot and 0 otherwise.

C. Performance Measures

We say that user \(i\) decodes if and only if he successfully decodes either in the first or second slot. Denote by \(P_i(s)\) the probability that \(i = 1,\ldots, K\) decodes. \(P(s) = \frac{1}{K} \sum_{i=1}^{K} P_i(s)\) the probability that a user chosen uniformly at random amongst \(i = 1,\ldots, K\) decodes, and \(P_+(s)\) probability that all users decode. We study two performance measures for our problems: the multicast rate:

\[
R^m = \frac{1}{2} \max_{s \geq 0} \{\log_2(1 + s) P(s)\},
\]

which is the expected number of bits received by a user chosen uniformly at random per time slot, and the outage rate:

\[
R^o = \frac{1}{2} \log_2(1 + s) \text{ with } s \text{ solution to } P_+(s) = 1 - \epsilon.
\]

which is the largest rate such that all users decode with probability at least \(1 - \epsilon\), with \(\epsilon\) some fixed reliability level. Both performance measures are interesting for different scenarios: \(R^m\) seems more appropriate when considering say video streaming where the goal is that on average most users receive enough information, whereas \(R^o\) seems more suited to applications such as broadcasting of safety information, where it is important that all users obtain the message.

D. The Block Model

To study large systems, we introduce a block model for \(\Gamma\). Users \(1,\ldots, K\) are partitioned in classes \(1,\ldots, C\), where \(c_i \in \{1,\ldots, C\}\) indicates the class of user \(i\), and \(c_0 = 0\) by convention. There are \(K\alpha_c\) users of class \(c\), with \(\alpha_c > 0\) the proportion of users of class \(c\) and \(\sum_{c=1}^{C} \alpha_c = 1\). Matrix \(\Gamma\) is a block matrix, so that the mean channel gains between two users depends solely on their class. Namely, \(\gamma_{i,j} = g_{\alpha_j, c_j}\) for all \(i, j = 0,\ldots, K\). Define \(G = \max_{c,c' \in 0,\ldots,C} g_{c,c'}\) the largest entry of \(\Gamma\), so that \(\max_{i,j=0,\ldots,K} \gamma_{i,j} = G\). This model can represent any network topology, as the number of classes may be arbitrary. We introduce the following natural assumption.

Assumption 1: Any class of users can be reached in two transmissions, so that for all \(c = 1,\ldots, C\) there exists \(c'\) such that \(g_{0,c} g_{c,c'} > 0\).

E. Baseline

We will compare the performance of this scheme to the most natural baseline which is the same scheme where only the first time slot is used. The performance of the baseline is recalled below.

Proposition 1: The performance of the baseline scheme is:

\[
R^m = \max_{s \geq 0} \{\log_2(1 + s) \sum_{c=1}^{C} \alpha_c e^{-\frac{V(s)}{\alpha_c}}\} = \mathcal{O}(1),
\]

\[
R^o = \log_2 \left(1 + \frac{1}{K} \sum_{c=1}^{C} \alpha_c e^{-\frac{V(s)}{\alpha_c}}\right) = \mathcal{O}\left(\frac{1}{K}\right).
\]

Proof. In the baseline scheme, user \(i\) successfully decodes if and only if \(|h_{0,i}|^2 \geq s\). Since \(|h_{0,i}|^2\) follows an exponential distribution with mean \(g_{0,i}\), we have:

\[
P_i(s) = \mathbb{P}(|h_{0,i}|^2 \geq s) = e^{-\frac{s}{g_{0,i}}},
\]

and averaging:

\[
P(s) = \frac{1}{K} \sum_{i=1}^{K} e^{-\frac{s}{g_{0,i}}} = \sum_{c=1}^{C} \alpha_c e^{-\frac{V(s)}{\alpha_c}}.
\]

which is the announced result.

