Stochastic differential equations related to random matrix theory

By

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Abstract

In this note we review recent results on existence and uniqueness of solutions of infinite-dimensional stochastic differential equations describing interacting Brownian motions on $\mathbb{R}^d$.

§1. Introduction

Let $X^N(t) = (X^N_j(t))_{j=1}^N$ be a solution of the stochastic differential equation (SDE)

\begin{equation}
\frac{dX^N_j(t)}{dt} = dB_j(t) + \frac{\beta}{2} \sum_{k=1, k \neq j}^N \frac{dt}{X^N_j(t) - X^N_k(t)}
\end{equation}

(1.1)

or the SDE with Ornstein-Uhlenbeck’s type drifts

\begin{equation}
\frac{dX^N_j(t)}{dt} = dB_j(t) - \frac{\beta}{4N} X^N_j(t)dt + \frac{\beta}{2} \sum_{k=1, k \neq j}^N \frac{dt}{X^N_j(t) - X^N_k(t)}
\end{equation}

(1.2)

where $B_j(t), j = 1, 2, \ldots, N$ are independent one-dimensional Brownian motions. These are called Dyson’s Brownian motion models with parameters $\beta > 0$ [4]. They were introduced to understand the statistics of eigenvalues of random matrix ensembles as
distributions of particle positions in one-dimensional Coulomb gas systems with log-potential.

The solution of (1.2) is a natural reversible stochastic dynamics with respect to $\tilde{\mu}^N_{\text{bulk},\beta}$:

$$ (1.3) \quad \tilde{\mu}^N_{\text{bulk},\beta}(d\mathbf{x}_N) = \frac{1}{Z} h_N(\mathbf{x}_N)^\beta e^{-\frac{\beta}{4N} \sum |x_N|^2} d\mathbf{x}_N, $$

where $d\mathbf{x}_N = dx_1 dx_2 \cdots dx_N$, $\mathbf{x}_N = (x_i) \in \mathbb{R}^N$, and

$$ h_N(\mathbf{x}_N) = \prod_{i<j} |x_i - x_j|. $$

Throughout, $Z$ denotes a normalizing constant. Gaussian ensembles are called Gaussian orthogonal/unitary/symplectic ensembles (GOE/GUE/GSE) according to their invariance under conjugation by orthogonal/unitary/symplectic groups, which correspond to the inverse temperatures $\beta = 1, 2$ and 4, respectively \[9, 2\]. It is natural to believe that the $N$-limit of the process $X^N(t)$ solves the infinite-dimensional stochastic differential equation (ISDE)

$$ (1.4) \quad dX^j(t) = dB^j(t) + \frac{\beta}{2} \lim_{r \to \infty} \sum_{k=1, k \neq j}^{\infty} \frac{dt}{X^j(t) - X^k(t)}. $$

The result was not proved rigorously until a few years ago when it was shown for $\beta = 2$ in \[17\], for $\beta = 1, 2, 4$ in \[8\], and for $\beta \geq 1$ in \[23\].

Set $Y^N_j(t) = N^{1/6}(X^N_j(t) - 2\sqrt{N})$, $j = 1, 2, \ldots, N$ for the solution $\mathbf{X}^N$ of (1.2). It has also been shown that the $N$-limit of the process $\mathbf{Y}^N(t)$ solves the ISDE

$$ (1.5) \quad dY^j(t) = dB^j(t) + \frac{\beta}{2} \lim_{r \to \infty} \left\{ \sum_{k=1, k \neq j}^{\infty} \frac{1}{Y^j(t) - Y^k(t)} - \int_{-r}^{r} \frac{\tilde{\rho}(x)dx}{x} \right\} dt, $$

with $\tilde{\rho}(x) = \pi^{-1} \sqrt{-x} 1(x < 0)$, for $\beta = 2$ \[17\] and for $\beta = 1, 2, 4$ \[8\].

One of the key parts of proving the above results is the existence and uniqueness of solutions of an ISDE of the form

$$ (1.6) \quad dX^j(t) = dB^j(t) - \frac{1}{2} \nabla \Phi(X^j(t)) dt - \frac{1}{2} \sum_{k=1, k \neq j}^{\infty} \nabla \Psi(X^j(t), X^k(t)) dt $$

with free potential $\Phi$ and interaction (pair) potential $\Psi$. In ISDEs (1.4) and (1.5), $\Psi$ is given by the log pair potential $-\beta \log |x - y|$. The present note is a short summary of results on existence and uniqueness of solutions for ISDE (1.6).
§ 2. Quasi-Gibbs measure

