On approximate controllability of a class of degenerate fractional order distributed systems

V E Fedorov\textsuperscript{1,2,3}, D M Gordievskikh\textsuperscript{4} and N V Filin\textsuperscript{1,3}

\textsuperscript{1} Mathematical Analysis Department, Mathematics Faculty, Chelyabinsk State University, 129 Kashirin Brothers St., 454001 Chelyabinsk, Russia
\textsuperscript{2} Laboratory of Functional Materials, South Ural State University, Lenin Av. 76, 454080 Chelyabinsk, Russia
\textsuperscript{3} Department of Differential Equations, N.N. Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences, 16 S. Kovalevskaya St., 620108 Yekaterinburg, Russia
\textsuperscript{4} Department of Physics, Mathematics and Information Technology Education, Shadrinsk State Pedagogical University, 3 Karl Liebknecht St., 641870 Shadrinsk, Russia

E-mail: kar@csu.ru

Abstract. The approximate controllability issues for a class of control systems, whose dynamics is described by an equation in a Banach space with a linear degenerate operator at the Riemann — Liouville fractional derivative, is investigated. Under the condition of \( p \)-boundedness of the pair of operators in the equation the control system is reduced to subsystems on two mutually complement subspaces. It was shown that the approximate controllability of the whole system is equivalent to the approximate controllability of the two subsystems. Criteria of the approximate controllability for the system and two subsystems are obtained. Analogous results are got on the approximate controllability for free time and for systems of the same form with a finite-dimensional input. The obtained criteria were applied to the investigation of the approximate controllability for a distributed system with polynomials of a differential with respect to the spatial variables self-adjoint elliptic operator and for a system of the Scott-Blair — Oskolkov type.

1. Introduction

We study the issues of the approximate controllability for distributed control systems of the form

\[ D^\alpha_t L x(t) = M x(t) + B u(t) + y(t), \]  \hspace{1cm} (1)

where \( D^\alpha_t \) is the Riemann — Liouville fractional derivative, \( \mathcal{X}, \mathcal{Y}, \mathcal{U} \) are Banach spaces, operators \( L, M : \mathcal{X} \to \mathcal{Y} \) are linear and closed, the operator \( L \) is degenerate, i.e. \( \ker L \neq \{0\} \), operator \( B : \mathcal{U} \to \mathcal{Y} \) is linear and continuous, \( u : [0, T] \to \mathcal{Y} \) is a control function, \( y : [0, T] \to \mathcal{Y} \) is a given function. Such evolution equations with a degenerate operator at the highest order derivative are often called degenerate. The specifics of degenerate evolution equations are such that the natural conditions for determining the initial state of the corresponding systems are not Cauchy type conditions, but the generalized conditions of Showalter — Sidorov \( (Px)^{(k)}(0) = x_k \), \( k = 0, 1, \ldots, m-1 \), \( m-1 < \alpha \leq m \in \mathbb{N} \). The projection \( P \) will be defined below.

Controllability (or approximate controllability) problems for systems of the form \( D^\alpha_t x(t) = S x(t) + B u(t) + y_1(t) \), which are solved with respect to the time derivative, having, possibly,
dependent on $t$ operators $S$, $B$, at $\alpha = 1$ were researched in [1, 2, 3, 4, 5], and by many other authors, see the surveys [6, 7]. The cases of fractional $\alpha$ were studied in the works [8, 9, 10] and others.

The controllability and approximate controllability of various degenerate systems (1) ($\ker L \neq \{0\}$) of the order $\alpha = 1$ were investigated in [11, 12, 13, 14, 15, 16]. For systems (1) of fractional order $\alpha$, not solved with respect to the fractional derivative, these issues were studied, for example, in the works [17, 18, 19]. The approximate controllability of various classes of essentially degenerate systems of the form (1), when $\ker L \neq \{0\}$, with the Gerasimov — Caputo and the Riemann — Liouville fractional derivatives was studied in papers [20, 21, 22, 23, 24, 25]. In these works the cases of a respectively bounded pair of operators $(L, M)$ (in [20, 21, 22]), where there exists an analytical resolving family of operators of the corresponding homogeneous equation, and of a respectively sectorial pair $(L, M)$ (in [23, 24, 25]), when such family there exists in a sector of the complex plain. Here we close this cycle of works by investigating a case of a respectively bounded pair of operators $(L, M)$ and the Riemann — Liouville fractional derivative, which has not yet been considered.

In the present paper after the second section with preliminaries, the definition of the approximate controllability in time $T$ were introduced for the system (1) and its subsystems on the degeneration subspace and on the phase space of the corresponding homogeneous equation. Here it was shown that the approximate controllability of the whole system is equivalent to the approximate controllability of the two subsystems. Criteria of the approximate controllability for the system and two subsystems were obtained in the fourth section. The next two sections are dedicated to the approximate controllability for free time and for systems of the same form with a finite-dimensional input. The obtained criteria were applied to the investigation of the approximate controllability for a distributed system with polynomials of a differential with respect to the spatial variables self-adjoint elliptic operator in the seventh section and for a system of the Scott-Blair — Oskolkov type in the eighth section. In the last section it is demonstrated that the density of the operator $B$ is not necessary for the approximate controllability of a control system even in the finite-dimensional case.

2. The generalized Showalter — Sidorov type problem

Let $X$, $Y$ be Banach spaces, $L(X; Y)$ be the Banach space of linear continuous operators from $X$ into $Y$, and $Cl(X; Y)$ be the set of all linear closed operators, densely defined in $X$, acting into $Y$, $L(X; X') := L(X), Cl(X; X') := Cl(X)$.

Hereafter $L \in L(X; Y)$, $\ker L \neq \{0\}$, $M \in Cl(X; Y)$ has a domain $D_M$ with the graph norm of the operator $M$. Denote $\rho^L(M):=\{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in L(Y; X)\}$, $\sigma^L(M) := \mathbb{C} \setminus \rho^L(M)$, $N_0 := \{0\} \cup N$, $R^L_\mu(M) := (\mu L - M)^{-1} L$, $L^L_\mu(M) := L(\mu L - M)^{-1}$.

The operator $M$ is called $(L, \sigma)$-bounded, if $\sigma^L(M) \subset \{\mu \in \mathbb{C} : |\mu| \leq a\}$ for some $a > 0$. If the operator $M$ is $(L, \sigma)$-bounded, then there exist the projectors

$$P = \frac{1}{2\pi i} \int_{|\mu|=a} R^L_\mu(M) d\mu \in L(X), \quad Q = \frac{1}{2\pi i} \int_{|\mu|=a} L^L_\mu(M) d\mu \in L(Y).$$

Let $X^0 := \ker P$, $X^1 := \text{im} P$, $Y^0 := \ker Q$, $Y^1 := \text{im} Q$, $M_k := M|_{D_M \cap X^k}$, $L_k := L|_{X^k}$, $k = 0, 1$.

Theorem 1 [26]. Let an operator $M$ be $(L, \sigma)$-bounded. Then

(i) $X = X^0 \oplus X^1, \ Y = Y^0 \oplus Y^1$;

(ii) $L_k \in L(X^k; Y^k), \ k = 0, 1, M_0 \in Cl(X^0; Y^0), \ M_1 \in Cl(X^1; Y^1)$;

(iii) there exist operators $M^{-1}_0 \in L(Y^0; X^0), \ L^{-1}_1 \in L(Y^1; X^1)$.