Once again \(i\) decodes successfully if and only if \(|h_{0,i}|^2 \geq s\), \(|h_{0,i}|^2\) follows an exponential distribution with mean \(g_{0,i}\) and \(h_{0,1}, \ldots, h_{0,K}\) are independent so that:

\[
P_+(s) = \mathbb{P}(|h_{0,i}|^2 \geq s, i = 1,\ldots, K) = \prod_{i=1}^{K} \mathbb{P}(|h_{0,i}|^2 \geq s) = \exp\left\{-s \sum_{i=1}^{K} \frac{1}{g_{0,i}}\right\} = \exp\left\{-sK \sum_{c=1}^{C} \frac{\alpha_c}{g_{0,c}}\right\}.
\]

The outage rate is \(\log_2(1 + s)\) with \(s\) solution to the equation \(P_+(s) = 1 - \epsilon\), so replacing yields the announced result.
III. Multicast Rate

In this section we study the multicast rate of the proposed scheme, and show that it undergoes a phase transition in the regime of a large number of users $K \to \infty$, so that the probability of success $P(s)$ becomes constant-by-parts and may be computed explicitly as a function of $g$ and $\alpha$.

A. Success probability

We first prove Proposition 2 a formula for the probability of success $P_i(s)$, which will serve as the backbone of our analysis. This result shows that the success probability $P_i(s)$ can be controlled by examining the fluctuations of $X_i(s)$, a sum of independent Bernoulli random variables.

**Proposition 2:** For any $i = 1, \ldots, K$ and $s \geq 0$ we have:

$$P_i(s) = 1 - (1 - e^{-\frac{s}{X_i(s)}}) \mathbb{E}\left(1 - \exp\left(-\frac{s}{X_i(s)}\right)\right),$$

where $Z_i(s), \ldots, Z_K(s)$ are independent random variables in $\{0, 1\}$ with $\mathbb{E}(Z_i(s)) = e^{-\frac{s}{X_i(s)}}$ and $X_i(s) = \sum_{j=1}^{K} Z_j(i) \gamma_{j,i}$.

**Proof.** Consider $h = (h_{i,j})_{i,j=0,\ldots,K}$ and $h' = (h'_{i,j})_{i,j=0,\ldots,K}$ the channel coefficients during the first and second time slot respectively. By assumption $h$ is independent from $h'$. Denote by $Z(s) = (Z_1(s), \ldots, Z_K(s))$ the outcome of the first time slot, where $Z_i(s) = 1$ if user $i$ decodes correctly at the first time slot. We may write $Z_i(s) \equiv \mathbb{1}\{|h_{0,i}|^2 \geq s\}$. Since $|h_{0,i}|^2$ has exponential distribution with mean $\gamma_{0,i}$, it follows that $\mathbb{E}(Z_i(s)) = e^{-\frac{s}{X_i(s)}}$. Furthermore, since $h_{0,1}, \ldots, h_{0,K}$ are independent, so are $Z_1(s), \ldots, Z_K(s)$.

Consider user $i$. Conditionally to the value of $Z(s)$, user $i$ does not decode successfully in the first phase if and only if $Z_i(s) = 0$. If $Z_i(s) = 0$, he does not decode successfully in the second phase if and only if $|\sum_{j \neq i} h'_{j,i} Z_j(s)|^2 = |\sum_{j=1}^{K} b_{j,i} Z_j(s)|^2 \leq s$, where $|\sum_{j=1}^{K} b_{j,i} Z_j(s)|^2$ has exponential distribution with mean $\sum_{j=1}^{K} \gamma_{j,i} Z_j(s) = X_i(s)$, since $Z(s)$ is independent of $h'$. Hence

$$P(i \text{ does not decode}|Z(s)) = (1 - Z_i(s))(1 - e^{-\frac{s}{X_i(s)}}).$$

Taking expectations over $Z(s)$ yields the result. $\Box$

B. Asymptotic behavior

We now analyze how the multicast rate scales in the regime of a large number of users. Define:

$$\beta_c = \max_{c'=1,\ldots,C} \left\{g_{0,c'} \mathbb{1}\{g_{0,c'} > 0\}\right\},$$

the largest value of $g_{0,c'}$, the mean channel gain from the station to class $c'$, where class $c'$ can communicate with class $c$ in the second phase i.e. $g_{0,c'} > 0$. Further define the minimum value $\beta^* = \min_{c=1,\ldots,C} \beta_c$. Theorem 2 shows that in the limit of a large number of users, for any user $i$, the success probability $P_i$ undergoes a phase transition at the value $\beta_c \ln K$. Namely transmissions are always successful above this threshold, and always unsuccessful below. This has 3 consequences:

- Our scheme transmits at rate $\frac{1}{2} \log_2(1 + \beta \ln K)$ with an arbitrarily high probability of success for any $\beta < \beta^*$. As $K \to \infty$, the multicast rate of our scheme scales as $\mathcal{O}(\ln \ln K)$ while the baseline yields a multicast rate of $\mathcal{O}(1)$. So considering two slots instead of one has a dramatic impact on performance. Considering more than two time slots does not seem to improve this scaling.
- To obtain an order-optimal rate, one can set $s = \beta \ln K$ for any $\beta < \beta^*$, so that optimizing over $s$ is not needed.

**Theorem 1:** (i) For any $\beta > 0$ and $i = 1, \ldots, K$ we have:

$$P_i(\beta \ln K) \to \begin{cases} 1 & \text{if } \beta < \beta_c, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) We have:

$$\hat{P}(\beta \ln K) \to \sum_{c=1}^{C} \alpha_c \mathbb{1}\{\beta < \beta_c\},$$

**Proof.** From Proposition 2 we have:

$$P_i(\beta \ln K) = 1 - (1 - e^{-\frac{\beta \ln K}{X_i(\beta \ln K)}})\mathbb{E}(\exp(-\frac{\beta \ln K}{X_i(\beta \ln K)})).$$

We have $K^{-\beta \ln K} \to 0 \text{ as } K \to \infty$ so that:

$$\lim_{K \to \infty} P_i(\beta \ln K) = \lim_{K \to \infty} \mathbb{E}(e^{-\frac{\beta \ln K}{X_i(\beta \ln K)}}).$$

The $\beta > \beta_c$ case. Assume that $\beta > \beta_c$, and we control the expectation of $X_i(\beta \ln K)$. Since $\mathbb{E}(Z_j(s)) = e^{-\frac{s}{X_j(\beta \ln K)}}$:

$$\mathbb{E}(\frac{X_i(\beta \ln K)}{\beta \ln K}) = \sum_{j=1}^{K} \gamma_{j,i} K^{-\frac{\beta \ln K}{\beta \ln K}} = \sum_{c=1}^{C} \alpha_c g_{0,c} K^{1 - \frac{\beta \ln K}{\beta \ln K}}$$

$$\leq \sum_{c=1}^{C} \alpha_c g_{0,c} K^{1 - \frac{\beta \ln K}{\beta \ln K}} \to 0$$

since for all $c$ either $g_{0,c} = 0$ or $g_{0,c} \leq \beta_c$, and $\beta > \beta_c$. Therefore, $\frac{X_i(\beta \ln K)}{\beta \ln K}$ converges to 0 in $L^1$ so that it converges to 0 in distribution as well. Since $x \mapsto e^{-\frac{x}{2}}$ is both continuous and bounded we get:

$$\lim_{K \to \infty} P_i(\beta \ln K) = \lim_{K \to \infty} \mathbb{E}(e^{-\frac{\beta \ln K}{X_i(\beta \ln K)}}) = 0.$$

The $\beta < \beta_c$ case. Now consider $\beta < \beta_c$. We control the moments of $X_i(s)$:

$$\mathbb{E}(X_i(s)) = \sum_{j=1}^{K} \gamma_{j,i} e^{-\frac{s}{\gamma_{j,i}}},$$

and since $Z_1(s), \ldots, Z_K(s)$ are independent:

$$\text{var}(X_i(s)) = \sum_{j=1}^{K} \gamma_{j,i}^2 e^{-\frac{s}{\gamma_{j,i}}} (1 - e^{-\frac{s}{\gamma_{j,i}}})$$

$$\leq G \sum_{j=1}^{K} \gamma_{j,i} e^{-\frac{s}{\gamma_{j,i}}} = GE(X_i(s)).$$
Apply Chebychev’s inequality:

\[
P\left(X_i \leq \frac{\mathbb{E}(X_i)}{2}\right) \leq \frac{\text{Var}(X_i)}{\mathbb{E}(X_i)^2} \leq \frac{4\text{Var}(X_i)}{\mathbb{E}(X_i)} \leq \frac{4G}{\mathbb{E}(X_i)}\]

using the previous bound. Hence

\[
P\left(X_i \geq \frac{\mathbb{E}(X_i)}{2}\right) \geq 1 - \frac{4G}{\mathbb{E}(X_i)}. \tag{1}
\]

