TWISTED STRONG MACDONALD THEOREMS AND ADJOINT ORBITS

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Abstract. The strong Macdonald theorems state that, for $L$ reductive and $s$ an odd variable, the cohomology algebras $H^*(L[z]/z^N)$ and $H^*(L[z,s])$ are freely generated, and describe the cohomological, $s$-, and $z$-degrees of the generators. The resulting identity for the $z$-weighted Euler characteristic is equivalent to Macdonald’s constant term identity for a finite root system. We calculate $H^*(p/z^Np)$ and $H^*(p[s])$ for $p$ a standard parahoric in a twisted loop algebra, giving strong Macdonald theorems that take into account both a parabolic component and a possible diagram automorphism twist. In particular we show that $H^*(p/z^Np)$ contains a parabolic subalgebra of the coinvariant algebra of the fixed-point subgroup of the Weyl group of $L$, and thus is no longer free. We also prove a strong Macdonald theorem for $H^*(b;S^s\mathfrak{n})$ and $H^*(b/z^N\mathfrak{n})$ when $b$ and $\mathfrak{n}$ are Iwahori and nilpotent subalgebras respectively of a twisted loop algebra. For each strong Macdonald theorem proved, taking $z$-weighted Euler characteristics gives an identity equivalent to Macdonald’s constant term identity for the corresponding affine root system. As part of the proof, we study the regular adjoint orbits for the adjoint action of the twisted arc group associated to $L$, proving an analogue of the Kostant slice theorem.

1. Introduction

Macdonald’s constant term identity states that if $\Delta$ is a reduced root system then

\[
\left[e^0\right] \prod_{\alpha \in \Delta^+} \prod_{i=1}^{N} (1 - q^{i-1}e^{-\alpha})(1 - q^i e^{\alpha}) = \prod_{i=1}^{l} \binom{N(m_i + 1)}{N}_q,
\]

where $m_1, \ldots, m_l$ is the list of exponents of $L$ and $\binom{a}{b}_q$ is the $q$-binomial coefficient. Macdonald presented the identity as a conjecture in [Ma82], and observed that it constitutes the untwisted case of a constant term identity for affine root systems. Further extensions (including a $(q,t)$-version) and proofs for individual affine root systems followed (see for instance [ZB85] [Hab86] [Ze87] [St88] [Ze88] [Ma88] [Gu90] [GG91] [Kad94]) until Cherednik gave a uniform proof of the most general version using double affine Hecke algebras [Ch95].

Suppose $\Delta$ is the root system of a semisimple Lie algebra $L$ with exponents $m_1, \ldots, m_l$. Prior to Cherednik’s proof, Hanlon observed in [Ha86] that the constant term identity would follow from a stronger conjecture:

\[
\text{The cohomology } H^*\left(L[z]/z^N\right) \text{ is a free super-commutative algebra with } N \text{ generators of cohomological degree } 2m_i + 1 \text{ for each } i = 1, \ldots, l, \text{ of which, for fixed } i, \text{ one has } z\text{-degree } 0 \text{ and the others have } z\text{-degree } Nm_i + j \text{ for } j = 1, \ldots, N - 1.
\]
Hanlon termed this the strong Macdonald conjecture, and gave a proof for $L = \mathfrak{sl}_n$. Feigin observed in [Fe91] that the identity of (1) and the theorem of (2) follow from:

The (restricted) cohomology $H^*(L[z, s])$ for $s$ an odd variable is a free super-commutative algebra with generators of tensor degree $2m_i + 1$ and $2m_i + 2$, $z$-degree $n$, for $i = 1, \ldots, l$ and $n \geq 0$, where tensor degree refers to combined cohomological and $s$-degree.

This version of the strong Macdonald conjecture corresponds to the $(q, t)$ version of the Macdonald constant term conjecture. However, an error was discovered in Feigin’s proof of (3). A complete proof of (2) and (3) was given by Fishel, Grojnowski, and Teleman [FGT08], using an explicit description of the relative cocycles combined with Feigin’s idea (a spectral sequence argument) to prove (2) from (3). The free algebra $H^*(L[z])$ (which can easily be calculated from the Hochschild-Serre spectral sequence) appears as a subalgebra of $H^*(L[z, s])$, and Fishel, Grojnowski, and Teleman also prove that if $\mathfrak{b}$ is the Iwahori subalgebra $\{ f \in L[z] : f(0) \in \mathfrak{b}_0 \}$ then $H^*(\mathfrak{b}[s])$ is the free algebra $H^*(\mathfrak{b}_0[s]) \otimes H^*(L[s]) \otimes H^*(L[z, s])$. In this case their proof does not yield explicit generating cocycles.

The purpose of this paper is to show that $H^*(\mathfrak{p}[s])$ is a free super-commutative algebra, and determine the degrees of the generators, when $\mathfrak{p}$ is a standard parahoric in the twisted loop algebra $L[z^{\pm 1}]^\sigma$, for $\sigma$ a (possibly trivial) diagram automorphism of $L$. Our proof is along the same lines as [FGT08]; in particular, we are able to give an explicit description of cocycles for the relative cohomology, and hence apply Feigin’s spectral sequence to determine the cohomology of the truncations $\mathfrak{p}/z^N\mathfrak{p}$ when $N$ is a multiple of the order of $\sigma$. Combined, our results for $L[z]^\sigma$ give an extension of the strong Macdonald theorems to match the affine version of Macdonald’s constant term identity. For a general parahoric, our calculation reveals that $H^*(\mathfrak{p}[s])$ is isomorphic to $H(\mathfrak{p}_0[s]) \otimes H^*(L[s]) \otimes H(L[z, s]^\sigma)$, and hence can be viewed as providing an interpolation between the two extremal results of Fishel, Grojnowski, and Teleman.

The algebras $H^*(\mathfrak{p}/z^N\mathfrak{p})$ also have an interesting description. As in [2], the algebras $H^*(L[z]^\sigma/z^N)$ are free, but this is no longer the case with a non-trivial parabolic component. The algebra $H^*(\mathfrak{p}/z^N\mathfrak{p})$ is isomorphic to $H^*(\mathfrak{g}_0) \otimes \overline{R}(L^\sigma, \mathfrak{g}_0) \otimes H^*(L[z]^\sigma/z^N)$, where $\mathfrak{g}_0 = \mathfrak{p}_0 \cap \overline{\mathfrak{p}_0}$ is the reductive component of the parabolic $\mathfrak{p}_0$, and $\overline{R}(L^\sigma, \mathfrak{g}_0)$ is the parabolic subalgebra of the coinvariant algebra of the Weyl group of $L^\sigma$. A classic theorem of Borel states that $\overline{R}(L^\sigma, \mathfrak{g}_0)$ is isomorphic to the cohomology algebra of the generalized flag variety $X$ corresponding to the Lie algebra pair $(L^\sigma, \mathfrak{p}_0)$ [Bo53c] [BG73]. The cohomology of $X$ is in turn isomorphic to the Lie algebra cohomology algebra $H^*(L^\sigma, \mathfrak{g}_0)$. If $\mathfrak{p}$ is a parahoric in an untwisted loop algebra, then it is not hard to show that $H^*(\mathfrak{p}/z^N\mathfrak{p}, \mathfrak{g}_0)$ is isomorphic to $H^*(L^\sigma, \mathfrak{g}_0)$, and hence in the simplest case our result gives a Lie algebraic proof of Borel’s theorem. This is not the first description of $H^*(L^\sigma, \mathfrak{g}_0)$ using Lie algebraic methods: the cohomology of $X$ can also be described using the Schubert cells, and a basis of $H^*(L^\sigma, \mathfrak{g}_0)$ dual to the Schubert cells has been worked out in Lie algebraic terms by Kostant [Ko63a]. However, this description omits the ring structure.

One intriguing consequence of Hanlon’s conjecture [2] is that $H^*(L[z]/z^N)$ is isomorphic as a vector space to $H^*(L)^{\otimes N}$. Since $L[z]/(z^N - t) \cong L[t]^{\otimes N}$ for $t \neq 0$, this means that while the structure of $L[z]/(z^N - t)$ changes dramatically as $t$ degenerates to zero, the cohomology is unchanged. Hanlon termed this “property M”, and conjectured that it holds not only for semisimple Lie algebras, but also for the nilpotent radical of a parabolic in a semisimple Lie algebra and the Heisenberg Lie algebras [Ha90]. Kumar gave counterexamples to property
M for the nilpotent radical of a parabolic [Ku99]. The conjecture for Heisenberg Lie algebras remains open, along with a number of other questions [Ha94] [HW03]. In the case of a parahoric in a twisted loop algebra $L[z^{\pm 1}]^\sigma$, if $t \neq 0$ then the truncation $p/(z^N - t)p$ is isomorphic to $L^\otimes N/k$, regardless of the parahoric component. Our calculation shows that the cohomology is unchanged for $L[z]^{\sigma}/(z^N - t)$ as $t$ degenerates to zero, but degenerates from $H^*(L)^{\otimes N/k}$ to $H^*(g_0) \otimes H^*(L^\sigma; g_0) \otimes H^*(L^\sigma) H^*(L)^{\otimes N/k}$ for a general parahoric truncation $p/(z^N - t)p$.

The proof of the strong Macdonald theorem in [FGT08] is based on a Laplacian calculation for $H^*(L[z,s])$ using the unique Kahler metric on the loop Grassmannian. The Laplacian calculation shows that the ring of harmonic forms is isomorphic to a ring of basic and invariant forms on the arc space $L[[z]]$. Kostant’s theorems about adjoint orbits in a reductive Lie algebra extend immediately from $L$ to $L[[z]]$, and can be used to determine the ring of basic and invariant forms on $L[[z]]$. In the case of the parahoric, the corresponding homogeneous space has many Kahler metrics. To follow the line of the proof in [FGT08], we show that there is a particular choice of Kahler metric that makes an analogous Laplacian calculation work. The ring of harmonic forms is isomorphic to (a ring similar to) the ring of basic and invariant forms on $p$. To calculate this ring, we study the adjoint orbits on the twisted arc space $L[[z]]^\sigma$, proving a slice theorem for twisted arcs in the regular semisimple locus, and an analogue of the Kostant slice theorem (the significant facts about adjoint orbits extend immediately to the arc space, but this is no longer the case when the diagram automorphism is involved).

Removing the super-notation, the cohomology ring of $p[s]$ is isomorphic to the cohomology ring of $p$ with coefficients in the symmetric algebra $S^*p^*$ of the restricted dual of $p$. Frenkel and Teleman have shown that $H^*(b; S^*n^*)$ is a free algebra (and determined the degrees of the generators) when $b$ and $n$ are Iwahori and nilpotent subalgebras respectively of an untwisted loop algebra [FT06]. We prove Frenkel and Teleman’s result in the twisted case and calculate the cohomology of the corresponding truncation $b/z^N n$. More generally, strong Macdonald theorems for different choices of coefficients might allow us to determine the cohomology of other truncations, such as $L[z]^\sigma/z^N$ when $N$ is not divisible by the order of $\sigma$. At the moment, this question appears to be open. The question of finding a Lie algebra analogue of the constant term identity for Koornwinder-Macdonald polynomials [Di96] also seems to be open.

1.1. Organization. Section 2 contains an overview of our cohomology results and the connection with the constant term identity. Section 3 contains the Laplacian calculation. Section 4 defines twisted jet and arc schemes. Section 5 contains theorems on adjoint orbits of twisted jet and arc groups. Section 6 contains the cohomology calculations of $H^*(p[s])$ and $H^*(b, S^*n^*)$, while Section 7 contains the spectral sequence argument for calculating the cohomology of the truncation $p/z^N p$.

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2. Cohomology of standard parahorics

2.1. Notation and terminology. We fix the following terminology and notation throughout the paper, except where explicitly stated. \( L \) will be a reductive Lie algebra with diagram automorphism \( \sigma \) of finite order \( k \). By definition \( L \) has a triangular decomposition \( L = \mathfrak{u}_0 \oplus \mathfrak{h} \oplus \mathfrak{u}_0 \) where the Cartan algebra \( \mathfrak{h} \) and nilpotent radicals \( \mathfrak{u}_0 \) and \( \mathfrak{u}_0 \) are \( \sigma \)-invariant, and such that \( \sigma \) permutes the simple roots corresponding to the Borel \( \mathfrak{h} \oplus \mathfrak{u}_0 \). We say that a Cartan, Borel, or nilpotent radical is compatible with \( \sigma \) if it appears in such a decomposition. The twisted loop algebra is the Lie algebra \( \mathfrak{g} = L[z^{\pm 1}]^{\tilde{\sigma}} \), where \( \tilde{\sigma} \) is the automorphism sending \( f(z) \mapsto \sigma(f(q^{-1}z)) \) for \( q \) a fixed \( k \)th root of unity. \( \mathfrak{g} \) can be written as

\[
\mathfrak{g} = \bigoplus_{i=0}^{k-1} L_a \otimes z^a \mathbb{C}[z^{\pm k}],
\]

where \( L_a \) is the \( q^a \)th eigenspace of \( \sigma \). If \( L \) is simple then each \( L_a \) is an irreducible \( L_0 \)-module. In particular if \( L \) is simple then \( L_0 \) is also simple; in general \( L_0 \) will be reductive. A reductive Lie algebra \( L \) has an anti-linear Cartan involution \( \tilde{\cdot} \) and a contragradient positive-definite
Hermitian form \{,\}. These two structures extend to the twisted loop algebra \(g\) so that for any grading of type \(d\), \(\overline{g}_n = g_{-n}\) and \(g_m \perp g_n\) when \(m \neq n\).

The root system of \(g\) can be described as follows. Let \(h\) be a Cartan subalgebra of \(L\) compatible with the diagram automorphism. Then \(h_0 := h^*\) is a Cartan in \(L_0\), and \(L_0\) has a set \(\alpha_1,\ldots,\alpha_l\) of simple roots which are projections of simple roots of \(L\). The roots of \(g\) can be described as \(\alpha + n\delta \in h^*_0 \times \mathbb{Z}\) where either \(\alpha\) is a weight of \(L\) with \(n \equiv a \mod k\), or \(\alpha = 0\) and \(n \neq 0\), and \(\delta\) comes from the rotation action of \(\mathbb{C}^*\) on \(g\). Assume that \(L\) is simple, and let \(\psi\) be either the highest weight of \(L_1\) (an irreducible \(L_0\)-module) if \(k > 1\), or the highest root of \(L\), if \(k = 1\). Then the set \(\alpha_0 = \delta - \psi, \alpha_1,\ldots,\alpha_l\) is a complete set of simple roots for \(g\). If \(L\) is reductive then we can choose a set of simple roots by decomposing \(L\) as a direct sum of \(\sigma\)-invariant simple subalgebras plus centre, and taking the simple root sets from each corresponding factor of \(g\).

The twisted loop algebra \(g\) can be given a \(\mathbb{Z}\)-grading by assigning degree \(d_i \geq 0\) to the positive root vector associated to \(\alpha_i\). In Kac’s terminology this is called a grading of type \(d\) [Ka83]. A parahoric subalgebra of \(g\) is a subalgebra of the form \(p = \bigoplus_{n \geq 0} g_n\), for some \(\mathbb{Z}\)-grading of \(g\) of type \(d\). A parahoric subalgebra contains a nilpotent subalgebra \(u = \bigoplus_{n \geq 0} g_n\). We will say that a parahoric is standard with respect to the choice of simple roots if it comes from a grading of type \(d\) such that \(d_i > 0\) whenever \(\alpha_i\) is of the form \(\delta - \psi\) for \(\psi \in h^*_0\). Suppose \(p\) is a standard parahoric. Let \(S = \{\alpha_i : d_i = 0\}\), and \(p_0\) be the standard parabolic subalgebra of \(L_0\) defined by

\[
p_0 = h_0 \bigoplus \bigoplus_{\alpha \in \Delta^+} (L_0)_{\alpha} \bigoplus \bigoplus_{\alpha \in \Delta^- \cap \mathbb{Z}[S]} (L_0)_{\alpha},
\]

where \(\Delta^+\) are the positive and negative roots of \(L_0\) with respect to the chosen simple roots. Then \(p = \{f \in g : f(0) \in p_0\}\), while \(u = \{f \in g : f(0) \in u_0\}\), where \(u_0\) is the nilpotent radical of \(p_0\). Note that in this context the nilpotent radical of an algebra \(\mathfrak{g}\) is defined to be the largest nilpotent ideal in \([\mathfrak{g}, \mathfrak{g}]\) (or equivalently the intersection of the kernels of all irreducible representations), so that \(u_0\) does not intersect the centre of \(L\). If \(p_0\) is a Borel, then \(p\) is called a standard Iwahori subalgebra.

The completion of a subalgebra \(K \subset g\) with respect to a \(\mathbb{Z}\)-grading is the algebra \(\hat{K} = \lim \leftarrow K/K^{(k)}\), where \(K^{(k)} = \bigoplus_{n > k} K_n\). If a parahoric subalgebra \(p\) is completed with respect to a grading of \(g\) of type \(d\), the result is a pro-Lie algebra \(\hat{p}\). The pro-algebra structure on \(\hat{p}\) is independent of the choice of grading. The dual of a pro-algebra \(\hat{p}\) will always refer to the continuous dual \(\hat{p}^* \cong \bigoplus \hat{p}_n^*\) with respect to the inverse limit topology. The continuous cohomology \(H^*_\alpha(\hat{p}; V)\) is defined similarly to the ordinary cohomology using a version of the Koszul complex with continuous cochains. We refer to [Fu86] for more details on continuous cohomology. In this case, continuous cohomology is the same as the restricted cohomology of [FGT08].

2.2. Exponents and diagram automorphisms. The exponents of \(L\) are integers \(m_1,\ldots,m_l\) such that \(H^*(L)\) is the free super-commutative algebra generated in degrees \(2m_1 + 1,\ldots,2m_l + 1\), where \(l\) is the rank of \(L\). Equivalently, we can define the exponents by saying that \((S^*L)^L\) is the free commutative algebra generated in degrees \(m_1 + 1,\ldots,m_l + 1\). Extend the action of \(\sigma\) to \(S^*L^*\) by \(\sigma(f)(z) = f(\sigma^{-1}z)\). This convention is chosen so that \(\sigma(\text{ad}^l(x)f) = \text{ad}^l(\sigma(x))\sigma(f)\) for all \(f \in S^*L^*\) and \(x \in L\). Let \(\mathfrak{M}\) be the ideal in \((S^*L)^L\) generated by elements of degree greater than zero. The diagram automorphism \(\sigma\) acts diagonalizably on the space \(\mathfrak{M}/\mathfrak{M}^2\).
of generators for \((S^*L^*)^L\), and consequently it is possible to find homogeneous generators of \(\mathbb{C}[Q]\) which are eigenvectors of \(\sigma\).

**Definition 2.1.** Choose a set of homogeneous generators for \((S^*L^*)^L\) which are eigenvectors of \(\sigma\). The exponents of \(L\) can be sorted into different sets \(m_1^{(a)}, \ldots, m_{i_a}^{(a)}\), \(a \in \mathbb{Z}_k\), by letting \(m_1^{(a)} + 1, \ldots, m_{i_a}^{(a)} + 1\) be the list of degrees of homogeneous generators of \((S^*L^*)^L\) with eigenvalue \(q^{-a}\) (note the negative exponent). We call the elements of these sets the exponents of \(L_a\).

Recall that if \(V\) is an \(L_0\)-module and \(\{h, e, f\}\) is a principal \(\mathfrak{sl}_2\)-triple in \(L_0\), then the generalized exponents of \(V\) are the eigenvalues of \(h/2\) on the subspace \(V^{L_0}\) fixed by the abelian subalgebra \(L_0^e\). The generalized exponents are always non-negative integers, and the dimension of \(V^{L_0}\) is equal to the dimension of the zero weight space of \(V\).

**Proposition 2.2.** The exponents of \(L_a\) are the generalized exponents of \(L_a\) as an \(L_0\)-module.

The proof of Proposition 2.2 will be given in Subsection 6.1. The generalized exponents of \(L_a\) are the same as the ordinary exponents, and \(l_0\) is the rank of \(L_0\), so there is no conflict in our terminology. In general \(l_a\) is the dimension of \(\mathfrak{h} \cap L_a\), where \(\mathfrak{h}\) is a Cartan compatible with \(\sigma\). If \(L\) is simple, then \(k\) is either 1 or 2, except when \(L = \mathfrak{so}(8)\) in which case \(k\) can be 3 and \(L_1\) is isomorphic to \(L_2\). As a result, the exponents of \(L_a\) are the same as the exponents of \(L_{-a}\). A principal \(\mathfrak{sl}_2\)-triple in \(L_0\) is also principal in \(L\) (see Lemma 5.9), so \(L^e\) is abelian and hence \(L_a^{L_0^e} = L_a^e\), simplifying the definition of generalized exponents in this case. The eigenvalues of \(h/2\) give a principal grading \(L_a = \bigoplus L_a^{(i)}\) of each \(L_a\) such that \(L = \bigoplus_i \bigoplus_a L_a^{(i)}\) is a principal grading for \(L\). The representation theory of \(\mathfrak{sl}_2\) then implies:

**Corollary 2.3.** The multiplicity of \(m\) in the list of exponents of \(L_a\) is \(\dim L_a^{(m)} - \dim L_a^{(m+1)}\), where \(L_a = \bigoplus L_a^{(i)}\) is a principal grading.

The exponents of \(L_a\) can be easily determined when \(L\) is simple, and are given in the following table:

| Type of \(L\) | \(k\) | Type of \(L_0\) | Exponents of \(L_0\) | Exponents of \(L_{\pm 1}\) |
|---------------|------|-----------------|---------------------|---------------------|
| \(A_{2n}\)    | 2    | \(B_n\)         | 1, 3, \ldots, 2n − 1 | 2, 4, \ldots, 2n    |
| \(A_{2n−1}\)  | 2    | \(C_n\)         | 1, 3, \ldots, 2n − 1 | 2, 4, \ldots, 2n − 2 |
| \(D_n\)       | 2    | \(B_{n−1}\)     | 1, 3, \ldots, 2n − 3 | \(n − 1\)           |
| \(E_6\)       | 2    | \(F_4\)         | 1, 5, 7, 11          | 4, 8                |
| \(D_4\)       | 3    | \(G_2\)         | 1, 5                | 3                   |

2.3. Cohomology of superpolynomials in a standard parahoric. Let \(\mathfrak{p} = \{f \in \mathfrak{g} : f(0) \in \mathfrak{p}_0\}\) be a standard parahoric in a twisted loop algebra \(\mathfrak{g}\), and let \(\hat{\mathfrak{p}}[s]\) denote the superpolynomial algebra in one odd variable with values in \(\hat{\mathfrak{p}}\). The cohomology of the super Lie algebra \(\hat{\mathfrak{p}}[s]\) can be calculated as in the ordinary case using the Koszul complex, so any grading on \(\hat{\mathfrak{p}}[s]\) induces a grading on \(H_{cts}(\hat{\mathfrak{p}}[s])\). In particular \(H_{cts}(\hat{\mathfrak{p}}[s])\) is graded by \(z\)-degree and by \(s\)-degree.

**Theorem 2.4.** Let \(m_1^{(a)}, \ldots, m_{i_a}^{(a)}\) denote the exponents of \(L_a\), and let \(r_1, \ldots, r_{i_a}\) denote the exponents of the reductive algebra \(\mathfrak{p}_0 \cap \overline{\mathfrak{p}_0}\), where \(\mathfrak{p}_0\) is a parabolic in \(L_0\). If \(\mathfrak{p}\) is the standard
parahoric \{ f \in L[[z]]^\sigma : f(0) \in p_0 \} then the cohomology ring \( H^*_\text{cts}(\hat{p}[s]) \) is a free supercommutative algebra generated in degrees given in the following table:

| Cohomological degree | s-degree | z-degree | Index set |
|----------------------|----------|----------|-----------|
| 2r_i + 1             | 0        | 0        | i = 1, \ldots, l_0 |
| r_i + 1              | r_i + 1  | 0        | i = 1, \ldots, l_0 |
| m_i^{(-a)} + 1       | m_i^{(-a)} + 1 | kn - a | n \geq 1, a = 0, \ldots, k - 1, i = 1, \ldots, l_a |
| m_i^{(-a)} + 1       | m_i^{(-a)} | kn - a | n \geq 1, a = 0, \ldots, k - 1, i = 1, \ldots, l_a |

To prove Theorem 2.4 we give an explicit description of a generating set of cocycles for the relative cohomology ring \( H^*_\text{cts}(\hat{p}, g_0; S^*\hat{p}^*) \), where \( g_0 = p_0 \cap \overline{p}_0 \). Choose a set of generators \( I_i^a \), \( a \in \mathbb{Z}_k, i = 1, \ldots, l_a \) for \((SL)^*\) such that \( I_i^a \) is an eigenvector of \( \sigma \) with eigenvalue \( q^{-a} \). Also choose a set of homogeneous generators \( R_1, \ldots, R_{l_0} \) for \((S^*g_0)^*\). The polynomial functions \( I_i^a \) on \( L \) induce functions \( \tilde{I}_i^a : L[[z]] \to \mathbb{C}[[z]] \), and the coefficients \( [z]^a \tilde{I}_i^a \) of \( z^n \) in \( \tilde{I}_i^a \) restrict to \( \hat{p} \)-invariant polynomial functions on \( \hat{p} \). Similarly, the polynomials \( R_i \) on \( p_0 \) can be pulled back via the quotient map \( p \to g_0 \) to \( \hat{p} \)-invariant polynomials on \( \hat{p} \). Finally, 1-cocycles can be constructed as follows. If \( J \) is a derivation of \( \hat{p} \) that kills \( g_0 \) and \( \phi \in S^k\hat{p}^* \) is \( \hat{p} \)-invariant then the tensor

\[
\hat{u} \otimes S^{k-1} \hat{p} \to \mathbb{C} : x \otimes s_1 \circ \ldots \circ s_{k-1} \mapsto \phi(Jx \circ s_1 \circ \ldots \circ s_{k-1}).
\]

is a cocycle (see Lemma 3.1).

