Quasi-permutable normal operators in octonion Hilbert spaces and spectra

Ludkovsky S.V.

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Abstract

Families of quasi-permutable normal operators in octonion Hilbert spaces are investigated. Their spectra are studied. Multiparameter semigroups of such operators are considered. A non-associative analog of Stone’s theorem is proved.

1 Introduction

The theory of bounded and unbounded normal operators over the complex field is classical and have found many-sided applications in functional analysis, differential and partial differential equations and their applications in the sciences [4, 11, 12, 14, 32]. Nevertheless, hypercomplex analysis is fast developing, because it is closely related with problems of theoretical and mathematical physics and of partial differential equations [2, 7, 9]. On the other hand, the octonion algebra is the largest division real algebra in which the complex field has non-central embeddings [3, 11, 13]. The octonion algebra also is intensively used in mathematics and various applications [5, 10, 8, 15, 16].

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Previously analysis over quaternion and octonions was developed and spectral theory of bounded normal operators and unbounded self-adjoint operators was described \[18, 19, 20, 21, 22\]. Their applications in partial differential equations were outlined \[23, 24, 25, 26, 27\]. This paper is devoted to families of quasi-permutable normal operators in octonion Hilbert spaces. Their spectra are studied. Multiparameter semigroups of such operators are considered. A non-associative analog of Stone’s theorem is proved.

Notations and definitions of papers \[18, 19, 20, 21, 22\] are used below. The main results of this article are obtained for the first time.

2 Quasi-permutability of normal operators

1. Definitions. If \(jA\) is a set of \(R\) homogeneous \(\mathcal{A}_v\) additive operators with \(\mathcal{A}_v\) vector domains \(D(\mathcal{A})\) dense in a Hilbert space \(X\) over the Cayley-Dickson algebra \(\mathcal{A}_v\), \(2 \leq v, \quad j \in \Lambda, \quad \Lambda\) is a set, then we denote by \(al_{\mathcal{A}_v}(jA : \ j \in \Lambda)\) a family of all operators \(B\) with \(\mathcal{A}_v\) vector domains in \(X\) obtained from \((jA : \ j \in \Lambda)\) by a finite number of operator addition, operator multiplication and left and right multiplication of operators on Cayley-Dickson numbers \(b \in \mathcal{A}_v\) or on \(bI\), where \(I\) denotes the unit operator on \(X\).

Let \(\mathcal{A}_v\) be two normal operators in a Hilbert space \(X\) over the Cayley-Dickson algebra \(\mathcal{A}_v\), \(2 \leq v\). Suppose that \(\mathcal{A}_v\) and \(\mathcal{A}_v\) are affiliated with a quasi-commutative von Neumann algebra \(\mathcal{A}\) over \(\mathcal{A}_v\) with \(2 \leq v \leq 3\). Let \(\mathcal{E}_v\) and \(\mathcal{E}_v\) be their \(\mathcal{A}_v\) graded projection valued measures defined on the Borel \(\sigma\)-algebra of subsets in \(\mathcal{A}_v\) (see also \(\S\)2 and \(\S\)I.2.58 and I.2.73 in \[28\]).

In this section the simplified notation \(E\) instead of \(\hat{E}\) will be used.

We shall say that two normal operators \(\mathcal{A}_v\) and \(\mathcal{A}_v\) quasi-permute if

\[
1E(\delta_1) 2E(\delta_2) = 2E(\delta_2)1E(\delta_1)
\]

for each Borel subsets \(\delta_1\) and \(\delta_2\) in \(\mathcal{A}_v\).

Operators \(\mathcal{A}_v\), \(\mathcal{A}_v\) and \(\mathcal{A}_v\) are said to have property \(P\) if they satisfy the following four conditions \((P1 - P4)\):

\((P1)\) they are normal,
(P2) they are affiliated with a von Neumann algebra $\mathcal{A}$ over either the quaternion skew field or the octonion algebra $\mathcal{A}_v$ with $2 \leq v \leq 3$ and

\[ (P3) \quad A = 1 A 2 A \]  

(P4) the family $\text{alg}_{\mathcal{A}_v}(I, A, A^*, 1 A, 1 A^*, 2 A, 2 A^*) =: Q(A, 1 A, 2 A) =: Q$ over $\mathcal{A}_v$ generated by these three operators is quasi-commutative, that is a von Neumann algebra $\text{cl}_{\text{alg}}\mathcal{A}_v(I, A, A^*, 1 A, 1 A^*, 2 A, 2 A^*) \subset L_q(X)$ contained in $L_q(X)$ is quasi-commutative for each bounded Borel subsets $\delta, \delta_1, \delta_2 \in \mathcal{B}(\mathcal{A}_v)$, where $2 \leq v \leq 3$.

It is possible to consider a common domain $\mathcal{D}^\infty(Q) := \cap_{T \in Q} \mathcal{D}^\infty(T)$ for a family of operators $Q$, where $\mathcal{D}^\infty(T) := \cap_{n=1}^\infty \mathcal{D}(T^n)$. Then the family $Q$ on $\mathcal{D}^\infty(Q)$ can be considered as an $\mathcal{A}_v$ vector space. Take the decomposition $Q = Q_0 i_0 \oplus Q_1 i_1 \oplus \ldots \oplus Q_{2^v-1} i_{2^v-1}$ of this $\mathcal{A}_v$ vector space with pairwise isomorphic real vector spaces $Q_0, Q_1, ..., Q_{2^v-1}$. Then as in §2.5 [29] for each operator $B \in Q$ we put

\[ (2) \quad B = \sum_j j B \text{ with } j B = \hat{\pi}^j(B) \in Q_j i_j \]

for each $j$, where $\hat{\pi}^j : Q \rightarrow Q_j i_j$ is the natural $\mathbb{R}$ linear projection, real linear spaces $Q_j i_j$ and $i_j Q_j$ are considered as isomorphic, so that

\[ (3) \quad \sum_{k=0}^{2^v-1} k \hat{T} = T. \]

If $E$ is an $\mathcal{A}_v$ graded projection valued measure on the Borel $\sigma$-algebra $\mathcal{B}(\mathcal{A}_v)$ for a normal operator $T \in Q$, for uniformity of this notation we put also

\[ (4) \quad k \hat{E}(dz).ty = \hat{\pi}^k E(dz).ty \]

for every vector $y \in X$ and $t = t_0 i_0 + \ldots + t_{2^v-1} i_{2^v-1} \in \mathcal{A}_v$, where $z \in \mathcal{A}_v$, $t_0, ..., t_{2^v-1} \in \mathbb{R}$, $E(dz).ty = E(dz).(ty)$.

