On recovering quadratic pencils with singular coefficients and entire functions in the boundary conditions

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The paper deals with a new type of inverse spectral problems for second-order quadratic differential pencils when one of the boundary conditions involves arbitrary entire functions of the spectral parameter. Although various aspects of the inverse spectral theory for the pencils have been of a special interest during the last decades, such settings were considered before only in the particular case of a Sturm–Liouville equation. We develop an approach covering also the quadratic dependence on the spectral parameter in the differential equation, which is based on the completeness and basisness of certain functional systems. By this approach, we obtain a uniqueness theorem and an algorithm for solving the inverse problem along with sufficient properties of the mentioned systems. The presented results give a universal tool for studying a number of important specific situations, including various Hochstadt–Lieberman-type inverse problems both on an interval and on geometrical graphs, which is illustrated as well.

KEYWORDS
analytical dependence on the spectral parameter, differential pencils, inverse spectral problems, partial inverse problems, singular coefficients, uniqueness theorem

MSC CLASSIFICATION
34A55, 34B05, 34B24

1 | INTRODUCTION

In the recent two decades, there essentially increased an interest in the inverse spectral theory for second-order quadratic differential pencils of the form

\[-y'' + [q(x) + 2\lambda p(x)]y = \lambda^2 y, \quad x \in (0, \pi),\]

(1)

under various types of boundary conditions including those polynomially dependent on the spectral parameter \(\lambda\) (see earlier studies\(^1\)–\(^{29}\) and references therein). Here, \(p(x)\) and \(q(x)\) are some, generally speaking, complex-valued functions in appropriate classes.

The inverse spectral theory deals with recovering operators from their spectral characteristics. The classical results in this direction (see earlier research\(^30\)–\(^37\)) relate to the Sturm–Liouville operator

\[\mathcal{L} y := -y'' + q(x)y = \lambda^2 y, \quad x \in (0, \pi),\]

(2)

which, in turn, is a particular case of (1) with \(p(x) = 0\) and with \(\lambda^2\) as a spectral parameter.
In particular, Borg\(^{30}\) established that the potential \(q(x)\) is uniquely determined by specifying the spectra of two boundary value problems for Equation (2) with one common boundary condition. For example, one can use the following two pairs of boundary conditions:

\[
y(0) = y^{(j)}(\pi) = 0, \quad j = 0, 1.
\]  

(Borg's result turned out to be a prototype of the future results also for pencil (1). Moreover, similarly to the case of the Sturm–Liouville operator (2), the specification of two spectra for (1) is equivalent to the specification of any other classical type of spectral data such as the Weyl function or one spectrum along with the corresponding weight numbers (see, e.g., earlier studies\(^{12,15,22,27,42}\), arises in modeling interactions between colliding relativistic particles in quantum mechanics,\(^1\) in mechanical systems vibrating in viscous media,\(^3\) and so on. We note that, technically, the term \(2\lambda p(x)y\) in (1) appears, for example, after separating variables in a string equation involving the first time-derivative of the unknown function, which is multiplied with a coefficient depending only on the spacial variable (see, e.g.,\(^{3,5}\)).

A majority of studies of the Sturm–Liouville operator as well as pencil (1) is devoted to the so-called regular case when \(q(x) \in L(0, \pi)\), while \(p(x) \in AC[0, \pi]\). However, during the last two decades, there appeared a lot of studies concerning the singular case \(q(x) \in W^{-1}_2[0, \pi]\) (see earlier studies\(^{38–44}\) and the references therein) and \(p(x) \in L_2(0, \pi)\), respectively, as soon as one deals with the pencil. The former means that \(q = \sigma^2\) for some \(\sigma \in L_2(0, \pi)\), where the derivative is understood in sense of distributions. The regularization of the differential expression in (2) requires introducing the quasi-derivative \(y^{[1]} := y' - \sigma(x)y\), after which \(\ell'\) takes the form

\[
\ell' y = -(y^{[1]})' - \sigma(x)y^{[1]} - \sigma^2(x)y
\]

whenever \(y, y^{[1]} \in AC[0, \pi]\) (see, e.g., Savchuk and Shkalikov\(^{39}\)). Thus, Equation (1) can be rewritten as

\[
\ell' y + 2\lambda p(x)y = \lambda^2 y,
\]

with such \(\ell'\), while the first derivative \(y'\) in (3) should be replaced with the quasi-derivative \(y^{[1]}\).

We note that considering the negative Sobolev space \(W^{-1}_2[0, \pi]\) allows one to cover various singular potentials that widely appear in quantum mechanics\(^{45}\) such as the Dirac delta-function or Coulomb-type singularities\(^{1,3}\). Moreover, such settings are often more convenient also from a purely mathematical point of view since they give treatment in a more general class. We note that inverse problems for pencil (1) with singular coefficients were studied in earlier studies\(^{12,15,29}\).

In the present paper, we focus on the singular case and study a new type of inverse spectral problems for pencil (1), which deals with special boundary conditions involving arbitrary transcendental entire functions of the spectral parameter \(f_1(\lambda)\) and \(f_2(\lambda)\):

\[
y(0) = 0, \quad f_1(\lambda)y^{[1]}(\pi) + f_2(\lambda)y(\pi) = 0.
\]  

Specifically, let \(\{\lambda_n\}_{n \geq 1}\) be the spectrum or a subspectrum of the boundary value problems (4) and (5) and denote by \(\omega_0\) the fractional part of the mean value

\[
\omega = \frac{1}{\pi} \int_0^\pi p(t)dt.
\]

Then the inverse problem is formulated as follows.

**Inverse Problem 1.** Given \(\{\lambda_n\}_{n \geq 1}\) and \(\omega_0\), recover the functions \(p(x)\) and \(\sigma(x)\).

In many concrete situations, the value \(\omega_0\) can be found from the asymptotics of \(\{\lambda_n\}_{n \geq 1}\), and thus, its specification is excessive. In the general case, however, it is impossible to say anything about behavior of the spectrum. Our strategy consists in establishing conditions under which the solution of inverse problem 1 is unique and can be obtained by an appropriate algorithm.

An advantage of considering boundary conditions of the form (5) is connected with a possibility of reducing many important classes of inverse problems to inverse problem 1. As illustration, in Sections 5 and 6, we reduce solving the Hochstadt–Lieberman-type inverse problems both on an interval and on a star-type graph to inverse problem 1. This
reduction gives us results that extend the corresponding ones in Yang and Zettl \(^\text{13}\) and Bondarenko, \(^\text{20}\) respectively, to the singular case.

