Energy-constrained diamond norms and their use in quantum information theory

M.E. Shirokov

Abstract

We consider the family of energy-constrained diamond norms on the set of Hermitian-preserving linear maps (superoperators) between Banach spaces of trace class operators. We prove that any norm from this family generates the strong (pointwise) convergence on the set of all quantum channels (which is more adequate for describing variations of infinite-dimensional channels than the diamond norm topology).

We obtain continuity bounds for information characteristics (in particular, classical capacities) of energy-constrained quantum channels (as functions of a channel) with respect to the energy-constrained diamond norms which imply uniform continuity of these characteristics with respect to the strong convergence topology.

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*Steklov Mathematical Institute, RAS, Moscow, email:msh@mi.ras.ru
1 Introduction

The diamond-norm distance between two quantum channels is widely used as a measure of distinguishability between these channels [21, Ch.9]. But the topology (convergence) generated by the diamond-norm distance on the set of infinite-dimensional quantum channels is too strong for analysis of real variations of such channels. Indeed, for any sequence of unitary operators $U_n$ strongly converging to the unit operator $I$ but not converging to $I$ in the operator norm the sequence of channels $\rho \mapsto U_n\rho U_n^*$ does not converge to the identity channel with respect to the diamond-norm distance. In general, the closeness of two quantum channels in the diamond-norm distance means, by Theorem 1 in [11], the operator norm closeness of the corresponding Stinespring isometries. So, if we use the diamond-norm distance then we take into account only such perturbations of a channel that corresponds to uniform deformations of the Stinespring isometry (i.e. deformations with small operator norm). As a result, there exist quantum channels with close physical parameters (quantum limited attenuators) having the diamond-norm distance equal to 2 [25].

To take into account deformations of the Stinespring isometry in the strong operator topology one can consider the strong convergence topology on the set of all quantum channels defined by the family of seminorms $\Phi \mapsto \|\Phi(\rho)\|_1, \rho \in \mathcal{S}(\mathcal{H}_A)$ [9]. The convergence of a sequence of channels $\Phi_n$ to a channel $\Phi_0$ in this topology means that $\lim_{n \to \infty} \Phi_n(\rho) = \Phi_0(\rho)$ for all $\rho \in \mathcal{S}(\mathcal{H}_A)$. In this paper we show that the strong convergence topology on the set of quantum channels is generated by any of the energy-constrained diamond norms on the set of Hermitian-preserving linear maps (superoperators) between Banach spaces of trace class operators (provided the input Hamiltonian satisfies the particular condition).
The energy-constrained diamond norms turn out to be effective tools for quantitative continuity analysis of information characteristics of energy-constrained quantum channels (as functions of a channel). They can be used instead of the diamond norm in standard argumentations, in particular, in the Leung-Smith telescopic method used in [12] for continuity analysis of capacities of finite-dimensional channels.

We obtain continuity bounds for information characteristics (in particular, classical capacities) of energy-constrained quantum channels with respect to the energy-constrained diamond norms. They imply, by the above-mentioned correspondence between these norms and the strong convergence, the uniform continuity of these characteristics with respect to the strong convergence topology.

2 Preliminaries

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space, $\mathfrak{B}(\mathcal{H})$ the algebra of all bounded operators with the operator norm $\| \cdot \|$ and $\mathfrak{T}(\mathcal{H})$ the Banach space of all trace-class operators in $\mathcal{H}$ with the trace norm $\| \cdot \|_1$. Let $\mathcal{S}(\mathcal{H})$ be the set of quantum states (positive operators in $\mathfrak{T}(\mathcal{H})$ with unit trace) [6, 21].

Denote by $I_\mathcal{H}$ the unit operator in a Hilbert space $\mathcal{H}$ and by $\text{Id}_\mathcal{H}$ the identity transformation of the Banach space $\mathfrak{T}(\mathcal{H})$.

If quantum systems $A$ and $B$ are described by Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ then the bipartite system $AB$ is described by the tensor product of these spaces, i.e. $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. A state in $\mathcal{S}(\mathcal{H}_{AB})$ is denoted $\rho_{AB}$, its marginal states $\text{Tr}_{\mathcal{H}_B} \rho_{AB}$ and $\text{Tr}_{\mathcal{H}_A} \rho_{AB}$ are denoted respectively $\rho_A$ and $\rho_B$.

The von Neumann entropy $H(\rho) = \text{Tr} \eta(\rho)$ of a state $\rho \in \mathcal{S}(\mathcal{H})$, where $\eta(x) = -x \log x$, is a concave lower semicontinuous function on $\mathcal{S}(\mathcal{H})$ taking values in $[0, +\infty]$. We will use the binary entropy $h_2(x) = \eta(x) + \eta(1-x)$ and the function $g(x) = (1+x)h_2\left(\frac{x}{1+x}\right) = (x+1) \log(x+1) - x \log x$.

The quantum relative entropy for two states $\rho$ and $\sigma$ is defined as

$$H(\rho \| \sigma) = \sum_i \langle i | \rho \log \rho - \rho \log \sigma | i \rangle,$$

where $\{|i\rangle\}$ is the orthonormal basis of eigenvectors of the state $\rho$ and it is assumed that $H(\rho \| \sigma) = +\infty$ if $\text{supp} \rho$ is not contained in $\text{supp} \sigma$ [14, 20].
The quantum mutual information (QMI) of a state $\rho_{AB}$ is defined as

$$I(A:B)_\rho = H(\rho_{AB}) \rho_A \otimes \rho_B = H(\rho_A) + H(\rho_B) - H(\rho_{AB}),$$

where the second expression is valid if $H(\rho_{AB})$ is finite \[13\]. Basic properties of the relative entropy show that $\rho \mapsto I(A:B)_\rho$ is a lower semicontinuous function on the set $\mathcal{S}(\mathcal{H}_{AB})$ taking values in $[0, +\infty]$.

A quantum channel $\Phi$ from a system $A$ to a system $B$ is a completely positive trace preserving superoperator $\mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_B)$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are Hilbert spaces associated with these systems \[6, 21\].

Denote by $\mathcal{F}(A, B)$ the set of all quantum channels from $A$ to $B$. There are different nonequivalent metrics on $\mathcal{F}(A, B)$. One of them is induced by the diamond norm

$$\|\Phi\|_\diamond = \sup_{\rho \in \mathcal{S}(\mathcal{H}_{AR})} \|\Phi \otimes \text{Id}_R(\rho)\|_1$$

of a Hermitian-preserving superoperator $\Phi : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_B)$ \[1\]. The latter coincides with the norm of complete boundedness of the dual map $\Phi^* : \mathcal{B}(\mathcal{H}_B) \rightarrow \mathcal{B}(\mathcal{H}_A)$ to $\Phi$ \[15\].

