Linearized gravity in flat braneworlds with anisotropic brane tension

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Abstract

We study the four-dimensional gravitational fluctuation on anisotropic brane tension embedded in braneworlds with vanishing bulk cosmological constant. In this setup, warp factors have two types (A and B) and we point out that the two types correspond to positive and negative tension brane, respectively. We show that volcano potential in the model of type A has singularity and the usual Newton’s law is reproduced by the existence of normalizable zero mode. While, in the case of type B, the effective Planck scale is infinite so that there is no normalizable zero mode.

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Motivated by the Randall and Sundrum model \cite{1, 2}, the localization of the four-dimensional gravity is widely investigated in the framework of warped braneworlds. In the original Randall-Sundrum braneworld of $AdS_5$ with an infinite extra dimension, massless gravity is trapped on the brane and massive gravity has continuous spectrum, consequently, the usual four-dimensional Newton's law can be seen by observer at large distance scale. Following remarkable development of $AdS/CFT$ correspondence \cite{3, 4, 5} that the gravitational theories in $AdS$ space are dual to conformal field theories on the boundary, the Randall-Sundrum model becomes attractive model in particle physics and cosmology. Moreover there have been many proposals to show how four-dimensional gravity emerges in warped braneworld with infinite extra dimensions \cite{6, 7, 8, 9}. For example, the cases of $dS_4$ brane and $AdS_4$ brane embedded in $AdS_5$ are investigated in detail \cite{10}. In the case of $dS_4$ brane, a massless bound state gravity exists. While in the case of $AdS_4$ brane, the massless bound state gravity don’t exist and massive gravity has discrete spectrum because the volcano potential blows up at the boundary as if box-like potential. Thus the behavior of gravity is governed by the volcano potential in Schrödinger equation for gravitational fluctuation.

In this letter, we consider the braneworlds with vanishing cosmological constant in contrast with the Randall-Sundrum model with negative bulk cosmological contrast. We study the fluctuation of gravity on the brane with anisotropic brane tension embedded in this braneworld, and show whether the usual Newton’s law is reproduced or not by investigating the volcano potential of this setup.

We consider a single $(3 + n)$-brane embedded in the $(5 + n)$-dimensional world. The brane is located at $y = 0$, where the $y$-direction is assumed to have $Z_2$ symmetry. Here the ansatz for $(5 + n)$-dimensional metric is written in the following form \cite{11}

$$
\begin{align*}
\mathit{ds}^2 &= a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu + c^2(y)\sum_{i=1}^n dz_i^2 + dy^2
\equiv g_{MN}dx^M dx^N.
\end{align*}
$$

Note that the usual four-dimensional spacetime preserves Poincâre invariance and the warp factor of four-dimensional spacetime is different from that of the extra $n$ dimensional space. It is assumed that $z$-directions are compactified in the same radius $R \sim M_{\text{Pl}}^{-1}$, where $M_{\text{Pl}}$ is Planck scale. We give Einstein equation of this setup, $\mathcal{R}_{MN} - \frac{1}{2}g_{MN}\mathcal{R} = -\kappa^2\Lambda g_{MN} - \kappa^2 T_{MN}$, where $\Lambda$ is the bulk cosmological constant and $T_{MN}$ is the energy-momentum tensor of the brane, and $1/\kappa^2$ is the higher dimensional gravitational constant which has mass dimension $3 + n$. Here it is assumed that the distribution of the brane tension of the brane with $(4 + n)$-dimensional spacetime is anisotropic and that the effects of matters on the brane are neglected \cite{11}. Hence the energy-momentum tensor of brane is given by

$$
T^M_N = \delta(y) \text{diag}(V,V,V,V^*,\cdots,V^*,0),
$$

2
where $V$ and $V^*$ represent the brane tension of four-dimensional spacetime and the one of extra $n$-dimensional space, respectively.

For $\Lambda = 0$, solving Einstein equation with metric of Eq.(1), warp factors have two types as follows \[11\]

**Type A**

$$a(y) = \left(1 - \left|\frac{y}{d}\right|^k\right), \quad c(y) = \left(1 - \left|\frac{y}{d}\right|^l\right),$$

and

**Type B**

$$a(y) = \left(1 + \left|\frac{y}{d}\right|^k\right), \quad c(y) = \left(1 + \left|\frac{y}{d}\right|^l\right),$$

where $d$ is the positive constant and the warp factors are normalized to be unity at $y = 0$. For both types, $k$ and $l$ must be determined by the following equations

$$4k + nl = 1, \quad 4k^2 + nl^2 = 1.$$  \hspace{1cm} (5)

