NOTES ON TWISTED EQUIVARIANT K-THEORY FOR C*-ALGEBRAS

YOSUKE KUBOTA

Abstract. In this paper, we study a generalization of twisted (groupoid) equivariant K-theory in the sense of Freed-Moore for \( \mathbb{Z}_2 \)-graded C*-algebras. It is defined by using Fredholm operators on Hilbert modules with twisted representations. We compare it with another description using odd symmetries, which is a generalization of van Daele’s K-theory for \( \mathbb{Z}_2 \)-graded Banach algebras. In particular, we obtain a simple presentation of the twisted equivariant K-group when the C*-algebra is trivially graded. It is applied for the bulk-edge correspondence of topological insulators with CT-type symmetries.

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1. INTRODUCTION

Recently, there has been an increasing interests in relations between K-theory and the theory of topological insulators, an area of solid state physics. According to Kitaev’s periodic table [Kit09], free fermion phases are topologically classified in 10 types, each of which corresponds to one of 2 complex K-groups and 8 real K-groups. This classification is formulated by Freed-Moore [FM13] in terms of twisted K-theory. A key idea is Wigner’s theorem [Wig59] (see also Section 1 of [FM13]), which asserts that a quantum symmetry is given by a linear/antilinear and even/odd projective representation of a groupoid. Freed and Moore introduce a generalized version of twisted equivariant K-theory of groupoids classifying these representations.

Here, twisted K-theory [DK70, Ros89, AS04, TXLG04, Kar08b, FHT11, FM13] is a kind of cohomology theory determined by spaces (or groupoids

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in general) and twists on it, that is, triplets \((\phi, c, \tau)\) where \(\phi\) and \(c\) are homomorphisms from \(G\) to \(\mathbb{Z}_2\) and \(\tau\) is a \(\phi\)-twisted central extension of \(G\) by \(T\). Roughly speaking, twisted K-theory associated to these data classifies \(\mathbb{Z}_2\)-graded \(\tau\)-projective representations of \(G\) whose \(\mathbb{Z}_2\)-grading and linearity are determined by \(c\) and \(\phi\). For a fixed twist \((\phi, c, \tau)\) of \(G\), the twisted K-group of the action groupoid \(G \ltimes X\) with respect to the pull-back of \((\phi, c, \tau)\) is a topological invariant of a \(G\)-space \(X\). It enables us to regard twisted K-theory as a functor from the category of locally compact Hausdorff \(G\)-spaces to the category of \(R(G)\)-modules.

In this paper, we investigate a canonical generalization of groupoid equivariant K-theory of \(C^*\)-algebras for twisted equivariant setting in the sense of [FM13]. After reviewing a classification of twists in this sense by the Čech cohomology groups in Section 2 we start with the generalization of Kasparov’s KK-theory [Kas80] in Section 3. A general framework for twisted equivariant KK-theory is introduced by Chabert-Echterhoff [CE01] as a bifunctor from the category of twisted \(G\)-\(C^*\)-algebras to the category of \(R(G)\)-modules. In contrast, we define the \(\Phi KK^G_{\epsilon,\tau}\)-group as a bifunctor from the category of \((\phi\)-twisted) \(G\)-\(C^*\)-algebras. The group \(\Phi KK^G_{\epsilon,\tau}(A, B)\) is evidently suitable as a generalization of the KK-group because it is actually isomorphic to a certain KKR-group in the sense of Moutuou [Mon14] (Proposition 3.12).

In Section 4 we define the twisted equivariant K-group as the twisted equivariant KK-group \(\Phi KK^G_{\epsilon,\tau}(\mathbb{R}, A)\). For example, when \(A\) is the continuous function algebra of a compact Hausdorff \(G\)-space \(X\), it is isomorphic to the set of homotopy classes of twisted \(G\)-equivariant families of Fredholm operators on \(X\), which is a generalization of Atiyah’s formulation of K-theory using Fredholm operators [Ati89, AS04]. Moreover, we obtain a generalization of the Green-Julg theorem (Theorem 4.10).

It is natural to expect that this new K-theory is presented by using “finite dimensional” objects such as vector bundles or projections instead of Fredholm operators. In general, it does not go on even if the groupoid is proper. However, when the groupoid and its central extension has enough finite dimensional representations, a desired presentation is given in Section 5 as a generalization of van Daele’s formulation of K-theory for \(\mathbb{Z}_2\)-graded Banach algebras. Here, the boundary map of the long exact sequence is also presented in a simple way by the exponential map. Moreover, we obtain a more simple presentation of twisted equivariant K-groups for ungraded \(C^*\)-algebras which is similar to Karoubi’s K-theory [Kar68, Kar08a].

This presentation of the twisted equivariant K-groups is applied for a topological classification of gapped Hamiltonians of fermionic quantum systems. The boundary map of the Toeplitz exact sequence gives the bulk-edge correspondence of topological insulators. In [Kub15], the author consider the coarse Mayer-Vietoris exact sequence for twisted equivariant K-groups of Roe algebras in order to prove the bulk-edge correspondence for quantum systems which is not translation-invariant such as quasi-crystals.

In this context, Kellendonk [Kel15] gives a detailed calculation of van Daele’s K-groups for certain \(\mathbb{Z}_2\)-graded Real \(C^*\)-algebras in connection with the classification of topological phases for each of 10 symmetry types in
Kitaev’s periodic table. We remark that these groups are the same thing as the description given in Theorem 5.14 of the twisted equivariant K-groups when we consider the group $A$ and a twist $(\phi, c, \tau)$ as in Example 2.6.

**Notations.** We use the following notations throughout this paper.

- We assume that (topological) groupoids $\mathcal{G} = (\mathcal{G}^0, \mathcal{G}^1, s, r)$ are second countable, locally compact and Hausdorff groupoids with a Haar system (Definition 2.2 of [Ren80]). A groupoid is proper if the map $(s, r) : \mathcal{G}^1 \to \mathcal{G}^0 \times \mathcal{G}^0$ is proper. We use the convention $\mathcal{G}^0_s := \{(s, r)^{-1}(x, y), \mathcal{G}^1_s \times_r \mathcal{G}^1$ i.e. $g \circ h$ is well-defined if $s(g) = r(h)$.
- For a pre-simplicial space $M_\bullet$ (see Section 2 of [Tu06]), we write $\tilde{\varepsilon}_i : M_n \to M_{n-1}$ ($i = 0, \ldots, n$) for the face maps induced from the unique increasing map $\varepsilon_i$ that avoids $i$.
- For a $C^*$-algebra $A$, let $A_\mathbb{R}$ denote the underlying real $C^*$-algebra. We use the same symbol $A_\mathbb{R}$ for the Real $C^*$-algebra $A_\mathbb{R} \otimes_\mathbb{C} C$.
- For a Hilbert space $\mathcal{H}$ (resp. a Hilbert $C^*$-module), let $K(\mathcal{H})$ and $\mathbb{B}(\mathcal{H})$ (resp. $L(\mathcal{H})$) denote the compact operator algebra and the bounded operator algebra on $\mathcal{H}$. We write $M_{p,q}$ for the (Real) matrix algebra $K(\mathbb{R}^n \oplus (\mathbb{R}^{op})^m)$.
- We write as $\alpha_g^A$ (or simply $\alpha_g$ unless it causes some confusion) for a group action on a $C^*$-algebra $A$. Similarly, we write as $u_g^E$ (or simply $u_g$) for a representation on a Hilbert $C^*$-module $E$.
- Throughout this paper we deal with $\mathbb{Z}_2$-graded $C^*$-algebras, that is, $C^*$-algebras together with the involutive $\ast$-automorphism $\gamma_A$ (or simply $\gamma$). Let $A^{even}$ and $A^{odd}$ denote the $+1$ and $-1$ eigenspaces of $\gamma$ respectively. We use the notation $[a, b]$ for the supercommutator $[a, b] := ab - (-1)^{|a||b|}ba$.
- For a complex number $\lambda$ (resp. a complex line bundle, an element in a Real $C^*$-algebra), we write as $\lambda_\mathbb{R} := \lambda$ and $i\lambda := \overline{\lambda}$. In the same way, we say a map between vector spaces is 0-linear (resp. 1-linear) if it is linear (resp. antilinear).
- Let $\mathbb{C}e_{n,m}$ denote the Real Clifford algebra associated to $\mathbb{R}^{n+m}$ together with the inner product $(\xi, \xi) = - (\xi_1)^2 - \cdots - (\xi_n)^2 + (\xi_{n+1})^2 + \cdots + (\xi_{n+m})^2$. That is, $\mathbb{C}e_{n,m}$ is the finite dimensional Real $C^*$-algebra generated by odd Real elements $f_1, \ldots, f_n$ and $e_1, \ldots, e_m$ with relations $[e_i, e_j] = 2\delta_{i,j}$, $[f_i, f_j] = -2\delta_{i,j}$ and $[e_i, f_j] = 0$. Let $C_{n,m}$ denote the Clifford group, that is, the subgroup of $\mathbb{C}e_{n,m}$ generated by $\{e_1, \ldots, e_m, f_1, \ldots, f_n\}$.

### 2. Extensions and Brauer groups on groupoids

In this section, we review relations between twists, Brauer groups and Čech cohomology groups and generalize them for twists in the sense of Freed-Moore [FM13]. Most arguments are based on the works on [KMRW98] and [Tu06].

First of all, we show the conclusion of this section.

**Theorem 2.1.** Let $G$ be a groupoid and let $\phi$ be a $\mathbb{Z}_2$-valued 1-cocycle of $G$. Then, the following three groups are canonically isomorphic:
(1) the \(\phi\)-twisted graded Brauer group \(\hat{\text{Br}}(\mathcal{G})\),
(2) the group \(\hat{\text{Tw}}(\mathcal{G})\) of twists on \(\mathcal{G}\) associated to \(\phi\),
(3) the Čech cohomology group \(H^1(\mathcal{G};\mathbb{Z}_2) \oplus H^2(\mathcal{G};\mathbb{T}_\phi)\).

Consequently, twists of \(\mathcal{G}\) are classified by
\[
(\phi, c, \tau) \in \prod_{\phi \in H^1(\mathcal{G};\mathbb{Z}_2)} \hat{H}^1(\mathcal{G};\mathbb{Z}_2) \oplus H^2(\mathcal{G};\mathbb{T}_\phi).
\]

Before the proof, we introduce the definition of each object appearing in the statement of Theorem 2.1.

The Čech cohomology groups for groupoids \(\mathcal{G}\) and \(\mathcal{G}\)-sheaves are the same thing as the one introduced in the Section 4 of [Tu06]. Let \(\mathcal{G}\) be an abelian group. By Proposition 5.2 of [Tu06], a \(\mathcal{G}\)-valued 1-cocycle \(\phi\) corresponds to a principal \(\mathcal{G}\)-bundle \(p_\phi : G^0_\phi \to G^0\) with \(\mathcal{G}\)-action. In other words, \(\phi\) determines a Hilsum-Skandalis map [HS87] from \(\mathcal{G}\) to \(\mathcal{G}\). By replacing \(\mathcal{G}\) with another groupoid \(G_\phi := \mathcal{G} \times G^0_\phi \times G\) which is Morita equivalent to \(\mathcal{G}\), we may assume that \(\phi\) is represented by a groupoid homomorphism \(\phi : G_\phi \to G\).

Let \(\epsilon\) denote the 2-cocycle on the abelian group \(\hat{H}^1(\mathcal{G};\mathbb{Z}_2)\) which takes value in \(\hat{H}^2(\mathcal{G};\mathbb{T}_\phi)\) determined by
\[
\epsilon(c_1, c_2)(g, h) := (-1)^{c_2(g) \cdot c_1(h)}
\]
and let \(\hat{H}^1(\mathcal{G};\mathbb{Z}_2) \oplus \hat{H}^2(\mathcal{G};\mathbb{T}_\phi)\) denote the central extension group corresponding to \(\epsilon\).

**Definition 2.2** (Definition 7.23 of [FM13]). Let \(\mathcal{G}\) be a groupoid and let \(\phi\) be a groupoid homomorphism from \(\mathcal{G}\) to \(\mathbb{Z}_2\).

- A \(\mathbb{Z}_2\)-grading of \(\mathcal{G}\) is a groupoid homomorphism \(c : \mathcal{G} \to \mathbb{Z}_2\).
- A \(\phi\)-twisted extension of \(\mathcal{G}\) is a principal \(\mathbb{T}\)-bundle \(\tau : (\mathcal{G}^\tau)^1 \to \mathcal{G}^1\)
together with the linear isomorphism
\[
\lambda_{g_2,g_1} : \phi(g_2) L^\tau_{g_2} \otimes L^\tau_{g_1} \to L^\tau_{g_2g_1}
\]
(where \(L^\tau := (\mathcal{G}^\tau)^1 \times_\mathbb{T} \mathbb{C}\)) such that the digram
\[
\begin{array}{ccc}
L^\tau_{g_3} \otimes L^\tau_{g_2g_1} & \xrightarrow{\text{id} \otimes \lambda_{g_2,g_1}} & L^\tau_{g_3g_2g_1} \\
\downarrow & & \downarrow \\
\phi(g_1) L^\tau_{g_3g_2} \otimes L^\tau_{g_1} & \xrightarrow{\lambda_{g_3,g_2}\otimes \text{id}} & L^\tau_{g_3g_2g_1}
\end{array}
\]

commutes.

We say that a triple \((\phi, c, \tau)\) as above is a twist of \(\mathcal{G}\).

**Example 2.3.** Let \(\tau\) be a central extension of \(\mathcal{G}\) by \(\mathbb{Z}_2\). Then, the associated bundle \(\mathcal{G}^\tau \times_\mathbb{T} \mathbb{Z}_2\) with respect to the inclusion \(\mathbb{Z}_2 \cong \{\pm 1\} \subset \mathbb{T}\) is a \(\phi\)-twisted central extension of \(\mathcal{G}\) by the product \((g, t) \cdot (h, s) := (gh, \phi(g)s t)\).

**Example 2.4.** Consider the group \(\mathcal{G} = \mathbb{Z}_2\) with the generator \(b\) and the group homomorphism \(\phi_\mathbb{R} := \text{id} : \mathbb{Z}_2 \to \mathbb{Z}_2\). Then, the isomorphism class of a \(\phi_\mathbb{R}\)-twisted central extension \(\tau\) of \(\mathbb{Z}_2\) by \(\mathbb{T}\) is determined by the square \(b^2 \in \mathbb{T}\) of a lift \(b\) of \(b\). By the relation \(b \cdot b^2 = b^2 \cdot b = b\bar{b}^2\), the scalar \(b^2\) must be \(\pm 1\). The corresponding two \(\phi_\mathbb{R}\)-twisted extensions are denoted by \(\tau_0^\mathbb{R}\) and
\[ \tau^0_0 \text{ respectively. For a central extension } \tau \text{ of } G \text{ by } \mathbb{Z}_2, \text{ let } \tau_R \text{ and } \tau_H \text{ denote central extensions } \tau \times \tau^0_0 \text{ and } \tau \times \tau^0_0 \text{ of } G_R := G \times \mathbb{Z}_2 \text{ respectively.}

**Example 2.5.** Let \( F_{n,m} \) denote the group \( \mathbb{Z}_2^{n+m} = \mathbb{Z}_2 \mathbf{f}_i \oplus \cdots \oplus \mathbb{Z}_2 \mathbf{f}_j \oplus \mathbb{Z}_2 \mathbf{e}_i \oplus \cdots \oplus \mathbb{Z}_2 \mathbf{e}_m \). Then, the canonical homomorphism from the Clifford group \( C_{n,m} \) to \( F_{n,m} \) mapping \( e_i \) and \( f_j \) to \( \mathbf{e}_i \) and \( \mathbf{f}_j \) respectively determines a central extension of \( F_{n,m} \) by \( \mathbb{Z}_2 \). We write \( \tau^0_{n,m} \) for this extension and \( c^0_{n,m} \) for the group homomorphism \( F_{n,m} \to \mathbb{Z}_2 \) given by \( c_{n,m}(\mathbf{e}_i) = c_{n,m}(\mathbf{f}_j) = 1 \).

