Calculus of extensive quantities

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Abstract. We show how a commutative monad gives rise to a theory of extensive quantities, including (under suitable further conditions) a differential calculus of such. The relationship to Schwartz distributions is discussed. The paper is a companion to the author’s “Monads and extensive quantities”, but is phrased in more elementary terms.

Introduction

Quantities of a given type distributed over a given space (say distributions of smoke in a given room) may often be added, and multiplied by real scalars – ideally, they form a real vector space. Lawvere stressed that the dependence of such vector spaces on the space over which the quantities in question are distributed, should be taken into account; in fact, the dependence is \textit{functorial}. The viewpoint leads to a distinction between two kinds of quantities: the functorality may be covariant, or it may be contravariant: In this context, the covariant quantity types are called \textit{extensive quantities}, and the contravariant ones \textit{intensive quantities}. This usage is an attempt to put mathematical precision into the use of these terms in classical philosophy of physics. Mass distribution is an extensive quantity; mass density is an intensive one. Lawvere observed that extensive and intensive quantities often come in pairs, with a definite pattern of mutual relationship, like the homology and cohomology functors on the category of topological spaces.

In \cite{14}, we showed how such a pattern essentially comes about, whenever one has a commutative monad \( T \) on a Cartesian Closed Category \( \mathcal{E} \) (where \( \mathcal{E} \) is meant to model some category of spaces, not specified further).

Such a monad is in particular a covariant endo-functor on \( \mathcal{E} \), so the emphasis in our theory is the covariant aspect: the extensive quantities. We attempt to push a theory of these as far as possible, with the intensive quantities in a secondary role.

This in particular applies to the \textit{differential} calculus of extensive quantities on the line \( R \), which will be discussed in the last Sections\( \S \) and\( \S \) here, we also discuss the relationship to the theory of Schwartz distributions of compact support (these have the covariant functoriality requested for extensive quantities, and is a basis for a classical version of a theory of extensive quantities).

The article \cite{15} by Reyes and the author develop some further differential calculus of extensive quantities, not only in dimension 1, as here; but it is couched in terms of the Schwartz (double dualization) paradigm, which we presently want to push in the background.
1 Monads and their algebras

The relationship between universal algebra, on the one hand, and monads on the category of sets on the other, became apparent in the mid 60s, through the work of Linton, Manes, Kleisli, and many others:

If \( T = (T, \eta, \mu) \) is such a monad, and \( X \) is a set, an element \( P \in T(X) \) may be interpreted as an \( X \)-ary operation on arbitrary \( T \)-algebras \( B = (B, \beta) \): if \( \phi : X \to B \) is an \( X \)-tuple of elements in \( B \), we can construct a single element \( (P, \phi) \in B \), namely the value of 

\[
\begin{array}{ccc}
T(X) & \xrightarrow{T(\phi)} & T(B) \\
& \xrightarrow{\beta} & B
\end{array}
\]

on the element \( P \in T(X) \). Then every morphism \( f : B \to C \) of \( T \)-algebras is a homomorphism with respect to the operation defined by \( P \). There is also a converse statement.

The monad-theoretic formulation of universal algebra can be lifted to symmetric monoidal closed categories \( E \) other than sets, provided one considers the monad \( T \) to be \( E \)-enriched\(^1\), in particular, it applies to \( E \)-enriched monads on any cartesian closed category \( E \).

Recall that for any functor \( T : \mathcal{E} \to \mathcal{E} \), one has maps \( \text{hom}(X, Y) \to \text{hom}(T(X), T(Y)) \), sending \( f \in \text{hom}(X, Y) \) to \( T(f) \in \text{hom}(T(X), T(Y)) \); the \( E \)-enrichment means that these maps not only can be defined for the hom-sets, but for the hom-objects, so that we get maps (the “strength” of \( T \))

\[
st_{X,Y} : Y^X \to T(Y)^{T(X)};
\]

\( Y^X \), as an object of \( \mathcal{E} \) carries more structure than the mere set of maps from \( X \) to \( Y \), e.g. it may carry some topology, if \( \mathcal{E} \) happens to be of topological nature.

This widening of the scope of monad theoretic universal algebra was documented in a series of articles by the author in the early 1970s, cf. \[6\], \[7\], \[8\], \[9\], \[10\]. In particular, the formulation makes sense for cartesian closed categories, which is the context of the present note. Via this formulation, it makes contact with functional analysis, because the basic logic behind functional analysis is the ability to form function spaces, even non-linear ones. For instance, the category \( \mathcal{E} \) of convenient vector spaces, and the smooth (not necessarily non-linear maps) is a cartesian closed category, where several theories of functional analysis have natural formulation, e.g. the theory of distributions (in the sense of Schwartz and others).

We expound here some aspects of the relationship between the theory of strong monads and functional analysis, by talking about \( \mathcal{E} \) as if it were just the category of sets, and where \( \mathcal{E} \)-enrichment therefore is automatic. (A rigorous account of these aspects for general cartesian closed categories \( \mathcal{E} \) is given in \[14\].) So our exposition technique here is in the spirit of the “naive” exposition of synthetic differential geometry, as given in \[13\], say.

One aim of the theory developed in \[14\], and expounded “synthetically” in the present note, is to document that the space \( T(X) \) may be seen as a space of extensive quantities (of some type) on \( X \), in the sense of Lawvere. So we prefer to talk about a \( P \in T(X) \) as an extensive quantity on \( X \), rather than as “an \( X \)-ary operation, operating on all \( T \)-algebras \( B \).

\(^1\)For the notion of monad, and algebras for a monad, the reader may consult \[2\].

\(^2\)For these notions, the reader is again referred to \[2\].
To make this work well, one must assume that the monad $T$ is commutative$, see Section and$. A main emphasis in Lawvere’s ideas is that the space of extensive quantities on $X$ depend (covariant) functorially on $X$; in the present context, functorality is encoded by the fact that $T$ is a functor. So for $f : X \to Y$, we have $T(f) : T(X) \to T(Y)$; when $T$ is well understood from the context, we may write $f_*$ for $T(f)$; this is a type of notation that is as old as the very notion of functor (recall homology!).

Thus, the semantics of $P \in T(X)$, as an $X$-ary operation an $T$-algebras $(B, \beta)$, may be rendered in terms of a pairing

$$\langle P, \phi \rangle := \beta(\phi(P)),$$

where $\phi : X \to B$ is a map (“an $X$-tuple of elements of $B$”). This semantic aspect of extensive quantities is essential in Section 9, where it is seen as the basis of a synthetic theory of Schwartz distributions.

For completeness, let us indicate how the pairing is defined without using individual elements $P \in T(X)$ and $\phi \in B^X$, as a map

$$T(X) \times B^X \to B,$$

namely utilizing the assumed enrichment of $T$ over $E$:

$$T(X) \times B^X \xrightarrow{T(X) \times st} T(X) \times T(B)^{T(X)} \xrightarrow{ev} T(B) \xrightarrow{\beta} B;$$

here, $ev$ denotes the “evaluation” map (part of the cartesian closed structure of $E$). Henceforth, we shall be content with using “synthetic” descriptions, utilizing elements.

We note the following naturality property of the pairing: for $f : X \to Y$, $P \in T(X)$ and $\psi : Y \to B$, we have

$$\langle f_*(P), \psi \rangle = \langle P, f^*(\psi) \rangle,$$

where $f^*(\psi) := \psi \circ f$. For, the left hand side is $\beta(\psi_* f_* P)$, and the right hand side is

$$\beta((f^*(\psi)), P) = \beta(\psi_*(\psi) f_*(P)),$$

but $(\psi \circ f)_* = \psi_* f_*$ since $T$ is a functor.

