ERROR ANALYSIS OF DISCONTINUOUS GALERKIN METHOD FOR THE TIME FRACTIONAL KDV EQUATION WITH WEAK SINGULARITY SOLUTION

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Abstract. In this work, the time fractional KdV equation with Caputo time derivative of order $\alpha \in (0, 1)$ is considered. The solution of this problem has a weak singularity near the initial time $t = 0$. A fully discrete discontinuous Galerkin (DG) method combining the well-known L1 discretisation in time and DG method in space is proposed to approximate the time fractional KdV equation. The unconditional stability result and $O(N^{-\min\{r\alpha, 2-\alpha\}} + h^{k+1})$ convergence result for $P^k$ ($k \geq 2$) polynomials are obtained. Finally, numerical experiments are presented to illustrate the efficiency and the high order accuracy of the proposed scheme.

1. Introduction. Scott Russell [24] in 1834 first observed the physical solution of the classical Korteweg-de Vries (KdV) equation, which is a typical dispersive nonlinear partial differential equation (PDE), and then the equation itself was later derived by Korteweg and de Vries [14] in 1895. Since then, the KdV equation has been widely applied in many fields to modelling a wide range of physical phenomena such as collision-free hydro-magnetic waves in a cold plasma, ion-acoustic waves, interaction of nonlinear waves [30], interfacial electrohydrodynamics [8], etc. The KdV equation is a model of the water wave, when the wave height is small compared to the water depth [29], the nonlinear equation will be reduced to the linear KdV equation.
When the next state of this physical phenomenon depends not only on its current state but also on its historical states (non-local property), then this phenomenon can be described using fractional differential equations. In recent years ordinary and partial differential equations of fractional order have been the focus of many studies because of their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics, and engineering [10, 28, 15, 26, 22, 17, 18, 13, 11, 23, 1, 2]. Fractional calculus in mathematics is a natural extension of integer-order calculus and gives a useful mathematical tool for modeling many processes in nature.

In this paper, we shall apply the DG method to the following time fractional KdV problem

\[
\begin{cases}
D_t^\alpha u(x, t) + \sigma u_{xxx}(x, t) = f(x, t), \quad \forall \ (x, t) \in \Omega \times (0, T], \\
u(x, 0) = u_0(x), \quad \text{for } x \in \Omega,
\end{cases}
\]

where \(\Omega = (a, b), \ 0 < \alpha < 1\) is a given constant describing the order of the fractional time, \(\sigma \geq 0\) is a given constant, and \(f, u_0\) are sufficiently smooth functions. The boundary condition in this paper is periodic boundary. In (1), the Caputo fractional derivative \(D_t^\alpha u\) is defined as

\[
D_t^\alpha u(x, t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} \frac{\partial u(x, s)}{\partial s} \, ds,
\]

where \(\Gamma\) is the Gamma function. The Caputo fractional derivative is considered because it has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables [23]. For more details of the mathematical properties of fractional derivatives and integrals, we refer the readers to the mentioned references.

For the sufficiently smooth solutions, there have a few numerical works in the literature to solve the time fractional KdV equation. Zhang [31] constructed an efficient unconditionally stable finite difference scheme to solve the linearized time fractional KdV equation where the nonlocal fractional derivative is evaluated via a sum-of-exponentials approximation for the convolution kernel. Chen and Sun [3] proposed a fully discrete scheme combining a Petrov-Galerkin spectral method for the spatial discretization and L1-approximation for the Caputo temporal derivative to solve the linearized time fractional KdV equation. Mustapha et al. [6, 21] presented the hybridizable discontinuous Galerkin (HDG) method for the spatial discretization of time fractional diffusion model. Wei et al. [27] presented a fully discrete numerical scheme for the time-fractional KdV-Burgers-Kuramoto equation, which is discretized by the difference scheme in time and the local discontinuous Galerkin method in space, and then gave the stability and convergence analysis. The key of the LDG method for KdV equation is to rewrite the equation into a first order system by introducing two auxiliary variables. LDG method can obtain higher convergence order for spatial, while the shortcomings of this method is computational cost larger whether you use explicit and implicit step to solve the fully discrete system by the LDG method. Cheng and Shu introduced an new DG method in [4], which can be applied without introducing any auxiliary variables or rewriting the original equation into a larger system for higher order spatial derivatives. So, the computation and the storage will be reduced obviously.

