Unconventional color superfluidity in ultra-cold fermions: Quintuplet pairing, quintuple point and pentacriticality

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We describe the emergence of color superfluidity in ultra-cold fermions induced by color-orbit and color-flip fields that transform a conventional singlet-pairing s-wave system into an unconventional non-s-wave superfluid with quintuplet pairing. We show that the tuning of interactions, color-orbit and color-flip fields transforms a momentum-independent scalar order parameter into an explicitly momentum-dependent tensor order parameter. We classify all unconventional superfluid phases in terms of the loci of zeros of their quasi-particle excitation spectrum in momentum space and we identify several Lifshitz-type topological transitions. Furthermore, when boundaries between phases are crossed, non-analyticities in the compressibility arise. We find a quintuple point, which is also pentacritical, where four gapless superfluid phases converge into a fully gapped superfluid phase.

Ultra-cold fermions with two internal states trapped in harmonic, box and optical lattice potentials can serve as simulators of models often found in condensed matter physics. Key examples are experimental simulations of the crossover from Bardeen-Cooper-Schrieffer (BCS) to Bose-Einstein Condensation (BEC) superfluidity [1–8], and more recently, experimental simulations of Fermi-Hubbard systems in optical lattices [9, 10], where an orbit field in the laboratory are being sought [30] using Yb via magnetic Fano-resonances or in 40K via orbital Fano-resonances [31, 32]. In this paper, we take advantage of the existing tunability in ultra-cold fermions to propose the existence of unconventional color superfluids with quintuplet pairing in the presence of color-orbit coupling. When s-wave interactions and color-flip fields are changed, the resulting phase diagram is very rich, containing a quintuple and pentacritical point, where the compressibility is non-analytic and four gapless superfluid phases converge into a fully gapped one.

To describe interacting three-color fermions under the influence of color-orbit and color-flip fields, we start with a general independent-particle Hamiltonian that results from the coupling to a spatially modulated chip or Raman beams [18, 19]

\[ H_0(k) = \varepsilon(k) 1 - h_x(k) \mathbf{J}_x - h_z(k) \mathbf{J}_z + b_z \mathbf{J}_z^2, \]

where \( \mathbf{J}_\ell \) are spin-one angular momentum matrices with \( \ell = \{x, y, z\} \). The reference kinetic energy \( \varepsilon(k) = k^2/(2m) + \eta \) is the same for all colors, \( h_x(k) = -\sqrt{2}\Omega \) is the color-orbit Rabi field, and \( h_z(k) = 2k_F k_z/(2m) + \delta \) is a momentum dependent Zeeman field along the \( z \)-axis, which is transverse to the momentum transfer direction along the \( x \)-axis, and \( b_z = k_z^2/(2m) - \eta \) is the quadratic color-shift term. Notice that \( h_z(k) \) contains the color-orbit coupling term \( 2k_F k_z/(2m) \) as well as a color-shift term controlled by the detuning \( \delta \). A similar hamiltonian was studied for spin-one bosons [20, 33].

The chip-atom or Raman-atom interaction Hamiltonian can be written in second-quantized notation as

\[ H_{CA} = \sum_k \Psi_k^\dagger H_0(k) \Psi_k \]

where the spinor creation operator is \( \Psi_k = [\psi_R^k, \psi_G^k, \psi_B^k] \), with \( \psi_c^k \) creating a fermion label by momentum \( k \) in color state \( c = \{R, G, B\} \) [20]. We use as units the Fermi energy \( E_F = k_F^2/(2m) \) and the Fermi momentum \( k_F = (2\pi^2 n)^{1/3} \), based on the total density of fermions \( n = 3k_F^3/(6\pi^2) \) with initial identical kinetic energies \( \varepsilon_k = k^2/(2m) \) for all three internal states. This means that our reference system is that with all parameters \( \eta, k_F, \Omega \) and \( \delta \) set to zero.
remain an eigenbasis, and thus are uncoupled. In this way, we add interactions between atoms in different inter-

colour-flip fields and atom-atom interactions is

\[ H = H_{CA} + H_{AA} - \mu \bar{N}, \]  

where \( \bar{N} = \sum c \bar{c} \psi_{c}^{\dagger}(k) \psi_{c}(k) \) represents the total number of particles. We focus on uniform superfluid phases with \( Q = 0 \) and order parameter tensor \( \Delta_{cc'} = -g_{cc'}(b_{cc'}(0))/V \), leading to the Hamiltonian

\[ H_{MF} = \frac{1}{2} \sum_{k} \Psi_{N}(k) \mathbf{H}_{MF}(k) \Psi_{N}(k) + V \sum_{c \neq c'} |\Delta_{cc'}|^2 + C(\mu) \]  

where the six-component Nambu spinor is \( \Psi_{N}(k) = \left[ \Psi_{R}^{\dagger}(k), \Psi_{G}^{\dagger}(k), \Psi_{B}^{\dagger}(k), \Psi_{R}(-k), \Psi_{G}(-k), \Psi_{B}(-k) \right] \).

