On Koopman-von Neumann Waves

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In this paper we study the classical Hilbert space introduced by Koopman and von Neumann in their operatorial formulation of classical mechanics. In particular we show that the states of this Hilbert space do not spread, differently than what happens in quantum mechanics. The role of the phases associated to these classical "wave functions" is analyzed in details. In this framework we also perform the analog of the two-slit interference experiment and compare it with the quantum case.
1 Introduction

In their standard formulation classical and quantum mechanics are written in two completely different mathematical languages: for example in classical mechanics observables are functions of a 2n-dimensional phase space, while in quantum mechanics they are self-adjoint operators acting on a Hilbert space. In the literature there are a lot of attempts to reformulate classical and quantum mechanics in similar forms, [1]-[3]. In this paper we shall concentrate on the work of Koopman and von Neumann (KvN) who proposed, in 1931-32, an operatorial formulation of classical mechanics, [3]-[4]. The starting point of their work is the possibility of defining an Hilbert space of complex and square integrable classical ”wave” functions $\psi(\varphi)$ such that $\rho(\varphi) \equiv |\psi(\varphi)|^2$ can be interpreted as a probability density of finding a particle at the point $\varphi = (q, p)$ of the phase space. This $\rho$ has to evolve in time according to the well-known Liouville equation:

$$i \frac{\partial}{\partial t} \rho(q, p) = \hat{H} \rho(q, p)$$  (1.1)

where $\hat{H}$ is the Liouville operator $\hat{H} = -i \partial_p H \partial_q + i \partial_q H \partial_p$ and $H$ is the Hamiltonian of the standard phase space. In order to obtain (1.1) Koopman and von Neumann postulated the same evolution for $\psi$:

$$i \frac{\partial}{\partial t} \psi(q, p) = \hat{H} \psi(q, p)$$  (1.2)

Since $\hat{H}$ contains only first order derivatives it is easy to check that from eq. (1.2) one can derive eq. (1.1). This is something which does not happen in quantum mechanics where the analogue of $\hat{H}$ is the Schrödinger operator which contains second order derivatives.

Now in quantum mechanics the complex character of the wave function is of fundamental importance for a lot of reasons: while the modulus of the wave function gives the probability density $\rho$, the phase of $\psi$ brings in also some physical information. In fact it is related to the mean value of the momentum operator $\hat{p}$ and it gives origin to the appearance of interference effects in two-slit experiments. To answer the question if phases of classical ”wave” functions $\psi(q, p)$ play an analogous role also in classical mechanics is the main goal of this paper which is organized as follows.

In section 2 we will briefly review the Koopman-von Neumann formalism and the associated functional formulation [3]. In section 3 we will present a very simple but pedagogical example that shows how, in the operatorial approach to classical mechanics, the lack of an uncertainty relation between $\hat{q}$ and $\hat{p}$ and the different form of $\hat{H}$ with respect to the Schrödinger operator $\hat{H}$ prevents the spreading of the wave functions. We shall also show that the phase and the modulus totally decouple in the equation of motion of $\psi$ and that the phases do not influence the expectation values of the observables of classical mechanics. In section 4, after spending some words about the abstract Hilbert space of classical mechanics, we shall underline how a lot of the
previous considerations are a consequence of the particular representation we have chosen. It is the representation where the classical ”wave” functions $\psi(q,p)$ are given by functions of $q$ and $p$. If we change representation the situation changes drastically. In particular if we use a representation where $p$ in $\psi(q,p)$ is replaced by a different variable $\lambda_p$, then phases begin to play a crucial role since they bring in physical information (e.g. the mean value of $\hat{p}$). In this representation $\psi(q,\lambda_p)$ and $\rho(q,\lambda_p)$ do not evolve in the same way, as $\psi(q,p)$ and $\rho(q,p)$ used to do, and there is, in a certain sense, a sort of spreading of the wave function. Since in this representation phases seem to play a physical role we can say that it is necessary to consider, for mathematical consistency, an Hilbert space made up with complex and not with real wave functions.

In quantum mechanics the complexity of the wave functions is one of the most important reasons for the interference effects. Therefore a question that arises quite naturally is: what happens in this operatorial approach to classical mechanics if we consider complex wave functions? In section 5 we shall propose the classical analog of the two-slit experiment and we will perform in details all the relevant calculations. We will show that the different form of the evolution operators in the quantum and in the classical case leads to totally different results: the known interference phenomenon at the quantum level and its non-appearance at the classical one. As we said above we tentatively link this difference to the different form of the evolution operators but more work has to be done to get a deeper understanding of the phenomenon.

2 Operatorial Approach to Classical Mechanics

In quantum mechanics, starting from the Schrödinger equation for the wave function $\psi(x,t)$,

$$
i\hbar \frac{\partial \psi(x,t)}{\partial t} = \hat{H} \psi(x,t) \Rightarrow \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x,t) + V(x)\psi(x,t)
$$

we have that the probability density $\rho(x,t) = |\psi(x,t)|^2$ satisfies a continuity equation of the form:

$$
\frac{\partial \rho}{\partial t} = -\text{div} \vec{j}
$$

where we have indicated with $j$ the probability density current:

$$
\vec{j} = -\frac{i\hbar}{2m} \left( \psi^* \nabla \psi - \psi \nabla \psi^* \right)
$$

Now if we write the wave function as $\psi = \sqrt{\rho} \exp[iS/\hbar]$ we discover immediately that the phase $S$ enters explicitly into the expression of the current probability density $\vec{j}$:

$$
\vec{j} = \frac{\rho \nabla S}{m}
$$
As a consequence the equation of evolution of \( \rho \) (2.2), couples the phase and the modulus square \( \rho \) of the wave functions. We can also notice from eq. (2.2) that in quantum mechanics the probability density \( \rho \) does not evolve in time with the same Schrödinger Hamiltonian \( \hat{H} \) which gives the evolution of the wave function \( \psi \).

The situation is completely different in the Hilbert space of classical mechanics. In fact, following Koopman and von Neumann, we postulate that the wave functions \( \psi(\varphi, t) = \psi(q, p, t) \) evolve in time with the Liouvillian operator:

\[
\hat{H} = -i\partial_{p_i} H \partial_{q_i} + i\partial_{q_i} H \partial_{p_i}
\]  

(2.5)

according to the following equation:

\[
i\frac{\partial}{\partial t} \psi = \hat{H} \psi \Rightarrow \frac{\partial}{\partial t} \psi = (-\partial_{p_i} H \partial_{q_i} + \partial_{q_i} H \partial_{p_i})\psi
\]  

(2.6)

We can think of (2.6) as the analogue of the quantum Schrödinger equation, i.e. as the fundamental equation governing the evolution of the vectors in the Hilbert space of classical mechanics. These vectors are the complex wave functions on the phase space obeying the normalizability condition \( \int dqdp \psi^*(q, p)\psi(q, p) = 1 \). If we take the complex conjugate of (2.6) we obtain:

\[
\frac{\partial}{\partial t} \psi^* = (-\partial_{p_i} H \partial_{q_i} + \partial_{q_i} H \partial_{p_i})\psi^*
\]  

(2.7)

i.e. \( \psi \) and \( \psi^* \) satisfy the same equation. Now if we multiply (2.6) by \( \psi^* \) and (2.7) by \( \psi \) and we sum the two equations we re- obtain eq. (1.1), i.e. the evolution of \( \rho(q, p) \equiv \psi^*(q, p)\psi(q, p) \) with the Liouvillian operator:

\[
\frac{\partial}{\partial t} \rho = (-\partial_{p_i} H \partial_{q_i} + \partial_{q_i} H \partial_{p_i})\rho \Rightarrow i\frac{\partial}{\partial t} \rho = \hat{H} \rho
\]  

(2.8)

So we have derived the standard Liouville equation (2.8) as a consequence of having postulated eq. (2.6). Moreover we notice that (2.8) does not couple the modulus square of \( \psi \), i.e. \( \rho \), with the phase of \( \psi \) differently from what happens in quantum mechanics, eq. (2.2) and (2.4).

If we define the scalar product between two wave functions \( \psi \) and \( \tau \) as \( \langle \psi|\tau \rangle = f d\varphi \psi^*(\varphi)\tau(\varphi) \) it is easy to show that \( \langle \psi|\hat{H}\tau \rangle = \langle \hat{H}\psi|\tau \rangle \), i.e. the Liouvillian \( \hat{H} \) is a self-adjoint operator; consequently the norm of the state \( \langle \psi|\psi \rangle = f d\varphi \psi^*(\varphi)\psi(\varphi) \) is conserved during the evolution and we can consistently interpret \( \rho(\varphi) = \psi^*(\varphi)\psi(\varphi) \) as a probability density in the phase space.

