Dedicated to the 70th anniversary of Peter Olver

NATURAL DIFFERENTIAL INVARIANTS AND EQUIVALENCE OF THIRD ORDER NONLINEAR DIFFERENTIAL OPERATORS

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Abstract. We give a description of the field of rational natural differential invariants for a class of nonlinear differential operators of the third order on a two dimensional manifold and show their application to the equivalence problem of such operators.

1. Introduction

In this paper, we continue to study rational differential invariants of differential operators on two-dimensional manifolds. Here we consider a class of nonlinear operators that in local coordinates $x_1, x_2$ have the following form:

$$A_w: f \mapsto \sum_{i,j,k} a_{ijk}(x,f) \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} + \sum_{i,j} a_{ij}(x,f) \frac{\partial^2 f}{\partial x_i \partial x_j}$$

$$+ \sum_{i,j} a_i(x,f) \frac{\partial f}{\partial x_i} + a_0(x,f)f, \quad (1)$$

where $x = (x_1, x_2)$, $f = f(x)$.

We call such operators as weakly nonlinear operators.

In paper [5], we found rational differential invariants of the 2nd order weakly nonlinear operators and used them to solve the local equivalence problem.

Here we use the methods of [5] to find rational differential invariants for the 3rd order weakly nonlinear operators and apply them to solve the local as well as the global equivalence problem.

Additionally we will assume that all coefficients $a(x,y)$ of these operators are smooth in $x$ and rational in $y$.

It is easy to see that the pseudogroup of local diffeomorphisms (in variables $x$) naturally acts in this class of operators.

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The main step in studies of such operators is a representation of operator $A_w$ as a pair $(A, f)$ (we call it as a (related pair)), where

$$A = \sum_{i,j,k} a_{ijk}(x, y) \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} + \sum_{i,j} a_{ij}(x, y) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i a_i(x, y) \frac{\partial}{\partial x_i} + a_0(x, y) \quad (2)$$

is a linear third order linear differential operator on the extended space $(x, y)$.

The procedure of descent allows us to get differential invariants of weakly nonlinear operators from invariants of related pairs and linear operators of the form

$$Af = \sum_{i,j,k} a_{ijk}(x, f) \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} + \sum_{i,j} a_{ij}(x, f) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i a_i(x, f) \frac{\partial}{\partial x_i} + a_0(x, f) \quad (3)$$

Using this procedure and a machinery of reduced Gröbner bases, we propose a method to find invariants of the 3rd order weakly nonlinear operators.

2. Notations

In this paper, we use the same notations as in [3, 4].

Let $M$ be $n$-dimensional manifold and let $\tau : TM \to M$ and $\tau^* : T^*M \to M$ be respectively tangent and cotangent bundles over $M$.

We will denote by $\text{Diff}_k(M)$ the module of linear $k$-th order operators, acting in $C^\infty(M)$. The corresponding vector bundle of such operators we denote by $\chi_k : \text{Diff}_k \to M$, thus $\text{Diff}_k(M)$ is also the module of smooth sections of $\chi_k$, $\text{Diff}_k(M) = C^\infty(\chi_k)$.

The group of diffeomorphisms of $M$ will be denoted by $\mathcal{G}(M)$ and the multiplicative group of nowhere vanishing functions on $M$ will be denoted as $\mathcal{F}(M)$.

We denote by $\Sigma^k(M)$ and $\Sigma_k(M)$ the modules of symmetric $k$-forms and $k$-vectors respectively.

Also, we denote by $\Omega^k$ the modules of exterior $k$-forms.

3. Linear differential operators on $M$

In this section, we recall the necessary facts from [3] on scalar linear third order differential operators on two-dimensional manifolds.

Let $M$ be 2-dimensional smooth oriented connected manifold and let

$$\pi : M \times \mathbb{R} \to M$$
Third order differential operators

be trivial line bundle. Denote by
\[ \pi_k : J^k(\pi) \to M \]
the bundles of \( k \)-jets of smooth sections of \( \pi \) (i.e. \( k \)-jets of smooth functions on \( M \)).

Let \( G(M) \) be the group of diffeomorphisms of the manifold \( M \).

Then, together with the action of \( G(M) \) on \( M \), we have also the actions \( G(M) \) in the bundles \( \pi_k \) by prolongations of diffeomorphisms.

The prolongations of diffeomorphisms \( \phi \in G(M) \) in the bundles \( \pi_k \) will be denoted by \( \phi^{(k)} \).

3.1. Let
\[ \chi_3 : \text{Diff}_3(M) \to M \]
be the bundle of the third order linear differential operators on \( M \) and
\[ \chi_{3,k} : J^k(\chi_3) \to M \]
be bundles of their \( k \)-jets of sections of \( \chi_3 \).