Similarly, if $B = (B, \beta)$ and $(C = (C, \gamma))$ are $T$-algebras, and $F : B \to C$ is a $T$-homomorphism$, we have, for $P \in T(X)$ and $\phi : X \to B$ that

$$F(\langle P, \phi \rangle) = \langle P, F(\phi) \rangle.$$ (4)

This is an immediate consequence of $F \circ \beta = \gamma \circ T(F)$, the equation expressing that $F$ is a $T$-homomorphism.

The terminal object of $E$ is denoted $1$. The object $T(1)$ plays a special role as the algebra of “scalars”, and we denote it also $R$; with suitable properties of $T$, it will in fact

$3$It should be stressed that to be commutative is a property of enriched (=strong) monads, and enrichment is a structure, not a property. However, for the case where $E$ is the category of sets, enrichment is automatic.

$4$later in this article, $T$-homomorphisms will be called “$T$-linear” maps.
carry a canonical commutative ring structure, see Section 7. Its multiplicative unit 1 is the
element picked out by $\eta_1$. For any $X \in \mathcal{E}$, we have a unique map $X \to 1$, denoted $!$ (when
$X$ is understood from the context). For $P \in T(X)$, we have canonically associated a scalar
tot$(P) \in T(1)$, the total of $P$, namely
\[
tot(P) := !_* (P).
\]
From uniqueness of maps to 1 follows immediately that $P$ and $f_* (P)$ have the same total,
for any $f : X \to Y$.

One example of a monad $T$ (on a suitable cartesian closed category of smooth spaces)
is where $T(X)$ is the space of Schwartz distributions of compact support; we return to
Schwartz distributions in [9] and they are only mentioned here as a warning, namely that
functorality is a strong requirement; thus for instance, a uniform distribution on the line can
never have a total. (The functorial properties of non-compact distributions are not under-
stood well enough presently.) In [11], the authors construct the free real vector space monad
$T$, in a category of suitable “convenient” spaces ([16]), by carving it, out by topological
means, from the monad of (compactly supported) Schwartz distributions.

The units $\eta_X : X \to T(X)$ will, in terms of Schwartz distribution theory, pick out the
Dirac distributions $\delta_x$; therefore, we shall allows ourselves the following doubling of nota-
tion: for $x \in X$, we write
\[
\eta_X(x) = \delta_x
\]
(with $X$ understood from the context, on the right hand side).

**Proposition 1**  Let $B = (B, \beta)$ be a $T$-algebra, and $\phi : X \to B$ a map. Then
\[
\langle \delta_x, \phi \rangle = \phi (x).
\]
In particular,
\[
\langle \delta_x, \eta_X \rangle = \delta_x.
\]

**Proof.** In elementfree terms, the first equation says that the composite
\[
X \xrightarrow{\eta_X} T(X) \xrightarrow{T(\phi)} T(B) \xrightarrow{\beta} B
\]
equals $\phi : X \to B$. And this holds, because by naturality of $\eta$, $T(\phi) \circ \eta_X = \eta_B \circ \phi$; but $\beta \circ \eta_B$
is the identity map on $B$, by the unitary law for the algebra structure $\beta$. The second equation
is then immediate.

An extensive quantity of the form $\delta_x$ has total 1 $\in T(1)$,
\[
tot(\delta_x) = 1.
\]
(5)

For, the composite map
\[
X \xrightarrow{\eta_X} T(X) \xrightarrow{T(!)} T(1) = R
\]
equals the composite

\[ X \overset{!}{\longrightarrow} 1 \overset{\eta}{\longrightarrow} T(1) = R, \]  

(6)

by naturality of \( \eta \) w.r.to \( ! : X \rightarrow 1 \), and this is the map taking value 1 for all \( x \in X \).

**Proposition 2** For the total of \( P \in T(X) \), we have

\[ \text{tot}(P) = \langle P, 1_X \rangle, \]

where \( 1_X \) denotes the function \( X \rightarrow R \) with constant value 1 \( \in R \).

**Proof.** The map \( 1_X \) is displayed in (6) above, so \( \langle P, 1_X \rangle \) is by definition the result of applying to \( P \in T(X) \) the composite

\[ T(X) \overset{T(!)}{\longrightarrow} T(1) \overset{T(\eta)}{\longrightarrow} T^2(1) \overset{\mu}{\longrightarrow} T(1) = R. \]

Now by one of the unit laws for a monad, the composite of the two last maps here is the identity map of \( T(1) \), so the displayed composite is just \( T(!) \); this is the map which to \( P \) returns the total of \( P \).

**Notation.** We attempted to make the notation as standard as possible. On three occasions, this forces us to have double notation, like the \( \eta \)-\( \delta \) doubling above, and later \( E \) (“expectation”) for \( \mu \), and an “integral” symbol for the pairing of extensive and intensive quantities. An exception to standard notation is that the exponential object \( B^X \) is denoted \( X \downarrow B \), to keep it online. (Other online notations have also been used, like \( [X, B] \) or \( X \rightarrow B \).)

## 2 Tensor product of extensive quantities

Let \( T \) be an algebraic theory, and let \( P \) and \( Q \) be an \( X \)-ary and a \( Y \)-ary operation of it. Then one can define, semantically, an \( X \times Y \)-ary operation \( P \otimes Q \): given an \( X \times Y \)-tuple \( \theta \) on the \( T \)-algebra \( B \); we think of \( \theta \) as an matrix of elements of \( B \) with \( X \) rows and \( Y \) columns. Now evaluate \( P \) on each of the \( Y \) columns; this gives a \( Y \)-tuple of elements of \( B \); then evaluate \( Q \) on this \( Y \)-tuple; this gives an element of \( B \). This element is declared to be the value of the \( X \times Y \)-ary operation \( P \otimes Q \) on \( \theta \).

One might instead first have evaluated \( Q \) on each of the rows, and then evaluated \( P \) on the resulting \( X \)-tuple; this would in general give a different result, denoted \( P \otimes Q \); the theory is called commutative if \( \otimes \) and \( \otimes \) agree.

This kind of tensor product was formulated monad theoretically, and without reference to the semantics involving \( T \)-algebras, in the author’s 1970-1972 papers, so as to be applicable for any strong monad on any cartesian closed category \( \mathcal{E} \); it takes the form of two maps\(^5\) natural in \( X \) and \( Y \in \mathcal{E} \),

\[ T(X) \times T(Y) \overset{\otimes}{\longrightarrow} T(X \times Y); \]

\(^5\)denoted in (6) by \( \psi_{X,Y} \) and \( \psi_{Y,X} \), respectively
the monad is called *commutative* if these two maps agree, for all $X$ and $Y$. Both $\otimes$ and $\bar{\otimes}$ make the functor $T$ a *monoidal* functor, in particular they satisfy a well known associativity constraint: $(P \otimes Q) \otimes S = P \otimes (Q \otimes S)$, modulo the isomorphisms induced by $(X \times Y) \times Z \cong X \times (Y \times Z)$, and similarly for $\bar{\otimes}$. (The “nullary” part that goes along with $\otimes$, is a map $1 \to T(1)$.)