Before going on we want to show how the Liouvillian \( \hat{H} \) arises in a natural way in the functional approach to classical mechanics described in ref. which represents the path integral counterpart of the KvN operatorial formulation. Let us ask ourselves: which is the probability of finding a particle at a point \( \varphi = (q^1, \ldots, q^n; p_1, \ldots, p_n) \) in phase space at time \( t \) if it was at \( \varphi_i \) at the initial time \( t_i \)? This probability is one if
\( \phi^a_i \) and \( \varphi^a \) are connected with a classical path \( \phi^a_{cl} \), i.e. a path that solves the classical equations of motion, and it is zero in all the other cases. So we can write:

\[
P(\varphi^a, t | \varphi^a_i, t_i) = \tilde{\delta}(\varphi^a - \phi^a_{cl}(t; \varphi_i))
\]  

(2.9)

where the RHS is a functional delta that forces us to stay on the classical path \( \phi^a_{cl} \) associated with the initial condition \( \varphi_i \). The functional delta can be rewritten as a delta on the Hamiltonian equations of motion \( \dot{\varphi}^a = \omega^{ab} \partial_b H(\varphi) \) via the introduction of a suitable functional determinant:

\[
\tilde{\delta}(\varphi^a - \phi^a_{cl}(t; \varphi_i)) = \tilde{\delta}(\dot{\varphi}^a - \omega^{ab} \partial_b H) \det(\partial_t \delta^a_b - \omega^{ac} \partial_c \partial_b H)
\]  

(2.10)

It can be shown, see eq. (3.51) of [5], that the determinant in the previous equation is formally independent of the fields \( \varphi \) and so it can be put equal to one if we are not interested in the study of nearby trajectories. Now if we exponentiate the Dirac delta in (2.10) via the introduction of auxiliary variables \( \lambda_a \) we get the following path integral:

\[
P(\varphi, t | \varphi_i, t_i) = \int \mathcal{D} \varphi \tilde{\delta}(\varphi^a - \omega^{ab} \partial_b H) = \int \mathcal{D} \varphi \mathcal{D} \lambda e^{i \int \mathcal{L} dt}
\]  

(2.11)

where the Lagrangian \( \mathcal{L} \) is:

\[
\mathcal{L} = \lambda_a \dot{\varphi}^a - \mathcal{H}
\]  

(2.12)

with

\[
\mathcal{H} = \lambda_a \omega^{ab} \partial_b H
\]  

(2.13)

From the kinetic part of the Lagrangian we can deduce the form of the commutators of the associated operatorial theory:

\[
[\varphi^a, \lambda_b] = i \delta^a_b \Rightarrow [\hat{q}^i, \hat{\lambda}_q] = [\hat{p}_i, \hat{\lambda}_p] = i \delta^i_j
\]  

(2.14)

The previous commutators can be realized considering \( \varphi^a \) as multiplicative operators and \( \lambda_b \) as derivative ones:

\[
\hat{\lambda}_b = -i \frac{\partial}{\partial \varphi^b} \Rightarrow \hat{\lambda}_q = -i \frac{\partial}{\partial q^j}, \quad \hat{\lambda}_p = -i \frac{\partial}{\partial p^j}
\]  

(2.15)

From now on we will indicate this representation as the Schrödinger representation of classical mechanics. Using (2.13) \( \mathcal{H} \) becomes the following operator:

\[
\hat{\mathcal{H}} = -i \omega^{ab} \partial_b \partial_a = -i \partial_{q_i} H \partial_{q_i} + i \partial_{q_i} H \partial_{p_i}
\]  

(2.16)

that is exactly the Liouvillian \( \hat{\mathcal{H}} \) of eq. (2.5). This confirms that the path integral (2.11) is the correct functional counterpart of the KvN operatorial theory.

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2All the commutators different from (2.14) are identically zero. In particular, differently from the quantum case, we have that \([q, p] = 0\) which implies that we can determine with an arbitrary precision the position and the momentum of a classical particle, like it happens in the standard phase space approach to classical mechanics.
In rederiving (2.11) we have started from the transition probability \( P(\varphi, t|\varphi_i, t_i) \) of going from \( \varphi_i \) to \( \varphi \) in a time interval \( t - t_i \). So we can say that the path integral (2.11) gives a kernel of evolution for the classical probability densities \( \rho(\varphi, t) \), in the sense that if we know the probability density at the initial time \( t_i \) we can derive the probability density \( \rho \) at any later time \( t \) via the standard relation:

\[
\rho(\varphi, t) = \int d\varphi_i P(\varphi, t|\varphi_i, t_i)\rho(\varphi_i, t_i)
\]

Let us remember that KvN postulated for \( \psi \) the same equation of motion (2.6) as for \( \rho \). As a consequence their evolution can be represented as:

\[
\psi(\varphi, t) = \int d\varphi_i K(\varphi, t|\varphi_i, t_i)\psi(\varphi_i, t_i)
\]

where the kernel of evolution \( K(\varphi, t|\varphi_i, t_i) \) has the same expression as the kernel of evolution \( P(\varphi, t|\varphi_i, t_i) \) for the densities \( \rho \). The reader may be puzzled that the same kernel propagates both \( \psi \) and \( |\psi|^2 \). We will see that there is no contradiction in this by working out in details the case of a free particle in appendix A.

### 3 Spreading and Phases of the Wave Functions

One of the most characteristic effects of quantum mechanics is the spreading of the wave functions during their time evolution. Let us consider a quantum system of a free particle in one dimension with Hamiltonian \( \hat{H} = -\hbar^2 \partial^2/\partial x^2 \) and initially described by a gaussian wave function:

\[
\psi(x) = \frac{1}{\sqrt{\sqrt{\pi}a}} \exp\left(-\frac{x^2}{2a^2} + i\frac{\hbar}{m}p_i x \right)
\]

It is easy to check that at the beginning the mean value of the position \( x \) is equal to zero and the uncertainty in the measurement of the position is: \( \langle \Delta x \rangle^2 = \frac{a^2}{2} \). At time \( t \) the wave function will be:

\[
\psi(x, t) = N \cdot \exp\left[-\frac{m}{2(ma^2 + i\hbar t)}\left(x - \frac{p_i t}{m}\right)^2 + i\frac{\hbar}{m}\left(p_i x - \frac{p_i^2}{2m}t\right)\right]
\]

We can note how the coefficient \( p_i \), which entered only into the phase of the wave function at time \( t = 0 \), managed to enter also the modulus of the wave function at time \( t > 0 \), see eq. (3.2). As a consequence the expectation value of the position \( x \) at time \( t \) depends explicitly on the original phase \( p_i \); in fact we have:

\[
\langle x(t) \rangle = \int dx \: x \cdot |\psi(x, t)|^2 = p_i t / m
\]
So we see that the information about the mean value of $x$ is carried by terms appearing in the original phase of $\psi$. For the mean square deviation of $x$ we get:

$$\langle \Delta x(t) \rangle^2 = (x - \bar{x}(t))^2 = \frac{a^2}{2} \left( 1 + \frac{t^2 \hbar^2}{m^2 a^4} \right)$$

(3.4)

So, at $t \to \infty$ the wave function will be totally delocalized:

$$\lim_{t \to \infty} \langle \Delta x(t) \rangle^2 = +\infty, \quad \forall a > 0$$

(3.5)

This effect is present also if we prepare an initial state very sharply peaked around the origin, in fact:

$$\lim_{a \to 0} \Delta (\langle x(t) \rangle^2) = \lim_{a \to 0} \left( \frac{a^2}{2} + \frac{t^2 \hbar^2}{2m^2 a^2} \right) = +\infty, \quad \forall t > 0$$

(3.6)

Note that the previous limit is $+\infty$ because of the presence of the parameter $a^2$ in the denominator. The relation (3.6) is not surprising: if $a \to 0$ at the beginning then we have a state perfectly localized in space, i.e. $\langle \Delta x \rangle^2 \to 0$ and, from the Heisenberg uncertainty relations, we can deduce that $\langle \Delta p \rangle^2 \to +\infty$. So in this case the initial momentum is completely undetermined and, consequently, even after an infinitesimal time interval, also the position of the particle becomes completely undetermined. These are the well-known quantum mechanical effects.

