Quantum Markov Processes  
(Correspondences and Dilations)

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Abstract

We study the structure of quantum Markov Processes from the point of view of product systems and their representations.

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1 Introduction

A quantum Markov process is a pair, \((\mathcal{M}, \{P_t\}_{t \geq 0})\), consisting of a von Neumann algebra \(\mathcal{M}\) and a semigroup \(\{P_t\}_{t \geq 0}\) of unital, completely positive, normal linear maps on \(\mathcal{M}\) such that \(P_0\) is the identity mapping on \(\mathcal{M}\) and such that the map \(t \to P_t(a)\) from \([0, \infty)\) to \(\mathcal{M}\) is continuous with respect to the \(\sigma\)-weak topology on \(\mathcal{M}\) for each \(a \in \mathcal{M}\). Over the years, there have been numerous studies wherein the authors “dilate” the Markov semigroup \(\{P_t\}_{t \geq 0}\) to an \(E_0\)-semigroup, in the sense of Arveson [2] and Powers [19], of endomorphisms \(\{\alpha_t\}_{t \geq 0}\) of a larger von

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Neumann algebra \( \mathcal{R} \). Depending on context, the process of dilation has taken different meanings. Here we mean the following: Suppose \( \mathcal{M} \) acts on a Hilbert space \( H \), then a quadruple \((K, \mathcal{R}, \{\alpha_t\}_{t \geq 0}, u_0)\), consisting of a Hilbert space \( K \), a von Neumann algebra \( \mathcal{R} \), an \( E_0 \)-semigroup \( \{\alpha_t\}_{t \geq 0} \) of *-endomorphisms of \( \mathcal{R} \) and an isometric embedding \( u_0 : H \to K \), will be called an \( E_0 \)-dilation of the quantum Markov process \( (\mathcal{M}, \{P_t\}_{t \geq 0}) \) (or, simply, of \( \{P_t\}_{t \geq 0} \)) in case for all \( T \in \mathcal{M} \), all \( S \in \mathcal{R} \) and all \( t \geq 0 \) the following equations hold

\[
P_t(T) = u_0^*\alpha_t(u_0Tu_0^*)u_0
\]

and

\[
P_t(u_0^*Su_0) = u_0^*\alpha_t(S)u_0.
\]

Our objective in this paper is to prove that if the Hilbert space \( H \) on which \( \mathcal{M} \) acts is separable, then such a dilation always exists.

What is novel in our approach is that we recognize the space of the Stinespring dilation of each \( P_t \) as a correspondence \( E_t \) over the commutant of \( \mathcal{M} \), \( \mathcal{M}' \). (All relevant terms will be defined below.) These correspondences are then assembled and “dilated” to a product system \( \{E(t)\}_{t \geq 0} \) of correspondences over \( \mathcal{M}' \), very similar to the product systems that Arveson defined in [2]. Then we describe \( \{P_t\}_{t \geq 0} \) explicitly in terms of what we call a “fully coisometric, completely contractive covariant representation” of \( \{E(t)\}_{t \geq 0} \), denoted \( \{T_t\}_{t \geq 0} \), in a fashion that derives immediately from our work in [3]. A bit more explicitly, but still incompletely, we find that \( \{P_t\}_{t \geq 0} \) may be expressed in terms of \( \{T_t\}_{t \geq 0} \) via the formula

\[
P_t(a) = \tilde{T}_t(I_{E(t)} \otimes a)\tilde{T}_t^*,
\]

where \( a \in \mathcal{M} \) and \( t \geq 0 \), where \( \tilde{T}_t \) is the operator from \( E(t) \otimes H \) to \( H \) defined by the equation \( \tilde{T}_t(\xi \otimes h) = T_t(\xi)h \). Then we dilate \( \{T_t\}_{t \geq 0} \) to what is called an isometric representation \( \{V_t\}_{t \geq 0} \) of \( \{E(t)\}_{t \geq 0} \) on a Hilbert space \( K \). If \( u_0 : H \to K \) is the embedding that goes along with \( \{V_t\}_{t \geq 0} \), the we find that \( T_t(\xi) = u_0^*V_t(\xi)u_0 \) for all \( \xi \in E(t) \) and that the \( E_0 \)-semigroup of endomorphisms \( \{\alpha_t\}_{t \geq 0} \) that want is given by the formula

\[
\alpha_t(R) = \tilde{V}_t(I_{E(t)} \otimes R)\tilde{V}_t^*,
\]

where \( R \) runs over the von Neumann algebra \( \mathcal{R} \) generated by \( \{\alpha_t(u_0Mu_0^*)\}_{t \geq 0} \). That is, \((K, \mathcal{R}, \{\alpha_t\}_{t \geq 0}, u_0)\) is the dilation of \((\mathcal{M}, \{P_t\}_{t \geq 0})\).

An important part of our analysis was inspired by Bhat’s paper [7]. Recently, Bhat and Skeide [8] have dilated a quantum Markov process \((\mathcal{M}, \{P_t\}_{t \geq 0})\) using a product system over the von Neumann algebra \( \mathcal{M} \) (ours is over \( \mathcal{M}' \)). The precise connection between their work and ours has still to be determined. However, what we find attractive about our approach is the close explicit connection between dilations of quantum Markov processes and the classical dilation theory of contraction operators on Hilbert space pioneered by B. Sz-Nagy (see [23]).
In the next section we develop the theory of correspondences over von Neumann algebras sufficiently so that we can link up with theory developed in \[13\] in which representations and dilations of \(C^*\)-correspondences are considered. We also show how what we call the Arveson correspondence \(E_P\) associated with the Stinespring dilation of a completely positive map \(P\) can be dilated to a bigger correspondence \(E\) in such a way that a certain completely contractive covariant representation of \(E\) that gives \(P\) is dilated to a fully coisometric, isometric representation of \(E\). This representation of \(E\) gives a “power” dilation of \(P\).

Then, in Section 3, we construct a “discrete” dilation \((K, \mathcal{R}, \{\alpha_t\}_{t \geq 0}, u_0)\) of the quantum Markov process \((\mathcal{M}, \{P_t\}_{t \geq 0})\). It is here, following ideas developed in Section 2, that we dilate the family \(\{E_P(t)\}_{t \geq 0}\) to a product system of correspondences \(\{E(t)\}_{t \geq 0}\) over \(\mathcal{M'}\). In Section 4, we show that if the Hilbert space on which \(\mathcal{M}\) acts is separable, then the dilation \((K, \mathcal{R}, \{\alpha_t\}_{t \geq 0}, u_0)\) we construct in Section 2 is, in fact, an \(E_0\)-dilation.

We adopt the standard notation that if \(A\) is a subset of a Hilbert space \(H\), then \([A]\) will denote the closed linear span of \(A\).

## 2 Dilations of Completely Positive Maps

Throughout, \(\mathcal{M}\) will denote a von Neumann algebra. While much of what we will have to say about von Neumann algebras can be formulated in a space-free fashion, it will be convenient to view \(\mathcal{M}\) as acting on a fixed Hilbert space \(H\). Thus, we will work inside \(\mathcal{B}(H)\), the bounded operators on \(H\). Also, throughout, \(P\) will denote a fixed completely positive, unital and normal map from \(\mathcal{M}\) to \(\mathcal{M}\). We need to call attention to specific features of the minimal Stinespring dilation of \(P\) \[22, 1, 4\].

Form the algebraic tensor product, \(\mathcal{M} \otimes H\) and define the sesquilinear form \(\langle \cdot, \cdot \rangle\) on this space by the formula

\[
\langle T_1 \otimes h_1, T_2 \otimes h_2 \rangle = \langle h_1, P(T_2^*T_1)h_2 \rangle,
\]

\(T_i \otimes h_i \in \mathcal{M} \otimes H\). The complete positivity of \(P\) guarantees that this form is positive semidefinite. Therefore, the Hausdorff completion of \(\mathcal{M} \otimes H\) is a Hilbert space, which we shall denote by \(\mathcal{M} \otimes_p H\). We shall not distinguish between an element in \(\mathcal{M} \otimes H\) and its image in \(\mathcal{M} \otimes_p H\). The formula

\[
\pi_P(S)(T \otimes h) := ST \otimes h,
\]

\(S \in \mathcal{M}, T \otimes h \in \mathcal{M} \otimes_p H\) defines a representation of \(\mathcal{M}\) on \(\mathcal{M} \otimes_p H\) that is normal because \(P\) is normal. Also, the formula

\[
W_P(h) := I \otimes h,
\]

\(h \in H\), defines an isometric imbedding of \(H\) in \(\mathcal{M} \otimes_p H\), and there results the fundamental equation

\[
P(T) = W_P^* \pi_P(T) W_P,
\]
T ∈ \mathcal{M}. It is an easy matter to check that \( M \otimes \pi \) is minimal in the sense that the smallest subspace of \( M \otimes \pi \) containing \( W \pi \) and reducing \( \pi \pi \) is all of \( M \otimes \pi \). Consequently, the triple \((\pi, M \otimes \pi, W \pi)\) is the unique minimal triple, \((\pi, K, W)\), up to unitary equivalence, such that

\[
P(T) = W^* \pi(T) W,
\]

\( T \in \mathcal{M} \). We therefore refer to \((\pi, M \otimes \pi, W \pi)\) as the Stinespring dilation of \( \pi \).

The adjoint \( W^* \) of the isometric embedding \( W \pi \) of \( \pi \) in \( M \otimes \pi \) has an explicit form that we will need throughout our analysis:

\[
W^*_\pi (X \otimes h) = P(X)h, \quad X \otimes h \in M \otimes \pi.
\]  

This is easy to see because

\[
\langle W^*_\pi (X \otimes h), k \rangle = \langle X \otimes h, W \pi k \rangle = \langle X \otimes h, I \otimes k \rangle = \langle h, P(X^*)k \rangle = \langle P(X)h, k \rangle.
\]

A space of critical importance for us will be the intertwining space,

\[
\mathcal{L}_\mathcal{M}(H, M \otimes \pi) := \{ X : H \to M \otimes \pi \mid XT = \pi \pi(T)X, T \in \mathcal{M} \}.
\]

That is, \( \mathcal{L}_\mathcal{M}(H, M \otimes \pi) \) is the space of operators that intertwine the identity representation of \( \mathcal{M} \) on \( H \) and \( \pi \pi \). This space turns out to be a \( W^* \)-correspondence over the commutant \( \mathcal{M}' \) of \( \mathcal{M} \). The notion of a \( W^* \)-correspondence is fundamental in this study, and therefore we pause to develop the terminology and to cite some important facts.

We follow Lance [12] for the general theory of Hilbert \( C^* \)-modules that we shall use. In particular, unless indicated to the contrary, a Hilbert module \( \mathcal{X} \) over a \( C^* \)-algebra \( A \), will be a right Hilbert \( C^* \)-module. We write \( \mathcal{L}(\mathcal{X}) \) for the space of continuous, adjointable \( A \)-module maps on \( \mathcal{X} \) (which we shall write on the left of \( \mathcal{X} \)) and we shall write \( \mathcal{K}(\mathcal{X}) \) for the space of (generalized) compact operators on \( \mathcal{X} \), i.e., \( \mathcal{K}(\mathcal{X}) \) is the span of the the rank one operators \( \xi \otimes \eta^* \), \( \xi, \eta \in \mathcal{X} \), where \( \xi \otimes \eta^*(\zeta) = \xi(\eta, \zeta) \).

**Definition 2.1** Let \( A \) and \( B \) be \( C^* \)-algebras. A \( C^* \)-correspondence from \( A \) to \( B \) is a Hilbert \( C^* \)-module \( \mathcal{X} \) over \( B \) endowed with the structure of a left module over \( A \) via a \( * \)-homomorphism \( \varphi : A \to \mathcal{L}(\mathcal{X}) \). A \( C^* \)-correspondence over \( A \) is simply a \( C^* \)-correspondence from \( A \) to \( A \).

When dealing with specific \( C^* \)-correspondences, \( \mathcal{X} \) from a \( C^* \)-algebra \( A \) to a \( C^* \)-algebra \( B \), it will be convenient to suppress the \( \varphi \) in formulas involving the left action and simply write \( a \xi \) or \( a \cdot \xi \) for \( \varphi(a) \xi \). This should cause no confusion in context.

\( C^* \)-correspondences should be viewed as generalized \( C^* \)-homomorphisms. Indeed, the collection of \( C^* \)-algebras together with (isomorphism classes of) \( C^* \)-correspondences is a category that contains (contravariantly) the category
of $C^*$-algebras and (conjugacy classes of) $C^*$-homomorphisms. Of course, for this to make sense, one has to have a notion of composition of correspondences and a precise notion of isomorphism. The notion of isomorphism is the obvious one: a bijective, bimodule map that preserves inner products. Composition is “tensoring”: If $\mathcal{X}$ is a $C^*$-correspondence from $A$ to $B$ and if $\mathcal{Y}$ is a correspondence from $B$ to $C$, then the balanced tensor product, $\mathcal{X} \otimes_B \mathcal{Y}$ is an $A,C$-bimodule that carries the inner product defined by the formula

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle_{\mathcal{X} \otimes_B \mathcal{Y}} := \langle \eta_1, \varphi((\xi_1, \xi_2)_{\mathcal{X}}) \eta_2 \rangle_{\mathcal{Y}}.$$  

The Hausdorff completion of this bimodule is again denoted by $\mathcal{X} \otimes_B \mathcal{Y}$ and is called the composition of $\mathcal{X}$ and $\mathcal{Y}$. At the level of correspondences, composition is not associative. However, if we pass to isomorphism classes, it is. That is, we only have an isomorphism $(\mathcal{X} \otimes \mathcal{Y}) \otimes \mathcal{Z} \simeq \mathcal{X} \otimes (\mathcal{Y} \otimes \mathcal{Z})$. It is worthwhile to emphasize here that while it often is safe to ignore the distinction between correspondences and their isomorphism classes, at times, as we shall see, the distinction is of critical importance.

If $\mathcal{N}$ is a von Neumann algebra and if $\mathcal{X}$ is a Hilbert $C^*$-module over $\mathcal{N}$, then $\mathcal{X}$ is called self-dual in case every continuous $\mathcal{N}$-module map $\Phi$ from $\mathcal{X}$ to $\mathcal{N}$ is implemented by an element of $\mathcal{X}$, i.e., in case there is an $\xi_\Phi \in \mathcal{X}$ so that $\Phi(\xi) = \langle \xi_\Phi, \xi \rangle$, $\xi \in \mathcal{X}$. There is a topological characterization of self-dual Hilbert $C^*$-modules over von Neumann algebras given in [6] that will be useful for us. To state it, recall that their $\sigma$-topology on a Hilbert $C^*$-module $\mathcal{X}$ over a von Neumann algebra $\mathcal{N}$ is the topology defined by the functionals

$$f(\cdot) := \sum_{n=1}^{\infty} w_n(\langle \eta_n, \cdot \rangle)$$

where the $\eta_n$ lie in $\mathcal{X}$, the $w_n$ lie in $\mathcal{N}_*$, and $\sum \|w_n\| \|\eta_n\| < \infty$. Baillet, Denizeau, and Havet proved that a Hilbert $C^*$-module $\mathcal{X}$ over a von Neumann algebra $\mathcal{N}$ is self-dual if and only if the unit ball in $\mathcal{X}$ is compact in the $\sigma$-topology [6, Proposition 1.7]. In [16], Paschke proved that if $\mathcal{X}$ is a self-dual Hilbert $C^*$-module over a von Neumann algebra $\mathcal{N}$, then $\mathcal{L}(\mathcal{X})$ is a von Neumann algebra, i.e., $\mathcal{L}(\mathcal{X})$ is a $C^*$-algebra which is also a dual space and which, therefore, may be represented faithfully on Hilbert space in such a way that the weak-$^*$ topology on $\mathcal{L}(\mathcal{X})$ coincides with the $\sigma$-weak topology on the image.

**Definition 2.2** Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras and let $\mathcal{X}$ be a Hilbert $C^*$-module over $\mathcal{N}$. Then $\mathcal{X}$ is called a Hilbert $W^*$-module over $\mathcal{N}$ in case $\mathcal{X}$ is self-dual. The module $\mathcal{X}$ is called a $W^*$-correspondence from $\mathcal{M}$ to $\mathcal{N}$ in case $\mathcal{X}$ is a self-dual $C^*$-correspondence from $\mathcal{M}$ to $\mathcal{N}$ such that the $*$-homomorphism $\varphi : \mathcal{M} \to \mathcal{L}(\mathcal{X})$ giving the left module structure on $\mathcal{X}$ is normal.

It is evident that the composition of $W^*$-correspondences is again a $W^*$-correspondence. The following proposition shows one way to construct $W^*$-correspondences.
Proposition 2.3 Let $H$ and $K$ be Hilbert spaces. Let $M$ be a von Neumann algebra on $K$, let $N$ be a von Neumann algebra on $H$, and let $\mathcal{Y} \subseteq B(H,K)$ be a $\sigma$-weakly closed linear space of operators. Suppose that $M \mathcal{Y} \mathcal{N} \subseteq \mathcal{Y}$ and that $\mathcal{Y}' \mathcal{Y} := \{Y^*Y \mid Y \in \mathcal{Y}\}$ is contained in $\mathcal{N}$. Then $\mathcal{Y}$ is a self-dual Hilbert $W^*$-module over $\mathcal{N}$ that has the structure of a $W^*$-correspondence from $M$ to $N$. Further, the $\sigma$-topology on $\mathcal{Y}$ coincides with the $\sigma$-weak topology on $\mathcal{Y}$ as a subspace of $B(H,K)$.

**Proof.** It is evident that $\mathcal{Y}$ has the structure of a $C^*$-correspondence from $M$ to $N$. The main point of the proposition is the assertion about self-duality and the topologies. The functionals $f$ defining the $\sigma$-topology are of the form $f(\cdot) := \sum_{n=1}^\infty w_n(\langle \eta_n, \cdot \rangle)$ where the $\eta_n$ lie in $\mathcal{Y}$, the $w_n$ lie in $N_\ast$, and $\sum \|w_n\| \|\eta_n\| < \infty$. Evidently, each of these is $\sigma$-weakly continuous. Conversely, given a functional on $\mathcal{Y}$ of the form $g(Y) = \langle Yh, k \rangle$, we may assume that $k$ is in the closed span of $\{Yh \mid Y \in \mathcal{Y}, h \in H\}$ and approximate $g$ in norm by functionals of the form

$$
\hat{g}(Y) = \sum_{m=1}^r \langle Yh, Ym(h')_m \rangle = \sum_{m=1}^r \langle Y^*Y(h_m)_m, (h'_m) \rangle.
$$

Each of these functionals is continuous in the $\sigma$-topology. Since the space of $\sigma$-continuous functionals is a Banach space [6, 1.2], the functional $Y \rightarrow \langle Yh, k \rangle$ is in this space, and so is every $\sigma$-weakly continuous functional on $\mathcal{Y}$. It follows that the two topologies coincide on $\mathcal{Y}$. Since the closed unit ball in $\mathcal{Y}$ is $\sigma$-weakly compact, it must be compact in the $\sigma$-topology. By [6, Proposition 1.7], $\mathcal{Y}$ is self-dual. ■

**Remark 2.4**

(i) The theory developed in [6] can be used to prove a converse to this result: Given a $W^*$-correspondence $\mathcal{Y}$ from $M$ to $N$, then there are faithful normal representations $\pi : M \rightarrow B(K)$ and $\rho : N \rightarrow B(H)$ and there is a linear map $\Phi : \mathcal{Y} \rightarrow B(H,K)$ such that $\Phi(\varphi(T)YS) = \pi(T)\Phi(Y)\rho(S)$ and $\rho((X,Y)Y') = \Phi(X)^*\Phi(Y')$ for all $X, Y \in \mathcal{Y}$, $T \in M$, and $S \in N$, and such that $\Phi$ is a homomorphism with respect to the $\sigma$-topology on $\mathcal{Y}$ and the $\sigma$-weak topology on $\Phi(\mathcal{Y})$. Thus, in a sense, the construction in Proposition 2.3 is universal.

(ii) Suppose $\mathcal{X}$ is a self-dual Hilbert $W^*$-module over a von Neumann algebra $\mathcal{N}$ and that $\pi : M \rightarrow \mathcal{L}(\mathcal{X})$ is a $C^*$-homomorphism on the von Neumann algebra $\mathcal{M}$. Then $\pi$ is normal if for every bounded net $\{A_\alpha\} \subseteq \mathcal{N}$, with $A_\alpha \rightarrow A$ weakly, every $g \in N_\ast$, and every $X, Y \in \mathcal{X}$, we have $g(\langle \pi(A_\alpha)X, Y \rangle) \rightarrow g(\langle \pi(A)X, Y \rangle)$. This follows from the fact that $\mathcal{L}(\mathcal{X})$ is the dual space of the tensor product $\mathcal{X} \otimes \mathcal{X}^* \otimes N_\ast$ equipped with the greatest cross norm [10, Proposition 3.10].

Proposition 2.5 Let $(\pi_P, M \otimes_P H, W_P)$ be the Stinespring dilation of a completely positive map $P$ on the von Neumann algebra $M$. Then $\mathcal{L}_M(H, M \otimes_P H)$ is a $\sigma$-weakly closed subspace of $B(H, M \otimes_P H)$ that is closed under left multiplication by $I \otimes M'$ and right multiplication by $M'$ and has the property that $\mathcal{L}_M(H, M \otimes_P H)^* \mathcal{L}_M(H, M \otimes_P H) \subseteq M'$. Thus $\mathcal{L}_M(H, M \otimes_P H)$ has the structure of a $W^*$-correspondence over $M'$.  

