THE MOTIVIC THOM-SEBASTIANI THEOREM FOR REGULAR AND FORMAL FUNCTIONS

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ABSTRACT. Thanks to Hrushovski-Loeser’s work on motivic Milnor fibers, we give a model-theoretic proof for the motivic Thom-Sebastiani theorem in the case of regular functions. Moreover, slightly extending of Hrushovski-Loeser’s construction adjusted to Sebag, Loeser and Nicaise’s motivic integration for formal schemes and rigid varieties, we formulate and prove an analogous result for formal functions. The latter is meaningful as it has been a crucial element of constructing Kontsevich-Soibelman’s theory of motivic Donaldson-Thomas invariants.

1. INTRODUCTION

Let $f$ and $g$ be holomorphic functions on complex manifolds of dimensions $d_1$ and $d_2$, having isolated singularities at $x$ and $y$, respectively. Define $f \oplus g$ by $f \oplus g(x, y) = f(x) + g(y)$. Let $F_{f,x}$ be the (topological) Milnor fiber of $(f, x)$, the same for $(g, y)$ and $(f \oplus g, (x, y))$. The original Thom-Sebastiani theorem [24] states that there exists an isomorphism between the cohomology groups

$$H^{d_1+d_2-1}(F_{f \oplus g, (x,y)}, \mathbb{Q}) \cong H^{d_1-1}(F_{f, x}, \mathbb{Q}) \otimes H^{d_2-1}(F_{g, y}, \mathbb{Q})$$

compatible with the monodromies. Steenbrink in [25] refined a conjecture on the Thom-Sebastiani theorem for the mixed Hodge structures, which was fulfilled later and independently by Varchenko [27] and Saito [22]. In the letters to A’Campo (1972) and to Illusie (1999), Pierre Deligne discussed the $\ell$-adic version for an arbitrary field (rather than complex numbers), in which he replaced the Milnor fibers by the nearby cycles and used Laumon’s construction of convolution product (cf. [16, Définition 2.7.2]); this work recently has been fully realized by Fu [8]. Furthermore, Denef-Loeser [5] and Looijenga [19] also provided proofs of the motivic version for motivic vanishing cycles in the case of fields of characteristic zero, from which the classical results were recovered without the hypothesis that $x$ and $y$ are isolated singularities.

We come back to the problem on the motivic Thom-Sebastiani theorem in the framework for the motivic Milnor fibers of formal functions. It has been likely a formally unsolved problem, but already used in Kontsevich-Soibelman’s theory of motivic Donaldson-Thomas invariants for non-commutative Calabi-Yau threefolds (see [15]). Using Temkin’s results on resolution of singularities of an excellent formal scheme [26] and Denef-Loeser’s formulas for the motivic Milnor fiber of a regular function [3, 6], Kontsevich and Soibelman introduce in [15] the motivic Milnor fiber of a formal function. The motivic Thom-Sebastiani theorem for formal functions

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that concerns this notion is a key to construct the motivic Donaldson-Thomas invariants. In fact, it has the same interpretation as Denef-Loeser’s and Looijenga’s local version (cf. [5], [19]) and a complete proof for it should be required. This is the main purpose of the present article.

The motivic Milnor fiber of a regular function may be described in terms of resolution of singularity, after the works of Denef-Loeser [3, 6, 7] and of Guibert-Loeser-Merle [10, 11, 12]. In particular, Guibert-Loeser-Merle had the refinement when applying this method to further extensions of the motivic Thom-Sebastiani theorem (see [10], [11], [12]). Recently, with the help of Hrushovski-Kazhdan’s motivic integration, Hrushovski and Loeser [14] even give a more flexible manner to describe the motivic Milnor fiber in terms of the data of the corresponding analytic Milnor fiber (introduced by Nicaise-Sebag [20]). An important application of this approach is our proof of the integral identity conjecture in [17]. Also in [17], a slight generalization of Hrushovski-Loeser’s construction [14] combined with Nicaise’s formula on volume Poincaré series [21] allows to interpret in the same way as in [14] the motivic Milnor fiber of a formal function. However, this method requires the restriction to studying over algebraically closed fields of characteristic zero (hence the hypothesis in the present work).

Our article is organized as follows. In Section 2, we recall some basic and essential backgrounds on the motivic Milnor fiber of a regular function, in which the local form of Denef-Loeser and Looijenga’s motivic Thom-Sebastiani theorem is included (Theorem 2.1), using the main references [3, 4, 5, 6, 7] and [19]. The local form states that

\[
S^\phi f_{x,y} = S^\phi f_x \ast S^\phi g_y,
\]

where \( S^\phi f_x \) is the motivic Milnor fiber of \( (f, x) \), \( S^\phi f_x := (-1)^{d_1-1}(S^\phi f_x - 1) \), the same for \( g_y \) and \( (f \oplus g, (x, y)) \), and \( \ast \) is the convolution product (cf. Subsection 2.3). Here, one does not need to assume that \( x \) and \( y \) are isolated singular points.

Using the tools from [13] and [14], recalled partly here in Section 4, we introduce a new proof for this formula in Section 5. Notice that the previous formula lives in the monodromic Grothendieck ring \( \hat{M}_k^\mu \), by a technical reason, however, our proof only runs in a localization of \( \hat{M}_k^\mu \).

In Section 3 we mark the highlights and the essences of motivic integration for special formal schemes, following [23], [18], [20], [21] and [17]. In particular, by [17], we show that Kontsevich-Soibelman’s motivic Milnor fiber of a formal function and Nicaise’s volume Poincaré series mention on the same thing and this can be also read off from the corresponding analytic Milnor fiber. Furthermore, we can use the model-theoretic tools recalled in Section 4 to describe the volume Poincaré series, hence the motivic Milnor fiber of a formal function. The formal version of the motivic Thom-Sebastiani theorem has the same form as the regular one but \( f \) and \( g \) replaced by formal functions \( \tilde{f} \) and \( \tilde{g} \), respectively (Theorem 3.3). It is proven in Section 4 using the development of tools in Section 3 as well as some analogous techniques in the proof of the regular version in Section 5.

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2. Preliminaries

Throughout the present article, we always assume that $k$ is an algebraically
closed field of characteristic zero.

2.1. Grothendieck rings of algebraic varieties. By definition, an algebraic $k$-
variety is a separated reduced $k$-scheme of finite type. Let $\text{Var}_k$ be the
category of algebraic $k$-varieties, its morphisms are morphisms of algebraic $k$-varieties. The
Grothendieck group $K_0(\text{Var}_k)$ is an abelian group generated by symbols $[X]$ for
objects $X$ in $\text{Var}_k$ subject to the relations $[X] = [Y]$ if $X$ and $Y$ are isomorphic in
$\text{Var}_k$, $[X] = [\emptyset] + [X \setminus Y]$ if $Y$ is Zariski closed in $X$. Moreover, $K_0(\text{Var}_k)$ is also a ring
with unit with respect to the cartesian product. Set $L := [\mathbb{A}_k^1]$ and denote by $M_k$
the localization of $K_0(\text{Var}_k)$ with respect to the multiplicative system \{L$^i$ | $i \in \mathbb{N}$\}.

Let $\mu_m$ (or $\mu_m(k)$) be the group scheme of $m$th roots of unity in $k$. Varying
$m \geq 1$ in $\mathbb{N}$, such schemes give rise to a projective system with respect to morphisms
$\mu_m \rightarrow \mu_n$ given by $\xi \mapsto \xi^n$, and its limit will be denoted by $\hat{\mu}$. A good $\mu_m$-action
on an object $X$ of $\text{Var}_k$ is a group action of $\mu_m$ on $X$ such that each orbit is contained in
an affine $k$-subvariety of $X$. A good $\hat{\mu}$-action on $X$ is a $\hat{\mu}$-action which factors
through a good $\mu_m$-action for some $m \geq 1$ in $\mathbb{N}$.

The $\hat{\mu}$-equivariant Grothendieck group $K^\hat{\mu}_0(\text{Var}_k)$ is an abelian group generated
by the iso-equivariant classes of varieties $[X, \sigma]$, with $X$ an algebraic $k$-variety, $\sigma$ a
good $\hat{\mu}$-action on $X$, modulo the conditions $[X, \sigma] = [Y, \sigma|_Y] + [X \setminus Y, \sigma|_{X \setminus Y}]$ if $Y$ is
Zariski closed in $X$ and $[X \times \mathbb{A}_k^n, \sigma] = [X \times \mathbb{A}_k^n, \sigma']$ if $\sigma, \sigma'$ lift the same $\hat{\mu}$-action on
$X$ to an affine action on $X \times \mathbb{A}_k^n$. In the present article we shall write $[X, \sigma]$ simply
by $[X]$, when the $\hat{\mu}$-action $\sigma$ is clear. Similarly as previous, $K^\hat{\mu}_0(\text{Var}_k)$ has a natural
ring structure due to the cartesian product. Let $M_k^\hat{\mu}$ denote $K^\hat{\mu}_0(\text{Var}_k)[L^{-1}]$, it is
the $\hat{\mu}$-equivariant version of $M_k$ above. Let $M_k^\hat{\mu, \text{loc}}$ be the localization of $M_k^\hat{\mu}$
with respect to the multiplicative family generated by the elements $1 - L^i$, with $i \geq 1$ in
$\mathbb{N}$. We shall also write $\text{loc}$ for the localization morphism $M_k^\hat{\mu} \rightarrow M_k^\hat{\mu, \text{loc}}$.

2.2. Motivic Milnor fiber. Let $X$ be a pure $d$-dimensional smooth $k$-variety, $f$
a non-constant regular function on $X$, and $x$ a closed point in the zero locus of $f$. Denote by $X_{x, m}$ (or $X_{x, m}(f)$) the set of arcs $\varphi(t)$ in $X(k[t]/(t^{m+1}))$ originated at
$x$ with $f(\varphi(t)) \equiv t^m \mod t^{m+1}$, which is a locally closed subvariety of $k$-variety
$X(k[t]/(t^{m+1}))$. Since $X_{x, m}$ is invariant by the $\hat{\mu}$-action on $X(k[t]/(t^{m+1}))$ given by
$\xi \cdot \varphi(t) = \varphi(\xi t)$, it defines an $\hat{\mu}$-equivariant class $[X_{x, m}]$ in $M_k^\hat{\mu}$. The motivic zeta
function of $f$ at $x$ is the formal series

$$Z_{f,x}(T) = \sum_{m \geq 1} [X_{x, m}]L^{-md}T^m$$

with coefficients in $M_k^\hat{\mu}$. By Denef-Loeser [8], $Z_{f,x}(T)$ is a rational function, i.e., a
$M_k^\hat{\mu}$-linear combination of 1 and products finite (possibly empty) of $L^aT^b/(1-L^aT^b)$
with $(a, b)$ in $\mathbb{Z} \times \mathbb{N}_{>0}$. Remark that we can take by [5] the limit $\lim_{T \rightarrow \infty}$ for rational
functions such that $\lim_{T \rightarrow \infty} \left( L^aT^b/(1 - L^aT^b) \right) = -1$. Then the motivic Milnor
fiber of $f$ at $x$ is defined as $-\lim_{T \to \infty} Z_{f, x}(T)$ and denoted by $S_{f,x}$. This is a virtual variety in $M_k^\phi$.