Among the equations satisfied is $\otimes \circ (\eta_X \times \eta_Y) = \eta_{X \times Y}$, which in the notation with $\delta$ reads: for $x \in X$ and $y \in Y$,

$$\delta_x \otimes \delta_y = \delta_{(x,y)}. \tag{7}$$

If $A = (A, \alpha)$ and $C = (C, \gamma)$ are $T$-algebras, it makes sense to ask whether a map $A \times X \to C$ is a $T$-homomorphism in the first variable, cf. [9]; and similarly it makes sense to ask whether a map $X \times A \to C$ is a $T$-homomorphism in the second variable. We shall use the term “$T$-linear map” as synonymous with $T$-homomorphism; this allows us to use the term $T$-bilinear for a map $A \times B \to C$ which is a $T$-homomorphism in each of the two input variables separately (where $A, B,$ and $C$ are $T$-algebras), and similarly “$T$-linear in the first variable”, etc.

If $f : X \times Y \to C$ is any map into a $T$-algebra $C$, it extends uniquely over $\eta_X \times \eta_Y$ to a map $T(X) \times T(Y) \to C$ which is $T$-linear in the first variable. Similarly $f$ extends uniquely over $X \times \eta_Y$ to a map $X \times T(Y) \to C$ which is $T$-linear in the second variable. However, a map $X \times Y \to C$ does not necessarily extend to a $T$-bilinear $T(X) \times T(Y) \to C$; for this, one needs commutativity of $T$.

## 3 Commutative monads

We henceforth consider a commutative monad $T = (T, \eta, \mu)$ on $\mathcal{E}$. The reader may have for instance the free-abelian-group monad in mind.

From [9], we know that commutativity of $T$ is equivalent to the assertion that $\otimes : T(X) \times T(Y) \to T(X \times Y)$ is $T$-bilinear, for all $X$ and $Y$. (In the non-commutative case, $\otimes$ will only be $T$-linear in the second variable, and $\bar{\otimes}$ will only be $T$-linear in the first variable.)

Then if $C = (C, \gamma)$ is a $T$-algebra, any map $f : X \times Y \to C$ extends uniquely over $\eta_X \times \eta_Y$ to a $T$-bilinear map $T(X) \times T(Y) \to C$. Since $f$ also extends uniquely over $\eta_X \times \eta_Y$ to a $T$-linear $T(X \times Y) \to C$, one may deduce that $\otimes : T(X) \times T(Y) \to T(X \times Y)$ is in fact a *universal* $T$-bilinear map out of $T(X) \times T(Y)$.

If $B = (B, \beta)$ is a $T$-algebra, and $X$ is an arbitrary object, $X \cap B$ carries a canonical “pointwise” $T$-algebra structure inherited from $\beta$. In the category of sets, this is just the “coordinatwise” $T$-algebra structure on $\Pi_X B$.

Let $(A, \alpha)$ and $(B, \beta)$ be $T$-algebras. If $\mathcal{E}$ has sufficiently many equalizers, there is a subobject $A \cap_T B$ of $A \cap B$, which in the set case consists of those maps $A \to B$ which happen to be $T$-linear. With $T$ commutative, the subobject $A \cap_T B \subseteq A \cap B$ is in fact a sub-$T$-algebra, cf. [8].

If $A = (A, \alpha)$, $B = (B, \beta)$, and $C = (C, \gamma)$ are $T$-algebras, a map $A \times B \to C$ is $T$-bilinear iff its transpose $A \to B \cap C$ is $T$-linear, and factors through the subalgebra $B \cap_T C$. 

6
In [14], Theorem 1, we prove that for a commutative monad $T$, the exponential adjoint of the pairing $T(X) \times (X \rhd B) \to B$ is a $T$-bilinear map $T(X) \to (X \rhd B) \rhd_T B$; so therefore also, we have

**Theorem 1** The pairing $T(X) \times (X \rhd B) \xrightarrow{\langle \cdot, \cdot \rangle} B$ is $T$-bilinear, for any $T$-algebra $B$.

### 4 Convolution

If $a : X \times Y \to Z$ is any map, and $P \in T(X)$, $Q \in T(Y)$, we may form $a_*(P \otimes Q) \in T(Z)$, called the *convolution* of $P$ and $Q$ along $a$, and denoted $P \ast_a Q$. Since $\otimes$ is $T$-bilinear and $a_*$ is $T$-linear, it follows that $P \ast_a Q$ depends in a $T$-bilinear way on $P, Q$. In diagram, convolution along $a$ is the composite

$$T(X) \times T(Y) \xrightarrow{\otimes} T(X \times Y) \xrightarrow{T(a)} T(Z).$$

The convolution along the unique map $1 \times 1 \to 1$ gives a multiplication on $R = T(1)$, which is commutative.

A consequence of the naturality of $\otimes$ w.r.to the maps $! : X \to 1$ and $! : Y \to 1$ is that

$$\text{tot}(P \otimes Q) = \text{tot}(P) \cdot \text{tot}(Q),$$

where the dot denotes the product in $R = T(1)$. Note that this product is itself (modulo the identification $T(1 \times 1) \cong T(1)$) a tensor product, $T(1) \times T(1) \to T(1 \times 1) \cong T(1)$.

We have, for $x \in X$ and $y \in Y$

$$\delta_x \ast_a \delta_y = \delta_{a(x,y)}.$$  \hspace{1cm} (9)

This follows from (7), together with naturality of $\eta$ w.r.to $a$, which in $\delta$-terms reads

$$T(a)(\delta_{(x,y)}) = \delta_{a(x,y)}.$$  \hspace{1cm} (9)

If $a$ is an associative operation $X \times X \to X$, it follows from properties of monoidal functors that the convolution along $a$, $T(X) \times T(X) \to T(X)$ is likewise associative. If $a$ is commutative, commutativity of convolution along $a$ will be a consequence, but here, one uses the assumption that the monad $T$ is commutative.

### 5 The space of scalars $R := T(1)$

The space $T(1)$ plays the role of the “ring” of scalars, or number line. It has a $T$-linear structure, since it is a $T$-algebra, and it carries a $T$-bilinear multiplication $m$, namely

$$T(1) \times T(1) \xrightarrow{\otimes} T(1 \times 1) \cong T(1),$$

7
which is commutative and associative. The multiplicative unit is picked out by \( \eta_1 : 1 \to T(1) = R \). Also \( R \) acts on any \( T(X) \), by

\[
T(1) \times T(X) \xrightarrow{\otimes} T(1 \times X) \cong T(X),
\]

(10)
generalizing the description of multiplication on \( R \); it is denoted just by a dot \( \cdot \). This action is likewise \( T \)-bilinear, and unitary and associative. These assertions follow from \( T \)-bilinearity of \( \otimes \), and the compatibility of \( \otimes \) with \( \eta \).

There are properties of \( T \) which will imply that \( T \)-algebras carry abelian group structure; in this case, \( R \) is a commutative ring, with the above \( m \) as multiplication, and any \( T(X) \) is an \( R \)-module, with \( T \)-linear maps being in particular \( R \)-linear, see Section 7.

