GLUING ACTION GROUPOIDS: DIFFERENTIAL OPERATORS AND FREDHOLM CONDITIONS

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Abstract. We prove some Fredholm conditions for many algebras of differential operators on particular classes of open manifolds, which include asymptotically Euclidean or asymptotically hyperbolic manifolds. Our typical result is that an operator $P$ is Fredholm if, and only if, it is elliptic and some limit operators $(P_\alpha)_{\alpha \in \mathcal{A}}$ are invertible. The operators $P_\alpha$ are right-invariant operators on amenable Lie groups $G_\alpha$, and are of the same type of $P$. To obtain this result, we consider algebras of differential operators that are generated by groupoids. We study a general gluing procedure for groupoids, and use it to construct a groupoid $\mathcal{G}$ by gluing reductions of action groupoids $(X_i \rtimes G_i)_{i \in I}$. We show that when each Lie group $G_i$ is amenable and acts trivially on $\partial X_i$, then the differential operators generated by $\mathcal{G}$ satisfy the aforementioned Fredholm conditions. Many classes of differential operators on open manifolds satisfy these conditions, and we give several examples.

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1. Introduction

The aim of this paper is to study some algebras of differential operators on open manifold that are “regular enough at infinity”. More specifically, we look for a full characterization of the Fredholm operators, meaning those that are invertible modulo a compact operator.
1.1. Background. Let $M_0$ be a smooth manifold and $P : H^s(M_0) \to H^{s-m}(M_0)$ a differential operator of order $m$, acting between Sobolev spaces. When $M_0$ is a closed manifold, it is well-known that $P$ is Fredholm if, and only if, it is elliptic [15]. This is a important result, that has many applications to partial differential equations, spectral theory and index theory. A great deal of work has been done to obtain such conditions when $M_0$ is not compact: in that case, being elliptic is no longer a sufficient condition to be Fredholm.

A possible approach to this problem is to consider manifolds $M_0$ which embed as the interior of a manifold with corner $M$, and differential operators which are “regular” near $\partial M$. This is the approach followed by Melrose, Schulze, Nistor and their collaborators: see for instance [9, 23, 26, 32]. A differential operator $P$ in this setting is Fredholm if, and only if, it is elliptic and a family of limit operators $(P_x)_{x \in \partial M}$ is invertible; we shall give more details below. The operators treated in this way include geometric operators on asymptotically Euclidean or asymptotically hyperbolic manifolds, as well as manifolds with cylindrical ends.

A slightly different approach has been followed by Georgescu, Iftimovici and their collaborators [12, 13, 21, 25]. These authors considered the smooth action of a Lie group $G$ on a compact manifold with corners $M$. The set of fundamental vector fields on $M_0 \subset M$, obtained by differentiating the action, generates an algebra $\text{Diff}_G(M_0)$ of differential operators acting on $C^\infty(M_0)$. If $M_0$ is diffeomorphic to $G$, and under some conditions on $G$ (such as amenability), one obtains Fredholm conditions for operators the $P \in \text{Diff}_G(M)$. In this setting, the limit operators $(P_x)_{x \in \partial M}$ are obtained as “translates at infinity” of $P$ under the action of $G$. This point of view allows the study of operators with singular coefficients, such as those occurring in the $N$-body problem.

1.2. Overview of the main results. Our paper takes a step toward unifying both approaches. Following Ammann, Lauter and Nistor [2, 18], we study an algebra of differential operators $\text{Diff}_G(M_0)$ that is generated by a groupoid $\mathcal{G}$ with units $M$. A particular case in when $\mathcal{G}$ is the action groupoid $G \times M$: in that case, one recovers the work of Georgescu, Iftimovici and others introduced above.

We introduce the class of boundary action groupoids, that are obtained by gluing a family of action groupoids $(G_i \times M_i)_{i \in I}$ (in a sense made precise below). We will show that this setting recovers many algebras of differential operators which were studied by Melrose, Schulze, Nistor and their collaborators. Moreover, we rely on recent results of Carvalho, Nistor and Qiao [4] to obtain the following Fredholm condition:

**Theorem 1.1.** Let $\mathcal{G}$ be a boundary action groupoid, and $P \in \text{Diff}_G^\infty(M_0)$. Assume that the action of $\mathcal{G}$ on $\partial M$ is trivial and that, for each $x \in \partial M$, the isotropy group $\mathcal{G}_x^\infty$ is amenable. Then $P : H^k(M_0) \to H^{k-m}(M_0)$ is Fredholm if, and only if

1. $P$ is elliptic, and
2. $P_x : H^k(\mathcal{G}_x^\infty) \to H^{k-m}(\mathcal{G}_x^\infty)$ is invertible for all $x \in \partial M$.

We will show that the assumptions of Theorem 1.1 are satisfied in many natural situations, for instance when one whishes to study geometric operators on asymptotically Euclidean, or asymptotically hyperbolic manifolds. The limit operators $P_x$ are right-invariant differential operators on the groups $\mathcal{G}_x^\infty$, of the same type of $P$. For example, if $P$ is the Laplacian on $M_0$, then $P_x$ is also the Laplacian for a right-invariant metric on the group $\mathcal{G}_x^\infty$.

Although it is not stated as such in this paper, Theorem 1.1 generalizes directly to algebras of pseudodifferential operators generated by $\mathcal{G}$. The assumption that $\mathcal{G}$ acts trivially on $\partial M$ is actually superfluous, and will be removed in a subsequent paper.
1.3. Contents of the paper. We start in Section 2 by reviewing some relevant facts about locally compact groupoids. We discuss the necessary facts about Lie groupoids and their algebroids, and give several important examples.

Section 3 introduces the main construction of the paper, which is the gluing of a family of locally compact groupoids \((G_i)_{i \in I}\). We give two different conditions that are sufficient to define a groupoid structure on the gluing \(G = \bigcup_{i \in I} G_i\), and show that \(G\) is Hausdorff if each \(G_i\) is Hausdorff. When each \(G_i\) is a Lie groupoid, we describe the Lie algebroid of the glued groupoid \(G\).

It is in Section 4 that we define the main object of this paper, which is the class of boundary action groupoids. We give some examples of boundary action groupoids which occur naturally when dealing with analysis on open manifolds. We then explain the construction of the algebra of differential operators generated by a Lie groupoid \(G\), and prove the Fredholm condition given by Theorem 1.1.

2. Lie groupoids and algebroids

The aim of this section is to recall some basic definitions and constructions regarding groupoids, and especially Lie groupoids. The reader may refer to Mackenzie’s books [19, 20] for more details about Lie groupoids in general, as well as historical comments.

2.1. Locally compact groupoids. Let us begin with the definition of a groupoid, as in [19, 31].

Definition 2.1. A groupoid is a small category in which every morphisms are invertible.

