ON MORI CHAMBER AND STABLE BASE LOCUS DECOMPOSITIONS

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ABSTRACT. The effective cone of a Mori dream space admits two wall-and-chamber decompositions called Mori chamber and stable base locus decompositions. In general the former is a non trivial refinement of the latter. We investigate, from both the geometrical and the combinatorical viewpoints, the differences between these decompositions. Furthermore, we provide a criterion to establish whether the two decompositions coincide for a Mori dream space of Picard rank two, and we construct an explicit example of a Mori dream space of Picard rank two for which the decompositions are different, showing that our criterion is sharp.

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1. INTRODUCTION

Mori dream spaces, introduced by Y. Hu and S. Keel in [HK00], are varieties whose total coordinate ring, called the Cox ring, is finitely generated. The birational geometry of a Mori dream space is encoded in its cone of effective divisors together with a chamber decomposition on it, called Mori chamber decomposition. Two effective divisors lie in the interior of the same Mori chamber if there is an isomorphism between the target spaces of the corresponding dominant rational maps making the obvious triangular diagram commutative.

The birational geometry of a Mori dream spaces can also be described via the Variation of Geometric Invariant Theory of its Cox ring. As proven in [HK00] and [ADHL15, Section 3.3.4] from this point of view GIT chambers correspond to Mori chambers.

The pseudo-effective cone of a projective variety with zero irregularity, so in particular of a Mori dream space, can be decomposed into chambers depending on the stable base locus of the corresponding linear series. Such decomposition called stable base locus decomposition in general is coarser than the Mori chamber decomposition.

Mori chamber and stable base locus decompositions of important classes of moduli spaces such as Hilbert schemes of points on surfaces [BCT3], [ARCHT3], Kontsevich spaces of stable maps [Che08, CC10, CC11], spaces of complete forms [Hue15, Mas18a, Mas18b], and moduli spaces of parabolic bundles [Muk05, AM16] have recently been studied in a series of papers.

In this paper, given a Mori dream space $X$, we aim to understand how far is the stable base locus decomposition of $\text{Eff}(X)$ from determining its Mori chamber decomposition. In Section 3 we produce examples of Mori dream spaces for which the two decompositions are different and we interpret them both from the geometric and the combinatorial viewpoints.

While producing examples of either non compact varieties or of varieties with Picard rank greater than of equal to three turns out to be fairly feasible, it is quite tricky to exhibit a normal $\mathbb{Q}$-factorial
projective Mori dream space of Picard rank two for which the Mori chamber decomposition is a non trivial refinement of the stable base locus decomposition. In Example 3.6 we construct such a Mori dream space, and at the best of our knowledge this is the fist example of a projective variety displaying this particular behavior appearing in the literature.

**Theorem 1.1.** Let $Z$ be the toric variety with Cox ring $K[T_1, \ldots, T_{11}]$ whose grading matrix and irrelevant ideal are the following

$$Q = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \end{bmatrix} \quad J_{\text{irr}}(Z) = \langle T_1, T_2 \rangle \cap \langle T_3, \ldots, T_{11} \rangle$$

and let $F, G$ be two general polynomials of degree $(2, 2)$ in the $T_i$. Then the ring

$$\frac{K[T_1, \ldots, T_{11}]}{(F, G)}$$

is the Cox ring a projective normal $\mathbb{Q}$-factorial Mori dream space $X \subset Z$ of Picard rank two. Furthermore, the Mori chamber decomposition of $\text{Eff}(X)$ consists of three chambers while its stable base locus decomposition consists of two chambers.

In Section 4 we focus on Mori dream spaces of Picard rank two. Recall that a Mori dream space can be recovered as a GIT quotient, with respect to a suitable polarization, of the spectrum of its Cox ring by a torus. In the fist part of Section 4 assuming that the Picard rank is two, we reach a simple description of the non semi-stable loci with respect to all the possible polarizations, and of the stable base loci of effective divisors in terms of the generators of the Cox ring. Thanks to these characterizations in Theorem 1.1 we get technical criteria on the non semi-stable loci aimed to establish whether the Mori chamber and the stable base locus decomposition of a given Mori dream space of Picard rank two coincide. As observed in Corollary 4.12 the irreducibility of the non semi-stable loci is a sufficient condition for the two decompositions to coincide.

In Theorem 1.13 we prove that under suitable inequalities, that need just the knowledge of the generators of the Cox ring in order to be checked, the two decompositions coincide. Furthermore, in Proposition 4.2 we get another criterion for the equality of the decompositions. The usefulness of these results lies in the fact that in general, even in Picard rank two, the stable base locus decomposition is considerably easier to compute than the Mori chamber decomposition.

Note that if $X$ is a projective Mori dream space of Picard rank two we can fix a total order on the classes in the effective cone: $w \leq w'$ if $w$ is on the left of $w'$. Given two convex cones $\lambda, \lambda'$ contained in the effective cone we will write $\lambda \leq \lambda'$ if $w \leq w'$ for any $w \in \lambda$ and $w' \in \lambda'$. Denote by $\{f_1, \ldots, f_r\}$ a minimal set of homogeneous generators for the Cox ring $\mathcal{R}(X)$ of $X$, and let $w_i = \deg(f_i)$ for any $i$.

The criteria in the Proposition 4.2 Theorem 4.13 and Corollary 4.14 can be summarized in the following statement.

**Theorem 1.2.** Let $X$ be a $\mathbb{Q}$-factorial Mori dream space with Picard rank two, $\{f_1, \ldots, f_r\}$ a minimal set of homogeneous generators for the Cox ring $\mathcal{R}(X)$, $w_i := \deg(f_i)$, and $\lambda_A$ be the ample chamber of $X$. Denote by $c$ the codimension of $X$ into its canonical toric embedding [ADHL15] Section 3.2.5. Define

$$h^+ := \#\{f_i : w_i \geq \lambda_A\} \quad \text{and} \quad h^- := \#\{f_i : w_i \leq \lambda_A\}$$

If one of the following two conditions is satisfied

(i) all the generators of $\mathcal{R}(X)$ appear in the walls of the stable base locus decomposition of $\text{Eff}(X)$,
(ii) $h^- > c$ and $h^+ > c$,

then the Mori chamber decomposition and the stable base locus decomposition of $\text{Eff}(X)$ coincide.

In particular, if $Z$ is a projective normal $\mathbb{Q}$-factorial toric variety with $\text{rk}(\text{Cl}(Z)) = 2$, and $X \subset Z$ is a projective normal $\mathbb{Q}$-factorial Mori dream hypersurface such that $\iota^*: \text{Cl}(Z) \to \text{Cl}(X)$ is an isomorphism, then the Mori chamber and the stable base locus decompositions of both $\text{Eff}(Z)$ and $\text{Eff}(X)$ coincide.

As observed in Remark 4.15 Theorem 1.1 shows that the bounds in Theorem 1.2 item (ii) can not be improved. Indeed in Theorem 1.1 we have $h^+ = c = 2$.

In Section 4.16 we show how Theorem 1.2 item (ii), together with the classification of Picard rank two varieties, with a torus action of complexity one in [FHN16] Theorem 1.1] immediately implies...
that the Mori chamber decomposition is equal to the stable base locus decomposition for this class of varieties. Note that, as shown by the examples in Section 4.1, Lemma 4.3 along with the proof of Theorem 4.8 provide a concrete method to compute Mori chamber decompositions.

We would like so stress that these results can also be useful in order to compute the Sarkisov factorization of a birational map \( X \to Y \) between two \( \mathbb{Q} \)-factorial Fano varieties of Picard rank one. Indeed, if there exists a Mori dream space \( Z \) of Picard rank two admitting a dominant morphism \( Z \to X \) then such a factorization is determined by a so called 2-ray game on \( Z \), and such a 2-ray game is in turn determined by the Mori chamber decomposition of \( \text{Eff}(Z) \). We refer to [Cor05, HM13, AZ16, Ahm17] for details on this topic and explicit examples.

Finally, in Section 4 we apply Theorem 1.2 item (i) to show that the Mori chamber decomposition of the blow-up \( G(r, n)_1 \) of the Grassmannian \( G(r, n) \), parametrizing \( r \)-planes in \( \mathbb{P}^n \), at a point coincides with its stable base locus decomposition, and can be described in terms of linear systems of hyperplanes containing the osculating spaces of \( G(r, n) \) at the blown-up point. This provides a positive answer to [MR18, Question 6.9].

All through the paper we will work over an algebraically closed field \( K \) of characteristic zero, and given a \( \mathbb{Q} \)-factorial Mori dream space \( X \) we will denote by \( \text{MCD}(X) \) and \( \text{SBLD}(X) \) respectively the Mori chamber decomposition and the stable base locus decomposition of its effective cone.

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### 2. Mori Chambers and Stable Base Locus Decompositions

Let \( X \) be a normal projective variety over an algebraically closed field of characteristic zero. We denote by \( N^1(X) \) the real vector space of \( \mathbb{R} \)-Cartier divisors modulo numerical equivalence. The *nef cone* of \( X \) is the closed convex cone \( \text{Nef}(X) \subset N^1(X) \) generated by classes of nef divisors. The *movable cone* of \( X \) is the convex cone \( \text{Mov}(X) \subset N^1(X) \) generated by classes of *movable divisors*. These are Cartier divisors whose stable base locus has codimension at least two in \( X \). The *effective cone* of \( X \) is the convex cone \( \text{Eff}(X) \subset N^1(X) \) generated by classes of *effective divisors*. We have inclusions \( \text{Nef}(X) \subset \text{Mov}(X) \subset \text{Eff}(X) \).

We will denote by \( N_1(X) \) be the real vector space of numerical equivalence classes of 1-cycles on \( X \). The closure of the cone in \( N_1(X) \) generated by the classes of irreducible curves in \( X \) is called the *Mori cone* of \( X \), we will denote it by \( \text{NE}(X) \).

A class \([C] \in N_1(X)\) is called *moving* if the curves in \( X \) of class \([C]\) cover a dense open subset of \( X \). The closure of the cone in \( N_1(X) \) generated by classes of moving curves in \( X \) is called the *moving cone* of \( X \) and we will denote it by \( \text{mov}(X) \). We refer to [Deb01, Chapter 1] for a comprehensive treatment of these topics.

We say that a birational map \( f : X \to X' \) to a normal projective variety \( X' \) is a *birational contraction* if its inverse does not contract any divisor. We say that it is a *small \( \mathbb{Q} \)-factorial modification* if \( X' \) is \( \mathbb{Q} \)-factorial and \( f \) is an isomorphism in codimension one. If \( f : X \to X' \) is a small \( \mathbb{Q} \)-factorial modification, then the natural pull-back map \( f^* : N^1(X') \to N^1(X) \) sends \( \text{Mov}(X') \) and \( \text{Eff}(X') \) isomorphically onto \( \text{Mov}(X) \) and \( \text{Eff}(X) \), respectively. In particular, we have \( f^*(\text{Nef}(X')) \subset \text{Mov}(X) \).

**Definition 2.1.** A normal projective \( \mathbb{Q} \)-factorial variety \( X \) is called a *Mori dream space* if the following conditions hold:

- \( \text{Pic}(X) \) is finitely generated, or equivalently \( h^1(X, \mathcal{O}_X) = 0 \),
- \( \text{Nef}(X) \) is generated by the classes of finitely many semi-ample divisors,
- there is a finite collection of small \( \mathbb{Q} \)-factorial modifications \( f_i : X \to X_i \), such that each \( X_i \) satisfies the second condition above, and \( \text{Mov}(X) = \bigcup_i f_i^*(\text{Nef}(X_i)) \).

