Brown spaces and the Golomb topology

Abstract

A Brown space is a topological space $X$ such that for all non-empty open subsets $U$ and $V$ of $X$, we have $\text{cl}_X(U) \cap \text{cl}_X(V) \neq \emptyset$. It is clear that Brown spaces are connected and not completely Hausdorff. Given $a, b \in \mathbb{N}$, whose greatest common divisor is 1, we consider the arithmetic progression $P_B(a, b) = \{b + an : n \in \mathbb{N} \cup \{0\}\}$. The family $B_G$ of all such arithmetic progressions is a base for a topology $\tau_G$ on $\mathbb{N}$. In this paper we show that for every $d \in \mathbb{N}$, the set $P_B(1, d)$ is a Brown space which is dense in $(\mathbb{N}, \tau_G)$. In particular, $(\mathbb{N}, \tau_G)$ is a Brown space. We also show that for each prime number $p$ and every natural number $c$, such that the greatest common divisor between $p$ and $c$ is 1, the set $P_B(p, c)$ is totally separated. We write some consequences of such result. For example the space $(\mathbb{N}, \tau_G)$ is not connected im kleinen at each of its points. This generalizes a result of Kirch AM. We also present a simpler proof of a result presented by Szczuka P. Some general properties of Brown spaces are also presented in this paper.

Introduction

We denote by $\mathbb{N}$ and $\mathbb{Z}$ by the sets of integers and of natural numbers, respectively, and we let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We also denote by $\mathbb{P}$ the set of prime numbers and consider that $\mathbb{P} \subset \mathbb{N}$. Given $a, b \in \mathbb{N}$, the symbol $(a, b)$ denotes the greatest common divisor of $a$ and $b$ and we consider the infinite arithmetic progressions

$$P(a, b) = \{b + an : n \in \mathbb{N}_0\} = b + a\mathbb{N}_0,$$

and

$$P_B(a, b) = \{b + an : n \in \mathbb{N}_0\} = b + a\mathbb{N}_0,$$

provided that $(a, b) = 1$.

For $a \in \mathbb{N}$ and $b \in \mathbb{Z}$ we also consider the infinite arithmetic progressions

$$P_f(a, b) = \{b + az : z \in \mathbb{Z}\} = b + a\mathbb{Z}$$

and $M(a) = \{an : n \in \mathbb{N}\}$.

Clearly $P(a, b) = P_B(a, b) \cap \mathbb{N}_0$ and $P_B(a, b) = P_B(a, b)$ if and only if $(a, b) = 1$. Note that $M(a) = P(a, a)$. In 1955 Furstenberg $\mathbb{N}$ showed that in the family $B_G = \{P_B(a, b) : (a, b) \in \mathbb{N} \times \mathbb{Z}\}$ is a base for a topology $\tau_f$ on $\mathbb{Z}$. The topological space $(\mathbb{Z}, \tau_f)$ is second countable and $\tau_f$, and hence metrizable. Moreover each basic set $P_B(a, b)$ is open and closed in $(\mathbb{Z}, \tau_f)$, so this space is zero-dimensional and not connected.

In 1959 and 1962 Golomb SW$^{4,5}$ showed in that the family $B_G = \{P_B(a, b) : (a, b) \in \mathbb{N} \times \mathbb{N} \land (a, b) = 1\}$

is a base for a topology $\tau_G$ on $\mathbb{N}$. Indeed $\tau_G = \{\emptyset\} \cup \{\{U \subset \mathbb{N} : \text{for each } b \in U \text{ there is } a \in \mathbb{N} \text{ such that } (a, b) = 1 \text{ and } P_B(a, b) \subset U\}$.

In,$^1$ first edition was published in 1970, $\tau_G$ is called the “relatively prime integer topology”, though in this paper, as well as in all the papers by Szczuka P.$^{5-10}$ we call $\tau_G$ the Golomb topology and the topological space $(\mathbb{N}, \tau_G)$ is called the Golomb space. It is known that $(\mathbb{N}, \tau_G)$ is second countable, $\tau_G$ and connected Theorems 2 and 3,$^4$ and Theorems 2 and 3.$^4$ Using the fact that for every $p \in \mathbb{P}$, the set $M(p)$ is closed in $(\mathbb{N}, \tau_G)$, Golomb SW$^4$ proved in both Theorem 1 & Theorem 1 that the set $\mathbb{P}$ is infinite. The proof of the connectedness of $(\mathbb{N}, \tau_G)$, as presented by Golomb SW,$^4$ uses Number Theory. As it is indicated in$^{4,5}$ “a proof of the connectedness of $(\mathbb{N}, \tau_G)$, without reference to Number Theory, was presented by Brown M$^{11}$ in the April 1953 meeting of the American Mathematical Society”, held in New York. Brown M$^{11}$ studied the space $(\mathbb{N}, \tau_G)$, though he did not publish his work. The abstract of his talk, published in$^{11}$ is the following one:

A countable connected Hausdorff space. The points are the positive integers. Neighborhoods are sets of integers $\{a + bx\}$, where $a$ and $b$ are relatively prime to each other $(x = 1, 2, 3, \ldots)$. Let $\{a + bx\}$ and $\{c + dx\}$ be two neighborhoods. It is shown that $bd$ is a limit point of both neighborhoods. Thus, the closures of any two neighborhoods have a nonvoid intersection. This is a sufficient condition that a space be connected.

This abstract served Clark PL,$^{12}$ in 2017, the authors of,$^{12}$ to coin the following term:

Definition 1.1 A Brown space is a topological space $X$ such that for all non-empty open subsets $U$ and $V$ of $X$, we have $\text{cl}_X(U) \cap \text{cl}_X(V) \neq \emptyset$. If $X$ is a topological space and $Y \subset X$, we say that $Y$ is a Brown space in $X$ if $Y$ is a subspace of $X$, is a Brown space.

The following result appears in [2, Proposition 6].

Theorem 1.2 Each Brown space $X$ is connected.

Proof. If $X$ is not connected, then there exist non-empty open and closed subsets $U$ and $V$ of $X$ such that $X = U \cup V$ and $U \cap V = \emptyset$. Then $\text{cl}_X(U) \cap \text{cl}_X(V) = U \cap V = \emptyset$, a contradiction to the fact that $X$ is a Brown space.

In this paper we will present some general properties of Brown spaces. We will give an explicit proof of the fact that $(\mathbb{N}, \tau_G)$ is a Brown space (Theorem 3.4). We will also show that, for every $d \in \mathbb{N}$, the subset $P_B(1, d)$ is a Brown space in $(\mathbb{N}, \tau_G)$ (Theorem 3.3). Since $(\mathbb{N}, \tau_G)$ is second countable, it is also Lindelöf. By the space $(\mathbb{N}, \tau_G)$ is not $T_1$.$^{12}$ Since every second countable space is Lindelöf, $^{12}$ every
Lindelöf and $T_2$ space is $T_{1.5}$, and hence $T_{1.5}$, the space $(\mathbb{N},\tau_2)$ is not $T_1$. Without using all these results from General Topology (that lead to the fact that every non-empty countable connected $T_2$ space is a one-point-set), in both Theorem 4 and Theorem 4, Golomb SW proved that $(\mathbb{N},\tau_0)$ is not $T_1$ by showing that for the closed set $M(2)$ in $(\mathbb{N},\tau_0)$ that do not contain the point $1$, there are no open subsets $U$ and $V$ in $(\mathbb{N},\tau_0)$ such that $1 \in U$, $M(2) \subset V$, and $U \cap V = \emptyset$.

Since compact $T_2$ spaces as well as locally compact and $T_2$ spaces are $T_{1.5}$, the space $(\mathbb{N},\tau_2)$ is not connected nor locally compact (compare with Theorem 5 and Theorem 5). The paper is divided in three sections. After this Introduction, in Section 2 we write the notation as well as some preliminary results that we will use in the paper. In this section we also write some general properties of Brown spaces. In Section 3 we write properties of the Golomb topology as well as of some subsets of it.

### Notation and preliminary results

In this paper we will use notation and results from both Number Theory and from General Topology: Concerning Number Theory, if $c,d \in \mathbb{Z}$ and $c \neq 0$, then the symbol $c \mid d$ means that there exists $a \in \mathbb{Z}$ such that $d = ca$. If $c,d \in \mathbb{Z}$ and $m \in \mathbb{N} - \{1\}$, then the symbol $c \equiv d(modm)$ means that $m \mid (c - d)$. The next result is proved in.

**Theorem 2.1** Let $a,b,q,r \in \mathbb{Z}$ be such that $a = bq + r$. Then $(a,b) = (b,r)$.

The following result was used in both [13, p. 169] and [1, p. 902] without proof. We will use it in Section 3 so, for completeness, we present a proof here.

