The Entries of a Refinement Equation and a Generalization of the Discrete Wave Equation

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Dedicated to the memory of Richard Askey

Abstract. We start by presenting a generalization of a discrete wave equation that is particularly satisfied by the entries of the matrix coefficients of the refinement equation corresponding to the multiresolution analysis of Alpert. The entries are in fact functions of two discrete variables and they can be expressed in terms of the Legendre polynomials. Next, we generalize these functions to the case of the ultraspherical polynomials and show that these new functions obey two generalized eigenvalue problems in each of the two discrete variables, which constitute a generalized bispectral problem. At the end, we make some connections to other problems.

1. Introduction

Let \( \hat{p}_j(t) \) be the Legendre polynomial of degree \( j \) orthonormal on \([-1, 1]\) with positive leading coefficient, i.e. \( \hat{p}_j(t) = k_j t^j + \text{lower degree terms with } k_j > 0 \) and for any two nonnegative integers \( k \) and \( l \) we have

\[
\int_{-1}^{1} \hat{p}_k(t)\hat{p}_l(t)\,dt = \begin{cases} 
0, & k \neq l; \\
1, & k = l.
\end{cases}
\]

In [8] the Alpert multiresolution analysis was studied in detail and important in this study was the integral

\[
f_{i,j} = \int_{0}^{1} \hat{p}_i(t)\hat{p}_j(2t - 1)\,dt,
\]

where \( \hat{p}_i \) is the orthonormal Legendre polynomial. These coefficients are entries in the refinement equation associated with this multiresolution analysis. The fact that the Legendre polynomials are involved in the above integral allowed the authors in [8] to obtian many types of recurrence formulas in \( i \) and \( j \) including a generalized eigenvalue problem in each of the indices. These two equations together give rise to a bispectral generalized eigenvalue problem. Here we consider the above integral where the Legendre polynomials are replaced by the ultraspherical polynomials \( \hat{p}_{i}^{(\lambda)} \) which allows the introduction of the parameter \( \lambda \) i.e.

\[
f_{i,j}^{(\lambda)} = \int_{0}^{1} \hat{p}_{i}^{(\lambda)}(t)\hat{p}_{j}^{(\lambda)}(2t - 1)(t(1-t))^{\lambda-1/2}\,dt.
\]

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In section 2 a vast generalization of the above integral is shown to give rise to a 2D wave equation and solutions to the special case of the above integral are plotted to show the oscillations. In section 3 the Legendre case above is analyzed and various properties of the coefficients \( f_{i,j} \) are derived. One point of this section is to derive the orthogonality property of these coefficients using that they come from special functions. In section 4 the Legendre polynomials are replaced by the ultraspherical polynomials and their scaled weight. While this allows the introduction of the parameter \( \lambda \) the connection to multiresolution analysis is lost due to the weight. Here it is shown that the coefficients \( f^{(\lambda)}_{i,j} \) satisfy a wave equation and also a bispectral generalized eigenvalue problem. Two proofs are given developing the generalized eigenvalue problem. One is based on the fact that the polynomials satisfy a Sturm-Liouville differential equation and the second follows from the formula for \( f^{(\lambda)}_{i,j} \) in terms of a \( 2F1 \) hypergeometric function. In section 5 connections are made to various other problems.

2. The 2D discrete wave equation

Let \( \{ P_n \}_{n=0}^\infty \) and \( \{ Q_n \}_{n=0}^\infty \) be two families of orthonormal polynomials or, equivalently, two families that obey the three-term recurrence relations

\[
a_{n+1}P_{n+1}(t) + b_nP_n(t) + a_nP_{n-1}(t) = tP_n(t), \quad n = 0, 1, 2, \ldots
\]

and

\[
c_{n+1}Q_{n+1}(t) + d_nQ_n(t) + c_nQ_{n-1}(t) = tQ_n(t), \quad n = 0, 1, 2, \ldots,
\]

where the coefficients \( a_n \) and \( c_n \) are positive and the coefficients \( b_n \) and \( d_n \) are real. If we assume that \( a_0 = c_0 = 0 \) then the initial conditions take the form

\[ P_0 = 1, \quad Q_0 = 1. \]

It should be stressed here that by imposing these particular initial conditions we implicitly assume that the corresponding orthogonality measures are probability measures.

In addition, suppose we are given a measure \( \sigma \) on \( \mathbb{R} \) with finite moments. Then, let us consider the coefficients

\[
u_{i,j} = \int_{\mathbb{R}} P_i(t)Q_j(\alpha t + \beta) d\sigma(t),
\]

where \( \alpha \neq 0 \) and \( \beta \) are complex numbers. It turns out that these coefficients constitute a solution of a generalized wave equation on the two dimensional lattice.

**Theorem 2.1.** We have that

\[
a_{i+1}u_{i+1,j} + b_iu_{i,j} + a_iu_{i-1,j} = \frac{c_{j+1}}{\alpha} u_{i,j+1} + \frac{d_j - \beta}{\alpha} u_{i,j} + \frac{c_j}{\alpha} u_{i,j-1}
\]

for \( i, j = 0, 1, 2, \ldots. \)

**Proof.** Let us introduce the following partial difference operators on the lattice

\[
\partial_{ii}^2 u_{i,j} := a_{i+1}u_{i+1,j} + b_iu_{i,j} + a_iu_{i-1,j}
\]

and

\[
\Delta_{jj}^2 u_{i,j} := c_{j+1}u_{i,j+1} + d_ju_{i,j} + c_ju_{i,j-1}.
\]
Then we have that
\[
\partial_{ii}^2 u_{i,j} = \int_{\mathbb{R}} \partial_{ii}^2 P_i(t)Q_j(\alpha t + \beta) d\sigma(t) = \int_{\mathbb{R}} t P_i(t)Q_j(\alpha t + \beta) d\sigma(t)
\]
\[
= \frac{1}{\alpha} \int_{\mathbb{R}} P_i(t)(\alpha t + \beta)Q_j(\alpha t + \beta) d\sigma(t) - \frac{\beta}{\alpha} \int_{\mathbb{R}} P_i(t)Q_j(\alpha t + \beta) d\sigma(t)
\]
\[
= \frac{1}{\alpha} \int_{\mathbb{R}} P_i(t)\Delta_{jj}^2(Q_j(\alpha t + \beta)) d\sigma(t) - \frac{\beta}{\alpha} u_{i,j}
\]
\[
= \frac{1}{\alpha} \partial_{jj}^2 u_{i,j} - \frac{\beta}{\alpha} u_{i,j}.
\]
Therefore, we have arrived at the discrete wave equation on the lattice
\[
(2.3) \quad \partial_{ii}^2 u_{i,j} = \frac{1}{\alpha} \partial_{jj}^2 u_{i,j} - \frac{\beta}{\alpha} u_{i,j},
\]
which can be easily transformed into (2.2).

