Periodic orbits near bifurcations of codimension two: Classical mechanics, semiclassics, and Stokes transitions

Henning Schomerus

Fachbereich Physik, Universität–Gesamthochschule Essen, D–45117 Essen, Germany
(Date: November 8, 2018)

We investigate classical and semiclassical aspects of codimension–two bifurcations of periodic orbits in Hamiltonian systems. A classification of these bifurcations in autonomous systems with two degrees of freedom or time–periodic systems with one degree of freedom is presented. We derive uniform approximations to be used in semiclassical trace formulas and determine also certain global bifurcations in conjunction with Stokes transitions that become important in the ensuing diffraction catastrophe integrals.

Pacs: 05.45.+b, 03.65.Sq, 03.20.+i

I. INTRODUCTION

Periodic–orbit theory aims at the semiclassical evaluation of energy levels of quantum systems and relates their spectral properties to periodic orbits of the corresponding classical system. For autonomous systems one considers the trace of the Green’s function $G(E)$ which determines also the density of states $d(E)$. For periodically driven systems the object of interest are the traces $\text{tr} E^n$ of the stroboscopic time–evolution operator over $n$ periods; they encode the so–called quasienegries of states that are stationary in the stroboscopic description. Both types of traces can be written as a sum of individual contributions of periodic orbits for chaotic (hyperbolic) systems or a sum over rational tori for integrable motion.

Recent semiclassical studies based on periodic–orbit theory were devoted to the neighbourhood, in the space of control parameters, of classical bifurcations. These are instances in which periodic orbits coalesce and are the mechanism how orbits are born or disappear, or change their configuration when the energy or an external parameter is varied. Bifurcations are ubiquitous in systems with a mixed phase space and pave the path from integrable to chaotic motion.

A collective treatment of the bifurcating orbits was found necessary, and even more the inclusion of predecessors of such orbits which live in complexified phase space and were termed ghosts. A collective contribution comes from an orbit cluster and can only far away from the bifurcation be written as a sum of individual contributions. Both types of contributions are an additive term in the periodic–orbit expansion of the trace in question.

The existing semiclassical (and most of the classical) studies focus on the generic bifurcations in the classification of Meyer and Bruno (see also [8]). These are the bifurcations that are typically encountered when one has only a single parameter at hand to steer the system through parameter space, or, equivalently, when one investigates the periodic–orbit families in a given autonomous system as a function of energy. In general one assigns a codimension to each type of bifurcation by counting the number of parameters to be controlled in order to encounter it in a general setting. (The class of bifurcations of a given codimension is enlarged when symmetries are imposed on the system.) The generic bifurcations are accordingly the bifurcations of codimension one. In each of these bifurcations there is a central orbit of period $n$, surrounded by one or two satellite orbits of period $nm$. The cases $m = 1, 2, 3, 4$ are the tangent, period–doubling, period–tripling, and period–quadrupling bifurcations, respectively. There are two types of period–quadrupling bifurcations (island chain and touch–and–go), but only one for all other $m$. All period–$m$ bifurcations with $m \geq 5$ follow the island chain pattern.

In transitional approximations for the collective contributions were derived that are only valid close to the (generic) bifurcation; far away from the bifurcation they give rise to individual contributions with the wrong amplitudes. Refs. [10, 12] give uniform approximations that are valid even far away from the bifurcation, where they asymptotically split into individual contributions with the correct amplitudes.

The present work is devoted to bifurcations of codimension two in autonomous Hamiltonian systems with two degrees of freedom or time–periodic Hamiltonian systems with one degree of freedom. The bifurcations are classified and their impact on semiclassical periodic–orbit theory is studied in detail. We derive uniform approximations of collective contributions to the semiclassical traces and discuss certain global bifurcations in conjunction with so–called Stokes transition.

Bifurcations of codimension two manifest themselves in one–parameter studies in certain sequences of generic bifurcations. Sadovskiï, Shaw, and Delos found that such sequences can be explained by normal–form theory, but did not attempt a classification with respect to the codimension. The classical part of the present study is very much inspired by these works.

It was indeed demonstrated in Ref. [27] that bifurcations of codimension two are frequently felt semiclassical.
cally and necessitate a collective treatment even when one steers the system through control space with less than two parameters. This implies that collective contributions of this kind will constitute a basic ingredient in a semiclassical trace formula for systems with a mixed phase space. They indeed played an essential rôle in the semiclassical determination of the quasienergies of the kicked top \([28]\).

The paper is organized as follows. In section II, we derive normal forms for the Hamiltonian that describe the bifurcations of codimension two. We find that they are organized by the multiplicity \(m\) in analogy to the situation for codimension one. The corresponding sequences of codimension one bifurcations involve a tangent bifurcation of period \(nm\), followed by a period–\(m\) bifurcation that involves another orbit of period \(n\). (The representative case studied in \([27]\) corresponds to \(m = 3\).

The normal forms and the sequences of codimension one bifurcations in the neighbourhood of the codimension two point in control space (technically spoken, the *unfolding* of the normal forms) are discussed and illustrated in section III.

With section IV we turn to aspects within semiclassical periodic–orbit theory and present the starting point for the derivation of collective contributions of bifurcating orbits, consisting of two–dimensional integrals over phase space or Poincaré surfaces of section that involve the generating function \(\hat{S}\) of the classical stroboscopic map.

Uniform approximations of these contributions are derived in section V. They involve a phase function \(\Phi\) and an amplitude function \(\Psi\). Normal forms of these are obtained from the corresponding Hamiltonians by non–canonical transformations and partial integrations. Investigating the influence from higher–order terms in the phase equips us with a sufficient number of coefficients to guarantee the right stationary–phase limit of the expressions, which are therefore truly uniform. [The cases \(m = 1, 2\) lead to standard diffraction integrals connected to the cusp and butterfly catastrophes, respectively. Among the large number of applications, the transitional approximations have been investigated in connection to bifurcations of closed orbits in \([29]\). Uniform approximations have not been derived there, however, and the canonical invariant determination of coefficients as well as Stokes transitions are also not discussed.]

In section VII we discuss certain global bifurcations that become important in the ensuing diffraction catastrophe integrals. They give rise to *Stokes transitions* in which the contribution of a ghost satellite is switched on or off. The ghosts and transitions arise when the integrals are analyzed using the method of steepest descent. Stokes transitions have been investigated in the context of diffraction integrals and asymptotic expansions before. A uniform approximation for an isolated transition is given in \([30]\) and has been applied for perturbed cat maps in \([31]\), which is the only treatment of this phenomenon in semiclassics that we know of. The Stokes transitions investigated there, however, occur far away from any other bifurcation and can be regarded as isolated. A transition requires special treatment when it occurs in the immediate neighbourhood of a usual ‘local’ bifurcation, a situation that is often encountered in mixed systems. The uniform approximations and normal forms derived here can also be employed to describe the Stokes transition of a period–\(nm\) ghost prior to a tangent bifurcation when the so–called ‘dominant’ orbit involved is real and of period \(n\). The complete sequence of local and global bifurcations that we can handle consists of the period–\(m\) bifurcation at the central orbit and tangent bifurcations of satellites, followed by Stokes transitions in which ghost satellites once more interact with the central orbit.

We conclude and point out open questions in section VII.

**II. NORMAL FORMS OF THE HAMILTONIAN FOR BIFURCATIONS OF CODIMENSION TWO**

**A. Objective**

The local bifurcations to be discussed are instances in which periodic orbits coalesce as parameters are varied. The types of bifurcations generically encountered in a given class of systems depends on the number of parameters varied, and the number of parameters typically needed to be controlled in order to find a particular type is called its codimension. Here we investigate bifurcations of codimension two in the class of periodically time–dependent Hamiltonian systems with one degree of freedom. In other words, we study families of Hamiltonians

\[ H(q, p, t; \varepsilon, a) \quad (1) \]

that depend on the two parameters \(\varepsilon\) and \(a\) and obey \(H(t) = H(t + 1)\), where the period is set to unity for convenience. In general these systems have no time–reversal nor any geometric symmetry. The discussion directly carries over to autonomous systems with two degrees of freedom since these can be reduced to one–parameter families of periodic systems with one degree of freedom by a standard procedure described e. g. in \([3]\).

**B. The bifurcation condition**

The periodic orbits show up as fixed points in iterations of the so–called stroboscopic map \((q, p) \rightarrow (q', p')\) which is induced by the evolution over one period. This map is area preserving. Its linearized version
(corresponding to a $2 \times 2$ matrix) hence obeys \( \det M = 1 \). Orbits that appear for the first time in the \( n_0 \)-th iteration are said to be of primitive period \( n_0 \). Such an orbit gives rise to \( n_0 \) fixed points in each \( n \)-step map with \( n = n_0r \), where \( r \) is an integer counting repetitions.

The eigenvalues \( \lambda_{1,2} \) of the linearized \( n_0 \)-step map \( M^{(n_0)} \) are reciprocal to each other. A stable orbit has unimodular eigenvalues and hence \( \text{tr} M^{(n_0)} = 2 \cos \omega \) with the real stability angle \( \omega \). An orbit is instable if the eigenvalues are real. There are two cases depending on the sign of the eigenvalues, \( \text{tr} M^{(n_0)} = \pm 2 \cosh \omega' \), with the real and by convention positive instability exponent \( \omega' \).

In general, an orbit bifurcates whenever the linearized \( n \)-step map \( M^{(n)} \) (with again \( n = n_0r \)) acts at the locus of the orbit in phase space in at least one direction as the identity map and hence obeys

\[
\text{tr} M^{(n_0)} = \text{tr} \left( M^{(n_0)} \right)^r = 2 ,
\]

or, equivalently,

\[
\text{tr} M^{n_0} = 2 \cos(2\pi l/m) ,
\]

where the integers \( l, m \) are taken as relatively prime. This bifurcation condition implies a discrete \( m \)-fold rotational symmetry \( C_m \) in the flow pattern around the bifurcating orbit. For \( m \geq 2 \) the orbit in question is a ‘central’ orbit on which ‘satellite’ orbits of primitive period \( n_0m \) contract at the bifurcation. For \( m = 1 \) there are two possibilities, the orbit is either involved as a satellite in a bifurcation with an orbit of smaller primitive period, or it takes part in an isochronous bifurcation with other orbits of same period. Turning these observations around, there is always a central periodic orbit of smallest primitive period \( n_0 \) among the bifurcating orbits which coalesce with satellites of period \( n_0m \). For that reason \( m \) is called the multiplicity.

