BERNSTEIN-NIKOLSKII INEQUALITIES AND RIESZ INTERPOLATION FORMULA ON COMPACT HOMOGENEOUS MANIFOLDS

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Abstract. Bernstein-Nikolskii inequalities and Riesz interpolation formula are established for eigenfunctions of Laplace operators and polynomials on compact homogeneous manifolds.

1. Introduction

Consider a trigonometric polynomial $T$ of one variable $t$ as a function on a unit circle $S$. For its derivatives of $T^{(k)}$ the so-called Bernstein and Bernstein-Nikolskii inequalities hold true

\[(1.1) \quad \| T^{(k)} \|_{L_p(S)} \leq n^k \| T \|_{L_p(S)} \]

and

\[(1.2) \quad \| T^{(k)} \|_{L_q(S)} \leq 3n^{k+1/p-1/q} \| T \|_{L_p(S)}, \]

where $n$ is the order of $T$ and $1 \leq p \leq q \leq \infty$. The constant 3 is not the best but the inequality is exact in the sense that for the Feyer kernel

$$ F_n(t) = \frac{1}{n+1} \left( \frac{\sin \frac{2n+1}{2} t}{2 \sin \frac{t}{2}} \right), t \in S, $$

one has

$$ \| F_n^{(k)} \|_{L_q(S)} = C_{p,q} n^{k+1/p-1/q} \| F_n \|_{L_p(S)}. $$

The Bernstein inequality (1.1) can be obtained as a consequence of the Riesz interpolation formula

\[(1.3) \quad \frac{dT(t)}{dt} = \frac{1}{4\pi} \sum_{k=1}^{2n} (-1)^{k+1} \frac{1}{\sin^2 \frac{t}{2k}} T(t+t_k), t \in S, t_k = \frac{2k-1}{2n} \pi. \]

If one will treat $T$ as an entire function of exponential type on $\mathbb{C}$ which is bounded on the real axis, then the Riesz interpolation formula can be written in the form

\[(1.4) \quad \frac{dT(t)}{dt} = \frac{n}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{(k-1/2)^2} T(t + \frac{\pi}{n} (k-1/2)), t \in \mathbb{R}. \]

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The Bernstein-Nikolskii inequality (1.2) is a consequence of the inequality (1.1) and the following inequality which is known as the Nikolskii inequality (1.5)

\[
\|T\|_{L^p(S)} \leq \max_{u \in S} \left( h \sum_{k=1}^{N} |T(kh - u)|^p \right)^{1/p} \leq (1 + nh) \|T\|_{L^p(S)},
\]

where \( h = \frac{2\pi}{N}, N \in \mathbb{N}, 1 \leq p \leq \infty \). Similar results hold true for the \( m \)-dimensional torus \( T^m = S \times \ldots \times S \). The inequalities (1.1)-(1.5) and their proofs can be found in [2], Ch. 4, and in [23], Ch. 2 and 3.

Trigonometric polynomials can be characterized as eigenfunctions of the Laplace operator on \( T^m \). On the other hand, if one considers the equivariant embedding of \( T^m \) into Euclidean space \( \mathbb{R}^{2m} \) (flat torus) then every trigonometric polynomial on \( T^m \) can be identified with a restriction to \( T^m \) of an algebraic polynomial in the ambient space in \( \mathbb{R}^{2m} \).

All results listed above are at the very core of the classical approximation theory. The goal of the present article is to obtain similar results for a compact homogeneous manifold \( M \).

Very deep generalizations of some ideas which intimately relate to Bernstein-Markov type inequalities were obtained by J. Bourgain [3], A. Brudnyi [7], [8], A. Carbery and J. Wright [10]. In particular, the results of A. Brudnyi can be used to obtain a version of our Theorem 3.2. An abstract approach to Bernstein inequality was suggested by A. Gorin [16].

The Bernstein inequality on compact homogeneous manifolds was developed and explored in [4], [5], [20], [24]-[27], [29]. In particular, the Bernstein inequality on spheres was considered in [13]-[11], [21]. The Bernstein-Nikolskii-type inequalities on compact symmetric spaces of rank one were considered in interesting papers [6], [12]. But as well as we know nobody considered generalizations of (1.5). In fact the inequality we prove (see (1.13) below) even more general than (1.5) and seems to be new even in the case of trigonometric polynomials on a torus. Our approach to the Bernstein inequality and the Riesz interpolation formula is closer to the classical one in the sense that we are using first-order differential operators instead of using the Laplace-Beltrami operator as it was done in [20]. Note that generalizations of Bernstein-Nikolskii inequalities to non-compact symmetric spaces will appear in our paper [28].

In what follows we introduce some very basic notions of harmonic analysis on compact homogeneous manifolds [19], Ch. II. More details on this subject can be found, for example, in [31], [32].

Let \( M, \dim M = m \), be a compact connected \( C^\infty \)-manifold. It says that a compact Lie group \( G \) effectively acts on \( M \) as a group of diffeomorphisms if

1) every element \( g \in G \) can be identified with a diffeomorphism \( g : M \to M \)

of \( M \) onto itself and

\[
g_1 g_2 \cdot x = g_1 \cdot (g_2 \cdot x), g_1, g_2 \in G, x \in M,
\]

where \( g_1 g_2 \) is the product in \( G \) and \( g \cdot x \) is the image of \( x \) under \( g \),

2) the identity \( e \in G \) corresponds to the trivial diffeomorphism

\[
e \cdot x = x,
\]

3) for every \( g \in G, g \neq e \), there exists a point \( x \in M \) such that \( g \cdot x \neq x \).
A group $G$ acts on $M$ transitivity if in addition to 1)-3) the following property holds:
4) for any two points $x, y \in M$ there exists a diffeomorphism $g \in G$ such that $g \cdot x = y$.

A homogeneous compact manifold $M$ is an $C^\infty$-compact manifold on which transitively acts a compact Lie group $G$. In this case $M$ is necessary of the form $G/K$, where $K$ is a closed subgroup of $G$. The notation $L_p(M), 1 \leq p \leq \infty$, is used for the usual Banach spaces $L_p(M, dx), 1 \leq p \leq \infty$, where $dx$ is an invariant measure.

Every element $X$ of the Lie algebra of $G$ generates a vector field on $M$ which we will denote by the same letter $X$. Namely, for a smooth function $f$ on $M$ one has

$$Xf(x) = \lim_{t \to 0} \frac{f(\exp tX \cdot x) - f(x)}{t}$$

for every $x \in M$. In the future we will consider on $M$ only such vector fields. Translations along integral curves of such vector field $X$ on $M$ can be identified with a one-parameter group of diffeomorphisms of $M$ which is usually denoted as $\exp tX, -\infty < t < \infty$. At the same time the one-parameter group $\exp tX, t \geq \infty$, can be treated as a strongly continuous one-parameter group of operators in a space $L_p(M), 1 \leq p \leq \infty$ which acts on functions according to the formula

$$f \to f(\exp tX \cdot x), t \in \mathbb{R}, f \in L_p(M), x \in M.$$  

The generator of this one-parameter group will be denoted as $D_{X,p}$ and the group itself will be denoted as $e^{tD_{X,p}}f(x) = f(\exp tX \cdot x), t \in \mathbb{R}, f \in L_p(M), x \in M$.

According to the general theory of one-parameter groups in Banach spaces [9], Ch. I, the operator $D_{X,p}$ is a closed operator in every $L_p(M), 1 \leq p \leq \infty$. In order to simplify notations we will often use notation $DX$ instead of $D_{X,p}$.

It is known ([18], Ch. V, proof of the Theorem 3.1,) that on every compact homogeneous manifold $M = G/K$ there exist vector fields $X_1, X_2, \ldots, X_d, d = \dim G$, such that the second order differential operator on $M$

$$X_1^2 + X_2^2 + \ldots + X_d^2, d = \dim G,$$

commutes with all $X_1, \ldots, X_d$. The corresponding operator in $L_p(M), 1 \leq p \leq \infty$, (1.7)

$$- \mathcal{L} = D_1^2 + D_2^2 + \ldots + D_d^2, D_j = D_{X_j}, d = \dim G,$$

commutes with all operators $D_j = D_{X_j}$. This operator $\mathcal{L}$ which is usually called the Laplace operator is involved in most of constructions and results of our paper.

In some situations the operator $\mathcal{L}$ is essentially the Laplace-Beltrami operator of an invariant metric on $M$. It happens for example in the following cases.

1) If $M$ is a $d$-dimensional torus and $-\mathcal{L}$ is the sum of squares of partial derivatives.

2) If the manifold $M$ is itself a group $G$ which is compact and semi-simple then $-\mathcal{L}$ is exactly the Laplace-Beltrami operator of an invariant metric on $G$ ([18], Ch. II, Exercise A4).

3) If $M = G/K$ is a compact symmetric space of rank one then the operator $-\mathcal{L}$ is proportional to the Laplace-Beltrami operator of an invariant metric on $G/K$. It follows from the fact that in the rank one case every second-order operator which
commutes with all invariant vector fields is proportional to the Laplace-Beltrami operator \( (\mathbb{L}, \text{Ch. II, Theorem } 4.11) \).

Let us stress one more time that in the present paper we use only the property that the operator \( L \) commutes with all other invariant vector fields on \( M \) and we do not explore its relations to the Laplace-Beltrami operator of the invariant metric.

