On the Remarkable Features of Binding Forms

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Abstract—Hilbert’s “Entscheidungsproblem” has given rise to a broad and productive line of research in mathematical logic, where the classification process of decidable classes of first-order sentences represent only one of the remarkable results. According to the criteria used to identify the particular classes of interest, this process was declined into several research programs, of which some of the most deeply investigated are the ones classifying sentences in prefix normal form in base of their prefix vocabulary.

Unfortunately, almost all of these approaches did not shed any light on the reasons why modal logic is so robustly decidable. Trying to answer to this question, Andréka, van Benthem, and Németi introduced the guarded fragment of first-order logic, which generalizes the modal framework by essentially retaining all its fundamental properties. They started, so, a completely new research program based on the way quantifications can be relativized. Although this approach succeeded in its original task, we cannot consider it satisfactory in spotting the reasons why some complex extensions of modal logic are well-behaved. In particular, by just using the related results, we are not able to derive the decidability of multi-agent logics for strategic abilities.

In this paper, aiming to lay the foundation for a more thorough understanding of some of these decidability questions, we introduce a new kind of classification based on the binding forms that are admitted in a sentence, i.e., on the way the arguments of a relation can be bound to a variable. We describe a hierarchy of first-order fragments based on the Boolean combinations of these forms, showing that the less expressive one is already incomparable with the guarded logic and related extensions. We also prove, via a new model-theoretic technique, that it enjoys the finite-model property and a PSPACE satisfiability problem.

I. INTRODUCTION

Since from the publication of the revolutionary negative solutions owed to Church \cite{11-13} and Turing \cite{4,5}, Hilbert’s original “Entscheidungsproblem” \cite{6} turned into a vast classification process looking for all those classes of first-order sentences having a decidable satisfiability \cite{7}. Depending upon the syntactic criteria used to identify the particular classes of interest \cite{8}, this process was declined into several research programs, among which we can mention, on one side, those limiting relation arities \cite{9} or free variables \cite{10,11}, and, on the other one, the classifying sentences in prefix normal form in base of their prefix vocabulary \cite{12}.

The research in this field has had a considerable impact, from both a theoretical and practical point of view, in numerous areas on the edge between mathematics and computer science, e.g., reverse mathematics \cite{13}, descriptive complexity, database theory, and formal verification, just to cite a few. However, almost all of the classic approaches did not shed any satisfactory light on a fundamental question: why are modal logic and derived frameworks, like those ones featuring fixpoint constructs, so robustly decidable \cite{14,15}? Trying to find a plausible answer, Andréka, van Benthem, and Németi introduced the guarded fragment of first-order logic \cite{16}, which generalizes the modal framework by essentially retaining all its model-theoretic and algorithmic properties. They started, so, a completely new research program based on the way quantifications can be relativized to some particular facts, avoiding the usual syntactic restrictions on quantifier patterns, relation arities, and number of variables. Pushing forward the idea that robust fragments of first-order logic owe their nice properties to some kind of guarded quantification, several extensions along this line of research were introduced in the literature, such as the clique guarded \cite{17}, loosely guarded \cite{18}, the clique guarded \cite{17,19}, the action guarded \cite{20,21}, and the guarded fixpoint logic \cite{22}. This classification program has also important applications in database theory and description logic, where it is relevant to evaluate a query against guarded first-order theories \cite{23}.

Only recently, ten Cate and Segoufin observed that the first-order translation of modal logic, beside the guarded nature of quantifications, presents another important peculiarity: negation is only applied to sentences or monadic formulas, \textit{i.e.}, formulas with a single free variable. Exploiting this observation, they introduced a new robust fragment of first-order logic, called unary negation \cite{24,25}, which extends modal logic, as well as other formalisms, like Boolean conjunctive queries, that cannot be expressed in terms of guarded quantifications. Since this new logic is incomparable to the guarded fragments, right after the original work, another logic was proposed, called guarded negation \cite{26}, unifying both the approaches of classification. Syntactically, there is no primary universal quantification and the use of negation is only allowed if guarded by an atomic relation. In terms of expressive power, this fragment forms a strict extension of both the logics on which it is based, while preserving the same desirable properties. However, it has to be noted that it is still incomparable to more complex extensions of the guarded fragment, such as the clique guarded one. This way of analyzing formulas founded on the guarded nature of negation has also remarkable applications to database theory, where it is well-known that complementation makes queries hard to evaluate \cite{27}.

Although these two innovative classification programs really succeeded in the original task to explain the nice properties of modal logic, we cannot consider them completely satisfactory with respect to the more general intent to identify the reasons why some of its complex extensions are so well-behaved. In particular, by just using the related model-theoretic and algorithmic results, we are not able to answer to the question about the decidability of several multi-agent logics for strategic abilities, such as the Alternating Temporal Logics ATL \cite{28,29} and ATL∗ \cite{30} and the one-goal fragment of Strategy Logic SL\cite{16} \cite{31-33}, which do not intrinsically embed such kinds of relativization. For example, consider the ATL
formula $\varphi = \lbrack a, b, c \rbrack \neg \psi$ over a game structure with the four agents $a$, $b$, $c$, and $d$. Intuitively, it asserts that agent $d$ has a strategy, which depends upon those chosen by the other three ones, ensuring that the property $\psi$ does not hold. Now, it is easy to observe that the underlying strategy quantifications can be represented as a prefix of the form $\forall d \exists$ coupled with the quaternary relation described by $\psi$. Therefore, $\varphi$ is interpretable neither as a decidable prefix-vocabulary class nor as a two-variable formula. Moreover, quantifications are not guarded and negation is applied to the property $\psi$ that cannot be considered as monadic. Another explicit example is given by the $SL[1g]$ sentence $[[x]](y)[z](a, x)(b, x)(d, y)(c, z)\psi$ asserting that, once $a$ and $b$ chose the common strategy $x$, agent $d$ can select its better response $y$ to ensure $\psi$, in a way that is independent of the behavior $z$ of $c$. Here, we have a more complex quantification prefix of the form $\forall \exists$ coupled with the quaternary relation $\psi$, in which two of its arguments are bound to the same variable. Also in this case, we are not able to cast this sentence in any of the decidable cases described by some of the classification programs already introduced in the literature. In particular, it is not guarded negation, since universal quantifications are used as primary construct, which is prohibited in that fragment.

In this paper, trying to lay the foundation for a more thorough understanding of these decidability questions, we introduce a new classification program based on the *binding forms* that are admitted in a sentence, i.e., on the way the arguments of a relation can be bound to a variable. To do this, similarly to the treatment of the attributes of a table in a database setting, we define a generalization of standard notions of language signature and relational structure, which makes syntactically explicit the arguments of interest. With more detail, every relation $r$ is associated with a set of arguments $\{a_1, \ldots, a_n\}$, which are bound to the variables via a binding form $(a_1, x_1) \cdots (a_n, x_n) r$ that replaces the usual writing $r(x_1, \ldots, x_n)$. Our notation, although perfectly equivalent to the classic one, allows to introduce and analyze, in an easier and natural way, a hierarchy of four fragments of first-order logic based on the Boolean combinations of these forms. In particular, we study the simplest fragment, called *one binding*, proving that it is already incomparable to clique guarded and guarded negation. Simple examples of one-binding sentences are given by the translation of the game properties described above: $\forall x \forall y \forall z \exists w(a, x)(b, y)(c, z)(d, w) \neg r$ and $\forall x \exists y \forall z(a, x)(b, x)(d, y)(c, z) r$, where in place of the temporal goal $\psi$ we have the relation $r$ whose arguments $\{a, b, c, d\}$ stand for the agents of the game. As main results, via a novel model-theoretic technique, we prove that our logic enjoys the finite-model property, both Craig’s interpolation and Beth’s definability, and a PSPACE satisfiability problem.

II. Preliminaries

Since from Codd’s pioneering work on the structure of relational databases [24], several kinds of first-order languages have been used to describe databases queries [33]. In particular, *first-order logic* (FOL, for short) has been established as the main theoretical framework in which to prove results about properties of query languages [36–38]. In this context, a table is usually represented as a mathematical relation between elements of a given domain, where its attributes are mapped to the indexes of that relation in a predetermined fixed way. Consequently, the attributes do not have any explicit matching element in the syntax of the language. In this paper, in order to introduce the *binding-form fragments* of FOL, we need to reformulate both the syntax and the semantics of the logic in a way that is much closer to the database setting. In particular, we explicitly associate a finite non-empty set of arguments to each relation. Those arguments are handled in the syntax via corresponding symbols. To this aim, we introduce an alternative version of classic *language signatures* and relational structures.

A. Language Signatures

A *language signature* is a mathematical object that simply describes the structure of all non-logical symbols composing a first-order formula. The typology we introduce here is purely relational, since we do not make use of constant or function symbols. Also, in our reasonings, we do not explicitly consider distinguished relations as equality, equivalences, or orders.

**Definition II.1 (Language Signature).** A language signature (LS, for short) is a tuple $\mathcal{L} = (\mathcal{L}_r, \mathcal{R}_l, \mathcal{A}_r)$, where $\mathcal{L}_r$ and $\mathcal{R}_l$ are the finite non-empty sets of argument and relation names and $\mathcal{A}_r$ are the finite non-empty sets of relations $\mathcal{A}_r : \mathcal{R}_l \rightarrow 2^{\mathcal{L}_r} \setminus \{\emptyset\}$ is the argument function mapping every relation $r \in \mathcal{R}_l$ to its non-empty set of arguments $\mathcal{A}_r(r) \subseteq \mathcal{A}_r$.

