EXACT RECOVERY OF COMMUNITY DETECTION IN DEPENDENT GAUSSIAN MIXTURE MODELS

ZHONGYANG LI AND SICHEN YANG

Abstract. We study the community detection problem on a Gaussian mixture model, in which (1) vertices are divided into \( k \geq 2 \) distinct communities that are not necessarily equally-sized; (2) the Gaussian perturbations for different entries in the observation matrix are not necessarily independent or identically distributed. We prove necessary and sufficient conditions for the exact recovery of the maximum likelihood estimation (MLE), and discuss the cases when these necessary and sufficient conditions give sharp threshold. Applications include the community detection on a graph where the Gaussian perturbations of observations on each edge is the sum of i.i.d. Gaussian random variables on its end vertices, in which we explicitly obtain the threshold for the exact recovery of the MLE.

1. Introduction

Community structures are very common in graphic models for natural and social phenomena. For instance, in structures of matter atoms form molecules in such a way that atoms in the same molecule have stronger connections compared to those in different molecules; in social networks individuals form groups so that there are more communications among individuals in the same group than individuals in different groups. The main goal for community detection is to determine the unknown group division of individuals based on observations of (random) connections between individuals. Identifying different communities in the stochastic block model is a central topic in many fields of science and technology; see [19, 9, 23, 13, 10, 11, 8, 15, 21, 18, 3, 20, 4, 2, 1] for an incomplete list.

In this paper we study the community detection problem for the Gaussian mixture model, in which \( n \) vertices, representing \( n \) individuals, belong to \( k \) \((k \geq 2)\) different communities, or groups. We observe a \( p \times 1 \) vector for each one of the \( n \) vertices, perturbed by a \( p \times 1 \) Gaussian vector with mean-0 entries that are assumed to be neither independent nor identically distributed. More precisely, each entry of the \( p \times n \) perturbation matrix is a multiple of a standard Gaussian random variable, while the variances of different entries are different; and moreover, different entries may have nontrivial co-variances. Given such an observation, we find the maximum likelihood estimation (MLE) for the community assignment, and study the probability that the MLE equals the true community assignment as the number of vertices \( n \to \infty \). If this probability tends to 1 as \( n \to \infty \), we say exact recovery occurs.

Heuristically, it is natural to conjecture that exact recovery may occur when the variances of the perturbations are small but does not occur when these variances are large. This is
indeed the case when the Gaussian perturbations for different observations are independent; see [17] for the case when the Gaussian perturbations are assumed to be i.i.d; and [16] for the case when the Gaussian perturbations are assumed to be independent but not identically distributed. A block-covariance model was studied in [7, 6], in which the columns of the \( p \times n \) observation matrix are assumed to be independent, while the entries of each column may have non-trivial covariance. Another dependent model called the Ising block model was studied in [5].

The major theme of the paper is to further investigate the conditions for the exact recovery when the Gaussian perturbations are assumed to be neither independent nor identically distributed. In this case it turns out that the exact recovery threshold depends on the maximal eigenvalue of the co-variance matrix of the Gaussian perturbations. In this paper, we obtain explicit necessary and sufficient conditions for the exact recovery of MLE when the number of vertices in each community is either known or unknown. As could be seen in Section 2, the results are similar for the two cases. To illustrate the general theory developed in the paper, we study an example in which \( n \) vertices of a graph are divided into two groups with size \( \alpha n \) and \( (1 - \alpha)n \), respectively; the observation is a weighted adjacency matrix of the graph with weight 1 for each edge joining two vertices in the same community, and weight 0 for each edge joining two vertices in distinct communities, perturbed by a \( n \times n \) matrix, each entry of which is the sum of the two i.i.d. centered Gaussian random variables associated to the two end vertices of the edge. In this model, different entries in the Gaussian perturbation matrix are not independent if they correspond to two edges sharing a vertex. By applying the general theory developed in this paper, we obtain the threshold for the MLE exact recovery regarding the variances of the Gaussian perturbation, such that if the Gaussian variance is below the threshold, a.s. exact recovery occurs in MLE, while if the Gaussian variance is the above the threshold, a.s. exact recovery does not occur in the MLE. Other examples when the Gaussian perturbation matrix consists of independent entries can be found in [17, 16]. The threshold in the dependent model turns out to be smaller than the threshold for the independent model; see [17].

The implement of the MLE is usually very slow; in some cases relaxing some constraints in MLE leads to the more efficient Semi-Definite Programming (SDP). However, most SDPs require the partition to be balanced or that, at least, the size of each group is known in advance. The MLE optimization discussed here has the advantage to attack the case of unbalanced sizes of groups, and the case when the size of each group is unknown. When the perturbations for the the entries of the observation matrix are assumed to be i.i.d. Gaussian, and using the average value of the observations associated to each group to approximated the expectation of the observation to the group, in this case the MLE becomes the K-means estimation. The SDP relaxation for K-means have been studied extensively, see for example [22]. We expect a similar SDP relaxation for K-means in the dependent case in which the inner product in the objective function should be defined with respect to the inverse of the covariance matrix; yet in this paper, we shall focus on the more fundamental MLE and try to investigate its statistical limit.

The organization of the paper is as follows: in Section 2, we rigorously define the model, introduce the MLE, and state the main results of the paper. In Section 3, we prove Theorem
2.10 and Corollary 3.3, which give sufficient conditions for the exact recovery of MLE when the number of vertices in each community is unknown. In Section 4, we prove Theorem 2.16, which gives a sufficient condition for the exact recovery of MLE when the number of vertices in each community is known. In Section 5, we prove Theorem 2.17 and Corollary 5.1, which give necessary conditions for the exact recovery of MLE when the number of vertices in each community is unknown. In Section 6, we prove Theorem 2.18 and Corollary 6.1, which give necessary conditions for the exact recovery of MLE when the number of vertices in each community is known. In Section 7, we study a specific dependent Gaussian mixture model and provide numerical results that verify our theoretical bounds provided by theorems in Section 2.

In Appendix A, we review facts about pseudo determinants and Moore-Penrose inverses for matrices. In Appendix B, we review results about the distribution of maximum of dependent Gaussian random variables.

2. Main Results

In this section, we rigorously define the Gaussian mixture model studied in the paper, introduce the maximum likelihood estimation (MLE) for the model and state the main results of the paper about the threshold for the exact recovery of MLE.

2.1. Dependent Gaussian Mixture Model. Let $n \geq k \geq 2$ be positive integers. Write $[n] = \{1, 2, \ldots, n\}$ be a set of $n$ vertices divided into $k$ different communities; and $[k] := \{1, \ldots, k\}$ be the set of communities. A mapping $x : [n] \rightarrow [k]$ which assigns a unique community represented by an integer in $[k]$ to each one of the $n$ vertices in $[n]$ is called a community assignment mapping. Let $\Omega$ be the set consisting of all the possible mappings from $[n]$ to $[k]$; i.e.

$$\Omega := \{x : [n] \rightarrow [k]\}.$$  

Each mapping in $\Omega$ is a community assignment mapping.

Let $p \geq 1$ be a positive integer. Define

$$\theta : \Omega \times [p] \times [k] \rightarrow \mathbb{R}$$

to be a function on the set $\Omega \times [p] \times [k]$ taking real values.

For a community assignment mapping $x \in \Omega$, let $A_x$ be a $p \times n$ matrix whose entries are given by

$$(A_x)_{i,j} = \theta(x, i, x(j)), \quad \forall i \in [p], j \in [n].$$

Let $\Sigma$ be a $pn \times pn$ matrix with real entries defined by

$$\Sigma := (\rho_{i,j;k,l})_{i,k\in[p],j,l\in[n]} \in \mathbb{R}^{pn \times pn}.$$

Let $P, Q$ be two $p \times n$ matrices. Define the inner product of $P, Q$ by

$$\langle P, Q \rangle = \sum_{i \in [p]} \sum_{j \in [n]} P_{i,j}Q_{i,j}.$$ 

The norm $\|P\|$ for a matrix $P$ is defined by

$$\|P\| = \sqrt{\langle P, P \rangle}.$$
Let \( W \in \mathbb{R}^{p \times n} \) satisfy
(1) each entry in \( W \) is a Gaussian random variable with mean 0; and
(2) for \( i, k \in [p] \) and \( j, l \in [n] \),
\[
\text{Cov} \left( (W)_{i,j}, (W)_{k,l} \right) = \rho_{i,j;k,l}
\]
Define a random observation matrix \( K_x \) by
(2.2) \[
K_x = A_x + W;
\]
Let \( y \in \Omega \) be the true community assignment mapping. Given the observation \( K_y \), the goal is to determine the true community assignment mapping \( y \).

2.2. Maximum Likelihood Estimation. We shall apply the maximum likelihood estimation (MLE) to identify the true community assignment mapping \( y \).

For \( i \in [k] \), let \( n_i \) be the number of vertices in community \( i \) under the mapping \( y \). More precisely, \( n_1, \ldots, n_k \) are positive integers satisfying
\[
\sum_{i=1}^{k} n_i = n.
\]
and
\[
|y^{-1}(i)| = n_i, \; \forall i \in [k].
\]
Let
\[
\Omega_{n_1,\ldots,n_k} := \{ x \in \Omega : |x^{-1}(i)| = n_i, \; \forall i \in [k] \}
\]
be the set of all the community assignment mappings such that there are exactly \( n_i \) vertices in the community \( i \), for each \( i \in [k] \).

Definition 2.1. Let
(2.3) \[
\hat{y} := \text{argmin}_{x \in \Omega} \sum_{i,k \in [p]; j,l \in [n]} (K_y - A_x)_{i,j} (\Sigma^\dagger)_{i,j;k,l} (K_y - A_x)_{k,l}
\]
and
(2.4) \[
\tilde{y} := \text{argmin}_{x \in \Omega_{n_1,\ldots,n_k}} \sum_{i,k \in [p]; j,l \in [n]} (K_y - A_x)_{i,j} (\Sigma^\dagger)_{i,j;k,l} (K_y - A_x)_{k,l}
\]
Here \( \Sigma^\dagger \) is the Moore-Penrose inverse of the matrix \( \Sigma \) as defined in Definition A.2.

Lemma 2.2. Let \( \hat{y} \) and \( \tilde{y} \) be defined as in Definition 2.1. Then \( \hat{y} \) is the MLE with respect to the observation \( K_y \) in \( \Omega \). \( \tilde{y} \) is the MLE with respect to the observation \( K_y \) in \( \Omega_{n_1,\ldots,n_k} \).

Proof. By definition, the MLE with respect to the observation \( K_y \) in \( \Omega \) (resp. \( \Omega_{n_1,\ldots,n_k} \)) should maximize the probability density of the observation \( K_y \) among all \( x \in \Omega \) (resp. \( x \in \Omega_{n_1,\ldots,n_k} \)). If the true community assignment mapping \( y = x \), we may consider \( K_y \) as a random vector with mean value \( A_x \), and co-variance \( \Sigma \).

Therefore if we let \( \mathcal{R} \simeq \mathbb{R}^{\text{rank}(\Sigma)} \) be the rank(\( \Gamma \))-dimensional subspace where \( W \) is supported, i.e.
\[
\mathcal{R} := \{ \Sigma^\dagger v : v \in \mathbb{R}^n \}.\]
Here $\Sigma^{1/2}$ is defined as in Definition A.5. With respect to the Lebesque measure on $A_x + \mathcal{R}$, the probability density of $K_y$ is

\[
(2.5) \quad \left(\frac{1}{\sqrt{(2\pi)^m \det^* \Sigma}}\right) e^{-\frac{1}{2} \sum_{i,k \in [p]; j,l \in [n]} (K_y - A_x)_{i,j} (\Sigma^\dagger)_{i,j;k,l} (K_y - A_x)_{k,l},}
\]

where $\det^*$ is the pseudo-determinant of a matrix as defined in (A.1). It is straightforward to check that the minimizer of

\[
G_\Sigma(x, y) := \sum_{i,k \in [p]; j,l \in [n]} (K_y - A_x)_{i,j} (\Sigma^\dagger)_{i,j;k,l} (K_y - A_x)_{k,l}
\]

is exactly the maximizer of the probability density (2.5). Then the lemma follows. □

2.3. Necessary and Sufficient Conditions for the Exact Recovery of MLE. The main results of the paper is about the necessary and sufficient conditions for the exact recovery of MLE. Before stating these conditions, we first introduce a few definitions. Definition 2.3 describes the sample space for community assignment mappings in which we implement the MLE. Instead of running the MLE among all the possible community assignment mappings, we make a natural regularity assumption requiring that the number of vertices in each community is at least $cn$ for $c \in (0, 1)$.

**Definition 2.3.** For each real number $c \in (0, 1)$, let

\[
\Omega_c := \left\{ x \in \Omega : \frac{|x^{-1}(i)|}{\sum_{j \in [k]} |x^{-1}(j)|} \geq c, \forall i \in [k] \right\},
\]

i.e. $\Omega_c$ consists of all the community assignment mappings such that the ratio of the number of vertices in each community to the total number of vertices is at least $c$.

An important observation is that the MLE can never distinguish two community assignment mappings obtained from each other by a composition with a $\theta$-preserving permutation of communities; see Lemma 3.1. Hence the best thing one can expect from the MLE is to recover the community assignment mapping up to equivalence defined by a composition with a $\theta$-preserving permutation of communities.