For $p \in N_0$ an operator $M$ is called $(L, p)$-bounded, if it is $(L, \sigma)$-bounded and $G^p \neq 0$, $G^{p+1} = 0$, where $G = M_0^{-1} L_0 \in L(X^0)$.
Let $\Gamma(\cdot)$ is the Euler gamma function,

$$J_\beta^\beta x(t) := \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} x(s) \, ds, \ t > 0,$$

is the Riemann — Liouville fractional integral of the order $\beta > 0$. $D_t^\alpha L_x(t) := D_{t}^\mu J_t^{m-\alpha} x(t)$ is the fractional Riemann — Liouville derivative of the order $\alpha > 0$. Let at $\alpha, \beta > 0$ the Mittag-Leffler function

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \ \ X_{\alpha,\beta}(t) := E_{\alpha,\beta}(L_1^{-1} M_1 t^\alpha) P.$$

Let $\alpha \in (m-1, m]$. Consider the generalized Showalter — Sidorov type problem

$$D_t^{\alpha-m+k} P x(0) = x_k, \quad k = 0, 1, \ldots, m-1,$$  

for the degenerate fractional equation

$$D_t^\alpha L x(t) = M x(t) + y(t), \quad t \in (0, T].$$

A solution of problem (2), (3) on $(0, T]$ is a function $x \in C((0, T]; D_M)$, such that $J_t^{m-\alpha} P x \in C^{m-1}((0, T]; Y)$, $J_t^{m-\alpha} L x \in C^m((0, T]; Y)$, and equalities (2), (3) hold.

**Theorem 2** [27]. Let an operator $M$ be $(L, p)$-bounded, $(D_t^\alpha G)^1 M_0^{-1} (I - Q) y \in C((0, T]; Y)$, $l = 0, 1, \ldots, p$, $y \in C((0, T]; Y)$, $x_k \in X^1$, $k = 0, 1, \ldots, m-1$. Then problem (2), (3) has a unique solution, it has the form

$$x(t) = \sum_{k=0}^{m-1} t^{\alpha-m+k} X_{\alpha,\alpha-m+k+1}(t) x_k + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) L_1^{-1} Q y(s) \, ds -$$

$$\sum_{l=0}^{p} (D_t^\alpha G)^l M_0^{-1} (I - Q) y(t).$$

3. Approximate controllability of a system and its subsystems

Let an operator $M$ be $(L, p)$-bounded, $U$ is a Banach space, $y \in C_{\alpha, G}([0, T]; Y)$, where

$$C_{\alpha, G}([0, T]; Y) := \{ y \in C([0, T]; Y) : (D_t^\alpha G)^l M_0^{-1} (I - Q) y \in C([0, T]; Y), l = 0, 1, \ldots, p \}.$$

The dynamics of a control system is described by the equation

$$D_t^\alpha L x(t) = M x(t) + B u(t) + y(t),$$

its initial state is determined by the conditions

$$D_t^{\alpha-m+k} P x(0) = x_k, \quad k = 0, 1, \ldots, m-1.$$

Control functions $u(\cdot)$ will be chosen from the space

$$C_{\alpha, P}([0, T]; U) := \{ u \in C([0, T]; U) : (D_t^\alpha)^l u \in C([0, T]; U), l = 0, 1, \ldots, p \}.$$
By means of the results of Theorem 1 problem (4), (5) may be reduced to the Cauchy type
problem
\[ D_t^{α-m+k}x^1(0) = x_k, \quad k = 0, 1, \ldots, m - 1, \]  
for the system of two equations
\[ D_t^αx^1(t) = Sx^1(t) + L_1^{-1}Q(Bu(t) + y(t)), \]  
\[ D_t^αGx^0(t) = x^0(t) + M_0^{-1}(I-Q)(Bu(t) + y(t)) \]  
on the subspaces \( X^1 \) and \( X^0 \) respectively, where \( S = L_1^{-1}M_1 \in L(X^1), \) \( x^1(t) = Px(t), \) \( x^0(t) = (I-P)x(t). \) Due to results of [27] a unique solution of problem (6), (7) is
\[ x^1(t) = \sum_{k=0}^{m-1}a^{α-m+k}x_{a,a-m+k+1}(t)x_k + \int_0^t(t-s)^{α-1}x_{a,a}(t-s)L_1^{-1}Q(Bu(s) + y(s))ds, \]
and the unique solution of equation (8) is the function
\[ x^0(t) = -\sum_{i=0}^{p}(D_t^αG)^iM_0^{-1}(I-Q)(Bu(t) + y(t)). \]

Let \( x(T;\bar{x};u) \) be the value at \( t = T \) of the solution of problem (4), (5) with the initial data \( \bar{x} = (x_0, x_1, \ldots, x_{m-1}) \) in (5) and with the control function \( u. \) Analogously, \( x^1(T;\bar{x};u) \) be the value at the time moment \( T \) of the solution of (6), (7), and \( x^0(T;u) \) be the value at \( t = T \) of the solution of equation (8).

System (4) is called approximately controllable for time \( T > 0, \) if for all \( ε > 0, \) \( \bar{x} \in X, \) \( \bar{x} = (x_0, x_1, \ldots, x_{m-1}) \in (X^1)^m \) in (5) there exists a control function \( u \in C_{α,p}([0,T];U), \) such that \( \|x(T;\bar{x};u) - \bar{x}\|_X \leq ε. \)

System (7) is called approximately controllable for time \( T > 0, \) if for all \( ε > 0, \) \( \bar{x} \in X^1, \) \( \bar{x} = (x_0, x_1, \ldots, x_{m-1}) \in (X^1)^m \) in (6) there exists a control \( u \in C_{α,p}([0,T];U), \) such that \( \|x^1(T;\bar{x};u) - \bar{x}\|_{X^1} \leq ε. \)

System (8) is called approximately controllable for time \( T > 0, \) if for all \( ε > 0, \) \( \bar{x} \in X^0 \) there exists a control \( u \in C_{α,p}([0,T];U), \) such that \( \|x^0(T;u) - \bar{x}\|_{X^0} \leq ε. \)

Let us show that we can control two systems (7) and (8) at the same time by the same function \( u(\cdot) \) successfully.

**Theorem 3.** Let an operator \( M \) be \((L,p)\)-bounded, \( y \in C_{α,G}([0,T];Y). \) Then system (4) is approximately controllable for time \( T, \) if and only if systems (7) and (8) are approximately controllable for time \( T. \)

**Proof.** The direct statement is obvious, since system (4) is decomposed into two subsystems (7) and (8) on mutually complementary subspaces. Let us prove the inverse statement. Let for all \( \bar{x} \in X^0, \) \( ε > 0 \) there exists a function \( u_0 \in C_{α,p}([0,T];U), \) such that
\[ \| -\sum_{i=0}^{p}(D_t^αG)^iM_0^{-1}(Bu_0(t) + y(t))|_{t=T} - \bar{x}\|_X \leq ε/3, \]
and
\[ \forall \bar{x} \in (X^1)^m \quad \forall \bar{x} \in X^1 \quad \forall ε > 0 \quad \exists u_1 \in C_{α,p}([0,T];U) \quad \|x^1(T;\bar{x};u_1) - \bar{x}\|_{X^1} \leq ε/3. \]

Denote \( τ = T - t, \) \( v(τ) = u_1(T - τ) \) at \( τ \in [0,T], \) and
\[ v(τ) = \left( \frac{τ}{δ} \right)^{α(p+1)} \frac{1}{Γ(α(p+1)+1)} \sum_{k=0}^{p} a_k τ^{αk} + \sum_{k=0}^{p} u_k τ^{αk}, \quad τ \in [0,δ], \]  
\( a_k, u_k \in U, \) \( k = 0, 1, \ldots, p. \)
Therefore, \((D_{q}G)^{l}M_{0}^{-1}Bu(t)|_{t=0} = G^{l}M_{0}^{-1}Bu_{l}\), set \(u_{l} = (D_{q}G)^{l}u_{0}(t)|_{t=T},\ l = 0, 1, \ldots, p\).