Here, the function \( C(\mu) = \frac{1}{2} \sum_{k} \varepsilon_{c}(\varepsilon_{c} - \mu) \) contains the term \( \xi_{c}(k) = \varepsilon_{c}(k) - \mu \) representing the residual kinetic energies, with \( \varepsilon_{c}(k) \) representing the diagonal matrix elements of \( H_{0}(k) \). By \( \varepsilon_{R}(k) = \varepsilon_{G}(k) = \varepsilon_{B}(k) = \varepsilon(k) + h_{c}(k) + b_{c} \); \( \varepsilon_{c}(k) = \varepsilon(k) \) and \( \varepsilon_{B}(k) = \varepsilon(k) + h_{c}(k) + b_{c} \). The \( 6 \times 6 \) Hamiltonian matrix is

\[ H_{MF}(k) = \begin{pmatrix} \Pi_{0}(k) & \Lambda \\ \Lambda^{\dagger} & -\Pi_{0}(-k) \end{pmatrix}, \]  

where the diagonal block matrix is \( \Pi_{0}(k) = H_{0}(k) - \mu \mathbf{1} \) and the off-diagonal block matrix is

\[ \Lambda = \begin{pmatrix} 0 & \Delta_{RG} & \Delta_{RB} \\ -\Delta_{RG} & 0 & \Delta_{GB} \\ -\Delta_{RB} & -\Delta_{GB} & 0 \end{pmatrix}, \]  

representing the order parameter tensor. In this work, we consider the simpler case where \( g_{RB} = g_{GB} = 0 \), and \( g_{RG} = g \), which leads to \( \Delta_{RG} = \Delta_{GB} = 0 \), and \( \Delta_{RB} = \Delta \), such that the order parameter tensor \( \Delta_{\omega} \) is characterized by a single complex scalar \( \Delta \). For example, this choice reflects the experimental condition of three internal states of trapped \(^{40}\text{K}\) in the vicinity of its s-wave Fano-Feshbach resonance near 200 Gauss, where states \( |R| = |0, -9/2 \rangle \) and \( |B| = |0, -7/2 \rangle \) interact, but state \( |G| = |0, -5/2 \rangle \) does not interact with any other state. This situation corresponds to a single-channel pairing color superfluid, where only \( R \) and \( B \) fermions experience attractive s-wave interactions.

The corresponding thermodynamic potential is

\[ Q_{MF} = -\frac{T}{2} \sum_{k} \ln \left\{ 1 + \exp \left[ -\frac{E_{j}(k)}{T} \right] \right\} + V \frac{1}{g_{RB}} \delta Q_{MF}/\delta \Delta_{RB} = 0 \]  

where \( j = \{1, \cdots, 6\} \) labels eigenenergies \( E_{j}(k) \) of \( H_{MF}(k) \) in Eq. [8]. Minimizing \( Q_{MF} \) with respect to \( \Delta_{RB} \) via \( \delta Q_{MF}/\delta \Delta_{RB} = 0 \) leads to the order parameter equation

\[ \frac{V}{g_{RB}} \Delta_{RB} = \frac{1}{2} \sum_{k} \sum_{j=1}^{3} \tanh \left( \frac{E_{j}(k)}{2T} \right) \frac{\partial E_{j}(k)}{\partial \Delta_{RB}}, \]  

In Fig. 1 we show eigenvalues \( E_{\alpha}(k) \) of \( H_{0}(k) \) versus momentum \( k_{z} \) for fixed momentum transfer \( k_{T} = 0.35k_{F} \) and quadratic color shift \( b_{z} = 0 \), that is, \( \eta = k_{T}^{2}/2m \). The cases with color-flip (Rabi) frequencies \( \Omega = 0 \) and \( \Omega = E_{F} \) are shown in a) and b), respectively. In a) states \( \{ R, G, B \} \) are not mixed: the dotted-red line corresponds to \( E_{\alpha}(k) \), the dashed-green line to \( E_{\gamma}(k) \), and the solid-blue line to \( E_{\delta}(k) \). In b) states \( \{ R, G, B \} \) are mixed: the dotted-magenta line corresponds to \( E_{\eta}(k) \), the dashed-yellow line to \( E_{\delta}(k) \), and the solid-cyan line to \( E_{\delta}(k) \).
and fixing the total number of particles via \( N = -\partial Q_{MF}/\partial \mu|_{T,V} \) leads to the number equation

\[
N = \frac{1}{2} \sum_k \left[ \sum_{j=1}^{3} \tanh \left( \frac{E_j(k)}{2T} \right) \frac{\partial E_j(k)}{\partial \mu} + 3 \right]. \tag{10}\]