2. **Lemma.** Let operators $A$, $B$ and $D$ have property $P$ and let $F$ be an $\mathcal{A}_v$ graded projection operator which quasi-permutes with $A$ so that $\mathcal{R}(F) \subset \mathcal{D}(A)$, where $\mathcal{D}(A) = \text{Domain}(A), \mathcal{R}(A) = \text{Range}(A)$. Suppose that $G$, $H$ and $J$ are the restrictions of $A$, $FB$ and $FD$ to $\mathcal{R}(F)$ respectively.
Then \( G, \ H \) and \( J \) are bounded operators so that \( H \) and \( J \) quasi-permute with \( G \). Moreover, \( H^* \) and \( J^* \) are the restrictions to \( \mathcal{R}(F) \) of \( B^*F^* \) and \( C^*F^* \) respectively, where

\[
(1) \ j^k(\hat{B}^*) = (-1)^{\kappa(j,k)+\eta(k)} k \hat{F}^j(\hat{B}^*) \quad \text{and} \\
(2) \ j^k(\hat{D}^*) = (-1)^{\kappa(j,k)+\eta(k)} k \hat{F}^j(\hat{D}^*)
\]

for each \( j, k \), with \( \kappa(j, k) = 0 \) for \( j = k \) or \( j = 0 \) or \( k = 0 \), \( \kappa(j, k) = 1 \) for \( j \neq k \geq 1 \), \( \eta(0) = 0 \), \( \eta(k) = 1 \) for each \( k \geq 1 \).

**Proof.** Note 2.5 and Theorems 2.29, 2.44 and Proposition 2.32 in [29] and Definitions 1 imply that in components the following formulas are satisfied:

\[
(3) \ \frac{j}{4} \hat{E}(\delta_1) = \frac{k}{4} \hat{E}(\delta_2) = (-1)^{\kappa(j,k)} k \hat{E}(\delta_1) = \frac{j}{4} \hat{E}(\delta_2) = \frac{k}{4} \hat{E}(\delta_1)
\]

for each \( j, k = 0, 1, 2, \ldots \), where \( \kappa(j, k) = 0 \) for \( j = k \) or \( j = 0 \) or \( k = 0 \), \( \kappa(j, k) = 1 \) for \( j \neq k \geq 1 \),

where \( \theta_\ell(x_j) \) is denoted by \( x_j \) for short, \( \theta_\ell^k : X_j \to X_k \) is an \( R \)-linear topological isomorphism of real normed spaces (see §§I.2.1 and I.2.73 in [28]). Suppose that \( x, y \in \mathcal{R}(F) \), hence \( x, y \in \mathcal{D}(B) \subset \mathcal{D}(B^*) \), since \( \mathcal{R}(F) \subset \mathcal{D}(A) \subset \mathcal{D}(B) \).

Therefore

\[
(4) \ < FBx; y > = < Bx; y > = < x; B^*y > = < x; F^*Bx > \quad \text{and} \\
(5) \ < j^k \hat{B}x_j; y_j > = < k \hat{B}x_j; y_j > = (-1)^{\kappa(j,k)+\eta(k)} < x_j; j^k \hat{F}^k(\hat{B}^*)y_j >
\]

If \( L = F^*B \big|_{\mathcal{R}(F)} \), then \( H^* = L \) and \( H = L^* \) by Formula (4). The operator \( L^* \) is closed, consequently, \( H \) is closed and \( \mathcal{D}(H) \subset \mathcal{R}(F) \). In view of the closed graph theorem for \( R \)-linear operators the operator \( H \) is bounded 1.8.6 [12]. This implies that the operator \( G \) is also bounded, since the operator \( A \) is normal and hence closed so that \( \mathcal{R}(F) \subset \mathcal{D}(A) \). In view of Theorems 2.27, 2.29 an 2.44 in [29] the operator \( A \) has an \( A_v \) graded projection valued measure. Take now \( x \in \mathcal{R}(F) \), hence \( Ax \in \mathcal{R}(F) \subset \mathcal{D}(A) \subset \mathcal{D}(B) \), since

\[
\hat{F}^k \hat{F}^j = (-1)^{\kappa(j,k)} k \hat{F}^j \hat{F}^k \quad \text{and} \quad \mathcal{D}(F) = \mathcal{D}(F)_0 \oplus \ldots \oplus \mathcal{D}(F)_{m_i} \oplus \ldots
\]

for each \( j, k \) and

\[
A = \int_{A_v} F(dt).t
\]

so that \( j^k \hat{A} \subset (-1)^{\kappa(j,k)} k \hat{A} j^k \hat{F} \) for each \( j, k \). Symmetric proof is for \( A \) and \( C \) instead of \( A \) and \( B \). The operators \( B^*B \) and \( C^*C \) belong to the family \( alg_{A_v}(J, A, A^*, B, B^*, C, C^*) \).
In view of Theorem I.3.23 [28] the spectra of \( B^*B = \int_{-\infty}^{\infty} B^*B F(dt).t^2 \) and \( D^*D = \int_{-\infty}^{\infty} D^*D F(dt).t^2 \) are real so that \( B^*B F \) and \( D^*D F \) are \( \mathcal{A}_v \) graded projection valued measures for \( B^*B \) and \( D^*D \) respectively on \( \mathcal{B}(\mathbb{R}) \subset \mathcal{B}(\mathcal{A}_v) \). Then from Formulas (2, 4) and 1(1, P1 – P4) we deduce that
\[
(7) \quad (j \hat{F}^j k \hat{B}) s A_{x_a} = (j \hat{F}^j k \hat{B}) \sum_{p,q} \{p \hat{D}^q \hat{A} + (-1)^{\kappa(p,q)} q \hat{D}^p \hat{B}] = \sum_{p,q} \sum_{i,j} \{j \hat{F}^j k \hat{B}^{p \hat{D}^q \hat{I}} + (-1)^{\kappa(p,q)}(j \hat{F}^j k \hat{B})\{q \hat{D}^p \hat{B]}} = (-1)^{\kappa(s,l)} s \hat{A}(j \hat{F}^j k \hat{B} x_n),
\]
since the set theoretic composition of operators is associative: \((FB)(DB) = F((BD)B), \) where \( l \) is such that \( i_j k \in \mathbb{R}_l \). Thus \( H \) and analogously \( J \) quasi-permute with \( G \), since the family \( \text{alg} \gamma. \text{alg}(I, A, A^*, 1A, 1A^*, 2A, 2A) \) is quasi-commutative. From Formulas (5, 6) we infer Equalities (1, 2).

3. Notation. Suppose that \( a, b \in \mathcal{A}_v \). If \( b_j \geq a_j \) for each \( j = 0, 1, 2, ..., 2^r - 1 \), this fact will be denoted by \( b \succeq a \). Then \( \mathcal{I}_{a,b} := \{ z \in \mathcal{A}_v : b \succeq z \succeq a \} \).

4.Lemma. Let operators \( A, B \) and \( D \) have property \( P \) and let \( F \) be an \( \mathcal{A}_v \) graded projection valued measure for \( A \), let also \( b \succeq a \in \mathcal{A}_v \). Then \( \mathcal{R}(F(\mathcal{I}_{a,b})) =: Y \) reduces both \( B \) and \( D \) and these operators restricted to \( Y \) are bounded and normal and they quasi-permute with the restriction \( |A|_Y \).

