Graph Isomorphism for Bounded Genus Graphs In Linear Time

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Abstract

For every integer \( g \), isomorphism of graphs of Euler genus at most \( g \) can be decided in linear time. This improves previously known algorithms whose time complexity is \( n^{O(g)} \) (shown in early 1980’s), and in fact, this is the first fixed-parameter tractable algorithm for the graph isomorphism problem for bounded genus graphs in terms of the Euler genus \( g \). Our result also generalizes the seminal result of Hopcroft and Wong in 1974, which says that the graph isomorphism problem can be decided in linear time for planar graphs.

Our proof is quite lengthly and complicated, but if we are satisfied with an \( O(n^3) \) time algorithm for the same problem, the proof is shorter and easier.

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1 Introduction

1.1 The Graph Isomorphism Problem

The graph isomorphism problem asks whether or not two given graphs are isomorphic. It is considered by many as one of the most challenging problems today in theoretical computer science. While some complexity theoretic results indicate that this problem might not be NP-complete (if it were, the polynomial hierarchy would collapse to its second level, see [4, 10, 20, 21, 56]), no polynomial time algorithm is known for it, even with extended resources like randomization or quantum computing.

On the other hand, there is a number of important classes of graphs on which the graph isomorphism problem is known to be solvable in polynomial time. For example, in 1990, Bodlaender [8] gave a polynomial time algorithm for the graph isomorphism problem for graphs of bounded tree-width. Many NP-hard problems can be solved in polynomial time, even in linear time, when input is restricted to graphs of tree-width at most $k$ [2, 9]. So, Bodlaender’s result may not be surprising, but the time complexity in [8] is $O(n^{k+2})$, and no one could improve the time complexity to $O(n^{O(1)})$ until quite recently [38]. This indicates that even for graphs of bounded tree-width, the graph isomorphism problem is not trivial at all.

Another important family of graphs is the planar graphs. In 1966, Weinberg [61] gave a very simple $O(n^2)$ algorithm for the graph isomorphism problem for planar graphs. This was improved by Hopcroft and Tarjan [26, 27] to $O(n \log n)$. Building on this earlier work, Hopcroft and Wong [28] published in 1974 a seminal paper, where they presented a linear time algorithm for the graph isomorphism problem for planar graphs.

There are some other classes of graphs on which the graph isomorphism problem is solvable in polynomial time. This includes minor-closed families of graphs [41, 48, 49], and graphs without a fixed graph as a topological minor [24]. A powerful approach based on group theory was introduced by Babai [3]. Based on this approach, Babai et al. [5] proved that the graph isomorphism problem is polynomially solvable for graphs of bounded eigenvalue multiplicity, and Luks [37] described his well-known group theoretic algorithm for the graph isomorphism problem for graphs of bounded degree. Babai and others [6, 7] investigated the graph isomorphism problem for random graphs.

1.2 Bounded Genus Graphs

Leaving the plane to consider graphs on surfaces of higher genus, the graph isomorphism problem seems much harder. In 1980, Filotti, Mayer [19] and Miller [40] showed that for every orientable surface $S$, there is a polynomial time algorithm for the graph isomorphism problem for graphs that can be embedded in $S$, but the time complexity is $n^{O(g)}$, where $g$ is the Euler genus of $S$. Lichtenstein [36] gives an $O(n^3)$ algorithm for the graph isomorphism problem for projective planar graphs. These works came out in the early 1980’s. These classes of graphs were extensively studied from other perspectives. For example, Grohe and Verbitsky [22, 23], who studied this problem from a logic point of view, made some interesting progress. However, no one could improve the time complexity in the last 30 years. This can be perhaps explained in the following way. We can rather easily reduce the problem to 3-connected graphs. For planar graphs, the famous result of Whitney tells us that embeddings of 3-connected graphs in the plane are (combinatorially) unique. This allows us to reduce the graph isomorphism problem to the map isomorphism problem, which is easier (see Hopcroft and Wong [28] and Theorem 2.1). But for every nonsimply connected surface $S$, there exist 3-connected graphs with exponentially many embeddings. This makes an essential difference between planar graphs and graphs in surfaces of higher genus. In addition, Thomassen [58] proved that it is NP-complete to determine Euler genus of a given graph.

A graph $G$ embedded in a surface $S$ has face-width or representativity at least $k$, $fw(G) \geq k$, if every non-contractible closed curve in the surface intersects the graph in at least $k$ points. This notion turns out to be of great importance in the graph minor theory of Robertson and Seymour, cf. [30], and in topological graph theory, cf. [47]. If an embedding of $G$ in $S$ is of face-width $k$, then we sometimes call this embedding
If $G$ is 3-connected and $\text{fw}(G) \geq 3$, then the embedding has properties that are characteristic for 3-connected planar graphs. The main property is that the faces are all simple polygons and that they intersect nicely — if two distinct faces are not disjoint, their intersection is either a single vertex or a single edge. Therefore such embeddings are sometimes called polyhedral embeddings.

The important property about 3-connected graphs that have a polyhedral embedding in a surface is the following in [31, 46].

**Lemma 1.1** Let $G$ be a 3-connected polyhedrally embeddable graph in a surface $S$ of Euler genus $g$. There is a function $f(g)$ such that $G$ has at most $f(g)$ different polyhedral embeddings in $S$.

In fact, in [31], the following was shown.

**Theorem 1.2** For each surface $S$, there is a linear time algorithm for the following problem: Given an integer $k \geq 3$ and a graph $G$, either find an embedding of $G$ in $S$ with face-width at least $k$, or conclude that $G$ does not have such an embedding. Moreover, if there is an embedding in $S$ of face-width at least $k$ and $G$ is 3-connected, the algorithm gives rise to all embeddings with this property. Furthermore, the number of such embeddings is at most $f(g)$, where $f(g)$ comes from Lemma 1.1.

We have to require the face-width of the embedding to be at least 3 in Theorem 1.2, since there are 3-connected graphs with exponentially many embeddings in any surface (other than the sphere). If we want to have a unique embedding in the surface of Euler genus $g$ (which is an analogue of Whitney’s theorem on the uniqueness of an embedding in the plane), then the face-width must be $\Theta(\log g / \log \log g)$. Sufficiency of this was proved in [43, 57], necessity in [1].

### 1.3 Our Main Result

Our main result of this paper is the following.

**Theorem 1.3** For every integer $g$, isomorphism of graphs of Euler genus at most $g$ can be decided in linear time.

Let us point out that the proof is quite lengthy and complicated, but if we are satisfied with an $O(n^3)$ time algorithm for the same problem, the proof becomes shorter and easier. In particular, the proof of Theorem 5.1, which is the most technical in our proof, becomes much simpler (we will mention this point in the proof of Theorem 5.1).

Theorem 1.3 is a generalization of the seminal result of Hopcroft and Wong [28] that says that there is a linear time algorithm for the graph isomorphism problem for planar graphs. As remarked above, the time complexity of previously known results for the graph isomorphism problem for graphs embeddable in a surface of the Euler genus $g$ is $n^{O(g)}$, and this was proved in the early 1980's. Theorem 1.3 is the first improvement in these 30 years, and the first fixed-parameter tractable result in terms of the Euler genus $g$ for the graph isomorphism problem of this class of graphs.

Let us point out that if we are satisfied with an $O(n^3)$ time algorithm for Theorem 1.3, the proof will be much easier and simpler. Indeed, it seems to us that the hard part of our proof will be significantly simplified (cf., proofs of Theorem 5.1 and Lemma 6.3).

In Section 2, we shall give overview of our algorithm. Before that, we give several basic definitions.

### 1.4 Basic Definitions

Before proceeding, we review basic definitions concerning our work.

For basic graph theoretic definitions, we refer the reader to the book by Diestel [16]. For the notions of topological graph theory we refer to the monograph by Mohar and Thomassen [47]. A separation $(A, B)$
is a pair of sets $G = A \cup B$ such that there are no edges between $A - B$ and $B - A$. The order of the separation $(A, B)$ is $|A \cap B|$. By an embedding of a graph in a surface $S$ we mean a 2-cell embedding in $S$, i.e., we always assume that every face is homeomorphic to an open disk in the plane. Such embeddings can be represented combinatorially by means of local rotation and signature. See [17] for details. The local rotation and signature define rotation system. We define the Euler genus of a surface $S$ as $2 - \chi(S)$, where $\chi(S)$ is the Euler characteristic of $S$. This parameter coincides with the usual notion of the genus, except that it is twice as large if the surface is orientable.

A graph $G$ embedded in a surface $S$ has face-width (or representativity) at least $\theta$ if every closed curve in $S$, which intersects $G$ in fewer than $\theta$ vertices and does not cross edges is contractible (null-homotopic) in $S$. Alternatively, the face-width of $G$ is equal to the minimum number of facial walks whose union contains a cycle which is non-contractible in $S$. It is known that if face-width of $G$ is at least two, then every face bounds a disk. See [17] for further details. Given a non-contractible curve in a non-orientable surface, there are two kinds of non-contractible curves; either orientation-preserving or not orientation-preserving.

Let $W$ be an embedding of $G$ in a surface $S$ (given by means of a rotation system and a signature). A surface minor is defined as follows. For each edge $e$ of $G$, $W$ induces an embedding of both $G - e$ and $G/e$ ($/$ means contraction). The induced embedding of $G/e$ is always in the same surface (unless $e$ is a loop), but the removal of $e$ may give rise to a face which is not homeomorphic to a disk, in which case the induced embedding of $G - e$ may be in another surface (of smaller genus). A sequence of contractions and deletions of edges results in a $W'$-embedded minor $G'$ of $G$, and we say that the $W'$-embedded minor $G'$ is a surface minor of the $W$-embedded graph $G$.

Let $K$ be a subgraph of $G$. A $K$-bridge $B$ in $G$ (or a bridge $B$ of $K$ in $G$) is a subgraph of $G$ which is either an edge $e \in E(G) \setminus E(K)$ with both endpoints in $K$, or it is a connected component of $G - K$ together with all edges (and their endpoints) between the component and $K$. The vertices of $B \cap K$ are the attachments of $B$. A vertex of $K$ of degree different from 2 in $K$ is called a branch vertex of $K$. A branch of $K$ is any path in $K$ (possibly closed) whose endpoints are branch vertices but no internal vertex on this path is a branch vertex of $K$. Every subpath of a branch $e$ is a segment of $e$. If a $K$-bridge is attached to a single branch $e$ of $K$, it is said to be local. Otherwise it is called stable. The number of branch vertices of $K$ is denoted by $\text{size}(K)$.

In this paper, we use the concept “cylinder”. Let $G$ be a graph embedded in a surface $S$. Let $C_1, C_2$ be non-contractible curves in the same homotopy in $S$ (which is not a sphere) that do not cross. Then a cylinder $W$ is an embedded subgraph of $G$ bounded by curves $C_1, C_2$. So $W$ can be considered as a plane graph with the outer face boundary $C'_1$, and with the inner face boundary face $C'_2$, such that the face $C'_i$ is obtained by cutting along this curve $C_i$ for $i = 1, 2$. Hence all the vertices of $G$ hitting the curve $C_i$ must be in the face $C'_i$ of the cylinder for $i = 1, 2$. Note that $C'_1$ and $C'_2$ could intersect, but since $C_1, C_2$ do not cross, we may assume that $C'_1$ is the outer face boundary and $C'_2$ is the inner face boundary.

### 1.5 2-connected components, Triconnected components and decomposition

In this paper, we want to work on 3-connected graphs. The importance of 3-connectivity stems from the fact that if a planar graph is 3-connected (triconnected), then it has a unique embedding on a sphere. Hence an efficient algorithm that decomposes a graph into triconnected components is sometimes useful as a subroutine in problems like planarity testing and planar graph isomorphism.

We now define this decomposition formally.

A biconnected component tree decomposition of a given graph $G$ consists of a tree-decomposition $(T, R)$ such that for every $tt' \in E(T)$, $R_t \cap R_{t'}$ consists of a single vertex and for every $t \in T$, $R_t$ consists of a 2-connected graph (i.e., block). $T$ is called a biconnected component tree.

Let $G$ be a 2-connected graph. A triconnected component tree decomposition of $G$ consist of a tree-decomposition $(T, R)$ such that for every $tt' \in E(T)$, $R_t \cap R_{t'}$ consists of exactly two vertices and for every $t \in T$, the torso $R_t$, which is obtained from $R_t$ by adding an edge between $R_t \cap R_{tt'}$ for all $tt' \in T$, consists
of a 3-connected graph (i.e., a 3-connected graph or a triangle or a $k$-bond for $k \geq 3$, i.e., two vertices with $k$ edges between them). $T$ is called a triconnected component tree.

The followings are known in [12]. Their algorithmic parts are from Hopcroft and Tarjan [25].

**Theorem 1.4** For any graph $G$, a biconnected component tree decomposition is unique. Moreover, there is an $O(n)$ time algorithm to construct a biconnected component tree decomposition.

**Theorem 1.5** For any 2-connected graph $G$, a triconnected component tree decomposition is unique. Moreover, there is an $O(n)$ time algorithm to construct a triconnected component tree decomposition.

2 Overview of our algorithm

Theorem 1.3 can be shown by two steps. The first step is our structural theorems. This is the most technical part. So let us give a sketch of our proof in the next subsection. The second step is concerning “map isomorphism” which will be detained in the following subsection.

2.1 Structural results and their proof techniques

Our main structural result is concerning a 3-connected graph $G$ that can be embedded in the surface $S$ of Euler genus $g$, but cannot be embedded in a surface $S'$ of Euler genus at most $g - 1$. Let us point that the standard arguments allow us to reduce to 3-connected graphs in linear time (see Section 7 for more details). Thus the main arguments in this paper deal with 3-connected graphs.

Below, if we say an embedding of $G$ then it means an embedding of $G$ in $S$ of Euler genus $g$.

If $G$ has a polyhedral embedding, then apply Theorem 1.2 to obtain all polyhedral embeddings in $O(n)$ time (there are at most $f(g)$ different polyhedral embeddings, where $f(g)$ comes from Lemma 1.1). This means that we can test graph isomorphism of two graphs $G_1, G_2$ if both $G_1$ and $G_2$ have polyhedral embeddings, because we have all different polyhedral embeddings of $G_1$ and $G_2$, respectively (Indeed, this is exactly the main result in [31]. Essentially, we can reduce the graph isomorphism problem to the “map isomorphism problem”, because one map of a polyhedral embedding of $G_1$ is map isomorphic to some map of a polyhedral embedding of $G_2$, if $G_1$ and $G_2$ are isomorphic. See Theorem 2.1). So the difficult case is when $G$ does not have a polyhedral embedding. So let us consider the following case:

**Case A.** $G$ does not have any polyhedral embedding, but has an embedding of face-width exactly two.

One difference between Case A and the polyhedral embedding case is that there may be exponentially many embeddings. Figure 1 illustrates an example on a torus that has exponentially many embeddings. To see this, degree four vertices could be embedded in two ways. So the embedding is more flexible and the flexibility of “bridges” is the main issue. But we can see from Figure 2 that if we cut along some two non-contractible curves of order two, then we obtain a ”thin” cylinder that contains all flexible bridges.

To be more precise, let us look at Figure 2. What we want is to take a curve $C_1$ hitting only $c,d$ and a curve hitting $C_2$ hitting only $e,f$. Then we obtain the graph bounded by $C_1$ and $C_2$, which is the ”cylinder” we want to take and which contains all flexible bridges. Then we want to recurse our algorithm to the rest of the graph. Note that all the non-contractible curves that are homotopic to $C_1$ (and $C_2$) and that hits exactly two vertices are in this “thin” cylinder. Moreover the rest of the graph can be embedded in a surface of smaller Euler genus.

This figure motivates us what to do. Specifically, concerning the structural result for Case A, we try to find, in $O(n)$ time, a constant-sized collection of pairs of subgraphs that contain all non-contractible curves that hit exactly two vertices in some embedding of face-width two, as follows:

**Structural Result:** There is a $q'(g)$ for some function $q'$ of $g$ such that

1. there are $q' \leq q'(g)$ pairs $(G'_1, L'_1), \ldots, (G'_d, L'_d) \in Q$,
Figure 1: Exponentially many embeddings on torus

Figure 2: Finding a constant-sized collection of pairs of subgraphs of $G$ that contain all non-contractible curves that hit exactly two vertices in some embedding of face-width two.
2. pairs \((G'_i, L'_i)\) are canonical in a sense that graph isomorphism would preserve these pairs (see more details at the end of Case A for the meaning of this item),

3. for all \(i\), \(G = G'_i \cup L'_i\) and \(|G'_i \cap L'_i| = 4\),

4. for all \(i\), \(G'_i\) can be embedded in a surface of Euler genus at most \(g - 1\),

5. for all \(i\), \(L'_i\) is a cylinder with the outer face \(F_1\) and the inner face \(F_2\) with the following property: there is a non-contractible curve \(C_j\) that hits exactly two vertices \(x_j, y_j\) in some embedding of \(G\) of face-width two for \(j = 1, 2\), and \(x_1, y_1\) are contained in \(F_1\) and \(x_2, y_2\) are contained in \(F_2\) (so \(L'_i\) attaches to the rest of the graph \(G'_i\) at vertices \(x_1, x_2, y_1, y_2\)),

6. an embedding of \(G\) of face-width two in \(S\) can be obtained from some embedding of \(G'_i\) in a surface of Euler genus at most \(g - 1\) and the embedding of the cylinder \(L'_i\) by identifying the respective copies of \(x_1, x_2, y_1\) and \(y_2\) in \(G'\) and \(L'\) (so \(G'_i\) also contains all the vertices \(x_1, x_2, y_1, y_2\) and they are on the border of \(G'_i\) and \(L'_i\), respectively),

7. for any non-contractible curve that hits exactly two vertices \(x, y\) in some embedding of \(G\) of face-width two, both \(x\) and \(y\) are contained in \(L'_i\) for some \(i\).

In Figure 2, the cylinder bounded by the non-contractible curve \(C_1\) hitting only \(c, d\) and the non-contractible curve \(C_2\) hitting only \(e, f\), is \(L'_i\), and the rest graph obtained by splitting \(c, d, e, f\) is \(G'_i\).

Remark for the non orientation-preserving case. We need to clarify difference between the orientation-preserving case and the non orientation-preserving case. In 1-7 above, we only deal with the orientation-preserving case. On the other hand, when we deal with the non orientation-preserving curve, there is one difference. Namely in 5, the definition of the cylinder is different. Figure 3 tells us what happens to the non-orientation-preserving curve. We first split \(a\) and \(b\) into \(a, a'\) and \(b, b'\) respectively. Then we flip the component containing \(a'\) and \(b'\). This is what happens in Figure 3.

Now suppose there is a non orientation-preserving curve \(C\) of order exactly two. Then it is straightforward to see that there is a face \(W\) that any non orientation-preserving curve of order exactly two that is homotopic to \(C\) must hit two vertices of \(W\) (see Figure 4). Following Figure 4, we cut along the curve through \(a\) and \(b\), and then split \(a\) and \(b\) into \(a, a'\) and \(b, b'\) respectively, and finally we flip the component containing \(a'\) and \(b'\), as in Figure 3. Then we obtain the situation as in Figure 5. Namely, we have a new face \(W'\) which is obtained from \(W\) by taking the part between \(a\) and \(b\), and the flipped part between \(b\) and \(a\) (i.e., the upper part between \(b'\) and \(a'\) in Figure 5). Then all non-contractible curves of order exactly two that are homotopic to \(C\) must hit two vertices of the resulting face \(W'\), with one vertex in the upper part between \(b'\) and \(a'\), and the other vertex in the lower part between \(a\) and \(b\).

Intuitively, what we need for 5 is to cut along \(e = x_1, d = y_1\), and to cut along \(c = x_2, f = y_2\) in Figures 4 and 5 with the condition that there is no non-contractible curve of order exactly two that hits two vertices of the face \(W'\), with one vertex in the upper part between \(e\) and \(c\) and the other vertex in the lower part between \(f\) and \(d\). Then what we obtain is the following:

\[5'\] \(L'_i\) is a planar graph with the outer face boundary \(W\) with four vertices \(x_1, y_2, x_2, y_1\) appearing in this order listed when we walk along \(W\), with the following property: there is a non-contractible curve \(C_j\) that hits exactly two vertices \(x_j, y_j\) in some embedding of \(G\) of face-width two for \(j = 1, 2\). See Figure 8 which will be explained later.

But all other points (1-4, 6,7) are the exactly same.

