GLOBAL OPTIMIZATION REDUCTION OF GENERALIZED MAFATTI’S PROBLEM

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Abstract. In this paper, we generalize Malfatti’s problem as a continuation of works [6, 7]. The problem has been formulated as a global optimization problem. To solve Malfatti’s problem numerically, we propose the co-called “Hill method” which is based on a heuristic approach. Some computational results for two and three-dimensional test problems are provided.

1. Introduction. In 1803, Malfatti (1737–1807) of the University Ferrara posed the problem of determining the three circular columns of marble of possibly different sizes which, when carved out of a right triangular prism, would have the largest possible total cross section [14]. This is equivalent to finding the maximum total area of three circles which can be packed inside a right triangle of any shape without overlapping. Malfatti gave the solution as three circles (the Malfatti circles) tangent to each other and to two sides of the triangle.

In [12], it was shown that the Malfatti circles were not optimal. The most common methods used for finding the best solutions to Malfatti’s problem were algebraic and geometric approaches [1, 11, 9]. In 1994 Zalgaller and Los [24, 13] proved that the greedy arrangement solves the Malfatti’s problem. Melissen conjectured in [15]:

Conjecture 1. The greedy arrangement has the largest total area among of $n$ ($n \geq 4$) non-overlapping circles in a triangle.

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Most recently, Malfatti’s problem was examined from the standpoint of the global optimization theory using the algorithm from [5-7]. In these works, Malfatti’s problem was formulated as a convex maximization problem and then the global optimality conditions developed by A.S. Strekalovsky [21] together with the Algorithm MAX [5] were applied to solve the original Malfatti’s problem for the case $n = 4$.

In this paper, we formulate Malfatti’s generalized problem and propose a global optimization method and an algorithm for its numerical solution. The paper is organized as follows. In Section 2 we present a formulation of Malfatti’s generalized problem as well as its reduction to a nonconvex optimization problem. Section 3 discusses the global optimization method and the algorithm. Numerical results are presented in Section 4.

2. Malfatti’s Generalized Problem. We generalize Malfatti’s problem as follows: how to pack $K$ non-overlapping balls of maximum total volume in a given bounded polyhedral set $D \subset \mathbb{R}^n$?

We introduce the following sets. Denote by $B(x^0, r)$ a ball with a center $x^0 \in \mathbb{R}^n$ and a radius $r \in \mathbb{R}$:

$$B(x^0, r) = \{ x \in \mathbb{R}^n \mid \| x - x^0 \| \leq r \}. \quad (1)$$

A bounded polyhedral set $D \subset \mathbb{R}^n$ is given by

$$D = \{ x \in \mathbb{R}^n \mid (a^i, x) \leq b_i, a^i \in \mathbb{R}^n, b_i \in \mathbb{R}, i = 1, m \}, \quad (2)$$

where $\langle \ , \ \rangle$ denotes the scalar product of two vectors in $\mathbb{R}^n$, $\| \cdot \|$ is the Euclidean norm, and int $D \neq \emptyset$.

Theorem 2.1. [6] $B(x^0, r) \subset D$ if and only if

$$\langle a^i, x^0 \rangle + r \| a^i \| \leq b_i, i = 1, m. \quad (3)$$

Proof. Necessity. Let $y \in B(x^0, r)$ and $y \in D$. The point $y \in B(x^0, r)$ can be easily presented as $y = x^0 + rh$, $h \in \mathbb{R}^n$, $\| h \| \leq 1$. It follows from the condition $y \in D$ that $\langle a^i, y \rangle \leq b_i, i = 1, m$ or, equivalently, $\langle a^i, x^0 \rangle + r \langle a^i, h \rangle \leq b_i, i = 1, m, \ \forall h \in \mathbb{R}^n$. Hence, we have

$$\langle a^i, x^0 \rangle + r \max_{\| h \| \leq 1} \langle a^i, h \rangle \leq b_i, i = 1, m,$$

