Boundary states in coset conformal field theories

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We construct various boundary states in the coset conformal field theory $G/H$. The $G/H$ theory admits the twisted boundary condition if the $G$ theory has an outer automorphism of the horizontal subalgebra that induces an automorphism of the $H$ theory. By introducing the notion of the brane identification and the brane selection rule, we show that the twisted boundary states of the $G/H$ theory can be constructed from those of the $G$ and the $H$ theories. We apply our construction to the $su(n)$ diagonal cosets and the $su(2)/u(1)$ parafermion theory to obtain the twisted boundary states of these theories.
1 Introduction

The recent developments in the construction of the boundary states in rational conformal field theories have revealed the rich structure of conformal field theories with boundaries \[1,2,3\]. It is now recognized that the rational boundary states are described by a non-negative integer matrix representation (NIM-rep) of the fusion algebra \[4,3,5\]. The situation changes, however, if we relax the condition of rationality on the boundary states. Many rational CFT’s are equipped with an extended symmetry such as the current algebra, and they are no more rational with respect to the Virasoro algebra. If we require only the conformal invariance on the boundary states, instead of the full chiral algebra, the classification problem gets much complicated, for which we have no generic answer.

In the context of string theory, boundary states give rise to D-branes. In order to have a consistent theory, D-branes have to keep (super) conformal invariance on the worldsheet. The conservation of the extended current algebra is an additional requirement, which is not necessary in general. The study of the conformal boundary states is therefore inevitable for the full understanding of the spectrum of D-branes.

An interesting approach to the construction of the conformal boundary states in the WZW models has been proposed in \[6\] \[^{1}\]. The strategy of \[6\] is to decompose the G WZW model into the H part and the coset G/H, where H is a subgroup of the group G

\[
G \sim G/H \times H. \tag{1.1}
\]

From this decomposition, we have several boundary conditions of the G theory. Adopting the usual boundary condition for both of the H and the G/H parts yields the ordinary boundary condition of the G theory. On the other hand, we can twist the boundary condition of the H part by an automorphism that leaves the Virasoro algebra invariant. Twisting the G/H part in the same way does not affect the boundary condition of the G theory. Taking the ordinary condition in the coset theory, however, gives the novel boundary condition of the G theory. This condition breaks the G current algebra while its conformal invariance is manifest. In \[6\], it has been shown that the conformal boundary state, not rational with respect to G, does exist for the case of G = SU(2). The boundary states in the coset theories are therefore useful building blocks in the construction of

\[^{1}\text{For the } c = 1 \text{ models, it is possible to take another approach } 7,8,9\text{.} \]
the conformal boundary states in the WZW models. Although the coset theory with boundaries has been studied from the sigma model point of view [6,10,11,12], the algebraic study such as [1,2,3] is necessary to explore the stringy regime of the theory.

In this paper, we give the general method to obtain the boundary states in the $G/H$ coset conformal theory. In particular, we show that a NIM-rep of the $G/H$ theory can be constructed from a pair of NIM-reps for the $G$ and the $H$ theories. In doing this, we introduce the notion of the brane identification and the brane selection rule, which are considered to be the boundary version of the field identification and the selection rule in the coset theory. We apply our method to the twisted boundary states of the $su(n)_1 \oplus su(n)_1/su(n)_2$ diagonal coset and the $su(2)_k/u(1)_k$ parafermion theory, and obtain the result consistent with that in [6].

The organization of the paper is as follows. In the next section, we review some results about the boundary states in rational conformal field theories, especially the WZW models. In Section 3, we give arguments for the existence of an automorphism of the boundary states, which is the dual of the automorphism of the current algebra. In Section 4, we give the rule to yield NIM-reps in the coset theory. We show that a pair of NIM-reps in the $G$ and the $H$ theories yield a NIM-rep in the $G/H$ theory after an appropriate identification of the states generated by the automorphism of the boundary states. In Section 5, we apply our method to several examples.

2 Boundary states in the WZW models

In this section, we review some basic results about the boundary states in the WZW models following to [1,2,3].

The most simple boundary condition for the current algebra is

$$J^a_n + \tilde{J}^a_{-n} = 0,$$  \hspace{1cm} (2.1)

where $J^a$ and $\tilde{J}^a$ represent the holomorphic and the anti-holomorphic parts of the algebra, respectively. The Ishibashi states $\{ |\lambda \rangle \} |\lambda \in \text{Spec}(G) \}$ are the building blocks of the boundary states [13]. Here, we denote by $\text{Spec}(G)$ the set of the integrable representations
of the algebra $g$ at level $k$, namely, $\text{Spec}(G) = P_k^+(g)$. We normalize $|\lambda\rangle$ as follows
\[
\langle \lambda | \tilde{q}^Hc | \lambda \rangle = \frac{1}{S_{0\lambda}} \chi_\lambda (-1/\tau) = \sum_{\mu \in \text{Spec}(G)} \frac{S_{\lambda \mu}}{S_{0\lambda}} \chi_\mu (\tau) = \chi_0 (\tau) + \cdots ,
\] (2.2)
where $\tilde{q} = e^{-2\pi i/\tau}$ and $H_c = \frac{1}{2} (L_0 + \tilde{L}_0 - \frac{c}{12})$ is the closed string Hamiltonian. ‘0’ stands for the vacuum representation. This normalization corresponds to the following scalar product in the space of the boundary states \cite{3}
\[
\langle \alpha | |\beta \rangle = \lim_{q \to 0} q^{c_2/24} \langle \alpha | \tilde{q}^Hc | \beta \rangle .
\] (2.3)
Here $c$ is the central charge of the theory and $q = e^{2\pi i/\tau}$.

The boundary condition (2.1) relates a representation $\lambda$ with $\bar{\lambda}$. Hence, $(\lambda)_L \otimes (\bar{\lambda})_R$ must exist in the closed string spectrum in order to have the Ishibashi state $|\lambda\rangle$. In the case of the charge-conjugation modular invariant $Z = \sum_{\lambda \in \text{Spec}(G)} \chi_\lambda \bar{\chi}_\lambda$, we obtain all the Ishibashi states $|\lambda\rangle, \lambda \in \text{Spec}(G)$. However, in the other cases, the set of the allowed Ishibashi states is in general different from $\text{Spec}(G)$. We denote this set by $\mathcal{E}$
\[
\mathcal{E} = \{ \lambda | (\lambda)_L \otimes (\bar{\lambda})_R \in \text{closed string spectrum} \} .
\] (2.4)

For the diagonal modular invariant $Z = \sum_{\lambda \in \text{Spec}(G)} \chi_\lambda \bar{\chi}_\lambda$, only the self-conjugate representations are allowed and $\mathcal{E} = \{ \lambda \in \text{Spec}(G) | \bar{\lambda} = \lambda \}$. The multiplicity of a representation $\lambda$ in $\mathcal{E}$ can be greater than 1, as is seen in the $D_{\text{even}}$ invariant of $\text{su}(2)$.

A generic boundary state $|\alpha\rangle$ satisfying the boundary condition (2.1) is a linear combination of the Ishibashi states
\[
|\alpha\rangle = \sum_{\lambda \in \mathcal{E}} \psi_\alpha^\lambda |\lambda\rangle .
\] (2.5)
We denote by $\mathcal{V}$ the set labelling the boundary states
\[
\mathcal{V} = \{ \alpha | \text{label of the boundary states} \} .
\] (2.6)

The annulus amplitude between two boundary states takes the form
\[
Z_{\alpha\beta} = \langle \beta | \tilde{q}^Hc | \alpha \rangle = \sum_{\lambda \in \mathcal{E}, \mu \in \text{Spec}(G)} \psi_\alpha^\lambda S_{\mu \lambda} S_{0 \lambda} \bar{\psi}_\beta^\lambda \chi_\mu (\tau) = \sum_{\mu \in \text{Spec}(G)} n_{\mu\alpha}^\beta \chi_\mu .
\] (2.7)
Here we denote the multiplicity of the representation $\mu$ in $Z_{\alpha \beta}$ by $n_{\mu \alpha \beta}$

$$n_{\mu \alpha \beta} = \sum_{\lambda \in \mathcal{E}} \psi_{\alpha \lambda} \frac{S_{\mu \lambda}}{S_{0 \lambda}} \bar{\psi}_{\beta \lambda} = \sum_{\lambda \in \mathcal{E}} \psi_{\alpha \lambda} \gamma_{\lambda}^{(\mu)} \bar{\psi}_{\beta \lambda},$$  \hspace{1cm} (2.8a)

In the matrix form, this can be written as

$$n_{\mu} = \psi \gamma^{(\mu)} \psi^\dagger,$$  \hspace{1cm} (2.8b)

where $(\psi)^{\lambda}_{\alpha} = \psi_{\alpha \lambda}$ and $(n_{\mu})^{\beta}_{\alpha} = n_{\mu \alpha \beta}$. Here we denote by $\gamma^{(\mu)}_{\lambda}$ the generalized quantum dimension

$$\gamma^{(\lambda)} = \text{diag}(\gamma^{(\lambda)}_{\rho}) = \text{diag} \left( \frac{S_{\lambda \rho}}{S_{0 \rho}} \right)_{\rho \in P^+_k}. \hspace{1cm} (2.9)$$

Clearly, $n_{\mu \alpha \beta}$ takes non-negative integer values for the consistent boundary conditions \[ 14 \]. Moreover, $n_{0 \alpha \beta} = \delta_{\alpha \beta}$ since the vacuum is unique, and $n_{\mu}^{T \mu}$ is related with $n_{\mu}$ via

$$n_{\mu}^{T \mu} = n_{\mu}^{\dagger} = (\psi \gamma^{(\mu)} \psi^\dagger)^{\dagger} = \psi (\gamma^{(\mu)})^{\dagger} \psi^\dagger = \psi \gamma^{(\mu)} \psi^\dagger = n_{\bar{\mu}}. \hspace{1cm} (2.10)$$

We call the set of the consistency conditions the Cardy condition

$$n_{\mu \alpha \beta} \in \mathbb{Z}_{\geq 0}, \quad n_{0 \alpha \beta} = \delta_{\alpha \beta} \ (n_0 = 1), \quad n_{\mu \beta}^{\alpha} = n_{\mu \alpha \beta} \quad (n_{\mu}^{T \mu} = n_{\bar{\mu}}), \hspace{1cm} (2.11)$$

and the boundary states satisfying the Cardy condition the Cardy states. It should be noted that the Cardy condition is only a necessary condition for consistency. There are many non-physical NIM-reps that do not correspond to any modular invariant \[ 5 \] \ (e.g., the tadpole NIM-rep of $su(2)$ \[ 13 \]).

