Dunkl Translations, Dunkl-Type BMO Space, and Riesz Transforms for the Dunkl Transform on $L^\infty$

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ABSTRACT. In this paper we will give some results on the support of Dunkl translations on compactly supported functions. Then we will define the Dunkl-type BMO space and Riesz transforms for the Dunkl transform on $L^\infty$ and prove the boundedness of the Riesz transforms from $L^\infty$ to the Dunkl-type BMO space under the assumption of the uniform boundedness of Dunkl translations. The proof and the definition in the Dunkl setting will be harder than in the classical case for the lack of some properties of Dunkl translations similar to those of classical translations. We will also extend Gallardo and Rejeb’s precise description of the support of Dunkl translations on characteristic functions to all nonnegative radial functions in $L^2(m_k)$.

KEY WORDS: Dunkl translations, Riesz transforms, Dunkl-type BMO space

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1. Introduction

Let $T$ be a bounded operator on $L^2(\mathbb{R}^N)$, and let $K$ be a function on $\mathbb{R}^N \times \mathbb{R}^N \setminus \{(x, x) : x \in \mathbb{R}^N\}$ such that, for any $f \in L^2(\mathbb{R}^N)$ with compact support,

$$ Tf(x) = \int_{\mathbb{R}^N} K(x, y)f(y) \, dy, \quad x \in \mathbb{R}^N \setminus \text{supp}(f), $$

where $K$ satisfies

$$ \int_{|x-y|>2|x-w|} |K(x, y) - K(w, y)| \, dy \leq C; \quad (1.1) $$

then $T$ is a bounded operator from $L^\infty$ to the BMO space, or the space of bounded mean oscillation functions. Let $K(x, y) = c_N(x_j - y_j)/|x-y|^{N+1}$, $j = 1, \ldots, N$. For any $\varepsilon > 0$, consider the truncation $K_\varepsilon$ defined by $K_\varepsilon(x, y) = K(x, y)$ if $|x - y| > \varepsilon$ and $K_\varepsilon(x, y) = 0$ if $|x - y| \leq \varepsilon$. For a bounded function $f$, the ordinary Riesz transform is defined by

$$ R_j(f)(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} (K_\varepsilon(x, y) - K_1(0, y))f(y) \, dy. $$

It is well known that the Riesz transform is bounded on $L^2(\mathbb{R}^N)$ and that the kernel $K(x, y)$ satisfies (1.1) and hence is a bounded operator from $L^\infty$ to the BMO space. In this paper we will extend similar results to the context of Dunkl theory.

In [2] the $L^p$-boundedness, $1 < p < \infty$, and the weak $L^1$ boundedness of the Riesz transforms for the Dunkl transform was proved by adapting classical $L^p$-theory of Calderón–Zygmund; so the Riesz transforms can be defined as bounded operators on $L^p$, $1 < p < \infty$, and weakly bounded operators on $L^1$ (see also [8] for the $L^p$ boundedness of the Dunkl–Riesz transforms with radial power weights). But there is no reasonable and consistent definition of Riesz transforms in terms of an integral on $L^1(m_k)$ in the Dunkl setting. Recently, Riesz transforms were defined in a weak sense on $L^1(m_k)$ (see [3]) by using a test function space containing a Poisson kernel, and it was shown in [3] and [6] that in the Dunkl setting, the real Hardy space $H^1_\Delta$ associated to the Dunkl Laplacian $\Delta$ can be characterized by Riesz transforms and also coincides with $H^1_{\text{atom}}$. 

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The following formula, which shows that the Dunkl translation operators $\tau_x$ are contractions on $L^2(m_k)$, is well known:

$$\|\tau_x f\|_{2,k} \leq \|f\|_{2,k}, \quad f \in L^2(m_k).$$

Assume the uniform $L^1$-boundedness of the Dunkl translations (see [7]). Then, by Riesz–Törin interpolation and the skew-symmetry of Dunkl translations, uniform $L^p$-boundedness ($1 \leq p \leq \infty$) is immediate, that is, for any root system $R$, any multiplicity function $k \geq 0$, and any $f \in L^p(m_k)$,

$$\|\tau_x f\|_{p,k} \leq C\|f\|_{p,k},$$

where $C$ is a constant independent of $x$. It has been known that this assumption holds for radial functions and the one-dimensional case, and hence for the case $G = Z^N_2$. Here we refer to this assumption as the uniform boundedness of Dunkl translations. There have been many results based on this long-standing conjecture, and it would be an excellent work if one could prove it. The authors of [11] proved the uniform boundedness of the spherical average of Dunkl translations, and as applications, they found that the uniform boundedness assumption can be avoided in the proof of some related results. But it is still inevitable for many other results, such as the $L^p$ boundedness of the Dunkl multiplier operator for $s > N/2$ (see [7; Theorem 8.1]) and Theorem 1.1 in this paper.

In this paper we will define the Dunkl-type BMO space and $\mathcal{R}_j f$, where $\mathcal{R}_j$ is the Riesz transform for the Dunkl transform, as Dunkl-type BMO functions for all $f \in L^\infty$. Then, under the assumption of the uniform boundedness of Dunkl translations, we will prove the boundedness of the Riesz transforms from $L^\infty$ to the Dunkl-type BMO space. This will also mean a half of duality of the Hardy space $H^1_\Delta$ and the Dunkl-type BMO space.