or

$$\langle a^i, x^0 \rangle + r \langle a^i, \frac{a^i}{\| a^i \|} \rangle \leq b_i, i = 1, m,$$

which yields

$$\langle a^i, x^0 \rangle + r \| a^i \| \leq b_i, i = 1, m.$$

Sufficiency. Let the condition (3) be satisfied, and on the contrary, assume that there exists $\tilde{y} \in B(x^0, r)$ such that $\tilde{y} \not\in D$. Clearly, there exists $\tilde{h} \in \mathbb{R}^n$ such that $\tilde{y} = x^0 + r\tilde{h}, \| \tilde{h} \| \leq 1$. Since $\tilde{y} \not\in D$, there exists $j \in \{1, 2, \ldots, m\}$ for which $\langle a^j, \tilde{y} \rangle > b_j$ or $\langle a^j, x^0 + r\tilde{h} \rangle = \langle a^j, x^0 \rangle + r\langle a^j, \tilde{h} \rangle > b_j$.

On the other hand, we have $\langle a^j, x^0 \rangle + r \| a^j \| > b_j$ which contradicts (3).

Denote by $u^1(z_1, z_2, \ldots, z_n)$, $u^2(z_{n+1}, \ldots, z_{2n})$, $\ldots$, $u^j(z_{j-1}z_{n+1}, \ldots, z_{jn})$, $j = 1, K$ centres of these balls inscribed in the polyhedral set $D$ defined by (2). Let $z_{Kn+1}, z_{K(n+1)}, \ldots, z_{K(n+1)}$ be their corresponding radii.

Now we are ready to formulate the Malfatti’s generalized problem:
\[
\max f(z) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)} \sum_{j=1}^{K} z_{Kn+j}^{n},
\]
\[
(a^i, u^j) + z_{Kn+j}\|a^i\| \leq b_i, i = \overline{1, m}, j = \overline{1, K},
\]
\[
\|u^i - u^j\|^2 \geq (z_{Kn+i} + z_{Kn+j})^2, i, j = \overline{1, K}, i \neq j, i, j \in \binom{K}{2},
\]
\[
z_{Kn+1} \geq 0, z_{Kn+2} \geq 0, \ldots, z_{K(n+1)} \geq 0,
\]
where \(\Gamma(\cdot)\) is the gamma-function.

The function \(f\) in (4) denotes a total volume of \(K\) balls. Conditions (5) define that all the balls are inscribed into a polyhedral set; meanwhile conditions (6) provide that the balls do not overlap.

3. Global Optimization Method and Algorithm. Problem (4)−(7) is a convex maximization problem over nonconvex sets and belongs to a class of global optimization. It can be verified that the problem is NP hard. The problem was solved globally by an algorithm in [7] which seems to us computationally expensive for a high dimensional case. For this purpose, to solve the problem we propose the “Hill method” which is heuristic.

The algorithm assumes elimination of several local minima from a set of local extrema by moving to a saddle-point like to the hill with the most gentle sloping. Literally, it means that a new local descent should start not from a neighborhood of a local minimum as it was done in the method by K.L. Teo [22], but from a saddle-point in the same neighborhood with the lowest function value. The idea is based on the same hypothesis as the MSBH method [11, 12], which has been proven to perform well when solving problems of global optimization with a large number of local extrema. The “Hill method” includes a systematically random multistart, which is a basis of all global optimization techniques. The algorithm of the method proposed consists of the multistart, the local descent (a specific version of the conjugate gradient method) and a phase of withdrawal from the local extremum to the “hill” for a new local descent. The full iterations procedure is described below (the Hessian matrix eigenvalues at Step 11 are computed approximately).

As a stop criterion of the algorithm proposed, one can use the number of extrema found by the local search or the elapsed computing time. All constraints of problem (4)−(7) have been penalized into a single objective function before we applied the “Hill method”. Then problem (4)−(7) reduces to a global optimization problem over a box constraints \(z \in \mathbb{Z} = \{z \in \mathbb{R}^n | z^l \leq z \leq z^g\}\), assume \(D \subset \mathbb{Z}\).