So far, the number of the independent Cardy states $|\mathcal{V}|$ is not specified. From now on, we assume that the number of the Cardy states is equal to the number of the Ishibashi states \[ 16 \]

$$|\mathcal{V}| = |\mathcal{E}| \quad \text{(assumption of completeness)}. \hspace{1cm} (2.12)$$

In other words, the boundary state coefficient $\psi$ is a square matrix. From the Cardy condition (2.11), $n_0 = \psi \psi^\dagger = 1$. For a square $\psi$, this means that $\psi$ is unitary. The situation is quite analogous to the Verlinde formula \[ 15 \]

$$\mathcal{N}_{\lambda \mu}^{\nu} = \sum_{\rho \in \text{Spec}(G)} \frac{S_{\lambda \rho} S_{\mu \rho}}{S_{0 \rho}} \bar{S}_{\nu \rho} = \sum_{\rho \in \text{Spec}(G)} S_{\mu \nu} \gamma^{(\lambda \rho)} \bar{S}_{\nu \rho}, \hspace{1cm} (2.13a)$$
where $N_{\lambda \mu \nu}$ is the fusion coefficient $(\lambda) \times (\mu) = \sum \nu N_{\lambda \mu \nu}(\nu)$. In the matrix form, this can be written as

$$N_{\lambda} = S \gamma(\lambda) S^\dagger,$$

(2.13b)

where $(N_{\lambda})_{\mu \nu} = N_{\lambda \mu \nu}$. From the associativity of the fusion algebra, one can show that $N_{\lambda}$ satisfies the fusion algebra

$$N_{\lambda} N_{\mu} = \sum_{\nu \in \text{Spec}(G)} N_{\lambda \mu \nu} N_{\nu},$$

(2.14)

which implies by the Verlinde formula that

$$\gamma(\lambda) \gamma(\mu) = \sum_{\nu \in \text{Spec}(G)} N_{\lambda \mu \nu} \gamma(\nu).$$

(2.15)

The generalized quantum dimension $\{\gamma(\lambda)\}$ is therefore a one-dimensional representation of the fusion algebra. If we use $\psi$ instead of $S$ in the Verlinde formula, we obtain $n_{\mu}$. Hence, $n_{\mu}$, as well as $N_{\mu}$, satisfies the fusion algebra

$$n_{\lambda} n_{\mu} = \sum_{\nu \in \text{Spec}(G)} N_{\lambda \mu \nu} n_{\nu}.$$

(2.16)

The Cardy condition (2.11) together with the assumption of completeness (2.12) implies that $\{n_{\lambda}\}$ forms a non-negative integer matrix representation (NIM-rep) of the fusion algebra.

For each set of the mutually consistent boundary states, we have a NIM-rep of the fusion algebra. However, the converse is in general not true. There are many ‘unphysical’ NIM-reps that do not correspond to any modular invariant [5]. The typical example is the tadpole NIM-rep $T_n$ of $su(2)_{2n-1}$ [4,3], which can be constructed by orbifolding the regular NIM-rep $A_{2n}$. The exponent $E(T_n)$ consists of only the even representations of $su(2)$ at level $2n - 1$. Hence, there is no modular invariant compatible with $E(T_n)$ since the level is odd. Although the spectrum of the diagonal modular invariant at level $2n - 1$ contains $E(T_n)$ as a subset, the overlap of the $T_n$ boundary states with the ordinary $A_{2n}$ yields the $su(2)$ character with irrational coefficients, which implies that the $T_n$ state is unphysical. This example shows that the Cardy condition is not a sufficient condition for consistency.

We give some examples of the Cardy states below.

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2.1 Untwisted states

Since the non-negative integer matrix $(N_{\lambda})_{\mu}^{\nu} = N_{\lambda \mu}^{\nu}$ satisfies the fusion algebra, $\{N_{\lambda}\}$ is a NIM-rep of the fusion algebra (regular representation). From the Verlinde formula (2.13), the corresponding diagonalization matrix $\psi$ is the modular transformation matrix $S$. The spectrum $E$ of the Ishibashi states coincides with Spec$(G)$. Hence the resulting Cardy states are those for the charge conjugation modular invariant. Since the $S$-matrix maps Spec$(G)$ to Spec$(G)$ itself, the label $\mathcal{V}$ of the Cardy states also coincides with Spec$(G)$. The Cardy states take the form

$$|\lambda\rangle = \sum_{\mu \in \text{Spec}(G)} S_{\lambda \mu} |\mu\rangle.$$  \hfill (2.17)

2.2 Twisted states

The simple Lie algebra $g$ has an outer automorphism $\omega$ for $g = A_l, D_l, E_6$ (see Table 1). Here, $r$ is the order of $\omega$ and we denote the representation of $g$ by its Dynkin label, namely, $\lambda = \lambda_1 \Lambda_1 + \cdots + \lambda_l \Lambda_l$.

We can use this outer automorphism $\omega$ of the horizontal subalgebra $g$ to twist the boundary condition of the current algebra $g^{(1)}$

$$J^a_n + \omega(J^{\bar{a}}_{-n}) = 0.$$ \hfill (2.18)

Since $\lambda \neq \omega(\lambda)$ for a generic representation $\lambda$, the spectrum $E$ of the Ishibashi states is

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Table 1: The diagram automorphism $\omega$ of the simple Lie algebra $g$. $r$ is the order of $\omega$ and $\{\lambda_1, \lambda_2, \cdots\}$ is the Dynkin label of the weight $\lambda$.

| $g$     | $\omega(\lambda)$ | $r$ |
|---------|--------------------|-----|
| $A_{2l}$ | $(\lambda_{2l}, \lambda_{2l-1}, \cdots, \lambda_1)$ | 2   |
| $A_{2l-1}$ | $(\lambda_{2l-1}, \lambda_{2l-2}, \cdots, \lambda_1)$ | 2   |
| $D_{l+1}$ | $(\lambda_1, \cdots, \lambda_{l-1}, \lambda_{l+1}, \lambda_l)$ | 2   |
| $E_6$    | $(\lambda_5, \lambda_4, \lambda_3, \lambda_2, \lambda_1, \lambda_6)$ | 2   |
| $D_4$    | $(\lambda_4, \lambda_2, \lambda_1, \lambda_3)$ | 3   |
restricted to
\[ E = P^k_+(g^{(1)}) = \{ \lambda \in P^k_+(g^{(1)}) \mid \omega(\lambda) = \lambda \}. \]  
(2.19)

(for simplicity, we consider only the charge-conjugation modular invariant). The Cardy states are labelled by the integrable representation of the twisted affine Lie algebra \( g^{(r)} \) associated with \( g \) and \( \omega \).

\[ \mathcal{V} = P^k_+(g^{(r)}). \]  
(2.20)

This can be understood as follows.

Let \( \{ |\lambda; \omega\rangle \mid \omega(\lambda) = \lambda \} \) be the Ishibashi states satisfying the boundary condition (2.18). The Cardy state \( |\alpha; \omega\rangle \) can be expressed in terms of \( |\lambda; \omega\rangle \)

\[ |\alpha; \omega\rangle = \sum_{\lambda \in \mathcal{E}} \psi_\alpha^\lambda |\lambda; \omega\rangle. \]  
(2.21)

Consider the annulus amplitude between \( |\alpha; \omega\rangle \) and the untwisted Cardy state \( |0\rangle = \sum_{\lambda \in \text{Spec}(\mathcal{G})} S_0\lambda |\lambda\rangle \)

\[ Z_{0, (\alpha; \omega)} = \langle 0| q^{H_c} |\alpha; \omega\rangle = \sum_{\lambda \in \mathcal{E}} S_0\lambda \psi_\alpha^\lambda \langle \langle \lambda| q^{H_c} |\lambda; \omega\rangle. \]  
(2.22)

In the open string channel, the boundary condition of the current \( J^a \) is twisted at the one end of the annulus that corresponds to \( |\alpha; \omega\rangle \). Hence, the current algebra in the open string channel is twisted to yield the twisted affine Lie algebra \( g^{(r)} \), and the annulus amplitude can be expressed in terms of the character of the current of \( g^{(r)} \). The modular transformation of the character of \( g^{(r)} \) is those for another twisted algebra \( \tilde{g}^{(r)} \) [17] (see Table 2). The overlap \( \langle \langle \lambda| q^{H_c} |\lambda; \omega\rangle \) of two Ishibashi states is therefore nothing but the character of

| Table 2: The modular transformation of the twisted affine Lie algebras |
|---------------------------------|
| \( g^{(r)} \) | \( A_2^{(2)} \) | \( A_{2l-1}^{(2)} \) | \( D_{2l+1}^{(2)} \) | \( E_6^{(2)} \) | \( D_4^{(3)} \) |
| \( \tilde{g}^{(r)} \) | \( A_2^{(2)} \) | \( D_{l+1}^{(2)} \) | \( A_{2l-1}^{(2)} \) | \( E_6^{(2)} \) | \( D_4^{(3)} \) |
In our normalization, we obtain \(^2\)

\[
\langle\lambda|\tilde{g}^H|\lambda;\omega\rangle = \frac{1}{S_{0\lambda}} \chi^\tilde{g}(\lambda;\omega)(-1/\tau) = \frac{1}{S_{0\lambda}} \sum_{\mu \in P^k_+(\tilde{g}(r))} \tilde{S}_{\lambda\mu} \chi^g(\mu;\omega)(\tau/\tau),
\]

(2.23)

where \(\tilde{S}\) is the modular transformation matrix between \(g(r)\) and \(\tilde{g}(r)\). \(\tilde{\lambda} \in P^k_+(\tilde{g}(r))\) is determined from \(\lambda \in P^k_{(+}(g(1))\) by comparing the modular anomaly. We display the concrete form of \(\tilde{\lambda}\) in Table 3.