Remark 2.2. It is often more useful to see a groupoid \(G\) as a set of objects \(G^{(0)}\) and a set of morphisms \(G^{(1)}\). We will often identify \(G\) with its set of morphisms \(G^{(1)}\). Any element \(g \in G\) has a domain \(d(g)\) and range \(r(g)\) in \(G^{(0)}\), as well as an inverse \(\iota(g) := g^{-1} \in G\). To every object \(x \in G^{(0)}\) corresponds a (unique) unit map \(u(x) \in G\). Finally, the product of two morphisms defines a map \(\mu\) from the set of composable arrows 
\[
G^{(2)} := \{(g, h) \in G^{(1)} \times G^{(1)}, d(g) = r(h)\}
\]
to \(G\). The groupoid \(G\) is completely determined by the pair \((G^{(0)}, G^{(1)})\), together with the five structural maps \(d, r, \iota, u\) and \(\mu\) [19, 31].

We now fix some notations for later. When \((g, h) \in G^{(2)}\), the product will be written simply as \(gh := \mu(g, h)\). We shall also write \(G \rightharpoonup M\) for a groupoid \(G\) with objects \(G^{(0)} = M\). Finally, let \(A \subset M\), and put \(G_A := d^{-1}(A)\) and \(G^A := r^{-1}(A)\). The groupoid \(G|_A := G^A \cap G_A\) will be called the reduction of \(G\) to \(A\).

Definition 2.3. A locally compact groupoid is a groupoid \(G \rightharpoonup M\) such that :

1. \(G\) and \(M\) are locally compact spaces, with \(M\) Hausdorff,
2. the structural morphisms \(d, r, \iota, u\) and \(\mu\) are continuous, and
3. \(d : G \to M\) is surjective and open.

Note that these conditions imply that \(r : G \to M\) is surjective and open as well. In this paper, we will not assume the space \(G\) to be Hausdorff, and we will always specify when it is so. We give several examples of groupoids in Subsection 2.3 below.
2.2. Lie groupoids and Lie algebroids. Lie groupoids are groupoids with a smooth structure. The manifolds we consider here may have corners, which occurs in many applications; for example, this is the case when one has to remove a singularity by blowing-up a submanifold [27, 33]. Thus, in our setting, a manifold $M$ is a second-countable space that is locally modelled on open subsets of $[0, \infty[^n$, with smooth coordinate changes [27]. Note that $M$ is not necessarily Hausdorff, unless stated explicitly. By a smooth manifold, we shall mean a manifold without corners.

**Definition 2.4.** A Lie groupoid is a groupoid $\mathcal{G} \rightrightarrows M$ such that

1. $\mathcal{G}$ and $M$ are manifolds with corners, with $M$ Hausdorff,
2. the structural morphisms $d, r, \iota$ and $u$ are smooth,
3. the range $d$ is a submersion, and
4. the product $\mu$ is smooth as well.

In particular, the Lie groupoids we use are locally compact and second countable spaces, but they are not necessarily Hausdorff (and many important examples, coming in particular from foliation theory [5], yield non Hausdorff groupoids). A slightly more general class of groupoids, also useful in applications, is that of continuous family groupoids, for which we assume smoothness along the fibers only, and continuity along the units [17, 30].

If $G$ is a Lie group (which is a particular example of Lie groupoid), then its tangent space over the identity element has a structure of Lie algebra, induced by the correspondence with right-invariant vector fields on $G$. The corresponding construction for a general Lie groupoid is that of a Lie algebroid [19, 20].

**Definition 2.5.** Let $M$ be a manifold with corners and $A \rightarrow M$ a smooth vector bundle. We say that $A$ is a Lie algebroid if there is a Lie algebra structure on the space of sections $\Gamma(A)$, together with a vector bundle morphism $\rho : A \rightarrow TM$ covering the identity, and such that the induced morphism $\rho : \Gamma(A) \rightarrow \Gamma(TM)$ is a Lie algebra morphism. In that case, the map $\rho$ is called the anchor of $A$.

**Example 2.6.** Let $G \rightrightarrows M$ be a Lie groupoid. Then $u : M \rightarrow G$ is an embedding and we can consider

$$\mathcal{A}G := (\ker d_\ast)|_M = \bigsqcup_{x \in M} T_xG_x,$$

which is a vector bundle over $M$. The smooth sections of $\mathcal{A}G$ are in one-to-one correspondence with the right-invariant vector fields on $\ker d_\ast$, which form a Lie algebra. This gives $\mathcal{A}G$ a Lie algebroid structure, whose anchor is given by $r_\ast$.

The definition of Lie algebroid morphism was given for instance in [19, 20].

**Definition 2.7.** Let $A \rightarrow M$ and $B \rightarrow N$ be two Lie algebroids. A morphism of Lie algebroids from $A$ to $B$ is pair $(\Phi, \phi)$ such that

1. $\phi : M \rightarrow N$ is a smooth map and $\Phi : A \rightarrow B$ a vector bundle morphism covering $\phi$,
2. $\Phi$ induces a Lie algebra morphism $\Gamma(A) \rightarrow \Gamma(B)$, and
3. the following diagram is commutative:

$$\begin{array}{ccc}
A & \xrightarrow{\Phi} & B \\
\downarrow{\rho_A} & & \downarrow{\rho_B} \\
TM & \xrightarrow{\phi_*} & TN,
\end{array}$$

with $\rho_A$ and $\rho_B$ the respective anchor maps.
A Lie algebroid $A \to M$ is said to be \textit{integrable} whenever there is a Lie groupoid $\mathcal{G} \rightrightarrows M$ such that $\mathcal{A}\mathcal{G}$ is isomorphic to $A$. Not every Lie algebroid is integrable: the relevant obstruction is discussed in [6]. However, some classical results of Lie algebra theory remain true in this more general case. In order to state those, we shall say that a Lie groupoid $\mathcal{G} \to M$ is $d$-connected (respectively $d$-simply-connected) if each of its $d$-fibers $\mathcal{G}_x$ is connected (respectively simply-connected), for every $x \in M$. A proof of the following two results may be found in [24, 28].

\textbf{Theorem 2.8 (Lie I).} Let $A \to M$ be a Lie algebroid. If $A$ is integrable, then there is a (unique) $d$-simply-connected groupoid integrating $A$.

\textbf{Theorem 2.9 (Lie II).} Let $\phi : A \to B$ be a morphism of integrable Lie algebroids, and let $\mathcal{G}$ and $\mathcal{H}$ be integrations of $A$ and $B$. If $\mathcal{G}$ is $d$-simply-connected, then there is a (unique) morphism of Lie groupoids $\Phi : \mathcal{G} \to \mathcal{H}$ such that $\phi = \Phi_*$.  

2.3. \textbf{Examples of Lie groupoids.} Let us now give a few common examples of Lie groupoids.

\textbf{Example 2.10 (Sets).} A set $M$ is a very simple example of groupoid, for which every arrow is a unit.

\textbf{Example 2.11 (Lie groups).} A Lie group is a Lie groupoid with only one object, and obvious structure maps. Its Lie algebroid is the Lie algebra of the group.

The following examples are more involved, and will be useful in what follows.

\textbf{Example 2.12 (The pair groupoid).} Let $M$ be a smooth manifold, and consider the Lie groupoid $\mathcal{G} = M \times M$ with structural morphisms as follow: the domain is $d(x,y) = y$, the range $r(x,y) = x$, and the product is given by $(x,y)(y,z) = (x,z)$. Thus $u(x) = (x,x)$ and $\iota(x,y) = (y,x)$. This example is called the \textit{pair groupoid} of $M$. The Lie algebroid of $\mathcal{G}$ is isomorphic to $TM$.