The sum over \( j \) involves only quasiparticle energies (\( j = \{1,2,3\} \)), because we used quasiparticle/quasihole symmetry to eliminate the quasihole energies. Using the relation \( V/g_{RB} = -mV/(4\pi a_s) + \sum_k 1/(2\epsilon_k) \), we express the bare coupling constant \( g_{RB} \) in terms of the scattering length \( a_s \) in the absence of the color-orbit and color-flip fields. We note also that Eqs. (8), (9) and (10) are only valid at low temperatures, that is, \( T \ll E_F \), as they do not include amplitude and phase fluctuations of the order parameter that become increasingly more important as temperature is raised from zero. Thus, in the remainder of the paper, we choose the specific value of \( T = 0.02E_F \) to illustrate the low temperature regime of phase diagrams, such that all qualitative changes occur in the experimental range \(-2 < 1/(k_F a_s) < 2\).

Of the three quasiparticle bands \( E_1(k), E_2(k) \) and \( E_3(k) \), only the lowest energy dispersion can have zeros and therefore be used to classify the emergent superfluid phases based on the type of nodal quasiparticles that emerge \[28\]. Thus, in Fig. 2 we plot the momentum space loci of \( E_2(k) = 0 \) versus \((k_x,k_z)\), where \( k_z \) represents a radial vector in the \( k_z \) plane, for zero color shift (detuning \( \delta = 0 \)) and zero quadratic color (Zeeman) shift \((b_z = 0)\) describing one of the simplest experimental situations that can be engineered for three internal states of \(^{40}\)K. We show only the first quadrant, because the loci have azimuthal symmetry in the \( k_yk_z \) plane and reflection symmetry in the \( k_z \) direction. This means that red dots along the \( k_z \) axis represent circles in the \( k_y \) plane, and that blue lines in the \( k_xk_z \) plane represent surfaces in three-dimensional momentum space \((k_x,k_y,k_z)\). In the top panels, we describe the normal phases \( N1, N2, N3 \), with one, two or three distinct Fermi surfaces, respectively. In the bottom panels, we show the nodal structure of the three superfluid phases that have a boundary with the normal state. The phase \( R1S0 \) has one ring and zero surface of nodes, the phase \( R1S1 \) has one ring and one surface of nodes, and the phases \( R2S1 \) have two rings and one surface of nodes. The phase \( R0S1^* \) (not shown in Fig. 2) is the limiting case where two-rings annihilate in momentum space at the equator of the surface of nodes.

In Fig. 2 we plot phase diagrams of \( \Omega/E_F \) versus scattering parameter \( 1/(k_F a_s) \) for \( b_z = 0 \) with \( k_T = 0.35k_F \) in Fig. 3a and with \( k_T = 0 \) in Fig. 3b. Normal phases \( N1, N2 \) and \( N3 \) are indicated by dark-gray, light-gray and white colors, respectively. Gapless superfluid phases are color coded as \( R1S0 \) (red), \( R1S1 \) (blue), \( R2S1 \) (green), \( R0S1^* \) (magenta), \( R0S1 \) (orange), \( R0S2 \) (cyan), \( R0S3 \) (brown), and the fully gapped \((FG)\) phase (yellow). In Fig. 3, where \( k_T \neq 0 \), the phase transitions from superfluid to normal are all continuous. However, in Fig. 3a, where \( k_T = 0 \), the transition from superfluid to normal phases are discontinuous, similar to the case of two internal states with \( k_T = 0 \) \[24,32\].

The transitions between superfluid phases are topological and of the Lifshitz-type \[40\], where the number of simply-connected residual Fermi surfaces change when the phase boundaries between neighboring superfluid phases are crossed. In Fig. 3a, there is a quintuple and pentacritical point, where the phases \( R1S0, R1S1, R2S1 \) and \( R0S1^* \) converge into a fully gapped \( FG \) phase.
If we start in the \(FG\) phase and circle the quintuple point counterclockwise, a ring emerges from \(k = 0\) at the \(FG/RS1^*\) boundary leading to phase \(RS1^0\), from there a surface of nodes arises from \(k = 0\) at the \(RS1^0/RS1\) boundary leading to phase \(RS1\), then another ring of nodes appears from \(k = 0\) at the \(RS1/RS2^1\) boundary. Furthermore, two rings annihilate at finite momentum where a surface of nodes exist at the \(RS2^1/RS1^*\) boundary leading to phase \(RS1^*\), and finally a full gap emerges at the \(RS1^*/FG\) boundary the residual surface of nodes disappear through \(k = 0\). The compressibility \(\kappa = n^2 (\partial n/\partial \mu)_{T,V}\), with \(n = N/V\), becomes \(\kappa = \kappa_c + \kappa_\rho |\bar{\mu} - \bar{\mu}_c|^{1/2}\) when the critical point is approached through phases with dominant nodal surfaces, or becomes \(\kappa = \kappa_c + \kappa_{RS1^0} |\bar{\mu} - \bar{\mu}_c|\) when the critical point is approached through the single ring phase \(RS1^0\). Here, \(\bar{\mu} = \mu/E_F\), \(\mu_c\) is the critical chemical potential, and \(\kappa_\rho\) is a phase dependent coefficient. In all cases, the derivative \((\partial \kappa/\partial \mu)_{T,V}\) is discontinuous at the critical point. Similarly, at the pentacritical point in Fig. 3b, the compressibility \(\kappa = \kappa_c + \kappa_{RS1} |\bar{\mu} - \bar{\mu}_c|/2\), when approached from gapless phases with surface nodes.