Using this fact, the annulus amplitude (2.22) can be written in the form

\[
Z_{\emptyset,\alpha;\omega} = \sum_{\lambda \in P^k_+(\tilde{g}(r))} \sum_{\mu \in P^k_+(g(1))} \psi_{\alpha}^\lambda \tilde{S}_{\lambda\mu} \chi^g(\mu;\omega).
\]

(2.24)

For the consistent boundary states, the coefficient of the character \(\chi^g(\mu;\omega)\) should be non-negative integer. Clearly, this condition is satisfied by setting \(\psi_{\alpha}^\lambda = \tilde{S}_{\alpha\lambda}\), where \(\alpha \in P^k_+(g(1))\). Hence, we set

\[
|\alpha;\omega\rangle = \sum_{\lambda \in P^k_+(g(1))} \tilde{S}_{\alpha\lambda}|\lambda;\omega\rangle, \quad \alpha \in P^k_+(g(1)).
\]

(2.25)

The consistency with the symmetric states \(|\beta\rangle\) other than \(|0\rangle\) readily follows since \(|\beta\rangle\) can be obtained from \(|0\rangle\) by the fusion in the open string channel [14]. In order to see

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\(^2\)For \(A^{(2)}_{2l}\), the arguments of the characters should be slightly modified.
the mutual consistency of the twisted Cardy states, we consider the annulus amplitude between two twisted states

\[ Z_{(\alpha;\omega)(\beta;\omega)} = \sum_{\lambda \in P_{k,\omega}^{+}} \frac{S_{0}}{S_{\lambda}^{\lambda}} \tilde{S}_{\lambda} \tilde{S}_{\lambda}^{\mu} (\tilde{S}_{0}^{\dagger})_{\lambda \beta} \chi_{\mu} = \sum_{\mu} N_{\mu \alpha}^{\omega \beta} \chi_{\mu}. \] (2.26)

For the consistency of the states, the coefficients \( N_{\mu \alpha}^{\omega \beta} \) should be non-negative integer (a NIM-rep of the fusion algebra). One can see that this number also appears in the annulus amplitude between \(|\alpha\rangle\) and \(|\beta;\omega\rangle\)

\[ Z_{\alpha(\beta;\omega)} = \sum_{\lambda \in P_{k,\omega}^{+}(g^{(1)})} \sum_{\mu \in P_{k}^{+}(g^{(2)})} S_{\lambda \alpha} \tilde{S}_{\lambda} \tilde{S}_{\lambda}^{\mu} (\tilde{S}_{0}^{\dagger})_{\lambda \beta} \chi_{\mu} = \sum_{\lambda, \mu} N_{\alpha \mu}^{\omega \beta} \chi_{\mu}. \] (2.27)

Hence, the mutual consistency of the twisted states follows from the consistency between the symmetric and the twisted states.

2.3 \( u(1)_{k} \)

The \( u(1)_{k} \) chiral algebra with \( k \in \mathbb{Z} \) has \( 2k \) primary fields. We label them by \( m \in \mathbb{Z}/2k\mathbb{Z} \)

\[ \text{Spec}(u(1)_{k}) = \mathbb{Z}/2k\mathbb{Z} = \{ m = 0, 1, \cdots, 2k - 1 \}. \] (2.28)

The modular transformation matrix reads

\[ S_{mm'} = \frac{1}{\sqrt{2k}} e^{-\frac{\pi i}{4} \delta_{mm'}}, \] (2.29)

and the fusion algebra has the form \((m) \times (m') = (m + m')\).

Let us consider the twisted boundary states in this theory. The outer automorphism of \( u(1) \) is the charge conjugation \( \omega_{c} : m \to -m \). The representations self-conjugate under \( \omega_{c} \) are \( m = 0 \) and \( k \). Hence, we obtain

\[ \mathcal{E} = \{ 0, k \}. \] (2.30)

Correspondingly, we have two Cardy states denoted by \( \alpha = \pm \)

\[ \mathcal{V} = \{ +, - \}. \] (2.31)

The boundary state coefficient \( \psi \) reads

\[ \psi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \] (2.32)
3 Automorphisms of boundary states

Let $\text{Aut}(g)$ be the group of the outer automorphism of a current algebra $g = g^{(1)}$. $\text{Aut}(g)$ contains a normal abelian subgroup $\mathcal{O}(g)$ which is isomorphic to the center $Z(G)$ of the group $G$

$$\mathcal{O}(g) \simeq Z(G), \quad \mathcal{O}(g) \ni A \mapsto b(A) = e^{-2\pi i A(\Lambda_0)} \in Z(G). \quad (3.1)$$

Here $\Lambda_0$ is the 0-th fundamental weight of $g$ and $b$ is a group isomorphism

$$b(AA') = b(A)b(A'), \quad A, A' \in \mathcal{O}(g). \quad (3.2)$$

$b(A)$ is a multiple of the identity within an irreducible representation of $g$. We denote the eigenvalue of $b(A)$ in the representation $\lambda$ by $b_{\lambda}(A)$

$$b(A)|\lambda\rangle = b_{\lambda}(A)|\lambda\rangle, \quad b_{\lambda}(A) = e^{-2\pi i (A(\Lambda_0), \lambda)}, \quad \lambda \in P^k(g). \quad (3.3)$$

The modular transformation matrix $S$ intertwines $\mathcal{O}(g)$ with $Z(G)$

$$S_{\lambda\mu} = S_{\lambda\mu} b_{\mu}(A), \quad A \in \mathcal{O}(g), \quad (3.4a)$$

which can be written in the form

$$AS = S b(A), \quad A_{\lambda\mu} = \delta_{A_{\lambda\mu}}, \quad b(A) = \text{diag}(b_{\lambda}(A)). \quad (3.4b)$$

Setting $\lambda = 0$ in this equation, we obtain

$$b_{\mu}(A) = \frac{S_{A_{\lambda\mu}}^{0\mu}}{S_{0\mu}} = \gamma_{\mu}^{(A0)}. \quad (3.5)$$

Therefore $b_{\mu}(A)$ is nothing but the generalized quantum dimension. The outer automorphism $A \in \mathcal{O}(g)$ acts on the fusion algebra as

$$N_{A\lambda\mu}^{\nu} = N_{\lambda\mu}^{A\nu} = N_{A^{-1}\nu}^{\lambda\mu}, \quad (3.6a)$$

which follows from eq.(3.4) and the Verlinde formula (2.13). In the matrix form, this can be written as

$$N_{A\lambda} = AN_\lambda = N_\lambda A. \quad (3.6b)$$
Setting $\lambda = 0$ again, we obtain

$$A = AN_0 = N_{A_0}. \quad (3.7)$$

Hence, the action of the outer automorphism $A$ is equivalent to the fusion with $A_0$ (simple currents [20]). This is the ‘$S$-dual’ of eq. (3.5).

The outer automorphism $A \in O(g)$ naturally acts on the label of the Cardy states $\mathcal{V}$, since $A = N_{A_0}$. We use the same symbol $A$ for its realization on

$$A = n_{A_0} = \psi(\gamma_{A_0})\psi^\dagger = \psi b(A)\psi^\dagger. \quad (3.8)$$

In the component form,

$$A_{\alpha}^\beta = \sum_{\lambda \in \mathcal{E}} \psi_{\alpha}^{\lambda} b_{\lambda}(A) \bar{\psi}_{\beta}^{\lambda}. \quad (3.9)$$

We can rewrite this as

$$A\psi = \psi b(A), \quad (3.10a)$$

$$\psi_{A\alpha}^{\lambda} \equiv \sum_{\beta \in \mathcal{E}} A_{\alpha}^\beta \psi_{\beta}^{\lambda} = \psi_{\alpha}^{\lambda} b_{\lambda}(A), \quad (3.10b)$$

where we define $A\alpha \in \mathcal{V}$ by

$$A_{\alpha}^\beta = \delta_{A\alpha,\beta}. \quad (3.11)$$

There are elements of $O(g)$ that leave $\mathcal{V}$ invariant. They form a subgroup of $O(g)$, which we call the stabilizer of $\mathcal{V}$ and denote by $S(\mathcal{V})$

$$S(\mathcal{V}) = \{A \in O(g) \mid A\alpha = \alpha \text{ for any } \alpha \in \mathcal{V}\}$$

$$= \{A \in O(g) \mid b_{\lambda}(A) = 1 \text{ for any } \lambda \in \mathcal{E}\}. \quad (3.12)$$

The action on $\mathcal{V}$ is caused by the quotient group $O(\mathcal{V}) \equiv O(g)/S(\mathcal{V})$, which we call the automorphism group of $\mathcal{V}$

$$O(\mathcal{V}) = O(g)/S(\mathcal{V}). \quad (3.13)$$

From (3.4) and $b^T = b$, we obtain

$$b(A) = SA^T S^\dagger. \quad (3.14)$$

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We consider the counterpart of this in \( V \), namely,

\[
\tilde{b}(A) = \psi A^T \psi^*, \quad A \in \mathcal{O}(\mathcal{E}) \subset \mathcal{O}(g),
\]  

(3.15)

where \( \mathcal{O}(\mathcal{E}) \) is the group of outer automorphisms of \( \mathcal{E} \) defined as

\[
\mathcal{O}(\mathcal{E}) = \{ A \in \mathcal{O}(g) | A(\mathcal{E}) = \mathcal{E} \}. \tag{3.16}
\]