6 Intensive quantities, and their action on extensive quantities

The space \( R = T(1) \) is a \( T \)-algebra, with a commutative \( T \)-bilinear commutative monoid structure (so in the additive case, Section 7 it is in particular a commutative ring). From general principles it follows that for any \( X \), the space \( X \sqcap R = X \sqcap T(1) \) inherits a \( T \)-algebra structure and a \( T \)-bilinear monoid structure; in fact \( \sqcap R \) is a contravariant functor with values in the category of monoids whose multiplication is \( T \)-bilinear. If \( f : X \to Y \), the map \( f \sqcap R : Y \sqcap R \to X \sqcap R \) preserves this structure. The map \( f \sqcap R \) is denoted \( f^* \), and is a kind of companion to the covariant \( f_* : T(X) \to T(Y) \). In the terminology of Lawvere [18], \( X \sqcap R \) is a space of intensive quantities on \( X \). Note that \( T \) only enters in the form of \( R = T(1) \).

The monoid \( X \sqcap R = X \sqcap T(1) \) acts on any space of the form \( X \sqcap T(Y) \), by a simple “pointwise” lifting of the action of \( T(1) \) on \( T(Y) \), described in (10), (with \( Y \) instead of \( X \)):

\[
\{X \sqcap T(1)\} \times \{X \sqcap T(Y)\} \cong X \sqcap \{T(1) \times T(Y)\} \xrightarrow{\otimes 1_Y} X \sqcap T(1 \times Y) \cong X \sqcap T(Y).
\]

(The monoid structure on \( R = T(1) \) is a special case.)

We shall describe an action of the monoid \( X \sqcap R \) on \( T(X) \). It has a special case the “multiplication of a distribution by a function” known from (Schwartz) distribution theory.

Notationally, we let the action be from the right, and denote it \( \vdash \).

\[
T(X) \times (X \sqcap R) \xrightarrow{\vdash} T(X).
\]

It is \( T \)-linear in the first variable; it is the \( 1 \)-\( T \)-linear extension over \( \eta_{X \times R} \) of a certain map \( X \times (X \sqcap R) \to T(X) \); in other words, we describe first \( P \vdash \phi \) for the case where...
\[ P = \eta(x) = \delta_x \text{ for some } x \in X, \text{ and where } \phi : X \to R \text{ is any function. Namely, we put} \]
\[ \delta_x \vdash \phi := \phi(x) \cdot \delta_x, \]
recalling that \( T(X) \) carries a (left) action \( \cdot \) by \( R = T(1) \) (of course “left” and “right” does not make any difference here, since the monoids in question are commutative).

Since \( T \)-linearity implies homogeneity w.r.to multiplication by scalars \( \lambda \) in \( R \), we have in particular that
\[ \lambda \cdot (\delta_x \vdash \phi) = (\lambda \cdot \delta_x) \vdash \phi \]
(11)
We shall prove that the action \( \vdash \) is unitary and associative. The unit of \( X \upharpoonright R \) is \( 1_X \), i.e. the function with constant value \( 1 \in R \). So we should prove \( P \vdash 1_X = P \). By \( T \)-linearity in the first variable, it is enough to see it when the input \( P \) from \( T(X) \) is of the form \( \delta_x \), so we should prove, for the unitary property,
\[ \delta_x \vdash 1_X = \delta_x \]
and similarly, for the associative property, it suffices to prove
\[ (\delta_x \vdash \phi) \vdash \psi = \delta_x \vdash (\phi \cdot \psi) \]
for \( \phi \) and \( \psi \) arbitrary functions \( X \to R \). The first equation then is a consequence of \( 1_X(x) = 1 \); unravelling similarly the second equation, one sees that the two sides are, respectively \( (\phi(x) \cdot \psi(x)) \cdot \delta_x \) (using (11)), and \( (\phi \cdot \psi)(x) \), and their equality is a consequence of the pointwise nature (and commutativity) of the multiplication in \( R \).

The following result serves in the Schwartz theory in essence as the definition of the action \( \vdash \) of intensive quantities on extensive ones. Recall that the multiplicative monoid of \( R \) acts on any \( T \)-algebra of the form \( T(Y) \), via \( \otimes_{1,Y} : T(1) \times T(Y) \to T(1 \times Y) \cong T(Y) \). The action is temporarily denoted \( \vdash \); for \( Y = 1 \), it is just the multiplication on \( R \). This action extends pointwise to an action \( \vdash \) of \( X \upharpoonright R \) on \( X \upharpoonright T(Y) \).

**Proposition 3** Let \( \phi \) be a function \( X \to R \), and let \( \psi \) be a function \( X \to T(Y) \). Then for any \( P \in T(X) \),
\[ \langle P \vdash \phi, \psi \rangle = \langle P, \phi \vdash \psi \rangle \in T(Y). \]

**Proof.** Since both sides of the claimed equation depend in a \( T \)-linear way on \( P \), it suffices to prove the equation for the case where \( P \) is \( \delta_x \) for some \( x \in X \). We calculate the left hand side:
\[ \langle \delta_x \vdash \phi, \psi \rangle = \langle \phi(x) \cdot \delta_x, \psi \rangle = \phi(x) \cdot (\delta_x, \psi) = \phi(x) \cdot \psi(x), \]
using Proposition[1] and the right hand side similarly calculates
\[ \langle \delta_x, \phi \vdash \psi \rangle = (\phi \vdash \psi)(x), \]
which is likewise \( \phi(x) \vdash \psi(x) \), because of the pointwise character of the action \( \vdash \) on \( X \upharpoonright T(Y) \).
Corollary 1  The pairing \( \langle P, \phi \rangle \) (for \( P \in T(X) \) and \( \phi \in X \sqcup R \)) can be described in terms of \( \vdash \) as follows:

\[
\langle P, \phi \rangle = \text{tot}(P \vdash \phi).
\]

Proof. Take \( Y = 1 \) (so \( T(Y) = R \)), and take \( \psi = 1_X \). Then

\[
\text{tot}(P \vdash \phi) = \langle P \vdash \phi, 1_X \rangle = \langle P, \phi \cdot 1_X \rangle
\]

using (2), and then the Proposition. But \( \phi \cdot 1_X = \phi \).

7 Additive structure

We shall in the present Section describe a simple categorical property of the monad \( T \), which will guarantee that “\( T \)-linearity implies additivity”, even “\( R \)-linearity” in the sense of a rig \( R \in E \) (“rig” = commutative semiring), namely \( R = T(1) \). This condition will in fact imply that \( E^T \) is an additive (or linear) category.

We begin with some standard general category theory, namely a monad \( T = (T, \eta, \mu) \) on a category which has finite products and finite coproducts. (No distributivity is assumed.) So \( E \) has an initial object \( 0 \). If \( T(\emptyset) \in E \) is a terminal object, then the object \( (T(\emptyset), \mu_0) \) is a zero object in \( E^T \), i.e. it is both initial and terminal. It is initial because \( T \), as a functor \( E \to E^T \), is a left adjoint, hence preserves initials; and since \( T(\emptyset) = 1 \), it is also terminal (the terminal object in \( E^T \) being \( 1 \in E \), equipped with the unique map \( T(1) \to 1 \) as structure). This zero object in \( E^T \) we denote \( 0 \). Existence of a zero object in a category implies that the category has distinguished zero maps \( 0_{A,B} : A \to B \) between any two objects \( A \) and \( B \), namely the unique map \( A \to B \) which factors through \( 0 \). For \( E^T \), we can even talk about the zero map \( 0_{X,B} : X \to B \), where \( X \in E \) and \( B = (B, \beta) \in E^T \), namely \( 0_{X,B} \in \eta_X \) followed by the zero map \( 0_{T(X),B} : T(X) \to B \). We have a canonical map \( X + Y \to T(X) \times T(Y) \); the composite \( X \to X + Y \to T(X) \times T(Y) \) is \( (\eta_X, 0_{X,T(Y)}) \) (here, the first map is the coproduct inclusion map). Similarly, we have a canonical map \( Y \to T(X) \times T(Y) \). Using the universal property of coproducts, we thus get a canonical map \( \Phi_{X,Y} : X + Y \to T(X) \times T(Y) \). It extends uniquely over \( \eta_{X+Y} : X + Y \to T(X + Y) \) to a \( T \)-linear map

