Full history recursive multilevel Picard approximations for ordinary differential equations with expectations

Christian Beck\textsuperscript{1}, Martin Hutzenthaler\textsuperscript{2}, Arnulf Jentzen\textsuperscript{3}, and Emilia Magnani\textsuperscript{4}

\textsuperscript{1} Department of Mathematics, ETH Zurich, Zürich, Switzerland; Faculty of Mathematics and Computer Science, University of Münster, Münster, Germany; e-mail: christian.beck@uni-muenster.de
\textsuperscript{2} Faculty of Mathematics, University of Duisburg-Essen, Essen, Germany; e-mail: martin.hutzenthaler@uni-due.de
\textsuperscript{3} Department of Mathematics, ETH Zurich, Zürich, Switzerland; Faculty of Mathematics and Computer Science, University of Münster, Münster, Germany; School of Data Science and Shenzhen Research Institute of Big Data, The Chinese University of Hong Kong, Shenzhen, China; e-mail: ajentzen@uni-muenster.de
\textsuperscript{4} Department of Computer Science, University of Tübingen, Tübingen, Germany; Department of Mathematics, ETH Zurich, Zürich, Switzerland; e-mail: emilia.magnani@uni-tuebingen.de

March 4, 2021

Abstract

We consider ordinary differential equations (ODEs) which involve expectations of a random variable. These ODEs are special cases of McKean–Vlasov stochastic differential equations (SDEs). A plain vanilla Monte Carlo approximation method for such ODEs requires a computational cost of order $\varepsilon^{-3}$ to achieve a root-mean-square error of size $\varepsilon$. In this work we adapt recently introduced full history recursive multilevel Picard (MLP) algorithms to reduce this computational complexity. Our main result shows for every $\delta > 0$ that the proposed MLP approximation algorithm requires only a computational effort of order $\varepsilon^{-(2+\delta)}$ to achieve a root-mean-square error of size $\varepsilon$. 
1 Introduction

It is a very challenging task in applied mathematics to design efficient approximation algorithms for high-dimensional partial differential equations (PDEs). Recently, significant progress has been made in this area of research through the development of so-called full history recursive multilevel Picard (MLP) approximation algorithms [19, 20, 25]. Up to now, MLP approximation algorithms are the only approximation algorithms in the scientific literature for which it has been rigorously proved that they can overcome the curse of dimensionality in the numerical approximation of second-order semilinear elliptic and parabolic PDEs with general time horizons in the sense that the number of computational operations needed to achieve a desired approximation accuracy $\varepsilon \in (0, \infty)$ grows at most polynomially in both the PDE dimension $d \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and the reciprocal $1/\varepsilon$ of the desired approximation accuracy $\varepsilon$. We also refer to [4, 5, 7, 21, 23, 24, 25, 26, 27] and the overview articles [6, 18] for computational problems in which MLP approximation algorithms have been shown to overcome the curse of dimensionality.

To develop a better understanding for the recently proposed MLP approximation methods, we aim within this article to extend the MLP approximation algorithms to the case of ordinary differential equations (ODEs) involving the expectations of random variables. To achieve a root-mean-square error of size $\varepsilon \in (0, \infty)$ a plain vanilla Monte Carlo method requires a computational cost of order $\varepsilon^{-3}$. We reduce this computational cost and show that for an arbitrarily small $\delta \in (0, \infty)$ the proposed MLP approximation schemes achieve an approximation accuracy of size $\varepsilon \in (0, \infty)$ with a computational effort of order $\varepsilon^{-(2+\delta)}$. More precisely, Theorem 3.5 below, the main result of this article, proves under suitable assumptions that for every $\varepsilon \in (0, 1]$, $\delta \in (0, \infty)$ the MLP approximation scheme achieves a root mean square error of at most $\varepsilon$ with a computational effort of order $\varepsilon^{-(2+\delta)}$. As an illustration of Theorem 3.5 below, we present now in the following result, Theorem 1.1, a simplified version of Theorem 3.5.

**Theorem 1.1.** Let $d \in \mathbb{N}$, $\delta \in (0, \infty)$, $T, L \in [0, \infty)$, $\Theta = \cup_{n=1}^{\infty} \mathbb{Z}^d$, $X \in C([0, T], \mathbb{R}^d)$, $\xi \in \mathbb{R}^d$, let $\|\cdot\|: \mathbb{R}^d \to [0, \infty)$ be a norm on $\mathbb{R}^d$, let $(S, \mathcal{S})$ be a measurable space, let $F: \mathbb{R}^d \times S \to \mathbb{R}^d$ be $(\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{S})/\mathcal{B}(\mathbb{R}^d)$-measurable, assume for all $x, y \in \mathbb{R}^d$, $s \in S$ that $\|F(x, s) - F(y, s)\| \leq L \|x - y\|$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $Z^\delta: \Omega \to S$, $\theta \in \Theta$, be i.i.d. random variables, assume that $\mathbb{E}[\|F(\xi, Z^0)\|^2] < \infty$, assume for all $t \in [0, T]$ that $X(t) = \xi + \int_0^t \mathbb{E}[F(X(r), Z^0)] dr$, let $\varepsilon^\theta: \Omega \to [0, 1]$, $\theta \in \Theta$, be independent $\mathcal{U}_{[0,1]}$-distributed random variables, assume that $(\varepsilon^\theta)_{\theta \in \Theta}$ and $(Z^\theta)_{\theta \in \Theta}$ are.

---

1. Introduction

2. Error analysis for multilevel Picard (MLP) approximations
   2.1 Setting
   2.2 Time-discrete Gronwall inequalities
   2.3 On random evaluations of random fields
   2.4 A priori bounds for solutions of ordinary differential equations
   2.5 Properties of MLP approximations
   2.6 Error estimates for MLP approximations

3. Complexity analysis for MLP approximation algorithms
   3.1 Computational cost analysis for MLP approximation algorithms
   3.2 Overall complexity analysis for MLP approximation algorithms

Contents

1 Introduction

2 Error analysis for multilevel Picard (MLP) approximations

3 Complexity analysis for MLP approximation algorithms

1 Introduction
independent, let \( X^\theta_{n,m} : [0, T] \times \Omega \to \mathbb{R}^d \), \( n, m \in \mathbb{N}_0 \), \( \theta \in \Theta \), satisfy for all \( n \in \mathbb{N}_0 \), \( m \in \mathbb{N} \), \( \theta \in \Theta \), \( t \in [0, T] \) that
\[
X^\theta_{n,m}(t) = \sum_{k=1}^{m_{n-1}} \frac{t}{m_{n-1}} \sum_{l=1}^{n-1} F(X^\theta_{l,m}(t), Z^{(\theta,l,k)}(t)) - F(X^\theta_{l-1,m}(t), Z^{(\theta,l,k)}(t)) + \left[ \sum_{k=1}^{m_n} F(\xi, Z^{(0,0,k)}) \right] + \xi,
\]
and for every \( n, m \in \mathbb{N} \) let \( RV_{n,m} \in \mathbb{N} \) be the number of realizations of random variables \( (Z^\theta)_{\theta \in \Theta} \) which are used to compute one realization of \( X^\theta_{n,m}(T) \) (cf. (93)). Then there exist \( \varepsilon \in \mathbb{R} \) and \( N = (N_\varepsilon)_{\varepsilon \in (0,1]} : \{0,1\} \to \mathbb{N} \) such that for all \( \varepsilon \in (0,1] \) it holds that \( RV_{N_\varepsilon,N_\varepsilon} \leq c \varepsilon^{-(2+\delta)} \) and \( (\mathbb{E}[\|X(T) - \mathcal{X}^\theta_{N_\varepsilon,N_\varepsilon}(T)\|^2])^{1/2} \leq \varepsilon \).

Theorem 1.1 is an immediate consequence of Theorem 3.5 in Section 3 below. Theorem 1.1 establishes under suitable conditions that for every \( \delta \in (0, \infty) \) there exists \( \varepsilon \in \mathbb{R} \) such that the solution \( X \in C([0,T],\mathbb{R}^d) \) of the differential equation \( X(t) = \xi + \int_0^t \mathbb{E}[F(X(r), Z^0)] \, dr \), \( t \in [0,T] \), (cf. Lemma 3.3), can be approximated by the recursive MLP approximation schemes in (1) with a root mean square error of size \( \varepsilon \in (0,1] \) and a computational effort that is bounded by \( c \varepsilon^{-(2+\delta)} \). The computational effort is quantified by the numbers \( RV_{n,m} \), \( n, m \in \mathbb{N} \). The function \( F : \mathbb{R}^d \times S \to \mathbb{R}^d \) is required to be \( (B(\mathbb{R}^d) \otimes S) / B(\mathbb{R}^d) \) -measurable and Lipschitz continuous in the first variable, uniformly in the second variable, and we assume that \( \mathbb{E}[\|F(\xi, Z^0)\|^2] < \infty \). We note that differential equations of the type \( X(t) = \xi + \int_0^t \mathbb{E}[F(X(r), Z^0)] \, dr \), \( t \in [0,T] \), can be considered as a special case of so-called McKean–Vlasov stochastic differential equations (SDEs). Numerical approximation methods for McKean–Vlasov SDEs have been widely developed in the scientific literature. In particular, in the article [12] Bossy and Talay carried out the approximation of McKean–Vlasov SDEs with simulations through interacting particles systems and time discretizations. Many other authors have contributed to this approach. In particular, we refer among others to [2, 8, 9, 10, 11, 16, 17, 32, 33, 34]. Moreover, alternative approximation methods for McKean–Vlasov SDEs relying on cubature formulas [14, 15], analytical expansions [22], or tamed Milstein schemes [3, 30] have been developed. For further numerical approximation methods for McKean–Vlasov SDEs we also refer, e.g., to [1, 13, 31]. The problems which are treated in these references are of course far more general and involved than the expectation ODEs which we consider in this article. To the best of our knowledge, Theorem 1.1 is the first result in the scientific literature which shows that solutions of the special class of McKean–Vlasov SDEs considered in this article can be approximated with a root-mean-square error of size \( \varepsilon \) with a computational cost of order \( \varepsilon^{-(2+\delta)} \).

The remainder of this article is organized as follows. Section 2 introduces multilevel Picard (MLP) approximation schemes for a special type of ODEs involving expectations of a random variable (see Setting 2.1 in Subsection 2.1) and provides \( L^2 \)-error estimates for the differences between the MLP approximations and the exact solution of the ODE under consideration. In Section 3 we combine the error analysis from Section 2 with suitable estimates for the computational effort for the proposed MLP approximation algorithms (see Lemma 3.1 below) to perform a complexity analysis for the proposed MLP approximation algorithms and we show in Theorem 3.5 that for every \( \delta \in (0, \infty) \), \( \varepsilon \in (0,1] \) it holds that MLP approximations can achieve a root mean square error of size \( \varepsilon \in (0,1] \) with a computational effort of size \( \varepsilon^{-(2+\delta)} \).

2 Error analysis for multilevel Picard (MLP) approximations

In this section we introduce in Setting 2.1 in Subsection 2.1 below MLP approximations (see (4) in Setting 2.1) for ODEs involving expectations (see (3) in Setting 2.1) and we provide in Proposition 2.16 in Subsection 2.6 an \( L^2 \)-error analysis for the proposed MLP approximation schemes. In particular, we establish in Proposition 2.16, the main result of this section, an upper bound for the root mean square error between solutions of ODEs involving expectations and the corresponding MLP approximations.

Our proof of Proposition 2.16 is heavily motivated by [25] and has, roughly speaking, four steps: first, (i) we perform a bias-variance decomposition of the mean square error between solutions of ODEs involving expectations and the
corresponding MLP approximations, second, (ii) we estimate the biases of the proposed MLP approximations, third, (iii) we estimate the variances of the proposed MLP approximations, and, finally, (iv) we obtain the final bound by applying a Gronwall type argument to the recursive inequalities obtained through combining the bias-variance decomposition (see (i) above) with the bias estimates (see (ii) above) and the variance estimates (see (iii) above).

Our proof of Proposition 2.16 relies on several auxiliary results, which we present in Subsections 2.2–2.6 below. In Subsection 2.2 we recall in Lemma 2.2 and Lemma 2.3 elementary and well-known time-discrete Gronwall inequalities. Proofs for the results in Lemma 2.2 and Lemma 2.3 in Subsection 2.2 below can be found, e.g., in [26, Corollary 2.2 and Lemma 3.12]. In Subsection 2.3 we exhibit in Lemmas 2.4–2.6 elementary and well-known results about random variables which arise from evaluating random fields at random indices. Proofs for Lemma 2.4 and Lemma 2.5 can, e.g., be found in [26, Lemma 2.14] and [25, Lemma 2.3]. We also include in this section a proof of Lemma 2.6, which is a slight modification of [26, Lemma 2.16]. In Subsection 2.4 we establish in Lemma 2.7 elementary a priori bounds for solutions of ODEs involving expectations. In Subsection 2.5 we present in Lemma 2.10 and Lemma 2.12 some fundamental measurability, integrability, and distribution properties of MLP approximations. The proofs of Lemma 2.10 and Lemma 2.12 employ the well-known and elementary results in Lemma 2.8, Lemma 2.9, and Lemma 2.11 below. Proofs for Lemma 2.8 and Lemma 2.9 can be found, e.g., in [26, Lemmas 3.4 and 3.5]. Note that Lemma 2.11 is a slightly modified version of [26, Lemma 3.7]. In Subsection 2.6 we recall some elementary and well-known results in Lemma 2.13–Lemma 2.15 below before proving Proposition 2.16. We refer, e.g., to [26, Lemma 3.10] and [26, Lemma 2.9] for proofs of Lemma 2.13 and Lemma 2.14. Lemma 2.15, which is a simple consequence of Tonelli’s theorem, is a slightly modified version of [26, Lemma 3.14].

