TOPOLOGY OF TROPICAL MODULI SPACES OF WEIGHTED STABLE CURVES IN HIGHER GENUS

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Abstract. Given integers $g \geq 0$, $n \geq 1$, and a vector $w \in (\mathbb{Q} \cap (0,1))^n$ such that $2g - 2 + \sum w_i > 0$, we study the topology of the moduli space $\Delta_{g,w}$ of $w$-stable tropical curves of genus $g$ with volume 1. The space $\Delta_{g,w}$ is the dual complex of the divisor of singular curves in Hassett’s moduli space of $w$-stable genus $g$ curves $\overline{M}_{g,w}$. When $g \geq 1$, we show that $\Delta_{g,w}$ is simply connected for all values of $w$. We also give a formula for the Euler characteristic of $\Delta_{g,w}$ in terms of the combinatorics of $w$.

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1. INTRODUCTION

Fix integers $g \geq 0$, $n \geq 1$ and a vector of rational weights $w \in (\mathbb{Q} \cap (0,1))^n$ satisfying

\begin{equation}
2g - 2 + \sum_{i=1}^{n} w_i > 0.
\end{equation}

We study the topology of the moduli space $\Delta_{g,w}$ of $w$-stable tropical curves of genus $g$ and volume 1. Here a $w$-stable tropical curve is a pair $(G, \ell)$ where $G$ is a $w$-stable graph (see (2.2)) and $\ell : E(G) \to \mathbb{R}_{\geq 0}$ is a length function; the volume is the sum of edge lengths in $G$, i.e., $\sum_{e \in E(G)} \ell(e)$. Following the work of Ulirsch [Uli15] and Chan, Galatius, and Payne [CGP21], the space $\Delta_{g,w}$ can be realized as the dual complex of the normal crossings divisor $\overline{M}_{g,w} \smallsetminus \overline{M}_{g,w}$ on Hassett’s moduli stack $\overline{M}_{g,w}$; here $\overline{M}_{g,w}$ denotes the dense open substack of $\overline{M}_{g,w}$ parameterizing smooth but not necessarily distinctly marked curves. Our first main theorem is that $\Delta_{g,w}$ is simply connected for $g \geq 1$.

Theorem 1.2. For any $g, n \geq 1$ and $w \in (\mathbb{Q} \cap (0,1))^n$, the space $\Delta_{g,w}$ is simply connected.

Our second result is a calculation of the Euler characteristic of $\Delta_{g,w}$ in terms of the top weight Euler characteristics of the moduli spaces $\overline{M}_{g,r}$ of smooth $r$-marked algebraic curves of genus $g$; see essential background on weight filtrations and mixed Hodge structures in Section 1.1. Set

$[n] := \{1, \ldots, n\}$. 
We call a partition \( P_1 \sqcup \cdots \sqcup P_r \vdash [n] \) \( w \)-admissible if \( \sum_{i \in P_j} w_i \leq 1 \) for all \( 1 \leq j \leq r \). Let \( N_{r,w} \) denote the number of \( w \)-admissible partitions of \([n]\) with \( r \) parts.

**Theorem 1.3.** Let \( W = W_1 \subset \cdots \subset W_{6g-6+2r} \subseteq H^*(\mathcal{M}_{g,r}, \mathbb{Q}) \) be the weight filtration of the rational singular cohomology of the moduli stack \( \mathcal{M}_{g,r} \) and denote by \( \chi^W_{6g-6+2r} \) the Euler characteristic of the top graded piece

\[
\text{Gr}^W_{6g-6+2r} H^*(\mathcal{M}_{g,r}; \mathbb{Q}) = W_{6g-6+2r}/W_{6g-7+2r}
\]

of the weight filtration. Then

\[
\chi(\Delta_{g,w}) = 1 - \sum_{r=1}^{n} N_{r,w} \cdot \chi^W_{6g-6+2r}(\mathcal{M}_{g,r}).
\]

A generating function for the numbers \( \chi^W_{6g-6+2r}(\mathcal{M}_{g,r}) \) is given in [CFGP19]; together with their result, Theorem 1.3 allows for the computer-aided calculation of \( \chi(\Delta_{g,w}) \) for arbitrary \( g \) and \( w \). In [CFGP19, Corollary 8.1], the authors give a closed form in the case when the number of marked points is large. For \( r > g + 1 \),

\[
\chi^W_{6g-6+2r}(\mathcal{M}_{g,r}) = (-1)^{r+1} \frac{(g + r - 2)!}{g!} B_g,
\]

where \( B_g \) is the \( g \)-th Bernoulli number, characterized by

\[
\frac{t}{e^t - 1} = \sum_{\ell=0}^{\infty} B_\ell \frac{t^\ell}{\ell!}.
\]

Substituting into Theorem 1.3 yields the following closed form.

**Corollary 1.4.** Given \( g \geq 0 \) and a weight vector \( w \) satisfying Equation 1.1 and that \( N_{r,w} = 0 \) for \( r \leq g + 1 \), the Euler characteristic of \( \Delta_{g,w} \) is

\[
\chi(\Delta_{g,w}) = 1 + \sum_{r=1}^{n} N_{r,w} (-1)^{r} \frac{(g + r - 2)!}{g!} B_g.
\]

Let \( S(m, r) \) denote the number of \( r \)-partitions of \([m]\) for \( m \geq 1 \) and \( r \geq 0 \); these are called the **Stirling numbers of the second kind**. Expanding the Bernoulli number \( B_g \) (see [Apo98]) in terms of Stirling numbers

\[
B_g = \sum_{\ell=0}^{g} (-1)^\ell \frac{\ell!}{\ell + 1} S(g, \ell),
\]

we obtain the following closed form for the Euler characteristic of \( \Delta_{g,w} \) for heavy/light weights.

