Extension criteria for homogeneous Sobolev spaces
of functions of one variable

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Abstract

For each $p > 1$ and each positive integer $m$ we give intrinsic characterizations of the restriction of the homogeneous Sobolev space $L^m_p(\mathbb{R})$ to an arbitrary closed subset $E$ of the real line.

We show that the classical one dimensional Whitney extension operator [52] is “universal” for the scale of $L^m_p(\mathbb{R})$ spaces in the following sense: for every $p \in (1, \infty]$ it provides almost optimal $L^m_p$-extensions of functions defined on $E$. The operator norm of this extension operator is bounded by a constant depending only on $m$. This enables us to prove several constructive $L^m_p$-extension criteria expressed in terms of $m^{th}$ order divided differences of functions.

Contents

1. Introduction. 2
2. Main Theorems: necessity. 6
   2.1. Divided differences: main properties. 7
   2.2. Proofs of the necessity parts of the main theorems. 7
3. The Whitney extension method in $\mathbb{R}$ and traces of Sobolev functions. 9
   3.1. Interpolation knots and their properties. 9
   3.2. Lagrange polynomials and divided differences at interpolation knots. 12
   3.3. Whitney $m$-fields and Hermite polynomials. 16
   3.4. Extension criteria in terms of sharp maximal functions: sufficiency. 19
4. A variational extension criterion for Sobolev jets. 22
5. The Main Lemma: from jets to Lagrange polynomials. 28
6. The variational extension criterion: sufficiency. 38
7. The Finiteness Principle for $L^m_\infty(\mathbb{R})$ traces: multiplicative finiteness constants. 43

References 45

Math Subject Classification 46E35
Key Words and Phrases Homogeneous Sobolev space, trace space, divided difference, extension operator.
This research was supported by Grant No 2014055 from the United States-Israel Binational Science Foundation (BSF).
1. Introduction.

In this paper we characterize the restrictions of homogeneous Sobolev functions of one variable to an arbitrary closed subset of the real line. For each $m \in \mathbb{N}$ and each $p \in (1, \infty]$, we consider $L_p^m(\mathbb{R})$, the standard homogeneous Sobolev space on $\mathbb{R}$. We identify $L_p^m(\mathbb{R})$ with the space of all real valued functions $F$ on $\mathbb{R}$ such that the $(m-1)$-th derivative $F^{(m-1)}$ is absolutely continuous on $\mathbb{R}$ and the weak $m$-th derivative $F^{(m)} \in L_p(\mathbb{R})$. $L_p^m(\mathbb{R})$ is seminormed by $\|F\|_{L_p^m(\mathbb{R})} = \|F^{(m)}\|_{L_p(\mathbb{R})}$.

In this paper we study the following

**Problem 1.1** Let $p \in (1, \infty]$, $m \in \mathbb{N}$, and let $E$ be a closed subset of $\mathbb{R}$. Let $f$ be a function on $E$. We ask two questions:

1. How can we decide whether there exists a function $F \in L_p^m(\mathbb{R})$ such that the restriction $F|_E$ of $F$ to $E$ coincides with $f$?
2. Consider the $L_p^m(\mathbb{R})$-seminorms of all functions $F \in L_p^m(\mathbb{R})$ such that $F|_E = f$. How small can these seminorms be?

We denote the infimum of all these seminorms by $\|f\|_{L_p^m(\mathbb{R})|_E}$; thus

$$\|f\|_{L_p^m(\mathbb{R})|_E} = \inf\{\|F\|_{L_p^m(\mathbb{R})}: F \in L_p^m(\mathbb{R}), F|_E = f\}. \quad (1.1)$$

We refer to $\|f\|_{L_p^m(\mathbb{R})|_E}$ as the *trace norm on $E$ of the function $f$* in $L_p^m(\mathbb{R})$. This quantity provides the standard quotient space seminorm in the *trace space* $L_p^m(\mathbb{R})|_E$ of all restrictions of $L_p^m(\mathbb{R})$-functions to $E$, i.e., in the space

$$L_p^m(\mathbb{R})|_E = \{f : E \to \mathbb{R} : \text{there exists } F \in L_p^m(\mathbb{R}) \text{ such that } F|_E = f\}.$$

Whitney [52] completely solved an analog of part 1 of Problem 1.1 for the space $C^m(\mathbb{R})$. Whitney’s extension construction [52] produces a certain *extension operator*

$$\mathcal{F}^{(Wh)}_{m,E} : C^m(\mathbb{R})|_E \to C^m(\mathbb{R}) \quad (1.2)$$

which *linearly and continuously* maps the trace space $C^m(\mathbb{R})|_E$ into $C^m(\mathbb{R})$. (See also Merrien [37].)

In fact the extension method developed by Whitney in [52] readily adapts to also provide a complete solution to Problem 1.1 for the space $L_p^m(\mathbb{R})$. Recall that $L_p^m(\mathbb{R})$ can be identified with the space $C^{m-1,1}(\mathbb{R})$ of all $C^{m-1}$-functions on $\mathbb{R}$ whose derivatives of order $m - 1$ satisfy a Lipschitz condition. In particular, the method of proof and technique developed in [52] and [37] lead us to the following well known description of the trace space $L_p^m(\mathbb{R})|_E$: A function $f \in L_p^m(\mathbb{R})|_E$ if and only if the following quantity

$$\mathcal{L}_{m,\infty}(f : E) = \sup_{S \subseteq E, |S| = m+1} |\Delta^m f[S]|$$

is *finite*. Here $\Delta^m f[S]$ denotes the $m^{th}$ order divided difference of $f$ on an $(m + 1)$-point set $S$.

Furthermore,

$$C_1 \mathcal{L}_{m,\infty}(f : E) \leq \|f\|_{L_p^m(\mathbb{R})|_E} \leq C_2 \mathcal{L}_{m,\infty}(f : E) \quad (1.3)$$

where $C_1$ and $C_2$ are positive constants depending only on $m$. (Recall that $\Delta^m f[S]$ coincides with the coefficient of $x^m$ in the Lagrange polynomial of degree at most $m$ which agrees with $f$ on $S$. See Section 2.1 for other equivalent definitions of divided difference and their main properties.)

We refer the reader to [33, 52] for further results in this direction.
There is an extensive literature devoted to a special case of Problem 1.1 where \( E \) consists of all the elements of a strictly increasing sequence \( \{x_i\}_{i=1}^\infty \) (finite, one-sided infinite, or bi-infinite). We refer the reader to the papers of Favard [17], Chui, Smith, Ward [9][10][50], de Boor [11-14], Fisher, Jerome [25], Golomb [28], Jakimovski, Russell [30], Kunkle [35], Pinkus [38], Schoenberg [40] and references therein for numerous results in this direction and techniques for obtaining them.

In particular, for the space \( L^m_\infty(\mathbb{R}) \) Favard [17] developed a powerful linear extension method (very different from Whitney’s method [52]) based on a certain delicate duality argument. Note that for any set \( E \) as above and every \( f : E \to \mathbb{R} \), Favard’s extension operator \( F_{m,E} \) yields an extension of \( f \) with the smallest possible seminorm in \( L^m_\infty(\mathbb{R}) \). (Thus, \( \|f\|_{L^m_\infty(\mathbb{R})} \leq \|F_{m,E}(f)\|_{L^m_\infty(\mathbb{R})} \) for every function \( f \) defined on \( E \).) Note also that Favard’s approach leads to the following slight refinement of (1.3):

\[
\|f\|_{L^m_\infty(\mathbb{R})} \sim \sup_{\ell_1 \leq i \leq \ell_2 - m} |\Delta^m f[x_1, ..., x_{i+m}]|.
\]

See Section 7 for more details.

Modifying Favard’s extension construction, de Boor [11-13] characterized the traces of \( L^m_\infty(\mathbb{R}) \)-functions to arbitrary sequences of points in \( \mathbb{R} \).

**Theorem 1.2** (17) Let \( p \in (1, \infty) \), and let \( \ell_1, \ell_2 \in \mathbb{Z} \cup \{\pm \infty\} \), \( \ell_1 + m \leq \ell_2 \). Let \( f \) be a function defined on a strictly increasing sequence of points \( E = \{x_i\}_{i=\ell_1}^{\ell_2} \). Then \( f \in L^m_p(\mathbb{R})_E \) if and only if the following quantity

\[
\overline{L}_{m,p}(f : E) = \left( \sum_{i=\ell_1}^{\ell_2 - m} (x_{i+m} - x_i) |\Delta^m f[x_1, ..., x_{i+m}]|^p \right)^{\frac{1}{p}}
\]

is finite. Furthermore, \( \|f\|_{L^m_p(\mathbb{R})_E} \sim \overline{L}_{m,p}(f : E) \) with constants of equivalence depending only on \( m \).

For a special case of this result, for sequences satisfying some global mesh ratio restrictions, see Golomb [28]. See also Estévez [16] for an alternative proof of Theorem 1.2 for \( m = 2 \).

Using a certain limiting argument, Golomb [28, Theorem 2.1] showed that Problem 1.1 for \( L^m_\infty(\mathbb{R}) \) and an arbitrary set \( E \subset \mathbb{R} \) can be reduced to the same problem, but for arbitrary finite sets \( E \). More specifically, his result (in an equivalent form) provides the following formula for the trace norm in \( L^m_p(\mathbb{R})_E \):

\[
\|f\|_{L^m_p(\mathbb{R})_E} = \sup\{ \|f\|_{L^m_p(\mathbb{R})_{E'}} : E' \subset E, \#E' < \infty \}.
\]

Let us remark that, by combining this formula with de Boor’s Theorem 1.2, we can obtain the following description of the trace space \( L^m_p(\mathbb{R})_E \) for an arbitrary closed set \( E \subset \mathbb{R} \).

**Theorem 1.3** (Variational extension criterion for \( L^m_p(\mathbb{R}) \)-traces) Let \( p \in (1, \infty) \) and let \( m \) be a positive integer. Let \( E \subset \mathbb{R} \) be a closed set containing at least \( m + 1 \) points. A function \( f : E \to \mathbb{R} \) can be extended to a function \( F \in L^m_p(\mathbb{R}) \) if and only if the following quantity

\[
\mathcal{L}_{m,p}(f : E) = \sup_{\{x_0, ..., x_n\} \subset E} \left( \sum_{i=0}^{n-m} (x_{i+m} - x_i) |\Delta^m f[x_1, ..., x_{i+m}]|^p \right)^{\frac{1}{p}}
\]

is finite. Here the supremum is taken over all all integers \( n \geq m \) and all strictly increasing sequences \( \{x_0, ..., x_n\} \subset E \) of \( n \) elements. Furthermore,

\[
\|f\|_{L^m_p(\mathbb{R})_E} \sim \mathcal{L}_{m,p}(f : E).
\]

The constants of equivalence (1.5) depend only on \( m \).
In the present paper we give a direct and explicit proof of Theorem 1.3 which does not use any limiting argument. Actually we show, perhaps surprisingly, that the very same Whitney extension operator \( T_{m,E}^{(Wh)} \) (see (1.2)) which was introduced in [52] for characterization of the trace space \( C^m(R)\mid E \), provides almost optimal extensions of functions belonging to \( L^p(R)\mid E \) for every \( p \in (1,\infty) \).

We also give another characterization of the trace space \( L^m_p(R)\mid E \) expressed in terms of \( L^p \)-norms of certain kinds of “sharp maximal functions” which are defined as follows:

For each \( m \in \mathbb{N} \), each closed set \( E \subset \mathbb{R} \) with \( \# E > m \), and each function \( f : E \to \mathbb{R} \) we let \((\Delta^m f)^\sharp_E\) denote the maximal function associated with \( f \) which is given by

\[
(\Delta^m f)^\sharp_E (x) = \sup_{\{x_0, \ldots, x_n\} \subset E \atop x_0 < x_1 < \ldots < x_n} \frac{|\Delta^{m-1} f[x_0, \ldots, x_{m-1}] - \Delta^{m-1} f[x_1, \ldots, x_m]|}{|x - x_0| + |x - x_m|}, \quad x \in \mathbb{R}.
\]  

(1.6)

**Theorem 1.4** Let \( p \in (1,\infty) \), \( m \in \mathbb{N} \), and let \( f \) be a function defined on a closed set \( E \subset \mathbb{R} \). The function \( f \in L^m_p(R)\mid E \) if and only if \((\Delta^m f)^\sharp_E \in L^p(R)\). Furthermore,

\[
\|f\|_{L^m_p(R)\mid E} \sim \|\Delta^m f\|^\sharp_E \|_{L^p(R)}
\]

with the constants in this equivalence depending only on \( m \) and \( p \).

Note that

\[
(\Delta^m f)^\sharp_E (x) \leq \sup_{S \subset E, \# S = m+1} \frac{|\Delta^m f[S]|}{\operatorname{diam}(S)} \leq 2(\Delta^m f)^\sharp_E (x) \quad \text{for all} \quad x \in \mathbb{R}.
\]  

(1.7)

(See property (2.2) below.) This inequality, together with Theorem 1.4 and the definition in (1.6) now imply the following explicit formulae for the trace seminorm of a function in the space \( L^m_p(R)\mid E \):

\[
\|f\|_{L^m_p(R)\mid E} \sim \left\{ \int_{\mathbb{R}} \sup_{\{x_0, \ldots, x_n\} \subset E \atop x_0 < x_1 < \ldots < x_n} \left( \frac{|\Delta^{m-1} f[x_0, \ldots, x_{m-1}] - \Delta^{m-1} f[x_1, \ldots, x_m]|^p}{|x - x_0|^p + |x - x_m|^p} \right) \, dx \right\}^{\frac{1}{p}}
\]

\[
\sim \left\{ \int_{\mathbb{R}} \sup_{S \subset E, \# S = m+1} \left( \frac{|\Delta^m f[S]|}{\operatorname{diam}(S)} \right)^p \, dx \right\}^{\frac{1}{p}}.
\]

We feel a strong debt to the remarkable papers of Calderón and Scott [7, 8] which are devoted to characterization of Sobolev spaces on \( \mathbb{R}^n \) in terms of classical sharp maximal functions. These papers motivated us to formulate and subsequently prove Theorem 1.4.

For analogs of Theorems 1.3 and 1.4 for the space \( L^1_p(\mathbb{R}^n) \), \( n \in \mathbb{N} \), \( n < p < \infty \), we refer the reader to [45, 47].

Our next new result, Theorem 1.5 below, states that there exists a solution to Problem 1.1 which depends linearly on the initial data, i.e., the functions defined on \( E \).

**Theorem 1.5** For every closed subset \( E \subset \mathbb{R} \), every \( p > 1 \) and every \( m \in \mathbb{N} \) there exists a continuous linear extension operator which maps the trace space \( L^m_p(R)\mid E \) into \( L^m_p(R) \). Its operator norm is bounded by a constant depending only on \( m \).
Let us recall something of the history of the previous results which led us to Theorem 1.5. We know that for each closed $E \subset \mathbb{R}$ the Whitney extension operator $\mathcal{F}^{(Wh)}_{m,E}$ maps $L^m_{\infty}(\mathbb{R})|_E$ into $L^m_{\infty}(\mathbb{R})$ with the operator norm $\|\mathcal{F}^{(Wh)}_{m,E}\|$ bounded by a constant depending only on $m$. As we have mentioned above, if $E$ is a sequence of points in $\mathbb{R}$, Favard’s linear extension operator also maps $L^m_{\infty}(\mathbb{R})|_E$ into $L^m_{\infty}(\mathbb{R})$, but with the operator norm $\|\mathcal{F}^{(Favard)}_{m,E}\| = 1$.

For $p \in (1, \infty)$ and an arbitrary sequence $E \subset \mathbb{R}$ Theorem 1.5 follows from [11, Section 4]. Luli [36] gave an alternative proof of Theorem 1.5 for the space $L^p_{\infty}(\mathbb{R})$ and a finite set $E$. In the multidimensional case the existence of corresponding linear continuous extension operators for the Sobolev spaces $L^p_{m}(\mathbb{R}^n)$, $n < p < \infty$, was proven in [45] ($m = 1$, $n \in \mathbb{N}$, $E \subset \mathbb{R}^n$ is arbitrary), [29] and [46] ($m = 2$, $n = 2$, $E \subset \mathbb{R}^2$ is finite), and [23] (arbitrary $m, n \in \mathbb{N}$ and an arbitrary $E \subset \mathbb{R}^n$). For the case $p = \infty$ see [5] ($m = 2$) and [19,20] ($m \in \mathbb{N}$).

In a forthcoming paper [49] we will present a solution to an analog of Problem 1.1 for the normed Sobolev space $W^m_p(\mathbb{R})$.

Let us briefly describe the structure of the present paper and the main ideas of our approach.

First we note that the equivalence (1.5) is not trivial even in the simplest case, i.e., for $E = \mathbb{R}$; in this case (1.5) tells us that for every $f \in L^m_{p}(\mathbb{R})$ and every $p \in (1, \infty)$

$$\|f\|_{L^m_{p}(\mathbb{R})} \sim \mathcal{L}_{m,p}(f : \mathbb{R})$$

(1.8)

with constants depending only on $m$. In other words, the quantity $\mathcal{L}_{m,p}(\cdot : \mathbb{R})$ provides an equivalent seminorm on $L^m_{p}(\mathbb{R})$. This characterization of the space $L^m_{p}(\mathbb{R})$ is known in the literature; see F. Riesz [39] ($m = 1$ and $1 < p < \infty$), Schoenberg [40] ($p = 2$ and $m \in \mathbb{N}$), and Jerome and Schumaker [32] (arbitrary $m \in \mathbb{N}$ and $p \in (1, \infty)$). Of course, the equivalence (1.8) implies the necessity part of Theorem 1.3. Nevertheless, for the reader’s convenience, in Section 2.2 we give a short direct proof of this result (together with the proof of the necessity part of Theorem 1.4).

In Section 3 we recall the Whitney extension method [52] for functions of one variable. We prove a series of auxiliary statements which enable us to adapt Whitney’s construction to extension of $L^m_{p}(\mathbb{R})$-functions. We then use this extension technique and a criterion for extension of Sobolev jets [47] to help us prove the sufficiency part of Theorem 1.3. (See Section 3.4.)

The sufficiency part of Theorem 1.3 is proven in Sections 4-6. One of the main ingredients of this proof is Theorem 4.1, a refinement of the extension criterion given in Theorem 3.16. Another important ingredient of the proof of the sufficiency is Main Lemma 5.1 which provides a certain controlled transition from Hermite polynomials of a function to its Lagrange polynomials. See Section 5.

In Section 6, with the help of these results, Theorem 4.1 and Main Lemma 5.1 we complete the proof of the sufficiency part of Theorem 1.3.

In Section 7 we discuss the dependence on $m$ of the constants $C_1, C_2$ in inequality (1.3). We interpret this inequality as a particular case of the Finiteness Principle for traces of smooth functions. (See Theorem 7.1.) We refer the reader to [4,5,18,21,44] and references therein for numerous results related to the Finiteness Principle.

For the space $L^m_{\infty}(\mathbb{R})$ the Finiteness Principle is equivalent to the following statement: there exists a constant $\gamma = \gamma(m)$ such that for every closed set $E \subset \mathbb{R}$ and every $f \in L^m_{\infty}(\mathbb{R})|_E$ the following inequality

$$\|f\|_{L^m_{\infty}(\mathbb{R})|_E} \leq \gamma \sup_{S \subset E, \#S = m+1} \|f|_S\|_{L^{m}_{\infty}(\mathbb{R})|_S}$$

(1.9)

holds. We can express this result by stating that the number $m + 1$ is a finiteness number for the space $L^m_{\infty}(\mathbb{R})$. We also refer to any constant $\gamma$ which satisfies (1.9) as a multiplicative finiteness constant for
the space \( L^m_\infty(\mathbb{R}) \). In this context we let \( \gamma^\alpha(L^m_\infty(\mathbb{R})) \) denote the infimum of all multiplicative finiteness constants for \( L^m_\infty(\mathbb{R}) \) for the finiteness number \( m+1 \).

One can easily see that \( \gamma^\alpha(L^1_\infty(\mathbb{R})) = 1 \). In Theorem 7.3 we show that

\[
\gamma^\alpha(L^2_\infty(\mathbb{R})) = 2 \quad \text{and} \quad (\pi/2)^{m-1} < \gamma^\alpha(L^m_\infty(\mathbb{R})) < (m-1) 9^m \quad \text{for every} \quad m > 2. \tag{1.10}
\]

The proof of (1.10) relies on results of Favard [17] and de Boor [11, 12] devoted to calculation of certain extension constants for the space \( L^m_\infty(\mathbb{R}) \). See Section 7 for more details.

Readers might find it helpful to also consult a much more detailed version of this paper posted on the arXiv [48].

Acknowledgements. I am very thankful to M. Cwikel for useful suggestions and remarks. I am grateful to Charles Fefferman, Bo’az Klartag and Yuri Brudnyi for valuable conversations. The results of this paper were presented at the 11th Whitney Problems Workshop, Trinity College Dublin, Dublin, Ireland. I am very thankful to all participants of this conference for stimulating discussions and valuable advice.

2. Main Theorems: necessity.

Let us fix some notation. Throughout the paper \( C, C_1, C_2, \ldots \) will be generic positive constants which depend only on \( m \) and \( p \). These symbols may denote different constants in different occurrences. The dependence of a constant on certain parameters is expressed by the notation \( C = C(m) \), \( C = C(p) \) or \( C = C(m, p) \). Given constants \( \alpha, \beta \geq 0 \), we write \( \alpha \sim \beta \) if there is a constant \( C \geq 1 \) such that \( \alpha/C \leq \beta \leq C \alpha \).

Given a measurable set \( A \subset \mathbb{R} \), we let \( |A| \) denote the Lebesgue measure of \( A \). If \( A \subset \mathbb{R} \) is finite, by \#\( A \) we denote the number of elements of \( A \).