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**Proof.** Evidently, if $X \in \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_R H)$ and $T \in \mathcal{M}'$, then $XT \in \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_R H)$. Indeed, if $S \in \mathcal{M}$, then $XTS = XST = \pi_p(S)XT$, showing that $XT \in \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_R H)$. Also, if $X, Y \in \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_R H)$, then $X^*Y \in \mathcal{M}'$ because for all $T \in \mathcal{M}$, $X^*YT = X^*\pi_p(T)Y = TX^*Y$. Thus, by Proposition 2.3 it remains to give the left action of $\mathcal{M}'$. On the face of it, this is evidently given by the formula $\varphi(T) = I \otimes T$, $T \in \mathcal{M}'$. However, the meaning of $I \otimes T$, $T \in \mathcal{M}'$, and the expression $I \otimes \mathcal{M}'$, need a little development. For $T \in \mathcal{M}'$, we prove that the algebraic tensor product $I \otimes T$ is bounded as follows: Observe that for $\sum_{i=1}^n S_i \otimes h_i \in M \otimes H$, we have

\[
\left\| (I \otimes T) \left( \sum_{i=1}^n S_i \otimes h_i \right) \right\|^2 = \langle I \otimes T \left( \sum_{i=1}^n S_i \otimes h_i \right), I \otimes T \left( \sum_{i=1}^n S_i \otimes h_i \right) \rangle
\]

\[
= \langle \left( \sum_{i=1}^n S_i \otimes Th_i \right), \left( \sum_{i=1}^n S_i \otimes Th_i \right) \rangle
\]

\[
= \sum_{i,j=1}^n \langle Th_i, P(S_j^*S_j)Th_j \rangle
\]

\[
= \sum_{i,j=1}^n \langle h_i, T^*P(S_j^*S_j)Th_j \rangle.
\]

However, since $P$ is completely positive the operator matrix $(P(S_j^*S_j))$ is a positive element in $M_n(\mathcal{M}')$ and so can be written as $C^*C$, for an element $C \in M_n(\mathcal{M}')$. Therefore, $(T^*P(S_j^*S_j)T) = T^*C^*CT = C^*T^*TC \leq \|T\|^2 C^*C$, where $\hat{T}$ is the $n$-fold inflation of $T$. Consequently, the last term in the displayed equation is dominated by

\[
\|T\|^2 \sum_{i,j=1}^n \langle h_i, T^*P(S_j^*S_j)Th_j \rangle = \|T\|^2 \left\| \sum_{i=1}^n S_i \otimes h_i \right\|^2
\]

Thus $I \otimes T$ extends to an element in $\pi_p(\mathcal{M})'$, which we continue to denote by $I \otimes T$. The collection of all these operators on $M \otimes_R H$ is denoted by $I \otimes \mathcal{M}'$. Evidently, the map $T \mapsto I \otimes T$ is a (not-necessarily-injective) normal $*$-homomorphism of $\mathcal{M}'$ onto its range. Nevertheless, we denote the range by $I \otimes \mathcal{M}'$ and note that $\mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_R H)$ is a left $\mathcal{M}'$ module through this homomorphism. Since $\mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_R H)$ is manifestly $\sigma$-weakly closed, the proof is completed by appeal to Proposition 2.3. ■

For our purposes, a drawback of $\mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_R H)$ is that it is a space of operators acting between two different Hilbert spaces. We want to “pull $\mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_R H)$ back” to $H$ using $W_p$ and the following device that is due to Arveson 11. Given $Y \in B(H)$, and $S \otimes h$ in the algebraic tensor product $\mathcal{M} \otimes H$, we set

\[
\Phi_Y(S \otimes h) := SY^*h,
\]
and extend $\Phi_Y$ by linearity to a densely defined linear map from $\mathcal{M} \otimes_P H$ to $H$ with domain $\mathcal{M} \otimes H$. We write $\mathcal{E}_P$ for the space of all operators $Y \in B(H)$ such that $\Phi_Y$ is continuous. (In this case, of course, we continue to write $\Phi_Y$ for the unique continuous extension to all of $\mathcal{M} \otimes_P H$.)

**Proposition 2.6** The space $\mathcal{E}_P$ is a linear space that is stable under left and right multiplication by elements from $\mathcal{M}'$, and the pairing $(Y, Z) := \Phi_Y \Phi_Z^*$ converts $\mathcal{E}_P$ into a $W^*$-correspondence that is isomorphic to $\mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_P H)$ under the map $Y \mapsto \Phi_Y^*$.

The advantage of $\mathcal{E}_P$ over $\mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_P H)$ is not only that $\mathcal{E}_P \subseteq B(H)$, but also, as we shall see shortly, given two completely positive maps $P$ and $Q$ on $\mathcal{M}$, the relations among $\mathcal{E}_P$, $\mathcal{E}_Q$, and $\mathcal{E}_{PQ}$ are easier to work with than those among $\mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_P H)$, $\mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_Q H)$, and $\mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_P \mathcal{M} \otimes_P H)$.

**Proof.** Evidently, $\mathcal{E}_P$ is a linear space. For $R \in \mathcal{M}'$ and $Y \in \mathcal{E}_P$, $\Phi_{R} = R^* \Phi_Y$, and so $\mathcal{E}_P \mathcal{M}' \subseteq \mathcal{E}_P$. For the other side, fix $Y \in \mathcal{E}_P$ and $T \in \mathcal{M}'$. Then for $(\sum_{i=1}^{n} S_i \otimes h_i) \in \mathcal{M} \otimes H$, $\Phi_{TY}(\sum_{i=1}^{n} S_i \otimes h_i) = \sum_{i=1}^{n} S_i Y^* T^* h_i = \Phi_{Y}(\sum_{i=1}^{n} S_i \otimes T^* h_i)$. Therefore $\Phi_{TY} = \Phi_Y(I \otimes T^*)$ on $\mathcal{M} \otimes H$, showing that $\Phi_{TY}$ is bounded; i.e., $\mathcal{M}' \mathcal{E}_P \subseteq \mathcal{E}_P$.

Next note that for $Y \in \mathcal{E}_P$, $S, T \in \mathcal{M}$, and $h \in H$,

$$\Phi_Y(\pi_P(T)(S \otimes h)) = \Phi_Y(T S \otimes h) = T S Y^* h = T \Phi_Y(S \otimes h);$$

i.e., $\Phi_Y \pi_P(T) = T \Phi_Y$. Taking adjoints, we conclude that $\Phi_Y^* \in \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_P H)$. Thus, from the properties of $\mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_P H)$, we know that the formula $(Y, Z) := \Phi_Z \Phi_Y^*$, defines an $\mathcal{M}'$-valued function on $\mathcal{E}_P$. In fact, it is clearly an $\mathcal{M}'$-valued sesquilinear form, since $\Phi_{RY} = R^* \Phi_Y$ and $\Phi_{TY} = \Phi_Y(I \otimes T^*)$, $R, T \in \mathcal{M}'$, and it is clearly positive semidefinite. It is definite because if $(Y, Y) = \Phi_Y \Phi_Y^* = 0$, then $\Phi_Y = 0$, and so $TY^* h = 0$ for all $T \in \mathcal{M}$ and $h \in H$.

Taking $T = I$ we conclude that $Y = 0$.

The map $Y \to \Phi_Y^*$ preserves inner products by definition. Further, it is a bimodule map since $(\Phi_{RY})^* = (R^* \Phi_Y)^* = \Phi_Y^* R$ and $(\Phi_{RY})^* = (\Phi_Y(I \otimes R^*))^* = (I \otimes R) \Phi_Y^*$, for all $R \in \mathcal{M}'$. Thus, to show that $\mathcal{E}_P$ is isomorphic to $\mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_P H)$ under this map, we need only show that it is onto. However, we assert that for all $X \in \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_P H)$, $X = \Phi_{(W^*_p X)}^*$. Indeed, for $h, h' \in H$, and $S \in \mathcal{M}$, the fact that $(I \otimes S)X = XS$ implies that

$$\Phi_{(W^*_p X)}(S \otimes h) = S X^* W p h = S X^*(I \otimes h) = X^*(S \otimes h).$$

This shows that $(W^*_p X)$ is in $\mathcal{E}_P$ and that $X = \Phi_{(W^*_p X)}^*$. The facts that $\mathcal{E}_P$ and $\mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_P H)$ are isomorphic and that $\mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_P H)$ is self-dual imply that $\mathcal{E}_P$ is self-dual. $\blacksquare$

**Definition 2.7** The $W^*$-correspondence $\mathcal{E}_P$ over $\mathcal{M}'$ associated with a normal, unital completely positive map $P$ on a von Neumann algebra $\mathcal{M}$ will be called the Arveson correspondence associated with $P$. 

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The following corollary is immediate from the self-duality of the spaces involved. We call attention to it because it will be used several times in the sequel.

**Corollary 2.8** If a subspace $\mathcal{Y}$ of $\mathcal{E}_P$ or of $\mathcal{L}_M(H,M \otimes \rho H)$ has zero annihilator, i.e., if $\mathcal{Y}^\perp = 0$, then $\mathcal{Y}$ is dense.

A special case of our analysis so far needs to be singled out. Suppose that $P = \alpha$ is a unital, normal, $\ast$-endomorphism of $M$. (All endomorphisms will be unital, normal, and preserve adjoints.) Then $M \otimes \alpha H$ is isomorphic to $H$ under the map $T \otimes h \rightarrow \alpha(T)h$, which is $W_\alpha^{-1} = W_\alpha^\ast$. Further, $\pi_\alpha$ is unitarily equivalent to $\alpha$. Thus, we may identify $\mathcal{E}_P = \mathcal{E}_\alpha$ directly with $\mathcal{L}_M(H,M \otimes P H)$ as in the following corollary of Proposition 2.6.

**Corollary 2.9** If $\alpha$ is a unital, normal, $\ast$-endomorphism of $M$, then $\mathcal{E}_\alpha = \{X \in B(H) \mid XT = \alpha(T)X, T \in M\}$ with the inner product $\langle X_1, X_2 \rangle = X_1^* X_2$.

The next lemma may seem like a technicality, but among other things, it establishes that the modules $\mathcal{L}_M(H,M \otimes P H)$ and $\mathcal{E}_P$ are nonzero. It plays other useful roles in the sequel.

**Lemma 2.10** In the setting of a normal, unital completely positive map $P$ on $M$ that we have been studying,

$$M \otimes P H = \bigvee \{\Phi_Y(H) \mid Y \in \mathcal{E}_P\} = \bigvee \{X(H) \mid X \in \mathcal{L}_M(H,M \otimes P H)\}.$$ 

**Proof.** Since $\pi_P$ is a normal $\ast$-representation, its kernel is of the form $Mq$ for a central projection $q$. Write $\pi'$ for the representation of $M$ that is reduction by $I - q$, i.e., $\pi'(S) = S(I - q)$. Then for $S \in M$, $\|\pi'(S)\| = \|\pi_P(S)\|$, so that $\pi'$ and $\pi_P$ are quasiequivalent. If $Q$ is the projection of $M \otimes P H$ onto $\bigvee \{X(H) \mid X \in \mathcal{L}_M(H,M \otimes P H)\}$, then for every $L \in \mathcal{L}_M(H,M \otimes P H)$, $L(H)$ is $\pi_P(M)$-invariant. Hence $Q \in \pi_P(M)'$. If $\pi_0$ is the reduction of $\pi_P$ to the range of $I - Q$, then, on the one hand, $\pi_0 \leq \pi_P$ and on the other, $\pi_0$ is disjoint from $\pi'$. Since $\pi'$ is quasiequivalent to $\pi_P$ we conclude that $\pi_0 = 0$, i.e., that $Q = 1$.

We next want to illuminate the relation between the composition of two completely positive maps on $M$ and the composition of their Arveson correspondences. This was worked out in the case when $M = B(H)$ by Arveson in [6, Theorem 1.12]. Given two normal, unital completely positive maps $P, Q : M \rightarrow M$, we shall write $m$ for the multiplication map from $\mathcal{E}_P \otimes M$ to $B(H)$. That is, $m(Y \otimes Z) = YZ$.

**Lemma 2.11** The range of $m$ is contained in $\mathcal{E}_{PQ}$.

**Proof.** First observe that if $Y \in \mathcal{E}_P$, if $A = (a_{ij})$ is a positive semidefinite element in $M_n(M)$, and if $h = (h_1, h_2, \ldots, h_n)$ is an $n$-tuple of elements from $H$, then

$$\left\| A^{1/2} (Y^* \otimes I) h \right\|^2 \leq \|\Phi_Y\|^2 \langle h_i, \sum P(a_{ij}) h_j \rangle.$$  

(2)
To see this, note that if $A$ is a diad, i.e., if $A$ has the form

$$A = (S_1, S_2, \ldots, S_n)^* (S_1, S_2, \ldots, S_n), \quad S_i \in \mathcal{M},$$

then the left-hand side of the inequality is simply $\| \sum S_i Y^* h_i \|^2$, while the right-hand side is $\| \Phi_Y \|^2 \| \sum S_i \otimes h_i \|^2$. So, the inequality is valid by definition. However, every non-negative $A \in M_n(\mathcal{M})$ is a sum of diads. (See [5, Lemma 3.11].)

So the inequality is valid as claimed.

Now fix $Y \in \mathcal{E}_P$, $Z \in \mathcal{E}_Q$, $\sum S_i \otimes h_i \in \mathcal{M} \otimes H$. Then

$$\left\| \sum S_i (Y^* h_i) \right\|^2 = \left\| \sum S_i (Z^* (Y^* h_i)) \right\|^2 \leq \| \Phi_Z \|^2 \left\| \sum S_i \otimes Q Z^* Y^* h_i \right\|^2$$

$$= \| \Phi_Z \|^2 \left\| \sum (Z^* (Y^* h_i), Q(S^*_i S_j) Y^* h_j) \right\|^2$$

$$= \| \Phi_Z \|^2 \| (Q(S^*_i S_j))^{1/2} (Y^* \otimes I) h \|^2$$

$$\leq \| \Phi_Z \|^2 \| \Phi_Y \|^2 \left\| \sum h_i, P(Q(S^*_i S_j)) Y^* h_j \right\|^2$$

$$= \| \Phi_Z \|^2 \| \Phi_Y \|^2 \left\| S \otimes P Q h \right\|^2.$$

This shows that $\Phi_{YZ}$ is bounded, and that $\| \Phi_{YZ} \| \leq \| \Phi_Z \| \| \Phi_Y \|$. Thus $Y Z \in \mathcal{E}_{PQ}$. $\blacksquare$

We also want to express $m$ in terms of the space $\mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_P H)$. For this purpose, fix two normal, unital, completely positive maps $P, Q : \mathcal{M} \rightarrow \mathcal{M}$. We define a map $\Psi : \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_P H) \otimes \mathcal{M} \rightarrow \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes Q) \otimes \mathcal{M} \rightarrow \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes Q \mathcal{M} \otimes_P H)$ by the formula $\Psi(X \otimes Y) = (I \otimes X) Y$, where $I \otimes X$ is the map from $\mathcal{M} \otimes Q H$ to $\mathcal{M} \otimes Q \mathcal{M} \otimes_P H$ given by the equation $I \otimes X (S \otimes h) = S \otimes X h$. We also define a map $V_0 : \mathcal{M} \otimes_P H \rightarrow \mathcal{M} \otimes Q \mathcal{M} \otimes_P H$ via the equation $V_0 (S \otimes h) = S \otimes I \otimes h$, and we define $V : \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_P H) \rightarrow \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes Q \mathcal{M} \otimes_P H)$ by the formula $V(X) = V_0 X$.

**Proposition 2.12** In the notation just established, $\Psi$ is an isomorphism of correspondences and $V$ is an isometry whose range is $\{ X \in \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes Q \mathcal{M} \otimes_P H) \mid X(H) \subseteq \mathcal{M} \otimes Q I \otimes_P H \}$. Further, if we write $U_P : \mathcal{E}_P \rightarrow \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes P H)$ for the isomorphism defined above, and similarly write $U_Q$ and $U_{PQ}$, then

$$U_{PQM} (U_P^{-1} \otimes U_Q^{-1}) = V^* \Psi,$$  \hfill (3)

showing that $m$ is coisometric and $m^*$ is isometric.

**Proof.** On the one hand, $\Psi(X \otimes Y)^* \Psi(X' \otimes Y') = Y^* (I \otimes X^*) (I \otimes X') Y' = Y^* (I \otimes X^* X') Y'$ - the inner product in $\mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes Q \mathcal{M} \otimes_P H)$. On the other hand, recall that the left action of $Z \in \mathcal{M}'$ on $\mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes Q H)$ is given by the equation $(I \otimes Z) Y, Y \in \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes Q H)$. Consequently, in $\mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_P H) \otimes \mathcal{M}', \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes Q H)$,

$$\langle X \otimes Y, X' \otimes Y' \rangle = \langle Y, (X, X') \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes P H) (Y', Y') \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes Q H) \rangle$$

$$= \langle Y, (I \otimes X^* X') \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes Q H) \rangle = Y^* (I \otimes X^* X') Y'.$$
Thus \( \Psi \) preserves the inner products.

To see that \( \Psi \) is a bimodule map, let \( S \in \mathcal{M}' \), \( X \in \mathcal{L}_{\mathcal{M}}(H, \mathcal{M} \otimes \mathcal{P} H) \), and \( Y \in \mathcal{L}_{\mathcal{M}}(H, \mathcal{M} \otimes \mathcal{Q} H) \). Then

\[
\Psi(S(X \otimes Y)) = \Psi((I \otimes S)X \otimes Y) = (I \otimes (I \otimes S))X)Y = (I \otimes I \otimes S)(I \otimes X)Y = S\Psi(X \otimes Y),
\]

while

\[
\Psi(X \otimes Y S) = (I \otimes X)(Y S) = ((I \otimes X)Y)S = \Psi(X \otimes Y)S.
\]

To see that \( \Psi \) is surjective, we use the fact that \( \mathcal{L}_{\mathcal{M}}(H, \mathcal{M} \otimes \mathcal{Q} \mathcal{M} \otimes \mathcal{P} H) \) is self-dual (see Proposition 2.3) and show that \( \text{Im} \Psi \perp = \{0\} \). Corollary 2.8, then, will yield the result. If \( Z \) annihilates \( \text{Im} \Psi \), then for every \( X \in \mathcal{L}_{\mathcal{M}}(H, \mathcal{M} \otimes \mathcal{P} H) \) and for every \( Y \in \mathcal{L}_{\mathcal{M}}(H, \mathcal{M} \otimes \mathcal{Q} H) \), \( Y^* (I \otimes X^*) Z = 0 \). Observe that \( (I \otimes X^*) Z \) is a map from \( H \) to \( \mathcal{M} \otimes \mathcal{Q} H \). By Lemma 2.11, \( \mathcal{M} \otimes \mathcal{Q} H \) is the span of \( Y(H), Y \in \mathcal{L}_{\mathcal{M}}(H, \mathcal{M} \otimes \mathcal{Q} H) \). Consequently, \( \cap \{ \ker Y^* \mid Y \in \mathcal{L}_{\mathcal{M}}(H, \mathcal{M} \otimes \mathcal{Q} H) \} = \{0\} \). Since \( Y^* (I \otimes X^*) Z = 0 \) for all such \( Y \), we conclude that \( (I \otimes X^*) Z = 0 \) for all \( X \in \mathcal{L}_{\mathcal{M}}(H, \mathcal{M} \otimes \mathcal{P} H) \). By Lemma 2.11 again, \( \cap \{ \ker X^* \mid X \in \mathcal{L}_{\mathcal{M}}(H, \mathcal{M} \otimes \mathcal{P} H) \} = \{0\} \), and so \( Z = 0 \).

Turning now to \( V \), note that it is an easy matter to check that \( V_0 \) is an isometry. Consequently, for \( X \in \mathcal{L}_{\mathcal{M}}(H, \mathcal{M} \otimes \mathcal{P} H) \), \( V^*(X)^* V(X) = X^* V_0^* V_0 X = X^* X \). So \( V \) is isometric. Also, it is evident from the definitions that \( V \) is a bimodule map and that its image is \( \{ X \in \mathcal{L}_{\mathcal{M}}(H, \mathcal{M} \otimes \mathcal{Q} \mathcal{M} \otimes \mathcal{P} H) \mid X(H) \subseteq \mathcal{M} \otimes \mathcal{Q} I \mathcal{P} H \} \). Thus, we are left to prove equation (5). To this end, let \( Y \in \mathcal{E}_{\mathcal{P}} \) and \( Z \in \mathcal{E}_{\mathcal{Q}} \). As an equation between maps from \( \mathcal{M} \otimes \mathcal{P} \mathcal{H} \) to \( H \), the relation \( \Phi_Z(I \otimes \Phi_Y)V_0 = \Phi_Y Z \) is immediate. Therefore, \( \Phi_Y Z = \Phi_Y^* (I \otimes \Phi_Y^*) \Phi_Z^* \). Since \( U_{\mathcal{P} \mathcal{Q} \mathcal{H}}(Y) = \Phi_Y^* \), \( U_{\mathcal{P} \mathcal{Q} \mathcal{H}}(Y Z) = \Phi_Z^* \) and \( m(Y \otimes Z) = Y Z \), we find that

\[
U_{\mathcal{P} \mathcal{Q} \mathcal{H}}(U_{\mathcal{P} \mathcal{Q} \mathcal{H}}^{-1}(I \otimes \Phi_Y^*) \Phi_Y^* \Phi_Z^*) = U_{\mathcal{P} \mathcal{Q} \mathcal{H}}(Y Z) = \Phi_Z^* = V_0^* \Psi(\Phi_Y^* \Phi_Z^*) = V^* (\Psi(\Phi_Y^* \Phi_Z^*)).
\]

\[
\text{Corollary 2.13} \quad \text{Under the hypotheses of Proposition 2.12,}
\]

\[
\bigvee \{ T(H) \mid T \in \mathcal{L}_{\mathcal{M}}(H, \mathcal{M} \otimes \mathcal{Q} \mathcal{M} \otimes \mathcal{P} H) \} = \mathcal{M} \otimes \mathcal{Q} \mathcal{M} \otimes \mathcal{P} H.
\]

\[
\text{Proof.} \quad \text{The span} \ \bigvee \{ T(H) \mid T \in \mathcal{L}_{\mathcal{M}}(H, \mathcal{M} \otimes \mathcal{Q} \mathcal{M} \otimes \mathcal{P} H) \} \ \text{contains the span} \ \bigvee \{ (I \otimes X)Y(H) \mid X \in \mathcal{L}_{\mathcal{M}}(H, \mathcal{M} \otimes \mathcal{P} H), Y \in \mathcal{L}_{\mathcal{M}}(H, \mathcal{M} \otimes \mathcal{Q} H) \}. \ \text{But using Lemma 2.11 twice, we see that this space is} \ \bigvee \{ (I \otimes X)(\mathcal{M} \otimes \mathcal{Q} H) \mid X \in \mathcal{L}_{\mathcal{M}}(H, \mathcal{M} \otimes \mathcal{P} H) \} = \mathcal{M} \otimes \mathcal{Q} \mathcal{M} \otimes \mathcal{P} H. \quad \square
\]

Although the product of two correspondences associated with completely positive maps does not coincide with the correspondence of the product of the maps, there are important special situations when they do. This, and more, is spelled out in the following proposition. (For a related result, see [3, Theorem 2.12].)
Proposition 2.14  (1) If \( \alpha \in \text{Aut}(\mathcal{M}) \), then \( \mathcal{E}_{\alpha^{-1}} = \mathcal{E}_\alpha^* \), and \( \mathcal{E}_\alpha \mathcal{E}_\alpha^* \) and \( \mathcal{E}_\alpha^* \mathcal{E}_\alpha \) are \( \sigma \)-weakly dense in \( \mathcal{M}' \). In particular, \( \mathcal{E}_\alpha \) is an \( \mathcal{M}' - \mathcal{M}' \) equivalence bimodule.