Let \( S \) be a closed set in \( \mathbb{R}^d \) such that \( 0 \in S \) and \( \overline{S^{\text{int}}} = S \), where \( S^{\text{int}} \) denotes the interior of \( S \). The configuration space \( \mathcal{M} \) of unlabelled particles is given by

\[
\mathcal{M} = \left\{ \xi : \xi \text{ is a nonnegative integer valued Radon measure in } S \right\} = \left\{ \xi(\cdot) = \sum_{j \in \mathbb{I}} \delta_{x_j}(\cdot) : \sharp\{j \in \mathbb{I} : x_j \in K\} < \infty, \text{ for any } K \text{ compact} \right\},
\]

where \( \mathbb{I} \) is a countable set and \( \delta_a \) is the Dirac measure at \( a \in S \). Thus \( \mathcal{M} \) is a Polish space with the vague topology. We also introduce a subset \( \mathcal{M}_{\text{s.i.}} \) of \( \mathcal{M} \):

\[
\mathcal{M}_{\text{s.i.}} = \{ \xi \in \mathcal{M} : \xi(\{x\}) \leq 1 \text{ for all } x \in S, \xi(S) = \infty \},
\]

that is, the set of configurations of an infinite number of particles without collisions. For Borel measurable functions \( \Phi : S \to \mathbb{R} \cup \{\infty\} \) and \( \Psi : S \times S \to \mathbb{R} \cup \{\infty\} \) and a given increasing sequence \( \{b_r\} \) of \( \mathbb{N} \), we introduce the Hamiltonian

\[
H_r(\xi) = H_r^{\Phi, \Psi}(\xi) = \sum_{x_j \in S_r} \Phi(x_j) + \sum_{x_j, x_k \in S_r, j < k} \Psi(x_j, x_k), \quad \xi = \sum_{j \in \mathbb{I}} \delta_{x_j},
\]

where \( S_r = \{x \in S : |x| < b_r\} \). We call \( \Phi \) a free potential, and call \( \Psi \) an interaction potential. Let \( \Lambda_r^m \) be the restriction of a Poisson random measure with intensity measure \( dx \) on \( \mathcal{M}^m_r \). We define maps \( \pi_r, \pi^c_r : \mathcal{M} \to \mathcal{M} \) such that \( \pi_r(\xi) = \xi(\cdot \cap S_r) \) and \( \pi^c_r(\xi) = \xi(\cdot \cap S^c_r) \). For two measures \( \nu_1, \nu_2 \) on a measurable space \( (\Omega, \mathcal{F}) \) we write \( \nu_1 \leq \nu_2 \) if \( \nu_1(A) \leq \nu_2(A) \) for any \( A \in \mathcal{F} \). We can now state the definition of a quasi-Gibbs measure [13] [14].

**Definition 2.1.** A probability measure \( \mu \) on \( \mathcal{M} \) is said to be a \((\Phi, \Psi)\)-quasi Gibbs measure if its regular conditional probabilities

\[
\mu_{r, \xi}^m(d\zeta) = \mu(d\zeta | \pi^c_r(\zeta) = \pi^c_r(\xi), \zeta(S_r) = m), \quad r, m \in \mathbb{N},
\]

satisfy that, for \( \mu \)-a.s. \( \zeta \),

\[
c^{-1}e^{-H_r(\eta)}\Lambda_r^m(\pi_{S_r} \in d\eta) \leq \mu_{r, \xi}^m(\pi_{S_r} \in d\eta) \leq ce^{-H_r(\eta)}\Lambda_r^m(\pi_{S_r} \in d\eta).
\]

Here, \( c = c(r, m, \xi) \) is a positive constant depending on \( r, m \), and \( \xi \).

It is readily seen that the quasi-Gibbs property is a generalized notion of the canonical Gibbs property. If \( \mu \) is a \((\Phi, \Psi)\)-quasi Gibbs measure, then \( \mu \) is also a \((\Phi + \Phi_{\text{loc.bdd}}, \Psi)\)-quasi Gibbs measure for any locally bounded measurable function \( \Phi_{\text{loc.bdd}} \). In this sense, the notion of “quasi-Gibbs” seems to be robust. Information about the free potential of \( \mu \) is determined from its logarithmic derivative [12].
A function \( f \) on \( \mathcal{M} \) is called a polynomial function if

\[ f(\xi) = Q(\langle \phi_1, \xi \rangle, \langle \phi_2, \xi \rangle, \ldots, \langle \phi_\ell, \xi \rangle) \]

with \( \phi_k \in C^\infty_c(S) \) and a polynomial function \( Q \) on \( \mathbb{R}^\ell \), where

\[ \langle \phi, \xi \rangle = \int_S \phi(x)\xi(dx) \]

and \( C^\infty_c(S) \) is the set of smooth functions with compact support. We denote by \( \mathcal{P} \) the set of all polynomial functions on \( \mathcal{M} \).

