Is the ISO(2, 1) Gauge Gravity equivalent to the Metric Formulation?

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Abstract

The quantization of the gravitational Chern-Simons coefficient is investigated in the framework of ISO(2, 1) gauge gravity. Some paradoxes involved are cured. The resolution is largely based on the inequivalence of ISO(2, 1) gauge gravity and the metric formulation. Both the Lorentzian scheme and the Euclidean scheme lead to the coefficient quantization, which means that the induced spin is not quite exotic in this context.
I. INTRODUCTION

Despite of its local triviality, (2+1)-dimensional Einstein gravity has been studied for a long time because it has still rich global structure [1] and can be considered as the physics of the (3+1)-dimensional cosmic strings [2]. Moreover, the geometry becomes nontrivial at the semiclassical level [3] where the gravitational Chern-Simons term (CST),

$$\mathcal{L}_{CS} = -\frac{1}{\kappa} \omega_a \wedge (d\omega^a + \frac{1}{3} \epsilon_{abc} \omega^b \wedge \omega^c),$$

(1)

is generated effectively by the fermion coupled to gravity [4]. Here $\kappa = 16\pi G$ and $\alpha$ is a coefficient with the dimension of length.

One of the most interesting and intricate things of this latter semiclassical system is the coefficient quantization paradox for the gravitational CST [5]. This problem is well addressed in the papers [6] and can be briefly summarized as follows.

The subject can be probed in two different contexts, the dreibein formulation and the metric formulation. In each context, it can be treated again in two ways, the Lorentzian method and the Euclidean method. This latter method is based on the analytic continuation procedure called the Wick rotation, which is conventionally adopted in most of the quantum field theories.

One important point in the dreibein formulation is to check whether the gauge symmetry $SO(2, 1)$ of Lorentzian system becomes $SO(3)$ upon the Wick rotation. This is not simple to answer since the internal gauge symmetry is not affected by such a transformation in the ordinary Yang-Mills theory. However in the dreibein formulation, ‘Euclideanization’ will be shown below to change the gauge symmetry. Then the third-homotopy argument [3,6] tells us that the coefficient is quantized in the Euclidean scheme but not in the Lorentzian method.

Since it is hard to say that those two schemes are not equivalent, this result gets us into a confusion. Another confusion comes from the metric formulation, where the local symmetry is known as the diffeomorphism. Since every diffeomorphism of $\mathbb{R}^3$ with the given boundary
conditions is homotopic to the identity, the coefficient need not be quantized irrespective of
the metric signature \[7\]. This result is in contradiction with that of gauge formulation since
those two formulations are conventionally believed to be equivalent.

An attempt to solve these paradoxes has been made in \[6\]. It was shown there that
the coefficient need not be quantized in the gauge formulation even through the Euclidean
method. In the proof, \(\theta\)-sector term \[8\] was introduced. The term has its own coefficient
and has no dynamical relevance due to its topological nature. Then through the Euclidean
argument, one gets the quantization condition on the \textbf{linear combination} of the coeffi-
cients of gravitational CST and \(\theta\)-sector term rather than on that of the former term alone.
Therefore for any value of the gravitational CS coefficient, one can adjust the coefficient of
\(\theta\)-sector term to satisfy the quantization condition. Consequently, one can conclude in all
cases that the gravitational CS coefficient need not be quantized.

However as is said in the paper \[3\], the appearance of the \(\theta\)-sector term and the physical
interpretation of its coefficient are unclear in the Einstein-Hilbert Lagrangian. Besides, the
new condition tells us that those two coefficients are mutually dependent. This might be
a paradox because there is no reason to think so in the Lorentzian argument. Another
important point to note is that the dreibein formulation and the metric formulation are \textbf{not}
equivalent at least for the above semiclassical system \[9\]. Therefore, we are left with only
one paradox: In the dreibein formulation, the Euclidean scheme and Lorentzian scheme give
different results for the coefficient quantization.

