Quantum Symmetric Pairs and the Reflection Equation

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August 21, 2006

Abstract

It is shown that central elements in G. Letzter’s quantum group analogs of symmetric pairs lead to solutions of the reflection equation. This clarifies the relation between Letzter’s approach to quantum symmetric pairs and the approach taken by M. Noumi, T. Sugitani, and M. Dijkhuizen.

We develop general tools to show that a Noumi-Sugitani-Dijkhuizen type construction of quantum symmetric pairs can be performed preserving spherical representations from the classical situation. These tools apply to the symmetric pair $F_{II}$ and to all symmetric pairs which correspond to an automorphism of the underlying Dynkin diagram. Hence Noumi-Sugitani-Dijkhuizen type constructions with desirable properties are possible for various symmetric pairs for exceptional Lie algebras.

1 Introduction

The reflection equation with spectral parameter first appeared in I. V. Cherednik’s work on factorizing scattering on the half line [Che84]. The present paper is devoted to the reflection equation without spectral parameter [KSS93] and its relation to quantum group analogs of symmetric pairs.

Let $\theta$ be an involutive automorphism of a complex semisimple Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}^\theta$ denote the fixed Lie subalgebra. The pair $(\mathfrak{g}, \mathfrak{g}^\theta)$ is a classical (infinitesimal) symmetric pair.

*Supported by the German Research Foundation (DFG)
In a series of papers \cite{Nou96, NS95, Dij96, DN98, DS99}, and references therein, analogs of \(U(\mathfrak{g}^\theta)\) inside the simply connected quantum universal enveloping algebra \(\tilde{U}_q(\mathfrak{g})\) were constructed for all classical symmetric pairs. The constructions were done case by case and depend on solutions of the reflection equation. The solutions of the reflection equation were given explicitly.

Taking a unifying approach, G. Letzter classified all maximal left coideal subalgebras \(\tilde{B} \subseteq \tilde{U}_q(\mathfrak{g})\) which specialize to \(U(\mathfrak{g}^\theta)\) as \(q\) goes to 1 \cite{Let02}. These coideal subalgebras can be given explicitly in terms of generators and relations \cite{Let03}. However, the reflection equation does not appear in this construction. In \cite[Section 6]{Let99} it was shown that the coideal subalgebras constructed via solutions of the reflection equation are contained in Letzter’s coideal subalgebras. The proof depends on the abstract characterization of quantum symmetric pairs.

In the present paper we consider the converse problem: Let \(\tilde{B} \subseteq \tilde{U}_q(\mathfrak{g})\) be one of Letzter’s coideal subalgebras.

1) Is there a solution of the reflection equation canonically associated to \(\tilde{B}\) which can be used for a Noumi-Sugitani-Dijkhuizen type construction of a subcoideal \(L \subseteq \tilde{B}\)?

In order to replace \(\tilde{B}\) by \(L\) when looking at invariants, we moreover ask the following question:

2) Let \(V\) be a type one representation of \(\tilde{U}_q(\mathfrak{g})\). Does the space of \(L\)-invariant elements in \(V\) coincide with the subspace of \(\tilde{B}\)-invariant elements?

In this paper question 1) is answered in the affirmative. It is shown that suitable elements in the center \(Z(\tilde{B})\) lead to solutions of the reflection equation. The main ingredient in the argument is an isomorphism between the \(q\)-deformed coordinate ring \(R_q[\mathfrak{g}]\) and the locally finite part of \(\tilde{U}_q(\mathfrak{g})\). This isomorphism is given by \(l\)-functionals and is the inverse of Caldero’s isomorphism \cite{Cal93} given by the Rosso form. The center \(Z(\tilde{B})\) is calculated in \cite{KL} where it is shown that the desired central elements exist. The knowledge of the center together with a suitable notion of minimal weight leads to solutions of the reflection equation also for exceptional Lie algebras.

The main motivation for question 1) is to gain a better understanding of the ring of invariants \(R_q[\mathfrak{g}]^\tilde{B} = \{ x \in R_q[\mathfrak{g}] \mid bx = \varepsilon(b)x \forall b \in \tilde{B} \}\). For the quantum symmetric pairs of classical type the solution of the reflection equation was used to obtain sets of generators of \(R_q[\mathfrak{g}]^\tilde{B}\) \cite{Nou96, DN98}. It is
planned to investigate a similar construction for general quantum symmetric pairs.

An affirmative answer to Question 2) on the other hand would show that all results about zonal spherical functions and Macdonald polynomials [Let03, Let04] also hold for the coideal $L$. In the present paper this is proved for the symmetric pair $FI I$. We also formulate a generalization of Question 2) where $L$ is replaced by the left coideal generated by any nontrivial element of the basis of the center $Z(\hat{B})$ determined in [KL]. This generalization is proved for all symmetric pairs which correspond to automorphisms of the underlying Dynkin diagram.

The paper is organized as follows. In Section 2 we fix notations and recall general facts about the universal $r$-form for $R_q[G]$. These facts are used to show that $l$-functionals give an isomorphism between $R_q[G]$ and the locally finite part of $U_q(g)$.

In Section 3.1 it is shown how suitable central elements of a coideal subalgebra of $\hat{U}_q(g)$ lead to solutions of the reflection equation. This result is applied to Letzter’s family of quantum symmetric pairs in 3.5. To this end Letzter’s construction of quantum symmetric pair coideal subalgebras and properties of their center are recalled in Sections 3.2 and 3.3 respectively. Moreover, in 3.4 a natural notion of minimal weights is introduced. A precise formulation of Question 2) above is given in Section 3.5 as Conjecture 1 and a formulation of its generalization is given as Conjecture 2.

In this paper, following [Let02] we consider left coideal subalgebras and the right adjoint action. In 3.7 the reflection equation obtained here is translated into the setting of right coideal subalgebras reproducing precisely the reflection equation [Dij96, (5.5)], [DS99, (6.4)].

In Section 4 we develop general results about the left coideal generated by a nontrivial element of the basis of $Z(\hat{B})$. These results allow us to prove Conjecture 1 for the symmetric pair of type $FI I$. Section 5 is devoted to symmetric pairs which correspond to an automorphism of the underlying Dynkin diagram. In this case weight considerations and a suitable projection map allow us to recover all generators of $\hat{B}$ inside the coideal generated by an element of the basis of $Z(\hat{B})$. This proves Conjecture 2 in this special case.

The author is very grateful to Gail Letzter for encouragement and many instructive discussions about quantum symmetric pairs. He also wishes to thank Jasper Stokman for his interest and helpful comments.
2 Quantum groups

We write \( \mathbb{C} \), \( \mathbb{Z} \), and \( \mathbb{N}_0 \) to denote the complex numbers, the integers, and the nonnegative integers, respectively.

2.1 Notations

Let \( \mathfrak{g} \) be a finite dimensional complex semisimple Lie algebra of rank \( n \) and \( \mathfrak{h} \subseteq \mathfrak{g} \) a fixed Cartan subalgebra. Let \( \Delta \subseteq \mathfrak{h}^* \) denote the root system associated with \((\mathfrak{g}, \mathfrak{h})\). Choose an ordered basis \( \pi = \{\alpha_1, \ldots, \alpha_n\} \) of simple roots for \( \Delta \) and let \( \Delta^+ \) (resp. \( \Delta^- \)) be the set of positive (resp. negative) roots with respect to \( \pi \). Identify \( \mathfrak{h} \) with its dual via the Killing form. The induced nondegenerate symmetric bilinear form on \( \mathfrak{h}^* \) is denoted by \((\cdot, \cdot)\). We write \( Q(\pi) \) for the root lattice and \( P(\pi) \) for the weight lattice associated to the root system \( \Delta \). Let \( \omega_i \in \mathfrak{h}^*, i = 1, \ldots, n \) be the fundamental weights with respect to \( \pi \) and \( P^+(\pi) \) denote the set of dominant weights, i.e. the \( \mathbb{N}_0 \)-span of \( \{\omega_i | i = 1, \ldots, n\} \). Let \( \leq \) denote the standard partial ordering on \( \mathfrak{h}^* \). In particular, \( \mu \leq \gamma \) if and only if \( \gamma - \mu \in \mathbb{N}_0 \pi \). Finally, let \( \mathcal{W} \) denote the Weyl group associated to the root system \( \Delta \) and let \( w_0 \in \mathcal{W} \) be the longest element in \( \mathcal{W} \) with respect to \( \pi \).

2.2 \( \hat{U}_q(\mathfrak{g}) \) and \( R_q[G] \)

Let \( \mathcal{C} = \mathbb{C}(q^{1/N}) \) denote the field of rational functions in one variable \( q^{1/N} \) where \( N \) is sufficiently large such that \( (\lambda, \mu) \in \frac{1}{N} \mathbb{Z} \) for all \( \lambda, \mu \in P(\pi) \). We consider here the simply connected quantum universal enveloping algebra \( \hat{U}_q(\mathfrak{g}) \) as the \( \mathcal{C} \)-algebra generated by elements \( \{x_i, y_i, \tau(\lambda) | i = 1, \ldots, n, \lambda \in P(\pi)\} \) and relations as given for instance in [Jos95, Section 3.2.9]. As usual we will write \( t_i = \tau(\alpha_i) \). For \( \alpha \in Q(\pi) \) and any subset \( M \subseteq \hat{U}_q(\mathfrak{g}) \) let

\[
M_\alpha := \{ u \in M \mid \tau(\lambda) u \tau(-\lambda) = q^{(\lambda, \alpha)} u \text{ for all } \lambda \in P(\pi) \}
\]

(1)

denote the set of elements of weight \( \alpha \) in \( M \). The algebra \( \hat{U}_q(\mathfrak{g}) \) has a Hopf algebra structure with counit \( \varepsilon \), coproduct \( \Delta \), and antipode \( \sigma \) as in [Jos95, Section 3.2.9]. In particular the coproduct satisfies

\[
\Delta x_i = t_i \otimes x_i + x_i \otimes 1, \quad \Delta y_i = 1 \otimes y_i + y_i \otimes t_i^{-1}, \quad \Delta t_i = t_i \otimes t_i.
\]

(2)
Recall that $\sigma^2$ is a Hopf algebra automorphism of $U_q(\mathfrak{g})$ and that

$$\sigma^2(a) = \tau(-2\rho)a\tau(2\rho)$$

(3)

where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ is half the sum of all positive roots of $\mathfrak{g}$.

As usual let $U_q(\mathfrak{n}^+), U_q(\mathfrak{n}^-), G^+$, and $G^-$ denote the subalgebras of $U_q(\mathfrak{g})$ generated by the sets $\{x_i | i = 1, \ldots, n\}$, $\{y_i | i = 1, \ldots, n\}$, $\{x_i t^{-1} | i = 1, \ldots, n\}$, and $\{y_i t_i | i = 1, \ldots, n\}$, respectively. Let moreover $U_q(b^+)$ and $U_q(b^-)$ denote the subalgebras of $U_q(\mathfrak{g})$ generated by the sets $\{x_i, \tau(\lambda) | i = 1, \ldots, n, \lambda \in P(\pi)\}$ and $\{y_i, \tau(\lambda) | i = 1, \ldots, n, \lambda \in P(\pi)\}$, respectively. Let moreover $\bar{T}$ for the multiplicative group generated by $\tau(\lambda)$, $\lambda \in P(\pi)$, and $\bar{U}$ for the subalgebra of $U_q(\mathfrak{g})$ generated by $\bar{T}$. The algebra $\bar{U}_q(\mathfrak{g})$ admits a triangular decomposition. More precisely, the multiplication map induces an isomorphism

$$U_q(\mathfrak{n}^-) \otimes \bar{U}^0 \otimes U_q(\mathfrak{n}^+) \rightarrow \bar{U}_q(\mathfrak{g}).$$

The same holds if one or both of $U_q(\mathfrak{n}^-)$ and $U_q(\mathfrak{n}^+)$ are replaced by $G^-$ and $G^+$, respectively.