(2) If \( \alpha \) is an endomorphism of \( \mathcal{M} \) and if \( Q \) is a normal, unital, completely positive map of \( \mathcal{M} \), then the multiplication map \( m : \mathcal{E}_\alpha \otimes_{\mathcal{M}'} \mathcal{E}_Q \to \mathcal{E}_{\alpha \circ Q} \), defined above, is an isomorphism.

(3) If \( \alpha \in \text{Aut}(\mathcal{M}) \) and if \( P \) is a normal, unital completely positive map, then the map \( m : \mathcal{E}_\alpha \otimes_{\mathcal{M}'} \mathcal{E}_P \to \mathcal{E}_{P \circ \alpha} \) is an isomorphism.

(4) If \( P \) and \( Q \) are conjugate normal, unital completely positive maps (i.e., if there is an automorphism \( \alpha \) of \( \mathcal{M} \) such that \( P = \alpha \circ Q \circ \alpha^{-1} \)), then

\[
\mathcal{E}_P \simeq \mathcal{E}_\alpha \otimes_{\mathcal{M}'} \mathcal{E}_Q \otimes_{\mathcal{M}'} \mathcal{E}_\alpha^*.
\]

In particular, \((\mathcal{M}', \mathcal{E}_P)\) and \((\mathcal{M}', \mathcal{E}_Q)\) are strongly Morita equivalent in the sense of [15].

We note for the sake of emphasis, that the tensor products described in equation (1) are realized through operator multiplication and adjunction. That is, if \( R \) and \( T \in \mathcal{E}_\alpha \) and \( S \in \mathcal{E}_Q \), then \( T^* \in \mathcal{E}_\alpha^* \) and \( R \otimes S \otimes T^* = RST^* \).

Proof. (1) By Corollary 2.9, \( \mathcal{E}_\alpha = \{ X \in B(H) \mid XT = \alpha(T)X, \, T \in \mathcal{M} \} \). The inner product is given by the formula \( \langle X_1, X_2 \rangle = X_1^*X_2 \). It follows easily that \( \mathcal{E}_{\alpha^{-1}} = \mathcal{E}_\alpha \) and that \( \mathcal{E}_\alpha^* \mathcal{E}_\alpha \subseteq \mathcal{M}' \). In fact, the \( \sigma \)-weak closure of \( \mathcal{E}_\alpha^* \mathcal{E}_\alpha \) is a 2-sided ideal in \( \mathcal{M}' \), and therefore is of the form \( q\mathcal{M}' \), for some central projection in \( \mathcal{M}' \). However, given \( X \in \mathcal{E}_{\alpha^{-1}} \), with polar decomposition \( X = V|X| \), we see that \( V \in \mathcal{E}_{\alpha^{-1}} \) and \( VV^* \) is the projection onto the closure of the range of \( X \). Since \( VV^* \in \mathcal{E}_{\alpha^{-1}} \mathcal{E}_\alpha = \mathcal{E}_\alpha \mathcal{E}_\alpha \subseteq q\mathcal{M}' \), the range of \( X \) is contained in the range of \( q \). However, Lemma 2.10 implies, now, that \( q = I \); i.e., that \( \mathcal{E}_\alpha^* \mathcal{E}_\alpha \) is \( \sigma \)-weakly dense in \( \mathcal{M}' \). Hence, \( \mathcal{E}_\alpha \) is a normal equivalence bimodule.

(2) From Proposition 2.12, we know in general that \( m \) is an isomorphism if \( \mathcal{M} \otimes_Q I \otimes_P H = \mathcal{M} \otimes_Q \mathcal{M} \otimes_P H \). If \( P = \alpha \) is an endomorphism of \( \mathcal{M} \), then for every \( T, S \in \mathcal{M} \) and \( h, k \in H \), we have

\[
\langle T \otimes h - I \otimes \alpha(T)h, S \otimes k \rangle_{\mathcal{M} \otimes Q H} = (T \otimes h, S \otimes k) - \langle I \otimes \alpha(T)h, S \otimes k \rangle = (h, \alpha(T^*S)k) - \langle \alpha(T)h, \alpha(S)k \rangle = 0.
\]

Hence \( T \otimes h = I \otimes \alpha(T)h \) and so \( \mathcal{M} \otimes Q H = I \otimes H = H \). Thus, \( \mathcal{M} \otimes_Q \mathcal{M} \otimes Q H = \mathcal{M} \otimes Q I \otimes Q H \), as required.

(3) As in (2), we need to show that \( \mathcal{M} \otimes \alpha I \otimes Q H \) coincides with \( \mathcal{M} \otimes Q \mathcal{M} \otimes \alpha H \). This is obvious, since for all \( S, T \in \mathcal{M} \), and \( h \in H \), \( S \otimes T \otimes h = S\alpha^{-1}(T) \otimes I \otimes h \).

(4) From (2) and (3), we know that \( \mathcal{E}_P \simeq \mathcal{E}_\alpha \otimes \mathcal{E}_Q \otimes \mathcal{E}_\alpha^* \), and (1) implies then that \( \mathcal{E}_P \otimes \mathcal{E}_\alpha \simeq \mathcal{E}_\alpha \otimes \mathcal{E}_Q \). In fact, (1) also asserts that \( \mathcal{E}_Q \) is an \( \mathcal{M}' - \mathcal{M}' \) equivalence bimodule and so \( \mathcal{E}_P \) is strongly Morita equivalent in the sense of [15].

Although \( \mathcal{E}_P \otimes \mathcal{E}_Q \not\simeq \mathcal{E}_{PQ} \), and especially, \( \mathcal{E}_{P}^\otimes \not\simeq \mathcal{E}_{P}^{\otimes n} \) in general, we shall soon see that it is possible to “dilate” \( \mathcal{E}_P \) to a correspondence \( \mathcal{F}_\alpha \) where \( \alpha \) is an endomorphism of the commutant an isomorphic copy of \( \mathcal{M}' \). We then have \( \mathcal{F}_\alpha^{\otimes n} \simeq \mathcal{F}_\alpha^{\otimes n} \), by part (2) of Proposition 2.14. This \( \alpha \), then, will turn out to be a
“dilation” of $P$. To effect this program, we require some of the technology from [3]. We generally adopt the terminology and notation of [3], but with some minor modifications because we are working in the category of von Neumann algebras and normal maps - representations, and completely positive maps.

**Definition 2.15** Let $\mathcal{E}$ be a $W^*$-correspondence over a von Neumann algebra $\mathcal{N}$ and let $H_0$ be a Hilbert space.

1. A completely contractive covariant representation of $\mathcal{E}$ in $B(H_0)$ is a pair $(T, \sigma)$, where
   (a) $\sigma$ is a normal $*$-representation of $\mathcal{N}$ in $B(H_0)$.
   (b) $T$ is a linear, completely contractive map from $\mathcal{E}$ to $B(H_0)$ that is continuous in the $\sigma$-topology of $\mathcal{A}$ on $\mathcal{E}$ and the $\sigma$-weak topology on $B(H_0)$.
   (c) $T$ is a bimodule map in the sense that $T(S\xi R) = \sigma(S)T(\xi)\sigma(R)$, $\xi \in \mathcal{E}$, and $S, R \in \mathcal{N}$.

2. A completely contractive covariant representation $(T, \sigma)$ of $\mathcal{E}$ in $B(H_0)$ is called isometric in case
   \[ T(\xi)^* T(\eta) = \sigma(\langle \xi, \eta \rangle), \]  
   for all $\xi, \eta \in \mathcal{E}$.

To lighten the terminology, we shall refer to an isometric, completely contractive, covariant representation simply as an isometric covariant representation. There is no problem doing this because it is easy to see that if one has a pair $(T, \sigma)$ satisfying all the conditions of part 1 of Definition 2.15, except possibly the complete contractivity assumption, but which is isometric in the sense of equation (5), then necessarily $T$ is completely contractive.

The theory developed in [3] applies here to prove that if a completely contractive covariant representation, $(T, \sigma)$, of $\mathcal{E}$ in $B(H_0)$ is given, then it determines a contraction $\tilde{T} : \mathcal{E} \otimes_\sigma H \to H$ defined by the formula $\tilde{T}(\eta \otimes h) := T(\eta)h$, $\eta \otimes h \in \mathcal{E} \otimes_\sigma H$. Here, $\mathcal{E} \otimes_\sigma H$ denotes the Hausdorff completion of the algebraic tensor product $\mathcal{E} \otimes H$ in the pre-inner product given by the formula $\langle \xi \otimes h, \eta \otimes k \rangle := \langle h, \sigma(\langle \xi, \eta \rangle)k \rangle$. (See [3, Lemma 3.5].) Also, there is an induced representation $\sigma^\mathcal{E} : \mathcal{L}(\mathcal{E}) \to B(\mathcal{E} \otimes_\sigma H)$ defined by the formula $\sigma^\mathcal{E}(S) := S \otimes I$ [3, Lemma 3.4]. Recalling that $\mathcal{L}(\mathcal{E})$ is a von Neumann algebra, it is not hard to see that $\sigma^\mathcal{E}$ is a normal representation. The operator $\tilde{T}$ and $\sigma^\mathcal{E}$ are related by the equation

\[ \tilde{T} \sigma^\mathcal{E} \circ \varphi = \sigma \tilde{T}. \]  

In fact we have the following lemma that is immediate from [3] and [4]. See, in particular, [3, Lemmas 3.4-3.6] and [4, Lemma 2.1].
Lemma 2.16. The map \((T, \sigma) \rightarrow \tilde{T}\) is a bijection between all completely contractive covariant representations \((T, \sigma)\) of \(E\) on the Hilbert space \(H\) and contractive operators \(\tilde{T} : \mathcal{E} \otimes \sigma H \rightarrow H\) that satisfy equation (6). Given such a \(\tilde{T}\) satisfying this equation, \(T\), defined by the formula \(T(\xi)h := \tilde{T}(\xi \otimes h)\), together with \(\sigma\) is a completely contractive covariant representation of \(E\) on \(H\). Further, \((T, \sigma)\) is isometric if and only if \(\tilde{T}\) is an isometry.

We note in passing that this lemma shows that the \(\sigma\)-weak continuity of \(T\) really depends only on the fact that \(\sigma\) is normal.

The map \(\Psi : \mathcal{L}(E) \rightarrow \mathcal{B}(H)\) defined, then, by the formula
\[
\Psi(S) := \tilde{T}\sigma^E(S)\tilde{T}^*,
\]
\(S \in \mathcal{L}(E)\), evidently is completely positive, normal, and contractive.

Definition 2.17 Given a completely contractive covariant representation \((T, \sigma)\) of \(E\) in \(\mathcal{B}(H)\), the map \(\Psi\) is called the completely positive extension of \((T, \sigma)\), and the representation \((T, \sigma)\) is called fully coisometric in case \(\Psi(I_E) = I_H\).

The terminology is reminiscent of the theory of a single contraction. A completely contractive covariant representation \((T, \sigma)\) is isometric precisely when \(\tilde{T}\) is an isometry. Likewise, it is fully coisometric precisely when \(\tilde{T}\) is a coisometry. The map \(\Psi\) is a normal \(*\)-representation precisely when \((T, \sigma)\) is isometric and it is a unital \(*\)-representation precisely when \((T, \sigma)\) is both isometric and fully coisometric. (We have, however, resisted the temptation to call \((T, \sigma)\) unitary in this case.)

Our next result, which is a variant of [13, Corollary 5.21], shows that a completely contractive covariant representation \((T, \sigma)\) can be dilated to an isometric covariant representation in the following sense.

Theorem and Definition 2.18 Let \(E\) be a \(W^*\)-correspondence over a von Neumann algebra \(N\) and let \((T, \sigma)\) be a completely contractive covariant representation of \(E\) on the Hilbert space \(H\). Then there is a Hilbert space \(K\) containing \(H\) and an isometric covariant representation \((V, \rho)\) of \(E\) on \(K\) such that if \(P\) is the projection of \(K\) onto \(H\), then

1. \(P\) commutes with \(\rho(N)\) and \(\rho(A)P = \sigma(A)P, A \in N\); and
2. for all \(\eta \in E\), \(V(\eta)^*\) leaves \(H\) invariant and \(PV(\eta)P = T(\eta)P\).

The representation \((V, \rho)\) may be chosen so that the smallest subspace \(K\) containing \(H\) that reduces \((V, \rho)\) is \(K\). When this is done, \((V, \rho)\) is unique up to unitary equivalence and is called the minimal isometric dilation of \((T, \sigma)\).

Further, if \((T, \sigma)\) is fully coisometric, the (unique minimal) isometric dilation \((V, \rho)\) is fully coisometric, too.

Proof. One can construct a proof following the steps leading to Theorem 3.3 and Corollary 5.21 in [13]. However, continuity issues must be dealt with along the way and one needs to observe that the ideal \(J\) discussed there plays
no role here. Rather than doing this, it is easier and it may be more revealing to appeal to Lemma 2.16 and simply write down the operator \( \tilde{V} \) and representation \( \rho \) that lead to the dilation \( (V, \rho) \) of \( (T, \sigma) \). The remaining details will be very easy to verify.

To this end, let \( \Delta = (I - \tilde{T}^* \tilde{T})^{1/2} \) and let \( D \) be its range. Then \( \Delta \) is an operator on \( \mathcal{E} \otimes_{\sigma} H \) and commutes with the representation \( \sigma^E \circ \varphi \) of \( \mathcal{N} \), by equation (\( \mathcal{E} \)). Write \( \sigma_1 \) for the restriction of \( \sigma^E \circ \varphi \) to \( D \). Let \( \sigma_2 = \sigma_1^E \circ \varphi \) on \( \mathcal{E} \otimes_{\sigma_1} D \), and let \( \sigma_3 = \sigma_2^E \circ \varphi \) on \( \mathcal{E} \otimes_{\sigma_2} (\mathcal{E} \otimes_{\sigma_1} D) \). It is easy to see that \( \sigma_3 \) is naturally unitarily equivalent to \( \sigma_1^E \circ \varphi_2 \) on \( \mathcal{E} \otimes_{\sigma_1} D \), where \( \varphi_2 \) is the representation of \( \mathcal{N} \) in \( \mathcal{L}(\mathcal{E} \otimes_{\sigma_2}) \) defined by the formula \( \varphi_2(a)(\xi \otimes \eta) = (\varphi(a)\xi) \otimes \eta \).

We shall identify them henceforth and in general, we write \( \sigma_{n+1} \) for \( \sigma_1^E \circ \varphi_n \) on \( \mathcal{E} \otimes_{\sigma_1} D \), where \( \varphi_n \) has its obvious meaning. It is evident that all the \( \sigma_n \) are normal. We let

\[
K = H \oplus D \oplus \bigoplus_{n=1}^{\infty} \mathcal{E} \otimes_{\sigma_n} \otimes_{\sigma_1} D
\]

and we let \( \rho = \sigma \oplus \sigma_1 \oplus \bigoplus_{n=1}^{\infty} \sigma_{n+1} \), i.e., thinking matricially, \( \rho = \text{diag}(\sigma, \sigma_1, \sigma_2, \ldots) \).

Then a moment’s reflection reveals that \( \rho \) is a normal representation of \( \mathcal{N} \) on \( K \) whose restriction to \( H \) is \( \sigma \), of course. Form \( \mathcal{E} \otimes_{\rho} K \) and define \( \tilde{V} : \mathcal{E} \otimes_{\rho} K \to K \) matricially as

\[
\begin{bmatrix}
\tilde{T} & 0 & 0 & \cdots \\
\Delta & 0 & 0 & \cdots \\
0 & I & 0 & \cdots \\
0 & 0 & I & \cdots \\
\vdots & 0 & 0 & I \\
& & & \ddots
\end{bmatrix}
\]

Of course the identity operators in this matrix really must be interpreted as the operators that identify \( \mathcal{E} \otimes_{\sigma_{n+1}} (\mathcal{E} \otimes_{\sigma_1} D) \) with \( \mathcal{E} \otimes^{(n+1)}_{\sigma_1} D \).

It is easily checked that \( \tilde{V} \) is an isometry and that the associated covariant representation \( (V, \rho) \) is an isometric dilation \( (T, \sigma) \). Moreover, it is easily checked that \( (V, \rho) \) is minimal, i.e., that the smallest subspace of \( K \) containing \( H \) and reducing \( (V, \rho) \) is \( K \). Further, if \( (T, \sigma) \) is fully coisometric, so that \( \tilde{T} \) is a coisometry, then so is \( \tilde{V} \) a coisometry and \( (V, \rho) \) is fully coisometric.

The proof of the uniqueness of \( (V, \rho) \) is the same as in the \( C^* \)-setting and is given in [13, Proposition 3.2].

Finally, to see that \( V \) is fully coisometric if \( T \) is, observe that if \( T \) is fully coisometric, then \( \tilde{T} \) is a coisometry as we noted earlier. Thus \( \tilde{T} \Delta^2 = 0 \). This implies that \( \tilde{T} \Delta = 0 \). Therefore, from the form of \( \tilde{V} \), we see that \( \tilde{VV}^* = I \), which proves that \( \tilde{V} \) is fully coisometric.

We shall use Theorem 2.18 only for the module \( \mathcal{L}_M(H, \mathcal{M} \otimes_{\rho} H) \) associated to a completely positive map \( P \) on a von Neumann algebra \( \mathcal{M} \) and only for the
special covariant representation \((T, \sigma)\) which identifies \(\mathcal{L}_M(H, M \otimes_P H)\) with \(\mathcal{E}_P\). However, we shall employ a picture of the dilation \((V, \rho)\) that is different from the one constructed in Theorem 2.18. It will play a critical role in our analysis of semigroups of completely positive maps.

The definition of \((T, \sigma)\) is simple: \(T\) maps \(\mathcal{L}_M(H, M \otimes_P H)\) to \(B(H)\) via the formula:

\[
T(X) := W_P^* X, \quad X \in \mathcal{L}_M(H, M \otimes_P H),
\]

and \(\sigma\) is the identity representation,

\[
\sigma(S) = S, \quad S \in \mathcal{M}'.
\]

Of course, \(\sigma\) is \(\sigma\)-weakly continuous. Also, a straightforward calculation shows that \(T\) is a bimodule map. To see that \(T\) is completely contractive, we appeal to [13, Lemma 3.5] and show that the linear transformation \(\tilde{T} : \mathcal{L}_M(H, M \otimes_P H) \otimes_{\sigma} H \to H\) defined by the formula

\[
\tilde{T}(\sum X_j \otimes h_j) = \sum W_P^* X_j h_j
\]

is contractive. However, this is immediate:

\[
\left\| \sum W_P^* X_j h_j \right\|^2 \leq \left\| \sum X_j h_j \right\|^2 = \sum \langle h_k, X_k^* X_j h_j \rangle = \left\| \sum X_j \otimes h_j \right\|^2.
\]

As we remarked after Lemma 2.16, \(T\) is continuous with respect to the \(\sigma\)-topology on \(\mathcal{L}_M(H, M \otimes_P H)\) and the \(\sigma\)-weak topology on \(B(H)\), and so \((T, \sigma)\) is a completely contractive representation of \(\mathcal{L}_M(H, M \otimes_P H)\) on \(H\).

Evidently, \(T\) is really the inverse of the map \(X \to \Phi_X^*\) that we used to identify \(\mathcal{L}_M(H, M \otimes_P H)\) with \(\mathcal{E}_P\) in Proposition 2.6. Indeed, using the notation of that proposition, we see that for \(Y \in \mathcal{E}_P\), \(T(\Phi_Y^*) = W_P^* \Phi_Y^* = (\Phi_Y W_P)^* = Y\). Now all this may look trivial. It appears that after identifying \(\mathcal{L}_M(H, M \otimes_P H)\) with \(\mathcal{E}_P\) we are simply studying the identity covariant representation of \(\mathcal{E}_P\). However, we need to emphasize that the heart of the matter lies in the fact that the inner product on \(\mathcal{E}_P\) is not the one coming from operator multiplication in \(B(H)\) (unless \(P\) is an endomorphism - see Corollary 2.9). Rather, it is defined through the map \(X \to \Phi_X^*\) (or through its inverse \(T\)) which identifies \(\mathcal{E}_P\) with \(\mathcal{L}_M(H, M \otimes_P H)\).

**Definition 2.19** The completely contractive covariant representation \((T, \sigma)\) of \(\mathcal{L}_M(H, M \otimes_P H)\), where \(T\) is defined by (7) and where \(\sigma\) is the identity representation, will be called the identity covariant representation of \(\mathcal{L}_M(H, M \otimes_P H)\).

As we noted above, and as we shall use to good effect, \((T, \sigma)\) really identifies \(\mathcal{L}_M(H, M \otimes_P H)\) with \(\mathcal{E}_P\) and when this identification is made, the maps \(T\) and \(\sigma\) are both the identity maps.
To present the model for the minimal isometric dilation \((V, \rho)\) of \((T, \sigma)\) with which we will work, we define, for \(0 \leq k < \infty\), maps

\[
\iota_k : \underbrace{\mathcal{M} \otimes_p \mathcal{M} \otimes_p \cdots \otimes_p \mathcal{M} \otimes_p}^{k \text{ times}} H \to \underbrace{\mathcal{M} \otimes_p \mathcal{M} \otimes_p \cdots \otimes_p \mathcal{M} \otimes_p}^{k+1 \text{ times}} H
\]

by the formula \(\iota_k(T_1 \otimes T_2 \otimes \cdots \otimes T_k \otimes h) = I \otimes T_1 \otimes T_2 \otimes \cdots \otimes T_k \otimes h\). Of course, \(\iota_0 = W_P\). Since \(P\) is unital, this map is a well defined isometry of \(H_k := \mathcal{M} \otimes_p \mathcal{M} \otimes_p \cdots \otimes_p \mathcal{M} \otimes_p H\) into \(H_{k+1} := \mathcal{M} \otimes_p \mathcal{M} \otimes_p \cdots \otimes_p \mathcal{M} \otimes_p H\). We write \(H_\infty\) for the Hilbert space inductive limit, \(\lim \limits_{\rightarrow} (H_k, \iota_k)\), and we write \(W_k\) for the canonical (isometric) embeddings of \(H_k\) into \(H_\infty\). Given \(X \in \mathcal{L}_M(H, \mathcal{M} \otimes_p H)\), we define \(X_k : H_k \to H_{k+1}\) by the formula \(X_k(T_1 \otimes T_2 \otimes \cdots \otimes T_k \otimes h) = T_1 \otimes T_2 \otimes \cdots \otimes T_k \otimes Xh\). A straightforward calculation using the fact that \(X\) intertwines the actions of \(\mathcal{M}\) on \(H\) and on \(\mathcal{M} \otimes_p H\) shows that \(X_k\) is bounded with \(\|X_k\| \leq \|X\|\). Further, the diagram

\[
\begin{array}{cccccccc}
H & \xrightarrow{\iota_0} & H_1 & \xrightarrow{\iota_1} & \cdots & \xrightarrow{\iota_{k-1}} & H_k & \xrightarrow{\iota_k} & H_{k+1} & \to & \cdots & \to & H_\infty \\
\downarrow & & \downarrow & & \ldots & & \downarrow & & \downarrow & & \ldots & & \downarrow \\
S & \xrightarrow{\iota_0} & I \otimes S & \xrightarrow{\iota_1} & \cdots & \xrightarrow{\iota_{k-1}} & I \otimes \cdots \otimes I \otimes S & \xrightarrow{\iota_k} & \cdots \\
\end{array}
\]

commutes and so defines an operator \(X_\infty \in \mathcal{B}(H_\infty)\). We shall see in a moment that the map \(X : H \to X_\infty\), which we shall call \(V\), is part of an isometric covariant representation of \(\mathcal{L}_M(H, \mathcal{M} \otimes_p H)\), \((V, \rho)\).