In this paper, we solve this paradox in the context of Poincaré gauge gravity. For that
purpose, we briefly review in the next section, the Poincaré gauge formulation of the (2+1)-
dimensional gravity with the gravitational CST. We discuss about its inequivalence with
the metric formulation, focussing on the resulting geometries a massive point source makes
in each formulation. Section 3 deals with the Euclidean method to show explicitly that
the internal gauge group \(ISO(2, 1)\) of the Poincaré gauge gravity changes to \(ISO(3)\) upon
Wick rotation. In section 4, it is shown that even the Lorentzian argument results in the
quantization condition for the gravitational CS coefficient. This amazing result is because
the homotopy map in this case should be further specified to respect the Lorentz structure. Section 5 concludes the paper discussing about the physical implications of our results and some future prospects on the subject.

II. (2+1)-DIMENSIONAL GRAVITY WITH THE GRAVITATIONAL CST

One interesting point in (2+1)-dimension is the existence of another formulation of gravity, i.e., Poincaré gauge gravity [10]. A characteristic feature of this formulation is that the Einstein-Hilbert Lagrangian can be obtained just from the Chern-Simons Lagrangian for the (2+1)-dimensional Poincaré group, ISO(2, 1):

\[-\frac{1}{\kappa} \mathcal{L}_{EH} = -\frac{1}{\kappa} < A \wedge (dA + \frac{2}{3} A \wedge A) > -\frac{1}{\kappa} e_a \wedge (2d\omega^a + \epsilon^{abc} \omega^b \wedge \omega^c), \tag{2}\]

where the Lie algebra valued one form \( A = e^a P_a + \omega^a J_a \) is the gauge connection and the nondegenerate invariant quadratic form \(<,>\) on the ISO(2, 1) group manifold is defined as \(< P_a, J_b > = \eta_{ab}, < J_a, J_b > = 0, < P_a, P_b > = 0\). In the above derivation, the following \( iso(2, 1) \) algebra was used.

\[ [P_a, J_b] = \epsilon_{ab}^c P_c, \quad [J_a, J_b] = \epsilon_{ab}^c J_c, \quad [P_a, P_b] = 0, \tag{3}\]

where \( \epsilon^{012} = -\epsilon_{012} = 1 \). The definition of the torsion and the curvature are manifest in its field strength components: \( F_{\mu\nu} \equiv \mathcal{T}^{a}_{\mu\nu} P_a + \mathcal{R}^{a}_{\mu\nu} J_a \).

When the matrix composed of the components \( e^a_{\mu} \) is invertible, the Lagrangian (2) is the same as that of the dreibein formulation. However, this needs not be the case because \( e^a_{\mu} \) are the gauge connection components. To define the invertible soldering form that geometrically relates the affine tangent space with the base manifold, we need another important ingredient, the Poincaré coordinates, \( \phi^a, (a = 0, 1, 2) \) [9,11]. These are nothing but the isovector components and are concerned with the affine nature of the group ISO(2, 1). We can define gauge invariant metric making use of the soldering form \( \mathcal{E}^{a}_{\mu} \equiv \mathcal{D}_{\mu} \phi^a \). Therefore in a specific gauge making \( \phi^a(x^\mu) = 0 \), the above Lagrangian (2) can be considered as that
of the dreibein formulation. Hereafter we use the terminologies, the dreibein formulation
and the Poincaré gauge formulation interchangeably.

On the other hand, if we restrict the gauge connection components to satisfy the torsion
free condition, \( T = de + \omega \wedge e = 0 \), the same Lagrangian (2) can be considered as that
of the metric formulation. The physics of this metric formulation is thoroughly studied in
the seminal paper [1]. For the case with a massive spinning point source, the geometry is
locally trivial but globally has the space-conical and time-helical structure. As can be noted
above, the essential difference between the dreibein formulation and the metric formulation
is concerned with the torsion free condition. However, despite of this difference, they result
in the same geometry both for the source free case and for the case with a massive spinning
point source [11].

