Weyl invariance with a nontrivial mass scale

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Abstract. A theory with a mass scale and yet Weyl invariant is presented. The theory is not invariant under all diffeomorphisms but only under transverse ones. This is the reason why Weyl invariance does not imply scale invariance in a free falling frame. Physical implications of this framework are discussed.

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1 Introduction

It is often stated that conformal invariance prevents a mass scale to appear in any quantum field theory [1]. When gravitation is dynamical, conformal invariance means Weyl invariance, that is, invariance of the theory under Weyl rescalings

\[ g_{\mu\nu}(x) \rightarrow \Omega^2(x) g_{\mu\nu}(x) \]  

(1.1)

This cherished belief is not true anymore when the theory is not invariant under the full group of diffeomorphisms, Diff but only under the volume preserving subgroup, which corresponds to transverse generators, TDiff. The action of unimodular gravity [3], for example, reads

\[ S_{UG} \equiv \int d^n x \mathcal{L}_{UG} \equiv -M_P^{n-2} \int |g|^{1/n} \left( R + \frac{(n-1)(n-2)}{4n^2} g^{\mu\nu} \nabla_{\mu} g \nabla_{\nu} g \right) \]  

(1.2)

and involves an explicit mass scale. This action is not Diff invariant as advertised; its symmetry is here reduced to

\[ H \equiv \text{TDiff} \ltimes \text{Weyl} \]  

(1.3)

The reason as to how this is at all possible is that when the theory is invariant under TDiff \subset Diff, even though the theory is Weyl invariant, the flat space limit is not necessarily scale invariant, because the Weyl dimension of the measure does not coincide with the scale dimension of \( d^n x \).

Consider, for example, the TDiff \ltimes Weyl coupling of a scalar field with gravity. There are two options. The first one is to consider matter as a Weyl singlet. The action reads

\[ S = \int d^n x \left\{ |g|^{1/n} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right\} \]  

(1.4)

The other option is to assign the usual nontrivial conformal weight to the scalar, that is

\[ \phi \rightarrow \Omega^{-1} \phi \]  

(1.5)

In that case, we only have to realize that the object

\[ |g|^{\frac{1}{2n}} \phi \]  

(1.6)

is a Weyl singlet, so that

\[ S = \int d^n x \left\{ |g|^{1/n} g^{\mu\nu} \nabla_\mu \left( g^{\frac{1}{2n}} \phi \right) \nabla_\nu \left( g^{\frac{1}{2n}} \phi \right) - V( g^{\frac{1}{2n}} \phi ) \right\} \]

\[ = \int d^n x \left\{ |g|^{2/n} \left( \frac{\nabla g}{g} \right)^2 + \frac{1}{n} \frac{\nabla g \cdot \nabla \phi}{g} + (\nabla \phi)^2 \right\} - V( g^{\frac{1}{2n}} \phi ) \]
is again Weyl invariant. In particular the mass term
\[-\frac{m^2}{2}|g|^\frac{1}{n}\phi^2\]  
(1.7)
is perfectly kosher.

In writing so, we have lost longitudinal diffeomorphisms in both cases, but we have gained Weyl invariance. Weyl Ward’s identity read in this case
\[g^{\mu\nu}\frac{\delta S}{\delta g^{\mu\nu}} = 0 \]  
(1.8)
where
\[\Theta_{\mu\nu} \equiv \frac{\delta S}{\delta g^{\mu\nu}} = |g|^{1/n} \left( \frac{1}{n} (\nabla \phi)^2 g_{\mu\nu} - \nabla_{\mu} \phi \nabla_{\nu} \phi \right) \]  
(1.9)
The point is that in spite of the fact that the action (1.4) reduces in a free falling frame to the flat space one, the corresponding “energy-momentum tensor”, \(\Theta_{\mu\nu}\), fails to do so. Actually, only its traceless piece is recovered; the contribution of the potential has disappeared completely. This “energy-momentum” does not generate the conserved energy-momentum in flat space. What happens is that Rosenfeld’s prescription [8] needs the coupling to gravity to be full Diff invariant in order for it to work properly; that is to reduce to Noether’s current in flat space.

Nevertheless, when gravity is dynamical according to (1.2) Bianchi identities ensure consistency (cf. [5] for a short review).