Recently, Stynes et al. [25] considered that the solution for the time fractional reaction-diffusion equation has a weak singularity near the initial time \(t = 0\) under some proper regularity and compatibility assumptions. The bounds on certain
derivatives of the solution satisfy
\[ \left| \frac{\partial^n u}{\partial t^q}(x,t) \right| \leq C(1 + t^{\alpha-q}) \text{ for } q = 0, 1, 2, \tag{3} \]
where \(0 < \alpha < 1\). Furthermore, if \(u_0 \in D(\mathcal{L}^{(q+3)/2}), f \in D(\mathcal{L}^{(q+1)/2})\) for each \(t \in [0, T]\), Huang et.al. [12] obtained
\[ \left| D_t^\alpha \left( \frac{\partial^n u}{\partial x^q} \right) \right| \leq C \text{ for } q \geq 0, \tag{4} \]
where the fractional power \(\mathcal{L}^\gamma\) of the operator \(\mathcal{L}\) (see, e.g., [9]) with each non-negative \(\gamma \in \mathbb{R}\) has domain
\[ D(\mathcal{L}^\gamma) := \left\{ g \in L^2(\Omega) : \sum_{i=1}^{\infty} \lambda_i^{2\gamma} \left| (g, \psi_i) \right|^2 < \infty \right\}. \]
See [16], for more regularity results. Most of existing works on L1 and L1-type approximations of Caputo derivative make unrealistic assumption that the solution is smooth at the initial time \(t = 0\). This leads to that the corresponding error estimates are always restrictive since often the sufficiently smooth solution is assumed.

The main purpose of this paper is to solve the problem (1) with the regularity assumption (3) by using a fully discrete L1-DG method. This method is based on L1 approximation with graded mesh in time and the DG method with uniform mesh in space. A detailed analysis for the stability and convergence of the fully discrete scheme is given. The scheme is unconditionally stable and convergent with order \(O(N^{-\min\{\alpha,2-\alpha\}} + h^{k+1})\) for \(P^k\) \((k \geq 2)\) polynomials. Finally, numerical experiments are present to demonstrate our theoretic result.

The structure of this paper is as follows. In Section 2, we present the fully discrete L1-DG scheme. In Section 3, the stability and convergence results of the proposed scheme are given. In section 4, we present numerical experiments to illustrate the accuracy of our proposed scheme. Finally, some conclusions are given in Section 5.

Notation. We use \(C\) to denote a generic constant that depends on the data of (1) but is independent of the mesh; it can take different values in different places. We write \(\| \cdot \|\) for the norms in \(L^2(\Omega)\). The \(L^2(\Omega)\) inner product is denoted by \((\cdot, \cdot)\).

2. Fully discrete L1-DG scheme. In this section, we will approximate the problem (1) using L1 approximation with graded mesh in time and the DG method with uniform mesh in space. The benefits of the time graded meshes for solving various types of fractional diffusion problems were investigated extensively by Mustapha and McLean, see for example [19, 20].

Let \(M\) and \(N\) be positive integers. Set \(h = l/M\) and \(x_{m+1/2} = mh\) for \(m = 0, 1, \ldots, M\). We define the mesh intervals \(I_m = [x_{m-1/2}, x_{m+1/2}]\) and \(x_m = (x_{m-1/2} + x_{m+1/2})/2\) for \(m = 1, 2, \ldots, M\).

Set \(t_n = T(n/N)^r\) for \(n = 0, 1, \ldots, N\), where the temporal mesh grading constant \(r \geq 1\) is chosen by the user. If \(r = 1\), then the mesh is uniform. Set \(\tau_n = t_n - t_{n-1}\) for \(n = 0, 1, \ldots, N\).

The fractional derivative \(D_t^\alpha u(x,t)\) defined in (2) can be discretized by following L1 scheme given in [25]
\[ D_t^\alpha u^n := \frac{d_{n,1}}{\Gamma(2-\alpha)} u^n - \frac{d_{n,n}}{\Gamma(2-\alpha)} u^0 + \frac{1}{\Gamma(2-\alpha)} \sum_{i=1}^{n-1} u^{n-i}(d_{n,i+1} - d_{n,i}), \tag{5} \]
where \(d_{n,i} := [(t_n - t_{n-i})^{1-\alpha} - (t_n - t_{n-i+1})^{1-\alpha}] / \tau_{n-i+1} \) for \( i \geq 1 \); in particular, \(d_{n,1} = \tau_{n-\alpha} \).

By the mean value theorem, it’s direct to check that

\[
d_{n,i+1} \leq d_{n,i} \text{ for } i \geq 1.
\] (6)

The truncation error \(\varphi^n(x)\) at time \(t = t_n\) is defined by

\[
\varphi^n(x) := D_t^\alpha u(x, t_n) - D_N^\alpha u(x, t_n)
\]

\[
= \sum_{i=0}^{n-1} \frac{1}{\Gamma(1-\alpha)} \int_{t_i}^{t_{i+1}} (t_n - s)^{-\alpha} \left[ \frac{\partial u}{\partial s}(x, s) - \frac{u(x, t_{i+1}) - u(x, t_i)}{\tau_{i+1}} \right] ds.
\] (7)

**Lemma 2.1.** [25, Lemma 5.2] Assume the solution of problem (1) satisfies (3), then there exists a constant \(C\) such that for all \(1 \leq n \leq N\) one has

\[
|\varphi^n(x)| \leq Cn^{-\min\{\alpha, 2-\alpha\}}.
\]