Given \( A, B \subset \mathbb{R} \), let

\[
diam A = \sup \{|a-a'| : a, a' \in A\} \quad \text{and} \quad dist(A, B) = \inf \{|a-b| : a \in A, b \in B\}.
\]

For \( x \in \mathbb{R} \) we also set \( \text{dist}(x, A) = \text{dist}([x], A) \). The notation

\[
A \to x \quad \text{will mean that} \quad \text{diam}(A \cup \{x\}) \to 0.
\]

Given \( M > 0 \) and a family \( \mathcal{I} \) of intervals in \( \mathbb{R} \) we say that covering multiplicity of \( \mathcal{I} \) is bounded by \( M \) if every point \( x \in \mathbb{R} \) is covered by at most \( M \) intervals from \( \mathcal{I} \).

Given a function \( g \in L_{1,\text{loc}}(\mathbb{R}) \) we let \( \mathcal{M}[g] \) denote the Hardy-Littlewood maximal function of \( g \):

\[
\mathcal{M}[g](x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |g(y)|dy, \quad x \in \mathbb{R}. \tag{2.1}
\]

Here the supremum is taken over all closed intervals \( I \) in \( \mathbb{R} \) containing \( x \).

By \( \mathcal{P}_m \) we denote the space of all polynomials of degree at most \( m \) defined on \( \mathbb{R} \). Finally, given a nonnegative integer \( k \), a \((k+1)\)-point set \( S \subset \mathbb{R} \) and a function \( f \) on \( S \), we let \( L_S[f] \) denote the Lagrange polynomial of degree at most \( k \) interpolating \( f \) on \( S \); thus

\[
L_S[f] \in \mathcal{P}_k \quad \text{and} \quad L_S[f](x) = f(x) \quad \text{for every} \quad x \in S.
\]
2.1. Divided differences: main properties.

In this section we recall several useful properties of the divided differences of functions. We refer
the reader to [13 Ch. 4, §7], [26] Section 1.3 and [48] Section 2.1 for the proofs of these properties.
Everywhere in this section \( k \) is a nonnegative integer and \( S = \{x_0, \ldots, x_k\} \) is a \((k + 1)\)-point subset of \( \mathbb{R} \). In (\( \star 1 \))-\( \star 3 \)) by \( f \) we denote a function defined on \( S \).

Then the following properties hold:

(\( \star 1 \)) \( \Delta^0 f[S] = f(x_0) \) provided \( S = \{x_0\} \) is a singleton.

(\( \star 2 \)) If \( k \in \mathbb{N} \) then

\[
\Delta^k f[S] = \Delta^k f[x_0, x_1, \ldots, x_k] = \left( \Delta^{k-1} f[x_1, \ldots, x_k] - \Delta^{k-1} f[x_0, \ldots, x_{k-1}] \right) / (x_k - x_0).
\]  

Furthermore,

\[
\Delta^k f[S] = \sum_{i=0}^{k} \frac{f(x_i)}{\omega'(x_i)} = \sum_{i=0}^{k} \frac{f(x_i)}{\prod_{j \in \{0, \ldots, k\}, j \neq i} (x_i - x_j)}
\]

where \( \omega(x) = (x - x_0) \ldots (x - x_k) \).

(\( \star 3 \)) We recall that \( L_S[f] \) denotes the Lagrange polynomial of degree at most \( k = \#S - 1 \) interpolating \( f \) on \( S \). Then the following equality

\[
\Delta^k f[S] = \frac{1}{k!} L_S^{(k)}[f]
\]

holds. Thus, \( \Delta^k f[S] = A_k \) where \( A_k \) is the coefficient of \( x^k \) of the polynomial \( L_S[f] \).

(\( \star 4 \)) Let \( k \in \mathbb{N} \), and let \( x_0 = \min\{x_i : i = 0, \ldots, k\} \) and \( x_k = \max\{x_i : i = 0, \ldots, k\} \). Then for every function \( F \in C^k[x_0, x_k] \) there exists \( \xi \in [x_0, x_k] \) such that

\[
\Delta^k F[x_0, x_1, \ldots, x_k] = \frac{1}{k!} F^{(k)}(\xi).
\]

(\( \star 5 \)) Let \( x_0 < x_1 < \ldots < x_k \), and let \( F \) be a function on \([x_0, x_k]\) with absolutely continuous derivative of order \( k - 1 \). Then

\[
|\Delta^k F[S]| \leq \frac{1}{(k - 1)!} \cdot \frac{1}{x_k - x_0} \int_{x_0}^{x_k} |F^{(k)}(t)| \, dt.
\]  

Furthermore, for every \( \{x_0, \ldots, x_m\} \subset \mathbb{R}, x_0 < \ldots < x_m \), and every \( F \in L^\infty_{(0, \infty)}(\mathbb{R}) \) the following inequality

\[
m! \cdot |\Delta^m F[x_0, \ldots, x_m]| \leq \|F\|_{L^\infty_{(0, \infty)}(\mathbb{R})}
\]

holds.

2.2. Proofs of the necessity parts of the main theorems.

(Theorem 2.3: Necessity) Let \( 1 < p < \infty \) and let \( f \in L^m_p(\mathbb{R}) \). Let \( F \in L^m_p(\mathbb{R}) \) be an arbitrary function such that \( F|_E = f \). Let \( n \geq m \) and let \( \{x_0, \ldots, x_n\} \subset E, x_0 < \ldots < x_n \). From (2.5), for every \( i, 0 \leq i \leq n - m \), we have

\[
A_i = (x_{i+m} - x_i) |\Delta^m f[x_i, \ldots, x_i+m]|^p = (x_{i+m} - x_i) |\Delta^m F[x_i, \ldots, x_i+m]|^p
\]

\[
\leq (x_{i+m} - x_i) \cdot \frac{1}{((m - 1))!} \cdot \frac{1}{x_{i+m} - x_i} \int_{x_i}^{x_{i+m}} |F^{(m)}(t)|^p \, dt = \frac{1}{((m - 1))!} \int_{x_i}^{x_{i+m}} |F^{(m)}(t)|^p \, dt.
\]
proving the necessity part of Theorem 1.3.

This inequality together with definition (1.4) implies that \( L_{m,p}(f : E) \leq 2 \|F\|_{L_p^m(\mathbb{R})} \).

Finally, taking the infimum in the right hand side of this inequality over all functions \( F \in L_p^m(\mathbb{R}) \) such that \( F|_E = f \), we conclude that

\[
L_{m,p}(f : E) \leq 2 \|f\|_{L_p^m(\mathbb{R})|_E}
\]

proving the necessity part of Theorem 1.4. \( \square \)

(Theorem 1.4: Necessity) Let \( p \in (1, \infty) \) and let \( f \in L_p^m(\mathbb{R})|_E \). Let \( F \in L_p^m(\mathbb{R}) \) be an arbitrary function such that \( F|_E = f \). Let \( S = \{x_0, \ldots, x_m\} \subset E \), \( x_0 < \ldots < x_m \), and let \( x \in \mathbb{R} \).

From (2.2) and (2.5), we have

\[
B = \frac{|\Delta^{m-1} f[x_0, \ldots, x_{m-1}] - \Delta^{m-1} f[x_1, \ldots, x_m]|}{|x - x_0| + |x - x_m|} = \frac{|\Delta^{m-1} F[x_0, \ldots, x_{m-1}] - \Delta^{m-1} F[x_1, \ldots, x_m]|}{|x - x_0| + |x - x_m|} \\
= \frac{|\Delta^m F[x_0, \ldots, x_m]|}{|x - x_0| + |x - x_m|} < (m-1)! \frac{1}{|x - x_0| + |x - x_m|} \int_0^x |F^{(m)}(t)| dt.
\]

Let \( I \) be the smallest closed interval containing \( S \) and \( x \). Clearly, \( |I| \leq |x - x_0| + |x - x_m| \) and \( I \supseteq [x_0, x_m] \). Hence,

\[
B \leq \frac{1}{(m-1)!} \frac{1}{|I|} \int_I |F^{(m)}(t)| dt \leq M[F^{(m)}](x). \quad \text{See (2.1)}.
\]

This inequality together with definition (1.6) implies that \( (\Delta^m f)_E^p(x) \leq M[F^{(m)}](x) \) on \( \mathbb{R} \). Hence,

\[
\| (\Delta^m f)_E^p \|_{L_p^m(\mathbb{R})} \leq M[F^{(m)}]|_{L_p^m(\mathbb{R})},
\]

so that, by the Hardy-Littlewood maximal theorem,

\[
\| (\Delta^m f)_E^p \|_{L_p^m(\mathbb{R})} \leq C(p) \|F^{(m)}\|_{L_p^m(\mathbb{R})} = C(p) \|F\|_{L_p^m(\mathbb{R})}.
\]

Taking the infimum in the right hand side of this inequality over all functions \( F \in L_p^m(\mathbb{R}) \) such that \( F|_E = f \), we finally obtain the required inequality

\[
\| (\Delta^m f)_E^p \|_{L_p^m(\mathbb{R})} \leq C(p) \|f\|_{L_p^m(\mathbb{R})|_E}.
\]

The proof of the necessity part of Theorem 1.4 is complete. \( \square \)
3. The Whitney extension method in \( \mathbb{R} \) and traces of Sobolev functions.

In this section we prove the sufficiency part of Theorem 1.4. Given a function \( F \in C^m(\mathbb{R}) \) and \( x \in \mathbb{R} \), we let

\[
T_x^m[F](y) = \sum_{k=0}^{m} \frac{1}{k!} F^{(k)}(x)(y-x)^k, \quad y \in \mathbb{R},
\]

denote the Taylor polynomial of \( F \) of degree \( m \) at \( x \).

Let \( E \) be a closed subset of \( \mathbb{R} \), and let \( P = \{ P_x : x \in E \} \) be a family of polynomials of degree at most \( m \) indexed by points of \( E \). (Thus \( P_x \in \mathcal{P}_m \) for every \( x \in E \).) Following [22], we refer to \( P \) as a Whitney \( m \)-field defined on \( E \).

We say that a function \( F \in C^m(\mathbb{R}) \) agrees with the Whitney \( m \)-field \( P = \{ P_x : x \in E \} \) on \( E \), if \( T_x^m[F] = P_x \) for each \( x \in E \). In that case we also refer to \( P \) as the Whitney \( m \)-field on \( E \) generated by \( F \) or as the \( m \)-jet generated by \( F \). We define the \( L_p^m \)-"norm" of the \( m \)-jet \( P = \{ P_x : x \in E \} \) by

\[
\| P \|_{m,p,E} = \inf \left\{ \| F \|_{L_p^m(\mathbb{R})} : F \in L_p^m(\mathbb{R}), T_x^{m-1}[F] = P_x \text{ for every } x \in E \right\}.
\]

We prove the sufficiency part of Theorem 1.4 in two steps. At the first step, given \( m \in \mathbb{N} \) we construct a linear operator which to every function \( f \) on \( E \) assigns a certain Whitney \((m-1)\)-field

\[
P^{(m,E)}[f] = \{ P_x \in \mathcal{P}_{m-1} : x \in E \}
\]

such that \( P_x(x) = f(x) \) for all \( x \in E \). We produce \( P^{(m,E)}[f] \) by a slight modification of Whitney’s extension construction [52]. See also [26, 27, 34, 37] where similar constructions have been used for characterization of traces of \( L_\infty^m(\mathbb{R}) \)-functions.

At the second step of the proof we show that for every \( p \in (1, \infty) \) and every function \( f : E \to \mathbb{R} \) such that \( (\Delta^m f)^\#_E \in L_p(\mathbb{R}) \) (see (1.6)) the following inequality

\[
\| P^{(m,E)}[f] \|_{m,p,E} \leq C(m, p) \| (\Delta^m f)^\#_E \|_{L_p(\mathbb{R})}
\]

holds. One of the main ingredients of the proof of (3.3) is a trace criterion for jets generated by Sobolev functions. See Theorem 3.16 below.

3.1. Interpolation knots and their properties.

Let \( E \subset \mathbb{R} \) be a closed subset, and let \( k \) be a non-negative integer, \( k \leq \#E \). Following [52] (see also [34, 37]), given \( x \in E \) we construct an important ingredients of our extension procedure, a finite set \( Y_k(x) \subset E \), which, in a certain sense, is “well concentrated” around \( x \). This set provides interpolation knots for Lagrange and Hermite polynomials which we use in our modification of the Whitney extension method.

We will need the following notion. Let \( A \) be a nonempty finite subset of \( E \), \( A \neq E \). Suppose that \( A \) contains at most one limit point of \( E \). We assign to \( A \) a point \( a_E(A) \in E \) in the closure of \( E \setminus A \) having the minimal distance to \( A \). More specifically:

(i) If \( A \) does not contain limit points of \( E \), the set \( E \setminus A \) is non-empty and closed, so that in this case \( a_E(A) \) is a point nearest to \( A \) on \( E \setminus A \). Clearly, in this case \( a_E(A) \notin A \);

(ii) Suppose there exists a (unique) point \( a \in A \) which is a limit point of \( E \). In this case we set \( a_E(A) = a \).
Note that in both cases 
\[
\text{dist}(a_E(A), A) = \text{dist}(A, E \setminus A).
\]

Now, let us construct a family of points \(\{y_0(x), y_1(x), \ldots, y_{n(x)}\}\) in \(E\), \(0 \leq n_k(x) \leq k\), using the following inductive procedure.

First, we put \(y_0(x) = x\) and \(Y_0(x) = \{y_0(x)\}\). If \(k = 0\), we put \(n_k(x) = 0\), and stop.

Suppose that \(k > 0\). If \(y_0(x) = x\) is a limit point of \(E\), we again put \(n_k(x) = 0\), and stop. If \(y_0(x)\) is an isolated point of \(E\), we continue the procedure.

We define a point \(y_1(x) \in E\) by \(y_1(x) = a_E(Y_0(x))\), and set \(Y_1(x) = \{y_0(x), y_1(x)\}\). If \(k = 1\) or \(y_1(x)\) is a limit point of \(E\), we put \(n_k(x) = 1\), and stop.

Let \(k > 1\) and \(y_1(x)\) is an isolated point of \(E\). In this case we put 
\[
y_2(x) = a_E(Y_1(x)) \quad \text{and} \quad Y_2(x) = \{y_0(x), y_1(x), y_2(x)\}.
\]

If \(k = 2\) or \(y_2(x)\) is a limit point of \(E\), we set \(n_k(x) = 2\), and stop. But if \(k > 2\) and \(y_2(x)\) is an isolated point of \(E\), we continue the procedure and define \(y_3\), etc.

At the \(j\)-th step of this algorithm we obtain a \(j + 1\)-point set \(Y_j(x) = \{y_0(x), \ldots, y_j(x)\}\).

If \(j = k\) or \(y_j(x)\) is a limit point of \(E\), we put \(n_k(x) = j\) and stop. But if \(j < k\) and \(y_j(x)\) is an isolated point of \(E\), we define a point \(y_{j+1}(x)\) and a set \(Y_{j+1}(x)\) by the formulae 
\[
y_{j+1}(x) = a_E(Y_j(x)) \quad \text{and} \quad Y_{j+1}(x) = \{y_0(x), \ldots, y_j(x), y_{j+1}(x)\}.
\]

Clearly, for a certain \(n = n_k(x)\), \(0 \leq n \leq k\), the procedure stops. This means that either \(n = k\) or, when \(n < k\), the points \(y_0(x), \ldots, y_{n-1}(x)\) are isolated points of \(E\), but 
\[
y_n(x) \quad \text{is a limit point of } \quad E.
\]

We also introduce points \(y_j(x)\) and sets \(Y_j(x)\) for \(n_k(x) \leq j \leq k\) by letting 
\[
y_j(x) = y_{n_j(x)}(x) \quad \text{and} \quad Y_j(x) = Y_{n_j(x)}(x).
\]

Note that, given \(x \in E\) the definitions of points \(y_j(x)\) and the sets \(Y_j(x)\) do not depend on \(k\), i.e., \(y_j(x)\) is the same point and \(Y_j(x)\) is the same set for every \(k \geq j\). This is immediate from (3.6).

In the next three lemmas, we describe several important properties of the points \(y_j(x)\) and sets \(Y_j(x)\).

**Lemma 3.1** Given \(x \in E\), the points \(y_j(x)\) and the sets \(Y_j(x)\), \(0 \leq j \leq k\), have the following properties:

(a) \(y_0(x) = x\) and \(y_j(x) = a_E(Y_{j-1}(x))\) for every \(1 \leq j \leq k\);

(b) Let \(n = n_k(x) \geq 1\). Then \(y_0(x), \ldots, y_{n-1}(x)\) are isolated points of \(E\).

Furthermore, if \(y \in Y_n(x)\) and \(y\) is a limit point of \(E\), then \(y = y_n(x)\).

In addition, if \(0 < n < k\), then \(y_n(x)\) is a limit point of \(E\);

(c) \(Y_j(x) = \min\{j, n_k(x)\} + 1\) for every \(0 \leq j \leq k\);

(d) For every \(j = 0, \ldots, k\), we have 
\[
[\min Y_j(x), \max Y_j(x)] \cap E = Y_j(x).
\]

Furthermore, the point \(y_j(x)\) is either minimal or maximal point of the set \(Y_j(x)\).
Proof. Properties (b)-(d) are immediate from the definitions of the points \( y_j(x) \) and sets \( Y_j(x) \).

Let us prove (a). We know that \( y_0(x) = x \) and, thanks to (3.4), \( y_j(x) = a_E(Y_{j-1}(x)) \) for every \( j = 1, \ldots, n_k(x) \). If \( n_k(x) < j \leq k \), then, by (3.6), \( Y_j(x) = Y_{n_k(x)}(x) \) for every \( j, n_k(x) \leq j \leq k \).

On the other hand, since \( n_k(x) < k \), the point \( y_{n_k(x)}(x) \) is a unique limit point of \( E \). See (3.5). From this, definition of \( a_E \) and (3.6), for every \( j, n_k(x) < j \leq k \), we have

\[
a_E(Y_{j-1}(x)) = a_E(Y_{n_k(x)}(x)) = y_{n_k(x)} = y_j(x),
\]

proving property (a) in the case under consideration. \( \Box \)

Lemma 3.2 Let \( x_1, x_2 \in E \) and let \( 0 \leq j \leq k \). If \( x_1 \leq x_2 \) then

\[
\min Y_j(x_1) \leq \min Y_j(x_2) \quad \text{and} \quad \max Y_j(x_1) \leq \max Y_j(x_2).
\]

Proof. We proceed by induction on \( j \). Because \( Y_0(x_1) = \{ x_1 \} \) and \( Y_0(x_2) = \{ x_2 \} \), we conclude that (3.8) holds for \( j = 0 \).

Suppose that (3.8) holds for some \( j, 0 \leq j \leq k - 1 \). Let us prove that

\[
\min Y_{j+1}(x_1) \leq \min Y_{j+1}(x_2).
\]

We recall that, thanks to (3.4), \( y_{j+1}(x_\ell) = a_E(Y_j(x_\ell)) \) for each \( \ell = 1, 2 \), and

\[
Y_{j+1}(x_\ell) = Y_j(x_\ell) \cup \{ y_{j+1}(x_\ell) \}.
\]

If \( Y_j(x_2) \) contains a limit point of \( E \), then \( y_{j+1}(x_2) \in Y_j(x_2) \) so that \( Y_{j+1}(x_2) = Y_j(x_2) \). This equality and assumption (3.8) imply that

\[
\min Y_{j+1}(x_1) \leq \min Y_j(x_1) \leq \min Y_j(x_2) = \min Y_{j+1}(x_2)
\]

proving (3.9) in the case under consideration.

Now, suppose that all points of \( Y_j(x_2) \) are isolated points of \( E \). In particular, from part (b) of Lemma 3.1 and definitions (3.5), (3.6), we have \( 0 \leq j \leq n_k(x_2) \). This inequality and part (c) of Lemma 3.1 imply that \( \# Y_j(x_2) = j + 1 \).

Consider two cases. First, let us assume that

\[
\min Y_j(x_1) < \min Y_j(x_2).
\]

Then, for each point \( a \in \mathbb{R} \) nearest to \( Y_j(x_2) \) on the set \( E \setminus Y_j(x_2) \), we have \( a \geq \min Y_j(x_1) \). This inequality, definition of \( a_E \) and (3.4) together yield \( a_E(Y_j(x_2)) = y_{j+1}(x_2) \geq \min Y_j(x_1) \).

Combining this inequality with (3.10) and (3.11), we obtain the required inequality (3.9).

Now, prove (3.9) whenever \( \min Y_j(x_1) = \min Y_j(x_2) \). This equality and the second inequality in (3.8) imply that

\[
Y_j(x_1) \subset I = [\min Y_j(x_2), \max Y_j(x_2)].
\]

In turn, (3.7) tells us that \( I \cap E = Y_j(x_2) \) proving that \( Y_j(x_1) \subset Y_j(x_2) \). Recall that in the case under consideration all points of \( Y_j(x_2) \) are isolated points of \( E \). Therefore, all points of \( Y_j(x_1) \) are isolated points of \( E \) as well. Now, using the same argument as for the set \( Y_j(x_2) \), we conclude that \( \# Y_j(x_1) = j + 1 = \# Y_j(x_2) \).

Thus \( Y_j(x_1) \subset Y_j(x_2) \) and \( \# Y_j(x_1) = \# Y_j(x_2) \). Hence, \( Y_j(x_1) = Y_j(x_2) \) proving (3.9) in the case under consideration.

In the same fashion we prove that \( \max Y_{j+1}(x_1) \leq \max Y_{j+1}(x_2) \).

The proof of the lemma is complete. \( \Box \)
Lemma 3.3 ([34] p. 231) Let \( x_1, x_2 \in E \), and let \( Y_k(x_1) \neq Y_k(x_2) \). Then for all \( 0 \leq i, j \leq k \) the following inequality
\[
\max \{|y_i(x_1) - y_j(x_1)|, |y_i(x_2) - y_j(x_2)|\} \leq \max \{i, j\} |x_1 - x_2|
\]
holds.

This lemma implies the following

Corollary 3.4 For every \( x_1, x_2 \in E \) such that \( Y_k(x_1) \neq Y_k(x_2) \) the following inequality
\[
\text{diam} \ Y_k(x_1) + \text{diam} \ Y_k(x_2) \leq 2k |x_1 - x_2|
\]
holds.

