Coherent and semiclassical states in a magnetic field in the presence of the Aharonov–Bohm solenoid

V G Bagrov1,2, S P Gavrilov3,4, D M Gitman3 and D P Meira Filho3

1 Department of Physics, Tomsk State University, 634050 Tomsk, Russia
2 Tomsk Institute of High Current Electronics, SB RAS, 634034 Tomsk, Russia
3 Institute of Physics, University of São Paulo, CP 66318, CEP 05315-970 São Paulo, SP, Brazil

E-mail: bagrov@phys.tsu.ru, gavrilovsergey@yahoo.com, gitman@dfn.if.usp.br and dmeira@dfn.if.usp.br

Received 17 August 2010, in final form 1 December 2010
Published 4 January 2011
Online at stacks.iop.org/JPhysA/44/055301

Abstract
A new approach to constructing coherent states (CS) and semiclassical states (SS) in a magnetic-solenoid field is proposed. The main idea is based on the fact that the AB solenoid breaks the translational symmetry in the xy-plane; this has a topological effect such that there appear two types of trajectories which embrace and do not embrace the solenoid. Due to this fact, one has to construct two different kinds of CS/SS which correspond to such trajectories in the semiclassical limit. Following this idea, we construct CS in two steps, first the instantaneous CS (ICS) and then the time-dependent CS/SS as an evolution of the ICS. The construction is realized for nonrelativistic and relativistic spinning particles both in (2 + 1) and (3 + 1) dimensions and gives a non-trivial example of SS/CS for systems with a nonquadratic Hamiltonian. It is stressed that CS depending on their parameters (quantum numbers) describe both pure quantum and semiclassical states. An analysis is represented that classifies parameters of the CS in such respect. Such a classification is used for the semiclassical decompositions of various physical quantities.

PACS numbers: 03.65.Ge, 03.65.Sq

1. Introduction
Quantum interaction of charged particles with the field of an infinitely long and infinitesimally thin magnetic solenoid (further the Aharonov–Bohm (AB) field) has been studied both theoretically and experimentally for many years. In spite of the fact that particle wavefunctions vanish on the solenoid line, the particles feel the presence of the AB solenoid [1]. This
phenomenon is called the AB effect and is interpreted as a possibility for locally trivial vector potentials to give rise to observable effects in a nontrivial topology. A number of theoretical works and convincing experiments were carried out to clarify the AB effect and prove its existence. By the middle of the 1980s, the AB effect in low energy physics had become a good instrument for investigating new physical phenomena, principally in condensed matter physics, where the AB ring had been the mainstay of mesoscopic physics research since its inception (see [2] for a general review). It was discovered that the effect is relevant to a number of physical problems, e.g. to anyons in high-$T_c$ superconductivity [3], electronic excitations in graphene with topological defects [4, 5], nanotubes [6], nonrelativistic scattering in Chern–Simons theory [7], theory of unparticles [8] and so on. AB vacuum polarization and AB radiation are relevant to cosmic string dynamics, see for example [9, 10].

A splitting of Landau levels in a superposition of the AB field and a parallel uniform magnetic field gives an example of the AB effect for bound states. In what follows, we call such a superposition the magnetic-solenoid field (MSF). Solutions of the Schrödinger equation with the MSF were first studied in [11]. Solutions of relativistic wave equations (Klein–Gordon and Dirac ones) with the MSF were first obtained in [12] and then used in [13] to study the AB effect in cyclotron and synchrotron radiations. On the basis of these solutions, the Green’s functions and the problem of the self-adjointness of Dirac Hamiltonians with the MSF were studied [14–18]. A complete spectral analysis for all the self-adjoint nonrelativistic and relativistic Hamiltonians with the MSF was performed in [19]. Recently, the interest in the MSF (and related multivortex examples) has been renewed in connection with planar physics problems and the quantum Hall effect [20]. It is important to stress that in contrast to the pure AB field case, where particles interact with the solenoid for a finite short time, moving in the MSF the particles interact with the solenoid permanently. This opens up more possibilities to study such an interaction and corresponds to a number of real physical situations. For example, recent fabrication of a graphene allows an experimental observation and application of effects with relativistic spinning particles under usual laboratory conditions [21].

In some cases, it is enough and, moreover, more adequate (and convenient) to use a semiclassical description of a physical system. Semiclassical states (SS) of the system provide such a description. Usually, such states are identified with different kinds of the so-called coherent states (CS). However, such a formal identification is known for systems with quadratic Hamiltonians, whereas in other cases both SS and CS construction and their identification are problematic. In addition to the well-known applications of SS/CS in quantum theory [22], there appeared recently new important applications to quantum computations, see, e.g., [23]. Constructing SS/CS for particles in the AB field and in the MSF is a nontrivial problem (which was an open problem until present), in particular, due to the nonquadratic structure of particle Hamiltonians with such fields. Besides numerous possible practical applications, constructing such states gives an important example of SS/CS for nonquadratic Hamiltonians and may, for example, answer an important theoretical question: to what extent the AB effect is of a pure quantum nature? In a sense constructing SS/CS is a complementary task to the path integral construction, which also is an open problem in the case of the particle in the MSF. One can suppose that SS/CS in the MSF are in a sense analogous to the ones in the pure magnetic field. In the latter case SS are identified with CS that are well known, see e.g. [24, 25]. In such CS, the mean values of particle coordinates move along classical trajectories. The latter trajectories are circles whose radii and center position (quantum numbers) label these quantum CS. Constructing CS for particles in the MSF, we will try to maintain the basic properties of already known CS for quadratic systems. In particular, such CS have to minimize uncertainty relations for some physical quantities (e.g. coordinates and momenta) at a fixed time instant.
and means of particle coordinates, calculated with respect to time-dependent CS, have to move along the corresponding classical trajectories. In addition, CS have to be labeled by quantum numbers that have a direct classical analog, let us say by phase-space coordinates. It is also desirable for time-dependent CS to maintain their form under time evolution.

It should be mentioned that some attempts to construct SS/CS for particles in the MSF are presented in [26]. However, the states constructed there do not obey the principal requirement for SS/CS, the corresponding means do not move along classical trajectories. In our recent article [27], we succeeded in constructing a principally different kind of CS for requirement for SS/case of relativistic spinless and nonrelativistic and relativistic spinning particles both in nonrelativistic spinless particles in the MSF. In this work we extend this construction to the nonrelativistic spinless and nonrelativistic and relativistic spinning particles both in (3+1) and (2+1) dimensions (dim). In addition, developing semiclassical approximation techniques, we constructed SS in the MSF on the base of the CS. The progress is related to a nontrivial observation that, in the problem under consideration, there are two kinds of SS/CS, those which correspond classical trajectories which embrace the solenoid and those which do not. It should be stressed that the identification of SS and CS in the MSF depends essentially on quantum numbers that label these states (on types and positions of the corresponding classical trajectories). Particles in constructed SS/CS move along classical trajectories, the states maintain their form under time evolution and form a complete set of functions, which can be useful in semiclassical calculations. In the absence of the AB field these states are reduced to the well known states in the case of a uniform magnetic field Malkin–Man’ko CS [24]. The constructed states give a non-trivial example of SS/CS for systems with a nonquadratic Hamiltonian. In addition, they allow one to treat the AB effect on the classical language, revealing an influence of the AB field on parameters of classical trajectories in the magnetic field. It should be noted that quantum motion of spin-1/2 Dirac fermions is qualitatively different in (3+1) and (2+1) dim. Since Dirac fermions in (2+1) dim (in particular, massless ones) describe single-electron dynamics in graphene, we devoted part of our study to their SS and CS in the MSF.

The paper is organized as follows. In section 2, we begin our consideration with the classical description of particle motion in the MSF. We recall that the MSF is a collinear superposition of a constant uniform magnetic field of strength \( B \) and the AB field (field of an infinitely long and infinitesimally thin solenoid with a finite constant internal magnetic flux \( \Phi \)). Setting the \( z \)-axis along the AB solenoid, the MSF strength takes the form \( B = (0, 0, B_z) \):

\[
B_c = B + \Phi \delta(x)\delta(y) = B + \frac{\Phi}{\pi r} \delta(r), \quad B = \text{const}, \quad \Phi = \text{const}.
\]  

(1)

We use the following electromagnetic potentials\(^5\) \( A^\mu \), assigned to the MSF (1): \( A^0 = A^3 = 0 \) and

\[
A^1 = -y \left( \frac{\Phi}{2\pi r^2} + \frac{B}{2} \right), \quad A^2 = x \left( \frac{\Phi}{2\pi r^2} + \frac{B}{2} \right).
\]  

(2)

In section 3, we briefly outline relativistic quantum mechanics of spinning particles in the MSF, introducing physical quantities important for our purposes. Here we use the so-called natural self-adjoint extensions of the corresponding Hamiltonians, which correspond to the zero-radius limit of the regularized case of a finite-radius solenoid. Explicit forms of relativistic and nonrelativistic quantum stationary states of spinning particles in the MSF are placed in appendix A. In section 4, subsection 4.1, we build instantaneous CS for nonrelativistic

\(^5\) We accept the following notations for four- and three-vectors: \( a = (a^\mu, \mu = 0, 1) = (a^0, \mathbf{a}), \mathbf{a} = (a^i, i = 1, 2, 3) = (a^0 = a_x, a^1 = a_y, a^3 = a_z), a_0 = -a^i \), in particular, for the spacetime coordinates: \( x^\mu = (\mathbf{t}, t) = (x, t) = (x, x^0 = c t), a^1 = a_x = x, a^2 = y, a^3 = z \), as well as cylindrical coordinates \( r, \psi \), in the \( xy \)-plane, such that \( x = r \cos \psi, y = r \sin \psi \) and \( r^2 = x^2 + y^2 \). Besides, \( \mathbf{dx} = dx^0 \mathbf{dx}, \mathbf{dx} = dx^1 dx^2 dx^3 \) and the Minkowski tensor \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \).
and relativistic spinning particles both in (2 + 1) and (3 + 1) dim, using some universal 
constructions. In subsection 4.2, we study the semiclassical approximation. On the basis 
of the CS, we construct SS, developing a technique of semiclassical calculations by means of 
various physical quantities. In section 5, we construct time-dependent CS for different kinds 
of particles, find and analyze trajectories of means. Some details of these calculations are 
placed in appendix B. We summarize and discuss the obtained results in section 6.

2. Classical motion in the MSF

As was mentioned in the introduction, our intention is to construct CS in the MSF. The basic 
expected properties of the CS are described in terms of the classical motion in the MSF.

That is why we start our exposition with this section, where we present a brief description 
of the classical motion of a charge $q = \pm e$ with mass $M$ in the MSF. Trajectories $x^\nu(s)$ are 
parametrized by the Minkowski interval $s$ and obey the Lorentz equations:

$$M^2 c^2 \ddot{x}^\nu = q F^\nu{\mu} \dot{x}_\mu,$$  \hspace{0.5cm} (3)

where $\dot{x}^\nu = dx^\nu/ds$, $F^\nu{\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu$. As follows from (1), the only nonzero components 
of $F^\nu{\mu}$ are $F^{21} = -F^{12} = B$. For trajectories that do not intersect the $z$-axis, we obtain 
from (3)

$$P_0 = \text{const}, \quad P_3 = \text{const}, \quad \dot{P}_1 = \epsilon \kappa P_2, \quad \dot{P}_2 = -\epsilon \kappa P_1, \quad \dot{P}_1^2 + P_2^2 = \dot{P}_3^2 = \text{const},$$ \hspace{0.5cm} (4)

where $P^\nu = M c \dot{x}^\nu = p^\nu - \frac{q}{c} A^\nu$ is the kinetic momentum, and $p^\nu$ is the 
generalized particle momentum, $p^\nu P_\nu = (Mc)^2$, $\kappa = |q B|/Mc^2$, $\epsilon = \text{sign}(q B)$. Thus, the total particle energy $E = c P^0$ is also an integral of motion. One can see that the general solution of (4) reads

$$ct = \frac{p_0}{Mc}, \quad z = -\frac{p_3}{Mc} s + z_0 = -\frac{cp_3}{p_0} t + z_0, \quad \dot{x} = x_0 + R \cos \psi,$$

$$y = y_0 - \epsilon R \sin \psi; \quad \psi = \kappa s + \psi_0 = \omega t + \psi_0, \quad \omega = \frac{|q B|}{p_0}. $$ \hspace{0.5cm} (5)

where $x_0$, $y_0$, $z_0$, $p_0$, $p_3$, $R$ and $\psi_0$ are integration constants. It follows from (5) that

$$(x - x_0)^2 + (y - y_0)^2 = R^2, \quad x_0 = R_c \cos \alpha, \quad y_0 = R_c \sin \alpha,$$

$$r^2 = x^2 + y^2 = R^2 + R_c^2 + 2 R R_c \cos(\psi + \epsilon \alpha), \quad R_c = \sqrt{x_0^2 + y_0^2}. $$ \hspace{0.5cm} (6)

Projections of particle trajectories on the $xy$-plane are circles of the radii $R_c$ with central points 
$(x_0, y_0)$ placed on the distance $R_c$ from the origin. Particle images on the $xy$-plane are rotating 
with the synchrotron frequency $\omega$. For an observer which is placed near the solenoid with 
z > 0, the rotation of the particle with $\epsilon = 1$ is clockwise, and for the particle with $\epsilon = -1$ is 
anticlockwise. Thus, equations of motion for the charge $-q$ can be obtained from equations 
of motion (5) with the charge $q$ by the substitution $p_0$ by $-p_0$. Along the $z$-axis the particle 
has a constant velocity $dz/dt = -cp_3/p_0$. We denote by $r_{\text{max}} = R + R_c$ the maximal possible 
moving off and by $r_{\text{min}} = |R - R_c|$ the minimal possible moving off of the particle from the 
$z$-axis.

We note that equations of motion for a nonrelativistic particle ($P^2 \ll (Mc)^2$) in the MSF 
follow from (5) setting $p_0 = Mc$. Then $ct = s$ and $\omega = \omega_{\text{NR}} = |q B|/Mc$, where $\omega_{\text{NR}}$ is the 
cyclotron frequency.

The square of the particle rotation energy is $E_\perp^2 = c^2 P_\perp^2$ and determines the radius $R$ as follows:

$$R^2 = E_\perp^2 (q B)^{-2}. $$ \hspace{0.5cm} (7)
Using (6), one can calculate the angular momentum projection $L_z$:

$$L_z = y p_1 - x p_2 = \frac{\epsilon M c}{kappa_1} (R_c^2 - R^2) + \frac{q \Phi}{2\pi c},$$

which is a (dependent) integral of motion.

The presence of the AB solenoid (the magnetic flux $\Phi$) breaks the translational symmetry in the $xy$-plane. In classical theory, this fact has only a topological effect; there appear two types of trajectories, we label them with an index $j = 0, 1$ such that $j = 1$ corresponds to $(R_c^2 - R^2) > 0$ (embraces the solenoid), and $j = 0$ corresponds to $(R_c^2 - R^2) < 0$ (does not embrace the solenoid), see figure 1.

Already in the classical theory, it is convenient to introduce dimensionless complex quantities $a_1$ and $a_2$ (containing the constant $\hbar$) as follows:

$$a_1 = \frac{i P_1 - \epsilon P_2}{\sqrt{2\hbar Mc} \sqrt{R}}, \quad a_2 = \frac{M c (x - i \epsilon y) - i P_1 - \epsilon P_2}{\sqrt{2\hbar Mc} \sqrt{R_c}},$$

They define physical quantities $R, R_c, x, y, P_\perp$ and $L_z$ as follows:

$$R^2 = \frac{2}{\gamma} a_1^* a_1, \quad R_c^2 = 2\gamma^{-1} a_2^* a_2, \quad (x - i \epsilon y) = \sqrt{2\gamma} (a_2 - a_1^*),$$

$$P_\perp = 2\gamma \hbar^2 a_1^* a_1, \quad L_z = \hbar (a_2^* a_2 - a_1^* a_1) + \frac{q \Phi}{2\pi c}.$$  

One can see that $a_1 \exp(i \omega t)$ and $a_2$ are complex (dependent) integrals of motion.

Another important dimensionless integral of motion $\lambda$ (in the classical theory $\lambda > 0$) reads

$$\lambda = \frac{p_0 + p_3}{Mc}.$$
Thus, we can choose the set \( x_0, y_0, z_0, \lambda, R \) and \( \psi_0 \) as six independent integrals of motion.

Often, it is convenient to use the light cone variables \( x_{\pm} \):

\[
    x_- = ct - z, \quad x_+ = ct + z \iff \frac{x_+ + x_-}{2}, \quad z = \frac{x_+ - x_-}{2}.
\]  

(13)

In terms of such variables, the general solution (5) takes the form

\[
    ct = \frac{1 + (\mu R)^2 + \lambda^2}{2\lambda^2} x_-, \quad z = \frac{1 + (\mu R)^2 - \lambda^2}{2\lambda^2} x_- + z_0,
\]

\[
    x = x_0 + R \cos \psi, \quad y = y_0 - \epsilon R \sin \psi; \quad \psi = \tilde{\omega} x_+ + \psi_0,
\]  

(14)

where \( x_- \) plays the role of the time.

### 3. Quantum mechanics with the MSF

In quantum theory, it is convenient to represent the magnetic flux \( \Phi \) of the AB solenoid via Dirac’s fundamental magnetic flux \( \Phi_0 = 2\pi c h / e \) as follows:

\[
    (\Phi / \Phi_0) \text{sign} B = l_0 + \mu \Rightarrow l_0 = [(\Phi / \Phi_0) \text{sign} B] \in \mathbb{Z},
\]

\[
    0 \leq \mu = (\Phi / \Phi_0) \text{sign} B - l_0 < 1,
\]  

(15)

where \( l_0 \) is an integer and the quantity \( \mu \) is called the mantissa of the magnetic flux. In fact, \( \mu \) determines all the quantum effects in the AB and MSF, see e.g. [13]. We note that definition (15) differs from the one \( \tilde{\mu} = (\Phi / \Phi_0) - [(\Phi / \Phi_0)] \) for the mantissa of \( \Phi \), which was used in some earlier works and which does not contain the factor \( \text{sign} B \). The quantities \( \mu \) and \( \tilde{\mu} \) are related as follows: \( \mu = \tilde{\mu}, B > 0; \mu = 1 - \tilde{\mu}, B < 0. \) It turns out that definition (15) is very convenient and allows one to write universal expressions for any mutual orientations of the uniform magnetic field and the AB flux.

