Orientation of good covers

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Abstract

We study systems of orientations on triples that satisfy the following so-called interiority condition:
\[ \circlearrowleft(ABD) = \circlearrowleft(BCD) = \circlearrowleft(CAD) = 1 \] implies \[ \circlearrowleft(ABC) = 1 \] for any \( A, B, C, D \). We call such an orientation a P3O (partial 3-order), a natural generalization of a poset, that has several interesting special cases. For example, the order type of a planar point set (that can have collinear triples) is a P3O; we denote a P3O realizable by points as \( p\text{-}P3O \).

If we do not allow \[ \circlearrowleft(ABC) = 0 \], we obtain a T3O (total 3-order). Contrary to linear orders, a T3O can have a rich structure. A T3O realizable by points, a \( p\text{-}T3O \), is the order type of a point set in general position.

In \cite{1} we defined a 3-order on pairwise intersecting convex sets; such a P3O is called a C-P3O. In this paper we extend this 3-order to pairwise intersecting good covers; such a P3O is called a GC-P3O. If we do not allow \( \circlearrowleft(ABC) = 0 \), we obtain a C-T3O and a GC-T3O, respectively.

The main result of this paper is that there is a \( p\text{-}T3O \) that is not a GC-T3O, implying also that it is not a C-T3O—this latter problem was left open in our earlier paper. Our proof involves several combinatorial and geometric observations that can be of independent interest. Along the way, we define several further special families of GC-T3O’s.

1 Introduction

Given some base set, a mapping \( \circlearrowleft \) from its ordered triples to \( \{\pm 1, 0\} \) is a \textit{partial orientation} if
\[ \circlearrowleft(ABC) = \circlearrowleft(CAB) = \circlearrowleft(BCA) = - \circlearrowleft(ACB) = - \circlearrowleft(BAC) = - \circlearrowleft(CBA) \] for every \( A, B, C \).

If \( \circlearrowleft \) never takes zero, then \( \circlearrowleft \) is a \textit{total orientation}. An orientation satisfies the \textit{interiority condition} if
\[ \circlearrowleft(ABD) = \circlearrowleft(BCD) = \circlearrowleft(CAD) = 1 \] implies \[ \circlearrowleft(ABC) = 1 \] for every \( A, B, C, D \).

If \( \circlearrowleft(ABD) = \circlearrowleft(BCD) = \circlearrowleft(CAD) = 1 \) or \( \circlearrowleft(ABD) = \circlearrowleft(BCD) = \circlearrowleft(CAD) = -1 \) for some \( A, B, C, D \), then we write \( D \in \text{conv}(ABC) \). See Figure 1(a). (In the definition of \( \text{conv}(ABC) \) the order of \( A, B, C \) is not relevant.)

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A total orientation that satisfies the interiority condition is a T3O (total 3-order), and a partial orientation is a P3O (partial 3-order). The notion T3O was introduced by Knuth [8] under the name *interior triple system*, according to Knuth “for want of a better name.” He noted that the orientation of a planar point set in general position is a T3O, while if we allow collinearity, it is a P3O. We say that a T3O (resp. P3O) that has a realization by a planar set of points is a p-T3O (resp. p-P3O). We denote the family of all P3O’s by P3O and, similarly, for its subfamilies, we use the calligraphic T3O, p-P3O, p-T3O, respectively.

In a companion paper [1], motivated by a lemma of Jobson et al. [6] (see also Lehel and Tóth [9]), we have defined an orientation on intersecting planar convex sets, as follows. If A ∩ B ∩ C ≠ ∅, then ∘(ABC) = 0. Otherwise, by [6], R² \ (A ∪ B ∪ C) has one bounded component, and its boundary has exactly one arc from each of the boundaries of A, B, and C. We defined ∘(ABC) = 1 if in cyclic counterclockwise order these arcs belong to A, B, C, and proved that ∘ satisfies the interiority condition, i.e., it is a 3-order. Denote the subfamily of T3O and P3O that have a realization by pairwise intersecting planar convex sets by C-T3O and C-P3O, respectively. In particular, if no three sets from a pairwise intersecting convex family have a common point (called a *holey family* in [1]), the orientation ∘ gives a C-T3O on them. In this paper, we extend the orientation ∘ to good covers, and denote the respective families by GC-T3O and GC-P3O. As convex sets are always good covers, C-P3O ⊆ GC-P3O ⊆ P3O. Both inequalities are strict: GC-P3O ≠ P3O follows from Theorem[1] while C-P3O ≠ GC-P3O because with good covers a certain five-point configuration can be realized whose p-P3O was shown not be a C-P3O in [1]; see Figure 2.

We have shown in [1] that C-T3O ⊆ p-T3O (implying also C-P3O ⊆ p-P3O), and p-P3O ⊆ C-P3O. In this paper we establish the strengthening p-T3O ⊆ C-T3O, i.e., that there is a 3-order that is realizable by points in general position, but not by pairwise intersecting convex sets. This follows from the following more general result.

**Theorem 1.** p-T3O ⊆ GC-T3O.

Our proof will first establish that the 3-order of some point set is not realizable by some special subfamily of good covers, and then gradually increase the complexity of this subfamily, while also making our point set larger, until we establish the theorem.
The rest of the paper is organized as follows. In Section 2 we define an orientation ∩ on good covers, and show that in a GC-T3O realization we can assume that the sets are pairwise once intersecting topological trees, which also helps proving that ∩ satisfies the interiority condition, thus a 3-order. In Section 3 we define the subfamilies of T3O that are of interest to us and depict their relations to each other in Figure 11. In Section 4 we determine which p-T3O’s are realizable by T-shapes. In Section 5 we establish an auxiliary lemma on the tangency graph of topological trees. In Section 6 we present the proof of Theorem 1. Finally, in Section 7 we pose some open problems.

2 Good covers and topological trees

A family of sets is a good cover if the intersection of any subfamily is contractible or empty [10]. For example, any family of convex sets is a good cover. Another example is any family of sets that pairwise intersect in at most one point, which can either be a crossing point or a tangency.

Given three pairwise intersecting sets, A, B, C, define ∩(A, B, C) as follows. If A ∩ B ∩ C ≠ ∅, then ∩(A, B, C) = 0. Otherwise, if there is a Jordan curve γ such that γ is the concatenation of three curves, γ_A, γ_B, γ_C, such that γ_A ⊂ A, γ_B ⊂ B and γ_C ⊂ C, then if γ_A, γ_B, γ_C follow each other around γ in counterclockwise order, define ∩(A, B, C) = 1, while if γ_A, γ_B, γ_C follow each other around γ in clockwise order, define ∩(A, B, C) = −1.

We will show that for good covers the above orientation ∩ is well-defined (see Claim 2) and satisfies the interiority condition (see Corollary 8), thus it is a 3-order. For an example that satisfies the premise of the interiority condition, see Figure 1(b), while no three of the four sets in Figures 3(a) and (b) satisfy the premise.

First we fix some notations. We make no distinction between an element and the set representing it. The restriction of the orientation ∩ to some elements X = {X_1, ..., X_n} from a family (point sets, good covers, etc.), is denoted by ∩(X_1, ..., X_n) or simply ∩(X). For brevity, when talking about orientations, we may even refer to ∩(X) simply as X if it leads to no confusion.

Claim 2. The orientation ∩ is well-defined.

This simple topological claim is probably already known but we provide a proof for completeness.

