Non-Hermitian gap closure and delocalization in interacting directed polymers

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We examine the interplay of band theory and non-Hermitian mechanics in a classical system of thermally fluctuating directed polymers subject to shear forces and experiencing a continuum periodic potential in 1+1D. The equilibrium polymer conformations are described by a mapping to a quantum system with a non-Hermitian Hamiltonian and with fermionic statistics generated by noncrossing interactions among polymers. Using molecular dynamics simulations and analytical calculations, we identify a localized and a delocalized phase of the polymer conformations, separated by a delocalization transition which corresponds (in the quantum description) to the breakdown of a band insulator when driven by an imaginary vector potential. We find the critical shear value and the critical exponent by which the shear modulus diverges in terms of the branch points in the complex-valued band structure at which the bandgap closes. We also investigate the combined effects of non-Hermitian delocalization and localization due to both periodicity and disorder, uncovering preliminary evidence that while disorder favours localization at high values, it encourages delocalization at lower values.

Introduction. Non-Hermitian operators [1, 2] have been extensively used to describe the dynamics of a variety of quantum [3–5] as well as classical systems [6–9]. They are ubiquitous in both exact and effective models of nature capturing gain/loss in open systems [4], dissipation [10], probability fluxes [11], sensitivity to boundary conditions [12–13], and various other rich behaviour restricted by assumptions of Hermiticity. They also enable the description of new kinds of phase transitions and topological classifications beyond the existing Hermitian framework for condensed matter [14–16]. For example, when non-Hermitian systems are periodic in space, their excitations are described by complex-valued band structures [17] which support uniquely non-Hermitian properties such as exceptional points (branch points) [18]. The physical implications of non-Hermitian band effects have been explored in a wide range of systems [19–23].

In systems of many bodies—such as spin waves, electrons, polymer chains, and vortex lines—generic thermodynamic phases are distinguished by the localization properties of probability densities throughout the bulk [24]. Here, non-Hermitian terms quantify the capacity of external forces and fields to generate fluxes of probability/information. Typically, delocalization transitions are caused when these terms cross a threshold value set by the confining effects of disorder and inter-particle interactions [11–25]. Such non-Hermitian delocalization has been extensively studied in thermally fluctuating lines (directed paths) under simultaneous tension and shear forces in the presence of randomly-positioned pinning sites [25, 28–32], which served as a model for magnetic vortices pinned to discrete columnar defects in superconductors [33–35]. However, the simultaneous interplay of non-Hermitian drive, thermal fluctuations, and confinement due to spatial periodicity, as opposed to disorder, remains poorly understood. Prior works on non-Hermitian delocalization of vortices in periodic lattices operated in the tight-binding limit [30–38], which is limited to the analysis of the ground state and fails to capture dependencies on the form of the continuous potential [39]. Field-theoretic studies, which in turn derived their Hamiltonian from the tight-binding limit, uncovered new thermodynamic phases in the Hermitian setting [40], whose non-Hermitian counterparts have been investigated using a mean-field model in 2+1-dimensions (the upper critical dimension) [41] but not in 1+1D. The form of the continuum equilibrium density profiles, the connection with non-Hermitian modification of band structures, and possibility of topological phenomena was not elucidated in...
these prior studies.

In this work, we demonstrate a manifestation of non-Hermitian band physics in a continuum statistical mechanical system. Specifically, we use classical molecular dynamics simulations and analytical calculations to study the effect of shear forces on directed polymers confined to two dimensions and experiencing a smoothly-varying periodic potential (Fig. 1). Non-crossing interactions among polymers render the system equivalent to a fermionic band-insulating state where individual polymers are localized to distinct grooves. Upon increasing the shear strength, the polymers collectively undergo a delocalization transition at a threshold shear beyond which their average equilibrium conformations are tilted and no longer align with the substrate.

By mapping our system to a 1D quantum Hamiltonian with non-Hermitian drive, we establish that the delocalization is triggered by a gap closure in the complex non-Hermitian band structure associated with the substrate potential in the presence of shear forces. Among our main results is the theoretical prediction and verification by simulation of the exact value of the critical shear at which the polymers delocalize in a continuum potential (Eq. (3)). We also find the critical exponent by which the shear modulus diverges in terms of the order of the branch point in the complex-valued non-Hermitian energy band. Finally, we also report preliminary evidence of a reentrant delocalization transition in the presence of both periodic potential and disorder, and explore possible connections with non-Hermitian topological pumps.

**Tilting and delocalization of fluctuating chains on periodic substrates.** We performed molecular dynamics simulations (see Methods) of thermally fluctuating chains of monomers stretched out, with a tension force, on a two-dimensional substrate potential (see Fig. 1 and Supplementary Video). The monomers repel each other with a short-range potential to emulate the constraint that the polymer chains cannot cross each other. The substrate potential, , is periodic along the direction and constant along the direction. The tension force, , is applied to the ends of each chain at the desired angle, , with respect to the grooves of the potential (the -axis). After an equilibration period, monomer positions can be aggregated over statistically independent time points to obtain equilibrium density profiles of the fluctuating polymers.

At equilibrium, a single polymer chain wanders across the simulation box with no preferred position (Fig. 2a). When the center-of-mass motion is subtracted, the equilibrium density profile displays an overall tilt (Fig. 2c) with a tilt angle that aligns with the force angle (Fig. 2g) as would be expected for a free chain under tension.