Since \(x \mapsto e^{-x^2/2}\) is increasing:

\[
\mathbb{E}(e^{-\beta \ln K / \ln |K|}) \geq \mathbb{E}(X_i(\beta \ln K)) \geq \mathbb{E}(X_i(\beta \ln K)) e^{-\frac{\beta \ln K}{2 \ln |K|}}.
\]

Consider \(\hat{c}\) such that \(g_{\hat{c}, c_i} > 0\) and \(g_{\hat{c}, \hat{c}} = \beta e_{\hat{c}}\). Then:

\[
\mathbb{E}\left(X_i(\beta \ln K)\right) = \sum_{c=1}^{C} \alpha_c g_{\hat{c}, c_i} K^{1 - \frac{\beta}{\beta \ln K}} \geq \frac{\alpha_c g_{\hat{c}, c_i} K^{1 - \frac{\beta}{\beta \ln K}}}{\beta \ln K} \to \infty.
\]

Replacing in (1) we deduce

\[
P\left(X_i(\beta \ln K) \geq \frac{\mathbb{E}(X_i(\beta \ln K))}{2}\right) \to 1,
\]

and \(e^{-\frac{\beta \ln K}{2 \ln |K|}} \to 1\) so that:

\[
\lim_{K \to \infty} P_i(\beta \ln K) = \lim_{K \to \infty} \mathbb{E}(e^{-\frac{\beta \ln K}{2 \ln |K|}}) = 1,
\]

which completes the proof of statement (i). Statement (ii) follows from the fact that \(P(s) = \frac{1}{K} \sum_{c=1}^{K} P_i(s)\).

**Corollary 2:** For any \(\beta < \beta^*\) we have:

\[
R^n \geq \frac{1 - o(1)}{2} \log_2 (1 + \beta \ln K), \quad K \to \infty.
\]

**C. Non asymptotic behavior**

We may state Theorem 3 a further result which gives a tractable, accurate approximation for \(1 - P_j(\beta \ln K)\) in the regime where \(P_j(\beta \ln K) \to 1\) i.e. whenever \(\beta < \beta_{e_i}\). Two main facts are worth mentioning:

- This approximation is very accurate even for modest size systems (say \(K \geq 50\)) as shown by our numerical experiments (see section \(\ref{section:numeric}\)).
- Due to its accuracy, it allows to find the optimal value of \(s\) given \(g\) and \(\alpha\) in a tractable manner, so that finite size systems can be dealt with efficiently.

**Theorem 3:** Consider \(\beta < \beta_{e_i}\), then:

\[
1 - P_j(\beta \ln K) \sim_{K \to \infty} 1 - \exp\left\{ \frac{\beta \ln K}{\sum_{c=1}^{C} \alpha_c g_{\hat{c}, c_i} K^{1 - \frac{\beta}{\beta \ln K}}\right\},
\]

**Proof.** Define \(f(x) = 1 - e^{-x/2}\). Throughout the proof consider \(K\) fixed, and define \(V = \frac{X_i(\beta \ln K)}{\beta \ln K}\).

From Proposition 2:

\[
1 - P_j(\beta \ln K) \sim_{K \to \infty} (1 - K^{-\frac{\beta}{\beta \ln K}}) \mathbb{E}(f(V)) \sim_{K \to \infty} \mathbb{E}(f(V)).
\]

since \(K^{-\frac{\beta}{\beta \ln K}} \to 0\). Define:

\[
v_K = \mathbb{E}(V) = \frac{\sum_{c=1}^{C} \alpha_c g_{\hat{c}, c_i} K^{1 - \frac{\beta}{\beta \ln K}}}{\beta \ln K} \to_{K \to \infty} \infty.
\]

since \(\beta < \beta^*\). To complete the proof, it suffices to show that

\[
\mathbb{E}(f(V)) \sim_{K \to \infty} f(v_K).
\]

Now consider \(\delta \in (0, 1)\) fixed and let us bound \(\mathbb{E}(f(V))\).