The quantum behavior of spinning (spin-1/2) relativistic particles in the MSF is described by Dirac wavefunctions \( \Psi \) that obey the Dirac equation with the electromagnetic potentials (2):

\[
    i\hbar \partial_\mu \Psi = \hat{H} \Psi, \quad \hat{H} = c\gamma^0(\gamma^1 \hat{P} + Mc),
\]  

(16)

where \( \gamma^\nu = (\gamma^0, \gamma^i) \), \( \gamma^i = (\gamma^k) \) are \( \gamma \)-matrices; \( \hat{P}^k = \hat{P}^k - \frac{i}{2} A^k \), \( \hat{P}^k = -i\hbar \partial_k \). Below, we consider the Dirac equation in \((2 + 1)\) dim, where \( k = 1, 2 \), and in \((3 + 1)\) dim, where \( k = 1, 2, 3 \).

It is natural that the solutions of the Dirac equation in \((2 + 1)\) dim and in \((3 + 1)\) dim have much in common. Nevertheless, the algebra of the Dirac \( \gamma \)-matrices for these cases is different as well as the spin description; all this implies, e.g., the well known fact that quantum mechanics of the spinning particle (both nonrelativistic and relativistic) in the presence of a uniform magnetic field is essentially different in \((2 + 1)\) dim and in \((3 + 1)\) dim. That is why we consider the problem under consideration both in \((2 + 1)\) dim and \((3 + 1)\) dim separately (the former case cannot be extracted from the latter one in a trivial manner).

It is also known that the AB effect in condensed matter physics, in particular planar physics, is important in the nonrelativistic case, \( E_{\text{2D}}^2 \ll Mc^2 \). Therefore, we pay a special attention to such a limit. In addition, the massless (which is, in a sense, equivalent to the ultrarelativistic case) Dirac equation in \((2 + 1)\) dim describes under some conditions the graphene physics. In such a case, the Fermi velocity \( v_F \approx c/300 \) plays the role of the effective velocity of light and has to substitute \( c \) in all the corresponding expressions. In this connection, we consider the ultrarelativistic limit in detail.
In (2+1) dim, a Dirac wavefunction $\Psi$ is a spinor dependent on $x^0, x^1$ and $x^2$, and there are two nonequivalent representations for $\gamma$-matrices:

$$\gamma^0 = \sigma^3, \quad \gamma^1 = i\sigma^2, \quad \gamma^2 = -i\sigma^1\zeta, \quad \zeta = \pm 1,$$

where $\sigma = (\sigma^i)$ are Pauli matrices. Choosing the 'polarizations' $\zeta = +1$, we describe 'spin-up' particles, and choosing $\zeta = -1$, we describe 'spin-down' particles. In (2+1) dim these are different particles. There exist 'spin-up' and 'spin-down' antiparticles. In contrast to the (3+1) dim case, particles and antiparticles in (2+1) dim have only one spin polarization state.

Stationary states of the Dirac equation with the MSF in (2+1) dim have the form

$$\Psi = \exp \left[-\frac{i}{\hbar} (cp0t) \right] \psi^{(\frac{\zeta}{2})}(x^\perp), \quad \zeta = \pm 1, \quad x^\perp = (0, x^1, x^2),$$

where spinors $\psi^{(\frac{\zeta}{2})}(x^\perp)$ are subjected to the equations

$$\begin{align*}
(\sigma \not{P}_\perp + M c \sigma^3)\psi^{(\frac{\zeta}{2})}(x^\perp) &= p_0\psi^{(\frac{\zeta}{2})}(x^\perp), \\
(\sigma^i \not{P}_\perp \sigma^i + M c^3)\psi^{(-\frac{\zeta}{2})}(x^\perp) &= p_0\psi^{(-\frac{\zeta}{2})}(x^\perp).
\end{align*}$$

We note that $cp_0 = E > 0$ for particles, and $cp_0 = -E < 0$ for antiparticle states.

One can see that

$$\psi^{(-\frac{\zeta}{2})}(x^\perp) = \sigma^2 \psi^{(\frac{\zeta}{2})}(x^\perp).$$

That is the reason why we will consider only the case $\zeta = 1$ in what follows. In such a case, a self-adjoint Hamiltonian $\hat{H}^\theta$ has the form

$$\hat{H}^\theta = c(\sigma \not{P}_\perp + M c \sigma^3).$$

Its domain $D^\theta_H$ depends essentially on the sign $\theta = \text{sign} \Phi = \pm 1$ of the magnetic flux; that is why the Hamiltonian has a label $\theta$.

In (3+1) dim, a Dirac wavefunction $\Psi$ is a bispinor dependent on $x^0, x^1, x^2$ and $x^3$. Then unlike (2+1) dim we are not restricted in the choice of the evolution parameter of time $x^0$ but we can also use the light-cone variable $x^\perp$. There is also an opportunity to build differently spinors adapting them to the nonrelativistic or ultrarelativistic limit. All this is described in detail in appendix A.

It should be recalled that all self-adjoint extensions of (2+1) and (3+1) Dirac Hamiltonians in the MSF were constructed in [15, 16, 19], see, also, [14]. The domains of (3+1) Dirac Hamiltonian in the MSF are trivial extensions of the corresponding domains mentioned in the (2+1) case, that is why we retain for them the same notation $D^\theta_H$ and use for the self-adjoint (3+1) Dirac Hamiltonian the same notation $\hat{H}^\theta$. Of course, in this case $\hat{H}^\theta = c\gamma^0 \left( \sum_{k=1,2,3} \gamma^k \hat{P}^k + M c \right)$. In addition, considering a regularized case of a finite-radius solenoid, it was demonstrated that the zero-radius limit yields two (depending on $\theta$) of self-adjoint extensions, with domains $D^\theta_H$. In contrast to the spinless case we both domains $D^\theta_H$ involve irregular but still square-integrable radial functions that do not vanish as $r \to 0$. In fact, this means that any wavefunction is completely determined by its values for $r > 0$. Its value in the point $r = 0$ can be set arbitrary. In appendix A, we represent solutions of equations (18) and (16) in (3+1) dim for both values of $\theta$ in the domains $D^\theta_H$.

Some important remarks should be made.

(1) In the case of spinning particles, some results essentially depend both on the mantissa of the magnetic flux $\mu$ and on the direction of the flux $\theta$. The latter dependence appears due to

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6 The domain of a spinless particle Hamiltonian in the MSF involves only regular radial functions as $r \to 0$, see [16]. Here we use the terms 'regular' and 'irregular' as $r \to 0$ in the following sense. We call a function 'regular' if it behaves as $r^\epsilon$ as $r \to 0$ with $\epsilon \geq 0$, and 'irregular' if $\epsilon < 0$.
the spin presence and is specific only for states with irregular radial functions. In such states there is a superstrong contact interaction between the magnetic moment of the particle and the solenoid flux. Namely, this interaction dependent on the spin presence and is specific only for states with irregular radial functions. In such states one can interpret the corresponding eigenvalues of the operators $\hat{\sigma}_z$ and $\hat{\sigma}_\perp$ as follows:

$$\hat{\sigma}_z = \hat{\sigma}_z + \frac{\gamma l}{\hbar}$$

and ($\alpha_\perp \hat{\sigma}_\perp$)$^2$, $\hat{\sigma}_\perp = \hat{\sigma}_\perp - \hbar \hat{\sigma}_x$, $\alpha_\perp = (\alpha_1^1, \alpha_2^2, 0)$, are self-adjoint on the domain $D_H^0$ and mutually commuting integrals of motion (all these operators commute with the Hamiltonian $\hat{H}$) [15, 16]. In (2 + 1) dim, the total angular momentum operator $\hat{J} = -i\hbar \hat{\sigma}_x + \hbar \hat{\sigma}_\perp^2/2$, which is a dimensional reduction of the operator $\hat{J}_z$ in (3 + 1) dim, and $(\sigma \hat{\sigma}_\perp)^2$ are self-adjoint on $D_H^0$ and mutually commuting integrals of motion [15, 16]. One can say that $c^2(\sigma \hat{\sigma}_\perp)^2$ and $\hat{J}_z$ in (2 + 1) dim are integrals of motion which play the role of a square of the transverse kinetic energy ($E^2_\perp$ in (7)) and the $z$-component of the total angular momentum ($L_z$ in (8)) in the case of a spinning particle, respectively. Then it is useful to define self-adjoint operators $\hat{R}^2$ and $\hat{R}_c^2$ by analogy with corresponding classical relations (7) and (8) as follows:

$$\hat{R}^2 = c^2(\sigma \hat{\sigma}_\perp)^2(qB)^{-2}, \quad \hat{R}_c^2 = c^2(\alpha \hat{\sigma}_\perp)^2(qB)^{-2}, \quad \hat{R}_c^2 - \hat{R}^2 = -2\epsilon\{(l_0 + \mu)\hbar - \epsilon \hat{J}\}|qB|^{-1}$$

in (2 + 1) dim, and

$$\hat{R}_c^2 - \hat{R}^2 = -2\epsilon\{(l_0 + \mu)\hbar - \epsilon \hat{J}_z\}|qB|^{-1}$$

in (3 + 1) dim.

One can find two types ($j = 0, 1$) of solutions of the Dirac equation which are common eigenvectors of operators $\hat{R}^2$ and $\hat{J}$ in (2 + 1) dim and operators $\hat{R}^2$ and $\hat{J}_z$ in (3 + 1) dim, see (A.1), (A.34) and (A.40) in appendix A, respectively. Such solutions have two quantum numbers $n_1$ and $n_2$ in common; we may be using for them the general notation $\Psi_{n_1, n_2}^{(j)}$. Note that eigenvalues of the operators $\hat{J}$ and $\hat{J}_z$ are the same, $J = J_z = \epsilon\hbar(l_0 - l + 1/2)$, where $l$ is an integer. Then, using an appropriate inner product on the $xy$-plane, see (A.12) in (2 + 1) dim and (A.31) in (3 + 1) dim, we obtain the mean of the operator $\hat{R}^2 - \hat{R}_c^2$:

$$\langle \Psi_{n_1, n_2}^{(j)} \hat{R}^2 - \hat{R}_c^2 \rangle = \frac{2\epsilon}{\gamma}(l + \mu),$$

in (2 + 1) dim, and

$$\langle \Psi_{n_1, n_2}^{(j)} \hat{R}_c^2 - \hat{R}^2 \rangle = \frac{2\epsilon}{\gamma}(l + \mu),$$

in (3 + 1) dim.

In the semiclassical limit the sign of the mean allows one to interpret the corresponding states as particle trajectories that embrace and do not embrace the solenoid. Namely, an orbit embraces the solenoid for $l \geq 0$ (type $j = 1$) and do not for $l \leq -1$ (type $j = 0$). This classification corresponds to the classical one introduced in the previous section, see equation (8) and figure 1. Trajectories with $l = 0, -1$ are situated closest to the solenoid.
For $\mu = 0$ there is no impact of the AB solenoid on the energy spectrum and the energy spectrum is given by the Landau formula. For $\mu \neq 0$, energies of states with $j = 1$ differ from the Landau levels, whereas energies with $j = 0$ coincide with the Landau levels, see (A.11) in appendix A.

4. Instantaneous CS on the $xy$-plane

4.1. Quantum states

In spite of the differences between stationary states of the spinning particle in $(2+1)$ dim and in $(3+1)$ dim, see (A.2), (A.37) and (A.39), in both dimensions one can build CS on the $xy$-plane in a similar manner, using some universal constructions. Let us introduce operators $\hat{a}_1, \hat{a}_2$, and $\hat{a}_1^\dagger, \hat{a}_2^\dagger$ that correspond to classical quantities $a_1, a_2$, and $a_1^*, a_2^*$:

$$
\hat{a}_1 = \frac{i \hat{P}_1 - \epsilon \hat{P}_2}{\sqrt{2\hbar Mc}} , \quad \hat{a}_2 = \frac{Mcx(x - i\epsilon y) - i \hat{P}_1 - \epsilon \hat{P}_2}{\sqrt{2\hbar Mc}} , \\
\hat{a}_1^\dagger = -\frac{i \hat{P}_1 + \epsilon \hat{P}_2}{\sqrt{2\hbar Mc}} , \quad \hat{a}_2^\dagger = \frac{Mcx(x + i\epsilon y) + i \hat{P}_1 - \epsilon \hat{P}_2}{\sqrt{2\hbar Mc}} .
$$

(25)

It should be noted that the operators $\hat{P}_1$ and $\hat{P}_2$ are symmetric but not self-adjoint on the domain $\mathcal{D}_{\vartheta}$. That is why one cannot consider $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$ as adjoint to $\hat{a}_1$ and $\hat{a}_2$, respectively. Nevertheless, the operators (25) play an important role in the further constructions.

Using properties of Laguerre functions, one can find the action of these operators on functions (A.5):

$$
\hat{a}_1 \Phi_{n_1,n_2,\sigma}(\varphi, \rho) = \sqrt{n_1} \Phi_{n_1-1,n_2,\sigma}(\varphi, \rho) , \quad \hat{a}_1^\dagger \Phi_{n_1,n_2,\sigma}(\varphi, \rho) = -\sqrt{n_1+1} \Phi_{n_1+1,n_2,\sigma}(\varphi, \rho) , \\
\hat{a}_2 \Phi_{n_1,n_2,\sigma}(\varphi, \rho) = \sqrt{n_2} \Phi_{n_1,n_2-1,\sigma}(\varphi, \rho) , \quad \hat{a}_2^\dagger \Phi_{n_1,n_2,\sigma}(\varphi, \rho) = -\sqrt{n_2+1} \Phi_{n_1,n_2+1,\sigma}(\varphi, \rho) ,
$$

(26)

where possible values of $n_1$ and $n_2$ depend on $m, l, \sigma$ and $j$ according to (A.5) and the functions $\Phi_{n_1,n_2,\sigma}$ are defined in (A.8).

Formal commutators between the operators $\hat{a}_1^\dagger, \hat{a}_1$, and $\hat{a}_2^\dagger, \hat{a}_2$ have the form

$$
[\hat{a}_1, \hat{a}_1^\dagger] = 1 + f , \quad [\hat{a}_2, \hat{a}_2^\dagger] = 1 - f , \quad [\hat{a}_1, \hat{a}_2] = f , \quad [\hat{a}_1, \hat{a}_1^\dagger] = 0 ,
$$

with a singular function $f = \Phi(\pi Br)^{-1} \delta(r) = 2(l_0 + \mu)\delta(\rho)$. All the solutions of the Dirac equation in $(2+1)$ and $(3+1)$ dim introduced in appendix A are expressed via functions (A.5) and describe states of spinning particles out of the solenoid, $r > 0$. In such a case, the function $f$ gives zero contributions and can be neglected. In turn, this means that on the domain $\mathcal{D}_{\vartheta}^0$ operators $\hat{a}_1^\dagger, \hat{a}_2^\dagger$ and $\hat{a}_1, \hat{a}_2$ behave as creation and annihilation operators. Then the quantities $x - i\epsilon y$ and $\hat{L}_z$ can be expressed in terms of the operators $\hat{a}_1^\dagger, \hat{a}_1$ and $\hat{a}_2^\dagger, \hat{a}_2$ as follows:

$$
(x - i\epsilon y) = \sqrt{2\gamma^{-1}}(\hat{a}_2 - \hat{a}_1^\dagger) , \quad -\frac{\epsilon}{\hbar} \hat{L}_z + l_0 + \mu = (\hat{N}_1 - \hat{N}_2) ,
$$

(27)

which is the same in $(2+1)$ and $(3+1)$ dim. The operator $c^2(\sigma \cdot \hat{P}_\perp)^2$ in the $(2+1)$ dim case and $c^2(\alpha_\perp \cdot \hat{P}_\perp)^2$ in the $(3+1)$ dim case have the following representations:

$$
c^2(\sigma \cdot \hat{P}_\perp)^2 = 2\hbar c|qB|[(\hat{N}_1 + (1 - \sigma^3)\epsilon)/2] \text{ in } (2+1) \text{ dim} ,
$$

$$
c^2(\alpha_\perp \cdot \hat{P}_\perp)^2 = 2\hbar c|qB|[(\hat{N}_1 + (1 - \Sigma_\epsilon)/2] \text{ in } (3+1) \text{ dim} ,
$$

(28)
where relations (26) and expression (A.11) are used. Then the operators $\hat{R}^2$ and $\hat{R}^2_\epsilon$ have the following form (see definitions (23)):

$$\hat{R}^2 = \gamma^{-1}(2\hat{N}_1 + 1 - \sigma^3\epsilon) \text{ in } (2 + 1) \text{ dim}, \quad \hat{R}^2_\epsilon = \gamma^{-1}(2\hat{N}_1 + 1 - \Sigma_\epsilon) \text{ in } (3 + 1) \text{ dim}, \quad \hat{R}^2_\epsilon = \gamma^{-1}(2\hat{N}_2 + 1) \text{ in } (2 + 1) \text{ and in } (3 + 1) \text{ dim}. \quad (29)$$

As was already mentioned in the introduction, our aim is to construct CS. One can formulate a definition of CS for systems with quadratic Hamiltonians, see [22]. Unfortunately, no general definition of CS for an arbitrary quantum system exists. In our case, with a nonquadratic Hamiltonian, defining CS, we would like to maintain the basic properties of already known CS for quadratic systems. First of all, these states have to minimize uncertainty relations for some physical quantities (e.g. coordinates and momenta) at any fixed time instant. Second, means of these quantities, calculated with respect to time-dependent CS, have to move along classical trajectories. It is also desirable for time-dependent CS to maintain their form under the time evolution. Here, it is supposed that time-dependent CS are solutions of the corresponding wave equation, Dirac or Pauli (and Klein–Gordon or Schrödinger equation in the case of a spinless particle). Thus, the problem of constructing the CS states is mainly reduced to a suitable choice of the form of the CS at a fixed time instant. We call such CS instantaneous CS (ICS) in what follows. In the case of quadratic systems, e.g. a non-relativistic nonquadratic Hamiltonian, defining CS, we would like to maintain the basic properties of particle Hamiltonians are nonquadratic, these operators are not exactly annihilation operators for both types of the functions $\Phi^{(j)}_{n_1, n_2, \sigma}$. Nevertheless, as is demonstrated below, one can construct some kind of ICS that have the above-described properties. These states are very close to eigenvectors of the introduced operators $\hat{a}_1, \hat{a}_2$ from (25). At the same time, these states maintain their form under the time evolution.