By [BCHM10, Corollary 1.3.2] smooth Fano varieties are Mori dream spaces. In fact, there is a larger class of varieties called log Fano varieties which are Mori dream spaces as well. By the work of M. Brion [Bri93] we have that \( \mathbb{Q} \)-factorial spherical varieties are Mori dream spaces. An alternative proof of this result can be found in [Per14, Section 4].
The collection of all faces of all cones $f_i^*(\operatorname{Nef}(X_i))$ in Definition 2.1 forms a fan which is supported on $\operatorname{Mov}(X)$. If two maximal cones of this fan, say $f_i^*(\operatorname{Nef}(X_i))$ and $f_j^*(\operatorname{Nef}(X_j))$, meet along a facet, then there exist a normal projective variety $Y$, a small modification $\varphi: X_i \rightarrow X_j$, and $h_i: X_i \rightarrow Y$ and $h_j: X_j \rightarrow Y$ small birational morphisms of relative Picard number one such that $h_j \circ \varphi = h_i$. The fan structure on $\operatorname{Mov}(X)$ can be extended to a fan supported on $\operatorname{Eff}(X)$ as follows.

**Definition 2.2.** Let $X$ be a Mori dream space. We describe a fan structure on the effective cone $\operatorname{Eff}(X)$, called the Mori chamber decomposition. We refer to [HK00 Proposition 1.11] and [Oka16 Section 2.2] for details. There are finitely many birational contractions from $X$ to Mori dream spaces, denoted by $g_i: X \rightarrow Y_i$. The set $\operatorname{Exc}(g_i)$ of exceptional prime divisors of $g_i$ has cardinality $\rho(X/Y_i) = \rho(X) - \rho(Y_i)$. The maximal cones $C$ of the Mori chamber decomposition of $\operatorname{Eff}(X)$ are of the form: $C_i = \langle g_i^*(\operatorname{Nef}(Y_i)), \operatorname{Exc}(g_i) \rangle$. We call $C_i$ or its interior $C_i^\circ$ a maximal chamber of $\operatorname{Eff}(X)$.

**Definition 2.3.** Let $X$ be a normal projective variety with finitely generated divisor class group $\operatorname{Cl}(X) := \operatorname{WDiv}(X)/\operatorname{PDiv}(X)$, in particular $h^1(X, \mathcal{O}_X) = 0$. The Cox sheaf and Cox ring of $X$ are defined as

$$\mathcal{R} := \bigoplus_{[D] \in \operatorname{Cl}(X)} \mathcal{O}_X(D) \quad \mathcal{R}(X) := \Gamma(X, \mathcal{R})$$

Recall that $\mathcal{R}$ is a sheaf of $\operatorname{Cl}(X)$-graded $\mathcal{O}_X$-algebras, whose multiplication maps are discussed in [ADHL13 Section 1.4]. In case the divisor class group is torsion-free one can just take the direct sum over a subgroup of $\operatorname{WDiv}(X)$, isomorphic to $\operatorname{Cl}(X)$ via the quotient map, getting immediately a sheaf of $\mathcal{O}_X$-algebras. Denote by $\hat{X}$ the relative spectrum of $\mathcal{R}$ and by $\overline{X}$ the spectrum of $\mathcal{R}(X)$. The $\operatorname{Cl}(X)$-grading induces an action of the quasitorus $H_X := \operatorname{Spec} \mathbb{C}[\operatorname{Cl}(X)]$ on both spaces. The inclusion $\mathcal{O}_X \rightarrow \mathcal{R}$ induces a good quotient $p_X: \hat{X} \rightarrow X$ with respect to this action. Summarizing we have the following diagram

$\begin{array}{ccc}
\hat{X} & \xrightarrow{p_X} & X \\
\downarrow & & \downarrow \\
\overline{X} & & \\
\end{array}$

to which we will refer as the Cox construction of $X$. In case $\mathcal{R}(X)$ is a finitely generated algebra the complement of $\hat{X}$ in the affine variety $\overline{X}$ has codimension $\geq 2$. This subvariety is the irrelevant locus and its defining ideal is the irrelevant ideal $\mathcal{I}_{\text{irr}}(X) \subseteq \mathcal{R}(X)$.

**Remark 2.4.** By [HK00 Proposition 2.9] a normal and $\mathbb{Q}$-factorial projective variety $X$ over an algebraically closed field $K$, with finitely generated Picard group is a Mori dream space if and only if $\mathcal{R}(X)$ is a finitely generated $K$-algebra. Furthermore, the following equality holds

$$\dim \mathcal{R}(X) = \dim X + \operatorname{rank} \mathcal{Cl}(X)$$

see for instance [ADHL15 Theorem 3.2.1.4].

Let $X$ be a normal $\mathbb{Q}$-factorial projective variety, and let $D$ be an effective $\mathbb{Q}$-divisor on $X$. The stable base locus $B(D)$ of $D$ is the set-theoretic intersection of the base loci of the complete linear systems $|sD|$ for all positive integers $s$ such that $sD$ is integral

$$B(D) = \bigcap_{s>0} B(sD).$$

Since stable base loci do not behave well with respect to numerical equivalence, we will assume that $h^1(X, \mathcal{O}_X) = 0$ so that linear and numerical equivalence of $\mathbb{Q}$-divisors coincide.

Then numerically equivalent $\mathbb{Q}$-divisors on $X$ have the same stable base locus, and the pseudo-effective cone $\overline{\operatorname{Eff}}(X)$ of $X$ can be decomposed into chambers depending on the stable base locus of the corresponding linear series called stable base locus decomposition, see [CJFG17 Section 4.1.3] for further details.

If $X$ is a Mori dream space, satisfying then the condition $h^1(X, \mathcal{O}_X) = 0$, determining the stable base locus decomposition of $\operatorname{Eff}(X)$ is a first step in order to compute its Mori chamber decomposition.
Remark 2.5. Recall that two divisors $D_1, D_2$ are said to be Mori equivalent if $B(D_1) = B(D_2)$ and the following diagram of rational maps is commutative

$$
\begin{array}{ccc}
\phi_{D_1} & \Xrightarrow{X} & \phi_{D_2} \\
X(D_1) & \xrightarrow{\sim} & X(D_2)
\end{array}
$$

where the horizontal arrow is an isomorphism. Therefore, the Mori chamber decomposition is a refinement of the stable base locus decomposition.

Let $X$ be a Mori dream space with Cox ring $\mathcal{R}(X)$ and grading matrix $Q$. The matrix $Q$ defines a surjection

$$Q : E \to \text{Cl}(X)$$

from a free, finitely generated, abelian group $E$ to the divisor class group of $X$. Denote by $\gamma$ the positive quadrant of $E_0 := E \otimes \mathbb{Q}$. Let $e_1, \ldots, e_r$ be the canonical basis of $E_0$. Given a face $\gamma_0 \leq \gamma$ we say that $i \in \{1, \ldots, r\}$ is a cone index of $\gamma_0$ if $e_i \in \gamma_0$. The face $\gamma_0$ is an $F$-face if there exists a point of $\bar{X} = \text{Spec}(\mathcal{R}(X))$ whose $i$-th coordinate is non-zero exactly when $i$ is a cone index of $\gamma_0$ [ADHL15, Construction 3.3.1.1]. The set of these points is denoted by $\bar{X}(\gamma_0)$.

Example 2.6. If $\mathcal{R}(X) = \frac{\mathbb{K}[T_1, \ldots, T_5]}{(T_1T_2 + T_3T_4T_5)}$ then $\gamma_0 = \text{cone}(e_1, e_4)$ is an $F$-face and

$$\bar{X}(\gamma_0) = \{(x_1, 0, 0, x_4, 0) \in \bar{X} : x_1x_4 \neq 0\}$$

On the other hand, cone$(e_1, e_3, e_4)$ is not an $F$-face.

Given the Cox construction of $X$ we denote by $X(\gamma_0) \subseteq X$ the image of $\bar{X}(\gamma_0)$, and given an $F$-face $\gamma_0$ its image $Q(\gamma_0) \subseteq \text{Cl}(X)_\mathbb{Q}$ is an orbit cone of $X$. The set of all orbit cones of $X$ is denoted by $\Omega$. Accordingly to [ADHL15, Definition 3.1.2.6] a class $\omega \in \text{Cl}(X)$ defines the GIT chamber

$$\lambda(\omega) := \bigcap_{\{\omega \in \Omega : \omega \supseteq \gamma\}} \omega$$

(2.7)

If $w$ is an ample class of $X$ the corresponding GIT chamber is the semi-ample cone of $X$. The variety $X$ can be reconstructed from the pair $(\mathcal{R}(X), \Phi)$ formed by the Cox ring together with a bunch of cones, consisting of certain subsets of the orbit cones [ADHL15, Definition 3.1.3.2]. According to [ADHL15, Example 3.1.3.6] every GIT chamber $\lambda$ defines a bunch of orbit cones

$$\Phi(\lambda) := \{\omega \in \Omega : \omega \supseteq \lambda\}$$

Given a class $w \in \text{Cl}(X)$ we denote by $\lambda^{\text{shl}}(w)$ the subset of $\text{Cl}(X)_\mathbb{Q}$ consisting of all classes which have the same stable base locus of $w$.

Proposition 2.8. Let $X$ be a normal variety with finitely generated Cox ring, bunch of orbit cones $\Phi$, and let $w \in \text{Cl}(X)$ be a class of $X$. Then

$$\lambda^{\text{shl}}(w) = \bigcap_{\{\omega \in \Phi : \omega \supseteq \omega\}} \omega \cap \bigcap_{\{\omega \in \Phi : \omega \supseteq \omega\}} \omega^c$$

(2.9)

Proof. Recall that, according to [ADHL15, Construction 3.2.1.3], the set of relevant faces $\text{rlv}(\Phi)$ is the set of faces of $\gamma$ which are mapped by $Q$ to elements of $\Phi$. Each relevant face $\gamma_0 \leq \gamma$ defines a subset $X(\gamma_0) \subseteq X$ consisting of all the points of $X$ whose $i$-th Cox coordinate is non-zero exactly when $i$ is a cone index of $\gamma_0$ [ADHL15, Construction 3.3.1.1]. By [ADHL15, Proposition 3.3.2.8] the stable base locus of a class $w$ is the union

$$B(w) := \bigcup_{\{\gamma_0 \in \text{rlv}(\Phi) : w \notin Q(\gamma_0)\}} X(\gamma_0)$$

Applying $Q$ to the elements of the set $\{\gamma_0 \in \text{rlv}(\Phi) : w \notin Q(\gamma_0)\}$ one gets the set $\{\omega \in \Phi : w \notin \omega\}$ and the former set is completely determined by the latter. We claim that two classes $w, w'$ define the same stable base locus if and only if the following holds

$$\{\omega \in \Phi : w \in \omega\} = \{\omega \in \Phi : w' \in \omega\}$$

(2.10)
Proof. It is a direct consequence of Proposition 2.8 and of the following equalities
\[
\lambda = \lambda^\mathrm{shl},
\]
where the first is by definition while the second is due to the fact that any two classes in the relative interior \(\lambda(w)\) determine the same chamber.

\[\lambda(w) \subseteq \lambda^\mathrm{shl}(w)\]

**Corollary 2.11.** Let \(X\) be a normal variety with finitely generated Cox ring, bunch of orbit cones \(\Phi\) and let \(w \in \text{Cl}(X)\) be a class of \(X\). Then the following inclusion holds
\[
\lambda(w) \subseteq \lambda^\mathrm{shl}(w)
\]

**Proof.** It is a direct consequence of Proposition 2.8 and of the following equalities
\[
\lambda(w) = \bigcap_{\{\omega \in \Phi : w \in \omega\}} \omega = \bigcap_{\{\omega \in \Phi : w \in \omega\}} \omega \cap \bigcap_{\{w \in \Phi : w \in \omega\}} \omega^C
\]
where the first is by definition while the second is due to the fact that any two classes in the relative interior \(\lambda(w)\) determine the same chamber. \(\square\)

**Remark 2.12.** Let \(X\) be a normal variety with finitely generated Cox ring, bunch of orbit cones \(\Phi(\lambda)\) and let \(\lambda_1, \lambda_2\) be two GIT chambers distinct from \(\lambda\). If
\[
\text{cone}(\lambda \cup \lambda_1) = \text{cone}(\lambda \cup \lambda_2)
\]
then \(\lambda^\mathrm{shl}(w_1) = \lambda^\mathrm{shl}(w_2)\) for any \(w_i \in \lambda_i^\circ\). This is an immediate consequence of the following equalities
\[
\{\omega \in \Phi(\lambda) : w_i \in \omega\} = \{\omega : \{w_i\} \cup \lambda \subseteq \omega\} = \{\omega : \lambda_1 \cup \lambda \subseteq \omega\} = \{\omega : \text{cone}(\lambda_1 \cup \lambda) \subseteq \omega\}.
\]

### 3. Examples

In this section we give examples of varieties for which the Mori chamber and the stable base locus decomposition do not coincide, and we analyze this phenomenon from both the geometrical and the combinatorial point of view.

