Non-Equilibrium Steady State of the Lieb-Liniger model: exact treatment of the Tonks Girardeau limit

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Aiming at studying the emergence of Non-Equilibrium Steady States (NESS) in quantum integrable models by means of an exact analytical method, we focus on the Tonks-Girardeau or hard-core boson limit of the Lieb-Liniger model. We consider the abrupt expansion of a gas from one half to the entire confining box, a prototypical case of inhomogeneous quench, also known as “geometric quench”. Based on the exact calculation of quench overlaps, we develop an analytical method for the derivation of the NESS by rigorously treating the thermodynamic and large time and distance limit. Our method is based on complex analysis tools for the derivation of the asymptotics of the many-body wavefunction, does not make essential use of the effectively non-interacting character of the hard-core boson gas and is sufficiently robust for generalisation to the genuinely interacting case.
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I. INTRODUCTION

The derivation of macroscopic physical laws that govern non-equilibrium phenomena from the underlying microscopic particle interactions described by quantum physics is one of the fundamental open questions of statistical and mathematical physics. In this quest the problem of quantum transport between two halves of a system, initially at equilibrium with different thermodynamic parameters and then joined together, is of central interest. Mathematical physics studies [1–7] followed by an increasing number of theoretical physics works [8–16] supported by the development of powerful numerical techniques for the simulation of quantum dynamics [17–24] and experimental advances in the observation of quantum many-body dynamics [25–28] have kept the attention on this problem alive and growing for decades.

In more general terms this physical protocol, more recently known also as “inhomogeneous quantum quench”, refers to the dynamics of an extended closed quantum system that is initially prepared in a macroscopically inhomogeneous state and let to evolve under a homogeneous Hamiltonian. A typical case of an inhomogeneous initial state is the one corresponding to the partitioning protocol, outlined above, where a system is initially split in two disconnected halves at thermal equilibrium and different temperature or particle density [8, 11, 12]. Other suitable choices include equilibrium (ground or thermal) states with a “local temperature” (or particle or energy density) profile varying from one value in the left half to a different one in the right, in either a smooth or a sharp way [29–31], or quantum gas expansion protocols, also called geometric quenches, in which a system of atoms expands from a smaller to a larger box [32].

A first understanding of quantum transport following an inhomogeneous quench can be drawn from the study of non-interacting systems [3, 9, 10, 13, 14, 33–35]. For such systems it has been shown that a Non-Equilibrium Steady State (NESS) emerges at large times, which provides a complete description of the asymptotic values of local observables, like the particle density and current. However, while results for non-interacting systems have been rigorously derived and known for years, the study of the effects of interactions, which are obviously expected to play a significant role in transport, turned out to be a hard problem and progress was slow. A breakthrough came recently from the study of integrable interacting systems [36, 37], which are characterised by the presence of an infinite number of local conservation laws. It was postulated that a Generalised Hydrodynamics (GHD) theory, based upon an infinite set of continuity equations associated to the conservation laws, provides an exact description of the NESS in such systems. The predictions of this conjecture have been tested numerically with great success both in spin chains and in quantum gas models solvable by Bethe Ansatz [38–42] and have been verified even experimentally [43]. Moreover it has paved the way for interesting extensions and applications [38–41, 44–52], as it essentially reduces the characterisation of macroscopic properties resulting from the complicated dynamics of quantum many-body systems to a relatively simple semiclassical description. At this point it is worth to recognise the important role played by integrability in understanding the physics of the quantum many-body problem in general, especially in the field of out-of-equilibrium dynamics [12, 53–60]. In contrast to non-interacting models and perturbative techniques that for many decades dominated any attempt to understand quantum statistical mechanics, in recent years integrable models have emerged as a powerful tool to unveil nontrivial and non-perturbative effects of interactions that are nowhere seen in
non-interacting models and cannot always be studied satisfactorily by non-interacting approximations or perturbative methods.

A rigorous proof or at least a better understanding of the mathematical reasons that explain the emergence of GHD are highly desirable for many reasons. First, even though the perfect agreement with numerical simulations and experimental verification of its predictions leave little space for doubts on its validity, GHD is still a conjecture and a proof would enable us to identify possible exceptional special cases. Second, the study of diffusion and even more of deviations from standard diffusive behaviour, a subject of central interest in quantum transport physics, can be based on next-to-leading order asymptotics (beyond Euler scale) of the dynamics after inhomogeneous quenches and therefore it would benefit essentially from exact results that would be intermediate steps of a proof of GHD. Such results would be useful for the verification of important results on diffusion that have already been derived from extensions of GHD [61–64]. Third, proving GHD can be an important step towards solving the more general problem of quantum many-body dynamics in integrable systems (and possibly beyond). In particular, it could pave the way for rigorously proving the “Quench Action” method[56, 65], which is the basis of GHD, or even extending it in such a way that it can be used also for the study of intermediate dynamics.

For these and many other reasons, several approaches to explain GHD have been developed. Most of these have focused on the derivation of a formula giving the expectation value of current operators in generalised Gibbs ensembles, on which GHD is crucially based [36, 66–69]. A general proof of this formula for quantum spin chains is now available based on algebraic Bethe Ansatz techniques [66, 68]. Another method could be based on form factor expansions [70–73] possibly by a suitable generalisation to the case of equilibrium states.

Starting with the case of effectively non-interacting dynamics presented in this work, we aim to develop a different approach: based on an exact analytical derivation of the asymptotics of local observables after a partitioning quench, we wish to demonstrate the emergence of the NESS dependent on the distance over time (ray) ratio and to verify a rigorous proof or at least a better understanding of the mathematical reasons that explain the emergence of the quench action: based on an exact analytical derivation of the asymptotics of local observables after a partitioning quench, we wish to demonstrate the emergence of the NESS dependent on the distance over time (ray) ratio and to verify a rigorous proof or at least a better understanding of the mathematical reasons that explain the emergence of the quench action.

Third, proving GHD can be an important step towards solving the more general problem of quantum many-body dynamics in integrable systems (and possibly beyond). In particular, it could pave the way for rigorously proving the “Quench Action” method[56, 65], which is the basis of GHD, or even extending it in such a way that it can be used also for the study of intermediate dynamics.

The paper is organised as follows. We first describe the mathematical problem: the model and quench protocol, as well as our objectives (sec. II). In an interlude serving as a warm-up example, we solve the single particle problem (sec. III), stressing the mathematical ideas that lead to the result and that will be later generalised to the many particle problem. Passing to the main part, we study the NESS for the Tonks-Girardeau gas expansion (sec. IV), avoiding use of free fermion techniques. Lastly, we discuss the main steps and essential ingredients of the method, evaluating the potential for generalisation to the genuinely interacting case (sec. V).

II. MODEL AND QUENCH PROTOCOL

The Lieb-Liniger model describes a one-dimensional Bose gas with point-like interactions. Its Hamiltonian is

$$H_{LL} = \int_{-L/2}^{+L/2} dx \left[ -\Psi^\dagger(x) \frac{\partial^2}{\partial x^2} \Psi(x) + c \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x) \right]$$

where $c$ is the interaction strength and the particle mass has been set to $m = 1/2$. We assume periodic boundary conditions $\Psi(+L/2) = \Psi(-L/2)$. The particle number and momentum operators are respectively

$$N = \int_{-L/2}^{+L/2} dx \Psi^\dagger(x) \Psi(x)$$
$$P = -i \int_{-L/2}^{+L/2} dx \Psi^\dagger(x) \frac{\partial}{\partial x} \Psi(x)$$

The Lieb-Liniger model is integrable i.e. its eigenstates are given by the Bethe Ansatz for any value of the interaction $c$. In the Tonks-Girardeau or hard-core boson limit $c \to \infty$ that we shall consider in this work, the Bethe states and Bethe equations simplify dramatically and the system becomes effectively equivalent to a system of free fermions.
This equivalence can be expressed also algebraically through a Jordan-Wigner-like transformation relating the boson operators $\Psi$ to hard-core boson operators $\Psi_{HB}$, which anti-commute under exchange of their positions, except for coinciding positions. In terms of these operators the Hamiltonian becomes non-interacting

$$H_{TG} = \int_{-L/2}^{+L/2} dx \left[ -\frac{d^2}{dx^2} \Psi_{HB}(x) \right]$$

The geometric quench problem we shall study is described as follows. We consider a gas of $N$ particles, initially restricted in an interval of length $L/2$, let us say in the right half of the system. We assume that the system lies in an equilibrium state $\rho_0$ of the (Lieb-Liniger or Tonks-Girardeau) Hamiltonian restricted in this interval with periodic boundary conditions $\Psi(+L/2) = \Psi(0)$.

