POLYNOMIAL LIE ALGEBRAS AND ASSOCIATED PSEUDOGROUP STRUCTURES IN COMPOSITE QUANTUM MODELS

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Abstract
Polynomial Lie (super)algebras $g_{pd}$ are introduced via $G_i$-invariant polynomial Jordan maps in quantum composite models with Hamiltonians $H$ having invariance groups $G_i$. Algebras $g_{pd}$ have polynomial structure functions in commutation relations, are related to pseudogroup structures $\exp V, V \in g_{pd}$ and describe dynamic symmetry of models under study. Physical applications of algebras $g_{pd}$ in quantum optics and in composite field theories are briefly discussed.

1. INTRODUCTION

The symmetry methods are fruitfully exploiting in quantum physics from the time of its origin (see, e.g., [1-6] and references therein). In particular, they provide an elegant treatment of physical tasks using a powerful formalism of Lie groups and algebras, especially generalized coherent states (GCS) and related techniques [2,4-6], when Hamiltonians $H$ of systems under study are linear forms

$$H = \sum_{\alpha=1}^{d} \lambda_{\alpha} F_{\alpha} + C, \quad [F_{\alpha}, F_{\beta}]_\pm = F_{\alpha} F_{\beta} - F_{\beta} F_{\alpha} = \sum_{\gamma=1}^{d} c_{\alpha\beta}^{\gamma} F_{\gamma} \equiv \psi_{\alpha\beta}^{1}(\{F_{\gamma}\}) \quad (1.1)$$

in generators $F_{\alpha}$ of $d(<\infty)$-dimensional Lie algebras $g^{D}$ of dynamic symmetry ($\lambda_{\alpha}$ are $c$-number coefficients and $[F_{\alpha}, C] = 0$). But for last years nonlinear models in many branches of quantum physics have called for different extensions of usual Lie algebras in Eq. (1.1) by means of 1) admitting infinite dimensions $d$ [7,8], 2) involving two types commutators $([,]_\pm)$ defining Lie superalgebras [7] and 3) using nonlinear structure functions $\psi_{\alpha\beta}^{1} (\{F_{\alpha}\})$ defining nonlinear or deformed Lie (super)algebras [7-11]. However, at present, these new Lie-algebraic structures, enabling to display important structure features of models with such generalized Hamiltonians (1.1), do not yield universal techniques for solving physical tasks due to the absence of simple ("finite") "disentangling" (Zassenhaus) and "multiplication" (Baker-Campbell- Hausdorff) formulas for their exponentials [5,7,10] which determine a high efficiency of group-theoretical methods [2,4]. Therefore, keeping in mind relevant extensions of these methods for new algebras, it is of importance to examine possibilities of adequate modifications of group-theoretical and Lie-algebraic techniques for single classes of models.
In the present paper, summarizing and developing results of the papers [5, 12,13], we discuss these problems for polynomial Lie (super)algebras $g_{pd}$ which describe dynamic symmetries of composite many-body models with Hamiltonians $H$ having invariance groups $G_i(H)$ and are obtained via generalized $G_i$-invariant Jordan maps [5].

2. POLYNOMIAL $G_i$-INVARIANT JORDAN MAPS AND POLYNOMIAL LIE (SUPER)ALGEBRAS IN MANY-BODY PHYSICS

As is known, in composite models of many-body physics whose Hamiltonians $H$ and quantum state spaces $L(H)$ are given in terms of boson-fermion operators $a_i, a_i^+, b_j, b_j^+$ ($[a_i, a_j^+] = \delta_{ij}, [b_j, b_j^+] = \delta_{ij}$), Lie-algebraic methods are introduced in a natural way via different boson-fermion maps [2–5,14]. Specifically, the Jordan map [5,14]: $(a_i, a_i^+, b_j, b_j^+) \mapsto F_\alpha$ giving generators $F_\alpha$ by quadratic forms in $a_i, a_i^+, b_j, b_j^+$ reduces quadratic (in field operators) Hamiltonians $H_0(a_i, a_i^+, b_j, b_j^+)$ to the form (1.1) [2–5].