We note that inverse problems involving transcendental entire functions in one boundary condition were studied before only for the Sturm–Liouville Equation (2), that is, when \(p(x) = 0\). An important such consideration relates to the so-called transmission eigenvalue problems, \(^\text{46–51}\) which have attracted much attention in connection with the inverse acoustic scattering problem. Transmission eigenvalue problems possess some concrete \(f_1(\lambda)\) and \(f_2(\lambda)\) in the boundary condition, which allows one to investigate the spectrum.

Arbitrary entire functions in the boundary condition, to the best of our knowledge, for the first time appeared in Yang et al.\(^\text{52}\) That required, however, making appropriate assumptions on some related functional systems in order to obtain a solution of the inverse problem. Further aspects of such problems were studied in earlier research, \(^\text{53,54}\) while their connection with partial inverse problems on geometrical graphs was established in Bondarenko.\(^\text{55}\) In our study, this range of questions is addressed for the first time to more general pencil (1) with nonlinear dependence on the spectral parameter.

The paper is organized as follows. In Section 2, we introduce necessary objects and provide auxiliary results. In Section 3, we obtain a uniqueness theorem and an algorithm for solving inverse problem 1; see Theorem 1 and Algorithm 1. In Section 4, we find alternative conditions on the subspectrum; see Theorem 2. In Section 5, the Hochstadt–Lieberman-type partial inverse problem on an interval is reduced to inverse problem 1. Finally, in Section 6, by this method, we obtain the solution of a partial inverse problem for pencil (1) on a star-type graph.

2 PRELIMINARIES

Consider the solutions \(S(x, \lambda)\) and \(C(x, \lambda)\) of Equation (4) satisfying the initial conditions

\[
S(0, \lambda) = 0, \ S^{[1]}(0, \lambda) = 1, \ C(0, \lambda) = 1, \ C^{[1]}(0, \lambda) = 0.
\]

These solutions exist, and they are unique since (4) can be rewritten as a first-order system with integrable coefficients; see earlier studies.\(^\text{39,42}\) By the same reason, \(S(\pi, \lambda), S^{[1]}(\pi, \lambda), C(\pi, \lambda),\) and \(C^{[1]}(\pi, \lambda)\) are entire functions. Further, we use the following representations being a consequence of Pronska\(^\text{42}\), Theorem 1 (see also Bondarenko and Gaidel\(^\text{29}\), Lemma 2.1):

\[
\begin{align*}
S(\pi, \lambda) &= \frac{\sin \pi (\lambda - \omega)}{\lambda} + \frac{1}{\pi} \int_0^\pi \mathcal{K}(t) \exp(i \lambda t) \, dt, \\
S^{[1]}(\pi, \lambda) &= \cos \pi (\lambda - \omega) + \int_0^\pi \mathcal{N}(t) \exp(i \lambda t) \, dt,
\end{align*}
\]

where \(\mathcal{K}, \mathcal{N} \in L_2(-\pi, \pi)\).

Put \(Z_0 = \mathbb{Z} \setminus \{0\}\). The zeros of the function \(S(\pi, \lambda)\) can be enumerated as \(\{\theta_k\}_{k \in \mathbb{Z}_o}\) counting their multiplicities in such way that they satisfy asymptotics

\[
\theta_k = k + \omega_0 + \kappa_k, \ k \in \mathbb{Z}_o, \ \{\kappa_k\}_{k \in \mathbb{Z}_o} \in \ell_2,
\]

see Pronska.\(^\text{42}\) Let us denote \(\theta_0 = 0\) and

\[
S_\theta = \{n \in \mathbb{Z} : \forall k < n \ \theta_k \neq \theta_n\}, \ m_{\theta,n} = \#\{k \in \mathbb{Z} : \theta_n = \theta_k\}.
\]

Without loss of generality, we assume that for \(n > 0, \ \theta_n \neq 0,\) and equal values in the sequence \(\{\theta_k\}_{k \in \mathbb{Z}}\) follow each other:

\[
\theta_n = \theta_{n+1} = \ldots = \theta_{n+m_{\theta,n}-1}, \ n \in S_\theta.
\]

We also introduce the notations

\[
e(x, \lambda) = \exp(i \lambda x), \ f^{(j)}(z) = \left. \frac{d^j}{d \lambda^j} f(\lambda) \right|_{\lambda = z}.
\]

Hereafter, we will use the following fact being a direct consequence of Lemma 2 in Buterin.\(^\text{56}\)
Proposition 1. Let \( \{ \theta_k \}_{k \in \mathbb{Z}_0} \) be an arbitrary sequence satisfying asymptotics (7). Then, the functional sequence 
\[ e^{\{C+n\} t, \theta_n} \{n \in \mathbb{Z}_0, v=0, m_{\theta,n}-1 \} \] is a Riesz basis in \( L_2(-\pi, \pi) \).

Consider the Weyl function \( M(\lambda) = -\frac{C(\pi, \lambda)}{S(\pi, \lambda)} \). In Bondarenko and Gaidel,\(^{29}\) the following inverse problem was studied:

**Inverse Problem 2.** Given \( \{ \theta_k \}_{k \in \mathbb{Z}_0} \) and the values

\[ M_{n+\nu} := \text{Res}_{\theta=\theta_n} (\theta - \theta_n)^\nu M(\theta), \quad n \in \mathbb{Z}_0, \quad \nu = \begin{cases} 0, & \text{if } m_{\theta,n} - 1, \theta_n \neq 0, \\
0, & \text{if } m_{\theta,n} - 2, \theta_n = 0, \\
0, & \text{if } m_{\theta,n} - 2, \theta_n = 0. \end{cases} \]

find \( p \) and \( q \).