3 Some facts about the strong convergence topology on the set of quantum channels

The strong convergence topology on the set $\mathcal{F}(A, B)$ of quantum channels from $A$ to $B$ is generated by the strong operator topology on the set of all linear bounded operators from the Banach space $\mathcal{T}(\mathcal{H}_A)$ into the Banach space $\mathcal{T}(\mathcal{H}_B)$ \[16\]. This topology is studied in detail in \[9\] where it is used for approximation of infinite-dimensional quantum channels by finite-dimensional ones. Separability of the set $\mathcal{S}(\mathcal{H}_A)$ implies that the strong convergence topology on the set $\mathcal{F}(A, B)$ is metrisable (can be defined by some metric). The convergence of a sequence $\{\Phi_n\}$ of channels to a channel $\Phi_0$ in this topology means that

$$\lim_{n \to \infty} \Phi_n(\rho) = \Phi_0(\rho) \text{ for all } \rho \in \mathcal{S}(\mathcal{H}_A).$$

We will use the following simple observations easily proved by using boundedness of the operator norm of all quantum channels.

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Lemma 1. The strong convergence topology on \( \mathcal{F}(A, B) \) coincides with the topology of uniform convergence on compact subsets of \( \mathcal{S}(\mathcal{H}_A) \).

Lemma 2. The strong convergence of a sequence \( \{\Phi_n\} \subset \mathcal{F}(A, B) \) to a channel \( \Phi_0 \) implies strong convergence of the sequence \( \{\Phi_n \otimes \text{Id}_R\} \) to the channel \( \Phi_0 \otimes \text{Id}_R \), where \( R \) is any system.

It is the strong convergence topology that makes the set \( \mathcal{F}(A, B) \) of all channels topologically isomorphic to a subset of states of a composite system (generalized Choi-Jamiolkowski isomorphism).

Proposition 1. Let \( R \cong A \) and \( \omega \) be a pure state in \( \mathcal{S}(\mathcal{H}_{AR}) \) such that \( \omega_A \) is a full rank state in \( \mathcal{S}(\mathcal{H}_A) \). Then the map \( \Phi \mapsto \Phi \otimes \text{Id}_R(\omega) \) is a homeomorphism from the set \( \mathcal{F}(A, B) \) equipped with the strong convergence topology onto the subset \( \{\rho \in \mathcal{S}(\mathcal{H}_{BR}) \mid \rho_R = \omega_R\} \).

If both systems \( A \) and \( B \) are infinite-dimensional then the set \( \mathcal{F}(A, B) \) is not compact in the strong convergence topology. Proposition 1 implies the following compactness criterion for subsets of quantum channels in this topology.

Proposition 2. A subset \( \mathcal{F}_0 \subseteq \mathcal{F}(A, B) \) is relatively compact in the strong convergence topology if and only if there exists a full rank state \( \sigma \) in \( \mathcal{S}(\mathcal{H}_A) \) such that \( \{\Phi(\sigma)\}_{\Phi \in \mathcal{F}_0} \) is a relatively compact subset of \( \mathcal{S}(\mathcal{H}_B) \).

Note that the "only if" part of Proposition 2 is trivial, since the relative compactness of \( \mathcal{F}_0 \) implies relative compactness of \( \{\Phi(\sigma)\}_{\Phi \in \mathcal{F}_0} \) for arbitrary state \( \sigma \) in \( \mathcal{S}(\mathcal{H}_A) \) by continuity of the map \( \Phi \mapsto \Phi(\sigma) \).

Proposition 2 makes it possible to establish existence of a channel with required properties as a limit point of a sequence of explicitly constructed channels by proving relative compactness of this sequence. For example, by this way one can generalize the famous Petz theorem for two non-faithful (degenerate) infinite rank states starting from the standard version of this theorem (in which both states are assumed faithful) [18 the Appendix].

4 The energy-constrained diamond norms

Let \( H_A \) be a positive (unbounded) operator in \( \mathcal{H}_A \) with dense domain treated as a Hamiltonian of quantum system \( A \). Then \( \text{Tr} H_A \rho \) is the (mean) energy
of a state $\rho \in \mathcal{S}(\mathcal{H}_A)$. Consider the family of norms

$$\|\Phi\|_E^E \doteq \sup_{\rho \in \mathcal{S}(\mathcal{H}_A), \text{Tr}H_\rho \leq E} \|\Phi \otimes \text{Id}_R(\rho)\|_1, \quad E > E_0^A, \quad (2)$$
onumber

on the set $\mathcal{L}(A,B)$ of Hermitian-preserving superoperators from $\mathcal{S}(\mathcal{H}_A)$ to $\mathcal{S}(\mathcal{H}_B)$, where $E_0^A$ is the infimum of the spectrum of $H_A$ and $R$ is an infinite-dimensional quantum system. They can be called energy-constrained diamond norms (briefly, ECD-norms). It is clear that formula (2) defines a seminorm on $\mathcal{L}(A,B)$, the implication $\|\Phi\|_E^E = 0 \Rightarrow \Phi = 0$ can be easily shown by the “convex mixture” arguments from the proof of part A of the following

**Proposition 3.** Let $\mathcal{F}(A,B)$ be the set of all channels from $A$ to $B$.

A) The convergence of a sequence $\{\Phi_n\}$ of channels in $\mathcal{F}(A,B)$ to a channel $\Phi_0$ with respect to any of the norms in (2) implies the strong convergence of $\{\Phi_n\}$ to $\Phi_0$, i.e. for any $E > E_0^A$ the following implication holds

$$\lim_{n \to +\infty} \|\Phi_n - \Phi_0\|_E^E = 0 \Rightarrow \lim_{n \to +\infty} \Phi_n(\rho) = \Phi_0(\rho) \quad \forall \rho \in \mathcal{S}(\mathcal{H}_A). \quad (3)$$

B) If the operator $H_A$ has discrete spectrum $\{E_k\}_{k \geq 0}$ of finite multiplicity such that $E_k \to +\infty$ as $k \to +\infty$ then ” $\iff$ “ holds in (3) for any $E > E_0^A$.

**Remark 1.** The assertions of Proposition 3 remain valid for any operator-norm bounded subset of $\mathcal{L}(A,B)$ (instead of $\mathcal{F}(A,B)$). It follows, in particular, that the strong operator topology on such subsets is generated by the single norm $\|\cdot\|_E^E$ provided that the corresponding operator $H_A$ satisfies the condition of part B.