This map, introducing collective dynamic variables $F_\alpha \in g_0^D$, is particularly fruitful when $H_0$ have (both continuous and discrete) invariance groups $G_i(H_0)$:

$$[H_0, G_i(H_0)] = 0 \implies [g_0^D, G_i(H_0)] = 0$$

(2.1)

and field operators are transformed with respect to certain (as a rule, fundamental) irreducible representations (IRs) of the groups $G_i(H_0)$ [5,15]. Then $F_\alpha \in g_0^D$ are quadratic vector $G_i(H_0)$- invariants; besides, by virtue of the construction and Eq. (2.1), invariant (Casimir and class) operators $C_k(G_i)$ and $C_k(g_0^D)$ determining IRs of $G_i$ and $g_0^D$ are functionally connected and their eigenvalues on spaces $L(H)$ are specified by certain common sets $[l_i] \equiv [l_0, l_1, \ldots]$ of invariant quantum numbers $l_i$ labeling extremal (lowest) vectors $|l_i\rangle$ of both $G_i$ and $g_0^D$ IRs. All that, in turn, entails spectral decompositions

$$L(H) = \sum_{[l_i]} \sigma([l_i]) L(|l_i\rangle)$$

(2.2)

of spaces $L(H)$ in direct sums of the subspaces $L(|l_i\rangle)$ which are invariant with respect to joint actions of algebras $g_0^D$ and groups $G_i$ being carrier-spaces of so-called factor-representations (isotypic components) [2] of both algebraic structures $G_i$ and $g_0^D$.

In the case of suitable groups $G_i$ decompositions (2.2) have the simple spectra ($\sigma([l_i]) = 1$), and, then, pairs $(G_i, g_0^D)$ say to act complementarily [15] on $L(H)$ and to form the Weyl-Howe dual pairs since such pairs were first considered in quantum mechanics by H. Weyl in the analysis of interrelations between unitary and permutation symmetries of $N$-electron systems [1], and their explicit mathematical characterization was given by R. Howe [16]. A physical importance of dual pairs is due to that they describe completely both invariance and dynamic symmetries of models under study (see, e.g., [15,5,12] and references therein).

The constructions above are generalized in a natural manner when extending quadratic Hamiltonians $H_0$ by $G_i$-invariant polynomials $H_I(a_i, a_i^+, b_j, b_j^+)$ of higher degrees which describe essentially nonlinear interactions [5,17]. Then generalized dual pairs $(G_i, g_{pd})$ of invariance groups $G_i$ and (describing dynamic symmetry) Lie-like (super)algebras $g^D = g_{pd}$ are obtained via $G_i$-invariant generalized Jordan maps [5]

$$(a_i, a_i^+, b_j, b_j^+) \mapsto (F_\alpha, V_\lambda, V_\lambda^+)$$

(2.3)

expanding the sets $\{F_\alpha \in g_0^D\}$ by some additional generators $V_\lambda, V_\lambda^+$ which are simultaneously elementary vector $G_i$-invariants and components of two mutually contragradient $g_0^D$- irreducible tensor operators $V, V^+$ given by homogeneous polynomials in $a_i, a_i^+, b_j, b_j^+$. (Note that in practice Hamiltonians $H_0, H_I$ may contain, besides
$g_0^D$-covariant operators, e.g., the Pauli matrices $\sigma_i(i)$ etc. that leads to appropriate modifications of Eq. (2.3) [5,12].) Then, by virtue of the vector invariant theory [1], the sets $\{F_\alpha, V_\lambda\}$ form finite-dimensional integrity bases [1] of associative algebras $A_\xi_i$ of $G_i$-invariants embedded in enveloping algebras $U(w(m))$ of the Weyl-Heisenberg (super)algebras $w(m)$ with generators $a_i, a_i^+, b_j, b_j^+$. Furthermore, endowing sets $\{F_\alpha, V_\lambda\}$ by commutators $[,]_\pm$, one gets (via a specific extension of the Ado’s theorem [2]) finite-dimensional Lie-like (super)algebras $g_{pd} = g^D$ of dynamic symmetry which, however, have polynomial structure functions $\psi_{\alpha\beta}(\{F_\alpha\})(p = \text{deg}(\psi))$ for commutators $[V_\alpha, V_\beta]_\pm, [V_\alpha, V_\beta^+]_\pm$ (the subscripts ”$\pm$” are determined by the ”fermion contents” of operators $V_\alpha$) and may be named as polynomial Lie (super)algebras. Emphasize an importance of the $G_i$-invariance of polynomials $H_\ell(\ldots)$ because, in general, it is impossible to get finite-dimensional algebras if cancelling this condition [5,12].