Remark 1. Let us observe that we only care about non-contractible curves of length two that are NOT separating, because the graph \(G\) is 3-connected. Moreover, if \(G\) is a cylinder with the boundaries \(C_1\)
Figure 3: Cutting along a non-orientation-preserving curve

Figure 4: Non orientation preserving curves of order two
and $C_2$ (so $G$ is obtained by gluing $C_1$ and $C_2$), then we have to do something else because in this case $G_i'$ could be empty but $G$ itself is $L_i'$. This is exactly the case when the surface $S$ is torus or the Kleinbottle, and moreover, cutting along a non-contractible curve of length two reduces the Euler genus by two (thus when $S$ is the Kleinbottle, $H$ neither is surface-separating nor hits only one crosscap). This “degenerated” case has to be dealt with separately, which is done in Theorem 5.3.

The proof for this structural result consists of the following two step solutions:

1. Find a set of “subgraphs(skeletons)” $F'$ in $G$ that can be extended to all the face-width two embeddings of $G$, in $O(n)$ time. Moreover, each subgraph $F' \in F'$ has bounded number of branch vertices (that only depends on Euler genus $g$). The important property of $F'$ is that each face-two embedding of $G$ can be obtained by extending some member in $F'$ (see below for more details).

Specifically subgraphs(skeletons) $F'$, together with some choice of “bridge” embeddings, give rise to all the face-width two embeddings of $G$.

2. Given a set of the subgraphs $F'$, we want to (in a canonical way) produce pairs $(G_i', L_i')$ (as above) that cover all the vertices that are contained in some non-contractible curve of order two in some embedding of $G$.

Let us give more details for (1) first. The idea is that any embedding of $G$ in the surface $S$ of Euler genus $g$ can be obtained as the following two-stage process.

1. We choose the subgraph $F'$ together with its embedding, in a set of (embedded) subgraphs $F'$ of $G$.

   So $F'$ can be thought of a “skeleton” for the embedding of $G$.

2. For every bridge of $F'$ in $G$, we choose a face of the embedding of $F'$ where to draw this bridge.

Below, we present the properties of the subgraph $F'$ and of the set $F'$, which we find in $O(n)$ time\(^2\) and are detailed in Lemma 3.3.

\(^2\) Finding the set $F'$ in $O(n)$ is also one of the most technical part. The proof was given in [31], but for the completeness, we give a proof in Section 5 and in Section 8.
1. For each $F' \in \mathcal{F}'$, $F'$ is in one of minimal (with respect to edge deletion and contraction) graphs of face-width two in $S$ of Euler genus $g$.

2. $|\mathcal{F}'| \leq l(g)$ for some function $l$ of $g$.

3. For each $F' \in \mathcal{F}'$, $\text{bsize}(F') \leq l'(g)$ for some function $l'$ of $g$.

4. For every embedding of $G$ of face-width two in $S$, there is a subgraph $F'$ (with its corresponding embedding $II$ of face-width two) in $\mathcal{F}'$ such that the embedding $II$ of $F'$ can be extended to this embedding of $G$.

Hence the embedding of $G$ can be seen as the embedding of $F'$, with some bridges embedded into faces of the embedding of $F'$.

5. Moreover, we can assume that every aforementioned bridge of $F'$ in $G$ is stable.

More details concerning (1) are described in Section 3.

Let us move to (2). We now try to (in a canonical way) produce pairs $(G_i', L'_i)$ that cover all vertices $Q$ that are contained in some non-contractible curve that hits exactly two vertices in some embedding of $G$ that extends the embedding of the skeleton $F' \in \mathcal{F}'$.

Here is a crucial observation.

Since the embedding of $F'$ is already of face-width two, all such vertices $Q$ are, in fact, in $F'$ (i.e., any non-contractible curve of order two has to hit two vertices of $F'$). See Sections 3 and 5 for more details.

Here, we need to bound the number of homotopy types. In Section 4 (see Lemma 4.1), it is shown that there are at most $f(g)$ homotopy classes to consider. More specifically, we show that curves from at most $f(g)$ homotopy classes may hit exactly two vertices of $F'$. So it remains to produce pairs $(G_i', L'_i)$ separately for one fixed graph $F' \in \mathcal{F}'$ and for one fixed homotopy class, which hereafter we assume.

The rest of arguments in (2) are detailed in Theorems 5.1 and 5.3. Here we give a sketch of proof of Theorem 5.1, which is one of the most technical parts in this paper. For simplicity, let us first focus on an orientation-preserving curve. Roughly, the argument goes as follows.

Phase 1. We try to find one such a non-contractible curve $C'$ (for some embedding $II'$ of $G$ that extends the embedding $II$ of $F'$). This is actually the most technical part of the proof in Theorem 5.1 see Claim 5.2. Indeed, in the proof of Theorem 5.1 we give a lengthy and involved proof to find such a non-contractible curve $C'$ in linear time.\footnote{If we are satisfied with an $O(n^3)$ algorithm for Theorem 1.3, then Phase 1 is much easier; we just guess these two vertices $x', y'$, and then add two “dummy vertices $z_1, z_2$ to both $G$ and $F$, such that both $z_1$ and $z_2$ are only adjacent to both $x'$ and $y'$. Let $G'$ be the resulting graph of $G$ and $F'$ be the resulting graph of $F$. Then we just need to figure out whether or not $G'$ has a face-two embedding that extends the embedding of $F'$. This can be done in linear time. See more details in Remark 3 right after Theorem 5.1.} Let $x', y'$ be the vertices of $F'$ that this curve hits.

Phase 2. Once we find such two vertices $x', y'$ from Phase 1, we cut the graph along this curve (i.e., split $x'$ and $y'$ into two copies $x'_1, x'_2$ and $y'_1, y'_2$, respectively, and split the incident edges into the “left” side and the “right” side, such that the “left” side of edges of $x'$ ($y'$, resp.) are only incident with $x'_1$ ($y'_1$, resp.)). See Figure 6. Let us remind the reader that at this moment, we only focus on the orientation-preserving curve.

We add the edges $x'_1y'_1$ and $x'_2y'_2$, and let $G'$ be the modified graph of $G$ after the cutting. If there is a cutvertex in the modified graph $G'$, then it would be a witness for face-width one in the aforementioned embedding (otherwise it would be also a cutvertex in $G$, a contradiction because $G$ is 3-connected). So we can confirm that $G'$ is 2-connected. Hence there are two disjoint paths $P_1, P_2$ between $(x'_1, y'_1)$ and $(x'_2, y'_2)$.
In Figure 2, if we cut the surface with a non-contractible curve hitting only \( a, b \) or \( a', b' \), then we can obtain two disjoint paths obtained by \( P_1, P_2 \).

**Phase 3.** We now apply Theorem 1.5 to \( G' \) to obtain a triconnected component tree decomposition \((T, R)\). Note that the triconnected component tree decomposition is unique by Theorem 1.5. Since \( G \) is 3-connected, it can be shown that for any \( a, b \) or \( a', b' \), then there would be a 2-separation in \( G' \) which would be also a 2-separation of \( G \), a contradiction to the 3-connectivity of \( G \). Note that edges \( x'_1 y'_1 \) and \( x'_2 y'_2 \) are present, so both \( x'_i \) and \( y'_i \) are in the same component for \( i = 1, 2 \).

This indeed implies that \( T \) is a path \( P \) with two endpoints \( a, b \) such that \( R_a \) contains both \( x'_1 \) and \( y'_1 \) and \( R_b \) contains both \( x'_2 \) and \( y'_2 \).

**Phase 4.** Take the vertex \( v \) of \( P \) such that \( \bigcup_{v \in P'} R_t \) induces a cylinder \( T_1 \) with \( x'_1, y'_1 \) in the outer face boundary \( C_1 \) and with \( v_1, v_2 \) in the inner face boundary \( C_2 \), subject to that \( P' \) is as long as possible, where \( P' \) is a subpath of \( P \) between \( a \) and \( v \), and \( v_1, v_2 \in R_a \cap R_{v'} \) with \( vv'' \in E(P) \) and \( v'' \notin P' \). Since \( \bigcup_{v \in P'} R_t \) induces a cylinder \( T_1 \), any non-contractible curve hitting only \( v_1, v_2 \) is in the same homotopy class as \( C' \).

Similarly, we take the vertex \( v' \) of \( P \) such that \( \bigcup_{v' \in P''} R_t \) induces a cylinder \( T_2 \) with \( y'_2, x'_2 \) in the outer face boundary \( C'_1 \) and with \( v'_1, v'_2 \) in the inner face boundary \( C'_2 \), subject to that \( P'' \) is as long as possible, where \( P'' \) is a subpath of \( P \) between \( b \) and \( v' \), and \( v'_1, v'_2 \in R_b \cap R_{v''} \) with \( v'v'' \in E(P) \) and \( v'' \notin P'' \). Again since \( \bigcup_{v' \in P''} R_t \) induces a cylinder \( T_2 \), any non-contractible curve hitting only \( v'_1, v'_2 \) is in the same homotopy class as \( C'' \).

Then the cylinder bounded by \( C_2 \) and \( C'_2 \) (which is union of the cylinders \( T_1 \) and \( T_2 \)) yields a desired pair \((G'_i, L'_i)\), where \( L_i \) is the cylinder.

**Correctness.** We now show that this choice allows us to be canonical; essentially this claim follows from the following two facts:

1. The facts that we took the extremal \( R_v, R_{v'} \), and

2. the triconnected component tree decomposition is unique by Theorem 1.5

It can be shown that if we start with a different non-contractible curve in the same homotopy class (as \( C' \)) that hits exactly two vertices, it is hidden somewhere in the cylinder we constructed, and we would find the same cylinder. This indeed allows us to work on the same graph that can be embedded in a surface.
Figure 7: The non orientation-preserving case

of smaller Euler genus, because for each homotopy class, we obtain the same graph $G_i$. Let us give more intuition from Figure 2. If we start with the curve hitting only $a$ and $b$, we would obtain the cylinder bounded by curves hitting $c, d$ and $e, f$, respectively. This cylinder certainly contains the curve $C'$ hitting $a'$ and $b'$. Even we start with the curve $C'$, we would obtain the same cylinder.

Remark 2. Let us briefly look at the non orientation-preserving case. As in Phase 1, suppose we find one such a non-contractible curve $C'$; let $x', y'$ be the vertices of $F'$ that this curve hits. As in Phase 2, we cut the graph along this curve (i.e., twisting the edges of one part of $x', y'$ by reversing their order in the embedding allows us to split the incident edges into two parts, so that we can define $x'_1, x'_2, y'_1, y'_2$. See Figures 4 and 5). In Phase 2, we obtain two disjoint paths $P_1, P_2$, but in this case, $P_1$ joins $x'_1$ and $y'_1$, and $P_2$ joins $x'_2$ and $y'_2$. See Figure 7. The rest of the arguments is the same. Note that the “cylinder” we shall find corresponds to Figure 8. Namely, we first follow $v_1$ to $v'_2$ along the face $W$, then walk from $v'_2$ to $v'_1$ through the non-contractible curve, then walk from $v'_1$ to $v_2$ through the face $W$, and finally walk from $v_2$ to $v_1$ through the non-contractible curve. Thus we can obtain $L'_i$ which is a planar graph with the outer face $W'$ with four vertices $v_1, v'_2, v'_1, v_2$ appearing in this order listed when we walk along $W$. This finishes Case A.

Case B. $G$ does not have any face-width two embedding, but has an embedding of face-width exactly one.

In this case, we also have a “two steps” solution, as in Case A. As for the first step, we also obtain a set of “skeletons”, as in Case A. Then for the second step, we try to obtain, in $O(n)$ time, the set of vertices $V_1$ of order $q(g)$ (for some function $q$ of $g$) in $O(n)$ time such that for each vertex $c \in V_1$, the following property holds;

there is an embedding of $G$ of face-width exactly one with a non-contractible curve $C$ hitting only $c$.

Moreover, none of the vertices in $G - V_1$ satisfies this property and we are canonical (we will clarify what this means later).

Let us look at the first step. To this end, we need the following result, Theorem 6.1 of Mohar [45] (see Theorem 3.4 later):
In $O(n)$, we can obtain a subgraph $F$ of $G$ that cannot be embedded in a surface of smaller Euler genus, but can be embedded in $S$. Moreover, $F$ is minimal with respect to this property (i.e., any deletion of an edge or a vertex of $F$ results in a graph that is embeddable in a surface of smaller Euler genus), and $\text{bsize}(F) \leq l''(g)$ for some function $l''$ of $g$.

We find all embeddings of $F$ $F'' = \{\hat{F}_1, \ldots, \hat{F}_l\}$ such that each of them can be extended to an embedding of $G$ in $O(n)$ time. This is possible since $l''(g)$ is a fixed constant only depending on $g$ (so $l$ is also a fixed constant only depending on $g$).

Moreover we also show a kind of the converse; For each face-width one embedding of $G$ in $S$, there is an embedding $\hat{F}_i$ of $F$ in $F''$ such that the embedding $\hat{F}_i$ can be extended to the embedding of $G$.

This can be shown by enumerating all the embeddings of $F$ in $S$ (this is possible since, again, $l''(g)$ is a fixed constant only depending on $g$). For more details, see Section 6.

Let us move to the second step. To this end, we first note that a non-contractible curve hitting exactly one vertex must be orientation-preserving (see Lemma 4.2). To find the vertex set $V_1$, here is a crucial observation.

Since the embedding of $F$ in $S$ is already of face-width one, all such vertices $V_1$ are in fact in $F$ (i.e., any non-contractible curve of order one has to hit one vertex of $F$). See Sections 3 and 6 for more details.

In Section 4 (see Lemma 4.1), it is shown that there are at most $f(g)$ homotopy classes to consider. More specifically, we can show that curves from at most $f(g)$ homotopy classes may hit exactly one vertex of $F$. So it remains to find such vertices separately for one fixed homotopy class and for one fixed embedding of $F$.

Our important step is the following; We will show in Lemma 6.2 that if we walk along a face $W$ in the embedding of $F$, there are no four branches $R_1, R_2, R_3, R_4$ appearing in this order listed when we walk along $W$, such that $R_1 = R_3$ and $R_2 = R_4$, i.e, $R_1$ appears twice in $W$ and $R_2$ appears twice in $W$ too. This implies that
we are canonical in the following sense; suppose there is a non-contractible curve $C'$ that hits exactly one vertex $v$ in a branch $P$ of $F$ in an embedding of $G$ that extends the embedding of $F$. Then $C'$ uniquely splits the incidents edges of $v$ into the “left” side and the “right side” (note that the curve $C$ is orientation-preserving).

This allows us to show the following, which will be proved in Lemma 6.3.

The stable bridges, together 3-connectivity of $G$, give $O(1)$ candidates for an intersection point of a non-contractible curve hitting exactly one vertex on every face of the embedding of $F$. Moreover, we are canonical.

This allows us to obtain the set of vertices $V_1$, as above, in $O(n)$ time.

2.2 How do the structural results help?

Our second step is about map isomorphism. Let us first mention that a map is a graph together with a (2-cell) embedding in some surface, and that map isomorphism between two maps is an isomorphism of underlying graphs which preserves the facial walks of the maps. For the map isomorphism problem for graphs embeddable in a surface $S$ of Euler genus at most $g$, we know the following result in [31], which we shall use.

**Theorem 2.1** For every surface $S$ (orientable or non-orientable), there is a linear time algorithm to decide whether or not two embedded graphs in $S$ represent isomorphic maps.

The key of our algorithm for Theorem 1.3 is that the first structural results allow us to reduce the graph isomorphism problem for bounded genus graphs to the map isomorphism problem, which can be done by Theorem 2.1.

2.3 Overview of our graph isomorphism algorithm

We now give an overview of our algorithm for Theorem 1.3. Suppose we want to test the graph isomorphism of two graphs $G_1, G_2$, both admit an embedding in a surface $S$ of Euler genus $g$.

Let us give overview of our algorithm.

**Step 1.** Making both $G_1$ and $G_2$ 3-connected.

Our first step is to reduce both graphs $G_1$ and $G_2$ to be 3-connected. This is quite standard in this literature, see [13, 35], so we omit details, which will be described in Section 7.

**Step 2.** Finding the minimum Euler genus of a surface $S$ for which both $G_1$ and $G_2$ can be embedded.

Our second step is to see if we can embed both $G_1$ and $G_2$ in a fixed surface. This can be done by a result of Mohar [44, 45].

**Theorem 2.2** (Mohar [44, 45]) For fixed $g$, there is a linear time algorithm to give either an embedding of a given graph $G$ in a surface of Euler genus $g$ or a minimal forbidden minor for the surface of Euler genus $g$ in $G$.

Alternatively, we can use a new linear time algorithm by Kawarabayashi, Mohar and Reed [33]. Hereafter, we assume that both $G_1$ and $G_2$ can be embedded in the surface of Euler genus $g$ (otherwise clearly $G_1$ and $G_2$ are not isomorphic).

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4It is trivial to do this in $O(n^2)$ time, as two embeddings are fixed (so we just guess which vertex of one graph can map to which vertex of the other vertex).
In fact, we would like to know the minimum Euler genus of a surface $S$ for which both $G_1$ and $G_2$ can be embedded. This can be done in linear time for fixed $g$, since we know that the upper bound of Euler genus of $G_1$ and of $G_2$ is at most $g$. Hence we just need to apply Theorem 1.2 to both $G_1$ and $G_2$ at most $g$ times. Therefore, after performing Theorem 2.2 at most $O(g)$ times, we may assume that both $G_1$ and $G_2$ can be embedded in the surface $S$ of the minimum Euler genus $g$.

**Step 3.** For $G = G_1, G_2$, if $G$ has a polyhedral embedding (including a planar embedding), then apply Theorem 1.2 to obtain all polyhedral embeddings in $O(n)$ times. Therefore, after performing Theorem 2.2 at most $O(g)$ times, we may assume that both $G_1$ and $G_2$ can be embedded in the surface $S$ of the minimum Euler genus $g$.

**Step 4.** For $G = G_1, G_2$, suppose $G$ does not have any polyhedral embedding, but has an embedding of face-width exactly two. Unfortunately in this case, we cannot enumerate all the embeddings as we did in Step 3, because in contrast with the case when $G$ has a polyhedral embedding, the number of embeddings of face-width exactly two is not quite bounded by a constant.

Instead, in $O(n)$ time, we enumerate at most $q'(g)$ (for some function $q'$ of $g$) different pairs of subgraphs $(G'_i, L'_i), \ldots, (G'_{q'}, L'_{q'}) \in \mathcal{Q}$ of $G$, as in Case A above, such that $G'_i$ can be embedded in a surface of Euler genus at most $g - 1$ and $L'_i$ can be embedded in a plane (since it is a cylinder). Moreover we are canonical, as discussed in Case A.

Then after Step 4, we apply our whole algorithm recursively to each of $G'_i, L'_i$ in the pair $(G'_i, L'_i)$ with "marked" vertices $x_1, y_1, x_2, y_2$ both in $G'_i$ and in $L'_i$. Note that we just need to apply Step 6 to $L'_i$. For more details, see Section 7.

**Step 5.** For $G = G_1, G_2$, if $G$ does not have any face-width two embedding, but has an embedding of face-width exactly one, then in $O(n)$ time, we obtain the set of vertices $V_1$ of order at most $q(g)$ (for some function $q$ of $g$) such that for each vertex $c \in V_1$, there is an embedding of $G$ of face-width exactly one and moreover there is a non-contractible curve $C$ that hits only $c$ in this embedding. Furthermore, there is no such a vertex in $G - V_1$ and we are canonical, as discussed in Case B. We shall show this in Theorem 6.3.

This allows us to create $q \leq q(g)$ different subgraphs $G_1, \ldots, G_q$ of $G$ of Euler genus at most $g - 1$ that can be obtained from $G$ by splitting each vertex of $V_1$ into the "right" side and the "left" side. Let us observe that at Step 5, we now that $C$ must be orientation-preserving (for otherwise, $G$ can be embedded in a surface of smaller Euler genus, see Lemma 4.2 due to Vitray 40.) Then we recursively apply our whole algorithm (from Step 1) to each of these graphs $G_1, \ldots, G_q$ with "marked" vertices in $V_1$.

**Step 6.** Testing graph isomorphism of embedded graphs.

When the current graph comes to Step 6, it comes from Step 3. Thus at the moment, we have either a planar embedding of a 3-connected graph or a polyhedral embedding of a 3-connected graph in some surface.

By Theorem 2.1, we can check map isomorphism of the embedding of some graph $G'_1$ and of the embedding of some other graph $G'_2$ in $O(n)$ time. Note that if $G'_1$ and $G'_2$ are map isomorphic, then $G'_1$ and $G'_2$ are isomorphic.