3.2. Lagrange polynomials and divided differences at interpolation knots.

In this section we present a series of important properties of Lagrange polynomials which we use later on in proofs of extension criteria.

Lemma 3.5 Let \( k \) be a nonnegative integer, and let \( P \in \mathcal{P}_k \). Suppose that \( P \) has \( k \) real distinct roots which lie in a set \( S \subset \mathbb{R} \). Let \( I \subset \mathbb{R} \) be a closed interval.

Then for every \( i, 0 \leq i \leq k \), the following inequality
\[
\max_i |P^{(i)}| \leq (\text{diam}(I \cup S))^{k-i} |P^{(k)}|
\]
holds.

Proof. Let \( x_j, j = 1, \ldots, k \), be the roots of \( P \), and let \( X = \{x_1, \ldots, x_k\} \). The lemma’s hypothesis tells us that \( X \subset S \). Clearly,
\[
P(x) = \frac{P^{(k)}}{k!} \prod_{i=1}^{k} (x - x_i), \quad x \in \mathbb{R},
\]
so that for every \( i, 0 \leq i \leq k \),
\[
P^{(i)}(x) = \frac{i!}{k!} P^{(k)} \sum_{X' \subset X, \#X' = k-i} \prod_{y \in X'} (x - y), \quad x \in \mathbb{R}.
\]
Hence,
\[
\max_i |P^{(i)}| \leq \frac{i!}{k!} \frac{k!}{i!(k-i)!} (\text{diam}(I \cup X))^{k-i} |P^{(k)}| \leq (\text{diam}(I \cup S))^{k-i} |P^{(k)}|
\]
proving the lemma.

We recall that, given \( S \subset \mathbb{R} \) with \( \#S = k + 1 \) and a function \( f : S \to \mathbb{R} \), by \( L_S[f] \) we denote the Lagrange polynomial of degree at most \( k \) interpolating \( f \) on \( S \).

Lemma 3.6 Let \( S_1, S_2 \subset \mathbb{R} \), \( S_1 \neq S_2 \), and let \( \#S_1 = \#S_2 = k + 1 \) where \( k \) is a nonnegative integer. Let \( I \subset \mathbb{R} \) be a closed interval. Then for every function \( f : S_1 \cup S_2 \to \mathbb{R} \) and every \( i, 0 \leq i \leq k \), we have
\[
\max_j |L_{S_1}^{(i)}[f] - L_{S_2}^{(i)}[f]| \leq (k + 1)! (\text{diam}(I \cup S_1 \cup S_2))^{k-i} A
\]
(3.12)

where
\[
A = \max_{S' \subset S_1 \cup S_2 \atop \#S' = k+2} |\Delta^{k+1} f[S']| \text{ diam } S'.
\]
(3.13)

12
Proof. Let $n = k + 1 - \#(S_1 \cap S_2)$; then $n \geq 1$ because $S_1 \neq S_2$. Let $\{Y_j : j = 0, ..., n\}$ be a family of $(k + 1)$-point subsets of $S$ such that $Y_0 = S_1$, $Y_n = S_2$, and $\#(Y_j \cap Y_{j+1}) = k$ for every $j = 0, ..., n - 1$. Let $P_j = L_{Y_j}[f]$, $j = 0, ..., n$. Then

$$\max_i |L^{(i)}_{S_1}[f] - L^{(i)}_{S_2}[f]| = \max_i |P^{(i)}_0 - P^{(i)}_n| \leq \sum_{j=0}^{n-1} \max_j |P^{(j)}_j - P^{(j)}_{j+1}|.$$ 

Note that each point $y \in Y_j \cap Y_{j+1}$ is a root of the polynomial $P_j - P_{j+1} \in \mathcal{P}_k$. Thus, if the polynomial $P_j - P_{j+1}$ is not identically 0, it has precisely $k$ distinct real roots which belong to the set $S_1 \cup S_2$. We apply Lemma 3.5 taking $P = P_j - P_{j+1}$ and $S = S_1 \cup S_2$, and obtain the following:

$$\max_i |P^{(j)}_j - P^{(j)}_{j+1}| \leq (\text{diam}(I \cup S_1 \cup S_2))^{k-1} |P^{(k)}_j - P^{(k)}_{j+1}|.$$ 

From (2.2) and (2.3), we have

$$|P^{(k)}_j - P^{(k)}_{j+1}| = |L^{(k)}_{Y_j}[f] - L^{(k)}_{Y_{j+1}}[f]| = k!|\Delta^k f| Y_j - \Delta^k f| Y_{j+1}| \leq k!|\Delta^{k+1} f| S^{(j)} \text{ diam } S^{(j)}$$

where $S^{(j)} = Y_j \cup Y_{j+1}$, $j = 0, ..., n - 1$. Hence,

$$\max_i |L^{(i)}_{S_1}[f] - L^{(i)}_{S_2}[f]| \leq k! (\text{diam}(I \cup S_1 \cup S_2))^{k-1} \sum_{j=0}^{n-1} |\Delta^{k+1} f| S^{(j)} \text{ diam } S^{(j)}. \tag{3.14}$$

Clearly, $S^{(j)} \subset S_1 \cup S_2$ and $\#S^{(j)} = k + 2$. Therefore, each summand of the sum in the right hand side of (3.14) is bounded by A (see (3.13)). This, (3.14) and inequality $n \leq k + 1$ imply (3.12) completing the proof of the lemma. \qed

**Lemma 3.7** Let $k$ be a nonnegative integer, $\ell \in \mathbb{N}$, $k < \ell$, and let $\mathcal{Y} = \{y^i\}_{j=0}^\ell$ be a strictly increasing sequence in $\mathbb{R}$. Let $I = [y_0, y_\ell]$, $S_1 = \{y_0, ..., y_k\}$, $S_2 = \{y_{\ell-k}, ..., y_\ell\}$, and let

$$S^{(j)} = \{y_j, ..., y_{k+j+1}\}, \quad j = 0, ..., \ell - k - 1. \tag{3.15}$$

Then for every function $f : \mathcal{Y} \rightarrow \mathbb{R}$, every $i$, $0 \leq i \leq k$, and every $p \in [1, \infty)$ the following inequality

$$\max_i |L^{(i)}_{S_1}[f] - L^{(i)}_{S_2}[f]|^p \leq C(k)^p (\text{diam } I)^{(k+i)p-1} \sum_{j=0}^{\ell-k-1} |\Delta^{k+1} f| S^{(j)}|^p \text{ diam } S^{(j)} \tag{3.16}$$

holds.

**Proof.** Repeating the proof of inequality (3.14), we obtain the following:

$$B = \max_i |L^{(i)}_{S_1}[f] - L^{(i)}_{S_2}[f]| \leq k! (\text{diam } I)^{k-1} \sum_{j=0}^{\ell-k-1} |\Delta^{k+1} f| S^{(j)} \text{ diam } S^{(j)}. \tag{3.17}$$

Let $I_j = [y_j, y_{k+j+1}]$. Then, thanks to (3.15), $\text{diam } I_j = \text{diam } S_j = y_{k+j+1} - y_j$. Furthermore, since $\{y^i\}_{j=0}^\ell$ is a strictly increasing sequence and $\#S_j = k + 2$, the covering multiplicity of the family $\{I_j : j = 0, ..., \ell - k - 1\}$ is bounded by $2k + 3$. Hence,

$$\sum_{j=0}^{\ell-k-1} \text{diam } S^{(j)} = \sum_{j=0}^{\ell-k-1} \text{diam } I_j = \sum_{j=0}^{\ell-k-1} |I_j| \leq (2k + 3)|I|. \tag{3.18}$$
We recall that $p$ such that $\text{diam}(S) < \delta$ so that $\lambda \max_{S < S} |\Delta^{k+1} f[S']| \text{diam}(S')$.

This inequality, Hölder’s inequality and (3.17) together imply that

$$B \leq (k!)^p (\text{diam} I)^{(k-i)p} \left(\sum_{j=0}^{\ell-k-1} \text{diam} S^{(j)}\right)^{p-1} \left(\sum_{j=0}^{\ell-k-1} |\Delta^{k+1} f[S^{(j)}]|^p \text{diam} S^{(j)}\right).$$

From this and (3.18) we have (3.16) proving the lemma.

□

**Lemma 3.8** Let $k$ be a nonnegative integer and let $1 < p < \infty$. Let $f$ be a function defined on a closed set $E \subset \mathbb{R}$ with $\#E > k + 1$. Suppose that

$$\lambda = \sup_{S \subset E, \#S = k+2} |\Delta^{k+1} f[S]| (\text{diam} S)^{\frac{1}{p}} < \infty. \quad (3.19)$$

Then for every limit point $x$ of $E$ and every $i, 0 \leq i \leq k$, there exists a limit

$$f_i(x) = \lim_{S \to x, S \subset E, \#S = k+1} L^{(i)}_S[f](x). \quad (3.20)$$

(Recall that the notation $S \to x$ means that $\text{diam}(S \cup \{x\}) \to 0$.)

Furthermore, let $P_x \in \mathcal{P}_k$ be a polynomial such that

$$P^{(i)}_x(x) = f_i(x) \quad \text{for every} \quad i, 0 \leq i \leq k. \quad (3.21)$$

Then for every $\delta > 0$ and every set $S \subset E$ such that $\#S = k + 1$ and $\text{diam}(S \cup \{x\}) < \delta$ the following inequality

$$\max_{[x-\delta, x+\delta]} |P^{(i)}_x - L^{(i)}_S[f]| \leq C \lambda \delta^{k+1-i-1/p}, \quad 0 \leq i \leq k, \quad (3.22)$$

holds. Here $C$ is a constant depending only on $k$.

*Proof*. Let $\delta > 0$ and let $S_1, S_2$ be two subsets of $E$ such that $\#S_j = k + 1$ and $\text{diam}(S_j \cup \{x\}) < \delta, j = 1, 2$. Hence, $S = S_1 \cup S_2 \subset I = [x-\delta, x+\delta]$. Lemma 3.6 tells us that

$$|L^{(i)}_{S_1}[f](x) - L^{(i)}_{S_2}[f](x)| \leq (k+1)! (\text{diam} I)^{k-i} \max_{S' \subset S, \#S' = k+2} |\Delta^{k+1} f[S']| \text{diam} S'.$$

Thanks to (3.19), $|\Delta^{k+1} f[S']| \leq \lambda (\text{diam} S')^{-\frac{1}{p}}$ for every $(k+2)$-point subset $S' \subset E$, so that

$$|L^{(i)}_{S_1}[f](x) - L^{(i)}_{S_2}[f](x)| \leq (k+1)! \lambda (2\delta)^{k-i} \max_{S' \subset S, \#S' = k+2} (\text{diam} S')^{1-1/p} \leq (k+1)! \lambda (2\delta)^{k-i} (2\delta)^{1-1/p}.$$

Hence,

$$|L^{(i)}_{S_1}[f](x) - L^{(i)}_{S_2}[f](x)| \leq C(k) \lambda \delta^{k+1-i-1/p}, \quad (3.23)$$

so that

$$|L^{(i)}_{S_1}[f](x) - L^{(i)}_{S_2}[f](x)| \to 0 \quad \text{as} \quad \delta \to 0.$$

(We recall that $p > 1$.) This proves the existence of the limit in (3.20).

Let us prove inequality (3.22). Thanks to (3.23), for every two sets $S, \bar{S} \in E$, with $\#S = \#\bar{S} = k + 1$ such that $\text{diam}(S \cup \{x\}), \text{diam}(S \cup \{x\}) < \delta$, the following inequality

$$|L^{(i)}_{S}[f](x) - L^{(i)}_{\bar{S}}[f](x)| \leq C(k) \lambda \delta^{k+1-i-1/p}$$
holds. Passing to the limit in this inequality whenever the set \( \tilde{S} \to x \) (i.e., \( \text{diam}(\tilde{S} \cup \{x\}) \to 0 \)), we obtain the following:

\[
|P_x^{(i)}(x) - L_S^{(i)}[f](x)| \leq C(k) \lambda \delta^{k+1-i-1/p}.
\]

See (3.20) and (3.21). Therefore, for each \( y \in [x - \delta, x + \delta] \), we have

\[
|P_x^{(i)}(y) - L_S^{(i)}[f](y)| = \left| \sum_{j=0}^{k-i} \frac{1}{j!} (P_x^{(i+j)}(x) - L_S^{(i+j)}[f](x)) (y - x)^j \right| \leq \sum_{j=0}^{k-i} \frac{1}{j!} |P_x^{(i+j)}(x) - L_S^{(i+j)}[f](x)| \delta^j \leq C(k) \lambda \sum_{j=0}^{k-i} \delta^{k+1-i-j-1/p} \delta^j \leq C(k) \lambda \delta^{k+1-i-1/p}.
\]

The proof of the lemma is complete. \( \square \)

**Lemma 3.9** Let \( k, p, E, f, \lambda \) and \( x \) be as in Lemma 3.8. Then for every \( i, 0 \leq i \leq k \),

\[
\lim_{S \to x, \#S = i+1} i! \Delta^i f[S] = f_i(x).
\]

**Proof.** Let \( \delta > 0 \) and let \( S \subset E \) be a finite set such that \( \#S = i + 1 \) and \( \text{diam}(S \cup \{x\}) < \delta \). Since \( x \) is a limit point of \( E \), there exists a set \( Y \subset E \cap [x - \delta, x + \delta] \) with \( \#Y = k + 1 \) such that \( S \subset Y \). Then, thanks to (3.22),

\[
\max_{[x-\delta, x+\delta]} |P_x^{(i)} - L_Y^{(i)}[f]| \leq C(k) \lambda \delta^{k+1-i-1/p}.
\]

Because the Lagrange polynomial \( L_Y[f] \) interpolates \( f \) on \( S \), we have \( \Delta^i f[S] = \Delta^i(L_Y[f])[S] \) so that, thanks to (2.4), there exists \( \xi \in [x - \delta, x + \delta] \) such that \( i! \Delta^i f[S] = L_Y^{(i)}[f](\xi) \).

This equality and (3.24) imply that

\[
|P_x^{(i)}(\xi) - i! \Delta^i f[S]| = |P_x^{(i)}(\xi) - L_Y^{(i)}[f](\xi)| \leq C(k) \lambda \delta^{k+1-i-1/p}.
\]

Hence,

\[
|f_i(x) - i! \Delta^i f[S]| = |P_x^{(i)}(x) - i! \Delta^i f[S]| \leq |P_x^{(i)}(x) - P_x^{(i)}(\xi)| + |P_x^{(i)}(\xi) - i! \Delta^i f[S]| \leq |P_x^{(i)}(x) - P_x^{(i)}(\xi)| + C(k) \lambda \delta^{k+1-i-1/p}.
\]

Since \( P_x^{(i)} \) is a continuous function and \( p > 1 \), the right hand side of this inequality tends to 0 as \( \delta \to 0 \) proving the lemma. \( \square \)

**Lemma 3.10** Let \( p \in (1, \infty) \), \( k \in \mathbb{N} \), and let \( f \) be a function defined on a closed set \( E \subset \mathbb{R} \) with \( \#E = k + 1 \). Suppose that \( f \) satisfies condition (3.19).

Let \( x \in E \) be a limit point of \( E \), and let \( S \) be a subset of \( E \) with \( \#S \leq k \) containing \( x \). Then for every \( i, 0 \leq i \leq k + 1 - \#S \),

\[
\lim_{S \to x, \#S = i+1} L_S^{(i)}[f](x) = f_i(x).
\]

**Proof.** For \( S = \{x\} \) the statement of the lemma follows from Lemma 3.8.

Suppose that \( \#S > 1 \). Let \( I_0 = [x - 1/2, x + 1/2] \) so that \( \text{diam} I_0 = 1 \). We prove that for every \( i, 0 \leq i \leq k - 1 \), the family of functions

\[
\{L_Y^{(i+1)}[f] : Y \subset I_0 \cap E, \#Y = k + 1\}
\]
is uniformly bounded on $I_0$ provided condition $\text{(3.19)}$ holds. Indeed, fix a subset $Y_0 \subset I_0 \cap E$ with $\#Y_0 = k + 1$. Lemma $3.6$ and $\text{(3.12)}$ together imply that for arbitrary $Y \subset I_0 \cap E$, $Y \neq Y_0$, with $\#Y = k + 1$, we have

$$
\max_{h_0} |L^{(i+1)}_Y[f] - L^{(i+1)}_{Y_0}[f]| \leq (k + 1)! (\text{diam}(I_0 \cup Y \cup Y_0))^{k-i-1} A = (k + 1)! A
$$

where $A = \max[|\Delta^{k+1}f[S']| \text{ diam}S' : S' \subset Y_0 \cup Y, \#S' = k + 2]$.

Therefore, thanks to $\text{(3.19)}$,

$$
\max_{h_0} |L^{(i+1)}_Y[f] - L^{(i+1)}_{Y_0}[f]| \leq (k + 1)! A \max_{S' \subset Y \cup Y_0, \#S' = k+2} (\text{diam}S')^{1-1/p} \leq (k + 1)! \lambda.
$$

Applying this inequality to an arbitrary set $Y \subset I_0 \cap E$ with $\#Y = k + 1$ and to every $i, 0 \leq i \leq k - 1$, we conclude that

$$
\max_{h_0} |L^{(i+1)}_Y[f]| \leq B_i \quad \text{where} \quad B_i = \max_{h_0} |L^{(i+1)}_{Y_0}[f]| + (k + 1)! \lambda. \quad \text{(3.25)}
$$

Fix $\varepsilon > 0$. Lemma $3.9$ tells us that there exists $\tilde{\delta} \in (0, 1/2]$ such that for an arbitrary set $V \subset E$, with $\text{diam}(V \cup \{x\}) < \tilde{\delta}$ and $\#V = i + 1$, the following inequality

$$
|i! \Delta^i f[V] - f_i(x)| \leq \varepsilon/2 \quad \text{(3.26)}
$$

holds. Let $S'$ be an arbitrary subset of $E$ such that $S \subset S'$, $\#S' = k + 1$, and

$$
\text{diam}((S' \setminus S) \cup \{x\}) < \delta = \min\{\tilde{\delta}, \varepsilon/(2B_i)\}. \quad \text{(3.27)}
$$

Recall that $\#S' - \#S = k + 1 - \#S \geq i$, so that there exists a subset $V \subset (S' \setminus S) \cup \{x\}$ with $\#V = i + 1$. Thanks to $\text{(2.4)}$, there exists $\xi \in [x - \delta, x + \delta]$ such that $i! \Delta^i(L_{S'}[f])[V] = L_{S'}^{(i)}[f](\xi)$.

On the other hand, because the polynomial $L_{S'}[f]$ interpolates $f$ on $V$, the divided difference $\Delta^i f[V] = \Delta^i(L_{S'}[f])[V]$ so that, $i! \Delta^i f[V] = L_{S'}^{(i)}[f](\xi)$.

This and $\text{(3.26)}$ imply that $|f_i(x) - L_{S'}^{(i)}[f](\xi)| \leq \varepsilon/2$.

It remains to note that, thanks to $\text{(3.25)}$ and $\text{(3.27)}$,

$$
|L_{S'}^{(i)}[f](\xi) - L_{S'}^{(i)}[f](x)| \leq \max_{[x-\delta, x+\delta]} \left|L_{S'}^{(i+1)}[f]\right| \cdot |x - \xi| \leq B_i \delta \leq B_i (\varepsilon/(2B_i)) = \varepsilon/2,
$$

so that

$$
|f_i(x) - L_{S'}^{(i)}[f](x)| \leq |f_i(x) - L_{S'}^{(i)}[f](\xi)| + |L_{S'}^{(i)}[f](\xi) - L_{S'}^{(i)}[f](x)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon
$$

proving the lemma. \( \square \)

### 3.3. Whitney \( m \)-fields and Hermite polynomials.

We turn to constructing of the Whitney \((m - 1)\)-field $\mathbb{P}^{(m,E)}$ mentioned at the beginning of Section 3. See $\text{(3.2)}$. Everywhere in this section we will assume that $f$ is a function on $E$ satisfying the following condition:

$$
\sup_{S \subset E, \#S = m+1} |\Delta^m f[S]| (\text{diam} S)^{\frac{1}{2}} < \infty. \quad \text{(3.28)}
$$
Let \( k = m - 1 \). Given \( x \in E \), let
\[
S_x = Y_k(x) = \{ y_0(x), \ldots, y_{n_k(x)}(x) \}
\]
and let
\[
s_x = y_{n_k(x)}.
\]
We recall that the points \( y_j(x) \) and the sets \( Y_j(x) \) are defined by formulae (3.4)-(3.6).

The next two propositions describe the main properties of the sets \( \{ S_x : x \in E \} \) and the points \( \{ s_x : x \in E \} \). These properties are immediate from Lemmas 3.1, 3.2 and Corollary 3.4.

**Proposition 3.11**

(i) \( S_x \subset E \), \( x \in S_x \) and \( \# S_x \leq m \) for every \( x \in E \). Furthermore,
\[
[\min S_x, \max S_x] \cap E = S_x;
\]
(ii) For every \( x_1, x_2 \in E \) such that \( S_{x_1} \neq S_{x_2} \) the following inequality
\[
diam S_{x_1} + diam S_{x_2} \leq 2 m |x_1 - x_2|
\]
holds;
(iii) If \( x_1, x_2 \in E \) and \( x_1 < x_2 \) then
\[
\min S_{x_1} \leq \min S_{x_2} \quad \text{and} \quad \max S_{x_1} \leq \max S_{x_2}.
\]

**Proposition 3.12**

(i) The point \( s_x \) belongs to \( S_x \) for every \( x \in E \). This point is either minimal or maximal point of the set \( S_x \);
(ii) All points of the set \( S_x \setminus \{ s_x \} \) are isolated points of \( E \) provided \( \# S_x > 1 \). If \( y \in S_x \) and \( y \) is a limit point of \( E \), then \( y = s_x \);
(iii) If \( \# S_x < m \) then \( s_x \) is a limit point of \( E \).