**Remark 2.2.** If we are given an equation of the form (2.2) then due to the Favard theorem the coefficients will uniquely determine the families \(\{P_n\}_{n=0}^{\infty}\) and \(\{Q_n\}_{n=0}^{\infty}\) of orthonormal polynomials. The measure \(\sigma\) is responsible for the initial state when \(j = 0\) and \(j\) can be thought of as a discrete time. Namely, for a solution of the form (2.1) to exist we need to satisfy the initial condition
\[
u_{i,0} = \int_{\mathbb{R}} P_i(t) d\sigma(t),
\]
which means that given initial function \(\nu_{i,0}\) of the discrete space variable \(i\), we need to find \(\sigma\). The latter problem is equivalent to a moment problem.

It is also worth mentioning here that another type of cross-difference equations on \(Z^2\) was recently discussed in [3] and the construction was based on multiple orthogonal polynomials. Type I Legendre-Angelesco multiple orthogonal polynomials also arise in the wavelet construction proposed by Alpert [7].

Next, we are going to consider a particular case of the above scheme. So, let \(P_n\) and \(Q_n\) be the same family of the orthonormal Legendre polynomials \(\hat{p}_n\), which verify the three-term recurrence relation
\[
\frac{(n+1)}{\sqrt{(2n+1)(2n+3)}} \hat{p}_{n+1}(t) + \frac{n}{\sqrt{(2n-1)(2n+1)}} \hat{p}_{n-1}(t) = t \hat{p}_n(t),
\]
where \(n = 0, 1, 2, \ldots\). Also set \(\sigma\) to be the Lebesgue measure on the interval \([0, 1]\). As a result, the coefficients (2.1) take the form
\[
(2.4) \quad f_{i,j} = \int_0^1 \hat{p}_i(t)\hat{p}_j(2t-1) dt.
\]

It is not so hard to see that the polynomials \(\hat{p}_j(2t-1)\) are orthogonal on the interval \([0, 1]\) with respect to the Lebesgue measure, consequently
\[
(2.5) \quad f_{i,j} = 0, \quad j > i = 0, 1, 2, \ldots.
\]
Since the coefficients of the three-term recurrence relation for the Legendre polynomials are explicitly known, the coefficients of equation (2.2) become explicit as well. The following Corollary can be found in [8].
Corollary 2.3. The function $f_{i,j}$ satisfies,

\[
\frac{j + 1}{\sqrt{(2j + 1)(2j + 3)}} f_{i,j+1} + f_{i,j} + \frac{j}{\sqrt{(2j - 1)(2j + 1)}} f_{i,j-1} = \frac{2(i + 1)}{\sqrt{(2i + 1)(2i + 3)}} f_{i+1,j} + \frac{2i}{\sqrt{(2i - 1)(2i + 1)}} f_{i-1,j}
\]

for $i, j = 0, 1, 2, \ldots$.

Below is the MATLAB generated graphical representation of some behavior of the solution $f_{i,j}$ to equation (2.6), which is a generalization of the discretized wave equation.

Figure 1. This picture demonstrates the moving wave. Here, one can see two graphs of the function $f = f(i) = f_{i,j}$ of the discrete space variable $i$ at the two different discrete times $j = 15$ and $j = 20$.

To sum up, we would like to point out here that the form (2.1) of solutions of the discrete wave equations is very useful for understanding the behavior of solutions because there are many asymptotic results for a variety of families of orthogonal polynomials.

3. Some further analysis of the coefficients $f_{i,j}$

In this section, we will obtain some properties of the coefficients $f_{i,j}$ based on the intuition and observations developed in [8]. In particular, we will rederive and expand upon some orthogonality properties of the coefficients $f_{i,j}$.

We begin with the following statement, which is based on formula (2.4) and some known properties of the Legendre polynomials.
Theorem 3.1. Let \(k\) and \(l\) be two nonnegative integer numbers. Then one has
\[
\sum_{j=0}^{\infty} f_{k,j} f_{l,j} = \begin{cases} 
0, & \text{if } k \text{ and } l \text{ are both even or both odd;} \\
1, & \text{if } k = l; \\
\frac{\sqrt{2k+1} \cdot \sqrt{2l+1}}{2^{k+l-1} (k-l) (k+l+1) \left(\frac{k}{2}\right)! \left(\frac{l}{2}\right)!} \left(\frac{2}{k+1}\right)^2, & \text{if } k \text{ is even and } l \text{ is odd.}
\end{cases}
\]

Proof. Without loss of generality, we can assume that \(k \leq l\). Next observe that due to (2.5) the left-hand side of formula (3.1) is truncated to,
\[
\sum_{j=0}^{\infty} f_{k,j} f_{l,j} = \sum_{j=0}^{k} f_{k,j} f_{l,j},
\]
which can be written as
\[
\sum_{j=0}^{k} f_{k,j} f_{l,j} = \sum_{j=0}^{k} \int_{0}^{1} \hat{p}_k(x) \hat{p}_j(2x - 1) \, dx \int_{0}^{1} \hat{p}_l(y) \hat{p}_j(2y - 1) \, dy.
\]
One can rewrite the expression in the following manner
\[
\sum_{j=0}^{k} f_{k,j} f_{l,j} = \int_{0}^{1} \hat{p}_l(y) \left( \int_{0}^{1} \hat{p}_k(x) \sum_{j=0}^{k} \hat{p}_j(2x - 1) \hat{p}_j(2y - 1) \, dx \right) \, dy.
\]
Since the Christoffel-Darboux kernel
\[
2 \sum_{j=0}^{k} \hat{p}_j(2x - 1) \hat{p}_j(2y - 1)
\]
is a reproducing kernel, we get
\[
\sum_{j=0}^{k} f_{k,j} f_{l,j} = 2 \int_{0}^{1} \hat{p}_k(y) \hat{p}_l(y) \, dy.
\]
Next recall that one can explicitly compute the quantity
\[
\int_{0}^{1} \hat{p}_k(y) \hat{p}_l(y) \, dy
\]
for any nonnegative integers \(k\) and \(l\). If \(k\) and \(l\) have the same parity the symmetry properties of the Legendre polynomials allow the above integral to be extended to the full orthonality interval \([-1,1]\] which gives the first two parts of the Theorem. The third case of formula (3.1) is a consequence of \([4, p.173, Art. 91, Example 2]\).

One can also compute the inner product of vectors \(f_{i,j}\) taken the other way.

Theorem 3.2. Let \(k\) and \(l\) be two nonnegative integer numbers. Then one has
\[
\sum_{i=0}^{\infty} f_{i,k} f_{i,l} = \begin{cases} 
0, & \text{if } k \neq l; \\
1/2, & \text{if } k = l.
\end{cases}
\]

Proof. Let \(n\) be a nonnegative integer. Then we can write
\[
\sum_{i=0}^{n} f_{i,k} f_{i,l} = \sum_{j=0}^{n} \int_{0}^{1} \hat{p}_i(x) \hat{p}_k(2x - 1) \, dx \int_{0}^{1} \hat{p}_i(y) \hat{p}_l(2y - 1) \, dy,
\]
which can be rewritten as follows
\[ \sum_{i=0}^{n} f_{i,k} f_{i,l} = \int_{-1}^{1} \hat{p}_k(2x - 1)\chi_{[0,1]}(x) \left( \sum_{i=0}^{n} \left( \int_{-1}^{1} \hat{p}_i(2y - 1)\chi_{[0,1]}(y)\hat{p}_i(y)dy \right) \hat{p}_i(x) \right) dx. \]