In the following, as running example, we use the simple LS $\mathcal{L} = (\{a, b\}, \{q, r, s\}, \mathcal{A}_r)$ having $q$ and $r$ as two binary relations over the arguments $\mathcal{A}_r(q) = \mathcal{A}_r(r) = \{a, b\}$ and $s$ as a monadic relation over the argument $\mathcal{A}_r(s) = \{a\}$.

B. Relational Structures

Given a language signature, we define the interpretation of all symbols by means of a relational structure, i.e., a carrier domain together with an association of each relation with an appropriate set of tuples on the elements of that domain. Since, in our framework, relations with the same arity may have different arguments, it is not sufficient for us to manage usual tuples as components of their interpretation. For this reason, we map a given relation $r$ to a set of *tuple functions* $t$ having $ar(r)$ as support and elements of the carrier domain as values.

**Definition II.2 (Relational Structure).** A relational structure over an LS $\mathcal{L} = (\mathcal{L}_r, \mathcal{R}_l, \mathcal{A}_r)$ (C-RS, for short) is a tuple $\mathcal{R} = (D_{\mathcal{R}}, t_l)$, where $D_{\mathcal{R}}$ is the non-empty set of arbitrary objects called *domain* and $t_l : \mathcal{R} \rightarrow 2^{\mathcal{L}_r} \setminus \{\emptyset\}$ is the relation function *mapping every relation* $r \in \mathcal{R}_l$ to a set $ar(r) \subseteq \mathcal{A}_r(r)$ of *tuple functions* $t \in \mathcal{R}_r$ from arguments of $r$ to elements of the domain.

The *order* of an RS $\mathcal{R}$ is given by the size $|\mathcal{R}| \equiv |D_{\mathcal{R}}|$ of its domain. A relational structure is *finite* iff it has finite order.

An example of a finite RS over the previous LS $\mathcal{L}$ is given by $\mathcal{R} = (\{(0, 1)\}, t_l)$, where $q^\mathcal{R} = \{(a \rightarrow 0, b \rightarrow 0), (a \rightarrow 0, b \rightarrow 1)\}$, $r^\mathcal{R} = \{(a \rightarrow 0, b \rightarrow 0), (a \rightarrow 1, b \rightarrow 1)\}$, and $s^\mathcal{R} = \{(a \rightarrow 1)\}$ are the interpretations of the three relations.
III. First-Order Logic

We start by describing a slightly different but completely equivalent formalization of both the syntax and the semantics of FOL according to the explained alternatives of language signature and relational structure. We also introduce a new family of fragments based on the kinds of binding forms that are admitted in a formula, i.e., on the ways an argument can be bound to a variable.

From now on, if not differently stated, we use $\mathcal{L} = (\mathcal{A}_r, \mathcal{R}_l, ar)$ to denote an a priori fixed LS. Also, $\mathcal{V}_r$ represents an enumerable non-empty set of variables. For sake of succinctness, to indicate the extension of $\mathcal{L}$ with $\mathcal{V}_r$, we adopt the composed symbol $\mathcal{L}(\mathcal{V}_r)$ in place of the tuple $(\mathcal{L}, \mathcal{V}_r)$.

A. Syntax

As far as the syntax of FOL concerns, the novelty of our setting resides in the explicit presence of arguments as atomic components of a formula. In particular, a variables $x$ cannot be directly applied to the index associated with an argument $a$ of a relation $r$ as in the usual writing $r(\ldots, x, \ldots)$, but a construct $(a, x)$, called binding, is required to link $x$ to $a$ in $r$.

**Definition III.1 (FOL Syntax).** FOL formulas over $\mathcal{L}(\mathcal{V}_r)$ are built by means of the following context-free grammar, where $a \in \mathcal{A}_r$, $r \in \mathcal{R}_l$, and $x \in \mathcal{V}_r$:

$$
\varphi \::= \neg \varphi \mid (\varphi \land \varphi) \mid (\varphi \lor \varphi) \mid \exists x. \varphi \mid \forall x. \varphi \mid (a, x).\varphi.
$$

$\mathcal{L}(\mathcal{V}_r)$-FOL denotes the enumerable set of all formulas over $\mathcal{L}(\mathcal{V}_r)$ generated by the above grammar.

Consider the LS described in Section II-A. Then, $\varphi_1 = \exists x\forall y \exists z ((a, x)(b, y)(q \lor \neg s) \land (a, y)(b, z)x)$, $\varphi_2 = \exists x(a, x)r$, and $\varphi_3 = (a, x)\forall y(b, y)q$ are three formulas over $\mathcal{L}(\{x, y, z\})$. Note that, by fixing an ordering on the arguments, it is possible to rewrite our FOL formulas in the classic syntax. For instance, $\varphi_1$ can be expressed by $\exists x\forall y \exists z ((q(x, y) \lor \neg s(x)) \land r(y, z))$, once it is assumed that $a < b$. On the other hand, every classic FOL formula can be easily translated in our syntax by means of numeric arguments that represent the positions in the relations. As an example, the transitivity property $\forall x\forall y \exists z ((\neg r(x, y) \lor \neg r(y, z)) \lor r(x, z))$ can be rewritten as follows: $\forall x\forall y \forall z (((1, x)(2, y) \lor r(x, z)) \lor (1, x)(2, z) \lor r(x, z))$. Observe that the parentheses surrounding the binary Boolean connectives are employed to enforce the property of unique parsing of a formula. However, in the following, we simplify the notation by avoiding their use in the unambiguous cases.

Usually, predicative logics, i.e., languages having explicit quantifiers, need a concept of free or bound placeholder to formally evaluate the meaning of their formulas. Placeholders are used, indeed, to enlighten particular positions in a syntactic expression that are crucial to the definition of its semantics. Classic formalizations of FOL just require one kind of placeholder represented by the variables on which the formulas are built. In our new setting, instead, both variables and arguments have this fundamental role. In particular, arguments are used to decouple variables from their association with a relation. Consequently, we need a way to check whether a variable is quantified or an argument is bound. To do this, we use the concept of free arguments/variables, i.e., the subset of $\mathcal{A}_r \cup \mathcal{V}_r$ containing all arguments that are free from binding and all variables occurring in some binding that are not quantified.

**Definition III.2 (Free Placeholders).** The set of free arguments/variables of an $\mathcal{L}(\mathcal{V}_r)$-FOL formula can be computed via the function free : $\mathcal{L}(\mathcal{V}_r)$-FOL $\rightarrow 2^{\mathcal{A}_r \cup \mathcal{V}_r}$ defined as follows:

1. $\text{free}(r) \triangleq ar(r)$, where $r \in \mathcal{R}_l$;
2. $\text{free}(\neg \varphi) \triangleq \text{free}(\varphi)$;
3. $\text{free}(\varphi_1 \varphi_2) \triangleq \text{free}(\varphi_1) \cup \text{free}(\varphi_2)$, where $\varphi_0 \in \{\land, \lor\}$;
4. $\text{free}((\forall x. \varphi) \setminus \{x\})$, where $\forall x \in \{\exists, \forall\}$;
5. $\text{free}((a, x)\varphi) \triangleq \begin{cases} \text{free}(\varphi) \setminus \{a\} \cup \{x\}, & \text{if } a \in \text{free}(\varphi); \\ \text{free}(\varphi), & \text{otherwise.} \end{cases}$

Observe that, free arguments are introduced in Item 1 and removed in Item 4. Moreover, free variables are introduced in Item 5 and removed in Item 4.

A formula $\varphi$ without free arguments (resp., variables), i.e., $\text{ar}(\varphi) \triangleq \text{free}(\varphi) \cap \mathcal{A}_r = \emptyset$ (resp., $\text{vr}(\varphi) \triangleq \text{free}(\varphi) \cap \mathcal{V}_r = \emptyset$), is named argument-closed (resp., variable-closed). If $\varphi$ is both argument- and variable-closed, it is referred to as a sentence.

Consider the formulas $\varphi_1$, $\varphi_2$, and $\varphi_3$ given above. It is easy to verify that $\text{free}(\varphi_1) = \emptyset$, $\text{free}(\varphi_2) = \text{ar}(\varphi_2) = \{b\}$, and $\text{free}(\varphi_3) = \text{vr}(\varphi_3) = \{x\}$. Thus, $\varphi_1$ is a sentence, $\varphi_2$ is variable-closed, and $\varphi_3$ is argument-closed.

B. Semantics

The semantics of FOL we introduce here is defined, as usual, w.r.t. an RS. In fact, the peculiarities of our setting just concern the interpretation of binding constructs and the non-standard evaluation of relations.

In order to formalize the meaning of a formula, we first need to describe the concept of assignment, i.e., a partial function $\chi \in \text{Ass}_D \triangleq (\mathcal{A}_r \cup \mathcal{V}_r) \rightarrow D$ mapping each placeholder in its domain to a value of an arbitrary set $D$, which is used to set a valuation of all the free arguments and variables. For a given placeholder $p \in \mathcal{A}_r \cup \mathcal{V}_r$ and a value $d \in D$, the notation $\chi[p \mapsto d]$ represents the new assignment defined on $\text{dom}(\chi[p \mapsto d]) \supseteq \text{dom}(\chi)[\cup\{p\}]$ that returns $d$ on $p$ and is equal to $\chi$ on the remaining part of its domain, i.e., $\chi[p \mapsto d](p) \equiv d$ and $\chi[p \mapsto d](p') \equiv \chi(p')$, for all $p' \in \text{dom}(\chi)[\cup\{p\}]$.