We note that if \( S(\pi, \lambda) \) and \( S^{(1)}(\pi, \lambda) \) are known, then one can find the input data of inverse problem 2. Indeed, it is easy to see that

\[ C(x, \lambda)S^{(1)}(x, \lambda) - C^{(1)}(x, \lambda)S(x, \lambda) = 1, \quad x \in [0, \pi]. \tag{9} \]

Using (9) in \( x = \pi \), we obtain

\[ M(\lambda)S^{(1)}(\pi, \lambda) = \frac{1}{S(\pi, \lambda)} - C^{(1)}(\pi, \lambda). \]

Relation (9) also yields \( S^{(1)}(\pi, \theta_n) \neq 0, n \in \mathbb{Z}_0 \). Then, we have

\[ M_{n+\nu} = \frac{1}{S^{(1)}(\pi, \theta_n)} \text{Res}_{\theta=\theta_n} (\theta - \theta_n)^\nu S(\pi, \theta), \quad n \in \mathbb{Z}_0, \quad \nu = \begin{cases} 0, & \text{if } m_{\theta,n} - 1, \theta_n \neq 0, \\
0, & \text{if } m_{\theta,n} - 2, \theta_n = 0. \end{cases} \tag{10} \]

Since \( \{ \theta_k \}_{k \in \mathbb{Z}_0} \) are the zeros of \( S(\pi, \lambda) \), we uniquely construct the input data of inverse problem 2. In Bondarenko and Gaidel,\(^{29}\) for inverse problem 2, a uniqueness theorem and an algorithm of solution were obtained in the singular case \( q \in W^{-1}_2(0, \pi) \) and \( p \in L_2(0, \pi) \). In the regular case \( q \in L_2(0, \pi) \) and \( p \in W^1_2(0, \pi) \), the corresponding results were obtained earlier in previous studies.\(^{6,14}\) Further, we reduce inverse problem 1 to inverse problem 2 and apply the results from Bondarenko and Gaidel.\(^{29}\)

## 3  |  A UNIQUENESS THEOREM AND AN ALGORITHM

In this section, we consider the characteristic function and the subspectrum of the boundary value problems (4) and (5). We introduce the designations which allow working with multiple values in the subspectrum. After constructing a special functional sequence \( \{ v_n \}_{n=0}^\infty \), we obtain a uniqueness theorem and an algorithm for inverse problem 1.

Denote as \( L(p, \sigma) \) the boundary value problem (4) and (5). Note that a number \( \lambda \) is an eigenvalue of the boundary value problem \( L(p, \sigma) \) if and only if it is a zero of the characteristic function

\[ \Delta(\lambda) = f_1(\lambda)\eta_1(\lambda) + f_2(\lambda)\eta_2(\lambda), \tag{11} \]

where we denoted \( \eta_1(\lambda) = S(\sigma, \lambda) \) and \( \eta_2(\lambda) = S^{(1)}(\sigma, \lambda) \). Consider a sequence \( \{ \lambda_k \}_{k \geq 1} \) such that \( \Delta(\lambda_k) = 0 \) and each \( \lambda_k \) occurs in the sequence not more times than its multiplicity as zero of \( \Delta(\lambda) \). We call \( \{ \lambda_k \}_{k \geq 1} \) subspectrum.

Now, following the scheme applied in Bondarenko,\(^{53}\) we construct certain functional sequence by the given subspectrum. Put \( \lambda_0 := 0 \) and introduce the notations

\[ \mathcal{S}_\lambda = \{ n \geq 0 : \lambda_n \neq \lambda_k \forall k : 0 \leq k < n \}, \quad m_{\lambda,n} = \#\{ k \geq 0 : \lambda_k = \lambda_n \}. \]

We make the following assumption on \( \{ \lambda_k \}_{k \geq 2} \) analogous to (8):

\[ \lambda_n = \lambda_{n+1} = \ldots = \lambda_{n+m_{\lambda,n}-1}, \quad n \in \mathcal{S}_\lambda. \]
Substituting (6) into (11), we obtain
\[
\lambda \Delta(\lambda) = \lambda f_1(\lambda) \left( \lambda \cos(\lambda - \omega) + \int_{-\pi}^{\pi} \mathcal{N}(t)e(t, \lambda) dt \right) + f_2(\lambda) \left( \sin(\lambda - \omega) + \int_{-\pi}^{\pi} \mathcal{K}(t)e(t, \lambda) dt \right).
\]  
(12)

Introduce the Hilbert space of complex-valued vector-functions
\[
\mathcal{H} = L_2(-\pi, \pi) \oplus L_2(-\pi, \pi).
\]

For \( g = [g_1, g_2] \) and \( h = [h_1, h_2] \), the scalar product and the norm in \( \mathcal{H} \) are given by the formulae
\[
(g, h)_{\mathcal{H}} = \int_{-\pi}^{\pi} \left( g_1(t)h_1(t) + g_2(t)h_2(t) \right) dt, \quad \|h\|_{\mathcal{H}} = \sqrt{(h, h)}.
\]

In particular, we have \( u(t) := \left[ \overline{\mathcal{N}(t)} \overline{\mathcal{K}(t)} \right] \in \mathcal{H} \).

Then, relation (12) yields
\[
(u(\cdot), v(\cdot, \lambda))_{\mathcal{H}} = \lambda \Delta(\lambda) + w(\lambda),
\]
where
\[
v(t, \lambda) = \left[ \lambda f_1(\lambda)e(t, \lambda) \quad f_2(\lambda)e(t, \lambda) \right],
\]
\[
w(\lambda) = -f_1(\lambda)\lambda \cos(\lambda - \omega) - f_2(\lambda)\sin(\lambda - \omega).
\]

Since \( \{\lambda_k\}_{k \geq 0} \) are zeros of \( \lambda \Delta(\lambda) \), we get
\[
(u(\cdot), v^{<\nu>(\cdot, \lambda_n)})_{\mathcal{H}} = w^{<\nu>(\lambda_n)}, \quad n \in \mathbb{S}_1, \quad \nu = 0, m_{\lambda,n} - 1.
\]
(13)

For \( n = 0 \) and \( \nu = 0 \), by the analyticity of \( S(\pi, \lambda) \) at \( \lambda = 0 \), we have a stronger relation
\[
\int_{-\pi}^{\pi} \mathcal{K}(t) dt = \sin \pi \omega;
\]
(14)

see (6). For \( n \in \mathbb{S}_1 \) and \( \nu = 0, m_{\lambda,n} - 1 \), we denote
\[
v_{n+\nu}(t) = \begin{cases} 
  v^{<\nu>(t, \lambda_n)}, & n + \nu > 0, \\
  [0, 1], & n = \nu = 0, \\
  w_{n+\nu} = \begin{cases} 
    w^{<\nu>(\lambda_n)}, & n + \nu > 0, \\
    \sin \pi \omega, & n = \nu = 0.
  \end{cases}
\end{cases}
\]

Obviously, when \( n \in \mathbb{S}_1 \) and \( \nu = 0, m_{\lambda,n} - 1 \), the index \( k = n + \nu \) runs over all values in \( \mathbb{N} \cup \{0\} \). Using the introduced notations, from (13) and (14), we obtain
\[
(u, v_k)_{\mathcal{H}} = w_k, \quad k \geq 0.
\]
(15)

Consider the following assumption on the sequence \( \{v_k\}_{k=0}^{\infty} \):

(C) The sequence \( \{v_k\}_{k=0}^{\infty} \) is complete in \( \mathcal{H} \).