Since the polynomials \( \hat{p}_i \) form an orthonormal basis in \( L^2([-1, 1], dt) \) we know that
\[ \sum_{i=0}^{n} \left( \int_{-1}^{1} \hat{p}_i(2y - 1)\chi_{[0,1]}(y)\hat{p}_i(y)dy \right) \hat{p}_i(x) \xrightarrow{L^2([-1,1], dt)} \hat{p}_l(2x - 1)\chi_{[0,1]}(x) \]
as \( n \to \infty \). As a result we arrive at the following relation
\[ \sum_{i=0}^{\infty} f_{i,k} f_{i,l} = \int_{-1}^{1} \hat{p}_k(2x - 1)\chi_{[0,1]}(x)\hat{p}_l(2x - 1)\chi_{[0,1]}(x) dx \]
\[ = \int_{-1}^{1} \hat{p}_k(2x - 1)\hat{p}_l(2x - 1) dx = \frac{1}{2} \int_{-1}^{1} \hat{p}_k(t)\hat{p}_l(t) dt, \]
which finally gives (3.2). □

As a consequence we can say a bit more about the asymptotic behavior of the coefficients \( f_{i,j} \).

**Corollary 3.3.** Let \( k \) be a fixed nonnegative integer number. Then
\[ f_{i,k} \to 0 \]
as \( i \to \infty \).

**Proof.** The statement immediately follows from the fact that the series
\[ \sum_{i=0}^{\infty} f_{i,k}^2 \]
converges. □

**Remark 3.4.** From (3.2) one gets that
\[ \sum_{i=0}^{\infty} f_{i,j}^2 = 1/2 \]
for any nonnegative \( j \). This means that the energy of the wave represented by \( f = f(i) = f_{i,j} \) is conserved over the discrete time \( j \).

**Remark 3.5.** The fact that \( f_{i,k} \) can be represented as a hypergeometric function allows a more precise asymptotic estimate; see formula (4.21).

To conclude this section we would like to mention a few relations for the coefficients \( f_{i,j} \) that were obtained in [8]. The first of the relations is a three-term recurrence relation in \( i \) and, therefore, it resembles the Sturm-Liouville problem that we usually get when solving the wave equation by separation of variables.

**Proposition 3.6 ([8]).** Let \( j \) be a fixed nonnegative integer number. Then the function \( f = f(i) = f_{i,j} \) of the discrete variable \( i \) satisfies the generalized eigenvalue
problem

\begin{equation}
\frac{i(i+1)(i+2)}{\sqrt{(2i+3)(2i+1)}} f_{i+1,j} + \frac{(i-1)(i+1)}{\sqrt{(2i+1)(2i-1)}} f_{i-1,j} =
\end{equation}

\[ = j(j+1) \left( \frac{i}{\sqrt{(2i+3)(2i+1)}} f_{i+1,j} + f_{i,j} + \frac{i+1}{\sqrt{(2i+1)(2i-1)}} f_{i-1,j} \right) \]

for \( i = 0, 1, 2, \ldots \).

The other one is a three-term recurrence relation in \( j \) and plays a similar role as \( (3.3) \) but in the discrete variable \( j \).

**Proposition 3.7** ([8]). Let \( i \) be a fixed nonnegative integer number. Then the function \( f = f(j) = f_{i,j} \) of the discrete variable \( j \) satisfies the generalized eigenvalue problem

\begin{equation}
\frac{j(j+1)(j+2)}{\sqrt{(2j+3)(2j+1)}} f_{i,j+1} + 3j(j+1)f_{i,j} + \frac{(j-1)(j)(j+1)}{\sqrt{(2j+1)(2j-1)}} f_{i,j-1} =
\end{equation}

\[ = i(i+1) \left( \frac{j}{\sqrt{(2j+3)(2j+1)}} f_{i+1,j} + f_{i,j} + \frac{j+1}{\sqrt{(2j+1)(2j-1)}} f_{i,j-1} \right) \]

for \( j = 0, 1, 2, \ldots \).

In what follows we will generalize \( (3.3) \) and \( (3.4) \) to the case of the ultraspherical polynomials and for one of those will employ an approach different from the one used in [8].

**4. The case of ultraspherical polynomials**

In this section we will carry over our findings from the case of the Legendre polynomials to the case of the family of the ultraspherical polynomials which include the Legendre polynomials as a special case.

Recall that for \( \lambda > -1/2 \) an ultraspherical polynomial \( \hat{p}_n^{(\lambda)}(t) \) is a polynomial of degree \( n \) that is the orthonormal polynomial with respect to the measure

\[ (1 - t^2)^{\lambda-1/2} dt. \]

In an analogous way to \( f_{i,j} \), let us consider the function of the discrete variables \( i \) and \( j \)

\begin{equation}
f_{i,j}^{(\lambda)} = \int_0^1 \hat{p}_i^{(\lambda)}(t) \hat{p}_j^{(\lambda)}(2t-1)(t(1-t))^{\lambda-1/2} dt
\end{equation}

and notice that

\[ f_{i,j} = f_{i,j}^{(1/2)}. \]

Also, it is worth mentioning that the polynomials \( \hat{p}_j^{(\lambda)}(2t-1) \) are orthogonal with respect to the measure

\[ (t(1-t))^{\lambda-1/2} dt. \]

Next since the ultraspherical polynomials satisfy the three-term recurrence relation

\[ \frac{1}{2} \sqrt{\frac{(n+1)(n+2\lambda)}{(n+\lambda)(n+\lambda+1)}} \hat{p}_{n+1}^{(\lambda)}(t) + \frac{1}{2} \sqrt{\frac{n(n+2\lambda-1)}{(n+\lambda-1)(n+\lambda)}} \hat{p}_{n-1}^{(\lambda)}(t) = t \hat{p}_n^{(\lambda)}(t) \]
the following corollary of Theorem 2.1 is immediate.

**Corollary 4.1.** The function \( f^{(\lambda)}_{i,j} \) satisfies

\[
\frac{1}{2} \sqrt{\frac{(j+1)(j+2\lambda)}{(j+\lambda)(j+\lambda+1)}} f^{(\lambda)}_{i,j+1} + \frac{1}{2} \sqrt{\frac{j(j+2\lambda-1)}{(j+\lambda-1)(j+\lambda)}} f^{(\lambda)}_{i,j} = \frac{1}{2} \sqrt{\frac{(i+1)(i+2\lambda)}{(i+\lambda)(i+\lambda+1)}} f^{(\lambda)}_{i+1,j} + \frac{1}{2} \sqrt{\frac{i(i+2\lambda-1)}{(i+\lambda-1)(i+\lambda)}} f^{(\lambda)}_{i-1,j}
\]

for \( i, j = 0, 1, 2, \ldots \).

As one can see from the above statement, the function \( f^{(\lambda)}_{i,j} \) is a solution of a discrete wave equation and Figure 2 demonstrates how the function changes with \( \lambda \) when \( j \) is fixed.

![Figure 2](image)

**Figure 2.** This picture shows the \( \lambda \)-evolution of the function \( f^{(\lambda)} = f^{(\lambda)}(i) = f^{(\lambda)}_{i,j} \) of the discrete space variable \( i \) when the discrete time \( j \) is fixed and \( j = 15 \).