**Proof.** A) The assumed density of the domain of $H_A$ in $\mathcal{H}_A$ implies density of the set $\mathcal{S}_0$ of states $\rho$ with finite energy $\text{Tr}H_\rho$ in $\mathcal{S}(\mathcal{H}_A)$. Hence, since the operator norm of all the superoperators $\Phi_n - \Phi_0$ is bounded, to prove the implication in (3) it suffices to show that $\lim_{n \to +\infty} \Phi_n(\rho) = \Phi_0(\rho)$ for any $\rho \in \mathcal{S}_0$ provided that $\lim_{n \to +\infty} \|\Phi_n - \Phi_0\|_E^E = 0$.

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1The value of $\text{Tr}H_\rho$ (finite or infinite) is defined as $\sup_n \text{Tr}\, P_n H_\rho$, where $P_n$ is the spectral projector of $H_A$ corresponding to the interval $[0, n]$.

2I am grateful to A. Winter who pointed me that formula (2) defines a real norm, see the Note Added at the end of the paper.

3It means that the topology generated on the set $\mathcal{F}(A,B)$ by any of the norms in (2) coincides with the strong convergence topology on $\mathcal{F}(A,B)$. 

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Let $\rho$ be a state in $\mathcal{S}_0$ and $\sigma$ a state such that $\text{Tr}H_A\sigma < E$. Then for sufficiently small $p > 0$ the energy of the state $\rho_p = (1 - p)\sigma + p\rho$ does not exceed $E$. Hence the left part of (3) implies

$$\lim_{n \to +\infty} \Phi_n(\rho_p) = \Phi_0(\rho_p) \quad \text{and} \quad \lim_{n \to +\infty} \Phi_n(\sigma) = \Phi_0(\sigma).$$

It follows that $\Phi_n(\rho)$ tends to $\Phi_0(\rho)$ as $n \to +\infty$.

B) By the assumption $H_A = \sum_{k=0}^{+\infty} E_k^A|\tau_k\rangle\langle\tau_k|$, where $\{|\tau_k\rangle\}_{k=0}^{+\infty}$ is the orthonormal basis of eigenvectors of $H_A$ corresponding to the nondecreasing sequence $\{E_k^A\}_{k=0}^{+\infty}$ of eigenvalues tending to $+\infty$. Let $P_n = \sum_{k=0}^{n-1} |\tau_k\rangle\langle\tau_k|$ be the projector on the subspace $\mathcal{H}_n$ spanned by the vectors $|\tau_0\rangle, \ldots, |\tau_{n-1}\rangle$.

Consider the family of seminorms

$$q_n(\Phi) = \sup_{\rho \in \mathcal{S}(\mathcal{H}_n \otimes \mathcal{H}_R)} \|\Phi \otimes \text{Id}_R(\rho)\|_1, \quad n \in \mathbb{N}, \quad (4)$$

on the set of Hermitian-preserving superoperators from $\mathcal{S}(\mathcal{H}_A)$ to $\mathcal{S}(\mathcal{H}_B)$. Note that the system $R$ in (1) may be $n$-dimensional. Indeed, by convexity of the trace norm the supremum in (1) can be taken over pure states $\rho$ in $\mathcal{S}(\mathcal{H}_n \otimes \mathcal{H}_R)$. Since the marginal state $\rho_R$ of any such pure state $\rho$ has rank $\leq n$, by applying local unitary transformation of the system $R$ we can put all these states into the set $\mathcal{S}(\mathcal{H}_n \otimes \mathcal{H}_n')$, where $\mathcal{H}_n'$ is any $n$-dimensional subspace of $\mathcal{H}_R$.

Let $\{\Phi_k\}$ be a sequence of channels strongly converging to a channel $\Phi_0$. By Lemmas 1 and 2 this implies that $\sup_{\rho \in \mathcal{S}(\mathcal{H}_n \otimes \mathcal{H}_R)} \|\Phi_k \otimes \text{Id}_R(\rho)\|_1$ tends to zero for any compact subset $\mathcal{E}$ of $\mathcal{S}(\mathcal{H}_{AR})$. Since $\dim \mathcal{H}_R = n$, the set $\mathcal{S}(\mathcal{H}_n \otimes \mathcal{H}_R)$ is compact and we conclude that

$$\lim_{k \to \infty} q_n(\Phi_k - \Phi_0) = 0 \quad \forall n \in \mathbb{N}. \quad (5)$$

Let $E \geq E_0^A$ and $\rho$ be a state in $\mathcal{S}(\mathcal{H}_{AR})$ such that $\text{Tr}H_A\rho_A \leq E$. Then the state $\rho_n = (1 - r_n)^{-1} P_n \otimes I_R \rho P_n \otimes I_R$, where $r_n = \text{Tr}(I_A - P_n)\rho_A$, belongs to the set $\mathcal{S}(\mathcal{H}_n \otimes \mathcal{H}_R)$. By using the inequality

$$\|\rho - \rho_n\|_1 \leq 2\|\langle I_A - P_n\rangle \otimes I_R \rho P_n \otimes I_R\|_1 + 2r_n \leq 4\sqrt{r_n} \leq 4\sqrt{E/E_n^A}$$

easily proved via the operator Cauchy-Schwarz inequality (see the proof of Lemma 11.1 in [3]) and by noting that $\text{Tr}H_A\rho_A \leq E$ implies $r_n \leq E/E_n^A$ we obtain

$$\|\rho - \rho_n\|_1 \leq 2\|\langle I_A - P_n\rangle \otimes I_R \rho P_n \otimes I_R\|_1 + 2r_n \leq 4\sqrt{r_n} \leq 4\sqrt{E/E_n^A}.$$
It follows that $\|\Phi_k - \Phi_0\|_E \leq q_n(\Phi_k - \Phi_0) + 8\sqrt{E/E_A}n$. Since $E_A^n \to +\infty$ as $n \to +\infty$, this inequality and (5) show that $\lim_{k \to \infty} \|\Phi_k - \Phi_0\|_E = 0$. □

5 Continuity bounds for information characteristics of quantum channels with respect to the ECD-norms

In this section we show that the ECD-norms can be effectively used for quantitative continuity analysis of information characteristics of energy-constrained quantum channels (as functions of a channel). They allow (due to Proposition 3) to prove uniform continuity of these characteristics with respect to the strong convergence of quantum channels.

In what follows we will consider quantum channels between given infinite-dimensional systems $A$ and $B$. We will assume that the Hamiltonians $H_A$ and $H_B$ of these systems are densely defined positive operators and that

$$\text{Tr}e^{-\lambda H_B} < +\infty$$

for all $\lambda > 0$. (6)

To formulate our main results introduce the function

$$F_{H_B}(E) = \sup_{\text{Tr}H_B\rho \leq E} H(\rho), \quad E \geq E^B_0,$$

where $E^B_0$ is the minimal eigenvalue of $H_B$. Properties of this function are described in Proposition 1 in [17], where it is shown, in particular, that

$$F_{H_B}(E) = \lambda(E)E + \log \text{Tr}e^{-\lambda(E)H_B} = o(E) \quad \text{as} \quad E \to +\infty,$$

where $\lambda(E)$ is determined by the equality $\text{Tr}H_Be^{-\lambda(E)H_B} = E\text{Tr}e^{-\lambda(E)H_B}$, provided that condition (6) holds.

