GEOMETRIC CHARACTERIZATIONS OF EMBEDDING THEOREMS

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Abstract. The embedding theorem arises in several problems from analysis and geometry. The purpose of this paper is to provide a deeper understanding of analysis and geometry with a particular focus on embedding theorems on spaces of homogeneous type in the sense of Coifman and Weiss. We prove that embedding theorems hold on spaces of homogeneous type if and only if geometric conditions, namely the measures of all balls have lower bounds, hold. As applications, our results provide new and sharp previous related embedding theorems for the Sobolev, Besov and Triebel-Lizorkin spaces.

1. Introduction

The purpose of this paper is to provide a deeper understanding of analysis and geometry with a particular focus on embedding theorems on spaces of homogeneous type which were introduced by Coifman and Weiss in the early 1970s, in [CW1]. The original motivation to introduce spaces of homogeneous type is to carry out the Calderon-Zygmund theory on locally compact abelian groups to a general geometric framework. Spaces of homogeneous type in the sense of Coifman and Weiss have become a standard setting for harmonic analysis related to maximal function, differentiation theorem, singular integrals, function spaces such as Hardy spaces and functions of bounded mean oscillation, and many others. As Meyer remarked in his preface to [DH], “One is amazed by the dramatic changes that occurred in analysis during the twentieth century. In the 1930s complex methods and Fourier series played a seminal role. After many improvements, mostly achieved by the Calderon–Zygmund school, the action takes place today on spaces of homogeneous type. No group structure is available, the Fourier transform is missing, but a version of harmonic analysis is still present. Indeed the geometry is conducting the analysis.”

We say that \((X,d,\mu)\) is a space of homogeneous type in the sense of Coifman and Weiss if \(d\) is a quasi-metric on \(X\) and \(\mu\) is a nonzero measure satisfying the doubling condition. More precisely, a quasi-metric \(d\) on a set \(X\) is a function \(d : X \times X \to [0, \infty)\) satisfying (i) \(d(x, y) = d(y, x) \geq 0\) for all \(x, y \in X\); (ii) \(d(x, y) = 0\) if and only if \(x = y\); and (iii) the quasi-triangle inequality: there is a constant \(A_0 \in [1, \infty)\) such that for all \(x, y, z \in X\),

\[
    d(x, y) \leq A_0[d(x, z) + d(z, y)].
\]

We define the quasi-metric ball by \(B(x, r) := \{y \in X : d(x, y) < r\}\) for \(x \in X\) and \(r > 0\). We say that a nonzero measure \(\mu\) satisfies the doubling condition if there is a constant \(C_\mu\) such
that for all \( x \in X \) and \( r > 0 \),
\[
(1.2) \quad \mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)) < \infty.
\]

We point out that the doubling condition (1.2) implies that there exist positive constants \( \omega \) (the upper dimension of \( \mu \)) and \( C \) such that for all \( x \in X \), \( \lambda \geq 1 \) and \( r > 0 \),
\[
(1.3) \quad \mu(B(x, \lambda r)) \leq C \lambda^\omega \mu(B(x, r)).
\]

Note that there is no differentiation structure on spaces of homogeneous type and even the original quasi-metric \( d \) may have no any regularity and quasi-metric balls, even Borel sets, may not be open. By the end of the 1970s, it was well recognized that much contemporary real analysis requires little structure on the underlying space. For instance, to obtain the maximal function characterizations for the Hardy spaces on spaces of homogeneous type, Macías and Segovia proved in [MS1] that one can replace the quasi-metric \( d \) by another quasi-metric \( d' \) on \( X \) such that the topologies induced on \( X \) by \( d \) and \( d' \) coincide, and \( d' \) has the following regularity property:
\[
(1.4) \quad |d'(x, y) - d'(x', y)| \leq C_0 d'(x, x')^\theta [d'(x, y) + d'(x', y)]^{1-\theta}
\]
for some constant \( C_0 \), some regularity exponent \( \theta \in (0, 1) \), and for all \( x, x', y \in X \). Moreover, if quasi-metric balls are defined by this new quasi-metric \( d' \), that is, \( B'(x, r) := \{ y \in X : d'(x, y) < r \} \) for \( r > 0 \), then the measure \( \mu \) satisfies the following property:
\[
(1.5) \quad \mu(B'(x, r)) \sim r.
\]

Note that property (1.5) is much stronger than the doubling condition. Macías and Segovia established the maximal function characterization for Hardy spaces \( H^p(X) \) with \( (1 + \theta)^{-1} < p \leq 1 \), on spaces of homogeneous type \((X, d', \mu)\) that satisfy the regularity condition (1.4) on the quasi-metric \( d' \) and property (1.5) on the measure \( \mu \); see [MS2].

The seminal work on these spaces \((X, d', \mu)\) that satisfy the regularity condition (1.4) on the quasi metric \( d' \) and property (1.5) on the measure \( \mu \) is the \( T(b) \) theorem of David–Journé–Semmes [DJS]. The crucial tool in the proof of the \( T(b) \) theorem is the existence of a suitable approximation to the identity provided by these geometric conditions given in (1.4) and (1.5). The construction of such an approximation to the identity is due to Coifman. More precisely, \( S_k(x, y) \), the kernel of the approximation to the identity \( S_k \), satisfies the following conditions: for some constants \( C > 0 \) and \( \varepsilon > 0 \),
\[
(i) \quad S_k(x, y) = 0 \text{ for } d'(x, y) \geq C 2^{-k}, \text{ and } \|S_k\|_{\infty} \leq C 2^k,
(ii) \quad |S_k(x, y) - S_k(x', y)| \leq C 2^{k(1+\varepsilon)}d'(x, x')^\varepsilon,
(iii) \quad |S_k(x, y) - S_k(x, y')| \leq C 2^{k(1+\varepsilon)}d'(y, y')^\varepsilon, \text{ and }
(iv) \quad \int_X S_k(x, y) \, d\mu(y) = 1 = \int_X S_k(x, y) \, d\mu(x).
\]

Let \( D_k := S_{k+1} - S_k \). In [DJS], the Littlewood–Paley theory for \( L^p(X) \), \( 1 < p < \infty \), was established; namely, if \( \mu(X) = \infty \) and \( \mu(B(x, r)) > 0 \) for all \( x \in X \) and \( r > 0 \), then for each
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$p$ with $1 < p < \infty$ there exists a positive constant $C_p$ such that

$$C_p^{-1} \|f\|_p \leq \left\| \left\{ \sum_k |D_k(f)|^2 \right\}^{\frac{1}{2}} \right\|_p \leq C_p \|f\|_p.$$  

The above estimates were the key tool for proving the $Tb$ theorem on $(X, d', \mu)$; see [DJS] for more detail. Later, almost all important results on Euclidean space such as the Calderón reproducing formula, test function spaces and distributions, the Littlewood–Paley theory, function spaces and the embedding theorems were carried over on $(X, d', \mu)$ that satisfy the regularity condition (1.4) on the quasi-metric $d'$ and property (1.5) on the measure $\mu$. See [H1, H2], [HS], [HL] and [DH] for more details.

To devote to the so-called first order calculus, systematic theories on metric measure spaces were developed since the end of the 1970s, see, for example, [Ch, HK, He1, He2]. Note that metric measure spaces are spaces of homogeneous type in the sense of Coifman and Weiss. For instance, Ahlfors $\omega$-regular space $(X, d, \mu)$ is such a metric measure space with $r^{\omega}/C \leq \mu(B(x, r)) \leq Cr^{\omega}$ for some $\omega > 0$ and $C$ independent of $x$ and $0 < r \leq \sup_{x,y \in X} d(x, y)$.

Ahlfors regular spaces are closely related to many questions such as complex analysis, singular integrals, the estimates of the bounds for heat kernels. See [A], [ARSW], [BCG] and [MMV] for more details.

Another important metric measure space is the Carnot-Carathéodory space. There are several equivalent definitions for the Carnot-Carathéodory distance, see [JS] and [NSW]. Here we mention Nagel and Stein’s work in [NS] on the Carnot-Carathéodory space. Let $M$ be a connected smooth manifold and $\{X_1, \cdots, X_k\}$ are $k$ given smooth real vector fields on $M$ satisfying Hörmander condition of order $m$, that is, these vector fields together with their commutators of order $\leq m$ span the tangent space to $M$ at each point. The most important geometric objects used on the Carnot-Carathéodory spaces are (i) a class of equivalent control distances constructed on $M$ via the vector fields $\{X_1, \cdots, X_k\}$; (ii) the volumes of balls satisfying the doubling property and the certain lower bound estimates. More precisely, $\mu(B(x, sr)) \sim s^{m+2}\mu(B(x, r))$ for $s \geq 1$ and $\mu(B(x, sr)) \sim s^4\mu(B(x, r))$ for $s \leq 1$. These conditions on the measure are weaker than property (1.5) but are still stronger than the original doubling condition (1.2). The Carnot-Carathéodory spaces, as natural model geometries, are closely related to hypoelliptic partial differential equations, subelliptic operators, CR geometry and quasiconformal mapping. See [FP], [F], [FL], [GN1, GN2], [K], [NSW], [SC] and [VSC].

In [HMY1], motivated by the work of Nagel and Stein, the Hardy spaces were developed on spaces of homogeneous type with a regular quasi-metric and a measure satisfying the above conditions. Moreover, in [HMY2] singular integrals and the Besov and Triebel-Lizorkin spaces were also developed on spaces of homogeneous type $(X, d, \mu)$ where the quasi-metric $d$ satisfies the Hölder regularity property (1.4) and the measure $\mu$ satisfies the doubling property together with the reverse doubling condition, that is, there are constants $\kappa \in (0, \omega]$. 
and $c \in (0, 1]$ such that

$$c\lambda^\kappa \mu(B(x, r)) \leq \mu(B(x, \lambda r))$$

(1.6)

for all $x \in X$, $0 < r < \sup_{x,y \in X} d(x, y)/2$ and $1 \leq \lambda < \sup_{x,y \in X} d(x, y)/2r$.

Recently, in [GKZ] and [HLW], by different approaches, the Besov and the Triebel-Lizorkin spaces on metric measure spaces $(X, d, \mu)$ with the measure $\mu$ satisfying doubling property only and the Hardy spaces on spaces of homogeneous type in the sense of Coifman and Weiss were developed, respectively.

However, whether the most important embedding theorems can be established on spaces of homogeneous type in the sense of Coifman and Weiss is still an open problem. Even this is open whenever $(X, d, \mu)$ is a metric measure space with the measure $\mu$ satisfying the doubling together with the reverse doubling conditions.