It is not so hard to see that it is possible to generalize (3.1) and (3.2) to the case of the ultraspherical polynomials.

**Theorem 4.2.** Let \( k \) and \( l \) be two nonnegative integer numbers. Then one has

\[
\sum_{j=0}^{\infty} f^{(\lambda)}_{k,j} f^{(\lambda)}_{l,j} = \frac{1}{2^{2\lambda}} \int_0^1 \hat{p}_k^{(\lambda)}(y) \hat{p}_l^{(\lambda)}(y)(y(1-y))^{\lambda-\frac{1}{2}} dy
\]

for any \( \lambda > -1/2 \) and

\[
\sum_{i=0}^{\infty} f^{(\lambda)}_{i,k} f^{(\lambda)}_{i,l} = \int_0^1 \hat{p}_k^{(\lambda)}(2x-1) \hat{p}_l^{(\lambda)}(2x-1)x^{2\lambda-1} \left( \frac{1-x}{1+x} \right)^{\lambda-\frac{1}{2}} dx
\]

provided that \( \lambda > 0 \).
Proof. As before we can assume that $k \leq l$ therefore,
\[
\sum_{j=0}^{\infty} f^{(\lambda)}_{k,j} f^{(\lambda)}_{i,j} = \sum_{j=0}^{k} f^{(\lambda)}_{k,j} f^{(\lambda)}_{i,j} = \\
\int_{0}^{1} \hat{p}^{(\lambda)}_{i}(y) \left( \int_{0}^{1} \hat{p}^{(\lambda)}_{k}(x) \sum_{j=0}^{k} p^{(\lambda)}_{j}(2x-1) p^{(\lambda)}_{j}(2y-1)(x(1-x))^{\lambda-\frac{1}{2}}dx \right) (y(1-y))^{\lambda-\frac{1}{2}}dy.
\]
Since the Christoffel-Darboux kernel
\[
2^{2\lambda} \sum_{j=0}^{k} p^{(\lambda)}_{j}(2x-1) p^{(\lambda)}_{j}(2y-1)
\]
is a reproducing kernel in the corresponding $L_{2}$-space, we get
\[
\sum_{j=0}^{\infty} f^{(\lambda)}_{k,j} f^{(\lambda)}_{i,j} = \frac{1}{2\lambda} \int_{0}^{1} \hat{p}^{(\lambda)}_{k}(y) \hat{p}^{(\lambda)}_{i}(y)(y(1-y))^{\lambda-\frac{1}{2}}dy.
\]
To prove the second equality, consider the following representation of the finite sum
\[
\sum_{i=0}^{n} f^{(\lambda)}_{i,k} f^{(\lambda)}_{i,l} = \int_{-1}^{1} \hat{p}^{(\lambda)}_{k}(2x-1) \chi_{[0,1]}(x) \frac{x^{\lambda-1/2}}{(1+x)^{\lambda-1/2}} P_{n}(x)(1-x^{2})^{\lambda-1/2}dx,
\]
where
\[
P_{n}(x) = \sum_{i=0}^{n} \int_{-1}^{1} \left( \hat{p}^{(\lambda)}_{i}(2y-1) \chi_{[0,1]}(y) \frac{y^{\lambda-1/2}}{(1+y)^{\lambda-1/2}} \hat{p}^{(\lambda)}_{i}(y)(1-y^{2})^{\lambda-1/2}dy \right) \hat{p}^{(\lambda)}_{i}(x)
\]
If $\lambda > 0$ then
\[
P_{n}(x) \searrow_{L_{2}([-1,1),(1-x^{2})^{\lambda-1/2}dx]} \hat{p}^{(\lambda)}_{i}(2x-1) \chi_{[0,1]}(x) \frac{x^{\lambda-1/2}}{(1+x)^{\lambda-1/2}}
\]
as $n \to \infty$. Next since the functional
\[
F(g) = \int_{-1}^{1} \hat{p}^{(\lambda)}_{k}(2x-1) \chi_{[0,1]}(x) \frac{x^{\lambda-1/2}}{(1+x)^{\lambda-1/2}} g(x)(1-x^{2})^{\lambda-1/2}dx,
\]
is continuous for $\lambda > 0$ we arrive at the following
\[
\sum_{i=0}^{\infty} f^{(\lambda)}_{i,k} f^{(\lambda)}_{i,l} = \int_{0}^{1} \hat{p}^{(\lambda)}_{k}(2x-1) \hat{p}^{(\lambda)}_{i}(2x-1) x^{2\lambda-1} \left( \frac{1-x}{1+x} \right)^{\lambda-\frac{1}{2}}dx.
\]
which completes the proof. \qed

Remark 4.3. It is not clear if it is possible to explicitly evaluate the integrals from (4.3) and (4.4) but it would be nice to find the expressions beyond the Legendre case.

The next step is to obtain an analog of Proposition 3.6 which will be a 1D-relation for the function $f^{(\lambda)}_{i,j}$ unlike (4.2).
Theorem 4.4. Let \( j \) be a fixed nonnegative integer number. Then the function \( f = f(i) = f_{i,j}^{(\lambda)} \) of the discrete variable \( i \) satisfies the generalized eigenvalue problem

\[
(4.5) \quad 2((i + \lambda)^2 - 1/4)(i + \lambda + \frac{3}{2}) \sqrt{\frac{i + 2\lambda}{(i + 1)(i + \lambda + 1)(\lambda + i)}} f_{i+1,j}^{(\lambda)} + \\
+ 2((i + \lambda)^2 - 1/4)(i + \lambda - \frac{3}{2}) \sqrt{\frac{i}{(i - 1 + \lambda)(i - 1 + 2\lambda)(\lambda + i)}} f_{i-1,j}^{(\lambda)} = \\
(j + \lambda - \frac{1}{2})(j + \lambda + \frac{1}{2}) [2(i + \lambda - 1/2) \sqrt{\frac{i + 2\lambda}{(i + 1)(i + \lambda + 1)(\lambda + i)}} f_{i+1,j}^{(\lambda)} + 4f_{i,j}^{(\lambda)} + \\
2(i + \lambda + 1/2) \sqrt{\frac{i}{(i - 1 + \lambda)(i - 1 + 2\lambda)(\lambda + i)}} f_{i-1,j}^{(\lambda)}],
\]

for \( i = 0, 1, 2, \ldots \) and, here, the number \((j + \lambda - \frac{1}{2})(j + \lambda + \frac{1}{2})\) is the corresponding generalized eigenvalue.