It is well known that condition (6) implies continuity of the von Neumann entropy on the subset of $\mathcal{S}(\mathcal{H}_B)$ determined by the inequality $\text{Tr}H_B\rho \leq E$ for any $E \geq E^B_0$ and attainability of the supremum in (7) at the Gibbs state $\gamma_B(E) = e^{-\lambda(E)H_B}/\text{Tr}e^{-\lambda(E)H_B}$ [20]. So, we have $F_{H_B}(E) = H(\gamma_B(E))$ for any $E > E^B_0$.

The function $F_{H_B}$ is increasing and concave on $[E^B_0, +\infty)$. Denote by $\hat{F}_{H_B}$ any upper bound for $F_{H_B}$ defined on $[0, +\infty)$ possessing the properties

$$\hat{F}_{H_B}(E) > 0, \quad \hat{F}_{H_B}'(E) > 0, \quad \hat{F}_{H_B}''(E) < 0 \quad \text{for all} \quad E > 0 \quad (9)$$
and
\[ \hat{F}_{H_B}(E) = o(E) \quad \text{as} \quad E \to +\infty. \] (10)

At least one such function \( \hat{F}_{H_B} \) always exists. It follows from (8) that one can use \( F_{H_B}(E + E_0^B) \) in the role of \( \hat{F}_{H_B}(E) \).

If \( B \) is the \( \ell \)-mode quantum oscillator with the Hamiltonian
\[ H_B = \sum_{i=1}^{\ell} \hbar \omega_i \left( a_i^+ a_i + \frac{1}{2} I_B \right), \]
where \( a_i \) and \( a_i^+ \) are the annihilation and creation operators and \( \omega_i \) is the frequency of the \( i \)-th oscillator [6, Ch.12] then one can show that
\[ F_{H_B}(E) \leq \hat{F}_{\ell,\omega}(E) = \ell \log E + E_0 + \ell, \] (11)
where \( E_0 = \frac{1}{2} \sum_{i=1}^{\ell} \hbar \omega_i \) and \( E_* = \left[ \prod_{i=1}^{\ell} \hbar \omega_i \right]^{1/\ell} \) and that upper bound (11) is \( \varepsilon \)-sharp for large \( E \) [19]. It is easy to see that the function \( \hat{F}_{\ell,\omega} \) possesses properties (9) and (10). So, it can be used in the role of \( \hat{F}_{H_B} \) in this case.

5.1 The output Holevo quantity of a channel

In this subsection we obtain continuity bound for the function \( \Phi \mapsto \chi(\Phi(\mu)) \) with respect to the ECD-norm, where \( \chi(\Phi(\mu)) \) is the Holevo quantity of the image of a given (discrete or continuous) ensemble \( \mu \) of input states under the channel \( \Phi \).

Discrete ensemble \( \mu = \{ p_i, \rho_i \} \) is a finite or countable collection \( \{ \rho_i \} \) of quantum states with the corresponding probability distribution \( \{ p_i \} \). The Holevo quantity of \( \mu \) is defined as
\[ \chi(\mu) = \sum_i p_i H(\rho_i \| \bar{\rho}(\mu)) = H(\bar{\rho}(\mu)) - \sum_i p_i H(\rho_i), \]
where \( \bar{\rho}(\mu) = \sum_i p_i \rho_i \) is the average state of \( \mu \) and the second formula is valid if \( H(\bar{\rho}(\mu)) < +\infty \). This quantity gives the upper bound for classical information obtained by recognizing states of the ensemble by quantum measurements [5]. It plays important role in analysis of information properties of quantum systems and channels [6, 21].

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4We use the natural logarithm.
Generalized (continuous) ensembles of quantum states are defined as Borel probability measures on the set of quantum states $[6, 8]$. The average state of a generalized ensemble $\mu$ is the barycenter of $\mu$ defined by the Bochner integral

$$\bar{\rho}(\mu) = \int \rho \mu(d\rho).$$

The Holevo quantity of a generalized ensemble $\mu$ is defined as

$$\chi(\mu) = \int H(\rho \| \bar{\rho}(\mu)) \mu(d\rho) = H(\bar{\rho}(\mu)) - \int H(\rho) \mu(d\rho),$$

where the second formula is valid under the condition $H(\bar{\rho}(\mu)) < +\infty [8]$.

The average energy of $\mu$ is given by the formula

$$E(\mu) = \text{Tr} H \bar{\rho}(\mu) = \int \text{Tr} H \rho \mu(d\rho),$$

where $H$ is the Hamiltonian of the system (the integral is well defined, since the function $\rho \mapsto \text{Tr} H \rho$ is lower semicontinuous). For a discrete ensemble $\mu = \{p_i, \rho_i\}$ we have $E(\mu) = \sum_i p_i \text{Tr} H \rho_i$.

For an ensemble $\mu$ of states in $\mathcal{S}(\mathcal{H}_A)$ its image $\Phi(\mu)$ under a quantum channel $\Phi : A \to B$ is defined as the ensemble corresponding to the measure $\mu \circ \Phi^{-1}$ on $\mathcal{S}(\mathcal{H}_B)$, i.e. $\Phi(\mu)\{\mathcal{S}_B\} = \mu\{\Phi^{-1}(\mathcal{S}_B)\}$ for any Borel subset $\mathcal{S}_B$ of $\mathcal{S}(\mathcal{H}_B)$, where $\Phi^{-1}(\mathcal{S}_B)$ is the pre-image of $\mathcal{S}_B$ under the map $\Phi$. If $\mu = \{p_i, \rho_i\}$ then this definition implies $\Phi(\mu) = \{p_i, \Phi(\rho_i)\}$.

**Proposition 4.** Let $\mu$ be a generalized ensemble of states in $\mathcal{S}(\mathcal{H}_A)$ with finite average energy $E(\mu)$. Let $\Phi$ and $\Psi$ be channels from $A$ to $B$ such that $\text{Tr} H B \Phi(\bar{\rho}(\mu)) \leq E$ and $\frac{1}{2} \|\Phi - \Psi\|_{\text{E}(\mu)} \leq \varepsilon$. If the Hamiltonian $H_B$ satisfies conditions (6) then

$$|\chi(\Phi(\mu)) - \chi(\Psi(\mu))| \leq \varepsilon(2t + r_\varepsilon(t)) F_{HB}(E_{\varepsilon t}) + 2g(\varepsilon r_\varepsilon(t)) + 2h_2(\varepsilon t)$$

(12)

for any $t \in (0, \frac{1}{2}]$, where $F_{HB}(E)$ is any upper bound for the function $F_{HB}(E)$ (defined in (7)) with properties (8) and (9), in particular $F_{HB}(E) = F_{HB}(E + E_0^B)$ and $r_\varepsilon(t) = (1 + t/2)/(1 - \varepsilon t)$.