\textbf{Example 2.13 (Actions groupoids).} Let $X$ be a smooth manifold and $G$ a Lie group acting on $X$ smoothly and from the right. The \textit{action groupoid} generated by this action is denoted by $X \rtimes G$. Its set of arrows is $X \times G$, together with the structural morphisms $r(x,g) := x$, $d(x,g) := x \cdot g^{-1}$ and $(x,h)(x \cdot h^{-1}, g) := (x,gh)$.

The Lie algebroid of $X \rtimes G$ is denoted by $X \rtimes \mathfrak{g}$. As a vector bundle, it is simply $X \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of $G$. Its Lie bracket is generated by the one of $\mathfrak{g}$; namely, if $\xi, \eta$ are constant sections of $X \rtimes \mathfrak{g}$ such that $\xi(x) = \xi$ and $\eta(x) = \eta$ for all $x \in \xi$, then $[\xi, \eta]_{\mathfrak{g} \times \xi}$ is the constant section on $\xi$ everywhere equal to $[\xi, \eta]_{\mathfrak{g}}$. The anchor $\rho : X \rtimes \mathfrak{g} \to TX$ is given by the \textit{fundamental vector fields} generated by the action:

$$\rho(x, \xi) = \left. \frac{d}{dt} \right|_{t=0} (x \cdot \exp(t\xi))$$

for all $x \in X$ and $\xi \in \mathfrak{g}$. The study of such groupoids relates to that of crossed-product algebras, which have been much studied in the literature [34] (see also [12, 25]).

\textbf{Example 2.14.} Let $M, N$ be manifolds with corners, and $f : M \to N$ a surjective submersion that preserves inward-pointing vectors. Assume that we have a Lie groupoid $\mathcal{H} \rightrightarrows N$. The \textit{fibered pull-back} of $\mathcal{H}$ along $f$ is defined by

$$f^\perp(\mathcal{H}) = \{(x,g,y) \in M \times \mathcal{G} \times M, d(g) = f(x), r(g) = f(y)\}$$

with units $M$. The domain and range are given by $d(x,g,y) = y$ and $r(x,g,y) = y$. The product is $(x,g,y)(y,g',y') := (x,gg',y')$. The groupoid $f^\perp(\mathcal{H})$ is a Lie groupoid, whose Lie algebroid is given by the \textit{thick pull-back}

$$f^\perp(\mathcal{A}\mathcal{H}) := \{ (\xi, X) \in \mathcal{A}\mathcal{H} \times TM, \rho(\xi) = f_*(X) \}.$$  

See [4, 19, 20] for more details.
3. Gluing groupoids

We describe in this section a procedure for gluing locally compact groupoids. This extends a construction of Gualtieri and Li that was used to classify the Lie groupoids integrating certain Lie algebroids [14] (see also [27]).

3.1. The gluing construction. Let $X$ be a locally compact Hausdorff space, covered by a family of open sets $(U_i)_{i \in I}$. For each $i \in I$, let $G_i \rightarrowtail U_i$ be a locally compact groupoid with domain $d_i$ and range $r_i$. Assume that we are given a family of isomorphisms between all the reductions

$$\phi_{ji} : G_i|_{U_i \cap U_j} \rightarrow G_j|_{U_i \cap U_j},$$

such that $\phi_{ij} = \phi_{-1}^{-1}$ and $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ on the common domains. Our aim is to glue the groupoids $G_i$ to build a groupoid $G \rightarrowtail X$ such that, for all $i \in I$,

$$G|_{U_i} \simeq G_i.$$

As a topological space, the groupoid $G$ is defined as the quotient

$$G = \bigsqcup_{i \in I} G_i / \sim,$$

where $\sim$ is the equivalence relation generated by $g \sim \phi_{ji}(g)$, for all $i,j \in I$ and $g \in G_i$. The set $G$ is locally compact for the quotient topology, and inherits its source and target maps from the groupoids $G_i$. Indeed, if $g \in G$ is the equivalence class of a $g_i \in G_i$, we define

$$d(g) = d_i(g_i) \quad \text{and} \quad r(g) = r_i(g_i).$$

The unit $u : X \rightarrow G$ and inverse maps are defined in the same way. Therefore, the subsets $G|_{U_i} = r^{-1}(U_i) \cap d^{-1}(U_i)$ are well defined, for each $i \in I$.

**Lemma 3.1.** For each $i \in I$, the quotient map $\pi_i : G_i \rightarrow G$ induces an homeomorphism (of topological spaces)

$$\pi_i : G_i \rightarrow G|_{U_i}.$$

*Proof.* The topology on $G$ is the coarsest one such that each quotient map $\pi_i$ is open and continuous, for every $i \in I$. Moreover, for any $i \in I$, the definition of the equivalence relation $\sim$ in Equation (1) implies that $\pi_i$ is injective. Therefore, the map $\pi_i$ is an homeomorphism onto its image, which is obviously contained in $G|_{U_i}$.

To prove that $\pi_i(G_i) = G|_{U_i}$, let $g \in G|_{U_i}$ be represented by an element $g_j \in G_j$, for $j \in I$. Then $g_j \in G_j|_{U_i \cap U_j}$, which is isomorphic to $G_i|_{U_i \cap U_j}$ through $\phi_{ij}$, thus $g$ also has a representative in $G_i$. This shows that $\pi_i(G_i) = G|_{U_i}$. ☐

In particular, Lemma 3.1 implies that the structural maps $d, r, u$ and $i$ are continuous, and that the domain and range maps $d, r : G \rightarrow X$ are open. With Remark 2.2 in mind, the only missing element to have a groupoid structure on $G$ is a well-defined product. Therefore, define the set of composable arrows by

$$G^{(2)} = \{(g,h) \in G, s(g) = t(h)\}.$$

A problem is that there are a priori no relation between the two groupoids $G_i$ and $G_j$, for $i \neq j$. Thus, if $(g_i, g_j) \in G^{(2)}$ with $g_i \in G_i$ and $g_j \in G_j$, then there is a priori no obvious way of defining the product $g_i g_j$ in $G$. A way around this issue is to introduce a *gluing condition*, so that any composable pair $(g, h) \in G^{(2)}$ is actually contained in a single groupoid $G_k$, for a $k \in I$.

**Definition 3.2.** We say that a family $(G_i \rightarrowtail U_i)_{i \in I}$ of locally compact groupoids satisfy the *weak gluing condition* if for every composable pair $(g, h) \in G^{(2)}$, there is an $i \in I$ such that both $g$ and $h$ have a representative in $G_i$. 
Another way to say this is that the family \((G_i)_{i \in I}\) should provide an open cover of the space of composable arrows \(G^{(2)}\).

**Lemma 3.3.** Assume that the family \((G_i)_{i \in I}\) satisfy the weak gluing condition. Then there is a unique groupoid structure on

\[
\mathcal{G} = \bigsqcup_{i \in I} G_i/\sim
\]

such that the projection maps \(\pi_i : G_i \to G|_{U_i}\) are isomorphisms of locally compact groupoids, for every \(i \in I\).