**Definition III.3 (FOL Semantics).** Let $\mathcal{R}$ be an $\mathcal{L}$-RS and $\varphi$ an $\mathcal{L}(\mathcal{V}_r)$-FOL formula. Then, for all assignments $\chi \in \text{Ass}_D$ with free$(\varphi) \subseteq \text{dom}(\chi)$, it holds that the relation $\mathcal{R}, \chi \models \varphi$ is inductively defined on the structure of $\varphi$ as follows:

1. For every relation $r \in \mathcal{R}_l$, it holds that $\mathcal{R}, \chi \models \varphi \iff \chi_{|\text{ar}(\varphi)} \in r(\varphi)$.
2. For all formulas $\varphi$, $\varphi_1$, and $\varphi_2$, it holds that:
   a) $\mathcal{R}, \chi \models \neg \varphi$ if not $\mathcal{R}, \chi \models \varphi$, that is $\mathcal{R}, \chi \not\models \varphi$;
   b) $\mathcal{R}, \chi \models \varphi_1 \land \varphi_2$ if $\mathcal{R}, \chi \models \varphi_1$ and $\mathcal{R}, \chi \models \varphi_2$;
   c) $\mathcal{R}, \chi \models \varphi_1 \lor \varphi_2$ if $\mathcal{R}, \chi \models \varphi_1$ or $\mathcal{R}, \chi \models \varphi_2$.
3. For a variable $x \in \mathcal{V}_r$ and a formula $\varphi$, it holds that:
   a) $\mathcal{R}, \chi \models \exists x. \varphi$ if there exists a value $d \in D$ such that $\mathcal{R}, \chi[x \mapsto d] \models \varphi$;
b) $\mathcal{R}, \chi \models \forall x. \varphi$ iff, for all values $d \in Dm$, it holds that $\mathcal{R}, \chi[x \mapsto d] \models \varphi$.

4) For an argument $a \in Ar$, a variable $x \in Vr$, and a formula $\varphi$, it holds that $\mathcal{R}, \chi \models (a, x) \varphi$ iff $\mathcal{R}, \chi[a \mapsto \chi(x)] \models \varphi$.

Intuitively, Items 2 and 3 define the classic semantics of Boolean connectives and first-order quantifiers, respectively. Item 1 instead, verifies an atomic relation $\mathcal{R}$. Item 1, instead, verifies an atomic relation $\mathcal{R}$. RS $\mathcal{R}$ and an $\mathcal{L}(Vr)$-FOL sentence $\varphi$, we say that $\mathcal{R}$ is a model of $\varphi$, in symbols $\mathcal{R} \models \varphi$, iff $\mathcal{R}, \emptyset \models \varphi$, where $\emptyset \in \text{AssDm}$ simply denotes the empty assignment. We also say that $\varphi$ is satisfiable iff there exists a model for it. Given two $\mathcal{L}(Vr)$-FOL formulas $\varphi_1$ and $\varphi_2$, we say that $\varphi_1$ implies $\varphi_2$, in symbols $\varphi_1 \Rightarrow \varphi_2$, iff $\mathcal{R}, \chi \models \varphi_1$ implies $\mathcal{R}, \chi \models \varphi_2$, for each $\mathcal{L}$-RS $\mathcal{R}$ and assignment $\chi \in \text{AssDm}$ with free($\varphi_1$), free($\varphi_2$) $\subseteq$ dom($\chi$). Moreover, we say that $\varphi_1$ is equivalent to $\varphi_2$, in symbols $\varphi_1 \equiv \varphi_2$, iff both $\varphi_1 \Rightarrow \varphi_2$ and $\varphi_2 \Rightarrow \varphi_1$ hold.

Consider again the formulas $\varphi_1$, $\varphi_2$, and $\varphi_3$ and the $\mathcal{L}$-RS $\mathcal{R}$ described in Section 11-B. We have that $\mathcal{R}, \emptyset \models \varphi_1$. Moreover, $\mathcal{R}, \chi \models \varphi_2$ and $\mathcal{R}, \chi \not\models \varphi_3$, where $\chi(x) = 0$ and $\chi(b) = 1$.

C. Fragments

We now describe a family of four syntactic fragments of FOL by means of a normal form where relations over the same set of arguments may be clustered together by a unique sequence of bindings. With more detail, we consider Boolean combinations of sentences in prenex normal form in which quantification prefixes are coupled with Boolean combinations of these relation clusters called binding forms. Each fragment is then characterized by a specific constraint on the possible combinations of binding forms.

A quantification prefix $\psi \in \text{Qu} \subseteq \{\exists x, \forall x : x \in Vr\}^*$ is a finite word on the set of quantifications $\{\exists x, \forall x : x \in Vr\}$ over $Vr$, in which each variable occurs at most once. It naturally induces an injective partial function $\psi : Vr \rightarrow [0, |\psi|]$ assigning to each variable $x \in \text{vr}(\psi)$ in its domain the position $\psi(x)$ in $[0, |\psi|]$ where it occurs in $\psi$. So, either $(\psi)_{\psi(x)} = \exists x$ or $(\psi)_{\psi(x)} = \forall x$. Similarly to a quantification prefix, a binding prefix $\beta \in \text{Bn} \subseteq \{(a, x) : a \in Ar \land x \in Vr\}^*$ is a finite word on the set of bindings $\{(a, x) : a \in Ar \land x \in Vr\}$ over Ar and Vr, in which each argument occurs at most once. By abuse of notation, $\beta$ can also be interpreted as a partial function $\beta : Ar \rightarrow Vr$ assigning to each argument $a \in \text{ar}(\beta)$ in its domain the variable $\beta(a) \in \text{vr}(\beta)$ it is bound to in $\beta$. As an example, the bindings $b_1 = (a, x)(b, y)$, $b_2 = (a, y)(b, z)$, and $\psi = \exists x \forall y \exists z$ are the binding and quantification prefixes occurring in the formula $\varphi = \psi(b_1(q \lor \neg s) \land b_2 x)$. Previously given, where $b_1(a) = x$, $b_1(b) = b_2(a) = y$, $b_2(b) = z$, and $\varphi(x) < \varphi(y) < \varphi(z)$.

In the following, by Boolean combination over a given set of elements $E$ we simply mean a syntactic expression obtained by the grammar $\phi := e \lor \phi \land \phi \lor \varphi$, where $\phi \in E$. The set of all these combinations is denoted by $\mathcal{B}(E)$. Moreover, wit : $\mathcal{B}(E) \rightarrow 2^{\mathcal{E}}$ is the function assigning to each combination $\phi \in \mathcal{B}(E)$ the family wit($\phi$) $\subseteq 2^E$ of subsets of $E$ that are witness for $\phi$, i.e., $E' \in \text{wit}(\phi)$ iff $E' \models \phi$, for all $E' \subseteq E$, where the relation $\models$ is classically interpreted. For instance, the Boolean combination $\phi = (e_1 \lor \neg e_2) \land (\neg e_1 \lor e_2)$ over $E = \{e_1, e_2\}$ has just 0 and E itself as possible witnesses.

Finally, a derived relation $\bar{r} \in \mathcal{R} \setminus \bigcup_{A \subset Ar} B_{\{r \in \mathcal{R} : \text{ar}(r) = A\}}$ is a Boolean combination over the set of relations in $\mathcal{R}$ having all the same arguments. An example for the LS $\mathcal{L}$ of Section 11-A is given by $q \land r$. On the contrary, $q \lor \neg \neg s$ is not a derived relation, since $\text{ar}(x) \neq \text{ar}(s)$.

Definition III.4 (Binding Fragments). Boolean binding formulas over $\mathcal{L}(Vr)$ are built by means of the following context-free grammar, where $\varphi \in \text{Qu}$, $b \in \text{Bn}$, and $\bar{r} \in \mathcal{R}$:

$\psi := \psi \psi | (\varphi \land \varphi) | (\varphi \lor \varphi) ;$
$\psi := \bar{r} \bar{r} | (\psi \land \psi) | (\psi \lor \psi) .$

$\mathcal{L}(Vr)$-FOL[bb] denotes the enumerable set of all formulas over $\mathcal{L}(Vr)$ generated by the principal rule $\varphi$. Moreover, the conjunctive, disjunctive, and one binding fragments of FOL (FOL[cb], FOL[db], and FOL[1b], for short) are obtained, respectively, by weaken the rule $\psi$ as follows: $\psi := \bar{r} \bar{r} | (\psi \land \psi)$, $\psi := \bar{r} \bar{r} | (\psi \lor \psi)$, and $\psi := \bar{r} \bar{r} .$

As an example, consider the FOL[bb] sentence $\exists x \forall y \exists z ((a, x)(b, y)(\neg (a, x) \lor (a, y)(b, 2) \lor (a, x)))$. It is not hard to see that it is equivalent to the formula $\varphi_1$ given above. In general, by applying a simple generalization of the classic translation procedure used to obtain a prenex normal form, it is always possible to transform a FOL formula in a FOL[bb] equivalent one, with only a linear blow-up in its length.

Theorem III.1 (Binding Normal Form). For each $\mathcal{L}(Vr)$-FOL-formula there is an equivalent $\mathcal{L}(Vr)$-FOL[bb] one.

An immediate consequence of the previous theorem is that FOL[bb] inherits all computational and model-theoretic properties of FOL. Consequently, the following holds.

Corollary III.1 (FOL[bb] Negative Properties). FOL[bb] does not enjoy the finite-model property and has an undeniable satisfiability problem.

This result spurred us to investigate the simplest fragment FOL[1b] that turns out to be well-behaved both from a model-theoretic and algorithmic point of view. Actually, we conjecture that FOL[cb] enjoys the same nice properties of FOL[1b].

Conjecture III.1 (FOL[cb] Positive Properties). FOL[cb] enjoys the finite-model property and a decidable satisfiability problem.

On the other hand, it is easy to prove that FOL[db] is not
Theorem III.2 (FOL[pb] Negative Property). FOL[pb] does not enjoy the finite-model property.