**Upper bound:** Since \(x \mapsto f(x)\) is decreasing:

\[
f(V) = f(V) \mathbb{1}\{V \leq (1 - \delta)v_K\} + f(V) \mathbb{1}\{V \geq (1 + \delta)v_K\}
\]

\[
\leq \mathbb{1}\{V \leq (1 - \delta)v_K\} + f((1 - \delta)v_K).
\]

So taking expectations and using Chernoff’s inequality, which is recalled as proposition \(\ref{proposition:chebychev} in appendix:

\[
\mathbb{E}(f(V)) \leq \mathbb{P}(V \leq (1 - \delta)v_K) + f((1 - \delta)v_K)
\]

\[
\leq e^{-\frac{\delta^2 v_K}{2}} + f((1 - \delta)v_K).
\]

Now using the facts \(v_K \to \infty\) so that \(f(v_K) \sim \frac{1}{v_K}\) and \(e^{-\frac{\delta^2 v_K}{2}} \to 0\) we get:

\[
\lim_{K \to \infty} \sup_{K \to \infty} \frac{\mathbb{E}(f(V))}{f(v_K)} \leq \frac{1}{1 - \delta}.
\]

**Lower bound:** Similarly:

\[
f(V) = f(V) \mathbb{1}\{V \leq (1 + \delta)v_K\} + f(V) \mathbb{1}\{V \geq (1 + \delta)v_K\}
\]

\[
\geq f((1 + \delta)v_K).
\]

So taking expectations and using Chernoff’s inequality, which is recalled as proposition \(\ref{proposition:chebychev} in appendix:

\[
f(V) \geq \mathbb{P}(V \leq (1 + \delta)v_K) f((1 + \delta)v_K)
\]

\[
\geq (1 - e^{-\frac{\delta^2 v_K}{2}}) f((1 + \delta)v_K).
\]

Using the facts \(v_K \to \infty\) so that \(f(v_K) \sim \frac{1}{v_K}\) and \(e^{-\frac{\delta^2 v_K}{2}} \to 0\) we get:

\[
\lim_{K \to \infty} \inf_{K \to \infty} \frac{\mathbb{E}(f(V))}{f(v_K)} \geq \frac{1}{1 + \delta}.
\]

Putting it together: The above holds for any \(\delta \in (0, 1)\). So we have proven that \(\delta\) arbitrarily small:

\[
\frac{1}{1 + \delta} \leq \lim_{K \to \infty} \sup_{K \to \infty} \frac{\mathbb{E}(f(V))}{f(v_K)} \leq \lim_{K \to \infty} \frac{\mathbb{E}(f(V))}{f(v_K)} \leq \frac{1}{1 - \delta}.
\]

Hence we have proven

\[
\mathbb{E}(f(V)) \sim_{K \to \infty} f(v_K).
\]

which concludes the proof. □
IV. OUTAGE RATE

We now turn to the outage rate, and we show that in the regime of a large number of users, the outage rate may be computed explicitly as a function of $g$ and $\alpha$. We further show that, while the outage rate of the baseline scheme vanishes when $K \to \infty$ our scheme guarantees a constant outage rate.

A. Success probability

As in the multicast case, we express the outage rate as a function of $X_1(s), \ldots, X_K(s)$ in Proposition 3.

Proposition 3: For any $s \geq 0$ we have:

$$P_A(s) = \mathbb{E}\left\{ \exp\left\{ -s \sum_{i=1}^{K} \frac{1 - Z_i(s)}{X_i(s)} \right\} \right\},$$

where $Z_1(s), \ldots, Z_K(s)$ are independent random variables in $\{0,1\}$ and $\mathbb{E}(Z_i(s)) = e^{-\frac{\gamma_i}{\sigma_i}}$ for $i = 1, \ldots, K$ and $X_i(s) = \sum_{j=1}^{K} Z_j(s)^{h_{j,i}}$.