The system can be generalized to include the gravitational CST (1). The term was first
introduced by S. Deser, R. Jackiw, and S. Templeton in the metric formulation to make the
(2+1)-dimensional geometry locally nontrivial [3]. Afterwards, the term was also shown to
be a possible term generated effectively at the semiclassical level by the radiative correction
of the fermion field coupled to gravity [1].

One can assume the same term to be generated in the Poincaré gauge formulation also,
although we don’t show it explicitly. This might happen because the spinor transforms under
only the subgroup \( SO(2, 1) \) and the CST can be generated as in other (2+1)-dimensional
gauge theories [12]. Apart from this naive argument, the term can be formulated in the
context of \( ISO(2, 1) \) gauge theory by adopting the following generalized quadratic form for
the Lie algebra \( iso(2, 1) \):

\[
<P_a, J_b >= \eta_{ab}, \quad <J_a, J_b >= \alpha \eta_{ab}, \quad <P_a, P_b >= 0.
\]

(4)

This is easily shown to be the most general quadratic form that is nondegenerate and
associative, i. e., has the cyclic property like the trace. Making use of this quadratic form,
one can get both the Einstein-Hilbert term (2) and the gravitational CST (1) through the
CSL [9].
One can guess from this enlarged Lagrangian that the geometry heavily depends on the formulation in consideration. In the metric formulation one use the torsion free condition. This means the gauge connection components $\omega^a{}_{\mu}$ and $e^a{}_{\mu}$ are not independent of each other. Indeed the former can be written in terms of the latter as

$$\omega^a{}_{b\mu} = -\partial_{[\mu}e^a{}_{\nu]}e_b{}^\nu + \partial_{[\mu|}e_{b\nu]}e^{a\nu} - e_c{}_{\sigma}[\partial_{[\mu}e^c{}_{\sigma]}e^{a\rho}e_b{}^\rho].$$

Due to this condition, the derivative order of the gravitational CST, (1), becomes third while the Einstein-Hilbert term (2) remains to be of the first order. This relative difference in the derivative order makes the gravitational CST dominant at the large momentum limit while the Einstein-Hilbert term relevant at the small momentum limit. This also provides the clue for the topologically massive nature of the graviton. In the asymptotic region, where the Einstein-Hilbert term becomes dominant, the geometry becomes that of pure Einstein-Hilbert case except the induced spin $\alpha m$ [13]:

$$ds^2 \sim -(dt - \frac{\kappa(\alpha m + \sigma)}{4\pi}d\theta)^2 + \frac{1}{r^{2\sigma}}(dr^2 + r^2 d\theta^2),$$

where $\sigma$ is the particle spin.

On the other hand in the gauge formulation, both terms, (1) and (2) are of the same derivative order. Therefore, the dynamics does not depend on the momentum scale at all. Specifically this case allows the exact solution, so the above asymptotic solution extends to all space except the source point. The geometry is locally flat with no topologically massive graviton even in the presence of the gravitational CST. This is the result of treating the connection components $e$ and $\omega$ as independent variables. Moreover, it opposes the usual belief that the effective Chern-Simons-like terms assure the topologically massive mode for the gauge particles. The only effect of the term is the induced spin [9]. This is in contrast with the case of the metric formulation, where it was shown that no exact stationary solution with the above asymptotically flat limit (1) is possible except the critical case ($\sigma + \alpha m = 0$) [14].

Another difference is that they choose, in the metric formulation, the negative sign of Einstein constant to avoid the repulsive force for the positive mass. Therefore, the deficit
angle $8\pi Gm = \kappa m/2$ is negative and there is no mass bound. However, there is no such problem in the gauge formulation because the geometry is locally flat. The positive deficit angle of our case results in a mass bound $1/4G$.

This inequivalence is in contrast with the (3+1)-dimensional case, where the Einstein gravity and Cartan theory are equivalent in the source free region. In fact, it is a general feature in the ‘Palatini’ type actions with an extra term, whose derivative order depends on the Palatini condition.