Proof. Consider the conjunction $\varphi = \varphi_{irr} \land \varphi_{unb} \land \varphi_{tron}$ of the following three FOL[pb] sentences, asserting that the binary relation $r$ on arguments $\{a, b\}$ is irreflexive, unbounded, and transitive: $\varphi_{irr} = \forall x (\varphi(x, x) \Rightarrow \lnot r)$, $\varphi_{unb} = \forall x \exists y (\varphi(x, y) \land \lnot r)$, and $\varphi_{tron} = \forall x \forall y (\varphi(x, y) \Rightarrow \lnot r \lor \varphi(y, x))$. It is easy to see that $\varphi$ has only infinite models, since it forces $r$ to be a strict partial order without upper bound.

Despite this negative result, we conjecture that the satisfiability problem for FOL[pb] is decidable. Actually, we ground our statement on the observation that classic undecidability proofs, as the reduction from the domino problem, seem to require both conjunctions and disjunctions of binding forms.

Conjecture III.2 (FOL[pb] Positive Property). FOL[pb] enjoys a decidable satisfiability problem.

D. Skolemization

Skolemization is a standard tool in model theory that, by replacing each occurrence of an existential variable $x$ with a functional symbols ranging over the universal variables from which $x$ depends, syntactically transform a formula while preserving its satisfiability. Following the same principle, we give a semantic notion of this concept that is at the base of our novel model-theoretic technique for FOL[1b]. With more detail, we define a machinery called Skolem Map that, taken an assignment over the universal variables of a quantification prefix, returns a new assignment over all variables, by complying with the functional dependences.

Let $\varphi$ be a quantification prefix. Then, $\exists \varphi(\varphi') \equiv \{ x \in vr(\varphi) : (\varphi')_x \equiv x \} \land \forall(\varphi') \equiv vr(\varphi) \setminus \exists(\varphi')$ denote the sets of existential and universal variables quantified in $\varphi$. Moreover, $<_{\varphi} \subseteq vr(\varphi) \times vr(\varphi)$ and $\sim_{\varphi} \subseteq vr(\varphi) \times \exists(\varphi)$ represent, respectively, the precedence ordering on the variables in $vr(\varphi)$ and the corresponding relation of functional dependence induced by $\varphi$, i.e., for all $x_1, x_2 \in vr(\varphi)$, it holds that $x_1 <_{\varphi} x_2$ iif $\varphi(x_1) < \varphi(x_2)$ and $\sim_{\varphi} \equiv <_{\varphi} \cap (\exists(\varphi) \times \exists(\varphi))$. For instance, if $\varphi = \exists x \forall y \exists z$ then $\exists(\varphi) = \{ x, z \}$, $\forall(\varphi) = \{ y \}$, $x <_{\varphi} y <_{\varphi} z$, and $y \sim_{\varphi} z$. In the following, we often indicate by $AsgD(\varphi) \equiv \{ x \in AsgD : dom(\chi) = P \}$ the set of all assignments defined on the subset of placeholders $P \subseteq Ar \cup Vr$.

Definition III.5 (Skolem Map). Let $\varphi$ be a quantification prefix and $D$ an arbitrary set. Then, a Skolem map for $\varphi$ over $D$ is a function $\delta \in SM_{D}(\varphi) \subseteq AsgD(\exists(\varphi) \Rightarrow AsgD(\forall(\varphi)))$ satisfying the following properties:

1) $\delta(\chi)(x) = x$, for all $\chi \in AsgD(\forall(\varphi))$ and $x \in \forall(\varphi)$;
2) $\delta(\chi_1)(x) = \delta(\chi_2)(x)$, for all $\chi_1, \chi_2 \in AsgD(\forall(\varphi))$ and $x \in \exists(\varphi)$ such that $\chi_1 : \Delta_\varphi(x) = \chi_2 : \Delta_\varphi(x)$, where $\Delta_\varphi(x) \equiv \{ x' \in \forall(\varphi) : x' \sim_{\varphi} x \}$.

Intuitively, Item 1 ensures that the Skolem map $\delta$ behaves as the identity function over all universal variables, while Item 2 forces the value $\delta(\chi)(x)$ associated with an existential variable $x$ to be function only of the values $\chi(x')$ associated with the universal variables $x' \in \Delta_\varphi(x)$ from which $x$ depends.

Consider the set $D = \{ 0, 1 \}$ and the quantification prefix $\varphi = \exists x \forall y \exists z$. Then, a possible Skolem map for $\varphi$ over $D$ is the function $\delta$ such that $\delta(\chi)(x) = 0$ and $\delta(\chi)(y) = \delta(\chi)(z) = y$. For all $\chi \in AsgD_{D}(\{ y \})$.

In the remaining part of this section, we a priori fix an $\mathcal{L}$-RS $\mathcal{R} = (Dm, r)$, an $\mathcal{L}(\mathcal{V})$-FOL formula $\psi$, a quantification prefix $\varphi \in Qu$ with $vr(\varphi) \subseteq vr(\psi)$, and an assignment $\chi \in AsgD_{Dm}$ with free($vr(\psi)$) $\subseteq dom(\chi)$.

A Skolem map can be seen as a tool to remove a quantification prefix $\varphi$ from a formula $\varphi \psi$ providing a suitable evaluation for some of the free variables in $\psi$. In fact, one can note a strict connection with the notion of satisfiability, as the following definition suggests.

Definition III.6 (Skolem Satisfiability). Let $\delta \in SM_{Dm}(\varphi)$ be a Skolem map for $\varphi$ over $Dm$. Then, $\mathcal{R}$ satisfies $\chi$ given $\delta$ and, in symbols $\mathcal{R}, \chi \models \delta \psi$, iff $\mathcal{R}, \chi' \models \psi$, for all $\chi' \in AsgD_{m}(dom(\chi) \cup vr(\psi))$ such that $\chi' \subseteq \chi$ and $\chi'_{vr(\psi)} \subseteq \lnot \psi$.

To better understand the meaning of this definition, consider again the $\mathcal{L}$-RS $\mathcal{R}$ described in Section I-B the previous formula $\varphi_1 = \varphi_2$ with $\varphi_2 = \exists x \forall y \exists z$ and $\psi = q_1 (q \land \lnot s) \land q_2 x$, and the Skolem map $\delta$ for $\varphi$ of the example above. Then, one can verify that $\mathcal{R}, \emptyset \models \delta \psi$.

Driven by the intuition behind Skolem maps, it is not hard to prove by structural induction that the standard notion of satisfiability is actually equivalent to the Skolem one.

Theorem III.3 (Skolem Satisfiability). $\mathcal{R}, \chi \models \varphi \psi$ iff there is a Skolem map $\delta \in SM_{Dm}(\varphi)$ for $\varphi$ over $Dm$ such that $\mathcal{R}, \chi \models \delta \psi$.

IV. EXPRESSIVENESS COMPARISON

In order to compare the expressive power of our family of logics, in particular FOL[1b], to the well-known clique guarded (FOL[cc], for short [17], [19] and guarded negation (FOL[GN], for short) [26] fragments of FOL, we shall study a suitable concept of bisimulation between RSs, for which the satisfiability of FOL[1b] turns out to be invariant.

A. Bisimulation

We now introduce a novel definition of bisimulation between RSs that result to be bisimilar iff every mapping between partial assignments can be extended to a single tuple functions in a way that preserves the interpretation of the relations.

In the following, by $\chi[P \mapsto d]$ we denote the assignment that agrees with $\chi$ on $dom(f) \setminus P$ and is equal to $d$ on all placeholders in $P$.

Definition IV.1 (One-Binding Bisimulation). Let $\mathcal{R}_1 = (Dm_1, r_1)$ and $\mathcal{R}_2 = (Dm_2, r_1)$ be two $\mathcal{L}$-RSs. Then, $\mathcal{R}_1$ is one-binding bisimilar to $\mathcal{R}_2$ iff there exists a total relation $\sim \subseteq \bigcup_{A \subseteq Ar} AsgD_{m}(A) \times AsgD_{m}(A)$ for which the following hold.

1) For all $A \subseteq Ar$, $A' \subseteq Ar \setminus A$, and pairs of assignments $\chi_1 \in AsgD_{m}(A)$ and $\chi_2 \in AsgD_{m}(A)$, if $\chi_1 \sim \chi_2$ then:
forth for all $d_i \in \text{Dom}_1$, there is $d_2 \in \text{Dom}_2$ such that $x_i[A' \rightarrow d_i, x_2[A' \rightarrow d_2]$;  
back for all $d_1 \in \text{Dom}_1$, there is $d_2 \in \text{Dom}_2$ such that $x_i[A' \rightarrow d_1, x_2[A' \rightarrow d_2]$.

2) For all $r \in \text{RL}$, $x_i \in \text{AsSg}_{\text{DSm}}(ar(r))$, and $x_2 \in \text{AsSg}_{\text{DSm}}(ar(r))$, with $x_i \sim x_2$, it holds that $x_i \in r^{\mathcal{S}_1}$ iff $x_2 \in r^{\mathcal{S}_2}$.

Consider the LS $\mathcal{L} = \{a, b\}$, \{x, ar\}, where $ar(x) = \{a, b\}$, and the two $\mathcal{L}$-RSs $\mathcal{R}_1 = (D_{m_1}, r_1)$ and $\mathcal{R}_2 = (D_{m_2}, r_2)$ given in Figure [1] where $D_{m_1} = \{0, 1, 2\}$, $D_{m_2} = \{1', 2'\}$, and $r_1^{\mathcal{S}_1} = D_{m_1} \cup \{1', 2'\}$, and $r_2^{\mathcal{S}_2} = D_{m_2} \cup \{(a \rightarrow 1', b \rightarrow 2')\}$. It is not hard to see that they are one-binding bisimilar. Indeed, the forth condition is trivially satisfied, since the first structure is contained into the second one. For the back condition, it is enough to observe that each pair of elements in relation (resp., not in relation) in $\mathcal{R}_2$ has a corresponding pair in $\mathcal{R}_1$.