**Theorem 2.2** Let $b \in \mathbb{Z}$. If, $p \in \mathbb{P}$ is such that $(b,p) = 1$, then for each $n,s \in \mathbb{N}_0$, $(pn + b, p^s) = 1$. (3)

**Proof.** If $s = 0$, then

$$(pn + b, p^s) = (pn + b, p^0) = (pn + b, 1) = 1, \quad \text{for each } n \in \mathbb{N}_0.$$ 

If $s = 1$, then since $(p,b) = 1$, by Theorem 2.1,

$$(pn + b, p^s) = (pn + b, p) = (p,b) = 1, \quad \text{for each } n \in \mathbb{N}_0.$$ (4)

Now assume that there exist $x_0 \in \mathbb{N} - \{1\}$ and $n_0 \in \mathbb{N}_0$ such that $g = (pn_0 + b, p^0) > 1$. Let $q \in \mathbb{P}$ be such that $q \mid g$. Then $q \mid (pn_0 + b)$ and $q \mid p^0$, so $q = p$. Hence $q \mid (pn_0 + b)$ and $q \mid p$, so $q \mid (pn_0 + b, p)$. This implies, using (4) with $n = n_0$, that $q \mid 1$, a contradiction. Thus $(pn + b, p^s) = 1$, for every $n \in \mathbb{N}_0$.

Concerning General Topology, given a topological space $X$ and $A \subset X$, we denote by $cl(A)$ and by $int(A)$ the closure and the interior of $A$ in $X$, respectively. In particular, for $A \subset \mathbb{N}$, the symbol $cl(A) = \mathbb{N}$ denotes the closure of $A$ in $(\mathbb{N},\tau_0)$. We consider the closed interval $[0,1]$ with its usual topology. Recall that a topological space $X$ is said to be $T_i$ if for each $x \in X$, the set $\{x\}$ is closed in $X$; $T_{0.5}$ Hausdorff if for every $x$, $y \in X$ such that $x \neq y$, there exist open sets $U$ and $V$ in $X$ so that $x \in U$, $y \in V$ and $U \cap V = \emptyset$; $T_1$ or completely Hausdorff if for every $x$, $y \in X$ with $x \neq y$, there exist open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $cl(U) \cap cl(V) = \emptyset$.

**Theorem 2.3** No Brown space $X$, with at least two points, is $T_{2.5}$.

By Theorem 2.3 no connected $T_1$, with at least two points, is a Brown space.

Let $X$ be a topological space and $x \in X$. We say that $X$ is indiscrete in $x$ or that $X$ is an indiscrete point of $X$ if the only open subset of $X$ that contains $x$ is $X$ itself. We say that $X$ is indiscrete if its topology is the indiscrete topology. Note that $X$ is indiscrete if and only if every point of $X$ is indiscrete. The following result is proved in [2, Proposition 6].

**Theorem 2.4** Let $X$ be a topological space. Then

- if $X$ contains an indiscrete point, then $X$ is a Brown space;
- if $X$ is a Brown space, then $X$ is regular if and only if $X$ is indiscrete.

Note that a $T_1$ space contains no indiscrete points. By Theorem 2.4, no connected $T_1$ space, with at least two points, is a Brown space. We also have that if $X$ is a connected regular space without indiscrete points, then $X$ is not a Brown space. Hence the converse of Theorem 1.2 is not true.

We say that a topological property $P$ is

- hereditary if for any space $X$ that has the property $P$, every subspace of $X$ also has the property $P$;
- multiplicative if for any family $\{X_s : s \in S\}$ of topological spaces with the property $P$, the Cartesian product $\prod_{s \in S}X_s$, with the product topology, also has the property $P$;
- factorizable if for any family $\{X_s : s \in S\}$ of topological spaces with the Cartesian product $\prod_{s \in S}X_s$, with the product topology, has the
property $P$ then each factor $X_s$ also has the property $P$.

**Theorem 2.5** Let $X$ and $Y$ be topological spaces and $f : X \to Y$ be a continuous and surjective function. If $X$ is a Brown space, then $Y$ is a Brown space.

