On Decoupling of Functions of Normal Vectors II

Pavel G. Grigoriev and Stanislav A. Molchanov
Department of Mathematics and Statistics
University of North Carolina at Charlotte
Charlotte, NC 28223, USA
thepavel@mail.ru

Abstract
A decoupling type inequality for a sum of functions of Gaussian vectors is established.

Key words: Decoupling, Gaussian vectors, Wick’s polynomials, Hermite’s polynomials.

1 Results

In [3] the following decoupling results were established.

Theorem 1.1. For a normally distributed random vector $\bar{Y} = (Y_i)_{i=1,\ldots,d}$ satisfying $EY_i = 0$, $EY_i^2 = 1$, $i = 1, \ldots, d$, we have

$$c^- \sum_{i=1}^d \|\varphi_i(Y_i)\|^2_2 \leq \left\| \sum_{i=1}^d \varphi_i(Y_i) \right\|^2_2 \leq c^+ \sum_{i=1}^d \|\varphi_i(Y_i)\|^2_2$$

for all measurable functions $\varphi_i : \mathbb{R} \to \mathbb{R}$ satisfying $E\varphi_i(Y_i) = 0$ with the constants $c^-$ and $c^+$ being the smallest and the largest eigenvalues of the correlation matrix of $\bar{Y}$. Moreover the constants are the best possible.

Theorem 1.2. Let $\bar{Y}_1 = (Y_{1,i})_{i=1,\ldots,d_1}$ and $\bar{Y}_2 = (Y_{2,j})_{j=1,\ldots,d_2}$ be standard normal vectors with the correlations

$$EY_{\alpha,i}Y_{\beta,j} = \begin{cases} 0, & \alpha = \beta, i \neq j \\ 1, & \alpha = \beta, i = j \\ \rho_{i,j}, & \alpha \neq \beta \end{cases}$$

Then

$$c_-(\|\varphi_1(\bar{Y}_1)\|^2_2 + \|\varphi_2(\bar{Y}_2)\|^2_2) \leq \left\| \varphi_1(\bar{Y}_1) + \varphi_2(\bar{Y}_2) \right\|^2_2 \leq c_+(\|\varphi_1(\bar{Y}_1)\|^2_2 + \|\varphi_2(\bar{Y}_2)\|^2_2)$$

holds for all measurable functions $\varphi_\alpha : \mathbb{R}^{d_\alpha} \to \mathbb{R}$ such that $E\varphi_\alpha(\bar{Y}_\alpha) = 0$, $\alpha = 1, 2$, with the constants $c_\pm = 1 \pm s^*$, where $s^*$ is the maximum singular value of the matrix $R = (\rho_{i,j})_{i=1,\ldots,d_1,j=1,\ldots,d_2}$. These constants cannot be improved.
These theorems refine the estimates used by Cherny et al. [1], [2]. Here we generalize these results further and prove

**Theorem 1.3.** Let \( \bar{Y}_\alpha = (Y_{\alpha,i})_{i=1,...,d_\alpha}, \alpha = 1,...,N, \) be standard normal vectors with the correlations

\[
EY_{\alpha,i}Y_{\beta,j} = \begin{cases} 
0, & \alpha = \beta, i \neq j \\
1, & \alpha = \beta, i = j \\
\rho_{\alpha,\beta}^{i,j}, & \alpha \neq \beta
\end{cases}
\]

Then

\[
C_+ \sum_{\alpha=1}^{N} \|\varphi_\alpha(\bar{Y}_\alpha)\|_2^2 \leq \left\| \sum_{\alpha=1}^{N} \varphi_\alpha(\bar{Y}_\alpha) \right\|_2^2 \leq C_- \sum_{\alpha=1}^{N} \|\varphi_\alpha(\bar{Y}_\alpha)\|_2^2
\]

for all measurable functions \( \varphi_\alpha : \mathbb{R}^{d_\alpha} \rightarrow \mathbb{R} \), satisfying \( E\varphi_\alpha(\bar{Y}_\alpha) = 0 \) for all \( \alpha = 1,...,N \). The constants \( C_\pm = 1 \pm \sigma_0 \), where \( \sigma_0 \) denotes the largest eigenvalue of the matrix \( S^* = (s_{\alpha,\beta}^{i,j})_{i,\beta \leq N} \) with \( s_{\alpha,\beta}^{i,j} \) being the maximum singular values of the matrices \( R_{\alpha,\beta} = (\rho_{\alpha,\beta}^{i,j})_{i=1,...,d_\alpha,j=1,...,d_\beta} \) for \( \alpha \neq \beta \) and \( s_{\alpha,\alpha}^{i,j} := 0 \).

Note that while Theorems 1.1 and 1.2 give the decoupling estimates with sharp constants, Theorem 1.3 provides just rough estimate with not the best constants (in particular \( C_- \) could be negative). However, in view of the applications described in Cherny et al. [1], [2] the upper bound in (1) is still interesting.

**2 Proof of Theorem 1.3**

The proof of Theorem 1.3 follows the framework used in [3]. We need to introduce notations used in [3] and borrowed from [4].