What happens in the operatorial version of classical mechanics? As we have seen in the previous section, the evolution in time of the wave functions is generated by the Liouvillian itself $\hat{\mathcal{H}} = -i \partial_p H \partial_q + i \partial_q H \partial_p$ which, in the particular case of a one-dimensional free particle, has the following simplified form:

$$\hat{\mathcal{H}} = -\frac{i}{m} \partial \frac{\partial}{\partial q}$$

(3.7)

The free Liouvillian is essentially the product of two commutative operators: an operator of multiplication $\hat{p}$ and an operator of derivation $-i \frac{\partial}{\partial q}$. So if we want to diagonalize the $\hat{\mathcal{H}}$ of eq. (3.7) we have to diagonalize simultaneously both $\hat{p}$ and $-i \frac{\partial}{\partial q}$. The eigenstates of $\hat{p}$ associated to an arbitrary real eigenvalue $p_0$ are the Dirac deltas $\delta(p - p_0)$; the eigenstates of $-i \frac{\partial}{\partial q}$ are instead the plane waves $\frac{1}{\sqrt{2\pi}} \exp[i\lambda q]$ and their correspondent eigenvalues are $\lambda_q$. So the eigenstates of the free Liouvillian (3.7) are just the product of the eigenstates of $\hat{p}$ and $-i \frac{\partial}{\partial q}$:

$$\tau_{\lambda_q p_0}(q, p) = \frac{1}{\sqrt{2\pi}} e^{i\lambda q} \delta(p - p_0)$$

(3.8)

We call $\lambda_q$ the eigenvalues of $-i \frac{\partial}{\partial q}$ since $-i \frac{\partial}{\partial q}$ is just a representation of the abstract Hilbert space operator $\hat{\lambda}_q$: see (2.15) and the next section.
and the associated eigenvalues are the product of the eigenvalues: \( E = \frac{\lambda_q p_0}{m} \). Now, suppose to take as initial wave function the following double gaussian in \( q \) and \( p \):

\[
\psi(q, p, t = 0) = \frac{1}{\sqrt{\pi ab}} \exp \left( -\frac{q^2}{2a^2} - \frac{(p - p_i)^2}{2b^2} \right)
\]

(3.9)

where \( a \) and \( b \) are related to our initial uncertainty in the knowledge of \( q \) and \( p \):

\[
(\Delta q)^2 = \frac{a^2}{2}, \quad (\Delta p)^2 = \frac{b^2}{2}.
\]

Note that, since in classical mechanics \( \hat{q} \) and \( \hat{p} \) commute, there isn’t any uncertainty relation. As a consequence \( a \) and \( b \) in (3.9) are two completely independent parameters and the product \((\Delta q)^2 \cdot (\Delta p)^2\) can assume arbitrary small values.

Now we can write the initial wave function (3.9) as a superposition of the eigenstates (3.8) of the Hamiltonian \( \hat{H} \) as:

\[
\psi(q, p, t = 0) = \int d\lambda_q d\lambda_p c(\lambda_q, \lambda_p) \tau_{\lambda_q \lambda_p}(q, p)
\]

(3.10)

where the coefficients \( c(\lambda_q, \lambda_p) \) are:

\[
c(\lambda_q, \lambda_p) = \int dq dp \tau^{*}_{\lambda_q \lambda_p}(q, p) \psi(q, p, t = 0) = \sqrt{\frac{a}{\pi b}} \exp \left( -\frac{\lambda_q^2 a^2}{2} - \frac{(p_0 - p_i)^2}{2b^2} \right)
\]

(3.11)

The wave function at \( t \) is given by:

\[
\psi(q, p, t) = \int d\lambda_q d\lambda_p c(\lambda_q, \lambda_p) \exp[-iEt] \tau_{\lambda_q \lambda_p}(q, p)
\]

\[
= \frac{1}{\sqrt{\pi ab}} \exp \left[ -\frac{1}{2a^2} \left( \frac{q - p_0}{m} \right)^2 - \frac{(p - p_i)^2}{2b^2} \right]
\]

(3.12)

Note that this \( \psi(t) \) is related to \( \psi(0) \) by the following equation:

\[
\psi(q, p, t) = \psi \left( q - \frac{pt}{m}, p, 0 \right) = \psi \left( q - \frac{\partial H}{\partial p} t, p + \frac{\partial H}{\partial q} t, 0 \right)
\]

(3.13)

The previous relation could be inferred also from the path integral (2.11). In fact in the case of a free particle, see appendix A, the kernel of propagation is correctly given by:

\[
K(\varphi, t|\varphi_i, 0) = \delta \left( q - q_i - \frac{p_i t}{m} \right) \delta (p - p_i)
\]

(3.14)

from which we obtain immediately:

\[
\psi(\varphi, t) = \int d\varphi_i K(\varphi, t|\varphi_i, 0) \psi(\varphi_i, 0) = \psi \left( q - \frac{pt}{m}, p, 0 \right)
\]

(3.15)
If we identify the modulus square of the wave function $|\psi(q,p,t)|^2 = \rho(q,p,t)$ with the probability density of finding a particle in a certain point of the phase space we have that (3.13) implies:

$$\rho(q,p,t) = \rho \left( q - \frac{\partial H}{\partial p} t, p + \frac{\partial H}{\partial q} t, 0 \right)$$  \hspace{1cm} (3.16)

This is an equation in perfect agreement with the Liouville theorem $\frac{d}{dt} \rho = 0$.

Let us now go back to eq. (3.12) and calculate the mean values of the dynamical variables at a generic time $t$. They are:

$$\bar{q} = \int dqdp q \cdot |\psi(q,p,t)|^2 = \frac{p_i t}{m}, \quad \bar{p} = \int dqdp p \cdot |\psi(q,p,t)|^2 = p_i$$ \hspace{1cm} (3.17)

Note that, differently from quantum mechanics, the information on the mean values of $\hat{p}$ is given by coefficients which appear in the modulus of the wave function. The mean square deviations are:

$$\overline{(\Delta q(t))^2} = \frac{a^2}{2} + \frac{b^2}{2} \frac{t^2}{m^2}, \quad \overline{(\Delta p(t))^2} = \frac{b^2}{2}$$ \hspace{1cm} (3.18)

Therefore also in classical mechanics if $b \neq 0$ we have $\lim_{t \to \infty} (\Delta q(t))^2 = +\infty$ and the wave function is totally delocalized. This is not strange if we consider that we are giving a statistical description of a set of particles with a momentum distributed in a gaussian way around $p_i$. This means that we can have particles with momenta both greater and smaller than $p_i$. These particles cause a dispersion in the wave function and, consequently, in the distribution of probability around the mean value of $q$.

If we instead consider the mechanics of a single particle we can measure exactly its position and momentum at the initial time. In this case the terms that parametrize the gaussians (3.1) go to zero ($a \to 0, b \to 0$) and the initial wave function is a good approximation for the double Dirac delta $\delta(q)\delta(p - p_i)$. At the generic time $t$ the state will be described by another couple of Dirac deltas: $\delta \left( q - \frac{p_i t}{m} \right) \delta(p - p_i)$. In this limiting case the variances are identically zero because, differently from the quantum case (3.6), in (3.18) the parameters $a^2$ and $b^2$ do not appear in the denominator:

$$\lim_{a,b \to 0} (\Delta q(t))^2 = \lim_{a,b \to 0} \left( \frac{a^2}{2} + \frac{b^2}{2} \frac{t^2}{m^2} \right) = 0, \quad \lim_{a,b \to 0} (\Delta p(t))^2 = \lim_{b \to 0} \frac{b^2}{2} = 0$$ \hspace{1cm} (3.19)

and the particle remains perfectly localized in the phase space at every instant of time $t$.

We feel that even this very simple and pedagogical example can be used to underline some very important differences between the quantum and the classical operatorial approaches which are:

1) in classical mechanics we can prepare an initial wave function that approximates a
double Dirac delta in $q$ and $p$ since $\hat{q}$ and $\hat{p}$ are commuting operators and so there is no uncertainty relation between them;

2) the classical dynamics given by $\mathcal{H}$ is such that, if we know with absolute precision the position and the momentum at $t = 0$, they remain perfectly determined at every instant of time $t$ and there is not any spreading, see eq. (3.19);

3) the knowledge about the average momentum of the classical particle is brought by terms appearing in the modulus of the wave function.