2.3. **The motivic Thom-Sebastiani theorem for regular functions.** In this subsection, we restate the motivic Thom-Sebastiani theorem for motivic Milnor fibers.

Let us recall the concept of convolution product from [5], [19] and [10]. Consider the Fermat varieties $F_0$ and $F_1$ in $\mathbb{G}_m^2$ defined by the equations $u^m + v^m = 0$ and $u^m + v^m = 1$, respectively. We endow with the standard $(\mu_m \times \mu_m)$-action on these varieties. If $X$ and $Y$ are algebraic $k$-varieties with $\mu_m$-action, one defines

$$[X] * [Y] = -[F_1^m \times (X \times Y)] + [F_0^m \times (X \times Y)]$$

where, for $i \in \{0, 1\}$,

$$F_i^m \times (X \times Y) = F_i^m \times (X \times Y)/\sim$$

with $(au, bv, x, y) \sim (u, v, ax, by)$ for any $a, b$ in $\mu_m$. The group scheme $\mu_m$ acts diagonally on $F_i^m \times (X \times Y)$. Passing to the projective limit that $M_k^\phi$ equals

$$\lim_{\xi} M_k^\xi$$

we get the convolution product $*$ on $M_k^\phi$. This product is commutative and associative (see for example [10]).

Let $f$ and $g$ be regular functions on smooth algebraic $k$-varieties $X$ and $Y$, respectively. Define $f \oplus g (x, y) = f(x) + g(y)$. For closed points $x$ in $X_0$ and $y$ in $Y_0$, we set

$$S_{f,x}^\phi = (-1)^{\dim X - 1}(S_{f,x} - 1), \quad S_{g,y}^\phi = (-1)^{\dim Y - 1}(S_{g,y} - 1).$$

**Theorem 2.1** ([5], [19]). The identity $S_{f \oplus g,(x,y)}^\phi = S_{f,x}^\phi * S_{g,y}^\phi$ holds in $M_k^\phi$.

**Remark 2.2.** In fact, in [5] and [19], one proved the motivic Thom-Sebastiani theorem in the framework of motivic vanishing cycles, which implies Theorem 2.1.

3. **The motivic Thom-Sebastiani formula for formal functions**

Let $X$ be a generically smooth special formal $k[[t]]$-scheme of relative dimension $d$, with reduction $X_0$ and structural morphism $\mathfrak{f}$. Let $x$ be a closed point of $X_0$.

3.1. **The motivic Milnor fiber of a formal function.** By [26] (see also [21]), there exists a resolution of singularities $\mathfrak{h} : \mathfrak{X} \to X$ of $X_0$. Let $\mathfrak{E}_i$, $i \in I$, be the irreducible components of $\mathfrak{X}$, and $E_i$ be the multiplicity of $\mathfrak{E}_i$ in $\mathfrak{X}$. We set $E_i = (\mathfrak{E}_i)_0$ for $i \in J$, $E_I = \bigcup_{i \in I} E_i$ and $E_i^0 = E_i \setminus \bigcup_{i \notin I} E_j$ for a nonempty subset $I$ of $J$. Let $\{U\}$ be a covering of $\mathfrak{X}$ by affine open subschemes with $U \cap E_i^0 \neq \emptyset$ such that, on this piece, $\mathfrak{f} \circ \mathfrak{h} = \mathfrak{u} \prod_{i \in I} y_i^{N_i}$, where $\mathfrak{u}$ is a unit, $y_i$ is a local coordinate defining $E_i$. Set $m_I := \gcd(N_i)_{i \in I}$. One can construct as in [27] an unramified Galois covering $\pi_I : \tilde{E}_i^0 \to E_i^0$ with Galois group $\mu_{m_I}$, which is given over $U \cap E_i^0$ by

$$\{(z, y) \in A_k^1 \times (U \cap E_i^0) : z^{m_I} = \mathfrak{u}(y)^{-1}\}.$$

$\tilde{E}_i^0$ is endowed with a natural $\mu_{m_I}$-action good over $E_i^0$ obtained by multiplying the $z$-coordinate with elements of $\mu_{m_I}$. We also restrict this covering over $E_i^0 \cap \mathfrak{h}^{-1}(x)$
and obtain a class, written as $[\tilde{E}_f^\ell \cap h^{-1}(x)]$, in $\mathcal{M}_k^\ell$. The **motivic Milnor fiber of the formal germ** $(X, x)$, or of $f$ at $x$, is defined to be the quantity

$$\sum_{\varnothing \neq I \subset J} (1 - \mathbb{L})|I|-1 [\tilde{E}_f^\ell \cap h^{-1}(x)]$$

in $\mathcal{M}_k^\ell$. We denote it by $S(\mathcal{X}, x)$ or by $S_{f, x}$. By [17, Lemma 5.7], using volume Poincaré series, $S_{f, x}$ is well defined, i.e., independent of the choice of the resolution of singularities $h$.

**Remark 3.1.** Let $\tilde{\mathcal{X}}_x$ denote the formal completion of $\mathcal{X}$ at $x$, and let $f_x$ be the structural morphism of $\tilde{\mathcal{X}}_x$, which is induced by $f$. We are able to use a resolution of singularity of $\mathcal{X}$ at $x$ to define the motivic Milnor fiber $S_{f, x}$. Then, it is clear that $S_{f, x} = S_{f_x, x}$.

### 3.2. Integral of a gauge form and volume Poincaré series.

#### 3.2.1. Stft formal schemes.

Assume that $\mathcal{X}$ is a separated generically smooth formal $k[[t]]$-scheme topologically of finite type and that the relative dimension of $\mathcal{X}$ is $d$.

One may regard $\mathcal{X}$ as the inductive limit of the $k[[t]]/(t^{m+1})$-schemes topologically of finite type $\mathcal{X}_m = (\mathcal{X}, \mathcal{O}_{\mathcal{X}} \otimes k[[t]]/t^{m+1})$ in the category of formal $k[[t]]$-schemes.

By Greenberg [9], there exists a unique $k$-scheme $\text{Gr}_m(\mathcal{X}_m)$ topologically of finite type, which is a separated generically smooth formal $S$-scheme $\text{Gr}_m(\mathcal{X}_m)$ topologically of finite type. We denote by $\pi_m$ the canonical projection $\text{Gr}(\mathcal{X}) \to \text{Gr}_m(\mathcal{X}_m)$. See more in [9] for some basic properties of the functor $\text{Gr}$.

By [23, 18], the motivic measure of a stable cylinder $A$ in $\text{Gr}(\mathcal{X})$ is the following

$$\mu(A) = |\pi_\ell(A)|\mathbb{L}^{-(d+1)}$$

for $\ell \in \mathbb{N}$ large enough. Let $\alpha : A \to \mathbb{Z} \cup \{\infty\}$ be a function on $A$ that takes only a finite number of values such that every fiber $\alpha^{-1}(m)$ is a stable cylinder in $\text{Gr}(\mathcal{X})$. Let $\omega$ be a gauge form on $\mathcal{X}_n$. By [2, Proposition 1.5] (see also [18]), there exists a canonical isomorphism $\Omega^d_{\mathcal{X}_n}(\mathcal{X}_n) \cong \Omega^d_{\mathcal{X}_n}(\mathcal{X}) \otimes \mathbb{K}(\mathcal{X})/k[[t]]$, thus there exist an $n$ in $\mathbb{N}$ and a differential form $\tilde{\omega}$ in $\Omega^d_{\mathcal{X}_n}(\mathcal{X})$ such that $\omega = t^{-n} \tilde{\omega}$. Let $\varphi$ be a point of $\text{Gr}(\mathcal{X})$ outside $\text{Gr}(\mathcal{X}_{\text{sing}})$. Then, we can regard it as a morphism of formal schemes $\text{Spf}(k[[t]]) \to \mathcal{X}$, or as a morphism of rings $\mathcal{O}_\mathcal{X}(\mathcal{X}) \to k[[t]]$. Thus it induces a morphism of rings $\bar{\varphi} : \varphi^* \Omega^d_{\mathcal{X}_n}(\mathcal{X}) \to k[[t]]$, which is a surjection. One defines

$$\text{ord}_\mathcal{X}(\varphi) = \text{ord}_\mathcal{X}(\bar{\varphi}(\varphi^* \omega)) \text{ and } \text{ord}_\mathcal{X}(\omega) = \text{ord}(\tilde{\omega}) - n.$$

The latter is independent of the choice of $\bar{\omega}$ (cf. [18]). Since $\omega$ is a gauge form, it follows from [18] Proof of 4.1.2 that $\text{ord}_\mathcal{X}(\omega)$ is an integer-valued function taking only a finite number of values and that its fibers are stable cylinder. Then one defines (cf. [23, 18])

$$\int_{\mathcal{X}_n} |\omega| := \sum_{m \in \mathbb{Z}} \mu(\{\varphi \in \text{Gr}(\mathcal{X}) \mid \text{ord}_\mathcal{X}(\omega)(\varphi) = m\}) \mathbb{L}^{-m} \in \mathcal{M}_k.$$
3.2.2. Special formal schemes. We consider the more general case where $\mathfrak{X}$ is a generically smooth special formal $k[[t]]$-schemes (see [11] for definition). Let $\mathfrak{Y} \to \mathfrak{X}$ be a Néron smoothing for $\mathfrak{X}$, i.e. a morphism of special formal $k[[t]]$-schemes, $\mathfrak{Y}$ adic smooth over $k[[t]]$, inducing an open embedding $\mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ with $\mathfrak{Y}_\eta \otimes_k k((t)) \to \mathfrak{X}_\eta \otimes_k k((t))$, $K$ for any finite unramified extension $K$ of $k((t))$. It exists by [21], furthermore, we are able to (and we shall from now on) choose $\mathfrak{Y}$ to be separated generically smooth formal $k[[t]]$-scheme topologically of finite type. Using [21 Propositions 4.7, 4.8], for any gauge form $\omega$ on $\mathfrak{X}_\eta$, we define

$$\int_{\mathfrak{X}_\eta} |\omega| := \int_{\mathfrak{Y}_\eta} |\omega| \in M_k.$$

For any $m$ in $\mathbb{N}_{>0}$, let $\mathfrak{X}(m) := \mathfrak{X} \otimes_{k[[t]]} k[[t^1/m]]$, $\mathfrak{X}_\eta(m) := \mathfrak{X}_\eta \otimes_{k((t))} k((t^{1/m}))$ and $\omega(m)$ the pullback of $\omega$ via the natural morphism $\mathfrak{X}_\eta(m) \to \mathfrak{X}_\eta$. The Néron smoothing $\mathfrak{Y} \to \mathfrak{X}$ for $\mathfrak{X}$ induces a Néron smoothing $\mathfrak{Y}(m) \to \mathfrak{X}(m)$ on $\mathfrak{Y}(m)$ and $\mathfrak{Y}(m)$ is also topologically of finite type, like $\mathfrak{Y}$. The canonical $\mu$-action on $Gr(\mathfrak{Y}(m))$ is given by $a \varphi(t^{1/m}) = \varphi(at^{1/m})$. It induces a $\mu_m$-action on $\int_{\mathfrak{X}_\eta} |\omega|$, thus we regard $\int_{\mathfrak{X}_\eta} |\omega|$ as an element of $M^\mu_k$.