**Corollary 1.5.** Given a heavy/light weight vector \( w = (1^{(n)}, \varepsilon^{(m)}) \) where \( n \geq g + 1 \), \( m > 0 \), and \( 0 < \varepsilon < 1/m \),

\[
\chi(\Delta_{g,w}) = 1 + \sum_{r=1}^{m} \sum_{\ell=0}^{g} (-1)^{n+r+\ell} \frac{(g + n + r - 2)!}{g!(\ell + 1)} S(m, r) S(g, \ell).
\]

Using this corollary above, we compute explicitly some of the Euler characteristics of \( \Delta_{g,(1^{(n)},\varepsilon^{(m)})} \) in Table 1.
1.1. **Motivation.** Throughout the paper, we work over the complex numbers $\mathbb{C}$. The Deligne-Mumford-Knudsen compactification $\overline{M}_{g,n}$ is a toroidal compactification of the moduli stack $M_{g,n}$; the toroidal structure comes from the fact that the boundary divisor $\overline{M}_{g,n} \setminus M_{g,n}$ has normal crossings. As a Deligne-Mumford stack, the rational cohomology of $M_{g,n}$ carries a mixed Hodge structure; see [Del74]. That is, there is a weight filtration

$$W_1 \subset \cdots \subset W_{6g-2n} = H^*(M_{g,n}; \mathbb{Q})$$

such that, for each $j$, the quotient

$$\text{Gr}_i^W H^j(M_{g,n}, \mathbb{Q}) = W_i \cap H^j(M_{g,n}; \mathbb{Q})/W_{i-1} \cap H^j(M_{g,n}; \mathbb{Q})$$

carries a pure Hodge structure of weight $i$. The top graded piece of the weight filtration of $M_{g,n}$ can be identified with the reduced homology of the dual complex of the divisor $\overline{M}_{g,n} \setminus M_{g,n}$, up to a degree shift. As discussed in [CGP21, CGP19], the dual complex of this divisor may be identified with the tropical moduli space $\Delta_{g,n}$, furnishing isomorphisms

$$\tilde{H}_{j-1}(\Delta_{g,n}; \mathbb{Q}) \cong \text{Gr}_i^W H^{6g-6+2n-j}(M_{g,n}; \mathbb{Q}).$$

In [Has03], Hassett introduced the moduli stack $\overline{M}_{g,w}$ as an alternative compactification of $M_{g,n}$: in $\overline{M}_{g,w}$, marked points are allowed to coincide if the sum of the corresponding entries of $w$ is no greater than 1. Thus $\overline{M}_{g,w}$ contains an open substack $M_{g,w}$ parameterizing smooth, but not necessarily distinctly marked algebraic curves of genus $g$, and we have the containments $M_{g,n} \subset M_{g,w} \subset \overline{M}_{g,w}$. Although the embedding $M_{g,n} \subset \overline{M}_{g,w}$ is no longer toroidal, $\overline{M}_{g,w} \setminus M_{g,w}$ is still a normal crossings divisor, and the dual complex of this divisor has a natural modular interpretation as the moduli space $\Delta_{g,w}$ of tropical $w$-stable curves of volume 1 and genus $g$; see [Uli15, CHMR14]. Therefore, one has isomorphisms

$$\tilde{H}_{j-1}(\Delta_{g,w}; \mathbb{Q}) \cong \text{Gr}_i^W H^{6g-6+2n-j}(M_{g,w}; \mathbb{Q}),$$

identifying the reduced rational homology of $\Delta_{g,w}$ with the top graded piece of the rational cohomology of $M_{g,w}$.

| $g$ | $m = 0$ | $m = 1$ | $m = 2$ | $m = 3$ | $m = 4$ |
|-----|--------|--------|--------|--------|--------|
| $n = 2$ | -5 | 2 | 0 | 2 |
| $n = 3$ | 3 | -3 | 9 | -15 |
| $n = 4$ | -5 | 19 | -53 | 163 |
| $n = 5$ | 25 | -95 | 385 | -1535 |

$\begin{array}{|c|c|c|c|c|c|}
\hline
\text{Table 1. Euler characteristics of } \Delta_{g,1(1)^{n},\varepsilon(m)} \text{ for } g = 0, 1, 2, 3 \text{ and some } (n, m) \\
\text{where } n \geq g + 1 \text{ and } m > 0. \text{ Note that when } g = 0, \text{ we start with } n = 2 \text{ since the space } \Delta_{0,1,\varepsilon(m)} \text{ is empty; when } g = 0, n = 2, m = 1, \Delta_{0,1,\varepsilon} \text{ is also empty.} \\
\hline
\text{g} & \text{m} & \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} \\
\hline
\text{0} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\text{1} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\text{2} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\text{3} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\text{4} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\text{5} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\text{6} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\end{array}$
1.2. Previous work. This work benefits from and builds on previous work of many authors on the topology of tropical moduli spaces, which we summarize here.

When \( g = 0 \), the complex \( \Delta_{0,w} \) may be identified with various objects whose homotopy types are known.

1. When \( w = (1^{(n)}) \), Vogtmann showed that \( \Delta_{0,n} \) is homotopic to a wedge of \( (n - 2)! \) spheres of dimension \( n - 4 \), by identifying it as the link of a vertex of a quotient simplicial complex by the outer automorphisms of a finitely generated free group; see [CV86, Vog90]. In [RW96], Robinson and Whitehouse gave a different proof of the same result, by contracting a large subcomplex \( X_{0,n} \) of \( \Delta_{0,n} \).