This restriction for \( A \) is necessary because the indices of \( \psi \) runs only \( \mathcal{E} \). It is not clear for the author whether \( \tilde{b}(A) \) is diagonal or not. We therefore assume that \( \tilde{b}(A) \) is diagonal. Actually, this holds for all the examples discussed later. Then the above equation reads

\[
\tilde{b}_\alpha(A) \delta_{\alpha\beta} = \sum_{\lambda,\mu \in \mathcal{E}} \psi^\lambda \langle A_\lambda \rangle \tilde{\psi}^\mu = \sum_{\mu \in \mathcal{E}} \psi^\lambda A^\mu \tilde{\psi}^\mu, \tag{3.17}
\]

or equivalently,

\[
\tilde{b}(A) \psi = \psi A^T, \quad \tilde{b}_\alpha(A) \psi^\lambda = \psi^\lambda A^\mu, \quad A \in \mathcal{O}(\mathcal{E}). \tag{3.18a}
\]

\[
\tilde{b}_\alpha(A) \psi^\lambda = \psi^\lambda A^\mu, \quad A \in \mathcal{O}(\mathcal{E}). \tag{3.18b}
\]

From the equations (3.10) and (3.18), we obtain the transformation properties of the Cardy states

\[
b(A)|\alpha\rangle = \sum_{\lambda \in \mathcal{E}} \psi^\lambda b_\lambda(A)|\lambda\rangle = \sum_{\lambda \in \mathcal{E}} \psi^\lambda A_\alpha^\lambda |\lambda\rangle = |A\alpha\rangle, \quad A \in \mathcal{O}(V), \tag{3.19a}
\]

\[
A|\alpha\rangle = \sum_{\lambda \in \mathcal{E}} \psi^\lambda A|\lambda\rangle = \sum_{\lambda,\mu \in \mathcal{E}} \psi^\lambda |\mu\rangle A_\mu^\lambda = \sum_{\lambda,\mu \in \mathcal{E}} \psi^\lambda |\mu\rangle \delta_{A_\mu,\lambda}
= \sum_{\mu \in \mathcal{E}} \psi^\mu A^\mu |\mu\rangle \sum_{\mu \in \mathcal{E}} \tilde{b}_\alpha(A) \psi^\mu |\mu\rangle = \tilde{b}_\alpha(A)|\alpha\rangle, \quad A \in \mathcal{O}(\mathcal{E}). \tag{3.19b}
\]

The center \( \tilde{b}(A) \in Z(G) \) induces a permutation of the Cardy states, which is an automorphism of \( V \). On the other hand, \( A \in \mathcal{O}(\mathcal{E}) \) measures the ‘charge’ (or the conjugacy class) of the Cardy states.

For a NIM-rep \( n_\lambda \) of the fusion algebra, there corresponds a graph whose vertices are labelled by the set \( \mathcal{V} \). We can identify the boundary states with the vertices of the graph. Then the automorphism group \( \mathcal{O}(V) \) is naturally interpreted as the automorphism of the graph, while \( \tilde{b}_\alpha \) represents a coloring of the graph.
3.1 Untwisted states

For the untwisted states, \( V = \mathcal{E} = \text{Spec}(G) \) and \( \psi = S \). The equations (3.10) and (3.18) reduce to eq.(3.4). The transformation property (3.19) of the Cardy states therefore reads

\[
b(A)|\alpha\rangle = |A\alpha\rangle, \quad A|\alpha\rangle = b_\alpha(A)|\alpha\rangle, \quad A \in \mathcal{O}(g).
\] (3.20)

3.2 Twisted states

For the twisted states, we have seen

\[
E = P_+^{k,\omega}(g^{(1)}) \simeq P_+^{k}(\tilde{g}^{(r)}), \quad V = P_+^{k}(g^{(r)}),
\]

\[
\psi_\alpha^\lambda = \tilde{S}_\alpha^\lambda.
\] (3.21)

From this form, it is natural to expect that

\[
\mathcal{O}(E) \simeq \mathcal{O}(\tilde{g}^{(r)}), \quad \mathcal{O}(V) \simeq \mathcal{O}(g^{(r)}).
\] (3.22)

We will show this is actually the case. We restrict ourselves to the case of \( g = A_{2l-1} \). The other \( g \)'s can be treated in the same way.

For \( g = A_{2l-1} \), \( g^{(r)} = A_{2l-1}^{(2)} \) and \( \tilde{g}^{(r)} = D_{l+1}^{(2)} \). The explicit form of \( E \) and \( V \) reads

\[
\mathcal{E} = P_+^{k,\omega}(A_{2l-1}^{(1)})
\]

\[
= \{(\lambda_1, \lambda_2, \ldots, \lambda_l, \lambda_2, \lambda_1) | 2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{l-1} + \lambda_l \leq k, \lambda_i \in \mathbb{Z}_{\geq 0} \}
\]

\[
\simeq P_+^{k}(D_{l+1}^{(2)}) = \{(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_l) | 2\tilde{\lambda}_1 + 2\tilde{\lambda}_2 + \cdots + 2\tilde{\lambda}_{l-1} + \tilde{\lambda}_l \leq k, \tilde{\lambda}_i \in \mathbb{Z}_{\geq 0} \},
\] (3.23)

\[
V = P_+^{k}(A_{2l-1}^{(2)}) = \{(\alpha_1, \alpha_2, \ldots, \alpha_l) | \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_l \leq k, \alpha_i \in \mathbb{Z}_{\geq 0} \}.
\]

First, note that

\[
\mathcal{O}(A_{2l-1}^{(1)}) = \{1, A, A^2, \ldots, A^{2l-1} \} \simeq \mathbb{Z}_{2l},
\] (3.24)

where the generator \( A \) acts on \( \lambda \) as

\[
A : (\lambda_1, \lambda_2, \ldots, \lambda_{2l-2}, \lambda_{2l-1}) \mapsto (\lambda_0, \lambda_1, \cdots, \lambda_{2l-2}),
\]

\[
\lambda_0 = k - (\lambda_1 + \cdots + \lambda_{2l-1}).
\] (3.25)

Clearly, the elements that leave \( \mathcal{E} \) invariant are only 1 and \( A^l \), hence

\[
\mathcal{O}(\mathcal{E}) = \{1, A^l\} \simeq \mathbb{Z}_2.
\] (3.26)
From eq. (3.23), one can see that $A^l$ induces the following action on $P^k(D^{(2)}_{l+1})$

$$A^l : (\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots, \tilde{\lambda}_{l-1}, \tilde{\lambda}_l) \mapsto (\tilde{\lambda}_{l-1}, \tilde{\lambda}_{l-2}, \cdots, \tilde{\lambda}_1, \tilde{\lambda}_0),$$

$$\tilde{\lambda}_0 = k - (2\tilde{\lambda}_1 + \cdots + 2\tilde{\lambda}_{l-1} + \tilde{\lambda}_l), \quad (3.27)$$

which is exactly the same as the action of the outer automorphism group $O(D^{(2)}_{l+1}) \cong Z_2$. Hence, we have verified the first equality of eq. (3.22), $O(E) \cong O(\tilde{g}^{(r)})$.

The center of $SU(2l)$ is also $Z_{2l}$, which is generated by $b(A)$. The eigenvalue $b_\lambda(A)$ of $b(A)$ on the representation $\lambda$ reads

$$b_\lambda(A) = \exp \left( -\frac{\pi i}{l} ((2l - 1)\lambda_1 + (2l - 2)\lambda_2 + \cdots + \lambda_{2l-1}) \right). \quad (3.28)$$

On $E$, this can be written as

$$b_\lambda(A) = \exp \left( -\frac{\pi i}{l} (2l\lambda_1 + 2l\lambda_2 + \cdots + 2l\lambda_{l-1} + l\lambda_l) \right) = (-1)^{\lambda_1}. \quad (3.29)$$

Hence, the stabilizer $S(V)$ consists of $\{1, A^2, \cdots, A^{2l-2}\}$ and $S(V) \cong Z_l$. The automorphism group $O(V)$ is therefore $O(V) = Z_{2l}/Z_l = \{1, A\} \cong Z_2$. One can identify this with the outer automorphism group of $A^{(2)}_{2l-1}$, which acts on $P^k(A^{(2)}_{2l-1})$ as

$$A : (\alpha_1, \alpha_2, \cdots, \alpha_{l-1}, \alpha_l) \mapsto (\alpha_0, \alpha_2, \cdots, \alpha_{l-1}, \alpha_l),$$

$$\alpha_0 = k - (\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_l). \quad (3.30)$$

Actually, from the formula (proved in Appendix)

$$\tilde{S}_{A\alpha,\lambda} = \tilde{S}_{\alpha\lambda}(-1)^{\lambda_1}, \quad A \in O(A^{(2)}_{2l-1}),$$

$$\tilde{S}_{A,\lambda} = (-1)^{\alpha_1 + 2\alpha_2 + \cdots + l\alpha_l} \tilde{S}_{\alpha,\lambda}, \quad \tilde{A} \in O(D^{(2)}_{l+1}), \quad (3.31a, b)$$

one can confirm that the action of $O(V)$ coincides with that of $O(A^{(2)}_{2l-1})$. Hence, we have verified the second equality of eq. (3.22), $O(V) \cong O(g^{(r)})$. Note that the above formula (3.31) also exhibits that $\tilde{b}(A)$ in eq. (3.18) is diagonal, namely, $\tilde{b}_\alpha(A) = (-1)^{\alpha_1 + 2\alpha_2 + \cdots + l\alpha_l}$.