3.2.3. Volume Poincaré series. Let $\mathfrak{X}$ be a generically smooth special formal $k[[t]]$-schemes, $x$ a closed point of $\mathfrak{X}_0$ and $\hat{\mathfrak{X}}_x$ the formal completion of $\mathfrak{X}$ at $x$. Denoting by $|x|$ the tube of $x$, namely the analytic Milnor fiber of $f$ at $x$ (cf. [20]), we have the canonical isomorphism $|x| \cong (\hat{\mathfrak{X}}_x)_\eta$. Set $|x|_m := x \times_{k((t))} k((t^{1/m}))$. Let us consider the volume Poincaré series of $(|x|, \omega)$, where $\omega$ is a gauge form on $|x|$, (cf. [21])

$$S(|x|, \omega; T) := \sum_{m \geq 1} \left( \int_{|x|_m} |\omega(m)| \right) T^m \in M^\mu_k[[T]].$$

Remark 3.2. More generally, the volume Poincaré series of separated generically smooth formal schemes topologically of finite type (resp. separated quasi-compact smooth rigid varieties) were introduced and studied first by Nicaise-Sebag in [20]. After that, Nicaise [21] studied these objects in the framework of generically smooth special formal schemes (resp. bounded smooth rigid varieties).

In practice, one may assume $\omega$ is $\hat{\mathfrak{X}}_x$-bounded, i.e., $\omega$ lies in the image of the natural map $\Omega^d_{\hat{\mathfrak{X}}_x} \to \Omega^d_{|x|}(\mathfrak{X}_x)$ (cf. [21 Definition 2.11]).

Since $k$ is an algebraically closed field, $S(|x|, \omega; T)$ is independent of the choice of the uniformizing parameter $t$. Indeed, let $t'$ be another uniformizing parameter for $k[[t]]$. Then $t' = at$, where $a = \alpha(t) \in k[[t]]$ and $\alpha(0) \in k^\times$. Since $k$ contains all roots, the $m$th roots of $\alpha$ are again in $k[[t]]$. This induces a canonical isomorphism of $\mathfrak{X}(t^{1/m}) \to \mathfrak{X}(t^{1/m})$, that implies the previous claim. By Nicaise [21 Corollary 7.13], if the gauge form $\omega$ is $\hat{\mathfrak{X}}_x$-bounded, this series $S(|x|, \omega; T)$ is a rational function.

Proposition 3.3. With the notation and the hypotheses as previous, the following identity holds in $M^\mu_k$.

$$S_{t, x} = -L^d \lim_{T \to \infty} \sum_{m \geq 1} \left( \int_{|x|_m} |\omega(m)| \right) T^m.$$
4.2. Measured categories

Proof. The identity is true in $M_k$ because of the definition of $S_0$ as well as Nicaise’s formula for $-\lim_{T \to \infty} S(\mathbb{K}[\omega; T])$ in [21 Proposition 7.36]. To see that it is true in $M_k^\mu$, we refer to the proof of Lemma 5.7 in [17]. \hfill \square

3.3. Statement of result for formal functions. Given integers $d_1 \geq 1$ and $d_2 \geq 1$. Let $f$ be a formal power series in $k[[x]]$ with $f(0) = 0$ and $g$ in $k[[y]]$ with $g(0) = 0$. Here $x = (x_1, \ldots, x_{d_1})$, $y = (y_1, \ldots, y_{d_2})$ and we write by the same symbol $0$ for the origin of $\mathbb{A}_k^{d_1}$, $\mathbb{A}_k^{d_2}$ or $\mathbb{A}_k^1$ (whenever necessary, e.g., in Section 6 however, we shall write $0_{d_i}$ for the origin of $\mathbb{A}_k^{d_i}$, $i \in \{1, 2\}$). Let us consider the following special formal $k[[t]]$-schemes

$$\mathcal{X} : = \text{Spf}(k[[t, x]]/(f(x) - t)),$$
$$\mathcal{Y} : = \text{Spf}(k[[t, y]]/(g(y) - t)),$$
$$\mathcal{X} \oplus \mathcal{Y} : = \text{Spf}(k[[t, x, y]]/(f(x) + g(y) - t)),$$

with structural morphisms $f$, $g$ and $f \oplus g$ induced by $f$, $g$ and $f \oplus g$, respectively. Set $S^0_{f, 0} : = (-1)^{d_1 - 1}(S_{f, 0} - 1)$ and the same for $g$ and $f \oplus g$. We now set up the statement of the motivic Thom-Sebastiani theorem for formal schemes and then prove it in the setting of $M_k^\mu$, using Hrushovski-Kahdan’s integration [13] via the work of Hrushovski and Loeser [14].

Theorem 3.4. The identity $S^0_{f \oplus g, (0, 0)} = S^0_{f, 0} \ast S^0_{g, 0}$ holds in $M_k^\mu_{k, \text{loc}}$.

The complete proof is given in Section 3.

4. Extension of Hrushovski-Loeser’s morphism

4.1. The theory $\text{ACVF}_{k((t))}(0, 0)$. We consider the theory $\text{ACVF}_{k((t))}(0, 0)$ of algebraically closed valued fields of equal characteristic zero that extend $k((t))$ (cf. [13]). Its sort $\text{VF}$ admits the language of rings, while the sort $\text{RV}$ is endowed with abelian group operations $\cdot$, $/$, a unary predicate $k^\times$ for a subgroup, a binary operation $+$ on $k = k^\times \cup \{0\}$. We also have an imaginary sort $\Gamma$ that is with a uniquely divisible abelian group. For a model $L$ of this theory, let $R_L$ (resp. $\mathfrak{m}_L$) denote its valuation ring (resp. the maximal ideal of $R_L$). The following are the “elementary” $L$-definable sets of $\text{ACVF}_{k((t))}(0, 0)$:

$$\text{VF}(L) = L, \quad \text{RV}(L) = L^\times/(1 + \mathfrak{m}_L), \quad \Gamma(L) = L^\times/R_L^\times, \quad k(L) = R_L/\mathfrak{m}_L.$$

In general, a definable subset of $\text{VF}^n(L)$ is a finite Boolean combination of set of the forms $\text{val}(f_1) \leq \text{val}(f_2)$ or $f_3 = 0$, where $f_i$ are polynomials with coefficients in $k((t))$. The same definition may apply to definable subsets of $\text{RV}^n(L)$, $\Gamma^n(L)$ or $k^n(L)$. Correspondingly, there are natural maps between these sets $\text{rv} : \text{VF} \to \text{RV}$, $\text{val} : \text{VF} \to \Gamma$, $\text{val}_{\text{rv}} : \text{RV} \to \Gamma$ and $\text{res} : R_L \to k(L)$. There is an exact sequence of groups

$$1 \to k^\times \to \text{RV} \xrightarrow{\text{val}_{\text{rv}}} \Gamma \to 0.$$

4.2. Measured categories (following [13]).
4.2.1. $\nu$-categories. Let $\mu_\nu \nu[n]$ be the category of $k(t)$-definable sets (or definable sets, for short) endowed with definable volume forms, up to $\nu$-equivalence. One may show that it is graded via the following subcategories $\mu_\nu \nu[n]$, $n \in \mathbb{N}$. An object of $\mu_\nu \nu[n]$ is a triple $(X, f, \varepsilon)$ with $X$ a definable subset of $\nu^\ell \times \nu^\ell'$, for some $\ell, \ell' \in \mathbb{N}$, $f : X \to \nu^n$ a definable map with finite fibers and $\varepsilon : X \to \nu$ a definable function; a morphism from $(X, f, \varepsilon)$ to $(X', f', \varepsilon')$ is a definable essential bijection $F : X \to X'$ such that

$$\varepsilon = \varepsilon' \circ F + \text{val}(\text{Jac}F)$$

away from a proper closed subvariety of $X$. Here, that $F : X \to X'$ is an essential bijection means that there exists a proper closed subvariety $Y$ of $X$ such that $F|_{X \setminus Y} : X \setminus Y \to X' \setminus F(Y)$ is a bijection (see [13, Subsection 3.8]).

Let $\mu_\nu \nu[bdd][n]$ be the full subcategory of $\mu_\nu \nu[n]$ whose objects are bounded definable sets with bounded definable forms $\varepsilon$. If considering $\varepsilon : X \to \nu$ as the zero function, we get the categories $\nu \nu$ and $\nu \nu[bdd]$ as well as $\nu \nu[bdd][n]$. In this case, the measure preserving property of a morphism $F$ is characterized by the condition $\text{val}(\text{Jac}F) = 0$, outside a proper closed subvariety.

Convention. For simplicity, we shall omit the symbol $f$ in the triple $(X, f, \varepsilon)$ if no possibility of confusion appears.

4.2.2. $\rho$-categories. Similarly, we consider the category $\mu_\rho \rho[n]$ graded by $\mu_\rho \rho[n]$, $n \in \mathbb{N}$. By definition, an object of $\mu_\rho \rho[n]$ is a triple $(X, f, \varepsilon)$ with $X$ a definable subset of $\rho^\ell$, for some $\ell \in \mathbb{N}$, $f : X \to \rho^n$ a definable map with finite fibers, and $\varepsilon : X \to \nu$ a definable function; a morphism $(X, f, \varepsilon) \to (X', f', \varepsilon')$ is a definable bijection $F : X \to X'$ such that

$$\varepsilon = \varepsilon' \circ F + \sum_{i=1}^{n} \text{val}_\nu(f_i)$$

away from a proper closed subvariety (the measure preserving property). The category $\mu_\rho \rho[bdd][n]$ is defined as the full subcategory of $\mu_\rho \rho[n]$ such that, for each object $(X, f, \varepsilon)$, $\text{val}_\nu(X)$ is a finite set. The category $\mu_\rho \rho[bdd]$ is defined as $\mu_\rho \rho$ with $\text{val}_\nu$-image of objects bounded below. In the case where, for each object $(X, f, \varepsilon)$ of one of the previous categories, taking $\varepsilon$ being the zero function, we get the subcategories $\nu \rho$, $\nu \rho[bdd]$ and $\nu \rho[bdd][n]$.

In the present article, we also consider $\rho$, a category defined exactly as $\nu \rho$ but the measure preserving property is not required for morphisms.