The goal of this paper is to answer these problems. Throughout the rest of the paper, we will work on the space of homogeneous type $(X, d, \mu)$ in the sense of Coifman and Weiss with $\mu(\{x\}) = 0$ and $0 < \mu(B(x, r)) < \infty$ for all $x \in X$ and $r > 0$. To characterize the embedding theorem on spaces of homogeneous type $(X, d, \mu)$, the crucial geometric condition is the following

**Definition 1.1.** Suppose that $(X, d, \mu)$ is a space of homogeneous type in the sense of Coifman and Weiss with the upper dimension $\omega$. The measure $\mu$ is said to have a lower bound if there is a constant $C$ such that

$$\mu(B(x, r)) \geq Cr^\omega$$

for each $x \in X$ and all $r > 0$. And $\mu$ has a locally lower bound if

$$\mu(B(x, r)) \geq Cr^\omega$$

for each $x \in X$ and all $0 < r \leq 1$.

We remark that the lower bound conditions on the measure were used in the classical cases. To see this, let $(M, g)$ be a complete non-compact Riemannian manifold of dimension $n$ having non-negative curvature. Let $\mu$ be the canonical Riemannian measure on $M$ and denote by $V(x, r)$ the volume of the ball of radius $r > 0$ centered at $x \in M$, i.e., $V(x, r) = \mu(B(x, r))$. Note also that any smooth $(n-1)$-submanifold (i.e., hypersurface of co-dimension 1) inherits a Riemannian measure which we will denote by $\mu_{n-1}$. It is well known that from the celebrated Bishop–Gromov comparison theorem (see, e.g., [C]), we have $V(x, 2r) \leq 2^n V(x, r)$. In this setting, the measure with lower bound condition is equivalent to the Sobolev-type inequality and related to the isoperimetric inequality and Poincaré’s inequality. See Theorem 3.1.1 and Theorem 3.1.2 in [SC] and see also [CKP]. In [HKT], it was proved that if the Sobolev embedding theorem holds in $\Omega \subset \mathbb{R}^n$, in any of all the possible cases, then $\Omega$ satisfies the measure density condition, i.e. there exists a constant $c > 0$ such that for all $x \in \Omega$ and all $0 < r \leq 1$

$$|B(x, r) \cap \Omega| \geq cr^n.$$
In Section 2, we will show the main result of this paper. Applications of this main result will be given in the last section.

2. Embedding Theorem

Before stating our main result, we recall the following remarkable orthonormal wavelet basis which was constructed recently byAuscher and Hytönen in [AH].

**Theorem 1** ([AH] Theorem 7.1, [HLW] Theorem 2.9 and Corollary 2.10). Let $(X, d, \mu)$ be a space of homogeneous type in the sense of Coifman and Weiss with quasi-triangle constant $A_0$. There exists an orthonormal wavelet basis $\{\psi^k_\alpha\}, k \in \mathbb{Z}, x^k_\alpha \in \mathcal{Y}^k$, of $L^2(X)$, having exponential decay

\[|\psi^k_\alpha(x)| \leq \frac{C}{\sqrt{\mu(B(x^k_\alpha, \delta^k))}} \exp \left(-\nu \left(\frac{d(x^k_\alpha, x)}{\delta^k}\right)^a\right),\]

Hölder regularity

\[|\psi^k_\alpha(x) - \psi^k_\alpha(y)| \leq \frac{C}{\sqrt{\mu(B(x^k_\alpha, \delta^k))}} \left(\frac{d(x, y)}{\delta^k}\right)^\eta \exp \left(-\nu \left(\frac{d(x^k_\alpha, x)}{\delta^k}\right)^a\right)\]

for $d(x, y) \leq \delta^k$, and the cancellation property

\[\int_X \psi^k_\alpha(x) d\mu(x) = 0, \quad \text{for } k \in \mathbb{Z} \quad \text{and} \quad \alpha^k \in \mathcal{Y}^k.\]

Moreover,

\[f(x) = \sum_{k \in \mathbb{Z}} \sum_{\alpha \in \mathcal{Y}^k} \langle f, \psi^k_\alpha \rangle \psi^k_\alpha(x)\]

in the sense of $L^2(X)$, $G_0(\beta', \gamma')$ for each $\beta' \in (0, \beta)$ and $\gamma' \in (0, \gamma)$ and the space $(G_0(\beta, \gamma))'$ of distributions.

Here $\delta$ is a fixed small parameter, say $\delta \leq 10^{-3} A_0^{-10}$, $a = (1 + 2 \log_2 A_0)^{-1}$, and $C < \infty$, $\nu > 0$ and $\eta \in (0, 1]$ are constants independent of $k$, $\alpha$, $x$ and $x^k_\alpha$. See [AH] and [HLW] for more notations and details of the proofs.

We now introduce the sequence spaces on spaces of homogeneous type in the sense of Coifman and Weiss as follows.

**Definition 2.1.** Suppose that $\omega$ is the upper dimension of $(X, d, \mu)$. For $-\infty < s < \infty$ and $0 < p, q \leq \infty$, we say that a sequence $\{\lambda^k_\alpha\}_{\alpha \in \mathcal{Y}^k, k \in \mathbb{Z}}$ belongs to $\dot{b}^p_q$ if

\[\|\{\lambda^k_\alpha\}\|_{\dot{b}^p_q} := \left\{ \sum_{k \in \mathbb{Z}} \delta^{-k\omega q} \left[ \sum_{\alpha \in \mathcal{Y}^k} \left( \mu(Q^k_\alpha)^{1/p-1/2} |\lambda^k_\alpha| \right)^p \right]^{q/p} \right\}^{1/q} < \infty\]

and a sequence $\{\lambda^k_\alpha\}_{\alpha \in \mathcal{Y}^k, k \in \mathbb{Z}}$ belongs to $\ddot{b}^p_q$ if

\[\|\{\lambda^k_\alpha\}\|_{\ddot{b}^p_q} := \left\{ \sum_{k \in \mathbb{Z}^+} \delta^{-k\omega q} \left[ \sum_{\alpha \in \mathcal{Y}^k} \left( \mu(Q^k_\alpha)^{1/p-1/2} |\lambda^k_\alpha| \right)^p \right]^{q/p} \right\}^{1/q} < \infty,\]
where $Q^k_\alpha$ are dyadic cubes in $\mathcal{Y}^k$.

A sequence $\{\lambda^k_\alpha\}_{\alpha \in \mathcal{Y}^k, k \in \mathbb{Z}}$ belongs to $f_{p}^{s, q}$ for $-\infty < s < \infty, 0 < p < \infty, 0 < q \leq \infty$ if

$$\left\| \left\{ \lambda^k_\alpha \right\} \right\|_{f_{p}^{s, q}} := \left\| \left\{ \sum_{k \in \mathbb{Z}} \sum_{\alpha \in \mathcal{Y}^k} \delta^{-kq} \left( \mu(Q^k_\alpha)^{-1/2} |\lambda^k_\alpha| \chi_{Q^k_\alpha}(x) \right) \right\}^{1/q} \right\|_{L^p(X)} < \infty,$$

and a sequence $\{\lambda^k_\alpha\}_{\alpha \in \mathcal{Y}^k, k \in \mathbb{Z}^+}$ belongs to $f_{p}^{s, q}$ if

$$\left\| \left\{ \lambda^k_\alpha \right\} \right\|_{f_{p}^{s, q}} := \left\| \left\{ \sum_{k \in \mathbb{Z}^+} \sum_{\alpha \in \mathcal{Y}^k} \delta^{-kq} \left( \mu(Q^k_\alpha)^{-1/2} |\lambda^k_\alpha| \chi_{Q^k_\alpha}(x) \right) \right\}^{1/q} \right\|_{L^p(X)} < \infty.$$

We remark that the above sequence spaces on $R^n$ were introduced by Frazier and Jawerth in [FJ] with $\delta = 2^{-1}$ and $\mu(Q^k_\alpha) = 2^{-kn}$.

The main result of this paper is the following embedding theorem.

**Theorem 2.2.** (i) Let $0 < p_i \leq \infty, 0 < q \leq \infty, i = 1, 2$ and $s_1 \leq s_2$ with $-\infty < s_1 - \omega/p_1 = s_2 - \omega/p_2 < \infty$. Then

$$\left\| \left\{ \lambda^k_\alpha \right\} \right\|_{b_{p_1}^{s_1, q}} \leq C \left\| \left\{ \lambda^k_\alpha \right\} \right\|_{b_{p_2}^{s_2, q}}$$

if and only if the measure $\mu$ has the lower bound and

$$\left\| \left\{ \lambda^k_\alpha \right\} \right\|_{b_{p_1}^{s_1, q}} \leq C \left\| \left\{ \lambda^k_\alpha \right\} \right\|_{b_{p_2}^{s_2, q}}$$

if and only if the measure $\mu$ has the locally lower bound.

(ii) Let $0 < p_i < \infty$ and $0 < q_i \leq \infty$ for $i = 1, 2$, and $s_1 \leq s_2$ with $-\eta < s_1 - \omega/p_1 = s_2 - \omega/p_2 < \eta$. Then

$$\left\| \left\{ \lambda^k_\alpha \right\} \right\|_{f_{p_1}^{s_1, q_1}} \leq C \left\| \left\{ \lambda^k_\alpha \right\} \right\|_{f_{p_2}^{s_2, q_2}}$$

if and only if $\mu$ has the lower bound and

$$\left\| \left\{ \lambda^k_\alpha \right\} \right\|_{f_{p_1}^{s_1, q_1}} \leq C \left\| \left\{ \lambda^k_\alpha \right\} \right\|_{f_{p_2}^{s_2, q_2}}$$

if and only if $\mu$ has the locally lower bound.