**Theorem 2.5.** Let \( p \) be a standard parahoric in \( g \), and let \( J \) be the derivation from Theorem 3.3. Then there is a metric on the Koszul complex such that the harmonic cocycles for \( H^*_\text{cts}(\hat{p}, g_0; S^*\hat{p}^*) \) form a free supercommutative ring generated by the cocycles in the following table:

| Cocycle description | Coh. deg. | Sym. deg | z-deg. | Index set |
|---------------------|-----------|----------|--------|-----------|
| \( R_i \)           | 0         | \deg R_i | 0      | i = 1, \ldots, l_0 |
| \( [z^{kn-a}]\tilde{I}_i^{-a} \) | 0         | \deg \tilde{I}_i^{-a} | kn - a | n \geq 1, i = 1, \ldots, l_{-a}, a = 0, \ldots, k - 1 |
| \( x \otimes s \mapsto [z^{kn-a}]\tilde{I}_j^{-a}(Jx \circ s) \) | 1         | \deg \tilde{I}_j^{-a} - 1 | kn - a | n \geq 1, i = 1, \ldots, l_{-a}, a = 0, \ldots, k - 1 |

Proving Theorem 2.5 is the main concern of the paper; the proof is finished in Subsection 6.2.

**Proof of Theorem 2.4 from Theorem 2.5.** Since the bracket of \( \hat{p}[s] \) is zero on the odd component, the Koszul complex for \( \hat{p}[s] \) reduces to the Koszul complex for \( p \) with coefficients in \( S^*\hat{p}^* \). Thus there is a ring isomorphism \( H^*_\text{cts}(\hat{p}[s]) \cong H^*_\text{cts}(\hat{p} ; S^*\hat{p}^*) \) in which \( H^{r-n-q}(\hat{p} ; S^q\hat{p}^*) \) corresponds to the cohomology classes in \( H^*_\text{cts}(\hat{p}[s]) \) of s-degree \( q \). This isomorphism preserves z-degree. The degree zero component of \( p \) is \( g_0 = p_0 \cap \overline{p}_0 \), a reductive algebra which is the quotient of \( p \) by the standard nilpotent subalgebra \( u \). It follows from the Hochschild-Serre spectral sequence (in particular Theorem 12 of [HS53]) that there is a ring isomorphism

\[
H^*_\text{cts}(\hat{p} ; S^*\hat{p}^*) \cong H^*(g_0) \otimes H^*_\text{cts}(\hat{p} ; g_0 ; S^*\hat{p}^*).
\]

Then Theorem 2.4 follows from the description of relative cohomology. \( \square \)
When \( p_0 = L_0 \), Theorem 2.4 states that the algebra \( H^*_{cts}(L[z, s])^{\tilde{\gamma}} \) is the free supercommutative algebra with generators in tensor degree \( 2m_i^{(a)} + 1 \) and \( 2m_i^{(a)} + 2 \), and z-degree \( nk + a \), for \( a = 0, \ldots, k - 1, i = 1, \ldots, l_a \), and \( n \geq 0 \). In addition the Hochschild-Serre spectral sequence implies that \( H^*(p_0[s]) \cong H^*(\tilde{p}_0) \otimes (S^*q_{\tilde{p}_0})^{p_0} \), where \( g_0 = p_0 \cap \tilde{p}_0 \), so \( H^*(p_0[s]) \) is isomorphic to the subalgebra of \( H^*_{cts}(\hat{p}[s]) \) of z-degree zero. In fact, the inclusion is the pullback map given by evaluation at zero, as can be seen from the explicit description of harmonic cocycles, so \( H^*_{cts}(\hat{p}[s]) \) is naturally isomorphic to \( H^*(p_0[s]) \otimes H^*(L_0[n]) H^*_{cts}(L[z, s])^{\tilde{\gamma}} \).

We can also ask for an explicit description of the relative cohomology groups \( H^*_{cts}(\hat{p}, g_0; S^*\hat{u}^*) \). In this case, we can only provide an answer when \( p \) is an Iwahori subalgebra—that is, a standard parahoric \( \{ f \in L[[z]]^p : f(0) \in p_0 \} \) where \( p_0 \) is a Borel subalgebra.

**Theorem 2.6.** Let \( \mathfrak{b} \) be an Iwahori subalgebra of \( \mathfrak{g} \), and let \( \mathfrak{n} \) be the nilpotent subalgebra. Let \( J \) be the derivation from Theorem 3.3. Then there is a metric on the Koszul complex such that the harmonic cocycles for \( H^*_{cts}(\hat{b}, \mathfrak{b}_0; S^*\hat{u}^*) \) form a free supercommutative ring generated by the cocycles in the following table:

| Cocycle description | Coh. deg. | Sym. deg. | z-deg. | Index set |
|---------------------|-----------|-----------|--------|-----------|
| \([z^{kn-a}]\bar{I}_i^{-a}\) | 0         | \(\deg I_i^{-a}\) | \(kn - a\) | \(n \geq 1, \ i = 1, \ldots, l_a, \ a = 0, \ldots, k - 1\) |
| \(x \otimes s \mapsto [z^{kn-a}]\bar{I}_j^{-a}(Jx \circ s)\) | 1         | \(\deg I_j^{-a} - 1\) | \(kn - a\) | \(n \geq 1, \ i = 1, \ldots, l_a, \ a = 0, \ldots, k - 1\) |

Theorem 2.6 can be used to calculate \( H^*_{cts}(\hat{b}; S^*\hat{n}) \) as in the proof of Theorem 2.4. With an appropriate degree shift, the cohomology ring \( H^*_{cts}(\hat{b}, \mathfrak{b}_0; S^*\hat{u}^*) \) can also be regarded as the \( \mathfrak{h} \)-invariant part of \( H^*_{cts}(\hat{n}[s]) \). The proof of Theorem 2.6 will be completed in Subsection 6.3.

### 2.4. Cohomology of the truncated algebra

If \( N \) is a multiple of \( k \) then \( z^N L[z]^{\tilde{\gamma}} \) is a subset of \( L[z]^{\tilde{\gamma}} \), and hence \( z^N p \) is an ideal of \( p \). Theorem 2.5 can be used to determine the cohomology of the finite-dimensional Lie algebra \( p/z^N p \).

Recall that the coinvaiant algebra of the Weyl group \( W(L_0) \) is the quotient of \( S^*\mathfrak{h}_0^* \) by the ideal generated by \( (S^*\mathfrak{h}_0^*)^{W(L_0)} \). We define \( \overline{W(L_0, g_0)} \) to be the graded algebra which is the quotient of \( (S^*\mathfrak{g}_0^*)^g_0 \) by the ideal generated by \( (S^*\mathfrak{L}_0^*)^{L_0} \), where \( S^*\mathfrak{L}_0^* \) acts on \( S^*\mathfrak{g}_0^* \) by restriction. By the Chevalley restriction theorem, \( \overline{W(L_0, g_0)} \) is isomorphic to the subalgebra of \( W(g_0)-\)invariants in the coinvariant algebra of \( W(L_0) \). It is well-known that the Poincare series for \( \overline{W(L_0, g_0)} \) with the symmetric grading is \( \prod_{i=1}^{l_0} (1 - q^{r_i+1})^{-1} \prod_{i=1}^{l_0} (1 - q^{m_i(0)+1}) \), where \( m_i^{(0)} \) refers to the exponents of \( L_0 \) and \( r_i \) refers to the exponents of \( g_0 \). The dimension of \( \overline{W(L_0, g_0)} \) is \( |W(L_0)|/\{W(g_0)|\} \).

**Theorem 2.7.** Let \( m_1^{(a)}, \ldots, m_{l_a}^{(a)} \) denote the exponents of \( L_a \), and let \( r_1, \ldots, r_{l_0} \) be the exponents of the reductive Lie algebra \( g_0 = p_0 \cap \tilde{p}_0 \). Let \( \overline{W(L_0, g_0)} \) denote the coinvaiant algebra, with a cohomological grading (resp. z-grading) defined by setting the cohomological degree (resp. z-degree) to twice (resp. \( N \) times) the symmetric degree.

If \( p \) is the standard parahoric \( \{ f \in L[[z]]^p : f(0) \in p_0 \} \) and \( N \) is a multiple of \( k \) then the cohomology algebra \( H^*(p/z^N p) \) is isomorphic to \( \overline{W(L_0, g_0)} \otimes \Lambda \), where \( \Lambda \) is the free supercommutative algebra generated in degrees given by the following table:
| Cohomological degree | z-degree | Index set |
|----------------------|----------|-----------|
| 2r_i + 1             | 0        | i = 1, ..., l_0 |
| 2m_i(a) + 1          | Nm_i(a) + nk + a | a = 0, ..., k - 1, i = 1, ..., l_a, 0 < nk + a < N |

As in the proof of Theorem 2.4, we have

$$H^*(p/z^Np) \cong H^*(g_0) \otimes H^*(p/z^Np, g_0),$$

so we only need to compute the relative cohomology. This will be done with a spectral sequence argument in Section 7 (see Proposition 7.6).

When the parabolic component is trivial, $H^*\left(L[z]^\sigma/z^N\right)$ is simply the free super-commutative algebra with one set of generators in cohomological degree 2m_i(a) + 1 and z-degree 0 for $i = 1, ..., l_0$, and another set of generators in cohomological degree 2m_i(a) + 1 and z-degree Nm_i(a) + nk + a, where $a = 0, ..., k - 1, i = 1, ..., l_a$, and $n$ such that $0 < nk + a < N$. Theorem 2.7 can be restated as saying that $H^*(p/z^Np)$ is the algebra $H^*(g_0) \otimes \overline{R}(L_0, g_0) \otimes H^*\left(L[z]^\sigma/z^N\right)$.

In Lemma 7.7, we prove that if $g$ is untwisted and $N = 1$ then $H^*(p/zp, g_0)$ is isomorphic to $H^*(L_0, g_0)$. This algebra is in turn isomorphic to the cohomology ring of the generalized flag variety corresponding to the pair $(L_0, p_0)$. The z-grading on $H^*(p/zp, g_0)$ corresponds to the holomorphic grading appearing in the Hodge decomposition. The fact that $\overline{R}(L_0, g_0)$ is isomorphic to $H^*(L_0, g_0)$ is a classic theorem of Borel ([Bo53], see Theorem 5.5 of [BGG73] for the parabolic case). Thus Theorem 2.7 can be seen as a generalization of Borel’s theorem.

We can compare the cohomology of $p/z^Np$ with the cohomology of more general truncations. If $P(z)$ is a polynomial in $z$, then $P(z^k)L[z]^\sigma$ is a subset of $L[z]^\sigma$, and hence $P(z^k)p$ is an ideal of $p$. We can assume that $P$ is monic, and write $P = z^d + P_0$, where $d$ is the degree of $P$ and $P_0$ contains lower degree terms. Suppose $x \in L_i$ for $i \geq 0$. Then $(z^d + P_0(z^k))xz^l$ is in $P(z^k)p$ if and only if either $i > 0$ or $x \in p_0$, so the dimension of $p/P(z^k)p$ is $d \cdot \dim L$.

**Lemma 2.8.** If $P$ and $Q$ are coprime then $p/P(z^k)Q(z^k)p \cong p/P(z^k)p \oplus p/Q(z^k)p$.

By Lemma 2.8, the study of $p/P(z^k)p$ reduces to the case where $P$ is the power of a linear factor. In the untwisted case, $L[z] \cong L[z - \alpha]$, so $L[z]/(z - \alpha)^N \cong L[z]/z^N$. However, in the twisted case this argument does not apply, since the automorphism $z \mapsto q^{-1}z$ is different from $z - \alpha \mapsto q^{-1}(z - \alpha)$. In particular:

**Lemma 2.9.** If $\alpha \neq 0$ then $p/(z^k - \alpha)p$ is isomorphic to $L$.

**Proof.** Let $\beta$ be a kth root of $\alpha$. Then evaluation at $\beta$ defines a morphism $p/(z^k - \alpha)p \rightarrow L$. Both $L$ and $p/(z^k - \alpha)p$ have the same dimension, so we just need to show that this map is onto. Given $x \in L$, write $x = \sum_{i=0}^{k-1} x_i$ where $x_i \in L_i$. Let $f = \sum_{i=1}^{k-1} x_i\beta^{-i}z^l + x_0\alpha^{-1}z^k$. Then $f(\beta) = x$. \[\square\]

The author does not know if an analogue of Lemma 2.9 holds for higher powers of $(z^k - \alpha)$. The main case of interest is $p/(z^N - t)p$, which can be regarded as a deformation of $p/z^Np$. Since $z^N/k - t$ splits into $N/k$ coprime linear factors, the algebra $p/(z^N - t)p$ is isomorphic to $L^\otimes N/k$ for $t \neq 0$. At $t = 0$, the algebra $p/z^Np$ has a large nilpotent ideal. Ignoring z-degrees, Theorem 2.7 tells us that $H^*\left(L[z]^\sigma/z^N\right) \cong H^*(L)^\otimes N/k$, so the cohomology of $L[z]^\sigma/(z^N - t)$ is independent of the value of $t$. On the other hand, Theorem 2.7 tells us that $H^*\left(p/(z^N - t)p\right)$ changes from $H^*(L)^\otimes N/k$ to $H^*(g_0) \otimes H^*(L_0, g_0) \otimes H^*\left(L[z]^\sigma/z^N\right)$ as $t$ degenerates to zero,
where \( H^*(L_0) \) acts on \( H^*(g_0) \) via pullback. Interestingly, the cohomology of \( p/z^k(z^N - t)p \) is unchanged as \( t \) degenerates to zero.

If \( p = b \) is an Iwahori and \( n \) is the nilpotent subalgebra, then a similar analysis can be performed for \( b/z^Nn \).

**Theorem 2.10.** Let \( m_1^{(a)}, \ldots, m_l^{(a)} \) denote the exponents of \( L \), let \( b \) be an Iwahori subalgebra of the twisted loop algebra \( g \), and let \( n \) be the nilpotent subalgebra. Then \( H^*(b/z^N n) \) is the free super-commutative algebra with a generator in cohomological degree \( 2m_1^{(a)} + 1 \) and z-degree \( Nm_1^{(a)} + nk + a \) for every \( a = 0, \ldots, k - 1 \), \( i = 1, \ldots, l_ao \), and \( n \) such that \( 0 < nk + a \leq N \), as well as \( l_0 \) generators of cohomological degree 1 and z-degree 0.

As with Theorem 2.7, the proof of Theorem 2.10 reduces via the Hochschild-Serre spectral sequence to the computation of the relative cohomology, which is also completed in Section 7 (see Proposition 7.8).

If \( P(z) \) is a polynomial of degree \( d \), then \( b/P(z^k)n \) has dimension \( d \cdot \dim L + l_0 \). Furthermore, \([b, P(z^k)h_0]\) is contained in \( P(z^k)n \), and there is a morphism \( b/P(z^k)n \to b/P(z^k)b \) with kernel \( P(z^k)h_0 \), so \( b/P(z^k)n \) is a central extension of \( b/P(z^k)b \) of rank \( l_0 \).

**Lemma 2.11.** If \( t \neq 0 \) then \( b/(z^N - t)b \) is isomorphic to \( L^{\otimes N/k} \oplus \mathbb{C}l_0 \), where the second summand is abelian.

**Proof.** \( b/(z^N - t)b \) is isomorphic to the direct sum of \( N/k \) copies of \( L \). If \( L \) is semisimple, then so is \( b/(z^N - t)b \), so all central extensions are trivial. The reductive case reduces to the semisimple case by splitting off the centre.

Thus \( H^*(b/(z^N - t)n) \) is also independent of \( t \) when z-degrees are disregarded.

### 2.5. The Macdonald constant term identity

**Theorem 2.12** (Cherednik). Let \( N \) be a multiple of \( k \), and let \( S_N \) be the set of real roots \( \alpha + n\delta \) of the twisted loop algebra \( g \) with \( 0 \leq n \leq N \), such that \( \alpha \) is a positive (resp. negative) root of \( L_0 \) if \( n = 0 \) (resp. \( n = N \)). Let \( \rho \) be the element of \( h_0 \) such that \( \alpha_i(\rho) = 1 \) for all simple roots \( \alpha_1, \ldots, \alpha_{l_0} \) of \( L_0 \), and let \( \rho_N = -N\rho + \delta^* \). Then

\[
[e^0] \prod_{\alpha \in S_N} (1 - e^{-\alpha}) = \prod_{\alpha \in S_N} (1 - q^{\epsilon(\alpha(\rho_N))})^{\epsilon(\alpha)},
\]

where \( \epsilon(\alpha) \) is the sign of \( \alpha(\rho_N) \).

Define a twisted \( q \)-binomial coefficient for \( a \in \mathbb{Z}_k \) and multiples \( N, M \) of \( k \) by

\[
\binom{N}{M}_{k,a} = \prod_{i+1 \leq j \leq \min(N, M) \mod k} (1 - q^i) \prod_{0 \leq j \leq \min(M, N) \mod k} (1 - q^i)^{-1}.
\]

The right-hand side of Theorem 2.12 can be simplified by extending an idea of [Ma82] from the untwisted case.
Lemma 2.13. The identity of Theorem 2.12 is equivalent to

\[ [e^0] \prod_{\alpha \in S_N} (1 - e^{-\alpha}) = \prod_{a \in \mathbb{Z}_k} \prod_{i=1}^{l_a} \left( \frac{N(m_i^{(a)} + 1)}{N} \right)_{k,a}. \]

Proof. Let \( \Delta_a \) be the set of weights of the \( L_a \)-module \( L_a \), and let \( \Delta_a^+ \) denote the subset of \( \alpha \in \Delta_a \) such that \( \alpha(\rho) > 0 \). If \( \theta \) is an arbitrary function from positive integers to a multiplicative group, then

\[
\prod_{\alpha \in \Delta_a^+} \frac{\theta(\alpha(\rho) + 1)}{\theta(\alpha(\rho))} = \prod_{i=1}^{l_a} \frac{\theta(m_i^{(a)} + 1)}{\theta(1)}.
\]

To prove this, note that the eigenvalues of \( \rho \) on \( L_a \) are integers giving the principal grading of \( L_a \), so the identity follows immediately from Corollary 2.3 by comparing the number of times \( \theta(m) \) occurs on the top versus the bottom.

Define

\[
A_a = \prod_{\alpha + n\delta \in S_N} \prod_{\alpha \in \Delta_a} (1 - q^{\alpha(\rho_N)})^{\epsilon(\alpha)},
\]

and set \( \theta_{-a}(m) = (1 - q^{Nm-a})(1 - q^{Nm-k-a}) \cdots (1 - q^{Nm-N+k-a}) \), for \( a \in \mathbb{Z}_k \) represented by one of \( 0, \ldots, k-1 \). Then

\[
A_0 = \prod_{\alpha \in \Delta_0^+} \prod_{n=0}^{N/k-1} (1 - q^{N\alpha(\rho) - nk})^{-1} \prod_{n=1}^{N/k} (1 - q^{N\alpha(\rho) + nk})
\]

while if \( a \neq 0 \) we have

\[
A_a = \prod_{\alpha \in \Delta_a^+} \prod_{n=0}^{N/k-1} (1 - q^{N\alpha(\rho) - nk - a})^{-1} (1 - q^{N\alpha(\rho) + nk + a}).
\]

In both cases,

\[
A_a = \prod_{\alpha \in \Delta_a^+} \theta_{-a}(\alpha(\rho))^{-1}\theta_a(\alpha(\rho) + 1).
\]

Even if \(-a \) and \( a \) are not congruent, \( L_a \) and \( L_{-a} \) are still isomorphic, so

\[
A_a A_{-a} = \prod_{\alpha \in \Delta_a^+} \theta_{-a}(\alpha(\rho))^{-1}\theta_a(\alpha(\rho) + 1)\theta_a(\alpha(\rho))^{-1}\theta_{-a}(\alpha(\rho) + 1).
\]

Hence the right hand side of Theorem 2.12 is equal to

\[
\prod_{a=0}^{k-1} A_a = \prod_{a \in \mathbb{Z}_k} \prod_{i=1}^{l_a} \frac{\theta_a(m_i^{(a)} + 1)}{\theta_a(1)},
\]

as required. \( \square \)

Let \( C^* \) be a chain complex with an additional grading \( C^* = \bigoplus C_n^* \). The weighted Euler characteristic of \( C^* \) is

\[
\chi(C^*; q) = \sum_{n,i} (-1)^i \dim C_n^i q^n.
\]
As in the unweighted case, the weighted Euler characteristic is invariant under taking homology. Let \( p = \{ f \in L[[z]]^\hat{g} : f(0) \in p_0 \} \) be a standard parahoric and \( g_0 = p_0 \cap \bar{p}_0 \). Theorem 2.12 can be proved by comparing the \( z \)-weighted Euler characteristic for the Koszul complex of the pair \( (p/z^N p, g_0) \) with the weighted Euler characteristic of the cohomology ring:

Proof. Write \( p_0 = g_0 \oplus u_0 \), for \( u_0 \) the nilpotent radical. Let \( K \) be a compact subgroup acting on \( L \) with complexified Lie algebra \( g_0 \), and let \( T \) be a maximal torus in \( K \) with complexified Lie algebra \( h_0 \). Let \( \pi_a \) denote the representation of \( K \) on \( L_a \), and let \( \phi \) and \( \bar{\phi} \) denote the representation of \( K \) on \( u_0 \) and \( \bar{u}_0 \) respectively. The weighted Euler characteristic of the Koszul complex is

\[
\chi(q) = \sum (-1)^i q^i \dim \left( \bigwedge^i (p/z^N p)^a \right)
\]

By orthogonality of traces of representations with respect to Haar measure,

\[
\chi(q) = \int_K \det(\mathbb{1} - \phi(k)) \det(\mathbb{1} - q^N \bar{\phi}(k)) \prod_{0 < n < N} \det(\mathbb{1} - q^n \pi_n(k)) dk.
\]

The integrand is conjugation invariant, so by the Weyl integral formula,

\[
\chi(q) = \frac{1}{|W(g_0)|} \int_T \det(\mathbb{1} - \phi(t)) \det(\mathbb{1} - q^N \bar{\phi}(t)) \prod_{0 < n < N} \det(\mathbb{1} - q^n \pi_n(t)) \prod_{\alpha \in \Delta(g_0)} (1 - e^{\alpha(t)}) dt
\]

\[
= \frac{1}{|W(g_0)|} [e^0] \prod_{\alpha \in \Delta^+(g_0)} (1 - e^\alpha) \cdot \Phi,
\]

where \( \Delta(g_0) \) is the root system of \( g_0 \) and

\[
\Phi = \prod_{\alpha \in \Delta^+(g_0)} (1 - e^{-\alpha})^{-1} (1 - q^N e^\alpha)^{-1} \prod_{\alpha \in S_N} (1 - e^\alpha) \prod_{0 < n < N} (1 - q^n)^{l_n}
\]

(note that the inverses divide into the other multiplicands). The coefficient of \( q^i \) in \( \Phi \) is (up to sign) the character of a \( g_0 \)-module, so \( \Phi \) is \( W(g_0) \) invariant. Now we use the identity

\[
(6) \sum_{w \in W(g_0)} \prod_{\alpha \in \Delta^+(g_0)} 1 - q^N e^{w\alpha} \prod_{i=1}^{l_0} \frac{1 - q^{N(r_i+1)}}{1 - q^N} = \prod_{i=1}^{l_0} \frac{1 - q^{N(r_i+1)}}{1 - q^N}
\]

found in [Ma82, Ma72] to get

\[
\chi(q) = \frac{1}{|W(g_0)|} \prod_{i=1}^{l_0} \frac{1 - q^N}{1 - q^{N(r_i+1)}} \cdot [e^0] \sum_{w \in W(g_0)} \prod_{\alpha \in \Delta^+(g_0)} 1 - q^N e^{w\alpha} \prod_{\alpha \in \Delta(g_0)} (1 - e^{\alpha}) \cdot \Phi
\]

\[
= \frac{1}{|W(g_0)|} \prod_{i=1}^{l_0} \frac{1 - q^N}{1 - q^{N(r_i+1)}} \cdot [e^0] \sum_{w \in W(g_0)} w \cdot \prod_{\alpha \in \Delta^+(g_0)} \frac{1 - q^N e^{\alpha}}{1 - e^{\alpha}} \prod_{\alpha \in \Delta(g_0)} (1 - e^{\alpha}) \cdot \Phi.
\]

Since the action of \( W(g_0) \) does not change the constant term, this last sum gives

\[
\chi(q) = \prod_{i=1}^{l_0} (1 - q^{N(r_i+1)})^{-1} \prod_{0 < n \leq N} (1 - q^n)^{l_n} \cdot [e^0] \prod_{\alpha \in S_N} (1 - e^{-\alpha}).
\]
On the other hand, Theorem 2.7 implies
\[
\chi(q) = \prod_{i=1}^{l_0} \left( 1 - q^{N(r_i+1)} \right)^{-1} \prod_{0<n\leq N} \prod_{i=1}^{l_n} \left( 1 - q^{N_{m_i}^{(n)}+n} \right)
\]
Identifying these two equations gives the identity of Lemma 2.13. □

Note that when \( p = b \) is an Iwahori the equivalence follows without using the Weyl integration argument or identity (6). The \( z \)-weighted Euler characteristic identity for \( H^*(b/\mathbf{z}^N\mathbf{n}, \mathfrak{h}_0) \), is similarly equivalent to the identity of Lemma 2.13.