Proof. Consider the pair of operators \( A \) and \( B \). Put \( nF := F|_{\mathcal{I}_{b(n),b(n)}} \) and \( nV = \mathcal{R}(F(\mathcal{I}_{b(n),b(n)})) \) with \( b(n)_j = ni_j \) for every \( n \in \mathbb{N} \) and each \( j = 0, 1, 2, ..., 2^r - 1 \). Then \( nV \subset n_{+1}V \) for each \( n \). Therefore, an \( \mathcal{A}_v \) vector subspace \( \bigcup_{n} nV =: V \) is dense in the Hilbert space \( X \) over the Cayley-Dickson algebra \( \mathcal{A}_v \), consequently, \( \lim_{n} nF = I \) in the strong operator topology. Each operator \( nA := A|_{nV} \) is bounded and normal and has the \( \mathcal{A}_v \) graded projection valued measure on the Borel \( \sigma \)-algebra \( \mathcal{B}(\mathcal{A}_v) \) of all Borel subsets in \( \mathcal{A}_v \) so that \( nF = F|_{nV} \) for each natural number \( n \). We consider the restriction \( nG := nFB|_{nV} \). It is known from Lemma 2, that each operator \( nG \) is bounded and quasi-permutates with \( nB \) so that
\[
(1) \quad \quad c^n = \hat{F}^j_k \hat{B}^j_k \hat{F} = (-1)^{\kappa(j,k)} \hat{F}^j_k \hat{B}^j_k \hat{F}
\]
for each \( j, k \), consequently,
\[
(2) \quad \quad \hat{F}(\delta) \hat{F}(\delta_1) \hat{F}(\delta_2) = (-1)^{\kappa(j,k)} \hat{F}(\delta) \hat{F}(\delta_2) \hat{F}(\delta_1)
\]
for each \( j, k \).
for each \( x \in nV_0 \) and \( \delta, \delta_1, \delta_2 \in \mathcal{B}(\mathcal{A}_v) \), where \( _nG \) and \( _nB \) denote \( \mathcal{A}_v \) graded projection valued measures for the operators \( _nG \) and \( _nB \) correspondingly.

Let now \( y \in \mathcal{D}(A)_0 \) and \( \delta \in \mathcal{B}(\mathcal{A}_v) \) be fixed, hence

\[
\lim_n \quad \frac{1}{n} \tilde{F}(\delta)(\frac{1}{n} \tilde{F}(\delta_1)) = \lim_n \quad \frac{1}{n} \tilde{F}(\delta)(\frac{1}{n} \tilde{F}(\delta_2)) = \frac{1}{n} \tilde{F}(\delta) \quad \frac{1}{n} \tilde{F}(\delta_1) x
\]

where \( i_s = \pm i_t, \psi(s, j, k) \in \{0, 1\} \) is an integer so that \( i_s(i_j i_k) = (-1)^{\psi(s, j, k)}(i_s i_j)i_k \).

If a vector \( x \in \bigcup_n nV_0 \) is given, then there exists a natural number \( m \) such that

\[
\lim_n \quad \frac{1}{n} \tilde{F}(\delta)(\frac{1}{n} \tilde{F}(\delta_1)) = \lim_n \quad \frac{1}{n} \tilde{F}(\delta)(\frac{1}{n} \tilde{F}(\delta_2))\]

for each \( n > m \), consequently,

\[
\lim_n \quad \frac{1}{n} \tilde{F}(\delta)(\frac{1}{n} \tilde{F}(\delta_1)) = \frac{1}{n} \tilde{F}(\delta)(\frac{1}{n} \tilde{F}(\delta_2)),
\]

where \( F(\mathcal{A}_v) = I, I \) denotes the unit operator. From Formulas (2 \& 5) and the inclusions \( \bigcup_n nV =: V \subset \mathcal{D}(A) \subset \mathcal{D}(B) \) it follows, that

\[
\tilde{F}(\delta)^j x s i_s = (-1)^{\xi(j, k, s)} \frac{1}{n} \tilde{F}(\delta) \quad \frac{1}{n} \tilde{F}(\delta_1) x s i_s
\]

for each \( x s i_s \in V \) and \( j, k, s = 0, 1, 2, ..., \) where \( \xi(j, k, s) \in \{0, 1\} \) is such integer number that \( i_j(i_k i_s) = (-1)^{\xi(j, k, s)}i_k(i_j i_s) \). From the formula \( i_j(i_k i_s) + i_k(i_j i_s) = 2i_sRe(i_j i_k) \) we get \( (-1)^{\xi(j, k, s)} = (-1)^{n(j, k)} \) for each \( j, k \) and \( s \), since an algebra \( \mathcal{A}_v \) over \( \mathbb{R} \) generated by \( i_j, i_k \) and \( i_s \) has an embedding into the octonion algebra which is alternative \( \mathfrak{I} \) (see also Formulas 4.24(7, 8) in [22]). Thus \( BV \subset V \) and \( _nB = BV \subset V \).

In view of Lemma 2 we have \( _nH^* = B^*|_{nV_nF(\mathcal{I} - b(n), b(n))} \) and from the proof above we get

\[
\tilde{F}(\delta)^j x s i_s = (-1)^{\xi(j, k, s)} \frac{1}{n} \tilde{F}(\delta) \quad \frac{1}{n} \tilde{F}(\delta_1),
\]

that is \( _nH^* \) quasi-permutates with \( _nF \).

In view of Lemma 2 we have \( _nH^* = B^*|_{nV_nF(\mathcal{I} - b(n), b(n))} \) and from the proof above we get

\[
\frac{1}{n} \tilde{F}(\delta)^j x s i_s = (-1)^{\xi(j, k, s)} \frac{1}{n} \tilde{F}(\delta) \quad \frac{1}{n} \tilde{F}(\delta_1),
\]

consequently, \( B(nV) \subset nV \) and \( B(nV) \subset nV \). Consider decomposition \( x = y + z \) with \( y \in nV \) and \( z \in nV^\perp \), then \( x \in \mathcal{D}(B) \) is equivalent to...
\( z \in \mathcal{D}(B) \). The latter inclusion implies \( z \in \mathcal{D}(B) \cap _n V \), if additionally \( x \in _n V \), then we get \(< B^* y; z \rangle = < y; Dz \rangle = 0 \), consequently, \( Bz \in _n V \) and this together with (7) leads to the inclusion \( _n FB \subset B _n F \), that is \( \hat{i} \hat{F}^k \hat{B} \subset (-1)^\kappa(j,k) k \hat{B} \hat{i} \hat{F} \). For any \( \mathbb{R} \)-linear spaces a sign in an inclusion does not play any role. Thus \( _n V \) reduces \( B \) and \( _n G \) into a normal operator \( _n Q = B|_{_n V} \).