2. When \( w \) is heavy/light, i.e. \( w = (1^{(n)}, \varepsilon^{(m)}) \) for \( \varepsilon < 1/m \), Cavalieri, Harer’s complex of curves \( C \)

3. When \( w \) has at least two weight-1 entries, Cerbu et al. in [CMP+20] showed that \( \Delta_{0,w} \) is homotopic to a wedge of spheres of possibly varying dimensions, by identifying a large contractible subcomplex and using known results on homotopy types of subspace arrangements. The authors also provided infinite families of \( w \) where \( \Delta_{0,w} \) is disconnected, and examples where \( \pi_1(\Delta_{0,w}) = \mathbb{Z}/2\mathbb{Z} \). In the latter scenario, the authors proved that the universal cover has the homotopy type of a wedge of spheres.

For higher values of \( g \), the following results are known.

1. When \( w = (1^{(n)}) \), Chan, Galatius, and Payne showed in [CGP19] that \( \Delta_{1,n} \) is homotopic to \( \frac{1}{2}(n-1)! \) spheres of dimension \( n - 1 \). In [Cha21], Chan independently showed that the reduced integral homology \( \tilde{H}_*(\Delta_{2,n}; \mathbb{Z}) \) is supported in the top two degrees and that a subcomplex of \( \Delta_{2,n} \) has torsion in high degrees. Chan also computed the reduced rational homology \( \tilde{H}_*(\Delta_{2,n}; \mathbb{Q}) \) for \( n \leq 8 \). For higher genera, Chan, Galatius, and Payne showed that \( \Delta_{g,n} \) is at least \( (n - 3) \)-connected [CGP19].

2. When \( w \) has at least two weight-1 entries, [CMP+20] leveraged a relation between \( \Delta_{0,w} \) and \( \Delta_{1,w} \) to prove that \( \Delta_{1,w} \) is homotopic to a wedge of spheres.

3. When \( w = (1^{(n)}, \varepsilon^{(m)}) \) is heavy/light, the same authors showed that \( \Delta_{1,w} \) is homotopic to \( \frac{1}{2}(n - 1)! n^m \) spheres of dimension \( n + m - 1 \).

Most recently, in [ACP19], Allcock, Corey, and Payne showed that \( \Delta_g \) and \( \Delta_{g,n} \) are simply connected for \( (g, n) \neq (0, 4), (0, 5) \). They give two proofs of this result; one relies on a cellular approximation theorem in dimension 1 for symmetric CW-complexes, and the other, suggested by A. Putman, uses Harer’s result [Har86] that Harvey’s complex of curves \( C_{g,n} \) is simply connected, together with the fact that \( \Delta_{g,n} \) is homeomorphic to the quotient of the complex \( C_{g,n} \) by the action of the pure mapping class group. In this paper, we use a similar technique as in the first proof of [ACP19] and deduce simple connectedness for \( \Delta_{g,w} \). The working framework in this paper is based on graph categories and symmetric \( \Delta \)-complexes, heavily used in [CGP21, CGP19]. The main tools are the contractibility criterion developed in [CGP19] and the Grothendieck group of varieties.

Remark 1.6. The tropical Hassett space \( \Delta_{g,w} \) is related to several complexes studied in the context of geometric group theory, notably Harvey’s complex of curves \( C_{g,n} \) [Har81] and Hatcher’s complex of sphere systems \( S_{g,n} \) [Hat95]. In either case, one can define a subcomplex
parameterizing \(w\)-stable collections, i.e., those collections of curves, respectively 2-spheres, whose dual graph is \(w\)-stable in the sense of (2.2) below. Then \(\Delta_{g,w}\) may be realized as the quotient of either subcomplex by a suitable group action. In the case of \(C_{g,n}\), the action is by the pure mapping class group, and in the case of \(S_{g,n}\), the action is by the quotient of the pure mapping class group by the normal subgroup generated by Dehn twists. A proof that either of these subcomplexes is simply connected would lead to an alternate proof of simply connectedness of \(\Delta_{g,w}\), via the approach of Putman mentioned above. We also refer any interested reader to the first version of this preprint on arXiv, where we employ the Seifert-van Kampen theorem for CW complexes inductively to prove the simple connectivity of \(\Delta_{g,w}\); see [KLSY20].

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2. Background

2.1. The graph categories \(\Gamma_{g,w}\). Given integers \(g \geq 0, n \geq 1\) and a vector \(w \in (\mathbb{Q} \cap (0,1])^n\) of rational numbers satisfying

\[
2g - 2 + \sum_{i=1}^{n} w_i > 0,
\]

we define a graph category \(\Gamma_{g,w}\). All graphs considered in this paper allow loops and parallel edges. First, a weighted \(n\)-marked graph \(G\) is a triple \(G = (G, m, h)\) consisting of a finite connected graph \(G\) together with a marking function \(m : [n] \to V(G)\) and a vertex weight function \(h : V(G) \to \mathbb{Z}_{\geq 0}\). We say \(G\) is \(w\)-stable if it satisfies the stability condition

\[
2h(v) - 2 + \text{val}(v) + \sum_{i \in m^{-1}(v)} w_i > 0,
\]

for all \(v \in V(G)\), where \(\text{val}(v)\) denotes the valence of \(v\) in \(G\), i.e., the number of half edges incident to \(v\); thus a loop contributes twice to the valence of its vertex. The genus of \(G\) is defined as

\[
g(G) := b_1(G) + \sum_{v \in V(G)} h(v),
\]

where \(b_1(G) = |E(G)| - |V(G)| + 1\) is the first Betti number of \(G\). The objects in \(\Gamma_{g,w}\) are \(w\)-stable weighted \(n\)-marked graphs of genus \(g\).