1Given two sets that intersect once in some neighborhood, their intersection point is called a tangency, or touching point, if it can be eliminated by a small perturbation of the sets.

2Note that for convex sets this definition coincides with the one from [1].
Suppose that we are given three pairwise intersecting sets $A, B, C$ with $A \cap B \cap C = \emptyset$. Define $A^-$ to be the unique connected component of $A \setminus (B \cup C)$ which has a common boundary with both $A \cap C$ and $A \cap B$ (such a component exists). Define $A'$ to be the connected component of $A \setminus C$ containing $A^-$. We define $B^-, C^-, B', C'$ similarly. Fix a Jordan curve $\gamma$ as in the definition of $\partial (ABC)$ with the following properties: it has 6 parts, one in each of $A^-, A \cap B, B^-, B \cap C, C^-, A \cap C$, in this order. It is easy to see that such a curve exists. Let $\gamma_A$ be the union of the first two parts, $\gamma_B$ the middle two parts, and $\gamma_C$ the last two parts. Thus $\gamma_A \subset A', \gamma_B \subset B', \gamma_C \subset C'$ (see Figure 4). Assume without loss of generality that $\gamma_A, \gamma_B, \gamma_C$ follow each other in counterclockwise order.

Now take any other Jordan curve $\gamma'$ as in the definition of $\partial (ABC)$, with three parts such that $\gamma'_A \subset A, \gamma'_B \subset B$ and $\gamma'_C \subset C$. We need to prove that in $\gamma'$ these three parts follow each other in the same order as in $\gamma$. To show this, we will continuously morph $\gamma'$ in several steps so that at the end it becomes $\gamma$, and the respective three parts morph into each other while remaining inside the required sets all the time, e.g., $\gamma'_A$ morphs to $\gamma$ so that it stays inside $A$ during the whole process. Here by morphing, we mean a continuous transformation such that the Jordan curve remains a Jordan curve during the whole process. This implies that the parts follow each other in the same order in the two curves, as required.

First, $\gamma'_A$ may have parts in other connected components of $A \setminus C$ besides $A'$. We ‘pull’ these back with a morph into $A \cap C$ without changing $\gamma'_B, \gamma'_C$ (see the dashed blue curve in Figure 4). This can be done as neither $\gamma'_B$, nor $\gamma'_C$ can go into these components. We do the same for $\gamma'_B$ and $\gamma'_C$. After this step $\gamma'_A \subset A'$ etc. (just like for $\gamma$).

Next, we morph $\gamma'$ so that the endpoints of $\gamma'_A$ end up on the boundary of $A'$, etc. For this, it is enough to move continuously the endpoints of the parts along $\gamma'$, without changing $\gamma'$ itself. To be able to do this we need that the part of $\gamma_A$ outside $A^-$ is completely inside $A \cap C$, which is guaranteed by the previous morphing step. The unfilled circles on the blue curve in Figure 4 are the new endpoints.

Next, we morph $\gamma'$ so that the endpoints of $\gamma'_A$ become equal to the endpoints of $\gamma_A$. We detail how to do this. Let $\delta$ be the common boundary part of $A'$ and $A \cap C$. Take the endpoint of $\gamma'$ and $\gamma$ on $\delta$ and the arc of $\delta$ connecting them. Now take a small enough vicinity of $\delta$, which we call $D_\delta$ (topologically an open
Figure 4: Proof of Claim 2: The stages of morphing $\gamma'$, the blue curve with squares at endpoints of its parts, to the green curve, then to the yellow curve, then finally to $\gamma$, the red curve with disks at endpoints of its parts.

disk). It is possible to morph $D_\delta$ onto itself so that the endpoint of $\gamma'$ is morphed into the endpoint of $\gamma$. There is even such a morphing in which the restriction to $D_\delta \cap A'$ (resp. to $D_\delta \cap (A \cap C)$) of the morphing is a morphing onto itself, i.e., no point crosses the boundary $\delta$ during the morphing. This morphing of $D_\delta$ restricted to $\gamma'$ gives the required morphing. We do the same for $\gamma'_B$ and $\gamma'_C$ (the curve we get is drawn green in Figure 4). Note that if we choose small enough vicinities then these three morphs do not affect each other.

Note that in the lenses defined by $\gamma'_A$ and the common boundary of $A^-$ and $A \cap C$, there cannot be parts of $\gamma'_B$ and $\gamma'_C$. Thus we can morph $\gamma'_A$ so that it lies in $A'$ without changing $\gamma'_B$ or $\gamma'_C$. Similarly, we can morph $\gamma'_B$ (resp. $\gamma'_C$) so that it lies in $B'$ (resp. $C'$) (the current curve is drawn yellow in Figure 4).

Note that the sets $A', B', C'$ are disjoint apart from their boundaries, so now we can easily morph $\gamma'_A$ into $\gamma_A$ inside $A'$ etc. to get the curve $\gamma$, finishing the proof.

We define a special subfamily of good covers, where each set is a topological tree.

A topological tree is an injective embedding of a (graph theoretic) tree, that has no degree two vertices, into the plane, such that vertices are mapped to points, and edges are mapped to simple curves. The images of the degree one vertices are called leaves, while the images of the vertices with degree at least three are called branching points.

A family of topological trees forms a good cover if every pair of trees intersect at most once. (This is not an if and only if condition, but it will be more convenient for us to work only with such families.) For two trees $A$ and $B$ we denote their intersection point by $A \cap B$, i.e., $A \cap B = \{A \cap B\}$. For three trees, $A$, $B$ and $C$, we have $\bigcap (A, B, C) = 0$ if and only if their pairwise intersection points coincide.

We need to introduce some notation. See Figure 5 for illustration of the next definition.

3The difference is that there is less space around the new math operator. The similarity can lead to no confusion, as these two operators denote practically the same thing.
**Definition 3.** Suppose that $X$ is a topological tree and the point $p_i$ is in $X$ for each $1 \leq i \leq k$.

Define $X[p_1, \ldots, p_k]$ to be the minimal connected subset of $X$ which contains $p_i$ for every $i$, and for which every connected component of $X \setminus X'$ has $p_i$ on its boundary for some $i$.

Define $X[p_1, p_2]$ to be the set of those points $p \in X$ for which $X[p-p_2]$ contains $p_1$.

If $A_1, \ldots A_k$ are topological trees that intersect $X$ once and $p_i = X \cap A_i$, then for brevity we can replace $p_i$ in the above notations with $A_i$. For example, $X[A_1-A_2] = X[p_1-A_2] = X[p_1-p_2].$

Now we are ready to prove our main structural tool.

**Proposition 4.** Assume that in a planar family:

§1 Every set is a topological tree.

§2 Every pair of sets intersects in exactly one point.

Then the following hold:

§3 The union of any three sets, $A, B, C$, without a common point, contains exactly one cycle, i.e., a Jordan curve. The interior of this cycle is called the 

cycle and is denoted by $\triangle(ABC)$. The boundary of $\triangle(ABC)$ consists of parts of $A, A \cap B, B \cap C, C \cap A$, in this order, if $\cap(ABC) = 1$. From the boundary of $\triangle(ABC)$, there can be subtrees of $A, B, C$ going inwards and outwards, these we call hairs. Hairs might have branchings on them but they are disjoint from each other. See Figure 6(a).

§4 Any four sets satisfy the interiority condition, thus $\cap$ is a 3-order.