3. THE LAPALACIAN CALCULATION AND THE SET OF HARMONIC FORMS

As in the overview, let \( p \) be a parahoric (in this case not necessarily standard) in the twisted loop algebra \( \hat{\mathfrak{g}} \), and \( u_0 \) the corresponding nilpotent. Choose a homogeneous basis \( \{ z_k \} \) for \( u \), and let \( \{ z^k \} \) be the dual basis of \( \mathbf{u}^* \). Let \( (V, \pi) \) be a \( \hat{p} \)-module. The Koszul complex for \( H^{\text{cts}}(\hat{p}; \mathfrak{g}_0; V) \) is the chain complex \((C^q, \partial)\) defined by
\[
C^q(\hat{p}; \mathfrak{g}_0; V) = \left( \bigwedge^q \mathbf{u}^* \otimes V \right)^{\mathfrak{g}_0}
\]
and
\[
\partial = \sum_{k \geq 1} \epsilon(z^k) \left( \frac{1}{2} \text{ad}_{\mathbf{u}}(z_k) + \pi(z_k) \right).
\]
If \( C^q \) is given a positive-definite Hermitian form then the cohomology \( H^* \) can be identified with the set \( \ker \square \) of harmonic forms, where \( \square = \partial \overline{\partial}^* + \overline{\partial}^* \partial \). The goal of this section is to calculate \( \ker \square \) for \( V = S^*\hat{p}^* \) and \( V = S^*\mathbf{u}^* \), in a metric that we will introduce.

**Lemma 3.1.** Let \( V \) be an \( \hat{p} \)-module, and \( J \) a derivation of \( p \) which annihilates \( \mathfrak{g}_0 \). If \( \phi \in \bigwedge^k \mathbf{u}^* \otimes V \) is \( \hat{p} \)-invariant then
\[
x_1 \wedge \cdots \wedge x_k \mapsto \phi(Jx_1, \ldots, Jx_k)
\]
is a cocycle in \( C^k(\hat{p}; \mathfrak{g}_0; V) \).

**Proof.** Let \( f \) be the cochain constructed as in equation (7). Then
\[
(\overline{\partial}f)(x_0, \ldots, x_k) = \sum_{i<j} (-1)^{i+j} f([x_i, x_j], \ldots, \tilde{x}_j, \ldots) + \sum_i (-1)^i x_i f(\ldots, \tilde{x}_i, \ldots)
\]
\[
= \sum_{i<j} (-1)^{i+j} \phi(J[x_i, x_j], Jx_0, \ldots) + \sum_{i} (-1)^i x_i \phi(Jx_0, \ldots)
\]
\[
= \sum_{i} (-1)^i (\text{ad}'(x_i) \phi)(Jx_0, \ldots, \tilde{x}_i, \ldots) + \sum_{i} (-1)^i x_i \phi(Jx_0, \ldots, \tilde{x}_i, \ldots),
\]
where the last equality follows from the fact that \( J \) is a derivation. If \( \phi \) is \( \hat{p} \)-invariant then the last line is zero, so \( f \) is a cocycle. That \( f \) is \( \mathfrak{g}_0 \)-invariant is clear from the \( \hat{p} \)-invariance and the fact that \( J \) annihilates \( \mathfrak{g}_0 \). □

**Definition 3.2.** A linear function \( \iota : \hat{p} \to \mathbf{u} \) defines a contraction operator
\[
\iota(f) : \bigwedge^p \mathbf{u}^* \otimes S^p \hat{p}^* \to \bigwedge^{p-1} \mathbf{u}^* \otimes S^{p+1} \hat{p}^*.
\]
A cochain \( \omega \in \bigwedge^* \mathbf{u}^* \otimes S^* \hat{p}^* \) is \( \mathbf{u} \)-basic if \( \iota(f)\omega = 0 \) for all \( f \) of the form \( y \mapsto \{ x, y \}, x \in \mathbf{u} \).
The main theorem of this section allows us to identify $H_{cts}^*(\hat{p}, g_0; S^*\hat{u}^*)$ with the ring of $u$-basic $\hat{p}$-invariant cochains.

**Theorem 3.3.** Let $p$ be a parahoric in a twisted loop algebra $g$ and let $u$ be the nilpotent subalgebra. Then there is a positive-definite Hermitian form on $C^* = C^*(\hat{p}, g_0; S^*\hat{u}^*)$ and a derivation $J$ of $p$ such that the harmonic forms in $C^*$ are closed under multiplication, and furthermore the map in Lemma 3.1 gives an isomorphism between the ring of $u$-basic $\hat{p}$-invariant elements of $\bigwedge^* \hat{u}^* \otimes S^*\hat{p}^*$ and the ring of harmonic forms.

Before proceeding to the proof, we note that Theorem 3.3 can be rephrased in a geometric manner. Let $\mathcal{P}$ and $\mathcal{N}$ be pro-Lie groups with Lie algebras $p$ and $u$ respectively. The space $p/\bigoplus_{n>0} g_n$ has the structure of an affine variety, so the pro-algebra $\hat{p}$ can be regarded as a scheme with coordinate ring $S^*\hat{p}^*$.

**Definition 3.4.** The tangent space $T\hat{p}$ is isomorphic to $\hat{p} \times \hat{p}$. Let $T_{>0}\hat{p}$ denote the subbundle of $T\hat{p}$ isomorphic to $\hat{p} \times u$, and $T_{>0}^*\hat{p}$ the continuous dual bundle of $T_{>0}$. Let $\Omega^*_0\hat{p}$ denote the ring of global sections of $T_{>0}^*\hat{p}$.

The bundle $T_{>0}\hat{p}$ contains all tangents to $\mathcal{N}$-orbits. We will say that an element of $\Omega^*_0\hat{p}$ is $\mathcal{N}$-basic if it vanishes on all tangents to $\mathcal{N}$-orbits.

With this terminology, we can identify the ring of $u$-basic $\hat{p}$-invariant cochains with the ring of $\mathcal{P}$-invariant $\mathcal{N}$-basic elements of $\Omega^*_0\hat{p}$.

Although Theorem 3.3 covers the main case of interest, a more natural result occurs if $S^*\hat{p}^*$ is replaced with $S^*\hat{u}^*$. An element $\omega$ of $\bigwedge^* \hat{u}^* \otimes S^*\hat{u}^*$ is $\hat{p}$-basic if $i(f)\omega = 0$ for all linear endomorphisms $f$ of $\hat{u}$ of the form $y \mapsto [x, y], x \in \hat{p}$.

**Theorem 3.5.** Let $p$ be a parahoric in a twisted loop algebra $g$, and let $u$ be the nilpotent subalgebra. Then there is a positive-definite Hermitian form on $C^*(\hat{p}, g_0; S^*\hat{u}^*)$ and a derivation $J$ (the same as in Theorem 3.3) such that the harmonic forms are closed under multiplication, and furthermore the map of Lemma 3.1 gives an isomorphism between the ring of $\hat{p}$-basic and invariant elements of $\bigwedge^* \hat{u}^* \otimes S^*\hat{u}^*$ and the ring of harmonic forms.

In geometric language, the ring of $\hat{p}$-basic and invariant cochains is the same as the ring of $\mathcal{P}$-basic and invariant algebraic forms on $u$.

For the proofs of Theorems 3.3 and 3.5, we assume that the underlying Lie algebra $L$ is semisimple, as this simplifies our Kahler metric construction. If $L$ is reductive then $L = [L, L] \oplus z$, where $z$ is the centre, and consequently $g = [g, g] \oplus z[z, z]$. It is easy to deduce the reductive case of Theorems 3.3 and 3.5 from the semisimple case by splitting off the centre (for instance we can extend $J$ to be the identity on $z[z]$). The proof we give actually holds in more generality. Let $g$ be a $\mathbb{Z}$-graded Lie algebra (such that $\dim g_n < +\infty$ for all $n$) with a conjugation (an anti-linear automorphism sending $g_n \mapsto g_{-n}$) and a contragradient positive-definite Hermitian form (satisfying $g_n \perp g_m$ for $m \neq n$). Let $b = \bigoplus_{n>0} g_n$ and $u = \bigoplus_{n>0} g_n$. Notice that $\overline{u} = g/b$ is an $b$-module. If $\overline{u}^b = (\overline{u}^b)$ then Theorems 3.3 and 3.5 hold for $C^*(b, g_0; S^*b^*)$ and $C^*(b, g_0; S^*\hat{u}^*)$.

3.1. Nakano's identity and the semi-infinite chain complex. For the purposes of this section, let $g$ be a $\mathbb{Z}$-graded Lie algebra with a conjugation, as in the last paragraph of the previous section (this time a contragradient metric is not required). Let $b$ and $u$ be the

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2 This is the condition that does not hold if $L$ has centre.
subalgebras $\bigoplus_{n \geq 0} \mathfrak{g}_n$ and $\bigoplus_{n > 0} \mathfrak{g}_n$ respectively. These working assumptions are based on the standard conventions in semi-infinite cohomology, see e.g. [FGZ86]. In this section we state a version of Nakano’s identity for the relative cohomology of $(\mathfrak{b}, \mathfrak{g}_0)$. This version of Nakano’s identity is a straight-forward generalization of a version for loop groups due to Teleman [Te95].

Let $V$ be a locally finite $\mathfrak{b}$-module, such that the action of $\mathfrak{g}_0$ extends to an action of the complex conjugate $\bar{\mathfrak{b}}$. Both the $\mathfrak{b}$ and the $\mathfrak{b}$ action will be denoted by $\pi$. The relative semi-infinite chain complex with coefficients in $\mathfrak{g}_0$ is a bicomplex $C^{*,*}(V)$ defined by

$$C^{-p,q}(V) = \left( \bigwedge^q \hat{\mathfrak{u}}^* \otimes \bigwedge^p \bar{\mathfrak{u}} \otimes V \right)^{\mathfrak{g}_0}.$$ 

There are truncated actions of $u$ on $\bar{\mathfrak{u}} = \mathfrak{g}/\mathfrak{b}$ and of $\bar{\mathfrak{u}}$ on $u = \mathfrak{g}/\mathfrak{b}$. Both will be denoted by $\tilde{\text{ad}}$. The bicomplex $C^{*,*}$ has two differentials $\partial$ and $D$, of degrees $(0, 1)$ and $(1, 0)$ respectively. $\partial$ is the differential for the Lie algebra cohomology of $(\mathfrak{b}, \mathfrak{g}_0)$ with coefficients in $\bigwedge^* \bar{\mathfrak{u}} \otimes V$, and can be explicitly defined as

$$\sum_{k>0} \epsilon(z^k) \left( \frac{1}{2} \text{ad}^t(z_k) + \tilde{\text{ad}}^t(z_k) + \pi(z_k) \right),$$

where $\{z_k\}_{k \geq 1}$ is a homogeneous basis of $u$ as before and $\epsilon$ is exterior multiplication. $D$ is the differential for the Lie algebra homology of $\bar{\mathfrak{u}}$ with coefficients in $\bigwedge^* \bar{\mathfrak{u}} \otimes V$, restricted to the $\mathfrak{g}_0 = \bar{\mathfrak{b}}/\bar{\mathfrak{u}}$ invariants. $D$ can be explicitly defined as

$$\sum_{k<0} \left( \frac{1}{2} \text{ad}(z_k) + \tilde{\text{ad}}(z_k) + \pi(z_k) \right) \iota(z^k),$$

where $z_{-k} = \bar{z}_k$ and $\iota$ is the contraction operator on $\bigwedge^* \bar{\mathfrak{u}}$. Note that $\iota$ is extended to $C^{*,*}(V)$ so as to respect super-commutativity, so $\iota(f) \alpha \otimes \beta \otimes v = (-1)^q \alpha \otimes \iota(f) \beta \otimes v$ for $\alpha \in \bigwedge^q \hat{\mathfrak{u}}^*$, $\beta \in \bigwedge^* \bar{\mathfrak{u}}$, and $v \in V$.

Let $d$ be the total differential $d = \partial + D$. A standard fact from semi-infinite cohomology is that $d^2 = \epsilon(\gamma)$, where $\gamma$ is the the semi-infinite cocycle defined by $\gamma|_{\mathfrak{g}_m \times \mathfrak{g}_n} = 0$ if $m + n \neq 0$ or $m = n = 0$, and otherwise by

$$\gamma(x, y) = \sum_{0 \leq n < k} \text{tr}_{\mathfrak{g}_n} (\text{ad}(x) \text{ad}(y))$$

for $x \in \mathfrak{g}_k$, $y \in \mathfrak{g}_{-k}$, and $k > 0$. Since $\gamma$ has type $(1, 1)$, the operator $\epsilon(\gamma)$ on $C^{*,*}$ should be interpreted in the semi-infinite sense, wherein $\epsilon(z^k \wedge z^{-l}) = \epsilon(z^k) \iota(z^{-l})$, for $k, l > 0$.

**Definition 3.6.** A Kahler metric for the pair $(\mathfrak{b}, \mathfrak{g}_0)$ is a Hermitian form $(,) \text{ on } \mathfrak{g}$ such that

- $(,) \text{ is positive-definite on } u \text{ and zero on } \mathfrak{g}_0,$
- $([x, y], z) = -(y, [x, z])$ for all $x \in \mathfrak{g}_0,$ and
- the fundamental form $\omega \in u^* \otimes \bar{u}^* \subset \bigwedge^2 \mathfrak{g}^*$ defined by $\omega(a, b) = -i(a, \bar{b})$ for $a \in u,$ $b \in \bar{u}$ is a cocycle.

Note that we can define a Kahler metric by giving the the restricted Hermitian form on $u$ and then extending to $\mathfrak{g}$ by zero on $\mathfrak{g}_0$ and by $(\bar{a}, \bar{b}) = (a, \bar{b})$ for $a, b \in u.$
Suppose that there is a Kahler metric for the \((b, g_0)\). Let \(L\) denote multiplication by the fundamental form \(\omega\), defined explicitly as

\[
L = -i \sum_{k \geq 1} \epsilon(z^k)t(z^{-k}),
\]

where the basis \(\{z_k\}\) is now required to be orthonormal in the Kahler metric. Let \(\Lambda = L^*\) be the adjoint of \(L\) on the complex \(C^{*,*}(\mathbb{C})\) with trivial coefficients, extended by \(\otimes \mathbb{C}\) on \(C^{*,*}(V)\). Letting \(H = [\Lambda, L]\), it is not hard to check that \(\{H, \Lambda, L\}\) is an \(\mathfrak{sl}_2\)-triple, and that \(H\) acts on \(C^{-p,q}\) by \(p - q\) (in other words, if the degree of \(C^{-p,q}\) is defined to be \(q - p\) then \(H\) acts by \(-\) deg). This \(\mathfrak{sl}_2\)-action is used in Hodge theory to prove Nakano’s identity. Teleman adapted this proof to give an algebraic version of Nakano’s identity for the loop algebra \([Te95]\). More generally, the same proof gives:

**Proposition 3.7 (Nakano’s identity).** Suppose there is a Kahler metric for \((b, g_0)\), and \(V\) has a contragradient positive-definite Hermitian form. Then in the induced metric on \(C^{*,*}(V)\) we have

\[
\square = \square + i[\epsilon(\gamma + \Theta), \Lambda],
\]

where \(\square\) is the \(\bar{\partial}\)-Laplacian, \(\square\) is the \(D\)-Laplacian, \(\gamma\) is the semi-infinite cocycle, and \(\Theta\) is the curvature form

\[
\Theta = \sum_{i,j \geq 1} z^{-i} \wedge z^j ([\pi(z^{-i}), \pi(z_j)] - \pi([z^{-i}, z_j])).
\]

On restricting to \(p = 0\), the complex \((C^{0,q}(V), \bar{\partial})\) becomes the Koszul complex for the Lie algebra cohomology of the pair \((b, g_0)\) with coefficients in \(V\). The curvature term \(i[\epsilon(\Theta), \Lambda]\) is straight-forwardly shown to be

\[
-\sum_{i,j \geq 1} \epsilon(z^i)t(z_j) ([\pi(z_i), \pi(z_{-j})] - \pi([z_i, z_{-j}]))
\]

on \(C^{0,q}(V)\), where \(\{z_i\}\) is a homogeneous basis orthonormal in the Kahler metric.

### 3.2. Kahler metrics for parahorics and the derivation \(J\).

Now we return to the case of a parahoric \(p\) in a twisted loop algebra \(g\). Recall that \(p = \bigoplus_{n \geq 0} g_n\) for some grading of type \(d\), and that for the purposes of the proof we are assuming that \(L\) is semisimple. Consequently there is a Kac-Moody algebra \(\tilde{g}\) associated to \(g\), and this Kac-Moody algebra has a standard non-degenerate invariant symmetric bilinear form \(\langle , \rangle\). The contragradient Hermitian form \(\{ , \}\) on \(g\) defines a symmetric invariant bilinear form \(\{ , \}^\gamma\), and this symmetric form extends to a scalar multiple of the standard invariant form on \(\tilde{g}\). The twisted loop algebra \(\tilde{g}\) is also graded by the root lattice of the Kac-Moody algebra associated to \(g\). Let \(\rho\) be a weight of the Kac-Moody defined on simple coroots by \(\rho(\alpha_i^\vee) = 0\) if \(d_i = 0\) and \(\rho(\alpha_i^\vee) = 1\) if \(d_i > 0\) (note that the \(\alpha_i^\vee\)'s are coroots of the associated Kac-Moody, not of the twisted loop algebra \(g\)). Let \(J\) be the derivation of \(p\) acting on root spaces \(g_{\alpha}\) as multiplication by \(2\langle \rho, \alpha \rangle\).

**Proposition 3.8.** Let \(\{ , \}\) be the contragradient positive definite Hermitian form on \(g\), normalized to match the standard invariant form on the associated Kac-Moody. Then \(J\) is positive-definite and \(\langle , \rangle = \{ J, \cdot \} = \{ , J \}\) is a Kahler metric for \((p, g_0)\) with fundamental form \(i\gamma\), where \(\gamma\) is the semi-infinite cocycle.
Proof. The only thing to prove is that $(\cdot,\cdot)$ has fundamental form $i\gamma$. We go about the proof somewhat backwards: let $\tilde{ad}_p$ denote the truncated action of $p \oplus \bar{p}$ on $p = g/\bar{u}$, and define

$$J' = \sum_{k \geq 1} \tilde{ad}_p(x_k)\tilde{ad}_p(x_k)^*,$$

where $\{x_k\}_{k \geq 1}$ is a homogeneous basis for $u$, orthonormal in the contragradient metric. Define $(\cdot,\cdot)' = \{J',\cdot\}$. Then $J$ is positive-semidefinite by definition, so $(\cdot,\cdot)'$ is a positive-semidefinite Hermitian form. Suppose $a \in g_n$, $b \in g_{-n'}$, $n,n' \geq 0$, and assume without loss of generality that $x_1, \ldots, x_m$ is a basis of $g_1 \oplus \ldots \oplus g_n$. Since $\tilde{ad}_p(x_k)^* = -\tilde{ad}_p(\bar{x}_k)$ we have

$$(a,b)' = \{J'a,\tilde{b}\} = \sum_{k=1}^m \{[\bar{x}_k,a], [\bar{x}_k,b]\} = -\sum_{k=1}^m \{[b,[a,\bar{x}_k]], \bar{x}_k\} = -\sum_{l=1}^n \text{tr}_{g_{-l}}(\text{ad}(b) \text{ad}(a)).$$

Now $\text{tr}_{g_{-l}}(\text{ad}(b) \text{ad}(a)) = \text{tr}_{g_{n-l}}(\text{ad}(a) \text{ad}(b))$, so $-i(a,b)' = i\gamma(a,b)$. Since $\gamma$ is a cocycle and $\{\cdot,\cdot\}$ is contragradient, it follows that $J'$ is a derivation. (It is possible to show that $J'$ is a derivation directly but it takes a little work).

Now $p$ is generated by $g_0$ and the root vectors $e_i$ with $d_i > 0$. $J'$ annihilates $g_0$, and if $d_i > 0$ then

$$J'e_i = \frac{[e_i, [f_i, e_i]]}{\{e_i, e_i\}} = 2(p, \alpha_i)e_i,$$

where $f_i = -\bar{c}_i$. It follows that $J' = J$, finishing the proof. \qed

More generally, if $g$ is a $\mathbb{Z}$-graded Lie algebra with conjugation and a contragradient positive-definite Hermitian form, then we can define a Kahler metric simply by using the operator $J'$ form the proof of Proposition 3.8. The hypothesis $\bar{u}^0 = 0$ is needed to ensure that the metric is positive-definite.

3.3. Calculation of the curvature term. If $S$ is a linear operator $\hat{u}^* \rightarrow \hat{p}^*$, define an operator $d_R(S)$ on $\bigwedge^* \hat{u}^* \otimes S^*\hat{p}^*$ by

$$d_R(S)(\alpha_1 \wedge \ldots \wedge \alpha_k \otimes b) = \sum_i (-1)^{i-1} \alpha_1 \wedge \ldots \alpha_i \wedge \alpha_{k-i} \otimes \sum_{j=0}^{k-i} S(\alpha_j) \circ b_i.$$

If $T$ is an operator $\hat{p}^* \rightarrow \hat{u}^*$, define a similar operator $d_L(T)$ by

$$d_L(T)(\alpha \otimes b_1 \circ \ldots \circ b_l) = \sum_i T(b_i) \wedge \alpha \otimes b_1 \circ \ldots \circ b_i \cdots \circ b_l.$$

Recall that truncated actions are denoted by $\tilde{ad}$, with subscripts denoting the appropriate truncated space. By abuse of notation, let $J^{-1}$ denote the inverse of the restriction of the derivation of Proposition 3.8 to $u$. We will also use $J^{-1}$ to denote the dual operator on $u^*$.

**Proposition 3.9.** Let $p$ be a parahoric subalgebra of a twisted loop algebra $g$. Let $V = S^*\hat{p}^*$ with the contragradient metric. The Laplacian on $C^*(V)$ with respect to the dual Kahler metric from Proposition 3.8 has curvature term

$$i[\epsilon(\gamma + \Theta),\Lambda] = \sum_{i>0} d_R\left(\tilde{ad}_p^t(x_i)\tilde{ad}_p(x_i)^*\right) d_R\left(\tilde{ad}_p(x_i)^{\dagger}J^{-1}\right).$$

where \( \{x_i\} \) is a basis for \( u \) orthonormal in the contragradient metric, and \( \hat{u}^* \) is considered as the subset of \( \hat{p}^* \) that is zero on \( g_0 \) (so that \( \text{ad}_p^* \) sends \( u^* \) to \( \hat{p}^* \)).

**Proof.** Let \( R = i[\epsilon(\Theta), \Lambda] \). The action \( \text{ad}_p^* \) acts as a derivation on the symmetric algebra \( S^*\hat{p}^* \), so by Equation (8), \( R \) is a second-order operator. This means that if \( \alpha_0, \ldots, \alpha_k \in \hat{u}^* \), \( b_0, \ldots, b_l \in \hat{p}^* \) then

\[
R(\alpha_0 \wedge \cdots \wedge \alpha_k \otimes b_0 \circ \cdots \circ b_l) = \sum_{i,j} (-1)^i R(\alpha_i \otimes b_j)\alpha_0 \cdots \hat{\alpha}_i \cdots \hat{b}_j \cdots b_l.
\]

In particular, \( R \) is determined by its action on \( \hat{u}^* \otimes \hat{p}^* \).

