The universal Mumford curve and its periods in formal geometry

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ABSTRACT

We construct the universal Mumford curve as a family of Mumford curves over a formal neighborhood of the locus of degenerated curves in the moduli space over \( \mathbb{Z} \) of pointed stable curves. Furthermore, we give explicit formulas of abelian differentials and period isomorphism for the universal Mumford curve.

1. Introduction

As an analog of the Schottky uniformization theory of Riemann surfaces [24], Mumford [18] established a uniformization theory for Mumford curves which are defined as stable curves over noetherian and complete local rings whose special fibers are degenerate, i.e., consisting of projective lines. There were many researches on this subjects, and in recent works of Ulirsch [25] and Poineau-Turchetti [22], by using tropical geometry and Berovich geometry over \( \mathbb{Z} \) [20, 21], a universal Mumford curve was given unifying Riemann surfaces and Mumford curves over \( p \)-adic rings. Considering the Raynaud correspondence [23] between formal and rigid analytic geometry, one can expect to construct such a curve in formal geometry over \( \mathbb{Z} \).

In this paper, we construct the universal Mumford curve over the formal neighborhood of the locus of degenerated curves in the moduli space over \( \mathbb{Z} \) of pointed stable curves. By our approach, one obtain computable local coordinates around this locus by which we can expand and study Teichmüller modular forms [10] and their variants [13, 14, 15], and we give an algebraic proof, applicable to the \( p \)-adic case, of variational formulas on abelian differentials and periods of degenerating curves [1, 8]. Furthermore, this formal neighborhood gives also a neighborhood of the locus of degenerate curves in complex geometry which can be analytically extended to the whole moduli space of stable pointed curves over \( \mathbb{C} \).

Our construction of the universal Mumford curve uses the theory of generalized Tate curves [10] which are universal families of curves with given degeneration data, and become Schottky uniformized Riemann surfaces and Mumford curves by specializing moduli and deformation parameters. By extending results of [11, 12] on the

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construction of Teichmüller groupoids, we obtain the universal Mumford curve by gluing generalized Tate curves comparing the associated parameters.

We proceed to explicit formulas of abelian differentials defined on the universal Mumford curve called universal differentials which was partly obtained and applied to the $p$-adic soliton theory in [9]. We show variational formulas of the universal differentials which imply an important fact that they are stable, namely they have only logarithmic poles on the special fiber. Furthermore, we describe the period isomorphism between the de Rham and Betti cohomology groups, and the associated Gauss-Manin connection on the universal Mumford curve. This description unifies the classical formulas for Riemann surfaces, and results of Gerritzen [7] and de Shalit [5, 6] for $p$-adic Mumford curves using Coleman integration [2, 3].

2. Generalized Tate curve

2.1. Schottky uniformization

A Schottky group $\Gamma$ of rank $g$ is defined as a free group with generators $\gamma_i \in PGL_2(\mathbb{C}) (1 \leq i \leq g)$ which map Jordan curves $C_i \subset \mathbb{P}^1_\mathbb{C} = \mathbb{C} \cup \{\infty\}$ to other Jordan curves $C_{i-1} \subset \mathbb{P}^1_\mathbb{C}$ with orientation reversed, where $C_{\pm 1}, \ldots, C_{\pm g}$ with their interiors are mutually disjoint. Each element $\gamma \in \Gamma - \{1\}$ is conjugate to an element of $PGL_2(\mathbb{C})$ sending $z$ to $\beta \gamma z$ for some $\beta \gamma \in \mathbb{C} \times \mathbb{C}$ with $|\beta \gamma| < 1$ which is called the multiplier of $\gamma$. Therefore, one has

$$\gamma(z) - \alpha \gamma = \beta \gamma(z) - \alpha' \gamma$$

for some element $\alpha, \alpha' \gamma$ of $\mathbb{P}^1_\mathbb{C}$ called the attractive, repulsive fixed points of $\gamma$ respectively. Then the discontinuity set $\Omega_\Gamma \subset \mathbb{P}^1_\mathbb{C}$ under the action of $\Gamma$ has a fundamental domain $D_\Gamma$ which is given by the complement of the union of the interiors of $C_{\pm 1}, \ldots, C_{\pm g}$. The quotient space $R_\Gamma = \Omega_\Gamma / \Gamma$ is a (compact) Riemann surface of genus $g$ which is called Schottky uniformized by $\Gamma$ (cf. [24]). Furthermore, by a result of Koebe, every Riemann surface of genus $g$ can be represented in this manner.

2.2. Generalized Tate curve

A (pointed) curve is called degenerate if it is a stable (pointed) curve and the normalization of its irreducible components are all projective (pointed) lines. Then the dual graph $\Delta = (V, E, T)$ of a pointed stable curve is a collection of 3 finite sets $V$ of vertices, $E$ of edges, $T$ of tails and 2 boundary maps

$$b : T \to V, \quad b : E \to (V \cup \{\text{unordered pairs of elements of } V\})$$

such that the geometric realization of $\Delta$ is connected and that $\Delta$ is stable, namely its each vertex has at least 3 branches. The number of elements of a finite set $X$ is denoted by $\# X$, and a (connected) stable graph $\Delta = (V, E, T)$ is called of $(g, n)$-type if $\text{rank}_\mathbb{Z} H_1(\Delta, \mathbb{Z}) = g$, $\# T = n$. Then under fixing a bijection $\nu : T \to \{1, \ldots, n\}$, which we
call a numbering of $T$, $\Delta = (V, E, T)$ becomes the dual graph of a degenerate $n$-pointed curve of genus $g$ such that each tail $h \in T$ corresponds to the $\nu(h)$th marked point. In particular, a stable graph without tail is the dual graph of a degenerate (unpointed) curve by this correspondence. If $\Delta$ is trivalent, i.e. any vertex of $\Delta$ has just 3 branches, then a degenerate $\sharp T$-pointed curve with dual graph $\Delta$ is maximally degenerate. An orientation of a stable graph $\Delta = (V, E, T)$ means giving an orientation of each $e \in E$. Under an orientation of $\Delta$, denote by $\pm E = \{ e, -e \mid e \in E \}$ the set of oriented edges, and by $v_h$ the terminal vertex of $h \in \pm E$ (resp. the boundary vertex of $h \in T$). For each $h \in \pm E$, denote by let $|h| \in E$ be the edge $h$ without orientation.