Note that the equations for the exponents are closely similar to the Kasner solutions appearing in the higher dimensional cosmology. The jump condition at $y = 0$ due to the delta-function leads to the relation between brane tensions as follows

$$\frac{V}{1 - k} = \frac{V^*}{1 - l} = \pm \frac{2}{\kappa^2 d},$$

where $+(-)$ corresponds to the type A(B). From the above equation, due to positive constant $d$, type A and type B correspond to positive tension branes and negative tension branes, respectively. This situation is similar to the Randall-Sundrum model with single brane because $- (+)$ in warp factor $e^{\mp 2|y|/\rho}$ obtained in the model corresponds to positive (negative) tension brane, where $\rho$ is the radius of $AdS$. From Eq.(6), anisotropic brane tensions are related through the number of $n$. Solving Eq.(6), we have

$$k = k_{1,2} = \frac{2 \pm \sqrt{n(n+3)}}{2(n+4)}, \quad l = l_{1,2} = \frac{n \mp 2\sqrt{n(n+3)}}{n(n+4)},$$

where the subscript 1 and 2 correspond to the choice of taking upper sign and lower sign, respectively. Thus the forms of warp factors depend on both $n$ and the choice of sign. From Eq.(7), for arbitrary positive integer $n \geq 1$, always $k_{1,2} < 1$ and $l_{1,2} < 1$.

To see the four-dimensional gravity on the brane, the fluctuation around the background metric is given by $a^2(y)\eta_{\mu\nu} + h_{\mu\nu}(x,y)$. Here the dependence of $z$-directions in $h_{\mu\nu}$ are neglected. Although the gravity in the bulk has massive KK-modes along $n$ compactified $z$-directions with Planck size, these dimensions are invisible at low energy. Namely, at large distance, it is considered that the fluctuations of $z$-directions are negligible at this stage. Later we discuss this point.
Figure 1: Volcano potential of type A

Setting $h_{\mu\nu}(x, y) = h_{\mu\nu}(x)e^{-n/2}(y)\hat{\psi}(y)$, the linearized equation of motion for transverse-traceless mode is given by

$$
\frac{d^2}{dy^2}\hat{\psi}(y) + \left[ \frac{m^2}{a^2} - \frac{8a''}{a} + 4 \left( \frac{a'}{a} \right)^2 - \frac{5}{2}n \frac{c''}{c} - \frac{1}{4}n(n-2) \left( \frac{c'}{c} \right)^2 
\right.
- 2\kappa^2V\delta(y) \left. \right] \hat{\psi}(y) = 0.
$$

Performing the change of variables, $\hat{\psi}(y) = \tilde{\psi}(u)(1 \mp |y|/d)^{1-k}$ and $(1 \mp |y|/d)^{1-k} = 1 \mp (1-k)|u|/d$, we can obtain familiar non-relativistic quantum mechanics problem as follows,

$$
\left[ -\frac{d^2}{du^2} + V(u) \right] \tilde{\psi}(u) = m^2\tilde{\psi}(u),
$$

where $m^2$ is the four-dimensional mass. The potential $V(u)$ in Schrödinger equation is so-called the volcano potential which can be written in terms of $k$ using Eq.(5)

$$
V(u) = -\frac{1}{4} \left( \frac{d}{1-k} \mp |u| \right)^2 \mp \frac{1-k}{d} \delta(u).
$$

Here upper sign (lower sign) corresponds to the type A(B) as shown in Figure.1(2). In the volcano potential of both types, the eigenvalue becomes positive definite value, $m^2 \geq 0$. This is because the bulk equation of Eq.(5) is expressed as $Q^\dagger Q\tilde{\psi} = m^2\tilde{\psi}$, where $Q = d/du \pm 2^{-1}(d/(1-k) \mp u)^{-1}$. Thus since there are no tachyon modes, the gravitational background becomes stable. Due to $k < 1$ for positive integer $n$, the delta function of type A becomes potential of attractive force via positive tension brane and the delta function of type B is potential of repulsive force via negative tension brane. In the case of type A shown in Figure.1, the $u$ coordinate runs from $-d/(1-k)$ to $d/(1-k)$, both side of the volcano potential goes negative infinity at $u = \pm d/(1-k)$. If we take the upper sign in Eq.(7), $k_1 > 0$ and $l_1 < 0$, then $c \to \infty$ at $y = \pm d$ (this implies that $u = \pm d/(1-k)$). On the other hand, taking the
lower sign, \( k_2 < 0 \) and \( l_2 > 0 \), \( a \to \infty \) at the point. This implies that the singularity at \( y = d \) becomes curvature singularity. However, in this case of \( n = 1, k_2 = 0 \) and \( l_2 = 1 \), the warped metrics have no singularity. In the case of type B shown in Figure 2, the \( u \) coordinate runs from \( -\infty \) to \( +\infty \), \( |V(u)| \to 0 \) for \( |u| \to \infty \). For arbitrary \( n \), the warped metrics have no singularity.