**Theorem 1.1.** Under the assumption of the uniform boundedness of Dunkl translations, the Riesz transforms for the Dunkl transform are bounded operators from $L^\infty$ to the Dunkl-type BMO space.

The part (ii) of the following theorem shows that the inclusion of the support of $\tau_x f$ given in [7; Theorem 1.7] (part (i) of the following theorem) is the coincidence when $k > 0$. Its preciseness has been proved for characteristic functions by Gallardo and Rejeb [9], and we extend the result to any nonnegative radial functions in $L^2(m_k)$ in this paper.

**Theorem 1.2.** If $f \in L^2(m_k)$ and $\text{supp } f \subseteq B(0, r)$, then, for any $x \in \mathbb{R}^N$,

(i) (see [7; Theorem 1.7])

$$\text{supp } \tau_x f(-\cdot) \subseteq \bigcup_{g \in G} B(gx, r);$$

(ii) if the multiplicity function $k$ is positive and $f$ is a nonnegative radial function in $L^2(m_k)$, $\text{supp } f = B(0, r)$, then

$$\text{supp } \tau_x f(-\cdot) = \bigcup_{g \in G} B(gx, r).$$

Part (i) of this theorem also means that the Dunkl translation of a function in $L^2(m_k)$ with compact support is compactly supported. This will be used in the proof of Theorem 1.1. However, unlike in classical analysis, the inclusion $\text{supp } f \subseteq \bigcup_{g \in G} B(gx, r)$, $x \in \mathbb{R}^N$, does not imply $\text{supp } \tau_x f \subseteq B(0, r)$, as will be shown in Section 2. For this reason, the proof of the boundedness of Riesz transforms from $L^\infty$ to the BMO space in the Dunkl setting differs from that in the classical case.

This paper is organized as follows. In Section 2 we present some definitions and fundamental results from Dunkl’s analysis. In Section 3 we prove Theorem 1.1(ii) and give more information about the support of Dunkl translations on compactly supported functions based on results of [7]. Section 4 is devoted to the Riesz transforms for the Dunkl transform. In Section 5 we define the
Dunkl-type BMO space and the Riesz transforms for the Dunkl transform on $L^\infty$ and prove the boundedness of the Riesz transforms from $L^\infty$ to the Dunkl-type BMO space. We first prove this for compactly supported functions and then for all functions in $L^\infty$, using a lemma which we give in the same section.

2. Preliminaries

For any $x$ and $y$ in Euclidean space $\mathbb{R}^N$, we denote by $\langle x, y \rangle = \sum_{j=1}^N x_j y_j$ the standard inner product associated with the norm $\|x\|$. For any nonzero vector $\alpha \in \mathbb{R}^N$, the reflection $\sigma_\alpha$ with respect to the hyperplane $\alpha^\perp$ orthogonal to $\alpha$ is defined as

$$\sigma_\alpha(x) = x - 2 \frac{\langle x, \alpha \rangle}{\|\alpha\|^2} \alpha.$$ 

A finite set $R \subset \mathbb{R}^N \setminus \{0\}$ is called a root system if $\sigma_\alpha(R) = R$ for any $\alpha \in R$. Given a root system $R$, the finite subgroup $G$ of $O(N)$ generated by the reflections $\sigma_\alpha$ is called the finite reflection group of the root system. We define a multiplicity function $k : R \to \mathbb{C}$ as a $G$-invariant function, that is, such that $k(\alpha) = k(\beta)$ if $\sigma_\alpha$ and $\sigma_\beta$ are conjugate. We assume that $k \geq 0$ in this paper. The Dunkl operators $T_\xi$, $\xi \in \mathbb{R}^N$, which were introduced in [5], are defined by the following deformations by difference operators of the directional derivatives $\partial_\xi$:

$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R} \frac{k(\alpha)}{2} \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}$$

$$= \partial_\xi f(x) + \sum_{\alpha \in R^+} k(\alpha) \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle},$$

where $R^+$ is any fixed positive subsystem of $R$. They commute pairwise and are skew-symmetric with respect to the $G$-invariant measure $d\nu_k(x) = h_k^2(x) \, dx$, where

$$h_k(x) = \prod_{\alpha \in R^+} |\langle \alpha, x \rangle|^{k(\alpha)}$$

and $m_k$ is a doubling measure, which means that there is a constant $C > 0$ such that

$$m_k(B(x, 2r)) \leq C m_k(B(x, r))$$

for $x \in \mathbb{R}^N$ and $r > 0$, where $B(x, r) = \{ y \in \mathbb{R}^N : \|x - y\| \leq r \}$. By $N = N + \sum_{\alpha \in R} k(\alpha)$ we denote the homogeneous dimension of the root system. Let $e_j$, $j = 1, \ldots, N$, be the canonical orthonormal basis in $\mathbb{R}^N$, and let $T_j = T_{e_j}$. The Dunkl Laplacian is defined by $\Delta = \sum_{j=1}^N T_j^2$. It commutes with the action of $G$, that is, $g \circ \Delta = \Delta \circ g$ for any $g \in G$, and has the following explicit expression:

$$\Delta f(x) = \Delta_{\text{eul}} f(x) + 2 \sum_{\alpha \in R^+} k(\alpha) \left( \frac{\langle \nabla_{\text{eul}} f, \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2} \right).$$

The operator $-\Delta$ is essentially self-adjoint and positive definite, and so $\Delta$ is the generator of the contraction semigroup $\{ e^{t\Delta} \}_{t \geq 0}$.