2.1 Setting

**Setting 2.1.** Let \( d \in \mathbb{N}, T, L \in [0, \infty), \Theta = \cup_{n=1}^\infty \mathbb{Z}^n, X \in \mathcal{C}([0, T], \mathbb{R}^d), \xi \in \mathbb{R}^d, \) let \( \| \cdot \| : \mathbb{R}^d \to [0, \infty) \) be the standard norm on \( \mathbb{R}^d, \) let \( (S, S) \) be a measurable space, let \( F : \mathbb{R}^d \times S \to \mathbb{R}^d \) be \( (\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{S})/\mathcal{B}(\mathbb{R}^d) \)-measurable, assume for all \( x, y \in \mathbb{R}^d, s \in S \) that

\[
\| F(x, s) - F(y, s) \| \leq L \| x - y \|,
\]

let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space, let \( Z^\theta : \Omega \to S, \theta \in \Theta, \) be independent and identically distributed (i.i.d.) random variables, let \( \varphi : \Omega \to [0, 1], \theta \in \Theta, \) be independent \( \mathcal{U}_{[0,1]} \)-distributed random variables, let \( \mathcal{R}^\theta = (\mathcal{R}^\theta_r)_{r \in [0, T]} : [0, T] \times \Omega \to [0, T], \theta \in \Theta, \) satisfy for all \( t \in [0, T], \theta \in \Theta \) that \( \mathcal{R}^\theta_t = \varphi(t), \) assume that \( (\varphi(t))_{\theta \in \Theta} \) and \( (Z^\theta)_{\theta \in \Theta} \) are independent, assume for all \( t \in [0, T] \) that \( \int_0^t \mathbb{E}\| F(X(r), Z^0) \| \, dr < \infty \) and

\[
X(t) = \xi + \int_0^t \mathbb{E}[F(X(r), Z^0)] \, dr,
\]

and let \( X_{n,m}^\theta : [0, T] \times \Omega \to \mathbb{R}^d, n, m \in \mathbb{N}_0, \theta \in \Theta, \) satisfy for all \( n \in \mathbb{N}_0, m \in \mathbb{N}, \theta \in \Theta, t \in [0, T] \) that

\[
X_{n,m}^\theta(t) = \sum_{l=1}^{n-1} \frac{t}{m^{n-l}} \sum_{k=1}^{m^{n-l}} \left( F(X_{l,m}^{(\theta,l,k)}(\mathcal{R}_t^{(\theta,l,k)}), Z^{(\theta,l,k)}_t) - F(X_{l-1,m}^{(\theta,l,k)}, Z^{(\theta,l,k)}_t) \right)
\]

\[
+ \frac{\int_{0}^{l} \mathbb{E}[F(X(r), Z^{(\theta,0,0)})]}{m^n} 
\]

\[
\sum_{k=1}^{m^n} F(X^{(\theta,0,0)}) + \xi.
\]

2.2 Time-discrete Gronwall inequalities

**Lemma 2.2.** Let \( N \in \mathbb{N} \cup \{ \infty \}, \alpha, \beta \in [0, \infty), (\epsilon_n)_{n \in \mathbb{N}_0 \cap [0, N]} \subseteq [0, \infty) \) satisfy for all \( n \in \mathbb{N}_0 \cap [0, N] \) that \( \epsilon_n \leq \alpha + \beta \sum_{k=0}^{n-1} \epsilon_k. \) Then it holds for all \( n \in \mathbb{N}_0 \cap [0, N] \) that \( \epsilon_n \leq \alpha(1 + \beta)^n \leq \alpha e^{\beta n} < \infty. \)

**Lemma 2.3.** Let \( \alpha, \beta \in [0, \infty), M \in (0, \infty), (\epsilon_n, k)_{n,k \in \mathbb{N}} \subseteq [0, \infty) \) satisfy for all \( n, k \in \mathbb{N}_0 \) that \( \epsilon_{n,k} \leq \frac{\alpha}{M^{-k} n^{k}} \beta \sum_{l=0}^{n-1} \frac{\epsilon_{l,k+1}}{M^{-l}}. \) Then it holds for all \( n, k \in \mathbb{N}_0 \) that \( \epsilon_{n,k} \leq \alpha(1 + \beta)^n M^{-k} n^{k} < \infty. \)
2.3 On random evaluations of random fields

Lemma 2.4. Let \((\Omega, \mathcal{F}, (S, \mathcal{S}), (E, \mathcal{E}))\) be measurable spaces, let \(U = (U(s))_{s \in S} = (U(s, \omega))_{(s, \omega) \in S \times \Omega} : S \times \Omega \to E\) be \((S \otimes \mathcal{F})/\mathcal{E}\)-measurable, and let \(X : \Omega \to S\) be \(\mathcal{F}/\mathcal{S}\)-measurable. Then it holds that \(U(X) = (U(X(\omega), \omega))_{\omega \in \Omega} : \Omega \to E\) is \(\mathcal{F}/\mathcal{E}\)-measurable.

Lemma 2.5. Let \((\Omega, \mathcal{F}, P)\) be a probability space, let \((S, \mathcal{S})\) be a separable metric space, let \(U = (U(s))_{s \in S} = (U(s, \omega))_{(s, \omega) \in S \times \Omega} : S \times \Omega \to [0, \infty)\) be a continuous random field, let \(X : \Omega \to S\) be a random variable, and assume that \(U\) and \(X\) are independent. Then it holds that \(U(X) = (\Omega \ni \omega \mapsto U(X(\omega), \omega) \in [0, \infty))\) is \(\mathcal{F}/\mathcal{B}(\mathbb{R}, \mathbb{R})\)-measurable and \(\mathbb{E}[U(X)] = \int_S \mathbb{E}[U(s)](X(P)_{(P,B(S)})(ds)\).

Lemma 2.6. Let \(d \in \mathbb{N}\), let \((\Omega, \mathcal{F}, \mathcal{P})\) be a probability space, let \(\|\| : \mathbb{R}^d \to [0, \infty)\) be a norm on \(\mathbb{R}^d\), let \((S, \mathcal{S})\) be a separable metric space, let \(U = (U(s))_{s \in S} = (U(s, \omega))_{(s, \omega) \in S \times \Omega} : S \times \Omega \to \mathbb{R}^d\) be a continuous random field, let \(X : \Omega \to S\) be a random variable, assume that \(U\) and \(X\) are independent, and assume that \(\int_S \mathbb{E}[\|\|U(s))||X(P)_{(P,B(S)}(ds) < \infty\).

(i) it holds that \(U(X) = (\Omega \ni \omega \mapsto U(X(\omega), \omega) \in \mathbb{R}^d)\) is \(\mathcal{F}/\mathcal{B}(\mathbb{R}^d)\)-measurable,

(ii) it holds that \((X(P)_{(P,B(S)}(\{s \in S : \mathbb{E}[\|\|U(s))|| = \infty\}) = 0, and

(iii) it holds that \(\mathbb{E}[\|\|U(X(\omega))|| = \infty\) and \(\mathbb{E}[U(X)] = \int_S \mathbb{E}[\|\|U(s))||X(P)_{(P,B(S)}(ds)\).

Proof of Lemma 2.6. Throughout this proof let \(\|\| : \mathbb{R}^d \to [0, \infty)\) be the standard norm on \(\mathbb{R}^d\), let \(Z : \Omega \to \Omega\) satisfy for all \(\omega \in \Omega\) that \(Z(\omega) = \omega\), let \(u_k = (u_k(s))_{s \in S} = (u_k(s, \omega))_{(s, \omega) \in S \times \Omega} : S \times \Omega \to \mathbb{R}, k \in \{1, 2, \ldots, d\}\), satisfy for all \(F \in \mathcal{S}, \omega \in \Omega\) that \((u_1(\omega, \omega), u_2(\omega, \omega), \ldots, u_d(\omega, \omega)) = U(s, \omega)\), and let \(u_i : s \times \Omega \to [0, \infty), i \in \{1, 2, \ldots, d\}\), satisfy for all \(s \in S, \omega \in \Omega, i \in \{1, 2, \ldots, d\}\) that \(u_i(s, \omega) = \max\{u_i(s), 0\}\) and \(u_i(s, \omega) = \max\{-u_i(s), 0\}\). Observe that for all \(i \in \{1, 2, \ldots, d\}\) it holds that \(u_i = u_i - u_i\). Moreover, note that the hypothesis that \(U\) is a continuous random field ensures that \(U\) is \((\mathcal{B}(S) \otimes \mathcal{S}_\Omega(|\{U(s) : s \in S\}|)/\mathcal{B}(\mathbb{R}^d)\)-measurable. Next note that the fact that \(\sigma\{\{U(s) : s \in S\}\} \subseteq \mathcal{F}\) ensures that \(Z = \mathcal{F}/\mathcal{S}_\Omega(|\{U(s) : s \in S\}|)-\mathcal{B}(\mathbb{R}^d)\)-measurable. The hypothesis that \(X\) is \(\mathcal{F}/\mathcal{B}(S)\)-measurable hence proves that \(\Omega \ni \omega \mapsto (X(\omega), Z(\omega)) = (X(\omega), \omega) \in S \times \Omega\) is \((\mathcal{B}(S) \otimes \mathcal{S}_\Omega(|\{U(s) : s \in S\}|)/\mathcal{B}(\mathbb{R}^d)\)-measurable. Combining this with the fact that \(U\) is \((\mathcal{B}(S) \otimes \mathcal{S}_\Omega(|\{U(s) : s \in S\}|)/\mathcal{B}(\mathbb{R}^d)\)-measurable establishes item (i). Furthermore, observe that the hypothesis that \(\int_S \mathbb{E}[\|\|U(s))||X(P)_{(P,B(S)}(ds) < \infty\) ensures that

\[
(X(P)_{(P,B(S)}(\{s \in S : \mathbb{E}[\|\|U(s))|| = \infty\}) = 0. \tag{5}
\]

This establishes item (ii). Moreover, note that Lemma 2.5 and the hypothesis that \(\int_S \mathbb{E}[\|\|U(s))||X(P)_{(P,B(S)}(ds) < \infty\) assure that

\[
\mathbb{E}[\|\|U(X))|| = \int_S \mathbb{E}[\|\|U(s))||X(P)_{(P,B(S)}(ds) < \infty. \tag{6}
\]

Next let \(a \in (0, \infty)\) satisfy that

\[
\|\|U(X))|| \leq a\|\|U(X))|| \tag{7}
\]

(cf., e.g., Kreyszig [29, Theorem 2.4-5]). Observe that (7), (6), the fact that for all \(i \in \{1, 2, \ldots, d\}\) it holds that \(|u_i| = u_i + u_i\), and the fact that for all \(i \in \{1, 2, \ldots, d\}\) it holds that \(|u_i(X))|| \leq \|\|U(X))|| imply that for all \(i \in \{1, 2, \ldots, d\}\) it holds that

\[
\mathbb{E}[u_i(X))| + \mathbb{E}[u_i(X))| = \mathbb{E}[u_i(X))| \leq \mathbb{E}[\|\|U(X))|| \leq a\mathbb{E}[\|\|U(X))|| < \infty. \tag{8}
\]

This and the hypothesis that \(\int_S \mathbb{E}[\|\|U(s))||X(P)_{(P,B(S)}(ds) < \infty\) ensure that for all \(i \in \{1, 2, \ldots, d\}\) it holds that

\[
\int_S (\mathbb{E}[u_i(s)) + u_i(s)))X(P)_{(P,B(S)}(ds) \leq a \int_S (\mathbb{E}[\|\|U(s))||X(P)_{(P,B(S)}(ds) < \infty. \tag{9}
\]
Lemma 2.5, (8), and the fact that for all \( i \in \{1, 2, \ldots, d\} \) it holds that \( u_i = \mathbf{u}_i - \mathbf{u}_i \) hence demonstrate that for all \( i \in \{1, 2, \ldots, d\} \) it holds that

\[
\mathbb{E}[u_i(X)] = \mathbb{E}[\mathbf{u}_i(X) - \mathbf{u}_i(X)] = \mathbb{E}[\mathbf{u}_i(X)] - \mathbb{E}[\mathbf{u}_i(X)] = \int_S \mathbb{E}[u_i(s)](X(\mathbb{P}(S)))\,ds - \int_S \mathbb{E}[u_i(s)](X(\mathbb{P}(S)))\,ds
\]

This implies that

\[
\mathbb{E}[U(X)] = \int_S \mathbb{E}[U(s)](X(\mathbb{P}(S)))\,ds.
\]

Combining this and (6) establishes item (iii). The proof of Lemma 2.6 is thus completed.

2.4 A priori bounds for solutions of ordinary differential equations

Lemma 2.7. Assume Setting 2.1. Then it holds for all \( t \in [0, T] \) that

\[
\|X(t) - \xi\| \leq T(\mathbb{E}[[F(\xi, \mathbb{Z}^0)]]^{1/2})e^{LT}.
\]

Proof of Lemma 2.7. Throughout this proof assume without loss of generality (w.l.o.g.) that \( \mathbb{E}[[F(\xi, \mathbb{Z}^0)]] < \infty \). Observe that (3) ensures that for all \( t \in [0, T] \) it holds that

\[
\|X(t) - \xi\| = \left\| \int_0^t \mathbb{E}[F(X(s), \mathbb{Z}^0)] \, ds \right\| \leq \int_0^t \|\mathbb{E}[F(X(s), \mathbb{Z}^0)]\| \, ds \leq \int_0^t \mathbb{E}[[F(X(s), \mathbb{Z}^0)]] \, ds.
\]

In addition, observe that the hypothesis that \( X \in C([0, T], \mathbb{R}^d) \) ensures that

\[
\int_0^T \|X(s) - \xi\| \, ds < \infty.
\]

Moreover, note that (2) and the triangle inequality ensure that for all \( x \in \mathbb{R}^d, s \in S \) it holds that

\[
\|F(x, s)\| \leq \|F(\xi, s)\| + \|F(x, s) - F(\xi, s)\| \leq \|F(\xi, s)\| + L\|x - \xi\|.
\]

This, (13), and Jensen’s inequality ensure that for all \( t \in [0, T] \) it holds that

\[
\|X(t) - \xi\| \leq \int_0^t \mathbb{E}[[F(X(s), \mathbb{Z}^0)]] \, ds \leq \int_0^t \mathbb{E}[[F(\xi, \mathbb{Z}^0)]] + L\|X(s) - \xi\| \, ds
\]

\[
= \int_0^t \mathbb{E}[[F(\xi, \mathbb{Z}^0)]] \, ds + L\int_0^t \|X(s) - \xi\| \, ds \leq T\mathbb{E}[[F(\xi, \mathbb{Z}^0)]] + L\int_0^t \|X(s) - \xi\| \, ds
\]

\[
= T\left(\mathbb{E}[[F(\xi, \mathbb{Z}^0)]]\right)^{1/2} + L\int_0^t \|X(s) - \xi\| \, ds \leq T(\mathbb{E}[[F(\xi, \mathbb{Z}^0)]]^{1/2}) + L\int_0^t \|X(s) - \xi\| \, ds.
\]

Combining this and (14) with Gronwall’s integral inequality implies that for all \( t \in [0, T] \) it holds that

\[
\|X(t) - \xi\| \leq T(\mathbb{E}[[F(\xi, \mathbb{Z}^0)]]^{1/2})e^{LT} \leq T(\mathbb{E}[[F(\xi, \mathbb{Z}^0)]]^{1/2})e^{LT}.
\]

This establishes (12). The proof of Lemma 2.7 is thus completed.
2.5 Properties of MLP approximations

Lemma 2.8. Let $d, N \in \mathbb{N}$, let $(\Omega, \mathcal{F}, P)$ be a probability space, let $X_k: \Omega \to \mathbb{R}^d$, $k \in \{1, 2, \ldots, N\}$, be independent random variables, let $Y_k: \Omega \to \mathbb{R}^d$, $k \in \{1, 2, \ldots, N\}$, be independent random variables, and assume for every $k \in \{1, 2, \ldots, N\}$ that $X_k$ and $Y_k$ are identically distributed. Then it holds that $(\sum_{k=1}^N X_k): \Omega \to \mathbb{R}^d$ and $(\sum_{k=1}^N Y_k): \Omega \to \mathbb{R}^d$ are identically distributed random variables.