**Definition 2.4.** For $x \in \Omega$, let $C(x)$ consist of all the $x' \in \Omega$ such that $x'$ can be obtained from $x$ by a $\theta$-preserving bijection of communities. More precisely, $x' \in C(x) \subset \Omega$ if and only if the following conditions hold

1. for $i, j \in [n]$, $x(i) = x(j)$ if and only if $x'(i) = x'(j)$; and
2. for $i \in [p]$ and $j \in [n]$, $\theta(x, i, x(j)) = \theta(x', i, x'(j))$.

Note that condition (1) above is equivalent of saying that there is a bijection $\eta : [k] \to [k]$, such that

\[
x = \eta \circ x'
\]

where $\circ$ denotes the composition of two mappings; (2) says that the bijection $\eta$ is $\theta$-preserving.
We define an equivalence relation on $\Omega$ as follows: we say $x, z \in \Omega$ are equivalent if and only if $x \in C(z)$. Let $\overline{\Omega}$ be the set of all the equivalence classes in $\Omega$. More precisely, 

$$\overline{\Omega} := \{ C(x) : x \in \Omega \}.$$ 

We also assume that $\theta$ satisfies the following assumption.

**Assumption 2.5.** Let $x, z \in \Omega$. If for any $i \in [p]$ and $j \in [n]$, 

$$\theta(x, i, x(j)) = \theta(z, i, z(j));$$

then $x \in C(z)$.

Assumption 2.5 actually says that for two community assignment mappings $x$ and $z$, if they are not equivalent then $\theta \circ x$ and $\theta \circ z$ are different. In other words, it assumes that $\theta$ can distinguish different equivalence classes in $\Omega$.

Define

$$L_\Sigma(x, y) := \sum_{i, k \in [p] ; j, l \in [n]} (A_x - A_y)_{i,j} (\Sigma^\dagger)_{i;j,k,l} (A_x - A_y)_{k,l}$$

**Definition 2.6.** For $i, j \in [k]$ and $x, z \in \Omega$, let $t_{i,j}(x, z)$ be the number of vertices in $[n]$ which are in community $i$ under the mapping $x$ and in community $j$ under the mapping $z$. More precisely, $t_{i,j}(x, z)$ is a nonnegative integer given by 

$$t_{i,j}(x, z) = |x^{-1}(i) \cap z^{-1}(j)|.$$ 

satisfying

$$\sum_{j \in [k]} t_{i,j}(x, z) = n_i(x); \quad \sum_{i \in [k]} t_{i,j}(x, z) = n_j(z);$$

Define 

$$S_{i,j}(x, z) := x^{-1}(i) \cap z^{-1}(j).$$

**Definition 2.7.** Define a set

$$B := \left\{ (t_{1,1}, t_{1,2}, \ldots, t_{k,k}) \in \{0, 1, 2, \ldots, n\}^{k^2} : \sum_{i=1}^{k} t_{i,j} = n_j \right\}.$$

For $\epsilon > 0$, define a set $B_\epsilon$ consisting of all the $(t_{1,1}, t_{1,2}, \ldots, t_{k,k}) \in B$ satisfying all the following conditions:

1. \( \forall i \in [k], \max_{j \in [k]} t_{j,i} \geq n_i - ne. \)
2. For $i \in [k]$, let $t_{w(i), i} = \max_{j \in [k]} t_{j,i}$. Then $w$ is a bijection from $[k]$ to $[k]$.
3. $w$ is $\theta$-preserving, i.e. for any $x \in \Omega$, $i \in [p]$ and $a \in [k]$, we have 

$$\theta(x, i, a) = \theta(w \circ x, i, w(a)).$$

**Assumption 2.8.** Assume $\epsilon \in (0, \frac{c}{2n_\epsilon})$, $x \in \Omega_{\frac{c}{2}}$ and $y \in \Omega_\epsilon$. Assume there exists $\Delta_1 > 0$ such that:
Let \( y_1, y_2 \in \Omega_{\frac{2c}{n}} \) and \( a, b \in [k] \) such that \( a \neq b \). Let \( i, j \in [n] \) such that \( i \in y_1^{-1}(a) \cap x^{-1}(b) \). Let \( y_2 : [n] \to [k] \) be defined as follows

\[
y_2(j) := \begin{cases} 
    b & \text{if } j = i \\
    y_1(j) & \text{if } j \in [n] \setminus \{i\}.
\end{cases}
\]

When

\[
(2.10)
\]

\[
(t_{1,1}(x, y_1), t_{1,2}(x, y_1), \ldots, t_{k,k}(x, y_1)) \in B_{\epsilon}
\]

such that for all \( i \in [k] \)

\[
t_{i,i} = \max_{j \in [k]} t_{j,i}(x, y_1);
\]

\( \epsilon \in (0, \frac{2c}{3k}) \); and \( y_1 \notin C(x) \); we have

\[
(2.11)
\]

\[
L_\Sigma(x, y_1) - L_\Sigma(x, y_2) \geq \Delta_1(1 + o(1)).
\]

where \( o(1) \to 0 \), as \( n \to \infty \).

**Assumption 2.9.** Assume \( \epsilon \in (0, \frac{2c}{3k}) \), \( x \in \Omega_{\frac{2c}{n}} \) and \( y \in \Omega_c \). Assume there exists \( \Delta_1 > 0 \) such that:

Let \( y_1^{(a,i)}, y_2 \in \Omega_{\frac{2c}{n}} \) and \( a, b \in [k] \) such that \( a \neq b \). Let \( i, j \in [n] \) such that \( i \in y_2^{-1}(b) \cap x^{-1}(b) \). Let \( y_1^{(a,i)} : [n] \to [k] \) be defined as follows

\[
y_1^{(a,i)}(j) := \begin{cases} 
    a & \text{if } j = i \\
    y_2(j) & \text{if } j \in [n] \setminus \{i\}.
\end{cases}
\]

When

\[
(2.12)
\]

\[
(t_{1,1}(x, y_1^{(a,i)}), t_{1,2}(x, y_1^{(a,i)}), \ldots, t_{k,k}(x, y_1^{(a,i)})) \in B_{\epsilon}
\]

such that for all \( i \in [k] \)

\[
t_{i,i} = \max_{j \in [k]} t_{j,i}(x, y_1);
\]

\( \epsilon \in (0, \frac{2c}{3k}) \); and \( y_1 \notin C(x) \); then we have

\[
(2.13)
\]

\[
\lim_{n \to \infty} \sum_{a \in [n]} \sum_{j \in [k]} e^{-\frac{L_\Sigma(x, y_1^{(a,i)}) - L_\Sigma(x, y_2)}{8}} = 0.
\]

Theorem 2.10 gives a sufficient condition for the exact recovery when the number of vertices in each community is unknown, and will be proved in Section 3.

**Theorem 2.10.** Assume \( y \in \Omega_c \) is the true community assignment mapping. Suppose that Assumptions 2.5 2.9 hold. Let \( \epsilon \in (0, \frac{2c}{3k}) \). If

\[
(2.14)
\]

\[
\lim_{n \to \infty} n \log k - \frac{1}{8} \min_{x : (t_{1,1}(x,y), \ldots, t_{k,k}(x,y)) \in B_{\epsilon} \cup B_{\epsilon}} L_\Sigma(x, y) = -\infty,
\]

then \( \lim_{n \to \infty} \Pr(\hat{y} \in C(y)) = 1 \).
Corollary 2.11. Assume \( y \in \Omega_c \) is the true community assignment mapping. Suppose that Assumptions 2.5 and 2.8 hold. Let \( \epsilon \in (0, \frac{2c}{3k}) \). If (2.21) holds, and for any constant \( \delta > 0 \) independent of \( n \),

\[
\lim_{n \to \infty} \log k + \log n - \frac{\Delta_1(1-\delta)}{8} = -\infty, (2.17)
\]

then \( \lim_{n \to \infty} \Pr(\hat{y} \in C(y)) = 1 \).

Definition 2.12. Define the distance function

\[
D_{\Omega}(x, y) = \sum_{i,j \in [k], i \neq j} t_{i,j}(x, y).
\]

for \( x, y \in \Omega \).

From Definition 2.12, it is straightforward to check that

\[
D_{\Omega}(x, y) = n - \sum_{i \in [k]} t_{i,i}(x, y)
\]

Assumption 2.13. Assume \( x, y_m, y_h \in \Omega \) such that

1. \( D_{\Omega}(y_m, y_h) = j \), where \( j \geq 2 \) is a positive integer; and
2. There exist \( u_1, \ldots, u_j \in [n] \), such that
   (a) \( y_m(v) = y_h(v) \), for all \( v \in [n] \setminus \{u_1, \ldots, u_j\} \); and
   (b) \( y_m(u_i) \neq y_h(u_i) = x(u_i) = y_m(u_{i-1}) \) for all \( i \in [j] \).
   (c) \( t_{1,1}(x, y_m), t_{1,2}(x, y_m), \ldots, t_{k,k}(x, y_m) \in B_{\epsilon} \) with \( \epsilon \in (0, \frac{2c}{3k}) \) and \( w(i) = i \).

Then

\[
L_\Sigma(x, y_m) - L_\Sigma(x, y_h) \geq j\Delta_2(1 + o(1)) (2.18)
\]

for some \( \Delta_2 > 0 \).

Theorem 2.16 gives a sufficient condition for the exact recovery when the number of vertices in each community is known, and will be proved in Section 4.

Assumption 2.14. Assume \( \epsilon \in (0, \frac{2c}{3k}) \), \( x \in \Omega_{\frac{2c}{3k}} \), and \( y \in \Omega_c \). Then for all \( x, y \in \Omega \), and

\[
(t_{1,1}(x, y), t_{1,2}(x, y), \ldots, t_{k,k}(x, y)) \in B \setminus B_{\epsilon}, (2.19)
\]

we have

\[
\|Q_r Y_x\|^2 \geq T(n) (2.20)
\]

where for \( Q_r \Gamma_x \in \mathbb{R}^{r \times 1} \), \( \|Q_r Y_x\|^2 \) is the sum of squares of its \( pn \) components.

Remark 2.15. When \( \Sigma \) is invertible \( Q_r = Q \), and (2.20) becomes

\[
\|Y_x\|^2 \geq T(n);
\]

given that \( Q \) is orthogonal.
Theorem 2.16. Suppose that Assumptions 2.14, 2.13 hold. If
\begin{equation}
\lim_{n \to \infty} n \log k - \frac{T(n)}{8\lambda_1(n)} = -\infty,
\end{equation}
and for any constant \( \delta > 0 \) independent of \( n \),
\begin{equation}
\lim_{n \to \infty} \log k + \log n - \frac{\Delta_2(1-\delta)}{8} = -\infty,
\end{equation}
then \( \lim_{n \to \infty} \Pr(\hat{y} \in C(y)) = 1 \).

Theorem 2.17 gives a necessary condition for the exact recovery when the number of vertices in each community is known, and will be proved in Section 5.

Recall that \( y \in \Omega \) is the true community assignment mapping satisfying \( |y^{-1}(i)| = n_i \), for all \( i \in [k] \). Let \( a \in [n] \). Let \( y^{(a)} \in \Omega \) be defined by
\begin{equation}
y^{(a)}(i) = \begin{cases} 
y(i) & \text{if } i \in [n], \text{ and } i \neq a \\
y^{(a)}(a) & \text{if } i = a.
\end{cases}
\end{equation}
such that
\( y(a) \neq y^{(a)}(a) \in [k] \).

Theorem 2.17. Let \( H \subseteq [n] \) be a subset of vertices. For \( a \in H \), let
\begin{equation}
\eta_a := \frac{2 \sum_{i,k \in [p]; j,l \in [n]} \left( A_{y^{(a)}} - A_y \right)_{i,j} (\Sigma^\dagger)_{i,j;k,l}(W)_{k,l}}{L_{\Sigma}(y^{(a)}, y)}
\end{equation}
Let \( \Phi_H \) be the covariance matrix for \( \{\eta_a\}_{a \in H} \) such that \( \lambda_0(n,H) \) is the minimal eigenvalue of \( \Phi_H \), and that there exists a constant \( \delta > 0 \) independent of \( n \) satisfying
\begin{equation}
\sqrt{2\lambda_0(n,H) \log |H|} > 1 + \delta;
\end{equation}
where \( |H| \) is the cardinality of \( H \). Then
\begin{equation}
\lim_{n \to \infty} \Pr(\hat{y} \in C(y)) = 0.
\end{equation}

Theorem 2.18 gives a necessary condition for the exact recovery when the number of vertices in each community is unknown, and will be proved in Section 6.

Let \( a, b \in [n] \) such that \( y(a) \neq y(b) \). Let \( y^{(ab)} \in \Omega_{n_1, \ldots, n_k} \) be the community assignment mapping defined by
\begin{equation}
y^{(ab)}(i) = \begin{cases} 
y(i) & \text{if } i \in [n] \setminus \{a,b\} \\
y(b) & \text{if } i = a \\
y(a) & \text{if } i = b
\end{cases}
\end{equation}
In other words, \( y^{(ab)} \) is obtained from \( y \) by exchanging \( y(a) \) and \( y(b) \).

