Semiclassical theory of magnetotransport through a chaotic quantum well

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(March 24, 2022)

We develop a quantitative semiclassical formula for the resonant tunneling current through a quantum well in a tilted magnetic field. It is shown that the current depends only on periodic orbits within the quantum well. The theory explains the puzzling evolution of the tunneling spectra near a tilt angle of 30° as arising from an exchange bifurcation of the relevant periodic orbits.

PACS numbers: 05.45.+b, 72.15.Gd, 73.20.Dx

The resonant tunneling diode (RTD) in a magnetic field tilted with respect to the tunneling direction, has been extensively studied in recent years as a simple experimental system which manifests the quantum signatures of classical chaos. The measured I-V characteristics show resonance peaks which evolve in a complex manner as magnetic field, \( B \), and tilt angle, \( \theta \), are varied. The existence and periodicity of these peaks in various parameter intervals has been associated with the existence of certain periodic orbits and their bifurcations. The link to quantum mechanics has been made by intuitive appeals to Gutzwiller oscillations of the density of states, scaling analyses of the exact quantum spectrum, and the numerical discovery of sequences of wavefunctions scarred by periodic orbits. However, previous to this work, it has not been shown that periodic orbits indeed determine the quantum tunneling oscillations in the semiclassical limit. Indeed, the analogy between the tilted well and another well-studied chaotic system, the hydrogen atom in magnetic field, has frequently been emphasized, however in diamagnetic hydrogen it has been shown that all orbits closed at the nucleus determine the absorption spectrum, not just the periodic orbits. Moreover, even assuming the importance of periodic orbits, the periodic orbit theory of the system is sufficiently complex that the contribution of specific periodic orbits and their bifurcations to the experimental I-V data has been controversial. Below we derive a quantitative semiclassical formula for the tunneling current which demonstrates that the current is dominated by periodic orbits and apply the formula to previously unpublished data which reveals an interesting exchange bifurcation involving four period-two orbits.

In an RTD under a bias voltage \( V \), tunneling current flows from the emitter state through the double-barriers confining the quantum well. The data presented are from an RTD with a 120nm wide well and experimental details are given in Ref. 2. When a large magnetic field (> 1T) is applied, the emitter state is quantized into the first few Landau levels. The electric field is normal to the barriers (\( \mathbf{E} = E \hat{z} \)), while the magnetic field is tilted in the \( y-z \) plane, \( \mathbf{B} = \cos \theta \hat{z} + \sin \theta \hat{y} \).

After tunneling into the well through the emitter barrier, an electron gains large kinetic energy before colliding with the collector barrier. Typically, the electron will traverse the well hundreds of times before tunneling out and will lose much of its energy due to optic phonon emission. Therefore, the tunneling is sequential and the resonances are substantially broadened by \( h/\tau_{\text{opt}} \), where the phonon emission time is \( \sim 0.1 \) ps. For describing this limit the Bethe tunneling hamiltonian formalism is inappropriate which greatly simplifies the problem. Using this approach and taking into account that the relative barrier widths are chosen so that charge accumulation in the well is negligible, one finds that the current is simply proportional to the tunneling rate through the emitter barrier, \( W_{c\rightarrow w} : j = n_e W_{c\rightarrow w}, \) where \( n_e \) is the surface concentration in the emitter layer.

\( W_{c\rightarrow w} \) can be calculated from the Fermi Golden Rule with the coupling matrix element between the wavefunctions \( \Psi_e \) and \( \Psi_w \), corresponding to the isolated emitter and isolated well respectively. In the limit when the height of the emitter barrier \( V_0 \) is much larger than the injection energy \( \epsilon_i \), the cyclotron energy \( \hbar \omega_c \) and the voltage drop across the barrier, the coupling matrix element can be simplified to:

\[
M_{nk} = \frac{\hbar^2}{m^*} \int dS \Psi_{n}^* (x, y, 0) \frac{\partial \Psi_{k}^w (x, y, z)}{\partial z} \bigg|_{z=0} \quad (1)
\]

where the integration is performed over the inner surface of the emitter barrier \( z = 0 \).