The bifurcations of codimension one have been classified by Meyer and Bruno \( [13, 15] \) (see the introductory section). They constitute the building blocks of the scenarios of higher codimension and will be illustrated together with those of codimension two in the next section. Recall that for each \( m \) there is exactly one type with the exception of \( m = 4 \) which allows for two variants.

\[ M = \begin{pmatrix} \frac{\partial q'}{\partial p} & \frac{\partial q'}{\partial q} \\ \frac{\partial p'}{\partial p} & \frac{\partial p'}{\partial q} \end{pmatrix} \]  

Form of the Hamiltonian in the vicinity of this orbit. Following Refs. \( [7,21,22,33] \), we aim at the reduction of the general expressions to certain simple normal forms by suitable canonical transformations. In that way one can identify the parameters that govern the distance to the bifurcation. They can be chosen such that \( \varepsilon = 0 \) brings us on a codimension one bifurcation and the codimension two scenario is encountered if in addition \( a = 0 \).

For codimension one the construction (that is carried out for that case in detail in \( [6] \) and is recapitulated below) leads to the Birkhoff normal forms

\[
h^{(m)}(q,p) - H_0 = \begin{pmatrix} m \end{pmatrix}. \tag{5}
\]

The quantity \( H_0 \) is a constant. These expressions are autonomous and display the \( m \)-fold symmetry even globally. The periodic orbits are mapped to fixed points \( \varepsilon = 0 \), \( \partial H / \partial q = 0 \) and are thus determined as roots of polynomials in \( p \) and \( q \). From Vieta’s relations between these roots (or locations of satellites) and the coefficients of the polynomials it follows that orbits collapse on the center as the lowest–order terms \( \sim \varepsilon \) (and \( \sim a \) for codimension two) of the fixed–point equations are steered to zero.

To describe the codimension two variants we have to include higher–order terms to account for additional satellites that approach the center and obtain the extended normal forms

\[
h^{(m)}(q,p) - H_0 = \begin{pmatrix} m \end{pmatrix}. \tag{7}
\]

The normal forms for \( m \geq 5 \) are the usual Birkhoff normal forms; they will, however, be investigated not only for small \( \varepsilon \) but also for small \( a \). The expressions for \( m \leq 4 \) go exactly one order beyond the Birkhoff normal forms.

\[ m \frac{\partial^2 H_0}{\partial q \partial p} = 0, \quad \frac{\partial^2 H_0}{\partial p \partial q} = 0 \]

The bifurcation condition \( \{4\} \) is reflected by the Hamiltonian flow around the central orbit; accordingly, the bifurcations can be investigated by studying the general C. Classification of normal forms

The bifurcation condition \( \{4\} \) is reflected by the Hamiltonian flow around the central orbit; accordingly, the bifurcations can be investigated by studying the general

D. Derivation of normal forms

In the derivation of the normal forms \( [6] \) and \( [33] \) the central orbit is placed into the origin of a local coordinate
system \((q,p)\) by a time–dependent canonical transformation. This is done in such a way that a Taylor expansion in \(q\) and \(p\) yields

\[
H(q, p, t) = H_0 + \frac{\omega}{2} q^2 + \frac{\sigma}{2} p^2 + O(3)
\]

(8)

with time–independent \(\omega\) and \(\sigma\). The remainder in the expansion indicates third and higher orders in \(q\) and \(p\). The new Hamiltonian will be \(n_0\)–periodic but otherwise as general as the one we started with. Hence it suffices to study the case \(n_0 = 1\).

One passes to a rotating coordinate system, where the angular frequency \(2\pi t/m\) is adopted to the motion around the center at the bifurcation, and examines the expansion of the Hamiltonian in a Taylor series in \(q\) and \(p\) as well as a Fourier series in \(t\). Most terms in the expansion can be removed by canonical transformations (specified below) up to a certain order in \(p, q\) and the deviations of the parameters \(\varepsilon, a\) that govern the distance to the bifurcation. The expansion is carried out up to a certain degree, requiring that no bifurcating orbits are added or qualitatively affected by the omitted higher–order terms.

1. The cases \(m = 1, 2\)

The linearized map \(M\) has degenerate eigenvalues 1 for \(m = 1\) and \(-1\) for \(m = 2\). This entails that the corresponding bifurcation scenarios of codimension one and two are essentially one–dimensional. The reason is that the linearized map is in these cases generically not diagonalizable: Otherwise one would have located the matrix \(M = \pm 1\) in the three–dimensional manifold \(Sp(2, \mathbb{R})\) of real \(2 \times 2\) matrices with \(\det M = 1\) [23]. With only two parameters at one’s disposal, however, one generically finds only such matrices with \(\text{tr} M = \pm 2\) for which the eigenspace is one–dimensional. The linearized map describes then a shear transformation and acts as the identity in only one direction.

The reasoning can be put onto another footing by resolving the apparent paradox that we can cast the identity in only one direction. The reasoning can be put onto another footing by

\[
G(\theta, I) = I\theta + I^{k/2} \sum_{l' = -\infty}^{\infty} \sum_{m'} G_{k'l'm'} \times \sin \left[ m' \theta + 2\pi \left( \frac{ml}{m} - l' \right) t + \phi_{klm} \right]
\]

(10)

with

\[
G_{k'l'm'} = \frac{V_{k'l'm'}}{\omega m' - 2\pi l'} = \frac{V_{k'l'm'}}{\varepsilon m' + 2\pi l'm/m - 2\pi l'}.
\]

(11)

After the transformation we switch back in our notation from \(J, \theta\) to \(I, \phi\). The coefficient \(G_{k'l'm'}\) diverges at the bifurcation if the resonance condition \(l' = 1/m\) is met. This affects, for instance, all \(t\)–dependent terms \((l' = m = 0)\). The remaining \(\phi\)–dependent terms are of type \(I^{k/2} \cos(\omega t - \phi_{kn})\). Here \(n = 1, 2, 3, \ldots\) is an integer since \(l\) and \(m\) are relatively prime, and \(k \geq nm\) as before. In the orders that appear in the normal forms the latter restriction admits only \(n = 1\). The \(\phi\)–dependent term of lowest order in \(I\) is generically of type \(I^{m/2} \cos(\omega t - \phi_{m1})\). A shift of \(\phi\) eliminates the constant in the cosine. If the coefficient of this term is not small then one can get rid of constants \(\phi_{k3}\) in higher orders \(k > m\) by a transformation of the form \(\phi = \theta + \sum_{k' = 1}^{\infty} g_{k'} I^{k'}\) with suitably chosen coefficients.

3. Further reduction for \(m = 4\)

Additional considerations are needed for \(m = 4\). The most general expression that goes one order beyond the Birkhoff normal form reads

For \(m \geq 3\) the central orbit is stable close to the bifurcation so that we can achieve \(\sigma = \omega\) in (8), entailing \(\text{tr} M = 2\cos \omega\). It is then convenient to use canonical polar coordinates [8]. In the rotating coordinate system the leading–order term in \(I\) takes the form \(\varepsilon I\) with \(\varepsilon = \omega - 2\pi m/3\) in agreement with the bifurcation condition (6). The expansion reads in detail

\[
H = H_0 + \varepsilon I + \sum_{k = 3}^{\infty} \sum_{l' = -\infty}^{\infty} \sum_{m'} V_{kl'm'} I^{k/2} \times \cos \left[ m' \phi + 2\pi \left( \frac{ml}{m} - l' \right) t + \phi_{klm} \right],
\]

(9)

where \(m'\) runs from \(-k, -k + 2, \ldots, k\) since only such terms arise from expressions of type \(q^k p^{k'}\). Let us assume that all \(t\) and \(\phi\) dependence is already eliminated up to a certain order \(I^{k/2}\). The majority of terms of this order are then removed by a canonical transformation to new coordinates \(J, \theta\) that is generated according to \(\partial G J = J, \partial G \theta = \phi\), \(H' = H - \partial G^2\) by the function

\[
\begin{align*}
G(\theta, I) &= I\theta + I^{k/2} \sum_{l' = -\infty}^{\infty} \sum_{m'} G_{k'l'm'} \times \sin \left[ m' \theta + 2\pi \left( \frac{ml}{m} - l' \right) t + \phi_{klm} \right]
\end{align*}
\]

(10)

with

\[
G_{k'l'm'} = \frac{V_{k'l'm'}}{\omega m' - 2\pi l'} = \frac{V_{k'l'm'}}{\varepsilon m' + 2\pi l'm/m - 2\pi l'}.
\]

(11)
The codimension two bifurcation is approached for vanishing $\varepsilon$ and $a$ (or $b$), while the other second-order coefficient $b$ (or $a$) is finite. Both cases are equivalent and mapped onto each other by a rotation about $\pi/4$. The normal form $H^{(4)}$ has been written down for small $a$. Since $b$ is finite we can eliminate two of the three third-order terms in (12) by a canonical transformation and achieve $d = e = 0$. [The corresponding generating function is of the simple form $G = \phi - (dI^2 \sin 4\theta)/(8b) - \varepsilon I^2/(8b)$ if corrections involving $\varepsilon$ and $a$ are neglected; the complete form is slightly more complicated.]

The fine-print in the derivation for $m = 4$ is that two orbits pretend to bifurcate also as we send $a \rightarrow b$. We identify now these orbits and show that for codimension two they are actually ghosts (complex solutions of the fixed-point equations) at a finite distance from the center. For simplicity we set $e = 0$; this does not affect the general line of reasoning. The satellites that concern us solve the fixed-point equation

$$
\frac{\partial H^{(4)}}{\partial \phi} = -4I^2(a - b + (c - d)I) \sin 4\phi = 0 \quad (13)
$$

by $I = I^{(0)} \equiv (b - a)/(c - d)$. The other fixed-point equation yields

$$
\cos 4\phi = -3\frac{c + d}{c - d} - \frac{c - d}{(a - b)^2} + 2\frac{a + b}{a - b} = C^{(0)} \quad (14)
$$

(These satellites are related by the reflection symmetry $\phi \rightarrow -\phi$ and undergo a pitchfork bifurcation as $|C^{(0)}| = 1$. The symmetry is broken if $e \neq 0$.) For reasons similar to those put forward for $m = 1, 2$, two parameters have to be controlled in order achieve $I^{(0)} = 0$, i.e., $a = b$: For the given parameter combination there is no $\phi$ dependence in the second order of $I$. But actually there are two independent terms involving $\phi$, namely $\cos 4\phi$ and $\sin 4\phi$ — one of them had been eliminated by a diagonalization that is again sensible only if the other has a non-vanishing coefficient. We already used two parameters, then, such that we must assume $\varepsilon$ and $a \approx b$ to be finite. This gives $|\cos 4\phi| \sim (a - b)^{-2} \gg 1$ and $\cos \phi \sim (a - b)^{-1/2} = O(I^{(0)}^{-1/2})$, and the Cartesian coordinates (14) indeed remain finite: As announced this shows that the orbits that appeared to bifurcate are complex solutions (with real $p$ and imaginary $q$) of the fixed-point equations and stay away from the center.