To verify “only if”-parts of Theorem 2.2, taking the sequence $\{\lambda^k_\alpha\}$ with $\lambda^k_{\alpha_0} = 1$ and $\lambda^k_k = 0$ with $(\alpha, k) \neq (k_0, \alpha_0)$ for $k_0, k \in \mathbb{Z}$ and $\alpha_0 \in \mathcal{Y}^{k_0}, \alpha \in \mathcal{Y}^k$, we then have

$$\left\| \left\{ \lambda^k_{\alpha_0} \right\} \right\|_{b_{p_1}^{s_1, q}}(x) = \delta^{-k_0s_1} \mu(Q^k_{\alpha_0})^{1/p_1-1/2}.$$

Similarly,

$$\left\| \left\{ \lambda^k_{\alpha_0} \right\} \right\|_{b_{p_2}^{s_2, q}}(x) = \delta^{-k_0s_2} \mu(Q^k_{\alpha_0})^{1/p_2-1/2}.$$

Therefore, if

$$\left\| \left\{ \lambda^k_\alpha \right\} \right\|_{b_{p_1}^{s_1, q}} \leq C \left\| \left\{ \lambda^k_\alpha \right\} \right\|_{b_{p_2}^{s_2, q}},$$

we should have $\delta^{-k_0s_2} \mu(Q^k_{\alpha_0})^{1/p_2-1/2} \geq C \delta^{-k_0s_1} \mu(Q^k_{\alpha_0})^{1/p_1-1/2}$ and this implies that $\mu(Q^k_{\alpha_0}) \geq C\delta^{k_0\omega}$, for any $k_0 \in \mathbb{Z}, \alpha_0 \in \mathcal{Y}^{k_0}$. Repeating the same proof implies that if

$$\left\| \left\{ \lambda^k_\alpha \right\} \right\|_{b_{p_1}^{s_1, q}} \leq C \left\| \left\{ \lambda^k_\alpha \right\} \right\|_{b_{p_2}^{s_2, q}},$$

then $\mu(Q^k_{\alpha_0}) \geq C\delta^{k_0\omega}$, for any $k_0 \in \mathbb{Z}^+, \alpha_0 \in \mathcal{Y}^{k_0}$.
The “only if” parts for $\hat{f}^q_p(X)$ and $f^q_p(X)$ can be verified similarly to get $\mu(Q^{k_0}) \geq C\delta^{k_0\omega}$ for any $k_0 \in \mathbb{Z}$, $\alpha_0 \in \mathcal{V}^{k_0}$ and $\mu(Q^{k_0}) \geq C\delta^{k_0\omega}$ for any $k_0 \in \mathbb{Z}^+$, $\alpha_0 \in \mathcal{V}^{k_0}$, respectively.

Finally, the lower bound conditions will follow from the geometric structure of $(X,d,\mu)$, namely the following propositions.

**Proposition 2.3.** Suppose that for every $\ell \in \mathbb{Z}$ and every $\alpha \in \mathcal{V}^\ell$,
\begin{equation}
\mu(Q^\ell_\alpha) \geq C\delta^{\ell\omega},
\end{equation}
where $C$ is a positive constant independent of $\ell$ and $\alpha$. Then we have
\begin{equation}
\mu(Q^k_\beta) \geq \tilde{C}\delta^{k\omega}
\end{equation}
for every $k \in \mathbb{Z}$ and every $\beta \in \mathcal{X}^k$, where $\tilde{C}$ is a positive constant independent of $k$ and $\beta$.

Assume the above proposition for the moment, we can obtain the following result for the lower bound of the measure of any balls in $X$, which provide the necessary condition for the embedding theorem.

**Proposition 2.4.** Suppose that for every $k \in \mathbb{Z}$ and every $\beta \in \mathcal{X}^k$,
\begin{equation}
\mu(Q^k_\beta) \geq C\delta^{k\omega},
\end{equation}
where $C$ is a positive constant independent of $k$ and $\alpha$. Then we have
\begin{equation}
\mu(B(x,r)) \geq \tilde{C}r^\omega
\end{equation}
for every $x \in X$ and every $r > 0$, where $\tilde{C}$ is a positive constant independent of $x$ and $r$.

Before proving Propositions 2.3 and 2.4, we recall the fundamental result of the construction of dyadic cubes by Hytonen and Kairema [HK].

**Theorem 2** ([HK] Theorem 2.2). Suppose that constants $0 < c_0 \leq C_0 < \infty$ and $\delta \in (0,1)$ satisfy
\begin{equation}
12A_0^3C_0\delta \leq c_0.
\end{equation}
Given a set of points $\{z^k_\alpha\}_\alpha$, $\alpha \in \mathcal{A}_k$, for every $k \in \mathbb{Z}$, with the properties that
\begin{equation}
d(z^k_\alpha, z^k_\beta) \geq c_0\delta^k (\alpha \neq \beta), \quad \min_{\alpha} d(x, z^k_\alpha) < C_0\delta^k,
\end{equation}
we can construct families of sets $\bar{Q}^k_\alpha \subseteq Q^k_\alpha \subseteq \overline{Q}^k_\alpha$ (called open, half-open and closed dyadic cubes), such that:
\begin{enumerate}
\item[(11)] $\bar{Q}^k_\alpha$ and $\overline{Q}^k_\alpha$ are the interior and closure of $Q^k_\alpha$, respectively;
\item[(12)] if $\ell \geq k$, then either $Q^\ell_\beta \subseteq Q^k_\alpha$ or $Q^k_\alpha \cap Q^\ell_\beta = \emptyset$;
\item[(13)] $X = \bigcup_{\alpha} Q^k_\alpha$ (disjoint union) for all $k \in \mathbb{Z}$;
\item[(14)] $B(z^k_\alpha, c_1\delta^k) \subseteq Q^k_\alpha \subseteq B(z^k_\alpha, C_1\delta^k)$, where $c_1 := (3A_0^2)^{-1}c_0$ and $C_1 := 2A_0C_0$;
\item[(15)] if $\ell \geq k$ and $Q^\ell_\beta \subseteq Q^k_\alpha$, then $B(z^\ell_\beta, C_1\delta^\ell) \subseteq B(z^k_\alpha, C_1\delta^k)$.
\end{enumerate}
The open and closed cubes $\tilde{Q}_\alpha^k$ and $Q_\alpha^k$ depend only on the points $z_\beta^\ell$ for $\ell \geq k$. The half-open cubes $Q_\alpha^k$ depend on $z_\beta^\ell$ for $\ell \geq \min(k, k_0)$, where $k_0 \in \mathbb{Z}$ is a preassigned number entering the construction.

Proof of Proposition 2.3. For each fixed $k \in \mathbb{Z}$ and $\beta \in \mathcal{X}^k$, we now consider the number of the children of the dyadic cube $Q_\beta^k$. Suppose now $Q_\beta^k$ has $M$ children and $M \geq 2$, say $Q_\beta^{k+1}, \ldots, Q_\beta^{k+1}$. Then from the construction of $\mathcal{Y}^k$ we get that there must be one of the $M$ children belongs to $\mathcal{X}^{k+1}$ sharing the same center with $Q_\beta^k$, and the other $M - 1$ children belong to $\mathcal{Y}^k$. Without lost of generality, we assume that $Q_\beta^{k+1} \in \mathcal{X}^{k+1}$ and $Q_\beta^{k+1}, \ldots, Q_\beta^{k+1} \in \mathcal{Y}^k$. Then from (2.5) we get that $\mu(Q_\beta^{k+1}) \geq C\delta^{(k+1)\omega}$ for $i = 2, \ldots, M$. As a consequence, we have

$$\mu(Q_\beta^k) \geq \sum_{i=2}^M \mu(Q_\beta^{k+1}) \geq C(M - 1)\delta^{(k+1)\omega} = \tilde{C}\delta^{k\omega},$$

where we set $\tilde{C} = C(M - 1)\delta^\omega$.

Suppose now $Q_\beta^k$ has only one child, say $Q_\beta^{k+1}$. Note that actually $Q_\beta^{k+1}$ is the same as $Q_\beta^k$. Suppose $Q_\beta^{k+1}$ has $M$ children in the $k + 2$ level, $M \geq 2$. Then using the previous argument we get that

$$\mu(Q_\beta^k) \geq C(M - 1)\delta^{(k+2)\omega} = \tilde{C}\delta^{k\omega},$$

where we set $\tilde{C} = C(M - 1)\delta^{2\omega}$.

If $Q_\beta^{k+1}$ also has only one child, say $Q_\beta^{k+2}$, then we further consider the number of the children of $Q_\beta^{k+2}$.

Thus, we now claim the following fact:

Suppose now we have a chain, $\{Q_\beta_0^k, Q_\beta_1^k, Q_\beta_2^k, \ldots, Q_\beta_n^k\}$, satisfying the condition that $Q_\beta^{k+1}$ is the only one child of $Q_\beta_0^k$, $Q_\beta^{k+2}$ is the only one child of $Q_\beta_1^k$, and so on. Here $k \in \mathbb{Z}$ and $\beta_i \in \mathcal{X}^{k+i}$, $i = 0, 1, 2, \ldots, n$.

Then there exists a positive constant $N$ such that for every possible chain satisfying the above conditions, we have $n \leq N$. In other words, $N$ is independent of $k$ and $\beta_0, \ldots, \beta_n$.

We now prove the above claim. Suppose $\{Q_\beta_0^k, Q_\beta_1^k, Q_\beta_2^k, \ldots, Q_\beta_n^k\}$ is an arbitrary chain satisfying the condition as in the claim.

First note that from (2.14) in Theorem 2, we have

$$B(y_\beta^k, c_1 \delta^k) \subseteq Q_\beta^k \subseteq B(y_\beta^k, C_1 \delta^k)$$

for $i = 0, 1, \ldots, n$, where $c_1 := (3A_3^2)^{-1}c_0$ and $C_1 := 2A_0C_0$.

Second, due to the condition of this chain, all the dyadic cubes $Q_\beta^{k+i}$ with $i = 1, 2, \ldots, n$ are the same as $Q_\beta^k$ and sharing the same center point. Thus, we have

$$B(y_\beta^k, c_1 \delta^k) \subseteq Q_\beta^k \subseteq B(y_\beta^k, C_1 \delta^k),$$

where $y_\beta^k$ coincides with $y_\beta^{k+n}$. As a consequence, we get that

$$c_1 \delta^k \leq C_1 \delta^{k+n}.$$
Note that from the setting of the space of homogeneous type \((X, d, \mu)\), we have the assumptions that \(\mu(\{x\}) = 0\) for every \(x \in X\) and that \(0 < \mu(B(x, r)) < \infty\) for every \(x \in X\) and \(r > 0\). Thus, the balls \(\{B(y_{\beta_0}^k, c_1 \delta^{k+i})\}\) shrinks to the center point \(y_{\beta_0}^k\) if \(i\) tends to infinity, and these balls must not be the same as \(\{y_{\beta_0}^k\}\) since there is no point mass.

This implies that
\[
n \leq \log_{\delta^{-1}} \left( \frac{C_1}{C_4} \right),
\]
which yields that the claim holds with \(N = \lfloor \log_{\delta^{-1}} \left( \frac{C_1}{C_4} \right) \rfloor + 1\).

Thus, for each fixed \(k \in \mathbb{Z}\) and \(\beta \in \mathcal{X}^k\), if the dyadic cube \(Q_{\beta_1}^k\) has only one child \(Q_{\beta_2}^{k+1}\), \(Q_{\beta_3}^{k+1}\) has only one child \(Q_{\beta_4}^{k+2}\) and so on, then this chain \(\{Q_{\beta_i}^k\}\) must be finite and has at most \(N\) cubes. That is, \(Q_{\beta_N}^{k+N}\) must have \(M\) children and \(M \geq 2\), say \(Q_{\gamma_{11}}^{k+N-1}, \ldots, Q_{\gamma_{1M}}^{k+N-1}\), and there is only one of these \(M\) children belonging to \(\mathcal{X}^{k+N}\), all the other \(M - 1\) children belonging to \(\mathcal{X}^{k+N}\).