**Definition 3.13**

Given a function \( f : E \to \mathbb{R} \) satisfying condition (3.28), we define the Whitney \((m - 1)\)-field \( P^{(m,E)}[f] = \{ P_x \in \mathcal{P}_{m-1} : x \in E \} \) as follows:

(i) If \( \# S_x < m \), part (iii) of Proposition 3.12 tells us that \( s_x \) is a limit point of \( E \). Then, thanks to (3.28) and Lemma 3.8 for every \( i, 0 \leq i \leq m - 1 \), there exists a limit
\[
f_i(s_x) = \lim_{S \to S_x} L^i_S(f)(s_x).
\]

We define a polynomial \( P_x \in \mathcal{P}_{m-1} \) as the Hermite polynomial satisfying the following conditions:
\[
P_x(y) = f(y) \quad \text{for every} \quad y \in S_x,
\]
and
\[
P_x^{(i)}(s_x) = f_i(s_x) \quad \text{for every} \quad i, 1 \leq i \leq m - \# S_x.
\]

(ii) If \( \# S_x = m \), we put
\[
P_x = L_{S_x}[f].
\]
The next lemma shows that the field \( \mathbb{P}_{m,E}^m[f] \) determined by Definition \[3.13\] is well defined.

**Lemma 3.14** For each \( x \in E \) there exists the unique polynomial \( P_x \) satisfying conditions \[3.35\] and \[3.36\] provided condition \[3.28\] holds.

**Proof.** In case (i) \((\#S_x < m)\) the existence and uniqueness of \( P_x \) satisfying \[3.35\] and \[3.36\] is immediate from [11, Ch. 2, Section 11]. See also formula \[3.40\] below. Clearly, in case (ii) \((\#S_x = m)\) property \[3.35\] holds as well, and \[3.36\] holds vacuously. □

Let us note that \( x \in S_x \) and \( P_x = f \) on \( S_x \) (see \[3.35\]) proving that

\[
P_x(x) = f(x) \quad \text{for every } x \in E. \tag{3.38}
\]

For the case \( m > 1 \) and \( \#S_x < m \), we give an explicit formula for the Hermite polynomials \( P_x, x \in E \), from Definition \[3.13\]. See [11, Ch. 2, Section 11].

Let \( n = \#S_x - 1 \) and let \( y_i = y_i(x), i = 0, \ldots, n \), so that \( S_x = \{y_0, \ldots, y_n\} \). See \[3.29\]. (Note also that in these settings \( s_x = y_n \).) In this case the Hermite polynomial \( P_x \) satisfying \[3.35\] and \[3.36\] can be represented as a linear combination of polynomials

\[
H_0, \ldots, H_n, \tilde{H}_1, \ldots, \tilde{H}_{m-n-1} \in \mathbb{P}_{m-1}
\]

which are uniquely determined by the following conditions:

(i) \( H_i(y_i) = 1 \) for every \( i, 0 \leq i \leq n \), and \( H_j(y_i) = 0 \) for every \( 0 \leq i, j \leq n, i \neq j \), and

\[
H_i(y_n) = \ldots = H_i^{(m-n-1)}(y_n) = 0 \quad \text{for every } i, 0 \leq i \leq n.
\]

(ii) \( \tilde{H}_j(y_i) = 0 \) for every \( 0 \leq i \leq n, 1 \leq j \leq m - n - 1 \), and for every \( 1 \leq j \leq m - n - 1 \),

\[
\tilde{H}_j^{(j)}(y_n) = 1 \quad \text{and} \quad \tilde{H}_j^{(\ell)}(y_n) = 0 \quad \text{for every } \ell, 1 \leq \ell \leq m - n - 1, \ell \neq j.
\]

The existence and uniqueness of \( H_i \) and \( \tilde{H}_j \), \( 0 \leq i \leq n, 1 \leq j \leq m - n - 1 \), are proven in [11, Ch. 2, Section 11]. It is also shown there that for every \( P \in \mathbb{P}_{m-1} \) the following unique representation

\[
P(y) = \sum_{i=0}^{n} P(y_i) H_i(y) + \sum_{j=1}^{m-n-1} P^{(j)}(y_n) \tilde{H}_j(y), \quad y \in \mathbb{R}, \tag{3.39}
\]

holds. In particular,

\[
P_x(y) = \sum_{i=0}^{n} f(y_i) H_i(y) + \sum_{j=1}^{m-n-1} f^{(j)}(y_n) \tilde{H}_j(y), \quad y \in \mathbb{R}. \tag{3.40}
\]

Clearly, \( P_x \) meets conditions \[3.35\] and \[3.36\].

Let \( I \subset \mathbb{R} \) be a bounded closed interval, and let \( C^m(I) \) be the space of all \( m \)-times continuously differentiable functions on \( I \). We norm \( C^m(I) \) by

\[
\|f\|_{C^m(I)} = \sum_{i=0}^{m} \max |f^{(i)}|.
\]

We will need the following important property of the polynomials \( \{P_x : x \in E\} \).

18
Lemma 3.15 Let $f$ be a function defined on a closed set $E \subset \mathbb{R}$ with $\#E > m + 1$, and satisfying condition (3.28). Let $I$ be a bounded closed interval in $\mathbb{R}$. Then for every $x \in E$

$$\lim_{S' \setminus S \to s} \|L_{S'}[f] - P_x\|_{C^m(I)} = 0.$$  

Proof. The lemma is obvious whenever $\#S_x = m$ because in this case $L_{S_x}[f] = P_x$. In particular, the lemma is trivial for $m = 1$.

Let now $m > 1$ and let $\#S_x < m$. In this case $P_x$ can be represented in the form (3.40). Because $s_x$ is a limit point of $E$ (see part (iii) of Proposition 3.12), Lemma 3.10 and (3.36) imply that

$$\lim_{S' \setminus S \to s} L_{S'}^{(j)}[f](s_x) = f_j(s_x) = P_x^{(j)}(s_x) \quad \text{for every} \quad 1 \leq i \leq m - n - 1. \quad (3.41)$$

Let $n = \#S_x - 1$ and let $S_x = \{y_0, ..., y_n\}$ where $y_i = y_i(x), \ i = 0, ..., n$.

Then, thanks to (3.39), for every set $S' \subset E$ with $\#S' = m$ such that $s_x \subset S'$, the polynomial $L_{S'}[f]$ has the following representation:

$$L_{S'}[f](y) = \sum_{i=0}^{n} f(y_i) H_i(y) + \sum_{j=1}^{m-n-1} L_{S'}^{(j)}[f](s_x) \tilde{H}_j(y), \ y \in \mathbb{R}.$$  

From this, (3.40) and (3.41), we have

$$\max_j \|L_{S'}[f] - P_x\| = \max_j \left| \sum_{j=1}^{m-n-1} (L_{S'}^{(j)}[f](s_x) - P_x^{(j)}(s_x)) \tilde{H}_j \right| \leq \sum_{j=1}^{m-n-1} \|L_{S'}^{(j)}[f](s_x) - P_x^{(j)}(s_x)\| \max_j |\tilde{H}_j|.$$  

Hence, $\max_j \|L_{S'}[f] - P_x\| \to 0$ as $S' \setminus S \to s_x$ provided $S_x \subset S' \subset E$ and $\#S' = m$.

Note that the uniform norm on $I$ and the $C^m(I)$-norm are equivalent norms on the finite dimensional space $P_m$. Therefore, convergence of $L_{S'}[f]$ to $P_x$ in the uniform norm on $I$ implies convergence of $L_{S'}[f]$ to $P_x$ in the $C^m(I)$-norm, proving the lemma. \qed

3.4. Extension criteria in terms of sharp maximal functions: sufficiency.

Let $f$ be a function on $E$ such that $(\Delta^m f)^\frac{1}{p}_E \in L_p(\mathbb{R})$. See (1.6) and (1.7).

Let us prove that $f$ satisfies condition (3.28). Indeed, let $S = \{x_0, ..., x_m\} \subset E$, $x_0 < ... < x_m$. Clearly, for every $x \in [x_0, x_m]$, we have $\text{diam}([x] \cup S) = \text{diam} S = x_m - x_0$, so that, thanks to (1.7),

$$|\Delta^m f[S]|^p \text{ diam } S = \left| \Delta^m f[S] \right|^p (\text{diam } S)^p \left( (x_m - x_0) \leq 2^p \left( (\Delta^m f)_E^\frac{1}{p}(x) \right)^p (x_m - x_0).$$

Integrating this inequality (with respect to $x$) over the interval $[x_0, x_m]$, we obtain the following:

$$|\Delta^m f[S]|^p \text{ diam } S \leq 2^p \int_{x_0}^{x_m} \left( (\Delta^m f)_E^\frac{1}{p}(x) \right)^p dx \leq 2^p \| (\Delta^m f)_E^\frac{1}{p} \|_{L_p(\mathbb{R})}.$$  

Hence,

$$\sup_{S \subset E, \#S = m+1} |\Delta^m f[S]| \left( (\text{diam } S) \right)^\frac{1}{p} \leq 2 \| (\Delta^m f)_E^\frac{1}{p} \|_{L_p(\mathbb{R})} < \infty$$

19
proving (3.28).

This condition and Lemma 3.14 guarantee that the Whitney \((m - 1)\)-field \(P^{(m,E)}[f]\) from Definition 3.13 is well defined.

Now, let us show that inequality (3.3) holds. Its proof relies on Theorem 3.16 below which provides a criterion for the restrictions of Sobolev jets.

For each family \(P = \{P_x : x \in E\}\) of polynomials we let \(P^\sharp_{m,E}\) denote a certain kind of a “sharp maximal function” associated with \(P\) which is defined by

\[
P^\sharp_{m,E}(x) = \sup_{a_1, a_2 \in E, a_1 \neq a_2} \frac{|P_{a_1}(x) - P_{a_2}(x)|}{|x - a_1|^m + |x - a_2|^m}, \quad x \in \mathbb{R}.
\]

**Theorem 3.16** (47) Let \(m \in \mathbb{N}, p \in (1, \infty),\) and let \(E\) be a closed subset of \(\mathbb{R}\). Suppose we are given a family \(P = \{P_x : x \in E\}\) of polynomials of degree at most \(m - 1\) indexed by points of \(E\).

Then there exists a \(C^{m-1}\)-function \(F \in L^p_{m}(\mathbb{R})\) such that \(T^{m-1}_{x}[F] = P_x\) for every \(x \in E\) if and only if \(P^\sharp_{m,E} \in L^p_{m}(\mathbb{R})\). Furthermore,

\[
\|P\|_{m,p,E} \sim \|P^\sharp_{m,E}\|_{L^p_{m}(\mathbb{R})}
\]

(3.42)

with the constants in this equivalence depending only on \(m\) and \(p\).

We recall that the quantity \(\|P\|_{m,p,E}\) is defined by (3.1).

**Lemma 3.17** Let \(f\) be a function on \(E\) such that \((\Delta^m f)^\sharp_{E} \in L^p_{m}(\mathbb{R})\). Then for every \(x \in \mathbb{R}\) the following inequality

\[
(P^{(m,E)}[f])^\sharp_{m,E}(x) \leq C(m) (\Delta^m f)^\sharp_{E}(x)
\]

(3.43)

holds.

**Proof.** Let \(x \in \mathbb{R}, a_1, a_2 \in E, a_1 \neq a_2,\) and let \(r = |x - a_1| + |x - a_2|\). Let \(\overline{S_j} = S_{a_j}\) and let \(s_j = s_{a_j},\) \(j = 1, 2.\) See (3.29) and (3.30). We know that \(a_j, s_j \in \overline{S_j}, j = 1, 2\) (see Propositions 3.11 and 3.12).

Suppose that \(\overline{S_1} \neq \overline{S}_2.\) From inequality (3.32), we have

\[
\text{diam } \overline{S_1} + \text{diam } \overline{S}_2 \leq 2 m |a_1 - a_2|.
\]

(3.44)

Fix an \(\varepsilon > 0.\) Lemma 3.15 tells us that for each \(j = 1, 2\) there exists an \(m\)-point subset \(S_j \subset E,\) such that \(\overline{S_j} \subset S_j,\) \(\text{diam}(\{s_j\} \cup (S_j \setminus \overline{S_j})) \leq r\) and and

\[
|P_{a_j}(x) - L_{S_j}[f](x)| \leq \varepsilon r^m/2^{m+1}.
\]

(3.45)

Recall that \(s_j \in \overline{S_j}, j = 1, 2,\) so that

\[
\text{diam } S_j \leq \text{diam } \overline{S}_j + \text{diam}(\{s_j\} \cup (S_j \setminus \overline{S}_j)) \leq \text{diam } \overline{S}_j + r.
\]

This inequality together with (3.44) imply that

\[
\text{diam } S_j \leq 2m |a_1 - a_2| + r, \quad j = 1, 2.
\]

(3.46)
Let $I$ be the smallest closed interval containing $S_1 \cup S_2 \cup \{x\}$. Since $a_j \in \overline{S_j} \subset S_j$, $j = 1, 2$, 
\[
\text{diam } I \leq |x - a_1| + |x - a_2| + \text{diam } S_1 + \text{diam } S_2
\]
so that, thanks to (3.46),
\[
\text{diam } I \leq |x - a_1| + |x - a_2| + 4m|a_1 - a_2| + 2r \leq (4m + 3)r.
\] (Recall that $r = |x - a_1| + |x - a_2|$.) From this inequality and inequality (3.45), we have
\[
|P_{a_1}(x) - P_{a_2}(x)| \leq |P_{a_1}(x) - L_{S_1}[f](x)| + |L_{S_1}[f](x) - L_{S_2}[f](x)| + |P_{a_2}(x) - L_{S_1}[f](x)| = J + \varepsilon r^m / 2^m
\]
where $J = |L_{S_1}[f](x) - L_{S_2}[f](x)|$.

Let us estimate $J$. We may assume that $S_1 \neq S_2$; otherwise $J = 0$. We apply Lemma 3.6 taking $k = m - 1$ and $i = 0$, and get
\[
J \leq \max_f |L_{S_1}[f] - L_{S_2}[f]| \leq m! (\text{diam } I)^{m-1} \max_{S' \subset S, \#S' = m+1} |\Delta^m f[S']| \text{ diam } S'
\]
where $S = S_1 \cup S_2$. This inequality together with (3.47) and (1.7) implies that
\[
J \leq C(m) r^{m-1} (\Delta^m f)_{E}^\sharp (x) \max_{S' \subset S, \#S' = m+1} \text{ diam } (\{x \cup S'\}) \leq C(m) r^m (\Delta^m f)_{E}^\sharp (x).
\]

We are in a position to prove inequality (3.43). We have:
\[
\frac{|P_{a_1}(x) - P_{a_2}(x)|}{|x - a_1|^m + |x - a_2|^m} \leq 2^m r^{-m} |P_{a_1}(x) - P_{a_2}(x)| \leq 2^m r^{-m} (J + \varepsilon r^m / 2^m)
\]
\[
\leq C(m) 2^m r^{-m} r^m (\Delta^m f)_{E}^\sharp (x) + \varepsilon = C(m) (\Delta^m f)_{E}^\sharp (x) + \varepsilon.
\]
Because $\varepsilon > 0$ is arbitrary, we conclude that
\[
\frac{|P_{a_1}(x) - P_{a_2}(x)|}{|x - a_1|^m + |x - a_2|^m} \leq C(m) (\Delta^m f)_{E}^\sharp (x)
\] (3.48)
provided $\overline{S}_1 \neq \overline{S}_2$. Clearly, this inequality also holds whenever $\overline{S}_1 = \overline{S}_2$ because in this case $P_{a_1} = P_{a_2}$.

Finally, taking the supremum in the left hand side of (3.48) over all $a_1, a_2 \in E$, $a_1 \neq a_2$, we obtain (3.43). The proof of the lemma is complete. \qed

We finish the proof of Theorem [1.4] as follows.

Let $f$ be a function on $E$ such that $(\Delta^m f)_{E}^\sharp \in L_p^m(\mathbb{R})$, and let $P^{(m,E)}[f] = \{P_x \in P_{m-1} : x \in E\}$ be the Whitney $(m-1)$-field from Definition 3.13. Lemma 3.17 tells us that
\[
\|(P^{(m,E)}[f])^\sharp\|_{m,E} \leq C(m) \|(\Delta^m f)_{E}^\sharp\|_{L_p^m(\mathbb{R})}.
\]
Combining this inequality with equivalence (3.42), we obtain inequality (3.3).

This inequality and definition (3.1) imply the existence of a function $F \in L_p^m(\mathbb{R})$ such that $T^{m-1}_x[F] = P_x$ on $E$ and
\[
\|F\|_{L_p^m(\mathbb{R})} \leq 2 \|(P^{(m.E)}[f])\|_{m.p,E} \leq C(m, p) \|(\Delta^m f)_{E}^\sharp\|_{L_p^m(\mathbb{R})}.
\] (3.49)

We also note that $P_x(x) = f(x)$ on $E$, see (3.38), so that
\[
F(x) = T^{m-1}_x[F](x) = P_x(x) = f(x), \quad x \in E.
\]
Thus $F \in L_p^m(\mathbb{R})$ and $F|_E = f$ proving that $f \in L_p^m(\mathbb{R})|_E$. Furthermore, thanks to (1.1) and (3.49),
\[
\|f\|_{L_p^m(\mathbb{R})} \leq \|F\|_{L_p^m(\mathbb{R})} \leq C(m, p) \|(\Delta^m f)_{E}^\sharp\|_{L_p^m(\mathbb{R})}.
\]
The proof of Theorem [1.4] is complete. \qed
4. A variational extension criterion for Sobolev jets.

One of the main ingredients of our proof of the sufficiency part of Theorem 1.3 (see Section 6) is the following refinement of Theorem 3.16.

**Theorem 4.1** Let \( m \in \mathbb{N} \), \( p \in (1, \infty) \), and let \( E \subset \mathbb{R} \) be a closed set. Suppose we are given a Whitney \((m - 1)\)-field \( \mathbf{P} = \{ P_x : x \in E \} \) defined on \( E \). There exists a \( C^{m-1} \)-function \( F \in L^p_E(\mathbb{R}) \) such that

\[
T^{m-1}_x[F] = P_x \text{ for every } x \in E
\]

if and only if the following quantity

\[
N_{m,p,E}(\mathbf{P}) = \sup \left\{ \sum_{j=1}^{k-1} \sum_{i=0}^{m-1} \frac{|P^{(j)}_{x_{j}}(x_j) - P^{(j)}_{x_{j+1}}(x_j)|^p}{(x_{j+1} - x_j)^{(m-1)p-1}} \right\}^{1/p}
\]

is finite. Here the supremum is taken over all integers \( k > 1 \) and all finite strictly increasing sequences \( \{x_1, \ldots, x_k\} \subset E \).

Furthermore, \( \|\mathbf{P}\|_{m,p,E} \sim N_{m,p,E}(\mathbf{P}) \) with the constants in this equivalence depending only on \( m \).

**Proof.** (Necessity.) Let \( \{x_j\}_{j=1}^k \) be a strictly increasing sequence in \( E \). Let \( \mathbf{P} = \{ P_x : x \in E \} \) be a Whitney \((m - 1)\)-field on \( E \), and let \( F \in L^p_E(\mathbb{R}) \) be a function satisfying condition (4.1). The Taylor formula with the reminder in the integral form tells us that for every \( x \in \mathbb{R} \) and every \( a \in E \) the following equality

\[
F(x) - T^{m-1}_a[F](x) = \frac{1}{(m-1)!} \int_a^x F^{(m)}(t) (x - t)^{m-1} dt
\]

holds.

Let \( 0 \leq i \leq m - 1 \). Differentiating this equality \( i \) times (with respect to \( x \)) we obtain the following:

\[
F^{(i)}(x) - (T^{m-1}_a[F])^{(i)}(x) = \frac{1}{(m-1-i)!} \int_a^x F^{(m)}(t) (x - t)^{m-1-i} dt.
\]

From this and (4.1), we have

\[
P^{(i)}_x(x) - P^{(i)}_a(x) = \frac{1}{(m-1-i)!} \int_a^x F^{(m)}(t) (x - t)^{m-1-i} dt \quad \text{for every } x \in E.
\]

Therefore, for every \( j \in \{1, \ldots, k - 1\} \) the following equality

\[
P^{(i)}_{x_j}(x_j) - P^{(i)}_{x_{j+1}}(x_j) = \frac{1}{(m-1-i)!} \int_{x_{j+1}}^{x_j} F^{(m)}(t) (x_j - t)^{m-1-i} dt
\]

holds. Hence,

\[
|P^{(i)}_{x_j}(x_j) - P^{(i)}_{x_{j+1}}(x_j)|^p \leq \frac{(x_{j+1} - x_j)^{(m-1-i)p}}{(m-1-i)!} \left( \int_{x_j}^{x_{j+1}} |F^{(m)}(t)| dt \right)^p
\]

holds.
proving that

\[
\frac{|P^{(i)}(x_j) - P^{(i)}_{x_{j+1}}(x_j)|^p}{(x_{j+1} - x_j)^{(m-i)p-1}} \leq \frac{(x_{j+1} - x_j)^{1-p}}{(m-1-i)!^p} \left( \int_{x_j}^{x_{j+1}} |F^{(m)}(t)|^p \, dt \right) \leq \frac{1}{((m-1-i)!)^p} \int_{x_j}^{x_{j+1}} |F^{(m)}(t)|^p \, dt.
\]

Consequently,

\[
\sum_{j=1}^{k-1} \sum_{i=0}^{m-1} \frac{|P^{(i)}(x_j) - P^{(i)}_{x_{j+1}}(x_j)|^p}{(x_{j+1} - x_j)^{(m-i)p-1}} \leq \sum_{j=1}^{k-1} \sum_{i=0}^{m-1} \frac{1}{((m-1-i)!)^p} \int_{x_j}^{x_{j+1}} |F^{(m)}(t)|^p \, dt = \left( \sum_{i=0}^{m-1} \frac{1}{((m-1-i)!)^p} \right) \int_{x_1}^{x_k} |F^{(m)}(t)|^p \, dt \leq e^p \|F\|_{L_p^m(\mathbb{R})}^p.
\]

Taking the supremum in the left hand side of this inequality over all finite strictly increasing sequences \(\{x_j\}_{j=1}^k\) in \(E\), and then the infimum in the right hand side over all function \(F \in L_p^m(\mathbb{R})\) satisfying (4.1), we obtain the required inequality \(N_{m,p,E}(P) \leq e \|P\|_{m,p,E}\).