Algebras $g_{pd}$ are extensions of Lie algebras $g_0^D$ and have the coset structure [10,11]:

$$g_{pd} = h + v, \quad h = g_0^D, \quad v = \text{Span}\{V_\alpha, V_\alpha^+\}; \quad [h, v] \subseteq v, \quad [v, v] \subset U(h) \quad (2.4)$$

that enables us to construct IRs of $g_{pd}$ starting from $h$-modules [5] ($U(h)$ are enveloping algebras of $h$, and hereon we omit the subscript ”$\pm$” in $[,]_\pm$). For example, extensions via (2.3) of the unitary algebras $u(m)$ by their $(C_n$-invariant) symmetric and $(SU(n)$-invariant) skew-symmetric tensor operators give two classes of polynomial oscillators (super)algebras (see Section 4) whereas such extensions of the symplectic algebras $sp(2m, R)$ by $SO(n)$-invariant skew-symmetric tensors yield polynomial deformations of the Lie algebra $u(m, m)$ [5]. Without dwelling on other general properties of algebras $g_{pd}$ we only note that they are close in their structure with $W_n$-algebras [10], can be enlarged (via repeated commutators) to certain graded infinite-dimensional Lie (super)algebras $g_{pd} = \sum_{r=0}^\infty g_r$, $[g_r, g_s] \subset g_{r+s}$, $g_{r>0} = U(h)(v)^r$, and their exponentials $\exp(g_{pd})$ generate (non-analytical) pseudogroup structures having, in general, no finite disentangling formulas [5]. And now we consider applications of concrete algebras $g_{pd}$.

3. POLYNOMIAL LIE ALGEBRAS $sl_{pd}(2)$ IN QUANTUM OPTICS

Simplest examples of polynomial Lie algebras are given by algebras $sl_{pd}(2) = \text{Span}(V_0, V_\pm)$, obtained via extending the unitary algebra $u(1) = \text{Span}(V_0)$ by generators $V_\pm$ and satisfying the commutation relations (CRs)

$$[V_0, V_\pm] = \pm V_\pm, \quad [V_-, V_+] = \psi_{p}^{-1}(V_0) \equiv \Psi^p(V_0 + 1) - \Psi^p(V_0), \quad [\Psi^p(R_0), V_0] = 0 \quad (3.1)$$

where $\Psi^p(\ldots)$ is the polynomial of degree $p$ in the variable $V_0$, $\Psi^p(R_0) = \Psi^p(V_0) - V_+ V_-$ is the $sl_{pd}(2)$ Casimir operator (with $R_0$ being the ”lowest weight operator”) and hereon we omit the identity operator symbol $I$ in expressions like $\Psi^p(V_0 + aI)$. As is seen from Eq. (3.1), algebras $sl_{pd}(2)$ are reduced to the Lie algebra $sl(2)$ when $p = 2$ and may be considered as its specific deformations; they are also obtained from certain $q$-deformed algebras by means of the Wigner- Inöüm contraction when $q \to 1$ [6].