The truncated action on \( V \) is isomorphic to the truncated action on \( V' = S^*\hat{p} \) via the contragradient metric. Let \( R' = i[\epsilon(\Theta'), \Lambda] \). If \( f \in \hat{u}^* \) and \( w \in \hat{p} \) then we claim that

\[
R'(f \otimes w) = \sum_{i > 0} \text{ad}_u^*(w)z_i \otimes \text{ad}_\hat{p}(z_i)\phi^{-1}(f),
\]

where \( \{z_i\} \) is any homogeneous basis of \( u \), \( \phi \) is the isomorphism \( \mathfrak{u} \rightarrow \hat{u}^* \) induced by the Kahler metric, and \( \mathfrak{u} \) is considered as a subset of \( \mathfrak{p} \), so that \( \text{ad}_\hat{p} \) maps from \( \mathfrak{u} \) to \( \hat{p} \). To prove this, let \( \{z_i\} \) be orthonormal with respect to the Kahler metric, and think about \( f = z^k \), \( w \) arbitrary. Observe that

\[
\text{ad}_\hat{p}(z)w = \sum_{s \geq 1} y^{-s}([z, w])y_{-s},
\]

where \( \{y_s\}_{s \geq 1} \) is a homogeneous basis of \( \mathfrak{p} \) and \( y_{-s} = \overline{y_s} \). So if \( z_{-j} \in \mathfrak{g}_{-m} \), then

\[
\text{ad}_\hat{p}(z_i)\text{ad}_\hat{p}(z_{-j})w = \sum_{s \geq 1} y^{-s}([z_i, [z_{-j}, w]])y_{-s},
\]

\[
\text{ad}_\hat{p}(z_{-j})\text{ad}_\hat{p}(z_i)w = \sum_{s \geq 1} y^{-s}([z_{-j}, [z_i, w]])y_{-s},
\]

\[
\text{ad}_\hat{p}([z_i, z_{-j}])w = \sum_{s \geq 1} y^{-s}([[z_i, z_{-j}], w]])y_{-s}.
\]

Consequently

\[
\left( \text{ad}_\hat{p}(z_i), \text{ad}_\hat{p}(z_{-j}) \right) - \text{ad}_\hat{p}([z_i, z_{-j}])w = \sum_{-m \leq n \leq 0} \sum_{y_{-s} \in \mathfrak{g}_n} y^{-s}([z_{-j}, [z_i, w]])y_{-s}.
\]

After removing the reference to \( m \) here, we get

\[
\left( \text{ad}_\hat{p}(z_i), \text{ad}_\hat{p}(z_{-j}) \right) - \text{ad}_\hat{p}([z_i, z_{-j}])w = \sum_{s \geq 1} \sum_{l \geq 1} y^{-s}([z_{-j}, z_l])z^l([z_i, w])y_{-s}.
\]

Now from Equation (8),

\[
R'(z^k \otimes w) = -\sum_{i > 0} z^i \otimes \left( \text{ad}_\hat{p}(z_i), \text{ad}_\hat{p}(z_{-k}) \right) - \text{ad}_\hat{p}([z_i, z_{-k}])w
\]

\[
= -\sum_{i, l, s > 0} z^i y^{-s}([z_{-k}, z_l])z^l([z_i, w])y_{-s}.
\]
Moving the \( w \) action from \( z_i \) to \( z^i \), the last expression becomes
\[
- \sum_{i,s>0} \tilde{\text{ad}}_u^t(w)z^i \otimes y^{-s}([z_{-k}, z_i])y_s = \sum_{i>0} \tilde{\text{ad}}_u^t(w)z^i \otimes \tilde{\text{ad}}_p(z_i)z_{-k}.
\]

The proof of the claim is finished by noting that this last expression is independent of the choice of basis \( \{z_i\} \) for \( u \) and that that \( z_{-k} = \phi^{-1}(z^k) \).

Now we can translate from \( V' \) to \( V \) using the isomorphism \( \psi : \hat{p} \to \hat{p}^* \) induced by the contragradient form. The operator \( J \) on \( u \) has a basis \( \{x_i\} \) of eigenvectors orthonormal in the contragradient metric. If \( Jx_i = \lambda_i x_i \) then \( \phi(x_i) = \lambda_i x^i \), and thus \( \psi \circ \phi^{-1}(x^i) = \lambda_i^{-1} x^i \). It follows that \( \psi \circ \phi^{-1} = J^{-1} \) on \( \hat{u}^* \). Next, \( \tilde{\text{ad}}_u^t(w)\psi(x) = -\tilde{\text{ad}}_p(x)\psi(w) \). Since \( \psi(x_i) = x^i \) we can conclude that
\[
R(f \otimes g) = - \sum_{i>0} \tilde{\text{ad}}_p^t(\bar{x}_i)g \otimes \tilde{\text{ad}}_p^t(x_i)J^{-1} f,
\]
where \( \tilde{\text{ad}}_p(\bar{x}_i) \) is regarded as a map from \( \hat{p}^* \) to \( \hat{u}^* \).

If \( S, T \in \text{End}(\hat{p}^*) \), let \( \text{Switch}(S, T) \) be the second order operator on \( \Lambda^* \hat{p}^* \otimes S^* \hat{p}^* \) sending \( \alpha \otimes \beta \mapsto T\beta \otimes S\alpha \). Note that \( \tilde{\text{ad}}_p^t(x_i)^* = -\tilde{\text{ad}}_p^t(\bar{x}_i) \). We have shown that \( R \) is the restriction of the operator
\[
\sum_i \text{Switch} (\tilde{\text{ad}}_p^t(x_i)J^{-1}, \tilde{\text{ad}}_p^t(x_i)^*)
\]
to \( \Lambda^* \hat{u}^* \otimes S^* \hat{u}^* \), where \( J^{-1} \) is zero on \( g_0 \). It is easy to see that \( \text{Switch}(S, T) = d_L(T)d_R(S) - (TS)^\wedge \), where \( (TS)^\wedge \) is the operator \( TS \) extended to \( \Lambda^* \hat{p}^* \) as a derivation. Also, \( d_L(T) = d_R(T^*J^{-1}) \), where \( T^* \) is the adjoint of \( T \) in the contragradient metric. Note that the \( J^{-1} \) term comes from the difference between the contragradient metric on the symmetric factor and the Kahler metric on the exterior factor. Finally we have
\[
R = \sum_i d_R \left( \tilde{\text{ad}}_p^t(x_i)J^{-1} \right)^* d_R \left( \tilde{\text{ad}}_p^t(x_i)J^{-1} \right) + \sum_i \left( \tilde{\text{ad}}_p^t(\bar{x}_i)\tilde{\text{ad}}_p^t(x_i)J^{-1} \right)^\wedge.
\]

Now \( \sum_i \tilde{\text{ad}}_p^t(\bar{x}_i)\tilde{\text{ad}}_p^t(x_i) \) is the negative of the dual of the derivation \( J \) on \( u \), while \( J^{-1} \) is the dual of the inverse of \( J \). Thus on \( \Lambda^* \hat{u}^* \), this second summand is simply \(-\deg \). But since we have chosen a Kahler metric with fundamental form \( i\gamma \), we have \( i[\epsilon(\gamma), \Lambda] = [L, \Lambda] = -H = \deg \), finishing the proof of the Proposition.

Similarly, given endomorphisms \( S \) and \( T \) of \( \hat{u}^* \) we can define operators \( d_R(S) \) and \( d_L(T) \) on \( \Lambda^* \hat{u}^* \otimes S^* \hat{u}^* \).

**Proposition 3.10.** Let \( p \) be a parahoric subalgebra of a twisted loop algebra \( g \), and let \( u \) be the nilpotent subalgebra. Let \( V = S^* \hat{u}^* \) with the contragradient metric. The Laplacian on \( C^*(V) \) with respect to the dual Kahler metric from Proposition 3.8 has curvature term
\[
i[\epsilon(\gamma + \Theta), \Lambda] = \sum_{i\geq 0} d_R \left( \tilde{\text{ad}}_u^t(y_i)J^{-1} \right)^* d_R \left( \tilde{\text{ad}}_u^t(y_i)J^{-1} \right),
\]
where \( \{y_i\}_{i\geq 0} \) is a basis for \( p \) orthonormal in the contragradient metric.

The proof of Proposition 3.10 is similar to the proof of Proposition 3.9. A proof of the analogous result for symmetrizable Kac-Moody algebras with the principal grading can be found in [S10].
3.4. Proof of Theorems 3.3 and 3.5. Once again let $J$ denote the operator on $\hat{u}^*$ which is the dual of the derivation $J$ on $u$. We give a proof of Theorem 3.3; the proof of Theorem 3.5 is identical. Let $J_\Delta$ denote the diagonal extension of $J$ to the exterior factor of $\bigwedge^\ast u^* \otimes S^n p^\ast$. The adjoint of $ad_u(x)$ in the Kahler metric is $-\hat{J}ad_u(x)J^{-1}$. Thus we can directly calculate that

$$D^* = -\sum_{i<0} \epsilon(x_i) \left( (J\hat{ad}_u(x_{-i})J^{-1})^\wedge + \hat{ad}_p(x_{-i}) Sym_i \right),$$

where $\{x_i\}$ is a basis of $u$ orthonormal in the contragradient metric and $x_{-i} = \pi_i$. On $C^0,q(V)$ the $D$-Laplacian is $\square = DD^*$, so the set of harmonic cocycles is the joint kernel of the operators $D^*$ above and $d_R \left( \hat{ad}_p(x_i)J^{-1} \right)$, $i \geq 1$. The kernel of $D^*$ on $C^0,q(V)$ is the joint kernel of the operators $(J\hat{ad}_u(x_i)J^{-1})^\wedge + \hat{ad}_p(x_i)^{Sym}$, $i \geq 1$. Now we have

$$d_R \left( \hat{ad}_p(x_i)J^{-1} \right) J_\Delta = J_\Delta d_R \left( \hat{ad}_u(x_i) \right) \quad \text{and} \quad \left( (J\hat{ad}_u(x_i)J^{-1})^\wedge + \hat{ad}_p(x_i)^{Sym} \right) J_\Delta = J_\Delta \left( \hat{ad}_u(x_i)^\wedge + \hat{ad}_p(x_i)^{Sym} \right).$$

Thus we see that $J_\Delta^{-1}$ identifies the set of harmonic cocycles with the joint kernels of the operators $d_R \left( \hat{ad}_p(x_i) \right)$, $i \geq 1$, and $\left( \hat{ad}_u(x_i)^\wedge + \hat{ad}_p(x_i)^{Sym} \right)$, $i \geq 1$. Since the elements of $C^0,q(V)$ are $g_0$-invariant by definition, the kernel of the latter family of operators is the set of $p$-invariant cochains. The kernel of the former family of operators is the set of $u$-basic cochains, finishing the proof.

4. Twisted arc and jet schemes and the twisted arc group

This section covers background material on twisted arc and jet schemes, and proves basic facts about the twisted arc group arising from a diagram automorphism. The material in this section will be used to study adjoint orbits in Section 5.

4.1. Twisted arc and jet schemes. By a variety, we mean a separated, reduced, but not necessarily irreducible, scheme of finite type over $\mathbb{C}$. The arc scheme $J_\infty X$ of a variety $X$ over $\mathbb{C}$ is a separated scheme of infinite type representing the functor $Y \mapsto \text{Hom}(Y \times \text{Spec} \mathbb{C}[z], X)$. Intuitively the arc scheme is the space of maps from the formal arc $\text{Spec} \mathbb{C}[z]$ into $X$. The $m$th jet scheme $J_m X$ ($0 \leq m < +\infty$) is a separated scheme of finite type over $\mathbb{C}$ representing the functor $Y \mapsto \text{Hom}(Y \times \text{Spec} \mathbb{C}[z]/z^m, X)$. If $m \leq n$ then there is a morphism $J_n X \to J_m X$, and $J_\infty X$ is the inverse limit of the jet schemes of $X$. The $\mathbb{C}$-points of $J_m X$ are $m$-jets, i.e. morphisms $\text{Spec} \mathbb{C}[z]/z^m \to X$. For example, $J_0 X = X$ and $J_1 X$ is the tangent scheme of $X$. If $X$ is the affine subset of $\mathbb{C}^n$ cut out by the equations $f_1 = \ldots = f_k = 0$ then $J_m X$ is the subscheme of $(\mathbb{C}[z]/z^m)^n$ cut out by the equations $f_i(x_1, \ldots, x_n) = 0$, $i = 1, \ldots, k$, where $x_i \in \mathbb{C}[z]/z^m$ and $(\mathbb{C}[z]/z^m)^n$ is regarded as the affine space of dimension $mn$. The association $V \mapsto J_m V$ is functorial, so if $G$ is an algebraic group then $J_m G$ is an algebraic group when $m < +\infty$, and a pro-group when $m = +\infty$. The arc scheme of $X$ is sometimes denoted by $X[[z]]$, but we use the notation $J_\infty X$ so that propositions can be stated uniformly for both arc and jet schemes.

The following well-known lemma is useful for working with jet schemes:

**Lemma 4.1 ([Mu01])**. If $X \to Y$ is etale then $J_m X = X \times_Y J_m Y$ for all $0 \leq m \leq +\infty$. 


Open immersions are etale, so if $U \subset X$ is an open subset then the pullback of $U$ via $J_mX \to X$ is equal to $J_mU$. In particular, $J_mX$ is covered by open subsets $J_mU$, where $U \subset X$ is an affine open (if $m = +\infty$ then we use the fact that the inverse limit of affine schemes is affine). If there is an etale map $X \to U \subset J$ instance the wording of Lemma 4.2.

and the truncation morphisms $J$ (Zariski) locally-trivial $A$ while $J$.

Let $U \to X$. Choose a fixed $m$ such that the pullback of $U$ over $U$ is isomorphic to the trivial $G$-bundle $U \times G$.

Now $J_m$ preserves etale maps (by Lemma 4.1 and the fact that etale maps are preserved under base change) and thus $J_mU \to J_mM$ is etale and surjective. The proof is finished by observing that $J_m$ preserves pullbacks (which follows from the definition of the pullback via the functor of points).

Proposition 4.3 on jet schemes has the following corollary:

Corollary 4.4. Suppose $X$ has a free $G$-action such that an etale-locally trivial quotient $X \to X/G$ exists. Then $J_m(X/G)$ is isomorphic to $J_mX/J_mG$, $0 \leq m < +\infty$, and $J_\infty(X/G)$ is isomorphic to $J_\infty(X/J_\infty G)$, where this last quotient is the pro-group quotient, i.e. the inverse limit of the quotients $J_mX/J_mG$, $0 \leq m < +\infty$.

If $X \to X/G$ is etale-locally trivial then it is also surjective, so by Corollary 4.4 and Lemma 4.2 the map $J_\infty X \to J_\infty X/J_\infty G$ is surjective. If $X$ is affine with a free $G$-action and $G$ is reductive then $X/G = X/G$, the GIT quotient, and $X \to X/G$ is etale-locally trivial by Luna’s slice theorem \cite{Lun73} (the theorem applies because all orbits under a free action are closed, see the discussion on page 53 of \cite{Bo91}). All the quotients we study will be of this type.

Now suppose that $X$ has an automorphism $\sigma$ of finite order $k$. This automorphism lifts to an automorphism $\tilde{\sigma}$ of the jet and arc schemes $J_mX$. Choose a fixed $k$th root of unity $q$, and let $m(q)$ denote the automorphisms of $\mathbb{C}[z]/z^n$ and $\mathbb{C}[[z]]$ induced by sending $z \mapsto qz$.

Definition 4.5. Let $\tilde{\sigma}$ denote the automorphism $\sigma \circ m(q)^{-1}$. The twisted jet (resp. arc) scheme $J_\sigma^mX$ is the equalizer of the morphisms $1_{J_mX}$ and $\tilde{\sigma}$ in the category of schemes.

In other words, if $m < +\infty$ then $J_\sigma^mX$ represents the functor $Y \mapsto \{f \in \text{Hom}(Y \times \text{Spec}\mathbb{C}[z]/z^n, X) : f \circ m(q) = \sigma \circ f\}$, while $J_\infty^\sigma X$ represents the functor $Y \mapsto \{f \in \text{Hom}(Y \times \text{Spec}\mathbb{C}[z]/z^n, X) : f \circ m(q) = \sigma \circ f\}$.

\[\text{The infinite-type schemes we work with are nice enough that they could be called “smooth” in their own right. However, we avoid this complication and only use smoothness for schemes of finite type. See for instance the wording of Lemma 4.2.}\]
$J_m^\sigma X$ is a closed subscheme of $J_m X$, and $(J_m X)^\sigma$ is separated for all $m$. Since Spec $\mathbb{C}[[z]]$ is the direct limit of schemes Spec $\mathbb{C}[z]/z^n$, it follows from the functor of points characterisation that $J_m^\sigma X$ is the inverse limit of schemes $J_m^\sigma X$, $0 \leq m < +\infty$. Since $J_m^\sigma X$ is a closed subscheme of $J_m X$, it is covered by the inverse images of the open subschemes $J_m U \subseteq J_m X$, for $U \subseteq X$ open affine. The inverse image of $J_m U$ in $J_m^\sigma X$ is the same as the inverse image of $\tilde{\sigma}(J_m U) = J_m \sigma(U)$. Thus the inverse image of $J_m U$ in $J_m^\sigma X$ is the same as in the inverse image of $J_m V$, where $V = U \cap \sigma(U) \cap \ldots \cap \sigma^{k-1}(U)$. By definition $\sigma(V) = V$, and $V$ is affine because $X$ is separated. Finally, the pullback of $J_m V$ to $J_m^\sigma U$ is $J_m^\sigma V$, and $J_m^\sigma V$ is affine. We conclude that $J_m^\sigma X$ is covered by open affines $J_m^\sigma U$ where $U \subseteq X$ runs through open affines such that $\sigma(U) = U$.

The following lemma is an immediate consequence of the definition of tangent and jet (resp. arc) schemes via functor of points.

Lemma 4.6. Let $\sigma_*$ be the automorphism induced by $\sigma$ on $TX$. Then the tangent scheme to $J_m^\sigma X$ is naturally isomorphic to the twisted jet (resp. arc) scheme $J_m^{\sigma_*}(TX)$ of the tangent scheme to $X$.

Using known results for finite-dimensional varieties, we can show that the twisted jet scheme of a smooth variety is also smooth.

Lemma 4.7. Let $0 \leq m < +\infty$. If $X$ is a smooth variety with a finite-order automorphism $\sigma$ then $J_m^\sigma X$ is a smooth variety. In addition, if $X$ and $Y$ are both smooth varieties with finite-order automorphisms $\sigma_X$ and $\sigma_Y$ and $X \to Y$ is a $\sigma$-equivariant smooth map then $J_m^\sigma X \to J_m^\sigma Y$ is smooth.

Proof. We can assume that $X$ is affine. Since $X$ is smooth, $J_m X$ is also a smooth variety. The twisted jet scheme $J_m^\sigma X$ is the fixed-point scheme of the finite group $\langle \tilde{\sigma} \rangle$. It is a well-known consequence of Luna’s slice theorem that the fixed-point variety of a reductive algebraic group acting on a smooth variety is also smooth. This also holds for the fixed-point scheme by Proposition 7.4 of [Fo73], so $J_m^\sigma X$ is smooth.\footnote{If $X$ is not smooth then the fixed-point scheme of a reductive group action can be non-reduced.}

Since $J_m^\sigma X$ is a smooth variety the tangent scheme is a vector bundle. By Lemma 4.6 $T_x J_m^\sigma X = (T_x J_m X)^{\tilde{\sigma}_*}$ and similarly $T_y J_m^\sigma Y = (T_y J_m Y)^{\tilde{\sigma}_*}$. If $X \to Y$ is smooth then $J_m X \to J_m Y$ is smooth by Lemma 4.2, hence $T J_m X \to T J_m Y$ is surjective on fibres, and it follows that $(T_x J_m X)^{\tilde{\sigma}_*} \to (T_y J_m Y)^{\tilde{\sigma}_*}$ is surjective. Since both $J_m^\sigma X$ and $J_m^\sigma Y$ are smooth, $J_m^\sigma X \to J_m^\sigma Y$ is a smooth map.

Note that $J_m^\sigma X$ is not necessarily irreducible, as $X^\sigma$ can be disconnected.

We also have the following analogue of Lemma 4.1.

Lemma 4.8. Let $0 \leq m \leq +\infty$. Suppose that $X$ and $Y$ have finite-order automorphisms $\sigma_X$ and $\sigma_Y$. If $X \to Y$ is an etale $\sigma$-equivariant map then $J_m^\sigma X = X^\sigma \times_{Y^\sigma} J_m^\sigma Y$.

Proof. By Lemma 4.1 $J_m X \cong X \times_Y J_m Y$. The automorphism $\tilde{\sigma}_X$ on $J_m X$ translates to the unique automorphism on the latter space which lies above $\sigma_X$ on $X$, $\sigma_Y$ on $Y$, and $\tilde{\sigma}_Y$ on $J_m Y$. The result follows from the functor of points characterisations of the twisted jet and arc schemes and the fibre product.

Finally, the jet structure distinguishes a subbundle of the tangent bundle of a jet or arc space.
Definition 4.9. If $X$ is a variety with finite-order automorphism $\sigma$, we let $T_{\text{const}}J_m^\sigma X$ denote the pullback $X^\sigma \times_{TX^\sigma} TJ_m^\sigma$, where $TJ_m^\sigma X \to TX^\sigma$ is the differential of the projection $J_m^\sigma X \to X^\sigma$ and $X^\sigma \to TX^\sigma$ is the zero section. Intuitively $T_{\text{const}}J_m^\sigma$ is the space of infinitesimal families of jets (resp. arcs) which are constant at $z = 0$.

4.2. Connectedness of the twisted arc group. In this section $G$ will be a connected algebraic group with Lie algebra $L$, such that the diagram automorphism $\sigma$ lifts to $G$ (for example, this occurs if $G$ is simply-connected). $H$ will be the torus corresponding to the chosen Cartan $\mathfrak{h}$.

We recall some basic facts about diagram automorphisms and the structure of $L$, using terminology and basic results from Chapter 9, Section 5 of [Ca05]. Let $\mathfrak{h}_i$ denote the $i$th eigenspace of $\sigma$ acting on $\mathfrak{h}$. By definition, there is a choice of simple roots $\alpha_1, \ldots, \alpha_l$ such that $\sigma$ permutes the corresponding coroots $h_{\alpha_i}$ and Chevalley generators $e_{\alpha_i}$. If $J$ is an orbit the $\sigma$-action on simple roots, let $\alpha_J = \frac{1}{|J|} \sum_{\alpha \in J} \alpha$. Then the set \{ $\alpha_J|_{\mathfrak{h}_0} : J$ is an orbit of $\sigma$ \} is a set of simple roots for $L_0$. Restriction to $\mathfrak{h}_0$ gives an isomorphism between the subgroup $W^\sigma$ (where $W$ is the Weyl group of $L$) and the Weyl group $W(L_0)$ of $L_0$. The simple generator $s_J$ of $W(L_0)$ given by reflection through $\alpha_J$ on $\mathfrak{h}_0$ corresponds to the element of $W^\sigma \subset W(L)$ which is the maximal element in the subgroup of $W(L)$ generated by reflection through the simple roots in $J$. In addition, we will need:

Lemma 4.10. If $N(H)$ is the normalizer of $H$ in $G$, then $N(H)^\sigma = N_{G^\sigma}(H^\sigma)$, the normalizer of $H^\sigma$ in $G^\sigma$. Consequently $W^\sigma = N(H)^\sigma / H^\sigma \subset W(L)$. Furthermore, the inclusion $W(L_0) \cong W^\sigma \hookrightarrow W(L)$ is length-preserving, in the sense that if $w \in W^\sigma$, then it is possible to get a reduced expression for $w$ by first taking a reduced expression $w = s_{J_1} \cdots s_{J_r}$ for $w$ in $W(L_0)$, and then replacing each $s_{J_i}$ with a reduced expression in $W(L)$.

Proof. For the first part, let $\rho \in \mathfrak{h}$ be the element such that $\alpha(\rho) = 1$ for all simple roots $\alpha$ of $L$. Then $\rho$ is regular in $\mathfrak{h}$ and belongs to $\mathfrak{h}_0$. Any element of $N_{G^\sigma}(H^\sigma)$ sends $\rho$ to another regular element of $\mathfrak{h}$, and hence belongs to $N(H)$.

For the second part, we refer to the proof of Proposition 9.17 of [Ca05].

We can use Lemma 4.10 to prove:

Lemma 4.11. Choose a Borel subgroup $B$ of $G$ containing $H$ and compatible with $\sigma$ and let $X = \overline{BB}$ be the big cell of the corresponding Bruhat decomposition. If $x \in G$ belongs to a Bruhat cell $BwB$ with $w \in W^\sigma$ then there is $g \in N(H)^\sigma$ such that $gx \in X$.

Proof. If we take for $g$ a representative of $w^{-1}$ in $N(H)^\sigma$, then $gBwB \subset \overline{BB}$.

Proposition 4.12. $G^\sigma$ is connected.

Proof. The connected component $(G^\sigma)^\circ$ of $G^\sigma$ is a connected reductive group with Lie algebra $L_0$. Since $\sigma$ permutes coroots, it is easy to see that $H^\sigma$ is a connected torus, and in fact is a Cartan in $(G^\sigma)^\circ$. As in Lemma 4.11 let $B$ be a Borel subgroup of $G$ containing $H$ and compatible with $\sigma$, and let $X$ be the corresponding big cell. If $g \in G^\sigma$ belongs to a Bruhat cell $BwB$ then $g \in BwB \cap \sigma(BwB)$, so $w \in W^\sigma$. By Lemma 4.10 every element of $N(H)^\sigma$ can be implemented by an element of $(G^\sigma)^\circ$. So by Lemma 4.11 we just need to prove that $G^\sigma \cap X$ is contained in $(G^\sigma)^\circ$.