Find \(a_{k}, k = 0, 1, \ldots, p\), such that \((D_{q}G)^{l}v(t)|_{t=\delta} = (D_{q}G)^{l}u_{0}(t)|_{t=T-\delta},\ l = 0, 1, \ldots, p\), from the corresponding system of linear algebraic equations with upper triangular matrix at the unknown vector, having nonzero elements on the main diagonal. Set \(u(t) = v(T-t), t \in [0, T]\), hence, \(u \in C_{\alpha, p}([0, T]; Y)\). For all \(t \in [T - \delta, T]\) we have \(\|u(t)\| \leq C\), where \(C > 0\) does not depend on \(\delta\). Choose a sufficiently small \(\delta > 0\), such that

\[
\varepsilon\delta^{-1} \geq 3 \max_{s \in [0, T]} s^{\alpha-1} \|X_{\alpha, \alpha}(s)L_{1}^{-1}QB\|_{\mathcal{L}(U, X)}(C + \max_{t \in [0, T]} (\|u_{1}(t)\|u_{t} + \|u_{0}(t)\|u_{t})),
\]

consequently, for every \(\hat{x} = \hat{x}^{0} + \hat{x}^{1}\)

\[
\|x(T; \tau; u) - \hat{x}\|_{X} \leq \|x^{0}(T; u_{0}) - \hat{x}^{0}\|_{X} + \|x^{1}(T; \tau; u_{1}) - \hat{x}^{1}\|_{X} + \|x^{1}(T; \tau; u) - x^{1}(T; \tau; u_{1})\|_{X}
\]

\[
\leq 2\varepsilon/3 + \delta \max_{s \in [0, T]} s^{\alpha-1} \|X_{\alpha, \alpha}(s)L_{1}^{-1}QB\|_{\mathcal{L}(U, X)} \max_{t \in [0, T]} (\|u_{1}(t)\|u_{t} + \|u(t)\|u_{t}) \leq \varepsilon.
\]

4. Approximate controllability conditions

Let \(Z\) be a Banach space, \(A\) be some set of indices, \(D_{\alpha} \subset Z\) for \(\alpha \in A\). Denote \(\text{span}\{D_{\alpha} : \alpha \in A\}\) the linear span of the union of sets \(D_{\alpha}, \alpha \in A\), and by \(\text{im}D_{\alpha} : \alpha \in A\) its closure in the space \(Z\) is denoted. Denote by \(\text{im}A\) the closure of the image \(\text{im}A\) of an operator \(A : D_{A} \rightarrow Z\).

**Lemma 1.** Let an operator \(M\) be \((L, p)\)-bounded, \(Qy \in C([0, T]; Y)\). Then system (7) is approximately controllable for time \(T\) if and only if \(\text{span}\{\text{im}X_{\alpha, \alpha}(s)L_{1}^{-1}QB : 0 < s < T\} = \mathcal{X}\).

**Proof.** The proof is similar to the proof of analogous assertion in [22], where the system with the Gerasimov — Caputo derivative is considered.

We can formulate the obtained result in the next form.

**Theorem 4.** Let \(A \in \mathcal{L}(X), \ z \in C([0, T]; X)\). Then the system \(D_{q}^{\alpha}x(t) = Ax(t) + Bu(t) + z(t)\) is approximately controllable for time \(T\) if and only if \(\text{span}\{\text{im}X_{\alpha, \alpha}(s^{\alpha}A)B : 0 < s < T\} = \mathcal{X}\).

**Remark 1.** Due to Lemma 1 the approximate controllability of (7) for time \(T\) implies its approximate controllability for any greater time \(T_{1} > T\).

**Lemma 2.** Let an operator \(M\) be \((L, p)\)-bounded, \(y \in C_{\alpha, G}([0, T]; Y)\). Then system (8) is approximately controllable for time \(T\), if and only if

\[\text{span}\{\text{im}G^{l}M_{0}^{-1}(I - Q)B : l = 0, 1, \ldots, p\} = \mathcal{X}\^0.\]

**Proof.** The assertion follows from the form of the solution of system (8):

\[x^{0}(t) = -\sum_{l=0}^{p} (D_{q}G)^{l}M_{0}^{-1}(I - Q)y(t) - \sum_{l=0}^{p} G^{l}M_{0}^{-1}(I - Q)B(D_{q}G)^{l}u(t).\]

**Remark 2.** Lemma 2 implies that an approximately controllable for time \(T\) system (8) is approximately controllable for any time \(T_{1} > 0\). Therefore, due to Remark 1 and Theorem 3 the approximate controllability of system (4) for time \(T\) implies its approximate controllability for any greater time \(T_{1} > T\).

The next criterion of the approximate controllability follows from Theorem 3, Lemma 1 and Lemma 2.

**Theorem 5.** Let an operator \(M\) be \((L, p)\)-bounded, \(y \in C_{\alpha, G}([0, T]; Y)\). Then system (4) is approximately controllable for time \(T\), if and only if

\[\text{span}\{\text{im}G^{l}M_{0}^{-1}(I - Q)B : l = 0, 1, \ldots, p\} = \mathcal{X}\^0, \ \text{span}\{\text{im}X_{\alpha, \alpha}(s)L_{1}^{-1}QB : 0 < s < T\} = \mathcal{X}\^1.\]

Let us formulate simple sufficient conditions of the approximate controllability for the considered systems.
Corollary 1. Let an operator $M$ be $(L,p)$-bounded, $y \in C_{\alpha,G}([0,T];\mathcal{Y})$. Then

(i) if $\operatorname{im}Q = \mathcal{Y}$, then system (7) is approximately controllable for any time $T$;

(ii) if $\operatorname{im}(I-Q)B = \mathcal{Y}$, then system (8) is approximately controllable for any time $T > 0$;

(iii) if $\operatorname{im}B = \mathcal{Y}$, then system (4) is approximately controllable for any time $T > 0$.

Proof. Note, that $\lim_{t \to +0} X_{\alpha,\alpha}(t)|_{X^1} = \Gamma(\alpha)^{-1}I|_{X^1}$ in the norm of the space $L(X^1)$, consequently, for small $t > 0$ the operators $X_{\alpha,\alpha}(t)|_{X^1} : X^1 \to X^1$, $L_1^{-1} : \mathcal{Y} \to X^1$ are homeomorphisms. Subsequently, for such $t$ the image $\operatorname{im}X_{\alpha,\alpha}(t)L_1^{-1}Q$ is dense in $X^1$, if and only if the image $\operatorname{im}Q$ is dense in the subspace $\mathcal{Y}$. Due to Lemma 1, we obtain assertion (i).

The equality $\operatorname{im}(I-Q)B = \mathcal{Y}$ and the continuity of $M_0^{-1}$ implies, that $\operatorname{im}M_0^{-1}(I-Q)B = M_0^{-1}[\mathcal{Y}] = D_{M_0}$. Since the domain $D_{M_0}$ is dense in $\mathcal{Y}$ and due to Lemma 2 we obtain statement (ii). Here we can take the control function in the form $u(t) = a_0 t^{a-m}$ with some $a_0 \in \mathcal{U}$, so that $D_{M_0}^a u(t) \equiv 0$.

Statements (i) and (ii) implies statement (iii) due to Theorem 3.

Remark 3. Due to Corollary 1 the issue of the approximate controllability is essential, if $\operatorname{im}B \neq \mathcal{Y}$ only.

5. Approximate controllability for free time

Let us consider the issues of the approximate controllability for free time of the considered systems. Suppose that in (4)

$$y \in C_{\alpha,G}([0,\infty);\mathcal{Y}) := \{y \in C([0,\infty);\mathcal{Y}) : (D_0^\alpha G)^l M_0^{-1}(I-Q)y \in C([0,\infty);\mathcal{Y}), l = 0,1,\ldots,p\}.$$

System (4) is called approximately controllable for free time, if for all $\varepsilon > 0$, $\hat{x} \in \mathcal{X}$, $\overline{\mathcal{Y}} = (x_0, x_1, \ldots, x_{m-1}) \in (X^1)^m$ in (5) there exist $T > 0$ and a control $u \in C_{\alpha,p}([0,T];\mathcal{U})$, such that $\|x(T;\overline{\mathcal{Y}},u) - \hat{x}\|_{X} \leq \varepsilon$.

System (7) is called approximately controllable for free time, if for all $\varepsilon > 0$, $\hat{x} \in X^1$, $\overline{\mathcal{X}} = (x_0, x_1, \ldots, x_{m-1}) \in (X^1)^m$ in (6) there exist $T > 0$, $u \in C_{\alpha,p}([0,T];\mathcal{U})$, such that $\|x^1(T;\overline{\mathcal{X}},u) - \hat{x}\|_{X^1} \leq \varepsilon$.

It is obvious, that approximately controllable for time $T > 0$ system is approximately controllable for free time, the inverse statement is not valid. So, sufficient conditions of the approximate controllability for time $T > 0$ of a control system are sufficient for its approximate controllability for free time. Necessary conditions of the approximate controllability for free time are necessary for the approximate controllability for time $T > 0$ of the same system.

