Joint measurements and Svetlichny’s inequality

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Abstract
We prove that Svetlichny’s inequality can be derived from the existence of joint measurements and the principle of no-signaling. Then we show that, on the basis of the assumption of quantum measurement, it would imply the breach of causality if the magnitude of violation of Svetlichny’s inequality exceeds the quantum bound.

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1. Introduction

Heisenberg’s uncertainty principle [1] forbids that non-commuting observables can be simultaneously measured with arbitrary accuracy. The extreme opposite of Heisenberg’s uncertainty is Einstein’s reality [2], and Bell’s inequality [3] originally derived just from the assumption of Einstein’s reality and the no-signaling principle. However, quantum mechanics allows unsharp simultaneous measurements of two non-commuting observables [4–6]. This means that if we tolerate an increase in the variance of these two non-commuting observables, we can perform their joint measurement. Here, joint measurement is defined in such a way that one measurement on a single physical system simultaneously produces values for more than one observable. If these two observables mutually commute, it is obvious that their joint measurements can be accomplished with projective quantum measurements; if these two observables do not mutually commute, in order to carry out the joint measurement, one has to adopt positive operator valued measures (POVMs) [7–9]. Essentially, the existence of joint measurements of some observables demands that a joint probability distribution of the values of these observables exist. Back in 1982, Fine [10] pointed out that the existence of a joint probability distribution for the values of the observables which are involved in Bell’s inequality must result in the satisfaction of Bell’s inequality. Along this line of thought, Andersson et al [7, 8] replaced Einstein’s reality with the joint measurements and showed that from

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joint measurements and the no-signaling principle, Bell’s inequality can also be derived. This indicates that the results of joint measurements result in the satisfaction of Bell’s inequality. They also found that Bell’s inequality naturally provides a tight bound on the sharpness for the joint measurements, and this bound was first derived by Busch [11].

The purpose of this work is to introduce joint measurements into the study of genuine multipartite correlations that cannot be reduced to mixtures of states in which a smaller number of subsystems are entangled, and to investigate the connection between them. The violation of Svetlichny’s inequality (SI) [12] is a confirmation of genuine multipartite correlations, and the magnitude of violation of SI is usually regarded as a measure of it. We show that, if the no-signaling principle is available and joint measurements are made on one particle of a $N$-particle quantum system, then the $N$-particle SI can be derived. This means that the existence of joint measurements must produce the absence of genuine multipartite correlations. Finally, we give a discussion about the quantum bound on the violation of SI.

2. Derivation of SI

In this section, we only give the derivation of SI by assuming the existence of joint measurements and the no-signaling principle, and present how to achieve joint measurements in the next section.

Derivation of three-particle SI. Three-particle SI [12] can distinguish between genuine three-particle correlations and two-particle correlations. This means that one can find a violation of SI inequality if and only if there exist genuine three-particle correlations in a three-particle setting. Consider three observers, Alice, Bob and Carol, who share three entangled qubits. Each of them can choose to measure one of two dichotomous observables. We denote $A$ and $A'$ as Alice’s measurement results when she performs the measurements $a$ and $a'$, respectively, and similarly $B$, $B'$, $b$ and $b'$ ($C$, $C'$, $c$ and $c'$) for Bob’s (Carol’s), and the measurement results of all observables can be $-1$ or $+1$. Then SI is expressed as [12]

$$S_3 \equiv |E(ABC) + E(ABC') + E(A'BC) - E(AB'C') - E(A'BC') - E(A'B'C') - E(A'B'C')| \leq 4,$$

(1)

where $S_3$ is the three-particle Svetlichny operator and $E(ABC)$ represents the expectation value of the product of the measurement outcomes of the observables $a$, $b$ and $c$. It was shown by Svetlichny [12] that quantum predictions can violate his inequality, and the maximum violation ($S = 4\sqrt{2}$) allowed in quantum mechanics can be achieved with GHZ states [13].