To this end, we must first define \(\rho\) through the following diagram, where \(S \in \mathcal{M}'\). The diagram

\[
\begin{array}{cccccccc}
H & \xrightarrow{\iota_0} & H_1 & \xrightarrow{\iota_1} & \cdots & \xrightarrow{\iota_{k-1}} & H_k & \xrightarrow{\iota_k} & \cdots \\
\downarrow & & \downarrow & & \ldots & & \downarrow & & \ldots \\
S & \xrightarrow{\iota_0} & I \otimes S & \xrightarrow{\iota_1} & \cdots & \xrightarrow{\iota_{k-1}} & H_k & \xrightarrow{\iota_k} & \cdots \\
\end{array}
\]

commutes and, therefore, defines an operator \(\rho(S)\) on \(H_\infty\). Note that

\[
W_0^* \rho(S) W_k = I \otimes \cdots \otimes I \otimes S,
\]

where, recall, \(W_k\) is the canonical embedding of \(H_k\) in \(H_\infty\). From this it is obvious that \(\rho\) is a normal representation of \(\mathcal{M}'\) on \(H_\infty\) that is reduced by each of the spaces \(W_k H_k\). In particular, note that \(W_0^* \rho(\cdot) W_0 = \sigma\).

If the diagrams that define \(V\) and \(\rho\), \((8)\) and \((9)\), resp., are combined in the obvious way, it becomes clear that for \(X \in \mathcal{L}_M(H, \mathcal{M} \otimes_p H)\) and \(S \in \mathcal{M}'\),

\[
V(XS) = (XS)_\infty = X_\infty \rho(S) = V(X) \rho(S)
\]
Therefore, we must, strictly speaking, identify $X_k : H_k \rightarrow H_{k+1}$ is defined by the formula $X_k(T_1 \otimes T_2 \otimes \cdots \otimes T_k \otimes h) = T_1 \otimes T_2 \otimes \cdots \otimes T_k \otimes Xh$ and similarly for $Y_k$. Consequently, we find that

$$X^*_k(T_1 \otimes T_2 \otimes \cdots \otimes T_{k+1} \otimes h) = T_1 \otimes T_2 \otimes \cdots \otimes X^*(T_{k+1} \otimes h)$$

because

$$\langle T_1 \otimes T_2 \otimes \cdots \otimes X^*(T_{k+1} \otimes h), S_1 \otimes S_2 \otimes \cdots \otimes S_k \otimes k \rangle = \langle X^*(T_{k+1} \otimes h), P(T^*_k P(\cdots) S_k)k \rangle = \langle T_{k+1} \otimes h, XP(T^*_k P(\cdots) S_k)k \rangle = \langle T_{k+1} \otimes h, (P(T^*_k P(\cdots) S_k) \otimes I) Xk \rangle \quad \text{(because } X \in \mathcal{L}_M(H, \mathcal{M} \otimes P H))$$

$$= \langle T_1 \otimes T_2 \otimes \cdots \otimes T_{k+1} \otimes h, S_1 \otimes S_2 \otimes \cdots \otimes S_k \otimes Xk \rangle = \langle T_1 \otimes T_2 \otimes \cdots \otimes T_{k+1} \otimes h, X_k(S_1 \otimes S_2 \otimes \cdots \otimes S_k \otimes k) \rangle.$$

Therefore,

$$X^*_k Y_k (T_1 \otimes T_2 \otimes \cdots \otimes T_k \otimes h) = X^*_k (T_1 \otimes T_2 \otimes \cdots \otimes T_k \otimes Yh)$$

$$= T_1 \otimes T_2 \otimes \cdots \otimes T_k \otimes X^* Yh$$

$$= W^*_k \rho(X^* Y) W_k (T_1 \otimes T_2 \otimes \cdots \otimes T_{k+1} \otimes h).$$

Thus $W^*_k \rho(X^* Y) W_k = X^*_k Y_k = W^*_k V(X)^* W_{k+1} W^*_k W_{k+1} V(Y) W_k$ for all $k$, from which it follows that $V(X)^* V(Y) = \rho((X, Y))$, i.e., that $(V, \rho)$ is isometric.

We now show that $(V, \rho)$ dilates $(T, \sigma)$ in the sense described in Theorem 2.18. Of course to do this, we must, strictly speaking, identify $H$ with the subspace $W_0 H$ of $H_\infty$. When this is done, the projection $P$ of $H_\infty$ on $H$ is $W_0 W^*_0$. We already have seen that $H = W_0 H$ reduces $\rho$ and that $\rho|H = \sigma$ as is required in part 1. of Theorem 2.18. Also note that for $X \in \mathcal{L}_M(H, \mathcal{M} \otimes P H)$,

$$W^*_0 X_{\infty} W_0 = \iota^*_0 X = T(X),$$

which is an evident consequence of the properties of inductive limits: For $h \in H$, $X_{\infty} W_0 h = W_1 Xh$, and $W^*_0 W_1 = \iota^*_0$. This of course means that $T(X) = W^*_0 V(X) W_0$, so that after identifying $H$ with $W_0 H$, through $W_0$, we see that $T(X) = PV(X)|H$, $X \in \mathcal{L}_M(H, \mathcal{M} \otimes P H)$, as required in part 2. of Theorem 2.18. But we also need to check that $V(X)^*$ maps $H$ into itself. Equivalently, we need to show that $V(X)$ maps $H_\infty \otimes H$ into itself.

For this purpose, it suffices to show that for each $k \geq 1$, $V(X)$ maps $W_k H_k \otimes H$ into $H_\infty \otimes H$. To show how this is done, but to keep the matters simple, we
show that \( V(X) \) maps \( W_2H_2 \odot H \) into \( H_\infty \odot H \). So let \( W_2 \sum T_i \otimes S_i \otimes h_i \) be an element of \( W_2H_2 \). To say this is orthogonal to \( H \) means that for all \( h \in H \),
\[
0 = \langle W_2 \sum T_i \otimes S_i \otimes h_i, W_0h \rangle \\
= \langle \sum T_i \otimes S_i \otimes h_i, I \otimes I \otimes h \rangle \\
= \langle \sum P(P(T_i)S_i)h_i, h \rangle;
\]
i.e., \( W_2 \sum T_i \otimes S_i \otimes h_i \in W_2H_2 \odot H \) if and only if \( \sum P(P(T_i)S_i)h_i = 0 \). Now assume that \( W_2 \sum T_i \otimes S_i \otimes h_i \in W_2H_2 \odot H \), let \( h \) be an element in \( H \), and compute:
\[
\langle V(X)W_2 \sum T_i \otimes S_i \otimes h_i, W_0h \rangle \\
= \langle \sum T_i \otimes S_i \otimes Xh_i, I \otimes I \otimes h \rangle_{H_3} \\
= \langle \sum Xh_i, P(S^*_i P(T^*_i)) \otimes h \rangle_{H_3} \\
= \langle \sum ((P(P(T_i)S_i) \otimes I)Xh_i, I \otimes h \rangle_{H_3} \\
= \langle \sum X(P(P(T_i)S_i))h_i, I \otimes h \rangle_{H_3} \\
= \langle X(\sum P(P(T_i)S_i))h_i, I \otimes h \rangle_{H_3} \\
= 0,
\]
since \( \sum P(P(T_i)S_i))h_i = 0 \). Thus \( (V, \rho) \) satisfies condition 2. in Theorem 2.18.

To show that this \( (V, \rho) \) is unitarily equivalent to the dilation of \( (T, \sigma) \) that is provided by Theorem 2.18, we appeal to Proposition 3.2 of [13] (which we stated as part of Theorem 2.18) and show that \( (V, \rho) \) is minimal; i.e., that there are no closed subspaces \( K \) properly contained between \( H \) and \( H_\infty \) that are invariant under the images of \( V \) and \( \rho \). So, suppose \( K \) is such a subspace, then for every \( X \in \mathcal{L}_M(H, M \otimes_P H) \), \( X_\infty(H) = V(X)(H) \) is contained in \( K \). Hence, in particular, \( X_\infty(H) = X_\infty(W_0H) = X(H) \subseteq K \). However, the span of \( \{ X(H) \mid X \in \mathcal{L}_M(H, M \otimes_P H) \} \) is \( M \otimes_P H \), by Lemma 2.10, and so we conclude that \( W_1H_1 = W_1(M \otimes_P H) \subseteq K \). This, in turn, implies that \( X_\infty(M \otimes_P H) \subseteq K \); i.e., that \( \langle I \otimes X \rangle(M \otimes_P H) \subseteq K \). Applying Lemma 2.10 again, we see that \( W_2H_2 = W_2(M \otimes_P M \otimes_P H) \) is contained in \( K \). Continuing in this manner, we find that \( W_kH_k \) is contained in \( K \) for every \( k \). Hence \( K = H_\infty \).

Since our special \( (V, \rho) \) is unitarily equivalent to the one provided by Theorem 2.18, we may infer that \( V \) is continuous with respect to the \( \sigma \)-topology on \( \mathcal{L}_M(H, M \otimes_P H) \) and the \( \sigma \)-weak topology on \( B(H) \).

We summarize our discussion of the identity representation of \( \mathcal{L}_M(H, M \otimes_P H) \) in the following theorem.

**Theorem 2.20** The maps \( V \) and \( \rho \) defined by the diagrams (5) and (6) together form an isometric covariant representation of \( \mathcal{L}_M(H, M \otimes_P H) \) that dilates the identity representation \( (T, \sigma) \) of \( \mathcal{L}_M(H, M \otimes_P H) \). Moreover, \( (T, \sigma) \) and \( (V, \rho) \) are fully coisometric.
Proof. The only thing that remains to be proved is the last statement about \((T, \sigma)\) and \((V, \rho)\) being fully coisometric. However, for this purpose, it suffices to show that \((T, \sigma)\) is fully coisometric, by Theorem 2.18. Recall that \(T\) maps \(\mathcal{L}_M(H, M \otimes p H) \otimes_\sigma H\) to \(H\) by the formula \(\hat{T}(X \otimes h) = W^*_p X h\).

To calculate \(\hat{T}^*\), simply observe that for \(X \in \mathcal{L}_M(H, M \otimes p H)\) and \(h \in H\), \((\hat{T}^* k, X \otimes h) = \langle k, \hat{T}(X \otimes h) \rangle = \langle k, W^*_p X h \rangle = \langle W_p k, X h \rangle\). However, by Lemma 2.10, \(\{X h \mid X \in \mathcal{L}_M(H, M \otimes p H), h \in H\}\) spans \(M \otimes p H\). So, if we let \(u : \mathcal{L}_M(H, M \otimes p H) \otimes H \to M \otimes p H\) be defined by the formula \(u(X \otimes h) = X h\), the \(u\) is a Hilbert space isomorphism such that \(\langle \hat{T}^* k, X \otimes h \rangle = \langle W_p k, X h \rangle = \langle u W_p k, X \otimes h \rangle\) for all \(k\) and all \(X \otimes h\). Thus \(\hat{T}^*\) is the isometry \(u^* W_p\), proving that \(\hat{T}\) is a coisometry and, therefore, that \((T, \sigma)\) is fully coisometric. \(\blacksquare\)

If \(\mathcal{N}\) is a von Neumann algebra and if \(\mathcal{E}\) is a \(W^*\)-correspondence over \(\mathcal{N}\), then we have seen how a completely contractive covariant representation \((T, \sigma)\) of \(\mathcal{E}\) on a Hilbert space \(H\) gives rise to a completely positive map \(\Psi = \Psi_T\) of \(\mathcal{E}(\mathcal{E})\) on \(H\). (See Definition 2.17.) However, equally important for our purposes is the related completely positive map \(\Theta = \Theta_T\) on the commutant of \(\sigma(\mathcal{N})\), \(\sigma(\mathcal{N})'\), that is described in the next proposition.

**Proposition 2.21** Let \(\mathcal{N}\) be a von Neumann algebra, let \(\mathcal{E}\) be a \(W^*\)-correspondence over \(\mathcal{N}\), and let \((T, \sigma)\) be a completely contractive covariant representation of \(\mathcal{E}\) on a Hilbert space \(H\). For \(S \in \sigma(\mathcal{N})'\), set

\[
\Theta(S) = \Theta_T(S) := \hat{T}(1_\mathcal{E} \otimes S)\hat{T}^*.
\]

Then \(\Theta\) is normal completely positive map from \(\sigma(\mathcal{N})'\) into itself that is unital if and only if \((T, \sigma)\) is fully coisometric. Further, if \((T, \sigma)\) is isometric, then \(\Theta\) is multiplicative, i.e., \(\Theta\) is an endomorphism of \(\sigma(\mathcal{N})'\), and, conversely, if \(\Theta\) is multiplicative, then the correspondence \(\mathcal{E}\) decomposes as the direct sum of two subcorrespondences, \(\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2\), so that \((T|\mathcal{E}_1, \sigma)\) is isometric and \((T|\mathcal{E}_2) = 0\).

**Proof.** Much of the proof may be dug out of [14]. See Lemma 2.3 there, in particular. Here are the particulars. First, recall the induced representation \(\sigma^\mathcal{E} : \mathcal{L}(\mathcal{E}) \to B(\mathcal{E} \otimes_\sigma H)\), \(\sigma^\mathcal{E}(X) = X \otimes I_H\). As Rieffel shows in Theorem 6.23 of [20], the commutant of \(\sigma^\mathcal{E}(\mathcal{L}(\mathcal{E}))\) is \(\mathbb{C}1_\mathcal{E} \otimes \sigma(\mathcal{N})'\), and of course the map \(S \to 1_\mathcal{E} \otimes S\) is a normal representation of \(\sigma(\mathcal{N})'\) onto \(\mathbb{C}1_\mathcal{E} \otimes \sigma(\mathcal{N})'\). Thus \(\Theta\) is a normal completely positive map from \(\sigma(\mathcal{N})'\) into \(B(H)\). The problem is to locate its range. This, however, is easy on the basis of equation (9). Given \(R \in \mathcal{N}\) and \(S \in \sigma(\mathcal{N})'\), that equation implies that

\[
\sigma(R)\Theta(S) = \sigma(R)\hat{T}(1_\mathcal{E} \otimes S)\hat{T}^* = \hat{T}\sigma^\mathcal{E} \circ \varphi(R)(1_\mathcal{E} \otimes S)\hat{T}^*
\]

\[
= \hat{T}(1_\mathcal{E} \otimes S)\sigma^\mathcal{E} \circ \varphi(R)\hat{T}^* = \hat{T}(1_\mathcal{E} \otimes S)\hat{T}^*\sigma(R)
\]

\[
= \Theta(S)\sigma(R),
\]

so \(\Theta(S) \in \sigma(\mathcal{N})'\).

Of course \(\Theta\) is unital if and only if \((T, \sigma)\) is fully coisometric.
As for the last assertion, the direct statement is proved as Lemma 2.3 of [4]. For the converse, suppose that \( \Theta \) is multiplicative. Then \( \tilde{T}T^* = \Theta(I) \) is a projection. Therefore, \( T^*\tilde{T} \) is a projection on \( \mathcal{E} \otimes_{\sigma} H \), call it \( q \). Since \( \Theta \) is multiplicative, we infer that \( q(1_{\mathcal{E}} \otimes S_1)q(1_{\mathcal{E}} \otimes S_2)q = q(1_{\mathcal{E}} \otimes S_1S_2)q \) for all \( S_1, S_2 \in \sigma(\mathcal{N})' \). This implies that \( q \in (\mathbb{C}1_{\mathcal{E}} \otimes \sigma(\mathcal{N})')' = \sigma^\mathcal{E}(\mathcal{L}(\mathcal{E})) \), by Rieffel’s theorem [20, Theorem 6.23] and the fact that \( \sigma^\mathcal{E} \) is a normal representation of the von Neumann algebra \( \mathcal{L}(\mathcal{E}) \). Thus, \( q = \sigma^\mathcal{E}(Q) \) for a projection \( Q \in \mathcal{L}(\mathcal{E}) \). If \( E_1 := QE \) and \( E_2 := (1_{\mathcal{E}} - Q)E \), then it is easy to see that \( (T|E_1, \sigma) \) is isometric, while \( T|E_2 = 0 \). We omit the details.

**Definition 2.22** Let \( E \) be a \( W^* \)-correspondence over a von Neumann algebra \( \mathcal{N} \) and let \((T, \sigma)\) be a completely contractive covariant representation of \( E \) on the Hilbert space \( H \), the normal, completely positive map \( \Theta_T : \sigma(\mathcal{N})' \rightarrow \sigma(\mathcal{N})' \) defined by equation (14) will be called the induced (completely positive) map on \( \sigma(\mathcal{N})' \). If \( T \) is isometric, then \( \Theta_T \) will be called the induced endomorphism of \( \sigma(\mathcal{N})' \).

If we apply Proposition 2.21 to the identity representation \((T, \sigma)\) of \( \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes P \mathcal{H}) \) or of \( \mathcal{E}_P \), for a completely positive map \( P \) on a von Neumann algebra \( \mathcal{M} \), we recapture \( P \). Specifically, we have

**Corollary 2.23** Let \( P \) be a normal, unital, completely positive map on the von Neumann algebra \( \mathcal{M} \), and let \((T, \sigma)\) be the identity representation of the Arveson correspondence \( \mathcal{E}_P \simeq \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes P \mathcal{H}) \) on \( H \). Then \( \Theta_T = P \).

**Proof.** We apply Proposition 2.21 with the von Neumann algebra \( \mathcal{N} \) identified with \( \mathcal{M}' \). So, for \( S \in \mathcal{M} \) and \( h \in H \), we have from the computations in the proof of Proposition 2.21 (the fact that \( \tilde{T}^* = u^*W_P \), so \( \tilde{T} = W_Pu \))

\[
\Theta(S)h = \tilde{T}(1_{\mathcal{E}} \otimes S)\tilde{T}^*h = W_P^*u(I_{\mathcal{E}} \otimes S)u^*W_Ph = P(S)h.
\]

We conclude this section with our principal dilation result for single completely positive maps. It is the key to our analysis of semigroups.

**Theorem 2.24** Let \( \mathcal{M} \) be a von Neumann algebra acting on a Hilbert space \( H \) and let \( P : \mathcal{M} \rightarrow \mathcal{M} \) be a normal, unital, completely positive map of \( \mathcal{M} \). Let \((T, \sigma)\) be the identity representation on \( H \) of the Arveson correspondence \( \mathcal{E}_P \), let \((V, \rho)\) be the minimal isometric dilation of \((T, \sigma)\) on the Hilbert space \( K \), and let \( W : H \rightarrow K \) be the associated embedding. If \( \mathcal{R} := \rho(\mathcal{M}')' \), then

1. \( W^*RW = \mathcal{M} \), so that \( \mathcal{M} \) is a corner of \( \mathcal{R} \) (and \( \mathcal{R}' \) is a normal homomorphic image of \( \mathcal{M}' \)),

2. \( W^*RW = \mathcal{M} \), so that \( \mathcal{M} \) is a corner of \( \mathcal{R} \) (and \( \mathcal{R}' \) is a normal homomorphic image of \( \mathcal{M}' \)).
2. $\Theta_V$ is a unital, normal $*$-endomorphism of $\mathcal{R}$, and

3. for every non-negative integer $n$,

$$P^n(T) = W^*\Theta_V^n(WTW^*)W$$

and

$$P^n(W^*SW) = W^*\Theta_V^n(S)W$$

for all $S \in \mathcal{R}$, and $T \in \mathcal{M}$.

Thus, the induced endomorphism $\Theta_V$ of $\mathcal{R}$ is a power dilation of $P$.

**Proof.** From Corollary 2.23, we know that $P$ is the induced completely positive map $\Theta_T$ on $\mathcal{M}$. Also, since $(V, \rho)$ is the minimal isometric dilation of $(T, \sigma)$ and $W$ is the embedding map, we know that $WH$ is invariant under $V(Y)^*$ for all $Y \in \mathcal{E}$ and $W^*V(Y)W = T(Y)$. Since $W^*\rho(S)W = \sigma(S)$ for all $S \in \mathcal{M}'$ by definition of $(V, \rho)$, and since $\sigma(S) = S$, $S \in \mathcal{M}'$, by definition of the identity representation, we see that

$$W^*RW = W^*\rho(M')'W = (W^*\rho(M'))' = (M')' = M.$$  \hspace{1cm} (11)

By Theorem 2.20, $(T, \sigma)$ and $(V, \rho)$ are fully coisometric, and so, by Proposition 2.21, $\Theta_V$ is a normal, unital, $*$-endomorphism of $\mathcal{R} = \rho(M')'$. Since $WH$ is invariant under $V(Y)^*$, $Y \in \mathcal{E}$, we see that for $Y \in \mathcal{E}$ and $k \in K$,

$$WW^*\hat{V}(Y \otimes (I - WW^*))k = WW^*V(Y)(I - WW^*)k = 0$$

so that $WW^*\hat{V}(I \otimes (I - WW^*)) = 0$. Therefore, $WW^*\Theta_V((I - WW^*)) = WW^*\hat{V}(I \otimes (I - WW^*))\hat{V}^* = 0$; i.e. $WW^*\Theta_V(WW^*) = WW^*$. Multiplying this equation on the left by $W^*$, we see that

$$W^*\Theta_V(WW^*) = W^*.$$  \hspace{1cm} (12)

Since $T(\cdot) = W^*V(\cdot)W$, it follows that $\hat{T} = W^*\hat{V}(I \otimes W)$. Consequently, for $L \in \mathcal{M}$,

$$P(L) = \hat{T}(I \otimes L)\hat{T}^* = W^*\hat{V}(I \otimes W)(I \otimes L)(I \otimes W^*)\hat{V}^*W = W^*\Theta_V(WLW^*)W.$$  \hspace{1cm} (13)

On the other hand, for $S \in \mathcal{R}$, we find from this equation and the fact that $W^*SW \in \mathcal{M}$ (by (11)) that

$$P(W^*SW) = W^*\Theta_V(WW^*SWW^*)W$$

$$= W^*\Theta_V(WW^*SWW^*)W = W^*\Theta_V(WW^*)\Theta_V(S)\Theta_V(WW^*)W$$

$$= W^*\Theta_V(S)W,$$

using equation (12).
To relate $P^2$ to $\Theta^2_V$, let $L \in \mathcal{M}$. Then, using equation (12) again, we find that

$$\begin{align*}
P^2(L) &= P(P(L)) = P(W^*\Theta_V(WLW^*)W) \\
&= W^*\Theta_V(WW^*\Theta_V(WLW^*)WW^*)W \\
&= W^*\Theta_V(WW^*)\Theta^2_V(WLW^*)\Theta_V(WW^*)W \\
&= W^*\Theta^2_V(WLW^*)W.
\end{align*}$$

Continuing in this manner, we find that $P^n(L) = W^*\Theta^n_V(WLW^*)W$ for all $L \in \mathcal{M}$.