Lemma 2.9. Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $(S, \delta)$ be a separable metric space, let $(E, d)$ be a metric space, let $U, V: S \times \Omega \to E$ be continuous random fields, let $X, Y: \Omega \to S$ be random variables, assume that $U$ and $X$ are independent, assume that $V$ and $Y$ are independent, assume for all $s \in S$ that $U(s)$ and $V(s)$ are identically distributed, and assume that $X$ and $Y$ are identically distributed. Then it holds that $U(X) = (U(X(\omega)), \omega)_{\omega \in \Omega}: \Omega \to E$ and $V(Y) = (V(Y(\omega), \omega))_{\omega \in \Omega}: \Omega \to E$ are identically distributed random variables.

Lemma 2.10 (Properties of MLP approximations). Assume Setting 2.1 and let $m \in \mathbb{N}$. Then

(i) for all $\theta \in \Theta$, $n \in \mathbb{N}_0$ it holds that $\mathcal{X}^\theta_{n,m}:[0,T] \times \Omega \to \mathbb{R}^d$ is a stochastic process with continuous sample paths,

(ii) for all $\theta \in \Theta$, $n \in \mathbb{N}_0$ it holds that $\mathcal{X}^\theta_{n,m}$ is $(\mathcal{B}([0,T]) \otimes \sigma_\Theta((e^{(\theta,i)}(\omega))_{\theta \in \Theta}, (Z^{(\theta,i)}(\omega))_{\theta \in \Theta})) / \mathcal{B}(\mathbb{R}^d)$-measurable,

(iii) for all $\theta \in \Theta$, $n \in \mathbb{N}$, $t \in [0,T]$ it holds that

\[
\{([N_0 \cap [0,n-1]] \times \mathbb{N}) \ni (l,k) \to \begin{cases} F(\xi, Z^{(\theta,0,k)}), & l = 0 \\ F(\mathcal{X}^{(\theta,0,l,k)}_{l,m}(\mathcal{R}_{\theta,l,k}(\theta)), Z^{(\theta,0,l,k)}) - F(\mathcal{X}^{(\theta,0,l-1,k)}_{l-1,m}(\mathcal{R}_{\theta,l-1,k}(\theta)), Z^{(\theta,0,l-1,k)}) & : l > 0 \end{cases}
\]

is an independent family of random variables, and

(iv) for all $n \in \mathbb{N}_0$, $t \in [0,T]$ it holds that $\Omega \ni \omega \to \mathcal{X}^\theta_{n,m}(t, \omega) \in \mathbb{R}^d$, $\theta \in \Theta$, are identically distributed random variables.

Proof of Lemma 2.10. We first prove item (i) by induction on $n \in \mathbb{N}_0$. For the base case $n = 0$ observe that the hypothesis that for all $\theta \in \Theta$, $t \in [0,T]$ it holds that $\mathcal{X}^\theta_{0,m}(t) = \xi$ demonstrates that for all $\theta \in \Theta$ it holds that $\mathcal{X}^\theta_{0,m}:[0,T] \times \Omega \to \mathbb{R}^d$ is a stochastic process with continuous sample paths. This establishes item (i) in the base case $n = 0$. For the induction step $N_0 \ni (n-1) \to n \in \mathbb{N}$ let $n \in \mathbb{N}$ and assume that for every $j \in \mathbb{N}_0 \cap [0,n)$, $\theta \in \Theta$ it holds that $\mathcal{X}^\theta_{j,m}:[0,T] \times \Omega \to \mathbb{R}^d$ is a stochastic process with continuous sample paths. Observe that the induction hypothesis, the fact that $Z^\theta$ is $\mathcal{F} / \mathcal{S}$-measurable, the fact that for all $\theta \in \Theta$ it holds that $\mathcal{R}^\theta:[0,T] \times \Omega \to [0,T]$ are stochastic processes, Lemma 2.4, and (4) prove that for all $\theta \in \Theta$ it holds that $\mathcal{X}^\theta_{n,m}:[0,T] \times \Omega \to \mathbb{R}^d$ is a stochastic process. This, the induction hypothesis, the fact that for all $\theta \in \Theta$ it holds that $\mathcal{R}^\theta:[0,T] \times \Omega \to [0,T]$ are stochastic processes with continuous sample paths, (2), and (4) ensure that for all $\theta \in \Theta$ it holds that $\mathcal{X}^\theta_{n,m}:[0,T] \times \Omega \to \mathbb{R}^d$ is a stochastic process with continuous sample paths. Induction thus establishes item (i). Next we prove item (ii) by induction on $n \in \mathbb{N}_0$. For the base case $n = 0$ observe that the hypothesis that for all $\theta \in \Theta$, $t \in [0,T]$ it holds that $\mathcal{X}^\theta_{0,m}(t) = \xi$ demonstrates that for all $\theta \in \Theta$ it holds that $\mathcal{X}^\theta_{0,m}:[0,T] \times \Omega \to \mathbb{R}^d$ is $(\mathcal{B}([0,T]) \otimes \sigma_\Theta((e^{(\theta,i)}(\omega))_{\theta \in \Theta}, (Z^{(\theta,i)}(\omega))_{\theta \in \Theta})) / \mathcal{B}(\mathbb{R}^d)$-measurable. This implies item (iii) in the base case $n = 0$. For the induction step $N_0 \ni (n-1) \to n \in \mathbb{N}$ let $n \in \mathbb{N}$ and assume that for every $j \in \mathbb{N}_0 \cap [0,n)$, $\theta \in \Theta$ it holds that $\mathcal{X}^\theta_{j,m}((\mathcal{B}([0,T]) \otimes \sigma_\Theta((e^{(\theta,i)}(\omega))_{\theta \in \Theta}, (Z^{(\theta,i)}(\omega))_{\theta \in \Theta})) / \mathcal{B}(\mathbb{R}^d)$-measurable. Note that the induction hypothesis, the fact that $\mathcal{F}: \mathbb{R}^d \times \mathcal{S} \to \mathbb{R}^d = (\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{S}) / \mathcal{B}(\mathbb{R}^d)$-measurable, the fact that for all $\theta \in \Theta$ it holds that $Z^\theta$ is $\mathcal{F} / \mathcal{S}$-measurable, (4), and Lemma 2.4 prove that for all $\theta \in \Theta$, $t \in [0,T]$ it
holds that
\[
\sigma_\Omega(\mathcal{X}_{n,m}^\theta(t)) \subseteq \sigma_\Omega\left(\left\{Z^{(\theta,l,k)}_{t} \mid k \in \{1,2,\ldots,m^n-1\}, t \in \mathbb{N}[1,n]\right\} \cup \left\{R^{(\theta,l,k)}_{t} \mid k \in \{1,2,\ldots,m^n-1\}, t \in \mathbb{N}[1,n]\right\} \cup \left\{Z^{(\theta,l,k)}_{t} \mid k \in \{1,2,\ldots,m^n-1\}, t \in \mathbb{N}[1,n]\right\} \cup \left\{Z^{(\theta,l,k)}_{t} \mid k \in \{1,2,\ldots,m^n-1\}, t \in \mathbb{N}[1,n]\right\} \cup \left\{Z^{(\theta,l,k)}_{t} \mid k \in \{1,2,\ldots,m^n-1\}, t \in \mathbb{N}[1,n]\right\}
\]

Moreover, observe that item (i) ensures that for all \( \theta \in \Theta \) it holds that \( \mathcal{X}_{n,m}^\theta(t) = \left\{X_{n,m}^\theta(t) \mid t \in [0,T] \right\} \subseteq \sigma_\Omega(\mathcal{X}_{n,m}^\theta(t)) \)\( /B(\mathbb{R}^d) \)-measurable. Combining this with (19) demonstrates that for all \( \theta \in \Theta \) it holds that \( \mathcal{X}_{n,m}^\theta(t) \subseteq \sigma_\Omega(\mathcal{X}_{n,m}^\theta(t)) \)\( /B(\mathbb{R}^d) \)-measurable. Induction thus establishes item (ii). Furthermore, observe that item (ii), the hypothesis that \( (Z^\theta)_{\theta \in \Theta} \) are independent, the hypothesis that \( (\mathcal{X}_{n,m}^\theta(t)) \) are i.i.d., and Theorem 2.4 prove item (iii). Next we prove item (iv) by induction on \( n \in \mathbb{N}_0 \). For the base case \( n = 0 \) observe that for all \( \theta \in \Theta \), \( t \in [0,T] \) it holds that \( \mathcal{X}_{0,m}^\theta(t) = \{X_{0,m}^\theta(t) \mid t \in [0,T] \} \subseteq \sigma_\Omega(\mathcal{X}_{0,m}^\theta(t)) \)\( /B(\mathbb{R}^d) \)-measurable. This establishes item (iv) in the base case \( n = 0 \). For the induction step \( n \geq 1 \rightarrow n \in \mathbb{N} \), let assume that for every \( j \in \mathbb{N}_0 \cap [0,n) \), \( t \in [0,T], \theta \in \Theta \) it holds that \( \mathcal{X}_{j,m}^\theta(t) \subseteq \sigma_\Omega(\mathcal{X}_{j,m}^\theta(t)) \)\( /B(\mathbb{R}^d) \)-measurable. Induction thus establishes item (iv).

**Lemma 2.11.** Assume Setting 2.1, let \( \theta \in \Theta, t \in [0,T] \), let \( U_1 : [0,t] \times \Omega \rightarrow \{0,\infty\} \) and \( U_2 : [0,t] \times \Omega \rightarrow \mathbb{R}^d \) be stochastic processes with continuous sample paths, assume for all \( i \in \{1,2\} \) that \( U_i \) and \( U_i(t) \) are independent, and assume that \( \int_0^t \mathbb{E}[\|U_i(r)\|] \, dr < \infty \). Then it holds for all \( i \in \{1,2\} \) that Borel\( \{0,t]\}(\{r \in [0,t] \mid \mathbb{E}[\|U_i(r)\|] = \infty\}) = 0, \mathbb{E}[\|U_1(R_i^\theta)\|] < \infty \), and
\[
t \mathbb{E}[U_i(R_i^\theta)] = \int_0^t \mathbb{E}[U_i(r)] \, dr.
\]

**Proof of Lemma 2.11.** Throughout this proof assume w.l.o.g. that \( T > 0 \) and \( t > 0 \). Observe that the hypothesis that \( R_i^\theta = \varepsilon_i^T \) implies that \( R_i^\theta \) is \( U_{1,0}\)-distributed. Combining this with the fact that \( U_1 \) is a stochastic process with continuous sample paths, the fact that \( U_1 \) and \( R_i^\theta \) are independent, and Lemma 2.5 assures that
\[
t \mathbb{E}[U_1(R_i^\theta)] = t \int_{[0,t]} \mathbb{E}[U_1(r)](R_i^\theta(P)[B(0,1,t)])(dr)
\]

\[
= t \int_{[0,t]} \mathbb{E}[U_1(r)](U_{1,0}(r))(dr) = t \int_0^t \mathbb{E}[U_1(r)] \, dr = \int_0^t \mathbb{E}[U_1(r)] \, dr.
\]
In addition, note that the fact that \( \mathcal{R}_t^0 \) is \( \mathcal{U}_{[0,1]} \)-distributed, the fact that \( U_2 \) is a stochastic process with continuous sample paths, the fact that \( U_2 \) and \( \mathcal{R}_t^0 \) are independent, the hypothesis that \( \int_0^t E[\|U_2(r)\|] \, dr < \infty \), and Lemma 2.6 ensure that \( Borel_{[0,1]}(\{r \in [0,t]: E[\|U_2(r)\|] = \infty\}) = 0 \), \( E[\|U_2(\mathcal{R}_t^0)\|] < \infty \), and

\[
t E[U_2(\mathcal{R}_t^0)] = t \int_{[0,1]} E[U_2(r)](\mathcal{R}_t^0(P) \mathcal{B}_{[0,1]})(dr) = t \int_{[0,1]} E[U_2(r)](U_{[0,1]})(dr) = \int_0^t E[U_2(r)] \, dr = \int_0^t E[U_2(r)] \, dr.
\]

Combining this with (22) establishes (21). The proof of Lemma 2.11 is thus completed.

\[\square\]

Lemma 2.12 (Expectations of MLP approximations). Assume Setting 2.1 and assume that \( E[\|F(\xi, Z^0)\|] < \infty \). Then

(i) for all \( n \in \mathbb{N}_0 \), \( m \in \mathbb{N} \), \( t \in [0,T] \), \( s \in [0,t] \) it holds that

\[
E[\|\mathcal{A}_{n,m}^0(s)\|] + t E[\|\mathcal{A}_{n,m}^0(\mathcal{R}_t^0)\|] \leq \frac{1}{n} E[\|\mathcal{A}_{n,m}^0(\mathcal{R}_t^0, Z^0)\|]
\]

and

(ii) for all \( n, m \in \mathbb{N} \), \( t \in [0,T] \) it holds that

\[
E[\|\mathcal{A}_{n,m}^0(t)\|] \leq \frac{1}{n} E[\|\mathcal{A}_{n,m}^0(\mathcal{R}_t^0, Z^0)\|].
\]

Proof of Lemma 2.12. Throughout this proof let \( m \in \mathbb{N} \). Observe that Lemma 2.11, items (i) and (ii) in Lemma 2.10, and the fact that for all \( n \in \mathbb{N} \) it holds that \( \mathcal{A}_{n,m}^0 \), \( Z^0 \), and \( \mathbf{v}^0 \) are independent demonstrate that for all \( n \in \mathbb{N}_0 \), \( t \in [0,T] \) it holds that

\[
t E[\|\mathcal{A}_{n,m}^0(\mathcal{R}_t^0)\|] + t E[\|\mathcal{A}_{n,m}^0(\mathcal{R}_t^0, Z^0)\|] = \int_0^t E[\|\mathcal{A}_{n,m}^0(r)\|] \, dr + \int_0^t E[\|\mathcal{A}_{n,m}^0(r, Z^0)\|] \, dr.
\]

Next we claim that for all \( n \in \mathbb{N}_0 \), \( t \in [0,T] \), \( s \in [0,t] \) it holds that

\[
E[\|\mathcal{A}_{n,m}^0(s)\|] + \int_0^t E[\|\mathcal{A}_{n,m}^0(r)\|] \, dr + \int_0^t E[\|F(\mathcal{A}_{n,m}^0(r), Z^0)\|] \, dr < \infty.
\]