Proof. Let $U$ and $V$ be non-empty open subsets of $Y$. Since $f$ is continuous and surjective, $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty open subsets of $X$. Hence, since $X$ is a Brown space, $cl_X(f^{-1}(U)) \cap cl_X(f^{-1}(V)) \neq \emptyset$. By continuity of $X$ we have $\emptyset \neq cl_X(f^{-1}(U)) \cap cl_X(f^{-1}(V)) = f^{-1}(cl_Y(U)) \cap f^{-1}(cl_Y(V)) = f^{-1}(cl_Y(U) \cap cl_Y(V))$.

Hence $cl_X(U) \cap cl_X(V) \neq \emptyset$, so $Y$ is a Brown space.

By Theorem 2.5 being a Brown space is a topological property. Moreover,

**Theorem 2.6** Being a Brown space is both a multiplicative and a factorizable property.

Proof. Let $\{X_s : s \in S\}$ be a family of non-empty topological spaces. Let $X = \prod_{s \in S} X_s$ and assume that $X$ has the product topology. For each $t \in S$ the projection $p_t : X \to X_t$ defined for any $x = (x_s)_s \in X$ by $p_t(x) = x_t$ is continuous and surjective. Hence if $X$ is a Brown space then, by Theorem 2.5, $X_t$ is a Brown space too. This shows that being a Brown space is a factorizable property.

Now assume that each $X_s$ is a Brown space. Let $U$ and $V$ be two non-empty open subsets of $X$. Fix $x \in U$ and $y \in V$ and assume that $B = \prod_{s \in S} B_s$ and $C = \prod_{s \in S} C_s$ are basic subsets of $X$ such that $x \in B \subset U$ and $y \in C \subset V$. For each $s \in S$, the sets $B_s$ and $C_s$ are non-empty and open in the Brown space $X_s$, so $cl_{X_s}(B_s) \cap cl_{X_s}(C_s) \neq \emptyset$. Then

$$cl_X(B) \cap cl_X(C) = \prod_{s \in S} cl_{X_s}(B_s) \cap \prod_{s \in S} cl_{X_s}(C_s) = \prod_{s \in S} (cl_{X_s}(B_s) \cap cl_{X_s}(C_s)) \neq \emptyset.$$  

Hence $\emptyset \neq cl_X(B) \cap cl_X(C) \subset cl_X(U) \cap cl_X(V)$, so $X$ is a Brown space. This shows that being a Brown space is a multiplicative property.

In Theorem 3.5 we will show that being a Brown space is not a hereditary property. If $X$ and $Y$ are topological spaces, $f : X \to Y$ is a quotient mapping and $X$ is a Brown space then, by Theorem 2.5, $Y$ is a Brown space too.

A topological space $X$ is said to be 

**hereditarily disconnected** if no non-empty connected subset of $X$ contains more than one point;

**totally separated** if for every $x, y \in X$ with $x \neq y$, there exist open sets $U$ and $V$ in $X$ such that $x \in U \subset X \cup V$ and $y \in V \subset X$ (and $U \cap V = \emptyset$);

**zero-dimensional** is $X$ is $T_1$ and has a base consisting of open and closed sets.

Hereditarily disconnected spaces are also called **totally disconnected**. In [14] it is shown that zero-dimensional spaces are hereditarily disconnected. In [1] it is proved that if $X$ is hereditarily disconnected and locally compact, then $X$ is zero-dimensional. Note that a space $X$ is hereditarily disconnected if and only if, for each $x \in X$, the component $C_x$ of $X$ that contains $x$ is a one-point-set, namely $\{x\}$. Note also that $X$ is totally separated if and only if, for every $x \in X$, the quasi-component $Q_x$ of $X$ that contains $x$ is a one-point-set, namely $\{x\}$. Since $C_x \subseteq Q_x$ for each $x \in X$, totally separated spaces are hereditarily disconnected.

Let $X$ be a topological space and $a, b \in X$. We say that $X$ is **almost connected im kleinen** at $x$, if for any open subset $U$ of $X$ such that $x \in U$, there exists an open and connected subset $U$ of $X$ such that $x \in U \subset \subset U$.

Let $X$ be a topological space and $x \in X$. We say that $X$ is **locally connected at** $X_s$, if for any open subset $U$ of $X$ such that $x \in U \subset \subset U$.

Let $X$ be a topological space and $x \in X$. We say that $X$ is **connected im kleinen** at $X_s$, if for any open subset $U$ of $X$ such that $x \in U \subset \subset U$.

Let $X$ be a topological space and $x \in X$. We say that $X$ is **connected** if and only if, for each $x, y \in X$, there exists a connected subset $U$ of $X$ such that $x \in U \subset \subset U$.