For a classical free particle it is easy to show that if we add a phase factor to an arbitrary initial wave function this phase factor does not pass into the real part during the evolution differently from what happens in quantum mechanics, see eq. (3.2). This can be proved as follows: every initial classical wave function

$$\psi(q, p) = F(q, p) \exp[iG(q, p)]$$

(3.20)
can always be written as a superposition of the eigenstates of the free Liouvillian (3.7) in the following way:

$$\psi(q, p) = \int dq dp_0 c(\lambda_q, p_0) \tau_{\lambda_q p_0}(q, p)$$

(3.21)

where the eigenstates $\tau_{\lambda_q p_0}(q, p)$ are given by eq. (3.3). So the coefficients $c(\lambda_q, p_0)$ are basically the Fourier transform, in the $q$ variable only, of $\psi(q, p_0)$:

$$c(\lambda_q, p_0) = \frac{1}{\sqrt{2\pi}} \int dq e^{-i\lambda_q q} F(q, p_0) e^{iG(q, p_0)}$$

(3.22)

At this point we can find the free evolution of the wave function in the usual way:

$$\psi(q, p, t) = \int dq dp_0 c(\lambda_q, p_0) \exp[-i\mathcal{E}t] \tau_{\lambda_q p_0}(q, p) =$$

$$= \int dq c(\lambda_q, p) \exp[i\lambda_q \left(q - \frac{pt}{m}\right)] = F \left(q - \frac{pt}{m}, p \right) \exp \left[iG \left(q - \frac{pt}{m}, p \right) \right]$$

(3.23)

and we obtain a result that is again in perfect agreement with the kernel of evolution (3.14). Therefore, in the case of a free particle, we have that for every initial wave function $\psi(q, p, t = 0) = F(q, p) \exp[iG(q, p)]$ the probability density $|\psi(q, p, t)|^2$ does not depend on the phase $G(q, p)$ not only at the beginning, but also at any later time; in fact from (3.23) we have that $|\psi(q, p, t)|^2 = F^2 \left(q - \frac{pt}{m}, p \right)$. This has some consequences also on the expectation values of the observables. If we assume that observables in classical mechanics are only the functions $O(\phi)$ or, in operatorial terms, only the operators $O(\hat{\phi})$ then it is easy to check that their expectation values do not depend on the phase $G$ of the wave function (3.20):

$$\langle O \rangle = \int d\phi F^*(\phi) \exp[-iG(\phi)] O(\phi) F(\phi) \exp[iG(\phi)] = \int d\phi F^*(\phi) O(\phi) F(\phi)$$

(3.24)
This is true also at later times $t$ because of the form (3.23) of the wave function at $t > 0$.

This independence from the phase $G$ would not happen if the observables were dependent also on $\lambda$: $\mathcal{O}(\hat{\varphi}, \hat{\lambda})$, because $\hat{\lambda}$ is a derivative operator and it would give

$$\langle \mathcal{O} \rangle = \int d\varphi F^*(\varphi) \exp[-iG(\varphi)] \mathcal{O}(\varphi, -i \frac{\partial}{\partial \varphi}) F(\varphi) \exp[iG(\varphi)] =$$

$$= \int d\varphi F^*(\varphi) Q(\varphi, F, F', G, G') F(\varphi)$$

(3.25)

So $\langle \mathcal{O} \rangle$ would be a function of $F(\varphi)$, its derivative $F'$ with respect to $\varphi$, but also of the phase $G(\varphi)$ and its derivative $G'$. These considerations can be extended from the free particle case to a physical system characterized by a generic Hamiltonian $\hat{\mathcal{H}}$. In fact the solution of the equation:

$$i \frac{\partial}{\partial t} \psi(q, p, t) = \hat{\mathcal{H}} \psi(q, p, t)$$

(3.26)

is given by [4]:

$$\psi(q, p, t) = \psi(\bar{q}(q, p, t), \bar{p}(q, p, t))$$

(3.27)

where $\bar{q}$ and $\bar{p}$ are the solutions of the equations:

$$\dot{\bar{q}}_j(q, p, t) = -\frac{\partial H(\bar{q}, \bar{p})}{\partial \bar{p}_j}, \quad \dot{\bar{p}}_j(q, p, t) = \frac{\partial H(\bar{q}, \bar{p})}{\partial \bar{q}_j}$$

(3.28)

with the initial conditions $\bar{q}_j(q, p, 0) = q^0_j, \quad \bar{p}_j(q, p, 0) = p^0_j$. So, according to eq. (3.27), the evolution of a classical system via the Liouvillian does not modify, in the Schrödinger representation, the functional form of $\psi$ provided we write it using the $\bar{q}, \bar{p}$. This has, as an immediate consequence, that if we take a wave function without any phase at the initial time, phases cannot be generated during the evolution:

$$\tilde{\mathcal{H}}: \psi(\text{without phases } t = 0) \longrightarrow \psi(\text{without phases } t)$$

(3.29)

In quantum mechanics instead, even if we start from a wave function that does not present phases, these ones will be created in general at later times via the operator $\tilde{\mathcal{H}}$:

$$\tilde{\mathcal{H}}: \psi(\text{without phases } t = 0) \longrightarrow \psi(\text{with phases } t)$$

(3.30)

This can be seen for example from (3.3) and (3.2). Even if we start with no phase $p_i = 0$, at time $t$ we get that the wave function (3.4) becomes:

$$\psi = N \cdot \exp \left[ -\frac{m}{2(\text{ma}^2 + \text{iht})} x^2 \right]$$

(3.31)

and it has a phase because of the term $\text{iht}$ in the denominator. All this is a consequence of the fact that the phase and the modulus of a wave function interact in quantum mechanics as one can see from standard text books, [8]. In fact writing

$$\psi(x) = A(x) \exp \left[ \frac{\text{i}}{\hbar} S(x) \right]$$

(3.32)
and equating the real and imaginary part of the Schrödinger equation (2.1) we obtain the following two equations for $A(x)$ and $S(x)$:

$$\begin{align}
\frac{\partial S}{\partial t} + \frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 + V &= \frac{\hbar^2}{2mA} \frac{\partial^2 A}{\partial x^2} \\
m \frac{\partial A}{\partial t} + \frac{\partial A}{\partial x} \frac{\partial S}{\partial x} + \frac{A}{2} \frac{\partial^2 S}{\partial x^2} &= 0
\end{align}$$

(3.33)

From (3.33) we see that $S$ and $A$ are coupled by their equations of motion.

In classical mechanics if we start from

$$\psi(q, p) = F(q, p) \exp[iG(q, p)]$$

(3.34)

we can insert it in eq. (2.6): $i\frac{\partial \psi}{\partial t} = \hat{H}\psi$ and, equating the real and imaginary part, we get:

$$i\frac{\partial F}{\partial t} = \hat{H}F, \quad i\frac{\partial G}{\partial t} = \hat{H}G$$

(3.35)

So we see that in classical mechanics the phase and the modulus decouple from each other. I owned this analysis to E. Gozzi [9] who has once summarized all this with the sentence: "What is quantum mechanics? Quantum mechanics is the theory of the interaction of a phase with a modulus".

As we have just proved, in classical mechanics there is at least one representation in which this interaction is completely lost and the evolution of the modulus is completely decoupled from the evolution of the phase. One may then think that it is useless to deal with complex wave functions if their phases do not bring in any physical information. This is true only if we decide to work with the Schrödinger representation. If, instead, we change representation, as we will do in the next section, and use the one where $\hat{p}$ is realized as the derivative with respect to $\lambda_p$, we shall then show that the mean value of $\hat{p}$ is related to the phase of the wave functions. So, if we want to be as general as possible and not just stick to the Schrödinger representation, we have to assume that the classical wave functions are complex objects.

4 Abstract Hilbert space and $(q, \lambda_p)$ representation

In the previous section we have restricted ourselves to the Schrödinger representation in which both $\hat{q}$ and $\hat{p}$ are realized as multiplicative operators, and we have worked out everything in this frame. What we want to do now is to construct the Hilbert space of classical mechanics from an abstract point of view independently of any particular representation. We can start observing that $\hat{q}$ and $\hat{p}$ can be considered as a complete set of commuting operators whose real eigenvalues can vary with continuity from $-\infty$ to $+\infty$:

$$\hat{q}|q, p\rangle = q|q, p\rangle; \quad \hat{p}|q, p\rangle = p|q, p\rangle$$

(4.1)
The eigenstates $|q,p\rangle$ form an orthonormal and complete set which can be used as a basis for our Hilbert space. The orthonormality and completeness relations are respectively given by:

$$\langle q', p'|q'', p''\rangle = \delta(q' - q'')\delta(p' - p''), \quad \int dqdp\, |q,p\rangle\langle q,p| = 1 \quad (4.2)$$