4.2.3. $\gamma$-categories. The category $\mu_\gamma[n]$ consists of pairs $(\Delta, l)$ with $\Delta$ a definable subset of $\gamma^n$ and $l : \Delta \to \nu$ a definable map. A morphism $(\Delta, l) \to (\Delta', l')$ is a definable bijection $\lambda : \Delta \to \Delta'$ which is liftable to a definable bijection $\text{val}_\gamma^{-1}\Delta \to \text{val}_\gamma^{-1}\Delta'$ such that

$$|x| + l(x) = |\lambda(x)| + l'(\lambda(x)).$$

The category $\mu_\gamma[bdd][n]$ is the full subcategory of $\mu_\gamma[n]$ such that, for each object $(\Delta, l)$ of $\mu_\gamma[bdd][n]$, there exists a $\gamma \in \Gamma$ with $\Delta \subset [\gamma, \infty)^n$. By definition, the categories $\mu_\gamma$ and $\mu_\gamma[bdd]$ are the direct sums $\bigoplus_{\gamma \geq 1} \mu_\gamma[n]$ and $\bigoplus_{\gamma \geq 1} \mu_\gamma[bdd][n]$, respectively. The subcategories whose objects are of the form $(\Delta, 0)$ will be denoted by $\nu \gamma$ and $\nu \gamma[bdd]$. 

4.3. Structure of $K(\mu_1^\mathrm{VF}^\mathrm{bdd})$. Let $\mathcal{C}$ be one of the categories in Subsection 4.2. Then, as in [13], we denote the Grothendieck semiring of $\mathcal{C}$ by $K_+(\mathcal{C})$ and the associated ring by $K(\mathcal{C})$. By [13], there is a natural morphism of
\begin{equation}
N : K_+(\mu_1^\mathrm{Res}) \otimes K_+(\mu_1^\mathrm{VF}^\mathrm{bdd}) \to K_+(\mu_1^\mathrm{VF}^\mathrm{bdd})
\end{equation}
constructed as follows. Note that two objects admitting a morphism $\lambda$ in $\mu_1^\mathrm{Res}[n]$ define the same element in $K_+(\mu_1^\mathrm{Res}[n])$, hence $\lambda$ lifts to a morphism in $\mu_1^\mathrm{VF}^\mathrm{bdd}[n]$ between their pullbacks. Thus there exists a natural morphism $K_+(\mu_1^\mathrm{Res}[n]) \to K_+(\mu_1^\mathrm{VF}^\mathrm{bdd})$ mapping the class of $(\Delta, l)$ to the class of $(\text{val}^{-1}(\Delta), \text{loval})$. Also, for each object $(X, f, \varepsilon)$ in $\mu_1^\mathrm{Res}[n]$, we may consider an étale map $\ell : X \to k^n$. By this, we have the natural morphism $K_+(\mu_1^\mathrm{Res}[n]) \to K_+(\mu_1^\mathrm{VF}^\mathrm{bdd})$ by sending the class of $(X, f, \varepsilon)$ to the class of $(X \times_{\ell, \text{res}} R^n, \text{pr}_1 \circ \varepsilon)$. In particular, if $X$ is Zariski open in $k^n$, then $X \times_{\ell, \text{res}} R^n$ is simply $\text{res}^{-1}(X)$.

**Theorem 4.1** (Hrushovski-Kazhdan [13]). The morphism $N$ is a surjection. Moreover, it also induces a surjective morphism $N$ between the associated rings.

The description of $N$, or more precisely, $N^{-1}$ modulo $\ker(N)$, in [13] and [14], is slightly more explicit and more intrinsic. Indeed, one first constructs the natural morphism
\[ K_+(\mu_1^\mathrm{Res}) \otimes K_+(\mu_1^\mathrm{VF}^\mathrm{bdd}) \to K_+(\mu_1^\mathrm{VF}^\mathrm{bdd}) \]
due to the inclusion Res $\subset$ VF and the valuation map val$_v$ (cf. [13] or [14]). This morphism is a surjection, its kernel is generated by $1 \otimes [\text{val}_v^{-1}(\gamma)|_1 - [\gamma]|_1 \otimes 1$, with $\gamma$ definable in $\Gamma$. The subscript 1 means that the classes are in degree 1. Secondly, the canonical morphism
\[ K_+(\mu_1^\mathrm{Res})[n] \to K_+(\mu_1^\mathrm{VF}^\mathrm{bdd})[n] \]
induced by the map $\text{Ob}_\mu^\Gamma \to \text{Ob}_\mu^\Gamma$ sending $(X, f, \varepsilon)$ to $(LX, Lf, Le)$, where $LX = X \times_{f, \text{rv}} (VF^\times)^n$, $Lf(a, b) = f(a, \text{rv}(b))$ and $Le(a, b) = \varepsilon(a, \text{rv}(b))$.

**Remark 4.2.** According to [13] Proposition 10.10, an element of $K_+(\mu_1^\mathrm{Res})$ may be written as a finite sum of elements of the form $[(X \times \text{val}_v^{-1}(\Delta), f, \varepsilon)]$. Furthermore, an argument in the proof of [13] Proposition 10.10 implies a fact that $[(X \times \text{val}_v^{-1}(\Delta), f, \varepsilon)] = [(X, f_0, 1)] \otimes [(\Delta, l)]$, where $f_0 : X \to \text{RV}^n$, $l : \Delta \to \Gamma$ are some definable functions.

4.4. Extending Hrushovski-Loeser’s construction.

4.4.1. The morphisms $h_m$ and $\tilde{h}_m$. From now on, we shall denote by $!K(\text{Res})$ the quotient of $K(\text{Res})$ subject to the relations $[\text{val}_v^{-1}(a)] = [\text{val}_v^{-1}(0)]$ for $a$ in $\Gamma$, and by $!K(\text{Res})[L^{-1}]$ the localization of $!K(\text{Res})[L^{-1}]$ with respect to the multiplicative family generated by $1 - [\mathbb{A}^1]^i$, $i \geq 1$. Let $m, n$ be in $\mathbb{N}$, $m \geq 1$, $(\Delta, l)$ in $\mu_1^\mathrm{Res}[n]$ and $e$ in $\Gamma$ with $me \in \mathbb{Z}$. Set $\Delta(m) = \Delta \cap (1/m\mathbb{Z})^n$, $\Delta_{l, e} = l^{-1}(e)$ and $\alpha_m(\Delta, l) = \sum_{e \in \Gamma, me \in \mathbb{Z}} \sum_{\gamma \in \Delta_{l, e}(m)} L^{-m(\gamma|_+ e)}(L - 1)^n$

\[ \sum_{e \in \mathbb{Z}} \sum_{\gamma \in \Delta_{l, e}/(m)} L^{-m(\gamma|_e)}(L - 1)^n. \]

It is clear that $\alpha_m(\Delta, l)$ is an element of $!K(\text{Res})[L^{-1}]_{\text{loc}}$, and moreover, $\alpha_m$ is independent of the choice of coordinates for $\Gamma^n$. Indeed, let $\lambda$ be the morphism in
Lemma 4.3. \( h \) the morphism \( \alpha ! \) their restriction, namely, \( \alpha ! \) of \( [13, \text{Lemma 9.5}] \). Since \( \ker(\alpha !) \) is contained in \( \ker(\beta !) \), we are able to construct a morphism \( \beta_m : K(\mu \Gamma^{\text{bdd}}) \to !K(\text{RES})[L^{-1}]_{\text{loc}} \). By using \([14]\), for any \( \Delta \) in \( \text{vol} \Gamma^{\text{bdd}} \), one sets \( \tilde{\alpha}_m(\Delta) = \sum_{\gamma \in \Delta} L^{-m|\gamma|(L - 1)n} \) and obtains a morphism of rings

\[
\tilde{\alpha}_m : K(\text{vol} \Gamma^{\text{bdd}}) \to !K(\text{RES})[L^{-1}]_{\text{loc}}.
\]

Thus we can consider \( \alpha_m \) as an extension of \( \tilde{\alpha}_m \); moreover, we have

\[
(4.2) \quad \alpha_m(\Delta, l) = \sum_{e \in \mathbb{Z}} \tilde{\alpha}_m(\Delta_{l,e/m}) L^{-e}.
\]

We are able to construct a morphism \( \beta_m : K(\mu \Gamma \text{RES}) \to !K(\text{RES})[L^{-1}]_{\text{loc}} \) by using Hrushovski-Loeser’s method. Thanks to Remark \([12]\) however, it suffices to define value under \( \beta_m \) of elements of the form \( [(X, f, 1)] \) with \( (X, f, 1) \) an object in \( \mu \Gamma \text{RES} \). Assume that \( f(X) \subset V_{\gamma_1} \times \cdots \times V_{\gamma_m} \), i.e., \( \text{val}_rf(x) = \gamma_i \) for every \( x \in X \). We set \( \beta_m(X, f, 1) = [X]|L^{-1}[1]^m|^{\gamma_i} \) if \( m\gamma \in \mathbb{Z}^n \) and \( \beta_m(X, f, 1) = 0 \) otherwise.

There are two steps to check that \( \ker(\tilde{\alpha}_m \otimes \beta_m) \) is contained in \( \ker(N_0) \), where \( N_0 \) is \( N \) reduced to the volume version (for the structure of \( K(\text{vol} \Gamma^{\text{bdd}}) \)). These steps correspond to the factorization of \( N_0 \) into \( K(\text{vol} \Gamma^{\text{bdd}}) \otimes K(\text{vol} \text{RES}) \to K(\text{vol} \Gamma^{\text{bdd}}) \) and \( K(\text{vol} \Gamma^{\text{bdd}}[n]) \to K(\text{vol} \Gamma^{\text{bdd}}[n]) \otimes [1]_{1, \text{res}^{1/1}} \). Hrushovski and Loeser \([14]\) passed these by direct computation. This can be applied to show that \( \ker(\alpha_m \otimes \beta_m) \) is contained in \( \ker(N) \). Consequently, we obtain from the tensor products \( \tilde{\alpha}_m \otimes \beta_m \) and \( \alpha_m \otimes \beta_m \) morphisms of rings

\[
\tilde{h}_m : K(\text{vol} \Gamma^{\text{bdd}}) \to !K(\text{RES})[L^{-1}]_{\text{loc}}
\]

and

\[
h_m : K(\mu \Gamma \text{VF}^{\text{bdd}}) \to !K(\text{RES})[L^{-1}]_{\text{loc}}.
\]

Moreover, there is a presentation of \( h_m \) in terms of \( \tilde{h}_m \) induced from \([12]\). Namely, we have the following lemma whose proof is trivial and left to the reader.

**Lemma 4.3.** \( h_m([(X, e)]) = \sum_{e \in \mathbb{Z}} \tilde{h}_m([e^{-1}(e/m)]) L^{-e} \) (in \( !K(\text{RES})[L^{-1}]_{\text{loc}} \)).