Then using the previous argument again we get that
\[
\mu(Q_{\beta_1}^k) \geq C(M-1)\delta^{(k+N+1)\omega} = \tilde{C}\delta^{k\omega},
\]
where we set \(\tilde{C} = C(M-1)\delta^{(N+1)\omega}\). \(\square\)

**Proof of Proposition 2.4.** Fix \(x \in X\) and \(r > 0\). We set \(\alpha = (1 + 2\delta^{-1})^{-1}\). Next we choose \(k \in \mathbb{Z}\) such that
\[
C_1 \delta^{k+1} \leq \alpha r < C_4 \delta^k,
\]
where \(C_1\) is the constant in Theorem 2. Then we have
\[
(1 - \alpha) r < r - C_4 \delta^{k+1},
\]
which implies that \(C_1 \delta^{k+1} < \alpha r\) and hence
\[
C_1 \delta^{k} < \frac{\alpha r}{\delta} = \frac{(1 - \alpha)}{2}r. \tag{2.16}
\]

Now note that \(B(x, \alpha r)\) must be cover by a union of at most \(M\) dyadic cubes in \(\mathcal{X}^k\), since 1) \(\cup_{\beta \in \mathcal{X}} Q_{\beta}^k = X\); 2) \(\alpha r\) is comparable to the sidelength of the cubes in \(\mathcal{X}^k\); 3) the doubling property of \(\mu\) implies that the space \((X, d)\) is geometrically doubling. Here we also point out that \(M\) is a positive constant independent of \(x\) and \(r\).

Suppose now \(Q_{\beta_1}^k, \ldots, Q_{\beta_n}^k\) is the dyadic cubes in \(\mathcal{X}^k\) such that \(\cup_{i=1}^n Q_{\beta_i}^n\) covers \(B(x, \alpha r)\) and that \(Q_{\beta_i}^k \cap B(x, \alpha r) \neq \emptyset\), \(n \leq M\). Then from (2.14) in Theorem 2, we have
\[
Q_{\beta_i}^k \subseteq B(y_{\beta_i}^k, C_1 \delta^k)
\]
for \(i = 0, 1, \ldots, n\).

We now point out that (2.16) implies that \(B(y_{\beta_i}^k, C_1 \delta^k)\) is contained in \(B(x, r)\) for every \(i = 0, 1, \ldots, n\). Hence, \(Q_{\beta_i}^k\) is contained in \(B(x, r)\) for every \(i = 0, 1, \ldots, n\). As a consequence,
we have
\[
\mu(B(x,r)) \geq \sum_{i=1}^{n} \mu(Q_{\beta_i}^k) \geq n \cdot C \delta^{k\omega} \geq n \cdot C \left( \frac{\alpha r}{C_1} \right)^\omega \geq \tilde{C} r^\omega,
\]
where the second inequality follows from (2.7), and \( \tilde{C} = C \left( \frac{\alpha}{C_1} \right)^\omega \).

\[\square\]

**Corollary 2.5.** Suppose that for every \( k \in \mathbb{Z} \) and \( k \geq 0 \), and for every \( \alpha \in \mathcal{Y}^k \),
\[
\mu(Q_{\alpha}^k) \geq C \delta^{k\omega},
\]
where \( C \) is a positive constant independent of \( k \) and \( \alpha \). Then we have
\[
\mu(B(x,r)) \geq \tilde{C} r^\omega
\]
for every \( x \in X \) and every \( r > 0 \), where \( \tilde{C} \) is a positive constant independent of \( x \) and \( r \).

**Proof.** We first claim that the condition (2.18) for every \( k \in \mathbb{Z} \) and \( k \geq 0 \), and for every \( \alpha \in \mathcal{Y}^k \) implies that
\[
\mu(Q_{\alpha}^\ell) \geq C \delta^{\ell\omega}
\]
for every \( \ell \in \mathbb{Z} \) and every \( \beta \in \mathcal{X}^\ell \).

Suppose the above claim holds, then by applying the result in Proposition 2.4, we obtain that (2.19) holds for every \( x \in X \) and every \( r > 0 \), where \( \tilde{C} \) is a positive constant independent of \( x \) and \( r \).

Now we prove the claim. First suppose that \( \ell \geq 0 \) and \( \beta \in \mathcal{X}^\ell \). Then following the same proof of Proposition 2.3, we obtain that
\[
\mu(Q_{\alpha}^\ell) \geq C \delta^{\ell\omega},
\]
where \( C \) is a positive constant independent of \( \ell \) and \( \beta \).

Next consider \( \ell = -1 \) and \( \beta \in \mathcal{X}^{-1} \). Note that all the decendent of the cube \( Q_{\alpha}^\ell \) are in \( \mathcal{X}^L \) with the level index \( L \geq 0 \). Following the same proof of Proposition 2.3 again, we obtain that
\[
\mu(Q_{\alpha}^\ell) \geq C \delta^{\ell\omega},
\]
where \( C \) is a positive constant independent of \( \ell \) and \( \beta \). Thus, the claim (2.20) holds for all \( \ell \geq -1 \).

By induction we obtain that the claim (2.20) holds for all \( \ell < -1 \) as well. This completes the proof of Corollary 2.5.

\[\square\]

We now show the “ if” parts of Theorem 2.2. Applying \( \mathcal{H}^{p_1} \)-inequality for \( \mathcal{H}^{p_1} \leq 1 \) yields
\[
\| \{ b_{\alpha}^{k} \} \|_{\mathcal{H}^{p_1}^{q}(X)} = \left\{ \sum_{k \in \mathbb{Z}} \delta^{-k \mathcal{s}_{1} q} \left[ \sum_{\alpha \in \mathcal{Y}^k} \left( \mu(Q_{\alpha}^k)^{1/p_1 - 1/2} | b_{\alpha}^{k} | \right)^{p_1} \right]^{q/p_1} \right\}^{1/q}
\leq \left\{ \sum_{k \in \mathbb{Z}} \delta^{-k \mathcal{s}_{1} q} \left[ \sum_{\alpha \in \mathcal{Y}^k} \left( \mu(Q_{\alpha}^k)^{1/p_1 - 1/2} \mu(Q_{\alpha}^{k \mathcal{s}_{1} q})^{1/p_2 - 1/2} | b_{\alpha}^{k} | \right)^{p_2} \right]^{q/p_2} \right\}^{1/q}
\]
where the lower bound condition \( \mu(Q_{\alpha_0}^k) \geq C\delta^{k_0}\omega \) together with the facts that \( \frac{1}{p_1} - \frac{1}{p_2} \leq 0 \) and \( s_1 - \omega/p_1 = s_2 - \omega/p_2 \) are used in the last inequality and equality, respectively.

Repeating the same proof implies that if \( \mu(Q_{\alpha}^k) \geq C\delta^{k}\omega \), for any \( k \in \mathbb{Z}^+ \), \( \alpha \in \mathcal{Y}^k \), then

\[
\|Q_{\alpha}^k\|_{b_1^{p_1,q}(X)} \leq C\|Q_{\alpha}^k\|_{b_2^{p_2,q}(X)}.
\]

To show the “if” part of Theorem 2.2 for sequence spaces in \( \tilde{f}^{s,q}_p(X) \), by the homogeneity of the norm \( \| \cdot \|_{\tilde{f}^{s,q}_p(X)} \), we may suppose \( \|Q_{\alpha}^k\|_{\tilde{f}^{s,q}_p(X)} = 1 \) without loss of generality. Since

\[
\|Q_{\alpha}^k\|_{\tilde{f}^{s,q}_p(X)} = \int_0^\infty t^{p_1-1} \mu\left( \left\{ x : \left\{ \sum_{j=\infty}^{\infty} \sum_{\tau \in \mathcal{Y}^{\gamma}} \delta^{-j\gamma q_n} \mu(Q_{\tau}^k)|\chi_{Q_{\tau}^k}(x)|^{q_1} \right\}^{1/q_1} > t \right\} \right) dt,
\]

the point of departure of the proof is to estimate the following distribution function

\[
\mu\left( \left\{ x : \left\{ \sum_{j=\infty}^{\infty} \sum_{\tau \in \mathcal{Y}^{\gamma}} \delta^{-j\gamma q_n} \mu(Q_{\tau}^k)|\chi_{Q_{\tau}^k}(x)|^{q_1} \right\}^{1/q_1} > t \right\} \right).
\]

To this end, note that by the orthogonality of wavelets \( \psi_{\alpha}^k \), we have

\[
(2.21) \quad |\lambda_{\alpha}^k| = \left| \sum_{k \in \mathbb{Z}} \sum_{\alpha \in \mathcal{Y}^k} \lambda_{\alpha}^k \langle \psi_{\alpha}^k, \psi_{\tau}^k \rangle \right|.
\]

To estimate \( \lambda_{\alpha}^k \), the key point is to replace the orthogonal estimate for \( \langle \psi_{\alpha}^k, \psi_{\tau}^j \rangle \) by the following almost orthogonal estimate (see [HLW] for the proof) for \( j, k \in \mathbb{Z}, \alpha \in \mathcal{Y}^k, \tau \in \mathcal{Y}^j \), \( \gamma > 0 \) and any \( \epsilon : 0 < \epsilon < \eta \),

\[
(2.22) \quad \left| \langle \psi_{\alpha}^k, \psi_{\tau}^j \rangle \right| \leq C\delta^{-k-j} \mu(Q_{\tau}^k)^{\frac{1}{2}} \mu(Q_{\alpha}^k)^{\frac{1}{2}} \frac{\mu(Q_{\tau}^k)^{\frac{1}{2}}}{\mu(Q_{\alpha}^k)^{\frac{1}{2}}} \left( \frac{\delta(k\wedge j)}{\delta(k\wedge j) + d(x_{\alpha}^k, x_{\tau}^j)} \right)^{\gamma}.
\]

We obtain that there exists a constant \( C \) such that

\[
(2.23) \quad \sum_{\tau \in \mathcal{Y}^j} \mu(Q_{\tau}^k)^{-1/2} |\lambda_{\alpha}^k| |\chi_{Q_{\tau}^k}(x)| \leq C\sum_{k \in \mathbb{Z}} \sum_{\tau \in \mathcal{Y}^j} \sum_{\alpha \in \mathcal{Y}^k} |\lambda_{\alpha}^k| |\chi_{Q_{\tau}^k}(x)|
\]

\[
\times \delta^{-k-j} \mu(Q_{\tau}^k)^{\frac{1}{2}} \frac{\mu(Q_{\alpha}^k)^{\frac{1}{2}}}{\mu(Q_{\tau}^k)^{\frac{1}{2}}} \left( \frac{\delta(k\wedge j)}{\delta(k\wedge j) + d(x_{\alpha}^k, x_{\tau}^j)} \right)^{\gamma}.
\]
Now we claim that for $r \leq 1$,

\[
(2.24) \quad \sum_{\tau \in \mathcal{Y}^j} \sum_{\alpha \in \mathcal{Y}_k} \frac{\mu(Q^k_\alpha)^{\frac{1}{2}}}{V_{\delta(k\wedge j)}(x^k_{\alpha}) + V_{\delta(k\wedge j)}(x^\tau_{\alpha}) + V(x^k_{\alpha}, x^\tau_{\alpha}) \left( \frac{\delta(k\wedge j)}{\delta(k\wedge j) + d(x^k_{\alpha}, x^\tau_{\alpha})} \right)^{\gamma}} |\lambda^k_\alpha| \chi^{Q^k_\alpha}(x) 
\]

\[
\leq C \delta^{k\omega(1-\frac{1}{r})} \mu(B)^{\frac{1}{r} - 1} \inf_{y \in \overline{B}} \left\{ M \left( \sum_{\alpha \in \mathcal{Y}_k} \mu(Q^k_\alpha)^{-r/2} |\lambda^k_\alpha|^r \chi^{Q^k_\alpha}(y) \right) \right\}^\frac{1}{r},
\]

where $B = B(x, \delta^{k\wedge j})$ and $M$ is the Hardy–Littlewood maximal function.