If condition (C) is fulfilled, we can find \( u = \left[ \overline{\mathcal{K}} \overline{\mathcal{N}} \right] \). Then, the functions \( S^{[1]}(\pi, \lambda) \) and \( S(\pi, \lambda) \) become known, and the input data of inverse problem 2 can be constructed uniquely.

Now, we are ready to obtain a uniqueness theorem for inverse problem 1. Along with \( L(p, \sigma) \), we consider another boundary value problem \( L(\tilde{p}, \tilde{\sigma}) \) of the same form but with other coefficients \( \tilde{p}, \tilde{\sigma} \in L_2(0, \pi) \). Let us agree that, if a symbol \( \alpha \) denotes an object related to \( L(p, \sigma) \), then the symbol \( \tilde{\alpha} \) with tilde will denote the analogous object related to \( L(\tilde{p}, \tilde{\sigma}) \).
Theorem 1. Let \((\lambda_n)_{n=1}^{\infty}\) and \((\overline{\lambda}_n)_{n=1}^{\infty}\) be subspectra of the boundary value problems \(L(p, \sigma)\) and \(L(\overline{p}, \overline{\sigma})\), respectively. Suppose that the sequence \((v_n)_{n=0}^{\infty}\) constructed by \((\lambda_n)_{n=1}^{\infty}\) satisfies condition (C). Then the equalities \((\lambda_n)_{n=1}^{\infty} = (\overline{\lambda}_n)_{n=1}^{\infty}\) and \(\omega_0 = \overline{\omega}_0\) yield \(p = \overline{p}, \sigma = \overline{\sigma}\).

Proof. I. Let us construct the sequences \((\tilde{v}_k)_{k=0}^{\infty}\) and \((\tilde{w}_k)_{k=0}^{\infty}\) by the subspectrum \((\overline{\lambda}_n)_{n=1}^{\infty}\) analogously to the sequences \((v_k)_{k=0}^{\infty}\) and \((w_k)_{k=0}^{\infty}\). Similar to (15), we have

\[
(\tilde{u}, \tilde{v}_k) = \tilde{w}_k, \quad k \geq 0, \quad \tilde{u} := \left[ \tilde{\mathcal{N}} \tilde{\mathbf{K}} \right].
\]

The condition \(\omega_0 = \overline{\omega}_0\) means that \(\omega = \overline{\omega} + m\) for some \(m \in \mathbb{Z}\). Since \((\lambda_n)_{n=1}^{\infty} = (\overline{\lambda}_n)_{n=1}^{\infty}\) and \(\omega = \overline{\omega} + m\), by the antiperiodicity of the functions \(\sin \pi x\) and \(\cos \pi x\), the equalities \(\tilde{v}_k = v_k\) and \(\tilde{w}_k = (-1)^m w_k\) hold. The latter means \((u - (-1)^m \tilde{u}, v_n) = 0\), and by condition (C), we arrive at \(u = (-1)^m \tilde{u}\) in \(\mathcal{H}\). Then, we have \(\mathcal{K} \equiv (-1)^m \tilde{\mathbf{K}}\) and \(\mathcal{N} \equiv (-1)^m \tilde{\mathcal{N}}\).

By formula (6), the identities

\[
S(x, \lambda) = (-1)^m \tilde{S}(x, \lambda), \quad S^{[1]}(x, \lambda) = (-1)^m \tilde{S}^{[1]}(x, \lambda)
\]

are fulfilled. Then, the corresponding sequences \((\theta_k)_{k \in \mathbb{Z}_0}\) and \((\overline{\theta}_k)_{k \in \mathbb{Z}_0}\), being the part of input data for inverse problem 2, coincide. Using (10) and the analogous relation for \(\tilde{M}_k\), we get \(M_k = \tilde{M}_k, k \in \mathbb{Z}_0\). By virtue of the uniqueness theorem\textsuperscript{29}, Theorem 2.9, for inverse problem 2, we have proved that \(p = \overline{p}\) in \(L_2(0, \pi)\) and \(q = \overline{q}\) in \(W^{-1}_2(0, \pi)\). We also have \(\omega = \overline{\omega}\).

II. Let us prove that \(\sigma = \overline{\sigma}\) as well. Since \(q = \overline{q}\) in \(W^{-1}_2(0, \pi)\), for some \(h \in \mathbb{C}\), we have \(\sigma = \overline{\sigma} + h\). Then, it is easy to see that

\[
S(x, \lambda) = \tilde{S}(x, \lambda), \quad S^{[1]}(x, \lambda) = \tilde{S}^{[1]}(x, \lambda) = hS(x, \lambda), \quad x \in [0, \pi], \quad \lambda \in \mathbb{C}.
\]

Substituting \(x = \pi\), we obtain \(\tilde{S}^{[1]}(\pi, \lambda) - S^{[1]}(\pi, \lambda) = hS(\pi, \lambda)\). Using this formula and (16) with \(m = 0\), we have \(hS(\pi, \lambda) \equiv 0\). Taking arbitrary \(\lambda \neq \theta_n\), we obtain \(h = 0\).

For the algorithm, we need a stronger assumption than (C):

(B) The sequence \((v_k)_{k=0}^{\infty}\) is an unconditional basis in \(\mathcal{H}\).

Under this condition, there exists the basis \((v_k)_{k=0}^{\infty}\) which is biorthogonal to \((v_k)_{k=0}^{\infty}\), and the series \(\sum_{k=0}^{\infty} (f, v_k)v_k^*\) converges to \(f\) for each \(f \in \mathcal{H}\); see Gohberg and Krein.\textsuperscript{57, Ch. VI} Based on this fact and the above computations, we obtain the following algorithm for solving inverse problem 1.

Algorithm 1. Given the subspectrum \((\lambda_n)_{n=1}^{\infty}\) and the value \(\omega_0\), in order to recover \(p\) and \(\sigma\), one should:

1. Construct \((v_k)_{k=0}^{\infty}\) and \((w_k)_{k=0}^{\infty}\) putting \(\omega := \omega_0\).
2. Find the sequence \((v_k)_{k=0}^{\infty}\) which is biorthogonal to \((v_k)_{k=0}^{\infty}\) in \(\mathcal{H}\), and construct

\[
u = \left[ \nu \right] = \sum_{k=0}^{\infty} \overline{w}_k v_k^*.
\]

3. Construct the functions \(S(x, \lambda)\) and \(S^{[1]}(x, \lambda)\) by (6).
4. Find \((\theta_k)_{k \in \mathbb{Z}_0}\) as the zeros of \(S(x, \lambda)\) and then \((M_k)_{k \in \mathbb{Z}_0}\) by formula (10).
5. Find \(p\) and \(q\) as the solution of inverse problem 2, see Algorithm 2.8 in Bondarenko and Gaidel.\textsuperscript{29}
6. Put \(\omega = \frac{1}{\pi} \int_0^\pi p(s) ds\). If \(\omega - \omega_0\) is odd, multiply by \(-1\) the functions \(S(x, \lambda)\) and \(S^{[1]}(x, \lambda)\) constructed in 3.
7. Choose any \(\tilde{\sigma} \in L_2(0, \pi)\) such that \(\tilde{\sigma}' = q\). Construct \(\tilde{S}^{[1]}(x, \lambda)\) corresponding to the boundary value problem \(L(p, \tilde{\sigma})\).