We shall show that after Step 6, we can, in $O(n)$ time, figure out whether or not $G_1$ and $G_2$ are isomorphic in Section 7.

We now discuss time complexity. Let us observe that in Steps 4 and 5, we create at most $q'(g), q(g)$ different subgraphs of $G_1$ and of $G_2$, respectively, and we recursively apply our whole algorithm again to each of these different subgraphs of $G_1$ and of $G_2$. However, when we recurse, we know that Euler genus of each subgraph already goes down by at least one. Also, note that in Step 3, we create at most $f(g)$ different subgraphs of $G_1$ and of $G_2$, respectively. Since $g$ is a fixed constant and in addition, we recurse
at most \( g \) times, therefore in our recursion process, we create at most \( w(g) \) different subgraphs of \( G_1 \) and of \( G_2 \) in total, for some function \( w \) of \( g \).

In Step 6, we can figure out all pairs of graphs \( (H_1, H'_1), \ldots \) with \( H_i \subseteq G_1 \) and \( H'_i \subseteq G_2 \), where both \( H_i \) and \( H'_i \) are graphs at Step 6, such that \( H_i \) and \( H'_i \) are isomorphic for all \( i \) (with respect to the marked vertices). This can be done in \( O(n) \) time by Theorem 2.1 since we create at most \( w(g) \) subgraphs of \( G_i \) for some function \( w \) of \( g \) in our recursion process (\( i = 1, 2 \)).

For each subgraph of \( G_i \) \( (i = 1, 2) \) in Step 6, we can easily go back to the reverse order of Steps 4 and 5 to come up with the original graphs \( G_1 \) and \( G_2 \) in \( O(n) \) time, because in both Steps 4 and 5, we only “split” a few vertices, and these vertices are all marked. Thus having known all pairs of graphs \( (H_1, H'_1), \ldots \) with \( H_i \subseteq G_1 \) and \( H'_i \subseteq G_2 \) such that \( H_i \) and \( H'_i \) are isomorphic for all \( i \) (with respect to the marked vertices), we can see if \( G_1 \) and \( G_2 \) are isomorphic in \( O(n) \) time.

In summary, we create only constantly many subgraphs in our recursion process. Since all of Steps 1-6 can be done in \( O(n) \) time, so the time complexity is \( O(n) \).

Steps 2, 3 and Step 6 are already described above. So it remains to consider Steps 1, 4 and 5, and the correctness of our algorithm. Some details of Step 1 will be given in Section 7, but this is all standard (see [13, 35]).

The rest of the paper is organized as follows. In Section 3, we give several facts about minimal embeddings of face-width \( k \), which are one key in our proof. In Section 4, we define homology in a surface, which is necessary in our proof. In Section 5, we deal with the case when a given graph has an embedding in a surface \( S \) with face-width exactly two (but does not have an embedding with face-width three). In Section 6, we deal with the case when a given graph has an embedding in a surface \( S \) with face-width exactly one (but does not have an embedding with face-width two). Finally in Section 7, we give several remarks for our algorithm for Theorem 1.3 including the correctness of our algorithm.

3 Minimal embedding of face-width \( k \) and minimal subgraph of face-width \( k \)

Recall that an embedding of a given graph is minimal of face-width \( k \), if it has face-width \( k \), but for each edge \( e \) of \( G \), the face-width of \( G - e \) and of \( G/e \) are both less than \( k \). By Theorems 5.6.1 and 5.4.1 in [47], any minimal embedding of face-width \( k \geq 2 \) has at most \( l'(g, k) \) vertices for some function \( l' \) of \( g, k \) (therefore there are only bounded number of minimal embeddings of face-width \( k \)). Most importantly, a given graph \( G \) has an embedding in the surface \( S \) with face-width at least \( k \) if and only if \( G \) contains a minimal embedding of face-width \( k \) as a surface minor.

Let us now state one result in [31].

**Theorem 3.1** Suppose \( g, l \) are fixed integers. Let \( H \) be a graph of order \( l \) that is embedded in a surface of Euler genus \( g \).

Given a graph \( G \) that has an embedding in \( S \), we can determine in \( O(n) \) time whether or not \( G \) has \( H \) as a surface minor of an embedding of \( G \) in \( S \).

For a completeness of our proof, we give a proof of Theorem 3.1 in the appendix.

We consider a family of minimal embeddings of face-width \( k \). From Theorem 3.1, we can obtain the following.

**Theorem 3.2** Suppose \( G \) can be embedded in a surface \( S \) of Euler genus \( g \) with face-width \( k \geq 2 \) (for fixed \( k \)). We can in \( O(n) \) find a family of graphs \( F = \{ F_1, \ldots, F_l \} \) with the following properties:

1. For all \( i \), the embedding \( H_i \) of \( F_i \) is a minimal embedding of face-width \( k \), and \( F_i \) (with the embedding \( H_i \)) is a surface minor of some embedding of \( G \) in \( S \) of face-width \( k \).
Let Lemma 3.3 contractions, except for the last statement of Lemma 3.3, which will be clarified right after Lemma 3.3.

1. For all embedding of $G$ of face-width at least $k$ in a surface $S$, there is a graph $F_i$ (with its corresponding embedding $II_i$ of face-width $k$) in $F$ such that this embedding of $G$ has $F_i$ (and its embedding $II_i$) as a surface minor.

The next lemma, which we stick to "subgraphs" instead of "minors", is easy to show by reversing the contractions, except for the last statement of Lemma 3.3 which will be clarified right after Lemma 3.3.

Lemma 3.3 Let $G$ be a graph that has an embedding in a surface $S$ of the Euler genus $g$.

Suppose $F = \{F_1, \ldots, F_l\}$ (with their corresponding embeddings $II_1, \ldots, II_l$, respectively) is a set of graphs having all minimal embeddings of face-width $k \geq 2$ (for fixed $k$) as in Theorem 3.2.

Then we can in $O(n)$ time find a family of subgraphs $F' = \{F'_1, \ldots, F'_l\}$ of $G$ such that for all $i$, $F'_i$ is obtained from the surface minor of $F_i$ by reversing the contractions. Moreover the following holds as well:

1. For all $i$, the embedding $II'_i$ of $F'_i$ (of face-width exactly $k$) in $S$ can be extended from $II_i$.
2. $l \leq N(g, k)$ (as in the second item of Theorem 3.2).
3. $\text{bsize}(F'_i) \leq l'(g, k)$ for all $i$, where $l'$ is some function of $g, k$.
4. The embedding $II'_i$ of $F'_i$ can be extended to an embedding of $G$ in $S$, by embedding each $F'_i$-bridge in some face of $F'_i$ (we call this embedding "the embedding of $F'_i$ can be extended to an embedding of $G".)$
5. For any embedding of $G$ of face-width at least $k$ in a surface $S$, there is a subgraph $F'_i$ (with its corresponding embedding $II'_i$ of face-width $k$) in $F'$ such that the embedding $II'_i$ of $F'_i$ can be extended to this embedding of $G$.

Remark: Let us clarify the last point which is the key in the algorithm given in [31]. Indeed, we can show the following.

For any embedding of $G$ in a surface $S$ having $H$ as a surface minor (with $|H| \leq h(g)$ for some function $h$ of $g$), there is a subgraph $H'$ (with its corresponding embedding $II'$ that is obtained from the surface minor of $H$ by reversing the contractions ($II'$ is also obtained from the embedding of the surface minor of $H$ by revising the contractions). Moreover the embedding of $G$ induces the embedding $II'$ of $H'$.

Furthermore, if a surface minor of a graph $H$ is guaranteed to exist in some embedding of $G$, the above subgraph $H'$ (that is obtained from a surface minor of $H$ by reversing the contractions) can be found in $O(n)$ time (so we can find the surface minor $H$ as well), even without knowing the actual embedding of $G$.

The hard part of the above remark is the algorithmic statement (i.e., even without knowing the actual embedding of $G$, we have to find $H'$ and its embedding in $O(n)$ time). Indeed, the rest follows trivially from the definition of the surface minor.

To show our algorithmic claim, we need the following result due to Mohar [44, 45] (see Theorem 6.1 in [45]).

\footnote{We apply this result to the surface of Euler genus exactly $g - 1$. Then we obtain Theorem 3.4.}
Theorem 3.4 Let $G$ be a graph that can be embedded in a surface $S$ of Euler genus $g$, but cannot be embedded in a surface of smaller Euler genus. Then in $O(n)$, we can obtain a subgraph $F$ of $G$ that cannot be embedded in a surface of smaller Euler genus, but can be embedded in $S$. Moreover, $F$ is minimal with respect to this property (i.e., any deletion of an edge or a vertex of $F$ results in a graph that is embeddable in a surface of smaller Euler genus), and $\text{bsize}(F) \leq l''(g)$ for some function $l''$ of $g$.

Let us give a proof of the algorithmic claim. Since the proof is almost identical to that of Theorem 3.2 we just give a sketch here.

Fix the embedding $II$ of the graph $H$, and fix one embedding $II''$ of $F$, where $F$ comes from Theorem 3.4. We first remark that ALL embeddings of $G$ we consider are obtained from some embedding of $F$ by adding all $F$-bridges to some faces of the embedding of $F$.

The main idea is the following: By the standard graph minor argument (finding irrelevant vertices, see [52]), for each embedding of $F$, we can modify $F$ so that the embedding of $F$ (and the branch vertices) is the same, but $F$ is contained in a small tree-width graph. The same is true for any surface minor $H$ we consider. So the proof goes as follows: Fix the endpoints of $F$. Apply the irrelevant vertices argument with respect to the existence of $F$ (and its fixed embedding) and the existence of $H$ (and its fixed embedding). Then we obtain a small tree-width subgraph $G'$ of $G$, but both $F$ and $H$ are contained in $G'$. We can then find both $H$ and $F$ (with their embeddings) in $G'$ in linear time by the standard dynamic programming.

Let us be more precise. We consider which branch vertices of $F$ in the embedding $II''$ can go to which face in the embedding $II$ of a surface minor of $H$. Again there are $u(g)$ ways to enumerate, since $\text{bsize}(F) \leq l''(g)$ and $|H| \leq h(g)$. Let us say a pattern for each way, and enumerate all patterns $P$.

Fix one pattern $P \in P$. As in the proof of Theorem 3.1 we shall try to find a surface minor of $H$ (with the embedding $II$) satisfying this pattern $P$. This can be done in $O(n)$ time by mimicking the proof of Theorem 3.1. Note that the proof for Theorem 3.1 presented in the appendix is the standard way of the graph minor technique, see [53].

Therefore, by examining all the patterns in $P$, we can enumerate all surface minors (of $H$ and its embedding $II$) satisfying some pattern in $P$.

Because $F$ in Theorem 3.4 cannot be embedded in a surface of smaller Euler genus, so any embedding of $G$ in $S$ induces a 2-cell embedding of $F$ in $S$. Hence there is one embedding of $F$ that can be extended to the embedding of $G$, and therefore this pattern in $P$ (with a surface minor of $H$) is covered by our enumerations. Thus for any embedding of $G$ in a surface $S$ that is guaranteed to have $H$ as a surface minor, there is a subgraph $H'$ (with its corresponding embedding $II'$) that is obtained from a surface minor of $H$ by reversing the minor operations (and the embedding $II'$ is also obtained from the embedding $II$ of $H$ by reversing the contractions), and moreover the embedding of $G$ induces the embedding $II'$ of $H'$. Furthermore, even without knowing the actual embedding of $G$, we can find such a subgraph $H'$ and its embedding $II'$ in $O(n)$ time (so we can find the surface minor $H$ as well). This proves our claim for our remark.

Let us mention one algorithmic result that is needed in this paper, see [44] [45].

Theorem 3.5 Let $G$ be a graph and $K$ be a subgraph of $G$. Suppose that $K$ has an embedding $II$ in a surface of Euler genus $g$. Then in $O(n)$, we can test whether or not the embedding $II$ of $K$ can be extended to an embedding of $G$ in $S$. If such an embedding exists, this algorithm can give an embedding of $G$ in $S$ that extends the embedding $II$ of $K$.

In the rest of our proof, given a subgraph $W$ of a 3-connected graph $G$ embedded in a surface $S$, we want all $W$-bridges to be stable. To do that, we need some “local” changes. Let us make it more precise.

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6We need to find a surface minor of $H$ with the embedding $II$ satisfying this pattern $P$. This requires to find a rooted subdivision. This problem is almost the same as the disjoint paths problem, instead of just finding a surface minor of $H$ only. But the proof given in Appendix for Theorem 3.1 works for this problem setting as well.

7An $O(n^2)$ time algorithm is very easy. The difficult part is to get it down to an $O(n^2)$ time algorithm.
Let $P$ be a branch of $W$ of length at least two, and let $Q$ be a path in $G$ with endpoints $x, y \in V(P)$ and otherwise disjoint from $W$. Let $W'$ be obtained from $W$ by replacing the path $xPy$ (the subpath of $P$ with endpoints $x$ and $y$) by $Q$; then we say that $W'$ is obtained from $W$ by rerouting $P$ along $Q$, or simply that $W'$ is obtained from $W$ by rerouting. Note that $P$ is required to have length at least two, and hence this relation is not symmetric. We say that the rerouting is proper if all the attachments of the $W'$-bridge that contains $Q$ belong to $P$. The following lemma is essentially due to Tutte. (For the proof, see [34, 47] for example).

**Lemma 3.6** Let $G, W$ be as above. Note that $G$ is 3-connected. Then there exists a subgraph $W'$ of $G$ obtained from $W$ by a sequence of proper reroutings such that every $W'$-bridge is stable. Moreover, $W'$ is still embedded in a surface $S$.

Note that the last statement trivially follows because proper reroutings only change a branch but do not changes any branch vertex. Note also that we can perform a sequence of proper reroutings in $O(n)$ time such that every $W'$-bridge is stable (see [29]). Hence we obtain the following useful result.

**Lemma 3.7** Let $G$ and the subgraphs $F'_1, \ldots, F'_l$ be as in Lemma 3.3 and suppose that $G$ is 3-connected. In $O(n)$ time, we can modify all the subgraphs $F'_1, \ldots, F'_l$ by a sequence of proper reroutings, so that every $F'_i$-bridge is stable for $i = 1, \ldots, l$.

4 Homotopy and non-contractible curve

In this section, we discuss homotopy on a surface. We now follow the notation in [39]. Let $S$ be a surface. A (closed) curve in $S$ is a continuous mapping from $S^1$ to $S$, where $S^1$ denotes the sphere. The curve is simple if it is a 1-1 mapping. A curve $\gamma$ will usually be identified with its image $\gamma(S^1)$ in $S$, particularly when considering topological properties: simple closed curves on $S$ correspond to subsets of $S$, homeomorphic to the sphere. If $G$ is a graph embedded in $S$ then any cycle in $G$ may also be viewed as a simple closed curve in $S$.

Two-sided simple closed curves are either bounding (i.e., $S \setminus \gamma(S^1)$ has two connected components) or non-bounding ($S \setminus \gamma(S^1)$ is connected). One-sided simple closed curves are always non-bounding. Recall that closed curves $\gamma_0, \gamma_1$ from $S^1$ to a surface $S$ are homotopic if there is a continuous mapping $H : S^1 \times [0, 1]$ to $S$ such that $H(s, 0) = \gamma_0(s)$ and $H(s, 1) = \gamma_1(s)$ for each $s \in S^1$. The mapping $H$ itself is called a homotopy between $\gamma_0$ and $\gamma_1$.

If for some $s_0 \in S^1$, $\gamma_0(s_0) = \gamma_1(s_0) = v_0$, and there is a homotopy $H$ between $\gamma_0$ and $\gamma_1$ such that $H(s_0, t) = v_0$ for all $t \in [0, 1]$ then the two curves are said to be homotopic relative to the point $v_0$. To distinguish these two types of homotopy we sometimes use the name free homotopy for the general case, and homotopy in $(S, v_0)$ for the case of homotopy relative to $v_0$. Homotopy gives rise to the equivalence relation, also termed homotopy, and the corresponding equivalence classes are called homotopy classes. The trivial homotopy class, for instance, is the class of the constant mapping.

**Lemma 4.1** Let $F$ be a subgraph of $G$ that is embedded in $S$ with face-width $l \geq 2$. We can specify at most $\text{bsize}(F)^l$ different nontrivial homotopy classes such that any noncontractible curve hitting exactly $l$ vertices of $G$ (in an embedding of $G$ that extends the embedding of $F$) must lie in one of these homotopy classes.

**Proof.** By the existence of the embedding of $F$, any noncontractible curve $\gamma$ hitting exactly $l$ vertices of $G$ (in an embedding of $G$ that extends the embedding of $F$) must hit exactly $l$ vertices of $F$. Let us consider this curve $\gamma$ in the embedding of $F$. Since each face of $F$ in this embedding bounds a disk (because face-width of this embedding is at least $l \geq 2$, see [47]), we can modify $\gamma$ so that it only hits branch vertices of $F$ (and moreover the resulting curve is homotopic to $\gamma$). There are at most $\text{bsize}(F)^l$ possible choices.
of $l$ branch vertices of $F$. Thus there are also at most $\text{bsize}(F)^l$ different nontrivial homotopy classes such that any noncontractible curve hitting exactly $l$ vertices of $G$ (in an embedding of $G$ that extends the embedding of $F$) must lie in one of these homotopy classes.

\[ \square \]

**Remark:** By the proof of Lemma 4.1, once an embedding of $F$ (of face-width $l$) is given, we can easily find, in $\text{bsize}(F)^l$ time, at most $\text{bsize}(F)^l$ different nontrivial homotopy classes such that any noncontractible curve hitting exactly $l$ vertices of $G$ (in an embedding of $G$ that extends the embedding of $F$) must lie in one of these homotopy classes.

We also give the following lemma which is useful in our proof when a given non-contractible curve is not orientation-preserving.

**Lemma 4.2** Let $G$ be a graph embedded in a non-orientable surface $S$ of the Euler genus $g$. If there is a non-contractible curve $C$ that is not orientation-preserving such that $C$ hits exactly one vertex, then $G$ can be embedded in a surface $S'$ of smaller Euler genus $g'$.

This lemma is essentially due to Vitray [60]. See more details in Robertson and Vitray [55]. The proof goes as follows. Suppose $x$ is the only vertex that is hit by $C$. Then $C$ divides the edges incident with $x$ into two parts. Twisting the edges of one part by reversing their order in the embedding transforms the embedding to an embedding in a surface $S'$ of smaller Euler genus $g'$. Thus the lemma follows.

5 **Face-Width Two Case**

Let $G$ be a 3-connected graph that can be embedded in a surface $S$ of Euler genus $g$. As mentioned in Lemma 1.1 there is a constant $f(g)$ such that every graph $G$ admits at most $f(g)$ polyhedral embeddings. In this lemma, as mentioned before, we have to require the face-width of the embedding to be at least 3 (i.e., polyhedral embedding), since there are 3-connected graphs with exponentially many non-polyhedral embeddings in any surface (other than the sphere). So if $G$ does not have a polyhedral embedding in $S$, then we cannot use Lemma 1.1.

We now restrict our attention to the face-width two embedding case, i.e., a given graph $G$ does not have a polyhedral embedding in a surface $S$ but has a face-width two embedding in $S$. Moreover we assume that $G$ cannot be embedded in a surface $S'$ of smaller Euler genus (hence the Euler genus of $S$ is positive).

By Theorem 3.2, we can in $O(n)$ time find all graphs $F = \{F_1, \ldots, F_l\}$ with their embeddings $II = \{II_1, \ldots, II_l\}$ of face-width two, respectively, with the following properties: each of them is a surface minor of an embedding of $G$, and each embedding of $F$ is a minimal embedding of face-width $k = 2$. Note that some two graphs in $F$ may be isomorphic, but their embeddings are different.

By Lemmas 3.3 and 3.7 (and the remark right after Lemma 3.3), we can in $O(n)$ time obtain a family of subgraphs $F' = \{F'_1, \ldots, F'_l\}$ of $G$ such that the following holds:

1. For all $i$, $F'_i$ is obtained from the surface minor $F_i$ by reversing the minor-operations.
2. For all $i$, each $F'_i$-bridge is stable.
3. For all $i$, $F'_i$ is embedded in $S$ of face-width two, and this embedding is extended from $II_i$.
4. $l \leq N(g)$ for some function $N$ of $g$.
5. $\text{bsize}(F'_i) \leq l'(g)$ for all $i$, where $l'$ is some function of $g$.
6. the embedding of $F'_i$ can be extended to an embedding of $G$ in $S$, by embedding each $F'_i$-bridge in some face of $F'_i$. 