Further, if $D \in \text{conv}(ABC)$, then

\[ \triangle(ABD) \cup \triangle(BCD) \cup \triangle(ACD) \cup D[A-B-C] \cup \{ \text{at most one-one hairpart of } A, B \text{ and } C \} \]

gives a partition of $\triangle(ABC) \cup \{ A \cap D, B \cap D, C \cap D \}$. In particular, $D[A, B, C] \subset \triangle(ABC)$, apart from its endpoints, $\{ A \cap D, B \cap D, C \cap D \}$.

§5 If $D, E \in \text{conv}(ABC)$ and the orientation on $A, B, C, D, E$ is realizable by five points in general position, then $D \cap E \in \triangle(ABC)$.

By §3 and §4 the orientation $\cap$ defined for good covers gives a P3O for any family satisfying the conditions §1 and §2. We call a P3O that is realizable this way a Tr-P3O. If in addition no three trees have a common intersection, then it is also called a Tr-T3O representation.

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4 Two trees are allowed to have multiple branches from their intersection point.

5 Note that for convex sets this definition coincides with the one from 1.
Remark 5. Note that because of the hairs several intuitive statements are false. For example, it is possible that $D[A,B,C] \subset \Delta(ABC)$ but $D \notin \text{conv}(ABC)$. See Figure 6(b). Also, if in §5 we only assume that $D,E \in \text{conv}(ABC)$, then it is possible that $D \cap E \notin \Delta(ABC)$. See Figure 6(c).

Proof of §3. As $A$ is a tree, there is exactly one path in $A$ between $A \cap B$ and $A \cap C$, and this cannot intersect $B$ or $C$. Similarly, there is one path in $B$ between $A \cap B$ and $B \cap C$, and there is one path in $C$ between $A \cap C$ and $B \cap C$. The union of these three paths gives the required Jordan curve.

Proof of §4. Assume that $\bigtriangledown(ABC) = \bigtriangledown(ABD) = \bigtriangledown(BCD) = \bigtriangledown(CAD) = 1$. To show that the interiority condition holds we need to prove that $\bigtriangledown(ABC) = 1$. First, we assume $\bigtriangledown(ABC) \neq 0$, i.e., $A \cap B \cap C = \emptyset$.

Consider $D' = D[A-B-C]$. Either $D'$ is a path, or a Y-shaped star.

In the former case, we can assume without loss of generality that $B \cap D \in D[A-C] = D'$, i.e., $B \cap D$ lies between $A \cap D$ and $C \cap D$ on $D$. From $\bigtriangledown(CAD) = 1$, we know on which side of $D'$ the hollow $\Delta(ACD)$ lies. See Figure 7(a).

Note that $\Delta(ABD) \subset \Delta(ACD)$ would imply $\bigtriangledown(ABD) = \bigtriangledown(ACD) \neq \bigtriangledown(CAD)$, contradicting our assumptions. Similarly, $\Delta(BCD) \not\subset \Delta(ACD)$.

This implies that none of the paths $B[D-A]$ and $B[D-C]$ can start from $B \cap D$ towards the interior of $\Delta(ACD)$, as otherwise they would need to intersect $\partial \Delta(ACD)$, either in $A$ or in $C$, which would give $\Delta(ABD) \subset \Delta(ACD)$ or $\Delta(BCD) \subset \Delta(ACD)$, respectively, contradicting our previous observation. See Figure 7(b).

Figure 6: $\Delta(ABC)$ and the hairs.

Figure 7: Proof of §4.
Figure 8: A triple intersection can be perturbed in two ways.

Figure 7(a). As both paths start from $B \cap D$ towards the exterior of $\Delta(ACD)$, we can pretend that $D'$ is a (degenerate) Y-shape such that its leaves in the counterclockwise order are $A \cap D, B \cap D, C \cap D$.

The same argument rules out the possibility that $D'$ is a Y-shape such that its leaves in the counterclockwise order are $A \cap D, C \cap D, B \cap D$.

Therefore, we can conclude that $D'$ needs to be a (possibly degenerate) Y-shape such that its leaves in the counterclockwise order are $A \cap D, B \cap D, C \cap D$. Denote the branching point of $D'$ by $D_y$ (where $D_y = B \cap D$ if $D'$ is degenerate). See Figure 7(b).

Denote the branching point of $A[B-C-D]$ by $A_y$, or if $A[B-C-D]$ is a path, then let $A_y$ stand for whichever of $A \cap B, A \cap C$ and $A \cap D$ lies in the middle of the path. In other words, $A_y$ is the point up to which $A[D-B]$ and $A[D-C]$ follow the same route starting from $A \cap D$. In particular, $A[D-A_y]$ does not contain $A \cap B$ or $A \cap C$ in its interior. We similarly define $B_y$ and $C_y$.

Now, walk along $\partial \Delta(ABD), \partial \Delta(BCD)$ and $\partial \Delta(ACD)$, starting always from $D_y$. Note that in the union of these three walks, During the walk around $\partial \Delta(ABD)$, we cover the part from $D_y$ to $A_y$, then we go from $A_y$ to $B_y$, then back to $D_y$. Around $\partial \Delta(BCD)$, we cover the part from $D_y$ to $B_y$, then we go from $B_y$ to $C_y$, then back to $D_y$. Finally, around $\partial \Delta(ACD)$, we cover the part from $D_y$ to $C_y$, then we go from $C_y$ to $A_y$, then back to $D_y$. Note that each part from $D'$ occurs there and back, while all the other parts are disjoint, apart from their endpoints. But this means that by eliminating the there-and-back parts, we get a walk from $A_y$ to $B_y$ to $C_y$, then back to $A_y$, that has the same orientation as the original walks, and contains $AB, BC$ and $AC$. That is, $\partial(ABC) = 1$, as claimed. This finishes the proof of the interiority condition if $\partial(ABC) \neq 0$.

Now assume for a contradiction that $\partial(ABC) = 0$, i.e., $A \cap B \cap C \neq \emptyset$. By $\Delta_A B \cap C$ is a single point; denote it by $x$. If we replace $A$ with $A[x-D]$, then $\partial(A, B, C, D)$ remains the same. We can similarly replace $B$ and $C$ with $B[x-D]$ and $C[x-D]$, respectively. This way each of $A, B$ and $C$ became a curve ending in $x$. But then by a slight perturbation of the curves in the vicinity of $x$ we can achieve that they pairwise intersect once but in three different points, such that $\partial(ABC)$ becomes $-1$ (see Figure 8).

But this would contradict the interiority condition for $\partial(ABC) \neq 0$, which we have already proved. This finishes the proof of the interiority condition if $\partial(ABC) = 0$.

The structural description that we have obtained in the $\partial(ABC) \neq 0$ case implies that $\Delta(ABC) = \Delta(ABD) \cup \Delta(BCD) \cup \Delta(ACD) \cup D'$ or \{the parts from $A_y$ to $A \cap D$, from $B_y$ to $B \cap D$, and from $C_y$ to $C \cap D$\}—this last part gives the three possible hairs. Since $D$ intersects $\Delta(ABC)$ exactly three times, this also implies $D[A, B, C] \subset \Delta(ABC)$.

\[\square\]

Proof of $\Delta$. Suppose for a contradiction that $x = D \cap E \not\in \Delta(ABC)$.
By \cite{4} \( D[A, B, C] \subset \mathcal{A}(ABC) \) and thus \( x \) falls in one connected component of \( D \setminus D[A, B, C] \) which implies that the three paths \( D[E-A], D[E-B], D[E-C] \) all go the same way from \( x \) until they reach \( \mathcal{A}(ABC) \).