**Example 3.1.** (Birational viewpoint) Consider a plane \(\Pi \subset \mathbb{P}^n\) and five general points \(p_1, \ldots, p_5 \in \Pi\). Let \(f : X \to \mathbb{P}^n\) be the blow-up of \(\mathbb{P}^n\) at \(p_1, \ldots, p_5\) with exceptional divisors \(E_1, \ldots, E_5\). Then the strict transform \(\tilde{\Pi} \subset X\) of \(\Pi\) is a del Pezzo surface of degree four. In particular \(\tilde{\Pi}\) is a Mori dream space.

Let \(e_1, \ldots, e_5\) be the classes of a line in the exceptional divisors, and \(l\) the pull-back a a general line in \(\mathbb{P}^n\). Let \(\tilde{C} \subset X\) be an irreducible curve. If \(\tilde{C}\) gets contracted by \(f\) the \(\tilde{C} \sim m_i e_i\) with \(m_i > 0\) for some \(i \in \{1, \ldots, 5\}\). Otherwise, we may write \(\tilde{C} \sim dl - m_1 e_1 - \cdots - m_5 e_5\), that is \(\tilde{C}\) is the strict transform of a curve \(C \subset \mathbb{P}^n\) of degree \(d\) having multiplicity \(m_i\) at \(p_i\) for \(i = 1, \ldots, 5\).

If \(d < m_1 + \cdots + m_5\) then \(C \subset \Pi\) and \(\tilde{C} \subset \tilde{\Pi}\). In this case we may write \(\tilde{C}\) as a linear combination with non-negative coefficients of \(e_1, \ldots, e_5, l - e_i - e_j, 2l - e_i - \cdots - e_5\) since these are the generators of \(\text{NE}(\tilde{\Pi})\). If \(d \geq m_1 + \cdots + m_5\) then we may write
\[
\tilde{C} \sim m_1(l - e_1) + \cdots + m_5(l - e_5) + (d - m_1 - \cdots - m_5)l
\]
where \(l - e_i = (l - e_i - e_j) + e_j\). Therefore,
\[
\text{NE}(X) = \langle e_i, l - e_i - e_j, 2l - e_i - \cdots - e_5 \rangle
\]
Now, let \(D \subset \mathbb{P}^n\) be the divisor given by the union of \(n - 2\) general hyperplanes containing \(\Pi\). For the strict transform \(\tilde{D} \subset X\) of \(D\) we have \(\tilde{D} \sim (n - 2)(H - E_1 - \cdots - E_5)\), and
\[
-K_X - \epsilon \tilde{D} \sim (n + 1 - \epsilon(n - 2))H - ((n - 1) - \epsilon(n - 2))(E_1 + \cdots E_5)
\]
where \(H\) is the pull-back of the hyperplane section of \(\mathbb{P}^n\) via the blow-up morphism. Now, note that
\[
(-K_X - \epsilon \tilde{D}) \cdot (l - e_i - e_j) = \epsilon(n - 2) - n + 3, \quad (-K_X - \epsilon \tilde{D}) \cdot (2l - e_i - \cdots - e_5) = 3\epsilon(n - 2) - 3n + 7,
\]
yields the fibration \( D \). On the other hand, \( \text{Mov} \) position of \( X \) and \( \text{Exc} \) appear in the expression of \( D \). Note that \( D_1 \) is nef, and since \( D_1^n > 0 \) it is also big. Therefore, \( D_1 \) is semi-ample and big. Now, consider a curve \( C \subset X \)
\[
\tilde{C} \sim \alpha_{1,2}(l - e_1 - e_2) + \cdots + \alpha_{4,5}(l - e_4 - e_5) + \beta \tilde{C}_{1,...,5} + \gamma_1 e_1 + \cdots + \gamma_5 e_5
\]
where \( \tilde{C}_{1,...,5} \sim 2l - e_1 - \cdots - e_5 \). Assume that \( D_1 : \tilde{C} = \gamma_1 + \cdots + \gamma_5 = 0 \). Hence we may write
\[
\tilde{C} \sim (2\beta + \alpha_{1,2} + \cdots + \alpha_{4,5})l - (\beta + \alpha_{1,2} + \cdots + \alpha_{4,5})e_1 - \cdots - (\beta + \alpha_{4,5})e_5
\]
Since \( 2\beta + \alpha_{1,2} + \cdots + \alpha_{4,5} < \beta + \alpha_{1,2} + \cdots + \alpha_{4,5} + \beta + \alpha_{4,5} \) we conclude that \( \tilde{C} \subset \tilde{\Pi} \).

Then, a large enough multiple of \( D_i \) induces a birational morphism \( f_{D_i} : X \to X_i \) contracting \( \tilde{\Pi} \) onto a \( \mathbb{P}^1 \) contained in \( X_i \), and whose exceptional locus is exactly \( \tilde{\Pi} \), that is \( \text{Exc}(f_i) = \tilde{\Pi} \). Indeed, \( D_i|_{\tilde{\Pi}} \) yields the fibration \( \tilde{\Pi} \to \mathbb{P}^1 \) induced by the linear system of conics in \( \Pi \) through \( p_1, \ldots, p_5 \).

Let \( f_{D_i} : X \to X_i \) and \( f_{D_j} : X \to X_j \) be the morphisms induced respectively by \( D_i \) and \( D_j \). From now on we will assume that \( n \geq 4 \) so that \( f_{D_i} \) is a small contraction. Then \( \text{Exc}(f_{D_i}) = \text{Exc}(f_{D_j}) = \tilde{\Pi} \). On the other hand, \( D_i \) and \( D_j \) give rise to two different flops

\[
\begin{array}{ccc}
X^- & \xrightarrow{\chi^-} & X \\
\downarrow & & \downarrow \chi^+ \\
X_i & & X_j
\end{array}
\]

Therefore, \( X^+ \) and \( X^- \) correspond to two different chambers \( C^+ \) and \( C^- \) of the Mori chamber decomposition of \( \text{Mov}(X) \). On the other hand, by Nakamaye’s theorem [Laz04] Theorem 10.3.5] the base locus of \( \chi^+ \) is \( \text{Exc}(f_{D_i}) \) and the base locus of \( \chi^- \) is \( \text{Exc}(f_{D_j}) \). These base loci are respectively the stable base loci \( D_i \) and \( D_j \).

Finally, since \( \text{Exc}(f_{D_i}) = \text{Exc}(f_{D_j}) = \tilde{\Pi} \) we conclude that \( C^- \cup C^+ \) is a unique chamber of the stable base locus decomposition of \( \text{Mov}(X) \).

More generally, for \( i = 1, \ldots, 5 \) we get five different chambers, all of them adjacent to \( \text{Nef}(X) \), of the Mori chamber decomposition of \( \text{Mov}(X) \) whose union gives a single chamber of its stable base locus decomposition. Furthermore, the stable base locus of a divisor in this chamber is exactly the surface \( \tilde{\Pi} \).

**Example 3.2.** (Cox rings viewpoint) Let \( X \) be the toric variety with Cox ring \( \mathcal{R}(X) := K[T_1, \ldots, T_5] \) whose grading matrix and irrelevant ideal are the following
\[
Q = \begin{bmatrix} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & -1 & 1 & -1 \end{bmatrix} \quad \mathcal{J}_{\text{irr}}(X) = \langle T_1, T_4 \rangle \cap \langle T_1, T_5 \rangle \cap \langle T_3, T_5 \rangle
\]
The degrees of the generators are displayed in the following picture together with the semi-ample cone \( \lambda_A \) which is the gray region

![Diagram](image)
The matrix $Q$ defines the following exact sequence where the right hand side $\mathbb{Z}^2$ is identified with the divisor class group of $X$

\[ 0 \to \mathbb{Z}^3 \to \mathbb{Z}^5 \xrightarrow{Q} \mathbb{Z}^2 \to 0 \]

Denote by $\gamma$ the positive quadrant of $\mathbb{Q}^5$. Since the Cox ring is a polynomial ring the $\mathfrak{S}$-faces are all the faces of $\gamma$ and the orbit cones are all their projections in $\mathbb{Q}^2$. It follows that the maximal GIT chambers of $X$ are all the 2-dimensional cones generated by pairs of consecutive rays. Recall that $\lambda_A$ is the GIT chamber corresponding to the semi-ample cone, generated by $w_1, w_5$. The corresponding bunch of orbit cones is

\[ \Phi(\lambda_A) := \{ \text{orbit cones } \omega \text{ such that } \omega^\circ \supset \lambda_A^\circ \} \]

In particular the only orbit cone of $\Phi$ which contains $w_2$ is the whole of $\mathbb{Q}^2$. Denote by $\lambda_{i,j}$ the cone determined by $\omega_i$ and $\omega_j$, and observe that $\text{cone}(\lambda_A \cup \lambda_{2,3}) = \text{cone}(\lambda_A \cup \lambda_{2,4}) = \mathbb{Q}^2$. Thus, by Remark 2.12 we conclude that $\lambda^{\text{ab}}(w_2) = \lambda_{2,3} \cup \lambda_{2,4}$.