7. For any embedding of $G$ of face-width exactly two in a surface $S$, there is a subgraph $F'_1$ (with its corresponding embedding $II'_1$ of face-width two) in $F'$ such that the embedding $II'_1$ of $F'_1$ can be extended to this embedding of $G$.

We now prove the following main result in this section.

**Theorem 5.1** Let $G, S, F'$ be as above. Suppose that $G$ does not have a face-width three embedding in $S$. Fix one graph $F' \in F'$ with the face-width two embedding in $S$. Fix one homotopy class $H$ of $S$.

Suppose furthermore that if $g = 2$, then $H$ either is surface-separating or hits only one crosscap. In $O(n)$ time, we can find two non-contractible curves $C_1, C_2$ in $H$ that hit exactly two vertices in some embedding of $G$ that extends the embedding of $F'$, with the following properties:

1. There is a cylinder with the outer face $F_1$ and the inner face $F_2$ with the following property: Suppose that $C_i$ hits only $x_i, y_i$ in some embedding of $G$ of face-width two for $i = 1, 2$. Then $x_1, y_1$ are contained in $F_1$ and $x_2, y_2$ are contained in $F_2$. Moreover, $F_i$ is obtained by cutting along $C_i$ for $i = 1, 2$.

2. For any embedding of $G$ that extends the embedding of $F'$, there is no curve $C'$ in $H$ such that $C'$ hits exactly two vertices $u, v$, and at least one of $u, v$ is outside the cylinder.

We note that the two curves $C_1$ and $C_2$ may share a vertex (or even two vertices).

**Remark 1.** The following proof is somewhat complicated and lengthy. In addition, one crucial lemma, which we call “Canonical Lemma” will be shown at the end of Section 6 because it is the most convenient for us to present, first, the proof of “Canonical Claim” in the proof of Lemma 6.3. Intuition behind the proof can be found in Figure 2. What we want is to take a curve for us to present, first, the proof of “Canonical Claim” in the proof of Lemma 6.3. Intuition behind the which we call “Canonical Lemma” will be shown at the end of Section 6, because it is the most convenient

Remark 2. If $g = 2$ and $H$ neither is surface-separating nor hits only one crosscap, then the following proof does not work. This is exactly the case when $x_1 = x_2$ and $y_1 = y_2$ (i.e, the cylinder is “degenerated” in this sense). We need Theorem 5.3.

**Remark 3.** If $H$ is not orientation-preserving, then a curve $\bar{C}$ in $H$ divides the edges incident with any vertex $x$ of $\bar{C}$ into two parts (See Figure 3). When we cut the graph $G$ along $\bar{C}$, we obtain the embedding of $G'$ in a surface $S'$ of smaller Euler genus $g'$ such that the vertices that are hit by $\bar{C}$ can be “splitted” into two vertices, and twisting the edges of one part of the vertex $x$ in $\bar{C}$ by reversing their order in the embedding of $G$ transforms to the embedding of $G'$ in a surface $S'$ of smaller Euler genus $g'$ (See Figures 3, 4 and 5).

As we have discussed “Remark for the non orientation-preserving case” in Overview of our algorithm, we need to change the cylinder. Namely, if we obtain two curves $C_1$ and $C_2$ in $H$ as above, after cutting along $C_1$ and $C_2$, we can obtain a planar graph with the outer face boundary $W$ that contains both the

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*As remarked below, we only have to guess two vertices in $F'$*
vertices in $C_1$ and in $C_2$ (instead of getting a cylinder as above), and moreover $x_1, y_2, x_2, y_1$ appear in this
order listed when we walk along $W$. We also include this case as a “cylinder”. See Figure 8.

\textbf{Proof.} Since this embedding of $F'$ is a face-width two embedding in $S$, if there is a non-contractible
curve $C$ in $H$ that hits exactly two vertices in an embedding of $G$ that extends the embedding of $F'$, then
$C$ must hit two vertices in $F'$.

We first show the following, which is trying to find just one non-contractible curve $\bar{C}$ in $H$:

\textbf{Claim 5.2} In $O(n)$ time, we can find a non-contractible curve $\bar{C}$ in $H$ that hits exactly two vertices in an
embedding $II$ of $G$ that extends the embedding of $F'$, if it exists.

\textbf{Proof.} In the following proof, we do not have to distinguish the "orientation-preserving" case and the
"not orientation-preserving" case for $H$, because we only need to find ONE non-contractible curve in $H$.

For two adjacent faces $W_1, W_2$ of $F'$ and for any two vertices $u, v$ in $V(W_1) \cap V(W_2)$ (but not in the
same component of $W_1 \cap W_2$), we can figure out whether or not there is a non-contractible curve $C'$ in $H$
that hits exactly two vertices $u, v$ in some embedding of $G$ that extends the embedding of $F'$ in $O(n)$ time,
as follows:

We just apply Theorem 3.5 to $K = F' + uw_1v + uw_2v$, the embedding $\bar{K}$ of $K$ and $G = G + uw_1v + uw_2v$, where $+$ means to add two paths $uw_1v$ and $uw_2v$ to $K$ and $G$, respectively
and moreover, $w_1$ must be embedded in $W_1$, while $w_2$ must be embedded in $W_2$, to obtain the
embedding $\bar{K}$ of $K$. See Figure 9.

Note that the cycle $uw_1vw_2u$ is in $H$. Since $\text{bsize}(F') \leq l'(g)$ for some function $l'$ of $g$, it remains to show that, given any two adjacent faces $W_1, W_2$ of $F'$,
in $O(n)$ time, we can find the vertices $u$ and $v$ in $V(W_1) \cap V(W_2)$.

Since $G$ is 3-connected (and since each $F'$-bridge is stable), any vertex in $V(W_1) \cup V(W_2)$, except for the
branch vertices of $F'$, is an attachment of some $F'$-bridge that is stable. It follows that:

(1) Each bridge having an attachment in some component of $W_1 \cap W_2$ must be embedded in $W_1 \cup W_2$.
Moreover, if some bridge $B$ has an attachment in some component of $W_1 \cap W_2$ but also has an attachment
in $W_1$ ($W_2$, resp.) that is not in any component of $W_1 \cap W_2$, then $B$ has to be embedded in $W_1$ ($W_2$, resp.).

Let us consider branches $R_1, \ldots, R_k$ of $W_1 \cap W_2$ for some $k$. Note that these branches are paths (some branch
vertex $v$ could be in $W_1 \cap W_2$ though). Again, since the embedding of $F'$ is a face-with two embedding in $S$ and since each $F'$-bridge is stable, it follows that:
Let us assume that \( W \) is in \( R_1 \) and \( v \) is in \( R_2 \). We now give the following “Canonical Lemma”, which is crucial in the proof of Claim [5.2]. Since we need some tool in Section [6] and in addition, it is the most convenient for us to present, first, the proofs of Lemma [6.2] and “Canonical Claim (1)” in the proof of Claim 5.2. Since we need some tool in Section 6 and in addition, it is the most convenient for us to present, first, the proofs of Lemma [6.3] the proof will be given at the end of Section 6.

**Canonical Lemma.** If there is a non-contractible curve \( C \) that hits only \( u \) and \( v \) in the embedding of \( G \) that extends the embedding of \( F' \), then all \( F' \)-bridges with at least one attachment outside \( R_1 \cup R_2 \) are uniquely placed into the “left” side and the “right side” (or into the “one” part and the “other” part, if \( C \) is non orientation-preserving) of \( C \) in \( W_1 \cup W_2 \). It follows that \( C \) uniquely splits the incidents edges of both \( u \) and \( v \) into the “left” side and the “right side” (if \( C \) is non orientation-preserving, then the “left” side and the “right side” are replaced by the “one” part and the “other” part).

Moreover, given \( W_1,W_2 \), in \( O(n) \) time, either we can place all \( F' \)-bridges \( B \) into the “left” side and the “right side” (or into the “one” part and the “other” part, if \( C \) is non orientation-preserving) of \( C \) in \( W_1 \cup W_2 \), or we can conclude that such a non-contractible curve \( C \) does not exist.

Assume that there is a non-contractible curve \( C \) that hits only \( u \) and \( v \) as in Canonical Lemma. Then we can place all \( F' \)-bridges \( B \) into the “left” side and the “right side” (or into the “one” part and the “other” part, if \( C \) is non orientation-preserving) of \( C \) in \( W_1 \cup W_2 \), in \( O(n) \) time, as claimed in Canonical Lemma. Let \( a_1,b_1 \) be the endvertices of \( R_1 \), and let \( a_2,b_2 \) be the endvertices of \( R_2 \), respectively. So they are branch vertices.

By the canonical lemma and since every vertex of \( R_1 \cup R_2 \) (except possibly for \( a_1,a_2,b_1,b_2 \)) is an attachment of an \( F' \)-bridge, we have the following:

There are at most two vertices \( a',b' \) in \( R_1 \) with \( a' \) closer to \( a_1 \) with the following properties:

- For each vertex \( z' \) between \( a_1 \) and \( a' \) (except for \( a_1,a' \)), there is a \( F' \)-bridge \( M \) that has an attachment between \( a_1 \) and \( a' \) such that \( M \) blocks a non-contractible curve of order exactly two that is homotopic to \( C \) and that contains \( z' \).
- Moreover, for each vertex \( z' \) between \( b_1 \) and \( b' \) (except for \( b_1,b' \)), there is a \( F' \)-bridge \( M \) that has an attachment between \( b_1 \) and \( b' \) such that \( M \) blocks a non-contractible curve of order exactly two that is homotopic to \( C \) and that contains \( z' \).

The same thing also holds for \( R_2 \). Let \( a'',b'' \) be the corresponding vertices of \( a',b' \) in \( R_2 \).

Thus we know that \( u \) must be between \( a' \) and \( b' \) and \( v \) must be between \( a'' \) and \( b'' \). Let \( P' \) be the subpath of \( R_1 \) between \( a' \) and \( b' \), and let \( P'' \) be the subpath of \( R_2 \) between \( a'' \) and \( b'' \). By Canonical Lemma, all \( F' \)-bridges that have an attachment in \( P' \cup P'' \) must have all attachments in \( P' \cup P'' \).

Let \( L \) be a plane graph obtained from \( P' \cup P'' \) together with all \( F' \)-bridges with all attachments in \( P' \cup P'' \). So we have an embedding in the cylinder so that \( P',P'' \) are two disjoint paths from the inner cycle to the outer cycle (we can find such a planar embedding in \( O(n) \) time using any planarity testing algorithm in \( O(n) \) time, say [27]).

We then find desired \( u,v \) in \( O(n) \) time by finding a two vertex cut that separates the inner cycle and the outer cycle of the cylinder. Note that two vertices in the vertex cut must consist of one vertex out of the paths \( P',P'' \), because \( P',P'' \) are two disjoint paths from the inner cycle to the outer cycle. Note also that there may be many choices for \( u,v \) in \( L \), but we only need one choice of \( u,v \). This proves Claim 5.2.

\[ \square \]
We now try to complete our proof of Theorem 5.1 by using a non-contractible curve $C$ obtained in Claim 5.2 in $O(n)$ time.

Suppose first that any curve in $H$ is orientation-preserving. Then any curve $C$ in $H$ tells us which side is “left” of $C$ and “right” of $C$. Suppose next that any curve in $H$ is not orientation-preserving. Then $C$ divides the edges incident with any vertex $x$ of $C$ into two parts (See Figure 3). We now cut the graph $G$ (with the embedding $II$) along $C$ to obtain the embedding $II'$ of $G'$ in a surface $S'$ of smaller Euler genus $g'$ such that the vertices that are hit by $C$ can be “split” into two vertices. If $C$ is orientation-preserving, we split $C$ into two $C_1$ and $C_2$ so that $C_1$ ($C_2$, resp.) has neighbors only in the “left” side of $C$ (right side of $C$, resp.). If $C$ is not orientation-preserving, twisting the edges of one part of any vertex $x$ of $C$ by reversing their order in the embedding $II$ of $G$ transforms to the embedding $II'$ of $G'$ in a surface $S'$ of smaller Euler genus $g'$ (see Figures 3, 4, and 5).

Suppose first that $C$ is orientation-preserving and hits two vertices $x, y$. We now cut the graph $G$ (with the embedding $II$) along $C$ to obtain an embedding of $G'$ in a surface $S'$ of smaller Euler genus $g'$ such that the vertices $x, y$ are “split” into two vertices $x_1, x_2$ and $y_1, y_2$, respectively, and moreover both $x_1$ and $y_1$ ($x_2$ and $y_2$, resp.) have neighbors only in the “left” side of $C$ (right side of $C$, resp.). See Figure 6.

We now add two edge $x_1y_1, x_2y_2$ and let $G'$ be the resulting graph. We first show:

(4) there are two vertex disjoint paths between $x_1, y_1$ and $x_2, y_2$.

Proof. For otherwise, there is a separation $(A', B')$ of order at most one in $G'$ such that $A'$ contains $x_1, y_1$ and $B'$ contains $x_2, y_2$. In this case, $A' \cap B'$ induces a non-contractible curve (in $H$) in some embedding of $G$ in the surface $S$, but this contradicts the fact that the embedding of $G$ is a face-width two embedding. Thus such two vertex disjoint paths exist.

Similarly, we show that

(5) $G'$ is 2-connected.

Proof. For otherwise, there is a separation $(A, B)$ of order at most one in $G'$. By our construction and since $G$ is 3-connected, at least one of $A - B$ and $B - A$ must contain at least two vertices of $x_1, x_2, y_1, y_2$, but this is not possible because of $x_1y_1, x_2y_2 \in E(G')$ and the existences of the two disjoint paths by (4). Thus $G'$ is 2-connected.

Suppose there is a non-trivial separation $(A, B)$ of order exactly two in $G'$ (i.e., $A - B \neq \emptyset$ and $B - A \neq \emptyset$). Since $G$ is 3-connected, so both $A$ and $B$ must contain at least one vertex of $x_1, x_2, y_1, y_2$. If $A$ contains at most one vertex of $x_1, y_1, x_2, y_2$, say $x_1$, then $x_1 \in A \cap B$, because $x_1y_1 \in E(G')$. But then $A \cap B$ is also a 2-separation in $G$, a contradiction. This implies that

(6) for any separation $(A, B)$ of order exactly two in $G'$, $A$ contains all of $x_1, y_1$ or all of $x_2, y_2$. Moreover, the two disjoint paths paths $P_1, P_2$ from $x_1, y_1$ to $x_2, y_2$ imply that $A \cap B$ consists of two vertices with one vertex in $P_1$ and the other in $P_2$.

We now apply Theorem 1.5 to $G'$ to obtain a triconnected component tree decomposition $(T, R)$. The important point here is that this triconnected component tree decomposition is unique by Theorem 1.5.

As shown in (6), for any $tt' \in T$, $R_t \cap R_{t'}$ must contain one vertex in $P_1$ and the other vertex in $P_2$. This indeed implies that $T$ is a path $P$ with two endpoints $a, b$ such that $R_a$ contains both $x_1$ and $y_1$ and $R_b$ contains both $x_2$ and $y_2$. Thus we have the following path decomposition: We have $R_1, R_2, \ldots, R_l$ for some integer $l \geq 1$ such that

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9In Figure 2 if we cut the surface with a non-contractible curve hitting only $a, b$ or $a', b'$, then we can obtain two disjoint paths obtained by $P_1, P_2$. 

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1. $R_1$ contains $x_1, y_1$ and $R_l$ contains $x_2, y_2$.

2. each $R_i$ has no separation $(A, B)$ with $R_{i-1} \cap R_i$ in $A$, $R_{i+1} \cap R_i$ in $B$ and $|A \cap B| = 2$, and

3. $|R_i \cap R_{i+1}| = 2$ for $i = 1, \ldots, l$.

We now try to test the following from $R_1$, and from $R_l$, respectively.

$R_i$ is a cylinder bounded by two cycles $C_i^u, C_i^v$ with $C_i^u$ containing $R_{i-1} \cap R_i$ and $C_i^v$ containing $R_{i+1} \cap R_i$.

This can be done in $O(n)$ time by the planarity testing.

Take the largest $j_1$ that satisfies the above criteria from $R_1$, and take the smallest $j_2$ that satisfies the above criteria from $R_l$. Then it is straightforward to see that $\bigcup_{i=1}^{j_1} R_i$ induces a cylinder $T_1$ with $x_1, y_1$ in the outer face boundary $U_1$ and with $v_1, v_2$ in the inner face boundary $U_2$, where $v_1, v_2 \in R_{j_1} \cap R_{j_1+1}$. Moreover any non-contractible curve hitting only $v_1, v_2$ is in the same homotopy class as $\hat{C}$.

Similarly, $\bigcup_{i=j_2}^l R_i$ induces a cylinder $T_2$ with $y_2, y_2$ in the outer face boundary $U_1'$ and with $v_1', v_2'$ in the inner face boundary $U_2'$, where $v_1', v_2' \in R_{j_2} \cap R_{j_2-1}$. Again any non-contractible curve hitting only $v_1', v_2'$ is in the same homotopy class as $\hat{C}$.

Then the cylinder bounded by $U_2$ and $U_2'$ is $L_i$, and hence we obtain a desired pair $(G_i, L_i)$.

**Canonical issue.** We now show that this choice allows us to be canonical, which shows the second assertion of Theorem 5.1. Let us give intuition from Figure 2. If we start with the curve hitting only $a$ and $b$, we would obtain the cylinder bounded by curves hitting $c, d$ and $e, f$, respectively. This cylinder certainly contains the curve $C'$ hitting $a'$ and $b'$. Even we start with the curve $C'$, we would obtain the same cylinder.

Essentially this canonical claim follows from the following three facts:

1. Canonical Lemma allows us to confirm that all $F_i$-bridges with at least one attachment in $R_1 \cup R_2$ and with at least one attachment outside $R_1 \cup R_2$ are uniquely placed into the “left” side and the “right side” (or into the “one” part and the “other” part, if $C$ is non orientation-preserving) of $C$ in $W_1 \cup W_2$. (see the definitions in the proof of Claim 5.2),

2. we take the extremal $R_{j_1}, R_{j_2}$, and

3. the triconnected component tree decomposition is unique by Theorem 1.5.

The first fact implies that if we can find one non-contractible curve in the same homotopy class (as $\hat{C}$) that hits exactly two vertices, then only flexible $F'$-bridges with at least one attachment in $R_1 \cup R_2$ are the ones with attachments all in $W_1 \cup W_2$. We can then show that if we start with a different non-contractible curve $C''$ in the same homotopy class (as $\hat{C}$) that hits exactly two vertices, it is hidden somewhere in the cylinder we constructed, and we would find the same cylinder. To this end, if $C''$ is contained in $W_1 \cup W_2$, then $C''$ would be in the cylinder we constructed, because we can confirm that all flexible $F'$-bridges are those with all attachments in $R_1 \cup R_2$ by Canonical Lemma, and moreover $C''$ must give rise to a 2-separation in the above proof of Claim 5.2 and hence we would obtain the same cylinder bounded by the same non-contractible curves.

Assume finally that $C''$ is not contained in $W_1 \cup W_2$. Much of the same things happens. If $C''$ is contained in some other faces $W_1', W_2'$, then again by Canonical Lemma, we can confirm that all flexible $F'$-bridges are those with all attachments in $R_1' \cup R_2'$, where $C''$ hit branches $R_1'$ and $R_2'$ that are in the intersection of $W_1'$ and $W_2'$. By our choice of the cylinder, $C''$ must give rise to a 2-separation in the above proof of Claim 5.2 and hence we would obtain two homotopic curves of order two, such that the cylinder bounded by these two curves must contain $C''$. In both cases, $C''$ is hidden somewhere in the cylinder we
constructed, and we would find the same cylinder because the above arguments can apply with $\hat{C}$ replaced by $C''$.

This indeed allows us to work on the same graph that can be embedded in a surface of smaller Euler genus, because for each homotopy class, we obtain the same graph $G_i$.

**Non-orientable case.** Finally, suppose that $\hat{C}$ is not orientation-preserving. Much of the same thing happens. Indeed, “left side” and “right side” can be replaced by “one part” and “the other part” (See Figure 3), and all the same arguments give rise to a cylinder bounded by $C_1$ and $C_2$ in $G$, but the definition of the cylinder is changed as in 5 in “Remark for the non orientation-preserving case” in Overview of our algorithm.

