ON REGULARIZABLE BIRATIONAL MAPS

JULIE DÉSERTI

ABSTRACT. Bedford asked if there exists a birational self map $f$ of the complex projective plane such that for any automorphism $A$ of the complex projective plane $A \circ f$ is not conjugate to an automorphism. In this article we give such a $f$ of degree 5.

1. INTRODUCTION

Denote by $\text{Bir}(\mathbb{P}^k_{\mathbb{C}})$ the group of all birational self maps of $\mathbb{P}^k_{\mathbb{C}}$, also called the $k$-dimensional Cremona group. Let $\text{Bir}_d(\mathbb{P}^k_{\mathbb{C}})$ be the algebraic variety of all birational self maps of $\mathbb{P}^k_{\mathbb{C}}$ of degree $d$. When $k = 2$ and $d \geq 2$ these varieties have many distinct components, of various dimensions ([6, 2]). The group $\text{Aut}(\mathbb{P}^k_{\mathbb{C}}) = \text{PGL}(k + 1, \mathbb{C})$ acts by left translations, by right translations, and by conjugacy on $\text{Bir}_d(\mathbb{P}^k_{\mathbb{C}})$. Since this group is connected, these actions preserve each connected component.

A birational map $f: \mathbb{P}^k_{\mathbb{C}} \dashrightarrow \mathbb{P}^k_{\mathbb{C}}$ is regularizable if there exist a smooth projective variety $V$ and a birational map $g: V \dashrightarrow \mathbb{P}^k_{\mathbb{C}}$ such that $g^{-1} \circ f \circ g$ is an automorphism of $V$. To any element $f$ of $\text{Bir}(\mathbb{P}^k_{\mathbb{C}})$ we associate the set $\text{Reg}(f)$ defined by

$$\text{Reg}(f) := \{A \in \text{Aut}(\mathbb{P}^k_{\mathbb{C}}) \mid A \circ f \text{ is regularizable}\}.$$ 

On the one hand Dolgachev asked whether there exists a birational self map of $\mathbb{P}^k_{\mathbb{C}}$ of degree $> 1$ such that $\text{Reg}(f) = \text{Aut}(\mathbb{P}^k_{\mathbb{C}})$. In [5] we give a negative answer to this question. More precisely we prove

**Theorem 1.1** ([5]). Let $f$ be a birational self map of $\mathbb{P}^k_{\mathbb{C}}$ of degree $\geq 2$.

The set of automorphisms $A$ of $\mathbb{P}^k_{\mathbb{C}}$ such that $\deg ((A \circ f)^n) \neq (\deg (A \circ f))^n$ for some $n > 0$ is a countable union of proper Zariski closed subsets of $\text{PGL}(k + 1, \mathbb{C})$.

In particular there exists an automorphism $A$ of $\mathbb{P}^k_{\mathbb{C}}$ such that $A \circ f$ is not regularizable.

On the other hand Bedford asked: does there exist a birational map $f$ of $\mathbb{P}^k_{\mathbb{C}}$ such that $\text{Reg}(f) = \emptyset$? We will focus on the case $k = 2$. According to [1] if $\deg f = 2$, then...

---

2010 Mathematics Subject Classification. 14J50, 14E07.

Key words and phrases. Cremona group, birational map, automorphisms of surfaces, regularization.

The author was partially supported by the ANR grant Fatou ANR-17-CE40- 0002-01 and the ANR grant Foliage ANR-16-CE40-0008-01.
Reg(f) \neq \emptyset$. What about birational maps of degree 3? Blanc proves that the set
\[ \{ f \in \text{Bir}_3(\mathbb{P}^2_C) \mid \text{Reg}(f) \neq \emptyset, \lim_{n \to +\infty} (\deg(f^n))^{1/n} > 1 \} \]
is dense in Bir$_3(\mathbb{P}^2_C)$ and that its complement has codimension 1 (see [3]). Blanc also gives a positive answer to Bedford question in dimension 2: if $\chi: \mathbb{P}^2_C \dashrightarrow \mathbb{P}^2_C$ is the birational map given by
\[ \chi: (x : y : z) \mapsto (xz^5 + (yz^2 + x^3)^2 : yz^5 + x^3z^3 : z^6) \]
then Reg(\chi) = \emptyset.