To solve a subproblem of the local minimum search, we implement a combination of two algorithms which enables us to retain superlinear convergence without the need to employ the procedure of the search space reduction when we touch the constraints. In the reduced gradient algorithm the auxiliary point is constructed at the \(k\)-th iteration:

\[
\tilde{z}_i^k = \begin{cases} 
  z_i^l, & z_i^k - \nabla f(z_i^k) < z_i^l, \\
  z_i^k - \nabla f(z_i^k), & z_i^l \leq z_i^k - \nabla f(z_i^k) \leq z_i^g, i = \overline{1, nK}, \\
  z_i^g, & z_i^k - \nabla f(z_i^k) > z_i^g 
\end{cases}
\]

Then we solve a one dimensional minimization problem to obtain \(z^{k+1}\).
\[ \text{Algorithm 1: “Hill” Algorithm} \]

\[
\min_{\alpha \in [0,1]} f(z^k + \alpha(z^k - z^{k+1})) = f(z^{k+1}), \quad \alpha \in [0,1].
\]

The conjugate gradient method is implemented with the use of a sinusoidal space known as the Gernet-Valentine transformation [10, 23, 8]. The pre-computed gradient of the function is transformed component-wise to ensure that variations satisfy the direct constraints.

\[
d^k = \frac{1}{2}(z^g - z^l) \cos(\arcsin(\frac{2z^k - z^g - z^l}{z^g - z^l})) \nabla f(z^k).
\]

A conjugate direction can be chosen according to the Polak-Polyak-Ribiere method [2, 16] \[ q^k = -d^k + \beta^k q^{k-1}, \quad \beta^k = \begin{cases} (d^k)^T(d^k - d^{k-1}) / \|d^k-1\|, & k \in K_{fr}, \\ 0, & k \notin K_{fr}, \end{cases} \]

where \( K_{fr} \) is the updated iteration index.

The variation step length is obtained by solving \( \min f(z(\alpha)) = f(z^{k+1}), \quad \alpha \geq 0, \)

where the variation is \( z(\alpha) = \frac{1}{2}(z^g - z^l) \cos\left(\arcsin\left(\frac{2z^k - z^g - z^l}{z^g - z^l}\right) + \alpha q^k\right) \).

It is well known that this version of the conjugate gradient algorithm has some shortcomings associated with the “stick to the border” effect. It can be eliminated if we periodically run the reduced gradient algorithm described above.
Proposed “Hill method” can be used for solving various problems of nonconvex optimization, not only for Malfatti’s generalized problem but also for such fields arisen in applications as packing problems, transportation problems, inventory control problems, see e.g. [19, 20, 17, 18].

4. Numerical Results. The algorithm proposed was tested on Malfatti’s generalized problems described below. The algorithm was coded using C-language. Problems (4)–(7) were solved numerically for the following feasible sets and given number of circles.

**Test Problem 1**
The set $D$ is given by:

\[
\begin{align*}
2x_2 - 5x_1 &\leq 10, \\
5x_1 + 9x_2 &\leq 45, \\
2x_1 - 3x_2 &\leq 18, \\
-3x_1 - x_2 &\leq 6.
\end{align*}
\]

The solution for $K = 3$ is $f^*_3 = 48.5424$ and circle parameters are presented in Table 1.

**Table 1. Test Problem 1 for $K = 3$.**

| $x_1^*$ | $x_2^*$ | $r^*$ |
|---------|---------|-------|
| 1.9011  | -0.2129 | 3.6336|
| 6.7104  | -0.0751 | 1.1775|
| 0.4961  | -4.5530 | 0.9282|

For $K = 4$ and $K = 5$, we have $f^*_4 = 50.9128$ and $f^*_5 = 51.4262$ respectively.