Finally, by structural induction, it is easy to prove that the existence of a one-binding bisimulation between two RSs implies their indistinguishability w.r.t. FOL [1b] sentences. In other words, FOL [1b] is invariant under one-binding bisimilarity.

**Theorem IV.1** (One-Binding Invariance). Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two one-binding bisimilar $\mathcal{L}$-RSs. Then, for all $\mathcal{L}$-FOL [1b] sentences $\varphi$, it holds that $\mathcal{R}_1 \models \varphi$ iff $\mathcal{R}_2 \models \varphi$.

**B. Expressiveness**

We now have the tool to compare the expressive power of FOL [1b] first w.r.t. its extensions FOL [cg] and FOL [db] and, successively, to the other fragments FOL [cg] and FOL [gn].

**Theorem IV.2** (FOL [1b] vs. FOL [cb] and FOL [db]). FOL [1b] is strictly less expressive than FOL [cb] and FOL [db].

*Proof.* Observe preliminary that, since FOL [1b] is a syntactic fragment of both FOL [cb] and FOL [db], they are at least as expressive as the former. Now, consider the one-binding bisimilar RSs $\mathcal{R}_3$ and $\mathcal{R}_4$ of Figure [1] By Theorem IV.1 (One-Binding Invariance), there is no FOL [1b] sentence that distinguishes them. On the contrary, for the FOL [cb] sentence $\varphi = \exists x \exists y \exists z ((a, x)(b, y) \wedge (a, z)(b, w)) \wedge (a, x)(b, w \neg)$, it holds that $\mathcal{R}_3 \not\models \varphi$. Thus, $\varphi$ has no FOL [1b] equivalent. To verify the thesis for FOL [db] too, it is enough to observe that there is a FOL [db] sentence equivalent to $\neg \varphi$.

We can now focus on FOL [cg] and FOL [gn]. To evaluate the comparisons, it is useful to do some preliminary observations. Consider the LS $\mathcal{L}$ of the previous example and the two simple $\mathcal{L}$-RSs $\mathcal{R}_4 = (D_{m_4}, r_4)$ and $\mathcal{R}_5 = (D_{m_4}, r_5)$ depicted in Figure 2 where $D_{m_4} = \{0\}$, $r_4^{\mathcal{S}_3} = \{(a \rightarrow 0, b \rightarrow 0)\}$, $D_{m_4} = \{0, 1\}$, and $r_5^{\mathcal{S}_4} = r_4^{\mathcal{S}_3} \cup \{(a \rightarrow 1, b \rightarrow 1)\}$. It is easy to observe that they are both clique-guarded and guarded-negation bisimilar, via the the morphisms $f$ and $g$ represented by the dashed arrows [17, 19, 26]. Now, let $\mathcal{L}'$ be a new LS extending the previous one with a monadic relation $p$ having $ar(p) = \{b\}$. Moreover, consider the two $\mathcal{L}'$-RSs $\mathcal{R}_5 = (D_{m_5}, r_{5b})$ and $\mathcal{R}_6 = (D_{m_5}, r_{5b})$, depicted in Figure 3 where $p^{\mathcal{S}_5} = p^{\mathcal{S}_4} = \{(b \rightarrow 3)\}$. It is not hard to see that they are one-binding bisimilar, since the element 2, when associated to the argument $a$ in $\mathcal{R}_5$, can be mapped to the element 0 in $\mathcal{R}_5$ in order to be sure that it has just an "x-successor" that does not satisfy $p$.

**Theorem IV.3** (FOL [1b] vs. FOL [cg] and FOL [gn]). FOL [1b] is incomparable with FOL [cg] and FOL [gn].

*Proof.* To show that FOL [cg] and FOL [gn] are not as expressive as FOL [1b], consider the clique-guarded and guarded-negation bisimilar RSs $\mathcal{R}_4$ and $\mathcal{R}_5$ described above. By known invariance theorems [17], [19], [26], there are neither FOL [cg] nor FOL [gn] sentences that distinguish them. On the contrary, for the FOL [1b] sentence $\varphi_1 = \forall x \forall y ((a, x)(b, y) \rightarrow r_4)$, it holds that $\mathcal{R}_3 \models \varphi_1$ and $\mathcal{R}_4 \not\models \varphi_1$. Thus, $\varphi_1$ has no FOL [cg] or FOL [gn] equivalent. For the converse, consider the one-binding bisimilar RSs $\mathcal{R}_5$ and $\mathcal{R}_6$ described above. By Theorem IV.1 there is no FOL [1b] sentence able to distinguish them. On the contrary, for the sentence $\varphi_2 = \exists x \forall y ((a, x)(b, y) \rightarrow (\exists z (a, y)(b, z) x \wedge (b, z)(p)))$, it holds that $\mathcal{R}_5 \models \varphi_2$ and $\mathcal{R}_6 \not\models \varphi_2$. Thus, $\varphi_2$ has no FOL [1b] equivalent. At this point, it remains just to observe that $\varphi_2$ is a guarded sentence. So, it belongs to both FOL [cg] and FOL [gn].

**V. Model-Theoretic Analysis**

We now come to the more technical part of this paper, in which we describe a novel model-theoretic tool for FOL [1b] that allows to obtain the results of finite-model property, Craig interpolation, and Beth definability. Moreover, it represents a key point in the decidability procedure given in the last section.

The tool is based on two characterization theorems that tightly correlate the concept of entanglement, a semantic property on the sets of Skolem maps $\delta$, associated with the subsentences $\varphi_i \delta_i \gamma_i$ of a given sentence of interest $\varphi$, to that of overlapping, a syntactic property on the structure of pairs of quantification and binding prefixes $(\varphi_i, \beta_i)$. Intuitively, on one hand, a set $\{\delta_i\}$ is entangled if the underlying quantifications force the corresponding derived relations $\{\gamma_i\}$ to be verified on a same tuple function $t$, i.e., in other words, if there exists a set of assignments $\{\chi_i \in \text{rng}(\delta_i)\}$ such that $t = \chi_i \circ \beta_i$, for all indexes $i$. On the other hand, a set $\{(\varphi_i, \beta_i)\}$ is overlapping if each argument is bound to an existential variable at most once and there is a strict total order $\prec$ between them that agrees, via $\{\delta_i\}$, with all functional dependences of the prefixes $\varphi_i$, which we describe as two sides of the same coin,
To have a deeper intuition behind the idea just discussed, we now describe four examples, built on the LS $\mathcal{L} = \{a, b, c\}$, \{q, r, ar\} with ar(q) = ar(r) = \{a, b, c\}, which covers some interesting cases of correlation among the concepts. Consider the $\mathcal{L}$-FOL sentence $\varphi_1 = \forall x \exists y \forall z. \varphi_2 = \forall y \exists z. \varphi_3 = \forall x \exists y \forall z$, and $b_1 = b_2 = b_3 = (a, x)(b, y)(c, z)$. It is not hard to see that $\varphi_1$ is unsatisfiable, since it requires the inconsistent derived relations $q$, $\rightarrow$, and $q \leftrightarrow r$ to hold on a same tuple function $t$. Indeed, independently of the Skolem maps $\delta_1$, $\delta_2$, and $\delta_3$ that one may associate with the quantification prefixes $\varphi_1$, $\varphi_2$, and $\varphi_3$, there are always three assignments $\chi_1 \in \text{rng}(\delta_1)$, $\chi_2 \in \text{rng}(\delta_2)$, and $\chi_3 \in \text{rng}(\delta_3)$; such that $t = x_1 \circ b_1 = x_2 \circ b_2 = x_3 \circ b_3$. So, $\delta_1$, $\delta_2$, and $\delta_3$ are necessarily entangled. This is because the pairs $(\varphi_1, \delta_1)$, $(\varphi_2, \delta_2)$, and $(\varphi_3, \delta_3)$ are overlapping, due to the fact that we can order the arguments as follows: $a \sim b \sim c$. Given this order, it is simple to find the tuple function $t$. First choose an arbitrary value for $a$. Then, compute the corresponding value for $b$ via $\delta_1$. Finally, extract from $\delta_2$ the value for $c$. Now, let $\varphi_2$ be the sentence obtained from $\varphi_1$ by replacing the prefixes $\varphi_3$ and $\delta_3$ with the following ones: $\varphi_3 = \forall x \exists y \forall z$ and $\delta_3 = \{a, x\}(b, y)(c, y)$. We have that $\varphi_2$ is satisfiable on a simple $\mathcal{L}$-RS of order 2, where the Skolem map $\delta_2$ is such that $\delta_2(\chi)(z) \neq \chi(y)$. Indeed, there are no assignments $\chi_2 \in \text{rng}(\delta_2)$ and $\chi_3 \in \text{rng}(\delta_3)$ for which $\chi_2 \circ b_2 = \chi_3 \circ b_3$, since $(\chi_2 \circ b_2)(b) \neq (\chi_2 \circ b_2)(c)$ but $(\chi_3 \circ b_3)(b) = (\chi_3 \circ b_3)(c)$. Hence, $\delta_2$ and $\delta_3$ are untangled. Therefore, there is no tuple function $t$ on which $\varphi_2$ may require the three derived relations all together. The syntactic reason here is that $(\varphi_2, \delta_2)$ and $(\varphi_3, \delta_3)$ does not allow to find the required order between the argument, since from them it follows that $c \sim c$. Similarly to the previous example, consider the sentence $\varphi_3$ drawn from $\varphi_1$ by substituting $\forall x \exists y \forall z$ for the prefix $\varphi_3$. Also in this case, $\varphi_3$ is satisfiable on a simple $\mathcal{L}$-RS of order 2, where the three Skolem maps are chosen, for instance, as follows: $\delta_1(\chi)(y) = \chi(x)$, $\delta_2(\chi)(z) = \chi(y)$, and $\delta_3(\chi)(x) \neq \chi(z)$. As a matter of fact, there are no assignments $\chi_1 \in \text{rng}(\delta_1)$, $\chi_2 \in \text{rng}(\delta_2)$, and $\chi_3 \in \text{rng}(\delta_3)$ such that $t = x_1 \circ b_1 = x_2 \circ b_2 = x_3 \circ b_3$, since the definitions of $\delta_1$ and $\delta_3$ require $\chi(a) = \chi(b) = \chi(c)$ while that of $\delta_2$ implies $\chi(a) \neq \chi(c)$. Here, the pairs $(\varphi_1, \delta_1)$, $(\varphi_2, \delta_2)$, and $(\varphi_3, \delta_3)$ are not overlapping, since we can obtain a cycle $a \sim b \sim c \sim a$. Finally, derive the sentence $\varphi_3$ from $\varphi_2$, where the prefix $\varphi_3$ is set to $\exists y \forall \forall y$. It is easy to see that $\varphi_3$ is again satisfiable, since we can choose one the two Skolem maps $\delta_2$ and $\delta_3$ in such a way that $\delta_2(\chi_2)(x) \neq \delta_3(\chi_3)(x)$, independently of the assignments $\chi_2$ and $\chi_3$. Consequently, there is no common tuple function $t$. Indeed, the argument $c$ is existential in both $(\varphi_2, \delta_2)$ and $(\varphi_3, \delta_3)$.