The morphisms in \(\Gamma_{g,w}\) are maps that factor as compositions of isomorphisms and edge contractions. To be precise, an isomorphism \(\varphi : G \to G'\) where \(G = (G, m, h)\) and \(G' = (G', m', h')\) is an isomorphism \(\varphi : G \to G'\) such that \(m' = \varphi \circ m\) and \(h' \circ \varphi = h\). An edge contraction \(c : G \to G/e\) of an edge \(e\) in \(G\) is given by removing \(e\) and identifying its two endpoints if \(e\) is not a loop, and by removing \(e\) and increasing the weight of its base vertex by one if \(e\) is a loop. We say two graphs have the same combinatorial type if they are isomorphic in \(\Gamma_{g,w}\).
We will say that $G'$ is an \textit{uncontraction} of a graph $G$ if $G'$ contracts to $G$ after a series of edge contractions. One can alternatively describe $\Gamma_{g,w}$ as the full subcategory of the category $\Gamma_{g,n}$ defined in [CGP19] whose objects are graphs in $\Gamma_{g,n}$ that satisfy (2.2); when $w = (1^{(n)})$, $\Gamma_{g,n} = \Gamma_{g,w}$. To work with a small category, we hereafter tacitly replace $\Gamma_{g,n}$ with a choice of skeleton thereof. This also induces a choice of skeleton of $\Gamma_{g,w}$ for all weight vectors $w$. Abusing notation, we will write $G \in \Gamma_{g,w}$ for $G \in \text{Ob}(\Gamma_{g,w})$. We now give the definition of the tropical moduli space $\Delta_{g,w}$ using $\Gamma_{g,w}$.

### 2.2. Description of $\Delta_{g,w}$ as a symmetric $\Delta$-complex.

The space $\Delta_{g,w}$ is the geometric realization of a symmetric $\Delta$-complex, in the sense of [CGP21]. Let $I$ be the category having one object for each finite set

$$[p] := \begin{cases} 
\{0, \ldots, p\} & \text{for } p \geq 0, \\
\emptyset & \text{for } p = -1.
\end{cases}$$

and morphisms consisting of all injections. A symmetric $\Delta$-complex $X$ is a functor $X : I^{\text{op}} \to \text{Sets}$, and a morphism of symmetric $\Delta$-complexes is a natural transformation of functors. For simplicity, we write $X_p$ for $X([p])$. There is a geometric realization functor associating a topological space to each symmetric $\Delta$-complex $X$, which we describe below. Each injection $\iota : [p] \to [q]$ induces a map on standard simplices $\iota_* : \sigma^p \to \sigma^q$, defined by

$$\iota_* \left( \sum_{i=0}^{p} t_i e_i \right) = \sum_{i=0}^{q} \left( \sum_{j \in \iota^{-1}(i)} t_j \right) e_i.$$ 

The geometric realization of $X$ is then defined as

$$|X| := \left( \coprod_{p \geq 0} X_p \times \sigma^p \right) / \sim$$

where the equivalence relation $\sim$ is generated by relations of the form $(X(\iota)(x), a) \sim (x, \iota_*(a))$, whenever $\iota \in \text{Hom}_I([p], [q])$, $x \in X_q$, and $a \in \sigma^p$. The $k$-skeleton of $|X|$ is

$$|X|^{(k)} = \left( \coprod_{p=0}^{k} X_p \times \sigma^p \right) / \sim$$

with the same equivalence relation.

We will use $\Delta_{g,w}$ both for the functor and its geometric realization when there is no confusion. We set

$$\Delta_{g,w}([p]) = \{ (G, \tau) \mid G \in \Gamma_{g,w}, |E(G)| = p + 1, \tau : E(G) \to [p] \text{ a bijection} \}/ \sim, $$

where $(G, \tau) \sim (G', \tau')$ if and only if there exists a $\Gamma_{g,w}$-isomorphism $\varphi : G \to G'$ such that the diagram

$$\begin{tikzcd}
E(G) \ar[rr, \varphi] \ar[d, \tau, two heads] & & E(G') \ar[d, \tau', two heads] \\
[p] & &
\end{tikzcd}$$

commutes. We put $[G, \tau]$ for the equivalence class of $(G, \tau)$. On morphisms, we define $\Delta_{g,w}$ as follows: given an injection $\iota : [p] \to [q]$ and $[G, \tau] \in \Delta_{g,w}([q])$, we set $H$ to be the graph obtained from $G$ by contracting all edges which are not labelled by $\tau^{-1}(\iota([p]))$. 

$$\begin{tikzcd}
G \ar[r, \varphi] \ar[rr, \tau] \ar[dr, \tau'] & G' \ar[d, \iota]\ar[ru, \iota'] \\
& [p]
\end{tikzcd}$$

The above diagram commutes if and only if $\tau^{-1}(\iota([p]))$ is a labelling of $G$.
and \( \pi : E(H) \to [p] \) to be the unique edge-labelling of \( H \) which preserves the order of the remaining edges. It is known that the order of contraction of edges does not affect the final result. Then \( \Delta_{g,w}(\iota)([G, \tau]) = [H, \pi] \).

**Remark 2.3.** There is an alternate definition of \( \Delta_{g,w} \) as a colimit of maps of simplices; see [CMP+20, Section 1]. We briefly summarize the intuition here.