**Example 2.6.** Let \( \mathcal{S} \) denote the finite abelian group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) with the generator \( C := (1,0) \) and \( T := (0,1) \). There are three proper subgroups \( \mathcal{C}, \mathcal{P} \) and \( \mathcal{T} \) of \( \mathcal{S} \) generated by \( C, CT \) and \( C T \). We fix two group homomorphisms \( \phi, \psi : \mathcal{S} \to \mathbb{Z}_2 \) defined by \( \phi_0(C) = \phi_0(T) = 1, c_3(C) = 1 \) and \( c_3(T) = 0 \). According to Lemma 6.17 and Proposition 6.4 of [FM13], every central extension \( \mathcal{A} \) of a subgroup \( \mathcal{A} \) of \( \mathcal{S} \) has a unique lift \( C, T \) of \( \mathcal{S} \) such that \( (CT)^2 = 1 \) and the pairs \( (A, \tau) \) are chosen by the choice of signs \( C^2 = \pm 1 \) and \( T^2 = \pm 1 \). We say that a CT-type symmetry is a quadruple \( (\mathcal{G}, \phi, c, \tau) \) where \( \mathcal{G} = \mathbb{G}_0 \times A \), \( (\phi, c) = (\phi_0, c_3) \circ \text{pr}_2 \) and \( \mathcal{G}^* = \mathbb{G}_0^* \times A^\tau \). For a CT-type symmetry, the pair \( (A, \tau_A) \) is called its CT-type.

For a groupoid \( \mathcal{G} \) and the Baer sum \( [L_1] + [L_2] := [L_1 \otimes L_2] \). Let \( \mathcal{G}(\mathcal{U}) \) denote the group of twists \( \text{Hom}(\mathcal{G}, \mathcal{S}) \otimes \varepsilon \text{ext}(\mathcal{G}, \mathcal{T}) \) associated to \( \phi \). Here, for \( c_1, c_2 \in \text{Hom}(\mathcal{G}, \mathcal{S}) \), \( \varepsilon(c_1, c_2) \) denotes the central extension \( \mathcal{G} \times \mathbb{Z}_2 \) of \( \mathcal{G} \) by \( \mathbb{Z}_2 \) with the product \( (g, i) \cdot (h, j) := (gh, i + j + c_2(g)c_1(h)) \).

The group \( \mathcal{G}(\mathcal{U}) \) is related to Čech cohomology groups by the following way. For a groupoid \( \mathcal{G} \) and an open covering \( \mathcal{U} \) of \( \mathcal{G}^0, \mathcal{G}(\mathcal{U}) \) denotes the groupoid given by \( \mathcal{G}(\mathcal{U})^0 := \coprod_{U \in \mathcal{U}} U \),

\[ \mathcal{G}(\mathcal{U})^1 := \{(y, g, x) \in \mathcal{G}(\mathcal{U})^0 \times \mathcal{G}^1 \times \mathcal{G}(\mathcal{U})^0 \mid s(g) = \pi(x), r(g) = \pi(y)\} \]

where \( \pi \) is the canonical projection from \( \mathcal{G}(\mathcal{U})^0 \) to \( \mathcal{G} \), and \( s \) and \( r \) are the third and first projections and \( (z, g, y) \cdot (y, h, x) := (z, gh, x) \). Remark that this groupoid is Morita equivalent to \( \mathcal{G} \).

**Proposition 2.7** (Proposition 5.2 and Proposition 5.6 of [Tu06]). The inductive limit

\[ \mathcal{G}(\mathcal{U}) := \lim \mathcal{G}(\mathcal{U}_{U_i}) \]

is isomorphic to \( \hat{H}^1(\mathcal{G}; \mathbb{Z}_2) \oplus \hat{H}^2(\mathcal{G}; \mathcal{T}) \).

Next, we turn to the classification of certain \( \phi \)-twisted \( \mathcal{G} \)-equivariant bundles of \( \mathcal{C}^* \)-algebras. Before that, we prepare terminologies of linear actions and representations of groupoids for the convenience of readers. Basic references are Section 13–14 of [FDS8] and [KMRW98].

Let \( X \) be a locally compact second countable Hausdorff space. A Banach bundle or a continuous field of Banach spaces over \( X \) is a topological space \( \mathcal{E} \) together with a continuous open surjection \( p : \mathcal{E} \to X \), continuous maps \( \|\cdot\| : \mathcal{E} \to \mathbb{R}^+, + : \mathcal{E} \times_X \mathcal{E} \to \mathcal{E} \) and \( \cdot : \mathbb{C} \times \mathcal{E} \to \mathcal{E} \) such that these operations determine the Banach space structure on each fiber \( \mathcal{E}_x := p^{-1}(x) \).
and any nets $\xi_j$ in $\mathcal{E}$ such that $p(\xi_j) \to x$ and $\|\xi_j\| \to 0$ converges to 0. For a Banach bundle $\mathcal{E}$, the space $\Gamma_0(X, \mathcal{E})$ of continuous sections vanishing at infinity is a Banach space which is embedded in $\prod_{x \in X} \mathcal{E}_x$. It is known that a Banach bundle $\mathcal{E}$ over a locally compact Hausdorff space $X$ has enough continuous cross-sections, that is, for any $\xi_x \in E_x$ there is a continuous section $\xi \in \Gamma(X, E)$ such that $\xi(x) = \xi_x$ (see Appendix C of [FD88]).

A (strongly continuous) $\phi$-twisted $G$-Banach bundle is a Banach bundle $\mathcal{E}$ over $\mathcal{G}^0$ together with a continuous bundle map $\alpha : s^*\mathcal{E} \to r^*\mathcal{E}$ such that each $\alpha_g : \mathcal{E}_{s(g)} \to \mathcal{E}_{r(g)}$ is a bounded linear map, the map $g \mapsto \|\alpha_g(\xi(s(g)))\|$ is continuous for any $\xi \in \Gamma_0(X, \mathcal{E})$ and $\alpha_g \alpha_h = \alpha_{gh}$, $\alpha_{id_x} = id_{\mathcal{E}_x}$. We say that a $\mathbb{R}$-linear action $\alpha$ of $G$ on a $\mathbb{Z}_2$-graded Banach bundle $\mathcal{E}$ is $\phi$-linear if each $\alpha_g$ is $\phi(g)$-linear and $c$-graded if each $\alpha_g$ is even if $c(g) = 0$ and odd if $c(g) = 1$.

In this paper, we mainly deal with $\mathcal{G}$-bundles of Hilbert spaces and $C^*$-algebras. A Hilbert bundle $\mathcal{H}$ over $X$ is a Banach bundle together with the inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times X \mathcal{H} \to \mathbb{C}$ such that $\|\xi\|^2 = \langle \xi, \xi \rangle$ for any $\xi \in \mathcal{H}$. It is shown in Proposition 2 of [Pat12] that the section space $\Gamma_0(X, \mathcal{E})$ has a canonical Hilbert $C_0(X)$-module structure and every Hilbert $C_0(X)$-module is isomorphic to the section space of a Hilbert bundle. We say that a linear action $\alpha$ of $G$ on a Hilbert bundle $\mathcal{H}$ over $\mathcal{G}^0$ is unitary if each $\alpha_g$ is a $\mathbb{C}$-linear unitary operator (i.e. $u_g$ is bijective and satisfies $(u_g\xi, u_g\eta) = (\xi, \eta)$ for any $\xi, \eta \in \mathcal{H}_g$).

**Definition 2.8.** A $(\phi, c, \tau)$-twisted $G$-Hilbert bundle is a Hilbert bundle $\mathcal{H}$ over $\mathcal{G}^0$ together with a $\mathbb{C}$-linear $c$-graded unitary action $u$ of $\mathcal{G}^\tau$ on $\mathcal{H}$ whose restriction on $\mathbb{T}$ is given by the complex multiplication. We say that a $(\phi, c, \tau)$-twisted $G$-Hilbert bundle is a $(\phi, c, \tau)$-twisted unitary representation of $G$ if the underlying Hilbert bundle is locally trivial whose structure group is the unitary group $U(\mathcal{H}_0)$ with the strong topology.

Here, we use the letter $\mathcal{H}_0$ for the abstract separable infinite dimensional Hilbert space $\ell^2\mathbb{N}$. By the Kasparov stabilization theorem, for any non-trivial Hilbert bundle $\mathcal{H}$ over a locally compact second countable space $X$, $\mathcal{H} \otimes \mathcal{H}_0$ is isomorphic to the trivial bundle $X \times \mathcal{H}_0$ (cf. Proposition 7.4 of [Lan95]). Therefore, for any $(\phi, c, \tau)$-twisted Hilbert $G$-bundle $\mathcal{H}$, $\mathcal{H} \otimes \mathcal{H}_0$ is a $(\phi, c, \tau)$-twisted unitary representation.

**Example 2.9.** Set

$$C_c(G, L^\tau) := \{ \xi \in C_c(\mathcal{G}^\tau) \mid \xi(t^{-1}g) = \phi(g)^t \xi(g) \text{ for any } t \in \mathbb{T} \},$$

which is a $\mathbb{Z}_2$-graded complex vector space by the scalar multiplication $(\lambda \cdot \xi)(g) := \phi(g)\lambda \xi(g)$ and the $\mathbb{Z}_2$-grading $(\gamma \xi)(g) := (-1)^{c(g)}\xi(g)$. Let $\phi L^2_{c,\tau}G$ be the Hilbert $C_0(\mathcal{G}^0)$-module given by the completion of $C_c(G)$ with respect to the inner product $\langle \xi, \eta \rangle (x) = \int \xi(g)\overline{\eta(g)}d\lambda_x(g)$. Then, the corresponding Hilbert bundle over $\mathcal{G}^0$ is a $(\phi, c, \tau)$-twisted $G$-Hilbert bundle whose fiber on $x$ is $L^2(Gx, L^\tau)$ with the left regular representation $u_g(\xi)(h) = \xi(g^{-1}h)$. Set $\phi L^2_{c,\tau}G \otimes \mathcal{H}_0$ and let $\phi L^2_{c,\tau}G$ denote the corresponding $(\phi, c, \tau)$-twisted unitary representation.

A $\phi$-twisted $G$-$C^*$-bundle is a $\phi$-twisted $G$-Banach bundle $(A, \alpha)$ together with continuous maps $\bigcdot : A \times X A \to A$ and $\ast : A \to A$ such that each $A_x$
is a C*-algebra and each $\alpha_g : A_{s(g)} \rightarrow A_{r(g)}$ is a *-homomorphism by these operations. The space of continuous sections vanishing at infinity $\Gamma_0(X, A)$ of a $\phi$-twisted $G$-C*-bundle $A$ is a $\phi$-twisted $G$-C*-algebra (which will be introduced in Definition 3.1) by the fiberwise *-operation, the sup norm and the canonical $G$-action.

We say that two $\phi$-twisted $G$-C*-algebras $A$ and $B$ are Morita-equivalent if there is a $\phi$-twisted $\mathbb{Z}_2$-graded $G$-equivariant Hilbert $B$-module $E$ (Definition 3.4) such that $\langle E, E \rangle = B$ and $\mathbb{K}(E) \cong A$. Two $\phi$-twisted $G$-C*-bundles are Morita-equivalent if the corresponding $G$-C*-algebras are Morita-equivalent.

**Definition 2.10.** We say that a $G$-C*-bundle $A$ is ($\mathbb{Z}_2$-graded) elementary if each fiber is isomorphic to the algebra of compact operators on $\mathbb{Z}_2$-graded separable Hilbert spaces and satisfies Fell’s condition, that is, for any $x \in G^0$ there is a neighborhood $U$ of $x$ and a section of rank 1 projections on $U$. The $\phi$-twisted Brauer group $\hat{\phi}\text{Br}(G)$ of $G$ is the group of Morita-equivalent classes of elementary $\phi$-twisted $G$-C*-bundles under the product given by the graded tensor product.

Note that the underlying C*-bundle over $G^0$ of a stable (i.e. $A \cong A \otimes \mathbb{K}$) elementary $G$-C*-bundle $A$ is locally trivial (see Theorem 10.7.15 of [Dix77]).

**Example 2.11.** Let $H$ be a $(\phi, c, \tau)$-twisted $G$-Hilbert bundle and $\mathcal{H} := \Gamma_0(G^0, H)$. Then, $\mathbb{K}(H)$ is isomorphic to the section algebra of a $\phi$-twisted $G$-C*-bundle (see 10.7.1 of [Dix77]). Hereafter we use the same symbol $\mathbb{K}(\mathcal{H})$ for this $\phi$-twisted $G$-C*-bundle. Moreover, it satisfies Fell’s condition because a local section of rank 1 projections in $\mathbb{K}(\mathcal{H})$ corresponds to a continuous non-zero local section of $H$. For two $(\phi, c, \tau)$-twisted $G$-C*-bundles $\mathcal{H}_1$ and $\mathcal{H}_2$, $\mathbb{K}(\mathcal{H}_1)$ and $\mathbb{K}(\mathcal{H}_2)$ are Morita-equivalent since $\mathbb{K}(\mathcal{H}_1, \mathcal{H}_2)$ is a $\mathbb{K}(\mathcal{H}_1)$-$\mathbb{K}(\mathcal{H}_2)$ imprimitivity bimodule. Therefore, we have a map $^\phi \text{Tw}(G) \rightarrow \hat{\phi}\text{Br}(G)$ given by $(c, \tau) \mapsto \mathbb{K}^G(c, \tau) : = \mathbb{K}(\mathcal{H}^G(c, \tau))$. This map is actually a group homomorphism since $\mathbb{K}(\mathcal{H}_1) \otimes \mathbb{K}(\mathcal{H}_2) \cong \mathbb{K}(\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2)$ and $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$ is a $(\phi, c, \tau)$-twisted Hilbert $G$-bundle where $c = c_1 + c_2$ and $\tau = \tau_1 + \tau_2 + \varepsilon(c_1, c_2)$ as is shown in Lemma 3.3 in Section 3.

**Proof of Theorem 2.1.** The rest part of the proof is constructing the inverse $\hat{\phi}\text{Br}(G) \rightarrow ^\phi \text{Tw}(G)$ of the homomorphism in Example 2.11 which is given in the same way as in Theorem 10.1. of [KMRW98]. We only remark that for a $\phi$-twisted elementary $G$-C*-bundle whose underlying C*-bundle is isomorphic to $C_0(G^0) \otimes \mathbb{K}(\mathcal{H}_0)$ (and hence the $G$-action is given by a groupoid homomorphism $\pi : G^1 \rightarrow \text{Aut}_{\text{qtm}} \mathbb{K}(\mathcal{H}_0)$), the groupoid $E(\pi) := \{(g, u) \in G \times \text{Aut}_{\text{qtm}} \mathcal{H}_0 \mid \pi(g) = \text{Ad}(u)\}$

is a $\phi$-twisted central extension of $G$ by $T$ and the canonical action of $E(\pi)$ on $C_0(G^0) \hat{\otimes} \mathcal{H}_0$ determines a $(\phi, c, \tau)$-twisted $G$-Hilbert bundle. Here $\text{Aut}_{\text{qtm}} \mathbb{K}(\mathcal{H}_0)$ denotes the group of linear/antilinear even *-automorphisms on $\mathbb{K}(\mathcal{H}_0)$ and $\text{Aut}_{\text{qtm}} \mathcal{H}_0$ denotes the group of linear/antilinear and even/odd unitary operators on $\mathcal{H}_0$.

For a $\phi$-twisted elementary $G$-C*-bundle $A$, we call the corresponding element in $H^1(G; \mathbb{Z}_2) \oplus_* H^2(G; T_\phi)$ is the Dixmier-Douady invariant of $A$. 


