The blow-up rate for a non-scaling invariant semilinear wave equations

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Abstract

We consider the semilinear wave equation

\[ \partial_t^2 u - \Delta u = f(u), \quad (x, t) \in \mathbb{R}^N \times [0, T), \]

with \( f(u) = |u|^{p-1}u \log a (2 + u^2) \), where \( p > 1 \) and \( a \in \mathbb{R} \). We show an upper bound for any blow-up solution of (1). Then, in the one space dimensional case, using this estimate and the logarithmic property, we prove that the exact blow-up rate of any singular solution of (1) is given by the ODE solution associated with (1), namely

\[ u'' = |u|^{p-1}u \log a (2 + u^2). \]

Unlike the pure power case (\( g(u) = |u|^{p-1}u \)) the difficulties here are due to the fact that equation (1) is not scale invariant.

MSC 2010 Classification: 35L05, 35B44, 35L71, 35L67, 35B40
Keywords: Semilinear wave equation, Blow-up, log-type nonlinearity.

1 Introduction

This paper is devoted to the study of blow-up solutions for the following semilinear wave equation:

\[
\begin{cases}
\partial_t^2 u = \Delta u + f(u), & (x, t) \in \mathbb{R}^N \times [0, T), \\
u(x, 0) = u_0(x) \in H^1_{loc,u}(\mathbb{R}^N), & \partial_t u(x, 0) = u_1(x) \in L^2_{loc,u}(\mathbb{R}^N),
\end{cases}
\]

(1.1)

where \( u(t) : x \in \mathbb{R}^N \to u(x, t) \in \mathbb{R} \) with focusing nonlinearity \( f \) defined by:

\[ f(u) = |u|^{p-1}u \log a (2 + u^2), \quad p > 1, \quad a \in \mathbb{R}. \]

(1.2)

The spaces \( L^2_{loc,u}(\mathbb{R}^N) \) and \( H^1_{loc,u}(\mathbb{R}^N) \) are defined by

\[ L^2_{loc,u}(\mathbb{R}^N) = \{ u : \mathbb{R}^N \to \mathbb{R} / \sup_{d \in \mathbb{R}^N} \left( \int_{|x-d| \leq 1} |u(x)|^2 dx \right) < +\infty \}, \]

\[ H^1_{loc,u}(\mathbb{R}^N) = \{ u : \mathbb{R}^N \to \mathbb{R} / \sup_{d \in \mathbb{R}^N} \left( \int_{|x-d| \leq 1} |u(x)| dx \right) < +\infty \}. \]
and
\[ H^1_{loc,u}(\mathbb{R}^N) = \{ u \in L^2_{loc,u}(\mathbb{R}^N), |\nabla u| \in L^2_{loc,u}(\mathbb{R}^N) \}. \]

We assume in addition that \( p > 1 \) and if \( N \geq 2 \), we further assume that
\[ p < p_c \equiv 1 + \frac{4}{N-1}. \]  

A semilinear wave equation with nonlinearity, including a logarithmic factor, has been introduced in various nonlinear physical models in the context of nuclear physics, wave mechanics, optics, geophysics etc ... see e.g. [3, 4].

The defocusing case has been studied in the mathematical literature and the first results are due to [44] where Tao proved a global well-posedness and scattering result for the three dimensional nonlinear wave equation \( \partial_t^2 u = \Delta u - |u|^4 u \log(2 + u^2) \), in the radial case. See also the work of Shih [43], where the method is refined to treat \( \partial_t^2 u = \Delta u - |u|^4 u \log^c (2 + u^2) \), for any \( c \in (0, \frac{1}{2}) \). Later, Roy extends in [41] the results (global well-posedness and scattering) to solutions to the log-log-supercritical equation \( \partial_t^2 u = \Delta u - |u|^4 u \log \left( \log(10 + u^2) \right) \), for \( c \) small, without any radial assumption. This series of works should be considered as a starting point for the understanding of the global behavior of the solutions in the Sobolev supercritical regime \( \partial_t^2 u = \Delta u - |u|^p u \), where \( p > 4 \). In this direction, we aim to give a light in the understanding of the superconformal range \( p > p_c \) related to the blow-up rate of the solution of equation (1.8) below.

Let us mention that the blow-up question for the semilinear heat equation \( \partial_t u = \Delta u + |u|^{p-1} u \log^a(2 + u^2) \) is studied by Duong-Nguyen-Zaag in [18]. More precisely, they construct for this equation a solution which blows up in finite time \( T \), only at one blow-up point \( a \), according to the following asymptotic dynamics:
\[ u(x,t) \sim \phi(t) \left( 1 + \frac{(p-1)|x-a|^2}{4p(T-t)\log(T-t)} \right)^{-\frac{1}{p-1}}, \quad \text{as } t \to T, \]  

where \( \phi(t) \) is the unique positive solution of the ODE
\[ \phi' = |\phi|^{p-1} \phi \log^a(2 + \phi^2), \quad \lim_{t \to T} \phi(t) = +\infty. \]  

Given that we have the same expression in the pure power nonlinearity case \( g(u) = |u|^{p-1}u \) with \( \phi(t) \) replaced by \( \kappa(T-t)^{-\frac{1}{p-1}} \) (see [9]), we see that the effect of the nonlinearity is all encapsulated in the ODE (1.5).

Equation (1.1) is well-posed in \( H^1_{loc,u} \times L^2_{log,u} \). This follows from the finite speed of propagation and the well-posedness in \( H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \). The existence of blow-up solutions \( u(t) \) of (1.1) follows from ODE techniques or the energy-based blow-up criterion by Levine [29] (see also [30, 42, 43]). More blow-up results can be found in Caffarelli and Friedman [10, 11], Kichenassamy and Littman [26, 27]. Numerical simulations of blow-up are given by Bizoñ and al. (see [5, 6, 7, 8]).
If \( u \) is an arbitrary blow-up solution of (1.1), we define (see for example Alinhac [1]) a 1-Lipschitz curve \( \Gamma = \{(x, T(x))\} \) such that the maximal influence domain \( D \) of \( u \) (or the domain of definition of \( u \)) is written as
\[
D = \{(x, t) \mid t < T(x)\}.
\]
\( \bar{T} = \inf_{x \in \mathbb{R}^N} T(x) \) and \( \Gamma \) are called the blow-up time and the blow-up graph of \( u \). A point \( x_0 \) is a non characteristic point if there are
\[
\delta_0 \in (0, 1) \text{ and } t_0 < T(x_0) \text{ such that } u \text{ is defined on } \mathcal{C}_{x_0, t_0}\delta_0 \cap \{t \geq t_0\}
\]
where \( \mathcal{C}_{x, t, \delta} = \{(x, t) \mid t < \bar{t} - \delta|x - \bar{x}|\} \).

In this paper, we study the blow-up rate of any singular solution of (1.1). Before going on, it is necessary to mention that the blow-up rate in the case with pure power nonlinearity
\[
\partial_t^2 u = \Delta u + |u|^{p-1} u, \quad (x, t) \in \mathbb{R}^N \times [0, T),
\]
was studied by Merle and Zaag in [31, 32, 33]. More precisely, they proved that if \( u \) is a solution of (1.8) with blow-up graph \( \Gamma : \{x \mapsto T(x)\} \) and \( x_0 \) is a non-characteristic point, then, for all \( t \in [\frac{3T(x_0)}{4}, T(x_0)] \),
\[
0 < \varepsilon_0(p) \leq (T(x_0) - t)^{\frac{2}{p+1}} \frac{\|u(t)\|_{L^2(B(x_0, T(x_0) - t))}}{(T(x_0) - t)^{\frac{2}{p}}} + (T(x_0) - t)^{\frac{2}{p} + 1} \left( \frac{\|\partial_t u(t)\|_{L^2(B(x_0, T(x_0) - t))}}{(T(x_0) - t)^{\frac{2}{p}}} + \frac{\|\partial_x u(t)\|_{L^2(B(x_0, T(x_0) - t))}}{(T(x_0) - t)^{\frac{2}{p}}} \right) \leq K,
\]
where the constant \( K \) depends only on \( p \) and on an upper bound on \( T(x_0), 1/T(x_0), \delta_0(x_0) \) and the initial data in \( H^1_{\text{loc},u}(\mathbb{R}^N) \times L^2_{\text{loc},u}(\mathbb{R}^N) \). Namely, the blow-up rate of any singular solution of (1.8) is given by the solution of the associated ODE \( u'' = |u|^{p-1} u \). Note that this result about the blow-up rate is valid in the subconformal and conformal case (\( 1 < p \leq p_c \)).

In a series of papers, Merle and Zaag [34, 35, 37, 38] (see also Côte and Zaag [12]) give a full picture of blow-up for solutions of equation (1.8) in one space dimension. Among other results, Merle and Zaag proved that characteristic points are isolated and that the blow-up set \( \{(x, T(x))\} \) is \( \mathcal{C}^1 \) near non-characteristic points and corner-shaped near characteristic points. In higher dimensions, the method used in the one-dimensional case does not remain valid because there is no classification of selfsimilar solutions of equation (1.8) in the energy space. However, in the radial case outside the origin, Merle and Zaag reduce to the one-dimensional case with perturbation and obtain the same results as for \( N = 1 \) (see [39] and also the extension by Hamza and Zaag in [25] to the Klein-Gordon equation and other damped lower-order perturbations of equation (1.8)). Later, Merle and Zaag could address the higher dimensional case in the subconformal case and prove the stability of the explicit selfsimilar solution with respect to the blow-up point and initial data (see [39, 40]). Considering the behavior of radial solutions at the origin, Donninger and Schörkhuber were able to prove the stability of the ODE solution.
\[ u(t) = \kappa_0(p)(T - t)^{-\frac{2}{p+1}} \] in the lightcone with respect to small perturbations in initial data, in a stronger topology (see [14, 15, 16, 17]). Their approach is based in particular on a good understanding of the spectral properties of the linearized operator in self-similar variables, operator which is not self-adjoint. Recently, by establishing a suitable Strichartz estimates for the critical wave equation in similarity variables, Donninger in [13] prove the stability of the solution of the ODE with respect to small perturbations in initial data, in the energy space. Let us also mention that Killip, Stovall and Vigan proved in [28] that in superconformal and Sobolev subcritical range, an upper bound on the blow-up rate is available. This was further refined by Hamza and Zaag in [24].

In [22, 23], using a highly non-trivial perturbative method, we could obtain the blow-up rate for the Klein-Gordon equation and more generally, for equation

\[ \partial_t^2 u = \Delta u + |u|^{p-1}u + f(u) + g(\partial_t u), \quad (x,t) \in \mathbb{R}^N \times [0,T), \] (1.10)

under the assumptions \(|f(u)| \leq M(1 + |u|^q)|v| \] and \(|g(v)| \leq M(1 + |v|)|v| \), for some \(M > 0\) and \(q < p \leq \frac{N+3}{N-1}\). In fact, we proved a similar result to (1.9), valid in the subconformal and conformal case. Let us also mention that in [19, 20, 21], the results obtained in [22, 23] were extended to the strongly perturbed equation (1.10) with \(|f(u)| \leq M(1 + |u|^p \log^{-a}(2 + u^2))\), for some \(a > 1\) though keeping the same condition in \(g\).

In the previous works [19, 20, 21, 22, 23], we consider a class of perturbed equations where the nonlinear term is equivalent to the pure power \(|u|^{p-1}u\) and we obtain the estimate (1.9). This is due to the fact that the dynamics is governed by the ODE equation:

\[ u'' = |u|^{p-1}u. \] Furthermore, our proof remains (non trivially) perturbative with respect to the homogeneous PDE (1.8), which is scale invariant.

This leaves unanswered an interesting question: is the scale invariance property crucial in deriving the blow-up rate?

In fact we had the impression that the answer was ”yes”, since the scaling invariance induces in similarity variables a PDE which is autonomous in the unperturbed case (1.8), and asymptotically autonomous in the perturbed case (1.10).