Note that if \( M = G/K \) is a compact symmetric space then the number \( d = \dim G \) of operators in the formula \( (\mathbb{L}) \) can be strictly bigger than the dimension \( m = \dim M \). For example on a two-dimensional sphere \( S^2 \) the Laplace-Beltrami operator \( L_{S^2} \) can be written as

\[
L_{S^2} = D_1^2 + D_2^2 + D_3^2,
\]

where \( D_i, i = 1, 2, 3, \) generates a rotation in \( \mathbb{R}^3 \) around coordinate axis \( x_i \):

\[
D_i = x_j \partial_k - x_k \partial_j,
\]

where \( j, k \neq i \).

The operator \( L \) is an elliptic differential operator which is defined on \( C^\infty(M) \) and we will use the same notation \( L \) for its closure from \( C^\infty(M) \) in \( L^p(M), 1 \leq p \leq \infty \). In the case \( p = 2 \) this closure is a self-adjoint positive definite operator in the space \( L^2(M) \). The spectrum of this operator is discrete and goes to infinity

\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots,
\]

where we count each eigenvalue with its multiplicity. For eigenvectors corresponding to eigenvalue \( \lambda_j \) we will use notation \( \varphi_j \), i.e.

\[
L \varphi_j = \lambda_j \varphi_j.
\]

Let \( \varphi_0, \varphi_1, \varphi_2, \ldots \) be a corresponding complete system of orthonormal eigenfunctions and \( E_\omega(\mathcal{L}), \omega > 0 \), be a span of all eigenfunctions of \( L \) whose corresponding eigenvalues are not greater \( \omega \).

In the rest of the paper the notations \( \mathbb{D} = \{D_1, \ldots, D_d\}, d = \dim G \), will be used for differential operators in \( L^p(M), 1 \leq p \leq \infty \), which are involved in the formula \( (\mathbb{L}) \).

**Definition 1.1.** We say that a function \( f \in L^p(M), 1 \leq p \leq \infty \), belongs to the Bernstein space \( B^p_\omega(\mathbb{D}) \), if and only if for every \( 1 \leq i_1, \ldots, i_k \leq d \) the following Bernstein inequality holds true

\[
\|D_{i_1} \ldots D_{i_k} f\|_p \leq \omega^k \|f\|_p, k \in \mathbb{N}.
\]

**Definition 1.2.** We say that a function \( f \in L^p(M), 1 \leq p \leq \infty \), belongs to the Bernstein space \( B^p_\omega(\mathcal{L}) \), if and only if for every \( k \in \mathbb{N} \) the following Bernstein inequality holds true

\[
\|\mathcal{L}^k f\|_p \leq \omega^k \|f\|_p, k \in \mathbb{N}.
\]

Since \( \mathcal{L} \) in the space \( L^2(M) \) is self-adjoint and positive-definite there exists a unique positive square root \( \mathcal{L}^{1/2} \). In this case the last inequality is equivalent to the inequality

\[
\|\mathcal{L}^{k/2} f\|_2 \leq \omega^{k/2} \|f\|_2, k \in \mathbb{N}.
\]

Note that at this point it is not clear if the Bernstein spaces \( B^p_\omega(\mathbb{D}), B^p_\omega(\mathcal{L}) \) are linear spaces. These facts will be established later in the Lemma 2.1 and Theorem 2.2.

The following Lemma was proved in \[25\] for any homogeneous manifold.
Lemma 1.3. There exists a constant $N(M)$ such that for any sufficiently small $r > 0$ there exists a set of points $\{x_i\}$ from $M$ such that
1) balls $B(x_i, r)$ are disjoint,
2) balls $B(x_i, 2r)$ form a cover of $M$,
3) multiplicity of the cover by balls $B(x_i, 4r)$ is not greater $N(M)$.

Definition 1.4. We will use notation $Z(r, N(M))$ for a set of points $\{x_i\} \subset M$ which satisfies the properties 1)- 3) from the last Lemma 1.1 and we will call such set a $(r, N(M))$-lattice of $M$.

Definition 1.5. We will use notation $Z_G(r, N(M))$ for a set of elements $\{g_r\}$ of the group $G$ such that the points $\{x_r = g_r \cdot o\}$ form a $(r, N(M))$-lattice in $M$ (here $\{o\} \subset M$ is the origin of $M$). Such set $Z_G(r, N(M))$ will be called a $(r, N(M))$-lattice in $G$.

Our main results are the following. In the section 2 we establish Riesz interpolation formula for Bernstein spaces $\mathcal{B}^p_n(\mathbb{D}), \mathbb{D} = \{D_1, ..., D_d\}, d = \dim G$, and use this formula to prove some basic properties of the Bernstein spaces.

Theorem 1.6. The following conditions are equivalent for any $1 \leq p \leq \infty$:
1) $f \in \mathcal{B}^p_n(\mathbb{D}), \mathbb{D} = \{D_1, ..., D_d\}, d = \dim G$,
2) for any $1 \leq i_1, ..., i_k \leq d$, any $1 \leq j \leq d$, and any functional $\psi^* \in L_p(M)^*$, $1 \leq p \leq \infty$, the function
$$\langle e^{tD_1}D_{i_1}...D_{i_k}f, \psi^* \rangle : \mathbb{R} \rightarrow \mathbb{R},$$
of the real variable $t$ has an extension to the complex plane $\mathbb{C}$ as an entire function of the exponential type at most $\omega$ and is bounded on the real line,
3) the following Riesz interpolation formula holds true
$$(1.12)\quad D_{i_1}...D_{i_k}f = \mathcal{R}^\omega_{i_1}...\mathcal{R}^\omega_{i_k}f, 1 \leq i_k \leq d,$$were
$$(1.13)\quad \mathcal{R}^\omega_i f = \frac{\omega}{\pi^2} \sum_{j \in \mathbb{Z}} \frac{(-1)^{j-1}}{(j-1/2)^2} e^{\omega(j-1/2)} D_i f,$$
were $e^{tD_X}f(x) = f(\exp tX \cdot x), t \in \mathbb{R}, f \in L_p(M), x \in M$, and convergence in $(1.13)$ is understood in the $L_p(M)$-sense.

It is also shown that if $M$ is equivariantly embedded into Euclidean space $\mathbb{R}^N$ and $P_n(M)$ is the set of restrictions to $M$ of polynomials in $\mathbb{R}^N$ of order $n$ then for any $f \in P_n(M)$ the following Riesz interpolation formula holds true
$$D_{i_1}...D_{i_k}f = \mathcal{R}^n_{i_1}...\mathcal{R}^n_{i_k}f, 1 \leq i_k \leq d.$$In particular
$$L f = \sum_{i,k=1}^d \mathcal{R}^n_{i_1} (\mathcal{R}^n_{i_k} f), f \in P_n(M).$$For example, in the case of the unit two-dimensional sphere $\mathbb{S}^2$ with the standard embedding into $\mathbb{R}^3$ the last formula means that the function $L f$ where $f$ is a polynomial can be calculated by using a combination of translations of $f$ with respect to rotations around coordinate axes. Using the Riesz interpolation formula we also prove Bernstein inequality for polynomials on compact homogeneous manifolds equivariantly embedded into Euclidean space.
In the section 3 we prove a Nikolskii-type inequality. Namely, we show that for any $1 \leq p \leq \infty$, any natural $l > m/p$, there exists a constant $C(M, l)$ such that for any $\omega > 0$ any $(r, N(G))$-lattice $Z_G(r, N(G)) \subset G$ with sufficiently small $r > 0$, and any $q \geq p$ the following inequalities hold true

$$\|f\|_q \leq r^{m/q} \sup_{g \in G} \left( \sum_{g_i \in Z_G(r, N(G))} \left( \|f(g_i g \cdot o)\|_p \right)^p \right)^{1/p} \leq C(M, l)r^{m/q - m/p} \left( 1 + (r\omega)^l \right) \|f\|_p, m = \dim M,$$

for all $f \in B^p_\omega(\mathbb{D})$. In particular, these inequalities hold true for polynomials in $P_n(M)$ with $\omega = n$.

Using these Nikolskii-type inequalities we prove that for any $1 \leq p, q \leq \infty$ the following equality holds true

$$B^p_\omega(\mathbb{D}) = B^q_\omega(\mathbb{D}) \equiv B^q_\omega(\mathbb{D}), \mathbb{D} = \{D_1, ..., D_d\}, d = \dim G,$$

which means that if the Bernstein-type inequalities (1.11) are satisfied for a single $1 \leq p \leq \infty$, then they are satisfied for all $1 \leq p \leq \infty$.

The inequalities (1.11) and (1) are used to obtain the following inequality of the Bernstein-Nikolskii-type

$$\|D_1 \cdots D_k f\|_q \leq C(M)\omega^{k + \frac{m}{p} - \frac{m}{q}} \|f\|_p, f \in B_\omega(\mathbb{D}), m = \dim M,$$

for a certain constant $C(M)$ and any $1 \leq p \leq q \leq \infty, 1 \leq i_1, ..., i_k \leq d, k \in \mathbb{N}, d = \dim G$.