The model can be easily modified to incorporate the Higgs mechanism, that is
\[S_H = \int d^n x \left\{ |g|^{1/n} g^{\mu\nu} \left( \nabla_{\mu} + i e |g|^{\frac{1}{2n}} A_{\mu} \right) \left( g^{\frac{1}{2n}} \phi^* \right) \left( g^{\frac{1}{2n}} A_{\nu} \right) - V(|g|^{\frac{1}{2n}} |\phi|) \right\} \]
\[= \int d^n x \left\{ |g|^{1/2} g^{\mu\nu} \left[ \frac{1}{2n} \nabla_{\mu} |\phi|^2 + \nabla_{\mu} \phi^* \right] \left[ \frac{1}{2n} \nabla_{\nu} |\phi|^2 + \nabla_{\nu} \phi^* \right] - \frac{\lambda}{4!} \left( |g|^{\frac{1}{n}} |\phi|^2 - v^2 \right)^2 \right\} \]  
(1.10)
where a quartic symmetry breaking potential has been incorporated in the last formula.

2 Spontaneous symmetry breaking

Let us summarize. If we are willing to break the full Diff symmetry to its transverse subgroup, we can earn in compensation Weyl invariance. This gauge invariance does not however imply scale invariance in a free falling frame; otherwise Weyl invariance would not be compatible with a nontrivial mass scale. That is in contrast with what happens in full Diff invariant theories [4, 7].

In other words, when the symmetry is reduced to TDiff, Weyl invariance in curved space does not necessarily reduce to scale invariance in flat space.

There are well-known notorious difficulties to break spontaneously any of these symmetries. Let us now study a (rather contrived) way to do that in our case.

Consider two symmetric tensor fields \(f_{\mu\nu}\) and \(\Lambda^{\alpha\beta}\) and the following Diff-invariant term in a lagrangian
\[L_T = \sqrt{\det f_{\mu\nu}} \left[ \Lambda^{\alpha\beta}(x) \left( f_{\alpha\beta} - g_{\alpha\beta} \right) + \mathcal{L}_D \right] \]  
(2.1)
where $\mathcal{L}_D$ is a Diff-invariant lagrangian, and

$$f^N_{\alpha\lambda} \equiv f^\beta_\alpha f^\gamma_\beta \ldots f^\sigma_\lambda \quad (\text{N times}) \quad (2.2)$$

The EM for $\Lambda$ imply that

$$f^N_{\alpha\beta} = g_{\alpha\beta} \quad (2.3)$$

and then

$$f = \det f_{\mu\nu} = g^{1 \over N} \quad (2.4)$$

On $\Lambda$-shell then the symmetry has been reduced to the subgroup $H$

$$L_T = |g|^{1 \over 2N} \mathcal{L}_D \quad (2.5)$$

In that way the Diff invariance will be broken to TDiff invariance, which will be enhanced to Weyl symmetry if the power of $N$ is chosen properly. Let us now work out a simple example of the breaking from Diff to TDiff.

$$S = \int d^n x \left\{ \sqrt{|\det f_{\mu\nu}|} \left[ \Lambda^\alpha\beta(x) (f^\alpha_{\alpha\beta} - g_{\alpha\beta}) + {1 \over 2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right] + \lambda {1 \over 4!} (\phi^2 - v^2)^2 \right\} \quad (2.6)$$

On shell this reduces to

$$S = \int d^n x \left\{ |g|^{1/n} \left[ -{1 \over 2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right] + \lambda {1 \over 4!} (\phi^2 - v^2)^2 \right\} \quad (2.7)$$

A spontaneous symmetry breaking solution can be found in the unimodular gauge where

$$|g| = 1 \quad (2.8)$$

$$S = \int d^n x \left\{ -{1 \over 2} \hat{g}^{\mu\nu} \nabla_\mu \hat{\phi} \nabla_\nu \hat{\phi} + \lambda {1 \over 4!} (\hat{\phi}^2 - v^2)^2 \right\} \quad (2.9)$$

Then, through Weyl rescalings

$$g_{\mu\nu} \equiv \Omega^2 \hat{g}_{\mu\nu} \quad (2.10)$$

we recover a whole Weyl orbit of solutions.

A similar analysis can be easily be worked out for the Higgs model.

### 3 Conclusions

We have discussed Weyl invariant theories that are not scale invariant in a free falling frame, where the space-time metric reduces to the flat one. This is due to the fact that our theories are not invariant under all diffeomorphisms, but only under transverse ones.

We have also discussed simple models in which the Diff invariance is spontaneously broken to TDiff.

A natural question at this point would be: what good is Weyl symmetry in this context and why would we want it?

The answer to that is manyfold.

For once, it forbids a cosmological constant, or rather, any constant term in the lagrangian does not couple to dynamical gravity.
Also the concept of a mass changes, insofar as it applies not really to the scalar field, but rather to the Weyl invariant combination $|g|^{\frac{1}{n}} \phi$. The scalar field $\phi$ has not physical meaning by itself.

The final dream is of course that if Weyl invariance could survive at the quantum level, this theory would be a finite one [6], although the one loop calculations made up to now lead to anomalies in the Weyl Ward identities [3].

More work is needed, however, in this and related issues.

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