Remark 1.2. Note that $\chi = (x + y^2, y) \circ (x, y + x^3)$ in the affine chart $z = 1$. Indeed Blanc example can be generalized as follows: the birational map given in the affine chart $z = 1$ by
\[ \chi_{n,p} = (x + y^n, y) \circ (x, y + x^p) = (x + (y + x^p)^n, y + x^p) \]
satisfies Reg($\chi_{n,p}$) = \emptyset.

In this article we prove that there exists a birational self map $\psi$ of $\mathbb{P}^2_C$ such that $\deg(\psi) < 6$ and Reg(\psi) = \emptyset:

**Theorem A.** If $\psi: \mathbb{P}^2_C \dashrightarrow \mathbb{P}^2_C$ is the birational map given by
\[ \psi: (x : y : z) \mapsto (x^2yz^2 - z^5 + x^5 : x^2(y^2z - z^3) : xz(x^2y - z^3)) \],
then Reg(\psi) = \emptyset.

**Acknowledgements.** I would like to thank Serge Cantat for many interesting discussions. I am also grateful to the referee who has led me to considerably improve the drafting of the article.

2. PROOF OF THEOREM A

Let $S$ be a smooth projective surface. Let $\phi: S \dashrightarrow S$ be a birational map. This map admits a resolution
\[ \begin{array}{ccc}
Z \ar@{^{(}->}[r]^-{\pi_2} & S \ar@{-->}@/_1pc/[l]^-{\pi_1}
\end{array} \]
where $\pi_1: Z \to S$ and $\pi_2: Z \to S$ are finite sequences of blow-ups. The resolution is minimal if and only if no $(-1)$-curve of $Z$ is contracted by both $\pi_1$ and $\pi_2$. The base-points Base($\phi$) of $\phi$ are the points blown-up by $\pi_1$, which can be points of $S$ or infinitely near points. The proper base-points of $\phi$ are called indeterminacy points of $\phi$ and form a set denoted Ind($\phi$). Finally we denote by Exc($\phi$) the set of curves contracted by $\phi$. 
Denote by \( b(\phi) \) the number of base-points of \( \phi \); note that \( b(\phi) \) is equal to the difference of the ranks of \( \text{Pic}(Z) \) and \( \text{Pic}(S) \) and thus equal to \( b(\phi^{-1}) \). Let us introduce the **dynamical number of the base-points of** \( \phi \)

\[
\mu(\phi) = \lim_{k \to +\infty} \frac{b(\phi^k)}{k}.
\]

Since \( b(\phi \circ \varphi) \leq b(\phi) + b(\varphi) \) for any birational self map \( \varphi \) of \( S \), \( \mu(\phi) \) is a non-negative real number. As \( b(\phi) = b(\phi^{-1}) \) one gets \( \mu(\phi^k) = |k \mu(\phi)| \) for any \( k \in \mathbb{Z} \). Furthermore if \( Z \) is a smooth projective surface and \( \varphi : S \to Z \) a birational map, then for all \( n \in \mathbb{Z} \)

\[-2b(\varphi) + b(\varphi^n) \leq b(\varphi \circ \varphi^n \circ \varphi^{-1}) \leq 2b(\varphi) + b(\varphi^n) ;
\]

hence \( \mu(\varphi) = \mu(\varphi \circ \varphi \circ \varphi^{-1}) \). One can thus state the following result:

**Lemma 2.1** (4). *The dynamical number of base-points is an invariant of conjugation. In particular if \( \phi \) is a regularizable birational self map of a smooth projective surface, then \( \mu(\phi) = 0 \).*

A base-point \( p \) of \( \phi \) is a **persistent base-point** if there exists an integer \( N \) such that for any \( k \geq N \)

\[
\begin{align*}
\diamond & \ p \in \text{Base}(\phi^k) \\
\diamond & \ p \not\in \text{Base}(\phi^{-k}).
\end{align*}
\]

Let \( p \) be a point of \( S \) or a point infinitely near \( S \) such that \( p \not\in \text{Base}(\phi) \). Consider a minimal resolution of \( \phi \)

\[
\begin{array}{c}
\pi_1 \bigg/ \ \ \pi_2 \\
S \quad \xrightarrow{\phi} \quad \xrightarrow{\phi} \quad S
\end{array}
\]