Due to the translational invariance in the \( x \)-direction, the classical dynamics within the well can be reduced to two degrees of freedom, \( y, \epsilon \), with an effective potential \( V (y, z) \). The tunneling rate, which is proportional to the square of the matrix element (\( \langle 0 | \), can be expressed in terms of the related Green function \( G(y_1, z_1 = 0; y_2, z_2 = 0; \epsilon) \). This Green function is then replaced by its semiclassical approximation (\( \mathbf{G} \)), which is determined by all classical trajectories connecting the points \( (y_1, 0); (y_2, 0) \). Defining \( y = (y_1 + y_2)/2, \Delta y = y_1 - y_2, \) it is convenient to introduce the Wigner transform of the emitter wavefunction, \( f_w (y, \epsilon) = \hbar^{-1} \int d\Delta y \Psi_e (y - \Delta y/2, 0) * \Psi^*_w (y + \Delta y/2, 0) \exp (i\epsilon \Delta y/h). \)

Since the emitter state \( \Psi_e \) involves only the few lowest single-particle levels, it can be calculated accurately using a variational approach.
We then obtain for the oscillatory part of $W_{e\rightarrow w}$:

$$W_{\text{osc}} = \int dp_y \int dy \, f_W (y, p_y) \sum_{\gamma} \frac{(p_{1y}^2 + p_{2y}^2) f}{(m^*)^2} \int d\Delta y \times \Re \left\{ \frac{8D_{1/2}}{2\sqrt{\pi \hbar}} \exp \left[ -\frac{t_{\gamma}}{\tau_{\text{opt}}} + i \frac{S_\gamma - p_y \Delta y}{\hbar} \right] \right\}$$

(2)

where $\gamma$ is the index of the classical trajectories, $S_\gamma \equiv S_{\gamma} (y - \Delta y, 0; y + \Delta y, 0; \varepsilon)$ is the action integral, $t_{\gamma}$ is the traversal time and $D_{\gamma}$ is the (complex) amplitude. The momenta $p_{1y}^2$ and $p_{2y}^2$ correspond to respectively the initial and final points of the trajectory $\gamma$. The factor $\exp (-t_{\gamma}/\tau_{\text{opt}})$ represents the effect of phonon emission and suppresses the contributions of trajectories longer than $\tau_{\text{opt}}$ (corresponding to $\sim 4$ bounces with the collector).

The next step depends on the properties of the emitter wavefunction. If this wavefunction were well-localized spatially on the scale of the electron wavelength within the well, then effectively it would provide a delta-function source and the resulting formula for the tunneling rate would involve all closed orbits starting exactly at the injection point, as in the atomic case. However, as noted above, the emitter state is a linear combination of the first few Landau levels and its Wigner function $f_W (y, p_y)$ has a typical spatial spread of order the effective magnetic length, $l_B \equiv \sqrt{\hbar/eB \cos \theta}$, centered at the “injection point” $y = y_i$, and falls off rapidly outside this interval. Hence $\Delta y \sim h/p_y \sim l_B \sim \sqrt{\hbar}$, and in the semiclassical limit $h \rightarrow 0$ one can expand $S_\gamma / h$ to second order in powers of $\Delta y$ near the action of the closed orbit. The function $S_\gamma (y, y)/h$ is rapidly varying on this spatial scale as $h \rightarrow 0$ and its stationary points correspond to the periodic orbits. Let $y_n$ be a point of contact with the emitter for a given isolated periodic orbit (labelled by the index $\mu$), then all nearby closed orbits can be represented by the correct order by quadratic expansion in $\delta y = y - y_n$. If $y_\mu$ is within $l_B$ ($\sim l_B^{1/2}$) of the injection point $y_i$, then this orbit will contribute substantially. In this case the quadratic expansion correctly approximates the value of $S_\gamma (y, y)$. If $|y_\mu - y_i| \gg l_B$ then this expansion is inaccurate near $y_i$, but the contribution of such orbits is negligible due to rapid variation of the phase $S_\gamma (y, y)$. One then finds

$$W_{\text{osc}} = \frac{16}{m^*} \sum_{\mu} \frac{p_{1y}^2}{\sqrt{m_{1y}^2 + m_{2y}^2 + 2}} \int dy \int dp_y f_W (y, p_y) \cos \left[ \frac{S_\mu}{h} - \frac{\pi n_\mu}{2} + Q_\mu (\delta y, \delta p_y) \right]$$