To show that $P^n(W^*SW) = W^*\Theta^n_V(S)W$ for all $S \in \mathcal{R}$, and all $n$, we need to generalize equation (12) to $W^*\Theta^n_V(WW^*) = W^*$, for all $n$. However, this is an easy induction, the general step of which is:

$$\begin{align*}
WW^*\Theta^{n+1}_V(WW^*) &= WW^*\Theta_V(WW^*)\Theta^n_V(WW^*) \\
&= WW^*\Theta_V((I - WW^*)\Theta^n_V(WW^*)) \\
&= WW^*\Theta^n_V(WW^*) + WW^*\Theta_V(I - WW^*)\Theta^n_V(WW^*) \\
&= WW^*.
\end{align*}$$

Thus $WW^*\Theta^n_V(WW^*) = WW^*$ for all $n$. Multiplying through on the left by $W^*$ gives the desired formula.

Using this, we see that since $W^*SW \in \mathcal{M}$ for all $S \in \mathcal{R}$, our earlier calculation gives

$$\begin{align*}
P^n(W^*SW) &= W^*\Theta^n_V(WW^*SWW^*)W \\
&= W^*\Theta^n_V(WW^*)\Theta^n_V(S)\Theta^n_V(WW^*)W \\
&= W^*\Theta^n_V(S)W.
\end{align*}$$

3 Semigroups of Completely Positive Maps

In this section, we focus on semigroups $\{P_t\}_{t \geq 0}$ of unital, normal, completely positive maps on our basic von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $H$. That is, we assume that $P_{t+s} = P_tP_s$, $s, t \geq 0$, and $P_0$ is the identity map on $\mathcal{M}$. We call $\{P_t\}_{t \geq 0}$ a completely positive semigroup on $\mathcal{M}$, or simply a cp semigroup, for short. We make no continuity assumptions on $\{P_t\}_{t \geq 0}$ in this section and, in fact, everything we say is true if the additive semigroup of non-negative real numbers is replaced by any totally ordered semigroup. Our goal is to dilate $\{P_t\}_{t \geq 0}$ to a semigroup of endomorphisms in much the same fashion that we did for a single completely positive map in Section 2. However, there is a complication that must be addressed.

Let $\mathcal{E}_t$ be the Arveson correspondence over $\mathcal{M}'$ associated with $P_t$, $t \geq 0$. As in Section 2, we shall view $\mathcal{E}_t$ as either a space of operators on $H$ or as the
space $L_\mathcal{M}(H, \mathcal{M} \otimes P_\tau, H)$. As we noted, the spaces $E_i$ need not “multiply”, i.e., $E_i \otimes E_s$ need not be isomorphic to $E_{i+s}$. So, we will have to “dilate” these to a family $\{E(t)\}_{t \geq 0}$ of $\mathcal{M}'$ correspondences such that $E(t) \otimes E(s) \simeq E(t+s)$. That is, we need to dilate these to a (discrete) product system over $\mathcal{M}'$ - a notion that is inspired by Arveson’s product systems in [2]. This we do following in outline arguments of Bhat in [3]. There are similarities also between our arguments and arguments in [3], but our correspondences are over $\mathcal{M}'$ as opposed to being over $\mathcal{M}$ and we cannot tap directly into their arguments. Once $\{E(t)\}_{t \geq 0}$ is constructed, we promote the identity representations of the $E_i$’s to completely contractive representations of the $E(t)$’s and then dilate these to isometric representations of the $E(t)$’s. These last representations will implement a semigroup of endomorphisms of a bigger von Neumann algebra in which $\mathcal{M}$ sits as a corner. The semigroup of endomorphisms will be the desired dilation of $\{P_t\}_{t \geq 0}$.

Let $\mathcal{P}(t)$ denote the collection of partitions of the closed interval $[0,t]$ and order these by refinement. For a $p \in \mathcal{P}(t)$, we shall write $p = \{0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t\}$. For such a $p$, we shall write

$$H_{p,t} := M \otimes P_{t_1} \otimes P_{t_2-t_1} \otimes \cdots \otimes P_{t_{n-1}-t_{n-2}} H.$$ 

Then it is easy to see that $H_{p,t}$ is a left $\mathcal{M}$-module via the formula $S \cdot (T_1 \otimes T_2 \otimes \cdots T_n \otimes h) : = (ST_1) \otimes T_2 \otimes \cdots T_n \otimes h, S \in \mathcal{M}, (T_1 \otimes T_2 \otimes \cdots T_n \otimes h) \in H_{p,t}$. Also, it is easy to see that $L_\mathcal{M}(H, H_{p,t})$ becomes an $\mathcal{M}'$-correspondence via the actions

$$(XR)h := X(Rh)$$

and

$$(RX)h := (I \otimes R)Xh,$$

$R \in \mathcal{M}', X \in L_\mathcal{M}(H, H_{p,t})$ and $h \in H$, where $I$ is the identity operator on $H_{p,t}$. The inner product is given by the formula $\langle X_1, X_2 \rangle := X_1^* X_2$. Note that the map $R \mapsto (X_1, RX_2) = X_1^* (I \otimes R)X_2$ is $\sigma$-weakly continuous, so that $L_\mathcal{M}(H, H_{p,t})$ is, indeed, an $\mathcal{M}'$-correspondence.

We shall write $L_t$ for the $\mathcal{M}'$-correspondence $L_\mathcal{M}(H, \mathcal{M} \otimes P_\tau, H)$. Then Proposition 2.12 shows that $L_\mathcal{M}(H, H_{p,t})$ is isomorphic to $L_{t-t_{n-1}} \otimes \cdots \otimes L_{t_1}$ as $\mathcal{M}'$-correspondences.

We next want to show that the Hilbert spaces $H_{p,t}$ and $\mathcal{M}'$-correspondences $L_\mathcal{M}(H, H_{p,t})$ form inductive systems so that we can take their direct limits. For this purpose, consider first the case when $p' : = \{0 = t_0 < t_1 < t_2 < \cdots < t_k < \tau < t_{k+1} < \cdots t_{n-1} < t_n = t\}$, a one point refinement of $p$. Then we obtain a Hilbert space isometry $v_0 : H_{p,t} \rightarrow H_{p',t}$ defined by the formula $v_0(T_1 \otimes T_2 \otimes \cdots T_n \otimes h) = T_1 \otimes \cdots \otimes T_k \otimes I \otimes T_{k+1} \otimes \cdots T_n \otimes h$ and an $\mathcal{M}'$-correspondence isometry $v : L_\mathcal{M}(H, H_{p,t}) \rightarrow L_\mathcal{M}(H, H_{p',t})$ defined by the formula $v(X) := v_0 \circ X$. The proof of these facts is a minor modification of the proof of Proposition 2.12 and so will be omitted.
Since every refinement of a partition can be obtained by a sequence of one-point refinements, it is clear that for every pair of partitions $(p, p')$, with $p'$ refining $p$, we have Hilbert space isometries $v_{0, p, p'} : H_{p, t} \rightarrow H_{p', t}$ and $M'$-correspondence isometries $v_{p, p'} : L_M(H, H_{p, t}) \rightarrow L_M(H, H_{p', t})$ so that $v_{0, p, p'}' \circ v_{0, p, p'} = v_{0, p, p'}'$ and $v_{p', p'}' \circ v_{p, p'} = v_{p', p'}$, when $p''$ refines $p'$ and $p'$ refines $p$. The Hilbert space isometry $v_{0, p, p'}$ simply sends a decomposable tensor $T_1 \otimes T_2 \otimes \cdots \otimes T_n \otimes h \in H_{p, t}$ to the decomposable tensor in $H_{p', t}$ obtained from $T_1 \otimes T_2 \otimes \cdots \otimes T_n \otimes h$ by inserting identity operators in those positions where new indices have been added to $p$ to obtain $p'$. The $M'$-correspondence isometry $v_{p, p'}$ is defined by the formula $v_{p, p'}(X) := v_{0, p, p'} \circ X$, $X \in L_M(H, H_{p, t})$.

We may thus form the direct limits

$$H_t := \lim_{\rightarrow} (H_{p, t}, v_{0, p, p'})$$

and

$$E(t) := \lim_{\rightarrow} (L_M(H, H_{p, t}), v_{p, p'})$$

Note that $H_t$ is a left $M$ module since each $H_{p, t}$ is and the maps $v_{0, p, p'}$ respect the action of $M$. It is also a left $M'$-module, since $M'$ acts on each $H_{p, t}$ via the formula $R(T_1 \otimes T_2 \otimes \cdots \otimes T_n \otimes h) = (I \otimes R)(T_1 \otimes T_2 \otimes \cdots \otimes T_n \otimes h) = T_1 \otimes T_2 \otimes \cdots \otimes T_n \otimes Rh$, $T_1 \otimes T_2 \otimes \cdots \otimes T_n \otimes h \in H_{p, t}$, $R \in M'$ and the maps $v_{0, p, p'}$ respect this action. It is now easy to see that $L_M(H, H_t)$ has the structure of an $M'$-correspondence. Indeed, the bimodule structure has just been indicated. One passes to the limit when writing $H_t = \lim_{\rightarrow} (H_{p, t}, v_{0, p, p'})$ in $L_M(H, H_t)$. Since each $L_M(H, H_{p, t})$ is an $M'$-correspondence in an obvious way, so is $L_M(H, H_t)$ via the limit of the inner products on the $L_M(H, H_{p, t})$.

**Lemma 3.1** Each $E(t)$ is isomorphic, as an $M'$-correspondence, to $L_M(H, H_t)$.

**Proof.** For each $p \in \mathcal{P}(t)$, we write $v_{0, p, \infty}$ for the canonical isometric embedding of $H_{p, t}$ in $H_t$. Since the $v_{0, p, p'}$ are $M$-module maps, so is $v_{0, p, \infty}$. Hence we obtain $M'$-correspondence isometries $v_{p, \infty} : L_M(H, H_{p, t}) \rightarrow L_M(H, H_t)$ by setting $v_{p, \infty}(X) = v_{0, p, \infty} \circ X$. However, for $p'$ finer than $p$, we have $v_{0, p', \infty} \circ v_{0, p, p'} = v_{0, p, \infty}$. Hence $v_{p', \infty} \circ v_{p, p'} = v_{p, \infty}$. Thus, by the universal properties of inductive limits, we obtain an $M'$-correspondence isometry $v : E(t) \rightarrow L_M(H, H_t)$. We need to show that $v$ is surjective. To this end, observe that if $P$ and $Q$ are two normal, unital, completely positive maps on $M$, then using Proposition 2.12 (and applying Lemma 2.10), we find that $\mathcal{F}(X(H))$ and $\mathcal{F}(Y(H)) \subseteq \mathcal{F}(X(H))$. Hence, $\mathcal{F}(Y(H)) \subseteq \mathcal{F}(X(H))$ and $\mathcal{F}(Y(H)) \subseteq \mathcal{F}(X(H))$. Consequently, given any $X \in L_M(H, H_t)$ satisfying $X'Y = 0$ for all $Y \in \mathcal{F}(E(t))$, we have $X = 0$. That is, the orthogonal complement of $\mathcal{F}(E(t))$ in $L_M(H, H_t)$ is zero. Since $L_M(H, H_t)$ is self-dual, by Proposition 2.3, we conclude that $v(E(t)) = L_M(H, H_t)$. □
If \( p_1 \in \Psi(t) \) and \( p_2 \in \Psi(s) \), then we shall write \( p_2 \lor p_1 + s \) for the following partition in \( \Psi(t+s) \):

\[
\{0 = s_0 < s_1 < \cdots < s_{m-1} < s_m (= s = t_0 + s) < t_1 + s < t_2 + s \cdots < t_{n-1} + s < t_n + s = t + s\},
\]

where \( p_1 = \{0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t\} \) and \( p_2 = \{0 = s_0 < s_1 < s_2 < \cdots < s_{m-1} < s_m = s\} \). Note the order in the definition of \( p_2 \lor p_1 + s \). The "concatination" of partitions is not commutative. It is designed to support the isomorphism of \( E(s) \otimes E(t) \) with \( E(t+s) \) that we are about to describe.

**Lemma 3.2** Let \( p_1 \in \Psi(t) \) and \( p_2 \in \Psi(s) \) and write \( p \) for \( p_2 \lor p_1 + s \). Then the map that sends \( X \otimes Y \in \mathcal{L}_M(H, H_{p_1,t}) \otimes \mathcal{L}_M(H, H_{p_2,s}) \) to \( (I_s \otimes X)Y \) in \( \mathcal{L}_M(H, H_{p,t+s}) \) extends to an isomorphism of \( \mathcal{M}'\)-correspondences, where \( I_s \) denotes the identity map on \( M \otimes P_{s_1} \otimes P_{s_2-r_1} \otimes \cdots \otimes P_{s_{m-1}} \otimes M \) and where \( p_2 = \{0 = s_0 < s_1 < s_2 < \cdots < s_{n-1} < s_m = s\} \). Further, this isomorphism induces a natural isomorphism of \( \mathcal{M}'\)-correspondences from \( E(t) \otimes E(s) \) onto \( E(t+s) \).

**Proof.** That the map \( X \otimes Y \to (I_s \otimes X)Y \) induces an isomorphism of \( \mathcal{M}'\)-correspondences from \( \mathcal{L}_M(H, H_{p_1,t}) \otimes \mathcal{L}_M(H, H_{p_2,s}) \) into \( \mathcal{L}_M(H, H_{p,t+s}) \) is essentially proved in Proposition 2.13. To see that the isomorphism is surjective, simply apply Corollary 2.13 (several times).

To get the isomorphism from \( E(t) \otimes E(s) \) onto \( E(t+s) \), we appeal to universal properties of inductive limits. Let \( p_1 \) and \( p_1' \) be partitions in \( \Psi(t) \), with \( p_1' \) finer than \( p_1 \), and let \( p_2 \) be a partition in \( \Psi(s) \). Write \( p = p_2 \lor p_1 + s \) and \( p' = p_2 \lor p_1' + s \). Also let \( \alpha_{p_1,p_2} \) be the isomorphism from \( \mathcal{L}_M(H, H_{p_1,t}) \otimes \mathcal{L}_M(H, H_{p_2,s}) \) onto \( \mathcal{L}_M(H, H_{p,t+s}) \) that sends \( X \otimes Y \) to \( (I_s \otimes X)Y \), and let \( \alpha_{p_1',p_2} \) be the similarly defined isomorphism from \( \mathcal{L}_M(H, H'_{p_1,t}) \otimes \mathcal{L}_M(H, H_{p_2,s}) \) onto \( \mathcal{L}_M(H, H'_{p,t+s}) \). Then we have the following diagram, which is easily seen to be commutative:

\[
\begin{array}{ccc}
\mathcal{L}_M(H, H_{p_1,t}) \otimes \mathcal{L}_M(H, H_{p_2,s}) & \xrightarrow{\alpha_{p_1,p_2}} & \mathcal{L}_M(H, H_{p,t+s}) \\
\downarrow_{\upsilon_{p_1,p_1'} \otimes I} & & \downarrow_{\upsilon_{p,p'}} \\
\mathcal{L}_M(H, H'_{p_1,t}) \otimes \mathcal{L}_M(H, H_{p_2,s}) & \xrightarrow{\alpha_{p_1',p_2}} & \mathcal{L}_M(H, H'_{p,t+s})
\end{array}
\]

In the limit, we obtain an isometry from \( E(t) \otimes \mathcal{L}_M(H, H_{p_2,s}) \) into \( E(t+s) \). A similar argument yields an isometry from \( E(t) \otimes E(s) \) into \( E(t+s) \). It is clear from the definition of this map that its image contains all the \( \mathcal{L}_M(H, H_{p,t+s}) \), where \( p \) is constructed as \( p_2 \lor p_1 + s \). (We shall view these spaces as contained in \( E(t+s) \) without reference to the isomorphic embeddings.) For a given partition \( p \in \Psi(s+t) \), we can refine it by adding \( s \) to get \( p' \), say. Then \( \mathcal{L}_M(H, H'_{p',t+s}) \) is contained in \( E(t+s) \) and contains a copy of \( \mathcal{L}_M(H, H_{p,t+s}) \). Hence the image contains all the \( \mathcal{L}_M(H, H_{p,t+s}) \) and so must be all of \( E(t+s) \).

**Remark 3.3** Given \( t, s, r \in (0, \infty) \) and partitions \( p_1 \in \Psi(t) \), \( p_2 \in \Psi(s) \), and \( p_3 \in \Psi(r) \), one can define an isomorphism of \( \mathcal{M}'\)-correspondences between
\[ \mathcal{L}_M(H, H_{p,t}) \otimes \mathcal{L}_M(H, H_{p,s}) \otimes \mathcal{L}_M(H, H_{r,t}) \] and \[ \mathcal{L}_M(H, H_{p,t+r}) \] in two different, but natural, ways, where \( p = p_3 \lor (p_2 + r) \lor (p_1 + s + r) \): In the first, we map the left hand side, \[ \mathcal{L}_M(H, H_{p,t}) \otimes \mathcal{L}_M(H, H_{p,s}) \otimes \mathcal{L}_M(H, H_{r,t}) \] to \[ \mathcal{L}_M(H, H_{p,t}) \otimes \mathcal{L}_M(H, H_{p,s}) \otimes \mathcal{L}_M(H, H_{r,t}) \], where \( p' = p_3 \lor (p_2 + r) \), and then to \( \mathcal{L}_M(H, H_{p,t+r}) \), while in the second, we map \[ \mathcal{L}_M(H, H_{p,t}) \otimes \mathcal{L}_M(H, H_{p,s}) \otimes \mathcal{L}_M(H, H_{r,t}) \] to \[ \mathcal{L}_M(H, H_{p,t+r}) \otimes \mathcal{L}_M(H, H_{p,s}) \otimes \mathcal{L}_M(H, H_{r,t}) \], where \( p'' = p_2 \lor (p_1 + s) \) and then to \[ \mathcal{L}_M(H, H_{p,t+r}) \]. These two ways of identifying \[ \mathcal{L}_M(H, H_{p,t}) \otimes \mathcal{L}_M(H, H_{p,s}) \otimes \mathcal{L}_M(H, H_{r,t}) \] and \[ \mathcal{L}_M(H, H_{p,t+r}) \] amount to nothing more than identifying \( X_1 \otimes X_2 \otimes X_3 \) with \( (I_{++} \otimes I_3) \circ (I_1 \otimes X_2) \circ X_3 \), as we may. Passing to the limit yields the natural isomorphisms

\[ (E(t) \otimes E(s)) \otimes E(r) \simeq E(t) \otimes (E(s) \otimes E(r)) \simeq E(t + s + r). \]

Our analysis to this point shows that if we set \( E(0) = \mathcal{M}' \), then \( \{E(t)\}_{t \geq 0} \) is a discrete product system in the sense of

**Definition 3.4** Let \( \mathcal{N} \) be a von Neumann algebra. A discrete product system over \( \mathcal{N} \) is simply a family \( \{E(t)\}_{t \geq 0} \) of \( W^* \)-correspondences over \( \mathcal{N} \) such that \( E(0) = \mathcal{N} \) and such that \( E(t + s) \simeq E(t) \otimes E(s) \) for all \( t, s \in [0, \infty) \).

The particular product system that we associated with the semigroup \( \{P_t\}_{t \geq 0} \) in the preceding paragraphs will be called the (discrete) product system of \( M' \)-correspondences associated with \( \{P_t\}_{t \geq 0} \).

A (completely contractive) covariant representation of a discrete product system \( \{E(t)\}_{t \geq 0} \) on a Hilbert space \( H \) is simply a family \( \{T_t\}_{t \geq 0} \) of completely contractive linear maps, where \( T_t \) maps from \( E(t) \) to \( \mathcal{B}(H) \) such that each \( T_t \) is continuous with respect to the \( \sigma \)-topology on \( E(t) \) and the \( \sigma \)-weak topology on \( \mathcal{B}(H) \), \( T_0 \) is a \( \ast \)-representation of \( E(0) = \mathcal{N} \) on \( H \), and such that \( T_t \otimes T_s = T_{t+s} \) (after identifying \( E(t + s) \simeq E(t) \otimes E(s) \)).

**Remark 3.5** It is useful to think of product systems as semigroups and then to view covariant representations as representations of such a semigroup. However, when working with any particular product system and representation, it frequently becomes necessary to make explicit the isomorphisms between \( E(t) \otimes E(s) \) and \( E(t + s) \) and then, of course, the formulas involving \( \{T_t\}_{t \geq 0} \) become correspondingly more complicated.

Note, too, that the definition of a covariant representation implies that \( T_t(a \xi b) = T_0(a) T_t(\xi) T_0(b) \) for all \( t \geq 0 \), \( \xi \in E(t) \), \( a, b \in \mathcal{N} \). Thus if \( \{T_t\}_{t \geq 0} \) is a covariant representation of the product system then for each \( t \), \( T_t, T_0 \) is a completely contractive covariant representation of \( E(t) \) in the sense of Definition 2.14.

**Definition 3.6** A covariant representation \( \{T_t\}_{t \geq 0} \) of a product system \( \{E(t)\}_{t \geq 0} \) is called isometric in case for each \( t \), \( T_t, T_0 \) is isometric in the sense of Definition 2.14. It is called fully coisometric in case for each \( t \), \( T_t, T_0 \) is fully coisometric in the sense of Definition 2.14.
Our next objective is to show how a fully coisometric covariant representation of a product system \( \{ E(t) \}_{t \geq 0} \) can be dilated to a fully coisometric and isometric representation of \( \{ E(t) \}_{t \geq 0} \).