The operators $\partial_\xi$ and $T_\xi$ are intertwined by the Laplace-type operator

$$V_k f(x) = \int_{\mathbb{R}^N} f(y) \, d\mu_x(y)$$

associated to a family $\{ \mu_x \mid x \in \mathbb{R}^N \}$ of probability measures with compact support, that is,

$$T_\xi \circ V_k = V_k \circ \partial_\xi.$$
Specifically, the support of $\mu_x$ is contained in the convex hull $\text{co}(G \cdot x)$, where $G \cdot x = \{ g \cdot x \mid g \in G \}$ is the orbit of $x$. For any Borel set $B$ and any $r > 0$ and $g \in G$, these probability measures satisfy

$$\mu_{rx}(B) = \mu_x(r^{-1}B), \quad \mu_{gx}(B) = \mu_x(g^{-1}B).$$

The Dunkl kernel $E(x, y)$ is defined by

$$E(x, y) = V_k(e^{x \cdot y})(x) = \int_{\mathbb{R}^d} e^{\langle \eta, y \rangle} d\mu_x(\eta).$$

It is a generalization of the exponential function $e^{x \cdot y}$. For any fixed $y \in \mathbb{R}^N$, the Dunkl kernel $E(x, y)$ is the unique analytic solution to the differential equation system

$$T_\xi f = (\xi, y)f, \quad f(0) = 1.$$ 

For $f \in L^1(m_k)$, the Dunkl transform is defined by

$$F(f)(\xi) = \frac{1}{c_k} \int_{\mathbb{R}^N} f(x)e^{-i\xi \cdot x} dm_k(x), \quad c_k = \int_{\mathbb{R}^N} e^{-x^2/2} dm_k(x).$$

Obviously, $F(\triangle f)(\xi) = -\|\xi\|^2 F(f)(\xi)$ and $F(e^{1/2}f) = e^{1/2} F(f), f \in L^2(m_k)$. It follows that

$$e^{t\triangle} f(x) = k_t \ast f = \int_{\mathbb{R}^N} h_t(x, y)f(y) dm_k(y),$$

where $k_t(x) = c_k^{-1}(2t)^{-N/2} e^{-x^2/(4t)}$ and the heat kernel is $h_t(x, y) = \tau_x k_t(-y)$.

Given $x \in \mathbb{R}^N$, the Dunkl translation operator $\tau_x$ is defined on $L^1(m_k)$ by

$$F(\tau_x f)(y) = E( ) \ast f(y), \quad y \in \mathbb{R}^N.$$ 

It can also be defined by

$$\tau_x f(y) = (V_k)_y(V_k)_x[(V_k)^{-1}(f)(x + y)].$$

Below we list some basic properties of Dunkl translatons:

1. $\tau_0 = I$ (identity);
2. $\tau_x f(y) = \tau_y f(x), x, y \in \mathbb{R}^N, f \in S(\mathbb{R}^N)$ (symmetry);
3. $\tau_x(f_\lambda) = (\tau_{\lambda^{-1}x}f)_\lambda, \lambda > 0, x \in \mathbb{R}^N, f \in S(\mathbb{R}^N)$;
4. $T_\xi(\tau_x f) = \tau_x(T_\xi f), x, \xi \in \mathbb{R}^N$;
5. $\int_{\mathbb{R}^N} \tau_x f(y) g(y) dm_k(y) = \int_{\mathbb{R}^N} f(y) \tau_{-x} g(y) dm_k(y), \quad x \in \mathbb{R}^N, f, g \in S(\mathbb{R}^N)$ (skew-symmetry).

The Dunkl translations can be defined on $L^p(m_k), 1 \leq p < \infty$, in the distributional sense due to the last formula. Further,

$$\int_{\mathbb{R}^N} \tau_x f(y) dm_k(y) = \int_{\mathbb{R}^N} f(y) dm_k(y), \quad x \in \mathbb{R}^N, f \in S(\mathbb{R}^N).$$

The following formula was first proved by Rösler [13] for Schwartz functions and then extended to all continuous radial functions in [4]:

$$\tau_x f(-y) = \int_{\mathbb{R}^N} (\tilde{f} \circ A)(x, y, \eta) d\mu_x(\eta), \quad x, y \in \mathbb{R}^N, \quad (2.2)$$
Thus, we have

\[ A(x, y, \eta) = \sqrt{\|x\|^2 + \|y\|^2 - 2\langle y, \eta \rangle} = \sqrt{\|x\|^2 - \|\eta\|^2 + \|y - \eta\|^2}. \]

It follows from the symmetry of Dunkl translations that (see [10])

\[ \tau_{-x} f(y) = \tau_y f(-x) = \tau_x f(-y), \quad x, y \in \mathbb{R}^N, \ f \in S(\mathbb{R}^N)_{rad}. \]

The Dunkl convolution of Schwartz functions is defined by

\[ (f * g)(x) = \int_{\mathbb{R}^N} f(y) \tau_x g(-y) \, dm_k(y); \]

it can be written as

\[ (f * g)(x) = \int_{\mathbb{R}^N} (Ff)(\xi)(Fg)(\xi) E(ix, \xi) \, dm_k(\xi). \]

The following are some basic properties of Dunkl convolution:

1. \( F(f * g) = Ff \cdot Fg; \)
2. \( F(f \cdot g) = Ff \cdot Fg; \)
3. \( f * g = g * f; \)
4. \( (f * g) * h = f * (g * h); \)
5. \( \|f * g\|_{2,k} \leq \|f\|_{1,k} \|g\|_{2,k}. \)

\( f \in L^1(m_k), \ g \in L^2(m_k). \)

3. Some Results on the Supports of Dunkl Translations

**Proof of Theorem 1.2.** (ii) It suffices to prove that

\[ \text{supp} \ \tau_x f(-\cdot) \supseteq \bigcup_{g \in G} B(gx, r). \]

Firstly, we will prove this for continuous nonnegative radial functions. Suppose that there exists a \( y \in \bigcup_{g \in G} B(gx, r) \) (that is, there also exists a \( g \in G \) such that \( \|y - g \cdot x\| \leq r \)) for which \( y \notin \text{supp} \ \tau_x f(-\cdot), \) which means the existence of an \( \varepsilon > 0 \) such that, for any \( z \in B(y, \varepsilon), \)

\[ 0 = \tau_x f(-z) = \int_{\mathbb{R}^N} \tilde{f}(\sqrt{\|x\|^2 + \|z\|^2 - 2\langle z, \eta \rangle}) \, d\mu_x(\eta); \]

then

\[ \tilde{f}(\sqrt{\|x\|^2 + \|z\|^2 - 2\langle z, \eta \rangle}) = 0 \quad \text{for any } \eta \in \text{supp} \mu_x. \]

According to the result of Gallardo and Rejeb (see [9]) that the orbit \( G \cdot x \) of \( x \) is contained in the support of \( \mu_x \) if \( k > 0, \) for the above \( g \) we can select \( \eta = g \cdot x, \) and then \( f(z - g \cdot x) = \tilde{f}(\|z - g \cdot x\|) = 0. \)

For any \( z_1 \in B(y - g \cdot x, \varepsilon), \) we have \( z_1 + g \cdot x \in B(y, \varepsilon), \) and so \( f(z_1) = f(z_1 + g \cdot x - g \cdot x) = 0, \)

which means that \( y - g \cdot x \notin \text{supp} f, \) and this leads to a contradiction to \( \text{supp} f = B(0, r). \)

Thus, for any nonnegative radial functions \( f \in L^2(m_k) \) with \( \text{supp} f = B(0, r), \) by the density of continuous functions with compact support \( B(0, r) \) in \( L^2(B(0, r), m_k), \) there exists a sequence of continuous nonnegative radial functions \( g_n \) with support \( B(0, r) \) such that \( f/2 \) can be approximated by \( g_n \) with respect to the \( L^2 \)-norm. So, for any nonnegative smooth function \( \varphi \) on \( \mathbb{R}^N \) with compact support, we have \( \int g_n \varphi \to \int f/2 \varphi. \) If \( (\text{supp} \varphi)^c \cap B(0, r) \neq \emptyset, \) then \( \int f \varphi > 0, \) where \( A^c \) stands for the interior of \( A \) for any \( A \subseteq \mathbb{R}^N. \) It follows that there exists a sufficiently large natural number \( L \) for which \( \int g_L \varphi < \int f \varphi. \) If \( (\text{supp} \varphi)^c \cap B(0, r) = \emptyset, \) then, for any \( n \in \mathbb{N}, \) we have \( \int g_n \varphi = \int f \varphi = 0. \)

So, for any nonnegative smooth function \( \varphi \) on \( \mathbb{R}^N \) with compact support, we have \( \int g_L \varphi \leq \int f \varphi. \) Thus, \( g_L \leq f \) a.e., and \( \int \tau_{-x} g_L \cdot \varphi \leq \int \tau_{-x} f \cdot \varphi \) by the positivity of Dunkl translations on radial
functions. The set \( D = (\text{supp } \tau_x f)^c \) is the largest open set with the property \( 0 = \int \tau_x f \cdot \varphi \) for any smooth function \( \varphi \) with compact support in \( D \). If \( \varphi \geq 0 \), then \( \int \tau_x g_L \cdot \varphi = 0 \). Thus, since \( \tau_x g_L \geq 0 \), it follows that
\[
\bigcup_{g \in G} B(gx, r) = \text{supp } \tau_x g_L \subseteq D^c = \text{supp } \tau_x f. \quad \square
\]

**Remark 3.1.** This theorem does not hold for \( k \geq 0 \). For example, given any nontrival finite reflection group \( G \), we can take \( k \equiv 0 \). Then \( \text{supp } \tau_x f(-\cdot) = B(x, r) \) when \( f = B(0, r) \), and this set obviously differs from \( \bigcup_{g \in G} B(gx, r) \), since \( G \) is nontrival. We refer to [9; Example 3.1] for more counterexamples.

**Corollary 3.2.** If \( f \in L^2(m_k), x \in \mathbb{R}^N \), and \( \text{supp } f \cap \bigcup_{g \in G} B(gx, r) = \emptyset \), then \( \text{supp } \tau_x f \cap B(0, r) = \emptyset \).

**Proof.** For any function \( g \in L^2(m_k) \) with \( \text{supp } g \subseteq B(0, r) \), it follows from Theorem 1.2(i) that
\[
\text{supp } \tau_x g \subseteq \bigcup_{g \in G} B(gx, r).
\]
By the skew symmetry of Dunkl translations, we have
\[
\int_{\mathbb{R}^N} \tau_x f(y) g(y) \, dm_k(y) = \int_{\mathbb{R}^N} f(y) \tau_x g(y) \, dm_k(y) = 0.
\]
Thus, \( \tau_x f(y) = 0 \), \( y \in B(0, r) \).