More specifically, suppose we find one such a non-contractible curve $\bar{C}$; let $x, y$ be the vertices of $F'$ that this curve hits. We cut the graph along this curve by twisting the edges of one part of $x, y$ of $\hat{C}$ by reversing their orders in the embedding $\bar{II}$ of $G$, which transforms to the embedding $\bar{II}$ of $G'$ in a surface $S'$ of smaller Euler genus $g'$. This allows us to split the incident edges of $x, y$ into two parts, so that we can define $x_1, x_2, y_1, y_2$. See Figures 3, 4 and 5.

As in (4), we obtain two disjoint paths $P_1, P_2$, but in this case, $P_1$ joins $x_1$ and $y_1$, and $P_2$ joins $x_2$ and $y_2$. See Figure 7. The rest of the arguments is the same. Note that the “cylinder” we shall find corresponds to Figure 8. Namely, we first follow $v_1$ to $v_2$ along the face $W$, then walk from $v_2$ to $v_1$ through the non-contractible curve, then walk from $v_1$ to $v_2$ through the face $W$, and finally walk from $v_2$ to $v_1$ through the non-contractive curve. Thus we can obtain $L'_1$ which is a planar graph with the outer face $W$ with four vertices $v_1, v'_2, v'_1, v_2$ appearing in this order listed when we walk along $W'$.

Thus $C_1$ and $C_2$ are as desired, and we can find $C_1, C_2$ in $O(n)$ time. $\square$

As mentioned in Remark 2 right after Theorem 5.1, the above proof for Theorem 5.1 does not work when $W = V(G)$ and $G'$ is a cylinder with the boundaries $C_1$ and $C_2$ (so $G$ is obtained from $G'$ by gluing $C_1$ and $C_2$). This is exactly the case when the surface $S$ is torus or the Kleinbottle, and moreover, cutting along a curve in $H$ reduces the Euler genus by two (thus when $S$ is the Kleinbottle, $H$ neither is surface-separating nor hits only one crosscap). In this case, we also need the following result.

**Theorem 5.3** Let $G, S, F'$ be as above, where $S$ is either torus or the Kleinbottle. Fix one graph $F' \in F'$ with the face-width two embedding in $S$. Fix one nontrivial homotopy class $H$ of $S$ that neither is a surface-separating nor hits only one crosscap.

Suppose that $G$ has an embedding of face-width exactly two that extends the embedding of $F'$, and that contains a non-contractible curve in $H$ that hits exactly two vertices. In $O(n)$ time, we obtain the unique circular chain decomposition of $G$: $B_1, B_2, \ldots, B_l$ for some integer $l \geq 1$ such that

1. each $B_i$ is a cylinder bounded by two cycles $C_1, C_2$ with $C_1$ containing $B_{i-1} \cap B_i$ and $C_2$ containing $B_{i+1} \cap B_i$,
2. each $B_i$ has no separation $(A, B)$ with $B_{i-1} \cap B_i$ in $A$, $B_{i+1} \cap B_i$ in $B$ and $|A \cap B| = 2$,
3. $|B_i \cap B_{i+1}| = 2$ for $i = 1, \ldots, l$, and
4. if $u \in B_j$ and $u \in B_i$ for $i < j$, then $u \in B_m$ for $m = i, \ldots, j$.

So this circular chain decomposition can be thought of a generalization of a triconnected component tree decomposition $(T, R)$ such that $T$ is a path $P$. If we identify two vertices of $R_a$ and two vertices of $R_b$, where $a, b$ are endpoints of $P$, then we obtain the above circular chain decomposition.

**Proof.** We follow the notation and the proof of Theorem 5.1 (in particular, $\bar{C}, G', x_1, x_2, y_1, y_2$ are as in the proof of Theorem 5.1 after the proof of (4)). Let us observe that (4)-6) in the proof of Theorem 5.1
are still true. Hence in $G'$, there are two disjoint paths $P_1, P_2$ such that $P_1$ ($P_2$, resp.) joins $x_1$ and $x_2$ ($y_1$ and $y_2$) or $x_1$ and $y_2$ ($y_1$ and $x_2$).

Let us add edges $x_1y_1, x_2y_2$ if they are not present in $G'$. Since $G$ is 3-connected, by the existence of two disjoint paths $P_1, P_2$, we have the following chain decomposition: $B_1, B_2, \ldots, B_l$ for some integer $l \geq 1$ such that

1. each $B_i$ is a cylinder bounded by two cycles $C_1, C_2$ with $C_1$ containing $B_{i-1} \cap B_i$ and $C_2$ containing $B_{i+1} \cap B_i$,
2. $B_1$ contains $x_1, y_1$ and $B_l$ contains $x_2, y_2$,
3. each $B_i$ has no separation $(A, B)$ with $(B_{i-1} \cap B_i)$ in $A$, $(B_{i+1} \cap B_i)$ in $B$ and $|A \cap B| = 2$,
4. $|B_i \cap B_{i+1}| = 2$ for $i = 1, \ldots, l-1$ and moreover $B_i \cap B_{i+1}$ consists of one vertex in $P_1$ and the other vertex in $P_2$, and
5. if $u \in B_j$ and $u \in B_i$ for $i < j$, then $u \in B_m$ for $m = i, \ldots, j$.

Note that this is a triconnected component tree decomposition $(T, R)$ such that $T$ is a path. Note also that this decomposition can be found in $O(n)$ time by Theorem 1.5 since $G' \cup \{x_1y_1, x_2y_2\}$ is 2-connected. Moreover this decomposition is unique.

By the uniqueness of the decomposition mentioned above, it follows that by identifying $x_1$ and $x_2$, and $y_1$ and $y_2$, we obtain the unique circular decomposition as in Theorem 5.3. In particular, for any non-contractible curve $C'$ in $H$ that hits exactly two vertices, we can obtain the above unique chain decomposition $B'_1, B'_2, \ldots, B'_l$ such that the two vertices in $C'$ are in one of $B_i \cap B_{i+1}$, and moreover for any $i = 1, \ldots, l-1$, $B'_i \cap B'_{i+1}$ corresponds to $B_{j+i} \cap B_{j+i+1}$ for some $j$. \hfill \Box

6 Face-Width One Case

Let $G$ be a 3-connected graph that can be embedded in a surface $S$ of Euler genus $g > 0$ and of face-width exactly one. In this section, we assume that $G$ can neither be embedded in a surface $S'$ of smaller genus $g'$ nor be embedded in the same surface $S$ with face-width at least two.

We first prove the following lemma.

**Lemma 6.1** Let $F''$ be a set of all embeddings of $F$ (as in Theorem 3.4) in $S$. For each face-width one embedding of $G$ in $S$, there is an embedding of $F$ in $F''$ such that each face in this embedding bounds a disk (with possibly some boundary vertices appearing twice or more), and moreover this embedding can be extended to the embedding of $G$.

**Proof.** Fix one face-width one embedding of $G$ in $S$. This embedding induces an embedding $II$ of $F$ which can be extended to the embedding of $G$. Since $F$ cannot be embedded in a surface of smaller Euler genus, it follows that each face is bounded by a disk of $II$ (with possibly some boundary vertices appearing twice or more). \hfill \Box

By Theorems 3.4 and 3.5 we can find a family of all embeddings of $F$ $F'' = \{\hat{F}_1, \ldots, \hat{F}_l\}$ such that each of them can be extended to an embedding of $G$ in $O(n)$ time, by taking all possible embeddings of $F$ in $S$ and then applying Theorem 3.5 with these embeddings (Note that $\text{bsize}(F) \leq l''(g)$ for some function $l''$ of $g$, so finding all possible embeddings of $F$ in $S$ can be done in constant time).

Therefore, by Lemmas 3.7 and 6.1 we can in $O(n)$ time obtain the following.

1. For all $i$, each $F$-bridge is stable in the embedding $\hat{F}_i$. 

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2. \( l \leq N''(g) \) for some function \( N'' \) of \( g \) (since \( \text{bSize}(F) \leq l''(g) \)).

3. The embedding \( \hat{F}_i \) can be extended to an embedding of \( G \) in \( S \), by embedding each \( F \)-bridge in some face of \( \hat{F}_i \).

4. For each face-width one embedding of \( G \) in \( S \), there is an embedding \( \hat{F}_i \) of \( F \) in \( F'' \) such that each face in this embedding bounds a disk (with possibly some boundary vertices appearing twice or more), and moreover the embedding \( \hat{F} \) can be extended to the embedding of \( G \).

Let us now prove the following simple, but important lemma, which tells us how we get “canonical”.

Lemma 6.2 Let \( \hat{F} \) be an embedding of \( F \) in \( S \). Suppose there are four branches \( R_1, R_2, R_3, R_4 \) appearing in this order listed in a face \( W \) of \( \hat{F} \), when we walk along \( W \), and \( R_1 \) and \( R_3 \) are same and \( R_2 \) and \( R_4 \) are same (See Figures 10 and 11).

If there is a non-contractible curve \( C_1 \) that hits only one vertex \( u \) in \( R_1 \) (and hence in \( R_3 \)) in \( \hat{F} \), then there is no non-contractible curve \( C_2 \), which is not homotopic to \( C_1 \) and which hits only one vertex \( v \) in \( R_2 \) (and hence in \( R_4 \)) in \( \hat{F} \).

Proof. Suppose for a contradiction that such two vertices \( u, v \) exist. We shall show that we can embed \( F \) in a surface of smaller Euler genus. This would be a contradiction to Theorem 3.4.

To this end, let us first remind that both \( C_1 \) and \( C_2 \) are orientation-preserving by Lemma 4.2. So we can split both \( u \) and \( v \) into two vertices \( u_1, u_2 \) and \( v_1, v_2 \), respectively, such that both \( u_1 \) and \( v_1 \) have neighbors (in \( F \)) that are “right-side” of \( C_1, C_2 \), respectively, while both \( u_2 \) and \( v_2 \) have neighbors (in \( F \)) that are “left-side” of \( C_1, C_2 \), respectively. We now add \( u_1u_2, v_1v_2 \).

Note that if we paste \( R_1 \) and \( R_3 \), and \( R_2 \) and \( R_4 \), we obtain a torus. We now delete \( u_1u_2, v_1v_2 \) from the embedding \( \hat{F} \) of \( F \). Then we destroy the torus from the embedding \( \hat{F} \), and let \( \hat{F}' \) be the resulting embedding. But then we can add a single-cross to the embedding \( \hat{F}' \) by adding \( u_1u_2, v_1v_2 \). This implies that the resulting embedding of \( \hat{F}' \) has smaller Euler genus. We can now contract edges \( u_1u_2, v_1v_2 \) to obtain the original graph \( F \). Then the resulting embedding is still in a surface of smaller Euler genus, a contradiction to the assumption on the minimum Euler genus embedding \( \hat{F} \). See Figures 10 and 11. 

We now mention the most important result in this section. The following lemma holds.
Lemma 6.3 Let $G$ be a 3-connected graph that can be embedded in a surface $S$ of Euler genus $g > 0$ and of face-width exactly one, but that can neither be embedded in a surface $S'$ of smaller Euler genus $g'$ nor be embedded in the same surface $S$ with face-width at least two.

Let $F$ be as above. Then $F$ contains all the vertices $V_1$ such that each vertex $u$ in $V_1$ is hit by a non-contractible curve that hits only $u$ in some face-width one embedding of $G$ in $S$. Moreover, we can uniquely split the incident edges of $u$ into the “left” side and the “right side”.

In addition there are at most $q(g)$ such vertices $V_1$ for some function $q(g)$ of $g$, no vertex in $V(G) - V_1$ is hit by a non-contractible curve of order exactly one in any embedding of $G$ in $S$, and we can find all such vertices $V_1$ in $O(n)$ time.

Proof. For each embedding of $G$ of face-width exactly one in $S$, as mentioned above, there is an embedding $\hat{F}_i$ of $F$ in $F''$ that can be extended to this embedding of $G$. Therefore, each non-contractible curve that hits exactly one vertex would hit a vertex in $F$. Thus all the vertices $V_1$ are in $F$.

Fix one embedding $\hat{F}_i$ of $F$ in $F''$. Since $l \leq N''(g)$, it remains to find in $O(n)$ time all the vertices $V_1'$ such that each vertex $u$ in $V_1'$ is hit by a non-contractible curve that hits only $u$ in an embedding of $G$ that extends the embedding $\hat{F}_i$, and moreover we are canonical.

In order to figure out whether or not, in some embedding of $G$ that extends the embedding $\hat{F}_i$, there is a non-contractible curve that hits exactly one vertex $u$ in a face $W$ of $\hat{F}_i$, we can do the following:

By Theorem 3.5 applied to $K = F + uvv'u$, the embedding $\hat{K}$ of $K$ and $G = G + uvv'u$, where $+$ means to add a path of the form $uvv'u$, $u$ is in $W$ and both $v$ and $v'$ are dummy vertices in the face $W$, and moreover $uvv'u$ induces a non-contractible cycle in the embedding $\hat{K}$ of $K$, we can figure out whether or not $u$ can be hit by a non-contractible curve that hits exactly one vertex $u$ in some embedding of $G$ that extends the embedding $\hat{F}_i$. See Figure 12.

It remains to show that there are actually at most $O(1)$ vertices $u$ in $W$ that we have to guess, and moreover we are canonical.

Let us consider one face $W$ of $F$. So $W$ consists of branches $E_1, \ldots, E_t$ (for some $t$ that depends on $g$). Suppose $E_1$ and $E_j$ are same. Then by Lemma 6.2 any of $E_2, \ldots, E_{j-1}$ is a different branch from $E_{j+1}, \ldots, E_t$. This tells one important observation, which follows from Lemma 6.2 and the fact that $F$ cannot be embedded in a surface of smaller Euler genus (by Theorem 3.4):

(1) We are canonical in the following sense; suppose there is a non-contractible curve $C'$ that hits exactly one vertex $v$ in a branch $P$ of $F$ in an embedding of $G$ that extends the embedding of $F$. Then $C'$
uniquely splits the incidents edges of $v$ into the “left” side and the “right side” and hence we can uniquely split $v$ into the “left” side and the “right side” (note that the curve $C$ is orientation-preserving).

Let us remind that since $G$ is 3-connected (and since each $F'$-bridge is stable), any vertex in $W$, except for the branch vertices of $F$, is an attachment of some $F$-bridge. Let $a, b$ be the two branch vertices of $E_1$ (and $E_j$). By (1), each $F$-bridge having an attachment in $E_1$ (and $E_j$) is uniquely placed either with an attachment in the “left side” $E_2, \ldots, E_{j-1}$ or with an attachment in the “right side” $E_{j+1}, \ldots, E_i$ or both. Then there is an edge $e = a'b'$ of $E_1$ (and $E_j$) with $a'$ closer to $a$ in $E_1$ with the following property: For each vertex $z'$ between $a$ and $a'$ (except for $a, a'$), there is a $F'$-bridge $M$ that has an attachment between $a$ and $a'$ such that $M$ blocks a non-contractible curve of order exactly one that is homotopic to $C$ and that contains $z'$. Moreover, for each vertex $z'$ between $b$ and $b'$ (except for $b, b'$), there is a $F'$-bridge $M$ that has an attachment between $b$ and $b'$ such that $M$ blocks a non-contractible curve of order exactly one that is homotopic to $C$ and that contains $z'$.

Since every vertex in $E_1$ (and $E_j$) is an attachment of some $F$-bridge that is stable, the only candidates for $u$ in $E_1$ (and $E_j$) are $a', b'$. Hence there are at most two choices for $u$ as above, and moreover, by (1) we can find such at most two vertices $u$ in $O(n)$ time. Note that by (1), we can uniquely split the incidents edges of $u$ into the “left” side and the “right side”, if such a desired curve $C$ that hits $u$ exists.

As promised Remark 1 right Theorem 5.1 in Section 5, we now show the following “Canonical Lemma”, which is crucial in the proof of Claim 5.2 and hence Theorem 5.1. We follow the proof of Lemma 6.3 together with that of Lemma 6.2 (see (1) in the proof of Lemma 6.3). Let us state it again (we shall follow the notations in the proof of Claim 5.2).

**Canonical Lemma.** If there is a non-contractible curve $C$ that hits only $u$ and $v$ in the embedding of $G$ that extends the embedding of $F'$, then all $F'$-bridges with at least one attachment in $R_1 \cup R_2$ and with at least one attachment outside $R_1 \cup R_2$ are uniquely placed into the “left” side and the “right side” (or into the “one” part and the “other” part, if $C$ is non orientation-preserving) of $C$ in $W_1 \cup W_2$. It follows that $C$ uniquely splits the incidents edges of both $u$ and $v$ into the “left” side and the “right side” (if $C$ is non orientation-preserving, then the “left” side and the “right side” are replaced by the “one” part and the “other” part).
Moreover, given $W_1, W_2$, in $O(n)$ time, either we can place all $F'$-bridges $B$ into the “left” side and the “right side” (or into the “one” part and the “other” part, if $C$ is non orientation-preserving) of $C$ in $W_1 \cup W_2$, or we can conclude that such a non-contractible curve $C$ does not exist.

**Proof.** Let us remind the reader that we are following the proof of Claim 5.2. In particular, we are assuming (1) and (2). Following (2), we now look at the case when a bridge $B$ has attachments only in components of $W_1 \cap W_2$ in the embedding of $F'$. We give the following crucial observation, which is analogue to Lemma 6.2:

(3) Suppose there are three branches $R_1, R_2, R_3$ appearing in this counter clockwise order listed when we walk along $W_1$, and suppose furthermore, there are three branches $R'_1, R'_2, R'_3$ in $W_2$ in this counter clockwise order listed when we walk along $W_2$ (see Figures 13 and 14), such that $R_i$ and $R'_i$ are the same for $i = 1, 2, 3$.

If there is a non-contractible curve $C_1$ that hits only two vertices, one $u$ in $R_1$ (and hence in $R'_1$) and the other $v$ in $R_2$ (and hence in $R'_2$) in some embedding of $G$, then there is no non-contractible curve $C_2$ that is not homotopic to $C_1$ and that hits only two vertices, one $u$ in $R_1$ (and hence in $R'_1$) and the other in $R_3$ (and hence in $R'_3$) in this embedding of $G$.

**Proof.** The proof is also identical to that of Lemma 6.2. For completeness, suppose for a contradiction that such a curve $C_2$ exists. We show that we can embed $G$ in a surface of smaller Euler genus. This can be easily achieved as in Figures 13 and 14. Note that two curves $C_1, C_2$ may be viewed as two non-contractible curves in the torus or in the Kleinbottle, but the new contractible curve is going through only one crosscap.

Then the resulting embedding of $G$ is in a surface of smaller Euler genus, a contradiction to the assumption on the minimum Euler genus embedding of $G$. \[ \square \]

Let us remind (1) in the proof of Lemma 6.3 which essentially follows from the fact that $F$ cannot be embedded in a surface of smaller Euler genus. (3) implies that if $F'$ cannot be embedded in a surface of smaller Euler genus, then Canonical Lemma would follow because there is no branch $R$ (other than $R_1, R_2$) in $W_1 \cup W_2$ such that $R$ appears in both $W_1$ and $W_2$ as in (3), and there is a non-contractible curve of order exactly two in the embedding of $F'$, which hits exactly one vertex in $R_1$ and exactly one vertex in $R$.

However, this is not true; namely $F'$ could be a graph that can be embedded in a surface of smaller Euler genus. For example, we can take $K_4$ in a projective plane. It is known (see Section 5.6 in [4]) that
$K_4$ is the only minimal face-width two embedding in a projective plane (with all faces size four), yet it can be surely embedded in a plane. So we cannot follow the proof of Lemma 6.2.

Instead, we first take $F$ from Theorem 3.4 which is a subgraph of $G$ that cannot be embed in a surface of smaller Euler genus. In addition, as the arguments before Lemma 6.2 we can find a family of all embeddings of $F$, $F'' = \{\tilde{F}_1, \ldots, \tilde{F}_i\}$ such that each of them can be extended to an embedding of $G$ in $O(n)$ time, and such that 1-4 above are satisfied. Let $F'' = F' \cup F$. Since $F$ cannot be embedded in a surface of smaller Euler genus, neither can $F''$. Moreover, applying Lemma 3.7 allows us to confirm that each $F''$-bridge is stable. Since the embedding of $F'$ can be extended to an embedding of $G$, for each $i$ we can define an embedding $\tilde{F}_i'$ of $F''$ in $S$ that is obtained from $\tilde{F}_i$ and the embedding of $F’$. Note that the embedding $\tilde{F}_i'$ is a face-width two embedding, because $F'$ is a subgraph of $F''$.