The presence of the AB flux breaks the translational symmetry in the $xy$-plane. That is why in the problem under consideration, there appear two types of CS; those which correspond to classical trajectories which embrace the solenoid and those which do not. Taking into account the classification of quantum states according to types $j = 1$ and $j = 0$, which depends on the sign of the mean value (24), we see that each of these CS must be constructed using stationary states of the same type.

It is convenient to pass from the functions $\Phi^{(j)}_{n_1, n_2, \sigma}$ (A.5) to new functions $\Phi^{(j)}_{z_1, z_2, \sigma}$ as follows:

$$\Phi^{(j)}_{z_1, z_2, \sigma}(\varphi, \rho) = \sum_l \Phi^{(j)}_{z_1, z_2, \sigma}(\varphi, \rho), \quad \Phi^{(j)}_{z_1, z_2, \sigma}(\varphi, \rho) = \sum_m \frac{z_1^{n_1} z_2^{n_2} \Phi^{(j)}_{n_1, n_2, \sigma}(\varphi, \rho)}{\sqrt{1 + n_1} \dim_{2\rho}}. \quad (30)$$

Here $z_1$ and $z_2$ are complex parameters, possible values of $n_1$ and $n_2$ depend on $m, \ell, \sigma$ and $j$ according to equation (A.5), and we set $\dim = 1$. The functions $\Phi^{(j)}_{z_1, z_2, \sigma}$ can be expressed via special functions $Y_\alpha$:

$$\Phi^{(0)}_{z_1, z_2, \sigma}(\varphi, \rho) = \exp \{i\epsilon [l_0 - l - (1 - \epsilon\sigma)/2] \varphi \} Y_{-\ell_1}(z_1, z_2, \rho), \quad \ell_1 = \ell - (1 - \epsilon\sigma)/2 + \mu,$$

$$\Phi^{(1)}_{z_1, z_2, \sigma}(\varphi, \rho) = \exp \{i\epsilon [l_0 - l - (1 - \epsilon\sigma)/2] \varphi \} + \pi [l - (1 - \epsilon)(1 + \sigma)/4]\} Y_\alpha(z_2, z_1, \rho),$$

$$Y_\alpha(z_1, z_2, \rho) = \frac{\sum_{m=0}^{\infty} \frac{z_1^{m+\alpha} z_2^{m+\alpha}}{\sqrt{1 + m} \dim_{1 + m + \alpha}} I_{m+\alpha, \alpha}(\rho)}{\sqrt{1 + m} \dim_{1 + m + \alpha}}. \quad (31)$$
With the help of the well-known sum,
\[
\sum_{m=0}^{\infty} \frac{z^m J_\alpha(m)}{(1 + m)^{1/2}} = z^{-1/2} \exp(z - x/2) J_\alpha(2\sqrt{xz}),
\]
where \(J_\alpha\) are the Bessel functions of the first kind, one can obtain the following representation for \(Y_\alpha\):
\[
Y_\alpha(z_1, z_2; \rho) = \exp \left( z_1 z_2 - \frac{\rho}{2} \right) (\sqrt{z_2/z_1})^{\alpha} J_\alpha(2\sqrt{z_1z_2\rho}). \tag{32}
\]

Using the functions \(\Phi^{(j)}_{\pm, z_1, z_2, \sigma}\), one can construct ICS on the \(xy\)-plane. In \((2 + 1)\) dim and in \((3 + 1)\) dim, ICS are constructed with the help of spinors described in appendix A by substituting the functions \(\Phi^{(j)}_{\alpha_1, \alpha_2, \sigma}\) for the function \(\Phi^{(j)}_{\pm, z_1, z_2, \sigma}\).

Thus, using equations (A.2), (A.4) and (28), we obtain ICS for massive, \(\xi = +1\), spinning particles on the \(xy\)-plane and in \((2 + 1)\) dim:
\[
\begin{align*}
\Psi^{(j)}_{\pm, z_1, z_2}(\varphi, \rho) & = \alpha^3 \left[ \pm \Pi_0 \left( \widehat{N}_1 \right) - \sigma \hat{P} \right] + M c \left[ u^{(j)}_{\pm, z_1, z_2, \pm 1}(\varphi, \rho),
\right. \\
& \left. - \mathcal{P}_\mu \left( \Pi_0 \left( \widehat{N}_1 \right) + M c \right) \Phi^{(j)}_{\pm, z_1, z_2, \pm \mp 1} \right],
\end{align*}
\]
where
\[
\Pi_0(\widehat{N}_1) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-\frac{\Pi_0^2}{4}} d\tau, \quad \Pi_0^2 = M c^2 + 2h \left| q B \right|/c \left[ \widehat{N}_1 + (1 - \sigma \epsilon)/2 \right]. \tag{33}
\]

For such states, we have
\[
\left( \psi^{(j)}_{\pm, z_1, z_2}, \psi^{(j)}_{\pm, z_1, z_2} \right)_D = 2 Mc \left( \Phi^{(j)}_{\pm, z_1, z_2, \pm 1}, \left[ \Pi_0 \left( \widehat{N}_1 \right) + M c \right] \Phi^{(j)}_{\pm, z_1, z_2, \pm \mp 1} \right)_H, \tag{34}
\]
where the inner product \(\left( \cdot, \cdot \right)_H\) is defined by (A.9).

According to (A.16)–(A.18), nonrelativistic ICS for \((2 + 1)\) spin-up particles have the form
\[
\Psi^{(j)}_{\pm, z_1, z_2}(\varphi, \rho) = u^{(j)}_{\pm, z_1, z_2, \pm 1}(\varphi, \rho), \tag{35}
\]
whereas for the spin-down particles they read
\[
\Psi^{(j)}_{\pm, z_1, z_2}(\varphi, \rho) = u^{(j)}_{\pm, z_1, z_2, \mp 1}(\varphi, \rho), \tag{36}
\]
According to (A.42), ICS on the \(xy\)-plane for \((3 + 1)\) nonrelativistic particles with a given spin polarization \(s = \pm 1\) have the form
\[
\begin{align*}
\Psi^{(j)NR}_{\pm, z_1, z_2, +1}(\varphi, \rho) & = \left( \Psi^{(j)NR}_{\pm, z_1, z_2, -1}(\varphi, \rho) \right)_{\mp 1},
\end{align*}
\]
where the inner product of four-component spinors \(\Psi\) and \(\Psi^\dagger\) on the \(xy\)-plane is defined in (A.31).
The inner products of such states have the form
\[
\langle \Psi_{(j+1), z_1, z_2}, \Psi_{(j+1), z_1, z_2} \rangle_D = \sum_{\sigma=\pm 1} \left( \Phi_{(j+1), z_1, z_2, \sigma}, \Phi_{(j+1), z_1, z_2, \sigma} \right)_\perp.
\]  
(39)

In the same manner, with the help of (A.22), one can obtain ICS for (2 + 1) massless \( \zeta = -1 \) fermions.

According to equation (A.37) for (3+1) relativistic spinning particles, ICS on the \( xy \)-plane have the form
\[
U_{z_1, z_2, \sigma}^{(j+1)}(\varphi, \rho) = \Phi_{z_1, z_2, \sigma}^{(j+1)}(\varphi, \rho) \left( \frac{v_\sigma}{\sqrt{1 - \sigma^2 u^2}} \right).
\]  
(40)

The inner product of such states on the \( xy \)-plane reads
\[
\left( U_{z_1, z_2, \sigma}^{(j+1)}, U_{z_1, z_2, \sigma}^{(j+1)} \right)_\perp = 2 \left( \Phi_{z_1, z_2, \sigma}^{(j+1)}, \Phi_{z_1, z_2, \sigma}^{(j+1)} \right)_\perp.
\]  
(41)

One can see that in all the cases, the inner product of ICS on the \( xy \)-plane is expressed via the matrix elements
\[
\left( \Phi_{z_1, z_2, \sigma}^{(j+1)}, \Phi_{z_1, z_2, \sigma}^{(j+1)} \right)_\perp = \delta_{jj} R_{\sigma}^{(j)}; \quad R_{\sigma}^{(0)} = Q_{-\mu_\sigma} \left( \sqrt{z_{1z_2}^2}, \sqrt{z_{1z_2}^2} \right), \quad R_{\sigma}^{(1)} = Q_{\mu_\sigma} \left( \sqrt{z_{1z_2}^2}, \sqrt{z_{1z_2}^2} \right), \quad \mu_{\sigma} = \mu - \frac{1}{2} \vartheta \epsilon (1 - \vartheta \epsilon);
\]  
(42)

where \( Q_{\alpha}(u, v) = Q_{\alpha}(u, v) + (v/u)^{\alpha} I_\alpha(2uv), \quad Q_{\alpha}^{-}(u, v) = \sum_{r=1}^{\infty} (v/u)^{\alpha r} I_{\alpha r}(2uv) \),

where \( I_{\alpha} \) are the modified Bessel functions of the first kind. We note that in contrast to the spinless case [27], the subindex \( \alpha \) in the functions \( Q_{\alpha}(u, v) \) can also take negative values \( -1 < \alpha < 0 \).

The importance of the obtained result should be noted. It turns out that all the means and matrix elements with respect to the ICS are expressed only via two functions \( Q_{\alpha}(u, v) \) and \( Q_{\alpha}^{-}(u, v) \). That is why the further study of such physical quantities is reduced to the analysis of these functions.

It follows from (26) that
\[
N_k \Phi_{z_1, z_2, \sigma}^{(j+1)}(\varphi, \rho) = \frac{z_1 \partial_{\vartheta}}{\vartheta} \Phi_{z_1, z_2, \sigma}^{(j+1)}(\varphi, \rho), \quad k = 1, 2
\]  
(43)

and
\[
a_1 \Phi_{z_1, z_2, \sigma}^{(j+1)}(\varphi, \rho) = z_1 [ \Phi_{z_1, z_2, \sigma}^{(j+1)}(\varphi, \rho) - (-1)^j \Phi_{z_1, z_2, \sigma}^{(j+1)}(\varphi, \rho)]_{\vartheta = 0(\vartheta \epsilon)/2}.
\]  
(44)

Equations (44) allow one to calculate the matrix elements
\[
\left( \Phi_{z_1, z_2, \sigma}^{(j+1)}, a_k \Phi_{z_1, z_2, \sigma}^{(j+1)} \right)_\perp = (a_k)_{z_1, z_2, \sigma}^{(j+1)}, \quad k = 1, 2
\]  
(45)

Results of such calculations are, for example,
\[
(a_1)_{z_1, z_2, \sigma}^{(0,0)} = z_1 Q_{-\mu_\sigma} \left( \sqrt{z_{1z_2}^2}, \sqrt{z_{1z_2}^2} \right), \quad (a_2)_{z_1, z_2, \sigma}^{(0,0)} = z_2 Q_{-\mu_\sigma} \left( \sqrt{z_{1z_2}^2}, \sqrt{z_{1z_2}^2} \right),
\]  
(46)
Using an appropriate inner product, see above, we define means of an operator \( \hat{F} \) with respect to the ICS on the \( xy \)-plane, \( (\hat{F})_{ij} \). Then, we consider important cases when a matrix operator \( \hat{F} \) is either the identity matrix \( I \) multiplied by a differential operator \( f \), \( \hat{F} = fI \) or \( \hat{F} = c^2(\sigma \hat{P}_\perp)^2 \) in \( (2+1) \) dim and \( \hat{F} = c^2(\alpha \hat{P}_\perp)^2 \) in \( (3+1) \) dim. Here, we can express \((\hat{F})_{ij}\) via the means \((\Phi^j_{z_i z_2 \sigma}, \hat{F}_\sigma \Phi^j_{z_i z_2 \sigma})\), where either \( \hat{F}_\sigma = f \) or \( \hat{F}_\sigma = c^2\hat{P}_\perp^2 - \epsilon hc|q|B|\sigma|. \) Thus, we obtain, for example, for \((3+1)\) relativistic and \((2+1)\) nonrelativistic spin-up particles the following expression (with the corresponding interpretations of the number \( \sigma \)):

\[
(\hat{F})_{ij} = \left(\Phi^j_{z_i z_2 \sigma}, \hat{F}_\sigma \Phi^j_{z_i z_2 \sigma}\right)_\perp.
\]

For \((2+1)\) massive \( \zeta = +1 \) relativistic particles we obtain

\[
(\hat{F})_{ij} = \left(\Phi^j_{z_i z_2 \sigma}, \hat{F}_\sigma [\Pi_0(z'_i \partial_{z_i}) + Mc] \Phi^j_{z_i z_2 \sigma}\right)_\perp |_{z_i = z'_i},
\]

whereas for \((2+1)\) massless \( \zeta = +1 \) fermions, we have

\[
(\hat{F})_{ij} = \sum_{\sigma = \pm 1} \left(\Phi^j_{z_i z_2 \sigma}, \hat{F}_\sigma \Phi^j_{z_i z_2 \sigma}\right)_\perp.
\]

Note that means \((\overline{F})_{ij}\) for \((3+1)\) nonrelativistic spinning particles and antiparticles at given \( s \) are expressed via means (47) for the \((2+1)\) nonrelativistic case according to (37).

Then, using (43) and notation (47), we obtain the means of operators \( \hat{N}_k, \ k = 1, 2 \), for example,

\[
(\overline{N_k})_{ij} = z_i \partial_{z_i} \ln |\Phi^j_{z_i z_2 \sigma}|_{|z_i| = z_i}
\]

for the \((3+1)\) particle and for nonrelativistic \((2+1)\) spin-up particles;

\[
(\overline{N_k})_{ij} = z_i \partial_{z_i} \ln \sum_{\sigma = \pm 1} |\Phi^j_{z_i z_2 \sigma}|_{|z_i| = z_i}
\]

for \((2+1)\) massless \( \zeta = +1 \) fermions; and

\[
(\overline{N_k})_{ij} = z_i \partial_{z_i} \ln \left|\Pi_0(z'_i \partial_{z_i}) + Mc\right|_{|z'_i| = z_i}
\]

for \((2+1)\) relativistic massive \( \zeta = +1 \) particles.

Using (46), we find that

\[
(x - i\epsilon y)_{ij} = \sqrt{2\gamma^{-1}}[(a_2)_{ij} - (a_1)_{ij}],
\]

where, for example,

\[
(a_1)_{(0)} = z_1 \Delta_{1-\mu_\sigma}(|z_1|, |z_2|), \quad (a_2)_{(0)} = z_2,
\]

\[
(a_1)_{(1)} = z_1, \quad (a_2)_{(1)} = z_2 \Delta_{\mu_\sigma}(|z_2|, |z_1|), \quad \Delta_{\sigma}(u, v) = \frac{Q_{\mu}(u, v)}{Q_{\mu}(v, u)}
\]

for the \((3+1)\) particle and for the nonrelativistic \((2+1)\) spin-up particle \((\sigma = +1)\) and antiparticle \((\sigma = -1)\), and

\[
(a_1)_{(0)} = z_1 \sum_{\sigma = \pm 1} Q_{1-\mu_\sigma}(|z_1|, |z_2|) \sum_{\sigma = \pm 1} Q_{1-\mu_\sigma}(|z_1|, |z_2|), \quad (a_2)_{(0)} = z_2,
\]

\[
(a_1)_{(1)} = z_1, \quad (a_2)_{(1)} = z_2 \sum_{\sigma = \pm 1} Q_{\mu_\sigma}(|z_2|, |z_1|)
\]

for \((2+1)\) massless \( \zeta = +1 \) fermions.

Note that one can get the means \((\overline{N_3})_{ij}\) and \((\overline{a_3})_{ij}\) for the case of a spinless particle from expressions (50), (54) and (42) at \( \mu_\sigma = \mu \), see [27].
4.2. Semiclassical approximation

Representations (50)–(52) allow us to connect means of $\hat{R}^2$ and $\hat{R}_z^2$ with the parameters $z_1$ and $z_2$. It follows from (29) that

$$\overline{(R^2)}_{(j)} = \gamma^{-1}[2(N_3)_{(j)} + 1 - \sigma \epsilon], \quad \overline{(R_z^2)}_{(j)} = \gamma^{-1}[2(N_3)_{(j)} + 1].$$

(56)

Note that these relations are valid in the case of spinless particles at $\sigma = 0$. We expect that in the semiclassical limit $(N_3)_{(j)} \approx |z_k|^2$. At the same time length scales defined by means of $\overline{(R^2)}_{(j)}$ and $\overline{(R_z^2)}_{(j)}$ have to be sufficiently large, which implies $|z_k|^2 \gg 1$ in the semiclassical limit.