3. Twisted equivariant $KK$-theory

In this section, we introduce the notion of twisted equivariant $KK$-theory for pairs of $\phi$-twisted $G$-$C^*$-algebras. We start with the definition of $\phi$-twisted actions of groupoids on $C^*$-algebras. For a locally compact Hausdorff space $X$, a $\mathbb{Z}_2$-graded $X$-$C^*$-algebra is a $\mathbb{Z}_2$-graded $C^*$-algebra $A$ together with a unital $\mathbb{Z}_2$-graded $*$-homomorphism $\tilde{C}_b(X) \to \mathbb{Z}M(A)$. For example, the section algebra $\Gamma_0(X,\mathcal{A})$ of a $C^*$-bundle over $X$ is a $X$-$C^*$-algebra. Conversely, by Proposition A.3 of [TXLG04], a $X$-$C^*$-algebra $A$ such that $C_0(X)A = A$ is isomorphic to the section algebra of an upper-semicontinuous $X$-$C^*$-bundle (a variation of $X$-$C^*$-bundle such that the norm $||\cdot|| : \mathcal{A} \to \mathbb{R}^+$ is upper-semicontinuous). Here, the fiber $A_x$ of the corresponding upper-semicontinuous $C^*$-bundle is given by

$$A_x := A/(\{f \in C_0(X) \mid f(x) = 0\}A).$$

Moreover, bundle maps $\mathcal{A} \to \mathcal{B}$ and pull-backs $f^*A$ by a continuous map $f : Y \to X$ of $C^*$-bundles correspond to $X$-$*$-homomorphisms ($*$-homomorphisms $\varphi : A \to B$ satisfying $\varphi(fa) = f\varphi(a)$) and the tensor product $C(Y \otimes_X A$ (Definition 1.5 and Definition 1.6 of [Kas88]) respectively.

**Definition 3.1** (Définition 3.4 of [LG99]). A $\phi$-twisted $\mathbb{Z}_2$-graded $G$-$C^*$-algebra is a $\mathbb{Z}_2$-graded $G^0$-$C^*$-algebra $A$ together with a $G^1$-$*$-homomorphism $\alpha : s^*A \to s^*A$ such that each $\alpha_g : A_\phi(q) \to A_{\tau(q)}$ is a $\phi(g)$-linear $\mathbb{Z}_2$-graded $*$-homomorphism satisfying $\alpha_g \circ \alpha_h = \alpha_{gh}$ and $\alpha_{id_x} = id_{A_x}$.

**Remark 3.2.** Let $A$ be a $\phi$-twisted $G$-$C^*$-algebra. Then, by the same action $\alpha$, the underlying Real $C^*$-algebra $A_\mathbb{R}$ is regarded as a Real $G$-$C^*$-algebra. Actually, the correspondence $a \oplus jb \mapsto (a + ib) \oplus (a - ib)$ gives an isomorphism between $A_\mathbb{R} \cong A \oplus jA$ (where $j$ is the imaginary unit of the Real $C^*$-algebra $A_\mathbb{R}$) and $p^*_\phi A$ as $G$-$C^*$-algebras. Moreover, the right $\mathbb{Z}_2$-action on $p^*_\phi A$ corresponds to the complex conjugation on $A_\mathbb{R}$.

**Example 3.3.** A Real $G$-$C^*$-algebra $(A, \alpha')$ is regarded as a $\phi$-twisted $G$-$C^*$-algebra by the action $\alpha_g(x) := \phi(g)\alpha'_g(x)$. In particular, for a Real $C^*$-algebra $A$, the $C^*$-algebra $C_b(G^0, A)$ together with the trivial $G$-action is a “trivial” $\phi$-twisted $G$-$C^*$-algebra. For simplicity of notation, we use the same letter $A$ for this $\phi$-twisted $G$-$C^*$-algebra. Conversely, a $\phi_\mathbb{R}$-twisted $\mathbb{G}_\mathbb{R}$-$C^*$-algebra is the same thing as a Real $G$-$C^*$-algebra.

**Definition 3.4.** Let $A$ be a $\mathbb{Z}_2$-graded $\phi$-twisted $G$-$C^*$-algebra. A $(\phi, c, \tau)$-twisted $G$-equivariant Hilbert $A$-module is a $\mathbb{Z}_2$-graded Hilbert $A$-module $E$ with the $(\phi, c, \tau)$-twisted representation of $G$, that is, a $\phi$-linear action $u : s^*E \to r^*E$ of $G^\tau$ such that $u|_{\tau}$ is given by the complex multiplication, $\langle u_g\xi, u_g\eta \rangle = \alpha_{\tau(g)}(\langle \xi, \eta \rangle)$ and $\gamma_E(u_g\xi) = (-1)^{c(g)}u_g\gamma_E(\xi)$ for any $g \in (G^\tau)^1$.

For example, a $(\phi, c, \tau)$-twisted $G$-equivariant Hilbert $\mathbb{R}$-module is the same thing as the section space of a $(\phi, c, \tau)$-twisted $G$-Hilbert bundle. In general, for a $\mathbb{Z}_2$-graded $\phi$-twisted Hilbert $G$-module $E$ over $A$ and a $(\phi, c, \tau)$-twisted $G$-Hilbert bundle $\mathcal{K}$, the tensor product $E \otimes \mathcal{K}$ is a $(\phi, c, \tau)$-twisted $G$-equivariant Hilbert $A$-module.

In general, the twisted $G$-equivariant Hilbert $A$-module structure on the interior tensor product of two Hilbert modules is given as the following
Lemma 3.5. Let $A_i$ be $\mathbb{Z}_2$-graded $\phi$-twisted $\mathcal{G}$-$C^*$-algebras, let $(E_i, u^i)$ be $(\phi, c_i, \tau_i)$-twisted $\mathcal{G}$-equivariant Hilbert $A_i$-modules for $i = 1, 2$ and let $\varphi : A_1 \to \mathcal{L}(E_2)$ be a $*$-homomorphism such that
\begin{equation}
\alpha^L_{a}\varphi(a) = (-1)^{|a| \cdot |c|} \varphi(\alpha^A_{a}(a))
\end{equation}
for any homogeneous $a \in (A_1)_{s(h)}$. Then,
\[ u_g(\xi \otimes \eta) = (-1)^{c_2(g) \cdot |\xi|} u_g^1 \xi \otimes u_g^2 \eta \]
determines a well-defined $(\phi, c, \tau)$-twisted Hilbert $G$-module structure on $E := E_1 \tilde{\otimes} A_1 E_2$ where $c := c_1 + c_2$ and $\tau := \tau_1 + \tau_2 + \epsilon(c_1, c_2)$.

Proof. First we observe that the above $u_g$ determines a well-defined linear map on $E \otimes (A_1)_{s(h)} \otimes F$. Actually, for any $g \in \mathcal{G}$, $\xi \in (E_1)_{s(h)}$ and homogeneous elements $a \in (A_1)_{s(h)}$ and $\eta \in (E_2)_{s(h)}$,
\[ (-1)^{c_2(g) \cdot |\xi|} u_g^1 \xi \otimes (u_g^2 \eta) = (-1)^{c_2(g) \cdot |\xi| + |a|} (u_g^1 \xi) \alpha_g(a) \otimes (u_g^2 \eta) \]
and
\[ (-1)^{c_2(g) \cdot |\xi|} (u_g^1 \xi) \otimes (u_g^2 (a \eta)) = (-1)^{c_2(g) \cdot |\xi|} (u_g^1 \xi) \otimes ((-1)^{c_2(g) \cdot |a|} \alpha_g(a) (u_g^2 \eta)) \]
for any homogeneous $\xi \in (E_1)_{s(h)}$ and $\eta \in (E_2)_{s(h)}$.

Now for any homogeneous $\xi_1 \in (E_1)_{s(h)}$ and $\eta_1 \in (E_2)_{s(h)}$,
\[ \langle u_g^1(\xi_1 \otimes \eta_1), u_g^1(\xi_2 \otimes \eta_2) \rangle = (-1)^{c_2(g) \cdot (|\xi_1| + |\xi_2|)} \langle u_g^2 \eta_1, u_g^2 \xi_1, u_g^2 \xi_2 \rangle \cdot (u_g^2 \eta_2) \]
\[ = (-1)^{c_2(g) \cdot (|\xi_1| + |\xi_2|)} \langle u_g^2 \eta_1, \alpha_g(\xi_1, \xi_2) \rangle \cdot (u_g^2 \eta_2) \]
\[ = \langle u_g^2 \eta_1, u_g^2 (\xi_1, \xi_2) \cdot \eta_2 \rangle \]
\[ = \alpha_g(\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle) \]
implies that the $G$-action on $E_1 \tilde{\otimes} A_1 \otimes E_2$ preserves the inner product and hence extends to $E_1 \tilde{\otimes} A \otimes E_2$. The projective representation $u$ is $(\phi, c, \tau)$-twisted because
\[ u_g u_h(\xi \otimes \eta) = (-1)^{c_2(h) \cdot |\xi| + c_2(g) \cdot |\eta|} u_g^1 u_h^1 \xi \otimes u_g^2 u_h^2 \eta \]
\[ = (-1)^{c_2(g) \cdot c_2(h)} \tau_1(g, h) \tau_2(g, h) u_{gh}(\xi \otimes \eta) \]
for any $\xi \in (E_1)_{s(h)}$ and $\eta \in (E_2)_{s(h)}$. \qed

Example 3.7. For a $(\phi, c, \tau)$-twisted unitary representation $(\mathcal{V}, \pi)$ of $\mathcal{G}$, we write $\tilde{(\mathcal{V}, \pi)}$ for the $(\phi, c, \tau + \epsilon(c, c))$-twisted unitary representation given by $\tilde{\varphi} = \mathcal{V}$ as Hilbert spaces and $\pi(g) := \gamma^c(g) \circ \pi(g)$. Then, $\varphi := \text{id} : \mathbb{K} (\mathcal{V}) \to \mathbb{K} (\tilde{\mathcal{V}})$ satisfies (3.6). In this case, we have $\mathcal{V}^* \tilde{\otimes} \mathbb{K} (\mathcal{V}) \tilde{\otimes} \mathbb{R} \cong \mathcal{V}^*$ as $\phi$-twisted $\mathbb{R} \otimes \mathbb{R}$ bimodules and $\hat{\mathcal{V}} \tilde{\otimes} \mathbb{K} (\mathcal{V}) \cong \mathbb{K} (\mathcal{V})$ as $\phi$-twisted $\mathbb{K} (\mathcal{V}) \otimes \mathbb{K} (\mathcal{V})$ bimodules.
In the same way as Corollary 6.22 of [Tu99], we obtain the Kasparov stabilization theorem for \((\phi, c, \tau)\)-twisted \(\mathcal{G}\)-equivariant Hilbert \(A\)-modules when the groupoid \(\mathcal{G}\) is proper. That is, for any \(\sigma\)-unital \(\phi\)-twisted \(\mathcal{G}\)-C*-algebra \(A\) and countably generated \((\phi, c, \tau)\)-twisted \(\mathcal{G}\)-equivariant Hilbert \(A\)-modules \(E\), there is a \(\mathcal{G}\)-equivariant unitary equivalence

\[
(3.8) \quad \phi \mathcal{H} c^G \tau_A \oplus E \cong \phi \mathcal{H} c^G \tau_A
\]

where \(\phi \mathcal{H} c^G \tau_A\) denotes the \((\phi, c, \tau)\)-twisted Hilbert \(\mathcal{G}\)-module \(\phi \mathcal{H} c^G \tau \otimes A\). In particular, \(\mathcal{H} \mathcal{G}_A \otimes \mathcal{V} \cong \mathcal{H} c^G \tau A\) for any \((\phi, c, \tau)\)-twisted representation \(\mathcal{V}\) of \(\mathcal{G}\).

**Definition 3.9.** Let \(A\) and \(B\) be \(\mathbb{Z}_2\)-graded \(\mathcal{G}\)-C*-algebras. A \((\phi, c, \tau)\)-twisted \(\mathcal{G}\)-equivariant Kasparov \(A\)-\(B\) bimodule is a triple \((E, \varphi, F)\) where

- a countably generated \((\phi, c, \tau)\)-twisted \(\mathcal{G}\)-equivariant Hilbert \(B\)-module \(E\),
- a \(\mathbb{Z}_2\)-graded \(*\)-homomorphism \(\varphi : A \to \mathbb{L}(E)\) satisfying (3.6),
- an odd self-adjoint operator \(F \in \mathbb{L}(E)\) such that \(\varphi(a)F = \mathcal{G}\)-continuous,

\([\varphi(a), F], \varphi(a)(F^2 - 1) \in \mathbb{K}(E)\) and \(r^* \varphi(a)((-1)^{\varphi(a)}(s^* F - r^* F) \in \mathbb{K}(r^* E)\).

We say that two \((\phi, c, \tau)\)-twisted \(\mathcal{G}\)-equivariant Kasparov \(A\)-\(B\) bimodules \((E, \varphi_1, F_1)\) and \((E, \varphi_2, F_2)\) are homotopic if there is a \((\phi, c, \tau)\)-twisted \(\mathcal{G}\)-equivariant Kasparov \(A\)-\(B\) 0, 1) bimodule \((\tilde{E}, \tilde{\varphi}, \tilde{F})\) such that each ev\(_i^*\)(\(\tilde{E}, \tilde{\varphi}, \tilde{F}\)) is unitary equivalent to \((E, \varphi_i, F_i)\) for \(i = 0, 1\).

**Definition 3.10.** We define \(\phi \mathrm{KK}^\mathcal{G}_{c, \tau}(A, B)\) the group of homotopy classes of \((\phi, c, \tau)\)-twisted \(\mathcal{G}\)-equivariant Kasparov \(A\)-\(B\) bimodules.

**Example 3.11.** For \(\tau \in H^2(\mathcal{G}, \mathbb{Z}_2)\), let \(G_R, \phi_R\) and \(\tau_R\) be as in Example 2.4. Then, the twisted equivariant KK-group \(\phi \mathrm{KK}^\mathcal{G}_{\tau_R}(A, B)\) is isomorphic to the twisted equivariant Real KK-group \(\mathrm{KK}^\mathcal{G}(A, B)\) by definition. In the same way, the group \(\phi \mathrm{KK}^\mathcal{G}_{\tau_R}(A, B)\) is isomorphic to the twisted Quaternionic KK-group \(\mathrm{KK}^\mathcal{G}(A, B)\).

**Proposition 3.12.** Let \(A\) and \(B\) be \(\phi\)-twisted \(\mathbb{Z}_2\)-graded \(\mathcal{G}\)-C*-algebras. Set \(\varpi := \tau + \epsilon(c, c)\). Then, there is a natural isomorphism

\[
\phi \mathrm{KK}^\mathcal{G}_{c, \varpi}(A, B) \cong \phi \mathrm{KK}^\mathcal{G}(A, B \otimes \phi \mathcal{H} c^G \tau接手).
\]

In particular, when \(A\) is a Real \(\mathcal{G}\)-C*-algebra,

\[
\phi \mathrm{KK}^\mathcal{G}_{c, \varpi}(A, B) \cong \mathrm{KK}^\mathcal{G}(A, (B \otimes \phi \mathcal{H} c^G \tau接手)).
\]

It reduces the twisted equivariant KK-group to the existing KK-group since \(\phi \mathrm{KK}^\mathcal{G}(A, B)\) is isomorphic to \(\mathrm{KK}^\mathcal{G} \otimes \mathcal{G}^\mathcal{G}(p_A^*, A, p_B^*)\) in the sense of Mouton [Mon14]. Here, the groupoid \(\mathcal{G} \times \mathcal{G}^\mathcal{G}\) is regarded as a Real groupoid by the \(\mathbb{Z}_2\)-action from the right. As a consequence, we obtain the 8-fold Bott periodicity for twisted equivariant KK-theory.