In this paper we prove that the answer is ”no” from the example on the non homogeneous PDE (1.8). In fact, our situation is different from (1.8), and (1.10). Indeed, the term like \(|u|^{p-1}u \log^a(2 + u^2)\) is playing a fundamental role in the dynamics of the blow-up solution of (1.1). More precisely, we obtain an analogous result to (1.9) but with a logarithmic correction as shown in (1.26) below. In fact, the bow-up rate is given by the solution of the following ordinary differential equation:

\[ u'' = |u|^{p-1}u \log^a(2 + u^2). \] Before handling the PDE, we first study the associated ODE to (1.11)

\[ v_T''(t) = |v_T(t)|^{p-1}v_T(t) \log^a(v_T^2(t) + 2), \quad v(T) = \infty, \] (1.11)

and show that the nonlinear term including the logarithmic factor gives raise to a different dynamic. In fact, thanks to Lemma A.2, we can see that the solution \(v_T\) satisfies

\[ \psi_T(t) \sim \kappa_a \psi_T(t), \text{ as } t \to T, \quad \text{where } \kappa_a = \left(\frac{2^{1-2a}(p+1)}{(p-1)^2-a}\right)^{\frac{1}{p-1}}, \] (1.12)
and
\[ \psi_T(t) = (T - t)^{-\frac{2}{p-1}} (-\log(T - t))^{-\frac{2}{p-1}}. \quad (1.13) \]

Let us introduce the following similarity variables, defined for all \( x_0 \in \mathbb{R}, T_0 \) such that \( 0 < T_0 \leq T(x_0) \) by:
\[ y = \frac{x - x_0}{T_0 - t}, \quad s = -\log(T_0 - t), \quad u(x, t) = \psi_{T_0}(t) w_{x_0, T_0}(y, s). \quad (1.14) \]

From (1.1), the function \( w_{x_0, T_0} \) (we write \( w \) for simplicity) satisfies the following equation for all \( y \in B, s > 0 \) and \( s \geq -\log T_0 \):
\[ \partial_s^2 w = \frac{1}{\rho} \text{div} (\rho \nabla w - \rho(y, \nabla w)y) + \frac{2a}{(p-1)s} y \cdot \nabla w - \frac{2p+2}{(p-1)^2} w + \gamma(s) w \\
- \left( \frac{p+3}{p-1} - \frac{2a}{(p-1)s} \right) \partial_s w - 2y \cdot \nabla \partial_s w + e^{-\frac{2a}{p-1}s} s^{\frac{1}{p-1}} f(\phi(s)w), \quad (1.15) \]
where \( \rho(y) = (1 - |y|^2)\alpha \),
\[ \alpha = \frac{2}{p-1} - \frac{N-1}{2} > 0, \quad (1.16) \]
\[ \gamma(s) = \frac{a(p + 5)(p - a - 1)}{(p-1)^2s} - \frac{a(p + a - 1)}{(p-1)^2s^2}, \quad (1.17) \]
and
\[ \phi(s) = e^{\frac{2a}{p-1}s} s^{\frac{1}{p-1}}. \quad (1.18) \]

This change of variables is associated to the nonlinear wave equation including a logarithmic nonlinearity (1.1). In fact, we have the same transformation as in the pure power case \( (g(u) = |u|^{p-1}u) \). In the new set of variables \( (y, s) \), the behavior of \( u \) as \( t \to T_0 \) is equivalent to the behavior of \( w \) as \( s \to +\infty \). Also, if \( T_0 = T(x_0) \), then we simply write \( w_{x_0} \) instead of \( w_{x_0, T(x_0)} \).

The equation (1.15) will be studied in the Hilbert space \( \mathcal{H} \)
\[ \mathcal{H} = \left\{ (w_1, w_2), \left| \int_B \left( w_1^2 + |\nabla w_1|^2 - |y, \nabla w_1|^2 \right) + w_2^2 \right| dy < +\infty \right\}, \]
where \( B = B(0, 1) \) stands for the unit ball of \( \mathbb{R}^N \) and throughout the paper.

Throughout this paper, \( C \) denotes a generic positive constant depending only on \( p, N \) and \( a \), which may vary from line to line. Also, we will use \( K \) to denote a generic positive constant depending only on \( p, N, a, \delta_0(x_0) \) and initial data which may vary from line to line. We write \( f(s) \sim g(s) \) to indicate \( \lim_{|s| \to \infty} \frac{f(s)}{g(s)} = 1 \). Furthermore, we denote by
\[ F(u) = \int_0^u f(v) dv = \int_0^u |v|^{p-1} v \log^a(v^2 + 2) dv. \quad (1.19) \]
As we mentioned earlier, the invariance of equation (1.8) under the scaling transformation \( u \mapsto u_{\lambda}(x,t) = \lambda^{-\frac{2}{p-1}} u(\lambda x, \lambda t) \) was crucial in the construction of the Lyapunov functional in similarity variables (see Antonini and Merle [2]). The fact that the equation (1.10) is not invariant under the last scaling transformation implies that the existence of a Lyapunov functional in similarity variables is far from being trivial (see [19, 20, 21, 22, 23]).

In this paper, we prove a polynomial (in \(s\)) space-time bound on the similarity variables’ version of the solution \( u \) of (1.1), valid in any dimensions in the subconformal case. However, our main contribution lays, in one space dimension. It consists in the construction of a Lyapunov functional in similarity variables for the problem (1.15) and the proof that the blow-up rate of any singular solution of (1.1) is given by the solution of the following ODE: \( u'' = |u|^{p-1} u \log^a (2 + u^2) \).

Let us give some details regarding our strategy in this paper.

First, we exploit some functional to obtain a rough estimate on the blow-up solution; namely a polynomial (in \(s\)) bound on the solution in similarity variables. The issue is how to handle the perturbative terms in (1.15). In fact, in order to control them, we view equation (1.15) as a perturbation of the case of a pure power nonlinearity (case where \(a = 0\) in (1.15)) with the following terms:

\[
\frac{2a}{(p-1)s}y \nabla w, \quad \gamma(s)w, \quad \frac{2a}{(p-1)s} \partial_s w \quad \text{and} \quad e^{-\frac{2a}{p-1} s^{\frac{2}{p-1}}} f(\phi(s)w). \quad (1.20)
\]

The first three terms are lower order terms which were already handled in the subconformal perturbative case treated in [23, 20]. However, since the nonlinear term \( e^{-\frac{2a}{p-1}s^{\frac{2}{p-1}}} f(\phi(s)w) \) depends on time \(s\), we expect the time derivatives to be delicate. Thanks to the fact that \( u f(u) - (p+1) \int_0^s f(v)dv \sim \frac{2a}{p+1} |u|^{p+1} \log^{a-1} (2 + u^2) \), as \( u \to \infty \), we construct a functional (in Section 2) satisfying this kind of differential inequality:

\[
\frac{dh(s)}{ds} \leq -\alpha \int_B (\partial_s w)^2 \frac{\rho(y)}{1 - |y|^2} dy + C \frac{h(s)}{s}, \quad (1.21)
\]

where \( \alpha \) is defined in (1.16), and this implies a polynomial estimate.

Now, we announce the following rough polynomial space-time estimate:

**Theorem 1.** Consider \( u \) a solution of (1.1) with blow-up graph \( \Gamma : \{ x \mapsto T(x) \} \) and \( x_0 \) a non characteristic point. Then, there exists \( t_0(x_0) \in [0,T(x_0)) \) and \( q = q(p,a,N) > 0 \) such that, for all \( T_0 \in (t_0(x_0), T(x_0)) \), for all \( s \geq -\log(T_0 - t_0(x_0)) \), we have

\[
\int_s^{s+1} \int_B \left( w^2(y,\tau) + (\partial_s w(y,\tau))^2 + |\nabla w(y,\tau)|^2 \right) dy d\tau \leq K_1 s^q, \quad (1.22)
\]

where \( w = w_{x_0,t_0} \), \( K_1 \) depends on \( p,a,\delta_0(x_0), T(x_0), t_0(x_0) \) and \( ||(u(t_0(x_0)), \partial_t u(t_0(x_0)))||_{H^1 \times L^2(B(x_0, \frac{T(x_0)-t_0(x_0)}{\delta_0(x_0)}))} \).
In the original variables, Theorem 1 implies the following:

**Corollary 2.** Consider $u$ a solution of (1.1) with blow-up graph $\Gamma: \{ x \mapsto T(x) \}$ and $x_0$ a non characteristic point. Then, there exists $t_0(x_0) \in [0, T(x_0))$ and $q = q(p, a, N) > 0$ such that, for all $t \in [t_0(x_0), T(x_0))$, we have

$$
\int_t^{T(x_0)-\frac{1}{L}(T(x_0)-t)} \int_{B(x_0, T(x_0)-\tau)} \frac{u^2(x, \tau)}{\psi_T(x_0)(\tau)(T(x_0)-\tau)^{8/7}} \, dx \, d\tau \leq K_2 \left( - \log(T(x_0)-t) \right)^q,
$$

and

$$
\int_t^{T(x_0)-\frac{1}{L}(T(x_0)-t)} \int_{B(x_0, T(x_0)-\tau)} \frac{|\nabla u(x, \tau)|^2 + (\partial_t u(x, \tau))^2}{\psi_T(x_0)(\tau)(T(x_0)-\tau)^{8/7}} \, dx \, d\tau \leq K_2 \left( - \log(T(x_0)-t) \right)^q.
$$

**Remark 1.1.** The estimates obtained in Theorem 1 and Corollary 2 do not seem to be optimal unfortunately. Indeed, we expect the solution of the PDE $u$ to be bounded by the solution of the ODE $\psi_T(x_0)$, as in the case $a = 0$. Accordingly, we conjecture that the right-hand sides in the inequalities in Theorem 1 and Corollary 2 to be constant.

Even though the rough estimate obtained seems bad, it is very useful to allow us to derive, in one space dimension, a Lyapunov functional for equation (1.15). More precisely, we use this polynomial estimate and the structure of the nonlinear term to construct a Lyapunov functional for equation (1.15) as a crucial step to derive the optimal estimate. Let us note that the method is valid only in one dimensional case and breaks down in higher dimensional case (see below in Remark 1.7). For that reason, Theorem 3 and Theorem 4 given below are valid only in the one dimensional case. Accordingly, in the rest of this paper, we consider the one dimensional case.

To state our main result, we start by introducing the following functionals,

$$
E_1(w(s), s) = \int_{-1}^{1} \left( \frac{1}{2} (\partial_s w)^2 + \frac{1}{2} (\partial_y w)^2 (1 - y^2) + \frac{p + 1}{(p - 1)^2} w^2 - e^{-\frac{2(p+1)}{p-1}s} \frac{2a}{s^{p-1}} F'(\phi w) \right) \rho(y) \, dy,
$$

$$
L_0(w(s), s) = E_1(w(s), s) - \frac{1}{s \sqrt{s}} \int_{-1}^{1} \partial_s w \rho(y) \, dy,
$$

where $F$ is defined by (1.19). Moreover, for all $s \geq \max(1, -\log T_0)$, we define the functional

$$
L(w(s), s) = \exp \left( \frac{p + 3}{\sqrt{s}} \right) L_0(w(s), s) + \theta e^{-s},
$$

where $\theta$ is a sufficiently large constant that will be determined later. We derive that the functional $L(w(s), s)$ is a decreasing functional of time for equation (1.15), provided that $s$ is large enough. Clearly, by (1.23) and (1.24), the functional $L(w(s), s)$ is a small perturbation of the natural energy $E_1(w(s), s)$.

Here is the statement of our main theorem in this paper.
Theorem 3. Consider a solution of (1.1) in one space dimension \((N = 1)\), with blow-up graph \(\Gamma: \{x \mapsto T(x)\}\), and \(x_0\) a non characteristic point. Then there exists \(t_1(x_0) \in [0,T(x_0))\) such that, for all \(T_0 \in (t_1(x_0), T(x_0)]\), for all \(s \geq -\log(T_0 - t_1(x_0))\), we have

\[
L(w(s + 1), s + 1) - L(w(s), s) \leq -\frac{2}{p - 1} \int_s^{s+1} \int_{-1}^1 (\partial_s w)^2 \frac{\rho(y)}{1 - y^2} dy d\tau,
\]

where \(w = w_{x_0,T_0}\) is defined in (1.14).

Remark 1.2. We have chosen to present our main result as Theorem 3 since the existence of a Lyapunov functional in similarity variables is far from being trivial and it represents the crucial step in this paper.

Remark 1.3. Since we crucially need a covering technique in our argument, in fact, we need a uniform version for \(x\) near \(x_0\) (see Theorem 3 below).

Remark 1.4. Let us note that our method breaks down in the case of a characteristic point, since in the construction of the Lyapunov functional in similarity variables, we use a covering technique in our argument which is not available at a characteristic point. At this moment, we do not know whether Theorem 3 continues to hold if \(x_0\) is a characteristic point.

As we said earlier, the existence of this Lyapunov functional \(L(w(s), s)\) together with a blow-up criterion for equation (1.15) make a crucial step in the derivation of the blow-up rate for equation (1.1). Indeed, with the functional \(L(w(s), s)\) and some more work, we are able to adapt the analysis performed in [31, 32, 33] for equation (1.8) and obtain the following result:

Theorem 4. (Blow-up rate for equation (1.1)).

Consider a solution of (1.1) in one space dimension \((N = 1)\), with blow-up graph \(\Gamma: \{x \mapsto T(x)\}\) and \(x_0\) a non characteristic point. Then there exist \(\hat{S}_2\) large enough such that

i) For all \(s \geq \hat{s}_2(x_0) = \max(\hat{S}_2, -\log \frac{T(x_0)}{4})\),

\[
0 < \varepsilon_0 \leq \|w_{x_0}(s)\|_{H^1((-1,1))} + \|\partial_s w_{x_0}(s)\|_{L^2((-1,1))} \leq K,
\]

where \(w_{x_0} = w_{x_0,T(x_0)}\) is defined in (1.14).

ii) For all \(t \in [t_2(x_0), T(x_0))\), where \(t_2(x_0) = T(x_0) - e^{-\hat{s}_2(x_0)}\), we have

\[
0 < \varepsilon_0 \leq \frac{1}{\psi_T(x_0)(t)} \frac{\|u(t)\|_{L^2(I(x_0,T(x_0)-t))}}{\sqrt{T(x_0) - t}} + \frac{\|\partial_s u(t)\|_{L^2(I(x_0,T(x_0)-t))}}{\sqrt{T(x_0) - t}} \leq K,
\]

where \(K = K(p,a,T(x_0),t_2(x_0),\|u(t_2(x_0))\|_{H^1 \times L^2(I(x_0,T(x_0)-t_2(x_0))})\),

\(\psi_T(x_0)(t)\) is defined in (1.13), \(I(x_0,t) = (x_0 + t, x_0 - t)\) and \(\delta_0(x)\) is defined in (1.7).
Remark 1.5. As in the pure power nonlinearity case \((1.8)\), the proof of Theorem 4 relies on four ideas (the existence of a Lyapunov functional, interpolation in Sobolev spaces, some critical Gagliardo-Nirenberg estimates and a covering technique adapted to the geometric shape of the blow-up surface). It happens that adapting the proof of \([32]\) given in the pure power nonlinearity case \((1.8)\) is straightforward. Therefore, we only present the key argument dedicated to the control of the 4th term in \((1.20)\), and refer to \([31, 32, 33]\) for the treatment of the terms appearing in the definition of \(E_1(w(s), s)\) defined in \((1.23)\) and refer to \([22, 23, 19, 20, 21]\) for the control of the three first terms of \((1.20)\) for the rest of the proof.