III. LOCAL BIFURCATION SCENARIOS

We discuss now in detail the bifurcations of codimension two that are described by the normal forms given in the preceding section. In each case the location of the periodic points, given as the solutions of the fixed-point equations

$$
\frac{\partial H}{\partial \theta} = 0, \quad \frac{\partial H}{\partial p} = 0 \quad (15)
$$

are investigated as the parameters are varied. Sequences of codimension one bifurcations are encountered if only one parameter is varied close to a codimension two point [21,22]. They are discussed here for fixed $a$ and variable $\varepsilon$ and are illustrated by contour plots of the normal forms. Unstable orbits appear there as saddles while stable orbits correspond to maxima or minima. In all these sequences there is a period–$m$ bifurcation at $\varepsilon = 0$ and a tangent bifurcation of satellites at the parameter combinations

| $m$ | tangent bifurcation of satellites |
|-----|---------------------------------|
| 1   | $\varepsilon = -\frac{1}{2\sqrt{3}}$ |
| 2   | $\varepsilon = \frac{1}{3} \frac{a}{b}$ |
| 3   | $\varepsilon = \frac{2}{3} \frac{a}{b}$ |
| 4   | $\varepsilon = -\frac{1}{2} \frac{a}{b}$ |
| 5   | $\varepsilon = -\frac{128}{675} \frac{a}{b}$ |
| 6   | $\varepsilon = -\frac{1}{2} \frac{a}{b}$ |
| $\geq 7$ | $\varepsilon \approx \frac{1}{2} \frac{a}{b}$ |

Global bifurcations that are of particular interest in the context of uniform approximations are discussed in section IV.

A. Tangent bifurcations ($m = 1$)

In a tangent bifurcation two orbits of the same primitive period coalesce. On one side of the bifurcation both orbits are ghosts, i.e., complex solutions of the fixed-point equations, and their coordinates and other characteristic quantities are related by complex conjugation. On the other side of the bifurcation both orbits are real, one of them being initially stable and the other unstable. The scenario is described by the normal form $h^{(1)}$ which accounts for two periodic orbits $\pm$ at coordinates $p_{\pm} = 0$ and

$$
q_{\pm} = \pm \sqrt{-\frac{1}{3} \frac{\varepsilon}{a}} \quad (17)
$$

Often one encounters a third orbit of identical period in close neighbourhood (in phase space) to the bifurcating orbits. This orbit must be taken into account, for instance, to obtain a reasonable semiclassical approximation. One has to work then with the extended normal form $H^{(1)}$. The fixed-point equation $\partial H/\partial \theta = 0$ is a real cubic polynomial in $q$ and has three solutions. The number of real solutions is determined by the sign of the discriminant

$$
D = \left(\frac{\varepsilon}{8 \, b}\right)^2 + \frac{1}{4 \, b} \left(\frac{1}{4 \, b}\right)^3 \quad (18)
$$
There are three real solutions for \( D < 0 \) and only one for \( D > 0 \) which is then accompanied by two complex ones. Tangent bifurcations are encountered at \( D = 0 \), that is \( \varepsilon = 0 \) or \( \varepsilon = -a^3/(4b^2) \).

A sequence of these two tangent bifurcations is depicted in Figure 1. The codimension two bifurcation is obtained when \( \varepsilon \) and \( a \) pass zero simultaneously. If this is done in such a way that the discriminant changes sign then the number of solutions changes from one to three in a pitchfork bifurcation. Such a bifurcation is even of codimension one if the system is time–reversal symmetric or has a reflection symmetry \( 1 \rightarrow 2 \).

**B. Period–doubling bifurcation \((m = 2)\) and tangent bifurcation of satellites**

In a period–doubling bifurcation the central orbit changes its stability by absorbing or emitting a satellite of double period. In the Birkhoff normal form \( h^{(2)} \) the central orbit sits at coordinates \( q_0 = p_0 = 0 \) and the satellite is represented by two fixed points

\[
p_1 = 0, \quad q_1 = \pm \sqrt{-\frac{\varepsilon}{2a}}. \tag{19}
\]

In the extended normal form \( H^{(2)} \) the central orbit lies again at \( q_0 = p_0 = 0 \), but there are now two satellites \( \pm \) with coordinates

\[
q_\pm^2 = \frac{1}{3} a \pm \sqrt{\frac{a^2}{9b^2} - \frac{1}{3} \varepsilon} \tag{20}
\]

A tangent bifurcation of the satellites is encountered at

\[
\varepsilon = \frac{a^2}{3b}, \tag{21}
\]

but the condition \( ab < 0 \) must be obeyed since otherwise both orbits are still ghosts with purely imaginary \( q \)-coordinates. A sequence of tangent bifurcation of the satellites and period–doubling bifurcation (with \( ab < 0 \)) is shown in Figure 2.

**C. Period–tripling bifurcation \((m = 3)\) and tangent bifurcation of satellites**

The situation for the period tripling is visualized in the sequence of contour plots in Figure 1c. Initially, a stable periodic orbit of period one is surrounded by its stability island. At a certain value of the control parameter two satellites of triple period come into existence via a tangent bifurcation. Then the inner (unstable) satellite approaches the central orbit, collides with it in the period tripling, and finally re–emerges on the other side. This scenario has been investigated, for instance, in the diamagnetic Kepler problem \[3\] and for the kicked top \[27\].

The Birkhoff normal form \( h^{(3)} \) describes the central orbit at \( I = 0 \) and the unstable satellite that is involved in the tripling. The \( \phi \)-coordinate of the satellite obeys \( \partial h^{(3)}/\partial \phi = -3aI^3/2\sin 3\phi = 0 \). Since a three-fold symmetry is implied by this equation it suffices to consider the second equation \( \partial h^{(3)}/\partial I = 0 \) on the \( p \)–axis after switching back to the coordinates \( p, q \), yielding \( \varepsilon p + 3a/\sqrt{8} ap^2 = 0 \). Hence the canonical radial coordinate of the satellite is \( I = p^2/2 = 4c^2/(9a^2) \).

In the extended normal form \( H^{(3)} \) the \( \phi \)-coordinates of the satellites once more obey \(-3aI^3/2\sin 3\phi = 0\). On the \( p \)–axis they satisfy now \( \varepsilon p + 3a/\sqrt{8} ap^2 + bp^3 = 0 \). This equation has three solutions,

\[
p_0 = 0, \quad p_\pm = -\frac{3a}{4\sqrt{2}b} \pm \sqrt{\frac{9a^2}{32b^2} - \varepsilon/b}. \tag{27}
\]

One in fact sees that the inclusion of the next–order term implies the existence of a further satellite. At \( \varepsilon = 9a^2/(32b) \) the satellites undergo a tangent bifurcation and for \( \varepsilon/b > 9a^2/(32b^2) \) both satellites are ghosts. For \( 0 < \varepsilon/b < 9a^2/(32b^2) \) both satellites are on the same side of the central orbit, while after the period tripling \( \varepsilon = 0 \) they lie opposite to each other. In the limit \( \varepsilon/b \rightarrow -\infty \) the satellites form a broken torus, well separated from the central orbit. When a second parameter is varied to achieve \( a = \varepsilon = 0 \), both satellites are contracted onto the central orbit in the codimension two bifurcation.

**D. Period–quadrupling bifurcation \((m = 4)\) and tangent bifurcations of satellites**

There are two variants of the period–quadrupling bifurcation depending on the magnitude of the coefficients \( a \) and \( b \) in the normal form \( h^{(4)} \). In both cases there are two satellites of quadruple period involved that lie at \( \sin 4\phi = 0 \) and are distinguished by the quantity \( \cos 4\phi = \pm 1 \equiv \sigma \). Their radial distance is given by \( I^{(\sigma=1)} = -\varepsilon/(4a) \) and \( I^{(\sigma=-1)} = -\varepsilon/(4b) \). In the “touch–and–go” case \( \varepsilon/b \rightarrow -\infty \) one satellite becomes a ghost while in turn a ghost solution becomes real and emerges from the central orbit. In the island–chain scenario \( \varepsilon/b \rightarrow -\infty \) and \( b \) unstable satellite becomes real and emerges from the central orbit. In the island–chain scenario \( \varepsilon/b \rightarrow -\infty \), \( a \) unstable and \( b \) on one side of the bifurcation and two real satellites on the other, one of them being stable and the other unstable.

The next–order terms in the extended normal form \( H^{(4)} \) involve three new parameters \( c, d \) and \( e \) and give rise to six satellites. For \( e = 0 \) there are two satellites on the lines \( \cos 4\phi = 1 \) and two on the lines \( \cos 4\phi = -1 \) as well as the two satellites discussed in the derivation of the normal form \( H^{(4)} \). There is a tangent bifurcation at \( 2a^2 = 3ec \), where the satellites on \( \cos 4\phi = 1 \) coalesce, and another one at \( 2b^2 = 3ed \) that involves the satellites on \( \cos 4\phi = -1 \). A great variety of possible configurations of all six satellites exists. Here we are, however, only concerned with the codimension
two bifurcation, described by $H^{(4)}$ and encountered for $\varepsilon = a = 0$. It involves only three satellites, those on the lines $\cos 4\phi = 1$ with radial coordinates

$$I^{(1)}_{\pm} = -\frac{1}{3} a \mp \sqrt{\frac{1}{9} a^2 - \frac{1}{3} \varepsilon} c$$

(22)

and that satellite with $\cos 4\phi = -1$ which is closer to the center and lies with $H^{(4)}$ at $I^{(1)} = -\varepsilon/(4b)$. Compared to the situation described by $H^{(4)}$ the second satellite on the line $\cos 4\phi = -1$ is shifted to infinity; the two satellites at $I = (b-a)/d$ have now angular coordinates $\cos 4\phi \approx -5$ and are therefore ghosts. (Certainly they eventually may become real at finite values of $\varepsilon$ and $a$, far away from the codimension two bifurcation and therefore out of the scope of the present work.) A tangent bifurcation is met at $\varepsilon = 2a^2/(3c)$ provided that $I = -a/(3c) > 0$, since the Cartesian coordinates $(H)$ are otherwise imaginary. Sequences of a tangent bifurcation at positive $I$ and the two variants of quadrupling bifurcations are shown in Figure 4.

### E. Period–m bifurcation with $m \geq 5$ and tangent bifurcations of satellites

The codimension one bifurcations for $m \geq 5$ follow the island–chain pattern already encountered for $m = 4$: There are two satellites that are ghosts on one side of the bifurcation and real on the other, one of them being stable and the other unstable. The stable and unstable periodic points form a chain similar to the broken rational tori that appear in almost integrable systems. Indeed, the $\phi$–dependent terms in the normal forms are of the form of a small perturbation in that situation.