Algebras $sl_{pd}(2)$ arise via the map (2.3) in nonlinear models of quantum optics where coset generators $V_\pm$ are interpreted as creation/ destruction operators of specific coherent structures (clusters) [12]. For example, Hamiltonians

$$H = H_0 + H_I = \sum_{i=0}^1 \omega_i a_i^+ a_i + g(a_i^+)^n(a_0^m + g^*(a_0)^m(a_1)^n, \quad m \leq n, \quad (3.2)$$

describing multiphoton processes of scattering on multimode Fock spaces $L_F \equiv L(H) = \text{Span}\{|n_i\} = \Pi_i|n_i!^{1/2}(a_i^+)^{n_i}|0 > \}$ ($g$ are coupling constants, $\omega_i$ are field modes
frequencies and $\hbar = 1$), have invariance groups $G_i(H) = C_n \otimes C_m \otimes \exp(i\lambda R_1)$ where $C_n = \{c_{kn} = \exp(i2\pi k/n) : a_i^+ \to c_{kn}a_i^+\}$, $R_1 = (ma_1^+a_1 + na_0^+a_0)/(m + n)$.

Then the map (2.3) given as follows

$$V_0 = (a_i^+a_1 - a_0^+a_0)/(m + n), \quad V_+ = (a_1^+)^m(a_0^+)^m, \quad V_- = (V_+)^+$$

reduces Eq. (3.2) to the form

$$H = aV_0 + gV_+ + g^*V_- + C, \quad [V_\alpha, C] = 0, \quad \alpha = n\omega_1 - m\omega_0, \quad C = R_1(\omega_1 + \omega_0)$$

and determines the generalized dual pairs $(G_i(H), sl_{pd}(2))$. The structure polynomials $\Psi^p(V_0)$ are determined with the help of Eqs. (3.3), the characteristic relation

$$(V_+V_- - \Psi^p(V_0))|_{L(H)} = 0$$

and defining relations for $a(i), a^+(i)$ [12]:

$$\Psi^p(V_0) = (nV_0 + R_1)^{(m)}(R_1 - mV_0 + m)^{(m)}, \quad p = m + n, \quad A^{(b)} = A(A - 1)...(A - b + 1)$$

(An extra dependence of $\Psi^p(V_0)$ on $R_1$ reflects functional interrelations between invariant operators of $G_i(H)$ and $sl_{pd}(2)$.) The subspaces $L([l_i]) = \text{Span}\{[l_i]; v \} \equiv V_+^+[l_i], V_0[l_i]; v = (l_0 + v)[l_i]; v, R_i[l_i]; v = l_i[l_i]; v, i = 0, 1, V_-[l_i]) = 0\}$ in Eq. (2.2) are generated by the lowest vectors $[l_i] = \{s!\kappa!\}^{-1/2}(a_0^+)^s(a_1^+)\kappa|0>, \kappa = 0, 1, ..., n - 1, \quad s = 0, 1, ..., l_0 = (\kappa - s)/(m + n), l_1 = (mk + ns)/(m + n); \kappa$ specifies IRs of discrete invariance subgroups $C_n$ and $s = d([l_i])$ for compact versions of $sl_{pd}(2)$ that is the case when $m \neq 0$ in (3.2). (Compact $su_{pd}(2)$) and non-compact $(su_{pd}(1,1))$ realizations of $sl_{pd}(2)$ algebras are distinguished depending on whether dimensions $d([l_i])$ of the spaces $L([l_i])$ are finite or infinite [12].) Eqs. (3.1), (3.4)-(3.6) yield requisites for developing both exact and approximate Lie-algebraic methods to solve physical tasks in models (3.2). We outline them following the papers [12,13].

Exact methods are based on using Eqs. (3.1) and their resemblances with defining relations for $sl(2)$. Thus, substituting Eq. (3.4) in the Heisenberg equations for cluster dynamic variables $V_\alpha(t)$ related to generators $V_\alpha$ one gets non-linear equations

$$i\frac{dV_0}{dt} = gV_+ - g^*V_-, \quad i\frac{dV_+}{dt} = -aV_+ - g^*\psi^{p-1}(V_0), \quad i\frac{dV_-}{dt} = aV_- + g\psi^{p-1}(V_0)$$

generalizing linear Bloch equations for $sl(2)$ and having in the cluster mean-field approximation $<|f(V_\alpha)| = f(<|V_\alpha|>)$ quasiclassical solutions expressed in terms of hyperelliptic (Abelian) functions [12] when direct extensions of $sl(2)$-algebraic techniques [4] are impossible for finding evolution $(U_H(t))$ and diagonalizing $(S)$ operators because relevant disentangling formulas for $\exp(\sum_i a_iV_i)$ and explicit expressions for matrix elements $<[l_i]; f|\exp(\sum_i a_iV_i)[|l_i]; v>$ are absent [12].