Denote their intersection point with \( \partial \mathcal{A}(ABC) \) by \( D_x \). Note that \( D_x \) is one of \( A \cap D, B \cap D, C \cap D \).

Similarly, the three paths \( E[D-A], E[D-B], E[D-C] \) all start the same way from \( x \). Denote their intersection point with \( \partial \mathcal{A}(ABC) \) by \( E_x \). This time there is no need to make any wise notes.

We claim that \( \mathcal{A} \) behaves essentially the same way on any good cover as on topological trees.

\begin{lemma}
The sets in any good cover, where at most one triple has a non-empty intersection, can be replaced by topological trees that pairwise intersect at most once, such that \( \mathcal{A} \) remains unchanged on all triples.
\end{lemma}

\begin{proof}
See Figure 9(a) for an illustration of the proof.
\end{proof}

Next, we prove that \( \mathcal{A} \) behaves essentially the same way on any good cover as on topological trees.

\section*{Lemma 6}
The sets in any good cover, where at most one triple has a non-empty intersection, can be replaced by topological trees that pairwise intersect at most once, such that \( \mathcal{A} \) remains unchanged on all triples.

\begin{proof}
See Figure 9(a) for an illustration of the proof.
\end{proof}

We can assume that every set intersects some other set from our family. For each set \( A \) from the good cover, and for each connected component \( A_i \) of those points that are only in \( A \) and in no other set, we do the following. On the boundary of \( A_i \) there are pairwise disjoint arcs which are on the boundary of some other set as well. We put a topological star inside \( A_i \) whose center is on one of these arcs, and there is a leaf on each of the rest of the arcs.

Now, assume that there is no triple intersection. For each non-empty intersection \( A \cap B \), we select a point \( p_{AB} \) inside it (\( p_{AB} \) will be the intersection point of the topological trees corresponding to \( A \) and
and draw non-crossing curves from \( p_{AB} \), one to every leaf of the earlier defined stars that are on the boundary of \( A \cap B \). Each such curve is added to the topological tree it touches at the boundary of \( A \cap B \).

If there is a triple intersection \( A \cap B \cap C \neq \emptyset \), then we treat \( (A \cap B) \cup (A \cap C) \cup (B \cap C) \) as one double intersection, and do the same as before, selecting a point \( p_{ABC} \) from \( A \cap B \cap C \) (and no other points from \( A \cap B, A \cap C, \) and \( B \cap C \), so in this case there are no points \( p_{AB}, p_{AC}, p_{BC} \)). In the remainder of the proof, we will not discuss this special triple intersection region in detail—all steps work for it the same way.

It is easy to see that we get topological trees, as all the points inside a set \( A \) are eventually connected. Denote this tree by \( T_A \subset A \). These topological trees intersect pairwise exactly once, in the point selected inside the intersection of their corresponding sets, i.e., \( T_A \cap T_B = p_{AB} \).

We are left to show that the orientations are preserved. As the trees satisfy §2, \( \mathcal{O} \) is well-defined on the trees by §3. Take some \( A, B, C \) and the (unique) Jordan curve \( \gamma \) such that \( \gamma_A \subset T_A, \gamma_B \subset T_B \) and \( \gamma_C \subset T_C \). As \( T_X \subset X \) for any set \( X \), also \( \gamma_A \subset A, \gamma_B \subset B \) and \( \gamma_C \subset C \), so the same \( \gamma \) shows that \( \mathcal{O}(T_A T_B T_C) = \mathcal{O}(ABC) \).

Remark 7. The condition that at most one triple has a non-empty intersection is necessary because we allow trees to intersect in at most one point. If, for example, \( A \cap B \cap C \neq \emptyset \) and \( A \cap B \cap D \neq \emptyset \) but \( A \cap B \cap C \cap D = \emptyset \), this obviously cannot be realized by trees that pairwise intersect at most once. See Figure 9(b) for such a good cover. But this is essentially the only obstruction—our proof can be modified in a straight-forward way to work also if we require that there are no four sets such that \( A \cap B \cap C \neq \emptyset \) and \( A \cap B \cap D \neq \emptyset \).

**Corollary 8.** The orientation \( \mathcal{O} \) is a 3-order on good covers.

**Proof.** We need to show that \( \mathcal{O} \) satisfies the interiority condition. Take four sets such that \( \mathcal{O}(ABD) = \mathcal{O}(BCD) = \mathcal{O}(CAD) = 1 \). In particular, these four sets can have at most one non-empty triple intersection, \( A \cap B \cap C \). Therefore, by Lemma 9 we can convert \( A, B, C, D \) to trees while preserving \( \mathcal{O} \), and so §4 implies that \( \mathcal{O}(ABC) = 1 \).

**Remark 9.** This also implies that \( \mathcal{O} \) is a 3-order on convex sets, reproving a key lemma from §1.

Denote a \( \mathcal{P}_3O/\mathcal{T}_3O \) realizable by good covers (resp. by topological trees), by \( \mathcal{G}C-\mathcal{P}_3O/\mathcal{G}C-\mathcal{T}_3O \) (resp. by \( \mathcal{T}3O \), and the respective subfamilies by calligraphic, as usual.

**Corollary 10.** \( \mathcal{G}C-\mathcal{T}_3O = \mathcal{T}3O \).

This implies that to establish Theorem 1, it is enough to prove \( \mathcal{P}3O \subseteq \mathcal{T}3O \).

### 3 Subfamilies of \( \mathcal{T}_3O \)

Here we define several subfamilies of good covers and point sets that are of interest to us. For completeness, we also include the already defined families. The containment relations among the respective \( \mathcal{T}_3O \) families are depicted in Figure 11.

- **\( \mathcal{P}_3O \)**: A total 3-order over any set, i.e., an orientation on the triples that satisfies the interiority condition.
Figure 10: Out of the 16 order types on 6 points, we could realize 15 as a $\gamma$-T3O.
• p-T3O: A 3-order on a planar point set in general position, where each triple is oriented depending on whether the points are in clockwise or counterclockwise position. This is more commonly known as the order type of the point set.

• fast-p-T3O: Special case of p-T3O where the points form a fast-growing point set. (Defined in Section 4.)

• conv-p-T3O: Special case of p-T3O where the points are in convex position. (All conv-p-T3O of some fixed size are isomorphic up to relabeling.)

• GC-T3O: A 3-order on a good cover of pairwise intersecting sets in the plane, such that no three of them have a point in common, where each triple is oriented according to \( \ominus \) defined in Section 2.

• C-T3O: Special case of GC-T3O where each set is convex. (A family of such convex sets is called a holey family and studied in detail in [1].)

• Tr-T3O: Special case of GC-T3O where each set is a topological tree.

• Y-T3O: Special case of Tr-T3O where each topological tree is a Y-shape, meaning that it has only one branching point, which is of degree three, i.e., it is an image of \( K_{1,3} \). We also require for every pair of topological trees that they either pairwise cross (as opposed to touch) or one of them ends in their intersection point.\(^6\)

• T-T3O: Special case of Y-T3O where each set is a T-shape, i.e., one horizontal line, and one vertical downward halfline, ending in an interior point of the horizontal line. Note that no lines or halflines coincide in the representation because any two T-shapes intersect in exactly one point.

