Abstract. This work is a study of p-adic multiple zeta values at roots of unity (pMZV\(\mu_N\)'s), the p-adic periods of the crystalline pro-unipotent fundamental groupoid of \(\mathbb{P}^1 - \{0, \mu_N, \infty\}\)/\(\mathbb{F}_q\). The main tool is new objects which we call p-adic pro-unipotent harmonic actions. In this part IV we define and study p-adic analogues of some elementary complex analytic functions which interpolate multiple zeta values at roots of unity such as the multiple zeta functions. The indices of pMZV\(\mu_N\)'s involve sequences of positive integers; in this IV-1, by considering an operation which we call localization (inverting certain integration operators) in the pro-unipotent fundamental groupoid of \(\mathbb{P}^1 - \{0, \mu_N, \infty\}\), and by using p-adic pro-unipotent harmonic actions, we extend the definition of pMZV\(\mu_N\)'s to indices for which these integers can be negative, and we study these generalized pMZV\(\mu_N\)'s.

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1. Introduction

1.1. Complex and \(p\)-adic multiple zeta values at roots of unity. This work is a study of \(p\)-adic multiple zeta values at roots of unity (\(pz\mu_N\)'s), the \(p\)-adic periods of the crystalline pro-unipotent fundamental groupoid (abbreviated \(\pi_1^{un,crys}\)) of \((\mathbb{P}^1 - \{0, \mu_N, \infty\})/\mathbb{F}_q\), with \(\mathbb{F}_q\) of characteristic \(p\) prime to \(N\) which contains a primitive \(N\)-th root of unity. They are \(p\)-adic analogues of the (complex) multiple zeta values at roots of unity (\(Mz\mu_N\)'s), which are the following numbers. Let \(\xi_N \in \mathbb{C}\) be a primitive \(N\)-th root of unity, \(n_1, \ldots, n_d \in \mathbb{N}^*\), and \(j_1, \ldots, j_d \in \mathbb{Z}/N\mathbb{Z}\) such that \((n_d, j_d) \neq (1, 0)\), and \((z_1, \ldots, z_{n_1}) = (\xi^{j_1}, 0, \ldots, 0, \ldots, \xi^{j_d}, 0, \ldots, 0)\), the associated multiple zeta value at \(N\)-th roots of unity is:

\[
\zeta(n_1; (\xi_N^{j_1}))_d = (-1)^d \sum_{0 < m_1 < \ldots < m_d} \frac{(\xi_N^{j_1})^{m_1} \cdots (\xi_N^{j_d})^{m_d}}{m_1^{n_1} \cdots m_d^{n_d}} \in \mathbb{C}
\]

where \(n = n_d + \ldots + n_1\) is called the weight of \((n_1; (\xi_N^{j_1}))_d\), and \(d\) is called its depth.

The \(pz\mu_N\)'s (Definition 3.4) are defined abstractly, without explicit formulas, as \(\mu\)'s found in part I appearing as relations between \(Ad\) of integrals. They are numbers \((\xi_N^{j_1}, \ldots, \xi_N^{j_d})\) of \((\mathbb{P}^1 - \{0, \mu_N, \infty\})/\mathbb{F}_q\), with \(\mathbb{F}_q\), \(d \in \mathbb{N}^* n_1, \ldots, n_d \in \mathbb{N}^*, j_1, \ldots, j_d \in \mathbb{Z}/N\mathbb{Z}\).

In the terminology above, the term "at roots of unity" is usually omitted if \(N = 1\).

1.2. Summary of parts I, II, III. In part I [J I-1] [J I-2] [J I-3], we have found a \(p\)-adic analogue of the series formula in [11] which has two particular features: it is explicit and it keeps track of the motivic Galois action on \(\pi_1^{un,crys}((\mathbb{P}^1 - \{0, \mu_N, \infty\})/\mathbb{F}_q)\).

In part II [J II-1] [J II-2] [J II-3], we have deduced from these formulas a version of the motivic Galois theory of \(pz\mu_N\)'s formulated in terms of series instead of being formulated as usual in terms of integrals.

In part III [J III-1] [J III-2], we have defined and studied a generalization of the notion of \(pz\mu_N\) at roots of unity of order divisible by \(p\).

In all the previous parts, our results led us to replace the \(pz\mu_N\)'s by some variants, equivalent to them in a certain sense, which we call the adjoint \(p\)-adic zeta values at \(N\)-th roots of unity (Definition 3.4), abbreviated \(Adpz\mu_N\)'s. Indeed, the results of [J I-2] [J I-3] show that the \(Adpz\mu_N\) are more directly adapted to explicit computations than the \(pz\mu_N\)'s. The main objects in the previous parts were some group actions called \(p\)-adic pro-unipotent harmonic actions found in [J II-2] and [J II-3].

The explicit formulas for \(pz\mu_N\)'s found in part I appear as relations between \(Adpz\mu_N\)'s and the following numbers called weighted multiple harmonic sums, with \(m \in \mathbb{N}^*\), \(d \in \mathbb{N}^* n_1, \ldots, n_d \in \mathbb{N}^*, j_1, \ldots, j_d+1 \in Z/N\mathbb{Z}\): \[
\text{har}_m (n_1; (\xi_N^{j_1}))_d = \sum_{0 < m_1 < \ldots < m_d < m} \frac{(\xi_N^{j_1})^{m_1} \cdots (\xi_N^{j_d})^{m_d}}{m_1^{n_1} \cdots m_d^{n_d}} \in \mathbb{Q}(\xi)
\]

1.3. Motivation for part IV. The \(Mz\)'s of depth one, i.e. the numbers [11] with \(N = 1\) and \(d = 1\), are the values of Riemann’s zeta function at positive integers, and special values of other classical functions appearing in analytic number theory. These functions have generalizations which depend on any \(d \in \mathbb{N}^*\) variables, defined some iterated series as in [11], as well as generalizations "at roots of unity" taking into account the numerators in the iterated series of [11].

\[\text{For complex or } p\text{-adic } Mz\mu_N\text{'s, } (n_1; (\xi_N^{j_1}))_d \text{ is an abbreviation of } \left(\frac{\xi_N^{j_1}}{n_1}, \ldots, \frac{\xi_N^{j_d}}{n_d}\right), \text{ whereas for multiple harmonic sums, it is an abbreviation of } \left(\frac{\xi_N^{j_1}}{n_1}, \ldots, \frac{\xi_N^{j_{d+1}}}{n_d}\right)\]
Since we constructed in part I, and studied in parts II and III, a p-adic analogue of the series expansions of $\text{MZV}_{\mu N}$’s, we want to know if p-adic analogues of such interpolating functions exist.

1.4. Motivation for parts IV-1. We take as starting point of this paper the most straightforward example of interpolation of the $\text{MZV}_{\mu N}$’s; namely, the multiple zeta functions at $N$-th roots of unity ($\text{MZF}_{\mu N}$’s) : for any $j_1, \ldots, j_d \in \mathbb{Z}/N\mathbb{Z}$ :

$$(s_1, \ldots, s_d) \in U_d \mapsto \sum_{0 < m_1 < \ldots < m_d} \frac{\left( \frac{\zeta_N^{s_1}}{\zeta_N} \right)^{m_1} \ldots \left( \frac{1}{\zeta_N} \right)^{m_d}}{m_1^{s_1} \ldots m_d^{s_d}} \in \mathbb{C}$$

where $U_d = \{(s_1, \ldots, s_d) \in \mathbb{C}^d \mid \text{Re}(s_{d-r+1} + \ldots + s_d) > r \text{ for all } r = 1, \ldots, d\}$. We want to know whether these functions have natural p-adic analogues interpolating $p\text{MZV}_{\mu N}$’s and, if this is the case, we want to study them.

The $\text{MZF}_{\mu N}$’s have a meromorphic continuation to $\mathbb{C}^d$, defined in [E], [M], [Z], [AET], [Go] for $N = 1$, and in [FKMT1] for any $N$. Their meromorphic continuation has singularities along certain hyperplanes, which are identified the most precisely in [AET] and [FKMT1].

One can then define values of these functions at tuples of integers of any sign, i.e. $\zeta\left(\frac{\xi_N^{j_1}}{\xi_N}, \ldots, \frac{\xi_N^{j_d}}{\xi_N}\right)$, for any $n_1, \ldots, n_d \in \mathbb{Z}$; this requires to remove a singularity. This can be done by considering the limit at tuples of integers along a certain direction [AET] [AT] [Ko] [O] [Sa1] [Sa2], or by a certain "renormalization" process [GZ], [MP] [GPZ], or by a certain "desingularization" process [FKMT1].

We note that the question of defining numbers $\zeta\left(\frac{\xi_N^{j_1}}{\xi_N}, \ldots, \frac{\xi_N^{j_d}}{\xi_N}\right)$, for any $n_1, \ldots, n_d \in \mathbb{Z}$ does not necessarily involves the meromorphic continuation of $\text{MZF}_{\mu N}$’s ; one can consider it only via the formulas of equation (1.1) ; we have to find a correct notion of regularizations, either for a generalization of the expression of $\text{MZV}_{\mu N}$’s as iterated series :

$$\zeta_{\text{reg}, \mathbb{C}}\left(\{n_i\}, \left(\frac{\xi_N^{j}}{\xi_N}\right)\right) = \lim_{m \to \infty} \sum_{0 < m_1 < \ldots < m_d < m} \frac{\left( \frac{\xi_N^{j}}{\xi_N} \right)^{m_1} \ldots \left( \frac{1}{\xi_N} \right)^{m_d}}{m_1^{n_1} \ldots m_d^{n_d}}$$

or for a generalization of the expression of $\text{MZV}_\mu$ as iterated integrals, which amounts to :

$$\zeta_{\text{reg}, \mathbb{C}}\left(\{n_i\}, \left(\frac{\xi_N^{j}}{\xi_N}\right)\right) = \lim_{z \to 1} \sum_{0 < m_1 < \ldots < m_d} \frac{\left( \frac{\xi_N^{j}}{\xi_N} \right)^{m_1} \ldots \left( \frac{1}{\xi_N} \right)^{m_d}}{m_1^{n_1} \ldots m_d^{n_d}}$$

In the end, there are several notions of $\zeta\left(\{n_i\}; \left(\frac{\xi_N^{j}}{\xi_N}\right)\right)$ for any $n_1, \ldots, n_d \in \mathbb{Z}$, interrelated in various ways. We would like to know if there are natural p-adic analogues of these values.