**Proof.** Let \((g,h) \in G^{(2)}\) be a composable pair. The weak gluing condition implies that there is an \(i \in I\) such that \(g\) and \(h\) have representatives \(g_i\) and \(h_i\) in \(G_i\). We thus define the product \(gh\) as the class of \(g_i h_i\) in \(G\), and we check at once that this does not depend of a choice of representative for \(g\) and \(h\). Lemma 3.1 and the definition of the structural maps on \(G\) imply that each \(\pi : G_i \to G|_{U_i}\) is an isomorphism of locally compact groupoids, for each \(i \in I\).

To show the uniqueness of the groupoid structure on \(G\), let us assume conversely that each map \(\pi_i : G_i \to G|_{U_i}\) is a groupoid isomorphism. Since the reductions \((G|_{U_i})_{i \in I}\) cover \(G\), the source, target, identity and inverse maps of \(G\) are prescribed by those of each \(G_i\). Moreover, the weak gluing condition implies that for each composable pair \((g,h) \in G^{(2)}\), both \(g\) and \(h\) lie in a same reduction \(G|_{U_i}\). Thus the product on \(G\) is also determined by those of each groupoid \(G_i\), for \(i \in I\). \(\square\)

**Definition 3.4.** The groupoid \(G\) of Lemma 3.3 defines the *gluing (or glued groupoid)* of a family of locally compact groupoids \((G_i)_{i \in I}\) satisfying the gluing condition. We denote it

\[
\mathcal{G} = \bigsqcup_{i \in I} G_i,
\]

when there is no ambiguity about the family of isomorphisms \((\phi_{ij})_{i,j}\) involved.

**Remark 3.5.** The glued groupoid can also be defined by a universal property. Assume we only have two groupoids \(G_1 \rightrightarrows U_1\) and \(G_2 \rightrightarrows U_2\), and set \(G_{12} := G_1|_{U_1 \cap U_2} \simeq G_2|_{U_1 \cap U_2}\). Then \(G = G_1 \cup G_2\) is the pushout of the inclusions morphisms \(j_i : G_{12} \hookrightarrow G_i\), for \(i = 1, 2\). It is the “smallest” groupoid such that there is a commutative diagram

\[
\begin{array}{ccc}
G & \leftarrow & G_{12} \\
\downarrow & & \downarrow j_2 \\
G_1 & \leftarrow & G_2.
\end{array}
\]

When we have a general family \((G_i)_{i \in I}\) satisfying the gluing condition, the glued groupoid can similarly be defined as the colimit relative to the inclusions \(G|_{U_i \cap U_j} \hookrightarrow G_i\), for all \(i, j \in I\).

**Remark 3.6.** It is possible for a family \((G_i)_{i \in I}\) to satisfy the weak gluing condition, even though there is a pair \((G_{i_0},G_{j_0})\) that do not satisfy the gluing condition, for some \(i_0,j_0 \in I\). For instance, let \(X\) be a locally compact, Hausdorff space and \(U_1, U_2\) two distinct open subsets in \(X\) with non-empty intersection \(U_{12}\). Let

\[
G_0 = X \times X, \quad G_1 = U_1 \times U_1 \quad \text{and} \quad G_2 = U_2 \times U_2
\]

be pair groupoids over \(X, U_1\) and \(U_2\) respectively. The family \((G_0,G_1,G_2)\) satisfies the weak gluing condition of Definition 3.2, and may be glued to obtain the groupoid \(G = X \times X = G_0\). However, the pair \((G_1,G_2)\) does not satisfy the weak gluing condition.
Lemma 3.7. Let \((G_i)\) be a family of groupoids satisfying the weak gluing condition. If each \(G_i\), for \(i \in I\), is a Hausdorff groupoid, then the gluing \(G = \bigcup_{i \in I} G_i\) is also Hausdorff.

Proof. Let \(g, h \in G\). There are two cases.

- Assume \(d(g) = d(h)\) and \(r(g) = r(h)\). Then, because of the gluing condition, there is an \(i \in I\) such that \(g\) and \(h\) are both in the Hausdorff groupoid \(G_{(i)}\).
- Otherwise, either \(d(g) \neq d(h)\) or \(r(g) \neq r(h)\). Let us assume the former. Then, since \(X\) is Hausdorff, there are open sets \(U, V \subseteq X\) such that \(d(g) \in U\), \(d(h) \in V\) and \(U \cap V = \emptyset\). Thus \(g \in G_U\) and \(h \in G_V\), which are disjoint open subsets of \(G\).

\[\square\]

We also introduce the strong gluing condition, which is often easier to check.

Definition 3.8. We say that the family \((G_i \rightrightarrows U_i)_{i \in I}\) of locally compact groupoids satisfy the strong gluing condition if, for each \(x \in X\), there is an \(i_x \in I\) such that

\[G_{(i_x)} \cdot x \subset U_{i_x}\]

for all \(i \in I\).

In other words, the orbit of a point through the action of \(G\) should always be induced by a single element of the family \((G_i)_{i \in I}\).

Lemma 3.9. Let \((G_i)_{i \in I}\) be a family of groupoids which satisfies the strong gluing condition. Then the family \((G_{(i)})_{i \in I}\) also satisfies the weak gluing condition.

Proof. Let \((g, h) \in G^{(2)}\), and assume that \(g\) has a representative \(g_i \in G_i\) and \(h\) a representative \(h_j \in G_j\). Let \(x = d(g) = r(h)\). The gluing condition implies that there is an \(i_x \in I\) such that \(G_i \cdot x \subset U_{i_x}\) and \(G_j \cdot x \subset U_{i_x}\). Thus \(r_i(g_i) \in U_{i_x}\), so \(g_i \in G_i|_{U_i \cap U_{i_x}}\). But there is an isomorphism

\[\phi_{i_x}: G_i|_{U_i \cap U_{i_x}} \to G_{i_x}|_{U_{i_x}}\]

so that \(g\) actually has a representative \(g_{i_x}\) in \(G_{i_x}\). The same arguments show that \(h\) also has a representative \(h_{i_x}\) in \(G_{i_x}\).

\[\square\]

We conclude this subsection with a condition for which a groupoid \(G \rightrightarrows X\) may be written as the gluing of its reductions. This definition was introduced by Gualtieri and Li for Lie algebroids [14].

Definition 3.10. Let \(G \rightrightarrows X\) be a topological groupoid over a locally compact Hausdorff space, and \((U_i)_{i \in I}\) an open cover of \(X\). Then \((U_i)_{i \in I}\) is said to be an orbit cover relative to \(G\) if for all \(x \in M\), there is an \(i_x \in I\) such that \(G \cdot x \subset U_{i_x}\).