Theorem 2.18. Let \( H_1, H_2 \subseteq [n] \) be two subsets of vertices. For \( a \in H_1, b \in H_2 \), let
\begin{equation}
\eta_{ab} := \frac{2 \sum_{i,k \in [p]; j,l \in [n]} \left( A_{y^{(ab)}} - A_y \right)_{i,j} (\Sigma^\dagger)_{i,j;k,l}(W)_{k,l}}{L_{\Sigma}(y^{(ab)}, y)}
\end{equation}
Let $\Psi_{H_1,H_2}$ be the covariance matrix for $\{\eta_{a,b}\}_{a \in H_1, b \in H_2}$ with minimal eigenvalue $\mu_0(n, H_1, H_2)$, such that there exists a constant $\delta > 0$ independent of $n$ satisfying
\[
\sqrt{2\mu_0(n, H_1, H_2)} \log(|H_1||H_2|) > 1 + \delta
\]
Then
\[
\lim_{n \to \infty} \text{Pr}(\hat{y} \in C(y)) = 0.
\]

Given Theorems 2.10, 2.16, 2.17, 2.18, we may wondering if there exist sharp threshold for the exact recovery. In other words, whether or not there exist certain conditions, such that exact recover occurs if and only if these conditions hold. Consider for example, when the number of vertices in each community is unknown. By Theorems 2.10 and 2.17, if
\[
\log |H| = (1 + o(1)) \log n;
\]
and
\[
\lambda_0(n, H) = \frac{4}{\Delta_1}
\]
Then condition (2.25) becomes
\[
\Delta_1 < 8(1 - \delta') \log n
\]
for some constant $\delta' > 0$, which is very close to the complement of condition (2.22) in Theorem 2.10. Note that $\lambda_0(n, H)$ is the minimal eigenvalue for the covariance matrix $\Phi_H$ of $\{\eta_a\}_{a \in H}$, and the diagonal entries of $\Phi_H$ is given by
\[
\mathbb{E}\eta_a^2 = \frac{4}{L^\Sigma(y^{(a)}, y)} \leq \frac{4}{\Delta_1}
\]
by Assumption (2.8). We can see as long as $\Phi_H$ is close to a diagonal matrix when $n \to \infty$, and when the identity holds in (2.12), we can have a sharp threshold for exact recovery. This is exactly the case of the example we discuss in Section 7.

3. Exact Recovery with Unknown Community Sizes

In this section, we prove Theorem 2.10 and its corollary (Corollary 3.3).

3.1. Proof of Theorem 2.10. Recall that $y \in \Omega_{n_1, \ldots, n_k}$ is the true community assignment mapping. Note that
\[
G_{\Sigma}(x, y) = \sum_{i,k \in [p], j,l \in [n]} (K_y)_{i,j} (\Sigma^\dagger)_{i,j;k,l} (K_y)_{k,l} + \sum_{i,k \in [p], j,l \in [n]} (A_x)_{i,j} (\Sigma^\dagger)_{i,j;k,l} (A_x)_{k,l}
\]
\[
-2 \sum_{i,k \in [p], j,l \in [n]} (A_x)_{i,j} (\Sigma^\dagger)_{i,j;k,l} (K_y)_{k,l}
\]
For each fixed observation $K_y$, $\sum_{i,k \in [p], j,l \in [n]} (K_y)_{i,j} (\Sigma^\dagger)_{i,j;k,l} (K_y)_{k,l}$ is fixed and independent of $x \in \Omega$. Therefore
\[
\hat{y} = \arg\min_{x \in \Omega} G_{\Sigma}(x, y)
\]
\[
= \arg\min_{x \in \Omega} \left[ \sum_{i,k \in [p], j,l \in [n]} (A_x)_{i,j} (\Sigma^\dagger)_{i,j;k,l} (A_x)_{k,l} - 2 (A_x)_{i,j} (\Sigma^\dagger)_{i,j;k,l} (K_y)_{k,l} \right]
\]
For \( x \in \Omega \), define
\[
(3.1) \quad f(x) := \sum_{i,k \in [p]; j,l \in [n]} (A_x)_{i,j} (\Sigma^\dagger)_{i,j;k,l} (A_x)_{k,l} - 2 (A_x)_{i,j} (\Sigma^\dagger)_{i,j;k,l} (K_y)_{k,l}
\]
Then
\[
(3.2) \quad f(x) - f(y) = \sum_{i,k \in [p]; j,l \in [n]} (A_x)_{i,j} (\Sigma^\dagger)_{i,j;k,l} (A_x)_{k,l} - 2 \left[(A_x)_{i,j} - (A_y)_{i,j}\right] (\Sigma^\dagger)_{i,j;k,l} (A_y)_{k,l} - 2 \left[(A_y)_{i,j} - (A_y)_{i,j}\right] (\Sigma^\dagger)_{i,j;k,l} (W)_{k,l}
\]
Then \( f(x) - f(y) \) is a Gaussian random variable with mean value
\[
\mathbb{E}(f(x) - f(y)) = L \Sigma(x,y);
\]
and variance
\[
\text{Var}(f(x) - f(y)) = 4L \Sigma(x,y).
\]
It is straightforward to check the following lemma.

**Lemma 3.1.** Let \( x, z \in \Omega \). If \( x \in C(z) \), then
\[
f(x) = f(z); \quad K_x = K_z.
\]
Define
\[
p(\hat{y}; \sigma) := \Pr(\hat{y} \in C(y)) = \Pr\left(f(y) < \min_{C(x) \in \Pi, \Pi(x) \neq C(y)} f(x)\right)
\]
Then
\[
1 - p(\hat{y}; \sigma) \leq \sum_{C(x) \in \Pi; C(x) \neq C(y)} \Pr(f(x) - f(y) \leq 0)
\]
\[
= \sum_{C(x) \in \Pi; C(x) \neq C(y)} \Pr_{\xi \sim \mathcal{N}(0,1)} \left(\xi \leq -\frac{L \Sigma(x,y)}{2}\right)
\]
\[
\leq \sum_{C(x) \in \Pi; C(x) \neq C(y)} e^{-\frac{L \Sigma(x,y)}{8}}.
\]

**Proof of Theorem 2.10.** Note that
\[
\sum_{C(x) \in \Pi \setminus \{C(y)\}} e^{-\frac{L \Sigma(x,y)}{8}} \leq I_1 + I_2
\]
where

\[ I_1 = \sum_{C(x) \in \prod_{\mathbb{S}_x^I}(t_{1,1}(x,y), \ldots, t_{k,k}(x,y)) \in [\mathcal{B} \setminus \mathcal{B}_f], C(x) \not= C(y)} e^{-L_S(x,y) \over 8} \]

and

\[ I_2 = \sum_{C(x) \in \prod_{\mathbb{S}_x^I}(t_{1,1}(x,y), \ldots, t_{k,k}(x,y)) \in \mathcal{B}, C(x) \not= C(y)} e^{-L_S(x,y) \over 8}. \]

By Lemma 3.2, when Assumption 2.14 holds, we have

\[ I_1 \leq k^n e^{-\min_{x, y} (t_{1,1}(x,y), \ldots, t_{k,k}(x,y)) \in [B \setminus B_e] L_S(x,y) } \]

When (2.21) holds, we obtain

\[ \lim_{n \to \infty} I_1 = 0. \]

Now let us consider \( I_2 \). Assume \( \epsilon \in (0, \frac{2n}{3}) \). Let \( w \) be the bijection from \([k]\) to \([k]\) as defined in Definition 2.7(3). Let \( y^* \in \Omega \) be defined by

\[ y^*(z) = w(y(z)), \forall z \in [n]. \]

Then \( y^* \in C(y) \). Moreover, \( x \) and \( y^* \) satisfies

\[ t_{i,i}(x, y^*) \geq n_i(y^*) - n \epsilon, \forall i \in [k]. \]

We consider the following community changing process to obtain \( x \) from \( y^* \).

1. If for all \((j, i) \in [k]^2\), and \( j \neq i \), \( t_{j,i}(x, y^*) = 0 \), then \( x = y^* \).
2. If (1) does not hold, find the least \((j_1, i_1) \in [k]^2\) in lexicographic order such that \( j_1 \neq i_1 \) and \( t_{j_1,i_1}(x, y^*) > 0 \). Choose an arbitrary vertex \( u_1 \in S_{j_1,i_1}(x, y^*) \). Define \( y_1 \in \Omega \) as follows

\[ y_1(z) = \begin{cases} j_1 & \text{if } z = u_1 \\ y^*(z) & \text{if } z \in [n] \setminus \{u_1\} \end{cases} \]

Then we have

\[ t_{j_1,i_1}(x, y_1) = t_{j_1,i_1}(x, y^*) - 1 \]

and

\[ t_{j_1,j_1}(x, y_1) = t_{j_1,j_1}(x, y^*) + 1. \]

Therefore, \( x, y_1 \) and \( y^* \) satisfies

\[ t_{i_1,i_1}(x, y_1) \geq n_{i_1}(y_1) - n \epsilon; \]

\[ t_{i_1,i_1}(x, y_1) \geq n_{i_1}(y^*) - n \epsilon; \]

\[ n_{i_1}(y_1) \geq n_{i_1}(y^*) - n \epsilon; \]

for all \( i \in [k] \). We deduce that whenever \( \{t_{i,j}(x, y^*)\}_{(i,j) \in [k]^2} \in \mathcal{B}_e \), \( \{t_{i,j}(x, y_1)\}_{(i,j) \in [k]^2} \in \mathcal{B}_e \).
In general, if we have constructed \( y_r \in \Omega \) (\( r \geq 1 \)) satisfying all the following conditions:

\[
\begin{align*}
  t_{i,i}(x, y_r) &\geq n_i(y_r) - \varepsilon; \\
  t_{i,i}(x, y_r) &\geq n_i(y^*) - \varepsilon; \\
  n_i(y_r) &\geq n_i(y^*) - \varepsilon;
\end{align*}
\]

(3.7)

for all \( i \in [k] \). We now construct \( y_{r+1} \in \Omega \) as follows.

(a) If for all \( (j, i) \in [k]^2 \), and \( j \neq i \), \( t_{j,i}(x, y_r) = 0 \), then \( x = y_r \); then the construction process stops at this step.
(b) If (a) does not hold, find the least \( (j_{r+1}, i_{r+1}) \in [k]^2 \) in lexicographic order such that \( j_{r+1} \neq i_{r+1} \) and \( t_{j_{r+1},i_{r+1}}(x, y_r) > 0 \). Choose an arbitrary vertex \( u \in S_{j_{r+1},i_{r+1}}(x, y_r) \). Define \( y_{r+1} \in \Omega \) as follows

\[
y_{r+1}(z) = \begin{cases} 
  j_{r+1} & \text{if } z = u_{r+1} \\
  y_r(z) & \text{if } z \in [n] \setminus \{u_{r+1}\}
\end{cases}
\]

Then it is straightforward to check that

\[
\begin{align*}
  t_{i,i}(x, y_{r+1}) &\geq n_i(y_{r+1}) - \varepsilon; \\
  t_{i,i}(x, y_{r+1}) &\geq n_i(y^*) - \varepsilon; \\
  n_i(y_{r+1}) &\geq n_i(y^*) - \varepsilon;
\end{align*}
\]

for all \( i \in [k] \).

Recall that the distance \( D_\Omega \) in \( \Omega \) is defined in Definition 2.12. From the constructions of \( y_{r+1} \) we have

\[ D_\Omega(x, y_{r+1}) = D_\Omega(x, y_r) - 1. \]

Therefore there exists \( h \in [n] \), such that \( y_h = x \).

Since any \( x \) in \( \mathcal{B}_c \) can be obtained from \( y \) by the community changing process described above, we have

\[
I_2 \leq \sum_{h=1}^{\infty} \sum_{u_1, \ldots, u_h \in [n]} \sum_{j_1 \ldots j_h \in [k]} e^{-L_\Sigma(y_{u_1}^{(u,j)}, y^*)} e^{-L_\Sigma(y_{j_1}^{(u,j)}, y^*)};
\]

\[
= \sum_{h=1}^{\infty} \sum_{u_1, \ldots, u_h \in [n]} \sum_{j_1 \ldots j_h \in [k]} \prod_{g=1}^{h} e^{-L_\Sigma(y_{j_g}^{(u,j)}, y^*)} e^{-L_\Sigma(y_{u_g}^{(u,j)}, y^*)};
\]

where \( u := (u_1, \ldots, u_h) \), \( j := (j_1, \ldots, j_h) \); and for \( g \in [h] \) \( y_{u_g}^{(u,j)} \in \Omega \) is defined by

\[
y_{u_g}^{(u,j)}(z) = \begin{cases} 
  j_g & \text{if } z = u_l, \text{ for } l \in [g] \\
  y^*(z) & \text{otherwise.}
\end{cases}
\]

where \( z \in [n] \); and \( y_{0}^{(u,j)} = y^* \). Then

\[
I_2 \leq \sum_{h=1}^{\infty} \max_{x,y_2 \in \Omega \text{ s.t. (2.14) holds}} \max_{x,y_2 \in \Omega \text{ s.t. (2.14) holds}} \sum_{a \in [n]} \sum_{j \in [k]} e^{-L_\Sigma(x,y^*,a,j)} e^{-L_\Sigma(x,y_2)}
\]

\[
\begin{bmatrix}
\end{bmatrix}^h
\]
When (2.15) holds, we obtain
\[
\lim_{n \to \infty} I_2 = 0 \tag{3.8}
\]
Then the theorem follows from (3.3) and (3.8).

**Proof of Corollary 2.11.** By Theorem 2.10, it suffices to show that Assumptions 2.8 and (2.22) implies Assumption 2.9. Indeed by (2.12) we have
\[
\sum_{a \in [n]} \sum_{j \in [k]} e^{-L_{\Sigma}(x,y)^{(n)}_{a,j}} \leq nke^{-\frac{\Delta_1}{\mathfrak{s}}}
\]
Then (2.15) follows from (2.22).