Let us observe that if there is a non-contractible curve $C''$ in $F'$ that is homotopic to $C$, then $C''$ must hit at least two vertices of $F''$. For each $i$, if there is still a non-contractible curve $C''$ in $F''$ that is homotopic to $C$, that is contained in $W_1 \cup W_2$ of the embedding of $F'$, and that hits exactly two vertices of $F''$, one in $R_1$ and the other in $R_2$, then there must exist a subpath $R_i' \subseteq R_i$ and a subpath $R_i'' \subseteq R_i$ such that both $R_i'$ and $R_i''$ are branches of $F''$, and $C''$ hits exactly one vertex in $R_i'$ and exactly one vertex in $R_i''$.

Since $F''$ cannot be embedded in a surface of smaller Euler genus, there is no closed curve $C_2$ as in (3) in the embedding $\tilde{F}_i'$ of $F''$, which implies that there is no branch $R_3$ (nor branch $R_5$) as in (3). Moreover, no face of $W_1$ appears twice in the embedding $\tilde{F}_i'$ of $F''$ since $\tilde{F}_i'$ is of face-width two (for $i = 1, 2$).

This implies that $F''$-bridges $B'$ that have at least one attachment in $R_1' \cup R_2'$ and at least one attachment in $R_1' \cup R_2'$, are uniquely placed into the “left” side and the “right side” (or into the “one” part and the “other” part, if $C$ is non orientation-preserving) of $C$ in $W_1 \cup W_2$. It follows that $C$ uniquely splits the incidences of both $u$ and $v$ into the “left” side and the “right side” (if $C$ is non orientation-preserving, then the “left” side and the “right side” are replaced by the “one” part and the “other” part). We can also place all $F''$-bridges $B'$ into the “left” side and the “right side” (or into the “one” part and the “other” part, if $C$ is non orientation-preserving) of $C$ in $W_1 \cup W_2$, in $O(n)$ time, by simply looking at each bridge. Note that for this argument, we only need the assumption “there is a non-contractible curve $C''$ that hits exactly two vertices of $F'$, one in $R_1$ and the other in $R_2$”, and no assumption on the vertices $u,v$ is needed. It follows that given $W_1, W_2$, in $O(n)$ time, either we can place all $F''$-bridges $B'$ into the “left” side and the “right side” (or into the “one” part and the “other” part, if $C$ is non orientation-preserving) of $C$ in $W_1 \cup W_2$, or we can conclude that such a non-contractible curve $C$ does not exist.

Since the embeddings of $F''$-bridges $B'$ induce the embeddings of all $F'$-bridges $B$ that have at least one attachment in $R_1 \cup R_2$ and at least one attachment outside $R_1 \cup R_2$, we can also place all $F'$-bridges
moreover, the graph induced by positions $(T, R)$. This way, we make sure that how we glue the component of $T$ does not contain the root. The vertices of $G$ we can define the subtree $T'$ rooted at $t$ is isomorphic to the subtree $T''$ rooted at $t'$, and moreover, the graph induced by $\bigcup_{t \in T''} R_t$ is isomorphic to the graph induced by $\bigcup_{t' \in T''} R_t$ with respect to the rooted vertex (see Theorem 5.8 in [13], or [35]).

Again, by Theorems [1.4 and 1.5], both the biconnected component tree decomposition and the triconnected component tree decomposition are unique. For our convenience, let us assume that both $T$ and $T'$ are rooted.

For each biconnected component $R_t$ of the biconnected component tree decomposition $(T, R)$, we assign colors to the adhesion sets of $R_t \cap R_{t'}$ for each $tt' \in T$. So each adhesion set receives a color and any two different adhesion sets receive different colors. This allows us to define a “rooted tree”, where for each edge $tt' \in T$ where $t$ is closer to the root, we can define the subtree $T''$ of $T$ that takes all nodes of $T$ that are in the component of $T - tt'$ that does not contain the root. The vertex of $G$ corresponding to $R_t \cap R_{t'}$ will tell us which node of $T''$ is the root. Thus the colored vertex can be thought of a way to tell the parent node of any subtree of $T$.

It follows that given two graphs $G_1$ and $G_2$ and their biconnected component tree decompositions $(T_{G_1}, R_{G_1})$ and $(T_{G_2}, R_{G_2})$, $G_1$ and $G_2$ are isomorphic if and only if for each $t \in T_{G_1}$ and its corresponding node $t' \in T_{G_2}$, the subtree $T''_{G_1}$ rooted at $t$ is isomorphic to the subtree $T''_{G_2}$ rooted at $t'$, and moreover, the graph induced by $\bigcup_{t \in T''} R_t$ is isomorphic to the graph induced by $\bigcup_{t' \in T''} R_t$ with respect to the rooted vertex (see Theorem 5.8 in [13], or [35]).

Similarly, for each triconnected component $R_t'$ of the triconnected component tree decomposition $(T', R')$, we assign colors to the adhesion sets of $R_t' \cap R_{t'}$ for $tt' \in T'$. So each adhesion set receives a color and any two different adhesion sets receive different colors. Note that $|R_t' \cap R_{t'}| = 2$ for each $tt' \in T'$. This, again, allows us to define a “rooted tree”, where for each edge $tt' \in T'$ where $t$ is closer to the root, we can define the subtree $T'''$ of $T'$ that takes all nodes of $T'$ that are in the component of $T' - tt'$ that does not contain the root. The vertices of $G$ corresponding to $R_t' \cap R_{t'}$ will tell us which node of $T'''$ is the root. Thus the colored vertices can be thought of a way to tell the parent node of any subtree of $T'$. Let us observe that $\{u, v\} = R_t' \cap R_{t'}$ receive the same color, but there is an “orientation” between $u$ and $v$. This way, we make sure that how we glue $R_t'$ and $R_{t'}$ together at $u, v$. (More precisely, we make sure that $u \in R_t'$ should not map to $v \in R_{t'}$, and $v \in R_{t'}$ should not map to $u \in R_t'$ either).

It follows that given two biconnected graphs $G_1$ and $G_2$ and their triconnected component tree decompositions $(T_{G_1}, R_{G_1})$ and $(T_{G_2}, R_{G_2})$, $G_1$ and $G_2$ are isomorphic if and only if for each $t \in T_{G_1}$ and its corresponding node $t' \in T_{G_2}$, the subtree $T'''_{G_1}$ rooted at $t$ is isomorphic to the subtree $T'''_{G_2}$ rooted at $t'$, and moreover, the graph induced by $\bigcup_{t \in T'''} R_t'$ is isomorphic to the graph induced by $\bigcup_{t' \in T'''} R_t'$ with respect to the rooted vertices (see Theorem 4.2 in [13], or [35]).
Let us observe that both in the biconnected component tree decomposition and in the triconnected component tree decomposition, there are at most \( g \) components that are not planar. This follows from the following fact:

If \( G = G_1 \cup G_2 \) with \(|G_1 \cap G_2| \leq 2\), Euler genus of \( G' \) is that of \( G_1' \) plus that of \( G_2' \), where \( G', G_1', G_2' \) are obtained from \( G, G_1, G_2 \) by adding the edge in \( G_1 \cap G_2 \), if it is not present (see [29]).

Let us first start with biconnected component tree decompositions of \( G_1, G_2 \) respectively. We first group all the “planar” biconnected components into one component \( G_{1,0} \). As remarked above, there are at most \( g \) non-planar biconnected components. Therefore, we can enumerate all biconnected components \( G_{i,j} \) that are not planar for \( j = 1, \ldots, l \leq g \) and for \( i = 1, 2 \).

For each non-planar biconnected component \( G_{i,j} \), we obtain a triconnected component tree decomposition. We then group all the “planar” triconnected components into one set \( G_{i,0}' \). As remarked above, there are at most \( g \) non-planar triconnected components. Therefore, we can enumerate all triconnected components \( G_{i,j}' \) that are not planar for \( j = 1, \ldots, l' \leq g \) and for \( i = 1, 2 \).

We now check, for each \( j \), whether or not \( G_{1,j}' \) and \( G_{2,j}' \) are isomorphic for \( j = 0, 1, \ldots, l' \) (with respect to the colored vertices). For each component of \( G_{i,0}' \), this can be done by Hopcroft and Wong [28]. Note that since the triconnected tree decompositions are the same, we know which components we have to compare. Assume for the moment that we can check isomorphism for \( G_{1,j}' \) and \( G_{2,j}' \) for \( j = 1, \ldots, l' \).

We now glue these graphs \( G_{i,j}' \) together at the colored vertices (with orientation), to obtain the original biconnected graph \( G_{i,k} \), and check whether or not \( G_{1,k} \) and \( G_{2,k} \) are isomorphic. This can be clearly done in \( O(n) \) time, because the abstract trees of the triconnected component tree-decompositions are the same, so we just need to look at the colored vertices.

Similarly, we now glue these graphs \( G_{i,k} \) together at the colored vertices, to obtain the original graph \( G_i \), and check whether or not \( G_1 \) and \( G_2 \) are isomorphic. This can be done in \( O(n) \) time, because, again, the abstract trees of the biconnected component tree-decompositions are the same, so we just need to look at the colored vertices.

Therefore, it remains to consider each “non-planar” 3-connected component of \( G_1 \) and of \( G_2 \), respectively. Hereafter, we may assume that both graphs \( G_1 \) and \( G_2 \) are 3-connected non-planar.

**Step 4.** In Step 4, let \( G, S, F' \) be as in Theorem 5.1. Let us fix one graph \( F_i' \in F' \) and its face-width two embedding. By Theorem 5.1 for any homotopy class \( H \), there is an \( O(n) \) time algorithm to find the cylinder, as in Theorem 5.1. We claim that if we only care about non-contractible curves that hit exactly two vertices in an embedding of \( G \) that extends the embedding of \( F_i' \), we only have to consider at most \( r(g) \) non-contractible curves (for some function \( r \) of \( g \)), each in different nontrivial homotopy classes. To see this, we first need the remark right after Lemma 4.1, i.e., \( \text{size}(F_i') \leq l'(g) \) (for some function \( l' \) of \( g \)). Then by Lemma 4.4, we just need to consider \( r(g) \) nontrivial homotopy classes, as claimed.

Since there are at most \( l \leq N(g) \) subgraphs of \( G \) in \( F' \) and since we only have to consider at most \( r(g) \) different nontrivial homotopy classes for non-contractible curves that hit exactly two vertices in the embedding of a graph in \( F' \) (for some function \( r \) of \( g \)), thus in \( O(n) \) time, we can enumerate the following pairs of subgraphs:

There is a \( q'(g) \) for some function \( q' \) of \( g \) such that

1. there are \( q' \leq q'(g) \) pairs \((G_1', L_1'), \ldots, (G_q', L_q')\),
2. for all \( i \), \( G = G_i' \cup L_i' \) and \(|G_i' \cap L_i'| = 4\),
3. for all \( i \), \( G_i' \) can be embedded in a surface of Euler genus at most \( g - 1 \),
4. pairs \((G_i', L_i')\) are canonical in a sense that graph isomorphism would preserve these pairs,
5. for all \( i \), \( L'_i \) is a cylinder with the outer face \( F_1 \) and the inner face \( F_2 \) with the following property: there is a non-contractible curve \( C_j \) that hits exactly two vertices \( x_j, y_j \) in some embedding of \( G \) of face-width two for \( j = 1, 2 \), and \( x_1, y_1 \) are contained in \( F_1 \) and \( x_2, y_2 \) are contained in \( F_2 \), where \( L'_i, C_1, C_2 \) are obtained from Theorem 5.1 (we do not distinguish between the orientation-preserving case and the non-orientation-preserving case. For the second case, we refer the reader to 5 in "Overview of our algorithm".).

6. an embedding of \( G \) of face-width two in \( S \) can be obtained from some embedding of \( G'_i \) in a surface of Euler genus at most \( g - 1 \) and an embedding of the cylinder \( L'_i \) by identifying respective copies of vertices \( x_1, x_2, y_1 \) and \( y_2 \) (thus \( G'_i \) also contains all the vertices of \( x_1, x_2, y_1, y_2 \) and they are on the border of \( G'_i \) and \( L'_i \), respectively),

7. for any non-contractible curve that hits exactly two vertices \( x, y \) in some embedding of \( G \) of face-width two, both \( x \) and \( y \) are contained in \( L'_i \) for some \( i \), and

8. for \( i_1 \neq i_2 \), either \( L'_{i_1} \) and \( L'_{i_2} \) come from different graphs in \( F' \), or \( L'_{i_1} \) and \( L'_{i_2} \) come from different nontrivial homotopy classes for the embedding of a single graph in \( F' \).

Note that the cylinder includes the case mentioned in Remark 3 right after Theorem 5.1 if the homotopy class is not orientation-preserving.

Thus after Step 4, we apply our whole algorithm recursively to each of \( G'_i, L'_i \) in the pair \((G'_i, L'_i)\) with colored vertices \( x_1, y_1, x_2, y_2 \) both in \( G'_i \) and in \( L'_i \). Note that we just need to apply Step 6 to \( L'_i \).

Note also that we may obtain the unique decomposition by vertex-two cuts as in Theorem 5.3 such that each piece is planar. In this case, we directly go to Step 6.

**Step 5.** In Step 5, we create a set \( G \) of subgraphs of \( G \) which is obtained from \( G \) by splitting each vertex \( v \) of \( V_1 \) into the "right" side and the "left" side, where \( V_1 \) comes from Lemma 6.3 Let us observe that at Step 5, as in Lemma 6.3, the "left" side and the "right" side can be uniquely determined and hence we are canonical. Since \( |V_1| \leq q(g) \), thus \( |G| \leq q(g) \). We apply our whole algorithm recursively to each graph in \( G \) and the colored "splitted" vertex \( v \).

**Time Complexity.** Let us observe that in Steps 4 and 5, we have created at most \( q'(g), q(g) \) subgraphs of \( G \), respectively, and we recursively apply our whole algorithm to each of these different subgraphs. On the other hand, each of these subgraphs can be embedded in a surface of Euler genus at most \( g - 1 \) and hence, we recurse at most \( g \) times. Thus in our recursion process, we create at most \( w(g) \) subgraphs in total (for some function \( w \) of \( g \)). Since all Steps 1-6 can be done in \( O(n) \) time, and since we only deal with constantly many subgraphs in our recursion process, so the time complexity of our algorithm for Theorem 1.3 is \( O(n) \), as claimed.

**Correctness.** It remains to show the correctness of our algorithm. Note that by Step 1, we may assume that the current graph is 3-connected.

Suppose we want to test the graph isomorphism of two 3-connected graphs \( G_1, G_2 \), both admit an embedding in a surface \( S \) of the Euler genus \( g \). We assume that this embedding is a minimum Euler genus embedding, i.e., neither \( G_1 \) nor \( G_2 \) can be embedded in a surface of smaller Euler genus \( g' \).

Suppose one (or both) of \( G_1 \) and \( G_2 \) has a polyhedral embedding in \( S \). As in Step 3, we apply Theorem 1.2 to both \( G_1 \) and \( G_2 \), and hence we obtain all polyhedral embeddings of both \( G_1 \) and \( G_2 \). Then for each of these polyhedral embeddings of \( G_1 \), and for each of these polyhedral embeddings of \( G_2 \), we just apply Theorem 2.1 to figure out whether or not these two embeddings are isomorphism. If there are embeddings of \( G_1 \) and of \( G_2 \) that are isomorphic, then we know that \( G_1 \) and \( G_2 \) are isomorphic. Otherwise, they are not.
Suppose none of $G_1$ and $G_2$ has a polyhedral embedding in $S$. Suppose first that one (or both) of $G_1$ and $G_2$ has a face-width two embedding in $S$. As above, in Step 4, we create $q'$ different pairs of subgraphs $G_1 = (G_{i,1}', L_{i,1}')$, ..., $(G_{q',1}', L_{q',1}')$ of $G_1$, and $q''$ different pairs of subgraphs $G_2 = (G_{1,2}', L_{1,2}'), ..., (G_{q'',2}', L_{q''}', 2)$ of $G_2$.

We claim that if $G_1$ and $G_2$ are isomorphic, we can create a pair $(G_{i,1}', L_{i,1}')$ of $G_1$ as above and a pair $(G_{j,2}', L_{j,2}')$ of $G_2$ as above, such that $G_{i,1}'$ and $G_{j,2}'$ are isomorphic and $L_{i,1}'$ and $L_{j,2}'$ are isomorphic. Let us fix the same embedding of $G_1, G_2$. Then as in Section 5 there is a graph $F_{i,1}' \subseteq F'$ with its embedding that can be extended to the embedding of $G_1$, and there is also a graph $F_{j,2}' \subseteq F'$ with its embedding that can be extended to the embedding of $G_2$. Note that $F_{i,1}'$ may not be the same graph as $F_{j,2}'$. However, we know that any non-contractible curve $C$ that hits exactly two vertices in $G_1$ (and in $G_2$) must hit two vertices of $F_{i,1}'$ (and $F_{j,2}'$). So if $g \neq 2$ or $g = 2$ but $H$ (a nontrivial homotopy class $H$) is as in Theorem 5.3 then by Theorem 5.1 we can find $L_1'$ in $G_1$ and $L_2'$ in $G_2$, respectively, such that every non-contractible curve in $H$ that hits exactly two vertices in $G_1$ and in $G_2$, respectively, is contained in $L_1'$ and $L_2'$, respectively. Moreover, both $L_1'$ and $L_2'$ are bounded by such curves, and both $L_1'$ and $L_2'$ are canonical. Therefore $L_1'$ is isomorphic to $L_2'$, and hence $G_1 - L_1'$ is isomorphic to $G_2 - L_2'$, as claimed.

In summary, we have the following:

If $G_1$ and $G_2$ are isomorphic, then $q' = q''$ and there is one pair of graphs $(G_{i,1}', L_{i,1}')$ in $G_1$ and the other pair of graphs $(G_{j,2}', L_{j,2}')$ in $G_2$ such that $G_{i,1}'$ and $G_{j,2}'$ are isomorphic and $L_{i,1}'$ and $L_{j,2}'$ are isomorphic. If $G_1$ and $G_2$ are not isomorphic, there are no such pair of graphs.

If $g = 2$ and $H$ is as in Theorem 5.3 we obtain the unique decomposition by vertex-two cuts as in Theorem 5.3 such that each piece is planar. Then we go to Step 6.

Suppose finally none of $G_1$ and $G_2$ has a face-width two embedding in $S$. By Lemma 6.3, we obtain the vertex set $V_i'$ in $G_1$ and the vertex set $V_j'$ in $G_2$, where $V_i'$ corresponds to $V_i$ in Lemma 6.3 for $i = 1, 2$. As above, we create subgraphs $G_i$ of $G_i$ which is obtained from $G_i$ by splitting each vertex of $V_i$ into the “right” side and the "left" side. Moreover, we are canonical.

By Lemma 6.3 it is straightforward to see the following;

If $G_1$ and $G_2$ are isomorphic, then $|V'_i| = |V''_i|$, and there is one graph in $G_1$ and the other graph in $G_2$ that are isomorphic. If $G_1$ and $G_2$ are not isomorphic, there are no such two graphs.

Finally, when the current graph comes to Step 6, it comes from either Step 3 or Step 4 (with the unique decomposition by vertex-two cuts as in Theorem 5.3 such that each piece is planar). In the second case, we just need to consider each piece of a 3-connected planar graph. Thus at the moment, we have either a planar embedding of a 3-connected graph or a polyhedral embedding of a 3-connected graph in a surface of Euler genus $g > 0$.

Since we already have all the polyhedral embeddings of $G_1'$ (which is a subgraph of $G_1$) and of $G_2'$ (which is a subgraph of $G_2$), if two graphs $G_1'$ and $G_2'$ are isomorphic at Step 6, there must exist an embedding of $G_1'$ and an embedding of $G_2'$ that represent isomorphic maps (with respect to some colored vertices). By Theorem 2.1 we can check map isomorphism of the embedding of $G_1'$ and the embedding of $G_2'$ in $O(n)$ time, and hence we can check whether or not $G_1'$ and $G_2'$ are isomorphic (with respect to some colored vertices) in $O(n)$ time.

Therefore in Step 6, we can figure out all pairs of subgraphs $(H_1, H_1'), ...$ with $H_i \subseteq G_1$ and $H_i' \subseteq G_2$, where both $H_i$ and $H_i'$ are graphs at Step 6, such that $H_i$ and $H_i'$ are isomorphic (with respect to some colored vertices) for all $i$. This can be done in $O(n)$ time by Theorem 2.1 since we create at most $w(g)$ subgraphs of $G_i$ for some function $w$ of $g$ in our recursion process ($i = 1, 2$).