The p-adic zeta function of Kubota and Leopoldt, which we will denote by $L_p$, is defined as a p-adic interpolation of the desingularized values of the Riemann zeta function at positive integers, using Kummer’s congruences. Coleman has proved in [Co] that, for all $n \in \mathbb{N}^*$ such that $n \geq 2$, we have $\zeta_{p,1}(n) = p^n L_p(n, \omega^{1-n})$, where $\omega$ is Teichmüller’s character, and $\zeta_{p,1}$ refers to $p\text{MZV}_{\mu N}$’s as denoted in §1.1. This implies that the map $n \in \mathbb{N}^* \subset \mathbb{Z}_p \mapsto p^{-n} \zeta_{p,1}(n) \in \mathbb{Q}_p$ is continuous with respect to $n$ on each class of congruence modulo $p - 1$, and can be extended to a continuous function on $\mathbb{Z}_p$, except for the class of congruence $n \equiv 1 \pmod{p - 1}$, where this is true instead for $n \mapsto (n - 1)\zeta_p(n)$. This property can be retrieved by the following formula, which is known, and is also a particular case of our formulas of part I for $p\text{MZV}_{\mu N}$’s :

$$\zeta_{p,1}(n) = \frac{1}{n - 1} \sum_{l_2 = 1}^{l_1} \binom{-n}{l_2} B_l \sum_{0 < m < p} \frac{p^{m+l-1}}{m^{n+l-1}}$$

This gives hope that $p\text{MZV}_{\mu N}$’s of higher depth might have some p-adic continuity properties and might be interpolated by a continuous function. However, the case of depth one is particular : the Frobenius of $\pi_1^{\text{un,crys}}(\mathbb{P}^1 - \{0, \mu_N, \infty\})$ can be described as a relation between certain p-adic series indexed as $\sum_{0 < m}$ and their variants restricted to $\sum_{0 < m}$ ; in higher depth, the Frobenius is much more complicated.

In higher depth, actually, some generalizations of the Kubota-Leopoldt $L$-function, based on the
desingularization of the meromorphic continuation of MZF$_{\mu_N}$’s, have been defined in [FKMT2]. Some of their values at tuples of positive integers are expressed in terms of $p$-adic iterated integrals on $\mathbb{P}^1 - \{0, \mu_{cp}, \infty\}$, with $c \in \mathbb{N}^*$ prime to $p$ (FKMT2, Theorem 3.41). However, studying the role of these functions in the question explained in §1.3 goes beyond the scope of this paper.

In [FKMT3], some $p$MZV$_{\mu_N}$’s at tuples of integers of any sign are defined in certain particular cases: the indices are $((n_i); (\xi_N^{\mu_i}))$ for any $n_1, \ldots, n_d \in \mathbb{Z}$, but $\xi_N^{1} \neq 1, \ldots, \xi_N^{n} \neq 1$. In terms of our notation of §1.1, they are generalizations of the values $\zeta_{p,-\infty}(w)$. Their definition relies on the theory of Coleman integration in the sense of [V].

1.5. Main ideas. In the formula (1.1), the exponent $n_i$ in the iterated series corresponds to the iteration $n_i - 1$ times of the operator $f \mapsto \int f \frac{dz}{z}$ in the definition of the iterated integral. Replacing this integration operator by its inverse gives similar series (provided it converges) with $n_i$ possibly negative. This has been used in several papers, including in [FKMT3], and we also have used it in [JI-2]. We are going to use again this idea here, but more systematically. We will use the term "localization of $\pi_1^\text{un,DR}(\mathbb{P}^1 - \{0, \mu_N, \infty\})" to refer to the inversion of the all the operators $f \mapsto \int f \omega$ with $\omega$ a differential form $\frac{dz}{z}$, $x \in \{0, \xi^1, \ldots, \xi^n\}$, on $\mathbb{P}^1 - \{0, \mu_N, \infty\}$.

If we consider an iterated integral as in (1.1) but on a variable path (in the sense of [CH]), instead of the path $[0,1] \to [0,1], t \mapsto t$ which is implicit in (1.1), we obtain functions called multiple polylogarithms [Go], characterized as solutions to a certain differential equation (Proposition-Definition 2.3). If we allow the inversion of integration operators, we will obtain "localized multiple polylogarithms", which will be $Q(\xi)$-linear combinations of products of iterated integrals by algebraic functions. This phenomenon already appeared implicitly in [JI-2], §4-§5, via the map "loc" which was used to define what we called the $p$-adic pro-unipotent $\Sigma$-harmonic action. Here, this phenomenon will be studied intrinsically, in particular its $p$-adic aspects. By this phenomenon, all the numbers obtained by considering localized iterated integrals at tangential base-points remain in the same algebra of periods: they are certain $Q(\xi)$-linear combinations of $p$MZV$_{\mu_N}$’s.

The $p$MZV$_{\mu_N}$’s are defined using the notion of Frobenius structure of a $p$-adic differential equation. Thus in order to look for a good meaning of a notion of $p$MZV$_{\mu_N}$’s at sequences of integers of any sign, we should show a compatibility between the localization and the Frobenius structure. We will see that imposing this compatibility makes things actually simpler in the $p$-adic case than in the complex case.

Our idea is to replace the Frobenius by what we called the harmonic Frobenius in [JI-2], and to use $p$-adic pro-unipotent harmonic actions. Although it is possible to define localized $p$-adic multiple polylogarithms by Coleman integration, their regularization at $z \to 1$ is not well-defined in general because they have a pole which is not logarithmic. This is the difficulty observed in [FKMT3]. We will see how to avoid it by replacing the Frobenius by the harmonic Frobenius.

In part 1, we have obtained formulas involving series, representing the $p$MZV$_{\mu_N}$’s. However, both the domains of summation and the summands were functions of the indices $n_d, \ldots, n_1$. Here, if we want to study these sums of series as functions of $n_d, \ldots, n_1$, we will make some changes of variables giving domains of summation independent of $n_d, \ldots, n_1$.

1.6. Outline. In §2, we define the "localization" of $\pi_1^{\text{un,DR}}(X)$ for $X$ equal to a punctured projective line over a field of characteristic zero and the KZ connection associated with it, on a neighborhood of 0 (Definition 2.7). This incorporates a notion of localized multiple polylogarithms (Definition 2.8). We define maps which encode different expressions of the "localized iterated integrals" in terms of algebraic functions and the iterated integrals (Proposition-Definition 2.10, Proposition-Definition 2.13). Then we define the analytic continuation of the localized multiple polylogarithms, in the complex setting (Definition 2.21) and in the $p$-adic setting (Definition 2.26). The $p$-adic setting is applied to $\pi_1^{\text{un,cryst}}(\mathbb{P}^1 - \{0, \mu_N, \infty\}/F_q)$.

In §3 we review the notion of $p$MZV$_{\mu_N}$’s (Definition 3.24, Definition 3.25, Definition 3.33), and the $p$-adic pro-unipotent $\Sigma$-harmonic action $\phi_{\Sigma}^\text{har}$ of [JI-2]. We define some "localized" variants of $\phi_{\Sigma}^\text{har}$ (Proposition-Definition 3.11) and the localized adjoint $p$MZV$_{\mu_N}$’s (Definition 3.12). In this new setting, the localized version of $\phi_{\Sigma}^\text{har}$ defined in [JI-2] will be now viewed as "localized at the source", and we now have two other "localized" variants of $\phi_{\Sigma}^\text{har}$, called, respectively, "localized at the target" and
'localized at the source and target'.

In §4, we explain briefly why the properties from [J I-3] describing formulas for the iteration of the harmonic Frobenius can be generalized to the setting of §3 (Proposition-Definition [J I]).

In §5, we bring together the localization built in §3 and the algebraic relations satisfied by pMZVµN's, which we studied in part II [J II-1] [J II-2] [J II-3] ; and we show that the localized adjoint pMZVµN's satisfy a variant of the adjoint double shuffle relations defined in [J II-1] (Proposition [5.3]).

The main theorem is a summary of the main properties of our localized p-adic multiple zeta values at roots of unity.

We refer to the notion of adjoint double shuffle relations defined in [J II-1].

Theorem IV-1
i) (Nature of localized AdpMZVµN's)
The localized Ad pMZVµN's are in the Q(ξN)-algebra generated by pMZVµN's. In particular, they are periods of the crystalline pro-unipotent fundamental groupoid of (P1 − {0, µN, ∞})/Fq.
The totally negative Ad pMZVµN's (in the sense of Definition [5.15]) are algebraic numbers, in Q(ξ); more precisely, they are in an algebra of functions defined explicitly in terms of polynomials of Bernoulli numbers, or, alternatively, in terms of prime multiple harmonic sums at negative indices. The vanishing of the odd Bernoulli numbers imply the vanishing of certain particular totally negative AdpMZVµN's.

ii) (Formulas)
The formulas of [J I-2] and [J I-3] for AdpMZVµN's can be extended into explicit formulas for the localized Ad pMZVµN's, involving extensions of the p-adic pro-unipotent harmonic action ∇har and the map of iteration of the harmonic Frobenius iterhar.

iii) (Algebraic relations)
The localized Ad pMZVµN's satisfy an extension of the adjoint double shuffle relations

We consider these properties as a justification of our definition of the localized p-adic ZMVs.

We note that this theorem can be extended in an obvious way to the p-adic multiple zeta values at roots of unity of order divisible by p defined in [J III-1].

We will prove later that the formulas of [J I-2] can be modified to be formulas with domains of summation independent of (l, (ni); (ξN)).

We will use this later to deduce that the adjoint pMZV's ζAd(l; n_d, . . . , n_1) (here N = 1) have some continuity properties with respect to n_1 and n_d viewed as p-adic integers.

In the next version of this paper, we will also define a notion of localized pMZVµN's, without the term adjoint. This will be done by using a generalization of the main equation found in [AETbis].

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The idea of inverting integration operators being standard, they may be some references missing and I apologize if this is the case.

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2. Localized complex and p-adic multiple polylogarithms

We formalize a notion of localization on a neighborhood of 0 of the De Rham pro-unipotent fundamental groupoid of and the KZ connection (§2.1), we define "localization maps" enabling to "compute" it (§2.2) and we discuss the analytic continuation of the localized multiple polylogarithms (§2.3) in the complex (§2.3.1) and p-adic (§2.3.2) settings. The p-adic aspects are restricted to the case of P1 − {0, µN, ∞}.

2.1. The De Rham pro-unipotent fundamental groupoid of P1 − {0 = z_0, z_1, . . . , z_r, ∞}, the KZ connection, and its localization on a neighborhood of the origin.

2.1.1. Review on π1,un,DR(P1 − {0 = z_0, z_1, . . . , z_r, ∞}) and ∇KZ. Let K be a field of characteristic zero, z_0, . . . , z_r ∈ K with z_0 = 0 and z_r = 1, and X = (P1 − {0, z_1, . . . , z_r, ∞})/K.
We review $\pi^\text{un,DR}_1(X)$ and the KZ connection (defined and described in [D]), and we define their 'localized' version on a neighborhood of 0.