Note that $U_q(\mathfrak{n}^+), U_q(\mathfrak{n}^-), G^+$, and $G^-$ are direct sums of their weight spaces. Using this decomposition into weight spaces and the triangular decomposition one can define projections

$$\pi_{\alpha, \beta} : \bar{U}_q(\mathfrak{g}) \cong U_q(\mathfrak{n}^-) \otimes \bar{U}^0 \otimes G^+ \rightarrow U_q(\mathfrak{n}^-)_\alpha \otimes \bar{U}^0 \otimes G^+_\beta$$

(4)

for all $\alpha, \beta \in Q(\pi)$. Moreover, define

$$\pi_\alpha : \bar{U}_q(\mathfrak{g}) \rightarrow \bar{U}_q(\mathfrak{g})_\alpha$$

(5)

to be the projection onto the weight space of weight $\alpha \in Q(\pi)$.

The following notation is used in Sections $\textbf{3.2}$ and $\textbf{5}$. For any multiindex $I = (i_1, \ldots, i_m)$, $1 \leq i_j \leq n$, define $y_I = y_{i_1} \cdots y_{i_m}$ and $x_I = x_{i_1} \cdots x_{i_m}$. Set $\text{wt}(I) = \alpha_{i_1} + \cdots + \alpha_{i_m}$. Let $\mathcal{I}$ be a set of multiindices such that $\{y_I | I \in \mathcal{I}\}$ is a basis of $U_q(\mathfrak{n}^-)$.

For $\mu \in P^+(\pi)$ let $V(\mu)$ denote the uniquely determined finite dimensional simple left $\bar{U}_q(\mathfrak{g})$-module of highest weight $\mu$. More explicitly, there exists a highest weight vector $v_\mu \in V(\mu) \setminus \{0\}$ such that

$$x_i v_\mu = 0, \quad \tau(\lambda) v_\mu = q^{(\mu, \lambda)} v_\mu \quad \forall \ i = 1, \ldots, n \text{ and } \lambda \in P(\pi).$$

(6)
For any weight vector \( v \in V(\mu) \) we will write \( \text{wt}(v) \) to denote its weight. We consider the dual space \( V(\mu)^* \) always with its natural right \( \tilde{U}_q(\mathfrak{g}) \)-module structure defined by \( v^* u(v) = v^*(uv) \) for all \( v^* \in V(\mu)^*, v \in V(\mu), \) and \( u \in \tilde{U}_q(\mathfrak{g}) \). We call an element \( v^* \) of the right \( \tilde{U}_q(\mathfrak{g}) \)-module \( V(\mu)^* \) a weight vector of weight \( \nu \in P(\pi) \) if \( v^* \tau(\lambda) = q^{(\nu,\lambda)} v^* \) for all \( \lambda \in P(\pi) \). Moreover, we write \( V(\mu)^* \) to denote the space spanned by all weight vectors of weight \( \nu \) in \( V(\mu)^* \).

The \( q \)-deformed coordinate ring \( R_q[G] \) is defined \[\text{Jos95}, 9.1.1\] to be the subspace of the linear dual \( U_q(\mathfrak{g})^* \) spanned by the matrix coefficients of the finite dimensional irreducible representations \( V(\mu), \mu \in P^+(\pi) \). For \( v \in V(\mu), v^* \in V(\mu)^* \) the matrix coefficient \( c_{v^*,v}^\mu(X) \) is defined by

\[
\quad c_{v^*,v}^\mu(X) = v^*(Xv).
\]

The linear span of matrix coefficients of \( V(\mu) \) will be denoted by \( C^{V(\mu)} \).

2.3 The universal \( r \)-form for \( R_q[G] \)

The existence of the Rosso form for \( U_q(\mathfrak{g}) \) is reflected in the fact that \( R_q[G] \) is a coquasitriangular Hopf algebra. Recall (e.g. \[KS97, \text{chapter 10}\]) that a coquasitriangular bialgebra over \( \mathbb{C} \) is a bialgebra \( A \) over \( \mathbb{C} \) equipped with a convolution invertible skew-pairing \( r : A \otimes A \rightarrow \mathbb{C} \) such that

\[
\quad m_{A^{\text{op}}} = r \ast m_A \ast \bar{r}.
\]

Here \( m_A : A \otimes A \rightarrow A \) denotes the multiplication map of the algebra \( A \), the symbol \( \ast \) denotes the convolution product and \( \bar{r} \) is the convolution inverse of \( r \). If \( (A, r) \) is a coquasitriangular bialgebra then \( r \) is called a universal \( r \)-form for \( A \). We quickly review the construction of the universal \( r \)-form for \( R_q[G] \) along the lines of \[Gai95, Jos95\] in our notational conventions.
Let $R_q[B^+]$ and $R_q[B^-]$ denote the restriction of $R_q[G] \subseteq \mathcal{U}_q(g)^*$ to $\mathcal{U}_q(b^+)$ and $\mathcal{U}_q(b^-)$, respectively. The Hopf algebra structure of $R_q[G]$ induces Hopf algebra structures on $R_q[B^+]$ and $R_q[B^-]$. Let $\Psi^+ : R_q[G] \to R_q[B^+]$ and $\Psi^- : R_q[G] \to R_q[B^-]$ denote the canonical projections.

By [Tan92, Section 2] there exists a unique skew-pairing $\varphi : \mathcal{U}_q(b^-) \times \mathcal{U}_q(b^+) \to \mathbb{C}$ such that

$$\varphi(y_i, x_j) = -\delta_{ij}/(q^{(\alpha_i, \alpha_i)/2} - q^{-(\alpha_i, \alpha_i)/2})$$

$$\varphi(\tau(\lambda), \tau(\mu)) = q^{-(\lambda, \mu)}.$$

By [Jos95, 9.4.7] the map $\varphi$ can be used to define Hopf algebra isomorphisms

$$\Phi^- : \mathcal{U}_q(b^+) \to R_q[B^-], \quad x \mapsto \varphi(\cdot, x)$$

$$\Phi^+ : \mathcal{U}_q(b^-) \to R_q[B^+], \quad y \mapsto \varphi(y, \cdot).$$

The universal $r$-form for $R_q[G]$ is defined to be the composition of maps

$$r : R_q[G] \otimes R_q[G] \xrightarrow{\Psi^+ \otimes \Psi^-} R_q[B^+] \otimes R_q[B^-] \xrightarrow{(\Phi^+)^{-1} \otimes (\Phi^-)^{-1}} \mathcal{U}_q(b^-) \otimes \mathcal{U}_q(b^+) \xrightarrow{\varphi} \mathbb{C}.$$

By [Gai95, 2.2.3], [Jos95, 9.4.7] the map $r$ is a universal $r$-form for $R_q[G]$. For the purpose of the present paper the following properties of $r$ are relevant.

**Proposition 1** Let $\lambda, \mu \in P^+(\pi)$ and let $\{v_i\} \subseteq V(\lambda)$ and $\{w_j\} \subseteq V(\mu)$ be weight bases with dual bases $\{v_i^*\} \subseteq V(\lambda)^*$ and $\{w_j^*\} \subseteq V(\mu)^*$, respectively. Then the following properties hold.

i) $r(c_{v_i^*, v_i}^\lambda, c_{w_j^*, w_j}^\mu) \neq 0 \implies \text{wt}(v_i) \geq \text{wt}(v_j) \text{ and } \text{wt}(w_j) \leq \text{wt}(w_j).$

ii) $r(c_{v_i^*, v_i}^\lambda, c_{w_j^*, w_j}^\mu) \neq 0 \implies \text{wt}(v_i) - \text{wt}(v_j) = \text{wt}(w_j) - \text{wt}(w_j).$

iii) $r(c_{v_i^*, v_i}^\lambda, c_{w_j^*, w_j}^\mu) = \varphi(\tau(\text{wt}(v_i)), \tau(\text{wt}(w_j))) = q^{-(\text{wt}(v_i), \text{wt}(w_j))}.$

iv) $r(\sigma(a), \sigma(b)) = r(a, b).$

**Proof:** Property i) follows from the fact that $r$ factors over $R_q[B^+] \otimes R_q[B^-]$. Property ii) follows from the properties of $\varphi$ [Tan92 Lemma 2.1.3]. To verify property iii) note that

$$\Phi^+(\tau(-\text{wt}(v_i))) = c_{v_i^*, v_i}^\lambda|_{R_q[B^+]},$$

$$\Phi^-(\tau(-\text{wt}(w_j))) = c_{w_j^*, w_j}^\mu|_{R_q[B^-]}.$$

The last statement follows from [Tan92 Lemma 2.1.2].
2.4 The locally finite part

Any Hopf algebra $H$ with antipode $\sigma$ acts on itself from the left and the right by the left and right adjoint action, respectively. Using Sweedler notation these actions are given explicitly as follows

$$\text{ad}_l(h)u = h_{(1)}u\sigma(h_{(2)}), \quad \text{ad}_r(h)u = \sigma(h_{(1)})uh_{(2)}, \quad u, h \in H.$$ 

The right locally finite part is defined by

$$F_r(H) = \{h \in H \mid \dim((\text{ad}_r H)h) < \infty\}.$$ 

Similarly, the left locally finite part $F_l(H)$ is defined using the left adjoint action $\text{ad}_l$ instead of $\text{ad}_r$. Note that for $u, x \in H$ one has $(\text{ad}_r u)x = \sigma((\text{ad}_l(\sigma^{-1}(u))\sigma^{-1}(x))$ and therefore

$$F_r(H) = \sigma(F_l(H)). \quad (7)$$

It was shown in [JL94, Thm. 4.10], [CA93] that the locally finite part of $\tilde{U}_q(g)$ can be written explicitly as

$$F_l(\tilde{U}_q(g)) = \bigoplus_{\mu \in P^+(\pi)} (\text{ad}_l \tilde{U}_q(g))\tau(-2\mu).$$

By (7) this implies

$$F_r(\tilde{U}_q(g)) = \bigoplus_{\mu \in P^+(\pi)} (\text{ad}_r \tilde{U}_q(g))\tau(2\mu).$$

The following observation will be frequently used in Section 4. For any Hopf algebra $H$ the right adjoint action is compatible with the coproduct in the following sense

$$\Delta(\text{ad}_r b)C = (\text{ad}_r b_{(2)})C_{(1)} \otimes \sigma(b_{(1)})C_{(2)}b_{(3)} \quad \text{for all } b, C \in H \quad (8)$$

In particular in view of (2) one obtains for the generators $x_i, y_i$ of $\tilde{U}_q(g)$ the relations

$$\Delta(\text{ad}_r x_i)C = (\text{ad}_r x_i)C_{(1)} \otimes t_i^{-1}C_{(2)} + (\text{ad}_r t_i)C_{(1)} \otimes t_i^{-1}C_{(2)}x_i$$
$$- C_{(1)} \otimes t_i^{-1}x_iC_{(2)}, \quad (9)$$

$$\Delta(\text{ad}_r y_i)C = (\text{ad}_r y_i)C_{(1)} \otimes C_{(2)}t_i^{-1} + C_{(1)} \otimes C_{(2)}y_i$$
$$- (\text{ad}_r t_i^{-1})C_{(1)} \otimes y_i t_i C_{(2)} t_i^{-1}. \quad (10)$$
for all $C \in \tilde{U}_q(g)$.

**Remark:** For $H = \tilde{U}_q(g)$ it was proved explicitly [JL95 5.3], [Jos95 7.1.6], [Let02 Thm. 5.1] that $F_l(H)$ is a left coideal of $H$. However, this result holds for any Hopf algebra $H$. It can be proved analogously to [HK03 Lemma 5].

### 2.5 $l$-functionals

We now apply the theory of $l$-functionals (e.g. [KS97 10.1.3]) to the coquasitriangular Hopf algebra $R_q[G]$. This will lead to an isomorphism between the matrix coefficients $C^V(\mu)$ and the corresponding component of the locally finite part.