### 3.2. Another corollary of Theorem 2.10

Since \(\Sigma\) is symmetric and positive definite, there exists an orthogonal matrix \(Q \in \mathbb{R}^{pn \times pn}\), such that
\[
\Sigma = Q^t \Lambda Q
\]
where
\[
\Lambda = \text{diag}[\lambda_1(n), \ldots, \lambda_r(n), 0, \ldots, 0];
\]
such that \(r \in [pn]\) is the rank of \(\Sigma\)
\[
\lambda_1(n) \geq \lambda_2(n) \geq \ldots \lambda_r(n) > 0
\]
are eigenvalues of \(\Sigma\). Let
\[
\Lambda_r : = \text{diag}[\lambda_1(n), \ldots, \lambda_r(n)];
\]
and assume that \(Q_r \in \mathbb{R}^{r \times pn}\), such that
\[
Q = \begin{pmatrix} Q_r \\ Q_r \end{pmatrix}
\]
Then by Proposition A.3, we obtain
\[
\Sigma^\dagger = Q_r^t \Lambda_r^{-1} Q_r. \tag{3.9}
\]
For \(x \in \Omega\), define \(\Upsilon_x, \Gamma_x \in \mathbb{R}^{pn \times 1}\) by
\[
[\Upsilon_x]_{n(i-1)+j} : = 2(A_x - A_y)_{i,j}; \quad \forall i \in [p], j \in [n]. \tag{3.10}
\]
\[
[\Gamma_x]_{n(i-1)+j} : = \frac{2(A_x - A_y)_{i,j}}{L_{\Sigma}(x,y)}; \quad \forall i \in [p], j \in [n]. \tag{3.11}
\]

**Lemma 3.2.** Assume that \(\theta, \Sigma\) satisfies Assumptions 2.14. Then for all the \(x,y \in \Omega\) such that (2.19) holds, we have
\[
L_{\Sigma}(x,y) \geq \frac{T(n)}{\lambda_1(n)}.
\]
where \(\lambda_1(n) > 0\) is the maximal eigenvalue of \(\Sigma\).
Proof. Then by (3.9) we have
\[
L \Sigma(x,y) = Y^t \Sigma^t Y = \|Q_r Y_x\|^2 - \left(\frac{Y^t Q_r^t \Lambda_r^{-1} Q_r Y_x}{\|Q_r Y_x\|^2}\right)
\]
By the min-max theorem for matrix eigenvalues, we have
\[
\frac{Y^t Q_r^t \Lambda_r^{-1} Q_r Y_x}{\|Q_r Y_x\|^2} \geq \frac{1}{\lambda_1(n)}
\]
Then the lemma follows from Assumption 2.14. □

Corollary 3.3. Assume \( y \in \Omega_c \) is the true community assignment mapping. Let \( \lambda_1(n) > 0 \) be the maximal eigenvalue of \( \Sigma \). Suppose that Assumptions 2.5, 2.14 and 2.8 hold. Let \( \epsilon \in (0, \frac{2c}{3k}) \). If (2.21) and (2.22) hold, then \( \lim_{n \to \infty} \Pr(\hat{y} \in C(y)) = 1 \).

Proof. The corollary follows from Lemma 3.2 and Theorem 2.10. □

4. Exact Recovery with Known Community Sizes

In this section, we prove Theorem 2.16.

For each \( x \in \Omega_{n_1, \ldots, n_k} \), let
\[
C^*(x) := C(x) \cap \Omega_{n_1, \ldots, n_k};
\]
i.e. \( C^*(x) \) consists of all the community assignment mappings in \( \Omega_{n_1, \ldots, n_k} \) that are equivalent to \( x \) in the sense of Definition 2.4. Let
\[
\overline{\Omega}_{n_1, \ldots, n_k} := \{ C^*(x) : x \in \Omega_{n_1, \ldots, n_k} \};
\]
i.e. \( \overline{\Omega}_{n_1, \ldots, n_k} \) consists of all the equivalence classes in \( \Omega_{n_1, \ldots, n_k} \).

By Lemma 3.1, we see that for \( x, z \in \Omega_{n_1, \ldots, n_k} \), if \( x \in C^*(z) \), then \( f(x) = f(z) \). Then
\[
1 - p(\bar{y}; \sigma) \leq \sum_{C^*(x) \in (\overline{\Omega}_{n_1, \ldots, n_k} \setminus \{C^*(y)\})} \Pr(f(x) - f(y) \leq 0)
\]
\[
= \sum_{C^*(x) \in (\overline{\Omega}_{n_1, \ldots, n_k} \setminus \{C^*(y)\})} \Pr_{\xi \in \mathcal{N}(0,1)} \left( \xi \geq \frac{\sqrt{L \Sigma(x,y)}}{2} \right)
\]
\[
\leq \sum_{C^*(x) \in (\overline{\Omega}_{n_1, \ldots, n_k} \setminus \{C^*(y)\})} e^{-\frac{L \Sigma(x,y)}{8}}
\]

Proof of Theorem 2.16. Let
\[
J := \sum_{C^*(x) \in (\overline{\Omega}_{n_1, \ldots, n_k} \setminus \{C^*(y)\})} e^{-\frac{L \Sigma(x,y)}{8}}
\]
By (4.1), it suffices to show that \( \lim_{n \to \infty} J = 0 \).

Let
\[
0 < \epsilon < \min \left( \frac{2c}{3k}, \min_{i,j \in [k], n_i \neq n_j} \left| \frac{n_i - n_j}{n} \right| \right)
\]
Note that

\[ J \leq J_1 + J_2, \]

where

\[ J_1 = \sum_{C^*(x) \in \prod_{n_1, \ldots, n_k} (t_{1,1}(x,y), \ldots, t_{k,k}(x,y)) \in (B, B^c), C^*(x) \neq C^*(y)} e^{-L_{\Sigma}(x,y)} \]

and

\[ J_2 = \sum_{C^*(x) \in \prod_{n_1, \ldots, n_k} (t_{1,1}(x,y), \ldots, t_{k,k}(x,y)) \in B, C(x) \neq C(y)} e^{-L_{\Sigma}(x,y)}. \] (4.2)

Under Assumption 2.14, by Lemma 3.2 we have

\[ 0 \leq J_1 \leq k^n e^{-\frac{T(o)}{8m_1(n)}}. \]

By (2.21), we have

\[ \lim_{n \to \infty} J_1 = 0. \] (4.3)

Under Assumption 2.13, we can show that

\[ \lim_{n \to \infty} J_2 = 0 \]

by the same argument as in the proof of Theorem 2.8 in [16]. Then the theorem follows.

\[ \square \]

5. No Exact Recovery with Unknown Community Sizes

In this section, we prove Theorem 2.17 and its corollary (Corollary 5.1).

5.1. Proof of Theorem 2.17. Note that

\[ 1 - p(\hat{y}; \sigma) \geq \Pr \left( \bigcup_{a \in [n]} \{ f(y^{(a)}) - f(y) < 0 \} \right) \]

Since any of the event \( \{ f(y^{(a)}) - f(y) < 0 \} \) implies \( \hat{y} \neq y \).

By (2.7) we obtain

\[ L_{\Sigma}(y^{(a)}, y) = \sum_{i,k \in p; j,l \in [n]} \left( A_{y^{(a)}} - A_y \right)_{i,j} (\Sigma^\dagger)_{i,j;k,l,1} \left( A_{y^{(a)}} - A_y \right)_{k,l}, \]

Then from (3.2) we have

\[ f(y^{(a)}) - f(y) = -2 \sum_{i,k \in p; j,l \in [n]} \left( A_{y^{(a)}} - A_y \right)_{i,j} (\Sigma^\dagger)_{i,j;k,l,1} (W)_{k,l} + L_{\Sigma}(y^{(a)}, y). \]

Let \( H \subseteq [n] \) be given as in the assumptions of the theorem. For \( a \in H \), \( f(y^{(a)}) - f(y) \) is a collection of Gaussian random variables with mean \( L_{\Sigma}(y^{(a)}, \hat{y}) \) and variance \( 4L_{\Sigma}(y^{(a)}, y) \).
Then we have
\[1 - p(\hat{y}; \sigma) \geq \Pr\left(\max_{a \in H}\{f(y) - f(y^{(a)})\} > 0\right) = \Pr\left(\max_{a \in H}\eta_a > 1\right)\]

By Corollary B.3, we have
\[
\lim_{n \to \infty} \Pr\left(\max_{a \in H}\eta_a > \sqrt{2\lambda_0(n, H) \log |H|(1 - \epsilon)}\right) = 1;
\]
where \(\lambda_0(n, H)\) is the minimal eigenvalue of \(\Phi_H\). Hence if (2.25) holds, we have (2.26). \(\square\)

5.2. A Corollary of Theorem 2.17. Note that \(\eta_a\) is a Gaussian random variable with \(\mathbb{E}\eta_a = 0\); and
\[
\mathbb{E}\eta_a^2 = \frac{4}{L_{\Sigma}(y^{(a)}, y)}.
\]
Moreover, for \(a_1, a_2 \in H\), and \(a_1 \neq a_2\), we have
\[
\mathbb{E}\eta_{a_1}\eta_{a_2} = \frac{4 \sum_{i, k \in [y], j, l \in [n]} \left(\mathbf{A}_{y^{(a_1)}} - \mathbf{A}_{y}\right)_{i,j} (\Sigma^\dagger)_{i,j;k,l} \left(\mathbf{A}_{y^{(a_2)}} - \mathbf{A}_{y}\right)_{k,l}}{L_{\Sigma}(y^{(a_1)}, y)L_{\Sigma}(y^{(a_2)}, y)}.
\]
Let \(\Gamma_H \in \mathbb{R}^{pn \times |H|}\) be defined as follows
\[
\Gamma_H := \left(\Gamma_{y^{(a_1)}}, \ldots, \Gamma_{y^{(a_h)}}\right)
\]
where \(h = |H|\) and \(H = \{a_1, \ldots, a_h\}\), and for \(x \in \Omega\), \(\Gamma_x\) is defined as in (3.11). Then the covariance matrix \(\Phi_H\) for \(\{\eta_a\}_{a \in H}\) is
\[
\Phi_H = \Gamma_H^\dagger \Sigma \Gamma_H.
\]
If \(\Sigma\) is invertible, we have
\[
\lambda_0(n, H) := \min_{\beta \in \mathbb{R}^h} \frac{\beta^\dagger \Gamma_H^\dagger \Sigma \Gamma_H \beta}{\beta^\dagger \beta} = \min_{\beta \in \mathbb{R}^h} \frac{\beta^\dagger \Gamma_H^\dagger \Sigma \Gamma_H \beta}{\beta^\dagger \beta} \cdot \frac{\beta^\dagger \Gamma_H \beta}{\beta^\dagger \beta} = \min_{\alpha \in \mathbb{R}^m} \frac{\alpha^\dagger \Sigma^\dagger \alpha}{\alpha^\dagger \alpha} \cdot \min_{\beta \in \mathbb{R}^h} \frac{\beta^\dagger \Gamma_H \beta}{\beta^\dagger \beta} = \sigma_0^2(n, H) \lambda_1(n)
\]
where \(\sigma_0(n, H)\) is the minimal singular value for \(\Gamma_H\). Then we have the following corollary:

Corollary 5.1. Let \(H \subseteq [n]\) be a subset of vertices such that \(|H| = h\). If \(\Sigma\) is invertible and
\[
\sqrt{2\sigma_0^2(n, H) \log h} > 1 + \delta; \quad \text{(5.4)}
\]
Then
\[
\lim_{n \to \infty} \Pr(\hat{y} \in C(y)) = 0.
\]
6. No Exact Recovery with Known Community Sizes

In this section, we prove Theorem 2.18 and its corollary (Corollary 6.1).

6.1. Proof of Theorem 2.18. Note that

\[ 1 - p(\tilde{y}; \sigma) \geq \Pr \left( \bigcup_{a, b \in [n], y(a) \neq y(b)} (f(y(ab)) - f(y) < 0) \right), \]

since any of the event \((f(y(ab)) - f(y) < 0)\) implies \(\tilde{y} \neq y\).