\footnote{The functions $h_2(x)$ and $g(x)$ are defined in Section 2.}
If \( B \) is the \( \ell \)-mode quantum oscillator then

\[
|\chi(\Phi(\mu)) - \chi(\Psi(\mu))| \leq \varepsilon(2t + r_\varepsilon(t)) \left[ \hat{F}_{\ell,\omega}(E) - \ell \log(\varepsilon t) \right] + 2g(\varepsilon r_\varepsilon(t)) + 2h_2(\varepsilon t)
\]

for any \( t \in (0, \frac{1}{2\varepsilon}] \), where \( \hat{F}_{\ell,\omega}(E) \) is defined in (11). Continuity bound (13) with optimal \( t \) is tight for large \( E \).

**Remark 2.** Condition (10) implies that \( \lim_{x \to +0} x \hat{F}_{H_B}(E/x) = 0 \). Hence, the right hand side of (12) tends to zero as \( \varepsilon \to 0 \).

**Remark 3.** It is easy to see that the right hand side of (12) attains minimum at some optimal \( t = t(E, \varepsilon) \). It is this minimum that gives proper upper bound for \( |\chi(\Phi(\mu)) - \chi(\Psi(\mu))| \).

**Proof.** Let \( \mu = \{p_i, \rho_i\}_{i=1}^{m} \) be a discrete ensemble of \( m \leq +\infty \) states and \( \hat{\rho}_{AC} = \sum_{i=1}^{m} p_i \rho_i \otimes |i\rangle\langle i| \) the corresponding cq-state (here \( \{|i\rangle\}_{i=1}^{m} \) is an orthonormal basis in a \( m \)-dimensional Hilbert space \( \mathcal{H}_C \)). Then \( \text{Tr}_{H_B} \hat{\rho}_A = E(\mu) \) and hence

\[
\|\Phi \otimes \text{Id}_C(\hat{\rho}_{AC}) - \Psi \otimes \text{Id}_C(\hat{\rho}_{AC})\|_1 \leq \|\Phi - \Psi\|^E(\mu).
\]

So, in this case the assertions of the proposition follow from Proposition 7 in [19], since the left hand side of (14) coincides with

\[
2D_0(\{p_i, \Phi(\rho_i)\}, \{p_i, \Psi(\rho_i)\}) = \sum_{i=1}^{m} p_i \|\Phi(\rho_i) - \Psi(\rho_i)\|_1.
\]

Let \( \mu \) be an arbitrary generalized ensemble. The construction from the proof of Lemma 1 in [8] gives the sequence \( \{\mu_n\} \) of discrete ensembles weakly\(^6\) converging to \( \mu \) such that \( \hat{\rho}(\mu_n) = \hat{\rho}(\mu) \) for all \( n \). Since the assumption \( \text{Tr}_{H_B} \hat{\Phi}(\hat{\rho}(\mu)), \text{Tr}_{H_B} \hat{\Psi}(\hat{\rho}(\mu)) \leq E \) implies \( H(\hat{\Phi}(\hat{\rho}(\mu)), H(\hat{\Psi}(\hat{\rho}(\mu)))) < +\infty, \)

Corollary 1 in [9] shows that

\[
\lim_{n \to \infty} \chi(\Phi(\mu_n)) = \chi(\Phi(\mu)) \quad \text{and} \quad \lim_{n \to \infty} \chi(\Psi(\mu_n)) = \chi(\Psi(\mu)).
\]

\(^6\)The weak convergence of a sequence \( \{\mu_n\} \) to a measure \( \mu_0 \) means that \( \lim_{n \to \infty} \int f(\rho) \mu_n(d\rho) = \int f(\rho) \mu_0(d\rho) \) for any continuous bounded function \( f \) on \( \mathcal{S}(\mathcal{H}) \) [2].
So, the validity of inequality (12) for the ensemble $\mu$ follows from the validity of this inequality for all the discrete ensembles $\mu_n$ proved before. Inequality (13) is a direct corollary of (12).

The tightness of continuity bound (13) follows from the tightness of continuity bound (22) for the Holevo capacity in this case (obtained from (13)).

By Remark 2 Propositions 3 and 4 imply the following Corollary 1. If the Hamiltonians $H_A$ and $H_B$ satisfy, respectively, the condition of Proposition 3B and condition (6) then for any generalized ensemble $\mu$ of states in $\mathcal{S}(H_A)$ with finite average energy the function $\Phi \mapsto \chi(\Phi(\mu))$ is uniformly continuous on the sets

$$ \mathfrak{F}_{\mu,E} = \{ \Phi \in \mathfrak{F}(A,B) \mid \text{Tr} H_B \Phi(\bar{\rho}(\mu)) \leq E \}, \quad E > E^B_0,$$

with respect to the strong convergence topology.

Remark 4. The uniform continuity with respect to the strong convergence topology means uniform continuity with respect to any metric generating this topology, in particular, with respect to any of the ECD-norms.

5.2 QMI at the output of n copies of a local channel

The following proposition is a corollary of Proposition 3B in [19] proved by the Leung-Smith telescopic trick used in [12] and Winter’s technique from [23]. It gives tight continuity bounds for the function $\Phi \mapsto I(B^n : C)_{\Phi^\otimes_n \otimes \text{Id}_C(\rho)}$ for any given $n$ and a state $\rho \in \mathcal{S}(H_A^\otimes \otimes H_C)$ with respect to the ECD-norm provided that the marginal states $\rho_{A_1}, ..., \rho_{A_n}$ have finite energy.