Next, we will use DG method proposed by Cheng and Shu in [4] to discretize space direction. The corresponding weak formulation of (1) is: Find \(u(\cdot, t) \in H^1(\Omega)\) for each \(t \in (0, T]\), such that

\[
\left\{ \begin{array}{l}
\int_{I_m} D_t^\alpha u v dx - \int_{I_m} u v_{xxx} dx + (\sigma u v_x - \sigma u_x v + \sigma u v) \big|_{m+\frac{1}{2}}^{m+\frac{1}{2}} \\
\quad \quad = \int_{I_m} fv dx, \; \forall \; v \in H^1(\Omega), \\
\int_{I_m} u(x, 0) v dx = \int_{I_m} u_0 v dx, \; \forall \; v \in H^1(\Omega).
\end{array} \right.
\] (8)

The piecewise polynomial space \(V_h^k\) as the space of polynomials of the degree up to \(k\) in each cell \(I_m\) is defined as

\[
V_h^k = \{v_h : v_h|_{I_m} \in P^k(I_m); \; 1 \leq m \leq M\}.
\]

We construct the following semi-discrete DG scheme: Find \(u_h(\cdot, t) \in V_h^k\) for each \(t \in (0, T]\), such that

\[
\left\{ \begin{array}{l}
\int_{I_m} D_t^\alpha u_h v_h dx - \int_{I_m} u_h v_{xxx} dx + \sigma(u_h v_{xx})|_{m+\frac{1}{2}} - \sigma(u_h v_{xx})|_{m-\frac{1}{2}} \\
-\sigma((u_h)_x v_{xx})|_{m+\frac{1}{2}} + \sigma((u_h)_x v_{xx})|_{m-\frac{1}{2}} + \sigma((u_h)_x v_{xx})|_{m+\frac{1}{2}} \\
-\sigma((u_h)_x v_{xx})|_{m-\frac{1}{2}} = \int_{I_m} fv_h dx, \; \forall \; v_h \in V_h^k, \\
\int_{I_m} u(x, 0) v_h dx = \int_{I_m} u_0 v_h dx, \; \forall \; v_h \in V_h^k,
\end{array} \right.
\] (9)

where \(\hat{u}_h, (\hat{u}_h)_x,\) and \((\hat{u}_h)_xx\) are numerical fluxes. These three fluxes are taken by

\[
\hat{u}_h = u^+_h, \; (\hat{u}_h)_x = (u_h)_x^+, \; (\hat{u}_h)_xx = (u_h)_xx.
\]
Next, in (9) we apply the L1 scheme (5) for the temporal discretisation:
\[
\begin{align*}
\int_{I_m}^{} D^\alpha_N u^n_h \, dx - \sigma \int_{I_m}^{} u^n_h(v_h)_{xxx} \, dx + \sigma ((u^n_h)^+(v_h)_{xx})_{m+\frac{1}{2}} \\
-\sigma ((u^n_h)^-(v_h)_{xx})_{m-\frac{1}{2}} - \sigma ((u^n_h)^+(v_h)_x)_{m+\frac{1}{2}} + \sigma ((u^n_h)^-(v_h)_x)_{m-\frac{1}{2}} \\
+ \sigma (u^n_h v_h^m)_{m+\frac{1}{2}} - \sigma (u^n_h v_h^m)_{m-\frac{1}{2}} = \int_{I_m}^{} f^n v_h \, dx, \ \forall \ v_h \in V^k_h,
\end{align*}
\]
(10)

Summing the equation (10) over \( m = 1, \ldots, M \) yields the fully discrete DG method:
\[
\begin{align*}
\int_{\Omega}^{} D^\alpha_N u^n_h v_h \, dx + \sigma \sum_{m=1}^M \left[ (u^n_h)^+(v_h)_{xx})_{m+\frac{1}{2}} - ((u^n_h)^+(v_h)_x)_{m+\frac{1}{2}} \right] \\
+ \sigma \sum_{m=1}^M \left[ (u^n_h)^+(v_h)_x)_{m-\frac{1}{2}} - ((u^n_h)^+(v_h)_x)_{m+\frac{1}{2}} \right] - \sigma \sum_{m=1}^M \int_{I_m}^{} u^n_h(v_h)_{xxx} \, dx \\
+ \sigma \sum_{m=1}^M \left[ (u^n_h v_h)_{m+\frac{1}{2}} - (u^n_h v_h)_{m-\frac{1}{2}} \right] = \int_{\Omega}^{} f^n v_h \, dx, \ \forall \ v_h \in V^k_h,
\end{align*}
\]
(11)

Denote
\[
\bar{v} = \frac{v^+ + v^-}{2}, \quad [v] = v^+ - v^-.
\]

Define the operator \( A \) by
\[
A(\xi, \eta) = -\sigma \sum_{m=1}^M \int_{I_m}^{} \xi \eta_{xxx} \, dx + \sigma \sum_{m=1}^M \left( \xi^+ \eta^- - \xi^- \eta^+ \right)_{m+\frac{1}{2}} \\
- \sigma \sum_{m=1}^M \left( \xi^+ \eta^- - \xi^- \eta^+ \right)_{m-\frac{1}{2}}, \ \forall \ \xi, \eta \in V^k_h.
\]