### A. Entangled Quantifications

We start by dealing with the semantic part of our tool. First, we introduce the notions of schemas and coupling maps that formalize, in a suitable way, the corresponding concepts of pairs of prefixes and set of Skolem maps explained in the informal comment above. Then, we describe a new notion of satisfiability of a sentence, given a coupling map, that generalizes the one for Skolem maps. Finally, we define the concept of entanglement and show how to use it to prove the finite-model property.

A schema $\sigma = (\sigma, b) \in \text{Sch} \triangleq \{(\sigma, b) \in Qn \times Bn : \nu r(\sigma) = \nu r(b)\}$ is a pair of two prefixes, one of quantification $\psi(\sigma) \triangleq \psi$ and the other of binding $\nu(b) \triangleq \nu$, on the sets of arguments $\sigma(\sigma) \triangleq \sigma(b)$ and same variables $\nu r(\sigma) \triangleq \nu r(b)$. By $\text{Sch}(A) \triangleq \{\sigma \in \text{Sch} : \sigma(\sigma) = A\}$ we denote the subset of schemas having argument set $A \subseteq A$. In what follows, we a priori fix a finite set of schemas $S \subseteq \text{Sch}$, its size $n \triangleq |S|$, and both the maximal numbers of existential variables $\hat{h} \triangleq \max_{\sigma \in S} |\exists r(\psi(\sigma))|$ and arguments $k \triangleq \max_{\sigma \in S} |\sigma(\sigma)|$.

**Definition V.1** (Coupling Map). Let $D$ be an arbitrary set. Then, a coupling map for $S$ over $D$ is a function $\gamma \in \text{CM}_{D}(S) \triangleq S \rightarrow_{\sigma} \text{SM}_{D}(\psi(\sigma))$ assigning to every schema $\sigma \in S$ a Skolem map $\gamma(\sigma) \in \text{SM}_{D}(\psi(\sigma))$ for $\psi(\sigma)$ over $D$.

Similarly to the Skolem satisfiability, we now introduce the coupling satisfiability. As a matter of technical convenience, we do not treat formulas explicitly, but use a suitable function $fr$ representing a set of possible sentences of the form $\phi b r$.

**Definition V.2** (Coupling Satisfiability). Let $R = (Dm, rl)$ be an $\mathcal{L}$-RS and $\gamma \in \text{CM}_{D}(S)$ a coupling map for $S$ over $Dm$. Moreover, let $fr \in \text{Fr}(S) \triangleq S \rightarrow_{\sigma} \{\hat{r} \in R : \sigma(\hat{r}) = fr(\sigma)\}$ be a formula function for $S$ assigning to each of its schemas $\sigma$ a derived relation $fr(\sigma)$ on the arguments of $\sigma$ itself. Then, $R$ satisfies $fr$ for $\gamma$, in symbols $R \models_{\gamma} fr$, iff $R \models_{\gamma(\sigma)} fr(\sigma)$, for all $\sigma \in S$.

Due to the marked similarity between Skolem and coupling satisfiability, the next result follows as an immediate corollary of Theorem III.13 (Skolem Satisfiability).

**Corollary V.1** (Coupling Satisfiability). Let $R = (Dm, rl)$ be an $\mathcal{L}$-RS and $fr \in \text{Fr}(S)$ a formula function for $S$. Then, $R \models_{\gamma} fr$, for all $\sigma = (\sigma, b) \in S$ with $\hat{r} = fr(\sigma)$, if there exists a coupling map $\gamma \in \text{CM}_{D}(S)$ for $S$ over $Dm$ such that $R \models_{\gamma} fr$.

We give here the fundamental definition of coupling entanglement that allows to verify the existence of a tuple function shared by the Skolem maps in the range of a coupling map.

**Definition V.3** (Coupling Entanglement). Let $\gamma \in \text{CM}_{D}(S)$ be a coupling map for $S$ over a set $D$. Then, the entanglement of $\gamma$ w.r.t. $S' \subseteq S \cap \text{Sch}(A)$ over $A' \subseteq A \subseteq Ar$ is the set of functions of $\gamma(\tau) \rightarrow_{Dm}$ to $\text{Ent}_{\gamma}(S', A') \triangleq \{t \in A' \rightarrow D : \forall \sigma \in S' \exists x \in \text{(rng}(\gamma(\sigma)) \cdot t = (\chi \circ b) \times \gamma(\tau)\}$. If $\text{Ent}_{\gamma}(S', A') \neq \emptyset$ then $\gamma$ is said to be entangled w.r.t. $S'$ over $A'$, otherwise, it is untangled.

As already mentioned in the informal description above, the
concept of entanglement indirectly induces a preorder among coupling maps. Such a preorder plays a key role in all proofs of model-theoretic and algorithmic properties for FOL[18].

Definition V.4 (Coupling Preorder). Let \(\gamma_1 \in \text{CM}_{D_1}(S)\) and \(\gamma_2 \in \text{CM}_{D_2}(S)\) be two coupling maps for \(S\) over the sets \(D_1\) and \(D_2\). Then, \(\gamma_1\) is at least as entangled as \(\gamma_2\), in symbols \(\gamma_2 \leq \gamma_1\), if and only if, whenever \(\gamma_2\) is entangled \(w.r.t.\) \(S' \subseteq S \cap \text{Sch}(A)\) over \(A' \subseteq A \subseteq \mathbb{A}\), so is \(\gamma_1\), i.e., \(\text{Ent}_{\gamma_1}(S', A') \neq \emptyset\) implies \(\text{Ent}_{\gamma_1}(S', A') \neq \emptyset\).

As stated in the next theorem, the main feature for such a preorder is that it is downward-bounded, namely it admits minimal elements. In particular, among those elements it is always possible to find a finite coupling model, i.e., a map whose range is composed only by Skolem maps defined over a finite domain. The proof of this property, reported in the next section, is one of the most important result we derive from the announced characterization theorems.

Theorem V.1 (Finite Minimal Coupling). There exists a coupling map \(\gamma^* \in \text{CM}_{D^*}(S)\) for \(S\) over a finite set \(D^*\) with \(|D^*| \leq n \cdot h^{2(n-k)}\) that is minimal \(w.r.t.\) \(\leq\).

Informally, the property of entanglement describes the degree of possible inconsistencies among the derived relations of a given sentence. Therefore, the lesser entangled a coupler map is, the easier a model for that sentence can be found. Next theorem actually formalize this intuitive idea, by showing that, once a formula function \(fr\) is satisfied by a coupling map \(\gamma_1\), it is satisfied by all coupling maps \(\gamma_2\) that are less entangled than \(\gamma_1\).

Theorem V.2 (Satisfiability Preservation). Let \(\gamma_1 \in \text{CM}_{D_1}(S)\) and \(\gamma_2 \in \text{CM}_{D_2}(S)\) be two coupling maps for \(S\) over the sets \(D_1\) and \(D_2\) such that \(\gamma_2 \leq \gamma_1\). Moreover, let \(fr \in \text{Fr}(S)\) be a formula function for \(S\). If there is an \(\mathcal{L}\)-RS \(R_1 = (D_1, r_{l_1})\) such that \(\forall_{\gamma_1} fr\) then there exists an \(\mathcal{L}\)-RS \(R_2 = (D_2, r_{l_2})\) such that \(\forall_{\gamma_2} fr\) as well.

By suitably exploiting the results stated in the previous two theorems, we can finally prove that FOL[18] has the finite-model property.

Theorem V.3 (Finite Model Property). Every \(\mathcal{L}(\forall r)\)-FOL[18] satisfiable sentence is finite satisfiable.