Remark 1.6. Since we crucially need a covering technique in the argument of the construction of the Lyapunov functional, our method breaks down in the case of a characteristic point and we are not able to obtain the sharp estimate as in the unperturbed case \((1.8)\).

Remark 1.7. It should be noted here that the restriction to a one dimensional space is due to the use of the embedding \(H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})\). Unfortunately, as we pointed in the construction of the Lyapunov functional, our method breaks down in the case of higher dimensions, and we are not able to obtain the sharp estimate as in the case of pure power nonlinearity \((1.8)\). However, as already stated in Theorem 1 above, we can derive a polynomial in \(s\) space-time estimate in higher dimension in the subconformal case \((1 < p < \frac{N+3}{N-1}).

Remark 1.8. Let us remark we can obtain the same blow-up rate for the more general equation
\[
\partial_t^2 u = \partial_x^2 u + |u|^{p-1} u \log^a(2 + u^2) + k(u), \quad (x, t) \in \mathbb{R} \times [0, T),
\]
under the assumption that \(|k(u)| \leq M(1 + |u|^p \log^b(2 + u^2))\), for some \(M > 0\) and \(b < a - 1\). More precisely, under this hypothesis, we can construct a suitable Lyapunov functional for this equation. Then, we can prove a similar result to \((1.26)\). However, the case where \(a - 1 \leq b < a\) seems to be out reach with our technics, though we think we may obtain the same rate as in the unperturbed case.

This paper is organized as follows: In Section 2, we obtain a rough control of the solution \(w\) in the subconformal case. In Section 3, in one space dimension and thanks to the result obtained, we prove that the functional \(L(w(s), s)\) is a Lyapunov functional for equation \((1.15)\). Thus, we get Theorem 3. Finally, applying this last theorem, we prove Theorem 4.

2 A polynomial bound for solution of equation \((1.15)\)

Consider \(u\) a solution of \((1.1)\) with blow-up graph \(\Gamma : \{x \mapsto T(x)\}\) and \(x_0\) a non characteristic point. This section is devoted to deriving a uniform version of Theorem 1 valid for \(x\) near \(x_0\). More precisely, this is the aim of this section.

**Theorem 1** Consider \(u\) a solution of \((1.7)\) with blow-up graph \(\Gamma : \{x \mapsto T(x)\}\) and \(x_0\) a non characteristic point. Then, there exists \(t_0(x_0) \in [0, T(x_0))\) and \(q = q(a, p, N) > 0\)
such that, for all } T_0 \in (t_0(x_0), T(x_0)], \text{ for all } s \geq \log(T_0 - t_0(x_0)) \text{ and } x \in \mathbb{R}^N \text{ where } |x - x_0| \leq \frac{\delta_0(x_0)}{s_0(x_0)}, \text{ we have}
\int_s^{s+1} \int_B \left( w^2(y, \tau) + |\nabla w(y, \tau)|^2 + (\partial_s w(y, \tau))^2 \right) dyd\tau \leq K_1 s^q, \quad (2.1)
where } w = w_{x,T^*(x)} \text{ is defined in } (1.14), \text{ with }
T^*(x) = T_0 - \delta_0(x_0)(x - x_0) \quad (2.2)
and } \delta_0(x_0) \text{ defined in } (1.7). \text{ Note that } K_1 \text{ depends on } p, a, N, \delta_0(x_0), T(x_0), t_0(x_0) \text{ and } \|u(t_0(x_0)), \partial_iu(t_0(x_0))\|_{H^1 \times L^2(B(x_0, \frac{T(x_0) - t_0(x_0)}{\delta_0(x_0)})�)}.

In order to prove this theorem, we need to construct a Lyapunov functional for equation (1.14). In order to do so, we start by introducing the following functionals:
\begin{align*}
E_N(w(s), s) &= \int_B \left( \frac{1}{2} (\partial_s w)^2 + \frac{1}{2} |\nabla w|^2 - \frac{1}{2} |y, \nabla w|^2 + \frac{p + 1}{(p - 1)^2} w^2 - e^{-\frac{2(p+1)}{p-1}} s^{\frac{2p}{p-1}} F(\phi w) \right) \rho(y) dy, \\
J_N(w(s), s) &= -\frac{1}{s} \int_B \partial_s w \rho(y) dy, \\
H_{N,m}(w(s), s) &= E_N(w(s), s) + m J_N(w(s), s),
\end{align*}
where } F \text{ is given by } (1.19) \text{ and } m > 0 \text{ is a sufficiently large constant that will be fixed later.}

As we see above, the target of this section is to prove, for some } m_0 \text{ large enough, that the energy } H_{m_0,N} \text{ satisfies the following inequality:
\begin{align*}
\frac{d}{ds} H_{m_0,N}(w(s), s) &\leq -\alpha \int_B (\partial_s w)^2 \rho(y) \left( \frac{1}{1 - |y|^2} + \frac{m_0(p + 3)}{2s} H_{m_0,N}(w(s), s) \right) + C e^{-2s}, \quad (2.4)
\end{align*}
which implies that } H_{m_0,N}(w(s), s) \text{ satisfies the following polynomial estimate:
\begin{align*}
H_{m_0,N}(w(s), s) &\leq K s^{\mu_0}, \quad (2.5)
\end{align*}
for some } K > 0 \text{ and } \mu_0 > 0.

In the remaining part of this section, we consider } u \text{ a solution of } (1.1) \text{ with blow-up graph } \Gamma : \{ x \rightarrow T(x) \} \text{ and } x_0 \text{ a non characteristic point. Let } T_0 \in (0, T(x_0)), \text{ for all } x \in \mathbb{R}^N \text{ such that } |x - x_0| \leq \frac{T_0}{\delta_0(x_0)}, \text{ where } \delta_0 \in (0, 1) \text{ is defined in } (1.7) \text{ and we write } w \text{ instead of } w_{x,T^*(x)} \text{ defined in } (1.14) \text{ with } T^*(x) \text{ given by } (2.2).

### 2.1 Classical energy estimates

In this subsection, we state two lemmas which are crucial for the construction of a Lyapunov functional. We begin with bounding the time derivative of } E_N(w(s), s) \text{ in the following lemma:
Lemma 2.1. For all \( s \geq \max(-\log T^*(x), 1) \), we have
\[
\frac{d}{ds} E_N(w(s), s) \leq -\frac{3\alpha}{2} \int_B (\partial_s w)^2 \frac{\rho(y)}{1 - |y|^2} dy + \frac{C}{s^{a+1}} \int_B |w|^{p+1} \log^a(2 + \phi^2 w^2) \rho(y) dy + \Sigma_1(s),
\]
where \( \Sigma_1(s) \) satisfies
\[
\Sigma_1(s) \leq C \int_B |\nabla w|^2 (1 - |y|^2) \rho(y) dy + \frac{C}{s^2} \int_B w^2 \rho(y) dy + C e^{-2s}.
\]

Proof: Multiplying (1.15) by \( \partial_s w \rho(y) \) and integrating over \( B \), we obtain
\[
\frac{d}{ds} E_N(w(s), s) = -2\alpha \int_B (\partial_s w)^2 \frac{\rho(y)}{1 - |y|^2} dy + 2p + 2 \frac{p+1}{p-1} e^{-\frac{2(p+1)s}{p-1}} \int_B F(\phi w) - \frac{\phi w f(\phi w)}{p+1} \rho(y) dy,
\]
\[
-2 \frac{2a}{p-1} e^{-\frac{2(p+1)s}{p-1}} \int_B (F(\phi w) - \frac{\phi w f(\phi w)}{2}) \rho(y) dy,
\]
\[
+ \gamma(s) \int_B w \partial_s w \rho(y) dy + \frac{2a}{(p-1)s} \int_B (\partial_s w)^2 \rho(y) dy,
\]
\[
+ \frac{2a}{(p-1)s} \int_B y \nabla w \partial_s w \rho(y) dy.
\]

Now, we control the terms \( \Sigma_1^1(s), \Sigma_1^2(s), \Sigma_1^3(s) \) and \( \Sigma_1^4(s) \). Note from (5.27), (A.25) and (A.26) that
\[
F(\phi w) - \frac{\phi w f(\phi w)}{p+1} \leq C + C \frac{\phi w}{s} f(\phi w),
\]
which implies, for all \( s \geq \max(-\log T^*(x), 1) \),
\[
\Sigma_1^1(s) \leq C e^{-\frac{2(p+1)s}{p-1}} \int_B \phi w f(\phi w) \rho(y) dy + C e^{-2s}.
\]

Let us recall, from the expression of \( \phi = \phi(s) \) defined in (1.18), that we have, for all \( s \geq \max(-\log T^*(x), 1) \),
\[
e^{-\frac{2(p+1)s}{p-1}} \phi w f(\phi w) = \frac{1}{s^a} |w|^{p+1} \log^a(2 + \phi^2 w^2).
\]

Thus, using (2.10) and (2.11), we obtain, for all \( s \geq \max(-\log T^*(x), 1) \),
\[
\Sigma_1^1(s) \leq \frac{C}{s^{a+1}} \int_B |w|^{p+1} \log^a(2 + \phi^2 w^2) \rho(y) dy + C e^{-2s}.
\]
Similarly, by (A.23) and (2.11), we obtain easily, for all $s \geq \max(-\log T^*(x), 1)$,

$$
\Sigma_i^2(s) \leq \frac{C}{s^{a+1}} \int_B |w|^{p+1} \log^a(2 + \phi^2 w^2) \rho(y) dy + Ce^{-2s}.
$$

(2.13)

By using the following basic inequality

$$ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2, \quad \forall \varepsilon > 0,$$

(2.14)

and the expression of $\gamma(s)$ defined in (1.17), we write, for all $s \geq \max(-\log T^*(x), 1)$

$$
\Sigma_i^3(s) + \Sigma_i^4(s) \leq \frac{1}{p - 1} \int_B (\partial_s w)^2 \frac{\rho(y)}{1 - |y|^2} dy + \frac{C}{s^2} \int_B (|\nabla w|^2 (1 - |y|^2) + w^2) \rho(y) dy.
$$

(2.15)

The result (2.6) and (2.7) follows immediately from (2.8), (2.12), (2.13) and (2.15), which ends the proof of Lemma 2.1.

Remark 2.1. By showing the estimate proved in Lemma 2.1 related to the so called natural functional $E_N(w(s), s)$, we have some nonnegative terms in the right-hand side of (2.6) and this does not allow to construct a decreasing functional (unlike the case of a pure power nonlinearity). The main problem is related to the nonlinear term

$$
\frac{1}{s^{a+1}} \int_B |w|^{p+1} \log^a(2 + \phi^2(s) w^2) \rho(y) dy = \frac{1}{s} \int_B we^{-\frac{2s}{p+1} s^{a+1} f(\phi(s)w) \rho(y) dy}.
$$

To overcome this problem, we adapt the strategy used in [22, 23, 19, 20, 21]. More precisely, by using the identity obtained by multiplying equation (1.1) by $w \rho(y)$, then integrating over $B$, we can introduce a new functional $H_{m,N}$, defined in (2.3) where $m > 0$ is sufficiently large and will be fixed such that $H_{m,N}$ satisfies a differential inequality similar to (1.21).

We are going to prove the following estimate on the functional $J_N(w(s), s)$.