**Table 2. Test Problem 1 for $K = 4$.**

| $x_1^*$ | $x_2^*$ | $r^*$ |
|---------|---------|-------|
| 1.9609  | -0.2849 | 3.6675|
| 6.7807  | -0.0898 | 1.1563|
| 0.4795  | -4.6023 | 0.8969|
| 0.3978  | 3.8828  | 0.7834|

**Table 3. Test Problem 1 for $K = 5$.**

| $x_1^*$ | $x_2^*$ | $r^*$ |
|---------|---------|-------|
| 1.9607  | -0.2849 | 3.6677|
| 6.7799  | -0.0899 | 1.1567|
| 0.4796  | -4.6020 | 0.8972|
| 0.3973  | 3.8822  | 0.7841|
| -0.3701 | -3.6201 | 0.4016|
Test Problem 2
The set $D$ is defined as:

$$
x_2 \leq 7,$$
$$7x_1 + 5x_2 \leq 56,$$
$$3x_2 - 7x_1 \leq 28,$$
$$-x_2 \leq 0.$$

The problem was solved for three, four and five circles. Table 4 shows that the greedy algorithm \[15\] works correctly for Problem 2 with $K = 3, 4, 5$. The computed best values of the total area are $f^*_3 = 46.2041$, $f^*_4 = 46.9827$, $f^*_5 = 47.7479$. The numerical results for Problem 2 are shown in Table 4 and at Fig. 2.

Table 4. Test Problem 2 for $K = 3, 4, 5$.

| $K$ | $x^*_1$ | $x^*_2$ | $r^*$ |
|-----|---------|---------|-------|
| 1   | 1.2601  | 3.4685  | 3.4685|
| 2   | 5.4923  | 1.2905  | 1.2905|
| 3   | -2.475  | 1.0056  | 1.0056|
| 4   | 5.2051  | 3.0559  | 0.4981|
| 5   | 3.8888  | 0.4980  | 0.4981|

Figure 2. Circles for $K = 3$ and $K = 5$ for test problem 2.
Test Problem 3
The set $D$ is described as follows:

\begin{align*}
  x_2 - 2x_1 &\leq 12, \\
  3x_1 - 2x_1 &\leq 24, \\
  x_1 + 4x_2 &\leq 32, \\
  7x_1 + 5x_2 &\leq 63.
\end{align*}

Numerical solutions are $f^*_3 = 60.9256$, $f^*_4 = 62.4909$, $f^*_5 = 63.697$.

Table 5. Test Problem 3 for $K = 3, 4, 5$.

| $x^*_1$ | $x^*_2$ | $r^*$  |
|---------|---------|--------|
| 0.7187  | 3.8509  | 3.8509 |
| 5.7749  | 1.6597  | 1.6597 |
| -3.8282 | 1.3422  | 1.3422 |
| 5.2807  | 3.9803  | 0.7129 |
| 7.7998  | 0.6176  | 0.6176 |

Figure 3. Circles placed into the test polygon 3 for $K = 3, 5$.

Test Problem 4
The set $D$ is given by the following system of equations:

\begin{align*}
  30x_1 + 70x_2 + 21x_3 &\leq 210, \\
  80x_1 + 56x_3 - 70x_2 &\leq 560, \\
  20x_2 + 6x_3 - 30x_1 &\leq 60, \\
  -80x_1 - 20x_2 + 16x_3 &\leq 160, \\
  x_3 &\leq 0,
\end{align*}

which is a four-angle pyramid with the vertices $(-2; 0; 0)$, $(7; 0; 0)$, $(0, -8; 0)$, $(0; 3; 0)$, $(0; 0; 10)$. The task is to inscribe four (since the pyramid base is a quadrangle) or five spheres inside the polyhedron. We obtain $f^*_4 = 75.322$, $f^*_5 = 76.6280$. Fig. 4 shows the polyhedron $D$ corresponding to Test Problem 4 and the spheres obtained.
Test Problem 5

This problem includes all the constraints from the previous problem and one more cutting of the plane satisfying \(20x_1 + 175x_2 + 168x_3 \leq 910\). The new vertices are \((-2; 0; 0), (7; 0; 0), (0, -8; 0), (0; 3; 0), (0; 0; 10), (0; 2; \frac{10}{3}, 3.5; 0; 5), (0; -2; 7.5)\).