Proof. We use the same notation as in the proof of Proposition 2. User $i$ successfully decodes if and only if $Z_i(s) = 1$ or $\sum_{j=1}^{K} Z_j(s)^{h_{j,i}} \geq s$. So $i$ decodes if and only if:

$$\left| \sum_{j=1}^{K} Z_j(s)^{h_{j,i}} \right|^2 \geq s(1 - Z_i(s)).$$

Now:

$$\mathbb{P}(\text{all decode} | Z(s))$$

$$= \mathbb{P}\left( \left| \sum_{j=1}^{K} Z_j(s)^{h_{j,i}} \right|^2 \geq s(1 - Z_i(s)), \forall i | Z(s) \right)$$

$$= \prod_{i=1}^{K} \mathbb{P}\left( \left| \sum_{j=1}^{K} Z_j(s)^{h_{j,i}} \right|^2 \geq s(1 - Z_i(s)) | Z(s) \right)$$

$$= \exp\left\{ -s \sum_{i=1}^{K} \frac{1 - Z_i(s)}{X_i(s)} \right\},$$

since conditional to $Z(s)$, $\sum_{j=1}^{K} Z_j(s)^{h_{j,i}}$ has exponential distribution with mean $X_i(s)$, and $h_{j,i}$ has independent entries. Averaging over $Z(s)$ yields the result. \hfill \Box

B. Asymptotic behavior

We now prove our main result concerning outage rate stated in Theorem 4. The main proof element is to show that, for any $s$ random variable $\sum_{i=1}^{K} \frac{1 - Z_i(s)}{X_i(s)}$ concentrates around its expectation when the number of users $K$ grows large. The following facts should be noted:

- The outage rate can be computed explicitly as a function of $g$ and $\alpha$ (see corollary 5) by a zero-finding method such as bisection or Newton-Raphson.
- In fact, this asymptotic result provides a very good approximation even for systems of modest size (say $K \geq 10$), as shown by numerical experiments (see section V).

Theorem 4: For any $s \geq 0$ we have:

$$P_A(s) \to K \to \infty \mathbb{P} \left\{ \sum_{i=1}^{K} \frac{1 - Z_i(s)}{X_i(s)} \right\} = K \to \infty \exp\left\{ -s \sum_{j=1}^{C} \frac{\alpha_j(1 - e^{-\frac{\gamma_j}{\sigma_j}})}{C - G(e^{-\frac{\gamma_j}{\sigma_j}})} \right\}.$$

Proof. We first prove the following fact:

$$\sum_{i=1}^{K} \frac{1 - Z_i(s)}{X_i(s)} \to K \to \infty \sum_{i=1}^{K} \frac{1 - e^{-\frac{\gamma_i}{\sigma_i}}}{\mathbb{E}(X_i(s))}.$$

We bound the error as follows:

$$\left| \sum_{i=1}^{K} \frac{1 - Z_i(s)}{X_i(s)} - \sum_{i=1}^{K} \frac{1 - e^{-\frac{\gamma_i}{\sigma_i}}}{\mathbb{E}(X_i(s))} \right|$$

$$\leq \sum_{i=1}^{K} \frac{1 - Z_i(s)}{X_i(s)} - \frac{1 - Z_i(s)}{\mathbb{E}(X_i(s))} + \sum_{i=1}^{K} \frac{Z_i(s) - e^{-\frac{\gamma_i}{\sigma_i}}}{\mathbb{E}(X_i(s))}.$$

We now prove that both terms go to 0 in probability.

First term. We have:

$$\sum_{i=1}^{K} \frac{1 - Z_i(s)}{X_i(s)} - \frac{1 - Z_i(s)}{\mathbb{E}(X_i(s))} \leq \sum_{i=1}^{K} \frac{X_i(s) - \mathbb{E}(X_i(s))}{X_i(s)\mathbb{E}(X_i(s))} = \sum_{i=1}^{K} \frac{|X_i(s) - \mathbb{E}(X_i(s))|}{X_i(s)\mathbb{E}(X_i(s))}.$$

Furthermore:

$$\mathbb{E}(X_i(s)) = K \sum_{c=1}^{C} \alpha_c g_{c, c} e^{-\frac{\gamma_c}{\sigma_c}} \geq m(s)K,$$

with

$$m(s) = \min_{c=1, \ldots, C} \max_{c'=1, \ldots, C} \left\{ \alpha_c g_{c, c'} e^{-\frac{\gamma_c}{\sigma_c}} \right\} > 0.$$

where $m(s) > 0$ from assumption 4.