It is clear from the definition that \((U_i)_{i \in I}\) is an orbit cover if, and only if, the family \((G|_{U_i})_{i \in I}\) satisfies the strong gluing condition of Definition 3.8. The family of isomorphisms \((\phi_{ij})\) between reductions is given by the identity maps on each \(G|_{U_i \cap U_j}\), for \(i, j \in I\). Hence:

Lemma 3.11. Let \(G \rightrightarrows X\) be a groupoid and \((U_i)_{i \in I}\) an orbit cover of \(X\) relative to \(G\). Then \(G\) is isomorphic to the gluing of its reductions \((G|_{U_i} \rightrightarrows U_i)_{i \in I}\).

\[\square\]

3.2. Gluing Lie groupoids. Let \(M\) be a manifold with corners, and \((U_i)_{i \in I}\) an open cover of \(M\). Let \((G_i)_{i \in I}\) be a family of Lie groupoids satisfying the weak gluing condition of Definition 3.2. Assume that the morphisms \(\phi_{ij}: G_i|_{U_i \cap U_j} \to G_j|_{U_i \cap U_j}\) are Lie groupoid morphisms, and let \(G := \bigcup_{i \in I} G_i\) be the glued groupoid over \(M\).

Lemma 3.12. If each \(G_i\), for \(i \in I\), is a Lie groupoid, then there is a unique Lie groupoid structure on \(G\) such that \(\pi_i: G \to G_i|_{U_i}\) is an isomorphism of Lie groupoids, for all \(i \in I\).
Proof. By Definition 3.4, the reductions \( \mathcal{G}|_{U_i} \cong \mathcal{G}_i \), for \( i \in I \), provide an open cover of \( \mathcal{G} \). Since each \( \mathcal{G}_i \) is a Lie groupoid, and all \( \phi_{ij} \) are smooth, this induces a manifold structure on \( \mathcal{G} \). Each structural map of \( \mathcal{G} \) coincides locally with a structural map of one of the groupoids \( \mathcal{G}_i \), hence is smooth. This gives the Lie groupoid structure. □

Remark 3.13. A similar statement holds when each \( \mathcal{G}_i \) is a continuous family groupoids, for all \( i \in I \): then \( \mathcal{G} \) is also a continuous family groupoid [17, 30].

To specify the Lie algebroid of \( \mathcal{G} \), we need first study the gluing of Lie algebroids. For each \( i \in I \), let \( A_i \rightarrow U_i \) be a Lie algebroid. Assume that there are Lie algebroid isomorphisms \( \psi_{ij} : A_i|_{U_i \cap U_j} \rightarrow A_j|_{U_i \cap U_j} \) covering the identity, such that \( \psi_{ij}^{-1} = \psi_{ji} \) and \( \psi_{ij} \psi_{jk} = \psi_{ik} \) on common domains. As vector bundles, the family \( (A_i)_{i \in I} \) is in particular a family of groupoids that satisfies the strong gluing condition of Definition 3.8 (the orbit of any \( x \in M \) is reduced to \( \{x\} \)). Thus, the gluing \( A = \bigcup_{i \in I} A_i \) is a smooth vector bundle on \( M \), with inclusion maps \( \pi_i : A_i \hookrightarrow A \).

Lemma 3.14. There is a unique Lie algebroid structure on \( A = \bigcup_{i \in I} A_i \) such that each map \( \pi_i : A_i \rightarrow A \) is a morphism of Lie algebroids.

Proof. By definition, the Lie algebroids \( A_i|_{U_i} \cong A_i \), for all \( i \in I \), provide an open cover of \( A \). Let \( X,Y \in \Gamma(A) \), and define \( [X,Y] \in \Gamma(A) \) by

\[
[X,Y]|_{U_i} := [X|_{U_i}, Y|_{U_i}],
\]

where \([\cdot,\cdot]_i\) is the Lie bracket on \( A_i \). Since \( A_i|_{U_i \cap U_j} \) and \( A_j|_{U_i \cap U_j} \) are isomorphic as Lie algebroids, the section \([X,Y]_i \) is well-defined on \( U_i \cap U_j \), for all \( i,j \in I \). This defines the Lie bracket on \( \Gamma(A) \). The anchor map is similarly defined as \( \rho(X)|_{U_i} := \rho_i(X|_{U_i}) \), with \( \rho_i \) the anchor map of \( A_i \). Because the family \( (A_i)_{i \in I} \) cover \( A \), this is the unique Lie algebroid structure on \( A \) such that each map \( \pi_i : A_i \rightarrow A \) is a Lie algebroid isomorphism. □

Lemma 3.15. Let \( (\mathcal{G}_i \cong U_i)_{i \in I} \) be a family of Lie groupoids satisfying the gluing condition, with isomorphisms \( \phi_{ij} : \mathcal{G}_i|_{U_i \cap U_j} \rightarrow \mathcal{G}_j|_{U_i \cap U_j} \). The Lie algebroid of the gluing \( \mathcal{G} = \bigcup_{i \in I} \mathcal{G}_i \) is isomorphic to the gluing of the family \( (\mathcal{A}G_i)_{i \in I} \), with Lie algebroid isomorphisms \( (\phi_{ij})_* : \mathcal{A}G_i|_{U_i \cap U_j} \rightarrow \mathcal{A}G_j|_{U_i \cap U_j} \), for \( i,j \in I \).

Proof. By definition of the quotient maps \( \pi_i : \mathcal{G}_i \rightarrow \mathcal{G} \), the map \( \pi_j^{-1} \circ \pi_i \) coincides with the isomorphism \( \phi_{ij} : \mathcal{G}_i|_{U_i \cap U_j} \rightarrow \mathcal{G}_j|_{U_i \cap U_j} \), for all \( i,j \in I \). Let \( \xi \in \mathcal{A}G_i|_{U_i \cap U_j} \). Then

\[
(\pi_j)_*(\xi) = (\pi_j)_* \circ (\pi_j^{-1} \circ \pi_i)_*(\xi) = (\pi_j)_* \circ (\phi_{ij})*_*(\xi) \in \mathcal{A}G_i|_{U_i \cap U_j}
\]

Let \( \Psi : \bigcup_{i \in I} \mathcal{A}G_i \rightarrow \mathcal{A}G \) bet the map given by \( \Psi(\xi) := \pi_i(\xi) \), whenever \( \xi \in \mathcal{A}G_i \). Equation (2) implies that \( \Psi \) induces a map from the quotient \( A = \bigcup_{i \in I} \mathcal{A}G_i \), which is the glued algebroid, to \( \mathcal{A}G \). Each map \( \pi_i : \mathcal{G}_i \rightarrow \mathcal{G} \) gives an isomorphism \( (\pi_i)_* : \mathcal{A}G_i \rightarrow \mathcal{A}G|_{U_i} \), so \( \Psi : A \rightarrow \mathcal{A}G \) is also a Lie algebroid isomorphism. □

4. Boundary action groupoids and Fredholm conditions

We shall study Fredholm conditions for algebras of differential operators generated by Lie groupoids \( \mathcal{G} \Rightarrow M \). To this end, we define the class of boundary action groupoids, which are obtained by gluing reductions of action groupoids. We will show that many examples of groupoids arising in analysis on open manifold belong to this class, and obtain Fredholm condition for the associated differential operators.
4.1. Boundary action groupoids. We remind the reader that orbit covers were introduced in Definition 3.10.

