Calculation of Wilson loops in 2-dimensional Yang-Mills theories

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Abstract

The vacuum expectation value of the Wilson loop functional in pure Yang-Mills theory on an arbitrary two-dimensional orientable manifold is studied. We consider the calculation of this quantity for the abelian theory in the continuum case and for the arbitrary gauge group and arbitrary lattice action in the lattice case. A classification of topological sectors of the theory and the related classification of the principal fibre bundles over two-dimensional surfaces are given in terms of a cohomology group. The contribution of \(SU(2)\)-invariant connections to the vacuum expectation value of the Wilson loop variable is also analyzed and is shown to be similar to the contribution of monopoles.

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1 Introduction

Yang-Mills theory in two dimensions has been object of intensive studies during almost two decades. It is known that the classical theory is trivial and the quantum theory has no propagating degrees of freedom. However, this theory possesses many interesting properties when formulated on topologically nontrivial spaces \[1, 2\] and also in the large \(N\) limit \((N\) is related to the rank of the gauge group \(G = SU(N)\) \[3\]). A two-dimensional Yang-Mills theory is almost topological in the sense that, for example, for compact spacetime manifolds of surface area (two-dimensional volume) \(V\) its partition function depends only on \(e^2 V\), where \(e\) is the gauge coupling constant. Also, as we will see, the vacuum expectation value of a Wilson loop depends only on the area of the region surrounded by the loop and is independent of the points where the loop is located. This is a manifestation of the invariance under the area preserving diffeomorphisms. It is believed that two-dimensional Yang-Mills theories share some of the important qualitative features of four-dimensional ones, in particular the area law behaviour. It was shown recently that in the strong coupling limit they can be related to lower-dimensional strings \[4\] and that they play special role in M-theory \[5\]. All these have led to the revival of interest in studying two-dimensional Yang-Mills theories. Rich mathematical structures, appearing in such theories, were studied in a number of papers \[1, 2, 6\] (see lecture \[7\] for a review and references). The partition function and the vacuum expectation values of Wilson loop functionals were calculated by many authors with various techniques \[8\] - \[10\]. Physical aspects of these theories, in particular the analysis of Polyakov loops and the \(\theta\)-vacua, were discussed in Refs. \[11\].

In the present article we focus on the study of the vacuum expectation values \(\langle T_\gamma \rangle\) of the Wilson loop functionals \(T_\gamma\) in the pure Yang-Mills theories with arbitrary gauge group \(G\) on two-dimensional orientable compact manifolds \(M\). Such theories are non-trivial due to non-trivial topology of \(M\). In the classical theory the Wilson loop functionals \[12\] form a natural (over)complete set of gauge invariant variables. It is believed that their vacuum expectation values play similar role in the quantum theory. The area law behaviour of this quantity is considered as an indication of the regime of confinement in the corresponding gauge theory with quarks. There are various techniques for the calculation of \(\langle T_\gamma \rangle\) in two-dimensional pure gauge theories \[8\] - \[10\]. Here we carry out such calculation by using different techniques and paying attention to some important details. In the abelian case \(G = U(1)\) we calculate the vacuum expectation value of the Wilson variable by expanding the functional integral over the topological sectors of the theory and integrating over fluctuations around the monopole solution in each sector. The motivation is based on an analogy with quantum mechanics which we will also discuss. It will be shown that the functional integral for \(\langle T_\gamma \rangle\) is indeed saturated by the sum of the contributions of the topological sectors. For this expansion one needs a classification of the topological sectors of the theory or, equivalently, the classification of the principal fibre bundles over two-dimensional surfaces, and we will give a solution of this problem in terms of cohomology groups. Next, we perform the calculation of the vacuum expectation value of the Wilson variable for an arbitrary gauge group using a lattice technique. The novelty here is in
carrying out the integration on the lattice in terms of loop variables associated to the plaquettes and homotopically non-trivial loops and imposing the relation between the loops on the lattice by inserting a certain \( \delta \)-function. An alternative technique of the calculation in terms of differential forms on the lattice, that works for the abelian case is also discussed.

Besides discussing the methods of calculation of the vacuum expectation value \( \langle T_\gamma \rangle \) of the Wilson variable we also focus on the analysis of the contributions of monopole solutions and invariant gauge connections [13], [14] to this quantity. Our aim is to understand the role of such configurations in the formation of the area law dependence of \( \langle T_\gamma \rangle \). The class of invariant connections plays important role in certain problems of field theory, like, for example, in the coset space dimensional reduction in multidimensional gauge theories [15], [16]. Besides many known monopole and instanton solutions, it also includes families of connections parametrized by a finite number of parameters. The expectation is that though this class of connections is rather restricted, it may nevertheless capture some important properties of the complete theory. As concrete examples we analyze the invariant connections in the Yang-Mills theory with certain gauge groups on the two-dimensional sphere \( S^2 \) and calculate the contribution of the family of invariant connections in the case of \( G = SU(2) \). Our observation is that the quadratic area dependence of the contribution of the monopoles and of the invariant connections to \( \ln \langle T_\gamma \rangle \) is crucial for the area law dependence of the complete function in a wide range of the area variable.

The plan of the article is as follows. In Sect. 2 we will give basic definitions and discuss the functional integral in gauge theory in the case when the space of connections consists of many components (sectors). The result on the classification of principle fibre bundles is also presented. In Sect. 3 we will calculate the vacuum expectation value \( \langle T_\gamma \rangle \) of the Wilson loop variable in the abelian case for arbitrary \( M \). In Sect. 4 \( \langle T_\gamma \rangle \) will be calculated on the lattice for an arbitrary gauge group and arbitrary lattice action. In Sect. 5 we will study the invariant connections and their contribution to \( \langle T_\gamma \rangle \) in the case \( M = S^2 \) and \( G = SU(2) \). In Sect. 6 a discussion of the role of monopoles and invariant connections for the area law behaviour of \( \langle T_\gamma \rangle \) will be presented.

## 2 Wilson loop variables and integration over connections

Let us first set some basic notations and definitions. In the present article we consider pure gauge theories on a two-dimensional orientable Riemannian manifold \( M \) with a gauge group \( G \). \( M \) plays the role of the Euclidean space-time, and we will mainly be considering the case when it is compact. By \( G \) we will denote the Lie algebra of \( G \). We use the formalism of principal fibre bundles and connections in them (see, for example, [13]) to describe the gauge theory [17], [18]. Let \( P(M,G) \) be a principal fibre bundle over the manifold \( M \) with the structure group \( G \). A potential \( A_\mu \) of the gauge theory is characterized by a connection in \( P(M,G) \) via a local section in a standard way. Namely,
let $s$ be a local section over a neighbourhood $U$ of $M$. Then $A = (ie)^{-1}s^*w$, where $e$ is the gauge coupling constant and $s^*w$ is the pull-back of the connection form $w$ with respect to this section (we use for $A$ the normalization adopted in field theory). The $G$-valued 1-form $A$ determines the components of the local gauge potential $A_\mu$ on $M$ in the gauge corresponding to the section $s$ through the relation $A = A_\mu dx^\mu$. If $U_i$ and $U_j$ are two overlapping neighbourhoods, then at $x \in U_{ij} = U_i \cap U_j$ the gauge potential forms $A^{(i)}$ and $A^{(j)}$, defined for $U_i$ and $U_j$ respectively, satisfy certain consistency conditions. These conditions basically mean that $A^{(i)}$ and $A^{(j)}$ are gauge equivalent. Vice versa, a set of forms $A^{(i)}$, defined on neighbourhoods $U_i$ of an atlas $\{U_i, \psi_i\}$ of the manifold $M$ and satisfying the consistency conditions, define a unique connection form $w$ on $P(M,G)$ (see, for example, [17]). We denote by $\mathcal{A}$ the space of connections in $P(M,G)$.

Let us fix a point $x_0$ in $M$ and consider based loops defined in a standard way as continuous mappings of the unit interval $I = [0,1]$ into $M$:

$$\gamma : I \to M, \quad s \in I \to \gamma(s) \in M$$

with $\gamma(0) = \gamma(1) = x_0$. The space of loops on $M$ based at $x_0$ will be denoted as $\Omega(M,x_0)$.

Let us consider a loop $\gamma$ which, for the sake of simplicity, is situated inside one neighbourhood and associate with it the element of $G$ called holonomy:

$$H_\gamma(A) = \mathcal{P} \exp \left(i e \oint_\gamma A\right),$$

(1)

where $e$ is the gauge coupling constant, $\mathcal{P}$ means path ordering and $A$ is the gauge 1-form, defining the potential. We will be interested in traced holonomies

$$T_\gamma(A) = \frac{1}{d_R} Tr R(H_\gamma(A)) = \frac{1}{d_R} \chi_R(H_\gamma(A))$$

(2)

called also Wilson loop variables [12]. Here $R$ is an irreducible representation of $G$, $d_R$ is its dimension and $\chi_R$ is its character. Let us denote by $\mathcal{T}$ the group of local gauge transformations, i.e. the group of smooth vertical automorphisms of $P$. It is known that a set of elements $T_\gamma$, associated to loops $\gamma \in \Omega(M,x_0)$ and satisfying certain relations, called Mandelstam conditions, suffices to separate points in $\mathcal{A}/\mathcal{T}$, representing classes of gauge equivalent configurations, and therefore enable to reconstruct all smooth gauge connections up to gauge equivalence [19]. Due to this property the Wilson loop functionals (2) form a natural set of gauge invariant functions of connections in the classical theory.

We will study the vacuum expectation value $< T_\gamma >$ of the Wilson loop functional defined as

$$< T_\gamma > = \frac{1}{Z(0)} Z(\gamma),$$

(3)

$$Z(\gamma) = \int \mathcal{D}A e^{-S(A)} T_\gamma(A),$$

(4)

where the Yang-Mills action is given by

$$S(A) = \frac{1}{4} \int_M Tr(F \wedge *F) = \int_M \frac{1}{8} Tr(F_{\mu
u}F^{\mu\nu}) \sqrt{\det g_{\mu\nu}} d^2x.$$
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The expression (4) and the measure $DA$ are understood in the heuristic sense adopted in quantum field theory. It is known that in two-dimensional Yang-Mills theories the measure can be defined in a rigorous way [20] (see also [10]), we will return to this issue in Sect. 4. The integral in (4) is taken over the space of connections $\mathcal{A}$ (we will not make distinction between connection forms in $P(M,G)$ and gauge forms $A$ on $M$). In general $\mathcal{A}$ may consist of a number of components (or sectors) $\mathcal{A}_\alpha$, labelled by elements $\alpha$ of an index set $\mathcal{B}$, $\alpha \in \mathcal{B}$. The functional integral is represented as a sum over the elements of $\mathcal{B}$, each term of the sum being the functional integral over the connections in $\mathcal{A}_\alpha$. This feature has an analog in quantum mechanics which we are going to discuss first.

Consider the functional integral describing propagation of a particle on a manifold $M$ from a point $x_0$ to $x_1$ during the time $T$:

$$K(T; x_0, x_1) = \langle x_1 | e^{-TH} | x_0 \rangle = \frac{1}{Z} \int_{x(0)=x_0}^{x(T)=x_1} D\alpha(t) \exp \left\{ - \int_{0}^{T} L(x, \dot{x}) dt \right\}.$$ 

Here we integrate over all trajectories $x(t)$ (in general, not only over smooth ones) going from $x_0$ to $x_1$. Let us focus on the case when the initial point and the final point coincide:

$$K(T; x_0, x_0) = \frac{1}{Z} \int_{x(0)=x_0}^{x(T)=x_0} D\alpha(t) \exp \left\{ - \int_{0}^{T} L(x, \dot{x}) dt \right\},$$ (6)

so that the integral goes over closed paths in $M$, i.e. over elements of $\Omega(M, x_0)$. If $M$ is not simply connected, i.e. the first homotopy group $\pi_1(M, x_0) \neq 0$, then $\Omega(M, x_0)$ consists of more than one connected components. The set of path connected components of $\Omega(M, x_0)$ is in one-to-one correspondence with elements of $\pi_1(M, x_0)$ [21]. So, the functional integral (3) becomes

$$K(T; x_0, x_0) = \sum_{\alpha \in \pi_1(M, x_0)} K^{(\alpha)}(T; x_0, x_0)$$ (7)

$$= \sum_{\alpha \in \pi_1(M, x_0)} \frac{1}{Z} \int_{x^{(\alpha)}(0)=x_0}^{x^{(\alpha)}(T)=x_0} D\alpha^{(\alpha)}(t) \exp \left\{ - \int_{0}^{T} L(x^{(\alpha)}, \dot{x}^{(\alpha)}) dt \right\},$$

and each integral in this sum is taken over paths $x^{(\alpha)}$ belonging to a class corresponding to $\alpha \in \pi_1(M, x_0)$.

As an example let us consider the case $M = S^1$ and let $\varphi$ be the angle parameter of the circle, $0 \leq \varphi < 2\pi$. The fundamental group $\pi_1(S^1)$ is isomorphic to the abelian group of integer numbers, $\pi_1(S^1) \cong \mathbb{Z}$, therefore the path-connected components of $\Omega(S^1, 0)$ are labelled by integers $n$ called winding number. The component with a winding number $n$ consists of loops which go $n$ times around the circle $S^1$ (counterclockwise for $n > 0$ and clockwise for $n < 0$). Thus, in this case

$$K(T; 0, 0) = \sum_{n=-\infty}^{\infty} \frac{1}{Z} \int_{\varphi^{(n)}(0)=0}^{\varphi^{(n)}(T)=0} D\varphi^{(n)}(t) \exp \left\{ - \int_{0}^{T} L(\varphi^{(n)}, \dot{\varphi}^{(n)}) dt \right\},$$

(see, for example, Ref. [22] for a study of the quantum rotor withing such approach).
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An alternative but equivalent way of treating the integral (7) is to consider it not over the paths on the manifold $M$ but on the universal covering space $\tilde{M}$ of $M$ [23]. Such space is simply connected and exists for every connected manifold $M$. In this case we integrate over paths in $\tilde{M}$. We fix an initial point $\tilde{x}_{0}^{(0)}$ which projects to the point $x_{0} \in M$, $p(\tilde{x}_{0}^{(0)}) = x_{0}$, where $p$ is the covering space map. Then we sum contributions over all paths which end at points $\tilde{x}_{0}^{(\alpha)}$ of the set $p^{-1}(x_{0})$, since they all project to $x_{0}$. It is known that elements of $p^{-1}(x_{0})$ are in 1-1 correspondence with elements $\alpha \in \pi_{1}(M, x_{0})$, therefore

$$K(T; x_{0}, x_{0}) = \sum_{\alpha \in \pi_{1}(M, x_{0})} \frac{1}{Z} \int_{\tilde{x}(0) = \tilde{x}_{0}^{(\alpha)}} D\tilde{x}(t) \exp \left\{ - \int_{0}^{T} L(\tilde{x}, \dot{\tilde{x}}) dt \right\}.$$ 

In the case $M = S^{1}$ the universal covering space $\tilde{M} = \mathbb{R}^{1}$ with the coordinate $\tilde{\varphi}$, $-\infty < \tilde{\varphi} < \infty$, $p(\tilde{\varphi}) = \tilde{\varphi} \mod 2\pi$ and

$$K(T; 0, 0) = \sum_{n = -\infty}^{\infty} \frac{1}{Z} \int_{\tilde{\varphi}(0) = 0} D\tilde{\varphi}(t) \exp \left\{ - \int_{0}^{T} L(\tilde{\varphi}, \dot{\tilde{\varphi}}) dt \right\}.$$ 

In principle, to each partial amplitude some weights can be assigned:

$$K(T; x_{0}, x_{0}) = \sum_{\alpha \in \pi_{1}(M, x_{0})} \chi(\alpha) K^{(\alpha)}(T; x_{0}, x_{0})$$

(cf. (7)). In Ref. [24] an analysis of these weights was carried out and it was shown that this arbitrariness leads only to an irrelevant overall phase factor for the total amplitude.