The group \( G(M) \) acts on operators \( A \in \text{Diff}_3(M) \) in natural way:
\[ \phi_* : A \to \phi_*(A) = \phi_* \circ A \circ \phi_*^{-1}, \]
where the morphism \( \phi_* : C^\infty(M) \to C^\infty(M) \) is defined by the formula
\[ \phi_* = (\phi^{-1})^* : h \mapsto h \circ \phi^{-1}. \]

3.2. Symbols. By the symbol \( \sigma_{3,A} \) of the operator \( A \), we means the equivalence class
\[ \sigma_{3,A} = A \mod \text{Diff}_2(M). \]

Let an operator \( A \in \text{Diff}_3(M) \) be represented in local coordinates \( x_1, x_2 \) of \( M \) in the form
\[ A = a_1 \partial_1^3 + 3a_2 \partial_1^2 \partial_2 + 3a_3 \partial_1 \partial_2^2 + a_4 \partial_2^3 \]
\[ + b_1 \partial_1^2 + 2b_2 \partial_1 \partial_2 + b_3 \partial_2^2 + c_1 \partial_1 + c_2 \partial_2 + a_0, \quad (4) \]
where the all coefficients are smooth functions of \( x = (x_1, x_2) \), \( \partial_1 \) and \( \partial_2 \) are \( \partial_{x_1} \) and \( \partial_{x_2} \) respectively.

Then the symbol \( \sigma_{3,A} \) is identified with the symmetric 3-vector
\[ \sigma_{3,A} = a_1 \partial_1^3 + 3a_2 \partial_1^2 \partial_2 + 3a_3 \partial_1 \partial_2^2 + a_4 \partial_2^3 \in \Sigma_3(M), \quad (5) \]
where dots and degrees are symmetric products of the vector fields \( \partial_1, \partial_2 \).

The symbol \( \sigma_{3,A} \) is regular (at point or in domain) if it as a homogeneous cubic polynomial on the cotangent bundle \( T^*(M) \) has distinct roots.

Denote by \( \Delta(\sigma_{3,A}) \) the discriminant of \( \sigma_{3,A} \), then
\[ \Delta(\sigma_{3,A}) = 6a_1a_2a_3a_4 - 4(a_1a_3^3 + a_4a_2^3) + 3a_2^2a_3^2 - a_1^2a_4^2. \]

Recall that the symbol \( \sigma_{3,A} \) has three distinct real roots if \( \Delta(\sigma_{3,A}) > 0 \) and one real and two complex roots if \( \Delta(\sigma_{3,A}) < 0 \).
Thus, we say that the operator $A$ is regular if its symbol $\sigma_{3,A}$ is regular or $\Delta(\sigma_{3,A}) \neq 0$

Moreover, we say that an operator $A$ is hyperbolic if $\Delta(\sigma_{3,A}) > 0$ and ultrahyperbolic if $\Delta(\sigma_{3,A}) < 0$.

Locally, the symbol of a hyperbolic operator can be presented as a symmetric product of pair wise linear independent vector fields $\sigma_{3,A} = X_1 \cdot X_2 \cdot X_3$, and therefore there are local coordinates $x_1, x_2$ such that

$$\sigma_{3,A} = (a\partial_1 + b\partial_2) \cdot \partial_1 \cdot \partial_2,$$

where $a$ and $b$ are smooth functions and $ab \neq 0$.

For the case of ultrahyperbolic operators we have $\sigma_{3,A} = X \cdot q$, where $X$ is a nonzero vector field and $q \in \Sigma_2$ is a positive symmetric 2-vector. Therefore, there are local coordinates $x_1, x_2$ such that

$$\sigma_{3,A} = (a\partial_1 + b\partial_2) \cdot (\partial_1^2 + \partial_2^2),$$

where $a$ and $b$ are smooth functions and $a^2 + b^2 > 0$.

3.3. **Wagner connections.** The following result due to Wagner [7, 3].

**Theorem 1.**

1. Let $A \in \text{Diff}_3(M)$ be a regular differential operator. Then there exist a unique linear connection $\nabla$ such that the symbol $\sigma_{3,A}$ is parallel with respect to $\nabla$.

2. The curvature tensor of $\nabla$ is zero.

We call this connection Wagner’s connection.

In the case of hyperbolic symbol, we choose local coordinates in $M$ such that $\sigma$ has form (6). Then the non zero Christoffel coefficients $\Gamma^i_{jk}$ of the Wagner connection are the following:

$$\Gamma^1_{11} = \frac{1}{3} (\ln \frac{b}{a^2})_{x_1}, \quad \Gamma^2_{22} = \frac{1}{3} (\ln \frac{a}{b^2})_{x_2},$$

$$\Gamma^1_{12} = \frac{1}{3} (\ln \frac{b}{a^2})_{x_2}, \quad \Gamma^2_{21} = \frac{1}{3} (\ln \frac{a}{b^2})_{x_1}.$$

In the ultrahyperbolic case (7) we have the following nonzero Christoffel coefficients:

$$\Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{6} (\ln(a^2 + b^2))_{x_2},$$

$$\Gamma^1_{21} = -\Gamma^2_{11} = \frac{a x_2 b - a b x_1}{a^2 + b^2}, \quad \Gamma^1_{22} = -\Gamma^2_{12} = \frac{a b x_2 - a x_2 b}{a^2 + b^2}.$$