As defined in the introduction, a \( w \)-stable tropical curve of volume 1 is a pair \((G, \ell)\) where \( G \in \Gamma_{g,w} \) and \( \ell : E(G) \to \mathbb{R}_{\geq 0} \) is a length function satisfying \( \sum_{e \in E(G)} \ell(e) = 1 \). If \( \ell(e) = 0 \) for some \( e \), then \( (G, \ell) \) is identified with \( (G/e, \ell|_{G/e}) \), where \( \ell|_{G/e} \) is the restriction of \( \ell \) to \( E(G/e) = E(G) \setminus \{e\} \). A point \( x \in \Delta_{g,w} \) can be identified with a \( w \)-stable tropical curve of volume 1. Suppose \( x \) is contained in a \( p \)-simplex corresponding to an edge-labelled graph \([G, \tau]\), with coordinates \((a_0, \ldots, a_p)\). Then \( x \) corresponds to the tropical curve \((G, \ell)\) where \( \ell(\tau^{-1}(i)) = a_i \) for each \( i \in [p] \). In general, the point \( x \) may be contained in more than one simplex, but there will be a unique simplex which contains \( x \) in its interior. This corresponds to the unique representative of the resulting tropical curve which has a strictly positive length function.

**Example 2.4.** Let \( g = 1 \) and \( w = (\varepsilon(3)) \) for \( 0 < \varepsilon < 1/3 \). The combinatorial types in \( \Gamma_{1,w} \) are edge contractions of the type shown in the interior of the left triangle in Figure 1. The 0-skeleton of \( \Delta_{1,w} \) is a single point; the 1-skeleton is three half-edges glued at the point. The whole space is a hollow tetrahedron, which is homeomorphic to \( S^2 \).

![Figure 1. The geometric realization of \( \Delta_{1,w} \) is the (hollow) tetrahedron on the right, obtained by folding the 2-simplex on the left along the interior grey lines.](image)

3. **Proof that \( \Delta_{g,w} \) is simply connected**

Let \( g \geq 1, n > 0, \) and fix a weight vector \( w \in (\mathbb{Q} \cap (0, 1])^n \). We will prove that \( \pi_1(\Delta_{g,w}) \) is trivial. Our strategy is to show that the 1-skeleton of \( \Delta_{g,w} \) is contained in a contractible subcomplex. We first recall briefly the contractibility criterion developed in Section 4 of [CGP19].

Let \( X \) be a symmetric \( \Delta \)-complex (representing both the functor and the geometric realization thereof when no confusion arises), and continue to denote by \( X \), the set \( X([i]) \).

**Definition 3.1.** A property on \( X \) is a subset of the vertices \( P \subseteq X_0 \).
Writing \( \text{Simp}(X) = \coprod_{p \geq 0} X_p \) for the set of all simplices of \( X \), and given a subset of vertices \( P \subseteq X_0 \), we define

\[
P(X) := \{ \sigma \in \text{Simp}(X) : \sigma \text{ has a vertex in } P \}.
\]

Elements of \( P(X) \) are called \( P \)-simplices. Given an integer \( i \geq 0 \), let \( X_{P,i} \) denote the subcomplex of \( X \) generated by the set of \( P \)-simplices with at most \( i \) non-\( P \) vertices. Write \( X_P = X_{P,\infty} \) to be the subcomplex generated by all \( P \)-simplices.

Let \( B \in \Gamma_{g,w} \) be the graph consisting of one edge connecting two vertices \( v_1 \) and \( v_2 \), such that \( h(v_1) = 1 \) and \( m^{-1}(v_1) = \emptyset \). Note that when \( g = 1 \) with \( \sum_{i=1}^n w_i > 1 \), or when \( g \geq 2 \), \( B \) is stable. Given a graph \( G \in \Gamma_{g,w} \), we call an edge \( e \in E(G) \) a 1-end if contracting all edges of \( G \) except for \( e \) yields \( B \).

**Lemma 3.2.** Let \( g = 1 \) with \( \sum_{i=1}^n w_i > 1 \), or \( g \geq 2 \) with arbitrary \( w \). Let \( X = \Delta_{g,w} \) and \( P,Q \subset X_0 \) be properties with \( P = \emptyset \) and \( Q = \{ B \} \). Then there is a deformation retract from \( (\Delta_{g,w})_Q \searrow (\Delta_{g,w})_{Q,0} \).

**Proof.** As is discussed in the proof of Theorem 1.1(2) in [CGP19], this lemma reduces to the fact that every graph has a canonical maximal uncontraction by 1-ends. Indeed, the argument given in [CGP19] applies in this case as well. \( \square \)

We consider the loop-weight locus of \( \Delta_{g,w} \) parametrizing tropical curves with loops or vertices of positive weight, denoted by \( \Delta_{g,w}^{lw} \), and show that it is contractible.

**Theorem 3.3.** The subcomplex \( \Delta_{g,w}^{lw} \) of \( \Delta_{g,w} \) is contractible.

**Proof.** We first note that \( \Delta_{g,w}^{lw} \) is indeed a subcomplex of \( \Delta_{g,w} \), as the property of having a loop or vertex of positive weight is closed under edge contraction. When \( g = 1 \) and \( \sum_{i=1}^n w_i \leq 1 \), the subcomplex \( \Delta_{g,w}^{lw} \) contains a single point, so the statement is trivially true. Let us assume for the rest of the proof that either \( g \geq 2 \) or \( \sum_{i=1}^n w_i > 1 \) when \( g = 1 \).