Now we assume that Alice performs a joint measurement of $a$ and $a'$, and denote the measurement results as $A_J$ and $A'_J$. We can obtain

$$E(A_J BC) + E(A'_J BC)$$

$$= P(A_J = A'_J = BC) + P(A_J = -A'_J = BC)$$

$$- P(A_J = A'_J = -BC) - P(A_J = -A'_J = -BC)$$

$$+ P(A'_J = A_J = BC) + P(A'_J = -A_J = BC)$$

$$- P(A'_J = A_J = -BC) - P(A'_J = -A_J = -BC)$$

$$= 2[P(A_J = A'_J = BC) - P(A'_J = A_J = -BC)]$$

$$\leq 2[P(A_J = A'_J = BC) + P(A'_J = A_J = -BC)]$$

$$= 2P(A_J = A'_J; bc),$$

(2)
where $P$ is the probability function and $P(A_J = A'_J; bc)$ represents the probability that Alice obtains $A_J = A'_J$ when Bob and Carol respectively choose to perform the measurements $b$ and $c$.

Similarly, we can obtain

$$E(A_JBC') - E(A'_JBC') = 2[P(A_J = -A'_J = BC') - P(-A'_J = A_J = -BC')]$$

$$\leq 2[P(A_J = -A'_J = BC') + P(-A'_J = A_J = -BC')]$$

$$= 2P(A_J = -A'_J; bc'),$$

(3)

$$E(A_JB'C) - E(A'_JB'C) \leq 2P(A_J = -A'_J; b'c),$$

(4)

and

$$E(A_JB'C') + E(A'_JB'C') \leq 2P(A_J = A'_J; b'c').$$

(5)

From equations (1)–(5), we obtain

$$S^I_3 \leq 2P(A_J = A'_J; bc) + 2P(A_J = A'_J; b'c')$$

$$+ 2P(A_J = -A'_J; bc) + 2P(A_J = -A'_J; b'c'),$$

(6)

where we denote $S^I_3$ as the three-particle Svetlichny operator concerned with joint measurements. Due to the no-signaling principle, the probability of Alice obtaining $A_J = A'_J$ or $A_J = -A'_J$ should be independent of the measurement choices of Bob and Carol, i.e.

$$P(A_J = A'_J; bc) = P(A_J = A'_J; b'c') = P(A_J = A'_J),$$

$$P(A_J = -A'_J; bc') = P(A_J = -A'_J; b'c)$$

$$= P(A_J = -A'_J).$$

(7)

From equations (6) and (7), we finally obtain $S^I_3 \leq 4$, i.e. the three-particle SI.

**Derivation of N-particle SI.** Suppose there are $N$ players who share $N$ particles; each of them performs dichotomous measurements on each of the $N$ particles. The measurement settings are represented by $x_1, x_2, \ldots, x_N$, respectively, with possible values 0, 1. The measurement results are represented by $A_1, A_2, \ldots, A_N$, respectively, and with possible values $-1, 1$. Then the $N$-particle SI can be expressed as [14]

$$S_N = \left| \sum_{[x_1]} v(x_1, x_2, \ldots, x_N) E(A_1A_2 \cdots A_N|x_1, x_2, \ldots, x_N) \right|$$

$$\leq 2^{N-1},$$

(8)

where $S_N$ is the $N$-particle Svetlichny operator, $[x_1]$ stands for an $N$-tuple $x_1, x_2, \ldots, x_N$, $E(A_1A_2 \cdots A_N|x_1, x_2, \ldots, x_N)$ represents the expectation value of the product of the measurement outcomes of the observables $x_1, x_2, \ldots, x_N$ and $v(x_1, x_2, \ldots, x_N)$ is a sign function given by

$$v(x_1, x_2, \ldots, x_N) = (-1)^{1 + \frac{1}{2}(k-1)},$$

(9)

where $k$ is the number of times index 1 appears in $(x_1, x_2, \ldots, x_N)$.