Theorem 6. Let an operator $M$ be $(L,p)$-bounded, $y \in C_{\alpha,G}([0,\infty);\mathcal{Y})$. Then system (4) is approximately controllable for free time, if and only if systems (7) and (8) are approximately controllable for free time.

Proof. As in Theorem 3 the inverse statement is not evident only, let us prove it. Let for all $\hat{x}^0 \in X^0$, $\varepsilon > 0$ there exist $T_0 > 0$, $w_0 \in C_{\alpha,p}([0,T_0];\mathcal{U})$, such that

$$\left\| - \sum_{l=0}^{p} (D_0^l G)^l M_0^{-1}(Bw_0(t) + y(t))|_{t=T_0} - \hat{x}^0 \right\|_{X} \leq \varepsilon/3,$$

and

$$\forall \varepsilon > 0 \exists \hat{x}^1 \in X^1 \forall \overline{\mathcal{X}} \in (X^1)^m \exists T_1 > 0 \exists u_1 \in C_{\alpha,p}([0,T_1];\mathcal{U}) \left\| x^1(T_1;\overline{\mathcal{X}},u_1) - \hat{x}^1 \right\|_{X^1} \leq \varepsilon/3.$$

Take in the construction of the control in the proof of Theorem 3 $T = T_1$ and $w_0(t) = w_0(t + T_0 - T)$ at $t \in \{0,T - T_0\}$, and obtain the required inequality.
Lemma 3. Let an operator $M$ be $(L,p)$-bounded, $Qy \in C([0,\infty);Y)$. Then system (7) is approximately controllable for free time, if and only if
\[
\overline{\text{span}}\{\text{im}X_{\alpha,a}(s)L_1^{-1}QB : s > 0\} = \mathcal{X}^1.
\] (9)

Proof. It is sufficient to consider only the approximate controllability of (7) from zero state ($\tau = 0$). Suppose that the system is not approximately controllable for free time from zero. Then the set of vectors
\[
\int_0^T (T-s)^{\alpha-1}X_{\alpha,a}(T-s)L_1^{-1}QBu(s)ds, \quad T > 0, \ u \in C_{a,p}([0,T];\mathcal{U}),
\]
is not dense in $\mathcal{X}^1$. Hence, due to the Hahn — Banach theorem there exists $f \in \mathcal{X}^{1*} \setminus \{0\}$, such that
\[
f \left( \int_0^T (T-s)^{\alpha-1}X_{\alpha,a}(T-s)L_1^{-1}QBu(s)ds \right) = 0
\]
at all $T > 0, \ u \in C_{a,p}([0,T];\mathcal{U})$.

At every $v$ from the Lebesgue — Bochner space $L_q(0,T;\mathcal{U}), \ 1 \leq q < \infty$, there exists a sequence $\{u_n\} \subset C_{a,p}([0,T];\mathcal{U}) \subset C_{a,p}([0,T];\mathcal{U})$, such that $\lim_{n \to \infty} u_n = v$ in $L_q(0,T;\mathcal{U})$. Therefore, we have at $\alpha \geq 1$
\[
\left| \int_0^T f((T-s)^{\alpha-1}X_{\alpha,a}(T-s)L_1^{-1}QB(u_n(s) - v(s))) ds \right| \leq
\]
\[
\|f\|_{\mathcal{X}^{1*}}T^{\alpha-1}E_{\alpha,a}\left(\|L_1^{-1}M_1\|_{\mathcal{L}(\mathcal{X}^1)}T^\alpha\right)\|L_1^{-1}QB\|_{\mathcal{L}(\mathcal{U};\mathcal{X})} \int_0^T \|u_n(s) - v(s)\|_\mathcal{U} ds \to 0
\]
as $n \to \infty$. At $\alpha \in (0,1)$
\[
\left| \int_0^T f((T-s)^{\alpha-1}X_{\alpha,a}(T-s)L_1^{-1}QB(u_n(s) - v(s))) ds \right| \leq
\]
\[
\|f\|_{\mathcal{X}^{1*}}E_{\alpha,a}\left(\|L_1^{-1}M_1\|_{\mathcal{L}(\mathcal{X}^1)}T^\alpha\right)\|L_1^{-1}QB\|_{\mathcal{L}(\mathcal{U};\mathcal{X})} \int_0^T \|u_n(s) - v(s)\|_\mathcal{U} ds =
\]
\[
\|f\|_{\mathcal{X}^{1*}}E_{\alpha,a}\left(\|L_1^{-1}M_1\|_{\mathcal{L}(\mathcal{X}^1)}T^\alpha\right)\|L_1^{-1}QB\|_{\mathcal{L}(\mathcal{U};\mathcal{X})} \int_0^T \|u_n(s) - v(s)\|_\mathcal{U} ds = \int_0^T \|u_n(s) - v(s)\|_\mathcal{U} ds \to 0
\]
as $n \to \infty$. Here $r \in (1,1/1 - \alpha)$.

Therefore, equality (10) is true for all $T > 0, \ u \in L_q(0,T;\mathcal{U})$. Take arbitrary $T > 0, \ t_0 \in (0,T), \ \delta \in (0,\min\{t_0,T-t_0\})$, consider $u_\delta(t) := w \in \mathcal{U}$ at $t \in [t_0 - \delta, t_0 + \delta], \ u_\delta(t) := 0$ for $t \in [0,T] \setminus [t_0 - \delta, t_0 + \delta]$. Hence, $u_\delta \in L_q(0,T;\mathcal{U}), \ q > 1$, by the continuity of the integrand
\[
0 = \frac{1}{2\delta} \int_{t_0 - \delta}^{t_0 + \delta} f( (T-s)^{\alpha-1}X_{\alpha,a}(T-s)L_1^{-1}QBw(s)) ds = f( \frac{(T-\xi)^{\alpha-1}X_{\alpha,a}(T-\xi)L_1^{-1}QBw(t_0 - \delta)}{2\delta} )
\]
at some $\xi \in (t_0 - \delta, t_0 + \delta)$. Consequently, $f \left( X_{\alpha,\alpha}(T - \xi)L_1^{-1}QBw \right) = 0$, passing to the limit as $\delta \to 0^+$ we obtain that $f \left( X_{\alpha,\alpha}(T - t_0)L_1^{-1}QBw \right) = 0$ for every $t_0 \in (0, T)$, $w \in \mathcal{U}$. Hence, due to condition (9) $f = 0$, we have the contradiction. Therefore, the system (7) is approximately controllable for free time.

The direct assertion of Lemma 3 follows from the integral form of the solution of equation (7) with zero initial data.

In other words, we have the next result.

**Theorem 7.** Let $A \in \mathcal{L}(\mathcal{X})$, $z \in C([0, \infty); \mathcal{X})$. Then the system $D^t \alpha x(t) = Ax(t) + Bu(t) + z(t)$ is approximately controllable for free time, if and only if $\text{span}\{\text{im}E_{\alpha,\alpha}(s^\alpha A) : s > 0\} = \mathcal{X}$.

**Lemma 4.** Let an operator $M$ be $(L, p)$-bounded, $y \in C_{\alpha,G}([0, \infty); \mathcal{Y})$. Then system (8) is approximately controllable for free time, if and only if it is approximately controllable for some time $T > 0$.

**Proof.** Indeed, we see that in Lemma 2 the necessary an sufficient conditions of the approximate controllability for time $T$ of system (8) do not depend on time $T$.

Lemmas 2, 3, 4 and Theorem 6 implies the next criterion of the approximate controllability of system (4).