To prove (2.24), for $x \in \chi^{Q^k_\alpha}$ first replacing $V_{\delta(k\wedge j)}(x^k_{\alpha}) + V_{\delta(k\wedge j)}(x^\tau_{\alpha}) + V(x^k_{\alpha}, x^\tau_{\alpha})$ and $\delta(k\wedge j) + d(x^k_{\alpha}, x^\tau_{\alpha})$ by $V_{\delta(k\wedge j)}(x^k_{\alpha}) + V_{\delta(k\wedge j)}(x) + V(x^k_{\alpha}, x)$ and $\delta(k\wedge j) + d(x^k_{\alpha}, x)$, respectively, and then taking sum over $\tau \in \mathcal{Y}^j$ together with the fact $r \leq 1$ yield

\[
\sum_{\tau \in \mathcal{Y}^j} \sum_{\alpha \in \mathcal{Y}_k} \frac{\mu(Q^k_\alpha)^{\frac{1}{2}}}{V_{\delta(k\wedge j)}(x^k_{\alpha}) + V_{\delta(k\wedge j)}(x^\tau_{\alpha}) + V(x^k_{\alpha}, x^\tau_{\alpha}) \left( \frac{\delta(k\wedge j)}{\delta(k\wedge j) + d(x^k_{\alpha}, x^\tau_{\alpha})} \right)^{\gamma}} |\lambda^k_\alpha| \chi^{Q^k_\alpha}(x) 
\]

\[
\leq C \left\{ \sum_{\alpha \in \mathcal{Y}_k} \mu(Q^k_\alpha)^{\frac{1}{2}} \left[ \int X \left( \sum_{\alpha \in \mathcal{Y}_k} \mu(Q^k_\alpha)^{-r/2} \chi^{Q^k_\alpha}(y) \right)^\frac{1}{r} \right] \sum_{\alpha \in \mathcal{Y}_k} \mu(Q^k_\alpha)^{-\frac{1}{2}} \left[ \int X \left( \frac{\delta(k\wedge j)}{\delta(k\wedge j) + d(y, x)} \right)^{\gamma} \right] \sum_{\alpha \in \mathcal{Y}_k} \mu(Q^k_\alpha)^{-\frac{1}{2}} |\lambda^k_\alpha|^r \chi^{Q^k_\alpha}(y) d\mu(y) \right\}^\frac{1}{r}. 
\]

Note that $C \delta^{k\omega} \leq \mu(Q^k_\alpha)$ and $r \leq 1$ imply that the last term above is bounded by

\[
C \delta^{k\omega(1-\frac{1}{r})} \left\{ \int X \left( \sum_{\alpha \in \mathcal{Y}_k} \mu(Q^k_\alpha)^{-1} \chi^{Q^k_\alpha}(y) \right)^\frac{1}{r} \right\} \sum_{\alpha \in \mathcal{Y}_k} \mu(Q^k_\alpha)^{-\frac{1}{2}} \left[ \int X \left( \frac{\delta(k\wedge j)}{\delta(k\wedge j) + d(y, x)} \right)^{\gamma} \right] \sum_{\alpha \in \mathcal{Y}_k} \mu(Q^k_\alpha)^{-\frac{1}{2}} |\lambda^k_\alpha|^r \chi^{Q^k_\alpha}(y) d\mu(y) \right\}^\frac{1}{r}. 
\]

For any $x' \in B$, we have

\[
H_1 \leq C \delta^{k\omega(1-\frac{1}{r})} \left\{ \mu(B)^{1-r} \int B \sum_{\alpha \in \mathcal{Y}_k} \mu(Q^k_\alpha)^{-\frac{1}{2}} |\lambda^k_\alpha|^r \chi^{Q^k_\alpha}(y) d\mu(y) \right\}^\frac{1}{r}. 
\]
\[
\leq C \delta^{k\omega(1-\frac{r}{2})} \mu(B)^{\frac{1}{2}-1} \left\{ M \left( \sum_{\alpha \in \mathcal{Y}^k} \mu(Q^k_{\alpha})^{-\frac{r}{2}} |\lambda^k_{\alpha}| \chi_{Q^k_{\alpha}}(x') \right) \right\}^{\frac{1}{2}}.
\]

To estimate the term \(H_2\), applying the doubling property on the measure \(\mu\) yields
\[
\int_{[\delta^{-(m+1)}B] \setminus [\delta^{-m}B]} \sum_{\alpha \in \mathcal{Y}^k} \delta^{(k\wedge j)} \left[ \frac{1}{V_{\delta^{(k\wedge j)}}(x) + V_{\delta^{(k\wedge j)}}(x') + V(y, x) \left( \frac{\delta^{(k\wedge j)}}{\delta^{(k\wedge j)}} + d(y, x) \right) \right]^{\gamma} \right] \mu(Q^k_{\alpha})^{-\frac{r}{2}} |\lambda^k_{\alpha}| \chi_{Q^k_{\alpha}}(y) d\mu(y)
\]
\[
\leq C \delta \mu(\delta^{-m}B)^{1-r} \int_{\delta^{-(m+1)}B} \sum_{\alpha \in \mathcal{Y}^k} \mu(Q^k_{\alpha})^{-\frac{r}{2}} |\lambda^k_{\alpha}| \chi_{Q^k_{\alpha}}(x') \mu(\delta^{-m}B)^{1-r} \sum_{\alpha \in \mathcal{Y}^k} \mu(Q^k_{\alpha})^{-\frac{r}{2}} |\lambda^k_{\alpha}| \chi_{Q^k_{\alpha}}(x') \]
\[
\leq C \delta \mu(\delta^{-m}B)^{1-r} \int_{\delta^{-(m+1)}B} \sum_{\alpha \in \mathcal{Y}^k} \mu(Q^k_{\alpha})^{-\frac{r}{2}} |\lambda^k_{\alpha}| \chi_{Q^k_{\alpha}}(x') \mu(\delta^{-m}B)^{1-r} \sum_{\alpha \in \mathcal{Y}^k} \mu(Q^k_{\alpha})^{-\frac{r}{2}} |\lambda^k_{\alpha}| \chi_{Q^k_{\alpha}}(x').
\]

If \(\gamma\) is chosen so that \(\gamma r - \omega(1-r) > 0\), then
\[
H_2 \leq C \delta^{k\omega(1-\frac{1}{r})} \mu(B)^{\frac{1}{2}-1} \left\{ M \left( \sum_{\alpha \in \mathcal{Y}^k} \mu(Q^k_{\alpha})^{-\frac{r}{2}} |\lambda^k_{\alpha}| \chi_{Q^k_{\alpha}}(x') \right) \right\}^{\frac{1}{2}}.
\]

This implies that
\[
\sum_{\tau \in \mathcal{Y}^j} \sum_{\alpha \in \mathcal{Y}^k} \mu(Q^j_{\tau})^{-1/2} |\lambda^j_{\tau}| \chi_{Q^j_{\tau}}(x)
\]
\[
\leq C \sum_{k \in \mathbb{Z}} \delta^{k\omega} \mu(B)^{\frac{1}{r}-1} \left\{ M \left( \sum_{\alpha \in \mathcal{Y}^k} \mu(Q^k_{\alpha})^{-\frac{r}{2}} |\lambda^k_{\alpha}| \chi_{Q^k_{\alpha}}(y) \right) \right\}^{\frac{1}{r}}.
\]

Taking infimum for \(x'\) over \(B\) implies the claim (2.24).

The estimate (2.23) together with the estimate (2.24) yields
\[
\sum_{\tau \in \mathcal{Y}^j} \mu(Q^j_{\tau})^{-1/2} |\lambda^j_{\tau}| \chi_{Q^j_{\tau}}(x)
\]
\[
\leq C \sum_{k \in \mathbb{Z}} \delta^{k\omega} \mu(B)^{\frac{1}{r}-1} \left\{ M \left( \sum_{\alpha \in \mathcal{Y}^k} \delta^{k\omega} \mu(B)^{\frac{1}{r}-1} \delta^{k\omega} \mu(B)^{\frac{1}{r}-1} \delta^{k\omega} \mu(B)^{\frac{1}{r}-1} \delta^{k\omega} \mu(B)^{\frac{1}{r}-1} \delta^{k\omega} \mu(B)^{\frac{1}{r}-1} \right) \right\}^{\frac{1}{r}}.
\]

Choosing \(r\) so that \(r < \min\{p_2, q_2, 1\}\) and denoting
\[
F_k(y) = \left\{ M \left( \sum_{\alpha \in \mathcal{Y}^k} \delta^{k\omega} \mu(Q^k_{\alpha})^{-\frac{r}{2}} |\lambda^k_{\alpha}| \chi_{Q^k_{\alpha}}(y) \right) \right\}^{\frac{1}{r}},
\]

we have
\[
\left\{ \sum_{j=-\infty}^{\infty} \sum_{\tau \in \mathcal{Y}^j} \delta^{-j\lambda_1} \left( \mu(Q^j_{\tau})^{-1/2} |\lambda^j_{\tau}| \chi_{Q^j_{\tau}}(x) \right) \right\}^{1/q_1}.\]
Applying the Fefferman–Stein vector-valued maximal function inequality [FS] for \( r < \min\{p_2, q_2\} \) together with the fact that \( \|\{\lambda_{\alpha}^k\}\|_{l^{p_2/q_2}(X)} = 1 \) yields

\[
\inf_{y \in B} F_k(y) \leq \inf_{y \in B} \left\{ \sum_{k \in \mathbb{Z}} (F_k(y))^{q_2} \right\}^{1/q_2}
\leq C \left\{ \mu(B)^{-1} \int_B \left( \sum_{k \in \mathbb{Z}} (F_k(x))^{q_2} \right)^{p_2/q_2} d\mu(x) \right\}^{1/p_2}
\leq C \mu(B)^{-1/p_2} \left\| \left( \sum_{k \in \mathbb{Z}} (F_k(y))^{q_2} \right)^{1/q_2} \right\|_{p_2}
\leq C \mu(B)^{-1/p_2} \left\| \left\{ \sum_{k \in \mathbb{Z}} \left( \sum_{\alpha \in \mathcal{A}^k} \delta^{-k \omega(x)} \mu(Q_{\alpha}^k)^{-1/2} |\lambda_{\alpha}^k| \chi_{Q_{\alpha}^k}(x) \right)^{q_2} \right\}^{1/q_2} \right\|_{p_2}
\leq C \mu(B)^{-1/p_2}.
\]