By (2.7) we obtain

\[ L_\Sigma(y(ab), y) = \sum_{i, k \in [p]; j, l \in [n]} \left( A_{y(ab)} - A_y \right)_{i, j} (\Sigma^\dagger)_{i, j; k, l} \left( A_{y(ab)} - A_y \right)_{k, l}. \]

Then from (3.2) we have

\[ f(y(ab)) - f(y) = -2 \sum_{i, k \in [p]; j, l \in [n]} \left( A_{y(ab)} - A_y \right)_{i, j} (\Sigma^\dagger)_{i, j; k, l} (W)_{k, l} + L_\Sigma(y(ab), y). \]

Let \(H_1, H_2 \subseteq [n]\) be given as in the assumptions of the theorem. For \(a \in H_1\) and \(b \in H_2\), \(f(y(ab)) - f(y)\) is a collection of Gaussian random variables with mean \(L_\Sigma(y(ab), y)\) and variance \(4L_\Sigma(y(ab), y)\). Then we have

\[ 1 - p(\tilde{y}; \sigma) \geq \Pr \left( \max_{a \in H_1, b \in H_2} \{ f(y) - f(y(ab)) \} > 0 \right) \]
\[ = \Pr \left( \max_{a \in H_1, b \in H_2} \eta_{ab} > 1 \right) \]

By Lemma B.2, we have

\[ \lim_{n \to \infty} \Pr \left( \max_{a \in H_1, b \in H_2} \eta_{ab} > \sqrt{2\mu_0(n, H_1, H_2) \log(|H_1||H_2|)(1 - \epsilon)} \right) = 1 \]

Hence if (6.1) holds, we have (2.29). \(\square\)

For \(a, b \in H\), \(\eta_{ab}\) is a Gaussian random variable with mean 0 and variance

\[ \mathbb{E}\eta_{ab}^2 = \frac{4}{L_\Sigma(y(ab), y)}. \]

Moreover, for \(a, c \in H_1\) and \(b, d \in H_2\), such that \((a, b) \neq (c, d)\), we have

\[ \mathbb{E}\eta_{ab}\eta_{cd} = \frac{4}{L_\Sigma(y(ab), y)L_\Sigma(y(cd), y)} \sum_{i, k \in [p]; j, l \in [n]} \left( A_{y(ab)} - A_y \right)_{i, j} (\Sigma^\dagger)_{i, j; k, l} \left( A_{y(cd)} - A_y \right)_{k, l}. \]

Let \(\Gamma_{H_1, H_2} \in \mathbb{R}^{pn \times |H_1||H_2|}\) be defined as follows

\[ \Gamma_{H_1, H_2} := \left( \Gamma_{y(a_1 h_1)}, \Gamma_{y(a_1 h_2)}, \ldots, \Gamma_{y(a_h h_1)} \right) \]

where \(h_i = |H_i|\) for \(i \in \{1, 2\}\), \(H_1 = \{a_1, \ldots, a_{h_1}\}\) and \(H_2 = \{b_1, \ldots, b_{h_2}\}\); for each \(x \in \Omega\), \(\Gamma_x\) is defined as in (3.11). Then the covariance matrix \(\Phi_{H_1, H_2}\) for \(\{\eta_{ab}\}_{a \in H_1, b \in H_2}\) is

\[ \Phi_{H_1, H_2} = \Gamma_{H_1, H_2}^t \Sigma \Gamma_{H_1, H_2}. \]
If $\Sigma$ is invertible, by similar computations as in (5.3), we obtain
$$
\mu_0(n, H_1, H_2) \geq \frac{\sigma_0^2(n, H_1, H_2)}{\lambda_1(n)}
$$
where $\sigma_0(n, H_1, H_2)$ is the minimal singular value for $\Gamma_{H_1, H_2}$. Then we have the following corollary:

**Corollary 6.1.** Let $H_1, H_2 \subseteq [n]$ be two subset of vertices such that $|H_1| = h_1$ and $|H_2| = h_2$. If $\Sigma$ is invertible and

$$
\sqrt{\frac{2\sigma_0^2(n, H_1, H_2) \log(h_1h_2)}{\lambda_1(n)}} > 1 + \delta; \quad (6.1)
$$

Then
$$
\lim_{n \to \infty} \Pr(\hat{y} \in C(y)) = 0.
$$

7. An Example

In this section, we study a specific Gaussian mixture model in which different entries in the Gaussian perturbation matrix are not independent. By applying the general theory developed in this paper, we obtain the threshold for the exact recovery of MLE in terms of the intensities of the Gaussian perturbation; see Theorems 7.4 and 7.5. Other examples when the Gaussian perturbation matrix consists of independent entries can be found in [17, 16].

Let $n$ be a positive integer. Let $\alpha \in (0, 1)$. Consider a graph consisting of $n$ vertices that are divided into two communities with size $\lfloor \alpha n \rfloor$, $n - \lfloor \alpha n \rfloor$, respectively. To each vertex $v \in [n]$, we associate an i.i.d. Gaussian random variable
$$
\xi_v \sim \mathcal{N}(0, s^2),
$$
where $s > 0$ is the standard deviation. Let $p = n$. For each community assignment mapping $x \in \Omega$ and $i, j \in [n]$, define
$$
(A_x)_{i,j} = \begin{cases} 
1 & \text{if } x(i) = x(j) \\
0 & \text{otherwise}
\end{cases}
$$

Define
$$
(W)_{i,j} := \xi_i + \xi_j.
$$
Then the covariance matrix $\Sigma$ for $W$ is given by
$$
(\Sigma)_{i,j;k,l} = \begin{cases} 
0 & \text{if } \{i, j\} \cap \{k, l\} = \emptyset \\
s^2 & \text{if } |\{i, j\} \cap \{k, l\}| = 1, \text{ and } i \neq j, \text{ and } k \neq l \\
2s^2 & \text{if } |\{i, j\} \cap \{k, l\}| = 1, \text{ and } i = j \text{ and } k \neq l \\
2s^2 & \text{if } |\{i, j\} \cap \{k, l\}| = 1, \text{ and } i \neq j \text{ and } k = l \\
2s^2 & \text{if } |\{i, j\} \cap \{k, l\}| = 2 \\
4s^2 & \text{if } i = j = k = l.
\end{cases}
$$
Then the rank of $\Sigma$ is $n$. It has an eigenvalue $2ns^2$ of multiplicity $(n-1)$, and an eigenvalue $4ns^2$ of multiplicity 1. Moreover,

$$\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)^t \in \mathbb{R}^{n^2 \times 1}$$

is an eigenvector corresponding to the eigenvalue $4ns^2$.

The Moore-Penrose inverse of $\Sigma$ is given by

$$\left(\Sigma^\dagger\right)_{i,j,k,l} = \begin{cases} 
-\frac{3}{4n^3s^2} & \text{if } \{i, j\} \cap \{k, l\} = \emptyset \\
\frac{n-3}{3n^3s^2} & \text{if } |\{i, j\} \cap \{k, l\}| = 1, \text{ and } i \neq j, \text{ and } k \neq l \\
\frac{2n-3}{3n^3s^2} & \text{if } |\{i, j\} \cap \{k, l\}| = 1, \text{ and } i = j \text{ and } k \neq l \\
\frac{2n-3}{3n^3s^2} & \text{if } |\{i, j\} \cap \{k, l\}| = 1, \text{ and } i \neq j \text{ and } k = l \\
\frac{4n-3}{3n^3s^2} & \text{if } i = j = k = l.
\end{cases}$$

Let $x, y \in \Omega$ and $i, j \in [n]$. Then

$$(A_x - A_y)_{i,j} = \begin{cases} 
0 & \text{if } x(i) = x(j) \text{ and } y(i) = y(j) \\
0 & \text{if } x(i) \neq x(j) \text{ and } y(i) \neq y(j) \\
-1 & \text{if } x(i) \neq x(j) \text{ and } y(i) = y(j) \\
1 & \text{if } x(i) = x(j) \text{ and } y(i) \neq y(j)
\end{cases}$$

Then

$$(7.1) \quad L_{\Sigma}(x, y) = U_1 + U_2 + U_3 + U_4;$$

where

$$U_1 = \sum_{i,j,k,l \in [n], x(i) \neq x(j), y(i) = y(j), x(k) \neq x(l), y(k) = y(l)} (\Sigma^\dagger)_{i,j,k,l}$$

$$U_2 = -\sum_{i,j,k,l \in [n], x(i) \neq x(j), y(i) = y(j), x(k) = x(l), y(k) \neq y(l)} (\Sigma^\dagger)_{i,j,k,l}$$

$$U_3 = -\sum_{i,j,k,l \in [n], x(i) = x(j), y(i) \neq y(j), x(k) \neq x(l), y(k) = y(l)} (\Sigma^\dagger)_{i,j,k,l}$$

$$U_4 = \sum_{i,j,k,l \in [n], x(i) = x(j), y(i) \neq y(j), x(k) = x(l), y(k) \neq y(l)} (\Sigma^\dagger)_{i,j,k,l}$$

Then we have

$$U_1 = -\frac{3}{4n^3s^2} \left|\{i, j, k, l \in [n] : x(i) \neq x(j), y(i) = y(j), x(k) \neq x(l), y(k) = y(l), \{i, j\} \cap \{k, l\} = \emptyset\}\right|$$

$$+ \frac{n-3}{4n^3s^2} \left|\{i, j, k, l \in [n] : x(i) \neq x(j), y(i) = y(j), x(k) \neq x(l), y(k) = y(l), |\{i, j\} \cap \{k, l\}| = 1, i \neq j, k \neq l\}\right|$$

$$+ \frac{2n-3}{4n^3s^2} \left|\{i, j, k, l \in [n] : x(i) \neq x(j), y(i) = y(j), x(k) \neq x(l), y(k) = y(l), |\{i, j\} \cap \{k, l\}| = 2, i \neq j\}\right|$$
\[ U_2 = \frac{3}{4n^3s^2} |\{i, j, k, l \in [n] : x(i) \neq x(j), y(i) = y(j), x(k) = x(l), y(k) \neq y(l), \{i, j\} \cap \{k, l\} = \emptyset\}| \\
- \frac{n - 3}{4n^3s^2} |\{i, j, k, l \in [n] : x(i) \neq x(j), y(i) = y(j), x(k) = x(l), y(k) \neq y(l), \{i, j\} \cap \{k, l\} = 1, i \neq j, k \neq l\}| \\
- \frac{2n - 3}{4n^3s^2} |\{i, j, k, l \in [n] : x(i) \neq x(j), y(i) = y(j), x(k) = x(l), y(k) \neq y(l), \{i, j\} \cap \{k, l\} = 2, i \neq j\}| \\
\]

\[ U_3 = \frac{3}{4n^3s^2} |\{i, j, k, l \in [n] : x(i) = x(j), y(i) \neq y(j), x(k) \neq x(l), y(k) = y(l), \{i, j\} \cap \{k, l\} = \emptyset\}| \\
- \frac{n - 3}{4n^3s^2} |\{i, j, k, l \in [n] : x(i) = x(j), y(i) \neq y(j), x(k) \neq x(l), y(k) = y(l), \{i, j\} \cap \{k, l\} = 1, i \neq j, k \neq l\}| \\
+ \frac{2n - 3}{4n^3s^2} |\{i, j, k, l \in [n] : x(i) = x(j), y(i) \neq y(j), x(k) = x(l), y(k) \neq y(l), \{i, j\} \cap \{k, l\} = 2, i \neq j\}| \\
\]

\[ U_4 = -\frac{3}{4n^3s^2} |\{i, j, k, l \in [n] : x(i) = x(j), y(i) = y(j), x(k) = x(l), y(k) = y(l), \{i, j\} \cap \{k, l\} = \emptyset\}| \\
+ \frac{n - 3}{4n^3s^2} |\{i, j, k, l \in [n] : x(i) = x(j), y(i) = y(j), x(k) = x(l), y(k) = y(l), \{i, j\} \cap \{k, l\} = 1, i \neq j, k \neq l\}| \\
+ \frac{2n - 3}{4n^3s^2} |\{i, j, k, l \in [n] : x(i) = x(j), y(i) = y(j), x(k) \neq x(l), y(k) \neq y(l), \{i, j\} \cap \{k, l\} = 2, i \neq j\}| \\
\]

Recall that \( y \) is the true community assignment mapping, in which all the vertices are divided into two communities of corresponding sizes \([\alpha n]\) and \(n - [\alpha n]\), respectively, for some \(\alpha \in (0, 1)\). Assume \(x \in \Omega\) is a community assignment mapping with \(m\) and \(n - m\) vertices in two communities, respectively, where \(m \in [n]\). Then we have

\[ (7.2) \quad t_{1,1}(x, y) + t_{2,1}(x, y) = [\alpha n]; \]

\[ (7.3) \quad t_{1,2}(x, y) + t_{2,2}(x, y) = n - [\alpha n]. \]

Then we obtain

\[ U_1 = \frac{1}{n^2s^2} \left[ t_{1,1}(x, y)t_{2,1}(x, y)(t_{1,1}(x, y) + t_{2,1}(x, y)) + t_{1,2}(x, y)t_{2,2}(x, y)(t_{1,2}(x, y) + t_{2,2}(x, y)) \right] \\
- \frac{3}{n^3s^2} \left[ t_{1,1}(x, y)t_{2,1}(x, y) + t_{1,2}(x, y)t_{2,2}(x, y) \right]^2 \]

\[ U_2 = -\frac{1}{n^2s^2} \left[ t_{1,1}(x, y)t_{2,1}(x, y)(t_{1,2}(x, y) + t_{2,2}(x, y)) + t_{1,2}(x, y)t_{2,2}(x, y)(t_{1,1}(x, y) + t_{2,1}(x, y)) \right] \\
+ \frac{3}{n^3s^2} \left[ t_{1,1}(x, y)t_{2,1}(x, y) + t_{1,2}(x, y)t_{2,2}(x, y) \right] \left[ t_{1,1}(x, y)t_{1,2}(x, y) + t_{2,1}(x, y)t_{2,2}(x, y) \right] \]

\[ U_3 = U_2. \]
Moreover, we can also write

\[ U_3 = -\frac{1}{n^2s^2} [t_{1,1}(x,y)t_{1,2}(x,y)(t_{2,1}(x,y) + t_{2,2}(x,y)) + t_{2,1}(x,y)t_{2,2}(x,y)(t_{1,1}(x,y) + t_{1,2}(x,y))] \\
+ \frac{3}{n^3s^2} [t_{1,1}(x,y)t_{2,1}(x,y) + t_{1,2}(x,y)t_{2,2}(x,y)] [t_{1,1}(x,y)t_{1,2}(x,y) + t_{2,1}(x,y)t_{2,2}(x,y)] \]

\[ U_4 = \frac{1}{n^3s^2} [t_{1,1}(x,y)t_{1,2}(x,y)(t_{1,1}(x,y) + t_{1,2}(x,y)) + t_{2,1}(x,y)t_{2,2}(x,y)(t_{2,1}(x,y) + t_{2,2}(x,y))] \\
- \frac{3}{n^3s^2} [t_{1,1}(x,y)t_{1,2}(x,y) + t_{2,1}(x,y)t_{2,2}(x,y)]^2 \]