The polymer conformations are markedly different when the system is prepared at a commensurate filling of one polymer per groove of the periodic potential. At small force angles, the wandering of polymers in the multi-chain system is suppressed and each chain is localized to a distinct groove (Fig. 2b) breaking ergodicity. The aggregated density profile shows that the chains remain vertical even at nonzero shear, except for a small amount of bending near the ends (Fig. 2b). Only at force angles larger than a threshold value, , do the chain conformations acquire a finite tilt (Fig. 2f), aligning with the applied force (Fig. 2g). At , not only do the chains display an abrupt tilt, they also drift back and forth across the substrate at equilibrium.

![Figure 2](image-url)
with no preferred center-of-mass location (Supplementary Video). This motion occurs via the diffusion of kinks that carry a chain over a potential peak to the adjacent valley (Fig. 2); the kink positions and motion must be coordinated across all chains to satisfy the noncrossing constraint [41].

In summary, the commensurate system exhibits two distinct equilibrium phases: a localized phase with untitled chain conformations confined to individual potential grooves, and a delocalized phase in which chains are tilted in the direction of the applied force and wander across the substrate. We will show that the existence of these phases can be explained by mapping the classical equilibrium system to the quantum probability evolution of noninteracting fermions experiencing a periodic potential [42] in the presence of a non-Hermitian term due to shear [25]. We find that the sudden delocalization transition is due to gap closure in the complex-valued band structure of a non-Hermitian Hamiltonian. This behaviour is then akin to the breakdown of a band into surface states of a finite crystal [50–52] and more recently the value of which is set by the shear force \( F_\theta \) and temperature \( T \) in the polymer system. When \( \text{Im}(k) = 0 \), the energies are real and form separate bands (shown in blue). As the shear force is increased \( \text{Im}(k) \) increases and the energies form complex-valued ellipses (grey dotted contours). At the critical value of \( \text{Im}(k) = \mu/a \) the ground-state ellipse meets the first excited band and a commensurate filled crystal is no longer a band insulator. This is the delocalization mechanism exhibited by the polymer system. While the complex energies shown here have been computed for the specific potential \( V(x) = V_0 \cos(2\pi x/a) \), with \( V_0 = 1 \), this behaviour is generic [54].

FIG. 3. The (complex-valued) energies, \( \varepsilon \), of the Hamiltonian \( \frac{p^2}{2m} + V(x) \), where \( V(x) \) is periodic, as a function of the imaginary component of the Bloch wave-vector, \( \text{Im}(k) \), the value of which is set by the shear force \( F_\theta \) and temperature \( T \) in the polymer system. When \( \text{Im}(k) = 0 \), the energies are real and form separate bands (shown in blue). As the shear force is increased \( \text{Im}(k) \) increases and the energies form complex-valued ellipses (grey dotted contours). At the critical value of \( \text{Im}(k) = \mu/a \) the ground-state ellipse meets the first excited band and a commensurate filled crystal is no longer a band insulator. This is the delocalization mechanism exhibited by the polymer system. While the complex energies shown here have been computed for the specific potential \( V(x) = V_0 \cos(2\pi x/a) \), with \( V_0 = 1 \), this behaviour is generic [54].

By Bloch’s theorem, \( \Psi(x) \) should be of the form \( e^{ikx}u_k(x) \). However, to ensure that \( \Psi(x) \) is physical (in particular that it obeys periodic boundary conditions) we must choose \( k \) to be complex such that \( \text{Im}(k) = g/h = F_\theta \beta T \), with the well-known mapping between the canonical partition function \( \tilde{Z} \sim e^{-\beta E} (\beta = 1/k_B T) \) and the Feynman propagator \( e^{S/h} \) for a quantum system with action \( S \) in imaginary time. The mapped quantum system has a Hamiltonian [20, 47],

\[
H = \sum_{n=1}^{N} \left( \frac{(p_n + ig)^2}{2m} + V(x_n) + \sum_{n' \neq n} |c| \delta(x_n - x_{n'}) \right),
\]

where \( m = F, g = F_\theta \), and \( p_n = -i\hbar \frac{\partial}{\partial x_n} = -\frac{i}{\hbar} \frac{\partial}{\partial x_n} \) (see Appendix [3] for derivation). This is a one-dimensional system of \( N \) quantum particles in a periodic potential acted upon by an imaginary ‘vector-potential’ term, \( ig \).

At large inter-polymer repulsion strength, \( |c| \rightarrow \infty \), the interactive term can be absorbed into the boundary conditions of the many-body wave function [42] using Girardeau’s mapping [43] which effectively maps bosons with contact repulsion to non-interacting fermions. The many-body Hamiltonian then becomes a sum of single-body terms, \( H = \sum_n \hat{H}_n(g) \) with \( \hat{H}_n(g) = \frac{(p_n + ig)^2}{2m} + V(x_n) \). Now, \( \Psi(x) = e^{i\Phi} \Psi' (x) \) is an eigenstate of \( \hat{H}(g) \) provided \( \Psi'(x) \) is an eigenstate of the shear-less Hamiltonian, \( \hat{H}(g = 0) = \frac{p^2}{2m} + V(x) \). (This is analogous to a gauge-transformation of the vector-potential [49].)