Find \(\sigma\) by the formula

\[
\sigma = \tilde{\sigma} + h, \quad h = \frac{\tilde{S}^{[1]}(x, \lambda) - S^{[1]}(x, \lambda)}{S(x, \lambda)}.
\]
Remark 1. In fact, for Algorithm 1, we need only the basis property of \( \{ v_k \}_{k=0}^{\infty} \), being weaker than condition (B). However, practically, it is easier to verify alternative conditions sufficient for (B); see Theorem 2 in the next section.

4 SUFFICIENT CONDITIONS FOR (C) AND (B)

In this section, we obtain conditions guarantying (C) and (B) that are easier to verify. Consider the following list of independent conditions:

(S) For \( n \in \mathbb{N} \), the functions \( f_1(\lambda) \) and \( f_2(\lambda) \) do not vanish simultaneously at \( \lambda_n \).

(C2) The functional sequence \( \{ e^{i \gamma \lambda}(t, \lambda_n) \}_{n \in \mathbb{Z}, \lambda_n = 0, m_{\lambda,n} - 1} \) is complete in \( L_2(-2\pi, 2\pi) \).

(A) For the subspectrum \( \{ \lambda_n \}_{n \geq 1} \), the following formulae hold: \( 1m \lambda_n = O(1) \), and \( m_{\lambda,n} = 1 \) for \( n \geq n_0 \).

(B2) The functional sequence \( \{ e^{i \gamma \lambda}(t, \lambda_n) \}_{n \in \mathbb{Z}, \lambda_n = 0, m_{\lambda,n} - 1} \) is a Riesz basis in \( L_2(-2\pi, 2\pi) \).

The main result of the section is the following theorem.

**Theorem 2.** (i) Under conditions (S) and (C2), condition (C) holds.

(ii) Under conditions (S), (A), and (B2), condition (B) holds.

By assumption (i) of this theorem, conditions (S) and (C2) are sufficient for fulfilling assumptions of Theorem 1. By assumption (ii), conditions (S), (A), and (B2) are sufficient for Algorithm 1. We also note that condition (S) appears naturally because if \( f_1(\lambda_n) = 0 \) and \( f_2(\lambda_n) = 0 \) at the same time, then the eigenvalue \( \lambda_n \) brings no information on the coefficients \( p \) and \( \sigma \).

To prove Theorem 2, we need several auxiliary results.

**Lemma 1.** Let us introduce the functions

\[
\begin{align*}
g(t, \lambda) &= \left[ \lambda \eta_1(\lambda) e(t, \lambda) - \eta_2(\lambda) e(t, \lambda) \right], \\
g_{n+\nu}(t) &= \begin{cases} 
 0 & n = 0, \\
 1 & \nu > 1, \\
 \left[ g^{\nu}(t, \lambda_n), n + \nu > 1, 
\right] & n = \nu = 0, 
\end{cases} \\
\end{align*}
\]

Under condition (S), the sequence \( \{ v_n \}_{n=0}^{\infty} \) is complete in \( \mathcal{H} \) whenever \( \{ g_n \}_{n=0}^{\infty} \) is complete in \( \mathcal{H} \).

Lemma 1 is proved analogously to Lemma 3.2 in Bondarenko.53

**Proposition 2.** Let \( G(\lambda) \) be an entire function such that

\[
|G(\lambda)| \leq C \exp(|1m \lambda|2\pi), \\
G \in L_2(\mathbb{R}), \\
G^{\nu}(\lambda) = 0, \\
\nu > 1, m_{\lambda,n} - 1.
\]

If condition (C2) holds, then \( G(\lambda) \equiv 0 \).

Proposition 2 easily follows from the Paley–Wiener theorem.

**Lemma 2.** Denote

\[
g^0(x, \lambda) = \left[ \exp(i\lambda x) \sin(\lambda - \omega) - \exp(i\lambda x) \cos(\lambda - \omega) \right].
\]

Let a sequence of distinct numbers \( \{ x_n \}_{n=0}^{\infty} \) be such that \( \{ \exp(i\tau_n x) \}_{n=0}^{\infty} \) is a Riesz basis in \( L_2(-2\pi, 2\pi) \). Then, \( g^0(x, \tau_n) \), \( n \geq 0 \), form a Riesz basis in \( \mathcal{H} \).

**Proof.** 1. First, suppose \( \omega = 0 \), then

\[
g^0_0(t) = \left[ \exp(i\tau_n t) \sin(\tau_n - \omega) - \exp(i\tau_n t) \cos(\tau_n - \omega) \right], \\
\quad n \geq 0.
\]

Let us prove completeness of the system \( \{ g^0_n \}_{n=0}^{\infty} \).

Consider \( h \in \mathcal{H} \) such that \( (h, g^0_0) = 0 \) for all \( n \geq 0 \). Introduce the function

\[
G(\lambda) = -\int_{-\pi}^{\pi} \left( h_1(t) \sin(\pi \lambda - \pi \omega) \right) \exp(i\lambda t) dt,
\]

(17)
which turns zero at $\lambda = \tau_n$. The function $G(\lambda)$ and the numbers $\lambda_n = \tau_n$, $n \geq 0$ satisfy the conditions of Proposition 2, and we get $G(\lambda) \equiv 0$. Substituting $\lambda = n \in \mathbb{Z}$ into (17), we obtain $h_2 = 0$, and as a consequence, $h_1 = 0$. Thus, we have proved $h = 0$, and the sequence $\{g_n^0\}_{n \geq 0}$ is complete in $H$.