For $m \geq 5$ the usual Birkhoff normal forms describe even the codimension two bifurcations: in addition to the orbits participating in the period–$m$ bifurcation of codimension one they also account for the satellites that are involved in the subsequent tangent bifurcations. In the case $m = 5$ one obtains three satellites at $\sin 5\phi = 0$ that satisfy on the $p$–axis

$$\varepsilon + a p^2 + \frac{5}{4\sqrt{2}} b p^3 = 0.$$  

(23)

As for $m = 1$ it is the discriminant

$$D = 2 \left( \frac{2}{5} \varepsilon b \right)^2 + \frac{2}{5} \frac{a}{b} \left( \frac{8}{15} a \right) \left( \frac{8}{15} a \right)^2$$

(24)

of the equation that governs the number of real solutions. There is the period–5 bifurcation at $\varepsilon = 0$ and a tangent bifurcation at $\varepsilon = -128a^3/(675b^2)$. That sequence is depicted in Figure 4.

In the case $m = 6$ one finds four satellites, two on each of the lines $\cos 6\phi = \pm 1 \equiv \sigma$ at

$$I^{(1)}_{\pm} = -\frac{1}{3} a \pm \sqrt{\frac{1}{9} a^2 - \frac{1}{3} \varepsilon} c.$$  

(25)

Tangent bifurcations take place at independent parameter combinations $\varepsilon = a^2/[(b+\sigma c)]$ (provided that the $I$–coordinate is not negative). A sequence with two tangent bifurcations at positive values of $I$ is shown in Figure 4. All four satellites approach the center in the codimension two bifurcation as $\varepsilon$ and $a$ are sent to zero.

For $m \geq 7$ there are even more satellites in the Birkhoff normal form than the four that participate in the codimension two scenario. In first order, the relevant satellites lie on the lines $\cos m\phi = \pm 1 \equiv \sigma$ at a radial distance

$$I^{(1)}_{\pm} = -\frac{1}{3} a \pm \sqrt{\frac{1}{9} a^2 - \frac{1}{3} \varepsilon} b.$$  

(26)

which is independent of $\sigma$. The $\phi$–dependent term induces a small correction of order $\sigma I^{m/2-2}$. Before the tangent bifurcations, which are encountered at almost identical values $\varepsilon \approx a^2/(3b)$, both satellite pairs have complex $I$. After the bifurcation both the inner as well as the outer orbits form island chains that are visible in phase space if the $I$–coordinate is positive. The corresponding sequence is shown in Figure 4.

### IV. Periodic Orbits in Semiclassical Approximations

Individual contributions of periodic orbits to semiclassical expressions of traces lose their validity close to bifurcations. A collective treatment of the involved orbits is then necessary. Here we discuss general aspects [1–4,8] in which uniform approximations that provide the required regularization are integrated in the next section.

Gutzwiller’s trace formula relates the trace of the retarded Green’s function

$$G(E) = \frac{1}{E + i0^+ - H}$$

(27)

of an autonomous system with Hamiltonian $H$ to the properties of the periodic orbits in the classical system and thus provides a semiclassical expression of the density of states

$$d(E) = -\frac{1}{\pi} \text{Im} \text{tr} G(E).$$

(28)

A suitable starting point for the derivation of periodic–orbit contributions is given by an integral over Poincaré surfaces of section $\Omega$ (see e.g. [4,40]),

$$\text{tr} G(E) = \sum_{n=1}^{\infty} \frac{1}{i\hbar} \int_{\Omega} dq_n dp_n \frac{1}{2\pi\hbar} \frac{1}{n} \frac{\partial^2 \hat{S}^{(n)}}{\partial q_n \partial p_n} \right|^{1/2} \times \exp \left[ \frac{i}{\hbar} \left( \hat{S}^{(n)}(q_n, p_n; E) - q_n p_n \right) - i \frac{\pi}{2} n \right].$$

(29)

This involves the generating function $\hat{S}^{(n)}(q_n, p_n; E)$ of the $n$th iterate of the Poincaré map $(q_n, p_n) \rightarrow (q_n, p_n)$ with
\[ \frac{\partial \hat{S}^{(n)}}{\partial q_n} = p_n , \quad \frac{\partial \hat{S}^{(n)}}{\partial p_0} = q_0 , \quad \frac{\partial \hat{S}^{(n)}}{\partial E} = T , \quad (30) \]

where \( T \) denotes the time elapsed along the trajectory. The integration domain \( \Omega \) might be disjunct and is also used to account for the different sheets of the generating function, i.e., the multivaluedness of \( \hat{S}^{(n)}(q_n, p_0; E) \). Finally, \( \nu \) is the Morse index.

A similar expression can also be obtained for periodically driven systems that are described stroboscopically by a time-evolution operator \( F \). For convenience we set the stroboscopic period to unity. \( F \) represents in general the unitary operator of a quantum map with a classical limit. The eigenstates of \( F \) are stroboscopically stationary, and the phases of the unimodular eigenvalues are called quasienergies. The quasienergy spectrum is encoded in the traces \( \text{tr} F^n \). With the Van Vleck propagator one obtains the expression

\[
\text{tr} F^n = \int_{\Omega} \frac{dq_n dp_0}{2\pi \hbar} \left| \frac{\partial^2 \hat{S}^{(n)}}{\partial q_n \partial p_0} \right|^{1/2} \exp \left[ \frac{i}{\hbar} \left( \hat{S}^{(n)}(q_n, p_0) - q_n p_0 \right) - \frac{i \pi \nu}{2} \right] \quad (31)
\]

where \( \hat{S}^{(n)}(q_n, p_0) \) is now the generating function of the \( n \)-step map with

\[
\frac{\partial \hat{S}^{(n)}}{\partial q_n} = p_n , \quad \frac{\partial \hat{S}^{(n)}}{\partial p_0} = q_0 , \quad (32)
\]

and the integration domain \( \Omega \) lies in phase space [again, accounting also for the multivaluedness of \( \hat{S}^{(n)}(q_n, p_0) \)].

We consider contributions to \( \text{tr} F^n \) or \( i\hbar \text{tr} G(E) \), the factor being introduced to facilitate a parallel investigation of both cases. Let us examine the contribution of an arbitrarily chosen region \( \Omega' \),

\[
C_{\Omega'} = \frac{1}{2\pi \hbar} \int_{\Omega'} dq' dp \Psi(q', p) \exp \left[ \frac{i}{\hbar} \Phi(q', p) - \frac{i \pi \nu}{2} \right] , \quad (33)
\]

with the notations \( q' \equiv q_n, p \equiv p_0 \). For the contributions to \( i\hbar \text{tr} G(E) \) the phase function \( \Phi \) and the amplitude function \( \Psi \) are

\[
\Phi(q', p) = \hat{S}^{(n)}(q', p; E) - q' p \quad (34)
\]

\[
\Psi(q', p) = \frac{1}{n} \frac{\partial \hat{S}^{(n)}}{\partial E} \left| \frac{\partial^2 \hat{S}^{(n)}}{\partial q' \partial p} \right|^{1/2} . \quad (35)
\]

For contributions to \( \text{tr} F^n \) they read

\[
\Phi(q', p) = \hat{S}^{(n)}(q', p) - q' p \quad (36)
\]

\[
\Psi(q', p) = \left| \frac{\partial^2 \hat{S}^{(n)}}{\partial q' \partial p} \right|^{1/2} . \quad (37)
\]

The main contributions to the integral \( C_{\Omega'} \) arise from regions \( \Omega' \) around the stationary points. These are given by \( \frac{\partial \Phi}{\partial q'} = 0 \) and \( \frac{\partial \Phi}{\partial p} = 0 \) which corresponds to

\[
\frac{\partial \hat{S}^{(n)}}{\partial q_n} = q_n , \quad \frac{\partial \hat{S}^{(n)}}{\partial p_0} = p_0 . \quad (38)
\]

The solutions are the periodic points of a given energy in the autonomous case or given stroboscopic period in the driven case. These regions come naturally into focus when one uses the steepest–descent method to find the leading–order term of an asymptotic expansion of the integral in \( \hbar \). To achieve this goal the integration variables are complexified and the initial contour is deformed such that the maxima lie at the solutions of \( \hat{S}^{(n)} \). From the maxima one follows paths of steepest descent of the integrand. The new contour has to originate from the original one by a continuous deformation without crossing singularities. In order to construct a contour that connects the original integration boundaries one automatically visits also some complex ‘ghost’ solutions of \( \hat{S}^{(n)} \). Only the ghosts that lie on the deformed contour are relevant. We come back to this point in the discussion of global bifurcations in section [4].

The size of the region \( \Omega' \) of almost stationary phase depends on \( \hbar \) and is determined by the condition that the typical variation of the phase \( \Phi(q', p)/\hbar \) over that region is of order \( 2\pi \). The region can contain several stationary points, i.e., phase space points of a number of periodic orbits, and each orbit may contribute with several points.

The region shrinks if \( \hbar \) is sent to zero while all parameters are fixed at general values (codimension zero), and finally splits into regions containing only a single periodic point. Each of these gives rise to a usual stationary–phase (sp) contribution

\[
C_{\Omega'}^{(sp)} = T_0 \exp \left[ \frac{i}{\hbar} \hat{S}^{(n)}(0) - i \frac{\mu}{2} \right] / n_0 \left| 2 - \text{tr} M^{(n)}(r) \right|^{1/2} . \quad (39)
\]

There are \( n_0 \) such contributions from each orbit. Moreover, repetitions of an orbit are regarded as independent here. Four canonical invariant characteristic quantities of periodic points enter this semiclassical expression, the primitive period \( T_0 \), the action \( S^{(n)} \), the stability factor \( \text{tr} M^{(n)} \), and the Maslov index \( \mu \). They also determine the uniform approximations to be derived in the next section.

In the periodically driven case, \( T_0 = n_0 \) is the primitive stroboscopic period that was introduced in section [4]. (Recall that we set the stroboscopic period to unity.) The action \( S \) is given by the value of Hamilton’s principal function, \( S = \int (p dq - H dt) \), or, equivalently, by the value of the phase

\[
S = \hat{S}^{(n)}(q, p) - qp \quad (40)
\]

at the periodic point. The linearized \( n \)-step map \( M^{(n)} \) was likewise introduced in section [1]. It is connected to
the second derivatives of \( \tilde{S}^{(n)} \) and involved in the expression through its trace

\[
\text{tr} \, M^{(n)} = 1 + \frac{\left( \frac{\partial^2 \tilde{S}^{(n)}}{\partial q \partial p} \right)^2 - \frac{\partial^2 \tilde{S}^{(n)}}{\partial q^2} \frac{\partial^2 \tilde{S}^{(n)}}{\partial p^2} }{\frac{\partial^2 \tilde{S}^{(n)}}{\partial q \partial p}} .
\]

Finally, there is the Maslov index \( \mu = \nu - \frac{1}{2} \text{sign} \, G \) where \( \nu \) is the Morse index and \( \text{sign} \, G \) denotes the difference in the number of positive and negative eigenvalues of the matrix

\[
G = \begin{pmatrix} \frac{\partial^2 \Phi}{\partial q^2} & \frac{\partial^2 \Phi}{\partial q \partial p} \\ \frac{\partial^2 \Phi}{\partial p \partial q} & \frac{\partial^2 \Phi}{\partial p^2} \end{pmatrix},
\]

involving second derivatives of \( \Phi = \tilde{S}^{(n)} - q'p \).