**Example 3.3.** (Fans viewpoint) Let $v_1, \ldots, v_5 \in \mathbb{Z}^3$ be a set of vectors which is Gale dual to the set $w_1, \ldots, w_5 \in \mathbb{Z}^2$ in Example 3.2. We can assume $v_1, \ldots, v_5$ to be the five columns of the following matrix

\[
\begin{bmatrix}
1 & 0 & 0 & -1 & -1 \\
0 & 1 & 0 & 1 & 2 \\
0 & 0 & 1 & 1 & 0
\end{bmatrix}
\]

These vectors generate the one dimensional cones of a fan $\Sigma$ whose cones are displayed in the following picture

![Fan Structure](image)

The toric variety $X = X(\Sigma)$ is the same as the one previously defined in Example 3.2. The displayed fan structure correspond to choosing the semi-ample chamber to be the GIT chamber $\lambda_A$ of the previous picture. The following are the fan structures of the toric varieties whose semi-ample cone is respectively $\text{cone}(w_1, w_4)$ and $\text{cone}(w_3, w_5)$

![Fan Structures](image)

**Example 3.4.** (Compactification) Let $Y$ be the toric variety with Cox ring $R := K[T_1, \ldots, T_6]$ whose grading matrix and irrelevant ideal are the following

\[
Q = \begin{bmatrix}
0 & 2 & 2 & 0 & 1 & 1 \\
0 & 3 & 1 & 1 & 0 & 1 \\
1 & 0 & 2 & 0 & 2 & 1
\end{bmatrix}
\]

\[ J_{\text{irr}}(Y) = \langle T_1, T_4 \rangle \cap \langle T_1, T_5 \rangle \cap \langle T_3, T_6 \rangle \cap \langle T_2, T_3, T_6 \rangle \cap \langle T_2, T_4, T_6 \rangle \]

The toric variety $Y$ is a completion of the variety $X$ in Example 3.2 obtained by adding the vector $(1, -4, -2)$ to the primitive generators of the one dimensional cones of $X$. The maximal cones of $Y$ have the following indices: $[1, 2, 3], [2, 3, 4], [2, 4, 5], [1, 2, 6], [2, 5, 6], [1, 3, 6], [3, 4, 6], [4, 5, 6]$. The variety $Y$ is $\mathbb{Q}$-factorial, non-Gorenstein, with $2K_Y$ Cartier. The effective cone of $Y$ has 16 maximal GIT chambers, and the moving cone of $Y$ has 3 maximal chambers. The columns of each of the following matrices generate a maximal GIT chamber of the moving cone, where the first one corresponds to the semi-ample cone

\[
\mathcal{A} = \begin{bmatrix}
1 & 1 & 2 & 2 \\
1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4
\end{bmatrix}
\]

\[ \mathcal{M}_1 = \begin{bmatrix}
1 & 1 & 2 & 2 \\
1 & 1 & 3 & 3 \\
1 & 2 & 2 & 4
\end{bmatrix}
\]

\[ \mathcal{M}_2 = \begin{bmatrix}
1 & 2 & 4 & 6 \\
1 & 1 & 3 & 3 \\
1 & 3 & 4 & 8
\end{bmatrix}
\]
The columns of each of the following matrices generate a maximal GIT chamber of the effective cone of $X$.

\[
\mathcal{C}_1 = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 3 \\ 1 & 0 & 4 \end{bmatrix} \quad \mathcal{C}_2 = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 1 & 0 & 2 \end{bmatrix}
\]

The stable base locus of a divisor which lies in the interior of either $\mathcal{C}_1$ or $\mathcal{C}_2$ is $V(T_2)$. Thus the union of these two chambers is a unique stable base locus chamber. On the other hand, if we denote by $X_i$ the projective toric variety whose semi-ample cone is given by $\mathcal{M}_i$, then from the point of view of $X_i$ the GIT chambers coincide with the stable base locus chambers.

**Example 3.5.** Consider the toric variety $Z = Z(\Lambda)$ with fan $\Lambda \subset \mathbb{Q}^3$ given in the following picture

![Toric Variety Diagram](image)

One can take for instance $v_1, \ldots, v_6$ to be the columns of the following matrix

\[
\begin{bmatrix} -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}
\]

Then $Z$ is a non-complete Mori dream space 3-fold with Picard number three and with Cox ring generated by six free variables with degrees given by the columns $w_1, \ldots, w_6$ of the following matrix

\[
\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 2 & -2 & 0 & -1 & 0 & 1 \end{bmatrix}
\]

The effective cone of $Z$ is the whole of $\mathbb{Q}^3$, it has 14 maximal GIT chambers and six of them are in the movable cone. Let us denote by $Z_j := Z(\lambda_j), j = 1, \ldots, 6$ the different models where $Z = Z_1$, and $\lambda_j, j = 1, \ldots, 6$ are the corresponding GIT chambers. The related fans are the following

![Fan Diagram](image)

and the possible flips are as follows

\[
Z_1 \overset{\eta_1}{\longrightarrow} Z_2 \overset{\eta_2}{\longrightarrow} Z_3 \overset{\eta_4}{\longrightarrow} Z_4 \overset{\eta_3}{\longrightarrow} Z_5 \overset{\eta_5}{\longrightarrow} Z_6
\]

In $Z$ we have that $\lambda_2, \lambda_3, \lambda_4$ are in distinct stable base locus chambers but $\lambda_5$ and $\lambda_6$ are inside the same SBL chamber. More precisely, if $w_j \in \lambda_j^\circ, j = 1, \ldots, 6$ then the stable base loci are

\[
\mathbf{B}(w_1) = \emptyset, \mathbf{B}(w_2) = l_{1,4}, \mathbf{B}(w_3) = l_{1,4} \cup l_{1,5}, \mathbf{B}(w_4) = l_{1,4} \cup l_{2,4}, \mathbf{B}(w_5) = \mathbf{B}(w_6) = l_{1,4} \cup l_{1,5} \cup l_{2,4}
\]
where $l_{i,j}$ is the curve in the intersection of the toric divisors $D_i, D_j$ corresponding to the vectors $v_i, v_j$ in the fan $\Lambda$.

Geometrically we see that the flip $\eta_6$ has $l_{2,5}$ as flipping curve, and $l_{2,5}$ does not exist in $Z_1$. Therefore, the flipping curve in the last flip is not visible from the point of view of $Z_1$.

Note that we can produce an example of a complete 3-fold $W$ with Picard number four such that $\text{MCD}(W) \neq \text{SBLD}(W)$ taking

$$v_8 = \frac{-v_1 \cdots - v_7}{6} = (0, 0, -1)$$

and including the relevant additional cones.

The following will be the leading example in the next section. Indeed, we will produce a complete Mori dream space $X$ of Picard rank two such that $\text{MCD}(X) \neq \text{SBLD}(X)$.

**Example 3.6.** (Fundamental example) Let $Z$ be the toric variety with Cox ring $K[T_1, \ldots, T_{11}]$ whose grading matrix and irrelevant ideal are the following

$$Q = \begin{bmatrix}
1 & 1 & 2 & 2 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1
\end{bmatrix} \quad \mathcal{J}_{irr}(Z) = \langle T_1, T_2 \rangle \cap \langle T_3, \ldots, T_{11} \rangle$$

Denote by $w_i \in \mathbb{Z}^2$ the degree of $T_i$. The following picture displays the degrees of the generators of the Cox ring together with the three maximal GIT chambers of the moving cone, where the shaded one is the ample cone.

Let $\overline{Z} = K^{11}$ be the spectrum of the Cox ring of $Z$ and let $\overline{X}$ be the affine subvariety defined by

$$\overline{X} = \{ F = G = 0 \} \subset \overline{Z}$$

where $F, G$ are general polynomial of degree $(2, 2)$ in the $T_i$. that is general linear combinations of the following monomials

$$\left\{ \begin{array}{l}
T_1^2 T_8^2 \quad T_1^2 T_9^2 \quad T_1^2 T_{10}^2 \quad T_2^2 T_8^2 \quad T_2^2 T_9^2 \quad T_2^2 T_{10}^2 \quad T_2^2 T_{11}^2 \\
T_3 T_8 \quad T_3 T_9 \quad T_3 T_{10} \quad T_3 T_{11} \quad T_4 T_8 \quad T_4 T_9 \quad T_4 T_{10} \quad T_4 T_{11} \\
T_5 T_8 \quad T_5 T_9 \quad T_5 T_{10} \quad T_5 T_{11} \quad T_6 T_8 \quad T_6 T_9 \quad T_6 T_{10} \quad T_6 T_{11}
\end{array} \right\}$$

(3.7)

It is immediate to check that for $F, G$ general enough $\overline{X}$ is irreducible and $\text{codim}_{\overline{X}}(\text{Sing}(\overline{X})) \geq 2$. Since a local complete intersection is Cohen-Macaulay by Serre’s criterion on normality we get that $\overline{X}$ is a normal variety. Let $p_Z : \hat{Z} \to Z$ be the characteristic space morphism of $Z$ and let $\hat{X} := \overline{X} \cap \hat{Z}$. The image of $\hat{X}$ via $p_Z$ is a subvariety $X$ of $Z$. Since $\overline{X}$ is irreducible and normal, and $X$ is a GIT quotient of $\overline{X}$ by a reductive group [Bri10 Theorem 1.24 (vi)] yields that $X$ is irreducible and normal as well. We claim that

$$\hat{X} = \text{Zariski closure of } p_Z^{-1}(X \cap Z') \text{ in } \hat{Z}$$

where $Z'$ is the smooth locus of $Z$. Indeed $Z'$ contains the open subset $Z''$ of $Z$ obtained by removing the union of all the toric subvarieties of the form $p_Z(V(T_i, T_j))$, for any pair of indices $i, j$. Since the Zariski closure of $p_Z^{-1}(X \cap Z'')$ in $\hat{Z}$ equals $\hat{X}$ the claim follows.

Observe that $\text{codim}_{\overline{X}}(\overline{X}, \hat{X}) \geq 2$. Thus if one can show that the pull-back $i^* : \text{Cl}(Z) \to \text{Cl}(X)$, induced by the inclusion, is an isomorphism then by [ADHL15 Corollary 4.1.1.5], it follows that the Cox ring of $X$ is

$$\mathcal{R}(X) = \frac{K[T_1, \ldots, T_{11}]}{I(\overline{X})}$$
Note that each of $w_{1,6}$, $w_{3,7}$ and $w_{7,11}$ is an orbit cone of $X$. In particular, the three maximal chambers $\lambda, \lambda, \lambda_A$ of the moving cone are GIT chambers. On the other hand, since $w_{1,2,7}$ is not an orbit cone, it follows that each orbit cone contains
\[ \{ \omega : \lambda_A \subseteq \omega \text{ and } \lambda' \subseteq \omega \} = \{ \omega : \lambda_A \subseteq \omega \text{ and } \lambda \nsubseteq \omega \} \]

Then $\lambda$ and $\lambda'$ are contained in the same SBL chamber. It remains to show that
\[ i^* : \text{Cl}(Z) \to \text{Cl}(X) \]
is an isomorphism. Note that the $K^* \times K^*$ action on $X \setminus (V(T_2) \cup V(T_8))$ is trivial, so if we remove the images of $V(T_2) \cup V(T_8)$ from $X$, the resulting variety is isomorphic to an affine space. Therefore, $\text{Cl}(X)$ is generated by the classes of the images of the two irreducible divisors $V(T_i) \cap X$, with $i \in \{2,8\}$, and $\rho(X) \leq 2$.

Note that crossing the wall corresponding to $w_1, w_2$ we get a morphism $f : Z \to \mathbb{P}^1$. Furthermore, $X \subset Z$ is not contained in any fiber of $f$, and hence $f$ restricts to a surjective morphism $f|_X : X \to \mathbb{P}^1$. This forces $\rho(X) \geq 2$. Finally, we conclude that the images of $V(T_2), V(T_8)$ form a basis of $\text{Cl}(X)$ and $\rho(X) = 2$.

**Remark 3.8.** All the computations of this section have been implemented in Magma and Maple programs. For convenience of the reader we include, as ancillary files in the arXiv version of the paper, the following files:

- *Readme.txt*, a text on how to use the remaining files;
- *SBLib.m*, the Magma library containing all the functions needed to verify our examples;
- *Examples.txt*, the examples.

For an optimized version, implemented in Maple and Singular, of some of the algorithms presented in this library see [Kei12]. The library *SBLib.m* contains nine commands which we briefly describe here.