**Corollary 2.** The torsion form $\theta$ of the Wagner connection is

$$\theta = \frac{1}{3} (\ln \frac{b^2}{a})_{x_1} dx_1 + \frac{1}{3} (\ln \frac{a^2}{b})_{x_2} dx_2$$

for the hyperbolic case and

$$\theta = \frac{a b x_2 - a x_2 b}{a^2 + b^2} dx_1 + \frac{a b x_1 - a x_1 b}{a^2 + b^2} dx_2 - \frac{1}{6} d(\ln(a^2 + b^2))$$

for the ultrahyperbolic case.
3.4. Symbols and quantization. Let $\Sigma = \oplus_{k \geq 0} \Sigma^k(M)$ be the graded algebra of symmetric differential forms and let $\nabla$ be the Wagner connection associated with a regular symbol from $\Sigma^3(M)$. Then the covariant differential

$$d_{\nabla} : \Omega^1(M) \longrightarrow \Omega^1(M) \otimes \Omega^1(M)$$

define derivation

$$d^s_{\nabla} : \Sigma^* \longrightarrow \Sigma^{*+1}$$

of degree one in graded symmetric algebra $\Sigma = \oplus_{k \geq 0} \Sigma^k(M)$. Namely, this derivation is defined by its action on generators, and we have

$$d^s_{\nabla}(\alpha_k) = d : C^\infty(M) \longrightarrow \Omega^1(M) = \Sigma^1,$$

$$d^s_{\nabla} : \Omega^1(M) = \Sigma^1 \longrightarrow \Omega^1(M) \otimes \Omega^1(M) \xrightarrow{\text{Sym}} \Sigma^2.$$

Let now $\alpha_k \in \Sigma_k(M)$. We define a differential operator $Q(\alpha_k) \in \text{Diff}_k(M)$ as follows:

$$Q(\alpha_k)(h) \overset{\text{def}}{=} \frac{1}{k!} \left\langle \alpha_k, (d^s_{\nabla})^k(h) \right\rangle,$$

where $h \in C^\infty(M)$, $(d^s_{\nabla})^k(h) \in \Sigma^k(M)$, and $\langle \cdot, \cdot \rangle$ is the standard convolution

$$\Sigma_k(M) \otimes \Sigma^k(M) \longrightarrow C^\infty(M).$$

Remark, that the value of the symbol of the derivation $d^s_{\nabla}$ at a covector $\theta$ equals to the symmetric product by $\theta$ into the module $\Sigma^*$. Therefore, the symbol of operator $Q(\alpha_k)$ equals $\alpha_k$ as the symbol of a composition of operators equals the composition of symbols.

We call differential operator $Q(\alpha_k)$ a quantization of symbol $\alpha_k$.

Let now $A \in \text{Diff}_3(M)$ and $\sigma_{3,A}$ be its symbol. Then operator

$$A - Q(\sigma_{3,A})$$

has order 2, and let $\sigma_{2,A}$ be its symbol.

Then operator $A - Q(\sigma_{3,A}) - Q(\sigma_{2,A})$ has order 1 and let $\sigma_{1,A}$ be its symbol. Thus we get subsymbols $\sigma_{i,A} \in \Sigma_i(M)$, $0 \leq i \leq 2$, such that

$$A = Q(\sigma_{(3)}(A)),$$

where

$$\sigma_{(3)}(A) = \sigma_{3,A} + \sigma_{2,A} + \sigma_{1,A} + \sigma_{0,A} \quad (8)$$

is the total symbol and

$$Q(\sigma_{(3)}(A)) = Q(\sigma_{3,A}) + Q(\sigma_{2,A}) + \ldots + Q(\sigma_{0,A}).$$
3.4.1. Coordinates. Let \( x_1, x_2 \) be local coordinates in a neighborhood \( \mathcal{O} \subset M \), where the symbol \( \sigma_{3,A} \) is regular. Denote by \( x_1, x_2, w_1, w_2 \) induced standard coordinates in the tangent bundle over \( \mathcal{O} \).

Then \( d\nabla(dx_k) = -\sum \Gamma^k_{ij}dx_i \otimes dx_j \), where \( \Gamma^k_{ij} \) are the Christoffel symbols of the Wagner connection \( \nabla \).

Thus, in coordinates \( x, w \) we have \( d\nabla(w_k) = -\sum \Gamma^k_{ij}w_iw_j \) and the derivation \( d\nabla \) has the form:

\[
d\nabla = \sum w_i \partial_{x_i} - \sum \Gamma^k_{ij}w_iw_j \partial_{w_k}.
\]

3.5. Universal differential operator. We define a total operator of third order \( \Box_3 : C^\infty(J^k\chi_3) \rightarrow C^\infty(J^{k+3}\chi_3) \) as it was done in [3].