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Non-Hermitian gap closure and critical shear. At zero shear, the eigenenergies of the Hermitian Hamiltonian are real and form disconnected energy bands. At finite shear, however, the energies of the eigenstates become complex-valued since the Hamiltonian is non-Hermitian, \( \hat{H}(g)^\dagger = \hat{H}^\dagger (-g) \neq \hat{H}(g) \). The complex energy bands, \( \varepsilon_n(k) \), can be regarded as Riemann sheets with successive sheets meeting each other at the branch points of the multi-valued function \( \varepsilon(k) \) [50, 54]. The nth sheet meets the \( (n + 1) \)th sheet at branch points which occur at \( k \) equal to \( \pm \frac{\pi}{a} + i \frac{\mu}{a} \), where the dimensionless
Rescaled critical force angle, $F\beta a \theta c$

Theory
- $V_0 = 0.239$, varying $\beta$
- $V_0 = 0.318$, varying $\beta$
- $\beta = 0.5$, varying $V_0$
- $\beta = 1$, $V_0 = 0.477$
- $\beta = 2$, $V_0 = 0.159$

Rescaled potential strength, $F\beta a^2 V_0$
- $V_0 = 0.239$, varying $\beta$
- $V_0 = 0.318$, varying $\beta$
- $\beta = 0.5$, varying $V_0$
- $\beta = 1$, $V_0 = 0.477$
- $\beta = 2$, $V_0 = 0.159$

FIG. 4. Critical force angles measured from simulations with different potential amplitudes and temperatures, rescaled by the energy and force scales governing the underlying Schrödinger equation (Eq. (14)). Symbols are labeled according to the parameter which is kept constant in distinct sets of simulations. The other parameters are $F = 20$, $a = 1$, $N = 10$ in simulation units. Error bars show uncertainty in the critical force estimate due to the finite sampling resolution of applied shear forces. Solid curve shows the theoretical prediction, Eq. (3).

number $\mu_n$, which quantifies the distance of the branch points from the real $k$ axis, will determine the value of the critical shear.

In the polymer system, commensurate filling ensures that the quantum particles completely fill the ground state energy band $\varepsilon_0(k)$. Fig. 3 shows that as the shear force $F\theta$, and therefore Im$(k)$, increases, the elliptical energy bands grow in size. At a critical value of shear force, $F\beta \theta$ equals the branch point value $\mu_0/a$ and the ground state energy band meets the higher band. The energy gap closes and the system is now a conductor that enables probability flows driven by the imaginary vector potential. These probability flows manifest themselves as tilts in the polymer density profiles, Fig. 2.

The critical angle of force, $\theta_c$, at which the polymers acquire a collective tilt (Fig. 2) is then predicted to be

$$\theta_c = \frac{\mu_0}{F\beta a}. \quad (3)$$

Notably, the value of $\mu_n$ depends on the details of the periodic potential and its dependence on the energy gap or the amplitude of the potential energy is non-universal \[55\] \[56\]. This subtlety is not captured by tight-binding studies of the complex band structure \[37\], which gloss over the details of the periodic potential and suggest that $\theta_c$ should scale with the energy gap in a universal manner.

Analysis of the underlying Schrödinger equation (see Appendix D1) shows that the gap closure of the complex band structure is governed by two dimensionless parameters: the dimensionless shear force, $F\beta a$ and the generalized potential strength, $V_0 F\beta a^2$ where $V_0$ is the potential amplitude. In Fig. 4 we report measurements of $\theta_c$ from molecular dynamics simulations in which the parameters $\beta$ and $V_0$ were varied. We find that delocalization thresholds from simulations covering a broad range of parameter values collapse onto a narrow region in the force angle-potential strength plane when rescaled according to the quantum mapping. Furthermore, the rescaled critical force angles are consistent with the theoretical prediction of the localization-delocalization transition—the branch point distance $\mu_0$ at the given potential amplitude (Eq. (3)).

The non-Hermitian band structure also determines the delocalization behavior for other filling densities $f = N/M$ (where $M$ is number of unit cells) of the polymers. When $f$ is non-integer the polymers are expected to be generically delocalized because there is no energy gap separating the last occupied single-particle state from the first unoccupied state. By contrast, when $f$ is an integer the system exhibits a transition from a localized to delocalized state at a critical force angle given by $\frac{\mu_0}{Fa}.

The delocalization transition due to closure of the non-Hermitian energy gap is also apparent in the behaviour of the wave functions. We can write the many-body wave function of the commensurate system as a real-valued Slater determinant of Wannier functions defined on the ground state band (see Appendix E). The Wannier functions of the $n$th band are also known to depend on the branch point locations $\mu_n$: their spatial profiles fall off as $\exp(-\mu_n x/a) \sim x$. This means that as long as the “gain factor” $e^{\frac{\phi}{\hbar}}$ is smaller than $e^{-\frac{\mu_0 a}{\hbar}}$ the wave functions remain localized, recovering the prediction for the critical angle, $\mu_n/a = g_c/\hbar = F\beta \theta_c$.