Lemma 2.2. For all $s \geq \max(-\log T^*(x), 1)$, we have

$$
\frac{d}{ds} J_N(w(s), s) \leq \frac{p + 3}{2s} E_{0,N}(w(s), s) - \frac{p + 7}{4s} \int_B (\partial_s w)^2 \rho(y) dy
$$

$$
- \frac{p - 1}{4s} \int_B (|\nabla w|^2 - (y, \nabla w)^2) \rho(y) dy - \frac{p + 1}{2(p - 1)s} \int_B w^2 \rho(y) dy
$$

$$
- \frac{p - 1}{2(p + 1)s^{a+1}} \int_B |w|^{p+1} \log^a(2 + \phi^2 w^2) \rho(y) dy + \Sigma_2(s),
$$

where $\Sigma_2(s)$ satisfies

$$
\Sigma_2(s) \leq \frac{C}{\sqrt{s}} \int_B (\partial_s w)^2 \frac{\rho(y)}{1 - |y|^2} dy + \frac{C}{s^{a+1}} \int_{-1}^1 |\nabla w|^2 (1 - |y|^2) \rho(y) dy
$$

$$
+ \frac{C}{s^{a+1}} \int_B w^2 \rho(y) dy + \frac{C}{s^{a+2}} \int_B |w|^{p+1} \log^a(2 + \phi^2 w^2) \rho(y) dy + Ce^{-2s}.
$$

(2.16)
Proof: Note that \( J_N(w(s), s) \) is a differentiable function and that we get for all \( s \geq \max(-\log T^*(x), 1) \),

\[
\frac{d}{ds} J_N(w(s), s) = -\frac{1}{s} \int_B (\partial_s w)^2 \rho(y)dy - \frac{1}{s} \int_B w \partial_s^2 \rho(y)dy + \frac{1}{s^2} \int_B w \partial_s \rho(y)dy.
\]

From equation (1.15), we obtain

\[
\frac{d}{ds} J_N(w(s), s) = \frac{1}{s} \int_B ((\nabla w)^2 - |y, \nabla w|^2) \rho(y)dy - \frac{2}{s} \int_B \partial_s w y, \nabla \rho(y)dy
\]

\[
- \frac{1}{s^{a+1}} \int_{-1}^1 |w|^{p+1} \log^a (2 + \phi^2 w^2) \rho(y)dy - \frac{2a}{(p - 1)s} \int_B w y, \nabla \rho(y)dy
\]

\[
+ \frac{1}{s} \left( \frac{p + 3}{p - 1} - 2N - \frac{2a + 1 - p}{(p - 1)s} \right) \int_B w \partial_s w \rho(y)dy - \frac{1}{s} \int_B (\partial_s w)^2 \rho(y)dy
\]

\[
+ \frac{1}{s} \left( \frac{2(p + 2)}{(p - 1)^2} - \gamma(s) \right) \int_B w^2 \rho(y)dy + \frac{4\alpha}{s} \int_B w \partial_s w \frac{|y|^2 \rho(y)}{1 - |y|^2} dy.
\]

According to the expressions of \( E_0(w(s), s) \), \( \phi(s) \) defined in (2.3) and (1.18) and the identity (2.11) with some straightforward computation, we obtain (2.16) where

\[
\Sigma_2(s) = \Sigma_1^1(s) + \Sigma_2^2(s),
\]

and

\[
\Sigma_1^1(s) = \frac{p + 3}{2} e^{\frac{2(p+1)s}{p-1} - \frac{2a}{p-1}} \int_B \left( F(\phi w) - \frac{\phi w f(\phi w)}{p + 1} \right) \rho(y)dy,
\]

\[
\Sigma_2^2(s) = -\frac{2}{s} \int_B \partial_s w y, \nabla \rho(y)dy - \frac{\gamma(s)}{s} \int_B w^2 \rho(y)dy
\]

\[
+ \frac{1}{s} \left( \frac{p + 3}{p - 1} - 2N + \frac{p - 1 - 2a}{(p - 1)s} \right) \int_B w \partial_s w \rho(y)dy
\]

\[
+ \frac{4\alpha}{s} \int_B w \partial_s w \frac{|y|^2 \rho(y)}{1 - |y|^2} dy - \frac{2a}{(p - 1)s^2} \int_B w y, \nabla \rho(y)dy.
\]

We are going now to estimate the different terms of (2.18). Thanks to (2.11) and (2.9), we conclude that for all \( s \geq \max(-\log T^*(x), 1) \)

\[
\Sigma_1^1(s) \leq \frac{C}{s^{a+2}} \int_B |w|^{p+1} \log^a (2 + \phi^2 w^2) \rho(y)dy + C e^{-2s}.
\]

(2.19)

By using the inequality (2.14) and (1.17), we conclude that for all \( s \geq \max(-\log T^*(x), 1) \),

\[
\Sigma_2^2(s) \leq \frac{C}{\sqrt{s}} \int_B (\partial_s w)^2 \frac{\rho(y)}{1 - |y|^2} dy + \frac{C}{s^{\sqrt{s}}} \int_B |\nabla w|^2 (1 - |y|^2) \rho(y)dy
\]

\[
+ \frac{C}{s^{\sqrt{s}}} \int_B w^2 \frac{\rho(y)}{1 - |y|^2} dy.
\]

(2.20)
Let us recall from [31] the following Hardy type inequality
\[ \int_B w^2 |y|^2 \rho(y) / (1 - |y|^2) \, dy \leq C \int_B |\nabla w|^2 (1 - |y|^2) \rho(y) \, dy + C \int_B w^2 \rho(y) \, dy. \] (2.21)
(see the appendix in [31] for a proof). Using (2.21) and the fact that \( \rho(y) / (1 - |y|^2) = \rho(y) + \frac{|y|^2 \rho(y)}{1 - |y|^2} \), we get
\[ \int_B w^2 \rho(y) / (1 - |y|^2) \, dy \leq C \int_B |\nabla w|^2 (1 - |y|^2) \rho(y) \, dy + C \int_B w^2 \rho(y) \, dy. \] (2.22)
Thus, it follows from (2.20) and (2.22) that for all \( s \geq \max(-\log T^*(x), 1) \),
\[ \Sigma_2^2(s) \leq \frac{C}{\sqrt{s}} \int_B (\partial_s w)^2 \frac{\rho(y)}{1 - |y|^2} \, dy + \frac{C}{s^{\sqrt{s}}} \int_B |\nabla w|^2 (1 - |y|^2) \rho(y) \, dy + \frac{C}{s^{\sqrt{s}}} \int_B w^2 \rho(y) \, dy. \] (2.23)
Consequently, collecting (2.18), (2.19) and (2.23), one easily obtains that \( \Sigma_2(s) \) satisfies (2.17), which ends the proof of Lemma 2.2.

### 2.2 Existence of a decreasing functional for equation (1.15)

In this subsection, by using Lemmas 2.1 and 2.2, we are going to construct a decreasing functional for equation (1.15). Let us define the following functional:
\[ N_{m,N}(w(s), s) = s^{-\frac{m(p+3)}{2}} H_{m,N}(w(s), s) + \sigma(m) e^{-s}, \] (2.24)
where \( H_{m,N} \) is defined in (2.3), and \( m \) and \( \sigma = \sigma(m) \) are constants that will be determined later.

We now state the following proposition:

**Proposition 2.3.** There exist \( m_0 > 1, \sigma_0 > 0, S_1 \geq 1 \) and \( \lambda_1 > 0 \), such that for all \( s \geq \max(-\log T^*(x), S_1) \), we have the following inequality:
\[ N_{m_0,N}(w(s + 1), s + 1) - N_{m_0,N}(w(s), s) \leq - \frac{2}{(p - 1) s^b} \int_s^{s+1} \int_B (\partial_s w)^2 \rho(y) \, dy \, d\tau \]
\[ - \frac{\lambda_1}{s^{b+1}} \int_s^{s+1} \int_B (|\nabla w|^2 - (y, \nabla w)^2) \rho(y) \, dy \, d\tau \]
\[ - \frac{\lambda_1}{s^{\alpha+b+1}} \int_s^{s+1} \int_B |w|^{p+1} \log^a (1 + \phi^2 \omega^2) \rho(y) \, dy \, d\tau \]
\[ - \frac{\lambda_1}{s^{b+1}} \int_s^{s+1} \int_B w^2 \rho(y) \, dy \, d\tau, \] (2.25)
where
\[ b = \frac{m_0(p+3)}{2}. \] (2.26)

Moreover, there exists \( S_2 \geq S_1 \) such that for all \( s \geq \max(-\log T^*(x), S_2) \), we have
\[ N_{m_0,N}(w(s), s) \geq 0. \] (2.27)
We now choose $S = S_1(m_0, a, p, N)$ large enough ($S_1 \geq 1$), so that for all $s \geq S_1$, we have
\[
\frac{m_0(p - 1)}{8(p + 1)} - \frac{C_0}{s} \geq 0, \quad \frac{1}{p - 1} - \frac{C_0 m_0}{s^{a+1}} \geq 0, \quad \frac{m_0(p - 1)}{4} - \frac{C_0 m_0}{s^{a+1}} - \frac{C_0}{s} \geq 0, \quad \frac{p + 7}{8} - \frac{C_0}{s^{a+1}} \geq 0, \quad \frac{m_0(p + 1)}{4(p - 1)} - \frac{C_0 m_0}{s^{a+1}} - \frac{C_0}{s} \geq 0.
\]
Then, we deduce that for all $s \geq \max(-\log T^*(x), S_1)$,
\[
\frac{d}{ds} H_{m_0,N}(w(s), s) \leq -\frac{2}{p - 1} \int_B (\partial_s w)^2 \rho(y) dy + \frac{m_0(p + 3)}{2s} H_{m_0,N}(w(s), s) + \lambda_0 \int_B \frac{w^{p+1} \log^a (2 + \phi^2(s) w^2)}{1 - |y|^2} \rho(y) dy
\]
By integrating in time between $s$ and $s + 1$ the inequality (2.31) and using (2.32), we easily obtain (2.25). This concludes the proof of the first part of Proposition 2.3.

We prove (2.27) here. The argument is the same as in the corresponding part in [22, 23, 19, 20, 21]. We write the proof for completeness. Arguing by contradiction, we assume that there exists $s_1 > 0$ such that for all $(y, s) \in B \times [1 + s_1, +\infty)$

\[
\frac{\partial}{\partial s} \log(w(y, s)) \leq -|\nabla w(y, s)|^2 - (y, \nabla w(y, s))^2 + w^2 \rho(y) dy - e^{-s} \left(\sigma - C_0(m_0 + 1) e^{-s}\right).
\]

We now choose $\sigma = C_0(m_0 + 1) e^{-s_1}$, so we have, for all $s \geq S_1$

\[
\sigma - C_0(m_0 + 1) e^{-s} \geq 0.
\]

By integrating in time between $s$ and $s + 1$ the inequality (2.31) and using (2.32), we easily obtain (2.25). This concludes the proof of the first part of Proposition 2.3.

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\[
\frac{\partial}{\partial s} \log(w(y, s)) \leq -|\nabla w(y, s)|^2 - (y, \nabla w(y, s))^2 + w^2 \rho(y) dy - e^{-s} \left(\sigma - C_0(m_0 + 1) e^{-s}\right).
\]

We now choose $\sigma = C_0(m_0 + 1) e^{-s_1}$, so we have, for all $s \geq S_1$

\[
\sigma - C_0(m_0 + 1) e^{-s} \geq 0.
\]

By integrating in time between $s$ and $s + 1$ the inequality (2.31) and using (2.32), we easily obtain (2.25). This concludes the proof of the first part of Proposition 2.3.

We prove (2.27) here. The argument is the same as in the corresponding part in [22, 23, 19, 20, 21]. We write the proof for completeness. Arguing by contradiction, we assume that there exists $s_1 > 0$ such that for all $(y, s) \in B \times [1 + s_1, +\infty)$

\[
\frac{\partial}{\partial s} \log(w(y, s)) \leq -|\nabla w(y, s)|^2 - (y, \nabla w(y, s))^2 + w^2 \rho(y) dy - e^{-s} \left(\sigma - C_0(m_0 + 1) e^{-s}\right).
\]

We now choose $\sigma = C_0(m_0 + 1) e^{-s_1}$, so we have, for all $s \geq S_1$

\[
\sigma - C_0(m_0 + 1) e^{-s} \geq 0.
\]
So, by (2.24), we have
\[ N_{m_0,N}(\tilde{w}^\delta_1(s), s) \geq -e^{-2(s+1)s} s^2 \int_B F(\phi \tilde{w}^\delta_1) \rho(y) dy. \]

Due to (A.24), we infer,
\[ N_{m_0,N}(\tilde{w}^\delta_1(s), s) \geq -Ce^{-2s} \int_B |\phi \tilde{w}^\delta_1| |\phi| dy - Ce^{-2s}. \]

(2.36)

Notice that, after a change of variables defined in (2.33), we find that
\[ T_\delta(s) = -\frac{1}{2} \int_B |\phi \tilde{w}^\delta_1| |\phi| dy - \frac{1}{2} \int_B |\phi \tilde{w}^\delta_1| |\phi| dy + \frac{1}{2} \int_B |\phi \tilde{w}^\delta_1| |\phi| dy. \]

Thus (2.27) holds. This concludes the proof of Proposition 2.3.

2.3 Proof of Theorem 1

We define the following time:
\[ t_0(x_0) = \max(T(x_0) - e^{-s_2}, 0). \]

(2.39)

According to the Proposition 2.3, we obtain the following corollary which summarizes the principle properties of \( N_{m_0,N}(w(s), s) \) defined in (2.24).