• \( \gamma \)-T3O: Special case of Y-T3O where each set is a simple curve, i.e., an image of a single edge, and these curves pairwise cross (as opposed to touch). We have depicted the representation of some p-T3O as \( \gamma \)-T3O in Figure 10.

• \( \ell \)-T3O: Special case of \( \gamma \)-T3O where each curve is a line, i.e., a collection of lines in general position (no two are parallel and no three pass through the same point).

Observation 11 (Goodman-Pollack [3]). Given \( n \) lines, \( \ell_1, \ldots, \ell_n \), ordered according to their slopes in clockwise circular order, and \( n \) points, \( p_1, \ldots, p_n \), in convex position, ordered in counterclockwise order, \( \ominus(\ell_1, \ldots, \ell_n) = \ominus(p_1, \ldots, p_n) \).

Corollary 12. \( \ell\)-T3O = conv-p-T3O.

\(^6\)Such intersections could also be elongated to form a crossing. We do allow the branching point of a Y-shape to also be its intersection point with another Y-shape; this is relevant when the intersection point is the branching point for both trees. For every Y-shape this can occur with at most one other Y-shape, as no three have a point in common.

\(^7\)Strictly speaking, T-shapes are not topological trees because they are unbounded, but it would be easy to turn them into Y-shapes by replacing the lines/halflines with sufficiently long segments.

\(^8\)Strictly speaking, curves are not Y-shapes because they do not have a branching point, but we could introduce one on each at an arbitrary place to turn them into Y-shapes.
Figure 11: Containment relations among different $\mathcal{T}3\mathcal{O}$'s. Most containments are true by definition, our main result, Theorem 1, is the non-containment between $p\mathcal{T}3\mathcal{O}$ and $GC\mathcal{T}3\mathcal{O}$. In most cases we did not prove that the containments are strict but we conjecture that all of them are. The equivalence $GC\mathcal{T}3\mathcal{O} = \text{Tr-}\mathcal{T}3\mathcal{O}$ is Corollary 11, $T-\mathcal{T}3\mathcal{O} = \text{fast-}p\mathcal{T}3\mathcal{O}$ is Proposition 13, and $\ell\mathcal{T}3\mathcal{O} = \text{conv-}p\mathcal{T}3\mathcal{O}$ is Corollary 12.

4 T-shapes and fast-growing point sets

Call an $n$-element point set $\mathcal{P}$ fast-growing if its points have an ordering $\{p_1, \ldots, p_n\}$ such that there is a permutation $\pi : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ for which $\odot(p_ip_jp_k) = -1$ if and only if $\pi(j) > \pi(i), \pi(k)$ for any $i < j < k$. Note that such a $\mathcal{P}$ has the same order type as the point set \{(i, N_{\pi(i)}) : 1 \leq i \leq n\} if $N_i$ is a sufficiently fast-growing function, like $2^i$. However, a fast-growing point set might also look very different; for example, all point sets in convex position are fast-growing, as their order type is the same as for the point set $p_i = (i, 2^i)$. Note that $p_1, p_n, p_{\pi(1)}$ and $p_{\pi(n)}$ are always (not necessarily different) extremal points of $\mathcal{P}$, and that the condition $\odot(p_ip_jp_k) = -1$ if and only if $\pi(j) > \pi(i), \pi(k)$ is equivalent to that any $p_{\pi(i)}$ "sees" the points $\{p_{\pi(j)} : j < i\}$ in the same order as the point $p_{\pi(n)}$ does, if $p_{\pi(i)}$ starts looking around from $p_{\pi(n)}$. Using this observation, it is easy to check that any point set on at most five points is fast-growing, while on six points out of the 16 p-T3O’s 12 are realizable; see Figure 12.

Denote the T3O’s that are representable by a fast-growing point set by fast-p- T3O. Corollary 12 implies that $\ell\mathcal{T}3\mathcal{O} \subset \text{fast-}p\mathcal{T}3\mathcal{O}$. This containment is strict because, for example, four points in non-convex position give a fast-p-T3O. The growth rates are also different: we have $(n - 1)! \ell\mathcal{T}3\mathcal{O}$’s, about $(n!)^2$ fast-p-T3O’s, and about $(n!)^4$ p-T3O’s on $n$ labeled elements [4] [5].

We can also give the following characterization.

Proposition 13. fast-p-$\mathcal{T}3\mathcal{O} = T-\mathcal{T}3\mathcal{O}$.

Proof. First, observe that the 3-orders defined by T-shapes are the same as defined by $\perp$-shapes and so it is enough to show that the latter are the same as fast-p-T3O’s.

Given a fast-p-T3O, we can realize it with a fast growing point set, and then for each point put a $\perp$-shape whose hat and leg meet at that point. It is easy to see that the orientation of this family of $\perp$-shapes is the same as for the point set, see Figure 13.

Conversely, given a T-T3O, realize it with $\perp$-shapes. Notice that changing the $y$-coordinates of the hats, while preserving their order, does not change the orientation of the $\perp$-shapes. Thus, we can change
Figure 12: Out of the 16 order types on 6 points, exactly 12 are realizable as a fast-p-T3O. We have numbered the points of these 12 such that the numbers indicate the order of the points from bottom to top. That is, the point numbered \( i \) can be mapped to \((\pi^{-1}(i), 2^i)\), where \( \pi \) is some appropriate permutation of \( \{1, \ldots, 6\} \). In particular, to check the correctness of a numbering, by our observation it is enough to verify that the point numbered with \( i + 1 \) “sees” the points assigned smaller numbers in the same order as the point numbered with \( i \) does, if they start looking around from the point numbered 6. For example, in case of the order type in the bottom-right corner, these orders are as follows: 1: \( \emptyset \), 2: 1, 3: 12, 4: 132, 5: 1342, 6: 15342. As each of these sequences are derived from the previous one by inserting the next number, this is the order type of a fast-growing point set.

A simple cases analysis shows that such a numbering is not possible in the remaining 4 cases, whose points are labeled with letters, thus those are not fast-p-T3O’s, and by Proposition 13, also not T-T3O’s.
the $y$-coordinates so that they form a fast-growing function. Take the meeting point of the hat and leg for each $\perp$-shape; these form a fast-growing point set. As before, it is easy to see that order type of this point set is the same as the orientation of the original family of $\perp$-shapes.

**Corollary 14.** $t\cdot T3\mathcal{O} \subseteq T\cdot T3\mathcal{O} \subseteq p\cdot T3\mathcal{O}$.

## 5 The number of tangencies in a family of $t$-intersecting trees

Before we can start the proof of Theorem 1, we need one more tool, in order to be able to switch from tangencies to crossings among the sets of some large enough family at the cost of losing some sets. The following lemma was recently shown by Keszegh and Pálvölgyi [7] in the special case when we have curves instead of trees; our proof uses several ideas from their proof.

**Lemma 15.** In the tangency graph of a family of topological trees that pairwise intersect in at most $t$ points, and no three trees have a common point, there is no $K_{t+3,c}$ for some large enough $c$ (depending only on $t$).