**Proof.** By Lemma 3.5 and Example 3.7, we have a one-to-one correspondence between \((\phi, c, \tau)\)-twisted Kasparov bimodules for \(A, B\) and \(\phi\)-twisted
Kasparov bimodules for \(A, B \otimes \phi K^G_{c,\tau}
\) given by
\[
(E, \varphi, F) \quad \mapsto \quad (E \otimes (\phi K^G_{c,\tau})^*, \varphi \otimes 1, F \otimes 1),
\]
\[
(E', \varphi', F') \quad \mapsto \quad (E', \varphi', F').
\]
Note that the operators \(F \otimes 1\) and \(F' \otimes 1\) satisfy the relations on group actions because \(L(E \otimes (\phi K^G_{c,\tau})^*) \cong L(E')\) and \(L(E' \otimes \phi K^G_{c,\tau} \otimes \phi K^G_{c,\tau}) \cong K(E')\).

When \(c = 0, \tau = 0\) and \(A\) is a Real \(\mathcal{G}\)-\(C^*\)-algebra, \(\phi KK^G(A, B) \cong \phi KK^G_\epsilon(p_\epsilon^* A, p_\epsilon^* B)\) is isomorphic to \(\phi KK^{\mathbb{Z}_2}(A, p_\epsilon^* B)\), which is isomorphic to \(KK^G(A, B_{\mathbb{R}})\) by Remark 7.2. \(\square\)

At the end of this section, we briefly summarize basic properties of the Kasparov product. Let \((E_1, \varphi_1, F_1)\) be a \((\phi, \tau_1, c_1)\)-twisted \(\mathcal{G}\)-equivariant Kasparov \(A_1, B_1 \otimes D\) bimodule and let \((E_2, \varphi_2, F_2)\) be a \((\phi, \tau_2, c_2)\)-twisted \(\mathcal{G}\)-equivariant Kasparov \(A_2, B_2 \otimes D\)-\(B_2\) bimodule. Set \(\tau = \tau_1 + \tau_2 + c_1 + c_2\) and \(c = c_1 + c_2\). We say that a \((\phi, c, \tau)\)-twisted \(\mathcal{G}\)-equivariant Kasparov \(A_1 \otimes A_2, B_1 \otimes B_2\) bimodule \((E, \varphi, F)\) is a Kasparov product \((E_1, \varphi_1, F_1) \otimes_B (E_2, \varphi_2, F_2)\) if \(E = E_1 \otimes_B E_2, \varphi(a_1 \otimes a_2) = \varphi_1(a_1) \otimes \varphi_2(a_2)\) and \(F\) satisfies
\[
o \text{the operator } F \text{ is an } F_2\text{-connection, that is, } FT_\xi = -1)^{\phi} T_\xi F_2 \text{ is in } \mathbb{K}(E_2, E), \text{ and } T_\xi F_2 - (1)^{\phi} F_2 T_\xi \in \mathbb{K}(E, E_2) \text{ for any } \xi \in E_1.
\]
(3.13) (where \(T_\xi(\eta) := \xi \otimes \eta\).
\(\circ\) the operator \(\varphi(a)[F, F_1 \otimes 1][\varphi(a)^*\) is positive modulo compact for any \(a \in A_1 \otimes A_2\).

It is straightforward to check that the Kasparov product always exists and induces the well-defined product of KK-groups
\[
\phi KK^G_{c_1, \tau_1}(A_1, B_1 \otimes D) \otimes \phi KK^G_{c_2, \tau_2}(A_2 \otimes D, B_2) \rightarrow \phi KK^G_{c, \tau}(A_1 \otimes A_2, B_1 \otimes B_2)
\]
which is associative by the same proof as Theorem 12 of [Ska84] (for Real KK-theory).

As a corollary, we obtain an isomorphism between \(\phi KK^G_{\epsilon, \tau}(A, B)\) and the group \(\phi KK^G_{\epsilon, \tau}(A, B)_{oh}\) of “operator homotopy” equivalence classes of \((\phi, \epsilon, \tau)\)-twisted \(\mathcal{G}\)-equivariant Kasparov \(A-B\) bimodules (cf. Subsection 18.5 of [Blu98]). Here, two \((\phi, c, \tau)\)-twisted \(\mathcal{G}\)-equivariant Kasparov \(A-B\) bimodules \((E_1, \varphi_1, F_1)\) are operator homotopic if there are degenerate \((\phi, c, \tau)\)-twisted \(\mathcal{G}\)-equivariant Kasparov \(A-B\) bimodules \((E'_1, \varphi_1, F'_1)\), an even unitary operator \(U : E_1 + E'_1 \rightarrow E_2 + E'_2\) and a norm continuous path \(F_t\) of operators on \(E_2 + E'_2\) such that \((E_2 + E'_2, \varphi_2 \oplus \varphi'_2, F_t)\) are \((\phi, c, \tau)\)-twisted \(\mathcal{G}\)-equivariant Kasparov \(A-B\) bimodules and \(F_0 = U^* (F_1 + F'_1) U, F_1 = F_2 + F'_2\).

4. Twisted Equivariant K-theory

In the same fashion as the untwisted case, it is natural to define the twisted equivariant K-group as a special case of the twisted equivariant KK-groups introduced in Section 3. In this section, we study basic properties of twisted equivariant K-groups. In particular, we introduce a generalization of the twisted crossed product for twists in the sense of [FM13] and prove the generalization of the Green-Julg theorem.

Hereafter, we assume that \(G\) is a proper groupoid.
Definition 4.1. Let $A$ be a $\phi$-twisted $\mathbb{Z}_2$-graded $\mathcal{G}$-$C^*$-algebra. The twisted equivariant $K$-group $\phi KK^G_{0,c,t}(\mathbb{R}, A)$ is defined as the twisted equivariant $K$-group $\phi KK^G_{0,c,t}(\mathbb{R}, A)$.

We remark that every element in $\phi KK^G_{0,c,t}(\mathbb{R}, A)$ is represented by a Kasparov bimodule of the form $[\phi \mathcal{C}_G^{c,t}(A), 1, F]$. Actually, a Kasparov bimodule $(E, \varphi, F)$ is equivalent to $(pE \oplus \phi \mathcal{C}_G^{c,t}, \varphi, Fp \oplus F_0)$ and $pE \oplus \phi \mathcal{C}_G^{c,t} \cong \phi \mathcal{C}_G^{c,t}$ by (3.3), where $p := \varphi(1)$ and $(\phi \mathcal{C}_G^{c,t}, \varphi, F_0)$ is a degenerate Kasparov bimodule such that $\varphi_0$ is unital.

This definition has another reasonable presentation, which is a generalization of $K$-theory for $\mathbb{Z}_2$-graded $C^*$-algebras introduced in [AK97, HK97]. Let $S$ be the $C^*$-algebra $\mathcal{C}_0(\mathbb{R})$ together with the $\mathbb{Z}_2$-grading $\gamma(f)(x) = f(-x)$ and the Real structure $\overline{f}(x) = f(x)$. The groupoid $\mathcal{G}$ acts on $S$ by $\alpha_g(f) = \gamma^c(g)(\phi(g)f)$.

Proposition 4.2. Let $A$ be a $\phi$-twisted $\mathbb{Z}_2$-graded $\mathcal{G}$-$C^*$-algebra. Then, the group $\phi KK^G_{0,c,t}(A)$ is isomorphic to the set of homotopy classes of $\mathcal{G}$-equivariant $\ast$-homomorphisms $[S, A \hat{\otimes} \phi \mathcal{C}_G^{c,t}]^\mathcal{G}$.

For a $\mathcal{G}$-$C^*$-algebra $A$, let $A^+$ denote the unitalization $A + C_b(\mathbb{R}) \subset M(A \oplus \mathbb{R})$ of $A$. Then, there is a one-to-one correspondence between $\mathcal{G}$-equivariant $\ast$-homomorphisms from $S$ to $A \hat{\otimes} \phi \mathcal{C}_G^{c,t}$ and unitaries $(A \hat{\otimes} \phi \mathcal{C}_G^{c,t})^+$ such that $u - 1 \in A \hat{\otimes} \phi \mathcal{C}_G^{c,t}$, $\gamma(u) = u^*$ and $\alpha_g(u) = \gamma^c(g)(\phi(g)u)$. Actually, a $\ast$-homomorphism $\varphi$ corresponds to a unitary $1 + \varphi(-\exp(\pi ix(1 + x^2)^{-1}) - 1)$.

Proof. The proof is given in the same way as in Theorem 4.7 of [Tro90]. By the functional calculus given in Theorem 3.2 of [Tro90], we obtain a $\mathbb{Z}_2$-graded submodule $E := \varphi(C_c(\mathbb{R})) \hat{\otimes} \phi \mathcal{C}_G^{c,t}$ of $\phi \mathcal{C}_G^{c,t}$ and a regular odd self-adjoint operator $D$ on $E$ given by $D(\varphi(f)\xi) := \varphi(xf)\xi$. Consequently we obtain a Kasparov $\mathbb{R}$-bimodule $[E, 1, D(1 + D^2)^{-1/2}]$. This correspondence gives a group homomorphism $[S, A \hat{\otimes} \phi \mathcal{C}_G^{c,t}]^\mathcal{G} \to \phi KK^G_{0,c,t}(\mathbb{R}, A)$.

In the same way as the proof of Theorem 4.7 of [Tro90], we can check that it is an isomorphism. We remark that for any Kasparov bimodule $(E, \varphi, F)$, we can replace $F$ with another Fredholm operator $F' = \int_{\mathcal{G}}^* \alpha_g(F_{s(g)})d\lambda^c(g)$ commuting with the $\mathcal{G}$-action since $\mathcal{G}$ is proper. \hfill $\square$

Now we generalize the Green-Julg theorem [Jul81] for twisted equivariant $K$-theory. First, we start with the definition of the crossed product for groupoid actions with general twists. Here, we say that an extension of a groupoid $\mathcal{G}$ by a group $N$ is a principal $N$-bundle $\mathcal{G}^\sigma$ over $\mathcal{G}^1$ with a groupoid structure which is compatible with the projection $\pi : \mathcal{G}^\sigma \to \mathcal{G}$ and the identification $\pi^{-1}(1_x) = 1_x \cdot N \cong N$ for each $x \in \mathcal{G}^0$. Denote $n_x := 1_x \cdot n$ for $n \in N$.

Definition 4.3. Let $\phi : \mathcal{G} \to \mathbb{Z}_2$ be a groupoid homomorphism and let $A$ be a $\mathcal{G}^0$-$C^*$-algebra. We say that a $(\phi, \sigma)$-twisted $\mathcal{G}$-action is a pair $(\mathcal{G}^\sigma, \alpha)$ where $\mathcal{G}^\sigma$ is a groupoid extension of $\mathcal{G}$ by a locally compact subgroup $N$ of $\mathcal{G}M(A)$ and $\alpha$ is a $\phi$-twisted action of $\mathcal{G}^\sigma$ on $A$ such that $\alpha_n = \text{Ad} u_n$ for any $u \in N$. 
In the same way as in Section 3, we get a family of $\phi$-linear $G^0$-automorphisms $\alpha = \{\alpha^\mu : s^*A[V^\mu] \to r^*A[V^\mu]\}_{\mu \in \Gamma^1}$ and a family $\sigma = \{\sigma^\nu : V^\nu \to N\}_{\nu \in \Gamma^2}$ such that
\[
\alpha^\nu_{gh} \circ \alpha^\mu_h = \text{Ad}(\sigma^\nu(g, h)r(g)) \circ \alpha^\nu_{gh},
\]
\[
\sigma^\nu_{gh}(g, h) = (\alpha^\nu_{gh})^*(\sigma^\nu(g, h)),
\]
\[
\sigma^\nu_{h}(1_{r(g)}, h) = 1,
\]
\[
\alpha^\nu_{1_x} = \text{id}_{\lambda_x},
\]
for each $\nu \in \Gamma^2$, $\kappa \in \Gamma^3$, $x \in G^0$ and $g \in G^1$. We say that $\sigma$ is a 2-cocycle associated to $\alpha$.

**Definition 4.4.** Let $c : G \to \mathbb{Z}_2$ be a groupoid homomorphism. For a $(\phi, \sigma)$-twisted $G$-action on a $C^*$-algebra $A$, the $(\phi, c, \sigma)$-twisted crossed product $G \ltimes_{\phi, c, \sigma}^A$ is the $\mathbb{Z}_2$-graded $C^*$-algebra $G \ltimes_{c, \sigma}^A$ together with the Real structure induced from the complex conjugation on $A_R$.

Here, for a $\sigma$-twisted $G$-$C^*$-algebra, the full graded twisted crossed product $G \ltimes_{c, \sigma}^A$ (see Section 2 of [CE01]) is given by the completion of the subalgebra $C_c(G, r^*A, \sigma)$ of $C_b(G^\sigma, r^*A)$ consisting of functions $f$ with compact support on $G$ such that $f(\tilde{g}u) = f(\tilde{g})\alpha_g(u^*) \in A_{r(\tilde{g})}$ with operations
\[
f_1 \ast f_2(\tilde{g}) := \int_{G^{\sigma}(\tilde{g})} (-1)^{c(\tilde{g}) \cdot |f_1(\tilde{h})|} f_1(\tilde{h})\alpha^A_{\tilde{h}}(f_2(\tilde{h}^{-1}\tilde{g}))d\lambda^\sigma(\tilde{g})(h),
\]
\[
f^*(\tilde{g}) := (-1)^{c(\tilde{g}) \cdot |f(\tilde{g})|} \alpha^A_{\tilde{g}}(f(\tilde{g}^{-1})^*),
\]
\[
\alpha_{\tilde{h}}(f)(\tilde{g}) := \alpha^A_{\tilde{h}}(f(\tilde{h}^{-1}\tilde{g})),
\]
\[
\gamma(f)(\tilde{g}) := (-1)^{c(\tilde{g})} \gamma_A(f(\tilde{g})),
\]
for $\tilde{g}, \tilde{h} \in G^\sigma$ by the maximal $C^*$-norm smaller than
\[
\|f\|_1 := \sup_{x \in G^0} \left\{ \int_{G^x} \|f(g)\| \, d\lambda^x(g), \int_{G^x} \|f(g)\| \, d\lambda_x(g) \right\}
\]
(note that $\|f(\tilde{g})\|$ is a function on $G$). Note that our definition is slightly different from Definition 5.1 of [Th12].

In Section 3.6 of [KS04], the reduced crossed product of a groupoid action is defined as the quotient of the full twisted crossed product with respect to the family of kernels of (pointwise) regular representations. The difference between full and reduced crossed products causes no problem for us because they always coincide when $G$ is proper.

For a $(\phi, c)$-twisted representation $u$ of $G^\sigma$ and a $*$-representation $\pi$ of $A$ on a Hilbert bundle $\mathcal{H}$ over $G^0$ which are twisted covariant (that is, it satisfies $u_g\pi(a)u^*_g = (-1)^{c(g)\cdot |\pi(a)|}\pi(\alpha_g(a))$ for any $g \in G^1$ and $a \in A_{r(g)}$), we have a unique representation $\tilde{\pi}$ of the full crossed product $G \ltimes_{c, \tau}^A$ on $p^\phi_0 \mathcal{H} \cong \mathcal{H}_R$.  

**Example 4.5.** When $\phi$ is trivial, then the $(\phi, c, \sigma)$-twisted crossed product is isomorphic to $(G \ltimes_{c, \sigma}^A)_R$, whose Real K-group is isomorphic to the K-group of $G \ltimes_{c, \sigma}^A$. 