Because \( p \) is not a base-point of \( \phi \) it corresponds via \( \pi_1 \) to a point of \( Z \) or infinitely near; using \( \pi_2 \) we view this point on \( S \) again maybe infinitely near and denote it \( \phi^\ast(p) \). For instance if \( S = \mathbb{P}_C^2, p = (1 : 0 : 0) \) and \( f \) is the birational self map of \( \mathbb{P}_C^2 \) given by

\[
(z_0 : z_1 : z_2) \mapsto (z_1 z_2 + z_0^2 : z_0 z_2 : z_2^2)
\]

the point \( f^\ast(p) \) is not equal to \( p = f(p) \) but is infinitely near to it. Note that if \( \varphi \) is a birational self map of \( S \) and \( p \) is a point of \( S \) such that \( p \not\in \text{Base}(\phi), \phi(p) \not\in \text{Base}(\phi) \), then \( (\varphi \circ \phi)^\ast(p) = \varphi^\ast(\phi^\ast(p)) \). One can put an equivalence relation on the set of points of \( S \) or infinitely near \( S \): the point \( p \) is **equivalent** to the point \( q \) if there exists an integer \( k \) such that \( (\phi^k)^\ast(p) = q \); in particular \( p \not\in \text{Base}(\phi^k) \) and \( q \not\in \text{Base}(\phi^{-k}) \). Remark that the equivalence class is the generalization of set of orbits for birational maps.

Let us give the relationship between the dynamical number of base-points and the equivalence classes of persistent base-points:
Proposition 2.2 \([4]\). Let \(S\) be a smooth projective surface. Let \(\phi\) be a birational self map of \(S\).

Then \(\mu(\phi)\) coincides with the number of equivalence classes of persistent base-points of \(\phi\). In particular \(\mu(\phi)\) is an integer.

This interpretation of the dynamical number of base-points allows to prove the following result that gives a characterization of regularizable birational maps:

Theorem 2.3 \([4]\). Let \(\phi\) be a birational self map of a smooth projective surface. Then \(\phi\) is regularizable if and only if \(\mu(\phi) = 0\).

2.1. Base-points of \(\psi\). The birational map

\[
\psi: (x : y : z) \to (x^2yz^2 - z^5 + x^5 : x^2y - z^3 : xz(x^2y - z^3))
\]

has only one proper base-point, namely \(p_1 = (0 : 1 : 0)\), and all its base-points are in tower that is: the nine base-points of \(\psi\) that we denote \(p_1, p_2, \ldots, p_9\) are such that \(p_i\) is infinitely near to \(p_{i-1}\) for \(2 \leq i \leq 9\). We denote by \(\pi: S \to \mathbb{P}^2_C\) the blow-up of the 9 base-points, and still write \(L_x\) (resp. \(C\)) the strict transform of the line \(L_x \subset \mathbb{P}^2_C\) of equation \(x = 0\) (resp. the curve of equation \(x^2y - z^3 = 0\)) which is contracted by \(\psi\). We denote by \(E_i \subset S\) the strict transform of the curve obtained by blowing up \(p_i\). The configuration of the curves \(E_1, E_2, \ldots, E_9, L_x\) and \(C\) is

\[\text{FIGURE 1.}\]

Two curves are connected by an edge if their intersection is positive. Let us write \(\psi_A = A \circ \psi\) where \(A\) is an automorphism of \(\mathbb{P}^2_C\). Because \(\pi\) is the blow-up of the base-points of \(\psi\), which are also the base-points of \(\psi_A\), the map \(\eta = \psi_A \circ \pi\) is a birational morphism \(S \to \mathbb{P}^2_C\) which is the blow-up of the base-points of \(\psi_A^{-1}\). In fact

\[
\begin{array}{ccc}
S & \xrightarrow{\pi} & \mathbb{P}^2_C \\
\downarrow{\psi_A} & & \downarrow{\eta} \\
\mathbb{P}^2_C & \xrightarrow{\psi_A} & \mathbb{P}^2_C
\end{array}
\]
is the minimal resolution of $\psi_A$.

The morphism $\eta$ contracts $L_x$ and $C$ as well as the union of eight other irreducible curves which are among the curves $E_1, E_2, \ldots, E_9$. The configuration of Figure 1 shows that $\eta$ contracts the curves $L_x, E_2, E_3, E_4, E_5, E_6, E_7, E_8, C$ following this order.