The De Rham pro-unipotent fundamental groupoid of $X$, denoted by $\pi^\text{un,DR}_1(X)$, is a groupoid in pro-affine schemes over $X$: its base-points are the points of $X$, the points of punctured tangent spaces $T_x - \{0\}$, $x \in \{0, t^1, \ldots, t^N, \infty\}$ called tangential base-points ([D], §15), and the canonical base-point $\omega^\text{DR}$ ([D], (12.4.1)).

All pro-affine schemes $\pi^\text{un,DR}_1(X, y, x)$ are canonically isomorphic as schemes to $\pi^\text{un,DR}_1(X, \omega^\text{DR})$, these isomorphisms being compatible with the groupoid structure ([D], §12). Thus, describing the groupoid $\pi^\text{un,DR}_1(X, y, x)$ is reduced to describing $\pi^\text{un,DR}_1(X, \omega^\text{DR})$, which is done in the next statement.

**Proposition-Definition 2.1.** Let $\mathfrak{c}$ be the alphabet $\{e_0, e_1, \ldots, e_z\}$, and let $\text{Wd}(\mathfrak{c})$ be the set of words over $\mathfrak{c}$ (including the empty word).

i) The shuffle Hopf algebra over $\mathfrak{c}$, denoted by $\mathfrak{O}^{\mathfrak{c},t}$, is the $\mathbb{Q}$-vector space $\mathfrak{Q}(\mathfrak{c}) = \mathfrak{Q}(e_0, e_1, \ldots, e_z)$, which admits $\text{Wd}(\mathfrak{c})$ as a basis, with the following operations:

a) the shuffle product $\eta : \mathfrak{O}^{\mathfrak{c},t} \otimes \mathfrak{O}^{\mathfrak{c},t} \rightarrow \mathfrak{O}^{\mathfrak{c},t}$ defined by, for all words $:(e_{z_{n+1}}, \ldots, e_{z_{n+1}}) \in \mathfrak{e}(e_{z_1}, \ldots, e_{z_1}) = \sum_{\sigma} e_{z_{\sigma^{-1}(1)}, \ldots, e_{z_{\sigma^{-1}(1)}}}$ where the sum is over permutations $\sigma$ of $\{1, \ldots, n \}$ such that $\sigma(n) < \ldots < \sigma(1)$ and $\sigma(n+1) < \ldots < \sigma(n+1)$.

b) the deconcatenation coproduct $\Delta^{\mathfrak{c},t} : \mathfrak{O}^{\mathfrak{c},t} \rightarrow \mathfrak{O}^{\mathfrak{c},t} \otimes \mathfrak{O}^{\mathfrak{c},t}$, defined by, for all words $\Delta^{\mathfrak{c},t}(e_{z_1}, \ldots, e_{z_1}) = \sum_{w=0}^n e_{z_{1}, \ldots, e_{z_{n+1}}} \otimes e_{z_{1}, \ldots, e_{z_{1}}}$.

c) the counit $\epsilon : \mathfrak{O}^{\mathfrak{c},t} \rightarrow \mathfrak{Q}$ sending all non-empty words to 0.

d) the antipode $S : \mathfrak{O}^{\mathfrak{c},t} \rightarrow \mathfrak{O}^{\mathfrak{c},t}$, defined by, for all words $S(e_{z_1}, \ldots, e_{z_1}) = (-1)^i e_{z_i}, \ldots, e_{z_1}.

ii) ([D], §12) The group scheme $\text{Spec}(\mathfrak{O}^{\mathfrak{c},t})$ is pro-unipotent and canonically isomorphic to $\pi^\text{un,DR}_1(X, \omega^\text{DR})$.

Since $\text{Spec}(\mathfrak{O}^{\mathfrak{c},t})$ is pro-unipotent, its points can be expressed in a canonical way in terms of the dual of the topological Hopf algebra $\mathfrak{O}^{\mathfrak{c},t}$. This is written in the next statement, in which, following a common abuse of notation, we denote in the same way the letters $e_z$ and their duals.

**Proposition-Definition 2.2.** i) Let $K(\langle \mathfrak{c} \rangle) = K(\langle e_{z_0}, \ldots, e_z \rangle)$ be the non-commutative $K$-algebra of power series over the variables $e_{z_0}, \ldots, e_z$, with coefficients in $K$. It is the completion of the universal enveloping algebra of the complete free Lie algebra over the variables $e_{z_0}, \ldots, e_z$. It thus has a canonical structure of topological Hopf algebra.

We write an element $f \in K(\langle \mathfrak{c} \rangle)$ as $f = \sum_{w \in \text{Wd}(\mathfrak{c})} f[w]w$, where $f[w] \in K$ for all $w$. We have

$$\begin{align*}
\text{Lie}(\mathfrak{O}^{\mathfrak{c},t})^\vee \otimes_{\mathfrak{Q}} K &= \{ f \in K(\langle \mathfrak{c} \rangle) \mid \forall w \neq 0, w' \neq 0 \in \text{Wd}(\mathfrak{c}), f[w \cdot w'] = 0 \} \\
&= \text{\{ primitive elements of } K(\langle \mathfrak{c} \rangle) \text{\} }
\end{align*}$$

The equation above is called the shuffle equation modulo products.

ii) We have a canonical isomorphism of topological Hopf algebras $\mathfrak{O}^{\mathfrak{c},t} \otimes_{\mathfrak{Q}} K(\langle \mathfrak{c} \rangle)^\vee = K(\langle \mathfrak{c} \rangle)$ and

$$\begin{align*}
\text{Spec}(\mathfrak{O}^{\mathfrak{c},t})(K) &= \{ f \in K(\langle \mathfrak{c} \rangle) \mid \forall w, w' \in \text{Wd}(\mathfrak{c}), f'[w \cdot w'] = f[w]f[w'], \text{ and } f[0] = 1 \} \\
&= \text{\{ grouplike elements of } K(\langle \mathfrak{c} \rangle) \text{\} }
\end{align*}$$

The equation above is called the shuffle equation.

We now review the connection $\nabla_{\text{KZ}}$ associated with $\pi^\text{un,DR}_1(X)$ and multiple polylogarithms, viewed first as power series.

**Proposition-Definition 2.3.** (follows from [D], §7.30 and §12)

i) The connection associated with $\pi^\text{un,DR}_1(X)$, called the Knizhnik-Zamolodchikov (for short, KZ) connection of $X$, is the connection on $\pi^\text{un,DR}_1(X, \omega^\text{DR}) \times X$ defined by $\nabla_{\text{KZ}} : f \mapsto df - \sum_{i=0}^d f_{x_{z_i}}(x_{z_i}) f_i$.

ii) The coefficients of its horizontal sections are iterated integrals of $\frac{dz_{z_i}}{z_{z_i}}$, $j = 0, \ldots, r$ (in the sense of Chen [C]), if $K \rightarrow \mathbb{C}$, and in the sense of Coleman [Go] if $K \rightarrow \mathbb{C}_p$ is unramified, and called multiple polylogarithms [Go].

Assume $K$ is embedded in $\mathbb{C}$ or $\mathbb{C}_p$ for $p$ a prime number. For $d \in \mathbb{N}^*$, $n_1, \ldots, n_d \in \mathbb{N}^*$, $j_1, \ldots, j_d \in \{1, \ldots, r\}$, let $\text{Li}(\langle n_i \rangle; (z_{j_i}))_d \in K[[z]]$ be the formal iterated integral of the sequence of differential
forms \( \frac{dz}{z} \cdots \frac{dz}{z} \) \( \frac{dz}{z} \cdots \frac{dz}{z} \) \( \frac{dz}{z} \cdots \frac{dz}{z} \). Then, for \( z \in K \) such that \( |z| < 1 \), we have:

\[
(2.3) \quad L_i^0 ((n_i); (z_j))_d(z) = \sum_{0 < m_1 < \ldots < m_d} \frac{(\frac{z}{n_1})^{n_1} \cdots (\frac{z}{n_d})^{n_d}}{m_1 \cdots m_d} \in K
\]

2.1.2. Localization of \((\pi^\infty_{1, \text{DR}}(\mathbb{P}^1 - \{0 = z_0, z_1, \ldots, z_r, \infty\}), \nabla_{KZ})\) on a neighborhood of zero. We now formalize the localization on a neighborhood of 0 of \( \pi^\infty_{1, \text{DR}}(X_K), \nabla_{KZ} \) (Definition 2.2).

Let \( A \) be a ring, and \( S \) a multiplicative subset of \( A \). The localization of \( A \) at \( S \) is the ring \( S^{-1} A \) representing the subfunctor of \( \text{Hom}(A, -) \) defined by the homomorphisms mapping \( S \) to units. Explicitly, \( S^{-1} A \) is the ring whose elements are sums of elements of the form \( x_1 y_1^{-1} x_2 y_2^{-1} \cdots x_i y_i^{-1} \), with \( x_i \in A \), \( y_i \in S \). The representability of the functor above is granted because it is continuous and satisfies the solution set condition. This notion is mostly usual when \( A \) is commutative, or if the \((A, S)\) satisfies Ore’s conditions, which are a weak variant of the commutativity assumption.

For us, localizing \( \pi^\infty_{1, \text{DR}}(X) \) will mean replacing \( \mathcal{O}^w.e \), the Hopf algebra of \( \pi^\infty_{1, \text{DR}}(X_K, \omega_{\text{DR}}) \), regarded as a ring whose multiplication is the concatenation of words, by its localization at the part of non-zero elements (which is multiplicative because it is an integral ring). We define a ring which will have a surjection onto the localization; this will be practical for writing some results.

**Definition 2.4.** Let \( e^\text{inv} \) be the alphabet \( \{e_0^\text{inv}, e_1^\text{inv}, \ldots, e_{z_r}^\text{inv}\} \). Let \( \epsilon \cup e^\text{inv} \) be the alphabet \( \{e_0, e_1, \ldots, e_r, e_0^\text{inv}, e_1^\text{inv}, \ldots, e_{z_r}^\text{inv}\} \).

Let \( \text{Wd}(\epsilon \cup e^\text{inv}) \) be the set of words over \( \epsilon \cup e^\text{inv} \).

Let \( K(\langle \langle \epsilon \cup e^\text{inv} \rangle \rangle) \) the non-commutative \( K \)-algebra of formal power series over the variables equal to the letters of \( \epsilon \cup e^\text{inv} \).