Let us return to the gauge theory. In this case the set of connected components of $\mathcal{A}$ is in 1-1 correspondence with the set of non-equivalent (i.e. which cannot be mapped one into another by a bundle isomorphism) principal $G$-bundles $P(M, G)$ over manifold $M$. Let us denote this set as $\mathcal{B}_{G}(M)$, i.e. $\mathcal{B} \cong \mathcal{B}_{G}(M)$. The problem of characterization of this set and classification of such bundles is considered in a number of books and articles. A method, which in many cases gives a solution to this problem and which we closely follow here, is discussed in lectures [25]. Relevant results from algebraic topology can be found in [20]-[28]. Here we only outline the construction. We would like to note that in fact the considerations and the results in the rest of this section are valid for a more general case, namely when $M$ is a CW-complex and not just a smooth manifold.

It is known that for each structure group there exists a universal principal $G$-bundle $E_{G} = P(BG, G)$ with the base $BG$. The total space $E_{G}$ of the universal bundle is $\infty$-universal, i.e. $\pi_{q}(E_{G}) = 0$ for all $q \geq 1$. The important property is that every principal $G$-bundle over $M$ can be obtained (up to equivalence) from $E_{G}$ as the induced bundle, i.e. $P = f^{*}(E_{G})$ for some mapping $f : M \rightarrow BG$ [27]. One can prove that $\mathcal{B}_{G}(M)$ is isomorphic to the space of all homotopy equivalent classes of mappings $f : M \rightarrow BG$,

$$\mathcal{B}_{G}(M) \cong [M ; BG].$$

By examining a certain exact homotopy sequence it can be shown that the base $BG$ of the universal bundle satisfies the property [24], [27]

$$\pi_{q}(BG) = \pi_{q-1}(G), \quad q \geq 1. \quad (8)$$
The question is how one can construct such universal bundles and characterize \([M; BG]\) in terms of objects which can be calculated in a relatively easy way. It turns out that the Eilenberg - MacLane spaces play important role for this problem because of their special homotopic properties. Such spaces are often denoted as \(K(\pi, n)\), where \(\pi\) is a group and \(n\) is a positive integer, and are defined as follows:

i) \(K(\pi, n)\) is path connected;

ii) \(\pi_q(K(\pi, n)) = \begin{cases} \pi, & \text{if } q = n, \\ 0, & \text{if } q \neq n. \end{cases}\)

If \(n \geq 1\) and \(\pi\) is Abelian, the space \(K(\pi, n)\) exists as a CW-complex and can be constructed uniquely up to a homotopy equivalence \([26]\). The property, which is crucially important for us, is the following:

\[ [M; K(\pi, n)] \cong H^n(M, \pi), \]

where \(H^n(M, \pi)\) is the \(n\)th singular cohomology group \([26], [27], [29]\). A simple example is \(K(\mathbb{Z}, 1) = S^1\).

Often the Eilenberg-Maclane spaces are infinite dimensional. The example important for us is the space \(CP^\infty\) which is a CW-complex. It is understood as the union (direct limit) of the complex projective spaces \(CP^n\) of the sequence \(CP_1 \subset CP_2 \subset \ldots [26]\). Then \(\pi_q(CP^\infty) = \lim_{j \to \infty} \pi_q(CP_j)\) and

\[ \pi_2(CP^\infty) = \mathbb{Z}, \quad \pi_q(CP^\infty) = 0 \quad \text{for} \quad q \neq 2. \]

Thus, \(CP^\infty = K(\mathbb{Z}, 2)\).

This space serves for the classification of principal fiber bundles with \(G = U(1)\). Indeed, one can construct the sequence of Hopf fibrations \(S^{2n+1} \to CP_n\) with the fibre \(U(1)\). It can be shown that the space \(S^\infty\) is \(\infty\)-universal and the bundle \(S^\infty = P(CP^\infty, U(1))\) is the universal bundle. Therefore, \(EU(1) = S^\infty, BU(1) = CP^\infty\) and

\[ B_{U(1)}(M) \cong [M; BU(1)] = [M; K(Z, 2)] \cong H^2(M; \mathbb{Z}), \quad (9) \]

Ref. [23]. This relation gives the classification of principal \(U(1)\)-bundles \(P(M, U(1))\) over \(M\). Note that the result (1) is valid for \(M\) of any dimension. For the case when \(M\) is a smooth manifold it was discussed in [30].

Now let us consider the generalization of this construction for the case of other gauge groups. The idea is that for classification of bundles with the base \(M\) with \(\dim M \leq n\) only homotopy groups of low dimensions are important. Then instead of \(BG\) one can use some other space \(BG_n\) which is related to it and which may be easier to construct and to study. To define \(BG_n\) let us introduce the notion of \(p\)-equivalence. Consider two path connected spaces \(X\) and \(Y\) and a continuous map \(f : X \to Y\) such that for all \(x \in X\) the induced map \(f_* : \pi_q(X, x) \to \pi_q(Y, f(x))\) is an isomorphism for \(0 < q < p\) and an epimorphism for \(q = p\). Such map \(f\) is called \(p\)-equivalence. Then, if there exists some space \(BG_n\) and a map \(f : BG \to BG_n\) which is \((n + 1)\)-equivalence, it can be proved that

\[ [M; BG] \cong [M; BG_n] \]
for any CW-complex $M$ with $\dim M \leq n$. 

Let us suppose that the structure group (gauge group) is connected, so that $\pi_0(G) = 0$. Due to Eq. (8) $\pi_1(BG) = 0$. Then we take $BG_2 = K(\pi_1(G), 2)$. We have 

\[
\begin{align*}
\pi_1(BG_2) &= \pi_1(K(\pi_1(G), 2)) = 0 \cong \pi_1(BG), \\
\pi_2(BG_2) &= \pi_2(K(\pi_1(G), 2)) = \pi_1(G) \cong \pi_2(BG), \\
\pi_q(BG_2) &= \pi_q(K(\pi_1(G), 2)) = 0 \quad \text{for} \quad q > 2.
\end{align*}
\]

Moreover, 

\[
[BG; BG_2] = [BG; K(\pi_1(G), 2)] \cong H^2(BG, \pi_1(G))
\]

and this assures the existence of a mapping $f : BG \to K(\pi_1(G), 2)$ which is 3-equivalence [26]. Thus, if $\dim M = 2$

\[
B_G(M) \cong [M; BG] \cong [M; BG_2] = [M; K(\pi_1(G), 2)] \cong H^2(M, \pi_1(G)). \tag{10}
\]

This gives us the classification of principal fibre bundles $P(M, G)$ over two-dimensional manifolds CW-complexes with the structure group $G$.

An equivalent classification of principal fibre bundles over a two-dimensional surface in terms of elements of the group $\Gamma$, specifying the global structure of $G$ through the relation $G = \tilde{G}/\Gamma$, where $\tilde{G}$ is the universal covering group of $G$, was given in Ref. [2].

For completeness, we present here a list of the first homotopy groups $\pi_1(G)$ for some Lie groups which are of interest in gauge theories:

1. Simply connected, $G = SU(n), Sp(n)$: $\pi_1(G) = 0$.
2. $G = SO(n)$, $n = 3$ and $n \geq 5$: $\pi_1(G) = \mathbb{Z}_2$.
3. $G = U(n)$: $\pi_1(G) = \mathbb{Z}$.

For $G = U(n)$ and $n = 1$ the gauge group is abelian and result (10) becomes the relation (4) discussed above.

### 3 Calculation of the Wilson loop for $G = U(1)$, continuous case

In this section we calculate the vacuum expectation value (3) of the Wilson loop variable in the abelian case. Since the surface $M$ has non-trivial topology the result is non-trivial. The functional integral will be evaluated by summing over the monopoles and integrating over the fluctuations around each monopole. The result will be shown to be exact by comparing with the result of the calculation of $< T_\gamma >$ on the lattice in Sect. 4. This means that the functional integral (4) is saturated by quasiclassical contributions.

As it was shown in the previous section, $U(1)$-bundles are classified by elements of the second cohomology group $H^2(M, \mathbb{Z})$. The class $c_1(P) \in H^2(M, \mathbb{Z})$, corresponding to
the bundle $P$, is known as the first (integer) Chern class \cite{13, 26, 28}. For a closed orientable 2-dimensional manifold $M \ H^2(M, Z) \cong Z$. For the bundle $P_n$, characterized by the label $n \in B \cong \mathbb{Z}$, the integer cohomology class can be represented by $eF(n)/(2\pi)$ with $F(n)$ being the curvature 2-form defined locally through $F(n) = dA(n)$, where $A(n)$ is the gauge 1-form given by a connection in $P_n$. The integral
\[
\frac{e}{2\pi} \int_M F(n) = n, \tag{11}
\]
does not depend on the choice of the connection and gives the first Chern number which is also called topological charge.

Let us turn to the calculation of the functional integral (4). We will consider the case of one simple loop $\gamma$ which is the boundary of a region $\sigma$, i.e. $\gamma = \partial \sigma$. So, we consider the case of homologically trivial loop. Generalization to the case of multiple loops is straightforward though rather cumbersome, see Ref. \cite{10}. Let us assume that the Wilson loop variable (2) is defined for the irreducible representation of $U(1)$ labelled by $\nu \in \mathbb{Z}$, i.e.
\[
T_{\gamma}(A) = \exp \left\{ i\nu \oint_{\gamma} A \right\}. \tag{12}
\]
Using the quantum mechanics analogy, discussed in the previous section, and the fact that $H^2(M; Z) \cong \mathbb{Z}$ we write
\[
Z(\gamma) = \sum_{n=-\infty}^{\infty} \int D A(n) e^{-S(A(n))} T_{\gamma}(A(n)) = \sum_{n=-\infty}^{\infty} Z^{(n)}(\gamma), \tag{13}
\]
where in each term $Z^{(n)}$ the integration goes over gauge potentials with the Chern number equal to $n$, i.e. over potentials given by connections in the fibre bundle $P_n$. We represent $A^{(n)} = \tilde{A}^{(n)} + a$, where $\tilde{A}^{(n)}$ is a potential with
\[
\int_M \tilde{F}^{(n)} = \frac{2\pi n}{e},
\]
$\tilde{F}^{(n)} = d\tilde{A}^{(n)}$ and $a$ is a 1-form with the zero Chern number. Thus the 2-form $F^{(n)}$ is equal to $F^{(n)} = \tilde{F}^{(n)} + f$, where $f = da$ globally, and satisfies Eq. (11). As $\tilde{A}^{(n)}$ we take instanton configurations, i.e. solutions of the equation of motion. In the literature they are often referred to as monopoles, and we will follow this terminology in the present paper.

Assume that the manifold $M$ is endowed with a Riemannian metric $g_{ij}$. Let us denote by $(\alpha, \beta)$ the scalar product of forms defined in the standard way:
\[
(\alpha, \beta) = \int_M \alpha \wedge * \beta \tag{14}
\]
for two $p$-forms $\alpha$ and $\beta$.

For the action (10) in the abelian case the equation of motion is $\delta F = 0$, where $\delta$ is the adjoint of the exterior derivative defined by $(\alpha, d\beta) = (\delta \alpha, \beta)$. Since any 2-form on
the two-dimensional manifold $M$ is proportional to the volume 2-form $\mu$, which locally is equal to
\[ \mu := \sqrt{\det g_{\mu \nu} dx^1 \wedge dx^2}, \] (15)
the 2-form $\tilde{F}^{(n)}$ can be written as
\[ \tilde{F}^{(n)} = \kappa^{(n)}(x) \mu. \] (16)
The equation of motion requires that
\[ \delta \tilde{F}^{(n)} = - * d * (\kappa^{(n)}(x) dv) = * d \kappa^{(n)} = 0, \]
i.e. $\kappa^{(n)}$ is constant. Then condition (11) gives
\[ \kappa^{(n)} = \frac{2\pi n}{V e}, \] (17)
where $V$ is the volume (area) of $M$.
The term $Z^{(n)}$ in Eq. (13) equals
\[ Z^{(n)}(\gamma) = \int_{\mathcal{A}(0)} Dae^{-S(\tilde{A}^{(n)} + a)} T^{(n)}(\tilde{A}^{(n)} + a). \]
Here the functional integration goes over connections with zero topological charge on the trivial bundle $P_0$. The abelian action can be written as
\[ S(\tilde{A}^{(n)} + a) = \frac{1}{2}(F, F) = \frac{1}{2}(\tilde{F}^{(n)} + f, \tilde{F}^{(n)} + f) \]
\[ = \frac{1}{2}(\tilde{F}^{(n)}, \tilde{F}^{(n)}) + (\tilde{F}^{(n)}, f) + \frac{1}{2}(f, f) \]
\[ = S(\tilde{A}^{(n)}) + S(a) \]
since the term linear in $f$ vanishes (recall that $f = da$ globally). In accordance with definition (12)
\[ T^{(n)}(\tilde{A}^{(n)} + a) = \exp \left\{ i \nu e \oint_\gamma (\tilde{A}^{(n)} + a) \right\} = \exp \left\{ i \nu e \oint_\gamma \tilde{A}^{(n)} \right\} \exp \left\{ i \nu e \oint_\gamma a \right\}, \]
and the expression for the vacuum expectation value of the Wilson loop variable factorizes:
\[ Z(\gamma) = \left[ \sum_{n = -\infty}^{\infty} Z^{(n)}_{\text{mon}}(\gamma) \right] Z_0(\gamma), \] (18)
\[ Z^{(n)}_{\text{mon}} = e^{-S(\tilde{A}^{(n)})} T^{(n)}(\tilde{A}^{(n)}), \] (19)
\[ Z_0(\gamma) = \int_{\mathcal{A}(0)} Da e^{-S(a)} T^{(n)}(a). \] (20)
Let us calculate first the monopole contribution (19). It turns out that for this we do not need explicit expressions for the monopole solutions $\tilde{A}^{(n)}$. Indeed, using Eqs. (16) and (17) one readily obtains that
\[
S(\tilde{A}^{(n)}) = \frac{1}{2} \int_M \tilde{F}^{(n)} \wedge \ast \tilde{F}^{(n)} = \frac{1}{2} \int_M \kappa^{(n)} \mu \wedge \ast \kappa^{(n)} \mu = \frac{1}{2} \left( \frac{2\pi n}{eV} \right)^2 V = \frac{2\pi^2 n^2}{e^2V}.
\]
If $\gamma = \partial \sigma$, where $\sigma$ is the interior of $\gamma$, and $\tilde{A}^{(n)}$ is regular in $\sigma$, then by using Stoke’s theorem we obtain that
\[
\oint_{\gamma} \tilde{A}^{(n)} = \oint_{\partial \sigma} \tilde{A}^{(n)} = \int_{\sigma} d\tilde{A}^{(n)} = \int_{\sigma} \tilde{F}^{(n)} = \kappa^{(n)} \int_{S} \mu = \frac{2\pi n}{eV} S,
\]
where we denoted the area of the region $\sigma$ by $S$. We will see that in fact for closed $M$ the final answer does not depend on whether we take the interior of the closed path $\gamma$ (with the area $S$) or its exterior (with the area $V - S$). The contribution of the monopoles to the function (18) is equal to
\[
Z_{\text{mon}}(\gamma) := \sum_{n = -\infty}^{\infty} Z^{(n)}(\gamma) = \sum_{n = -\infty}^{\infty} \exp \left( -\frac{2\pi^2 n^2}{e^2V} + i \frac{2\pi n}{V} \nu S \right) \quad (21)
\]
For the purpose of comparison with the result of the lattice calculation and in order to have an expression with clearer symmetry properties we present Eq. (21) in another form by using the Poisson summation formula
\[
\sum_{n = -\infty}^{\infty} f(n) = \sum_{l = -\infty}^{\infty} \int_{-\infty}^{\infty} dz f(z) e^{2\pi ilz} \quad (22)
\]
with
\[
f(z) = \exp \left( -\frac{2\pi^2 z^2}{e^2V} + i \frac{2\pi z}{V} \nu S \right).
\]
By a straightforward calculation we obtain the following result
\[
Z_{\text{mon}}(\gamma) = \sum_{l = -\infty}^{\infty} \int_{-\infty}^{\infty} dz \exp \left( -\frac{2\pi^2 z^2}{e^2V} + 2\pi i \frac{S}{V} \nu z + 2\pi ilz \right)
\]
\[
= e^{\sqrt{\frac{V}{2\pi}}} \sum_{l = -\infty}^{\infty} \exp \left[ -\frac{e^2V}{2} \left( \frac{S}{V} \nu + l \right)^2 \right] \quad (23)
\]
We see that $Z_{\text{mon}}$ is a function of $e^2V$ and $S/V$. Now it is clear that it is invariant under the transformation $S \rightarrow (V - S)$.