It is obvious that
\[
A(\xi, \xi) = -\sigma \sum_{m=1}^M \int_{I_m}^{} \xi \xi_{xxx} \, dx + \sigma \sum_{m=1}^M \left( \xi^+ \xi^- - \xi^- \xi^+ \right)_{m+\frac{1}{2}} \\
- \sigma \sum_{m=1}^M \left( \xi^+ \xi^- - \xi^- \xi^+ \right)_{m-\frac{1}{2}} \\
= \sigma \sum_{m=1}^M \left[ \frac{1}{2}(\xi_x)^2 - \xi \xi_{xx} \right]_{m+\frac{1}{2}} + \sigma \sum_{m=1}^M \left( \xi^- \xi^- - \xi_+ \xi_+ \right)_{m+\frac{1}{2}} \\
+ \sigma \sum_{m=1}^M \left( \xi_+ \xi_+ - \xi^- \xi^- \right)_{m-\frac{1}{2}} \\
= \sigma \sum_{m=1}^M \left[ \frac{1}{2}(\xi^- \xi^-)_{m+\frac{1}{2}} - \frac{1}{2}(\xi^+ \xi^+)_{m-\frac{1}{2}} - (\xi^+ \xi^-)_{m+\frac{1}{2}} + (\xi^+ \xi^-)_{m-\frac{1}{2}} \right]
\]
\[ \sigma \sum_{m=1}^{M} \left[ \frac{1}{2} (\xi_x^+ \xi_x^-)_{m+\frac{1}{2}} + \frac{1}{2} (\xi_x^- \xi_x^+)_{m-\frac{1}{2}} - (\xi_x^+ \xi_x^-)_{m+\frac{1}{2}} \right] \]

\[ = \sigma \sum_{m=1}^{M} \left[ \frac{1}{2} (\xi_x^- \xi_x^-)_{m+\frac{1}{2}} + \frac{1}{2} (\xi_x^+ \xi_x^+)_{m+\frac{1}{2}} - (\xi_x^- \xi_x^-)_{m+\frac{1}{2}} \right] \]

\[ = \frac{\sigma}{2} \sum_{m=1}^{M} |\xi_x| \left|_{m+\frac{1}{2}} \right. \]

where the second equation from the bottom is obtained by the periodic boundary condition \((\xi_x^+)_\frac{1}{2} = (\xi_x^-)_\frac{1}{2}\). Then the fully discrete L1-DG method (11) can be written as:

\[
\begin{cases}
(D_N^\alpha u_h^n, v_h) + A(u_h^n, v_h) = (f^n, v_h), \forall v_h \in V_h^k, \\
(u_h^0, v_h) = (u_0, v_h), \forall v_h \in V_h^k.
\end{cases}
\]

### 3. Stability and convergence analysis

In this section, we will examine the stability property and convergence property of the scheme (13) we just proposed.

Our stability result will be presented in the following general framework. Suppose that for each \(n\), the function \(\mu^n \in V_h^k\) satisfies

\[
\begin{cases}
(D_N^\alpha \mu^n, v_h) + A(\mu^n, v_h) = (g^n, v_h), \forall v_h \in V_h^k, \\
(\mu^0, v_h) = (\phi, v_h), \forall v_h \in V_h^k.
\end{cases}
\]

A stability bound for (14) will be present in the next Lemma.

**Lemma 3.1.** The solution \(\mu^n\) of the numerical scheme (14) satisfies

\[
\|\mu^n\| \leq \frac{\sigma}{2} \left[ \Gamma(2-\alpha)\|g^n\| + d_{n,n} \|\mu^0\| + \sum_{i=1}^{n-1} (d_{n,i} - d_{n,i+1}) ||\mu^{n-i}|| \right],
\]

for \(n = 1, 2, \ldots, N\).

**Proof.** Letting \(v_h = \mu^n\) in equation (14), we have

\[
(D_N^\alpha \mu^n, \mu^n) + A(\mu^n, \mu^n) = (g^n, \mu^n).
\]

By equation (12), one has

\[
(D_N^\alpha \mu^n, \mu^n) + \frac{\sigma}{2} \sum_{m=1}^{M} [(\mu^n)_x]^2 \left|_{m+\frac{1}{2}} \right. = (g^n, \mu^n).
\]

By \(\sigma \geq 0\) and \([(\mu^n)_x]^2 \geq 0\), (16) leads to

\[
(D_N^\alpha \mu^n, \mu^n) \leq (g^n, \mu^n).
\]

Recalling the definition \(D_N^\alpha \mu^n\) of (5) and employing a Cauchy-Schwarz inequalities implies that

\[
\frac{d_{n,1}}{\Gamma(2-\alpha)} \|\mu^n\|^2 \leq \|g^n\| \|\mu^n\|
\]

\[+ \frac{1}{\Gamma(2-\alpha)} \left[ d_{n,n} \|\mu^0\| + \sum_{i=1}^{n-1} (d_{n,i} - d_{n,i+1}) ||\mu^{n-i}|| \right] \|\mu^n\|,
\]

which is equivalent to (15). \(\square\)
As in [25], define the positive real numbers $\theta_{n,j}$, for $n = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, n - 1$, by

$$
\theta_{n,n} = 1, \quad \theta_{n,j} = \tau_n^{\alpha} (d_{n,i} - d_{n,i+1}) \theta_{n-i,j}.
$$

Observe that (6) implies $\theta_{n,j} > 0$ for all $n, j$.