We note that in the pure quantum case, as a characteristic quantum scale of the rotational motion we can take the quantity

$$E_{\text{quant}}^2 = 2|qB|/\hbar c = 2M^2c^3|B|/B_0, \quad B_0 = M^2c^3/|q|\hbar,$$

where $B_0 = m^2c^3/e\hbar \approx 4.4 \times 10^{13}$ G is the critical magnetic field above which the nonlinearity of QED becomes actual. The corresponding length scale is

$$R_{\text{quant}} = \sqrt{2\gamma^{-1}} = \sqrt{2B_0/|B|} \lambda_C, \quad \lambda_C = \hbar/Mc.$$

(57)

For the angular momentum projection $J$, a characteristic quantum scale is obviously $\hbar$. For a given energy, i.e. for a given $\overline{(R^2)}_{(j)}$, the quantity $(J_z)_{(j)}$ is proportional to $\overline{(R_z^2)}_{(j)}$ due to (23) and, therefore, can be characterized by the corresponding length scale $R_{\text{quant}}$. Note that the $R_{\text{quant}}$ is much larger than the Compton length $\lambda_C$ if the magnetic field $B$ is weak, $B_0/|B| \gg 1$.

Thus, the conditions $|z_k|^2 \gg 1$ correspond to

$$\overline{(R^2)}_{(j)}, \quad \overline{(R_z^2)}_{(j)} \gg R_{\text{quant}}.$$

At the same time, in the quantum case, the dimensionless quantities $|z_k|^2$ are of the order 1. We see that the semiclassical decompositions are adequate namely in the case of strong enough magnetic fields (e.g. pulsar magnetic fields $B$ for which $B_0/|B| \sim 10^{-2}$ and $R_{\text{quant}} \ll \lambda_C$).

We expect that the sign of the difference

$$d_{(j)} = \sqrt{\overline{(R^2)}_{(j)}} - \sqrt{\overline{(R_z^2)}_{(j)}}$$

(58)

is related to the trajectory type in the classical limit. One can see that such a limit implies the following conditions:

$$|d_{(j)}| \gg R_{\text{quant}} \sim \|z_1\| - |z_2| \gg 1.$$

In particular, for states with $j = 0$, we have $|z_1| \ll |z_2|$, and for states with $j = 1$, we have $|z_1| \gg |z_2|$. We note that in both cases the corresponding functions $Q_a(u, v)$ are calculated at $|u| > |v| \gg 1$.

There exist all the derivatives $\partial_u [(v/u)^\mu \ast I_a(2uv)]$, the series $Q_a^\ast(u, v)$ (42) converges and the series $\sum_{\nu=1}^{\infty} \partial_v [(v/u)^\mu \ast I_a(2uv)]$ converges uniformly on the half-line $0 < \Re v < \infty$. Thus, one can write a differential equation for $Q_a^\ast(u, v)$:

$$\partial_u Q_a^\ast(u, v) = 2e[(v/u)^\mu I_a(2uv) + Q_a^\ast(u, v)].$$

A solution of this equation, which corresponds to (42), reads

$$Q_a^\ast(u, v) = 2 e^{i\epsilon} \int_0^v e^{-i\theta} (\bar{u}/u)^\mu I_a(2u\bar{v}) \bar{v} d\bar{v}.$$
Then, using the asymptotic of $\tilde{Q}_j(u, v)$, we represent this solution as follows:

$$\tilde{Q}_j(u, v) = e^{u/v} \tilde{Q}_j(u, v), \quad \tilde{Q}_j(u, v) = [1 - T(u, v)],$$

$$T(u, v) = 2 e^{-u^2} \int_v^{\infty} e^{-\tilde{v}^2} (\tilde{v}/u)^a I_a(2u\tilde{v}) \tilde{v} d\tilde{v}. \quad (59)$$

Then

$$Q_j(u, v) = e^{u/v} \tilde{Q}_j(u, v), \quad \tilde{Q}_j(u, v) = [1 - T(u, v) + e^{-u^2-v^2}(v/u)^a I_a(2uv)]. \quad (60)$$

and we can calculate, for example, the means $(50)$:

$$\overline{(N_k)}_{(j)} = \left| z_{1j} \right|^2 + z_k \tilde{z}_{k} \ln \tilde{R}_{\sigma}^{(j)} \bigg|_{z_j = z_k},$$

$$\tilde{R}_{\sigma}^{(0)} = \tilde{Q}_{1-\mu_\sigma} \left( \sqrt{z_{1j}^2 + \sqrt{z_{2j}^2}}, \sqrt{z_{1j}^2 + \sqrt{z_{2j}^2}} \right), \quad \tilde{R}_{\sigma}^{(1)} = \tilde{Q}_{\mu_\sigma} \left( \sqrt{z_{1j}^2 + \sqrt{z_{2j}^2}}, \sqrt{z_{1j}^2 + \sqrt{z_{2j}^2}} \right). \quad (61)$$

These means can be represented explicitly in a real form, taking into account that

$$z_k \partial_{z_{k}} \ln \tilde{R}_{\sigma}^{(0)} \bigg|_{z_j = z_k} = \frac{\delta_{k,1} \left| z_{1j} \right| \partial_{u} \tilde{Q}_{1-\mu_\sigma}(u, v) + \delta_{k,2} \left| z_{2j} \right| \partial_{v} \tilde{Q}_{1-\mu_\sigma}(u, v)}{2 \tilde{Q}_{1-\mu_\sigma}(u, v)} \bigg|_{u = \left| z_1 \right|, v = \left| z_2 \right|},$$

$$z_k \partial_{z_{k}} \ln \tilde{R}_{\sigma}^{(1)} \bigg|_{z_j = z_k} = \frac{\delta_{k,1} \left| z_{1j} \right| \partial_{u} \tilde{Q}_{\mu_\sigma}(u, v) + \delta_{k,2} \left| z_{2j} \right| \partial_{v} \tilde{Q}_{\mu_\sigma}(u, v)}{2 \tilde{Q}_{\mu_\sigma}(u, v)} \bigg|_{u = \left| z_1 \right|, v = \left| z_2 \right|}.$$

We stress that means $(61)$ allow the limit $\mu_\sigma \to 0$. Thus, the contribution due to the AB field can be easily isolated.

Using power decomposition of the function $\tilde{v}^{a+1} I_a(2u\tilde{v}) e^{-2u\tilde{v}}$ near the point $\tilde{v} = v$ for an estimation of the integral $T(u, v)$ in $(59)$ and asymptotics of the function $I_a(2uv)$, one can see that $\left| z_{k} \right|^2 \gg z_k \partial_{z_k} \ln \tilde{R}_{\sigma}^{(j)}$ for $|v| \gg |u| \gg 1$. Thus, we obtain the semiclassical expansions:

$$|z_{1j}|^2 = \frac{v}{2} (R_1^{(j)}) + \cdots, \quad |z_{2j}|^2 = \frac{v}{2} (R_2^{(j)}) + \cdots, \quad |z_k|^2 \gg 1, \quad (62)$$

which connect means of $\tilde{R}_1$ and $\tilde{R}_2$ with the parameters $z_1$ and $z_2$. Thus, if $|\left| z_1 \right| - \left| z_2 \right| | \gg 1$, equations $(62)$ imply the following relations:

$$|z_{1j}| \ll |z_{2j}|, \quad j = 0; \quad |z_{1j}| \gg |z_{2j}|, \quad j = 1. \quad (63)$$

It should be noted that relations $(63)$ have nothing to do with conditions of the applicability of the semiclassical expansions $(62)$. Obtaining the latter expansions we have supposed that $|z_{1j}| \lesssim |z_{2j}|$ for states with $j = 0$, and $|z_{1j}| \gtrsim |z_{2j}|$ for states with $j = 1$. Therefore, relations $(62)$ between means of $\tilde{R}_1$ and $\tilde{R}_2$ and parameters $z_1$ and $z_2$ take place even if a definite relation between sign$(\left| z_1 \right| - \left| z_2 \right|)$ and $j$ is absent.

Retaining only leading terms in decompositions $(62)$, we reproduce the corresponding classical relations $(10)$ with $|z_{1j}| = |a_1|$ and $|z_{2j}| = |a_2|$. In other words, one can say that the classical relations $(10)$ correspond to the leading approximation for sufficiently large radii. Thus, the leading approximation in the semiclassical expansions corresponds to the classical limit. Next-to-leading terms define physical quantities in the semiclassical approximation. These terms depend on the space dimension and particle spin.

Let us consider the semiclassical approximation retaining next-to-leading and next-next-to-leading terms. If $|v| \gg |u| \gg 1$, one can approximate the integral $T(u, v)$ in $(59)$ by a power series in $u/v$ as follows:

$$T(u, v) = (v/u)^a I_a(2uv) e^{-u^2-v^2} (1 + u/v + \cdots).$$

Then, using the asymptotic of $I_a(2uv)$, we obtain from $(60)$

$$\tilde{Q}_j(u, v) = 1 - \frac{(u/v)^{1-a}}{2 \sqrt{\pi u v}} e^{-(v-u)^2}. \quad (64)$$
which implies that
\[
\partial_u \hat{Q}_\alpha(u, v) \approx -\frac{\partial_v \hat{Q}_\alpha(u, v)}{Q_\alpha(u, v)} \approx \left(\frac{u}{v}\right)^{1/2-a} e^{-(v-u)^2}, \quad |v| \gg |u| \gg 1. \tag{65}
\]

Thus, for semiclassical states corresponding to orbits situated far enough from the solenoid, i.e. for \(|z_1| - |z_2| \gg 1\), the terms \(z_2 \partial_z \ln R^{(j)}_\gamma|_{z_1=2}\) are small as \(\exp(-|z_1|^2 - |z_2|^2)\). Then the semiclassical expansions (62) in the next-to-leading approximation read
\[
|z_1|^2 \approx \frac{\gamma}{2} (R^{(2)}_\gamma)_j - (1 - \sigma \epsilon)/2, \quad |z_2|^2 = \frac{\gamma}{2} (R^{(2)}_\gamma)_j - 1/2, \quad |z_1| - |z_2| \gg 1.
\]

In the most interesting case when a semiclassical orbit is situated near the solenoid, such that the condition \(|v| - |u| \ll 1\) holds, the influence of the AB solenoid (due to \(\mu \neq 0\)) on the orbits is not small. In such a case
\[
T(u, v) = \frac{1}{2} - \frac{v-u}{\sqrt{\pi}} + \frac{\alpha + 1/2}{2\sqrt{\pi}u} + O(|v-u|^3) + O(|u|^{-2}),
\]
\[
\hat{Q}_\alpha(u, v) \approx \frac{1}{2} + \frac{v-u}{\sqrt{\pi}} - \frac{\alpha - 1/2}{2\sqrt{\pi}u},
\]
\[
\partial_u \hat{Q}_\alpha(u, v) \approx -\frac{\partial_v \hat{Q}_\alpha(u, v)}{\hat{Q}_\alpha(u, v)} \approx \frac{2}{\sqrt{\pi}} \left(1 - \frac{v-u}{\sqrt{\pi}} + \frac{\alpha - 1/2}{2\sqrt{\pi}u}\right),
\]
such that, for example, means (61) are
\[
(N_k)_j \approx |z_k|^2 + (-1)^k \left|\frac{|z_k|}{\sqrt{\pi}} - (1/2)^k + 2 \left|\frac{|z_1| - |z_2|}{\sqrt{\pi}}\right| + \frac{1 - \mu_\sigma}{2\pi}\right|, \quad |z_1| - |z_2| \ll 1. \tag{67}
\]

Thus, for \(|z_1| - |z_2| \ll 1\), all \(\mu\)-dependent contributions to the means \((N_k)_j\) are of the order 1, which is natural for a pure quantum case. Next-to-leading contributions to the means (of order \(|z_1|^2\)) that do not depend on \(\mu\) are much bigger. These semiclassical contributions appear since each \(j\)-type ICS includes only half of eigenfunctions of the operator \(\hat{J}_z\). It follows from (67) that in the leading approximation
\[
(N_1)_j - (N_2)_j \approx - (1)^{j+1} \frac{|z_1| + |z_2|}{\sqrt{\pi}}, \quad |z_1| - |z_2| \ll 1. \tag{68}
\]

At the same time, relations (62) yield in the semiclassical approximation
\[
|z_1|^2 \approx \frac{\gamma}{2} (R^{(2)}_\gamma)_j + (-1)^j \sqrt{\frac{\gamma}{2\pi} (R^{(2)}_\gamma)_j},
\]
\[
|z_2|^2 \approx \frac{\gamma}{2} (R^{(2)}_\gamma)_j - (-1)^j \sqrt{\frac{\gamma}{2\pi} (R^{(2)}_\gamma)_j}, \quad |z_1| - |z_2| \ll 1.
\]

Then, using (68), we obtain
\[
(R^{(2)}_\gamma)_j - (R^{(2)}_\gamma)_j \approx - (1)^{j+1} \sqrt{\frac{2}{\pi \gamma} (R^{(2)}_\gamma)_j}, \quad |z_1| - |z_2| \ll 1,
\]
which implies that
\[
(J)_j - \frac{g \Phi}{2\pi c} \approx \epsilon (-1)^j \frac{\hbar}{\sqrt{\pi}} \sqrt{\frac{2}{\pi \gamma} (R^{(2)}_\gamma)_j + \sqrt{R^{(2)}_\gamma}_j}, \quad R^{(2)}_{\text{quant}} = 1.
\]

Then, the quantity \(d_{(j)}\) (58) is
\[
d_{(j)} \approx (1)^{j+1} \sqrt{\frac{2}{\pi \gamma}}, \quad |z_1| - |z_2| \ll 1. \tag{70}
\]
Thus, for $|z_1| - |z_2| \ll 1$, and in the semiclassical approximation, the mean minimal possible moving off $|d_{(j)}|$ of the particle from the solenoid line is of order $R_{\text{quant}}$; in particular, $d_{(j)} < 0$ for states with $j = 0$, and $d_{(j)} > 0$ for states with $j = 1$, independently on the sign of the difference $|z_1| - |z_2|$.

Equations (43) and (60) allow us to calculate variances of the operators $\hat{N}_k$:

$$\text{Var}_j(N_k) = \left(N_k^2\right)_{(j)} - \left((N_k)_{(j)}\right)^2.$$  

In the semiclassical approximation, we have

$$\text{Var}_j(N_k) \approx |z_k|^2, \quad |z_1| - |z_2| \gg 1;$$

$$\text{Var}_j(N_k) \approx (1 - 1/\pi)|z_k|^2, \quad |z_1| - |z_2| \ll 1. \quad (71)$$

Thus, standard deviations of $\bar{R}^2$ and $\bar{R}_c^2$ in the semiclassical ICS are of the same order for any value $|z_1| - |z_2|$, namely

$$\delta_j(R^2) = \sqrt{\text{Var}_j(R^2)} \sim R_{\text{quant}}\sqrt{\bar{R}^2_{(j)}}, \quad \delta_j(R^2) = \sqrt{\text{Var}_j(R^2)} \sim R_{\text{quant}}\sqrt{\bar{R}^2_{(j)}}.$$  

In this case, the typical spreads of the radii $R$ and $R_c$ are given by the standard deviations

$$\delta_j(R) = \delta_j(R^2)\left(\bar{R}^2_{(j)}\right)^{-1/2} \sim R_{\text{quant}}, \quad \delta_j(R_c) = \delta_j(R^2)\left(\bar{R}^2_{(j)}\right)^{-1/2} \sim R_{\text{quant}}. \quad (72)$$

For $|z_1| - |z_2| \ll 1$, the difference $\bar{R}^2_{(j)} - \left(\bar{R}^2_{(j)}\right)$ is of the order of the standard deviation of $\bar{R}^2 - \bar{R}^2_{(j)}$, which is $\delta_j(R^2) + \delta_j(R^2)$. Therefore, the mean angular momentum (69) is of the order of the $J_z$ standard deviation, and for $|z_1| - |z_2| \ll 1$, the quantum scale of angular momentum is much greater than $\hbar$. In this case $|d_{(j)}|$ is of the order the $\delta_j(R) + \delta_j(R_c) \sim R_{\text{quant}}$.

We note that ICS in the manner of Malkin–Man’ko [24] (the case $\Phi = 0$) can be associated with the superposition of the $j = 1$ and $j = 0$ states for $\mu = 0$, which already includes all the eigenfunctions of $J_z$. The inner product on the $xy$-plane between such states is the sum $\mathcal{R}_{\sigma}^{(0)} + \mathcal{R}_{\sigma}^{(1)}$ at $\mu_j = 0$. Using equations (8,511.1) from [28], we find

$$\left(\mathcal{R}_{\sigma}^{(0)} + \mathcal{R}_{\sigma}^{(1)}\right)_{|\mu_j = 0} = \exp\left(z_1'z_1' + z_2'z_2'\right) \implies \left(\mathcal{R}_{\sigma}^{(0)} + \mathcal{R}_{\sigma}^{(1)}\right)_{|\mu_j = 0} = 1.$$  

Therefore, in such ICS, $(N_k) = |z_k|^2$. Similar mutual compensations take place in the expressions for $\mathcal{R}_{\sigma}^{(0)}$ and $\mathcal{R}_{\sigma}^{(1)}$ for $\mu \neq 0$ in the semiclassical limit, $|z_k|^2 \gg 1$. For example, for $|z_1| - |z_2| \ll 1$, we obtain

$$\mathcal{R}_{\sigma}^{(0)} + \mathcal{R}_{\sigma}^{(1)} = 1 + \mathcal{O}(|z_k|^{-2}).$$

One can see that in states that are superpositions between different $j$, leading corrections to means $(N_k) = |z_k|^2$ (67) disappear in the semiclassical approximation.

If $|z_1|$ and $|z_2|$ differ essentially, i.e. $|z_1| - |z_2| \gg 1$, one may believe that next-to-leading terms in $\mathcal{R}_{\sigma}^{(j)}$, given by (64), remain uncompensated. That is not true. To see this, one has to take into account that next-to-leading terms in $\mathcal{R}_{\sigma}^{(0)} + \mathcal{R}_{\sigma}^{(1)}$ are due to contributions from (64) and from leading terms in $\tilde{Q}_\sigma(u, v)$ for $|u| \gg |v|$:  

$$\tilde{Q}_\sigma(u, v) = \frac{1}{2\sqrt{\pi uv}}(v/u)^\sigma e^{-(v-u)^2}, \quad |u| \gg |v| \gg 1. \quad (73)$$

We recall that for ICS, the domain $|u| > |v|$ is not a classical one even if $|z_k|^2 \gg 1$.