We also prove the following embeddings which describe relations between Bernstein spaces $B_\omega(\mathbb{D}), \mathbb{D} = \{D_1, ..., D_d\}, d = \dim G$, and eigen spaces $E_\lambda(\mathcal{L})$ for $-\mathcal{L} = D_1^2 + D_2^2 + \cdots + D_d^2, d = \dim G$,

$$B_\omega(\mathbb{D}) \subset E_{\omega_2 d}(\mathcal{L}) \subset B_{\omega_\sqrt{d}}(\mathbb{D}), d = \dim G, \omega > 0.$$  

These embeddings obviously imply the equality

$$\bigcup_{\omega > 0} B_\omega(\mathbb{D}) = \bigcup_j E_{\lambda_j}(\mathcal{L}),$$

which means that a function on $M$ satisfies a Bernstein inequality (1.11) in a norm of $L_p(M), 1 \leq p \leq \infty$, if and only if it is a linear combination of eigenfunctions of $\mathcal{L}$.

As a consequence we obtain the following inequalities

$$\|\mathcal{L}^k \varphi\|_p \leq (d\omega)^{2k} \|\varphi\|_p, k \in \mathbb{N}, d = \dim G,$$

for every $\varphi \in E_\omega(\mathcal{L}), 1 \leq p \leq \infty$.

Note that in the case of homogeneous manifolds of rank one a better constant for such inequality was given by A. Kamzolov [20].

Another consequence of our Bernstein-Nikolskii inequality is the following estimate for every $\varphi \in E_\omega(\mathcal{L})$

$$\|\mathcal{L}^k \varphi\|_q \leq C(M)\omega_2^{k + \frac{m}{p} - \frac{m}{q}} \|\varphi\|_p, k \in \mathbb{N}, m = \dim M, d = \dim G, 1 \leq p \leq q \leq \infty,$$

for a certain constant $C(M)$ which depends just on the manifold. At the end of the paper we establish the following relations

$$P_n(M) \subset B_n(\mathbb{D}) \subset E_{n_2 d}(\mathcal{L}) \subset B_{n_\sqrt{d}}(\mathbb{D}), d = \dim G, n \in \mathbb{N},$$
and
\[ \bigcup_n \mathbf{P}_n(M) = \bigcup \mathbf{B}_\omega(\mathcal{D}) = \bigcup_j \mathbf{E}_{\lambda_j}(\mathcal{L}), n \in \mathbb{N}, \]
where \( \mathbf{P}_n(M) \) is the space of polynomials. Note that the embedding
\[ (1.16) \quad \mathbf{P}_n(M) \subset \mathbf{B}_\infty(\omega), \]
was proved by D. Ragozin [29].

2. Bernstein inequality and Riesz interpolation formula on compact homogeneous manifolds

We assume that \( A \) is a generator of one-parameter group of isometries \( e^{tA} \) in a Banach space \( E \) with the norm \( \| \cdot \| \). The Bernstein space \( \mathbf{B}_\omega(A), \omega > 0 \), is introduced as a set of all vectors \( f \) in \( E \) for which
\[ (2.1) \quad \| A^k f \| \leq \omega^k \| f \|, k \in \mathbb{N}. \]

Let’s introduce the operator
\[ \mathcal{R}_\omega^A f = \frac{\omega}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{(k-1/2)^2} e^{(\pi(k-1/2)A)} f = \mathcal{R}_\omega^A f, \]
Since \( \| e^{tA} f \| = \| f \|, f \in E \), and since the following identity holds
\[ (2.2) \quad \frac{\omega}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{1}{(k-1/2)^2} = \omega, \]
the operator \( \mathcal{R}_\omega^A, \omega > 0 \), is a bounded operator in \( E \) and
\[ (2.3) \quad \| \mathcal{R}_\omega^A f \| \leq \omega \| f \|, f \in E. \]

**Lemma 2.1.** The following conditions are equivalent:

1) \( f \in \mathbf{B}_\omega(A) \);
2) for any functional \( \psi^* \) from the dual space \( E^* \) and for any \( n \in \mathbb{N} \) the function
\[ (2.4) \quad F_n(t) = \langle e^{tA} A^n f, \psi^* \rangle : \mathbb{R} \rightarrow \mathbb{R}, \]
has an extension to the complex plane \( \mathbb{C} \) as an entire function of the exponential type at most \( \omega \) and is bounded on the real line;
3) the following Riesz interpolation formula holds true
\[ (2.5) \quad A^n f = (\mathcal{R}_\omega^A)^n f, n \in \mathbb{N}. \]

**Proof.** Let us assume that \( f \in \mathbf{B}_\omega(A) \). According to a general theory of one-parameter groups of operators in Banach spaces [9], Ch.1,
\[ \frac{d}{dt} e^{tA} f = Ae^{tA} f. \]
Since
\[ \frac{d}{dt} F(t) = \frac{d}{dt} \langle e^{tA} f, \psi^* \rangle = \langle \frac{d}{dt} e^{tA} f, \psi^* \rangle = \langle A e^{tA} f, \psi^* \rangle, \]
it implies that if \( f \in \mathbf{B}_\omega(A) \) then for any functional \( \psi^* \in E^* \) the scalar function
\[ F(z) = \langle e^{zA} f, \psi^* \rangle \]
is entire because its Taylor series at \( t = 0 \) is the series
\[ (2.6) \quad F(z) = \langle e^{zA} f, \psi^* \rangle = \sum_{l=0}^{\infty} \frac{z^l}{l!} \langle A^l f, \psi^* \rangle \]
which converges because of the estimate
\[ |\langle A^t f, \psi^* \rangle| \leq \|\psi^*\| \| A^t f \| \leq \omega^t \|\psi^*\| \|f\|. \]

The last estimate also implies the inequality
\[ (2.7) \quad |F(z)| \leq e^{z|\omega|} \|\psi^*\| \|f\|, \quad \psi^* \in E^*, \]
which shows that \( F(t) = \langle e^{tA} f, \psi^* \rangle \) has an extension to the complex plane \( \mathbb{C} \) as an entire function of the exponential type at most \( \omega \).

Moreover, because the group \( e^{tA} \) is an isometry group we have for any real \( t \) the inequality
\[ |F(t)| = |\langle e^{tA} f, \psi^* \rangle| \leq \|\psi^*\| \|e^{tA} f\| \leq \|\psi^*\| \|f\|, \]
which shows that the function \( F \) is bounded on the real line. Thus, we proved that if \( f \in \mathcal{B}_\omega(A) \) then the function \( F_{n-1}(t) = \langle e^{tA} A^{n-1} f, \psi^* \rangle \), \( n \in \mathbb{N} \), the following classical Riesz interpolation formula takes place
\[
\frac{d}{dt} F_{n-1}(t) = \frac{\omega}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{(k - 1/2)^2} F_{n-1}(t + \frac{\pi}{\omega}(k - 1/2)).
\]

Using the same arguments as above we can write this formula in the following form
\[
\langle A e^{tA} A^{n-1} f, \psi^* \rangle = \frac{\omega}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{(k - 1/2)^2} \langle e^{(t+\frac{\pi}{\omega}(k-1/2))A} A^{n-1} f, \psi^* \rangle.
\]

For \( t = 0 \) it gives
\[ (2.8) \quad \langle A^n f, \psi^* \rangle = \frac{\omega}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{(k - 1/2)^2} \langle e^{(\frac{\pi}{\omega}(k-1/2))A} A^{n-1} f, \psi^* \rangle. \]

Since the last formula holds for any functional \( \psi^* \in E^* \) it proves the equality
\[ (2.9) \quad A^n f = \mathcal{R}_A^n A^{n-1} f, \quad n \in \mathbb{N}, \]
for every function for which the property 2) holds. But then
\[ A^n f = A(A(...Af)) = \mathcal{R}_A^\omega (\mathcal{R}_A^\omega (... \mathcal{R}_A^\omega f)) = (\mathcal{R}_A^\omega)^n f, \quad n \in \mathbb{N}. \]

The implication 2) \( \rightarrow 3) \) is proved.

To finish the proof of the Theorem we have to show that 3) implies 1). But this fact easily follows from the formulas \( (2.5) \) and \( (2.3) \). \( \square \)

As a consequence of this Theorem we obtain the following Corollary.

**Corollary 2.1.** If \( A \) is a generator of one-parameter group of isometries \( e^{tA} \) in a Banach space \( E \) then the Bernstein spaces \( \mathcal{B}_\omega(A) \) are linear and closed for every \( \omega > 0 \).

We return to a homogeneous manifold \( M \) and we are going to use notations which were developed in the Introduction.
Theorem 2.2. The set $\mathcal{B}_p^\nu(\mathbb{D}), \mathbb{D} = \{D_1, ..., D_d\}, d = dimG, 1 \leq p \leq \infty$, has the following properties:

1) it is invariant under every $D_\nu, 1 \leq \nu \leq d$;
2) it is a linear subspace of $L_p(M)$;
3) it is a closed subspace of $L_p(M)$.