In local coordinates this operator has the form

\[
\Box_3 = 6 \sum_{\alpha, 0 \leq |\alpha| \leq 3} \frac{u^\alpha}{\alpha!} \left( \frac{d}{dx} \right)^\alpha,
\]

where \( \alpha = (\alpha_1, \alpha_2) \), \( |\alpha| = \alpha_1 + \alpha_2 \), \( (d/dx)^\alpha = (d/dx_1)^{\alpha_1}(d/dx_2)^{\alpha_2} \), and \( d/dx_1, d/dx_2 \) are total derivatives.

**Theorem 3.** The operator \( \Box_3 \) commutes with the action of the group \( G(M) \) on the jet bundles.

3.6. Natural differential invariants of regular operators. By natural differential invariants of order \( k \) we mean a function on \( J^k(\chi_3) \) which are \( G(M) \)-invariant and rational along fibers of the projection \( \chi_{3,k} \).

**Theorem 4.** If \( I \) is a natural differential invariant of order \( \leq k \) for differential operators of the third order, then \( \Box_3(I) \) is a natural differential invariant of the order \( \leq (k + 3) \) for these operators.

We say that two natural differential invariants \( I_1, I_2 \) are in general position if

\[
\hat{d}I_1 \wedge \hat{d}I_2 \neq 0,
\]

where \( \hat{d} \) is the total differential, and denote by \( \mathcal{O}(I_1, I_2) \subset J^\infty(\chi_3) \) the open domain, where condition (10) holds.

**Theorem 5.** Let natural differential invariants \( I_1, I_2 \) are in general position and let

\[
J^\alpha = \Box_3(P^\alpha),
\]

where \( \alpha = (\alpha_1, \alpha_2) \) and \( 0 \leq |\alpha| \leq 3 \).

Then the field of natural differential invariants for differential operators of the third order in the domain \( \mathcal{O}(I_1, I_2) \) is generated by invariants \( I_1, I_2, J^\alpha \) and all their Tresse derivatives

\[
\frac{d^l J^\alpha}{dI_1^{\alpha_1}dI_2^{\alpha_2}},
\]
where \( l = l_1 + l_2 \).

Thus, to obtain the field of natural differential invariants of linear regular differential operators of order 3, it is enough to have two natural differential invariants \( I_1, I_2 \) in general position.

The invariants can be obtained by various methods.

For example, as the invariant \( I_1 \) one can take the free term \( u_0 \) of the universal operator \( \Box_3 \), or \( I_1 \) is natural invariant such that \( I_1(A) \) is the natural convolution of the torsion form \( \theta \) of the Wagner connection and the subsymbol \( \sigma_{1,A} \).

As the natural invariant \( I_2 \) one can take, for example, the invariant \( \Box_3(I_1) \).

3.6.1. Let \( F_k^{(3)} \) be the field of all natural differential invariants of order \( \leq k \) of linear scalar differential operators of order \( \leq 3 \) on \( M \).

The \( G(M) \)-action on \( M \) is transitive and, therefore, the set of all such invariants forms an \( \mathbb{R} \)-field \( F_k^{(3)} \).

The natural projections \( \chi_{k,l} : J^k(\chi) \to J^l(\chi), \ k > l \), define the embeddings \( F_l^{(3)} \hookrightarrow F_k^{(3)} \) and their inductive limit \( F_*^{(3)} \) is the field of natural differential invariants of linear differential operators on the manifold \( M \).

Remark that the \( G(M) \)-action on differential operators satisfies the conditions of the Lie-Tresse theorem (see [2]) and, therefore, the field \( F_*^{(3)} \) separates regular \( G(M) \)-orbits.

4. LINEAR DIFFERENTIAL OPERATORS ON \( M \times \mathbb{R} \)

Consider now linear differential operators \( A \) of the third order on the manifold \( J^0(\pi) = M \times \mathbb{R} \), that satisfy the following condition:

\[
(A - A(1))(y) = 0,
\]

where \( y \) is the fibrewise coordinate on \( J^0(\pi) \).

In local coordinates \( (x_1, x_2) \) on \( M \), we have representation (2) of operators of such type:

\[
A = \sum_{i,j,k} a_{ijk}(x, y) \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} + \sum_{i,j} a_{ij}(x, y) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i a_i(x, y) \frac{\partial}{\partial x_i} + a_0(x, y).
\]

The module of these operators we will denote by \( \text{Diff}_3(M, \mathbb{R}) \) and by

\[
\zeta : \text{Diff}_3(M, \mathbb{R}) \longrightarrow M \times \mathbb{R}
\]

we will denote the corresponding bundle of these operators.
As before, elements of the module of $\text{Diff}_3(M, \mathbb{R})$ are just sections of the bundle $\zeta$ with the correspondence

$$\text{Diff}_3(M, \mathbb{R}) \equiv C^\infty(\zeta), \quad A \mapsto s_A,$$

where $s_A(a, y) = A_{(a,y)}$ and $(a, y) \in M \times \mathbb{R}$.