**Definition 2.2.** We call \( d\mu \in L^1_{loc}(S \times \mathcal{M}, \mu^{[1]}(\cdot)) \) the logarithmic derivative of \( \mu \) if

\[ \int_{S \times \mathcal{M}} d\mu(x, \eta) f(x, \eta) d\mu^{[1]}(x, \eta) = -\int_{S \times \mathcal{M}} \nabla_x f(x, \eta) d\mu^{[1]}(x, \eta) \]

is satisfied for \( f \in C^\infty_c(S) \). Here \( \mu^{[k]} \) is the Campbell measure of \( \mu \)

\[ \mu^{[k]}(A \times B) = \int_A \mu_x(B) \rho^k(x) dx, \quad A \in \mathcal{B}(S^k), B \in \mathcal{B}(\mathcal{M}), \]

\( \mu_x \) is the reduced Palm measure conditioned at \( x \in S^k \)

\[ \mu_x = \mu \left( \cdot - \sum_{j=1}^k \delta_{x_j} \middle| \xi(x_j) \geq 1 \text{ for } j = 1, 2, \ldots, k \right), \]

and \( \rho^k \) is the \( k \)-correlation function for \( k \in \mathbb{N} \).

Quasi-Gibbs measures inherit the following property from canonical Gibbs measures [19, Lemma 11.2]. Let \( \mathcal{T}(\mathcal{M}) \) be the tail \( \sigma \)-field

\[ \mathcal{T}(\mathcal{M}) = \bigcap_{r=1}^\infty \sigma(\pi^r) \]

and let \( \mu^\xi_{\text{Tail}} \) be the regular conditional probability defined as

\[ \mu^\xi_{\text{Tail}} = \mu(\cdot | \mathcal{T}(\mathcal{M}))(\xi). \]

Then the following decomposition holds:

\[ \mu(\cdot) = \int_{\mathcal{M}} \mu^\xi_{\text{Tail}}(\cdot) \mu(d\xi). \]

Furthermore, there exists a subset \( \mathcal{M}_0 \) of \( \mathcal{M} \) satisfying \( \mu(\mathcal{M}_0) = 1 \) and, for all \( \xi, \eta \in \mathcal{M}_0 \):

\[ \mu^\xi_{\text{Tail}}(A) \in \{0, 1\} \quad \text{for all } A \in \mathcal{T}(\mathcal{M}), \]

\[ \mu^\xi_{\text{Tail}}(\{\zeta \in \mathcal{M} : \mu^\xi_{\text{Tail}} = \mu^\xi_{\text{Tail}}\}) = 1, \]

\[ \mu^\xi_{\text{Tail}} \text{ and } \mu^\eta_{\text{Tail}} \text{ are mutually singular on } \mathcal{T}(\mathcal{M}) \text{ if } \mu^\xi_{\text{Tail}} \neq \mu^\eta_{\text{Tail}}. \]
§ 3. General theory of solutions of ISDEs

A polynomial function \( f \) on \( \mathcal{M} \) is a local function, that is, a function satisfying \( f(\xi) = f(\pi_r(\xi)) \) for some \( r \in \mathbb{N} \). When \( \xi \in \mathcal{M}_m^r \), \( m \in \mathbb{N} \cup \{0\} \) and \( \pi_r(\xi) \) is represented by \( \sum_{j=1}^m \delta_{x_j} \), we can regard \( f(\xi) = f(\sum_{j=1}^m \delta_{x_j}) \) as a permutation invariant smooth function on \( S_r^m \). For \( f, g \in \mathcal{P} \), define
\[
\mathbb{D}(f, g)(\xi) = \frac{1}{2} \sum_{j=1}^\infty \nabla_{x_j} f(\xi) \cdot \nabla_{x_j} g(\xi).
\]

For a probability \( \mu \) on \( \mathcal{M} \), we denote by \( L^2(\mathcal{M}, \mu) \) the space of square integrable functions on \( \mathcal{M} \) with the inner product \( \langle \cdot, \cdot \rangle_\mu \) and the norm \( \| \cdot \|_{L^2(\mathcal{M}, \mu)} \). We consider the bilinear form \( (\mathcal{E}^\mu, \mathcal{P}^\mu) \) on \( L^2(\mathcal{M}, \mu) \) defined by
\[
\mathcal{E}^\mu(f, g) = \int_{\mathcal{M}} \mathbb{D}(f, g) d\mu, \quad \mathcal{P}^\mu = \{ f \in \mathcal{P} : \| f \|_1^2 < \infty \},
\]
where \( \| f \|_1^2 = \mathcal{E}^\mu(f, f) + \| f \|_{L^2(\mathcal{M}, \mu)}^2 \).

We make the following assumptions

(A.0) \( \mu \) has a locally bounded \( n \)-correlation function \( \rho^n \) for each \( n \in \mathbb{N} \).