The distance between orbits \( G \cdot x \) and \( G \cdot y \) is defined by (see [6])
\[
d_G(x, y) = \min_{g \in G} \|g \cdot y - x\|. \quad (3.1)
\]
For any fixed point \( x \) and a ball \( B(x, r) \) with center \( x \), we set \( B^* = B(x, 2r) \) and \( Q^* = \bigcup_{g \in G} gB^* \).

Given any \( y \in B(x, r) \), if \( z \in \mathbb{R}^N \setminus Q^* \), then (see [2])
\[
d_G(x, z) > 2\|y - x\|. \quad (3.2)
\]

**Theorem 3.3.** Let \( f \in L^p(m_k), 1 \leq p < \infty \), be a radial function. If \( \text{supp } f \cap B(0, r) = \emptyset \), then, for any \( x \in \mathbb{R}^N \),

\[
\text{supp } \tau_x f(-\cdot) \cap \bigcup_{g \in G} B(gx, r) = \emptyset. \quad (3.3)
\]

**Proof.** First, we prove the theorem for continuous radial functions. It is easy to see that
\[
\max_{g \in G} \|g \cdot x - y\| \geq A(x, y, \eta) \geq d_G(x, y) \quad (3.4)
\]
for any \( x, y \in \mathbb{R}^N \) and \( \eta \in \co(G \cdot x) \). Given any continuous radial function \( f \) with support contained in \( B(0, r)^c \), if
\[
\tau_x f(-y) = \int_{\mathbb{R}^N} (\tilde{f} \circ A)(x, y, \eta) \, d\mu_x(\eta) \neq 0,
\]
then \( \max_{g \in G} \|g \cdot x - y\| \geq r \). Therefore, \( \text{supp } \tau_x f(-\cdot) \cap \bigcup_{g \in G} B(gx, r) = \emptyset \). By the density of continuous functions in \( L^p(m_k) \) and the continuity of Dunkl translations on \( L^p(m_k) \) for radial functions, (3.2) can be extended to any radial functions in \( L^p(m_k) \).
\[\square\]
Remark 3.4. One may expect that \( \text{supp} \tau_x f(-\cdot) \cap \bigcup_{g \in G} B(gx, r) = \emptyset \), but this is not correct even for a characteristic function in the case of a general finite reflection group \( G \). An argument similar to that in Corollary 3.2 shows that this also means that \( \text{supp} f \subseteq \bigcup_{g \in G} B(gx, r) \) does not imply \( \text{supp} \tau_x f \subseteq B(0, r) \), unlike in the classical case.

As an immediate consequence of the theorem, the condition of Corollary 4.1 in [7] can be weakened for radial functions.

Corollary 3.5. Suppose that \( \| g \cdot x - y \| < 1 \) for all \( g \in G \) and \( x, y \in \mathbb{R}^N \). Let \( f \) be a radial function in \( L^p(m_k) \), \( 1 \leq p < \infty \), such that \( f(z) = 0 \) for all \( z \in B(0, 1) \). Then \( \tau_x f(y) = 0 \).

Remark 3.6. This theorem cannot be extended to nonradial functions, because the Stone–Weierstrass theorem does not hold on \( B(0, r)^\circ \), and there is no result on the support of the distribution associated to \( \tau_x f(y) \) more precise than that in [1].

4. Riesz Transforms for the Dunkl Transform

The Riesz transforms \( \mathcal{R}_j \) in the Dunkl setting are defined by

\[
\mathcal{R}_j(f)(x) = \lim_{\varepsilon \to 0} d_k \int_{|y| \geq \varepsilon} \tau_x f(-y) \frac{y_j}{|y|^p} \, dm_k(y), \quad f \in S(\mathbb{R}^N),
\]

where \( j = 1, \ldots, N \), \( d_k = 2^{N/2} \Gamma((N + 1)/2)/\sqrt{\pi} \), and \( p_k = N + 1 \). It was proved in [14] that

\[
F(\mathcal{R}_j f)(\xi) = -i \frac{\xi_j}{|\xi|} (Ff)(\xi), \quad j = 1, \ldots, n,
\]

and \( \mathcal{R}_j \) is a bounded operator on \( L^2(m_k) \). Clearly,

\[
\mathcal{R}_j f = -T_{e_j} (-\Delta)^{-1/2} f = -\lim_{\varepsilon \to 0, M \to \infty} c \int_{\varepsilon}^{M} T_{e_j} e^{it\Delta} f \, \frac{dt}{\sqrt{t}},
\]

and the integral converges for \( f \in L^2(m_k) \). It is obvious that the Riesz transforms commute with the Dunkl translations. If \( f \in L^2(m_k) \) has compact support, then, as shown in [2], for all \( x \in \mathbb{R}^N \) such that \( g \cdot x \in \mathbb{R}^N \setminus \text{supp}(f) \) for any \( g \in G \), we have

\[
\mathcal{R}_j(f)(x) = \int_{\mathbb{R}^N} K_j(x, y) f(y) \, dm_k(y),
\]

where

\[
K_j(x, y) = d_k \left\{ K_j^{(1)}(x, y) + \sum_{\alpha \in \mathbb{R}^+} \frac{k(\alpha \omega_j)}{p_k - 2} K_j^{(\alpha)}(x, y) \right\},
\]