The proof of the necessity is complete. \(\square\)

(Sufficiency.) Let \(P = \{P_x : x \in E\}\) be a Whitney \((m-1)\)-field defined on \(E\) such that

\[\lambda = N_{m,p,E}(P) < \infty.\]

See (4.2). Thus, for every strictly increasing sequence \(\{x_j\}_{j=1}^k\) in \(E\) the following inequality

\[\sum_{j=1}^{k-1} \sum_{i=0}^{m-1} \frac{|P^{(i)}(x_j) - P^{(i)}_{x_{j+1}}(x_j)|^p}{(x_{j+1} - x_j)^{(m-i)p-1}} \leq \lambda^p \tag{4.3}\]

holds. Our aim is to prove the existence of a function \(F \in L_p^m(\mathbb{R})\) such that \(T_{x_j}^{m-1}[F] = P_x\) for every \(x \in E\) and \(\|F\|_{L_p^m(\mathbb{R})} \leq C(m) \lambda\).

We construct \(F\) with the help of the classical Whitney extension method \([51]\). It is proven in \([47]\) that this method provides an almost optimal extension of the restrictions of Whitney \((m-1)\)-fields generated by Sobolev \(W_p^m(\mathbb{R}^n)\)-functions. In this paper we will use a special one dimensional version of this method suggested by Whitney in \([52]\) Section 4).

Because \(E\) is a closed subset of \(\mathbb{R}\), the complement of \(E\), the set \(\mathbb{R} \setminus E\), can be represented as a union of a certain finite or countable family

\[\mathcal{J}_E = \{J_k = (a_k, b_k) : k \in \mathcal{K}\}\]

of pairwise disjoint open intervals (bounded or unbounded). Thus, \(a_k, b_k \in E \cup \{-\infty, +\infty\}\) for all \(k \in \mathcal{K}\),

\[\mathbb{R} \setminus E = \bigcup\{J_k = (a_k, b_k) : k \in \mathcal{K}\} \quad \text{and} \quad J_k \cap J_{k'} = \emptyset \quad \text{for every} \quad k', k'' \in \mathcal{K}, k' \neq k''.\]

To each interval \(J \in \mathcal{J}_E\) we assign a polynomial \(H_J \in \mathcal{P}_{2m-1}\) as follows:

1. Let \(J = (a, b)\) be an unbounded open interval, i.e., either \(a = -\infty\) and \(b\) is finite, or \(a\) is finite and \(b = +\infty\). In the first case (i.e., \(J = (a, b) = (-\infty, b)\)) we set \(H_J = P_b\), while in the second case (i.e., \(J = (a, b) = (a, +\infty)\)) we set \(H_J = P_a\).
(2) Let $J = (a, b) \in \mathcal{F}_E$ be a bounded interval so that $a, b \in E$. In this case we define the polynomial $H_J \in \mathcal{P}_{2m-1}$ as the Hermite polynomial satisfying the following conditions:

$$H_J^{(i)}(a) = P_a^{(i)}(a) \quad \text{and} \quad H_J^{(i)}(b) = P_b^{(i)}(b) \quad \text{for all} \quad i = 0, \ldots, m - 1. \quad (4.5)$$

The existence and uniqueness of the polynomial $H_J$ follows from [1] Ch. 2, Section 11].

Finally, we define the extension $F$ by the formula:

$$F(x) = \begin{cases} P_y(x), & x \in E, \\ \sum_{J \in \mathcal{F}_E} H_J(x) \chi_J(x), & x \in \mathbb{R} \setminus E. \end{cases} \quad (4.6)$$

We note that inequality (4.3) implies the following property of the Whitney field $\mathcal{P} = \{P_x : x \in E\}$: for every $x, y \in E$ and every $i, 0 \leq i \leq m - 1$, we have

$$|P_y^{(i)}(x) - P_y^{(i)}(y)| \leq \lambda |x - y|^{m-i-1/p}.$$ 

Recall that $p > 1$, which implies that

$$P_x^{(i)}(x) - P_y^{(i)}(x) = o(|x - y|^{m-1-i}) \quad \text{provided} \quad x, y \in E \quad \text{and} \quad 0 \leq i \leq m - 1. \quad (4.7)$$

Whitney [52] proved that for every $(m - 1)$-field $\mathcal{P} = \{P_x : x \in E\}$ satisfying (4.7), the extension $F$ defined by formula (4.6) is a $C^{m-1}$-function on $\mathbb{R}$ which agrees with $\mathcal{P}$ on $E$, i.e.,

$$F^{(i)}(x) = P_x^{(i)}(x) \quad \text{for all} \quad x \in E \quad \text{and} \quad i = 0, \ldots, m - 1. \quad (4.8)$$

Let us show that $F \in L_p^m(\mathbb{R})$ and $\|F\|_{L_p^m(\mathbb{R})} \leq C(m) \lambda$. Our proof of these properties of the function $F$ relies on the following description of $L_p^1(\mathbb{R})$-functions.

**Theorem 4.2** Let $p > 1$ and let $\tau > 0$. Let $G$ be a continuous function on $\mathbb{R}$ satisfying the following condition: There exists a constant $A > 0$ such that for every finite family $\mathcal{I} = \{I = [u_I, v_I]\}$ of pairwise disjoint closed intervals of diameter at most $\tau$ the following inequality

$$\sum_{I = [u_I, v_I] \in \mathcal{I}} \frac{|G(u_I) - G(v_I)|^p}{(v_I - u_I)^{p-1}} \leq A$$

holds. Then $G \in L_p^1(\mathbb{R})$ and $\|G\|_{L_p^1(\mathbb{R})} \leq CA^{1/p}$ where $C$ is an absolute constant.

**Proof.** The Riesz theorem [39] tells us that $G \in L_p^1(\mathbb{R})$. See also [32]. For $\tau = \infty$ inequality $\|G\|_{L_p^1(\mathbb{R})} \leq CA^{1/p}$ follows from [2] Theorem 2] and [3] Theorem 4]. (See also a description of Sobolev spaces obtained in [3] § 4, 3]). For the case $0 < \tau < \infty$ we refer the reader to [47] Section 7].

We will also need the following auxiliary lemmas.

**Lemma 4.3** Let $J = (a, b) \in \mathcal{F}_E$ be a bounded interval, and let $H_J \in \mathcal{P}_{2m-1}$ be the Hermite polynomial satisfying (4.5). Then for every $n \in \{0, \ldots, m\}$ and every $x \in [a, b]$ the following inequality

$$|H_J^{(n)}(x)| \leq C(m) \min \{Y_1(x), Y_2(x)\}$$
holds. Here

\[ Y_1(x) = |P_a^{(n)}(x)| + \left\{ \sum_{i=0}^{m-1} \frac{|P_b^{(i)}(b) - P_a^{(i)}(b)|}{(b - a)^{m-i}} \right\} \cdot (x - a)^{m-n} \]

and

\[ Y_2(x) = |P_b^{(n)}(x)| + \left\{ \sum_{i=0}^{m-1} \frac{|P_b^{(i)}(a) - P_a^{(i)}(a)|}{(b - a)^{m-i}} \right\} \cdot (b - x)^{m-n}. \]

Proof. Definition (4.5) implies the existence of constants \( \gamma_m, \gamma_{m+1}, \ldots, \gamma_{m-1} \in \mathbb{R} \) such that

\[ H_j(x) = P_a(x) + \sum_{k=m}^{2m-1} \frac{1}{k!} \gamma_k (x - a)^k \quad \text{for every} \quad x \in [a, b]. \]

Hence, for every \( n \in \{0, ..., m \} \) and every \( x \in [a, b] \),

\[ H_j^{(n)}(x) = P_a^{(n)}(x) + \sum_{k=m}^{2m-1} \frac{1}{(k-n)!} \gamma_k (x - a)^{k-n}. \]

In particular,

\[ H_j^{(n)}(b) = P_a^{(n)}(b) + \sum_{k=m}^{2m-1} \frac{1}{(k-n)!} \gamma_k (b - a)^{k-n} \]

which together with (4.5) implies that

\[ \sum_{k=m}^{2m-1} \gamma_k \frac{(b - a)^{k-n}}{(k-n)!} = P_b^{(n)}(b) - P_a^{(n)}(b), \quad \text{for all} \quad n = 0, ..., m - 1. \]

Thus, the tuple \((\gamma_m, \gamma_{m+1}, \ldots, \gamma_{m-1})\) is a solution of the above system of \( m \) linear equations with respect to \( m \) unknowns. Whitney [52] proved the existence of constants \( K_{k,i} \), \( k = m, ..., 2m - 1, \ i = 0, ..., m - 1 \), depending only on \( m \), such that

\[ \gamma_k = \sum_{i=0}^{m-1} K_{k,i} \frac{P_b^{(i)}(b) - P_a^{(i)}(b)}{(b - a)^{k-i}}, \quad k = m, ..., 2m - 1. \]

This representation enables us to estimate \( H_j^{(n)} \) as follows: Thanks to (4.10),

\[ |\gamma_k| \leq C(m) \sum_{i=0}^{m-1} \frac{|P_b^{(i)}(b) - P_a^{(i)}(b)|}{(b - a)^{k-i}}, \quad k = m, ..., 2m - 1, \]

and, thanks to (4.9),

\[ |H_j^{(n)}(x)| \leq |P_a^{(n)}(x)| + \sum_{k=m}^{2m-1} \frac{1}{(k-n)!} |\gamma_k| (x - a)^{k-n} \quad \text{for every} \quad x \in [a, b]. \]
Lemma 4.4 Let $J$ be a bounded interval. Then for every $x \in [a, b]$ the following inequality holds.

$$|H_{j}^{(m)}(x)| \leq |P_{a}^{(n)}(x)| + C(m) \sum_{k=m}^{2m-1} \sum_{i=0}^{m-1} \frac{|P_{b}^{(i)}(b) - P_{a}^{(i)}(b)|}{(b-a)^{k-i}} (x-a)^{k-n}$$

$$= |P_{a}^{(n)}(x)| + C(m) \sum_{i=0}^{m-1} \sum_{k=m}^{2m-1} \frac{|P_{b}^{(i)}(b) - P_{a}^{(i)}(b)|}{(b-a)^{k-i}} (x-a)^{k-n}$$

$$= |P_{a}^{(n)}(x)| + C(m) \sum_{i=0}^{m-1} |P_{b}^{(i)}(b) - P_{a}^{(i)}(b)| \frac{(x-a)^{m-n}}{(b-a)^{m-i}} \sum_{k=m}^{2m-1} (x-a)^{k-m}$$

$$\leq |P_{a}^{(n)}(x)| + C(m) m \sum_{i=0}^{m-1} |P_{b}^{(i)}(b) - P_{a}^{(i)}(b)| \frac{(x-a)^{m-n}}{(b-a)^{m-i}}$$

proving that $|H_{j}^{(n)}(x)| \leq C(m)m Y_{1}(x)$ for all $x \in [a, b]$. Interchanging the roles of $a$ and $b$, we show that $|H_{j}^{(n)}(x)| \leq C(m) Y_{2}(x)$ on $[a, b]$ proving the lemma. \hfill \Box

**Lemma 4.4** Let $J = (a, b) \in \mathcal{J}_{E}$ be a bounded interval. Then for every $x \in [a, b]$ the following inequality

$$|H_{j}^{(m)}(x)| \leq C(m) \min \left\{ \sum_{i=0}^{m-1} \frac{|P_{b}^{(i)}(b) - P_{a}^{(i)}(b)|}{(b-a)^{m-i}}, \sum_{i=0}^{m-1} \frac{|P_{b}^{(i)}(a) - P_{a}^{(i)}(a)|}{(b-a)^{m-i}} \right\}$$

holds.

**Proof.** The proof is immediate from Lemma 4.3 because $P_{a}$ and $P_{b}$ belong to $\mathcal{P}_{m-1}$. \hfill \Box

**Lemma 4.5** Let $I$ be a finite family of pairwise disjoint closed intervals $I = [u_{I}, v_{I}]$ such that $(u_{I}, v_{I}) \subset \mathbb{R} \setminus E$ for every $I \in \mathcal{I}$. Let $F$ be the function defined by (4.6). Then

$$\sum_{I=[u_{I}, v_{I}] \in \mathcal{I}} \frac{|F^{(m-1)}(v_{I}) - F^{(m-1)}(u_{I})|^{p}}{(v_{I} - u_{I})^{p-1}} \leq C(m)^{p} \lambda^{p} \cdot (4.11)$$

**Proof.** Let $I = [u_{I}, v_{I}] \in \mathcal{I}$. Since $(u_{I}, v_{I}) \subset \mathbb{R} \setminus E$, there exist an interval $J = (a, b) \in \mathcal{J}_{E}$ containing $(u_{I}, v_{I})$. (Recall that the family $\mathcal{J}_{E}$ is defined by (4.4)). The extension formula (4.6) tells us that $F|_{J} = H_{J}$. This property and Lemma 4.4 imply that

$$|F^{(m-1)}(u_{I}) - F^{(m-1)}(v_{I})| = |H_{j}^{(m-1)}(u_{I}) - H_{j}^{(m-1)}(v_{I})| \leq (\max_{I} |H^{(m)}|)(v_{I} - u_{I})$$

$$\leq C(m)(v_{I} - u_{I}) \sum_{i=0}^{m-1} \frac{|P_{b}^{(i)}(a) - P_{a}^{(i)}(a)|}{(b-a)^{m-i}} \cdot (v_{I} - u_{I})$$

Hence,

$$V_{I} \leq C(m)^{p} m^{p} (v_{I} - u_{I}) \sum_{i=0}^{m-1} \frac{|P_{b}^{(i)}(a) - P_{a}^{(i)}(a)|^{p}}{(b-a)^{(m-i)p}} \cdot (4.12)$$

For every $J = (a, b) \in \mathcal{J}_{E}$ by $I_{J}$ we denote a subfamily of $\mathcal{I}$ defined by

$$I_{J} = \{ I \in \mathcal{I} : I \subset [a, b] \}.$$
Let $\mathcal{J} = \{ J \in \mathcal{I}_E : I_J \neq \emptyset \}$. Then, thanks to \eqref{4.12}, for every $J = (a_J, b_J) \in \mathcal{J}$
\[
Q_J = \sum_{I = [u_I, v_I] \in I_J} V_I \leq C(m)^p \left( \sum_{I \in I_J} \text{diam } I \right) \left( \sum_{i=0}^{m-1} \frac{|P_{a_J}^i(a_J) - P_{b_J}^i(a_J)|^p}{(b_J - a_J)^{(m-i)p}} \right).
\]

We know that the intervals of the family $I_J$ are pairwise disjoint (because the intervals of $I$ have this property). Hence,
\[
Q_J \leq C(m)^p (b_J - a_J) \left( \sum_{i=0}^{m-1} \frac{|P_{a_J}^i(a_J) - P_{b_J}^i(a_J)|^p}{(b_J - a_J)^{(m-i)p}} \right) = C(m)^p \sum_{i=0}^{m-1} \frac{|P_{a_J}^i(a_J) - P_{b_J}^i(a_J)|^p}{(b_J - a_J)^{(m-i)p-1}}.
\]

Finally,
\[
Q = \sum_{I = [u_I, v_I] \in \mathcal{J}} V_I = \sum_{J = (a_J, b_J) \in \mathcal{J}} Q_J \leq C(m)^p \sum_{J = (a_J, b_J) \in \mathcal{J}} \sum_{i=0}^{m-1} \frac{|P_{a_J}^i(a_J) - P_{b_J}^i(a_J)|^p}{(b_J - a_J)^{(m-i)p-1}}.
\]

Note that the family $\tilde{\mathcal{J}}$ consist of pairwise disjoint intervals. Therefore, thanks to assumption \eqref{4.3}, $Q \leq C(m)^p \lambda^p$, completing the proof of the lemma. \quad \square

**Lemma 4.6** Let $I = \{ I = [u_I, v_I] \}$ be a finite family of closed intervals such that $u_I, v_I \in E$ for each $I \in I$. Suppose that the open intervals $\{ (u_I, v_I) : I \in I \}$ are pairwise disjoint. Then inequality \eqref{4.11} holds.

**Proof.** Because $F$ agrees with the Whitney $(m - 1)$-field $P = \{ P_x : x \in E \}$, see \eqref{4.3}, we have $F^{(m-1)}(x) = P_x^{(m-1)}(x)$ for every $x \in E$. Hence,
\[
A = \sum_{I = [u_I, v_I] \in I} \frac{|F^{(m-1)}(u_I) - F^{(m-1)}(v_I)|^p}{(v_I - u_I)^{p-1}} = \sum_{I = [u_I, v_I] \in I} \frac{|P_{u_I}^{(m-1)}(u_I) - P_{v_I}^{(m-1)}(v_I)|^p}{(v_I - u_I)^{p-1}}.
\]

Because the intervals $\{ (u_I, v_I) : I \in I \}$ are pairwise disjoint, assumption \eqref{4.3} implies that $A \leq \lambda^p$ proving the lemma. \quad \square

We are in a position to finish the proof of the sufficiency. Let $I$ be a finite family of pairwise disjoint closed intervals. We introduce the following notation: given an interval $I = [u, v], u \neq v$, we put
\[
Y(I; F) = \frac{|F^{(m-1)}(u) - F^{(m-1)}(v)|^p}{(v - u)^{p-1}}.
\]

We put $Y(I; F) = 0$ whenever $u = v$, i.e., $I = [u, v]$ is a singleton.

To each interval $I \in I$ we assign three intervals $I^{(1)}, I^{(2)}, I^{(3)}$ as follows:

Let $I = [u_I, v_I] \in I$ be an interval such that $I \cap E \neq \emptyset$ and $\{ u_I, v_I \} \notin E$. Thus either $u_I$ or $v_I$ belongs to $\mathbb{R} \setminus E$. Let $u'_I$ and $v'_I$ be the points of $E$ nearest to $u_I$ and $v_I$ on $I \cap E$ respectively. Then $[u'_I, v'_I] \subset [u_I, v_I]$. Let
\[
I^{(1)} = [u_I, u'_I], \quad I^{(2)} = [u'_I, v'_I] \quad \text{and} \quad I^{(3)} = [v'_I, v_I].
\]

Note that $u'_I, v'_I \in E$ and $(u_I, u'_I), (v'_I, v_I) \in \mathbb{R} \setminus E$ provided $u_I \notin E$ and $v_I \notin E$. Furthermore,
\[
Y(I; F) \leq 3^p \left\{ Y(I^{(1)}; F) + Y(I^{(2)}; F) + Y(I^{(3)}; F) \right\}.
\]
If \( I \in \mathcal{I} \) and \((u_t, v_t) \subset \mathbb{R} \setminus E\), or \( u_t, v_t \in E \), we put \( I^{(1)} = I^{(2)} = I^{(3)} = I \).

Clearly,
\[
A(F; \mathcal{I}) = \sum_{I \in \mathcal{I}} Y(I; F) \leq 3^\rho \sum_{I \in \mathcal{I}} \{ Y(I^{(1)}; F) + Y(I^{(2)}; F) + Y(I^{(3)}; F) \}
\]
proving that
\[
A(F; \mathcal{I}) \leq 3^\rho \sum_{I \in \mathcal{I}} Y(I; F).
\tag{4.13}
\]

Here \( \mathcal{I} = \bigcup \{ \mathcal{I}_j : j = 1, 2, 3 \} \) where \( \mathcal{I}_j = \bigcup \{ I^j : I \in \mathcal{I} \}, j = 1, 2, 3 \).

We know that for each \( I = [u_t, v_t] \in \mathcal{I} \) either \((u_t, v_t) \in \mathbb{R} \setminus E\), or \( u_t, v_t \in E \), or \( u_t = v_t \) (and so \( Y(I; F) = 0 \)). Furthermore, the open intervals \( \{(u_t, v_t) : I \in \mathcal{I} \} \) are pairwise disjoint. This property of \( \mathcal{I} \), inequality \( (4.13) \), Lemma \( 4.5 \) and Lemma \( 4.6 \) imply that \( A(F; \mathcal{I}) \leq C(m)^\rho \lambda^p \).

Because \( I \) is an arbitrary finite family of pairwise disjoint closed intervals, the function \( G = F^{(m-1)} \) satisfies the hypothesis of Theorem \( 4.2 \). This theorem tells us that the function \( F^{(m-1)} \in L_p^1(\mathbb{R}) \) and \( \|F^{(m-1)}\|_{L_p^1(\mathbb{R})} \leq C(m) \lambda \). This implies that \( F \in L_p^m(\mathbb{R}) \) and \( \|F\|_{L_p^m(\mathbb{R})} \leq C(m) \lambda \) proving the sufficiency part of Theorem \( 4.4 \) \( \square \)

The proof of Theorem \( 4.4 \) is complete. \( \square \)

5. The Main Lemma: from jets to Lagrange polynomials.

This section is devoted to the second main ingredient of our proof of the sufficiency part of Theorem \( 1.3 \) the Main Lemma \( 5.1 \) Let \( E \subset \mathbb{R} \) be a closed set with \( \# E \geq m + 1 \). Let \( \lambda > 0 \) and let \( f \) be a function on \( E \) satisfying condition \( (3.28) \), i.e.,
\[
\sup_{S \subset E, \# S = m+1} |\Delta^m f[S]| (\text{diam} S)^{\frac{1}{2}} \leq \lambda.
\]

In Section 3 we have proved that in this case the Whitney field \( P^{m,E}[f] = \{ P_x \in P_{m-1} : x \in E \} \) satisfying conditions \( (3.34)-(3.37) \) of Definition \( 3.13 \) is well defined. The Main Lemma \( 5.1 \) below provides a controlled transition from the (Hermite) polynomials \( \{ P_x : x \in E \} \) of the function \( f \) to its Lagrange polynomials.