The diffeomorphism group $G(M)$ acts by prolongation $\phi \mapsto \phi^{(0)}$ on the manifold $J^0(\pi) = M \times \mathbb{R}$, preserves the function $y$, and therefore acts in the bundle $\zeta$ as well as in the $k$-jet bundles $\zeta^k$:

$$J^k(\zeta) \to M \times \mathbb{R}.$$

4.1. **Linear differential operators** $A_f$. Given an operator $A \in \text{Diff}_3(M, \mathbb{R})$ and a function $f \in C^\infty(M)$ we define the operator

$$A_f = s_f^* \circ A \circ \pi^* \in \text{Diff}_3(M)$$

as the operator that corresponds to the restriction of the section $s_A$ to the graph of the function $f$. Here $s_f: M \to M \times \mathbb{R}$ is the section of the bundle $\pi$ that corresponds to the function $f$.

In local coordinates, we get the above representation (3) for this type of operators:

$$A_f = \sum_{i,j,k} a_{ijk}(x, f) \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} + \sum_{i,j} a_{ij}(x, f) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i a_i(x, f) \frac{\partial}{\partial x_i} + a_0(x, f).$$

4.2. **Weakly nonlinear differential operators** $A_w$. Define now the space of weakly nonlinear operators of the third order $\text{Diff}^w_3(M)$ as the space of differential operators $A_w$ on $C^\infty(M)$ of the form

$$A_w(f) = A_f(f),$$

where $A \in \text{Diff}_3(M, \mathbb{R})$ and $f \in C^\infty(M)$.

**Proposition 6.** The mappings

$$\Theta: \text{Diff}_3(M, \mathbb{R}) \times C^\infty(M) \longrightarrow \text{Diff}_3(M), \quad \Theta: (A, f) \mapsto A_f,$$

and

$$\Theta_w: \text{Diff}_3(M, \mathbb{R}) \longrightarrow \text{Diff}^w_3(M), \quad \Theta_w: A \mapsto A_w,$$

are natural in the following sense:

$$\phi_* (A, f) = (\phi_*^{(0)}(A), \phi_* (f)),$$

$$\phi_* (A_f) = (\phi_*^{(0)}(A))_{\phi_*(f)},$$

and

$$\phi_* (A_w) = (\phi_*^{(0)}(A))_w,$$

for all diffeomorphisms $\phi \in G(M)$.

**Proof.** The proof is almost verbatim repetition of the proof of Proposition 1 in [5].
5. Related pairs and their invariants

The above proposition has the following consequences:

- By differential $G(M)$-invariants of weakly nonlinear operators we will mean differential $G(M)$-invariants of operators in $\text{Diff}_3(M, \mathbb{R})$, that are $G(M)$-invariant functions on jet spaces $J^k(\zeta)$.

- In what follows, we will require that operators $A \in \text{Diff}_3(M, \mathbb{R})$ under consideration are rational in $y$ (as sections of the bundle $\zeta$). Also, by differential invariants of rational weakly nonlinear operators we mean $G(M)$-invariant functions on jet spaces $J^k(\zeta)$ that are rational along fibres of the projections $\pi \circ \zeta_k: J^k(\zeta) \to M$.

- The group $G(M)$ acts transitively on the manifold $M$ and therefore rational differential invariants of order $\leq k$ for rational weakly nonlinear operators form a field $F_{w,k}$. We have the embedding $\zeta_{k,l}^*: F^w_k \to F^w_l$, if $k \geq l$, where $\zeta_{k,l}: J^k(\zeta) \to J^l(\zeta)$ are the natural projections, and $F^w = \bigcup_{k \geq 0} F^w_k$ is the field of all rational differential invariants of rational weakly nonlinear operators.

We consider the following vector bundles over manifold $J^0(\pi) = M \times \mathbb{R}$:

1. jet bundles of functions on manifold:
   $$\pi_{l,0}: J^l(\pi) \to J^0(\pi),$$

2. jet bundles of operators:
   $$\zeta_k: J^k(\zeta) \to M \times \mathbb{R},$$

3. the Whitney sum of vector bundles $\zeta_k$ and $\pi_{l,0}$
   $$r_{k,l}: \mathbb{R}P^{k,l} = \zeta_k \oplus_{M \times \mathbb{R}} \pi_{l,0} \to M \times \mathbb{R}.$$

We call the last bundle as the bundle of related pairs.

Elements of the total space of this bundle are related pairs

$$\left([A]_{(x,y)}, [f]_l\right)$$

consisting of $k$-jet $[A]_l(x,y)$ of operator $A \in \text{Diff}_3(M, \mathbb{R})$ at the point $(x, y) \in M \times \mathbb{R}$ and $l$-jet $[f]_l$ of function $f \in C^\infty(M)$ at the point $x \in M$ under condition that $f(x) = y$.