Critical exponent of diverging shear modulus. We can define the shear modulus, $\Gamma$, of the system as the applied stress, $F\theta/a$, divided by the resulting strain, $\phi$. Fig. 3 shows that for a commensurate system, $\Gamma = F\theta/a \phi$ equals $F/a$ at $\theta \gg \theta_c$. As $\theta$ approaches $\theta_c$ from above, $\phi$ decreases faster than $\theta$ so that $\theta/\phi$, and hence $\Gamma$, diverges as $\theta \to \theta_c$. To find the exponent of divergence we estimate $\phi$ by the integrated current, $j = \frac{\partial p}{\partial \tau}$ where $p(x, \tau)$ is the polymers’ probability density (See Appendix G). We find that $\phi$ is proportional to a sum of factors of the form $\text{Im} (\varepsilon) e^{-i\beta \text{Re}(\varepsilon) L}$. As $\theta$ approaches $\theta_c$ for a commensurate system the imaginary part of the energy is known to behave like $|\theta - \theta_c|^{1/2}$ since the branch point is of order one \[50\]. Therefore, we expect $\Gamma$ to diverge as $\sim |\theta - \theta_c|^{-1/2}$ for large system sizes.

Rotated frame and quantized transport. An alternative description of the evolving conformations of the equilibrium system along the tension direction is obtained by rotating the $\tau$ axis to align with the net force. In this rotated frame, the potential energy grooves are no longer aligned with $\tau$ and the corresponding quantum system becomes time-dependent, opening up the possibility of topological phenomena due to spacetime-periodic poten-
The energy functional in this frame is

$$E = \sum_{n=1}^{N} \int_0^L d\tau \left( \frac{F}{2} (\partial_{x} x_n)^2 + V(x_n + \theta \tau) + \sum_{n' \neq n} \delta(x_n - x_{n'}) \right).$$

The quantum Hamiltonian for each particle then acquires a time-dependence in the potential,

$$\hat{H} = \frac{p^2}{2m} + V(x + igt/m).$$

The problem now reduces to a non-Hermitian topological pump — a sliding potential with (imaginary) speed $v = ig/m = i\theta$. In the quantum adiabatic approximation, such a sliding potential gives rise to a topologically protected transport [59] which is quantized and proportional to the Chern number [60]. Since the potential is of the trivial form, $V(x + \alpha t)$, the Chern number evaluates to +1 (in contrast to the non-trivial Chern numbers in polymer systems in Ref. [44]). Each polymer contour, on average, shifts to the right by one lattice step for each period in the $\tau$ direction, which corresponds to the polymer profiles following the (tilted) grooves. This matches the commensurate conformations observed at low force angles in simulations.

At higher force angles, the polymers realign themselves vertically to follow the tension direction rather than the potential grooves. The delocalization transition therefore corresponds to a non-Hermitian breakdown of the adiabatic topological pump. Although we do not have a quantitative understanding of the transition, one possibility is that the sharp breakdown of adiabaticity occurs because the time-modulated system encounters a Floquet exceptional point [61] (see Appendix [I] for more details).

Reentrant localization due to disorder. The present model of fluctuating lines under shear was originally introduced to study the competition between non-Hermitian delocalization and Anderson localization [25]. We now investigate the interplay of our band-insulator model with a periodic potential [37] have shown that small amounts of disorder reduced the real-valued energy gap between bands along the Re($\varepsilon$) axis without affecting the extended nature of the Bloch eigenstates near the band edges. As a result, we expect low levels of disorder to introduce localization, which corresponds to the Chern number [60]. Since the potential is of the trivial form, $V(x + \alpha t)$, the Chern number evaluates to +1 (in contrast to the non-trivial Chern numbers in polymer systems in Ref. [44]). Each polymer contour, on average, shifts to the right by one lattice step for each period in the $\tau$ direction, which corresponds to the polymer profiles following the (tilted) grooves. This matches the commensurate conformations observed at low force angles in simulations.

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These observations can be explained by considering the separate effects of disorder on the bandgap and on the localization properties of the single-particle eigenstates. Numerical studies of the lattice Hatano-Nelson model with a periodic potential [37] have shown that small amounts of disorder reduced the real-valued energy gap between bands along the Re($\varepsilon$) axis without affecting the extended nature of the Bloch eigenstates near the band edges. As a result, we expect low levels of disorder to introduce localization, which corresponds to the Chern number [60]. Since the potential is of the trivial form, $V(x + \alpha t)$, the Chern number evaluates to +1 (in contrast to the non-trivial Chern numbers in polymer systems in Ref. [44]). Each polymer contour, on average, shifts to the right by one lattice step for each period in the $\tau$ direction, which corresponds to the polymer profiles following the (tilted) grooves. This matches the commensurate conformations observed at low force angles in simulations.

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order to shift the delocalization transition to smaller shear values due to the reduction of the bandgap. For a given shear value, however, all single-particle eigenstates are expected to become localized at high-enough disorder due to Anderson localization, and the many-body fermionic ground state would also be localized even in the absence of an energy gap between unoccupied and occupied states \(62\). Higher values of tilt are necessary to drive the non-Hermitian delocalization of the single-particle eigenstates at large disorder, leading to an increase in the threshold shear value. The non-monotonic dependence of the critical force angle on disorder strength is consistent with a switch in the dominant localization mechanism, from band-insulator physics at low disorder to Anderson localization of single-particle eigenstates at high disorder. This mechanism could serve as a non-Hermitian version of a reentrant localization transition, reminiscent of similar phenomena in Hermitian systems \(63, 64\).