Now as an algebraic variety, $X \cong U \times H \times U$, where $U$ is the unipotent radical of $B$. The action of $\sigma$ on $X$ translates to the action of $\sigma$ on each factor. Let $\mathfrak{u}$ be the Lie algebra of $U$. The exponential map for nilpotent Lie algebras is bijective, so $U^\sigma$ is the unipotent subgroup
corresponding to the nilpotent Lie algebra $\mathfrak{n}$. In particular $U^\sigma$ is connected, and similarly with $\overline{U}^\sigma$. We conclude that $X^\sigma = G^\sigma \cap X$ is connected. □

Using the fact that the exponential map for nilpotent (resp. pro-nilpotent) Lie algebras is bijective, we immediately get the following corollary.

**Corollary 4.13.** If $0 \leq m < +\infty$ then $J_m^\sigma G$ is a connected algebraic group with Lie algebra $L[z]/z^m$. Similarly $J_\infty^\sigma G$ is a connected pro-algebraic group with Lie algebra $L[[z]]^\sigma$.

As a scheme the Lie algebra of $J_m^\sigma G$ can be identified with $J_m^\sigma L$.

The following proposition will be crucial in the next section, since it proves that $J_m^\sigma(G/H)$ is a $J_m^\sigma G$-homogeneous space.

**Proposition 4.14.** $J_m^\sigma(G/H) \cong J_m^\sigma G/J_m^\sigma H$, where the latter space is either the group quotient if $0 \leq m < +\infty$, or the pro-group quotient if $m = +\infty$.

**Proof.** $G \to G/H$ is an etale-locally trivial principal bundle, so $J_m(G/H) \cong J_mG/J_mH$. There is an inclusion $J_m^\sigma G/J_m^\sigma H \hookrightarrow (J_mG/J_mH)^\sigma$. To prove the proposition, we will show that this inclusion is surjective for all $m < +\infty$. If $m < +\infty$ then biregularity follows from bijectivity because $(J_mG/J_mH)^\sigma$ will be a homogeneous space. Biregularity for $m = +\infty$ follows from the universal property of inverse limits.

Define $\alpha : J_mG \to J_mG$ by $g \mapsto g^{-1} \tilde{\sigma}(g)$. To show that the inclusion is surjective we need to show that every element of $(J_mG/J_mH)^\sigma$ has a representative $x \in J_mG$ such that $\alpha(x) = e$. The map $\alpha$ has a number of nice properties. First, the fibres of $\alpha$ are left $J_m^\sigma G$-cosets. Second, $g \in J_mG$ represents an element of $(J_mG/J_mH)^\sigma$ if and only if $\alpha(g) \in J_mH$. Third, if $\alpha(g) \in J_mH$ and $h \in J_mH$ then $\alpha(gh) = \alpha(g)\alpha(h)$. By these last two properties, we will have $(J_mG/J_mH)^\sigma = J_m^\sigma G/J_m^\sigma H$ if and only if $\alpha(J_mG) \cap J_mH = \alpha(J_mH)$.

Our proof depends on the Bruhat geometry of $G$, so pick a Borel subgroup $B \subset G$ compatible with $\sigma$. Let $X = \overline{BB}$ be the big cell. Suppose $x \in J_mG$ and $\alpha(x) \in J_mH$. Writing $x(0) = b_0wb_1$, we get $\alpha(x(0)) = b_1^{-1}w^{-1} \alpha(b_0) \sigma(w) \sigma(b_1) \in H$. But $\alpha(b_0) \in B$, so $wB \cap B\sigma(w)B \neq \emptyset$, and thus $w$ belongs to $W^\sigma$. Consequently there is $g_0 \in G^\sigma$ such that $g_0x(0) \in X$, implying that $g_0x \in J_mX$. Since $\alpha(x) = \alpha(g_0x)$ for $g_0 \in G^\sigma$, we just need to show that $\alpha(J_mX) \cap J_mH$ is contained in $\alpha(J_mH)$.

The space $X$ is isomorphic to $\overline{U} \times B$ via the multiplication map, where $\overline{U}$ is the unipotent subgroup of $B$. Thus we can write any element of $J_mX$ uniquely as $a(z)b(z)$, where $a(z) \in J_m\overline{U}$ and $b(z) \in J_mB$. Suppose $\alpha(a(z)b(z)) = h(z) \in J_mH$. Since $\alpha(a(z)b(z)) = b(z)^{-1} \alpha(a(z)) \tilde{\sigma}(b(z))$, we see that $\alpha(a(z)) = b(z)h(z)\tilde{\sigma}(b(z))^{-1} \in J_mB$. Since $\alpha(a(z)) \in J_m\overline{U}$, this implies that $\alpha(a(z)) = e$ and consequently $\alpha(b(z)) = h(z)$. To finish the proof, observe that $B \cong U \times H$ via the multiplication map, where $U$ is the unipotent subgroup of $B$. Writing $b(z) = b'(z)h'(z)$ for $b'(z) \in J_mU$ and $h'(z) \in J_mH$, we get $\alpha(b(z)) = h'(z)^{-1} \alpha(b'(z)) \tilde{\sigma}(h'(z))$, and hence $\alpha(b'(z))$ can be written as an element of $J_mH$. This implies that $\alpha(b'(z)) = e$, finishing the proof, since $\alpha(h'(z)) = h(z)$. □

5. Slice theorems for the adjoint action

We continue to use the notation from Section 4. In particular, $G$ is a connected algebraic group with Lie algebra $L$ such that $\sigma$ extends to $G$, and the Lie algebra of $J_m^\sigma G$ is identified with $J_m^\sigma L$. In addition, we fix a standard parabolic subalgebra $\mathfrak{p}_0 \subset L_0$, and let $\mathfrak{p}_m = \{ f \in J_m^\sigma : f(0) \in \mathfrak{p}_0 \}$. Note that $\mathfrak{p}_\infty$ is the completion of a standard parahoric in $L[z^{\pm 1}]^\sigma$, which we also denote by $\mathfrak{p}$. We let $\mathcal{P}_m$ be the connected algebraic (resp. pro-algebraic) subgroup
of $J_m^\sigma G$ corresponding to $p_m$, and $\mathcal{N}_m$ be the nilpotent (resp. pro-nilpotent) radical of $P_m$. The reductive factor $p_0 \cap \overline{p_0}$ of $P_0$ is denoted by $g_0$.

In this section we prove two slice theorems for the adjoint action of $P_m$ on $p_m$. The first is an analogue of the well-known slice theorem for regular semisimple elements in $L$, and is given in Subsection 5.1. The second is an analogue of the Kostant slice theorem, and is given in Subsection 5.2. These theorems will be used in the next section to determine the $P_\infty$-invariant $\mathcal{N}_\infty$-basic elements of $\Omega^*_\infty p_\infty$.

The slice theorems are stated in terms of the GIT quotients $Q := L//G$ (ie. $Q$ is the affine variety with coordinate ring $\mathbb{C}[Q] = (S^* L^*)^G$) and $R := p_0//P_0$. Recall that $\mathbb{C}[Q]$ is a free algebra generated by homogeneous elements in degrees $m_1 + 1, \ldots, m_t + 1$, where $l$ is the rank of $L$ and $m_1, \ldots, m_t$ are the exponents. A similar result holds for $\mathbb{C}[R]$.

**Lemma 5.1.** Let $u_0$ the nilpotent radical of $p_0$, so that $p_0 = g_0 \oplus u_0$. If $f \in \mathbb{C}[R]$ then $f(x, y) = f(x, 0)$ for all $x \in g_0, y \in u_0$. Consequently, if $M$ is the Levi subgroup of $P_0$ then $R \cong g_0//M \cong h_0//W(g_0)$, where $W(g_0)$ is the Weyl group of $g_0$, and $\mathbb{C}[R]$ is a free algebra generated by homogeneous elements in degrees given by the exponents of $g_0$.

**Proof.** The set of regular elements $h_0^\sigma$ is dense in $h_0$. Since $[g_0, x] + h_0 = g_0$ for any element $x \in h_0^\sigma$, the set $Mh_0^\sigma$ is dense in $g_0$. Let $N_0$ be the unipotent subgroup corresponding to $u_0$. If $x$ belongs to $h_0^\sigma$ then $N_0x = x + u_0$. Since $N_0$ is normal in $P_0$, this property extends to any $x \in Mh_0^\sigma$. So if $f$ is invariant then $f(x, y) = f(n(x, 0)) = f(x, 0)$ for $x$ in an open dense subset of $p_0 \cap \overline{p_0}$. \hfill $\Box$

5.1. **The regular semisimple slice.** Let $L^{rs} \subset L$ be the subset of regular semisimple elements. $L^{rs}$ is an affine open subset of $L$ (its complement is the vanishing set of a single $G$-invariant function) and consequently the image $Q^r$ of $L^{rs}$ in $Q = L//G$ is open. The well-known regular semisimple slice theorem states that there is a commutative square

\[(9)\]

\[
\begin{array}{ccc}
G/H \times_W h^r & \longrightarrow & L^{rs} \\
\downarrow & & \downarrow \\
\h^r/W & \longrightarrow & Q^r
\end{array}
\]

where $W$ is the Weyl group of $L$ and $h^r$ is the set of regular elements in $h$. The notation $G/H \times_W h^r$ denotes the quotient of $G/H \times h^r$ under the free action of $W = N(H)/H$ acting by right multiplication on $G/H$ and by the adjoint action on $h^r$. Both horizontal maps are isomorphisms. The top horizontal map is given by multiplication, while the bottom horizontal map is projection to $Q$.

Since $\sigma$ is an automorphism, the sets $L^{rs}$ and $h^r$ are closed under $\sigma$ and we can apply $J_m^\sigma$ to both spaces. The image $R^r$ of $p_0 \cap L^r_0$ in $R$ is open, since it’s complement is the zero set of a single $P_0$-invariant function. As usual, let $P_\infty/J_\infty^\sigma H$ denote the pro-group quotient. Similarly $P_\infty/J_\infty^\sigma H \times_{W(g_0)} J_\infty^\sigma h^r$ will denote the pro-group quotient of $P_\infty/J_\infty^\sigma H \times J_\infty^\sigma h^r$ by $W(g_0)$, and $J_\infty^\sigma h^r/W(g_0)$ denotes the pro-group quotient of $J_m^\sigma h^r$ by $W(g_0)$. We have the following analogue of Equation (9) for twisted jet and arc schemes.
**Theorem 5.2.** Let $0 \leq m \leq +\infty$. Then there is a commutative diagram

\[
\begin{array}{ccc}
P_m/J_m^r H \times_{W(g_0)} J_m^r \mathfrak{h}^r & \longrightarrow & P_m \cap J_m^r L^{rs} \\
\downarrow & & \downarrow \\
(J_m^r \mathfrak{h}^r) / W(g_0) & \longrightarrow & R^r \times_{Q^r} J_m^r Q^r
\end{array}
\]

in which the horizontal maps are isomorphisms, with the top map induced by multiplication and the bottom map induced from the two projections $J_m^r \mathfrak{h}^r / W(g_0) \to J_m^r Q^r$ and $\mathfrak{h}_0^r / W(g_0) \cong R^r$.

To prove Theorem 5.2, we start with the case $m = 0$ (likely well-known, but we give the proof for completeness).

**Lemma 5.3.** There is a commutative diagram

\[
\begin{array}{ccc}
P_0 / H^\sigma \times_{W(g_0)} \mathfrak{h}_0^r & \longrightarrow & P_0 \cap L_0^{rs} \\
\downarrow & & \downarrow \\
\mathfrak{h}_0^r / W(g_0) & \longrightarrow & R^r
\end{array}
\]

in which both horizontal maps are isomorphisms. The top horizontal map is induced by multiplication, while the bottom horizontal map is induced by the projection $\mathfrak{h}_0 \to R$.

**Proof.** That the bottom map is an isomorphism comes from Lemma 5.1.

The Weyl groups of $\mathfrak{g}_0$ and $L_0$ can be expressed in terms of $M$ and $G^\sigma$ as $W(\mathfrak{g}_0) = N_M(H^\sigma \cap \mathcal{M}) / (H^\sigma \cap \mathcal{M})$ and $W(L_0) = N_{G^\sigma}(H^\sigma) / H^\sigma$. Using the Bruhat decomposition for $G^\sigma$ and $\mathcal{M}$ simultaneously, as well as the Levi decomposition for $P_0$, it is possible to show that $N_{G^\sigma}(H^\sigma) \cap P_0 \subset N_M(H^\sigma \cap \mathcal{M})$. The resulting inclusion $N_{G^\sigma}(H^\sigma) \cap P_0 / H^\sigma \subset N_M(H^\sigma \cap \mathcal{M}) / H^\sigma \cap \mathcal{M}$ is an isomorphism.

Now the commutative diagram in Equation (9) can be extended by adding the commutative square

\[
\begin{array}{ccc}
P_0 / H^\sigma \times_{W(g_0)} \mathfrak{h}_0^r & \longrightarrow & P_0 \cap L_0^{rs} \\
\downarrow & & \downarrow \\
G^\sigma / H^\sigma \times_{W(L_0)} \mathfrak{h}_0^r & \longrightarrow & L_0^{rs}
\end{array}
\]

in which the vertical maps are the natural inclusions. To show that the left vertical map is injective take two elements $([p], x)$ and $([p'], x')$ which are equal in the codomain. This means that there is $w \in N_{G^\sigma}(H^\sigma)$ with $[pw^{-1}] = [p']$ and $wx_0 = x'_0$. The former condition implies that $w \in P_0 \cap N_{G}(H)$, so $[w] \in W(L_0)$ represents an element of $W(\mathfrak{g}_0)$, and $([p], x) = ([p'], x')$ in $P_0 / H_0 \times_{W(\mathfrak{g}_0)} \mathfrak{h}_0^r$.

Since the bottom map of Equation (11) is an isomorphism, we just need to show that $\mathfrak{p}_0 / H^\sigma \times_{W(g_0)} \mathfrak{h}_0^r$ maps onto $\mathfrak{p}_0 \cap L_0^{rs}$. Suppose $x \in \mathfrak{p}_0$ is semisimple in $L_0$. Since diagonalizability is preserved by restriction to an invariant subspace and by descent to a quotient by an invariant subspace, we can write $x = x_0 + x_1$, where $x_0$ is a semisimple element of $\mathfrak{p}_0 \cap \mathfrak{p}_0$ and $x_1 \in \mathfrak{u}_0$. Conjugating $x_0$ by an element of the Levi factor $\mathcal{M}$ to be in $\mathfrak{h}_0$, we can assume that $x \in \mathfrak{h}_0$, a Borel subalgebra of $L_0$ contained in $\mathfrak{p}_0$. Thus the problem is reduced to showing that $\mathfrak{h}_0 \cap L_0^{rs} \subset B_0 \mathfrak{h}_0^r$. Given $x$ in the former set, take $g \in G^\sigma$ such that $gx = y \in \mathfrak{h}_0^r$. Then
Proposition 9.18 of [Ca05]. All the roots of \( L \) are self-normalizing, \( g^{-1}w \in B_0 \) and \( x = (g^{-1}w)(w^{-1}y) \in B_0b_0^0. \)

We need two facts about diagram automorphisms and the structure of \( L \). We use the convention from Section 4 to express the simple roots \( \{\alpha_J\} \) of \( L_0 \) in terms of simple roots \( \{\alpha\} \) of \( L \).

Lemma 5.4. \( h_0 \cap h^r = h_0^r \), the set of elements in \( h_0 \) which are regular in \( L_0 \). Similarly, \( L_0 \cap L^{rs} = L_0^{rs} \).

Proof. The restriction map \( h^r \to h_0^r \) sends roots of \( L \) to positive multiples of roots of \( L_0 \) by Proposition 9.18 of [Ca05]. All the roots of \( L_0 \) are covered by this map, so \( h_0 \cap h^r = h_0^r \). An element \( x \in L_0 \) is semisimple in \( L_0 \) if and only if it is semisimple in \( L \). If it is semisimple in \( L_0 \) then it can be conjugated to an element of \( h_0 \), so the statement for \( L_0 \) follows from the statement for \( h_0 \).

Lemma 5.5. There is a parabolic \( p' \) of \( L \) preserved by \( \sigma \) such that \( p' \cap L_0 = p_0 \). If \( m \) is the standard reductive factor of \( p' \) then \( m \cap L_0 = g_0 \), the reductive factor of \( p_0 \), and \( W(m)^\sigma = W(g_0) \), where both are regarded as subgroups of \( W(L) \).

Proof. Let \( S \) be the subset of simple roots \( \{\alpha_J\} \) determining \( p_0 \) and let \( S' \) be the subset of simple roots of \( L \) which appear in some \( \sigma \)-orbit \( J \) for \( \alpha_J \in S \). Let \( p' \) be the parabolic subalgebra determined by \( S' \). Clearly \( p' \) is \( \sigma \)-invariant. By Lemma 4.10, an element \( w \in W^{\sigma} \) belongs to \( W(g_0) \) if and only if it has a reduced expression consisting of reflections through simple roots in \( S' \), which is exactly the condition that \( w \) belongs to \( W(m) \). If \( \alpha_J \) is a simple root of \( L_0 \), then the corresponding positive Chevalley generator \( e_J \) is a linear combination of the positive Chevalley generators corresponding to the simple roots of \( L \) in \( J \), and similarly for the negative Chevalley generator \( f_J \). Since \( p_0 \) is generated as a Lie algebra by \( h_0 \), all the \( e_J \)'s, and the \( f_J \)'s such that \( \alpha_J \in S \), it follows that \( p_0 \subset p' \cap L_0 \). Since \( p' \) is a parabolic subalgebra of \( L_0 \), it follows that \( p' \cap L_0 = p_0 \). Similarly \( m \cap L_0 = g_0 \).

The real form \( h_R \) of \( h \) is the real subspace where all roots take real values, or equivalently the real span of the coroots. If \( x \in h \) let \( Re x \) be the projection of \( x \) to \( h_R \) under the (real-linear) splitting \( h = h_R \oplus i h_R \). Note that \( Re \sigma x = \sigma Re x \) and \( Re wx = w Re x \) for all \( w \in W \).

Proof of Theorem 5.2. First we show that the bottom map of Equation (10) is an isomorphism. Let \( p' \) be the parabolic of \( L \) over \( p_0 \), as in Lemma 5.5. We start by proving that \( J_m^\sigma h^r/W(g_0) \cong J_m^\sigma h^r/W(m) \), where \( W(m) \) is the Weyl group of the reductive factor of \( p' \). Since \( J_m^\sigma h^r/W(m) \) is smooth when \( m < +\infty \) by Lemma 4.7, it is sufficient to prove that the map is bijective. By Corollary 4.4, \( J_m^\sigma h^r/W(m) \cong J_m^\sigma h^r/W(m) \), so every element of \( J_m^\sigma h^r/W(m) \) is represented by an element of \( f \in J_m^\sigma h^r \) such that \( w\sigma(f) = f \) for some \( w \in W(m) \). Let \( S' \) be the set of simple roots determining \( p' \), let \( \Delta' \) be the set of all roots of \( m \), and let \( D = \{ x \in h_R : \alpha(x) \neq 0, \alpha \in \Delta' \} \). The connected components of \( D \) are of the form \( C \times \mathbb{R}^r \), where \( C \) is an open Weyl chamber of \( m \) and \( r = \dim h_R - |S'| \). Consequently \( W(m) \) acts transitively and freely on the connected components of \( D \), so we can assume that \( Re f(0) \in D_0 = \{ x \in h_R : \alpha(x) > 0, \alpha \in S' \} \). But \( S' \) is \( \sigma \)-invariant, so \( D_0 \) is also \( \sigma \)-invariant, and thus \( Re \sigma f(0) = \sigma Re f(0) \in D_0 \). Since \( Re f(0) = w\sigma Re f(0) \), this implies that \( w = e \) and consequently \( f \in J_m^\sigma h^r \). Thus the map \( J_m^\sigma h^r \to J_m^\sigma h^r/W(m) \) is surjective. Suppose
$f, g \in J_m^\sigma h^r$ are equal in $J_m^\sigma (h^r/W(m))$. Then there is $w \in W(m)$ such that $wf = g$. Since $f(0), g(0) \in h^r_0$, we have $\sigma(w)f(0) = g(0) = wf(0)$, and consequently $\sigma(w) = w$. Thus $f$ and $g$ are related by an element of $W(m) \cap W^\sigma = W(g_0)$.

As a special case of the above argument, we have $(h^r/W(m))^\sigma \cong h^r_0/W(g_0) = R^r$. Consequently $J_m^\sigma (h^r/W(m))$ maps to $R^r$ via evaluation at zero, and we conclude that the map $J_m^\sigma h^r/W(g_0) \to R^r \times Q^r J_m^\sigma Q^r$ factors through the isomorphism to $J_m^\sigma (h^r/W(m))$. Since $h^r/W(m) \to h^r/W(L)$ is etale, the space $J_m^\sigma (h^r/W(m))$ is isomorphic to $R^r \times Q^r J_m^\sigma Q^r$ by Lemma 4.8.

We have shown that the bottom map of Equation (10) is an isomorphism, so we just need to do the same for the top map. Consider the case when $p_0 = L_0$, so that $P_m = J_m^\sigma G$ and $W(g_0) = W(L_0)$. Combining Corollary 4.7 (note that $H$ is reductive so that $G/H$ is affine) and the isomorphism $(G/H) \times_W h^r \to L^r$, we get an isomorphism $J_m(G/H) \times_W J_m h^r \to J_m L^r$, where the former space is the quotient (resp. pro-quotient). The automorphism $\tilde{\sigma}$ on $J_m L^r$ translates to the diagonal action of $J_m(G/H) \times W(L) J_m h^r$, and we can show that this isomorphism identifies $J_m^\sigma L^r$ with $J_m^\sigma (G/H) \times_W J_m^\sigma h^r$ by a similar argument to the proof of Lemma 5.3. Namely, $(f, g) \in J_m(G/H) \times J_m h^r$ represents an element of $J_m^\sigma L^r$ if and only if there is $w \in W(L)$ such that $[\tilde{\sigma}(f)]w^{-1} = [f]$ and $w\tilde{\sigma}(g) = g$. Assuming that $Re g(0)$ is in the open Weyl chamber we get that $w = e$ and thus $[f] \in J_m^\sigma (G/H)$, $g \in J_m^\sigma h^r$. Similarly, any two elements of $J_m^\sigma (G/H) \times J_m^\sigma h^r$ with the same image in $J_m L^r$ are $W(L_0)$-translates. Finally we can apply Proposition 4.14 to replace $J_m^\sigma (G/H)$ with $J_m^\sigma G/J_m^\sigma H$.

Now for the general case look at the square

$$
P_m/J_m^\sigma H \times_{W(g_0)} J_m^\sigma h^r \longrightarrow p_m \cap J_m^\sigma L^r.
$$

The group quotient (resp. pro-group quotient) $P_m/J_m^\sigma H$ is a closed subscheme of $J_m^\sigma G/J_m^\sigma H$. As in Lemma 5.3, both vertical maps are inclusions and consequently the top horizontal map is injective. Every $x \in J_m^\sigma L^r$ can be written as $gy$ for $g \in J_m^\sigma G$ and $y \in J_m^\sigma h^r$. If $x \in p_m$ then $x(0) \in p_0$, after which Lemma 5.3 implies that there is $w \in W(L_0)$ such that $g(0)w^{-1} \in P_0$. Consequently $gw^{-1} \in P_m$ and $(gw^{-1}, wy)$ maps to $x$, so the top map is surjective as required. \qed

5.2. Arcs in the regular locus. Let $L^{reg}$ denote the open subset of regular elements in $L$, ie. the set of elements $x$ such that the stabilizer $L^x$ has dimension equal to the rank $l$ of $L$. Note that $L^{reg}$ is $\sigma$-invariant. Kostant famously proved that the map $L^{reg} \to Q$ is surjective and smooth, and furthermore is a $G$-orbit map, in the sense that every fibre is a single $G$-orbit [Ko63b]. The proof uses the Kostant slice, an affine subspace $\nu \subset L^{reg}$ of the form $e + L^l$, where $\{h, e, f\}$ is a principal $\mathfrak{sl}_2$-triple. Kostant showed that $\nu$ intersects each regular $G$-orbit in a unique point, and that $\nu \to L^{reg} \to Q$ is an isomorphism. The following theorem extends this idea to jet and arc groups.

**Theorem 5.6.** There is a Kostant slice $\nu$ of $L$ which is $\sigma$-invariant and such that $\nu^m$ is a Kostant slice for $L_m$. If $\nu$ is such a slice then $J_m^\sigma \nu \to J_m^\sigma Q^\sigma$ is an isomorphism for all $0 \leq m \leq +\infty$, and every $J_m^\sigma G$-orbit in $J_m^\sigma L^{reg}$ intersects $J_m^\sigma \nu$ in a unique point.

At $m = 0$, Theorem 5.6 implies that $Q^\sigma = L_0//G^\sigma$.

For Kostant’s smoothness result it is possible to incorporate a parabolic component.
Theorem 5.7. The map $p_m \cap J_m^* L^\text{reg} \to R \times_{Q^*} J_m^* Q$ is a surjective $P_m$-orbit map for all $0 \leq m \leq +\infty$, and is smooth for $0 \leq m < +\infty$.