The group $G(M)$ acts by prolongations in the bundles $r_{k,l}$ and by invariants of this action (or invariants of related pairs) we mean functions on the total space $\mathbb{C}P^{k,l} = J^k(\zeta) \oplus_{M \times \mathbb{R}} J^l(\pi)$ that are $G(M)$-invariant and rational along fibers of the projection $\pi \circ r_{k,l}: \mathbb{R}P^{k,l} \to M$.

Because of transitivity of $G(M)$-action on $M$, all such functions are completely determined by their values on fibre $(\pi \circ r_{k,l})^{-1}(a)$ at a base
point $a \in M$. Therefore, $\mathcal{G}(M)$-invariants of related pairs form an $\mathbb{R}$-field $\mathbf{F}_{k,l}$, that is a subfield of the filed $\mathbf{Q}_{k,l}$ of all rational functions on the fibre $(\pi \circ r_{k,l})^{-1}(a)$.

The natural projections $\mathbb{C} \mathbb{P}^{k',l'} \to \mathbb{C} \mathbb{P}^{k,l}$, where $k \leq k'$, $l \leq l'$, give us embeddings of fields $\mathbf{F}_{k,l} \subset \mathbf{F}_{k',l'}$, $\mathbf{Q}_{k,l} \subset \mathbf{Q}_{k',l'}$ and we define the fields $\mathbf{F}_l$, $\mathbf{Q}_l$ by the inductive limits:

$$\mathbf{F}_l = \bigcup_{k \geq 0} \mathbf{F}_{k,l}, \quad \mathbf{Q}_l = \bigcup_{k \geq 0} \mathbf{Q}_{k,l}.$$ 

Remark that $\mathbf{F}_0 = \mathbf{F}^w$ is just the field of rational differential invariants of rational weakly nonlinear differential operators.

Below we will discuss various methods of finding invariants, but first of all we remark that the vector field $\partial_y$ on $J^0(\pi) = M \times \mathbb{R}$ is an invariant of the $\mathcal{G}(M)$-action. Therefore its $l$-th prolongation $\partial_y^{(l)}$ on $J^l(\pi)$ is also $\mathcal{G}(M)$-invariant. The same is valid for the total derivation $\frac{d}{dy}$ that acts in $J^\infty(\zeta)$. All together they define $\mathcal{G}(M)$-invariant derivation $\nabla$ in the fields $\mathbf{Q}_l$, as well as in $\mathbf{F}_l$, where

$$\nabla(\alpha \beta) = \frac{d\alpha}{dy} \beta + \alpha \partial_y^{(l)}(\beta),$$

$\alpha \in \mathbf{Q}_0$ and $\beta$ is a function on $J^l(\pi)$.

6. Construction of invariants

At first we consider $y$ as a parameter and identify operators $A \in \text{Diff}_3(M, \mathbb{R})$ with 1-parametric family $A_y$ of operators in $\text{Diff}_3(M)$.

Remark that $y$ is a $\mathcal{G}(M)$-invariant. Therefore, for any invariant $I \in \mathbf{F}^{(3)}_k$ of linear differential operators on manifold $M$, the function $\hat{\Theta}_l: J^k(\zeta) \to \mathbb{R}$, where

$$\hat{\Theta}_l([A]^k_{[x,y]}) = I([A]^k_x)$$

is a $\mathcal{G}(M)$-invariant too.

Thus we get a mapping

$$\mathbf{F}^{(3)}_k \longrightarrow \mathbf{F}_0 = \mathbf{F}^w, \quad I \mapsto \hat{\Theta}_l,$$

that immediately gives us invariants of weakly nonlinear operators.

Moreover, application of the invariant derivation $\nabla$ essentially increases their amount.

As we have seen, Proposition 6, the mapping

$$\Theta_l: \mathbb{R} \mathbb{P}^{l,l} \longrightarrow J^l(\chi), \quad \Theta_l: ([A]^l_{[x,y]}, [f]^l_{x,x}) \mapsto [A]^l_j,$$

commutes with the $\mathcal{G}(M)$-action. Therefore,

$$\Theta^*_l(I) \in \mathbf{F}_l,$$

i.e., it is an invariant of related pairs for any invariant $I \in \mathbf{F}^{(3)}_l$. 
6.0.1. **Descent procedure.**

To get invariants of weakly nonlinear operators from invariants of related pairs we consider the following descent procedure for invariants

\[ \mathbf{F}_l \rightarrow \mathbf{F}^w. \]

Let \( I_0 \in \mathbf{F}_l \) be an invariant of related pairs and let \( I_1 = \nabla(I_0) \in \mathbf{F}_l, \ldots, I_{i+1} = \nabla(I_i) \in \mathbf{F}_l \) be its invariant derivatives. Remark that the transcendence degree of the field \( \mathbb{Q}_l \) over \( \mathbb{Q}_0 \) equals \( N = \dim \pi_{l,0} \).

Therefore, ([8]), there are polynomial relations between rational functions \( I_0, \ldots, I_N \). Denote by \( J(I) \subset \mathbb{Q}_0[X_0, \ldots, X_N] \) the ideal of these relations.