Nevertheless, substituting “pseudogroup” (cf. [9]) representations

$$U_H(t) = \sum_{f=-\infty}^{\infty} V^f \alpha^H(V_0; t), \quad S = \sum_{f=-\infty}^{\infty} V^f S_f(V_0)$$

for $U_H(t)$ and $S$ (with $V^{-k} = V^k [\prod_{l=0}^{k-1} \Psi^p(V_0 - l)]^{-1}$ due to Eq. (3.5)) or expansions

$$|E([l_i]; f) = A_f \prod_j (V_+ - \Lambda_j f(V_0))[|l_i]) = A_f \prod_j (V_+ - \kappa_j f)[|l_i]) = \sum Q_v(E_f)[|l_i]; v$$
for energy eigenstates \(|E([l_i]; f)\) in the diagonalizing scheme \(SHS = \hat{H}(V_0)\) and the time-dependent Schrödinger equation \(i\hbar dU_H(t)/dt = HU(t)\) and using Eqs. \((3.1), (3.5)\), one gets finite-difference and differential-difference equations determining (together with unitarity conditions) "coefficients" \(S_f(Y_0), u_f^q(V_0; t)\), diagonal Hamiltonian forms \(\hat{H}(V_0)\), amplitudes \(Q_v(E_f)\) and energy spectra \(\{E([l_i]; f)\}\) \([12,13]\). (Note that two first equalities in \((3.9)\) realize, in fact, the algebraic Bethe ansatz \([17]\) for wave functions \(\Psi([l_i], v)\) in terms of the \(sl_{pd}(2)\) generators \([13]\).) These equations define new (nonclassical) orthogonal functions in both discrete and continuous variables which are simultaneously related to solutions of singular differential equations yielded by using two conjugate differential realizations of generators \(V_\alpha [12,13]\): \(V_+ = z, V_0 = zd/dz + l_0, V_- = \frac{1}{z^{-1}}\Psi(zd/dz + l_0)\) and \(V_+ = d/dz, V_0 = zd/dz + l_0, V_- = \Psi(zd/dz + l_0)(dz/dz)^{-1}\) generalizing the Bargmann or GCS representations \([2,4]\) for \(sl(2)\) generators. However, at present, simple analytical expressions for these special functions are absent in general cases though some integral representations were found for them with the help of a specific "dressing" of \(sl(2)\)-solutions of certain auxiliary exactly solvable tasks \([12]\).

Approximate methods developed in \([12,13]\) are based on realizations of generators \(V_\alpha\) as special elements of extended enveloping algebras \(U_b(sl(2))\) of the \(sl(2)\) algebra via a generalized Holstein-Primakoff map given on each subspace \(L([l_i])\) as follows \([12]\)

\[
V_0 = Y_0 + l_0 \pm J, \quad V_+ = Y_+[\Phi(Y_0)]^{1/2}, \quad \Phi(Y_0) = \frac{\Psi_p(V_0 + 1)}{\Psi_0^2(V_0 + 1)}, \quad V_- = (V_+)^\dagger
\]  

(3.10)

where \(Y_0\) and \(\Psi^2(Y_0) = (J \pm Y_0)(\pm J + 1 - Y_0)\) are generators and the structure polynomials of the \(su(2)/su(1, 1)\) algebras, \(\mp J\) are lowest weights of their IRs realized on \(L([l_i])\) (upper/lower signs correspond to \(su_{pd}(2)/su_{pd}(1, 1)\) versions of \(sl_{pd}(2)\)).