**Main Lemma 5.1** Let \( k \in \mathbb{N} \), \( \epsilon > 0 \), and let \( X = \{ x_1, ..., x_k \} \subset E \), \( x_1 < ... < x_k \), be a sequence of points in \( E \). There exist a positive integer \( \ell \geq m \), a finite strictly increasing sequence \( V = \{ v_1, ..., v_\ell \} \) of points in \( E \), and a mapping \( H : X \rightarrow 2^V \) such that:

- \( \bullet 1 \) For every \( x \in X \) the set \( H(x) \) consists of \( m \) consecutive points of the sequence \( V \). Thus,
\[
H(x) = \{ v_{j_1(x)}, ..., v_{j_2(x)} \}
\]
where \( 1 \leq j_1(x) \leq j_2(x) = j_1(x) + m - 1 \leq \ell \);

- \( \bullet 2 \) \( x \in H(x) \) for each \( x \in X \). In particular, \( X \subset V \).

Furthermore, given \( i \in \{ 1, ..., k - 1 \} \) let \( x_i = v_{x_i} \) and \( x_{i+1} = v_{x_{i+1}} \). Then \( 0 < x_{i+1} - x_i \leq 2m \).

- \( \bullet 3 \) Let \( x', x'' \in X \), \( x' < x'' \). Then
\[
\min H(x') \leq \min H(x'') \quad \text{and} \quad \max H(x') \leq \max H(x'')
\]
(●4) For every \( x', x'' \in X \) such that \( H(x') \neq H(x'') \) the following inequality holds:
\[
diam H(x') + diam H(x'') \leq 2(m + 1) |x' - x''|
\] (5.1)

(●5) For every \( x, y \in X \) and every \( i, 0 \leq i \leq m - 1 \), we have
\[
|P_x^{(i)}(y) - L_{H(x)}^{(i)}[f](y)| < \varepsilon.
\] (5.2)

Proof. We proceed by steps.

STEP 1. At this step we introduce the sequence \( V \) and the mapping \( H \).

We recall that, given \( x \in E \), by \( S_x \) and \( s_x \) we denote a subset of \( E \) and a point in \( E \) whose properties are described in Propositions 3.11 and 3.12. In particular, \( |S_x| \leq m \). Let
\[
S_X = \bigcup_{x \in X} S_x \quad \text{and let} \quad n = |S_X|.
\] (5.3)

Clearly, one can consider \( S_X \) as a finite strictly increasing sequence of points \( \{u_i\}_{i=1}^n \) in \( E \). Thus
\[
S_X = \{u_1, \ldots, u_n\} \quad \text{and} \quad u_1 < u_2 < \ldots < u_n.
\] (5.4)

If \( |S_x| = m \) for every \( x \in X \), we set \( V = S_X \) and \( H(x) = S_x, x \in X \). In this case the required properties (●1)-(●5) of the Main Lemma are immediate from Propositions 3.11 and 3.12.

However, in general, the set \( X \) may have points \( x \) with \( |S_x| < m \). For those \( x \) we construct the required set \( H(x) \) by adding to \( S_x \) a certain finite set \( \tilde{H}(x) \subset E \). In other words, we define \( H(x) \) as
\[
H(x) = \tilde{H}(x) \cup S_x, \quad x \in X,
\] (5.5)

where \( \tilde{H}(x) \) is a subset of \( E \) such that \( \tilde{H}(x) \cap S_x = \emptyset \) and \( |\tilde{H}(x)| = m - |S_x| \).

Finally, we set
\[
V = \bigcup \{H(x) : x \in X\}.
\] (5.6)

We construct \( \tilde{H}(x) \) by picking \( (m - |S_x|) \) points of \( E \) in a certain small neighborhood of \( s_x \). Propositions 3.11, 3.12, and Lemma 3.15 enable us to prove that this neighborhood can be chosen so small that \( V \) and \( H \) will satisfy conditions (●1)-(●5) of the Main Lemma.

We turn to the precise definition of the mapping \( H \).

First, we set \( \tilde{H}(x) = \emptyset \) whenever \( |S_x| = m \). Thus,
\[
H(x) = S_x \quad \text{provided} \quad |S_x| = m.
\] (5.7)

Note, that in this case \( P_x = L_{S_x} \) \( L_{H(x)} \) (see (3.37)), so that (5.2) holds vacuously.

Let us define the sets \( H(x) \) for all points \( x \in X \) such that \( |S_x| < m \).

We recall that part (iii) of Proposition 3.12 tells us that
\[
\text{for each } x \in X \text{ with } |S_x| < m \text{ the point } s_x \text{ is a limit point of } E.
\] (5.8)

In turn, part (i) of this proposition tells us that
\[
either s_x = \min S_x \text{ or } s_x = \max S_x.
\]
Let 
\[ Z_X = \{ s_x : x \in X, \#S_x < m \}. \]

Then, thanks to (5.8),

\[ \text{every point } z \in Z_X \text{ is a limit point of } E. \quad (5.9) \]

Given \( z \in Z_X \), let

\[ K(z) = \{ x \in X : s_x = z \}. \quad (5.10) \]

The following lemma describes main properties of the sets \( K(z), z \in Z_X \).

**Lemma 5.2** Let \( z \in Z_X \). Suppose that \( K(z) \neq \{ z \} \). Then the following properties hold:

- (\( \Box \)) 1 The set \( K(z) \) lies on one side of \( z \), i.e.,
  \[ \text{either } \max K(z) \leq z \text{ or } \min K(z) \geq z; \quad (5.11) \]

- (\( \Box \)) 2 If \( \min K(z) \geq z \), then for every \( r > 0 \) the interval
  \[ (z - r, z) \text{ contains an infinite number of points of } E. \quad (5.12) \]

If \( \max K(z) \leq z \) then each interval \( (z, z + r) \) contains an infinite number of points of \( E \);

- (\( \Box \)) 3 If \( \min K(z) \geq z \) then
  \[ [z, y] \cap X \subset K(z) \text{ for every } y \in K(z). \quad (5.13) \]

Furthermore,

\[ \min S_y = z \text{ for all } y \in K(z), \quad (5.14) \]

and

\[ S_x \subset S_y \text{ for every } y \in K(z) \text{ and every } x \in [z, y] \cap E. \quad (5.15) \]

Correspondingly, if \( \max K(z) \leq z \) then \( [y, z] \cap X \subset K(z) \) for each \( y \in K(z) \). Moreover, \( S_x \subset S_y \) for every \( y \in K(z) \) and every \( x \in [y, z] \cap E \). In addition, \( \max S_y = z \text{ for all } y \in K(z) \);

- (\( \Box \)) 4 Assume that \( \min K(z) \geq z \). Let \( \bar{z} = \max K(z) \). Then
  \[ K(z) = [z, \bar{z}] \cap X. \quad (5.16) \]

Furthermore, in this case \( K(z) \subset S_{\bar{z}} \).

In turn, if \( \max K(z) \leq z \) then \( K(z) = [\bar{z}, z] \cap X \) where \( \bar{z} = \min K(z) \). In this case \( K(z) \subset S_{\bar{z}} \);

- (\( \Box \)) 5 \( \#K(z) \leq m \).

**Proof.** (\( \Box \)) 1 Suppose that (5.11) does not hold so that there exist \( z', z'' \in K(z) \) such that \( z'' < z < z' \). Thanks to (5.10), \( z = s_{z''} = s_{z'} \). We also know that \( z', s_{z'} \in S_{z'} \), see part (i) of Proposition 3.11 and part (i) of Proposition 3.12. This property and (5.11) tell us that

\[ [z, z'] \cap E = [s_{z'}, z'] \cap E \subset [\min S_{z'}, \max S_{z'}] \cap E = S_{z'}. \quad (5.17) \]
Part (i) of Proposition 3.11 also tells us that \( \#S_z \leq m \) proving that the interval 
\[(z, z') \] contains at most \( m \) points of \( E \). (5.18)

In the same way we show that the interval \((z'', z')\) contains a finite number of points of \( E \) proving that \( z \) is an isolated point of \( E \). On the other hand, \( z \in \mathbb{Z} \) so that, thanks to (5.9), \( z \) is a limit point of \( E \), a contradiction.

(2) Let \( z' \in K(z), z' \neq z \). Then \( z < z' \) so that, thanks to (5.18), the interval \((z, z')\) contains at most \( m \) points of \( E \). But \( z \) is a limit point of \( E \), see (5.9), so that the interval \((z - r, z)\) contains an infinite number of points of \( E \). This proves (5.12).

In the same fashion we prove the second statement of part (2).

(3) Let \( \min K(z) \geq z \), and let \( y \in K(z), y \neq z \). Prove that \( s_x = z \) for every \( x \in [z, y] \cap E \). We know that \( z = s_y < y \). Furthermore, property (5.17) tells us that 
\[ [z, y] \cap E \subset S_y. \] (5.19)

We recall that, thanks to (5.9), \( z \) is a limit point of \( E \) so that \( S_z = \{z\} \). Part (iii) of Proposition 3.11 tells us that \( z = \min S_z \leq \min S_x \) and \( \max S_x \leq \max S_y \), so that
\[ S_x \subset [z, \max S_y] \cap E. \] (5.20)

In particular, \( z \leq \min S_y \). But \( z = s_y \in S_y \), so that \( z = \min S_y \) proving (5.14).

In turn, thanks to (3.31),
\[ \min S_y, \max S_y] \cap E = [z, \max S_y] \cap E = S_y. \]

This and (5.20) imply that \( S_x \subset S_y \) for every \( x \in [z, y] \cap E \) proving (5.15).

Moreover, thanks to (5.8), if \( \#S_x < m \) then \( s_x \) is a limit point of \( E \). But \( s_x \in S_x \subset S_y \), therefore, part (ii) of Proposition 3.12 implies that \( s_x = s_y = z \).

If \( \#S_x = m \) then \( S_x = S_y \) (because \( S_x \subset S_y \) and \( \#S_y \leq m \)). Hence, \( z = s_y \in S_x \). But \( z \) is a limit point of \( E \) which together with part (ii) of Proposition 3.12 implies that \( s_x = z = s_y \).

Thus, in all cases \( s_x = z \) proving property (3) of the lemma in the case under consideration. Using the same ideas we prove the second statement of the lemma related to the case \( \max K(z) \leq z \).

(4) From (5.13), we have \([z, \bar{z}] \cap X \subset K(z) \).

On the other hand, \( K(z) \subset [z, \bar{z}] \) because \( z \leq \min K(z) \) and \( \bar{z} = \max K(z) \). Since \( K(z) \subset X \), see (5.10), \( K(z) \subset [z, \bar{z}] \cap X \) proving (5.16).

Furthermore, thanks to (5.19) (with \( y = \bar{z} \)), it follows that \([z, \bar{z}] \cap E \subset S_{\bar{z}} \) proving that
\[ K(z) = [z, \bar{z}] \cap X \subset [z, \bar{z}] \cap E \subset S_{\bar{z}}. \]

In the same way we prove the last statement of part (4) related to the case \( z \geq \max K(z) \).

(5) We recall that \( \#S_x \leq m \) for every \( x \in E \), see part (i) of Proposition 3.11. Part (4) of the present lemma tells us that \( K(z) \subset S_y \) where \( y = \max K(z) \) or \( y = \min K(z) \). Hence \( \#K(z) \leq \#S_y \leq m \).

The proof of the lemma is complete. \( \square \)

Let us fix a point \( z \in \mathbb{Z} \) and define the set \( H(x) \) for every \( x \in K(z) \). Thanks to property (5.11), it suffices to consider the following three cases:

Case (★1). Suppose that
\[ K(z) \neq \{z\} \text{ and } \min K(z) \geq z. \] (5.21)
Let \( y \in K(z) \). Thus \( y \in X \) and \( s_y = z \); we also know that \( y \geq z \). Then property (5.13) tells us that \([z,y] \cap X \subset K(z)\).

We also note that \( K(z) = [z, \bar{z}] \cap X \) where \( \bar{z} = \max K(z) \), and \( K(z) \subset S_z \), see part (\( \Box \)) of Lemma 5.2. Furthermore, part (\( \Box \)) of this lemma tells us that \( \#K(z) \leq m \).

Let us fix several positive constants which we need for definition of the sets \( \{H(x) : x \in X\} \).

We recall that \( x = \{x_1, \ldots, x_k\} \) and \( x_1 < \ldots < x_k \). Let \( I_X = [x_1, x_k] \).

We also recall that inequality (3.23) holds, and \( s_x = z \) provided \( x \in K(z) \). This enables us to apply Lemma 3.15 to the interval \( I = I_X \) and the point \( x \in K(z) \). This lemma tells us that

\[
\lim_{S' \subset E, \#S' = m} \|L_{S'}[f] - P_x\|_{C^m(I_X)} = 0.
\]

Thus, there exists a constant \( \delta_x = \delta_x(\varepsilon) > 0 \) satisfying the following condition: for every \( m \)-point set \( S' \) such that \( S_x \subset S' \subset E \) and \( S' \setminus S_x \subset (z - \bar{\delta}_x, z + \bar{\delta}_x) \) we have

\[
|P^{(i)}_x(y) - L^{(i)}_{S_x}[f](y)| < \varepsilon \quad \text{for every } i = 0, \ldots, m - 1, \quad \text{and every } y \in I_X.
\]

We recall that \( S_X = \{u_1, \ldots, u_n\} \) is the set defined by (5.3) and (5.4). Let

\[
\tau_X = \frac{1}{4} \min_{i=1}^{n-1} (u_{i+1} - u_i).
\]

Thus,

\[
|x - y| \geq 4 \tau_X \quad \text{provided } x, y \in S_X, x \neq y.
\]

Finally, we set

\[
\delta_z = \min \{\tau_X, \min_{x \in K(z)} \bar{\delta}_x\}.
\]

Clearly, \( \delta_z > 0 \) (because \( K(z) \) is finite).

Definition (5.24) implies the following: Let \( x \in K(z) \). Then for every \( i, \ 0 \leq i \leq m - 1 \), and every \( m \)-point set \( S' \) such that \( S_x \subset S' \subset E \) and \( S' \setminus S_x \subset (z - \delta_z, z + \delta_z) \), we have

\[
|P^{(i)}_x(y) - L^{(i)}_{S_x}[f](y)| < \varepsilon, \quad y \in X.
\]

Inequality (5.12) tells us that the interval \((z - \delta_z, z)\) contains an infinite number of points of \( E \). Let us pick \( m - 1 \) distinct points \( a_1 < a_2 < \ldots < a_{m-1} \) in \((z - \delta_z, z) \cap E \) and set

\[
W(z) = \{a_1, a_2, \ldots, a_{m-1}\}.
\]

Thus,

\[
W(z) = \{a_1, a_2, \ldots, a_{m-1}\} \subset (z - \delta_z, z) \cap E.
\]

In particular,

\[
z - \tau_X < a_1 < a_2 < \ldots < a_{m-1} < z \quad \text{(because } \delta_z \leq \tau_X)\).
\]

Let \( x \in K(z) \), and let \( \ell_x = \#S_x \). We introduce the set \( \tilde{H}(x) \) as follows: we set

\[
\tilde{H}(x) = \emptyset \quad \text{and } H(x) = S_x \quad \text{provided } \ell_x = m.
\]
If \( \ell_x < m \), we define \( \widetilde{H}(x) \) by letting
\[
\widetilde{H}(x) = \{a_{\ell_x}, a_{\ell_x+1}, ..., a_{m-1}\}.
\] (5.29)

Clearly, \( \#\widetilde{H}(x) + \#S_x = m \).

Then we define \( H(x) \) by formula (5.5), i.e., we set \( H(x) = \widetilde{H}(x) \cup S_x \). This definition, property (5.26) and inequality (5.25) imply that for every \( y \in X \) and every \( i, 0 \leq i \leq m - 1 \), the following inequality
\[
|p_x^{(i)}(y) - L_{H(x)}^{(i)}[f](y)| < \varepsilon
\] (5.30)
holds. Furthermore, property (5.15) tells us that \( S_{x'} \subset S_{x''} \) provided \( x', x'' \in K(z) \), \( x' < x'' \). This property and definition (5.29) imply that
\[
\min H(x') \leq \min H(x'') \text{ for every } x', x'' \in K(z), x' < x''.
\] (5.31)

Let us also note the following property of \( H(x) \) which directly follows from its definition: Let
\[
\widehat{H}(x) = \widetilde{H}(x) \cup \{s_x\}.
\] (5.32)
Then
\[
[\min H(x), \max H(x)] = [\min S_x, \max S_x] \cup [\min \widetilde{H}(x), \max \widetilde{H}(x)]
\] (5.33)
and
\[
[\min S_x, \max S_x] \cap [\min \widetilde{H}(x), \max \widetilde{H}(x)] = \{s_x\}.
\] (5.34)

*Case (★2).* \( K(z) \neq \{z\} \) and \( \max K(z) \leq z \).

Using the same approach as in *Case (★1)*, see (5.21), given \( x \in K(z) \), we define a corresponding constant \( \delta_z \), a set \( W(z) = \{a_1, ..., a_{m-1}\} \) and sets \( \widetilde{H}(x) \) and \( H(x) \). More specifically, we pick a strictly increasing sequence
\[
W(z) = \{a_1, a_2, ..., a_{m-1}\} \subset (z, z + \delta_z) \cap E.
\]
In particular, this sequence has the following property:
\[
z < a_1 < a_2 < ... < a_{m-1} < z + \tau_X.
\] (5.35)
(Recall that \( \tau_X \) is defined by (5.22).) Then we set \( \widetilde{H}(x) = \emptyset \) and \( H(x) = S_x \) if \( \ell_x = m \), and
\[
\widetilde{H}(x) = \{a_1, a_2, ..., a_{m-\ell_x}\} \text{ if } \ell_x < m.
\] (5.36)
(Recall that \( \ell_x = \#S_x \).) Finally, we define the set \( H(x) \) by formula (5.5).

As in *Case (★1)*, see (5.21), our choice of \( \delta_z \), \( \widetilde{H}(x) \) and \( H(x) \) provides inequality (5.30) and properties (5.31), (5.33) and (5.34).

*Case (★3).* \( K(z) = \{z\} \).

Note that in this case \( z \in X \) is a limit point of \( E \) and \( S_z = \{z\} \). This enables us to pick a subset \( W(z) = \{a_1, ..., a_{m-1}\} \subset E \) with \( \#W(z) = m - 1 \) such that either (5.27) or (5.35) hold.
We set \( \widetilde{H}(z) = W(z) \). Thus, in this case the set \( H(z) \) is defined by formula (5.5) with
\[
\widetilde{H}(z) = \{a_1, a_2, \ldots, a_{m-1}\},
\]
i.e., \( H(z) = \{z, a_1, a_2, \ldots, a_{m-1}\} \).

It is also clear that properties (5.33), (5.34) hold in the case under consideration. Moreover, our choice of the set \( W(z) \) provides inequality (5.30) with \( x = z \) and \( H(x) = \{z, a_1, a_2, \ldots, a_{m-1}\} \).

We have defined the set \( H(x) \) for every \( x \in X \). Then we define the set \( V \) by formula (5.6). Clearly, \( V \) is a finite subset of \( E \). Let us enumerate the points of this set in increasing order: thus, we represent \( V \) in the form
\[
V = \{v_1, v_2, \ldots, v_{\ell}\}
\]
where \( \ell \) is a positive integer and \( \{v_j\}_{j=1}^{\ell} \) is a strictly increasing sequence of points in \( E \).

**STEP 2.** At this step we prove two auxiliary lemmas which describe a series of important properties of the mappings \( \widetilde{H} \) and \( H \).

**Lemma 5.3** (i) For each \( x \in X \) the following inclusion
\[
\widetilde{H}(x) \subset (s_x - \tau_X, s_x + \tau_X)
\]
holds. (Recall that \( \widetilde{H}(x) = \widetilde{H}(x) \cup \{s_x\} \), see (5.32).)

(ii) The following property
\[
[\min H(x), \max H(x)] \subset \min S(x) - \tau_X, \max S(x) + \tau_X
\]
holds for every \( x \in X \).

**Proof.** Property (i) is immediate from (5.27), (5.32), (5.33). In turn, property (ii) is immediate from (5.33), (5.5) and (5.38). \( \square \)

**Lemma 5.4** Let \( x, y \in X \). Suppose that \( \#S_x < m \) and
\[
[\min \widetilde{H}(x), \max \widetilde{H}(x)] \cap [\min H(y), \max H(y)] \neq \emptyset.
\]
Then \( s_x = s_y \).

**Proof.** Part (iii) of Proposition 3.12 tells us that the point \( s_x \) is a limit point of \( E \). Thus, if \( y = s_x \), part (ii) of Proposition 3.12 implies that \( s_y = y \) so that in this case the lemma holds.

Let us prove the lemma for \( y \neq s_x \). To do so, we assume that \( s_x \neq s_y \).

First, we prove that \( s_x \notin [\min S_y, \max S_y] \). Indeed, suppose that \( s_x \in [\min S_y, \max S_y] \). In this case part (i) of Proposition 3.11 tells us that
\[
s_x \in [\min S_y, \max S_y] \cap E = S_y.
\]
But \( s_x \) is a limit point of \( E \), so that, thanks to part (ii) of Proposition 3.12, \( s_x = s_y \) which contradicts our assumption \( s_x \neq s_y \).

Thus, \( s_x \notin [\min S_y, \max S_y] \) so that \( s_x \neq \min S_y \) and \( s_x \neq \max S_y \). On the other hand, thanks to (5.3), the points \( s_x, \min S_y, \max S_y \) belong to \( S_X \). Therefore, thanks to (5.23),
\[
|s_x - \min S_y| \geq 4\tau_X \quad \text{and} \quad |s_x - \max S_y| \geq 4\tau_X.
\]
Hence,
\[ \text{dist}(s_x, [\min S_y, \max S_y]) \geq 4\tau_X. \]

On the other hand, part (i) and part (ii) of Lemma 5.3 tell us that \( \widehat{H}(x) \subset (s_x - \tau_X, s_x + \tau_X) \) and
\[ [\min H(y), \max H(y)] \subset [\min S_y - \tau_X, \max S_y + \tau_X]. \]

Hence,
\[ [\min \widehat{H}(x), \max \widehat{H}(x)] \cap [\min H(y), \max H(y)] = \emptyset. \]

This contradicts (5.39) proving that the assumption \( s_x \neq s_y \) does not hold. \( \square \)

**STEP 3.** We are in a position to prove properties (\( \bullet 1 \))-(\( \bullet 5 \)) of the Main Lemma 5.1.

- **Proof of property (\( \bullet 1 \)).** This property is equivalent to the following statement:

\[ [\min H(x), \max H(x)] \cap V = H(x) \quad \text{for every} \quad x \in X. \quad (5.40) \]

Let us assume that (5.40) does not hold for certain \( x \in X \), and show that this assumption leads to a contradiction.