**Theorem 7.** Let \( b_1, \ldots, b_r \in \mathbb{Q}_0[X_0, \ldots, X_N] \) be the reduced Gröbner basis in the ideal \( J(I) \) with respect to the standard lexicographic order. Then the coefficients of polynomials \( b_i \) are natural invariants of weakly nonlinear operators.

**Proof.** The action of the diffeomorphism group preserves the ideal \( J(I) \) as well as the lexicographic order. The reduced Gröbner basis in an ideal with respect to the lexicographic order is unique ([1]), and therefore, the action preserves elements of the basis and their coefficients. \( \square \)

7. **Example** \( n = 1, k = 3 \)

Let \( M = \mathbb{R} \).

Then in coordinate \( x \) on \( M \), an operator \( A \in \text{Diff}_3(M) \) has the form

\[ A = a_3 \partial^3 + a_2 \partial^2 + a_1 \partial + a_0, \]

where the all coefficients are smooth functions on \( x \) and \( \partial = \partial_x \).

The symbol \( a_3 \partial^3 \) of \( A \) defines an invariant connection on the line \( \mathbb{R} \) with the Christoffel coefficient \( \Gamma \):

\[ \Gamma = -\frac{a_3'}{3a_3}, \]

the total symbol \( \sigma_{(3)} \) of \( A \) with respect to the Wagner connection \( \nabla \) is the following:

\[ \sigma_{(3)} = \sigma_3 + \sigma_2 + \sigma_1 + \sigma_0, \]

where

\[ \begin{align*}
\sigma_3 &= a_3 \partial^3, \\
\sigma_2 &= (a_2 - a_3') \partial^2, \\
\sigma_1 &= (a_1 - \frac{2(a_3')^2a_3 + 3a_3' - a_3^2a_3}{9a_3} - (a_2 - a_3') \frac{a_3'}{3a_3}) \partial, \\
\sigma_0 &= a_0
\end{align*} \quad (12) \]

and the natural splitting of the operator \( A \) is the following:

\[ A = \widehat{\sigma_{(3)}} = \widehat{\sigma_3} + \widehat{\sigma_2} + \widehat{\sigma_1} + \widehat{\sigma_0}, \quad (13) \]
where the differential operators $\hat{\sigma}_k$ are the quantizations of the symbols $\sigma_k$, see [3]:

$$
\hat{\sigma}_3 = a_3 \partial^3 + a'_3 \partial^2 + \frac{2(a'_3)^2 a_3 + 3a_3}{9a_3} \partial,
$$

$$
\hat{\sigma}_2 = (a_2 - a'_3)(\partial^2 + \frac{a'_3}{3a_3} \partial),
$$

$$
\hat{\sigma}_1 = (a_1 - \frac{2(a'_3)^2 a_3 + 3a'_3 - a''_3 a_3}{9a_3} - (a_2 - a'_3) \frac{a'_3}{3a_3} \partial),
$$

$$
\hat{\sigma}_0 = a_0.
$$

Therefore, we get from (12) the following $G(M)$-invariants of operators $A$:

$$I_0 = a_0,$$

$$I_1 = \langle \sigma_1, da_0 \rangle = \left( a_1 - \frac{2(a'_3)^2 a_3 + 3a'_3 - a''_3 a_3}{9a_3} - (a_2 - a'_3) \frac{a'_3}{3a_3} \partial \right) a_0,$$

$$I_2 = \langle \sigma_2, da_0^2 \rangle = (a_2 - a'_3)(a_0')^2,$$

$$I_3 = \langle \sigma_3, da_0^3 \rangle = a_3(a_0')^3.$$

Now take an operator $A \in \text{Diff}_3(M, \mathbb{R})$.

$$A = a_3(x, y) \partial_x^3 + a_2(x, y) \partial_x^2 + a_1(x, y) \partial_x + a_0(x, y),$$

$$A_f = a_3(x, f) \partial_x^3 + a_2(x, f) \partial_x^2 + a_1(x, f) \partial_x + a_0(x, f), \quad f \in C^\infty(M).$$

Then, the corresponding invariants of relates pairs are the following:

$$I_0 = a^0,$$

$$I_1 = \left( a_1 - \frac{2(a_3 + a_3 y f')^2 a_3 + 3(a_3 + a_3 y f')}{9a_3} \right),$$

$$I_2 = \left( a_2 - (a_3 + a_3 y f') \frac{(a_3 + a_3 y f')}{3a_3} \right)(a_0 + a_0 y f'),$$

$$I_3 = a_3(a_0 + a_0 y f')^3.$$

From the expression of $I_3$, we get

$$f' = ((I_3/a_3)^{1/3} - a_{0,x})/a_{0,y}.$$

Substituting it into $I_2$, we get a relation of the form

$$KI_2 + K_1 I_3^{1/3} + K_2 I_3^{2/3} + K_3 I_3 + K_0 = 0,$$
where
\[ \frac{K_1}{K}, \frac{K_2}{K}, \frac{K_3}{K}, \frac{K_4}{K} \]
are invariants of operators \( A \in \text{Diff}_3(M, \mathbb{R}) \), i.e. invariants of weakly nonlinear operators.