Let $\Delta = (V, E, T)$ be a stable graph. Fix an orientation of $\Delta$, and take a subset $E$ of $\pm E \cup T$ whose complement $E_\infty$ satisfies the condition that
\[
\pm E \cap E_\infty \cap \{ -h \mid h \in E_\infty \} = \emptyset,
\]
and that $v_h \neq v_{h'}$ for any distinct $h, h' \in E_\infty$. We attach variables $x_h$ for $h \in E$ and $y_e = y_{-e}$ for $e \in E$. Let $R_\Delta$ be the $\mathbb{Z}$-algebra generated by $x_h$ ($h \in E$), $1/(x_e - x_{-e})$ ($e, -e \in E$) and $1/(x_h - x_{h'})$ ($h, h' \in E$ with $h \neq h'$ and $v_h = v_{h'}$), and let
\[
A_\Delta = R_\Delta[(y_e \ (e \in E))], \quad B_\Delta = A_\Delta \left[ \prod_{e \in E} y_e^{-1} \right].
\]
According to [10, Section 2], we construct the universal Schottky group $\Gamma$ associated with oriented $\Delta$ and $E$ as follows. For $h \in \pm E$, put
\[
\phi_h = \begin{pmatrix} x_h & x_{-h} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & y_h \end{pmatrix} \begin{pmatrix} x_h & x_{-h} \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{x_h - x_{-h}} \left\{ \begin{pmatrix} x_h & -x_h x_{-h} \\ 1 & -x_{-h} \end{pmatrix} - \begin{pmatrix} x_h & -x_h x_{-h} \\ 1 & -x_{-h} \end{pmatrix} y_h \right\},
\]
where $x_h$ (resp. $x_{-h}$) means $\infty$ if $h$ (resp. $-h$) belongs to $E_\infty$. This gives an element of $\text{PGL}_2(B_\Delta) = \text{GL}_2(B_\Delta)/B_\Delta^\times$ which we denote by the same symbol, and satisfies
\[
\frac{\phi_h(z) - x_h}{z - x_h} = y_h \frac{\phi_h(z) - x_{-h}}{z - x_{-h}} \quad (z \in \mathbb{P}^1),
\]
where $\text{PGL}_2$ acts on $\mathbb{P}^1$ by linear fractional transformation.

For any reduced path $\rho = h(1) \cdot h(2) \cdots h(l)$ which is the product of oriented edges $h(1), ..., h(l)$ such that $v_{h(i)} = v_{-h(i+1)}$, one can associate an element $\rho^*$ of $\text{PGL}_2(B_\Delta)$ having reduced expression $\phi_{h(l)} \phi_{h(l-1)} \cdots \phi_{h(1)}$. Fix a base vertex $v_b$ of $V$, and consider the fundamental group $\pi_1(\Delta, v_b)$ which is a free group of rank $g = \text{rank}_\mathbb{Z} H_1(\Delta, \mathbb{Z})$. Then the correspondence $\rho \mapsto \rho^*$ gives an injective anti-homomorphism $\pi_1(\Delta, v_b) \to \text{PGL}_2(B_\Delta)$ whose image is denoted by $\Gamma_\Delta$.

It is shown in [10, Section 3] and [11, 1.4] (see also [16, Section 2] when $\Delta$ is trivalent and has no loop) that for any stable graph $\Delta$, there exists a stable pointed curve $C_\Delta$ of genus $g$ over $A_\Delta$ which satisfies the following properties:
We review the construction of $C_\Delta$ obtained by substituting $y_e = 0$ ($e \in E$) in $R_\Delta$. The dual graph is the collection of $P_v := \mathbb{P}^1_{R_\Delta}$ ($v \in V$) by identifying the points $x_e \in P_{v_1}$ and $x_{-e} \in P_{v_{-1}}$ ($e \in E$), where $x_h$ denotes $\infty$ if $h \in \mathcal{E}_\infty$.

$C_\Delta$ gives rise to a universal deformation of degenerate pointed curves with dual graph $\Delta$. More precisely, $C_\Delta$ satisfies the following: For a noetherian and normal complete local ring $R$ with residue field $k$, let $C$ be a pointed Mumford curve over $R$, namely a stable pointed curve over $R$ with nonsingular generic fiber such that the closed fiber $C \otimes_R k$ is a degenerate pointed curve with dual graph $\Delta$, in which all double points and marked points are $k$-rational. Then there exists a ring homomorphism $A_\Delta \rightarrow R$ giving $C_\Delta \otimes_{A_\Delta} R \cong C$.

$C_\Delta \otimes_{A_\Delta} B_\Delta$ is smooth over $B_\Delta$ and is Mumford uniformized (cf. [18]) by $\Gamma$.

Take $x_h$ ($h \in \mathcal{E}$) as complex numbers such that $x_e \neq x_{-e}$ and that $x_h \neq x_{h'}$ if $h \neq h'$ and $y_h = y_{h'}$, and take $y_e$ ($e \in E$) as sufficiently small nonzero complex numbers. Then $C_\Delta$ becomes a pointed Riemann surface which is Schottky uniformized by the Schottky group $\Gamma$ over $\mathbb{C}$ obtained from $\Gamma$.

We review the construction of $C_\Delta$ given in [10, Theorem 3.5]. Let $T_\Delta$ be the tree obtained as the universal cover of $\Delta$, and denote by $\mathcal{P}_{T_\Delta}$ be the formal scheme as the union of $\mathbb{P}^1_{\Delta}$’s indexed by vertices of $T_\Delta$ under the $B_\Delta$-isomorphism by $\phi_e$ ($e \in E$). Then it is shown in [10, Theorem 3.5] that $C_\Delta$ is the formal scheme theoretic quotient of $\mathcal{P}_{T_\Delta}$ by $\Gamma_\Delta$.

**3. Universal Mumford curve**

**3.1. Comparison of deformations**

Let $\Delta = (V,E,T)$ be a stable graph which is not trivalent. Then there exists a vertex $v_0 \in V$ which has at least 4 branches. Take two elements $h_1, h_2$ of $\pm E \cup T$ such that $h_1 \neq h_2$ and $v_{h_1} = v_{h_2} = v_0$, and let $\Delta' = (V',E',T')$ be a stable graph obtained from $\Delta$ by replacing $v_0$ with an oriented (nonloop) edge $h_0$ such that $v_{h_1} = v_{h_2} = v_{h_0}$ and that $v_h = v_{-h_0}$ for any $h \in \pm E \cup T - \{h_1, h_2\}$ with $v_h = v_0$. Put $e_i = |h_i|$ for $i = 0, 1, 2$. Then we have the following identifications:

$$V = V' - \{v_{-h_0}\} \quad (\text{in which } v_0 = v_{h_0}), \quad E = E' - \{e_0\}, \quad T = T'.$$

**Theorem 3.1.** The generalized Tate curves $C_\Delta$ and $C_{\Delta'}$ associated with $\Delta$ and $\Delta'$ respectively are isomorphic over $R_{\Delta'}((s_{e_0}))[[s_e (e \in E)]]$, where

$$\frac{x_{h_1} - x_{h_2}}{s_{e_0}}, \quad \frac{y_{e_i}}{s_{e_0}s_{e_1}} \quad (i = 1, 2 \text{ with } h_i \notin T), \quad \frac{y_e}{s_e} \quad (e \in E - \{e_1, e_2\})$$
\[ \text{belong to } (A_{\Delta'})^\times \text{ if } h_1 \neq -h_2, \text{ and} \]

\[ \frac{x_{h_1} - x_{h_2}}{s_{e_0}}, \quad \frac{y_e}{s_e} \quad (e \in E) \]

\[ \text{belong to } (A_{\Delta'})^\times \text{ if } h_1 = -h_2. \]

**Remark 3.2.** From the properties of generalized Tate curves, one can see that the assertion holds in the category of complex geometry when \( x_{h_1} - x_{h_2}, y_e \) and \( s_e \) are taken to be sufficiently small complex numbers.