4.4.2. The morphism \( h \). We also use the morphisms in \([14]\) Subsection 8.5] with their restriction, namely, \( \alpha : K(\text{vol} \Gamma^{\text{bdd}}) \to !K(\text{RES})[L^{-1}] \) and \( \beta : K(\text{vol} \text{RES}) \to !K(\text{RES})[L^{-1}] \). By definition, \( \beta([X]) = [X], \alpha([\Delta]) = \chi(\Delta)(L - 1)^n \) if \( \Delta \) is a definable subset of \( \Gamma^n \), where \( \chi \) is the o-minimal Euler characteristic in the sense of \([13]\) Lemma 9.5. Since \( \ker(\alpha \otimes \beta) \) is contained in \( \ker(N_0) \), it gives rise to a morphism of rings

\[
K(\text{vol} \Gamma^{\text{bdd}}) \to !K(\text{RES})[L^{-1}].
\]

The composition of it with the localization \( !K(\text{RES})[L^{-1}] \to !K(\text{RES})[L^{-1}]_{\text{loc}} \) will be denoted by \( h \).}

**Proposition 4.4.** The formal series \( Z'(X, e)(T) := \sum_{m \geq 1} h_m([(X, e)]) T^m \) is a rational function. Moreover, we have \( \lim_{T \to \infty} Z'(X, e)(T) = -h([X]) \).

**Proof.** It is similar to the proof of \([14]\) Proposition 8.5.1. \( \square \)
4.5. Endowing with a μ-action and the morphisms hₘ, h̃ₘ and h. First, let us recall [14, Proposition 4.3]. Define a series \( \{t_m\}_{m \geq 1} \) by setting \( t_1 = t, \) \( t_{m+1} = \frac{t_m}{m} \), \( n \geq 1 \). For a \( k((t)) \)-definable set \( X \) over RES, we may assume \( X \subset V_{i_1/m} \times \cdots \times V_{i_n/m} \) for some \( n, m \) and \( i_j \)'s. It is endowed with a natural action \( \delta \) of \( \mu_m \). Now the \( k((t^{1/m})) \)-definable function

\[
(x_1, \ldots, x_n) \mapsto (x_1/\text{rv}(t^1_m), \ldots, x_n/\text{rv}(t^n_m))
\]

maps \( X \) to a constructible subset \( Y \) of \( A^n_k \), where \( Y \) is endowed with a \( \mu_m \)-action induced from \( \delta \). The correspondence \( X \to Y \) in its turn defines a morphism of rings

\[
!K(\text{RES})[\mathbb{L}^{-1}] \to !K(V_0(\text{Var}_k))[\mathbb{L}^{-1}] \quad \text{(13, Lemma 10.7, 14, Proposition 4.3.1)}. \]

Here, by definition, \( !K(V_0(\text{Var}_k)) \) is the quotient of \( K_0^\mu(\text{Var}_k) \) by identifying all the classes \( \left[ G, \sigma \right] \) with \( \sigma \) a \( \mu \)-action on \( G \) induced by multiplication by roots of 1. The previous morphism together with the natural one \( !K(\text{Var}_k)[\mathbb{L}^{-1}] \to \mathcal{M}^\mu_k \) induces the following morphisms of rings, both are denoted by \( \Theta \),

\[
!K(\text{RES})[\mathbb{L}^{-1}] \to \mathcal{M}^\mu_k \quad \text{and} \quad !K(\text{RES})[\mathbb{L}^{-1}]_{\text{loc}} \to \mathcal{M}^\mu_{k, \text{loc}}.
\]

We now define ring morphisms \( h_m := \Theta \circ h, \) \( h_m := \Theta \circ h \) with the same target \( \mathcal{M}^\mu_{k, \text{loc}} \). In fact, while \( h_m \) has the source \( K(\mu^1 \text{VF}^{\text{bdd}}) \), \( h_m \) and \( h \) starts from \( K(\text{volVF}^{\text{bdd}}) \). Similar to Lemma 4.3 and Proposition 4.4 we get

Lemma 4.5. \( h_m(\gamma(X, \varepsilon)) = \sum_{e \in \mathbb{Z}} h_m((\varepsilon^{-1}(e/m)))\mathbb{L}^{-e} \) (in \( \mathcal{M}^\mu_{k, \text{loc}} \)).

Proposition 4.6. The formal series \( Z(X, \varepsilon)(T) := \sum_{m \geq 1} h_m(\gamma(X, \varepsilon))T^m \) is a rational function. Moreover, we have \( \lim_{T \to \infty} Z(X, \varepsilon)(T) = -h(\gamma([X])) \).

4.6. Description of the motivic Milnor fibers.

4.6.1. Regular case. Let \( \gamma \) be in \( \Gamma \). A definable subset \( X \) of \( \text{VF}^\varepsilon \times \text{RV}^{\varepsilon} \) is \( \gamma \)-invariant if, for any \( (x, x') \in \text{VF}^\varepsilon \times \text{RV}^{\varepsilon} \) and any \( (y, y') \in \text{VF}^\varepsilon \times \text{RV}^{\varepsilon} \) with \( \text{val}(y) \geq \gamma \), both \( (x, x') \) and \( (x, x') + (y, y') \) simultaneously belong to either \( X \) or the complement of \( X \) in \( \text{VF}^\varepsilon \times \text{RV}^{\varepsilon} \). By [14, Lemma 3.1.1], any bounded definable subset of \( \text{VF}^\varepsilon \) that is closed in the valuation topology is \( \gamma \)-invariant for some \( \gamma \) in \( \Gamma \).

Assume that \( X \) is a \( \gamma \)-invariant definable subset of \( \text{VF}^n \times \text{RV}^{\varepsilon} \), where \( \gamma \) is in \( (1/m)\mathbb{Z} \subset \Gamma \). By [13, X(k((t^{1/m})))], the pullback of some definable subset \( X[m; \gamma] \) of \( (k[[t^{1/m}]]/\mathbb{L}^{\varepsilon})^n \times \text{RV}^{\varepsilon} \) and the projection \( X[m; \gamma] \to \text{VF}^n \) is a finite-to-one map. If \( \gamma' \) is in \( \Gamma \) with \( \gamma' \geq \gamma \), the equality \( [X[m; \gamma']] = [X[m; \gamma]]\mathbb{L}^{nm(\gamma' - \gamma)} \) holds in \( !K(\text{RES}^n) \), thus \( [X[m; \gamma]]\mathbb{L}^{-nm\gamma} \) in \( !K(\text{RES})[\mathbb{L}^{-1}] \) is independent of the choice of \( \gamma \) large enough. For brevity, we shall write \( \bar{X}[m] \) for the quantity \( [X[m; \gamma]]\mathbb{L}^{-nm\gamma} \) as well as for its image under \( \Theta \).

Proposition 4.7. (i) For \( X \) as previous, \( h_m([X]) = \text{loc}([\bar{X}[m]]) \).

(ii) Let \( f \) be a nonzero function on a \( d \)-dimensional smooth connected \( k \)-variety \( X, x \) a point of \( f^{-1}(0) \). Let \( \pi \) be the reduction map \( X(R) \to X(k) \). Set

\[
X := \{ x \in X(R) \mid \pi(x) = x, \text{rv}(f(x)) = \text{rv}(t) \}.
\]

Then \( h([X]) = \text{loc}(S_{f, x}) \).

(iii) For any \( \gamma \) in \( \Gamma \), \( h(\gamma_1) = 1 \) and \( h(\gamma_1) = \mathbb{L} \). (Note that \( \gamma_1 \) and \( \gamma_1 \) are the open and closed disks of valuative radius \( \gamma \).)
Proof. (i) See Hrushovski-Loeser [14].

(ii) We use [14] Corollary 8.4.2 for proving (ii). Since $X$ is 2-invariant (it is in fact $\gamma$-invariant for any $\gamma > 1$ in $\Gamma$),

$$X[m; 2] = \left\{ \varphi \in \mathcal{X}(k[t^{1/m}]/(t^2)) \mid \varphi(0) = x, \text{rv}(f(\varphi)) = \text{rv}(t) \right\}.$$  

The condition $\text{rv}(f(\varphi)) = \text{rv}(t)$ is equivalent to $f(\varphi) \equiv t \mod t^{(m+1)/m}$, thus $X[m; 2]$ is defensibly isomorphic via the map $t^{1/m} \mapsto t$ to

$$\left\{ \varphi \in \mathcal{X}(k[t]/(t^{m+1})) \mid \varphi(0) = x, f(\varphi) \equiv t^m \mod t^{m+1} \right\} \times \mathbb{A}^{(m-1)d_1}.$$  

We get $h_m([X]) = \text{loc}([X_{0,m}]|L^{-md_1})$ and the conclusion follows.

(iii) Assume $\gamma = a/b$ with $a, b$ in $\mathbb{Z}$ and $(a, b) = 1$. Then $h_m([a/b]) = L^{-ma}$ if $n = mb$ and $h_n([a/b]) = 0$ otherwise, thus $h([a/b]) = 1$. Also, $h_m([a/b]) = L^{-ma+1}$ if $n = mb$ and $h_n([a/b]) = 0$ otherwise, thus $h([a/b]) = L$. \qed

4.6.2. Formal case. Let $X$ be a rigid $k((t))$-variety which is the generic fiber of a special formal $k[[t]]$-scheme $\mathfrak{X}$, let $\omega$ be a gauge form on $X$. We set $\mathfrak{X} := \text{X} \otimes_{k[[t]]} k[[t]]^{1/bk}$ and $\mathfrak{X} := X \otimes_{k((t))} k((t))^{1/bk}$. The integer-valued function $\text{ord}_X(\omega)$ on $\mathfrak{X}$ was already recalled in (1.22). Using the same way, one may define a rational-valued function $\text{ord}_\mathfrak{X}(\omega)$ on $\mathfrak{X}$, where $\mathfrak{X}$ is the pullback of $\omega$ via a the natural morphism $\mathfrak{X} \to X$. We denote this rational-valued function by $\text{val}_\omega$.

Theorem 4.8. Let $\mathfrak{X}$ be a relatively $d$-dimensional special formal $k[[t]]$-scheme with structural morphism $f$. Let $\mathfrak{X}_n,f_n$ (resp. $\mathfrak{X}_n(m),f_n$) be a version of $\mathfrak{X}_n$ (resp. $\mathfrak{X}_n(m)$) in which $f_n(x) = t$ is replaced by $\text{rv}f_n(x) = \text{rv}(t)$ (resp. $f_n(x) \equiv t \mod t^{(m+1)/m}$).

Then, for any gauge form $\omega$ on $\mathfrak{X}_n$,

(i) $h_m([\mathfrak{X}_n,\text{val}_\omega]) = \text{loc}(L^d \int_{\mathfrak{X}_n(m)} |\omega(m)|)$,

(ii) $h_m([\mathfrak{X}_n,\text{val}_\omega]) = \text{loc}(L^d \int_{\mathfrak{X}_n(m),f_n} |\omega(m)|)$,

(iii) $h([\mathfrak{X}_n,f_n]) = \text{h}([\mathfrak{X}_n])$.

As a consequence, for a closed point $x$ of $\mathfrak{X}_n$ and a gauge form $\omega'$ on $|x|$, (iv) $h_m([x,\text{val}_{\omega'}]) = \text{loc}(L^d \int_{|x|,f_n} |\omega'(m)|)$,

(v) $h_m([x,\text{val}_{\omega'}]) = \text{loc}(L^d \int_{|x|,f_n} |\omega'(m)|)$,

(vi) $h([x,f_n]) = \text{h}([x]) = \text{loc}(S_{[x],f_n})$.