**Theorem 8.** Let an operator $M$ be $(L, p)$-bounded, $y \in C_{\alpha,G}([0, \infty); \mathcal{Y})$. Then system (4) is approximately controllable for free time, if and only if

$$\text{span}\{\text{im}G^t M_0^{-1}(I - Q)B : t = 0, 1, \ldots, p\} = \mathcal{X}^0$$

$$\text{span}\{\text{im}X_{\alpha,\alpha}(s)L_1^{-1}QB : s > 0\} = \mathcal{X}^1.$$

### 6. Approximate controllability with finite-dimensional input

Let $y : [0, T] \to \mathcal{Y}$, $b_i \in \mathcal{Y}$, $i = 1, 2, \ldots, n$. Consider the control system

$$D^t \alpha Lx(t) = Mx(t) + \sum_{i=1}^{n} b_i u_i(t) + y(t),$$  \hspace{1cm} (11)

where $u_i : [0, T] \to \mathbb{R}$ at some $T > 0$, $i = 1, 2, \ldots, n$. It is a partial case of system (4). Indeed, here we have $\mathcal{U} = \mathbb{R}^n$, $u = (u_1, u_2, \ldots, u_n)$, $Bu(t) = \sum_{i=1}^{n} b_i u_i(t)$. It is clear that $B \in \mathcal{L}(\mathbb{R}^n; \mathcal{Y})$.

Such control systems are called systems with a finite-dimensional input.

Under the condition of $(L, p)$-boundedness of the operator $M$ in according to Theorem 1 equation (11) can be reduced to the system of the two equations

$$D^t \alpha x^1(t) = Sx^1(t) + L_1^{-1} \sum_{i=1}^{n} b_i^1 u_i(t) + L_1^{-1} y^1(t),$$  \hspace{1cm} (12)

$$D^t \alpha Gx^0(t) = x^0(t) + M_0^{-1} \sum_{i=1}^{n} b_i^0 u_i(t) + M_0^{-1} y^0(t).$$  \hspace{1cm} (13)

Here $b_i^1 = Q b_i$, $b_i^0 = (I - Q) b_i$, $i = 1, 2, \ldots, n$, $x^1(t) = P x(t)$, $x^0(t) = (I - P) x(t)$, $y^1(t) = Q y(t)$, $y^0(t) = (I - Q) y(t)$, $t \geq 0$. The solution of problem (5) for equation (11) has the form

$$x(t) = \sum_{k=0}^{m-1} t^{\alpha - m + k} X_{\alpha,\alpha - m + k + 1}(t)x_k + \int_{0}^{t} (t - s)^{\alpha - 1} X_{\alpha,\alpha}(t - s)L_1^{-1} \left( \sum_{i=1}^{n} b_i^1 u_i(s) + y^1(s) \right) ds - \sum_{i=0}^{p} (D^t \alpha G)^i M_0^{-1} \left( \sum_{i=1}^{n} b_i^0 u_i(t) + y^0(t) \right).$$  \hspace{1cm} (14)

$$.$$  \hspace{1cm} (15)
Here (14) is the unique solution of the Cauchy type problem for equation (12), and (15) is the unique solution of equation (13). Control functions $u = (u_1, \ldots, u_n)$ will be chosen from the space $C_{\alpha,p}([0,T]; \mathbb{R}^n)$.

**Theorem 9.** Let an operator $M$ be $(L, p)$-bounded, $y \in C_{\alpha,G}([0,T]; \mathcal{Y})$. Then

(i) system (13) is approximately controllable for free time, if and only if $X^0 = D_{M_0}$ has a dimension not greater than $n(p+1)$, and span $\left\{ \sum_{l=0}^{p} (D_l^\alpha G)^l M_0^{-1} b_i^0, i = 1, 2, \ldots, n \right\} = X^0$;

(ii) system (12) is approximately controllable for time $T$, if and only if

$$\text{span}\{X_{\alpha,\alpha}(s)L^{-1}_1 Q b_i, 0 < s < T, i = 1, 2, \ldots, n\} = \mathcal{X}^1;$$

(iii) system (11) is approximately controllable for free time, if and only if $X^0 = D_{M_0}$ has a dimension not greater than $n(p+1)$,

$$\text{span}\left\{ \sum_{l=0}^{p} (D_l^\alpha G)^l M_0^{-1} b_i^0, i = 1, 2, \ldots, n \right\} = X^0,$$

$$\text{span}\{X_{\alpha,\alpha}(s)L^{-1}_1 Q b_i, 0 < s < T, i = 1, 2, \ldots, n\} = \mathcal{X}^1.$$

**Proof.** Due to Lemma 1, Lemma 2, Theorem 5 and the form of solutions (14), (15) we obtain the statements of this theorem. From the equality $\text{span}\left\{ \sum_{l=0}^{p} (D_l^\alpha G)^l M_0^{-1} b_i^0, i = 1, 2, \ldots, n \right\} = X^0$ it follows that the subspace $X^0$ has a finite dimension. Therefore, this subspace is closed, $X^0 = \text{span}\left\{ \sum_{l=0}^{p} (D_l^\alpha G)^l M_0^{-1} b_i^0, i = 1, 2, \ldots, n \right\} \subset D_{M_0},$ so $X^0 = D_{M_0}$.

**Remark 4.** We have $\mathcal{X}^0 = M[D_{M_0}] = \text{span}\left\{ \sum_{l=0}^{p} (D_l^\alpha J)^l b_i^0, i = 1, 2, \ldots, n \right\}$, where $J = L_0 M_0^{-1}$.

Consequently, the subspace $\mathcal{X}^0$ is also finite-dimensional.

Analogously it is easy to obtain the next criteria of the approximate controllability of systems with finite-dimensional input.

**Theorem 10.** Let an operator $M$ be $(L, p)$-bounded, $y \in C_{\alpha,G}([0,\infty); \mathcal{Y})$. Then

(i) system (13) is approximately controllable for free time, if and only if $X^0 = D_{M_0}$ has a dimension not greater than $n(p+1)$, and span $\left\{ \sum_{l=0}^{p} (D_l^\alpha G)^l M_0^{-1} b_i^0, i = 1, 2, \ldots, n \right\} = X^0$;

(ii) system (12) is approximately controllable for free time, if and only if

$$\text{span}\{X_{\alpha,\alpha}(s)L^{-1}_1 Q b_i, s > 0, i = 1, 2, \ldots, n\} = \mathcal{X}^1;$$

(iii) system (11) is approximately controllable for free time, if and only if $X^0 = D_{M_0}$ has a dimension not greater than $n(p+1)$, and

$$\text{span}\left\{ \sum_{l=0}^{p} (D_l^\alpha G)^l M_0^{-1} b_i^0, i = 1, 2, \ldots, n \right\} = X^0,$$

$$\text{span}\{X_{\alpha,\alpha}(s)L^{-1}_1 Q b_i, s > 0, i = 1, 2, \ldots, n\} = \mathcal{X}^1.$$

**Remark 5.** If an operator $M$ is $(L, p)$-bounded, $y \in C_{\alpha,G}([0,\infty); \mathcal{Y})$, then the approximate controllability for free time of system (13) implies that $M_0 \in \mathcal{L}(X^0, \mathcal{Y}^0)$. Indeed, it follows from the equality $D_{M_0} = X^0$ and the closedness of the operator $M_0$. 


7. A class of distributed control systems

Let \( n \geq m \), \( P_n(\lambda) = \sum_{i=0}^{n} c_i \lambda^i \), \( Q_m(\lambda) = \sum_{j=0}^{m} d_j \lambda^j \), \( c_i, d_j \in \mathbb{R} \), \( i = 0, 1, \ldots, n \), \( j = 0, 1, \ldots, m \),\( c_n \neq 0, d_m \neq 0 \). Suppose that \( \Omega \subset \mathbb{R}^d \) is a bounded region with sufficiently smooth boundary \( \partial \Omega \), operator pencil \( \Lambda, B_1, B_2, \ldots, B_r \) is regularly elliptic [28], where

\[
\Lambda u = \sum_{|q| \leq 2r} a_q(s) \frac{\partial^{q} u}{\partial s_1^{q_1} \partial s_2^{q_2} \ldots \partial s_d^{q_d}}, \quad a_q \in C^\infty(\Omega),
\]

\[
B_l u = \sum_{|q| \leq r_l} b_q(s) \frac{\partial^{q} u}{\partial s_1^{q_1} \partial s_2^{q_2} \ldots \partial s_d^{q_d}}, \quad b_q \in C^\infty(\partial \Omega), l = 1, 2, \ldots, r,
\]

\( q = (q_1, q_2, \ldots, q_d) \in \mathbb{N}_0^d, |q| = q_1 + \ldots + q_d \). Define an operator \( \Lambda_1 \in \mathcal{C}(L_2(\Omega)) \) with a domain \( D_{\Lambda_1} = H^{2r}(\Omega) \) [28] by the equality \( \Lambda_1 u = \Lambda u \). Suppose that \( \Lambda_1 \) is a selfadjoint operator, therefore, the spectrum \( \sigma(\Lambda_1) \) of \( \Lambda_1 \) is real and discrete [28]. Moreover, suppose that \( \sigma(\Lambda_1) \) is bounded from the right and does not contain zero point, \( \{ \varphi_k : k \in \mathbb{N} \} \) is orthonormal (in the sense of the inner product \( \langle \cdot, \cdot \rangle \) in \( L_2(\Omega) \)) system of eigenfunctions of \( \Lambda_1 \), numbered in nonincreasing order of their eigenvalues \( \{ \lambda_k : k \in \mathbb{N} \} \) taking into account their multiplicity.