For $i = 1, \ldots, K$, $X_i(s)$ is a sum of $K$ random variables bounded by $G$, so that Hoeffding’s inequality yields:

$$\mathbb{P}(|X_i(s) - \mathbb{E}(X_i(s))| \geq G\sqrt{K \ln K}) \leq \frac{1}{K^2}.$$

Using a union bound over $i = 1, \ldots, K$:

$$\mathbb{P}(\max_{i=1, \ldots, K} |X_i(s) - \mathbb{E}(X_i(s))|) \geq G\sqrt{K \ln K}) \leq \frac{1}{K} \to 0.$$

Therefore the event

$$A = \left\{ \max_{i=1, \ldots, K} |X_i(s) - \mathbb{E}(X_i(s))| \leq G\sqrt{K \ln K} \right\}$$

holds with probability $1 - \frac{1}{K}$.
occurs with high probability when $K \to \infty$. Consider $K$ large enough so that $m(s)K - \sqrt{K \ln K} \geq 0$. If $A$ occurs:

$$\sum_{i=1}^{K} \frac{|X_i(s) - \mathbb{E}(X_i(s))|}{X_i(s)\mathbb{E}(X_i(s))} \leq \frac{K\sqrt{K \ln K}}{m(s)K(m(s)K - \sqrt{K \ln K})} \rightarrow_0 0.$$

**Second term** Compute the second moment of the second term:

$$\mathbb{E}\left(\left(\sum_{i=1}^{K} \frac{Z_i(s) - e^{-\frac{1}{m(s)}}}{\mathbb{E}(X_i(s))}\right)^2\right) = \sum_{i=1}^{K} \frac{1}{\mathbb{E}(X_i(s))^2} \leq \frac{1}{m(s)^2K} \rightarrow_0 0,$$

since $\mathbb{E}(Z_i(s)) = e^{-\frac{1}{m(s)}}$ and $Z_1(s), \ldots, Z_K(s)$ are independent. So the second term goes to 0 in probability, since $L^2$ convergence implies convergence in probability.

Putting it together We have proven that:

$$\sum_{i=1}^{K} \frac{1 - Z_i(s)}{X_i(s)} \stackrel{p}{\rightarrow} \sum_{i=1}^{K} \frac{1 - e^{-\frac{1}{m(s)}}}{\mathbb{E}(X_i(s))} = \sum_{c=1}^{C} \frac{\alpha_c(1 - e^{-\frac{1}{m(s)}})}{\sum_{c'=1}^{C} \alpha_{c'}g_{c',c}e^{-\frac{1}{m(s)c'}}}.$$

Since convergence in probability implies convergence in distribution and $x \mapsto e^{-x}$ is continuous and bounded on $[0, \infty)$ we obtain the result:

$$P_s(s) \rightarrow \exp\left\{-s \sum_{c=1}^{C} \frac{\alpha_c(1 - e^{-\frac{1}{m(s)}})}{\sum_{c'=1}^{C} \alpha_{c'}g_{c',c}e^{-\frac{1}{m(s)c'}}}\right\}.$$

**Corollary 5:** When $K \to \infty$, the outage rate converges $\frac{1}{2} \log_2(1 + s)$ where $s$ is the unique solution to:

$$s \sum_{c=1}^{C} \frac{\alpha_c(1 - e^{-\frac{1}{m(s)}})}{\sum_{c'=1}^{C} \alpha_{c'}g_{c',c}e^{-\frac{1}{m(s)c'}}} = \ln\left(\frac{1}{1-\epsilon}\right).$$

and for $\epsilon \approx 0$ we have $s \approx 0$ and a Taylor development gives:

$$s \approx \sqrt{-\frac{\ln\left(\frac{1}{1-\epsilon}\right)}{\sum_{c=1}^{C} \frac{\alpha_c}{\sum_{c'=1}^{C} \alpha_{c'}g_{c',c}e^{-\frac{1}{m(s)c'}}}}}.$$

V. NUMERICAL EXPERIMENTS

In this section we present numerical experiments and show that our theoretical analysis provides accurate predictions of the system’s behavior. For each figure, three curves are presented: "baseline" is the performance of the baseline scheme, “simulation” is the exact performance of the proposed scheme obtained by simulation, and “approx” is the analytical approximation of the proposed scheme’s performance. For the multicast rate the approximation is given by Theorem 3 and for the outage rate, the approximation is given by Theorem 4.