We do not impose any further conditions on the initial state: it can be either the ground state $|\Phi_{GS,0}\rangle$, an excited state $|\Phi_0(\mu)\rangle$, a thermal state $e^{-\beta H_0}$ or other diagonal ensemble of eigenstates, i.e. it is described by a density matrix of the form

$$\hat{\rho}_0 = Z_0^{-1} \sum_{\mu:BA_0} \rho_0(\mu) |\Phi_0(\mu)\rangle \langle \Phi_0(\mu)|$$

where $|\Phi_0(\lambda)\rangle$ are eigenstates of $H_0$ with rapidities $\lambda$ satisfying Bethe equations corresponding to system size $L/2$ (BA$_0$) and

$$Z_0 \equiv \sum_{\mu:BA_0} \rho_0(\mu) |\Phi_0(\mu)\rangle \langle \Phi_0(\mu)|$$

so that $\hat{\rho}_0$ satisfies the normalisation condition

$$\text{Tr} \hat{\rho}_0 = \sum_{\mu:BA_0} \langle \Phi_0(\mu)|\hat{\rho}_0|\Phi_0(\mu)\rangle = 1$$

We then let the initial state evolve in time under the Hamiltonian $\hat{H}$ which is the same as $H_0$ except that it is defined in an interval of length $L$, again with periodic boundary conditions $\Psi(+L/2) = \Psi(-L/2)$.

Our goal is to calculate the asymptotics of local observables $\hat{O}(x, t)$ in the thermodynamic limit where both $N$ and $L$ tend to infinity with fixed non-zero average density $N/L = n$ and at large times $t$ and distances $x$ keeping the ratio $x/t$ fixed. We do not impose constraints on the observables $\hat{O}$: they may be single-point observables at position $x$ or multi-point correlation functions at positions $x + r_1, x + r_2, \ldots, x + r_n$ at fixed separations $r_i$. The order of limits we consider is first the thermodynamic limit and then the large time and distance limit. The quantity we wish to calculate is therefore

$$\lim_{x,t \to \infty} \lim_{x/t = \xi \text{ fixed}} \langle \hat{O}(x, t) \rangle = \lim_{x,t \to \infty} \lim_{x/t = \xi \text{ fixed}} N, L \to \infty \text{ fixed} \quad \text{Tr} \left\{ \hat{\rho}_0 e^{+i\hat{H}_t} e^{-iP_x} \hat{O} e^{+iP_x} e^{-i\hat{H}_t} \right\}$$

where $\hat{O} \equiv \hat{O}(0, 0)$.

We wish to show that a Non-Equilibrium Steady State (NESS) emerges in this double limit and to provide a complete description of this state. This means that our first objective is to show that in the above limit the observable can be expressed as an expectation value in a statistical ensemble $\hat{\rho}_{\text{NESS}}$ that is diagonal in the post-quench eigenstate basis, i.e. of the form

$$\hat{\rho}_{\text{NESS}} = Z_{\text{NESS}}^{-1} \sum_{\lambda:BA} \rho_{\text{NESS}}(\lambda) |\Phi(\lambda)\rangle \langle \Phi(\lambda)|$$

where $|\Phi(\lambda)\rangle$ are the eigenstates of $H$ with rapidities $\lambda$ satisfying the Bethe equations corresponding to system size $L$ (BA). The state $\hat{\rho}_{\text{NESS}}$ will be different for each space-time ray, for which reason we will denote it as $\hat{\rho}_{\text{NESS}, \xi}$ to show that it depends explicitly on the ratio $\xi = x/t$. Therefore what we wish to show is that

$$\lim_{x,t \to \infty} \lim_{x/t = \xi \text{ fixed}} N, L \to \infty \text{ fixed} \langle \hat{O}(x, t) \rangle = \lim_{x,t \to \infty} \lim_{x/t = \xi \text{ fixed}} N, L \to \infty \text{ fixed} \quad \text{Tr} \left\{ \hat{\rho}_{\text{NESS}, \xi} \hat{O} \right\}$$

(3)
where the expectation value is

$$\text{Tr} \left\{ \hat{\rho}_{\text{NESS}, \xi} \hat{O} \right\} = Z_{\text{NESS}}^{-1} \sum_{\lambda: BA} \rho_{\text{NESS}}(\lambda; \xi) \langle \Phi(\lambda) | \hat{O} | \Phi(\lambda) \rangle$$

The second objective is to determine this state in terms of the initial state and the space-time ratio $\xi$, i.e. to express $\rho_{\text{NESS}}(\lambda; \xi)$ as a function of $\rho_0(\lambda)$ and $\xi$.

Expanding in the pre- and post-quench eigenstate bases, we can write the time evolution of local observables in (1) as

$$\langle \hat{O}(x, t) \rangle = Z_0^{-1} \text{Tr} \left\{ \sum_{\mu: BA_0} \rho_0(\mu) \frac{\langle \Phi_0(\mu) | \hat{O} | \Phi_0(\mu) \rangle}{\langle \Phi_0(\mu) | \Phi_0(\mu) \rangle} \sum_{\lambda, \lambda': BA} \frac{\langle \Phi(\lambda') | \hat{O} | \Phi(\lambda) \rangle}{\langle \Phi(\lambda') | \Phi(\lambda') \rangle} e^{+iHt} e^{-iP_{x} \hat{O}} e^{+iP_{x} \hat{O}} e^{-iHt} \right\} = Z_0^{-1} \sum_{\mu: BA} \rho_0(\mu) \frac{\langle \Phi(\lambda') | \hat{O} | \Phi(\lambda) \rangle}{\langle \Phi(\lambda') | \Phi(\lambda') \rangle} \frac{M^{*}(\lambda'; \mu) M(\lambda; \mu)}{\mathcal{N}_0(\mu) \mathcal{N}(\lambda) \mathcal{N}(\lambda')} \langle \Phi(\lambda') | \hat{O} | \Phi(\lambda) \rangle$$

where we defined the norms of post- and pre-quench eigenstates as

$$\mathcal{N}(\lambda) \equiv \langle \Phi(\lambda) | \Phi(\lambda) \rangle$$

$$\mathcal{N}_0(\lambda) \equiv \langle \Phi_0(\lambda) | \Phi_0(\lambda) \rangle$$

since in their standard form the Bethe Ansatz eigenstates are not automatically normalised, and their overlaps as

$$\mathcal{M}(\lambda; \mu) \equiv \langle \Phi(\lambda) | \Phi(\mu) \rangle$$

Since we expect that the result we wish to show is valid for a general local observable and a general initial state diagonal in the pre-quench basis, the proof should not generally rely on special properties of the matrix elements $\langle \Phi(\lambda') | \hat{O} | \Phi(\lambda) \rangle$ or of the amplitudes $\rho_0(\mu)$. Motivated by this observation, the strategy we shall pursue is to analyse the asymptotics of any pre-quench eigenstate time evolved under the post-quench Hamiltonian. More specifically, we shall consider the quantity

$$e^{+iP_{x} - iHt} | \Phi_0(\mu) \rangle = \sum_{\lambda: BA} e^{+iP_{x} - iE(\lambda)t} \frac{\langle \Phi(\lambda) | \Phi(\lambda) \rangle}{\langle \Phi(\lambda) | \Phi(\lambda) \rangle} | \Phi_0(\mu) \rangle$$

$$= \sum_{\lambda: BA} e^{+iP_{x} - iE(\lambda)t} \mathcal{M}(\lambda; \mu) \mathcal{N}(\lambda) | \Phi(\lambda) \rangle$$

where we have introduced a resolution of the identity in the post-quench basis. What we aim to show is that, as long as coordinate space wave-functions are concerned, the following asymptotic relation holds

$$\lim_{x / t \to \xi, \xi \to \infty} \lim_{n: \text{fixed}} e^{+iP_{x} - iHt} | \Phi_0(\mu) \rangle \sim \lim_{N \to \infty} \lim_{n: \text{fixed}} | F(\mu; x / t) e^{+iP_{x} - iE(\mu)t} | \Phi(\mu) \rangle$$

that is, any pre-quench eigenstate time-evolved by $t$ and space-translated by $x$ can be replaced in the thermodynamics limit by a post-quench eigenstate with the same rapidity distribution. The equivalence in the above equation is meant to hold asymptotically in the double limit, as long as the operator or state that acts from the left is local.

If this equation holds then (4) gives

$$\lim_{x / t \to \xi, \xi \to \infty} \lim_{n: \text{fixed}} \langle \hat{O}(x, t) \rangle = \lim_{n: \text{fixed}} \sum_{\mu: BA} \frac{\rho_0(\mu)}{\mathcal{N}_0(\mu)} | F(\mu; x / t) |^2 \langle \Phi(\mu) | \hat{O} | \Phi(\mu) \rangle$$

which is indeed of the form (3) with the NESS given by

$$\frac{\rho_{\text{NESS}}(\lambda; x / t)}{Z_{\text{NESS}}} = \frac{\rho_0(\lambda)}{\mathcal{Z}_0 \mathcal{N}_0(\lambda)} | F(\lambda; x / t) |^2$$

therefore our goal will have been achieved.
III. SINGLE PARTICLE PROBLEM

Before we study the problem of derivation of the NESS in the Tonks-Girardeau limit of the Lieb-Liniger model, it is instructive to study the single particle problem first, which offers a pedagogic exposition of the necessary mathematical tools. The method presented here is essentially the same as that of [77] applied to a similar problem.