Discussion. We have demonstrated a non-Hermitian delocalization transition in a statistical mechanical system whose localized state is owed to a bandgap in a periodic substrate potential. Our work shows that even purely classical equilibrium settings harbour non-Hermitian physics which can be utilized to render the problem exactly solvable. The directed polymer system studied here serves as a test-bed for exploring non-Hermitian statistical physics which is straightforward to describe and visualize. It is amenable to mathematical and computational analysis and can be generalized to more complex potentials, interactions and disorder models \(65\). The uncovering of an adiabatic pump and branch point-enabled physics in the directed polymer system suggests the possibility of nontrivial topological dynamics especially in the higher dimensional variant of the system. Promising future directions include exploring many-body localization phenomena \(60\) and realizing topological band physics driven by thermal fluctuations \(44\) that is fundamentally non-Hermitian \(14, 17\).

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Appendix A: Simulation methods

We implemented Langevin dynamics simulations of discretized polymer chains using a modified version of the open-source simulation software HOOMD-Blue \(67\), with modifications made to enable the addition of periodic potentials of arbitrary phase. Each polymer is approximated as a chain of 200 particles of mass \(m\), connected by stiff harmonic springs with equilibrium length \(l_0\), implemented as a bond potential \(V_{\text{bond}}(r) = K(r - l_0)^2/2\) where \(r\) is the distance between adjacent particles on the chain and \(K\) is a stiffness constant. The noncrossing constraint is enforced by adding a stiff contact interaction between all pairs of particles in the system, with pair potential \(V_{\text{contact}}(r) = K(r - l_0)^2\) for separations \(r < l_0\). For simplicity, the same stiffness coefficient is used for both potentials.

The tension is implemented by applying the requisite forces on the first and last particles of each polymer chain in the desired shear angle relative to the vertical direction. To prevent the finite-length chains from drifting vertically, the first particle of each chain is confined to a \(\tau\) coordinate of zero with a deep and narrow harmonic potential well; the well does not constrain the horizontal motion of the particles. The substrate potential energy per unit length of the chain, \(V(x) = V_0 \cos(2\pi x/a)\), is implemented by adding a position-dependent potential energy of magnitude \(l_0 V(x)\) to each particle.

In all our simulations, we set \(m = 1\) and \(a = 1\) to set the mass and length scales. The time scale is implicitly defined by setting \(K = 10000\) in simulation units for the bond and contact stiffnesses across all simulations. We also set \(l_0 = 0.2\), so chains with 200 particles have an equilibrium length of 40\(a\). The simulation box has periodic boundary conditions along the \(x\) direction with dimension \(L_x = 10a\) (ten repetitions of the periodic potential) and the system size in the \(\tau\) direction is set to be much larger than the chain length.

The equilibrium behavior of the system is simulated by using the built-in Langevin integrator of HOOMD-Blue, which introduces random forces on each particle that replicate the effect of a finite temperature \(T\). Langevin dynamics requires the introduction of drag forces on each particle, \(F_{\text{drag},i} = -\gamma \mathbf{v}_i\) proportional to the instantaneous velocity \(\mathbf{v}_i\) of the \(i\)th particle. The value of the drag coefficient affects the transient dynamics as equilibrium is approached, but is not expected to affect the equilibrium properties. We choose a drag coefficient \(\gamma = 0.5\) for our simulations. The time step is chosen to be 0.001 in simulation units. All simulations are run for \(H = 10^7\) time steps or more. To aid the evolution to equilibrium conformations, the system is “annealed” by starting the simulation at a temperature of 1.5\(T\) and ramping the temperature down to the desired value \(T\) over the first \(H/2\) time steps. Equilibrium density profiles are then built up by sampling particle positions during the latter \(H/2\) time steps in intervals of \(10^4\) time steps.

Disorder in the substrate potential is implemented by
adding \( n_d \) cosine potentials with random amplitudes \( \alpha_i \), wave numbers \( p_i \), and phases \( \phi_i \) to \( V(x) \):

\[
V_{\text{disorder}}(x) = \sum_{i=1}^{n_d} \alpha_i \cos \left( \frac{2\pi p_i}{L} x + \phi_i \right),
\]

where \( \alpha_i \) are drawn from a uniform distribution chosen to generate the desired RMS amplitude \( \sigma_d \). \( \phi_i \) are drawn uniformly from the interval \([0, 2\pi]\), and \( p_i \) are drawn uniformly from the range \( 2 \leq p_i \leq 20 \).

**Appendix B: Derivation of the Schrödinger equation**

Consider any one of the N polymer chains, \( x(\tau') \). Suppose the chain is pinned at two points, \( \tau' = 0 \) and \( \tau' = \tau \), so that \( x(0) = x_0 \) and \( x(\tau) = x_\tau \). The energy of the fragment between the points is denoted by \( E[x; 0, \tau] \). The partition function of this fragment can be written as a path integral over all paths obeying the pinning constraints,

\[
\Psi(x, \tau, x_0, 0, \tau) = \int_{(x_0, 0)}^{(x, \tau)} Dx \exp(-\beta E[x; 0, \tau]), \quad (B1)
\]

where \((\beta = 1/k_B T)\). The energy of the polymer is local, \( E[x; 0, \tau + \epsilon] = E[x; 0, \tau] + E[x; \tau, \tau + \epsilon] \) for any \( \epsilon \). This implies the useful recursive relation,

\[
\Psi(x, \tau, x_0, \tau + \epsilon) \sim \int dx_x \Psi(x_x, x_\tau + x_x, \epsilon) \Psi(x_\tau + x_x, x_0, \tau) \quad (B2)
\]