Now, to prove $\{g_n^0\}_{n \geq 0}$ is a Riesz basis, it is sufficient to establish the two-side inequality

$$M_1 \sum_{n=0}^{N_0} |b_n|^2 \leq \left\| \sum_{n=0}^{N_0} b_n g_n^0 \right\|_H \leq M_2 \sum_{n=0}^{N_0} |b_n|^2$$

with some fixed constants $M_1$ and $M_2$ for any $N_0 \in \mathbb{N}$, $b_n \in \mathbb{C}$. For this purpose, we compute

$$(g_n^0, g_k^0)_H = \left( \sin \pi \tau_n \sin \pi \tau_k + \cos \pi \tau_n \cos \pi \tau_k \right) \int_{-\pi}^{\pi} \exp(i \tau_n t) \exp(i \tau_k t) dt.$$

Replacing $\sin z$ and $\cos z$ by combinations of exponential functions, after manipulations, we obtain

$$(g_n^0, g_k^0)_H = \frac{1}{2} (\exp(i \tau_n t), \exp(i \tau_k t))_{L_2(-2\pi, 2\pi)}.$$

Hence, it follows that

$$\left\| \sum_{n=0}^{N_0} b_n g_n^0 \right\|_H = \frac{1}{\sqrt{2}} \left\| \sum_{n=0}^{N_0} b_n \exp(i \tau_n t) \right\|_{L_2(-2\pi, 2\pi)}.$$

Since $\{\exp(i \tau_n x)\}_{n \geq 0}$ is a Riesz basis, the latter yields (18).

II. Consider the case of arbitrary $\omega$. Introduce the sequences

$$\alpha_n = \tau_n - \omega, \, \tilde{g}_n(x) = \left[ \exp(i \alpha_n x) \sin \pi \alpha_n - \exp(i \alpha_n x) \cos \pi \alpha_n \right], \, n \geq 0.$$

By part I, if $\{\exp(i \alpha_n t)\}_{n \geq 0}$ is a Riesz basis, so is $\{\tilde{g}_n\}_{n \geq 0}$. Note that multiplication by the function $\exp(-i\omega t)$ does not change the Riesz basis property. By this reason, $\{\exp(i \alpha_n t)\}_{n \geq 0}$ is a Riesz basis, and in turn, $g_n^0(t) = \exp(i\omega t)\tilde{g}_n(t)$, $n \geq 0$, constitute a Riesz basis.

**Proof of Theorem 2.** (i) Assume that (S) and (C2) hold. Let us prove the completeness of $\{g_n\}_{n=0}^\infty$ in $H$. Consider $h = [h_1 \, h_2] \in H$ such that $(h, g_k)_H = 0$, $k \geq 0$. Construct the function

$$G(\lambda) = \int_{-\pi}^{\pi} (h_1(t) \lambda \eta_1(\lambda) - h_2(t) \eta_2(\lambda))e(t, \lambda) dt.$$

Then, $G(\lambda)$ satisfies the conditions of Proposition 2, and $G(\lambda) \equiv 0$.

Substituting $\lambda = \theta_n$ into $G(\lambda)$, after differentiation, we obtain

$$\sum_{j=0}^{\nu} \int_{-\pi}^{\pi} h_2(t) \eta_2^{\nu-j}(\theta_n) e_2^{\nu-j}(t, \theta_n) dt = 0, \, n \in \mathbb{N}_0, \, \nu = 0, m_{\theta,n} - 1.$$

By induction, since $\eta_2(\theta_n) \neq 0$, we get

$$\int_{-\pi}^{\pi} e^{\nu-j}(t, \theta_n) h_2(t) dt = 0, \, n \in \mathbb{N}_0, \, \nu = 0, m_{\theta,n} - 1.$$

Using Proposition 1, we arrive at $h_2 = 0$, and hence, $h_1 = 0$. These identities mean the completeness of $\{g_n\}_{n=0}^\infty$ in $H$. By Lemma 1, we have proved (C).
(ii) Now, we prove that (S), (A), and (B2) yield (B). From formula (6), it follows that
\[ g(t, \lambda) = g^0(t, \lambda) + \exp(i\lambda t) \int_{-\pi}^{\pi} \exp(i\lambda \xi) f(\xi) d\xi, \quad f = [f_1, f_2] \in \mathcal{H}. \]

Referring to condition (A), we have \(|\exp(i\lambda_n t)| = O(1), n \geq n_0\). Since \{\exp(i\lambda_n \xi)\}_{n \geq n_0}

is a part of the Riesz basis in (B2), for \(j = 0, 1\), we have \(\{f_j^n \exp(i\lambda_n \xi) f_j(\xi) d\xi\}_{n \geq n_0} \in \ell_2\). Combining these formulæ, we obtain
\[
\left\{ \left\| \exp(i\lambda_n t) \int_{-\pi}^{\pi} \exp(i\lambda_n \xi) f(\xi) d\xi \right\|_{\mathcal{H}} \right\}_{n \geq n_0} \in \ell_2. \tag{19}
\]

Put \(r_n = \lambda_n\ for\ n \geq n_0\), while for the other \(n \in \mathbb{N}\), the numbers \(r_n\ are chosen such that \(r_n \neq r_k\ for all n \neq k\). Then, the system \{\exp(ir_n t)\}_{n \geq 0}\ differs only by a finite number of elements from the Riesz basis in (B2). It is easy to see that this system is complete; see corollary from Theorem 2 in Levin.58, Appendix III By Freiling and Yurko,34, Proposition 1.8.5 the system \{\exp(ir_n t)\}_{n \geq 0}\ is a Riesz basis as a complete sequence being quadratically close to a Riesz basis. Then, the system \{g_n^0\}_{n \geq 0}\ constructed in Lemma 2 is also a Riesz basis.

In part (i) of the theorem, we proved the completeness of the system \{g_n\}_{n \geq 0}. From formula (19), it follows that \(\{g_n\}_{n \geq 0}\) is quadratically close to \(\{g_n^0\}_{n \geq 0}\). Then, the system \{g_n\}_{n \geq 0}\ is a Riesz basis in \(\mathcal{H}\).