Consider a distributed control system

\[
B_l \Lambda^k u(s, t) = 0, \quad k = 0, 1, \ldots, n - 1, \quad l = 1, 2, \ldots, r, \quad (s, t) \in \partial \Omega \times (0, T],
\]

\[
D_t^\alpha P_n(\Lambda) x(s, t) = Q_m(\Lambda) x(s, t) + (\mu - \Lambda) u(s, t), \quad (s, t) \in \Omega \times (0, T],
\]

in which the initial state is defined by the conditions

\[
D_t^{\alpha - m + k} P_n(\Lambda) x(s, 0) = y_k(s), \quad s \in \Omega.
\]

Here \( \mu \in \mathbb{R} \). Put

\[
\mathcal{X} = \{ v \in H^{2r}(\Omega) : B_l \Lambda^k v(s) = 0, \quad k = 0, 1, \ldots, n - 1, \quad l = 1, 2, \ldots, r, \quad x \in \partial \Omega \},
\]

\[
\mathcal{Y} = L_2(\Omega), \quad L = P_n(\Lambda) \in \mathcal{L}(\mathcal{X}; \mathcal{Y}), \quad M = Q_m(\Lambda) \in \mathcal{L}(\mathcal{Y}; \mathcal{X}), \quad U = H^{2r}(\Omega), \quad B = \mu - \Lambda \in \mathcal{L}(U; \mathcal{Y}).
\]

Then system (16), (17) has form (4). As can be shown directly, if the spectrum \( \sigma(\Lambda_1) \) does not contain common roots of the polynomials \( P_n(\lambda) \) and \( Q_m(\lambda) \), then operator \( M \) is \((L, 0)-\)bounded. At the same time

\[
P = Q = \sum_{P_n(\lambda_k) \neq 0} \langle \cdot, \varphi_k \rangle \varphi_k, \quad \mathcal{X}^0 = \mathcal{Y}^0 = \text{span} \{ \varphi_k : P_n(\lambda_k) = 0 \},
\]

\( \mathcal{X} \) is the closure of \( \text{span} \{ \varphi_k : P_n(\lambda_k) \neq 0 \} \) in the space \( \mathcal{X} \), \( \mathcal{Y}^1 \) is the closure of the same span in \( L_2(\Omega) \),

\[
M_0^{-1} = \sum_{P_n(\lambda_k) = 0} \frac{\langle \cdot, \varphi_k \rangle \varphi_k}{Q_m(\lambda_k)}, \quad L_0 = 0, \quad G = 0, \quad L_1^{-1} = \sum_{P_n(\lambda_k) \neq 0} \frac{\langle \cdot, \varphi_k \rangle \varphi_k}{P_n(\lambda_k)},
\]

\[
X_{\alpha, \alpha}(t) = \sum_{P_n(\lambda_k) \neq 0} E_{\alpha, \alpha} \left( \frac{Q_m(\lambda_k)t^\alpha}{P_n(\lambda_k)} \right) \langle \varphi_k, \cdot \rangle \varphi_k, \quad B = \sum_{k=1}^{\infty} (\mu - \lambda_k) \langle \varphi_k, \cdot \rangle \varphi_k,
\]

\[
M_0^{-1}(I - Q) B = \sum_{P_n(\lambda_k) = 0} \frac{\mu - \lambda_k}{Q_m(\lambda_k)} \langle \cdot, \varphi_k \rangle \varphi_k,
\]

(19)
\[ X_{\alpha,\alpha}(t)L_1^{-1}QB = \sum_{P_n(\lambda_k) \neq 0} \frac{\mu - \lambda_k}{P_n(\lambda_k)} E_{\alpha,\alpha} \left( \frac{Q_m(\lambda_k)s^\alpha}{P_n(\lambda_k)} \right) \langle \varphi_k, \varphi_k \rangle. \]  

(20)

Remark 6. It is easy to show, that in the case of \((L, 0)\)-boundedness of an operator \(M\) the generalized Showalter — Sidorov type problem is equivalent to the problem

\[ D_t^{\alpha-m-k}Lx(0) = y_k, \quad k = 0, 1, \ldots, m - 1, \]

with \(y_k \in \mathcal{Y}^1\). Here conditions (18) have such form. With the determination of the control system initial state by these conditions, the definitions of the approximate controllability can be modified by the natural way with using \(\overline{\mathcal{Y}} = (y_0, y_1, \ldots, y_{m-1}) \in (\mathcal{Y}^1)^m\) instead of \(\overline{x} = (x_0, x_1, \ldots, x_{m-1}) \in (\mathcal{X}^1)^m\).

We will say about nongenerate subsystem (7) and degenerate subsystem (8) of control system (16), (17) as a partial case of system (4), without explicitly writing out these subsystems.

**Theorem 11.** Let \(n \geq m\), the spectrum \(\sigma(\Lambda_1)\) do not contain common roots of the polynomials \(P_n(\lambda)\) and \(Q_m(\lambda)\). 

(i) If \(\mu \in \sigma(\Lambda)\), \(P_n(\mu) \neq 0\), then nondegenerate subsystem (7) is not approximately controllable for free time, and degenerate subsystem (8) is approximately controllable for any time \(T > 0\).

(ii) If \(\mu \in \sigma(\Lambda)\), \(P_n(\mu) = 0\), then degenerate subsystem (8) is not approximately controllable for free time, and nondegenerate subsystem (7) is approximately controllable for any time \(T > 0\).

(iii) If \(\mu \notin \sigma(\Lambda)\), then system (16), (17) is approximately controllable for any time \(T > 0\).

Proof. In case (i) we have \(\overline{\mathcal{Y}^1} \neq \mathcal{Y}^1\), but we see, that the subspace span\(\{\varphi_k : \lambda_k = \mu\}\) is not attainable for trajectories of subsystem (7) due to (20). In case (ii) \(\overline{\mathcal{Y}^1}(I - Q)B = \mathcal{Y}^0\) and (19) implies that the subspace span\(\{\varphi_k : \lambda_k = \mu\}\) is not attainable for trajectories of subsystem (8). Finally, in case (iii) we have \(\overline{\mathcal{Y}^1} = \mathcal{Y}^1\) and due to Corollary 1 system (16), (17) is approximately controllable for any time \(T > 0\).

Let \(P_1(\lambda) = 1 - \lambda\), \(Q_1(\lambda) = \lambda\), \(Au = \Delta u\), \(r = 1\), \(B_1 = I\), \(\Omega = (0, \pi)\). Then three cases of Theorem 9 are \(\mu \in \mathbb{N} \setminus \{1\}\), \(\mu = 1\) and \(\mu \notin \mathbb{N}\) respectively.

Now consider the similar system with finite-dimensional system, which is described by the equation

\[ D_t^n P_n(\lambda)x(s, t) = Q_m(\lambda)x(s, t) + \sum_{i=1}^{n} b_i(s)u_i(t), \quad (s, t) \in \Omega \times (0, T], \]

(21)

endowed by the boundary conditions (16).