Using the universal $r$-form one defines $l$-functionals in the dual Hopf algebra $R_q[G]^\circ$ by

$$l^+(a) := r(\cdot, a), \quad l^-(a) := r(\sigma(a), \cdot),$$

$$l(a) := l^-(\sigma^{-1}(a_{(1)}))l^+(a_{(2)}) = r(a_{(1)}, \cdot)r(\cdot, a_{(2)}),$$

$$\tilde{l}(a) := l^+(a_{(1)})l^-(\sigma^{-1}(a_{(2)})) = r(\cdot, a_{(1)})r(\cdot, a_{(2)}).$$

for $a \in R_q[G]$. Note that by definition

$$l^+(a) = (\Phi^-)^{-1} \circ \Psi^-(a) \in \tilde{U}_q(b^+),$$

$$l^-(a) = (\Phi^+)^{-1} \circ \Psi^+(\sigma(a)) \in \tilde{U}_q(b^-)$$

and therefore $l(a), \tilde{l}(a) \in \tilde{U}_q(g)$ and

$$r(a, b) = l^+(b)(a) = \varphi(l^-(\sigma^{-1}(a)), l^+(b)).$$

Using the fact that the universal $r$-form is a skew-pairing one can determine the coproduct of the $l$-functionals. One obtains [KS97 10.1.3, Prop. 11]

$$\Delta l^+(a) = l^+(a_{(1)}) \otimes l^+(a_{(2)}), \quad \Delta l^-(a) = l^-(a_{(1)}) \otimes l^-(a_{(2)}),$$

$$\Delta l(a) = l(a_{(2)}) \otimes l^-(\sigma^{-1}(a_{(1)}))l^+(a_{(3)}) = l(a_{(2)}) \otimes \sigma(l^-(a_{(1)}))l^+(a_{(3)}),$$

$$\Delta \tilde{l}(a) = l^+(a_{(1)})l^-(\sigma^{-1}(a_{(3)})) \otimes \tilde{l}(a_{(2)}) = l^+(a_{(1)})\sigma(l^-(a_{(3)})) \otimes \tilde{l}(a_{(2)}).$$

The $l$-functionals $l$ and $\tilde{l}$ are compatible with the right and left adjoint action, respectively, in the following way [KS97 10.1.3, Prop. 11]

$$\text{ad}_r(u)(l(a)) = (a_{(1)}\sigma(a_{(3)}))(u) l(a_{(2)}), \quad (12)$$

$$\text{ad}_l(u)(\tilde{l}(a)) = (\sigma(a_{(1)})a_{(3)})(u) \tilde{l}(a_{(2)}). \quad (13)$$

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These formulas imply in particular that for all \( a \in R_q[G] \)
\[
l(a) \in F_r(\mathcal{U}_q(\mathfrak{g})), \quad \bar{l}(a) \in F_r(\mathcal{U}_q(\mathfrak{g})).
\]
The following proposition is a generalization of a result proved for \( \mathfrak{sl}(r+1) \) and \( \mathfrak{sp}(2r) \) in [HS98, Lemma 4.7]. The proof given here is significantly simpler.

**Proposition 2** For any \( \mu \in P^+(\pi) \) the \( l \)-functionals \( l \) and \( \bar{l} \) define isomorphisms of vector spaces
\[
l : C^V(\mu) \rightarrow (\text{ad}_{\mathcal{U}_q(\mathfrak{g})})\tau(-2w_0\mu), \quad \bar{l} : C^V(\mu) \rightarrow (\text{ad}_{\mathcal{U}_q(\mathfrak{g})})\tau(-2\mu)
\]
Moreover, \( l \) and \( \bar{l} \) are compatible with the right and left adjoint action, respectively, in the following way
\[
l(\sigma(u(2))au(1)) = (\text{ad}_r u)(a), \quad \bar{l}(u(2)\sigma(u(1))) = (\text{ad}_l u)\bar{l}(a),
\]
for all \( u \in \mathcal{U}_q(\mathfrak{g}), a \in R_q[G] \).

**Proof:** Formulas (15) follow immediately from (12) and (13).

For any dominant integral weight \( \mu \in P^+(\pi) \) let \( v_{w_0\mu} \in V(\mu) \) denote a lowest weight vector and let \( v^*_{w_0\mu} \in V(\mu)^* \) be a lowest weight vector in the dual right \( \mathcal{U}_q(\mathfrak{g}) \)-module such that \( v^*_{w_0\mu}(v_{w_0\mu}) = 1 \). In \( \mathcal{U}_q(\mathfrak{g}) \) using Proposition 1 one verifies
\[
\tau(-2w_0\mu) = l(c_{v^*_{w_0\mu}, v_{w_0\mu}}).
\]
To shorten notation define \( c := c_{v^*_{w_0\mu}, v_{w_0\mu}} \) and define a right action \( \text{ad}_r \) of \( \mathcal{U}_q(\mathfrak{g}) \) on \( R_q[G] \) by
\[
(\text{ad}_r u)a = \sigma(u(2))au(1).
\]
Then by (13) and (16) the map \( l \) maps \( (\text{ad}_r \mathcal{U}_q(\mathfrak{g})))c \) onto the right \( \mathcal{U}_q(\mathfrak{g}) \)-module \( (\text{ad}_r(\mathcal{U}_q(\mathfrak{g})))\tau(-2w_0\mu)) \). Moreover, by [JL94 Thm. 3.5] one has
\[
\dim((\text{ad}_r \mathcal{U}_q(\mathfrak{g})))\tau(-2w_0\mu)) = \dim(\text{End}(V(\mu))) = \dim(C^V(\mu))
\]
and therefore \( (\text{ad}_r \mathcal{U}_q(\mathfrak{g})))c = C^V(\mu) \) and \( l|_{C^V(\mu)} \) is also injective. This proves the first isomorphism in (14). The second isomorphism is obtained analogously. \( \blacksquare \)

**Remark:** P. Caldero showed that the Rosso form induces an isomorphism \( (\text{ad}_l \mathcal{U}_q(\mathfrak{g})))\tau(-2\mu) \rightarrow C^V(\mu) \) [Cal93]. Using the properties of \( \bar{l} \) and of the Rosso form one can verify that \( \bar{l} \) is the inverse of this isomorphism.
3 Reflection equation for quantum symmetric pairs

3.1 Coideal subalgebras and a reflection equation

Let $B \subseteq \tilde{U}_q(g)$ be a left coideal subalgebra with center $Z(B)$. In the previous section we saw that $(\text{ad}_r \tilde{U}_q(g))\tau(-2w_0\mu)$ is spanned by $l$-functionals. This can now be used to show that any element in $Z(B) \cap (\text{ad}_r \tilde{U}_q(g))\tau(-2w_0\mu)$ gives rise to a solution of the reflection equation.

For any $\mu \in P^+(\pi)$ let $N = \dim V(\mu)$ and let $\{v_1, \ldots, v_N\}$ be a basis of $V(\mu)$ with dual basis $\{v_1^*, \ldots, v_N^*\}$. To shorten notation define $c_j^i := c_{v_i^*, v_j}$ for $i, j = 1, \ldots, N$ and

$$R_{kl}^{ij} := r(c_k^i, c_l^j).$$

By Proposition 2 any element $C \in (\text{ad}_r \tilde{U}_q(g))\tau(-2w_0\mu)$ can be written in the form

$$C = \sum_{i,j} J_{ij}^l(c_j^i)$$

for some coefficients $J_{ij}^l \in \mathbb{C}$.

**Proposition 3** Given $\mu \in P^+(\pi)$, a left coideal subalgebra $B \subseteq \tilde{U}_q(g)$ and an element

$$C \in Z(B) \cap (\text{ad}_r \tilde{U}_q(g))\tau(-2w_0\mu).$$

Then the matrix $J = (J_{ij}^l)_{i,j=1,\ldots,N}$ defined by (17) satisfies the reflection equation

$$J_1 \tilde{R}J_2R = R_{21}J_2\tilde{R}_{21}J_1$$

where $\tilde{R}_{kl}^{ij} = r(c_k^i, \sigma(c_l^j))$, the lower index $21$ denotes conjugation with the twist $P : v \otimes w \mapsto w \otimes v$, and $J_{ij}^{21} = \delta_{ji}J_{i}^{i}, J_{21}^{ij} = \delta_{ik}J_{i}^{j}$.

**Proof:** As $\Delta B \subseteq \tilde{U}_q(g) \otimes B$ relation (11) implies

$$\sum_{i,j} \sigma(l^-(c_m^i))J_i^l(l^+(c_n^j) - J_m^l) \in B^+, \quad \text{for all } m, n = 1, \ldots, N,$$
where $B^+ = B \cap \ker(\varepsilon)$. By the $\text{ad}_r(B)$-invariance of $C$ this implies

$$\text{ad}_r \left( \sum_i J_i^n \sigma(l^-(c_m^i)) - \sigma^{-1}(l^+(c_n^i))J_i^m \right) (C) = 0.$$ 

Using relation (12) one obtains

$$\sum_{a,b,i,j,k,l} [J_i^a r(c_i^a, \sigma(c_j^b))J_j^b r(c_m^j, c_a^k) - r(c_i^a, c_j^b)J_j^b r(c_a^k, \sigma(c_i^a))J_i^a] l(c_b^a) = 0$$

for all $m, n = 1, \ldots, N$. As $l$ is injective and the $c_a^a$ are linearly independent this yields the desired formula. ■

**Remark:** Formula (18) coincides with the reflection equation [Dij96, Equation (5.5)] up to replacing $\tilde{R}$ by $R^{-1}$. In Section 3.7 we will see how to obtain [Dij96, Equation (5.5)] by a suitable translation from left to right coideal subalgebras.

### 3.2 Quantum symmetric pairs

We now wish to apply the above observation to the family of quantum symmetric pairs constructed and investigated by Gail Letzter in a series of papers [Let99], [Let02], [Let03]. Hence we first recall the construction and some properties of quantum symmetric pairs.

Let $\theta : g \to g$ be an involutive Lie algebra automorphism, i.e. $\theta^2 = \text{id}$, which is maximally split with respect to $\mathfrak{h}$ and the chosen triangular decomposition of $g$ in the sense of [Let02, Section 7]. Then $\theta$ induces an involution $\Theta$ of the root system $\Delta$ of $g$ and thus an automorphism of $\mathfrak{h}^*$. Define $\pi_\Theta = \{\alpha_i \in \pi | \Theta(\alpha_i) = \alpha_i\}$. Let $p$ be the permutation of $\{1, \ldots, n\}$ such that

$$\Theta(\alpha_i) \in -\alpha_{p(i)} + Z\pi_\Theta \text{ for all } \alpha_i \notin \pi_\Theta$$

(19) and $p(i) = i$ if $\alpha_i \in \pi_\Theta$.

Let $\mathcal{M} \subseteq \dot{U}_q(g)$ denote the Hopf subalgebra generated by $x_i$, $y_i$, $t_i^{\pm 1}$ for $\alpha_i \in \pi_\Theta$. Define a multiplicative subgroup $T_\Theta' \subseteq \dot{U}_q(g)$ by

$$T_\Theta' = \{\tau(\lambda) \mid \lambda \in P(\pi) \text{ and } \Theta(\lambda) = \lambda\}.$$
By definition the subalgebra $\tilde{B} \subseteq \tilde{U}_q(\mathfrak{g})$ is generated by $\mathcal{M}$, $T'_\Theta$, and elements $B_i$ for $\alpha_i \in \pi \setminus \pi_\Theta$ defined by

$$B_i = y_i t_i + d_i \tilde{\theta}(y_i) t_i + s_i t_i$$

(20)

for suitable $d_i, s_i \in \mathcal{C}$, $d_i \neq 0$, and where

$$\tilde{\theta}(y_i) = (\text{ad}_r x_{i_1}) \cdots (\text{ad}_r x_{i_1})(t_{p(i)}^{-1} x_{p(i)})$$

for suitable $\alpha_{i_1}, \ldots, \alpha_{i_m} \in \pi_\Theta$. For more details, in particular for the choice of $d_i, s_i$, and $\alpha_{i_1}, \ldots, \alpha_{i_m}$ consult [Let02, Section 7]. For the purpose of the present paper it suffices to know that $\tilde{\theta}(y_i)$ is $(\text{ad}_r \mathcal{M}^+)$-invariant.