Proof. To make all the formulas shorter and, more importantly transparent, let us introduce the following operators

\[
(4.6) \quad A_i = 2(i + \lambda + \frac{3}{2}) \sqrt{\frac{i + 2\lambda}{(i + 1)(i + \lambda + 1)(\lambda + i)}} E_+ \\
+ 2(i + \lambda - \frac{3}{2}) \sqrt{\frac{i}{(i - 1 + \lambda)(i - 1 + 2\lambda)(\lambda + i)}} E_- \\
= a_{i+1} E_+ + a_{i-1} E_-
\]

and

\[
(4.7) \quad B_i = 4I + 2(i + \lambda - 1/2) \sqrt{\frac{i + 2\lambda}{(i + 1)(i + \lambda + 1)(\lambda + i)}} E_+ \\
+ 2(i + \lambda + 1/2) \sqrt{\frac{i}{(i - 1 + \lambda)(i - 1 + 2\lambda)(\lambda + i)}} E_- \\
= 4I + b_{i+1} E_+ + b_{i-1} E_-,
\]

where \( I \) is the identity operator and \( E_+, E_- \) are the forward and backward shift operators on \( i \), respectively. With these notations, equation (4.5) can be rewritten as

\[
(4.8) \quad (i(i + 2\lambda) + \lambda^2 - 1/4)A_{ij}f_{i,j}^{(\lambda)} = (j(j + 2\lambda) + \lambda^2 - 1/4)B_{ij}f_{i,j}^{(\lambda)}
\]

or

\[
(4.9) \quad i(i + 2\lambda)A_{ij}f_{i,j}^{(\lambda)} + (\lambda^2 - 1/4)(A_i - B_i)f_{i,j}^{(\lambda)} = j(j + 2\lambda)B_i f_{i,j}^{(\lambda)}.
\]

Notice that

\[
(4.10) \quad A_i - B_i = -4I + 4 \sqrt{\frac{i + 2\lambda}{(i + 1)(i + \lambda + 1)(\lambda + i)}} E_+ \\
- 4 \sqrt{\frac{i}{(i - 1 + \lambda)(i - 1 + 2\lambda)(\lambda + i)}} E_-.
\]
Thus after two integration by parts we have
\begin{equation}
(4.11) \quad \frac{d}{dt}(t(1-t))^{\lambda+1/2} \frac{d}{dt}B_i f_i^{(\lambda)}(2t - 1) + j(j + 2\lambda)(t(1-t))^{\lambda-1/2}B_i f_i^{(\lambda)}(2t - 1) = 0.
\end{equation}

As is known [12], the ultraspherical polynomials satisfy the differential equation
\[ B_j f_j^{(\lambda)}(2t - 1) = \frac{1}{4} (\lambda + \frac{3}{2}) B_j f_j^{(\lambda)}(2t - 1) = 0. \]

Using the first equation in [12, equation (4.7.28)] gives
\[ \frac{d}{dt}(t(1-t))^{\lambda+1/2} \frac{d}{dt}B_i f_i^{(\lambda)}(2t - 1) + j(j + 2\lambda)(t(1-t))^{\lambda-1/2}B_i f_i^{(\lambda)}(2t - 1) = 0. \]

Now
\[ (i + 1)(i + 1 + 2\lambda)2(i + \lambda + \frac{3}{2}) - i(i + 2\lambda)2(i + \lambda + \frac{3}{2}) \equiv 4(\lambda^2 - 1/4) = 0, \]

it follows that
\[ (j(j + 2\lambda)B_i - i(i + 2\lambda)A_i - (\lambda^2 - 1/4)(A_i - B_i)) f_i^{(\lambda)}(t) = 0. \]

We note that
\[ B_i f_i^{(\lambda)}(t) = 4(1 + \frac{\lambda - 3/2}{i + 1})a_i + 4(1 + \frac{\lambda - 3/2}{i + 2\lambda - 1})a_i. \]

The substitution of these relations in (4.12) leads to the following
\[ \frac{d}{dt}B_i f_i^{(\lambda)}(t) = 4(1 + \frac{\lambda - 3/2}{i + 1})a_i + 4(1 + \frac{\lambda - 3/2}{i + 2\lambda - 1})a_i. \]

Using the first equation in [12] equation (4.7.28)] gives
\[ \frac{d}{dt}B_i f_i^{(\lambda)}(t) = 2\frac{(i + \lambda - 1)a_i}{i}(t \frac{d}{dt}B_i f_i^{(\lambda)}(t) - iB_i f_i^{(\lambda)}(t)). \]
so we find

\[
\frac{d}{dt} B_i \hat{p}_i^{(\lambda)}(t) = 4 \frac{d}{dt} (1 + t + \frac{\lambda - 3/2}{t + 1}) \hat{p}_i^{(\lambda)}(t) - 8 a_i \frac{\lambda - 3/2}{(i + 1)(i + 2\lambda - 1)} \frac{d}{dt} \hat{p}_{i-1}^{(\lambda)}(t)
\]

\[
= 4(\lambda - 1/2) \hat{p}_i^{(\lambda)}(t) + 4(1 + t) \frac{d}{dt} \hat{p}_i^{(\lambda)}(t).
\]

Thus we have

\[
((1 - t) \frac{d}{dt} - (\lambda + 1/2)) \frac{d}{dt} B_i \hat{p}_i^{(\lambda)}(t)
\]

\[
= 4((1 - t^2) \frac{d^2}{dt^2} - (2\lambda + 1)t \frac{d}{dt} - (\lambda^2 - 1/4)) \hat{p}_i^{(\lambda)}(t)
\]

and the result follows. \(\square\)

**Remark 4.5.** At first, we can see that equation (4.5) has the form

\[
\tilde{A}_i f_{i,j}^{(\lambda)} = (j + \lambda - 1/2)(j + \lambda + 1/2) B_i f_{i,j}^{(\lambda)},
\]

where

\[
\tilde{A}_i = (i + \lambda - 1/2)(i + \lambda + 1/2) A_i,
\]

the operators \(A_i\) and \(B_i\) are given by (4.6) and (4.7), respectively. At second, the above-given proof shows that the three-term recurrence relation (4.5) is a consequence of the fact that ultraspherical polynomials are eigenfunctions of a second order differential operator of a specific form. However, there is another way to see the validity of equation (4.5).

We first prove the following Lemma

**Lemma 4.6.** The following representation holds

\[
f_{i,j}^{(\lambda)} = \begin{cases} 0, & i < j; \\ \frac{1}{2^j \pi^{1/2}} \sqrt{\frac{\Gamma((\lambda+1),(2)\lambda_i)(2\lambda_i)}{\Gamma((\lambda)_j((\lambda+1)_j)} (\lambda + \frac{1}{2})^j \lambda)^{j-1/2}, & i \geq j. \end{cases}
\]

**Proof.** Write

\[
f_{i,j}^{(\lambda)} = k_{i,j,\lambda} \int_0^1 p_i^{(\lambda)}(t) p_j^{(\lambda)}(2t - 1)(t(1 - t))^{\lambda-1/2} dt,
\]

where \(p_n^{(\lambda)}\) is the monic orthogonal polynomial and

\[
k_{i,j,\lambda} = \frac{\Gamma((\lambda + 1))}{\Gamma(\lambda + \frac{1}{2}) \sqrt{\pi}} \lambda_i (\lambda + 1)_i \lambda_j j! (2\lambda)_j \lambda^{j-1/2}.
\]