### 3.3 $u(1)_k$

The modular transformation matrix of the $u(1)_k$ theory (2.29) has the following symmetry

$$S_{m+1,m'} = S_{mm'} e^{-\pi i m'}. \quad (3.32)$$
This is reminiscent of the relation (3.34), and we regard $A : m \mapsto m + 1$ as the generator of the automorphism group
\[ O(u(1)_k) = \{1, A, \cdots, A^{2^k-1}\} \simeq \mathbb{Z}_{2^k}. \] (3.33)

The 'center' is defined in the same way
\[ b_m(A) = e^{-\frac{\pi i}{k} m}. \] (3.34)

Let us consider the twisted Cardy states
\[ |\pm\rangle = \frac{1}{\sqrt{2}}(|0; \omega_c\rangle \pm |k; \omega_c\rangle). \] (3.35)

The automorphism group $O(\mathcal{E})$ is clearly \{1, $A^k$\} $\simeq \mathbb{Z}_2$, under which $|\pm\rangle$ transforms as
\[ A^k|\pm\rangle = \frac{1}{\sqrt{2}}(|k; \omega_c\rangle \pm |0; \omega_c\rangle) = \pm|\pm\rangle. \] (3.36)

Since $b_k(A) = -1$, the stabilizer $S(V)$ consists of \{1, $A$, $A^2$, $\cdots$, $A^{2^k-2}$\}. Hence, $O(V) = \mathbb{Z}_{2^k}/\mathbb{Z}_k = \{1, A\} \simeq \mathbb{Z}_2$, which acts on $|\pm\rangle$ as
\[ b(A)|\pm\rangle = \frac{1}{\sqrt{2}}(|0; \omega_c\rangle \mp |k; \omega_c\rangle) = |\mp\rangle. \] (3.37)

### 4 Twisted boundary states in coset theories

#### 4.1 Preliminaries

Corresponding to the algebra embedding $h \subset g$, a representation $\lambda$ of $g$ is decomposed in terms of the representations of $h$ as follows
\[ (\lambda) \mapsto \oplus_\mu (\lambda; \mu) \otimes (\mu), \quad \lambda \in \text{Spec}(G), \mu \in \text{Spec}(H). \] (4.1)

The spectrum of the $G/H$ coset theory is composed of all the possible combination $(\lambda; \mu)$
\[ \text{Spec}(G/H) = \{(\lambda; \mu) \mid \lambda \in \text{Spec}(G), \mu \in \text{Spec}(H), b_\lambda = b_\mu \} / (A\lambda; A\mu) \sim (\lambda; \mu). \] (4.2)
Here the relation \((A\lambda; A\mu) \sim (\lambda; \mu), A \in \mathcal{O}(h)\) is the field identification, \(^3\) and \(b_\lambda = b_\mu\) is the selection rule for the common center of \(G\) and \(H\). To be precise, we should write the projected weight as \(P\lambda\), instead of \(\lambda\), using the projection matrix \(P\). For simplicity, we omit this \(P\), since it is obvious from the context whether \(P\) should be appended or not.

In this paper, we restrict ourselves to the case that all the identification orbit have the same length \(N_0\)

\[ N_0 = |\{(A\lambda; A\mu) | A \in \mathcal{O}(h)\}| = |\mathcal{O}(h)|. \tag{4.3} \]

In particular, there is no fixed point in the field identification

\[(A\lambda; A\mu) \neq (\lambda; \mu), \text{ for any } \lambda \in \text{Spec}(G), \mu \in \text{Spec}(H), A \in \mathcal{O}(h). \tag{4.4} \]

The character of the coset theory is the branching function \(\chi_{(\lambda;\mu)}\) of the algebra embedding \(h \subset g\). From the branching rule (4.1), we obtain

\[ \chi^G_\lambda = \sum_{\mu, b_\lambda = b_\mu} \chi_{(\lambda;\mu)} \chi^H_\mu. \tag{4.5} \]

The modular transformation of the coset characters can be written as

\[ \chi_{(\lambda;\mu)}(-1/\tau) = \sum_{(\lambda';\mu') \in \text{Spec}(G/H)} S_{(\lambda;\mu)(\lambda';\mu')} \chi_{(\lambda';\mu')}(\tau), \tag{4.6} \]

\[ S_{(\lambda;\mu)(\lambda';\mu')} = N_0 S^G_{\lambda;\mu} S^H_{\lambda';\mu'}. \]

This \(S\)-matrix has several properties necessary for a consistent theory. First, \(S_{(\lambda;\mu)(\lambda';\mu')}\) does not depend on the representative of the field identification orbit. Namely,

\[ S_{(A\lambda; A\mu)(\lambda';\mu')} = N_0 S^G_{A\lambda,\lambda'} S^H_{A\mu,\mu'} = N_0 S^G_{\lambda,\lambda'} S^H_{\mu,\mu'} b_\lambda'(A)b_\mu'(A)^{-1} = S_{(\lambda;\mu)(\lambda';\mu')} . \tag{4.7} \]

Here we used the property \(^{3}\) for \(S^G\) and \(S^H\). The last equality follows from the

\(^3\)We do not consider the maverick cosets \([21,22]\), for which additional field identifications are necessary.
selection rule $b_\nu = b_\mu$. Next, most importantly, $S_{(\lambda\mu)(\nu';\mu')}$ is unitary

$$
\sum_{(\lambda';\mu')\in \text{Spec}(G/H)} S_{(\lambda\mu)(\nu';\mu')} \bar{S}_{(\lambda'';\mu'')(\lambda';\mu')} = \frac{1}{N_0} \sum_{\lambda', \mu'} \frac{1}{N_0} \sum_{A \in \mathcal{O}(h)} b_\lambda(A)b_\mu(A)^{-1} \times N_0^2 \bar{S}_{\lambda\mu'} \bar{S}_{\nu;\mu''} S_{\lambda';\mu'} \bar{S}_{\lambda'';\mu''}
$$

(4.8)

$$
\delta_{\lambda \mu} \delta_{\lambda' \mu'} = \delta_{(\lambda\mu)(\nu';\mu')}.
$$

Here we used our assumption of no fixed points to rewrite the sum

$$
\sum_{(\lambda';\mu')\in \text{Spec}(G/H)} \rightarrow \frac{1}{N_0} \sum_{\lambda', \mu'} \frac{1}{N_0} \sum_{A \in \mathcal{O}(h)} b_\lambda(A)b_\mu(A)^{-1}.
$$

(4.9)

The projection operator introduced above takes account of the selection rule. We parametrized the center of $H$ by the elements $A$ of $\mathcal{O}(h)$ using the isomorphism $Z(H) \simeq \mathcal{O}(h)$.

The fusion algebra of the coset theory can be obtained via the Verlinde formula

$$
\mathcal{N}_{(\lambda\mu)(\nu';\mu')}^{(\lambda'';\mu'')} = \sum_{(\rho, \sigma)\in \text{Spec}(G/H)} \frac{S_{(\lambda\mu)(\rho,\sigma)} S_{(\nu';\mu')(\rho,\sigma)} \bar{S}_{(\lambda'';\mu'')(\rho,\sigma)}}{\bar{S}_{(0,0)(\rho,\sigma)}}
$$

(4.10a)

$$
= \frac{1}{N_0} \sum_{\rho, \sigma} \frac{1}{N_0} \sum_{A \in \mathcal{O}(h)} b_\rho(A)b_\sigma(A)^{-1} \times N_0^2 \frac{S_{\lambda\mu'} S_{\nu;\mu''} \bar{S}_{\lambda'';\mu''} S_{\lambda';\mu'} \bar{S}_{\lambda'';\mu''}}{S_{\rho \sigma}}
$$

(4.10b)

$$
= \sum_{A \in \mathcal{O}(h)} \mathcal{N}_{A_{\lambda' \mu'}}^{A_{\lambda'' \mu''}}.
$$

In the matrix form,

$$
\mathcal{N}_{(\lambda\mu)} = \sum_{A \in \mathcal{O}(h)} \mathcal{N}_{A_{\lambda}} \otimes \mathcal{N}_{A_{\mu}}
$$

(4.10b)

where both the columns and the rows are restricted to $\text{Spec}(G/H)$. 17
4.2 Boundary states

The boundary condition of the coset theory $G/H$ follows from that of the current algebras $G$ and $H$. If we adopt the untwisted boundary condition (2.1) for $G$ and $H$, we have the untwisted boundary condition for $G/H$. The resulting boundary states are the untwisted Cardy states

$$|\lambda; \mu\rangle = \sum_{(\lambda'; \mu') \in \text{Spec}(G/H)} S_{(\lambda\mu)(\lambda'; \mu')} |\lambda'; \mu'\rangle$$, \hspace{1cm} (4.11)

where $|\lambda; \mu\rangle$ is the Ishibashi state for the primary field $(\lambda; \mu)$ normalized in the same way as before

$$\langle (\lambda; \mu) | \tilde{q}^H | (\lambda; \mu) \rangle = \frac{1}{S_{(0,0)(\lambda\mu)}(\lambda\mu)} \chi_{(\lambda\mu)}(-1/\tau).$$ \hspace{1cm} (4.12)

Suppose that the current algebra $G$ admits an automorphism $\omega$ of the horizontal algebra and that $\omega$ induces an automorphism of $H$. Then both the current algebras $G$ and $H$ can be twisted and we have the twisted boundary condition of the coset theory $G/H$. We have seen in the previous sections that there exists a NIM-rep of the fusion algebra for each set of the mutually consistent boundary states satisfying the Cardy condition. The regular NIM-rep $N_{\lambda}$ corresponds to the untwisted boundary states, while for the twisted states we have a non-trivial NIM-rep. We should find a non-trivial NIM-rep of the fusion algebra (4.10) for the twisted Cardy states in the coset theory.

Finding a NIM-rep is nothing but finding a diagonalization matrix $\psi$ of the fusion algebra. For the regular NIM-rep, $\psi$ coincides with the modular transformation matrix $S$, which is related with those of $G$ and $H$ as follows

$$S_{(\lambda\mu)(\lambda'; \mu')} = N_0 S^G_{\lambda\lambda'} \bar{S}^H_{\mu\mu'}.$$ \hspace{1cm} (4.13)

This form suggests the following expression for the twisted boundary states in the coset theory

$$\psi_{(\alpha; \beta)}^{(\lambda\mu)} = N \psi^G_\alpha \psi^H_\beta,$$ \hspace{1cm} (4.14)

where $\psi^G$ and $\psi^H$ are the boundary state coefficients for the twisted Cardy states in $G$ and $H$, respectively, and $N$ is an integer that divides $N_0$. We shall show that this actually realizes a NIM-rep of the fusion algebra (4.10) of the coset theory.
The label \((\alpha; \beta)\) of the boundary states is composed of those of the current algebra theories. However, not all the combination of \(\alpha \in \mathcal{V}^G\) and \(\beta \in \mathcal{V}^H\) is allowed, since \((\lambda; \mu)\) should belong to the spectrum (1.2) of the coset theory.