Example 4.6. When $\mathcal{G}$ acts on the Real $C^*$-algebra $\mathbb{R}$ trivially, the crossed product $\mathcal{G} \ltimes_{\tau} \mathbb{R}$ is called the $(\phi,c,\tau)$-twisted groupoid $C^*$-algebra and denoted by $\hat{\mathcal{C}}_{c,\tau}^{\phi,\tau}$. For example, the twisted groupoid $C^*$-algebra of $\mathcal{G}$ with respect to a real twist $(\phi_{\mathbb{R}},c,\tau_{\mathbb{R}})$ is isomorphic to $\mathcal{M}_2(\mathcal{C}^{\phi}_{c,\tau}(\mathcal{G};\mathbb{R}))$ where $\mathcal{C}^{\phi}_{c,\tau}(\mathcal{G};\mathbb{R})$ is the twisted graded groupoid $C^*$-algebra with coefficient in $\mathbb{R}$. For a general $\mathbb{Z}_2$-valued 1-cocycles $c$ on $\mathcal{G}$, we define the stable twisted groupoid $C^*$-algebra as $\hat{\mathcal{C}}_{c,\tau}^{\phi} := \hat{\mathcal{C}}_{c,\tau}^{\phi,\tau} \mathcal{G}$. When $c$ is represented by a groupoid homomorphism, it is isomorphic to $\hat{\mathcal{M}}_{1,1}(\phi^{\mathcal{G}}_{c,\tau})$. For example, when $(A,\tau)$ is a CT-type as in Example 2.6, then $\hat{\mathcal{C}}_{c,\tau}^{\phi}A$ is isomorphic to a Clifford algebra as indicated in Table 1 below.

| $\mathcal{A}$ | $\mathcal{G}$ | $\mathcal{T}$ | $\mathcal{C}$ | $\mathcal{D}$ |
|---------------|---------------|---------------|---------------|---------------|
| $\mathcal{T}^2$ | 1             | 1             | 1             | 1             |
| $\hat{\mathcal{C}}_{c,\tau}^{\phi}A$ | $\mathcal{C}_2$ | $\mathcal{C}_2$ | $\mathcal{C}_2$ | $\mathcal{C}_2$ |
| $\hat{\mathcal{C}}_{c,\tau}^{\phi}A$ | $\mathcal{C}_2$ | $\mathcal{C}_2$ | $\mathcal{C}_2$ | $\mathcal{C}_2$ |
| $\mathcal{K}_{c,\tau}^{\phi}A$ | $\mathcal{K}_{c,\tau}^{\phi}A$ | $\mathcal{K}_{c,\tau}^{\phi}A$ | $\mathcal{K}_{c,\tau}^{\phi}A$ | $\mathcal{K}_{c,\tau}^{\phi}A$ |
| Cartan | $\mathcal{A}$ | $\mathcal{A}$ | $\mathcal{A}$ | $\mathcal{A}$ |

Table 1. The 10-fold way and Clifford algebras

Example 4.7. Let $\mathcal{G}_{\mathbb{R}}$, $\phi_{\mathbb{R}}$ and $\tau_{\mathbb{R}}$ be as in Example 2.6. For a Real $\mathcal{G}$-C$^*$-algebra $A$ and $c \in \text{Hom}(\mathcal{G},\mathbb{Z}_2)$, the twisted crossed product $\mathcal{G}_{\mathbb{R}} \ltimes_{\tau_{\mathbb{R}}}^c A$ is isomorphic to $\mathcal{M}_2(\mathcal{G} \ltimes_{c,\tau} A)$. In the same way, the crossed product $\mathcal{G}_{\mathbb{R}} \ltimes_{\tau_{\mathbb{R}}}^c A$ is isomorphic to the quaternionic crossed product $\mathcal{G} \ltimes_{c,\tau}^H A := (\mathcal{G} \ltimes_{c,\tau} A) \hat{\otimes}_{\mathbb{R}}^H$. When $\mathcal{G}$ acts on the Real $\mathcal{G}$-algebras.

Example 4.8. Let $F_{n,m}$ and $(e^0_{n,m},r^0_{n,m})$ be as in Example 2.5. The real graded group $C^*$-algebra $C_{n,m}^{e_{n,m},r_{n,m}}(F_{n,m})$ is isomorphic to the complex Clifford algebra $\mathcal{M}_2(C_{n,m})$. Moreover, let $\mathcal{G}$ be a groupoid, let $(\phi,c,\tau)$ be a twist on $\mathcal{G}$ and let $A$ be a $\phi$-twisted $\mathcal{G}$-C$^*$-algebra. Set $\mathcal{G}' := \mathcal{G} \ltimes F_{n,m}$ and

$$(\phi,c',\tau') := (\phi,c,n,m,\tau) := (\phi \times 0, c \times e_{n,m}, \tau \times 0_{n,m}).$$

Then, $\mathcal{G}' \ltimes_{c',\tau'}^\phi A$ is isomorphic to $(\mathcal{G} \ltimes_{c,\tau} A) \hat{\otimes} C_{n,m}$.

Lemma 4.9. The crossed product $\mathcal{G} \ltimes_{c,\tau}^\phi A$ is Morita equivalent to $\mathcal{G} \ltimes_0 (A \hat{\otimes} \mathbb{K}_{c,\tau}^-)_{\mathbb{R}}$, where as Real $C^*$-algebras.

Proof. First, we reduce the problem for the case that $c = 0$. Let $\mathcal{V}$ be the $\mathbb{Z}_2$-graded vector space $\mathbb{R}$ with the Real $c$-twisted representation of $\mathcal{G}$ given by $\text{Ad}(1 \otimes e)^c(g)$ where $e := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then, we have a $*$-isomorphism

$$\mathcal{G} \ltimes_{c,\tau}^\phi (A \hat{\otimes} \mathbb{K}(\mathcal{V})) \to \mathcal{G} \ltimes_{c,\tau}^\phi (A \hat{\otimes} \mathbb{K}(\mathbb{R})) \cong (\mathcal{G} \ltimes_{c,\tau}^\phi A) \hat{\otimes} \mathbb{K}(\mathbb{R})$$

given by $f(\tilde{g}) \mapsto (1 \otimes e)^c(g)f(\tilde{g})$ for any $f \in C_{c,\tau}(\mathcal{G}, r^*A, \tau)$.

Next, we assume that $c = 0$. Recall that two $C^*$-algebras $A$ and $B$ are Morita equivalent if and only if there is a $C^*$-algebra $D$ and a projection $p \in M(D)$ such that $pDp \cong A$, $(1-p)D(1-p) \cong B$ and $DpD := \text{spau}\{d_1pd_2 \mid d_i \in D\}$ is dense in $D$. (See [BGR77].) Actually, $pD(1-p)$ is a $A$-$B$
imprimitivity bimodule and conversely $D := \begin{pmatrix} A & E \\ E^* & B \end{pmatrix}$ satisfies the above properties.)

Let $B$ be the $C^*$-algebra $A \hat{\otimes} \mathcal{K}(\phi \mathcal{H}^{-\tau}_G \oplus \mathbb{R}) = \begin{pmatrix} A & \hat{\otimes} \mathcal{K}^{-\tau}_G \\ A & \hat{\otimes} \mathcal{H}^{-\tau}_G \end{pmatrix}$ with the $(\phi, \sigma)$-twisted $G$-action

$$\alpha_g \left( \begin{array}{cccc} a \otimes x & b \otimes \xi^* \\ c \otimes \eta & d \end{array} \right) = \begin{pmatrix} \alpha_g(a) \otimes u_g x u_g^* & \alpha_g(b) \otimes (u_g \xi)^* \\ \alpha_g(c) \otimes u_g \eta & \alpha_g(d) \end{pmatrix}$$

where $\sigma$ is the $\mathcal{U}M(B)$-valued 2-cocycle given by $\sigma(g, h) = \begin{pmatrix} 1 & 0 \\ 0 & \tau(g, h) \end{pmatrix}$.

Set $D := G \ltimes_\sigma B$. Then, the projection $p := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in $M(D)$ satisfies $pDP \cong G \ltimes \phi \left( A \hat{\otimes} \mathcal{K}^{-\tau}_G \right)$ and $(1 - p)D(1 - p) \cong G \ltimes \phi A$. Moreover, since $D$ contains a dense subspace $C_c(G) B$ and $BC_c(G)$, $DpD$ contains a dense subset $C_c(G) BpBC_c(G) = C_c(G) BpBC_c(G)$ in $D$.

**Theorem 4.10.** Let $G$ be a proper groupoid. Then there is an isomorphism

$$\phi^K G_{e, c, \tau}(A) \cong \text{KR}_e (G \ltimes \phi_{c, -\tau} A),$$

where $\tau := \tau + \epsilon(c, e)$.

**Proof.** The Green-Julg theorem is generalized for proper groupoids in Proposition 6.25 of [Tu99]. By Proposition 3.12 $\phi^K G_{e, \tau}(\mathbb{R}, A)$ is isomorphic to the Real $KK$-group $\mathcal{KK}G(\mathbb{R}, (A \hat{\otimes} \mathcal{K}^{-\tau}_G)_{\mathbb{R}})$. Now the conclusion follows from the usual Green-Julg theorem for Real $C^*$-algebras and Lemma 4.9. \qed

**Corollary 4.11.** Let $G' := G_{n,m}$ and $(\phi, c', \tau') := (\phi, c_{n,m}, \tau_{n,m})$ be as in Example 4.8. For any $(\phi, c')$-twisted $G^*$-algebra $A$, we have an isomorphism

$$\phi^K G_{e, c', \tau'}(A) \cong \phi^K G_{m-n, c, \tau}(A).$$

In particular, $\phi^K G_{e, c', \tau'}(A) \cong \text{KR}_{m-n, c, \tau}(A)$ for any Real $G^*$-algebra $A$.

**Proof.** It follows from Example 4.6, Example 4.8 and the Green-Julg theorem 4.10. Note that $\tau_{n,m} + \epsilon(c_{n,m}, c_{n,m}) = \tau_{m,n}$ for Clifford twits. \qed

**Corollary 4.12.** Let $(G, \phi, c, \tau)$ be a CT-type symmetry as in Example 2.6 with the CT-type $(A, \tau)$ and let $A$ be a $\mathbb{Z}_2$-graded Real $\mathcal{H}_0^\tau$-algebra. Then, the twisted equivariant K-group $\phi^K G_{e, c, \tau}(A)$ is isomorphic to the equivariant K-group $K_{\mathcal{H}_0^\tau}(A)$. Here, $K_{\mathcal{H}_n^\tau}$ is one of $K_n$ ($n = 0, 1$) or $\text{KR}_n$ ($n = 0, \ldots, 7$) as indicated in Table 7.

**Proof.** It follows from the Green-Julg theorem 4.10 and Example 4.6. Since $A$ acts on $A$ trivially, the crossed product $G \ltimes_{\phi_{c, \tau}} A$ is isomorphic to $(G_0 \ltimes_{\tau_0} A) \hat{\otimes} (\mathcal{H}_c^* \tau A)$. \qed

5. The Group $\phi^K G_{e, c, \tau}(A)_{vD}$

In this section, we introduce another formulation of twisted equivariant K-theory based on van Daele’s definition of K-theory for $\mathbb{Z}_2$-graded Banach
algebras. This definition is related to Karoubi’s K-theory \cite{Kar88, Kar08} in the case that the C*-algebra is trivially graded.

Throughout this section, we assume that \(\mathcal{G}\) is a proper groupoid with the compact orbit space such that \(\mathcal{H}\) is AFGP for any \((\phi, c, \tau)\). Here we say that a (twisted) representation \(\mathcal{H}\) is AFGP (approximately finitely generated projective) if there is a \(\mathcal{G}\)-invariant approximate unit of projections \((p_n)\) in \(\mathbb{K}(\mathcal{H})\) (Definition 5.14 of \cite{TXLG04}). By Lemma 3.1 of \cite{HR98}, we can replace it with an increasing approximate unit of projections. In other words, \(\mathcal{H}\) is AFGP if and only if it is decomposed into a direct sum of finite dimensional representations.

**Remark 5.1.** A reasonable criterion for \(\mathcal{H}\) to be AFGP is given in Theorem 6.14 of \cite{EM09}; it holds if and only if every irreducible representation \(\mathcal{C}_G^\tau\) is contained in the fiber of a finite dimensional representation of \(\mathcal{G}\) (see also Theorem 6.15 of \cite{EM09}). For example, this condition is satisfied for a translation groupoid \(\mathcal{G} \times X\) if \(\mathcal{G}\) is compact.

Note that \(\mathcal{H}\) is AFGP if and only if so is \(\mathcal{H}\) and there is a finite dimensional \((\phi, c, \tau)\)-twisted representation of \(\mathcal{G}\) since \(\mathcal{C}_G^\tau \cong \mathcal{H} \otimes V\) (cf. Proposition 5.25 of \cite{TXLG04}). It is conjectured in Subsection 5.7 of \cite{TXLG04} that \(\mathcal{G}\) has a \(\tau\)-twisted representation if and only if \(\mathcal{G}\) is the image of \(\tilde{H}^2(\mathcal{G}; \mathbb{Q}/\mathbb{Z}\phi)\). It is known to be true in the case that \(\mathcal{G}\) is a compact group (\(\mathcal{G}\) always has a \(\tau\)-twisted \(\mathcal{G}\)-vector bundle), \(\mathcal{G}\) is a compact space (the Grothendieck-Serre theorem \cite{Gro95, AS04}) and \(\mathcal{G}\) is a Renault-Deaconu groupoid \cite{FG15} but still open in general.

Let \(A\) be a unital \(\phi\)-twisted \(\mathbb{Z}_2\)-graded \(\mathcal{G}\)-C*-algebra. We say that an element \(a \in A\) is graded \(\mathcal{G}\)-invariant with respect to the \(\mathbb{Z}_2\)-grading \(c\) on \(\mathcal{G}\) if \(\alpha_g(a) = \gamma^c(g)(a)\) for any \(g \in \mathcal{G}\). Let \(\phi F_c^G(A)\) denote the set of graded \(\mathcal{G}\)-invariant odd symmetries of \(A\) (remark that \(\phi F_c^G(A)\) is possibly empty, for example when \(A\) is trivially graded).

For a finite dimensional \((\phi, c, \tau)\)-twisted unitary representation \(V\) of \(\mathcal{G}\), set \(\phi F_c^G(A) := \phi F_c^G(A \otimes \mathbb{K}(V))\). For a fixed element \(e \in \phi F_c^G(A)\), set \(\phi K_{c,\tau}^G(A) := \lim_{\to \gamma} \pi_0 \phi F_c^G(A)\), where \(\gamma\) runs over all finite dimensional \((\phi, c, \tau)\)-twisted unitary representations. Here, the inductive limit is taken with respect to the map \(\pi_0 F_c^G(A) \to \pi_0 F_c^G(A)\) for \(V \subset \mathcal{W}\) given by \(x \mapsto x \oplus e \otimes 1_{\mathcal{V}}\). The direct sum \((x, y) \mapsto x \oplus y\) determines a map from \(\phi F_c^G(A) \times \phi F_c^G(A)\) to \(\phi F_c^G(A)\), which induces a map \(\phi K_{c,\tau}^G(A) \times \phi K_{c,\tau}^G(A)\). Note that we obtain the same group by taking limits on finite direct sums of \(\mathcal{H}\). By the same reasoning as Proposition 2.7 of \cite{VD88a}, it is associative and commutative. Consequently \(\phi K_{c,\tau}^G(A)\) has the structure of abelian semigroup with the unit \([e]\).

**Proposition 5.2.** Assume that there is an element \(e \in \phi F_c^G(A)\) and an even \(\mathcal{G}\)-invariant unitary \(v \in A\) such that \(v e v^* = -e\). Then \(\phi K_{c,\tau}^G(A)\) is an abelian group and independent of the choice of such \(e\) up to isomorphism.