The four quantities are almost of the same form in the autonomous case. The primitive period \( T_0 \) is the smallest period \( T \) [see eq. (33)] after which one comes back to the initial point when starting somewhere on the trajectory of the periodic orbit. The orbit shows up as a fixed point in all \( n \)-step maps on the surface of section where, as before, \( n = n_0r \), and \( r = T/T_0 \) is an integer counting repetitions. The quantity \( \hat{S} \) is again given by (41) but corresponds now to the reduced action \( S = \hat{p} \hat{q} \) \( dq \).

The linearized map on the surface of section and the stability factor are connected to the generating function in the same way as before, and the Maslov index is again connected to the Morse index and the matrix (12) by

\[
\mu = \nu - \frac{1}{2} \text{sign} \, G.
\]

**V. UNIFORM APPROXIMATIONS**

**A. Breakdown of stationary phase near bifurcations**

From the bifurcation condition (4) it follows that the individual contribution (33) of an orbit blows up close to a bifurcation and even diverges right at \( \text{tr} \, M = 2 \). The assumption under which a stationary–phase approximation is reasonable is then no longer fulfilled: It does not suffice to expand the generating function up to second order around the trajectory,

\[
\Phi(q',p) = S_0 - \frac{\omega}{2} q'^2 - \frac{\sigma}{2} p^2 + \mathcal{O}(3),
\]

since the coalescing orbits cannot be separated even in the limit \( h \to 0 \). In the two–parameter family of Hamiltonians (1) the stationary–phase approximation is in danger close to bifurcations of codimension one and two. In practical applications with small but finite values of \( h \), one typically even visits regions in which bifurcations of higher codimension are felt. Along the same line, bifurcations of codimension two are even felt when only a single parameter (frequently, none at all) is varied; this observation is indeed our principal incentive.

In this section we derive normal forms for the phase function \( \Phi \) and the amplitude function \( \Psi \) in (33) that supersede the quadratic form (13) and yield regular expressions as a substitute for the stationary–phase result (33) in regions affected by bifurcations of codimension two.

**B. Normal forms for phase and amplitude**

To each normal form of the Hamiltonian there is a corresponding expression for the generating function which carries over to the phase function \( \Phi \). This function has as many stationary points as the Hamiltonian. The simplest functional form that can be achieved is identical to the normal form of the Hamiltonian, but with altered coefficients. (Observe, however, that although this form of \( \Phi(q',p) \), when expressed by canonical polar coordinates, obeys again rotational symmetries \( C_m \), this is no longer the case for the map generated by it.) To illustrate this identification we note that the normal forms effectively describe the integrable dynamics of an autonomous system with one degree of freedom. In action–angle variables \( J, \psi \) the evolution over a time interval of duration one is generated by

\[
\tilde{S}(J, \psi') = S_0 + (\psi' + 2\pi n)J - H(J),
\]

such that \( J = J' \) and \( \psi = \psi' - \omega(J) \mod 2\pi \); this gives rise to branches of the generating function that are here enumerated by \( n \). We circumvent this obstacle by considering the map in Cartesian coordinates, generated by \( \tilde{S}(q',p) \), and appealing to the canonical invariance of the leading–order term in the \( h \)--expansion that we are looking for. [This does not affect the option to shift finally to canonical polar coordinates in the integral expression (33).] The transformation to these variables yields new coefficients: for instance, while the second derivatives of \( H \) around a stable orbit involve the bare stability angle \( \omega \), we are led for the generating function to the relation (41) with \( \text{tr} \, M = 2 \cos \omega \). The coefficients in \( \Phi \) are therefore related, but not identical to those in the normal forms of the Hamiltonian, although we will not reflect this in a change of notation; however, minus signs will be introduced following a convention that is motivated by eq. (41).

In section (4) we observed that the normal forms describe in some cases additional orbits that do not participate in the bifurcations of codimension two. In addition we will see that there are in general not enough coefficients to yield independent actions and semiclassical amplitudes in the stationary–phase approximation. To overcome these restrictions one has to consider the influence of higher–order terms in the normal forms on the classical properties of the orbits. These terms can be eliminated in the region of almost stationary phase by non–canonical
transformations. Such transformations can also be used to get rid of the additional non–bifurcating orbits described by the original normal form. The Jacobian of the transformation enters the amplitude function Ψ, which can be simplified further by partial integrations. The procedure is sketched in section V D. It results in the following normal forms for Φ and Ψ, which are obtained by inserting the normal forms for Φ and Ψ prescribed by the original normal form. The Jacobian of the transformation enters the amplitude function Ψ, which can be simplified further by partial integrations. The procedure is sketched in section V D. It results in the following normal forms for Φ and Ψ.

\[
\begin{array}{|c|c|}
\hline
m & \Phi^{(m)}(q', p) - S_0 \\
\hline
1 & -\varepsilon q^2 - aq^4 - bq^6 - \frac{9}{2}p^2 \\
2 & -\varepsilon q^2 - aq^4 - bq^6 - \frac{7}{2}p^2 \\
3 & -\varepsilon I - aI^3/2 \cos 3\phi - bI^2 \\
4 & -\varepsilon I - aI^2(1 + \cos 4\phi) - bI^2(1 - \cos 4\phi) - cI^3(1 + \cos 4\phi) \\
5 & -\varepsilon I - aI^2 - bI^5/2 \cos 5\phi - \varepsilon C^I I^3/2 \cos 5\phi \\
6 & -\varepsilon I - aI^2 - bI^3 - cI^3 \cos 6\phi - \varepsilon dI^2 \cos 6\phi \\
\geq 7, odd & -\varepsilon I - aI^2 - bI^3 - I^{3/2}(cI + \varepsilon d) \cos m\phi \\
\geq 8, even & -\varepsilon I - aI^2 - bI^3 - I^{3/2}(cI + \varepsilon d) \cos m\phi \\
\hline
\end{array}
\]

\[\text{(45)}\]

In \(\Phi^{(1,2)}\) we demand |σ| = 1. The normal forms for \(m \geq 3\) are expressed in canonical polar coordinates \(I, \phi\) defined in equation (41). Note that terms show up in the expressions for \(m \geq 5\) that cannot be expressed ‘perturbatively’ as \(q''q^k\) in Cartesian coordinates (41). The substitutions \(m\phi = 2\psi\) for \(m\) even and \(m\phi = \psi\) for \(m\) odd provide a potentially useful regularization.

Collective contributions of bifurcating periodic orbits are obtained by inserting the normal forms for Φ and Ψ into (33).

Numerically useful expressions in terms of Taylor series can be found for some of the integrals in \(\Phi^{(m)}\) \((m = 1, 2)\) and \(\Psi^{(m)}\) \((m = 3)\). The integrals for \(m = 1, 2\) can also be easily evaluated by the method of steepest descent (cf. section I B) since they are essentially one–dimensional. For \(m \geq 3\), however, a two–dimensional steepest–descent manifold might be quite difficult to construct. It is perhaps more convenient to deform only the \(I\) coordinate into the complex, yielding a simple steepest–descent contour for each fixed, real \(\phi\), and then to perform the \(\phi\)–integral of finite range.

C. Determination of coefficients

The coefficients in the normal forms have to be expressed by the classical quantities \(S, \text{tr}\, M\) of the orbits in order to obtain a contribution that is invariant under canonical transformations \([1, 12, 21, 34, 35]\). This is achieved by examination of the stationary–phase limit \(h \to 0\) of (33) while all other coefficients are fixed. With definition (34), each stationary point gives rise to a contribution

\[
C^{(sp)} = \frac{\Psi}{\sqrt{|\text{det}\,G|}} \exp \left[ \frac{i}{h} \Phi - \frac{i}{2} \left( \nu - \frac{1}{2} \text{sign} G \right) \right],
\]

which determines the parameters by comparison to (33). The collective contributions are then not only applicable in the immediate neighbourhood of the bifurcation, but also far away, where they split into a sum of isolated contributions of Gutzwiller type (39) with correct amplitudes and phases. Consequently they constitute uniform approximations. (Transitional approximations of the type mentioned in the introduction are obtained if one uses \(\Psi = 1\) instead.)

In detail, the properties of the central orbit determine \(\varepsilon\) and \(S_0\), since the stationary point in the origin gives

\[
C^{(sp)} = \exp \left[ iS_0/h - i\frac{1}{2} \nu \left( \sigma + \text{sign} \varepsilon \right) \right] / |\varepsilon| \quad \text{for} \quad m = 1, 2 \quad \text{recall that} \quad |\sigma| = 1 \quad \text{and} \quad C^{(sp)} = \exp \left[ iS_0/h - i\frac{1}{2} \nu \left( \sigma + \text{sign} \varepsilon \right) \right] / |\varepsilon| \quad \text{for} \quad m \geq 3.
\]

The remaining coefficients of the phase function are uniquely determined by the actions of the satellites. It turns out to be helpful to use an ansatz where the coefficients are expressed by scaled positions on a radial line connecting the satellites with the central orbit. For even \(m\) with two real satellites on such a line, for instance, we put them on scaled positions at \(x_1 = \pm 1, x_2 = \pm y\), corresponding to

\[
\frac{d\Phi}{dx} = Ax(x^2 - 1)(x^2 - y^2),
\]

integrate to obtain \(\Phi(x)\) and determine \(y\) from the (scale invariant) ratio

\[
\frac{S_1 - S_0}{S_2 - S_0} = \frac{1 - 3y^2}{y^4(y^2 - 3)}.
\]

Without restriction we can demand \(0 \leq y \leq 1\); then there is exactly one solution. The factor \(A\) follows from the absolute value of \(S_1 - S_0\), and the scale of \(x\) is fixed by knowledge of \(\varepsilon\). In the case of complex satellites they are placed at \(x = \pm 1 \pm iy\), i. e., \(\Phi = Ax^2[x^4 + 3x^2(y^2 - 1) + 3(y^2 + 1)^2]\), and \(y\) is obtained from

\[
\frac{\text{Re} \, S_1 - S_0}{\text{Im} \, S_1} = \frac{1 + 9y^2 - 9y^4 - y^6}{16y^9}.
\]

There is a solution with \(|y| < 1\) and another one with \(|y| > 1\). The right choice takes into consideration whether the ghost with \(\text{Im} \, S > 0\) lies on the steepest–descent
contour connecting the integration boundaries or not; see section [V].