In Bondarenko53, Lemma 3.1 under condition (S), the following relations were obtained:
\[
\eta_j^{cv>} (\lambda_n) = (-1)^{j-1} \sum_{k=0}^{y} C_{n,k} f_j^{cv>} (\lambda_n), \quad n \in \mathbb{S}_0, \quad j = 1, 2, \quad g_n = C_{n,0} v_n.
\]

where \(j = 1, 2,\) and \(C_{n,0} \neq 0\). By condition (A), for \(n \geq n_0\), we have
\[
\eta_j (\lambda_n) = (-1)^{j-1} C_{n,0} f_j (\lambda_n), \quad j = 1, 2, \quad g_n = C_{n,0} v_n.
\]

Put \(C_{n,0} = 1\ for\ n \in \mathbb{S}_0\) and consider the functions \(v_n = C_{n,0}^{-1} v_n,\ n \geq 0\). Obviously, the system \{\tilde{v}_n\}_{n \geq 0},\ being complete and quadratically close to \{g_n\}_{n \geq 0},\ is a Riesz basis. It is known that a Riesz basis is an unconditional; see Gohberg and Krein.57, Theorem 2.2 Then, \(v_n = C_{n,0} \tilde{v}_n,\ n \geq 0,\ constitute an unconditional basis.

\[\square\]

5 | APPLICATION TO A PARTIAL INVERSE PROBLEM ON AN INTERVAL

Let us apply the obtained results to studying the inverse problem of recovering the pencil on the half of an interval by one spectrum. We study the boundary value problem
\[
y'' + q(x)y + 2\lambda p(x)y = \lambda^2 y, \quad x \in (0, 2\pi), \tag{20}
\]
\[
y(0) = y(2\pi) = 0, \tag{21}
\]
where \(p \in L_2(0, 2\pi)\) and \(q \in W^{-1}_2(0, 2\pi)\). Considering arbitrary \(\sigma \in L_2(0, 2\pi)\) such that \(q = \sigma^2\), we rewrite (20) in the form (4). From (7), we have that the eigenvalues \{\mu_k\}_{k \in \mathbb{Z}_0}\ of the boundary value problems (20) and (21) satisfy the asymptotics
\[
\mu_k = \frac{k}{2} + \frac{1}{2\pi} \int_0^{2\pi} p(t) dt + \kappa_k, \quad k \in \mathbb{Z}_0, \quad \{\kappa_k\}_{k \in \mathbb{Z}_0} \in \ell_2. \tag{22}
\]

Note that since the quasi-derivative is absent in boundary conditions (21), the eigenvalues \{\mu_k\}_{k \in \mathbb{Z}_0}\ do not depend on the choice of \(\sigma \in L_2(0, 2\pi)\) up to a constant function.

We assume that the coefficients \(p\) and \(q\) are known on \((\pi, 2\pi)\). In Yang and Zettl,13 in the regular case of \(p \in W^1_2(0, 2\pi)\) and \(q \in L_2(0, 2\pi)\), the following inverse problem was studied.
**Inverse Problem 3.** Given \( \{\mu_k\}_{k \in \mathbb{Z}_0} \) along with \( p \) and \( q \) on \((\pi, 2\pi)\), recover the coefficients \( p \) and \( q \) on the interval \((0, \pi)\).

For the first time, inverse problem 3 in the regular case was studied by Buterin.\(^9\) He proved a uniqueness theorem and obtained an algorithm for solving the inverse problem for the boundary conditions possessing the spectral parameter. For the Dirichlet boundary conditions, the uniqueness theorem was proved by Yang and Zettl,\(^13\) but under an additional assumption of the spectrum simplicity. In the present paper, by reducing inverse problem 3 to inverse problem 1, we prove the uniqueness theorem in the singular case without assuming the simplicity of the spectrum.

For \( x \in (\pi, 2\pi) \), we take arbitrary \( \sigma(x) \) such that \( q = \sigma' \) and introduce the solution \( \psi(x, \lambda) \) of (20) satisfying the initial conditions

\[
\psi(2\pi, \lambda) = 0, \quad \psi^{(1)}(2\pi, \lambda) = 1.
\]

A number \( \lambda \) is an eigenvalue of problems (20) and (21) if and only if the functions \( \psi(x, \lambda) \) and \( S(x, \lambda) \) are linearly dependent. The latter holds whenever

\[
\psi^{(1)}(\pi, \lambda)S(\pi, \lambda) - S^{(1)}(\pi, \lambda)\psi(\pi, \lambda) = 0.
\]

It is clear that \( \psi(\pi, \lambda) \) and \( \psi^{(1)}(\pi, \lambda) \) are known entire functions. Then, the eigenvalues \( \{\mu_k\}_{k \in \mathbb{Z}_0} \) of (20) and (21) coincide with the eigenvalues of (4) and (5), where we put \( f_1(\lambda) = -\psi(\pi, \lambda) \) and \( f_2(\lambda) = \psi^{(1)}(\pi, \lambda) \). This fact and the results from previous sections allow us to obtain a uniqueness theorem. We use the same agreement regarding the symbols \( a \) and \( \bar{a} \) as it was done before Theorem 1.

**Theorem 3.** Let \( q = \bar{q} \) in \( W^{-1}_2(\pi, 2\pi), p = \bar{p} \) in \( L_2(\pi, 2\pi) \), and \( \{\mu_k\}_{k \in \mathbb{Z}_0} = \{\mu_k\}_{k \in \mathbb{Z}_0} \). Then, the identities \( q \equiv \bar{q} \) and \( p \equiv \bar{p} \) hold on \((0, 2\pi)\).

**Proof.** Putting \( f_1(\lambda) = -\psi(\pi, \lambda), f_2(\lambda) = \psi^{(1)}(\pi, \lambda) \), we consider two boundary value problems \( L(p, \sigma) \) and \( L(\bar{p}, \bar{\sigma}) \). Then, the sequences \( \{\lambda_k\}_{k \geq 1} = \{\mu_k\}_{k \in \mathbb{Z}_0} \) and \( \{\bar{\lambda}_k\}_{k \geq 1} = \{\bar{\mu}_k\}_{k \in \mathbb{Z}_0} \) are their spectra, respectively.

Taking into account formula (22), one can uniquely reconstruct \( \omega_0 \) by the spectrum \( \{\mu_k\}_{k \in \mathbb{Z}_0} \) along with the known mean value of \( p(x) \) on \((\pi, 2\pi)\). By the conditions of the theorem, the latter yields \( \omega_0 = \bar{\omega}_0 \). Let us prove that condition (C) holds. Then, the statement of the theorem will follow from Theorem 1.

The functions \( \psi(\pi, \lambda) \) and \( \psi^{(1)}(\pi, \lambda) \) cannot turn into zero for the same \( \lambda \), otherwise \( \psi(\pi, \lambda) \) would be a trivial solution. Thus, condition (S) is fulfilled. By extending the segment in Proposition 1 to \((-2\pi, 2\pi)\) and substituting \( \{\theta_n\}_{n \in \mathbb{Z}_0} = \{\mu_n\}_{n \in \mathbb{Z}_0} \), we assert condition (B2), which is stronger than (C2). By virtue of Theorem 2, condition (C) holds.