**Proposition 5.** Let $\Phi$ and $\Psi$ be channels from $A$ to $B$, $C$ be any system and $\rho$ a state in $\mathcal{S}(H_A^\otimes \otimes H_C)$ such that $E_A \doteq \max_{1 \leq k \leq n} \{ \text{Tr} H_A \rho_{A_k} \}$ is finite. Let $\frac{1}{2} \| \Phi - \Psi \|_{E_A^1} \leq \varepsilon$, $\text{Tr} H_B \Phi(\rho_{A_k}), \text{Tr} H_B \Psi(\rho_{A_k}) \leq E_k$ for $k = 1, n$ and

$$\Delta^n(\Phi, \Psi, \rho) \doteq \left| I(B^n : C)_{\Phi^\otimes_n \otimes \text{Id}_C(\rho)} - I(B^n : C)_{\Psi^\otimes_n \otimes \text{Id}_C(\rho)} \right|.$$

If the Hamiltonian $H_B$ satisfies conditions (6) then

$$\Delta^n(\Phi, \Psi, \rho) \leq 2n\varepsilon(2t + r_\varepsilon(t)) \hat{F}_{H_B} \left( \frac{E}{\varepsilon t} \right) + 2ng(\varepsilon r_\varepsilon(t)) + 4nh_2(\varepsilon t) \quad (15)$$

for any $t \in (0, \frac{1}{2\varepsilon}]$, where $E = n^{-1} \sum_{k=1}^n E_k$, $\hat{F}_{H_B}(E)$ is any upper bound for the function $F_{H_B}(E)$ (defined in (7)) with properties (9) and (11), in particular, $\hat{F}_{H_B}(E) = F_{H_B}(E + E_0)$ and $r_\varepsilon(t) = (1 + t/2)/(1 - \varepsilon t)$.

\footnote{The functions $h_2(x)$ and $g(x)$ are defined in Section 2.}
If $B$ is the $\ell$-mode quantum oscillator then
\[
\Delta^n(\Phi,\Psi,\rho) \leq 2n\varepsilon(2t + r_\varepsilon(t))\left[\hat{F}_\ell,\omega(E) - \ell \log(\varepsilon t)\right] + 2ng(\varepsilon r_\varepsilon(t)) + 4nh_2(\varepsilon t),
\]
where $\hat{F}_\ell,\omega(E)$ is defined in (11). Continuity bound (16) with optimal $t$ is tight for large $E$ (for any given $n$).

**Remark 5.** All the assertions of Proposition 5 remain valid for the quantum conditional mutual information (see Proposition 3B in [19]).

Since condition (10) implies $\lim_{x \to +0} x\hat{F}_H(E/x) = 0$, the right hand side of (15) tends to zero as $\varepsilon \to 0$. Hence Propositions 3 and 5 imply the following

**Corollary 2.** If the Hamiltonians $H_A$ and $H_B$ satisfy, respectively, the condition of Proposition 3B and condition (6) then for any $n \in \mathbb{N}$ and any state $\rho$ in $\mathcal{S}(H_A \otimes H_C)$ such that $\max_{1 \leq k \leq n} \{\text{Tr} H_A \rho A_k\} < +\infty$ the function $\Phi \mapsto \mathcal{I}(B^n:C)\Phi \otimes 1_{\text{Id}} C(\rho)$ is uniformly continuous on the sets
\[
\mathfrak{F}_{\rho,E} = \left\{ \Phi \in \mathfrak{F}(A,B) \left| \frac{1}{n} \sum_{k=1}^{n} \text{Tr} H_B \Phi(\rho A_k) \leq E \right\}, \quad E > E^B_0,
\]
with respect to the strong convergence topology.

### 5.3 Classical capacities of infinite-dimensional energy-constrained channels

When we consider transmission of classical information over infinite-dimensional quantum channels we have to impose the energy constraint on states used for coding information. For a single channel $\Phi : A \to B$ the energy constraint is determined by the linear inequality
\[
\text{Tr} H_A \rho \leq E,
\]
where $H_A$ is the Hamiltonian of the input system $A$. For $n$ copies of this channel the energy constraint is given by the inequality
\[
\text{Tr} H_A^n \rho^{(n)} \leq nE,
\]
\footnote{see Remark 4}
where $\rho^{(n)}$ is a state of the system $A^n$ ($n$ copies of $A$) and

$$H_{A^n} = H_A \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes H_A$$  \hspace{1cm} (18)$$

is the Hamiltonian of the system $A^n$. An operational definition of the classical capacity of an infinite-dimensional energy-constrained quantum channel can be found in [7]. If only nonentangled input encoding is used then the ultimate rate of transmission of classical information through the channel $\Phi$ with the constraint (17) on mean energy of a code is determined by the Holevo capacity

$$C_\chi(\Phi, H_A, E) = \sup_{\text{Tr} H_A \bar{\rho} \leq E} \chi(\{p_i, \Phi(\rho_i)\}), \quad \bar{\rho} = \sum_i p_i \rho_i,$$  \hspace{1cm} (19)$$

(the supremum is over all input ensembles $\{p_i, \rho_i\}$ such that $\text{Tr} H_A \bar{\rho} \leq E$). By the Holevo-Schumacher-Westmoreland theorem adapted to constrained channels ([7, Proposition 3]), the classical capacity of the channel $\Phi$ with constraint (17) is given by the following regularized expression

$$C(\Phi, H_A, E) = \lim_{n \to +\infty} n^{-1} C_\chi(\Phi^{\otimes n}, H_{A^n}, nE),$$

where $H_{A^n}$ is defined in (18). If $C_\chi(\Phi^{\otimes n}, H_{A^n}, nE) = nC_\chi(\Phi, H_A, E)$ for all $n$ then

$$C(\Phi, H_A, E) = C_\chi(\Phi, H_A, E),$$  \hspace{1cm} (20)$$

i.e. the classical capacity of the channel $\Phi$ coincides with its Holevo capacity. Note that (20) holds for many infinite-dimensional channels [6]. Recently it was shown that (20) holds if $\Phi$ is a gauge covariant or contravariant Gaussian channel and $H_A = \sum_{ij} \epsilon_{ij} a_i^\dagger a_j$ is a gauge invariant Hamiltonian (here $[\epsilon_{ij}]$ is a positive matrix) [3, 4].

The following proposition presents estimates for differences between the Holevo capacities and between the classical capacities of channels $\Phi$ and $\Psi$ with finite energy amplification factors for given input energy, i.e. such that

$$\sup_{\text{Tr} H_A \rho \leq E} H_B \Phi(\rho) \leq kE \quad \text{and} \quad \sup_{\text{Tr} H_A \rho \leq E} H_B \Psi(\rho) \leq kE$$  \hspace{1cm} (21)$$

for given $E \geq E_0^A$ and finite $k = k(E)$. Note that any channels produced in a physical experiment satisfy condition (21).