For \( k = 0,1,... \) we define Wick’s polynomials : \( x^k \) : by the extension

\[
\exp \left( ax - \frac{a^2}{2} \right) = \sum_{k=0}^{\infty} a^k : x^k :.\]

(Wick’s polynomials are specially normalized Hermite’s polynomials used in mathematical physics. We find these notations convenient for multidimensional case.)

Let \( \bar{k} = (k_1, \ldots, k_d) \) be a \( d \)-dimensional vector of non-negative integers. Set

\[
|\bar{k}| := k_1 + \cdots + k_d,
\]

\[
\bar{k}! := k_1!k_2! \cdots k_d!.
\]

\[
\bar{a}^k := a_1^{k_1}a_2^{k_2} \cdots a_d^{k_d}, \quad \text{for } \bar{a} := (a_i)_{i=1,...,d} \in \mathbb{R}^d.
\]

For a vector variable \( \bar{x} = (x_k)_{k=1}^{d} \) we define multidimensional Wick’s polynomial by

\[
: \bar{x}^k : := \prod_{i=1}^{d} : x_i^{k_i} :.
\]
It is well known (see e.g. [4]) that for a standard \(d\)-dimensional normal vector \(\bar{Y}\) the system \(\{k^{-1/2} : \bar{Y}^k :\}_{k \in \mathbb{Z}_0^d}\) is an orthonormal bases in the \(L_2\) space generated by all square-integrable \(f(\bar{Y})\). So for each \(\alpha = 1, \ldots, N\), we have

\[
\varphi_\alpha(\bar{Y}_\alpha) = \sum_{k \in \mathbb{Z}_0^d} a_{\alpha,k} \bar{Y}_\alpha^k = \sum_{n=0}^{\infty} \sum_{k \in \mathbb{Z}_0^d, |k|=n} a_{\alpha,k} \bar{Y}_\alpha^k \]

and therefore

\[
\sum_{\alpha=1}^{N} \varphi_\alpha(\bar{Y}_\alpha) = \sum_{\alpha=1}^{N} \sum_{n=0}^{\infty} \sum_{k_\alpha \in \mathbb{Z}_0^{d_\alpha}, |k_\alpha|=n} a_{\alpha,k_\alpha} \bar{Y}_\alpha^{k_\alpha} = \sum_{n=0}^{\infty} \sum_{\alpha=1}^{N} P_n(\bar{Y}_\alpha),
\]

where \(P_n(\bar{Y}_\alpha) := \sum_{k_\alpha \in \mathbb{Z}_0^{d_\alpha}, |k_\alpha|=n} a_{\alpha,k_\alpha} \bar{Y}_\alpha^{k_\alpha} \). In particular,

\[
\sum_{\alpha=1}^{N} \|\varphi_\alpha(\bar{Y}_\alpha)\|^2 = \sum_{n=0}^{\infty} \sum_{\alpha=1}^{N} \|P_n(\bar{Y}_\alpha)\|^2.
\]

It is well-known that \(P_{n_1}(\bar{Y}_\alpha)\) and \(P_{n_2}(\bar{Y}_\beta)\) are orthogonal whenever \(n_1 \neq n_2\) (see e.g. [3]). Consequently, to prove [1] for arbitrary \(\varphi_\alpha\) it suffices to prove it for \(\phi_\alpha(\bar{Y}_\alpha) = P_n(\bar{Y}_\alpha)\) for each \(n = 0, 1, \ldots\).

For a fixed \(n\), we have

\[
\left\| \sum_{\alpha=1}^{N} P_n(\bar{Y}_\alpha) \right\|^2 = \sum_{\alpha=1}^{N} \|P_n(\bar{Y}_\alpha)\|^2 + 2 \sum_{\alpha<\beta} \mathbb{E}[P_n(\bar{Y}_\alpha)P_n(\bar{Y}_\beta)]. \quad (2)
\]

Fix a pair \(\alpha < \beta\). Let \(R^{\alpha,\beta} = U\Sigma V^T\) be the singular value decomposition of the matrix \(R^{\alpha,\beta}\) (recall that here \(\Sigma\) is a \(d_\alpha \times d_\beta\) diagonal matrix whose diagonal entries are the singular values of \(R^{\alpha,\beta}\) and \(U, V\) are orthogonal matrices of corresponding sizes). Let \(\bar{Z}_1 := U\bar{Y}_\alpha\) and \(\bar{Z}_2 := V\bar{Y}_\beta\) (all the vectors are assumed being columns). This transformation can be written in the block matrix form

\[
\begin{pmatrix}
\bar{Z}_1 \\
\bar{Z}_2
\end{pmatrix} =
\begin{pmatrix}
U & 0 \\
0 & V
\end{pmatrix}
\begin{pmatrix}
\bar{Y}_\alpha \\
\bar{Y}_\beta
\end{pmatrix}
\]

with the obviously orthogonal \((d_\alpha + d_\beta) \times (d_\alpha + d_\beta)\) transformation matrix. It follows that