Eqs. \((3.10)\) enable us to re-write restrictions \(H_{[l_i]}\) of Eqs. \((3.4)\) on \(L([l_i])\) and basis vectors \([|l_i]; v\) in terms of \(Y_0\):

\[
H_{[l_i]}([Y_0]) = aY_0 + gY_+\sqrt{\Phi(Y_0)} + g^*\sqrt{\Phi(Y_0)}Y_- + \tilde{C}, \quad |[l_i]; v\rangle = \mathcal{N}(J, v)(Y_+)^v|[l_i]\rangle
\]  

(3.11)

(with \(\tilde{C} = C([l_i]) + a(l_0 \pm J)\) and to use the formalism \([4]\) of the \(SL(2)\) GCS

\[
|[l_i]; v; \xi\rangle = S_Y|[l_i]; v\rangle = \sum_{f \geq 0} S_f^Y(\xi)|[l_i]; f\rangle, \quad S_Y(\xi) = \exp(\xi Y_+ - \xi^* Y_-)
\]  

(3.12)

(with \(S_f^Y(\xi = r e^{i\theta})\) being the \(SL(2)\) Wigner \(D\)-function) for examining models \((3.4)\) in \(SL(2)\)-cluster (taking into account mode correlations in \((3.2)\) quasiclassical approximations \([13]\). Specifically, \(SL(2)\) GCS determine quasiclassical energy functionals

\[
\mathcal{H}^{eq}([l_i]; v; \xi) = \langle[l_i]; v; \xi|H_{[l_i]}|[l_i]; v; \xi\rangle = \tilde{C} + a(v \mp J)c(2r) - [ge^{-i\theta} + g^* e^{i\theta}] \sum_{f \geq 0} |S_{f^*}(\xi)S_{f^*+1}(\xi)||[\Psi^p(l_0 + 1 + f)]|^{1/2}
\]  

(3.13)

(with \(c(r) = \cos r/\cosh r\) for \(su(2)/su(1, 1)\)) or their mean-field approximations

\[
\mathcal{H}^{\text{mfa}}([l_i]; v; \xi) = H_{[l_i]}([l_i]; v; \xi)[Y_l([l_i]; v; \xi)]) = \tilde{C} + a(v \mp J)c(2r) - [ge^{-i\theta} + g^* e^{i\theta}] (J \mp v)s(2r)|\Phi((\mp J + v)c(2r))|^{1/2}
\]  

(3.14)

(with \(c(r) = \cos r/\cosh r, s(r) = \sin r/\sinh r\) for \(su(2)/su(1, 1)\) which can be applied in standard calculation schemes \([2,4,13]\).

For instance, inserting Eqs. \((3.13)\) and \((3.14)\) in the stationarity conditions

\[
a) \frac{\partial \mathcal{H}([l_i]; v; \xi)}{\partial \theta} = 0, \quad b) \frac{\partial \mathcal{H}([l_i]; v; \xi)}{\partial r} = 0
\]  

(3.15)
one finds approximate eigenfunctions $|E_{\text{even}}/\text{cmf}(l_i); v\rangle = S_V(\xi_0)^{|l_i|} |l_i; v\rangle$ and appropriate eigenenergies $E^q(|l_i); v\rangle = H^q(|l_i); v; \xi_0)$. $E_{\text{cmf}}(|l_i); v\rangle = H_{\text{cmf}}(|l_i); v; \xi_0)$ where $\xi_0 = r_0 g/|g|$ and values $r_0$ depend on $g, a, l_i$ and are determined by real solutions of algebraic equations obtained from Eq. (3.15b) with $v = 0$ [13].

The energy functionals (3.13), (3.14) may be also used for a quasiclassical analysis of the $SL(2)$-cluster dynamics described by the classical Hamiltonian equations (cf. [4])

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \quad H = \langle z(t); [l_i]|H|[l_i]; z(t) \rangle$$

(3.16)

for the canonical parameters $p = \langle z(t); [l_i]|Y_0|[l_i]; z(t) \rangle = \mp Jc(2r), q = \theta$ of the $SL(2)$ GCS $[[l_i]; z(t) = r \exp(-i\theta)] = S_V(\xi = -z(t)|[l_i])$ determining “principal” parts in the evolution operators $U_H(t) = \exp(i\alpha(t)Y_0 S_V(\xi(t))$ for initial states $|\phi(0)\rangle = |[l_i]\rangle[13]$. Similarly, this approach yields nonlinear quasiclassical Bloch-type equations

$$\dot{y} = \frac{1}{2} \nabla \times \nabla \mathcal{C}, \quad y = (y_1, y_2, y_0), \quad \mathcal{C} = \pm y_0^2 + y_1^2 + y_2^2, \quad \nabla = (\partial/\partial y_1, \partial/\partial y_2, \partial/\partial y_0)$$