We obtain

\[
\left\{ \sum_{j=-\infty}^N \sum_{\tau \in \mathcal{A}^j} \delta^{-js_{1q_1}} \left( \mu(Q_{\tau}^j)^{-1/2} |\lambda_{\tau}^j| \chi_{Q_{\tau}^j}(x) \right)^{q_1} \right\}^{1/q_1}
\leq C \left\{ \sum_{j=-\infty}^N \left( \sum_{k \in \mathbb{Z}} \delta^{k-j|m} \delta^{k\omega(x)-1} \delta^{-js_{1q_1}} \mu(B)^{1/p_2-1} \right)^{q_1} \right\}^{1/q_1}.
\]

The crucial point here is that we can choose \( r \) such that \( \frac{1}{r} - 1 = \frac{1}{p_2} \) and hence \( \mu(B)^{\frac{1}{p_2}-1} = 1 \).

Note that \( r = \frac{p_2}{1+p_2} < p_2 \). It suffices to show that \( \hat{j}_{p_1}^{s_{1q_1}} \equiv \hat{j}_{p_2}^{s_{2q_2}} \) for any \( q_2 > r = \frac{p_2}{1+p_2} \) since \( \hat{j}_{p_2}^{s_{2q_2}} \equiv \hat{j}_{p_2}^{s_{2q_2}} \) holds for any \( 0 < q < q_2 \). Under these assumptions with \( q_2 > r = \frac{p_2}{1+p_2} \), we have

\[ (2.25) \]

\[
\left\{ \sum_{j=-\infty}^N \sum_{\tau \in \mathcal{A}^j} \delta^{-js_{1q_1}} \left( \mu(Q_{\tau}^j)^{-1/2} |\lambda_{\tau}^j| \chi_{Q_{\tau}^j}(x) \right)^{q_1} \right\}^{1/q_1}
\leq C \left\{ \sum_{j=-\infty}^N \left( \sum_{k \in \mathbb{Z}} \delta^{k-j|m} \delta^{k\omega(x)-1} \delta^{-js_{1q_1}} \right)^{q_1} \right\}^{1/q_1}
\leq C \delta \left( \sum_{j=-\infty}^N \sum_{k \in \mathbb{Z}} \delta^{k-j|m} \delta^{k\omega(x)-1} \right)^{q_1}^{1/q_1}
\leq C \delta \frac{N}{m},
\]
where we choose \( \epsilon \) so that \(-\eta < -\epsilon < s_1 - \frac{\omega}{p_1} < \epsilon < \eta\). On the other hand, applying Hölder inequality with \( \frac{2^*}{q_1} > 1 \) and \( \frac{2^*}{q_1} \) inequality with \( \frac{2^*}{q_1} \leq 1 \) implies that

\[
\left\{ \sum_{j=N+1}^{\infty} \sum_{\tau \in \mathcal{I}} \delta^{-j_1 q_1} \left( \mu(Q^j_\tau)^{-1/2} |\lambda^j_\tau| |\chi_{Q^j_\tau}(x)\right) \right\}^{1/q_1}
\]

\[
(2.26) = \left\{ \sum_{j=N+1}^{\infty} \delta^j (s_2 - s_1) q_1 \left( \sum_{\tau \in \mathcal{I}} \delta^{-j_2 q_2} \mu(Q^j_\tau)^{-1/2} |\lambda^j_\tau| |\chi_{Q^j_\tau}(x)\right) \right\}^{1/q_1}
\]

\[
\leq C \delta^{N\left(\frac{\omega}{p_2} - \frac{\omega}{p_1}\right)} \left\{ \sum_{j=N+1}^{\infty} \sum_{\tau \in \mathcal{I}} \delta^{-j_2 q_2} \left( \mu(Q^j_\tau)^{-1/2} |\lambda^j_\tau| |\chi_{Q^j_\tau}(x)\right) \right\}^{1/q_2},
\]

where we use the fact that \( s_2 - s_1 = \frac{\omega}{p_2} - \frac{\omega}{p_1} > 0 \). From (2.25) and (2.26), it follows that

\[
\| |\lambda^j_\tau| |_{p_1}^{q_1}(X) p_1 \sum_{N=-\infty}^{\infty} \int_{2C\delta^{-\omega(N+1)/p_1} 2^{1/q_1}}^{t_p - 1} \times \mu \left( \left\{ \sum_{j=N+1}^{\infty} \sum_{\tau \in \mathcal{I}} \delta^{-j_1 q_1} \left( \mu(Q^j_\tau)^{-1/2} |\lambda^j_\tau| |\chi_{Q^j_\tau}(x)\right) \right\}^{1/q_1} > t \right) dt \]

\[
\leq p_1 \sum_{N=-\infty}^{\infty} \int_{2C\delta^{-\omega(N+1)/p_1} 2^{1/q_1}}^{t_p - 1} \times \mu \left( \left\{ \sum_{j=N+1}^{\infty} \sum_{\tau \in \mathcal{I}} \delta^{-j_2 q_2} \left( \mu(Q^j_\tau)^{-1/2} |\lambda^j_\tau| |\chi_{Q^j_\tau}(x)\right) \right\}^{1/q_2} > 2^{-1/q_2} (t/2) \right) dt \]

\[
\leq p_1 \int_{0}^{\infty} t_p^{p_1 - 1} \times \mu \left( \left\{ \sum_{j=-\infty}^{\infty} \sum_{\tau \in \mathcal{I}} \delta^{-j_2 q_2} \left( \mu(Q^j_\tau)^{-1/2} |\lambda^j_\tau| |\chi_{Q^j_\tau}(x)\right) \right\}^{1/q_2} > C \delta^{N\left(\frac{\omega}{p_2} - \frac{\omega}{p_1}\right)} 2^{-1/q_2} (t/2) \right) dt \]

\[
\leq p_2 \int_{0}^{\infty} t_p^{p_2 - 1} \times \mu \left( \left\{ \sum_{j=-\infty}^{\infty} \sum_{\tau \in \mathcal{I}} \delta^{-j_2 q_2} \left( \mu(Q^j_\tau)^{-1/2} |\lambda^j_\tau| |\chi_{Q^j_\tau}(x)\right) \right\}^{1/q_2} > Cu \right) ) du \]

\[
\leq C \| |\lambda^j_\tau| |_{p_2}^{q_2}(X).\]

The proof of the “if”-part of Theorem 2.2 for \( \tilde{f}^{s,q}(X) \) is concluded.
The similar proof yields that the locally lower bound condition on the measure \( \mu \) implies \( f_{p_1 q_1}^{s_1, q_1} \rightarrow f_{p_2 q_2}^{s_2, q_2} \). In fact, we have

\[
(2.27) \quad \left\{ \sum_{j=0}^{N} \sum_{\tau \in \mathcal{Y}_j} \delta^{-j s_1 q_1} \left( \mu(Q_\tau^j)^{-1/2} |\lambda_\tau^j| \chi_{Q_\tau^j}^j(x) \right)^{q_1} \right\}^{1/q_1} \leq C \delta^{-N \omega_{p_1}}
\]

and for \( N \geq -1, \)

\[
(2.28) \quad \left\{ \sum_{j=N+1}^{\infty} \sum_{\tau \in \mathcal{Y}_j} \delta^{-j s_1 q_1} \left( \mu(Q_\tau^j)^{-1/2} |\lambda_\tau^j| \chi_{Q_\tau^j}^j(x) \right)^{q_1} \right\}^{1/q_1} \leq \delta^{N \left( \frac{\omega_{p_2}}{p_2} - \frac{\omega_{p_1}}{p_1} \right)} \left\{ \sum_{j=N+1}^{\infty} \sum_{\tau \in \mathcal{Y}_j} \delta^{-j s_2 q_2} \left( \mu(Q_\tau^j)^{-1/2} |\lambda_\tau^j| \chi_{Q_\tau^j}^j(x) \right)^{q_2} \right\}^{1/q_2}.
\]

We write

\[
\| \lambda_\tau^j \|_{f_{p_1 q_1}^{s_1, q_1}(X)} = p_1 \int_0^{2C^{2^{1/q_1}}} t^{p_1-1} \mu \left\{ x : \left\{ \sum_{j=0}^{\infty} \sum_{\tau \in \mathcal{Y}_j} \delta^{-j s_1 q_1} \left( \mu(Q_\tau^j)^{-1/2} |\lambda_\tau^j| \chi_{Q_\tau^j}^j(x) \right)^{q_1} \right\}^{1/q_1} > t \right\} dt
\]

\[
+ p_1 \sum_{N=0}^{\infty} \int_{2C^{N-\omega(N+1)/p_1} 2^{1/q_1}}^{2C^{N-\omega N/p_1} 2^{1/q_1}} t^{p_1-1} \mu \left( \left\{ x : \left\{ \sum_{j=0}^{\infty} \sum_{\tau \in \mathcal{Y}_j} \delta^{-j s_1 q_1} \left( \mu(Q_\tau^j)^{-1/2} |\lambda_\tau^j| \chi_{Q_\tau^j}^j(x) \right)^{q_1} \right\}^{1/q_1} > t \right\} dt \right.
\]

\[
:= H + I.
\]

We only need to estimate \( H \) since the proof for \( I \) is the same as the above. By (2.28) with \( N = -1, \)

\[
\left\{ \sum_{j=0}^{\infty} \sum_{\tau \in \mathcal{Y}_j} \delta^{-j s_1 q_1} \left( \mu(Q_\tau^j)^{-1/2} |\lambda_\tau^j| \chi_{Q_\tau^j}^j(x) \right)^{q_1} \right\}^{1/q_1} \leq \delta^{\left( \frac{\omega_{p_1}}{p_1} - \frac{\omega_{p_2}}{p_2} \right)} \left\{ \sum_{j=0}^{\infty} \sum_{\tau \in \mathcal{Y}_j} \delta^{-j s_2 q_2} \left( \mu(Q_\tau^j)^{-1/2} |\lambda_\tau^j| \chi_{Q_\tau^j}^j(x) \right)^{q_2} \right\}^{1/q_2}.
\]