Set $B_i = y_i t_i$ for $\alpha_i \in \pi_\Theta$ and set $M^+ = \mathcal{M} \cap U_q(n^+)$. Recall the definition of $I$ given in Section 2.2. Given a multiindex $I = (i_1, \ldots, i_m)$ in $\mathcal{I}$, set $B_I = B_{i_1} \cdots B_{i_m}$. By [Let02, Section 7], we have a direct sum decomposition

$$\tilde{B} = \oplus_{I \in \mathcal{I}} B_I \mathcal{M}^+ T'_\Theta.$$  

(21)

It is sometimes convenient to replace the generators $\{B_i | \alpha_i \in \pi \setminus \pi_\Theta\}$ by a different set of generators $\{C_i | \alpha_i \in \pi \setminus \pi_\Theta\}$ obtained by interchanging $y_i t_i$ and $x_i$. More precisely, the new set of generators is obtained via the following map. By [Let03, Theorem 3.1], [Let, (1.22)] there exists an $\mathcal{C}$-linear algebra antiautomorphism $\kappa_B$ of $\tilde{U}_q(\mathfrak{g})$ defined by

$$\kappa_B(x_i) = c_B y_i t_i, \quad \kappa_B(y_i) = c_B^{-1} t_i^{-1} x_i, \quad \kappa_B(\tau(\lambda)) = \tau(\lambda)$$

for suitable nonzero $c_B \in \mathcal{C}$, such that

$$\kappa_B(\tilde{B}) = \tilde{B}.$$  

Note that in the above references $\kappa_B$ is defined to be conjugate linear in order to obtain a $*$-structure on $\tilde{B}$. However, linearity suffices for the present purposes. Note moreover, that by definition the antipode and the coproduct of $\tilde{U}_q(\mathfrak{g})$ satisfy

$$\kappa_B(\sigma(u)) = \sigma^{-1}(\kappa_B(u)), \quad \Delta(\kappa_B(u)) = (\kappa_B \otimes \kappa_B) \Delta(u)$$

(22)

for all $u \in \tilde{U}_q(\mathfrak{g})$. In particular, one gets

$$\kappa_B((\text{ad}_r u) C) = (\text{ad}_r(\sigma^{-1} \circ \kappa_B(u))) \kappa_B(C)$$

(23)
for all $u, C \in \tilde{U}_q(g)$, and hence
\[ \kappa_B \left( (\text{ad}_r \tilde{U}_q(g))\tau(2\mu) \right) = (\text{ad}_r \tilde{U}_q(g))\tau(2\mu) \] (24)
for all $\mu \in P^+(\pi)$.

The subalgebra generated by $M$ and $T'_0$ of $\tilde{B}$ is invariant under $\kappa_B$. Hence $\tilde{B}$ is also generated by $M, T'_0$, and elements $C_i = c_i B_i$ for $\alpha_i \in \pi \setminus \pi_\Theta$. Note that $\tilde{B}$ is generated by $M, T'_0, \Theta, \kappa_B$.

The center of quantum symmetric pair coideal subalgebras

In order to apply Proposition 3 to quantum symmetric pairs we recall some results of [KL] on the center $Z(\tilde{B})$ of the coideal subalgebras $\tilde{B} \subseteq \tilde{U}_q(g)$.

Let $W(\pi_\Theta) \subseteq W$ denote the subgroup generated by the simple reflections corresponding to the simple roots in $\pi_\Theta$. Note that $W(\pi_\Theta)$ is the Weyl group corresponding to the root system generated by $\pi_\Theta$. Let $w'_0 \in W(\pi_\Theta)$ denote the longest element in $W(\pi_\Theta)$. As shown in [KL] Theorem 8.5] the center $Z(\tilde{B})$ of $\tilde{B}$ has a basis $\{d_\mu | \mu \in P_{Z(\tilde{B})}\}$ indexed by the set

\[ P_{Z(\tilde{B})} = \{ \mu \in P^+(\pi) | \Theta(\mu) = \mu + w_0\mu - w'_0\mu \} . \]

The set $P_{Z(\tilde{B})}$ is explicitly determined in [KL] Proposition 9.1]. Note that $P_{Z(\tilde{B})} = -w_0 P_{Z(\tilde{B})}$. By [KL] Theorem 8.3] the central elements $d_\mu$ for $\mu \in P_{Z(\tilde{B})}$ satisfy

\[ d_\mu \in Y^\mu + \sum_{\gamma < 2\mu} \tilde{U}_q(g)_{-\gamma} \] (25)

where $Y^\mu \in (\text{ad}_r \tilde{U}_q(g))\tau(2\mu)$ is a highest weight vector with respect to the right adjoint action of spherical weight $2\tilde{\mu} = \mu - \Theta(\mu)$.  

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The center $Z(\hat{B})$ is invariant under the algebra antiautomorphism $\kappa_B$. Recall also from [KL, Theorem 8.3] that

$$d_\mu \in v_\mu + \sum_{\nu < \mu} (\text{ad}_r \check{U}_q(\mathfrak{g}))\tau(2\nu)$$

(26)

for some nonzero $v_\mu \in (\text{ad}_r \check{U}_q(\mathfrak{g}))\tau(2\mu)$ which is uniquely determined up to scalar multiplication. From (24) and the fact that \{d_\mu | \mu \in P_Z(\check{B})\} is a basis of $Z(\check{B})$ one obtains

$$\kappa_B(d_\mu) \in a_\mu v_\mu + \sum_{\nu < \mu} (\text{ad}_r \check{U}_q(\mathfrak{g}))\tau(2\nu)$$

(27)

for some $a_\mu \in \mathbb{C} \setminus \{0\}$. As $\kappa_B^2 = \text{id}$ one even gets $a_\mu \in \{-1, 1\}$. Hence, replacing $d_\mu$ by $(d_\mu + a_\mu \kappa_B(d_\mu))/2$ we may assume that

$$\kappa_B(d_\mu) = a_\mu d_\mu.$$  

(28)

Define $X^\mu = a_\mu \kappa_B(Y^\mu)$. Then by (23) the element $X^\mu \in (\text{ad}_r \check{U}_q(\mathfrak{g}))\tau(2\mu)$ is a lowest weight vector of weight $-2\check{\mu}$ with respect to the right adjoint action. Hence (25) and (28) imply

$$d_\mu \in X^\mu + \sum_{\gamma < 2\check{\mu}} \check{U}_q(\mathfrak{g})_{\gamma}.$$  

(29)

### 3.4 Minimal weights in $P_{Z(\hat{B})}$

By Proposition 3 one can find solutions of the reflection equation via elements in $Z(\hat{B}) \cap (\text{ad}_r \check{U}_q(\mathfrak{g}))\tau(-2w_0\mu)$. The following notion of minimality will lead to the desired central elements. Note that this definition is slightly more general than the definition of minimality in say [Hum72, Exercise 13.13].

**Definition 1** Let $\mathfrak{g}$ be simple. A nonzero weight $\mu \in P^+(\pi)$ is called minimal if it satisfies the following property: If $\nu \in P^+(\pi)$ and $\nu < \mu$ then $\nu = 0$.

Note that $\mu \in P^+(\pi)$ is minimal if and only if $-w_0\mu$ is minimal. As shown in [KL, Corollary 8.4] it follows immediately from (26) that for minimal $\mu \in P_{Z(\hat{B})}$ we may assume

$$d_\mu \in (\text{ad}_r \check{U}_q(\mathfrak{g}))\tau(2\mu).$$  

(30)
In Tables 1 and 2 we have listed all minimal weights and all minimal weights in \( P_{Z(\mathfrak{g})} \) using the labeling of simple roots of [Hum72, 11.4]. Symmetric pairs are labeled as in [Ara62, 5.11]. Following [Let03, Section 7], however, the parameter \( p \) occurring in Araki’s list will be denoted by \( r \), and as before \( n = \text{rank}(\mathfrak{g}) \). Table 1 is obtained by inspection from the table in [Hum72, p. 69] and Table 2 then follows from [KL, Proposition 9.1].

\[
\begin{array}{c|c|c|c}
A_n & \omega_1, \ldots, \omega_n, \omega_1+\omega_n & E_7 & \omega_1, \omega_2 \\
B_n & \omega_1, \omega_2 & E_8 & \omega_8 \\
C_n & \omega_1, \omega_2 & F_4 & \omega_4 \\
D_n & \omega_1, \omega_2, \omega_{n-1}, \omega_n & G_2 & \omega_1 \\
E_6 & \omega_1, \omega_2, \omega_6 & & \\
\end{array}
\]

Table 1: Minimal weights in \( P^+(\pi) \)

AI, AII  
AI\(\text{III}, \text{AIV} \)  
B\(n \)  
C\(n \)  
(DI, case 1), \( r \) even  
(DI, case 1), DII, \( r \) odd  
\( (\text{DI}, \text{case } 1), \text{(DIII, case 1), } n \) even  
\( (\text{DI}, \text{case } 2,3), \text{(DIII, case 1), } n \) odd  
EI, EIV  
EII, EIII  
E\(7 \)  
E\(8 \)  
F\(4 \)  
G\(2 \)  

\[
\begin{array}{c|c|c|c|c|c}
& & & & & \\
& & & & & \\
\omega_1+\omega_n & & & & & \\
\omega_1, \ldots, \omega_n, \omega_1+\omega_n & & & & & \\
\omega_1, \omega_n & & & & & \\
\omega_1, \omega_2 & & & & & \\
\omega_1, \omega_2, \omega_{n-1}, \omega_n & & & & & \\
\omega_1, \omega_2 & & & & & \\
\omega_1, \omega_2, \omega_{n-1}, \omega_n & & & & & \\
\omega_1, \omega_2 & & & & & \\
\omega_1, \omega_2, \omega_6 & & & & & \\
\omega_1, \omega_2 & & & & & \\
\omega_8 & & & & & \\
\omega_4 & & & & & \\
\omega_1 & & & & & \\
\end{array}
\]

Table 2: Minimal weights in \( P_{Z(\mathfrak{g})} \)
3.5 Invariants

Assume now that \( \mu \in P_{Z(\breve{B})} \) and let \( L_\mu \subseteq \breve{U}_q(\mathfrak{g}) \) denote the left coideal generated by the central element \( d_{-w_0\mu} \). As \( \hat{B} \subseteq \hat{U}_q(\mathfrak{g}) \) is a left coideal one has \( L_\mu \subseteq \hat{B} \). If \( \mu \in P_{Z(\breve{B})} \) is minimal then (30) holds also with \( \mu \) replaced by \( -w_0\mu \). In this case let again \( J \) denote the \( N \times N \)-matrix defined by \( d_\mu = \sum_{i,j} J^i_j l_i(e_j^\mu) \). By Proposition 3 the matrix \( J \) is a solution of the reflection equation (18). Let \( L^+ \) and \( L^- \) denote the \( N \times N \) matrix with entries \( l^+(e_j^\mu) \) and \( l^-(e_j^\mu) \), respectively. By construction \( L_\mu \) is the linear span of the matrix entries of \( \sigma(L^-)^tJ^t(L^+)^t \). Up to conventional changes explained in Section 3.7 the coideal \( L_\mu \) is of the type considered by M. Noumi, T. Sugitani, and M. Dijkhuizen.

For any left \( \hat{U}_q(\mathfrak{g}) \)-module \( M \) and any subset \( L \subseteq \hat{U}_q(\mathfrak{g}) \) define \( M^L = \{ m \in M \mid am = \varepsilon(a)m \forall a \in L \} \). We conjecture that \( L_\mu \) is big enough in the following sense.

Conjecture 1 Assume that \( \mathfrak{g} \) is simple and that \( \mu \in P_{Z(\breve{B})} \) is minimal. Then \( V(\nu)^{\breve{B}} = V(\nu)^{L_\mu} \) for all \( \nu \in P^+(\pi) \).