Proof. Since \(\varphi\) is satisfiable, there is an \(\mathcal{L}\)-RS \(R = (Dm, r_l)\) such that \(R \models \varphi\). This implies the existence of a witness for \(\varphi\) satisfied by \(R\), i.e., a set of subsentences \(F \in \text{wit}(\varphi)\) such that \(R \models \phi\) for all \(\phi \in F\). Now, let \(S \models \{ (\varphi, b) \in \text{Sch} : \exists r \in Rl. \varphi r \in F \} \) be the set of schemas for \(F\) and \(fr \in \text{Fr}(S)\) the formula function for \(S\) assigning to each of its schemas \(\sigma\) a derived relation \(\hat{r} = fr(\sigma)\) such that \(\varphi \hat{r} \in F\). W.l.o.g., we can assume that \(fr\) covers \(F\), i.e., for each sentence \(\phi \in F\), there is a schema \(\sigma = (\varphi, b) \in S\) such that \(\phi = \varphi \hat{r}\) with \(\hat{r} = fr(\sigma)\). Indeed, we can prevent two sentences from sharing the same schema by simply renaming their variables. At this point, by Corollary V.1 (Coupling Satisfiability), we derive the existence of a coupling map \(\gamma \in \text{CM}_{Dm}(S)\) such that \(R \models fr\). Moreover, by Theorem V.1 (Finite Minimal Coupling), there is a coupling map \(\gamma^* \in \text{CM}_{D^*}(S)\) over a finite set \(D^*\) with \(|D^*| \leq n \cdot h^{2(n-k)}\) that is minimal \(w.r.t.\) \(\leq\). Consequently, \(\gamma^* \leq \gamma\).

Now, by Theorem V.2 (Satisfiability Preservation), there is an \(\mathcal{L}\)-RS \(R^* = (D^*, r_l')\) such that \(R^* \models \varphi, fr\). Thus, again by Corollary V.1 we have that \(R^* \models \phi\), for all \(\phi \in F\). Hence, \(R^* \models \varphi\). To conclude the proof, it is enough to observe that \(R^*\) has order \(O(n \cdot h^{2(n-k)})\).

B. Entanglement Characterization

We can now deal with the syntactic part of our tool. To introduce the concept of overlapping used to state the characterization theorems, we make use of two suitable graphs defined on the structural properties of a given sets of schemas.

Let \(\sigma \in \text{Sch}\) be a schema. Then, \(\exists(\sigma) \models \{ a \in ar(\sigma) : \hat{b}(\sigma)(a) \in \exists(\psi(\sigma)) \}\) and \(\forall(\sigma) \models ar(\sigma) \setminus \exists(\sigma)\) denote the sets of existential and universal arguments in \(\sigma\), i.e., the arguments \(a \in ar(\sigma)\) that are associated with an existential or universal variable \(\psi(\sigma)(a)\) in the quantification prefix \(\psi(\sigma)\) via the binding prefix \(\hat{b}(\sigma)\). Moreover, \(\exists_{\sigma} \subseteq \text{ar}(\sigma) \times \text{ar}(\sigma)\) and \(\forall_{\sigma} \subseteq \forall(\sigma) \times \exists(\sigma)\) represent, respectively, the collapsing equivalence and the functional dependence induced by \(\sigma\), i.e., for all \(a_1, a_2 \in \text{ar}(\sigma)\), it holds that \(a_1 \exists_{\sigma} a_2\) if and only if \(\psi(\sigma)(a_1) = \psi(\sigma)(a_2)\), and, for all \(a_1 \in \forall(\sigma)\) and \(a_2 \in \exists(\sigma)\), it holds that \(a_1 \forall_{\sigma} a_2\) if and only if \(\psi(\sigma)(a_1) \sim \psi(\sigma)(a_2)\).

The vertices of the above mentioned graphs are arguments coupled in the schemas in which they occur, as the following description reports. Let \(A' \subseteq A \subseteq \mathbb{A}\) be two sets of arguments and \(S' \subseteq S \cap \text{Sch}(A)\) a set of schemas. Then, \(\text{Ar}_{S'} \models \{ (\sigma, a) \in S' \times A' : a \in \exists(\sigma) \}\) denotes the set of extended arguments for \(S'\) over \(A'\), i.e., the pairs \(e \models (\sigma, a) \in \text{Ar}_{S'}\) of a schema \(\sigma(e)\) \(\subseteq S'\) and one of its arguments \(a(e)\) \(\in \text{ar}(\sigma) \cap A'\). Also, \(\exists_{S'} \models \{ (\sigma, a) \in \text{Ar}_{S'} : a \in \exists(\sigma) \}\) and \(\forall_{S'} \models \text{Ar}_{S'} \setminus \exists_{S'}\) represent the set of existential and universal extended arguments.

We now introduce the first graph, which is used to represent the relation among the arguments that are somehow collapsed by the structure of the schemas, i.e., intuitively, they are forced to assume the same value. The collapsing graph for \(S'\) over \(A'\) is the symmetric directed graph \(\text{C}_{S'} \models \text{Ar}_{S'} \setminus \exists_{S'}\) with the extended arguments as vertices and the edge relation defined as follows: \(\exists_{S'} \models \{ (e_1, e_2) : (e_1, e_2) \in \text{Ar}_{S'} \sim \text{Ar}_{S'} : (\sigma(e_1) = \sigma(e_2) \Rightarrow a(e_1) \sim a(e_2)) \} +\).

![Figure 4. Collapsing graph C1](image)

![Figure 5. Collapsing graph C2](image)

In Figure 4 we depict the collapsing graph \(C_1 = \text{C}_{S_1}\), where \(S = \{ \sigma_1 = (\varphi_1, b_1), \sigma_2 = (\varphi_2, b_2), \sigma_3 = (\varphi_3, b_3) \}\) and \(A = \{ a, b, c \}\) are the set of schemas and arguments of the sentences \(\varphi_1, \varphi_2, \text{and } \varphi_3\) given in the intuitive discussion above. Here, the dots represents the extended arguments obtained by the
intersection of rows and columns. Moreover, we are omitting the transitive closure, since \( \equiv_{S}^{D} \) is an equivalence relation. In Figure 5 we report the collapsing graph \( C_{S} \) for the sentence \( \varphi_{2} \). Observe that the edge between the vertices \((\sigma_{3}, b)\) and \((\sigma_{3}, c)\) is due to the binding \( b_{3} \).

At this point, we define a property that precisely describe the case in which two existential arguments are forced to assume the same value. A collapsing graph \( C_{S}^{D} \) is conflicting if there exist two extended arguments \( e_{1}, e_{2} \in S_{D} \) such that \( e_{1} \equiv_{S}^{D} e_{2} \) and if \( \sigma(e_{1}) = \sigma(e_{2}) \) then \( a(e_{1}) \not\equiv a(e_{2}) \).

As an example, the graph \( C_{1} \) is conflicting when it is referred to the sentence \( \varphi_{4} \). This is due to the fact that the argument \( c \) is existentially quantified in both \( \sigma_{2} \) and \( \sigma_{3} \).

The second graph we introduce keeps track of the chains of functional dependences between arguments that cross all the schemas. The dependence graph for \( S' \) over \( A' \) is the directed graph \( D_{S}^{D} = (\text{Ar}_{S}, \text{Dep}_{S}^{D}) \) with the extended arguments as vertexes and the edge relation defined as follows: \( \text{Dep}_{S}^{D} = \equiv_{S}^{D} \cap \{ (e_{1}, e_{2}) \in \text{Ar}_{S} \times \text{Ar}_{S} : \sigma(e_{1}) = \sigma(e_{2}) \land a(e_{1}) \not\equiv a(e_{2}) \} \).

Figure 6. Dependence graph \( D_{1} \). Figure 7. Dependence graph \( D_{2} \). Figure 8. Dependence graph \( D_{3} \).

In Figures 6, 7, and 8 we report the dependence graphs \( D_{i} = D_{S}^{D} \) corresponding to the sentences \( \varphi_{1}, \varphi_{2}, \) and \( \varphi_{3} \), respectively. The black arrows represent the functional dependences inside a single schema, while the gray ones are obtained by the composition with the collapsing relation. Note that \( D_{1} \) is acyclic, so, we can construct an order among the extended arguments that agrees with all functional dependences of the schemas. On the contrary, in \( D_{2} \), there is a loop on \((\sigma_{2}, c)\) due to the structures of \( \sigma_{2} \) and \( \sigma_{3} \). Finally, also \( D_{3} \) is not acyclic, due to cycle among \((\sigma_{3}, a), (\sigma_{1}, b), \) and \((\sigma_{2}, c)\).

**Definition V.5 (Overlapping Schemas).** A set of schemas \( S' \subseteq S \cap \text{Sch}(A) \) is overlapping over a set of arguments \( A' \subseteq A \subseteq \text{Ar} \) iff the collapsing graph \( C_{S}^{D} \) is not conflicting and the dependence graph \( D_{S}^{D} \) is acyclic.

We can now state the two characterization theorems at the base of all results we provide for FOL[18]. They essentially move along two opposite directions. Intuitively, the first shows that, if a set of schemas is overlapping, every coupling map is necessarily entangled over that set. The second, instead, shows that, if a set of schemas is not overlapping, it is always possible to find a finite untangled coupling map over such a set.

**Theorem V.4 (Entangled Coupling Map).** For all coupling maps \( \gamma \in \text{CM}_{D}(S) \) for \( S \) over an arbitrary set \( D \), if \( S' \subseteq S \cap \text{Sch}(A) \) is overlapping over \( A' \subseteq A \subseteq \text{Ar} \) then \( \gamma \) is entangled w.r.t. \( S' \) over \( A' \).

**Theorem V.5 (Untangled Coupling Map).** There exists a coupling map \( \gamma^{*} \in \text{CM}_{D}^{*}(S) \) for \( S \) over a finite set \( D^{*} \) with \( |D^{*}| \leq n \cdot h \cdot 2^{(n-k)!} \) such that if \( S' \subseteq S \cap \text{Sch}(A) \) is not overlapping over \( A' \subseteq A \subseteq \text{Ar} \) then \( \gamma^{*} \) is untangled w.r.t. \( S' \) over \( A' \).