Proof. To prove the first part of the Theorem we have to show that if $f \in \mathcal{B}_p^\nu(\mathbb{D}), 1 \leq p \leq \infty$, and $g = D_\nu f$, for a $1 \leq \nu \leq d$, then the following inequality holds true

$$
(2.10) \|D_{i_1}...D_{i_k}g\|_p \leq \omega^k\|g\|_p, g = D_\nu f,
$$
for any $1 \leq i_1, i_2, ..., i_k \leq d$. First we are going to show that if $f \in \mathcal{B}_p^\nu(\mathbb{D}), 1 \leq p \leq \infty$, and $g = D_\nu f, 1 \leq \nu \leq d$, then for any $D_{i_1}, 1 \leq i_1 \leq d$, the following inequality holds

$$
(2.11) \|D_{i_1}g\|_p \leq \omega\|g\|_p.
$$

If $f \in \mathcal{B}_p^\nu(\mathbb{D})$, then for any $D_{\nu_1}, D_{\nu_2}, 1 \leq j, \nu \leq d$ and $g = D_{\nu_1}f$ the inequality $\|D_{\nu_1}g\|_p = \|D_{\nu_1}D_{\nu_2}f\|_p \leq \omega^{j+1}\|f\|_p = \omega^j\|(\omega\|g\|_p), l \in \mathbb{N}$, takes place. But then for any $z \in \mathbb{C}$ we have

$$
\|e^{zD_{i_1}}g\|_p = \left\| \sum_{l=0}^{\infty} \left( zD_{i_1} \right)^l f \right\|_p \leq \omega\|f\|_p \sum_{l=0}^{\infty} \frac{|z|^{l}\omega^l}{l!} = \omega|z|^\omega\|f\|_p, 1 \leq i_1 \leq d.
$$

As in the Lemma 2.1 it implies that for any functional $\psi^*$ on $L_p(M), 1 \leq p \leq \infty$, the scalar function

$$
F(z) = \langle e^{zD_{i_1}}g, \psi^* \rangle, 1 \leq i_1 \leq d,
$$
is an entire function of exponential type $\omega$. Moreover, since $e^{tD_{i_1}}$ is an isometry group in the space $L_p(M), 1 \leq p \leq \infty$, this function $F(t)$ is bounded on the real line

$$
|F(t)| = |\langle e^{tD_{i_1}}g, \psi^* \rangle| \leq \|\psi^*\||e^{tD_{i_1}}g\|_p \leq \|\psi^*\||g\|_p,
$$
for any functional $\psi^*$ on the space $L_p(M)$.

The same arguments which were used in the previous Lemma show the identity

$$
\frac{d}{dt}F(t) = \frac{d}{dt}\langle e^{tD_{i_1}}g, \psi^* \rangle = \langle \frac{d}{dt}e^{tD_{i_1}}g, \psi^* \rangle = \langle e^{tD_{i_1}}D_{i_1}g, \psi^* \rangle, 1 \leq i_1 \leq d.
$$

An application of the classical Bernstein inequality ( see [2], Ch. IV) to the function $F(t)$ in the uniform norm on the real line gives the inequality

$$
\sup_{t \in \mathbb{R}} |\frac{d}{dt}F(t)| \leq \omega \sup_{t \in \mathbb{R}} |F(t)|.
$$

In our notations it takes the form

$$
\sup_{t \in \mathbb{R}} |\langle e^{tD_{i_1}}D_{i_1}g, \psi^* \rangle| = \sup_{t \in \mathbb{R}} \left| \frac{d}{dt}\langle e^{tD_{i_1}}g, \psi^* \rangle \right| \leq \omega\|\psi^*\||g\|_p.
$$

By selecting $t = 0$ and a functional $\psi^*$ for which

$$
\langle D_{i_1}g, \psi^* \rangle = \|D_{i_1}g\|_p, \|\psi^*\|_p = 1,
$$
we obtain the inequality $\|D_{i_1}g\|_p \leq \omega\|g\|_p$. Now, suppose that we proved the inequality

$$
(2.12) \|D_{i_{k-1}}...D_{i_1}g\|_p \leq \omega^{k-1}\|g\|_p, g = D_\nu f,
$$
functions of one variable gives
\[ L_\omega \text{function of exponential type} \]

At this point we can repeat all the previous arguments to obtain
\[ \|D_{i_d} h\|_p \leq \omega \|h\|_p, \]

which along with the induction assumption (2.12) gives the desired inequality
\[ \|D_{i_k} \ldots D_{i_1} g\|_p = \|D_{i_d} h\|_p \leq \omega \|h\|_p = \omega \|D_{i_k} \ldots D_{i_1} g\|_p \leq \omega^k \|g\|_p. \]

The first part of the Theorem is proved.

To prove the second part of the Theorem it is enough to show that a function \( f \) belongs to the space \( B^\omega_p(\mathbb{D}) \), \( 1 \leq p \leq \infty \), if and only if for any \( 1 \leq i_1, \ldots, i_k \leq d \), any \( 1 \leq j \leq d \), and any functional \( \psi^* \in L_p(M)^* \) the function
\[
\langle \psi^*, e^{tD_j}D_{i_1} \ldots D_{i_k} f \rangle : \mathbb{R} \to \mathbb{R},
\]

of the real variable \( t \) is an entire function of the exponential type \( \omega \).

Suppose that \( f \in B^\omega_p(\mathbb{D}) \), \( 1 \leq p \leq \infty \), then for any function \( g = D_{i_1} \ldots D_{i_k} f, 1 \leq i_1, \ldots, i_k \leq d \), and any \( 1 \leq j \leq d \) the series
\[
e^{D_j} g = \sum \frac{\{zD_j\}^r}{r!} g
\]
is convergent in \( L_p(M) \) and represents an abstract entire function. Since \( \|D_j^r g\|_p \leq \omega^{k+r} \|f\|_p \) we have the estimate
\[
\|e^{D_j} g\|_p = \left\| \sum_{r=0}^\infty \frac{\{zD_j\}^r}{r!} g \right\|_p \leq \omega^k \|f\|_p \sum_{r=0}^\infty \frac{|z|^r \omega^r}{r!} = \omega^k |z|^\omega \|f\|_p,
\]

which shows that the function (2.14) has exponential type \( \omega \). Since \( e^{tD_j} \) is a group of isometries, the abstract function \( e^{tD_j} g \) is bounded by \( \omega^k \|f\|_p \). It implies that for any functional \( \psi^* \) on \( L_p(M), 1 \leq p \leq \infty \), the scalar function
\[
F(z) = \langle \psi^*, e^{zD_j} g \rangle
\]
is entire because it is defined by the series
\[
(2.15) \quad F(z) = \langle \psi^*, e^{zD_j} g \rangle = \sum_{r=0}^\infty \frac{z^r \langle \psi^*, D_j^r g \rangle}{r!}
\]
and because \( |\langle \psi^*, D_j^r g \rangle| \leq \omega^{k+r} \|\psi^*\| \|f\|_p \) we have
\[
(2.16) \quad |F(z)| \leq e^{|z| \omega^k} \|\psi^*\| \|f\|_p.
\]

For real \( t \) we also have \( |F(t)| \leq \omega^k \|\psi^*\| \|f\|_p \). Thus, we proved the if \( f \in B^\omega_p(\mathbb{D}), 1 \leq p \leq \infty \), then the function (2.13) is an entire function of the exponential type \( \omega \).

To prove the inverse statement let us note that the fact that \( f \) belongs to the space \( B^\omega_p(\mathbb{D}), 1 \leq p \leq \infty \), means in particular that for any \( 1 \leq j \leq d \) and any functional \( \psi^* \) on \( L_p(M), 1 \leq p \leq \infty \), the function \( F(z) = \langle \psi^*, e^{zD_j} f \rangle \) is an entire function of exponential type \( \omega \) which is bounded on the real axis \( \mathbb{R}^1 \). Since \( e^{tD_j} \) is a group of isometries in \( L_p(M) \), an application of the Bernstein inequality for functions of one variable gives
\[
\|\langle \psi^*, e^{tD_j} D_j^r f \rangle\|_{C(\mathbb{R}^1)} = \left\| \left( \frac{d}{dt} \right)^m \langle \psi^*, e^{tD_j} f \rangle \right\|_{C(\mathbb{R}^1)} \leq \omega^m \|\psi^*\| \|f\|_p, m \in \mathbb{N}.
\]
The last one gives for $t = 0$

$$\left| \langle \psi^*, D_j^m f \rangle \right| \leq \omega^m \|\psi^*\| \|f\|_p.$$  

Choosing $h$ such that $\|\psi^*\| = 1$ and

(2.17) $$\langle \psi^*, D_j^m f \rangle = \|D_j^m f\|_p$$

we obtain the inequality

(2.18) $$\|D_j^m f\|_p \leq \omega^m \|f\|_p, m \in \mathbb{N}.$$  

It was the first step of induction. Now assume that we already proved that the fact that $f$ belongs to the space $\mathbf{B}^p_\omega(\mathbb{D}), 1 \leq p \leq \infty$, implies the inequality

$$\|D_{i_1}...D_{i_k} f\|_p \leq \omega^k \|f\|_p$$

for any choice of indices $1 \leq i_1, i_2, ..., i_k \leq d$. Then we can apply our first step of induction to the function $g = D_{i_1}...D_{i_k}$. It proves if for any $1 \leq i_1, ..., i_k \leq d$, any $1 \leq j \leq d$, and any functional $\psi^* \in L_p(M)^*$ the function

(2.19) $$\langle \psi^*, e^{tD_j} D_{i_1}...D_{i_k} f \rangle : \mathbb{R} \to \mathbb{R},$$

of the real variable $t$ is an entire function of the exponential type $\omega$ then $f \in \mathbf{B}^p_\omega(\mathbb{D}), 1 \leq p \leq \infty$. Thus the second part of the Theorem 2.2 is proved.