Therefore, we obtain

\[
H \leq p_1 \int_0^{2C^{2^{1/q_1}}} t^{p_1-1} \mu \left\{ x : \left\{ \sum_{j=0}^{\infty} \sum_{\tau \in \mathcal{Y}_j} \delta^{-j s_2 q_2} \left( \mu(Q_\tau^j)^{-1/2} |\lambda_\tau^j| \chi_{Q_\tau^j}^j(x) \right)^{q_2} \right\}^{1/q_2} > \delta^{\left( \frac{\omega_{p_1}}{p_1} - \frac{\omega_{p_2}}{p_2} \right)} t \right\} dt
\]

\[
\leq \frac{p_1}{p_2} \left( 2C^{\left( \frac{\omega_{p_1}}{p_1} - \frac{\omega_{p_2}}{p_2} \right) 2^{1/q_1}} \right)^{p_1-p_2} \int_0^{2C^{\left( \frac{\omega_{p_1}}{p_1} - \frac{\omega_{p_2}}{p_2} \right) 2^{1/q_1}}} t^{p_2-1} \mu \left\{ x : \left\{ \sum_{j=0}^{\infty} \sum_{\tau \in \mathcal{Y}_j} \delta^{-j s_2 q_2} \left( \mu(Q_\tau^j)^{-1/2} |\lambda_\tau^j| \chi_{Q_\tau^j}^j(x) \right)^{q_2} \right\}^{1/q_2} > t \right\} dt.
\]
\[ \leq C\|\lambda_2^\frac{1}{p_2}\|_{L^{p_2,q_2}(\Omega)}^2. \]

The proof of Theorem 2.2 is complete.

### 3. Applications

We would like to point out that Theorem 2.2 shows that geometric conditions on the measure only play a crucial role for the embedding theorem. As an application of Theorem 2.2, we provide new embedding theorems of the Besov and Triebel-Lizorkin spaces on spaces of homogeneous type in the sense of Coifman and Weiss. For this purpose, we first recall test functions and distributions on \((X, d, \mu)\), space of homogeneous type in the sense of Coifman and Weiss.

**Definition 3.1.** (Test functions, [HLW]) Fix \(x_0 \in X, r > 0, \gamma > 0\) and \(\beta \in (0, \eta)\) where \(\eta\) is the regularity exponent from Theorem 2. A function \(f\) defined on \(X\) is said to be a test function of type \((x_0, r, \beta, \gamma)\) centered at \(x_0 \in X\) if \(f\) satisfies the following three conditions.

(i) (Size condition) For all \(x \in X\),

\[ |f(x)| \leq C \frac{1}{V_r(x_0) + V(x, x_0)} \left( \frac{r}{r + d(x, x_0)} \right)^\gamma. \]

(ii) (Hölder regularity condition) For all \(x, y \in X\) with \(d(x, y) < (2A_0)^{-1}(r + d(x, x_0))\),

\[ |f(x) - f(y)| \leq C \left( \frac{d(x, y)}{r + d(x, x_0)} \right)^\beta \frac{1}{V_r(x_0) + V(x, x_0)} \left( \frac{r}{r + d(x, x_0)} \right)^\gamma. \]

We denote by \(G(x_0, r, \beta, \gamma)\) the set of all test functions of type \((x_0, r, \beta, \gamma)\). The norm of \(f\) in \(G(x_0, r, \beta, \gamma)\) is defined by

\[ \|f\|_{G(x_0, r, \beta, \gamma)} := \inf \{ C > 0 : \text{(i) and (ii) hold} \}. \]

For each fixed \(x_0\), let \(G(\beta, \gamma) := G(x_0, 1, \beta, \gamma)\). It is easy to check that for each fixed \(x_1 \in X\) and \(r > 0\), we have \(G(x_1, r, \beta, \gamma) = G(\beta, \gamma)\) with equivalent norms. Furthermore, it is also easy to see that \(G(\beta, \gamma)\) is a Banach space with respect to the norm on \(G(\beta, \gamma)\).

For \(0 < \beta < \eta\) and \(\gamma > 0\), let \(\hat{G}(\beta, \gamma)\) be the completion of the space \(G(\eta, \gamma)\) in the norm of \(G(\beta, \gamma)\). For \(f \in \hat{G}(\beta, \gamma)\), define \(\|f\|_{\hat{G}(\beta, \gamma)} := \|f\|_{G(\beta, \gamma)}\). Finally, let \(G_0(\beta, \gamma) = \{ f \in G(\beta, \gamma) : \int_X f(x) d\mu(x) = 0 \}\) and \(\hat{G}_0(\beta, \gamma) = \{ f \in \hat{G}(\beta, \gamma) : \int_X f(x) d\mu(x) = 0 \}\).

**Definition 3.2.** (Distributions) The **distribution space** \((\hat{G}_0(\beta, \gamma))'\) is defined to be the set of all linear functionals \(\mathcal{L}\) from \(\hat{G}_0(\beta, \gamma)\) to \(\mathbb{C}\) with the property that there exists \(C > 0\) such that for all \(f \in \hat{G}_0(\beta, \gamma)\),

\[ |\mathcal{L}(f)| \leq C\|f\|_{\hat{G}(\beta, \gamma)}. \]

Similarly, \((\hat{G}(\beta, \gamma))'\) is defined to be the set of all linear functionals \(\mathcal{L}\) from \(\hat{G}(\beta, \gamma)\) to \(\mathbb{C}\).

The Besov and Triebel-Lizorkin spaces on \((X, d, \mu)\) are defined as follows.
Definition 3.3. Suppose that $|s| < \eta$ and $\omega$ is the upper dimension of $(X, d, \mu)$. Let $\psi_k^\alpha$ be a wavelet basis constructed in [AH]. For $\beta \in (0, \eta)$, $\gamma > 0$ and $\max\left(\frac{\omega}{\omega+\eta}, \frac{\omega}{\omega+\eta+s}\right) < p \leq \infty$ and $0 < q \leq \infty$, the Besov space $\dot{B}^{s,q}_p(X)$ is the collection of all $f \in (\dot{G}(\beta, \gamma))'$ such that the sequence $\{\langle \psi_k^\alpha, f \rangle\}$ belongs to $\dot{b}^{s,q}_p(X)$ and

$$\|f\|_{\dot{B}^{s,q}_p(X)} := \|\{\langle \psi_k^\alpha, f \rangle\}\|_{\dot{b}^{s,q}_p(X)}.$$ 

The Besov space $B^{s,q}_p(X)$ is the collection of all $f \in (\dot{G}(\beta, \gamma))'$ such that the sequence $\{\langle \psi_k^\alpha, f \rangle\}$ belongs to $b^{s,q}_p(X)$ and

$$\|f\|_{B^{s,q}_p(X)} := \|\{\langle \psi_k^\alpha, f \rangle\}\|_{b^{s,q}_p(X)}.$$ 

For $\beta \in (0, \eta)$, $\gamma > 0$ and $\max\left(\frac{\omega}{\omega+\eta}, \frac{\omega}{\omega+\eta+s}\right) < p < \infty$, $\max\left(\frac{\omega}{\omega+\eta}, \frac{\omega}{\omega+\eta+s}\right) < q \leq \infty$, the Triebel–Lizorkin space $\dot{F}^{s,q}_p(X)$ is the collection of all $f \in (\dot{G}(\beta, \gamma))'$ such that the sequence $\{\langle \psi_k^\alpha, f \rangle\}$ belongs to $\dot{f}^{s,q}_p(X)$ and

$$\|f\|_{\dot{F}^{s,q}_p(X)} := \|\{\langle \psi_k^\alpha, f \rangle\}\|_{\dot{f}^{s,q}_p(X)}.$$ 

The Triebel–Lizorkin space $F^{s,q}_p(X)$ is the collection of all $f \in (G(\beta, \gamma))'$ such that the sequence $\{\langle \psi_k^\alpha, f \rangle\}$ belongs to $f^{s,q}_p(X)$ and

$$\|f\|_{F^{s,q}_p(X)} := \|\{\langle \psi_k^\alpha, f \rangle\}\|_{f^{s,q}_p(X)}.$$ 

We remark that it is routing to verify the above definition is independent of the choice of the wavelet constructed in [AH]. We leave the details to the reader.

We now prove the following

Theorem 3.4. (i) Let $\max\left\{\frac{\omega}{\omega+\eta}, \frac{\omega}{\omega+\eta+s}\right\} < p_i \leq \infty$, $0 < q \leq \infty$, $i = 1, 2$ and $s_1 \leq s_2$ with $-\eta < s_1 - \omega/p_1 = s_2 - \omega/p_2 < \eta$. Then

$$\|f\|_{\dot{B}^{s_1,q}_{p_1}} \leq C\|f\|_{\dot{B}^{s_2,q}_{p_2}}$$

if and only if the measure $\mu$ has the lower bound and

$$\|f\|_{\dot{B}^{s_1,q}_{p_1}} \leq C\|f\|_{\dot{B}^{s_2,q}_{p_2}}$$

if and only if the measure $\mu$ has the locally lower bound.

(ii) Let $\max\left\{\frac{\omega}{\omega+\eta}, \frac{\omega}{\omega+\eta+s}\right\} < p_i < \infty$ and $\max\left\{\frac{\omega}{\omega+\eta}, \frac{\omega}{\omega+\eta+s}\right\} < q_i \leq \infty$ for $i = 1, 2$, and $s_1 \leq s_2$ with $-\eta < s_1 - \omega/p_1 = s_2 - \omega/p_2 < \eta$. Then

$$\|f\|_{\dot{F}^{s_1,q}_{p_1}} \leq C\|f\|_{\dot{F}^{s_2,q}_{p_2}}$$

if and only if $\mu$ has the lower bound and

$$\|f\|_{\dot{F}^{s_1,q}_{p_1}} \leq C\|f\|_{\dot{F}^{s_2,q}_{p_2}}$$

if and only if $\mu$ has the locally lower bound.
The proof will follow from Theorem 2.2. More precisely, taking \( f(x) = \psi_{\alpha_0}^{k_0} \) gives
\[
\| f \|_{B_p^{s,q}(X)} = \delta^{-k_0s_1} \mu(Q_{\alpha_0}^{k_0})^{1/p_1 - 1/2}.
\]
Similarly,
\[
\| f \|_{B_p^{s',q}(X)} = \delta^{-k_0s_2} \mu(Q_{\alpha_0}^{k_0})^{1/p_2 - 1/2}.
\]
Therefore, if
\[
\| f \|_{B_p^{s_1,q}} \leq C \| f \|_{B_p^{s_2,q}},
\]
we should have \( \delta^{-k_0s_2} \mu(Q_{\alpha_0}^{k_0})^{1/p_2 - 1/2} \geq C \delta^{-k_0s_1} \mu(Q_{\alpha_0}^{k_0})^{1/p_1 - 1/2} \) and this implies that \( \mu(Q_{\alpha_0}^{k_0}) \geq C \delta^{k_0\omega} \), for any \( k_0 \in \mathbb{Z} \) and \( \alpha_0 \in \mathcal{A}^{k_0} \). Repeating the same proof implies that if
\[
\| f \|_{B_p^{s_1,q}} \leq C \| f \|_{B_p^{s_2,q}},
\]
then \( \mu(Q_{\alpha_0}^{k_0}) \geq C \delta^{k_0\omega} \), for any \( k_0 \in \mathbb{Z}^+ \), \( \alpha_0 \in \mathcal{A}^{k_0} \).