\[
K_j^{(1)}(x, y) = \int_{\mathbb{R}^N} \frac{y_j - \eta_j}{A p_k(x, y, \eta)} \, d\mu_x(\eta),
\]

\[
K_j^{(\alpha)}(x, y) = \frac{1}{(y, \alpha)} \int_{\mathbb{R}^N} \left[ \frac{1}{A p_k(x, y, \eta)} - \frac{1}{A p_{k-2}(x, \sigma_\alpha(y), \eta)} \right] \, d\mu_x(\eta), \quad \alpha \in \mathbb{R}^+,
\]

and \( K_j(x, y) \) satisfies the condition

\[
\int_{d_G(x, z) > 2||y - x||} |K_j(z, x) - K_j(z, y)| \, dm_k(z) \leq C. \tag{4.1}
\]

The authors of [2] also proved that \( \mathcal{R}_j \) is a bounded operator on \( L^p(m_k) \), \( 1 < p < \infty \), by using this Calderón–Zygmund condition in the Dunkl setting.
Consider a $C^\infty$ function $\tilde{\varphi}_{n,\varepsilon}$ on $\mathbb{R}$ such that

- $\tilde{\varphi}_{n,\varepsilon}$ is odd;
- $\tilde{\varphi}_{n,\varepsilon}$ is supported in $\{ t \in \mathbb{R}; |t| \geq \varepsilon \}$;
- $\tilde{\varphi}_{n,\varepsilon} = 1$ on $\{ t \in \mathbb{R}; t \geq \varepsilon + 1/n \}$;
- $|\tilde{\varphi}_{n,\varepsilon}| \leq 1$.

We set

$$
\tilde{\phi}_{n,\varepsilon}(t) = \int_{-\infty}^{t} \frac{\tilde{\varphi}_{n,\varepsilon}(u)}{|u|^{p_k-1}} \, du \quad \text{and} \quad \phi_{n,\varepsilon}(y) = \tilde{\phi}_{n,\varepsilon}(||y||), \quad t \in \mathbb{R}, \ y \in \mathbb{R}^N.
$$

Clearly, $\phi_{n,\varepsilon}$ is a $C^\infty$ radial function and

$$
\lim_{n \to +\infty} \tilde{\varphi}_{n,\varepsilon}(||y||) = 1, \quad y \in \mathbb{R}^N, \ ||y|| > \varepsilon.
$$

Let $K_j^{(\varepsilon,n)}(x,y) = d_k T_j \tau_x(\phi_{n,\varepsilon})(-y)$. Here the action of $\tau_x$ on $\phi_{n,\varepsilon}$ is defined in the sense of distributions. Then it follows from the proof of [2; Proposition 3.2] that

$$
\mathcal{R}_j(f)(x) = \lim_{\varepsilon \to 0} \lim_{n \to +\infty} \int_{\mathbb{R}^N} K_j^{(\varepsilon,n)}(x,y)f(y) \, dm_k(y),
$$

where

$$
T_j \tau_x(\phi_{n,\varepsilon})(-y) = \int_{\mathbb{R}^N} \frac{(\eta_j - y_j)\tilde{\varphi}_{n,\varepsilon}(A(x,y,\eta))}{A_{\varepsilon,n}(x,y,\eta)} \, d\mu_x(\eta)
$$

$$
+ \sum_{\alpha \in \mathbb{R}_+} k(\alpha) \alpha \int_{\mathbb{R}^N} \frac{\tilde{\phi}_{n,\varepsilon}(A(x,\sigma,\eta,\eta)) - \tilde{\phi}_{n,\varepsilon}(A(x,y,\eta))}{(y,\alpha)} \, d\mu_x(\eta)
$$

and $\int_{\mathbb{R}^N} K_j^{(\varepsilon,n)}(x,y)f(y) \, dm_k(y)$ is to be understood as the principal value of a singular integral for $f \in L^2(m_k)$, because supp $T_j \phi_{n,\varepsilon} \subseteq (B(0,\varepsilon))^c$ does not necessarily imply supp $T_j \tau_x(\phi_{n,\varepsilon})(-\cdot) \subseteq (\bigcup_{y \in G} B(gx,\varepsilon))^c$, as shown in Section 2.

For any $f \in L^2(m_k)$ with compact support, if $\mathcal{R}_j^*$ is the adjoint operator of $\mathcal{R}_j$, then

$$
\mathcal{R}_j^*(f)(y) = \lim_{\varepsilon \to 0} \lim_{n \to +\infty} \int_{\mathbb{R}^N} K_j^{(\varepsilon,n)}(x,y)f(x) \, dm_k(x).
$$

Since $\mathcal{R}_j = -\mathcal{R}_j^*$, we have

$$
\mathcal{R}_j(f)(y) = -\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \int_{\mathbb{R}^N} K_j^{(\varepsilon,n)}(x,y)f(x) \, dm_k(x). \tag{4.2}
$$

If $y \in \mathbb{R}^N$ satisfies $G_y \cap \text{supp } f = \emptyset$, then $d_G(x,y) > 0$ for all $x \in \text{supp } f$. It follows from (3.4) that, for any $0 < \varepsilon < d_G(x,y)$,

$$
\varepsilon < A(x,y,\eta), \quad \varepsilon < A(x,\sigma,\eta,\eta), \quad x \in \text{supp}(f), \ \eta \in \text{co}(G \cdot x).
$$