The approach presented here to obtain the coefficients of \( \Phi \) works also for the other normal forms. Moreover, the stationary–phase result is a linear combination of the coefficients of the amplitude function \( \Psi \), which are therefore easily obtained by comparison with the semiclassical amplitudes in (24). For \( m \geq 5 \) there is a symmetry–related pair of ‘spurious’ ghost satellites (analogous to those already discussed for \( m = 4 \)), which are negligible not only since they do not lie on a steepest–descent contour (see again section [V]), but also because a ‘magic’ cancellation in course of the derivation (see below) entails \( \Psi = 0 \) at their positions. Fortunately, exactly one extra term in \( \Psi \) shows up in these cases which can be used to achieve this suppression. It seems reasonable that one uses this approach also in the case \( m = 4 \), where an extra coefficient is also at one’s disposal.

D. Derivation of normal forms

We perform now the reduction of the original normal forms for \( \Phi \) (identical in appearance to those of \( H \)) to the forms listed above and obtain in parallel the expressions of \( \Psi \). The remainder consists of higher orders in \( I \) as well as \( a, \varepsilon \). Once more we invoke Vieta’s relations and regard the coefficients as certain orders of the typical actions \( I \) and perform a partial integration of the first term that yields a term \( \sim \hbar Q^l = 6 \). This reduces the amplitude function to its normal form \( \Psi^{(2)} \).

In the case \( m = 3 \) we have to consider the influence of the terms \( cI^{5/2} \cos 3\phi + dI^3 \). We can safely use \( \varepsilon = O(I^4) \) and \( \sigma = O(I^{1/2}) \) as upper bounds in orders of the typical distance \( I \) of the satellites to the center, which is in turn connected with the size of the region of almost stationary phase. Note that this does not impose a restriction on the relative order of these parameters as long as they are small enough. Having this in mind, it is easy to see that the elimination of the extra term can be achieved by a transformation \( I = J + A J^{3/2} \cos 3\psi \) and \( \phi = \psi + B J^{1/2} \sin 3\psi \). The corresponding coefficients \( A \) and \( B \) give rise to a cancellation of the order \( J^{1/2} \) in the Jacobian. This order also does not show up in the original \( \Psi \), since it can expressed by terms of the form \( q^m p^k \) and obeys a three-fold symmetry. Hence, the order \( J^{1/2} \) is absent in \( \Psi^{(3)} \). Higher orders are again eliminated by partial integrations, carried out now with respect to \( I \). The integration over \( \phi \) suppresses terms that are odd in \( \phi \), such as \( \sin 3\phi \). Note that one could work alternatively with the form \( \Psi^{(3)} = 1 + \alpha I + \beta I^{3/2} \cos 3\phi \); the equivalence to \( \Psi^{(3)} \) is again worked out by a partial integration.

Only three of the six satellites described by \( H^{(3)} \) (12) are involved in the codimension two bifurcation with pealing to the splitting lemma of catastrophe theory. In the region of almost stationary phase the higher–order terms act as a perturbation and can be eliminated by substituting \( q' = Q + AQ^2 + BQ^3 \) with suitably chosen coefficients. The Jacobian of the transformation involves \( dq' = dQ(1 + 2AQ + 3BQ^2) \) and gives the normal form \( \Psi^{(4)}(q', p) \) announced above. The two additional coefficients are determined by the remaining stability factors \( tr M \) of the satellites. Corrections to \( \Phi \) of even higher order would carry over to higher–order terms in \( \Psi \). They involve additional coefficients and on first sight allow for ambiguities, but can be eliminated by successive partial integrations. The term of highest order is written as

\[
Q^l \sim Q^{l-3} \frac{d\Phi^{(1)}}{dQ} + \text{terms of order } l - 1, l - 2. \tag{51}
\]

The partial integration of the first term gives an order \( hQ^{l-1} \) and a boundary contribution that vanishes for \( l \geq 4 \). In the course of this procedure the constant 1 and the coefficients \( \alpha \) and \( \beta \) in \( \Psi^{(4)} \) acquire next–to–leading order corrections in \( h \) that can be discarded and reflect canonically non–invariant properties of the orbits.

For \( m = 2 \) the higher–order terms have to obey the reflection symmetry \( q' \rightarrow -q' \) and are on the \( q' \)-axis of the form \( dq^{18} + eq^{10} \). They are eliminated by \( q^2 = Q^2 + AQ^4 + BQ^6 \). The Jacobian involves only even orders of \( Q \). Terms of order \( l \geq 6 \) can be eliminated in \( \Psi \) by writing

\[
Q^l \sim Q^{l-5} \frac{d\Phi^{(2)}}{dQ} + \text{terms of order } l - 2, l - 4 \tag{52}
\]

and performing a partial integration of the first term that yields a term \( \sim hQ^{l-6} \). This reduces the amplitude function to its normal form \( \Psi^{(2)} \). The auxiliary variables \( I, J, A, B \) are not to be determined from explicit expansions of Hamiltonians or generating functions but rather from the actions and stability properties of the orbits as explained above; otherwise no canonically invariant result would be obtained.

For \( m = 1, 2, 3 \) all the periodic points described by the original normal forms are involved in the bifurcation. For \( m = 1 \) there is at least one real orbit which we place into the origin by a shift of \( q' \). This results in the normal form \( \Phi^{(1)} \). The three coefficients \( \varepsilon, a, b \) as well as the value \( S_0 = \Phi^{(1)}(0, 0) \) are fixed by the three actions of the orbits and one of the stability factors. From the normal form follows \( \Psi = 1 \), the other stability factors are therefore not yet independent. At a little distance to the bifurcation, however, higher–order terms in the Hamiltonian or the phase function act as a perturbation, and the implied relations between the classical quantities of the orbits are no longer valid. For \( m = 1 \) we consider terms of type \( c q^5 + dq^6 \). Terms involving \( p \) effectively do not alter the final expression and can be discarded by applying to the splitting lemma of catastrophe theory. In the region of almost stationary phase the higher–order terms act as a perturbation and can be eliminated by substituting \( q' = Q + AQ^2 + BQ^3 \) with suitably chosen coefficients. The Jacobian of the transformation involves \( dq' = dQ(1 + 2AQ + 3BQ^2) \) and gives the normal form \( \Psi^{(4)}(q', p) \) announced above. The two additional coefficients are determined by the remaining stability factors \( tr M \) of the satellites. Corrections to \( \Phi \) of even higher order would carry over to higher–order terms in \( \Psi \). They involve additional coefficients and on first sight allow for ambiguities, but can be eliminated by successive partial integrations. The term of highest order is written as

\[
Q^l \sim Q^{l-3} \frac{d\Phi^{(1)}}{dQ} + \text{terms of order } l - 1, l - 2. \tag{51}
\]

The partial integration of the first term gives an order \( hQ^{l-1} \) and a boundary contribution that vanishes for \( l \geq 4 \). In the course of this procedure the constant 1 and the coefficients \( \alpha \) and \( \beta \) in \( \Psi^{(4)} \) acquire next–to–leading order corrections in \( h \) that can be discarded and reflect canonically non–invariant properties of the orbits.

For \( m = 2 \) the higher–order terms have to obey the reflection symmetry \( q' \rightarrow -q' \) and are on the \( q' \)-axis of the form \( dq^{18} + eq^{10} \). They are eliminated by \( q^2 = Q^2 + AQ^4 + BQ^6 \). The Jacobian involves only even orders of \( Q \). Terms of order \( l \geq 6 \) can be eliminated in \( \Psi \) by writing

\[
Q^l \sim Q^{l-5} \frac{d\Phi^{(2)}}{dQ} + \text{terms of order } l - 2, l - 4 \tag{52}
\]

and performing a partial integration of the first term that yields a term \( \sim hQ^{l-6} \). This reduces the amplitude function to its normal form \( \Psi^{(2)} \).
m = 4. The normal form $\Phi^{(4)}$ given above has been given a deeper foundation in the discussion of the corresponding Hamiltonian $H^{(4)}$. The outer satellite on the lines $\cos 4\psi = -1$ is shifted to infinity and is infinitely unstable, $|\text{tr } M| = \infty$. The two satellites at radial distance $I = (b - a)/c$ have $\cos 4\psi \approx -5$. They are consequently ghosts with real actions and do not contribute in the stationary–phase limit (for a deeper foundation see section IX); moreover, they are quite far away from the bifurcating orbits. We assume for that reason that their influence is negligible. This leaves us with a normal form that is again completely determined by the actions and one stability factor. We assume that the non–bifurcating satellites remain negligible even under the influence of higher–order terms. The corresponding amplitude normal form $\Psi^{(4)}$ is irreducible under further partial integrations, but has one more coefficient than needed to account for independent stability factors of the satellites. In analogy to the situation to be discussed for $m \geq 5$ we can use this coefficient to yield $\Psi = 0$ at the position of the unwanted ghosts in favor of an additional suppression.

For $m = 5$ there are three bifurcating satellites but not enough coefficients in the original normal form to account for independent actions. Observe that a scaling transformation $I = A J$ does not affect the values of the phase function at the stationary points. Only the two combinations $\varepsilon^2/a$ and $\varepsilon^5/2b$ enter these values which have to match the three actions of the satellites. (The coefficients are determined uniquely if one takes the stability factors into consideration.) Independent actions are admissible after allowing for higher–order terms which are once more removed by a transformation in order to yield no spurious additional stationary points. The next–order term, conveniently expressed as $+\varepsilon c I^3/2 I^3$, can be eliminated by a transformation

$$I = J + c J^{3/2} \cos 5\psi \quad (53)$$

$$\phi = \psi - \frac{c}{2} J^{1/2} \sin 5\psi \quad (54)$$

which is similar to the one for $m = 3$, but now the order $J^{1/2}$ survives in the Jacobian,

$$\Psi = 1 - c J^{1/2} \cos 5\psi + \mathcal{O}(J) \quad (55)$$

The transformation gives rise to the term $-\varepsilon c J^{3/2} \cos 5\psi$ in $\Phi$ and provides us with the additional coefficient $c$. [For illustrative purposes we wish to mentioned here that the coefficient $\varepsilon c$ in $\Phi$ can be treated as $\mathcal{O}(J^{*3/2})$, where $J^*$ gives the order of the distance of the satellites to the central orbit. This order is related to the coefficients $\varepsilon$ and $a$ by application of Vieta’s relations to the stationary–point equation $\partial \Phi / \partial I = 0$.] The next–order corrections to $\Phi$ give even one coefficient more in $\Psi$ than necessary for independent stability factors of the bifurcating orbits. There is, however, an extra pair of satellites at $I = I^{(0)} = -\varepsilon c/b$ that even approaches the center as $\varepsilon \to 0$. The angular coordinate, however, obtains a large imaginary part since it obeys $(I^{(0)})^{1/2} \cos 5\phi = 1$, where a term of order $a$ has been dropped. Indeed this yields in leading order (53) a vanishing $\Psi$ and encourages us to use the extra coefficient to accomplish suppression of the unwanted ghosts.