As an illustration of this general calculation let us consider a concrete example of $M = S^2$ with $A^{(n)}$ being the Dirac monopoles. Consider a standard description of the
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principal \(U(1)\)-bundles \(P(S^2, U(1))\) (see, for example, [18], [21]). We cover the sphere \(S^2\), parametrized by angles \(\vartheta\) and \(\varphi\) \((0 \leq \vartheta \leq \pi, 0 \leq \varphi < 2\pi)\), with two charts, \(U_1\) and \(U_2\), chosen as \(U_1 = S^2 - \{S\}\) (the sphere without the southern pole) and \(U_1 = S^2 - \{N\}\) (the sphere without the northern pole):

\[
U_1 = \{0 \leq \vartheta < \pi, 0 \leq \varphi < 2\pi\},
\]

\[
U_2 = \{0 < \vartheta \leq \pi, 0 \leq \varphi < 2\pi\}.
\]

The fiber \(F = U(1) \cong S^1\) is parametrized by \(e^{i\psi}\). Thus

\[
\pi^{-1}(U_1) = U_1 \times U(1) = \{\vartheta, \varphi, \lambda_+(\vartheta, \varphi) = e^{i\psi_1}\},
\]

\[
\pi^{-1}(U_2) = U_2 \times U(1) = \{\vartheta, \varphi, \lambda_-(\vartheta, \varphi) = e^{i\psi_2}\},
\]

where \(\pi\) is the canonical projection. The bundle is characterized by the transition function \(\lambda_{12}(\vartheta, \varphi) = \lambda_2(\vartheta, \varphi)\lambda_1^{-1}(\vartheta, \varphi)\) which is a mapping from \(U_{12} = U_1 \cap U_2\) to the gauge group \(G = U(1)\). In our case it is enough to consider the transition function only on the equator, i.e. the mapping \(\tilde{\lambda}_{12} : S^1 \to U(1) \cong S^1\). Such mappings are labelled by integers \(n \in \mathbb{Z}\) and are of the form \(\tilde{\lambda}_{12}^{(n)}(\varphi) = e^{in\varphi}\). This implies that the principal bundles \(P(S^2, U(1))\) are labelled by integers \(n\) (in accordance with the general result (3)), and \(\psi_2 = \psi_1 + n\varphi\).

Connections in the bundles are given by

\[
\omega = -\frac{i}{2\pi}d\psi_2 + ieA_\mu^{(1)}dx^\mu \quad \text{in} \quad \pi^{-1}(U_1),
\]

\[
\omega = -\frac{i}{2\pi}d\psi_2 + ieA_\mu^{(2)}dx^\mu \quad \text{in} \quad \pi^{-1}(U_2),
\]

and \(A_\vartheta = A_\varphi^2, A_\varphi = A_\varphi^2 - n/(2\pi)\). The monopole solution is described by

\[
\tilde{A}_n^{(1)} = \frac{n}{2e}(1 - \cos \vartheta)d\varphi, \quad \tilde{A}_n^{(2)} = -\frac{n}{2e}(1 + \cos \vartheta)d\varphi
\]

and

\[
\tilde{F}_n = d\tilde{A}_n^{(1)} = d\tilde{A}_n^{(2)} = \frac{n}{2e} \sin \vartheta d\vartheta \wedge d\varphi = \frac{2\pi n}{Ve}\mu
\]

in accordance with Eq. (17). In Eqs. (26), (27) the lower index \(n\) of the forms \(\tilde{A}_n^{(i)}\) and \(\tilde{F}_n\) is the label of the fibre bundle in which the corresponding connection is defined.

Now let us turn to the calculation of the contribution \(Z_0(\gamma)\) given by Eq. (20). The action

\[
S(a) = \frac{1}{2} \int_M f \wedge \ast f = \frac{1}{2} \int_M (da, da) = \frac{1}{2} \int_M (a, \delta da).
\]

For further discussion we need to consider de Rham currents [32] or weak forms, according to the terminology of Ref. [33]. Let us denote by \(W^p(M)\) the space of weak \(p\)-forms on \(M\) and remind the definition. Generalizing the notion of distribution a weak \(p\)-form \(K\) is a linear functional on the space \(D^p(M)\), i.e. \(K : D^p(M) \to R\), where \(D^p(M)\) is the space of test \(p\)-forms with compact support. The space \(W^p(M)\) includes functionals corresponding
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to $C^\infty$ $p$-forms, and we will denote the space of such forms by $\Lambda^p(M)$. If $w \in \Lambda^p(M)$, then the corresponding functional is defined as

$$w[u] := (w, u) \equiv \int_M w \wedge u,$$

$u \in D^p(M)$. Because of this often the notation $(K, u)$ for $K[u]$ is used, where $K \in W^p(M)$. Some basic operations on forms are extended to weak forms in an obvious way

$$dK[u] = K[\delta u], \quad \delta K[u] = K[du], \quad \Delta K[u] = K[\Delta u],$$

etc., where $\Delta = d\delta + \delta d$ is the Laplacian. If $M$ is compact, which is the case under consideration in this section, $D^p(M) = \Lambda^p(M)$.

For our calculation we introduce the weak 1-form $J_\gamma$, characterizing the loop $\gamma$ in the following way:

$$J_\gamma[u] = \oint_\gamma u,$$

$u \in \Lambda^1(M)$. Since $\oint_\gamma u = \oint_\gamma u_{\nu} dx^\nu$, formally we can write

$$J_\gamma[u] = \int_M J_\gamma \wedge u = \int_M u_{\nu}(x) J_\gamma^\nu(x) \mu,$$

where $\mu$ is the volume 2-form and the contravariant components $J_\gamma^\nu$ are given by

$$J_\gamma^\nu(y) = \int_\gamma dx^\nu \frac{1}{\sqrt{\det g_{\mu\nu}}} \delta(y - x).$$

We also introduce the weak 2-form $J_\sigma$ which characterizes the surface $\sigma$, the interior of the closed path $\gamma$, as follows:

$$J_\sigma[v] = \int_\sigma v,$$

$v \in \Lambda^2(M)$. Since $\gamma = \partial \sigma$, for $u \in \Lambda^1(M)$ we have:

$$J_\sigma[du] = \int_\sigma du = \oint_{\partial \sigma} u = \oint_\gamma u = J_\gamma[u].$$

From this relation and Eqs. (28) we conclude that $\delta J_\sigma[u] = J_\sigma[du] = J_\gamma[u]$, i.e.

$$J_\gamma = \delta J_\sigma.$$ (31)

It is easy to see that formally

$$J_\sigma = h_\sigma(x) \mu,$$ (32)

where $h_\sigma(x)$ is the characteristic function of the region $\sigma$ given by

$$h_\sigma(x) = \begin{cases} 
1, & x \in \sigma, \\
0, & x \notin \sigma. 
\end{cases}$$ (33)
Note that in fact $J_\sigma$ belongs to the space $\mathcal{H}^2(M)$ of quadratically integrable 2-forms, i.e. 2-forms for which the norm $||J_\sigma||$, defined by (14), is finite.

The contribution of fluctuations can be written as

$$Z_0(\gamma) = \int_{A^{(0)}} D a \exp \left[ -\frac{1}{2} (a, \delta a) + i\nu e (J_\gamma, a) \right].$$

(34)

The operator $\delta d$ is not invertible on the space of gauge 1-forms $a$. First of all there are zero modes due to the gauge invariance of the theory, that is the symmetry under gauge transformations $a \rightarrow a' = a + d\xi$. One deals with this problem in the standard way by inserting the Faddeev-Popov $\delta$-function, which gives rise to the gauge-fixing term, and integrating over the gauge group [34]. Here we consider the covariant gauges, so that the additional factor equal to $\exp \left[ -\frac{1}{2\alpha} (\delta a, \delta a) \right]$, where $\alpha$ is the gauge fixing parameter, appears in Eq. (34). With this addition the quadratic operator does not have gauge zero modes anymore. We choose $\alpha = 1$, the Feynman gauge. As a result Eq.(34) becomes

$$Z_0(\gamma) = \int_{A^{(0)}} D a \exp \left[ -\frac{1}{2} (a, \Delta a) + i\nu e (J_\gamma, a) \right],$$

(35)

where $\Delta$ is the Laplacian. Let us remind that since we are considering the abelian case the Faddeev-Popov determinant gives an irrelevant constant factor in (35).

The Laplacian $\Delta$ may have further zero modes related to symmetries of the manifold $M$. The zero modes $\alpha_0$ are harmonic 1-forms on $M$, i.e. they satisfy $\Delta \alpha_0 = 0$. We will denote the space of harmonic $p$-forms on $M$ as $\text{Harm}^p(M)$. For example, if $M$ is the two-dimensional torus $T^2$, parametrized by the angles $(\phi_1, \phi_2)$ ($0 \leq \phi^i < 2\pi$), there are two independent zero modes $\alpha_{01} = d\phi_1/(2\pi)$ and $\alpha_{02} = d\phi_2/(2\pi)$. Any 1-form $\alpha = c_1\alpha_{01} + c_2\alpha_{02}$, where $c_1$ and $c_2$ are constants, is harmonic. It cannot be eliminated by gauge transformation $\alpha \rightarrow \alpha + d\xi$ because, as it is easy to see using, for example, the Fourier expansion, such function $\xi$ cannot be defined on the torus.

The existence of such zero modes is related to the invariance of the action $S(a)$ with respect to the transformation $a \rightarrow a' = U_c a := a + c_0 \alpha_0$, where $\alpha_0$ are zero modes. We assume that they are normalized by $(\alpha_{0i}, \alpha_{0j}) = \delta_{ij}$. According to the Hodge decomposition theorem (see, for example, [35])

$$\mathcal{H}^p(M) = \text{Harm}^p(M) \oplus (\text{Harm}^p)_\perp,$$

where $\mathcal{H}^p(M)$ denotes the space of quadratically integrable $p$-forms and $(\text{Harm}^p)_\perp$ is the orthogonal compliment of $\text{Harm}^p(M)$ with respect to the scalar product (14). Let us insert the unit

$$1 = \int \prod_i dc_i \prod_i \delta (\alpha_{0i}, U_c a).$$

After standard and simple manipulations we arrive to the expression

$$Z_0(\gamma) = \int \prod_i dc_i \int_{A^{(0)}} D a \exp \left[ -\frac{1}{2} (a, \Delta a) + i\nu e (J_\gamma, a) \right] \prod_i \delta ((\alpha_{0i}, a)),$$
Here we used the property of orthogonality

\[(\alpha_{0i}, J_{\gamma}) = (\alpha_{0i}, \delta J_{\sigma}) = (d\alpha_{0i}, J_{\sigma}) = 0,\]

following from (31), and the fact that harmonic forms are closed and co-closed.

On the space \((\text{Harm}^p)_{\perp}\) the Laplacian \(\Delta\) is invertible. Thus one can define the Green operator \(\tilde{\Delta}^{-1} : \Lambda^p(M) \to (\text{Harm}^p)_{\perp}\) as

\[\tilde{\Delta}^{-1} = \left(\Delta|_{(\text{Harm}^p)_{\perp}}\right)^{-1} \circ \Pi_p,\]

where \(\Pi_p\) is the projector of \(\mathcal{H}^p(M)\) onto \((\text{Harm}^p)_{\perp}\) \([35]\). In particular, \(\Pi_1 = 1 - \sum_i \alpha_{0i}\alpha_{0i}^*\), where

\[(\alpha_{0i}\alpha_{0i}^*) (\beta) = \alpha_{0i} \int \alpha_{0i} \wedge *\beta.\]  

(37)

The Green operator satisfies

\[\Delta \tilde{\Delta}^{-1} = \tilde{\Delta}^{-1} \Delta = \Pi_1 \equiv 1 - \sum_i \alpha_{0i}\alpha_{0i}^*.\]

Using this operator one can perform formally the Gaussian functional integration in (36) and obtain the naive expression

\[Z_0(\gamma) = \mathcal{N} \exp \left[ -\frac{\nu^2}{2} \epsilon^2 \Gamma(\gamma) \right],\]

\[\Gamma(\gamma) = (J_{\gamma}, (J_{\gamma})^{-1} J_{\gamma}),\]  

(38)

where constant \(\mathcal{N}\) accumulates all factors independent of \(\gamma\), they are irrelevant for our result. From (29) it may seem that the inner product of the forms in (38) is singular. We will show that in fact a well defined value can be assigned to it. For this let us consider the following regularized version of Eq. (38):

\[\Gamma(\gamma) = \lim_{\epsilon \to 0} (J_{\gamma}, (J_{\gamma})^{-1} J_{\gamma(\epsilon)}),\]  

(39)

where the closed path \(\gamma(\epsilon)\) is some small deformation of the closed path \(\gamma\) which does not have common points with \(\gamma\) and \(\gamma(0) = \gamma\) (we assume that the manifold \(M\) and the path \(\gamma\) are such that such deformation is possible). The product of two weak forms is not well defined in general. However, we will see that in our case Eq. (39) makes sense due to the facts that \(J_{\gamma} = \delta J_{\sigma}\), Eq. (31), and \(J_{\sigma} \in \mathcal{H}^2(M)\). We understand (35) as

\[\Gamma(\gamma) = \lim_{\epsilon \to 0} (J_{\sigma}, d\tilde{\Delta}^{-1} \delta J_{\sigma(\epsilon)}),\]  

(40)

with \(\gamma(\epsilon) = \partial \sigma(\epsilon)\).