The partition function can now be evaluated iteratively. For small \( \epsilon \) we can expand the left hand side as \( \Psi(x, \tau, x_0, \tau) + \epsilon \partial_x \Psi(x, x_0, \tau) + O(\epsilon^2) \). We also perform the expansion,

\[
\Psi(x_x + x_\tau, x_0, \tau) \approx \left( 1 + x_x \partial_x + \frac{x_x^2}{2} \partial_x^2 \right) \Psi(x, x_0, \tau). \quad (B3)
\]

Finally as the field does not change appreciably during the evolution by \( \epsilon \) we replace \( E[x; \tau, \tau + \epsilon] \) by its mean value to get,

\[
\Psi(x, x_\tau + x_x, \epsilon) \sim e^{-\beta \epsilon x_x (-\epsilon - \theta)^2 + V(x_x)} \quad (B4)
\]

Plugging everything in, performing the Gaussian integral, and discarding \( O(\epsilon^2) \) terms gives the following convection-diffusion equation,

\[
\frac{i\hbar}{\beta} \frac{\partial \Psi(x, \tau)}{\partial \tau} = \left( \frac{1}{2\beta F} \frac{\partial^2}{\partial x^2} - \theta \frac{\partial}{\partial x} - V(x) \right) \Psi(x, \tau). \quad (B5)
\]

Now we redefine variables by mapping \( \tau = it, \beta = \frac{1}{\hbar}, F = m, \) and \( F\theta = g \). This gives us the time-dependent Schrödinger equation.

\[
i\hbar \frac{\partial \Psi(x, t)}{\partial t} = \left( \frac{(p + ig)^2}{2m} + \frac{g^2}{2m} + V(x) \right) \Psi(x, t). \quad (B6)
\]

We absorb the constant \( \frac{g^2}{2m} \) in \( V(x) \) to recover the Hamiltonian given in Eq. (2). (Shifting all the energy eigenvalues \( \varepsilon_n \) by a real constant has no physical effect.)

**Appendix C: Ground-state dominance**

Building on the framework provided by the mapping to quantum mechanics, the solution of (B5) can be expanded as

\[
\Psi(x, \tau) = \sum c_n e^{-\beta \varepsilon_n \tau} \Psi_n(x), \quad (C1)
\]

where

\[
- \frac{1}{2\beta F} \frac{\partial^2}{\partial x^2} + \theta \frac{\partial}{\partial x} + \beta V(x) \right) \Psi_n(x) = \beta \varepsilon_n \Psi_n(x) \quad (C2)
\]

for some (complex-valued) quasi-energy eigenvalue \( \varepsilon_n \). As the polymer propagates from \( \tau = 0 \) to \( \tau = L \), the exponential factors decay if the energy has a positive real part. Far from the boundary then, the polymer’s profile is dominated by the ground-state wave function of the system, \( \Psi(x, \tau) \sim e^{-\beta \varepsilon_0 \tau} \Psi_0(x) \).

One should also consider the contribution to the partition function of a polymer being pinned at \( x(L) = x_L \) and propagating downwards. We will write this partition function contribution as,

\[
\bar{\Psi}(x, x_L, \tau) \sim \int_{(x_L, L)}^{(x, \tau)} Dx \exp(-\beta E[x; \tau, L]). \quad (C3)
\]

\( \bar{\Psi} \) obeys Eq. (B5) with a negative left hand side. That is, it satisfies

\[
- \frac{\partial \bar{\Psi}(x, \tau)}{\partial \tau} = \left( \frac{1}{2\beta F} \frac{\partial^2}{\partial x^2} - \theta \frac{\partial}{\partial x} - \beta V(x) \right) \bar{\Psi}(x, \tau). \quad (C4)
\]

And therefore, \( \bar{\Psi}(x, \tau) = \sum c_n e^{-\beta \varepsilon_n (L-x)} \Psi_n(x) \).

The full partition function then is,

\[
Z = \int dx \bar{\Psi}(x, \tau) \Psi(x, \tau), \quad (C5)
\]

and we identify \( \Psi(x, \tau) \bar{\Psi}(x, \tau) \sim \Psi^2(x, \tau) \) to be the probability density of the coordinate \( x \). This argument can be easily generalized to multiple polymers.
Appendix D: Dimensionless quantities governing the phase transition

One can transform $x$ to the dimensionless $r = x/a$ (where $a$ is the lattice constant) in Eq. (C2). On substituting $\Psi_n(r) = \Psi'_n(r) e^{F_\beta \theta a r}$ we get,

$$-\frac{1}{2} \frac{\partial^2}{\partial r^2} \Psi'_n(r) + V_0 F \beta^2 a^2 \frac{V(r)}{V_0} \Psi'_n(r) = \left( \varepsilon_n F \beta^2 a^2 - \frac{(F \beta \theta a)^2}{2} \right) \Psi'_n(r),$$  \hspace{1cm} (D1)

where $V_0$ is the amplitude of the periodic potential. The system is then governed by two dimensionless quantities: the dimensionless shear force, $F \beta \theta a$ and the generalized potential strength, $V_0 F \beta^2 a^2$.