If we denote the integral in equation (4.14) as \(I^{(1)}\) we find using the representation

\[
p_i^{(\lambda)}(t) = 2^i \frac{(\lambda + \frac{1}{2})_i}{(i + 2\lambda)_i} \sqrt{\frac{(\lambda)_i((\lambda+1)_i}{2^j (2\lambda)_j i!} F_1 (-i, i + 2\lambda; \frac{1 - t}{2})
\]

and set

\[
I^{(1)} = 2^{i+j} (-1)^j \sqrt{\frac{(\lambda + \frac{1}{2})_i}{(i + 2\lambda)_i} F_1 \frac{(\lambda + \frac{1}{2})_i}{(j + 2\lambda)_j} F_1},
\]

so

\[
I^{(1)} = 2^{i+j} (-1)^j \sqrt{\frac{(\lambda + \frac{1}{2})_i}{(i + 2\lambda)_i} F_1 \frac{(\lambda + \frac{1}{2})_i}{(j + 2\lambda)_j} F_1}.
\]
Another Proof of Theorem 4.4. To see this use the contiguous relation (see [2, equation (2.5.15)])

\[ 2b(c - b)(b - a - 1)_{2} F_{1} \left( \frac{a - 1}{c}, \frac{b + 1}{2} \right) 
- (b - a)(b + a - 1)(2c - b - a - 1)_{2} F_{1} \left( \frac{a}{c}, \frac{1}{2} \right) 
- 2a(b - c)(b - a + 1)_{2} F_{1} \left( \frac{a + 1}{c}, \frac{b - 1}{2} \right) = 0, \]
which with \(a = -i + j\), \(b = i + j + 2\), and \(c = 2j + 2\lambda + 1\) yields the relation

\[
(2i + 2\lambda - 1)(j + 2\lambda + 1)\sqrt{(i + 2\lambda)(i + 1)(i + \lambda + 1)(i + 1 - j)} f^{(\lambda)}_{i+1,j} - (2j + 2\lambda - 1)(2j + 2\lambda + 1) f^{(\lambda)}_{i,j} \\
+ (2i + 2\lambda + 1)(i - j - 1)\sqrt{(i + 1 + \lambda)(i - 1 + 2\lambda)(i + 1)} (i + 2\lambda - 1) f^{(\lambda)}_{i-1,j} = 0.
\]

The latter relation leads to (4.5). □

A generalized eigenvalue problem can also be found for \(i\) fixed. To this end we need to use the relation

\[
\begin{align*}
2F_1\left(\begin{smallmatrix} -n, b \\ c \\ x \end{smallmatrix} \right) &= \frac{(b)_n}{(c)_n} (-x)^n 2F_1\left(\begin{smallmatrix} -n, -c - n + 1 \\ -b - n + 1 \\ 1/x \end{smallmatrix} \right).
\end{align*}
\]

Therefore we find

\[
2F_1\left(\begin{smallmatrix} -i + j, i + j + 2\lambda \\ 2j + 2\lambda + 1 \\ 1/2 \end{smallmatrix} \right) = \frac{(i + j + 2\lambda)_{i-j}}{(2j + 2\lambda + 1)_{i-j}} (-2)^{j-i} 2F_1\left(\begin{smallmatrix} -i + j, -i - j - 2\lambda \\ -2i - 2\lambda + 1 \\ 2 \end{smallmatrix} \right).
\]

Following the steps used to obtain the recurrence formula for \(j\) fixed in the second proof we find that

\[ c_j f^{(\lambda)}_{i,j-1} + d_j f^{(\lambda)}_{i,j+1} + e_j f^{(\lambda)}_{i,j} = 0, \]

where

\[ c_j = -2(i + j + 2\lambda - 1)(i - j + 1)(2j + 2\lambda + 1)(j + \lambda + 1), \]

\[ d_j = -4(i - j - 1)(i + j + 2\lambda + 1)(j + \lambda + 1 - \frac{1}{2}) \sqrt{j(j + 1)(j + \lambda - 1)(j + \lambda + 1)} \]

and

\[ e_j = -2(2i + 2\lambda + 1)(2i + 2\lambda - 1)(j + \lambda + 1) \sqrt{j(j + \lambda)(j + \lambda - 1)} \]

\[ + 6(2j + 2\lambda - 1)(2j + 2\lambda + 1)(j + \lambda + 1) \sqrt{j(j + \lambda - 1)(j + \lambda)} \]

Since

\[ (i + j + 2\lambda + 1)(i - j + 1) = (i + \lambda + \frac{1}{2})(i + \lambda - \frac{1}{2}) - (j + \lambda + \frac{1}{2})(j + \lambda + \frac{3}{2}) \]

the above recurrence can be recast as the generalized eigenvalue equation

\[ A_j f^{(\lambda)}_{i,j} = (i + \lambda + \frac{1}{2})(i + \lambda - \frac{1}{2}) B_j f^{(\lambda)}_{i,j}, \]
where the operator $\hat{A}_j$ is the second order difference operator

$$\hat{A}_j = (2j + 2\lambda + 1)(2j + 2\lambda - 1)\left(3(j + \lambda + 1)I + (j + \lambda + 1/2)\hat{E}_+ + (j + \lambda - 3/2)(j + \lambda + 1)\sqrt{(j + 2\lambda - 1)/(j + \lambda)(j + \lambda - 1)}\hat{E}_-\right),$$

for $i = 0, 1, 2, \ldots$ and where the operators $\hat{A}_j$ and $\hat{B}_j$ are given by (4.19) and (4.20), respectively. Also, here, $(i + \lambda + 1/2)(i + \lambda - 1/2)$ is the corresponding generalized eigenvalue.

Theorem 4.7. Let $i$ be a fixed nonnegative integer number. Then the function $f = f(j) = f^{(\lambda)}_{i,j}$ of the discrete variable $j$ satisfies the generalized eigenvalue problem

$$\hat{A}_j f^{(\lambda)}_{i,j} = (i + \lambda + 1/2)(i + \lambda - 1/2)\hat{B}_j f^{(\lambda)}_{i,j}$$

for $i = 0, 1, 2, \ldots$ and where the operators $\hat{A}_j$ and $\hat{B}_j$ are given by (4.19) and (4.20), respectively. Thus we have just proved the following statement.

Theorem 4.8. For sufficiently large $i$ the following formula holds

$$f^{(\lambda)}_{i,j} = k_j \frac{\cos\left(\pi\left(j + \frac{3}{2} - \frac{i}{2} + \frac{1}{2}\right)\right)}{\sqrt{\pi i^{3/2}}} + O\left(\frac{1}{i^{3/2}}\right),$$

where

$$k_j = \frac{1}{2^{i+1-2\lambda}}\sqrt}\frac{(2\lambda)_j}{j!(\lambda)_j(\lambda + 1)_j\Gamma(2\lambda)}\Gamma(2j + 2\lambda + 1)(\lambda + 1/2)_j.$$

Proof. According to Lemma 4.6 for $i \geq j$ we have

$$f^{(\lambda)}_{i,j} = \frac{1}{2^{i-j+1}}\sqrt\frac{i!(\lambda + 1)_j(2\lambda)_j}{j!(\lambda)_j(\lambda + 1)_j}\frac{(i + 2\lambda)_j}{(i + j + 2\lambda)_j}\frac{1}{(i - j)!}\frac{1}{2}\left(\frac{-i + j, i + j + 2\lambda, 1/2}{2j + 2\lambda + 1}\right).$$

Then, since

$$\sqrt\frac{i!(\lambda + 1)_j(2\lambda)_j}{(\lambda)_i(i - j)!} = \sqrt\frac{1}{\lambda^2(2\lambda)}\Gamma(2j + \lambda)(1 + O(1/i)).$$
formula (4.21) follows from [9 formula (36)].