Using (53), (54) and the representation

$$\Delta_\sigma(u, v) = 1 - \sigma_\sigma(u, v), \quad \sigma_\sigma(u, v) = (v/u)^\sigma I_\sigma(2uv) e^{-u^2-v^2}/\tilde{Q}_\sigma(u, v),$$

$$|u| \gg |v| \gg 1,$$
we find
\[
(x - i\epsilon y)(_{(j)} = \sqrt{2\gamma^{-1} [\langle a_{2j} \rangle - \langle a_{1j} \rangle]}.
\]
\[
\langle a_{2j} \rangle - \langle a_{1j} \rangle = z_2 - z_2^i \Delta_{1-\mu}(\{z_1, z_2\}) = z_2 - z_2^i + z_2^i \sigma_{1-\mu}(\{z_1, z_2\}),
\]
\[
(\langle a_{2j} \rangle - \langle a_{1j} \rangle) = z_2 \Delta_{\mu}(\{z_2, z_1\}) - z_2^i = z_2 - z_2^i - z_2 \sigma_{\mu}(\{z_2, z_1\}).
\]
(74)

With the help of equations (74) we calculate the variance of \((x + y)\) in \(j\)-type states:
\[
\text{Var}_j(x + y) = \langle |x - i\epsilon y|^2 \rangle_{(j)} - \langle |x - i\epsilon y| \rangle_{(j)}^2
= 2\gamma^{-1} [\langle N_{2j} \rangle + \langle N_{2j} \rangle^1 + 1 - \langle a_{1j} \rangle^2 - \langle a_{2j} \rangle^2].
\]
(75)

Let us consider the semiclassical limit \(|z_k|^2 \gg 1\) for ICS with \(j = 0\) and \(|z_1| \gtrsim |z_2|\) and for
ICS with \(j = 1\) and \(|z_1| \gtrsim |z_2|\); in both such cases \(|v| \gtrsim |u| \gg 1\). In this case, using the above
results, one can verify that corrections to the classical expression \([\langle a_{2j} \rangle - \langle a_{1j} \rangle] = z_2 - z_2^i\)
are small. In particular, using equations (64) and (66), and asymptotics of \(I_\mu(2uv)\), in the
next-to-leading approximation we obtain the following result
\[
d_u(u, v) = \frac{1}{2\pi u v} (\epsilon v/u) e^{-(v-u)^2}, \quad |v| \gg |u|;
\]
\[
d_u(u, v) = \frac{1}{\pi u} \left( 1 - \frac{2v - u}{\sqrt{\pi}} + \frac{\alpha - 1/2}{\sqrt{\pi u}} \right), \quad |v - u| \ll 1.
\]
(76)

Thus, equations (74) match with the ones (61) in the classical limit, and due to (61), (65) and
(66), in the semiclassical approximation, the variances (75) are relatively small:
\[
\text{Var}_j(x + y) \approx 2\gamma^{-1}, \quad |z_2| \gg |z_1|\text{ for } j = 0, \quad |z_1| \gg |z_2|\text{ for } j = 1;
\]
\[
\text{Var}_0(x + y) \approx \frac{4|z_1|^3/2}{\sqrt{\pi \gamma}}, \quad \text{Var}_j(x + y) \approx \frac{4|z_2|^3/2}{\sqrt{\pi \gamma}}, \quad |z_1| - |z_2| \ll 1.
\]
(77)
Note that in the case of the spinless particle $\alpha > 0$ (see [27]), while $\alpha$ can also take negative values $-1 < \alpha < 0$ in the case of the spinning particle. Thus, e.g., for $|z_1 z_2| \ll 1$, $\bar{\alpha}_2^{(j)} - \bar{\alpha}_1^{(j)}$ differs essentially from $z_2 - z_1^*$. We have $\Delta_\alpha(u, v) = v^2(\alpha + 1)^{-1}$. Then,

$$
\overline{\bar{\alpha}_2^{(j)}} - \overline{\bar{\alpha}_1^{(j)}} \approx z_2, \quad \overline{\bar{\alpha}_2^{(j)}} - \overline{\bar{\alpha}_1^{(j)}} \approx -z_1^*.
$$

(78)

In this case, using (43) and (42), we obtain \(\text{Var}_j(x + y) \approx 2y^{-1}\). However, it is big in comparison with small $|z_k|^2$.

If $|u| \gg |v|$, we deal with the quantum case even for big $|z_k|^2$. Here $\Delta_\alpha(u, v) = v/u \to 0$ which gives a justification for relations (78). In addition, in the quantum case, we have $(\overline{\mathcal{N}_k^{(j)}})^{\perp} \sim 1$, and, at the same time, contributions to $(\overline{\mathcal{N}_k^{(j)}})$ that depend on $z_k$ are much smaller than 1. That is why means $(\overline{\mathcal{R}_2^{(j)}})$ and $(\overline{\mathcal{R}_2^{(j)}}, \gamma^{(j)})$, which are expressed via $(\overline{\mathcal{N}_k^{(j)}})$ by equation (56), depend only slightly on $z_k$. In this case the variances

$$\text{Var}_1(x + y) \approx 2y^{-1}|z_1|^2, \quad \text{Var}_2(x + y) \approx 2y^{-1}|z_2|^2$$

are much bigger than squares of the corresponding means $|(x - i\varepsilon y)^{(j)}|^2$.

Let us consider uncertainty relations in the semiclassical ICS. Let $\hat{F}_1$ and $\hat{F}_2$ be two self-adjoint operators satisfying the commutation relation $[\hat{F}_1, \hat{F}_2] = i\delta_{12}$, where $\delta_{12}$ is a symmetric operator with a real mean $\langle \delta_{12} \rangle$. In this case the uncertainty relation

$$\text{Var}(F_1) \text{Var}(F_2) \geq \frac{1}{4} \langle \delta_{12} \rangle
$$

holds, see e.g. [30]. Adopting this general relation to our particular cases, we obtain

$$
\text{Var}_j(\mathcal{P}_1^2) \text{Var}_j(x + y) \geq \hbar^2|\mathcal{P}_1 + i\varepsilon \mathcal{P}_2^{(j)}|^2,
$$

$$
\text{Var}_j(L_z) \text{Var}_j(x + y) \geq \frac{\hbar^2}{4}|(x - i\varepsilon y)^{(j)}|^2.
$$

(79)

Here $|(x - i\varepsilon y)^{(j)}|$ is given by (74) and $|(\mathcal{P}_1 + i\varepsilon \mathcal{P}_2^{(j)}|^2$ can be represented with the help of (25) as

$$
(\mathcal{P}_1 + i\varepsilon \mathcal{P}_2^{(j)})^2 = 2\gamma\hbar^2|(\alpha_1^{(j)})|^2.
$$

The variances $\text{Var}_j(\mathcal{P}_1^2)$ and $\text{Var}_j(L_z)$ can be expressed via $\text{Var}_j(\mathcal{N}_k)$ (71):

$$\text{Var}_j(\mathcal{P}_1^2) = (2\gamma\hbar^2)^2 \text{Var}_j(\mathcal{N}_k), \quad \text{Var}_j(L_z) = \hbar^2 \text{Var}_j(\mathcal{N}_1 - \mathcal{N}_2)
$$

and $\text{Var}_j(x + y)$ are given by (77).

We note that $|(\bar{\alpha}_1^{(j)})|^2 \approx |z_1|^2$ for any $|z_1| - |z_2|$, and for the sake of definiteness, we suppose that $|z_1^* - z_2| = |z_1 - z_2|$ for $|z_1| - |z_2| \ll 1$. Then, using (71) and (77), we see that for $|z_1| - |z_2| \gg 1$ the products of the variances from (79) are close to their possible minimal values:

$$\text{Var}_j(\mathcal{P}_1^2) \text{Var}_j(x + y) \approx 4\gamma^2|\mathcal{P}_1 + i\varepsilon \mathcal{P}_2^{(j)}|^2, \quad \text{Var}_j(L_z) \text{Var}_j(x + y) \approx \hbar^2|\mathcal{P}_1 + i\varepsilon \mathcal{P}_2^{(j)}|^2,
$$

and for $|z_1| - |z_2| \ll 1$ these variances are much bigger than the means $\hbar^2|\mathcal{P}_1 + i\varepsilon \mathcal{P}_2^{(j)}|^2$ and $(\hbar^2/4)|x - i\varepsilon y|^2$, respectively.

Of course, the AB effect is global. However, there exists a difference how this effect manifests itself in the pure AB field and in the MSF. In the latter case, there exists a possibility of characterizing especially constructed quantum states with respect to their ‘closeness’ to the AB solenoid. Namely the CS have, in a sense, such characteristics. The closer such states are to the solenoid, the more they are affected by it.
5. Time-dependent CS

On the basis of ICS discussed above, one can construct already time-dependent CS (we call them simply CS in what follows) as solutions of the corresponding nonstationary wave equations. One ought to mention that CS for the nonrelativistic spinless particle in the MSF were constructed in our recent work [27]. Below, we will construct CS for nonrelativistic and relativistic spinning particles in (2 + 1) and (3 + 1) dim.

5.1. Nonrelativistic particles

In (2 + 1) dim, the quantum behavior of the nonrelativistic spin-up particle (antiparticle) is governed by the Pauli equation (A.15), where the Hamiltonian can be represented as follows:

\[ \hat{H}_{\pm}^{NR} = \hbar \omega_{NR} [\hat{N} + (1 - \sigma \epsilon)/2]_{\sigma = \pm 1} \]

Solutions \( \Psi_{\pm}^{NR}(t, \mathbf{r}) \) of such an equation are

\[ \Psi_{\sigma}^{up}(t, \mathbf{r}) = \mathcal{N} \exp\{-i[\omega_{NR}(\sigma - \epsilon)/2]t\} \Phi_{\sigma}(t, \mathbf{r}, \epsilon) \psi_{\sigma} \]

respectively, for \( \pm \) cases, where \( \mathcal{N} \) is the normalization constant, \( \psi_{\sigma} \) is given by (A.3), and functions \( \Phi_{\sigma} \) are solutions of the following equation:

\[ i\hbar \partial_{t} \Phi_{\sigma} = \sigma \omega_{NR} \hat{N} \Phi_{\sigma} \]

(80)

One can obey (80) setting \( \Phi_{\sigma}(t, \mathbf{r}, \epsilon) = \Phi_{z_{1}, z_{2}, \sigma}^{(j)}(\mathbf{r}, \epsilon) \) where \( z_{1} \) is a complex function of time \( t \). Then,

\[ i\hbar \partial_{z_{1}} \Phi_{z_{1}, z_{2}, \sigma}^{(j)}(\mathbf{r}, \epsilon) = \sigma \omega_{NR} \Phi_{z_{1}, z_{2}, \sigma}^{(j)}(\mathbf{r}, \epsilon), \quad \dot{z}_{1} = dz_{1}/dt \]

Substituting (81) into (80), we find \( \dot{z}_{1} = \sigma \omega_{NR} z_{1} \), where (43) is used. It is convenient to write a solution for \( z_{1}(t) \) as follows:

\[ z_{1}(t) = -|z_{1}| \exp(-i\sigma \psi), \quad \psi = \omega_{NR} t + \psi_{0} \]

(82)

where \( |z_{1}| \) is a given constant. Thus, the functions

\[ \Psi_{\sigma}^{(j)up}_{CS}(t, \mathbf{r}) = \mathcal{N} \exp\{ -i \omega_{NR}(\sigma - \epsilon)/2 t \} \Phi_{\sigma}^{(j)}_{z_{1}(t), z_{2}, \sigma}^{(j)}(\mathbf{r}, \epsilon) \psi_{\sigma} \]

(83)

are solutions of the (2 + 1) Pauli equation for the spin-up particle. At the same time they have special properties that allow us to treat them as CS and even SS under certain conditions.

Consider the (2 + 1) Pauli equation (A.15) for spin-down particles. The corresponding Hamiltonian reads

\[ \hat{H}_{\pm}^{NR} = \hbar \omega_{NR} [\hat{N} + (1 - \sigma \epsilon)/2]_{\sigma = \mp 1} \]

Solutions \( \Psi_{\sigma}^{down}(t, \mathbf{r}) \) of such an equation have the form

\[ \Psi_{\sigma}^{down}(t, \mathbf{r}) = \mathcal{N} \exp\{ -i \omega_{NR}(\sigma + \epsilon)/2 t \} \Phi_{-\sigma}^{(j)}(t, \mathbf{r}, \epsilon) \psi_{-\sigma} \]

where \( \sigma = +1 \) for the particle and \( \sigma = -1 \) for the antiparticle. Similar to the spin-up case, one can construct CS as follows:

\[ \Psi_{\sigma}^{(j)down}_{CS}(t, \mathbf{r}) = \mathcal{N} \exp\{ -i \omega_{NR}(\sigma + \epsilon)/2 t \} \Phi_{\sigma}^{(j)}_{z_{1}(t), z_{2}, -\sigma}^{(j)}(\mathbf{r}, \epsilon) \psi_{-\sigma} \]

(84)

We note that means \( (\mathcal{F})_{(j)} \) in such CS are reduced to \( (\mathcal{F})_{(j)} \) given by (47).

In (3 + 1) dim, one can find CS for nonrelativistic spinning particles, with a given spin polarization \( s \). Such CS obey the nonrelativistic Dirac equation with the Hamiltonian

\[ \hat{H}_{\pm} = \left[ \alpha_{\mu} \hat{P}_{\mu} + \beta_{s} \right]/2M \]

(see appendix A) and have the form

\[ \Psi_{\sigma}^{(j)NR}_{CS, s, \sigma}(x) = \exp\left\{ -i \frac{\beta_{s}}{\hbar} \left( \frac{p_{s}^{z} \sigma t}{2M} + p_{s} z \right) \right\} \Psi_{\sigma}^{(j)NR}_{CS, s, \sigma}(t, \mathbf{r}) \]

(85)

\[ \Psi_{\sigma}^{(j)NR}_{CS, s, \sigma}(t, \mathbf{r}) = \begin{pmatrix} \Psi_{\sigma}^{(j)up}_{CS, s, \sigma}(t, \mathbf{r}) \\ 0 \end{pmatrix}, \quad \Psi_{\sigma}^{(j)NR}_{CS, s, -\sigma}(t, \mathbf{r}) = \begin{pmatrix} 0 \\ \Psi_{\sigma}^{(j)down}_{CS, s, \sigma}(t, \mathbf{r}) \end{pmatrix} \]
where representation (37) is used. If we consider only physical observables \( \hat{F} \) that do not depend on \( z \), then means \( \langle \hat{F} \rangle_{(j)} \) in CS (85) are expressed via the corresponding means on the \( xy \)-plane, i.e. via the corresponding means (47) for \( (2+1) \) dim particles.

5.2. Relativistic particles in \((3+1)\) dim

In \((3+1)\) dim, we consider the Dirac equation in the light-cone variables, such that \( x_− \) plays the role of time. Solutions \( \Psi \) of such an equation have the form

\[
\Psi_{\lambda,\sigma}(x) = \mathcal{N} \exp \left\{ -\frac{i}{2\hbar} \left[ \lambda M c x_+ + \left( \frac{M c}{\lambda} + \hbar \tilde{\omega} (1 - \sigma \epsilon) \right) x_- \right] \right\} \Phi_{\lambda,\sigma}(x_-,\varphi,\rho),
\]

where

\[
\frac{i}{\hbar} \frac{\partial \Phi(x_-)}{\partial x_-} = \tilde{\omega} \Phi(x_-), \quad \Phi(x_-) = \Phi_{(j)} z_1(x_-),
\]

see appendix A.

One can obey (86) setting

\[
\Phi(x_-) = \Phi_{(j)} z_1(x_-),
\]

where \( z_1(x_-) \) is a complex function of the time \( x_- \). Substituting (87) into (86), taking into account

\[
\frac{i}{\hbar} \frac{d z_1}{dx_-} = \tilde{\omega} z_1,
\]

and (43), we find

\[
\frac{d z_1}{dx_-} = \tilde{\omega} z_1, \quad z_1(x_-) = -|z_1| \exp(-i\psi), \quad \psi = \tilde{\omega} x_- + \psi_0,
\]

where \(|z_1|\) and \( \psi_0 \) are assumed to be some constants. Thus, we have a set of solutions of the Dirac equation in the following form:

\[
\Psi_{CS,\sigma}(x) = \mathcal{N} \exp \left\{ -\frac{i}{2\hbar} \left[ \lambda M c x_+ + \left( \frac{M c}{\lambda} + \hbar \tilde{\omega} (1 - \sigma \epsilon) \right) x_- \right] \right\} \times \Phi_{z_1(x_-),z_2,\sigma}(\varphi,\rho) \left( \frac{z_1,\sigma}{\nu \nu} \right),
\]

We interpret these solutions as CS with the light-cone time \( x_- \) evolution.

Let us deal with physical observables \( \hat{F} \) that do not depend of \( x_+ \), which is natural for the axial symmetry of the problem under consideration. Matrix elements of such observables in CS (89) (we use the inner product (A.30) on the hypersurface \( x_- = \text{const} \)) take the form

\[
\langle \Psi_{CS,\sigma}^{(j)} | \hat{F} | \Psi_{CS,\sigma}^{(j')} \rangle_{x_-} = \frac{(4\pi)^2 \hbar}{\gamma M c} \delta_{\lambda',\sigma} \delta(\lambda' - \lambda) \langle \Phi_{z_1(z_2),z_2,\sigma}^{(j')} | \hat{F} | \Phi_{z_1(z_2),\sigma}^{(j)} \rangle_{z_1(z_2)},
\]

where the inner product \( (\cdot,\cdot) \) is given by equation (A.9). That is why means \( \langle \hat{F} \rangle_{(j)} \) in such CS are expressed via \( (\hat{F})_{(j)} \) given by (47).