By Lemma 3.2, there is a deformation retract from \( (\Delta_{g,w})_Q \searrow (\Delta_{g,w})_{Q,0} \). Observe that \( (\Delta_{g,w})_Q \) is exactly the loop-weight locus \( \Delta_{g,w}^{lw} \). Indeed, if \( G \in (\Delta_{g,w})_Q \), then \( G \) is a contraction of a tropical curve with an 1-end, so \( G \) must have a loop or vertex with positive weight; if \( G \in \Delta_{g,w} \), then \( G \) has a nontrivial uncontraction by 1-ends, so \( G \) is in the closure of a \( Q \)-simplex. On the other hand, the subcomplex \( (\Delta_{g,w})_{Q,0} \) is the closure of the locus of tropical curves that have the following combinatorial type: the graph with a central vertex \( v \) and \( g \) bridges to vertices of weight 1, with all markings concentrated on \( v \). So the geometric realization of \( (\Delta_{g,w})_{Q,0} \) is the quotient of a \((g-1)\)-simplex by the action of its automorphism group, which is contractible. Therefore, the locus \( \Delta_{g,w}^{lw} \) is contractible. \( \square \)

Next, we show the slightly larger subcomplex \( \Delta_{g,w}^{mlw} \) that contains \( \Delta_{g,w}^{lw} \) is again contractible.

**Theorem 3.4.** The subcomplex \( \Delta_{g,w}^{mlw} \) of \( \Delta_{g,w} \) parametrizing tropical curves with loops, vertices of positive weight, or multiple edges is contractible.

**Proof.** Note that contracting an edge in a graph with multiple edges results in a graph with loops or multiple edges, so \( \Delta_{g,w}^{mlw} \) is a subcomplex. Our proof is parallel to the proof of Theorem 6.1 of [ACP19], which relies on the fact that if \( Y \) is contractible and \( f : Y \to Z \) is continuous, then the inclusion of \( Z \) into the mapping cone of \( f \) is a homotopy equivalence. The subcomplex \( \Delta_{g,w}^{mlw} \) is obtained from the subcomplex \( \Delta_{g,w}^{lw} \) as an iterated mapping cone through a series of maps from quotients of spheres \( S^{p-1}/\text{Aut}(G) \), where \( G \) ranges over all
graphs with multiple edges. Such quotients are contractible by [ACP19, Proposition 5.1], so the inclusion of $\Delta_{g,w}^{lw}$ into $\Delta_{g,w}^{mlw}$ is a homotopy equivalence.

Finally, we prove the main theorem of the section.

**Theorem 3.5.** Let $g, n \geq 1$ and $w \in (\mathbb{Q} \cap (0, 1))^n$. Then $\Delta_{g,w}$ is simply connected.

**Proof.** Observe the 1-skeleton of $\Delta_{g,w}$ is contained in the contractible subcomplex $\Delta_{g,w}^{mlw}$. By Theorem 3.1 of [ACP19], there is a surjection from $\pi_1(\Delta_{g,w}^{(1)}, x) \to \pi_1(\Delta_{g,w}, x)$; that is, $\pi_1(\Delta_{g,w}, x)$ is generated by loops in the 1-skeleton. Since $\Delta_{g,w}^{(1)}$ is contained in a contractible subcomplex, all loops in $\Delta_{g,w}^{(1)}$ are homotopically trivial in $\Delta_{g,w}$, and we conclude that $\pi_1(\Delta_{g,w}, x)$ is trivial.

\[ \square \]

4. The Euler characteristic of $\Delta_{g,w}$

Let $M_{g,w}$ be the coarse moduli space of $\mathcal{M}_{g,w}$. In this section we exhibit a useful decomposition of the class $[M_{g,w}]$ in terms of classes $[M_{g,r}]$ in the Grothendieck group of varieties. Using the fact that the virtual Poincaré polynomial is an Euler-Poincaré characteristic, this allows us to deduce the formula of Theorem 1.3.

**4.1. The Grothendieck group of varieties and Euler-Poincaré characteristics.** Let $k$ be a field. We denote by $K_0(\text{Var}/k)$ the Grothendieck group of varieties. This group is the quotient of the free abelian group on $k$-varieties by relations of the form

\[ [X] = [X \setminus Y] + [Y], \]

when $Y$ is a closed subvariety of $X$. Such relation are called the *cut-and-paste* relations. The additive identity is $[\emptyset]$. An *Euler-Poincaré characteristic* of $K_0(\text{Var}/k)$ is a group homomorphism

\[ \chi : K_0(\text{Var}/k) \to A \]

to an abelian group $A$. That is, for any closed subvariety $Y$ of $X$,

\[ \chi([X]) = \chi([Y]) + \chi([X \setminus Y]). \]

See [Cra04, Loe09]. Specializing to $k = \mathbb{C}$, one example of an Euler-Poincaré characteristic is given by the *virtual Poincaré polynomial*, which is the group homomorphism $K_0(\text{Var}/\mathbb{C}) \to \mathbb{Z}[t]$ defined by the formula

\[ P_X(t) = \sum_{m=0}^{2d} (-1)^m \chi_c^m(X)t^m, \]

where $d = \dim X$ and

\[ \chi_c^m(X) := \sum_{j=0}^{2d} (-1)^j \dim \text{Gr}_m^W H^j_c(X; \mathbb{Q}). \]
4.2. The stratification of $M_{g,w}$. Let $g \geq 0$, $n \geq 1$ and $w \in (\mathbb{Q} \cap (0,1))^n$ such that

$$2g - 2 + \sum_{i=1}^{n} w_i > 0.$$ 

To describe a stratification of $M_{g,w}$, we say that a set partition of $[n]$

$$\mathcal{P} = P_1 \sqcup \cdots \sqcup P_r \vdash [n]$$

is $w$-admissible if $\sum_{i \in P_j} w_i \leq 1$ for all $1 \leq j \leq r$. Given such a partition, we write $\mathcal{P} \vdash_w [n]$. We set $N_{r,w}$ to be the number of $w$-admissible partitions of $[n]$ with $r$ parts.