**Proof.** Suppose for a contradiction that there is a family of topological trees as in the lemma such that its tangency graph contains $K_{t+3,c}$ as a subgraph. Call the $t+3$ trees that form one part of this bipartite graph *red*, and the $c$ trees that form the other part of the bipartite graph *blue*. Each red tree is cut by the other $t+2$ red trees into at most $1 + (t + 2)t$ parts, and by the pigeonhole principle there are at least $c_2 = c/(1 + (t + 2)t)^{t+3}$ blue trees that intersect the same part from each red tree. Choosing these red parts, from now on we assume that we have a family of red and blue trees whose tangency graph contains a $K_{t+3,c_2}$ and the red trees are pairwise disjoint. After small local redrawing operations of the blue trees we can also assume that each tangency point is a leaf of the respective blue tree. Furthermore, from each blue tree we keep only the edges that participate in some path connecting two tangency points (of this tree and one of the $t + 3$ red trees).

Thus, the blue trees each have exactly $t + 3$ leaves, corresponding to tangency points, which implies that they have at most $t + 1$ branching points. As no three trees have a common point, each of these branching points can belong to at most one other blue tree. Define a graph on the $c_2$ blue trees where two trees are connected by an edge if they intersect in a branching point of one of them. Because of our
previous observation, this graph can have at most $c_2(t+1)$ edges. Therefore, we can select an independent set of size $c_3 = c_2/(t+2)$ from it by Turán’s theorem.

There are only bounded many options (depending on $t$) how such a blue tree can be embedded in the plane. Formally, if we label the leaves with the red trees touching at that leaf and put two blue trees in the same equivalence class whenever they are topologically equivalent (including the labels) and also have the same rotation system (i.e., the edges leaving each branching point come in the same circular order), then we get a bounded number of equivalence classes. Thus, if $c_3$ is big enough, then there are two blue trees in the same equivalent class—denote them by $B$ and $B'$.

To summarize, we are given two trees, $B$ and $B'$, both with $t+3$ labeled leaves, such that $B$ and $B'$ are topologically equivalent (including their labels) and their rotation systems are also the same. We are left to show that such a pair of trees intersect in at least $t+1$ points, as this gives the desired contradiction. We show this by induction on the number of leaves of $B$ and $B'$. For $t+1 = 0$ this trivially holds. Next, we assume that $t \geq 0$ and that for $t-1$ the statement holds, and show that it also holds for $t$.

First, we claim that there are two leaves, $v_1$ and $v_2$, of $B$ such that after removing from $B$ the path $P$ connecting $v_1$ and $v_2$ (without removing the branching points that lie on $P$), we get a connected graph (a tree). To see that, take any branching point $s$ of $B$ (which exists as we have $t+3 \geq 3$ leaves) and take a branching point $q$ which is farthest from $s$ in $B$ in the combinatorial distance (i.e., the connecting path has the largest possible number of branching points); by definition, $q$ is equal to $s$ if there are no other branching points. The at least two edges incident to $q$ that not towards $s$ must lead to leaves, as $q$ was a farthest branching point. It is easy to see that any two of these leaves are good choices for $v_1$ and $v_2$. Take also the corresponding leaves $v_1', v_2'$ and branching point $q'$ in $B'$ and the path $P'$ connecting them. See Figure 14.

Next, we show that $P$ must intersect $B'$ or $P'$ must intersect $B$. Assume on the contrary. Observe that the endpoints of $P$ and $P'$ are tangency points with the same pair of red trees. $P$, $P'$ and the two red trees together partition the plane into two connected regions, such that $q$ and $q'$ are on their common boundary, a Jordan curve, which we denote by $C$. Take a third leaf, $v_3$, of $B$ and a path $Q$ connecting $q$ and $v_3$. Similarly, let $Q'$ be the path of $B'$ connecting $q'$ with the leaf $v_3'$ corresponding to $v_3$. Due to the indirect assumption, both $Q$ and $Q'$ are disjoint from $C$. As the rotation system around $q$ and $q'$ is the same, they leave $C$ towards a different side of $C$, and therefore must end (in $v_3$ and $v_3'$) on a different side of $C$. However, $v_3$ and $v_3'$ are connected by a red tree that is disjoint from $C$, a contradiction.
Thus, without loss of generality, we can assume that \( P \) and \( B' \) intersect in a point \( p \). Notice that \( p \) cannot be equal to \( q \), as it is a branching point of \( B' \). If \( p \) is on the path of \( B \) connecting \( v_1 \) with \( q' \), then we delete this path from \( B \) and the path connecting \( v_1' \) with \( q' \) in \( B' \). Otherwise, we delete the path of \( B \) connecting \( v_2 \) with \( q \) and the path of \( B' \) connecting \( v_2' \) with \( q \). In both cases, the two trees we get each have \( t - 1 \) leaves, they do not intersect anymore in \( p \), and they are also topologically equivalent (with the leaves labeled and the rotation system preserved), thus we can apply induction to find \( t \) intersection points between them. Together with \( p \), there are \( t + 1 \) intersection points between \( B \) and \( B' \), a contradiction.

6 Proof of Theorem 1

Now we are ready to prove Theorem 1. Our starting point is the following geometric observation.

**Proposition 16.** Let \( A \) and \( B \) be two disjoint (topological) halflines. Let \( X \) and \( Y \) be two topological trees that have one intersection point. Assume further that both \( X \) and \( Y \) intersect both \( A \) and \( B \) in exactly one point, these four intersection points are all different, and none of them are branching points of the trees. Assume also that \( X[A-B] \) and \( Y[A-B] \) leave each of the two halflines on the same side, i.e., they touch the halflines from the same direction.

If \( \circ(XYA) \neq \circ(XYB) \), then \( X \cap Y = X[A,B] \cap Y[A,B] \) and \( X \cap Y \) cannot be a crossing point between \( X[A-B] \) and \( Y[A-B] \).

**Proof.** We can add a curve to \( A \) connecting \( A \) and \( B \) far enough from \( X \) and \( Y \) to create an intersection point of \( A \) and \( B \). Since \( \circ(XYA) \neq \circ(XYB) \), without loss of generality, \( \circ(XAB) = \circ(XYA) = \circ(XBY) \), i.e., \( X \in \text{conv}(ABY) \). From \( X[A,B,Y] \subset \Delta(ABY) \), which implies \( X \cap Y = X[A,B] \cap Y[A,B] \).

To see the second part, assume on the contrary that \( X \cap Y = X[A-B] \cap Y[A-B] \) is a crossing point of \( X \) and \( Y \). Then \( \circ(XYA) = \circ(XYB) \), a contradiction. See Figure 15(a).

**Lemma 17.** Suppose that we are given two halflines, \( A \) and \( B \), and a family of topological trees \( P \) which is a Tr-T3O. If any pair \( X,Y \in P \) satisfies the conditions of Proposition 16 with \( A \) and \( B \), then some \( Q \subset P, |Q| \geq |P|/2 \) has a T-T3O representation.

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\(^9\)However, it can be a touching point between them.
Proof. We can assume that $A$ and $B$ are vertical halflines, and that $X = X[A, B]$, i.e., $X[A|B] = X[B|A] = \emptyset$ for all $X \in \mathcal{P}$, as these parts anyhow do not take part in any intersections. By Proposition 10, $X[A\mathbin{-}B]$ and $Y[A\mathbin{-}B]$ are pairwise non-crossing. After a local redrawing around the touching points, we can also assume that there are no touchings. Thus, we can assume that each $X[A\mathbin{-}B]$ is horizontal and each $X$ is contained inside the strip bounded by the lines of these halflines. We can also assume that $A$ is downward infinite. $B$ is either also downward infinite, or upward infinite. As each $X \in \mathcal{P}$ is inside the strip, we can cover both cases by simply assuming that both $A$ and $B$ are lines bounding the strip. We can order the elements of $\mathcal{P}$ by the heights of $X[A\mathbin{-}B]$, i.e., define $X < Y$ if the $y$-coordinate of $A \cap X = A \cap Y$ is less than the $y$-coordinate of $B \cap X = B \cap Y$.