\[
\Phi_{X,Y} : T(X + Y) \to T(X) \times T(Y),
\]

and \( \Phi \) is natural in \( X \) and in \( Y \). We say that \( T : E \to E^T \) takes binary coproducts to products if \( \Phi_{X,Y} \) is an isomorphism (in \( E \) or equivalently in \( E^T \)) for all \( X, Y \) in \( E \). Note that the definition presupposed that \( T(\emptyset) = 1 \); it is the zero object in \( E^T \), so that if \( T \) takes binary coproducts to products, it in fact takes finite coproducts to products, in a similar sense. So we can also use the phrase “\( T \) takes finite coproducts to products” for this property of \( T \).

We define an “addition” map in \( E^T \); it is a map \( + : T(X) \times T(X) \to T(X) \), namely the composite

\[
\begin{array}{ccc}
T(X) \times T(X) & \xrightarrow{\Phi_{X,Y}^{-1}} & T(X + X) \xrightarrow{T(\nabla)} T(X)
\end{array}
\]
where $\nabla : X + X \to X$ is the codiagonal. So in particular, if $i_n$ denotes the $i$th inclusion ($i = 1, 2$) of $X$ into $X + X$, we have

$$id_{TX} = \begin{bmatrix} TX \xrightarrow{T(i_1)} T(X + X) \xrightarrow{\Phi_{X^2}} TX \times TX \xrightarrow{+} TX \end{bmatrix}. \quad (12)$$

Note that this addition map is $T$-linear. Under the identification $T(X) \cong T(\{X\} \times 0) \cong T(X) \times 1$, the equation (12) can also be read: $T(\{!\}) : T(\emptyset) \to T(X)$ is right unit for $+$, and similarly one gets that it is a left unit.

We leave to the reader the easy proof of associativity and commutativity of the map $+: T(X) \times T(X) \to T(X)$. It follows that $T(X)$ acquires structure of an abelian monoid in $\mathcal{E}^T$ (and also in $\mathcal{E}$).

For an abelian monoid $A$ in any category, we may ask whether $A$ is an abelian group or not (so there is a “minus” corresponding to the $+$); existence of such “minus” is a property of $A$, not an added structure. If $T$ is a monad which takes finite coproducts to products, it makes sense to ask whether the canonical monoid structure which $T$-algebras in this case have, is actually an abelian group structure; it is therefore a property on such $T$, not an added structure. We shall henceforth assume this property, since we need “minus” (= difference) for differential calculus.

In [14], we proved that

**Proposition 4** Every $T$-linear map $T(X) \to T(Y)$ is compatible with the abelian group structure.

We again assume that $T$ is a commutative monad. Recall that we then have the $T$-bilinear action $T(X) \times T(1) \to T(X)$. It follows from the Proposition that it is additive in each variable separately.

We have in particular the $T$-bilinear commutative multiplication $m : T(1) \times T(1) \to T(1)$, likewise bi-additive, $m(x + y, z) = m(x, z) + m(y, z)$, or in the notation one also wants to use,

$$(x + y) \cdot z = x \cdot z + y \cdot z,$$

so that $R = T(1)$ carries structure of a commutative ring. We may summarize:

**Proposition 5** Each $T(X)$ is a module over the ring $R = T(1)$; each $T$-linear map $T(X) \to T(Y)$ is an $R$-module morphism.

It is more generally true that $T$-linear maps $A \to B$ (for $A$ and $B \in \mathcal{E}^T$) are $R$-module maps. We shall not use this fact.

The property of $T$ that it “takes finite coproducts to products” accounts for a limited aspect of contravariance for extensive quantities: if a space $X$ is a coproduct, $X = X_1 + X_2$, the isomorphism $T(X_1 + X_2) \cong T(X_1) \times T(X_2)$ implies that an extensive quantity $P \in T(X)$ gives rise to a pair of extensive quantities $P_1 \in T(X_1)$ and $P_2 \in T(X_2)$, which one may reasonably may call the restrictions of $P$ to $X_1$ and $X_2$, respectively. Now, restriction is a contravariant construction, and applies as such, for intensive quantities, along any map. For
extensive quantities, and for monads $T$ of the special kind studied here, it applies only to quite special maps, namely to the inclusion maps into finite coproducts, like $X_1 \to X_1 + X_2$.

We leave to the reader to philosophize over the extent to which, given a distribution of smoke in a given room, it makes sense to talk about “the distribution of this quantity of smoke, restricted to the lower half of the room”.

(For distributions in the Schwartz sense, one may construct some further “restriction” constructions (restriction to open subsets); this is an aspect of the fact that the corresponding intensive quantities (= smooth functions) admit an “extension” construction from closed subsets.)

For $u \in R$, we have the translation map $\alpha^u : R \to R$ given by $x \mapsto x + u$. If $P \in T(R)$, we have thus also $\alpha^u_*(P) \in T(R)$.

We have the following reformulation of the translation maps in terms of convolution along the addition map $+: R \times R \to R$:

**Proposition 6** For any $P \in T(R)$ and $a \in R$,

$$\alpha^a_*(P) = \delta_a * P = P * \delta_a.$$

**Proof.** The second equation follows from commutativity of $+$. To see the first equation, we observe that both sides of $\alpha^a_*(P) = \delta_a * P$ depend $T$-linearly on $P$; so it suffices to prove this equation for the case where $P$ is of the form $\delta_b$ for $b \in R$. But $\alpha^a_*(\delta_b) = \delta_{a+b} = \delta_{b+a}$.

In particular, we see that $\delta_0$ is a neutral element for convolution, $\delta_0 * P = P * \delta_0 = P$.

8 Differential calculus of extensive quantities on $R$

We attempt in this Section to show how some differential calculus of extensive quantities $\in T(R)$ may be developed on equal footing with the standard differential calculus of intensive quantities (meaning here: functions defined on $R$). For this, we assume that the monad $T$ on $E$ has the properties described in Section 7 so in particular, $R$ is a commutative ring. To have some differential calculus going for such $R$, one needs some further assumption.

Consider a commutative ring $R$. Assume $D \subseteq R$ is a subset satisfying the following “KL’‘-axiom:

for any $f : R \to R$, there exists a unique $f' : R \to R$ such that for all $x \in R$

$$f(x + d) = f(x) + d \cdot f'(x) \quad \text{for all } d \in D. \quad (13)$$

Example: 1) models of synthetic differential geometry, with $D$ the set of $d \in R$ with $d^2 = 0$ (the simple “Kock-Lawvere” axiom says (cf. e.g. [11]) a little more than this, namely it also asks that any function $f : D \to R$ extends to a function $f : R \to R$.)

2) Any commutative ring, with $D = \{d\}$ for one single invertible $d \in R$. In this case, for given $f$, the $f'$ asserted by the axiom is the function

$$f'(x) = \frac{1}{d} \cdot (f(x + d) - f(x)).$$

12
the standard difference quotient.