III. EUCLIDEAN METHOD

Now let us go into the main point of this paper, the quantization of $\alpha$. Making use of the analytic continuation method, we first work in the Euclidean space, where the variation of the action under large gauge transformation is easy to visualize as some topological entity [3]. In the context of the Poincaré gauge gravity, we show that the symmetry $ISO(2,1)$ becomes $ISO(3)$ upon Wick rotation to the Euclidean space. We also check whether the Lagrangian in the Euclidean space is real or imaginary. Only the real action can result in the quantization of $\alpha$ coefficient when it is exponentiated with the extra quantum coefficient $i/\hbar$ (see Witten in [10]).

The transfer to the Euclidean space is performed by just changing Poincaré coordinates $\phi^a$ and gauge connections $\omega^a, e^a$ to the Euclidean version. This automatically induces the external spacetime coordinates transfer through the definition of the metric. The transfer relations between those two versions are given by $i \phi^0 \Rightarrow \tilde{\phi}^3, \phi^i \Rightarrow \tilde{\phi}^i, -\omega^0 \Rightarrow \tilde{\omega}^3, i \omega^i \Rightarrow \tilde{\omega}^i, i e^0 \Rightarrow \tilde{e}^3$ and $e^i \Rightarrow \tilde{e}^i$, where the tilde ‘~’ over variables denotes the Euclidean version. These relations are based on the relations between the two versions of the soldering form $i \mathcal{E}^0_\mu \Rightarrow \tilde{\mathcal{E}}^3_\mu$ and $\mathcal{E}^1_\mu \Rightarrow \tilde{\mathcal{E}}^i_\mu$.

The above transfer relations amount to the change of the internal gauge symmetry;

$$A = \omega^a J_a + e^0 P_a = \omega^0 J_0 + \omega^i J_i + e^0 P_0 + e^i P_i$$

$$\Rightarrow -\tilde{\omega}^3 J_0 - i\tilde{\omega}^3 J_i - i\tilde{\omega}^3 P_0 + \tilde{e}^i P_i = \tilde{A},$$ (7)
where one can read off the Euclidean generators $\tilde{J}_3 = -J_0 = J^0$, $\tilde{J}_i = -i J_i$, $\tilde{P}_3 = -i P_0 = i P^0$, $\tilde{P}_i = P_i$ satisfying $iso(3)$ algebra. This means that the internal Wick rotation of the affine coordinate $\phi^a$ effectively induces the metric transformation from Lorentzian to Euclidean and changes the gauge symmetry from $ISO(2,1)$ to $ISO(3)$. This is the crucial difference from the conventional gauge theory where the transfer of the external spacetime to Euclidean does not affect the internal space and the gauge group.

According to the above transfer relations, we are led to the Lagrangian transfer: $i L_{EH} \Rightarrow \tilde{L}_{EH}$, $-L_{CS} \Rightarrow \tilde{L}_{CS}$. Therefore, only CST remains to be real upon the transfer. Here, one should note that this internal Wick’s rotation may change the topological CSL but does not change the partition function because the analytic continuation of our concern is performed in the functional space only.

We next check whether the variation of the Euclidean action under a large gauge transformation gives some topological invariants concerned with the third homotopy structure of the transformation group [15]. Under the large transformation $A \rightarrow A' = U^{-1} A U + U^{-1} dU$,

$$\delta \tilde{L} = -\frac{1}{3} < U^{-1} dU \wedge U^{-1} dU \wedge U^{-1} dU >_E$$

$$= \frac{\alpha}{6} \varepsilon_{abc} (\Lambda^{-1} d\Lambda)^a \wedge (\Lambda^{-1} d\Lambda)^b \wedge (\Lambda^{-1} d\Lambda)^c$$

$$+ \frac{i}{2} \varepsilon_{abc} (\Lambda^{-1} dq)^a \wedge (\Lambda^{-1} d\Lambda)^b \wedge (\Lambda^{-1} d\Lambda)^c,$$

where the $<,>_E$ is the Euclidean inner product and $\Lambda \in SO(3)$ together with $q \in T(3)$ constitutes the Poincaré element $U$.