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9The gauge invariance condition for $H_A$ can be replaced by the condition (18) in [4].
Proposition 6. Let $\Phi$ and $\Psi$ be quantum channels from $A$ to $B$ satisfying condition (21) and $\frac{1}{2}\|\Phi - \Psi\|_E \leq \varepsilon$. If the Hamiltonian $H_B$ satisfies conditions (6) then

\[ |C_\chi(\Phi, H_A, E) - C_\chi(\Psi, H_A, E)| \leq \varepsilon(2t + r_\varepsilon(t))\widehat{F}_{H_B}(kE) + 2g(\varepsilon r_\varepsilon(t)) + 2h_2(\varepsilon t) \]

and

\[ |C(\Phi, H_A, E) - C(\Psi, H_A, E)| \leq 2\varepsilon(2t + r_\varepsilon(t))\widehat{F}_{H_B}(kE) + 2g(\varepsilon r_\varepsilon(t)) + 4h_2(\varepsilon t) \]

for any $t \in (0, \frac{1}{2\varepsilon}]$, where $r_\varepsilon(t) = (1 + t/2)/(1 - \varepsilon t)$ and $\widehat{F}_{H_B}(E)$ is any upper bound for the function $F_{H_B}(E)$ (defined in (7)) with properties (9) and (10), in particular, $\widehat{F}_{H_B}(E) = F_{H_B}(E + E_{B0})$.

If $B$ is the $\ell$-mode quantum oscillator then the right hand sides of (22) and (23) can be rewritten, respectively, as follows

\[ \varepsilon(2t + r_\varepsilon(t))\left[\widehat{F}_{\ell,\omega}(kE) - \ell \log(\varepsilon t)\right] + 2g(\varepsilon r_\varepsilon(t)) + 2h_2(\varepsilon t) \]

and

\[ 2\varepsilon(2t + r_\varepsilon(t))\left[\widehat{F}_{\ell,\omega}(kE) - \ell \log(\varepsilon t)\right] + 2g(\varepsilon r_\varepsilon(t)) + 4h_2(\varepsilon t), \]

where $\widehat{F}_{\ell,\omega}(E)$ is defined in (11). In this case continuity bound (22) with optimal $t$ is tight for large $E$ while continuity bound (23) is close-to-tight (up to the factor 2 in the main term).

Proof. Inequality (22) follows from definition (19) and Proposition 4.

To prove inequality (23) note that

\[ C_\chi(\Phi^\otimes n, H_{A^n}, nE) = \sup \chi(\{p_i, \Phi^\otimes n(\rho_i)\}), \]

where the supremum is over all ensembles $\{p_i, \rho_i\}$ of states in $\mathcal{S}(H_{A^n}^\otimes)$ with the average state $\bar{\rho}$ such that $\text{Tr}H_{A^n}\bar{\rho}_{A_j} \leq E$ for all $j = 1, n$. This can be easily shown by using symmetry arguments and the following well known property of the Holevo quantity:

\[ \frac{1}{n} \sum_{j=1}^n \chi(\{q_i^j, \sigma_i^j\}) \leq \chi\left(\left\{\frac{q_i^j}{n}, \sigma_i^j\right\}\right) \]

10The functions $h_2(x)$ and $g(x)$ are defined in Section 2.
for any collection \( \{ q_i^1, \sigma_i^1 \}, \ldots, \{ q_i^n, \sigma_i^n \} \) of discrete ensembles.

Since condition (21) implies

\[
\operatorname{Tr} H_B \Phi(\bar{\rho}_A) \leq kE \quad \text{and} \quad \operatorname{Tr} H_B \Psi(\bar{\rho}_A) \leq kE, \quad j = 1, n,
\]

for any ensemble \( \{ p_i, \rho_i \} \) satisfying the above condition, continuity bound (23) is obtained by using the representations

\[
\chi(\{ p_i, \Lambda^n(\rho_i) \}) = I(B^n : C)_{\Lambda^n \otimes \text{Id}_C(\hat{\rho})}, \quad \Lambda = \Phi, \Psi, \quad \text{where} \quad \hat{\rho}_{AC} = \sum_i p_i \rho_i \otimes |i\rangle \langle i|,
\]

Proposition 5 and the corresponding analog of Lemma 12 in [12].

If \( B \) is the \( \ell \)-mode quantum oscillator and \( \hat{F}_{H_B} = \hat{F}_{\ell,\omega} \) then we can estimate the right hand sides of (22) and (23) from above by using the inequality

\[
\hat{F}_{\ell,\omega}(E/x) \leq \hat{F}_{\ell,\omega}(E) - \ell \log x \quad \text{valid for any positive } E \text{ and } x \leq 1.
\]

The tightness of continuity bound (22) can be shown assuming that \( \Phi \) is the identity channel from the \( \ell \)-mode quantum oscillator \( A \) to \( B = A \) and \( \Psi \) is the completely depolarizing channel with the vacuum output state. These channels satisfy condition (21) with \( k = 1 \) and

\[
|C_\chi(\Phi, H_A, E) - C_\chi(\Psi, H_A, E)| = \sup_{\operatorname{Tr} H_A \rho \leq E} H(\rho) = F_{H_A}(E).
\]

By Lemma 2 in [19] in the case \( \hat{F}_{H_B} = \hat{F}_{\ell,\omega} \) the main term of (22) can be made not greater than \( \varepsilon[\hat{F}_{\ell,\omega}(E) + o(\hat{F}_{\ell,\omega}(E))] \) for large \( E \) by appropriate choice of \( t \). So, the tightness of continuity bound (22) follows from (24), since \( ||\Phi - \Psi||_E^E \leq ||\Phi - \Psi||_F = 2 \) and \( \lim_{E \to +\infty}(\hat{F}_{\ell,\omega}(E) - F_{H_A}(E)) = 0 \).

The above example also shows that the main term in continuity bound (23) is close to the optimal one up to the factor 2, since \( C(\Phi, H_A, E) = C_\chi(\Phi, H_A, E) \) and \( C(\Psi, H_A, E) = C_\chi(\Psi, H_A, E) \) for the channels \( \Phi \) and \( \Psi \).

The operational definition of the entanglement-assisted classical capacity of an infinite-dimensional energy-constrained quantum channel is given in [7, 10]. By the Bennett-Shor-Smolin-Thaplyal theorem adapted to constrained channels ([10 Theorem 1]) the entanglement-assisted classical capacity of an infinite-dimensional channel \( \Phi \) with the energy constraint (17) is given by the expression

\[
C_{ea}(\Phi, H_A, E) = \sup_{\operatorname{Tr} H_A \rho \leq E} I(\Phi, \rho),
\]

16
where $I(\Phi, \rho) = I(B : R)_{\Phi \otimes \text{Id}_R(\hat{\rho})}$, $\hat{\rho}_A = \rho$, rank$\hat{\rho} = 1$, is the quantum mutual information of the channel $\Phi$ at a state $\rho$. Proposition 5 with $n = 1$ implies the following

**Proposition 7.** Let $\Phi$ and $\Psi$ be quantum channels from $A$ to $B$ and $\frac{1}{2} \| \Phi - \Psi \|_E^E \leq \varepsilon$, where $\| \cdot \|_E$ is the ECD-norm defined in (2).