Finally, we have a technical corollary which we will need in the next section. Recall the definition of $T_{\text{const}}$ from the previous section, and define $T_{>0} p_m$ to be the subbundle of $T p_m$ of the form $p_m \times u_m$ where $u_m$ is the nilpotent subalgebra of $p_m$, i.e. the subset of elements $f \in p_m$ with $f(0) \in u_0$, the nilpotent radical of $p_0$.

Corollary 5.8. Let $0 \leq m \leq +\infty$. The differential of the map $p_m \to R \times_{Q^*} J_m^* Q$ induces a bundle map $T_{>0} p_m \to R \times_{Q^*} T_{\text{const}} J_m^*$. Over $p_m \cap J_m^* L^\text{reg}$ the bundle map is surjective on fibres.

To prove Theorem 5.6 we start by proving some simple facts about regular elements in $L_0$, using Kostant's characterisation of regular elements (Proposition 0.4 of [Ko63d]) in $L$: if $x = y + z$ is the Jordan decomposition of $x$, so that $y$ is semisimple, $z$ is nilpotent, and $[y, z] = 0$, then $x$ is regular if and only if $z$ is a principal nilpotent in the reductive subalgebra $L^y$. Note that, by definition, a nilpotent element of a reductive algebra $L$ is required to be in $[L, L]$, and if $z$ is a nilpotent in $L$ commuting with a semisimple element $y$, then $z$ is also a nilpotent in $L^y$.

Lemma 5.9. $L^\text{reg} \cap L_0 = L_0^\text{reg}$, the set of regular elements in $L_0$.

Proof. Suppose $x$ in $L_0$ has Jordan decomposition $x = y + z$ in $L$. Then $x = y + z$ is also the Jordan decomposition in $L_0$, and in particular $y$ and $z$ are in $L_0$. Now by conjugating by an element of $G^\sigma$ we can assume that $y \in h_0$, and in fact that $y$ is in the closed Weyl chamber corresponding to the Borel $L_0 \cap b$, where $b$ is the Borel in $L$ compatible with $\sigma$. Since the simple roots of $L$ project to positive multiples of the simple roots of $L_0$, $y$ is also in the closed Weyl chamber of $L$ corresponding to $b$. Let $S$ be the set of simple roots $\alpha_j$ for $L_0$ that are zero on $y$, and similarly let $S'$ be the set of simple roots for $L$ that are zero on $y$. Since $y$ is in the closed Weyl chamber, the stabilizer $L_0^y$ (respectively $L^y$) is the reductive Lie algebra $h_0 \oplus \bigoplus_{\alpha \in \mathbb{Z}[S]} (L_0)_\alpha$ (respectively $h \oplus \bigoplus_{\alpha \in \mathbb{Z}[S]} L_\alpha$). Now $x$ is regular in $L_0$ (respectively $L$) if and only if $z$ is a principal nilpotent in $L_0^y$ (respectively $L^y$). Every nilpotent element of $L_0^y$ is contained in a Borel, and all Borels are conjugate, so we can conjugate $z$ by an element of $(G^\sigma)^y$ to get $z$ contained in the Borel $L_0^y \cap b$ (since it does not have a component in the centre, $z$ will in fact be in the nilpotent radical of $L_0^y \cap b$). By Theorem 5.3 of [Ko59], $z$ is a principal nilpotent in $L_0^y$ if and only if the component of $z$ in $(L_0)_\alpha$ is non-zero for all $\alpha \in S$. But by the construction of the simple Chevalley generators of $L_0$, this is equivalent to the component of $z$ in $L_\alpha$ being non-zero for all $\alpha \in S'$. So $z$ is a principal nilpotent in $L_0^y$ if and only if $z$ is a principal nilpotent in $L^y$, and hence $x$ is regular in $L_0$ if and only if $x$ is regular in $L$. \[\square\]

We also need the following standard technical lemma.

Lemma 5.10. Let $q$ be a $\mathbb{Z}_{>0}$-graded Lie algebra, and let $n$ denote the ideal $\bigoplus_{k>0} q_k$. Suppose $y$ is an element of $q_0$, and that $r \subset n$ is a graded subspace such that $n = [n, y] \oplus r$. Then for every $x$ in the completion $\hat{n}$ there is $g$ in the pro-nilpotent group $\exp(\hat{n})$ such that $g(y + x) \in y + \hat{r}$.

Proof. Let $\{x_i\}$ be the sequence in $\hat{n}$ with $x_0 = x$ and $x_{i+1} = \exp(-z_i)(y + x_i) - y$, where $z_i \in \hat{n}$ is chosen so that $x_i = [z_i, y] + r_i$ for $r_i \in \hat{r}$. Since $\exp(-z_i)(y + x_i) = y + r_i - [z_i, x_i]$, we can show by induction that $z_i$ and the component of $x_i$ in $[n, y]$ are both zero below degree
i + 1. Hence the element $g = \cdots \exp(-z_2) \exp(-z_1) \exp(-z_0)$ is a well-defined element of $\exp(\mathfrak{n})$, and $g(y + x)$ is contained in $y + \mathfrak{r}$ as desired.

Proof of Theorem 5.6. If $m < +\infty$ then there is a homomorphism of Lie groups $J_m^G \to J_{m-1}^G$, so the induced map $J_m^G \to J_{m-1}^G$ on Lie algebras preserves semisimple (resp. nilpotent) elements. We say that an element of $J_m^G$ is pro-semisimple (resp. pro-nilpotent) if the image of the element is semisimple (resp. nilpotent) in $J_m^G$ for every $m < +\infty$. Just as in the finite-dimensional case, every element of $J_m^G$ can be written uniquely as $y + z$ where $y$ is pro-semisimple, $z$ is pro-nilpotent, and $[y, z] = 0$.

If $y \in J_m^G$ is semisimple (resp. pro-semisimple) then $y(0)$ is semisimple in $L$, and hence $L = L^{y(0)} + [L, y(0)]$. It follows from Lemma 5.10 that there is $y \in J_m^G$ such that $gy = y(0) + z$, where $z \in J_m^G y(0)$ and $z(0) = 0$. Since $z$ is nilpotent (resp. pro-nilpotent), uniqueness of the Jordan decomposition implies that $z = 0$.

More generally, if $x$ is an arbitrary element of $J_m^G$ then there is $g \in J_m^G$ such that $gx = y + z$, where $y \in L_0$ is semisimple and $z \in J_m^G y(0)$ is nilpotent (resp. pro-nilpotent). In particular $e = z(0)$ is nilpotent in $L_0^y$, so pick an $\mathfrak{sl}_2$-triple $\{h, e, f\}$ in $L_0^y$ containing $e$. Then $L^y = L^{y(0)} \oplus [L^y, e]$, so applying Lemma 5.10 again there is $g' \in J_m^G y$ such that $g'y = e + J_m^G y = J_m^G (e + L^{y(0)})$ and $g'(0)z(0) = e$.

Using this canonical form, we move on to the proof of the theorem statement. Pick a principal $\mathfrak{sl}_2$-triple $\{h, e, f\}$ in $L_0$. By Lemma 5.9 $\{h, e, f\}$ is also principal in $L$, so $\nu = e + L^j$ is a Kostant slice in $L$ invariant under $\sigma$, and $\nu^\sigma = e + L^j$ is a Kostant slice in $L_0$. It follows immediately that $J_m^\sigma \nu \to J_m^\sigma Q$ is an isomorphism, and also that $Q^\sigma = L_0^\sigma / G^\sigma$. Since $J_m^\sigma L^\reg \to J_m^\sigma Q$ is $J_m^G$-invariant, each orbit in $J_m^\sigma L^\reg$ can intersect $J_m^\sigma \nu$ at most once. So we just need to show that the multiplication map $J_m^\sigma G \times J_m^\sigma \nu \to J_m^\sigma L^\reg$ is surjective, or equivalently that every fibre of the map $J_m^\sigma L^\reg \to J_m^\sigma Q$ is a $J_m^G \sigma$-orbit.

The projection $L^\reg \to Q$ is smooth and every fibre is a $G$-orbit, so the multiplication map $G \times \nu \to L^\reg$ is surjective and smooth. Hence by Lemma 4.2 the multiplication map $J_m^G \times J_m^\nu \to J_m^L^\reg$ is surjective. Suppose $x_1$ and $x_2$ are two points of $J_m^L^\reg$ with the same value in $J_m^G Q$. Using the $m = 0$ case and the canonical form above, we can assume that $x_1(0) = x_2(0) = y + e'$, where $y$ is semisimple in $L_0$ and $e'$ is a principal nilpotent in $L_0^j$, and that $x_1$ and $x_2$ are in $y + J_m^\nu$, where $\nu$ is the Kostant slice $e' + L^{y(0)}$ in $L_0^j$. Since $x_1$ and $x_2$ have the same image in $J_m Q$, there is $g \in J_m G$ such that $gx_1 = x_2$. Multiplication by $g$ preserves Jordan decomposition, so $g \in (J_m G)^y$. The subgroup $G^y$ is a connected reductive subgroup of $G$ by Lemma 5, page 353 of [Ko63b], and the exponential map is a bijection for nilpotent (resp. pro-nilpotent) groups, so $(J_m G)^y = G^y \cdot \exp(z J_m L^y) = J_m G^y$, the connected subgroup of $J_m L$ with Lie algebra $J_m L^y$. Hence $x_1 - y$ and $x_2 - y$ are in the same regular $J_m G^y$-orbit of $J_m (L^y)^\reg$. But $x_1 - y$ and $x_2 - y$ belong to $J_m^\nu \subseteq J_m \nu$, which we have already observed intersects each $J_m G^y$-orbit exactly once, implying that $x_1 = x_2$ as desired.

Theorem 5.6 implies that the map $J_m^\sigma L^\reg \to J_m^\sigma Q$ is surjective for $0 \leq m \leq +\infty$, and smooth for $0 \leq m < +\infty$. To prove Theorem 5.7 we need to account for the parabolic component. Recall that $\mathfrak{g}_0 = \mathfrak{p}_0 \cap \mathfrak{p}_0$.

Lemma 5.11. The projection $\mathfrak{p}_0 \cap L^\reg \to R$ is a surjective smooth $\mathcal{P}_0$-orbit map. In addition, if $g \in G^\sigma$ fixes an element of $\mathfrak{p}_0 \cap L^\reg$ then $g$ belongs to $\mathcal{P}_0$.

Proof. Let $\mathfrak{h}_0$ be a Borel of $L_0$ contained in $\mathfrak{p}_0$ and compatible with $\mathfrak{h}_0$. Let $\mathfrak{u}_0$ be the unipotent radical of $\mathfrak{p}_0$, so that $\mathfrak{g}_0 = \mathfrak{p}_0 / \mathfrak{u}_0$. Finally let $\Delta$ be the set of roots of $L_0$, and
let $S \subset \Delta$ be the set of simple roots. Similarly let $S_0 \subset S$ be the set of simple roots of $g_0$ corresponding to the Borel $b_0 \cap g_0$, and let $\Delta_0 = \Delta \cap \mathbb{Z}[S_0]$ be the set of roots of $g_0$.

Now suppose $y \in p_0$ is semisimple in $L_0$, and let $y = y_0 + y_1$ where $y_0 \in g_0$ and $y_1 \in u_0$. Then $u_0 = [u_0, y_0] \oplus u_0^{\text{reg}}$, so by Lemma 5.10 there is $p \in P_0$ such that $py = y_0 + z$, where $z \in u_0^{\text{reg}}$. Since $py$ is semisimple, we conclude that $z = 0$, and ultimately that $y$ is conjugate by $P_0$ to an element of $b_0$.

Every element $x \in p_0$ can be written as $x = y + z$ where $y, z \in p_0$, $y$ is semisimple in $L_0$, $z$ is nilpotent in $L_0$, and $[y, z] = 0$. By the previous paragraph, it is possible to conjugate $x$ by an element of $P_0$ so that $y \in b_0$. We can then conjugate $x$ by an element of $P_0$ so that $z$ belongs to $b_0$. Assume $x$ is given with $y \in b_0$ and $z \in b_0'$. By a dimension argument, $b_0'$ is a Borel for the reductive Lie algebra $L_0'$. The corresponding simple roots are the indecomposable elements $S_y$ of $\Delta_0^+ \cap \alpha(y) = 0$. Similarly $b_0' \cap g_0'$ is a Borel for $g_0'$, and the simple roots are the elements of $S_y \cap \Delta_0$. The element $x$ is regular in $L_0$ if and only if it is a principal nilpotent in $L_0'$, which is true if and only if the projection to $(L_0)_\alpha$ is non-zero for all $\alpha \in S_y$. If this latter condition holds then the image of $x$ in $g_0 = p_0/ u_0$ is regular in $g_0$. The projection $p_0 \to g_0$ is $P_0$-equivariant, so we conclude that the projection sends regular elements of $L_0$ to regular elements of $g_0$.

Conversely, if $x \in b_0^{\text{reg}}$ then we can conjugate $x$ by an element of the subgroup of $g_0$ to be of the form $y + z$ where $y \in b_0$ and $z \in b_0'$ is a principal nilpotent. This means that the projection of $z$ to $(L_0)_\alpha$ is non-zero for every $\alpha \in S_y \cap \Delta_0$. Let $z'$ be an element of $L_0$ such that the projection of $z'$ to $(L_0)_\alpha$ is non-zero if $\alpha \in S_y \setminus \Delta_0$, and is zero otherwise. Then $x + z'$ is a regular element of $L_0$ which projects $x$. Using equivariance again, we conclude that the projection $p_0 \to g_0$ induces a surjection $p_0 \cap L_0^{\text{reg}} \to g_0^{\text{reg}}$. The map $g_0^{\text{reg}} \to R$ is a smooth surjection, and $p_0 \to g_0$ is smooth, so we conclude that $p_0 \cap L_0^{\text{reg}} \to R$ is a $P_0$-orbit map.

Now suppose $x_1$ and $x_2$ in $p_0 \cap L_0^{\text{reg}}$ map to the same element of $R$. As in the third paragraph, we can assume without loss of generality that $x_i = y_i + z_i$ with $y_i \in b_0$ and $z_i$ a principal nilpotent element of $L_0^y$ contained in $b_0^y$. In addition, the images of $x_1$ and $x_2$ in $g_0$ are conjugate by an element of $P_0$, so in particular we can assume that $y_1 = y_2$. Thus $z_1$ and $z_2$ are both principal nilpotents of $L_0^y$ contained in $b_0^y$, and hence are conjugate by an element of the Borel subgroup of $b_0^y$. We conclude that the projection $p_0 \cap L_0^{\text{reg}} \to R$ is a $P_0$-orbit map.

For the last part of the lemma, we again assume that $x \in p_0 \cap L_0^{\text{reg}}$ is of the form $y + z$ with $y \in b_0$ and $z \in b_0'$. If $x$ is regular then $L_0^y = (L_0^y)^2$ is contained in $b_0 \subset p_0$. By Proposition 14, page 362 of [Ko63b], $(G^n)^c$ is connected, and hence a subgroup of $P_0$.

Proof of Theorem 5.7. Suppose $x_1, x_2 \in p_m \cap J_m^a L^{\text{reg}}$ have the same image in $R \times_{Q^a} J_m^a Q$. By Theorem 5.6 there is $g \in J_m^a G$ such that $gx_1 = x_2$, while by Lemma 5.11 there is $p_0 \in P_0$ such that $p_0 x_1(0) = x_2(0)$. Thus $p_0^{-1} g(0)$ fixes $x_1(0) \in p_0 \cap L_0^{\text{reg}}$, so $g \in p_m$ by Lemma 5.11 and $p_m \cap J_m^a L^{\text{reg}} \to R \times_{Q^a} J_m^a Q$ is a $P_m$-orbit map.

To show surjectivity, observe that $p_m \cap J_m^a L^{\text{reg}} = (p_0 \cap L_0^{\text{reg}}) \times_{L_0} J_m^a L^{\text{reg}}$. A point of $R \times_{Q^a} J_m^a Q$ is determined by a pair of points $x \in R$ and $y \in J_m^a Q$ which have the same image in $Q^a$. Given a point specified in this manner, choose $x' \in p_0 \cap L_0^{\text{reg}}$ mapping to $x$ and $y' \in J_m^a L^{\text{reg}}$ mapping to $y$. Since $x$ and $y$ have the same image in $Q^a$, there is $g \in G^a$ such that $gy(0) = x$. Then $gy$ belongs to $p_m$ and maps to the point $(x, y) \in R \times_{Q^a} J_m^a Q$.

Since $p_m \cap J_m^a L^{\text{reg}} \to R \times_{Q^a} J_m^a Q$ is a $P_m$-orbit map, to show smoothness it is enough to show that the map $Tp_m \cap J_m^a L^{\text{reg}} \to T(R \times_{Q^a} J_m^a Q)$ is surjective. This follows from a
similar argument to the last paragraph. As mentioned in the proof of Theorem 5.6 if \( \nu_0 \) is a Kostant slice in \( L_0 \) then \( C' \times \nu_0 \to L_0^{\text{reg}} \) is smooth and surjective, so \( T C' \times T \nu_0 \to TL_0^{\text{reg}} \) is also surjective, and hence if two elements of \( TL_0^{\text{reg}} \) have the same image in \( TQ' \) then they are conjugate by an element of \( TG' \). Surjectivity of \( p_0 \cap L_0^{\text{reg}} \to R \) and \( J_m^0 L_0^{\text{reg}} \to J_m^0 Q \) follows from Lemma 5.11 and Theorem 5.6.

**Proof of Corollary 5.8.** \( R \times Q' T_{\text{const}} J_m^0 Q \) is isomorphic to the pullback \( R \times TQ' T J_m^0 Q \), where the map \( R \to TQ' \) is the composition of the zero section \( R \to TR \) with the differential \( TR \to TQ' \). The restriction of the differential \( Tp_m \to TR \) to \( T_{>0} p_m \) factors through the zero section \( R \to TR \), so the image of \( T_{>0} p_m \) is contained in \( R \times TQ' T J_m^0 Q \). To show that this bundle map is surjective on fibres, observe that, in the argument for smoothness in the proof of Theorem 5.7, if \( x \in TR \) is a zero tangent vector, then we can pick \( x' \in Tp_0 \cap TL_0^{\text{reg}} \) mapping to \( x \) which is also a zero tangent vector, and hence the resulting point of \( Tp_m \) will be contained in \( T_{>0} p_m \). □

### 6. Calculation of Parahoric Cohomology

In this section we finish the proofs of Theorems 2.5 and 2.6 and Proposition 2.2. We continue to use the notation of Section 5.

#### 6.1. Proof of Proposition 2.2

Pick a principal \( sl_3 \)-triple \( \{h,e,f\} \) in \( L_0 \), and note that \( \{h,e,f\} \) is principal in \( L \) by Lemma 5.9. We need to show that the eigenvalues of \( h/2 \) on \( L_0 \) agree with the subset of the exponents defined in Definition 2.1. Let \( L = \bigoplus L^{(i)} \) denote the principal grading of \( L \) induced by the eigenspace decomposition of \( h/2 \). Then \( m \geq 0 \) appears in the list of exponents of \( L \) with multiplicity \( \dim (L^{(m)})^e \).

Let \( \nu \) denote the Kostant slice \( f + L^e \). As previously mentioned, Kostant’s theorem states that the restriction map \( \mathbb{C}[Q] \to \mathbb{C}[\nu] \) is an isomorphism. Actually, a stronger statement is true. Identity \( \mathbb{C}[\nu] \) with polynomials on \( L^e \) in the obvious way. Filter \( \mathbb{C}[\nu] \) by setting \( \mathbb{C}[\nu]_m \) to be the subring of polynomials on \( \bigoplus_{i=0}^m (L^{(i)})^e \). Choose homogeneous generators for \( \mathbb{C}[Q] = (S^* L^e)^G \) and let \( \mathbb{C}[Q]_m \) be the subring generated by generators of degree at most \( m + 1 \). Then, by Theorem 7, page 381 of [Ko63b], the restriction map gives an isomorphism between \( \mathbb{C}[Q]_m \) and \( \mathbb{C}[\nu]_m \). Furthermore, if \( I \) is a generator of degree \( m + 1 \) then the restriction of \( I \) to \( \nu \) takes the form \( f + I_0 \) where \( f \) is in the dual space of \( (L^{(m)})^e \) and \( I_0 \in \mathbb{C}[\nu]_{m-1} \) does not have constant term.

The automorphism \( \sigma \) acts on both \( \mathbb{C}[\nu] \) and \( \mathbb{C}[Q] \), preserving the filtration in both cases, and the restriction map is \( \sigma \)-equivariant. As before, let \( \mathcal{M} \) denote the ideal in \( (S^* L^e)^G \) containing all elements of degree greater than zero, so that \( \mathcal{M}/\mathcal{M}^2 \) is the space of generators. By definition, the multiplicity of \( m \) as an exponent is the multiplicity of \( q^{-a} \) as an eigenvalue of \( \sigma \) acting on the degree \( m + 1 \) subspace of \( \mathcal{M}/\mathcal{M}^2 \). By the previous paragraph, this is equal to the multiplicity of \( q^{-a} \) as an eigenvalue of \( \sigma \) acting on the dual space of \( (L^{(m)})^e \), or equivalently the dimension of \( q^a \) as an eigenvalue of \( \sigma \) acting on \( (L^{(m)})^e \) itself.

#### 6.2. Proof of Theorem 2.5

Let \( \Omega_{\text{const}}^s R \times Q' T_{\text{const}} J_m^0 Q \) denote the sections of \( \bigwedge^* R \times Q' T_{\text{const}} J_m^0 Q \), where \( T_{\text{const}} J_m^0 Q \) is the dual bundle to \( T_{\text{const}} J_m^0 Q \). Similarly, let \( \Omega_{>0}^s p_m \) denote the sections of \( \bigwedge^* T_{>0} p_m \), where \( T_{>0} p_m \) is the dual bundle to \( T_{>0} p_m \). As per Theorem 3.3, we want to calculate the algebra of \( \mathcal{P}_{\infty} \)-invariant \( \mathcal{N}_{\infty} \)-basic elements of \( \Omega_{>0}^s p_{\infty} \).
Proposition 6.1. Pullback via the bundle map $T_{>0} p_m \to R \times Q^e T_{\text{const}} J_m^\alpha$ gives an isomorphism from the algebra $\Omega^*_\text{const} R \times Q^e J_m^\alpha Q$ to the algebra of $\mathcal{P}_m$-invariant $\mathcal{N}_m$-basic elements of $\Omega^*_0 p_m$.

Proof. Every section of $\Omega^*_0 p_m$ is a pullback from $\Omega^*_0 p_m$ for some $m < +\infty$. By Corollary 5.8 the pullback map is injective, so it is enough to prove surjectivity when $m < +\infty$.

Let $T^s_m$ denote the open subset $p_m \cap J_m^\alpha T^s$ of $p_m$. We start by showing that the pullback map is an isomorphism from $\Omega^*_\text{const} R \times Q^e J_m^\alpha Q$ to the algebra of $\mathcal{P}_m$-invariant $\mathcal{N}_m$-basic elements of $\Omega^*_0 p_m$. By Theorem 5.2, $T^s_m$ is isomorphic to $\mathcal{P}_m/J_m^\alpha H \times W(\mathfrak{g}_0) J_m^\alpha h^\alpha$. By Proposition 4.3:

$$T \left( \mathcal{P}_m/J_m^\alpha H \times W(\mathfrak{g}_0) J_m^\alpha h^\alpha \right) \cong T \mathcal{P}_m/J_m^\alpha H \times W(\mathfrak{g}_0) T J_m^\alpha h^\alpha.$$

$\mathcal{P}_m/\mathcal{N}_m$ is isomorphic to the connected reductive subgroup of $\mathcal{P}_m$ corresponding to the subalgebra $\mathfrak{g}_0 \subset J_m^\alpha L$. We work for a moment in the analytic category. Suppose $\gamma_t$ is a curve in $T^s_m$ representing an element of $T_{>0} p_m$, so that the image $t_\gamma$ of $\gamma_t$ in $\mathfrak{g}_0$ is constant. There are curves $\alpha_\gamma$ and $\beta_\gamma$ in $\mathcal{P}_m$ and $J_m^\alpha h^\alpha$ respectively such that $\alpha_\gamma t_\gamma = \gamma_t$. Let $\overline{\alpha}_\gamma$ denote the image of $\alpha_\gamma \in \mathcal{P}_m/\mathcal{N}_m$. Then $\overline{t}_\gamma = \overline{\alpha_\gamma t}_\gamma(0)$ is a constant curve in $\mathfrak{g}_0$, so $\overline{\alpha}_\gamma^{-1} \overline{\alpha}_\gamma t_\gamma(0)$ is a constant curve in $\mathfrak{h}_0^\alpha$. This implies that $\overline{\alpha}_\gamma^{-1} \overline{\alpha}_\gamma t_\gamma(0) \in w H^\alpha$, for some $w \in N(H)^\alpha$, from which we can conclude that $\overline{\alpha}_\gamma^{-1} \overline{\alpha}_\gamma t_\gamma(0) = w t_\gamma(0)$, so $\overline{\alpha}_\gamma t_\gamma(0)$ is constant, and hence represents a element of $T_{\text{const}} J_m^\alpha h^\alpha$. Since $w^{-1} \overline{\alpha}_\gamma^{-1} \overline{\alpha}_\gamma t_\gamma \in H^\alpha$, the curves $\overline{\alpha}_\gamma t_\gamma$ and $\overline{\alpha}_\gamma^{-1} \overline{\alpha}_\gamma t_\gamma$ are equal in $\mathcal{P}_m/J_m^\alpha H$.