(3.17)

which are equivalent in the mean-field approximation (3.14) to those obtained from Eqs. (3.7) ($y_i = \langle |l_i; z|{Y_i}|[l_i]; z \rangle, \mathbf{A} \times \mathbf{B}$ is the vector product symbol).

So, Eqs. (3.11)-(3.17) yield $SL(2)$-cluster quasiclassical solutions of spectral and evolution problems which take into account quantum correlations of interacting subsystems though they do not describe quantum dynamics exactly unlike the case of $sl(2)$-linear Hamiltonians [13].

4. POLYNOMIAL OSCILLATOR LIE ALGEBRAS in ANALYSIS of COMPOSITE FIELD MODELS

Another natural area of applications of algebras $g_{sd}$ is an algebraic analysis [5,18] of composite field models with internal (gauge) symmetries [2] which generalizes basic ideas of the paraquantization [3]. The simplest example of such analysis (but without introducing algebras $g_{sd}$) was given in [18] by using models (3.2) with $m = 0 = \omega_0$ to describe resonance states in particle physics; later it was generalized on multimode cases and applied to study multiphoton processes in quantum optics [5,10].

It was shown that operators $V^+ = (a_1^+)^n$ describe $C_n$-invariant n-particle kinematic clusters which display unusual (para)statistics and correspond to generalized asymptotically free fields realized on the Fock spaces $L_F$. Specifically, $V^+$-clusters satisfy noncanonical CRs (3.1) and multi-linear relations [5]:

$$ad_{V^+}v = 0, \quad ad_V v^+ \equiv [V, V^+], \quad V = (a_1)^n = (V^+)^+$$

(4.1)

generalizing (for $n \geq 3$) trilinear parastatistical Green’s relations [3]. Furthermore, the subspaces $L([l_0 = \kappa/n])$ in (2.2) describe coherent mixtures of constant numbers $\kappa$ of uncoupled particles $a_1^+$ and of varying in time numbers $N_V$ of $V$-clusters. However, operators $N_V$ have not standard (for (para)field) bilinear in $V, V^+$ forms (cf. [2,3]) but they can be expressed as nonlinear functions in the bilineals $V^+V, VV^+$ [5]:

$$N_V = (E_{11} - C(R_0))/n, \quad E_{11} = a_1^+ a_1 = nV_0 = \varphi(V^+V), \quad C([l_0]) = nl_0 = \kappa$$

(4.2)

as it follows from Eqs. (3.1), (3.5). Therefore, at best the quantities $V^+$ can be set in correspondence only to parafield quanta [3] (when $n = 2$) rather than to any asymptotically free particles [5]. Nevertheless, one can construct operators $W^+ = W^+(\{V_i\}), W = (W^+)^+$ obeying canonical CRs $[W, W^+] = 1$ and having the standard
number operators $N_W = W^+W = N_V$. Specifically, in [5,18,12] two equivalent forms were found for $W^+$:

$$W^+ = V^+ \sum_{\nu \geq 0} c_{\nu}(V^+)\nu(V) = V^+ \left[ V_0 - R_0 + 1 \right]^{1/2} \left[ (E_{11} + n) \right]$$

(4.3)

where the second form is, in fact, a modification of the (inverted) map (3.10) [12].