Definition 4.1. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. The groupoid $\mathcal{G}$ is a boundary action groupoid if

1. there is a dense, open and $\mathcal{G}$-invariant subset $U \subset M$ such that $\mathcal{G}|_U \cong U \times U$,
2. there is an open cover $(U_i)_{i \in I}$ of $M$, and an $i_0 \in I$ such that $U_{i_0} = U$,
3. for all $i \neq i_0$, there is a Hausdorff manifold $X_i$, a Lie group $G_i$ acting smoothly on $X_i$ on the right, and an open subset $V_i \subset X_i$ diffeomorphic to $U_i$, such that

$$\mathcal{G}|_{U_i} \cong (X_i \times G_i)|_{V_i}.$$

Remark 4.2. As a dense $\mathcal{G}$-orbit in $M$, the subset $U$ in Definition 4.1 is uniquely determined by $\mathcal{G}$. When $\mathcal{G}$ is a boundary action groupoid, we will always denote by the letter $U$ its unique open, dense and $\mathcal{G}$-invariant subset. The subset $U$ is often simply the interior $\overset{\circ}{U} = M \setminus \partial M$.

One of the main points of this definition is to give a better understanding of how $\mathcal{G}|_M$ and $\mathcal{G}|_{\partial M}$ are glued together near the boundary. In particular:

Lemma 4.3. Boundary action groupoids are Hausdorff.

Proof. We keep the notations of Definition 4.1 above. Note that all $(X_i \times G_i)|_{V_i}$ are Hausdorff groupoids (as subsets of the Hausdorff spaces $X_i \times G_i$). Since $(U_i)_{i \in I}$ is an orbit cover, Lemma 3.11 states that $\mathcal{G}$ is obtained by gluing Hausdorff groupoids. The result then follows from Lemma 3.7.

4.2. Examples. We will show here that many groupoids occurring in the study of analysis on singular manifolds are boundary action groupoids. We will explain in Subsection 4.4 how this class of groupoids allows to obtain Fredholm conditions for many interesting differential operators. Our examples are based on the following result:

Theorem 4.4. Let $M$ be a manifold with corners, and $M_0 := M \setminus \partial M$. Let $A \to M$ be a Lie algebroid, such that the anchor map $\rho$ induces an isomorphism $A|M_0 \cong TM_0$ and $\rho(A|_{\partial M}) \subset T(\partial M)$. Then there is a unique Lie groupoid $\mathcal{G} \rightrightarrows M$ integrating $A$, such that $\mathcal{G}|_{M_0} \cong M_0 \times M_0$ and $\mathcal{G}|_{\partial M}$ is $d$-connected.

Proof. The existence of such a groupoid has been proven by Debord [8] and Nistor [28]. If $\mathcal{H}$ is another groupoid satisfying the assumptions of Theorem 4.4, then $\mathcal{H}|_{\partial M}$ and $\mathcal{G}|_{\partial M}$ are $d$-connected integrations of $A|_{\partial M}$, so Theorem 2.9 states that they are isomorphic. The main result in [28] implies that $\mathcal{G}$ is then isomorphic to $\mathcal{H}$.

The groupoid $\mathcal{G}$ in Theorem 4.4 will be called the maximal integration of $A$. Based on this Theorem, we give several examples of boundary action groupoids which occur naturally in the context of analysis on open manifolds: see [4, 26] for more details.

Example 4.5 (0-groupoid). Consider $G_n := \mathbb{R}_+^* \times \mathbb{R}^{n-1}$, where $\mathbb{R}_+^*$ acts by dilation on $\mathbb{R}^{n-1}$. The right action of $G_n$ upon itself extends uniquely to an action on $X_n := \mathbb{R}_+ \times \mathbb{R}^{n-1}$, by setting

$$(x_1, \ldots, x_n) \cdot (t, \xi_2, \ldots, \xi_n) = (tx_1, x_2 + x_1\xi_2, \ldots, x_n + x_1\xi_n).$$

The Lie algebra of fundamental vector fields for this action (recall Example 2.13) is the one spanned by $(x_1\partial_1, \ldots, x_1\partial_n)$ on $X_n$. 
To generalize this setting, let $M$ be a manifold with boundary and let $\mathcal{V}_0$ be the Lie algebra of all vector fields on $M$ vanishing on $\partial M$. In a local coordinate system $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ near $\partial M$, we have

$$\mathcal{V}_0 = \text{Span}(x_1 \partial_1, \ldots, x_1 \partial_n),$$

as a $C^\infty(M)$-module.

It follows from Serre-Swan’s Theorem that there is a unique Lie algebroid $A_0 \hookrightarrow M$ such that the anchor map induces an isomorphism $\Gamma(A_0) \cong \mathcal{V}_0$. The 0-groupoid $\mathcal{G}_0 \rightrightarrows M$ is the maximal integration of $A_0$, as given by Theorem 4.4: it is the natural space for the Schwarz kernels of differential operators that are induced by asymptotically hyperbolic metrics on $M_0$ [22, 26].

**Theorem 4.6.** The 0-groupoid $\mathcal{G}_0 \rightrightarrows M$ is a boundary action groupoid. Moreover, for each $p \in \partial M$, there is a neighborhood $U$ of $p$ in $M$, and an open set $V \subset \mathbb{R}_+ \times \mathbb{R}^{n-1}$, such that

$$\mathcal{G}_0|_U \cong (X_n \ltimes \mathcal{G}_n)|_V.$$

**Proof.** For each $p \in \partial M$, there is a neighborhood $U$ of $p$ in $M$ that is diffeomorphic to an open subset $V \subset \mathbb{R}_+ \times \mathbb{R}^{n-1}$, through $\phi : U \to V$. The diffeomorphism $\phi$ maps $\partial U$ to $\partial V$, so $\phi_*(\mathcal{V}_0(U)) = \mathcal{V}_0(V)$. This implies that there is an isomorphism $A_0(U) \cong A_0(V)$ covering $\phi$. Both groupoids $\mathcal{G}_0|_U$ and $(X_n \ltimes \mathcal{G}_n)|_V$ are maximal integrations of $A_0(U) \cong A_0(V)$, so Theorem 4.4 implies that $\mathcal{G}_0|_U \cong (X_n \ltimes \mathcal{G}_n)|_V$.