- **Ffaces**: computes the $\mathfrak F$-faces of an ideal $I$ of a polynomial ring according to [ADHL15, Remark 3.1.1.11].
- **Eff**: computes the effective cone of a family of vectors in a rational vector space according to [ADHL15, Definition 2.2.2.5]. It takes as input the grading matrix whose columns are the relevant vectors.
- **Mov**: computes the moving cone of a family of vectors in a rational vector space according to [ADHL15, Definition 2.2.2.5]. It takes as input the grading matrix whose columns are the named vectors.
- **OrbitCones**: computes the orbit cones as projections of the $\mathfrak F$-faces according to [ADHL15, Proposition 3.1.1.10]. It takes as input a pair consisting of the set of $\mathfrak F$-faces together with the grading matrix.
- **GitChamber**: computes the GIT chamber defined by a class $w$ according to [ADHL15, Definition 3.1.2.6]. It takes as input a pair consisting of the set of orbit cones together with a class $w$.
- **GitFan**: computes the GIT quasi-fan, that is the collection of all the git cones, of the set of orbit cones according to [Kei12, Algorithm 8]. It takes as input a pair consisting of the set of orbit cones together with a class $w$.
- **BunchCones**: computes a bunch of orbit cones defined by a GIT chamber $\lambda$ according to [ADHL13, Example 3.1.3.6]. It takes as input a pair consisting of the set of orbit cones together with a class $w$ in the relative interior of $\lambda$.
- **SameSbl**: decides whether two classes $w_1, w_2$ have the same stable base locus according to Proposition 2.8. It takes as input a triple consisting of a bunch of orbit cones together with the two classes $w_1$ and $w_2$.
- **FindTriples**: determines all the triples $(\lambda_A, \lambda_1, \lambda_2)$ of GIT chambers such that $\lambda_1$ and $\lambda_2$ are contained in the same stable base locus chamber of the variety whose semiample chamber is $\lambda_A$. It works according to Proposition 2.8 and takes as input a triple consisting of the grading matrix, the set of orbit cones and the GIT fan.
4. Mori dream spaces of Picard rank two

Let $X$ be a Mori dream space with divisor class group $\text{Cl}(X)$ of rank two. Since $X$ is a projective variety its effective cone is pointed. Moreover $\text{Cl}(X)$ has rank two so that we can fix a total order on the classes in the effective cone $w \leq w'$ if $w$ is on the left of $w'$. Given two convex cones $\lambda, \lambda'$ contained in the effective cone we will write

$$\lambda \leq \lambda' \quad \text{if } w \leq w' \text{ for any } w \in \lambda \text{ and } w' \in \lambda'$$

Denote by $\{f_1, \ldots, f_r\}$ a minimal set of homogeneous generators for the Cox ring $\mathcal{R}(X)$ of $X$, and let $w_i = \text{deg}(f_i)$ for any $i$.

**Proposition 4.1.** Let $X$ be a projective $\mathbb{Q}$-factorial toric variety with Picard rank two. Then the Mori chamber and the stable base locus decompositions of $\text{Eff}(X)$ coincide.

**Proof.** Let $\lambda_A$ be the semi-ample cone of $X$ and let $\lambda', \lambda''$ be two distinct maximal GIT chambers of $X$.

According to (2.10) it suffices to show that there exists an orbit cone of the bunch which contains one of $\lambda_A \cup \lambda', \lambda_A \cup \lambda''$ but not the other. Since $X$ is complete the effective cone is pointed, so that, since the Picard rank is two, one can order the GIT chambers of $X$.

Assuming $\lambda' \leq \lambda''$, we have three possibilities: either $\lambda' \leq \lambda_A \leq \lambda''$, or $\lambda_A \leq \lambda' \leq \lambda''$, or $\lambda' \leq \lambda'' \leq \lambda_A$. Since $X$ is toric each pair of degrees of generators of the Cox ring span an orbit cone. Thus in the first two cases we can find an orbit cone which contains $\lambda_A \cup \lambda'$ but not $\lambda_A \cup \lambda''$, while in the last case we can find an orbit cone which contains $\lambda_A \cup \lambda''$ but not $\lambda_A \cup \lambda'$.

The following is our first simple criterion implying the equality of the two chamber decompositions for a $\mathbb{Q}$-factorial Mori dream space of Picard rank two.

**Proposition 4.2.** Let $X$ be a projective $\mathbb{Q}$-factorial Mori dream space with Picard rank two. If all the generators of $\mathcal{R}(X)$ appear in the walls of the stable base locus decomposition of $\text{Eff}(X)$ then the Mori chamber and the stable base locus decompositions of $\text{Eff}(X)$ coincide.

**Proof.** By (2.7) the Mori chamber decomposition is a subdivision of $\text{Eff}(X)$ whose walls are given by some of the generators of $\mathcal{R}(X)$, and it is a refinement of the stable base locus decomposition. Since, by hypothesis all the generators appear as walls of the stable base locus decomposition such a refinement must be trivial.

Now, we develop some technical results in order to describe the semi-stable loci corresponding to the GIT chambers of the Mori chamber decomposition of a Mori dream space of Picard rank two.

**Lemma 4.3.** Let $X$ be a Mori dream space with Picard rank two, $\lambda \subseteq \text{Cl}_\mathbb{Q}(X)$ a maximal GIT chamber of $X$, and $\overline{X}^{\text{ss}}(\lambda)$ the corresponding subset of semi-stable points of $X$. Then the following holds

$$\overline{X} \setminus \overline{X}^{\text{ss}}(\lambda) = V(f_i : w_i \leq \lambda) \cup V(f_i : \lambda \leq w_i)$$

**Proof.** By [ADHL15, Theorem 3.1.2.8] we have $\overline{X}^{\text{ss}}(\lambda) = \overline{X}^{\text{ss}}(w)$ for any $w \in \lambda^\circ$, where the second semi-stable locus is the complement of the zero set of all the homogeneous sections of the Cox ring whose degree is a positive multiple of $w$. If we choose such a class $w \in \lambda^\circ$ so that $w < w_i$ for any $w_i \in \lambda^\circ$ then each monomial in $f_1, \ldots, f_r$ of degree $nw$ must contain at least one $f_i$ with $w_i \leq \lambda$. Thus the inclusion

$$V(f_i : w_i \leq \lambda) \subseteq \overline{X} \setminus \overline{X}^{\text{ss}}(\lambda)$$

follows. The analogous inclusion for $V(f_i : \lambda \leq w_i)$ can be proved in a similar way.

To prove the opposite inclusion observe that if $\overline{x} \in \overline{X}$ is a point which does not belong to the union $V(f_i : w_i \leq \lambda) \cup V(f_i : \lambda \leq w_i)$, then there exist two sections $f_i$, with $w_i \leq \lambda$, and $f_j$, with $\lambda \leq w_j$, each of which does not vanish on $\overline{x}$.

Take non-negative $a, b \in \mathbb{Z}$ such that $aw_i + bw_j \in \lambda^\circ$. Since $f_i^a f_j^b$ is a homogeneous element of the Cox ring of degree $aw_i + bw_j \in \lambda^\circ$ which does not vanish on $\overline{x}$, the point $\overline{x}$ is in $\overline{X}^{\text{ss}}(\lambda)$. 

**Lemma 4.4.** Let $X$ be a Mori dream space with Picard rank two and let $\lambda, \lambda' \subseteq \text{Cl}_\mathbb{Q}(X)$ be two maximal distinct GIT chambers of $X$ with $\lambda \leq \lambda'$. Then the following inclusion is strict

$$V(f_i : w_i \leq \lambda') \subsetneq V(f_i : w_i \leq \lambda)$$
Proof. Assume that the equality $V(f_i : w_i \leq \lambda') = V(f_i : w_i \leq \lambda)$ holds. By hypothesis the inclusion $V(f_i : \lambda \leq w_i) \subseteq V(f_i : \lambda' \leq w_i)$ holds. Thus by Lemma 1.3 there would be an inclusion $\overline{X}^{ss}(\lambda') \subseteq \overline{X}^{ss}(\lambda)$. By [ADHL15, Theorem 3.1.2.8] the latter inclusion would imply $\lambda \subseteq \lambda'$, a contradiction. □

Recall that a Mori dream space $X$ is a good quotient of its characteristic space $\widehat{X} = \overline{X}^{ss}(\lambda_A)$, and denote by $p_{\lambda_A} : \widehat{X} \to X$ the good quotient map. The following simple characterization of stable base loci will be fundamental for the rest of the paper.

Lemma 4.5. If $\lambda \leq \lambda_A$ then the stable base locus of a class $w \in \lambda$ is

\begin{equation}
B(w) = p_{\lambda_A}(\widehat{X} \setminus \overline{X}^{ss}(\lambda)) = p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \leq \lambda))
\end{equation}

If $\lambda \geq \lambda_A$ then the stable base locus of a class $w \in \lambda$ is

\begin{equation}
B(w) = p_{\lambda_A}(\widehat{X} \setminus \overline{X}^{ss}(\lambda)) = p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \geq \lambda))
\end{equation}

Proof. In order to prove (4.6) just note that the first equality holds by definition while the second equality is due to Lemma 4.3 and the fact that $\lambda \leq \lambda_A$. Clearly (4.7) can be proved using a completely analogous argument. □

The following is the main technical tool of the paper.

Theorem 4.8. Let $X = X(\lambda_A)$ be a $\mathbb{Q}$-factorial Mori dream space with Picard rank two corresponding to the maximal chamber $\lambda_A$ of the Mori chamber decomposition of $\text{Eff}(X)$. If for any $\lambda' \leq \lambda \leq \lambda_A$ we have

\begin{equation}
V(f_i : w_i \leq \lambda') \setminus V(f_i : w_i \leq \lambda) \subseteq V(f_i : w_i \leq \lambda') \setminus V(f_i : w_i \geq \lambda_A)
\end{equation}

and for any $\lambda_A \leq \lambda \leq \lambda'$ we have

\begin{equation}
V(f_i : w_i \geq \lambda) \setminus V(f_i : w_i \leq \lambda_A) \subseteq V(f_i : w_i \geq \lambda) \setminus V(f_i : w_i \leq \lambda')
\end{equation}

then the Mori chamber and the stable base locus decompositions of $\text{Eff}(X)$ coincide.

Furthermore, if for any $\lambda \leq \lambda_A \leq \lambda'$ we have that

\begin{equation}
V(f_i : w_i \leq \lambda') \setminus (V(f_i : w_i \leq \lambda_A) \cup V(f_i : w_i \geq \lambda_A))
\end{equation}

is different from

\begin{equation}
V(f_i : w_i \geq \lambda') \setminus (V(f_i : w_i \leq \lambda_A) \cup V(f_i : w_i \geq \lambda_A))
\end{equation}

then the stable base loci of $\text{Eff}(X)$ are convex.

Proof. Let $\lambda$ be a maximal non-ample GIT chamber of $X$. Assume that $\lambda \leq \lambda_A$, where $\lambda_A$ is the ample cone of $X$. By (4.6) in Lemma 4.5 the stable base locus of a class $w \in \lambda$ is

\begin{equation}
B(w) = p_{\lambda_A}(\widehat{X} \setminus \overline{X}^{ss}(\lambda)) = p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \leq \lambda))
\end{equation}

Now let $\lambda' \leq \lambda_A$ be any maximal GIT chamber distinct from $\lambda$ and $\lambda_A$. Without loss of generality we can assume that $\lambda' \leq \lambda$ then by Lemma 4.4 we deduce the following

\begin{align*}
V(f_i : w_i \leq \lambda_A) & \subseteq V(f_i : w_i \leq \lambda) \supseteq V(f_i : w_i \leq \lambda') \\
V(f_i : w_i \geq \lambda_A) & \supseteq V(f_i : w_i \geq \lambda) \subseteq V(f_i : w_i \geq \lambda')
\end{align*}

where all the inclusions are strict. Taking the intersection with $\widehat{X}$ is equivalent to remove from $V(f_i : w_i \leq \lambda) \subseteq V(f_i : w_i \leq \lambda')$ the common subset $V(f_i : w_i \leq \lambda_A)$ and their intersection with $V(f_i : w_i \geq \lambda_A)$. So hypothesis (4.9) yields that

\begin{equation}
\widehat{X} \cap V(f_i : w_i \leq \lambda) \neq \widehat{X} \cap V(f_i : w_i \leq \lambda')
\end{equation}

Since $X$ is $\mathbb{Q}$-factorial, the good quotient $p_{\lambda_A} : \widehat{X} \to X$ is geometric [ADHL15, Corollary 1.6.2.7]. It follows that the images of the above sets via $p_{\lambda_A}$ remain distinct in $X$ and thus that $B(w) \neq B(w')$ for any $w \in \lambda'$ and $w' \in \lambda''$.