**Corollary 2.4. (Estimate on \( N_{m_0,N}(w(s), s) \)).** There exists \( t_0(x_0) \in [0, T(x_0)] \) such that, for all \( T_0 \in (t_0(x_0), T(x_0)] \), for all \( s \geq -\log(T_0 - t_0(x_0)) \) and \( x \in \mathbb{R}^N \) where \( |x - x_0| \leq \frac{1}{\delta_0(x_0)} \), we have
\[ 0 \leq N_{m_0,N}(w(s), s) \leq N_{m_0,N}(w(\hat{s}_0), \hat{s}_0), \]
\[ \int_s^{s+1} \int_B \left( |\nabla w(y, \tau)|^2 (1 - |y|^2) + \frac{(\partial_y w(y, \tau))^2}{1 - |y|^2} + w^2(y, \tau) \right) \rho(y) dy d\tau \leq C \left( 1 + N_{m_0,N}(w(\hat{s}_0), \hat{s}_0) \right) s^{k+1}, \]

(2.40)

(2.41)

where \( w = w_{x,T^*(x)} \) is defined in (1.14), with \( T^*(x) \) given in (2.2) where \( \hat{s}_0 = -\log(T^*(x) - t_0(x_0)) \) and \( b \) is defined in (2.26).
Proposition 3.1. Consider $u$ a solution of (1.14) with blow-up graph $\Gamma : \{x \mapsto T(x)\}$ and $x_0$ a non characteristic point. Then, there exists $t_0(x_0) \in [0, T(x_0))$ and $q_1 = q_1(p, a, N) > 0$ such that, for all $t_0 \in (t_0(x_0), T(x_0)]$, for all $s \geq -\log(T_0 - t_0(x_0))$ and $x \in \mathbb{R}$ where $|x - x_0| \leq \frac{\tilde{\delta}(x_0)}{2}$, we have

$$
\|w(s)\|_{H^1((-1, 1))} + \|\partial_s w(s)\|_{L^2((-1, 1))} \leq K_2 s^{q_1},
$$

where $w = w_{x, T^*(x)}$ is defined in (1.14), with $T^*(x)$ given in (2.14), $K_2$ depends on $p, a, \delta_0(x_0), T(x_0), t_0(x_0)$ and $\|u(t_0(x_0)), \partial_t u(t_0(x_0))\|_{H^1 \times L^2(I(x_0, T(x_0) - t_0(x_0)))}$.

Remark 3.1. By using the Sobolev’s embedding in one dimension space and the above proposition, we can deduce that

$$
\|w(s)\|_{L^\infty((-1, 1))} \leq K s^{q_1}, \quad \text{for all} \quad s \geq -\log(T^*(x) - t_0(x_0)).
$$

3 Proof of Theorem 3 and Theorem 4

In this section, we consider the one space dimensional case ($N = 1$). We prove Theorem 3 and Theorem 4 here. Before doing that, since we consider the one space dimensional case and thanks to Theorem 1, we first prove a polynomial estimate. This section is divided into three parts:

- In subsection 3.1 we prove a polynomial estimate.
- In subsection 3.2 we state a general version of Theorem 4 uniform for $x$ near $x_0$ and prove it.
- In subsection 3.3 we prove Theorem 4

3.1 Polynomial estimate

Proposition 3.1. Consider $u$ a solution of (1.14) with blow-up graph $\Gamma : \{x \mapsto T(x)\}$ and $x_0$ a non characteristic point. Then, there exists $t_0(x_0) \in [0, T(x_0))$ and $q_1 = q_1(p, a, N) > 0$ such that, for all $t_0 \in (t_0(x_0), T(x_0)]$, for all $s \geq -\log(T_0 - t_0(x_0))$ and $x \in \mathbb{R}$ where $|x - x_0| \leq \frac{\tilde{\delta}(x_0)}{2}$, we have

$$
\|w(s)\|_{H^1((-1, 1))} + \|\partial_s w(s)\|_{L^2((-1, 1))} \leq K_2 s^{q_1},
$$

where $w = w_{x, T^*(x)}$ is defined in (1.14), with $T^*(x)$ given in (2.14), $K_2$ depends on $p, a, \delta_0(x_0), T(x_0), t_0(x_0)$ and $\|u(t_0(x_0)), \partial_t u(t_0(x_0))\|_{H^1 \times L^2(I(x_0, T(x_0) - t_0(x_0)))}$.

Remark 3.1. By using the Sobolev’s embedding in one dimension space and the above proposition, we can deduce that

$$
\|w(s)\|_{L^\infty((-1, 1))} \leq K s^{q_1}, \quad \text{for all} \quad s \geq -\log(T^*(x) - t_0(x_0)).
$$
Proof of Proposition 3.1: We proceed in 2 steps:

- In step 1, we use the covering technique and the Sobolev’s embedding in two dimensions (space-time) to conclude a polynomial estimate related to the $L^{p+2}(-1,1)$ norm of $w(s)$.

- In step 2, by exploiting the result obtained in step 1 and the fact that $N_{m,0,1}(w(s),s)$ (defined in (2.24)) is a decreasing functional, we easily conclude the estimate (3.1).

**Step 1:** By using Theorem 1', we get for all $s \geq -\log(T^*(x) - t_0(x_0))$,

$$\int_s^{s+1} \int_{-1}^{1} \left( (\partial_s w(y,\tau))^2 + (\partial_y w(y,\tau))^2 + w^2(y,\tau) \right) dyd\tau \leq K_1 s^q. \quad (3.3)$$

Now, we use the Sobolev’s embedding in two dimensions (space-time) and (3.3) to conclude a polynomial estimate related to the $L^{p+2}(-1,1)$ norm of $w(s)$. Indeed, for all $s \geq -\log(T^*(x) - t_0(x_0))$, by using the mean value theorem, we derive the existence of $\sigma(s) \in [s, s+1]$ such that

$$\int_{-1}^{1} |w(y,\sigma(s))|^{p+2} dy = \int_{s}^{s+1} \int_{-1}^{1} |w(y,\tau)|^{p+2} dyd\tau. \quad (3.4)$$

Let us write the identity for all $s \geq -\log(T^*(x) - t_0(x_0))$,

$$\int_{-1}^{1} |w(y,s)|^{p+2} dy = \int_{-1}^{1} |w(y,\sigma(s))|^{p+2} dy + \int_{\sigma(s)}^{s} \frac{d}{d\tau} \int_{-1}^{1} |w(y,\tau)|^{p+2} dyd\tau. \quad (3.5)$$

By combining (3.4), (3.5) and (2.14), we infer for all $s \geq -\log(T^*(x) - t_0(x_0))$,

$$\int_{-1}^{1} |w(y,s)|^{p+2} dy \leq \int_{s}^{s+1} \int_{-1}^{1} |w(y,\tau)|^{p+2} dyd\tau + C \int_{s}^{s+1} \int_{-1}^{1} |w(y,\tau)|^{2p+2} dyd\tau$$

$$+ C \int_{s}^{s+1} \int_{-1}^{1} (\partial_s w(y,\tau))^2 dyd\tau. \quad (3.6)$$

By using Sobolev’s inequalities in two dimension (space time) and (3.3), we conclude that

$$\int_{s}^{s+1} \int_{-1}^{1} |w(y,\tau)|^{2p+2} dyd\tau \leq C \left( \int_{s}^{s+1} \int_{-1}^{1} \left( (\partial_s w)^2 + (\partial_y w)^2 + w^2 \right) dyd\tau \right)^{p+1} \leq K_1 s^{q(p+1)}. \quad (3.7)$$

Due to the classical inequality $x^{p+2} \leq 1 + x^{2p+2}$, for all $x \geq 0$, we have

$$\int_{s}^{s+1} \int_{-1}^{1} |w(y,\tau)|^{p+2} dyd\tau \leq C + C \int_{s}^{s+1} \int_{-1}^{1} |w(y,\tau)|^{2p+2} dyd\tau. \quad (3.8)$$

By combining (3.6), (3.7), (3.8) and (3.3), we deduce for all $s \geq -\log(T^*(x) - t_0(x_0))$, that

$$\int_{-1}^{1} |w(y,s)|^{p+2} dy \leq K_1 s^{q(p+1)}. \quad (3.9)$$
Step 2: From (A.23), (2.11), this yields
\[
e^{-\frac{2(p+1)x}{p-1}} s^{\frac{2a}{p-1}} \int_{-1}^{1} F(\phi w) \rho(y) dy \leq C \int_{-1}^{1} |w(y,s)|^{p+1} dy + Ce^{-2s}. \tag{3.10}
\]
To estimate the right-hand side in the inequality (3.10), we consider two cases:

Case 1: the case where \(a \geq 0\).

From this inequality \(2 + x^2 y^2 \leq (2 + x^2)(2 + y^2)\), for all \(x, y \in \mathbb{R}\), and the fact that \(\log^a\) is an increasing function on the interval \([2, \infty)\), we conclude that
\[
\log^a(2 + x^2 y^2) \leq (\log(2 + x^2) + \log(2 + y^2))^a. \tag{3.11}
\]
Using the inequality \((X + Y)^a \leq C(X^a + Y^a)\), for all \(X, Y \in \mathbb{R}_+\) and (3.11), we obtain
\[
\log^a(2 + x^2 y^2) \leq C \log^a(2 + x^2) + C \log^a(2 + y^2). \tag{3.12}
\]
By combining (3.12) and the inequality \(\log^a(2 + z^2) \leq C + C|z|\), for all \(z \in \mathbb{R}\), we conclude that
\[
\log^a(2 + x^2 y^2) \leq C \log^a(2 + x^2) + C + C|y|. \tag{3.13}
\]
Hence, by taking into account (3.13) and (1.18), we deduce that
\[
\frac{1}{s^a} \int_{-1}^{1} |w|^{p+1} \log^a(2 + \phi^2 w^3) dy \leq C \int_{-1}^{1} |w|^{p+1} dy + \frac{C}{s^a} \int_{-1}^{1} |w|^{p+2} dy. \tag{3.14}
\]
Therefore, using (3.9), (3.10), (3.14) and Jensen’s inequality, we get
\[
e^{-\frac{2(p+1)x}{p-1}} s^{\frac{2a}{p-1}} \int_{-1}^{1} F(\phi w) \rho(y) dy \leq K s^{q(p+1)}. \tag{3.15}
\]

Case 2: the case where \(a < 0\).

Using (3.10), we get
\[
e^{-\frac{2(p+1)x}{p-1}} s^{\frac{2a}{p-1}} \int_{-1}^{1} F(\phi w) \rho(y) dy \leq C \int_{-1}^{1} |w(y,s)|^{p+1} dy + Ce^{-2s}. \tag{3.16}
\]
By Jensen’s inequality and (3.9), we conclude that
\[
e^{-\frac{2(p+1)x}{p-1}} s^{\frac{2a}{p-1}} \int_{-1}^{1} F(\phi w) \rho(y) dy \leq K s^{q(p+1)-a}. \tag{3.17}
\]
Thanks to (3.15) and (3.17), we deduce for all \(a \in \mathbb{R}\), \(s \geq -\log(T^*(x) - t_0(x_0))\),
\[
e^{-\frac{2(p+1)x}{p-1}} s^{\frac{2a}{p-1}} \int_{-1}^{1} F(\phi w) \rho(y) dy \leq K s^{q(p+1)+|a|}. \tag{3.18}
\]
Now, we use (2.40), (2.42), (2.26), the fact that \(b + 1 = q\) and the definition of \(N_{m_0,1}(w(s), s)\) defined in (2.24), to conclude for all \(s \geq -\log(T^*(x) - t_0(x_0))\),
\[
H_{m_0,1}(w(s), s) \leq K s^b \leq K s^a, \tag{3.19}
\]
where $H_{m_0,1}$ is defined in (2.3). Thanks to (3.18) and the definition of $H_{m_0,1}(w(s), s)$, we deduce for all $s \geq -\log(T^*(x) - t_0(x_0))$,

$$
\int_{-1}^{1} \left( (\partial_x w)^2 + (\partial_y w)^2 (1 - y^2) + w^2 \right) \rho(y) \, dy \leq K s^{q(p+1)+|a|}.
$$

(3.20)

Note that the estimate (3.20) implies 3.1 (take $q_1 = q(p+1) + |a|$) but just in $(-\frac{1}{2}, \frac{1}{2})$. By using the covering technique, we extend this estimate from $(-\frac{1}{2}, \frac{1}{2})$ to $(-1, 1)$, we refer the reader to Merle and Zaag 32 (unperturbed case) and Hamza and Zaag 22 (perturbed case). This concludes the Proposition 3.1.

### 3.2 A Lyapunov functional

In this subsection, our aim is to construct a Lyapunov functional for equation (1.15). Note that this functional is far from being trivial and makes our main contribution. More precisely, thanks to the rough estimate obtained in the Proposition 3.1, we derive here that the functional $L(w(s), s)$ defined in (1.24) is a decreasing functional of time for equation (1.15), provided that $s$ is large enough.

Let us remark that in Section 2, we construct a Lyapunov functional $N_{m_0,1}(w(s), s)$ defined in (2.24), but we obtain just a rough estimate because the multiplier is not bounded. Nevertheless, the multiplier related to the functional $L(w(s), s)$ is nonnegative and bounded. Then, as we said above, the natural energy $E_1(w(s), s)$ defined in (1.23) is a small perturbation of $L(w(s), s)$.