The analysis above was generalized in [5] by means of: 1) extending on the case of "m" modes using $C_n$-invariant Hamiltonians

$$H = C + \sum_{i=1}^{m} \omega_i a_i^+ a_i + \sum_{1 \leq i_1 \leq \ldots \leq m} [g_{i_1 \ldots i_n} V^+_{i_1 \ldots i_n} + g^*_{i_1 \ldots i_n}], V^+_{i_1 \ldots i_n} = a^+_i \ldots a^+_n; \quad (4.4)$$

2) involving both boson ($a_i$) and fermion ($b_i$) variables; 3) considering Hamiltonians with non-Abelian invariance groups $G_i = SU(n)$ (obtained from (4.4) by the substitutions: $a_i^+ a_i \rightarrow (a_i^+ \cdot a_i) \equiv \sum_{j=1}^{n} a_j^+ a_j$, $V^+_{i_1 \ldots i_n} \rightarrow X^+_{i_1 \ldots i_n} \equiv \sum_{j=1}^{n} \epsilon_{j_1 \ldots j_n} a_j^+ \ldots a_j^+$). These procedures determine generalized dual pairs ($C_n, osc_{pd}(m; n)$) and ($SU(n), osc_{pd}(m; n)$) where polynomial oscillator (super)algebras $osc_{V}(m; n)$ and $osc_{X}(m; 1^n)$ are extensions of the unitary algebras $u(m) = Span\{E_{ij} = a_i^+ a_j\}$ and $Span\{E_{ij} = (a_i^+ \cdot a_j)\}$ by their symmetric ($V^+, V_-$) and skew-symmetric ($X^+, X_-$) tensor operators.

The operators $X^+, X_-$ and $V^+, V_-$ satisfy non-canonical CRs with right sides depending on $E_{ij}$ (and on the $SU(n)$ Casimir operators for $osc_{X}(m; 1^n)$) and obey (due to the invariant theory) certain extra "bootstrap" relations (of the type $V_{1 ... 1} V_{2 ... 2} = V_{12 ... 12}$, $i^2_{1212} = 0$, $i^2_{1212} = \delta_{12}$, $W_a^+ = (W_{a^+})^+$ (4.5)

(with coefficients $f_{i1 \ldots in}(\ldots)$ determined from finite-difference equations due to $[W_a^+, W_b^+] = \delta_{ab}$) is, in general, sufficiently difficult because of relations abovementioned between $V/X$-clusters.

5. CONCLUSION

In conclusion we briefly point out some of prospects of developing results obtained. For instance, formal constructions of Sections 2, 4 may be generalized by involving into consideration $q$-deformed oscillators and other invariance groups $G_i$. It is also of interest to examine infinite-dimensional algebras $\hat{g}_{pd}$ related to $g_{pd}$ along lines of standard studies

[7,8] and to construct non-Fock realizations of the $g_{pd}$ IRs (cf. [11]) extending arbitrary $h$-modules with the help of coset generators.

It is of very importance to develop exact methods of Section 3 since they outline ways of generalizing standard group-theoretical techniques for solving both spectral and evolution tasks with dynamic symmetry algebras $g_{pd}$. Specifically, one may use techniques of $q$-deformed algebras and $q$-special functions [6] (due to interrelations between $sl_{pd}(2)$ and $q$-deformed algebras) as well as pseudogroup and braided geometry concepts [9,10] (due to Eqs. (3.8),(3.9)) for determining new classes of special functions related to $g_{pd}$. At the same time quasiclassical approximations obtained may be considered as specific asymptotics of exact solutions that opens a possibility to use the techniques of asymptotic expansions [19] for finding latters. On the other hand, solutions of the
nonlinear Eqs. (3.7) enable to determine operator analogs of Abelian functions which are, probably, related to quantization problems on algebraic varieties [12].

Results of Section 4 provide an effective analysis of composite models with internal $G_i$-symmetries at algebraic and quasi-particle levels, when obtaining explicit expressions for $f_\ldots(\ldots)$ in Eqs. (4.5) and examining the limit ”$m \to \infty$”. Furthermore, involving into consideration spatiotemporal symmetries, one can construct appropriate ”physical” composite fields and relevant Hamiltonians (Lagrangians) for them in terms of ”quanta” $W_a$ (cf. [2,3,5,10]).

The author thanks A. Odzijewicz for interest in the work, useful discussions and remarks and V.N. Tolstoy for stimulating discussions. A partial support of the Russian Foundation for Basic Research (under grant No 96-02 18746a) is acknowledged.

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