The connection between the abstract vectors $|\psi\rangle$ and the wave functions $\psi(q,p)$ is given by the relation $\langle q,p|\psi\rangle = \psi(q,p)$. In this basis the operators $\hat{q}$ and $\hat{p}$ are diagonal:

$$\langle q', p'|\hat{q}|q'', p''\rangle = q'\delta(q' - q'')\delta(p' - p''); \quad \langle q', p'|\hat{p}|q'', p''\rangle = p'\delta(q' - q'')\delta(p' - p'') \quad (4.3)$$

while the operators $-i\frac{\partial}{\partial q}\left(-i\frac{\partial}{\partial p}\right)$ defined by the relations:

$$\langle q', p'|-i\frac{\partial}{\partial q}\left(-i\frac{\partial}{\partial p}\right)|\psi\rangle = -i\frac{\partial}{\partial q}\left(-i\frac{\partial}{\partial p}\right)\langle q', p'|\psi\rangle \quad (4.4)$$

are self-adjoint. From (1.2)-(4.4) it is easy to check that:

$$\langle q', p'\left[q, -i\frac{\partial}{\partial q}\right]|\psi\rangle = \langle q', p'|i|\psi\rangle, \quad \langle q', p'\left[p, -i\frac{\partial}{\partial p}\right]|\psi\rangle = \langle q', p'|i|\psi\rangle \quad (4.5)$$

while all other commutators are zero. Because of the completeness of $\langle q', p'|$ and the arbitrariness of $|\psi\rangle$ we have that (4.3) can be turned into the purely operatorial relations: $[\hat{q}, -i\frac{\partial}{\partial q}] = i$ and $[\hat{p}, -i\frac{\partial}{\partial p}] = i$ and so we can identify, from eq. (2.14), $\hat{\lambda}_q = -i\frac{\partial}{\partial q}$ and $\hat{\lambda}_p = -i\frac{\partial}{\partial p}$. Now it is easy to show that the Liouville equation (2.4) is nothing else than a particular representation of the abstract Liouville equation:

$$i\frac{\partial}{\partial t}|\psi, t\rangle = \hat{\lambda}_a\omega^{ab}\partial_b H|\psi, t\rangle \quad (4.6)$$

obtained using as basis the eigenfunctions of $q$ and $p$. If we consider as Hamiltonian in the standard phase space the following one: $H = \frac{p^2}{2m} + V(q)$, then the Liouville equation (4.6) becomes:

$$i\frac{\partial}{\partial t}|\psi, t\rangle = \left[\hat{\lambda}_q\frac{\hat{p}}{m} - \hat{\lambda}_p\partial_q V(q)\right]|\psi, t\rangle \quad (4.7)$$

Projecting the previous equation onto the basis $\langle q,p\rangle$ we easily obtain:

$$i\frac{\partial}{\partial t}\langle q,p|\psi, t\rangle = \langle q,p|\hat{\lambda}_q\frac{\hat{p}}{m}|\psi, t\rangle - \langle q,p|\hat{\lambda}_p\partial_q V(q)|\psi, t\rangle =$$

$$= -i\frac{\hat{p}}{m}\frac{\partial}{\partial q}\langle q,p|\psi, t\rangle + i\partial_q V(q)\frac{\partial}{\partial p}\langle q,p|\psi, t\rangle \quad (4.8)$$
that is equivalent to the usual Liouville equation:

$$\frac{\partial}{\partial t} \psi(q, p, t) = \left[ -\frac{p}{m} \frac{\partial}{\partial q} + \partial_q V(q) \frac{\partial}{\partial p} \right] \psi(q, p, t) \quad (4.9)$$

The \(|q, p\rangle\) basis is not the only one for the Hilbert space of classical mechanics. A very important representation [10] is the one in which we consider as basis the simultaneous eigenstates of \(\hat{q}\) and \(\hat{\lambda}_p\) which, according to (2.14), are commuting operators:

$$\hat{q}|q, \lambda_p\rangle = q|q, \lambda_p\rangle; \quad \hat{\lambda}_p|q, \lambda_p\rangle = \lambda_p|q, \lambda_p\rangle \quad (4.10)$$

Sandwiching the second relation in (4.10) with the bra \(\langle q', p'|\) we obtain

$$-i \frac{\partial}{\partial p'} \langle q', p'|q, \lambda_p\rangle = \lambda_p \langle q', p'|q, \lambda_p\rangle \quad (4.11)$$

The solution of this differential equation is:

$$\langle q', p'|q, \lambda_p\rangle = \frac{1}{\sqrt{2\pi}} \delta(q - q') e^{ip'\lambda_p} \quad (4.12)$$

The states \(|q, \lambda_p\rangle\) form a complete set of orthonormal eigenstates, i.e. an alternative basis for the vectors of our classical Hilbert space. In this basis we have:

$$\langle q, \lambda_p|\psi\rangle = \int dq dp |q, \lambda_p \rangle \langle q', p|\psi\rangle \quad (4.13)$$

which, via (4.12), gives:

$$\psi(q, \lambda_p) = \frac{1}{\sqrt{2\pi}} \int dp e^{-ip\lambda_p} \psi(q, p) \quad (4.14)$$

i.e. the wave functions in the new basis are related to the ones in the Schrödinger representation by means of a Fourier transform\(^4\). In this new representation we have for the \(\hat{p}\) operator:

$$\langle q, \lambda_p|\hat{p}|\psi\rangle = \int dq dp' \langle q, \lambda_p|\hat{p}|q', p\rangle \langle q', p|\psi\rangle = \frac{1}{\sqrt{2\pi}} \int dp' e^{-ip'\lambda_p} \psi(q, p') =$$

$$= \frac{1}{\sqrt{2\pi}} i \frac{\partial}{\partial \lambda_p} \int dp e^{-ip\lambda_p} \langle q, p|\psi\rangle = i \frac{\partial}{\partial \lambda_p} \langle q, \lambda_p|\psi\rangle \quad (4.15)$$

while for \(\hat{\lambda}_p\) we have:

$$\langle q, \lambda_p|\hat{\lambda}_p|\psi\rangle = \lambda_p \langle q, \lambda_p|\psi\rangle \quad (4.16)$$

Summarizing (4.10)-(4.16), we can say that in this representation we have to consider \(\hat{p}\) as a derivative operator: \(\hat{p} = i \frac{\partial}{\partial \lambda_p}\) and \(\hat{\lambda}_p\) as a multiplicative one. This is simply

\(^4\)We indicate the wave functions in the new basis with the same symbol \(\psi\) for notational simplicity.
a different realization of the usual commutation relation: $[\hat{p}, \hat{\lambda}_p] = i$. Using $\langle q, \lambda_p |$ to sandwich eq. (4.7) we get that the Louville equation becomes:

$$
i \frac{\partial}{\partial t} \psi(q, \lambda_p) = \frac{1}{m} \frac{\partial}{\partial q} \frac{\partial}{\partial \lambda_p} \psi(q, \lambda_p) - \lambda_p \lambda_p V(q) \psi(q, \lambda_p)$$

(4.17)

We shall now show that a lot of the results of the previous section were in a certain sense representation-dependent. In fact in the new representation, since the momentum $\hat{p}$ has become an operator of derivation, we have that the information about its mean value is brought in by the phase of the wave function similarly to what happens in quantum mechanics. For example the double gaussian state of eq. (3.9) becomes the following one in the new basis:

$$\psi(q, \lambda_p, t = 0) = \sqrt{\frac{b}{\pi a}} \exp \left( -\frac{q^2}{2a^2} \right) \exp \left( -\frac{\lambda_p^2 b^2}{2} - i p_i \lambda_p \right)$$

(4.18)

We obtain it by just applying formula (4.14) and, after the Fourier transform, the wave function which was real in the Schrödinger representation becomes complex. The mean values of $\hat{q}$ and $\hat{p}$ are obviously the same as before:

$$\bar{q} = \langle \psi | \hat{q} | \psi \rangle = 0, \quad \bar{p} = \langle \psi | \hat{p} | \psi \rangle = \langle \psi \left| i \frac{\partial}{\partial \lambda_p} \right| \psi \rangle = p_i$$

(4.19)

but now we see that elements appearing in the phases of the wave functions, like $p_i$ in (4.18), begin to play an important role since they are linked with the mean values of physical observables like $\hat{p}$.

Let us now make the evolution of (4.18) under the Liouvillian of a free particle. This Liouvillian in the new representation is given by:

$$\hat{H} = \frac{1}{m} \frac{\partial^2}{\partial q \partial \lambda_p}$$

(4.20)

Its eigenstates associated with the eigenvalues $p\lambda_q/m$ are:

$$\tau_{\lambda_q, p}(q, \lambda_p) = \frac{1}{2\pi} \exp[i\lambda_q q - i\lambda_p p]$$

(4.21)

Expanding then the wave function (4.18) in terms of the $\tau_{\lambda_q, p}$ above and making the evolution of the system, we obtain at time $t$:

$$\psi(q, \lambda_p, t) = N \cdot \exp \left[ -\frac{q^2}{2a^2} - \frac{p_i^2}{2b^2} - \frac{1}{2} \frac{\lambda_p^2 m a^2 b^2 + i q t b^2 + i p_i m a^2}{a^2 b^2 (m^2 a^2 + t^2 b^2)} \right]$$

(4.22)

From the previous formula we see how the factor $p_i$ which at time $t = 0$ entered only the phase factor, see (4.18), has passed at time $t$ also into the real part of the wave
function, exactly as in the quantum case we studied before. The expectation values and the variances of $q$ and $p$ are still given by eqs. (3.17)-(3.18):

$$\bar{q} = \int dq d\lambda_p \, q \cdot |\psi(t)|^2 = \frac{p_i t}{m}, \quad \bar{p} = p_i$$

and:

$$\langle \Delta q \rangle^2 = \frac{a^2}{2} + \frac{b^2}{2m^2}, \quad \langle \Delta p \rangle^2 = \frac{b^2}{2}$$