A similar situation is encountered for $m = 6$: Only three independent actions of four bifurcating satellites can be modeled with the original normal form, but higher–order terms give rise to corrections that lead to the given normal forms of phase and amplitude. An extra coefficient is again present to suppress the ghost pair at $I = -\varepsilon d/c$, and the Jacobian of the transformation turns out to be once more in favor of such a strategy.

Enough coefficients for the actions are present in the original normal form of $\Phi$ for $m \geq 7$, but the expression yields more stationary points than desired. The $\phi$–dependent terms of highest order in $I$ can be eliminated in favor of terms of lower order by successive substitutions $I = J + A J^l \cos m\phi$. This procedure can be carried out in parallel to substitutions $I = J + B J^l$ that aim at the elimination of $\phi$–independent terms. The coefficients of the remaining terms reflect the values at the stationary point up to the order that yields them as independent from each other and allows also for independent stability factors through the expression for $\Psi$. In the derivation one encounters again a Jacobian that vanishes at the location of the unwanted ghost pair at $I = -\varepsilon d/c$. We should note that the coefficient $c$ is here of order $a$, or, equivalently, $\varepsilon^{1/2}$.

VI. STOKES TRANSITIONS

A. Preliminary remarks

We mentioned in section XIV that a steepest–descent contour has also maxima in complexified phase space which correspond to ghost solutions of the stationary–point equations. A helpful rule in that respect is that only ghosts with $\text{Im } \Phi > 0$ can lie on the steepest–descent contour. Moreover, satellites that disappear in period–$m$ bifurcations of codimension one with even $m$ are afterwards ‘self–conjugated’ ghosts, that is, map onto themselves under complex conjugation, and have therefore real classical quantities (in canonical polar coordinates they have real $\phi$ and $I < 0$); accordingly, they do not contribute. The ghosts immediately beyond a period–$m$ bifurcation of codimension one with odd $m \geq 5$ have almost real action, a small imaginary part only being introduced from higher orders, and also do not contribute. For the remaining ghosts, however, it cannot be avoided to construct the contour in order to find out whether they are relevant or not (although the majority of relevant ghosts will be close to reality, i.e., about to bifurcate).

A steepest–descent contour consists of different sheets. The phase $\text{Re } \Phi$ in the exponent of the integrand is constant on each sheet and thus given by its value at the
periodic point, that is, the real part of its action. For
general combinations of the control parameters (codi-
mension zero) there will be only one orbit lying on each
sheet, though it is possible that it does so with more
than one of the points along its trajectory. Imagine now
that for one combination of the parameters a ghost lies
on the steepest–descent contour while for another one
it does not. The ghost is denoted by + in the follow-

\[ q'_{\pm} = -\frac{3a}{8} \pm \sqrt{\frac{9a^2}{64b^2} - \frac{1}{2}}. \quad (57) \]

(A tangent bifurcation is now encountered at \( \varepsilon = 9a^2/32b \). Two orbits coalesce also at \( \varepsilon = 0 \), but both
remain real there due to the construction.) The orbit at
the origin has the action \( S_0 \), while

\[ S_{\pm} = S_0 - q^2 \left( \frac{\varepsilon}{2} + \frac{a}{4} q_{\pm} \right) \quad (58) \]

for the satellites. The Stokes transition takes place at

\[ \varepsilon = \frac{3}{16} (3 + \sqrt{3}) \frac{a^2}{b}. \quad (59) \]

Figure 8 displays the integration contour in the complex
\( q' \)–plane for \( \varepsilon = 3(3 + \sqrt{3})/16, a = b = 1 \) together with
the equipotential lines of \(| \exp[i\Phi] |\) (or, equivalently, of
\( \Im \Phi \)). The plot demonstrates the well–known existence
of the connection and is characteristic even for arbitrary
sets of parameters that fulfill \( (54) \) since the shape of the
contour is fully determined by the combination \( \varepsilon b/a^2 \).
The contour expands linearly with a scaling of \( q \), such
that we can achieve, for instance, \( a = b \), and does not change if \( \Phi^{(1)} \) is multiplied by a real constant, which
allows to set \( b = 1 \).

For even \( m \) the problem is mapped on the case \( m = 2 \).
We already determined the locations \( (20) \) of the orbits
for the Hamiltonian normal form \( H^{(2)} \). By convention,
coefficients changed sign in the definition of \( \Phi^{(2)} \), but
the coordinates are not affected by this. The satellites have
the actions

\[ S_{\pm} = S_0 + \frac{a}{3b} \varepsilon - \frac{2}{27} b^2 + 2b \left( \frac{1}{9b^2} - \frac{1}{3b} \right)^{3/2}. \quad (60) \]

From the condition \( \Re S_{\pm} = S_0 \) one finds a Stokes tran-
sition of complex ghosts for \( a = 0 \) if \( \varepsilon b > 0 \). (No transition
is encountered at \( \varepsilon = \frac{3}{4} \frac{a^2}{b} \) or for \( a = 0 \), but \( \varepsilon b < 0 \), since
the radicand is positive then). As for \( m = 1, 3 \) there is
only one scale–invariant parameter combination \( \varepsilon b/a^2 \),
but for each value there are now two variants of the con-
tour depending on \( \text{sign}(ab) \). The Stokes transition at
\( a = 0 \) involves a ghost with a nonvanishing imaginary
part of the action. Figure 10 shows how the integration
contour changes in the complex \( q' \)–plane due to the tan-
gent bifurcation and the Stokes transition. The plots
prove the existence of a path that connects both sheets
at constant \( \Re \Phi \).

Different situations with \( \Re S_{\pm} = S_0 \) appear, however,
for \( \varepsilon = \frac{1}{3} \frac{a^2}{b} \). No Stokes transition happens there because
the satellite involved is real for \( ab < 0 \) and a ghost with
real action and real \( I < 0 \) for \( ab > 0 \). Figure 11 confirms
that indeed the contours are separated by another ‘real
ghost’ (dashed contour).

For \( m = 5 \) the phase function on the \( p^- \) axis is a poly-
nomial of degree five. It appears on first sight that the three
satellites could be arranged in such a way that the sheet
of the ghost \(+\) is separated from the sheet of the central
orbit \( 0 \) by the remaining real satellite \( 1 \). One easily finds,
however, that the sheets are separated for the particular
phase function only if the real parts of the three roots \( p_k, p_1 \),
\( p_1 \) of equation (23) all have the same sign, which in turn
can be ruled out by a careful inspection of equation (23):
From Vieta’s relations it follows that otherwise the coeffi-
cient of the linear term \( \sim [p_+p_-+(p_++p_-)p_1] \) would not
vanish. Note that the order \( T^{3/2} \) in \( \Phi^{(3)} \) indeed results in
a nonvanishing coefficient but is considered, as usually,
only as a perturbation and does not alter the situation
qualitatively. The existence of a steepest–descent con-
nection of the ghost and the central orbit at the Stokes
transition is then guaranteed also for \( m = 5 \). The tran-
sition takes place at the value of \( t = 75\varepsilon b^2/(4a^2) \) that is
the solution of
\[
640 + \frac{1520 t}{3} + \frac{925 t^2}{9} + t^3 = 0
\]
(61)
close to \( t = -97.6566 \). To derive this equation we first
reduce the expression for \( \text{Re} S - S_0 \) with help of the fixed–
point equation
\[
8t + 75r^2(4+5r) = 0
\]
(62)
for the scaled variable \( r \sim p \) to the form \( \text{Re} [300r^2 + t(8–10r + 75r^2)] = 0 \). Introducing \( r = (x+iy) \) here and in
the fixed–point equation and splitting the latter into real
and imaginary part, one can solve then for \( y^2 = 8x/5 + 3x^2
\) and \( t = 15x(2 + 5x^2) \) and obtains the cubic equation
\( 8 + 60x + 125x^2 + 75x^3 = 0 \). Its roots give three values
for \( t \) which all solve (22). Only one of the roots, however,
fulfills \( y^2 > 8x/5+3 > 0 \); this is the one that corresponds
to the approximate value of \( t \) given above.

For odd \( m \geq 7 \) there are four different satellites on each
line. They are generically grouped in pairs that lie on
opposite sides of the center, and the situation is similar to
\( m = 2 \) with slightly broken reflection symmetry. Indeed
it follows from the derivation of the normal forms that the
odd terms originating from higher–order perturbations
are negligible for small \( \varepsilon \) and \( a \); the results for \( m = 2 \) are
then directly applicable to the present case.

VII. CONCLUSIONS

We studied bifurcations of codimension two in Hamilton-
ian systems that are either autonomous and have two
degrees of freedom or periodic with one degree of free-
dom. The normal forms derived in section I and dis-
cussed in section III show that the typical sequences
of codimension one bifurcations in the neighbourhood (in
parameter space) of the bifurcation of codimension two
consists of a period–\( m \) bifurcation at a central orbit fol-
lowed by a tangent bifurcation in which satellites become
ghosts.

Additional generic scenarios are encountered in the
presence of symmetries \([10,21]\). Isochronous pitchfork
bifurcations are the most important addition of codimen-
sion one in the case of time–reversal or reflection symme-
tries; they will also show up in the neighbourhood of
codimension two bifurcations in these systems.

Only a small number of the bifurcations of codimension
one and two correspond to (special cases of) a so–called
elementary catastrophe due to Thom (see e. g. \([17,18]\)).
These appear in many different contexts and describe, for
instance, bifurcations of codimension up to four in maps
that are not restricted by area preservation. We use the
usual names and symbols and further denote each Hamil-
tonian bifurcation type by \( \langle m_k \rangle \), where \( m \) is the multi-
plicity and \( k = 1, 2 \) the codimension. The fold \( A_2 \) corre-
sponds to the tangent bifurcation \( (1_1) \). The cusp \( A_3 \)
is \( (1_2) \), and \( (2_1) \) is a cusp with a reflection symmetry. \( (2_2) \)
is a butterfly \( A_5 \) with reflection symmetry. The period–
tripling bifurcation \( (3_1) \) corresponds to a version of the
elliptic umbilic \( D_5^+ \). All other normal forms describe
catastrophes that would be of much higher codimension
without area preservation. Especially for the cases \( m \geq 3 \)
one has to rely on higher–order perturbation theory. It
implies that i) for a given codimension the class of bi-
furcations in Hamiltonian systems is considerably larger
and ii) although this can be circumvented by considering
a normal form of much higher codimension from ordinary
catastrophe theory, these normal forms have then again
to be restricted: Points that correspond to the trajectory
of one and the same orbit lie on the same height (energy
or action). The classical perturbation theory takes care
of this and in addition gives the right codimension.