For the classical symmetric pairs and a suitable choice of a minimal \( \mu \in P^+(\pi) \) the coideals \( L_\mu \) have been explicitly constructed. In this case the claim of the conjecture holds by [NS95, Theorem 1] and [Let02, Theorem 7.7]. In Section 4 we will obtain general results aiming at a proof of Conjecture 1. Although the general case remains open these results are strong enough to verify Conjecture 1 for the symmetric pair \( FII \).

To obtain solutions of the reflection equation via Proposition 3 it is necessary that \( d_\mu \) satisfies (30) and hence it is natural to assume that \( \mu \) is minimal. However, one could ask if Conjecture 1 also holds for not necessarily minimal \( \mu \in P_{Z(\breve{B})} \). This also allows us to consider irreducible symmetric pairs \((\mathfrak{g}, \mathfrak{g}^\theta)\) with not necessarily simple \( \mathfrak{g} \).

Conjecture 2 Assume that \((\mathfrak{g}, \mathfrak{g}^\theta)\) is irreducible and that \( \mu \in P_{Z(\breve{B})} \). Then \( V(\nu)^{\breve{B}} = V(\nu)^{L_\mu} \) for all \( \nu \in P^+(\pi) \).

In Section 5 the claim of Conjecture 2 is verified for all quantum symmetric pairs with \( \pi_\Theta = \emptyset \). In particular, Conjecture 2 is seen to hold in the case where \( \mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}'' \) for some simple \( \mathfrak{g}' \) and \( \theta \) interchanges the two isomorphic components.
Remark: M. Noumi constructed quantum symmetric pairs of type AI and AII using the $R$-matrix in the vector representation $V(\omega_1)$ and a reflection equation [Nou96, (2.7)] different from (18) and (35). However, $\omega_1 \notin P_{Z(\tilde{B})}$ for AI and AII. The construction in 3.1 only yields a solution of the reflection equation for the $R$-matrix of $V(\omega_1 + \omega_n)$. Conjectures 1 and 2 claim that with respect to its invariants the coideal constructed here is equivalent to the coideal constructed by Noumi. Noumi’s reflection equation occurs for all classical symmetric pairs apart from AIII [NS95]. However, for $g$ of type $B$, $C$, or $D$ the vector representation $V(\omega_1)$ is self dual and hence Noumi’s reflection equation can be translated into (18) in these cases.

In view of the above remarks it is one of the conceptual insights of the present paper that Noumi-Sugitani-Dijkhuizen type constructions can be performed with one and the same reflection equation uniformly for all symmetric pairs.

3.6 First properties of $L_\mu$

The following two observations will be frequently used in Sections 4 and 5.

Lemma 1 For any $\mu \in P_{Z(\tilde{B})}$ the left coideal $L_\mu$ is $(\text{ad}_l M)$-invariant.

Proof: For $b \in M \cap \ker(\varepsilon)$ and $C \in Z(\tilde{B})$ one has analogously to (8)

$$0 = \Delta((\text{ad}_l b)C) = b(1)C(1)\sigma(b(3)) \otimes (\text{ad}_l b(2))C(2).$$

In particular for $b = x_i$, $\alpha_i \in \pi_{\Theta}$, and $C = d_{-w_0^\mu}$ one obtains

$$C(1) \otimes (\text{ad}_l x_i)C(2) = t_i^{-1}(x_i C(1) - C(1)x_i) \otimes C(2) \in \tilde{U}_q(g) \otimes L_\mu.$$

Hence $L_\mu$ is $(\text{ad}_l x_i)$-invariant. Similarly one obtains that $L_\mu$ is $(\text{ad}_l y_i)$-invariant and $(\text{ad}_l t_i)$-invariant if $\alpha_i \in \pi_{\Theta}$. ■

Lemma 2 For any $\mu \in P_{Z(\tilde{B})}$ the left coideal $L_\mu$ is invariant under the algebra antiautomorphism $\kappa_{\tilde{B}}$.

Proof: This follows from (28) and the second formula of (22). ■
3.7 Left versus right

In this section the result from [3.1] is translated into the setting of right coideal subalgebras. This reproduces the reflection equation [Dij96, (5.5)]. The results of the present section are not relevant for the rest of the paper.

Using the antipode of $\tilde{U}_q(\mathfrak{g})$ there are two ways to turn a left $\tilde{U}_q(\mathfrak{g})$-module $V$ into a right $\tilde{U}_q(\mathfrak{g})$-module. One can define a right action on $V$ by

$$v \triangleleft u := \sigma(u)v \quad \forall u \in \tilde{U}_q(\mathfrak{g}), v \in V$$

or by

$$v \triangleleft_- u := \sigma^{-1}(u)v \quad \forall u \in \tilde{U}_q(\mathfrak{g}), v \in V.$$ 

We will denote the vectorspace $V$ endowed with the right module structures (31) and (32) by $V^+_r$ and $V^-_r$, respectively. For finite dimensional $V$ the right $\tilde{U}_q(\mathfrak{g})$-modules $V^+_r$ and $V^-_r$ are isomorphic, however, the isomorphism is not given by the identity map of the underlying vectorspace $V$.

This little observation explains why the reflection equation (18) slightly differs from the reflection equation [Dij96, (5.5)]. To be more precise, note that by (15) the map

$$V(\lambda)^* \otimes V(\lambda)_r^+ \to (\text{ad}_r \tilde{U}_q(\mathfrak{g}))(\tau(2\lambda)), \quad v^*_i \otimes v_j \mapsto l(c^i_j)$$

is an isomorphism of right $\tilde{U}_q(\mathfrak{g})$-modules. For any $N \times N$-matrix $J$ let $B_J \subseteq \tilde{U}_q(\mathfrak{g})$ denote the left coideal subalgebra generated by the matrix entries of

$$\sigma(L^-)^t J^t (L^+)^t.$$

With this notation the proof of Proposition 3 also proves the following generalization of a right version of [Nou96, Prop. 2.3], [DS99, Prop. 6.5].

**Proposition 4** For any $N \times N$-matrix $J$ the element

$$\sum_{i,j} J^t_{ij} v^*_i \otimes v_j \in V(\lambda)^* \otimes V(\lambda)_r^+$$

is $B_J$-invariant if and only if $J$ satisfies the reflection equation (18).
Proof: In view of the isomorphism all that needs to be checked is that \( \sum_{i,j}^l c_i^j \) is (\( \text{ad}_r B_J \))-invariant if and only if \( J \) satisfies \( (18) \). This has indeed been verified in the proof of Proposition \( 3 \). ■

To obtain the reflection equation from \([Dij96, (5.5)], [DS99, (6.4)]\) one has to write the invariant element as an element in \( V(\lambda)^* \otimes V(\lambda)_-^* \). Note that there is an isomorphism of right \( \hat{U}_q(g) \)-modules given by

\[
\eta_+ : V(\lambda)_-^* \to V(\lambda)_+^*, \quad v_\nu \mapsto q^{(\lambda-\nu,2\rho)} v_\nu
\]

where \( v_\nu \in V(\lambda)_\nu \) denotes a weight vector of weight \( \nu \) in the left \( \hat{U}_q(g) \)-module \( V(\lambda) \) and \( \rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \). Indeed, by \( (3) \) for any homogeneous element \( a \in \hat{U}_q(g)_\mu \) one has

\[
v_\lambda \prec_a = \sigma^{-1}(a) v_\lambda = q^{(\mu,2\rho)} \sigma(a) v_\lambda = q^{(\mu,2\rho)} v_\lambda \prec_+ a
\]

and therefore

\[
\eta_+(v_\lambda \prec_- a) = q^{(-\mu,2\rho)} \sigma^{-1}(a) v_\lambda = v_\lambda \prec_+ a.
\]

For any \( N \times N \)-matrix \( \bar{J} \) define \( \bar{B}_J \) where \( J_i^j := q^{(\lambda-\text{wt}(v_j),2\rho)} \bar{J}_i^j \). The following proposition states that exchange of left and right by means of the antipode \( \sigma \) transforms the situation considered here precisely into the situation considered in \([DS99]\) for the special case of Grassmannians.

**Proposition 5** For any \( N \times N \)-matrix \( \bar{J} \) the element

\[
C := \sum_{i,j} \bar{J}_i^j v_i^* \otimes v_j \in V(\lambda)^* \otimes V(\lambda)_-^*
\]

is \( \bar{B}_J \)-invariant if and only if \( \bar{J} \) satisfies the reflection equation

\[
\bar{J}_1 R^{-1} \bar{J}_2 R = R_{21} \bar{J}_2 R_{21}^{-1} \bar{J}_1.
\]

Moreover, let \( B_J^D \subseteq \hat{U}_q(g) \) denote the right coideal subalgebra generated by the coefficients of the \( N \times N \)-matrix \( \sigma(L^+) \bar{J} L^- \). Then \( \bar{B}_J = \sigma(B_J^D) \).

**Proof**: Note that

\[
(id \otimes \eta_-) \left( \sum_{i,j} \bar{J}_i^j v_i^* \otimes v_j \right) = \sum_{i,j} \bar{J}_i^j v_i^* \otimes v_j \in V(\lambda)^* \otimes V(\lambda)_+^*
\]
where $J^i_j = q^{(\lambda - \text{wt}(v_j), 2\rho)}\tilde{J}^i_j$. Thus by Proposition 4 the element $C$ is $\tilde{B}_f$-invariant if and only if $J$ satisfies the reflection equation (18). Thus $C$ is $\tilde{B}_f$-invariant if and only if $\tilde{J}$ satisfies

\[
\sum_{i,j,k,l} q^{(\lambda - \text{wt}(v_j), 2\rho)} \tilde{J}^n_i r(c^j_i, \sigma(c^h_j)) q^{(\lambda - \text{wt}(v_j), 2\rho)} \tilde{J}^k_j r(c^l_m, c^k_n) - r(c^h_j, c^k_n) q^{(\lambda - \text{wt}(v_j), 2\rho)} \tilde{J}^n_i r(c^l_m, c^k_n) = 0
\] (36)

for all $a, b, m, n$. Note that (3) implies

\[
q^{(\lambda - \text{wt}(v_j), 2\rho)} c^j_i = q^{(\lambda - \text{wt}(v_j), 2\rho)} \sigma^{-2}(c^j_i).
\] (37)

Using (37) relation (36) is seen to be equivalent to

\[
\sum_{i,j,k,l} q^{(\lambda - \text{wt}(v_j), 2\rho)} q^{(\lambda - \text{wt}(v_j), 2\rho)} \tilde{J}^n_i r(c^j_i, \sigma^{-1}(c^h_j)) \tilde{J}^k_j r(c^l_m, c^k_n) - q^{(\lambda - \text{wt}(v_j), 2\rho)} r(c^h_j, c^k_n) q^{(\lambda - \text{wt}(v_j), 2\rho)} \tilde{J}^n_i r(c^l_m, c^k_n) = 0
\]

for all $a, b, m, n$. Using Proposition iv) one verifies that the above equation is equivalent to

\[
\sum_{i,j,k,l} \tilde{J}^n_i r(\sigma(c^j_i), (c^h_j)) \tilde{J}^k_j r(c^l_m, c^k_n) - r(c^h_j, c^k_n) \tilde{J}^n_i r(\sigma(c^k_n), c^l_m) = 0
\]

for all $a, b, m, n$. This is just the desired equation (35) written explicitly.