By (7.1), we obtain

\[ L_\Sigma(x, y) = -\frac{3}{n^3s^2} [t_{1,1}(x,y)t_{1,2}(x,y) + t_{1,2}(x,y)t_{2,1}(x,y) - t_{1,1}(x,y)t_{2,1}(x,y) - t_{1,2}(x,y)t_{2,2}(x,y)]^2 \]

\[ + \frac{3}{n^3s^2} [t_{1,1}(x,y)t_{1,2}(x,y) - t_{1,2}(x,y)t_{2,1}(x,y)] [t_{1,1}(x,y) + t_{1,2}(x,y) - t_{2,1}(x,y) - t_{2,2}(x,y)] \]

By (7.2) and (7.3), we have

\[ L_\Sigma(x, y) = -\frac{3}{n^3s^2} [t_{1,1}(x,y) - t_{2,2}(x,y)]^2 [t_{1,1}(x,y) - t_{2,2}(x,y) + n - 2|\alpha n|]^2 \\
+ \frac{3}{n^3s^2} [t_{1,1}(x,y)(n - |\alpha n|) - |\alpha n|t_{2,2}(x,y)] [2(t_{1,1}(x,y) - t_{2,2}(x,y)) + n - 2|\alpha n|] \\
+ \frac{3}{n^3s^2} [(t_{2,2}(x,y))^2 - (t_{1,1}(x,y))^2 - t_{2,2}(x,y)(n - |\alpha n|) + |\alpha n|t_{1,1}(x,y)] (2|\alpha n| - n) \]

When the \( n \) vertices are divided into two equal-size communities, i.e. when \( \alpha = \frac{1}{2} \) and \( n \) is even, we have

\[ L_\Sigma(x, y) = -\frac{3}{n^3s^2} [t_{1,1}(x,y) - t_{2,2}(x,y)]^4 + \frac{3}{n^3s^2} [t_{1,1}(x,y) - t_{2,2}(x,y)]^2 \]

In this case we see exact recovery can never occur whenever \( s > 0 \), since \( f(x) = f(y) \) a.s. as long as \( t_{1,1}(x, y) = t_{2,2}(x, y) \).

When \( \alpha \neq \frac{1}{2} \). Without loss of generality, assume \( \alpha > \frac{1}{2} \). Let

\[ A : = t_{1,1}(x,y) + t_{2,2}(x,y); \]

\[ B : = t_{1,1}(x,y) - t_{2,2}(x,y). \]

Then we have

\[ L_\Sigma(x, y) = \frac{1}{n^3s^2} \left[ (nA - 3B^2)(B + (1 - 2\alpha)n)^2 + nB^2(n - A) \right] \]

**Proposition 7.1.** Assume \( \alpha \in (\frac{1}{2}, \frac{2}{3}) \). Then \( L_\Sigma(x, y) = 0 \) if and only if \( x \in C(y) \).

**Proof.** The following cases might occur

1. \( nA - 3B^2 > 0 \), then \( L_\Sigma = 0 \) if and only if
   \[ B + (1 - 2\alpha)n = 0; \text{ and } n - A = 0. \]
which corresponds to
\[(7.7) \quad t_{1,1}(x, y) = \alpha n; \text{ and } t_{1,2}(x, y) = (1 - \alpha)n\]
hence \(x = y\).

(2) \(nA - 3B^2 = 0\) if and only if one of the following two cases occurs
(a) \(n = A\); then we have (7.7) and \(x = y\).
(b) \(B = 0\): then \(A = 0\) since \(nA = 2B^2\). Then we have
\[t_{1,1}(x, y) = t_{2,2}(x, y) = 0, \quad \text{and} \]
\[t_{2,1}(x, y) = \alpha n; \quad \text{and} \]
\[t_{1,2}(x, y) = (1 - \alpha)n\]
Hence \(x \in C(y)\).

(3) \(nA - 3B^2 < 0\): then \(L_{\Sigma}(x, y) = 0\) if and only if
\[nB^2(n - A) = (3B^2 - nA)(B + (1 - 2\alpha)n)^2\]
Note that
\[B + (1 - 2\alpha n) = t_{1,2}(x, y) - t_{2,1}(x, y); \quad n - A = t_{1,2}(x, y) + t_{2,1}(x, y);\]
Hence we have
\[|B + (1 - 2\alpha n)| \leq |n - A|; \quad \text{and} \quad |B + (1 - 2\alpha n)| \leq \alpha n\]
Similarly from (7.4), (7.5) we obtain
\[|B| \leq A; \quad \text{and} \quad |B| \leq \alpha n\]
Then
\[(7.8) \quad nB^2(n - A) - (3B^2 - nA)(B + (1 - 2\alpha)n)^2\]
\[\geq n(n - A)\alpha \left( nA - \left(3 - \frac{1}{\alpha}\right)B^2\right)\]
\[\geq n^2A(n - A)\alpha(2 - 3\alpha) \geq 0\]
When \(\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right)\), identity holds if and only if \(n = A\), then \(x = y\); or \(A = 0\); then \(x \in C(y)\). 

\[\square\]

**Proposition 7.2.** Assume \(\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right)\) and \(\epsilon \in \left(0, \frac{1 - \alpha}{2}\right)\). When
\[(t_{1,1}(x, y), t_{1,2}(x, y), t_{2,1}(x, y), t_{2,2}(x, y)) \in B \setminus B_\epsilon,\]
we have
\[L_{\Sigma}(x, y) \geq C_{\alpha, \epsilon} \frac{n}{s^2}\]
where \(C_{\alpha, \epsilon} > 0\) is a constant depending only on \(\alpha\) and \(\epsilon\), but independent of \(s\) and \(n\).
Proof. When \( \epsilon \in (0, \frac{1-\alpha}{2}) \) and \( x \in B \setminus B_\epsilon \), by definition 2.7, we obtain
\[
A = t_{1,1}(x, y) + t_{2,2}(x, y) \geq \epsilon n;
\]
\[
n - A = t_{1,2}(x, y) + t_{2,1}(x, y) \geq \epsilon n.
\]

From (7.6) and (7.8) we obtain that when \( nA - 3B^2 < 0 \)
\[
L_\Sigma(x, y) \geq \frac{A(n - A)\alpha(2 - 3\alpha)}{ns^2} \geq \alpha(2 - 3\alpha)\epsilon^2 \frac{n}{s^2}
\]
When \( nA = 3B^2 \geq 0 \), the following cases might occur

(1) \( B^2 \leq (\alpha - \frac{1}{2})^2 nA \): then
\[
L_\Sigma(x, y) \geq \frac{1}{ns^2} (nA - 3B^2)(B + (1 - 2\alpha)n)^2
\]
\[
\geq \frac{1}{ns^2} A(2n - \sqrt{nA})^2 \left( \alpha - \frac{1}{2} \right)^2 \left[ 1 - \left( \alpha - \frac{1}{2} \right)^2 \right]
\]
\[
\geq \left( \alpha - \frac{1}{2} \right)^2 \left[ 1 - \left( \alpha - \frac{1}{2} \right)^2 \right] \epsilon(1 + \epsilon)^2 \frac{n}{s^2}
\]

(2) \( B^2 > (\alpha - \frac{1}{2})^2 nA \): then
\[
L_\Sigma(x, y) \geq \frac{1}{ns^2} B^2(n - A) \geq \frac{1}{ns^2} \left( \alpha - \frac{1}{2} \right)^2 A(n - A) \geq \epsilon^2 \left( \alpha - \frac{1}{2} \right)^2 \frac{n}{s^2}
\]
Let
\[
C_{\alpha, \epsilon} = \min \left\{ \alpha(2 - 3\alpha)\epsilon^2, \left( \alpha - \frac{1}{2} \right)^2 \left[ 1 - \left( \alpha - \frac{1}{2} \right)^2 \right] \epsilon(1 + \epsilon)^2, \epsilon^2 \left( \alpha - \frac{1}{2} \right)^2 \right\}
\]
Then the lemma follows. \( \square \)

7.1. Sharp Threshold of MLE when the Number of Vertices in Each Community is Unknown.

Proposition 7.3. Let \( \alpha > \frac{1}{2} \). Let \( x, y_1, y_2 \in \Omega \) satisfying (2.14) and (2.13). Moreover, let \( y \) be the true community assignment mapping. Assume
\[
(7.9) \quad (t_{1,1}(x, y), t_{1,2}(x, y), t_{2,1}(x, y), t_{2,2}(x, y)) \in B_\epsilon
\]
and
\[
D_\Omega(y_1) = D_\Omega(y_2) - 1.
\]
Then
\[
L_\Sigma(x, y_1) - L_\Sigma(x, y_2) = \frac{(2\alpha - 1)^2}{s^2} + O \left( \frac{1}{ns^2} + \frac{\epsilon}{s^2} \right)
\]
Proof. For \( i \in \{1, 2\} \), let
\[
A_i = t_{1,i}(x, y_i) + t_{2,2}(x, y_i)
\]
\[
B_i = t_{1,1}(x, y_i) - t_{2,2}(x, y_i).
\]
By (2.13), we obtain
\[ A_2 = A_1 + 1; \text{ and } |B_2 - B_1| = 1. \]

By (7.6), we have
\[ L_\Sigma(x, y_1) - L_\Sigma(x, y_2) = \frac{1}{n^3 s^2} \left[ (nA_1 - 3B_1^2) (B_1 + (1 - 2\alpha)n)^2 + nB_1^2 (n - A_1) 
- (nA_2 - 3B_2^2) (B_2 + (1 - 2\alpha)n)^2 - nB_2^2 (n - A_1) \right] \]

When (7.9) holds, under the assumptions of the proposition, we have
\[ (t_{1,1}(y, y), t_{1,2}(y, y), t_{2,1}(y, y), t_{2,2}(y, y)) \in B_\epsilon \text{ for } y \in \{1, 2\}. \]
Hence
\[ n - 2\epsilon n \leq A_i \leq n \]
\[ (2\alpha - 1)n - \epsilon n \leq B_i \leq (2\alpha - 1)n + \epsilon n \]

If \( B_2 = B_1 + 1 \), we obtain
\[ L_\Sigma(x, y_1) - L_\Sigma(x, y_2) = \frac{1}{n^3 s^2} \left[ -(2\alpha - 1)^2 n^3 + 2 ((2\alpha - 1)A + (12\alpha^2 - 10\alpha + 1)B) n^2 - 18B^2(2\alpha - 1)n + 12B^3 + O(n^3) \right] \]

When (7.10) and (7.11) holds, we have
\[ L_\Sigma(x, y_1) - L_\Sigma(x, y_2) = \frac{1}{n^3 s^2} \left[ (2\alpha - 1)^2 n^3 + O(\epsilon n^3 + n^2) \right] \]

Similarly if \( B_2 = B_1 - 1 \), we have
\[ L_\Sigma(x, y_1) - L_\Sigma(x, y_2) = \frac{1}{n^3 s^2} \left[ (2\alpha - 1)^2 n^3 + O(\epsilon n^3 + n^2) \right] \]

Then the lemma follows.

**Theorem 7.4.** Let \( \alpha \in \left( \frac{1}{2}, \frac{3}{2} \right) \). Assume there exists a constant \( \delta > 0 \), such that
\[ s < \frac{(1 - \delta)(2\alpha - 1)}{\sqrt{8 \log n}} \]

Then \( \lim_{n \to \infty} \Pr(\hat{y} \in C(y)) = 1. \)

**Proof.** Let \( \epsilon \in \left( 0, \frac{1 - \alpha}{2} \right) \). By Proposition 7.2, when (7.12) holds
\[ \lim_{n \to \infty} n \log k - \frac{1}{8} \min_{x \in (t_{1,1}(x, y), \ldots, t_{2,2}(x, y)) \in B \setminus B_k} L_\Sigma(x, y) \leq \lim_{n \to \infty} n \log 2 - \frac{C_{\alpha, s} \epsilon n \log n}{(1 - \delta)^2(2\alpha - 1)^2} = -\infty. \]

Moreover,
\[ \lim_{n \to \infty} \log 2 + \log n - (1 + \delta) \log n = -\infty \]

Then the theorem follows from Theorem 2.10.