We consider a particle in a box of length $L/2$, initially lying at one of the eigenstates $|\Phi_0(q)\rangle$ of the Hamiltonian $H_0 = -\partial_x^2$ with periodic boundary conditions at the points $x = 0$ and $L/2$. At $t = 0$ we extend the length of the box from $L/2$ to $L$ and let the initial state evolve under the Hamiltonian $H = -\partial_x^2$ with periodic boundary conditions at the points $x = -L/2$ and $+L/2$. Our strategy for the derivation of the NESS is based on the study of the asymptotics of the time evolved coordinate space wave-function of an arbitrary pre-quench eigenstate $|\Phi_0(q)\rangle$.

The coordinate space wave-functions of pre-quench eigenstates are

$$ \langle x | \Phi_0(q) \rangle = e^{iqx} \Theta(x) $$

with the quantised momenta $q$ given by

$$ q = 2\pi n_0/(L/2) = 4\pi n_0/L, \quad n_0 \in \mathbb{Z} \quad (7) $$

while the post-quench eigenstates are

$$ \langle x | \Phi(k) \rangle = e^{ikx} $$

with the quantised momenta $k$ given by

$$ k = 2\pi n/L, \quad n \in \mathbb{Z} \quad (8) $$

The norms of pre-quench eigenstates $N_0(q)$ and of post-quench eigenstates $N(k)$ are

$$ N_0(q) = L/2 $$
$$ N(k) = L $$

respectively.

We are interested in the asymptotic form of the wavefunction in the infinite length limit $L \to \infty$ at large distance $x$ and time $t$ with fixed ratio $\xi = x/t$. More specifically, we will calculate the asymptotics of the quantity

$$ \mathcal{K}(q; z; x, t) \equiv \langle z | e^{iPx - iHt} | \Phi_0(q) \rangle \quad (9) $$

where $P = -i\partial_x$ is the momentum operator, which generates space translations. Knowing $\mathcal{K}(q; z; x, t)$ we can derive the asymptotics of any local observable in any initial state that is diagonal in the pre-quench basis.

Introducing a resolution of the identity in the post-quench basis we have

$$ e^{iPx - iHt} | \Phi_0(q) \rangle = \sum_{k = 2\pi n_0/L \atop n \in \mathbb{Z}} e^{iP(k)x - iE(k)t} \frac{\langle \Phi(k) | \Phi(k) \rangle | \Phi_0(q) \rangle}{\langle \Phi(k) | \Phi(k) \rangle} \quad (10) $$

where

$$ \mathcal{M}(k; q) \equiv \langle \Phi(k) | \Phi_0(q) \rangle $$
$$ \mathcal{N}(k) \equiv \langle \Phi(k) | \Phi(k) \rangle $$

are the overlaps between pre- and post-quench eigenstates and the norm square of the latter respectively, and

$$ E(k) = k^2 \quad (11) $$
$$ P(k) = k \quad (12) $$

are the energy and momentum eigenvalues.
We therefore write conditions. But we could also see it as an independent discrete variable and consider
we can write each term of the sum as an integral over \( k \)

\[
M(s) = \int_0^{L/2} dx \ e^{i(q-k)x} = \frac{i}{k-q} (e^{i(q-k)L/2} - 1)
\]

Using the pre- and post-quench quantisation conditions (7) and (8), we find that the overlaps acquire a different form as functions of the momenta in the even and odd \( n \) sectors

\[
\mathcal{M}(k; q) = \begin{cases} 
\frac{(L/2)\delta_{n,2n_0}}{(L/2)\delta_{k,q}} & \text{if } n = \frac{kL}{2\pi} \text{ is even} \\
-2i(L/2\pi)/(n-2n_0) = -2i/(k-q) & \text{if } n \text{ is odd}
\end{cases}
\]

This means that in the thermodynamic limit \( L \to \infty \) we should integrate separately in the two sectors. It is convenient to redefine the overlap \( \mathcal{M}(k; q) \) as a function of an additional index \( s \) taking values +1 if \( n \) is even and −1 if \( n \) is odd. Note that the index \( s \) is itself a function of the discrete momentum variable \( k \) which is a solution of the quantisation conditions. But we could also see it as an independent discrete variable and consider \( k \) as a continuous variable, since \( s \) is the only information about the discreteness of the momentum that might be relevant in the thermodynamic limit.

We therefore write

\[
\mathcal{M}(k; q) = \begin{cases} 
\mathcal{M}_+(k; q) & \text{if } n = \frac{kL}{2\pi} \text{ is even} \\
\mathcal{M}_-(k; q) & \text{if } n \text{ is odd}
\end{cases}
\]

and from (13)

\[
\mathcal{M}_s(k; q) = \begin{cases} 
(L/2)\delta_{k,q} & \text{if } s = +1 \\
-2i/(k-q) & \text{if } s = -1
\end{cases}
\]

Moreover in the odd sector \( \mathcal{M}_-(k; q) \) develops a pole at \( k = q \), a value that is of course absent in the discrete set of \( k \) values for odd \( n \). This implies that the corresponding sum cannot be directly expressed as an integral along the real \( k \) axis, but the correct prescription for how to avoid the pole must be specified. This can be done using a standard complex analysis trick, as explained below.

Substituting (13) into (10) and evaluating the even sector sum we obtain

\[
K(q; z; x, t) = \sum_{s=\pm} \sum_{k=\pm 2\pi n/L \atop n \text{ even/odd}} e^{iP(k)x-iE(k)t} \frac{\mathcal{M}_s(k; q)}{N(k)} \langle z|\Phi(k) \rangle
\]

\[
= \frac{e^{iP(q)x-iE(q)t}}{2} (\langle z|\Phi(q) \rangle + \frac{1}{L} \sum_{k=\pm 2\pi n/L \atop n \text{ odd}} e^{iP(k)x-iE(k)t} \left( \frac{-2i}{k-q} \right) \langle z|\Phi(k) \rangle)
\]

In order to write the odd sector sum \( S_- \) as an integral in the infinite system size limit \( L \to \infty \), we first introduce a function of \( k \) that has simple poles of residue \( 1/(2\pi i) \) at the points \( k = 2\pi n/L \) for all odd integers \( n = 2\ell + 1 \) so that we can write each term of the sum as an integral over \( k \) along a small circle \( C_\ell = C(2\pi(2\ell+1)/L, \epsilon) \) around each of these points. The suitable function is

\[
F_-(k) = \frac{L}{4\pi} \frac{1}{e^{iKL/2} + 1}
\]

and we obtain

\[
S_- = \frac{1}{L} \sum_{k=\pm 2\pi n/L \atop n \text{ odd}} e^{iP(k)x-iE(k)t} \left( \frac{-2i}{k-q} \right) \langle z|\Phi(k) \rangle
\]

\[
= \frac{e^{iP(k)x-iE(k)t}}{4\pi} \int_{C_\ell} \frac{dk}{2\pi} \frac{1}{e^{ikL/2} + 1} e^{iP(k)x-iE(k)t} \left( \frac{-2i}{k-q} \right) \langle z|\Phi(k) \rangle
\]
Next, using the analyticity of the integrand everywhere else on the real $k$ axis except at $k = q$ where it is singular, we merge the circles into another contour composed of two straight lines $C_{\pm} = \mathbb{R} \pm i\varepsilon$ running just above and just below the real $k$ axis and a small circle $C_p = C(q, \varepsilon)$ in order to subtract the contribution of the pole at $k = q$

\[
S_- = -\int_{C_- - C_p} \frac{dk}{4\pi} \frac{1}{\text{e}^{ikL/2} + 1} e^{iP(k)x - iE(k)t} \left( \frac{-2i}{k - q} \right) \langle z|\Phi(k) \rangle
\]

\[
= -\left(\int_{-\infty - i\varepsilon}^{+\infty - i\varepsilon} - \int_{-\infty + i\varepsilon}^{+\infty + i\varepsilon}\right) \frac{dk}{4\pi} \frac{1}{\text{e}^{ikL/2} + 1} e^{iP(k)x - iE(k)t} \left( \frac{-2i}{k - q} \right) \langle z|\Phi(k) \rangle +
\]

\[
+ 2\pi i \frac{1}{4\pi} \frac{1}{\text{e}^{qL/2} + 1} e^{+iP(q)x - iE(q)t} \langle z|\Phi(q) \rangle \text{ Res}_{k=q} \left( \frac{-2i}{k - q} \right)
\]