Now, assume that $\lambda_A \leq \lambda$. Then (4.7) in Lemma 4.5 yields that the stable base locus of a class $w \in \lambda$ is

\begin{equation}
B(w) = p_{\lambda_A}(\widehat{X} \setminus \overline{X}^{ss}(\lambda)) = p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \geq \lambda))
\end{equation}
In this case if $\lambda'$ is a maximal chamber distinct from $\lambda$ such that $\lambda_A \leq \lambda \leq \lambda'$ Lemma 1.3 yields the following strict inclusions

$$V(f_i : w_i \geq \lambda_A) \subsetneq V(f_i : w_i \geq \lambda) \subsetneq V(f_i : w_i \geq \lambda')$$
$$V(f_i : w_i \leq \lambda_A) \supsetneq V(f_i : w_i \leq \lambda) \supsetneq V(f_i : w_i \leq \lambda')$$

To conclude it is enough to argue as in the previous case using (4.10) instead of (4.9).

Summing up we showed that any pair of distinct GIT chambers lying on the same side of $\lambda_A$ gives two different stable base locus chambers. Therefore, the Mori chamber decomposition of $\text{Eff}(X)$ coincide with its stable base locus decomposition.

Finally, an analogous argument shows that if $\lambda' \leq \lambda_A \leq \lambda'$ and our last hypothesis holds then for any $w \in \lambda$ and $w' \in \lambda'$ we have $B(w) = B(w')$, and hence the stable base locus chambers of $\text{Eff}(X)$ are convex.

**Remark 4.11.** Let us consider the Mori dream space $X$ in Example 3.6. Note that (3.7) yields

$$V(f_i : w_i \geq \lambda_A) = \{T_1 = T_2 = T_3 = T_4 = T_5 = T_6 = T_{11} = 0\}$$
$$V(f_i : w_i \leq \lambda_A) = \{T_5 = T_3 = T_9 = T_{10} = T_{11} = 0\}$$
$$V(f_i : w_i \leq \lambda') = \{T_1 = T_2 = T_8 = T_9 = T_{10} = T_{11} = 0\} \cup \{T_7 = T_8 = T_9 = T_{10} = T_{11} = 0\}$$

and $V(f_i : w_i \geq \lambda_A) \supset \{T_1 = T_2 = T_3 = T_9 = T_{10} = T_{11} = 0\}$ contains a component of the set $V(f_i : w_i \leq \lambda')$.

In what follows we work out some interesting consequences of Theorem 1.8

**Corollary 4.12.** Let $X = X(\lambda_A)$ be a $\mathbb{Q}$-factorial Mori dream space with Picard rank two corresponding to the maximal chamber $\lambda_A$ of the Mori chamber decomposition of $\text{Eff}(X)$. If for any maximal chamber $\lambda$ we have that $V(f_i : w_i \leq \lambda)$ and $V(f_i : w_i \geq \lambda)$ are irreducible then the Mori chamber and the stable base locus decompositions of $\text{Eff}(X)$ coincide.

**Proof.** Without loss of generality we may assume that $\lambda' \leq \lambda \leq \lambda_A$. Since $V(f_i : w_i \leq \lambda')$ is irreducible either $V(f_i : w_i \geq \lambda_A) \supset V(f_i : w_i \leq \lambda')$ or $V(f_i : w_i \geq \lambda_A) \cap V(f_i : w_i \leq \lambda')$ is a closed subset of $V(f_i : w_i \leq \lambda')$.

Assume that $V(f_i : w_i \geq \lambda_A) \supset V(f_i : w_i \leq \lambda')$. Then since $V(f_i : w_i \geq \lambda_A) \subseteq V(f_i : w_i \geq \lambda_A)$ we get that $X^{\text{ss}}(\lambda_A) \subseteq X^{\text{ss}}(\lambda')$, and [ADHL15 Theorem 3.1.2.8] yields that $\lambda' \not\leq \lambda_A$, a contradiction. Therefore, $V(f_i : w_i \geq \lambda_A)$ intersects $V(f_i : w_i \leq \lambda')$ in a closed subset, and to conclude it is enough to apply Theorem 1.8.

Now, we are ready to prove the main result of the paper.

**Theorem 4.13.** Let $X$ be a $\mathbb{Q}$-factorial Mori dream space with Picard rank two, $\{f_1, \ldots, f_r\}$ a minimal set of homogeneous generators for the Cox ring $\mathcal{R}(X)$, $w_i := \deg(f_i)$, and $\lambda_A$ be the ample chamber of $X$. Denote by $c$ the codimension of $X$ into its canonical toric embedding [ADHL15 Section 3.2.5]. Define

$$h^+ := \#\{f_i : w_i \geq \lambda_A\} \quad \text{and} \quad h^- := \#\{f_i : w_i \leq \lambda_A\}$$

If $h^- > c$ and $h^+ > c$, then the Mori chamber and the stable base locus decomposition of $\text{Eff}(X)$ coincide.

**Proof.** Consider $p_{\lambda_A} : \hat{X} \to X$, let $\overline{X}$ be the total space of $X$, and $\overline{\mathbb{A}} \cong \mathbb{A}^r$ be the affine space with coordinates given by the $f_i$. Let $\lambda', \lambda$ be two chambers of the Mori chamber decomposition of $\text{Eff}(X)$ as in the following picture

```
\begin{tikzpicture}
  \draw[->] (0,0) -- (1,1) node [above] {$\lambda$};
  \draw[->] (0,0) -- (-1,1) node [above] {$\lambda'$};
  \draw[->] (0,0) -- (0,1) node [right] {$\lambda_A$};
\end{tikzpicture}
```
Recall that by (4.6) in Lemma 4.5 the stable base loci of classes \( w \in \lambda, w' \in \lambda' \) are given respectively by
\[
\begin{align*}
B(w) &= p_{\lambda_A}(\tilde{X}(\lambda)) = p_{\lambda_A}(\tilde{X} \cap V(f_i : w_i \leq \lambda)) \\
B(w') &= p_{\lambda_A}(\tilde{X}(\lambda')) = p_{\lambda_A}(\tilde{X} \cap V(f_i : w_i \leq \lambda'))
\end{align*}
\]
and the non semi-stable locus of \( \lambda_A \) is
\[
V(f_i : w_i \leq \lambda_A) \cup V(f_i : w_i \geq \lambda_A)
\]
Now, \( \tilde{X} \subset \mathbb{A}^r \) has dimension \( \dim(X) + 2 \), and hence any irreducible component of the intersection \( \tilde{X} \cap V(f_i : w_i \leq \lambda') \) has dimension greater than or equal to \( \dim(X) + 2 - h' \), where \( h' = \# \{ f_i : w_i \leq \lambda' \} \).
Assume that an irreducible component of \( \tilde{X} \cap V(f_i : w_i \leq \lambda') \) is contained in \( \tilde{X} \cap V(f_i : w_i \geq \lambda_A) \).
Then such component must be contained in
\[
V(f_i, f_j : w_i \leq \lambda', w_j \geq \lambda_A)
\]
which has dimension \( r - h' - h^+ \). This forces \( h^+ \leq c \), a contradiction with our hypothesis. Now, to conclude that \( \lambda, \lambda' \) are two different stable base locus chambers it is enough to recall that Lemma 4.4 yields \( V(f_i : w_i \leq \lambda) \not\subset V(f_i : w_i \leq \lambda') \).
When \( \lambda_A \leq \lambda \leq \lambda' \) we argue in a completely analogous way, and then to conclude it is enough to apply Theorem 4.13.

The following is the first immediate consequence of Theorem 4.13.

**Corollary 4.14.** Let \( Z \) be a projective normal \( \mathbb{Q} \)-factorial toric variety with \( \text{rk}(\text{Cl}(Z)) = 2 \), and \( X \subseteq Z \) a projective normal \( \mathbb{Q} \)-factorial Mori dream hypersurface such that \( \iota^* : \text{Cl}(Z) \to \text{Cl}(X) \) is an isomorphism. Then the Mori chamber and the stable base locus decompositions of both \( \text{Eff}(Z) \) and \( \text{Eff}(X) \) coincide.

**Proof.** For a toric variety the claim follows from Proposition 4.11 and it is also an immediate consequence of Theorem 4.13 with \( c = 0 \). In general, following the notation in the proof of Theorem 4.13 there are always at least two generators in the sets \( \{ f_i : w_i \geq \lambda_A \}, \{ f_i : w_i \leq \lambda_A \} \) otherwise \( \lambda_A \) would be a chamber of \( \text{Eff}(X) \setminus \text{Mov}(X) \). Since \( c = \text{codim}_Z(X) = 1 \) we conclude by Theorem 4.13.

**Remark 4.15.** Theorem 4.13 is sharp. Indeed, the Mori dream space in Example 3.6 has three Mori chamber but just two stable base locus chambers. In this example \( h^+ = c = 2 \).

**Remark 4.16.** An intrinsic quadric is a normal \( \mathbb{Q} \)-factorial projective Mori dream space with Cox ring defined by a single quadratic relation. Smooth intrinsic quadrics with small Picard rank have been studied recently in [FHN15]. By Corollary 4.13 the Mori chamber and the stable base locus decomposition of the effective cone of an intrinsic quadric of Picard rank two coincide.

4.16. **Picard rank two varieties with a torus action of complexity one.** Recall that a variety with a torus action of complexity one is a normal complete algebraic variety \( X \) with an effective action of a torus \( T \) such that the biggest \( T \)-orbits are of codimension one in \( X \).

**Proposition 4.17.** Let \( X \) be smooth rational projective variety of Picard rank two that admits a torus action of complexity one. Then the Mori chamber and the stable base locus decomposition of \( \text{Eff}(X) \) coincide.

**Proof.** By [FHN16] Theorem 1.1 any smooth rational projective variety of Picard rank two with a torus action of complexity one, with just one exception, is a Mori dream hypersurface in its canonical toric embedding. Therefore, with the exception of the variety No. 13 in the statement of [FHN16] Theorem 1.1] the claim follows directly from Corollary 4.11.

On the hand, the Cox ring of the exceptional variety \( X \) has eight generators \( T_1, \ldots, T_8 \), with \( \text{deg}(T_1) = \text{deg}(T_3) = \text{deg}(T_5) = \text{deg}(T_7) \), and \( \text{deg}(T_2) = \text{deg}(T_4) = \text{deg}(T_6) = \text{deg}(T_8) \). Therefore, both the Mori chamber and the stable base locus decomposition of \( \text{Eff}(X) \) consist of a single chamber which is indeed the nef cone of \( X \).