For each subgraph of $G_i$ ($i = 1, 2$) in Step 6, we can easily go back to the reverse order of Steps 4 and 5 to come up with the original graphs $G_1$ and $G_2$ in $O(n)$ time, because in both Steps 4 and 5, we only “split” a few vertices, and these vertices are all colored so that we can identify two graphs.
Since we are canonical at Steps 4 and 5, thus having known all pairs of graphs \((H_1, H'_1), \ldots, (H_i, H'_i)\) with \(H_i \subseteq G_1\) and \(H'_i \subseteq G_2\) such that \(H_i\) and \(H'_i\) are isomorphic for all \(i\) (with respect to all colored vertices), we can see if \(G_1\) and \(G_2\) are isomorphic in \(O(n)\) time. □

References

[1] D. Archdeacon, Densely embedded graphs, *J. Combin. Theory Ser. B*, **54** (1992), 13–36.

[2] S. Arnborg and A. Proskurowski, Linear time algorithms for NP-hard problems restricted to partial \(k\)-trees, *Discrete Appl. Math.*, **23** (1989), 11–24.

[3] L. Babai, Monte Carlo algorithms in graph isomorphism testing, Univ. Montreal Tech. Rep. DMS 79-10, 1979. [http://people.cs.uchicago.edu/~laci/lasvegas79.pdf](http://people.cs.uchicago.edu/~laci/lasvegas79.pdf)

[4] L. Babai and S. Moran, Arthur - Merlin games: a randomized proof system and a hierarchy of complexity classes, *J. Comp. Syst. Sci.*, **36** (1988), 254–276.

[5] L. Babai, D.Y. Grigoryev and D. Mount, Isomorphism of graphs with bounded eigenvalue multiplicity, *Proceedings of the 14th ACM Symposium on Theory of Computing (STOC’82)*, (1982), 310–324.

[6] L. Babai, P. Erdős and S. Selkow, Random graph \(i\)-isomorphism, *SIAM J. Comput.*, **9**, 628–635, (1980).

[7] L. Babai and L. Kucera, Canonical labelling of graphs in linear average time, *Proceedings of the 20th Annual IEEE Symposium on Foundation of Computing (FOCS’79)*, (1979), 39–46.

[8] H. L. Bodlaender, Polynomial time algorithm for graph isomorphism and chromatic index on partial \(k\)-trees, *J. Algorithm*, **11** 631–643, (1990).

[9] H. L. Bodlaender, A linear-time algorithm for finding tree-decomposition of small treewidth, *SIAM J. Comput.*, **25** (1996) 1305–1317.

[10] R. Boppana, J. Hastad and S. Zachos, Does co-NP have short interactive proofs? *Information Processing Letters*, **25** (1987), 127–132.

[11] C. Chekuri and J. Chuzhoy, Polynomial Bounds for the Grid-Minor Theorem, *Proceedings of the 46th ACM Symposium on Theory of Computing (STOC’82)*, (2014), and [http://arxiv.org/abs/1305.6577](http://arxiv.org/abs/1305.6577) 2013.

[12] W. H. Cunningham and J. Edmonds, A combinatorial decomposition theory, *Canadian J. Math.*, **32** (1980), 734–765.

[13] S. Datta, N. Limaye, P. Nimbhorkar, T. Thierauf, and F. Wagner, Planar graph isomorphism is in log-space, *IEEE Conference on Computational Complexity(CCC’09)* 2009, 203–214.

[14] E. D. Demaine, F. Fomin, M. Hajiaghayi, and D. Thilikos, Subexponential parameterized algorithms on bounded-genus graphs and \(H\)-minor-free graphs, *J. ACM*, **52** (2005), 1–29.

[15] H. Djidjev and J. H. Reif, An efficient algorithm for the genus problem with explicit construction of forbidden subgraphs, *Proceedings of the 23rd ACM Symposium on Theory of Computing (STOC’91)*, (1991), 337–347.

[16] R. Diestel, *Graph Theory*, 2nd Edition, Springer, 2000.

[17] R. Diestel, K. Yu. Gorbunov, T. R. Jensen, and C. Thomassen, Highly connected sets and the excluded grid theorem, *J. Combin. Theory Ser. B*, **75** (1999), 61–73.

[18] L.S. Filotti, G.L. Miller and J. Reif, On determining the genus of a graph in \(O(n^{O(g)})\) steps, *Proceedings of the 11th ACM Symposium on Theory of Computing (STOC’79)*, (1979), 27–37.

[19] L.S. Filotti and J. N. Mayer, A polynomial time algorithm for determining the isomorphism of graphs of fixed genus, *Proceedings of the 12th ACM Symposium on Theory of Computing (STOC’80)*, (1980), 236–243.

[20] O. Goldreich, S. Micali, A. Wigderson, Proofs that yield nothing but their validity or all languages in NP have zero-knowledge proof systems, *J. ACM*, **38** (1991), 691–729.
[21] S. Goldwasser and M. Sipser, Private coins versus public coins in interactive proof systems, *Proceedings of the 18th ACM Symposium on Theory of Computing (STOC’86)*, (1986), 59–68.

[22] M. Grohe, Isomorphism testing for embeddable graphs through definability, *Proceedings of the 32th ACM Symposium on Theory of Computing (STOC’99)*, (1999), 63–72.

[23] M. Grohe and O. Verbitsky, Testing graph isomorphism in parallel by playing a game, *33rd International Colloquium on Automata, Languages and Programming (ICALP)*, (2006), 3–14.

[24] M. Grohe and D. Marx, Structure theorem and isomorphism test for graphs with excluded topological subgraphs, *Proceedings of the 44th ACM Symposium on Theory of Computing (STOC’12)*, (2012), 173–192.

[25] J. E. Hopcroft and R. Tarjan, Dividing a graph into triconnected components, *Siam J. Comput.*, 2 (1973), 135–158.

[26] J. E. Hopcroft and R. Tarjan, Isomorphism of planar graphs (working paper), In R. E. Miller and J. W. Thatcher, editors, *Complexity of Computer Computations*. Plenum Press, 1972.

[27] J. E. Hopcroft and R. Tarjan, A $v \log v$ algorithm for isomorphism of triconnected planar graphs, *J. Comput. System Science*, 7 (1973), 323–331.

[28] J. E. Hopcroft and J. Wong, Linear time algorithm for isomorphism of planar graphs, *Proceedings of the 6th ACM Symposium on Theory of Computing (STOC’74)*, (1974), 27–37.

[29] M. Juvan, J. Marincˇek and B. Mohar, Elimination of local bridges, *Math. Slovaca*, 47 (1997), 85–92.

[30] K. Kawarabayashi and B. Mohar, Some recent progress and applications in graph minor theory, *Graphs Combin.*, 23 (2007) 1–46.

[31] K. Kawarabayashi and B. Mohar, Graph and Map isomorphism and all polyhedral embeddings in linear time, *Proceedings of the 40th ACM Symposium on Theory of Computing (STOC’08)*, (2008), 471–480.

[32] K. Kawarabayashi and Y. Kobayashi, Linear min-max relation between the treewidth of $H$-minor-free graphs and its largest grid minor, *Proceedings of the 29th International Symposium on Theoretical Aspects of Computer Science (STACS’12)*, (2012), 278–289.

[33] K. Kawarabayashi, B. Mohar and B. Reed, A simpler linear time algorithm for embedding graphs into an arbitrary surface, *Proceedings of the 49th Annual Symposium on Foundations of Computer Science (FOCS 2008)*, (2008), 771–780.

[34] K. Kawarabayashi, S. Norine, R. Thomas, P. Wollan, $K_6$ minors in 6-connected graphs of bounded tree-width, [arXiv:1203.2171](http://arxiv.org/abs/1203.2171).

[35] J.P. Kukluk, L.B. Holder, and D.J. Cook, Algorithm and experiments in testing planar graphs for isomorphism, *Journal of Graph Algorithms and Applications*, 8, 2004.

[36] D. Lichtenstein, Isomorphism of graphs embeddable on the projective plane, *Proceedings of the 12th ACM Symposium on Theory of Computing (STOC’80)*, (1980), 218–224.

[37] E. Luks, Isomorphism of graphs of bounded valance can be tested in polynomial time, *J. Comput. System Sciences*, 25 (1982), 42–65.

[38] D. Lokshtanov, M. Pilipczuk, M. Pilipczuk, and S. Saurabh, Fixed-parameter tractable canonization and isomorphism test for graphs of bounded treewidth, to appear in *Proceedings of the 55th Annual Symposium on Foundations of Computer Science (FOCS 2014)*.

[39] A. Malnic and B. Mohar, Generaeting locally acyclic triangulations of surfaces, *J. Combin. Theory Ser. B*, 56 (1992), 157–164.

[40] G. Miller, Isomorphisms of graphs which are pairwise $k$-separable, *Information and Control*, 56 (1983), 21–33.

[41] G. Miller, Isomorphism of $k$-contractible graphs. A generalization of bounded valence and bounded genus graphs, *Information and Control*, 56, (1983), 1–20.

[42] G. Miller, Isomorphism testing for graphs of bounded genus, *Proceedings of the 12th ACM Symposium on Theory of Computing (STOC’80)*, (1980), 225–235.
8 Appendix

In this section, we give a proof of Theorem 3.1 (which was actually given in [31], but we give the proof again). We need some definitions.

A tree decomposition of a graph $G$ is a pair $(T, R)$, where $T$ is a tree and $R$ is a family $\{R_t | t \in V(T)\}$ of vertex sets $R_t \subseteq V(G)$, such that the following two properties hold:

1. For every vertex $x \in V(G)$, there is a node $t \in T$ with $x \in R_t$.
2. For every edge $(x, y) \in E(G)$, there is a node $t \in T$ with $x, y \in R_t$. 

[43] B. Mohar, Uniqueness and minimality of large-face-width embeddings of graphs, *Combinatorica*. 15 (1995), 541–556.

[44] B. Mohar, Embedding graphs in an arbitrary surface in linear time, *Proceedings of the 28th ACM Symposium on Theory of Computing (STOC’96)*, (1996), 392–397.

[45] B. Mohar, A linear time algorithm for embedding graphs in an arbitrary surface, *SIAM J. Discrete Math.*, 12 (1999), 6–26.

[46] B. Mohar and N. Robertson, Flexibility of polyhedral embeddings of graphs in surfaces, *J. Combin. Theory Ser. B*, 83 (2001), 38–57.

[47] B. Mohar and C. Thomassen, *Graphs on Surfaces*, Johns Hopkins University Press, Baltimore, MD, 2001.

[48] I.N. Ponomarenko, The isomorphism problem for classes of graphs that are invariant with respect to contraction, *Zap. Nauchen. Sem. Leningrad. Osdel. Mat. Inst. Steklov. (LOMI)*, 174 (1988), 147–177, in Russian.

[49] I. N. Ponomarenko, The isomorphism problem for classes of graphs, *Dokl. Akad. Nauk SSSR*, 304 (1989) 552–556; Engl. transl. in *Soviet Math. Dokl.*, 39 (1989), 119–122.

[50] B. Reed, Tree width and tangles: a new connectivity measure and some applications, in “Surveys in Combinatorics, 1997 (London)”, London Math. Soc. Lecture Note Ser. 241, Cambridge Univ. Press, Cambridge, 1997, 87–162.

[51] N. Robertson and P. D. Seymour, Graph minors. V. Excluding a planar graph, *J. Combin. Theory Ser. B*, 41 (1986), 92–114.

[52] N. Robertson and P.D. Seymour, Graph minors VII. Disjoint paths on a surface, *J. Combin. Theory Ser. B* 45 (1988) 212–254.

[53] N. Robertson and P. D. Seymour, Graph minors XIII. The disjoint paths problem, *J. Combin. Theory Ser. B*, 63 (1995), 65–110.

[54] N. Robertson, P. D. Seymour and R. Thomas, Quickly excluding a planar graph, *J. Combin. Theory Ser. B*, 62 (1994), 323–348.

[55] N. Robertson, R. P. Vitray, Representativity of surface embeddings, in: “Paths, Flows, and VLSI-Layout,” B. Korte, L. Lovász, H. J. Prömel, and A. Schrijver Eds., Springer-Verlag, Berlin, 1990, pp. 293–328.

[56] U. Schöning, Graph isomorphism is in the low hierarchy, *J. Comput. System Sciences*, 37 (1988), 312–323.

[57] P. D. Seymour and R. Thomas, Uniqueness of highly representative surface embeddings, *J. Graph Theory*, 23 (1996), 337–349.

[58] C. Thomassen, The graph genus problem is NP-complete, *J. Algorithms*, 10 (1988), 458–476.

[59] C. Thomassen, A simpler proof of the excluded minor theorem for higher surfaces. *J. Combin. Theory Ser. B*, 70 (1997), 306–311.

[60] R. Vitray, Representativity and flexibility of drawings of graphs on the projective plane, Ph. D Thesis, 1992, Ohio State University.

[61] H. Weinberg, A simple and efficient algorithm for determining isomorphism of planar triply connected graphs, *Circuit Theory*, 13 (1966), 142-148.
(W1) \( \bigcup_{t \in V(T)} R_t = V(G) \), and every edge of \( G \) has both ends in some \( R_t \).

(W2) If \( t, t', t'' \in V(T) \) and \( t' \) lies on the path in \( T \) between \( t \) and \( t'' \), then \( R_t \cap R_{t''} \subseteq R_{t'} \).

The width of a tree decomposition \((T, R)\) is \( \max\{|R_t| \mid t \in V(T)\} - 1 \), and the tree width of \( G \) is defined as the minimum width taken over all tree decompositions of \( G \). The adhesion of our decomposition \((T, R)\) for \( tt' \in T \) is \( R_t \cap R_{t'} \).

One of the most important results about graphs whose tree-width is large is the existence of a large grid minor or, equivalently, a large wall. Let us recall that an \((r, g)\)-grid minor, then it has an \((r, g)\)-wall. Let us recall that an \((r, g)\)-grid minor, and conversely, if \( G \) has an \((a \times b)\)-grid minor, then it has an \((a \times b)\)-wall.

We can also define an \((a \times b)\)-wall in a natural way, so that an \((a \times b)\)-wall is the same as an \((r \times r)\)-wall. It is easy to see that if \( G \) has an \((a \times b)\)-wall, then it has an \((\lfloor \frac{a}{2} \rfloor \times b)\)-grid minor, and conversely, if \( G \) has an \((a \times b)\)-grid minor, then it has an \((a \times b)\)-wall.

The main result in \cite{51} says the following (see also \cite{17, 32, 50, 54}).

**Theorem 8.1** For every positive integer \( r \), there exists a constant \( f(r) \) such that if a graph \( G \) is of tree-width at least \( f(r) \), then \( G \) contains an \((a \times b)\)-wall.

Very recently, Chekuri and Chuzhoy \cite{11} gives a polynomial upper bound for \( f(r) \). The best known lower bound on \( f(r) \) is of order \( \Theta(r^2 \log r) \), see \cite{54}.

Let \( H \) be an \((a \times b)\)-wall in \( G \). If \( G \) is embedded in a surface \( S \), then we say that the wall \( H \) is flat if the outer cycle of \( H \) bounds a disk in \( S \) and \( H \) is contained in this disk. The following theorem follows from Demaine et al. (Theorem 4.3) \cite{14}, together with Thomassen \cite{59} (see Proposition 7.3.1 in \cite{17}).

**Theorem 8.2** Suppose \( G \) is embedded in a surface with Euler genus \( g \). For any \( l \), if \( G \) is of tree-width at least \( 400l g^{3/2} \), then it contains a flat \( l \)-wall. If there is no flat \( l \)-wall in \( G \), then tree-width of \( G \) is less than \( 400l g^{3/2} \).

Let \( G \) be a graph that can be embedded in a surface \( S \) of Euler genus \( g \) and of face-width \( k \).

The proof of Theorem 3.1 consists of the following two steps.

1. If \( G \) is of tree-width \( w \) for fixed \( w \) (which only depends on \( g, k \)), then we apply the dynamic programming technique of Arnborg and Proskurowski \cite{2} to obtain in \( O(n) \) time the graph \( H \) and its embedding as a surface minor of some embedding of \( G \) in \( S \).

2. On the other hand, if \( G \) is of tree-width at least \( w \), then we keep deleting “irrelevant” vertices in \( G \) to obtain a graph \( G' \) of tree-width at most \( w \) (and moreover, \( G' \) does not have such an irrelevant vertex).

For the second, we need the following result. For the proof, see \cite{33, 47}. Define a vertex \( v \) of \( G \) to be an irrelevant vertex if \( G \) has a surface minor of a minimal embedding of face-width \( k \) if and only if \( G - v \) has. Given a planar graph \( H \), face-distance in \( H \) of two vertices \( x, y \in H \) is the minimal value of \( |H \cap C| \) taken over all curves \( C \) in \( H \) that link \( x \) to \( y \) and that meet \( H \) only in vertices in this embedding of \( H \).
Theorem 8.3 Suppose that $G$ contains a planar subgraph $Q$ and that $C$ is the outer cycle of a planar embedding of $Q$. Suppose also that for every vertex in $Q - C$, all its neighbors in the graph $G$ are contained in $Q$. Then every vertex $v$ of $Q$, which is of face-distance in $Q$ at least $k$ from all the vertices of the outer cycle $C$, is irrelevant.

By Theorem 8.2, if $G$ does not contain a vertex $v$ as in Theorem 8.3, then tree-width of $G$ is less than $400kq^{3/2}$. Thus by setting $w = 400kq^{3/2}$, it remains to show the above two points. The first point will be discussed in Subsection 8.1, while the second point will be discussed in Subsection 8.2.

8.1 Bounded tree-width case

Our algorithm needs to test whether or not a given graph $G$ is of bounded tree-width. This can be done in linear time by the algorithm of Bodlaender [9].

Theorem 8.4 For every fixed $l$, there is a linear time algorithm to determine whether or not a given graph $G$ is of tree-width at most $l$. Moreover, if this is the case, then the algorithm gives a tree-decomposition of tree-width at most $l$.

We need to use some tools from the graph minor theory in [53].

A rooted graph is an undirected graph $G$ with a set $R(G) \subseteq V(G)$ of vertices specified as roots and an injective mapping $\rho_G : R(G) \rightarrow \mathbb{N}$ assigning a distinct positive integer label to each root vertex. Isomorphisms of rooted graphs are defined in the obvious way, i.e., roots must be mapped to roots with the same label.

We say that a rooted graph $H$ is a minor of a rooted graph $G$ if there is a mapping $\phi$ (a model of $H$ in $G$) that assigns to each vertex $v \in V(H)$ a connected subgraph $\phi(v) \subseteq G$ and to each edge $e \in E(H)$ an edge $\phi(e)$ in $G$ such that the following holds:

1. The subgraphs $\phi(v)$ ($v \in V(H)$) are pairwise vertex-disjoint connected subgraphs of $G$.
2. Each edge $\phi(e)$ ($e \in E(H)$) is disjoint from all other edges $\phi(e')$ ($e' \in E(H)$) and intersects $\bigcup_{v \in V(H)} \phi(v)$ only at its endvertices.
3. If $u, v \in V(H)$ are the endpoints of $e \in E(H)$, then $\phi(e)$ is incident in $G$ with a vertex in $\phi(u)$ and with a vertex in $\phi(v)$.
4. For every $v \in R(H)$, $\phi(v)$ contains the vertex $u \in R(G)$ such that $\rho_G(u) = \rho_H(v)$.

The folio of a (rooted) graph $G$ is the set of all rooted minors of $G$. Clearly, the folio is closed under isomorphism, i.e., if rooted graphs $H$ and $H'$ are isomorphic and $H$ is in the folio of $G$, then $H'$ is in the folio as well. Note that there are $2^{\left|H(G)\right|}/2$ possible undirected graphs on $R(G)$. If $\delta$ is an integer, then the $\delta$-folio of $G$ contains every model $H$ of $G$ with $|V(H)| \leq \delta$. Obviously, every graph in the $\delta$-folio has at most $\delta$ vertices.

The folio of a graph $G$ relative to a set $Z \subseteq V(G)$ is the $2^{|Z|}$-folio of the rooted graph $G'$, where $G'$ is isomorphic as unrooted graphs, but $R(G') = Z$.

By a surface folio $\mathcal{F}$ of a rooted graph $G$ that can be embedded in a surface $S$ of Euler genus $g$, we mean that each model $Z_i$ in $\mathcal{F}$ is in a folio of $G$ and moreover, one embedding $II_i$ such that each face is homeomorphic to a disk and each $Z_{ij}$-bridge can be embedded in a face of $II_i$, is associated with $Z_i$. Moreover, the embedding $II_i$ of $Z_i$ can be extended to an embedding of $G$ in $S$. Note that there may be two models $Z_i, Z_j$ in $\mathcal{F}$ that are isomorphic, but their embeddings in $S$ are different.