To understand the emergence of momentum dependence in the order parameter, we write \(\mathbf{H}_{MF}(k)\) as

\[
\mathbf{H}_{MF}(k) = \begin{pmatrix}
\tilde{\mathbf{H}}_D(k) & \tilde{\mathbf{A}} \\
\tilde{\mathbf{A}}^\dagger & -\tilde{\mathbf{H}}_F(-k)
\end{pmatrix}
\]

in the mixed-color basis \(\alpha = \{\uparrow, 0, \downarrow\}\), where \(\mathbf{\Phi}_k = \left[\phi_{\uparrow0}(k), \phi_{0\downarrow}(k), \phi_{\downarrow\uparrow}(k)\right]\) is related to the \(\{R, G, B\}\) basis \(\Psi_k\) via a unitary transformation \(\Phi_k = \Psi_k^\dagger U(k)\). The matrix elements of the 3 x 3 blocks are \(\tilde{\mathbf{H}}_{D,A,B}(k) = \tilde{\mathbf{E}}_\alpha(k)\delta_{\alpha\beta}\), with \(\tilde{\mathbf{E}}_\alpha(k) = \tilde{\mathbf{E}}_\alpha(k) - \mu\) and \(\Lambda_{\alpha\beta} = \Lambda_{\beta\alpha}(k)\).

The matrix \(\tilde{\mathbf{A}}\) describing the order parameter tensor \(\Delta_{\alpha\beta}(k)\) is momentum dependent in contrast to the original matrix \(\Lambda\), which is independent of momentum. The order parameter tensor becomes

\[
\Delta_{\alpha\beta}(k) = \Delta \left[ u_{\alpha R}(k)u_{\beta B}(-k) - u_{\alpha B}(k)u_{\beta R}(-k) \right],
\]

where \(u_{\alpha\gamma}(k)\) are matrix elements of \(\mathbf{U}(k)\) and represent the \(R\) and \(B\) components of the eigenvector amplitudes \(u_{\alpha}(k) = [u_{\alpha R}(k), u_{\alpha G}(k), u_{\alpha B}(k)]\). The property \(\Delta_{\alpha\beta}(k) = -\Delta_{\beta\alpha}(k)\) guarantees that the diagonal elements \(\Delta_{\alpha\alpha}(k)\) have odd parity due to the Pauli principle. Also, \(\Delta_{\alpha\beta}(k)\) has nine components and can be written in the basis of total pseudo-spin \(S\) and total pseudo-spin projection \(m_s\) with singlet \((S = 0)\), triplet \((S = 1)\) and quintuplet \((S = 2)\) sectors. This is achieved by writing \(\Delta_{S_m}(k) = M_{S_m}^{\alpha\beta}\Delta_{\alpha\beta}(k)\), where \(M_{S_m}^{\alpha\beta}\) is a tensor of generalized Clebsch-Gordon coefficients 29.

For instance, in Fig. 3a \((k_F \neq 0)\), when \(\Omega \neq 0\), all superfluid phases have three vanishing order parameter components \(\Delta_{00}(k) = \Delta_{\Phi\Phi}(k) = \Delta_{\Psi\Psi}(k) = 0\), while the six remaining non-vanishing components can be obtained from \(\Delta_{00}(k)\) and \(\Delta_{\Phi\Psi}(k)\) via the symmetry relations: \(\Delta_{00}(k) = \Delta_{00}(k)\); \(\Delta_{0\Phi}(k) = -\Delta_{\Phi0}(k)\); \(\Delta_{1\Phi}(k) = -\Delta_{\Psi\Psi}(k)\); \(\Delta_{1\Psi}(k) = -\Delta_{0\Psi}(k)\). For all superfluid phases of Fig. 3b with \(\Omega \neq 0\), only the quintuplet sector \((S = 2)\) has non-vanishing components, thus leading to very unconventional color pairing, beyond the singlet and triplet channels of spin-1/2 fermions in condensed matter physics. When \(\Omega = 0\), as discussed earlier, the system trivializes since there is no color-mixing and no momentum dependence in the order parameter.