**Proof.** We prove the theorem when \( \Delta \) has no tail from which the assertion in general case follows. Over a certain open subset of \( \{(x_h, y_e) \in \mathbb{C}^\frac{\mathbb{Z}}{E} \times \mathbb{C}^E \} \) with sufficiently small absolute values \( |x_{h_1} - x_{h_2}| \) and \( |y_e| \), \( C_{\Delta} \) gives a deformation of the degenerate curve with dual graph \( \Delta' \). Hence by the universality of generalized Tate curves, there exists an injective homomorphism

\[ (*) \quad (R_{\Delta} \otimes \mathbb{C})[[y_e \ (e \in E)]] \hookrightarrow (R_{\Delta'} \otimes \mathbb{C})((s_0))[[s_e \ (e \in E)]] \]

which induces an isomorphism \( C_{\Delta} \cong C_{\Delta'} \) such that the degenerations of \( C_{\Delta} \) given by \( x_{h_1} - x_{h_2} \to 0 \) and \( y_e \to 0 \ (e \in E) \) correspond to those of \( C_{\Delta'} \) given by \( s_{e_0} \to 0 \) and \( s_e \to 0 \) respectively. Since these two curves are Mumford uniformized, a result of Mumford (cf. [18, Corollary 4.11]) implies that the uniformizing groups \( \Gamma_{\Delta} \) and \( \Gamma_{\Delta'} \) of \( C_{\Delta} \) and \( C_{\Delta'} \) respectively are conjugate over the quotient field of \( (R_{\Delta'} \otimes \mathbb{C})((s_0))[[s_e]] \).

Denote by \( \iota : \Gamma_{\Delta} \sim \Gamma_{\Delta'} \) the isomorphism defined by this conjugation. Since eigenvalues are invariant under conjugation and the cross ratio

\[ [a, b; c, d] = \frac{(a - c)(b - d)}{(a - d)(b - c)} \]

of 4 points \( a, b, c, d \) is invariant under linear fractional transformation, one can see the following:

(A) For any \( \gamma \in \Gamma_{\Delta} \), the multiplier of \( \gamma \) is equal to that of \( \iota(\gamma) \) via the injection \((*)\).

(B) For any \( \gamma_i \in \Gamma_{\Delta} \) \((1 \leq i \leq 4)\), the cross ratio \([a_1, a_2; a_3, a_4] \) of the attractive fixed points \( a_i \) of \( \gamma_i \) is equal to that of \( \iota(\gamma_i) \) via the injection \((*)\).

We consider the case that \( h_1 \neq -h_2 \). Put

\[ A_1 = R_{\Delta} \left[ \frac{y_e}{x_{h_1} - x_{h_2}} \quad (i = 1, 2), \quad y_e \ (e \in E - \{e_1, e_2\}) \right] \]

whose quotient field is denoted by \( \Omega_1 \), and let \( I_1 \) be the kernel of the natural surjection \( A_1 \to R_{\Delta} \). Then from (A) and (B) as above and results in [10, Section 1], we will show that the isomorphism \((*)\) descends to \( A_1 \cong A_{\Delta'} \), where the variables are related as in the first case of the assertion. We take a local coordinate \( \xi_h \) on \( P_{v(h)} = \mathbb{P}^1 \) as

\[ \xi_h(z) = \begin{cases} 
  z - x_h & \text{(if } h \not\in \mathcal{E}_\infty), \\
  1/z & \text{(if } h \in \mathcal{E}_\infty). 
\end{cases} \]
For \( z \in \mathbb{P}^1(\Omega_1) \) with \( \xi_h(z) \in I_1 \) and \( h' \in \pm E - \{h\} \) with \( v_{-h'} = v_h \), by the calculation in the proof of [10, Lemma 1.2], one can see that \( \xi_{h'}(\phi_{h'}(z)) \in I_1 \). Hence for \( \gamma \in \Gamma_\Delta \) with reduced expression \( \phi_{h(1)} \cdots \phi_{h(l)} \), if we take \( h \in \pm E - \{-h(l)\} \) with \( v_h = v_{-h(l)} \) and \( z \in \mathbb{P}^1(\Omega_1) \) with \( \xi_h(z) \in I_1 \), then by [10, Lemma 1.2], the attractive fixed point \( a \) of \( \gamma \) is given by \( \lim_{n \to \infty} \gamma^n(z) \), and hence \( \xi_{h(l)}(a) \in I_1 \). For each \( v \in V \), fix a path \( \rho_v \) in \( \Delta \) from the base point \( v_0 \) to \( v \). By the assumption on \( v_0 \), one can take distinct oriented edges \( h_3, h_4 \in \pm E - \{h_1, h_2\} \) with terminal vertex \( v_0 \), and \( \rho_i \in \pi_1(\Delta, v_0) \) (\( 1 \leq i \leq 4 \)) with reduced expression \( \cdots h_i \). Then the attractive fixed points \( \gamma_i = (\rho_{v_0})^{-1} \cdot (\rho_i^* \cdot \rho_{v_0}^*) \) satisfy that \( [a_1, a_3; a_2, a_4] \in (x_{h_1} - x_{h_2}) \cdot (A_1)^\times \). Furthermore, by Proposition 1.4 and Theorem 1.5 in [10], the attractive fixed points \( \gamma_i \) of \( i(\gamma_i) \) satisfy that \( [a'_1, a'_3; a'_2, a'_4] \in s_0 \cdot (A_{(\Delta, \tau_i)})^\times \), and hence from (B), we have the comparison of \( x_{h_1} - x_{h_2} \) and \( s_0 \). Since \( h_1 \) is not a loop by the assumption, the comparison of \( y_1/(x_{h_1} - x_{h_2}) \) and \( s_1 \) follows from applying (B) to \( \gamma_i = (\rho_{v_0})^{-1} \cdot (\rho_i^* \cdot \rho_{v_0}) \) (\( 1 \leq i \leq 4 \)), where \( \rho_i \in \pi_1(\Delta, v_0) \) has reduced expression:

\[
\begin{align*}
\rho_1 &= \cdots h_2, \\
\rho_2 &= \cdots h_3, \\
\rho_3 &= \cdots h_5 \cdot h_1, \\
\rho_4 &= \cdots h_6 \cdot h_1,
\end{align*}
\]

for distinct oriented edges \( h_5, h_6 \) with terminal vertex \( v_{-h_1} \). Similarly, we have the comparison of \( y_2/(x_{h_1} - x_{h_2}) \) (resp. \( y_e \) (\( e \in E - \{\text{loops}\} \)) and \( s_2 \) (resp. \( s_e \)), and further if \( e \in E \) is a loop, then the comparison of \( y_e \) and \( s_e \) follows from applying (A) to \( \gamma = (\rho_{v_e})^{-1} \cdot \rho_e \cdot \rho_{v_e}^* \). Therefore, the injection gives rise to \( A_1 \cong A_\Delta \) under which we have:

\[
\frac{x_{h_1} - x_{h_2}}{s_{e_0}}, \quad \frac{y_{e_i}}{s_{e_0} s_{e_i}} (i = 1, 2), \quad \frac{y_e}{s_e} (e \in E - \{e_1, e_2\}) \in (A_{\Delta'})^\times.
\]

One can show the assertion in the case that \( h_1 = -h_2 \) similarly. □

3.2. Construction of the universal Mumford curve

For nonnegative integers \( g, n \) such that \( 2g - 2 + n > 0 \), denote by \( \overline{M}_{g,n} \) the moduli stack over \( \mathbb{Z} \) of \( n \)-pointed stable curves of genus \( g \), and by \( M_{g,n} \) its substack classifying \( n \)-pointed proper smooth curves of genus \( g \) [4]. Then by definition, there exits the universal stable curve \( C_{g,n} \) over \( \overline{M}_{g,n} \).

**Theorem 3.3.** Let \( D_{g,n} \) be the closed substack of \( \overline{M}_{g,n} \) consisting of \( n \)-pointed degenerate curves of genus \( g \), and denote by \( N_{g,n} \) the formal completion of \( \overline{M}_{g,n} \) along \( D_{g,n} \). Then there exists an algebraization \( \mathcal{N}^\text{alg}_{g,n} \) of \( N_{g,n} \), namely \( \mathcal{N}^\text{alg}_{g,n} \) is a scheme containing \( D_{g,n} \) as its closed subset such that \( N_{g,n} \) is the formal completion of \( \mathcal{N}^\text{alg}_{g,n} \) along \( D_{g,n} \), and the fiber of \( C_{g,n} \) over \( N_{g,n} \) gives a universal family of generalized Tate curves.

**Proof.** Let \( \Delta = (V, E, T) \) be a stable graph of \( (g, n) \)-type, and take a system of coordinates on \( P_v = \mathbb{P}^1_{R_\Delta} (v \in V) \) such that \( x_h = \infty \) (\( h \in E_\infty \)) and that \( \{0, 1\} \subset P_v \).
is contained in the set of points given by \( x_h (h \in E \text{ with } \nu_h = \nu) \). Under this system of coordinates, one has the generalized Tate curve \( C_\Delta \) whose closed fiber \( C_\Delta \otimes_{A_\Delta} R_\Delta \) gives a family of degenerate curves over the open subspace of

\[
S = \{ (p_h \in P_{\nu_h})_{h \in \pm E, \pm T} \mid p_h \neq p_{h'} (h \neq h', \nu_h = \nu_{h'}) \}
\]
defined as \( p_e \neq p_{-e} (e \in E) \). Therefore, by taking another system of coordinates and comparing the associated generalized Tate curves with the original \( C_\Delta \) as in Theorem 3.1, \( C_\Delta \) can be extended over the algebraization of the formal completion of \( S \) in \( \overline{\mathcal{M}_{g,n}} \). Furthermore, by Theorem 3.1, one can glue generalized Tate curves \( C_\Delta \) over \( N_{g,n} \) for various stable graphs \( \Delta \) of \((g,n)\)-type, and hence the assertion follows. □

**Definition 3.4.** We call this family of generalized Tate curves the \( n \)-pointed *universal Mumford curve* of genus \( g \).

**Theorem 3.5.**

(1) For a complete nonarchimedean valuation field \( K \), denote by \( \mathcal{M}^{an}_{g,n/K} \) the \( K \)-analytic space associated with \( \mathcal{M}_{g,n} \). Then its \( K \)-analytic subspace associated with \( N^\text{alg}_{g,n} - D_{g,n} \) consists of all \( n \)-pointed Mumford curves over \( K \) of genus \( g \).

(2) Denote by \( \mathcal{M}^{an}_{g,n} \) the complex analytic space associated with \( \mathcal{M}_{g,n} \). Then \( \mathcal{M}^{an}_{g,n} \) is obtained from its subspace associated with \( N^\text{alg}_{g,n} - D_{g,n} \) by analytic extension.

**Proof.** The assertion (1) follows from Theorem 3.3 and the universality of generalized Tate curves reviewed in 2.2 (P2). The assertion (2) follows from the result of Koebe reviewed in 2.1. □

### 4. Universal differentials and periods

#### 4.1. Universal differentials on a generalized Tate curve

Let \( \Delta = (V, E, T) \) be a stable graph, and the notation be as in 2.2.

**Proposition 4.1.** Let \( \phi \) be a product \( \phi_{h(1)} \cdots \phi_{h(l)} \) with \( \nu_{-h(i)} = \nu_{h(i+1)} \) (1 \( \leq i \leq l - 1 \) which is reduced in the sense that \( h(i) \neq -h(i+1) \) (1 \( \leq i \leq l - 1 \)), and put \( y_\phi = y_{h(1)} \cdots y_{h(l)} \).

1. One has \( \phi(z) - x_{h(1)} \in y_{h(1)} \left( \mathcal{R}_\Delta \left[ z, \prod_{h \in \pm E} (z - x_h)^{-1} \right] [y_e (e \in E)] \right) \).

2. If \( a \in A_\Delta \) satisfies \( a - x_{-h(l)} \in A^x_\Delta \), then \( \phi(a) - x_{h(1)} \in I \). Furthermore, if \( a' - x_{-h(l)} \in A^x_\Delta \), then \( \phi(a) - \phi(a') \in (a - a')y_\phi A^x_\Delta \).

3. One has \( \frac{d\phi(z)}{dz} \in y_\phi \left( \mathcal{R}_\Delta \left[ \prod_{h \in \pm E} (z - x_h)^{-1} \right] [y_e (e \in E)] \right) \).