We now prove (26) by induction on \( n \in \mathbb{N}_0 \). For the base case \( n = 0 \) observe that the hypothesis that for all \( t \in [0,T] \) it holds that \( \mathcal{A}_{n,m}^0(t) = \xi \) and the hypothesis that \( E[\|F(\xi, Z^0)\|] < \infty \) imply that for all \( t \in [0,T] \), \( s \in [0,t] \) it holds that

\[
E[\|\mathcal{A}_{n,m}^0(s)\|] + \int_0^t E[\|\mathcal{A}_{n,m}^0(r)\|] \, dr + \int_0^t E[\|F(\mathcal{A}_{n,m}^0(r), Z^0)\|] \, dr \leq \|\xi\| + t \|\xi\| + t E[\|F(\xi, Z^0)\|] < \infty.
\]

This establishes (26) in the base case \( n = 0 \). For the induction step \( \mathbb{N}_0 \ni (n - 1) \rightarrow n \in \mathbb{N} \) let \( n \in \mathbb{N} \) and assume that for all \( j \in \mathbb{N}_0 \cap [0,n) \), \( t \in [0,T] \), \( s \in [0,t] \) it holds that

\[
E[\|\mathcal{A}_{n,m}^0(s)\|] + \int_0^t E[\|\mathcal{A}_{n,m}^0(r)\|] \, dr + \int_0^t E[\|F(\mathcal{A}_{n,m}^0(r), Z^0)\|] \, dr < \infty.
\]

Note that (4) and the triangle inequality ensure that for all \( t \in [0,T] \), \( s \in [0,t] \) it holds that

\[
E[\|\mathcal{A}_{n,m}^0(s)\|] \leq \sum_{l=1}^{n-1} \sum_{k=1}^{m^n} \left( E[\|F(\mathcal{A}_{n,m}^0(\mathcal{R}_t^0, Z^0), (0,l,k))\|] + E[\|F(\mathcal{A}_{n,m}^0(\mathcal{R}_t^0, Z^0), (0,l,k))\|]\right) + \frac{s}{m^n} \sum_{k=1}^{m^n} \left( E[\|F(\xi, Z^0, (0,l,k))\|]\right) + \|\xi\|.
\]

9
Furthermore, observe that the hypothesis that $\mathbb{E}[\|F(\xi, Z^0)\|] < \infty$ and the hypothesis that $(Z^0)_{\theta \in \Theta}$ are identically distributed random variables assure that for all $k \in \mathbb{Z}$ it holds that
\[
\mathbb{E}[\|F(\xi, Z^{(0,k)})\|] = \mathbb{E}[\|F(\xi, Z^0)\|] < \infty.
\] (30)

Moreover, observe that Lemma 2.11, the hypothesis that $(Z^0)_{\theta \in \Theta}$ are i.i.d., the hypothesis that $(\tau^0)_{\theta \in \Theta}$ are independent, the hypothesis that $(Z^0)_{\theta \in \Theta}$ and $(\tau^0)_{\theta \in \Theta}$ are independent, and items (i), (ii), and (iv) in Lemma 2.10 demonstrate that for all $i, j, k \in \mathbb{Z}$, $l \in \mathbb{N}_0$, $t \in [0, T]$, $s \in [0, t]$ it holds that
\[
s \mathbb{E}[\|F(X^0_{t,m}(r), Z^{(0,j,k)})\|] = \int_0^s \mathbb{E}[\|F(X^0_{t,m}(r), Z^{(0,j,k)})\|] \text{d}r = \int_0^s \mathbb{E}[\|F(X^0_{t,m}(r), Z^0)\|] \text{d}r.
\] (31)

Combining this, (28), (29), and (30) establishes that for all $t \in [0, T]$, $s \in [0, t]$ it holds that
\[
\mathbb{E}[\|X^0_{n,m}(s)\|] \leq \left( \sum_{i=1}^{n-1} \frac{1}{m^{n-1}} \left[ \int_0^s \mathbb{E}[\|F(X^0_{t,m}(r), Z^0)\|] \text{d}r + \int_0^s \mathbb{E}[\|F(X^0_{t-1,m}(r), Z^0)\|] \text{d}r \right] \right) + \frac{s}{m^n} \sum_{k=1}^{m^n} \mathbb{E}[\|F(\xi, Z^0)\|] + \|\xi\|.
\]
(32)

Hence, we obtain that for all $t \in [0, T]$ it holds that
\[
\int_0^t \mathbb{E}[\|X^0_{n,m}(r)\|] \text{d}r \leq t \sup_{s \in [0, t]} \mathbb{E}[\|X^0_{n,m}(s)\|] \leq t \left( \sum_{i=1}^{n-1} \int_0^t \mathbb{E}[\|F(X^0_{t,m}(r), Z^0)\|] \text{d}r + 2T \mathbb{E}[\|F(\xi, Z^0)\|] + \|\xi\| \right) < \infty.
\] (33)

Next note that (2) and the triangle inequality imply that for all $x \in \mathbb{R}^d$, $s \in S$ it holds that
\[
\|F(x, s)\| \leq \|F(\xi, s)\| + \|F(x, s) - F(\xi, s)\| \leq \|F(\xi, s)\| + L\|x - \xi\|.
\] (34)

This, the triangle inequality, (33), and the hypothesis that $\mathbb{E}[\|F(\xi, Z^0)\|] < \infty$ assure that for all $t \in [0, T]$ it holds that
\[
\int_0^t \mathbb{E}[\|F(X^0_{n,m}(r), Z^0)\|] \text{d}r \leq \int_0^t \mathbb{E}[\|F(\xi, Z^0)\|] \text{d}r + L\int_0^t \mathbb{E}[\|X^0_{n,m}(r) - \xi\|] \text{d}r
\leq T \mathbb{E}[\|F(\xi, Z^0)\|] + L\int_0^t \mathbb{E}[\|X^0_{n,m}(r)\|] \text{d}r + L\int_0^t \mathbb{E}[\|\xi\|] \text{d}r
\leq T \mathbb{E}[\|F(\xi, Z^0)\|] + L\int_0^t \mathbb{E}[\|X^0_{n,m}(r)\|] \text{d}r + LT \|\xi\| < \infty.
\] (35)

This, (32), and (33) establish that for all $t \in [0, T]$, $s \in [0, t]$ it holds that
\[
\mathbb{E}[\|X^0_{n,m}(s)\|] + \int_0^s \mathbb{E}[\|X^0_{n,m}(r)\|] \text{d}r + \int_0^t \mathbb{E}[\|F(X^0_{n,m}(r), Z^0)\|] \text{d}r < \infty.
\] (36)
Induction thus proves (26). Combining (25) and (26) hence establishes item (i). Next observe that (4), (26), items (i), (ii), and (iv) in Lemma 2.10, the hypothesis that \((Z^0)_{\theta \in \Theta}\) are i.i.d., the hypothesis that \((\nu^0)_{\theta \in \Theta}\) are i.i.d., the hypothesis that \((Z^0)_{\theta \in \Theta}\) and \((\nu^0)_{\theta \in \Theta}\) are independent, and Lemma 2.9 ensure that for all \(n \in \mathbb{N}, t \in [0, T]\) it holds that
\[
\mathbb{E}[\lambda_{n,m}^0(t)] = \sum_{i=1}^{n-1} \frac{t}{m^{n-1}} \sum_{k=1}^{m^{n-1}} \left( \mathbb{E}[F(\lambda_{t,m}^{0,l,k}(R^0_{i,t,k}), Z^{0,l,k})] - \mathbb{E}[F(\lambda_{t-1,m}^{0,l,k}(R^0_{i,t,k}), Z^{0,l,k})] \right) + \xi
\]
\[
= \frac{t}{m^n} \sum_{k=1}^{m^n} \mathbb{E}[F(\xi, Z^{0,0,k})] + \xi
\]
\[
= t \sum_{i=1}^{n-1} \left( \mathbb{E}[F(\lambda_{t,m}^{0,l,k}(R^0_{t}), Z^0)] - \mathbb{E}[F(\lambda_{t-1,m}^{0,l,k}(R^0_{t}), Z^0)] \right) + t \mathbb{E}[F(\xi, Z^0)] + \xi
\]
\[
= \xi + t \mathbb{E}[F(\lambda_{t-1,m}^{0,l,k}(R^0_{t}), Z^0)].
\]

Lemma 2.11, items (i) and (iv) in Lemma 2.10, the fact that for all \(n \in \mathbb{N}_0\) it holds that \(\lambda_{n,m}^0, Z^0, \) and \(\nu^0\) are independent, and (26) hence imply that for all \(n \in \mathbb{N}, t \in [0, T]\) it holds that
\[
\mathbb{E}[\lambda_{n,m}^0(t)] = \xi + \int_0^t \mathbb{E}[F(\lambda_{n-1,m}(r), Z^0)] \, dr.
\]

This establishes item (ii). The proof of Lemma 2.12 is thus completed. \(\square\)

### 2.6 Error estimates for MLP approximations

**Lemma 2.13.** Let \(n \in \mathbb{N}\), let \((\Omega, F, P)\) be a probability space, let \(X_1, X_2, \ldots, X_n: \Omega \to \mathbb{R}\) be independent random variables, and assume for all \(i \in \{1, 2, \ldots, n\}\) that \(\mathbb{E}[|X_i|] < \infty\). Then it holds that
\[
\text{Var} \left( \sum_{i=1}^n X_i \right) = \mathbb{E}\left[ \mathbb{E}\left[ \sum_{i=1}^n X_i \right]^2 - \sum_{i=1}^n \mathbb{E}[X_i]^2 \right] = \sum_{i=1}^n \text{Var}(X_i).
\]

**Lemma 2.14.** Let \((\Omega, F, \mu)\) be a measure space and let \(f: \Omega \to [0, \infty]\) be \(F/\mathcal{B}([0, \infty])\)-measurable. Then it holds that
\[
\left( \int f(\omega) \mu(\,d\omega) \right)^2 \leq \mu(\Omega) \int |f(\omega)|^2 \mu(\,d\omega).
\]

**Lemma 2.15.** Let \(T \in [0, \infty), k \in \mathbb{N}\), and let \(U: [0, T] \to [0, \infty]\) be \(\mathcal{B}(\{0, T\})/\mathcal{B}([0, \infty])\)-measurable. Then it holds that
\[
\int_0^T \frac{(T-t)^{k-1}}{(k-1)!} \int_0^t U(r) \, dr \, dt = \int_0^T \frac{(T-t)^{k-1}}{(k-1)!} U(t) \, dt.
\]

**Proof of Lemma 2.15.** Observe that Tonelli’s theorem assures that
\[
\int_0^T \frac{(T-t)^{k-1}}{(k-1)!} \int_0^t U(r) \, dr \, dt = \int_0^T \int_0^T \frac{(T-t)^{k-1}}{(k-1)!} U(r) \, dt \, dr
\]
\[
= \int_0^T \int_0^T \frac{(T-t)^{k-1}}{(k-1)!} U(r) \, dt \, dr
\]
\[
= \int_0^T \int_r^T \frac{(T-t)^{k-1}}{(k-1)!} \, dt \, U(r) \, dr = \int_0^T \frac{(T-t)^{k-1}}{k!} U(t) \, dt.
\]

The proof of Lemma 2.15 is thus completed. \(\square\)
Proposition 2.16. Assume Setting 2.1. Then it holds for all \( n \in \mathbb{N}_0, m \in \mathbb{N} \) that
\[
\left( \mathbb{E} \left[ \|X(T) - X^0_{n,m}(T)\|^2 \right] \right)^{1/2} \leq T \left( \mathbb{E} \left[ \|F(\xi, Z^0)\|^2 \right] \right)^{1/2} \left( 1 + 2LT \right)n e^{(LT+m/2)}m^{n/2}. \tag{43}
\]

Proof of Proposition 2.16. Throughout this proof assume w.l.o.g. that \( T > 0 \) and \( \mathbb{E}[\|F(\xi, Z^0)\|^2] < \infty \), let \( C \in [0, \infty) \) satisfy that
\[
C = T \left( \mathbb{E} \left[ \|F(\xi, Z^0)\|^2 \right] \right)^{1/2} e^{LT}, \tag{44}
\]
let \( \zeta_i \in \mathbb{R}, i \in \{1, 2, \ldots, d\} \), satisfy that \( (\zeta_1, \zeta_2, \ldots, \zeta_d) = \xi \), let \( \chi_{i,n,m}^\theta : [0, T] \times \Omega \to \mathbb{R}, n, m \in \mathbb{N}_0, \theta \in \Theta, i \in \{1, 2, \ldots, d\} \), satisfy for all \( n \in \mathbb{N}_0, m \in \mathbb{N}, \theta \in \Theta \) that \( (\chi_{i,n,m}^\theta, \chi_{2,n,m}^\theta, \ldots, \chi_{d,n,m}^\theta) = \chi_{n,m}^\theta \), let \( f_i : \mathbb{R}^d \times S \to \mathbb{R}, i \in \{1, 2, \ldots, d\} \), satisfy for all \( x \in \mathbb{R}^d, s \in S \) that
\[
(f_1(x,s), f_2(x,s), \ldots, f_d(x,s)) = F(x,s), \tag{45}
\]
and let \( m \in \mathbb{N} \). Observe that (4) establishes that for all \( n \in \mathbb{N}_0, \theta \in \Theta, t \in [0, T], i \in \{1, 2, \ldots, d\} \) it holds that
\[
\chi_{i,n,m}^\theta(t) = \frac{1}{m} \sum_{k=1}^{m} f_i(\xi, Z_{\theta, k}^0, t) - \sum_{k=1}^{m} f_i(\xi, Z_{\theta, k}^0, 0) + \zeta_i.
\tag{46}
\]
Furthermore, note that (44) assures that
\[
C^2 = T^2 \mathbb{E} \left[ \|F(\xi, Z^0)\|^2 \right] e^{2LT} \geq T^2 \left( \mathbb{E} \left[ \|F(\xi, Z^0)\|^2 \right] \right).
\tag{47}
\]
In addition, note that the hypothesis that \( \mathbb{E}[\|F(\xi, Z^0)\|^2] < \infty \) and Jensen’s inequality assure that
\[
(E[\|F(\xi, Z^0)\|])^2 \leq E[\|F(\xi, Z^0)\|^2] < \infty. \tag{48}
\]
This, Lemma 2.12, (2), and (3) demonstrate that for all \( t \in [0, T], n \in \mathbb{N} \) it holds that
\[
\|E[\chi_{n,m}^0(t)] - X(t)\| \leq \left\| \xi + \int_0^t E[F(\chi_{n-1,m}(r), Z^0)] \, dr - \xi - \int_0^t E[F(X(r), Z^0)] \, dr \right\|
\]
\[
= \left\| \int_0^t E[F(\chi_{n-1,m}(r), Z^0)] - E[F(X(r), Z^0)] \right\| \, dr
\]
\[
= \left\| \int_0^t E[F(\chi_{n-1,m}(r), Z^0)] - F(X(r), Z^0) \right\| \, dr
\]
\[
\leq \int_0^t E[\|F(\chi_{n-1,m}(r), Z^0) - F(X(r), Z^0)\|] \, dr
\]
\[
\leq \int_0^t E[|L||\chi_{n-1,m}(r) - X(r)|] \, dr = L \int_0^t E[|\chi_{n-1,m}(r) - X(r)|] \, dr.
\tag{49}
\]
This, Lemma 2.14, and Jensen’s inequality imply that for all \( t \in [0, T], n \in \mathbb{N} \) it holds that
\[
\|E[\chi_{n,m}^0(t)] - X(t)\|^2 \leq L^2 \int_0^t E[|\chi_{n-1,m}(r) - X(r)|] \, dr \leq L^2 T \int_0^t (E[|\chi_{n-1,m}(r) - X(r)|])^2 \, dr
\]
\[
\leq L^2 T \int_0^t (E[|\chi_{n-1,m}(r) - X(r)|])^2 \, dr \leq L^2 T \int_0^t (E[|\chi_{n-1,m}(r) - X(r)|])^2 \, dr. \tag{50}
\]
In addition, observe that Lemma 2.13, (48), item (i) in Lemma 2.12, item (iii) in Lemma 2.10, and (46) imply that for all $t \in [0, T], n \in \mathbb{N}$ it holds that

\[
\mathbb{E}[[X_{n,m}^0(t) - \mathbb{E}[X_{n,m}^0(t)]]^2] = \mathbb{E}\left[\sum_{i=1}^{d} [X_{i,n,m}^0(t) - \mathbb{E}[X_{i,n,m}^0(t)]]^2\right] = \sum_{i=1}^{d} \text{Var}(X_{i,n,m}^0(t))
\]

\[= \sum_{i=1}^{d} \sum_{l=1}^{n-m-1} \text{Var}\left(\frac{t}{m^{n-l}} \left[f_i(\chi_{t,m}^{0,l,k}(R_{t}^{0,l,k}), Z^{0,l,k}) - f_i(\chi_{t-1,m}^{0,l-k}(R_{t}^{0,l,k}), Z^{0,l,k})\right]\right) + \text{Var}\left(\frac{t}{m^n} \sum_{k=1}^{m} f_i(\xi, Z^{(0,0,k)})\right).\] (51)