Thanks to definition (5.6), if (5.40) does not hold then there exist \( y \in X \) and \( u \in H(y) \) such that
\[ u \in [\min H(x), \max H(x)] \setminus H(x). \quad (5.41) \]

Prove that \( \# S_x < m \). Indeed, otherwise, \( S_x = H(x) \) (see (5.7)). In this case (3.31) implies that
\[ [\min H(x), \max H(x)] \cap E = H(x) \quad \text{so that} \quad [\min H(x), \max H(x)] \setminus H(x) = \emptyset. \]

This contradicts (5.41) proving that \( \# S_x < m \).

Furthermore, property (3.31) tells us that \( [\min S_x, \max S_x] \cap E = S_x \subset H(x) \) so that
\[ u \notin [\min S_x, \max S_x]. \quad (5.42) \]

From this and (5.33), we have
\[ u \in [\min \widehat{H}(x), \max \widehat{H}(x)]. \quad (5.43) \]

We conclude that the hypothesis of Lemma 5.4 holds for \( x \) and \( y \) because \( \# S_x < m \), \( u \in H(y) \) and (5.43) holds. This lemma tells us that \( s_x = s_y \).

Let \( z = s_x \) so that \( x, y \in K(z) \), see (5.10). Clearly, \( K(z) \neq \{z\} \); otherwise \( x = y \) contradicting (5.41).

We know that either \( \min K(z) \geq z \) (i.e., \( z \) satisfies the condition of Case (\( \star 1 \)) of **STEP 1**, see (5.21)), or \( \max K(z) \leq z \) (Case (\( \star 2 \)) of **STEP 1** holds).

Suppose that \( \min K(z) \geq z \), see (5.21). Then \( \min S_x = s_x = z \) so that
\[ [\min S_x, \max S_x] = [z, \max S_x]. \quad (5.44) \]

Moreover, thanks to (5.29) and (5.32),
\[ \widehat{H}(x) = \{a_{\ell_1}, a_{\ell_1+1}, ..., a_{m-1}\} \quad \text{and} \quad \widehat{H}(x) = \{a_{\ell_1}, ..., a_{m-1}, z\}. \quad (5.45) \]

Here \( \ell_x = \# S_x \) and \( a_1, ..., a_{m-1} \) are \( m - 1 \) distinct points of \( E \) satisfying inequality (5.27).

Properties (5.42) and (5.44) tell us that \( u \neq z = s_x \). From this, (5.43) and (5.45), we have \( u \in [a_{\ell_1}, z] \).

On the other hand, \( u \in H(y) \). Because \( y \in K(z) \), definitions (5.5) and (5.29) imply that
\[ H(y) = \{a_{\ell_1}, ..., a_{m-1}\} \cup S_y. \]
Hence, \( u^2 \) property (\( \bullet \) 1) of the Main Lemma is complete.

- **Proof of property (\( \bullet \) 2).** Part (i) of Proposition \( \text{(3.12)} \) tells us that \( x \in S_x \) for every \( x \in E \). In turn, definition \( (5.5) \) implies that \( S_x \subseteq H(x) \) so that \( x \in H(x) \). Hence, \( x \in V \) for each \( x \in X \), see \( (5.6) \), proving that \( X \subseteq V \).

Let us prove that \( 0 < \kappa_i + 1 - \kappa_i \leq 2m \) provided \( x_i = v_{\kappa_i} \) and \( x_i + 1 = v_{\kappa_i + 1} \). The first inequality is obvious because \( x_i < x_{i+1} \) and \( V = \{ v \}^\ell_{j=1} \) is a strictly increasing sequence.

Our proof of the second inequality relies on the following fact:

\[
V \cap [x_i, x_{i+1}] \subseteq H(x_i) \cup H(x_{i+1}).
\] (5.46)

Indeed, let \( v \in V \cap [x_i, x_{i+1}] \). Then definition \( (5.6) \) implies the existence of a point \( \bar{x} \in X \) such that \( H(\bar{x}) \ni v \). Hence, \( v \leq \max H(\bar{x}) \).

Suppose that \( \bar{x} < x_i \). In this case property (\( \bullet \) 3) of the Main Lemma \( (5.1) \) (which we prove below) tells us that \( \max H(\bar{x}) \leq \max H(x_i) \) so that \( x_i \leq v \leq \max H(x_i) \). Hence, \( v \in [\min H(x_i), \max H(x_i)] \) (because \( x_i \in H(x_i) \)). We also know that \( v \in V \). This and property \( (5.40) \) (which is equivalent to property (\( \bullet \) 1) of the Main Lemma) imply that

\[
v \in [\min H(x_i), \max H(x_i)] \cap V = H(x_i).
\]

In the same way we show that \( v \in H(x_{i+1}) \) provided \( \bar{x} > x_{i+1} \), and the proof of \( (5.46) \) is complete.

Because \#\( H(x_i) \) = \#\( H(x_{i+1}) \) = \( m \), property \( (5.46) \) tells us that the interval \( [x_i, x_{i+1}] \) contains at most \( 2m \) points of the set \( V \). This implies the required second inequality \( \kappa_i + 1 - \kappa_i \leq 2m \) completing the proof of part (\( \bullet \) 2) of the Main Lemma.

- **Proof of property (\( \bullet \) 3).** Let \( x', x'' \in X \), \( x' < x'' \). Prove that

\[
\min H(x') \leq \min H(x'').
\] (5.47)

We recall that

\[
H(x') = \overline{H}(x') \cup S_{x'} \quad \text{and} \quad H(x'') = \overline{H}(x'') \cup S_{x''},
\]

see \( (5.5) \). Here \( \overline{H} \) is the set defined by formulae \( (5.28), (5.29), (5.36) \) and \( (5.37) \). Part (iii) of Proposition \( (5.11) \) tells us that \( \min S_{x'} \leq \min S_{x''} \). Since \( s_{x''} \in S_{x''} \), we have

\[
\min S_{x'} \leq \min S_{x''} \leq s_{x''} .
\] (5.48)

We proceed the proof of \( (5.47) \) by cases.

**Case 1.** Assume that \( \min S_{x'} < s_{x''} \).

Because the points \( \min S_{x'} \) and \( s_{x''} \) belong to the set \( S_X \) (see \( (5.3) \)), inequality \( (5.23) \) implies that

\[
s_{x''} - s_{x'} > 4\tau_X .
\] (5.49)

On the other hand, part (i) of Lemma \( (5.3) \) tells us that

\[
\overline{H}(x'') \subseteq \overline{H}(x'') \subseteq (s_{x''} - \tau_X, s_{x''} + \tau_X).
\]

(Recall that \( \overline{H}(x'') = \overline{H}(x'') \cup \{ s_{x''} \} \), see \( (5.32) \).) From this inclusion and \( (5.49) \), we have

\[
\min \overline{H}(x'') > s_{x''} - \tau_X > \min S_{x'} + 3\tau_X > \min S_{x''} .
\]

36
This and (5.48) implies us that
\[
\min H(x'') = \min (\tilde{H}(x')) = \min \{\min \tilde{H}(x'), \min S_{x''}\} \\
\geq \min S_{x''} \geq \min (\tilde{H}(x') \cup S_{x'}) = \min H(x')
\]
proving (5.47) in the case under consideration.

**Case 2.** Suppose that min $S_{x''} = s_{x''}$, and consider two cases.

**Case 2.1:** $\# S_{x''} = m$. Then $H(x') = S_{x''}$, see (5.7), which together with (5.48) implies that
\[
\min H(x') \leq \min S_{x''} \leq \min S_{x'} = \min H(x'')
\]
proving (5.47).

**Case 2.2:** $\# S_{x''} < m$. In this case part (iii) of Proposition 3.12 tells us that $s_{x''}$ is a limit point of $E$. We know that $s_{x''} = \min S_{x''}$ so that the point $\min S_{x''}$ is a limit point of $E$ as well. Hence, $\min S_{x''} = s_{x''}$, see part (ii) of Proposition 3.12.

Thus, $s_{x''} = s_{x''} = \min S_{x''}$. Let $z = s_{x''} = s_{x''}$. We know that $z = \min S_{x''} \leq x' < x''$, so that $x', x'' \in K(z)$, see (5.10). In particular, $K(z) \neq \{z\}$ proving that the point $z$ satisfies the condition of Case ($\star 1$) of STEP 1, see (5.21). In this case inequality (5.29) tells us that $\min H(x') \leq \min H(x'')$ proving (5.47) in Case 2.2.

Thus, (5.47) holds in all cases. Since each of the sets $H(x')$ and $H(x'')$ consists of $m + 1$ consecutive points of the strictly increasing sequence $V = \{v_i\}_{i=1}^{m}$ and $\min H(x') \leq \min H(x'')$, we conclude that $\max H(x') \leq \max H(x'')$.

The proof of property ($\star 3$) is complete.

**Proof of property ($\star 4$).** Let $x', x'' \in X$, $x' \neq x''$, and let $H(x') \neq H(x'')$.

Part (i) of Lemma 5.11 and definition (5.3) tell us that $x' \in S_{x'}$, $x'' \in S_{x''}$ and $S_{x'}, S_{x''} \in S_X$. Hence, $x', x'' \in S_X$ so that, thanks to (5.23), $|x' - x''| \geq 4 \tau_X$. In turn, definition (5.5) implies that
\[
\operatorname{diam} H(x') \leq \operatorname{diam} \tilde{H}(x') + \operatorname{diam} S_{x'} \quad \text{and} \quad \operatorname{diam} H(x'') \leq \operatorname{diam} \tilde{H}(x'') + \operatorname{diam} S_{x''}.
\]
Furthermore, part (i) of Lemma 5.12 tells us that
\[
\max \{\operatorname{diam} \tilde{H}(x'), \operatorname{diam} \tilde{H}(x'')\} \leq 2 \tau_X < |x' - x''|.
\]
Hence,
\[
\operatorname{diam} H(x') \leq \operatorname{diam} S_{x'} + |x' - x'| \quad \text{and} \quad \operatorname{diam} H(x'') \leq \operatorname{diam} S_{x''} + |x' - x'|. \tag{5.50}
\]

Prove that $S_{x'} \neq S_{x''}$. Indeed, suppose that $S_{x'} = S_{x''}$ and prove that this equality contradicts to the assumption that $H(x') \neq H(x'')$.

If $\# S_{x'} = \# S_{x''} = m$ then $S_{x'} = H_{x'}$ and $S_{x''} = H_{x''}$, see (5.7), which implies the required contradiction $H(x') = H(x'')$.

Let now $\# S_{x'} = \# S_{x''} < m$. In this case $s_{x'}$ is the *unique* limit point of $E$ which belongs to $S_{x'}$, see part (ii) of Proposition 3.12. A similar statement is true for $s_{x''}$ and $S_{x''}$. Hence $s_{x'} = s_{x''}$.

Let $z = s_{x'} = s_{x''}$. Thus, $x', x'' \in K(z)$, see (5.10). Since $x' \neq x''$, we have $K(z) \neq \{z\}$, so that either the point $z$ satisfies the condition of Case ($\star 1$) of STEP 1 (see (5.21)) or the condition of Case ($\star 2$) of STEP 1 holds. Furthermore, since $\# S_{x'} = \# S_{x''} < m$, the sets $H(x'), \tilde{H}(x'')$ are determined by the formula (5.29) or (5.36) respectively. In both cases the definitions of the sets $H(x'), \tilde{H}(x'')$ depend
only on the point $z$ (which is the same for $x'$ and $x''$ because $z = s_{x'} = s_{x''}$) and the number of points in the sets $S_{x'}$ and $S_{x''}$ (which of course is also the same because $S_{x'} = S_{x''}$).

Thus, $H(x') = H(x'')$ proving that

\[ H(x') = \overline{H}(x') \cup S_{x'} = \overline{H}(x'') \cup S_{x''} = H(x''), \]

a contradiction. This contradiction proves that $S_{x'} \neq S_{x''}$. In this case part (ii) of Proposition 3.11 tells us that

\[ \text{diam} S_{x'} + \text{diam} S_{x''} \leq 2 m |x' - x''|. \]

Combining this inequality with (5.50), we obtain the required inequality (5.1) proving the property (●4) of the Main Lemma.

- **Proof of property (●5).** In the process of constructing of the sets $H(x)$, $x \in X$, we have noted that in all cases of STEP 1 (Case (★1) (see (5.21)), Case (★2), Case (★3)) inequality (5.30) holds for all $y \in X$ and all $i$, $0 \leq i \leq m - 1$. This inequality coincides with inequality (5.2) proving property (●5) of the Main Lemma.

The proof of Main Lemma 5.1 is complete. $\Box$

### 6. The variational extension criterion: sufficiency.

In this section we prove the sufficiency part of Theorem 1.3. Let $E$ be a closed subset of $\mathbb{R}$ with $\#E \geq m + 1$, and let $f$ be a function on $E$ such that $\lambda = \mathcal{L}_{m,p}(f : E) < \infty$. See (1.4). This enables us to make the following

**Assumption 6.1** For every integer $n \geq m$ and every strictly increasing sequence of points $\{x_0, ..., x_n\}$ in $E$, the following inequality

\[ \sum_{i=0}^{n-m} (x_{i+m} - x_i) |\Delta^m f[x_i, ..., x_{i+m}]|^p \leq \lambda^p \]  

(6.1)

holds.

Our aim is to prove that there exists a function $F \in \mathcal{L}_p^m(\mathbb{R})$ such that $F|_E = f$ and $\|F\|_{\mathcal{L}_p^m(\mathbb{R})} \leq C(m,\lambda)$. Clearly, thanks to (6.1),

\[ \sup_{S \subset E, \#S = m+1} |\Delta^m f[S]| (\text{diam} S)^{\frac{i}{p}} \leq \lambda, \]

proving that inequality (5.28) holds. As we have shown in Section 3, in this case the Whitney field $\mathcal{P}^{(m,E)}[f] = \{P_x \in \mathcal{P}_{m-1} : x \in E\}$ introduced in Definition 3.13 is well defined.

We prove the existence of the function $F$ with the help of Theorem 4.1 which we apply to the field $\mathcal{P}^{(m,E)}[f]$. To enable us to do this, we first have to check that the hypothesis of this theorem holds, i.e., we must show that for every integer $k > 1$ and every strictly increasing sequence $\{x_j\}_{j=1}^k$ in $E$ the following inequality

\[ \sum_{j=1}^{k-1} \sum_{i=0}^{m-1} |P_{x_{j+1}}^{(i)}(x_j) - P_{x_j}^{(i)}(x_j)|^p \leq C(m)^p \lambda^p \]  

(6.2)

holds.
Our proof of inequality (6.2) relies on Main Lemma 5.1. More specifically, we fix $\varepsilon > 0$ and apply this lemma to the set $X = \{x_1, ..., x_k\}$ and the Whitney $(m-1)$-field $P^{(m,E)}[f]$. The Main Lemma 5.1 produces a finite strictly increasing sequence $V = \{v_j\}_{j=1}^{\ell}$ in $E$ and a mapping $H : X \to 2^V$ which to every $x \in X$ assigns $m$ consecutive points of $V$ possessing properties (•1)-(•5) of the Main Lemma.

Using these objects, the sequence $V$ and the mapping $H$, in the next two lemmas we prove the required inequality (6.2).

**Lemma 6.2** Let

$$A^+ = \sum_{j=1}^{k-1} \sum_{i=0}^{m-1} \frac{|L^{(i)}_{H(x_j)}[f](x_j) - L^{(i)}_{H(x_{j+1})}[f](x_j)|^p}{(x_{j+1} - x_j)^{(m-i)p-1}}. \tag{6.3}$$

Then $A^+ \leq C(m)^p \lambda^p$. (We recall that Assumption 5.1 holds for the function $f$.)

**Proof.** Let $I_j$ be the smallest closed interval containing $H(x_j) \cup H(x_{j+1})$, $j = 1, ..., k - 1$. Clearly, $\text{diam} I_j = \text{diam}(H(x_j) \cup H(x_{j+1}))$ and $x_{j+1}, x_j \in I_j$ (because $x_j \in H(x_j)$ and $x_{j+1} \in H(x_{j+1})$, see property (•2) of the Main Lemma 5.1). Furthermore, property (•3) of the Main Lemma 5.1 implies that

$$I_j = [\min H(x_j), \max H(x_{j+1})]. \tag{6.4}$$

Let us compare $\text{diam} I_j$ with $x_{j+1} - x_j$ whenever $H(x_j) \neq H(x_{j+1})$. In this case Properties (•2) and (•4) of the Main Lemma 5.1 tell us that $x_j \in H(x_j)$, $x_{j+1} \in H(x_{j+1})$, and

$$\text{diam} H(x_j) + \text{diam} H(x_{j+1}) \leq 2(m + 1)(x_{j+1} - x_j).$$

Hence,

$$\text{diam} I_j = \text{diam}(H(x_j) \cup H(x_{j+1})) \leq \text{diam} H(x_j) + \text{diam} H(x_{j+1}) + (x_{j+1} - x_j) \leq 2(m + 1)(x_{j+1} - x_j) + (x_{j+1} - x_j) = (2m + 3)(x_{j+1} - x_j).$$

This inequality and definition (6.3) imply that

$$A^+ \leq C(m)^p \sum_{j=1}^{k-1} \sum_{i=0}^{m-1} \frac{\max_{I_j} |L^{(i)}_{H(x_j)}[f] - L^{(i)}_{H(x_{j+1})}[f]|^p}{(\text{diam} I_j)^{(m-i)p-1}}. \tag{6.5}$$

Note that each summand in the right hand side of inequality (6.5) equals zero if $H(x_j) = H(x_{j+1})$. Therefore, in our estimates of $A^+$, without loss of generality, we may assume that

$$H(x_j) \neq H(x_{j+1}) \quad \text{for all} \quad j = 1, ..., k - 1. \tag{6.6}$$

Property (•1) of the Main Lemma 5.1 tells us that for every $j = 1, ..., k$, the set $H(x_j)$ consists of $m$ consecutive points of the strictly increasing sequence

$$V = \{v_1, ..., v_\ell\} \subset E. \tag{6.7}$$

Let $n_j$ be the index of the minimal point of $H(x_j)$ in the sequence $V$. Thus

$$H(x_j) = \{v_{n_j}, ..., v_{n_j+m-1}\}, \quad j = 1, ..., k - 1.$$
In particular, thanks to (6.4),
\[ I_j = [v_{n_j}, v_{n_j+1+m-1}], \quad j = 1, \ldots, k - 1. \]  
(6.8)

Let us apply Lemma [3.7] (with \( k = m - 1 \)) to the sequence \( Y = \{v_i\}_{i=n_j}^{n_j+1+m-1} \), the sets
\[ S_1 = H(x_j) = \{v_{n_j}, \ldots, v_{n_j+1+m-1}\}, \quad S_2 = H(x_{j+1}) = \{v_{n_{j+1}}, \ldots, v_{n_{j+1}+m-1}\}, \]
and the closed interval \( I = I_j = [v_{n_j}, v_{n_j+1+m-1}] \). Also, let \( S_j^{(n)} = \{v_n, \ldots, v_{n+m}\}, \quad n \leq n \leq n_{j+1} - 1 \).

In these settings Lemma [3.7] tells us that
\[
\max_{I_j} |L_{H(x_j)}^{(i)}(f) - L_{H(x_{j+1})}^{(i)}(f)|^p \leq ((m + 1)!)^p (\text{diam} I_j)^{(m-i)p-1} \sum_{n=n_j}^{n_{j+1}} |\Delta^m f[S_j^{(n)}]|^p \text{ diam } S_j^{(n)}.
\]

This inequality and (6.5) imply that
\[
A^+ \leq C(m)^p \sum_{j=1}^{k-1} \sum_{n=n_j}^{n_{j+1}} (v_{n+m} - v_n) |\Delta^m f[v_n, \ldots, v_{n+m}]|^p.
\]

Let us prove that each interval \( I_j \) contains at most 4\( m \) elements of the sequence \( V \), i.e.,
\[ n_{j+1} + m - n_j \leq 4m \quad \text{for every} \quad j = 1, \ldots, k - 1. \]
(6.9)

Indeed, let \( x_j = v_{x_j} \) and \( x_{j+1} = v_{x_{j+1}} \). Property (●2) of Main Lemma [5.1] implies that \( 0 < x_{j+1} - x_j \leq 2m \) and \( x_j \in H(x_j), x_{j+1} \in H(x_{j+1}) \). But \( \#H(x_j) = \#H(x_{j+1}) = m \) so that
\[
n_j \leq x_j \leq n_j + m - 1 \quad \text{and} \quad n_{j+1} \leq x_{j+1} \leq n_{j+1} + m - 1.
\]

These inequalities imply (6.9) because
\[
n_{j+1} + m - n_j \leq n_{j+1} + m - x_j + m - 1 \leq n_{j+1} - (x_{j+1} - 2m) + 2m - 1 \leq 4m.
\]

Property (6.9) enables us to estimate \( A^+ \) as follows: Let \( \mathcal{I} = \{I_j : j = 1, \ldots, k - 1\} \). Property (●3) of Main Lemma [5.1] and (6.6) imply that
\[
\{v_{n_j}\}_{j=1}^{k-1} \quad \text{is a strictly increasing subsequence of the sequence} \ V.
\]  
(6.10)

In turn, from (6.8), (6.9) and (6.10), it follows that every interval \( I_{j_0} \in \mathcal{I} \) has common points with at most 8\( m \) intervals \( I_j \in \mathcal{I} \). This leads us to the following property of the family \( \mathcal{I} \): there exist subfamilies \( \mathcal{I}_\nu \subset \mathcal{I}, \nu = 1, \ldots, \nu \), with \( \nu \leq 8m + 1 \), each consisting of pairwise disjoint intervals, such that \( \mathcal{I} = \cup \{\mathcal{I}_\nu : \nu = 1, \ldots, \nu\} \). The existence of subfamilies \( \{\mathcal{I}_\nu\} \) with these properties is immediate from the next well known statement from graph theory (see, e.g. [31]): Every graph can be colored with one more color than the maximum vertex degree.