Proof. We prove (i). By Lemma 4.3

$$h_m([\mathfrak{X}_n,\text{val}_\omega]) = \sum_{e \in \mathbb{Z}} h_m([\text{val}_\omega^{-1}(e/m)])L^{-e}. \quad (4.3)$$

By [14] Lemma 3.1.1], for each $e$ in $\mathbb{Z}$, there exists a $\gamma_{e,m}$ in $\Gamma$ such that $\text{val}_\omega^{-1}(e/m)$ is $\gamma_{e,m}$-invariant. Thus it follows from Proposition 4.7(i) that

$$h_m([\text{val}_\omega^{-1}(e/m)]) = \text{loc}(\text{val}_\omega^{-1}(e/m)[m]) = \text{loc}(\text{val}_\omega^{-1}(e/m)[m; \gamma'])L^{-md'+d} \quad (4.4)$$

for any $\gamma' \geq \gamma_{e,m}$ in $(1/m)\mathbb{Z} \subset \Gamma$.

Let $\mathfrak{X} \to \mathfrak{X}$ be a Néron smoothening for $\mathfrak{X}$ with $\mathfrak{X}$ relatively $d$-dimensional $k[[t]]$-formal scheme topologically of finite type. It is obvious that $\mathfrak{X}_n = \mathfrak{X}_n$ since $k[[t]]$ is henselian, so we can regard $\text{val}_\omega$ as a function on $\mathfrak{X}_n$. As the function
denote it by \textit{morphisms of rings} $M_m$ holds in $\mu$. Then
\[ Z = \{ (x, y) \in m^{d_1 + d_2} \mid \text{rv}(f(x) + g(y)) = \text{rv}(t) \} . \]

This is a bounded 2-invariant definable subset of $\text{VF}^{d_1 + d_2}$. By Proposition 4.7(ii), $h([Z]) = \text{loc}(S_{f \oplus g,(0,0)})$ in $M^{\mu}_{\text{d,loc}}$. Let us decompose $Z$ into a disjoint union of sets $X$, $Y$ and $Z \ast$ subject to conditions $\text{val}(f(x) < \text{val}(g(y))$, $\text{val}(f(x) > \text{val}(g(y))$ and $\text{val}(f(x) = \text{val}(g(y))$, respectively. In the sequel, we are going to compute $h([X])$, $h([Y])$, $h([Z \ast])$ and conclude.

Write $X = \{ (x, y) \in m^{d_1 + d_2} \mid \text{rv}(f(x)) = \text{rv}(t) \}$ as the product of the definable sets $X' := \{ x \in m^{d_1} \mid \text{rv}(f(x)) = \text{rv}(t) \}$ and $m^{d_2} = [0]^{d_2}_{1}$. Proposition 4.7 the items (ii) and (iii), gives $h([X_1]) = \text{loc}(S_{f,0})$ and $h([m^{d_2}]) = 1$, thus $h([X]) = \text{loc}(S_{f,0})$ in $M^{\mu}_{k,\text{d,loc}}$. Similarly, we also have $h([Y]) = \text{loc}(S_{g,0})$ in $M^{\mu}_{k,\text{d,loc}}$.

Set $Z_0^* = \{ (x, y) \in Z \ast \mid \text{val}(f(x) = 1) \}$, $Z_{1}^* = \{ (x, y) \in Z \ast \mid 0 < \text{val}(f(x) < 1 \}$, then $Z^* = Z_1^* \sqcup Z_{2}^*$. For our goal we introduce the following definable set
\[ Z_0 := \{ (x, y) \in m^{d_1 + d_2} \mid \text{val}(f(x) + g(y)) > 1, -\text{rv}(f(x) = \text{rv}(g(y) = \text{rv}(t)) . \]

We shall consider the identity $[Z^*] = ([Z_1^*] - [Z_0]) + ([Z_{2}^*] + [Z_0])$ in $K(\text{vol} \text{VF}^{\text{bdd}})$. 

Proposition 5.1. For \( m \geq 1 \), the equality

\[
\tilde{h}_m ([Z^*_1] - [Z_0]) = -\text{loc} \left( [X_{0,m}(f)] * [X_{0,m}(g)] L^{-m(d_1 + d_2)} \right)
\]

holds in \( \mathcal{M}_{k,\text{loc}}^\mu \). Moreover, also in this ring \( \mathcal{M}_{k,\text{loc}}^\mu \), we have

\[
h ([Z^*_1] - [Z_0]) = -\text{loc} (S_{f,0} * S_{g,0}).
\]

Lemma 5.2. The following hold in \( \mathcal{M}_{k,\text{loc}}^\mu \):

(i) \( \tilde{h}_m ([Z^*_1]) = \text{loc} ([X_{m,0}(f) \times X_{m,0}(g) \times \mu_m \times \mu_m F_1^m] L^{-m(d_1 + d_2)}) \);

(ii) \( \tilde{h}_m ([Z_0]) = \text{loc} ([X_{m,0}(f) \times X_{m,0}(g) \times \mu_m \times \mu_m F_0^m] L^{-m(d_1 + d_2)}) \).

Proof. (i) Since \( Z^*_1 \) is 2-invariant, we consider \( Z^*_1[m; 2] \) which equals

\[
\begin{cases}
(\varphi, \psi) \in \left( \frac{k[t_{1/m}]}{t^2} \right)_{d_1 + d_2} & (\varphi(0), \psi(0)) = (0, 0), \text{valf}(\varphi) = \text{valg}(\psi) = 1 \\
\text{ordf}(\varphi) = \text{ordg}(\psi) = m & f(\varphi) + g(\psi) \equiv t \mod t^{(m+1)/m} \\
\text{ordf}(\varphi) = \text{ordg}(\psi) = m & f(\varphi) + g(\psi) \equiv t^m \mod t^{m+1}
\end{cases}
\]

We claim that there is a canonical isomorphism between

\[
V := \left\{ (\varphi, \psi) \in \left( \frac{tk[t]}{t^{m+1}} \right)_{d_1 + d_2} \mid \text{ordf}(\varphi) = \text{ordg}(\psi) = m, f(\varphi) + g(\psi) \equiv t^{m+1} \mod t^{m+1} \right\}
\]

and

\[
X_{0,m}(f) \times X_{0,m}(g) \times \mu_m \times \mu_m F_1^m.
\]

Indeed, we define a map \( X_{0,m}(f) \times X_{0,m}(g) \times F_1^m \to V \) that sends \((\varphi(t), \psi(t); a, b)\) to \((\varphi(at), \psi(bt))\). It then induces a well defined morphism on the quotient

\[
\xi : X_{0,m}(f) \times X_{0,m}(g) \times \mu_m \times \mu_m F_1^m \to V.
\]

We also define a morphism

\[
\eta : V \to X_{0,m}(f) \times X_{0,m}(g) \times \mu_m \times \mu_m F_1^m
\]

given by \( \eta(\varphi(t), \psi(t)) = (\varphi((acf\varphi)^{-1/m}t), \psi((acg\psi)^{-1/m}t); (ac f \varphi)^{1/m}, (ac g \psi)^{1/m}) \).

It is clear that \( \xi \) and \( \eta \) are inverse of each other and the claim follows. Consequently, by Proposition 4.7(i),

\[
\tilde{h}_m ([Z^*_1]) = \text{loc} \left( [X_{0,m}(f) \times X_{0,m}(g) \times \mu_m \times \mu_m F_1^m] L^{-m(d_1 + d_2)} \right).
\]

(ii) Similarly as previous, since \( Z_0 \) is 2-invariant, \( Z_0[m; 2] \) is isomorphic to

\[
\begin{cases}
(\varphi, \psi) \in \left( \frac{tk[t]}{t^{m+1}} \right)_{d_1 + d_2} & \text{ord}(f(\varphi) + g(\psi)) > m \\
-f(\varphi) \equiv g(\psi) \equiv t^m \mod t^{m+1}
\end{cases}
\]

Also as above, we are able to prove that the constructible set

\[
\begin{cases}
(\varphi, \psi) \in \left( \frac{tk[t]}{t^{m+1}} \right)_{d_1 + d_2} & \text{ord}(f(\varphi) + g(\psi)) > m \\
-f(\varphi) \equiv g(\psi) \equiv t^m \mod t^{m+1}
\end{cases}
\]

is isomorphic to \( X_{0,m}(f) \times X_{0,m}(g) \times \mu_m \times \mu_m F_0^m \), thus (ii) is proven. \( \square \)
Proof of Proposition 5.1. By Lemma 5.2 and by definition of convolution product (cf. Subsection 2.3) we get $h_m (\{Z^*_1\} - [Z_0]) = - \text{loc} ([X_{0,m}(f)] * [X_{0,m}(g)] \mathbb{L}^{-m(d_1 + d_2)}).$ By a property of the Hadamard product, namely,

$$- \lim_{T \to \infty} \sum_{m \geq 1} [X_{0,m}(f)] \mathbb{L}^{-md_1 + d_2} T^m$$

we deduce that $h (\{Z^*_1\} - [Z_0]) = - \text{loc} (S_{f,0} * S_{g,0})$ in $M_{k, \text{loc}}^\mu.$ ⊠

5.2. Integral over $\Gamma$. Let $D$ be a definable subset of $\Gamma$. A function $\nu : D \to M_{k, \text{loc}}^\mu$ is called definable if $D$ may be partitioned into finitely many disjoint definable subsets $D_i$, $i \in I$, such that $\nu |_{D_i}$ is constant $c_i \in M_{k, \text{loc}}^\mu$ for every $i \in I$. Then, we define the integral of $\nu$ over $D$, which takes value in $M_{k, \text{loc}}^\mu$, as follows

$$\int_{\gamma \in D} \nu (\gamma) = \int_{\gamma \in D} \nu (\gamma) d\chi := \sum_{i \in I} c_i \chi (D_i).$$

Here, $\chi$ is the o-minimal Euler characteristic defined in [13] Lemma 9.5 followed by the localization morphism.

5.3. Completion of the proof of Theorem 2.1. In this subsection, we shall prove that $h (\{Z^*_1\} + [Z_0]) = 0$ in $M_{k, \text{loc}}^\mu$, thus finish the proof of (5.1).

5.3.1. Computation of $h (\{Z^*_1\})$ and $h ([Z_0])$. Let $\pi_{< 1}$ denotes the definable function $Z^*_1 \to (0, 1) \subset \Gamma$ mapping $(x, y)$ to $\text{valf}(x)$, and let $\nu : (0, 1) \to M_{k, \text{loc}}^\mu$ be the function defined by

$$\nu (\gamma) = h ([\pi_{< 1}^{-1}(\gamma)]).$$

Lemma 5.3. The function $\nu$ is definable.