We say that $X$ is **almost connected im kleinen** if $X$ is almost connected im kleinen at each of its points;

$X$ is **locally connected** if $X$ is locally connected at each of its points.

Clearly, if $X$ is locally connected at $x$, then $X$ is connected im kleinen at $x$. The converse of this implication is not true. We construct an example in $\mathbb{R}^2$, with the usual topology. For each $i \in \mathbb{N}$, let $q_i = \left(\frac{1}{i}, 0\right) \in \mathbb{R}^2$ and let $L_{i,n}$ be the straight line segment from $q_{i-1}$ to $q_i$. For each $(i, n) \in \mathbb{N} \times \mathbb{N}$, we consider

$$p_{i,n} = \frac{1}{i + 1} \cdot \frac{1}{n} \in \mathbb{R}^2,$$

where $y = 1 - \frac{1}{i}$ is the equation of the straight line in $\mathbb{R}^2$ that contains the points $p_{i,n}$ and $q_i$. For each $i \in \mathbb{N}$, we define

$$X_i = \bigcup_{n \in \mathbb{N} \cup \{0\}} L_{i,n}.$$  

Then $X_i = cl_X(\bigcup_{n \in \mathbb{N} \cup \{0\}} X_i)$ is a topological space which is compact, connected, connected im kleinen at $(0, 0)$ and not locally connected at $(0, 0)$. The space $X_i$, called the **infinite broom** in [5].

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Now assume that $X$, is connected im kleinen at $(0,0)$, and not locally connected at $(0,0)$. In\textsuperscript{16} it is shown that if a topological space $X$ is connected im kleinen at each of its points, then $X$ is locally connected. Let

$$
Y = \left( \bigcup_{n=1}^{\infty} L_{n,\theta} \right) - \left\{ y_{i} : i \in \mathbb{N} \right\} \subseteq X_{a}.
$$

Note that $X_{a}$ is almost connected im kleinen at any point $p \in Y$ and not connected im kleinen at such point $p$. Note also that if a $T_{3}$ space is connected im kleinen at $x \in X_{a}$, then it is almost connected im kleinen at $x$.

**Properties of the golomb space**

In the space $(\mathbb{N}, \tau_{G})$ a non-empty subset $U$ of $\mathbb{N}$ is open if and only if, for every $b \in U$, there exists $a \in \mathbb{N}$ such that $(a, b) = 1$ and $P_{G}(a, b) \subseteq U$ (compare with). Thus $U$ is infinite. In particular any subset of $\mathbb{N}$ with non-empty interior in $(\mathbb{N}, \tau_{G})$ is infinite. Let $a, x \in \mathbb{N}$. In Szczuka\textsuperscript{17} presented some results that involve the set $\text{cl} (P(a, x))$. In the she showed that if $x_{1} = x (\text{mod} \, a)$, $x_{1} \leq a$ and $\langle a, x \rangle = 1$, then $P(a, x_{1}) \subseteq \text{cl} (P(a, x))$. In\textsuperscript{8} she showed that if $a$ and $x$ are odd and $\langle a, x \rangle = 1$, then $\text{cl} (P_{G}(a, x)) = \text{cl} (P_{G}(2a, x))$.

We show the following result.

**Theorem 3.1** Let $a, x \in \mathbb{N}$ be so that $\langle a, x \rangle = 1$. Then $M(a) = \text{cl} (P_{G}(a, x))$.

**Proof.** Let $ab \in M(a)$ and let $W$ be an open subset of $(\mathbb{N}, \tau_{G})$ such that $ab \in W$. There exists $d \in \mathbb{N}$ such that $(d, ab) = 1$ and $P_{G}(d, ab) \subseteq W$. Assume first that $d = 1$. Then $P_{G}(a, x) = \{ x + n : n \in \mathbb{N}_{\geq 0} \}$ so if $x \leq ab$, then $ab \in P_{G}(a, x) \subseteq \text{cl} (P_{G}(a, x))$. If $ab < x$ then, since $P_{G}(d, ab)$ is infinite, there is $n \in \mathbb{N}_{\geq 0}$ such that $dn + ab \geq x$, so $ab + n \in P_{G}(ab, x) \subseteq W \cap P_{G}(a, x)$. This shows that $ab \in \text{cl} (P_{G}(a, x))$.