In Fig. 3b, we show the mixed-color particle (+) and hole (−) energies \(\pm \xi_\Phi(k)\), \(\pm \xi_\Psi(k)\), \(\pm \xi_\Psi(k)\) versus momentum \(k_x\) for the normal phase \(N2\), where \(\Delta = 0\), with \(\Omega = 0.79\) and \(1/(k_Fa_s) = -1.5\). In Fig. 4b, we show the quasi-particle (positive) energies \(E_1(k)\) (dotted gray), \(E_2(k)\) (dashed orange) \(E_3(k)\) (black solid) and quasi-hole (negative) energies \(E_4(k)\) \(E_5(k)\) \(E_6(k)\) \(E_7(k)\) \(E_8(k)\) \(E_9(k)\) \(E_{10}(k)\) (gray dotted) energies for the superfluid phase \(RS1\), with \(\Omega/E_F = 0.79\) and \(1/(k_Fa_s) = 0.23\). In Fig. 4a, we show the order parameter components \(\Delta_{\Phi\Phi}(k)\) (dotted green), \(\Delta_{\Phi\Psi}(k)\) (solid blue), \(\Delta_{\Psi\Psi}(k)\) (dotted gray), and \(\Delta_{\Psi\Phi}(k)\) (dashed red). The gaps in the excita-
tion spectrum of Fig. 1. occur at momentum locations where $\Delta_{22}(k)$ lift the degeneracies of the particle and hole energies shown in Fig. 1. In Fig. 1, we show the components $\Delta_{22}(k)$ (solid brown), $\Delta_{21}(k)$ (dot-dashed magenta), $\Delta_{20}(k) = 0$, $\Delta_{21}(k)$ (dotted orange), $\Delta_{22}(k)$ (dashed purple) in quintuplet sector ($S = 2$).

In conclusion, we proposed the existence of unconventional color superfluids with quintuplet pairing in the presence of color-orbit coupling for ultra-cold fermions. When s-wave interactions and color-flip fields are changed, we found that the resulting phase diagram is very rich, containing a quintuple and pentacritical point, where the compressibility is non-analytic and four gapless superfluid phases converge into a fully gapped one.

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Supplementary Material

Unconventional color superfluidity in ultra-cold fermions: Quintuplet pairing, quintuple point and pentacriticality

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Heating effects within Raman schemes have so far precluded studies of ultra-cold fermions with spin-orbit coupling at temperatures $T < 0.3E_F$, where $E_F$ is the system’s Fermi energy. However, new techniques using radio-frequency chips are under development to produce spin-orbit and color-orbit couplings without the use of Raman lasers and thus without heating the system. For ultra-cold fermions with three colors (internal states) $\{R, G, B\}$, a generic situation can be created via radio-frequency chips or Raman lasers leading to the independent particle Hamiltonian matrix

$$H_0(k) = \begin{pmatrix}
\varepsilon_R(k) & \Omega_{RG} & \Omega_{RB} \\
\Omega_{RG}^* & \varepsilon_G(k) & \Omega_{GB} \\
\Omega_{RB}^* & \Omega_{GB}^* & \varepsilon_B(k)
\end{pmatrix},$$

(13)

where $\varepsilon_c(k) = (k - k_c)^2/(2m) + \eta_c$ represents the energy of internal color state $c = \{R, G, B\}$ after net momentum transfer $k_c$, and $\eta_c$ is a reference energy of the atom at internal state $c$. The matrix elements $\Omega_{c'c}$ represent Rabi frequencies between atomic color states $c$ and $c'$.

In the main text, we investigate a simpler experimental situation by setting the Rabi frequency $\Omega_{RB} = 0$ in Eq. (13), indicating that there is no coupling between states $R$ and $B$. In addition, we consider that the Rabi frequencies associated with the transitions from states $R$ to $G$ and from $R$ to $B$ to be real and equal, that is, $\Omega_{RG} = \Omega_{RG}^* = \Omega_{GB} = \Omega_{GB}^*$. Furthermore, we assume that momentum transfers occur only in states $R$ and $B$, such that $k_R = k_T \hat{x}$, $k_G = 0$, and $k_B = -k_T \hat{x}$, where $k_T$ is the magnitude of the momentum transferred to the atom by photons or radio-frequency fields. Lastly, we can define an energy reference via the sum $\sum_c \eta_c = \eta$, leading to internal energies $\eta_R = -\delta$, $\eta_G = \eta$ and $\eta_B = +\delta$, where $\delta$ represents the detuning. Under these considerations the Hamiltonian matrix $H_0(k)$ shown above acquires the form described in Eq. (1) of the main text, when written in terms of the spin-one angular momentum matrices $J_\ell$ with $\ell = \{x, y, z\}$.