Proof. Via the definable bijection $(x, y) \mapsto (x, y, \text{valf}(x))$, we may regard $Z^*_1$ as a definable subset of $m^{d_1 + d_2} \times (0, 1)$. Consider the surjective morphism of rings

$$N_0 : K (\text{vol}^{\text{bdd}}) \otimes K (\text{volRES}) \to K (\text{volVF}^{\text{bdd}})$$

induced by $N$ in [13]. There exist definable subsets $W_i$ of $\text{RES}^i$ and $\Delta_i$ of $\Gamma^i \times (0, 1)$, $0 \leq i \leq d_1 + d_2$, with $N_0 \left( \sum_{i=0}^{d_1 + d_2} [\Delta_i] \otimes [W_{d_1 + d_2 - i}] \right) = [Z^*_{< 1}].$ By definition of $\alpha$, $\beta$ (cf. [13], (6.3)), $(\alpha \otimes \beta) \left( \sum_{i=0}^{d_1 + d_2} [\Delta_i] \otimes [W_{d_1 + d_2 - i}] \right) = \sum_{i=0}^{d_1 + d_2} \chi (\Delta_i) w_{d_1 + d_2 - i}$, where $w_{d_1 + d_2 - i} := [W_{d_1 + d_2 - i}] (\mathbb{L} - 1)^i.$ Similarly, for $\gamma \in (0, 1)$, there are definable subsets $W_{\gamma,i}$ of $\text{RES}^i$, $\Delta_{\gamma,i}$ of $\Gamma^i \times \{ \gamma \}$ with $N_0 \left( \sum_{i=0}^{d_1 + d_2} [\Delta_{\gamma,i}] \otimes [W_{\gamma,d_1 + d_2 - i}] \right) = [\pi_{< 1}^{-1}(\gamma)].$

Also, $(\alpha \otimes \beta) \left( \sum_{i=0}^{d_1 + d_2} [\Delta_{\gamma,i}] \otimes [W_{\gamma,d_1 + d_2 - i}] \right) = \sum_{i=0}^{d_1 + d_2} \chi (\Delta_{\gamma,i}) w_{\gamma,d_1 + d_2 - i}$, where $w_{\gamma,d_1 + d_2 - i} := [W_{\gamma,d_1 + d_2 - i}] (\mathbb{L} - 1)^i.$ We claim that $w_i = w_{\gamma,i}$ in $K (\text{RES})$. Indeed, the image of $W_i$ (resp. $W_{\gamma,i}$) in $K (\text{volVF}^{\text{bdd}})$ is $[W_i \times_{\text{\ell}, \text{res}} R^i]$ (resp. $[W_{\gamma,i} \times_{\text{\ell}, \text{res}} R^i]$), where $\ell : W_i \to k^i$ and $\ell_{\gamma,i} : W_{\gamma,i} \to k^i$ are étale maps, $R = \{ \tau \in \text{VF} | \text{val}(\tau) \geq 0 \}.$ The unique difference between $W_i \times_{\ell, \text{res}} R^i$ and $W_{\gamma,i} \times_{\ell_{\gamma,i}, \text{res}} R^i$ is that the former admits the condition $0 < \text{valf}(x) < 1$ while the latter satisfies $\text{valf}(x) = \gamma$. Thus
\[ W_i = [W_{γ,i}] \text{ in } K(\text{RES}). \] Consequently, \((α ⊗ β)\left(\sum_{i=0}^{d_1+d_2} [∆_{γ,i}] \otimes [W_{γ,i}] \right) = \sum_{i=0}^{d_1+d_2} χ(∆_{γ,i})w_{d_1+d_2}\). Since \(h\) induces from \(α ⊗ β\), we have the following

\[ (5.2) \quad h([Z_{<1}]) = \sum_{i=0}^{d_1+d_2} χ(∆_{i})θ_i, \quad h([π_{<1}^{-1}(γ)]) = \sum_{i=0}^{d_1+d_2} χ(∆_{γ,i})θ_i, \]

where \(θ_i := Θ(w_{d_1+d_2-i})\).

For \(0 ≤ i ≤ d_1 + d_2\), we identify \(Γ_1 × (0,1)\) with a subset of \(Γ^{d_1+d_2} × (0,1)\) in an obvious manner. Let \(pr_2\) be the second projection \(Γ^{d_1+d_2} × (0,1) \rightarrow (0,1)\) and \(D_i := pr_2(∆_i)\). Then \((0,1) = \bigcup_{i=0}^{d_1+d_2} D_i\). Moreover, for any \(γ\) in \((0,1)\), \(∆_{γ,i}\) is a fiber of the definable map \(∆_i \rightarrow D_i\), all the fibers of this map are definably isomorphic. Thus \(χ(∆_{i}) = χ(D_i)χ(∆_{γ,i})\). It and \((5.2)\) show that, on \(D_i\),

\[ (5.3) \quad ν(γ) = h([π_{<1}^{-1}(γ)]) = \sum_{i=0}^{d_1+d_2} χ(∆_{i})χ(D_i)^{-1}θ_i, \]

which proves the definability.

\[ \square \]

**Corollary 5.4.** \(\int_{γ ∈ (0,1)} h([π_{<1}^{-1}(γ)]) = h([Z_{<1}])\).

**Proof.** By definition as well as by \((5.2)\) and \((5.3)\),

\[ \int_{γ ∈ (0,1)} ν(γ) = \sum_{i=0}^{d_1+d_2} ν|_{D_i}χ(D_i) = \sum_{i=0}^{d_1+d_2} χ(∆_{i})χ(D_i)^{-1}θ_1χ(D_i) = h([Z_{<1}]) \]

\[ \square \]

Let \(π_0\) be the function \(Z_0 \rightarrow (1,∞) ⊂ Γ\) that sends \((x,y)\) to \(\text{val}(f(x) + g(y))\). Similarly as previous, we are able to prove the following

**Corollary 5.5.** The function \((1,∞) \rightarrow M_{k,\text{loc}}^µ\) given by \(h([π_{<1}^{-1}(γ)])\) is definable, moreover, \(\int_{γ ∈ (1,∞)} h([π_{<1}^{-1}(γ)]) = h([Z_0])\).

**5.3.2.** Conclusion. Let \(A\) be the annulus \(\{τ ∈ VF \mid 0 < \text{val}(τ) < 1\}\) and \(p_{<1}\) the function \(Z_{<1} \rightarrow A\) mapping \((x,y)\) to \(f(x)\). Then \(π_{<1} = p_{<1} \circ \text{val}\). The fiber over \(τ \in A\) of \(p_{<1}\) is the following

\[ (5.4) \quad p_{<1}^{-1}(τ) = \{(x,y) ∈ M^{d_1+d_2} \mid f(x) = τ, g(y) = −τ + t\}. \]

As for each \(γ\) in \((0,1)\), all the fibers \(p_{<1}^{-1}(τ), \tau \in \text{val}^{-1}(γ)\), are definably isomorphic, since the description \((5.3)\), it implies that the equalities

\[ [π_{<1}^{-1}(γ)] = \int_{τ ∈ \text{val}^{-1}(γ)} [p_{<1}^{-1}(τ)] = [\text{val}^{-1}(γ)][p_{<1,γ}] \]

hold in \(K(\text{volVF}^{\text{bdd}})\), where \([p_{<1,γ}^{-1}(τ)]\) is the constant function \([p_{<1}^{-1}(τ)]\) on \(\text{val}^{-1}(γ)\).

By Corollary 5.3.

\[ (5.5) \quad h([Z_{<1}]) = \int_{γ ∈ (0,1)} h([π_{<1}^{-1}(γ)]) = \int_{γ ∈ (0,1)} h([\text{val}^{-1}(γ)])h([p_{<1,γ}^{-1}(τ)]). \]

**Lemma 5.6.** \(h([p_{<1,γ}^{-1}(τ)])\) is independent of the choice of \(γ\) in \((0,1)\).
Proof. Using (5.4), namely,
\[ p_{<1}^{-1}(\tau) = \{ x \in m^{d_1} \mid f(x) = \tau \} \times \{ y \in m^{d_2} \mid g(y) = -\tau + t \}, \]
it suffices to prove \( h(\{ x \in m^d \mid f(x) = \tau \}) = h(\{ x \in m^d \mid f(x) = t \}) \) for any regular function \( f : A_k^d \to A_k^d \), vanishing at the origin of \( A_k^d \) and for any \( \gamma \) in \((0,1)\). Equivalently, it suffices to prove \( h(\{ x \in m^d \mid rvf(x) = rv(\tau) \}) = h(\{ x \in m^d \mid rvf(x) = rv(\gamma) \}) \) for \( \gamma = a/b \) in \((0,1)\) with \( a \) and \( b \) coprime integers, \( a < b \). Indeed, if \( m \) is not divisible by \( b \), then \( h_m(\{ x \in m^d \mid rvf(x) = rv(\gamma) \}) = 0 \). Otherwise, say, \( m = bs \), then
\[ \tilde{h}_{bs}(\{ x \in m^d \mid rvf(x) = rv(\gamma) \}) = \tilde{h}_{bs}(\{ x \in m^d \mid rvf(x) = rv(t) \}), \]
because, by a simple geometric computation, both sides are equal to \([X_{0,as}(f)]L^{-asd}\). This equality then implies the lemma.

Using (5.5) and Lemma 5.6, we get \( h([Z_{<1}]) = \left( \int_{\gamma \in (0,1)} h([val^{-1}(\gamma)]) \right) h([p_{<1}^{-1}]) \). Similarly as in the proofs of Lemma 5.3 and Corollary 5.4, we can easily show that \( \int_{\tau \in (0,1)} h([val^{-1}(\gamma)]) = h([A]) = -1 \). Thus
\[ (5.6) \quad h([Z_{<1}]) = -h([p_{<1}^{-1}]) \quad (\gamma \in (0,1)). \]

Denote by \( B \) the set \( \{ \tau \in VF \mid val(\tau) > 1 \} \) and consider the function \( p_0 : Z_0 \to B \) defined by \( p_0(x,y) = f(x) + g(y) \). Then, we have \( p_0 = p_0 \circ val \), moreover, the fiber over \( \tau \in B \) of \( p_0 \) equals
\[ p_0^{-1}(\tau) = \{ (x,y) \in m^{d_1}m^{d_2} \mid f(x) + g(y) = \tau, rvf(x) = rvg(y) = rv(t) \} = \{ (x,y) \in m^{d_1}m^{d_2} \mid f(x) = ct, g(y) = -ct + \tau, c \in 1 + m \}. \]
Similarly as in the proof of Lemma 5.6, we can show that \( h([p_0^{-1}(\tau)]) \) is independent of \( \tau \in B \) and, moreover, that
\[ (5.7) \quad h([p_{<1,1}^{-1}]) = h([p_0^{-1}(\tau)]) \]
for any \( \gamma \) in \((0,1)\) and any \( \tau \) in \( B \). An analogue of Lemma 5.3 and Corollary 5.4 gives rise to the formula
\[ (5.8) \quad h([Z_0]) = h([B])h([p_0^{-1}(\tau)]) = h([p_0^{-1}(\tau)]) \quad (\tau \in B). \]
Finally, it follows from (5.6), (5.8), and (5.7) that \( h([Z_{<1}] + [Z_0]) = 0 \) in \( M_{\text{loc}}^0 \). This together with Proposition 5.1 proves (5.1).