Without losing generality, we assume that the first player makes a joint measurement of $x_1 = 0$ and $x_1 = 1$ on the first particle, with the results $A'_1$ and $A'_1$. We note that the
summation in equation (8) can be expressed as

\[ S_{N}' = \left| \sum_{\{x_i\}'} [v(x_1 = 0, x_2, \ldots, x_N)E(A_1^f A_2 \cdots A_N|x_1 = 0, x_2, \ldots, x_N) \\
+ v(x_1 = 1, x_2, \ldots, x_N)E(A_1^f A_2 \cdots A_N|x_1 = 1, x_2, \ldots, x_N)] \right| \]

\[ = \left| \sum_{\{x_i\}'} v(x_1 = 0, x_2, \ldots, x_N)\left[ E(A_1^f A_2 \cdots A_N|x_1 = 0, x_2, \ldots, x_N) \\
+ (-1)^k E(A_1^f A_2 \cdots A_N|x_1 = 1, x_2, \ldots, x_N) \right] \right|, \] (10)

where \( k' \) denotes the number of times index 1 appears in \((x_2, \ldots, x_N)\), \( S_{N}' \) is the \(N\)-particle Svetlichny operator concerned with joint measurements and \(\{x_i\}'\) stands for an \((N - 1)\)-tuple \(x_2, \ldots, x_N\). There are \(2^{N-1}\) terms in the summation \(\sum_{\{x_i\}'}\). If the number of times index 1 appears in \((x_2, \ldots, x_N)\) is even, i.e. \(k'\) is even, we can obtain

\[ v(x_1 = 0, x_2, \ldots, x_N)[E(A_1^f A_2 \cdots A_N|x_1 = 0, x_2, \ldots, x_N) \\
+ (-1)^k E(A_1^f A_2 \cdots A_N|x_1 = 1, x_2, \ldots, x_N)] \leq \left| E(A_1^f A_2 \cdots A_N|x_1 = 0, x_2, \ldots, x_N) \\
+ E(A_1^f A_2 \cdots A_N|x_1 = 1, x_2, \ldots, x_N) \right| \\
= \left| 2[P(A_1^f = A_1^f = A_2 \cdots A_N|x_2, \ldots, x_N) \\
- P(A_1^f = -A_1^f = A_2 \cdots A_N|x_2, \ldots, x_N)] \right| \\
\leq 2[P(A_1^f = A_1^f = A_2 \cdots A_N|x_2, \ldots, x_N) \\
+ P(A_1^f = -A_1^f = A_2 \cdots A_N|x_2, \ldots, x_N)] \\
= 2P(A_1^f = A_1^f |x_2, \ldots, x_N), \] (11)

where \(P(A_1^f = A_1^f |x_2, \ldots, x_N)\) represents the probability that the joint measurement results of the first player satisfy \(A_1^f = A_1^f\) when the measurement setting of other players is \(x_2, \ldots, x_N\). Similarly, if \(k'\) is odd, we can obtain

\[ v(x_1 = 0, x_2, \ldots, x_N)[E(A_1^f A_2 \cdots A_N|x_1 = 0, x_2, \ldots, x_N) \\
+ (-1)^k E(A_1^f A_2 \cdots A_N|x_1 = 1, x_2, \ldots, x_N)] \leq \left| E(A_1^f A_2 \cdots A_N|x_1 = 0, x_2, \ldots, x_N) \\
- E(A_1^f A_2 \cdots A_N|x_1 = 1, x_2, \ldots, x_N) \right| \\
= \left| 2[P(A_1^f = -A_1^f = A_2 \cdots A_N|x_2, \ldots, x_N) \\
- P(A_1^f = -A_1^f = A_2 \cdots A_N|x_2, \ldots, x_N)] \right| \\
\leq 2[P(A_1^f = -A_1^f = A_2 \cdots A_N|x_2, \ldots, x_N) \\
+ P(A_1^f = -A_1^f = A_2 \cdots A_N|x_2, \ldots, x_N)] \\
= 2P(A_1^f = -A_1^f |x_2, \ldots, x_N). \] (12)