Proposition 4.1. In the Grothendieck group of $k$-varieties $K_0(\text{Var}/k)$,

$$[M_{g,w}] = \sum_{r=1}^{n} N_{r,w} [M_{g,r}].$$

Proof. The locus $M_{g,w}$ parameterizes irreducible smooth curves of genus $g$ with $n$ markings, such that whenever $\sum_{i \in S} w_i \leq 1$ for some $S \subseteq [n]$, the markings indexed by $S$ are allowed to coincide. Given a $w$-admissible partition

$$\mathcal{P} = P_1 \sqcup \cdots \sqcup P_r \vdash_w [n],$$

we define

$$Z_{\mathcal{P}} := \{(C, p_1, \ldots, p_n) \in M_{g,w} \mid p_i = p_j \text{ if and only if } i, j \in P_s \text{ for some } s \in [r]\}.$$ 

Then $Z_{\mathcal{P}} \cong M_{g,r}$. As $\mathcal{P}$ ranges over all $w$-admissible partitions of $[n]$, the loci $Z_{\mathcal{P}}$ form a locally closed stratification of $M_{g,w}$. Hence in the Grothendieck group, by [Mus, Proposition 1.1], we have

$$[M_{g,w}] = \sum_{\mathcal{P} \vdash_w [n]} [Z_{\mathcal{P}}] = \sum_{r=1}^{n} \sum_{\mathcal{P} \vdash_w [n]} [Z_{\mathcal{P}}] = \sum_{r=1}^{n} N_{r,w} [M_{g,r}],$$

as claimed. \hfill \Box

Remark 4.2. In [BH05, Section 4], Bini and Harer computed the Euler characteristics of $M_{g,n}$ for $2g - 2 + n > 0$. Our decomposition in Proposition 4.1 can be used to give the Euler characteristic of $M_{g,w}$ in terms of those of $M_{g,r}$ for $0 < r \leq n$.

We now record two corollaries which amount to the calculation of the numbers $N_{r,w}$ for special values of $w$.

Corollary 4.3. Let $w$ be heavy/light, i.e. $w = (1^{(n)}, \varepsilon^{(m)})$ for $m \geq 2$ and $0 < \varepsilon < 1/m$ satisfying $2g - 2 + n \geq 0$. Then

$$[M_{g,w}] = \sum_{r=1}^{m} S(m, r) [M_{g,n+r}],$$

where $S(m, r)$ is the Stirling number of the second kind, or the number of $r$-partitions of an $m$-set.
Furthermore, the \(m\)-restricted Stirling number of the second kind for \(n, r \geq 1\) is defined to be the number of partitions of an \(n\)-set into \(r\) nonempty subsets, each of which has at most \(m\) elements, and is denoted by
\[
\left\{ \begin{array}{c} n \\ r \end{array} \right\}_{\leq m}.
\]
For the generating function and other recurrence relations of the \(m\)-restricted Stirling numbers; see [KLM16]. Then we have the following.

**Corollary 4.4.** Let \(w = ((1/m)^{(n)})\) for \(n > m > 1\) such that \(2g - 2 + n/m > 0\). Then
\[
[M_{g,w}] = \sum_{r=\left\lceil \frac{n}{m} \right\rceil}^{n} \left\{ \begin{array}{c} n \\ r \end{array} \right\}_{\leq m} [M_{g,r}].
\]

**Corollary 4.5.** Let \(w = ((1/2)^{(n)})\) for \(n \geq 1\) such that \(2g - 2 + n/2 > 0\). Then
\[
[M_{g,w}] = \sum_{r=\left\lceil \frac{n}{2} \right\rceil}^{n} \prod_{i=0}^{n-r-1} \left( \begin{array}{c} n-2i \\ 2 \end{array} \right) \frac{n-r-1}{(n-r)!} [M_{g,r}].
\]

**Proof.** A \(((1/2)^{(n)})\)-admissible partition having \(r\) parts must consist of exactly \((n-r)\) subsets of size two and singletons otherwise. Therefore, we have
\[
\left\{ \begin{array}{c} n \\ r \end{array} \right\}_{\leq 2} = \prod_{i=0}^{n-r-1} \left( \begin{array}{c} n-2i \\ 2 \end{array} \right) \frac{n-r-1}{(n-r)!}.
\]
Plug this into Corollary 4.4, we obtain the desired expression. \(\Box\)

### 4.3. The Euler characteristics of \(\Delta_{g,w}\) and \(\mathcal{M}_{g,w}\). We can now exploit the additivity of Euler-Poincaré characteristics and the connection between \(\mathcal{M}_{g,w}\) and \(\Delta_{g,w}\) to prove Theorem 1.3. For a complex algebraic variety (or stack) \(X\) of dimension \(d\), let \(\chi^{tw}\) be the top weight Euler characteristic, defined as
\[
\chi^{tw}(X) := \sum_{i=0}^{2d} (-1)^i \dim \text{Gr}^{W}_{2d-i} H^{i}(X; \mathbb{Q}),
\]
and for any space \(Y\), let \(\tilde{\chi}(Y)\) be the reduced Euler characteristic. Recall from [CGP21, Theorem 5.8] that the isomorphism
\[
\text{Gr}^{W}_{6g-6+2n} H^{6g-6+2n-k}(\mathcal{M}_{g,n}; \mathbb{Q}) \cong \tilde{H}_{k-1}(\Delta_{g,n}; \mathbb{Q})
\]
is a special case of the isomorphism
\[
\text{Gr}^{W}_{2d} H^{2d-k}(\mathcal{X}; \mathbb{Q}) \cong \tilde{H}_{k-1}(\Delta(\mathcal{X} \subset \overline{\mathcal{X}}); \mathbb{Q})
\]
whenever \(\mathcal{X}\) is a smooth and separated \(d\)-dimensional DM stack over \(\mathbb{C}\), \(\overline{\mathcal{X}}\) is a smooth normal crossings of \(\mathcal{X}\), and \(\Delta(\mathcal{X} \subset \overline{\mathcal{X}})\) is the dual complex of the normal crossings divisor \(\overline{\mathcal{X}} \smallsetminus \mathcal{X}\).