A) If the Hamiltonian $H_A$ satisfies condition (8) then

$$|C_{ea}(\Phi, H_A, E) - C_{ea}(\Psi, H_A, E)| \leq 2\varepsilon(2t + \varepsilon_r(t))\hat{F}_{H_A}(\frac{E}{\varepsilon t})$$

(25) + $2g(\varepsilon_r(t)) + 4h_2(\varepsilon t)$

for any $t \in (0, \frac{1}{2\varepsilon}]$, where $\varepsilon_r(t) = (1 + t/2)/(1 - \varepsilon t)$ and $\hat{F}_{H_A}(E)$ is any upper bound for the function $F_{H_A}(E)$ (defined in [7]) with properties (9) and (10), in particular, $\hat{F}_{H_A}(E) = F_{H_A}(E + E_0^A)$.

If $A$ is the $\ell$-mode quantum oscillator then the right hand side of (25) can be rewritten as follows

$$2\varepsilon(2t + \varepsilon_r(t))\left[\hat{F}_{\ell,\omega}(E) - \ell \log(\varepsilon t)\right] + 2g(\varepsilon_r(t)) + 4h_2(\varepsilon t),$$

where $\hat{F}_{\ell,\omega}(E)$ is defined in (11). In this case continuity bound (25) with optimal $t$ is tight for large $E$.

B) If the channels $\Phi$ and $\Psi$ satisfy condition (21) and the Hamiltonian $H_B$ satisfies condition (9) then

$$|C_{ea}(\Phi, H_A, E) - C_{ea}(\Psi, H_A, E)| \leq 2\varepsilon(2t + \varepsilon_r(t))\hat{F}_{H_B}(\frac{kE}{\varepsilon t})$$

(26) + $2g(\varepsilon_r(t)) + 4h_2(\varepsilon t)$

for any $t \in (0, \frac{1}{2\varepsilon}]$, where $\hat{F}_{H_B}(E)$ is any upper bound for the function $F_{H_B}(E)$ with properties (9) and (10), in particular, $\hat{F}_{H_B}(E) = F_{H_B}(E + E_0^B)$.

If $B$ is the $\ell$-mode quantum oscillator then the right hand side of (26) can be rewritten as follows

$$2\varepsilon(2t + \varepsilon_r(t))\left[\hat{F}_{\ell,\omega}(kE) - \ell \log(\varepsilon t)\right] + 2g(\varepsilon_r(t)) + 4h_2(\varepsilon t),$$

where $\hat{F}_{\ell,\omega}(E)$ is defined in (11). In this case continuity bound (26) with optimal $t$ is tight for large $E$. 17
Note that continuity bound (25) holds for arbitrary channels \( \Phi \) and \( \Psi \).

Proof. A) Let \( \mathcal{H}_R \cong \mathcal{H}_A \) and \( H_R \) be an operator in \( \mathcal{H}_R \) unitarily equivalent to \( H_A \). For any state \( \rho \) satisfying the condition \( \text{Tr} H_A \rho \leq E \) there exists a purification \( \hat{\rho} \in \mathcal{S}(\mathcal{H}_{AR}) \) such that \( \text{Tr} H_R \hat{\rho} \leq E \). Since

\[
I(\Phi, \rho) = I(B:R)_\sigma \quad \text{and} \quad I(\Psi, \rho) = I(B:R)_\varsigma,
\]

where \( \sigma = \Phi \otimes \text{Id}_R(\hat{\rho}) \) and \( \varsigma = \Psi \otimes \text{Id}_R(\hat{\rho}) \) are states in \( \mathcal{S}(\mathcal{H}_{BR}) \) such that \( \text{Tr} H_R \sigma_R, \text{Tr} H_R \varsigma_R \leq E \) and \( \| \sigma - \varsigma \|_1 \leq \| \Phi - \Psi \|_E \), Proposition 2 in [19] with trivial \( C \) shows that the value of \( |I(\Phi, \rho) - I(\Psi, \rho)| \) is upper bounded by the right hand side of (25).

B) Continuity bound (26) is obtained similarly, since in this case we have \( \text{Tr} H_B \sigma_B, \text{Tr} H_B \varsigma_B \leq kE \).

The specifications concerning the cases when either \( A \) or \( B \) is the \( \ell \)-mode quantum oscillator follow from the inequality \( \hat{F}_{\ell,\omega}(E/x) \leq \hat{F}_{\ell,\omega}(E) - \ell \log x \) valid for any positive \( E \) and \( x \leq 1 \).

The tightness of the continuity bounds (25) and (26) can be shown assuming that \( \Phi \) is the identity channel from the \( \ell \)-mode quantum oscillator \( A \) to \( B = A \) and \( \Psi \) is the completely depolarizing channel with the vacuum output state. It suffices to note that \( C_{\text{ea}}(\Phi, H_A, E) = 2F_{H_A}(E) \) and \( C_{\text{ea}}(\Psi, H_A, E) = 0 \) and to repeat the arguments from the proof of Proposition 6.

\[ \square \]

If a function \( \hat{F}_H \) satisfies condition (10) then \( \lim_{x \to +0} x \hat{F}_H(E/x) = 0 \). So, Propositions 3, 6 and 7 imply the following observations.

**Corollary 3.** Let \( \mathfrak{F}(A,B) \) be the set of all quantum channels from \( A \) to \( B \) equipped with the strong convergence topology (described in Section 3).

A) If the Hamiltonians \( H_A \) and \( H_B \) satisfy, respectively, the condition of Proposition 3B and condition (6) then the functions

\[
\Phi \mapsto C_X(\Phi, H_A, E), \quad \Phi \mapsto C(\Phi, H_A, E) \quad \text{and} \quad \Phi \mapsto C_{\text{ea}}(\Phi, H_A, E)
\]

are uniformly continuous\(^{11} \) on any subset of \( \mathfrak{F}(A,B) \) consisting of channels with bounded energy amplification factor (for the given input energy \( E \)).

B) If the Hamiltonian \( H_A \) satisfies conditions (6) then the function \( \Phi \mapsto C_{\text{ea}}(\Phi, H_A, E) \) is uniformly continuous on \( \mathfrak{F}(A,B) \).

\(^{11}\) see Remark 4
Note Added: After posting the first version of this paper I was informed by A.Winter that he and his colleagues independently have come to the same "energy bounded" modification of the diamond norm. I am grateful to A.Winter for sending me a draft of their paper [24], which complements the present paper by detailed study of the ECD-norm from the physical point of view and by continuity bounds for the quantum and private classical capacities of energy-constrained channels. I hope it will appear soon.

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