In the quantum system, the finite action condition requires $U$ tend to constant in the asymptotic region. This means the field configuration $U$ can be considered as a map from $S^3$ to $ISO(2,1)$. In the eq. (8), only the first term in components can survive upon integration over $S^3$ since it is renewed as $\alpha Tr (\Lambda^{-1} d\Lambda)^3 / 6$. This counts the Brouwer degree (winding number) from $S^3$ (the compactified Euclidean space) to $S^3/\mathbb{Z}^2 \sim SO(3)$. The integration of second term over $S^3$ vanishes because it is a total divergence. Indeed, one can easily see this making use of the torsion free equation and the zero curvature equation for the null gauge distribution, $e = \Lambda^{-1} dq$, $\omega = \Lambda^{-1} d\Lambda$. Therefore, the invariance of the system under
the large transformation requires the quantization of CST coefficient. In fact, considering
the matter interaction, we note that extra coefficient $-1/\kappa$ is necessary for the whole gauge
field part. Therefore, the coefficient of CST becomes $-\alpha/\kappa$ which is to be quantized as
$-2\pi \alpha/h \kappa \in \mathbb{Z}$. However, the coefficient of the Einstein term, $1/\kappa$ itself has no reason to be
quantized as we saw above.

IV. LORENTZIAN METHOD

The above Euclidean result seems doubtful because it is in contradiction with the naive
Lorentzian argument that $\pi_3(ISO(2, 1)) = 0$ implies the failure of CST coefficient quan-
tization. To understand this mystery we work in the Lorentzian spacetime directly. We
first ask the compatibility of the Lorentzian structure on the $S^3$. The answer is given by
the Poincaré-Hopf theorem; a connected orientable compact manifold admits a hyperbolic
geometry if and only if its Euler character vanishes [16]. Since the Euler character of $S^3$
vanishes, both the Euclidean structure and the Lorentzian structure can be constructed on
it.

This raises us another question. What is the physical implication of working in Lorentzian
$S^3$ or Euclidean $S^3$? On the Lorentzian $S^3$, no timelike curve has end points [16]. This might
be understood as every timelike curve on $S^3$ should be closed. Indeed, the Lorentz structure
on the $S^3$ means that we have a parametrization map from $S^3$ to $\mathbb{R}$. This map specifies the
value of time parameter for each point of $S^3$. It is plausible to assume this parametrization
map to be continuous otherwise we cannot define the hypersurface on which the initial
physical data are set. The assumption forbids those timelike curves to proceed on $S^3$ without
overlapping. Therefore for any point of $S^3$, we have a closed timelike curve passing through
it. One needs not worry about the chronology violation because the parameter length of the
closed timelike curve is very large compared with the ordinary domain of physics and it is
not a relevant thing in our present analysis.

This peculiar spacetime structure can be realized by the Hopf bundle [15], that is, the
fibration of $S^3$ into nontrivial (timelike) $S^1$ bundle over (spacelike) $S^2$. We stress here that for each point on $S^2$, the time direction along the fiber $S^1$ and any spacelike direction over $S^2$ cannot be interchanged by any physical process due to the regulation of $SO(2,1)$. (We see the analogy in $\mathbb{R}^3$ case, where also both structures can be installed. The Lorentzian structure in this case means the foliation of the manifold $\mathbb{R}^3$ with the family of spacelike slices. This sort of foliation is also possible for the Euclidean structure. However in the Lorentzian structure, no physical process can interchange the line on those two dimensional slices with the line characterizing the remaining one dimension.) Physics definitely tells timelikeness from the spacelikeness. Therefore, given the Lorentzian structure, the topology $S^3$ should be further specified as the Hopf fibrated $S^3$ where the fiber and the base manifold should be discriminated. This latter condition is crucial in our argument and we denote this specification as $^HS^3$.