First, from the field identification \((A\lambda; A\mu) \sim (\lambda; \mu), A \in \mathcal{O}(h)\), it should be satisfied that
\[
\psi_{(\alpha;\beta)}^{(A\lambda; A\mu)} = \psi_{(\alpha;\beta)}^{(\lambda;\mu)}, \quad A \in \mathcal{O}(\mathcal{E}^H).
\] (4.15)
Here we restrict the identification group of the spectrum to \(\mathcal{O}(\mathcal{E}^H)\) since \(\psi^H\) is defined only in \(\mathcal{E}^H\). Correspondingly, the length of the identification orbit is shorter than \(N_0\). We set the integer \(N\) in (4.14) to the length of this orbit
\[
N = |\{(A\lambda; A\mu) | A \in \mathcal{O}(\mathcal{E}^H)\}|.
\] (4.16)
The requirement (4.15) implies that
\[
\psi_{(\alpha;\beta)}^{(\lambda;\mu)} = \psi_{(\alpha;\beta)}^{(A\lambda; A\mu)} = N\psi^G_{\alpha} \overline{\psi}^H_{\beta} A_{\mu} = N\psi^G_{\alpha} \overline{\psi}^H_{\beta} b_{\alpha}(A)b_{\beta}(A)^{-1} = \psi_{(\alpha;\beta)}^{(\lambda;\mu)} \tilde{b}_{\alpha}(A)\tilde{b}_{\beta}(A)^{-1},
\] (4.17)
where we used the relation (3.18) for \(\psi^G\) and \(\psi^H\). Hence we have to require \(\tilde{b}_{\alpha} = \tilde{b}_{\beta}\) for \((\alpha; \beta)\) to be a label of the boundary state.

Next, from the selection rule \(b_{\lambda} = b_{\mu}\) for \((\lambda; \mu)\), we obtain
\[
\psi_{(\alpha;\beta)}^{(\lambda;\mu)} = \psi_{(\alpha;\beta)}^{(\lambda;\mu)} b_{\lambda}(A)b_{\mu}(A)^{-1} = N\psi^G_{\alpha} \overline{\psi}^H_{\beta} \mu = \psi_{(A\alpha; A\beta)}^{(\lambda;\mu)},
\] (4.18)
where we used the relation (3.10). Therefore we should identify \((A\alpha; A\beta), A \in \mathcal{O}(\mathcal{V}^H)\), with \((\alpha; \beta)\). Together with the above result, we define the set \(\mathcal{V}^{G/H}\) of the labels of the twisted boundary states in the coset theory as follows
\[
\mathcal{V}^{G/H} = \{(\alpha; \beta) | \alpha \in \mathcal{V}^G, \beta \in \mathcal{V}^H, \tilde{b}_{\alpha} = \tilde{b}_{\beta} \}/(A\alpha; A\beta) \sim (\alpha; \beta).
\] (4.19)
We can see the structure of \(\mathcal{V}^{G/H}\) is exactly parallel to that of \(\text{Spec}(G/H)\). Namely, we have the counterpart of the field identification and the selection rule in \(\mathcal{V}^{G/H}\). We call the identification \((A\alpha; A\beta) \sim (\alpha; \beta)\) and the selection rule \(\tilde{b}_{\alpha} = \tilde{b}_{\beta}\) for the boundary states the brane identification and the brane selection rule, respectively.
With this definition at hand, we can check that the boundary state coefficients (4.14) actually realizes a NIM-rep of the fusion algebra. In order to do that, we restrict ourselves to the case that the length of the identification orbit \( \{(A\alpha; A\beta) | A \in \mathcal{O}(V^H)\} \) is also given by \( N (4.16) \). This is not so restrictive and holds for all the examples discussed below. Let us first check the unitarity of \( \psi_{(\alpha; \beta)}^{(\lambda; \mu)} \)

\[
\sum_{(\lambda; \mu) \in E_{G/H}} \psi_{(\alpha; \beta)}^{(\lambda; \mu)} \bar{\psi}_{(\alpha'; \beta')}^{(\lambda; \mu)} = \sum_{(\lambda; \mu)} N^2 \overline{\psi}^G_{\alpha} \overline{\psi}^G_{\beta} \psi^G_{\lambda} \psi^G_{\mu} \\
= \frac{1}{N} \sum_{\lambda, \mu} \left( \frac{1}{N} \sum_{A \in \mathcal{O}(V^H)} b_{\lambda}(A) b_{\mu}(A)^{-1} \right) \\
\times N^2 \overline{\psi}^G_{\alpha} \overline{\psi}^G_{\beta} \psi^G_{\lambda} \psi^G_{\mu} \\
= \sum_{A \in \mathcal{O}(V^H)} \sum_{\lambda, \mu} \overline{\psi}^G_{\alpha \lambda} \overline{\psi}^G_{\beta \lambda} \psi^G_{\mu} \\
= \sum_{A \in \mathcal{O}(V^H)} \delta_{A\alpha, \alpha'} \delta_{A\beta, \beta'}.
\]

(4.20)

From this calculation, one can see that \( \psi_{(\alpha; \beta)}^{(\lambda; \mu)} \) is unitary unless there are fixed points in the brane identification \( (A\alpha; A\beta) \sim (\alpha; \beta) \). If, for example, there is a fixed point \( A\alpha = \alpha, A\beta = \beta, \) for \( A^2 = 1 \), we have the result \( \sum_{(\lambda; \mu)} \psi_{(\alpha; \beta)}^{(\lambda; \mu)} \bar{\psi}_{(\alpha'; \beta')}^{(\lambda; \mu)} = 2 \) instead of 1. This is the situation familiar in the field identification of the coset theory and we need some resolution of the fixed point in order to have a consistent theory [22].

The NIM-rep associated with \( \psi_{(\alpha; \beta)}^{(\lambda; \mu)} \) can be obtained in the same way as above

\[
n_{(\lambda; \mu)(\alpha; \beta)}^{(\alpha'; \beta')} = \sum_{(\rho; \sigma) \in E_{G/H}} \psi_{(\alpha; \beta)}^{(\rho; \sigma)} S_{(\lambda; \mu)(\rho; \sigma)} \psi_{(\alpha'; \beta')}^{(\rho; \sigma)} \\
= \frac{1}{N} \sum_{\rho, \sigma} \left( \frac{1}{N} \sum_{A \in \mathcal{O}(V^H)} b_{\rho}(A) b_{\sigma}(A)^{-1} \right) \\
\times N^2 \psi_{G\rho}^A \psi_{G\rho}^A \psi_{H\sigma}^A \psi_{H\sigma}^A \\
= \sum_{A \in \mathcal{O}(V^H)} n_{A\lambda, \alpha}^G n_{A\mu, \beta}^H.
\]

(4.21a)

In the matrix form,

\[
n_{(\lambda; \mu)} = \sum_{A \in \mathcal{O}(V^H)} n_{A\lambda, \alpha}^G \otimes n_{A\mu, \beta}^H.
\]

(4.21b)

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where both the columns and the rows are restricted to $V^{G/H}$. Since the components of $n^G$ and $n^H$ are non-negative integers, $n_{(\lambda, \mu)}$ is also non-negative integer matrix. Hence $n_{(\lambda, \mu)}$ is a NIM-rep of the fusion algebra, if there is no fixed point in the brane identification.

5 Examples

In this section, we apply the methods developed in the previous section to obtain the twisted boundary states in various coset theories.

5.1 $su(3)_k \oplus su(3)_l/su(3)_{k+l}$

The diagonal coset of $su(3)$ is the simplest example that admits the twisted boundary conditions. We consider the case of $su(3)_1 \oplus su(3)_1/su(3)_2$, although our methods can be applied to the other levels.

Let us start with the twisted boundary states in the $su(3)$ theory at level $k$. The automorphism $\omega$ of the horizontal subalgebra acts on the weight of $su(3)$ as follows

$$\omega : (\lambda_1, \lambda_2) \mapsto (\lambda_2, \lambda_1). \quad (5.1)$$

The spectrum $\mathcal{E}$ invariant under $\omega$ reads

$$\mathcal{E} = P^k_+ \omega(A_1^{(1)}) = \{(\lambda_1, \lambda_1) \mid 2\lambda_1 \leq k, \lambda_1 \in \mathbb{Z}_{\geq 0}\}. \quad (5.2)$$

The boundary states are labelled by the integrable representation $\alpha$ of the twisted affine Lie algebra $A_2^{(2)}$

$$\mathcal{V} = P^k_+(A_2^{(2)}) = \{\alpha = (\alpha_1) \mid 2\alpha_1 \leq k, \alpha \in \mathbb{Z}_{\geq 0}\}, \quad (5.3)$$

and take the form

$$|\alpha; \omega\rangle = \sum_{\lambda \in \mathcal{E}} \tilde{S}_{\alpha \lambda} |\lambda; \omega\rangle. \quad (5.4)$$

Here $\tilde{S}$ is the modular transformation matrix of $A_2^{(2)}$

$$\tilde{S}_{\lambda \mu} = \frac{2}{\sqrt{k + 3}} \sin \left(\frac{2\pi}{k + 3}(\lambda_1 + 1)(\mu_1 + 1)\right), \quad \lambda, \mu \in P^k_+(A_2^{(2)}), \quad (5.5)$$
and $\tilde{\lambda}$ is defined as $\tilde{\lambda}_1 = \lambda_1$.