For simplicity, denote $X[A\mathbin{-}B]$ by $X^-$ and the rest of $X$ by $X^0 = X \setminus X[A\mathbin{-}B]$. Note that $X^0$ need not be connected at all. If $X < Y < Z$, then $X^- \cap Y^0 = \emptyset$ or $Y^0 \cap Z^- = \emptyset$. Indeed, otherwise $X \cap Y = X^- \cap Y^0$ and $Y \cap Z = Y^0 \cap Z^-$ and then $X$ and $Z$ cannot intersect since $Y^-$ separates $X^0$ from $Z^-$ and $Z^0$ from $X^-$, a contradiction. See Figure 13(b). Thus we can divide $\mathcal{P}$ into two groups:

$$\mathcal{P}^\perp = \{Y \in \mathcal{P} : \text{for } \forall X \in \mathcal{P} \text{ such that } X < Y \text{ we have } X^- \cap Y^0 = \emptyset\},$$

$$\mathcal{P}^\top = \{Y \in \mathcal{P} : \text{for } \forall Z \in \mathcal{P} \text{ such that } Z > Y \text{ we have } Z^- \cap Y^0 = \emptyset\}.$$ 

Without loss of generality, assume that $|\mathcal{P}^\top| \geq |\mathcal{P}|/2$.

Let the lowest member of $\mathcal{P}^\top$ be $X_0$, the second lowest be $X_1$ and $\mathcal{P}_0^\top = \mathcal{P}^\top \setminus \{X_0\}$. As $X_1^- \cap X_0^- = \emptyset$, $X_0$ remains completely below $X_1$ in the strip and thus for the rest of the trees in $\mathcal{P}_0^\top$ to intersect $X_0$, they all have to go below $X_1^-$, i.e., for every $X \in \mathcal{P}_0^\top$, $X \neq X_1$, we have that $X^0$ intersects $X_1^-$. Moreover, for $X, Y \in \mathcal{P}_0^\top$, $X^- \cap Y^0$ if and only if $X < Y$. For each $X \in \mathcal{P}_0^\top$, delete from it all parts that are not in $X^-$ or in the shortest path connecting $X^0 \cap X_0$ with $X^-$. Notice that in this drawing for every $X \in \mathcal{P}_0^\top$, $X^0$ is a curve with one endpoint on $X^-$ and its part below $X_1^-$ is disjoint from all other trees in $\mathcal{P}_0^\top$ and ends in $X \cap X_0$. Thus $X_0$ can be easily redrawn to be a curve going close to the original tree, connecting $X_0 \cap A$ with $X_0 \cap B$ and touching each $X \in \mathcal{P}_0^\top$ in the same order (and in the same point) as before. See Figure 13(c) for the drawing at this stage, where the original $X_0$ is shown dotted.

Now the trees in $\mathcal{P}^\top$ can be easily redrawn as T-shapes, where $X_0$ becomes a horizontal segment and for an $X \in \mathcal{P}_0^\top$ the hat of the respective T-shape is $X^-$, while the leg is a redrawning of the curve $X^0$ as a vertical segment.

\[ \square \]

**Lemma 18.** For any finite point set $\mathcal{P}$ and for every $\alpha > 0$ there is a point set $\mathcal{P}_\alpha$ such that for every $Q \subset \mathcal{P}_\alpha$ for which $|Q| \geq \alpha|\mathcal{P}_\alpha|$, there is a $Q' \subset Q$ such that $\mathcal{O}(Q') = \mathcal{O}(\mathcal{P})$, i.e., $\mathcal{P}$ and $Q'$ have the same order type.

**Corollary 19.** If $p\cdot T3O \subseteq X\cdot T3O$, where for any $X\cdot T3O$ is any subset of $T3O$, then for every $\alpha > 0$ there is a point set $\mathcal{P}_\alpha$ such that for every $Q \subset \mathcal{P}_\alpha$ for which $|Q| \geq \alpha|\mathcal{P}_\alpha|$, we have that $\mathcal{O}(Q) \notin X\cdot T3O$.

The proof of Lemma 18 follows from the powerful Multidimensional Szemerédi theorem of Furstenberg and Katznelson [2]. Indeed, we can pick $\mathcal{P}_\alpha$ to be a very large grid, and any $\alpha$-dense subset of $\mathcal{P}_\alpha$ will contain a grid of any given size (fixed before choosing the size of the large grid), that in turn will contain every order type up to some given size. In fact, the copy $Q'$ of $\mathcal{P}$ can even be chosen to be homothetic to $\mathcal{P}$ if $\mathcal{P}$ was a subset of some grid. However, the size of $\mathcal{P}_\alpha$ obtained this way will be enormous. Below we include a self-contained proof that gives a much better bound on $|\mathcal{P}_\alpha|$, and a $Q'$ that is almost homothetic to $\mathcal{P}$ in the sense that it can be the homothet of some arbitrarily slightly perturbed copy of $\mathcal{P}$ (the perturbation cannot be fixed but the extent of the perturbation can).
Proof of Lemma 18: We can assume $\alpha < 1$, since the statement holds for $\alpha = 1$ by setting $P_\alpha = P$ and is vacuous for $\alpha > 1$. Let $n = |P|$ and $P_1 = P$. Let $P_{i+1}$ be the point set obtained from $P$ by replacing each $p \in P$ with a small copy of $P_i$ close to $p$, such that for any $p, q, r \in P$ and closely placed points $p', q', r' \in P_{i+1}$, we have $\mathcal{O}(pqr) = \mathcal{O}(p'q'r')$. Starting with $P_1$ we repeat this process $k - 1$ times to get $P_k$ with $N = n^k$ points.

Notice that for every $2 \leq i \leq k$, if some subset of $P_i$ contains a point from each of the $n$ groups of $P_{i-1}$'s in $P_i$, then it contains $\mathcal{O}(P)$. 

Assume that $Q$ is a subset of $P$ that does not contain $\mathcal{O}(P)$. We need to show that $Q$ must be smaller than $\alpha|P|$. First, by the previous observation $Q$ must completely avoid one of the groups of $P_{k-1}$ in $P_k$. Then the rest of the groups are homothetic to $P_{k-1}$, and thus in each of them $Q$ must completely avoid one of the groups of $P_{k-2}$. By a repeated application of this argument, in each step we find that $Q$ avoids $\frac{1}{n}$th of the remaining points and thus altogether $Q$ can have at most $(\frac{n-1}{n})k \cdot N$ points. Thus $\frac{|Q|}{|P|} \leq (\frac{n-1}{n})^k = (1 - \frac{1}{n})^k < e^{-k/n} < \alpha$ if $k$ is large enough, we are done. \hfill $\square$

Proposition 20: $p$-T3O $\not\subset$ Y-T3O.