In what follows we apply the techniques developed in this section to compute the stable base locus decomposition which by Proposition 4.17 coincide with the Mori chamber decomposition, of the effective cones of the varieties in [FHN16] Theorem 1.1].
Example 4.18. (No. 6 in FHN16 Theorem 1.1) In this case $X$ is a variety of dimension $m + 3$ with Cox ring given by
\[ \mathcal{R}(X) \cong k[T_1, \ldots, T_6, S_1, \ldots, S_m] \]
with $m \geq 1$, and grading matrix
\[
\begin{pmatrix}
0 & 2c + 1 & a & b & c & 1 & 1 & \ldots & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & \ldots & 0
\end{pmatrix}
\]
with $a, b, c \geq 0$, $a < b$ and $a + b = 2c + 1$. Here we develop the case $0 < a < c$, when $a = 0$ or $a = c$ a similar argument will work. Therefore, MCD($X$) is a possibly trivial coarsening of the following decomposition

\[
\begin{array}{c}
\begin{array}{c}
\lambda_1 \\
\vdots \\
\lambda_5
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
w_1 \\
w_3 \\
w_5 \\
w_4 \\
w_2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\lambda_2 \\
\lambda_3 \\
\lambda_4
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
S_1, \ldots, S_m
\end{array}
\end{array}
\]
where $\lambda_1 = \lambda_A$ is the ample cone of $X$. Note that (1.6) in Lemma 4.5 yields
\[
\begin{align*}
B(w) &= p_{\lambda_A}(\hat{X} \cap V(f_i : w_i \leq \lambda_2)) = p_{\lambda_A}(\hat{X} \cap \{T_4 = T_5 = T_3 = T_1 = 0\}) & \text{if } w \in \lambda_2; \\
B(w) &= p_{\lambda_A}(\hat{X} \cap V(f_i : w_i \leq \lambda_3)) = p_{\lambda_A}(\hat{X} \cap \{T_3 = T_1 = T_5 = T_6 = 0\}) & \text{if } w \in \lambda_3; \\
B(w) &= p_{\lambda_A}(\hat{X} \cap V(f_i : w_i \leq \lambda_4)) = p_{\lambda_A}(\hat{X} \cap \{T_1 = T_3 T_4 + T_5^2 T_6 = 0\}) & \text{if } w \in \lambda_4; \\
B(w) &= p_{\lambda_A}(\hat{X} \cap V(f_i : w_i \leq \lambda_5)) = p_{\lambda_A}(\hat{X} \cap \{T_1 = T_3 T_4 + T_5^2 T_6 = 0\}) & \text{if } w \in \lambda_5.
\end{align*}
\]
Therefore, MCD($X$) = SBLD($X$) = \{\lambda_A, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}.

Example 4.19. (No. 8 in FHN16 Theorem 1.1) In this case $X$ is a variety of dimension $m + 3$ with Cox ring given by
\[ \mathcal{R}(X) \cong k[T_1, \ldots, T_6, S_1, \ldots, S_m] \]
with $m \geq 2$, and grading matrix
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \ldots & a_m
\end{pmatrix}
\]
with $0 \leq a_2 \leq \cdots \leq a_m$ and $a_m > 0$. We develop the case $0 < a_2 < \cdots < a_m$, the same argument will work in the remaining cases as well. Therefore, MCD($X$) is a possibly trivial coarsening of the following decomposition

\[
\begin{array}{c}
\begin{array}{c}
\lambda_m \\
\vdots \\
\lambda_1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
w_1, \ldots, w_6 \\
w_{m+1} \\
w_{m+5} \\
w_{m+6}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\lambda_{m-1} \\
\lambda_8 \\
\lambda_7
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
S_1, \ldots, S_m
\end{array}
\end{array}
\]
where $\lambda_m = \lambda_A$ is the ample cone of $X$. Note that (1.7) in Lemma 4.5 yields
\[
\begin{align*}
B(w) &= p_{\lambda_A}(\hat{X} \cap V(f_i : w_i \geq \lambda_j)) = p_{\lambda_A}(\hat{X} \cap \{S_j = \cdots = S_1 = T_1 T_2 + T_3 T_4 + T_5 T_6 = 0\}) & \text{if } w \in \lambda_j, \text{ for } j = 1, \ldots, m - 1. \text{ Therefore, MCD($X$) = SBLD($X$) = \{\lambda_A, \lambda_{m-1}, \ldots, \lambda_1\}.}
\end{align*}
\]
For all the other varieties listed in FHN16 Theorem 1.1, with the exception of the varieties No. 3 and No. 12 for generic parameters, arguing similarly we get that the Mori chamber decomposition of the variety coincide with the one of the ambient toric variety which is given by the corresponding grading matrix in FHN16 Theorem 1.1. In the following we study the two exceptional cases.
Example 4.20. (No. 3 in [FHN16 Theorem 1.1]) In this case $X$ is a 3-fold with Cox ring given by
\[ \mathcal{R}(X) \cong \frac{k[T_1, \ldots, T_6]}{(T_1T_2T_3^2 + T_4T_5 + T_6^2)} \]
with $m \geq 2$, and grading matrix
\[
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 2 - a & a & 1
\end{pmatrix}
\]
with $a \geq 1$. Therefore, in the case $a \geq 3$ the MCD$(X)$ is a possibly trivial coarsening of the following decomposition

\[ w_1, w_2 \]
\[ \lambda_4 \]
\[ \lambda_3 \]
\[ \lambda_2 \]
\[ \lambda_1 \]
\[ w_3 \]
\[ w_4 \]

where $\lambda_4 = \lambda_A$ is the ample cone of $X$. Note that (1.17) in Lemma 1.3 yields
\[
\begin{align*}
B(w) &= p_{\lambda_A}(\hat{X} \cap V(f_i : w_i \geq \lambda_3)) = p_{\lambda_A}(\hat{X} \cap \{T_6 = T_3 = T_4 = 0\}) \quad \text{if } w \in \lambda_3; \\
B(w) &= p_{\lambda_A}(\hat{X} \cap V(f_i : w_i \geq \lambda_2)) = p_{\lambda_A}(\hat{X} \cap \{T_6 = T_3 = T_4 = 0\}) \quad \text{if } w \in \lambda_2; \\
B(w) &= p_{\lambda_A}(\hat{X} \cap V(f_i : w_i \geq \lambda_1)) = p_{\lambda_A}(\hat{X} \cap \{T_4 = 0\}) \quad \text{if } w \in \lambda_1.
\end{align*}
\]

Therefore, MCD$(X) = \text{SBLD}(X) = \{\lambda_A, \lambda_2 \cup \lambda_3, \lambda_1\}$.

In the case $a = 2$, MCD$(X)$ is a possibly trivial coarsening of the following decomposition

\[ w_1, w_2 \]
\[ \lambda_4 \]
\[ \lambda_3 \]
\[ \lambda_2 \]
\[ \lambda_1 \]
\[ w_3 \]
\[ w_4 \]

where $\lambda_4 = \lambda_A$ is the ample cone of $X$. Note that (1.17) in Lemma 1.3 yields
\[
\begin{align*}
B(w) &= p_{\lambda_A}(\hat{X} \cap V(f_i : w_i \geq \lambda_2)) = p_{\lambda_A}(\hat{X} \cap \{T_6 = T_3 = T_4 = 0\}) \quad \text{if } w \in \lambda_2; \\
B(w) &= p_{\lambda_A}(\hat{X} \cap V(f_i : w_i \geq \lambda_3)) = p_{\lambda_A}(\hat{X} \cap \{T_6 = T_3 = T_4 = 0\}) \quad \text{if } w \in \lambda_3.
\end{align*}
\]

Therefore, MCD$(X) = \text{SBLD}(X) = \{\lambda_A, \lambda_3\}$.

In the case $a = 1$, MCD$(X)$ is a possibly trivial coarsening of the following decomposition

\[ w_1, w_2 \]
\[ \lambda_4 \]
\[ \lambda_3 \]
\[ \lambda_2 \]
\[ \lambda_1 \]
\[ w_3 \]
\[ w_4, w_5, w_6 \]

where $\lambda_4 = \lambda_A$ is the ample cone of $X$. Now (1.17) in Lemma 1.3 yields
\[
B(w) = p_{\lambda_A}(\hat{X} \cap V(f_i : w_i \geq \lambda_2)) = p_{\lambda_A}(\hat{X} \cap \{T_3 = 0\}) \quad \text{if } w \in \lambda_2.
\]

Therefore, MCD$(X) = \text{SBLD}(X) = \{\lambda_A, \lambda_2\}$.

Example 4.21. (No. 12 in [FHN16 Theorem 1.1]) In this case $X$ is a variety of dimension $m + 2$ with Cox ring given by
\[ \mathcal{R}(X) \cong \frac{k[T_1, \ldots, T_5, S_1, \ldots, S_m]}{(T_1T_2T_3^2 + T_4T_5 + T_6^2)} \]
with $m \geq 2$, and grading matrix
\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & \ldots & 0 \\
0 & 2c & a & b & c & 1 & \ldots & 1
\end{pmatrix}
\]
with $0 \leq a \leq c \leq b$ and $a + b = 2c$. Therefore, in the case $0 < a < c < b$, MCD($X$) is a possibly trivial coarsening of the following decomposition

$$w_{m+1}, \ldots, w_{m+5}$$

where $\lambda_5 = \lambda_A$ is the ample cone of $X$. Note that, by Lemma 1.3, yields

$$B(w) = p_{\lambda_A}(\hat{X} \cap V(f_i : w_i \geq \lambda_4)) = p_{\lambda_A}(\hat{X} \cap \{T_4 = T_5 = T_3 = T_1 = 0\})$$ if $w \in \lambda_4$;

$$B(w) = p_{\lambda_A}(\hat{X} \cap V(f_i : w_i \geq \lambda_3)) = p_{\lambda_A}(\hat{X} \cap \{T_5 = T_3 = T_1 = 0\})$$ if $w \in \lambda_3$;

$$B(w) = p_{\lambda_A}(\hat{X} \cap V(f_i : w_i \geq \lambda_2)) = p_{\lambda_A}(\hat{X} \cap \{T_3 = T_1 = T_5 = 0\})$$ if $w \in \lambda_2$;

$$B(w) = p_{\lambda_A}(\hat{X} \cap V(f_i : w_i \geq \lambda_1)) = p_{\lambda_A}(\hat{X} \cap \{T_1 = T_3 T_4 + T_2^2 = 0\})$$ if $w \in \lambda_1$.

Therefore, MCD($X$) = SBLD($X$) = $\{\lambda_A, \lambda_1, \lambda_2 \cup \lambda_3, \lambda_4\}$. If there is an equality in any of the inequalities $0 < a < c < b$, then some $w_j$ coincide and the corresponding chambers collapse as in Example 4.20. For instance, if $a = 0$ then $w_1 = w_3$ and the chamber $\lambda_1$ does not exist.

5. **Grassmannians blow-ups**

Let $G(r, n)$ be the Grassmannian parametrizing $r$-planes in $\mathbb{P}^n$, and $G(r, n)_k$ the blow-up of $\mathbb{P}^n$ at $k$ general points. These blow-ups have been studied in [MR18]. In particular the stable base locus decomposition given in [MR18, Theorem 1.3] is the Mori chamber decomposition of $\text{chamber}$ for each Grassmannian $G$ that is not $G$-reductive affine algebraic group $G$.

For instance, any toric variety is a spherical variety with $\Lambda$ : $\text{set}$, and as a consequence of Proposition 4.2 we will answer positively to [MR18] Question 6.9 which ask whether the decomposition given in [MR18, Theorem 1.3] is the Mori chamber decomposition of $\text{Eff}(G(r, n)_1)$.

**Definition 5.1.** A **spherical variety** is a normal variety $X$ together with an action of a connected reductive affine algebraic group $G$, a Borel subgroup $B \subseteq G$, and a base point $x_0 \in X$ such that the $B$-orbit of $x_0$ in $X$ is a dense open subset of $X$.

Let $(X, G, B, x_0)$ be a spherical variety. We distinguish two types of $B$-invariant prime divisors: a **boundary divisor** of $X$ is a $B$-invariant prime divisor on $X$, a **color** of $X$ is a $B$-invariant prime divisor that is not $G$-invariant.