In second quantization, the Hamiltonian matrix $H_0(k)$ becomes the chip-atom (Raman-atom) Hamiltonian

$$H_{CA} = \sum_k \Psi_k^* H_0(k) \Psi_k,$$

(14)
where the spinor creation operator is \( \Psi^\dagger_{c} = \left[ \psi_R^c(k), \psi_G^c(k), \psi_B^c(k) \right] \), with \( \psi^\dagger_c(k) \) creating a fermion labelled by momentum \( k \) and color \( c \in \{ R, G, B \} \). The Hamiltonian \( H_{CA} \) can be diagonalized via the rotation \( \Phi(k) = U(k) \Psi(k) \), which connects the three-component spinor \( \Psi(k) \) in the original color basis to the three-component spinor \( \Phi(k) \) representing the mixed color basis, that is, the basis of eigenstates. The matrix \( U(k) \) is unitary and satisfies the relation \( U^\dagger(k)U(k) = 1 \). The diagonalized Hamiltonian matrix is

\[
H_D(k) = U(k)H_0(k)U^\dagger(k),
\]

with matrix elements \( H_{D,\alpha\beta}(k) = \mathcal{E}_\alpha(k)\delta_{\alpha\beta} \), where \( \mathcal{E}_\alpha(k) \) are the eigenvalues of \( H_0(k) \) in Eq. 11 of the main text. The three-component spinor in the mixed-color eigenbasis is \( \Phi(k) = \left[ \phi^R_{\alpha}(k), \phi^G_{\alpha}(k), \phi^B_{\alpha}(k) \right] \), where \( \phi^c_{\alpha}(k) \) is the creation operator of a fermion with eigenenergy \( \mathcal{E}_\alpha(k) \) and mixed-color label \( \alpha \). The unitary matrix

\[
U(k) = \begin{pmatrix}
    u_{R}(k) & u_{G}(k) & u_{B}(k) \\
    u_{R}^T(k) & u_{G}^T(k) & u_{B}^T(k) \\
    u_{R}^T(k) & u_{G}^T(k) & u_{B}^T(k)
\end{pmatrix}
\]

has rows that satisfy the normalization condition \( \sum_\alpha |u_{\alpha c}(k)|^2 = 1 \), where \( \alpha = \{ \uparrow, \downarrow, \uparrow, \downarrow \} \).

When \( b_k = 0 \), the Hamiltonian matrix \( H_0(k) \) in Eq. 11 of the main text reduces to that of spin-one fermions under a momentum-dependent magnetic field. In this case, the eigenvalues of \( H_0(k) \) are

\[
\mathcal{E}_\alpha(k) = \varepsilon(k) - m_\alpha |h_{\text{eff}}(k)|,
\]

with \( m_\alpha = \{ +1, 0, -1 \} \). Here, the reference kinetic energy \( \varepsilon(k) = k^2/(2m) + \eta \) is the same for all colors with \( \eta = k^2/2m \), and the effective momentum-dependent magnetic field amplitude is

\[
|h_{\text{eff}}(k)| = \sqrt{|h_x(k)|^2 + |h_z(k)|^2},
\]

where \( h_x(k) = -\sqrt{2}\Omega \) is the color-flip Rabi field, and \( h_z(k) = 2k_T k_z/(2m) + \delta \) is a momentum dependent Zeeman field along the z-axis. As mentioned above, the label \( \alpha \) describes mixed color states induced by color-orbit coupling via the color-dependent momentum transfer \( k_T \) and by color-flip fields via the Rabi frequency \( \Omega \).

Adding attractive contact interactions \(-g_{cc'}\delta(r-r')\) of strength \( g_{cc'} > 0 \) between internal states \( c \neq c' \), and focusing on uniform superfluid phases \( Q = 0 \) with order parameter tensor \( \Delta_{cc'} = -g_{cc'}\langle b_{cc'}(0)/V \rangle \), leads to the mean-field Hamiltonian

\[
H_{\text{MF}} = \frac{1}{2} \sum_k \Psi^\dagger N_{c}(k)H_{\text{MF}}(k)\Psi_{N_{c}}(k) + V \sum_{c \neq c'} |\Delta_{cc'}|^2 \frac{1}{b_{cc'}} + C(\mu)
\]

described in Eq. 6 of the main text. Here, \( \Psi^\dagger N_{c}(k) = \left[ \Psi^R_{c}(k), \Psi^G_{c}(k), \Psi^B_{c}(k), \Psi^R_{-c}(k), \Psi^G_{-c}(k), \Psi^B_{-c}(k) \right] \) is a six-component Nambu spinor, and \( b_{cc'}(0) \) is the pairing operator with zero center of mass momentum. As indicated in the main text, we particularized our discussion to the case where s-wave interactions exist only between states \( \{ R \} \) and \( |B \rangle \), such that the \( |G \rangle \) state does not interact with the other two. In this situation, the order parameter tensor \( \Delta_{cc'} \) is represented by a single scalar \( \Delta_{RB} = \Delta \).