Consider $u$ a solution of (1.1) with blow-up graph $\Gamma : \{x \mapsto T(x)\}$ and $x_0$ a non characteristic point. Let $T_0 \in (t_0(x_0), T(x_0)]$. For all $x \in \mathbb{R}$ such that $|x-x_0| \leq \frac{T_0-t_0(x_0)}{\delta_0(x_0)}$, we write $w$ instead of $w_{x,T^*(x)}$ defined in (1.14) with $T^*(x)$ given by (2.2). Thanks to estimate (3.1), we can improve estimate (2.6) related to the control of the time derivative of the functional $E_1(w(s), s)$. More precisely, we prove the following lemma:

**Lemma 3.2.** For all $s \geq -\log(T^*(x) - t_0(x_0))$, we have

$$
\frac{d}{ds} E_1(w(s), s) \leq -\frac{3}{p-1} \int_{-1}^{1} (\partial_x w)^2 \frac{\rho(y)}{1-y^2} \, dy
$$

$$
+ \frac{K \log s}{s^{a+2}} \int_{-1}^{1} |w|^{p+1} \log^a (2 + \phi^2 w^2) \rho(y) \, dy
$$

$$
+ \frac{C}{s^q} \int_{-1}^{1} (\partial_y w)^2 (1 - y^2) \rho(y) \, dy + \frac{C}{s^q} \int_{-1}^{1} w^2 \rho(y) \, dy + Ce^{-s}.
$$

(3.22)

**Proof:** Since we consider the one space dimension and by using the additional information obtained in Subsection 3.1, we are going to refine the estimate related to $\Sigma_1^1(s)$ and $\Sigma_2^2(s)$ defined in (2.8). Let us mention that the estimate (2.15) related to $\Sigma_1^3(s) + \Sigma_1^4(s)$ defined in (2.8) is acceptable and does not need any improvement. More precisely, we write

$$
\Sigma_1^1(s) + \Sigma_2^2(s) = \frac{2p + 2}{p-1} e^{-\frac{2(p+1)s}{p-1}} s^{\frac{2p}{p-1}} \int_{-1}^{1} \left( F(\phi w) - \frac{\phi(s)wf(\phi w)}{p+1} \right) \rho(y) \, dy
$$

(3.23)
\[- \frac{2a}{p - 1} e^{- \frac{2(p+1)x}{p-1} s^{p-1} - \frac{2a}{p-1} - \int_{-1}^{1} (F(\phi w) - \frac{\phi w f(\phi w)}{2}) \rho(y) dy.\]

We attempt to group the main terms together. A straightforward computations implies that

\[\Sigma_1^1(s) + \Sigma_2^1(s) = \chi_1(s) + \chi_2(s),\] (3.23)

where

\[\chi_1(s) = \frac{a}{(p+1)s^{a+1}} \int_{A_1(s)} |w|^{a+1} \log^a(2 + \phi^2 w^2) \left(1 - \frac{4s}{(p-1) \log(2 + \phi^2 w^2)}\right) \rho(y) dy,\] (3.24)

\[\chi_2(s) = \frac{2e^{-2(p+1)s}}{p-1} s^{p-1} - \int_{A_2(s)} \left(\frac{a}{s} F_2(\phi w) - \frac{a}{s} F_1(\phi w)\right) \rho(y) dy,\] (3.25)

\[F_1(x) = - \frac{2a}{(p+1)^2} |x|^{p+1} \log^{a-1}(2 + x^2),\] (3.26)

and

\[F_2(x) = F(x) - \frac{xf(x)}{p+1} - F_1(x).\] (3.27)

We would like now to find an estimate for the term \(\chi_1(s)\). For this, for all \(s \geq -\log(T^*(x) - t_0(x_0))\), we divide \((-1,1)\) into two parts

\[A_1(s) = \{y \in (-1,1) \mid \phi(s) w^2(y, s) \leq 1\}\] and \(A_2(s) = \{y \in (-1,1) \mid \phi(s) w^2(y, s) \geq 1\}\). (3.28)

Accordingly, we write \(\chi_1(s) = \chi_1^1(s) + \chi_1^2(s)\), where

\[\chi_1^1(s) = \frac{a}{(p+1)s^{a+1}} \int_{A_1(s)} |w|^{a+1} \log^a(2 + \phi^2 w^2) \left(1 - \frac{4s}{(p-1) \log(2 + \phi^2 w^2)}\right) \rho(y) dy,\] (3.29)

\[\chi_1^2(s) = \frac{a}{(p+1)s^{a+1}} \int_{A_2(s)} |w|^{a+1} \log^a(2 + \phi^2 w^2) \left(1 - \frac{4s}{(p-1) \log(2 + \phi^2 w^2)}\right) \rho(y) dy.\] (3.30)

Note that, by using the definition of the set \(A_1(s)\) given in (3.28), we get, for all \(s \geq -\log(T^*(x) - t_0(x_0))\),

\[|w|^{p+1} \log^a(2 + \phi^2 w^2) \leq C_{\phi} \left(\frac{4s}{p-1}\right) (s) \log^a(2 + \phi(s)) \leq C e^{-s}.\] (3.31)

From (3.31) and the fact that

\[1 - \frac{4s}{(p-1) \log(2 + \phi^2 w^2)} \leq C,\] (3.32)
we get
\[ \chi_1^1(s) \leq C e^{-s}. \] (3.33)
Next, by using the definition of the set \( A_2(s) \) defined in (3.28), we write for all \( s \geq -\log(T^*(x) - t_0(x_0)) \),
\[ 1 - \frac{4s}{(p-1) \log(2 + \phi^2 w^2)} = \frac{1}{\log(2 + \phi^2 w^2)} \left( \log(2 + \phi^2 w^2) - \frac{4s}{p-1} \right). \] (3.34)
Here, the estimate proved in Subsection 3.1 is crucial to conclude. More precisely, by exploiting the expression of \( \phi \) given in (1.13) and the estimate (3.2), we conclude that
\[ \log(2 + \phi^2 w^2) - \frac{4s}{p-1} \leq K \log s. \] (3.35)
Also, by using the definition of the set \( A_2(s) \) defined in (3.28), we can write for all \( s \geq -\log(T^*(x) - t_0(x_0)) \),
\[ 1 - \frac{4s}{(p-1) \log(2 + \phi^2 w^2)} \leq K \frac{\log s}{s}. \] (3.37)
Adding (3.37) and (3.30), we have
\[ \chi_2^1(s) \leq \frac{K \log s}{s^{a+2}} \int_{-1}^1 |w|^{p+1} \log^a(2 + \phi^2 w^2) \rho(y) dy. \] (3.38)
Note that, by using the fact \( \chi_1(s) = \chi_1^1(s) + \chi_2^1(s) \), (3.33) and (3.38), we get
\[ \chi_1(s) \leq \frac{K \log s}{s^{a+2}} \int_{-1}^1 |w|^{p+1} \log^a(2 + \phi^2 w^2) \rho(y) dy + C e^{-s}. \] (3.39)
Finally, it remains only to control the term \( \chi_2(s) \). Note from (A.25) and (A.26) that
\[ \frac{1}{s} |F_1(\phi w)| + |F_2(\phi w)| \leq C + \frac{\phi w}{s^2} f(\phi w). \] (3.40)
By (3.25), (3.40) and (2.11), we have, for all \( s \geq -\log(T^*(x) - t_0(x_0)) \),
\[ \chi_2(s) \leq \frac{C}{s^{a+2}} \int_{-1}^1 |w|^{p+1} \log^a(2 + \phi^2 w^2) \rho(y) dy + C e^{-2s}. \] (3.41)
The result (3.21) derives immediately from (2.8), (2.15), (3.39), (3.41), and the identity (3.23), which ends the proof of Lemma 3.2.

With Lemmas 2.2 and 3.2, we are in a position to state and prove Theorem 3', which is a uniform version of Theorem 3 for \( x \) near \( x_0 \).
Theorem 3 (Existence of a Lyapunov functional for equation (1.15))
Consider \( u \) a solution of (1.1) with blow-up graph \( \Gamma : \{ x \mapsto T(x) \} \) and \( x_0 \) a non characteristic point. Then there exists \( t_1(x_0) \in [0, T(x_0)) \) such that, for all \( T_0 \in (t_1(x_0), T(x_0)] \), for all \( s \geq -\log(T_0 - t_1(x_0)) \) and \( x \in \mathbb{R} \), where \( |x - x_0| \leq \frac{e^{-s}}{w(x_0)} \), we have

\[
L(w(s + 1), s + 1) - L(w(s), s) \leq -\frac{2}{p - 1} \int_{s}^{s+1} \int_{-1}^{1} (\partial_s w)^2 \frac{\rho(y)}{1 - y^2} dy dr,
\]
where \( w = w_{x,T^*(x)} \) and \( T^*(x) \) is defined in (2.2).

Proof of Theorem 3: By exploiting the definition of \( L_0(w(s), s) \) in (1.23), we can write easily

\[
\frac{d}{ds} L_0(w(s), s) = \frac{d}{ds} E_1(w(s), s) + \frac{1}{\sqrt{s}} \frac{d}{ds} J_1(w(s), s) - \frac{1}{2s\sqrt{s}} J_1(w(s), s),
\]
where \( J_1(w(s), s) = \frac{1}{s} \int_{-1}^{1} w \partial_s w \rho(y) dy \). Lemmas 2.2 and 3.2 and the following inequality

\[
\frac{1}{2s^2\sqrt{s}} \int_{-1}^{1} w \partial_s w \rho(y) dy + \frac{p + 3}{2s^3} \int_{-1}^{1} \partial_y w \rho(y) dy \leq \frac{C}{s^2} \int_{-1}^{1} (\partial_s w)^2 \rho(y) dy + \frac{C}{s^2} \int_{-1}^{1} w^2 \rho(y) dy,
\]
allows to prove that for all \( s \geq -\log(T^*(x) - t_0(x_0)) \), we have

\[
\frac{d}{ds} L_0(w(s), s) \leq -(\frac{3}{p - 1} - \frac{C}{s}) \int_{-1}^{1} (\partial_s w)^2 \frac{\rho(y)}{1 - y^2} dy + \frac{p + 3}{2s\sqrt{s}} L_0(w(s), s)
- \frac{1}{s\sqrt{s}} \frac{(p + 1)}{2(p - 1)} - \frac{C}{\sqrt{s}} \int_{-1}^{1} w^2 \rho(y) dy
- \frac{1}{s\sqrt{s}} \frac{(p + 7)}{4} - \frac{C}{\sqrt{s}} \int_{-1}^{1} (\partial_s w)^2 \rho(y) dy
- \frac{1}{s\sqrt{s}} \frac{(p - 1)}{4} - \frac{C}{\sqrt{s}} \int_{-1}^{1} (\partial_y w)^2 (1 - y^2) \rho(y) dy
- \frac{1}{s^{a+\frac{3}{2}}} \frac{(p - 1)}{4} - \frac{K}{s} \int_{-1}^{1} |w|^{p+1} \log(2 + \varphi^2 w^2) \rho(y) dy
- \frac{C}{\sqrt{s}} \int_{-1}^{1} w^{p+1} \log(2 + \varphi^2 w^2) \rho(y) dy
+ C e^{-2s}\sqrt{s} + Ce^{-s}.
\]

Again, choosing \( S_3 > -\log(T(x_0) - t_0(x_0)) \) large enough, this implies that for all \( s \geq \max(-\log(T^*(x) - t_0(x_0)), S_3) \), we have

\[
\frac{d}{ds} L_0(w(s), s) \leq -\frac{2}{p - 1} \int_{-1}^{1} (\partial_s w)^2 \frac{\rho(y)}{1 - y^2} dy + \frac{p + 3}{2s\sqrt{s}} L_0(w(s), s) + Ce^{-s}.
\]
Recalling that,

\[
L(w(s), s) = \exp\left(\frac{p + 3}{\sqrt{s}}\right) L_0(w(s), s) + \theta e^{-s},
\]

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we get from straightforward computations

\[
\frac{d}{ds} L(w(s), s) = \frac{-p + 3}{2s\sqrt{s}} \exp \left( \frac{p + 3}{\sqrt{s}} \right) L_0(w(s), s) + \exp \left( \frac{p + 3}{\sqrt{s}} \right) \frac{d}{ds} L_0(w(s), s) - \theta e^{-s}. 
\]

Therefore, estimates (3.44) and (3.45) lead to the following crucial estimate:

\[
\frac{d}{ds} L(w(s), s) \leq -\frac{2}{p-1} \exp \left( \frac{p + 3}{\sqrt{s}} \right) \left( \int_{-1}^{1} (\partial_s w)^2 \frac{\rho(y)}{1 - y^2} dy + \left( C \exp \left( \frac{p + 3}{\sqrt{s}} \right) - \theta \right) e^{-s} \right).
\]

(3.46)

Since we have \(1 \leq \exp \left( \frac{p + 3}{\sqrt{s}} \right) \leq \exp \left( \frac{p + 3}{\sqrt{s_3}} \right)\), we then choose \(\theta\) large enough, so that \(C - \theta \leq 0\), which yields, for all \(s \geq \max(-\log(T^*(x) - t_0(x_0)), S_3)\),

\[
\frac{d}{ds} L(w(s), s) \leq -\frac{2}{p-1} \left( \int_{-1}^{1} (\partial_s w)^2 \frac{\rho(y)}{1 - y^2} dy \right) - \left( C \exp \left( \frac{p + 3}{\sqrt{s}} \right) - \theta \right) e^{-s}.
\]

A simple integration between \(s\) and \(s + 1\) ensures the result (3.42), where

\[
t_1(x_0) = \max(T(x_0) - e^{-s_3}, t_0(x_0)).
\]

(3.47)

This concludes the proof of Theorem 3. \hfill \blacksquare

We now claim the following lemma:

**Lemma 3.3.** There exists \(S_4 \geq S_3\) such that, if \(L(w(s_3), s_3) < 0\) for some \(s_3 \geq \max(S_4, -\log(T^*(x) - t_1(x_0)))\), then \(w\) blows up in some finite time \(s_4 > s_3\).