We can see $\eta: S \to \mathbb{P}^2_C$ as a sequence of nine blow-ups in the same way as we did for $\pi$. We denote by $q_1, q_2, \ldots, q_9$ the base-points of $\psi_A^{-1}$ (or equivalently the points blown up by $\eta$) so that $q_1 \in \mathbb{P}^2_C$ and $q_i$ is infinitely near to $q_{i-1}$ for $2 \leq i \leq 9$. We denote by $D \subset \mathbb{P}^2_C$ (resp. $C' \subset \mathbb{P}^2_C$) the line contracted by $\psi_A^{-1}$ which is the image by $A$ of the line $y = 0$ (resp. of the conic $z^2 - xy = 0$). We denote by $F_i \subset S$ the strict transform of the curve obtained by blowing up $q_i$. Because of the order of the curves contracted by $\eta$ we get equalities between $L_x, C, E_1, E_2, \ldots, E_9$ and $D, C', F_1, F_2, \ldots, F_9$ as follows

\[
\begin{align*}
C &= F_1 \\
E_8 &= F_2 \\
E_6 &= F_4 \\
E_4 &= F_6 \\
E_2 &= F_8 \\
E_9 &= L_x \\
E_7 &= F_3 \\
E_5 &= F_5 \\
E_3 &= F_7 \\
L_x &= F_9
\end{align*}
\]

**Figure 2.**

In particular we see that the configuration of the points $q_1, q_2, \ldots, q_9$ is not the same as that of the points $p_1, p_2, \ldots, p_9$. Saying that a point $m$ is proximate to a point $m'$ if $m$ is infinitely near to $m'$ and that it belongs to the strict transform of the curve obtained by blowing up $m'$ the configurations of the points $p_i$ and $q_i$ are

\[
\begin{align*}
p_1 &\rightarrow p_2 \rightarrow p_3 \leftarrow p_4 \rightarrow p_5 \leftarrow p_6 \rightarrow p_7 \leftarrow p_8 \rightarrow p_9 \\
q_1 &\leftarrow q_2 \leftarrow q_3 \leftarrow q_4 \leftarrow q_5 \leftarrow q_6 \leftarrow q_7 \leftarrow q_8 \leftarrow q_9
\end{align*}
\]

**Figure 3**

We will prove that for any integer $i > 0$ the point $p_3$ belongs to $\text{Base}(\psi_A^i)$ and does not belong to $\text{Base}(\psi_A^{-i})$. It implies that $\mu(\psi_A) > 0$ and that $\psi_A$ is not regularizable.

Denote by $k$ the lowest positive integer such that $p_1$ belongs to $\text{Base}(\psi_A^{-k})$. If no such integer exists we write $k = \infty$. For any $1 \leq i < k$ the point $p_1$ does not belong to $\text{Base}(\psi_A^{-i})$
so $\psi_A$ and $\psi_A^{-1}$ have no common base-point. As a consequence the set of base-points of the
map $\psi_A^{i+1} = \psi_A \circ \psi_A^i$ is the union of the base-points of $\psi_A^i$ and of the points $(\psi_A^j)^* (p_j)$ for
$1 \leq j \leq 9$. Since the map $\psi_A^{i+1}$ is defined at $p_1$ the point $(\psi_A^i)^* (p_j)$ is proximate to the point
$(\psi_A^{-i})^* (p_k)$ if and only if $p_j$ is proximate to $p_k$. Proceeding by induction on $i$ we get the
following assertions:

- for any $1 \leq i \leq k$ integer $\text{Base}(\psi_A^i) = \{(\psi_A^{-m})^* (p_j) | 1 \leq j \leq 9, 0 \leq m \leq i - 1\}$;
- for any $0 \leq -\ell \leq k$ the configuration of the points \{(\psi_A^{\ell})^* (p_j) | 1 \leq j \leq 9\} is given by