In the proof, for the corresponding boundary value problems (4) and (5) and its spectrum \( \{\lambda_n\}_{n \geq 1} \), we obtained (S) and (B2). The fulfillment of condition (A) obviously follows from (22). Then, by Theorem 2, condition (B) holds, and we can apply Algorithm 1 for recovery of \( p \) and \( q \) on \((0, \pi)\).

### 6 PARTIAL INVERSE PROBLEM ON A STAR-TYPE GRAPH

In this section, we briefly illustrate application of the results obtained for inverse problem 1 to studying a partial inverse problem on the graph. Consider the star-type graph \( G \) with \( m \) edges \( e_j, j = 1, m \), of equal length \( \pi \). Parametrize each edge \( e_j \) with the variable \( x_j \in [0, \pi] \) such that \( x_j = 0 \) corresponds to the boundary vertex of \( e_j \), while \( x_j = \pi \) corresponds to the internal vertex. We study the following boundary value problem for the quadratic differential pencil:

\[
\ell_j y_j + 2\lambda p_j(x_j) y_j = \lambda^2 y_j, \quad x_j \in (0, \pi), \quad j = 1, m, \quad (23)
\]

\[
y_j(0) = 0, \quad j = 1, m, \quad (24)
\]

\[
y_1(\pi) = y_2(\pi) = \ldots = y_m(\pi), \quad \sum_{j=1}^m y_j^{(1)}(\pi) = 0, \quad (25)
\]

where \( \ell_j y_j := -\left(y_j^{(1)}\right)' - \sigma_j(x_j) y_j^{(1)} - \sigma_j^2(x_j) y_j \) with \( p_j, \sigma_j \in L_2(0, \pi), j = 1, m \). Here, condition (24) is the Dirichlet boundary conditions, while (25) is the standard matching condition imposed at the common vertex of the edges \( x_j = \pi \).
Let $S_j(x, \lambda)$ be the solution of the $j$-th equation in (23) under the initial conditions $S_j(0, \lambda) = 0$, $S_j^{[1]}(0, \lambda) = 1$. Then, a number $\lambda$ is an eigenvalue of problems (23)–(25) if and only if it is a zero of the characteristic function

$$\Delta(\lambda) = \sum_{j=1}^{m} S_j^{[1]}(\pi, \lambda) \prod_{k=1, k \neq j}^{m} S_k(\pi, \lambda).$$

Using the corresponding representations (6) for $S_j(x, \lambda)$ and $S_j^{[1]}(x, \lambda)$, analogously to Lemma 1 in Bondarenko, one can prove that the set $\Lambda$ of the eigenvalues of the boundary value problems (23)–(25) with the account of algebraic multiplicities has the form

$$\Lambda = \{\lambda_0\} \bigcup \{\lambda_{nk} \}_{n \in \mathbb{Z}_0, k = 1, m}, \quad \lambda_{nk} = n + \beta_k + r_{nk}, \ r_{nk} = o(1), \ n \in \mathbb{Z}_0, k = 1, m,$

(26)

where $\beta_k$, $k = 1, m$, are the zeroes of the function

$$P(\lambda) = \sum_{j=1}^{m} \cos \pi(\lambda - \omega_j) \prod_{k=1, k \neq j}^{m} \sin \pi(\lambda - \omega_k), \ \omega_k := \frac{1}{\pi} \int_{0}^{\pi} p_j(t) dt,$$

lying in the strip $0 \leq \Re \lambda < 1$. Moreover, for $k = 1, m$, we have $\{r_{nk}\}_{n \in \mathbb{Z}_0} \in l_2$ if the following condition holds:

(D) $\beta_k \neq \beta_j$ for $k \neq j$, while $k, j = 1, m$.

Consider a subspectrum $\{\lambda_n\}_{n \geq 1} \subseteq \Lambda$ of the boundary value problems (23)–(25). Assume that the pencil coefficients $p_j$ and $\sigma_j$ are known for $j = 1, m$. We study the following partial inverse problem.

**Inverse Problem 4.** Given a subspectrum $\{\lambda_n\}_{n \geq 1}$, the fractional part of the number $\omega_1$, and the coefficients $p_j, \sigma_j$ for $j = 1, m$, recover $p_1$ and $\sigma_1$.

For quadratic differential pencils in the regular case, partial inverse problems on graphs were studied in earlier studies. Here, reducing inverse problem 4 to inverse problem 1, we demonstrate our approach to studying partial inverse problems on graphs in the singular case.

Putting $p := p_1, \sigma := \sigma_1$, and $S(x, \lambda) = S_1(x, \lambda)$, we obtain that $\{\lambda_n\}_{n \geq 1}$ is a subspectrum of the boundary value problems (4) and (5) with the known entire functions

$$f_1(\lambda) = \prod_{k=2}^{m} S_k(\pi, \lambda), \ f_2(\lambda) = \sum_{k=2}^{m} S_k^{[1]}(\pi, \lambda) \prod_{l=2, l \neq k}^{m} S_l(\pi, \lambda).$$

(27)

This allows us to formulate a uniqueness theorem for inverse problem 4 using Theorems 1 and 2. We use the same agreement regarding the symbols $\sigma$ and $\tilde{\sigma}$ as it was done before Theorem 1.

**Theorem 4.** Let $\sigma_j = \tilde{\sigma}_j$ and $p_j = \tilde{p}_j$ for $j = 1, m$. Suppose that for the functions determined in (27) and for a subspectrum $\{\lambda_n\}_{n \geq 1}$ conditions (S) and (C2) hold. Then, the equalities $\{\lambda_n\}_{n \geq 1} = \{\lambda_n\}_{n \geq 1}$, $(\omega_1 \bmod 1) = (\tilde{\omega}_1 \bmod 1) \ y i e l d \sigma_1 = \tilde{\sigma}_1$ and $p_1 = \tilde{p}_1$.

According to Theorem 2, if the subspectrum $\{\lambda_n\}_{n \geq 1}$ additionally satisfies conditions (A) and (B2), we can apply Algorithm 1 for solving inverse problem 4. For the fulfillment of (A) and (B2), it is sufficient that the subspectrum has a specific form

$$\{\lambda_n\}_{n \geq 1} = \{\lambda_1\} \bigcup \{\lambda_{nk_1}\}_{n \in \mathbb{Z}_0} \bigcup \{\lambda_{nk_2}\}_{n \in \mathbb{Z}_0}, \ 1 \leq k_1 < k_2 \leq m,$$

and condition (D) holds (see asymptotics (26)).

**Remark 2.** Analogously one can study other types of boundary conditions and other partial inverse problems, including the ones from earlier studies.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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