(A.1) There exist upper semi-continuous functions \( \Phi_0 : S \to \mathbb{R} \cup \{\infty\} \) and \( \Psi_0 : S \times S \to \mathbb{R} \cup \{\infty\} \) that are locally bounded from below, and \( c > 0 \) such that
\[
c^{-1}\Phi_0(x) \leq \Phi(x) \leq c\Phi_0(x), \quad c^{-1}\Psi_0(x, y) \leq \Psi(x, y) \leq c\Psi_0(x, y).
\]

(A.2) There exists a \( T > 0 \) such that for each \( R > 0 \)
\[
\liminf_{r \to \infty} \operatorname{Erf}\left(\frac{r}{(r + R)T}\right) \int_{|x| \leq r + R} \rho^1(x) dx = 0,
\]
where \( \operatorname{Erf}(t) = (2\pi)^{-1/2} \int_t^\infty e^{-x^2/2} dx \).

Note that \( \mathcal{P}^\mu = \mathcal{P} \) and \( (\mathcal{E}^\mu, \mathcal{P}^\mu) = (\mathcal{E}, \mathcal{P}) \) under condition (A.0).

Theorem 3.1 ([12][13][14][11][16]). Suppose that \( \mu \) is a \( (\Phi, \Psi) \)-quasi Gibbs measure satisfying (A.0) and (A.1). Then
(i) \( (\mathcal{E}, \mathcal{P}) \) is closable and its closure \( (\mathcal{E}^\mu, \mathcal{D}^\mu) \) is a quasi regular Dirichlet form and there exists the diffusion process \( (\Xi(t), P^\xi_t) \) associated with \( (\mathcal{E}^\mu, \mathcal{D}^\mu) \).
(ii) Furthermore, assume conditions (A.2) and (A.3):
(A.3) \( \operatorname{Cap}^\mu((\mathcal{M}_{S,1})^c) = 0 \) and \( \operatorname{Cap}^\mu(\xi(\partial S) \geq 1) = 0 \),
where \( \operatorname{Cap}^\mu \) is the capacity of the Dirichlet form. If there exists a logarithmic derivative \( d^\mu \), then there exists \( \mathcal{M} \subset \mathcal{M} \) such that \( \mu(\mathcal{M}) = 1 \), and for any \( \xi = \sum_{j \in \mathbb{N}} \delta_{x_j} \in \mathcal{M} \), there exists an \( S^\mathbb{N} \)-valued continuous process \( \mathbf{X}(t) = (X_j(t))_{j=1}^\infty \) satisfying \( \mathbf{X}(0) = \mathbf{x} = (x_j)_{j=1}^\infty \) and
\[
dX_j(t) = dB_j(t) + \frac{1}{2} d^\mu\left(X_j(t), \sum_{k \neq j} \delta_{X_k(t)}\right) dt, \quad j \in \mathbb{N}.
\]
Let $l$ be a label map from $M_{s.i.}$ to $S^N$, that is, for each $\xi \in M_{s.i.}$, $l(\xi) = (l(\xi)_j)_{j=1}^{\infty} \in S^N$ satisfies $\xi = \sum_{j=1}^{\infty} \delta_{l(\xi)_j}$. The map $l$ can be lifted to the map from $C([0, \infty), M_{s.i.})$ to $C([0, \infty), S^N)$. For $\Xi \in C([0, \infty), M_{s.i.})$ we put

$$\Xi^\diamond m(t) = \sum_{j=m+1}^{\infty} \delta_{X_j(t)}$$

for each $m \in \mathbb{N}$, where $(X_j)_{j=1}^{\infty} = l(\Xi) \in C([0, \infty), S^N)$. We make the following assumption.

(A4) There exists a subset $M_{SDE}$ of $M_{s.i.}$ such that $P_{\mu}(\xi \in M_{SDE}) = 1$ for any $\xi \in M_{SDE}$, and for each $\Xi \in C([0, \infty), M_{SDE})$ and each $m \in \mathbb{N}$,

\begin{equation}
Y_j^{(m)}(t) = dB_j(t) - \frac{1}{2} \sum_{k=1, k \neq j}^{m} \nabla \Psi(Y_j^{(m)}(t), Y_k^{(m)}(t))dt - \frac{1}{2} \int_{\mathcal{B}} \nabla \Psi(Y_j^{(m)}(t), X(t))\Xi^\diamond m(dX)dt, \quad 1 \leq j \leq m,
\end{equation}

\begin{equation}
Y_j^{(m)}(0) = l(\Xi(0))_j, \quad 1 \leq j \leq m,
\end{equation}

has a unique strong solution $Y^{(m)} = (Y_1^{(m)}, Y_2^{(m)}, \ldots, Y_m^{(m)})$.

We also make the following assumptions about the probability measure $\mu$

(A5) For each $r, T \in \mathbb{N}$, there exists a positive constant $c$ such that

$$\int_{S} \text{Erf} \left( \frac{|x| - r}{\sqrt{cT}} \right) \rho_1(x)dx < \infty.$$

(A6) The tail $\sigma$-field $\mathcal{T}(M)$ is $\mu$-trivial, that is, $\mu(A) \in \{0, 1\}$ for $A \in \mathcal{T}(M)$.