Now let us consider the large gauge transformation of the CS Lagrangian in the Lorentzian spacetime. The same logic as in the Euclidean argument takes us to the classification problem of all those transformations (mappings from $^HS^3$ to the $SO(2,1)$ group manifold) according to their homotopy structure. Here, it is impossible to define the global time over the whole space $S^2$, that is, one cannot fix a gauge globally for the Hopf bundle \[15\]. Instead, we divide the sphere $S^2$ into two overlapping regions $S^2_+$ and $S^2_-$ which excludes the South pole and the North pole respectively. On each region its own global time can be defined and they are connected with each other in the overlapping region by some $SO(2)$ transformation as in the Dirac monopole case. This means the whole spacetime topology $^HS^3$ is divided into two parts $S^2_+ \times S^1$ and $S^2_- \times S^1$. On the other hand, the topology of $SO(2,1)$ group manifold is two dimensional hyperboloid $H^2$ fibered with $S^1$ \[17\].

It is easy to see the homotopy structure of those mappings from $^HS^3$ to $SO(2,1)$ is nontrivial. Indeed, one can construct a mapping of the nontrivial Brouwer degree. Since $S^2_\pm$ and $H^2$ are homeomorphic, that is, topologically equivalent, one can define two one-to-one maps $\varphi_+: S^2_+ \times S^1 \to SO(2,1)$ and $\varphi_-: S^2_- \times S^1 \to SO(2,1)$ (timelike direction along $S^1$ is provided with the elements of spacial rotation $SO(2)$ and the spacelike direction over $S^1$ is provided with the elements of spacial rotation $SO(2)$ and the spacelike direction over
the regions $S^2_\pm$ is endowed with the boosting elements of $SO(2,1)$. It is straightforward to combine those maps to define a new map $\varphi : H S^2 \to SO(2,1)$. This map is two-to-one and is homotopically nontrivial, as is obvious from its components maps $\varphi_\pm$. Consequently, we arrive at the same conclusion as in the Euclidean argument that the coefficient $\alpha$ should be quantized.

This Minkowski argument is viable for $SU(2)$ gauge theory also. For that case, one can give one-to-one map from $H S^3$ to $SU(2) \simeq S^3$ by considering the Hopf fibration of $SU(2)$. This map is obviously of the Brouwer degree 1 and it is not homotopically equivalent to the constant map (Brouwer degree 0) as is assured by Hopf degree theorem [15].

V. CONCLUSIONS

So far, we have discussed about the coefficient quantization for the gravitational CST and solved some paradoxes involved in this subject. This resolution is largely based on the inequivalence between the metric formulation and the dreibein formulation in $(2+1)$-dimension. Since those two formulations are different, they need not accord to each other on the result for the coefficient quantization problem. We also showed it wrong to say that due to the vanishing third homotopy of $ISO(2,1)$, the coefficient need not be quantized in the Lorentzian scheme of the gauge formulation. Indeed when we analyze the homotopy structure in the Lorentzian scheme, we should exclude those homotopy maps that mixes the timelike curves with the spacelike curves. This restriction gives us further specified definition of the third homotopy and results in the coefficient quantization in the Lorentzian scheme. Therefore in the context of gauge gravity, the coefficient of the gravitational CST should be quantized both in the Lorentzian scheme and in its analytically continued method, i.e., the Euclidean scheme. Furthermore as is discussed above, this result does not contradict with that of the metric formulation, where the coefficient need not be quantized both in the Lorentzian scheme and in the Euclidean scheme. Meanwhile we confirmed the naive conjecture that the gauge symmetry $ISO(2,1)$ should be changed to $ISO(3)$ upon the Wick
rotation to the Euclidean space.