Appendix E: The full many-body system

The full many-body problem is easily solved once the single-body wave functions above are known. Using Girardeau’s mapping [48], the many-body Hamiltonian, Eq. (2) reduces to a sum of single-body Hamiltonians provided the many-body wave function obeys the constraint, $\Psi(\vec{x}) = \Psi(x_1, x_2, ..., x_N) = 0$ whenever any $x_i = x_j$. This constraint is satisfied by the Slater determinant of the single-body wave functions (the Bloch waves with complex momenta).

To get real-valued solutions, we note that since the Hamiltonian, Eq. (2), is real, its normalized eigenstates $\Psi_n(\vec{x})$ are either real or come in complex conjugate pairs. Thus we just need to ensure that the Slater determinant has either real eigenstates or pairs of complex conjugate ones.

For a filled band it is useful to define the solution in terms of Wannier functions rather than Bloch waves. The descriptions are equivalent since the determinant of a matrix of solutions is invariant on multiplication with a Unitary matrix. The Slater determinant in terms of the single-body Wannier functions, $\Phi_X(x)$, is

$$S(\vec{x}) = \frac{1}{\sqrt{N!}} \sum_{\sigma \in S_N} \text{sgn}(\sigma) \prod_{i=1}^N \Phi_X(\sigma(x_i)),$$ \hspace{1cm} (E1)

where the Wannier functions, $\Phi_X(x)$, are related to the Bloch functions, $\Psi_k(x)$, by

$$\Phi_X(x) = \frac{1}{\sqrt{N}} \sum_k e^{-i k X} \Psi_k(x); X \in \{a, 2a, \ldots, Ma\}.$$ \hspace{1cm} (E2)

While the above defined functions are not unique (due to the freedom in choice of global phase for the Bloch functions) a unique set of real-valued Wannier functions can always be found [54].

Finally, since the polymers are distinguishable we restrict the domain of the constructed wave-function:

$$\Psi(\vec{x}) = \begin{cases} \sqrt{N!} S(\vec{x}) & \text{if } \vec{x} \in \mathcal{R}_0 \\ 0 & \text{elsewhere,} \end{cases}$$ \hspace{1cm} (E3)

where $\mathcal{R}_0$ is defined by the inequalities, $x_1 < x_2 < \ldots < x_N$ (mod $Ma$), and is the physical region allowed in our non-crossing problem [72]. This redefining does not interfere with the wave function being an eigenstate since it still satisfies the eigenvalue equation and is still continuous. (The derivative of the wave function is allowed to be discontinuous because of the singular terms, $\delta(x_i - x_j)$, in the Hamiltonian.)

Note that while Girardeau’s fermion-to-boson mapping requires the determinant $S(\vec{x})$ to be multiplied with an antisymmetry factor, $A(\vec{x}) \in \{\pm 1\}$, to render it symmetric [48], this is not required since our wave function is non-zero only in $\mathcal{R}_0$.

Appendix F: Computation of band-structure and the critical angle

The band structure, $\varepsilon_n(k)$ at complex $k$ can be computed using any electronic-structure calculations software such as [70] as used in [71].

In our case, we used the fact that exact solutions for the cosine potential, $V(x) = V_0 \cos(2\pi x/a)$ are known in terms of the Mathieu functions [72]. In Fig. 3 we compute the Floquet exponent (which is the same as Im$(k)$ up to numerical factors) for different complex values of the energy eigenvalue $\varepsilon$. For Fig. 4 we use the fact that $\varepsilon_n(k)$ is real for $k$ in the line segment joining $\pi a^{-1} + i \frac{\mu L}{a}$ to $\pi a^{-1} - i \frac{\mu L}{a}$ [59]. To find $\mu$ for any fixed value of potential strength, $V_0$, we then had to increase the value of Im$(k)$ (with Re$(k)$ fixed to $\pi a^{-1}$) until the corresponding value of $\varepsilon$ became complex-valued.

Appendix G: Critical exponent

Using Appendix C we can write the normalized many-body probability density over the polymer chain coordinates $\vec{x} = (x_1, x_2, \ldots, x_N)$ at any $\tau$ as,

$$p(\vec{x}; \tau) = \Psi(\vec{x}, \tau) \overline{\Psi}(\vec{x}, \tau),$$

$$= \sum_{n,m} c_n \tilde{c}_m e^{-\beta(\varepsilon_n - \varepsilon_m) \tau} e^{-\beta \varepsilon_m L} \Psi_n(\vec{x}) \Psi_m(\vec{x}).$$ \hspace{1cm} (G1)

Here, $\Psi_n(\vec{x})$ are the normalized many-body eigenstates of the Hamiltonian, Eq. (2), which can be written as
a modified Slater determinant of the single-body eigenstates. $\varepsilon_n$ denotes the many-body energy which will be the sum of the single-body energies comprised in the eigenstate.