Both the theorems we have just stated form our characterization result on the correlation between the concepts of entanglement and that of overlapping, from which we can easily derive the existence of a finite minimal coupling.

**Proof of Theorem V.5** By Theorem V.3 (Untangled Coupling Map), there is a coupling map \( \gamma^{*} \in \text{CM}_{D}^{*}(S) \) for \( S \) over a finite set \( D^{*} \) with \( |D^{*}| \leq n \cdot h \cdot 2^{(n-k)!} \) such that if \( \gamma^{*} \) is entangled w.r.t. \( S' \subseteq S \cap \text{Sch}(A) \) over \( A' \subseteq A \subseteq \text{Ar} \) then \( \gamma^{*} \) is overlapping over \( A' \). Now, such a coupling map is necessarily minimal, i.e., \( \gamma^{*} \not\leq \gamma \), for all coupling maps \( \gamma \in \text{CM}_{D}(S) \) for \( S \) over an arbitrary set \( D \). Indeed, by Theorem V.4 (Entangled Coupling Map), \( \gamma \) is entangled w.r.t. \( S' \) over \( A' \) as well.

As a further model-theoretic result, a constructive version of the Craig interpolation property for FOL[18] can be provided. Here, due to the lack of space, we just give a quick intuition behind the proof of the basic case in which the input formulas are of the form \( \varphi_{1} = \varphi_{1} \land \hat{\varphi}_{1} \) and \( \varphi_{2} = \varphi_{2} \land \hat{\varphi}_{2} \) with \( \varphi_{1} \rightarrow \varphi_{2} \). Indeed, it is not hard to see that the formula \( \varphi = \varphi_{1} \lor \hat{\varphi}_{2} \) with \( \hat{\varphi}_{2} \) being the classic propositional interpolating between \( \hat{\varphi}_{1} \) and \( \hat{\varphi}_{2} \), is the interpolating formula for \( \varphi_{1} \) and \( \varphi_{2} \). A proof of the general case deeply relies on the characterizing theorems. By means of classic techniques, the Beth definability property can be also obtained from the Craig interpolation one.

**Theorem V.6 (Craig Interpolation and Beth Definability).** FOL[18] enjoys both Craig interpolation and Beth definability.

**VI. SATISFIABILITY**

We finally provide a PSPACE deterministic algorithm for the solution of the satisfiability problem for FOL[18], which can be interpreted as a satisfiability-modulo-theory procedure. Indeed, by means of a syntactic preprocessing based on the concept of overlapping schemas, the search for a model of a FOL[18] sentence is reduced to that of a sequence of Boolean formulas over the set of derived relations. The correctness of such an approach is crucially based on the fundamental characterization of the entanglement property previously discussed. It is interesting to observe that the procedure is independent from the size of the model pointed out in Theorem V.3.

To understand the main idea behind the algorithm, it is useful to describe it through a simple one-round two-player turn-based game between the existential player, called Eloise, willing to show that a sentence \( \varphi \) is satisfiable, and the universal player, called Abelard, trying to do exactly the opposite. First, Eloise choses a witness \( F \in \text{wit}(\varphi) \) for \( \varphi \) seen as a Boolean combination of simpler subsentences of the form \( \varphi R \). In this way, she identifies a formula function \( \text{fr} = \{ (\varphi, b) \in \text{Sch} \rightarrow \hat{\varphi} \in \text{RI} : \varphi R = F \} \) that describes \( F \) by associating each derived relation with the corresponding schema. Then, Abelard choses a subset of overlapping schemas \( S \subseteq \text{dom(fr)} \cap \text{Sch}(A) \) over a set of arguments \( A \subseteq \text{Ar} \).
At this point, Eloise wins the play iff the Boolean formula $\psi = \bigwedge_{\sigma \in S} fr(\sigma)$, obtained as the conjunction of all the derived relations associated with the schemas in $S$, is satisfiable. If this is not the case, Abelard has spotted a subset $\{\psi \land fr(\sigma) : (\varphi, b) \in S\}$ of the witness $\mathcal{W}$ that requires to verify the unsatisfiable property $\psi$ on a certain valuation of arguments in $\mathcal{W}$. Thus, F cannot have a model. Consequently, the sentence $\varphi$ is satisfiable iff Eloise has a winning strategy for this game.

**Algorithm 1: FOL[18] Satisfiability Checker.**

| signature | sat : $\mathcal{L}(Vr).\text{-FOL[18]} \rightarrow \{t, f\} |
|---|---|
| function | sat($\varphi$) |
| 1 | foreach $F \in \text{wit}(\varphi)$ do |
| 2 | fr ← $\{(\varphi, b) \in \text{Sch} \Rightarrow \widehat{\varphi} \in \mathcal{RI} : \varphi \land \widehat{\varphi} \in F\}$ |
| 3 | $i \leftarrow f$ |
| 4 | foreach $S \subseteq \text{dom}(fr) \cap \text{Sch}(A)$ with $A \subseteq \text{Ar}$ do |
| 5 | if $S$ is-overlapping-over $A$ then |
| 6 | if $\text{wit}(\bigwedge_{\sigma \in S} fr(\sigma)) = \emptyset$ then |
| 7 | $i \leftarrow \mathsf{t}$ |
| 8 | if $i = f$ then |
| 9 | return $t$ |
| 10 | return $f$ |

We can now describe the pseudo-code of the algorithm. The deterministic counterpart of Eloise’s choice is the selection of a witness $F \in \text{wit}(\varphi)$ in the loop at Line 1, which is followed by the computation of the corresponding formula function $fr$. At each iteration, a flag $i$ is also set to $f$, with the aim to indicate that $F$ is not inconsistent (a witness is consistent until proven otherwise). After this, we find the deterministic counterpart of Abelard’s choice, implemented by the combination of the three sets to the computed $F$ and a conditional statement at Lines 4 and 5, where a subset of overlapping schemas $S \subseteq \text{dom}(fr) \cap \text{Sch}(A)$ over a set of arguments $A \subseteq \text{Ar}$ is selected. This is done in order to verify the inconsistency of the conjunction $\bigwedge_{\sigma \in S} fr(\sigma)$ at Line 6. If this check is positive then the flag $i$ is switched to $t$. Once all choices for Abelard are analyzed, the computation reaches Line 8, where it is verified whether an inconsistency was previously found. In the negative case, the algorithm terminates by returning $t$, with the aim to indicate that a good guess for Eloise is possible. In the case all witnesses are analyzed, finding for each of them an inconsistency, Eloise has no winning strategy. Thus, the algorithm ends by returning $f$.

At this point, it remains to evaluate the complexity of the algorithm w.r.t. the length of the sentence $\varphi$. First, observe that it only requires a Boolean flag $i$, three sets $F$, $S$, and $A$ and a function $fr$, whose sizes are all linear in the input. Moreover, the membership at Line 1 is verifiable in $\text{PTIME}$, the emptiness of the witness set at Line 6 can be computed in $\text{NPTIME}$, and the check for the overlapping property at Line 5 can be easily executed in $\text{PSPACE}$. Consequently, the whole complexity is $\text{PSPACE}$.

**Theorem VI.1 (Decidability of Satisfiability).** The satisfiability problem for FOL[18] is $\text{PSPACE}$.

**VII. Discussion**

Trying to understand the reasons why some power extensions of modal logic are decidable, we introduced and studied a new family of FOL fragments based on the combinations of binding forms admitted in a sentence. In other words, we provided an innovative criterion to classify FOL formulas focused on the associations between arguments and variables. One of the main features of this classification is the avoiding of usual syntactic restrictions on quantifier patterns, number of variables, relation arities, and relativization of quantifications. Therefore, it represents a completely new framework in which to study model-theoretic and algorithmic properties for particular extensions of modal logic, like SL [31].

We analyzed the expressiveness of the introduced fragments, showing that the simplest one, called one binding (FOL[18]), is already incomparable with other important sublogics of FOL, such as the clique guarded and the guarded negation. Moreover, we proved that it enjoys the finite-model property, a $\text{PSPACE}$ satisfiability problem, and a constructive version of Craig’s interpolation and Beth’s definability. To do this, exploiting the fine structure of binding forms, we developed a technical characterization of first-order quantifications, which can be considered of interest by its own.

An important and immediate application of the nice properties of FOL[18] is the obtaining of similar results for the one-goal fragment of SL [32, 33]. Indeed, from a high-level point of view, every SL[16] sentence has a FOL[18] corresponding one, with same quantification and binding prefixes, in which the inner LTL temporal properties are replaced by suitable derived relations having the agents as arguments. The elements of the domain of quantification represent, therefore, the strategies of the game. Now, since we proved that a FOL[18] sentence just requires a finite model to be satisfied, it is enough to have a finite number of strategies for the corresponding SL[16] sentence. Consequently, the game only needs finitely many actions, i.e., the model is bounded. By means of a similar idea, the same property for automata over game structures [29] and, thus, for the alternating $\mu$CALCULUS can be also derived. Finally, observe that a proof of our conjecture about FOL[CB] immediately results in a proof of a related open problem about the conjunctive-goal fragment of SL [41].

As future works, besides a deeper study of the conjunctive and disjunctive fragments (FOL[CB] and FOL[DB]), there are several other directions that could be investigated. First, it would be interesting to analyze the model-checking complexity for FOL[18] and its connection with fragments of classic query languages. Then, we conjecture that some of its extensions, like those ones with equality, counting quantifiers, or fixpoint constructs, still preserve the same good model-theoretic and algorithmic properties. Finally, as a long term project, it would be nice to come up with a wider framework in which the incomparable fragments FOL[CG], FOL[GN], and FOL[18] can be simply seen as particular cases of a unique decidable logic.
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