Suppose that \( _n G \) is the canonical \( \mathcal{A}_v \) graded projection valued measure for \( _n G \) and \( B F \) is the canonical \( \mathcal{A}_v \) graded projection valued measure for \( B \), hence \( B F|_{_n V} = _n G F \) for each \( n \in \mathbb{N} \). If \( x \in \bigcup_n _n V \), there exists a natural number \( m \) so that

\[
(9) \quad j \hat{E}(\delta_1)(k \hat{F}(\delta_2)x_s i_s) = j \hat{E}_n \hat{F}(\delta_1)(k \hat{F}_{_n}(\delta_2)x_s i_s)
\]

\[
= (-1)^\xi(j,k,s) k \hat{F}(\delta_2)(j \hat{F}_{_n}(\delta_1)x_s i_s)
\]

for each Borel subsets \( \delta_1 \) and \( \delta_2 \) in \( \mathcal{A}_r \), since the restriction of \( A \) to \( _n V \) and \( _n G \) quasi-permute for all \( n \) in accordance with Lemma 2. On the other hand, the \( \mathcal{A}_v \) vector space \( V \) is dense in \( X \), consequently, \( B F \) and \( F \) quasi-permute:

\[
(10) \quad j B \hat{F}(\delta_1) k \hat{F}(\delta_2)x_0 = (-1)^\kappa(j,k) k \hat{F}(\delta_2) j B \hat{F}(\delta_1)x_0
\]

for each \( j, k = 0, 1, 2, \ldots \) and \( x_0 \in X_0 \).

If now \( F \) is an \( \mathcal{A}_v \) graded projection valued measure described in this lemma, then Formula (10) implies

\[
(11) \quad k \hat{F} j B \subseteq (-1)^\kappa(j,k) j B k \hat{F}
\]

for each \( j, k = 0, 1, 2, \ldots, 2^n - 1 \), consequently, \( \mathcal{R}(F) \) reduces \( B \) and \( B|_{\mathcal{R}(F)} \) is a normal operator with \( \mathcal{R}(F) \subset \mathcal{D}(B) \), since \( \mathcal{R}(F) \subset \mathcal{D}(A) \subset \mathcal{D}(B) \). This restriction \( B|_{\mathcal{R}(F)} \) is bounded by the closed graph theorem 1.8.6 \[12\]. Moreover, the restrictions of \( A \) and \( B \) to \( \mathcal{R}(F) \) quasi-permute. Analogous proof is valid for the pair \( A \) and \( C \) instead of \( A \) and \( B \).

**5. Theorem.** If operators \( A \), \( B \) and \( D \) satisfy property \( P \), then \( B \) and \( D \) quasi-permute so that

\[
(1) \quad j B k \hat{D} = (-1)^\kappa(j,k) k \hat{D} j B
\]
for each \(j, k\). Moreover,

\[
\hat{l} \hat{A} = \sum_{j, k, i: j \neq k} (i \hat{B}^k \hat{D} + (-1)^{\kappa(j, k)}k \hat{B}^j \hat{D})
\]

for each \(l\).

**Proof.** Consider the canonical \(A_v\) graded projection valued measure \(E\) for a normal operator \(A\) (see Definition 1). Then we put \(\nu^F := E(I_{a,b})\) with \(a_j = -ni_j\) and \(b_j = ni_j\) for each \(j\). From Theorems 2.27, 2.29 and 2.44 in [29] and §4 above we know that

\[
Ax = \int_{A_v} dA \chi(t) tx \quad \forall x \in \mathcal{D}(A) \quad \text{and}
\]

\[
Bx = \int_{A_v} dB \chi(t) tx \quad \forall x \in \mathcal{D}(B) \quad \text{and}
\]

\[
Dx = \int_{A_v} dD \chi(t) tx \quad \forall x \in \mathcal{D}(D),
\]

where \(A, B, D\) denote \(A_v\) graded projection valued measures for \(A\), \(B\) and \(D\) respectively. Then the condition \(A = BD\) gives

\[
A = \int_{A_v} dA \chi(t) t \int_{A_v} dD \chi(u) u x.
\]

To operators \(A\), \(B\) and \(D\) normal functions \(h_A\), \(h_B\) and \(h_D\) correspond so that \(h_A = h_B h_D\). On the other hand, to the operators \(A^* A\) and \(B^* B\) and \(D^* D\) non-negative self-adjoint functions \(|h_A|^2\), \(|h_B|^2\) and \(|h_D|^2\) correspond (see Proposition 2.32 in [29]). These operators \(A\) and \(B\) and \(D\) are normal so that they satisfy the identities \(A^* A = D^* B^* D = D^* B B^* D = A A^* = B D D^* B^* = B^* D B^*\) and \(B^* B = B^* B\) and \(D^* D = D D^*\).

In view of Theorems 2.29, 2.44 and Proposition 2.32 and Remark 2.43 in [29] to the \(A_v\) graded projection operator \(A \chi(\delta)\) a homomorphism \(\phi\) a (real) characteristic function \(\phi(A \chi(\delta)) = \chi_{\delta}\) of a subset \(\delta \subset \Lambda\) counterpose so that \(\chi_{\delta} = \omega(\chi_\delta)\). Therefore, Theorem 2.23 and Lemma 2.21 in [29], Formulas (3 - 6) and Conditions (P1 - P4) imply that their projection operators satisfy the equality

\[
B E(\delta_1) D E(\delta_2) = D E(\delta_2) B E(\delta_1)
\]

for each Borel subsets \(\delta_1\) and \(\delta_2\) in \(A_v\). In view of Lemma 4 \(R(\nu^F)\) reduces \(B\) and \(D\) and the restrictions of these operators to \(R(\nu^F)\) are bounded normal operators. On the other hand, \(\bigcup_{n=1}^\infty R(\nu^F)\) is dense in the Hilbert space
X over the Cayley-Dickson algebra \( \mathcal{A}_v \). Therefore, we infer from Formulas (3 - 7), that \( jB \) and \( kD \) satisfy Formulas (1, 2) for each \( j, k \), since

\[
j_B \hat{E}(\delta_1) k_D \hat{E}(\delta_2) = (-1)^{\kappa(j,k)} k_D \hat{E}(\delta_2) j_B \hat{E}(\delta_1)
\]

for every Borel subsets \( \delta_1 \) and \( \delta_2 \) in \( \mathcal{A}_v \) and for each \( j, k \).

6. Corollary. Suppose that operators \( A, B \) and \( D \) are self-adjoint and satisfy property \((P)\). Then \( BD = DB \).

Proof. This follows immediately from Theorems 2.27, 2.29 and 2.44 in [29] and Formulas 5(1 - 3), since spectra of self-adjoint operators are contained in the real field \( \mathbb{R} \) and the latter is the center of the Cayley-Dickson algebra \( \mathcal{A}_v \) so that \( t = t_0 \in \mathbb{R} \) in Formulas 5(1, 2), that is \( j = k = 0 \) only.

7. Lemma. Let operators \( B, D \) and \( A \) have property \( P \), let also \( B = T_B U_B, \ D = T_D U_D \) and \( A = T U \) be their canonical decompositions with positive self-adjoint operators \( T_B, T_D \) and \( T \) and unitary operators \( U_B, U_D \) and \( U \) respectively. Then \( T_B T_D = T_D T_B = T \) and \( U_B U_D = U \) and \( jU_B kU_D = (-1)^{\kappa(j,k)} jU_B kU_D \) for each \( j, k \), moreover, \( T_B U_D = U_D T_B \) and \( T_D U_B = U_B T_D \).