\[
\begin{array}{cccccccc}
(p_1)^* & \cdots & (p_9)^* & (p_1)^* & \cdots & (p_9)^* & (p_1)^* & \cdots & (p_9)^* \\
(p_2)^* & \cdots & (p_9)^* & (p_1)^* & \cdots & (p_9)^* & (p_1)^* & \cdots & (p_9)^* \\
(p_3)^* & \cdots & (p_9)^* & (p_1)^* & \cdots & (p_9)^* & (p_1)^* & \cdots & (p_9)^* \\
(p_4)^* & \cdots & (p_9)^* & (p_1)^* & \cdots & (p_9)^* & (p_1)^* & \cdots & (p_9)^* \\
(p_5)^* & \cdots & (p_9)^* & (p_1)^* & \cdots & (p_9)^* & (p_1)^* & \cdots & (p_9)^* \\
(p_6)^* & \cdots & (p_9)^* & (p_1)^* & \cdots & (p_9)^* & (p_1)^* & \cdots & (p_9)^* \\
(p_7)^* & \cdots & (p_9)^* & (p_1)^* & \cdots & (p_9)^* & (p_1)^* & \cdots & (p_9)^* \\
(p_8)^* & \cdots & (p_9)^* & (p_1)^* & \cdots & (p_9)^* & (p_1)^* & \cdots & (p_9)^* \\
(p_9)^* & \cdots & (p_9)^* & (p_1)^* & \cdots & (p_9)^* & (p_1)^* & \cdots & (p_9)^* \\
\end{array}
\]

Hence the point $p_3$ belongs to $\text{Base}(\psi_A^i)$ for any $1 \leq i \leq k$.

If $k = \infty$, then $p_3$ belongs to $\text{Base}(\psi_A^i)$ for any $i > 0$ and by definition of $k$ the point $p_1$
does not belong to $\text{Base}(\psi_A^{-i})$ for any $i > 0$, and so neither $p_3$. We can thus assume that $k$
is a positive integer.

Assume that $q_1$ belongs to $\text{Base}(\psi_A^i)$ for some $1 \leq i \leq k - 1$. Then $q_1$ is equal to
$(\psi_A^{-m})^* (p_j)$ for some $0 \leq m \leq k - 2$ and $1 \leq j \leq 9$. This implies that $p_j$ belongs to
$\text{Base}(\psi_A^{m+1})$ which is impossible because $m + 1 \leq k - 1$. Hence $q_1$ does not belong to
$\text{Base}(\psi_A^i)$ for any $1 \leq i \leq k - 1$.

We thus see that $\psi_A^{-1}$ has no common base-point with $\psi_A^i$ for $1 \leq i \leq k - 1$. In particular
if $B$ denotes $\text{Base}(\psi_A^{-1}) \cap \text{Base}(\psi_A^k)$, then

$$B = \{(\psi_A^{-(k-1)})^* (p_j) | 1 \leq j \leq 9\} \cap \{q_j | 1 \leq j \leq 9\}.$$ 

Let us remark that $p_1$ belongs to $\text{Base}(\psi_A^k)$ and $p_1$ does not belong to $\text{Base}(\psi_A^{-(k-1)})$; as
a result $(\psi_A^{-k-1})^* (p_1)$, which is a base-point of $\psi_A^k$, is also a base-point of $\psi_A^{-1}$. The set $B$
is thus not empty.

The configurations of the two sets of points \{(\psi_A^{-k-1})^* (p_j) | 1 \leq j \leq 9\} and \{q_j | 1 \leq j \leq 9\} imply that $q_1 = (\psi_A^{-(k-1)})^* (p_1)$.

Moreover either $B = \{q_1\}$, or $B = \{q_1, q_2\}$. Indeed $(\psi_A^{-(k-1)})^* (p_3)$ is proximate to
$(\psi_A^{-(k-1)})^* (p_2)$ and $(\psi_A^{-(k-1)})^* (p_1)$ whereas $q_3$ is proximate to $q_2$ but not to $q_1$.