**Theorem and Definition 3.7** Let \( \{ E(t) \}_{t \geq 0} \) be a discrete product system over a von Neumann algebra \( \mathcal{N} \) and let \( \{ T_t \}_{t \geq 0} \) be a fully coisometric covariant representation of \( \{ E(t) \}_{t \geq 0} \) on a Hilbert space \( H \). Then there is another Hilbert space \( K \), an isometry \( u_0 \) mapping \( H \) into \( K \), and fully coisometric, isometric covariant representation \( \{ V_t \}_{t \geq 0} \) of \( E \) on \( K \) so that

1. \( u_0^* V_t(\xi) u_0 = T_t(\xi) \) for all \( \xi \in E(t) \), \( t \geq 0 \); and

2. For \( \xi \in E(t) \), \( t \geq 0 \), \( V_t(\xi)^* \) leaves \( u_0(H) \) invariant.

The smallest subspace of \( K \) containing \( u_0(H) \) and reducing \( V_t(\xi) \) for every \( \xi \in E(t) \), \( t \geq 0 \), is all of \( K \). If \( \{ \{ V'_t \}_{t \geq 0}, u'_0, K' \} \) is another triple with same properties as \( \{ \{ V_t \}_{t \geq 0}, u_0, K \} \), then there is a Hilbert space isomorphism \( W \) from \( K \) to \( K' \) such that \( W V_t(\xi) W^{-1} = V'_t(\xi) \) for all \( \xi \in E(t) \) and \( t \geq 0 \), and \( W \circ u_0 = u'_0 \). We therefore call the triple, \( \{ \{ V_t \}_{t \geq 0}, u_0, K \} \), the minimal isometric dilation of \( \{ T_t \}_{t \geq 0} \).

**Proof.** For \( 0 \leq t < s \), we write \( U_{t,s} \) for the isomorphism from \( E(t) \otimes E(s-t) \) to \( E(s) \). Then the associativity of tensor products implemented through these isomorphisms coupled with the identification of \( E(t) \otimes E(s) \otimes E(r) \) with \( E(t+s) \) imply that \( U_{s,t} \circ U_{t,s} \otimes I_{t-s} = U_{t,r} \). Further, for any \( t \), we write \( T_t \) for the operator from \( E(t) \otimes T_0 \) to \( H \) defined by the formula \( T_t(\xi \otimes h) = T_t(\xi) h \). (See Lemma 2.16 and the discussion surrounding it.) For \( 0 \leq t < s \), we define \( u_{t,s} \) from \( E(t) \otimes T_0 \) to \( E(s) \otimes T_0 \) by the formula

\[
u_{t,s} := (U_{t,s} \otimes I_H)(I_{E(t)} \otimes \tilde{T}_{s-t}^*).
\]

Observe that each space \( E(t) \otimes T_0 \) is a left \( \mathcal{N} \)-module and that the \( u_{t,s} \) are \( \mathcal{N} \)-module maps.

We claim that each \( u_{t,s} \) is an isometry. Indeed, since \( U_{t,s} \) is a Hilbert module isomorphism, \( U_{t,s} \otimes I_H \) is a Hilbert space isomorphism, i.e., a unitary, and so

\[
u_{t,s} u_{t,s} = (I_{E(t)} \otimes \tilde{T}_{s-t})(U_{t,s} \otimes I_H)(I_{E(t)} \otimes \tilde{T}_{s-t}^*)
\]

\[
= (I_{E(t)} \otimes \tilde{T}_{s-t}^*)(I_{E(t)} \otimes \tilde{T}_{s-t})
\]

\[
= I_{E(t)} \otimes \tilde{T}_{s-t}^* \tilde{T}_{s-t}.
\]

However, this last term is the identity on \( E(t) \otimes T_0 \) because \( \{ T_t \}_{t \geq 0} \) is assumed to be fully coisometric.

Further observe that the composition properties of the \( U_{t,s} \) coupled with the fact that \( \{ T_t \}_{t \geq 0} \) is a covariant representation imply that for \( 0 \leq t < s < r \), \( u_{t,s} u_{s,r} = u_{t,r} \). Hence, if we agree to set \( u_{t,t} \) equal to the identity on \( E(t) \otimes T_0 \) for each \( t \), then \( \{ \{ E(t) \otimes T_0 \} \}_{t \geq 0}, \{ u_{t,s} \}_{0 \leq t \leq s} \) is an inductive system of Hilbert
Thus, for each \( u_s \) \( s \geq 0 \) on the union of the ranges of the \( u_s \), simply note that for \( s_1 > s_2 \geq 0 \), \( t, s \in \mathbb{N} \) canonical embeddings. Note that the \( u_t \)'s are isometries and \( \mathcal{N} \)-module maps.

To construct the dilation \( \{ V_t \}_{t \geq 0} \), we begin by defining \( V_t \) on the range of each \( u_s \) by the formula

\[
V_t(\xi) \cdot u_s(\eta \otimes h) := u_{t+s}(U_{t,t+s}(\xi \otimes \eta) \otimes h),
\]

where \( t, s \geq 0 \), \( \xi \in E(t) \), \( \eta \in E(s) \), and \( h \in H \). To see that \( V_t(\xi) \) is well-defined on the union of the ranges of the \( u_s \), simply note that for \( s_1 > s_2 \geq 0 \), \( t, s \in \mathbb{N} \), \( \xi \in E(t) \), and \( h \in H \), we have

\[
V_t(\xi)u_{s_1}u_{s_2,s_1}(\eta \otimes h) = V_t(\xi)u_{s_2}(\eta \otimes h).
\]

The \( \mathcal{N} \)-module structure on \( K \) is just that afforded by \( V_0 \). That is, for \( a \in \mathcal{N} \), \( V_0(a) \cdot u_s(\eta \otimes h) = u_s(a\eta \otimes h) \), \( \eta \otimes h \in E(s) \otimes T_0 H \). So, to show that every other \( V_t \) extends to all of \( K \), and yields an isometric representation of the \( \mathcal{N} \)-correspondence \( E(t) \), we first simply compute to see that for \( \xi_1, \xi_2 \in E(t) \), and \( \eta \otimes h, \zeta \otimes k \in E(s) \otimes T_0 H \), we have

\[
\langle V_t^*(\xi_1)V_t(\xi_2)u_s(\eta \otimes h), u_s(\zeta \otimes k) \rangle = \langle u_{t+s}(\xi_2 \otimes \eta \otimes h), u_{t+s}(\xi_1 \otimes \eta \otimes k) \rangle
\]

\[
= \langle \xi_2 \otimes \eta \otimes h, \xi_1 \otimes \eta \otimes k \rangle = \langle \eta \otimes h, \xi_2, \xi_1 \rangle \eta \otimes k = \langle \eta \otimes h, V_0(\xi_2, \xi_1)u_s(\eta \otimes k) \rangle.
\]

This shows that on the range of each \( u_s \) \( (V_t, V_0) \) is an isometric covariant representation of \( E(t) \). Thus on the range of each \( u_s \), \( V_t(\xi) \) is a bounded operator with norm bounded by \( \|\xi\| \). Hence, \( V_t(\xi) \) extends to all of \( K \) as a bounded operator. Further, this equation shows that if we denote the projection of \( K \) onto the range of \( u_s \) by \( Q_s \), i.e., if we let \( Q_s = u_su_s^* \), then

\[
Q_s(V_t(\xi_2)^*V_t(\xi_1))Q_s = Q_sV_0(\xi_2, \xi_1)Q_s.
\]

Since this so for all \( s \), it follows that \( V_t(\xi_2)^*V_t(\xi_1) = V_0(\xi_2, \xi_1) \) on all of \( K \). Thus, for each \( t \), \( (V_t, V_0) \) is an isometric covariant representation of \( E(t) \).

To show that \( \{ V_t \}_{t \geq 0} \) satisfies the semi-group property, let \( t = t_1 + t_2 \), let \( \xi_1 \in E(t_1) \), \( \xi_2 \in E(t_2) \), and let \( \eta \otimes h \in E(s) \otimes T_0 H \). Then on the one hand we have

\[
V_t(U_{t_1,t_2}(\xi_1 \otimes \xi_2))u_s(\eta \otimes h) = u_{t+s}(U_{t_1,t+s}(U_{t_1,t_2}(\xi_1 \otimes \xi_2) \otimes \eta) \otimes h),
\]

while on the other we have

\[
V_t(\xi_1)V_t(\xi_2)u_s(\eta \otimes h) = V_t(\xi_1)u_{t+s}U_{t_2,t_2+s}(\xi_2 \otimes \eta) \otimes h)
\]

\[
= u_{t+s}(U_{t_1,t+s}(\xi_1 \otimes U_{t_2,t_2+s}(\xi_2 \otimes \eta)) \otimes h).
\]

By Remark 7.2, we conclude that \( V_t(U_{t_1,t}(\xi_1 \otimes \xi_2)) = V_{t_1}(\xi_1)V_{t_2}(\xi_2) \). Ignoring the \( U_{t,t+s} \) when identifying \( E(t) \otimes E(s) \) with \( E(t+s) \), we obtain the desired result: \( V_t(\xi_1 \otimes \xi_2) = V_{t_1}(\xi_1)V_{t_2}(\xi_2) \).
Next, we show that each $V_t$ is continuous with respect to the $\sigma$-topology on $E(t)$ and the $\sigma$-weak topology on $B(K)$. For this, observe that for $\xi, \xi_1 \in E(t)$, $\eta, \xi_2 \in E(s)$, and $h, k \in H$, we have $(u_t(\xi)u_s(\eta \otimes h), u_t(\xi_1 \otimes \eta \otimes h), u_t(\xi_2 \otimes \eta \otimes h)) = (h, T_0(\xi \otimes \eta, \xi_1 \otimes \xi_2))k = (h, T_0(\eta, \xi_1 \otimes \xi_2))k$. Thus, for each $s \geq 0$, the map $\xi \mapsto V_t(\xi)|u_s(E(s) \otimes T_0 H)$ has the desired continuity properties. Since the union of the ranges of the $u_s$ is dense and since $\|V_t(\xi)\| \leq \|\xi\|$, we conclude that $V_t$ is continuous with respect to the $\sigma$-topology on $E(t)$ and the $\sigma$-weak topology on $B(K)$.

To see that $u^*_0 V(t) u_0 = T_t(\xi)$, i.e., to see that $\{V_t\}_{t \geq 0}$ dilates $\{T_t\}_{t \geq 0}$, simply note that for $h, h' \in H$, $t > 0$ and $\xi \in E(t)$, we have $(u^*_0 V(t) u_0(h), h') = (u_t(\xi \otimes h), u_0(h')) = (\xi \otimes (T^*_t h'), E(t) \otimes H) = (\xi \otimes h, T^*_t h')$.

To check that $V_t(\xi)^*, \xi \in E(t)$, leaves $u_0(H)$ invariant, first note that the computation just completed shows that for $\zeta \in E(r)$, $r \geq 0$, and $h \in H$, $u^*_0 u_r(\zeta \otimes h) = T^*_r(h)$. Hence, for $\xi \in E(t)$, $\eta \in E(s)$, and $h \in H$, $u^*_0 V(t) u_0(\eta \otimes h) = u^*_0 u^*_0 V(t)(\eta \otimes h) = T^*_r(h) = T^*_r(h) = u^*_0 V(t) u_0(\eta \otimes h)$. Since this holds for all $s \geq 0$, we see that $u^*_0 V(t) = u^*_0 V(t)$.

To see that $\{V_t\}_{t \geq 0}$ is fully coisometric because $\{T_t\}_{t \geq 0}$ is, we need to show that $\tilde{V}_t$ is a coisometry for each $t$. Since $\{V_t\}_{t \geq 0}$ is isometric, each $\tilde{V}_t$ is an isometry. Hence, all we need to do is to show that the range of each $\tilde{V}_t$ is dense. For this, it suffices to show that for every $s \geq 0$ the span of $\{V_t(\xi)|u_s(\eta \otimes h) : \xi \in E(t), h \in H\}$ equals $u^*_0 V(t) u_0(\eta \otimes h)$.

From what we have shown so far, it is clear that the smallest subspace of $K$ that contains $u_0(H)$ and reduces every $V_t(\xi)$ is all of $K$. The uniqueness of $\{V_t\}_{t \geq 0}$ up to unitary equivalence is proved just as in Proposition 3.2 of [13], and so will be omitted here.

**Remark 3.8** It is worthwhile pointing out that the relation $u^*_0 V(t) u_0 = T_t(\xi)$ in the preceding theorem is equivalent to the relation $u^*_0 \tilde{V}_t(I \otimes u_0) = \tilde{T_t}$. Further, the invariance of $u_0(H)$ under $V_t(\xi)^*$ is equivalent to the equation $u_0 u^*_0 \tilde{V}_t(I \otimes u_0) = u_0 u^*_0 \tilde{V}_t$. These assertions are immediate from the proof.

We return to our semigroup, $\{P_t\}_{t \geq 0}$, of completely positive maps on the von Neumann algebra $\mathcal{M}$ and to the associated product system $\mathcal{M}'$-correspondences $\{E(t)\}_{t \geq 0}$ that we constructed at the outset of this section. Our next objective, Theorem 3.9, is to show that there is a fully coisometric, completely contractive covariant representation $\{T_t\}_{t \geq 0}$ of $\{E(t)\}_{t \geq 0}$ on $H$ (the Hilbert space of $\mathcal{M}$) so that $\{P_t\}_{t \geq 0}$ can be represented by the formula

$$P_t(S) = \tilde{T_t}(I_{E(t)} \otimes S)\tilde{T^*_t}.$$
For this purpose, recall that for a partition \( p \in \mathcal{P}(t) \), the Hilbert space \( H_{p,t} \) is defined by equation (15). Let \( \{ T_t \}_{t \geq 0} \) be the discrete product system of \( \mathcal{M} \)-correspondences constructed from \( \{ P_t \}_{t \geq 0} \) as above, and let \( \{ \tilde{T}_t \}_{t \geq 0} \) be defined by equation (16).

**Theorem and Definition 3.9** Let \( \{ E(t) \}_{t \geq 0} \) be the discrete product system of \( \mathcal{M} \)-correspondences constructed from \( \{ P_t \}_{t \geq 0} \) as above, and let \( \{ T_t \}_{t \geq 0} \) be defined by equation (15). Then \( \{ T_t \}_{t \geq 0} \) is a fully coisometric, completely contractive covariant representation of \( \{ E(t) \}_{t \geq 0} \) such that

\[
P_t(S) = \tilde{T}_t(I_{E(t)} \otimes S)\tilde{T}_t^*,
\]

for all \( t \geq 0 \) and all \( S \in \mathcal{M} \). We call \( \{ T_t \}_{t \geq 0} \) the identity representation of \( \{ E(t) \}_{t \geq 0} \).

**Proof.** Since \( T_t \) is given by left multiplication by an operator between Hilbert spaces of norm at most one, viz. \( T_t \), \( T_t \) is completely contractive. To check that \( \{ T_t \}_{t \geq 0} \) is multiplicative, we identify \( E(t+s) \) with \( E(t) \otimes E(s) \) as above and proceed to show that under this identification, \( T_{t+s} = T_t \otimes T_s \). For this purpose, let \( p_1 \) be a partition in \( \mathcal{P}(t) \) and let \( p_2 \) be a partition in \( \mathcal{P}(s) \). Also, fix \( X_1, X_2 \) in \( \mathcal{L}(H_{p_1,t}) \) and \( X_2 \in \mathcal{L}(H_{p_2,t}) \). Then the map that sends \( X_1 \otimes X_2 \) to \( (I_s \otimes X_1)X_2 \) (where \( I_s \) denotes the identity map on \( M \otimes p_{s_1} \otimes \cdots \otimes p_{s_{j-1}} M \), with \( p_2 = \{ 0 = s_0 < s_1 < s_2 < \cdots < s_{j-1} < s_j = s \} \)) carries \( \mathcal{L}(H_{p_1,t}) \otimes \mathcal{L}(H_{p_2,t}) \) to \( \mathcal{L}(H, H_{p,t+s}) \), where \( p = p_2 \lor p_1 + s \).

To show that \( T_{t+s} = T_t \otimes T_s \), it follows from equation (15) and the properties of direct limits that we need only check that \( \tilde{T}_t^*(I_s \otimes X_1)X_2 = \tilde{T}_t^*(X_1)X_2 \).

For this purpose, consider the element \( S_1 \otimes S_2 \otimes \cdots \otimes S_j \otimes T_1 \otimes T_2 \otimes \cdots \otimes T_n \otimes k \).
in $H_{p,t+s}$ (where $p$ and $p_2$ have just been defined and $p_1 = \{0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t\}$). Then

$$t^*_p(S_1 \otimes S_2 \otimes \cdots \otimes T_n \otimes k) = P_{t-t_n}(P_{t_n-t_{n-2}}(\cdots (P_{s-s_{j-1}}(\cdots (P_{s_1}(S_2) \cdots T_n) k
= t^*_p((P_{s-s_{j-1}}(\cdots (P_{s_1}(S_2)) \cdots (S_j) T_1 \otimes T_2 \otimes \cdots \otimes T_n \otimes k) \).

Hence, for $X_i(h) \in H_{p,t_i}$, and $S_1, S_2, \ldots, S_j$ in $\mathcal{M}$,

$$t^*_p(S_1 \otimes S_2 \otimes \cdots \otimes S_j \otimes X_i(h)) = t^*_p((P_{s-s_{j-1}}(\cdots (P_{s_1}(S_2)) \cdots (S_j) X_i(h)) \).

Since $X_i$ is an $\mathcal{M}$-module map, this equation can be rewritten as

$$t^*_p(S_1 \otimes S_2 \otimes \cdots \otimes S_j \otimes X_i(h)) = t^*_p X_i((P_{s-s_{j-1}}(\cdots (P_{s_1}(S_2)) \cdots (S_j) h),

as was required.

To see that each $T_i$ is continuous with respect to the $\sigma$-topology on $E(t)$ and the $\sigma$-weak topology on $B(H)$, take $X \in \mathcal{L}_M(H,H_t)$, and $h', h' \in E(t)$ and note that $(T_i X) h', h' = \langle X(h), t_i h' \rangle$. Since $t_i(h') \in H_t = \{ Y(h) \mid Y \in \mathcal{L}_M(H,H_t), h \in H \}$, and since for $Y \in \mathcal{L}_M(H,H_t)$ and $k \in H$, $(X(h), Y(k)) = \langle h, \langle X, Y \rangle k \rangle$, the desired continuity is evident.

Finally, we must verify equation (16). For $t = 0$, the equation is clear, so we always work with a fixed $t > 0$. For $k \in H$, $T^*_i k \in \mathcal{L}_M(H,H_t) \otimes \mathcal{M}'$ and so we may write $T^*_i k = \sum X_i \otimes h_i$, $X_i \in \mathcal{L}_M(H,H_t)$, and $h_i \in H$. Then, for $Y \in \mathcal{L}_M(H,H_t)$ and $h \in H$, $(\sum X_i(h_i)) Y(h) = (\sum X_i(h_i)) Y(h) = (\sum X_i(h_i)) Y(h)$. Since $t_i(h') \in H_t$, this equation implies that $T_i X_i(h_i) = t_i(k)$. Consequently, $T_i(I \otimes S) T^*_i k = T_i(I \otimes S) \sum X_i \otimes h_i = T_i \sum X_i \otimes S h_i = t_i(\sum X_i \otimes S h_i)$. Since the $X_i$ are $\mathcal{M}$-module maps, this last equation equals $t_i^*(\sum X_i \otimes S h_i) = t_i^* S t_i(k)$. To evaluate $t_i^* S t_i(k)$, recall that $t_i$ is the natural embedding of $H$ into $H_t$, when $H_t$ is viewed as the inductive limit of the $H_{p,t_i}$. Using the relations among $t_i$, the $v_{0,p,t}$, and the $v_{0,p,\infty}$ established above, it suffices choose a partition $p \in \mathcal{P}(t)$, and evaluate $t^*_p S t_p(k)$. Suppose, then that $p = \{0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t\}$. Then $S t_p(k) = S(I \otimes I \otimes \cdots I \otimes k) = S \otimes I \otimes \cdots \otimes k$, and $t^*_p S t_p(k) = P_{t-t_{n-1}}(P_{t_{n-1}-t_{n-2}}(\cdots (P_{t_1}(S) I) \cdots) I) k = P_I(S) k$, by the semigroup property of $\{P_I\}_{i \geq 0}$. Thus $t_i^* S t_i(k) = P_I(S) k$ for all $p \in \mathcal{P}(t)$, and so, in the limit, $t_i^* S t_i(k) = P_I(S) k$.

Of course, setting $S = I$ in equation (16) shows that $\{T_i\}_{i \geq 0}$ is fully coisometric.

This result is really not special to our given Markov semigroup $\{P_t\}_{t \geq 0}$; the next result is a converse which shows that completely contractive representations of product systems of correspondences over a von Neumann algebra $\mathcal{N}$ always define completely positive semigroups on $\mathcal{N}$.
Theorem 3.10 Let $\mathcal{N}$ be a von Neumann algebra acting on a Hilbert space $H$, let $\{E(t)\}_{t \geq 0}$ be a discrete product system of $\mathcal{N}$-correspondences, and let $\{T_t\}_{t \geq 0}$ be a completely contractive representation of $\{E(t)\}_{t \geq 0}$ on $H$. For $S \in \mathcal{N}'$ and $t \geq 0$, define

$$\Theta_t(S) = \tilde{T}_t(I_{E(t)} \otimes S)\tilde{T}_t^*.$$ 

Then $\{\Theta_t\}_{t \geq 0}$ is a semigroup of normal, contractive, completely positive maps on $\mathcal{N}'$. Further, $\{\Theta_t\}_{t \geq 0}$ is unital if and only if $\{T_t\}_{t \geq 0}$ is fully coisometric, and $\{\Theta_t\}_{t \geq 0}$ is a semigroup of $*$-endomorphisms if $\{T_t\}_{t \geq 0}$ is isometric.

Proof. Most of the result is proved in Proposition 2.21. We simply need to note that $T_0$ is a normal $*$-representation of $\mathcal{N}$ on $H$ and that for each $t \geq 0$, $(T_t, T_0)$ is a completely contractive covariant representation of $E(t)$ on $H$. All that really requires attention is the fact that $\{\Theta_t\}_{t \geq 0}$ is a semigroup, i.e., that $\Theta_{t+s} = \Theta_t \circ \Theta_s$. However, the multiplicativity of $\{T_t\}_{t \geq 0}$ implies that for $s, t \geq 0$, $T_{t+s} = T_t (I_{E(t)} \otimes T_s)$ and from this we see immediately that $\Theta_{t+s} = \Theta_t \circ \Theta_s$. $\blacksquare$

We note in passing that if the $\Theta_t$ are multiplicative, then by Proposition 2.21, the $E(t)$ decompose into the direct sum $E(t) = E(t') \oplus E(t'')$ so that $T_t|E(t')$ is isometric, while $T_t|E(t'')$ is zero. The multiplicativity of the $\Theta_t$ forces relations among the $q_t$, where $q_t$ is the projection of $E(t)$ onto $E(t')$, but we shall not dwell on these here.

In the presence of Theorems 3.7 and 3.9, we are able to state and prove our dilation result, which is a semigroup analogue of Theorem 2.24.