**Proof.** Since the assertion (2) is proved in [10, Lemma 1.2], we will prove (1) and (3).
Put
\[ \phi = \begin{pmatrix} a_\phi & b_\phi \\ c_\phi & d_\phi \end{pmatrix}. \]

Since
\[ \begin{pmatrix} \alpha & -\alpha \beta \\ 1 & -\beta \end{pmatrix} \begin{pmatrix} \gamma & -\gamma \delta \\ 1 & -\delta \end{pmatrix} = (\gamma - \beta) \begin{pmatrix} \alpha & -\alpha \beta \\ 1 & -\delta \end{pmatrix}, \]
a_\phi, b_\phi, c_\phi and d_\phi are elements of \( A_\Delta \) whose constant terms are \( x_{h(1)} t, -x_{h(1)} x_{-h(l)} t, t \) and \( -x_{-h(l)} t \) respectively, where
\[ t = \frac{\prod_{s=2}^l (x_{h(s)} - x_{-h(s-1)})}{\prod_{s=1}^l (x_{h(s)} - x_{-h(s)})} \in A_\Delta^\times. \]

Then \( c_\phi z + d_\phi = t(z - x_{-h(l)}) + \cdots \), and hence
\[ \phi(z) - x_{h(1)} \in R_\Delta \left[ z, \prod_{h \in \pm E} (z - x_h)^{-1} \right] \left[ \left( y_e (e \in E) \right) \right]. \]

In order to prove (1), we may assume that \( l = 1 \), and then \( \phi(z) = x_{h(1)} \) under \( y_h(1) = 0 \). Therefore, the assertion (1) holds. The assertion (3) follows from
\[ \frac{d\phi(z)}{dz} = \frac{\det(\phi)}{(c_\phi z + d_\phi)^2} = \frac{\prod_{s=1}^l \det(\phi_{h(s)})}{(c_\phi z + d_\phi)^2} = \frac{\prod_{s=1}^l y_{h(s)}}{(c_\phi z + d_\phi)^2}, \]
and the above calculation. \( \Box \)

For a stable graph \( \Delta = (V, E, T) \), we define universal differentials on a generalized Tate curve \( C_\Delta \). Let \( \Gamma_\Delta = \text{Im}(\pi_1(\Delta, v_b) \to PGL_2(B_\Delta)) \) be the universal Schottky group as above. Then it is shown in [10, Lemma 1.3] that each \( \gamma \in \Gamma_\Delta \setminus \{1\} \) has its attractive (resp. repulsive) fixed points \( \alpha \) (resp. \( \alpha' \)) in \( \mathbb{P}^1_{B_\Delta} \) and its multiplier \( \beta \in \sum_{e \in E} A_\Delta \cdot y_e \) which satisfy
\[ \frac{\gamma(z) - \alpha}{z - \alpha} = \beta \frac{\gamma(z) - \alpha'}{z - \alpha'}. \]

Fix a set \( \{ \gamma_1, \ldots, \gamma_g \} \) of generators of \( \Gamma_\Delta \), and for each \( \gamma_i \), denote by \( \alpha_i \) (resp. \( \alpha_{-i} \)) its attractive (resp. repulsive) fixed points, and by \( \beta_i \) its multiplier. Then under the assumption that there is no element of \( \pm E \cap E_{\infty} \) with terminal vertex \( v_b \), for each \( 1 \leq i \leq g \), we define the associated universal differential of the first kind as
\[ \omega_i = \sum_{\gamma \in \Gamma_\Delta \setminus \{ \gamma_i \}} \left( \frac{1}{z - \gamma(\alpha_i)} - \frac{1}{z - \gamma(\alpha_{-i})} \right) dz. \]

Assume that \( \{ h \in \pm E \cap E_{\infty} \mid v_h = v_{b} \} = \emptyset \), \( \{ t \in T \mid v_t = v_b \} \neq \emptyset \).
Then for each $t \in T$ with $v_t = v_b$ and $k > 1$, we define the associated universal differential of the second kind as

$$
\omega_{t,k} = \begin{cases} 
\sum_{\gamma \in \Gamma_\Delta} \frac{d\gamma(z)}{(\gamma(z) - x_t)^k} & (x_t \neq \infty), \\
\sum_{\gamma \in \Gamma_\Delta} \gamma(z)^k d\gamma(z) & (x_t = \infty).
\end{cases}
$$

Furthermore, put $T_\infty = \{ t \in T \cap \mathcal{E}_\infty \mid v_t = v_b \}$ whose cardinality is 0 or 1, and take a maximal subtree $T_\Delta$ of $\Delta$, and for each $t \in T$, take the unique path $p_t = h(1) \cdots h(l)$ in $T_\Delta$ from $v_t$ to $v_b$, and put $\phi_t = \phi_{h(l)} \cdots \phi_{h(1)}$. Then for each $t_1, t_2 \in T$ with $t_1 \neq t_2$, we define the associated universal differential of the the third kind as

$$
\omega_{t_1, t_2} = \sum_{\gamma \in \Gamma_\Delta} \left( \frac{d\gamma(z)}{\gamma(z) - \phi_{t_1}(x_{t_1})} - \frac{d\gamma(z)}{\gamma(z) - \phi_{t_2}(x_{t_2})} \right),
$$

where $\phi_{t_i}(x_{t_i}) = \infty$ if $t_i \in T_\infty$.

**Theorem 4.2.**

1. For each $1 \leq i \leq g$, $\omega_i$ is a regular differential on $C_\Delta \otimes_{A_\Delta} B_\Delta$ (cf. [17, §3]).
2. For each $t \in T$ with $v_t = v_b$ and $k > 1$, $\omega_{t,k}$ is a meromorphic differential on $C_\Delta \otimes_{A_\Delta} B_\Delta$ which has only pole (of order $k$) at the point $p_t$ corresponding to $t$.
3. For each $t_1, t_2 \in T$ such that $t_1 \neq t_2$, $\omega_{t_1, t_2}$ is a meromorphic differential on $C_\Delta \otimes_{A_\Delta} B_\Delta$ which has only (simple) poles at the points $p_{t_1}$ (resp. $p_{t_2}$) corresponding to $t_1$ (resp $t_2$) with residue 1 (resp. $-1$).
4. Take $x_h$ ($h \in \pm E \cup T$) and $y_e$ ($e \in E$) be complex numbers as in 2.2 (P4). Then $\omega_i, \omega_{i,k}, \omega_{i_1, i_2}$ are abelian differentials on the Riemann surface $R_\Gamma$, where $\Gamma$ is the Schottky group obtained from $\Gamma_\Delta$.