**Definition 3.2.** Let $\mu$ be a probability measure on $M$ and let $\Xi(t)$ be an $M$-valued process. We say that $\Xi(t)$ satisfies the $\mu$-absolute continuity condition if $\mu \circ \Xi(t)^{-1}$ is absolutely continuous with respect to $\mu$ for all $t > 0$. We say that an $S^N$-valued process $X(t)$ satisfies the $\mu$-absolute continuity condition if $u(X(t))$ satisfies the $\mu$-absolute continuity condition, where $u$ is the map from $S^N$ to $M$ defined by $u((x_j)_{j=1}^{\infty}) = \sum_{j=1}^{\infty} \delta_{x_j}$.

Then we have the following theorem.

**Theorem 3.3** ([19]). Suppose that the assumptions in Theorem 3.1 are satisfied. Furthermore assume (A4)–(A6). Then, for $\mu$-a.s. $\xi$, ISDE (1.6) with $X(0) = l(\xi)$ has a strong solution satisfying the $\mu$-absolute continuity condition, and that pathwise uniqueness holds for ISDE (1.6) with the $\mu$-absolute continuity condition.
§ 4. Applications

Theorems 3.1 and 3.3 can be applied to quite general class of ISDEs. In this section we give some important examples.

**Example 4.1 (Canonical Gibbs measures).** Let $S = \mathbb{R}^d$, $d \in \mathbb{N}$. Assume that $\Phi = 0$ and that $\Psi_0$ is a super stable and regular in the sense of Ruelle [22], and is smooth outside the origin. Let $\mu$ be a canonical Gibbs measure with the interaction $\Psi_0$. Then its logarithmic derivative is

\[
\begin{align*}
\frac{d\mu}{\mu}(x, \sum_{k:k \neq j} \delta_{y_k}) &= - \sum_{k=1, k \neq j}^{\infty} \nabla \Psi_0(x - y_k).
\end{align*}
\]

Assume that (A.2) is satisfied. In the case $d \geq 2$, there exists a diffusion process associated with $\mu$ and the labeled process solves

\[
\begin{align*}
\frac{dX_j(t)}{dt} = dB_j(t) - \frac{1}{2} \sum_{k=1, k \neq j}^{\infty} \nabla \Psi_0(X_j(t) - X_k(t))dt.
\end{align*}
\]

In the case $d = 1$, $\Psi_0$ needs to be sufficient repulsive at the origin to satisfy (A.3).

Assume that (A.5) is satisfied and that, for each $n \in \mathbb{N}$, there exist positive constants $c, c'$ satisfying

\[
\begin{align*}
\sum_{r=1}^{\infty} \frac{\int_{|x| > r} \rho^1(x)dx}{r^c} &< \infty, \\
\sum_{i,j=1}^{d} \left| \frac{\partial^2}{\partial x_i \partial x_j} \Psi_0(x) \right| &\leq \frac{c'}{(1 + |x|)^{c' + 1}},
\end{align*}
\]

for all $|x| \geq 1/n$. In [19] Theorem 3.3 it was proved that, for $\mu$-a.s. $\xi$, ISDE (1.2) with $X(0) = l(\xi)$ has a strong solution satisfying the $\mu_{\text{Tail}}^{\xi}$-absolute continuity condition, and that pathwise uniqueness holds for ISDE (1.6) with the $\mu_{\text{Tail}}^{\xi}$-absolute continuity condition.

**Example 4.2 (Sine random point fields).** Let $\tilde{\mu}_{\text{bulk,}\beta}^N$ be the probability measure defined in (1.3). We denote by $\mu_{\text{bulk,}\beta}^N$ the distribution of $\sum_{j=1}^{N} \delta_{x_j}$ under $\tilde{\mu}_{\text{bulk,}\beta}^N$. For $\beta > 0$ the existence of the limit of $\mu_{\text{bulk,}\beta}^N$ as $N \to \infty$ was shown in Valkó-Virág [24]. We denote the limit by $\mu_{\text{bulk,}\beta}$. In particular, when $\beta = 2$, $\mu_{\text{bulk,}2}$ is the determinantal point process (DPP) with the sine kernel

\[
\begin{align*}
K_{\text{sin,}2}(x, y) &= \frac{\sin(x - y)}{\pi(x - y)},
\end{align*}
\]
and when $\beta = 1, 4$, it is a quaternion determinantal point process [2]. It was shown that $\mu_{\text{bulk}, \beta}$ for $\beta = 1, 2, 4$ is a quasi-Gibbs measure in [13], and that its logarithmic derivative is

$$d\mu \left( x, \sum_{k: k \neq j} \delta_{y_k} \right) = \beta \lim_{r \to \infty} \sum_{k: k \neq j, |y_k| < r} \frac{1}{x - y_k}$$

(4.6)

in [12]. In [19] Theorem 3.1 it was shown that for $\mu_{\text{bulk}, \beta}$-a.s. $\xi$, ISDE (1.4) with $X(0) = \mathcal{I}(\xi)$ has a strong solution satisfying the $\mu_{\text{bulk}, \beta, \text{Tail}}$-absolute continuity condition, and that pathwise uniqueness holds for ISDE (1.4) with the $\mu_{\text{bulk}, \beta, \text{Tail}}$-absolute continuity condition. In the case $\beta = 2$, the facts that $T(\mathcal{M})$ is $\mu_{\text{bulk}, 2}$-trivial and $\mu_{\text{bulk}, 2, \text{Tail}} = \mu_{\text{bulk}, 2}$ were shown in [15].