It is convenient to reformulate the KZ equation \( \nabla_{KZ}(L) = 0 \) as a fixed-point equation:

**Definition 2.5.** Let the integration operator \( K[[z]][\log(z)](\langle \epsilon \rangle) \to K[[z]][\log(z)](\langle \epsilon \rangle) : \)

\[
\text{Int}_{KZ} : L \mapsto \int_1^z \left( \frac{dz'}{z'} e_0 + \frac{dz'}{z'} e_1 L \right)
\]

Let \( L \in K[[z]][\log(z)](\langle \epsilon \rangle) \) whose coefficient of \( z^0 \) is \( \exp(e_0 \log(z)) \). We have the equivalence \( \nabla_{KZ}(L) = 0 \iff \text{Int}_{KZ}(L) = L \). These conditions are also equivalent to saying that \( L \) is the non-commutative generating series of multiple polylogarithms in the sense of Proposition-Definition 2.8. This way to formulate the KZ equation gives rise to the following definition of its localized variant:

**Definition 2.6.** Let

\[
\text{Int}^\text{loc}_{KZ} : L \mapsto \left( e_0^\text{inv} + \frac{d}{dz} + e_1^\text{inv} (z - 1) \frac{d}{dz} \right) L + \int_1^z \left( \frac{dz'}{z'} e_0 + \frac{dz'}{z'} - 1 e_1 \right) L
\]

We say that the equation \( \text{Int}^\text{loc}_{KZ}(L) = L \) is the localized version of the equation \( \nabla_{KZ}(L) = 0 \).

**Definition 2.7.** The localization on a neighborhood of 0 of \( \pi^\infty_{1, \text{DR}}(X_K), \nabla_{KZ} \) is the data of the \( K \)-algebra \( K(\langle \langle \epsilon \cup e^\text{inv} \rangle \rangle) \), the inclusion \( \pi^\infty_{1, \text{DR}}(X, \omega_{\text{DR}})(K) \subset K(\langle \langle \epsilon \cup e^\text{inv} \rangle \rangle) \), and the operator \( \text{Int}^\text{loc}_{KZ} \).

**Proposition-Definition 2.8.** i) The localized KZ equation has a unique solution \( L_0^\text{loc} \) such that \( L \in K[[z]][\log(z)](\langle \langle \epsilon \cup e^\text{inv} \rangle \rangle) \) whose coefficient of \( z^0 \) is \( \exp(e_0 \log(z)) \).

We call localized \( p \)-adic multiple polylogarithms the coefficients \( L_0^\text{loc}(z) \subset K[[z]][\log(z)] \).

ii) \( L_0^\text{loc} \) viewed as an element of \( K(\langle \langle \epsilon \cup e^\text{inv} \rangle \rangle) \) descends to an element of the \( (K(\langle \epsilon \rangle) - \{0\})^{-1} K(\langle \langle \epsilon \cup e^\text{inv} \rangle \rangle) \), and further to \( (K(\langle \epsilon \rangle) - \{0\})^{-1} K(\langle \epsilon \rangle)/\text{I_comm} \), where \( \text{I_comm} \) is the ideal generated by the relations \( e_z^0 e_z^1 = e_z^1 e_z^0 = 1 \).

**Proof.** Clear. \( \square \)

The next statement is a generalization of the expression of the power series expansions of multiple polylogarithms in terms of multiple harmonic sums ([Go], equation (1)):
Proposition-Definition 2.9. i) We call localized multiple harmonic sums the following numbers, for \( d \in \mathbb{N}^* n_1, \ldots, n_d \in \mathbb{Z}, j_1, \ldots, j_{d+1} \in \mathbb{Z}/\mathbb{N} \mathbb{Z}, m \in \mathbb{N}^* : 
\begin{align*}
\mathfrak{h}_m ((n_i); (\xi^N_i))_d &= \sum_{0 < m_1 < \ldots < m_d < m} \frac{(\frac{z_{j_1}}{z_{j_d}})^{m_1} \ldots (\frac{z_{j_{d+1}}}{z_{j_d}})^{m_d} (\frac{1}{z_{j_1+1}})^m}{m_1^{n_1} \ldots m_d^{n_d}} \nend{align*}
and weighted localized multiple harmonic sums the numbers 
\begin{align*}
\operatorname{har}_m \left( (n_i); (\xi^N_i) \right)_d &= m^{n_1 - \tilde{n}_1 + \ldots + n_d - \tilde{n}_d} \mathfrak{h}_m ((n_i); (\xi^N_i))_d 
\end{align*}

\begin{align*}
\text{ii)} \text{ Assume that } K \text{ is embedded in } \mathbb{C} \text{ or in } \mathbb{C}_p. \text{ For all } n_d, \tilde{n}_d, \ldots, n_1, \tilde{n}_1 \in \mathbb{N}^*, j_1, \ldots, j_{d+1} \in \mathbb{Z}/\mathbb{N} \mathbb{Z}, \text{ for all } z \in K, |z| < 1, \text{ the series below is absolutely convergent and we have : } 
\begin{align*}
\operatorname{Li}_{0}^{\text{loc}} \left[ \epsilon_0^{n_d-1} (e_0^{\text{inv}})^{\tilde{n}_d} e_{z_{j_d}} \ldots \epsilon_0^{n_1-1} (e_0^{\text{inv}})^{\tilde{n}_1} e_{z_{j_1}} \right] &= \sum_{0 < m_1 < \ldots < m_d} \frac{(\frac{z_{j_1}}{z_{j_d}})^{m_1} \ldots (\frac{z_{j_{d+1}}}{z_{j_d}})^{m_d}}{m_1^{n_1 - \tilde{n}_1} \ldots m_d^{n_d - \tilde{n}_d}} 
\end{align*}
\end{align*}
The coefficients of \( \operatorname{Li}_{0}^{\text{loc}} [w] \) with \( w \) of the form \( \tilde{w} e_x^{\text{inv}}, z \in \{z_0, z_1, \ldots, z_r\} \), are equal to 0.

2.2. Computation of the localization. We define two "localization maps", expressing the localization of \((x^{\text{inv}, \text{DR}}(\mathbb{P}^1 - \{0 = z_0, z_1, \ldots, z_r, \infty\}), \nabla_{\mathbb{K}Z})\) on a neighborhood of zero, in terms of iterated integrals and algebraic functions. For certain statements, we restrict for simplicity the study to the localization at the multiplicative part generated by \( e_0 \).

2.2.1. The localization map for multiple polylogarithms \( \operatorname{loc}^f \). We write the coefficients of \( \operatorname{Li}_{0}^{\mathbb{K}Z} \) as \( \mathbb{Q} \)-linear combinations of products of algebraic functions on \( \mathbb{P}^1 - \{0, \mu_N, \infty\} \) by coefficients \( \operatorname{Li}_{0}^{\mathbb{K}Z}[w] \) with \( w \in \mathcal{W}(e_{X_K}) \). In the next statement, we view \( \operatorname{Li}_{0}^{\text{loc}} \) as a map \( \mathcal{O}^{m,e}_{X} \to \mathcal{C}[[z]] \), and we view \( \operatorname{Li} \) as a map \( \mathcal{O}^{m,e} \to \mathcal{C}[[z]] \).

Proposition-Definition 2.10. Assume \(-1 \in \{z_1, \ldots, z_r\} \) (otherwise replace \( X \) by \( X' = (\mathbb{P}^1 - \{0, z_1, \ldots, z_r, -1, \infty\})/K \)). There exists a map 
\begin{align*}
\operatorname{loc}^f : \mathcal{O}^{m,e}_{\operatorname{loc}} \to \Gamma(X, \mathcal{O}_X) \otimes \mathcal{O}^{m,e}
\end{align*}
such that we have 
\begin{align*}
\operatorname{Li}^0_{0} &= (\operatorname{Li}^0_0 \circ \Gamma(X, \mathcal{O}_X)) \circ \operatorname{loc}^f
\end{align*}

Proof. Let us call weight the number of letters of a word \( w \) over the alphabet \( e \cup e^{\text{inv}} \). By induction on the weight, we are reduced to prove that for \( x \in \{0, z_1, \ldots, z_n\} \), and \( w \) a word over \( e \), and let \( n \in \mathbb{N}^* \). Then \( \int_0^{\frac{dz}{x - x'}} \frac{dz'}{(z - z')^n} \operatorname{Li}[w](z') \) is a \( \Gamma(X, \mathcal{O}_X) \)-linear combination of multiple polylogarithms.

If \( x \neq 0 \), we write \( \int_0^{\frac{dz}{x - x'}} \frac{dz'}{(z - z')^n} = \frac{1}{-x} \sum_{l \geq 0} (-1)^l \binom{n}{l} \left( \frac{1}{x} \right)^l = \frac{1}{-x} \sum_{l \geq 0} (-1)^l (-n - 1 - l)^l \binom{n}{l} \right) \)

We use that \( (-n - 1)^l \) is a polynomial function of \( l \).

If \( x = 0 \), integration by parts the KZ equation and induction on the weight reduces us to the case of \( w \) is of weight 1, in which case \( \operatorname{Li}[w](z') = \log(z' - x') \) with \( x' \in \{0, z_1, \ldots, z_r\} \); the result is then proved by another integration by parts.

2.2.2. The localization map for multiple harmonic sums \( \operatorname{loc}^\mathbb{C} \). We review from [J1-2] the map \( \operatorname{loc}^\mathbb{C} \) giving an expression of the localized multiple harmonic sums (in the sense of Definition 2.9) in terms of multiple harmonic sums and some polynomials of the upper bound of the domain of iterated summation.

Definition 2.11. Let \( S \) be a subset of \( \mathbb{N} \).

i) A connected partition of \( S \) is a partition of \( S \) into segments.

ii) An increasing connected partition of \( S \) is a connected partition of \( S \) with an order \( < \) on the corresponding set of parts of \( S \), such that if \( C < C' \), we have \( j < j' \) in \( \mathbb{N} \) for all \( j \in C \) and \( j' \in C' \).

iii) For a part \( P \) of \( \{1, \ldots, d\} \) and an increasing connected partition of \( P \) and the connected components of \( \{1, \ldots, d\} - P \).

iv) Let \( \partial S \) be the set of \( x \in S \) such that \( x + 1 \notin S \) or \( x - 1 \notin S \).

We apply the previous definitions to define a way to represent localized words which is adapted to our purposes.
Definition 2.12. Let \( w = (t_0, \ldots, t_l) \in \mathbb{Z}^d \).

i) Let \( \text{Sign}^+ (w) = \{ i \in [1, \ldots, d] \mid t_i < 0 \} \), and \( \text{Sign}^- (w) = \{ i \in [1, \ldots, d] \mid t_i \geq 0 \} \).

ii) Let \( r(w) \in \mathbb{N} \) be the number of connected components of \( \text{Sign}^-(w) \) in the sense of Definition 2.11, and we denote these connected components by \( [I_1(w) + 1, J_1(w) - 1], \ldots, [I_{r(w)}(w) + 1, J_{r(w)}(w) - 1] \), with \( I_1(w) < J_1(w) < I_2(w) < J_2(w) < \ldots < I_{r(w)}(w) < J_{r(w)}(w) \). We also write \( J_0 = 0 \) and \( I_{r(w)+1} = d + 1 \).

iii) Let us write \( t_i = n_i \) if \( t_i > 0 \), and \( t_i = -l_i \) if \( t_i \geq 0 \).