**Proof.** By Proposition 4.1, $\omega_i$ are differentials on $\mathcal{P}_{\Gamma_\Delta}$, and for any $\delta \in \Gamma_\Delta$,

$$
\omega_i(\delta(z)) = \sum_{\gamma \in \Gamma_\Delta / \langle \gamma_i \rangle} \left( \frac{1}{\delta(z) - \gamma(\alpha_i)} - \frac{1}{\delta(z) - \gamma(\alpha_{-i})} \right) d\delta(z)
$$

$$
= \sum_{\gamma \in \Gamma_\Delta / \langle \gamma_i \rangle} \left( \frac{\gamma(\alpha_i) - \gamma(\alpha_{-i})}{(\delta(z) - \gamma(\alpha_i))(\delta(z) - \gamma(\alpha_{-i}))} \right) d\delta(z)
$$

$$
= \sum_{\gamma \in \Gamma_\Delta / \langle \gamma_i \rangle} \left( \frac{(\delta^{-1}\gamma)(\alpha_i) - (\delta^{-1}\gamma)(\alpha_{-i})}{(z - (\delta^{-1}\gamma)(\alpha_i))(z - (\delta^{-1}\gamma)(\alpha_{-i}))} \right) dz
$$

$$
= \omega_i(\delta(z)).
$$
Therefore, by the construction of $C_\Delta$ reviewed in 2.2, $\omega_i$ give rise to differentials on $C_\Delta$ which are regular outside $\bigcup_{e \in E} \{ y_e = 0 \}$, and hence the assertions (1) follows. One can prove (2)–(3) similarly, and we prove (4). As is stated in 2.1, $R_\Gamma$ is given by the quotient space $\Omega_\Gamma/\Gamma$. Under the assumption on complex numbers $x_h$ and $y_e$, it is shown in [24] that $\sum_{y \in \Gamma} |\gamma'(z)|$ is uniformly convergent on any compact subset in $\Omega_\Gamma - \cup_{\gamma \in \Gamma} \gamma(\infty)$, and hence the assertion holds for $\omega_{t,k}$. If $a \in \Omega_\Gamma - \cup_{\gamma \in \Gamma} \gamma(\infty)$, then $\lim_{n \to \infty} \gamma_i^{+\infty}(a) = \alpha_{\pm i}$, and hence

$$d \left( \int_a \sum_{\gamma \in \Gamma} \frac{d\gamma(\zeta)}{\gamma(\zeta) - z} \right) = \sum_{\gamma \in \Gamma} \left( \frac{1}{z - (\gamma \gamma_i)(a)} - \frac{1}{z - \gamma(a)} \right) dz$$

$$= \sum_{\gamma \in \Gamma/\Gamma} \lim_{n \to \infty} \left( \frac{1}{z - (\gamma \gamma^n_i)(a)} - \frac{1}{z - (\gamma \gamma^n_i)(a)} \right) dz$$

$$= \omega_i(z).$$

Therefore, $\omega_i$ is absolutely and uniformly convergent on any compact subset in $\Omega_\Gamma$, and hence is an abelian differential on $\Omega_\Gamma/\Gamma$. □

4.2. Stability of universal differentials

For a vertex $v \in V$, denote by $C_v$ the corresponding irreducible component of $C_\Delta \otimes_A \Delta R_\Delta$. Then $P_v = \mathbb{P}_1 R_\Delta$ is the normalization of $C_v$.

**Theorem 4.3.**

1. For each $1 \leq i \leq g$, let $\phi_{h_i(1)} \cdots \phi_{h_i(l_i)}$ $(h_i(j) \in \pm E)$ be the unique reduced product such that $v_{-h_i(j)} = v_{h_i(j+1)}$ and $h_i(1) \neq -h_i(l_i)$ which is conjugate to $\gamma_i$. Then for each $v \in V$, the pullback $(\omega_i|_{C_v})^* \omega_i|_{C_v}$ to $P_v$ is given by

$$\left( \sum_{v_{h_i(j)} = v} \frac{1}{z - x_{h_i(j)}} - \sum_{v_{-h_i(l)} = v} \frac{1}{z - x_{-h_i(l)}} \right) dz.$$ 

2. For each $v \in V$, $(\omega_{t,k}|_{C_v})^*$ is given by $\frac{dz}{(z - x_e)^k}$ if $v = v_e$, and is 0 otherwise.

3. Denote by $\rho_{t_j} = h_j(1) \cdots h_j(l_j)$ the unique path from $v_{t_j} (t_j \in T)$ to $v_b$ in $T_\Delta$. Then for each $v \in V$, $\omega_{t_1, t_2|C_v}$ is given by

$$\left( \sum_{v_{h_i} = v} \frac{1}{z - x_{h_i}} - \sum_{v_{-h_i} = v} \frac{1}{z - x_{-h_i}} \right) dz,$$

where $h, k$ runs through \{ $t_1, h_1(1), ..., h_1(l_1), -t_2, -h_2(1), ..., -h_2(l_2)$ \}.

**Proof.** For the proof of (1), we may assume that $\gamma_i = \phi_{h_i(1)} \cdots \phi_{h_i(l_i)}$. Let $\gamma$ be an element of $\Gamma_\Delta$. Then by Proposition 4.1 (2), putting $y_e = 0$ ($e \in E$),

$$\frac{1}{z - \gamma(\alpha_i)} - \frac{1}{z - \gamma(\alpha_{-i})} = \frac{\gamma(\alpha_i) - \gamma(\alpha_{-i})}{(z - \gamma(\alpha_i))(z - \gamma(\alpha_{-i}))}.$$
is called a stable differential \( \{ \text{cf. [4]} \} \).

**Theorem 4.5.**

1. For each \( 1 \leq i \leq g \), \( \omega_i \) is a regular stable differential on \( C_\Delta \). Furthermore, \( \{ \omega \}_{1 \leq i \leq g} \) gives a basis of \( H^0 \left( C_\Delta, \omega_{C_\Delta/A_\Delta} \right) \).

2. For each \( t \in T \) with \( v_t = v_k \) and \( k > 1 \), \( \omega_{t,k} \) is a meromorphic stable differential on \( C_\Delta \) which has only pole (of order \( k \)) at the point \( p_t \) corresponding to \( t \).

3. For each \( t_1, t_2 \in T \) with \( t_1 \neq t_2 \), \( \omega_{t_1,t_2} \) is a meromorphic stable differential on \( C_\Delta \) which has only (simple) poles at the points \( p_{t_1} \) (resp. \( p_{t_2} \)) corresponding to \( t_1 \) (resp \( t_2 \)) with residue 1 (resp. -1).