Now let us study the operator \(d\tilde{\Delta}^{-1} \delta\) and prove that it is equal to the projector \(\Pi_2 \equiv 1 - w_{0i}w_{0i}^*\) on \(L^2(M)\), where \(w_{0i}\) are harmonic 2-forms and \(w_{0i}^*\) is the functional corresponding to \(w_{0i}\) in the sense explained above (see Eq. (37)). Indeed, since the Green operator commutes with all linear operators which commute with the laplacian \(\Delta\), in
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particular \( \delta \tilde{\Delta}^{-1} = \tilde{\Delta}^{-1} \delta \) and \( d\tilde{\Delta}^{-1} = \tilde{\Delta}^{-1} d \), the following chain of operator relations holds on \( \Lambda^2(M) \):

\[
d\tilde{\Delta}^{-1} \delta = d\delta \tilde{\Delta}^{-1} = (\Delta - \delta d) \tilde{\Delta}^{-1}
\]

\[
= \Pi_2 - \delta \tilde{\Delta}^{-1} d = \Pi_2.
\]

The last equality follows from the fact that the operator \( d \) acts trivially on \( \Lambda^2(M) \) (recall that \( \text{dim} M = 2 \)). The property (41) shows that the operator \( d\tilde{\Delta}^{-1} \delta \) is smooth enough and its action can be extended to \( H^2(M) \).

Let us return to the calculation of \( \Gamma(\gamma) \) in Eq. (40). Using Eq. (41) we obtain that

\[
\Gamma(\gamma) = \lim_{\epsilon \to 0} (J_\sigma, \Pi_2 J_{\sigma(\epsilon)}) = (J_\sigma, (1 - w_0^i w_0^i) J_\sigma)
\]

\[
= (J_\sigma, J_\sigma) - ||(J_\sigma, w_0^i)||^2.
\]

(42)

The form \( \tilde{w}_0 = \mu/\sqrt{V} \), where \( \mu \) is the volume 2-form (15), is the only (properly normalized) zero mode (harmonic) form in \( \Lambda^2(M) \). Indeed, \( d\tilde{w}_0 = 0 \) and using the properties of the adjoint of the exterior derivative and of the Hodge * operation we verify that

\[
\delta \tilde{w}_0 = *d* \tilde{w}_0 = *d \frac{1}{\sqrt{V}} = 0.
\]

This is in accordance with the fact that the space of harmonic \( p \)-forms \( Harm^p(M) \) is isomorphic to the cohomology group \( H^p(M; R) \) [35]. In the case under consideration \( p = 2, \text{dim} M = 2 \),

\[
Harm^2(M; R) \cong H^2(M; R) \cong R,
\]

\( \text{dim Harm}^2(M; R) = 1 \), and, thus, all harmonic 2-forms are proportional to \( \tilde{w}_0 \).

Using definition (31) we write

\[
(J_\sigma, \tilde{w}_0) = \int_\sigma \tilde{w}_0 = \frac{1}{\sqrt{V}} \int_\sigma \mu = \frac{S}{\sqrt{V}}.
\]

(43)

With the help of Eqs. (32), (33) we calculate the first term in (42):

\[
(J_\sigma, J_\sigma) = \int_M h_\sigma(x) \mu \wedge *h_\sigma(x) \mu = \int_M h_\sigma^2(x) \mu = \int_\sigma \mu = S.
\]

(44)

Combining formulas (38), (42) and results (33), (44) we obtain the final expression for the contribution of fluctuations around the monopoles:

\[
Z_0(\gamma) = \mathcal{N} e^{-\frac{e^2}{2} \nu^2 \frac{S(V - S)}{V}}.
\]

(45)

Note that this expression is invariant under the transformation \( S \to (V - S) \), the same as \( Z_{\text{mon}}(\gamma) \). Using (23) and (45) we obtain the result

\[
Z(\gamma) = Z_{\text{mon}}(\gamma) Z_0(\gamma) = \mathcal{N}' \sum_{l=-\infty}^{\infty} \exp \left[ -\frac{e^2}{2V} \left( \frac{S}{V} \nu + l \right)^2 - \frac{e^2}{2} \nu^2 \frac{S(V - S)}{V} \right].
\]
The final expression for the expectation value of the Wilson loop variable for a homologically trivial loop $\gamma$ for the gauge group $G = U(1)$ is
\[
<T_{\gamma}> = \frac{\sum_{l=-\infty}^{\infty} \exp \left[ -\frac{e^2}{2} V \left( \frac{2}{V} l + \frac{1}{V} \nu \right)^2 - \frac{e^2}{2} \frac{S(V-S)}{V} \right]}{\sum_{l=-\infty}^{\infty} \exp \left[ -\frac{e^2}{2} V l^2 \right]}.
\] (46)

It can be shown that for homologically non-trivial loops $<T_{\gamma}> = 0$ (see also [10]). Some discussion of the properties of the expression (46) will be given at the end of Sect. 4 where it appears as a particular case of the general formula for an arbitrary gauge group.

Let us analyze the quantity
\[
E(e^2V, S/V) := -\ln <T_{\gamma}>
\]
which in a theory with fermions characterizes the potential of the interaction. $E(e^2V, S/V)$ can be read off from Eq. (46) and is plotted in Fig. 1 as a function of $S/V$ for two values of $e^2V$ (for $\nu = 1$).

In the same plot we also show the contribution $E_{\text{mon}}(e^2V, S/V)$ of the abelian monopoles calculated from Eq. (21). We see from Fig. 1 (also from the exact formula (23)) that the dependence of $E_{\text{mon}}(e^2V, S/V)$ on $S/V$ at small $S/V$ is quadratic almost till $S/V = 0.5$, where the curve reaches its maximum. This, being combined with the contribution of the fluctuations, gives the linear dependence (the area law) almost for all $S/V$ in the interval $0 < S/V < 0.5$. The area law (linear dependence on $S/V$) for the Wilson loop in the pure Yang-Mills theory is considered as an indication of the regime of confinement in the corresponding Yang-Mills gauge theory with quarks. Thus the quadratic behaviour of the monopole contributions $E_{\text{mon}}(e^2V, S/V)$ seems to be an important feature which gives rise to the linear dependence of the complete function.

In Sect. 5 we will study the analogous function for the contribution of invariant connections.

4 **Lattice calculation of the Wilson loop for arbitrary gauge group**

In this section we study a general pure gauge theory (5) on a two-dimensional orientable compact manifold $M$. We will calculate the functional integral (4) again but now for the case when the gauge group is an arbitrary compact Lie group. As before we consider the case of a single simple loop $\gamma$. We assume that $\gamma$ divides $M$ into two regions $\sigma_1, \sigma_2$ of genera $r_1, r_2$ and areas $S_1, S_2$ respectively. The genus of $M$ is $r = r_1 + r_2$ and its area $V = S_1 + S_2$ (see an example in Fig. 2).

Let us consider lattices on the manifold $M$ [30]. A lattice can be viewed as a CW-complex $\Lambda = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$, where $\Lambda_0, \Lambda_1$ and $\Lambda_2$ are finite sets whose elements are 0-cells $s$ (sites), 1-cells $l$ (links) and 2-cells $p$ (plaquettes) respectively. For the purpose of computation of the expectation value of $T_{\gamma}$, we consider only lattices compatible with $\gamma$, i.e. lattices such that $\gamma$ is an 1-chain on them. The incidence functions $I(s, l), I(l, p)$ for pairs of cells of correlative dimension take values 0, 1 or −1 if the first cell is not a face, is a positive face or is a negative face of the second cell respectively. The topology of a
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continuum surface, modelled by the lattice, is encoded in the discrete structure given by $\Lambda$ and $I$. The area of the plaquette $p$ is denoted by $|p|$. The “lattice spacing” $a$ is defined as the minimum length such that every plaquette is confined in a circle of the radius $a$.

To calculate the functional integral (4) we will need a system of independent loops of the lattice $\Lambda$. For this we fix a site $s_0$ of $\Lambda_0$, and for each plaquette $p$ assign a connecting path $c_p$ from $s_0$ to $p$ and define the loop $\gamma_p = c_p \partial p c_p^{-1}$. In addition we choose a set of homotopically non-trivial loops $a_i, b_i i = 1, \ldots, r$ which play the role of the generators of the first homotopy group of $M$ [29], [31], [37]. The index $i$ labels the handles of $M$ (see Fig. 3 for an illustration).

At this point it is useful to think of $M$ as being represented by a polygon with certain identifications of its sides. The system of loops $\{\gamma_p, a_i, b_i\}$ is not independent. More precisely, it can be shown that there exists an appropriate choice of paths $c_p$ and an ordering of the plaquettes so that the following relation is true:

$$\prod_p \gamma_p = \prod_i a_i b_i a_i^{-1} b_i^{-1}. \quad (47)$$

The idea of the construction is rather simple and is illustrated in Fig. 4. The choice of the connecting paths $c_p$ is essentially determined by the ordering of the plaquettes and is not unique. It can be shown that final results do not depend on this ambiguity.

Using the results of [20] (see also [10]) one can give a well-defined meaning to the heuristic measure $D_A$ in Eq. (4). It can be understood as the measure $d\mu_0$ constructed out of copies of the Haar measure on the group $G$. This is possible if the action can be written as a cylindrical function on the space $A/T$. The standard Yang-Mills action $S(A)$ is not well defined on the lattice. Hence one has to use some regularized action $S_\Lambda(A)$ on the lattice $\Lambda$, calculate $Z_\Lambda(\gamma)$, which is a lattice analog of (4), and then take the limit $Z(\gamma) = \lim_{a \to 0} Z_\Lambda(\gamma)$, where $a$ is the lattice spacing of $\Lambda$.

In this article we consider a class of lattice actions with the property

$$e^{-S_\Lambda(A)} = \prod_{p \in \Lambda} e^{-S_p(H_{\gamma_p}(A))}, \quad (48)$$

where the product is taken over all plaquettes of $\Lambda$.

The function $S_p$ is a real function over $G$ which satisfies the following three conditions:

1) $S_p(g) = S_p(g^{-1})$ for $g \in G$;
2) it reaches the absolute minimum on the identity element;
3) $\lim_{a \to 0} S_p(H_{\gamma_p}(A))|p|^{-1} = \frac{1}{2} Tr(F_{\mu\nu}(x))^2$.

Two main examples, which we have in mind, are:

a) the Wilson action [12]:

$$S_p(g) = 1 - \frac{1}{d_F} Re \chi_F(g),$$

where $F$ denotes the fundamental representation of $G$;
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b) the heat-kernel action \[ S_p(g) = - \ln \sum_R d_R X_R(g) e^{-\frac{2}{c_2(R)} |p|}, \] (49)

where \( c_2(R) \) is the value of the Casimir operator for the representation \( R \). When \( G = U(1) \) the heat-kernel action is also called Villain action. With actions of this type the integrand in (4) is a cylindrical function.

The standard way to proceed would be the following. Taking into account relation (47) we can construct a system of independent loops \( \beta_j, j = 1, \ldots, N \), where \( N = \text{(number of plaquettes)} + 2r - 1 \). Let us introduce the notation \( g_{\beta_j} \equiv H_{\beta_j}(A) \), where the holonomy is defined by Eq. (1). The measure \( d\mu_0 \) is given by \( d\mu_0 = \prod_{j=1}^{N} dg_{\beta_j} \), where \( dg_{\beta_j} \) is the Haar measure on \( G \). Further, one expresses each loop \( \gamma_p \) or \( a_i \), or \( b_i \) in terms of \( \beta_j \), like \( \gamma_p = \prod_j \beta_j \), so that \( H_{\gamma_p}(A) = \prod_j g_{\beta_j} \), and considers functions \( \nu_p(g_{\beta_1}, \ldots, g_{\beta_N}) = \exp \left[ - S_p \left( H_{\gamma_p}(A) \right) \right] \)
on \( G^N \) (direct product of \( N \) copies of \( G \)). Since the lattice is compatible with \( \gamma \) we can also decompose \( \gamma \) over the independent loops \( \beta_j \): \( \gamma = \prod_j \beta_j \).

In this way one obtains that

\[
Z(\gamma) = \lim_{\mu_0 \to 0} Z_\Lambda(\gamma),
\]

\[
Z_\Lambda(\gamma) = \int_{A/T} d\mu_0 e^{-S_\Lambda(A)} T_\gamma([A])
\]

\[
= \int_{G^N} \prod_j dg_{\beta_j} \left( \prod_{p \in \Lambda} \nu_p(g_{\beta_1}, \ldots, g_{\beta_N}) \right) \frac{1}{d_R} \chi_R \left( \prod_{j \in \gamma} g_{\beta_j} \right).
\] (50)

Here, as in Eq. (2), we assume that the Wilson loop variable is in the irreducible representation \( R \) of \( G \). Actually the integration in (50) goes over a certain closure \( A/T \), see details in Ref. [10], [20].

In this article we will use another approach to the calculation of integral (50). We begin the calculation by changing it to another form. The variables \( \{g_q\} \equiv \{g_{\gamma_p}, g_{a_i}, g_{b_i}\} \) will be treated as independent ones and relation (47) will be imposed by inserting the \( \delta \)-function

\[
\Delta[\{g_q\}] = \delta_G(\prod_{p \in \Lambda} g_{\gamma_p}, \prod_i g_{a_i} g_{b_i} g_{a_i}^{-1} g_{b_i}^{-1})
\]

where \( \delta_G(g, h) \) is the Dirac distribution on the group \( G \) with the main property

\[
\int_G dg \delta_G(g, h) f(g) = f(h).
\]

Then

\[
Z_\Lambda(\gamma) = \int_{G^{N+1}} \left( \prod_i dg_{a_i} dg_{b_i} \right) \left( \prod_{p \in \Lambda} dg_{\gamma_p} e^{-S_p(H_{\gamma_p}(A))} \right) \Delta[\{g_q\}] \frac{1}{d_R} \chi_R \left( \prod_{q \in \gamma} g_q \right).
\] (51)
Now we use the fact that the action, given by Eq. (48), can be expressed in terms of generalized Fourier coefficients. We can write
\[ e^{-S_p(g)} = \sum_{R_p} d_{R_p} \chi_{R_p}(g) B_{R_p}(p), \]  
(52)
where the coefficients \( B_{R_p}(p) \) are equal to
\[ B_{R_p}(p) = \int dg e^{-S_p(g)} \chi^*_{R_p}(g), \]
and compute the functional integral (51):
\[
Z_\Lambda(\gamma) = \int_{G^{N+1}} \left( \prod_i d g_{a_i} d g_{b_i} \right) \left( \prod_{p \in \Lambda} d g_{\gamma_p} \right)
\times \left( \prod_{p \in \Lambda} \sum_{R_p} d_{R_p} \chi_{R_p}(g_{\gamma_p}) B_{R_p}(p) \right) \Delta[\{g_q\}] \frac{1}{d_R} \chi_R \left( \prod_{q \in \gamma} g_q \right) 
\]
(53)
\[
= \int_{G^{N+1}} \left( \prod_i d g_{a_i} d g_{b_i} \right) \left( \prod_{p \in \Lambda} d g_{\gamma_p} \right)
\times \left( \sum_{\{R_p\} \subset \Lambda} \left( \prod_{p \in \Lambda} B_{R_p}(p) \right) \int_{G^{N+1}} \left( \prod_i d g_{a_i} d g_{b_i} \right) \left( \prod_{p \in \Lambda} d g_{\gamma_p} \right) \right) 
\times \left( \prod_{p \in \Lambda} d_{R_p} \chi_{R_p}(g_{\gamma_p}) \right) \Delta[\{g_q\}] \frac{1}{d_R} \chi_R \left( \prod_{q \in \gamma} g_q \right) .
\]
Here we interchanged the product over plaquettes and the summation over representations, so that the last expression is a sum over "colorations" of the surface, i.e. configurations or sets \( \{R_p\} \) of irreducible representations of the gauge group such that to each plaquette \( q \) there is an \( R_q \in \{R_p\} \) associated to it. The integrals over \( g_{\gamma_p} \) are non-zero only for certain configurations, this depends on the topology of the surface.