For technical reasons which will appear in \( \S \) 3, we are actually going to replace the localized multiple harmonic sums (Proposition-Definition 2.14) by a variant whose domain of summation involves both strict and large inequalities. Indeed, the following variant of multiple harmonic sums appears in a natural way in the computation on \( \text{pMZV}_{\mu_N} \)'s.

Definition 2.13. Take the notations of Proposition-Definition 2.14 and Definition 2.12; let

\[
\left( \tilde{h}_m((n_i); (\xi^j)) \right)_d = \sum_{(m_1, \ldots, m_d) \in \tilde{\Delta}_w} \left( \frac{\xi^j}{\xi^l} \right)^{m_1} \cdots \left( \frac{1}{\xi^l} \right)^{m_d} m_1^d \cdots m_d^d
\]

where \( w = ((n_i); (\xi^j))_d \) and
\( \tilde{\Delta} = \{ (m_1, \ldots, m_d) \in \mathbb{N}^d \mid 0 \leq m_{I_1(w)} < \ldots < m_{I_{r(w)}} \} \).

Proposition-Definition 2.14. There exists a unique sequence \( (B_{(m_i); (\xi^j)}) \) of elements of \( \mathcal{Q}(\xi) \) such that, for all \( m, m' \) we have

\[
\sum_{m < m'} \sum_{l_{m' - 1}} \left( \frac{\xi^j}{\xi^l} \right)^{m_1} \cdots \left( \frac{1}{\xi^l} \right)^{m_d} m_1^d \cdots m_d^d = \sum_{\delta = 0}^{\infty} B_{\delta} \left( B_{\delta} \right)^{m_1} \cdots \left( B_{\delta} \right)^{m_d}
\]

Proof. The existence of these coefficients as well as formulas for the \( m \) can be obtained by induction on \( d \), and by considering the two following equalities, valid for \( l \in \mathbb{N}^* \) :

\[
\sum_{m_{I_1(w)}} T^{m_{I_1(w)}} = \left( T^{1/\delta} \right)^{(I_1(w) - 1) l} B_{\delta} T^{l - 1 - \delta} T^{l - 1 - \delta}.
\]

Proposition-Definition 2.15. Let the map \( \text{loc} : \mathcal{O}_m \rightarrow \mathcal{O} \) defined recursively as follows : let \( [C, D] \) be a connected component of \( \text{Sign}^-(w) \). Then applying equation (2.15) gives an expression of the form \( \tilde{h}_m(w) = \sum_{w'} h_m(w')P_{w'}(m) \) with, for all \( w' \), depth \( (w') \) \( < \text{depth}(w) \). We define loc \( (w) \) as \( \sum_{w'} \text{loc}(w')(\frac{1}{\gamma} P_{w'}) \). Then, we have : \( \tilde{h}_m(w) = (h(m) \times \text{eval}_m)(\text{loc}(w)) \)

where \( \text{h}(m) \times \text{eval}_m \) is defined as \( \text{h}(m) \otimes \text{eval}_m \) composed with the multiplication of tensor components.

In other terms, a localized multiple harmonic sum \( \tilde{h}_m(w) \) is a \( \mathcal{Q}(\xi) \)-linear combination of products of (non-localized) multiple harmonic sums by polynomials of \( m \).

Example 2.16. Below, \( l_1, l_2 \in \mathbb{N} \), and \( n_1, n_2 \in \mathbb{N}^* \).

i) Depth one and \( N = 1 \) :

\[
\tilde{h}_m(n_2, -l_1) = \left\{ \sum_{\delta = 1}^{\mathfrak{f}(l_1) + 1} \mathfrak{f}(l_1) m_1 \mathfrak{f}(n_2 - \delta_1) \text{ if } l_1 \geq l_2 \right\}
\]

\[
\tilde{h}_m(-l_2, n_1) = \left\{ \sum_{\delta = 1}^{\mathfrak{f}(l_2) + 1} \mathfrak{f}(l_2) m_1 \mathfrak{f}(n_1 - \delta_1) \text{ if } l_1 \leq l_2 \right\}
\]

ii) Depth two and \( N = 2 \) :

\[
\tilde{h}_m(n_2, -l_1) = \left\{ \sum_{\delta_1 = 1}^{\mathfrak{f}(l_1) + 1} \mathfrak{f}(l_1) m_1 \mathfrak{f}(n_2 - \delta_1) \text{ if } l_1 \leq l_2 \right\}
\]

\[
\tilde{h}_m(-l_2, n_1) = \left\{ \sum_{\delta_1 = 1}^{\mathfrak{f}(l_2) + 1} \mathfrak{f}(l_2) m_1 \mathfrak{f}(n_1 - \delta_1) \text{ if } l_2 \leq l_1 \right\}
\]

The next definitions can be used to give a close formula for the map \( \text{loc} \).

Definition 2.17. Let \( \text{SignPart}^+ \) be the set of couples \( (\text{Sign}(w), P^-) \) where \( w \) is as in Definition 2.12 and \( P^- \) is a connected partition of \( \text{Sign}^-(w) \). We define a map \( T : \text{SignPart}^+ \rightarrow \{ \text{Finite trees} \} \) by
sending $(w, P^-)$ to the tree defined recursively by:

a) the root of the tree is labeled by $(S^-(w), S^+(w))$

b) consider a vertex of the tree labeled by a couple of parts $(E^-, E^+)$ of $\{1, \ldots, d\}$. If $E^- \neq \emptyset$ and $E^+ \neq \emptyset$, then, for each part $P \subset \partial S^+(w)$, we draw an arrow starting from $V$ to a new vertex $V'$, and we label $V'$ by the couple $(E^+ - P, P)$.

Example 2.18. Below, we choose for all examples the connected partition of $\text{Sign}^-(w)$ made of singletons.

i) In depth one, the two trivial trees $(1)^-$ and $(1)^+$

ii) In depth two, we have the two trivial trees $(12)^-$ and $(12)^+$, as well as

$$
\begin{array}{ccc}
(1)^+(2)^- & & (1)^-(2)^+ \\
(1)^+ & & (1)^- \\
(2)^+ & & (2)^-
\end{array}
$$

iii) In depth three, we have the two trivial trees $(123)^+$ and $(123)^-$, as well as

$$
\begin{array}{ccc}
(1)^-(23)^+ & & (12)^+(3)^- \\
(2)^-(3)^+ & & (12)^+ \\
(3)^+ & & (3)^- \\
(1)^+ & & (1)^-
\end{array}
$$

Proposition 2.19. (informal version) A close formula for the map $\text{loc}^\Sigma$ can be written as a sum over the set of paths from the root to the leaves (sequences of nodes $(N_1, \ldots, N_r)$ such that $N_1$ is the root, $N_r$ is a leaf and, for each $i$, $N_{i+1}$ is a son of $N_i$).

Proof. Induction on $d$. \hfill \square

The explicit version of this proposition formula will appear in the next version of this text, and in the next version of [J1-2].

2.2.3. Correspondence between the two localizations. The localization maps $\text{loc}^I$ defined in Proposition-Definition 2.10 of §2.2.1, and $\text{loc}^\Sigma$ defined in Proposition-Definition 2.15 of §2.2.2 can be related to each other, via the power series expansions of localized multiple polylogarithms in terms of localized multiple harmonic sums (Proposition-Definition 2.9).

2.3. Multiple polylogarithms, localization and analytic continuation. We now define analytic continuations of the localized multiple polylogarithms of Proposition-Definition 2.8. We have to make a distinction between the complex setting (§2.3.1) and the $p$-adic setting (§2.3.2).

2.3.1. In the complex setting. We now assume that $K$ is embedded in $\mathbb{C}$. By [D], there is an isomorphism of comparison between $\pi_1^{\text{un}, \text{DR}}(X) \times \mathbb{C}$ and the Betti realization of $\pi_1^{\text{un}}(X_K) \times \mathbb{C}$, and the coefficients of this isomorphism are iterated path integrals in the sense of [CH].
Definition 2.20. (Goncharov, [Go]) Let $\gamma$ be a path on $X(C)$ in the generalized sense where the extremities of $\gamma$ are not necessarily points of $X(C)$ but can also be tangential base-points. The multiple polylogarithms are the following functions $\left( j_1, \ldots, j_n \in \{0, \ldots, r\} \right)$:
\[
\text{Li}(\gamma)(e_{j_1} \ldots e_{j_n}) = \int_{t_{j_n}=0}^{1} \gamma^*(\frac{dt_{j_n}}{t_{j_n} - z_{j_n}}) \int_{t_{j_{n-1}}=0}^{t_{j_n}} \ldots \gamma^*(\frac{dt_{2}}{t_{2} - z_{2j}}) \int_{t_{1}=0}^{t_{2}} \gamma^*(\frac{dt_{1}}{t_{1} - z_{1j}})
\]
Then
\[
\text{Li}(\gamma) = 1 + \sum_{z_{j_n}, \ldots, z_{j_1} \in \{0, t_1, \ldots, t_N\}} \text{Li}(\gamma)(e_{j_n} \ldots e_{j_1}) e_{j_n} \ldots e_{j_1} \in \pi_1^{\text{unDR}}(X, b, a)(C)
\]
where $a$ and $b$ are the extremities of $\gamma$.

Definition 2.21. Let $w$ be a localized word. Let $\gamma$ be a path on $(\mathbb{P}^1 - \{0, z_1, \ldots, z_r, \infty\}))(C)$ in the previous sense. We write $\text{loc}^t(w) = \sum_{w'} F_{w'} \otimes w'$.
Let $\text{Li}_{\text{loc}}[w] = \sum_{w'} F_{w'}(z) \text{Li}(w')$ where $z$ is the endpoint of $\gamma$.

2.3.2. In the $p$-adic setting for $\mathbb{P}^1 - \{0, \mu_N, \infty\}$. The notion of crystalline pro-unipotent fundamental groupoid $(z_1^{\text{un,crys}})$ has been defined with three different points of view in [D] §11, [CL] and [SH1], [SH2]. In our simple example, the three points of view are equivalent and we follow [D] §11.

We go back to the notations of §1.1: $p$ is a prime number, $N \in \mathbb{N}^*$ is prime to $p$, $\xi_N$ is a primitive $N$-th root of unity in $\mathbb{Q}_p$. We apply §2.1 and §2.2 in the case where $K = Q_p(\xi_N)$, $r = N$, and $(z_1, \ldots, z_r) = (\xi_N, \ldots, \xi_N^N)$, thus $X = (\mathbb{P}^1 - \{0, \mu_N, \infty\})/K$. According to [D], $\pi_1^{\text{un,crys}}(\mathbb{P}^1 - \{0, \mu_N, \infty\}/K)$ is the data of $\pi_1^{\text{unDR}}(\mathbb{P}^1 - \{0, \mu_N, \infty\}/K)$ plus the Frobenius structure of the KZ connection. The next definitions refer to Coleman integration as in [Co], [Bes], [V]. They depend on the choice of a determination of the $p$-adic logarithm. The alphabet $c$ of the previous paragraphs is now $\{c_0, c_1, \ldots, c_{q_N}\}$ and we denote it by $c_{0, \mu_N}$.