The Hamiltonian matrix \( H_{\text{MF}}(k) \) in Eq. 6 of the main text has six eigenvalues, which we order as \( E_1(k) > E_2(k) > E_3(k) > E_4(k) > E_5(k) > E_6(k) \). These eigenvalues exhibit quasiparticle/quasihole symmetry in momentum space for any value of detuning \( \delta \) and Rabi frequency \( \Omega \), which means \( E_0(k) = -E_1(-k) \), \( E_5(k) = -E_2(-k) \) and \( E_4(k) = E_3(-k) \). However, each eigenergy \( E_j(k) \) has well defined parity only when \( \delta = 0 \), in which case \( E_j(k) = E_j(-k) \) has even parity. In the limit of zero color-flip field \( \Omega \to 0 \), with \( k_T = 0 \) or \( k_T \neq 0 \), the excitation spectrum can be obtained analytically as the bands describing states \( \{ R, G, B \} \) do not mix, in which case, state \( |G \rangle \) is completely inert to pairing. When \( \Omega = 0 \), a spin-gauge symmetry relates trivially the states with \( k_T = 0 \) and \( k_T \neq 0 \), that is, \( k_T \) can be gauged away. This leads to quasiparticle energies \( E_j(k) = E_j(-k) = \sqrt{\epsilon^2_k + |\Delta|^2} \), where \( E_3(k) \) reduces to \( |\xi_k| \), with \( \xi_k = \epsilon_k - \mu \) being the independent particle energy. Thus, s-wave pairing occurs only between \( R \) and \( B \) states, while there is no pairing involving the \( G \) state, as expected. The only physical role played by state \( G \) is to contribute to the total density of fermions, and thus to affect the chemical potential \( \mu \). Furthermore, when \( \Omega \to 0 \), the number of particles in each band is conserved separately and color-orbit coupling can be gauged away, leading to an inert band \( G \) and to standard BCS-BEC crossover phenomena in the superfluid phase for bands \( R \) and \( B \).

To analyze the excitation spectrum \( E_j(k) \), we need to determine self-consistently the values of the order parameter amplitude \( \Delta_{RB} = \Delta \) and the chemical potential \( \mu \). For this purpose, we write the thermodynamic potential \( Q = \int_0^\beta \left[ \sum_k \frac{1}{2} \left( \varepsilon(k) - \mu \right)^2 + \sum_c |\Delta_{cc'}|^2 \frac{1}{b_{cc'}} \right] d\epsilon_k \)
\[-T \ln Z, \text{ where } Z = \int \Pi_\nu D \left[ \psi^\dagger_\nu(k), \psi_\nu(k) \right] \exp \left[ -S \right] \text{ is the grand-canonical partition function written in terms of the action } S. \text{ At the mean-field level the action is} \]

\[ TS_{MF} = -\frac{1}{2} \sum_{n,k} \Psi^\dagger_n(k)G^{-1}\Psi_n(k) + V \frac{|\Delta_{RB}|^2}{g_{RB}} + C(\mu), \tag{20} \]

where \( G^{-1}(i\omega_n, k) = [i\omega_n 1 - H_{MF}(k)] \) is the inverse of the resolvent (Green) matrix \( G(i\omega_n, k) \). Here, \( \omega_n = (2n+1)\pi T \) is the fermionic Matsubara frequency and \( T \) is the temperature. A Gaussian integration over the fermionic fields \( \{ \Psi^\dagger_n(k), \Psi_n(k) \} \) leads to the thermodynamic potential \( Q_{MF} \) shown in Eq. (8) of the main text, from which the self-consistent order parameter and number equations can be obtained as Eqs. (10) and (11) of the main text.