This is so since they are observable quantities and, consequently, they are independent of the representation we are using. In the new representation making a wave function of the form (4.18) well-localized both in $q$ and in $\lambda_p$ means sending $a \to 0, b \to \infty$. In this limiting case we have that $\langle \Delta q \rangle^2 \to \infty$ at every instant of time $t > 0$ and so there is a sort of spreading of the wave function. This is not surprising. In fact if the initial wave function is very peaked around $q = \lambda_p = 0$ then we know precisely the values of $q$ and $\lambda_p$, instead of the values of $q$ and $p$. From the commutator $[\hat{p}, \hat{\lambda_p}] = i$ we can derive the following uncertainty relation $\Delta p \cdot \Delta \lambda_p \geq 1/2$, where $\Delta p$ and $\Delta \lambda_p$ are the square roots of the mean square deviations. So, if we determine with absolute precision $\lambda_p$, as we do with the limit $b \to \infty$, the momentum $p$ is completely undetermined. Consequently also the position $q$ at every instant $t > 0$ is completely undetermined, because it follows the classical equations of motion $\dot{q} = p/m$. This fact has, as an immediate consequence, the spreading of $q$ and the complete delocalization of the wave function at every instant following the initial one.\footnote{The usual mechanics of the single particle can be reproduced also in this representation but we have to take the limit $a \to 0, b \to 0$, i.e. we have to use for $\lambda_p$ a plane wave of the type $e^{ip_\lambda \lambda_p}$.}

Another aspect that we can study is the continuity equation. We have seen in the second section that the continuity equation in the Schrödinger representation is nothing else than the usual Liouville equation for the probability density $\rho$. What happens in the other representation we have studied in eqs. (4.10)-(4.16)? According to what we have already seen, the Liouville equation for $\psi$ is:

$$i \frac{\partial}{\partial t} \psi(q, \lambda_p) = \frac{1}{m} \frac{\partial}{\partial q} \frac{\partial}{\partial \lambda_p} \psi(q, \lambda_p) - \lambda_p V'(q) \psi(q, \lambda_p)$$

while the one for the $\psi^*(q, \lambda_p)$ is the complex conjugate:

$$-i \frac{\partial}{\partial t} \psi^*(q, \lambda_p) = \frac{1}{m} \frac{\partial}{\partial q} \frac{\partial}{\partial \lambda_p} \psi^*(q, \lambda_p) - \lambda_p V'(q) \psi^*(q, \lambda_p)$$

From (4.25) and (4.26) we can obtain the equation for $\rho(q, \lambda_p) = \psi^*(q, \lambda_p) \psi(q, \lambda_p)$. It is of the form:

$$\frac{\partial}{\partial t} \rho(q, \lambda_p) + J = 0$$

where:

$$J = i \frac{1}{m} \left( \psi^* \frac{\partial}{\partial q} \frac{\partial}{\partial \lambda_p} \psi - \psi \frac{\partial}{\partial q} \frac{\partial}{\partial \lambda_p} \psi^* \right)$$
So we notice that in this case $\rho(q,\lambda_p)$ evolves with an equation that is completely different from the Liouville equation and there is no manner to write $J$ in terms only of $\rho$. Moreover, if we write $\psi(q,\lambda_p) = \sqrt{\rho} \exp[iS(q,\lambda_p)]$, the phase $S(q,\lambda_p)$ will enter explicitly $J$ and the equation of $\rho$, i.e. we have a situation very similar to that of quantum mechanics where phases and modulus are coupled in the equations of motion.

Another aspect that the $(q,\lambda_p)$ representation of classical mechanics and the standard quantum one have in common is that, even if we prepare a real wave function of $q$ and $\lambda_p$ at the initial time $t = 0$, phases will be created in general by $\hat{H}$ during the evolution. This can be seen by means of our usual example. In fact, if we put $p_i = 0$, we have from eqs. (4.18) and (4.22) that:

$$\sqrt{\frac{b}{\pi a}} \exp\left(-\frac{q^2}{2a^2} - \frac{\lambda_p^2 b^2}{2}\right) \longrightarrow N \cdot \exp\left[-\frac{q^2}{2a^2} - \frac{1}{2} \frac{(\lambda_p ma^2 b^2 + i\alpha t b^2)^2}{a^2 b^2 (m^2 a^2 + t^2 b^2)}\right]$$

(4.29)

that is:

$$\hat{H} : \psi(q,\lambda_p, t = 0 \text{ without phases}) \longrightarrow \psi(q,\lambda_p, t \text{ with phases})$$

(4.30)

Since the $(q,\lambda_p)$ representation has all these features in common with quantum mechanics it is not a case that this representation turns out to be the one where the transition to quantum mechanics is best understood.

### 5 Two-slit experiment

Having formulated classical mechanics in the same mathematical language of quantum mechanics, we think it may be useful to analyze the two-slit experiment in the classical KvN formalism and compare it with its quantum analogue. This kind of experiment is central in quantum mechanics and its mystery is best summarized in these words of Feynman: "The question is, how does [the two-slit experiment] really work? What machinery is actually producing this thing? Nobody knows any machinery. The mathematics can be made more precise; you can mention that they are complex numbers, and a couple of other minor points which have nothing to do with the main idea. But the deep mystery is what I have described, and no one can go any deeper today". As Feynman mention in the lines above one could think that the interference effects are there because of the complex nature of the wave functions. Then it is natural to check what happens in the classical KvN case where, as we showed in the previous section, the wave functions have to be complex. We will actually show that, despite the complex nature of these wave functions, interference effects do not appear. This confirms, as Feynman suspected, that the mystery of quantum mechanics is deeper than that.

If we want to describe a classical two-slit experiment we have to deal with a two dimensional problem. Let us call $y$ the axis along which our beam propagates and $x$ the orthogonal axis. We suppose that $y = 0$ is the starting coordinate of our beam.
The centers of the two slits $\Delta_1$ and $\Delta_2$ are placed respectively at $x_A$ and $-x_A$ on a first plate which has coordinate $y_F$ along the $y$ axis. The final screen is placed at $y_S$ like in the figure below.

![Diagram of two slits and first plate with coordinate $y_F$ and final screen at $y_S$.]

To simplify the problem we make the assumption\textsuperscript{[6]} that the motion of the particle along the $y$ direction is known precisely. This means that at the initial time we know with absolute precision the position and the momentum of the particle, for example $y(0) = 0$, $p_y(0) = p_y^0$. With this prescription we are sure that the beam will arrive at the two slits after a time $t_F = y_F m / p_y^0$ and at the final screen after a time $t_S = y_S m / p_y^0$. In this way we can concentrate ourselves only on the behaviour of the particles along the $x$-axis. Suppose we consider, along $x$, a double-gaussian wave function with an arbitrary phase factor $G(x, p_x)$:

$$
\psi(x, p_x, t = 0) = \frac{1}{\sqrt{\pi a b}} \exp\left[ -\frac{x^2}{2a^2} - \frac{p_x^2}{2b^2} + iG(x, p_x) \right] \quad (5.1)
$$

We assume $a$ and $b$ sufficiently large, i.e. the initial classical wave function sufficiently spread, in order to allow the beam to arrive at both slits. The evolution of the wave function will be via the free kernel of propagation $\text{[3.14]}$ up to the time $t_F = y_F m / p_y^0$ that is when the beam arrives at the first plate. The wave function at the time $t_F$ will be:

$$
\psi(x, p_x, t_F) = \frac{1}{\sqrt{\pi a b}} \exp\left[ -\frac{1}{2a^2} \left( x - \frac{p_x y_F}{p_y^0} \right)^2 - \frac{p_x^2}{2b^2} + iG(x - \frac{p_x y_F}{p_y^0}, p_x) \right] \quad (5.2)
$$

Let us suppose that the width of the two slits is $2\delta$, then the particles which at time $t_F$ are outside of the two intervals $\Delta_1 = (x_A - \delta, x_A + \delta)$ and $\Delta_2 = (-x_A - \delta, -x_A + \delta)$ are absorbed by the first plate and they don’t arrive at the final screen at all. Using Feynman’s words again: ”All particles which miss the slit[s] are captured and removed from the experiment\textsuperscript{[12]}”. Therefore the wave function just after $t_F$ can be rewritten in a compact way using a series of $\theta$-Heavyside functions:

$$
\psi(x, p_x, t_F + \epsilon) = N \psi(x, p_x, t_F) \left[ C_1(x) + C_2(x) \right] \quad (5.3)
$$

\textsuperscript{6}A similar assumption was made by Feynman in his analysis of the diffraction experiment $\text{[12]}$.\textsuperscript{17}
where $C_1(x) = \theta(x - x_\Delta + \delta) - \theta(x - x_\Delta - \delta)$ is the function that parametrizes the slit $\Delta_1$, $C_2(x) = \theta(x + x_\Delta + \delta) - \theta(x + x_\Delta - \delta)$ is the one that parametrizes the slit $\Delta_2$ and $N$ is a suitable normalization factor chosen in such a way that: $\int dx dp_x |\psi(x, p_x, t_\epsilon + \epsilon)|^2 = 1$.