In order to prove the part 3 of the Theorem 2.2 we assume that a sequence $f_k \in \mathbf{B}^p_\omega(\mathbb{D}), 1 \leq p \leq \infty$, converges in $L_p(M)$ to a function $f$. Because of the Bernstein inequality for any $1 \leq j \leq d$ the sequence $D_j f_k$ will be fundamental in $L_p(M)$. Note, that since the operator $D_j$ is a generator of a strongly continuous group of operators in the space $L_p(M)$ it is closed ([9], Ch. 1). It shows that the limit of the sequence $D_j f_k$ is the function $D_j f$ and because of it the following inequality holds

$$\|D_j f\|_p \leq \omega \|f\|_p.$$  

By repeating these arguments we can show that if a sequence $f_k \in \mathbf{B}^p_\omega(\mathbb{D})$ converges in $L_p(M)$ to a function $f$ then the Bernstein inequality

$$\|D_{i_1}...D_{i_k} f\|_p \leq \omega^k \|f\|_p, 1 \leq p \leq \infty,$$

for $f$ holds true. The Theorem 2.2 is proved.

Consider a compact symmetric space $M = G/K$, where $G$ is a compact Lie group. It is known ([32], Ch. IV) that every compact Lie group can be considered as a closed subgroup of the orthogonal group $O(\mathbb{R}^N)$ of a certain Euclidean space $\mathbb{R}^N$. This fact allows to identify $M$ with an orbit of a unit vector $v \in \mathbb{R}^N$ under action of a subgroup of the orthogonal group $O(\mathbb{R}^N)$ in $\mathbb{R}^N$. In this case $K$ will be the stationary group of $v$. Such embedding of $M$ into $\mathbb{R}^N$ is called equivariant.

We choose an orthonormal basis in $\mathbb{R}^N$ for which the first vector is the vector $v$.

$e_1 = v, e_2, ..., e_N$. Let $\mathbf{P}_n(M)$ be the space of restrictions to $M$ of all polynomials in $\mathbb{R}^N$ of degree $n$. This space is closed in the norm of $L_p(M), 1 \leq p \leq \infty$, which is constructed with respect to the $G$-invariant measure on $M$.

Let $T$ be the quasi-regular representation of $G$ in the space $L_p(M), 1 \leq p \leq \infty$. In other words, if $f \in L_p(M), g \in G, x \in M$, then

$$(T(g)f)(x) = f(g^{-1}x).$$
Lie algebra $\mathfrak{g}$ of the group $G$ is formed by $N \times N$ skew-symmetric matrices $X$ for which $\exp tX \in G$ for all $t \in \mathbb{R}$. The scalar product in $\mathfrak{g}$ is given by the formula

$$<X_1, X_2> = \frac{1}{2} tr(X_1 X_2^t) = -\frac{1}{2} tr(X_1 X_2), X_1, X_2 \in \mathfrak{g}.$$ 

Let $X_1, X_2, ..., X_d$ be an orthonormal basis of $\mathfrak{g}$, $\dim \mathfrak{g} = d$, and $D_1, D_2, ..., D_d$ be the corresponding infinitesimal operators of the quasi-regular representation of $G$ in $L_p(M), 1 \leq p \leq \infty$.

**Theorem 2.3.** If $M$ is equivariantly embedded into $\mathbb{R}^N$ then for any $1 \leq p \leq \infty$ the following inclusion holds true

$$(2.20)\quad P_n(M) \subset \mathcal{B}_{\zeta}^p(\mathbb{D}).$$

**Proof.** The general theory of skew-symmetric matrices (see [15], Ch. 3) implies that for any skew symmetric matrix $X$ each element of the matrix $\exp tX$ is a linear combination of the functions $\cos t\theta, \sin t\theta$ for certain real numbers $\theta_1, ..., \theta_{[n/2]}$, for which

$$\|X\|^2 = \sum_{i=1}^{[n/2]} \theta_i^2.$$ 

It shows that for any point $x \in M$ and any basis vector $e_i, 1 \leq i \leq N$, in $\mathbb{R}^N$ the coordinate function

$$x_i(\exp tX \cdot x) = \langle \exp tX \cdot x, e_i \rangle$$

is also a linear combination of $\cos t\theta, \sin t\theta$ whose coefficients are smooth functions of $x$. Thus we conclude that every function $x_i(\exp tX \cdot x)$ has extension to the complex plane $\mathbb{C}$ as entire function of exponential type $\leq \max|\theta_i| \leq \|X\|$.

Since for every $f \in P_n(M)$ and every basis vector $X_j, j = 1, ..., d; \|X_j\| = 1$, the function $f(\exp tX_j \cdot x), x \in M$, is a polynomial of degree $n$ in $x_i(\exp tX_j \cdot x)$ we obtain that for any functional $\psi^*$ on $L_p(M), 1 \leq p \leq \infty$, the function

$$F_{X_j}(t) = \langle f(\exp tX_j \cdot x), \psi^* \rangle, x \in M, \|X_j\| = 1,$$

has exponential type at most $n$ and according to the classical Riesz interpolation formula it gives

$$(2.21)\quad \frac{dF_{X_j}(t)}{dt} = \frac{n}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{(k-1/2)^2} F_{X_j} \left( t + \frac{\pi}{n} (k - 1/2) \right), t \in \mathbb{R}.$$ 

Since

$$\frac{dF_{X_j}(t)}{dt} = \left\langle \frac{d}{dt} f(\exp tX_j \cdot x), \psi^* \right\rangle = \left\langle D_j f(\exp tX_j \cdot x), \psi^* \right\rangle$$

we have

$$\left\langle D_j f(\exp tX_j \cdot x), \psi^* \right\rangle = \frac{n}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{(k-1/2)^2} \left\langle f \left( \exp \left( t + \frac{\pi}{n} (k - 1/2) X_j \cdot x \right) \right), \psi^* \right\rangle,$$

where $t \in \mathbb{R}, x \in M$. For $t = 0$ it gives

$$\left\langle D_j f(x), \psi^* \right\rangle = \frac{n}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{(k-1/2)^2} \left\langle f \left( \exp \left( \frac{\pi}{n} (k - 1/2) X_j \cdot x \right) \right), \psi^* \right\rangle.$$
Because this formula holds true for any functional \( \psi^* \) it implies
\[
D_j f(x) = \frac{n}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{(k-1/2)^2} f \left( \exp \left( \frac{\pi}{n} (k-1/2) X_j \cdot x \right) \right), \quad x \in M.
\]
Since
\[
e^{tD_X} f(x) = f(\exp tX_j \cdot x), \quad t \in \mathbb{R}, f \in L_p(M), x \in M,
\]
it gives
\[
D_j f(x) = \frac{n}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{(k-1/2)^2} e^{t_k D_j} f(x), \quad t_k = \frac{\pi}{n} (k-1/2), x \in M.
\]
Because
\[
\frac{n}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{(k-1/2)^2} = n,
\]
and because \( \| e^{t_k D_j} f \|_p = \| f \|_p \), we obtain the Bernstein inequality
\[
(2.22) \quad \| D_j f \|_p \leq n \| f \|_p, 1 \leq p < \infty.
\]
Since the set of polynomials \( P_n(M) \) is invariant under translations and closed \([29]\) it is invariant under all operators \( D_1, D_2, ..., D_d \), \( d = dimG \). It is clear that by using invariance of \( P_n(M) \) and the inequality \((2.22)\) we obtain the desired inequality
\[
\| D_{j_1} ... D_{j_k} f \|_p \leq n^k \| f \|_p, k \in \mathbb{N}.
\]
The Theorem is proved. \( \square \)

As a consequence of this Theorem and the Lemma 2.1 we obtain the following Corollary.

**Corollary 2.2.** If \( M \) is equivariantly embedded into \( \mathbb{R}^N \) then for any polynomial \( f \in P_n(M) \) the following Riesz interpolation formula holds
\[
D_{j_1} D_{j_2} ... D_{j_k} f(x) = R_{j_1}^n R_{j_2}^n ... R_{j_k}^n f(x),
\]
where
\[
R_j^n f(x) = \frac{n}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{(k-1/2)^2} e^{t_k D_j} f(x), \quad t_k = \frac{\pi}{n} (k-1/2), \quad x \in M, 1 \leq j \leq d.
\]

The next interpolation inequality for generators of one-parameter strongly continuous groups of operators in Banach spaces will be used in the following section.