The following stability result for (14) from Lemma 3.1 is deduced by these constants $\theta_{n,j}$.

**Lemma 3.2.** The solution of discrete scheme (14) satisfies

$$
\|\mu^n\| \leq \|\mu^0\| + \Gamma(2 - \alpha) T_n^{\alpha} \sum_{j=1}^n \theta_{n-j} \|g^n\|
$$

for $n = 1, 2, \ldots, N$.

**Proof.** One can obtain these bounds similarly to [25, Lemma 4.2].

The next result bounds a weighted sum of the $\theta_{n,j}$ in [25, Lemma 4.3] that will be needed in our error analysis.

**Lemma 3.3.** Let the parameter $\eta$ satisfy $\eta \leq \alpha r$. Then for $n = 1, 2, \ldots, N$, one has

$$
\sum_{j=1}^n j^{-\eta} \theta_{n,j} \leq \frac{T_n^{\alpha} N^{-\eta}}{1 - \alpha}.
$$

**Theorem 3.4** (Stability). The solution $\mu^n$ of the numerical scheme (14) satisfies

$$
\|\mu^n\| \leq \|\mu^0\| + \frac{\Gamma(2 - \alpha) T_n^{\alpha}}{1 - \alpha} \max_{1 \leq j \leq n} \|g^n\| \text{ for } 1 \leq n \leq N.
$$

**Proof.** By Lemma 3.3 one has

$$
\sum_{j=1}^n \theta_{n,j} \leq \frac{T_n^{\alpha}}{1 - \alpha},
$$

then the result follows from Lemma 3.2 immediately.

In order to estimate the error, the following projection $P$ into $V_h^k$ given in [4] will be used. For $k \geq 3$, we can choose $P$ such that for any $w$, $Pw$ satisfies

$$
\int_{I_m} wv_h \, dx = \int_{I_m} Pwv_h \, dx, \quad \forall v_h \in V_h^{k-3},
$$

and

$$
Pw^+ = w^+, \quad (Pw)^+_x = w^+_x, \quad (Pw)^-_{xx} = w^-_{xx},
$$

at all $x_{m+1/2}$. For $k = 2$, we can choose $P$ such that for any $w$, $Pw$ satisfies

$$
Pw^+ = w^+, \quad (Pw)^+_x = w^+_x, \quad (Pw)^-_{xx} = w^-_{xx},
$$

at any $x_{m+1/2}$.

For all these projections, the following inequality given in [5] holds

$$
\|Pw - w\| \leq C h^{k+1},
$$

where the constant $C$ depends on $k$ and the standard Sobolev $k + 1$ semi-norm $|w|_{k+1}$ of the smooth function $w$. 
Lemma 3.2 and (24) to get

\[ (D^a_N e^n, v_h) + A(e^n, v_h) = (D^a_N e^n, v_h) + A(e^n, v_h) - (\varphi^n, v_h) \]  

(20)

for all \( v_h \in V^k_h \), where \( \varphi^n \) is defined in (7).

**Theorem 3.5** (Error estimate). Assume the exact solution of problem (1) satisfy (3) and (4). Let \( u \) be the exact solution of (8) and \( u_h \) be the numerical solution of (9). If we impose a periodic boundary condition, and use \( V^k_h \) space with \( k \geq 2 \), then we have the following error estimate:

\[ \|u^n - u^n_h\| \leq C \left( h^{k+1} + N^{-\min\{r, 2-d\}} \right), \]

(21)

where \( C \) is a constant independent of \( h \) and \( N \).

**Proof.** Choosing \( v_h = e^n \) in (20) yields

\[ (D^a_N e^n, e^n) + A(e^n, e^n) = (D^a_N e^n, e^n) + A(e^n, e^n) - (\varphi^n, e^n). \]

(22)

For \( k = 2 \), we have \( \int_{I_m} e^n e^n_{xxx} \ dx = 0 \). For \( k \geq 3 \), we also have \( \int_{I_m} e^n e^n_{xxx} \ dx = 0 \) by (17). Furthermore, from (18) one has \( (e^n)^{+} = P(u^n)^{+} - (u^n)^{+} = 0 \), \( (e^n)^{+} = P(u^n)_{x}^{+} - (u^n)^{+} = 0 \) at each \( x_{m+1}/2 \). Thus

\[
A(\varepsilon^n, e^n) = \sum_{m=1}^{M} \left[ \left( (\varepsilon^n)^{+} (\varepsilon^n)^{+}_{xx} - (\varepsilon^n)^{+}_{x} (\varepsilon^n)^{+}_{xx} \right)_{m+\frac{1}{2}} - \left( (\varepsilon^n)^{+} (\varepsilon^n)^{+}_{xx} - (\varepsilon^n)^{+}_{x} (\varepsilon^n)^{+}_{xx} \right)_{m-\frac{1}{2}} - \sum_{m=1}^{M} \int_{I_m} e^n e^n_{xxx} \ dx \right. \\
\left. - \sum_{m=1}^{M} \int_{I_m} e^n e^n_{xxx} \ dx \right] = 0.
\]