To verify the second statement, note that (37) implies

\[
\sum_{i,j} \sigma(l^-(c^l_m)) J^i_j l^+(c^n_j) = \sum_{i,j} \sigma(l^-(c^l_m)) J^i_j \sigma^2(l^+(c^n_j)) q^{(\lambda - \text{wt}(v_j), 2\rho)} = \sigma \left( \sum_{i,j} \sigma(l^+(c^h_j)) J^i_j l^-(c^l_m) \right) q^{(\lambda - \text{wt}(v_j), 2\rho)}.
\]

\[\square\]

### 4 Towards a proof of Conjecture 1

Throughout this section we assume the $g$ is simple. The proof of the stronger Conjecture 2 in the case $g = g' \oplus g'$ is contained in Section 5.
4.1 Coproduct of central elements

Recall the definition of the projection maps \( \pi_{\alpha,\beta} \) and \( \pi_{\alpha} \) for \( \alpha, \beta \in Q(\pi) \) from Section 2.2. By the following Lemma the coproduct of the central element \( d_\mu \) for \( \mu \in P_{Z(\B)} \) can be in part read off the element \( Y^\mu \) defined by (25).

**Lemma 3** Let \( \mu \in P_{Z(\B)} \) and \( \alpha, \beta \in Q(\pi) \). Then

\[
(\pi_{-(\mu-w_0\mu-\alpha),\beta} \otimes \text{id}) \Delta(d_\mu) \in (\pi_{-(\mu-w_0\mu-\alpha),\beta} \otimes \pi_{\mu-w_0\mu-\alpha,\beta}) \Delta(Y^\mu) + \sum_{\eta < w_0'\mu-\mu+\alpha+\beta} U_q(n^-)_{-(\mu-w_0\mu-\alpha)} \tilde{U}^0 G_\beta^+ \otimes \tilde{U}_q(g)_{-\eta}.
\]

**Proof:** Recall from (25) that \( d_\mu \in Y^\mu + \sum_{\gamma < 2\tilde{\mu}} \tilde{U}_q(g)_{-\gamma} \) and \( Y^\mu \in \tilde{U}_q(g)_{-2\tilde{\mu}} \).

Note moreover that \( \Delta \tilde{U}_q(g)_\nu \subseteq \sum_{\zeta+\xi=\nu} \tilde{U}_q(g)_\zeta \otimes \tilde{U}_q(g)_\xi \). Hence using \( 2\tilde{\mu} = w_0'\mu - w_0\mu \) one obtains

\[
(\pi_{-(\mu-w_0\mu-\alpha),\beta} \otimes \text{id}) \Delta(Y^\mu) = (\pi_{-(\mu-w_0\mu-\alpha),\beta} \otimes \pi_{\mu-w_0\mu-\alpha-\beta}) \Delta(Y^\mu)
\]

\[
(\pi_{-(\mu-w_0\mu-\alpha),\beta} \otimes \text{id}) \Delta(\tilde{U}_q(g)_{-\gamma}) \subseteq \sum_{\eta < w_0'\mu-\mu+\alpha+\beta} U_q(n^-)_{-(\mu-w_0\mu-\alpha)} \tilde{U}^0 G_\beta^+ \otimes \tilde{U}_q(g)_{-\eta}
\]

for any \( \gamma < 2\tilde{\mu} \). \( \blacksquare \)

4.2 Finding one generator \( B_i \) in \( L_\mu \)

Recall that for any \( \mu \in P_{Z(\B)} \) the symbol \( L_\mu \) denotes the left coideal generated by the central element \( d_{-w_0\mu} \in Z(\B) \).

**Proposition 6** Let \( \mu \in P_{Z(\B)} \) such that \((\mu, \alpha_j) = 0 \) for all \( \alpha_j \in \pi_\Theta \) and \( 2(\mu, \alpha_i)/(\alpha_i, \alpha_i) = 1 \) for some \( \alpha_i \notin \pi_\Theta \). Then \( B_i \tau(-\Theta(\mu)-\mu) \in L_\mu \) and \( C_i \tau(-\Theta(\mu)-\mu) \in L_\mu \).

**Proof:** Consider the central element \( d_{-w_0\mu} \) which lies in \((\text{ad}, \tilde{U}_q(g))\tau(-2w_0\mu)\) up to terms of lower filter degree, and its highest weight component \( Y^{-w_0\mu} \).

Note that \((\mu, \alpha_j) = 0 \) for all \( \alpha_j \in \pi_\Theta \) implies \( w_0'\mu = \mu \). Hence, using \( \mu \in P_{Z(\B)} \), one obtains \( \Theta(w_0\mu) = \mu \) and thus

\[
-2w_0\mu = -w_0\mu - \Theta(-w_0\mu) = \mu - w_0\mu.
\]
By [JL94, 4.8] there is an isomorphism of right $U_q(n^-)$-modules

$$(\text{ad}_r U_q(n^-)) \tau(-2w_0 \mu) \cong V(\mu)^*, \quad (\text{ad}_r u) \tau(-2w_0 \mu) \mapsto v_{w_0 \mu}^* u$$ (39)

where $u \in U_q(n^-)$ and $v_{w_0 \mu}^* \in V(\mu)^*$ is nonzero on the lowest weight vector $v_{w_0 \mu}$ and vanishes on all other weight vectors of $V(\mu)$. The isomorphism (39) implies that

$$Y^{-w_0 \mu} = (\text{ad}_r y_i)(\text{ad}_r \tilde{Y}) \tau(-2w_0 \mu)$$

for some $\tilde{Y} \in U_q(n^-)_{- (\mu - w_0 \mu - \alpha_i)}$. Moreover, using $2(\mu, \alpha_i)/(\alpha_i, \alpha_i) = 1$ and the isomorphism (39) one sees that the weight space with respect to the left adjoint action $((\text{ad}_r U_q(n^-)) \tau(-2w_0 \mu))_{- (\mu - w_0 \mu - 2\alpha_i)}$ is empty. Hence, by (10) one obtains

$$(\tau_{- (\mu - w_0 \mu - \alpha_i), 0} \otimes \tau_{- \alpha_i}) \Delta(Y^{-w_0 \mu})$$

$$= (\text{ad}_r \tilde{Y}) \tau(-2w_0 \mu) \otimes \tau(-w_0 \mu - \mu + \alpha_i) y_i - q^{-\alpha_i} \tau(-w_0 \mu - \mu + \alpha_i) y_i \tau(-w_0 \mu - \mu + \alpha_i)$$

$$= (1 - q^{2(\alpha_i, \alpha_i)})(\text{ad}_r \tilde{Y}) \tau(-2w_0 \mu) \otimes \tau(-w_0 \mu - \mu + \alpha_i) y_i$$

$$= (1 - q^{2(\alpha_i, \alpha_i)})(\text{ad}_r \tilde{Y}) \tau(-2w_0 \mu) \otimes \tau(-w_0 \mu - \Theta(\mu)) y_i t_i$$

Lemma 3 for $\alpha = \alpha_i$, $\beta = 0$ and the coideal property for $L_\mu$ now imply that

$$L_\mu \cap \left( y_i t_i \tau(-\Theta(\mu) - \mu) + \sum_{\eta < \alpha_i} \hat{U}_q(g)_{-\eta} \right) \neq 0.$$ 

Hence by [21] there exists $m \in \mathcal{M} T_\Theta'$ such that $B_\mu \tau(-\Theta(\mu) - \mu) + m \in L_\mu$. Using the coideal property again one obtains $B_\mu \tau(-\Theta(\mu) - \mu) \in L_\mu$. The relation $C_\mu \tau(-\Theta(\mu) - \mu) \in L_\mu$ now follows from Lemma 2.

4.3 Finding generators $x_j, y_j \in L_\mu T'_\Theta \cap \mathcal{M}$

Proposition 7 Let $\mu \in P_{Z(B)}$ and $\alpha_i \in \pi \setminus \pi_\Theta$ and $\tau(\lambda) \in T'_\Theta$. Assume that $B_\mu \tau(\lambda) \in L_\mu$ and $(\alpha_i, \alpha_j) \neq 0$ for some $\alpha_j \in \pi_\Theta$. Then $x_j \tau(\lambda + \alpha_i + \Theta(\alpha_i)) \in L_\mu$ and $y_j t_j \tau(\lambda + \alpha_i + \Theta(\alpha_i)) \in L_\mu$.

Proof: Consult the list [Ara62, 5.11] of irreducible symmetric pairs for simple $\mathfrak{g}$ and note that $(\alpha_i, \alpha_j) \neq 0$ implies

$$2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j) = -1$$ (40)

unless the symmetric pair is of one of the following types:
1. (CII, case 2) and \(i = n, j = n - 1\).

2. \(BI/BII\) and \(r = i = n - 1, j = n\).

In case 1. by \([\text{Let03}, \text{Section 7}]\) one has \(B_n = y_n t_n + [(\text{ad}_r x_{n-1}^2)t_{n-1}^{-1} x_n] t_n\). Hence, using (5) and the relation \((\alpha_n, \alpha_{n-1}) = - (\alpha_{n-1}, \alpha_{n-1})\) one computes

\[
(\pi_{\alpha_n + \alpha_n} \otimes \text{id}) \Delta B_n = (q^{2(\alpha_{n-1}, \alpha_{n-1})} - 1) [(\text{ad}_r x_{n-1}) t_{n-1}^{-1} x_n] t_n \otimes t_{n-1}^{-2} x_{n-1}.
\]

The coideal property of \(L_\mu\) now implies

\[
x_{n-1} \tau(\lambda + \alpha_n + \Theta(\alpha_n)) = x_{n-1} \tau(\lambda - 2 \alpha_{n-1}) \in L_\mu.
\]

The relation \(y_{n-1} t_{n-1} \tau(\lambda + \alpha_n + \Theta(\alpha_n)) \in L_\mu\) follows from Lemma 2. Interchanging \(n\) and \(n - 1\) one obtains the claim of the Lemma in the case 2. Hence from now on we may assume (40) to hold.

Note that \(t_{p(i)}^{-1} x_{p(i)}\) generates a finite dimensional \(\text{ad}_r(\mathcal{M})\)-module \(V(\mathcal{M}, i)\) of lowest weight \(\Theta(\alpha_i)\) with respect to the right adjoint action. The relation (40) implies that

\[
V(\mathcal{M}, i)_{-\Theta(\alpha_i) - 2\alpha_j} = \{0\}.
\]

and that there exists \(\tilde{X} \in \mathcal{M}^+\) such that

\[
B_i = y_i t_i + c \left( (\text{ad}_r x_j)(\text{ad}_r \tilde{X})(t_{p(i)}^{-1} x_{p(i)}) \right) t_i + s_i t_i.
\]

Using (4), (11), and the abbreviation \(t = \tau(\Theta(\alpha_i) + \alpha_j)\) one obtains

\[
(\pi_{-\Theta(\alpha_i) - \alpha_j} \otimes \text{id})(B_i \tau(\lambda)) = [\text{ad}_r \tilde{X}] t_{p(i)}^{-1} x_{p(i)} [t_i \tau(\lambda) \otimes [\sigma(x_j) t - q^{\alpha_j, \Theta(\alpha_i) + \alpha_j} t \sigma(x_j)] t_i \tau(\lambda)] = - [(\text{ad}_r \tilde{X}) t_{p(i)}^{-1} x_{p(i)}] t_i \tau(\lambda) \otimes t_j^{-1} x_j t_i t_j \tau(\lambda)(1 - q^{2(\alpha_j, \Theta(\alpha_i) + \alpha_j)}).
\]

Note that (40) implies \((\alpha_j, \Theta(\alpha_i) + \alpha_j) = (\alpha_j, \alpha_i + \alpha_j) + (\alpha_j, \alpha_i) \neq 0\). Hence the coideal property of \(L_\mu\) implies \(x_j t_j \tau(\lambda + \alpha_i + \Theta(\alpha_i)) \in L_\mu\). The relation \(y_j t_j \tau(\lambda + \alpha_i + \Theta(\alpha_i)) \in L_\mu\) now follows from Lemma 2. ■
4.4 Obtaining $\mathcal{M}$-invariance

By [Let02 (7.5)] for any $\alpha_i \in \pi \setminus \pi_\Theta$ one can write

$$\Theta(\alpha_i) + \alpha_p(i) = \sum_{\alpha_j \in \pi_\Theta} n_{ij} \alpha_j$$

for some uniquely determined nonpositive integers $n_{ij}$. Define a subset of $\pi_\Theta$ by

$$\pi_{\Theta,i} = \{\alpha_j \in \pi_\Theta | n_{ij} \neq 0\}$$

and let $\mathcal{M}_i$ denote the subalgebra of $\mathcal{M}$ generated by \{\begin{align*}x_j, y_j, t_j, t_j^{-1} | \alpha_j \in \pi_{\Theta,i}\end{align*}\}.