Moreover, note that the hypothesis that $(Z_\theta)_{\theta \in \Theta}$ are i.i.d. and the fact that for all $Y \in \mathcal{L}^1(\mathbb{P}; \mathbb{R})$ it holds that $\text{Var}(Y) \leq \mathbb{E}[|Y|^2]$ ensure that for all $i \in \{1, 2, \ldots, d\}, n \in \mathbb{N}$ it holds that

\[
\text{Var}\left(\frac{t}{m^n} \sum_{k=1}^{m} f_i(\xi, Z^{(0,0,k)})\right) = \frac{t^2}{m^{2n}} \text{Var}\left(\sum_{k=1}^{m} f_i(\xi, Z^{(0,0,k)})\right) = \frac{t^2m^n}{m^{2n}} \text{Var}(f_i(\xi, Z^0)) \leq \frac{t^2}{m^n} \mathbb{E}[|f_i(\xi, Z^0)|^2]. \] (52)

This and (45) imply that for all $t \in [0, T], n \in \mathbb{N}$ it holds that

\[
\sum_{i=1}^{d} \text{Var}\left(\frac{t}{m^n} \sum_{k=1}^{m} f_i(\xi, Z^{(0,0,k)})\right) \leq \sum_{i=1}^{d} \frac{t^2}{m^n} \mathbb{E}[|f_i(\xi, Z^0)|^2] = \frac{t^2}{m^n} \mathbb{E}\left[\sum_{i=1}^{d} |f_i(\xi, Z^0)|^2\right] \leq \frac{t^2}{m^n} \mathbb{E}[|F(\xi, Z^0)|^2]. \] (53)

In addition, observe that items (i), (ii), and (iv) in Lemma 2.10, the hypothesis that $(Z_\theta)_{\theta \in \Theta}$ are i.i.d., the hypothesis that $(\tau_\theta)_{\theta \in \Theta}$ are independent, the fact that for all $Y \in \mathcal{L}^1(\mathbb{P}; \mathbb{R})$ it holds that $\text{Var}(Y) \leq \mathbb{E}[|Y|^2]$, and Lemma 2.9 imply that for all $i \in \{1, 2, \ldots, d\}, t \in [0, T], n \in \mathbb{N}, l \in \mathbb{N} \cap [1, n]$ it holds that

\[
\sum_{k=1}^{m^n-1} \text{Var}\left(\frac{t}{m^{n-l}} \left[f_i(\chi_{t,m}^{0,l,k}(R_{t}^{0,l,k}), Z^{0,l,k}) - f_i(\chi_{t-1,m}^{0,l-k}(R_{t}^{0,l,k}), Z^0)\right]\right) = m^{n-1} \text{Var}\left(\frac{t}{m^{n-l}} \left[f_i(\chi_{t,m}^{0}(R_{t}^{0}), Z^0) - f_i(\chi_{t-1,m}^{0}(R_{t}^{0}), Z^0)\right]\right) = \frac{m^{n-1}t^2}{m^{2(n-l)}} \mathbb{E}[|f_i(\chi_{t,m}^{0}(R_{t}^{0}), Z^0) - f_i(\chi_{t-1,m}^{0}(R_{t}^{0}), Z^0)|^2]. \] (54)
This and (45) ensure that for all \( t \in [0, T] \), \( n \in \mathbb{N} \), \( l \in \mathbb{N} \cap [1, n) \) it holds that
\[
\sum_{i=1}^{n} \sum_{k=1}^{m-1} \text{Var} \left( \frac{t}{m^{n-l}} \left[ f_i \left( \mathcal{X}_{l,m}^{(0,l,k)}(\mathcal{R}_{t}^{(0,l,k)}), Z^{(0,l,k)} \right) - f_i \left( \mathcal{X}_{l-1,m}^{(0,l,k)}(\mathcal{R}_{t}^{(0,l,k)}), Z^{(0,l,k)} \right) \right] \right) \\
\leq \sum_{i=1}^{d} \frac{n^{2}}{m^{n-l}} \mathbb{E} \left( \left\| f_i \left( \mathcal{X}_{l,m}^{0}(\mathcal{R}_{t}^{0}), Z^{0} \right) - f_i \left( \mathcal{X}_{l-1,m}^{0}(\mathcal{R}_{t}^{0}), Z^{0} \right) \right\| ^2 \right) \\
= \frac{n^{2}}{m^{n-l}} \mathbb{E} \left[ \sum_{i=1}^{d} \left\| f_i \left( \mathcal{X}_{l,m}^{0}(\mathcal{R}_{t}^{0}), Z^{0} \right) - f_i \left( \mathcal{X}_{l-1,m}^{0}(\mathcal{R}_{t}^{0}), Z^{0} \right) \right\| ^2 \right] \\
= \frac{n^{2}}{m^{n-l}} \mathbb{E} \left[ \left\| F \left( \mathcal{X}_{l,m}^{0}(\mathcal{R}_{t}^{0}), Z^{0} \right) - F \left( \mathcal{X}_{l-1,m}^{0}(\mathcal{R}_{t}^{0}), Z^{0} \right) \right\| ^2 \right].
\]
Combining this, (51), and (53) ensures that for all \( t \in [0, T] \), \( n \in \mathbb{N} \) it holds that
\[
\mathbb{E} \left[ \left\| \mathcal{X}_{n,m}^{0}(t) - \mathcal{X}_{n,m}^{0}(t) \right\| ^2 \right] = \sum_{l=1}^{n-1} \sum_{i=1}^{m-1} \sum_{k=1}^{m-1} \text{Var} \left( \frac{t}{m^{n-l}} \left[ f_i \left( \mathcal{X}_{l,m}^{0}(\mathcal{R}_{t}^{0}), Z^{0} \right) - f_i \left( \mathcal{X}_{l-1,m}^{0}(\mathcal{R}_{t}^{0}), Z^{0} \right) \right] \right) \\
= \sum_{l=1}^{n-1} \sum_{i=1}^{m-1} \sum_{k=1}^{m-1} \left[ \text{Var} \left( \frac{t}{m^{n-l}} f_i \left( \mathcal{X}_{l,m}^{0}(\mathcal{R}_{t}^{0}), Z^{0} \right) \right) + \text{Var} \left( \frac{t}{m^{n-l}} f_i \left( \mathcal{X}_{l-1,m}^{0}(\mathcal{R}_{t}^{0}), Z^{0} \right) \right) \right] \\
= \sum_{l=1}^{n-1} \sum_{i=1}^{m-1} \left[ \text{Var} \left( \frac{t}{m^{n-l}} f_i \left( \mathcal{X}_{l,m}^{0}(\mathcal{R}_{t}^{0}), Z^{0} \right) \right) + \text{Var} \left( \frac{t}{m^{n-l}} f_i \left( \mathcal{X}_{l-1,m}^{0}(\mathcal{R}_{t}^{0}), Z^{0} \right) \right) \right] \\
\leq \sum_{l=1}^{n-1} \frac{n^{2}}{m^{n-l}} \mathbb{E} \left[ \left\| F \left( \mathcal{X}_{l,m}^{0}(\mathcal{R}_{t}^{0}), Z^{0} \right) - F \left( \mathcal{X}_{l-1,m}^{0}(\mathcal{R}_{t}^{0}), Z^{0} \right) \right\| ^2 \right] + \frac{T^{2}}{m^{n-l}} \mathbb{E} \left[ \left\| F(\xi, Z^{0}) \right\| ^2 \right].
\]
Furthermore, note that (2), the fact that for all \( x, y \in \mathbb{R}^d \) it holds that \( \left\| x + y \right\| ^2 \leq 2(\left\| x \right\| ^2 + \left\| y \right\| ^2) \), items (i), (ii), and (iv) in Lemma 2.10, the hypothesis that \( (Z^{0})_{\theta \in \Theta} \) are i.i.d., the hypothesis that \( (\mathcal{X}^{0})_{\theta \in \Theta} \) are i.i.d., the hypothesis that \( (Z^{0})_{\theta \in \Theta} \) and \( (\mathcal{X}^{0})_{\theta \in \Theta} \) are independent, and Lemma 2.9 assure that for all \( t \in [0, T] \), \( n \in \mathbb{N} \) it holds that
\[
\sum_{l=1}^{n-1} \frac{n^{2}}{m^{n-l}} \mathbb{E} \left[ \left\| F \left( \mathcal{X}_{l,m}^{0}(\mathcal{R}_{t}^{0}), Z^{0} \right) - F \left( \mathcal{X}_{l-1,m}^{0}(\mathcal{R}_{t}^{0}), Z^{0} \right) \right\| ^2 \right] \\
\leq \sum_{l=1}^{n-1} \frac{n^{2}}{m^{n-l}} \mathbb{E} \left[ \left\| L \right\| ^2 \left\| \mathcal{X}_{l,m}^{0}(\mathcal{R}_{t}^{0}) - \mathcal{X}_{l-1,m}^{0}(\mathcal{R}_{t}^{0}) \right\| ^2 \right] \\
\leq \frac{2L^{2}T^{2}}{m^{n-l}} \left( \mathbb{E} \left[ \left\| \mathcal{X}_{l,m}^{0}(\mathcal{R}_{t}^{0}) - X(\mathcal{R}_{t}^{0}) \right\| ^2 \right] + \mathbb{E} \left[ \left\| \mathcal{X}_{l-1,m}^{0}(\mathcal{R}_{t}^{0}) - X(\mathcal{R}_{t}^{0}) \right\| ^2 \right] \right).
\]
The hypothesis that \( (\mathcal{X}^{0})_{\theta \in \Theta} \) are independent, the hypothesis that \( (\mathcal{X}^{0})_{\theta \in \Theta} \) and \( (Z^{0})_{\theta \in \Theta} \) are independent, items (i) and
(ii) in Lemma 2.10, and Lemma 2.11 hence imply that for all \( t \in [0, T], n \in \mathbb{N} \) it holds that

\[
\sum_{l=1}^{n-1} \frac{l^2}{m^n-1} \mathbb{E} \left[ \| F(\mathcal{X}_{l,m}^0(\mathcal{R}_l^0), Z) - F(\mathcal{X}_{l-1,m}^0(\mathcal{R}_l^0), Z) \|^2 \right]
\leq \sum_{l=1}^{n-1} \frac{2L^2T}{m^n-1} \left( \mathbb{E}[\| \mathcal{X}_{l,m}^0(\mathcal{R}_l^0) - X(\mathcal{R}_l^0) \|^2] + \mathbb{E}[\| \mathcal{X}_{l-1,m}^0(\mathcal{R}_l^0) - X(\mathcal{R}_l^0) \|^2] \right)
\leq \sum_{l=1}^{n-1} \frac{2L^2T}{m^n-1} \left( \int_0^t \mathbb{E}[\| \mathcal{X}_{l,m}^0(r) - X(r) \|^2] \, dr + \int_0^t \mathbb{E}[\| \mathcal{X}_{l-1,m}^0(r) - X(r) \|^2] \, dr \right)
\leq \sum_{l=1}^{n-1} \frac{2L^2T}{m^n-1} \left( \int_0^t \mathbb{E}[\| \mathcal{X}_{l,m}^0(r) - X(r) \|^2] \, dr + \int_0^t \mathbb{E}[\| \mathcal{X}_{l-1,m}^0(r) - X(r) \|^2] \, dr \right). \tag{58}
\]

This and (56) imply that for all \( t \in [0, T], n \in \mathbb{N} \) it holds that

\[
\mathbb{E}[\| \mathcal{X}_{n,m}^0(t) - \mathbb{E}[\mathcal{X}_{n,m}^0(t)] \|^2]
\leq \sum_{l=1}^{n-1} \frac{l^2}{m^n-1} \mathbb{E}[\| F(\mathcal{X}_{l,m}^0(\mathcal{R}_l^0), Z) - F(\mathcal{X}_{l-1,m}^0(\mathcal{R}_l^0), Z) \|^2] + \frac{T^2}{m} \mathbb{E}[\| F(\xi, Z) \|^2] \tag{59}
\]