To prove that $\mathcal{G}_0$ is a boundary action groupoid, let $(U_i)_{i=0}^n$ be an open cover of $\partial M$, such that each $\mathcal{G}_0|_{U_i}$ is isomorphic to a reduction of $X_n \ltimes \mathcal{G}_n$, for all $i = 1, \ldots, n$. Let $U_0 = M_0$. Then $(U_i)_{i=0}^n$ is an orbit cover of $M$ that satisfies the assumptions of Definition 4.1. □

**Example 4.7.** Example 4.5 can be slightly generalized by replacing $\mathcal{V}_0$ with a Lie algebra $\mathcal{V} \subset \Gamma(TM)$ such that, for any point $p \in \partial M$, there is an $n$-tuple $\alpha \in \mathbb{N}^n$ and a local coordinate system $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ near $p$ with

$$\mathcal{V} = \text{Span}(x_1^{\alpha_1} \partial_1, \ldots, x_1^{\alpha_n} \partial_n).$$

If $\alpha_1 = 1$ and every $\alpha_i \geq 1$, for $i = 2, \ldots, n$, then the maximal integration $\mathcal{G} \rightrightarrows M$ of $\mathcal{V}$ is again a boundary action groupoid. Indeed, consider the action of $\mathbb{R}^n_+$ on $\mathbb{R}^{n-1}$ given by

$$t \cdot (x_2, \ldots, x_n) = (t^{\alpha_2}x_2, \ldots, t^{\alpha_n}x_n),$$

and form the semidirect product $G_\alpha = \mathbb{R}^n_+ \rtimes_{\alpha} \mathbb{R}^{n-1}$ given by this action. As in Example 4.5, the right action of $G_\alpha$ upon itself extends uniquely to an action on $X_n := \mathbb{R}_+ \times \mathbb{R}^{n-1}$, by setting

$$(x_1, \ldots, x_n) \cdot (t, \xi_2, \ldots, \xi_n) = (tx_1, x_2 + t^{\alpha_2}\xi_2, \ldots, x_n + t^{\alpha_n}\xi_n).$$

An argument analogous to that of Theorem 4.6 shows that $\mathcal{G}$ is obtained by gluing reductions of actions groupoids $X_n \ltimes G_\alpha$, for some $n$-tuples $\alpha \in \mathbb{N}^n$.

**Example 4.8** (Scattering groupoid). Let $\mathbb{S}_+^n$ be the stereographic compactification of $\mathbb{R}^n$. Consider the action of $\mathbb{R}^n_+$ upon itself by translation, and extend it to $\mathbb{S}_+^n$ in the only possible way, by a trivial action on $\partial \mathbb{S}_+^n$. The action groupoid $\mathcal{G}_{sc} = \mathbb{S}_+^n \ltimes \mathbb{R}^n$ has been much studied in the literature, and relates to the study of the spectrum of the $N$-body problem in Euclidean space [12, 25].

As in Example 4.8, we can generalize this setting to any manifold with boundary $M$. Let $\mathcal{V}_0$ be the Lie algebra of vector fields on $M$ which are tangent to the boundary, and let $x \in C^\infty(M)$ be a defining function for $\partial M$. We define the Lie algebra of scattering vector fields on $M$ as $\mathcal{V}_{sc} := x\mathcal{V}_0$. In a local coordinate system $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ near $\partial M$, we have

$$\mathcal{V}_{sc} = \text{Span}(x_1^2 \partial_1, x_1 \partial_2, \ldots, x_1 \partial_n),$$
as a $C^\infty(M)$-module. One can check that, when $M = S^n_+$ as above, then $\mathcal{V}_\omega$ is the Lie algebra of fundamental vector fields induced by the action of $\mathbb{R}^n$ on $S^n_+$.

As in Example 4.5, there is a unique Lie algebroid $A_{sc} \to M$ whose sections are isomorphic to $\mathcal{V}_\omega$ through the anchor map. The scattering groupoid $\mathcal{G}_{sc} \rightrightarrows M$ is the maximal integration of $A_{sc}$, and it generates the algebra of differential operators on $M_0$ that are induced by asymptotically Euclidean metrics [22, 26]. The proof of Theorem 4.6 can be adapted to this context to give:

**Theorem 4.9.** The scattering groupoid $\mathcal{G}_{sc} \rightrightarrows M$ is a boundary action groupoid. Moreover, for each $p \in \partial M$, there is a neighborhood $U$ of $p$ in $M$, and an open set $V \subset S^n_+$, such that

$$\mathcal{G}_{sc}|_{\partial U} \simeq (S^n_+ \times \mathbb{R}^n)|_V.$$

4.3. **Differential operators and Sobolev spaces.** As in Subsection 4.1, let $\mathcal{G}$ be a boundary action groupoid with units $M$, and $U \subset M$ its unique dense $\mathcal{G}$-orbit. We recall in this subsection how $\mathcal{G}$ generates an algebra of differential operators on $U$, following [2, 29]. See also the recent papers [10, 11].

Let $\mathcal{A}\mathcal{G}$ be the Lie algebroid of $\mathcal{G}$, and let $\text{Diff}(\mathcal{G})$ be the universal enveloping algebra of $\Gamma(\mathcal{A}\mathcal{G})$. Any element $P \in \text{Diff}(\mathcal{G})$ can be written locally as

$$P = \sum_{|\alpha| \leq m} a_\alpha X_1^{\alpha_1} \cdots X_n^{\alpha_n},$$

where $X_1, \ldots, X_n$ is a local basis of $\Gamma(\mathcal{A}\mathcal{G})$, and the coefficients $a_\alpha$ are functions in $C^\infty(M)$. The smallest integer $m$ such that $P$ may be written as in Equation (3) is the order of $P$.

**Remark 4.10.** We highlighted in Example 2.6 that the smooth sections of $\mathcal{A}\mathcal{G}$ is in one-to-one correspondence with the space of right-invariant vector fields on $\mathcal{G}$ that are tangent to the fibers $\mathcal{G}_x$, for all $x \in M$. Correspondingly, each element $P \in \text{Diff}(\mathcal{G})$ can be identified with a smooth, right-invariant families $(P_x)_{x \in M}$, where each $P_x$ is a differential operator on $\mathcal{G}_x$ [2, 29].

The anchor map $\rho : \Gamma(\mathcal{A}\mathcal{G}) \to \Gamma(TM)$ extends as a morphism to the space of differential operators : $\rho : \text{Diff}(\mathcal{G}) \to \text{Diff}(M)$. Since differential operators on $M$ are also differential operators on $U$, there is an algebra morphism

$$\pi_0 : \text{Diff}(\mathcal{G}) \to \text{Diff}(U),$$

which is called the *vector representation*. Note that $\pi_0$ is not injective in general (see Remark 4.15 below). The image of $\pi_0$ is denoted by $\text{Diff}_G(U)$. The set of operators of order lesser than $m$ will be written as $\text{Diff}_G^m(U)$, for any $m \in \mathbb{N}$.

The same construction can be achieved for operators acting between sections of vector bundles $E, F$ on $M$. In that case, one has to replace the coefficients $a_\alpha$ in Equation (3) by morphisms $a_\alpha \in \text{Hom}(E; F)$. Since this contains no additional difficulties and would only add further notations, we will only concern ourselves with scalar differential operators in what follows.

**Example 4.11.** Let $M$ be a manifold with boundary and $\mathcal{G}_0 \rightrightarrows M$ the 0-groupoid, as introduced in Example 4.5. Let $M_0 := M \setminus \partial M$. The differential operators $P \in \text{Diff}_{\mathcal{G}_0}(M_0)$ are those that can be written

$$P = \sum_{|\alpha| \leq m} a_\alpha (x_1 \partial_1)^{\alpha_1} \cdots (x_1 \partial_n)^{\alpha_n},$$

in any local coordinate system $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ near any point of $\partial M$. Here the coefficients $a_\alpha$ are smooth functions on $M$. The algebra $\text{Diff}_{\mathcal{G}_0}(M_0)$ contains the geometric operators (Laplacian, Dirac) associated to asymptotically hyperbolic metrics on $M_0$ [2, 22, 26].
Example 4.12. If $G_{sc} \equiv M$ is the scattering groupoid of Example 4.8, then the differential operators $P \in \text{Diff}_{G_{sc}}(M_0)$ are those that can be written

$$P = \sum_{|\alpha| \leq m} a_\alpha (x_i \partial_i)^{\alpha_1} \ldots (x_i \partial_i)^{\alpha_n},$$

in any local coordinate system $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ near any point of $\partial M$. The algebra $\text{Diff}_{G_0}(M_0)$ contains the geometric operators associated to asymptotically Euclidean metrics on $M_0$ [2, 22, 26, 33].