We solve the Schrödinger equation of linearized gravity with volcano potential for type A and B. For zero mode wavefunction, we obtain

\[
\tilde{\psi}_0(u) \propto \left( \frac{d}{1-k} \mp |u| \right)^{\frac{1}{2}}. 
\]

where the jump condition of delta function is imposed. The wavefunction of the type A (upper sign) is normalizable over limited range \( |u| \leq d/(1-k) \). On the other hand, the wavefunction of the type B (lower sign) is non-normalizable because of \( \tilde{\psi}_0(u) \to \infty \) for \( |u| \to \infty \). Namely, in the case of type B, the normalizable zero mode wavefunction cannot exist. In the case of type A, the normalizable zero mode wavefunction is given by

\[
\tilde{\psi}_0(u) = \left( \frac{d}{1-k} \mp |u| \right)^{\frac{1}{2}}. 
\]

For massive mode wavefunction \( m^2 > 0 \), from Eqs. (11) and (12), the wavefunction of the gravity is written in terms of the superposition of Bessel functions as follows,

\[
\tilde{\psi}_m^\pm(u) = N_m^\pm \left( \frac{d}{1-k} \mp |u| \right)^{1/2} \times \left\{ \begin{array}{c}
Y_i \left( \frac{md}{1-k} \right) J_0 \left( m \left( \frac{d}{1-k} \mp |u| \right) \right) + Y_0 \left( m \left( \frac{d}{1-k} \mp |u| \right) \right) \\
J_i \left( \frac{md}{1-k} \right) \end{array} \right\}, 
\]

where upper sign (lower sign) corresponds to the type A(B) and \( N_m^\pm \) is normalization factor, and the jump condition of delta function is imposed. Thus massive modes
are continuous, namely, these aren’t bound states. From Eq.(13), the normalization factor of type A is expressed as

$$N_m^- = \left[ 2 \int_0^{u_0} dt \left\{ -\frac{Y_1(mu_0)}{J_1(mu_0)}J_0(t) + Y_0(t) \right\} \right]^{1/2},$$

(14)

where $u_0 = d/(1 - k)$. The normalization factor of type B is given by

$$N_m^+ = \frac{\sqrt{m} J_1(mu_0)}{\sqrt{J_1^2(mu_0) + Y_1^2(mu_0)}}.$$

(15)

The difference between the normalization factor of type A and B comes from the range of $u$-coordinate.

We are interested in the realization of the four-dimensional Newton’s law in the framework of model considered here. In the type A, the gravitational potential via zero mode is given by $V \propto G_N/r$ at large distance scale, where $G_N \sim \kappa^2 |\tilde{\psi}(0)|^2/R^n = \kappa^2(1 - k)/(R^nd)$. The contribution of massive mode gives small correction to the potential, it is given by $\int_0^{\infty} dm |\tilde{\psi}_m(0)|^2e^{-mr}/r$. In the type B, since there is no normalizable zero mode, the effective four-dimensional Planck scale is infinite. Thus the four-dimensional Newton’s law cannot be reproduced.

As mentioned previously, the $z$-dependence of wavefunction $h_{\mu\nu}(x, y)$ is neglected here. If the $z$-dependence is imposed, the wavefunctions of $z$-directions $\varphi(z)$ become plane waves with KK-mode expansion, namely, $\varphi(z) \sim \sum_p e^{ip_jz_j}/R^j$, where $p_j \in \mathbb{Z}$ and $j = 1, \cdots, n$. Furthermore, the eigenvalue of Eq. (9) must be replaced in $m^2 \to m^2 + \sum |p|^2/R^2$, thus the eigenvalues have both continuous modes and discrete KK-modes. Consequently, the volcano potential of Eq. (10) is not modified even if the fluctuations of $z$-directions are taken into account. Although the corrections to Newton’s law via massive modes have the contributions of discrete KK-modes, the contributions are sufficiently suppressed at large distance $r \gg M_{pl}^{-1}$.

In conclusion, we considered a single $(3 + n)$-brane embedded in $(5 + n)$-dimensional world with vanishing cosmological constant. It assumed that the brane has anisotropic brane tension and that the warp factor of four-dimensional space-time $a(y)$ is different from one of $n$-dimensional spaces $c(y)$. In this setup, warp factors have two types (A and B) and we pointed out that the type A and the type B correspond to the positive tension brane and the negative tension brane, respectively. The volcano potential of the type A has singularity, $y$-coordinate is effectively truncated. At this stage we don’t have quantitative to say about the physical implications of this singularity, we will describe it elsewhere. Investigating the linearized gravity around background metric, there is a normalizable zero mode wavefunction, consequently, the usual four-dimensional gravity is reproduced. On the other hand, the volcano potential of type B has no singularity, and it is shown that there is no normalizable zero mode. Since the effective four-dimensional Planck scale is infinite, the usual four-dimensional gravity cannot be reproduced. In contrast with
Randall-Sundrum model, we proposed a simple model in which the localization of
gravity is realized in braneworld with not $\Lambda < 0$ but $\Lambda = 0$, where $\Lambda$ is the bulk
cosmological constant. Thus the setup with warped geometry is completely different
from the setup with non-warped geometry Finally we describe a comment. Since the
setup considered here doesn’t include the fluctuations of gravity corresponding to
all coordinates, four-dimensional tensor structure cannot be explicitly reproduced.
Furthermore these fluctuations play an important role in brane stabilization. In the
future this point will be investigated elsewhere.

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