Following the same method in the spinless case, one can construct CS that are solutions of the Klein–Gordon equation.
5.3. $t$ and $x_-$ evolution of mean values

Let us calculate means $\langle x \rangle_{(j)}$ and $\langle y \rangle_{(j)}$ in the nonrelativistic CS constructed above. These means are expressed via the means $\langle x - i\epsilon y \rangle_{(j)}$, which have the form (53) and (54). Taking into account equation (82), one can see that means $\langle x \rangle_{(j)}$ and $\langle y \rangle_{(j)}$ are moving along a circle on the $xy$-plane with the cyclotron frequency $\omega_{\text{Cycl}}$, i.e. the trajectory of the means has the classical form. The same equations allow one to find a mean radius $\langle R \rangle_{(j)}$ of such a circle and the distance $\langle R \rangle_{(j)}$, between its center and the origin:

$$
\langle R \rangle_{(0)} = \sqrt{2\gamma^{-1}|z_1| \Delta_1 \mu_\sigma(|z_1|, |z_2|)}, \quad \langle R \rangle_{(0)} = \sqrt{2\gamma^{-1}|z_1|}.
$$

(90)

Note that for the spinless particle $\mu_\sigma = \mu$.

In the general case, the quantities $\langle R^2 \rangle_{(j)}$ and $\langle R^2 \rangle_{(j)}$ do not coincide with the corresponding quantities $\langle R^2 \rangle_{(j)}$ and $\langle R^2 \rangle_{(j)}$ given by equation (56). The latter quantities are expressed in terms of means of square of the transverse kinetic energy and $\hat{J}_z$ in the classical limit. The interpretation of $\langle R \rangle$ as the classical radius follows from equation (62). At the same time, the mean radius of the orbits coincides with the classical radius $R = \sqrt{2\hbar/M\omega}|z_1|$. The interpretation of $\langle R \rangle$ as the classical radius follows from equation (62) in the classical limit. States with $j = 0$ correspond to orbits that do not embrace the AB solenoid (which corresponds to $|z_1| \gtrapprox |z_2|$ in the classical limit). For such orbits, $\langle R \rangle_{(1)} < R$, where the quantity $R = \sqrt{2\hbar/M\omega}|z_1|$ is interpreted as a distance between the AB solenoid and the orbit center (see the classical limit of equation (62)).

One can see that the standard deviations $\delta_j(R)$, $\delta_j(R_c)$ and $\delta_j(x + y) = \sqrt{\text{Var}_j(x + y)}$ in CS (72) and (77) are relatively small for the semiclassical orbits situated far from the solenoid, i.e. for $||z_1| - |z_2|| \gg 1$. In this case the CS are mainly concentrated near classical orbits. In the most interesting case when a semiclassical orbit is situated near the solenoid, such that the condition $||z_1| - |z_2|| \ll 1$ holds, the standard deviation $\delta_j(x + y)$ increases significantly, $\delta_j(x + y) = \delta'(R) \approx 2\pi^{-1/2}\gamma^{-1/2}|z_1|^{5/4}$, while the standard deviations $\delta_j(R)$ and $\delta_j(R_c)$ remain relatively small. In this case $R \approx R_c$; however, $\langle R \rangle_{(1)} < R_c$, as it has to be for the semiclassical orbits. Thus, the standard deviation $\delta_j(x + y)$ of particle positions near classical orbits is relatively large at $R \approx R_c$, such that $\delta'(R) \gg |R - (R_{(1)}), |(\langle R \rangle_{(0)} - R_c)|$. We show the corresponding spreads in figure 2 (where $R_c = \langle R \rangle_{(1)}$ and $R = \langle R \rangle_{(0)}$).

We stress that for $\mu \neq 0$, relations between CS/SS parameters of particle trajectory in a constant uniform magnetic field differ from the classical ones due to the presence of the AB solenoid. Above we have demonstrated this, considering the radius $R$ (related to the energy of particle rotation) and the distance $R_c$ (related to particle angular momentum). Such relations do not feel the presence of the AB solenoid for $\mu = 0$, and even for $\mu \neq 0$, in the classical limit.

For relativistic particles in $(3+1)$ dim, we consider means $\langle x - i\epsilon y \rangle_{(j)}$ in CS (89) on the hypersurface $x_- = \text{const}$. Such means are reduced to the means $\langle x - i\epsilon y \rangle_{(j)}$, represented above by expressions (53) and (54). Relations (90) and (56) remain true. Here, however, the evolution is parametrized by the light-cone time $x_-$, via the function $z_1(x_-)$ given by equation
One can see that means $\langle x \rangle_{(j)}$ and $\langle y \rangle_{(j)}$ are rotating along circles on the $xy$-plane with the synchrotron frequency $\omega$, i.e. their trajectories have the classical form (14).

5.4. Ultrarelativistic particles in $(2+1)\dim$

In subsection 5.2 we constructed relativistic CS in $(3+1)\dim$, and in subsection 5.3, we demonstrated that in such CS the means have the classical form (14). We succeeded in doing this using light-cone parametrization of the evolution via the function $z_1(x_-)$ given by equation (88). Such a parametrization is possible only in the relativistic case in $(3+1)\dim$. Indeed, using (A.35), we can represent equation (A.26) for eigenfunctions of $\hat{P}_0 + \hat{P}_3$ with the eigenvalues $\lambda$ in the form of the first order Schrödinger-like equation (A.36), where $x_-$ plays the role of time and the operator $\hat{H}_{x-} = \hat{Q}^2 (2\lambda M c)^{-1}$ plays the role of the Hamiltonian. The Hamiltonian $\hat{H}_{x-}$ is quadratic with respect to the momentum operators. In the cases in $(2+1)\dim$ considered above, the light-cone variables $x_\perp$ cannot be introduced. Then we have to use the time $t$ parametrization of the evolution. This is why we cannot construct CS as exact solutions of the Dirac equation. This is a consequence of the fact that in the case under consideration, the Dirac Hamiltonian is not quadratic in momenta and the corresponding ICS do not maintain their form in the course of the evolution. Below we consider an example of such an evolution of ICS. We take massless $\zeta = +1$ fermions in $(2+1)\dim$ with the Hamiltonian $\hat{H}^\zeta = c\sigma \hat{P}_\perp$.

One can see (using the results of appendix A) that ICS (38) obey the following relation:

$$\hat{H}^\zeta \Psi^{(j,+1)}_{\zeta_1,\zeta_2}(\varphi, \rho) = \pm c \hat{\Pi}_0 \Psi^{(j,+1)}_{\zeta_1,\zeta_2}(\varphi, \rho), \quad (91)$$
where $\hat{N}_0 = \Pi_0(\hat{N}_1)$ is given by (33) at $\sigma = -\theta$. Then a formal solution of the Dirac equation, with ICS (38) as an initial condition, reads
\[ \Psi_{j+1, z_1 z_2}^{(j+1)}(t, \varphi, \rho) = \exp \left[ \frac{ic}{\hbar} \hat{N}_0 t \right] \Psi_{j, z_1 z_2}^{(j+1)}(\varphi, \rho). \] (92)

We call such a solution quasi-CS in what follows. As usual, we define means of an operator $\hat{F}$ in quasi-CS (92) by $(\hat{F})_{(j, k)}$:
\[ (\hat{F})_{(j, k)} = \left( \Psi_{j, z_1 z_2}^{(j+1)}(\varphi, \rho), \hat{F} \Psi_{j, z_1 z_2}^{(j+1)}(\varphi, \rho) \right)_D. \] (93)

One can see that in the semiclassical limit the means
\[ (x(t) - ie y(t))_{(j, k)} = \sqrt{2\pi}^{-1} \left| (a_2(t))_{(j, k)} - (a_1(t))_{(j, k)} \right| \] (94)
are moving along the corresponding classical trajectories. Since the operator $\hat{a}_2$ commutes with $\hat{N}_1$, the mean $(a_2(t))_{(j, k)}$ does not depend on time and coincides with its initial value, $(a_2(t))_{(j, k)} = (a_2)_j$, the latter is given by equation (55). Calculating the mean $(a_1(t))_{(j, k)}$, we find that
\[ (a_1(t))_{(0, k)} = \left( z_1 \sum_{\sigma = \pm 1} \exp \left[ \pm i \Omega \left( \frac{|z_1| + |z_2|}{2\sqrt{|z_1|}} \right) \right] Q_{1-\mu_\sigma}(|z_1|, |z_2|) \right) \sum_{\sigma = \pm 1} Q_{1-\mu_\sigma}(|z_1|, |z_2|), \] (95)
\[ (a_1(t))_{(1, k)} = \left( z_1 \sum_{\sigma = \pm 1} \exp \left[ \pm i \Omega \left( \frac{|z_1| + |z_2|}{2\sqrt{|z_1|}} \right) \right] Q_{\mu_\sigma}(|z_2|, |z_1|) \right) \sum_{\sigma = \pm 1} Q_{\mu_\sigma}(|z_2|, |z_1|), \]

(see appendix B) where the frequency operator $\Omega(\hat{N}_1)$ is given by equation (B.8).

Now let $|z_1|^2 \gg 1$, which corresponds to the semiclassical limit. Using relations (59) and (60), we represent (95) in the form
\[ (a_1(t))_{(0, k)} = \left( z_1 \sum_{\sigma = \pm 1} \exp \left[ \pm i \Omega t \right] Q_{1-\mu_\sigma}(|z_1|, |z_2|) \right) \sum_{\sigma = \pm 1} Q_{1-\mu_\sigma}(|z_1|, |z_2|), \]
\[ (a_1(t))_{(1, k)} = \left( z_1 \sum_{\sigma = \pm 1} \exp \left[ \pm i \Omega t \right] Q_{\mu_\sigma}(|z_2|, |z_1|) \right) \sum_{\sigma = \pm 1} Q_{\mu_\sigma}(|z_2|, |z_1|), \]
\[ \Omega = \Omega \left( |z_1|^2 + \frac{|z_1|}{2} \frac{d}{d|z_1|} \right). \]

The semiclassical expansions of the operator $\hat{\Omega}$ are
\[ \hat{\Omega} \approx \omega(|z_1|) \left[ 1 - \frac{1}{4|z_1| \frac{d}{d|z_1|}} + O(|z_1|^{-2}) \right], \]
\[ \omega(|z_1|) = c|q| B |\mathcal{E}|^{-1}(|z_1|), \quad \mathcal{E}(|z_1|) = \sqrt{2\pi} c|q| B |z_1|. \]

such that the standard deviation of $\hat{\Omega}$ is of the order $\omega(|z_1|)|z_1|^{-1}$. Here $\mathcal{E}(|z_1|)$ is the mean energy in quasi-CS, in the classical limit. Taking into account next-to-leading corrections to $\exp \left[ \pm i \Omega t \right]$, and using decompositions (64) and (66), we obtain $(a_1(t))_{(j, k)} = (a_1)_j e^{i\omega_j t}$, where
\[ \omega_j = \omega(|z_1|) \left[ 1 + O(|z_1|^{-2}) \right], \quad |z_1| - |z_2| \gg 1; \]
\[ \omega_j = \omega(|z_1|) \left[ 1 + \frac{(-1)^j}{2\sqrt{|z_1|}} + O(|z_1|^{-2}) \right], \quad |z_1| - |z_2| \ll 1. \] (96)

In such an approximation, $\omega(|z_1|)$ coincides with the classic synchrotron frequency $\omega$ given by equation (5).
Thus, in the classical limit, means $\langle x \rangle_{(j, \pm)}$ and $\langle y \rangle_{(j, \pm)}$ in the quasi-CS are rotating along circles on the $xy$-plane with the synchrotron frequency $\omega$. The greater the $|z_1|^2$ the smaller the spreading of the mean trajectories. One can see that additional (due to the next corrections) modifications of quasi-CS are essential only for evolution time that is much greater than the classical rotation period.

The mean radius of a trajectory is
\[
\sqrt{\langle R^2 \rangle_{(j, \pm)}} = \sqrt{\left[ 2(N_1)_{(j, \pm)} + 1 + \vartheta \epsilon \right] \gamma^{-1}}.
\]
The next-to-leading corrections to this relation can be found with the help of equations (51), (65) and (67). We note that in each approximation the classical relation between the rotation frequency and the radius holds, such that $\tilde{\omega}_j \sqrt{\langle R^2 \rangle_{(j, \pm)}} = c$ (we recall that in the graphene case $c$ means the effective velocity of light, that is, the Fermi velocity $v_F$).

6. Summary and discussion

A new approach to constructing CS/SS in the MSF is proposed. The main idea is based on the fact that the AB solenoid breaks the translational symmetry in the $xy$-plane; this has a topological effect such that there appear two types of circular trajectories which embrace and do not embrace the solenoid. Due to this fact, one has to construct two different kinds of CS/SS, which correspond to such trajectories in the semiclassical limit. Following this idea, we construct CS in two steps, first the instantaneous CS (ICS) and then the time-dependent CS/SS as an evolution of the ICS.

The approach is realized for nonrelativistic and relativistic spinning particles that allow us to build CS in both $(2 + 1)$ and $(3 + 1)$ dim, using some universal constructions, and gives a non-trivial example of SS/CS for systems with a nonquadratic Hamiltonian.

It is stressed that CS depending on their parameters (quantum numbers) describe both pure quantum and semiclassical states. An analysis is represented that classifies parameters of the CS in such respect. Such a classification is used for the semiclassical decomposition of various physical quantities.

In the pure quantum case, the mean values depend significantly on the particle spin and on the mantissa $\mu$ and are quite different from the corresponding classical values. In the semiclassical approximation, relations between CS/SS parameters and parameters that characterize classical trajectories are established. In the general case these relations differ from the ones in the pure magnetic field and such a distinction can be treated as the AB effect in the CS/SS. The classical relations correspond to the leading approximation for sufficiently large radii. Thus, the leading approximation in the semiclassical expansions corresponds to the classical limit. Next-to-leading terms define physical quantities in the semiclassical approximation. These terms depend on the space dimension and particle spin.

The following properties of the constructed time-dependent CS/SS should be stressed.

(a) In the nonrelativistic case, both in $(2 + 1)$ and $(3 + 1)$ dim, the time-dependent CS in each time instant retain the form of the corresponding ICS. The mean trajectories in such CS coincide with the classical ones, whereas the particle distributions are concentrated near the classical trajectories in the semiclassical approximation. In the presence of the AB solenoid, the spread of particle positions near the classical trajectory depends essentially on the mutual disposition between the trajectory and the solenoid. Such a spread is growing for trajectories situated near the AB solenoid. It should be noted that namely due to the bounded character of particle motion in the MSF, particle positions are essentially sensitive to the topological effect of breaking the translational symmetry in the $xy$-plane.
due to the presence of the AB solenoid. Thus, in spite of the well-known fact that the AB effect is global, in the MSF quantum states can be classified with respect to their ‘closeness’ to the AB solenoid.

(b) In the relativistic case, in $(3 + 1)$ dim, CS are constructed in the light-cone variables, where the evolution is parametrized by the light-cone time $x_−$. Such time-dependent CS obey all the properties as CS from the previous item (a).

(c) In $(2 + 1)$ dim, we constructed time-dependent SS for massless fermions. Such a problem can be related to the graphene physics. We call the constructed SS quasi-CS since they retain the ICS form with time evolution in the next-to-leading semiclassical approximation. In such an approximation, the classical relation between the rotation frequency and the radius holds; the rotation frequency coincides with the classic synchrotron frequency in the leading approximation. We stress on a principal difference between $(2 + 1)$ dim and $(3 + 1)$ dim, in $(2 + 1)$ dim. The Dirac Hamiltonian is not quadratic in momenta, whereas in $(3 + 1)$ dim it is. Namely this fact is responsible for the destruction of ICS in the course of the evolution.

Acknowledgments

The work of VGB is partially supported by the Russian Science and Innovations Federal Agency under contract no 02.740.11.0238 and Russia President grant SS-3400.2010.2. The work of SPG is supported by FAPESP/Brasil and the program Bolsista CAPES/Brasil. SPG thanks the University of São Paulo for hospitality. DMG acknowledges the permanent support of FAPESP and CNPq. DPMF thanks CNPq for support.