For $k = 1$, $\mathcal{E} = \{(0, 0)\}$ and $\mathcal{V} = P^{k=1}(A_2^{(2)}) = \{(0)\}$. Hence there exists only one twisted state

$$|0; \omega\rangle = \tilde{S}_{00}|(0, 0); \omega\rangle = |(0, 0); \omega\rangle.$$  \hspace{1cm} (5.6)

For $k = 2$, $\mathcal{E} = \{(0, 0), (1, 1)\}$ and $\mathcal{V} = P^{k=2}(A_2^{(2)}) = \{(0), (1)\}$. We have two twisted states

$$|\alpha; \omega\rangle = \sum_{\lambda=0,1} \tilde{S}_{\alpha\lambda}|(\lambda, \lambda); \omega\rangle, \quad \alpha = 0, 1,$$  \hspace{1cm} (5.7)

where $\tilde{S}$ takes the form

$$\tilde{S} = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin \frac{2\pi}{5} & \sin \frac{\pi}{5} \\ \sin \frac{\pi}{5} & -\sin \frac{2\pi}{5} \end{pmatrix}.$$  \hspace{1cm} (5.8)

Since the diagonal action of $\omega$ on $su(3)_1 + su(3)_1$ induces the automorphism of $su(3)_2 \subset su(3)_1 + su(3)_1$, we have the twisted boundary condition in the coset theory $su(3)_1 + su(3)_1/su(3)_2$. Since $A_2^{(2)}$ has no outer automorphism, $\mathcal{O}(\mathcal{E}) = \mathcal{O}(\mathcal{V}) = \{1\}$. Hence both the brane identification and the brane selection rule is trivial. We therefore obtain two twisted boundary states in the coset theory

$$|(0, 0; 0); \omega\rangle = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin \frac{2\pi}{5} & \sin \frac{\pi}{5} \\ \sin \frac{\pi}{5} & -\sin \frac{2\pi}{5} \end{pmatrix},$$  \hspace{1cm} (5.9)

$$|(0, 0; 1); \omega\rangle = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin \frac{2\pi}{5} & \sin \frac{\pi}{5} \\ \sin \frac{\pi}{5} & -\sin \frac{2\pi}{5} \end{pmatrix}.$$  \hspace{1cm} (5.10)

### 5.2 $su(4)_k \oplus su(4)_l/su(4)_{k+l}$

The diagonal coset of $su(4) = A_3$ can be treated in the same way as $su(3)$.

The automorphism $\omega$ acts on the weight of $su(4)$ as

$$\omega : (\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_3, \lambda_2, \lambda_1).$$  \hspace{1cm} (5.11)

Hence the spectrum $\mathcal{E}$ reads

$$\mathcal{E} = P^k\omega(A_3^{(1)}) = \{(\lambda_1, \lambda_2, \lambda_1) \mid 2\lambda_1 + \lambda_2 \leq k; \lambda_i \in \mathbb{Z}_{\geq 0}\}.$$  \hspace{1cm} (5.12)

The boundary states are labelled by the integrable representation $\alpha$ of the twisted affine Lie algebra $A_3^{(2)}$

$$\mathcal{V} = P^k(A_3^{(2)}) = \{\alpha = (\alpha_1, \alpha_2) \mid \alpha_1 + 2\alpha_2 \leq k, \alpha \in \mathbb{Z}_{\geq 0}\}.$$  \hspace{1cm} (5.13)
and take the form

$$|\alpha; \omega\rangle = \sum_{\lambda \in \mathcal{E}} \tilde{S}_{\alpha \lambda} |\lambda; \omega\rangle.$$  \hspace{1cm} (5.13)

Here $\tilde{S}$ is the modular transformation matrix between $A_3^{(2)}$ and $D_3^{(2)} \simeq A_3^{(2)}$.

For $k = 1$, $\mathcal{E} = \{(0,0,0), (0,1,0)\}$ and $\mathcal{V} = \{(0,0), (1,0)\}$. Hence we have two twisted boundary states, for which the coefficients read

$$\tilde{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \hspace{1cm} (5.14)$$

For $k = 2$, $\mathcal{E} = \{(0,0,0), (0,2,0), (1,0,1), (0,1,0)\}$ and $\mathcal{V} = \{(0,0), (2,0), (0,1), (1,0)\}$. We therefore have four twisted boundary states, for which the coefficients read

$$\tilde{S} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & 1 & \sqrt{3} \\ 1 & 1 & 1 & -\sqrt{3} \\ 1 & 1 & -2 & 0 \\ -\sqrt{3} & -\sqrt{3} & 0 & 0 \end{pmatrix}. \hspace{1cm} (5.15)$$

Since $\mathcal{O}(A_3^{(2)}) = \{1, A\} \simeq \mathbb{Z}_2$, we need the brane identification in the coset theory $su(4)_1 \oplus su(4)_1 / su(4)_2$. The length of the identification orbit is 2. The generator $A$ of $\mathcal{O}(A_3^{(2)})$ acts on $\mathcal{V}$ as

$$A : (\alpha_0, \alpha_2) \mapsto (\alpha_0, 2\alpha_2), \quad \alpha_0 = k - (\alpha_1 + 2\alpha_2). \hspace{1cm} (5.16)$$

On the other hand, $\mathcal{O}(D_3^{(2)}) = \{1, \tilde{A}\} \simeq \mathbb{Z}_2$ acts on $\mathcal{E}$ as

$$\tilde{A} : (\lambda_0, \lambda_2, \lambda_1) \mapsto (\lambda_1, \lambda_0, \lambda_1), \quad \lambda_0 = k - (2\lambda_1 + \lambda_2). \hspace{1cm} (5.17)$$

From the formula (3.31), we obtain

$$\tilde{b}_a(\tilde{A}) = (-1)^{\alpha_1}. \hspace{1cm} (5.18)$$

Putting these facts together, we can write down the set $\mathcal{V}^{G/H}$ of the label of the twisted boundary states for $su(4)_1 \oplus su(4)_1 / su(4)_2$

$$\mathcal{V}^{G/H} = \{((0,0), (0,0); (0,0)), ((0,0), (0,0); (2,0)), ((0,0), (0,0); (0,1)), ((0,0), (1,0); (1,0))\}. \hspace{1cm} (5.19)$$

The boundary state coefficients can be calculated by the formula (4.14) and coincide with $\tilde{S}$ for $k = 2$ (5.15).
5.3 \( su(2)_k/u(1)_k \)

The \( su(2)_k/u(1)_k \) parafermion theory \((PF_k)\) is the simplest example including the \( u(1) \) factor. This theory is equivalent with the \( su(k)_1 \oplus su(k)_1/su(k)_2 \) theory. Therefore we can check the validity of our procedure by comparing the result with that obtained above.

The spectrum of the parafermion theory reads

\[
\text{Spec}(PF_k) = \left\{ (l;m) \mid l = 0,1,\cdots,k, m \in \mathbb{Z}/2k\mathbb{Z}, l \equiv m \mod 2 \right\}/(k - l;m + k) \sim (l;m),
\]

(5.20)

where \( l \in P^k(A_1^{(1)}) \) stands for the integrable representation of \( su(2)_k \) while \( m \in \text{Spec}(u(1)_k) \) is the irreducible representation of \( u(1)_k \).

Although the charge conjugation \( \omega_c \) is an inner automorphism in \( su(2) \), it induces an outer automorphism \( \omega_c \) of \( u(1) \subset su(2) \). Hence \( \omega_c \) is an outer automorphism of the coset theory and we obtain the boundary states twisted by \( \omega_c \) \[6\]. Since \( \omega_c \) is inner in \( su(2) \), we use the untwisted boundary states \(|l\rangle\) for the \( su(2) \) sector \[4\]

\[
|l\rangle = \sum_{l'} S^{su(2)}_{ll'} |l'\rangle,
\]

(5.21)

where the modular transformation matrix reads

\[
S^{su(2)}_{ll'} = \sqrt{\frac{2}{k + 2}} \sin \left( \frac{\pi}{k + 2}(l + 1)(l' + 1) \right).
\]

(5.22)

The \( u(1) \) part is described by the twisted boundary states \(|\pm\rangle\) \[33\].

The brane identification and the brane selection rule are applied in the same way as the previous examples. However, there is a subtlety for \( k \in 2\mathbb{Z}_{\geq 0} \). Since the center \( b \) of \( SU(2) \) acts on \( m \in \text{Spec}(u(1)_k) \) as \((-1)^m\), the Ishibashi state \(|m = k;\omega_c\rangle\) transforms as \( b|m = k;\omega_c\rangle = (-1)^k|m = k;\omega_c\rangle\). For odd \( k \), this induces the automorphism of the boundary states \(|\pm\rangle \rightarrow |\mp\rangle\). For even \( k \), however, the action of \( b \) leaves \(|\pm\rangle \) invariant and the automorphism group \( O(\mathcal{V}^{u(1)}) \) is trivial (the stabilizer \( S(\mathcal{V}^{u(1)}) \) coincides with \( O(su(2)_k) \simeq \mathbb{Z}_2 \)). The set \( \mathcal{V}^{PF}_k \) of the label of the twisted boundary states in the parafermion theory

\[\text{To be precise, we have to twist the boundary states by the charge conjugation. We omit it since it does not affect the coefficient } S^{su(2)} \text{ of the boundary states.}\]
therefore reads

\[ \mathcal{V}^{P_{\text{odd}} \ k} = \{(l; (-1)^l) \mid l = 0, 1, \cdots, k/(k - l); + \} \sim (l; -) \]

\[ = \{(l; (-1)^l) \mid l = 0, 1, \cdots, (k - 1)/2 \} \quad \text{for odd } k \] (5.23a)

and

\[ \mathcal{V}^{P_{\text{even}} \ k} = \{(l; (-1)^l) \mid l = 0, 1, \cdots, k \} \sim (l; \pm) \]

\[ = \{(l; (-1)^l) \mid l = 0, 1, \cdots, k/2 \} \quad \text{for even } k \] (5.23b)

The set \( \mathcal{V}^{P_k} \) appears to have the same structure irrespective of the parity of \( k \). However, for even \( k \), \( (l; (-1)^l) \sim (k - l; (-1)^l) \) and we have a fixed point \( (k/2; (-1)^{k/2}) \) in the brane identification. In order to have a complete set of the boundary states, we have to resolve the fixed point. This is possible because we have one additional Ishibashi state \( |(k/2; k/2); \omega_c\rangle \rangle \) for even \( k \) and we obtain the resolved set

\[ \tilde{\mathcal{V}}^{P_{\text{even}} \ k} = \{(l; (-1)^l) \mid l = 0, 1, \cdots, k/2 - 1 \} \cup \{(k/2; (-1)^{k/2})_{\pm}\}. \] (5.23c)

The boundary state coefficients follow from the formula (4.14). For the ordinary states, we obtain

\[ |(l; (-1)^l); \omega_c\rangle = \sum_{l' \text{ even}} \sqrt{2} S^{su(2)}_{ll'} |(l'; 0); \omega_c\rangle\rangle, \quad l < \frac{k}{2}; \] (5.24a)

while for the ‘fractional’ states we have

\[ |(k/2; (-1)^{k/2})_{\pm}; \omega_c\rangle = \sum_{l' \text{ even}} \frac{1}{\sqrt{2}} S^{su(2)}_{ll'} |(l'; 0); \omega_c\rangle\rangle \pm \frac{1}{\sqrt{2}} |(k/2; k/2); \omega_c\rangle\rangle. \] (5.24b)

This reproduces the result obtained in [6].