Collective contributions to semiclassical traces were de-
erived that involve normal forms for a phase function \( \Phi \)
and an amplitude function \( \Psi \). The expressions involve
just as many coefficients as are determined by the ac-
tions and stability properties of the bifurcating orbits,
including a suppression of certain unwanted ghosts for
\( m \geq 4 \). The expressions constitute uniform approxima-
tions: They are also valid far away from the bifurcation
and asymptotically take the form of a sum of isolated
contributions \([8]\). The uniform approximations display
Stokes transitions in which the ghost satellites interact
once more with the central orbit and leave the steepest–
descent integration contour.

The validity of the approximations given here is lim-
ited if additional orbits become important or ‘unwanted’
ghosts become real; bifurcations of even higher codimen-
sion are then to be studied. The basic steps would be
the same as in the present study: Derivation of Hamilton-
ian normal forms that account for all bifurcating orbits;
reduction of normal forms to get rid of non–bifurcating
orbits and to account for independent stabilities and ac-
tions. An important open question is concerned with the
complexity of periodic–orbit clusters typically en-
countered in the quest of resolving spectra when one approaches the semiclassical limit. This requires knowledge of the dynamics up to the Heisenberg time $\sim 1/\hbar$ and involves a competition of increasing resolution in phase space and proliferation of periodic orbits.

One might also be concerned about cascades, which are sequences of bifurcations of a certain orbit of period $n$ at differing values of $\text{tr} M^{(n)} = 2 \cos \omega$ (cf. section II B). The most prominent example is the basic building block of period–doubling cascades: an orbit of period $2n$ is born at an orbit of period $n$ in a period–doubling bifurcation ($\text{tr} M^{(2n)} = 2$) and period–doubles itself at $\text{tr} M^{(2n)} = -2$. A huge variety of cascades exists, however, since bifurcations happen whenever the stability angle $\omega$ is a rational multiple of $2\pi$. Bifurcations in a cascade cannot be encountered simultaneously in parameter space, since this would imply a singular change in the linearized map. For that reason cascades cannot be regarded as unfoldings of bifurcations of higher codimension. The bifurcations in an unfolding show up simultaneously in a given iteration of the map; the cascades involve bifurcations that appear in distinct iterations. One could study, for instance, those cascades that arise from the iteration of the map generated by a normal form, and ask the question whether situations exist in which the orbits in the cascade must be treated collectively; it would be indeed nice to see that one can do without. An argument in favor of this expectation has been given in [7].

ACKNOWLEDGMENTS

The author thanks F. Haake, C. Howls, J. Keating, D. Sadovskii, and M. Sieber for helpful discussions. Support by the Sonderforschungsbereich ‘Unordnung und große Fluktuationen’ of the Deutsche Forschungsgemeinschaft is gratefully acknowledged.

[1] M. C. Gutzwiller, J. Math. Phys. 12, 343 (1971).
[2] M. C. Gutzwiller, Chaos in Classical and Quantum Mechanics (Springer, New York, 1990).
[3] M. Tabor, Physica D 6, 195 (1983).
[4] G. Junker and H. Leschke, Physica D 56, 135 (1992).
[5] M. V. Berry and M. Tabor, Proc. R. Soc. Lond. A 349, 101 (1976).
[6] M. V. Berry and M. Tabor, J. Phys. A 10, 371 (1977).
[7] A. M. Ozorio de Almeida and J. H. Hannay, J. Phys. A 20, 5873 (1987).
[8] A. M. Ozorio de Almeida, Hamiltonian Systems: Chaos and Quantization (Cambridge University Press, Cambridge, 1988).
[9] M. Kuś, F. Haake, and D. Delande, Phys. Rev. Lett. 71, 2167 (1993).
[10] M. Sieber, J. Phys. A 29, 4715 (1996).
[11] H. Schomerus and M. Sieber, J. Phys. A 30, 4537 (1997).
[12] M. Sieber and H. Schomerus, chaodynamics/9708013 accepted for publication in J. Phys. A (1997).
[13] K. R. Meyer, Trans. Am. Math. Soc. 149, 95 (1970).
[14] A. D. Bruno, Math. USSR Sbornik 12, 271 (1970).
[15] A. D. Bruno, preprint Nr. 18, Inst. Prikl. Mat. Akad. Nauk SSSR, Moskau (in Russian) (1972).
[16] R. J. Rimmer, Memoirs of the AMS 272, American Mathematical Society, Providence, Rhode Island.
[17] M. Golubitsky and I. Stewart, Physica D 24, 391 (1987).
[18] M. A. M. de Aguiar, C. P. Malta, M. Baranger, and K. T. R. Davies, Ann. Phys. (N. Y.) 180, 167 (1987).
[19] M. A. M. de Aguiar and C. P. Malta, Physica D 30, 413 (1988).
[20] A. M. Ozorio de Almeida and M. A. M. de Aguiar, Physica D 41, 391 (1990).
[21] D. A. Sadovskii, J. A. Shaw, and J. B. Delos, Phys. Rev. Lett. 75, 2120 (1995).
[22] D. A. Sadovskii and J. B. Delos, Phys. Rev. E 54, 2033 (1996).
[23] H. Poincaré, New Methods of Celestial Mechanics (Dover, New York, 1957), Vol. III.
[24] G. D. Birkhoff, Dynamical Systems (American Mathematical Society, New York, 1927).
[25] A. Deprit, Celest. Mech. 1, 12 (1969).
[26] V. I. Arnol’d, Geometrical Methods of the Theory of Ordinary Differential Equations, Vol. 250 of Series of Comprehensive Studies in Mathematics (Springer, New York, 1988).
[27] H. Schomerus, Europhys. Lett. 38, 423 (1997).
[28] H. Schomerus and F. Haake, Phys. Rev. Lett. 79, 1022 (1997); a more detailed exposition is in preparation.
[29] J. Main and G. Wunner, Phys. Rev. A 55, 1743 (1997).
[30] M. V. Berry, Proc. R. Soc. Lond. A 422, 7 (1989).
[31] P. A. Boasman and J. P. Keating, Proc. R. Soc. Lond. A 449, 629 (1995).
[32] K. R. Meyer and G. R. Hall, Introduction to Hamiltonian Dynamical Systems and the N-Body Problem, Vol. 90 of Applied Mathematical Sciences (Springer, New York, 1992).
[33] J.-M. Mao and J. B. Delos, Phys. Rev. A 45, 1746 (1992).
[34] S. Tomsovic, M. Grinberg, and D. Ullmo, Phys. Rev. Lett. 75, 4346 (1995).
[35] D. Ullmo, M. Grinberg, and S. Tomsovic, Phys. Rev. E 54, 136 (1996).
[36] F. J. Wright, J. Phys. A 13, 2913 (1980).
[37] T. Poston and I. N. Stewart, Catastrophe Theory and its Applications (Pitman, London, 1978).
[38] M. V. Berry and C. Upstill, in Progress in Optics, edited by E. Wolf (North-Holland, Amsterdam, 1980), Vol. VIII, pp. 257–346.
FIG. 1. Contour plots of the normal form $H^{(1)}$ as the parameters are steered to cross two tangent bifurcations close to the bifurcation point of codimension two. Initially, only one orbit is present. Two new orbits are born in a tangent bifurcation. One of them approaches the first orbit, and both annihilate in an inverse tangent bifurcation. A similar scenario exists in which stable orbits are unstable and vice versa; it is obtained by reversing the sign of $\sigma$.

FIG. 2. The typical sequence for $m = 2$ of a tangent bifurcation of period–two satellites and a period–doubling bifurcation is illustrated by contour plots of $H^{(2)}$. As for $m = 1$ there exists a similar scenario for the opposite sign of $\sigma$ in which the stability of orbits is changed. The tangent bifurcation would not be encountered in real phase space if the satellites meet at a negative value of $I$.

FIG. 3. The contour plots of the normal form $H^{(3)}$ display a sequence of a tangent bifurcation of satellites and a period–tripling bifurcation.

FIG. 4. The two sequences of contour plots of $H^{(4)}$ display a tangent bifurcation of satellites followed by a period–quadrupling bifurcation. The latter is encountered in its island–chain version above; below we have the touch–and–go scenario.
FIG. 5. The bifurcation scenario close to the codimension two point with $m = 5$, consisting of a tangent bifurcation of satellites and a period-5 bifurcation.

FIG. 6. Varying a parameter close to the codimension two point with $m = 6$, one might observe two pairs of satellites being born in a tangent bifurcation at positive $I$, as displayed here; the satellites closer to the center disappear in a subsequent period-6 bifurcation. The island chain that is left over here could be also steered to the center by letting $\alpha$, and then once more $\varepsilon$ change its sign.

FIG. 7. For $m \geq 7$ the resonant $\phi$-dependence of the normal form is weak, and tangent bifurcations of satellite pairs happen at almost identical values. Subsequently the inner island chain collapses onto the center and disappears. The remaining chain might follow, as explained for $m = 6$.

FIG. 8. Path of steepest descent $\text{Re} \Phi^{(1)}(q, 0) = S_0$ (thick line). The parameters are chosen to fulfill the condition for a Stokes transition. The transition indeed takes place since the contour connects the subdominant ghost with the dominant central orbit. The thin lines are the equipotential lines of $|\exp[i\Phi]|$ (or $\text{Im} \Phi$).
FIG. 9. Sequence of steepest–descent contours for $\Phi^{(2)}(q, 0)$ displaying a tangent bifurcation and a Stokes transition. Again there is a connection of the ghost satellite and the central orbit as the condition for a Stokes transition is fulfilled. Dashed lines indicate steepest–descent paths that are not needed to connect the integration boundaries. A trick can be played with these pictures to envision the situation for the tangent bifurcation at negative $I$: The plots are rotated by 90 degrees (which corresponds to inverting the sign of $a$), and the contour is picked that originally connects $\pm i\infty$ (see also Fig. 10).

FIG. 10. No Stokes transition happens for ghosts with real $I < 0$ in the representative case $m = 2$. The steepest–descent contours from the ghost and the central orbit (full lines) are separated by the (dashed) contour of another ghost at real, but negative $I$. Only the contour that visits the central orbit is needed to connect the integration boundaries.