**Lemma 2.4.** If \( A \) generates a \( C_0 \)-one-parameter group of operators \( e^{tA} \) such that \( \| e^{tA} f \| = \| f \| \) then for every \( n \geq 2 \) there exists a \( C(n) \) such that for all \( \varepsilon > 0 \) all \( 1 \leq m \leq n-1 \) and all \( f \) in the domain of \( A^n \)
\[
(2.23) \quad \| A^m f \| \leq \varepsilon^{n-m} \| A^n f \| + \varepsilon^{-m} C(n) \| f \|.
\]

**Proof.** According to the Hille-Phillips-Yosida theorem \([9], \text{Ch. I}\), the assumptions of the Lemma imply
\[
\| (I + \varepsilon A)^{-1} \| \leq 1
\]
and the same for the operator \((I - \varepsilon A)\). Then
\[
\| f \| \leq \| (I + \varepsilon A) f \|
\]
and the same for the operator \((I - \varepsilon A)\). It gives
The geodesic starting at $x$ and $(3.3)$ \[ \|D\| \sum_{\|D\|} \|D\| \leq \|A^2f\| + 2\|f\|. \]

So, for any $f$ from the domain of $A^2$ we have inequality

\[ \|Af\| \leq \varepsilon\|A^2f\| + 2/\varepsilon\|f\|, \varepsilon > 0. \]

The general case can be proved by induction. □

3. Two Nikolskii-type inequalities on compact homogeneous manifolds

Let $M = G/K$, $\dim M = n, \dim G = d$, be a homogeneous manifold which we consider with invariant Riemannian metric and corresponding Riemannian measure $dx$. The $B(x, \rho)$ will denote a ball whose center is $x \in M$ and radius is $\rho > 0$. Denote by $T_x(M)$ the tangent space of $M$ at a point $x \in M$ and let $\exp_x : T_x(M) \to M$ be the exponential geodesic map i. e. $\exp_x(u) = \gamma(1), u \in T_x(M)$ where $\gamma(t)$ is the geodesic starting at $x$ with the initial vector $u : \gamma(0) = x, \frac{d\gamma(0)}{dt} = u$. Since the manifold $M$ is compact there exists a positive $\rho_M$ such that the exponential map is a diffeomorphism of a ball of radius $\rho < \rho_M$ in the tangent space $T_x(M)$ onto the ball $B(x, \rho)$ for every $x \in M$. We consider only coordinate systems on $M$ which are given by the exponential map.

We fix a cover $B = \{B(y_\nu, r_0)\}$ of $M$ of finite multiplicity $N(M)$ (see Lemma 1.1)

\[ M = \bigcup B(y_\nu, r_0), \]

where $B(y_\nu, r_0)$ is a ball at $y_\nu \in M$ of radius $r_0 \leq \rho_M$, and consider a fixed partition of unity $\Psi = \{\psi_\nu\}$ subordinate to this cover. The Sobolev spaces $W_p^k(M), k \in \mathbb{N}, 1 \leq p < \infty$, are introduced as the completion of $C_0^\infty(M)$ with respect to the norm

\[ \|f\|_{W_p^k(M)} = \left( \sum_{\nu} \|\psi_\nu f\|_{W_p^k(B(y_\nu, r_0))}^p \right)^{1/p}. \]

Any two such norms are equivalent. We consider the system of vector fields $\mathbb{D} = \{X_1, ..., X_d\}, d = \dim G$, on $M = G/K$, which was described in the Introduction. Since vector fields $\mathbb{D} = \{X_1, ..., X_d\}$ generate the tangent space at every point of $M$ and $M$ is compact it is clear that the Sobolev norm (3.2) is equivalent to the norm

\[ \|f\|_p + \sum_{j=1}^{k} \sum_{\|D_{ij}\|, \|D_{ij}\|} \|D_{ij}f\|_p, 1 \leq p \leq \infty. \]

Using the closed graph Theorem and the fact that every $D_i$ is a closed operator in $L_p(M), 1 \leq p < \infty$, it is easy to show that the norm (3.3) is equivalent to the norm

\[ \|f\|_p + \sum_{\|D_{ij}\|, \|D_{ij}\|} \|D_{ij}f\|_p, 1 \leq p \leq \infty. \]

In other words, there exist constants $c_0(\mathbb{D}, B, \Psi, k), C_0(\mathbb{D}, B, \Psi, k)$ such that

\[ c_0(\mathbb{D}, B, \Psi, k) \|f\|_{W_p^k(M)} \leq \|f\|_{k,p} \leq C_0(\mathbb{D}, B, \Psi, k) \|f\|_{W_p^k(M)}. \]

Since the Laplace operator $\mathcal{L}$ which is defined in (1.7) is an elliptic operator on a compact manifold $M$ the regularity theorem for $\mathcal{L}$ means in particular (30), Ch.
I and Ch. III), that the norm of the Sobolev space $W^k_p(M), k \in \mathbb{N}, 1 \leq p < \infty$, is equivalent to the graph norm $\|f\|_p + \|\mathcal{L}^k f\|_p$. Thus, there exist two constants $c_1(\mathcal{L}, B, \Psi, k), C_1(\mathcal{L}, B, \Psi, k)$ such that

$$c_1(\mathcal{L}, B, \Psi, k) \|f\|_{W^k_p(M)} \leq \|f\|_p + \|\mathcal{L}^k f\|_p \leq C_1(\mathcal{L}, B, \Psi, k) \|f\|_{W^k_p(M)}.$$  

(3.6)  

In what follows $o \in M$ will denote the "origin" of the homogeneous manifold $M$ which corresponds to the coset defined by the subgroup $K$ in the representation $M = G/K$. Note that since $G$ acts on $M$, every function on $M$ can be treated as a function on $G$ according to the formula

$$f(g) = f(g \cdot o), g \in G.$$  

**Theorem 3.1.** For any $1 \leq p \leq \infty$, any natural $l > m/p, m = \dim M$, there exists a constant $C(M, l)$ such that for any $(r, N(G))$-lattice $Z_G(y_\nu, r, N(G)) \subset G$ with sufficiently small $r > 0$, any $\omega > 0$ and any $q \geq p$ the following inequalities hold

$$\|f\|_q \leq C(M, l)^{m/q - m/p} \left(1 + (r\omega)^l\right) \|f\|_p,$$

for all $f \in \mathcal{B}^\infty_\ell(\mathbb{D})$.

**Proof.** Our nearest goal is to prove the right-hand side of the inequality (6.1). We fix a sufficiently small ball $B(o, r), 0 < r < r_0$, in the tangent space $T_o M$ at the origin $o \in M$ and an $(r, N(M))$-lattice $\{g_i\} = Z_G(r, N(M)) \subset G$, such that translations $g_i \cdot B(o, r) = B(x_i, r), x_i = g_i \cdot o$, of the ball $B(o, r)$ are disjoint. We are going to use the following form of the Sobolev inequality (see [1], Lemma 5.15, or [22], Corollary 3.5.12)

$$|\phi(x)| \leq C_1(m, l) \sum_{0 \leq j \leq l} r^{j-m/p} \|\phi\|_{W^j_p(B(x_i, r))}, l > m/p,$$

(3.8)  

where $x \in B(x_i, r/2), \phi \in C^\infty(B(x_i, r))$.

We apply the inequality (3.8) to a function $\psi_\nu f$ from the formula (3.2) which gives the Sobolev norm. Since $\psi_\nu$ is a partition of unity and since $N(M)$ is a number of balls in the cover $B(y_\nu, r_0)$ which intersect each other we obtain for $x_i \in B(y_\nu, r_0)$

$$r^m |f(x)|^p = r^m \left|\sum_\nu \psi_\nu f(x_i)\right|^p \leq$$

(3.9)  

$$(N(M))^p \sum_\nu \left(r^m |\psi_\nu f(x_i)|\right)^p \leq C_1(m, l)(N(M))^p \sum_\nu \sum_{0 \leq j \leq l} r^{jp} \|\psi_\nu f\|^p_{W^j_p(B(x_i, r))},$$

where $l > m/p, 1 \leq p \leq \infty$. Summation over $i$ gives the inequality

$$\sum_i \left(r^m |f(x_i)|\right)^p \leq C_1(m, l)(N(M))^p \sum_\nu \sum_{0 \leq j \leq l} r^{jp} \sum_i \|\psi_\nu f\|^p_{W^j_p(B(x_i, r))}, l > m/p.$$
Since the balls $B(x_i, r)$ are disjoint and the support of $\psi$ is a subset of $B(y, r_0)$, we obviously have

$$\sum_i \|\psi f\|_{W^l_p(B(x_i, r))}^p \leq \|\psi f\|_{W^l_p(B(y, r_0))}^p, 0 \leq j \leq l.$$  

(3.10)

Thus we obtain that for any given $l > m/p$ there exists a constant $C_5(M, l) > 0$, such that for any $(r, N(M))$-lattice $\{g_i\} = Z_G(r, N(M)) \subset G$, the following inequality holds true for $1 \leq p \leq \infty$

$$ \left( \sum_i \left( \frac{r^{m/p}}{r} \|f(x_i)\| \right)^p \right)^{1/p} \leq C_5(M, l) N(M) \left( \|f\|_p + \sum_{j=1}^l r^{j/p} \left( \sum_{\nu} \|\psi f\|_{W^{l/p}_p(B(y, r_0))}^p \right)^{1/p} \right).$$

Since the norms $(3.2)$ and $(3.4)$ are equivalent we obtain the inequality

$$ \left( \sum_i \left( \frac{r^{m/p}}{r} \|f(x_i)\| \right)^p \right)^{1/p} \leq C_6 \left( \|f\|_p + \sum_{j=1}^l \sum_{0 \leq k_1, \ldots, k_j \leq d} r^j \|D_{k_1} \cdots D_{k_j} f\|_p \right), l > m/p,$$