Proof. The decompositions in the conditions of this lemma are particular cases of that of Theorem I.3.37 [28]. Consider the canonical \( \mathcal{A}_v \) graded resolutions of the identity \( E^B \) and \( E^D \) of operators \( B \) and \( D \) respectively. In view of Theorem 5

\[
j E^B(\delta_1) k E^D(\delta_2) = (-1)^{\kappa(j,k)} k E^D(\delta_2) j E^B(\delta_1)
\]

for every Borel subsets \( \delta_1 \) and \( \delta_2 \) in \( \mathcal{A}_v \) and each \( j, k \). We put \( F(dw,dz) = E^B(dw) E^D(dz) \), hence \( F(dw,dz) \) is a \( 2^{v+1} \) parameter \( \mathcal{A}_v \) graded resolution of the identity so that \( F_{ik}(\delta_1, \delta_2) x_k = E^B(\delta_1)(E^D)_{ik}(\delta_2) \) for each vector \( x_k \in X_k \) and every \( k \) and we put

\[
G := \int_{\mathcal{A}_v^2} dF(w,z) \cdot wz,
\]

where \( dF(w,z) \) is another notation of \( F(dw,dz) \), \( w, z \in \mathcal{A}_v \) (see also §I.2.58 [28]). This operator \( G \) is normal, since the quaternion skew field is associative.
and the octonion algebra is alternative and \((wz)(wz)^* = |wz|^2 = |w|^2|z|^2\) for each \(w, z \in A_v\) with \(2 \leq v \leq 3\). Then we get

\[
B = \int_{A_v^2} dF(w, z).w = \int_{A_v} dE^B(w).w
\]

and

\[
D = \int_{A_v^2} dF(w, z).z = \int_{A_v} dE^D(z).z,
\]

consequently,

\[
A = BD \quad \text{and} \quad j^kB^kD = (-1)^{\kappa(j,k)}k^jD^kB \quad \text{for each} \quad j, k,
\]

and hence

\[
\sum_{j,k: \; i_ji_k = i_l} [j^kB^kD + (-1)^{\kappa(j,k)}kB^jD] \subseteq i^lG
\]

for every \(l\). Therefore, \(A = G\), since a normal operator is maximal.

Then one can consider the function \(u(w, z) := \frac{wz}{|wz|}\) for \(wz \neq 0\), while \(u(w, z) = 1\) if \(wz = 0\), where \(w, z \in A_v\). The operator

\[
U := \int_{A_v^2} dF(w, z).u(w, z)
\]

is unitary, since \(|u(w, z)| = 1\) for each \(w\) and \(z\), the operator

\[
T := \int_{A_v^2} dF(w, z).|wz|
\]

is positive and self-adjoint, since

\[
< xT; x > := \int_{A_v^2} < x dF(w, z).|wz|; x > \geq 0
\]

for each \(x \in D(T)\) (see Proposition 2.35 [29]). On the other hand, \(u(w, z)|wz| = |wz|u(w, z) = wz\), since the algebra \(A_v\) is alternative for \(v \leq 3\), hence

\(TU = UT = G = A\) by Theorem 2.44 [29]. Moreover, we deduce from Theorem 2.44 [29] that the operators

\[
U_B := \int_{A_v^2} dF(w, z).u(w)
\]

and

\[
U_D := \int_{A_v^2} dF(w, z).u(z)
\]

are unitary and the the operators

\[
T_B := \int_{A_v^2} dF(w, z).|w|
\]

and

\[
T_D := \int_{A_v^2} dF(w, z).|z|
\]
are positive and self-adjoint, where \( u(w) := w/|w| \) if \( w \neq 0 \), also \( u(w) = 1 \) if \( w = 0 \). Since \( |w||z| = |wz| \) for each \( w \) and \( z \in A_v \) with \( v \leq 3 \), the inclusion follows

\[
T_B T_D \subseteq \int_{A_v^2} dF(w, z).|wz| = T.
\]

The functions \( u(w) \) and \( u(z) \) are bounded and \( u(w)u(z) = u(z)u(w) = u(w, z) \) on \( A_v^2 \), consequently,

\[
U_B U_D = \int_{A_v^2} dF(w, z).u(w, z) = U \text{ so that}
\]

\[
jU_B kU_D = (-1)^{\kappa(j,k)} kU_B jU_D \text{ for each } j, k. \text{ This implies that } A = UT = U_B U_D T = (U_B T_B)(U_D T_D) \text{, consequently, } U_D T U_B^* = T_B T_D.
\]

This means that the operators \( T \) and \( T_B T_D \) are unitarily equivalent, hence the operator product \( T_B T_D \) is self-adjoint. A self-adjoint operator is maximal, consequently, \( T = T_B T_D \) and similarly \( T = T_D T_B \). The real field \( \mathbb{R} \) is the center of the Cayley-Dickson algebra \( A_v \) for each \( v \geq 2 \), the real and complex fields are commutative, hence

\[
T_B U_D = \int_{A_v^2} dF(w, z).(|w|u(z)) = U_D T_B \text{ and}
\]

\[
T_D U_B = \int_{A_v^2} dF(w, z).(|z|u(w)) = U_B T_D.
\]

8. Notation. Let \( \Omega \) denote the set of all \( n \)-tuples \( x = (x_1, ..., x_m, x_{m+1}, ..., x_n) \) such that \( x_1, ..., x_m \) are non-negative integers, while \( x_{m+1}, ..., x_n \) are non-negative real numbers with \( \sum_{j=1}^n x_j > 0 \). Relative to the addition \( x + y = (x_1 + y_1, ..., x_n + y_n) \) this set \( \Omega \) forms a semi-group.

9. Theorem. Suppose that \( \{B^x : x \in \Omega\} \) is a weakly continuous semi-group of normal operators, that is satisfying the following conditions:

1) \( B^x \) is a normal operator acting on a Hilbert space \( X \) over the Cayley-Dickson algebra \( A_v \) for each element \( x \in \Omega \);

2) \( B^x B^y = B^{x+y} \) for each \( x, y \in \Omega \);

3) the \( A_v \) valued scalar product \( \langle B^x f; g \rangle \) is continuous in \( x \in \Omega \) for each marked \( f, g \in D := \bigcap_{x \in \Omega} D(B^x) \);

4) a family \( alg_{A_v} \{I, B^x, (B^x)^* : x \in \Omega\} \) is over the algebra \( A_v \) with \( 2 \leq v \leq 3 \). Then a unique \( 2n \)-parameter \( A_v \) graded resolution \( \{a_1, ..., a_n, b_1, ..., b_n\} \bar{F} : a, b \in \mathbb{R} \)
\( \Omega \) of the identity exists so that \((a,b) \hat{F} = 0\) if a negative coordinate \(a_k < 0\) exists for some \(k = 1, \ldots, n\), moreover,

\[
(5) \quad B^x = \int_{\mathbb{R}^n} d_{(a,b)} \hat{F}_x \{ a^x \exp[x_1 M_1(b_1)] \cdots \exp[x_n M_n(b_n)] \},
\]

where

\[
a^x = \prod_{k=1}^n a_k^{x_k},
\]

\( M_s : \mathbb{R}^n \rightarrow \mathcal{S}_v := \{ z \in \mathcal{A}_v : |z| = 1, Re(z) = 0 \} \) is a Borel function for each \( s \), \( a = (a_1, \ldots, a_n) \).