The decomposition of \( \gamma \) is given in terms of loops located in one of the regions whose border is \( \gamma \), say \( \sigma_1 \). Then,
\[ \gamma = \prod_p (1) \gamma_p \prod_i (1) a_i b_i a_i^{-1} b_i^{-1} \]
where the superscript \( (1) \) means the restriction to \( \sigma_1 \) (see Fig. 3).

To perform the \( g_{\gamma_p} \)-integration we use a procedure of "lattice reduction". If plaquettes \( p_1 \) and \( p_2 \) share a link which does not belong to \( \gamma \) (in this case they are in the same \( \sigma_i \) component), then we shift the group variable \( g_{p_1} \to g_{p_1} g_{p_2}^{-1} \) and use the invariance of the
measure. In this way \( g_{p_2} \) disappears from the arguments of the \( \delta \)-function \( \Delta \) and of \( \chi_R \), and we can integrate it out obtaining

\[
\int dg_{p_2} d_{R_{p_1}} \chi_{R_{p_1}}(g_{p_1} g_{p_2}^{-1}) d_{R_{p_2}} \chi_{R_{p_2}}(g_{p_2}) = d_{R_{p_1}} \chi_{R_{p_1}}(g_{p_1}) \delta_{R_{p_1} R_{p_2}}. \tag{54}
\]

The effect is that both plaquettes are forced to carry the same representation in which case the common link is “erased” and the two plaquettes merge into one plaquette while the structure of the integrand in (53) remains the same. We continue integrating over the plaquette variables in this way until we arrive to a lattice consisting only of two plaquettes \( P_1 \) and \( P_2 \) in representations \( R_1 \) and \( R_2 \) correspondingly. These two plaquettes can be envisioned by cutting the surface \( M \) through the lines \( \gamma \) and \( a_i, b_i \).

Thus we obtain that

\[
Z_A(\gamma) = \sum_{R_1, R_2} \left( \prod_{p \in \sigma_1} B_{R_1}(p) \right) \left( \prod_{p \in \sigma_2} B_{R_2}(p) \right) \Gamma(\sigma_1, \sigma_2), \tag{55}
\]

where

\[
\Gamma(\sigma_1, \sigma_2) = \int_{G_2^{r+2}} \left( \prod_i (dg_{a_i} dg_{b_i}) \right) (dg_1 dg_2) d_{R_1} \chi_{R_1}(g_1) d_{R_2} \chi_{R_2}(g_2) \times \delta_G(g_2 g_1, \prod_i g_{a_i} g_{b_i} g_{a_i}^{-1} g_{b_i}^{-1}) \frac{1}{d_R} \chi_R\left(g_1 \prod_i g_{a_i} g_{b_i} g_{a_i}^{-1} g_{b_i}^{-1}\right). 
\]

The \( g_2 \)-integration and the shift \( g_1 \rightarrow (g_1 \prod_i g_{a_i} g_{b_i} g_{a_i}^{-1} g_{b_i}^{-1}) \) lead to

\[
\Gamma(\sigma_1, \sigma_2) = \int_{G_2^{r+1}} \left( \prod_i (dg_{a_i} dg_{b_i}) \right) d_{R_1} \chi_{R_1}\left(g_1 \prod_i g_{a_i} g_{b_i} g_{a_i}^{-1} g_{b_i}^{-1}\right) \times d_{R_2} \chi_{R_2}\left(g_1^{-1} \prod_i g_{a_i} g_{b_i} g_{a_i}^{-1} g_{b_i}^{-1}\right) \frac{1}{d_R} \chi_R(g_1).
\]

Using the property

\[
\int dg \chi_R(g h g^{-1} k) = d_R^{-1} \chi_R(h) \chi_R(k)
\]

and Eq. (54) the integration over \( a_i, b_i \) can be fulfilled, and one obtains

\[
\int_{G_2^{r+1}} \left( \prod_i (dg_{a_i} dg_{b_i}) \right) d_{R_1} \chi_{R_1}\left(g_1 \prod_i g_{a_i} g_{b_i} g_{a_i}^{-1} g_{b_i}^{-1}\right) = \int_{G_1^{r+1}} (dg_{a_i}) d_{R_1} r_{R_1} \left(g_1 \prod_i g_{a_i}\right) \prod_i \chi_{R_1}(g_{a_i}^{-1}) = d_{R_1}^{1-2r_1} \chi_{R_1}(g_1),
\]
so that

$$
\Gamma(\sigma_1, \sigma_2) = \int_G dg_1 d^{1-2r_1} \chi_{R_1}(g_1) d^{1-2r_2} \chi_{R_2}(g_1^{-1}) \frac{1}{d_R} \chi_R(g_1)
$$

$$
= \frac{1}{d_R} d_R^{1-2r_1} d_R^{1-2r_2} \{R_1, R, R_2\},
$$

(56)

where \{R_1, R, R_2\} is the number of times the representation \(R_2\) is contained in \(R_1 \otimes R\) (fusion number).

Substituting (56) into (55) we obtain the final expression

$$
Z_A(\gamma) = \sum_{R_1, R_2} \left( \prod_{p \in \sigma_1} B_{R_1}(p) \right) \left( \prod_{p \in \sigma_2} B_{R_2}(p) \right) \frac{1}{d_R} d_R^{1-2r_1} d_R^{1-2r_2} \{R_1, R, R_2\}.
$$

The vacuum expectation value of the Wilson loop variable in the two-dimensional gauge theory on the lattice is equal to

$$
<T_\gamma> = \frac{\sum_{R_1, R_2} \left( \prod_{p \in \sigma_1} B_{R_1}(p) \right) \left( \prod_{p \in \sigma_2} B_{R_2}(p) \right) d_R^{1-2r_1} d_R^{1-2r_2} \{R_1, R, R_2\}}{d_R \sum_{R_3} \left( \prod_{p \in \Lambda} B_{R_3}(p) \right) d_R^{2-2r}}.
$$

To obtain the continuum limit one should, in general, compute the coefficients \(B_R(p)\) and take their infinitesimal area limit. This is the case if we work, for example, with the Wilson action. If we use instead the heat-kernel action (49) we have

$$
B_R(p) = e^{-\frac{c_2(R)}{2} |p|},
$$

and the vacuum expectation value

$$
<T_\gamma> = \frac{\sum_{R_1, R_2} \tilde{B}_{R_1}^{S_1} \tilde{B}_{R_2}^{S_2} d_R^{1-2r_1} d_R^{1-2r_2} \{R_1, R, R_2\}}{d_R \sum_{R_3} \tilde{B}_{R_3}^V d_R^{2-2r}},
$$

(57)

where \(\tilde{B}_R = \exp(-c_2(R)/2)\), is independent of the lattice spacing, so that it gives already the continuum limit. Of course, as one can check, in the limit \(a \to 0\) the results of the calculations with any action of the type (48) coincide (see [10] for examples).

Let us now consider the above result in the abelian case. For abelian groups \(d_R = 1\) and the topology of the surface is irrelevant. For \(G = U(1)\) the irreducible representations are labelled by integers \(n \in \mathbb{Z}\). If the region \(\sigma_1\) carries \(R_1 = n\), then from the definition of \{\(R_1, R, R_2\)\} in Eq. (56) \(R_2 = n + \nu\), where \(\nu\) labels the irreducible representation for which the Wilson loop variable is calculated (see [12]). Since \(B_n = \exp(-c_2 n^2/2)\), we have that

$$
<T_\gamma> = \frac{\sum_{n=-\infty}^{\infty} B_n^{S_1} B_{n+\nu}^{S_2}}{\sum_{n=-\infty}^{\infty} B_n^V} = \frac{\sum_{n=-\infty}^{\infty} e^{-S_1 c_2 n^2/2} e^{-S_2 c_2 (n+\nu)^2/2}}{\sum_{n=-\infty}^{\infty} e^{-c_2 n^2/2}}.
$$
Denoting $S_1$ and $S_2$ as $S$ and $V - S$ we obtain the following expression

$$
<T_\gamma> = \exp \left\{ -\frac{e^2 V \nu^2 S}{2 V} \left( 1 - \frac{S}{V} \right) \right\} \sum_{n=-\infty}^{\infty} e^{-e^2 V (n+S \nu/V)^2/2} \sum_{n=-\infty}^{\infty} e^{-e^2 V n^2/2},
$$

which coincides with the result (46) in Sect. 3.

Another derivation of the formula for $< T_\gamma >$ which uses the differential calculus of the lattice is given in the Appendix.

A few comments are relevant here:

- Eq. (58) is obviously invariant under the transformation $S \leftrightarrow V - S$.
- Although we have considered compact surfaces only, the same technique applies to the non-compact case. The main difference is that in the non-compact case one of the regions, say, $\sigma_1$ is infinite and the irreducible representation corresponding to it has to be trivial. This case can also be obtained as the limit $V \to \infty$ of the compact one. Performing the limit in Eq. (58) we see that the expectation value of the Wilson loop functional for the Euclidean plane $\mathbb{R}^2$ or any non-compact surface has the typical area law behaviour,

$$
<T_\gamma> = e^{-e^2 \pi \nu^2 S},
$$

where $S$ denotes the area of the region surrounded by the loop $\gamma$.
- For finite $V$ the same behaviour is obtained in the strong coupling limit $e^2 V \gg 1$. That is, in this limit the region with the smaller area dominates.
- In the weak coupling limit $e^2 V \ll 1$

$$
<T_\gamma> \sim \exp \left\{ -\frac{(e^2 V) \nu^2 S}{2 V} \left( 1 - \frac{S}{V} \right) \right\}.
$$

This behaviour can be deduced from the expression

$$
<T_\gamma> = \exp \left\{ -\frac{(e^2 V) \nu^2 S}{2 V} \left( 1 - \frac{S}{V} \right) \right\} \frac{1 + 2 \sum_{l=1}^{\infty} e^{-\frac{2\pi l S}{e^2 V}} \cos(2\pi l \frac{S}{V})}{\sum_{l=-\infty}^{\infty} e^{-\frac{2\pi l \nu}{e^2 V}}}
$$

which follows from Eq. (58) after applying the Poisson summation formula (22). This coincides with the contribution (15) of the trivial sector with zero topological charge, as it can be expected, since in the weak coupling limit the contributions of the monopoles are exponentially supressed, see Eq. (21).

For non-abelian groups properties of the representations of the particular group play an explicit role in (57). As an example let us consider the case $G = SU(2)$ with $\gamma$ in the
fundamental representation. We label the irreducible representations by \( j = 0, \frac{1}{2}, 1, \ldots \) as usual. Then \( d_j = 2j + 1 \) and \( c_2(j) = j(j + 1) \), and the vacuum expectation value of the Wilson loop variable is equal to

\[
<T_j> = \exp \left\{ -\frac{e^2 V}{8} S \left( 1 - \frac{S}{V} \right) \sum_{m=-\infty;m\neq0,-1}^{\infty} \frac{m^{1-2r_1}(m+1)^{1-2r_2}e^{-\frac{2\pi}{V}(m+\frac{V}{2})^2}}{m^2-2r_1e^{-\frac{2\pi}{8}m^2}} \right\}.
\]

5 Wilson loop variables for invariant connections.

In Sect. 3, while calculating the vacuum expectation average of the Wilson loop functional in the abelian case, we studied the contribution of instantons. In the present section we will calculate Wilson loop variables for a special class of connections called invariant connections. The result of the calculation will be compared with the exact formula derived above, and this will allow us to understand how much of the information is captured by the invariant connections. Our interest in this class of connections is motivated by the fact that the analogous calculation can be carried out in Yang-Mills theory in any dimension provided, of course, that non-trivial invariant connections exist.

First we discuss the definition of invariant connections and their construction.

Invariant connections in a principal fibre bundle are connections which are invariant under the action of a group of transformations on the bundle. In field theory models this situation occurs when some group \( K \) acts on the space-time manifold (the base of the fibre bundle) and its action can be lifted to the action of a subgroup of the group of automorphisms. Then \( K \) is called symmetry group. Let \( P(M,G) \) be a principal fibre bundle with the structure group \( G \) over the base \( M \) and \( L_k, k \in K \) be an action of the symmetry group in \( P \). Since \( K \) acts as a subgroup of the group of automorphisms of \( P \), the following relation holds:

\[
(L_k p)g = L_k(pg) \quad \text{for all} \quad k \in K, \quad g \in G, \quad p \in P,
\]

where multiplication by an element of \( G \) from the right denotes the canonical action of the structure group on the principal fibre bundle. Let us denote by \( O_k \) the corresponding action of the symmetry group on the base manifold. It is defined in a natural way by

\[
O_k (\pi(p)) = \pi(L_k p), \quad k \in K, \quad p \in P,
\]

where \( \pi \) is the canonical projection in \( P \). For our purposes it is enough to consider the case when the action of \( K \) on \( M \) is transitive. More general situation, when \( M \) consists of orbits of the action of \( K \) occurs, for example, in the problem of the coset space dimensional reduction [13] (see Ref. [14] for a review).

Now we are ready to give the definition. A connection in \( P \) is said to be invariant with respect to transformations of the group \( K \) if its connection form \( w \) satisfies

\[
L_k^* w = w
\]
for all \( k \in K \). In case of gauge theory the relevant question is what is the property of the gauge potential given by an invariant form. Such potentials were introduced in Refs. [14] and were called symmetric potentials. Initially they were used for the construction of Ansätze for solutions of equations of motion. It can be shown that condition (61) implies that for any \( k \in K \) there exists a gauge tranformation \( g_k(x) \in G \) such that the gauge potential \( A_\mu \), corresponding to the invariant connection form \( w \), satisfies

\[
(O_k A)_\mu = g_k(x)^{-1} A_\mu(x) g_k(x) + \frac{1}{ie} g_k(x)^{-1} \partial_\mu g_k(x)
\]  

(62)

for all \( k \in K \), where the l.h.s. is the field obtained by the space-time transformation. This formula means that the symmetric potential is invariant under transformations from \( K \) up to a gauge transformation.