Definition 2.22. (Furusho [F1] for $N = 1$, Yamashita [Y] for any $N$).
We fix a determination $\log_p$ of the $p$-adic logarithm. Let $\text{Li}_{p,\text{KZ}}$ be the non-commutative generating series of Coleman functions on $X$ which satisfies $\nabla_{\text{KZ}} \text{Li}_{p,\text{KZ}} = 0$ and $\text{Li}_{p,\text{KZ}}(z) \sim e^{c_0 \log_p(z)}$.

The next definition is a generalization of a definition in [FKMT3].

Definition 2.23. Let $w$ be a localized word. We write $\text{loc}^t(w) = \sum_{w'} F_{w'} \otimes w'$.
Let $\text{Li}_{p,\text{KZ}}[w] = \sum_{w'} F_{w'} \text{Li}_{p,\text{KZ}}[w']$.

3. Localized adjoint $p$-adic multiple zeta values at roots of unity
We review the definition of $p$-adic multiple zeta values at roots of unity, (§3.1), of the pro-unipotent $\Sigma$-harmonic action (§3.2) and we define and study the localized adjoint $p$-adic multiple zeta values at roots of unity (§3.3, §3.4).

3.1. Review on $p\text{MZV}_{\mu_N}$’s and $\text{AdpMZV}_{\mu_N}$’s. We review definitions of $p$-adic multiple zeta values at roots of unity.

Definition 3.1. ([DC], §5) Let $\tau$ be the action of $G_m(K)$ on $K\langle e_{X_k} \rangle$, that maps $(\lambda, f) \in G_m(K) \times K\langle e \rangle$ to $\sum_{w \in \mathcal{W}(e)} \lambda^{\text{weight}(w)} f[w]w$.

In the next definition, we adopt this convention, which is different from conventions used by some other authors.

Convention 3.2. For $\alpha \in \mathbb{N}^*$, the Frobenius iterated $\alpha$ times is $\phi(p^\alpha)\phi^{\alpha}$ where $\phi$ is the Frobenius in the sense of [D], §13.6, and, for each $\alpha \in -\mathbb{N}^*$, the Frobenius iterated $\alpha$ times is $\phi^{-\alpha}$ is in the sense of [D], §11.

Notation 3.3. Let $\Pi_{1,0} = \pi_1^{\text{un,DR}}(X, \mathbb{I}_{1,0})$.
A first point of view on the notion of $p\text{MZV}_{\mu_N}$’s uses the canonical De Rham paths evoked in §2.1.1:
Definition 3.4. (general definition in [1,1]. Definition 2.2.5; anterior particular cases: \( N = 1 \), \( \alpha = 1 \), Deligne, Arizona Winter School, 2002 (unpublished); \( N \in \{1,2\} \), \( \alpha = 1 \), Deligne and Goncharov [DC] §5.28; \( N = 1 \), \( \alpha = -1 \) Ünver [U2], §1; any \( N \) and \( \alpha = 1 \) \( \log(q)/\log(p) \), Yamashita [Y], Definition 3.1; any \( N \) and \( \alpha = -1 \) Ünver [U2], §2.2.3.

If \( \alpha \in \mathbb{N}^* \), let \( \Phi_{p,\alpha} = \tau(p^\alpha)\phi^\alpha\left(\frac{1}{\epsilon_{\alpha}}, 1_{\mathbb{P}}\right) \in \Pi_{1,0}(K) \); if \( \alpha \in -\mathbb{N}^* \), let \( \Phi_{p,\alpha} = \phi^\alpha\left(\frac{1}{\epsilon_{\alpha}}, 1_{\mathbb{P}}\right) \in \Pi_{1,0}(K) \).

For any \( \alpha \in \mathbb{N}^* - \mathbb{N}^* \), the \( p \)-adic multiple zeta values at roots of unity are the numbers \( \zeta_{p,\alpha}(\{n_i\}; \{e_{\xi_i}\}) = \Phi_{p,\alpha}[e_0^n_{\alpha-1}e_{\xi_1} \ldots e_0^{n_d-1}e_{\xi_d}] \), \( d \in \mathbb{N}^* \), and \( n_1, \ldots, n_d \in \mathbb{N}^* \), and \( j_1, \ldots, j_d \in \{1, \ldots, N\} \).

For all objects \( * \) above, and \( \alpha = \log(q)/\log(p) \), let \( *_{p,\alpha} = *_{p,\alpha} \).

The second point of view on the notion of \( p \text{-MZV}_{\mu_N} \)'s (which is the general and conceptual one whereas the previous one is more ad hoc) relies on Coleman integration.

Definition 3.5. \((N=1: \{\mathbb{F}_p\} \text{ Definition 2.17; any } N \text{ Yamashita (Y) Definition 2.4})

Let \( \Phi_{p,KZ} \) be the unique element of \( \Pi_{1,0}(K) \) which is invariant by the Frobenius. The numbers \( \zeta_{p,KZ}(\{n_i\}; \{e_{\xi_i}\}) = \Phi_{p,KZ}[e_0^n_{\alpha-1}e_{\xi_1} \ldots e_0^{n_d-1}e_{\xi_d}] \in K \) are called \( p \)-adic multiple zeta values at roots of unity.

In \([J1]3\), we have denoted by \( \Phi_{p,-\infty} = \Phi_{p,KZ} \), \( \zeta_{p,-\infty} = \zeta_{p,KZ} \) and we also defined the following variant. Below, the group law \( \circ_{\mathfrak{O}} \) on \( \Pi_{1,0} \) is the group law denoted by \( \circ \) in \([DQ]\), §5.12.

Definition 3.6. \([J1]3\) Let \( \Phi_{p,\alpha} \) be the inverse of \( \Phi_{p,KZ} \) for the group law \( \circ_{\mathfrak{O}} \). The numbers \( \zeta_{p,\alpha}(\{n_i\}; \{e_{\xi_i}\}) = \Phi_{p,\alpha}[e_0^n_{\alpha-1}e_{\xi_1} \ldots e_0^{n_d-1}e_{\xi_d}] \in K \) are called \( p \)-adic multiple zeta values at roots of unity.

In the Definitions 3.4, 3.5 and 3.6 we are actually adopting a terminology which differs from the terminologies in other works: the \( p \)-adic multiple zeta values at roots of unity for \( \alpha = -1 \) are called cyclotomic \( p \)-adic multiple zeta values in \([U2]\), those for \( \alpha = \log(q)/\log(p) \) or \( \alpha = -\infty \) are called \( p \)-adic multiple \( L \)-values in \([Y]\).

For any \( \alpha, \alpha' \in \mathbb{Z} \cup \{\pm \infty\} - \{0\} \), \( \zeta_{p,\alpha} \) and \( \zeta_{p,\alpha'} \) can be expressed in terms of each other: for certain particular \( \alpha \), this is written in \([U2]\), Theorem 2.14, and in \([Y]\); and this is expressed in terms of \( p \)-adic pro-unipotent harmonic actions in \([J1]3\). We have also defined:

Definition 3.7. \([J1]1\) For \( \alpha \in \mathbb{N}^* \), the numbers \( \zeta_{p,\alpha}^{\Lambda}(l; \{n_i\}; \{e_{\xi_i}\}) = \sum_{i=1}^{\mathbb{N}} \xi^{-\tau(p^\alpha)}(z \mapsto \xi^l)(\Phi_{p,\alpha}[e_0^{n_1}e_{\xi_1} \ldots e_0^{n_d}e_{\xi_d}] \in K) \).

We call these numbers the adjoint \( p \)-adic multiple zeta values at \( N \)-th roots of unity (\( \text{AdpMZV}_{\mu_N} \)'s).

In the particular case of \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \), these are called adjoint \( p \)-adic multiple zeta values (\( \text{AdpMZV} \)'s) and are the numbers \( \zeta_{p,\alpha}^{\Lambda}(l; \{n_i\}) = (\Phi_{p,\alpha}[e_0^{n_1}e_{\xi_1} \ldots e_0^{n_d}e_{\xi_d}] \in K) \).

3.2. Review on the \( p \)-adic pro-unipotent \( \Sigma \)-harmonic action. This paragraph is a preliminary for the definition of localized \( \text{AdpMZV}_{\mu_N} \)'s in §3.4, it is a review on definitions in \([J1]2\) §4, §5. We adopt the notations of \([J1]2\).

Below, \( \text{loc}^\vee \) is the dual of the map \( \text{loc}^\Sigma \) defined in Proposition-Definition 2.13. Let \( K \langle \{e_0^{1,1}, e_{\xi_1}, \ldots, e_{\xi_N}\} \rangle \) be the set of linear maps \( Q(e_0^{1,1}, e_{\xi_1}, \ldots, e_{\xi_N}) \) where \( Q(e_0^{1,1}, e_{\xi_1}, \ldots, e_{\xi_N}) \) is the localization of the non-commutative ring \( Q(e_0, e_{\xi_1}, \ldots, e_{\xi_N}) \) equipped with the concatenation product at the multiplicative part generated by \( e_0 \). The variant \( K \langle \{e_0^{1,1}, e_{\xi_1}, \ldots, e_{\xi_N}\} \rangle \) \{\mathbb{N}, (n_i), e_{\xi_i}, \alpha(Z/NZ)^{1+}\} = \prod_{d \in \mathbb{N}} \langle (n_i), e_{\xi_i}, \alpha(Z/NZ)^{1+}\} \rangle'_d \) contains, for each \( m \in \mathbb{N} \), the generating sequence \( \text{har}_{m}^\text{loc} \) of localized multiple harmonic sums \( \text{har}_{m}^\text{loc} \). Below, the subscript \( S \) denotes a condition on the \( p \)-adic valuations of the coefficients defined in \([J1]2\).