Recalling (22), we have

\[ (D^a_N e^n, e^n) + A(e^n, e^n) = (\chi^n, e^n), \]

(23)

where \( \chi^n = D^a_N e^n - \varphi^n \). From the definitions of \( \varphi^n \) and \( e^n \) we get

\[ \|\chi^n\| \leq \|D^a_N e^n\| + \|\varphi^n\| \]

\[ \leq Ch^{k+1} |D^a_N u^n|_{k+1, \Omega} + ||\varphi^n|| \]

\[ \leq Ch^{k+1} |D^a_t u^n - \varphi^n|_{k+1, \Omega} + ||\varphi^n|| \]

\[ \leq Ch^{k+1} \left( |D^a_t u|_{k+1, \Omega} + Cn^{-\min\{r, 2-d\}} \right) + Cn^{-\min\{r, 2-d\}} \]

(24)

where Lemma 2.1 is used. Now (23) is a particular case of (14), so we can invoke Lemma 3.2 and (24) to get

\[ \|e^n\| \leq \|e^0\| + \tau^n \Gamma(2 - \alpha) \sum_{j=1}^{n} \theta_{n,j} \|\chi^j\| \]
\[ \leq ||\varepsilon^0|| + C\tau^n \Gamma(2 - \alpha) \sum_{j=1}^{n} \theta_{n,j} \left( h^{k+1} + j^{-\min\{\rho \alpha, 2 - \alpha\}} \right) \]
\[ \leq C h^{k+1} + CT^\alpha \left( h^{k+1} + N^{-\min\{\rho \alpha, 2 - \alpha\}} \right) \]
\[ \leq C \left( h^{k+1} + N^{-\min\{\rho \alpha, 2 - \alpha\}} \right), \]
where we used \( ||\varepsilon^0|| = ||\varepsilon^0|| \leq ch^k + 1 \) and Lemma 3.3 with \( \eta = 0 \) for the \( h^{k+1} \) term and \( \eta = \min\{\rho \alpha, 2 - \alpha\} \) for the \( N^{-\min\{\rho \alpha, 2 - \alpha\}} \) term.

Combining this bound and (19), we get (21).

**Remark 1.** When \( k < 2 \), numerical experiments in [4] show that our scheme is not consistent.

### 4. Numerical experiments

Next, three numerical examples based on the \( P^2 \) and \( P^3 \) polynomials are presented to verify our theoretical findings and illustrate the validity of our scheme.

**Example 4.1.** Consider the problem (1) with \( \sigma = 1, \Omega = (0, 2\pi), T = 1, u_0 = \sin x, \) and
\[ f = -E_\alpha (-t^\alpha) (\sin x + \cos x) + \frac{6t^{3-\alpha}}{\Gamma(4 - \alpha)} \sin x - t^3 \cos x. \]

The exact solution of this problem is
\[ u(x, t) = E_\alpha (-t^\alpha) \sin x + t^3 \sin x, \]
where \( E_\alpha(z) = \sum_{j=0}^{\infty} z^j / \Gamma(j\alpha + 1) \) is the generalised Mittag-Leffler function [23, Section 1.2]. This solution displays typical layer behaviour at \( t = 0 \).

| Table 1. \( L^\infty(L^2) \) errors and orders of convergence on temporal direction for Example 4.1 with \( r = (2 - \alpha)/\alpha \). |
|---|---|---|---|---|---|---|
| \( N=32 \) | \( N=64 \) | \( N=128 \) | \( N=256 \) | \( N=512 \) | \( N=1024 \) |
| \( \alpha = 0.4 \) | 3.0496E-2 | 1.1101E-2 | 3.9235E-3 | 1.3578E-3 | 4.6379E-4 | 1.5729E-4 |
| | 1.45671 | 1.50161 | 1.53071 | 1.54981 | 1.56001 |
| \( \alpha = 0.6 \) | 3.8341E-2 | 1.5127E-2 | 5.8825E-3 | 2.2665E-3 | 8.6831E-4 | 3.3157E-4 |
| | 1.34171 | 1.36261 | 1.37591 | 1.38421 | 1.38881 |
| \( \alpha = 0.8 \) | 5.9953E-2 | 2.6607E-2 | 1.1728E-2 | 5.1485E-3 | 2.2540E-3 | 9.8512E-4 |
| | 1.17201 | 1.18171 | 1.18781 | 1.19161 | 1.19411 |

**Example 4.2.** Consider the problem (1) with \( \sigma = 1, \Omega = (0, 2\pi), T = 1, u_0 = 0, \) and
\[ f = \left[ \Gamma(1 + \alpha) + \frac{6t^{3-\alpha}}{\Gamma(4 - \alpha)} \right] \sin x - (t^\alpha + t^3) \cos x. \]

The exact solution of problem is
\[ u(x, t) = (t^\alpha + t^3) \sin x, \]
and periodic boundary conditions are used. This solution also displays typical layer behaviour at \( t = 0 \).