**Proof.** The proof is given in the same way as in Proposition 3 of \cite{VD88a}. Actually, if there is another \(f \in \phi F_c^G(A)\) and even \(\mathcal{G}\)-invariant unitary \(w\)
such that \( wfw^* = -f \), the map
\[
\text{Ad} \begin{pmatrix} 1 & f \\ 0 & w^* \end{pmatrix} \begin{pmatrix} 1 + fe & 1 - fe \\ 1 - fe & 1 + fe \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}
\]
duces an isomorphism \( \phi \mathbb{K} G_{c,\tau}(A) \to \phi \mathbb{K} G_{c,\tau}(A) \).

**Remark 5.3.** Let \( u \) be an even \( G \)-invariant unitary in \( A \otimes \mathbb{K}(V) \) commuting with \( e \). Then, \( \text{Ad} u \) acts trivially on \( \phi \mathbb{K} G_{c,\tau}(A) \) since \( usu^*e = (u \oplus u^*)(s \oplus e)(u \oplus u^*) \) and there is a homotopy \( (u \oplus 1)R_t(u^* \oplus 1)R_t \) of even \( G \)-invariant unitaries commuting with \( e \oplus e \) connecting \( u \oplus u^* \) and 1 (where \( R_t \) are the 2-by-2 rotation matrices). Consequently, the isomorphism in Proposition 5.2 is independent of the choice of unitaries \( v \). Moreover, when \( e \) and \( f \) are anticommutative, we have a simple presentation \( \frac{1}{2} \text{Ad}(1 + ef) \) of the isomorphism \( \phi \mathbb{K} G_{c,\tau}(A) \equiv \phi \mathbb{K} G_{c,\tau}(A) \).

Let \( \hat{\mathbb{M}}_{2,2}^g \) be the \( \mathbb{Z}_2 \)-graded real Hilbert space \( \mathbb{R} \oplus \mathbb{R}^{op} \oplus \mathbb{R} \oplus \mathbb{R}^{op} \) with the \( e \)-twisted representation \( u_g^\xi = u^g(\phi(g)\xi) \) of \( G \). Then, the \( \mathbb{Z}_2 \)-graded \( e \)-twisted \( C^* \)-algebra \( \hat{\mathbb{M}}_{2,2}^g := \mathbb{K}(\hat{\mathbb{R}}_{c,2}^2) \) is Morita-equivalent to \( \mathbb{R} \) and has a graded \( G \)-invariant odd symmetry \( v \). Here,
\[
u = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.
\]

We simply write \( \hat{\mathbb{M}}_{2,2}^g(A) \) for the tensor product \( \hat{\mathbb{M}}_{2,2}^g \otimes A \).

**Definition 5.4.** The twisted equivariant \( K \)-theory group \( \phi \mathbb{K} G_{c,\tau}(A)_{vD} \) is defined by the abelian group \( \text{Ker}(\phi \mathbb{K} G_{c,\tau}(\hat{\mathbb{M}}_{2,2}^g(A^{+}))_{v}) \to \phi \mathbb{K} G_{c,\tau}(\hat{\mathbb{M}}_{2,2}^g))_{v} \).

We remark that when \( A \) is unital \( A^+ \) is isomorphic to the direct sum \( A \oplus C_0(S^0) \) and hence \( \phi \mathbb{K} G_{c,\tau}(A)_{vD} \) is isomorphic to \( \phi \mathbb{K} G_{c,\tau}(\hat{\mathbb{M}}_{2,2}^g(A))_{v} \).

**Lemma 5.5.** There is a natural isomorphism \( \phi \mathbb{K} G_{1,c,\tau}(A)_{vD} \equiv \phi \mathbb{K} G_{1}(A \otimes \phi \mathbb{K} G_{c}^{c,\tau}), \) where \( c := \tau + e(c,c) \).

**Proof.** For a \( (a, c, \tau) \)-twisted representation \( V \), let \( \hat{V} \) be as in Example 3.7. The identity \( \phi \)-homomorphism \( A \otimes \mathbb{K}(V) \to A \otimes \mathbb{K}(\hat{V}) \) gives a homeomorphism \( \varphi_* : \phi F_c(A \otimes \mathbb{K}(V)) \to \phi F(A \otimes \mathbb{K}(\hat{V})) \). Since \( \mathbb{K}(V) \) is Morita equivalent to \( \phi \mathbb{K} G_{c}^{c,\tau}, \) the inductive limit \( \lim_{\text{ind}} \pi_0 \phi F(A \otimes \mathbb{K}(\hat{V})) \) is isomorphic to \( \phi \mathbb{K} G_{1}(A \otimes \phi \mathbb{K} G_{c}^{c,\tau})_{vD} \). Therefore, \( \varphi_* \) gives a desired isomorphism.

**Lemma 5.6.** If \( s, t \in \phi F_c^G(V)(A) \) satisfies \( \|s - t\| < 2 \), there is a \( G \)-invariant even self-adjoint element \( h \in A \otimes \mathbb{K}(V) \) such that \( \|h\| < 1 \), \( th\bar{t} = -h \) and \( s = \exp(2\pi i h t) \). Moreover, \( s \) and \( t \) are homotopic in \( \phi F_c^G(V)(A) \).

**Proof.** Since \( \|st - 1\| < 2 \), \( -1 \) is not contained in the spectrum of the \( G \)-invariant even unitary \( st \). Hence \( h := \log st \) with respect to the branch \( \{-i\mathbb{R}_{>0}\} \) is well-defined self-adjoint element in \( A \otimes \mathbb{K}(V) \) such that \( \|h\| < 1 \) and \( th\bar{t} = \log(t(st)t) = \log(st)^* = -\log st = -h \). Now, \( \tau \mapsto \exp(2\pi i rh\bar{t})t \) gives a continuous path connecting \( s \) and \( t \) in \( \phi F_c^G(V)(A) \).
Lemma 5.7. The functor $\hat{\phi}K_{1,c,\tau}\omega\vdash D$ has the following properties.

1. The correspondence $A \mapsto \hat{\phi}K_{1,c,\tau}(A)\vdash D$ is a covariant functor from the category of $\phi$-twisted $\mathbb{Z}_2$-graded $\mathcal{G}$-$C^*$-algebras and $\ast$-homomorphisms to the category of abelian groups.

2. If two $\mathcal{G}$-equivariant $\ast$-homomorphisms $\varphi$ and $\psi$ are homotopic, then $\varphi_* = \psi_* : \hat{\phi}K_{1,c,\tau}(A)\vdash D \rightarrow \hat{\phi}K_{1,c,\tau}(B)\vdash D$.

3. For any $\phi$-twisted representations $\mathcal{V}_1 \subset \mathcal{V}_2$ of $\mathcal{G}$, the inclusion $\mathbb{K}(\mathcal{V}_1) \rightarrow \mathbb{K}(\mathcal{V}_2)$ induces the isomorphism

$$f_{\mathcal{V}_1,\mathcal{V}_2}^A : \hat{\phi}K_{1,c,\tau}(A \otimes \mathbb{K}(\mathcal{V}_1))\vdash D \rightarrow \hat{\phi}K_{1,c,\tau}(A \otimes \mathbb{K}(\mathcal{V}_2))\vdash D.$$

4. A short exact sequence $0 \rightarrow I \xrightarrow{i} A \xrightarrow{s} A/I \rightarrow 0$ of $\phi$-twisted $\mathcal{G}$-$C^*$-algebras induces an exact sequence

$$\hat{\phi}K_{1,c,\tau}(I)\vdash D \rightarrow \hat{\phi}K_{1,c,\tau}(A)\vdash D \rightarrow \hat{\phi}K_{1,c,\tau}(A/I)\vdash D.$$

Proof. The assertion (1) and (2) follows from the definition.

When $\mathcal{V}_1$ and $\mathcal{V}_2$ are finite dimensional, (3) follows by definition. Therefore, $f_{\mathcal{V}_1,\mathcal{V}_2}^A$ is also isomorphic by continuity of the functor $\hat{\phi}K_{1,c,\tau}\omega\vdash D$, that is, $\lim_{\mathcal{V}_1,\mathcal{V}_2} \hat{\phi}K_{1,c,\tau}\omega\vdash D \cong \mathcal{K}_{1,c,\tau}(\lim_{\mathcal{V}_1,\mathcal{V}_2})\vdash D$. It can be checked in the same way as the case of $K$-groups for $\mathcal{G}$-$C^*$-algebras by Lemma 5.6 (see for example Theorem 6.3.2 of [RLL00]). Moreover, for any $\phi$-twisted representation $\mathcal{V}$ of $\mathcal{G}$, $f_{\mathcal{V}_1,\mathcal{V}_2}^A = f_{\mathcal{V}_1,\mathcal{V}_2}^A \mathbb{K}(\mathcal{V})$ since $\mathcal{H}_\mathcal{G} \cong \mathcal{V} \otimes \mathcal{H}_\mathcal{G}$. Finally, $f_{\mathcal{V}_1,\mathcal{V}_2}^A$ is isomorphic for general $\mathcal{V}_1$ and $\mathcal{V}_2$ since the diagram

$$\begin{array}{ccc}
\hat{\phi}K_{1,c,\tau}(A \otimes \mathbb{K}(\mathcal{V}_1))\vdash D & \xrightarrow{f_{\mathcal{V}_1,\mathcal{V}_2}^A} & \hat{\phi}K_{1,c,\tau}(A \otimes \mathbb{K}(\mathcal{V}_1 \oplus \mathcal{V}_2))\vdash D \\
\phi & \downarrow & \phi \\
\hat{\phi}K_{1,c,\tau}(A \otimes \mathbb{K}(\mathcal{V}_1 \oplus \mathcal{V}_2))\vdash D & \rightarrow & \hat{\phi}K_{1,c,\tau}(A \otimes \mathbb{K}(\mathcal{V} \oplus \mathcal{H}_\mathcal{G}))\vdash D \\
\phi & \downarrow & \phi \\
\phi & \rightarrow & \phi \\
\end{array}$$

commutes, $\mathfrak{X}$ implies $\mathcal{V}_2 \oplus \mathcal{H}_\mathcal{G} \cong \mathcal{H}_\mathcal{G}$.

To see (4), it suffices to check that $\text{Ker} \pi_* = \text{Im} \iota_*$. We may assume that $A$ is unital. Let $s$ be a super $\mathcal{G}$-equivariant odd symmetry in $A$ and let $t_1, t_2, \ldots, t_n$ such that $\|s_{t_1} - s_{t_2}\| < 2$. Therefore, it suffices to show that for $\tilde{s} \in \phi F_{\mathcal{G},\mathcal{V}}(\mathbb{M}_{2,2}(A))$ and $s' \in \phi F_{\mathcal{G},\mathcal{V}}(\mathbb{M}_{2,2}(A/I))$ such that $\|\pi(\tilde{s}) - s'\| < 2$, there is a lift $\tilde{s}' \in \phi F_{\mathcal{G},\mathcal{V}}(\mathbb{M}_{2,2}(A/I))$ of $s$ such that $\tilde{s}$ and $\tilde{s}'$ are homotopic. This follows from Lemma 5.6 and the fact that every even $\mathcal{G}$-invariant self-adjoint $h \in \mathbb{M}_{2,2}(A/I) \otimes \mathbb{K}(\mathcal{V})$ anticommuting with $s$ has an even $\mathcal{G}$-invariant self-adjoint lift $\tilde{h} \in \mathbb{M}_{2,2}(A/I) \otimes \mathbb{K}(\mathcal{V})$ anticommuting with $\tilde{s}$. \hfill $\square$

Lemma 5.7 asserts that the functor $\hat{\phi}K_{1,c,\tau}\omega\vdash D$ is equivariantly stable, homotopy invariant and half-exact. Hence, the same argument as Proposition 4.1 of [Cum84] can be applied for $\hat{\phi}K_{1,c,\tau}\omega\vdash D$ and we obtain the long exact sequence

$$\cdots \rightarrow \hat{\phi}K_{1,c,\tau}(SI)\vdash D \rightarrow \hat{\phi}K_{1,c,\tau}(SA)\vdash D \rightarrow \hat{\phi}K_{1,c,\tau}(SA/I)\vdash D \rightarrow \hat{\phi}K_{1,c,\tau}(I)\vdash D \rightarrow \hat{\phi}K_{1,c,\tau}(A)\vdash D \rightarrow \hat{\phi}K_{1,c,\tau}(A/I)\vdash D.$$
Here the boundary map \( \partial \) is given in the same way as Proposition 4.7 of [VD88a].

Moreover, this exact sequence is extended for negative direction. Let \( S^{p,q} \) denote the Real \( C^* \)-algebra \( C_0(\mathbb{R}^{p+q}) \) together with the Real structure \( \mathcal{F}(\xi,\eta) = f(\xi,-\eta) \).

**Proposition 5.8.** There is a natural isomorphism between \( \phi^*K^{G}_{1,c,\tau}(A)_{vD} \) and \( \phi^*K^{G}_{1,c,\tau}(S^{1,1}A)_{vD} \).

**Proof.** Let \( \mathcal{T}_{0,R} \) be the Real reduced Toeplitz algebra \( C^*(s-1) \) where \( s \in \mathbb{R}(\ell^2_\mathbb{R}Z_{\geq 0}) \) is the unilateral shift operator \( (s\xi)_i := \xi_{i-1} \). Then, the boundary homomorphism \( \partial : K_{1,c,\tau}(S^{1,1}A)_{vD} \to K_{1,c,\tau}(A)_{vD} \) associated to the exact sequence

\[
0 \to A \hat{\otimes} K \to A \hat{\otimes} \mathcal{T}_{0,R} \to S^{0,1}A \to 0
\]

gives a natural homomorphism from \( \phi^*K^{G}_{1,c,\tau}(S^{1,1}A)_{vD} \) to \( \phi^*K^{G}_{1,c,\tau}(\omega)_{vD} \). Moreover, applying Proposition 4.3 of [Cun84] for the functor \( \phi^*K^{G}_{1,c,\tau}(A \hat{\otimes} \omega)_{vD} \), we obtain \( \phi^*K^{G}_{1,c,\tau}(A \hat{\otimes} \mathcal{T}_{0,R})_{vD} = 0 \) and consequently \( \partial \) is an isomorphism. \( \square \)

Let \( \phi^*K^{G}_{1+p-q,c,\tau}(A)_{vD} \) denote the group \( \phi^*K^{G}_{1,c,\tau}(S^{p,q}A)_{vD} \), which depends only on the difference \( p-q \). Together with Proposition 5.8, we obtain the following.

**Corollary 5.9.** Let \( 0 \to I \to A \to A/I \to 0 \) be an exact sequence of \( \phi \)-twisted \( G \)-\( C^* \)-algebras. Then, we obtain the long exact sequence

\[
\cdots \to \phi^*K^{G}_{2,c,\tau}(A/I)_{vD} \to \phi^*K^{G}_{1,c,\tau}(I)_{vD} \to \phi^*K^{G}_{1,c,\tau}(A)_{vD} \to \phi^*K^{G}_{1,c,\tau}(A/I)_{vD} \\
\to \phi^*K^{G}_{0,c,\tau}(A)_{vD} \to \phi^*K^{G}_{0,c,\tau}(A/I)_{vD} \to \phi^*K^{G}_{1,c,\tau}(I)_{vD} \to \cdots
\]

Finally we relate \( \phi^*K^{G}_{1,c,\tau}(\omega)_{vD} \) with Definition 4.3. For a \( \phi \)-twisted \( \mathbb{Z}_2 \)-graded \( G \)-\( C^* \)-algebra \( A \), let \( M_s(A) \) denote the stable multiplier algebra \( M(A \hat{\otimes} K_G) \) and let \( Q_s(A) \) denote the stable corona algebra \( M_s(A)/(A \hat{\otimes} K_G) \).

**Lemma 5.10.** Let \( A \) be a \( \phi \)-twisted \( \mathbb{Z}_2 \)-graded \( G \)-\( C^* \)-algebra. Then, we have \( \phi^*K^{G}_{s,c,\tau}(M_s(A))_{vD} = 0 \) for any \( s \in \mathbb{Z} \).