Appendix A. Quantum stationary states

A.1. $(2 + 1)$ dim

In $(2 + 1)$ dim, the total angular momentum operator $\hat{J} = -i\hbar\partial_\phi + \hbar\sigma^3/2$, which is a dimensional reduction of the operator $\hat{J}_z$ in $(3 + 1)$ dim ($z$-component of the total angular momentum operator given by (22)), is self-adjoint on $D_{\theta\hat{H}}$ and commutes with $\hat{H}^\theta$. There exist common eigenvectors $\psi^{(j)}_{n_1,n_2}(\phi, \rho)$ of operators $\hat{H}^\theta$ and $\hat{J}$:

$$\hat{H}^\theta \psi^{(j)}_{n_1,n_2}(\phi, \rho) = c_p \psi^{(j)}_{n_1,n_2}(\phi, \rho), \quad c_p = \pm E, \quad E = \sqrt{(Mc^2)^2 + E^2_\perp},$$

$$\hat{J} \psi^{(j)}_{n_1,n_2}(\phi, \rho) = J \psi^{(j)}_{n_1,n_2}(\phi, \rho), \quad J = \epsilon \hbar(l_0 - l + 1/2), \quad j = 1, 2.$$  \hspace{1cm} (A.1)

It is convenient to use the following representation:

$$\psi^{(j)}_{n_1,n_2}(\phi, \rho) = \sigma^3(p_0 - \sigma \hat{P}_\perp) + Mc u^{(j)}_{n_1,n_2}(\phi, \rho),$$

$$u^{(j)}_{n_1,n_2}(\phi, \rho) = \sum_{\sigma=\pm 1} c_\sigma \Phi^{(j)}_{n_1,n_2,\sigma}(\phi, \rho) v_\sigma,$$  \hspace{1cm} (A.2)

where

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$  \hspace{1cm} (A.3)

and $c_\sigma$ are some constants. The columns $u^{(j)}_{n_1,n_2}$ are solutions of the eigenvalue problem

$$c^2(\sigma \hat{P}_\perp)^2 u^{(j)}_{n_1,n_2}(\phi, \rho) = E^2_\perp u^{(j)}_{n_1,n_2}(\phi, \rho).$$  \hspace{1cm} (A.4)

We note that the relation $(\sigma \hat{P}_\perp)^2 = \hat{P}_\perp^2 - \epsilon \hbar c^{-1} |qB| \sigma^3$ holds, which gives a relation to the energy spectrum for the spinless case.
The functions $\Phi^{(j)}_{\alpha_{1} \alpha_{2} \gamma} (\psi, \rho)$ have the form

$$\Phi^{(j)}_{\alpha_{1} \alpha_{2} \gamma} (\psi, \rho) = \mathcal{N} \exp \left[ i \left\{ l_{0} - l + (1 - \epsilon \sigma)/2 \right\} \psi \right] L_{n_{1} \gamma} (\rho),$$

with

$$n_{1} = m, \quad n_{2} = m - l + (1 - \epsilon \sigma)/2 - \mu,$$

$$l = l - (1 + \epsilon)(1 + \sigma)/2,$$

$$\mathcal{N} = \pi^{1/2} [l - (1 - \epsilon)(1 + \sigma)/4] L_{n_{1} \gamma} (\rho),$$

$$n_{1} = m + l - (1 - \epsilon \sigma)/2 + \mu, \quad n_{2} = m,$$

$$(1 - \epsilon \sigma)/2 < l < \infty; \quad \rho = \frac{\gamma}{2}. \quad (A.5)$$

Here $L_{n_{1} n_{2}} (\rho), m \geq 0$, are Laguerre functions that are related to the Laguerre polynomials $L_{n}^{m} (\rho)$ (see [28]) as follows:

$$L_{m+n+1}^{m} (\rho) = \sqrt{\frac{\Gamma (m + 1)}{\Gamma (m + \alpha + 1)}} e^{-\rho/2} \rho^{\alpha/2} L_{m}^{\alpha} (\rho), \quad L_{m}^{\alpha} (\rho) = \frac{1}{m!} \rho^{\alpha} \frac{d^m}{d \rho^m} e^{-\rho} \rho^{m+\alpha} \quad (A.6)$$

and $\mathcal{N}$ are normalization constants. For any real $\alpha > -1$, the functions $L_{m+n}^{\alpha} (\rho)$ form a complete orthonormal set on the semiaxis $\rho \geq 0$:

$$\int_{0}^{\infty} L_{m+n} (\rho) L_{m+n} (\rho') d \rho = \delta_{m,n}, \quad \sum_{m=0}^{\infty} L_{m+n} (\rho) L_{m+n} (\rho') = \delta (\rho - \rho'). \quad (A.7)$$

Thus, the domains $D_{\rho}^{m} \alpha$ are described completely by the asymptotic behavior of the functions from equations (A.5).

We define the functions $\Phi^{(j)}_{\alpha_{1} \alpha_{2} \gamma} \Phi^{(j)}_{\alpha_{1} \alpha_{2} \gamma}$ associated with transformation (26) as follows:

$$\Phi^{(j)}_{\alpha_{1} \alpha_{2} \gamma} (\psi, \rho) = \mathcal{N} \exp \left[ i \left\{ l_{0} - l + s_{1} + s_{2} + (1 - \epsilon \sigma)/2 \right\} \psi \right] L_{n_{1} n_{2}} (\rho),$$

$$\Phi^{(j)}_{\alpha_{1} \alpha_{2} \gamma} (\psi, \rho) = \mathcal{N} \exp \left[ i \left\{ l_{0} - l + s_{1} + s_{2} + (1 - \epsilon \sigma)/2 \right\} \psi \right.$$

$$\left. + \pi [l + s_{1} + s_{2} - (1 - \epsilon)(1 + \sigma)/4] L_{n_{1} n_{2}} (\rho), \right.$$

$$s_{1} = 0, \pm 1, \quad s_{2} = 0, \pm 1. \quad (A.8)$$

There appear new functions $\Phi^{(j)}_{\alpha_{1} \alpha_{2} \gamma} \Phi^{(j)}_{\alpha_{1} \alpha_{2} \gamma}$ with $n_{2} = m + 1 + (\theta - \sigma) \epsilon/2 - \mu$ and $\Phi^{(j)}_{\alpha_{1} \alpha_{2} \gamma} \Phi^{(j)}_{\alpha_{1} \alpha_{2} \gamma}$ with $n_{1} = m + (\sigma - \bar{\theta}) \epsilon/2 + \mu$. Such functions were not defined by equations (A.5). In addition, for $n_{1} = 0$ or $n_{2} = 0$, one has to bear in mind that

$$\delta_{1} \Phi^{(j)}_{\alpha_{1} \alpha_{2} \gamma} (\psi, \rho) \bigg|_{m=0} = 0, \quad \delta_{2} \Phi^{(j)}_{\alpha_{1} \alpha_{2} \gamma} (\psi, \rho) \bigg|_{m=0} = 0. \quad (A.9)$$

This allows us to interpret $\Phi^{(j)}_{\alpha_{1} \alpha_{2} \gamma} \Phi^{(j)}_{\alpha_{1} \alpha_{2} \gamma}$ and $\Phi^{(j)}_{\alpha_{1} \alpha_{2} \gamma} \Phi^{(j)}_{\alpha_{1} \alpha_{2} \gamma}$ as vacuum states.

Let us define an inner product of two functions $f (\psi, \rho)$ and $g (\psi, \rho)$ as

$$(f, g)_{\perp} = \frac{1}{2 \pi} \int_{0}^{\infty} d \rho \int_{0}^{2 \pi} d \psi f^{*} (\psi, \rho) g (\psi, \rho). \quad (A.10)$$

With respect to such an inner product, functions (A.5) form an orthogonal set

$$\left( \Phi^{(j)}_{\alpha_{1} \alpha_{2} \gamma}, \Phi^{(j)}_{\alpha_{1} \alpha_{2} \gamma} \right) = | \mathcal{N} |^{2} \delta_{\alpha_{1} \alpha_{2} \gamma} \delta_{\alpha_{1} \alpha_{2} \gamma}. \quad (A.11)$$

Moreover, these functions form a complete orthogonal set in $L_{2} (R^{2})$.

The energy spectrum of the self-adjoint Hamiltonians (21) can be represented with the help of the eigenvalues $E_{\perp}^{2}$ as

$$E_{\perp}^{2} = 2 \hbar c \left[ q B \right] \left[ n_{1} + (1 - \epsilon \sigma)/2 \right]. \quad (A.12)$$
where, depending on the \( \vartheta \) and \( \epsilon \), the quantum number \( n_1 \) takes its possible values according to (A.5). In the general case, eigenvalues \( E \) of the Hamiltonian \( \hat{H}_\vartheta \) are expressed via \( E_{\vartheta \epsilon}^2 \) for \( \sigma = +1 \) or \( \sigma = -1 \), according to equation (A.1), that is why \( E_{\vartheta \epsilon}^2 \) are labeled by the subscript \( \sigma \). Irregular at the origin radial functions appear in the domain \( D_{\vartheta \epsilon} \) for \( l = 0 \) and \( \sigma = -1 \), and in the domain \( D_{\vartheta \epsilon}^\perp \) for \( l = 0 \) and \( \sigma = +1 \).

We note that depending on \( \vartheta \epsilon \), energy levels of states having irregular radial functions coincide or differ from the Landau levels. In any case the difference always depends on \( \mu \) only. Energies of states with \( j = 1 \) differ from the Landau levels, whereas energies with \( j = 0 \) coincide with the Landau levels. If \( E_{\vartheta \epsilon}^2 \) are non-zero, the complete set of eigenvectors \( \psi_{n_1, n_2}^{(j)} \) is given by equation (A.2), where constants \( c_{1, \vartheta} \) are arbitrary, e.g. either \( c_{+1} \neq 0 \) or \( c_{-1} \neq 0 \). However, if \( E_{\vartheta \epsilon}^2 = 0 \), a completeness of the eigenvectors implies a special choice of \( c_{1, \vartheta} \), namely \( c_{-1} \neq 0 \) if \( \epsilon = -1 \) and \( c_{+1} \neq 0 \) if \( \epsilon = +1 \). In this case only the negative energy solutions (antiparticles with \( cp_0 = -M \)) are possible if \( \epsilon = -1 \) and only positive energy solutions (particles with \( cp_0 = +M \)) are possible if \( \epsilon = +1 \). They coincide with the corresponding spinors \( u_{n_1, n_2}^{(j)}(\psi, \rho) \) up to a normalization constant. This is a manifestation of the well-known asymmetry of the energy spectrum of \( (2 + 1) \) Dirac particles in the uniform magnetic field. We see that the same asymmetry takes place in the presence of the AB field. For particles, we select \( c_{+1} \neq 0 \), \( c_{-1} = 0 \), and then their energy spectrum is \( E = \sqrt{(Mc^2)^2 + E_{\vartheta \epsilon}^2} \), for antiparticles \( c_{+1} = 0 \), \( c_{-1} \neq 0 \), and then their energy spectrum is \( E = \sqrt{(Mc^2)^2 + E_{\vartheta \epsilon}^2} \).

If we define the inner product of spinors \( \psi(\psi, \rho) \) and \( \psi'(\psi, \rho) \) as follows:

\[
(\psi, \psi')_D = \frac{1}{2\pi} \int_0^\infty d\rho \int_0^{2\pi} d\varphi \psi^{\dagger}(\psi, \rho) \psi'(\psi, \rho),
\]

then the inner product of eigenvectors \( \psi_{n_1, n_2}^{(j)} \) has the form

\[
(\psi_{n_1, n_2}^{(j)}, \psi_{n_1, n_2}^{(j)})_D = 2Mc^2|c_{\vartheta}(\sigma)|^2 \left( \phi_{n_1, \vartheta \epsilon}^{(j)\dagger} \phi_{n_1, \vartheta \epsilon}^{(j)} \right)_\perp,
\]

where the inner product \((., .)_\perp\) is given by equation (A.10). With respect to the introduced inner product, eigenvectors (A.2) form an orthogonal set for any \( \vartheta \).

With the help of eigenvectors (A.2), we obtain the following solutions of the Dirac equation with a given energy \( cp_0 = \pm \xi \) in \( (2 + 1) \) dim:

\[
\Psi^{(j)}_{p_0, m, j}(t, \mathbf{r}) = \exp \left[ \frac{i}{\hbar} (cp_0 t) \right] \psi_{n_1, n_2}^{(j)}(\psi, \rho).
\]

We believe that nonrelativistic motion is described by the corresponding Pauli equation in \( (2 + 1) \) dim:

\[
i\hbar \partial_t \Psi_{\pm} = \pm \hat{H}^{NR}_{\pm} \Psi_{\pm}, \quad \hat{H}^{NR}_{\pm} = (\sigma \cdot \hat{\mathbf{P}})_\perp^2/2M.
\]

Solutions of such an equation can be obtained from (A.14) in the nonrelativistic limit. They have only one component, taking into account that in \( (2 + 1) \) dim there is only one spin polarization. Let us consider, for example, spin-up particles (\( \zeta = +1 \)). Then we obtain from (A.2)

\[
\Psi^{(j)\text{up}}_{\pm, m, l}(t, \mathbf{r}) = \exp \left[ \frac{i}{\hbar} \xi^{NR}_{\pm} t \right] \Phi_{n_1, n_2, \pm \mp}(\psi, \rho) \psi_{\pm}^{\dagger},
\]

where \( \xi^{NR}_{\pm} = E_{\vartheta \epsilon}^{(j)\perp(\pm 1)}/2Mc^2 \). The corresponding inner product of these solutions reads

\[
(\Psi^{(j)\text{up}}_{\pm, m, l}, \Psi^{(j)\text{up}}_{\pm, m, l})_D = \left( \Phi_{n_1, n_2, \pm \mp}^{(j)} \right)_\perp.
\]
With the help of relation (20), we obtain solutions that describe spin-down particles:

$$
\psi^{(j)\downarrow}_{\pm,m,l}(t, \mathbf{r}) = \exp \left( \frac{i}{\hbar} \mathcal{H}^{\downarrow}(\mathbf{r}) \right) \psi^{(j)\downarrow}_{\pm,m,l}(\varphi, \rho),
$$

$$
\left( \psi^{(j)\uparrow}_{\pm,m,l} \psi^{(j)\downarrow}_{\pm,m,l} \right)_D = \left( \Phi^{(j)\downarrow}_{n_1,n_2,\mp 1}, \Phi^{(j)\downarrow}_{n_1,n_2,\pm 1} \right)_\perp.
$$

(A.18)

It is worthwhile to make the following remark. Usually, in the nonrelativistic limit, negative energy solutions (those which after the charge conjugation operation represent wavefunctions of particles with the opposite charge \(-q\), and which are, in fact, antiparticles) of the Dirac equation are not considered. It is supposed that all the information about the quantum motion of the antiparticles can be extracted from particle motion. The latter is not true in the case under consideration (for the Dirac equation with the MSF in \((2 + 1)\) dim). Here energy spectra for particles and antiparticles are quite different. This asymmetry was the reason for us to represent explicitly, even in the nonrelativistic limit, the negative energy solutions \(\psi^{(j)\uparrow}_{\pm,m,l}(t, \mathbf{r})\) and \(\psi^{(j)\downarrow}_{\pm,m,l}(t, \mathbf{r})\), which correspond to spin-up and -down antiparticles.

For massless \(\xi = +1\) particles and antiparticles (in what follows we call all such particles simply fermions) in \((2 + 1)\) dim, the self-adjoint Dirac Hamiltonian is \(\hat{H}^\xi = c\sigma \bar{\mathbf{p}}_\perp\). Its eigenvalues are \(c p_0 = \pm \mathcal{E}\), where \(\mathcal{E} = \mathcal{E}_\perp (\mathcal{E}_\parallel, 0)\) (the eigenvalues \(\mathcal{E}_\perp (\mathcal{E}_\parallel, 0)\) are given by equation (A.11)). The corresponding eigenvectors of \(\hat{H}^\xi\) have the form

$$
\psi^{(j)\mp 1}_{\pm,m} = \exp \left( \frac{\pm i}{\hbar} \mathcal{H}^{\mp 1} \right) \psi^{(j)\mp 1}_{\pm,m},
$$

$$
\psi^{(j)\mp 1}_{\pm,m,l} = \exp \left( \frac{\pm i}{\hbar} \mathcal{H}^{\mp 1}_{\pm,m,l} \right) \psi^{(j)\mp 1}_{\pm,m,l},
$$

$$
\left( \psi^{(j)\mp 1}_{\pm,m,l} , \psi^{(j)\mp 1}_{\pm,m,l} \right)_D = \left( \Phi^{(j)\mp 1}_{n_1,n_2,\mp 1}, \Phi^{(j)\mp 1}_{n_1,n_2,\pm 1} \right)_\perp.
$$

(A.20)

In addition, there are nontrivial zero-mode (\(\mathcal{E} = 0\)) solutions:

$$
\psi^{(j)\mp 1}_{0,m} = \exp \left( \frac{\pm i}{\hbar} \mathcal{H}^{\mp 1}_{0,m} \right) \psi^{(j)\mp 1}_{0,m},
$$

$$
\left( \psi^{(j)\mp 1}_{0,m,l} , \psi^{(j)\mp 1}_{0,m,l} \right)_D = \left( \Phi^{(j)\mp 1}_{0,n_1,n_2}, \Phi^{(j)\mp 1}_{0,n_1,n_2} \right)_\perp_{m=0}.
$$

(A.21)

As follows from (20), for massless \(\xi = -1\) particles, the corresponding eigenvectors can be represented as

$$
\psi^{(j)\pm 1}_{\pm,m}(t, \mathbf{r}) = \exp \left( \frac{\pm i}{\hbar} \mathcal{E} t \right) \sigma^2 \psi^{(j)\pm 1}_{\mp,m}(t, \mathbf{r}),
$$

$$
\psi^{(j)\pm 1}_{0,m}(t, \mathbf{r}) = \sigma^2 \psi^{(j)\pm 1}_{0,m}(t, \mathbf{r}).
$$

(A.22)

The inner products of these solutions coincide with the ones of solutions \(\psi^{(j)\pm 1}_{\pm,m,l}\) given by equation (A.20):

$$
\left( \psi^{(j)\mp 1}_{\pm,m,l} , \psi^{(j)\mp 1}_{\pm,m,l} \right)_D = \left( \psi^{(j)\mp 1}_{\pm,m,l} , \psi^{(j)\mp 1}_{\pm,m,l} \right)_D ,
$$

$$
\left( \psi^{(j)\mp 1}_{0,n_1,n_2} , \psi^{(j)\mp 1}_{0,n_1,n_2} \right)_D = \left( \psi^{(j)\mp 1}_{0,n_1,n_2} , \psi^{(j)\mp 1}_{0,n_1,n_2} \right)_D.
$$
A.2. (3 + 1) dim

Here we consider the Dirac equation (16) in (3 + 1) dim. Let us introduce projection operators \( \hat{P}_{(\pm)} \) and two kinds of Dirac bispinor \( \Psi_{(\pm)} \):

\[
\hat{P}_{(\pm)} = (1 \pm \alpha^3)/2, \quad (\hat{P}_{(\pm)})^\dagger = \hat{P}_{(\pm)}, \quad (\hat{P}_{(\pm)})^2 = \hat{P}_{(\pm)},
\]

\[
\hat{P}_{(+)} \hat{P}_{(-)} = 0, \quad \hat{P}_{(+)} + \hat{P}_{(-)} = I,
\]

where \( I \) is the unit 4 \( \times \) 4 matrix, such that any \( \Psi \) can be represented as \( \Psi = \Psi_{(+)} + \Psi_{(-)} \). Then, the Dirac equation (16) is reduced to the following set of equations:

\[
(\hat{P}_0 + \hat{P}_3) \Psi_{(+)} = \hat{Q} \Psi_{(-)}, \quad (\hat{P}_0 - \hat{P}_3) \Psi_{(-)} = \hat{Q} \Psi_{(+)},
\]

\[
\tilde{Q} = (\alpha_1 \hat{P}_\perp) + M c \gamma^0, \quad \hat{P}_0 = i \hbar \partial_0,
\]

where \( \alpha' = \gamma^0 \gamma^1 \). Due to the axial symmetry of the problem, it is convenient to use the following representation for \( \gamma \)-matrices (see [9]):

\[
\gamma^0 = \text{diag}(\sigma^3, -\sigma^3), \quad \gamma^1 = \text{diag}(i \sigma^2, -i \sigma^2), \quad \gamma^2 = \text{diag}(-i \sigma^1, i \sigma^1), \quad \gamma^3 = \text{antidiag}(-I, I),
\]

(A.24)

where \( I \) is the unit 2 \( \times \) 2 matrix. Nevertheless, expressions for \( \alpha^3, \Sigma_z \) and \( \gamma^5 \) are the same in representation (A.24) and in the standard representation.