**Proof.** In view of Lemma 5 each operator \( B^x \) has the decomposition \( B^x = T^x U^x = U^x T^x \) with a positive self-adjoint operator \( T^x \) and a unitary operator \( U^x \). Since \( \{ B^x : x \in \Omega \} \) is a semi-group, the relations \( T^x T^y = T^{x+y} \) and \( U^x U^y = U^{x+y} \) are valid for each elements \( x, y \in \Omega \). That is, \( \{ T^x : x \in \Omega \} \) and \( \{ U^x : x \in \Omega \} \) are semi-groups of positive self-adjoint operators and unitary operators correspondingly.

If \( y^x = (0, \ldots, y^x_{m+1}, \ldots, y^x_n) \in \Omega \) are elements of the semi-group \( \Omega \) such that \( y^x = \frac{y^2 + y^3}{2} \), \( s = 1, 2, 3 \), \( f \) is a vector in a domain \( D \), then

\[
\| B^{y^1} f \|^2 = \langle B^{y^1} f, B^{y^1} f \rangle = \langle B^{y^2/2} B^{y^3/2} f, B^{y^2/2} B^{y^3/2} f \rangle \\
= \langle (B^{y^2/2})^* B^{y^2/2} f, (B^{y^2/2})^* B^{y^2/2} f \rangle \leq \| (B^{y^2/2})^* B^{y^2/2} f \| \| B^{y^2/2} \| B^{y^2/2} f \|
\]

by Cauchy-Schwartz’ inequality I.2.4(1) [28]. On the other hand,

\[
\| (B^{y^2/2})^* B^{y^2/2} f \|^2 = \langle (B^{y^2/2})^* B^{y^2/2} f, (B^{y^2/2})^* B^{y^2/2} f \rangle = \langle B^{y^2} f, B^{y^2} f \rangle = \| B^{y^2} f \|^2,
\]

since the semi-group \( \{ B^x : x \in \Omega \} \) is commutative and an operator \( B^x \) is normal for each \( x \in \Omega \). Thus the inequality

\[
\| B^{y^1} f \| \leq \| B^{y^2} f \| \| B^{y^3} f \|
\]

follows. This implies that the function \( q(y) := \| B^y f \| \) is convex and bounded in the variable \( y_p \) in any bounded segment \([\alpha, \beta] \subset (0, \infty)\), when other variables \( y_q \) with \( q \neq p \) are zero, \( p = m+1, \ldots, n \), since the exponential \( e^t \) and the natural logarithmic functions \( \ln(t) \) are convex and bounded on each segment \([\gamma, \delta] \subset (0, \infty)\) and \( \ln q(y^1) \leq \ln q(y^2) + \ln q(y^3) \).

Evidently, a commutative group \( \hat{\Omega} \) exists for the semi-group \( \Omega \) such that \( \Omega \subset \hat{\Omega} \subset \mathbb{R}^n \) and the function \( q(y) \) can be extended on \( \hat{\Omega} \) so that \( q(0) = \| f \| \).
and \( q(-y) = q(y) \) for \( y \in \Omega \). If \( q \) is continuous on \( \Omega \), its extension on \( \hat{\Omega} \) can be chosen continuous, since \( \hat{\Omega} \) is a completely regular topological space, i.e. \( T_1 \) and \( T_{3.5} \) (see [29]).

If \( \Omega \) is a group the function \( q(y) \) is positive definite, that is by the definition for each \( \lambda_1, ..., \lambda_k \in \mathbb{R} \oplus \mathbb{R}^i =: C_1 \) and \( y^1, ..., y^k \in \Omega \) the inequality

\[
\sum_{j,l} \lambda_j \bar{\lambda}_l q(y^j - y^l) \geq 0
\]

is valid, but this inequality follows from the formula

\[
\sum_{j,l} \lambda_j \bar{\lambda}_l q(y^j - y^l) = \| \sum_j \lambda_j B^{-j} f \| ^2
\]

and since \( \|x\| \geq 0 \) for each \( x \in X \).

Particularly, for elements \( x^k := (0, ..., x_k, 0, ..., 0) \) in the semi-group \( \Omega \) the mapping \( < T^{x^k} f; f > \) is continuous in \( x^k \) for each marked vector \( f \in D \). Indeed, for \( k = 1, ..., m \) this is evident, since \( x^k \in \mathbb{N} \) takes values in the discrete space in this case. If \( k = m + 1, ..., n \) one can use the formula \( < T^{x^k} f; f > = < B^{x^k/2} f, B^{x^k/2} f > = \| B^{x^k/2} f \| ^2 \) which implies that \( < T^{x^k} f; f > \) is a bounded convex function of \( x^k \) in every finite interval \([\alpha, \beta] \subset (0, \infty)\), when \( f \in D \) is a marked vector (see Theorem 2.29 and Formula 2.44(5) [29]).

Denote by \( s, t, E \) the canonical \( A_v \) graded resolution of the identity for \( T^{e_s} \), where \( e_s = (0, ..., 0, 1, 0, ...) \) denotes the basic vector with coordinate 1 at \( s \)-th place and zeros otherwise, \( t, s \in \mathbb{R} \). By the conditions of this theorem operators \( T^{e_s} \) and \( T^{e_p} \) commute for each \( s, p = 1, ..., n \), since

\[
(6) \ T^{e_s} T^{e_p} = T^{e_s+e_p} = T^{e_p} T^{e_s}.
\]

Due to Theorem 2.42 [29] the equality

\[
(7) \ j s, t, E k p, t, E = (-1)^{\nu(j, k)} k p, t, E j s, t, E
\]

is satisfied for each \( j, k \) and every \( s, p \), with \( t, s, t, p, p \in \mathbb{R} \). This implies that

\[
(8) \ (t_1, ..., t_n) E = 1, t, E ..., n, t, E
\]

is an \( n \)-parameter \( A_v \) graded resolution of the identity. Each operator \( T^{e_s} \) is positive, hence \( s, t, E = 0 \) for every \( t < 0 \), consequently, \((t_1, ..., t_n) E = 0 \) if \( t < 0 \) for some \( s = 1, ..., n \).

We now consider the operators

\[
(9) \ A^p x := \int_0^{\infty} ... \int_0^{\infty} d (t_1, ..., t_n) E . (t^{p_1}_1 ... t^{p_n}_n x),
\]

13
where \( p = (p_1, \ldots, p_n) \in \Omega \), \( x \in X \) for which the integral converges. We certainly have

\[
\int_0^\infty \cdots \int_0^\infty d(t_1, \ldots, t_n)E.(t_1^{p_1} \cdots t_n^{p_n}x) = \int_0^\infty \cdots \int_0^\infty (t_1^{p_1} \cdots t_n^{p_n})d(t_1, \ldots, t_n)E.x,
\]

since \( t_j^{p_j} \in \mathbb{R} \) for each \( j \) and \((t_1, \ldots, t_n)E\) is a real linear operator. If \( p_s \in \mathbb{Z}/2 \) for each \( s \), then \( T^p = T^{e_1p_1} \cdots T^{e_np_n} \subseteq A^p \), consequently, \( T^p = A^p \), since a self-adjoint operator is maximal.