Invariant connections were first studied by Wang [39] who proved a theorem which gives a complete characterization of such connections (see [13]). We will follow Ref. [16] in the description of this construction.

The fact that the symmetry group acts transitively on the base means that \( M \) is a coset space \( K/H \), where \( H \) is a subgroup of \( K \) called the isotropy group. \( K \) acts on \( K/H \) in the canonical way: \( O_k[k_1] = [k k_1] \), where points \( x \in M \) are understood as classes \( x = [k] \equiv kH \) of \( K/H \). Then the origin \( o \) of \( M = K/H \), that is the class containing the unit of \( K \), \( o = [e] = H \), is stable under the action of the elements from \( H \). It follows immediately from Eq. (60) that transformations \( L_h \) with \( h \in H \) act vertically on \( P \). Thus, for each \( p \in P \) there exists a mapping \( \chi_p : H \to G \) defined by \( L_h p = p \chi_p(h) \) for any \( h \in H \). It can be shown that this mapping is in fact a group homomorphism. The existing symmetry allows to carry out the reduction of the initial fibre bundle to its subbundle over the origin \( o \) such that the homomorphisms \( \chi_p \) are the same for points of this subbundle (we will denote them by \( \chi \)). Invariant connection forms \( w \) in \( P \) are characterized by their values in this subbundle and are in a one-to-one correspondence with some mapping \( \phi \) which we will describe right now. Let \( G, K \) and \( H \) be the Lie algebras of the groups \( G, K \) and \( H \) respectively. If \( H \) is a closed compact subgroup of \( K \), the case we have in mind, then the homogeneous space \( K/H \) is reductive, i.e. exists the decomposition

\[
K = H + M
\]  

(63)

with \( \text{ad}(H)M = M \), where \( \text{ad} \) denotes the adjoint action of the group on its Lie algebra. The mapping \( \phi \) maps from \( M \) into \( G \) and is equivariant, i.e. satisfies the condition

\[
\phi(\text{ad}(h)X) = \text{ad}(\chi(h))\phi(X)
\]  

(64)

for all \( h \in H \) and \( X \in M \). An explicit formula for an invariant form in terms of \( \chi \) and \( \phi \) can be also given. For our purposes it is more convenient to write such formula for the pull-back of \( w \) to the base \( M \) with respect to a (local) section \( s \). Let us denote such form on \( M \) by \( A \), \( A = s^*w \). Of course, if \( w \) is the invariant form, then \( A = A_\mu dx_\mu \), where \( A_\mu \) is the corresponding symmetric potential satisfying (62). Let \( \theta \) be the canonical left-invariant 1-form on the Lie group \( K \) with values in \( \mathcal{K} \), and \( \bar{\theta} \) is its pull-back to \( K/H \). If
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\( k(x) \in K \) is a representative of the class \( x \in K/H \), then \( \bar{\theta} = k(x)^{-1}dk(x) \). We decompose the 1-form \( \bar{\theta} \) into the \( \mathcal{H} \)- and \( \mathcal{M} \)-components in accordance with (63): \( \bar{\theta} = \bar{\theta}_\mathcal{H} + \bar{\theta}_\mathcal{M} \). Then it can be shown that for a given homomorphism \( \chi \) all invariant connections are described by

\[
A = \frac{1}{ie} \left( \chi(\bar{\theta}_\mathcal{H}) + \phi(\bar{\theta}_\mathcal{M}) \right),
\]

where the mapping \( \phi \) satisfies the equivariant condition (64). Here we denoted by the same letter \( \chi \) the homomorphism from \( \mathcal{H} \) to \( \mathcal{G} \) induced by the group homomorphism. Eq. (65) is essentially the result of the Wang theorem [13].

We see that the problem of construction of \( K \)-invariant connections on a homogeneous space \( M = K/H \) is reduced to the construction of the mappings \( \phi : \mathcal{M} \rightarrow \mathcal{G} \). For this we need to solve the constraints (64). For our purpose it is more convenient to consider the same constraint in the infinitesimal form

\[
\phi(Ad(Y)X) = Ad(\chi(Y))\phi(X), \quad Y \in \mathcal{H}, \quad X \in \mathcal{M},
\]

where \( Ad \) denotes the adjoint action of the algebra. Due to the algebraic structure inherited by the vector subspace \( \mathcal{M} \subset K \) \( Ad(h) \) is given by the Lie algebra bracket: \( Ad(Y)X = [Y,X] \). Similarly, if \( X_1, X_2 \in \mathcal{G}, \) \( Ad(X_1)X_2 = [X_1, X_2] \), where now the brackets denote the multiplication in the Lie algebra \( \mathcal{G} \). This case takes place in the r.h.s. of Eq. (66).

An effective technique for solving constraint (66) was developed in Refs. [40] (see also [16]). The main idea is to consider condition (64) as the intertwining condition and the mapping \( \phi \) as the intertwining operator which intertwines representations of \( \mathcal{H} \) in the vector spaces \( \mathcal{M} \) and \( \mathcal{G} \). Then the general structure of \( \phi \) is given by Schur’s lemma. One has to decompose the vector spaces \( \mathcal{M} \) and \( \mathcal{G} \) into irreducible representations with respect to the actions \( Ad(\mathcal{H}) \) and \( Ad(\chi(\mathcal{H})) \) of the algebra \( \mathcal{H} \) respectively and choose pairs of equivalent representations. In practice it is easier to work not with the original compact Lie algebras and subspaces but with their complexified versions. In this case Schur’s lemma tells that the intertwining operator \( \phi \) is the identity between the subspaces carrying equivalent irreducible representations of \( \mathcal{H} \) and is zero between the subspaces carrying non-equivalent ones.

As concrete examples we will consider the Yang-Mills theory on the two-dimensional sphere \( S^2 \) with the gauge groups \( G = U(1), \) \( SU(2) \) and \( SO(3) \). The sphere is realized as a coset space \( S^2 = SU(2)/U(1) \). First let us construct the 1-forms \( \bar{\theta}_\mathcal{H} \) and \( \bar{\theta}_\mathcal{M} \) which appear in Eq. (63). As in Sect. 3 we introduce the angles \( \vartheta \) and \( \varphi \), parametrizing the sphere, and the neighbourhoods \( U_1 \) and \( U_2 \) described by (24) and (25). As generators of \( K = SU(2) \) we take \( Q_j = \tau_j/2 \) \((j = 1, 2, 3)\), where \( \tau_j \) are the Pauli matrices. Let the subgroup \( H = U(1) \) be generated by \( Q_3 \). Then the one-dimensional algebra \( \mathcal{H} \) is spanned by \( Q_3 \), and the vector space \( \mathcal{M} \) in (63) is spanned by \( Q_1 \) and \( Q_2 \). Consider now the decomposition of the algebra \( \mathcal{K} \), which is the complexification of the Lie algebra of the group \( K = SU(2) \). We denote by \( e_\alpha \) and \( e_{-\alpha} \) the root vectors and by \( h_\alpha \) the corresponding
Cartan element of this algebra and take
\[ e_{\pm \alpha} = \tau_{\pm} = \frac{1}{2}(\tau_1 \pm i\tau_2), \quad h_{\alpha} = \tau_3 \]
with
\[ Ad(h_{\alpha}) (e_{\pm \alpha}) = [h_{\alpha}, e_{\pm \alpha}] = \pm 2e_{\pm \alpha}. \]
Then \( H = Ch_{\alpha}, \mathcal{M} = Ce_{\alpha} + Ce_{-\alpha} \) and the decomposition of the vector space \( \mathcal{M} \) into irreducible invariant subspaces of \( \mathcal{H} \) is described by the following decomposition of the representations:
\[ \mathbb{2} \rightarrow (2) + (-2), \quad (67) \]
where in the r.h.s. we indicated the eigenvalues of \( Ad(h_{\alpha}) \), and the space \( \mathcal{M} \) of reducible representation is indicated by its dimension in the l.h.s.

We choose the local representatives \( k^{(j)} (j = 1, 2) \) of points of the neighbourhood \( U_j \) of the coset space \( S^2 = SU(2)/U(1) \) as follows
\[ k^{(1)}(\vartheta, \varphi) = e^{-i\varphi \frac{\tau_3}{2}} e^{i\vartheta \frac{\tau_2}{2}} e^{-i\varphi \frac{\tau_1}{2}}, \quad k^{(2)}(\vartheta, \varphi) = e^{i\varphi \frac{\tau_3}{2}} e^{i(\vartheta - \pi) \frac{\tau_2}{2}} e^{-i\varphi \frac{\tau_1}{2}}. \]
The functions \( k^{(i)} : U_i \rightarrow SU(2) \) can also be viewed as local sections of the principal fibre bundle \( K = P(K/H, H) \) over the base \( K/H = S^2 \) with the structure group \( H = U(1) \). By straightforward computation one obtains the forms \( \bar{\theta}_H \) and \( \bar{\theta}_\mathcal{M} \):
\[ \bar{\theta}^{(i)} = (k^{(i)})^{-1} dk^{(i)} = \bar{\theta}^{(i)}_H + \bar{\theta}^{(i)}_\mathcal{M}, \]
\[ \bar{\theta}^{(1)}_H = \frac{i \tau_3}{2} (1 - \cos \vartheta) d\varphi, \quad (68) \]
\[ \bar{\theta}^{(1)}_\mathcal{M} = \frac{i \tau_1}{2} (-\sin \varphi d\vartheta - \sin \vartheta \cos \varphi d\varphi) + \frac{i \tau_2}{2} (\cos \varphi d\vartheta - \sin \vartheta \sin \varphi d\varphi), \]
\[ \bar{\theta}^{(2)}_H = -\frac{i \tau_3}{2} (1 + \cos \vartheta) d\varphi, \]
\[ \bar{\theta}^{(2)}_\mathcal{M} = \frac{i \tau_1}{2} (\sin \varphi d\vartheta - \sin \vartheta \cos \varphi d\varphi) + \frac{i \tau_2}{2} (\cos \varphi d\vartheta + \sin \vartheta \sin \varphi d\varphi). \]

Before using the general formula \((63)\) for the invariant gauge connections one has to specify the gauge group and the embedding \( \chi(H) \subset G \). We consider three examples.

Example 1. \( G = U(1) \). Let us realize elements of \( G \) by unimodular complex numbers \( \exp(it) \), where \( t \) is real. The group homomorphisms \( \chi : H \rightarrow G \) are labelled by integers \( n \) and given by
\[ \chi_n (e^{\frac{\tau_3}{2} t}) = e^{-i \frac{\pi}{2} t}. \]
The corresponding algebra homomorphisms are determined by \( \chi_n(h_{\alpha}) = n \). The one-dimensional vector space \( G = \text{Lie}(G) \) is the space of an irreducible representation of \( \chi(H) \) characterized by zero eigenvalue of \( Ad(\chi(h_{\alpha})) \). Taking decomposition \((67)\) of \( \mathcal{M} \) into account it is easy to see that intertwining condition \((66)\) fulfills only for \( \phi = 0 \). Thus, all
invariant connections are labelled by the integer \( n \) and the gauge potential is given by

\[
A^{(i)}_n = \frac{1}{ie^2} \chi_n(\tilde{\theta}_n^{(i)}), \quad i = 1, 2;
\]

\[
A^{(1)}_n = \frac{n}{2e} (1 - \cos \vartheta) d\varphi,
\]

\[
A^{(2)}_n = -\frac{n}{2e} (1 + \cos \vartheta) d\varphi,
\]

where the lower indices of \( A^{(1)}_n \) and \( A^{(2)}_n \) correspond to the labels of the homomorphism \( \chi_n \). These are the expressions for the Dirac-Wu-Yang abelian monopole \([29]\), and \( n \) characterizes the monopole charge. Monopoles with different \( n \) are described by connections in non-equivalent fibre bundles. In accordance with our discussion in Sect. 2 \( \mathcal{B}_U(1)(S^2) \cong \mathbb{Z} \), i.e. such bundles are indeed classified by integers. We have shown that all of them appear as particular invariant connections.

**Example 2.** \( G = SU(2) \). Let \( E_\alpha, E_{-\alpha} \) and \( H_\alpha \) be the root vectors and the Cartan element of the algebra \( \mathcal{G} = A_1 \), which appears as complexification of the Lie algebra of \( G \). We assume that they are given by the same combinations of the Pauli matrices as the corresponding elements of complexified Lie algebra of \( K \) described above. The group homomorphism \( \chi : H = U(1) \to G = SU(2) \) is given by the expression

\[
\chi \left( e^{i \tau_3^3} \alpha_3 \right) = e^{i \kappa \alpha_3} = \cos(\kappa \alpha_3^3) + i \tau_3 \sin(\kappa \alpha_3^3),
\]

and it is easy to check that this definition is consistent if \( \kappa \) is integer. Therefore the homomorphism is again labelled by \( n \in \mathbb{Z} \). The induced algebra homomorphism is given by

\[
\chi_n(h_\alpha) = nH_\alpha.
\]

The three-dimensional space \( \mathcal{G} \) of the adjoint representation of \( A_1 \) decomposes into three 1-dimensional irreducible invariant subspaces of \( \chi_n(\mathcal{H}) \) and the decomposition is characterized by the following decomposition of representations:

\[
3 \to (0) + (2n) + (-2n)
\]

(in brackets we indicate the eigenvalues of \( \text{Ad}(\chi_n(h_\alpha)) \)).

Let us now compare decompositions \([67]\) and \([72]\). For \( n \neq \pm 1, 0 \) there are no equivalent representations in the decomposition of \( \mathcal{M} \) and \( \mathcal{G} \) and the intertwining operator \( \phi : \mathcal{M} \to \mathcal{G} \) is zero. It also turns out to be zero for \( n = 0 \). In these cases according to \([62]\)

\[
A^{(1)}_n = \frac{n}{2e} \tau_3 (1 - \cos \vartheta) d\varphi,
\]

\[
A^{(2)}_n = -\frac{n}{2e} \tau_3 (1 + \cos \vartheta) d\varphi.
\]

If \( n = 1 \) or \( n = -1 \) the results are more interesting. Let us consider the case \( n = 1 \) first. Comparing \([67]\) and \([72]\) we see that there are pairs of representations with the same
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eigenvalues and, therefore, the intertwining operator \( \phi \) is non-trivial. It is determined by its action on the basis elements of \( \mathcal{M} \):

\[
\phi(e_{\alpha}) = f_1 E_\alpha, \quad \phi(e_{-\alpha}) = f_2 E_{-\alpha},
\]

where \( f_1, f_2 \) are complex numbers. The fact that the initial groups and algebras are compact implies a reality condition \[16\] which tells that \( f_1 = f_2^* \). Thus, the operator \( \phi \) and the invariant connection are parametrized by one complex parameter \( f_1 \) (we will suppress its index from now on). Using Eqs. (68), (69), (71) and (74) we obtain from (65) that

\[
A_1^{(1)} = \frac{1}{2e} \left( \begin{array}{cc} (1 - \cos \vartheta) d\varphi & f e^{-i\varphi} (-id\vartheta - \sin \vartheta d\varphi) \\ f^* e^{i\varphi} (id\vartheta - \sin \vartheta d\varphi) & -(1 - \cos \vartheta) d\varphi \end{array} \right),
\]

\[
A_1^{(2)} = \frac{1}{2e} \left( \begin{array}{cc} -(1 + \cos \vartheta) d\varphi & f e^{i\varphi} (-id\vartheta - \sin \vartheta d\varphi) \\ f^* e^{-i\varphi} (id\vartheta - \sin \vartheta d\varphi) & (1 + \cos \vartheta) d\varphi \end{array} \right).
\]

The curvature form \( F = dA + \frac{i}{2} [A, A] \) is described by the unique expression on the whole sphere and is equal to

\[
F_n = -\frac{1}{2e} \tau_3 \left( |f|^2 - 1 \right) \sin \vartheta d\vartheta \wedge d\varphi.
\]

The action for such configuration is equal to

\[
S_{inv}(f) = \frac{\pi}{2e^2 R^2} \left( |f|^2 - 1 \right)^2,
\]

where \( R \) is the radius of the sphere. Due to the \( K \)-invariance any extrema of the action found within the subspace of invariant connections is also an extremum in the space of all connections \[15\]. From Eq. (78) we see that there are two types of extrema in the theory: the maximum at \( f = 0 \) and the minima at \( f \) satisfying \( |f| = 1 \). Only one of them, the trivial extremum, was found in Ref. \[41\] as a spontaneous compactification solution in six-dimensional Kaluza-Klein theory.