Let \( a \in [n] \). Define \( y^{(a)} \in \Omega \) as in (2.23). By (7.6), we have
\[ \begin{array}{l}
(1) \text{ If } t_{1,1}(y^{(a)}, y) = \alpha n \text{ and } t_{2,2}(y^{(a)}, y) = (1 - \alpha)n - 1, \\
(7.13) \quad L_\Sigma(y^{(a)}, y) = \frac{1}{s^2 n^3} \left[ (2\alpha - 1)^2 n^3 - 4(3\alpha - 1)(\alpha - 1)n^2 - 6(2\alpha - 1)n - 3 \right] 
\end{array} \]
(2) If \( t_{1,1}(y^{(a)}, y) = \alpha n - 1 \) and \( t_{2,2}(y^{(a)}, y) = (1 - \alpha)n \),

\[
L(y^{(a)}, y) = \frac{1}{s-n^3} \left[ (2\alpha - 1)^2 n^3 - 4\alpha(3\alpha - 2)n^2 + 6(2\alpha - 1)n - 3 \right]
\]

(7.14)

For \( a, i, j \in [n] \), we have

\[
(A_{y^{(a)}} - A_{y})_{i,j} = \begin{cases} 
0 & \text{if } a \notin \{i, j\} \\
0 & \text{if } i = j = a \\
-1 & \text{if } i = a, j \neq a, y(i) = y(j) \\
-1 & \text{if } i \neq a, j = a, y(i) = y(j) \\
1 & \text{if } i = a, j \neq a, y^{(a)}(i) = y^{(a)}(j) \\
1 & \text{if } i \neq a, j = a, y^{(a)}(i) = y^{(a)}(j)
\end{cases}
\]

By (2.24), we obtain

\[
\eta_a = \eta_{a,1} + \eta_{a,2}.
\]

where

\[
\eta_{a,1} := \frac{2\sum_{k \in [n]: j, l \in [n], j \neq a} (A_{y^{(a)}} - A_{y})_{a,j} (\Sigma^\dagger)_{a,j;k,l} (\xi_k + \xi_l)}{L_{\Sigma}(y^{(a)}, y)}
\]

and

\[
\eta_{a,2} := \frac{2\sum_{i,k \in [n]: j \in [n]: i \neq a} (A_{y^{(a)}} - A_{y})_{i,a} (\Sigma^\dagger)_{i,a;k,l} (\xi_k + \xi_l)}{L_{\Sigma}(y^{(a)}, y)}
\]

Hence we have

\[
\eta_{a,1} = \frac{2}{L_{\Sigma}(y^{(a)}, y)} \left[ -\sum_{k \in [n]: j \in [n]: j \neq a, y(j) = y(a)} (\Sigma^\dagger)_{a,j;k,l} (\xi_k + \xi_l) + \sum_{k,l \in [n]: j \in [n]: i \neq a, y(j) = y^{(a)}(a)} (\Sigma^\dagger)_{a,j;k,l} (\xi_k + \xi_l) \right]
\]

where

\[
\alpha_1 = -\sum_{j \in [n]: \{a\}, y(j) = y(a)} (\Sigma^\dagger)_{a,j:a,a} (\xi_a + \xi_a) + \sum_{j \in [n]: \{a\}, y(j) = y^{(a)}(a)} (\Sigma^\dagger)_{a,j:a,a} (\xi_a + \xi_a)
\]

\[
\alpha_2 = -\sum_{j \in [n]: \{a\}, y(j) = y(a)} (\Sigma^\dagger)_{a,j:a,j} (\xi_a + \xi_j) + \sum_{j \in [n]: \{a\}, y(j) = y^{(a)}(a)} (\Sigma^\dagger)_{a,j:a,j} (\xi_a + \xi_j)
\]

\[
\alpha_1 = \begin{cases} 
\frac{2n-3}{2n^3 s^2} \xi_a [-2\alpha - 1] & \text{If } y(a) = 1 \\
\frac{2n-3}{2n^3 s^2} \xi_a [2\alpha - 1] & \text{If } y(a) = 2
\end{cases}
\]

\[
\alpha_2 = \begin{cases} 
\frac{2n-3}{4n^3 s^2} \left[ -2\alpha - 1 \right] \xi_a - \sum_{j \in [n]: \{a\}, y(j) = y(a)} \xi_j & \text{If } y(a) = 1 \\
\frac{2n-3}{4n^3 s^2} \left[ 2\alpha - 1 \right] \xi_a - \sum_{j \in [n]: \{a\}, y(j) = y(a)} \xi_j & \text{If } y(a) = 2
\end{cases}
\]
\[
\alpha_3 = - \sum_{j,l \in [n] \setminus \{a\}, y(j) = y(a), j \neq l} (\Sigma^t)_{a,j,a,l}(\xi_a + \xi_l) + \sum_{j,l \in [n] \setminus \{a\}, y(j) = y(a), j \neq l} (\Sigma^t)_{a,j,a,l}(\xi_a + \xi_l) \\
= \begin{cases} 
\frac{n-3}{4n^3s^2} \left( -(2\alpha - 1)n + 1 \right) (n - 2) \xi_a + 2 \sum_{l \in [n] \setminus \{a\}, y(l) = y(a)} \xi_l - (2\alpha - 1)n \sum_{l \in [n] \setminus \{a\}} \xi_l & \text{if } y(a) = 1 \\
\frac{n-3}{4n^3s^2} \left( [(2\alpha - 1) + 1] (n - 2) \xi_a + 2 \sum_{l \in [n] \setminus \{a\}, y(l) = y(a)} \xi_l + (2\alpha - 1)n \sum_{l \in [n] \setminus \{a\}} \xi_l & \text{if } y(a) = 2
\end{cases}
\]

\[
\alpha_4 = - \sum_{j \in [n], j \neq a, y(j) = y(a)} (\Sigma^t)_{a,j,j,a}(\xi_j + \xi_a) + \sum_{j \in [n], j \neq a, y(j) = y(a)} (\Sigma^t)_{a,j,j,a}(\xi_j + \xi_a) = \alpha_2
\]

\[
\alpha_5 = - \sum_{k,j \in [n] \setminus \{a\}, y(j) = y(a), k \neq j} (\Sigma^t)_{a,j,k,a}(\xi_k + \xi_a) + \sum_{k,j \in [n] \setminus \{a\}, y(j) = y(a), k \neq j} (\Sigma^t)_{a,j,k,a}(\xi_k + \xi_a) = \alpha_3
\]

\[
\alpha_6 = - \sum_{j \in [n] \setminus \{a\}, y(j) = y(a)} (\Sigma^t)_{a,j,j,j}(\xi_j + \xi_j) + \sum_{j \in [n] \setminus \{a\}, y(j) = y(a)} (\Sigma^t)_{a,j,j,j}(\xi_j + \xi_j) = \frac{2n - 3}{2n^3s^2} \left[ - \sum_{j \in [n] \setminus \{a\}, y(j) = y(a)} \xi_j + \sum_{j \in [n] \setminus \{a\}, y(j) = y(a)} \xi_j \right]
\]

\[
\alpha_7 = - \sum_{j,l \in [n] \setminus \{a\}, y(j) = y(a), l \neq j} (\Sigma^t)_{a,j,l,l}(\xi_j + \xi_l) + \sum_{j,l \in [n] \setminus \{a\}, y(j) = y(a), l \neq j} (\Sigma^t)_{a,j,l,l}(\xi_j + \xi_l)
\]

Then

- If \( y(a) = 1 \):
  \[
  \alpha_7 = \frac{n - 3}{4n^3s^2} \left( n - 2 \right) \left( - \sum_{j \in [n] \setminus \{a\}, y(j) = y(a)} \xi_j + \sum_{j \in [n] \setminus \{a\}, y(j) = y(a)} \xi_j \right) \\
  + 2 \sum_{l \in [n] \setminus \{a\}, y(l) = y(a)} \xi_l - (2\alpha - 1)n \sum_{l \in [n] \setminus \{a\}} \xi_l
  \]

- If \( y(a) = 2 \):
  \[
  \alpha_7 = \frac{n - 3}{4n^3s^2} \left( n - 2 \right) \left( - \sum_{j \in [n] \setminus \{a\}, y(j) = y(a)} \xi_j + \sum_{j \in [n] \setminus \{a\}, y(j) = y(a)} \xi_j \right) \\
  + 2 \sum_{l \in [n] \setminus \{a\}, y(l) = y(a)} \xi_l + (2\alpha - 1)n \sum_{l \in [n] \setminus \{a\}} \xi_l
  \]

\[
\alpha_8 = - \sum_{k,j \in [n] \setminus \{a\}, y(j) = y(a), k \neq j} (\Sigma^t)_{a,j,k,j}(\xi_k + \xi_j) + \sum_{k,j \in [n], y(j) = y(a), k \neq j} (\Sigma^t)_{a,j,k,j}(\xi_k + \xi_j) = \alpha_7
\]
By symmetry we have \( \eta \mid \mid \mid \)\( \eta \) such that for all \( i,j \):

\[
\alpha_g = - \sum_{k,j,l \in [n] \setminus \{a\}, y(j) = y(a), k \neq j, l \neq j} (\Sigma^\dagger_{a,j:k,l}(\xi_k + \xi_l)) + \sum_{k,j,l \in [n] \setminus \{a\}, y(j) = y(a), k \neq j, l \neq j} (\Sigma^\dagger_{a,j:k,l}(\xi_k + \xi_l))
\]

\[
= \begin{cases}
- \frac{3}{4n^4s^2} \left[ 4(n - 2) \sum_{k \in [n] \setminus \{a\}, y(k) = y(a)} \xi_k - (2n - 2)(2\alpha - 1)n \sum_{k \in [n] \setminus \{a\}} \xi_k \right] & \text{if } y(a) = 1 \\
- \frac{3}{4n^4s^2} \left[ 4(n - 2) \sum_{k \in [n] \setminus \{a\}, y(k) = y(a)} \xi_k + (2n - 2)(2\alpha - 1)n \sum_{k \in [n] \setminus \{a\}} \xi_k \right] & \text{if } y(a) = -1.
\end{cases}
\]

By symmetry we have \( \eta_{a,2} = \eta_{a,1} \), and therefore \( \eta_a = 2\eta_{a,1} \). Then

1. If \( y(a) = 1 \):

\[
\eta_a = 2 \left( -\frac{\xi_a}{2\alpha - 1} + \frac{2\alpha}{n(2\alpha - 1)^2} \sum_{k \in [n] \setminus \{a\}, y(k) \neq y(a)} \xi_k + \frac{2\alpha - 2}{n(2\alpha - 1)^2} \sum_{j \in [n] \setminus \{a\}, y(j) = y(a)} \xi_j \right) \left( 1 + O \left( \frac{1}{n} \right) \right)
\]

2. If \( y(a) = 2 \):

\[
\eta_a = 2 \left( \frac{\xi_a}{2\alpha - 1} + \frac{2 - 2\alpha}{n(2\alpha - 1)^2} \sum_{k \in [n] \setminus \{a\}, y(k) \neq y(a)} \xi_k - \frac{2\alpha}{n(2\alpha - 1)^2} \sum_{j \in [n] \setminus \{a\}, y(j) = y(a)} \xi_j \right) \left( 1 + O \left( \frac{1}{n} \right) \right)
\]

**Theorem 7.5.** Let \( \alpha \in \left( \frac{2}{3}, \frac{2}{1} \right) \), if there exists a constant \( \delta > 0 \) independent of \( n \) such that

\[
s > \frac{(1 + \delta)(2\alpha - 1)}{\sqrt{8\log n}},
\]

then

\[
\lim_{n \to \infty} \Pr(\hat{y} \in C(y)) = 0.
\]

i.e. a.s. exact recovery does not occur.

**Proof.** Choose \( |H| = h = \frac{n}{(\log n)^\tau} \) such that \( H = \{a_1, \ldots, a_h\} \). By (5.1), (7.13), (7.14)

\[
\mathbb{E}n^2_a = \frac{4}{L_{\Sigma(y^{(a)}, y)}} = \frac{4s^2}{(2\alpha - 1)^2} \left( 1 + O \left( \frac{1}{n} \right) \right)
\]

Moreover, for any \( a_1, a_2 \in [n] \), if \( a_1 \neq a_2 \),

\[
\mathbb{E}n_{a_1}n_{a_2} = O \left( \frac{1}{n} \right)
\]

Let \( \frac{4s^2}{(2\alpha - 1)^2} I_h \) be the \( h \times h \) diagonal matrix with diagonal entries \( \frac{4s^2}{(2\alpha - 1)^2} \). Then the covariance matrix \( \Phi_H \) for \( \{\eta_a\}_{a \in H} \) satisfies

\[
Z := \Phi_H - \frac{4s^2}{(2\alpha - 1)^2} I_h
\]

such that for all \( i, j \in [h] \),

\[
|Z_{i,j}| \leq \frac{C}{n}
\]

where \( C > 0 \) is a constant independent of \( n \). By the Weyl’s inequality / Lidski’s inequality, we have

\[
\left| \lambda_0(n, H) - \frac{4s^2}{(2\alpha - 1)^2} \right| \leq \|Z\|_{op} \leq \|Z\|_{F} \leq \frac{Ch}{n}
\]
Community Detection in Dependent Gaussian Mixture Model

Figure 7.1. Recovery probability \( \Pr(\hat{y} \in C(y)) \) for low-level noise \( s = (1 - \delta)(2\alpha - 1) \sqrt{2 \log n} \) \((\alpha = \frac{16}{25}, \delta = \frac{1}{2})\), with the number of vertices in each community unknown.

where \( \|Z\|_{op} \) is the operator norm of \( Z \); and \( \|Z\|_F \) is the Frobenius norm of \( Z \). If (7.15) holds, we have

\[
\lambda_0(n, H) \geq \frac{(1 + \delta)^2}{2 \log n} - \frac{C}{(\log n)^2}
\]

then (2.25) holds, and the theorem follows from Theorem 2.17.

From Theorems 7.4 and 7.5, we can see that the threshold for exact recovery in terms of the intensities of Gaussian random perturbations associated to vertices is \( (2\alpha - 1) \sqrt{8 \log n} \). It turns out that this threshold is of order \( O \left( \frac{1}{\sqrt{\log n}} \right) \), that is smaller than \( O \left( \sqrt{\frac{n}{\log n}} \right) \) threshold for the exact recovery in terms of the intensities of Gaussian perturbations when the Gaussian perturbations are i.i.d. and associated to each edge; see [17].