Take a partition of the Euclidean space \( \mathbb{R}^n \) into a countable family of bounded parallelepipeds \( J_k = \prod_{j=1}^n [a_j, b_j] \) so that they may intersect only by their boundaries: \( J_k \cap J_l = \partial J_k \cap \partial J_l \) for each \( k \neq l \in \mathbb{N} \), \( \bigcup_{k=1}^\infty J_k = \mathbb{R}^n \).

We put \( Y^k := \mathcal{R} (\hat{E}(J_k)) \), where \( \hat{E}(\delta) \) is the \( \mathcal{A}_v \) graded spectral measure corresponding to \( \iota E, \ \delta \in \mathcal{B}(\mathbb{R}^n), \ t \in \mathbb{R}^n \). Then the restriction \( B^s|_{Y^k} \) of \( B^s \) to \( Y^k \) is a bounded self-adjoint operator. If \( x, y \in \Omega \) are elements of the semi-group so that \( y_s \geq x_s \) and \( y_s \in \mathbb{Z}/2 \) for each \( s = 1, \ldots, n \), then \( D(T^y) \subseteq D(T^x) \), since \( T^y = T^x T^{y-x} \). Therefore, \( f \in D(A^y) = D(T^y) \subseteq D(T^x) \) for each \( f \in Y^k \), consequently, \( Y^k \subseteq \mathcal{D} \) for each natural number \( k \in \mathbb{N} \).

If \( f \in Y^k \oplus Y^l \) and \( g \in Y^l \), then

\[
\lim_{y \to x} <(T^y - A^y)(f + g); (f + g)> = <(T^x - A^x)(f + g); (f + g)> = 0,
\]

since \( T^y = A^y \) for each \( y \in (\mathbb{Z}/2)^n \cap \Omega \) and the \( \mathcal{A}_v \) valued scalar products \(< T^x f; f > \) and \(< A^x f; f > \) are continuous in each component \( x_s \) of \( x \). In the same manner we get \(< (T^x - A^x)f; f > = 0 \) and \(< (T^x - A^x)g; g > = 0 \), consequently, \(< (T^x - A^x)f; g > = 0 \). The \( \mathcal{A}_v \) vector space \( \bigcup_{k=1}^\infty Y^k \) is dense in the Hilbert space \( X \) over the Cayley-Dickson algebra \( \mathcal{A}_v \), hence \( T^x f^k = (A^x|_{Y^k})f^k = A^x f^k \) for each vector \( f^k \in Y^k \). This means that each \( Y^k \) reduces the operator \( T^x \) to \( (A^x|_{Y^k}) \), consequently, \( T^x = A^x \). From this it follows that the \( \mathcal{A}_v \) valued scalar product \(< T^x f; g > \) is continuous in \( x \in \Omega \) for each marked vectors \( f \in \mathcal{D} \) and \( g \in X \).

Consider the sub-semi-group \( \Omega_s := \{ x : x = x^s := (0, \ldots, 0, x_s, 0, \ldots) \in \Omega \} \), where \( s = 1, \ldots, n \), also we suppose that \( \hat{\mathcal{E}}(\{0\}) = 0 \), where \( \hat{\mathcal{E}}(\delta) \) is the \( \mathcal{A}_v \) graded projection valued measure corresponding to \( s, \delta \in \mathcal{B}(\mathbb{R}) \). This implies that the operator \( T^{x^s} \) has not the zero eigenvalue. Take arbitrary marked vectors \( f \in \mathcal{D} \) and \( g \in D(T^{y^s}) \). Then using the triangle inequality
we deduce that

\[ |< (U^{x'}) -U^{y'} > f; T^{y'} g > | = |< (U^{x'} -U^{y'}) T^{y'} f; g > | = |< (U^{x'} T^{y'} -U^{y'} T^{y'}) f; g > +< U^{x'} (T^{y'} - T^{x'}) f; g > | \leq |< (B^{x'} - B^{y'} ) f; g > | + \| (T^{y'} - T^{x'}) f \| \| g \| . \]

But the limits are zero \( \lim_{x' \to y'} < (B^{x'} - B^{y'}) f; g > = 0 \) due to suppositions of this theorem and \( \lim_{x' \to y'} \| (T^{y'} - T^{x'}) f \| = 0 \), since \( T^{x} = A^{x} \) and \( A^{x} \) has the integral representation given by Formula (9). Thus the limit

\[ \lim_{x' \to y'} < (U^{x'} - U^{y'}) f; h > = 0 \]

is zero for each \( f \in \mathcal{D} \) and \( h \in \mathcal{R}(T^{y'}) \). On the other hand, \( \mathcal{D} \) is dense in \( X \), since \( \bigoplus_{k=1}^{\infty} Y^{s} \) is dense in \( X \). The family \( U^{x} \) of unitary operators is norm bounded by the unit 1, consequently, \( \lim_{x' \to y'} < (U^{x'} - U^{y'}) f; h > = 0 \) for each \( f, h \in X \) and hence the semi-group \( \{ U^{x'} : x' \in \Omega \} \) is weakly continuous. The semi-group \( \{ U^{x'} : x' \in \Omega \} \) of unitary operators can be extended to a weakly continuous group of unitary operators putting \( U^{-x'} = (U^{x})^{\ast} \) and \( U^{0} = I \). This one-parameter commutative group of unitary operators is also strongly continuous, since

\[ \| (U^{x'} - U^{y'}) f \|^{2} = < (U^{x'} - U^{y'}) f; (U^{x'} - U^{y'}) f > = < (U^{x'} - U^{y'})^{\ast} (U^{x'} - U^{y'}) f; f > = < 2I - U^{x'-y'} - U^{y'-x'} f; f > = < (U^{0} - U^{x'-y'}) f; f > + < (U^{0} - U^{y'-x'}) f; f > . \]

In view of Theorem I.3.28 \[28\] there exists a unique \( \tilde{A}_{s} \) graded projection valued measure \( \tilde{F} \) so that

\[ (10) \quad < U(x') f; h > = \int_{-\infty}^{\infty} < \tilde{F}(db_{s}) \exp(x_{s} M_{s}(b_{s}) b_{s}) f; h > \]

for each \( f, h \in \mathcal{D}(Q^{s}) \), where

\[ (11) \quad < Q^{s} f, h > = \int_{-\infty}^{\infty} b_{s} < \tilde{F}(db_{s}) f; h > \]

for each \( f, h \in \mathcal{D}(Q^{s}) \),

\[ (12) \quad \mathcal{D}(Q^{s}) = \{ f : f \in X; \| Q^{s} f \|^{2} = \int_{-\infty}^{\infty} < \tilde{F}(db_{s}), b_{s}^{2} f; f > < \infty \} , \]