Theorem 3.11 Let $\mathcal{M}$ be a von Neumann algebra acting on the Hilbert space $H$ and let $\{P_t\}_{t \geq 0}$ be a semigroup of normal, unital, completely positive maps on $\mathcal{M}$ such that $P_0$ is the identity mapping on $\mathcal{M}$. Further, let $\{E(t)\}_{t \geq 0}$ be the product system of $\mathcal{M}'$-correspondences associated with $\{P_t\}_{t \geq 0}$, let $\{T_t\}_{t \geq 0}$ be the identity representation of $\{E(t)\}_{t \geq 0}$ on $H$ and let $\{\{V_t\}_{t \geq 0}, u_0, K\}$ be the minimal isometric dilation of $\{T_t\}_{t \geq 0}$. We write $\rho$ for $V_0$, thereby obtaining a normal $*$-homomorphism of $\mathcal{M}'$ into $\mathcal{B}(K)$ and we set $R$ equal to $\rho(\mathcal{M}')$.

Then $u_0^*R u_0 = \mathcal{M}$, and if we define $\{\alpha_t\}_{t \geq 0}$ by the formula

$$\alpha_t(S) = \tilde{V}_t(I_{E(t)} \otimes S)\tilde{V}_t^*,$$

$S \in R$, $t \geq 0$, then $\{\alpha_t\}_{t \geq 0}$ is a semigroup of unital, normal, $*$-endomorphisms of $R$ such that for $t \geq 0$, $S \in R$, and $T \in \mathcal{M}$,

$$P_t(u_0^*S u_0) = u_0^*\alpha_t(S)u_0$$

and

$$P_t(T) = u_0^*\alpha_t(u_0 Tu_0^*)u_0.$$  

Proof. Since $\{V_t\}_{t \geq 0}$ is a completely contractive covariant representation of $\{E(t)\}_{t \geq 0}$ on $K$, we know that $V_0 = \rho$ is a normal $*$-homomorphism of $\mathcal{M}'$ on $K$. Further, from Theorem 3.10, with $\mathcal{N} = \mathcal{M}'$, we see that $\{\alpha_t\}_{t \geq 0}$ is a
some observations due to D. SeLegue in [21, Section 2.7] when $M$ use salient features of [4, 5] and [21]. The first proposition is a generalization of (somewhat technical) lemmas and propositions. Basically, we will distill for our

is a quantum Markov process. Our arguments will be broken into a series of

offer somewhat different proofs.

{normal, unital, completely positive maps

in the sense that for all

with the weak topology on bounded subsets of a von Neumann algebra, our

semigroup of normal $*$-endomorphisms of $R (= \rho(M')^\prime)$ that are unital because

$\{V_t\}_{t \geq 0}$ is fully coisometric.

By the definition of $V_0$ in the proof of Theorem [6.7] we see that $\rho(a)u_s(\eta \otimes h) = V_0(a)u_s(\eta \otimes h) = u_s(\alpha h h)$, for all $a \in M'$, $s \geq 0$, and $\eta \otimes h \in E(s) \otimes H$. This implies that the range of each $u_s$ reduces $\rho(M')$ and, in particular that the restriction of $\rho(M')$ to $u_0(H)$ is unitarily equivalent to the identity representation of $M'$. Specifically, since $\rho(a)u_0(h) = u_0(\alpha h)$, for all $a \in M'$, $h \in H$, $a = u_0^*\rho(a)u_0$, $a \in M'$. Thus, $M' = u_0^*\rho(M')u_0$, so by the double commutant theorem, $M = u_0^*\rho(M')u_0 = u_0^*Ru_0$. Moreover, from Theorem 3.9 and equation (16), we see that for all $S \in R$,

$$
P_t(u_0^*Su_0) = \bar{T}_t(I_{E(t)} \otimes u_0^*Su_0)\bar{T}_t^* \begin{array}{l}
= u_0^*\bar{V}_t(I_{E(t)} \otimes u_0)(I_{E(t)} \otimes u_0^*Su_0)(I_{E(t)} \otimes u_0^*)\bar{V}_t^* u_0 \\
= u_0^*\bar{V}_t(I_{E(t)} \otimes u_0^*Su_0u_0^*)\bar{V}_t^* u_0 \\
= u_0^*u_0u_0^*\bar{V}_t(I_{E(t)} \otimes u_0^*Su_0)(I \otimes S)(I_{E(t)} \otimes u_0^*)\bar{V}_t^* u_0^*u_0^*u_0 \\
= u_0^*u_0u_0^*\bar{V}_t(I \otimes S)\bar{V}_t^* u_0^*u_0^*u_0 \\
= u_0^*u_0u_0^*\alpha_t(S)u_0u_0^*u_0 \\
= u_0^*\alpha_t(S)u_0 \end{array}$$

where the second and fifth equations are justified by Remark [6.8] and where the fourth and sixth equations are justified by the fact that the final projection of $u_0$ is $u_0u_0^*$. Equation (18) can be verified similarly, or directly from equation (17). ■

4 Minimality and Continuity

Our goal in this section is to show that under the hypothesis of separability on the Hilbert space $H$ and the hypothesis of weak continuity on $\{P_t\}_{t \geq 0}$ in Theorem 3.11, the Hilbert space $K$ that is produced there is separable and the semigroup $\{\alpha_t\}_{t \geq 0}$ is weakly continuous. That is, $\{\alpha_t\}_{t \geq 0}$ will be an $E_0$-semigroup. Therefore, throughout this section, we make the blanket assumption that our underlying Hilbert space $H$ is separable and that our semigroup of normal, unital, completely positive maps $\{P_t\}_{t \geq 0}$ on $M$ is (weakly) continuous in the sense that for all $T \in M$ and vectors $h_1, h_2 \in H$, the function $t \rightarrow \langle P_t(T)h_1, h_2 \rangle$ is continuous. Note that since the $\sigma$-weak topology coincides with the weak topology on bounded subsets of a von Neumann algebra, our continuity assumption on $\{P_t\}_{t \geq 0}$ is tantamount to assuming that $(M, \{P_t\}_{t \geq 0})$ is a quantum Markov process. Our arguments will be broken into a series of (somewhat technical) lemmas and propositions. Basically, we will distill for our

use salient features of [4, 3] and [21]. The first proposition is a generalization of some observations due to D. SeLegue in [21] Section 2.7 when $M = B(H)$. We offer somewhat different proofs.
Proposition 4.1 \((21)\) Under our standing assumptions on \(\{P_t\}_{t \geq 0}\), the following assertions are valid:

1. The map \(t \to P_t(X)\) is strongly continuous for all \(X \in \mathcal{M}\); i.e., for all \(h \in H\), \(t \to P_t(X)h\) is continuous from \([0, \infty)\) to \(H\).

2. Given a sequence \(\{X_n\} \subseteq \mathcal{M}\) that converges in the weak operator topology to \(X\), and given a sequence \(\{t_n\}_{n=1}^\infty \subseteq [0, \infty)\) converging to \(t\), the sequence of operators \(\{P_{t_n}(X_n)\}_{n=1}^\infty\) converges to \(P_t(X)\) in the weak operator topology, i.e., \(\{P_t\}_{t \geq 0}\) is jointly continuous in the weak operator topology.

**Proof.** For 1., first note that it suffices to prove the assertion when \(X = U\) is unitary and it suffices to show that \(P_t(U)h \to Uh\) for every \(h \in H\) as \(t \to 0\). Then observe that for any vector \(h \in H\), \(\lim_{t \to 0} \|P_t(U)h\| = \|h\|\). For not, then the \(\liminf\|P_t(U)h\|\) is strictly less than \(\|h\|\). Since \(\langle P_t(U)h, Uh \rangle \leq \|P_t(U)h\|\), the \(\liminf\|P_t(U)h, Uh\|\) is strictly less than \(\|h\|\), also. However, by our hypothesis on \(\{P_t\}_{t \geq 0}\), \(\langle P_t(U)h, Uh \rangle \to \langle Uh, Uh \rangle = \|h\|\).

Thus \(\lim_{t \to 0} \|P_t(U)h\|\) must be \(\|h\|\). But then we see that for all \(h \in H\), \(\|P_t(U)h - Uh\|^2 = \|P_t(U)h - Uh\| = \|P_t(U)h\|^2 - 2\Re(Uh, P_t(U)h) + \|Uh\|^2\) tends to zero, as \(t \to 0\), as required.

For 2., observe that the normality of \(P_t\) means that there is a unique bounded map \(\Psi_t\) such that \(P_t = \Psi_t^*\). The uniqueness and the fact that \(\{P_t\}_{t \geq 0}\) is a semigroup imply the same is true for \(\{\Psi_t\}_{t \geq 0}\), i.e., \(\Psi_{t+s} = \Psi_t \Psi_s\). The continuity of \(\{P_t\}_{t \geq 0}\) in the weak operator topology and the fact that \(\{P_t\}_{t \geq 0}\) is uniformly bounded imply that \(\omega \circ P_t(X)\) is continuous in \(t\) for all \(X \in \mathcal{M}\), and all \(\omega \in \mathcal{M}_\omega\).

If we write the pairing between \(\mathcal{M}_\omega\) and \(\mathcal{M}\) by \((\cdot, \cdot)\) as we shall, then this means that \(\langle \Psi_t(\omega), X \rangle\) is continuous in \(t\) for all \(X\); i.e., \(\{\Psi_t\}_{t \geq 0}\) is weakly continuous on \(\mathcal{M}_\omega\). But \(\mathcal{M}_\omega\) is separable and so by \([10]\) Corollary 3.1.8, the semigroup \(\{\Psi_t\}_{t \geq 0}\) is strongly continuous on \(\mathcal{M}_\omega\), i.e., for all \(\omega \in \mathcal{M}_\omega\), \(\|\Psi_t(\omega) - \Psi_s(\omega)\| \to 0\), as \(t \to s\). This means, in particular, that if \(\omega_h\) is the vector state associated with the vector \(h \in H\), then \(\|\omega_h \circ P_t - \omega_h \circ P_s\| \to 0\) as \(t \to s\). So, if \(\{X_n\}_{n=1}^\infty\) is a sequence in \(\mathcal{M}\) that converges weakly to \(X \in \mathcal{M}\), and if \(t_n \to t\), then

\[
|\langle P_{t_n}(X_n)h, h \rangle - \langle P_t(X)h, h \rangle| = |\omega_h \circ P_{t_n}(X_n) - \omega_h \circ P_t(X)|
\leq |\omega_h \circ P_{t_n}(X_n) - \omega_h \circ P_t(X)| + |\omega_h \circ P_t(X_n) - \omega_h \circ P_t(X)|
\]

\[
\leq \|\omega_h \circ P_{t_n} - \omega_h \circ P_t\| \|X_n\| + |\langle P_t(X_n - X)h, h \rangle|.
\]

Since the norms of the \(X_n\) are uniformly bounded by the uniform boundedness principle, this inequality shows that \(P_{t_n}(X_n) \to P_t(X)\) in the weak operator topology, as required.

**Proposition 4.2** Under our standing separability and continuity assumptions, the Hilbert space \(K\) in Theorem 3.1 is separable.

**Proof.** Recall from the proof of Theorem 3.1 that \(K\) is defined to be the inductive limit \(\lim\bigcup_t E(t) \otimes T_{t_0} H, u_{t_0}\). Since the sequence of spaces, \(\{E(n) \otimes T_{t_0} H\}_{n \geq 0}\), is cofinal in \(\{E(t) \otimes T_{t_0} H\}_{t \geq 0}\), it suffices to show each space \(E(t) \otimes T_{t_0} H\)
is separable. However, each space $E(t)$ is isomorphic to $L_M(H, H_t)$ by Lemma 3.3. So, if we can show that $H_t$ is separable, then $E(t)$ will be separable in the $\sigma$-topology (which is the same as the $\sigma$-weak topology by Proposition 2.3). But then, of course, $E(t) \otimes_{T_0} H$ will be spanned by a sequence $\{X_n \otimes h_m\}_{m,n \geq 0}$, where the $X_n$ run through a countable set that is dense in $E(t)$ in the $\sigma$-topology and the $h_m$ run through a countable dense set of $H$, and so $E(t) \otimes_{T_0} H$ will be separable. Thus we need to show $H_t$ is separable.

Now $H_t$ is, itself, an inductive limit $\lim\limits_{t \in t\mathbb{Q} \cap [0, t]}(H_{p,t}, v_{0,p,p'})$ where $p$ and $p'$ range over $\mathfrak{P}(t)$, and $p'$ refines $p$. The normality of the $P_t$ enables one to see that each $H_{p,t}$ is separable and the weak continuity of $\{P_t\}_{t \geq 0}$ enables one to replace $\mathfrak{P}(t)$ with a countable (but not, strictly speaking, cofinal) subset. From these two observations the separability of $H_t$ follows. Here are the details.

To see that $H_{p,t}$ is separable, first observe that $M \otimes_{P_t} H$ is separable for any $t$. For this, it suffices to show that if $\{T_n\}_{n \geq 0}$ is any sequence that is strongly dense in the unit ball of $M$ and if $\{h_n\}_{n \geq 0}$ is any dense sequence of vectors in $H$, then any decomposable tensor, $T \otimes h$, with $T$ in the unit ball of $M$, is in the closure of $\{T_n \otimes h_n\}_{n \geq 0}$. So, passing to subsequences, if necessary, assume that $T_n \rightarrow T$ strongly and that $h_n \rightarrow h$. Then $T_n \otimes h_n - T \otimes h = (T_n - T) \otimes h + T \otimes (h_n - h)$. However, $\|T_n - T\| \otimes h = \langle h, P_t((T_n - T)^*(T_n - T))h\rangle \rightarrow 0$ because $T_n \rightarrow T$ strongly and $P_t$ is normal. On the other hand, $\|T \otimes (h_n - h)\|^2 = \langle h_n - h, P_t(T^*T)(h_n - h)\rangle \rightarrow 0$ because $h_n \rightarrow h$ in $H$. Finally, since $h_n \rightarrow h$ in $H$ and since $T_n$, $T$, and their images under $P_t$ are all bounded in norm by 1, we see that $\|(T - T_n) \otimes (h - h_n)\|^2 = \langle h - h_n, P_t((T_n - T)^*(T_n - T))(h - h_n)\rangle \leq 4\|h - h_n\|^2 \rightarrow 0$. This shows that $T_n \otimes h_n \rightarrow T \otimes h$ as required. Now the proof that each $H_{p,t}$ is separable is proved by iterating this argument.

Let $\mathfrak{P}_0(t)$ be the collection of those partitions $p \in \mathfrak{P}(t)$ whose points lie in $t\mathbb{Q} \cap [0, t]$. Observe that $\mathfrak{P}_0(t)$ is countable and write $\tilde{H}_t$ for the union $\bigcup\{v_{0,p,\infty}(H_{p,t}) \mid p \in \mathfrak{P}_0(t)\}$. Then $\tilde{H}_t$ is the countable union of separable Hilbert spaces and so its closure is separable. We will show that its closure is all of $H_t$. For this purpose, it suffices to show that if $p$ is an arbitrary partition in $\mathfrak{P}(t)$, then $v_{0,p,\infty}(H_{p,t})$ is in the closure of $\tilde{H}_t$. This, in turn, will be clear if we can show that if $p = \{0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t\}$ and if $\{p_m\}_{m \geq 0} = \{\{0 = s(m)_0 < s(m)_1 < s(m)_2 < \cdots < s(m)_{n-1} < s(m)_n = t\}\}_{m \geq 0}$ is a sequence of partitions in $\mathfrak{P}_0(t)$ such that $\lim_{m \rightarrow \infty} s(m)_k = t_k$ for every $k$, then for every $n$-tuple $T_1, T_2, \cdots, T_n \in M$ and every $h \in H$,

$$\lim_{m \rightarrow \infty} v_{0,p_m,\infty} T_1 \otimes_{p_m(m)} T_2 \otimes_{p_m(m-1)} \cdots \otimes_{p_m(m-n+1)} h = v_{0,p,\infty} T_1 \otimes_{p_m} T_2 \otimes_{p_m-1} \cdots \otimes_{p_m-n+1} h.$$

To verify this equation, it suffices to assume that $s(m)_k = t_k$ for all $m$ and for all $k$ but one. So, in fact, it is enough to verify the desired limit when $p = \{0 = t_0 < t_1 < t_2 = t\}$ and when each $p_m$ is of the form $\{0 = s(m)_0 <
relative to $v^{3.8}$.) If we let $p_{u}$ multiplicative $\in q$ $i.e.$, the remark shows that $(\in V)$ Theorem 3.11 is weakly continuous. However, this is not necessary for the definition of minimality. That is, minimality makes sense without assuming that $\{\alpha_{t}\}_{t\geq 0}$ is weakly continuous. Here, minimality will be used to show that the $\{\alpha_{t}\}_{t\geq 0}$ we constructed in Theorem 3.11 is weakly continuous.

Lemma 4.3 With the notation of Section 2, we have, for all $t \geq 0$,

1. $\tilde{V}_{t}^{*}u_{0} = (I_{t} \otimes u_{0})\tilde{T}_{t}^{*}$, where $I_{t}$ denotes the identity operator on $E(t)$.

2. $\sqrt{\{ (I_{t} \otimes X)\tilde{T}_{t}^{*}h \mid X \in M, h \in H \} = \mathcal{E}_{t} \otimes_{M'} H}$.

3. $\sqrt{\{ (I_{t} \otimes Y)\tilde{V}_{t}^{*}h \mid Y \in \mathcal{R}, h \in H \} = \mathcal{E}_{t} \otimes_{M'} K}$.

4. $\sqrt{\{ \alpha_{t}(Y)u_{0}h \mid Y \in \mathcal{R}, h \in H \} = \sqrt{\{ V_{t}(X)k \mid X \in \mathcal{E}_{t}, k \in K \}$

Proof. (1) This is an easy consequence of Remark 3.8. Indeed, from the first part of that remark, we know that $u_{0}^{*}V_{t}(I \otimes u_{0}) = \tilde{T}_{t}$. So, $(I \otimes u_{0})^{*}V_{t}^{*}u_{0} = \tilde{T}_{t}^{*}$. Therefore, $(I \otimes u_{0}^{*})\tilde{V}_{t}^{*}u_{0} = (I \otimes u_{0})\tilde{T}_{t}^{*}$. On the other hand, the second part of the remark shows that $(I \otimes u_{0}^{*})V_{t}^{*}u_{0} = (I \otimes u_{0}^{*})\tilde{V}_{t}^{*}(u_{0}u_{0}^{*})u_{0} = \tilde{V}_{t}^{*}(u_{0}u_{0}^{*})u_{0} = \tilde{V}_{t}^{*}u_{0}$, so that $\tilde{V}_{t}^{*}u_{0} = (I_{t} \otimes u_{0})\tilde{T}_{t}^{*}$.

(2) Here, $\mathcal{E}_{t}$ is embedded as a subspace of $E(t)$ and so $\mathcal{E}_{t} \otimes H$ is contained in $E(t) \otimes H$. To show the desired equality, first note that if $h \in H$ and if $p \in \mathcal{P}(t)$, then $\iota_{p}(h) = I \otimes I \otimes \cdots \otimes I \otimes h$. (See the discussion after Remark 3.8.) If we let $p_{0} = \{ 0 = t_{0} < t_{1} = t \}$, then in the notation developed between Remark 3.8 and Theorem 3.9 we have $v_{0,p_{0}}(I \otimes h) = \iota_{p}(h)$. So, for $S \in \mathcal{M}$, $S_{t}(h) = S \otimes I \otimes \cdots \otimes I \otimes h = v_{0,p_{0}}(S \otimes h)$. By identifying $\mathcal{M} \otimes P_{t}$ with a
subspace of $H_{p,t}$, we write $S_{t_i}(h) \in \mathcal{M} \otimes P, H, S \in \mathcal{M}$. Since this holds for all partitions $p \in \mathcal{P}(t)$, we have

$$S_{t_i}(h) \in \mathcal{M} \otimes P, H,$$

for all $S \in \mathcal{M}$ and $t \geq 0$. Now fix an element of $E(t) \otimes H$ that is orthogonal to $\mathcal{E}_t \otimes H$ and write it as $\sum X_i \otimes h_i, X_i \in E(t), h_i \in H$. Then for every $X \in \mathcal{E}_t$ and $k \in H$, $0 = \langle X \otimes k, \sum X_i \otimes h_i \rangle = \sum \langle k, X^* X_i, h_i \rangle = \langle X(k), \sum X_i(h_i) \rangle$. Since $\mathcal{V}(X(h) \mid X \in \mathcal{E}_t(H, \mathcal{M} \otimes P, H) = \mathcal{E}_t, h \in H \rangle = \mathcal{M} \otimes P, H$, by Lemma 2.11, we see that

$$\sum X_i(h_i) \in (\mathcal{M} \otimes P, H)^{\perp}. \quad (20)$$

However, we have just shown above that $\mathcal{M} \otimes P, H \subseteq \mathcal{M} \otimes P, H$. Hence, for $S \in \mathcal{M}$ and $h \in H$,

$$\langle (I \otimes S) \tilde{T}^*_i h, \sum X_i \otimes h_i \rangle = \langle \tilde{T}^*_i h, \sum X_i \otimes S^* h_i \rangle
= \langle h, i^*_t(\sum X_i(S^* h_i)) \rangle = \langle h, i^*_t(S^* \sum X_i(h_i)) \rangle
= \langle S_{t_i}(h), \sum X_i(h_i) \rangle = 0,$$

where the last equation follows from (13) and (20), and the preceding one follows from the fact that the $X_i$ are $\mathcal{M}$-module maps, i.e., they commute with elements of $\mathcal{M}$. This equation thus shows that $[(I \otimes \mathcal{M}) \tilde{T}^*_i H]$ is contained in $\mathcal{E}_t \otimes H$. For the reverse containment, fix an element $\sum X_i \otimes h_i, X_i \in \mathcal{E}_t, h_i \in H$, that is in $\mathcal{E}_t \otimes H$ but orthogonal to $[(I \otimes \mathcal{M}) \tilde{T}^*_i H]$. Then for every $S \in \mathcal{M}$ and $h \in H$, the last equation shows that $0 = \langle (I \otimes S) \tilde{T}^*_i h, \sum X_i \otimes h \rangle = \langle S_{t_i}(h), \sum X_i(h) \rangle$. This shows that $\sum X_i(h)$ is orthogonal to $\mathcal{M} \otimes P, H$. Since $X_i \in \mathcal{E}_t, X_i(h_i) \in \mathcal{M} \otimes P, H$ for all $i$ and, so, $\sum X_i(h) = 0$. However, $\langle \sum X_i(h_i), \sum X_j(h_j) \rangle = \sum_{i,j} \langle X_i(h_i), X_j(h_j) \rangle = \| \sum X_i(h) \|^2 = 0$, and so $\sum X_i \otimes h_i = 0$ as was to be proved.