Full problem (4)–(7) for \(K = 3\) is formulated as:

\[
\max f_1 = \frac{4\pi}{3}(z_{10}^3 + z_{11}^3 + z_{12}^3), \quad (8)
\]

subject to:

\[
\begin{align*}
30z_1 + 70z_2 + 21z_3 + z_{10}\sqrt{6241} & \leq 210 \\
80z_1 - 70z_2 + 56z_3 + z_{10}\sqrt{14436} & \leq 560 \\
-30z_1 + 20z_2 + 6z_3 + z_{10}\sqrt{1336} & \leq 60 \\
-80z_1 - 20z_2 + 16z_3 + z_{10}\sqrt{7056} & \leq 160 \\
20z_1 175z_2 + 168z_3 + z_{10}\sqrt{59249} & \leq 910 \\
-3z_3 + z_{10} & \leq 0
\end{align*}
\]

\[
\begin{align*}
30z_4 + 70z_5 + 21z_6 + z_{11}\sqrt{6241} & \leq 210 \\
80z_4 - 70z_5 + 56z_6 + z_{11}\sqrt{14436} & \leq 560 \\
-30z_4 + 20z_5 + 6z_6 + z_{11}\sqrt{1336} & \leq 60 \\
-80z_4 - 20z_5 + 16z_6 + z_{11}\sqrt{7056} & \leq 160 \\
20z_4 175z_5 + 168z_6 + z_{11}\sqrt{59249} & \leq 910 \\
-3z_6 + z_{11} & \leq 0
\end{align*}
\]
$30z_7 + 70z_8 + 21z_9 + z_{12}\sqrt{6241} \leq 210$
$80z_7 - 70z_8 + 56z_9 + z_{12}\sqrt{14436} \leq 560$
$-30z_7 + 20z_8 + 6z_9 + z_{12}\sqrt{1336} \leq 60$
$-80z_7 - 20z_8 + 16z_9 + z_{12}\sqrt{7056} \leq 160$
$20z_7 + 175z_8 + 168z_9 + z_{12}\sqrt{59249} \leq 910$
$-z_9 + z_{12} \leq 0$

$$(z_1 - z_4)^2 + (z_2 - z_5)^2 + (z_3 - z_6)^2 \geq (z_{10} + z_{11})^2$$
$$(z_1 - z_7)^2 + (z_2 - z_8)^2 + (z_3 - z_9)^2 \geq (z_{10} + z_{12})^2$$
$$(z_4 - z_7)^2 + (z_5 - z_8)^2 + (z_6 - z_9)^2 \geq (z_{11} + z_{12})^2$$

$z_7 \geq 0, z_8 \geq 0, z_9 \geq 0.$

This problem has 12 variables and 26 constraints. For four and five spheres we derived $f_3^* = 66.9876, f_4^* = 68.9889, f_5^* = 70.8907$, which can be seen at Fig. 5.

Figure 5. Spheres placed into test polyhedron 5

5. **Conclusion.** The 200 year old Malfatti’s problem has been generalized as a circle packing problem from a viewpoint of global optimization. To solve the problem, we proposed the co-called “Hill method” which improves the current local solution and move towards the global one. To illustrate theoretical speculations, we verified the performance of the algorithm proposed on some test problems. In all cases, the numerical solutions found by the proposed algorithm proves experimentally Melissen’s conjecture [15]. A generalization of Malfatti’s problem for $n$ dimensional space with arbitrary finite number of balls as a nonconvex optimization problem can be, in future, solved by different optimization methods and techniques. The problem can be considered also as a general circle packing problem in operations research.
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