Proof. Fix $\alpha = 1/128$. The construction $P$ will consist of a $P_\alpha$ such that for every $Q \subset P_\alpha$ for which $|Q| \geq \alpha|P_\alpha|$, we have $\mathcal{O}(Q) \not\subset T$-T3O, and three more points, $A, B, C$, such that $P_\alpha \subset \text{conv}(ABC)$. We can assume by applying a suitable affine transformation (that leaves $\mathcal{O}(P_\alpha)$ the same) that $P_\alpha$ is flattened so that for any two points $X, Y \in P_\alpha$ the points $A$ and $B$ lay on different sides of the $XY$ line, i.e., $o(XYA) \neq o(XYB)$.

Assume on the contrary that $\mathcal{O}(P)$ has a realization with Y-shapes. By §5 every intersection occurs inside the closure of $\mathcal{A}(ABC)$, so we can assume that all Y-shapes are inside $\mathcal{A}(ABC)$. As each of $A, B, C$ are Y-shapes, their union consists of $\partial \mathcal{A}(ABC)$ and possibly some inward growing hairs. We cover each of $A$ and $B$ by at most four simple curves, called sections (which can overlap), such that one end of each section is on $\partial \mathcal{A}(ABC)$ and is either a vertex of $\mathcal{A}(ABC)$ or on a branching point. Note that a branching point of $A$ and $B$ is either on $\partial \mathcal{A}(ABC)$ (it can also coincide with one of the vertices) or in the interior of $\mathcal{A}(ABC)$, in which case it is connected to a vertex of $\partial \mathcal{A}(ABC)$ by a curve; in the latter case, the connecting curve will be covered by two sections. Any pair of sections, one of $A$ and the other of $B$, can be simultaneously extended into topological halflines outside $\mathcal{A}(ABC)$, such that they are pairwise disjoint, apart possibly from the intersection point $\cap B$, where they can touch, but not cross. As no Y-shape from $P_\alpha$ can contain $A \cap B$, we can redraw these topological halflines in a small neighborhood of $A \cap B$ so that they are disjoint.

Now we partition $P_\alpha$ into at most 64 groups depending on which of the four sections of $A$ and $B$ they intersect (4 · 4 options), and from which directions (2 · 2 options). By a slight redrawing, we can assume that no branching point of a Y-shape from $P_\alpha$ falls on $A$ or $B$. Recall that due to the flattening of $P_\alpha$, we have $o(XYA) \neq o(XYB)$ for every two different $X, Y \in \{A, B\}$, thus the elements of any group satisfy the conditions of Proposition 16. Fix a group that contains at least $|P_\alpha|/64$ elements, and apply Lemma 17 to redraw half of its elements (thus at least $|P_\alpha|/128$ elements) as T-shapes. On the other hand, by definition of $P_\alpha$, no 1/128 fraction of $|P_\alpha|$ gives a T-T3O, a contradiction. \hfill $\square$

We note that in the proof of Proposition 20 we used only that $A$ and $B$ are realized as Y-shapes, the rest of the topological trees could be arbitrary.

From Proposition 20 we can prove the following.
**Conjecture 25.**

**Problem 23.**

*Problem 22.* Determine the smallest number of points which are not representable as a special case is when we also require that each edge of each tree needs to be a segment between two points that the branching points of the trees also need to be from $T$. Another interesting, and probably easier, special case is when we also require that each edge of each tree needs to be a segment between two points of $P$.

**Proposition 21.** $p$-$T3O \not\in Tr$-$T3O$.

**Proof.** We will prove the following. If $P_0 = \{p_1, \ldots, p_n\}$ is a point set such that $\mathcal{C}(P_0) \not\in Y$-$T3O$, then there is a point set $P_n$ such that $\mathcal{C}(P_n) \not\in Tr$-$T3O$. Fix first a sequence of numbers $N_1, N_2, \ldots$ which grows very fast but is otherwise arbitrary. Now denote by $P_i$ the point set obtained from $P_0$ by placing $N_j$ points close to the point $p_j$ for all $1 \leq j \leq i$ (the points are placed arbitrarily in the close vicinity of $p_j$).

In particular $P_n$ has $\sum_{i=1}^{n} N_i$ points. Our strategy is to prove for each $i$, going from $n$ to 1, that if $\mathcal{C}(P_i)$ has a $Tr$-$T3O$ representation where the points $\{p_i+1, \ldots, p_n\}$ are represented by crossing Y-shapes, then $\mathcal{C}(P_i)$ has a Tr-$T3O$ representation where the points $\{p_i, \ldots, p_n\}$ are represented by crossing Y-shapes.

Thus it follows that $\mathcal{C}(P_n) \in Tr$-$T3O$ would imply $\mathcal{C}(P_0) \in Y$-$T3O$, contradicting our assumption.

So fix $i$ and let us assume that $\mathcal{C}(P_i)$ has a $Tr$-$T3O$ representation where the points $\{p_{i+1}, \ldots, p_n\}$ are represented by crossing Y-shapes. Denote the collection of the $N_i$ trees that correspond to points that are in the vicinity of $p_i$ by $T_i$, and the remaining $N_0 := N_1 + \cdots + N_{i-1} + n - i$ trees that correspond to points that are not in the vicinity of $p_i$ by $T_0$.

Now we want to show that we can redraw at least one tree $X$ from $T_i$, while leaving all the trees from $T_0$ unchanged, so that the set of their induced 3-order, $\mathcal{C}(T_0 \cup \{X\})$, remains the same. Note that this system does not include any triples that contain at least two trees from $T_i$. This means that we can ignore the intersection points between two trees from $T_i$. Therefore, we can assume that every leaf of a tree $X$ from $T_i$ is an intersection point with a tree from $T_0$, by pruning parts of $X$, if necessary. While members of $T_i$ might become disjoint, it still holds that the members of $T_0 \cup \{X\}$ are pairwise intersecting.

If every tree from $T_i$ has at least four leaves, intersecting a tree of $T_0$ in each of them then we fix four such leaves for each $T_i$. Divide $T_i$ into $N_i$ groups depending on which 4 trees they intersect in their four fixed leaves. By Lemma [15] no group can have more than $c$ trees (where $c$ is as in the lemma). Thus if we choose $N_i \geq c(N_i)$ then we get a contradiction.

Thus there is a tree from $T_i$ which has at most three leaves, this is a redrawing of the original $T_i$ as a Y-shape crossing every tree from $T_0$, exactly as we wanted.

As $Tr$-$T3O = GC$-$T3O$ by Corollary [10] this finishes the proof of Theorem [1]

7 Open problems

**Problem 22.** Determine the smallest number of points which are not representable as a GC-$T3O$, as a C-$T3O$, etc.

**Problem 23.** Determine the containment relations among the subfamilies of $T3O$. Is our diagram in Figure [11] complete or did we miss something?

**Problem 24.** Determine the number/growth rate of $GC$-$T3O$, C-$T3O$, etc., on $n$ elements.

Finally, we would like to pose the following strengthening of the thrackle conjecture.

**Conjecture 25.** Suppose that we are given in the plane $n$ points, $P$, and $m$ topological trees that pairwise intersect exactly once such that each leaf of each tree is from $P$. We conjecture that $m \leq n$.

Note that if each tree consists of a single edge, then we get back the original thrackle conjecture. Instead of trees, our conjecture might even hold for forests. We could also weaken our conjecture by requiring that the branching points of the trees also need to be from $P$. Another interesting, and probably easier, special case is when we also require that each edge of each tree needs to be a segment between two points of $P$. 20
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