Due to the no-signaling principle, the probability of the first player obtaining \(A_1^f = A_1^f\) or \(A_1^f = -A_1^f\) should be independent of the measurement choices of other players, i.e.

\[ P(A_1^f = A_1^f |x_2, \ldots, x_N) = P(A_1^f = A_1^f), \quad P(A_1^f = -A_1^f |x_2, \ldots, x_N) = P(A_1^f = -A_1^f). \] (13)
The number of terms with even $k'$ in the summation $\sum_{\{x_i\}'}$ in equation (10) is $2^{N-2}$ $\left(\binom{N-1}{0} + \binom{N-1}{2} + \cdots \right)$, and there are also $2^{N-2}$ terms with odd $k'$ in the summation $\sum_{\{x_i\}'}$. So from equation (10), we finally obtain the $N$-particle SI

\[ S_N' \equiv \sum_{\{x_i\}'} v(x_1 = 0, x_2, \ldots, x_N) \left[ E(A_1' A_2 \cdots A_N | x_1 = 0, x_2, \ldots, x_N) \right. \]

\[ + \left. (-1)^k E(A_1' A_2 \cdots A_N | x_1 = 1, x_2, \ldots, x_N) \right| \]

\[ \leq \left| \left( \binom{N-1}{0} + \binom{N-1}{2} + \cdots \right) \cdot 2P(A_1' = A_2') \right. \]

\[ + \left. \left( \binom{N-1}{1} + \binom{N-1}{3} + \cdots \right) \cdot 2P(A_1' = -A_2') \right| \]

\[ = 2^{N-1}. \]

(14)

3. Bound on the sharpness of joint measurements and the maximal violation of SI

If $a$ and $a'$ both denote the usual projective quantum measurements, they can be described by projector collections of $\{E(a), E(-a)\}$ and $\{E(a'), E(-a')\}$, respectively, where $E(\pm a) = \frac{1}{2}[I \pm a \cdot \sigma]$. $E(\pm a') = \frac{1}{2}[I \pm a' \cdot \sigma]$. $a$ and $a'$ are the directions of the measurements $a$ and $a'$, respectively, and $\sigma$ is the Pauli operator. Suppose $a$ and $a'$ do not mutually commute; we cannot perform a joint measurement of $a$ and $a'$ by carrying out a projective quantum measurement, since the observables $a$ and $a'$ do not share eigenstates. However, quantum mechanics allows us to perform joint unsharp measurements of these two observables, and the unsharp measurements for these two observables can be described as POVMs. We can describe the unsharp measurements of $a$ and $a'$, respectively, as \[ E_{\eta_1}(\pm a) = \frac{1}{2}[I \pm \eta_1 a \cdot \sigma], \]

\[ E_{\eta_2}(\pm a') = \frac{1}{2}[I \pm \eta_2 a' \cdot \sigma], \]

where $0 < \eta_1, \eta_2 \leq 1$ quantifies the sharpness of the joint measurements of $a$ and $a'$. For a joint measurement of $\{E_{\eta_1}(\pm a)\}$ and $\{E_{\eta_2}(\pm a')\}$, the only constraint on $a, a', \eta_1$ and $\eta_2$ is \[ |\eta_1 a + \eta_2 a'| + |\eta_1 a - \eta_2 a'| \leq 2. \]

(15)

(16)

If we take $\eta_1 = \eta_2 = \eta$, which means that the measurements of $\{E_{\eta}(\pm a)\}$ and $\{E_{\eta}(\pm a')\}$ have equal sharpness, then the above condition can be expressed as

\[ \eta(|a + a'| + |a - a'|) \leq 2. \]

(17)

For the case of $a \perp a'$, $|a + a'| + |a - a'|$ takes its maximal value of $2\sqrt{2}$, so from equation (17) we know that quantum mechanics allows joint unsharp measurements of any observables $a$ and $a'$ as long as the equal sharpness $\eta$ is less than or equal to $\frac{\sqrt{2}}{2}$.