The point $(\psi_A^{-(k-1)})^* (p_3)$ is thus a point infinitely near to $q_1$ in the second neighborhood
which is maybe infinitely near to $q_2$ but not equal to $q_3$. Recalling that $\eta$ is the blow up of
$q_1, q_2, \ldots, q_9$ the point $(\eta^{-1} \circ \psi_A^{-(k-1)})^* (p_3)$ corresponds to a point that belongs, as a proper
or infinitely near point, to one of the curves $F_1, F_2 \subset S$. So $(\pi \circ \eta^{-1} \circ \psi_A^{-(k-1)})^* (p_3)$ is a point
infinitely near to $p_3$. For any $1 \leq i \leq k$ the point $p_3$ does not belong to $\text{Base}(\psi_A^{-i})$; therefore
there is no base-point of $\psi_A^{-i}$ which is infinitely near to $p_3$. As a result $(\psi_A^{-(k-1)})^* (p_3)$ does not
belong to Base(ψ_A^−i) and p_3 does not belong to Base(ψ_A^{−(k+i)}). Moreover (ψ_A^{−(k+i)})(p_3) is infinitely near to (ψ_A^−i)(p_3). Choosing i = k we see that (ψ_A^{−2k})(p_3) is infinitely near to (ψ_A^{-k})(p_3) which is infinitely near to p_3. Continuing like this we get
\[ \forall i \geq 1 \quad p_3 \notin \text{Base}(ψ_A^{-i}).\]
To get the result it remains to show that p_3 belongs to Base(ψ_A^i) for any i \geq 1. Reversing the order of ψ_A and ψ_A^{-1} we prove as previously that
\[ \forall i \geq 1 \quad q_3 \notin \text{Base}(ψ_A^i).\]
Let us now see that
\[ (\forall i \geq 1 \quad q_3 \notin \text{Base}(ψ_A^i)) \Rightarrow (\forall i \geq 1 \quad p_3 \in \text{Base}(ψ_A^i)).\]
For i = 1 it is obvious. Assume i > 1; let us decompose
\[ \circ \psi_A^i \text{ into } \psi_A^{-1} \circ \psi_A.\]
\[ \circ \pi : S \rightarrow \mathbb{P}_C^2 \text{ into } \pi_{12} \circ \pi_{39} \text{ where } \pi_{12} : Y \rightarrow \mathbb{P}_C^2 \text{ is the blow up of } p_1, p_2 \text{ and } \pi_{39} : S \rightarrow Y \text{ is the blow up of } p_3, p_4, \ldots, p_9,\]
\[ \circ \eta : S \rightarrow \mathbb{P}_C^2 \text{ into } \eta_{12} \circ \eta_{39} \text{ where } \eta_{12} : Z \rightarrow \mathbb{P}_C^2 \text{ is the blow up of } q_1, q_2 \text{ and } \eta_{39} : S \rightarrow Z \text{ is the blow up of } q_3, q_4, \ldots, q_9.\]
Note that η_{39} contracts F_0, F_8, \ldots, F_3 onto the point Z ∋ q_3 \notin Base(ψ_A^{-1} \circ η_{12}). Consider the system of conics of \mathbb{P}_C^2 passing through p_1, p_2 and p_3. Denote by Λ its lift on Y; it is a system of smooth curves passing through q_3 with movable tangents and dimΛ = 2. The strict transform of Λ on S is a system of curves intersecting E_3 at a general movable point. The map η_{39} contracts the curves L_x, E_2, E_3, E_4, E_5, E_6, E_7. As the curve E_3 is contracted and is not the last one, the image of the system by η_{39} passes through q_3 with a fixed tangent corresponding to the point q_4. Since q_3 \notin Base(ψ_A^{-1} \circ η_{12}) the image of Λ ⊂ Y by ψ_A^{-1} \circ η \circ (π_{39})^{-1} has a fixed tangent at the point (ψ_A^{-1} \circ η_{12})(q_3). As a consequence p_3 belongs to Base(ψ_A^{-1} \circ η \circ (π_{39})^{-1}) and thus to Base(ψ_A^{-1} \circ η \circ (π_{39})^{-1} \circ (π_{12})^{-1}).

**REFERENCES**

[1] E. Bedford and K. Kim. Dynamics of rational surface automorphisms: linear fractional recurrences. J. Geom. Anal., 19(3):553–583, 2009.
[2] C. Bisi, A. Calabri, and M. Mella. On plane Cremona transformations of fixed degree. J. Geom. Anal., 25(2):1108–1131, 2015.
[3] J. Blanc. Dynamical degrees of (pseudo)-automorphisms fixing cubic hypersurfaces. Indiana Univ. Math. J., 62(4):1143–1164, 2013.
[4] J. Blanc and J. Déserti. Degree growth of birational maps of the plane. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 14(2):507–533, 2015.
[5] S. Cantat, J. Déserti, and J. Xie. Three chapters on Cremona groups. arXiv:2007.13841.
[6] D. Cerveau and J. Déserti. Transformations birationnelles de petit degré, volume 19 of Cours Spécialisés. Société Mathématique de France, Paris, 2013.
[7] J. Diller. Cremona transformations, surface automorphisms, and plane cubics. *Michigan Math. J.*, 60(2):409–440, 2011. With an appendix by Igor Dolgachev.

Université Côte d’Azur, Laboratoire J.-A. Dieudonné, UMR 7351, Nice, France

Email address: deserti@math.cnrs.fr