(3)

where

$$Q_\mu = \frac{2m_{1y}^2 (\delta y)^2 + (m_{2y}^2 - m_{1y}^2) \delta y \delta p_y - m_{12}^2 (\delta p_y)^2}{m_{1y}^2 + m_{2y}^2 + 2},$$

$\delta p_y = p_y - (p_y)_i$, the integer $n_\mu$ is the topological index of the periodic orbit, and the $2 \times 2$ monodromy matrix $M = (m_{ij})$ is calculated at the emitter barrier. Therefore we have shown that the tunneling current depends only on the periodic orbits.

The summation in (3) is performed over all isolated periodic orbits, both stable and unstable. Near a stable orbit, the classical motion is regular and the trajectories are confined to invariant tori in the phase space. One can still obtain discrete quantized energy levels from such regular motion by insisting that action integrals over the torus be quantized. This leads to a sequence of eigenfunctions localized on the tori of the stable islands. In contrast, the classical motion near an unstable orbit is chaotic with no locally conserved actions. As is now well-known, such motion cannot be semiclassically quantized to yield discrete energy levels. Therefore one would expect a qualitative difference between the contributions of stable and unstable orbits in Eq. (3). This difference, can be displayed explicitly by performing the summation over repetitions of the primitive periodic orbits. This summation can be performed exactly, yielding:

$$W = \frac{8}{m^*} \sum_{\mu} (p_\mu) \sum_{\ell} \Delta \left( \frac{T_\mu}{\tau_{\text{eff}}}, \frac{S_\mu (\varepsilon_\ell)}{\hbar} - \frac{\pi n_\mu}{2} \right) \int dp_y \int dy \, f_W (y, p_y) g_\ell^{\mu, +} (y, p_y)$$

(4)

where $\Delta (\sigma, \rho) = \frac{\sinh (\sigma)}{\cosh (\sigma) - \cos (\rho)}$, and the index $+ or -$ denotes stable or unstable orbits. The quantity $h/\tau_{\text{eff}}$ is an effective level-broadening which differs in the two cases.

(i) Stable orbits Here we expect the level-broadening to arise only due to phonon emission and indeed we find $\tau_{\text{eff}} = \tau_{\text{opt}}$. If phonon scattering is absent, $\tau_{\text{eff}} \rightarrow \infty$. Noting that $\lim_{\sigma \rightarrow 0} \Delta (\sigma, \rho) = 2\pi \delta (\rho)$, one finds that the stable orbit contribution to $W_{\text{osc}}$ is proportional to the sum $\sum_\mu \delta (\varepsilon - \varepsilon_\mu, \ell)$. The energies $\varepsilon_{\mu, \ell}$ are determined from the semiclassical quantization condition of the longitudinal energy $\varepsilon_\ell = \varepsilon - \hbar \omega^{\mu, +}_\ell (\ell + 1/2)$ along the periodic orbit $S_\mu (\varepsilon_\ell, \ell) = 2\pi \hbar (n + \frac{m_\mu}{4})$. Due to the harmonic approximation the quantization of the transverse oscillations around the PO simply yields equally spaced levels with spacing $\hbar \omega^{\mu, +}_\ell$, where the frequency $\omega^{\mu, +}_\ell = \phi_\mu / T_\mu$, with $\phi_\mu$ the winding number and $T_\mu$ the period of the orbit. We find that the coefficient functions in Eq. (3) are the Wigner transforms of the harmonic oscillator wavefunctions corresponding to these transverse modes:

$$g_\ell^{\mu, +} (y, p_y) \equiv (-1)^\ell L_\ell (2 | Q_\mu |) \exp (-| Q_\mu |)$$

(5)