The localized \( p \)-adic pro-unipotent \( \Sigma \)-harmonic action defined in \([J1]2\), §5 is a map:

\( \{\text{loc}_{\text{har}}^\Sigma \} : (K \langle \{e_0^{1,1}, e_{\xi_1}, \ldots, e_{\xi_N}\} \rangle \times \text{Map}(\mathbb{N}, K \langle \{e_0^{1,1}, e_{\xi_1}, \ldots, e_{\xi_N}\} \rangle) \rightarrow \text{Map}(\mathbb{N}, K \langle \{e_0^{1,1}, e_{\xi_1}, \ldots, e_{\xi_N}\} \rangle) \)

In this paper, we will call it the \( p \)-adic pro-unipotent \( \Sigma \)-harmonic action localized at the source. The \( p \)-adic pro-unipotent \( \Sigma \)-action defined in \([J1]2\) is the map:

\( \circ_{\text{har}}^\Sigma : (K \langle \{e_0^{1,1}, e_{\xi_1}, \ldots, e_{\xi_N}\} \rangle \times \text{Map}(\mathbb{N} \times K \langle \{e_0^{1,1}, e_{\xi_1}, \ldots, e_{\xi_N}\} \rangle) \rightarrow \text{Map}(\mathbb{N} \times K \langle \{e_0^{1,1}, e_{\xi_1}, \ldots, e_{\xi_N}\} \rangle) \)
defined as $c_{\text{har}}^\Sigma = (c_{\text{har}})^\text{loc} \circ (\text{id} \times \mu^\Sigma)$. We have, in the sense of Proposition-Definition 3.9, $\text{har}_{\mu^N} = \text{har}_{\mu^N}(c_{\text{har}}^\Sigma)^\text{loc} \circ (\text{id} \times \mu^\Sigma) = \text{har}_{\mu^N}(c_{\text{har}}^\Sigma)^\text{loc} \circ \text{har}_{\mu^N}^\text{loc}$.

Example 3.8. (Proposition-Definition 3.9, §5) $N = 1, d = 2 : \text{har}_{\mu^N}(n_1, n_2) = \text{har}_{\mu^N}(n_1, n_2) + 1_{i.t.} \geq 0 \prod_{i.t.}^2 \left( \left( \begin{array}{c} n_i \\ l_i \end{array} \right) \right) \mu_{\text{loc}}(n_1 + l_1) + \sum_{1 \leq t < n_2} \sum_{l \geq t} \mu_{\text{loc}}(n_2 - t, l) \mu_{\text{loc}}(n_1 + t, l).

3.3. Localized $p$MIZ$\mu_N$’s : the point of view of Frobenius-invariant paths. The problem which we want to tackle is to define $p$MZ$\mu_N$’s at indices $(\{n_i\}; \{\xi^j\})_d$ such that $n_1, \ldots, n_d$ are not necessarily $> 0$.

In this paragraph, we consider the notion of $p$MZ$\mu_N$’s in the sense of Coleman integration (Definition 3.3).

A partial solution to our problem is already given by Furusho, Komori, Matsumoto and Tsumura, using Vologodsky’s version of Coleman integration [V]. We reformulate it with our terminologies.

Proposition-Definition 3.9. (Furusho, Komori, Matsumoto, Tsumura, [FKMT3]) Assume that $\xi_{N}^i \neq 1, \ldots, \xi_{N}^j \neq 1$. Then $\text{Li}_{p,KZ}^\text{loc} ((n_i); \{\xi^j\})_d(z)$ has an asymptotic expansion in $K[\log_p(1 - z)]$ when $z \to 1$.
The constant coefficient of this power series expansion is a generalized $p$-adic multiple zeta value at $N$th roots of unity.

In [FKMT3], only the inversion of the integration operator associated with $e_0$ is considered, whereas here we invert the integration operators associated with all letters $e_{z_i}$ (§2). Thus, the definition above can be extended to our more general framework.

However, even with this generalization, the answer is only partial. For indices which do not necessarily satisfy the hypothesis $\xi_{N}^i \neq 1, \ldots, \xi_{N}^j \neq 1$, we have a different asymptotic expansion:

Lemma 3.10. Each function $\text{Li}_{p,KZ}^\text{loc}^{\text{w}}(n_i); \{\xi^j\})_d(z)$ admits, when $z \to 1$ and $z \in K$, an asymptotic expansion in the ring $K[\log_p(1 - z)]$. Proof. This follows from Definition 2.23 and auxiliary results to Furusho’s definition of $p$MZVs ([F1], Theorem 2.13 to Theorem 2.18, and Theorem 3.15).

It may be tempting to define a notion of regularized $p$MZ$\mu_N$’s by regularizing brutally the asymptotic expansion above and taking the constant term with respect to both $\log(1 - z)$ and $1/\log(1 - z)$. However, this definition is not relevant. It would imply that $\zeta_{p,KZ}(-n_i)$ is zero for all $n \in N^\ast$, whereas we expect non-zero values in odd weights, corresponding to the values at negative integers of the Riemann zeta function. These observations motivate to consider the point of view on $p$MZ$\mu_N$’s in terms of canonical De Rham paths (Definition 3.3), which we will do in the next paragraph.

3.4. Localized $Ad_p$MZ$\mu_N$’s : the point of view of canonical De Rham paths and pro-unipotent harmonic actions. As in the previous papers, we are going to replace the Frobenius by the harmonic Frobenius in the sense of [J1-2], to use the $p$-adic pro-unipotent harmonic actions, and to replace $p$MZ$\mu_N$’s by adjoint $p$MZ$\mu_N$’s.

What we want to define is numbers $\zeta_{p,a}^{Ad} \left( \sum_{1 \leq t < n_2} \sum_{l \geq t} \mu_{\text{loc}}(n_2 - t, l) \mu_{\text{loc}}(n_1 + t, l) \right)$ with $n_1, \ldots, n_d$ of any sign.
We are going to see that this approach will give us a solution to the problem observed in §3.3, defining implicitly.

The map $(\sigma^\Sigma)_{\text{loc}}$ mentioned in §3.2 is defined by lifting an equation involving multiple harmonic sums. We now define a $p$-adic pro-unipotent $\Sigma$-harmonic action involving a localization both at the source and at the target, by a similar procedure.

**Proposition-Definition 3.11.** Let the map
\[
\hat{\sigma}^\Sigma_{\text{loc,loc}} : K\langle\langle e_{0,j}\rangle\rangle \times \text{Map}(\mathbb{N}, K\langle\langle e_{0,j}\rangle\rangle^\Sigma) \rightarrow \text{Map}(\mathbb{N}, K\langle\langle e_{0,j}\rangle\rangle^\Sigma)
\]
called the $p$-adic pro-unipotent $\Sigma$-harmonic action localized at the source and at the target be the map defined by extending the following procedure, used for defining $(\sigma^\Sigma)_{\text{loc}}$, to localized multiple harmonic sums : we consider a multiple harmonic sum, whose domain of summation is defined by inequalities of the form $0 < m_1 < \ldots < m_d < p^m$. We write the Euclidean division of each $m_i$ by $p^a : m_i = p^au_i + n_i$ and we express the domain of summation in terms of the $u_i$’s and $r_i$’s. Then, we write $m_i^{-n_i} = r_i^{-n_i}(\sum_{l_i \geq 0} (-n_i)^l_i)\left(\frac{\zeta}{\zeta_{p}^\Sigma}\right)^{l_i}$. The map $\hat{\sigma}^\Sigma_{\text{loc,loc}}$ is the natural essentialization of the equation relating localized multiple harmonic sums which appears.

Let the $p$-adic pro-unipotent $\Sigma$-harmonic action localized at the target be the map $\hat{\sigma}^\Sigma_{\text{loc,loc}} = \hat{\sigma}^\Sigma_{\text{loc,loc}} \circ (\text{id} \times \text{loc}^\Sigma)$. We have :
\[
\text{har}_{p^a,\text{loc}} = \hat{\sigma}^\Sigma_{\text{loc,loc}} \circ \text{har}_{p^a,\text{loc}}(\zeta_{p}^\Sigma)
\]

**Proof.** Similar to the previous statements from [1][2], §5 reviewed in §3.2. □

We can now recuperate the localized AdpMZVµ’s, as the coefficients of the term of depth $0$ in the $p$-adic pro-unipotent $\Sigma$-harmonic action localized at the target.

**Definition 3.12.** Let $\alpha \in \mathbb{N}^\ast$. For any localized word $(l; (n_i), (\xi^j))_{\text{loc}}$, let us consider expression of $\text{har}_{p^a,\text{loc}}((n_i), (\xi^j))_{\text{loc}}$ in terms of $m$, $\text{har}_m$, and $\text{har}_{p^a}$ given by equation [3][7].

Let $\xi_{\text{loc}}(l; (n_i), (\xi^j))_{\text{loc}}$ be the coefficient of $\xi^m\text{har}_m(0)$ (where $j$ is the unique element of $\mathbb{Z}/N\mathbb{Z}$ such that such a term appears in the expression).

The weight of an index $(l; n_1, n_2)$ is $l + n_1 + n_2$.

We now focus on a particular case :

**Definition 3.13.** The totally negative AdpMZVµ’s are the numbers $\zeta_{p^a,\text{loc}}^\text{Ad}(l; (n_i), (\xi^j))$ with $n_i \leq 0$ for all $i$.

The next proposition is an analogue of the fact that the desingularized values of multiple zeta functions at tuples of negative integers are rational numbers, having a natural expression as polynomials of Bernoulli numbers.

**Proposition 3.14.** The totally negative AdpMZVµ’s are elements of $\mathbb{Q}(\xi)$.

*They can be non-zero only if $1 \leq l + n_1 + \ldots + n_d \leq n_1 + \ldots + n_d + d$, i.e. $1 - (n_1 + \ldots + n_d) \leq l \leq d$. 

**Proof.** Follows directly from Proposition-Definition 2[1][4] and Definition 3.12. □

In the next statement, we write formulas for some examples of the totally negative localized AdpMZVµ’s for $\mathbb{P}^1 - \{0, 1, \infty\}$.

**Example 3.15.** Depth one and two, $N = 1$. Let $n_1, n_2 \in \mathbb{N}$.

\[
\zeta_{p^a,\text{loc}}^\text{Ad}(l + n_1; -n_1) = \begin{cases} 
(p^a)^{-n_1}b^{n_1}_l = \frac{(n_1 + 1)}{n_1 + 1}b_{n_1 + 1 - l} & \text{if } 1 \leq l \leq n_1 + 1 \\
0 & \text{otherwise}
\end{cases}
\]
\[
\zeta_{p^a,\text{loc}}^\text{Ad}(l + n_2 + n_1; -n_2, -n_1) = \begin{cases} 
(p^a)^{-n_2 - n_1}b^{n_2, n_1}_{l_1} = \sum_{l_1 = 1}^{n_1 + 1} \frac{(n_1 + 1)}{n_1 + 1} \frac{(l + n_2 + 1)}{l + n_2 + 1}b_{n_1 + 1 - l_1}b_{l_1 + n_1 + 1 - l_1} & \text{if } 1 \leq l \leq 2 + n_1 + n_2 \\
0 & \text{otherwise}
\end{cases}
\]
4. Localized iteration of the harmonic Frobenius

The map of iteration of the Σ-harmonic Frobenius introduced in [11-3] gives a canonical expression of multiple harmonic sums of the form har_ρ^m in terms of multiple harmonic sums of the form har_ρ^m, for (α_0, α_l) ∈ (N^*)^2, built by using sums of series. We review it (§4.1) and explain briefly its generalization to the "localized" framework of §3.