8. EQUIVALENCE OF WEAKLY NONLINEAR OPERATORS

Let \( z \) be a natural differential invariant of weakly nonlinear operators and \( A \in \text{Diff}_3(M, \mathbb{R}) \).

Then the values \( z(A, y_0) = s^*_A(z) \) is a function rational in \( y \) with coefficients in \( C^\infty(M) \). Values \( z(A, y_0) \) of this function for a given value \( y_0 \) is a smooth function on \( M \).

We say that the operator \( A \) is in general position if for any point \( a \in M \) there are: natural invariants \( z_1, z_2 \), a value \( y_0 \) of \( y \), and a neighborhood \( U \subset M \), \( a \in U \), such that the mapping
\[ Z_{A,y_0}: U \to D \subset \mathbb{R}^2, \quad Z_{A,y_0}: x \mapsto (z_1(A, y_0)(x), z_2(A, y_0)(x)) \]
is a local diffeomorphism.

We say, that the mapping \( Z_{A,y_0} \) is a natural chart of the operator \( A \) and the functions \( z_1(A, y_0), z_2(A, y_0) \) are natural coordinates on \( U \).

We call the atlas of these charts \( \{ U_\alpha, \phi_\alpha: U_\alpha \to D_\alpha \subset \mathbb{R}^2 \} \) natural if coordinates \( \phi_\alpha = (z_1^\alpha(A, y_0), z_2^\alpha(A, y_0)) \) are given by distinct invariants \( (z_1^\alpha(A, y_0), z_2^\alpha(A, y_0)) \neq (z_1^\beta(A, y_0), z_2^\beta(A, y_0)) \), when \( \alpha \neq \beta \).

We denote by \( D_{\alpha \beta} = \phi_\alpha(U_\alpha \cap U_\beta) \) and we assume that domains \( D_\alpha \) and \( D_{\alpha \beta} \) are connected and simply connected.

Let \( A_\alpha = \phi_\alpha(A|_{U_\alpha}) \), \( A_{\alpha \beta} = \phi_\alpha(A|_{U_\alpha \cap U_\beta}) \) be the images of the operator \( A \) in these coordinates.

Then \( \phi_{\alpha \beta}(A_{\alpha \beta}) = A_{\beta \alpha} \), where \( \phi_{\alpha \beta} = \phi_\beta \circ \phi_\alpha^{-1} : D_{\alpha \beta} \to D_{\beta \alpha} \) are the transition mappings.

Take now two operators \( A, A' \in \text{Diff}_3(M, \mathbb{R}) \) and consider their natural charts \( Z_{A,y_0} \) and \( Z_{A',y'_0} \).

The \( \mathcal{G}(M) \)-equivalence of these operators means that the local diffeomorphism \( Z_{A',y'_0}^{-1} \circ Z_{A,y_0} \) does not depend on choice of \( y_0 \) and \( y'_0 \). In this case we will say that these charts are coordinated.

Summarizing we get the following result.

**Theorem 8.** Let operators \( A, A' \in \text{Diff}_3(M, \mathbb{R}) \) be in general position, then they are \( \mathcal{G}(M) \)-equivalent if and only if the following conditions hold:

1. the mappings \( \{ \phi'_\alpha: U'_\alpha \to D_\alpha \} \),
   where \( \phi'_\alpha = (z_1'^\alpha(A', y'_0), z_2'^\alpha(A', y'_0)) \) and \( U'_\alpha = (\phi'_\alpha)^{-1}(D_\alpha) \), constitute a natural atlas for the operator \( A' \),
2. the charts \( \phi_\alpha \in \{ \phi_\alpha: U_\alpha \to D_\alpha \} \) and \( \phi'_\alpha \in \{ \phi'_\alpha: U'_\alpha \to D_\alpha \} \) are coordinated,
3. \( \phi'_{\alpha \beta} = \phi_{\alpha \beta} \).
\( (4) \ A_\alpha = \phi'_\alpha\left(A'|_{U_\alpha}\right) \text{ and } A_{\alpha\beta} = \phi'_{\alpha\beta}\left(A'|_{U_{\alpha\beta}}\right). \)

Proof. Any diffeomorphism \( \psi: M \to M \) such that \( \psi^*_\ast(A) = A' \) transform natural atlas to the natural one and because of \( \psi^{*-1}(z(A)) = z(\psi_\ast(A)) \), for any natural invariant \( z \), this diffeomorphism has the form of the identity map in the natural coordinates. \( \square \)

**Corollary 9.** Let operators \( A \) and \( A' \in \text{Diff}_3(M, \mathbb{R}) \) be in general position. Then the weakly nonlinear differential operators \( A_w \) and \( A'_w \) are \( G(M) \)-equivalent if and only if the operators \( A \) and \( A' \) are \( G(M) \)-equivalent.

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