We consider three scenarios:

(a) A single class $C = 1$, with $\alpha = (1), g_{0,1} = 46$ dBm and $g_{1,1} = 23$ dBm.

(b) Two classes $C = 2$, with $\alpha = (0.5, 0.5), g_{0,1} = 46$ dBm, $g_{0,2} = 0$, $g_{1,1} = g_{2,2} = 23$ dBm and $g_{1,2} = g_{2,1} = 13$ dBm.

(c) A cell of radius 250 m, with $K$ users drawn uniformly at random in this area. The mean channel gains are taken as $\gamma_{i,j} = \rho_i - 128 - 36.4 \log_{10}(d_{i,j})$ dBm where $d_{i,j}$ is the distance between $i$ and $j$, expressed in kilometers, and $\rho_i = 46$ dBm if $i = 0$ and $\rho_i = 23$ dBm otherwise. For this scenario, the results are averaged over 100 random realizations.

Scenario (a) is the homogeneous case where all users are close to the station and to each other, scenario (b) is a case where users of class 1 are close to the station, and users of class 2 are far away from the station, so that in order to receive data they require users of class 1 to act as a relay. Finally scenario (c) represents the case where users arrive uniformly in a cell area, which seems like an acceptable model for a real system in high load (i.e. when there are many active users). For all scenarios $\epsilon = 10^{-2}$. For the multicast rate, figure 1 presents the multicast rate $R^m$ as a function of the number of users $K$ (when $s$ is chosen optimally), while figure 2 presents the multicast rate as a function of $s$, for various values of $K$. For outage rate, figure 3 presents the outage rate as a function of the number of users $K$, while figure 4 presents the outage probability $P_+(s)$ as a function of $s$ for system size $K = 100$.

**Fig. 1:** Multicast rate $R^m$ versus number of users $K$.

These numerical experiments point out three facts:

- The proposed scheme is vastly superior to the baseline scheme as soon as there are more than a few users, as predicted by our asymptotic analysis.
The proposed approximations/asymptotic expressions derived above predict the performance of the proposed scheme very accurately, even for systems of modest size, say $K \geq 40$ for the multicast rate and $K \geq 15$ for the outage rate.

As a consequence, setting the parameter $s$ can be done in a simple and tractable manner to obtain good practical performance for systems of modest size.
and

\[ P \left( \sum_{i=1}^{K} Y_i \leq (1 - \delta) y_K \right) \leq e^{-\frac{y_K \delta^2}{3}}. \]

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The graph shows the relationship between error probability and SNR (linear) for three different datasets:

- **Baseline**: Represented by magenta circles. This dataset follows a linear trend with a constant slope.
- **Simulation**: Represented by blue squares. This dataset also follows a linear trend with a slightly different slope compared to the baseline.
- **Approximation**: Represented by red crosses. This dataset is an approximation of the baseline and simulation datasets, showing a similar trend but with some deviation.

The x-axis represents the SNR (linear) ranging from $10^{-2}$ to $10^0$, and the y-axis represents the error probability ranging from $10^{-4}$ to $10^0$. The graph illustrates how error probability increases with decreasing SNR, with the baseline and simulation datasets converging at higher SNR values.
The graph shows the multicast rate (bits / channel use) vs. SNR (linear) with three different methods:

- **Baseline** represented by a circular data point line.
- **Simulation** represented by a square data point line.
- **Approx** represented by an 'x' data point line.

The x-axis represents the SNR (linear) ranging from 0 to 6, while the y-axis represents the multicast rate ranging from 0 to 0.4.