4.1. Review of the map of iteration of the harmonic Frobenius. Let Λ and a be formal variables. For n ∈ N^*, let pr_n : K(⟨⟨ε⟩⟩) → K(⟨⟨ε⟩⟩) be the map of "projection onto the terms of weight n" i.e. the sequence (pr_n)_{n ∈ N} is characterized by : for all f ∈ K(⟨⟨ε⟩⟩), and λ ∈ K^*, τ(λ)(f) = Σ_{n∈N} pr_n(f)λ^−n.

Let (α_0, α_l) ∈ (N^*)^2 such that, \( \alpha_0|\alpha_l \).

Below, we are using notations of [11-3]. There exists an explicit map, the Σ-harmonic iteration of the Frobenius

\[
\text{(iter}_\text{har})_{\Sigma}^{a, \Lambda} : (K(⟨⟨ε⟩⟩))_{\Sigma} \rightarrow K[[\Lambda^*]][a](⟨⟨ε⟩⟩)_{\Sigma}
\]

such that, the map (iter^a_\text{har})_{har} : (K(⟨⟨ε⟩⟩))_{har} \rightarrow (K(⟨⟨ε⟩⟩))_{har} defined as the composition of \( \text{iter}_\text{har, Σ}^{a, \Lambda} \) by the reduction modulo (a - \( \alpha_0 \), \( \Lambda - p^\alpha_0 \)), satisfies,

\[
\text{har}_q^a = \text{iter}_\text{har, Σ}^{a, \Lambda} (\text{har}_q^{\alpha_0})
\]

This was used in [11-3] to study the iterated Frobenius as a function of its number of iterations, with the application to have a natural indirect explicit computation of the pMZV_μ_N’s associated with Frobenius-invariant paths (Definition 3.5, Definition 3.6).

4.2. Generalization to the localized setting.

Definition 4.1. Let the localized iteration of the Σ-harmonic Frobenius be the map (iter^a_\text{har})_{har} composed with the map loc^\alpha, dual of the map loc^\Sigma of Proposition-Definition 2.15.

Alternatively, we can construct this map by generalizing the procedure of [11-3] for defining the localized iteration of the Σ-harmonic Frobenius. Taking a multiple harmonic sums whose domain of summation is of the form 0 < m_1 < \ldots < m_d < q^\alpha, we introduce the new parameters \( v_1, \ldots, v_d \) equal respectively to the \( q \)-adic valuations of \( m_1, \ldots, m_d \); we write \( m_i = q^{v_i}(qu_i + r_i) \) with \( r_i \in \{1, \ldots, q^\alpha - 1\} \). We rewrite the domain of summation defined by inequalities \( m_1 < \ldots < m_d \) in terms of \( v_i \)'s, \( u_i \)'s and \( r_i \)'s, and sum over these new variables.

In the next version of this paper, we will use this map to define a generalization to the localized setting of the AdpMZV_μ_N’s in the sense of Definition 3.5 and Definition 3.6 i.e. the pMZV_μ_N’s in the sense of Coleman integration.

5. Localization and algebraic relations

5.1. Generalities. In this paragraph we take the context of §2.1 : \( \pi_1^\text{un, DR}(X) \) where \( X = (\mathbb{P}^1 \setminus \{0 = z_0, z_1, \ldots, z_r, \infty\})/K, \) \( K \) being a field of characteristic zero.

5.1.1. Localization and shuffle equation. The shuffle equation (2.2) is satisfied by multiple polylogarithms (Proposition-Definition 2.3) : namely, we have for all words w, w' on the alphabet ε, \( \text{Li}[w \cdot w'] = \text{Li}[w]\text{Li}[w'] \). If we apply \( (z - z_i)^{l_{ij}} \) a certain number of times to this equation (where \( i \in \{0, \ldots, r\} \)), since this operator is a derivation, we obtain a variant of the shuffle relation which applies to certain localized multiple polylogarithms. The right-hand side of the relation obtained in this way is encoded by the following generalization of the deconcatenation coproduct \( \Delta_\text{dec} \) below, which is well-defined on a quotient of the space of words.

Definition 5.1. i) Let \( i \in \{0, \ldots, r\} \). Let \( e_{z_i}^\mathbb{Q}(\mathbb{Q}(\epsilon)) \) be the \( \mathbb{Q} \)-vector space freely generated by words of the form \( e_i w \) with w a word over \( \epsilon \) and \( l \in \mathbb{Z} \). Let \( I_{\text{loc}} \) be the ideal of \( e_{z_i}^\mathbb{Q}(\mathbb{Q}(\epsilon)) \) generated by the relations

\[
\sum_{(w_1, w_2)} |w_1'w_2' = w_1' \otimes w_2' = \sum_{(w_1, w_2)} | w_1w_2 = e_i w' \sum_{m=0}^l (i \choose m) e_0^{-m} w_1 \otimes e_0^{-(l-m)} w_2 \text{ for } w' \text{ word over } e_{0, i, j, N} \text{ and } l \in \mathbb{N}.
\]
ii) Let $\Delta_{loc}: \mathcal{Q}(e_0,e_0^{-1},e_{\mu_N})/I_{loc} \to \mathcal{Q}(e_0,e_0^{-1},e_{\mu_N})/I_{loc} \otimes \mathcal{Q}(e_0,e_0^{-1},e_{\mu_N})/I_{loc}$ be the linear map defined by, for all words $w$ over $e_{\mu_N}$: $\Delta_{loc}(e_{\mu_N}^{-m}w) = \sum_{(m_1,m_2) \in \mathbb{N}^2} \sum_{i=0}^{\infty} (l_i) e_{\mu_N}^{-m_1}w_1 \otimes e_{\mu_N}^{-((l_i)}w_2$

**Definition 5.2.** Let $K(e_{\mu_N}^{-m})$ be the set of linear maps $e_{\mu_N}^{-m}\mathcal{Q}(\mathcal{e}) \to K$.

5.1.2. **Localization quasi-shuffle relation.** The quasi-shuffle relation $[\mathcal{H}]$ is a consequence of the fact that a product of two sets $\{m_1, \ldots, m_d\} \in \mathbb{N}^d$ with $0 < m_1 < \ldots < m_d < m$ and $\{m'_1, \ldots, m'_d\} \in \mathbb{N}^d$ with $0 < m'_1 < \ldots < m'_d < m$ can be written canonically as a disjoint union of sets of the same type in $\mathbb{N}^r$, $r \in \{\max(d,d'), \ldots, d + d'\}$. For example, if $d = d' = 1$:

\[
\begin{array}{c}
\{m_1 \in \mathbb{N} | 0 < m_1\} \times \{m'_1 \in \mathbb{N} | 0 < m'_1\} = \left( \{(m_1, m'_1) \in \mathbb{N}^2 | 0 < m_1 < m'_1 < m\} \right)
\end{array}
\]

The quasi-shuffle relation of $\text{MZV}_{\mu_N}$'s is obtained by applying this equality to multiple harmonic sums and taking the limit $m \to \infty$ in $\mathbb{C}$. Example: $\zeta(n)\zeta(n') = \zeta(n,n') + \zeta(n',n) + \zeta(n+n')$.

It can be encoded in the form $h_m(w)h_m(w') = h_m(w \ast w')$ where $\ast$ is a bilinear map $\mathcal{O}^{\mu,\text{crys},\mu_N} \times \mathcal{O}^{\mu,\text{crys},\mu_N} \to \mathcal{O}^{\mu,\text{crys},\mu_N}$ called the quasi-shuffle product $[\mathcal{H}]$.

The following statement is clear; it should also be a priori already known and appear in several works, although we do not have references:

**Proposition-Definition 5.3.** There exists an explicit bilinear map $\ast_{loc}: e_0^{-1}\mathcal{O}^{\mu,\text{crys},\mu_N} \times e_0^{-1}\mathcal{O}^{\mu,\text{crys},\mu_N} \to e_0^{-1}\mathcal{O}^{\mu,\text{crys},\mu_N}$ (where the factor $e_0^{-1}$ means the localization at the multiplicative part generated by $e_0$ for the concatenation product), the localized quasi-shuffle product, such that, for localized multiple harmonic sums in the sense of Definition 2.9 we have, for all $w, w'$ localized words as in that Definition,

\[
h_m(w)h_m(w') = h_m(w \ast_{loc} w')
\]

5.2. **Application to localized $\text{AdpMZV}_{\mu_N}$'s.** We consider now the context of §2.3.2 and §3 : $\pi_{\text{un.crys}}^{1,\text{crys}}(\mathbb{P}^1 - \{0, \mu_N, \infty\})$.

In [JIT-1] we defined a notion of adjoint double shuffle relations, satisfied by the $\text{AdpMZV}_{\mu_N}$'s of 3.7. This includes a notion of adjoint quasi-shuffle relations. In [JIT-2], we have showed that the adjoint quasi-shuffle relations of $\text{AdpMZV}_{\mu_N}$'s can be retrieved by the formulas of part I involving the pro-unipotent harmonic actions. Here is a variant for the localized $\text{AdpMZV}_{\mu_N}$'s introduced in 3.12

**Proposition 5.4.** The localized $\text{AdpMZV}_{\mu_N}$'s satisfy a canonical family of polynomial equations which generalizes the adjoint quasi-shuffle relations of [JIT-1], and which we call the localized adjoint quasi-shuffle relations.

**Proof.** Same with the proof in [JIT-2] of the fact that we can retrieve the fact that $\text{AdpMZV}_{\mu_N}$'s satisfy the adjoint quasi-shuffle relations from the fact that multiple harmonic sums satisfy the quasi-shuffle equation and the equation relating $\text{AdpMZV}_{\mu_N}$'s and multiple harmonic sums involving the $p$-adic pro-unipotent harmonic action $\hat{h}_{\mu_N}$.

Here, let us write the localized quasi-shuffle relation for the multiple harmonic sums $\text{har}_{p,m}(w)\text{har}_{p,m}(w') = \text{har}_{p,m}(w \ast_{loc} w')$; then, we use that $\text{har}_{p,m}$'s have an expression in terms of $\text{har}_m$'s and certain power series of variable $m$ whose coefficients are written in terms of localized $\text{AdpMZV}_{\mu_N}$'s : equation 3.11.

By the linear independence of the $\text{har}_m$'s over the ring of overconvergent power series expansion of $m \in \mathbb{N} \subset \mathbb{Z}_p$, this implies a family of polynomial equations satisfied by the localized $\text{AdpMZV}_{\mu_N}$'s which we call as in the statement.

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