The latter curve projects to a constant curve in $\mathcal{P}_m/\mathcal{N}_m$, and since $\mathcal{P}_m \cong \mathcal{P}_m/\mathcal{N}_m \times \mathcal{N}_m$, is tangent to a left $\mathcal{N}_m$-coset in $\mathcal{P}_m$. Since $\mathcal{N}_m$ is normal, every left $\mathcal{N}_m$-coset is a right $\mathcal{N}_m$-coset. We conclude that over $T^s_m$, the map $T_{>0} p_m$ is isomorphic to the subbundle $T_{\mathcal{N}_m} \mathcal{P}_m/J_m^\alpha H \times W(\mathfrak{g}_0) T_{\text{const}} J_m^\alpha h^\alpha$ of $T \left( \mathcal{P}_m/J_m^\alpha H \times W(\mathfrak{g}_0) J_m^\alpha h^\alpha \right)$, where $T_{\mathcal{N}_m} \mathcal{P}_m/J_m^\alpha H$ is the subbundle of tangents to $\mathcal{N}_m$-orbits.

Recall the proof of Theorem 5.2 that $J_m^\alpha h^\alpha/W(\mathfrak{g}_0)$ is isomorphic to $J_m^\alpha h^\alpha/W(\mathfrak{m})$, so $T_{\text{const}}$ of the former space is well-defined. By Proposition 4.3 again, $T(J_m^\alpha h^\alpha/W(\mathfrak{g}_0)) \cong T(J_m^\alpha h^\alpha/W(\mathfrak{m}))$, so $T_{\text{const}}(J_m^\alpha h^\alpha/W(\mathfrak{g}_0))$ is a subbundle of $T(J_m^\alpha h^\alpha/W(\mathfrak{g}_0))$. A tangent vector $v \in T(J_m^\alpha h^\alpha/W(\mathfrak{g}_0))$ represents an element of $T_{\text{const}}(J_m^\alpha h^\alpha/W(\mathfrak{g}_0))$ if and only if the projection of $v(0)$ to $\mathfrak{h}_0^\alpha/W(\mathfrak{g}_0) \cong (h^\alpha/W(\mathfrak{m}))^\sigma$ is a zero tangent vector, where $v(0)$ is the image of $v$ in $T\mathfrak{h}_0^\alpha$. Since $\mathfrak{h}_0^\alpha \to \mathfrak{h}_0^\alpha$ is etale, this is true if and only if $v(0)$ is a zero tangent vector, so $T_{\text{const}}(J_m^\alpha h^\alpha/W(\mathfrak{g}_0)) \cong (T_{\text{const}} J_m^\alpha h^\alpha)/W(\mathfrak{g}_0)$. Similarly the isomorphism $J_m^\alpha h^\alpha/W(\mathfrak{g}_0) \cong R^x \times Q^e J_m^\alpha Q$ sends $T_{\text{const}} J_m^\alpha h^\alpha/W(\mathfrak{g}_0)$ to $R^x \times Q^e T_{\text{const}} J_m^\alpha h^\alpha$ (see the proof of Corollary 5.8).

Applying Theorem 5.2, we want to show that the bundle map

$$T_{\mathcal{N}_m} \mathcal{P}_m/J_m^\alpha H \times W(\mathfrak{g}_0) T_{\text{const}} J_m^\alpha h^\alpha \to T_{\text{const}} J_m^\alpha h^\alpha/W(\mathfrak{g}_0)$$

induced by projection on the second section gives an isomorphism from $\Omega^*_\text{const} J_m^\alpha h^\alpha/W(\mathfrak{g}_0)$ to the ring of $\mathcal{P}$-invariant $\mathcal{N}$-basic sections of $\bigwedge^* T_{\mathcal{N}_m} \mathcal{P}_m/J_m^\alpha H \times W(\mathfrak{g}_0) T_{\text{const}} J_m^\alpha h^\alpha$.

By pulling back to $\mathcal{P}_m/J_m^\alpha H \times J_m^\alpha h^\alpha$, we can identify the ring of $\mathcal{P}_m$-invariant $\mathcal{N}_m$-basic sections of $\bigwedge^* T_{\mathcal{N}_m} \mathcal{P}_m/J_m^\alpha H \times W(\mathfrak{g}_0) T_{\text{const}} J_m^\alpha h^\alpha$ with a subring of the $\mathcal{P}_m$-invariant $\mathcal{N}_m$-basic sections of $\bigwedge^* T_{\mathcal{N}_m} \mathcal{P}_m/J_m^\alpha H \times T_{\text{const}} J_m^\alpha h^\alpha$. This latter ring is isomorphic to the ring $\Omega^*_\text{const} J_m^\alpha h^\alpha$ with pullback via projection on the second factor. An element of $\Omega^*_\text{const} J_m^\alpha h^\alpha$ descends to a section over $\mathcal{P}_m/J_m^\alpha H \times W(\mathfrak{g}_0) J_m^\alpha h^\alpha$ if and only if it is $W(\mathfrak{g}_0)$-equivariant. The splitting $T J_m^\alpha h^\alpha = J_m^\alpha h^\alpha \times \mathfrak{g}_0 \oplus T_{\text{const}} J_m^\alpha h^\alpha$ allows us to identify the $W(\mathfrak{g}_0)$-module $\Omega^*_\text{const} J_m^\alpha h^\alpha$ with the subalgebra of differential forms which vanish on $J_m^\alpha h^\alpha \times \mathfrak{g}_0$. There is a similar splitting for $T J_m^\alpha h^\alpha/W(\mathfrak{g}_0)$, and thus a similar identification for $\Omega^*_\text{const} J_m^\alpha h^\alpha/W(\mathfrak{g}_0)$. The differential $T J_m^\alpha h^\alpha \to T J_m^\alpha h^\alpha/W(\mathfrak{g}_0)$ preserves this splitting, so the pullback map

\[\text{In contrast, there is no such identification for the } \mathcal{P}_m \text{-module } \Omega^*_0 p_m.\]
\[ \Omega^* \rightarrow \Omega^*/W(\mathfrak{g}_0) \rightarrow \Omega^*_{\text{const}} J^\sigma m \mathfrak{h}^r \] agrees with the pullback map on differential forms. Thus pullback gives an isomorphism from \( \Omega^*_{\text{const}} J^\sigma m \mathfrak{h}^r / W(\mathfrak{g}_0) \) to the \( W(\mathfrak{g}_0) \)-equivariant elements of \( \Omega^*_{\text{const}} J^\sigma m \mathfrak{h}^r \) (see, e.g., Theorem 1 of [Bis98]), and this implies that all \( \mathcal{P}_m \)-invariant \( \mathcal{N}_m \)-basic sections of \( \bigwedge^* T_{\mathcal{N}_m} \mathcal{P}_m / J^\sigma m H \times W(\mathfrak{g}_0) T^*_{\text{const}} J^\sigma m \mathfrak{h}^r \) come from pullback on the second factor.

To finish the proof, let \( \mathfrak{p}^\text{reg} = \mathfrak{p} \cap J^\sigma m L^\text{reg} \), and let \( \phi \) denote the map \( \mathfrak{p}^\text{reg} \rightarrow R \times Q \sigma \). By Theorem 5.7, \( \phi \) is smooth and surjective. Hence if \( f \) is a regular function defined on an open dense subset of \( R \times Q \sigma J^\sigma m Q \) such that \( \phi^* f \) extends to \( \mathfrak{p}^\text{reg} \), then \( f \) has a unique extension to \( R \times Q \sigma \). Suppose \( \omega \in \Omega^*_m \mathfrak{p}^\text{reg} \) is \( \mathcal{P}_m \)-invariant and \( \mathcal{N}_m \)-basic. Then there exists \( \alpha \in \Omega^*_{\text{const}} R^r \times Q \sigma J^\sigma m Q \) such that \( \phi^* \alpha = \omega \) over \( \mathfrak{p}_m^\text{reg} \). We can write \( \alpha = \sum f_i \alpha_i \), where the \( \alpha_i \)'s are elements of \( \Omega^*_{\text{const}} R^r \times Q \sigma J^\sigma m Q \) which are linearly independent in fibres, and the \( f_i \)'s are functions on \( R^r \times Q \sigma J^\sigma m Q \). Since the bundle map is surjective on fibres, the pullbacks \( \phi^* \alpha_i \) are linearly independent in fibres. Since \( \phi^* \alpha = \sum_i \phi^* f_i \alpha_i \) extends to \( \mathfrak{p}^\text{reg} \), the functions \( \phi^* f_i \) must extend to \( \mathfrak{p}^\text{reg} \), and consequently \( \alpha \) extends to \( R \times Q \sigma J^\sigma m Q \). The pullback \( \phi^* \alpha \) agrees with \( \omega \) on an open dense subset, so every \( \mathcal{P}_m \)-invariant \( \mathcal{N}_m \)-basic element of \( \Omega^*_m \mathfrak{p}^\text{reg} \) is the pullback of an element of \( \Omega^*_{\text{const}} R^r \times Q \sigma J^\sigma m Q \) as desired.

### Proof of Theorem 2.5

Let \( I_i^a \) and \( R_i \) be generators for \( \mathbb{C}[Q] \) and \( \mathbb{C}[R] \) as in the statement of Theorem 2.5. Choose coordinates \( \{ y_{ia} \} \) for \( Q \) such that pullback of \( y_{ia} \) via the projection \( L \rightarrow Q \) is \( I_i^a \). Similarly, choose coordinates \( \{ r_i \} \) for \( R \) such that the pullback of \( r_i \) via the projection \( p_0 \rightarrow R \) is \( R_i \). Note that the coordinates \( \{ y_{ia} \} \) with a fixed coordinate correspond to the subspace \( Q_a \) of \( Q \) on which \( \sigma \) acts as multiplication by \( q^a \) (by previously established convention, this means that \( \sigma y_{ia} = q^{-a} y_{ia} \)). Consider the \( r_i \)'s as functions on \( R \times Q \sigma J^\sigma m Q \), and let \( \tilde{y}_{ia} \) denote the induced map \( J^\sigma m Q \rightarrow Q \). Then the coordinate ring of \( R \times Q \sigma J^\sigma m Q \) is the free ring generated by the \( r_i \)'s and the functions \( z^{kn-a} \tilde{y}_{ia} \) for \( a = 0, \ldots, k-1 \) and \( n \geq 1 \). Consequently the ring \( \Omega^*_{\text{const}} R \times Q \sigma J^\sigma m Q \) is the free super-commutative ring generated by the above generators for the coordinate ring, along with the restrictions of the differential forms \( d[z^{kn-a} \tilde{y}_{ia}] \) to \( T_{\sigma} \Omega^*_{\text{const}} R \times Q \sigma J^\sigma m Q \rightarrow T_{\sigma} \mathfrak{p} \), again for \( a = 0, \ldots, k-1 \) and \( n \geq 1 \). Theorem 2.5 then follows from Theorem 3.3.

### 6.3. Proof of Theorem 2.6

The proof of Theorem 2.5 can be simplified and used to prove that pullback via the map \( \mathfrak{p}_0 \rightarrow R \times Q \sigma J^\sigma m Q \) gives an isomorphism between algebraic forms on \( R \times Q \sigma J^\sigma m Q \) and \( \mathcal{P}_m \)-basic and invariant forms on \( \mathfrak{p}_m \). When \( \mathfrak{p}_0 = L_0 \), this can be proven without Theorem 5.2. Namely, if \( \nu \) is a Kostant slice in \( L \), then, as previously mentioned, \( G \times \nu \rightarrow L^{\text{reg}} \) is surjective and smooth. By Lemma 4.7 and Theorem 5.6, the multiplication map \( J^\sigma m G \times J^\sigma m \nu \rightarrow J^\sigma m L^{\text{reg}} \) is surjective for all \( m \), and smooth for \( m < +\infty \). Since \( J^\sigma m \nu \) is isomorphic to \( J^\sigma m Q \), identification of algebraic forms on \( J^\sigma m Q \) with \( J^\sigma m G \)-basic and invariant algebraic forms on \( J^\sigma m L \) follows by pulling back to \( J^\sigma m G \times J^\sigma m \nu \).

This idea can be adapted to determine the algebra of \( \mathcal{B} \)-basic and invariant forms on \( \hat{\mathfrak{n}} \), where \( \mathfrak{b} \) is an Iwahori subalgebra, \( \mathcal{B} \) is the subgroup corresponding to the completion \( \hat{\mathfrak{b}} \), and \( \hat{\mathfrak{n}} \) is the completion of the nilpotent subalgebra of \( \mathfrak{b} \). More specifically, let \( \mathfrak{b}_m \) be the image of \( \hat{\mathfrak{b}} \) in \( J^\sigma m L \), let \( \mathcal{B}_m \) be the corresponding connected subgroup of \( J^\sigma m G \), and let \( \mathfrak{n}_m \) be the image of \( \mathfrak{n} \) in \( J^\sigma m L \). If \( X \) is a variety with finite order automorphism \( \sigma \), and \( p \in X \), let \( J^\sigma m_{\sigma^p X} \) denote the subscheme \( \{ f \in J^\sigma m X : f(0) = p \} \) of \( J^\sigma m X \) with a fixed base point.
Proposition 6.2. There is a map $\mathfrak{n}_m \rightarrow J_{m,0}^\sigma Q$, and pullback via this map gives an isomorphism between the ring of algebraic forms on $J_{m,0}^\sigma Q$ and the ring of $\mathcal{B}_m$-basic and invariant algebraic forms on $\mathfrak{n}_m$.

Proof. Once again it is sufficient to give the proof for $m < +\infty$. Let $e$ be a principal nilpotent of $L_0$, contained in $\mathfrak{n}_0$. Recall that $G^e$ is a connected subgroup of $\mathcal{B}_0$. Let $(J_{m}^\sigma G)_e$ denote the connected subgroup $\{ f \in J_{m}^\sigma G : f(0) \in G^e \}$ of $J_{m}^\sigma G$ with Lie algebra $\{ f \in J_{m}^\sigma L : f(0) \in L_0^e \}$. Since $f \in \mathfrak{n}_m$ belongs to $J_{m}^\sigma L_{reg}$ if and only if $f(0)$ is a principal nilpotent in $\mathfrak{n}_0$, and all principal nilpotents in $\mathfrak{n}_0$ are conjugate by an element of $\mathcal{B}_0$, it follows that the map

$$\mathcal{B}_m \times (J_{m}^\sigma G)_e J_{m,0}^\sigma L \rightarrow \mathfrak{n}_m \cap J_{m}^\sigma L_{reg}$$

is an isomorphism. Consequently $\mathcal{B}_m$-basic and invariant forms on $\mathfrak{n}_m \cap J_{m}^\sigma L_{reg}$ correspond to $(J_{m}^\sigma G)_e$-basic and invariant forms on $J_{m,0}^\sigma L$.

The projection $L \rightarrow Q$ sends $e$ to zero, so the restriction of $J_{m}^\sigma L \rightarrow J_{m}^\sigma Q$ to $J_{m,0}^\sigma L$ factors through $J_{m,0}^\sigma Q$. Choose a principal $\mathfrak{sl}_2$-triple $\{ h, e, f \}$ in $L_0$ containing $e$, and let $\nu = e + Lf$. The isomorphism $J_{m}^\sigma \nu \rightarrow J_{m}^\sigma Q$ identifies $J_{m,0}^\sigma \nu$ with $J_{m,0}^\sigma Q$, and every $(J_{m}^\sigma G)_e$-orbit on $J_{m,0}^\sigma L$ intersects $J_{m,0}^\sigma \nu$ in a unique point. Consequently the map $J_{m,0}^\sigma L \rightarrow J_{m,0}^\sigma Q$ is a surjective smooth $(J_{m}^\sigma G)_e$-orbit map. It follows that the multiplication map $(J_{m}^\sigma G)_e \times J_{m,e}^\sigma \nu \rightarrow J_{m,e}^\sigma L$ is smooth and surjective. We conclude that the pullback map from algebraic forms on $J_{m,e}^\sigma L$ to algebraic forms on $(J_{m}^\sigma G)_e \times J_{m,e}^\sigma \nu$ is injective, and thus pullback via the map $J_{m,e}^\sigma L \rightarrow J_{m,0}^\sigma Q$ gives an isomorphism between algebraic forms on $J_{m,0}^\sigma Q$ and $(J_{m}^\sigma G)_e$-basic and invariant forms on $J_{m,e}^\sigma L$.

Thus every $\mathcal{B}_m$-basic and invariant form on $\mathfrak{n}_m \cap J_{m}^\sigma L_{reg}$ is the pullback of a form from $J_{m,0}^\sigma Q$. Since $\mathfrak{n}_m \cap J_{m}^\sigma L_{reg}$ is dense in $\mathfrak{n}_m$, the proposition follows. □

This proof does not extend to nilpotent subalgebras of other parahorics, as $\mathfrak{u} \cap L_{reg}[z]$ is non-empty only in the Borel case. As in the proof of Theorem 2.5, Theorem 3.5 and Proposition 6.2 together imply Theorem 2.6.

7. Spectral sequence argument for the truncated algebra

In this section we finish the proof of Theorem 2.7 using a spectral sequence argument. Recall from Section 3 the definition of the operators $d_R(S)$ and $d_L(T)$ on $\bigwedge^* \hat{\mathfrak{p}} \otimes S^m \hat{\mathfrak{p}}$. The operator $d_R(S)$ is a generalized interior product, while $d_L(T)$ is a generalized exterior derivative. Hence we have the following version of Cartan’s identity:

Lemma 7.1. $d_R(S)d_L(T)+d_L(T)d_R(S) = (ST)^{Sym} + (TS)^\wedge$, where $(ST)^{Sym}$ is the extension of $ST$ to the symmetric factor as a derivation, and $(TS)^\wedge$ is the extension of $TS$ to the exterior factor as a derivation.

Let $P : \hat{\mathfrak{p}}^* \rightarrow \hat{\mathfrak{p}}^*$ be the dual of multiplication by $z^N$ on $\hat{\mathfrak{p}}$. Define $Q : \hat{\mathfrak{p}}^* \rightarrow \hat{\mathfrak{p}}^*$ by $(Qf)(x) = f(\frac{x}{z})$, where $\bar{x}$ is the projection to $z^N\hat{\mathfrak{p}}$ using the splitting $\hat{\mathfrak{p}} = (z^N\hat{\mathfrak{p}}) \oplus (\hat{\mathfrak{p}}/z^N\hat{\mathfrak{p}})$ suggested by the root grading. Note that $PQ = 1$, while $QP$ is projection to $(z^N\hat{\mathfrak{p}})^*$ using the corresponding splitting of $\hat{\mathfrak{p}}^*$. Thus $(d_R(P)d_L(Q) + d_L(Q)d_R(P))\omega = (n+q)\omega$ if $\omega \in \bigwedge^*(\hat{\mathfrak{p}}/z^N\hat{\mathfrak{p}})^* \otimes \bigwedge^*(z^N\hat{\mathfrak{p}})^* \otimes S^m \hat{\mathfrak{p}}$. Then $d_R(P)^2 = 0$, and we can use Cartan’s identity to show that

$$0 \rightarrow \bigwedge^*(\hat{\mathfrak{p}}/z^N\hat{\mathfrak{p}})^* \rightarrow \bigwedge^* \hat{\mathfrak{p}}^* \xrightarrow{d_R(P)} \bigwedge^{*-1} \hat{\mathfrak{p}}^* \otimes S^1 \hat{\mathfrak{p}} \xrightarrow{d_R(P)} \ldots$$
is exact. Further, $d_R(P)$ commutes with the Lie algebra cohomology operator $\bar{\partial}$ with coefficients in $S^*\hat{\mathfrak{p}}^*$. Since $d_R(P)$ is $\mathfrak{p}$-equivariant and preserves the subset of cochains which vanish on $\mathfrak{g}_0$, we can restrict to the relative cochain complex to get an exact sequence

$$0 \longrightarrow (\wedge^*(\hat{\mathfrak{u}}/z^N\hat{\mathfrak{p}})^*)^{\mathfrak{g}_0} \longrightarrow K^{*,0} \xrightarrow{d_R(P)} K^{*,1} \rightarrow \ldots,$$

where $K^{*,*}$ is the bigraded algebra $(\wedge^*\hat{\mathfrak{u}}^* \otimes S^*\hat{\mathfrak{p}}^*)^{\mathfrak{g}_0}$ graded by tensor (ie. combined exterior and symmetric) degree and symmetric degree, regarded as a bicomplex with differentials $\bar{\partial}$ (the Lie algebra cohomology differential for $\hat{\mathfrak{u}}$ with coefficients in $S^*\hat{\mathfrak{p}}^*)$ and $d_R(P)$. Note that both $\bar{\partial}$ and $d_R(P)$ are derivations of the algebra structure.

**Lemma 7.2.** Give $K^{*,*}$ a $z$-grading by taking the usual $z$-degree for the exterior factor, and $z$-degree + $N$ on $\hat{\mathfrak{p}}^*$ for the symmetric factor. This $z$-grading descends to $H^*(\text{total } K^{*,*})$, and there is an isomorphism $H^*(\mathfrak{p}/z^N\mathfrak{p}, \mathfrak{g}_0) \rightarrow H^*(\text{total } K^{*,*})$ which preserves $z$-degrees.

**Proof.** We have just shown that there is a chain map from the Koszul complex for $(\mathfrak{p}/z^N\mathfrak{p}, \mathfrak{g}_0)$ to total $K^{*,*}$. Consider the spectral sequence induced by the column-wise filtration of $K^{*,*}$, i.e. the descending filtration where the $p$th level contains all elements of $K^{a,b}$ with $a \geq p$. The $E_1$-term of this spectral sequence is

$$E_1^{p,q} = \begin{cases} (\wedge^p(\hat{\mathfrak{u}}/z^N\hat{\mathfrak{p}})^*)^{\mathfrak{g}_0} & q = 0 \\ 0 & q > 0 \end{cases},$$

with differential the restriction of $\bar{\partial}$. Hence

$$E_2^{p,q} = \begin{cases} H^p(\mathfrak{p}/z^N\mathfrak{p}, \mathfrak{g}_0) & q = 0 \\ 0 & q > 0 \end{cases}.$$

It follows from naturality of the spectral sequence that the induced map $H^*(\mathfrak{p}/z^N\mathfrak{p}, \mathfrak{g}_0) \rightarrow H^*(\text{total } K^{*,*})$ is an isomorphism. The $z$-degrees on $K^{*,*}$ are preserved by $\bar{\partial}$ and $d_R(P)$, so the $z$-grading descends to homology and likewise is preserved by the isomorphism. \hfill $\square$

To calculate $H^*(\text{total } K^{*,*})$, consider the spectral sequence of the bicomplex $K^{*,*}$ induced by the row-wise filtration, i.e. the descending filtration where the $p$th level contains all elements of $K^{a,b}$ with $b \geq p$. This spectral sequence has $E_1^{p,q} = H^p_{\text{cts}}(\mathfrak{p}, \mathfrak{g}_0; S^q\hat{\mathfrak{p}})$ with differential $d_R(P)$ (note that the order of the degrees is swapped compared to $K^{*,*}$, so $p$ is symmetric degree and $q$ is tensor degree). Thus $E_1^{*,*}$ is a freely generated differential super-commutative algebra, with generating cocycles explicitly described in Theorem 2.5 as follows. If $r_1, \ldots, r_i$ is a list of exponents for $\mathfrak{g}_0$ then there is a generator in $E_1^{r_1+1, r_i+1}$ represented by a cocycle $R_i$. If $m_1^{(-a)}, \ldots, m_{i-a}^{(-a)}$ is a list of twisted exponents then there is a generator in $E_1^{m_1^{(-a)}+1, m_{i-a}^{(-a)}+1}$ for every $n \geq 1$, represented by a cocycle $f_i^{nk-a} = [z^{nk-a}]\tilde{f}_i^{(-a)}$, and a generator in $E_1^{m_1^{(-a)}+1, m_{i-a}^{(-a)}+1}$ for every $n \geq 1$, represented by a cocycle $\omega_i^{nk-a} = J_\Delta d[z^{nk-a}]\tilde{f}_i^{(-a)}$. Since $d_R(P)$ is a derivation, we just need to determine its action on these generators. By degree considerations, $d_R(P)$ kills the generators $R_i$ and $f_i^{nk-a}$. Note that $f_i^0 = [z^0]\tilde{f}_i^{(0)}$ lies in $E_1^{*,*}$, as it belongs to the algebra $\mathbb{C}[R]$ generated by the $R_i$’s (apply Theorem 5.6 with $m = 0$). If the reductive algebra $L$ splits as a direct sum $L = J \oplus \bigoplus L^{(i)}$, where $J$ is the centre and the $L^{(i)}$’s are $\sigma$-invariant simple components, then we can assume that the generators $f_i^{(-a)}$ of $(S^*L^i)^L$ used to construct the cochains $f_i^{nk-a}$ belong either to $S^* J^*$ or to $(S^*(L^{(i)}))^L$ for some $i$. With this assumption we have:
Lemma 7.3. The differential $d_R(P)$ on $K^*\ast$ sends $\omega_i^{nk-a}$ to a non-zero scalar multiple of $f_i^{nk-a-N}$ if $nk-a \geq N$, and to zero otherwise.