\[
\mathbb{E}[P_n(\bar{Y}_\alpha)P_n(\bar{Y}_\beta)] = \mathbb{E}[P_n(\bar{Z}_1)P_n(\bar{Z}_2)] = \sum_{k_\alpha \in \mathbb{Z}_0^{d_\alpha}} \sum_{|k_\alpha|=n} \sum_{k_\beta \in \mathbb{Z}_0^{d_\beta}} \sum_{|k_\beta|=n} a_{\alpha,k_\alpha} a_{\beta,k_\beta} \mathbb{E}[\bar{Z}_1^{k_\alpha}] \bar{Z}_2^{k_\beta}\]

\[
\sum_{k_\alpha \in \mathbb{Z}_0^{d_\alpha}} \sum_{|k_\alpha|=n} \sum_{k_\beta \in \mathbb{Z}_0^{d_\beta}} \sum_{|k_\beta|=n}
\]

Note that the covariance structure of \((\bar{Z}_1, \bar{Z}_2)\) is relatively simple, its covariance matrix is

\[
\begin{pmatrix}
I_{d_\alpha} & \sum \Sigma^T \\
\sum \Sigma & I_{d_\beta}
\end{pmatrix}
\]
Without loss of generality we assume that \( d_\alpha \geq d_\beta \), i.e. the vectors \( \bar{Y}_\alpha \) are ordered according to their dimensions. Let us agree that the index vectors of different dimensions are equal \((\bar{k}_\alpha = \bar{k}_\beta)\) if the shorter vector \( \bar{k}_\beta \) coincides with the first \( d_\beta \) entries of \( k_\alpha \) and the other entries of \( k_\alpha \) are zeros. Also let us denote by \( \tilde{s} \) the vector of the diagonal entries of \( \Sigma \) (i.e. the singular values of \( R^{\alpha,\beta} \)).

Using Lemma 3.1 from [3] (for the case of diagonal \( R \), see also formula (3.5) in [3]) we conclude

\[
E : \tilde{Z}^{\bar{k}_\alpha}_1 : \tilde{Z}^{\bar{k}_\beta}_2 := \left\{ \begin{array}{ll}
0, & \bar{k}_\alpha \neq \bar{k}_\beta \\
k_\beta! \tilde{s}^{\bar{k}_\beta}, & \bar{k}_\alpha = \bar{k}_\beta
\end{array} \right.
\]

Using (4) we continue (3) as follows

\[
E[P_n(\bar{Y}_\alpha)P_n(\bar{Y}_\beta)] = \sum_{\bar{k}_\beta \in \mathbb{Z}^d, \bar{k}_\alpha \in \mathbb{Z}^d: |\bar{k}_\beta| = n, k_\alpha = k_\beta} a_{\alpha,k_\alpha} a_{\beta,k_\beta} \tilde{s}^{\bar{k}_\beta}.
\]

Taking into account that the singular values lie in \([0, 1]\) (because they are certain correlations) we can estimate

\[
|E[P_n(\bar{Y}_\alpha)P_n(\bar{Y}_\beta)]| \leq s_{\alpha,\beta}^* \sum_{\bar{k}_\beta \in \mathbb{Z}^d, \bar{k}_\alpha \in \mathbb{Z}^d: |\bar{k}_\beta| = n, k_\alpha = k_\beta} |a_{\alpha,k_\alpha} a_{\beta,k_\beta}| \leq s_{\alpha,\beta}^* \left( \sum_{\bar{k}_\alpha \in \mathbb{Z}^d: |\bar{k}_\alpha| = n} a_{\alpha,k_\alpha}^2 \right)^{1/2} \left( \sum_{\bar{k}_\beta \in \mathbb{Z}^d: |\bar{k}_\beta| = n} a_{\beta,k_\beta}^2 \right)^{1/2} = s_{\alpha,\beta}^* \|P_n(\bar{Y}_\alpha)\|_2 \|P_n(\bar{Y}_\beta)\|_2
\]

Recalling (2) we conclude

\[
\left\| \sum_{\alpha=1}^N P_n(\bar{Y}_\alpha) \right\|_2^2 - \sum_{\alpha=1}^N \|P_n(\bar{Y}_\alpha)\|_2^2 \leq 2 \sum_{\alpha<\beta} s_{\alpha,\beta}^* \|P_n(\bar{Y}_\alpha)\|_2 \|P_n(\bar{Y}_\beta)\|_2.
\]

By standard linear algebra argument we have that the right hand side is bounded by \( \sigma_0 \sum_{\alpha=1}^N \|P_n(\bar{Y}_\alpha)\|_2^2 \), where \( \sigma_0 \) is the largest eigenvalue of the matrix \( S^* \). We conclude that for a fixed \( n \) the estimate (11) holds for \( \phi_\alpha(\bar{Y}_\alpha) = P_n(\bar{Y}_\alpha) \), \( \alpha = 1, \ldots, N \), with \( C_\pm = 1 \pm \sigma_0 \). As it was already pointed out this means that we have (1) for all \( \phi_\alpha \) with the same constants. \( \square \)

References

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