Similar situation takes place for \( \kappa = -1 \). Again there exists a 1-parameter family of invariant connections parametrized by a complex parameter, say \( h \), analogous to \( f \). The action possesses two extrema: at \( h = 0 \) and for \( |h| = 1 \).

It turns out that the potentials (73), (75) and (76) are related to known non-abelian monopole solutions in this theory. Namely, for \( n \neq \pm 1 \) and for \( n = \pm 1 \) with \( f = 0 \) these expressions coincide with the monopole solutions with the monopole number \( \kappa = n \). In fact the solution with \( \kappa = n > 0 \) can be transformed to the solution with \( \kappa = -n \) by the gauge transformation \( A \rightarrow S^{-1} AS \) with the constant matrix \( S = -i\tau_1 \). Eqs. (73) and Eqs. (75) and (76) with \( f = 0 \) describe all monopoles in the \( SU(2) \) gauge theory \[12\] as it was shown in \[13\], \[14\] all of them, except the trivial configuration with \( n = 0 \), are unstable. This is in accordance with the topological classification of monopoles \[15\] (see also \[44\]) by elements of \( \pi_1(G) \). This also agrees with the fact that there is only one bundle (up to equivalence) with the base space \( S^2 \) and the structure group \( SU(2) \).
The latter statement follows from our discussion in Sect. 2. Indeed, in the case under consideration \(\mathcal{B}_{SU(2)}(S^2) \cong H^2(S^2, \pi_1(SU(2))) = 0\) since \(\pi_1(SU(2)) = 0\).

Thus, all the monopoles are described by connections in the trivial principal fibre bundle \(P(S^2, SU(2))\) and can be represented by a unique form on the whole sphere \([4 6]\). This is indeed the case. Namely there exist gauge transformations, different for \(U_1\) and \(U_2\) patches, so that the tranformed potentials coincide. Let us demonstrate this for the case \(n = 1\). In fact this property is true for the whole family of the invariant connections \((75), (76)\). The group elements of these gauge transformations of the potentials on \(U_1\) and \(U_2\) are

\[
V_1 = i \begin{pmatrix} \cos \frac{\varphi}{2} & e^{-i\varphi} \sin \frac{\varphi}{2} \\ e^{i\varphi} \sin \frac{\varphi}{2} & -\cos \frac{\varphi}{2} \end{pmatrix},
\]

\[
V_2 = i \begin{pmatrix} e^{i\varphi} \cos \frac{\varphi}{2} & \sin \frac{\varphi}{2} \\ \sin \frac{\varphi}{2} & -e^{-i\varphi} \cos \frac{\varphi}{2} \end{pmatrix}.
\]

By calculating

\[
A_1^{(i)'} = V_i^{-1} A_1^{(i)} V_i + \frac{1}{ie} V_i^{-1} dV_i
\]

for \(i = 1\) and \(i = 2\) one can easily check that the transformed potentials are equal to each other and are given by

\[
A_1^{(1)'} = A_1^{(2)'} = \frac{1}{2e} (\tau_+ c_+ + \tau_- c_- + \tau_3 c_3),
\]

where

\[
c_+ = c_-^* = e^{-i\varphi} \left[ -\cos \vartheta + \left( f \cos^2 \frac{\vartheta}{2} - f^* \sin^2 \frac{\vartheta}{2} \right) \right] \sin \vartheta d\varphi
\]

\[
+ i(-1 + f \cos^2 \frac{\vartheta}{2} + f^* \sin^2 \frac{\vartheta}{2} )d\vartheta
\]

\[
c_3 = \left( 1 - \frac{f + f^*}{2} \right) \sin^2 \vartheta d\varphi - i \frac{f - f^*}{2} \sin \vartheta d\vartheta.
\]

Note that in general expressions \((73), (76)\) and \((79)\) the phase of the complex parameter \(f\) can be rotated by residual gauge transformations which form the group \(U(1)\).

For \(f = 0\) this formula gives the known expression for the \(\kappa = 1\) \(SU(2)\)-monopole \([10]\):

\[
A_1^{(1)'} = A_1^{(2)'} = \frac{1}{4e} \begin{pmatrix} (1 - \cos 2\vartheta) d\varphi & e^{-i\varphi}(-2i d\vartheta - \sin 2\vartheta d\varphi) \\ e^{i\varphi}(2i d\vartheta - \sin 2\vartheta d\varphi) & - (1 - \cos 2\vartheta) d\varphi \end{pmatrix}.
\]

Of course, the forms \((73)\) and \((81)\) do not have singularities on the whole sphere. For \(f = f^* = 1\) the forms \((80)\) vanish. This shows that this configuration, which is also the extremum of the action, describes the trivial case of the \(SU(2)\)-monopole with \(\kappa = 0\). Note that in the original form the potentials \((73)\) and \((76)\) do not seem to be trivial. Of course, one can check that they are pure gauges and correspond to the flat connection. Vanishing of the gauge field \((77)\) for this value of \(f\) confirms this.
The picture we obtained is the following. For different homomorphisms \( \chi_n : H \to G \) we constructed different invariant connections given by Eqs. \((73)\), \((75)\) and \((76)\). For \( n \neq \pm 1 \) or \( n = \pm 1 \) with \( f = 0 \) the connection describes the \( SU(2) \)-monopole solution with the monopole number \( \kappa = n \). All \( SU(2) \)-monopoles on \( S^2 \) are reproduced in this way. As it was said above the solutions with numbers \( \kappa \) and \( (-\kappa) \) are gauge equivalent. In addition, there is a continuous 1-parameter family of invariant connections which passes through the configurations describing the \( SU(2) \)-monopoles with numbers \( \kappa = -1 \), \( \kappa = 0 \) and \( \kappa = 1 \) in the space of all connections of the theory. Connections from this family are described by Eqs. \((73)\), \((75)\), \((79)\) and \((80)\). Not all of these connections are gauge inequivalent. Classes of gauge equivalent invariant connections are labelled by values of \(|f|\). Thus, \(|f| = 0\) corresponds to the class of the \( \kappa = 1 \) monopole. The monopole with \( \kappa = -1 \) can be obtained from it by the gauge transformation with the constant matrix \( S = -i\tau_1 \), as it was explained above, and, hence, belongs to the same gauge class. Connections with \(|f| = 1\) form the class describing the monopole with \( \kappa = 0 \).

We will study the Wilson loop functional for this family of invariant connections shortly. But before let us consider the third example.

Example 3. \( G = SO(3) \). Let us recall that \( SO(3) = SU(2)/Z_2 \) and \( \pi_1(SO(3)) = Z_2 \). According to the discussion in Sect. 2

\[ \mathcal{B}_{SO(3)}(S^2) \cong H^2(S^2, Z_2) \cong Z_2, \]

so there are two non-equivalent principal fibre bundles with the base \( S^2 \) and the structure group \( SO(3) \): the trivial bundle \( P = S^2 \times SO(3) \) and the non-trivial one. This is in accordance with the topological classification of monopole solutions considered in Refs. \[15\], \[14\].

For the description of \( G = SO(3) \) it is convenient to continue using \( 2 \times 2 \) matrices with the identification of elements \( g \) and \( (-g) \). In this case from the explicit form of the group homomorphisms, Eq. \((70)\) it follows that \( 2\kappa \) must be integer. As in the previous example the invariant connection is given by Eqs. \((73)\) for integer \( n \neq \pm 1 \) and half-integer \( n \) and by Eqs. \((75)\), \((76)\) and \((79)\) for \( n = \pm 1 \). For \( \kappa = n \in Z \) we have the same set of \( SU(2) \)-monopole solutions with integer monopole number and the 1-parameter family of invariant connections as before. They are connections in the trivial bundle \( P = S^2 \times SO(3) \). For half-integer \( \kappa = n + 1/2, n \in Z \) the intertwining operator \( \phi = 0 \). Eqs. \((73)\) with \( n \) being substituted by \( n + 1/2 \) describe monopoles with half-integer number \( \kappa = n + 1/2 \). By analyzing these potentials at the equator of the sphere one can easily check that they are connections in the non-trivial bundle \( P(S^2, SO(3)) \). Only the monopoles with \( \kappa = 0 \) and \( \kappa = 1/2 \) are stable \[13\], \[14\]. The former belongs to the trivial and the latter belongs to the non-trivial topological sector of the theory, which in their turn correspond to the trivial and non-trivial \( SO(3) \)-bundles over \( S^2 \) respectively.

See \[17\] for more examples and alternative descriptions of the invariant connections.

Let us return to the example with \( M = S^2 \) and the gauge group \( G = SU(2) \) and calculate the contribution of the invariant connections to the functional integral giving
the vacuum expectation value of the Wilson loop functional. For this we consider a family of loops \( \gamma(\vartheta_0) \) on \( S^2 \) labelled by the angle \( \vartheta_0 \) given by

\[
\gamma(\vartheta_0) = \{(\vartheta_0, \varphi'), \vartheta_0 = \text{const}, 0 \leq \varphi' < 2\pi\}.
\]

These closed paths are parallel to the equator and are parametrized by the polar angle \( \varphi' \). Here for definiteness we consider the case when \( \gamma(\vartheta_0) \) lie in the neighbourhood \( U_1 \), i.e. \( 0 \leq \vartheta_0 \leq \pi \).

First we calculate the holonomy

\[
H_{\gamma(\vartheta_0)}(A) = \mathcal{P} \exp \left[ ie \int_{\gamma(\vartheta_0)} A \right],
\]

where we denoted \( A_1^{(1)} \), given by Eq. (75), as \( A \). For a fixed value of \( \vartheta_0 \) we introduce the element

\[
U(\varphi) = \mathcal{P} \exp \left[ ie \int_{\eta(\varphi; \vartheta_0)} A_{\varphi}(\vartheta_0, \varphi')d\varphi' \right],
\]

where we indicated explicitly the dependence of the \( \varphi \)-component of the gauge potential on the coordinates on the sphere. The path \( \eta(\varphi; \vartheta_0) \) on \( S^2 \) is defined by

\[
\eta(\varphi; \vartheta_0) = \{(\vartheta_0, \varphi'), \vartheta_0 = \text{const}, 0 \leq \varphi' < \varphi\}
\]

Of course, \( \eta(2\pi; \vartheta_0) = \gamma(\vartheta_0) \) and \( H_{\gamma(\vartheta_0)}(A) = U(2\pi) \). It is easy to check that \( U(\varphi) \) satisfies the matrix equation

\[
\frac{dU(\varphi)}{d\varphi} = ieU(\varphi)A_{\varphi}(\vartheta_0, \varphi), \quad (83)
\]

From Eq. (73) we see that the dependence of \( A \) on \( \varphi \) is quite simple: \( \varphi \) enters only through \( \exp(\pm i\varphi) \) in the off-diagonal terms. This suggests that it can be compensated by the unitary transformation with a matrix \( T(\varphi) \):

\[
A_{\varphi}(\vartheta_0, \varphi) = T(\varphi)A_{\varphi}(\vartheta_0, 0)T^{-1}(\varphi).
\]

Indeed the matrix \( T(\varphi) \), equal to

\[
T(\varphi) = \begin{pmatrix}
e^{-i\varphi} & 0 \\
0 & e^{i\varphi}
\end{pmatrix},
\]

realizes the necessary transformation. It is convenient to re-write Eq. (83) for the matrix \( V(\varphi) := U(\varphi)T(\varphi) \).

We get

\[
\frac{dV(\varphi)}{d\varphi} = ieV(\varphi)A_{\varphi}(\vartheta_0, 0) + V(\varphi)T^{-1}(\varphi)\frac{dT(\varphi)}{d\varphi} = ieV(\varphi)M(\vartheta_0), \quad (84)
\]
where the matrix $M$ does not depend on $\varphi$ and is equal to

$$M(\vartheta_0) = A_\varphi(\vartheta_0, 0) - \frac{1}{2e} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

Using Eq. (75) we re-write it as

$$M = C_i(\vartheta_0) \tau_i$$

with

$$C_1(\vartheta_0) = -\frac{1}{4e} (f + f^*) \sin \vartheta_0, \quad C_2(\vartheta_0) = -\frac{i}{4e} (f - f^*) \sin \vartheta_0, \quad C_3(\vartheta_0) = -\frac{1}{2e} \cos \vartheta_0.$$  

The solution of Eq. (84) can now be easily found. One obtains

$$V(\varphi) = e^{ieM(\vartheta_0)\varphi}$$

and from this calculates the holonomy (82):

$$H_{\gamma}(\vartheta_0) = V(2\pi)T^{-1}(2\pi) = -e^{2i\pi eM(\vartheta_0)}. \quad (85)$$

The traced holonomy is equal to

$$T_{\gamma}(\vartheta_0) = -\frac{1}{2} Tr e^{2\pi i C_i(\vartheta_0)\tau_i} = -\cos (2\pi e|C|) = -\cos \left( \pi \sqrt{\cos^2 \vartheta_0 + |f|^2 \sin^2 \vartheta_0} \right)$$

$$= -\cos \left( \pi \sqrt{1 + \frac{4S}{V} \left( 1 - \frac{S}{V} \right) (|f|^2 - 1)} \right), \quad (86)$$

where $S = 2\pi R^2 (1 - \cos \vartheta_0)$ is the area of the surface surrounded by the loop $\gamma(\vartheta_0)$ and $V = 4\pi R^2$ is the total area of the two-dimensional sphere with the radius $R$. For $|f| = 1$ $T_{\gamma}(\vartheta_0) = 1$ as it should be for the flat connection. Note that expression (86) is invariant under the transformation $S \rightarrow (V - S)$.

We would like to mention that result (85) can also be obtained using an alternative technique based on the representation of $H_{\gamma}(A)$ in terms of a functional integral over the Grassman variables [48] (see also [49]).