Tsai [23] proved the existence and uniqueness of solutions of ISDE (1.4) for $\beta \geq 1$ by a different method. Thus it is conjectured that $\mu_{\text{bulk}, \beta}$ is a quasi-Gibbs measure and has a logarithmic derivative of the form (4.6) for $\beta \geq 1$.

**Example 4.3** (Airy random point fields). We denote by $\mu_{\text{soft}, \beta}^N$ the distribution of $\sum_{j=1}^N \delta_{N^{1/\alpha}(x_j - 2 \sqrt{N})}$ under $\hat{\mu}_{\text{bulk}, \beta}^N$. For $\beta > 0$, the existence of the limit of $\mu_{\text{soft}, \beta}^N$ as $N \to \infty$ was shown in Ramírez-Rider-Virág [21]. We denote the limit by $\mu_{\text{soft}, \beta}$. In particular, when $\beta = 2$, $\mu_{\text{soft}, 2}$ is the DPP with the Airy kernel

$$K_{\text{Ai}, 2}(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y},$$

(4.7)

where $\text{Ai}$ denotes the Airy function and $\text{Ai}'$ its derivative [9]. When $\beta = 1, 4$, it is a quaternion determinantal point process [2]. In the cases $\beta = 1, 2, 4$, it has been proved that the random point field is quasi-Gibbsian [14], and that its logarithmic derivative is

$$d\mu \left( x, \sum_{k: k \neq j} \delta_{y_k} \right) = \beta \lim_{r \to \infty} \sum_{k: k \neq j, |y_k| < r} \frac{1}{x - y_k} - \int_{-r}^r \hat{\rho}(x)dx \frac{\hat{\rho}(x)dx}{-x},$$

and for $\mu_{\text{soft}, \beta}$-a.s. $\xi$, ISDE (1.5) with $X(0) = \mathcal{I}(\xi)$ has a strong solution satisfying the $\mu_{\text{soft}, \beta, \text{Tail}}$-absolute continuity condition, and pathwise uniqueness holds for ISDE (1.5) with the $\mu_{\text{soft}, \beta, \text{Tail}}$-absolute continuity condition [18] Theorem 2.3]. In the case $\beta = 2$ the facts that $T(\mathcal{M})$ is $\mu_{\text{soft}, 2}$-trivial and that $\mu_{\text{soft}, 2, \text{Tail}} = \mu_{\text{soft}, 2}$ were shown in [15].

Determining whether $\mu_{\text{soft}, \beta}$ has the quasi-Gibbs property for general $\beta$ and finding its logarithmic derivative is (4.8) are interesting and important problems.

**Example 4.4** (Bessel random point field). Let $S = [0, \infty)$ and $1 \leq \alpha < \infty$. Let $\mu_{\text{hard}, 2}$ be the determinantal point process with Bessel kernel

$$K_{J_\alpha}(x, y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J_\alpha'((\sqrt{y}) - \sqrt{x}J_\alpha'(\sqrt{x})J_\alpha(\sqrt{y})}{2(x - y)},$$

(4.9)
In [6] it was shown that $\mu_{\text{hard},2}$ is a quasi-Gibbs measure and that the related process is the unique strong solution of the ISDE

$$dX_j(t) = dB_j(t) + \left\{ \frac{\alpha}{2X_j(t)} + \sum_{k=1, k\neq j}^{\infty} \frac{1}{X_j(t) - X_k(t)} \right\} dt$$

with the $\mu_{\text{hard},2}$-absolute continuity condition.

**Example 4.5** (Ginibre random point field). Let $S = \mathbb{R}^2$ be identified as $\mathbb{C}$. Let $\mu_{\text{Gin}}$ be the DPP with the kernel $K_{\text{Gin}} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined by

$$K_{\text{Gin}}(x, y) = \frac{1}{\pi} e^{-|x|^2/2 - |y|^2/2} e^{xy}.$$

In [13] it was shown that $\mu_{\text{Gin}}$ is a quasi-Gibbs measure, and in [12] that the related process is a solution of the ISDE

$$dX_j(t) = dB_j(t) - X_j(t)dt + \lim_{r \to \infty} \sum_{k : k \neq j} \frac{X_j(t) - X_k(t)}{|X_j(t) - X_k(t)|^2} dt.$$

The pathwise uniqueness of solutions of (4.11) with the $\mu_{\text{Gin}}$-absolute continuity condition was shown in [19].