**Remark 4.9.** Formula (4.21) along with the fact that $f_{i,j}^{(\lambda)} = 0$ for $i < j$ show that the moving wave behavior of the solution demonstrated in Figure 1 is also characteristic for the solution $f_{i,j}^{(\lambda)}$ of the discrete wave equation (4.2) for any $\lambda > -1/2$.

Another useful asymptotic is when $i = k_1t$ and $j = k_2t$ where $k_1 > k_2$ are fixed and $t$ is large.

**Theorem 4.10.** For $k_1t$ and $k_2t$ integers with $k_1 > k_2 > 0$, and $\sqrt{2k_2/k_1} > 1$

(4.23) $f_{k_1t,k_2t}^{(\lambda)} = \frac{c(\epsilon, \lambda)}{2^{k_1+1}(k_1t)^{1/2}} \left( \frac{1 + \hat{b}(\epsilon)}{\epsilon - \hat{b}(\epsilon)} \right)^{(k_1-k_2)t} \left( \frac{1 + 2\epsilon - \hat{b}(\epsilon)}{1 + \epsilon} \right)^{(k_1+k_2)t+2\lambda} (1+O(1/t)),$

where

(4.24) $c(\epsilon, \lambda) = \epsilon^\lambda \frac{1}{\sqrt{\pi(1-\epsilon^2)(2\epsilon^2 - 1)^{3/2}}},$

$\epsilon = k_2/k_1$, and $\hat{b}(\epsilon) = \sqrt{2\epsilon^2 - 1}$.

**Proof.** In this case the representation given by equation (4.18) is most convenient. An application of the transformation T3 in [11] yields

$$
\begin{align*}
\mathbf{2F1} \left( \begin{array}{c}
-i+j, -i-j+2\lambda \\
-2t-2\lambda+1
\end{array}; 2 \right) \\
= \frac{2^{-j-1}(i-j)!}{(i+j+2\lambda+1)_{i-j-1}} \mathbf{2F1} \left( \begin{array}{c}
-i+j+1, i+j+2\lambda+1 \\
2
\end{array}; 1/2 \right) \\
= -\frac{2^{i+j+2\lambda-1}(i-j)!}{(i+j+2\lambda+1)_{i-j-1}} \mathbf{2F1} \left( \begin{array}{c}
-i+j+1, -i-j+2\lambda+1 \\
2
\end{array}; 1/2 \right)
\end{align*}
$$

where Euler’s transformation has been used to obtain the last equality. Thus with the use of the duplication formula for the $\Gamma$ function it follows

(4.25) $f_{i,j}^{(\lambda)} = d_{i,j} \mathbf{2F1} \left( \begin{array}{c}
i-j+1, -i-j+2\lambda+1 \\
2
\end{array}; 1/2 \right),$

where

$$
d_{i,j} = (-1)^{i-j+1}2^{i+2\lambda-1} \sqrt{(i+j)(j+i)\Gamma(2\lambda+j)i!\Gamma(2\lambda+i)} / j!\Gamma(2\lambda+i).
$$

This becomes

$$
d_{k_1t,k_2t} = (-1)^{i-j+1}2^{i+2\lambda-1} \left( \frac{2\lambda}{j^{\lambda-2}} \right)^{1/2} (1 + O(1/i)) \\
= (-1)^{(k_1-k_2)t+1}2^{k_2t+2\lambda-1} \left( \frac{k_2}{k_1} \right)^\lambda (k_1t)(1 + O(1/t)).
$$

The hypergeometric function on the right hand side of equation (4.25) is in the form to use the type B formulas in [11] and leads to considering the hypergeometric function $\mathbf{2F1} \left( \begin{array}{c}
\epsilon_1w+1, -w-2\lambda+1 \\
2
\end{array}; 1/2 \right)$ where $\epsilon_1w$ is an integer. Equation (4.4) in [11]
shows that the saddle points occur at 

\[ \frac{1 + \epsilon_1}{2} \pm \sqrt{\left(\frac{1 + \epsilon_1}{2}\right)^2 - 2\epsilon_1}. \]

If the discriminant is positive both saddles are real and equation (4.9) in [11] yields

\[ \sum_2 F_1 \left( \frac{\epsilon_1 w + 1, -w - 2\lambda + 1}{2} \right) \]

\[ = \frac{(-1)^k w^{k+1}}{w^2 \sqrt{\pi \epsilon_1 b(\epsilon_1)}} \left( \frac{1 + \epsilon_1 + b(\epsilon_1)}{1 - \epsilon_1 - b(\epsilon_1)} \right) \frac{(3 - \epsilon_1 - b(\epsilon_1)) w + 2\lambda}{2w + 4\lambda + \frac{1}{2}} (1 + O(1/w)), \]

where

\[ (4.27) \quad b(\epsilon_1) = \sqrt{(1 + \epsilon_1)^2 - 8\epsilon_1}. \]

With \( \epsilon_1 = \frac{k_1 - k_2}{k_1 + k_2} \) and \( w = (k_1 + k_2) t \) the above equations yield (4.23). \( \square \)

Remark 4.11. When the discriminant is negative, the two saddle points are conjugates of each other and so in this case equation (4.7) in [11] is used to obtain the asymptotics for \( \sum_2 F_1 \left( - \frac{\epsilon_1 t + 1, -t - 2\lambda + 1}{2} \right) \) which then are used to obtain the asymptotics of \( f_{i,j}^{(\lambda)} \).

We finish this section with a couple of Lemmas where we start with the recurrence formulas. Write the recurrence formula in equation (4.15) as

\[ a_{i,j} f_{i+1,j}^{(\lambda)} + b_{i,j} f_{i,j}^{(\lambda)} + c_{i,j} f_{i-1,j}^{(\lambda)} = 0, \]

and the recurrence formula in \( j \) as

\[ \hat{a}_{i,j} f_{i,j+1}^{(\lambda)} + \hat{b}_{i,j} f_{i,j}^{(\lambda)} + \hat{c}_{i,j} f_{i,j-1}^{(\lambda)} = 0, \]

with \( i \geq j \geq 0. \)

We can now prove the following simple statement.

Proposition 4.12. Given \( a_{i,j}, b_{i,j}, c_{i,j} \) and \( \lambda > -1/2 \). For each \( j > 0 \) the unique solution of equation (4.28) with initial conditions

\[ f_{j-1,j} = 0, \quad f_{j,j} = \int_0^1 \tilde{p}_j^{(\lambda)}(t) \tilde{p}_j^{(\lambda)}(2t - 1)(t(1 - t))^{\lambda - 1/2} dt \]

is the function

\[ f_{i,j} = I_{i,j}^{(\lambda)} := \int_0^1 \tilde{p}_j^{(\lambda)}(t) \tilde{p}_j^{(\lambda)}(2t - 1)(t(1 - t))^{\lambda - 1/2} dt. \]

If \( j = 0, \lambda > -1/2, \) and \( \lambda \neq 1/2 \) then \( f_{0,0} = I_{0,0}^{(\lambda)} \) gives the unique solution \( f_{i,0} = I_{i,0}^{(\lambda)} \). If \( \lambda = 1/2 \) then the initial conditions \( f_{0,0} = I_{0,0}^{(1/2)} \) and \( f_{1,0} = I_{1,0}^{(1/2)} \) are needed to give \( f_{i,j} = I_{i,j}^{(1/2)} \).