6. Proof of Theorem 3.4

It is Theorem 4.18 that completely interprets the role of the morphisms \( h_m \) and \( h \) in understanding the motivic Milnor fiber of a formal function from the non-archimedean geometry point of view. Motivated by these, to prove Theorem 3.4, also as the proof of the regular version (Section 5), we work on analytic Milnor fibers (in the sense of [20]) considered as definable sets in the theory ACVF\(_k(\mu)(0,0)\).
6.1. Using arguments in Section 5. Let \( Z \) be the analytic Minor fiber \([0,0]\) of \( f + g \) at the origin \((0,0)\) of \( \mathbb{A}^{d_1}_k \times \mathbb{A}^{d_2}_k \), namely,

\[
Z = \{ (x, y) \in \mathbb{m}^{d_1+d_2} \mid f(x) + g(y) = t \}.
\]

(To indicate precisely the origin of \( \mathbb{A}^{d_1}_k \), if necessary, we write \( 0_{d_1} \) instead of \( 0 \).)

It induces immediately from Theorem 4.8 that \( h([Z]) = \text{loc} \left( S_{\begin{array}{c} 1 \end{array}_2, \begin{array}{c} 0 \end{array}} \right) \) in \( \mathcal{M}^h_{k, \text{loc}} \).

Write \( Z \) as a disjoint union of definable subsets \( \mathcal{X} \) and \( Z^* \) respectively defined by \( \text{val} f(x) < \text{val} g(y) \), \( \text{val} f(x) > \text{val} g(y) \) and \( \text{val} f(x) = \text{val} g(y) \). Also by Theorem 4.8 we have \( h([\mathcal{X}]) = \text{loc} \left( S_{1,0} \right) \) and \( h([Z^*]) = \text{loc} \left( S_{0,0} \right) \) in \( \mathcal{M}^h_{k,\text{loc}} \).

To continue, we modify slightly \( Z^* \) into \( \check{Z}^* \), where

\[
\check{Z}^1 = \{ (x, y) \in \mathbb{m}^{d_1+d_2} \mid \text{rv}(f(x) + g(y)) = \text{rv}(t), \text{val} f(x) = \text{val} g(y) \},
\]

and note that \( h([\check{Z}]) = h([0])_{d_1+d_2} \cdot [Z^*] = h([Z^*]) \) in \( \mathcal{M}^h_{k,\text{loc}} \) since \( h([0]) = 1 \).

Now, decompose \( \check{Z}^1 \) into a disjoint union of \( \check{Z}_1^1 = \{ (x, y) \in \check{Z}^1 \mid \text{val} f(x) = 1 \} \) and \( \check{Z}_{<1}^1 = \{ (x, y) \in \check{Z}^1 \mid 0 < \text{val} f(x) < 1 \} \). Similarly as in Section 5 we use the definable set

\[
\check{Z}_0^1 := \{ (x, y) \in \mathbb{m}^{d_1+d_2} \mid \text{val} f(x) + g(y) > 1, -\text{rv} f(x) = \text{rv} g(y) = \text{rv}(t) \}
\]

and present \( [Z^1] \) as the sum \( ([Z_1^1] - [Z_0^1]) + ([Z_{<1}^1] + [Z_0^1]) \) in \( K(\text{volVF}_{\text{bdd}}) \). As in Subsection 5.3 we also obtain \( h([Z_{<1}^1] + [Z_0^1]) = 0 \). In the sequel, we shall prove that \( h([Z_1^1] - [Z_0^1]) = -\text{loc} \left( S_{1,0} * S_{0,0} \right) \) and the proof of Theorem 5.4 is completed.

6.2. Using motivic integral via Subsections 3.2, 4.4, 4.5, 4.6, Section 5

In this subsection, we prove the following

**Proposition 6.1.** With the previous notation, we have

\[
h([Z_1^1] - [Z_0^1]) = -\text{loc} \left( S_{1,0} * S_{0,0} \right).
\]

Let \( Z_1 \) (resp. \( Z_2 \)) be a Néron smoothening for the formal completion of \( \mathcal{X} \) at \( 0_{d_1} \) (resp. for the formal completion of \( \mathcal{Y} \) at \( 0_{d_2} \)) with \( Z_1 \) and \( Z_2 \) smooth, topologically of finite type over \( k[[t]] \). For any integer \( m \geq 1 \), let \( \text{Gr}(Z(m))_{\text{rv}} \) be the space defined as \( \text{Gr}(Z(m)) \) but \( f(x) = t \) replaced by \( f(x) \equiv t \mod t^{(m+1)/m} \), where \( f \) is the structural morphism of \( Z \). For \( i \in \{1, 2\} \), let \( \omega_i \) be a bounded gauge form on \( \mid 0_{d_i} \mid \) (remark that \( 0_{d_1} = 3_1, 0_{d_2} = 3_2, 0 \), and, for any integer \( e_i \), set

\[
\Phi(Z_i(m), \omega_i(m), e_i) := \mu(\{ \varphi \in \text{Gr}(Z_i(m))_{\text{rv}} \mid \text{ord}_{Z_i(m)}(\omega_i(m))(\varphi) = e_i \}),
\]

which is an element of \( \mathcal{M}^h_k \), by the \( \mu_m \)-action \( a_m \varphi(t) := \varphi(a t) \). By definition,

\[
\int_{0_{d_i}[m, \text{rv}]} |\omega_i(m)| = \sum_{e_i \in Z} \Phi(Z_i(m), \omega_i(m), e_i)L^{-e_i},
\]

for \( i \in \{1, 2\} \), where the sum runs over a finite set as \( \omega_i \) is a gauge form (see [18]). One thus deduces that

\[
\left( \int_{0_{d_1}[m, \text{rv}]} |\omega_1(m)| \right) * \left( \int_{0_{d_2}[m, \text{rv}]} |\omega_2(m)| \right) = \sum_{e_1, e_2 \in Z} \Phi(Z_1(m), \omega_1(m), e_1) * \Phi(Z_2(m), \omega_2(m), e_2)L^{-e_1 + e_2}.
\]
For $e_1, e_2$ in $\Gamma$, let $Z^{\dagger}_{1,e_1,e_2}$ (resp. $Z^{\dagger}_{0,e_1,e_2}$) be the subset of $Z^{\dagger}_{1}$ (resp. $Z^{\dagger}_{0}$) such that $\text{val}_{\omega_1}(x) = e_1$ and $\text{val}_{\omega_2}(y) = e_2$. For $e$ in $\Gamma$, set $Z^{\dagger}_{1,e} := \bigcup_{e_1+e_2=e} Z^{\dagger}_{1,e_1,e_2}$ and $Z^{\dagger}_{0,e} := \bigcup_{e_1+e_2=e} Z^{\dagger}_{0,e_1,e_2}$.

**Lemma 6.2.** For any integer $m \geq 1$, for any $e_1, e_2$ in $\Gamma$ with $me_1, me_2 \in \mathbb{Z}$,

$$
\hat{h}_m([Z^{\dagger}_{1,e_1,e_2}] - [Z^{\dagger}_{0,e_1,e_2}]) = -\text{loc}(L^{d_1 + d_2} \Phi(Z_1(m), \omega_1(m), me_1) * \Phi(Z_2(m), \omega_2(m), me_2)).
$$

**Proof.** Since $Z_1$ is topologically of finite type, there exist a convergent power series $\tilde{f}$ in $k\{x\}$ vanishing at 0 (hence a $k[[t]]$-scheme $X = \text{Spec}(k[[t]]/[\tilde{f}(t)-t])$) such that

$$\text{Gr}_t(Z_1(k)) = \{ \varphi \in (X \otimes_{k[[t]]} [k[t]/t^2]) (k[t]/t^2) \mid \text{val}(\varphi) > 0 \} \ni \{ \varphi \in (tk[t]/t^2)^{d_1} \mid \tilde{f}(\varphi) = t \}$$

for $t \in \mathbb{N}$. Similarly, $\text{Gr}_t(Z_2(k)) \cong \{ \varphi \in (tk[t]/t^2)^{d_2} \mid \tilde{G}(\varphi) = t \}$ for some $\tilde{g}$ in $k\{y\}$ with $\tilde{g}(0) = 0$. Thus, $L^{d_1} \Phi(Z_1(m), \omega_1(m), me_1)$ equals $L^{-md_1}$ times

$$\left[ \{ \varphi \in (tk[t]/t^2)^{d_1} \mid \tilde{f}(\varphi) \equiv t^m \text{ mod } t^{m+1}, \text{ord}_{Z_1}(\omega_1)(\varphi) = me_1 \} \right],$$

and $L^{d_2} \Phi(Z_2(m), \omega_2(m), me_2)$ equals $L^{-md_2}$ times

$$\left[ \{ \psi \in (tk[t]/t^2)^{d_2} \mid \tilde{g}(\psi) \equiv t^m \text{ mod } t^{m+1}, \text{ord}_{Z_2}(\omega_2)(\psi) = me_2 \} \right],$$

for $\ell \in \mathbb{N}$ large enough. At this time, we may use the arguments in the proof of Lemma 5.2 and Proposition 5.1 hence conclusion. \hfill \Box

**Lemma 6.3.** For any integer $m \geq 1$,

$$-h_m([[0_{d_1}, \text{val}_{\omega_1}]) * h_m([0_{d_2}, \text{val}_{\omega_2}])] = h_m([[Z^\dagger_{1}, \text{val}_{\omega_1} + \text{val}_{\omega_2}]) - ([Z^\dagger_{0}, \text{val}_{\omega_1} \oplus \text{val}_{\omega_2}]),$$

where, by definition, $\text{val}_{\omega_1} \oplus \text{val}_{\omega_2}(x,y) = \text{val}_{\omega_1}(x) + \text{val}_{\omega_2}(y)$.

**Proof.** Applying (6.1) and Lemmas 6.2 and 6.5 we get

$$-h_m([[0_{d_1}, \text{val}_{\omega_1}]) * h_m([0_{d_2}, \text{val}_{\omega_2}])] = -h_m([Z^\dagger_{1,e/m}] - [Z^\dagger_{0,e/m}]) \subseteq h_m([Z^\dagger_{1,e/m}] - [Z^\dagger_{0,e/m}]) \subseteq h_m([Z^\dagger_{1,e/m}] - [Z^\dagger_{0,e/m}])$$

where the lemma is proven. \hfill \Box

**Proof of Proposition 6.1 and Theorem 3.4.** Thanks to Lemma 6.3 Proposition 4.6 and Theorem 4.8 we have $h([Z^\dagger_1] - [Z^\dagger_0]) = -h([Z^\dagger_1] - [Z^\dagger_0]$ as desired. The proof of Theorem 3.4 is deduced from Proposition 6.1 and Subsection 6.1. \hfill \Box
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