Now assume that $a \geq 2$. By showing that $P_{G}(d, ab) \cap P_{G}(a, x) \neq \emptyset$ we will obtain that $W \cap P_{G}(a, x) \neq \emptyset$. To prove that $P_{G}(d, ab) \cap P_{G}(a, x) \neq \emptyset$, assume first that $d = 1$. Then $ab + x \in P_{G}(a, x)$ and $ab + x = 1 \cdot x + ab \in P_{G}(d, ab)$. Now assume that $d \geq 2$. Since $d \geq 2$ and $a \geq 2$, an element $z \in P_{G}(d, ab) \cap P_{G}(a, x)$ satisfies the system of congruences

$$z = ab \pmod{a} \quad \text{and} \quad z = x \pmod{a}. \quad (5)$$

Hence, an element in $P_{G}(d, ab) \cap P_{G}(a, x)$ is a solution of the system $(5)$. Conversely, every solution of the system $(5)$ is an element of $P_{G}(d, ab) \cap P_{G}(a, x)$. Let us show, then, that the system $(5)$ has a solution. If $(d, a) \neq 1$, then there exists $p \in \mathbb{P}$ such that $p | d$ and $p | a$. Then $p | d$ and $p | ab$, so $(d, ab) \neq 1$, a contradiction. Hence $(d, a) = 1$ and by the Chinese Remainder Theorem, the system $(5)$ has a solution. This shows that $P_{G}(d, ab) \cap P_{G}(a, x) \neq \emptyset$. Hence $W \cap P_{G}(a, x) \neq \emptyset$ and then $ab \in \text{cl} (P_{G}(a, x))$.

**Corollary 3.2** For every $d \in \mathbb{N}$, the set $P_{G}(1, d)$ is dense in $(\mathbb{N}, \tau_{G})$.

**Proof.** Let $d \in \mathbb{N}$. Put $a = 1$. Then $(a, d) = 1$, $M(a) = \mathbb{N}$ and, by Theorem 3.1, $\mathbb{N} = M(a) \subseteq \text{cl} (P_{G}(a, d))$, so $P_{G}(a, d) = P_{G}(1, d)$ is dense in $(\mathbb{N}, \tau_{G})$.

**Theorem 3.3** For every $d \in \mathbb{N}$, the set $P_{G}(1, d)$ is a Brown space in $(\mathbb{N}, \tau_{G})$. In particular, $P_{G}(1, d)$ is connected.

**Proof.** Let $U$ and $V$ be two non-empty open subsets of $P_{G}(1, d)$. Since $P_{G}(1, d)$ is open in $(\mathbb{N}, \tau_{G})$, the sets $U$ and $V$ are open in $(\mathbb{N}, \tau_{G})$. Let $x \in U$ and $y \in V$. There then exist $a, b \in \mathbb{N}$ such that $(a, x) = (b, y) = 1$ and $P_{G}(a, x) \subseteq U$ and $P_{G}(b, y) \subseteq V$. If $a = 1$, then $P_{G}(a, x) = \{ x + n : n \in \mathbb{N}_{\geq 0} \}$. Since $P_{G}(b, y)$ is infinite, there is $n_{0} \in \mathbb{N}$ such that $y + nb_{0} \geq x$ and then

$$y + nb_{0} \in P_{G}(a, x) \cap P_{G}(b, y) \subseteq U \cap V \cap P_{G}(a, x) \subseteq \text{cl}(U) \cap \text{cl}(V) \subseteq P_{G}(1, d) = \text{cl}(P_{G}(1, d)) \cap \text{cl}(U \cap V) \subseteq \text{cl}(U \cap V) \cap P_{G}(1, d) = \text{cl}(P_{G}(1, d)) \cap \text{cl}(U \cap V).
$$

Hence $\text{cl}(P_{G}(1, d)) \cap \text{cl}(U \cap V) \neq \emptyset$. Similarly, if $b = 1$ we have $\text{cl}(P_{G}(1, d)) \cap \text{cl}(U \cap V) \neq \emptyset$. Now assume that $a \geq 2$ and $b \geq 2$.