(3.11)

where $C_6 = C_6(M, \mathbb{D}, B, \Psi, l, N(M))$. Because every $D_{k}, k = 1, \ldots, d$, is a generator of a one-parameter isometric group of bounded operators in $L_p(M)$, the interpolation inequality $(2.23)$ can be used and then the last two inequalities imply the following estimate

$$ \left( \sum_i \left( \frac{r^{m/p}}{r} \|f(x_i)\| \right)^p \right)^{1/p} \leq C_7 \left( \|f\|_p + r^l \sum_{0 \leq k_1, \ldots, k_l \leq d} \|D_{k_1} \cdots D_{k_l} f\|_p \right), l > m/p,$$

where $C_7 = C_7(M, \mathbb{D}, B, \Psi, l, N(M))$. For $f \in B^\alpha_p(\mathbb{D})$ it gives

$$ \left( \sum_i \left( \frac{r^{m/p}}{r} \|f(x_i)\| \right)^p \right)^{1/p} \leq C_8 \left( 1 + (r\omega)^l \right) \|f\|_p, l > m/p,$$

where $C_8 = C_8(M, \mathbb{D}, B, \Psi, l, N(M))$. Applying this inequality to a translated function $f(h \cdot x), h \in G$, and using invariance of the measure $dx$ we obtain for $f \in B^\alpha_p(\mathbb{D})$

$$\sup_{h \in G} \left( \sum_i \left( \frac{r^{m/p}}{r} \|f(h \cdot x_i)\| \right)^p \right)^{1/p} \leq C_8 \left( 1 + (r\omega)^l \right) \|f\|_p, l > m/p.$$  

(3.12)

This inequality implies the right-hand side of the inequality $(3.1)$. 

(3.1)
To prove the left-hand side of the (3.1) we introduce the following neighborhood of the identity in the group $G$

$$Q_{4r} = \{ g \in G : g \cdot o \in B(o, 4r) \}.$$ 

According to the following formula which holds true for any continuous function $f$ on $M$

$$\int_M f(x)dx = \int_G f(g \cdot o)dg,$$

we have the following estimate for the characteristic function $\chi_B$ of the ball $B(o, 4r)$

$$(4r)^m \approx \int_{B(o, 4r)} dx = \int_M \chi_B(x)dx = \int_G \chi_B(g \cdot o)dg = \int_{Q_{4r}} dg.$$ 

Since every ball in our cover is a translation of the fixed ball $B(o, 4r)$ and these balls form a cover of $M$ the $G$-invariance of the measure $dx$ gives

$$\int_M |f(x)|^q dx \leq \sum_{g_i \in ZG(r, N(G))} \int_{g_i B(o, 4r)} |f(x)|^q dx \leq \sum_{g_i \in ZG(r, N(G))} \int_{B(o, 4r)} |f(g \cdot y)|^q dy =$$

$$\int_{Q_{4r}} \sum_{g_i \in ZG(r, N(G))} |f(g_i h \cdot o)|^q dh \leq (4r)^m \sup_{g \in G} \sum_{g_i \in ZG(r, N(G))} |f(g_i g \cdot o)|^q,$$

where $f \in L_q(M), 1 \leq q \leq \infty$, $m = \dim M$. Next, using the inequality

$$\left( \sum a_i^q \right)^{1/q} \leq \left( \sum a_i^p \right)^{1/p},$$

which holds true for any $a_i \geq 0, 1 \leq p \leq q \leq \infty$, we obtain the following inequality

$$\|f\|_q \leq 4^m r^{m/q} \sup_{g \in G} \left( \sum_{g_i \in ZG(r, N(G))} \langle |f(g_i g \cdot o)|^q \rangle \right)^{1/q} \leq$$

$$4^m r^{m/q} \sup_{g \in G} \left( \sum_{g_i \in ZG(r, N(G))} \langle |f(g_i g \cdot o)|^p \rangle \right)^{1/p} =$$

$$(3.13) \quad 4^m r^{m/q - m/p} \sup_{g \in G} \left( \sum_{g_i \in ZG(r, N(G))} \left( r^{m/p} |f(g_i g \cdot o)| \right)^p \right)^{1/p}.$$ 

From these inequalities and the observation, that for the element $g = g_i^{-1} h g_i$ the expression

$$\sum_{g_i \in ZG(r, N(G))} \left( r^{m/p} |f(g_i g \cdot o)| \right)^p$$

becomes the expression

$$\sum_{g_i \in ZG(r, N(G))} \left( r^{m/p} |f(h g_i \cdot o)| \right)^p,$$

we obtain the left-hand side of the inequality (3.1). The Theorem 3.1 is proved.

□
This Theorem is used to prove the following result.

**Theorem 3.2.** There exists a constant $C(M)$ such that for any $1 \leq p \leq q \leq \infty$ the following inequality holds true for all $f \in B_p^\infty(\mathbb{D})$

\[ \|f\|_q \leq C(M)\omega^{\frac{m}{q} - \frac{m}{p}} \|f\|_p, \quad m = \text{dim}\mathbb{D}. \]

**Proof.** The Theorem 3.1 imply that for any $1 \leq p \leq \infty$, any natural $l > m/p$ there exists a constant $C(M,l) > 0$ such that for any sufficiently small $r > 0$, any $\omega > 0$ and any $q \geq p$ the following inequality holds true

\[ \|f\|_q \leq C(M,l)\omega^{m/q - m/p}(1 + (r\omega)^l)\|f\|_p, \]

for all $f \in B_p^\infty(\mathbb{D})$. We make the substitution $t = r\omega$ into this inequality to obtain

\[ \|f\|_q \leq C(M,l)\eta_{p,q}(t)\omega^{m/p - m/q}\|f\|_p, \quad l > m/p, \]

where

\[ \eta_{p,q}(t) = t^{m/q - m/p}(1 + t^l), \quad t \in (0, \infty), \quad t = r\omega. \]

Since $l$ can be any number greater than $m/p$ and $p \geq 1$, we fix the number $l = 2m$. At the point

\[ t_{m,p,q} = \frac{\alpha}{2m - \alpha} \in (0, 1), \]

where $0 < \alpha = m/p - m/q < 1$, the function $\eta_{p,q}$ has its minimum, which is

\[ \eta_{p,q}(t_{m,p,q}) = \frac{1}{(1 - \beta)^{1 - \beta}} \leq 2, \]

where $\beta = \alpha/2m$. For a given $m \in \mathbb{N}, \omega > 0, 1 \leq p \leq q \leq \infty$, we can find corresponding $t_{m,p,q}$ using the formula \((3.15)\) and then can find the corresponding $r > 0$ as $r = r_{m,p,q,\omega} = t_{m,p,q}/\omega$. For such $r$ one can find a cover of the same multiplicity $N(M)$. For this cover we will have the inequality \((3.14)\). The Theorem is proved.

\[ \square \]

**Theorem 3.3.** For any $1 \leq p \leq q \leq \infty$ the following equality holds true

\[ B_p^\infty(\mathbb{D}) = B_p^\infty(\mathbb{D}) \equiv B_\omega(\mathbb{D}). \]

**Proof.** First we show that

\[ B_p^\infty(\mathbb{D}) \subset B_p^\infty(\mathbb{D}), 1 \leq p \leq q \leq \infty. \]

Since $B_p^\infty(\mathbb{D})$ is invariant under every operator $D_i, 1 \leq i \leq d$, it is enough to show that if $f \in B_p^\infty(\mathbb{D})$, then for any $1 \leq j \leq d, k \in \mathbb{N}$,

\[ \|D_j^k f\|_q \leq \omega^k\|f\|_q, 1 \leq p \leq q \leq \infty. \]

Because $f \in B_p^\infty(\mathbb{D})$ and this set is invariant under all operators $D_i$, the Theorem 3.1 gives that there exists a constant $C_{p,q}$ such that for any $z \in \mathbb{C}$

\[ \|e^{zD} f\|_q = \left\| \sum_{l=0}^{\infty} \frac{(z^l D^l f)}{l!} \right\|_q \leq C_{p,q}e^{\|z\|_q}\|f\|_p. \]

It implies that for any functional $\psi^*$ on $L_q(M), 1 \leq q \leq \infty$, the scalar function

\[ F(z) = \langle e^{zD} f, \psi^* \rangle, \]
is an entire function of exponential type $\omega$. At the same time it is bounded on the real axis $\mathbb{R}$ by the constant $\|\psi^*\| f$. The classical Bernstein inequality gives

$$\sup_{t \in \mathbb{R}} |\langle e^{tD_j} D_j^k f, \psi \rangle| = \sup_{t \in \mathbb{R}} \left| \frac{d}{dt} \right|^k \langle e^{tD_j} f, \psi \rangle \leq \omega^k \|\psi^*\| f, m \in \mathbb{N}.$$ 

When $t = 0$ we obtain

$$|\langle D_j^k f, \psi \rangle| \leq \omega^k \|\psi^*\| f.$$ 

Choosing $\psi^*$ such that $\|\psi^*\| = 1$ and $\langle D_j^k f, \psi \rangle = \|D_j^k f\|$ we get the inequality

$$\|D_j^k f\| \leq \omega^k \|f\|, k \in \mathbb{N}.$$ 

To prove an embedding which is opposite to (3.16) we use the fact that $M$ is compact and because of this the $L^\infty(M)$-norm dominates any $L^p(M)$-norm with $1 \leq p < \infty$. It gives the following inequality for any $f \in B^\omega(\mathbb{D}), 1 \leq p \leq \infty$

$$\|e^{zD_j} f\|_p = \left\| \sum_{l=0}^{\infty} (z^l D_j^l f) / l! \right\|_p \leq e^{\|z\| f\|_\infty},$$

which implies that for any functional $\psi$ on $L^p(M), 1 \leq p \leq \infty$, the scalar function

$$F(z) = \langle e^{zD_j} f, \psi \rangle,$$

is an entire function of exponential type $\omega$ which is bounded on the real axis $\mathbb{R}$ by the constant $\|\psi^*\| f$. At this point we can use the same arguments which were used above. The Theorem is proved.

\[\square\]

4. RELATIONS BETWEEN $B_\omega(\mathbb{D}), E_\omega(\mathcal{L})$ AND $P_n(M)$

We keep the same notations as above.