Combining this, e.g., the bias-variance type decomposition of the mean square error in [28, Lemma 2.2], (50), (47), and item (i) in Lemma 2.12 demonstrates that for all \( t \in [0, T], n \in \mathbb{N} \) it holds that

\[
\mathbb{E}[\| \mathcal{X}_{n,m}^0(t) - X(t) \|^2] = \mathbb{E}[\| \mathcal{X}_{n,m}^0(t) - \mathbb{E}[\mathcal{X}_{n,m}^0(t)] \|^2] + \| \mathbb{E}[\mathcal{X}_{n,m}^0(t)] - X(t) \|^2
\leq \sum_{l=1}^{n-1} \frac{2L^2T}{m^n-1} \left( \int_0^t \mathbb{E}[\| \mathcal{X}_{l,m}^0(r) - X(r) \|^2] \, dr + \int_0^t \mathbb{E}[\| \mathcal{X}_{l-1,m}^0(r) - X(r) \|^2] \, dr \right)
+ \frac{T^2}{m} \mathbb{E}[\| F(\xi, Z) \|^2] + L^2T \int_0^t \mathbb{E}[\| \mathcal{X}_{n-1,m}^0(r) - X(r) \|^2] \, dr
\leq \sum_{l=1}^{n-1} \frac{2L^2T}{m^n-1} \left( \int_0^t \mathbb{E}[\| \mathcal{X}_{l,m}^0(r) - X(r) \|^2] \, dr \right) + \sum_{l=1}^{n-2} \frac{2L^2T}{m^{n-(t+1)}} \left( \int_0^t \mathbb{E}[\| \mathcal{X}_{l,m}^0(r) - X(r) \|^2] \, dr \right)
+ \frac{C^2}{m^n} + L^2T \int_0^t \mathbb{E}[\| \mathcal{X}_{n-1,m}^0(r) - X(r) \|^2] \, dr
\leq \sum_{l=1}^{n-1} \frac{2L^2T}{m^n-1} \left( \int_0^t \mathbb{E}[\| \mathcal{X}_{l,m}^0(r) - X(r) \|^2] \, dr \right) + \sum_{l=1}^{n-1} \frac{2L^2T}{m^{n-(t+1)}} \left( \int_0^t \mathbb{E}[\| \mathcal{X}_{l,m}^0(r) - X(r) \|^2] \, dr \right) + \frac{C^2}{m^n}
\leq \frac{C^2}{m^n} + \sum_{l=1}^{n-1} \frac{4L^2T}{m^{n-(t+1)}} \left( \int_0^t \mathbb{E}[\| \mathcal{X}_{l,m}^0(r) - X(r) \|^2] \, dr \right). \tag{60}
\]

Next let \( \epsilon_{n,k} \in [0, \infty], n, k \in \mathbb{N}_0 \), satisfy for all \( n, k \in \mathbb{N}_0 \) that

\[
\epsilon_{n,0} = \mathbb{E}[\| \mathcal{X}_{n,m}^0(T) - X(T) \|^2] \quad \text{and} \quad \epsilon_{n,k+1} = \frac{1}{T^{k+1}} \int_0^T \frac{(T-t)^k}{k!} \mathbb{E}[\| \mathcal{X}_{n,m}^0(t) - X(t) \|^2] \, dt. \tag{61}
\]
Note that (60) and (61) imply that for all $n \in \mathbb{N}$ it holds that
\[
\epsilon_{n,0} = \mathbb{E}\big[\|X_{n,m}(T) - X(T)\|^2\big] 
\leq \frac{C^2}{m^n} + \sum_{l=0}^{n-1} \frac{4L^2T^2}{m^{n-(l+1)}T^l} \left( \int_0^T \mathbb{E}\big[\|X_{n,m}^0(r) - X(r)\|^2\big] \, dr \right) = \frac{C^2}{m^n(0)} + 4L^2T^2 \sum_{l=0}^{n-1} \frac{\epsilon_{l,1}}{m^{n-(l+1)}}.
\] (62)
Moreover, observe that for every $n \in \mathbb{N}_0$, $k \in \mathbb{N}$ with $T \wedge T' = \min(T, T')$, $k \wedge k$, $(U(r))_{r \in [0,T]} \wedge (E[\|X_{n,m}^0(r) - X(r)\|^2])_{r \in [0,T]}$ in the notation of Lemma 2.15) and (61) demonstrate that for all $n \in \mathbb{N}_0$, $k \in \mathbb{N}$ it holds that
\[
\frac{1}{T^k} \left( \int_0^T \frac{(T-t)^{(k-1)}}{(k-1)!} \int_0^t \mathbb{E}\big[\|X_{n,m}^0(r) - X(r)\|^2\big] \, dr \, dt \right) = \frac{T^k}{k!} \left( \int_0^T \frac{(T-t)^{(k-1)}}{(k-1)!} \int_0^t \mathbb{E}\big[\|X_{n,m}^0(r) - X(r)\|^2\big] \, dr \, dt \right) = T\epsilon_{n,k+1}.
\] (63)
This, (60), and (61) imply that for all $n, k \in \mathbb{N}$ it holds that
\[
\epsilon_{n,k} = \frac{1}{T^k} \left( \int_0^T \frac{(T-t)^{(k-1)}}{(k-1)!} \mathbb{E}\big[\|X_{n,m}^0(t) - X(t)\|^2\big] \, dt \right) 
\leq \frac{C^2}{T^km^n} \left( \int_0^T \frac{(T-t)^{(k-1)}}{(k-1)!} \, dt \right) + \sum_{l=0}^{n-1} \frac{4L^2T^k}{m^{n-(l+1)}T^l} \left( \int_0^T \frac{(T-t)^{(k-1)}}{(k-1)!} \int_0^t \mathbb{E}\big[\|X_{n,m}^0(r) - X(r)\|^2\big] \, dr \, dt \right) = \frac{C^2}{T^km^n} + 4L^2T^k \sum_{l=0}^{n-1} \frac{\epsilon_{l,k+1}}{m^{n-(l+1)}}.
\] (64)
Furthermore, note that Lemma 2.7 and (44) prove that for all $t \in [0,T]$ it holds that
\[
\|X(t) - \xi\|^2 \leq T^2 \mathbb{E}\big[\|F(\xi, Z^0)\|^2\big]c^2 = C^2.
\] (65)
The fact that for all $t \in [0,T]$ it holds that $X_{0,m}(t) = \xi$ and (61) hence assure that
\[
\epsilon_{0,0} = \mathbb{E}\big[\|X(T) - \xi\|^2\big] = \|X(T) - \xi\|^2 \leq C^2 = \frac{C^2}{m^0(0)}.
\] (66)
Moreover, observe that (61), (65), and the fact that for all $t \in [0,T]$ it holds that $X_{0,m}(t) = \xi$ ensure that for all $k \in \mathbb{N}$ it holds that
\[
\epsilon_{0,k} = \frac{1}{T^k} \int_0^T \frac{(T-t)^{(k-1)}}{(k-1)!} \mathbb{E}\big[\|X(t) - \xi\|^2\big] \, dt \leq \frac{C^2}{T^k} \int_0^T \frac{(T-t)^{(k-1)}}{(k-1)!} \, dt = \frac{C^2T^k}{k!} = \frac{C^2}{m^0(k)!}.
\] (67)
Combining this, (62), (64), and (66) demonstrates that for all $n, k \in \mathbb{N}_0$ it holds that
\[
\epsilon_{n,k} \leq \frac{C^2}{m^n(k!)} + 4L^2T^2 \left[ \sum_{l=0}^{n-1} \frac{\epsilon_{l,k+1}}{m^{n-(l+1)}} \right] = \frac{C^2m^k}{m^{n+k}(k!)} + 4L^2T^2 \left[ \sum_{l=0}^{n-1} \frac{\epsilon_{l,k+1}}{m^{n-(l+1)}} \right].
\] (68)
Lemma 2.3 (applied with $\alpha \wedge C^2\epsilon^m$, $\beta \wedge 4L^2T^2$, $M \wedge m$, $(\epsilon_{n,k})_{n,k \in \mathbb{N}_0}$, $(\epsilon_{n,k})_{n,k \in \mathbb{N}_0}$ in the notation of Lemma 2.3) therefore proves that for all $n, k \in \mathbb{N}_0$ it holds that
\[
\epsilon_{n,k} \leq \frac{C^2\epsilon^m(1 + 4L^2T^2)^n}{m^{n+k}}.
\] (69)
This and (61) imply that for all $n \in \mathbb{N}$ it holds that
\[ E[\|X(T) - X^0_{n,m}(T)\|^2] = \epsilon_{n,0} \leq C^2e^m(1 + 4L^2T^2)n/m^n. \] (70)

The fact that for all $x, y \in [0, \infty)$ it holds that $\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}$ and (44) hence demonstrate that for all $n \in \mathbb{N}$ it holds that
\[ \left( E[\|X(T) - X^0_{n,m}(T)\|^2] \right)^{1/2} \leq C\epsilon^{n/2}(\sqrt{1 + 4L^2T^2})n/m^{n/2} \leq \frac{C(1 + 2LT)^n\epsilon^{n/2}}{m^{n/2}} \]
\[ = \frac{T(E[\|F(\xi, Z)\|^2])^{1/2}(1 + 2LT)^n\epsilon^{(LT + m/2)}}{m^{n/2}}. \] (71)

The proof of Proposition 2.16 is thus completed. □

3 Complexity analysis for MLP approximation algorithms

In this section we provide in Theorem 3.5 in Subsection 3.2 below an overall complexity analysis for the MLP approximations introduced in Setting 2.1 in Subsection 2.1 above.

Our proof of Theorem 3.5 combines the error analysis in Proposition 2.16 with the bound for the computational cost of the proposed MLP approximations Lemma 3.1 in Subsection 3.1 and the elementary results in Lemmas 3.2–3.4. Proofs for Lemma 3.1 and Lemma 3.2 can be found, e.g., in [4, Lemma 3.14] and [5, Lemma 4.3] and Lemma 3.4 is a slightly modified version of [5, Lemma 4.2].

3.1 Computational cost analysis for MLP approximation algorithms

Lemma 3.1. Let $(RV_{n,m})_{n \in \mathbb{N}_0, m \in \mathbb{N} \subset \mathbb{N}}$ satisfy for all $n, m \in \mathbb{N}$ that $RV_{0,m} = 0$ and $RV_{n,m} \leq m^n + \sum_{t=1}^{n-1}[m^{n-t}(1 + RV_{t,m} + RV_{t-1,m})]$. Then it holds for all $n, m \in \mathbb{N}$ that $RV_{n,m} \leq (3m)^n$.

Lemma 3.2. Let $\alpha \in [1, \infty)$. Then it holds for all $k \in \mathbb{N}$ that $\sum_{n=1}^{k}(an)^n \leq 2(\alpha k)^k$.

3.2 Overall complexity analysis for MLP approximation algorithms

Lemma 3.3. Let $d \in \mathbb{N}$, $T, L \in [0, \infty)$, $\xi \in \mathbb{R}^d$, $X \in C([0, T], \mathbb{R}^d)$, let $\|\cdot\|: \mathbb{R}^d \to [0, \infty)$ be a norm on $\mathbb{R}^d$, let $(S, S)$ be a measurable space, let $E: \mathbb{R}^d \times S \to \mathbb{R}^d$ be $(B(\mathbb{R}^d) \otimes S)/B(\mathbb{R}^d)$-measurable, assume for all $x, y \in \mathbb{R}^d$, $s \in S$ that $\|F(x, s) - F(y, s)\| \leq L\|x - y\|$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $Z: \Omega \to S$ be a random variable, and assume that $E[\|F(\xi, Z)\|^2] < \infty$. Then

(i) for all $t \in [0, T]$ it holds that $[0, t] \times \Omega \ni (r, \omega) \mapsto F(X(r), Z(\omega)) \in \mathbb{R}^d$ is $(B([0, t]) \otimes \mathcal{F})/B(\mathbb{R}^d)$-measurable,

(ii) for all $t \in [0, T]$ it holds that $E[\|F(X(t), Z)\|^2] < \infty$, and

(iii) for all $t \in [0, T]$ it holds that $\int_0^T E[\|F(X(r), Z)\|^2] dr < \infty$.

Proof of Lemma 3.3. Throughout this proof let $t \in [0, T]$. Note that the hypothesis that for all $x, y \in \mathbb{R}^d$, $s \in S$ it holds that $\|F(x, s) - F(y, s)\| \leq L\|x - y\|$, the hypothesis that $F: \mathbb{R}^d \times S \to \mathbb{R}^d$ is $(B(\mathbb{R}^d) \otimes S)/B(\mathbb{R}^d)$-measurable, the hypothesis that $Z \in F(S)/S$-measurable, and the hypothesis that $X \in C([0, T], \mathbb{R}^d)$ assure that $[0, t] \times \Omega \ni (r, \omega) \mapsto F(X(r), Z(\omega)) \in \mathbb{R}^d$ is $(B([0, t]) \otimes \mathcal{F})/B(\mathbb{R}^d)$-measurable. This establishes item (i). In addition, note that the assumption that for all $x, y \in \mathbb{R}^d$, $s \in S$ it holds that $\|F(x, s) - F(y, s)\| \leq L\|x - y\|$, and the triangle inequality ensure that for all $x \in \mathbb{R}^d$, $s \in S$ it holds that
\[ \|F(x, s)\| \leq \|F(\xi, s)\| + \|F(x, s) - F(\xi, s)\| \leq \|F(\xi, s)\| + L\|x - \xi\|. \] (72)
In addition, observe that the hypothesis that $E[\|F(\xi, Z)\|^2] < \infty$ and Jensen’s inequality assure that
\[
(E[\|F(\xi, Z)\|])^2 \leq E[\|F(\xi, Z)\|^2] < \infty.
\] (73)
Combining this, (72), and the hypothesis that $X \in C([0, T], \mathbb{R}^d)$ implies that
\[
E[\|F(X(t), Z)\|] \leq E[\|F(\xi, Z)\| + L\|X(t) - \xi\|] = E[\|F(\xi, Z)\|] + L\|X(t) - \xi\| < \infty.
\] (74)
This establishes item (ii). Moreover, observe that (74), the assumption that for all $x, y \in \mathbb{R}^d$, $s \in S$ it holds that $\|F(x, s) - F(y, s)\| \leq L\|x - y\|$, and the fact that for all $a, b \in \mathbb{R}^d$ it holds that $\|a\| - \|b\| \leq \|a - b\|$ assure that for all $r, s \in [0, t]$ it holds that
\[
\begin{align*}
&|E[\|F(X(r, Z)\| - E[\|F(X(s, Z)\|)]| = |E[\|F(X(r, Z)\| - F(X(s, Z))]|| \\
&\leq |E[\|F(X(r, Z) - F(X(s, Z))|] = |E[L\|X(r) - X(s)\|] = L\|X(r) - X(s)\|.
\end{align*}
\] (75)
Combining this with the hypothesis that $X \in C([0, T], \mathbb{R}^d)$ ensures that $[0, t] \ni r \mapsto E[\|F(X(r), Z)\|] \in \mathbb{R}$ is continuous. This ensures that $\int_0^t \mathbb{E}[\|F(X(r), Z)\|] dr < \infty$. This establishes item (iii). This completes the proof of Lemma 3.3. □