We now evaluate the contribution of the pole at $k = q$, which is a simple pole due to the factor $1/(k - q)$ since the rest of the integrand is analytic at that point. The factor $1/(\text{e}^{qL/2} + 1)$ equals $1/2$ for any value of $q$ because, by satisfying the pre-quench quantisation conditions, $q$ is equal to $4\pi n_0/L$ with $n_0 \in \mathbb{Z}$. Substituting into the last equation we obtain

\[
S_- = -\left(\int_{-\infty - i\varepsilon}^{+\infty - i\varepsilon} - \int_{-\infty + i\varepsilon}^{+\infty + i\varepsilon}\right) \frac{dk}{4\pi} \frac{1}{\text{e}^{ikL/2} + 1} e^{iP(k)x - iE(k)t} \left( \frac{-2i}{k - q} \right) \langle z|\Phi(k) \rangle +
\]

\[
+ \frac{1}{2} e^{+iP(q)x - iE(q)t} \langle z|\Phi(q) \rangle
\]

Lastly, we take the infinite system size limit $L \to \infty$. We notice that

\[
\lim_{L \to \infty} \frac{1}{\text{e}^{kL/2} + 1} = \begin{cases} 
1 & \text{if } \text{Im}(k) > 0 \\
0 & \text{if } \text{Im}(k) < 0
\end{cases}
\]

Therefore the integral along the line below the real axis vanishes in this limit, while the remaining integral along the line above the real axis gives

\[
\lim_{L \to \infty} S_- = \int_{-\infty + i\varepsilon}^{+\infty + i\varepsilon} \frac{dk}{2\pi} e^{iP(k)x - iE(k)t} \left( \frac{-i}{k - q} \right) \langle z|\Phi(k) \rangle +
\]

\[
+ \frac{1}{2} e^{+iP(q)x - iE(q)t} \langle z|\Phi(q) \rangle
\]

Substituting the final result into (14), we obtain

\[
\lim_{L \to \infty} \mathcal{K}(q; z; x, t) = e^{+iP(q)x - iE(q)t} \langle z|\Phi(q) \rangle + \int_{-\infty + i\varepsilon}^{+\infty + i\varepsilon} \frac{dk}{2\pi} e^{iP(k)x - iE(k)t} \left( \frac{-i}{k - q} \right) \langle z|\Phi(k) \rangle \tag{17}
\]

Overall what we have achieved is to calculate the $L \to \infty$ limit of the sum over all modes in $\mathcal{K}(q; z; x, t)$ as an expression involving an integral on a line parallel to the real axis with the precise prescription on how to avoid the momentum pole at $k = q$.

Having taken the thermodynamic limit, we now proceed to derive the asymptotics at large $x, t$ keeping the ratio $x/t \equiv \xi$ fixed. We focus on the second term that involves the integral. The asymptotics of such integrals can be found easily using another complex analysis trick. Provided that the real part of the exponent of $e^{+iP(k)x - iE(k)t} = e^{+i(iP(k)x - iE(k)t)}$ is negative for all $k$ along the integration contour, the integral decays when $x, t \to \infty$. If it is non-negative for part of the contour, then in order to calculate its asymptotics it is sufficient to deform the contour into the region of the complex plane where it becomes negative. In doing so we will have to take into account if the contour crosses any singularity of the integrand, in which case we will have to subtract its contribution. The wave-function $\langle z|\Phi(k) \rangle$, being a simple plane wave, is analytic in $k$, so the only singularity that may be crossed is the pole at $k = q$.

The condition that the exponent has a negative real part gives

\[
\text{Re} \left[ P(k + i\varepsilon)x - iE(k + i\varepsilon)t \right] = \text{Re} \left[ i(P(k) + i\varepsilon P'(k)) x - i(E(k) + i\varepsilon E'(k)) t \right] = (-P'(k)\xi + E'(k)) \varepsilon t < 0
\]
where we made use of the analyticity of $P(k)$ and $E(k)$ on the real axis. We therefore find that for all $k$ such that

\[
\frac{E'(k)}{P'(k)} < \xi
\]

(which in the present case means for $2k < \xi$) we keep the original positive shift $\epsilon > 0$ and the integral in this interval vanishes as $x, t \to \infty$. On the other hand, in the interval where $k$ satisfies the inequality

\[
\frac{E'(k)}{P'(k)} > \xi
\]

(i.e. for $2k > \xi$ in the present case) we should deform the contour from above the real axis to below so that $\epsilon < 0$ and the condition is met. If the value $k = q$ lies within the interval where the latter inequality holds, then the momentum pole at $k = q$ is crossed during the deformation and its residue should be subtracted. As the deformed integral vanishes, the asymptotics is given precisely by the pole contribution. Note that in the above inequalities the quantity

\[
v(k) \equiv \frac{E'(k)}{P'(k)} = \frac{dE}{dP}
\]

can be recognized as the group velocity of the free particle excitations as given by their dispersion relation.

In the simple case where the group velocity $v(k)$ is a monotonically increasing function of the momentum $k$ (as in the present case where $v(k) = 2k$), the inequalities $v(k) \leq \xi$ reduce to $k \leq k_*(\xi)$ where the threshold value $k_*(\xi)$ is given by the equation

\[
v(k)(\xi)) = \xi
\]

whose solution in the present case is $k_*(\xi) = \xi/2$.

Following the above observations, we write

\[
\int_{-\infty+i\epsilon}^{+\infty+i\epsilon} \frac{dk}{2\pi} e^{iP(k)x - iE(k)t} \left( \frac{-i}{k - q} \right) \langle z | \Phi(k) \rangle
\]

\[
= \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} \frac{dk}{2\pi} \Theta(\xi - v(k)) e^{iP(k)x - iE(k)t} \left( \frac{-i}{k - q} \right) \langle z | \Phi(k) \rangle
\]

\[
+ \int_{-\infty-i\epsilon}^{+\infty-i\epsilon} \frac{dk}{2\pi} \Theta(v(k) - \xi) e^{iP(k)x - iE(k)t} \left( \frac{-i}{k - q} \right) \langle z | \Phi(k) \rangle
\]

\[
+ \int_{k_*(\xi) - i\epsilon}^{k_*(\xi) + i\epsilon} \frac{dk}{2\pi} e^{iP(k)x - iE(k)t} \left( \frac{-i}{k - q} \right) \langle z | \Phi(k) \rangle
\]

\[
- 2\pi i \frac{1}{2\pi} \Theta(v(q) - \xi) e^{iP(q)x - iE(q)t} \langle z | \Phi(q) \rangle \text{Res}_{k=q} \left( \frac{-i}{k - q} \right)
\]

(18)

Note that the integrals of the first and second line of (18) decay exponentially with $x, t$, while that of the third line decays algebraically and is the one that determines the scaling of decaying corrections to the asymptotics. Therefore we obtain

\[
\int_{-\infty+i\epsilon}^{+\infty+i\epsilon} \frac{dk}{2\pi} e^{iP(k)x - iE(k)t} \left( \frac{-i}{k - q} \right) \langle z | \Phi(k) \rangle
\]

\[
|z|, t \to \infty, x/t = \xi \text{ fixed}
\]

\[
- \Theta(v(q) - \xi) e^{iP(q)x - iE(q)t} \langle z | \Phi(q) \rangle
\]

Substituting our result into (17) we find the rather simple and elegant formula

\[
\lim_{L \to \infty} K(q; z, x, t) \left. \frac{1}{x/t = \xi \text{ fixed}} \right|_{x/t = \xi \text{ fixed}} \Theta(v(q) - \xi) e^{iP(q)x - iE(q)t} \langle z | \Phi(q) \rangle
\]
which can also be written as
\[
\lim_{x \to \pm \infty} \lim_{L \to \infty} e^{-iP(q)x + iE(q)t} K(q; z; x, t) = \Theta(\xi - v(q)) \langle z | \Phi(q) \rangle
\]
which concludes our calculation.

This formula directly leads to the conclusion that for a general initial state that is diagonal in the pre-quench basis and for a general local observable \( \hat{O} \), the large distance and time asymptotics in the thermodynamic limit is given by a NESS that is diagonal in the post-quench basis. Indeed we have
\[
\lim_{x/t \to \pm \infty} \lim_{L \to \infty} \langle \hat{O}(x, t) \rangle = \lim_{x/t \to \pm \infty} \lim_{L \to \infty} \text{Tr} \left\{ \hat{\rho}_0 e^{iHt} e^{-iPq} \hat{O} e^{iPq} e^{-iHt} \right\}
\]
\[
= \lim_{x/t \to \pm \infty} \lim_{L \to \infty} Z_0^{-1} \sum_{q=2 \pi n_0/(L/2)} \int_{-L/2}^{+L/2} dy dy' N_0^{-1}(q) \rho_0(q) \Phi^*(q; y'; x, t) K(q; y; x, t) \langle y' | \hat{O} | y \rangle
\]
\[
= Z_0^{-1} \int_{-\infty}^{+\infty} \frac{dq}{2\pi} \rho_0(q) \Theta(\xi - v(q)) \int_{-\infty}^{+\infty} dy dy' \langle \Phi(q) | y' \rangle \langle y' | \hat{O} | y \rangle \langle y | \Phi(q) \rangle
\]
\[
= Z_0^{-1} \int_{-\infty}^{+\infty} \frac{dq}{2\pi} \rho_0(q) \Theta(\xi - v(q)) \langle \Phi(q) | \hat{O} | \Phi(q) \rangle
\]
from which we see that the NESS is described by the momentum distribution
\[
\rho_{\text{NESS}}(q; \xi) \propto \rho_0(q) \Theta(\xi - v(q))
\]
Note that \( \rho_{\text{NESS}}(q; \xi) \) is space-time dependent through the ray parameter \( \xi = x/t \), being equal to the initial momentum distribution \( \rho_0(q) \) for all momenta \( q \) such that the corresponding group velocity is \( v(q) < \xi \) and to zero otherwise. This is what is expected on the basis of the simple semiclassical picture of ballistically moving quasiparticles: at any ray \( \xi \) the corresponding NESS is the result of the mixing of quasiparticles originating from either the left or the right half side with the corresponding momentum distribution i.e. \( \rho_{\text{NESS}}(q; \xi) \propto \rho_{OL}(q) \Theta(v(q) - \xi) + \rho_{OR}(q) \Theta(\xi - v(q)) \). In the present case \( \rho_{OL}(q) = 0 \) and \( \rho_{OR}(q) = \rho_0(q) \), since initially all particles are on the right side and the left side is empty. The latter formula which extends our result for the gas expansion to the more general case of gas mixing, can be derived straightforwardly following the same steps as above for a suitable initial state.