An interesting consequence of this coefficient quantization in the gauge formulation is that the induced spin is not exotic because its value involves the gravitational CS coefficient $\alpha$. This is in contrast with the inherent spin, which can be fractional in (2+1)-dimensional space. Moreover, the gravitational CST does not play any role in the fractional statistics either. Indeed, it is the Einstein-Hilbert term that dynamically realizes the spin-statistics theorem through the Aharonov-Bohm effect. Therefore any kind of spin, whether it is inherent or induced, gives the same form of Aharonov-Bohm phase upon the interchange of two identical particles carrying that spin [19]. The induced spin does not give rise to the exotic phase since it is not exotic, while the inherent spin results in the exotic phase. Specifically in the metric formulation, the induced spin also can be exotic producing anyonic behavior since the coefficient of the gravitational CST needs not be quantized there (Deser in [13]).

Throughout the study, we learned another lesson: It is misconceived that the CST assures a way of generating mass on the gauge particle. This is possible only in the case when the dynamical terms of mutually different derivative order are involved. In the metric formulation of (2+1)-dimensional gravity, the situation can be met through the torsion free condition. However in the dreibein formulation, this condition cannot be satisfied. In this sense, it is misnomer to call the gravitational CST as the topological mass term. This terminology makes sense only in the metric formulation. It would be interesting to study the cosmological consequence of the difference between those two formulation, on the ground of cosmic string.

The (2+1)-dimensional gravity has been studied in several contexts. J. H. Horne and E. Witten showed that the topologically massive model of [3] is equivalent to the Chern-Simons theory for the group $SO(3, 2)$ and therefore is finite and exactly soluble [20]. The equivalence is in the sense that a solution of the $SO(3, 2)$ gauge gravity, with its gauge specifically fixed so that the connection components $e^a_{\mu}$ form an invertible matrix, can be a solution of the topologically massive gravity and vice versa. This means that although those Lagrangians of the two formulations look mutually different, they result in the same geometry.
In this case, what puzzles us is that the homotopy analysis lead to the coefficient quantization for the gravitational CST both in the Lorentzian scheme and Euclidean scheme. However, one can resolve this problem as follows: In the analysis of the topologically massive system, one usually use the diffeomorphism as its symmetry. But as one may note from its sense of the equivalence with the $SO(3, 2)$ gauge gravity, this is quite reduced symmetry. Indeed, the diffeomorphism, on shell, can be thought of as some specific form of the conformal transformation [20] but the reverse is not true. Therefore, the $SO(3, 2)$ gauge gravity can be more than the topologically massive gravity. We don’t know about the true internal symmetry of this latter system. If it is the diffeomorphism, the coefficient need not be quantized. If it is a larger symmetry containing the diffeomorphism, there can be other possibilities.

E. W. Mielke and P. Baekler generalized the topologically massive model by adding a new translational Chern-Simons term $\sim \epsilon^{\mu\nu\rho} e^a_{\mu} T_{a\nu\rho}$ [21]. In collaboration with F. W. Hehl, they further generalized their model by the cosmological constant term and analyzed its dynamical structure [22]. They showed that their model gives a nontrivial geometry even without the torsion free condition. This seems to be in contradiction with our derivative order argument. However, in their generalized topologically massive model, the gauge symmetry is unclear to us. With a specific set of the coefficients, their model might be considered rather as (anti-)de Sitter gauge gravity, where the locally nontrivial ((anti-)de Sitter) structure is assured by the cosmological constant term (Witten in [10]).

The coefficient quantization has been also dealt with in several contexts. M. Henneaux and C. Teitelblim showed that the Dirac magnetic monopole leads to the quantization of the topological mass even in the abelian CS theory [23]. Quantization even occurs to the mass: A. Zee showed that the mass of a particle, around the gravitational analog of Dirac’s magnetic monopole, should be quantized [24]. Similar quantization was shown to occur around the spinning cosmic string by P. O. Marzur [25]. This seems more likely to happen in the presence of the gravitational CST because the induced spin itself, on which the energy depends, is quantized. However, this last case should be further investigated because we do
not know the meaning of the CST in the cosmic string physics [12]. It will be also interesting to probe the cases of the (anti-)de Sitter group. Since the Poincaré group can be considered as the vanishing cosmological constant limit of the (anti-)de Sitter group, such a coefficient quantization also should occur in the (anti-)de Sitter gauge gravity [26].

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