To study the well-posedness of differential equation, we need to introduce appropriate Sobolev spaces on $U$. Since $AG | U \simeq TU$, any metric on $AG$ induces a metric $g_0$ on $U$; such metrics will be said to be compatible with $G$. The compactness of $M$ implies that each compatible metrics is complete, and all of them are Lipschitz equivalent. Moreover, the associated Riemannian manifold $(U, g_0)$ is of bounded geometry [1, 2].

There are many equivalent definitions of Sobolev spaces on $(U, g_0)$, which are discussed in details in [1]. For example, if $\Delta$ is the Laplacian on $L^2(U, g_0)$, one may define

$$H^s(U) := \mathcal{D}((1 + \Delta)^{\frac{s}{2}}),$$

for any $s \in \mathbb{R}$, and endowe $H^s(U)$ with the graph topology. As shown in [1], any operator $P \in \text{Diff}_{G}^m(U)$ extends as a bounded operator from $H^s(U)$ to $H^{s-m}(U)$, for any $s \in \mathbb{R}$.

4.4. Fredholm conditions. Let $G \equiv M$ be a boundary action groupoid, with $U$ its unique dense $G$-orbit. Our aim in this Subsection is to obtain a characterization of Fredholm operators in $\text{Diff}_{G}(U)$. Examples 4.5 to 4.8 show that this setting includes many interesting cases of differential operators.

The following class of Lie groupoids was introduced in [4] to obtain Fredholm conditions similar to the ones we seek.

Definition 4.13. Let $G \equiv M$ be a Lie groupoid. The groupoid $G$ is a stratified submersion groupoid if there is a family of $G$-invariant open sets $(U_i)_{i=0}^k$ such that

$$\emptyset = U_{-1} \subset U_0 \subset U_1 \subset \ldots \subset U_k = M,$$

and each $S_i := U_i \setminus U_{i-1}$ is a manifold with corners, such that there exists a Lie group bundle $G_i \rightarrow B_i$, together with a tame submersion $f_i : S_i \rightarrow B_i$ of manifolds with corners and an isomorphism

$$G_{S_i} = f_i^*(G_i).$$

Recall that a differential operator $P : \Gamma(E) \rightarrow \Gamma(F)$ is called elliptic if its principal symbol $\sigma(P) \in \Gamma(T^*M)$ is invertible outside the zero-section [15]. The following Fredholm condition is one of the main results of [4].

Theorem 4.14. Let $G \equiv M$ be a Hausdorff stratified submersion Lie groupoid with filtration $(U_i)_{i=0}^k$. Assume that $U_0$ is dense with $G_{U_0} \simeq U_0 \times V_0$, and that all isotropy groups $G_0^x$ are amenable, for each $x \in M$. Then, for any $s \in \mathbb{R}$ and any operator $P \in \text{Diff}_{G}^m(U_0)$, the following statements are equivalent:

1. the operator $P : H^s(U_0) \rightarrow H^{s-m}(U_0)$ is Fredholm, and
2. the operator $P$ is elliptic and for each $x \in M \setminus U_0$, the operator $P_x : H^s(G_0^x) \rightarrow H^{s-m}(G_0^x)$ is invertible.

Remark 4.15. When $G$ is Hausdorff, a result of Khoshkam and Skandalis [16] implies that the vector representation $\pi_0 : \text{Diff}(G) \rightarrow \text{Diff}(U)$ is injective. This is essential to obtain the equivalence in Theorem 4.14, and this is why the assumption of $G$ being Hausdorff is important.
Similar characterizations of Fredholm operators were obtained in different contexts in [7, 9, 12, 23, 32], to cite a few examples. The operators $P_x$, for $x \in M \setminus U$, are called limit operators of $P$.

**Theorem 4.16.** Let $G \rightrightarrows M$ be a boundary action groupoid, and $U \subset M$ its unique dense $G$-orbit. Assume that the action of $G$ on $F := M \setminus U$ is trivial, and that for all $x \in \partial M$, the group $G^*_x$ is amenable. Let $P$ be an operator in $\text{Diff}^n_F(U)$. Then for all $s \in \mathbb{N}$, the operator $P : H^s(U) \to H^{s-m}(U)$ is Fredholm if, and only if:

1. $P$ is elliptic,
2. $P_x : H^k(G^*_x) \to H^{k-m}(G^*_x)$ is invertible for all $x \in F$.

Under the assumptions of Theorem 4.16, the characterization of Fredholm operators in $\text{Diff}_G(U)$ reduces to the study of right-invariant operators $P_x$ on the amenable groups $G^*_x$, for $x \in M \setminus U$. It should be emphasized that, if $P$ is a geometric operator (Dirac, Laplacian...) for a compatible metric on $M_0$, then each $P_x$ is an operator of the same type, induced by a right-invariant metric on the amenable group $G^*_x$ [18].

**Proof of Theorem 4.16.** First, according to Lemma 4.3, the groupoid $G$ is Hausdorff. Let $(U_i)_{i \in \mathbb{N}}$ be an orbit cover satisfying the conditions of Definition 4.1. Since $(U_i \cap F)_{i \geq 1}$ is an orbit cover for $G_F$, this groupoid is the gluing of the family $(G_{U_i \cap F})_{i \geq 1}$. Moreover, the action of $G$ on $F$ is trivial, so each $G_{U_i \cap F}$ is isomorphic to $(U_i \cap F) \times G_i$, for every $i \geq 1$.

These local trivializations show that $G_F$ is a Lie group bundle over each connected component of $F$. Thus, the groupoid $G$ is a stratified submersion groupoid with two stratas: $U \subset M$. Since all isotropy groups $G^*_x$ are amenable, for $x \in M$, the result follows from Theorem 4.14.

**Example 4.17.** The scattering groupoid $G_{sc}$ of Example 4.8 satisfies the assumptions of Theorem 4.16. When $P \in \text{Diff}^{I_G}_{sc}(M_0)$, the limit operators $P_{x}$ are translation-invariant operators on $\mathbb{R}^n$. In that case, the operator $P_x$ is simply a Fourier multiplier on $C^\infty(\mathbb{R}^n)$, whose invertibility is easy to study: see [3].

Theorem 4.16 extends straightforwardly to pseudodifferential operators and operators acting between sections of vector bundles. Note that when the action of $G$ on $\partial M$ is not trivial, the groupoid $G$ may not be a stratified submersion groupoid anymore. Nevertheless, we will show in a subsequent paper that Theorem 4.16 remains valid in that setting. The proof is more involved and requires to study the representations of the groupoid $G$.

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