15
$M_s(b_s)$ is a Borel function from $\mathbb{R}$ into the purely imaginary unit sphere $S_v := \{ z \in A_v : |z| = 1, \Re(z) = 0 \}$. Then we put $\hat{s}E(da_s, db_s) = \hat{s}E(da_s) \hat{s}F(db_s)$, where

$$\int \hat{s}E(\delta_1) \hat{s}F(\delta_2) = (-1)^{\alpha(j,k)} \int \hat{s}E(\delta_1) \hat{s}F(\delta_2)$$

for each $j, k$ and Borel subsets $\delta_1, \delta_2 \in \mathcal{B}(\mathbb{R})$. Then an operator $P^{x^s}$ exists prescribed by the formula:

$$P^{x^s} = \int_{-\infty}^{\infty} \int_{0}^{\infty} \hat{s}E(da_s, db_s) \cdot [a_s^{x^s} \exp(x_s M_s(b_s) b_s)].$$

This implies the inclusion $B^{x^s} \subseteq P^{x^s}$, but a normal operator is maximal, consequently, $B^{x^s} = P^{x^s}$ for each $s$ and $x^s \in \Omega$.

Suppose now that $\hat{s}E(\{0\}) \neq 0$, consider the null space $N^s := \ker(B^{x^s})$ of $B^{x^s}$. To each $A_v$ graded projection valued measure $\hat{s}E(\delta)$ associated with the family $alg_{A_v}(I, B^x, (B^x)^*)$ a real valued characteristic function in $\mathcal{N}(\Lambda, \mathbb{R})$ corresponds, where $\delta \in \mathcal{B}(\mathbb{R}^2)$, consequently, $N^s$ is an $A_v$ vector subspace in $X$. Let $X = N^s \oplus K^s$, hence $K^s$ is an $A_v$ vector space, since $N^s$ is the $A_v$ vector subspace of the $A_v$ Hilbert space $X$. Take the restrictions $B^{x^s}|_{N^s} =: B^{x^s,N}$ and $B^{x^s}|_{K^s} =: B^{x^s,K}$ of $B^{x^s}$ to $N^s$ and $K^s$ correspondingly.

This implies that the semi-group of normal operators $\{B^{x^s,K} : x^s \in \Omega\}$ possesses the property that none of the operators $B^{x^s,K}$ has zero eigenvalue. From Formula (13) it follows, that there exists a two-parameter resolution $s,K \hat{s}E$ of the identity so that

$$B^{x^s,K} = \int_{-\infty}^{\infty} \int_{0}^{\infty} \hat{s}E(da_s, db_s) \cdot [a_s^{x^s} \exp(x_s M_s(b_s) b_s)].$$

Define an $A_v$ graded projection value measure $s,N \hat{s}E$ so that

$$\mathcal{F}\mathcal{E}(dt_s, dq_s) = s,N;a_s,b_s \hat{s}E$$

so that $s,N;a_s,b_s \hat{s}E = 0$ for $a_s < 0$ and $s,N;a_s,b_s \hat{s}E = I$ when $a_s \geq 0$. Since $B^{x^s,N}(N^s) = \{0\}$, the integral representation follows:

$$B^{x^s,K} = \int_{-\infty}^{\infty} \int_{0}^{\infty} \hat{s}E(da_s, db_s) \cdot [a_s^{x^s} \exp(x_s M_s(b_s) b_s)].$$

Now it is natural to put $\hat{s}E(da_s, db_s) = s,N \hat{s}E(da_s, db_s) \oplus s,K \hat{s}E(da_s, db_s)$ for an $A_v$ graded projection valued measure on $X$, that induces the formula:

$$B^{x^s} = \int_{-\infty}^{\infty} \int_{0}^{\infty} \hat{s}E(da_s, db_s) \cdot [a_s^{x^s} \exp(x_s M_s(b_s) b_s)].$$
In accordance with Theorem 5

\[ q \hat{E}(\delta_1) k \hat{E}(\delta_2) = (-1)^{\kappa(j,k)} q \hat{E}(\delta_2) k \hat{E}(\delta_1) \]

for each \( s, q = 1, \ldots, n \) and \( j, k = 0, 1, \ldots, 2^v - 1 \) and every \( \delta_1, \delta_2 \in \mathcal{B}(\mathbb{R}^2) \), particularly, for \( j = k = 0 \) i.e. \( s \hat{E}(\delta_1) \) and \( q \hat{E}(\delta_2) \) commute. Then we put

\[ (a,b) \hat{F} = \int_{-\infty}^{a_1} \int_{-\infty}^{b_1} \cdots \int_{-\infty}^{a_n} \int_{-\infty}^{b_n} 1 \hat{E}(dt_1, dq_1) \cdots \hat{E}(dt_n, dq_n), \]

hence \( (a,b) \hat{F} \) is an \( \mathcal{A}_v \) graded resolution of the identity, for which

\[ d(a,b) \hat{F} \{ a^x \exp[x_1 M_1(b_1)b_1] \cdots \exp[x_n M_n(b_n)b_n] \} = \]

\[ 1 \hat{E}(da_1, db_1) \exp(x_1 M_1(b_1)b_1) \cdots \hat{E}(da_n, db_n) \exp(x_n M_n(b_n)b_n) a^x, \]

since the semi-groups \( \{ B^x : x \in \Omega \} \) and \( \{ T^x : x \in \Omega \} \) and \( \{ U^x : x \in \Omega \} \) are commutative, the real field \( \mathbb{R} \) is the center of the Cayley-Dickson algebra \( \mathcal{A}_v \), for each \( v \geq 2 \), the fields \( \mathcal{A}_0 = \mathbb{R} \) and \( \mathcal{A}_1 = \mathbb{C} \) are commutative, \( a_s \in \mathbb{R} \) and \( x_s \in \mathbb{R} \) for each \( s = 1, \ldots, n \). For the operators

\[ (17) \quad P^x = \int_{\mathbb{R}^n} d(a,b) \hat{F} \{ a^x \exp[x_1 M_1(b_1)b_1] \cdots \exp[x_n M_n(b_n)b_n] \}, \]

where

\[ a^x = \prod_{k=1}^{n} a_k^{x_k}, \]

\( M_s : \mathbb{R}^n \to \mathcal{S}_v := \{ z \in \mathcal{A}_v : |z| = 1, \text{Re}(z) = 0 \} \) is a Borel function for each \( s \), the inclusion follows \( B^x \subseteq P^x \) for each \( x \in \Omega \), since \( B^x = B^{x_1} \cdots B^{x_n} \), where the operators \( B^{x_1}, \ldots, B^{x_n} \) pairwise commute. But a normal operator is maximal, consequently, \( B^x = P^x \) for each \( x \in \Omega \). A uniqueness of the resolution \( (a,b) \hat{F} \) of the identity follows from uniqueness of \( s \hat{F} \) and \( q \hat{E} \) for each \( s \).

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