IV. TONKS-GIRARDEAU LIMIT OF THE LIEB-LINIGER MODEL

Having clearly explained the mathematical tools in the single case particle case, we can now proceed to the whole system fermions. As already explained, in the Tonks-Girardeau limit the model is effectively equivalent to non-interacting fermions. Based on the exact mapping to free fermions, in principle the calculation of any observable reduces through Wick’s theorem to the calculation of the fermionic two-point correlation function \( \langle \Psi^\dagger_F(x, t) \Psi_F(x + r, t) \rangle \), i.e., essentially to the single-particle problem we have just solved, apart from complications arising from the non-trivial expression of bosonic observables in terms of hard-core boson or fermion operators. For the purpose of generalising our approach to the genuinely interacting case, however, we will perform the calculation without taking advantage of Wick’s theorem, confronting the mathematical challenges of the many-body problem.

We start by writing the exact coordinate space eigenfunctions, which are
\[
\langle z | \Phi(\lambda) \rangle = \frac{1}{\sqrt{N!}} \text{det} \left[ \exp \left( i \lambda_j x_i \right) \right] \prod_{j > i} \text{sign} (x_j - x_i)
\]
where the rapidities \( \lambda \) in the system of length \( L \) are quantised following the equations
\[
\exp \left( i \lambda_j L \right) = (-1)^{N-1} \text{ for all } j = 1, 2, \ldots, N
\]
These are nothing but the Bethe equations, which in this limit reduce to the above decoupled form, and can be solved easily to give
\[
\lambda_j = \begin{cases} \frac{2\pi n_j}{L} & \text{for } N \text{ odd} \\ \frac{2\pi n_j}{L} \left( n_j + \frac{1}{2} \right) & \text{for } N \text{ even} \end{cases}
\]
\[
\text{for } n_j \in \mathbb{Z}
\]
Note that both the eigenfunctions and the rapidity eigenvalues take the same form as for a system of free fermions, apart from the odd-even selection rule on the values of the rapidities and the presence of the sign product in (19), which guarantees symmetry of the wavefunctions under exchange of the bosonic particles and accounts for the difference between hard-core bosons and fermions.

The energy and momentum eigenvalues corresponding to an eigenstate with rapidities $\lambda$ are respectively

$$E(\lambda) = \sum_{i=1}^{N} c(\lambda_i) = \sum_{i=1}^{N} \lambda_i^2$$

$$P(\lambda) = \sum_{i=1}^{N} p(\lambda_i) = \sum_{i=1}^{N} \lambda_i$$

The quantisation conditions for the half system are

$$\exp \left(i \mu_j L/2 \right) = (-1)^{N-1} \quad \text{for all} \quad j = 1, 2, \ldots, N$$

with solutions

$$\mu_j = \begin{cases} \frac{2\pi}{L} n_j & \text{for} \ N \ odd \\ \frac{2\pi}{L} (n_j + \frac{1}{2}) & \text{for} \ N \ even \end{cases} \quad n_j \in \mathbb{Z}$$

As before, we will focus on the derivation of the asymptotics of pre-quench eigenstates $|\Phi_0(\mu)\rangle$ time evolved under the Tonks-Girardeau Hamiltonian $H_{TG}$ and projected on a local basis. We therefore wish to calculate the asymptotics of the quantity

$$K(\mu; z; x, t) \equiv \langle z | e^{+iPx-iHt} | \Phi_0(\mu) \rangle$$

Introducing a resolution of the identity in the post-quench basis we have

$$e^{+iPx-iHt} | \Phi_0(\mu) \rangle = \sum_{\lambda: \text{BA}} e^{+iP(\lambda)x-iE(\lambda)t} \frac{\langle \Phi(\lambda) | \langle \Phi(\lambda) \rangle}{\langle \Phi(\lambda) | \langle \Phi(\lambda) \rangle} | \Phi_0(\mu) \rangle$$

$$= \sum_{\lambda: \text{BA}} e^{+iP(\lambda)x-iE(\lambda)t} \frac{\mathcal{M}(\lambda; \mu)}{\mathcal{N}(\lambda)} | \Phi(\lambda) \rangle$$

(24)

where the sum runs over all solutions of the Bethe equations for $N$ particles in length $L$, and the eigenstate overlaps and norms are

$$\mathcal{M}(\lambda; \mu) \equiv \langle \Phi(\lambda) | \Phi_0(\mu) \rangle$$

$$\mathcal{N}(\lambda) \equiv \langle \Phi(\lambda) | \Phi(\lambda) \rangle$$

From (19) it is straightforward to calculate the norm $\mathcal{N}(\lambda)$

$$\mathcal{N}(\lambda) = L^N$$

(25)

A. Initial state overlaps

The first problem we have to address is the calculation of the overlaps $\mathcal{M}(\lambda; \mu)$. From the coordinate space form of the eigenstate wave-functions (19) we find

$$\mathcal{M}(\lambda; \mu) = \int_0^{L/2} dx \langle \Phi(\lambda) | x \rangle \langle x | \Phi_0(\mu) \rangle$$

$$= \frac{1}{N!} \int_0^{L/2} dx \det_{i,j} \left[ \exp(-i\lambda_j x_i) \right] \det_{k,\ell} \left[ \exp(i\mu_k x_k) \right] \left( \prod_{j > i} \text{sign}(x_j - x_i) \right)^2$$

$$= \frac{1}{N!} \int_0^{L/2} dx \det_{i,j} \left[ \exp(-i\lambda_j x_i) \right] \det_{k,\ell} \left[ \exp(i\mu_k x_k) \right]$$
The last expression can be evaluated easily using the Andréief identity [78]
\[
\int \prod_{n=1}^{N} d\mu(x_n) \det_{i,j} [f_j(x_i)] \det_{k,l} [g_l(x_k)] = N! \det_{i,j} \left[ \int d\mu(x) f_i(x) g_j(x) \right]
\] (26)
which reduces the multiple integral of the product of two determinants to the determinant of a matrix whose elements are single-variable integrals. Using this identity we find
\[
\mathcal{M}(\lambda; \mu) = \det_{i,j} \left[ \int_0^{L/2} dx \exp (i(\mu_i - \lambda_j)x) \right]
\] (27)
The application of the Andréief identity for the calculation of eigenstate overlaps for this type of quench problems and the reduction to the single-particle expression is a property that is valid for any effectively non-interacting model as pointed out in [79], which studied the same problem in the spin-1/2 XX chain.

Note that the integral appearing in the last expression is precisely the same as in the overlap of the single-particle problem (13)
\[
\int_0^{L/2} dx \exp (i(\mu - \lambda)x) = \frac{i}{\lambda - \mu} \left( \exp (i(\mu - \lambda)L/2) - 1 \right)
\] (28)
In order to evaluate the thermodynamic limit later on, we would like to express the overlaps as continuous non-oscillatory functions of the rapidities $\lambda_i$, which can be done by eliminating $L$ using the quantisation conditions. The quantised values of $\mu$ and $\lambda$ as dictated by (22) and (20) are not generally the same as in the single-particle problem. If $N$ is odd then $\mu_i$ and $\lambda_j$ are integer multiples of $2\pi/(L/2)$ and $2\pi/L$ respectively, exactly as before (13). If however $N$ is even then they are half-integer multiples, which means that they can never be equal to each other (the difference of two integers cannot be $\frac{1}{2}$). Let us focus on this case of even $N$. We find
\[
\int_0^{L/2} dx \exp (i(\mu - \lambda)x) = \frac{1}{\lambda - \mu} \times \begin{cases} 
-1 - i & \text{if } n = \frac{N}{L} = \frac{1}{2} \text{ is even} \\
+1 - i & \text{if } n \text{ is odd}
\end{cases}
\] (29)
We therefore need again to introduce an index $s_j$ to distinguish between even and odd sectors ($s_j = \pm 1$) separately for each rapidity variable $\lambda_j$ and redefine the overlaps as functions of both the rapidity vectors $\lambda, \mu$ and an additional vector of discrete indices $s$ with each of the $s_j$ taking one of the two values depending on $\lambda_j$. We finally find that, for even $N$
\[
\mathcal{M}_s(\lambda; \mu) = \det_{i,j} \left[ \frac{i + s_j}{\lambda_j - \mu_i} \right]
\] (29)
Considering the variables $\lambda$ as continuous, we observe that, analogously to the single-particle case, the overlaps $\mathcal{M}_s(\lambda; \mu)$ have simple poles when any of the rapidities $\lambda_i$ tends to any of the $\mu_j$ i.e. they exhibit $N$-dimensional poles at $\lambda \to \mu$ and at all permutations of components of $\mu$. As in the single particle calculation, these poles are of “kinematical” type and reflect the fact that in the thermodynamic limit the complete set of particle momenta is conserved.