Similarly, if $V$ is an $R$-module, we say that it satisfies KL, if for any $f : R \to V$, there exists a unique $f' : R \to V$ such that (13) holds for all $x \in R$.

In either case, we may call $f'$ the derivative of $f$.

It is easy to see that any commutative ring $R$ is a model, using $\{d\}$ as $D$, as in Example 2) (and then also, any $R$-module $V$ satisfies then the axiom); this leads to some calculus of finite differences. Also, it is true that if $E$ is the category of abstract sets, there are no non-trivial models of the type in Example 1); but, on the other hand, there are other cartesian closed categories $E$ (e.g. certain toposes containing the category of smooth manifolds, cf. e.g. [11]), and where a rather full fledged differential calculus for intensive quantities emerges from the KL-axiom.

We assume that $R = T(1)$ satisfies the KL-axiom (for some fixed $D \subseteq R$), and also that any $R$-module of the form $T(R)$ does so.

Proposition 7 (Cancelling universally quantified $ds$) If $V$ is an $R$-module which satisfies KL, and $v \in V$ has the property that $d \cdot v = 0$ for all $d \in D$, then $v = 0$.

Proof. Consider the function $f : R \to V$ given by $t \mapsto t \cdot v$. Then for all $x \in R$ and $d \in D$

$$f(x + d) = (x + d) \cdot v = x \cdot v + d \cdot v,$$

so that the constant function with value $v$ will serve as $f'$. On the other hand, $d \cdot v = d \cdot 0$ by assumption, so that the equation may be continued,

$$= x \cdot v + d \cdot 0$$

so that the constant function with value $0 \in V$ will likewise serve as $f'$. From the uniqueness of $f'$, as requested by the axiom, then follows that $v = 0$.

We are now going to provide a notion of derivative $P'$ for any $P \in T(R)$. Unlike differentiation of distributions in the sense of Schwartz, which is defined in terms of differentiation of test functions $\phi$, our construction does not mention test functions, and the Schwartz definition $\langle P', \phi \rangle := -\langle P, \phi' \rangle$ comes in our treatment out as a result, see Proposition 13 below.

For $u = 0, P - \alpha^u(P) = 0 \in T(R)$. Assuming that the $R$-module $T(R)$ is KL, we therefore have for any $P \in T(R)$ that there exists a unique $P' \in T(R)$ such that for all $d \in D$,

$$d \cdot P' = P - \alpha^d(P).$$

Since $d \cdot P'$ has total 0 for all $d \in D$, it follows that $P'$ has total 0.

Differentiation is translation-invariant: using

$$\alpha^t \circ \alpha^s = \alpha^{t+s} = \alpha^s \circ \alpha^t,$$

it is easy to deduce that

$$\langle \alpha^t(P) \rangle' = \langle \alpha^t \rangle_*(P').$$

(14)
Proposition 8 Differentiation of extensive quantities on \( R \) is a \( T \)-linear process.

Proof. Let temporarily \( \Delta : T(R) \to T(R) \) denote the differentiation process. Consider a fixed \( d \in D \). Then for any \( P \in T(R) \), \( d \cdot \Delta(P) = d \cdot P' \) is \( P - \alpha_d^P \); it is a difference of the two \( T \)-linear maps, namely the identity map on \( T(R) \) and \( \alpha_d^P = T(\alpha_d) \), and as such is \( T \)-linear.

Thus for each \( d \in D \), the map \( d \cdot \Delta : T(R) \to T(R) \) is \( T \)-linear. Now to prove \( T \)-linearity of \( \Delta \) means, by monad theory, to prove equality of two maps \( T^2(R) \to T(R) \); and since \( d \cdot \Delta \) is \( T \)-linear, as we proved, it follows that the two desired maps \( T^2(R) \to T(R) \) become equal when post-composed with the map "multiplication by \( d' \)": \( T(R) \to T(R) \). Since \( d \in D \) was arbitrary, it follows from KL axiom for the \( R \)-module \( T(X) \) that the two desired maps are equal, proving \( T \)-linearity.

The structure map \( T(R) \to R \) of the \( T \)-algebra \( R = T(1) \) is \( \mu_1 : T^2(1) \to T(1) \). Just as \( \eta \) plays a special role, with \( \eta(x) \) being the Dirac distribution \( \delta_x \), the structure maps for \( T \)-algebras play a role that sometimes deserves an alternative notation and name; thus in particular \( \mu_1 : T(R) \to R \) plays in the context of probability distributions the role of \textit{expectation}, see \cite{14}, and we shall here again allow ourselves a doubling of notation and terminology:

\[
E(P) := \mu_1(P),
\]

the \textit{expectation} of \( P \in T(R) \). It is a scalar \( \in R \).

Note that for \( a \in R \),

\[
E(\delta_a) = a;
\]

since \( \delta_a \) is \( \eta_R(a) = \eta_{T(1)}(a) \), and \( E = \mu_1 \), this is a consequence of the monad law that \( \mu_X \circ \eta_{T(X)} \) is the identity map of \( T(X) \) for any \( X \), in particular for \( X = 1 \).

Proposition 9 Let \( P \in T(R) \). Then

\[
E(P') = -\text{tot}(P).
\]

Proof. The Proposition say that two maps \( T(R) \to R \) agree, namely \( E \circ \Delta \) and \( -\text{tot} \), where \( \Delta \), as above, is the differentiation process \( P \mapsto P' \). Both these maps are \( T \)-linear, so it suffices to prove that the equation holds for the case \( P = \delta_x \), so we should prove

\[
E(\delta_x') = -\text{tot}(\delta_x).
\]

By the principle of cancelling universally quantified \( ds \) (Proposition\cite{7}), it suffices to prove that for all \( d \in D \) that

\[
d \cdot E(\delta_x') = -d \cdot \text{tot}(\delta_x).
\]

The right hand side is \( -d \), by \cite{5}. The left hand side is

\[
E(d \cdot \delta_x') = E(\delta_x - \alpha_d \delta_x)
\]

\[
= E(\delta_x - \delta_{x+d})
\]

\[
= E(\delta_x) - E(\delta_{x+d}) = x - (x + d) = -d.
\]
by (15). This proves the Proposition.

The differentiation process for functions, as a map \( R \ni V \rightarrow R \ni V \), is likewise \( T \)-linear, but this important information cannot be used in the same way as we used \( T \)-linearity of the differentiation \( T(R) \rightarrow T(R) \), since, unlike \( T(R) \), \( R \ni V \) (not even \( R \ni R \)) is not known to be freely generated by elementary quantities like the \( \delta_i \)'s.

Recall that if \( F : V \rightarrow W \) is an \( R \)-linear map between KL modules

\[
F \circ \phi = (F \circ \phi)'
\]

for any \( \phi : R \rightarrow V \).

One can generalize the differentiation of extensive quantities on \( R \) to a differentiation of extensive quantities on any space \( X \) equipped with a vector field. The case made explicit is where the vector field is \( (x, d) \rightarrow x + d \) (or \( \frac{\partial}{\partial x} \)) on \( R \).