**Proof.** We only show that the latter assertion in (1) since the remains follow from Theorems 3.1 and 3.2. For each \( i \), take the product \( \phi_{h_i(1)} \cdots \phi_{h_i(l_i)} \) of \( \phi_h \) \( (h \in \pm E) \) as in Theorem 4.3 (1). Then by this theorem, \( \omega_i |_{C_{\phi_{h_i(1)}}} \) has simple pole at \( x_{h_i(1)} \) with residue 1, and hence \( \{ \omega_i \}_{1 \leq i \leq g} \) gives a basis of \( H^0 \left( C_\Delta, \omega_{C_\Delta/A_\Delta} \right) \) since \( \{ \gamma_i \} \) is a basis of \( H_1(\Delta, Z) \cong \Gamma_\Delta / [\Gamma_\Delta, \Gamma_\Delta] \). \( \square \)

### 4.3. Periods and Gauss-Manin connection

**Definition 4.6.** The formal Schottky space \( S_{g,n} \) is defined as the covering space of
\( \mathcal{N}_{g,n}^{\text{alg}} - D_{g,n} \) which classifies generators of uniformizing groups associated with points on \( \mathcal{N}_{g,n}^{\text{alg}} - D_{g,n} \). Then by Theorem 3.3, there exists the universal Mumford curve \( \pi : \mathcal{C}_{g,n} \to \mathcal{S}_{g,n} \) with generators of the uniformizing group.

We describe the sheaf
\[
\mathcal{H}_{1}^{\text{dR}} (\mathcal{C}_g / \mathcal{S}_{g,n}) = \pi^* (\Omega^{\bullet}_{\mathcal{C}_g / \mathcal{S}_{g,n}})
\]
on \mathcal{S}_{g,n} of the first relative de Rham cohomology groups, and the associated period map, where \( \mathcal{C}_g \) denotes \( \mathcal{C}_{g,n} \) without marked points. Denote by \( \Delta_0 \) the stable graph of \((g,n)\)-type consisting of one vertex and \(g\) loops, and fix generators \( \rho_1, \ldots, \rho_g \) of \( \pi_1(\Delta_0) \).

Then \( \{\rho_i\}_{1 \leq i \leq g} \) gives a system of basis of \( H^1(\Delta, \mathbb{Z}) \) for stable graphs \( \Delta = (V,E,T) \) of \((g,n)\)-type which is compatible with contraction of nonloop edges in \( E \), and hence there exist generators \( \gamma_i \) \((1 \leq i \leq g)\) of \( \Gamma_\Delta \) corresponding to \( \rho_i \). Therefore, by Theorem 3.3, the associated universal differentials \( \omega_i \) (resp. \( \omega_{t,k} \) \((t \in T, k > 1)\)) of the first (resp. second) kind are glued to differentials on \( \mathcal{C}_g / \mathcal{S}_{g,n} \) of the first (resp. second) kind which we denote by the same symbols.

**Theorem 4.7.**

1. The above differentials \( \omega_i \) \((1 \leq i \leq g)\) of the first kind make a basis of the sheaf \( \pi^* (\Omega^{\bullet}_{\mathcal{C}_g / \mathcal{S}_{g,n}}) \) on \( \mathcal{S}_{g,n} \).

2. The Jacobian of \( \mathcal{C}_g / \mathcal{S}_{g,n} \) becomes the Mumford abelian scheme \([19]\) whose multiplicative periods on \( \text{Spec}(\mathcal{A}_\Delta) \) is given in \([10, 3.10]\) as
\[
P_{ij} = \prod \psi_{ij}(\gamma) \quad (1 \leq i, j \leq g),
\]
where \( \gamma \) runs through all representatives of \( \langle \gamma_i \rangle \backslash \Gamma_\Delta / \langle \gamma_j \rangle \) and
\[
\psi_{ij}(\gamma) = \begin{cases} 
\beta_i & (i = j, \gamma \in \langle \gamma_i \rangle), \\
\frac{\alpha_i - \gamma(\alpha_j)(\alpha_i - \gamma)(\alpha_j)}{\alpha_i - \gamma(\alpha_j)(\alpha_i - \gamma)} & (\text{otherwise}).
\end{cases}
\]

**Proof.** The assertion (1) follows from Theorem 3.3 by gluing \( \omega_i \) on \( \mathcal{C}_\Delta \) given in Theorem 4.5 (1), and the assertion (2) follows from Theorem 3.3 by gluing the Jacobians of \( \mathcal{C}_\Delta \) given in \([10, \text{Theorem 3.13}]\). \( \square \)

**Theorem 4.8.** Assume that \( n > 0 \). Then there exists a subsheaf \( \mathcal{H}^{0,1} \) of
\[
\mathcal{H}_{1}^{\text{dR}} (\mathcal{C}_g / (\mathcal{S}_{g,n} \otimes \mathbb{Q}))
\]
with basis \( \{\eta_{ij}\}_{1 \leq j \leq g} \) which satisfies the following:

1. We have a direct sum decomposition
\[
\mathcal{H}_{1}^{\text{dR}} (\mathcal{C}_g / (\mathcal{S}_{g,n} \otimes \mathbb{Q})) = \pi^* (\Omega^{\bullet}_{\mathcal{C}_g / (\mathcal{S}_{g,n} \otimes \mathbb{Q})}) \oplus \mathcal{H}^{0,1}.
\]

2. The Gauss-Manin connection
\[
\nabla : \mathcal{H}_{1}^{\text{dR}} (\mathcal{C}_g / (\mathcal{S}_{g,n} \otimes \mathbb{Q})) \to \mathcal{H}_{1}^{\text{dR}} (\mathcal{C}_g / (\mathcal{S}_{g,n} \otimes \mathbb{Q})) \otimes \Omega_{\mathcal{S}_{g,n}}
\]
satisfies
\[ \nabla(\omega_i) = \sum_{j=1}^{g} \eta_j \otimes (dP_{ij}/P_{ij}), \ \nabla(\eta_i) = 0 \quad (1 \leq i \leq g). \]
(3) The decomposition in (1) gives the Hodge decomposition under regarding \( C_g/S_{g,n} \) as a family of Riemann surfaces.

Proof. We regard \( C_g \) as the universal family of Mumford curves over \( S_{g,n} \otimes \mathbb{Q} \). Then as is shown in [5, Proposition 2.5], relative differentials of the second kind modulo exact differentials on \( C_g/(S_{g,n} \otimes \mathbb{Q}) \) gives a subsheaf \( H^0,1 \) of \( H^1_{dR}(C_g/(S_{g,n} \otimes \mathbb{Q})) \) which is locally free of rank \( g \) and satisfies (1).