One can see that

\[
\tilde{Q}_\perp^2 = M^2 c^2 + \tilde{Q}_\perp^2, \quad \tilde{Q}_\perp^2 = (\alpha_1 \hat{P}_\perp)^2 = \hat{P}_\perp^2 - 4 \hbar c^{-1}|qB| \Sigma_z,
\]

(A.25)

where \( \Sigma_z = \text{diag}(\sigma^3, \sigma^3) \). In the MSF, the operators \( \hat{P}_0 + \hat{P}_3, \hat{P}_0 - \hat{P}_3 \) and \( \tilde{Q} \) mutually commute, such that (as follows from (A.23)) bispinors \( \Psi_{(\pm)} \) obey the same equations:

\[
(\hat{P}_0^2 - \hat{P}_3^2 - \tilde{Q}_\perp^2) \Psi_{(\pm)}(x) = 0.
\]

(A.26)

Representing \( \Psi_{(\pm)} \) via spinors \( \psi \) and \( \chi \),

\[
\Psi_{(-)} = \frac{1}{2} \begin{pmatrix} \psi \\ -\sigma^3 \psi \end{pmatrix}, \quad \Psi_{(+)} = \frac{1}{2} \begin{pmatrix} \chi \\ \sigma^3 \chi \end{pmatrix},
\]

(A.27)

we find the following equations for the spinors:

\[
\left[ \hat{P}_0^2 - \hat{P}_3^2 - (\hat{P}_\perp^2 - \epsilon |qB| \hbar c^{-1}) - M^2 c^2 \right] \psi(x) = 0.
\]

(A.28)

\[
(\hat{P}_0 + \hat{P}_3) \chi(x) = (\sigma^3 \sigma \hat{P}_\perp + M c) \psi(x).
\]

(A.29)

We note that both solutions \( \Psi_{(-)} \) and \( \Psi_{(+)} \) enter into a complete set of functions on the hypersurface \( t = \text{const} \).

The inner product of Dirac bispinors on the light-cone hypersurface \( x_- = \text{const} \) has the form

\[
(\Psi, \Psi')_{x_-} = \int \Psi^\dagger \Psi' d\mathbf{x} = \frac{2\pi}{\gamma} \int (\Psi_{(-)}^\dagger)_{x_+} d\mathbf{x}_+,
\]

(A.30)

see [29], where the inner product of four-component spinors \( \Psi \) and \( \Psi' \) on the \( xy \)-plane is defined as

\[
(\Psi, \Psi')_{D} = \frac{1}{2\pi} \int_0^\infty d\rho \int_0^{2\pi} d\phi \Psi^\dagger(\phi, \rho) \Psi'(\phi, \rho).
\]

(A.31)

It is expressed only in terms of the components \( \Psi_{(-)} \). At the same time, a complete set of functions on the hypersurface \( x_- = \text{const} \) consists only of \( \Psi_{(-)} \).
In the case under consideration, the operators \( \hat{P}_0, \hat{P}_3, J_z = \hat{L}_z + \Sigma_z/2 \) and a spin operator \( \hat{S}_z \) (\( z \)-component of a polarization pseudovector):

\[
\hat{S}_z = \frac{i}{2} (\hat{H}_x + \Sigma_z \hat{H}_y) = \gamma^0 \Sigma_z M c^2 - \gamma^5 e \hat{P}^3,
\]

are mutually commuting integrals of motion (all these operators commute with the Hamiltonian \( \hat{H}^0 \)) [15, 16]. In addition, the set \( \hat{P}_0, \hat{P}_3, J_z, \Sigma_z \) and \( \hat{Q}_z^2 \) represents mutually commuting operators, which, at the same time, commute with \( \alpha^1 \). This fact allows one to find solutions \( \Psi_{(-)} \) that are eigenvectors for the latter set. To this end one has to subject spinors \( \psi \) to the following equations:

\[
(\hat{P}_0 + \hat{P}_3) \psi(x) = \lambda M c \psi(x), \quad \hat{J}_z \psi(x) = J_z \psi(x), \quad \sigma^3 \psi(x) = \sigma \psi(x), \quad \sigma^2 \psi(x) = \sigma^2 \psi(x),
\]

where \( \sigma^2 \) is given by (A.11). Thus, we obtain for \( \Psi_{(-)} \)

\[
(\hat{P}_0 + \hat{P}_3) \Psi_{(-)} = \lambda M c \Psi_{(-)}, \quad \hat{J}_z \Psi_{(-)} = J_z \Psi_{(-)}, \quad J_z = \epsilon \hbar (l_0 - l + 1/2),
\]

\[
c^2 \hat{Q}_z^2 \Psi_{(-)} = \epsilon^2 \sqrt{\epsilon} \Psi_{(-)}, \quad \Sigma_z \Psi_{(-)} = \sigma \Psi_{(-)}, \quad \sigma = \pm 1.
\]

In the light-cone variables (13), we have the following representation:

\[
\hat{P}_0 + \hat{P}_3 = 2i\hbar \frac{\partial}{\partial x_+}, \quad \hat{P}_0 - \hat{P}_3 = 2i\hbar \frac{\partial}{\partial x_-}.
\]

Then, we can represent equation (A.26) for eigenfunctions of \( \hat{P}_0 + \hat{P}_3 \) with the eigenvalues \( \lambda \) in the form of the first order Schrödinger-like equation

\[
\left[ 2i\hbar \lambda M c \frac{\partial}{\partial x_-} - \hat{Q}_z^2 \right] \Psi_{(-)\lambda}(x) = 0,
\]

and we find a complete set of solutions \( \Psi_{(-)} \) in the following form:

\[
\Psi_{(-)\lambda, m, l, \sigma}^{(j)} = \exp \left\{ -\frac{i}{\gamma M c} \left[ \lambda M c x_+ + \left( \frac{M c}{\lambda} + \hbar \bar{c} (1 - \sigma \epsilon) \right) x_- \right] \right. \\
\left. -i\bar{c} x_- \right\} \Phi_{n_1, n_2, \sigma}(\varphi, \rho) \left( \frac{\nu_\sigma}{-\sigma \nu_\sigma} \right),
\]

where \( \bar{c} \) is given by (14), \( \nu_\sigma \) by equations (A.3), \( \Phi_{n_1, n_2, \sigma}^{(j)} \) by (A.5) and \( \lambda > 0 \) for particles and \( \lambda < 0 \) for antiparticles. We note that the quantum number \( \lambda \) is associated with the corresponding classical quantity \( \lambda \) from (12).

We note that the spin integral of motion \( \hat{S}_z \) does not commute with \( \alpha^1 \) such that solutions (A.37) are not eigenvectors of \( \hat{S}_z \). One can use the operator \( \Sigma_z \) instead of \( \hat{S}_z \) to characterize the spin polarization. In spite of the fact that \( \left[ \Sigma_z, \hat{H} \right] \neq 0 \) and, therefore, \( \Sigma_z \) is not an integral of motion with respect to the \( t \)-evolution, \( \Sigma_z \) is an integral of motion with respect to the evolution in the light-cone ‘time’ \( x_- \). That is why the ‘spin polarization’ \( \sigma \) of solutions (A.37) is conserved with the time \( x_- \). Taking all this into account, one can calculate the light-cone inner product (A.30) of solutions (A.37):

\[
\left( \Psi_{(-)\lambda, m, l, \sigma}^{(j)}, \Psi_{(-)\lambda', m, l, \sigma}^{(j)} \right)_{x_-} = \frac{4\pi^2 \hbar}{\gamma M c} \delta_{\sigma', \sigma} \delta(\lambda' - \lambda) \left( \Phi_{n_1, n_2, \sigma}^{(j)}, \Phi_{n_1, n_2, \sigma}^{(j)} \right)_+,
\]

where the inner product \( \left( , \right)_+ \) is given by equation (A.10).
Let us consider the quantum motion of spinning particles in the nonrelativistic limit. To this end it is more convenient, instead of solutions (A.37), to use another set of solution $\Psi_s(x)$:

$$\Psi_s(x) = \exp \left[ -\frac{i}{\hbar} (cp_0 t + p_3 z) \right] \psi_p(x), \quad s = \pm 1,$$

$$\Psi_s(x_{\perp}) = N \left( \frac{[1 + (p^3 + s\tilde{M})/M]\psi_{p,s}(x_{\perp})}{[1 - 1 + (p^3 + s\tilde{M})/M]\psi_{p,s}(x_{\perp})} \right),$$

where $\tilde{M} = \sqrt{M^2 + (p_3)^2}$. These solutions are eigenvectors of mutually commuting integrals of motion $\hat{P}_0$, $\hat{P}_3$, $\hat{J}_z$ and $\hat{S}_z$:

$$\hat{P}_0 \Psi_s(x) = \psi_p(x), \quad \hat{P}_3 \Psi_s(x) = \psi_p(x),$$

$$\hat{J}_z \Psi_s(x) = J_z \psi_p(x), \quad J_z = \hbar (l_0 - l + 1/2), \quad \hat{S}_z \Psi_s = s\tilde{M} \Psi_s.$$

The spinors $\psi_{p,s}(x_{\perp})$ obey the equation

$$\left( \sigma \cdot \vec{P}_{\perp} + s\tilde{M} \sigma^3 \right) \psi_{p,s}(x_{\perp}) = p_0 \psi_{p,s}(x_{\perp}).$$

One can see that at fixed $s$ and $p^3$, equation (A.41) is similar to equation (18) in $(2 + 1)$ dim such that its solutions will be used in what follows.

In the nonrelativistic limit, the spin operator $\hat{S}_z$ is reduced to $\hat{S}_{z,NR}^0 = \gamma^0 \Sigma_1 M c^2$ and $\tilde{M} = \hat{M}$. Then for $s = +1$ equation (A.41) coincides with equation (18). We remark that $\psi_{p_0,1}(x_{\perp}) = \psi_{p_0,1}(x_{\perp})$. As a result, we obtain wavefunctions of nonrelativistic spinning particles from equation (A.39):

$$\Psi_{s,m,l,j}^{(j)NR}(x) = \exp \left[ -\frac{i}{\hbar} \left( \frac{(p_3)^2 t}{2\tilde{M} + p_3 z} \right) \right] \psi_{p,m,l}^{(j)NR}(t,\vec{r}),$$

$$\Psi_{s,m,l,j}^{(j)NR}(t,\vec{r}) = \begin{cases} \psi_{p,m,l}^{(j)NR}(t,\vec{r}) & s = \pm 1 \\ 0 & s = 0 \end{cases},$$

where spinors $\psi_{p,m,l}^{(j)up}$ and $\psi_{p,m,l}^{(j)down}$ are respectively solutions (A.16) and (A.18) of the Pauli equation in $(2 + 1)$ dim with the MSF. Thus, wavefunctions of nonrelativistic spinning particles (antiparticles) in $(3 + 1)$ dim obey the nonrelativistic Dirac equation with the Hamiltonian $\hat{H}_{NR}^0 = (\hat{P}_{\perp}^2 + \hat{P}_3^2)/2\tilde{M}$.

**Appendix B. Mean $(a_{l1}(t))^{(j,\pm)}$**

To study the mean $(a_{l1}(t))^{(j,\pm)}$, one has to calculate the matrix element $\langle \Psi_{s,m,l,j}^{(j)NR}(t,\vec{r}), \hat{a}_l^\dagger \Psi_{s,m,l,j}^{(j)NR}(t,\vec{r}) \rangle_D$. The latter can be reduced to a matrix element with respect to the initial ICS $\Psi_{s,m,l,j}^{(j)NR}(\vec{r}, \rho)$ as follows:

$$\left( \Psi_{s,m,l,j}^{(j)NR}(t,\vec{r}), \hat{a}_l^\dagger \Psi_{s,m,l,j}^{(j)NR}(t,\vec{r}) \right)_D = \left( \Psi_{s,m,l,j}^{(j)NR}(\vec{r}, \rho), \hat{a}_l^\dagger \Psi_{s,m,l,j}^{(j)NR}(\vec{r}, \rho) \right)_D,$$

where

$$\hat{a}_l^\dagger (t) \approx \exp \left[ \pm \frac{i\gamma^0}{\hbar} \hat{P}_l(t) \right] \hat{a}_l^\dagger \exp \left[ \pm \frac{i\gamma^0}{\hbar} \hat{P}_l(t) \right].$$

The operator $\hat{a}_l^\dagger (t)$ obeys the equation

$$\frac{d\hat{a}_l^\dagger (t)}{dt} = -\frac{i\gamma^0}{\hbar} [\hat{a}_l^\dagger (t), \hat{P}_l].$$

32
Let us consider the commutator $[\hat{a}_1^*(t), \hat{\Pi}_0]$. First, we write
\[
[\hat{a}_1^*(t), \hat{\Pi}_0] = \left[\hat{a}_1^*(t), \hat{\Pi}_0^{-1}\right] + \hat{\Pi}_0^2 [\hat{a}_1^*(t), \hat{\Pi}_0^{-1}].
\]
(\text{B.4})

where the identity $[\hat{a}_1^*(t), \hat{\Pi}_0] = \left[\hat{a}_1^*(t), \hat{\Pi}_0^{-1}\right]$ is used. Then, we represent the commutator $[\hat{a}_1^*(t), \hat{\Pi}_0^{-1}]$ as follows:
\[
[\hat{a}_1^*(t), \hat{\Pi}_0^{-1}] = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \left(\hat{a}_1^*(t) - \hat{b}_1^*(t^2)\right) e^{-\hat{\Pi}_0^{-1} t^2} \, dt.
\]
(\text{B.5})

The operator $\hat{b}_1^*(t^2)$ obeys the equation
\[
\frac{d\hat{b}_1^*(t^2)}{dt^2} = \left[\hat{b}_1^*(t^2), \hat{\Pi}_0\right] = -\frac{2\hbar q B}{c} \hat{b}_1^*(t^2)
\]
(\text{B.6})

and coincides with $\hat{a}_1^*(t)$ at $t^2 = 0$. Such a solution of equation (\text{B.6}) reads
\[
\hat{b}_1^*(t^2) = \hat{a}_1^*(t) \exp(-2\hbar q B |t^2/c).
\]
(\text{B.7})

Substituting (\text{B.7}) into (\text{B.5}) and calculating the integral, we obtain
\[
[\hat{a}_1^*(t), \hat{\Pi}_0^{-1}] = \hat{a}_1^*(t) \left[\hat{\Pi}_0^{-1} - \left(\hat{\Pi}_0^2 + 2\hbar q B /c \right)^{-1/2}\right].
\]

Using this result in (\text{B.4}), we find
\[
[\hat{a}_1^*(t), \hat{\Pi}_0] = \hat{a}_1^*(t) \left(\hat{\Pi}_0 - \sqrt{\hat{\Pi}_0^2 + 2\hbar q B /c}\right)
\]

such that equation (\text{B.3}) has the following solution:
\[
\hat{a}_1^*(t) = \hat{a}_1^* e^{\Omega(\hat{N}_1)t}, \quad \Omega(\hat{N}_1) = 2q B \left(\sqrt{\hat{\Pi}_0^2 + 2\hbar q B /c} + \hat{\Pi}_0\right)^{-1},
\]
(\text{B.8})

where $\Omega(\hat{N}_1)$ can be interpreted as the frequency operator. With an account taken of (\text{B.8}) in equations (\text{B.1}), and using relations (46) and (49), we finally obtain expression (95).

References

[1] Aharonov Y and Bohm D 1959 Phys. Rev. 115 485
[2] Olariu S and Popescu I I 1985 Rev. Mod. Phys. 47 339
[3] Aharonov Y and Bohm D 1959 The Aharonov–Bohm Effect (Berlin: Springer)
[4] Chen Y H, Wilczek F, Witten E and Halperin B I 1989 Phys. Rev. Lett. 62 1071
[5] Charlier J-Ch, Blase X and Roche S 2007 Rev. Mod. Phys. 79 677
[6] Hagen C R 1985 Phys. Rev. D 31 848
[7] Boz M, Fainberg V and Pak N K 1995 Phys. Lett. A 207 1
[8] Olaford M G, March-Russel J and Wilczek F 1989 Nucl. Phys. B 328 140
[9] Olaford M G and Wilczek F 1989 Phys. Rev. Lett. 62 1071
[10] de Sousa Gerbert P and Jackiw R 1989 Commun. Math. Phys. 124 229
[11] Bordag M and Kirsten K 1999 Phys. Rev. D 60 105019

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J. Phys. A: Math. Theor. 44 (2011) 055301
[27] Bagrov V G, Gavrilov S P, Gitman D M and Meira Filho D P 2008 Problems of Modern Theoretical Physics ed V Epp (Tomsk: Tomsk State University Press) p 57

[28] Gradshtein I S and Ryzhik I M 1994 Tables of Integrals, Series and Products (New York: Academic)

[29] Gitman D M, Shachmatov V M and Shvartsman Sh M 1975 Sov. Phys. J. E 8 43

[30] Davidov A S 1976 Quantum Mechanics 2nd edn (Oxford: Pergamon)