B. Eigenstate summation through multivariable version of Cauchy’s integral formula

The next step is to perform the summation over post-quench eigenstates. This can be done again rigorously using the same contour integral trick as in the single particle case, the only difference being that we now need a multivariable version of it. For odd $N$, the rapidities $\lambda_j$ are integer multiples of $2\pi/L$ as in the previously studied single particle case, and we have already seen that the even sector should be treated differently from the odd sector, due to the exceptional property that in the even sector the overlaps are simply proportional to a Kronecker delta. Let us therefore focus now on the case of even $N$, which is different. In this case we still have to distinguish between the two sectors, but the even sector is not exceptional and can be treated in the same way as the odd. As before, to pick the values of rapidities allowed by the quantisation conditions (20), we introduce a suitable function $F_s(\lambda)$ that has
simple poles of residue 1/(2\pi i) when each of the \( \lambda_j \) is a half-integer multiple of 2\pi/L, and in addition distinguishes between odd and even integers through the corresponding index \( s_j \). More specifically, we choose

\[
F_s(\lambda) \equiv \left( -\frac{L}{4\pi} \right)^N \prod_{i=1}^{N} \frac{1}{18^i e^{\lambda_i L/2} + 1} \quad \text{for even } N
\]

(30)

We can easily verify that (30) considered as a function of \( \lambda_j \) has poles at 2\pi/L\( (n_j + \frac{1}{2}) \) with \( n_j \) even or odd for \( s_j = \pm1 \) respectively.

We can now use a multivariable version of the residue theorem to express the multiple summation over allowed rapidities, as an integral over a continuous function of the \( \lambda \)'s. More specifically, given a function \( F(\lambda) = 1/\prod_{i=1}^{N} f_i(\lambda) \) such that each of the \( f_i(\lambda) \) has a simple zero at a single point \( \lambda = \lambda^\ast \), then for any multi-dimensional contour \( C = C_1 \times C_2 \times \cdots \times C_N \) encircling this point and any function \( g(\lambda) \) of the rapidities \( \lambda \) that is analytic inside the contour \( C \), we have (see e.g. appendix A of [80])

\[
\oint_{C} \frac{d^N \lambda}{(2\pi i)^N} F(\lambda) g(\lambda) = \frac{g(\lambda^\ast)}{\det (\partial f_i / \partial \lambda^j) \bigg|_{\lambda=\lambda^\ast}}
\]

(31)

This can be generalised to the case of more than one zeroes, i.e. when the functions \( f_i(\lambda) \) have simple zeroes at each one of a set of points \( \{ \lambda_j^\ast \} \), in which case the right hand side of the above equation should be replaced by a sum over all these points with appropriate signs, determined by the direction of integration resulting from the contour deformation. If on the other hand the function \( g(\lambda) \) has additional poles inside the contour \( C \), not coinciding with any of the points \( \{ \lambda_j^\ast \} \), then the contours in the above formula should be modified to exclude these additional poles, or equivalently their residues must be subtracted off.

Using this method, the sum over post-quench eigenstates can be written in integral form as follows

\[
\sum_{\lambda_1, \lambda_2, \ldots} = \frac{1}{N!} \sum_{s} \oint_{C} d^N \lambda F_s(\lambda) \ldots
\]

where the contours \( C_i \) in each of the above integrals enclose the real rapidity axis where all allowed rapidities lie, i.e. should be chosen to consist of two straight lines one just above and one just below the real axis. However, because the overlaps \( M_s(\lambda; \mu) \) have multi-dimensional poles at \( \lambda = \mu \) and permutations, these points must be excluded by subtracting from each contour small circles around them or equivalently by subtracting the corresponding residues. The \( 1/N! \) factor accounts for permutations of the order of the rapidities \( \lambda \) which correspond to identical Bethe states, and the sum over the discrete indices \( s \) accounts for all combinations of the even and odd sectors for each of the rapidities.

We can therefore write (24) as

\[
e^{iPx - iHt} |\Phi_0(\mu)\rangle = \sum_{\lambda_1, \lambda_2, \ldots} \frac{e^{iP(\lambda)x - iE(\lambda)t} M_s(\lambda; \mu)}{N(\lambda)} |\Phi(\lambda)\rangle
\]

\[
= \frac{1}{N!} \sum_{s} \oint_{C} d^N \lambda F_s(\lambda) e^{iP(\lambda)x - iE(\lambda)t} M_s(\lambda; \mu) N(\lambda) |\Phi(\lambda)\rangle
\]

(32)

Note that the poles at \( \lambda = \mu \) and permutations are between those of \( F_s(\lambda) \) i.e. the rapidity combinations allowed by the full system’s Bethe equations, and do not coincide with any of them for even \( N \). This is because, as mentioned above, the allowed rapidities \( \lambda \) and \( \mu \) are half-integer multiples of 2\pi/L and 2\pi/(L/2) respectively, so they can never be equal to each other. Also note that all other constituents of the integrand in (32) i.e. the function \( e^{iP(\lambda)x - iE(\lambda)t} \) and the state \( |\Phi(\lambda)\rangle \) itself, are analytic functions for real rapidities and do not introduce any other singularities. Lastly, the dependence on the discrete multi-variable \( s \) is limited to \( F_s(\lambda) \) and \( M_s(\lambda; \mu) \) only.

At this step we can also evaluate the sum over the discrete indices \( s \). By first absorbing the product in (30) and the sum over \( s \) into the determinant (29), using the properties

\[
\prod_{i=1}^{N} f_i \det (A_{ij}) = \det (f_i A_{ij})
\]

and

\[
\sum_{s} \det (A_{ij}(s_j)) = \det \left( \sum_{s} A_{ij}(s) \right)
\]
we obtain

\[
\sum_s F_s(\lambda) M_s(\lambda; \mu) = \sum_s \left( -\frac{L}{4\pi} \right)^N \prod_{j=1}^N \frac{1}{\text{det}_{i,j} \left[ \frac{i + s_j}{\lambda_j - \mu_i} \right]}
\]

and after substitution into (32)

\[
e^{-iPz-iHt}|\Phi_0(\mu)\rangle = \frac{1}{N!} \oint_C \frac{d^N \lambda}{(2\pi)^N} e^{-iP(\lambda)x-iE(\lambda)t} \prod_{j=1}^N \frac{1-e^{i\lambda_j L/2}}{1+e^{i\lambda_j L}} \text{det}_{i,j} \left[ \frac{i}{\lambda_j - \mu_i} \right] |\Phi(\lambda)\rangle
\]

Note that in the last formula the function \( F_s(\lambda) \) appears to be replaced by a function with poles at the eigenvalues of both the even and odd sector eigenstates, which might suggest that the splitting into these two sectors is not that relevant in the end. However, the expression that we would obtain if we followed all earlier steps except for the splitting would be different and incorrect, even though still characterised by the same pole structure.

C. Multivariable Kinematical Pole Residue

Before we analyse the asymptotics of our contour integral formula in the thermodynamic and large space and time limit, let us focus on the multi-dimensional kinematical pole at \( \lambda \to \mu \) and calculate its residue. As anticipated based on the single particle calculation, this pole gives the only non-vanishing contribution of the contour integral in the above double limit.

Starting from (29), the residue of the overlap \( M_s(\lambda; \mu) \) at \( \lambda \to \mu \) (and at any permutation \( \mu_\pi \) of \( \mu \)) can be easily evaluated by expanding the determinant as a sum over permutations and noticing that exactly one of them and only that contributes a non-vanishing residue. Explicitly, we obtain

\[
\text{Res}_{\lambda=\mu} M_s(\lambda; \mu) = \sum_{\text{all perm. } \pi} (-1)^{[\pi]} \text{Res}_{\lambda=\mu} \prod_{j=1}^N \left( \frac{i + s_j}{\lambda_j - \mu_{\pi_j}} \right)
\]

\[= (-1)^N \prod_{j=1}^N (i + s_j)
\]  

(34)

The residue at \( \lambda = \mu_\pi \) is the same for all permutations \( \pi \), except for a sign which is equal to the signature of the permutation.