As is stated in 2.2 (P4), \( C_g \) gives a family of Riemann surfaces of genus \( g \), and hence by the algebraicity of the Gauss-Manin connection \( \nabla \), there exist sections \( \eta_j \) (\( 1 \leq j \leq g \)) of \( H^0,1 \) determined by the condition that \( \int_{\gamma_i} \eta_j \) is the Kronecker delta \( \delta_{ij} \), and by Theorem 4.7, \( \{\eta_j\}_j \) satisfies (2). By Proposition 4.1, for the universal differentials \( \omega_t,k \) (\( t \in T, k > 1 \)) of the second kind, \( \int_{\gamma_i} \omega_t,k \) are expressed as elements of \( S_{g,n} \otimes \mathbb{Q} \), and hence \( \omega_t,k \) are linear sums of \( \eta_j \) in \( H^0,1 \). Furthermore, by Theorems 4.3 (2) and 4.5 (2), \( \omega_t,k \) have vanishing annular residues which implies that \( H^0,1 \) is spanned by \( \{\omega_t,k\}_{t,k} \), and hence by \( \{\eta_j\}_j \). Therefore, the assertion (3) also holds.

We give a local explicit formula of the above \( \eta_j \) on \( \text{Spec}(A_{\Delta_0} \otimes \mathbb{Q}) \), where \( \Delta_0 \) be the above stable graph consisting of one vertex.

**Theorem 4.9.** Assume that \( n > 0 \), and take an tail \( t_0 \) of \( \Delta_0 \). Then each \( \eta_j \in H^0,1 \) defined in Theorem 4.8 is a linear sum of \( \omega_{t_0,k} \) (\( 2 \leq k \leq g+1 \)) over \( A_{\Delta_0} \otimes \mathbb{Q} \).

Proof. We may assume \( t_0 \not\in E_\infty \). Then by Proposition 4.1, \( \int_{\gamma_i} \omega_{t_0,k} \) (\( k > 1 \)) is expressed as an element of \( A_{\Delta_0} \otimes \mathbb{Q} \) whose constant term is
\[ \int_{\infty}^{\alpha_i} \frac{dz}{(z-x_{t_0})^k} = \frac{1}{(-k+1)(\alpha_i-x_{t_0})^{k-1}}. \]
In order to prove the assertion, it is enough to show that \( \Omega = \det(\int_{\gamma_i} \omega_{t_0,j+1})_{1 \leq i,j \leq g} \) is an invertible element of \( A_{\Delta_0} \otimes \mathbb{Q} \). By assumption, \( \alpha_i - \alpha_j \in A_{\Delta_0}^\times \) for \( i \neq j \), and hence the Vandermonde determinant
\[ \det\left(\frac{1}{(\alpha_i-x_{t_0})^{j}}\right)_{1 \leq i,j \leq g} = \prod_{i=1}^{g} \frac{1}{(\alpha_i-x_{t_0})} \prod_{1 \leq i<j \leq g} \frac{\alpha_i - \alpha_j}{(\alpha_i-x_{t_0})(\alpha_j-x_{t_0})} \]
belongs to \( A_{\Delta_0}^\times \). Therefore, \( \Omega \) becomes an invertible element of \( A_{\Delta_0} \otimes \mathbb{Q} \). □

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References

[1] M. Bainbridge and M. Möller, The Deligne-Mumford compactification of the real multiplication locus and Teichmüller curves in genus 3, Acta Math. 208 (2012) 1–92.

[2] R. Coleman, Dilogarithms, regulators, and p-adic L-functions, Invent. Math. 69 (1982) 171–208.

[3] R. Coleman and E. de Shalit, P-adic regulators on curves and special values of p-adic L-functions, Invent. Math. 93 (1988) 239–266.

[4] P. Deligne, D. Mumford, The irreducibility of the space of curves of given genus, Publ. Math. IHES 36 (1969) 75–109.

[5] E. de Shalit, Differentials of the second kind on Mumford curves, Israel J. of Math. 71 (1990) 1–16.

[6] E. de Shalit, Coleman integration versus Schneider integration on semistable curves, Doc. Math. Extra Volume Coates (2006) 325–334.

[7] L. Gerritzen, Periods and Gauss-Manin connection for families of p-adic Schottky groups, Math. Ann. 275 (1986) 425–453.

[8] X. Hu and C. Norton, General variational formulas for abelian differentials, to appear in IMRN.

[9] T. Ichikawa, P-adic theta functions and solutions of the KP hierarchy, Comm. Math. Phys. 176 (1996) 383–399.

[10] T. Ichikawa, Generalized Tate curve and integral Teichmüller modular forms, Amer. J. Math. 122 (2000) 1139–1174.

[11] T. Ichikawa, Teichmüller groupoids and Galois action, J. reine angew. Math. 559 (2003) 95–114.

[12] T. Ichikawa, Teichmüller groupoids, and monodromy in conformal field theory, Commun. Math. Phys. 246 (2004) 1–18.

[13] T. Ichikawa, Klein’s formulas and arithmetic of Teichmüller modular forms, Proc. Amer. Math. Soc. 146 (2018) 5105–5112.

[14] T. Ichikawa, An explicit formula of the normalized Mumford form, arXiv:1812.08331.

[15] T. Ichikawa, Chern-Simons invariant and Deligne-Riemann-Roch isomorphism, to appear in Trans. Amer. Math. Soc.
[16] Y. Ihara, H. Nakamura, On deformation of maximally degenerate stable marked curves and Oda’s problem, J. reine angew. Math. 487 (1997) 125–151.

[17] Yu. Manin, V. Drinfeld, Periods of $p$-adic Schottky groups, J. Reine Angew. Math. 262/263 (1972) 239–247.

[18] D. Mumford, An analytic construction of degenerating curves over complete local rings, Compos. Math. 24 (1972) 129–174.

[19] D. Mumford, An analytic construction of degenerating abelian varieties over complete rings, Compos. Math. 24 (1972) 239–272.

[20] J. Poineau, La droite de Berkovich sur $\mathbb{Z}$, Astérisque 334 (2010) xii+284.

[21] J. Poineau, Espaces de Berkovich sur $\mathbb{Z}$: étude locale. Invent. Math. 194 (2013) 535–590.

[22] J. Poineau and D. Turchetti, Schottky spaces and universal Mumford curves over $\mathbb{Z}$, https://poineau.users.lmno.cnrs.fr/Textes/MumfordZ.pdf.

[23] M. Raynaud, Géométrie analytique rigide d’après Tate, Kiehl... Mémoires de la S. M. F., tome 39–40 (1974) 319-327.

[24] F. Schottky, Über eine specielle Function, welche bei einer bestimmten linearen Transformation ihres Arguments unverändert bleibt, J. reine angew. Math. 101 (1887) 227–272.

[25] M. Ulirsch, A non-archimedean analogue of Teichmüller space and its tropicalization, arXiv:2004.07508.