**Proof.** If suffices to show that \( [s] = 0 \) for any \( s \in \phi^*F^{G}_{c,V}(\mathbb{M}_{2,2}^\infty((S^{p,q}M_s(A))^+)) \).

Let \( V_n : \mathbb{R} \to \ell^2\mathbb{Z} \) be isometries onto \( C \cdot \delta_n \). We identify \( \mathcal{H}_{G_A} \otimes \ell^2\mathbb{Z}_{>0} \) and \( \mathcal{H}_{G_A} \otimes \ell^2\mathbb{Z}_{>0} \) with \( \mathcal{H}_{G_A} \) by a unitary equivalence. Since the bilateral shift operator is homotopic to the identity in \( U\mathbb{B}(\ell^2\mathbb{Z}) \), there is a homotopy

\[
\sum_{n<0} V_n s V_n^* + s + \sum_{n>0} V_n v_0 V_n^* \sim \sum_{n<0} V_n s V_n^* + v_0 + \sum_{n>0} V_n v_0 V_n^*
\]

in \( \phi^*F^{G}_{c,V}(\mathbb{M}_{2,2}^\infty((S^{p,q}M_s(A))^+)) \). This implies that

\[
\left\lfloor \sum_{n<0} V_n s V_n^* \right\rfloor + [s] + \left\lfloor \sum_{n>0} V_n v_0 V_n^* \right\rfloor = \left\lfloor \sum_{n<0} V_n s V_n^* \right\rfloor + [v_0] + \left\lfloor \sum_{n>0} V_n v_0 V_n^* \right\rfloor
\]

and hence \( [s] = [v_0] = 0 \) in \( \phi^*K^{G}_{1+p-q,c,\tau}(M_s(A)) \). \( \square \)

**Theorem 5.11.** There is a natural isomorphism \( \phi^*K^{G}_{0,c,\tau}(SA) \to \phi^*K^{G}_{1,c,\tau}(A)_{vD} \).
Proof. This proof is based on the idea commented in Section 2 of [Roc04]. First we claim that \( \phi \KK^G_{\infty}(\mathbb{R}, A) \) is isomorphic to the twisted equivariant K-group \( \KK^G_{1,c,r}(Q_s(A))_{\nu_D} \). Let \( s \) be an element in \( \phi \FK^G_{c,V}(Q_s(A)) \). When we choose a self-adjoint lift \( \hat{s} \) in \( \mathbb{M}_2(\mathbb{M}_2(A)) \sim K(V) \) of \( s \), then \( \hat{s} \) is a Kasparov \( R \)-\( A \) bimodule. The correspondence \( [s] \mapsto \hat{s} \) determines a well-defined group homomorphism \( \Phi_A : \phi \KK^G_{1,c,r}(Q_s(A))_{\nu_D} \rightarrow \phi \KK^G_{c,V}(\mathbb{R}, A)_{\nu_D} \).

Surjectivity of \( \Phi_A \) follows from the Kasparov stabilization theorem (3.3) and the fact that \( \hat{s} \) is a Kasparov bimodule \( E, 1, F \) such that \( \hat{s} \) is homotopy to a super \( G \)-invariant odd symmetry in the space of Fredholm operators in \( L(\hat{s}G_{c,A} \otimes E) \). By [3.3], we may replace \( E \) with \( \hat{s}G_{c,A} \). Hence the assumption means that \( [s] + [\pi(F)] \) is in the image of \( \pi_s : \phi \KK^G_{1,c,r}(M_s(A))_{\nu_D} \rightarrow \phi \KK^G_{c,V}(Q_s(A))_{\nu_D} \) and we obtain \( [s] = 0 \) by Lemma 5.10.

Consider the map between long exact sequences

\[
\begin{array}{ccccccc}
\bigoplus_{c} \mathbb{K}_G & \rightarrow & \bigoplus_{c} M_s(A) & \rightarrow & \bigoplus_{c} Q_s(A) & \rightarrow & \bigoplus_{c} \mathbb{K}_G \\
\bigoplus_{F} \bigoplus_{c} \mathbb{K}_G & \rightarrow & \bigoplus_{F} \bigoplus_{c} M_s(A) & \rightarrow & \bigoplus_{F} \bigoplus_{c} Q_s(A) & \rightarrow & \bigoplus_{F} \bigoplus_{c} \mathbb{K}_G
\end{array}
\]

and apply Lemma 5.10. Then we obtain that the canonical inclusion \( SQ_s(A) \rightarrow Q_s(SA) \) induces an isomorphism \( \phi \KK^G_{1,c,r}(Q_s(A))_{\nu_D} \cong \phi \KK^G_{1,c,r}(Q_s(SA))_{\nu_D} \).

Finally we obtain the isomorphism

\[
\Psi_A : \phi \KK^G_{c,V}(\mathbb{R}, SA) \xrightarrow{\phi s^{-1}} \phi \KK^G_{1,c,r}(Q_s(A))_{\nu_D} \xrightarrow{\partial} \phi \KK^G_{1,c,r}(A \otimes \mathbb{K}_G)_{\nu_D}
\]

such that all of these homomorphisms are natural. \( \square \)

As a consequence of Theorem 5.11 we obtain the isomorphism

\[
\beta_A := \Psi_A \circ \beta \circ \Psi_A^{-1} : \phi \KK^G_{1,c,r}(A)_{\nu_D} \rightarrow \phi \KK^G_{1,c,r}(SA \otimes C\ell_{0,1})_{\nu_D}.
\]

where \( \beta := [C_0(\mathbb{R}) \otimes C\ell_{0,1}, 1, x(1 + x^2)^{-1/2}e] \in \KK^G(\mathbb{R}, S\ell_{0,1}) \) is the Bott element. On the other hand, a natural isomorphism \( \Xi_A \) from \( \phi \KK^G_{1,c,r}(A)_{\nu_D} \) to \( \phi \KK^G_{1,c,r}(SA \otimes C\ell_{0,1})_{\nu_D} \) is given in Theorem 2.14 of [VD88] as \( \Xi_A[s] := [z(s,t)] \) where

\[
\begin{align*}
\beta(s,t) & := \Ad((1 + ve)/\sqrt{2}) \circ \Ad \nu(s,t) \circ \Ad \nu(v,-t)(e), \\
\nu(x,t) & := \cos(\pi t/2) + xe \sin(\pi t/2).
\end{align*}
\]

Actually, these isomorphisms coincide. To see this, let \( s \) be an odd \( c \)-graded \( G \)-invariant symmetry in \( Q_s(A) \otimes K(V) \) and let \( \hat{s} \) be a self-adjoint lift of \( s \) in
and the operator \( e \cos(\pi t) + \tilde{s} \sin(\pi t) \) satisfies (3.13) with respect to the Kasparov product \( [\mathcal{C}_G^{c,r}(A), 1, \tilde{s}] \otimes_{\mathbb{R}} \beta \). Consequently we obtain \( \Phi_{\mathcal{S}A \otimes \mathcal{C}_0,1}[\tilde{s}] = (\beta \otimes_{\mathbb{R}} (\Phi_A[\tilde{s}]), \) which implies \( \Xi = \beta_A \) by naturality of \( \Xi \).

Moreover, there is another canonical identification of these groups. When we write a c-graded \( G \)-invariant odd symmetry in \( (\mathcal{M}_{2,2}(A) \otimes \mathbb{K}(V)) \otimes \mathcal{C}_0,1 \) as \( s_0 + s_1 e \) where \( e \) is the odd self-adjoint generator of \( \mathcal{C}_0,1 \), then \( s_0 \) and \( s_1 \) are self-adjoint element such that \( s_0 \) is odd, \( s_1 \) is even and satisfy \( s_0 s_1 = s_1 s_0 \) and \( s_0^2 + s_1^2 = 1 \). Therefore, by Proposition 4.2, the correspondence \( (s_1 + i s_0) \mapsto (s_0 + e s_1) \) induces the natural isomorphism \( \Theta_A \) from \( \phi \mathcal{K}_0^G(A) \) to \( \phi \mathcal{K}_1^G(A \otimes \mathcal{C}_0,1)e \) (which turns out to be isomorphic to \( \phi \mathcal{K}_1^{c,r}(A \otimes \mathcal{C}_0,1) \) by Proposition 5.2).

**Lemma 5.12.** The isomorphisms \( \Psi_A \otimes \mathcal{C}_0,1 \circ \beta \) and \( \Theta_A \) coincide.

**Proof.** By the above argument, we have \( \Psi_A \otimes \mathcal{C}_0,1 \circ \beta = \partial \circ \beta_A \circ \Phi_A = \partial \circ \Xi_A \).

For a c-graded \( G \)-invariant odd Fredholm operator \( F \in M_2(A) \otimes \mathbb{K}(V) \), the composition \( \partial \circ \Xi_A \) is calculated in Proposition 3.4 of [VDS88] as

\[
\partial \circ \Xi_A[F] = [- \text{Ad} \left( \frac{1}{\sqrt{2}}(1 + ve) \right)(\sin \pi F + e \cos \pi F)].
\]

On the other hand, \( \Theta_A[- \exp(\pi i F)] = [- \text{Ad} \left( \frac{1}{\sqrt{2}}(1 + ve) \right)(\cos \pi F + i \sin \pi F)] \) by definition and Remark 5.3. \( \square \)

**Proposition 5.13.** There is an exact sequence

\[
\phi \mathcal{K}_1^{G,A}(A)_{c,t} \to \phi \mathcal{K}_1^{G,A}(A/I)_{c,t} \to \phi \mathcal{K}_0^{G,A}(A)
\]

were the boundary map given by \( \partial[s] = - \exp(\pi i \tilde{s}) \) where \( \tilde{s} \) is a self-adjoint element in \( A \) such that \( \pi(\tilde{s}) = s \).

**Proof.** Consider the following diagram

\[
\begin{array}{cccccc}
\phi \mathcal{K}_1^{G,A}(A)_{c,t} & \dashrightarrow & \phi \mathcal{K}_1^{G,A}(A/I)_{c,t} & \dashrightarrow & \phi \mathcal{K}_0^{G,A}(A) \\
\downarrow \beta_A & & \downarrow \beta_{A/I} & \uparrow \Theta^{-1} & \uparrow \Theta^{-1} \\
\phi \mathcal{K}_1^{G,A}(SA \otimes \mathcal{C}_0,1)_{c,t} & \dashrightarrow & \phi \mathcal{K}_1^{G,A}(SA/A \otimes \mathcal{C}_0,1)_{c,t} & \dashrightarrow & \phi \mathcal{K}_0^{G,A}(A \otimes \mathcal{C}_0,1)_{c,t} & \dashrightarrow & \phi \mathcal{K}_0^{G,A}(A \otimes \mathcal{C}_0,1)_{c,t} \end{array}
\]

where the second row is the long exact sequence associated with the exact sequence \( 0 \to I \otimes \mathcal{C}_0,1 \to A \otimes \mathcal{C}_0,1 \to A/I \otimes \mathcal{C}_0,1 \to 0 \).
Since $\Theta_I^{-1} \circ \partial \circ \beta_{Q_s(I)} = \beta^{-1}_e \circ S \Phi_I \circ c_{0,1} \circ \beta_{Q_s(I)} = \Phi_A$, we obtain $\Theta_I^{-1} \circ \partial \circ \beta_{A/I} = \Phi_I \circ \mu_\ast$ since the diagram

$$
\begin{array}{ccc}
\phi K^G_{1,c,\tau}(A/I)_{\nu D} & \longrightarrow & \phi K^G_{1,c,\tau}(I) \\
\mu_\ast & \downarrow & \\
\phi K^G_{1,c,\tau}(Q_s(I))_{\nu D} & \longrightarrow & \phi K^G_{1,c,\tau}(I)
\end{array}
$$

commutes. For an odd self-adjoint $c$-graded $G$-invariant symmetry $s \in \hat{M}_{2,2}(A/I) \otimes \mathbb{K}(V)$, we have $\Phi_I \circ \mu_\ast([s]) = [-\exp(\pi i \mu(s))]$ under the identification in Proposition [12] where $\tilde{s}$ is an odd self-adjoint $c$-graded $G$-invariant lift of $s$ in $\hat{M}_{2,2} A \otimes \mathbb{K}(V)$.

Let $A$ be a trivially graded unital $\phi$-twisted $G$-$C^*$-algebra. We say that an element $x$ in $A$ is $c$-twisted $G$-invariant if $\alpha_g(x) = (-1)^{c(g)}(x)$ for any $g \in G$. For a finite dimensional $(\phi, c, \tau)$-twisted unitary representation $V$, let $\phi T_{c,\gamma}(A)$ denote the space of $c$-twisted $G$-invariant symmetries in $A \otimes \mathbb{K}(V)$. Here we write $V^\circ$ for the unitary representation of $G$ obtained by forgetting the $\mathbb{Z}_2$-grading of $V$. In the same way as the definition of $\phi K^G_{c,\tau}(A)_c$, the inductive limit $\phi K^G_{0,c,\tau}(A) := \varinjlim_{\gamma} \pi_0 \phi T_{c,\gamma}(A)$ with respect to the inclusion $\phi T_{c,\gamma}(A) \subset \phi T_{c,\gamma \oplus I}(A)$ given by $s \mapsto s \oplus \gamma_W$ (here $\gamma_W$ is the grading operator of $W$) has a structure of abelian group. Similarly, let $\phi U_{c,\gamma}(A)$ denote the space of unitaries in $A \otimes \mathbb{K}(V)$ such that $\alpha_g(u) = u$ if $c(g) + \phi(g) = 0$ and $\alpha_g(u) = u^\ast$ if $c(g) + \phi(g) = 1$. Set $\phi K^G_{1,c,\tau}(A) := \varinjlim_{\gamma} \phi U_{c,\gamma}(A)$ with respect to the inclusion $\phi U_{c,\gamma}(A) \subset \phi U_{c,\gamma \oplus I}(A)$ given by $u \mapsto u \oplus 1_W$.

For a nonunital $\phi$-twisted $G$-$C^*$-algebra $A$, the $\phi K^G_{c,\tau}(A)$ is defined by the kernel of $\phi K^G_{c,\tau}(A^+) \to \phi K^G_{c,\tau}(C_b(G^0))$.

**Theorem 5.14.** Let $A$ be a trivially graded $\phi$-twisted $G$-$C^*$-algebra.

1. The group $\phi K^G_{0,c,\tau}(A)$ is isomorphic to $\phi K^G_{c,\tau}(A)_{\nu D}$.
2. The group $\phi K^G_{1,c,\tau}(A)$ is isomorphic to $\phi K^G_{0,c,\tau}(A)_{\nu D}$.
3. Let $0 \to I \to A \to A/I \to 0$ be an exact sequence of $\phi$-twisted trivially graded $C^*$-algebras. Then there is an exact sequence

$$
\phi K^G_{0,c,\tau}(A) \to \phi K^G_{0,c,\tau}(A/I) \overset{\partial}{\to} \phi K^G_{1,c,\tau}(I) \to \phi K^G_{1,c,\tau}(A)
$$

where the boundary map $\partial$ is given by $\partial[s] = [-\exp(\pi i \tilde{s})]$ where $\tilde{s}$ is a self-adjoint lift of $s$.

**Proof.** We may assume $A$ is unital. The isomorphism $\phi K^G_{c,\tau}(A \otimes C_{0,1})_c \cong \phi K^G_{0,c,\tau}(A)$ is induced from the bijection $\phi F_{c,\gamma}(A \otimes C_{0,1}) \to \phi T_{c,\gamma}(A)$ given by

$$
\begin{align*}
s &= s_0 e + s_1 & \mapsto & s' = s_0 \gamma_W + s_1.
\end{align*}
$$

Here, the decomposition $s' = s_0 \gamma + s_1$ is given by the $\pm 1$ component of the involution $\text{Ad} \gamma_W$. This map determines a well-defined bijection because $s_0 e + s_1$ is in $\phi F_{c,\gamma}(A \otimes C_{0,1})$ if and only if $s_0$ is even, $s_1$ is odd, $s_1^* = s_i$, $s_0 s_1 = s_1 s_0$, $s_0^2 + s_1^2 = 1$, $\alpha_g(s_0) = s_0$ and $\alpha_g(s_1) = (-1)^{c(g)} s_1$. 