Then the same argument as in the proof of [2; Proposition 3.2] yields

$$
\lim_{\varepsilon \to 0} \lim_{n \to +\infty} K_j^{(\varepsilon,n)}(x,y) = K_j(x,y)
$$

and

$$
\mathcal{R}_j(f)(y) = -\int_{\mathbb{R}^N} K_j(x,y)f(x) \, dm_k(x), \quad G_y \cap \text{supp } f = \emptyset \tag{4.3}
$$

with the aid of the dominated convergence theorem.
5. The Dunkl-Type BMO Space and the Proof of Theorem 1.1

The study of the Dunkl-type BMO space dates back to [12], where this space was defined for the one-dimensional case. Here we will define the Dunkl-type BMO space for the multidimensional case.

Given a function \( f \in L^1_{\text{loc}}(m_k) \) and a ball \( B(x,r) \), we denote \( B_r \equiv B(0,r) \). Let \( f_{B_r}(x) \) be the average of \( \tau_x f \) on \( B_r \):

\[
f_{B_r}(x) = \frac{1}{m_k(B_r)} \int_{B_r} \tau_x f(y) \, dm(y).
\]

**Definition 5.1.** The Dunkl-type BMO space is the space of all functions in \( L^1_{\text{loc}}(m_k) \) satisfying \( \| f \|_{*,k} < \infty \), where

\[
\| f \|_{*,k} = \sup_{r > 0, x \in \mathbb{R}^N} \frac{1}{m_k(B_r)} \int_{B_r} |\tau_x f(y) - f_{B_r}(x)| \, dm(y).
\]

We can consider BMO as the quotient of the above space by the space of constant functions; then \( \| \cdot \|_{*,k} \) is a norm.

**Proof of Theorem 1.1.** Given a compactly supported function \( f \) in \( L^\infty \), thanks to Theorem 1.1(i), \( \tau_x f \) is compactly supported. We write \( \tau_x f = g_1 + g_2 \), where \( g_1 = (\tau_x f) \chi_{B_{2r}} \) and \( g_2 = (\tau_x f) \chi_{(B_{2r})^c} \). This decomposition is unique, because, as shown in Section 2, \( \text{supp} \ f \subseteq \bigcup_{y \in G} B(gx,r) \) does not imply \( \text{supp} \ \tau_x f \subseteq B(0,r) \). For any \( y \in B_r \), we have \( Gy \cap \text{supp} \ g_2 = \emptyset \). Thus, by (3.2), (4.1), (4.2), and the assumption of the uniform boundedness of Dunkl translations,

\[
|\mathcal{R}_j g_2(y) - \mathcal{R}_j g_2(0)| = \left| \int_{\mathbb{R}^N} (\mathcal{K}_j(z,y) - \mathcal{K}_j(z,0))g_2(z) \, dm(z) \right|
\]

\[
= \left| \int_{(B_{2r})^c} (\mathcal{K}_j(z,y) - \mathcal{K}_j(z,0))\tau_x f(z) \, dm(z) \right|
\]

\[
\leq \int_{d_G(0,z) > 2|y|} |\mathcal{K}_j(z,y) - \mathcal{K}_j(z,0)| \, dm(z)||\tau_x f||_{\infty}
\]

\[
\leq C\| f \|_{\infty}.
\]

Using again the uniform boundedness assumption and the \( L^2 \) boundedness of the Riesz transform and (2.1), we obtain

\[
\frac{1}{m_k(B_r)} \int_{B_r} |\mathcal{R}_j g_1| \leq \left( \frac{1}{m_k(B_r)} \int_{B_r} |\mathcal{R}_j g_1|^2 \right)^{1/2} \leq \left( \frac{1}{m_k(B_r)} \int |(\tau_x f) \chi_{B_{2r}}|^2 \right)^{1/2}
\]

\[
= \left( \frac{1}{m_k(B_r)} \int_{B_{2r}} |\tau_x f|^2 \right)^{1/2} \leq C\| \tau_x f \|_{\infty} \leq C\| f \|_{\infty}.
\]

Therefore,

\[
\frac{1}{m_k(B_r)} \int_{B_r} |\mathcal{R}_j \tau_x f(y) - \mathcal{R}_j g_2(0)| \, dm_k(y)
\]

\[
\leq \frac{1}{m_k(B_r)} \int_{B_r} |\mathcal{R}_j g_1(y)| \, dm_k(y) + \frac{1}{m_k(B_r)} \int_{B_r} |\mathcal{R}_j g_2(y) - \mathcal{R}_j g_2(0)| \, dm_k(y) \leq C\| f \|_{\infty}.
\]

Now we extend the definition of the Riesz transforms for the Dunkl transform to the whole space \( L^\infty \). For any function \( f \in L^\infty \), we define

\[
\mathcal{R}_j(f)(y) = -\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^N} (\mathcal{K}_j^{(\varepsilon,n)}(z,y) - \mathcal{K}_j^{(\varepsilon,n)}(z,0)) f(z) \, dm_k(z).
\]
For any $\varepsilon > 0$ and any $y \in \mathbb{R}^N$, there exists a sufficiently large $r > 2\varepsilon$ such that $y \in B_r$. We write $f = f_1 + f_2$, where $f_1 = f \chi_{B_{2r}}$ and $f_2 = f \chi_{(B_{2r})^c}$. Then $f_1$ belongs to $L^2(m_k)$, and so $\mathcal{R}_j f_1(y)$ converges almost everywhere. For any natural number $n$ larger than $1/\varepsilon$ and any $z \in \text{supp} f_2$,

$$d_G(z, 0) > 2r > \varepsilon + \frac{1}{n}, \quad d_G(z, y) > d_G(z, 0) - \|y\| > r > \varepsilon + \frac{1}{n}.$$  