§ 5. Remarks

In the previous section we gave some examples of DPPs that are not canonical Gibbs measures but quasi-Gibbs measures. It is expected that quite general DPPs have the quasi-Gibbs property. We thus present examples of DPPs related to random matrix theory or non-colliding Brownian motions, whose quasi-Gibbs property have not been shown.

**Example 5.1** (Pearcey process). Consider $2N$ noncolliding Brownian motions, in which all particles start from the origin and $N$ particles end at $\sqrt{N}$ at time $t = 1$, and the other $N$ particles end at $-\sqrt{N}$ at $t = 1$. We denote the system by $(X_1^N(t), \ldots, X_{2N}^N(t))$, $0 \leq t \leq 1$. When $N$ is very large, there is a cusp at $x_0^N = 0$ when $t_0 = \frac{1}{2}$, that is, before time $t_0$ particles are in one interval with high probability, while after time $t_0$ they are separated into two intervals by the origin. We denote the distribution

$$\sum_{j=1}^{2N} \delta_{2^{3/2}(2N)^{1/4}X_j^N(t)}(\frac{1}{2})$$
on \( \mathcal{M} \) by \( \mu^N_{\text{pearcey}} \). It was proved in Adler-Orantin-von Moerbeke \[1\] that
\[
\mu^N_{\text{pearcey}} \rightarrow \mu_{\text{pearcey}}, \quad \text{weakly as } N \rightarrow \infty
\]
and that \( \mu_{\text{pearcey}} \) is the DPP \( K_{\text{pearcey}}(x, y) \) given by
\[
K_{\text{pearcey}}(x, y) = \frac{P(x)Q''(y) - P'(x)Q'(y) + P''(x)Q(y)}{x - y}, \quad x, y \in \mathbb{R},
\]
with
\[
Q(y) = \frac{i}{2\pi} \int_{-i\infty}^{i\infty} e^{-u^{4}/4-uy} \, du \quad \text{and} \quad P(x) = \frac{1}{2\pi i} \int_C e^{v^{4}/4+vx} \, dv,
\]
where the contour \( C \) is given by the ingoing rays from \( \pm\infty e^{i\pi/4} \) to 0 and the outgoing rays from 0 to \( \pm\infty e^{-i\pi/4} \). These integrals are known as Pearcey’s integrals \[20\].

Example 5.2 (Tacnode process). Consider two groups of non-colliding pinned Brownian motions \( (X^N_1(t), \ldots, X^N_2N(t)) \) in the time interval \( 0 \leq t \leq 1 \), where one group of \( N \) particles starts and ends at \( \sqrt{N} \) and the other group of \( N \) particles starts and ends at \( -\sqrt{N} \). The distribution \( (N^{1/6}X^N_1(1), N^{1/6}X^N_2(1), \ldots, N^{1/6}X^N_{2N}(1/2)) \) on the Weyl chamber of type \( A_{2N-1} \)
\[
\mathbb{W}_{2N} = \left\{ x = (x_1, x_2, \ldots, x_{2N}) : x_1 < x_2 < \cdots < x_{2N} \right\},
\]
is given by
\[
m^N_{\text{tac}}(dX_{2N}) = \frac{1}{Z} \left[ \det_{1 \leq i, j \leq 2N} \left( e^{-2|x_i - a_j|^2} \right) \right]^2,
\]
where \( a_j = -\sqrt{N} \) for \( 1 \leq j \leq N \) and \( a_j = \sqrt{N} \) for \( N + 1 \leq j \leq 2N \). We denote the distribution of \( \sum_{j=1}^{2N} \delta_{N^{1/6}x_j} \) under \( m^N_{\text{tac}} \) by \( \mu^N_{\text{tac}} \). It was proved in Delvaux-Kuijlaars-Zhang \[3\] and Johansson \[7\] that
\[
\mu^N_{\text{tac}} \rightarrow \mu_{\text{tac}}, \quad \text{weakly as } N \rightarrow \infty
\]
and that \( \mu_{\text{tac}} \) is the DPP with the correlation kernel
\[
K_{\text{tac}}(x, y) \equiv L_{\text{tac}}(x, y) + L_{\text{tac}}(-x, -y), \quad x, y \in \mathbb{R},
\]
where
\[
L_{\text{tac}}(x, y) = K_{\text{Ai,2}}(x, y)
\]
\[
+ 2^{1/3} \int_{(0, \infty)^2} \, dudv \, \text{Ai}(y + 2^{1/3}u)R(u, v)\text{Ai}(x + 2^{1/3}v)
\]
\[
- 2^{1/3} \int_{(0, \infty)^2} \, dudv \, \text{Ai}(-y + 2^{1/3}u)\text{Ai}(u + v)\text{Ai}(x + 2^{1/3}v)
\]
\[
- 2^{1/3} \int_{(0, \infty)^3} \, dudvdw \, \text{Ai}(-y + 2^{1/3}u)R(u, v)\text{Ai}(v + w)\text{Ai}(x + 2^{1/3}w).
\]
Here, $R(x, y)$ is the resolvent operator for the restriction of the Airy kernel to $[0, \infty)$, that is, the kernel of the operator

\begin{equation}
R = (I - K_{Ai})^{-1}K_{Ai}
\end{equation}

on $L^2[0, \infty)$.