According to quantum measurement theory, there is an essential property of the unsharp measurement of $E_{\eta}(\pm a) = \frac{1}{2}[I \pm \eta a \cdot \sigma]$. For any state $\rho$, the average value of the measurement results of $\{E_{\eta}(\pm a)\}$ is proportional to the expectation value of the corresponding sharp measurement, i.e.

\[ \text{Tr}[\rho E_{\eta}(a)] - \text{Tr}[\rho E_{\eta}(-a)] = \eta \text{Tr}[\rho a \cdot \sigma]. \]

(18)

We return to Svetlichny’s experiment. There are $N$ players who share $N$ particles; the $i$th player can choose measuring one of two observables $a_i$ and $a_i'$ on the $i$th particle.
The observable $a_i$ ($a'_i$) can be described by projector collections of $\{E(a_i), E(-a_i)\}$ ($\{E(a'_i), E(-a'_i)\}$). We confine all unit vectors $a_i$ and $a'_i$ to lie in the $x$–$y$ plane and define $a_i = \cos \alpha_i \hat{i} + \sin \alpha_i \hat{j}$ and $a'_i = \cos \alpha'_i \hat{i} + \sin \alpha'_i \hat{j}$, where $\alpha_i$ ($\alpha'_i$) is the angle between the unit vector $a_i$ ($a'_i$) and the $x$ axis. For these $N$ particles in the GHZ state of $\frac{1}{\sqrt{2}} (|↑⟩^\otimes N \pm |↓⟩^\otimes N)$, we choose the measurement protocol [14]

$$(\alpha_1, \alpha_2, \ldots, \alpha_N) = \left(\frac{\pi}{4}, 0, \ldots, 0\right),$$

$$(\alpha'_1, \alpha'_2, \ldots, \alpha'_N) = \left(\frac{3\pi}{4}, \frac{\pi}{2}, \ldots, \frac{\pi}{2}\right);$$

(19)

then we can achieve the maximal value $2^{N-1}\sqrt{2}$ of the $S_N$, i.e. the maximal violation of SI [14].

Now we assume that the first player makes a joint unsharp measurement of $a_1$ and $a'_1$ on the first particle, with equal sharpness $\eta$. From equation (18), we can obtain $S_N' = \eta S_N$.

The condition of equation (17) means that, as long as the equal sharpness $\eta$ satisfies $\eta \leq \frac{\sqrt{2}}{2}$, quantum mechanics allows the existence of joint unsharp measurements of any observables $a$ and $a'$. The derivation in the previous section shows that the existence of joint measurements of two observables $a_1$ and $a'_1$ must demand that $S_N' \leq 2^{N-1}$ or the no-signaling principle will be breached. So we can conclude that the quantum bound of $S_N$ is necessarily not greater than $2^{N-1}\sqrt{2}$. From the above calculation, we demonstrate that this bound can be achieved in quantum mechanics with the GHZ state of $\frac{1}{\sqrt{2}} (|↑⟩^\otimes N \pm |↓⟩^\otimes N)$ and the measurement protocol of equation (19).

4. Conclusion

The feasibility of joint measurements of some observables implies that there must be a joint probability distribution of the values of these observables, so we can derive SI from the existence of the joint measurements and the no-signaling principle. This means that, if we do not breach the no-signaling principle, the results of joint measurements must produce the satisfaction of SI. Quantum mechanics allows joint unsharp measurement of any observables as long as the equal sharpness fulfills $\eta \leq \frac{\sqrt{2}}{2}$. Thus, the quantum bound of $S_N$ is necessarily not greater than $2^{N-1}\sqrt{2}$; otherwise we would obtain $S_N' > 2^{N-1}$ and this implies the breach of the no-signaling principle.

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