Moreover the value of the function \( F_s(\lambda) \) at \( \lambda = \mu \) and permutations can be obtained taking into account that the rapidities \( \mu \) satisfy the half system Bethe equations (22) for even \( N \)

\[
F_s(\mu) = \left( -\frac{L}{4\pi} \right)^N \prod_{i=1}^N \frac{1}{\text{det}_{i,j} \left[ \frac{1}{i \pm s_i \mu_j L/2 + 1} \right]}
\]

\[= \left( -\frac{L}{4\pi} \right)^N \prod_{i=1}^N \frac{1}{\text{det}_{i,j} (-1)^{N-1} + 1}
\]

\[= \left( -\frac{L}{4\pi} \right)^N \prod_{i=1}^N \frac{1}{-i s_i + 1}
\]

(35)

Notice that there is no oscillating phase left in the last expression. By combining (34) and (35) we obtain

\[
\text{Res}_{\lambda=\mu} F_s(\mu) M_s(\lambda; \mu) = \left( \frac{L}{4\pi} \right)^N \prod_{i=1}^N \frac{1 + s_i}{-i s_i + 1}
\]

\[= i^N \left( \frac{L}{4\pi} \right)^N
\]  

(36)
Notice that this expression is independent of \( s \), therefore the corresponding discrete index sum becomes trivial. From the above we conclude that the contribution of the kinematical poles to the multiple contour integral (32) is given by the simple result

\[
\frac{1}{N!} \sum_{s} \text{Res}_{\lambda=\mu} F_{s}(\lambda)e^{+iP(\lambda)\overline{x-iE(\lambda)t}} \frac{M_{s}(\lambda; \mu)}{N(\lambda)}|\Phi(\lambda)| = e^{iP(\mu)\overline{x-iE(\mu)t}}|\Phi(\mu)|
\]

where the \( N! \) multiplicity of the poles has been taken into account.

### D. Asymptotics of the integral in the thermodynamic limit

We are now ready to calculate the asymptotic form of the contour integral representation (32) in the thermodynamic limit. As expected, this step will result in an \( N \)-dimensional line integral in the positive direction, with a specific prescription on how to pass around the kinematical poles. As we have seen in the single particle case, this prescription is going to be crucial at the next step, which is the derivation of the asymptotics in the large \( x, t \) limit.

Starting from (32), we split each of the contours composing the multi-dimensional contour in \( C \) into their three parts: the straight line \( \mathbb{R} - i\epsilon \) below the real axis in the positive direction (\( C_- \)), the straight line \( \mathbb{R} + i\epsilon \) above the real axis in the negative direction (\( C_+ \)) and the circles \( C(\mu_i, \epsilon) \) centred at each of the points \( \mu_i \) with a small radius \( \epsilon \) in the negative direction (collectively denoted as \( C_p \)), where in all cases \( \epsilon > 0 \). With this decomposition of the contours, expanding the integral we obtain a sum of \( 3^N \) cross-terms, that is the sum of all possible combinations of integrals of the \( N \) rapidity variables along the three different parts of the contour. We will now show that in the thermodynamic limit, only \( 2^N \) terms survive, specifically those of the above terms that do not involve any integral along the first contour part \( C_- = \mathbb{R} - i\epsilon \). To see why this is true, we simply have to examine the behaviour of the function \( F_{s}(\lambda) \) when at least one of the rapidities lies along \( C_- \) in the limit \( L \to \infty \): we find that this limit results in an exponentially decaying factor that suppresses all such terms, while all others remain finite and even simplify.

Indeed, let use consider \( F_{s}(\lambda) \) given by (30) as a function of one of the rapidities, \( \lambda_i \). Firstly, we notice that

\[
\lim_{L \to \infty} \frac{1}{i\epsilon_i e^{i\lambda_i L/2} + 1} = \begin{cases} 1 & \text{if } \text{Im}(\lambda_i) > 0 \\ 0 & \text{if } \text{Im}(\lambda_i) < 0 \end{cases}
\]

independently of the value of \( s_i \). Actually it is much more useful to analyse the asymptotics towards these limits, which are exponentially decaying functions of \( L \) in both cases

\[
\frac{1}{i\epsilon_i e^{i\lambda_i L/2} + 1} \sim_{L \to \infty} \begin{cases} 1 + O(e^{-\epsilon L/2}) & \text{if } \lambda_i \in \mathbb{R} + i\epsilon \\ O(e^{-\epsilon L/2}) & \text{if } \lambda_i \in \mathbb{R} - i\epsilon \end{cases}
\]

If \( \lambda_i \in \mathbb{R} \) the limit is generally indeterminate and depends on the value of \( \lambda_i \), but when it is equal to one of the \( \mu_i \)’s we have already seen in (35) that it tends to the finite value \( 1/(-i\epsilon_i + 1) \) independently of \( L \). From these observations, we immediately see that in the limit \( L \to \infty \) the function \( F_{s}(\lambda) \) has a finite value plus \( O(e^{-\epsilon L/2}) \) corrections if all of the rapidities \( \lambda \) are either on \( C_- \) or at one of the points \( \mu_j \) (i.e. at the kinematical poles), while it decays as \( O(e^{-\epsilon n L/2}) \) if \( n \) out of the \( N \) rapidities \( \lambda \) are on \( C_+ \). Moreover, the surviving \( 2^N \) terms can be recast in the simpler form

\[
e^{+iPx-iHt}|\Phi_0(\mu)| = \frac{1}{N!} \left( \frac{L}{2} \right)^N \sum_{s} \prod_{i=1}^{N} \left( \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{d\lambda_i}{2\pi} + \frac{1}{1-i\epsilon_i} \int_{C_p} \frac{d\lambda_i}{2\pi} \right) e^{iP(\lambda)\overline{x-iE(\lambda)t}} \frac{M_{s}(\lambda; \mu)}{N(\lambda)}|\Phi(\lambda)| + O(e^{-\epsilon L/2})
\]

(37)

Note that we have used the minus signs \((-1)^N\) to reverse the integration direction in both \( C_+ \) and \( C_p \), so that we now have line integrals along \( \mathbb{R} + i\epsilon \) running in the positive direction and circular integrals around the poles also in the positive direction.

Obviously, we would arrive at an equivalent but more explicit formula if we started from the simplified version (33) and applied the same analysis. Explicitly, using the asymptotics

\[
\frac{1 - e^{+i\lambda_i L/2}}{1 + e^{+i\lambda_i L/2}} \sim_{L \to \infty} \begin{cases} 1 + O(e^{-\epsilon L/2}) & \text{if } \lambda_i \in \mathbb{R} + i\epsilon \\ O(e^{-\epsilon L/2}) & \text{if } \lambda_i \in \mathbb{R} - i\epsilon \end{cases}
\]

we obtain the final result

\[
e^{+iPx-iHt}|\Phi_0(\mu)| = \frac{1}{N!} \int_{C'} \frac{d^N\lambda}{(2\pi)^N} e^{+iP(\lambda)\overline{x-iE(\lambda)t}} \det_{i,j} \frac{1}{\lambda_j - \mu_i} |\Phi(\lambda)| + O(e^{-\epsilon L/2})
\]

(38)
where for brevity we denote the integration measure as

\[
\int_{C'} \frac{d^N \lambda}{(2\pi)^N} \equiv \prod_{i=1}^{N} \left( \int_{-\infty+\iota r}^{+\infty+\iota r} + \oint_{C_p} \right) \frac{d\lambda_i}{2\pi}
\]

and used also that \((-1)^N = 1\) for even \(N\).

It is important that we do not simply derive the limit \(L \to \infty\), but also the exact asymptotics, i.e. the exponential bound on corrections to this limit. This is crucial for taking the thermodynamic limit \(N, L \to \infty\) at fixed density \(n = N/L\) as it allows us to show that the two sides of (38) have the same thermodynamic limit, even without ever taking \(N \to \infty\) explicitly, which would result in the multiple rapidity integral turning into the less practical form of a functional integral. In other words, we have shown that the thermodynamic limit of \(e^{iP x - iH t}|\Phi_0(\mu)\rangle\) reduces to

\[
\lim_{N, L \to \infty, n : \text{fixed}} e^{iP x - iH t}|\Phi_0(\mu)\rangle = \lim_{N \to \infty} \frac{1}{N!} \int_{C'} \frac{d^N \lambda}{(2\pi)^N} e^{iP(\lambda)x - iE(\lambda)t} \det_{i,j} \left[ \frac{i}{\lambda_j - \mu_i} \right] |\Phi(\lambda)\rangle
\]

which is one of our main results. Not only is this limit simpler than the original, but also the functional form of the integrand has been greatly simplified. In particular the awkward dependence on \(L\) of the original expression, which was most responsible for the highly oscillatory behaviour of the integrand, has been completely eliminated. Lastly let us remark that, as can be easily verified, the multi-dimensional residues at the kinematical poles are equal to the values derived in the previous step.