Beyond the double slit we will propagate the $\psi$ cut-off functions $C$, suitable normalization factor chosen in such a way that: $x \in \Delta$ we expect that the wave function $\psi$ become different from zero also outside the intervals $\Delta_1$ or $\Delta_2$. Since there is no limitation in the momentum along the $x$-axis we expect that the wave function $\psi$ will spread along $x$ and, while time passes, it will become different from zero also outside the intervals $\Delta_1$ and $\Delta_2$.

Using the kernel of evolution for free particles (3.14) we can obtain from (5.3) the wave function at time $t_s = y_s m/p_y^0$ that is the time when the beam arrives at the final screen is:

$$\psi(x, p_x, t_s) = N \cdot \exp\left[ -\frac{1}{2a^2} \left( x - \frac{p_x y_s}{p_y^0} \right)^2 - \frac{p_x^2}{2b^2} \right] \cdot \exp\left[ iG \left( x - \frac{p_x y_s}{p_y^0}, p_x \right) \right] \cdot \left\{ C_1(x - \bar{a} p_x) + C_2(x - \bar{a} p_x) \right\} \quad (5.4)$$

where $\bar{a} = (y_s - y_\epsilon)/p_y^0$. The probability density to find a particle in a certain point $x$ on the last screen

$$P(x) = \int_{-\infty}^{\infty} dp_x |\psi(x, p_x, t_s)|^2 \quad (5.5)$$

We have to integrate over $p_x$ because we are interested in the number of particles that arrive at the final plate independently of their momentum. At this point we notice a first important property: even starting from an initial wave function with an arbitrary phase factor $G(x, p_x)$, at time $t_s$ we have for the entire wave function a common phase factor of the form $G(x - \frac{p_x y_s}{p_y^0}, p_x)$, see eq. (5.4). So $G$ will disappear completely in the evaluation of the modulus square and, consequently, in the $P(x)$ of eq. (5.3). Therefore the phase $G$ of the initial wave function (5.1) cannot have any observable consequence in the figure on the final screen.

The second important thing to notice is that, because of the properties of the $\theta$-functions, we have that the cut-off term $C_1 + C_2$ in (5.4) is idempotent:

$$(C_1 + C_2)^2 = C_1 + C_2 \quad (5.6)$$

Therefore we can rewrite (5.5) as:

$$P(x) = \int_{-\infty}^{\infty} dp_x |\psi(x, p_x, t_s)|^2 = N \cdot \left[ \int_{-\infty}^{\infty} dp_x F^2(x, p_x, t_s) C_1(x - \bar{a} p_x) + \int_{-\infty}^{\infty} dp_x F^2(x, p_x, t_s) C_2(x - \bar{a} p_x) \right] \quad (5.7)$$

where $N$ is a normalization factor and $F$ is given by:

$$F(x, p_x, t_s) \equiv \exp\left[ -\frac{1}{2a^2} \left( x - \frac{p_x y_s}{p_y^0} \right)^2 - \frac{p_x^2}{2b^2} \right] \quad (5.8)$$
Let us now re-arrange the arguments inside the \( \theta \)-functions of the \( C_1 \) and the \( C_2 \) as follows:

\[
C_1(x - \bar{a}p_x) = \theta\left(-p_x + \frac{x - x_A + \delta}{\bar{a}}\right) - \theta\left(-p_x + \frac{x - x_A - \delta}{\bar{a}}\right) \\
C_2(x - \bar{a}p_x) = \theta\left(-p_x + \frac{x + x_A + \delta}{\bar{a}}\right) - \theta\left(-p_x + \frac{x + x_A - \delta}{\bar{a}}\right)
\]  

(5.9)

Remembering the properties of the \( \theta \)-Heavyside functions it is easy to realize that when \( p_x \) is not in one of the two intervals: \( D_1 = \left[\frac{x - x_A - \delta}{\bar{a}}, \frac{x - x_A + \delta}{\bar{a}}\right] \) or \( D_2 = \left[\frac{x + x_A - \delta}{\bar{a}}, \frac{x + x_A + \delta}{\bar{a}}\right] \) there is no contribution to the modulus square. Therefore, apart from the normalization coefficient \( N \), we have that the final plot \( P(x) \) given by eq. (5.7) can be written as:

\[
P(x) = \int_{-\infty}^{\infty} dp_x |\psi(x,p_x,t_S)|^2 = N \left[ \int_{D_1} dp_x F^2(x,p_x,t_S) + \int_{D_2} dp_x F^2(x,p_x,t_S) \right]
\]  

(5.10)

where \( F \) is the function of eq. (5.8).

Now let us keep open only the first slit \( \Delta_1 \) and repeat the previous calculations. We can propagate the initial wave function \( \psi(x,p_x,t_F) \) up to the time \( t_F \) when the system is again described by the \( \psi(x,p_x,t_F) \) of eq. (5.2). The difference is that now we have to parametrize only the first slit \( \Delta_1 \). Therefore the second cut-off function \( C_2 \) is identically zero. Since \( C_1 \) itself is an idempotent function we can repeat the same steps as before, eqs. (5.3)-(5.10), freezing everywhere \( C_2 \) to zero. As final result we obtain the following probability on the last screen:

\[
P(x) = K \int_{D_1} dp_x F^2(x,p_x,t_S)
\]  

(5.11)

where \( F \) is again given by eq. (5.8). In the same manner keeping open only the slit \( \Delta_2 \) we will obtain that:

\[
P(x) = K \int_{D_2} dp_x F^2(x,p_x,t_S)
\]  

(5.12)

So, comparing (5.10) with (5.11) and (5.12), it is clear that when we keep open both slits \( \Delta_1 + \Delta_2 \) the total probability is the sum of the probabilities of having kept open first the slit \( \Delta_1 \) and then the slit \( \Delta_2 \). The first integral in (5.10) is then the probability for the particle to pass through the slit \( \Delta_1 \), while the second integral is the probability to pass through the slit \( \Delta_2 \). So, even if we start from complex wave functions in the classical Hilbert space, every interference effect disappears. This is very clear from Figure 1 which shows the plot of the \( P(x) \) of eq. (5.10) with the particular numerical values \( y_0/p_0^0 = 2, \ a = b = 1, \ x_A = 1, \ \delta = 0.1 \).

We will now perform a similar calculation at the quantum level and compare it with the previous classical experiment. In order to get an analytic result we will do
a simplification, that is we will assume that the motion along \( y \) is the same classical motion analyzed before. The reason for this assumption is that otherwise we would not be able to determine the time \( t_F \) at which the wave function arrives on the plate with the two slits. Along \( x \) instead we will assume that the motion is fully quantum mechanical. So our overall approach to the quantum case is actually a "semiclassical" approach. Nevertheless this will be sufficient to see the difference with the purely classical case we have analyzed previously.

In our semiclassical approach at the initial time the system along the \( y \)-axis is described by a double Dirac delta \( \delta(y)\delta(p_y - p^0_y) \) and this double Dirac delta evolves in time with the Liouvillian. In this way we know which is the time the particles arrive at the two slits. It is exactly the same as before. Along the other axis, \( x \), we consider instead an initial wave function given by:

\[
\psi(x) = \sqrt{\frac{1}{\sqrt{\pi a}}} \exp\left(-\frac{x^2}{2a^2}\right)
\]

(5.13)

With this choice the mean value of both \( x \) and \( p_x \) is zero at the initial time as in the classical case described by eq. (5.1). Making the above wave function evolve in time via the quantum Schrödinger operator, at time \( t_F \) we obtain:

\[
\psi(x, t_F) = \sqrt{\frac{ma}{\sqrt{\pi (ma^2 + i\hbar t_F)}}} \exp\left[-\frac{1}{2} \frac{mx^2}{ma^2 + i\hbar t_F}\right]
\]

(5.14)

Let us parametrize the two slits by means of the same series of \( \theta \)-Heavyside functions we have used in the classical case:

\[
C_1(x) = \theta(x - x_A + \delta) - \theta(x - x_A - \delta), \quad C_2(x) = \theta(x + x_A + \delta) - \theta(x + x_A - \delta)
\]

(5.15)

Just after the wave function has passed the plate with the two slits we have that:

\[
\psi(x, t_F + \epsilon) = \bar{N} \cdot \exp\left(-\frac{1}{2} \frac{m(x - x_F)^2}{ma^2 + i\hbar t_F}\right)[C_1(x) + C_2(x)]
\]