**Theorem 12.** Let \(n \geq m\), the spectrum \(\sigma(\Lambda_1)\) do not contain common roots of the polynomials \(P_n(\lambda)\) and \(Q_m(\lambda)\), \(b_i \in L_2(\Omega), i = 1, 2, \ldots, n\). Control system (16), (21) is approximately controllable for time \(T\), if and only if

\[ \text{span}\{\varphi_k : P_n(\lambda_k) = 0\} = \text{span}\left\{ \sum_{P_n(\lambda_k) = 0} \frac{b_i(s)u_i(t)}{Q_m(\lambda_k)\varphi_k}, i = 1, 2, \ldots, n \right\}, \]

(22)

and

\[ \text{span}\left\{ \sum_{P_n(\lambda_k) \neq 0} E_{\alpha,\alpha} \left( \frac{Q_m(\lambda_k)s^\alpha}{P_n(\lambda_k)} \right) \langle b_i, \varphi_k \rangle \varphi_k, i = 1, 2, \ldots, n, 0 < s < T \right\}. \]

System (16), (21) is approximately controllable for free time, if and only if equality (22) hold and

\[ \text{span}\{\varphi_k : P_n(\lambda_k) \neq 0\} = \]
Let for the distributed control system

\[ v(s, t) = 0, \quad (s, t) \in \partial \Omega \times (0, T], \tag{23} \]

\[ D^\alpha_t (1 - \chi \Delta) v(s, t) = \nu \Delta v(s, t) - (\tilde{v} \cdot \nabla)v(s, t) - (v \cdot \nabla)\tilde{v}(s, t) - r(s, t) + (Bu)(s, t), \quad (s, t) \in \Omega \times (0, T], \tag{24} \]

\[ \nabla \cdot v(s, t) = 0, \quad (s, t) \in \Omega \times (0, T], \tag{25} \]

the initial state is determined by the conditions

\[ D^\alpha_t - m + k v(s, 0) = v_k(s), \quad s \in \Omega, \quad k = 0, 1, \ldots, m - 1, \tag{26} \]

where \( m - 1 < \alpha \leq m \in \mathbb{N} \), as before. At \( \alpha = 1, \nu > 0 \) it is the Oskolkov’s model of the dynamics of the viscoelastic Kelvin — Voigt fluid [29] in the linear approximation, at \( \nu = 0 \) it is the linearized Scott-Blair system [30]. Here \( \Omega \subset \mathbb{R}^n \) is a bounded region with a smooth boundary \( \partial \Omega \). The parameter \( \chi \in \mathbb{R} \) characterizes elastic properties of the medium, and \( \nu \in \mathbb{R} \) describes its viscosity. Unknown functions are \( v = (v_1, v_2, \ldots, v_n) \) (the velocity) and \( r = (r_1, r_2, \ldots, r_n) \) (the pressure gradient). The term \( Bu \) will be defined below.

Denote \( L_2 := (L_2(\Omega))^n, H^1 := (W^1_0(\Omega))^n, H^2 := (W^2_0(\Omega))^n. \) The closure of \( \mathcal{L} = \{ v \in (C^0_\infty(\Omega))^n : \nabla \cdot v = 0 \} \) in \( L_2 \) will be denoted by \( H^1_\sigma \), and the closure in the norm \( H^1 \) will be denoted by \( H^1_\pi \). Moreover, \( H^2_\sigma := H^1_\sigma \cap H^2 \), \( H^2_\pi \) is the orthogonal complement for \( H^1_\sigma \) in the space \( L_2, \Sigma : L_2 \to H^1_\sigma, \Pi = I - \Sigma \) are the respective othoprojectors.

The operator \( A = \Sigma \Delta, \) extended to a closed operator in \( H^1_\sigma \) with the domain \( H^2_\sigma \), has a real, negative, discrete, spectrum with finite multiplicity, which is condensed only to \( -\infty \) [31]. Denote by \( \{ \lambda_k \} \) its eigenvalues, numbered in nonincreasing order taking into account their multiplicity, and \( \{ \varphi_k \} \) is the orthogonal system of the respective eigenfunctions, which form the basis in \( H^1_\sigma \) [31].

Define the operator \( D : H^2_\sigma \to L_2 \), acting by the rule \( v \to -(\tilde{v} \cdot \nabla)v(s, t) - (v \cdot \nabla)\tilde{v}(s, t) \) at a fixed \( \tilde{v} \in H^1_\pi \).

Taking into account the incompressibility equation (25), set \( \mathcal{X} = H^2_\sigma \times H^1_\pi, \mathcal{Y} = L_2 = H\sigma \times H_\pi, \)

\[ L = \begin{pmatrix} I - \chi A & 0 \\ -\chi \Pi \Delta & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{X}; \mathcal{Y}), \quad M = \begin{pmatrix} \nu A + \Sigma D & 0 \\ \nu \Pi \Delta + \Pi D & -I \end{pmatrix} \in \mathcal{L}(\mathcal{X}; \mathcal{Y}). \tag{27} \]

**Lemma 5.** Let \( \chi \neq 0, \chi^-1 \notin \sigma(A), \nu \in \mathbb{R}, \) operators \( L \) and \( M \) be defined by (27). Then the operator \( M \) is \( (L, 0) \)-bounded, there are projectors

\[ P = \begin{pmatrix} I & 0 \\ \Pi \Delta (I - A)^{-1} (\nu I + \chi \Sigma D) + \Pi D & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} I & 0 \\ -\chi \Pi \Delta (I - \chi A)^{-1} & 0 \end{pmatrix}. \tag{28} \]
Here we show on the simplest example that the density of the image im$\Pi$ exists at sufficiently great $|\mu|$ since $(I - \chi A)^{-1} \in \mathcal{L}(H_\sigma; \mathbf{H}_2^0)$, $(I - \chi A)^{-1} A, (I - \chi A)^{-1} \Sigma D \in \mathcal{L}(\mathbf{H}_2^0)$, $(I - \chi A)^{-1} A - (I - \chi A)^{-1} \Sigma D \in \mathcal{L}(\mathbf{H}_2^0)$ at $|\mu| > \|\nu(I - \chi A)^{-1} A + (I - \chi A)^{-1} \Sigma D\|_{\mathcal{L}(\mathbf{H}_2^0)}$.

Consequently, $\Pi(\mu \chi A + \nu \Delta + D)(\mu I - \chi A) - \nu A - \Sigma D)^{-1} \in \mathcal{L}(\mathbf{H}_\sigma; \mathbf{H}_\pi)$ and $(\mu L - M)^{-1} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})$ and the operator $M$ is $(L, \sigma)$-bounded.

Due to (29) we have

$$(\mu L - M)^{-1} = \begin{pmatrix} \Pi(\mu \chi A + \nu \Delta + D)(\mu I - \chi A)^{-1} A - (I - \chi A)^{-1} \Sigma D)^{-1} & 0 \\ (I - \chi A)^{-1} A - (I - \chi A)^{-1} \Sigma D)^{-1} - \chi \Pi \Delta & 0 \end{pmatrix},$$

$L(\mu L - M)^{-1} = \begin{pmatrix} (\mu I - \nu (I - \chi A)^{-1} A - (I - \chi A)^{-1} \Sigma D)^{-1} & 0 \\ -\chi \Pi \Delta (I - \chi A)^{-1} (\mu I - \nu A(I - \chi A)^{-1} - \Sigma D(I - \chi A)^{-1})^{-1} & 0 \end{pmatrix}.$

Using the expansion in the Laurent series of the operator-valued functions, we obtain the projectors (28) by integrating. Therefore, $L(I - P) = 0$, $L_0 = 0$, $G = 0$. Thus, operator $M$ is $(L, 0)$-bounded.

Therefore, $\mathcal{X}_0 = \ker P = \mathbf{H}_\sigma = \ker Q = \mathcal{Y}_0$,

$$\mathcal{X}_1 = \text{im} P = \{(w, \Pi \Delta (I - \chi A)^{-1} (\nu I + \chi \Sigma D) w + \Pi D w) : w \in \mathbf{H}_2^0\},$$

$$\mathcal{Y}_1 = \text{im} Q = \{(w, -\chi \Pi \Delta (I - \chi A)^{-1} w) : w \in \mathbf{H}_\sigma\}.$$

Note that (26) are the generalized Showalter — Sidorov conditions.