This, (6.5) and inequality \( \nu \leq 8m + 1 \) imply that \( A^+ \leq C(m)^p \max_{\nu} A_\nu : \nu = 1, \ldots, \nu \) where
\[
A_\nu = \sum_{j : I_j \in \mathcal{I}_\nu} \sum_{n=n_j}^{n_{j+1}} (v_{n+m} - v_n) |\Delta^m f[v_n, \ldots, v_{n+m}]|^p.
\]

40
Since the intervals of each family \( I_v, \nu = 1, \ldots, \nu, \) are \textit{pairwise disjoint}, the following inequality

\[
A_{\nu} \leq \sum_{n=1}^{\ell-m} (v_{n+m} - v_n) |\Delta^m f [v_n, \ldots, v_{n+m}]|^p
\]

holds. (We recall that \( \ell = \# V, \) see [6.7].)

Applying Assumption 6.1 to the right hand side of this inequality, we prove that \( A_{\nu} \leq \lambda^p \) for every \( \nu = 1, \ldots, \nu. \) Hence, \( \lambda^+ \leq (m)^p \lambda^p \) completing the proof of the lemma. \( \square \)

**Lemma 6.3** Inequality (6.2) holds for every integer \( k > 1 \) and every strictly increasing sequence \( \{x_1, \ldots, x_k\} \subset E. \)

**Proof.** Let us replace the Hermite polynomials \( \{P_j, : j = 1, \ldots, k\} \) in the left hand side of inequality (6.2) with the corresponding Lagrange polynomials \( L_{H(x_j)} \). We will do this with the help of property (5) of the Main Lemma (5.1).

For every \( j = 1, \ldots, k - 1, \) and every \( i = 0, \ldots, m - 1, \) we have

\[
|P_{x_j}(x_i) - L^{(i)}_{H(x_j)} f(x_i)| \leq |P_{x_j}(x_i) - L^{(i)}_{H(x_j)} f(x_i)| + |L^{(i)}_{H(x_j)} f(x_i) - L^{(i)}_{H(x_{j+1})} f(x_i)|
\]

\[
+ |L^{(i)}_{H(x_{j+1})} f(x_i) - P_{x_{j+1}}(x_i)|.
\]

Property (5) of the Main Lemma (see (5.2)) tells us that

\[
|P_{x_j}(x_i) - L^{(i)}_{H(x_j)} f(x_i)| \leq 2\varepsilon
\]

proving that

\[
|P_{x_j}(x_i) - P_{x_{j+1}}(x_i)| \leq |L^{(i)}_{H(x_j)} f(x_i) - L^{(i)}_{H(x_{j+1})} f(x_i)| + 2\varepsilon.
\]

Hence,

\[
|P_{x_j}(x_i) - P_{x_{j+1}}(x_i)|^p \leq 2^p |L^{(i)}_{H(x_j)} f(x_i) - L^{(i)}_{H(x_{j+1})} f(x_i)|^p + 4^p \varepsilon^p. \tag{6.11}
\]

Let \( A_1 \) be the left hand side of inequality (6.2), i.e.,

\[
A_1 = \sum_{j=1}^{k-1} \sum_{i=0}^{m-1} \frac{|P_{x_{j+1}}(x_i) - P_{x_j}(x_i)|^p}{(x_{j+1} - x_j)^{(m-i)p-1}}, \tag{6.12}
\]

and let

\[
A_2 = 4^p \sum_{j=1}^{k-1} \sum_{i=0}^{m-1} (x_{j+1} - x_j)^{1-(m-i)p}.
\]

Applying inequality (6.11) to each summand in the right hand side of (6.12), we obtain the following inequality: \( A_1 \leq 2^p A^+ + \varepsilon^p A_2. \) Recall that \( A^+ \) is the quantity defined by (6.3). Lemma 6.2 tells us that \( A^+ \leq C(m)^p \lambda^p. \) Hence \( A_1 \leq C(m)^p \lambda^p + \varepsilon^p A_2. \) But \( \varepsilon \) is an arbitrary positive constant, so that \( A_1 \leq C(m)^p \lambda^p \) proving (6.2) and the lemma. \( \square \)

We are in a position to complete the proof of Theorem 1.3.

**Proof of the sufficiency part of Theorem 1.3** Definition (4.2) and Lemma 6.3 imply that

\[
N_{m,p,E} \left( \mathcal{P}_{[m,E]} f \right) \leq C(m) \lambda = C(m) \mathcal{L}_{m,p}(f : E).
\]
See (1.4). (We recall that \( \lambda = \mathcal{L}_{m,p}(f : E) \).) This inequality and the sufficiency part of Theorem 4.1 imply that

\[
\| P^{(m,E)}[f] \|_{m,p,E} \leq C(m) N_{m,p,E} \left( P^{(m,E)}[f] \right) \leq C(m) \mathcal{L}_{m,p}(f : E).
\]

We recall that the quantity \( \| \cdot \|_{m,p,E} \) is defined by (5.1). This definition and the above inequality imply the existence of a function \( F \in L^m_p(\mathbb{R}) \) such that \( T_{x}^{m-1}[F] = P_{x} \) on \( E \), and

\[
\| F \|_{L^m_p(\mathbb{R})} \leq 2 \| P^{(m,E)}[f] \|_{m,p,E} \leq C(m) \mathcal{L}_{m,p}(f : E).
\]  

(6.13)

We know that \( P_{x}(x) = f(x) \) for every \( x \in E \). (See (3.34)-(3.37).) Therefore,

\[
F(x) = T_{x}^{m-1}[F](x) = P_{x}(x) = f(x) \quad \text{for all} \quad x \in E.
\]

Thus, \( F \in L^m_p(\mathbb{R}) \) and \( F|_E = f \) proving that \( f \in L^m_p(\mathbb{R})|_E \). Furthermore, definition (1.1) and inequality (6.13) imply that

\[
\| f \|_{L^m_p(\mathbb{R})|_E} \leq \| F \|_{L^m_p(\mathbb{R})} \leq C(m) \mathcal{L}_{m,p}(f : E)
\]

proving the sufficiency. \( \square \)

The proof of Theorem 1.3 is complete. \( \square \)

**Remark 6.4** Given a function \( f \) on \( E \), let us indicate the main steps of our extension algorithm suggested in Sections 3 and 4:

**Step 1.** We construct the family of sets \( \{ S_x : x \in E \} \) and the family of points \( \{ s_x : x \in E \} \) satisfying conditions of Proposition 3.11 and Proposition 3.12.

**Step 2.** At this step we construct the Whitney \( (m-1) \)-field \( P^{(m,E)}[f] = \{ P_x \in \mathcal{P}_{m-1} : x \in E \} \) satisfying conditions (i), (ii) of Definition 3.13.

**Step 3.** We define the extension \( F \) by formula (4.6).

We denote the extension \( F \) by \( F = \text{Ext}_E(f : L^m_p(\mathbb{R})) \). Clearly, \( F \) depends on \( f \) linearly proving that \( \text{Ext}_E(\cdot : L^m_p(\mathbb{R})) \) is a linear extension operator. Theorem 1.3 states that its operator norm is bounded by a constant depending only on \( m \). \( \triangledown \)

**Remark 6.5** In [48] Section 4.4] we give an alternative proof of Theorem 1.2 based on the extension algorithm described in Section 3 and Section 4 of the present paper.

We note that, for the case of sequences, this extension method can be simplified considerably. Indeed, we prove in [48] Remark 3.13] that for each \( x \in E \), the set \( S_x \) consist of \( m \) consecutive terms of the sequence \( E \). In this case part (ii) of Definition 3.13 tells us that \( P_x \) coincides with the Lagrange polynomial \( L_{s_i} \) interpolating \( f \) on \( S_x \).

In turn, this property immediately implies a variant of the Main Lemma for sequences, see [48] Lemma 4.13], where we set \( H(x) = S_x \), for every \( x \in E \). The required properties of the sets \( \{ H(x) : x \in E \} \) for this case are immediate from Proposition 3.11.

We also note that if \( E = \{ x_i \}_{i=\ell}^{\ell_2} \) is a strictly increasing sequence of points in \( \mathbb{R} \), and \( f \) is a function on \( E \), the extension \( F = \text{Ext}_E(f : L^m_p(\mathbb{R})) \) is a piecewise polynomial \( C^{m-1} \)-function which coincides with a polynomial of degree at most \( 2m - 1 \) on each interval \( (x_i, x_{i+1}) \). This enables us to reformulate this property of \( F \) in terms of Spline Theory as follows: **The extension \( F \) is an interpolating \( C^{m-1} \)-smooth spline of order \( 2m \) with knots \( \{ x_i \}_{i=\ell}^{\ell_2} \).**

Details are spelled out in [48]. \( \triangledown \)
7. The Finiteness Principle for $L^m_\infty(\mathbb{R})$ traces: multiplicative finiteness constants.

Let $m \in \mathbb{N}$. Everywhere in this section we assume that $E$ is a closed subset of $\mathbb{R}$ with $\#E \geq m + 1$. We will discuss equivalence (1.3) which states that

$$\|f\|_{L^m_\infty(\mathbb{R}))} \sim \sup_{S \subseteq E, \#S = m + 1} |\Delta^m f[S]|$$

(7.1)

for every function $f \in L^m_\infty(\mathbb{R}))|_E$. The constants in this equivalence depend only on $m$.

We can interpret this equivalence as a special case of the following Finiteness Principle for the space $L^m_\infty(\mathbb{R})$.

**Theorem 7.1** Let $m \in \mathbb{N}$. There exists a constant $\gamma = \gamma(m) > 0$ depending only on $m$, such that the following holds: Let $E \subseteq \mathbb{R}$ be a closed set, and let $f : E \to \mathbb{R}$.

For every subset $E' \subseteq E$ with at most $N = m + 1$ points, suppose there exists a function $F_{E'} \in L^m_\infty(\mathbb{R})$ with the seminorm $\|F_{E'}\|_{L^m_\infty(\mathbb{R})} \leq 1$, such that $F_{E'} = f$ on $E'$.

Then there exists a function $F \in L^m_\infty(\mathbb{R})$ with the seminorm $\|F\|_{L^m_\infty(\mathbb{R})} \leq \gamma$ such that $F = f$ on $E$.

**Proof.** The result is immediate from equivalence (7.1) and definition (1.1). Details are spelled out in [48, Section 5].

We refer to the number $N = m + 1$ as a finiteness number for the space $L^m_\infty(\mathbb{R})$. Clearly, the value $N(m) = m + 1$ in the finiteness Theorem 7.1 is sharp; in other words, Theorem 7.1 is false in general if $N = m + 1$ is replaced by some number $N < m + 1$.

We note that the Finiteness Principle also holds for the space $L^m_\infty(\mathbb{R}^n)$ for all $m, n \in \mathbb{N}$; in this case a corresponding number $N$ and a constant $\gamma$ depend only on $n$ and $m$. See [43] for the case $m = 2$, $n \in \mathbb{N}$, and [18] for the general case of $m, n \in \mathbb{N}$. It is also shown in [18] that a similar finiteness principle holds for the space $W^m_\infty(\mathbb{R}^n)$.

Theorem 7.1 implies the following: For every function $f \in L^m_\infty(\mathbb{R})|_E$, we have

$$\|f\|_{L^m_\infty(\mathbb{R}))} \leq \gamma(m) \sup_{S \subseteq E, \#S = m + 1} \|f|_S\|_{L^m_\infty(\mathbb{R}))} .$$

(7.2)

(Clearly, the converse inequality is trivial and holds with $\gamma(m) = 1$.) We refer to any constant $\gamma = \gamma(m)$ which satisfies (7.2) as a multiplicative finiteness constant for the space $L^m_\infty(\mathbb{R})$.

The following natural question arises:

**Question 7.2** What is the sharp value of the multiplicative finiteness constant for $L^m_\infty(\mathbb{R})$, i.e., the infimum of all multiplicative finiteness constants for $L^m_\infty(\mathbb{R})$ for the finiteness number $N = m + 1$?

We denote this sharp value of $\gamma(m)$ by $\gamma^\#$($L^m_\infty(\mathbb{R}))$). Thus,

$$\gamma^\#(L^m_\infty(\mathbb{R})) = \sup \frac{\|f\|_{L^m_\infty(\mathbb{R}))}}{\sup\{|f|_S\|_{L^m_\infty(\mathbb{R}))} : S \subseteq E, \#S = m + 1\} .$$

(7.3)

where the supremum is taken over all closed sets $E \subseteq \mathbb{R}$ with $\#E \geq m + 1$, and all $f \in L^m_\infty(\mathbb{R})|_E$.

The next theorem answers to Question 7.2 for $m = 1, 2$ and provides lower and upper bounds for $\gamma^\#(L^m_\infty(\mathbb{R}))$ for $m > 2$. These estimates show that $\gamma^\#(L^m_\infty(\mathbb{R}))$ grows exponentially as $m \to +\infty$.

**Theorem 7.3** (i) $\gamma^\#(L^1_\infty(\mathbb{R})) = 1$ and $\gamma^\#(L^2_\infty(\mathbb{R})) = 2$;

(ii) For every $m \in \mathbb{N}, m > 2$, the following inequalities

$$\left(\frac{\pi}{2}\right)^{m-1} < \gamma^\#(L^m_\infty(\mathbb{R})) < (m - 1) 9^m$$

hold.
The proof of this theorem relies on works \([11, 12, 17]\) devoted to calculation of a certain constant \(K(m)\) related to optimal extensions of \(L^m_{\infty}(\mathbb{R})\)-functions. This constant is defined by

\[
K(m) = \sup \frac{\|f\|_{L^m_{\infty}(\mathbb{R})}}{\max\{m! |\Delta^m f[x_i, ..., x_{i+m}]| : i = 1, ..., n\}} \tag{7.4}
\]

where the supremum is taken over all \(n \in \mathbb{N}\), all strictly increasing sequences \(X = \{x_1, ..., x_{m+n}\} \subset \mathbb{R}\) and all functions \(f \in L^m_{\infty}(\mathbb{R})\).

The constant \(K(m)\) was introduced by Favard \([17]\). (See also \([11, 12]\).) Favard \([17]\) proved that \(K(2) = 2\), and de Boor found efficient lower and upper bounds for \(K(m)\).

We prove that \(\gamma^d(L^m_{\infty}(\mathbb{R})) = K(m)\), see Lemma \([7.7]\) below. This formula and aforementioned results of Favard and de Boor imply the required lower and upper bounds for \(\gamma^d(L^m_{\infty}(\mathbb{R}))\) in Theorem \([7.3]\).

We will need a series of auxiliary lemmas.

**Lemma 7.4** Let \(S \subset \mathbb{R}, \#S = m + 1\), and let \(f : S \to \mathbb{R}\). Then \(\|f\|_{L^m_{\infty}(\mathbb{R})} = m! |\Delta^m f[S]|.\)

*Proof.* Let \(A = \min S, B = \max S\). Let \(F \in L^m_{\infty}(\mathbb{R})\) be an arbitrary function such that \(F|_S = f\).

Inequality \((2.6)\) tells us that \(m! |\Delta^m f[S]| = m! |\Delta^m F[S]| \leq \|F\|_{L^m_{\infty}(\mathbb{R})}\).

Taking the infimum in this inequality over all functions \(F \in L^m_{\infty}(\mathbb{R})\) such that \(F|_S = f\), we obtain the inequality \(m! |\Delta^m f[S]| \leq \|f\|_{L^m_{\infty}(\mathbb{R})}\).

Let us prove the converse inequality. Let \(F = L_S[f]\). Then

\[
m! |\Delta^m f[S]| = |L^{(m)}_S[f]| = \|F\|_{L^m_{\infty}(\mathbb{R})} \quad (\text{see } (2.3)).
\]

But \(F|_S = L_S[f]|_S = f\), so that \(\|f\|_{L^m_{\infty}(\mathbb{R})} \leq \|F\|_{L^m_{\infty}(\mathbb{R})} = m! |\Delta^m f[S]|\) proving the lemma. \(\square\)

**Lemma 7.5** Let \(E = \{x_1, ..., x_{m+n}\} \subset \mathbb{R}, n \in \mathbb{N}\), be a strictly increasing sequence, and let \(f\) be a function on \(E\). Then

\[
\max_{S \subset E, \#S = m+1} |\Delta^m f[S]| = \max_{i=1, ..., n} |\Delta^m f[x_i, ..., x_{i+m}]|.
\]

*Proof.* The lemma is immediate from the following property of divided differences (see \([26\), p. 15]): Let \(S \subset E, \#S = m+1\). There exist \(\alpha_1 \geq 0, i = 1, ..., n\) with \(\alpha_1 + ... + \alpha_n = 1\) such that

\[
\Delta^m f[S] = \sum_{i=1}^{n} \alpha_i \Delta^m f[x_i, ..., x_{i+m}] \quad \square
\]

**Lemma 7.6** Let \(E \subset \mathbb{R}\) be a closed set, and let \(f \in L^m_{\infty}(\mathbb{R})|_E\). Then

\[
\|f\|_{L^m_{\infty}(\mathbb{R})|_E} = \sup_{E' \subset E, \#E' < \infty} \|f|_{E'}\|_{L^m_{\infty}(\mathbb{R})|_{E'}}.
\]

*Proof.* We recall the following well known fact: for every closed bounded interval \(I \subset \mathbb{R}\) a ball in the space \(L^m_{\infty}(I)\) is a precompact subset in the space \(C(I)\). The lemma readily follows from this statement. We leave the details to the interested reader. \(\square\)

**Lemma 7.7** For every \(m \in \mathbb{N}\) the following equality \(\gamma^d(L^m_{\infty}(\mathbb{R})) = K(m)\) holds.
Proof. The inequality $K(m) \leq \gamma^k(L^m_\infty(\mathbb{R}))$ is immediate from Lemma 7.4, Lemma 7.5 and definition (7.4). The converse inequality directly follows from Lemma 7.6 and definition (7.3). Details are spelled out in [48, Section 6]. □

Proof of Theorem 7.3 The equality $\gamma^k(L^1_\infty(\mathbb{R})) = 1$ is immediate from the well known fact that a function satisfying a Lipschitz condition on a subset of $\mathbb{R}$ can be extended to all of $\mathbb{R}$ with preservation of the Lipschitz constant.

As we have mentioned above, $K(2) = 2$ (Favard [17]). de Boor [11][12] proved that

$$
\left(\frac{\pi}{2}\right)^{m-1} < \theta_m \leq K(m) \leq \Theta_m < (m - 1) 9^m \quad \text{for each } m > 2.
$$

(7.5)

Here

$$
\theta_m = \left(\frac{\pi}{2}\right)^{m+1} \left\{ \sum_{j=-\infty}^{\infty} \left(\frac{(-1)^j}{(2j+1)}\right)^{m+1} \right\} \quad \text{and} \quad \Theta_m = \left(\frac{2^{m-2}}{m}\right) + \sum_{i=1}^{m} \binom{m}{i} \left(\frac{m - 1}{i - 1}\right) 4^{m-i}.
$$

On the other hand, Lemma 7.7 tells us that $\gamma^k(L^m_\infty(\mathbb{R})) = K(m)$ which together with the equality $K(2) = 2$ and inequalities (7.5) implies statements (i) and (ii) of the theorem.

The proof of Theorem 7.3 is complete. □

We finish the section with two remarks.

Remark 7.8 One can readily see that $\theta_1 = 1$ and $\theta_2 = 2$. This together with part (i) of Theorem 7.3 implies that $\theta_m = K(m)$ for $m = 1, 2$.

In turn, thanks to (7.5), $\theta_m \leq K(m)$ for $m > 2$. de Boor [11] proved this inequality by showing that for a function $f(i) = (-1)^i$ defined on $\mathbb{Z}$, the following two properties hold:

(i) $m! |\Delta^m f[i, \ldots, i + m]| = 2^m$ for every $i \in \mathbb{Z}$; (ii) $\|f\|_{L_\infty^m(\mathbb{R})|_{\mathbb{Z}}} = \theta_m 2^m$.

Furthermore, it is shown that the Euler spline $E_m : \mathbb{R} \to \mathbb{R}$ (a classical object of Spline Theory introduced by Schoenberg [41]) provides an optimal (in $L^m_\infty(\mathbb{R})$) extension of $f$ from $\mathbb{Z}$ to all of $\mathbb{R}$.

We refer the reader to [11, Section 3], [41, Lecture 4], [48, Section 6.2] for more detail. □

Remark 7.9 In fact we know very little about the values of the constants $\gamma^k(\cdot)$ and its analogs for the spaces $L^m_\infty(\mathbb{R}^n)$ and $W^m_\infty(\mathbb{R}^n)$. Apart from the results related to $\gamma^k(L^m_\infty(\mathbb{R}))$ which we present in this section, there is perhaps only one other result in this direction. It is due to Fefferman and Klartag [24]. They studied the behavior of the sharp multiplicative finiteness constant for the space $W^2_\infty(\mathbb{R}^2)$ defined by

$$
\gamma^k(N; W^2_\infty(\mathbb{R}^2)) = \sup \frac{\|f\|_{W^2_\infty(\mathbb{R}^2)|_E}}{\sup\{\|f|_S\|_{W^2_\infty(\mathbb{R}^2)|_{S'}} : S \subset E, \#S \leq N\}}.
$$

It is proven in [24] that, after a certain renormalization of the space $W^2_\infty(\mathbb{R}^2)$, the following statement holds: \textit{there exists an absolute constant } $c > 0$ \textit{such that } $\gamma^k(N; W^2_\infty(\mathbb{R}^2)) > 1 + c$ \textit{for every positive integer } $N$. □

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45
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