In this two examples, the temporal accuracy is test by \( L^\infty(L^2) \) norm, which is defined by
\[ \|u(x, t_n) - u_n^h\|_{L^\infty(L^2)} = \max_{0 \leq n \leq N} \|u(x, t_n) - u_n^h\|. \]
Table 2. Errors and orders of convergence on space direction for Example 4.1 with $\alpha = 0.4$.

| Polynomial | $M$ | $\|u - u_h\|_{L^2}$ | Order | $\|u - u_h\|_{L^\infty}$ | Order |
|------------|-----|----------------------|-------|--------------------------|-------|
| $P^2$      | 5   | 5.3831E-01           | -     | 3.2328E-01               | -     |
|            | 10  | 7.8579E-02           | 2.7762| 4.7729E-02               | 2.7598|
|            | 20  | 9.9319E-03           | 2.9840| 6.2124E-03               | 2.9416|
|            | 40  | 1.1426E-04           | 3.1196| 7.5845E-04               | 3.0340|
| $P^3$      | 5   | 1.7236E-02           | -     | 1.3819E-02               | -     |
|            | 10  | 1.1399E-03           | 3.9184| 8.7589E-04               | 3.9798|
|            | 15  | 2.2712E-04           | 3.9406| 1.7695E-04               | 3.9667|
|            | 20  | 7.2979E-05           | 3.9418| 6.1408E-04               | 3.9070|

Table 3. $L^\infty(L^2)$ errors and orders of convergence on temporal direction for Example 4.2 with $r = (2 - \alpha)/\alpha$.

| $N$=32 | $N$=64 | $N$=128 | $N$=256 | $N$=512 | $N$=1024 |
|--------|--------|--------|--------|--------|--------|
| $\alpha = 0.4$ | 2.6605E-2 | 9.8042E-3 | 3.4860E-3 | 1.2119E-3 | 4.1549E-4 |
|         | 1.4402 | 1.4918 | 1.5243 | 1.5444 | 1.5510 |
| $\alpha = 0.6$ | 3.0086E-2 | 1.2002E-2 | 4.6980E-3 | 1.8177E-3 | 6.9850E-4 |
|         | 1.3258 | 1.3531 | 1.3699 | 1.3797 | 1.3836 |
| $\alpha = 0.8$ | 4.1374E-2 | 1.8226E-2 | 7.9818E-3 | 3.4836E-3 | 1.5178E-3 |
|         | 1.1827 | 1.1912 | 1.1961 | 1.1985 | 1.1992 |

Table 4. Errors and orders of convergence on space direction for Example 4.2 with $\alpha = 0.4$.

| Polynomial | $M$ | $\|u - u_h\|_{L^2}$ | Order | $\|u - u_h\|_{L^\infty}$ | Order |
|------------|-----|----------------------|-------|--------------------------|-------|
| $P^2$      | 5   | 3.8931E-01           | -     | 2.3483E-01               | -     |
|            | 10  | 5.6563E-02           | 2.7829| 3.4418E-02               | 2.7704|
|            | 20  | 7.1139E-03           | 2.9911| 4.4696E-03               | 2.9449|
|            | 40  | 7.7979E-04           | 3.1894| 5.4210E-04               | 3.0435|
| $P^3$      | 5   | 1.2812E-02           | -     | 1.0564E-02               | -     |
|            | 10  | 8.3809E-03           | 3.9342| 6.7270E-04               | 3.9731|
|            | 15  | 1.6615E-04           | 3.9552| 1.2940E-04               | 4.0071|
|            | 20  | 5.3162E-05           | 3.9564| 4.3749E-05               | 3.9578|

Theorem 3.5 predicts the rate of convergence of this error to be $O(h^{k+1} + N^{-\min\{r\alpha,2-\alpha\}})$. In Table 1 and Table 3 we show the $L^\infty(L^2)$ errors and the associated orders of convergence for Example 4.1 and Example 4.2 with different $\alpha$, and the mesh grading exponent $r = (2 - \alpha)/\alpha$. In these Tables we have taken $M = 500$ to eliminate the error caused by spatial discretization. From Tables 1 and 3, we see that the temporal accuracy is order $2 - \alpha$, which is consistent with our theoretical findings.