(3) From (1) we may write

$$(I \otimes \mathcal{R}) \tilde{V}_i^* u_0 H = (I \otimes \mathcal{R})(I \otimes u_0) \tilde{T}^*_i H = (I \otimes \mathcal{R} u_0) \tilde{T}^*_i H.$$

As we noted in the proof of Theorem 3.11, $u_0 u_0^*$ lies in $\mathcal{R}$. Consequently, $\mathcal{R} u_0 = \mathcal{R} u_0 u_0^* \mathcal{R} u_0 = \mathcal{R} u_0 \mathcal{M}$. So, using (2), we conclude that

$$[(I \otimes \mathcal{R}) \tilde{V}_i^* u_0 H] = [(I \otimes \mathcal{R} u_0)(I \otimes \mathcal{M})(I \otimes u_0) \tilde{T}^*_i H]
= [(I \otimes \mathcal{R} u_0)(\mathcal{E}_t \otimes H)] = \mathcal{E}_t \otimes [\mathcal{R} u_0 H].$$

Now let $p$ be the projection of $K$ onto $[\mathcal{R} u_0]$. Then $p \in \mathcal{R} = \rho(\mathcal{M}'') = \rho(\mathcal{M}')$; i.e. $p = \rho(p_0)$ for some projection $p_0 \in \mathcal{M}'$. However, $pK$ contains $u_0 H$ and $\rho(p_0)$ acts on $u_0 H$ by $\rho(p_0) u_0 h = u_0 p_0 h$. Hence $p_0 = I$ and so $p = I$. Thus $[\mathcal{R} u_0 H] = K$ and so $[(I \otimes \mathcal{R}) \tilde{V}_i^* u_0 H] = \mathcal{E}_t \otimes K$.

(4) The last assertion follows from the previous one and the definition of $\alpha_t$:

$$[\alpha_t(\mathcal{R}) u_0 H] = [\tilde{V}_i(I \otimes \mathcal{R}) \tilde{V}_i^* u_0 H] = [\tilde{V}_i(\mathcal{E}_t \otimes K)] = [V_i(\mathcal{E}_t)K].$$
Lemma 4.4 Let $q_t$ be the projection from $K$ onto $[\alpha_t(R)u_0H]$. Then $q_t$ lies in $\mathcal{R}$ and $q_t\alpha_t(q_s)$ is the projection onto $[V_{t+s}(\mathcal{E}_t \otimes \mathcal{E}_s)K]$.

Proof. The previous lemma shows that $q_t$ is the projection of $K$ onto $[V_t(\mathcal{E}_t)K]$. Thus $q_t$ lies in $\mathcal{R} = \rho(M')'$. Also, the range of $q_t$, which is $[\alpha_t(R)u_0H]$ is clearly invariant under $\alpha_t(R)$; i.e., $q_t \in \alpha_t(R)'$. Thus, in particular, $q_t$ commutes with $\alpha_t(q_s)$; so we see that $q_t\alpha_t(q_s)$ is a projection. We need to show that $q_t\alpha_t(q_s)$ has range $[V_{t+s}(\mathcal{E}_t \otimes \mathcal{E}_s)K]$. For this purpose, observe that the range of $\alpha_t(q_s)$ is $\tilde{V}_t(I \otimes q_s)\tilde{V}_t^* K = \tilde{V}_t(I \otimes q_s)(E(t) \otimes K) = \tilde{V}_t(E(t) \otimes q_s(K)) = \tilde{V}_t(E(t) \otimes [V_s(\mathcal{E}_s)K]) = [V_t(E(t))V_s(\mathcal{E}_s)K]$. Clearly, $[V_t(\mathcal{E}_t)V_s(\mathcal{E}_s)K] \subseteq [V_t(E(t))V_s(\mathcal{E}_s)K] \cap [V_t(\mathcal{E}_t)K]$. We claim that, in fact, the two subspaces coincide. To see this, let $w$ be the isometric embedding of $M \otimes P_t H$ into $H_t$, view $M \otimes P_t H$ as a subspace of $H_t$ and view $\mathcal{E}_t$ as a subspace of $E(t)$ (i.e. omit reference to the canonical embeddings.) Also, identify $\mathcal{L}_M(H, H_1)$ with $E(t)$ and $\mathcal{L}_M(H, M \otimes P_t H)$ with $E_t$, as we have throughout this paper. Then the map $p$ on $E(t) = \mathcal{L}_M(H, H_t)$ defined by the formula $p(X) = wu^* \circ X, X \in E(t)$, is a projection in $\mathcal{L}(E(t))$ with range $E_t$. For elements $V_t(X_1)V_s(Y_1)k_i, i = 1, 2$, in $[V_t(E(t))V_s(\mathcal{E}_s)K]$, we have

$$\langle V_t(X_1)V_s(Y_1)k_1, V_t(pX_2)V_s(Y_2)k_2 \rangle = \langle \tilde{V}_t(X_1 \otimes V_s(Y_1)k_1), \tilde{V}_t(pX_2 \otimes V_s(Y_2)k_2) \rangle = \langle X_1 \otimes V_s(Y_1)k_1, pX_2 \otimes V_s(Y_2)k_2 \rangle = \langle V_s(Y_1)k_1, p(X_1)^*X_2V_s(Y_2)k_2 \rangle = \langle p(X_1) \otimes V_s(Y_1)k_1, X_2 \otimes V_s(Y_2)k_2 \rangle = \langle V_t(p(X_1))V_s(Y_1)k_1, V_t(X_2)V_s(Y_2)k_2 \rangle.$$

Thus, the map $V_t(X)V_s(Y)k \to V_t(p(X))V_s(Y)k$ is selfadjoint and, therefore, is the orthogonal projection from $[V_t(E(t))V_s(\mathcal{E}_s)K]$ onto $[V_t(\mathcal{E}_t)V_s(\mathcal{E}_s)K]$. Write $q$ for this projection. A similar computation shows that the projection from $[V_t(E(t))K]$ onto $[V_t(\mathcal{E}_t)K]$ is given by the formula $V_t(X)k \to V_t(p(X))k$. However, this projection is just the restriction of $q_t$ to $[V_t(E(t))K]$. We can then restrict $q_t$ further to $[V_t(E(t))V_s(\mathcal{E}_s)K]$ and then the restricted image will clearly be $[V_t(E(t))V_s(\mathcal{E}_s)K] \cap [V_t(\mathcal{E}_t)K]$ (because $\alpha_t(q_s)$ commutes with $q_t$). Also, by the definition of $q$, this restriction of $q_t$ is just $q$ and so its image is $[V_t(\mathcal{E}_t)V_s(\mathcal{E}_s)K]$. Thus the range of $q_t\alpha_t(q_s)$ is $[V_t(\mathcal{E}_t)V_s(\mathcal{E}_s)K] = [V_{t+s}(\mathcal{E}_t \otimes \mathcal{E}_s)K]$. ■

Now let $p = \{0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t\}$ be a partition in $\mathcal{P}(t)$, write $q_s$ for the projection onto $[\alpha_s(R)u_0H]$, as in the last lemma, and set

$$q_{p,t} := q_{t-t_{n-1}} \alpha_{t-t_{n-1}}(q_{t_{n-1}-t_{n-2}}) \cdots \alpha_{t_2-t_1}(q_{t_1}).$$

Repeated use of the last lemma shows that $q_{p,t}(K) = [V_t(\mathcal{E}_t \otimes \mathcal{E}_{t-t_{n-1}} \otimes \cdots \otimes \mathcal{E}_{t_1}K) = [V_t(\mathcal{L}_M(H, H_{p,t}))K]$. Thus, it is clear that the $q_{p,t}$ increase as the partitions $p$ are refined and since $E(t) = \lim \mathcal{L}_M(H, H_{p,t})$, we see that they converge strongly to the projection onto $[V_t(E(t))K]$; call it $\mathcal{F}_t$. However, since $\tilde{V}_t$ is a coisometry, we see that $K = [\tilde{V}_t(E(t) \otimes K) = [V_t(E(t))K]$. Thus, $\mathcal{F}_t = I$, $t \geq 0$.
Observe that \( q_t \) is the same projection defined by Arveson in Section 3 of [4]. He uses a slightly different indexing scheme for the partitions that enter into his \( q_{p,t} \), but a moment’s reflection reveals that his \( q_{p,t} \) are the same as ours.

**Proposition 4.5** The semigroup of endomorphisms, \( \{\alpha_t\}_{t \geq 0} \), of \( R \) is minimal.

**Proof.** As Arveson indicates in §3 of [3] (see page 575, in particular), since we have shown that the projections \( q_t \) are all equal to \( I \), it remains to show that \( \alpha_t(u_0 u_0^* \alpha_{t_0} (u_0 u_0^* u_0^* u_0^* u_0^* \alpha_{t_0} (u_0 u_0^*)) u_0 h \mid a_i \in \mathcal{M}, h \in H, \text{and } t_i \geq 0 \}

and let \( R_+ \) be the von Neumann algebra generated by \( \{\alpha_t(u_0 u_0^* R u_0 u_0^*) \mid t \geq 0\} \). Then, as Arveson shows in Proposition 3.14 of [3], \( p_+ \) is the largest projection in the center of \( R_+ \) that dominates \( u_0 u_0^* \) and, as he shows in Theorem B of [3], because \( \{\alpha_t\}_{t \geq 0} \) is minimal, \( p_+ = I \). (Note: In the proof of [3], Theorem B, Arveson assumes that \( R \) is a factor. However, this assumption is not necessary for the implications spelled out there that we have used.) Thus we have

**Corollary 4.6** The von Neumann algebra \( R \) is generated by \( \{\alpha_t(u_0 u_0^* R u_0 u_0^*) \mid t \geq 0\} \) and \( K \) is the span \( \bigvee \{\alpha_{t_1}(u_0 a_1 u_0^*) \alpha_{t_2}(u_0 a_2 u_0^*) \cdots \alpha_{t_n}(u_0 a_n u_0^*) u_0 h \mid a_i \in \mathcal{M}, h \in H, \text{and } t_i \geq 0 \} \).

We let \( \mathcal{A} \) denote the \( C^* \)-algebra generated by \( \{\alpha_t(u_0 u_0^* R u_0 u_0^*) \mid t \geq 0\} \). Then \( \mathcal{A} \) is a translation invariant \( C^* \)-subalgebra of \( R \) that generates \( R \) as a von Neumann algebra. To show that \( \{\alpha_t\}_{t \geq 0} \) is weakly continuous on \( R \) we show first that it is weakly continuous on \( \mathcal{A} \) and then promote the weak continuity there to all of \( R \). For this purpose, we begin with the following result proved by SeLegue [21] in the context when \( \mathcal{M} = \mathcal{B}(H) \). Our proof is somewhat different.

**Proposition 4.7** ([21], Proposition 2.27) For every \( T \in \mathcal{M} \), \( \alpha_t(u_0 T u_0^*) \rightarrow u_0 T u_0^* \) in the strong operator topology as \( t \rightarrow 0+ \).

**Proof.** Fix \( T \in \mathcal{M} \) and \( k \in K \) and then

\[
\| (\alpha_t(u_0 T u_0^*) - u_0 T u_0^*) k \|^2 = (\langle (\alpha_t(u_0 T u_0^*) - u_0 T u_0^*) (\alpha_t(u_0 T u_0^*) - u_0 T u_0^*) k, k \rangle = \langle \alpha_t(u_0 T u_0^*) k, k \rangle - \langle \alpha_t(u_0 T u_0^*) u_0 T u_0^* k, k \rangle
\\ - \langle \alpha_t(u_0 T u_0^*) u_0 T u_0^* k, k \rangle + \langle u_0 T u_0^* k, k \rangle
\]

to realize that it suffices to show that \( \alpha_t(u_0 T u_0^*) \rightarrow u_0 T u_0^* \) in the weak operator topology as \( t \rightarrow 0+ \) for every \( T \in \mathcal{M} \). However, since we have shown that
{αℓ}t≥0 is minimal and since {αℓ}t≥0 is uniformly bounded, we may apply Corollary 4.4 to assert that it suffices to show that
\[ \langle αℓ(u0Tu0)k1, k2 \rangle \to \langle u0Tu0k1, k2 \rangle \]
for all \( T ∈ M \) and all vectors \( k_i \) of the form \( αℓ(u0a1u0^∗)αℓ2(u0a2u0^∗)⋯αℓn(u0anu0^∗)u0h_1, h_1 ∈ H, a_i ∈ M, \) and \( t_i ≥ 0 \). Note, too, that if any \( t_j = 0 \), then \( αℓ(u0a1u0^∗)αℓj(u0a2u0^∗)⋯αℓn(u0anu0^∗)u0h \) lies in \( H \) and so we may assume for the discussion that every \( t_i > 0 \). Also, let \( t \) be the minimal number among the \( t_i \)’s. Then we may write \( αℓ(u0a1u0^∗)αℓ2(u0a2u0^∗)⋯αℓn(u0anu0^∗)u0h \) as \( αℓ(⋯)u0h \). That is, \( αℓ(u0a1u0^∗)αℓ2(u0a2u0^∗)⋯αℓn(u0anu0^∗)u0h \) is in the cyclic subspace \( [αℓ(R)]u0h \). Thus, we may assume that \( k_1 = αℓ(R)u0h_1 \) and that \( k_2 = αs(L)u0h_2 \), where \( R \) and \( L \) are in \( R \), the \( h_i \) are in \( H \) and \( r, s > 0 \). We need to show that for \( T ∈ M \),
\[ \langle αℓ(u0Tu0^∗)αr(R)u0h_1, αs(L)u0h_2 \rangle \to \langle u0Tu0^∗αr(R)u0h_1, αs(L)u0h_2 \rangle \]
as \( t → 0^+ \). For this, we may assume at the outset that the \( t \)’s under consideration are all less than \( r \) and \( s \). Further, since \( αℓ(u0u0^∗) ≥ u0u0^∗ \) as we saw in the proof of Theorem 3.11, we find that
\[ \langle αℓ(u0Tu0^∗)αr(R)u0h_1, αs(L)u0h_2 \rangle = \langle αℓ(αs−1(L^∗)u0Tu0^∗αr−1(R))u0h_1, u0h_2 \rangle = \langle αℓ(u0u0^∗αs−1(L^∗)u0Tu0^∗αr−1(R))u0h_1, u0h_2 \rangle = \langle αℓ(u0P_{s−1}(u0^∗L^∗u0)u0^∗u0Tu0P_{r−1}(u0^∗Ru0))u0h_1, u0h_2 \rangle = \langle P_{r−1}(P_{s−1}(u0^∗L^∗u0))TP_{r−1}(u0^∗Ru0))h_1, h_2 \rangle. \]
By the first assertion in Proposition 4.1, the functions \( t → P_{s−1}(u0^∗L^∗u0) \) and \( t → P_{r−1}(u0^∗Ru0) \) are strongly continuous. Consequently, the function \( t → P_{s−1}(u0^∗L^∗u0)TP_{r−1}(u0^∗Ru0) \) is weakly continuous. Therefore, applying the second assertion in Proposition 4.1, we see that
\[ \langle P_{r−1}(P_{s−1}(u0^∗L^∗u0))TP_{r−1}(u0^∗Ru0))h_1, h_2 \rangle \to \langle u0Tu0^∗αr(R)u0h_1, αs(L)u0h_2 \rangle, \]
completing the proof. ■

Let \( R_0 := \{ R ∈ R \mid \lim_{t→0^+} αr(R) = R \) in the strong operator topology\}. Then, since \( \{αℓ\}_{t≥0} \) is a semigroup of \(*\)-endomorphisms of \( R \), \( R_0 \) is easily seen to be a \(*\)-subalgebra of \( R \). In fact, since \(|αℓ(R)k − Rk| ≤ |αℓ(S)k − Sk| + |αℓ(R − S)k − (R − S)k| ≤ 2∥R − S∥∥k∥ + |αℓ(S)k − Sk| \), we see that any \( R \) in the norm closure of \( R_0 \) already is in \( R_0 \). Thus, \( R_0 \) is a \( C^*\)-algebra. This \( C^*\)-algebra contains \( u_0Mu_0^* \) by the preceding proposition. But also, since each \( α_ℓ \) is a normal \(*\)-endomorphism of \( R \), we see that \( R_0 \) contains \( α_ℓ(u_0Mu_0^∗) \) for all \( r ≥ 0 \). Indeed, the proposition shows that for each \( T ∈ M \), \( α_ℓ(u_0Tu0^∗) → u0Tu0^∗ \) strongly as \( t → 0^+ \). Therefore, since \( α_ℓ \) is normal, \( α_ℓ(u_0Tu0^∗) \) → \( α_ℓ(u_0Tu0^∗) \) strongly as \( t → 0^+ \). Since \( α_ℓ(α_ℓ(u_0Tu0^∗)) = α_ℓ(α_ℓ(u_0Tu0^∗)) \), we see that \( α_ℓ(α_ℓ(u_0Tu0^∗)) \) → \( α_ℓ(u_0Tu0^∗) \) strongly as \( t → 0^+ \). Thus, \( R_0 \) is weakly dense in \( R \) by Corollary 4.4. We are therefore well on our way to showing that \( R_0 = R \). For this, we need the following lemma.
Lemma 4.8 The \( \alpha_t \), for strictly positive \( t \), are jointly faithful on \( \mathcal{R} \), i.e.,
\[
\bigcap \{ \ker(\alpha_t) \mid t > 0 \} = \{0\}.
\]

Proof. The kernel of each \( \alpha_t \) is a 2-sided, \( \sigma \)-weakly closed ideal in \( \mathcal{R} \). Thus so is \( \bigcap \{ \ker(\alpha_t) \mid t > 0 \} \). Hence, we may write \( \bigcap \{ \ker(\alpha_t) \mid t > 0 \} = q\mathcal{R} \) for some central projection \( q \) in \( \mathcal{R} \). Since for \( R \in \mathcal{A} \) we have \( \alpha_t(R) \to R \) strongly as \( t \to 0+ \), we conclude that \( \mathcal{A} \cap q\mathcal{R} = \{0\} \). Since \( \mathcal{A} \) generates \( \mathcal{R} \) as a von Neumann algebra by Corollary 4.6, we conclude that \( q = 0 \). □

We have arrived at the main theorem of the paper.

Theorem 4.9 Let \( (\mathcal{M}, \left\{ P_t \right\}_{t \geq 0}) \) be a quantum Markov process and assume that \( \mathcal{M} \) acts on a separable Hilbert space \( H \). Then the discrete dilation \( (K, \mathcal{R}, \left\{ \alpha_t \right\}_{t \geq 0}, u_0) \) constructed from \( \left\{ P_t \right\}_{t \geq 0} \) in Theorem 3.1 is an \( E_0 \)-dilation; i.e., \( \left\{ \alpha_t \right\}_{t \geq 0} \) is weakly continuous.

Proof. By Proposition 4.3 \( K \) is separable and so the predual of \( \mathcal{R}, \mathcal{R}_* \), is separable as a Banach space. We will write the pairing between \( \mathcal{R} \) and \( \mathcal{R}_* \) as \( \langle \cdot, \cdot \rangle \), i.e., \( \langle \omega, R \rangle = \omega(R), \omega \in \mathcal{R}_*, R \in \mathcal{R} \), as we did in Proposition 4.1. However, here we write \( \Psi_t \) for the pre-adjoint of \( \alpha_t \), i.e., \( \Psi_t(\omega) = \omega \circ \alpha_t \) for all \( \omega \in \mathcal{R}_* \); so \( \langle \Psi_t(\omega), R \rangle = \langle \omega, \alpha_t(R) \rangle \). Since the \( \sigma \)-weak topology on \( \mathcal{B}(K) \) agrees with the weak operator topology on bounded subsets, we see from the discussion following Proposition 4.7 that for all \( \omega \in \mathcal{R}_* \) and all \( R \in \mathcal{R}_0 \) the function \( t \to \langle \omega, \alpha_t(R) \rangle \) is continuous. However, if \( R \in \mathcal{R} \) we may find a sequence \( \left\{ R_n \right\}_{n=1}^{\infty} \) in \( \mathcal{R}_0 \) that converges weakly to \( R \). Consequently, the function \( t \to \langle \omega, \alpha_t(R) \rangle \) is the pointwise limit of the sequence of continuous functions \( t \to \langle \omega, \alpha_t(R) \rangle \). Therefore \( t \to \langle \omega, \alpha_t(R) \rangle \) is measurable for each \( \omega \in \mathcal{R}_* \) and each \( R \in \mathcal{R} \). That is, the semigroup of linear maps \( \left\{ \Psi_t \right\}_{t \geq 0} \) on \( \mathcal{R}_* \) is weakly measurable with respect to the duality between \( \mathcal{R}_* \) and \( \mathcal{R} \) (See [11] Definition 3.5.4.). Since \( \mathcal{R}_* \) is separable, Theorem 3.5.3 of [11] implies that \( t \to \langle \Psi_t(\omega) \rangle \) is strongly measurable as an \( \mathcal{R}_* \)-valued function. Thus, in the terminology of [11] Chapter 10], \( \left\{ \Psi_t \right\}_{t \geq 0} \) is a strongly measurable semigroup of linear maps on \( \mathcal{R}_* \). But then, Theorem 10.2.3 of [11] shows that at least for \( t \) strictly larger than zero, the function \( t \to \Psi_t \) is strongly continuous; i.e., for each \( \omega \in \mathcal{R}_* \), the \( \mathcal{R}_* \)-valued function on \( (0, \infty) \), \( t \to \Psi_t(\omega) \), is continuous with respect to the norm topology on \( \mathcal{R}_* \). To extend the continuity to all of \( [0, \infty) \), let \( \overline{\mathcal{R}}_* \) be the closed linear span \( \bigvee \{ \Psi_t(\mathcal{R}_*) \mid t > 0 \} \). If \( \overline{\mathcal{R}}_* \) is not all of \( \mathcal{R}_* \), then there is an \( R \in \mathcal{R} \) such that \( \langle \omega, R \rangle = 0 \) for all \( \omega \in \overline{\mathcal{R}}_* \). This means that for all \( t > 0 \) and all \( \omega \in \mathcal{R}_* \), \( \langle \omega, \alpha_t(R) \rangle = \langle \Psi_t(\omega), R \rangle = 0 \). Thus, \( R \) is in the kernels of all the \( \alpha_t \). However, Lemma 4.3 implies that \( R = 0 \). Thus, \( \mathcal{R}_* \) is all of \( \mathcal{R}_* \). Now we can appeal to Theorem 10.5.5 of [11] to conclude that \( \lim_{t \to 0^+} \| \Psi_t(\omega) - \omega \| = 0 \). Consequently, for all \( \omega \in \mathcal{R}_* \) and \( R \in \mathcal{R} \) we see that \( \langle \omega, \alpha_t(R) \rangle = \langle \Psi_t(\omega), R \rangle \to \langle \omega, R \rangle \) as \( t \to 0^+ \), which is what we wanted to prove. □

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