If $\delta$ is an integer, then the $\delta$-surface-folio of $G$ can be defined in the same way as $\delta$-folio.

It is known that the folio relative to bounded number of vertices can be determined in polynomial time if the tree-width is bounded.
Theorem 8.5 (See [2, 53]) For integers \( w \) and \( l \), there exists a \( (w + l)^{O(w + l)} O(n) \) time algorithm for computing the folio relative to a set of \( l \) vertices in graphs of tree-width \( w \). In particular, if \( w \) and \( l \) are fixed, there exists a linear-time algorithm.

We prove the following analogue of Theorem 8.5 for the surface-folio.

Theorem 8.6 For integers \( w \) and \( l \), there exists a \( w^{O(w)} O(n) \) time algorithm for computing the surface-folio relative to a set of \( l \leq w \) vertices \( Z \) in graphs of tree-width \( w \). In particular, if \( w \) and \( l \) are fixed, there exists a linear-time algorithm.

Proof. Our algorithm follows the standard dynamic programming approach of Arnborg and Proskurowski [2]. So let us give just a sketch. As in [2], we may assume that each degree in \( T \) is at most three.

Given a tree-decomposition \((T, R)\), the dynamic programming approach of Arnborg and Proskurowski [2] assumes that \( T \) is a rooted tree whose edges are directed away from the root. We fix the root node \( t \) and assume that \( Z \) is in \( R_t \).

For \( t_1 t'_1 \in E(T) \) (where \( t_1 \) is closer to the root than \( t'_1 \)), define \( S(t_1, t'_1) = R_{t_1} \cap R_{t'_1} \) and \( G(t_1, t'_1) \) to be the induced subgraph of \( G \) on vertices \( \bigcup R_s \), where the union runs over all nodes of \( T \) that are in the component of \( T - t_1 t'_1 \) that does not contain the root. The algorithm of Arnborg and Proskurowski starts at all the leaves of \( T \) and then we have to compute the following:

For every \( t_1 t'_1 \in E(T) \) (where \( t_1 \) is closer to the root than \( t'_1 \)), we compute the \( w^w \)-surface-folio relative to \( S(t_1, t'_1) \) in \( G(t_1, t'_1) \).

Note that since \( |S(t_1, t'_1)| \leq w \), the size of surface-folio relative to \( S(t_1, t'_1) \) is at most \( w^w \).

If \( t'_1 \) is a leaf, we can compute the \( w^w \)-surface-folio relative to \( S(t_1, t'_1) \) in \( G(t_1, t'_1) \) by a brute force in \( O(w^w) \) time.

We assume we have this information for each child \( t'_2 \) and \( t'_3 \) of \( t'_1 \). A simple brute force solution goes as follows. We try combining the \( w^w \)-surface-folio relative to \( S(t'_1, t'^2_2) \) in \( G(t'_1, t'^2_2) \) and the \( w^w \)-surface-folio relative to \( S(t'_1, t'^3_2) \) in \( G(t'_1, t'^3_2) \), together with each model in the \( w^w \)-surface-folio relative to \( S(t_1, t'_1) \) in \( R_{t_1} \). We can easily check if these three objects are consistent to represent a model in the \( w^w \)-surface-folio relative to \( S(t_1, t'_1) \) in \( G(t_1, t'_1) \). For each model in the \( w^w \)-surface-folio relative to \( S(t_1, t'_1) \) in \( R_{t_1} \), we keep such a solution. The number of combinations to consider is \( w^w \times w^w \times w^w \), and each can be checked in \( O(w^w) \) time, so the total time needed to compute the information for \( t_1 \) is \( O(w^w) \).

When we come to the root \( t \), we can compute the \( w^w \)-surface-folio relative to \( Z \). Since each iteration can be done in \( O(w^w) \) time, thus we can compute the \( w^w \)-surface-folio relative to \( Z \) in \( O(n) \) time. □

8.2 Bounding tree-width

We first give the following result shown in [33].

Theorem 8.7 Let \( G \) be a graph with minimum degree at least 2 with at most \( 4n \) edges. Let \( d > 8g \cdot 2^{16\sqrt{g}} \) and \( \epsilon = d^{-6} \). Then we can find in linear time one of the following:

1. A vertex set \( Z \) of at least 5\( \epsilon n \) vertices of degree 2, each of which has the same pair of neighbors as at least one other vertex in \( Z \).
2. An induced matching \( M \) in \( G \) containing at least \( \epsilon n \) edges.
3. A minor \( G' \) of \( G \) which is a forbidden minor for the surface \( S \) of Euler genus \( g \).
We are given a graph $G$ on a surface $S$ with Euler genus $g$. We want to bound its tree-width by deleting many vertices at once, and our goal is to do this in linear time. Moreover, we want that the deleted vertex set $U$ is irrelevant. Recall that a cycle $C$ in $G$ in a surface $S$ is called flat if $C$ bounds an open disc $D(C)$ in $S$. We say that a vertex $v \in G$ is $k$-nested, if there are $k$ disjoint cycles $C_1, \ldots, C_k$ such that $D(C_k) \supseteq \cdots \supseteq D(C_1)$, and $v$ is contained in the disk $D(C_1)$. Therefore, following Theorem 8.3, a vertex in $G$ is irrelevant if $v$ is $k$-nested in $G$. Let us restate our result here.

**Lemma 8.8** Given a graph $G$ that can be embedded in a surface $S$ with Euler genus $g$, for fixed $g, k$, there is a linear time algorithm to find a vertex set $X \subseteq V(G)$, such that each vertex in $X$ is irrelevant. Moreover, tree-width of the resulting graph $G - X$ is less than $400kg^{3/2}$.

**Proof.** Here is a description. Hereafter, we assume that $G$ has minimum degree at least 2.

**Step 1.** Find a sequence of graphs $G = G_0, G_1, \ldots, G_b$ such that $G_i$ is obtained from $G_{i-1}$ by either contracting an induced matching $M_i$ with at least $\epsilon|G_{i-1}|$ edges for some small but constant $\epsilon > 0$, or deleting a stable set of $\epsilon|G_{i-1}|$ vertices, each of degree 2. In the second case, every deleted vertex has the same neighbors as another vertex of degree 2 in the stable set. In addition, we add an edge between two neighbors of each vertex $x$ in the stable set.

In both cases, the resulting graph $G_i$ is a minor of $G_{i-1}$.

This step can be done as discussed in Theorem 8.7. We may assume that the third output in Theorem 8.7 would not happen.

We keep doing it $b$ steps, where $b$ is minimum integer such that $G_b$ has fewer than $B$ vertices for some absolute constant $B$. Then $b \leq \log_{414} n$ and the sum of the sizes of all $G_i$ is $O(n)$.

At each step $i$, we can either find a desired induced matching or a desired stable set in time $O(|G_i|)$ as explained in Theorem 8.7. Note that since $G$ can be embedded into the surface $\Sigma$ of Euler genus $g$, we never get the third outcome of Theorem 8.7.

**Step 2.** Apply a brute force algorithm to find irrelevant vertices of $G_b$. Since $|G_b| < B$, this can be done in constant time. Let $G'_b$ be the subgraph of $G_b$ obtained from $G_b$ by deleting irrelevant vertices. Since $G'_b$ has no vertex that is $k$-nested, so $G'_b$ has tree-width less than $400kg^{3/2}$ by Theorem 8.2.

We recursively apply Step 3 for $i = b, b-1, \ldots$.

Let $G_{i+1}$ be the graph obtained in the previous iteration. $G'_i$ is a subgraph of $G_{i+1}$ with the following properties:

1. $G'_{i+1}$ is embedded into a surface $\Sigma'$ of Euler genus $g$.
2. $G'_{i+1}$ does not have a vertex that is irrelevant.
3. Each vertex in $V(G_{i+1}) - V(G'_{i+1})$ is irrelevant.

The purpose of Step 3 is to start with $G'_i$, and then to construct a graph $G'_i$ satisfying the above properties for $i$ in $O(|G_i|)$ time. A short computation implies that if we can do it in $O(|G_i|)$ time for each $i$, Step 2 can be done in $O(n)$ time. Note that by the above properties, $G'_i$ is of tree-width at most $400kg^{3/2}$ by Theorem 8.2.

**Step 3.** We shall find a vertex set $X$ that consists of irrelevant vertices in the graph $G''_i$ in time $O(|G_i|)$, where $G''_i$ can be obtained from $G'_{i+1}$ by uncontracting the induced matching, or adding a stable set of $\epsilon|G_i|$ vertices each of degree 2, as in Theorem 8.7. Then output the graph $G'_i = G''_i - X$.

This step is crucial. It consists of several phases. Let us first observe the following:

If a vertex $x$ is irrelevant for $i$, then $x$ is irrelevant for $i' < i$.  

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In order to show this observation, we must prove that after deleting irrelevant vertices in \( G_i \), all the previously deleted vertices are also irrelevant in \( G_i \). We now argue that this is, indeed, true. Suppose not. In this case, we may assume that in the current graph \( G_i \), each of all the previously deleted vertices is \( k \)-nested, but when we delete an irrelevant vertex \( v \) from \( G_i \), there is a vertex \( w \) which was deleted previously, such that \( w \) is not \( k \)-nested in \( G_i - v \). Let \( C_1, \ldots, C_k \) be the \( k \) nested cycles surrounding \( w \) in \( G_i \), and let \( C'_1, \ldots, C'_k \) be the \( k \) nested cycles surrounding \( v \) in \( G_i \). Assume that \( v \) is in one of \( C_1, \ldots, C_k \), say \( C_l \). Let us assume \( l \geq k/2 \), as the other case is identical.

If we cannot reroute \( C_l \) using \( C'_l \), this means that \( C'_l \) hits both \( C_{l-1} \) and \( C_{l+1} \). Inductively, it can be shown that if we cannot reroute \( C_{l-j+1}, \ldots, C_l, \ldots, C_{l+j-1} \) using \( C'_1, \ldots, C'_j \), this means that \( C'_j \) hits both \( C_{l-j} \) and \( C_{l+j} \). However, we can reroute \( C_{2l-1}, \ldots, C_{l}, \ldots, C_1 \) using \( C_1', \ldots, C'_k \). So, there are \( k \) nested cycles in \( G_i - v \) surrounding \( w \), a contradiction. Thus the observation holds.

This observation implies that we only need to consider the graph \( G''_i \) to construct the subgraph \( G'_i \) of \( G_i \).

First, if there is a stable set of \( \epsilon |G_i| \) vertices in \( G_i \), each of degree 2, and \( G_{i+1} \) is obtained from \( G_i \) by deleting this stable set, then, since every vertex in the stable set has the same neighbors as at least one vertex in the stable set, and moreover, the edge in its neighbors is added to \( G_i \), it is easy to see that the resulting graph \( G''_i \) has no vertex that is \( k \)-nested, and hence we are done, as we just output \( G''_i \).

So we may assume that \( G_i \) has an induced matching \( M_i \) of order \( \epsilon |G_i| \). Recall that \( G''_i \) is the graph obtained from \( G'_{i+1} \) by uncontracting the matching \( M_i \) restricted to the graph \( G'_{i+1} \). First, let us observe that tree-width of \( G''_i \) is at most twice of that of \( G'_{i+1} \) (since the uncontraction increases tree-width by factor 2). So it follows that \( G''_i \) is of tree-width \( w \) at most \( 800 k g^{3/2} \). Thus let us keep in mind that we are only working on the tree-width bounded graph \( G''_i \), and we just need to find a desired set \( X \) as in Lemma 8.8 in \( G''_i \).

We now show how to obtain the graph \( G'_i \) from \( G''_i \). Recall that \( G''_i \) is embedded into a surface \( S \) of Euler genus \( g \).

By Theorem 8.4 we can obtain a tree-decomposition \((T, R)\) of \( G''_i \) of width \( w \). As in the proof of Theorem 8.6 we may assume that each degree in \( T \) is at most three. We fix the root note \( t \). Thus \( T \) is a rooted tree. As above, for \( t_1 t'_1 \in E(T) \) (where \( t_1 \) is closer to the root than \( t'_1 \)), define \( S(t_1, t'_1) = R_{t_1} \cap R_{t'_1} \) and \( G(t_1, t'_1) \) to be the induced subgraph of \( G \) on vertices \( \bigcup R_s \), where the union runs over all nodes of \( T \) that are in the component of \( T - t_1 t'_1 \) that does not contain the root.

The main idea in the rest of the proof is the following:

For each \( R_t \) in the tree-decomposition \((T, R)\), if we can compute the \( w^\text{w} \)-surface-folio relative to \( R_t \) in \( G \), then we can find all the irrelevant vertices in \( R_t \).

Indeed, if an irrelevant vertex is contained in one graph \( R_t \), then we can detect it by finding the \( w^\text{w} \)-surface-folio relative to \( R_t \).

So our algorithm will do the following two things simultaneously: constructing the \( w^\text{w} \)-surface-folio relative to \( R_t \) and deleting an irrelevant vertex is contained in \( R_t \).

We are now ready to describe our algorithm here. Because we need to compute the \( w^\text{w} \)-surface-folio relative to \( R_{t'} \) in \( G \) for each \( t' \in T \), thus we need to consider the two phases; working from the leaves, and working from the root.

**Phase 1.** Working from the leaves.

We first work from the leaves of the tree-decomposition. For all the leaves of \( T \), we can find all the irrelevant vertices in constant time, as each leaf has at most \( w \leq 800 k g^{3/2} \) vertices.

Let us look at a node \( t' \in T \). Let \( F_{t_1} \) be the \( w^\text{w} \)-surface-folio relative to \( S(t'_i, t_i) \) in \( G(t'_i, t_i) \) for \( i = 1, 2 \), where \( t_1, t_2 \) are the children of \( t' \). For each model \( F \in F_{t_i} \) and for each model \( F' \in F_{t_2} \), we compute the \( w^\text{w} \)-surface-folio relative to \( S(t''_i, t') \) in \( R_{t'} \cup F \cup F' \), where \( t'' \) is the parent of \( t' \) (if \( t' \) is a leaf of \( T \), then
\(F_t = F_t^1 = F_t^2 = \emptyset\). This can be easily done in \(O(|R_{t'} \cup F \cup F'|^{R_{t'} \cup F \cup F'}(t', t')\) time by a simple brute force. So at this moment, we can compute the \(w^w\)-surface-folio relative to \(S(t', t')\) in \(G(t', t')\). Then we delete all the the irrelevant vertices in \(R_{t'}\) in \(O(|R_{t'} \cup F \cup F'|^{R_{t'} \cup F \cup F'})\) time by again a simple brute force. After deleting the irrelevant vertices in \(R_{t'}\), we update the \(w^w\)-surface-folio relative to \(S(t'', t')\) in \(R_{t'} \cup F \cup F'\), in time \(O(|R_{t'} \cup F \cup F'|^{R_{t'} \cup F \cup F'}(R_{t'} \cup F \cup F')\).

Since \(|F_t|, |F_{t'}|, |R_{t'}|\) are all bounded in terms of \(k, g\), thus in order to compute the \(w^w\)-surface-folio relative to \(S(t'', t')\) and delete all the irrelevant vertices in \(R_{t'}\), it only takes \(O(f_1(k, g)|R_{t'}|)\) time in total for some function \(f_1\) of \(k, g\). Then we look at the parent of \(t''\), and so on.

By doing this procedure, we can reach the root node \(t\) from all the leaves. When we perform this algorithm at the root node \(t\), we can delete all the irrelevant vertices in \(R_t\), because we can compute the \(w^w\)-surface-folio relative to \(R_t\) in \(G\).

In each iteration, the time complexity is \(O(f_1(k, g)|R_{t'}|)\) for each \(t' \in T\). Thus in total, we can do Phase 1 in time \(O(f_1(k, g)n)\), which is linear with respect to \(n\).

This finishes the phase 1. Note that at the moment, we can detect all the irrelevant vertices in the root \(R_0\), but we may not be able to detect all the irrelevant vertices in other nodes \(R_t\). This is because we need the information about the \(w^w\)-surface-folio relative to \(R_t\) in \(G\). So far, for each \(t' \in T\), we only get the information about the \(w^w\)-surface-folio relative to \(R_{t'}\) in \(G(t', t_1)\) and \(G(t', t_2)\) where \(t_1, t_2\) are the children of \(t'\).

**Phase 2.** Working from the root in the resulting graph.

After the first phase, we need to work from the root. Let \(G'\) be the resulting graph from the phase 1, and let \((T, R)\) be the resulting tree-decomposition of \(G'\). Note that this tree-decomposition has still tree-width at most \(w \leq 800k^3/2\).

We now work from the root \(t\) to the leaves of \((T, R)\). For each \(t'\), we need to compute the \(w^w\)-surface-folio relative to \(S(t', t')\) in \((G - G(t', t')) \cup S(t', t')\), where \(t''\) is the parent of \(t'\). As in Phase 1, we are done with the root \(t\). Suppose \(t' \neq t\). Note also that the \(w^w\)-surface-folio \(F_{t'}\) relative to \(S(t', t_1)\) in \(G(t', t_1)\) is already computed by Step 1, where \(t_1\) is the child of \(t''\) with \(t_1 \neq t'\). Suppose we know the \(w^w\)-surface-folio \(F_{t''}\) relative to \(S(t'', t'')\) in \((G - G(t'', t'')) \cup S(t'', t'')\), where \(t''\) is the parent of \(t''\).

For each model \(F \in F_{t_1}\) and for each model \(F' \in F_{t_2}\), we compute the \(w^w\)-surface-folio relative to \(S(t', t')\) in \(R_{t'} \cup F \cup F'\). This can be easily done in \(O(|R_{t'} \cup F \cup F'|^{R_{t'} \cup F \cup F'}(R_{t'} \cup F \cup F')}\) time by a simple brute force. Note that we have already deleted the irrelevant vertices in \(R_{t'}\) because \(t''\) is the parent of \(t'\). We also note that at this moment, together with two \(w^w\)-surface-folios relative to \(R_{t'}\) in \(G(t', t'_1)\) and in \(G(t', t'_2)\) (computed in Phase 1), where \(t'_1, t'_2\) are the children of \(t'\), we can compute the \(w^w\)-surface-folio relative \(R_{t'}\). Then we delete all the the irrelevant vertices in \(R_{t'}\) by using the \(w^w\)-surface-folio relative to \(S(t', t')\) in \(R_{t'} \cup F \cup F'\), together with two \(w^w\)-surface-folios relative to \(R_{t'}\) in \(G(t', t'_1)\) and in \(G(t', t'_2)\) (computed in Phase 1), in \(O(|R_{t'}| + 3w^w)|R_{t'}|^{3w^w})\) time. Since \(|F_{t_1}|, |F_{t_2}|, |R_{t'}|\) are all bounded in terms of \(k, g\), thus in order to compute the \(w^w\)-surface-folio relative to \(S(t', t')\) in \((G - G(t', t')) \cup R_{t'}\), and delete all the irrelevant vertices in \(R_{t'}\), it only takes \(O(f_2(k, g)|R_{t'}|)\) time in total for some function \(f_2\) of \(k, g\). After deleting the irrelevant vertices in \(R_{t'}\), we update the \(w^w\)-surface-folio relative to \(S(t', t')\). This can be also done in \(O(f_2(k, g)|R_{t'}|)\) time, by following the above arguments.

Then we look at the children of \(t'\), and so on.

We keep applying this procedure until we reach all the leaves. Then for each \(t' \in T\), we can find the \(w^w\)-surface-folio relative to \(R_{t'}\) in \(G\), and detect all the irrelevant vertices in all the nodes \(R_{t'}\).

In each iteration of Phase 2, the time complexity is \(O(f_2(k, g)|R_{t'}|)\) for each \(t' \in T\). Thus in total, we can do Phase 2 in time \(O(n \times f_2(k, g))\), which is linear with respect to \(n\).

This completes the description of the algorithm.

\[\square\]

As observed above, all the irrelevant vertices in \(G''_t\) are deleted in the above algorithm.
In summary, we can, in time $O(|G''|)$, find a vertex set $X$ in $G''$ such that each vertex in $X$ is irrelevant in $G''$, and the graph $G'_i = G''_i - X$ has no irrelevant vertex. So $G'_i$ is of tree-width at most $400k^3/2$ by Theorem 8.2, and hence $G'_i$ is as desired.

Thus Step 3 is done and this completes the proof of Lemma 8.8. □