Since $L_\ell$ is Laguerre polynomial and $Q_\mu$ is defined in (3).
(ii) **Unstable orbits.** For unstable periodic orbits we find that the "effective" relaxation rate $1/\tau^\mu_{\text{eff}} = 1/\tau_{\text{opt}} + (\ell + 1/2) \lambda_\mu$, where $\lambda_\mu$ is the Lyapunov exponent near the orbit $\mu$. Hence $\tau_{\text{eff}}$ is finite and equal to the Lyapunov time when $\tau_{\text{opt}} \to \infty$. Therefore, instability acts as a sort of intrinsic level-broadening and the contribution of unstable POs to (4) never approaches a delta function providing individual levels. Instead this contribution describes the well-known clustering of levels responsible for Gutzwiller oscillations of the density of states within a distance $\ell$ to the tunneling rate is given by the function

$$g^\mu_{\ell-}(y, p_y) = (-1)^\ell \Re \{ L_\ell (2iQ_\mu) \exp(iQ_\mu) \times \left( 1 + i \frac{\sin (S_\mu/h - \pi n_\mu/2)}{\sinh (T_\mu/\tau^\mu_{\text{eff}})} \right) \}$$

(6)

corresponding to Wigner functions **averaged** over the eigenstates of the cluster.

It is not clear in this case how to introduce the cut-off value $\ell_{\text{max}}$, however for each PO the contributions to $W_{\text{osc}}$ decay exponentially $\sim \exp(-\lambda_\mu T_\mu \ell)$ with increasing $\ell$, and we find that for our case the main contribution to the tunneling rate is given by the $\ell = 0$ term.

Above we have argued that among the POs which reach the emitter only those with points of contact within a distance $\sim l_B$ from the injection point contribute strongly. Our formulas (4),(5),(6) permit us now to propose a precise criterion: a PO is "accessible" to tunneling electrons if the emitter Wigner function centered on $y = y_i$, $p_y = 0$ and of spatial width $\sim l_B$ overlaps the functions $g^\mu_{\ell\pm}$ centered at the contact points $y_i\mu\pm(p_y\mu)\pm$. The localization length of the wavefunctions $\Psi^\mu_{\mu,\ell}$ corresponding to the functions $g^\mu_{\ell\pm}$ can be shown by an extension of our above analysis to be $\ell_\mu = \sqrt{2h/M_\mu^2}/\sqrt{|4 - T^2|/|M_\mu^2|}$. The spatial scale associated with an unstable orbit is of the same form but instead of Gaussian decay of the wavefunctions at $\delta y > \ell_\mu$ one finds rapid oscillations $\Psi^\mu_{\mu,\ell} \sim \exp(\mp 2\ell_\mu)$.

The relations (3), (4), (6) constitute a precise semiclassical expression for the tunneling rate in terms of the contributions of the distinct periodic orbits which reach the emitter barrier ("emitter orbits"). We will apply them now to a specific parameter regime in the tilted well to compare qualitatively and quantitatively to the experimentally-observed I-V characteristics.

In the recent periodic orbit theory of Ref. 5, the semiclassical theory at $\theta = 0$ the only resonances observed in the I-V characteristic are associated with Bohr-Sommerfeld quantization of the (1,1) orbit which traverses the well with zero cyclotron energy. When $\theta \neq 0$ such period-one "traversing" orbits still exist in some regions of the $B - V$ parameter space and define a background frequency of resonance peaks. However now additional resonances appear corresponding to doubling or tripling of the frequency of peaks of $\theta = 30\degree$. These new peaks are associated with the existence of period-two and period-three orbits which appear and disappear as a result of bifurcations. Here we focus on the peak-doubling in the interval $29\degree < \theta < 34\degree$, where, as was shown in Ref. 5, there are four most relevant orbits, denoted by $(1,2)$, $(1,2)\ast$, $(1,2)_{1}^{\ast}$, $(1,2)_{2}^{\ast}$. For $\theta < 30\degree$, the starred and unstarred orbits are paired in the sense that they eventually become identical and disappear in inverse tangent bifurcations.