By Theorem 3.1, $M(a) \subseteq \text{cl}(P_{G}(a, x))$ and $M(b) \subseteq \text{cl}(P_{G}(b, y))$, so in particular, $\text{cl}(P_{G}(a, x)) \cap \text{cl}(P_{G}(b, y)) \subseteq P_{G}(1, d)$. Since $a \geq 2$ and $b \geq 2$, we have $d < ab$, so $ab \in P_{G}(1, d)$. Then

$$ab \in \text{cl}(P_{G}(a, x)) \cap \text{cl}(P_{G}(b, y)) \subseteq \text{cl}(U) \cap \text{cl}(V) \subseteq P_{G}(1, d) = \text{cl}(P_{G}(1, d)) \cap \text{cl}(U \cap V).$$

Hence $\text{cl}(P_{G}(1, d)) \subseteq \text{cl}(U \cap V) \neq \emptyset$, so $P_{G}(1, d)$ is a Brown space in $(\mathbb{N}, \tau_{G})$. Since Brown spaces are connected, $P_{G}(1, d)$ is connected.

Now we present an explicit proof of what Brown M\textsuperscript{17} claimed in assertion (B) of Section 1.

**Theorem 3.4** $(\mathbb{N}, \tau_{G})$ is a Brown space. In particular, $(\mathbb{N}, \tau_{G})$ is connected and it is not $T_{1}$.

**Proof.** Since $P_{G}(1, 1) = \mathbb{N}$ the result follows from Theorem 3.3 and the fact that Brown spaces are not $T_{1}$.

Now we prove the following result.

**Theorem 3.5** Let $c, p \in \mathbb{P}$ be such that $p \in \mathbb{P}$ and $\langle p, c \rangle = 1$. Take $a, b \in P_{G}(p, c)$ such that $a < b$ and $m, n \in \mathbb{N}_{\geq 0}$ so that $a = pm + x, b = pn + x$ and $0 \leq m < n$. Then

$$U = \bigcup_{i=0}^{m} P_{G}(p^{i+1}, p^{i} + c) \quad \text{and} \quad V = \bigcup_{j=m+1}^{n} P_{G}(p^{j-1}, p^{j} + c) \quad (6)$$

are open subsets of $(\mathbb{N}, \tau_{G})$ such that $a \in U, b \in V, U \cap V = \emptyset$.

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and $P_d(p, c) = U \cup V$. In particular, $P_d(p, c)$ is not connected.

Proof. Since $p \in \mathbb{P}$ and $(p, c) = 1$, by Theorem 2.2,
$$(p^{\pm_1}, pi + c) = 1, \quad \text{for each } i \in \mathbb{N}_0.$$ Then we can consider the sets $U$ and $V$ indicated in (6), and they are open in $(\mathbb{N}, \tau_G)$. Since $0 \leq m < n < 2^i \leq p^a$, we have $m + 1 \leq n \leq p^a - 1$, so
$$a = pm + c = p^{\pm_1}(0) + pm + c \in P_d(p^{\pm_1}, pm + c) = U$$
and
$$b = pn + c = p^{\pm_1}(0) + pn + c \in P_d(p^{\pm_1}, pn + c) = V.$$ Hence $a \in U$ and $b \in V$, so $U$ and $V$ are infinite. Now assume that $a \in U \cap V$. Let $i \in \{0, 1, \ldots, m\}$ and $j \in \{m + 1, m + 2, \ldots, p^a - 1\}$ be such that $z \in P_d(p^{\pm_1}, pi + c)$ and $z \in P_d(p^{\pm_1}, pj + c)$. Since $p^{\pm_1} > 1$, it follows that
$$z = pi + (\text{mod} p^{\pm_1}) \text{ and } z = pj + (\text{mod} p^{\pm_1}).$$ Hence $pi + c = pj + (\text{mod} p^{\pm_1})$, so $pi = pj (\text{mod} p^{\pm_1})$. This implies that $i = j (\text{mod} p^{\pm_1})$. However, since the set $\{0, 1, \ldots, m, m + 1, \ldots, p^a - 1\}$ is a complete system of reminders modulus $p^a$ and $i \leq m < m + 1 < j$, we have $i \neq j (\text{mod} p^{\pm_1})$. From this contradiction we infer that $U \cap V = \varnothing$.