**Proof:** The argument is the same as the similar part in Proposition 2.3 in this paper. \hfill \blacksquare

### 3.3 Proof of Theorem 4

In this subsection, we prove Theorem 4. Note that the lower bound follows from the finite speed of propagation and the wellposedness in \(H^1 \times L^2\). For a detailed argument in the similar case of equation (1.8), see Lemma 3.1 (page 1136) in [32].

We consider \(u\) a solution of (1.1) which is defined under the graph of \(x \mapsto T(x)\), and \(x_0\) a non characteristic point. Let

\[
t_2(x_0) = \max(T(x_0) - e^{-s_4}, t_1(x_0)).
\]

(3.48)

Given some \(T_0 \in (t_2(x_0), T(x_0)]\), for all \(x \in \mathbb{R}\) is such that \(|x - x_0| \leq \frac{T_0 - t_2(x_0)}{\delta_0(x_0)}\), where \(\delta(x_0)\) is defined in (1.7), we aim at bounding \(\|(w, \partial_s w)(s)\|_{H^1 \times L^2((-1,1))}\) for \(s\) large.

As in [23, 20], by combining Theorem 3 and Lemma 3.3 we get the following bounds:

**Corollary 3.4.** (Bound on \(L_0(w(s), s)\)). For all \(T_0 \in (t_2(x_0), T(x_0)]\), for all \(s \geq -\log(T_0 - t_2(x_0))\) and \(x \in \mathbb{R}\) where \(|x - x_0| \leq \frac{e^{-s}}{\delta_0(x_0)}\), we have

\[
-C \leq L_0(w(s), s) \leq CL_0(w(\bar{s}_2), \bar{s}_2) + C,
\]

(3.49)
where \( \hat{s}_2 = -\log(T^*(x) - t_2(x_0)) \).

Moreover, for all \( s \geq -\log(T^*(x) - t_2(x_0)) \), we have

\[
\int_s^{s+1} \int_{-1}^{1} (\partial_s w)^2 \rho(y) \frac{1}{1-y^2} dy ds \leq K, \tag{3.50}
\]

where \( K = K(a, p, T^*(x), \| (u(t_2), u_t(t_2)) \|_{H^1 \times L^2(I(x_0, t_2(x_0)))}), \ C = C(a, p) \) and \( \delta_0(x_0) \in (0, 1) \) is defined in \((1.7)\).

Remark 3.2. Using the definition of \((1.14)\) of \( w_{x,T^*(x)} = w \), we write easily

\[
L_0(w(\hat{s}_2), \hat{s}_2) \leq \hat{K}_1, \tag{3.51}
\]

where \( \hat{K}_1 = \hat{K}_1(T(x_0) - t_2(x_0), \| (u(t_2(x_0)), \partial_t u(t_2(x_0))) \|_{H^1 \times L^2(I(x_0, T(x_0) - t_2(x_0)))}) \).

Starting from these bounds, the proof of Theorem \([3]\) is similar to the proof in \([31, 32]\) except for the treatment of the nonlinear terms and of the perturbation terms. In our opinion, handling these terms is straightforward in all the steps of the proof, except for the first step, where we bound the time averages of the \( L_{p+1}^{p+1}((-1, 1)) \) norm of \( w \). For that reason, we only give that step and refer to \([31, 32]\) for the remaining steps in the proof of Theorem \([3]\). This is the step we prove here.

**Proposition 3.5.** For all \( s \geq 1 - \log(T^*(x) - t_2(x_0)) \),

\[
\int_{s}^{s+1} \int_{-1}^{1} e^{-2(p+1)s} \frac{d}{d\tau} F(\phi) \rho(y) dy d\tau \leq K. \tag{3.52}
\]

Proof: For \( s \geq 1 - \log(T^*(x) - t_2(x_0)) \), let us work with time integrals between \( s_1 \) et \( s_2 \) where \( s_1 \in [s-1, s] \) and \( s_2 \in [s+1, s+2] \). By integrating the expression \((1.23)\) of \( L_0(w(s), s) \) in time between \( s_1 \) and \( s_2 \), where \( s_2 > s_1 > -\log(T^*(x) - t_2(x_0)) \), we obtain:

\[
\int_{s_1}^{s_2} L_0(w(s), s) ds = \int_{s_1}^{s_2} \int_{-1}^{1} \left( \frac{1}{2} (\partial_s w)^2 + \frac{p+1}{(p-1)^2} w^2 - e^{-2(p+1)s} \frac{d}{d\tau} F(\phi) \rho(y) \right) dy ds \\
+ \frac{1}{2} \int_{s_1}^{s_2} \int_{-1}^{1} (\partial_y w)^2 (1 - y^2) \rho(y) dy ds - \int_{s_1}^{s_2} \frac{1}{s} \int_{-1}^{1} \partial_s w \rho(y) dy ds. \tag{3.53}
\]

By multiplying the equation \((1.15)\) by \( w(\rho(y)) \) and integrating both in time and in space over \((-1, 1) \times [s_1, s_2] \) we obtain the following identity, after some integration by parts:

\[
\left[ \int_{-1}^{1} (w \partial_s w + \frac{5-p}{2(p-1)} w^2) \rho(y) dy \right]_{s_1}^{s_2} = \int_{s_1}^{s_2} \int_{-1}^{1} (\partial_s w)^2 \rho(y) dy ds \tag{3.54}
- \int_{s_1}^{s_2} \int_{-1}^{1} (\partial_y w)^2 (1 - y^2) \rho(y) dy ds - \frac{2p+2}{(p-1)^2} \int_{s_1}^{s_2} \int_{-1}^{1} w^2 \rho(y) dy ds
\]

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By combining the identities (3.53), (3.54) and exploiting (3.55), we obtain

\begin{equation}
\int_{s_1}^{s_2} \int_{-1}^{1} e^{-\frac{2(p+1)s}{p-1}yw} w f(\phi w) \rho(y) dy ds - \frac{4}{p-1} \int_{s_1}^{s_2} \int_{-1}^{1} w \partial_s w \frac{y^2 \rho(y)}{1 - y^2} dy ds
\end{equation}

Note that, by using the identity (3.27), we get

\begin{equation}
\int_{s_1}^{s_2} \int_{-1}^{1} y \partial_s w \partial_s w \rho(y) dy ds + \frac{2a}{p-1} \int_{s_1}^{s_2} \int_{-1}^{1} 1 s \partial_s w \rho(y) dy ds
\end{equation}

We claim that Proposition 3.5 follows from the following Lemma where we control the space-time integral of the nonlinear term of \(w\) and all the terms on the right-hand side of the relation (3.56) in terms of the left-hand side:

**Lemma 3.6.** For all \(s \geq 1 - \log(T^*(x) - t_3(x_0))\), for some \(t_3(x_0) \in [t_2(x_0), T(x_0))\), for all \(\varepsilon > 0\),

\begin{equation}
\int_{-1}^{1} |w|^{p+1} \rho(y) dy \leq K + C \int_{-1}^{1} e^{-\frac{2(p+1)s}{p-1}yw} \frac{2a}{s^{2p-1}} F(\phi w) \rho(y) dy,
\end{equation}

\(27\)
\[
\begin{align*}
\int_{-1}^{1} e^{-\frac{2(p+1)x}{p-1}} s^{\frac{2a}{p-1}} F(\phi w) \rho(y) dy & \leq K + C \int_{-1}^{1} |w|^{p+1} \rho(y) dy, \\
\int_{s_1}^{s_2} \int_{-1}^{1} |y \partial_y w \partial_s w| \rho(y) dy ds & \leq \frac{K}{\varepsilon} + K \varepsilon \int_{s_1}^{s_2} \int_{-1}^{1} e^{-\frac{2(p+1)x}{p-1}} s^{\frac{2a}{p-1}} F(\phi w) \rho(y) dy ds, \\
\sup_{s \in [s_1, s_2]} \int_{-1}^{1} w^2(y, s) \rho(y) dy & \leq \frac{K}{\varepsilon} + K \varepsilon \int_{s_1}^{s_2} \int_{-1}^{1} e^{-\frac{2(p+1)x}{p-1}} s^{\frac{2a}{p-1}} F(\phi w) \rho(y) dy ds, \\
\int_{s_1}^{s_2} \int_{-1}^{1} w \partial_s w \frac{y^2 \rho(y)}{1-y^2} dy ds & \leq \frac{K}{\varepsilon} + K \varepsilon \int_{s_1}^{s_2} \int_{-1}^{1} e^{-\frac{2(p+1)x}{p-1}} s^{\frac{2a}{p-1}} F(\phi w) \rho(y) dy ds,
\end{align*}
\]

Indeed, from (3.56) and this Lemma, we deduce that
\[
\int_{s_1}^{s_2} \int_{-1}^{1} e^{-\frac{2(p+1)x}{p-1}} s^{\frac{2a}{p-1}} F(\phi w) \rho(y) dy ds \leq \frac{K}{\varepsilon} + (K \varepsilon + \frac{C}{s_1}) \int_{s_1}^{s_2} \int_{-1}^{1} e^{-\frac{2(p+1)x}{p-1}} s^{\frac{2a}{p-1}} F(\phi w) \rho(y) dy ds.
\]

Now, we can use the fact that \( s_1 \geq -1 - \log(T^*(x) - t_3(x_0)) \geq -1 - \log(T(x_0) - t_3(x_0)) \) and we choose \( T(x_0) - t_3(x_0) \) small enough, so that
\[
\frac{C}{s_1} \leq \frac{1}{-1 - \log(T(x_0) - t_3(x_0))} \leq C \varepsilon.
\]

If we choose \( \varepsilon \) small enough so that \( \frac{C}{s_1} \lesssim \frac{1}{4} \) and \( K \varepsilon \lesssim \frac{1}{4} \), we obtain
\[
\int_{s_1}^{s_2} \int_{-1}^{1} e^{-\frac{2(p+1)x}{p-1}} s^{\frac{2a}{p-1}} F(\phi w) \rho(y) dy ds \leq K.
\]

Since \([s, s+1] \subset [s_1, s_2]\), we derive (3.52).

It remains to prove Lemma 3.6.
Proof of Lemma 3.6. We first deal with the estimate (3.57) and (3.58). First, we divide \((-1, 1)\) into two parts \(A_1(s)\) and \(A_2(s)\) defined in (3.28).

Note that, by using the definition of the set \(A_1(s)\) defined in (3.28) and the expression of \(\phi\) defined in (1.13), we get,

\[
|w(y, s)|^{p+1} \leq \phi^{-\frac{1}{p+1}}(s) \leq Ce^{-s} \leq C, \quad \forall y \in A_1(s).
\]

(3.67)

From (A.23), (2.11) and the expression of \(\phi\) defined in (1.13), this yields

\[
e^{-\frac{2(p+1)s}{p-1}} F(\phi w) \leq C e^{-2s} + \frac{C}{s^a} |w(y, s)|^{p+1} \log(2 + \phi^2 w^2) \leq C, \quad \forall y \in A_1(s).
\]

(3.68)

Next, by using the definition of the set \(A_2(s)\) introduced in (3.28), the expression of \(\phi\) defined in (1.13) and the estimate (3.2) proved in Section 2, we conclude

\[
K^{-1} s \leq \log(2 + \phi^2 w^2) \leq K s.
\]

(3.69)

From (A.23), (2.11) and (3.69), this yields

\[
\frac{C}{K} |w(y, s)|^{p+1} \leq C + Ce^{-\frac{2(p+1)s}{p-1}} F(\phi w) \leq C + CK |w(y, s)|^{p+1}, \quad \forall y \in A_2(s).
\]

(3.70)

Adding (3.67), (3.68) and (3.70), we conclude that (3.70) is still valid, for all \(y \in (-1, 1)\). Therefore, the estimates (3.57) and (3.58) follow immediately from (3.70) after integration over \((-1, 1)\).

Thanks to (3.57) and (3.58), we can adapt with no difficulty the proof in the unperturbed case [31, 32] (up to some very minor changes), in order to get the proof of the estimates (3.59), (3.60), (3.61), (3.62) and (3.63). Also, by using (3.57) and the Hardy inequality (2.22), we easily conclude (3.64) and (3.65).

Finally, it remains only to control the terms \(A_5\) and \(A_6\). Note from (A.23), (A.25) and (A.26) that

\[
|F_1(\phi w)| + |F_2(\phi w)| \leq C + C \frac{F(\phi w)}{s}.
\]

(3.71)

The result (3.66) follows immediately from (3.71). This concludes the proof of Lemma 3.6 and Proposition 3.5 too.