Finally we calculate the quantity

$$< T_{\gamma}(\vartheta_0) >_{inv} = \frac{Z_{inv}(\gamma(\vartheta_0))}{Z_{inv}(0)}; \quad (87)$$

characterizing the contribution of invariant connections to the vacuum expectation value of the Wilson loop functional. The quantity $Z_{inv}(\gamma(\vartheta_0))$ is given by the formula

$$Z_{inv}(\gamma(\vartheta_0)) = \int df df^* e^{-S_{inv}(f)} T_{\gamma}(\vartheta_0)(A)$$

$$= -\int df df^* e^{-2\pi^2 (|f|^2 - 1)^2} \cos \left( \pi \sqrt{1 + \frac{4S}{V} \left( 1 - \frac{S}{V} \right) (|f|^2 - 1)} \right) \quad (88)$$
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and is an analog of the functional integral $Z(\gamma)$, Eq. (4). It mimics the “path-integral quantization” of the gauge model where the configuration space (the space of connections) is finite-dimensional and consists of $SU(2)$-invariant connections. The integral takes into account the contribution of the monopoles with $\kappa = 0, \pm 1$ and fluctuations around them along the invariant direction. The action for the invariant connection in the formula above is given by Eq. (78).

Of course, the complete contribution of the invariant connections, described by Eqs. (75), (76) or Eq. (79), to the true functional integral (4) is different because it includes also contributions of all fluctuations around them. Calculation of the complete contribution is beyond the scope of the present article. In the next section we are going to discuss which part of the true vacuum expectation value $\langle T_\gamma(\vartheta_0) \rangle$ is captured by $\langle T_\gamma(\vartheta_0) \rangle_{\text{inv}}$.

We would like to make a few remarks. In Ref. [49] it was argued that the Faddeev-Popov determinant on invariant connections turns out to be zero, hence the contribution of such connections to the functional integral vanishes. This issue was analyzed in [50] in a different context, namely the authors considered similar “quantization” for $SU(2)$-invariant connections in $(3 + 1)$-dimensional Ashtekar’s gravity. They found that zeros of the Faddeev-Popov determinant, which are responsible for the vanishing of the determinant, are cancelled by the contribution of the delta functions of constraints and the path integral measure is regular on invariant connections.

The plot of the energy $E_{\text{inv}}(e^2V, S/V) \equiv - \ln \langle T_\gamma(\vartheta_0) \rangle_{\text{inv}}$ as a function of $S/V$ for two values of $e^2V$ is given in Fig. 5. The analysis of Eqs. (87) and (88) shows that this function has the quadratic behaviour in $S/V$ for $0 < S/V < (S/V)_* < 1/2$ (the value of $(S/V)_*$ depends on $(e^2V)$ and is about 0.35 for the plots in Fig. 5). Recall that we found similar behaviour for the contribution of the abelian monopoles in Sect. 3.

6 Discussion of the results

In the present article we studied the vacuum expectation value $\langle T_\gamma \rangle$ of the Wilson loop variable in two-dimensional pure Yang-Mills theories. Firstly we discussed the calculation of this quantity by various techniques. In the abelian case we expanded the functional integral giving $\langle T_\gamma \rangle$ in a sum over topological sectors of the theory. They are in 1-1 correspondence with non-equivalent principal fibre bundles $P(M, G)$. As a by-product we obtained a classification of such bundles in terms of elements of the cohomology group $H^2(M, \pi_1(G))$, Eq. (10). In fact, this classification is valid even for the case when $M$ is a two-dimensional CW-complex. The integrals over connections in each topological sector were calculated semiclassically by expanding the action around the monopole configuration, representing this sector, and performing the Gaussian integration over the fluctuations. Of course, since the theory is abelian, the semiclassical approximation is exact. This calculation was done entirely within the geometrical description of gauge theories in terms of gauge connections in principal fibre bundles and for an arbitrary compact two-dimensional space-time $M$. This approach also allowed us to separate the contribution of the monopoles.
In the case of an arbitrary gauge group $G$ the calculation was carried on the lattice. We performed it directly in terms of plaquette variables (holonomies for loops associated to the plaquettes). Relation (47) between the loops for a given two-dimensional manifold induces the corresponding relation between the plaquette variables and the holonomies for homotopically non-trivial loops which was imposed by inserting the $\delta$-function. As far as we know, such technique was not used before for the calculation of $< T_\gamma >$. Of course, our result (58) agrees with the results of previous calculations [8] - [10].

General expression for $< T_\gamma >$, Eq. (58), being specified to the gauge group $G = U(1)$, reproduces the result of Sect. 3 thus proving that the functional integral for the abelian case is indeed saturated by the sum over the topological sectors and does not contain any contributions different from the instanton contributions and fluctuations around them (including, of course, the trivial perturbative sector).

In accordance with results of previous studies, we confirm that $< T_\gamma >$ has the area law behaviour which is a characteristic feature of two-dimensional pure Yang-Mills theories. Namely, $E(e^2 V, S/V) = -\ln < T_\gamma >$ is linear in $S/V$, where $S$ is a region surrounded by the loop $\gamma$, $V$ is the total area of the surface and $e$ is the gauge coupling, in the interval $0 < S/V < (S/V)_*$ (remember that $M$ is a compact surface). The range where the function $E$ is almost linear depends on the value of $e^2 V$, and the larger $e^2 V$, the closer to $1/2 (S/V)_*$ is. Thus for $e^2 V = 20 (S/V)_* \approx 0.45$. The question we analyzed in this article is the contribution of the monopoles and the invariant connections to $E(e^2 V, S/V)$.

The example with $G = U(1)$ shows that the contribution of the monopoles depends quadratically on $S/V$ for $0 < S/V < (S/V)_*$. The linear behaviour of $E(e^2 V, S/V)$ for $(S/V) \ll 1$ comes from the contribution of fluctuations. However, for $S/V \sim (S/V)_*$, higher corrections in $S/V$ already become important and lead to the deviation from the area law behaviour. The resaturation of the area law for the complete function $E(e^2 V, S/V)$ is the result of exact cancellation of the quadratic terms in $S/V$ in the sum of the contributions of the monopoles and fluctuations. In this sense the quadratic dependence of the contribution of monopoles $E_{\text{mon}}(e^2 V, S/V)$ is an indicator of the area law behaviour of the complete function.

Similar analysis can be carried out in the non-abelian case. For $G = SU(2)$ and $G = SO(3)$ the plots of $E(e^2 V, S/V)$ and $E_{\text{mon}}(e^2 V, S/V)$ are qualitatively the same as in Fig. 1 and we do not present them in this article. The only difference is the interval of the values of the functions. For $E_{\text{mon}}(e^2 V, S/V)$ we took just the sum $\sum_n \exp \left( -S(A^{(n)}) \right)$ of the leading quasiclassical contributions of the non-abelian monopole solutions found in Sect. 5, where $S(A^{(n)})$ is the action on these monopole solutions. We did not carry out the detailed analysis of contributions of fluctuations around the monopoles in these cases. Apparently the similarity with the abelian case arises from the fact that in a special gauge (say, $A_1 = 0$) the action of the theory becomes quadratic and we can expect the factorization of the contribution of fluctuations as in Eq. (18). We argue that again the quadratic behaviour of $E_{\text{mon}}(e^2 V, S/V)$ in $S/V$ for $0 < S/V < (S/V)_*$, serves as an indicator of the power law behaviour.

Finally, we studied the class of invariant connections. In the case when the space-time is the two-dimensional sphere, $M = S^2$, and for the gauge groups $G = U(1), SU(2)$ and
In this section we present an alternative technique for the calculation of \( \langle T_\gamma \rangle \) in terms of differential forms on the lattice \([51], [52]\) which is applicable for abelian gauge groups. Here we consider the case \( G = U(1) \).

Lattice \( q \)-forms are functions defined on the \( q \)-cells of a two-dimensional lattice \( \Lambda \) with values in an abelian group \( K \). The set of such forms is denoted by \( \Lambda^q(\Lambda, K) \), \( q = 0, 1, 2 \).

The loop \( \gamma \) is characterized by the integer valued 1-form \( J_\gamma \) whose values \( J_\gamma(l) \) give the number of times the loop passes through the link \( l \in \Lambda_1 \) (taking into account the orientation). The gauge potential is characterized by the 1-form \( \theta \) whose values are real numbers modulo \( 2\pi \). It is related to the gauge potential and the variables introduced in Sect. 4 through

\[
        g_l = e^{i\theta(l)} = e^{ie \int_l A}, \quad g_{\gamma_p} = \prod_{l \in p} g_l, \quad (A1)
\]

where \( l \) and \( p \) denote links and plaquettes of \( \Lambda \) respectively.
The operators of the continuous case have their discrete counterparts on the lattice. Let us introduce the exterior derivative operator $d$ and its adjoint $\delta$ which map $q$-forms to $q+1$ and $q-1$ forms respectively:

$$d : \Lambda^q(\Lambda, K) \longrightarrow \Lambda^{q+1}(\Lambda, K), \quad \delta : \Lambda^q(\Lambda, K) \longrightarrow \Lambda^{q-1}(\Lambda, K).$$

The action of $d$ and $\delta$ is defined by:

$$df(c_{q+1}) = \sum_{c_q} I(c_{q+1}, c_q) f(c_q),$$

$$\delta f(c_{q-1}) = \sum_{c_q} I(c_q, c_{q+1}) f(c_q),$$

where $f \in \Lambda^q(\Lambda, K)$, $c_q \in \Lambda_q$ and $I$ is the incidence function introduced in Sect. 4. The operator $d$ is zero on 2-forms and $\delta$ is zero on 0-forms. Let us introduce now the scalar product of two $q$-forms $f$ and $h$ as

$$<f, h> = \sum_{c_q} f(c_q) h(c_q).$$

It is easy to verify that $\delta$ and $d$ are adjoint to each other with respect to the scalar product: $<f, dh> = <\delta f, h>$. The Laplacian operator $\Delta$, mapping $q$-forms into $q$-forms, is defined in a standard way as $\Delta = \delta d + d \delta$. It is self-adjoint, $<f, \Delta g> = <\Delta f, g>$, and positive-semidefinite:

$$<f, \Delta f> = <df, df> + <\delta f, \delta f> = \|df\|^2 + \|\delta f\|^2.$$

One can easily check that the plaquette variable $g_{\gamma p}$ in Eq. (A1) is equal to

$$g_{\gamma p} = e^{i d \theta(p)}.$$

It can be shown that alternatively to the loop variables and the measure used in Sect. 4, one can take $\theta$ as the integration variables and $D \theta \equiv \prod_{l} d(\theta(l))$ as the measure. Then the path integral (51) can be written as

$$Z_{\Lambda}(\gamma) = \int D\theta \prod_{p} \exp\{-S_p(g_{\gamma p})\} \exp(i < J_{\gamma}, \theta>).$$

The action can be expressed through the Fourier coefficients $B_n(p)$ as in Eq. (52). Remember that now we are considering the case $G = U(1)$, so the irreducible representations are labelled by integers. Since the action is symmetric $B_{-n}(p) = B_n(p))$. Then we have

$$Z_{\Lambda}(\gamma) = \int D\theta \prod_{p} \sum_{n_p} B_{n_p}(p) \exp(-i n_p d \theta(p)) \exp(i < J_{\gamma}, \theta>)$$

$$= \int D\theta \prod_{\{n_p\}} \prod_{p} [B_{n_p}(p) \exp(-i n_p d \theta(p))] \exp(i < J_{\gamma}, \theta>)$$

$$= \int D\theta \prod_{s} (\prod_{p} B_{s(p)}(p)) \exp(-i < s, d \theta > + i < J_{\gamma}, \theta>)$$

$$= \int D\theta \prod_{s} (\prod_{p} B_{s(p)}(p)) \exp(-i < \delta s - J_{\gamma}, \theta>).$$
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where we have written the sum over configurations \( \{ n_p \} \) (a configuration is a set of integers \( n_p \) associated to the plaquettes) as a sum over integer valued 2-forms \( s \in \Lambda^2(\Lambda, \mathbb{Z}) \). These 2-forms characterize surfaces on \( M \). By integrating over \( \theta(l) \) we obtain that

\[
Z_\Lambda(\gamma) = \sum_s (\prod_p B_{s(p)}(p)) \delta(\delta s - J_\gamma) = \sum_{s: \delta s = J_\gamma} (\prod_p B_{s(p)}(p)).
\]

In the sum above only the 2-forms which correspond to the surfaces whose border is the loop \( \gamma \) contribute. The expectation value of the Wilson loop finally reads:

\[
<T_\gamma> = \frac{\sum_{s: \delta s = J_\gamma} (\prod_p B_{s(p)}(p))}{\sum_{s: \delta s = 0} (\prod_p B_{s(p)}(p))}.
\]

For the Villain action (see Eq. (49)) we obtain

\[
<T_\gamma> = \frac{\sum_{s: \delta s = J_\gamma} \exp \left\{ -\frac{e^2}{2} \sum_p (s(p))^2 |p| \right\}}{\sum_{s: \delta s = 0} \exp \left\{ -\frac{e^2}{2} \sum_p (s(p))^2 |p| \right\}}.
\]

In the compact case due to the condition \( \delta s = 0 \) only the surfaces which wrap the whole lattice \( n \) times \( (n \in \mathbb{Z}) \), i.e. whose 2-forms satisfy \( s(p) = n \) for all \( p \in \Lambda_2 \), contribute to the denominator. If the loop \( \gamma \) is simple and is the boundary of a region in \( M \), then the condition \( \delta s = J_\gamma \) implies that the only forms which give non-zero contribution are those which have the following two properties: 1) \( s \) is constant inside the regions bordered by \( \gamma \), i.e. \( s(p) = s(p') \) if both \( p \) and \( p' \) lie either in the interior or in the exterior of \( \gamma \); 2) \( s(p) - s(p') = \pm 1 \) (the sign depends on the relative orientation) when one of the plaquettes \( p \) and \( p' \) lies in the interior of the loop \( \gamma \) and the other lies in the exterior of it. With this results (46) and (58) are recovered.
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Figure 1: $E(e^2 V, S/V)$ (dashed line) and $E_{mon}(e^2 V, S/V)$ (solid line) as functions of $S/V$ for $e^2 V = 10$ (lower lines) and $e^2 V = 20$ (upper lines) in the abelian gauge theory.
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Figure 2: A loop $\gamma$ on a surface $M$ with genus $r = 3$.

Figure 3: The system $\{\gamma_p, a_i, b_i\}$ on the 2-torus. The lattice here is a minimal one with two plaquettes only. By cutting the surface through the lines $a_i, b_i$ we obtain the polygon of the figure. The relations are: $p_2p_1 = a_2b_2a_2^{-1}b_2^{-1}a_1b_1^{-1}b_1^{-1}$ and $\gamma = p_1b_1a_1b_1^{-1}a_1^{-1}$. 
Figure 4: A lattice with 4 plaquettes on a torus and a choice of connecting paths such that $aba^{-1}b^{-1} = \gamma_{p_1}\gamma_{p_2}\gamma_{p_3}\gamma_{p_4}$.
Figure 5: Contribution of the invariant connections $E_{\text{inv}}(e^2V, S/V)$ as a function of $S/V$ for $e^2V = 10$ (lower line) and $e^2V = 20$ (upper line) in the $SU(2)$ Yang-Mills theory.