The evolution of these orbits is represented by the four colored lines in Fig. 1. We recall (2) that under experimental conditions the classical mechanics depends only on two parameters $\beta = 2Bv_0/E$ (where $v_0$ is the velocity corresponding to the total energy: $v_0 = \sqrt{2\epsilon/m^*}$) and on the tilt angle $\theta$. As $\beta$ increases from zero these four $(1,2)$ orbits appear in cusp bifurcations and then disappear pairwise at higher $\beta$ in the inverse tangent bifurcations already mentioned. The hatched region denotes the semiclassical width of the emitter state, while the gray-scale regions denote the semiclassical widths defined by the $g^\mu$ (evaluated here at $B = 8 \text{ T}$) for the most accessible of these orbits for a given value of $\theta$. Wherever these regions overlap the semiclassical formula (4) will predict peak-doubling regions to appear in the $B - V$ parameter space for the corresponding values of $\beta$, as seen in Fig. 2.

A fascinating feature of the classical dynamics, noted in Ref. 5, occurs near $\theta = 30\degree$ for the experimental parameters of Ref. 2. The four $(1,2)$ orbits undergo an exchange bifurcation so that for $\theta > 30\degree$ the $(1,2)_{1}$ is paired with $(1,2)_{2}^{\ast}$, whereas the $(1,2)_{2}^{\ast}$ orbit is now paired with the $(1,2)_{2}$. For $\theta \approx 30\degree$ the $(1,2)_{2}^{\ast}$ orbit is accessible for over its entire interval of existence, $4.3 < \beta < 8$, and the $(1,2)_{1}$ is accessible in the overlapping interval $7.9 < \beta < 10.9$. This situation manifests itself in a large and continuous region of peak-doubling in the experimental data. By $\theta = 31\degree$, the pairing of the $(1,2)$ orbits has been interchanged (Fig. 1b): the $(1,2)_{1}^{\ast} - (1,2)_{2}^{\ast}$ pair exists at lower $\beta$ (higher voltage), and is moving away from the semiclassical accessibility region, whereas the $(1,2)_{1}$ orbit has become most accessible. Thus we expect the peak-doubling region to split into two smaller regions (Fig. 2b) and the amplitude of the oscillations to be weaker in the low-magnetic-field, high-voltage region (Fig. 2a). By $\theta = 34\degree$, the $(1,2)_{2}$ orbit has become inaccessible and these oscillations are no longer seen in Figs. 2d and 2e. In contrast, the $(1,2)_{1}$ orbit remains accessible and still produces strong oscillations in the data (Figs. 2d,e).

In Figs. 2c and 2f, we show the results of the semiclassical theory at $\theta = 31\degree$ and $\theta = 34\degree$. The semiclassical calculation of the tunneling current using formulae (1), (2), (5) is in good agreement with the experimental data.

We acknowledge helpful conversations with Gregor Hackenbroich, Tania Monteiro and John Delos. The work
of A.D.S. and E.N. was supported by NSF grant DMR-9215065.

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FIG. 1. (color) Bifurcation diagrams for the relevant period-two orbits at (a) before ($\theta = 29^\circ$) and (b) after ($\theta = 31^\circ$ and $34^\circ$) the exchange bifurcation. The shading represents the calculated localization lengths of the wavefunctions localized near the most relevant $(1, 2)$ periodic orbits (see text). The hatched region denotes the semiclassical width of the emitter state.

FIG. 2. (a),(d) Samples of experimental resonant tunneling $I - V$ traces, from which peak positions are determined and (b),(e) plotted versus total magnetic field, to be compared with (c),(f) results of the semiclassical calculation. Note at (a-c) $\theta = 31^\circ$ the onset of the separation of the large region of peak-doubling immediately after the exchange bifurcation and, at (d-f) $\theta = 34^\circ$, the peak-doubled region at lower magnetic field created by the exchange bifurcation is almost invisible due to much weaker coupling of the corresponding orbit $(1, 2)_2$ to the emitter (see Fig. 1c). The I-V traces (a), (d) evidence this by the disappearance of the oscillations at higher bias voltages.