The “only if” parts for \( \hat{F}_p^{s,q}(X) \) and \( F_p^{s,q}(X) \) can be verified similarly.

To show the the “if” parts, by the definition of \( \hat{B}_p^{s,q} \) and Theorem 2.2,
\[
\| f \|_{\hat{B}_p^{s_1,q}} = \| \{ f, \psi_{\alpha_0}^k \} \|_{\hat{B}_p^{s_1,q}} \leq C \| \{ f, \psi_{\alpha_0}^k \} \|_{\hat{B}_p^{s_2,q}} = C \| f \|_{\hat{B}_p^{s_1,q}}.
\]
The proofs for \( B_p^{s,q}(X), \hat{F}_p^{s,q}(X) \) and \( F_p^{s,q}(X) \) are same and we omit the details of the proofs.

The second application of Theorem 2.2 is to give the necessary and sufficient conditions for the classical weighted Besov and Triebel-Lizorkin spaces, in particular, the weighted Sobolev spaces on \( \mathbb{R}^n \). We now recall these spaces.

Let \( \psi \) be a non-negative function in \( \mathcal{S} \) on \( \mathbb{R}^n \) such that supp\( \psi \) = \{ \( 1/2 \leq |x| \leq 2 \) \}, \( \psi(x) \geq 0 \) for \( 1/2 \leq |x| \leq 2 \) and \( \sum_{j=-\infty}^{\infty} \psi(2^{-j}x) = 1 \) for \( |x| \neq 0 \). Let \( \psi_j, j = 0, \pm 1, \pm 2, \ldots \), and \( \Psi \) be functions in \( \mathcal{S} \) given by
\[
\hat{\psi}_j(x) = \psi(2^{-j}x), \quad \hat{\Psi}(x) = 1 - \sum_{j=1}^{\infty} \hat{\psi}_j(x).
\]

Weighted Besov and Triebel spaces are defined as follows. For \( -\infty < s < \infty \) and \( 0 < p, q \leq \infty \),
\[
B_p^{s,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}': \| f \|_{B_p^{s,q}} = \| \hat{\Psi} * f \|_{p,w} + \left\{ \sum_{j=1}^{\infty} (2^{js}\| \psi_j * f \|_{p,w})^q \right\}^{1/q} < \infty \right\}
\]
and
\[
\hat{B}_p^{s,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}_\infty': \| f \|_{\hat{B}_p^{s,q}} = \left\{ \sum_{j=-\infty}^{\infty} (2^{js}\| \psi_j * f \|_{p,w})^q \right\}^{1/q} < \infty \right\}.
\]
For \( -\infty < s < \infty \), \( 0 < p < \infty \) and \( 0 < q \leq \infty \),
\[
F_p^{s,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}': \| f \|_{F_p^{s,q}} = \| \hat{\Psi} * f \|_{p,w} + \left\{ \sum_{j=1}^{\infty} (2^{js}\| \psi_j * f \|_{p,w})^q \right\}^{1/q} < \infty \right\}
\]
and
\[ \hat{F}^{s,q}_{p,w}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}' \mid \|f\|_{F^{s,q}_{p,w}} = \left\| \left\{ \sum_{j=-\infty}^{\infty} (2^j |\psi_j * f|)^q \right\}^{1/q} \right\|_{p,w} < \infty \right\}. \]

We would like to point out that \( F^{s,q}_{p,w}(\mathbb{R}^n) \) and \( \hat{F}^{s,q}_{p,w}(\mathbb{R}^n) \) are the weighted Sobolev spaces.

Suppose that weights \( w \) satisfy Muckenhoupt \( A_p \) condition, \( 1 < p \leq \infty \). The embedding theorem for weighted Besov and Triebel spaces with \( A_p \) weights is the following

**Theorem 3.5.** (i) If \( -\infty < s_0 \leq s \leq \infty, 0 < p_0 \leq p \leq p_1 \leq \infty \) and \( s_0 - n/p_0 = s_1 - n/p_1 \), then \( B^{s_0,q}_{p_0,w} \hookrightarrow B^{s_1,q}_{p_1,w} \) if and only if \( w(B(x,r)) \geq cr^n \) for all \( x \) and \( 0 < r \leq 1 \), and \( \hat{B}^{s_0,q}_{p_0,w} \hookrightarrow \hat{B}^{s_1,q}_{p_1,w} \) if and only if \( w(B(x,r)) \geq cr^d \) for all \( x \) and \( 0 < r < \infty \).

(ii) If \( -\infty < s_0 < s_0 \leq s_0 \leq \infty, 0 < p_0 < p_1 \leq \infty \) and \( s_0 - n/p_0 = s_1 - n/p_1 \), then \( F^{s_0,q_1}_{p_0,w} \hookrightarrow F^{s_1,q_2}_{p_1,w} \) if and only if \( w(B(x,r)) \geq cr^d \) for all \( x \) and \( 0 < r \leq 1 \), and \( \hat{F}^{s_0,q_1}_{p_0,w} \hookrightarrow \hat{F}^{s_1,q_2}_{p_1,w} \) if and only if \( w(B(x,r)) \geq cr^d \) for all \( x \) and \( 0 < r < \infty \).

The “if” parts of this theorem were proved in [B]. To show the “only if” parts, it suffices to consider \( 1 < p, q < \infty \) and \( -1 < s < 1 \). To this end, we first recall the wavelet basis on \( \mathbb{R}^n \) given in [M].

**Theorem 3.6.** There exist \( 2^n - 1 \) functions \( \psi_1, \cdots, \psi_q \) having the following two properties:
\[ |\partial^\alpha \psi_i(x)| \leq C_N (1 + |x|)^{-N} \]
for every multi-index \( \alpha \in \mathbb{N}^n \) such that \( |\alpha| \leq r \), each \( x \in \mathbb{R}^n \) and every \( N \geq 1 \);
\[ \int x^\alpha \psi_i(x) dx = 0, \]
for \( |\alpha| \leq r \) and \( 1 \leq i \leq 2^n - 1 \). Moreover, the functions \( 2^{nj/2} \psi_i(2^j x - k), 1 \leq i \leq q, k \in \mathbb{Z}^n, j \in \mathbb{Z}, \) form an orthonormal basis of \( L^2(\mathbb{R}^n) \).

For each \( j \in \mathbb{Z} \), and \( k \in \mathbb{Z}^n \), let \( Q(j, k) \) denote the dyadic cube defined by \( 2^j x - k \in [0,1)^n \). Now we introduce the sequence spaces as follows.

**Definition 3.7.** We say that a sequence \( \{\lambda^j_k\} \) belongs to \( \hat{b}^{s,q}_{p,w} \) if
\[ \|\{\lambda^j_k\}\|_{\hat{b}^{s,q}_{p,w}} := \left\{ \sum_{j \in \mathbb{Z}} \delta^{-s} \left[ \sum_{k \in \mathbb{Z}^n} \left( w(Q(j,k)) \right)^{1/p-1/2} |\lambda^j_k|^{p} \right]^{q/p} \right\}^{1/q} < \infty \]
and a sequence \( \{\lambda^j_k\}_{j \in \mathbb{Z}^+, k \in \mathbb{Z}^n} \) belongs to \( b^{s,q}_{p,w} \) if
\[ \|\{\lambda^j_k\}\|_{b^{s,q}_{p,w}} := \left\{ \sum_{j \in \mathbb{Z}^+} \delta^{-s} \left[ \sum_{k \in \mathbb{Z}^n} \left( w(Q(j,k)) \right)^{1/p-1/2} |\lambda^j_k|^{p} \right]^{q/p} \right\}^{1/q} < \infty, \]
where \( Q(j,k) \) are dyadic cubes in \( \mathbb{R}^n \).

A sequence \( \{\lambda^j_k\}_{j \in \mathbb{Z}, k \in \mathbb{Z}^n} \) belongs to \( \hat{f}^{s,q}_{p,w} \) if
\[ \|\{\lambda^j_k\}\|_{\hat{f}^{s,q}_{p,w}} := \left\| \left\{ \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \delta^{-s} \left( w(Q(j,k))^{-1/2} |\lambda^j_k| \chi_{Q(j,k)}(x) \right) \right\}^{q/|q|} \right\|_{L^p(X)} < \infty, \]
and a sequence \( \{ \lambda_k^j \}_{j \in \mathbb{Z}^+, k \in \mathbb{Z}^n} \) belongs to \( f_{p,w}^{s,q} \) if

\[
\| \{ \lambda_k^j \} \|_{f_{p,w}^{s,q}} := \left\| \left\{ \sum_{j \in \mathbb{Z}^+} \sum_{k \in \mathbb{Z}^n} \delta^{-k_s q} \left( w(Q(j, k))^{-1/2} | \lambda_k^j | \chi_{Q(j, k)}(x) \right) \right\}^{1/q} \right\|_{L^p(X)} < \infty.
\]

It is a routing business to verify that \( f \in B_{p,w}^{s,q} \) if and only if \( \{ (f, 2^{n j/2} \psi_i(2^j \cdot k)) \} \in b_{s,q}^{p,w} \). Therefore, if \( B_{s,q}^{p,w} \hookrightarrow B_{s,q}^{p_1,w} \) then \( b_{s,q}^{p,w} \hookrightarrow b_{s,q}^{p_1,w} \). By Theorem 2.2, \( w(B(x, r)) \geq C r^n \) for all \( x \in \mathbb{R}^n \) and \( 0 < r \leq 1 \). The other proofs are similar and we omit the details.

We would like to point out that, in general, the \( A_p \) condition can not imply the lower bound property. To see this, let \( w_{\alpha, \beta}(x) \) equal \( |x|^\alpha \) if \( |x| \leq 1 \) and \( |x|^{\beta} \) if \( |x| > 1 \). It is easy to check that if \(-n < \beta < \alpha < n(p-1)\), then \( w_{\alpha, \beta} \in A_p \). However, the following inequality

\[ w(B(x, r)) \geq C r^n \]

can not hold for any fixed constant \( C \).

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