Let us calculate the explicit form of the boundary states for \( k = 3, 4 \), and compare it with the results for the diagonal coset. For \( k = 3 \), \( \mathcal{E} = \{(0; 0), (2; 0)\} \) and \( \mathcal{V} = \{(0; +), (1; -)\} \). In this basis, the boundary state coefficient \( \psi \) reads

\[ \psi = \frac{2}{\sqrt{5}} \left( \frac{\sin \frac{\pi}{5}}{\sin \frac{\pi}{5}} \sin \frac{2\pi}{5} - \frac{2\pi}{5} \right). \] (5.25)

This coincides with the result [3,9] for the \( su(3) \) diagonal coset after the identification \( |(0, 0; (1, 1))\rangle = -|((2, 0))\rangle \), \( |(0, 0; 0)\rangle = |(2; 0)\rangle \) and \( |(0, 0; 1)\rangle = |((0; 0))\rangle \).

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For $k = 4$, $\mathcal{E} = \{(0;0), (2;0), (4;0), (2;2)\}$ and $\mathcal{V} = \{(0;+), (1;−), (2;+)_\pm\}$. The boundary state coefficient reads

$$\psi = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 2 & 1 & 0 \\ \sqrt{3} & 0 & -\sqrt{3} & 0 \\ 1 & -1 & 1 & \sqrt{3} \\ 1 & -1 & 1 & -\sqrt{3} \end{pmatrix}, \quad (5.26)$$

which again coincides with the result (5.15) for the $su(4)$ diagonal coset after an appropriate identification of the states.

6 Summary

In this paper, we have developed the method for constructing the twisted boundary states in the $G/H$ coset conformal field theory. In the way analogous to the field identification and the selection rule of the coset theory, we introduce the notion of the brane identification and the brane selection rule which act on the set of the boundary states. We have shown that the twisted boundary states of the $G/H$ theory follow from those of the $G$ and the $H$ theories making use of these rules. As a check of our procedure, we have treated in detail the $su(n)_1 \oplus su(n)_1/su(n)_2$ theory and the $su(2)_k/u(1)_k$ parafermion theory, which are equivalent with each other, and have obtained the consistent results. Also, we have seen that our boundary states for the parafermion theory reproduces the results obtained in [6].

In this paper, we have restricted ourselves to the charge-conjugation (or the diagonal) modular invariant. It is interesting to extend our analysis to other non-trivial modular invariants. The minimal models have the description as the coset theory, namely, $su(2)_k \oplus su(2)_1/su(2)_{k+1}$. One can easily verify that our procedure, in particular the formula (4.14), yields all the boundary states of the minimal models obtained in [3], by appropriately extending the brane identification and the selection rule to the $D$ and the $E$ type modular invariants. The related problem is the issue of the unphysical NIM-reps. We can formally construct a NIM-rep of the $G/H$ theory starting from an unphysical NIM-rep of the $G$ (or $H$) theory. It is interesting to determine whether the resulting NIM-rep is physical or not, although it is likely to be unphysical.
It is also interesting to apply our method to the super coset theories, especially the Kazama-Suzuki models [24].

We have seen there are fixed points in the brane identification, and we should resolve them to obtain the consistent theory. This phenomena is the brane version of the field identification fixed points. Hence, in order to have a deep understanding of this, it will be necessary to extend our analysis to the case of the coset theory with the field identification fixed points.

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A Modular transformation matrix of \((A_{2l-1}^{(2)}, D_{l+1}^{(2)})\)

In this Appendix, we show the transformation property of the modular transformation matrix between \(A_{2l-1}^{(2)}\) and \(D_{l+1}^{(2)}\) under the action of the outer automorphism group of the algebras. The derivation is exactly parallel to the case of the untwisted affine Lie algebras (see, for example, §14.6 of [23]).

The modular transformation of the characters of the twisted affine Lie algebra \(A_{2l-1}^{(2)}\) gives rise to those of \(D_{l+1}^{(2)}\), and vice versa. The integrable representations of these algebras at level \(k\) are labelled as follows

\[
\begin{align*}
P^k_{+}(A_{2l-1}^{(2)}) &= \{ (\lambda_1, \lambda_2, \cdots, \lambda_l) \mid \lambda_1 + 2\lambda_2 + \cdots + 2\lambda_l \leq k; \lambda_i \in \mathbb{Z}_{\geq 0} \}, \\
\lambda &= (\lambda_1, \lambda_2, \cdots, \lambda_l).
\end{align*}
\]

The modular transformation matrix \(\tilde{S}_{\lambda\mu}\) for \(\lambda \in P^k_{+}(A_{2l-1}^{(2)}), \mu \in P^k_{+}(D_{l+1}^{(2)})\), takes the form

\[
\tilde{S}_{\lambda\mu} = \frac{j^2}{\sqrt{2}(k+2l)^{\frac{1}{2}}} \sum_{w \in W(C_l)} \epsilon(w) e^{-\frac{2\pi i}{k+2l}(w(\bar{\lambda} + \bar{\rho}), \bar{\mu} + \bar{\rho})}.
\]

Here the sum is taken over the Weyl group of \(C_l\), which is the horizontal subalgebra of \(A_{2l-1}^{(2)}\). We denote by \(\bar{\lambda}\) the finite part of \(\lambda\), which is expressed by the fundamental weights \(\Lambda_i\) of \(C_l\) as follows

\[
\bar{\lambda} = \lambda_1\Lambda_1 + \cdots + \lambda_l\Lambda_l.
\]

\(\bar{\mu}\) is a weight of \(C_l\) determined from \(\mu\) via

\[
\bar{\mu} = 2\mu_1\Lambda_1 + \cdots + 2\mu_{l-1}\Lambda_{l-1} + \mu_l\Lambda_l.
\]

The Weyl vectors, \(\bar{\rho}\) and \(\bar{\rho}\), are defined as

\[
\bar{\rho} = \Lambda_1 + \cdots + \Lambda_l, \\
\bar{\rho} = 2\Lambda_1 + \cdots + 2\Lambda_{l-1} + \Lambda_l.
\]

The outer automorphism group of \(A_{2l-1}^{(2)}\) is \(\mathbb{Z}_2\). The generator \(A \in O(A_{2l-1}^{(2)})\) acts on \(\lambda \in P^k_{+}(A_{2l-1}^{(2)})\) as

\[
A : (\lambda_1, \lambda_2, \cdots, \lambda_l) \mapsto (\lambda_0, \lambda_2, \cdots, \lambda_l), \\
\lambda_0 = k - (\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_l).
\]
One can show this can be written in the form
\[ \overline{A \lambda} = k \Lambda_1 + w_A(\bar{\lambda}), \] (A.7)
where \( w_A \in W \) is an element of the Weyl group for which \( \epsilon(w_A) = -1 \). Applying this formula to \( \lambda + \rho \), we obtain
\[ \overline{A \lambda + \rho} = (k + 2l) \Lambda_1 + w_A(\bar{\lambda} + \bar{\rho}), \] (A.8)
where we used \( A \rho = \rho \). Substituting this to (A.2) yields
\[ \tilde{S}_{A \lambda, \mu} = \frac{i^{l^2}}{\sqrt{2}} (k + 2l)^{-\frac{l}{2}} \sum_{w \in W(C_1)} \epsilon(w) e^{-\frac{2\pi i}{k+2l}(w \rho \lambda A, \bar{\mu} + \bar{\rho})} e^{-2\pi i (w \Lambda_1, \bar{\mu} + \bar{\rho})} \cdot \epsilon(w_A) e^{-2\pi i \mu l}. \] (A.9)

From the definition of \( \tilde{\mu} \), it can be shown that
\[ (w \Lambda_1, \tilde{\mu}) = \frac{\mu_l}{2} \mod \mathbb{Z} \text{ for any } w \in W, \] (A.10)
and we obtain the result
\[ \tilde{S}_{A \lambda, \mu} = \frac{i^{l^2}}{\sqrt{2}} (k + 2l)^{-\frac{l}{2}} \sum_{w \in W(C_1)} \epsilon(w) e^{-\frac{2\pi i}{k+2l}(w \rho \lambda A, \bar{\mu} + \bar{\rho})} \cdot \epsilon(w_A) e^{-2\pi i \mu l}. \] (A.11)

The outer automorphism of \( D^{(2)}_{l+1} \) is \( \mathbb{Z}_2 \) and generated by \( \tilde{A} \)
\[ \tilde{A} : (\mu_1, \cdots, \mu_{l-1}, \mu_l) \mapsto (\mu_{l-1}, \mu_{l-2}, \cdots, \mu_1, \mu_0), \]
\[ \mu_0 = k - (2\mu_1 + \cdots + 2\mu_{l-1} + \mu_l). \] (A.12)
The action on \( \tilde{S} \) can be calculated in the same way as above. The result reads
\[ \tilde{S}_{A \lambda, \mu} = (-1)^{\lambda_1 + 2\lambda_2 + \cdots + l \lambda_l} \tilde{S}_{A \mu}. \] (A.13)

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