**Proposition 8** Let $\mu \in P_{Z(B)}$ and $\alpha_i \in \pi \setminus \pi_\Theta$ and $\tau(\lambda) \in T_\Theta$. Assume that $B_i \tau(\lambda) \in L_\mu$. Then

$$V(\nu)^{L_\mu} \subseteq V(\nu)^{\mathcal{M}_i}$$

for any $\nu \in P^+(\pi)$.

**Proof**: Assume $\alpha_m \in \pi_{\Theta,i}$. There exist uniquely determined pairwise distinct simple roots $\alpha_i = \alpha_{i_0}, \alpha_{i_1}, \ldots, \alpha_{i_k} = \alpha_m$ such that $(\alpha_{i_j}, \alpha_{i_{j'}}) \neq 0$ if and only if $|j-j'| = 1$. For any $v \in V(\nu)^{L_\mu}$ we prove $x_m v = 0 = y_m v$ by induction on $k$. Proposition 7 implies $x_{i_j} \tau(\lambda + \alpha_i + \Theta(\alpha_i)) \in L_\mu$ and $y_{i_j} \tau(\lambda + \alpha_i + \Theta(\alpha_i)) \in L_\mu$, and hence the claim holds for $k = 1$. Moreover, by Lemma 1 one has

$$\text{ad}_t(x_{i_1} x_{i_2} \ldots x_{i_k}) x_{i_1} \tau(\lambda + \alpha_i + \Theta(\alpha_i)) \in L_\mu. \quad (42)$$

By induction hypothesis $v$ is invariant under $x_{i_1}, \ldots, x_{i_{k-1}}$. Hence (42) implies

$$x_{i_1} x_{i_2} \ldots x_{i_{k-1}} x_{i_k} v = 0. \quad (43)$$

Moreover, as the $\alpha_{i_j}$ are pairwise distinct and $v$ is invariant under $y_{i_1}$ one obtains

$$y_{i_1} x_{i_2} \ldots x_{i_{k-1}} x_{i_k} v = 0. \quad (44)$$

Relations (43), (44) and the induction hypothesis imply

$$x_{i_2} \ldots x_{i_{k-1}} x_{i_k} v = t_{i_1} x_{i_2} \ldots x_{i_{k-1}} x_{i_k} v = q^{(\alpha_{i_1}, \alpha_{i_2})} x_{i_2} \ldots x_{i_{k-1}} x_{i_k} v$$

and hence $x_{i_2} \ldots x_{i_{k-1}} x_{i_k} v = 0$. Repeating the above argument one obtains $x_{i_k} v = 0$. The relation $y_{i_k} v = 0$ is obtained in a similar manner. ■
4.5 The example FII

We apply the results of the previous subsections to prove Conjecture 1 for the quantum symmetric pair of type FII. Let $\mu = \omega_4$ be the only minimal weight in $P_{Z(B)}$, see Table 2. Note that $\Theta(\omega_4) = -\omega_4$ and hence Proposition 6 implies $B_4, C_4 \in L_\mu$. By Proposition 8 one obtains the claim of Conjecture 1.

Note that we have not proved that $x_i, y_i \in L_{\mu}T'_\Theta$ for $i = 1, 2$. However, Proposition 7 implies that $x_3, y_3, t_3 \in L_{\mu}T_\Theta(\alpha_1 + 2\alpha_2 + 3\alpha_3)$.

Remark: Note that in Propositions 6, 7, and 8 we have not assumed that $\mu \in P_{Z(B)}$ is minimal. However, the assumption $2(\mu, \alpha_i) = (\alpha_i, \alpha_i)$ of Proposition 6 doesn’t hold for general $\mu \in P_{Z(B)}$. In general the assumption $(\mu, \alpha_j) = 0$ for all $\alpha_j \in \pi_\Theta$ doesn’t even hold for all minimal $\mu \in P_{Z(B)}$. This is the reason why our arguments are not yet good enough to prove Conjectures 1 and 2 in general.

5 The proof of Conjecture 2 for $\pi_\Theta = \emptyset$

Let as before $L_{-w_0\mu}$ denote the left coideal generated by the central element $d_\mu$. For any $\alpha_i \in \pi$ we want to show that there exists $t \in T'_\Theta$ such that $B_it \in L_{-w_0\mu}$. For any multiindex $I$ recall the notation $x_I, y_I$ from Section 2.2.

5.1 One dimensional weight spaces of $V(\mu)^*$

Fix $\mu \in P^+(\pi)$ and let $\mathcal{I}_\mu$ denote a set of multiindices, including the empty multiindex $I_0 = ()$, such that $\{v_\mu^*x_I | I \in \mathcal{I}_\mu\}$ is a basis of $V(\mu)^*$. Here $v_\mu^* \in V(\mu)^*$ is defined by $v_\mu^*(v_\mu) = 1$ and $v_\mu^*(v) = 0$ for all $v \in V(\mu)$ of weight unequal $\mu$. In short, $v_\mu^*$ is a highest weight vector of the right $U_q(\mathfrak{g})$-module $V(\mu)^*$. Note that $\{v_{-\mu}^*y_I | I \in \mathcal{I}_\mu\}$ is a basis of $V(-w_0\mu)^*$ where $v_{-\mu}^*$ denotes a lowest weight vector of $V(-w_0\mu)^*$. In general, given $\mu \in P^+(\pi)$, there are many possible choices for $\mathcal{I}_\mu$. However, certain multiindices always have to be contained, as expressed by the following Lemma.

Lemma 4 Let $\mu \in P^+(\pi)$ and let $I = (i_1, \ldots, i_k)$ be a multiindex with the following properties:

1) $(\mu, \alpha_{i_r}) \neq 0$ if and only if $r = 1$. 

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2) \((\alpha_i, \alpha_i) \neq 0\) if and only if \(|r-s| = 1\).

Then \(I \in \mathcal{I}_\mu\). Furthermore, the multiindices \(I_l := (i_1, \ldots, i_l)\) for \(1 \leq l \leq k\) satisfy the relations

\[
\dim(V(\mu)_{\mu-wt(I_l)}) = 1 = \dim(V(-w_0\mu)_{\mu+wt(I_l)}). \tag{45}
\]

If moreover \(\mu \in P_{Z(\hat{B})}\) then

\[
\dim(V(\mu)_{\Theta(\mu-wt(I_l))}) = 1. \tag{46}
\]

for all \(l = 1, \ldots, k\).

**Proof:** Recall [Jos95, Proposition 4.2.8 (iii)] that the right module \(V(\mu)^*\) is generated by the highest weight vector \(v_\mu^*\) and relations

\[
v_\mu^* y_i = 0, \quad v_\mu^* x_i^{2(\mu, \alpha_i)/(\alpha_i, \alpha_i)+1} = 0, \quad v_\mu^* t_i = q^{(\mu, \alpha_i)} v_\mu^*.
\]

Using this presentation the first statement of the lemma and the left hand side of (45) are proved by induction on \(k\). The right hand side of (45) follows from the fact that there is a nondegenerate pairing of \(\hat{U}_q(\mathfrak{g})\)-modules between \(V(\mu)\) and \(V(-w_0\mu)\).

Recall [Let02, Section 7] that \(\Theta = -w_0^\prime d\) where \(d\) is a diagram automorphism. Hence for any \(\mu \in P^+(\pi)\) one has \(\Theta(w_0^\prime w_0\mu) \in P^+(\pi)\). To verify (46) note moreover that \(w_0^\prime(\mu - \Theta(\mu)) = \mu - \Theta(\mu)\) for all \(\mu \in P^+(\pi)\) and hence \(\Theta(w_0^\prime w_0\mu) = \mu\) if \(\mu \in P_{Z(\hat{B})}\). Hence it suffices to show that

\[
\dim(V(\mu)_{\nu}) = \dim(V(\Theta(w_0^\prime w_0\mu))_{\Theta(\nu)}^*) \tag{47}
\]

for all \(\mu \in P^+(\pi)\) and \(\nu \in P(\pi)\). Moreover, it suffices to verify (47) in the classical situation. By slight abuse of notation we denote the irreducible left \(\mathfrak{g}\)-module of highest weight \(\mu\) also by \(V(\mu)\). To verify (47) note that the vectorspace \(V(\mu)\) with the \(\mathfrak{g}\)-module structure defined by \(g \cdot v = \theta(g)v\) is isomorphic to \(V(\Theta(w_0^\prime w_0\mu))\). Indeed, the weight vector \(v_{w_0^\prime w_0\mu} \in V(\mu)\) of weight \(w_0^\prime w_0\mu\) for the old \(\mathfrak{g}\)-module structure is the highest weight vector for the new \(\mathfrak{g}\)-module structure \(\cdot\) and it is of weight \(\Theta(w_0^\prime w_0\mu)\). \(\blacksquare\)
5.2 A weight estimate

Before making use of the above Lemma we take another close look at the structure of the central element $d_\mu$. Recall (26) and write $v_\mu \in (\text{ad}_r \tilde{U}_q(\mathfrak{g}))\tau(2\mu)$ in the form

$$v_\mu = \sum_{I,J \in I_\mu} a_{IJ}(\text{ad}_r x_I)(\text{ad}_r y_J)\tau(2\mu).$$  \hspace{1cm} (48)

Lemma 5 Assume $\pi_\Theta = \emptyset$ and $\mu \in P_{Z(\tilde{B})}$. Then the following relation holds

$$v_\mu \in \sum_{-\Theta(\zeta) + \xi \leq 2\tilde{\mu}} (\text{ad}_r U_q(n^+)_\zeta)(\text{ad}_r U_q(n^-)_-\xi)\tau(2\mu).$$ \hspace{1cm} (49)

In particular, the coefficients $a_{IJ}$ in (48) vanish if $\text{wt}(J) - \Theta(\text{wt}(I)) \leq 2\tilde{\mu}$.

Proof: The relation $d_\mu \in \sum_{\nu \leq 2\tilde{\mu}} \tilde{U}_q(\mathfrak{g})\tau(\nu)$ together with $d_\mu \in \tilde{B} = \sum B_J T_\Theta^J$. Hence, using the explicit form (20) of the generators $B_i$, one obtains $d_\mu \in \sum_{-\Theta(\zeta) + \xi \leq 2\tilde{\mu}} U_q(n^+)_\zeta \tilde{U}_q(n^-)_-\xi$ which by [KL, Lemma 2.2] and (26) implies (49). □

5.3 Yet another projection

In the proof of the following lemma and subsequent arguments we will make use of a projection

$$\pi_{IJ} : (\text{ad}_r \tilde{U}_q(\mathfrak{g}))\tau(2\mu) \to C(\text{ad}_r x_I)(\text{ad}_r y_J)\tau(2\mu)$$

onto the space spanned by $(\text{ad}_r x_I)(\text{ad}_r y_J)\tau(2\mu)$. More precisely, for any element $a = \sum_{I',J' \in I_\mu} a_{I',J'}(\text{ad}_r x_{I'}) (\text{ad}_r y_{J'})\tau(2\mu)$ of $(\text{ad}_r \tilde{U}_q(\mathfrak{g}))\tau(2\mu)$ and any $I, J \in I_\mu$ we define $\pi_{IJ}(a) = a_{IJ}(\text{ad}_r x_I)(\text{ad}_r y_J)\tau(2\mu)$. Note that the formulas (9) and (10) imply that

$$\Delta((\text{ad}_r x_I)(\text{ad}_r y_J)\tau(2\mu)) \subseteq \sum_{\text{wt}(I') \leq \text{wt}(I)} C(\text{ad}_r x_{I'}) (\text{ad}_r y_{J'})\tau(2\mu) \otimes \tilde{U}_q(\mathfrak{g}).$$ \hspace{1cm} (50)

We now show that there are sufficiently many nonvanishing coefficients for which the bound of Lemma 5 is attained. We use the notation of Lemma 4 and assume that $\mu \in P_{Z(\tilde{B})}$. By (46) for any $I_l \in 1, \ldots, k$, there exists a uniquely determined $T_{I_l} \in I_\mu$ such that $\mu - \text{wt}(T_{I_l}) = \Theta(\mu - \text{wt}(I_l))$. Note that by definition $\text{wt}(I_l) - \Theta(\text{wt}(T_{I_l})) = 2\tilde{\mu}$. 28
Lemma 6 Assume \( \pi_0 = \emptyset \) and \( \mu \in P_{Z(B)} \) and let \( I = (i_1, \ldots, i_k) \) be a multiindex satisfying 1) and 2) of Lemma 4. For any \( l=1, \ldots, k \) the coefficient \( a_{\mathbf{I}_l} \) in (48) does not vanish.