**Proposition 10** Let \( P \in T(R) \) and \( Q \in T(R) \). Then

\[
(P \ast Q)’ = P’ \ast Q = P \ast Q’.
\]

**Proof.** By commutativity of convolution, it suffices to prove that \( (P \ast Q)' = P' \ast Q \). Both sides depend in a \( T \)-bilinear way on \( P \) and \( Q \), so it suffices to see the validity for the case where \( P = \delta_x \) and \( Q = \delta_b \). To prove \((\delta_x \ast \delta_b)' = \delta_x' \ast \delta_b \), it suffices to prove that for all \( d \in D \),

\[
d \cdot (\delta_x \ast \delta_b)' = d \cdot \delta_x' \ast \delta_b,
\]

and both sides come out as \( \delta_{a+b} - \delta_{a+b+d} \), using that \( * \) is \( R \)-bilinear.

**Primitives of extensive quantities**

We noted already in Section 1 that \( P \) and \( f_\alpha(P) \) have same total, for any \( P \in T(X) \) and \( f : X \rightarrow Y \). In particular, for \( P \in T(R) \) and \( d \in D \), \( d \cdot P' = P - \alpha_d(P) \) has total 0, so cancelling the universally quantified \( d \) we get that \( P' \) has total 0.

A *primitive* of an extensive quantity \( Q \in T(R) \) is a \( P \in T(R) \) with \( P' = Q \). Since any \( P' \) has total 0, a necessary condition that an extensive quantity \( Q \in T(R) \) has a primitive is that \( \text{tot}(Q) = 0 \). Recall that primitives, in ordinary 1-variable calculus, are also called “indefinite integrals”, whence the following use of the word “integration”:

**Integration Axiom.** Every \( Q \in T(R) \) with \( \text{tot}(Q) = 0 \) has a unique primitive.

(For contrast: for intensive quantities \( \phi \) on \( R \) (so \( \phi : R \rightarrow R \) is a function), the standard integration axiom is that primitives always exist, but are not unique, only up to an additive constant.)

By \( R \)-linearity of the differentiation process \( T(R) \rightarrow T(R) \), the uniqueness assertion in the Axiom is equivalent to the assertion: if \( P' = 0 \), then \( P = 0 \). (Note that \( P' = 0 \) implies that \( P \) is invariant under translations \( \alpha_d(P) = P \) for all \( d \in D \).) The reasonableness of this latter assertion is a two-stage argument: 1) if \( P' = 0 \), \( P \) is invariant under arbitrary translations
\( \alpha_u(P) = P.2 \) if \( P \) is invariant under all translations, and has compact support, it must be 0. (Implicitly here is: \( R \) itself is not compact.)

In standard distribution theory, the Dirac distribution \( \delta_a \) (where \( a \in R \)) has a primitive, namely the Heaviside “function”; but this “function” has not compact support - its support is a half line \( \subseteq R \).

On the other hand, the integration axiom provides a (unique) primitive for a distribution of the form \( \delta_a - \delta_b \), with \( a \) and \( b \) in \( R \). This primitive is denoted \( [a, b] \), the “interval” from \( a \) to \( b \); thus, the defining equation for this interval is

\[
[a, b]' = \delta_a - \delta_b.
\]

Note that the phrase “interval from \( a \) to \( b \)” does not imply that we are considering an ordering \( \leq \) on \( R \) (although ultimately, one wants to do so).

**Proposition 11** The total of \( [a, b] \) is \( b - a \).

**Proof.** We have

\[
tot([a, b]) = -E([a, b]) = -E(\delta_a - \delta_b)
\]

by Proposition 9 and the fact that \( [a, b] \) is a primitive of \( \delta_a - \delta_b \)

\[
= -E(\delta_a) + E(\delta_b) = b - a,
\]

by (15).

It is of some interest to study the sequence of extensive quantities

\[
[-a, a], \ [-a, a] \ast [-a, a], \ [-a, a] \ast [-a, a] \ast [-a, a], \ ...
\]

they have totals \( 2a, (2a)^2, (2a)^3, ... \); in particular, if \( 2a = 1 \), this is a sequence of probability distributions, approaching a Gauss normal distribution (the latter, however, has presently no place in our context, since it does not have compact support).

### 9 Extensive quantities and Schwartz distributions

Recall from (2) that \( P \in T(X) \) gives rise to an \( X \)-ary operation on any \( T \)-algebra \( B = (B, \beta) \), via \( \langle P, \phi \rangle := \beta(\phi, (P)) \), where \( \phi \in X \sqcap B \). The pairing is thus a map

\[
T(X) \times (X \sqcap B) \xrightarrow{-, -} B.
\]

which is \( T \)-bilinear (cf. Theorem 1 or 14). We may take the exponential transpose of the pairing; this is then a map \( \tau_X : T(X) \to (X \sqcap B) \sqcap_T B \).

The synthetic rendering of Schwartz distribution theory is that \( (X \sqcap R) \sqcap_T R \) “is” the space of Schwartz distributions of compact support: \( X \sqcap R \) is the “space of test functions” (not necessarily of compact support), and \( (X \sqcap R) \sqcap_T R \) is the space of \( T \)-linear functionals


\[X \ni R \to R\] on the space of such test functions. (In some well adapted models \(E\) of SDG, and for suitable \(T\), this can be proved to be an object whose set of global sections is in fact the standard Schwartz distributions of compact support on \(X\), if \(X\) is a smooth manifold; cf. [19] Proposition II.3.6 (Theorem of Que and Reyes).)

**Remark.** Consider a commutative ring object in a sufficiently cocomplete cartesian closed category \(E\). Let \(T\) be the (strong) monad which to \(X\) associates the free \(R\)-module on \(X\). Thus in particular \(R = T(1)\). The monad \(X \mapsto (X \ni R) \ni_T R\) is in general not a commutative monad, so cannot agree with \(T\), although in some cases, the monad map \(\tau: T \to (\_ \ni R) \ni_T R\) is monic. In [1], it is proved that \(T(X)\) for a special case (convenient vector spaces) can be “carved out” of \((X \ni R) \ni_T R\) by topological means. Other investigations, e.g. in [12], and a Theorem of Waelbroeck, describe a class of spaces \(X\) which “perceive” \(\tau_X\) to be an isomorphism.

To make contact with classical theory and intuition, we introduce, for the third time, a doubling of notation (this one is actually quite classical); for \(P \in T(X)\) and \(\phi \in X \ni B\) (where \(B\) is a \(T\) algebra), we write

\[
\int_X \phi(x) \, dP(x) := \langle P, \phi \rangle \in B,
\]

with \(x\) a dummy variable ranging over \(X\). Thus Proposition 2 may be rendered

\[
tot(P) = \int_X 1 \, dP(x).
\]

For \(B = T(1) = R\) and \(P \in T(R) = T^2(1)\), the \(\int\)-notation will help to motivate the use of the terminology “expectation of \(P\)” for \(\mu_1(P)\); let \(\phi: R \to R\) be the identity map. Then

\[
\langle P, \phi \rangle = \mu_1(\phi_*(P)) = \mu_1(P),
\]

since \(\phi_*\) is the identity map of \(T(R)\). On the other hand

\[
\int_X \phi(x) \, dP(x) = \int_X x \, dP(x),
\]

since \(\phi(x) = x\); this is the standard “integral” expression for expectation \(E(P)\) for a probability distribution \(P\) on \(R\).