**Lemma 4.6.** Let \(\mathcal{X}\) be a smooth, separated DM stack over \(\mathbb{C}\) and let \(\overline{\mathcal{X}}\) be a smooth normal crossings compactification of \(\mathcal{X}\). Then
\[
\chi^{tw}(\mathcal{X}) = -\tilde{\chi}(\Delta(\mathcal{X} \subset \overline{\mathcal{X}})).
\]
Proof. Let $d = \dim \mathcal{X}$. We have

$$
\chi^{tw}(\mathcal{X}) = \sum_{i=0}^{2d} (-1)^i \dim \text{Gr}_{2d-i}^W H^i(\mathcal{X}; \mathbb{Q})
= \sum_{i=0}^{2d} (-1)^i \dim \tilde{H}_{2d-i-1}(\Delta(\mathcal{X} \subset \overline{\mathcal{X}}); \mathbb{Q})
= -\chi(\Delta(\mathcal{X} \subset \overline{\mathcal{X}})).
$$

We now observe that the weight 0 compactly supported Euler characteristic is a motivic invariant; indeed the following lemma is proven upon realizing that for a complex algebraic variety $X$, we have that $\chi^0_c(X) = P_X(0)$, where $P_X$ is the virtual Poincaré polynomial.

**Lemma 4.7.** The weight 0 compactly supported Euler characteristic

$$
\chi^0_c: K_0(\text{Var}/\mathbb{C}) \to \mathbb{Z}
$$

is an Euler-Poincaré characteristic.

We can now prove Theorem 1.3, restated here.

**Theorem 1.2.** Let $W = W_1 \subset \cdots \subset W_{6g-6+2r} \subseteq H^*(\mathcal{M}_{g,r}; \mathbb{Q})$ be the weight filtration of the rational singular cohomology of the moduli stack $\mathcal{M}_{g,r}$ and denote by $\chi^W_{6g-6+2r}$ the Euler characteristic of the top graded piece

$$
\text{Gr}_{6g-6+2r}^W H^*(\mathcal{M}_{g,r}; \mathbb{Q}) = W_{6g-6+2r}/W_{6g-7+2r}
$$

of the weight filtration. Then

$$
\chi(\Delta_{g,w}) = 1 - \sum_{r=1}^{n} N_{r,w} \chi^W_{6g-6+2r}(\mathcal{M}_{g,r}).
$$

**Proof of Theorem 1.3.** By Proposition 4.1 and Lemma 4.7, we have

$$
\chi^0_c(M_{g,w}) = \sum_{r=1}^{n} N_{r,w} \chi^0_c(M_{g,r}).
$$

By [Beh04, Proposition 36] and [Edi10, Theorem 4.40], the coarse moduli scheme $\mathcal{X} \to X$ of a DM stack $\mathcal{X}$ induces an isomorphism of rational cohomology $H^*(\mathcal{X}; \mathbb{Q}) \cong H^*(X; \mathbb{Q})$, which is an isomorphism of mixed Hodge structures. Therefore, $\chi^0_c(\mathcal{X}) = \chi^0_c(X)$ and $\chi^{tw}(\mathcal{X}) = \chi^{tw}(X)$. For a smooth Deligne-Mumford stack $\mathcal{X}$ of dimension $d$, the Poincaré duality pairing

$$
H^j_c(\mathcal{X}; \mathbb{Q}) \times H^{2d-j}(\mathcal{X}; \mathbb{Q}) \to \mathbb{Q}
$$

induces a perfect pairing of graded pieces

$$
\text{Gr}_m^W H^j_c(\mathcal{X}; \mathbb{Q}) \times \text{Gr}_{2d-m}^W H^{2d-j}(\mathcal{X}; \mathbb{Q}) \to \mathbb{Q}
$$
for $0 \leq m \leq 2j$; see [PS08, Theorem 6.23]. Thus we can write
\[
\chi^0_c(\mathcal{X}) = \sum_{j=0}^{2d} (-1)^j \dim \text{Gr}^W_{2d} H^{2d-j}(\mathcal{X}; \mathbb{Q})
\]
\[
= \sum_{j=0}^{2d} (-1)^{2d-j} \dim \text{Gr}^W_{2d} H^{2d-j}(\mathcal{X}; \mathbb{Q}).
\]
In particular, it follows that
\[
\chi^0_c(\mathcal{X}) = \chi^{tw}(\mathcal{X}).
\]
Since $\mathcal{M}_{g,r}$ and $\mathcal{M}_{g,w}$ are smooth Deligne-Mumford stacks and $\Delta_{g,w} = \Delta(\mathcal{M}_{g,w} \subset \overline{\mathcal{M}}_{g,w})$, the result now follows from Lemma 4.6 and the fact that $\overline{\chi}(\Delta_{g,w}) = \chi(\Delta_{g,w}) - 1$. \hfill \Box

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