E. Asymptotics at large distances & times

Our last step is the derivation of the asymptotics at large distances and times. At this point it is convenient to apply the projection to the local basis \(\langle z|\), which has been implicitly assumed from the beginning, and use the coordinate space wavefunction of the Bethe states (19). The last result (39) now reads

\[
\lim_{N, L \to \infty, n : \text{fixed}} \langle z|e^{iP x - iH t}|\Phi_0(\mu)\rangle = \lim_{N \to \infty} \frac{1}{N!} \int_{C'} \frac{d^N \lambda}{(2\pi)^N} e^{iP(\lambda)x - iE(\lambda)t} \det_{i,j} \left[ \frac{i}{\lambda_j - \mu_i} \right] \langle z|\Phi(\lambda)\rangle
\]

Of course we can immediately see that, using the Andréief identity (26), the last expression gives

\[
\lim_{N, L \to \infty, n : \text{fixed}} \langle z|e^{iP x - iH t}|\Phi_0(\mu)\rangle = \lim_{N \to \infty} \prod_{j<i} \sign(z_j - z_i) \det_{i,j} \int_{C'} \frac{d^N \lambda}{(2\pi)^N} e^{iP(\lambda)x - iE(\lambda)t} \frac{i}{\lambda - \mu_i} e^{i\lambda z_j}
\]

which reduces the problem to precisely the single particle problem. Therefore the asymptotic analysis is exactly the same as before, giving

\[
\lim_{\xi, t \to \infty, n : \text{fixed}} \lim_{N \to \infty} \langle z|e^{iP(\mu)x + iE(\mu)t}|\Phi_0(\mu)\rangle = \lim_{N \to \infty} \prod_{i=1}^{N} \Theta(\xi - v(\mu_i)) \langle z|\Phi(\mu)\rangle
\]

(40)

Alternatively, if we do not want to rely on the Andréief identity, we can repeat the steps of the asymptotic analysis for the multivariate case. First of all, considering each integral separately it is clear that the real part of the exponent of the exponential function switches sign at the point \(\lambda_s(\xi)\) that is the solution of the equation

\[
v(\lambda_s) = \xi
\]

(41)

where

\[
v(\lambda) = \frac{E'(\lambda)}{P'(\lambda)}
\]

is the group velocity. Therefore in order to handle the \(x, t \to \infty\) limit, we deform the integration contours so that they always lie in the region where the real part of the exponent is negative, taking into account the contribution
of the kinematical pole if it is eventually crossed due to the deformation. Explicitly, the contour of the line integral becomes

\[ \int_{-\infty+ie}^{+\infty+ie} \frac{d\lambda}{2\pi} = \left( \int_{-\infty+ie}^{+\infty+ie} + \int_{-\infty+ie}^{+\infty-ie} \right) \frac{d\lambda}{2\pi} - \oint_{C_p} \frac{d\lambda}{2\pi} \Theta(v(\lambda) - \xi) \]

and combined with the earlier contribution of the circular integral around the pole

\[ \left( \int_{-\infty+ie}^{+\infty+ie} + \oint_{C_p} \right) \frac{d\lambda}{2\pi} = \left( \int_{-\infty+ie}^{+\infty+ie} + \int_{-\infty+ie}^{+\infty-ie} \right) \frac{d\lambda}{2\pi} + \oint_{C_p} \frac{d\lambda}{2\pi} \Theta(\xi - v(\lambda)) \]

We therefore split the contour \( C' \) into two parts: the stair-like contour \( C''_{\lambda_*(\xi)} \equiv (-\infty + ie, \lambda_*(\xi) + ie] \cup [\lambda_*(\xi) - ie, +\infty - ie) \) and the usual circular contour around the pole \( C_p \) now coming with a conditional measure \( \Theta(\lambda_*(\xi) - \lambda) = \Theta(\xi - v(\lambda)) \). With this decomposition of the contours, expanding the multiple integral we obtain a sum of 2\( N \) cross-terms, that are all possible combinations of integrals of the \( N \) rapidity variables along the two different parts of the contour. If the threshold value \( \lambda_*(\xi) \) is chosen as the solution of (41), then all terms involving at least one integral along \( C''_{\lambda_*(\xi)} \) decay in the large distance and time limit, since the integrand decays exponentially (line integrals parallel to the real axis) or at most algebraically (short line integral perpendicular to the real axis). Therefore the only surviving contribution comes from the cross-term that contains only circular integrals around the kinematical pole as factors, which is equal to the corresponding pole residue calculated earlier. In this way we find again our final result (40).

Obviously it is only choosing the appropriate value for \( \lambda_*(\xi) \) that guarantees that the asymptotics in the combined limit is given by the single cross-term corresponding to the multidimensional kinematical pole contribution. If \( \lambda_*(\xi) \) is chosen correctly then the next to leading order corrections must decay. As in the previous section where we evaluated the thermodynamic limit, these corrections come from all cross-terms between \( (N - 1) \) factors corresponding to pole residues and one factor corresponding to a line integral along the stair-shaped contour \( C''_{\lambda_*(\xi)} \)

\[ \sum_{j=1}^{N} e^{ip(\mu_j) x - e(\mu_j) t} \prod_{i \neq j} \Theta(\lambda_*(\xi) - \mu_i) \int_{C''_{\lambda_*(\xi)}} \frac{d\lambda}{2\pi} e^{ip(\lambda) x - e(\lambda) t} \langle z | \Phi(\mu_1, \ldots, \mu_{i-1}, \lambda, \mu_{i+1}, \ldots, \mu_N) \rangle \]

Therefore the condition for \( \lambda_*(\xi) \) can be phrased differently as the condition for which the above expression vanishes in the combined limit. The \( \lambda \)-independent coefficients of this sum that appear before the integral are oscillatory but non-decaying. Since the coordinate space wavefunction \( \langle z | \Phi \rangle \) essentially is a determinant, i.e. a sum of products of factors of which only one is \( \lambda \) dependent, for each of these terms the integral can be performed independently from the others terms. As a consequence, the asymptotics of the integral is the same for each of these terms and the same as above, therefore the threshold value is indeed given by (41).

Eq. (40) is our final result for the asymptotics of \( K(\mu; z; x, t) \) in the combined limit, which as explained in sec. II establishes the emergence of the NESS and provides a complete characterisation of it in terms of initial state information.

V. DISCUSSION

In this work we have developed a new analytical method for the derivation of asymptotics of local observables after a geometric quench, a special but representative type of inhomogeneous quench, focusing for the moment on the non-interacting case. From the detailed calculation presented here, we can draw certain conclusions that are instructive on how to proceed to the more interesting and highly non-trivial interacting case.

First of all, the derivation of the asymptotics does not rely significantly on special characteristics of the initial state or the observable considered. The only necessary information about these that we used is that the initial state is diagonal in the pre-quench basis and the observables are local. This suggests that our strategy of considering the asymptotics of each pre-quench eigenstate after time evolution under the post-quench Hamiltonian and projection onto the local basis should be suitable for the interacting case as well.

Our method was mainly based on analyticity properties of the pre- and post-quench eigenstate overlaps, especially on the kinematical poles at equal momenta. Their presence is a direct consequence of the step-like form of the initial
state inhomogeneity as reflected in momentum space and it is a characteristic feature of any inhomogeneous quench of this type (including “domain wall” states and smooth step-like initial density or temperature profiles). The necessity for decomposition of the post-quench eigenstates into parity sectors is also a general characteristic of such quenches related to the fact that the initial state is not symmetric under space reflections. We therefore see that the only relevant information about the initial state, beyond these general features, is the residue of the kinematical poles, which is much less information than the full functional form of the overlaps and can even be inferred from general thermodynamic properties in the two halves of the system.

Whereas the effectively non-interacting character of the system can be exploited to reduce the problem to a single particle quantum mechanics problem from which any observable can in principle be derived using Wick’s theorem, we have shown that the mathematical steps used in the single particle problem can be extended and applied successfully also in the many particle case. The main point where we have taken advantage of the non-interacting character of the problem was in the calculation of the initial state overlaps by means of the Andréief identity, which undoubtedly is the crucial starting point of the method. However, as already mentioned in the introduction, for the geometric quench protocol these overlaps are explicitly known for any value of the interaction by other means.

On the other hand, in the non-interacting case, summing over the quantised momentum eigenvalues in order to perform the expansion in the post-quench basis was done relatively easily as these are explicitly known and simply equidistant. This is no longer true in the interacting case, where the quantisation conditions are given by the complicated set of Bethe equations. However, the complex analysis method for writing the sum over eigenstates as a contour integral, completely circumvents the problem of explicit knowledge of the eigenvalues, therefore it is equally suitable for the analysis of the general case.

These observations are good indications that our method can be generalised to the genuinely interacting case, which will be the subject of subsequent work.

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