Proof. For \( j > 0, a_{i,j} \neq 0 \) for \( i \geq j \) so the result follows from equation (4.28). For \( j = 0 \) and \( \lambda \neq 1/2, c_{0,0} = 0 \neq a_{0,0} \) so that only \( f_{0,0} \) is needed to compute \( f_{1,0}. \)

The remaining \( f_{i,j} \) are computed in the standard fashion from equation (4.28). For the last case when \( \lambda = 1/2, a_{0,0} = 0 = b_{0,0} \) so \( f_{2,0} = \frac{c_{1,0}}{a_{1,0}} f_{0,0} \) and \( f_{3,0} = \frac{c_{2,0}}{a_{2,0}} f_{1,0}. \)

The remaining \( f_{i,0} \) are computed in the same way using the fact that \( a_{i,0} \neq 0 \) for \( i > 0. \)

Similarly, for the recurrence in \( j \) we have the following.
Proposition 4.13. Given $a_{i,j}$, $b_{i,j}$, $c_{i,j}$ and $\lambda > -1/2$. For each $i > 0$ the unique solution of equation (4.29) with initial conditions $f_{j,j+1} = 0$ and $f_{j,j} = f_{i,j}^{(\lambda)}$ is $f_{i,j} = f_{i,j}^{(\lambda)}$.

Since $\hat{c}_{i,j}$, $\hat{b}_{i,j}$, and $\hat{a}_{i,j}$ are not equal to zero for $i \geq j$ the result follows from equation (4.29).

5. Connections to other problems

Recall that it is said that a function $\Psi(x, \lambda)$ is a solution of a bispectral problem if it satisfies the following

$$
A\Psi(x, \lambda) = g(\lambda)\Psi(x, \lambda)
$$

$$
B\Psi(x, \lambda) = f(x)\Psi(x, \lambda),
$$

where $A$, $B$ are some operators and $f$, $g$ are some functions [6]. It is shown in [10] that if $A$ and $B$ are tridiagonal operators then the solutions of the corresponding bispectral problem are related to the Askey-Wilson polynomials.

The problem we are dealing with in this paper is the following generalization of a bispectral problem:

$$
A\Psi(x, \lambda) = g(\lambda)B\Psi(x, \lambda)
$$

$$
C\Psi(x, \lambda) = f(x)D\Psi(x, \lambda),
$$

where $A$, $B$, $C$, and $D$ are tridiagonal operators. Namely, Theorems 4.4 and 4.7 tell us that $f_{i,j}^{(\lambda)}$ is a solution of this generalized bispectral problem. Actually, it would be nice to find a characterization of such bispectral problems similar to what was done in [10]. It would also be interesting to study the consistency relations for the system (4.13), (4.12) and those relations will constitute a nonlinear system of difference equations on the coefficients of (4.13), (4.12).

Another link that is worth mentioning here is the relation to spectral transformations. To this end, let us consider two families $p_j^{(1/2)}(t)$ and $p_j^{(3/2)}(t)$ of ultrasperical polynomials. That is, we consider the two measures on $[-1,1]$

$$
d\mu_{1/2}(t) = dt, \quad d\mu_{3/2}(t) = (1-t^2)dt,
$$

which are clearly related in the following manner

$$
d\mu_{1/2}(t) = \frac{d\mu_{3/2}(t)}{(1-t^2)}.
$$

In this case, one usually says that $d\mu_{1/2}(t)$ is the Geronimus transformation of $d\mu_{3/2}(t)$ of the second order or $d\mu_{1/2}(t)$ is the inverse quadratic spectral transform of $d\mu_{3/2}(t)$ (for instance, see [11]). Also, we have that

$$
p_i^{(1/2)}(t) = \alpha(1, i)p_i^{(3/2)}(t) + \alpha(2, i)p_{i-1}^{(3/2)}(t) + \alpha(3, i)p_{i-2}^{(3/2)}(t),
$$

where $\alpha(1, i)$, $\alpha(2, i)$, and $\alpha(3, i)$ are some coefficients. If we introduce the coefficients

$$
f_i^{(1/2,3/2)} = \int_{-1}^{1} p_i^{(1/2)}(t)p_j^{(3/2)}(t)(1-t^2)dt
$$

then, due to Theorem 2.1, they satisfy the discrete wave equation and we also have

$$
p_i^{(1/2)}(t) = f_i^{(1/2,3/2)}(t)^2 p_i^{(3/2)}(t) + f_i^{(1/2,3/2)}(t)^2 p_{i-1}^{(3/2)}(t) + f_i^{(1/2,3/2)}(t)^2 p_{i-2}^{(3/2)}(t).
$$
Formula (5.2) shows that in the sequence \( f_{i,j}^{(1/2,3/2)} \) when \( j \) is fixed there are at most three nonzero coefficients. Moreover, it follows from [2, Theorem 7.1.4] that \( f_{i,i-1}^{(1/2,3/2)} = 0 \). In other words, in this case we have a localized wave and below is the simulation.

![Figure 3](image)

**Figure 3.** This picture shows three graphs of the function \( f = f^{(1/2,3/2)}(i) = f_{i,j}^{(1/2,3/2)} \) of the discrete space variable \( i \) at the three different discrete times \( j = 15, j = 30, \) and \( j = 45 \).

The phenomenon of localized waves is related to the fact that the measures are related to one another thorough spectral transformations. Still, one can define even more general coefficients

\[
    f_{i,j}^{(\lambda,\mu)} = \int_{-1}^{1} p_i^{(\lambda)}(t)p_j^{(\mu)}(t)(1-t^2)^{\mu-1/2} dt
\]

and, as before, they form a solution to a wave equation. Moreover, these coefficients are known explicitly [2, Section 7.1] and are called the connection coefficients. It will be shown in a forthcoming paper that the family \( f_{i,j}^{(\lambda,\mu)} \) is also a solution of a bispectral problem of the form [5,1].

Furthermore, even more general coefficients previously appeared in different contexts. In particular, the coefficients

\[
    f_{i,j,k}^{(1,1,1)} = \frac{2}{\pi} \int_{-1}^{1} p_i^{(1)}(t)p_j^{(1)}(t)p_k^{(1)}(t)(1-t^2)^{1/2} dt,
\]

where the polynomials \( p_i^{(1)}(t) \) are the monic Chebyshev polynomials of second kind, count Dyck paths [5]. From our perspective, the function \( f_{i,j,k}^{(1,1,1)} \) is a simultaneous solution of three wave equations. That is, if we fix one of the discrete variables \( i, j, \) or \( k \), then Theorem [2,1] leads to a wave equation in each of these cases. Thus, it leads to a multidimensional generalization of our findings and some relations between the numbers of Dyck paths of certain orders.

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