**Lemma 3.4.** Let $T, L, C \in [0, \infty)$, $\alpha \in [1, \infty)$, $\mathcal{R} \in N_0$, $(\gamma_n)_{n \in \mathbb{N}} \subseteq [0, \infty)$, $(\epsilon_n)_{n \in \mathbb{N}} \subseteq [0, \infty)$, assume for all $n \in \mathbb{N}$ that $\gamma_n \leq (\alpha n)^n$ and $\epsilon_n \leq C e^{n/2}(1 + 2LT)^n n^{-n/2}$, and let $N = \{N_{\epsilon} : (0, 1] \to [0, \infty) \text{ satisfy for all } \epsilon \in (0, 1) \}

Then

(i) for all $\epsilon \in (0, 1]$ it holds that $N_{\epsilon} < \infty$ and

(ii) for all $\epsilon \in (0, 1]$, $\delta \in (0, \infty)$ it holds that $\sup_{n \in [N_{\epsilon}, \infty) \cap \mathbb{N}} \gamma_n \leq \epsilon$ and

\[
\sup_{n \in [N_{\epsilon}, \infty) \cap \mathbb{N}} \gamma_n \leq \sup_{n \in \mathbb{N}} \left[ \frac{\alpha(n + (\mathcal{R} + 1))^{(n + (\mathcal{R} + 1))}}{n^{n(1+\delta)}} e^{n(1+\delta)}(1 + 2LT)^n (2 + 2\delta) \right] \max\{1, C^{2+2\delta}\} e^{-(2 + 2\delta)} < \infty.
\] (77)

**Proof of Lemma 3.4.** Throughout this proof let $a_{\delta} \in [0, \infty)$, $\delta \in (0, \infty)$, satisfy for all $\delta \in (0, \infty)$ that

\[
a_{\delta} = C^{2 + 2\delta} \sup_{n \in \mathbb{N}} \left[ \frac{\alpha(n + (\mathcal{R} + 1))^{(n + (\mathcal{R} + 1))}}{n^{n(1+\delta)}} e^{n(1+\delta)}(1 + 2LT)^n (2 + 2\delta) \right].
\] (78)

Observe that the fact that for all $t \in (0, \infty)$ it holds that $\ln(t) \leq t - 1$ ensures that

\[
\limsup_{n \to \infty} \left[ \ln(C e^{n/2}(1 + 2LT)^n n^{-n/2}) \right] = \limsup_{n \to \infty} \left[ \ln(C) + \frac{n}{2} + n \ln(1 + 2LT) - \frac{n}{2} \ln(n) \right] \leq \limsup_{n \to \infty} \left[ \ln(C) + \frac{n}{2} + 2nLT - \frac{n}{2} \ln(n) \right] = \limsup_{n \to \infty} \left[ n \ln(n) \left( \frac{1}{2nln(n)} + \frac{1}{2 \ln(n)} \right) \right] = -\infty.
\] (79)

This and the fact that $\lim_{n \to \infty} e^{r} = 0$ imply that

\[
0 \leq \limsup_{n \to \infty} \left[ C e^{n/2}(1 + 2LT)^n n^{-n/2} \right] = \limsup_{n \to \infty} \left[ \exp \left( \ln(C e^{n/2}(1 + 2LT)^n n^{-n/2}) \right) \right] = 0.
\] (80)
This and (76) ensure that for all $\varepsilon \in (0, 1]$ it holds that $N_\varepsilon < \infty$. This establishes item (i). Next note that the fact that for all $t \in (0, \infty)$ it holds that $\ln(t) \leq t - 1$ ensures that for all $\delta \in (0, \infty)$ it holds that
\[
\limsup_{n \rightarrow \infty} \left[ \ln \left( \frac{[\alpha(n + 1) + 1][n + \varepsilon]}{n^{n+\varepsilon}} \right) e^{n(1+\delta)(1 + 2LT)^{n(2+2\delta)}} \right] = \limsup_{n \rightarrow \infty} \left[ (n + 1 + \varepsilon) \ln(n + 1 + \varepsilon) - n(1 + \delta) \ln(n) + n(1 + \delta) + n(2 + 2\delta) \ln(1 + 2LT) \right]
\]
\[
\leq \limsup_{n \rightarrow \infty} \left[ n \ln(n) \left( \frac{(n + 1 + \varepsilon) \ln(n + 1 + \varepsilon)}{n \ln(n)} + \frac{(n + 1 + \varepsilon) \ln(n + 1 + \varepsilon)}{n \ln(n)} - (1 + \delta) + \frac{(1 + \delta)}{\ln(n)} + \frac{(2 + 2\delta)2LT}{\ln(n)} \right) \right]
\]
\[
= -\infty.
\]
This and the fact that \( \lim_{n \rightarrow -\infty} e^t = 0 \) imply that for all $\delta \in (0, \infty)$ it holds that
\[
0 \leq \limsup_{n \rightarrow \infty} \left[ \frac{[\alpha(n + 1) + 1][n + \varepsilon]}{n^{n+\varepsilon}} e^{n(1+\delta)(1 + 2LT)^{n(2+2\delta)}} \right] = 0.
\]
Combining this and (78) implies that for all $\delta \in (0, \infty)$ it holds that
\[
a_\delta = C^{2+2\delta} \sup_{n \in \mathbb{N}} \left[ \frac{[\alpha(n + 1) + 1][n + \varepsilon]}{n^{n+\varepsilon}} e^{n(1+\delta)(1 + 2LT)^{n(2+2\delta)}} \right] < \infty.
\]
In addition, observe that (76), the fact that for all $\varepsilon \in (0, 1]$ it holds that $N_\varepsilon < \infty$, and the hypothesis that for all $n \in \mathbb{N}$ it holds that $\varepsilon_n \leq C e^{n/2}(1 + 2LT)^n n^{-n/2}$ assure that for all $\varepsilon \in (0, 1]$ it holds that
\[
\sup_{n \in [N_\varepsilon, \infty) \cap \mathbb{N}} \varepsilon_n \leq \sup_{n \in [N_\varepsilon, \infty) \cap \mathbb{N}} \left[ Ce^{n/2}(1 + 2LT)^n n^{-n/2} \right] \leq \varepsilon.
\]
Next let $E \subseteq (0, 1]$ satisfy that $E = \{ \varepsilon \in (0, 1]: N_\varepsilon > 1 \}$. Note that (76) ensures that for all $\varepsilon \in E$ it holds that
\[
C e^{(N_\varepsilon - 1)/2}(1 + 2LT)^{(N_\varepsilon - 1)(N_\varepsilon - 1)^{-((N_\varepsilon - 1)/2)}} > \varepsilon.
\]
This implies that for all $\varepsilon \in E$ it holds that
\[
\frac{C}{\varepsilon} e^{(N_\varepsilon - 1)/2}(1 + 2LT)^{(N_\varepsilon - 1)(N_\varepsilon - 1)^{-((N_\varepsilon - 1)/2)}} > (N_\varepsilon - 1)^{(N_\varepsilon - 1)/2}.
\]
This, the hypothesis that for all $n \in \mathbb{N}$ it holds that $\gamma_n \leq (\alpha n)^n$, (83), and the fact that for all $K \in \mathbb{N}$, $\beta \in [1, \infty)$ it holds that $\sup_{n \in [1, K] \cap \mathbb{N}} (\beta n)^n = (\beta K)^K$ imply that for all $\varepsilon \in E$, $\delta \in (0, \infty)$ it holds that
\[
\sup_{n \in [1, N_\varepsilon + 1)] \cap \mathbb{N}} \gamma_n \leq \sup_{n \in [1, N_\varepsilon + 1] \cap \mathbb{N}} (\alpha n)^n = \frac{[\alpha(N_\varepsilon + 1)](N_\varepsilon + 1)}{(N_\varepsilon - 1)(1 + \delta)} = \frac{[\alpha(N_\varepsilon + 1)](N_\varepsilon + 1)}{(N_\varepsilon - 1)(1 + \delta)} \leq \frac{C^{2+2\delta} e^{N_\varepsilon - 1)(1 + \delta)}}{N_\varepsilon - 1)(1 + \delta)} \leq C^{2+2\delta} e^{2(2+2\delta)} \sup_{n \in [2, \infty) \cap \mathbb{N}} \left[ \frac{[\alpha(n + 1)](n + 1)}{(n - 1)(n - 1)(1 + \delta)} e^{n(1+\delta)(1 + 2LT)^{n(2+2\delta)}} \right]
\]
\[
= C^{2+2\delta} e^{2(2+2\delta)} \sup_{n \in \mathbb{N}} \left[ \frac{[\alpha(n + 1)](n + 1)}{n^{n+\varepsilon}} e^{n(1+\delta)(1 + 2LT)^{n(2+2\delta)}} \right] = a_\delta e^{-(2+2\delta)} < \infty.
\]
Moreover, observe that the hypothesis that for all \( n \in \mathbb{N} \) it holds that \( \gamma_n \leq (\alpha n)^n \) and the fact that for all \( K \in \mathbb{N} \), \( \beta \in [1, \infty) \) it holds that \( \sup_{n \in [1, K]} \beta^n n^\gamma_n \leq (\beta K)^K \) assures that for all \( \varepsilon \in (0, 1) \setminus E \), \( \delta \in (0, \infty) \) it holds that

\[
\sup_{n \in [1, N_{\mathcal{R}} + 3]} \gamma_n = \sup_{n \in [1, N_{\mathcal{R}} + 1]} \gamma_n \leq \sup_{n \in [1, N_{\mathcal{R}} + 1]} (\alpha n)^n = [\alpha (\mathcal{R} + 1)]^{(\mathcal{R} + 1)} \leq \varepsilon^{-(2 + 28\beta)}[\alpha (\mathcal{R} + 1)]^{(\mathcal{R} + 1)} < \infty.
\] (88)

Next observe that (83) implies that for all \( \delta \in (0, \infty) \) it holds that

\[
a_\delta = C^{2 + 28} \sup_{n \in \mathbb{N}} \left[ \frac{\alpha (n + \mathcal{R} + 1)^{(n + \mathcal{R} + 1)}}{n^{(1 + \delta)}} \right] (1 + 2LT)^n (1 + 2LT)^n (1 + 2LT)^n < \infty.
\] (89)

In addition, observe that for all \( \delta \in (0, \infty) \) it holds that

\[
[\alpha (\mathcal{R} + 1)]^{(\mathcal{R} + 1)} \leq \sup_{n \in \mathbb{N}} \left[ \frac{\alpha (n + \mathcal{R} + 1)^{(n + \mathcal{R} + 1)}}{n^{(1 + \delta)}} \right] (1 + 2LT)^n (1 + 2LT)^n (1 + 2LT)^n < \infty.
\] (90)

Combining this, (87), (88), and (89) establishes that for all \( \varepsilon \in (0, 1) \), \( \delta \in (0, \infty) \) it holds that

\[
\sup_{n \in [1, N_{\mathcal{R}} + 3]} \gamma_n \leq \varepsilon^{-(2 + 28\beta)} \max \left\{ a_\delta, [\alpha (\mathcal{R} + 1)]^{(\mathcal{R} + 1)} \right\} \leq \varepsilon^{-(2 + 28\beta)} \max \left\{ 1, C^{2 + 28\beta} \right\} \sup_{n \in \mathbb{N}} \left[ \frac{\alpha (n + \mathcal{R} + 1)^{(n + \mathcal{R} + 1)}}{n^{(1 + \delta)}} \right] (1 + 2LT)^n (1 + 2LT)^n (1 + 2LT)^n < \infty.
\] (91)

This and (84) establish item (ii). The proof of Lemma 3.4 is thus completed.

**Theorem 3.5.** Let \( d \in \mathbb{N} \), \( \mathcal{R} \in \mathbb{N}_0 \), \( T, L \in [0, \infty) \), \( \Theta = \cup_{\xi=1}^\infty \mathbb{Z}^d \), \( \xi \in \mathbb{R}^d \), let \( \|\| : \mathbb{R}^d \to [0, \infty) \) be a norm on \( \mathbb{R}^d \), let \( S, r \sigma \) be a measurable space, let \( F : \mathbb{R}^d \times S \to \mathbb{R}^d \) be \( (B(\mathbb{R}^d) \otimes S)/B(\mathbb{R}^d) \)-measurable, assume for all \( x, y \in \mathbb{R}^d \), \( s \in S \) that \( \|F(x, s) - F(y, s)\| \leq L \|x - y\| \), let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space, let \( Z^0 : \Omega \to S \), \( \theta \in \Theta \), be i.i.d. random variables, assume that \( \mathbb{E}[\|F(\xi, Z^0)\|^2] < \infty \), let \( \mathbb{V}^\theta : \Omega \to [0, 1] \), \( \theta \in \Theta \), be independent \( \mathcal{U}_{[0,1]} \)-distributed random variables, let \( \mathcal{R}^\theta : [0, T] \times \Omega \to [0, T] \), \( \theta \in \Theta \), satisfy for all \( t \in [0, T] \), \( \theta \in \Theta \) that \( \mathcal{R}^\theta_t = \mathcal{V}^\theta_t \), assume that \( (\mathcal{V}^\theta)_{\theta \in \Theta} \) are independent, let \( \mathcal{X}^\theta_{n,m} : [0, T] \times \Omega \to \mathbb{R}^d \), \( n, m \in \mathbb{N}_0 \), \( \theta \in \Theta \), satisfy for all \( n \in \mathbb{N}_0 \), \( m \in \mathbb{N} \), \( \theta \in \Theta \), \( t \in [0, T] \) that

\[
\mathcal{X}^\theta_{n,m}(t) = \left[ \sum_{l=1}^{n-1} \frac{t}{m^{n-l}} \sum_{k=1}^{m^{n-l}} F\left( \mathcal{X}^\theta_{l,m}(\mathcal{R}^\theta_{l-1,m}(\mathcal{R}^\theta_{l-2,m}(\cdots (\mathcal{R}^\theta_{l-n+1,m}(\mathcal{X}^\theta_{l-n,m}(\xi, Z^0, Z^0, \ldots, Z^0))))))) \right) \right] \quad (92)
\]

and let \( \mathcal{R}^\theta_{n,m} \) satisfy for all \( n, m \in \mathbb{N} \) that \( \mathcal{R}^\theta_{0,m} = 0 \) and

\[
\mathcal{R}^\theta_{n,m} \leq m^n + \sum_{l=1}^{n-1} \left( m^{n-l} (1 + \mathcal{R}_{l,m} + \mathcal{R}_{l-1,m}) \right). \quad (93)
\]

Then
(i) there exists a unique \( X \in C([0, T], \mathbb{R}^d) \) such that for all \( t \in [0, T] \) it holds that \( X(t) = \xi + \int_0^t \mathbb{E}[F(X(r), Z^0)] \, dr \) (cf. Lemma 3.3) and

(ii) there exist \( c = (c_\varepsilon)_{\varepsilon \in (0, \infty)} : (0, \infty) \to [0, \infty) \) and \( N = (N_\varepsilon)_{\varepsilon \in (0, 1)} : (0, 1) \to \mathbb{N} \) such that for all \( \delta \in (0, \infty) \), \( \varepsilon \in (0, 1) \) it holds that \( \sum_{n=1}^{N_\varepsilon} \| R_{n,n} \| \leq c_\varepsilon \varepsilon^{-(2+\delta)} \) and \( \sup_{n \in [N_\varepsilon, \infty)} \mathbb{E}[\| X(T) - X_{n,n}(T) \|^2]^{1/2} \leq \varepsilon \).