Proof of Theorem 4. Thanks to (3.52), (3.53) and (3.49), we deduce, for all \(s \geq -\log(T^*(x) - t_2(x_0))\)

\[
\int_s^{s+1} \int_{-1}^{1} \left( (\partial_s w)^2 + (\partial_y w)^2 (1 - y^2) + w^2 \right) \rho(y) dy d\tau \leq K.
\]

(3.72)

By using the covering technique (we refer the reader to Merle and Zaag [32] (pure power case) and Hamza and Zaag [22]), we conclude

\[
\int_s^{s+1} \int_{-1}^{1} \left( (\partial_s w)^2 + (\partial_y w)^2 + w^2 \right) dy d\tau \leq K.
\]

(3.73)
Similarly to the proof of Proposition 3.1 (Step 1), we get
\[
\int_{-1}^{1} |w(y,s)|^{p+2} dy \leq K. \tag{3.74}
\]
By (3.74), (3.58) and Jensen’s inequality, we infer
\[
\int_{-1}^{1} e^{-2(p+1)s} s^{2n} F(w) dy \leq K. \tag{3.75}
\]
Finally, the definition of \(L_0(w(s), s)\) given in (1.23) and the estimate (2.40) imply
\[
\int_{-1}^{1} \left( (\partial_s w)^2 + (\partial_y w)^2 (1 - y^2) + w^2 \right) \rho(y) dy \leq K. \tag{3.76}
\]
Once again, by using the covering technique, we deduce (1.26). This concludes the proof of Theorem 4. \(\blacksquare\)

### A Some elementary lemmas.

Let \(f, F, F_2\) be the functions defined in (1.2), (1.19) and (3.27). Clearly, we have

**Lemma A.1.** Let \(q > 1\),
\[
\int_{0}^{u} |v|^{q-1} v \log^a(2 + v^2) dv \sim \frac{|u|^{q+1}}{q+1} \log^a(2 + u^2), \quad \text{as}\quad |u| \to \infty, \tag{A.1}
\]
\[
F(u) \sim \frac{uf(u)}{p+1} \quad \text{as}\quad |u| \to \infty, \tag{A.2}
\]
\[
F_2(u) \sim \frac{Cuf(u)}{\log^2(2 + u^2)} \quad \text{as}\quad |u| \to \infty. \tag{A.3}
\]

**Proof.** An integration by parts yields, for any \(q > 1\) and \(a \in \mathbb{R}\),
\[
\int_{0}^{u} |v|^{q-1} v \log^a(2 + v^2) dv = \frac{|u|^{q+1}}{q+1} \log^a(2 + u^2) - \frac{2a}{q+1} \int_{0}^{u} \frac{|v|^{q+1}v}{2+v^2} \log^{a-1}(2 + v^2) dv. \tag{A.4}
\]
From the fact that,
\[
\left| \frac{|v|^{q+1}v}{2+v^2} \log^{a-1}(2 + v^2) \right| \leq C + C|v|^q \log^{a-1}(v^2 + 2), \quad \forall v \in \mathbb{R},
\]
we can write
\[
\left| \int_{0}^{u} \frac{|v|^{q+1}v}{2+v^2} \log^{a-1}(2 + v^2) dv \right| \leq C + C|u|^{q+1} \log^{a-1}(2 + u^2), \quad \forall u \in \mathbb{R}. \tag{A.5}
\]
From (A.4) and (A.5), one easily obtain
\[ \int_0^u |v|^{p-1} v \log^a(2 + v^2) dv \sim \frac{|u|^{q+1}}{q + 1} \log^a(2 + u^2), \quad \text{as} \quad |u| \to \infty, \]
which ends the proof of the estimates (A.1). Note that (A.2) is trivial from (A.1) and the definition of \( f \) given in (1.19).

It remains to prove (A.3). Note that it easily follows from (A.4) that
\[
F(u) = \int_0^u f(v) dv = \frac{|u|^{p+1}}{p + 1} \log^a(2 + u^2) - \frac{2a}{p + 1} \int_0^u |v|^{p-1} v \log^{a-1}(2 + v^2) dv \\
+ \frac{4a}{p + 1} \int_0^u \frac{|v|^{p-1} v}{2 + v^2} \log^{a-1}(2 + v^2) dv. \tag{A.6}
\]

Once again, by integrating by parts, we obtain
\[
\int_0^u |v|^{p-1} v \log^{a-1}(2 + v^2) dv = \frac{|u|^{p+1}}{p + 1} \log^{a-1}(2 + u^2) - \frac{2a - 2}{p + 1} \int_0^u |v|^{p-1} v \log^{a-2}(2 + v^2) dv \\
+ \frac{4a - 4}{p + 1} \int_0^u \frac{|v|^{p-1} v}{2 + v^2} \log^{a-2}(2 + v^2) dv. \tag{A.7}
\]

Therefore, (A.6), (A.7), (3.27) and (3.26), imply that
\[
F_2(u) = F_1^{1/2}(u) + F_2^{1/2}(u), \tag{A.8}
\]

where
\[
F_1^{1/2}(u) = \frac{4a(a - 1)}{(p + 1)^2} \int_0^u |v|^{p-1} v \log^{a-2}(2 + v^2) dv, \tag{A.9}
\]
\[
F_2^{1/2}(u) = \frac{4a}{p + 1} \int_0^u \left( \frac{2a - 2}{p + 1} \frac{|v|^{p-1} v}{2 + v^2} \log^{a-2}(2 + v^2) - \frac{|v|^{p-1} v}{2 + v^2} \log^{a-1}(2 + v^2) \right) dv. \tag{A.10}
\]

Let us find an equivalent to \( F_2 \). By exploiting the following estimates
\[
\left| \frac{2a - 2}{p + 1} \frac{|v|^{p-1} v}{2 + v^2} \log^{a-2}(2 + v^2) - \frac{|v|^{p-1} v}{2 + v^2} \log^{a-1}(2 + v^2) \right| \leq C + C|v|^{p-\frac{1}{2}},
\]
one easily obtains
\[
|F_2^{1/2}(u)| \leq C + C|u|^{p-\frac{1}{2}}. \tag{A.11}
\]
The result (A.3) immediately follows from (A.1) (A.8), (A.9) and (A.11), which ends the proof of Lemma (A.1).

The following lemma shows the asymptotic behavior of the solution of the associated ODE
\[
v''(t) = |v(t)|^{p-1} x \log^a(2 + v^2(t)), \quad v(0) = A > 0 \quad \text{and} \quad v'(0) = B > 0. \tag{A.12}
\]
Lemma A.2. The problem (A.12) has one positive solution. Moreover, there exist \( T < \infty \), such that the solution \( \psi \) satisfies the following asymptotic:

\[
v(t) \sim \kappa_a (T - t)^{-\frac{2}{p-1}} |\log(T - t)|^{-\frac{a}{p-1}}, \quad \text{as } t \to T,
\]

where \( \kappa_a = \left( \frac{2^{1-2a(p+1)}}{(p-1)^{2-a}} \right)^{\frac{1}{p-1}} \).

Proof. The uniqueness and local existence of (A.12) are derived by the Cauchy-Lipschitz property. Let \( T \) be the maximum time of the existence of the positive solution, i.e. \( v(t) \) exists for all \( t \in [0, T) \). We now prove that \( T < +\infty \). By contradiction, we suppose that the solution exists on \( [0, +\infty) \). By multiplying equation (A.12) by \( v'(t) \) and integrating with respect to time on \( (0, t) \), we obtain

\[
(v'(t))^2 = 2F(v(t)) + C,
\]

where \( F \) is defined in (1.19). Using (A.12), we conclude that \( v' \) is an increasing function, so for all \( t \in [0, +\infty) \) we have \( v'(t) \geq v'(0) > 0 \). Then, (A.14) becomes

\[
v'(t) = \sqrt{2F(v(t)) + C}.
\]

Using the fact that \( v'(t) > 0 \) and \( v(t) > 0 \), we deduce that

\[
\lim_{t \to +\infty} \int_0^t v'(s) \frac{ds}{\sqrt{2F(v(s)) + C}} = \lim_{t \to +\infty} \int_0^t ds = +\infty.
\]

Let us mention that

\[
F(v) \sim \frac{v^{p+1}}{p+1} \log^a(v^2 + 2), \quad \text{as } v \to \infty,
\]

and \( \int_0^t v'(s) \frac{ds}{v^{\frac{2}{p+1}}(s) \log^a(v^2(s) + 2)} \) is bounded. Thus, the contradiction follows.

Let us now prove (A.13). By integrating (A.15) with respect to time \( (0, t) \), we obtain

\[
T - t = \int_{v(t)}^{+\infty} \frac{du}{\sqrt{2F(u(t)) + C}}.
\]

Thus, for all \( \delta \in (0, \frac{p-1}{2}) \), there exist \( t_\delta \) such that for all \( t \in (t_\delta, T) \), we have

\[
\int_{v(t)}^{+\infty} \frac{du}{u^{\frac{2}{p+1}+\delta}} \leq T - t \leq \int_{v(t)}^{+\infty} \frac{du}{u^{\frac{2}{p+1}-\delta}}.
\]

This implies for all \( t \in (t_\delta, T) \) that:

\[
C^{-1}(T - t)^{-\frac{1}{p+1}} \leq v(t) \leq C(T - t)^{-\frac{1}{p+1}},
\]

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from which we have
\[ \log v(t) \sim -\frac{2}{p-1} \log(T-t) \quad \text{as} \quad t \to T, \]
and
\[ \log(v^2 + 2) \sim -\frac{4}{p-1} \log(T-t) \quad \text{as} \quad t \to T. \tag{A.18} \]

Hence, by using (A.15), (A.18) and (A.16), we obtain
\[ \frac{v'(t)}{v^{\frac{p+1}{2}}(t)} \sim \sqrt{\frac{2}{p+1}} \left( \frac{4}{p-1} \right)^{\frac{p}{2}} |\log(T-t)|^{\frac{p}{2}}, \quad \text{as} \quad t \to T. \tag{A.19} \]

By integrating over \((t,T)\), we have
\[ 2^{p-1}v^{\frac{p}{p+1}}(t) \sim \sqrt{\frac{2}{p+1}} \left( \frac{4}{p-1} \right)^{\frac{p}{2}} \int_t^T |\log(T-v)|^{\frac{p}{2}} dv \]
\[ \sim \sqrt{\frac{2}{p+1}} \left( \frac{4}{p-1} \right)^{\frac{p}{2}} (T-t) |\log(T-t)|^{\frac{p}{2}} \quad \text{as} \quad t \to T. \tag{A.20} \]

Using (A.20), we see after straightforward calculations that
\[ v(t) \sim \left( \frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p+1}} \left( \frac{4}{p-1} \right)^{-\frac{n}{p+1}} (T-t)^{-\frac{2}{p-1}} |\log(T-t)|^{-\frac{n}{p-1}} \quad \text{as} \quad t \to T. \]

This concludes the proof of (A.13).

By integrating by parts (see Lemma A.1), we can write
\[ uf(u) - (p+1)F(u) \sim \frac{2a}{p+1} |u|^{p+1} \log^{a-1}(2 + u^2), \quad \text{as} \quad |u| \to \infty, \tag{A.21} \]
where \(f\) and \(F\) defined respectively in (1.2) and (1.19). More precisely, we have for all \(u \in \mathbb{R}\)
\[ \left| uf(u) - (p+1)F(u) - \frac{2a}{p+1} |u|^{p+1} \log^{a-1}(2 + u^2) \right| \leq C + C|u|^{p+1} \log^{2a-2}(2 + u^2). \tag{A.22} \]

Thanks to (A.21) and (A.22), we will give the first and the second order terms in the expansion of the nonlinearity \(F(x)\) defined in (1.19), when \(|x|\) is large enough. More precisely, we now state the following estimates

**Lemma A.3.** For all \(s \geq 1\), for all \(z \in \mathbb{R}\),
\[ C^{-1} \phi(s)zf(\phi(s)z) \leq C + F(\phi(s)z) \leq C(1 + \phi(s)zf(\phi(s)z)) \tag{A.23} \]
\[ F(\phi(s)z) \leq C + C|\phi(s)z|^{p+1}, \tag{A.24} \]
\[ F_1(\phi(s)z) \leq C + C\frac{\phi(s)z}{s} f(\phi(s)z), \quad (A.25) \]
\[ F_2(\phi(s)z) \leq C + C\frac{\phi(s)z}{s^2} f(\phi(s)z), \quad (A.26) \]

where \( \phi, F, F_1 \) and \( F_2 \) are given in (1.18), (1.19), (3.26), (3.27), and
\[
\bar{p} = \begin{cases} 
  p + 1 & \text{if } N = 1, 2, \\
  p + \frac{2}{N-2} - \frac{2}{N-1} & \text{if } N \geq 3.
\end{cases} \quad (A.27)
\]

**Proof.** Note that (A.23) obviously follows from (A.2). Similarly, by taking into account the inequality \( \log^a(2 + u^2) \leq C + C|u|^\bar{p} - p \) and (A.2) we conclude (A.24). In order to derive estimates (A.25) and (A.26), considering the first case \( z^2\phi(s) \geq 4 \), then the case \( z^2\phi(s) \leq 4 \), we would obtain (A.25) and (A.26) by using (A.1), (A.2) and (A.3). This ends the proof of Lemma A.3. \( \blacksquare \)

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