Next, we will check the spatial accuracy. We choose $N$ big enough to eliminate the error caused by temporal discretization. We take $N = 1000$ and $N = 10000$ for...
$P^2$ and $P^3$ polynomials, respectively. In Tables 2 and 4 we also present the $L^2(\Omega)$ errors, $L^\infty(\Omega)$ errors and the associated orders of convergence at $t = 1$ for $\alpha = 0.4$ and the mesh grading $r = (2 - \alpha)/\alpha$. From 2 and Table 4, we see that the $O(h^{k+1})$ spatial accuracy is derived. The result supports the predicted rates of convergence.

The Figure 1 is the numerical solution for Example 4.1 with $\alpha = 0.4$. From figure 1, we observe an initial layer for the numerical solution at initial time $t = 0$, in agreement with the exact solution.

Example 4.3. We consider the problem (1) with $\sigma = 1$, $\Omega = (0, 2\pi)$, $T = 1$, $u_0 = \sin x$, and $f = 0$.

The exact solution of the Example 4.3 is unknown, so the order of convergence in the computed solutions are estimated by the two-mesh principle [7]. Now we describe this method. Let $u_n^h$ with $0 \leq n \leq N$ and $1 \leq m \leq M$ be the solution computed by our scheme (13). Then consider a second mesh that is also uniform in the spatial direction with $M$ mesh intervals, and in temporal direction is defined by

$$t_n = (n/(2N))^r \text{ for } 0 \leq n \leq 2N.$$

The solution computed on this mesh is denoted $z_n^h$ with $0 \leq n \leq 2N$ and $1 \leq m \leq M$. Then the $L^2$ error at the final time $T = 1$ is defined by

$$D_{M,N} = \|u_N^h - z_{2N}^h\|,$$
and they are used to compute the estimate rate of convergence
\[ \log_2 \left( \frac{D_{M,N}^r}{D_{M,2N}^r} \right). \]
In these Example $M = 500$ is taken to eliminate the error caused by spatial discretization. In Tables 5 and 6 we show the $L^2(\Omega)$ errors and the associated orders of convergence for $\alpha = 0.4$, $\alpha = 0.6$, $\alpha = 0.8$ and several values of the mesh grading exponent $r$. The numerical results are in agreement with Theorem 3.5: they show that $r \geq (2 - \alpha)/\alpha$ yields the optimal rate of convergence $O(N^{-2-\alpha})$.

**Table 5.** $L^2$ errors and orders of convergence on temporal direction for Example 4.3 with $r = (2 - \alpha)/\alpha$.

| $\alpha$ | N=64   | N=128  | N=256  | N=512  | N=1024 |
|----------|--------|--------|--------|--------|--------|
| 0.4      | 2.4959E-3 | 8.4989E-4 | 2.8741E-4 | 9.6635E-5 | 3.2367E-5 |
|          | 1.5542  | 1.5641  | 1.5720  | 1.5784  |        |
| 0.6      | 6.0125E-3 | 2.3383E-3 | 8.9915E-4 | 3.4367E-4 | 1.3092E-4 |
|          | 1.3624  | 1.3788  | 1.3875  | 1.3923  |        |
| 0.8      | 1.0359E-2 | 4.6808E-3 | 2.0899E-3 | 1.2645E-4 | 4.0879E-4 |
|          | 1.1460  | 1.1633  | 1.1736  | 1.1803  |        |

**Table 6.** $L^2$ errors and orders of convergence on temporal direction for Example 4.3 with $r = 2(2 - \alpha)/\alpha$.

| $\alpha$ | N=64   | N=128  | N=256  | N=512  | N=1024 |
|----------|--------|--------|--------|--------|--------|
| 0.4      | 6.0380E-3 | 2.2285E-3 | 7.2999E-4 | 2.4866E-4 | 8.4023E-5 |
|          | 1.5110  | 1.5371  | 1.5536  | 1.5633  |        |
| 0.6      | 1.1214E-2 | 4.4769E-3 | 1.7480E-3 | 6.7438E-4 | 2.5839E-4 |
|          | 1.3248  | 1.3567  | 1.3741  | 1.3839  |        |
| 0.8      | 1.5602E-2 | 7.0291E-3 | 3.1157E-3 | 1.3694E-3 | 5.9926E-4 |
|          | 1.1503  | 1.1737  | 1.1859  | 1.1923  |        |

5. **Concluding remarks.** In this paper, we have presented and analyzed a fully discrete L1-DG scheme for the time fractional KdV equation, whose solution has a weak singularity at initial time $t = 0$. The scheme is based on L1 approximation with graded mesh for the time Caputo fractional derivative and the DG method with uniform mesh for the spatial discretization. The scheme is unconditional stable and convergent with $O(N^{-\min\{r\alpha,2-\alpha\}} + h^{k+1})$ for $P^k$ ($k \geq 2$) polynomials. Numerical results demonstrated the accuracy of our proposed scheme. Unlike the traditional LDG method, the method in this paper is applied without introducing any auxiliary variables or rewriting the original equation into a first order system. Since all the historical data for $u_h$ must be stored, one of the future works along this direction is to investigate proper orthogonal decomposition (POD) method which allow to reduce the storage requirement by overcoming the so-called “global dependence” problem.
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