Now we show that $U \cup V = P_d(p, c)$. If $z \in U$, then there exists $i \in \{0, 1, \ldots, m\}$ such that $z \in P_d(p^{\pm_1}, pi + c)$. Let $z_0 \in \mathbb{N}_0$ be such that $z = p^{\pm_1}z_0 + pi + c$. Then $z = p(p^{\pm_1}z_0 + i) + c \in P_d(p, c)$. Similarly, if $z \in V$, there is $j \in \{m + 1, m + 2, \ldots, p^a - 1\}$ such that $z \in P_d(p^{\pm_1}, pj + c)$. Let $z_1 \in \mathbb{N}_0$ be such that $z = p^{\pm_1}z_1 + pj + c$. Then $z = p(p^{\pm_1}z_1 + j) + c \in P_d(p, c)$. This shows that $U \cup V = P_d(p, c)$. To prove the other inclusion, let $z \in P_d(p, c)$. Then there exists $k \in \mathbb{N}_0$ such that $z = pk + c$. Using the Division Algorithm we obtain $s, t \in \mathbb{N}_0$ such that $k = p^s + t$ and $0 \leq t < p^a$. Hence
$$z = pk + c = p(p^s + t) + c = p^{\pm_1}s + pi + c.$$ Since $0 \leq t < p^a$ and $0 \leq m < p^a$, we obtain
$$t \in \{0, 1, \ldots, p^a - 1\} \cup \{m + 1, m + 2, \ldots, p^a - 1\}.$$ Hence, by (7),
$$z \in \left( \bigcup_{i=0}^{p^{a-1}} P_d(p^{\pm_1}, pi + c) \right) \cup \left( \bigcup_{j=m+1}^{p^{a-1}} P_d(p^{\pm_1}, pj + c) \right) = U \cup V.$$ This shows that $P_d(p, c) \subset U \cup V$. Hence, $U \cup V = P_d(p, c)$. This completes the first part of the proof. Since $P_d(p, c)$ has been written as the union of two non-empty open subsets of $(\mathbb{N}, \tau_G)$, which are disjoint, the set $P_d(p, c)$ is not connected.

Corollary 3.6 Let $c, p \in \mathbb{N}$ be such that $p \in \mathbb{P}$ and $(p, c) = 1$. Take $a, b \in P_d(p, c)$ such that $a \neq b$. Then there exist open sets $U$ and $V$ in $(\mathbb{N}, \tau_G)$ such that $a \in U$, $b \in V$, $U \cap V = \varnothing$ and $P_d(p, c) = U \cup V$.

Corollary: Brown spaces and the Golomb topology.

This completes the first part of the proof. Since $P_d(p, c)$ has been written as the union of two non-empty open subsets of $(\mathbb{N}, \tau_G)$, which are disjoint, the set $P_d(p, c)$ is not connected.
Proof. Assume first that $P(a,c)$ is connected. If $\Theta(a) \not\subseteq \Theta(c)$, then $a > 1$ and there exists $p \in \Theta(a) - \Theta(c)$. Hence $p \in \mathbb{P}$, $p \mid a$ and $p \nmid c$, so $\langle p,c \rangle = 1$. Since $p \mid a$, we have $P(a,c) \subseteq P_p(a,c)$. By Theorem 3.7, $P_p(a,c)$ is hereditarily disconnected, so $P(a,c)$ is a one-point-set, a contradiction. This shows that $\Theta(a) \subseteq \Theta(c)$.

Now assume that $\langle a,c \rangle = 1$ and that $P^c_p(a,c) = P(a,c)$ is connected. Then, by the first part of the theorem, $\Theta(a) \subseteq \Theta(c)$. If $a \geq 2$, then there is $p \in \mathbb{P}$ such that $p \mid a$. Then $p \mid c$, contradicting that $\langle a,c \rangle = 1$. Hence $a = 1$. In Theorem 3.3 we proved that $P_p(1,c)$ is connected.

In Theorem 3.3 Szczuka P. showed the following result:

**Theorem 3.11** Let $a,c \in \mathbb{N}$. The arithmetic progression $P(a,c)$ is connected in $(\mathbb{N},\tau_G)$ if and only if $\Theta(a) \subseteq \Theta(c)$. In particular:

1. the progression $\{an : n \in \mathbb{N}\}$ is connected in $(\mathbb{N},\tau_G)$
2. if $\langle a,c \rangle = 1$, then $P_p(a,c)$ is connected in $(\mathbb{N},\tau_G)$ if and only if $a=1$.

The proof of the “only if” part of Theorem 3.11 is much simpler is we know that the progressions $P_p(a,c)$ are hereditarily disconnected, as we presented in the proof of Theorem 3.10. In it is claimed that the fact that each set $P_p(1,c)$ is connected is obvious.

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**Conflicts of interest**

Authors declare that there is no conflict of interest.

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