The isomorphism (2) is also given in the same way as above. For a \( \mathbb{Z}_2 \)-graded \( \phi \)-twisted unital \( G \)-\( C^\ast \)-algebra \( A \), let \( \phi U_{c, V}(A) \) be the space of unitaries in \( A \otimes K(V) \) such that \( \gamma(u) = u^* \) and \( \alpha_g(u) = \gamma(\phi(g)+\phi(g)) \) for any unitary \( u \) and \( \alpha_g \). Then, by Proposition 6.12 and Lemma 5.12, \( \phi K^G_{0, c, \tau}(A) \cong \lim_{\gamma, \gamma} \pi_0 \phi U_{c, V}(A) \). Now, the isomorphism \( \phi K^G_{0, c, \tau}(A \otimes C_{\ell 0,1}) \) is applied by Thiang [Thi15] for the study of topological insulators in \( [\text{Kar08a}] \), which is applied by Thiang [Thi15] for the study of topological insulators. Now we generalize it for twisted equivariant K-theory for trivially \( \mathbb{Z}_2 \)-graded \( \phi \)-twisted unital \( G \)-\( C^\ast \)-algebras. A \( (\phi, \tau) \)-twisted \( (n, m) \)-Karoubi triple is a triplet \( (W, \Gamma_1, \Gamma_2) \) where \( W \) is a \( \mathbb{Z}_2 \)-equivariant Hilbert \( A \)-module with a unitary \( \phi \)-twisted \( G \)-equivariant \( * \)-homomorphism \( \varphi : C_{\ell n, m} \rightarrow L(W) \) and \( \Gamma_1, \Gamma_2 \) are symmetries on \( W \) which determine a \( \mathbb{Z}_2 \)-grading on \( W \) compatible with \( \varphi \) and the \( G \)-action. A Karoubi triple \( (W, \Gamma_1, \Gamma_2) \) is trivial if \( \Gamma_1 = \Gamma_2 \). Let \( \phi K^G_{(n, m), \tau}(A)_{\text{Kar}} \) denote the quotient of the semigroup of homotopy classes of Karoubi triples (product is given by the direct sum) by the subsemigroup of trivial triples.

**Corollary 5.15.** Let \( A \) be a Real \( G \)-\( C^\ast \)-algebra. Then, the group \( \phi K^G_{(n, m), \tau}(A)_{\text{Kar}} \) is isomorphic to \( \phi K^G_{m-n, \tau}(A) \). In particular, for a real twist \( (\phi_\mathbb{R}, 0, \tau_\mathbb{R}) \) of \( G \), \( \phi K^G_{(n, m), \tau}(A)_{\text{Kar}} \) is isomorphic to \( K\mathbb{R}_{m-n, \tau}(A) \).

**Proof.** Let \( G' = \mathbb{R}_{n, m} \) and \( (\phi, c', \tau') = (\phi, c_{n, m}, \tau_{n, m}) \) be as in Example 4.8. Now, we get a homomorphism

\[
F_A : \phi K^G_{(n, m), \tau}(A) \rightarrow \phi K^G_{(n, m), \tau}(A)_{\text{Kar}}
\]

mapping \( s \in \phi G'_{c', \tau'}(A) \) to \([V \otimes A, s, \gamma]\).

For a \( (n, m) \)-Karoubi triple \( [W, \Gamma_1, \Gamma_2] \), we write \( W_2 \) for the \( (\phi, c', \tau') \)-twisted \( G' \)-equivariant Hilbert \( A \)-module \( W \) with the \( \mathbb{Z}_2 \)-grading \( \Gamma_2 \). It is embedded in \( \phi G'_{c', \tau'}(A) \) by (3.8). Since \( W \) is \( \mathbb{Z}_2 \)-graded \( \phi \)-twisted \( G' \)-equivariant Hilbert \( A \)-module, the projection \( P \) onto \( W_2 \) is compact by Lemma 6.3 of [Kar80] and hence \( p_n P p_n \rightarrow P \). Consequently, \( W_2 \) is embedded into \( p_n(\phi G'_{c', \tau'}(A) = A \otimes V \), where \( V := p_n(\phi G'_{c', \tau'}(A) \). It immediately implies that \( F_A \) is surjective. Moreover, we can check injectivity by applying this argument for degenerate Karoubi triples and Hilbert \( A \otimes C[0,1] \)-modules given by homotopies of Karoubi triples.
Now, the conclusion follows from Corollary 4.11 and Theorem 5.14.

The following corollary immediately follows from Theorem C orollary 4.12 and Theorem 5.14.

**Corollary 5.16.** Let \((G, \phi, c, \tau)\) be a CT-type symmetry as in Example 2.6 whose CT-type corresponds to \(KF_n\) as indicated in Table 1. Let \(A\) be a (trivially graded) Real \(G_0\)-\(C^*\)-algebra. Then;

1. The group \(\phi K^G_{0,c,\tau}(A)\) is isomorphic to \(KF_n^G(A)\).
2. The group \(\phi K^G_{-1,c,\tau}(A)\) is isomorphic to \(KF^G_{n-1}(A)\).
3. Let \(0 \to I \to A \to A/I \to 0\) be an exact sequence of \(\phi\)-twisted trivially graded \(C^*\)-algebras. Then there is an exact sequence

\[
\phi K^G_{0,c,\tau}(A) \to \phi K^G_{0,c,\tau}(A/I) \xrightarrow{\partial} \phi K^G_{-1,c,\tau}(I) \to \phi K^G_{-1,c,\tau}(A)
\]

where the boundary map \(\partial\) is given by \(\partial[s] = [-\exp(\pi i \tilde{s})]\) where \(\tilde{s}\) is a self-adjoint lift of \(s\).

**Example 5.17.** Let \(A\) be a Real \(C^*\)-algebra and let

\[
0 \to A \otimes \mathbb{K} \to A \otimes \mathcal{T}_\mathbb{R} \to A \otimes C(T^{0,1}) \to 0
\]

be the Toeplitz extension associated to \(A\). Then, the boundary homomorphism \(\partial = -\exp(\pi i\tilde{s})\) from \(\phi K^A_{0,c,\tau}(A \otimes C(T^{0,1})) \cong KF_n(A) \oplus KF_{n-1}(A)\) to \(\phi K^A_{0,c,\tau}(A) \cong KF_{n-1}(A)\) is given by \(0 \oplus \text{id}_{KF_{n-1}(A)}\).

In the theory of topological insulators and topological superconductors, gapped Hamiltonians of \(d\)-dimensional quantum systems with the CT symmetry \((A, \tau)\) are classified by the group \(\phi K^A_{d,c,\tau}(C(T^{0,d}))\) and the corresponding edge Hamiltonians are classified by \(\phi K^A_{1,c,\tau}(C(T^{0,d-1}))\). The homomorphism \(\partial\) gives a mathematical proof of the bulk-edge correspondence (cf. [BCR15]). The use of twisted equivariant K-theory for the study of topological phases is discussed in [Kub15].

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**References**

[AS04] Michael Atiyah and Graeme Segal, *Twisted K-theory*, Ukr. Mat. Visn. **1**(2004), no. 3, 287–330.

[Ati89] M. F. Atiyah. *K-theory*, Second, Advanced Book Classics, Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, ISBN 0-201-09394-4, (1989). Notes by D. W. Anderson.

[BCR15] Chris Bourne, Alan L. Carey, and Adam Rennie, *A noncommutative framework for topological insulators*, preprint, arXiv:1509.07210[math-ph], 2015.

[BGR77] Lawrence G. Brown, Philip Green, and Marc A. Rieffel, *Stable isomorphism and strong Morita equivalence of C*-algebras*, Pacific J. Math. **71** (1977), no. 2, 349–363.

[Bla98] Bruce Blackadar. *K-theory for operator algebras*, Second, Mathematical Sciences Research Institute Publications, vol. 5, Cambridge University Press, Cambridge, ISBN 0-521-63532-2, (1998).
NOTES ON TWISTED EQUIVARIANT K-THEORY FOR C*-ALGEBRAS

[CE01] Jérôme Chabert and Siegfried Echterhoff, Twisted equivariant $KK$-theory and the Baum-Connes conjecture for group extensions, $K$-Theory 23 (2001), no. 2, 157–200.

[Cun84] Joachim Cuntz, $K$-theory and $C^*$-algebras, Algebraic $K$-theory, number theory, geometry and analysis (Bielefeld, 1982), 1984, pp. 55–79, Springer, Berlin.

[Dix77] Jacques Dixmier, $C^*$-algebras, North-Holland Publishing Co., Amsterdam-New York-Oxford, ISBN 0-7204-0762-1, (1977). Translated from the French by Francis Jellett, North-Holland Mathematical Library, Vol. 15.

[DK70] P. Donovan and M. Karoubi, Graded Brauer groups and $K$-theory with local coefficients, Inst. Hautes Études Sci. Publ. Math. 38 (1970), 5–25.

[EM09] Heath Emerson and Ralf Meyer, Equivariant representable $K$-theory, J. Topol. 2 (2009), no. 1, 123–156.

[FD88] J. M. G. Fell and R. S. Doran. Representations of $*$-algebras, locally compact groups, and Banach $*$-algebraic bundles. Vol. 1, Pure and Applied Mathematics, vol. 125, Academic Press, Inc., Boston, MA, ISBN 0-12-252721-6, (1988).

[FG15] Carla Farsi and Elizabeth Gillaspy, Twists over Renault-Deaconu groupoids and twisted vector bundles, preprint, arXiv:1505.00364 [math.OA], 2015.

[FH11] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman, Loop groups and twisted $K$-theory I, J. Topol. 4 (2011), no. 4, 737–798.

[FM13] Daniel S. Freed and Gregory W. Moore, Twisted equivariant matter, Ann. Henri Poincaré 14 (2013), no. 8, 1927–2023.

[Gro95] Alexander Grothendieck, Le groupe de Brauer. I. Algèbres d’Azumaya et interprétations diverses, Séminaire Bourbaki, Vol. 9, 1995, pp. Exp. No. 290, 199–219, Soc. Math. France, Paris.

[HK97] Nigel Higson and Gennadi Kasparov, Operator $K$-theory for groups which act properly and isometrically on Hilbert space, Electron. Res. Announc. Amer. Math. Soc. 3 (1997), 131–142 (electronic).

[HR98] Jacob v. B. Hjelmborg and Mikael Rørdam, On stability of $C^*$-algebras, J. Funct. Anal. 155 (1998), no. 1, 153–170.

[HS87] Michel Hilsum and Georges Skandalis, Morphismes $K$-orientés d’espaces de feuilles et fonctorialité en théorie de Kasparov (d’après une conjecture d’A. Connes), Ann. Sci. École Norm. Sup. (4) 20 (1987), no. 3, 325–390.

[Jul81] Pierre Julg, $K$-théorie équivariante et produits croisés, C. R. Acad. Sci. Paris Sér. I Math. 292 (1981), no. 13, 629–632.

[Kar08a] Max Karoubi. $K$-theory: An introduction, Classics in Mathematics, Springer-Verlag, Berlin, ISBN 978-3-540-79889-7, (2008).

[Kar08b] Max Karoubi. Twisted $K$-theory—old and new, $K$-theory and noncommutative geometry, 2008, pp. 117–149, Eur. Math. Soc., Zürich.

[Kar68] Max Karoubi. Algèbres de Clifford et $K$-théorie, Ann. Sci. École Norm. Sup. (4) 1 (1968), 161–270.

[Kas80] G. G. Kasparov, The operator $K$-functor and extensions of $C^*$-algebras, Izv. Akad. Nauk SSSR Ser. Mat. 44 (1980), no. 3, 571–636, 719.

[Kas88] G. G. Kasparov, Equivariant $KK$-theory and the Novikov conjecture, Invent. Math. 91 (1988), no. 1, 147–201.

[Kei15] Johannes Kellendonk, On the $C^*$-algebraic approach to topological phases for insulators, preprint, arXiv:1509.06271 [math.KT], 2015.

[Ki09] Alexei Kitaev, Periodic table for topological insulators and superconductors, AIP Conference Proceedings 1134 (2009), no. 1.

[KMRW98] Alexander Kumjian, Paul S. Muhly, Jean N. Renault, and Dana P. Williams, The Brauer group of a locally compact groupoid, Amer. J. Math. 120 (1998), no. 5, 901–954.

[KS04] Mahmood Khoshkam and Georges Skandalis, Crossed products of $C^*$-algebras by groupoids and inverse semigroups, J. Operator Theory 51 (2004), no. 2, 255–279.
Y. Kubota, *Controlled topological phases and the bulk-edge correspondence*, preprint [arXiv:1511.05314[math-ph]](https://arxiv.org/abs/1511.05314), 2015.

E. C. Lance, *Hilbert $C^*$-modules, A toolkit for operator algebras*, London Mathematical Society Lecture Note Series, vol. 210, Cambridge University Press, Cambridge, ISBN 0-521-47910-X, (1995).

Pierre-Yves Le Gall, *Théorie de Kasparov équivariante et groupoïdes. I, K-Theory* 16 (1999), no. 4, 361–390.

El-kâïoum M. Moutuou, *Equivariant KK-theory for generalised actions and Thom isomorphism in groupoid twisted K-theory*, J. K-Theory 13 (2014), no. 1, 83–113.

Alan L. T. Paterson, *The stabilization theorem for proper groupoids*, Houston J. Math. 38 (2012), no. 1, 245–264.

Jean Renault. *A groupoid approach to $C^*$-algebras*, Lecture Notes in Mathematics, vol. 793, Springer, Berlin, ISBN 3-540-09977-8, (1980).

M. Rørdam, F. Larsen, and N. Laustsen. *An introduction to K-theory for $C^*$-algebras*, London Mathematical Society Student Texts, vol. 49, Cambridge University Press, Cambridge, ISBN 0-521-78334-8; 0-521-78944-3, (2000).

John Roe, *Paschke duality for real and graded $C^*$-algebras*, Q. J. Math. 55 (2004), no. 3, 325–331.

Jonathan Rosenberg, *Continuous-trace algebras from the bundle theoretic point of view*, J. Austral. Math. Soc. Ser. A 47 (1989), no. 3, 368–381.

Georges Skandalis, *Some remarks on Kasparov theory*, J. Funct. Anal. 184 (1984), no. 3, 337–347.

Guo Chuan Thiang, *On the K-theoretic classification of topological phases of matter*, Ann. Henri Poincaré (2015), 1–38.

Jody Trout, *On graded K-theory, elliptic operators and the functional calculus*, Illinois J. Math. 44 (2000), no. 2, 294–309.

Jean-Louis Tu, *Groupoid cohomology and extensions*, Trans. Amer. Math. Soc. 358 (2006), no. 11, 4721–4747 (electronic).

Jean-Louis Tu, *La conjecture de Novikov pour les feuilletages hyperboliques*, K-Theory 16 (1999), no. 2, 129–184.

Jean-Louis Tu, Ping Xu, and Camille Laurent-Gengoux, *Twisted K-theory of differentiable stacks*, Ann. Sci. École Norm. Sup. (4) 37 (2004), no. 6, 841–910.

A. Van Daele, *K-theory for graded Banach algebras. I*, Quart. J. Math. Oxford Ser. (2) 39 (1988), no. 154, 185–199.

A. Van Daele, *K-theory for graded Banach algebras. II*, Pacific J. Math. 134 (1988), no. 2, 377–392.

Eugene P. Wigner. *Group theory and its application to the quantum mechanics of atomic spectra*, Expanded and improved ed. Translated from the German by J. J. Griffin. Pure and Applied Physics. Vol. 5, Academic Press, New York-London, (1959).