Proof. The generator $\omega_i^{nk-a}$ can be rewritten as $d_L(J)f_i^{nk-a}$. Both $d_R(P)$ and $d_L(J)$ preserve the subalgebras $(S^a(L(i))^*)^L$ and $S^a_\ast$, so we can assume that $L$ is either simple or abelian. Since $d_R(P)f_i^{nk-a}=0$, we can use Lemma 7.3 to get $d_R(P)\omega_i^{nk-a}=(PJ)^{Sym}f_i^{nk-a}$. As an element of the dual of $S^a_\ast(J,P)$, $(PJ)^{Sym}f_i^{nk-a}$ is defined by

\[
x_1 \circ \cdots \circ x_{m_i(-a)+1} \mapsto \sum_j [z^{nk-a}] f_i^{(-a)}(x_1 \circ \cdots \circ J z^N x_j \circ \cdots).
\]

Suppose $L$ is abelian. Then, as noted after the statement of Theorem 3.3, we can assume that $J$ is the identity, so $(PJ)^{Sym}f_i^{nk-a}=f_i^{nk-a-N}$ as required.

This leaves the case that $L$ is simple, in which case $J$ is defined as the derivation of $\hat{\mathfrak{p}}$ acting on weight spaces $\mathfrak{g}_\alpha$ as multiplication by $\langle \rho, \alpha \rangle$, where $\rho$ is the weight of the associated Kac-Moody satisfying $\rho(\alpha_i^\vee)=1$ if $d_i>0$ in the grading of type $d$ determining $\mathfrak{p}$, and $\rho(\alpha_i^\vee)=0$ otherwise. Following the Kac convention in [Ka83], the Kac-Moody associated to $\mathfrak{g}$ is $\hat{\mathfrak{g}}=\mathfrak{g} \oplus \mathfrak{C}c \oplus \mathfrak{C}d$, where $c$ is central and $d$ acts by $z^\frac{d}{d_i}$. The roots of $\hat{\mathfrak{g}}$ belong to the dual of the Cartan $\mathfrak{h}_0 \oplus \mathfrak{C}c \oplus \mathfrak{C}d$, and are defined similarly to the roots of $\mathfrak{g}$, with $d_i$ replacing $\delta$. If $\alpha_0 = d^* - \psi, \alpha_1, \ldots, \alpha_l$ is a list of simple roots for $\hat{\mathfrak{g}}$, then the associated coroots are $\alpha_i^\vee = c^* - \psi_0, \alpha_i^\vee, \ldots, \alpha_l^\vee$, where $\psi_0$ is either $\psi^\vee$ in the untwisted case, or the element of $\mathfrak{h}_0$ such that $\langle x, \psi_0 \rangle = \psi(x)$ in the twisted case. The standard non-degenerate invariant form $\langle , \rangle$ for $\hat{\mathfrak{g}}$ satisfies $\langle \beta_0, c \rangle = \langle \beta_0, d \rangle = \langle c, c \rangle = \langle d, d \rangle = 0$ and $\langle c, d \rangle \neq 0$.

Write $\rho = \rho_0 + A c^*$ for some $\rho_0 \in \mathfrak{h}_0^*$. If $x_j \in \mathfrak{g}_\alpha$ in the above equation then $z^N x_j \in \mathfrak{g}_{\alpha+Nd^*}$, so $J z^N x_j = z^N (N\langle \rho, d^* \rangle + J)x_j$. Since $\langle \rho, d^* \rangle = A(c^*, d^*)$, we have

\[
d_R(P)\omega_i^{nk-a} = \begin{cases} \left(N\langle m_i^{(-a)} + 1 \rangle c^*, d^* \right) f_i^{nk-a-N} & nk > N \\ 0 & nk \leq N \end{cases}.
\]

Take a basis $\{x_{a,i}\}$ for $\mathfrak{g}_\alpha$, and let $x_{a,i}$ be the dual basis. Then $J_{Sym} x_{a,i} = \langle \rho, \alpha \rangle x_{a,i} = (\langle \rho_0, \alpha \rangle + A(c^*, \alpha)) x_{a,i}$. There is $\rho_0 \in \mathfrak{h}_0$ such that $\alpha(\rho_0) = \langle \rho_0, \alpha \rangle$ for all roots $\alpha$, so $ad^*(\rho_0) x_{a,i} = -\langle \rho_0, \alpha \rangle x_{a,i}$. Hence on the subring of $\mathfrak{h}_0$-invariant functions of $S^a_\ast \mathfrak{p}^*$, $J_{Sym}$ agrees with the derivation which sends $x_{a,i}$ to $A(c^*, \alpha)x_{a,i}$. The product $\langle c^*, d^* \rangle$ is equal to $\langle c^*, d^* \rangle$ times the $z$-degree of $x_{a,i}$. We conclude that $J_{Sym} f_i^{nk-a-N} = A(c^*, d^*) (nk-a-N) f_i^{nk-a-N}$, and consequently that

\[
d_R(P)\omega_i^{nk-a} = A(c^*, d^*) \left(N\langle m_i^{(-a)} + 1 \rangle + nk-a-N \right) f_i^{nk-a-N}
\]

if $N \leq nk-a$. Since $nk-a-N \geq 0$, the coefficient is non-zero as required. \hfill \Box

We now have a situation parallel to when we defined $d_R(P)$. Let $V_0$ be the free vector space spanned by basis elements $v_i^{nk+a}$ for $n \geq 0$, $a = 0, \ldots, k-1$, and $i = 1, \ldots, l_0$. For any integer $m$, let $V_m$ be the subspace of $V_0$ spanned by the $v_i^{nk+a}$'s with $nk+a \geq m$. Identify $\bigwedge^* V_1 \otimes S^a V_0$ with a subalgebra of $E_i^{*\ast}$ by sending $v_i^{nk+a}$ to $f_i^{nk+a}$ in the symmetric term and to $\omega_i^{nk+a}$ in the exterior term. Let $P'$ be the linear map $V_0 \to V_0$ sending $v_i^{nk+a}$ to $v_i^{nk+a-N}$ if $nk+a \geq N$, and to zero otherwise. Let $Q'$ be the operator $V_0 \leftrightarrow V_1$ sending $v_i^{nk+a} \mapsto v_i^{nk+a+N}$. Then $P'Q' = 1$, while $Q'P'$ is projection to $V_N$. So $d_R(P')d_L(Q') + d_L(Q')d_R(P')$ acts as multiplication by the combined symmetric degree and exterior $V_N$-degree. By Lemma 7.3 the differential on $E_i^{*\ast}$ restricts to $d_R(P')$ on $\bigwedge^* V_1 \otimes S^a V_0$, and hence the homology of the
differential on this subspace is the subalgebra $\Lambda_1$ of the $E_2$-term generated by $\omega_i^{nk+a}$'s with $0 < nk + a < N$. To get the whole $E_2$-term, recall:

**Lemma 7.4.** $(S^* g_0^*)^0$ is a free $(S^* L_0^L)^{a_0}$-module.

**Proof.** Let $S = (S^* g_0^*)^0$ and $A = (S^* L_0^L)^{a_0}$. Restriction to the Cartan $h_0$ gives isomorphisms $S \cong (S^* h_0^*)^W(g_0)$ and $A \cong (S^* h_0^*)^W(L_0^L)$, so $A$ is a subalgebra of $S$. By the Chevalley-Shephard-Todd theorem \[Ch55\] \[Ko63b\], there is a subset $H_0 \subset S^* h_0^*$ such that $S^* h_0^* \cong A \otimes H_0$ as a $W(L_0^L)$-module, where the isomorphism is given by multiplication, and $H_0$ is isomorphic to the regular representation. Hence $S \cong A \otimes H$ where $H = H_0^W(g_0)$.

The algebra $\mathbb{C}[Q^p]$ generated by the $f_i^p$'s is a subalgebra of $\bigwedge^* V_1 \otimes S^* V_0$. Note that $d_R(P^P)$ is $\mathbb{C}[Q^p]$-linear, since it kills $\mathbb{C}[Q^p]$ and is a derivation. The $E_1$-term is isomorphic to the base extension $\mathbb{C}[R] \otimes_{\mathbb{C}[Q^p]} \bigwedge^* V_1 \otimes S^* V$, with differential given by the base extension $1 \otimes d_R(P^P)$ of $d_R(P^P)$. Freeness implies that the $E_2$-term is $\mathbb{C}[R] \otimes_{\mathbb{C}[Q^p]} \Lambda_1$. Since the action of $\mathbb{C}[Q^p]$ on $\Lambda_1$ sends everything of symmetric degree $> 0$ to zero, the $E_2$-term is isomorphic to $\mathbb{R}(L_0, g_0) \otimes \Lambda_1$.

**Lemma 7.5.** The spectral sequence collapses at the $E_2$-term. Consequently the graded algebra of $H^*(\text{total } K^{*,*})$ with respect to the row-wise filtration is isomorphic to $\mathbb{R}(L_0, g_0) \otimes \Lambda_1$.

**Proof.** $\Lambda_1$ is a free algebra with a generator $\omega_i^{nk-a} \in E_2^{m_i^{(-a)}_{-a}}$ for every twisted exponent $m_i^{(-a)}_{-a}$ of $L$ and $n$ such that $0 < nk - a < N$. The subring $\mathbb{R}(L_0, g_0)$ lies in bidegrees $(a, a)$, so the entire $E_2$-term is contained in bidegrees $(a, b)$ with $a \leq b$. Suppose more generally that the $E_r$-term is contained in bidegrees $(a, b)$ with $a \leq b$, and is generated in bidegrees $(a, a + 1)$ and $(a, a)$. The $E_2$-term differential has bidegree $(2, -1)$, and thus annihilates $\mathbb{R}(L_0, g_0)$ and the generators $\omega_i^{nk-a}$. The same argument works for higher $E_r$-terms as well.

Now we just need to determine the ring structure of $H^*(\text{total } K^{*,*})$. The row-wise filtration of $K^{*,*}$ is the descending filtration where $F^p K^{*,*} = \bigoplus_{r \geq p} K^{r,*}$. Likewise $F^p H^*(\text{total } K^{*,*})$ is the subspace of homology classes which have a representative cocycle in $F^p K^{*,*}$. If $k \in K^{q,p}$ is such that $\partial k = d_R(P^P)k = 0$, then $k$ determines a homology class $[k]$ in $F^p H^*(\text{total } K^{*,*})$. Referring to the construction of the spectral sequence of a filtered differential module (see, e.g., pages 34-37 in \[MC01\]), we also see that $k$ determines a persistent element of the spectral sequence, i.e. $k$ represents an element in each $E_1^{p,q}$ (once again, note that the degrees are swapped between $K^{*,*}$ and $E^{*,*}$) that is killed by the $r$th differential, and the homology class of this element corresponds to the element represented by $k$ in $E^{p,q}_{r+1}$. The projection $F^p H^{p,q}(\text{total } K^{*,*}) \rightarrow E^{p,q}_{\infty}$ sends $[k]$ to the element represented by $k$ in $E_{\infty}$. Finally, when $E_1^{p,q}$ is identified with $H^q(K^{*,*}, \partial)$ the element of $E_1$ represented by $k$ is simply the homology class represented by $k$ in $H^q(K^{*,*}, \partial)$, and consequently the same is true of the identification of $E_2$ with $H^*(H^*(K^{*,*}, \partial), d_R(P^P))$. Note that this would not necessarily be true if $k$ was not homogeneous.

We know that the $E_2$-term is generated by classes represented by elements $R_i, i = 1, \ldots, l_0$ and $\omega_i^{nk+a}, i = 1, \ldots, l_a$ and $0 < nk + a < N$ in $K^{*,*}$. Let $\Lambda$ denote the subalgebra of $K^{*,*}$ generated by the elements $\omega_i^{nk+a}, i = 1, \ldots, l_a, 0 < nk + a < N$. By Theorem 2.5 and Lemma 7.3 $\mathbb{C}[R] \otimes \Lambda \subset K^{*,*}$ is annihilated by both $\partial$ and $d_R(P^P)$. Hence there is a homomorphism $\mathbb{C}[R] \otimes \Lambda \rightarrow H^*(\text{total } K^{*,*})$. Since $\partial \omega_i^N = 0$, Lemma 7.3 implies that the image of $f_i^p$ in $H^*(\text{total } K^{*,*})$ is zero, so the homomorphism $\mathbb{C}[R] \otimes \Lambda \rightarrow H^*(\text{total } K^{*,*})$ descends to a map.
$\mathbb{R}(L_0, g_0) \otimes \Lambda \to H^*({\text{total } K^*})$. By Lemma 7.5 and the argument of the last paragraph, this map is a bijection. We record this calculation in the following proposition.

**Proposition 7.6.** Let $\mathbb{R}(L_0, g_0)$ denote the algebra of Lemma 7.4 graded by symmetric degree. Give $\mathbb{R}(L_0, g_0)$ a cohomological grading by doubling the symmetric grading, and a $z$-grading by multiplying the symmetric grading by $N$. Then $H^*(p/zNp, g_0)$ is isomorphic to $\mathbb{R}(L_0, g_0) \otimes \Lambda$, where $\Lambda$ is the free algebra generated in cohomological degree $2m_1^a + 1$, $z$-degree $Nn_k^a + nk + a$, for $a = 0, \ldots, k-1$, $i = 1, \ldots, l_0$, and $n$ such that $0 < nk + a < N$.

Consider the untwisted case where $n = 1$. In this case, $p/zp$ is the semi-direct product $p_0 \ltimes L_0/p_0$, where $L_0/p_0$ has Lie bracket equal to zero. Then Proposition 7.6 implies that $H^*(p_0 \ltimes L_0/p_0, g_0)$ is isomorphic to $\mathbb{R}(L_0, g_0)$. The following Lemma implies that Proposition 7.6 actually gives a direct Lie algebra proof of Borel’s theorem that $\mathbb{R}(L_0, g_0)$ is isomorphic to $H^*(L_0, g_0)$. Note that the $z$-grading on $H^*(p_0 \ltimes L_0/p_0, g_0)$ is half the cohomological grading, and thus corresponds to the holomorphic grading on $H^*(L_0, g_0)$.

**Lemma 7.7.** Let $p_0 \ltimes L_0/p_0$ be the semi-direct product where $L_0/p_0$ has Lie bracket equal to zero. The cohomology ring $H^*(p_0 \ltimes L_0/p_0, g_0)$ is isomorphic to $H^*(L_0, g_0)$.

**Proof.** Let $X$ be the generalized flag variety $G^\sigma/P_0$, where $P_0$ is the parabolic subgroup of $G^\sigma$ corresponding to $p_0$. The complex-valued de Rham complex of $X$ can be realized as the relative Koszul complex

$$C^* (L_0, g_0; C^\infty(K; \mathbb{C})) = \left( \bigwedge^*(L_0/g_0)^* \otimes C^\infty(K; \mathbb{C}) \right)_{g_0},$$

where $K$ is a compact form of $X$. The de Rham differential $d$ translates to the Lie algebra cohomology boundary operator for $(L_0, g_0)$. Let $u_0$ be the nilpotent radical of $p_0$. The holomorphic structure on $X$ gives the de Rham complex a bigrading, which can be written in terms of $C^*(L_0, g_0; C^\infty(K; \mathbb{C})$ as

$$C^{p,q}(C^\infty(K; \mathbb{C})) = \left( \bigwedge^p u_0^* \otimes \bigwedge^q u_0^* \otimes C^\infty(K; \mathbb{C}) \right)_{g_0},$$

where $p$ is the holomorphic degree, and $q$ is the anti-holomorphic degree. The differential $d = \partial + \bar{\partial}$, where $\partial$ and $\bar{\partial}$ are the holomorphic and anti-holomorphic differentials respectively. On $C^*\cdot \cdot$, $\partial$ is the Lie algebra cohomology differential of $u_0$ with coefficients in $\bigwedge^* u_0^* \otimes C^\infty(K; \mathbb{C})$, where $u_0$ is the $u_0$-module $L_0/\overline{p}_0$. Similarly $\bar{\partial}$ is the Lie algebra cohomology differential of $u_0$ with coefficients in $\bigwedge^* u_0^* \otimes C^\infty(K; \mathbb{C})$. The Kahler identities then imply that the Laplacian $dd^* + d^*d$ of $d$ with respect to a Kahler metric is equal to twice the Laplacian $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. In particular the two differentials give the same cohomology.

A theorem of Chevalley-Eilenberg implies that the de Rham complex is quasi-isomorphic to the subcomplex $C(L_0, g_0; \mathbb{C})$ of equivariant forms [CE-ER]. Since $K$ acts by holomorphic maps on $X$, the same is true of the de Rham complex with the anti-holomorphic differential. Hence the Kahler identities imply that the cohomology of $C^*(L_0, g_0; \mathbb{C})$ is the same with respect to either $d$ or $\bar{\partial}$. Finally $(C(L_0, g_0; \mathbb{C}), \partial)$ can be identified with the Koszul complex for the Lie algebra cohomology of $H^*(p_0 \ltimes L_0/p_0, g_0)$. \qed

To finish the section, we observe that if $p$ is an Iwahori, then a similar spectral sequence calculation can be made with $S^*p^*$ replaced by $S^*\hat{u}$. In this case the spectral sequence will
converge to \( H^*(\mathfrak{p}/z^N n, \mathfrak{h}_0) \), while the \( E_1 \)-term of the spectral sequence is the free supercommutative algebra \( H^*_{dys}(\mathfrak{p}, \mathfrak{h}_0; S^*\hat{u}) \) generated by elements \( f^{nk-a}_i \in E_1^{m_i^{(a)}+1,m_i^{(a)}+1} \) and \( \omega^{nk-a}_i \in E_1^{m_i^{(a)}+1, m_i^{(a)}+1} \) for every \( n \geq 1, a = 0, \ldots, k - 1 \), and \( i = 1, \ldots, l_a \). The differential on the \( E_1 \)-term sends \( \omega^{nk-a}_i \) to \( f^{nk-a}_{i-N} \) if \( nk - a > N \), and to zero otherwise. Thus the \( E_2 \)-term will be the free algebra generated by the \( \omega^{nk-a}_i \)'s with \( 0 < nk - a \leq N \). Since the algebra is free, the isomorphism on graded algebras lifts to give:

**Proposition 7.8.** Let \( \mathfrak{b} \) be an Iwahori subalgebra of \( \mathfrak{g} \), and let \( \mathfrak{n} \) be the nilpotent subalgebra. Then \( H^*(\mathfrak{b}/z^N n, \mathfrak{h}_0) \) is a free algebra generated in cohomological degree \( 2m_i^{(a)} + 1 \), \( z \)-degree \( Nm_i^{(a)} + nk + a \), for \( a = 0, \ldots, k - 1, i = 1, \ldots, l_a \), and \( n \) such that \( 0 < nk + a \leq N \).

**References**

[BGG73] I.N. Bernstein, I.M. Gelfand, and S.I. Gelfand. Schubert cells and cohomology of the spaces \( G/P \). Russ. Math. Surv. 28 (1), 1973.

[Bo53] A. Borel. *Sur la cohomologie des espaces fibres principaux et des espaces homogenes des groupes de Lie compacts*. Ann. of Math. 57 (2), pp. 115-207, 1953.

[Bo91] A. Borel. *Linear algebraic groups*. 2nd ed, Springer-Verlag, Graduate texts in mathematics 126, 1991.

[Ch95] I. Cherednik. *Double Affine Hecke Algebras and Macdonald’s Conjectures*. Annals of Mathematics, vol. 141, no. 1, 191-216, 1995.

[Ch55] C. Chevalley. *Invariants of finite groups generated by reflections*. American Journal of Mathematics, vol. 77, no. 4, pp. 778-782, 1955.

[CE48] C. Chevalley and S. Eilenberg. *Cohomology Theory of Lie Groups and Lie Algebras*. Transactions of the American Mathematical Society, vol. 63, no. 1, pp. 85-124, 1948.

[Di96] J. F. van Diejen. *Self-dual Koornwinder-Macdonald polynomials*. Inventiones Mathematicae 126 (2), pp. 319-339, 1996.

[Fe91] B.I. Feigin. *Differential operators on the moduli space of G-bundles over curves and Lie algebra cohomology*. Special functions (Okayama, 1990), ICM-90 Satel l. Conf. Proc., Springer, Tokyo, pp. 97-105, 1991.

[FGZ85] D. Bressoud and D. Zeilberger. *A proof of Andrews’ q-Dyson conjecture*. Discrete Mathematics 54 (2), pp. 201-224, 1985.

[FG06] E. Frenkel and C. Teleman. *Self-extensions of Verma modules and differential forms on opers*. Compositio Mathematica 142, pp. 477-500, 2006.

[FT06] E. Frenkel and C. Teleman. *Self-extensions of Verma modules and differential forms on opers*. Compositio Mathematica 142, pp. 477-500, 2006.

[GG91] F. Garvan and G. Gonnet. *Macdonald’s constant term conjectures for exceptional root systems*. Bulletin of the American Mathematical Society (New Series) 24 (2), pp. 343-347, 1991.

[Gu90] R. Gustafson. *A generalization of Selberg’s beta integral*. Bull. Amer. Math. Soc. (N.S.) 22 (1), pp. 97-105, 1990.

[Hab86] L. Habseiger. *La q-conjecture de Macdonald-Morris pour G_2*. C.R. Acad. Sci. Paris Ser. I Math 303, pp. 211-213, 1986.

[Ha86] P. Hanlon. *Cyclic homology and the Macdonald conjectures*. Inventiones Mathematicae, v. 86, no. 1, 1986.
[Ha90] P. Hanlon. Some conjectures and results concerning the homology of nilpotent Lie algebras. Adv. Math., v. 84, no. 1, 91-134, 1990.
[Ha94] P. Hanlon. Combinatorial problems concerning Lie algebra homology. In Formal Power Series and Algebraic Combinatorics, Series in Discrete Mathematics and Theoretical Computer Science 24, American Mathematical Society: 1994.
[HW03] P. Hanlon and M. Wachs. On the property M conjecture for the Heisenberg Lie algebra. J. Combin. Theory Ser. A 99, no. 2, 219-231, 2002.
[HS53] G. Hochschild and J.-P. Serre. Cohomology of Lie algebras. Annals of Mathematics 57 (3), pp. 591-603, 1953.
[Ka83] V.G. Kac. Infinite-dimensional Lie algebras / an introduction. 1st ed, Birkhauser, Progress in Mathematics 44, 1983.
[Kad94] K.W.J. Kadell. A proof of the q-Macdonald-Morris conjecture for BC_n. Memoirs of the AMS 108, 1994.
[Ko59] B. Kostant. The Principal Three-Dimensional Subgroup and the Betti Numbers of a Complex Simple Lie Group. American Journal of Mathematics 81 (4), pp. 973-1032, 1959.
[Ko63a] B. Kostant. Lie Algebra Cohomology and Generalized Schubert Cells. Annals of Mathematics 77 (1), pp. 72-144.
[Ko63b] B. Kostant. Lie Group Representations on Polynomial Rings. American Journal of Mathematics 85 (3), pp. 327-404, 1963.
[Ku99] S. Kumar. Homology of certain truncated Lie algebras. Contemporary Mathematics 248, pp. 309-325, 1999.
[Lu73] D. Luna. Slices etales. Bull. Soc. Math. France 33, pp. 81-105, 1973.
[Ma82] I.G. Macdonald. Some Conjectures for Root Systems. SIAM Journal on Mathematical Analysis, 13 (6), pp. 988-1007, 1982.
[Ma72] I.G. Macdonald. The Poincare series of a Coxeter group. Math. Annalen 199, pp. 161-174, 1972.
[Ma88] I. G. MacDonald. A new class of symmetric functions. Publ. I.R.M.A. Strasbourg 372/S-20, 1988.
[MC01] J. McCleary. A User’s Guide to Spectral Sequences. 2nd ed., Cambridge University Press, Cambridge studies in advanced mathematics 58, 2001.
[Mu01] M. Mustata. Jet schemes of locally complete intersection canonical singularities. Invent. Math. 145 (3), pp. 397-424, 2001.
[Sl10] W. Slofstra. A Brylinski filtration for affine Kac-Moody algebras. arXiv: 1012.2095.
[St88] J.R. Stembridge. A short proof of Macdonald’s conjecture for the root systems of type A. Proceedings of the American Mathematical Society 102 (4), pp. 777-786, 1988.
[Te95] C. Teleman. Lie algebra cohomology and the fusion rules. Communications in Mathematical Physics, 172 (2), pp. 265-311, 1995.
[Ze87] D. Zeilberger. A proof of the G_2 case of Macdonald’s root system-Dyson conjecture. SIAM Journal on Mathematical Analysis 18, pp. 880-883, 1987.
[Ze88] D. Zeilberger. A unified approach to Macdonald’s root-system conjectures. SIAM Journal on Mathematical Analysis 19 (4), 1988.

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