For instance, any toric variety is a spherical variety with $B = G$ equal to the torus. For a toric variety there are no colors, and the boundary divisors are the usual toric invariant divisors.

Set $\Lambda := \{I \subset \{0, \ldots, n\}, |I| = r + 1\}$ and $N := \Lambda + 1$. Define the **Hamming distance** on $\Lambda$ as

$$d(I, J) = |I| - |I \cap J| = |J| - |I \cap J|$$

for each $I, J \in \Lambda$. Note that, with respect to this distance, the diameter of $\Lambda$ is $r + 1$. We consider the Grassmannian $G(r, n)$ in the usual Plücker embedding $G(r, n) \subset \mathbb{P}^N$.

For each pair $I = \{i_0 < \cdots < i_{r-1}\}, J = \{j_0 < \cdots < j_{r+1}\} \subset \{0, \ldots, n\}$ with $|I| = r, |J| = r + 2$, define a quadratic polynomial

$$F_{IJ} = \sum_{t=0}^{r+1} (-1)^t p_{i_0 \cdots i_{r-1} j_t} p_{j_1 \cdots j_{r+1}}$$

Then the ideal of $K[p_I, I \in \Lambda]$ generated by the $F_{IJ}$ is the ideal defining $G(r, n) \subset \mathbb{P}^N$ [Sha13, Section I.4]. We denote by $G(r, n)_1$ the blow-up of $G(r, n)$ at $p = \langle e_0, \ldots, e_r \rangle$, where $\{e_0, \ldots, e_n\}$ is the canonical basis of $K^{n+1}$, by $H$ the pull-back to $G(r, n)_1$ of the hyperplane section of $\mathbb{P}^N$, and by $E$ the exceptional divisor of the blow-up.
Proposition 5.3. In the polynomial ring $K[S, T_I, I \in \Lambda]$ consider the ideal $\mathfrak{I}$ generated by the Plücker relations \([52]\) in the coordinates $T_I$. Then

$$R(\mathcal{G}(r,n)_1) \cong \frac{K[S, T_I, I \in \Lambda]}{\mathfrak{I}}$$

and the degree of the variable $T_I$ in $Cl(\mathcal{G}(r,n)_1) = \mathbb{Z}[H] + \mathbb{Z}[E]$ is $(1, -d(I, \{0, \ldots, r\}))$, where $deg(S) = (0,1)$.

Proof. By [MR18, Proposition 4.1] under the action of the reductive group

the blow-up $\mathcal{G}(r,n)_1$ is a spherical variety. We consider the Borel subgroup $B \subset G$ of matrices with upper triangular blocks. Consider the divisors $D_0, \ldots, D_{r+1}$ in $\mathcal{G}(r,n)$ defined as $D_j := \{[\Sigma] \in \mathcal{G}(r,n) : \Sigma \cap \Gamma_j \neq \emptyset\}$, where

$$\Gamma_0 = \langle e_{r+1}, \ldots, e_n \rangle; \quad \Gamma_1 = \langle e_0, e_{r+1}, \ldots, e_{n-1} \rangle; \quad \vdots \quad \Gamma_{r+1} = \langle e_0, \ldots, e_r, e_{r+1}, \ldots, e_{n-r} \rangle; \quad \Gamma' = \langle 0, \ldots, e_r \rangle.$$

Pulling-back these divisors via the blow-up map we obtain divisors in $\mathcal{G}(r,n)_1$. For sake of simplicity we will use the same notation for divisors in $\mathcal{G}(r,n)$ and their pull-backs in $\mathcal{G}(r,n)_1$.

Now, note that $\mathcal{G}$ preserves the dimension of the intersection of a given subspace of $\mathbb{P}^n$ with $\Gamma_0$ and with $\Gamma'$. Therefore $\mathcal{G} \cdot D_0 = D_0$ and $\mathcal{G} \cdot E = E$ that, $D_0$ and $E$ are boundary divisors. Note also that each $D_j$ is a $B$-invariant but not a $G$-invariant prime divisor, and therefore $D_1, \ldots, D_{r+1}$ are colors.

In order to determine the $G$-orbit of $D_1, \ldots, D_{r+1}$ we have to describe these divisors explicitly as zeros of the Plücker polynomials in the Plücker coordinates.

In $\mathcal{G}(r,n)$ the divisor $D_0$ is given by $D_0 = V(p_{0,\ldots,r})$. Indeed, if $q \in D_0$ then $q = [\Sigma]$ with $\Sigma \cap \Gamma_0 \neq \emptyset$ and therefore it can be represented with a matrix whose first row is of the form $(0, \ldots, 0, a_{r+1}, \ldots, a_n)$.

This implies that $p_{0,\ldots,r}(q) = 0$. Conversely, if $p_{0,\ldots,r}(q) = 0$ then the most left $(r+1) \times (r+1)$ submatrix of any matrix representing $q$ has zero determinant, therefore there is another representation of $q$ such that the first row has the following form $(0, \ldots, 0, a_{r+1}, \ldots, a_n)$, and we conclude that $q \in D_0$.

Similarly, setting $I_j = \{e_0, \ldots, e_{j-1}, e_{r+1}, \ldots, e_{n-j}\}$ for $j = 0, \ldots, r+1$, we have $D_j = V(p_{I_j})$. Note that $d(I_j, I_0) = j$ for any $j$. More generally, one can consider the prime divisor $D_I := V(p_I)$ for any $I \in \Lambda$. We claim that the linear span of the orbit $G \cdot D_I$ is given by

$$\dim (G \cdot D_I) = \langle \langle D_j; d(J, I_0) = j \rangle, j = 0, \ldots, r+1 \rangle$$

(5.4)

Note that given $I, I' \in \Lambda$ with distance one and such that the non shared indexes, say $i \in I \setminus I'$, $i' \in I' \setminus I$ are in $\{0, \ldots, r\}$ we can find a $g \in \mathcal{G}$ such that $g(e_i) = e_{i'}$ and $g(e_j) = e_j$ for $j \neq i, i'$. The same holds if the non shared indexes are in $\{r+1, \ldots, n\}$, and we get that

$$\langle D_j; d(J, I_0) = j \rangle \subset G \cdot D_I, j = 0, \ldots, r+1$$

Since the $D_I$ such that $d(J, I_0) = j, j = 0, \ldots, r+1$, give a generating set of $H^0(\mathcal{G}(r,n), O_{\mathcal{G}(r,n)}(1))$ we get (5.4). Now, let $S$ and $T_I$ be the canonical sections associated respectively to $E$ and $D_I$. By [ADHL15, Theorem 4.5.4.6] $S, T_I, I \in \Lambda$ are generators of $R(\mathcal{G}(r,n)_1)$. Furthermore, by [Ris17, Lemma 7.2.1] for any $I \in \Lambda$ we have that

$$\text{mult}_{\langle e_0, \ldots, e_r \rangle} D_I = 1 + \dim (\langle e_i, i \notin I \rangle \cap \langle e_0, \ldots, e_r \rangle) = |\{\{0, \ldots, n\} \setminus I \cap \{0, \ldots, r\}\}| = d(I, I_0)$$

Therefore, if $\deg(S) = (0,1)$ the degree of the other generators of $R(\mathcal{G}(r,n)_1)$ in $\text{Pic}(\mathcal{G}(r,n)_1) = \mathbb{Z}[H] \oplus \mathbb{Z}[E]$ is given by $\deg(T_I) = (1, -d(I, I_0))$.

The matrix representing this grading has size $2 \times (N + 1)$ and is of the following form

$$A = \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 & 1 & \ldots & 1 \\ 1 & 0 & -1 & \ldots & -1 & -2 & \ldots & -(r+1) \end{pmatrix}$$
Our next aim is to find relations among the generators of \( \mathcal{R}(G(r,n)_1) \). Note that for each pair \( I = \{i_0 < \cdots < i_{r-1}\}, J = \{j_0 < \cdots < j_{r+1}\} \subset \{0, \ldots, n\} \) with \( |I| = r, |J| = r + 2 \), the polynomial

\[
G_{IJ} = \sum_{t=0}^{r+1} (-1)^t T_{i_0 \ldots i_{r-1} j_t} T_{j_{r+1} \ldots j_{r+1}}
\]

is homogeneous of degree \( 2, -|I \setminus J_0| - |J \setminus J_0| \). Let \( \mathcal{J} \subset K[T_I, I \in \Lambda] \) be the ideal generated by the \( G_{IJ} \). Since \( \frac{K[T_I, I \in \Lambda]}{\mathcal{J}} \) is the homogeneous coordinate ring of \( G(r,n) \), then

\[
\dim(K[T_I, I \in \Lambda]/\mathcal{J}) = (r + 1)(n - r) + 1
\]

and Remark 2.4 yields

\[
\dim(K[S, T_I, I \in \Lambda]/\mathcal{J}) = (r + 1)(n - r) + 2 = \dim(G(r,n)_1) + \text{rank}(\text{Pic}(G(r,n)_1)) = \dim(\mathcal{R}(G(r,n)_1))
\]

where we denote by \( \mathcal{J} \) the ideal generated by the polynomials \( G_{IJ} \) in \( K[S, T_I, I \in \Lambda] \). We conclude that there are no further relations in \( \mathcal{R}(G(r,n)_1) \) and hence \( \mathcal{R}(G(r,n)_1) = \frac{K[S, T_I, I \in \Lambda]}{\mathcal{J}} \) as claimed. \( \square \)

Now, we are ready to compute the Mori chamber decomposition of \( \text{Eff}(G(r,n)_1) \).

**Proposition 5.5.** Let \( G(r,n)_1 \) be the blow-up of the Grassmannian \( G(r,n) \) at a point. Then we have \( \text{Eff}(G(r,n)_1) = \langle E, H - (r+1)E \rangle \), \( \text{Nef}(G(r,n)_1) = \langle H, H - E \rangle \) and

\[
\text{Mov}(G(r,n)_1) = \begin{cases} 
\langle H, H - rE \rangle & \text{if } n = 2r + 1; \\
\langle H, H - (r+1)E \rangle & \text{if } n > 2r + 1.
\end{cases}
\]

Furthermore, \( \text{MCD}(G(r,n)_1) \) and \( \text{SBLD}(G(r,n)_1) \) coincide and their walls are given by the divisors \( E, H, H - E, \ldots, H - (r+1)E \) as represented in the following picture

\[
\begin{array}{c}
\text{E} \\
\text{C}_{-1} \\
\text{C}_0 = \text{Nef}(G(r,n)_1) = \langle H, H - E \rangle, \\
\text{C}_i = (H - iE, H - (i+1)E) \text{ for } i = 1, \ldots, r, \\
\text{H - (r+1)E} \\
\end{array}
\]

where with the notation \( C_i = (H - iE, H - (i+1)E) \) we mean that the ray spanned by \( H - (i+1)E \) belongs to \( C_i \) but the ray spanned by \( H - iE \) does not, and similarly with the notation \( C_{-1} = \{E, H\} \) we mean that the ray spanned by \( E \) belongs to \( C_{-1} \) but the ray spanned by \( H \) does not.

**Proof.** The claims on the effective, nef and movable cones follow from [MRTS] Theorem 1.3. Furthermore, by [MRTS] Theorem 1.3 the decomposition displayed in the statement is the stable base locus decomposition of \( \text{Eff}(G(r,n)_1) \). Now, by Proposition 5.3 all the generators of \( \mathcal{R}(G(r,n)_1) \) appear in the walls of the stable base locus decomposition of \( \text{Eff}(G(r,n)_1) \), and then Proposition 4.2 yields that the Mori chamber and the stable base locus decomposition of \( \text{Eff}(G(r,n)_1) \) coincide. \( \square \)

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