So, for the above $\varepsilon$ and $n$, we have

$$A(z, 0, \eta) > \varepsilon + \frac{1}{n}, \quad A(z, y, \eta) > \varepsilon + \frac{1}{n}, \quad \eta \in \text{co}(G \cdot x),$$

and

$$K_j^{(\varepsilon, n)}(z, 0) = K_j(z, 0), \quad K_j^{(\varepsilon, n)}(z, y) = K_j(z, y).$$

Therefore,

$$\mathcal{R}_j(f_2)(y) = - \int_{(B_{2r})^c} (K_j(z, y) - K_j(z, 0)) f(z) \, dm_k(z).$$

This integral converges, since

$$\left| \int_{(B_{2r})^c} (K_j(z, y) - K_j(z, 0)) f(z) \, dm_k(z) \right| \leq \int_{d_G(0, z) > 2\|y\|} |K_j(z, y) - K_j(z, 0)| \, dm_k(z) \|f\|_{\infty} \leq C \|f\|_{\infty}, \quad y \in B_r.$$  

Thus, the above definition (5.1) for the Riesz transforms for the Dunkl transform on $L^\infty$ makes sense for any $y \in \mathbb{R}^N$ and coincides with (4.2) for compactly supported functions in $L^2(m_k)$ as Dunkl-type BMO functions, since these two formulas differ by a constant.

Under the assumption of the uniform boundedness of Dunkl translations, for any $x \in \mathbb{R}^n$, $r > 0$, and $f \in L^\infty$, we have $\tau_x f \in L^\infty$. We write $\tau_x f = g_1 + g_2$, where $g_1 = (\tau_x f) \chi_{B_{2r}}$ and $g_2 = (\tau_x f) \chi_{(B_{2r})^c}$. Then

$$\mathcal{R}_j(g_1)(y) = - \lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^N} K_j^{(\varepsilon, n)}(z, y) g_1(z) \, dm_k(z) + \mathcal{R}_j(g_1)(0),$$

and for any $y \in B_r$,

$$\mathcal{R}_j(g_2)(y) = - \int_{(B_{2r})^c} (K_j(z, y) - K_j(z, 0)) \tau_x f(z) \, dm_k(z).$$

The same argument as for compactly supported functions gives

$$\frac{1}{m_k(B_r)} \int_{B_r} |\mathcal{R}_j \tau_x f(y) - \mathcal{R}_j(g_1)(0)| \, dm_k(y)$$

$$= \frac{1}{m_k(B_r)} \int_{B_r} \left| \mathcal{R}_j((\tau_x f) \chi_{B_{2r}})(y) + \int_{(B_{2r})^c} (-K_j(z, y) + K_j(z, 0)) \tau_x f(z) \, dm_k(z) \right| \, dm_k(y)$$

$$\leq C \|f\|_{\infty}.$$  

The following lemma will then imply the boundedness of the Riesz transforms for the Dunkl transform from $L^\infty$ to the Dunkl-type BMO space.

**Lemma 5.2.** Under the assumption of the uniform boundedness of Dunkl translations, for any $f \in L^\infty$ and any fixed $x \in \mathbb{R}^N$ and $r > 0$, $\mathcal{R}_j \tau_x f(y)$ and $\tau_x \mathcal{R}_j f(y)$ differ by a constant independent of $y$ for $y \in B_r$.  

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Proof. This statement is obvious for compactly supported functions in $L^\infty$ and implies that, for any such function $f$,

$$
\frac{1}{m_k(B_r)} \int_{B_r} \left| [\mathcal{R}_j, \tau_y] f(y) - \frac{1}{m_k(B_r)} \int_{B_r} [\mathcal{R}_j, \tau_y] f \right| \, dm_k(y) = 0, \tag{5.2}
$$

where $[X,Y] := XY - YX$.

If $\{D_i\}$ is a countable open cover of $\mathbb{R}^N$, where each $D_i$ is bounded, then, using partition of unity, we can write any function $f \in L^\infty$ as $\sum_{i=1}^{\infty} f_i$, where $\text{supp} \, f_i \subset D_i$. Let $g := [\mathcal{R}_j, \tau_y] f$. Then

$$
g = \sum_{i=1}^{\infty} g^{(i)}, \quad g^{(i)} = [\mathcal{R}_j, \tau_y] f_i,
$$

and by (5.1)

$$
\frac{1}{m_k(B_r)} \int_{B_r} \left| g(y) - \frac{1}{m_k(B_r)} \int_{B_r} g \right| \, dm_k(y)
\leq \sum_{i=1}^{\infty} \frac{1}{m_k(B_r)} \int_{B_r} \left| g^{(i)}(y) - \frac{1}{m_k(B_r)} \int_{B_r} g^{(i)} \right| \, dm_k(y) = 0.
$$

So, $g(y) = [\mathcal{R}_j, \tau_y] f(y)$ is a constant independent of $y$ for $y \in B_r$.

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