Proof of Theorem 3.5. Throughout this proof let \( \| \cdot \| : \mathbb{R}^d \to [0, \infty) \) be the standard norm on \( \mathbb{R}^d \), let \( a, b \in (0, \infty) \) satisfy for all \( x \in \mathbb{R}^d \) that \( a \| x \| \leq \| x \| \leq b \| x \| \) (cf., e.g., Kreyszig [29, Theorem 2.4-5]), let \( C \in (0, \infty) \) satisfy that \( C = ba^{-1}e^{a^{-1}bL^T T(\mathbb{E}[\| F(\xi, Z^0) \| ^2])^{1/2}} \), and let \( N : (0, 1) \to \mathbb{N} \cup \{ \infty \} \) satisfy for all \( \varepsilon \in (0, 1) \) that

\[ N_\varepsilon = \min \left\{ n \in \mathbb{N} : \sup_{m \in [n, \infty)} \left[ Ce^{m/2}(1 + 2a^{-1}bL^T)^n m^{-m/2} \right] \leq \varepsilon \right\} \cup \{ \infty \}. \tag{94} \]

Observe that the assumption that for all \( x, y \in \mathbb{R}^d, s \in S \) it holds that \( \| F(x, s) - F(y, s) \| \leq L \| x - y \| \) ensures that for all \( x, y \in \mathbb{R}^d \) it holds that

\[ \| \mathbb{E}[F(x, Z^0)] - \mathbb{E}[F(y, Z^0)] \| = \| \mathbb{E}[F(x, Z^0) - F(y, Z^0)] \| \leq \| \mathbb{E}[F(x, Z^0) - F(y, Z^0)] \| \leq L \| x - y \|. \tag{95} \]

This ensures that there exists a unique \( X \in C([0, T], \mathbb{R}^d) \) such that for all \( t \in [0, T] \) it holds that

\[ X(t) = \xi + \int_0^t \mathbb{E}[F(X(r), Z^0)] \, dr \] \tag{96}

(cf., e.g., Teschl [35, Section 2]). This establishes item (i). Next observe that item (i) in Lemma 3.4 (applied with \( T \cap T, L \cap a^{-1}bL, C \cap C, \mathfrak{R} \cap \mathfrak{R} \) in the notation of Lemma 3.4) assures that for all \( \varepsilon \in (0, 1) \) it holds that \( N_\varepsilon < \infty \). Next note that the fact that for all \( x \in \mathbb{R}^d \) it holds that \( a \| x \| \leq \| x \| \leq b \| x \| \) and the assumption that for all \( x, y \in \mathbb{R}^d, s \in S \) it holds that \( \| F(x, s) - F(y, s) \| \leq L \| x - y \| \) ensure that for all \( x, y \in \mathbb{R}^d, s \in S \) it holds that

\[ \| F(x, s) - F(y, s) \| \leq b \| F(x, s) - F(y, s) \| \leq b \| x - y \| \leq b a^{-1} \| x - y \|. \tag{97} \]

This, the fact that for all \( x \in \mathbb{R}^d \) it holds that \( a \| x \| \leq \| x \| \leq b \| x \| \), the fact that \( C = ba^{-1}e^{a^{-1}bL^T T(\mathbb{E}[\| F(\xi, Z^0) \| ^2])^{1/2}} \), (93), Lemma 3.1, and Proposition 2.16 imply that for all \( n \in \mathbb{N} \) it holds that \( RV_{n,n} \leq (3n)^n \) and

\[ \left( \mathbb{E}[\| X(T) - X_{n,n}(T) \|^2] \right)^{1/2} \leq \frac{1}{a} \left( \mathbb{E}[\| X(T) - X_{n,n}(T) \|^2] \right)^{1/2} \]
\[ \leq \frac{1}{a} \left( \mathbb{E}[\| F(\xi, Z^0) \|^2] \right)^{1/2} \leq \frac{1}{a} \left( 1 + 2a^{-1}bL \right)^n e^{(a^{-1}bL+n/2)} \]
\[ \leq \frac{b}{a} \left( \mathbb{E}[\| F(\xi, Z^0) \|^2] \right)^{1/2} \leq \frac{b}{a} \left( 1 + 2a^{-1}bL \right)^n e^{a^{-1}bL} \]
\[ = ba^{-1}e^{a^{-1}bL T(\mathbb{E}[\| F(\xi, Z^0) \| ^2])^{1/2}} \left( 1 + 2a^{-1}bL \right)^n e^{a^{-1}bL} \]
\[ = C(1 + 2a^{-1}bL^T e^{a^{-1}bL} n^{-n/2}) \]
\[ = C(1 + 2a^{-1}bL^T e^{a^{-1}bL} n^{-n/2}) \]

Lemma 3.4 (applied for every \( n \in \mathbb{N} \) with \( T \cap T, L \cap a^{-1}bL, C \cap C, \alpha \leq 3, \mathfrak{R} \cap \mathfrak{R}, \gamma_n \cap (3n)^n, \epsilon_n \cap (\mathbb{E}[\| X(T) - X_{n,n}(T) \|^2])^{1/2} \) in the notation of Lemma 3.4) hence proves that for all \( \varepsilon \in (0, 1) \), \( \delta \in (0, \infty) \) it holds that

\[ \sup_{n \in [N_\varepsilon, \infty)} \mathbb{E}[\| X(T) - X_{n,n}(T) \|^2] \leq \varepsilon \] \tag{99}
and
\[
\sup_{n \in [1, N_{\epsilon} + N]} (3n)^n \leq \sup_{n \in N} \left[ \frac{[3(n + N + 1)](n + N + 1)}{n^{n(1+\delta)}} e^n(1+\delta)(1 + 2a^{-1}bLT)^n(2+2\delta) \right] \max\{1, C^{2+2\delta}\} \varepsilon^{-(2+2\delta)} < \infty. \quad (100)
\]

This and (93), Lemma 3.1, Lemma 3.2, and the fact that for all \( K \in \mathbb{N}, \beta \in [1, \infty) \) it holds that \( \sup_{n \in [1, K]} (\beta n)^n = (\beta K)^K \) ensure that for all \( \varepsilon \in (0, 1], \delta \in (0, \infty) \) it holds that
\[
\sum_{n=1}^{N_{\epsilon} + N} RV_{n,n} \leq \sum_{n=1}^{N_{\epsilon} + N} (3n)^n \leq 2[3(N_{\epsilon} + N)](N_{\epsilon} + N) = 2 \sup_{n \in [1, N_{\epsilon} + N]} (3n)^n \leq 2 \sup_{n \in [1, N_{\epsilon} + N]} \left[ \frac{[3(n + N + 1)](n + N + 1)}{n^{n(1+\delta)}} e^n(1+\delta)(1 + 2a^{-1}bLT)^n(2+2\delta) \right] \max\{1, C^{2+2\delta}\} \varepsilon^{-(2+2\delta)} < \infty. \quad (101)
\]

This and (99) establish item (ii). The proof of Theorem 3.5 is thus completed. \( \square \)

References

[1] Agarwal, A., and Pagliarani, S. A Fourier-based Picard-iteration approach for a class of McKean–Vlasov SDEs with Lévy jumps. Stochastics (2020), 1–33.

[2] Antonelli, F., and Kohatsu-Higa, A. Rate of convergence of a particle method to the solution of the McKean–Vlasov equation. Ann. Appl. Probab. 12, 2 (2002), 423–476.

[3] Bao, J., Reiissing, C., Ren, P., and Stockinger, W. First-order convergence of Milstein schemes for McKean–Vlasov equations and interacting particle systems. Proc. A 477, 2245 (2021), 20200258, 27.

[4] Beck, C., Gonon, L., and Jentzen, A. Overcoming the curse of dimensionality in the numerical approximation of high-dimensional semilinear elliptic partial differential equations. arXiv:2003.00596 (2020), 50 pages.

[5] Beck, C., Hornung, F., Hutzenthaler, M., Jentzen, A., and Kruse, T. Overcoming the curse of dimensionality in the numerical approximation of Allen–Cahn partial differential equations via truncated full-history recursive multilevel Picard approximations. J. Numer. Math. 28, 4 (2020), 197–222.

[6] Beck, C., Hutzenthaler, M., Jentzen, A., and Kuckuck, B. An overview on deep learning-based approximation methods for partial differential equations. arXiv:2012.12348 (2020), 22 pages.

[7] Becker, S., Braunwarth, R., Hutzenthaler, M., Jentzen, A., and von Wurstemberger, P. Numerical simulations for full history recursive multilevel Picard approximations for systems of high-dimensional partial differential equations. Commun. Comput. Phys. 28, 5 (2020), 2109–2138.

[8] Belomestny, D., and Schoenmakers, J. Projected particle methods for solving McKean–Vlasov stochastic differential equations. SIAM J. Numer. Anal. 56, 6 (2018), 3169–3195.

[9] Belomestny, D., Szpruch, L., and Tan, S. Iterative multilevel density estimation for mckean-vlasov sdes via projections. arXiv:1909.11717 (2019), 22 pages.

[10] Bossy, M., and Jourdain, B. Rate of convergence of a particle method for the solution of a 1d viscous scalar conservation law in a bounded interval. Ann. Probab. 30, 4 (2002), 1797–1832.

[11] Bossy, M., and Talay, D. Convergence rate for the approximation of the limit law of weakly interacting particles: application to the Burgers equation. Ann. Appl. Probab. 6, 3 (1996), 818–861.
[12] Bossy, M., and Talay, D. A stochastic particle method for the McKean–Vlasov and the Burgers equation. *Math. Comput.* 66, 217 (1997), 157–192.

[13] Chassagneux, J.-F., Crisan, D., and Delarue, F. Numerical method for FBSDEs of McKean–Vlasov type. *Ann. Appl. Probab.* 29, 3 (2019), 1640–1684.

[14] Chaudru de Raynal, P., and Garcia Trillos, C. A cubature based algorithm to solve decoupled McKean–Vlasov forward-backward stochastic differential equations. *Stoch. Process. Their Appl.* 125, 6 (2015), 2206–2255.

[15] Crisan, D., and McMurray, E. Cubature on Wiener space for McKean–Vlasov sdes with smooth scalar interaction. *Ann. Appl. Probab.* 29, 1 (2019), 130–177.

[16] Dos Reis, G., Engelhardt, S., and Smith, G. Simulation of McKean–Vlasov SDEs with super-linear growth. *arXiv:1808.05530v4* (2020), 43 pages.

[17] Dos Reis, G., Smith, G., and Tankov, P. Importance sampling for McKean–Vlasov SDEs. *arXiv:1803.09320* (2018), 29 pages.

[18] E, W., Han, J., and Jentzen, A. Algorithms for solving high dimensional PDEs: From nonlinear Monte Carlo to machine learning. *arXiv:2008.13333* (2020), 40 pages.

[19] E, W., Hutzenthaler, M., Jentzen, A., and Kruse, T. Multilevel Picard iterations for solving smooth semilinear parabolic heat equations. *Accepted in SN Partial Differential Equations and Applications, arXiv:1607.03295* (2016), 19 pages.

[20] E, W., Hutzenthaler, M., Jentzen, A., and Kruse, T. On multilevel Picard numerical approximations for high-dimensional nonlinear parabolic partial differential equations and high-dimensional nonlinear backward stochastic differential equations. *J. Sci. Comput.* 79, 3 (2019), 1534–1571.

[21] Giles, M. B., Jentzen, A., and Welti, T. Generalised multilevel Picard approximations. *arXiv:1911.03188* (2019), 61 pages.

[22] Gobet, E., and Pagliarani, S. Analytical approximations of non-linear SDEs of McKean–Vlasov type. *J. Math. Anal. Appl.* 466, 1 (2018), 71–106.

[23] Hutzenthaler, M., Jentzen, A., and Kruse, T. Overcoming the curse of dimensionality in the numerical approximation of parabolic partial differential equations with gradient-dependent nonlinearities. *Accepted in Found. Comp. Math.*, *arXiv:1912.02571* (2019), 33 pages.

[24] Hutzenthaler, M., Jentzen, A., Kruse, T., and Nguyen, T. A. Multilevel Picard approximations for high-dimensional semilinear second-order PDEs with Lipschitz nonlinearities. *arXiv:2009.02484* (2020), 37 pages.

[25] Hutzenthaler, M., Jentzen, A., Kruse, T., Nguyen, T. A., and von Wurstemberger, P. Overcoming the curse of dimensionality in the numerical approximation of semilinear parabolic partial differential equations. *Proc. A* 476, 2244 (2020), 25 pages.

[26] Hutzenthaler, M., Jentzen, A., and von Wurstemberger, P. Overcoming the curse of dimensionality in the approximative pricing of financial derivatives with default risks. *Electron. J. Probab.* 25 (2020), 73 pages.

[27] Hutzenthaler, M., and Kruse, T. Multilevel Picard approximations of high-dimensional semilinear parabolic differential equations with gradient-dependent nonlinearities. *SIAM J. Numer. Anal.* 58, 2 (2020), 929–961.

[28] Jentzen, A., and von Wurstemberger, P. Lower error bounds for the stochastic gradient descent optimization algorithm: Sharp convergence rates for slowly and fast decaying learning rates. *J. Complexity* 57 (2020), 16 pages.
[29] Kreyszig, E. *Introductory functional analysis with applications*. John Wiley & Sons, New York-London-Sydney, 1978.

[30] Kumar, C., and Neelima. On explicit Milstein-type scheme for McKean–Vlasov stochastic differential equations with super-linear drift coefficient. *arXiv:2004.01266* (2020), 35 pages.

[31] Neelima, Biswas, S., Kumar, C., Dos Reis, G., and Reisinger, C. Well-posedness and tamed Euler schemes for McKean–Vlasov equations driven by Lévy noise. *arXiv:2010.08585* (2020), 33 pages.

[32] Reisinger, C., and Stockinger, W. An adaptive Euler–Maruyama scheme for McKean SDEs with super-linear growth and application to the mean-field FitzHugh–Nagumo model. *arXiv:2005.06034* (2020), 28 pages.

[33] Szpruch, L., Tan, S., and Tse, A. Iterative multilevel particle approximation for McKean–Vlasov SDEs. *Ann. Appl. Probab. 29*, 4 (2019), 2230–2265.

[34] Talay, D., and Vaillant, O. A stochastic particle method with random weights for the computation of statistical solutions of McKean–Vlasov equations. *Ann. Appl. Probab. 13*, 1 (2003), 140–180.

[35] Teschl, G. *Ordinary differential equations and dynamical systems*, vol. 140 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2012.