In [3, 7] it was also shown that

\[ \Xi_N(t) \equiv \sum_{j=1}^{2N} \delta_{N^{1/6}X_j(\frac{1}{2} + N^{-1/3}t)} \to \Xi(t), \quad \text{as } N \to \infty, \]

in the sense of finite-dimensional distributions, where $\Xi(t)$ is a reversible process with reversible measure $\mu_{\text{tac}}$. We expect that $\Xi(t)$ is the diffusion process associated with the Dirichlet form $(\mathcal{E}^{\mu_{\text{tac}}}, \mathcal{D}^{\mu_{\text{tac}}})$.

References

[1] Adler, M., Orantin, N. and von Moerbeke, P., Universality for the Pearcey process, Physica D 239 (2010), 924–941.
[2] Anderson, G. W., Guionnet, A. and Zeitouni, O., An Introduction to Random Matrices, Cambridge university press, 2010.
[3] Delvaux, S., Kuijlaars, B.J. and Zhang, L., Critical Behavior of Nonintersecting Brownian motions at a Tacnode, Comm. Pure Appl. Math. 64 (2011), 1305–1383.
[4] Dyson, F. J., A Brownian-motion model for the eigenvalues of a random matrix, J. Math. Phys. 3 (1962), 1191–1198.
[5] Fukushima, M., Oshima, Y. and Takeda, M., Dirichlet forms and symmetric Markov processes, 2nd ed., Walter de Gruyter, 2011.
[6] Honda, R. and Osada, H., Infinite-dimensional stochastic differential equations related to the Bessel random point fields, Stochastic Processes and their Applications 125 (2015), 3801–3822.
[7] Johansson, K., Non-colliding Brownian motions and the extended Tacnode process, Comm. Math. Phys., 269 (2012), 571–609.
[8] Kawamoto, Y. and Osada, H., Finite particle approximations of interacting Brownian motions in infinite dimensions and SDE gaps, (in preparation).
[9] Mehta, M. L., Random Matrices. 3rd edition, Amsterdam: Elsevier, 2004
[10] Osada, H., Dirichlet form approach to infinite-dimensional Wiener processes with singular interactions, Commun. Math. Phys., 176 (1996), 117–131.
[11] Osada, H., Tagged particle processes and their non-explosion criteria, J. Math. Soc. Japan, 62 (2010), 867–894.
[12] Osada, H., Infinite-dimensional stochastic differential equations related to random matrices, Probability Theory and Related Fields, 153 (2012), 471–509.
[13] Osada, H., Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials, Ann. of Probab., 41 (2013), 1–49.
[14] Osada, H., Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials II : Airy random point field, *Stochastic Processes and their Applications*, 123 (2013), 813–838.

[15] Osada, H. and Osada, S., Discrete approximations of determinantal point processes on continuous space : tree representations and tail triviality, (preprint) arXiv:1517677 [math.PR].

[16] Osada, H. and Tanemura, H., Cores of Dirichlet forms related to Random Matrix Theory, * Proc. Jpn. Acad., Ser. A*, 90 (2014), 145–150.

[17] Osada, H. and Tanemura, H., Strong Markov property of determinantal processes with extended kernels, *Stochastic Processes and their Applications*, 126 (2016), 186–208.

[18] Osada, H. and Tanemura, H., Infinite-dimensional stochastic differential equations arising from Airy random point fields, (preprint) arXiv:1408.0632 [math.PR].

[19] Osada, H. and Tanemura, H., Infinite dimensional stochastic differential equations and tail σ-fields, (preprint) arXiv:1412.8674 [math.PR].

[20] Pearcey, T., The structure of an electromagnetic field in the neighbourhood of a cusp of a caustic, *Phil. Mag.* 37 (1946), 311–317.

[21] Ramírez, J.A., Rider, B. and Virág, B., Beta ensembles, stochastic Airy spectrum, and a diffusion, *Journal of the American Mathematical Society*, 24 (2011), 919–944.

[22] Ruelle, D., Superstable interactions in classical statistical mechanics, *Commun. Math. Phys.* 18 (1970), 127–159.

[23] Tsai, Li-Cheng, Infinite dimensional stochastic differential equations for Dyson’s model, *Probability Theory and Related Fields*, DOI 10.1007/s00440-015-0672-2.

[24] Valkó, B. and Virág, B., Continuum limits of random matrices and the Brownian carousel, *Inventions* 177 (2009), 463–508.