Let $\mathcal{U} = \mathcal{Y}$, $B$ is the identical operator, i.e. $(Bu)(s, t) = u(s, t)$. Then $\text{im} B = L_2 = \mathbf{H}_\sigma \times \mathbf{H}_\pi$ is dense in $\mathcal{X} = \mathbf{H}_2^0 \times \mathbf{H}_\pi$, hence, control system (23)–(25) is approximately controllable for every time $T$ due to Corollary 1.

Let $\mathcal{U} = \mathbb{R}^n$, $b_i \in L_2$, $i = 1, 2, \ldots, n$, $u(s, t) = \sum_{i=1}^{n} b_i(s) u_i(t)$, i.e. (23)–(25) is a system with a finite-dimensional input. Then this system is not controllable even for free time, since $\mathcal{X}_0 = \mathbf{H}_\pi$ is not finite-dimensional and the conditions of Theorem 12 are not satisfied.

### 9. Simple control system

Here we show on the simplest example that the density of the image im$B$ of the operator $B$ at the control function is not necessary for the approximate controllibility. Consider a control system (not distributed)

$$D_t^a x^2(t) = x^1(t),$$

$$0 = x^2(t) - u(t).$$

Here $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$, $\mathcal{U} = \mathbb{R}$,

$$L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \quad B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$
Therefore, for every $\mu \in C$ $(\mu L - M)^{-1} = \begin{pmatrix} -1 & -\mu \\ 0 & -1 \end{pmatrix}$, $(\mu L - M)^{-1}L = L(\mu L - M)^{-1} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$, $P = Q = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $X^1 = Y^1 = \{0\}$, $X^0 = Y^0 = \mathbb{R}^2$, $M_0 = M_0^{-1} = I$, $L = L_0 = G$, $G^2 = 0$. Thus, the operator $M$ is $(L,1)$-bounded. The image of the operator $B$ is not dense in $Y$, moreover, $\text{im}M_0^{-1}(I - Q)B = \{0\} \times \mathbb{R}$, but

$$\text{span}\{\text{im}M_0^{-1}(I - Q)B, \text{im}GM_0^{-1}(I - Q)B\} = \text{span}\{\{0\} \times \mathbb{R}, \mathbb{R} \times \{0\}\} = \mathbb{R}^2$$

and system (30) is approximately controllable in any time $T > 0$.

10. Conclusion
For the considered class of distributed control systems with a linear degenerate operator at the Riemann — Liouville fractional derivative we obtain the criteria of the approximate controllability for time $T$, approximate controllability for free time in the cases of infinite-dimensional and finite-dimensional inputs. The obtained results are applied to the research of several specific control systems.

Acknowledgments
The work is supported by Act 211 of Government of the Russian Federation, contract 02.A03.21.0011, and the Russian Foundation of Basic Research, grant 19-41-450001.

References
[1] Kalman R E, Ho Y C and Narendra K C 1963 Controllability of linear dynamical systems Contributions to Differential Equations 1 189–213
[2] Krasovskii N N 1964 On the theory of controllability and observability of linear dynamic systems J. of Applied Mathematics and Mechanics 28 1–14
[3] Fattorini H O 1967 On complete controllability of linear systems J. of Differential Equations 3 391–402
[4] Kurzhanskiy A B 1969 Towards controllability in Banach spaces Differential Equations 5 1715–18
[5] Triggiani R 1975 Controllability and observability in Banach space with bounded operators SIAM J. on Control 13 452–91
[6] Curtain R F 1997 The Salamon — Weiss class of well-posed infinite dimensional linear systems: a survey IMA J. of Mathematical Control and Information 14 207–23
[7] Sholokhovich F A 1998 On controllability of linear dynamical systems News of Ural State University 10 103–26
[8] Debbouche A and Baleanu D 2011 Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems Computers and Mathematics with Applications 62 1442–50
[9] Chalishajar D N, Malar K and Karthikeyan K 2013 Approximate controllability of abstract impulsive fractional neutral evolution equations with infinite delay in Banach spaces Electronic J. of Differential Equations 275 1–21
[10] Zhou Y 2016 Fractional Evolution Equations and Inclusions: Analysis and Control (Amsterdam: Elseiver)
[11] Fedorov V E and Ruzakova O A 2002 Controllability of linear Sobolev type equations with relatively $p$-radial operators Russian Math. 7 54–7
[12] Fedorov V E and Ruzakova O A 2002 One-dimensional controllability of Sobolev linear equations in Hilbert spaces Differential Equations 38 1216–18
[13] Fedorov V E and Ruzakova O A 2003 Controllability in dimensions of one and two of Sobolev-type equations in Banach spaces Mathematical Notes 74 583–92
[14] Fedorov V E and Shkil’yar B 2012 Exact null controllability of degenerate evolution equations with scalar control Sbornik: Mathematics 203 1817–36
[15] Piekhanova M V and Fedorov V E 2013 Optimal Control for Degenerate Distributed Systems (Chelyabinsk: South Ural State University)
[16] Piekhanova M V and Fedorov V E 2014 On controllability of degenerate distributed systems Ufa Mathematical Journal 6 (2) 77–96
[17] Mahmudov N I 2013 Approximate controllability of fractional Sobolev-type evolution equations in Banach spaces Abstract and Applied Analysis 2013 1D 502839
[18] Fečkan M, Wang J and Zhou Y 2014 Controllability of fractional functional evolution equations of Sobolev type via characteristic solution operators J. of Optimization Theory and Applications 156 79–95
[19] Wang J, Fečkan M and Zhou Y 2017 Approximate controllability of Sobolev type fractional evolution systems with nonlocal conditions Evolution Equations and Control Theory 6 471–86
[20] Fedorov V E, Gordievskikh D M and Baybulatova G D 2017 Controllability of a class of weakly degenerate fractional order evolution equations AIP Conf. Proc. 1907 ID 020009
[21] Fedorov V E, Gordievskikh D M and Turov M M 2018 Infinite-dimensional and finite-dimensional ε-controllability for a class of fractional order degenerate evolution equations Chelyabinsk Physical and Mathematical Journal 3 (1) 5–26
[22] Fedorov V E and Gordievskikh D M 2018 Approximate controllability of strongly degenerate fractional order system of distributed control IFAC-PapersOnLine 51 (32) 675–80
[23] Fedorov V E, Gordievskikh D M, Baleanu D and Taş 2019 Criterion of the approximate controllability of a class of degenerate distributed systems with the Riemann — Liouville derivative Mathematical Notes of NEFU 26 (2) 41–59
[24] Baleanu D, Fedorov V E, Gordievskikh D M and Taş K 2019 Approximate controllability of infinite-dimensional degenerate fractional order systems in the sectorial case Mathematics 7 (8)
[25] Avilovich A S, Gordievskikh D M and Fedorov V E 2020 Issues of unique solvability and approximate controllability for linear fractional order equations with a Hölderian right-hand side Chelyabinsk Physical and Mathematical Journal 5 (1) 5–21
[26] Sviridyuk G A and Fedorov V E 2003 Linear Sobolev Type Equations and Degenerate Semigroups of Operators (Utrecht, Boston: VSP)
[27] Fedorov V E, Plekhanova M V and Nazhimov R R 2018 Degenerate linear evolution equations with the Riemann — Liouville fractional derivative Siberian Mathematical J. 59 (1) 136–46
[28] Triebel H 1977 Interpolation Theory. Function Spaces. Differential Operators (Berlin: VEB Deutscher Verlag der Wissenschaften)
[29] Oskolkov A P 1987 Initial-boundary value problems for equations of motion of Kelvin — Voigt fluids and Oldroyd fluids Trudy Matematicheskogo Instituta Steklova 179 126–64
[30] Blair G S 1944 A Survey of General and Applied Rheology (London: Pitman & Sons)
[31] Ladyzhenskaya O A 1969 The Mathematical Theory of Viscous Incompressible Flow (New York, London, Paris, Montreux, Tokyo, Melbourne: Gordon and Breach)