The map \(\tau_X : T(X) \to (X \ni R) \ni_T R\) is not necessarily monic; in the case of classical Schwartz distributions, it is monic, which allows the classical theory to identify extensive quantities in \(T(X)\) with elements in \((X \ni R) \ni_T R\), and so one avoids having to mention \(T(X)\) explicitly; the notion of extensive quantity on \(X\) is thus made dependent on the notion of intensive quantity (test function) on \(X\). It is, however, easy to give examples of \(T\)s where there are not sufficiently many “test functions” \(X \to R\) (with \(R = T(1)\)) to make \(\tau_X : T(X) \to (X \ni R) \ni_T R\) injective, whence one motivation for the study of \(T\), independent of the introduction of \(X \ni R\).
The injectivity of \( \tau_X \) may be expressed: “To test equality of \( P \) and \( Q \) in \( T(X) \), it suffices to test, for arbitrary functions \( \phi : X \to R \), whether \( \langle P, \phi \rangle = \langle Q, \phi \rangle ' \), whence the name test function.

If \( \tau_X \) is not injective, there are not sufficiently many such test functions \( X \to R \), but there are enough, if we allow test functions with arbitrary \( T \)-algebras \( B = (B, \beta) \) as their codomain. We have in fact

**Proposition 12** For any \( X \in \mathcal{E} \), there exists a \( T \)-algebra \( B \) so that \( \tau_X : X \to (X \triangleleft B) \triangleleft T B \) is monic.

**Proof.** Take \( B = T(X) \); then the map \( e : (X \triangleleft T(X)) \triangleleft T(X) \to T(X) \) given by “evaluation at \( \eta_X : X \to T(X) \)” is left inverse for \( \tau_X \). For, if \( P \in T(X) \), then to say \( e(\tau_X(P)) = P \) is equivalent to saying

\[
\langle P, \eta_X \rangle = P. \tag{17}
\]

Since the structure map of the \( T \)-algebra \( T(X) \) is \( \mu_X \), the definition (2) of the pairing \( \langle P, \eta_X \rangle \) gives \( \langle P, \eta_X \rangle = \mu_X(\eta_X(P)) \) (where \( \eta \) here is short for \( \eta_X \)). However, \( \eta_X \) is just another notation for \( T(\eta) \), and \( \mu_X \circ T(\eta) \) is the identity map on \( T(X) \) by one of the monad laws. So we get \( P \) back when we apply this map to \( P \).

Thus, instead of identifying an extensive quantity on \( X \) by its action on arbitrary test functions \( \phi : X \to R \), we identify it by its action on one single test function, namely the function \( \eta_X : X \to T(X) \).

I conjecture that \( T(X) \) is actually the end \( \int_{B \in T} (X \triangleleft B) \triangleleft T B \).

Here is an important relationship between differentiation of extensive quantities on \( R \), and of functions \( \phi : R \to T(X) \); such functions can be differentiated, since \( T(X) \) is assumed to be KL as an \( R \)-module. (In the Schwartz theory, this relationship, with \( X = 1 \), serves as definition of derivative of distributions.)

**Proposition 13** For \( P \in T(R) \) and \( \phi \in R \triangleleft T(X) \), one has

\[
\langle P', \phi \rangle = -\langle P, \phi' \rangle.
\]

**Proof.** We are comparing two maps \( T(R) \times (R \triangleleft T(X)) \to T(X) \), both of which are \( T \)-linear in the first variable. Therefore, it suffices to prove the equality for the case of \( P = \delta_t \); in fact, by \( R \)-bilinearity of the pairing, it suffices to prove that for any \( t \in R \) and \( d \in D \), we have

\[
\langle d \cdot (\delta_t)' \rangle = -\langle \delta_t, d \cdot \phi' \rangle.
\]

The left hand side is \( \langle \delta_t - \alpha_t^d(\delta_t), \phi \rangle \), and using bi-additivity of the pairing, this gives \( \phi(t) - ((\alpha_t^d)'(\phi)(t) = \phi(t) - \phi(t + d) \), which is \( -d \cdot \phi'(t) \).

Proposition 13 can be seen as a special case, with \( X = 1 \) (thus \( T(X) = R \)), and with \( \phi \) the identity function \( R \to R \). We use the “integral” notation. Thus \( x \) denotes the identity function on \( R \). So

\[
E(P') = \int_R x \, dP'(x) = -\int_R (x)' \cdot dP(x) = -\int_1 dP(x),
\]

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the middle equality by Proposition 13. This, however, is $-\text{tot}(P)$, by Proposition 2.

The relationship between differentiation of extensive and intensive quantities on $R$ expressed in Proposition 13 may be given a more “compact” formulation, using (17). For then we have, for $P \in T(R)$, that

$$P' = \langle P, \eta \rangle = -\langle P, \eta' \rangle.$$

Thus in particular, knowledge of $\eta'$ gives knowledge of $P'$ for any $P \in T(R)$. It also gives knowledge of $\phi'$ for any $\phi : R \to V$, with $V$ a $T$-algebra which is KL module. For, any such $\phi$ extends over $\eta_R$ to a (unique) $T$-linear $F : T(R) \to V$, so $\phi = F \circ \eta_R$; therefore

$$\phi' = (F \circ \eta_R)' = F \circ \eta_R',$

using (16).

Let us calculate $\eta' : R \to T(R)$ explicitly; we have for any $d \in D$ and $x \in R$ (writing $\eta$ for $\eta_R$)

$$d \cdot \eta'(x) = \eta(x+d) - \eta(x) = \delta_{x+d} - \delta_x = -d \cdot (\delta_x'),$$

so cancelling the universally quantified $d$, we get for any $x \in R$ that

$$\eta'(x) = -(\delta_x').$$

The first differentiation refers to differentiation of functions, the second to differentiation of distributions; it is tempting to write the former with a Newton dot; then we get $\eta^*(x) = -(\delta_x')$.

The following depends on the Leibniz rule for differentiating a product of two functions; so this is not valid under the general assumptions of this Section, but needs the further assumption of Example 2, namely that $D$ consists of $d \in R$ with $d^2 = 0$, as in synthetic differential geometry. We shall then use “test function” technique to prove

**Proposition 14** For any $P \in T(R)$ and $\phi \in R \cap R$,

$$(P \vdash \phi)' = P' \vdash \phi + P \vdash \phi'.$$

**Proof.** It suffices to prove that for the “universal” test function $\eta = \eta_X : X \to T(X)$, we have

$$\langle \langle P \vdash \phi', \eta \rangle = \langle P' \vdash \phi, \eta \rangle + \langle P \vdash \phi', \eta \rangle.$$

We calculate:

$$\langle \langle P \vdash \phi', \eta \rangle = -\langle P \vdash \phi, \eta' \rangle \quad \text{(by Proposition 13)}$$

$$= -\langle P, \phi \vdash \eta' \rangle \quad \text{(by Proposition 3)}$$

$$= -\langle P, (\phi \vdash \eta)' - \phi' \vdash \eta \rangle$$

using that Leibniz rule applies to any bilinear pairing, like $\vdash$,

$$= -\langle P, (\phi \vdash \eta)' \rangle + \langle P, \phi' \vdash \eta \rangle$$

$$= \langle P', \phi \vdash \eta \rangle + \langle P, \phi' \vdash \eta \rangle$$

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using Proposition 13 on the first summand
\[ = \langle P' \vdash \phi, \eta \rangle + \langle P \vdash \phi', \eta \rangle \]

using Proposition 3 on each summand
\[ = \langle P' \vdash \phi + P \vdash \phi', \eta \rangle \]

In other words, (replacing \( \dashv \) by \( \cdot \)), the proof looks formally like the one from books on distribution theory, but does not depend on “sufficiently many test functions with values in \( R \)’.

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