Proof: We proceed by induction on \( l \). Note that for the empty multiindex \( I_0 = () \) relation (29) implies \( a_{I_0} \neq 0 \). Assume now that \( a_{I_l} \neq 0 \) for some \( l \). Recall that by definition \( B_i = y_i t_i + t_{p(i)}^{-1} x_{p(i)} t_i + s_i t_i \) for any \( \alpha_i \in \pi \).

We will use the relation (ad
\[ r
\]
\[ y_i t_i \])\( v_{\mu} = s_i v_{\mu} \) (51) to show that \( a_{I_l} \neq 0 \). Recall the definition of the coefficients \( a_{IJ} \) in (48) and note that

\[
(\text{ad}_r y_i t_i)(v_{\mu}) \in (\text{ad}_r t_i) \sum_{I,J \in I_{\mu}} a_{IJ}(\text{ad}_r x_I)(\text{ad}_r y_J) \tau(2\mu) + \bigcap_{wt(J)-\Theta(wt(I))=2\bar{\mu}+\alpha_i} \ker(\pi_{IJ})
\]

\[
\subseteq \sum_{wt(J)-\Theta(wt(I))=2\bar{\mu}+\alpha_i} a'_{IJ}(\text{ad}_r x_I)(\text{ad}_r y_J) \tau(2\mu) + \bigcap_{wt(J)-\Theta(wt(I))=2\bar{\mu}+\alpha_i} \ker(\pi_{IJ}) \quad (52)
\]

for some \( a'_{IJ} \in \mathcal{C} \). Note moreover that in view of conditions 1) and 2) of Lemma 4 the assumption \( a_{I_l} \neq 0 \) implies \( a'_{I_l} \neq 0 \). Similarly one calculates

\[
(\text{ad}_r t_{p(i)}^{-1} x_{p(i)} t_i)(v_{\mu}) \sum_{I,J \in I_{\mu}} a_{IJ}(\text{ad}_r t_{p(i)}^{-1} x_{p(i)} t_i)(\text{ad}_r x_I)(\text{ad}_r y_J) \tau(2\mu)
\]

\[
= \sum_{I,J \in I_{\mu}} a''_{IJ}(\text{ad}_r x_I)(\text{ad}_r y_J) \tau(2\mu) \quad (53)
\]

for some \( a''_{IJ} \in \mathcal{C} \). Note here, that \( a''_{I_l} \neq 0 \) if and only if \( a_{I_l} \neq 0 \). Finally by Lemma 4 both \( v_{\mu} \) and \( (\text{ad}_r t_i)(v_{\mu}) \) are contained in

\[
\bigcap_{wt(J)-\Theta(wt(I))=2\bar{\mu}+\alpha_i} \ker(\pi_{IJ})
\]

In view of (51) the relations (52) and (53) imply that \( a'_{I_l} = -a''_{I_l} \). Now \( a_{I_l} \neq 0 \) follows from the properties of \( a'_{I_l} \) and \( a''_{I_l} \) noted above. \( \blacksquare \)
5.4 Finding $B_i \in L_{-2w_0\mu}T'_\Theta$

We are now in a position to prove that for any $\mu \in P_{Z(\bar{B})}$ the right coideal $L_{-2w_0\mu}$ contains sufficiently many generators of $\bar{B}$.

**Proposition 9** Assume that $\pi_\Theta = \emptyset$ and $\mu \in P_{Z(\bar{B})}$. Then for any $\alpha_i \in \pi$ there exists $t(i) \in T'_\Theta$ such that $B_i t(i) \in L_{-2w_0\mu}$.

**Proof:** As $\pi_\Theta = \emptyset$ we can choose $I = (i_1, \ldots, i_k)$ with $i_k = i$ such that assumptions 1) and 2) of Lemma 4 hold. Note that $\text{wt}(I_{k-1}) - \Theta(\text{wt}(\bar{T}_k)) = 2\bar{\mu} - \alpha_k$. In view of (48) and Lemma 5 the relation (50) implies

$$\text{(54)}$$

Using formulas (9) and (10) and assumptions 1) and 2) of Lemma 4 again one obtains

$$\text{(55)}$$

where $a := (\alpha_i, \alpha_{i_1} + \cdots + \alpha_{i_{k-1}})$ and $t := \tau(2\mu - \alpha_{i_1} - \cdots - \alpha_{i_{k-1}})$. Note that if $(\mu, \alpha_i) \neq 0$ then by construction $k = 1$ and hence $a = 0$ and $t = \tau(2\mu)$. If on the other hand $(\mu, \alpha_i) = 0$ then $a = (\alpha_i, \alpha_{i_{k-1}}) \neq 0$. One obtains

$$\text{(56)}$$

In any case, (32), (53), (34), and Lemma 4 imply that there exists

$$b \in L_{-w_0\mu} \cap (y_i t \tau(-\text{wt}(\bar{T}_k)) + \sum_{\nu \geq 0} \bar{U}_q(\mathfrak{g})_\nu)$$

Hence $B_i t(i) + m \in L_{-w_0\mu}$ for some $m \in \mathfrak{c} T'_\Theta$ and $t(i) = \tau(2\mu - \text{wt}(I_k) - \text{wt}(\bar{T}_k))$. Note that the relation $\text{wt}(I_k) - \Theta(\text{wt}(\bar{T}_k)) = 2\bar{\mu}$ implies $t(i) = \tau(\mu - \text{wt}(I_k) + \Theta(\mu - \text{wt}(I_k))) \in T'_\Theta$. Applying the coideal property of $L_{-w_0\mu}$ to the element $B_i t(i) + m$ one obtains $B_i t(i) \in L_{-w_0\mu}$. 

**Corollary 1** If $\pi_\Theta = \emptyset$ then Conjecture 3 holds.
Proof: By the above Proposition the coideal $L_\mu$ contains elements $B_i t(i)$ for all $\alpha_i \in \pi$ and some $t(i) \in T_\Theta$. If $\varepsilon(B_i) = 0$ then any element in $V(\nu)^{L_\mu}$ is also invariant under $B_i$. Note that $\varepsilon(B_i) = 0$ holds for all $\alpha_i$ such that $\Theta(\alpha_i) = -\alpha_{p(i)} \neq -\alpha_i$. By [Let03, Theorem 7.1(iii)] one has

$$B_i B_{p(i)} - B_{p(i)} B_i = \frac{t_i t_{p(i)}^{-1} - t_{p(i)}^{-1} t_i}{q^{(\alpha_i, \alpha_i)/2} - q^{-(\alpha_i, \alpha_i)/2}}$$

and hence $B_i$, $B_{p(i)}$, $t_i t_{p(i)}^{-1}$, and $t_{p(i)}^{-1} t_i$ generate a subalgebra isomorphic to $U_q(\mathfrak{sl}_2)$. Therefore invariance under $B_i$ and $B_{p(i)}$ implies invariance under $t_i t_{p(i)}^{-1}$. Hence $V(\nu)^{L_\mu} \subseteq V(\nu)^{T_\Theta}$. Proposition 9 now implies that any element in $V(\nu)^{L_\mu}$ is also invariant under all $B_i$. ■

References

[Ara62] S. Araki, On root systems and an infinitesimal classification of irreducible symmetric spaces, J. Math. Osaka City Univ. 13 (1962), 1–34.

[Cal93] P. Caldero, Éléments ad-finis de certains groupes quantiques, C. R. Acad. Sci. Paris (I) 316 (1993), 327–329.

[Che84] I.V. Cherednik, Factorizing particles on a half-line and root systems, Theoret. Math. Phys. 61 (1984), 977–983.

[Dij96] M.S. Dijkhuizen, Some remarks on the construction of quantum symmetric spaces, Acta Appl. Math. 44 (1996), no. 1-2, 59–80.

[DN98] M.S. Dijkhuizen and M. Noumi, A family of quantum projective spaces and related q-hypergeometric orthogonal polynomials, Transactions of the AMS 350 (1998), 3269–3296.

[DS99] M.S. Dijkhuizen and J.V. Stokman, Some limit transitions between BC type orthogonal polynomials interpreted on quantum complex Grassmannians, Publ. Res. Inst. Math. Sci. 35 (1999), 451–500.

[Gai95] D. Gaitsgory, Existence and uniqueness of the $R$-matrix in quantum groups, J. Algebra 176 (1995), 653–666.
[HK03] I. Heckenberger and S. Kolb, Podleś’s quantum sphere: dual coalgebra and classification of covariant first order differential calculus, J. Algebra 263 (2003), 193–214.

[HS98] I. Heckenberger and K. Schmüdgen, Classification of bicovariant differential calculi on the quantum groups $SL_q(n+1)$ and $Sp_q(2n)$, J. reine angew. Math. 502 (1998), 141–162.

[Hum72] J. E. Humphreys, Introduction to Lie algebras and representation theory, Springer-Verlag, New York, 1972.

[JL94] A. Joseph and G. Letzter, Separation of variables for quantized enveloping algebras, Amer. J. Math. 116 (1994), 127–177.

[JL95] ———, Verma module annihilators for quantized enveloping algebras, Ann. Sci. Éc. Norm. Sup. 28 (1995), 493–526.

[Jos95] A. Joseph, Quantum groups and their primitive ideals, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1995.

[KL] S. Kolb and G. Letzter, The center of quantum symmetric pair coideal subalgebras, Preprint, math.QA/0602638.

[KS97] A. Klimyk and K. Schmüdgen, Quantum groups and their representations, Springer-Verlag, Heidelberg, 1997.

[KSS93] P.P. Kulish, R. Sasaki, and C. Schwiebert, Constant solutions of reflection equations and quantum groups, J. Math. Phys. 34 (1993), no. 1, 286–304.

[Let] G. Letzter, Invariant differential operators for quantum symmetric spaces, Mem. Amer. Math. Soc., in press (available online http://www.math.vt.edu/people/letzter/invdiff.pdf).

[Let99] ———, Symmetric pairs for quantized enveloping algebras, J. Algebra 220 (1999), 729–767.

[Let02] ———, Coideal subalgebras and quantum symmetric pairs, New directions in Hopf algebras (Cambridge), MSRI publications, vol. 43, Cambridge Univ. Press, 2002, pp. 117–166.
[Let03] ———, Quantum symmetric pairs and their zonal spherical functions, Transformation Groups 8 (2003), 261–292.

[Let04] ———, Quantum zonal spherical functions and Macdonald polynomials, Adv. Math. 189 (2004), 88–147.

[Nou96] M. Noumi, Macdonald’s symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces, Adv. Math. 123 (1996), 16–77.

[NS95] M. Noumi and T. Sugitani, Quantum symmetric spaces and related $q$-orthogonal polynomials, Group theoretical methods in physics (Singapore) (A. Arima et. al., ed.), World Scientific, 1995, pp. 28–40.

[Tan92] T. Tanisaki, Killing forms, Harish-Chandra isomorphisms and universal $R$-matrices for quantum algebras, Infinite Analysis, World Scientific, 1992, pp. 941–962.

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