(5.16)

Using now the kernel of propagation [12] for a quantum free particle which is given by:

\[
K(x_b, t_b | x_a, t_a) = \left[\frac{2\pi \hbar(t_b - t_a)}{m}\right]^{-1/2} \exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)}\right)
\]

(5.17)

we get that at time \( t_S \) the wave function is:

\[
\psi(x, t_S) = \bar{N}_1 \int_{-\infty}^{+\infty} dx_F \exp\left[\frac{im(x - x_F)^2}{2\hbar(t_S - t_F)} - \frac{m(x_F)^2}{2(ma^2 + i\hbar t_F)}\right][C_1(x_F) + C_2(x_F)]
\]

(5.18)

where \( \bar{N}_1 \) is a new normalization constant. Differently from the classical case, the quantum kernel of propagation is not a simple Dirac delta and the previous integral
cannot be done explicitly. Anyway we can employ the properties of the \( \theta \)-functions in order to rewrite (5.18) as:

\[
\psi(x, t_s) = \bar{N}_1 \left\{ \int_{x_A - \delta}^{x_A + \delta} dx_F \exp \left[ \frac{im(x - x_F)^2}{2h(t_s - t_F)} - \frac{mx_F^2}{2(\bar{h}(t_s - t_F))} \right] + \right.
\]

\[
\left. + \int_{-x_A - \delta}^{-x_A + \delta} dx_F \exp \left[ \frac{im(x - x_F)^2}{2h(t_s - t_F)} - \frac{mx_F^2}{2(\bar{h}(t_s - t_F))} \right] \right\} \]  

(5.19)

In (5.19) we have two integrals of the same function over the two different intervals \( \Delta_1 = (x_A - \delta, x_A + \delta) \) and \( \Delta_2 = (-x_A - \delta, -x_A + \delta) \). The results will be two complex numbers \( \psi_1 \) and \( \psi_2 \) with different phases. So, differently from the classical case (5.4), the quantum wave function on the final screen \( \psi(x, t_s) \) has not a common phase factor and as a consequence the relative phases of \( \psi_1 \) and \( \psi_2 \) will play a crucial role in giving interference effects. In fact if we re-write the final wave function as:

\[
\psi(x, t_s) = \bar{N}_1 \left[ \psi_1(x, t_s) + \psi_2(x, t_s) \right] \]  

(5.20)

the probability on the last screen is given by the modulus square of \( \psi(x, t_s) \):

\[
P(x, t_s) = |\psi_1(x, t_s)|^2 + |\psi_2(x, t_s)|^2 + \psi_1^*(x, t_s)\psi_2(x, t_s) + \psi_1(x, t_s)\psi_2^*(x, t_s) \]  

(5.21)

Note that the last two terms in the previous formula are not identically zero. If we make a plot of \( P(x, t_s) \) as a function of \( x \) we see the evidence of the interference typical of quantum mechanics with a central maximum and a series of secondary maxima. This can be seen from Figure 2 which is the plot of \( P(x) \) in the case \( t_s = 2, t_F = 1, m = a = 1, \bar{h} = 1, \delta = 0.1 \) for two different distances of the slits: \( 2x_A = 1 \) and \( 2x_A = 2 \) respectively. We can note the presence of six minima in the first case and of twelve minima in the second one. This in perfect agreement with the well-known relation that the distance \( \Delta x \) between two successive maxima or minima in an interference figure is inversely proportional to the distance \( 2x_A \) between the slits. Therefore, even considering a quantum evolution only along the \( x \)-axis, the quantum wave functions create interference effects and the final result is a series of maxima and minima, like in the real experiment.

Summarizing the results of this section we can say that if we make the evolution along the \( x \) axis with the Schrödinger Hamiltonian \( \hat{H} \), even starting from a real wave function, like (5.13), phases will appear during the evolution in a non trivial way and they will contribute to create interference effects. Instead in the evolution along \( x \) with the Liouvillian \( \hat{H} \), even starting from a complex wave function, like (5.1), the phase appears as a common factor for the entire \( \psi \) on the final screen and so it does not contribute to \( |\psi|^2 \) and it has not observable consequences. We feel that the two different behaviours are basically due to the different forms of the evolution operators in the classical and in the quantum case.
6 Conclusions

In this paper we have shown, by means of simple and pedagogical examples, some of the differences between the operatorial approaches to classical and quantum mechanics. While in quantum mechanics the phases and the modulus of the wave functions are always coupled, eq. (3.33), in classical mechanics we can find a representation, the \((q,p)\) one, where we can decouple them. Their being coupled in some representation of classical mechanics, like the \((q,\lambda_p)\) one, is only an apparent phenomenon. We think that this is the real profound feature of quantum mechanics which makes it different from classical mechanics: that in quantum mechanics there is no way to decouple the phases from the modulus by just going to a proper representation like we can do in classical mechanics [9]. Of course we have not given a general proof of this in the sense that we have not really proven that in quantum mechanics this proper representation does not exist, but we strongly feel this is the case [13].

In this paper we have also performed the classical analog of the two-slit experiment and we have seen that interference effects do not appear. We have proved that by performing the detailed calculations giving the motion of the classical ”wave functions” in the two-slit experiment. That was just a particular example and we do not know if it works in the same way in general like, for example, in the multiple-slit case or in other phenomena where at the quantum level there is interference. To do the detailed calculations in all these cases may turn out to be even more difficult than in the two-slit case. To by-pass those calculations we would like to find a general proof of the absence of interference effects in the KvN formalism. For sure in this general proof a crucial role will be played by the different form that the classical \(\hat{H}\) and the quantum \(\hat{\Psi}\) have. Another crucial role may be played by the universal symmetries present in classical mechanics and discovered in [5]. Those symmetries may be trigger a sort of superselection mechanism which may prevent the interference from appearing. All these features may be also those that, being absent in quantum mechanics, forbid the decoupling of the phase and the modulus. Work is in progress [13] on these problems.

A Appendix

In this appendix we will show that there is no contradiction in the postulate of KvN of having for \(\psi\) the same evolution as for \(\rho = |\psi|^2\). The kernel \(P(\varphi, t|\varphi_i, t_i)\) for \(\rho\) is just (2.9) which is a Dirac delta. So, postulating this to be the same as the kernel \(K(\varphi, t|\varphi_i, t_i)\) of propagation of the \(\psi\), in the case of a free particle we have:

\[
K(\varphi, t|\varphi_i, t_i = 0) = \delta(q - q_i - \frac{p_i t}{m})\delta(p - p_i)
\]  

(A.1)
Let us now use this expression to check what we get for the kernel of $\rho$ knowing that $\rho(t) = |\psi(t)|^2$.

$$\rho(t) = \psi^*(t)\psi(t) = \int d\varphi_i K^*(\varphi, t|\varphi_i, 0)\psi^*(\varphi_i, 0) \cdot \int d\varphi'_i K(\varphi, t|\varphi'_i, 0)\psi(\varphi'_i, 0) =$$

$$= \int dq_i dp_i \delta \left( q - q_i - \frac{p_i t}{m} \right) \delta(p - p_i)\psi^*(q_i, p_i, 0) \cdot \int dq'_i dp'_i \delta \left( q - q'_i - \frac{p'_i t}{m} \right) \delta(p - p'_i)\psi(q'_i, p'_i, 0)$$

(A.2)

Now we can use the properties of the Dirac deltas to rewrite:

$$\rho(t) = \int dq_i dp_i dq'_i dp'_i \delta \left( q - q_i - \frac{p_i t}{m} \right) \delta(p - p_i)\delta(p_i - p'_i) \cdot \delta \left( q_i - q'_i + (p_i - p'_i) \frac{t}{m} \right) \psi^*(q_i, p_i, 0)\psi(q'_i, p'_i, 0)$$

(A.3)

The integrals over the primed variables can be done explicitly:

$$\int dq'_i dp'_i \delta(p_i - p'_i)\delta \left( q_i - q'_i + (p_i - p'_i) \frac{t}{m} \right) \psi^*(q_i, p_i, 0)\psi(q'_i, p'_i, 0) = \rho(\varphi_i, 0)$$

(A.4)

Substituting (A.4) into (A.3) we have finally:

$$\rho(t) = \int dq_i dp_i K(\varphi, t|\varphi_i, 0)\rho(\varphi_i, 0)$$

(A.5)

From this relation we get that the kernel of propagation of the $\rho$ is the same as the one of the $\psi$ and this proves that there is no contradiction in the KvN postulate.

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Figure Caption

Fig. 1: Classical Two-Slit Experiment.

Fig. 2: Quantum Two-Slit Experiment.
Figure 1: Classical Two-Slit Experiment
Figure 2: Quantum Two-Slit Experiment