The transitions between normal and superfluid phases are continuous for \( \Omega \neq 0 \) and \( k_T \neq 0 \) and are discontinuous for \( \Omega \neq 0 \) and \( k_T = 0 \) in the cases of the phase diagrams shown in Figs. 3a and 3b of the main text, respectively. For three-color states with single channel interaction \( g_{RB} = g \) and \( k_T = 0 \), the mixing of color states induced by the color-flip field (Rabi frequency) \( \Omega \) does not smooth out the transition into a continuous one. This occurs because \( \Omega \) mix states \( |R \rangle, |G \rangle \) and \( |B \rangle \), such that the only non-vanishing components of the order parameter tensor \( \Delta_{ab}(k) \) in the mixed color basis, Eq. (22) of the main text, are momentum independent and equal to \( \Delta_{\bar{0}0}(k) = \Delta_{0\bar{0}}(k) = |\Delta_{RB}\text{ sgn}[\Omega]|/\sqrt{2} \). Thus, there is an energy cost for pairing the \( |0 \rangle \) and \( |\uparrow\rangle \) or \( |0 \rangle \) and \( |\downarrow\rangle \) with zero center of mass momentum, similar to what happens for s-wave pairing for spin-1/2 systems in a Zeeman field. This conclusion is supported by analysis of the thermodynamic potential \( Q \). The thermodynamic potential difference between the superfluid and normal states is \( \delta Q = Q_QN = a|\Delta|^2 + b|\Delta|^4 + c|\Delta|^6 \) near the normal-superfluid phase boundary, where the parameters \( b \) and \( c \) are always positive, leading to a discontinuous transition. Such discontinuity, occurs at the Clogston limit obtained by balancing the magnetic energy \( h_s \chi_{xx} h_s/2 \) and the condensation energy \( \gamma|\Delta|^2 \), leading to the phase boundary \( \Omega = |\Delta|/\chi_{xx} \), where \( |\Delta| \) jumps discontinuously to zero.

The order parameter \( \Delta_{ab}(k) \), defined in Eq. (12) of the main text, has nine components and can be written in the basis of total pseudo-spin \( S \) and total pseudo-spin projection \( m_s \) with singlet \( (S = 0) \), triplet \( (S = 1) \) and quintuplet \( (S = 2) \) sectors. This is achieved by writing \( \Delta_S_{m_s}(k) = M_{a\beta}^{m_s} \Delta_{a\beta}(k) \), where \( M_{a\beta}^{m_s} \) is a tensor of generalized Clebsch-Gordon coefficients. The explicit forms of these matrix elements are

\[ \tilde{\Delta}_{00}(k) = \frac{1}{\sqrt{3}} \Delta_{\bar{0}\bar{0}}(k) - \frac{1}{\sqrt{3}} \Delta_{\bar{0}0}(k) + \frac{1}{\sqrt{3}} \Delta_{0\bar{0}}(k), \tag{21} \]

for the singlet sector with \( S = 0 \), with \( m_s = 0 \);

\[ \tilde{\Delta}_{11}(k) = \frac{1}{\sqrt{2}} \Delta_{\bar{0}\bar{0}}(k) - \frac{1}{\sqrt{2}} \Delta_{\bar{0}0}(k), \]
\[ \tilde{\Delta}_{10}(k) = \frac{1}{\sqrt{2}} \Delta_{\bar{0}\bar{0}}(k) - \frac{1}{\sqrt{2}} \Delta_{\bar{0}0}(k), \]
\[ \tilde{\Delta}_{1\bar{1}}(k) = \frac{1}{\sqrt{2}} \Delta_{0\bar{0}}(k) - \frac{1}{\sqrt{2}} \Delta_{0\bar{0}}(k), \tag{22} \]

for the triplet sector \( S = 1 \), with \( m_s = \{ +1, 0, -1 \} = \{ 0, 1, \bar{1} \} \), and

\[ \tilde{\Delta}_{22}(k), = \Delta_{\bar{0}\bar{0}}(k), \]
\[ \tilde{\Delta}_{21}(k) = \frac{1}{\sqrt{2}} \Delta_{\bar{0}\bar{0}}(k) + \frac{1}{\sqrt{2}} \Delta_{\bar{0}0}(k), \]
\[ \tilde{\Delta}_{20}(k) = \frac{1}{\sqrt{6}} \Delta_{\bar{0}\bar{0}}(k) + \frac{\sqrt{2}}{3} \Delta_{0\bar{0}}(k) \frac{1}{\sqrt{6}} \Delta_{\bar{0}0}(k), \]
\[ \tilde{\Delta}_{2\bar{1}}(k) = \frac{1}{\sqrt{2}} \Delta_{0\bar{0}}(k) + \frac{1}{\sqrt{2}} \Delta_{\bar{0}0}(k), \]
\[ \tilde{\Delta}_{2\bar{2}}(k) = \Delta_{\bar{0}\bar{0}}(k), \tag{23} \]

for the quintuplet sector \( S = 2 \), with \( m_s = \{ +2, +1, 0, -1, -2 \} = \{ 2, 1, 0, \bar{1}, \bar{2} \} \).

This concludes the supplementary material provided for the study of unconventional color superfluid phases of three-color fermions in the presence of color-orbit, color-flip and tunable interactions in a single s-wave channel. The physics described here may be investigated in fermionic isotopes of Lithium, Potassium and Yterbium, when three internal states are trapped and interactions are adjusted via Fano-Feshbach resonances.