**Theorem 4.1.** The following equality takes place

$$\|\mathcal{L}^{k/2} f\|_2^2 = \sum_{1 \leq i_1, \ldots, i_k \leq d} \|D_{i_1} \cdots D_{i_k} f\|_2^2,$$

which implies the following embeddings

$$B_{\sqrt{\omega/d}}(\mathbb{D}) \subset E_\omega(\mathcal{L}) = B_\omega^2(\mathcal{L}) \subset B_{\sqrt{\pi}}(\mathbb{D}),$$

and in particular the following equality

$$\bigcup_{\omega} B_\omega(\mathbb{D}) = \bigcup_{j} E_{\lambda_j}(\mathcal{L}).$$

**Proof.** Since the spectrum of $\mathcal{L}$ is discrete and of finite multiplicity, the space $E_\omega(\mathcal{L})$ is finite dimensional and the norm of $\mathcal{L}$ on this space is exactly $\omega$. It gives the embedding

$$E_\omega(\mathcal{L}) \subset B_\omega^2(\mathcal{L}).$$
Conversely, let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ...$ be the set of eigenvalues of $L$ listed with multiplicities and $\varphi_0, \varphi_1, \varphi_2, ...$ be a corresponding complete system of orthonormal eigenfunctions. Assume that

$$\lambda_m \leq \omega < \lambda_{m+1}.$$  

If a function $f$ belongs to the space $B_\omega^2(L)$ and the Fourier series

$$f = \sum_{j=0}^{\infty} c_j \varphi_j$$  

contains terms with $j \geq \lambda_{m+1}$, then

$$\lambda_{m+1}^{2k} \sum_{j=m+1}^{\infty} |c_j|^2 \leq \sum_{j=m+1}^{\infty} |\lambda_j c_j|^2 \leq \|L^k f\|^2 \leq \omega^{2k} \|f\|^2,$$

which implies

$$\sum_{j=m+1}^{\infty} |c_j|^2 \leq \left(\frac{\omega}{\lambda_{m+1}}\right)^{2k} \|f\|^2.$$  

In the last inequality the fraction $\omega/\lambda_{m+1}$ is strictly less than 1 and $k$ can be any natural number. It shows that the series (4.3) does not contain terms with $j \geq m + 1$, i.e. function $f$ belongs to $E_\omega(L)$. We proved the inclusion

$$B_\omega^2(L) \subset E_\omega(L),$$

which gives along with (4.2) the equality

$$B_\omega^2(L) = E_\omega(L).$$

The operator

$$-L = D_1^2 + ... + D_d^2$$

commutes with every $D_j$ (see the explanation before the formula (1.7) in the Introduction). The same is true for $L^{1/2}$. But then

$$\|L^{1/2} f\|_2^2 = \langle L^{1/2} f, L^{1/2} f \rangle = \langle L f, f \rangle = \sum_{j=1}^{d} \langle D_j^2 f, f \rangle = \sum_{j=1}^{d} \|D_j f\|_2^2,$$

$$\|L f\|_2^2 = \|L^{1/2} L^{1/2} f\|_2^2 = \sum_{j=1}^{d} \|D_j L^{1/2} f\|_2^2 = \sum_{j=1}^{d} \|L^{1/2} D_j f\|_2^2 = \sum_{j,k=1}^{d} \|D_j D_k f\|_2^2.$$

From here by induction on $k$ we obtain (4.1). It proves the formula (4.1), which implies the rest of the Theorem. Indeed, if $f \in B_\omega^2(L)$ we obtain that $f \in B_\omega^2(\mathbb{D}) = B_\omega(\mathbb{D})$ because

$$\|D_{i_1}...D_{i_k} f\|_2 \leq \left( \sum_{1 \leq i_1, ..., i_k \leq d} \|D_{i_1}...D_{i_k} f\|_2^2 \right)^{1/2} = \|L^{1/2} f\|_2 \leq \omega^k \|f\|_2.$$  

Thus

$$B_\omega^2(L) \subset B_\omega^2(\mathbb{D}) = B_\omega(\mathbb{D}).$$
On the other hand, if \( f \) belongs to \( B_{\omega/\sqrt{d}}(\mathbb{D}) = B_{\omega/\sqrt{d}}(\mathbb{B}) \) then
\[
\|L^{k/2}f\|_2 = \left( \sum_{1 \leq i_1, \ldots, i_k \leq d} \|D_{i_1} \ldots D_{i_k}f\|_2^2 \right)^{1/2} \leq \omega^k \|f\|_2^2,
\]
which together with (4.3) gives the embedding \( B_{\omega/\sqrt{d}}(\mathbb{D}) \subset E_\omega(L) = B_2^2(L) \).

**Theorem 4.2.** If \( M \) is equivariantly embedded into \( \mathbb{R}^N \) then the following equality holds true
\[
\bigcup_n P_n(M) = \bigcup_\omega B_\omega(\mathbb{D}).
\]

**Proof.** Note that since \( \mathcal{L} \) commutes with all operators of the form \( D_X \) where \( X \) is any invariant vector field on \( M \) (see the explanation before the formula (1.7) in the Introduction) it commutes with the action of \( G \) in the space \( L_2(M) \). Indeed if \( g \) is an element of \( G \) then the action
\[
x \mapsto g \cdot x, x \in M,
\]
is the same as a translation along integral curve \( \exp tx, t \in \mathbb{R} \), for an appropriate invariant vector field on \( M \). (22), Ch. XV, Theorem 8). The corresponding action of \( G \) in the space \( L_2(M) \) is given by the formula (see the Introduction)
\[
e^{tD_X} f(x) = f(\exp tX \cdot x), t \in \mathbb{R}, x \in M, f \in C^\infty(M).
\]
Thus, we have
\[
\mathcal{L}e^{tD_X}f = \mathcal{L} \sum \frac{(tD_X)^k f}{k!} = \sum \frac{(tD_X)^k \mathcal{L}f}{k!} = e^{tD_X}\mathcal{L}f.
\]
It shows that if \( \varphi \) is an eigenfunction with eigenvalue \( \lambda \) then the same is true for \( e^{tD_X} \varphi \) because
\[
(4.5) \quad \mathcal{L}(e^{tD_X} \varphi) = e^{tD_X} \mathcal{L} \varphi = \lambda (e^{tD_X} \varphi).
\]
It implies that all eigen spaces \( E_\omega(L) \) are invariant under action of \( G \) in the space \( L_2(M) \).

We are going to show that if \( f \in B_p^\omega(\mathbb{D}) \) for a \( 1 \leq p \leq \infty, \omega > 0 \), then \( f \) is a polynomial on \( M \). Since \( B_p^\omega(\mathbb{D}) = B_2^2(\mathbb{D}) \), \( 1 \leq p \leq \infty, \omega > 0 \), we obtain the inequality
\[
\|L^m f\|_2 = \sum_{1 \leq i_1, \ldots, i_k \leq d} \|D_{i_1}^2 \ldots D_{i_k}^2 f\|_2 \leq (\omega^2d)^k \|f\|_2, m = \dim M,
\]
which shows that \( f \) belongs to the space \( B_{\omega/\sqrt{d}}^\omega(L) \). Since by (4.3) \( B_2^2(\mathbb{D}) \) we obtain that \( f \) belongs to \( E_{\omega/\sqrt{d}}(L) \).

The space \( E_{\omega/\sqrt{d}}(L) \) is finite dimensional and according to (4.3) is invariant under the action of \( G \) in \( L_2(M) \). This fact implies that all translates of every function \( f \in B_p^\omega(\mathbb{D}), 1 \leq p \leq \infty, \omega > 0 \), belong to a finite dimensional space \( E_{\omega/\sqrt{d}}(L) \). In the terminology of [17], [18] it means that every function \( f \in B_p^\omega(\mathbb{D}), 1 \leq p \leq \infty, \omega > 0 \), is a \( G \)-finite vector of the quasi-regular representation of \( G \) in \( L_2(M) \) and by a result of S. Helgason [17], §3, such functions are restrictions of polynomials. In other words we proved the embedding
\[
\bigcup_{\omega > 0} B_{\omega}(\mathbb{D}) \subset \bigcup_{n \in \mathbb{N}} P_n(M).
\]
Since the Theorem 2.3 implies the opposite embedding we obtain the desired result. 

Theorem is proved. □

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