The theoretical results in Theorems 7.4 and 7.5 are verified perfectly by our numerical results, where we let \( 2 \leq n \leq 20 \), and we let \( \alpha = \frac{16}{25} \in \left( \frac{1}{2}, \frac{2}{3} \right) \), \( \delta = \frac{1}{2} > 0 \), and we let \( s = (1 - \delta)(2\alpha - 1) \sqrt{2 \log n} \) when verifying Theorem 7.4 and \( s = (1 + \delta)(2\alpha - 1) \sqrt{2 \log n} \) when verifying Theorem 7.5. We repeat running for 1000 times for each value of \( n \). The convergence is pretty fast for small values of \( n \) as we could see in Figures 7.1 and 7.2 verifying Theorems 7.4 and 7.5 respectively.

We also implement the numerical experiments with known community sizes, and the results are similar as shown in Figures 7.3 and 7.4. In particular, Figure 7.4 for large noise is of interest to us because it is possible that the recovery probability is not converging to zero when we have the extra community size information.
Figure 7.2. Recovery probability $\Pr(\hat{y} \in C(y))$ for high-level noise $s = (1+\delta)(2\alpha-1)\sqrt{2\log n}/\alpha (\alpha = 16/25, \delta = 1/2)$, with the number of vertices in each community unknown.

Figure 7.3. Recovery probability $\Pr(\check{y} \in C(y))$ for low-level noise $s = (1-\delta)(2\alpha-1)\sqrt{32\log n}/\alpha (\alpha = 16/25, \delta = 1/2)$, with the number of vertices in each community known.
8. Future Work

This work serves as a starting point for considering a more general regime than most current literature on community detection, where they focus on the case that $\rho_{i,j,k,l}$ is zero when $j \neq l$, so $\Sigma$ is actually a sparse matrix in such a case.

And sourcing from $\Sigma$, we have a series of future work, from the perspective of statistical estimator, numerical computation, and theoretical guarantees. We emphasize that these three perspectives are intertwined with each other and usually a future work in one perspective is also related to the other two.

From the perspective of statistical estimator, we could ask the following questions: If $\Sigma$ is not given but we know $\Sigma$ depends only on the community assignment of each vertex, or some simple feature of vertex e.g. high degree vertex or low degree vertex, or there is some sparsity/low-rank structure on it, how could we infer $\Sigma$ from the data? A relevant question here is that if only some elements of $\Sigma$ is given, what is the best estimator and what is the theoretical guarantees of it? Some of these questions seem to be challenging currently since the error in estimating $\Sigma$ may be magnified by a multiple up to the condition number of $\Sigma$, and there is not evidence to guarantee the condition number of $\Sigma$ is small in practice.

From the perspective of numerical computation, we could ask what is the best way to compute the MLE for relatively large $n$, usually in semi-definite programming, and what is the theoretical guarantees for that. This is not straightforward, as we need to kill the first-order term in the objective function if we hope to use similar techniques in [3]. Using slack variables helps, but it is hard to build theoretical guarantees for such estimators.
From the perspective of theory, we could ask what are results for almost exact recovery. Also, what is the theoretical guarantees for more general noise, like subGaussian noise, subexponential noise, or even some adversary wrong data or missing data. Such question is also interesting in practice, because there are a variety types of noises in practice, and we need to understand what is the theoretical guarantee before we adopt certain estimators.

**Appendix A. Pseudo Determinant and Moore-Penrose Inverse**

**Definition A.1.** Let $A$ be an $n \times n$ square matrix. The pseudo-determinant of $A$ is defined by

$$
\det^* A = \lim_{\alpha \to 0} \frac{\det(A + \alpha I)}{\alpha^{n - \text{rank}(A)}}
$$

Note that when $A$ is invertible, we have

$$
det A = \det^* A
$$

**Definition A.2.** Let $A \in \mathbb{R}^{m \times n}$ be an $m \times n$ matrix with real entries. If $A^+ \in \mathbb{R}^{n \times m}$ satisfies all the following conditions

1. $AA^+ A = A$; and
2. $A^+ AA^+ = A^+$; and
3. $(AA^+)^t = AA^+$; and
4. $(A^+ A)^t = A^+ A$

then $A^+$ is called a Moore-Penrose inverse of $A$. The Moore-Penrose inverse of $A \in \mathbb{R}^{m \times n}$ exists and is unique; see [24, 14].

Again if $A$ is nonsingular (i.e. invertible), we have

$$
A^+ = A^{-1},
$$

where $A^{-1}$ is the regular inverse of $A$.

**Proposition A.3.** (Explicit construction of Moore-Penrose inverse for a symmetric matrix) Let $A \in \mathbb{R}^{m \times m}$ be an $m \times m$ symmetric matrix with real entries such that the rank of $A$ is $r$. Assume

$$
A = PD P^t,
$$

where

- $P \in \mathbb{R}^{m \times m}$ is an orthogonal matrix and $P^t$ is its transpose; and
- $D \in \mathbb{R}^{m \times m}$, such that the top left $r \times r$ block of $D$ is a diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_r$ (eigenvalues of $A$); and all the other entries of $D$ are 0. More precisely, let

$$
D_r = \text{diag}(\lambda_1, \ldots, \lambda_r);
$$

then

$$
D = \begin{pmatrix}
D_r & 0 \\
0 & 0
\end{pmatrix}.
$$
Assume

\[ P = (P_r, \overline{P}_r); \]

where \( P_r \in \mathbb{R}^{m \times r} \) and \( \overline{P}_r \in \mathbb{R}^{m \times (m-r)} \). Then

\[ A^+ = P_rD_r^{-1}(P_r)^\dagger \]

For a general matrix \( A \in \mathbb{C}^{m \times n} \), we may construct its Moore-Penrose inverse similarly by using the singular value decomposition.

**Proposition A.4.** Let \( A \in \mathbb{R}^{m \times m} \) be a symmetric positive definite matrix. Let \( \lambda_1 \) be the maximal eigenvalue of \( A \); let \( \mu_0 \) be the minimum of all the nonzero eigenvalues of \( A^+ \). Then

\[ \mu_0 \lambda_1 = 1. \]

**Proof.** Check directly from Definition A.2. \( \square \)

**Definition A.5.** Let \( A \in \mathbb{C}^{n \times n} \) be a positive semi-definite matrix, then there is exactly one positive semi-definite matrix \( B \) such that

\[ A = B^* B \]

where \( B^* \) is the conjugate transpose of \( B \). Then we define

\[ A^\frac{1}{2} := B. \]

**Appendix B. Maximum of Dependent Gaussian Random Variables**

Throughout this section, we let \( X_1, \ldots, X_N \) be \( N \) Gaussian random variables with 0 mean. Let

\[ M_N := \max_{i \in [N]} X_i. \]

**Lemma B.1.** *(Theorem A.1 in [16])* Let \( \epsilon \in (0, 1) \). Then

\[ \Pr \left( M_N > (1 + \epsilon) \sqrt{2 \max_{i \in [N]} \text{Var}(G_i) \log N} \right) \leq N^{-\epsilon} \]

**Lemma B.2.** Define

\[
\begin{align*}
E_i &= \mathbb{E}(X_i|X_1, \ldots, X_{i-1}) \\
R_i &= X_i - E_i \\
\sigma_i^2 &= \mathbb{E}R_i^2 \\
\sigma^2 &= \min_{i \in [N]} \{\sigma_i^2 : \sigma_i^2 > 0\}.
\end{align*}
\]

(B.1)

(B.2)

Then for each \( \epsilon > 0 \),

\[ \lim_{N \to \infty} \Pr \left( M_N \geq \sigma \sqrt{2 \log N (1 - \epsilon)} \right) = 1. \]

**Proof.** When \( \sigma^2 := \min_{i \in [N]} \sigma_i^2 \), the theorem was proved as in Theorem 3.4 in [12]. The proof works as well if we define \( \sigma^2 \) as in (B.2). \( \square \)
Corollary B.3. Let $X_1, \ldots, X_N$ be $N$ Gaussian random variables with 0 mean. Let $\Phi$ be the covariance matrix for $X_1, \ldots, X_N$ such that $\lambda_0(N)$ is the minimal eigenvalue of $\Phi$. Then

$$\lim_{N \to \infty} \Pr \left( M_N \geq \sqrt{2\lambda_0(N) \log N (1 - \epsilon)} \right) = 1.$$ 

Proof. Let 

$$X = (X_1, \ldots, X_N)^t; \quad R = (R_1, \ldots, R_N)^t$$

where $(\cdot)^t$ denotes the transpose, and $\{R_i\}_{i \in [N]}$ are defined as in (B.1). Then 

$$R = BX;$$

where $B \in \mathbb{R}^{N \times N}$ is a lower-triangular matrix with diagonal entries 1. The covariance matrix of $R$ is given by 

$$\mathbb{E}RR^t = \mathbb{E}BB^t = \mathbb{E}[\sigma_1^2, \ldots, \sigma_N^2]$$

Assume 

$$B^t = (\beta_1, \ldots, \beta_N).$$

Then for $i \in [N]$ such that $\sigma_i^2 > 0$, 

$$\sigma_i^2 = \beta_i^t \Phi \beta_i \geq \frac{\beta_i^t \Phi \beta_i}{\beta_i^t \beta_i} \geq \min_{\beta \in \mathbb{R}^N} \frac{\beta^t \Phi \beta}{\beta^t \beta} \geq \lambda_0(N).$$

Then the corollary follows from Lemma B.2. \qed

Acknowledgements. ZL’s research is supported by National Science Foundation grant 1608896 and Simons Foundation grant 638143.

References

[1] E. Abbe. Community detection and stochastic block models: Recent developments. Journal of Machine Learning Research, 18:1–86, 2018.

[2] E. Abbe and C. Sandon. Community detection in general stochastic block models: Fundamental limits and efficient recovery algorithms. 2015 IEEE 56th Annual Symposium on Foundations of Computer Science, pages 670–688, 2015.

[3] Emmanuel Abbe, Afonso S. Bandeira, and Georgina Hall. Exact recovery in the stochastic block model. IEEE Transactions on Information Theory, 62:471–487, 2016.

[4] Afonso S. Bandeira, Nicolas Boumal, and Vladislav Voroninski. On the low-rank approach for semi-definite programs arising in synchronization and community detection, 2016.

[5] Q. Berthet, P. Rigollet, and P. Srivastava. Exact recovery in the ising block model. Annals of Statistics, 47:1805–1834, 2019.

[6] F. Bunea, C. Giraud, X. Luo, M. Royer, and N. Verzelen. Model assisted variable clustering: Minimax-optimal recovery and algorithms. Annals of Statistics, 48:117–137, 2020.

[7] F. Bunea, C. Giraud, M. Royer, and N. Verzelen. Pecok: a convex optimization approach to variable clustering. 2016. arXiv:1606.05100.

[8] Aurelien Decelle, Florent Krzakala, Cristopher Moore, and Lenka Zdeborová. Asymptotic analysis of the stochastic block model for modular networks and its algorithmic applications. Physical Review E, 84(6), Dec 2011.
[9] A.P. Dempster, N.M. Laird, and D.B. Rubin. Maximum likelihood from incomplete data via the em algorithm. *Journal of the Royal Statistical Society, Series B*, 39:1–38, 1977.

[10] M. E. Dyer and A. M. Frieze. The solution of some random np-hard problems in polynomial expected time. *Journal of Algorithms*, 10:451–489, 1989.

[11] C. Fraley and A. Raftery. Model-based clustering, discriminant analysis, and density estimation. *Journal of American Statistical Association*, 97:611–631, 2002.

[12] J.A. Hartigan. Bounding the maximum of dependent random variables. *Electronic Journal of Statistics*, 8:3126–3140, 2014.

[13] P. W. Holland, K. B. Laskey, and S. Leinhardt. Stochastic blockmodels: First steps. *Social Networks*, 5:109–137, 1983.

[14] M. James. The generalized inverse. *The Mathematical Gazette*, 62:109–114, 1978.

[15] F. Krzakala, C. Moore, E. Mossel, J. Neeman, A. Sly, L. Zdeborova, and P. Zhang. Spectral redemption in clustering sparse networks. *Proceedings of the National Academy of Sciences*, 110(52):20935–20940, Nov 2013.

[16] Z. Li. Exact recovery of community detection in k-community gaussian mixture models. 2020. [https://arxiv.org/abs/2009.01185](https://arxiv.org/abs/2009.01185).

[17] Z. Li. Exact recovery of community detection in k-partite graph models. 2020. [arXiv:1910.04320](https://arxiv.org/abs/1910.04320).

[18] L. Massoulié. Community detection thresholds and the weak Ramanujan property. *Proceedings of the 46th Annual ACM Symposium on Theory of Computing*, pages 694–703, 2014.

[19] J.B. McQueen. Some methods for classification and analysis of multivariate observations. *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability*, pages 281–297, 1967.

[20] Andrea Montanari and Subhabrata Sen. Semidefinite programs on sparse random graphs and their application to community detection, 2015.

[21] Elchanan Mossel, Joe Neeman, and Allan Sly. A proof of the blockmodel threshold conjecture. *Combinatorica*, 38:665–708, 2018.

[22] J. Peng and Y. Wei. Approximating k-means-type clustering via semidefinite program-ming. *SIAM J. on Optimization*, 18:186–205, 2015.

[23] D. Pollard. Strong consistency of k-means clustering. *Ann. Statist.*, pages 135–140, 1981.

[24] R.M. Pringle and A.A. Rayner. *Generalized Inverse Matrices with Applications to Statistics*. Griffin, 1971.