The tilt of the polymer chains, $\phi$, can be measured by the integrated current over the bulk. Away from the boundary, at $\tau = \frac{L}{2}$, we have

$$\phi \sim \int \frac{\partial \Theta(x, \tau)}{\partial \tau} |_{\tau = L/2},$$

$$= -\beta \sum_{n,m} c_n \tilde{c}_m (\varepsilon_n - \varepsilon_m) e^{-\beta(\varepsilon_n + \varepsilon_m)L/2} \int d\vec{x} \Psi_n(\vec{x}) \Psi_m(\vec{x}).$$

\[\text{(G2)}\]

$\phi$ is expected to be proportional to $\theta$ for incommensurate filling. For commensurate filling we expect,

$$\phi \propto \begin{cases} 0 & \text{if } \theta < \theta_c \\ \theta & \text{if } \theta > \theta_c \\ (\theta - \theta_c)^\eta & \text{if } \theta \sim \theta_c \end{cases} \quad \text{(G3)}$$

for some critical exponent $\eta$ which we find to be $\frac{1}{2}$ in the following.

Since the Hamiltonian, Eq. (2), is real, its normalized eigenstates $\Psi_n(\vec{x})$ are either real (with $\Re(k) = 0$ or $\pm \vec{z}$) or come in complex conjugate pairs (with oppositely signed $\Re(k)$). In the former case, the corresponding coefficient, $c_n$, in Eq. (G1) will be taken to be real-valued and in the latter case, the coefficients of $\Psi_n(\vec{x})$ and $\Psi^*_n(\vec{x})$ will be taken to be complex-conjugates. This will ensure that $p(\vec{x}; \tau)$ is real.

For single-body states, $\int d\vec{x} \Psi_n(x) \Psi_m(x)$ is equal to $+1$ only if either $a)$ $\Psi_n(x)$ is real and $n = m$, or $b)$ $\Psi_n(x)$ is complex-valued and $\Psi_m(x) = \Psi^*_n(x)$. In all other cases it evaluates to zero due to the orthogonality of the eigenstates.

For many-body states, we first note that any integral of the form

$$\langle O \rangle_1 = \int_{\vec{x} \in R_0} d\vec{x} \Psi_n(\vec{x}) O(\vec{x}) \Psi_m(\vec{x})$$

\[\text{(G4)}\]

where $\Psi_n(\vec{x})$ and $\Psi_m(\vec{x})$ are of the form given in Eq. (E3) and $O(\vec{x})$ is any permutation invariant function is equal to

$$\langle O \rangle_2 = \int_{\vec{x} \in D} d\vec{x} S_n(\vec{x}) O(\vec{x}) S_m(\vec{x})$$

\[\text{(G5)}\]

where $S_n(\vec{x})$ and $S_m(\vec{x})$ are the corresponding Slater determinants and $D$ is the full domain, $0 \leq x_i \leq M a$. The proof is straightforward: $D$ can be divided into $N!$ regions each with a specific ordering of $x_1, x_2, \ldots$. In the region $R_0 \subset D$, the integral for $\langle O \rangle_2$ just gives $\frac{\langle O \rangle_1}{N!}$.

To evaluate the integral in any of the other $N!$ regions specified by $P(x_1) < P(x_2) < \ldots P(x_N)$ with $P \in S_N$, we first apply $P^{-1}$ to each dummy variable $x_i$. This does not change either the measure, $d\vec{x}$, or $O(\vec{x})$, or the product $S_n(\vec{x}) S_m(\vec{x})$ since the signs cancel each other out. The only change then is the region of integration turning into $R_0$. Thus the integral over all the sectors is the same giving, $\langle O \rangle_2 = N! \langle O \rangle_1 = \langle O \rangle_1$.

$$\int d\vec{x} \Psi_n(\vec{x}) \Psi_m(\vec{x}),$$

for many-body states too then, is equal to $+1$ only if $a)$ for every complex-valued single-body state in $\Psi_n(\vec{x})$, $\Psi_m(\vec{x})$ contains the corresponding complex-conjugate state, and $b)$ for every real-valued single-body state in $\Psi_n(\vec{x})$, $\Psi_m(\vec{x})$ should also have the same state. In other words, $\Psi_n(\vec{x})$ and $\Psi_m(\vec{x})$ should be complex-conjugates.

We are now ready to tackle Eq. (G2). When $\Psi_n(\vec{x})$ is real, $m$ is forced to equal $n$ and the term vanishes by the canceling of energies. We can combine the other cases in complex conjugate pairs to get,

$$\phi = 4\beta \sum_n \text{Im}(c_n \tilde{c}_m) \text{Im}(\varepsilon_n) e^{-\beta \Re(\varepsilon_n)L}. \quad \text{(G6)}$$

where $m$ is defined by $\Psi_m(\vec{x}) = \Psi^*_n(\vec{x})$. We will be assuming $L > Ma > a$ which allows us to consider only the terms with small $\Re(\varepsilon)$.

For commensurate filling, $N/M = \text{integer}$, the groundstate energy is real and the term goes to zero. When $\theta$ is less than $\theta_c$, such that a finite gap $\varepsilon_{\text{gap}}$ is present, all other terms are diminished by a factor of $e^{-\beta \varepsilon_{\text{gap}} L}$ which goes to zero as $L \rightarrow \infty$. This is true even when $Ma \rightarrow \infty$. Hence, we get $\phi = 0$.

We deal with incommensurate filling and commensurate filling at $\theta \sim \theta_c$ together. Only the contributions from states lying in the bands occupied by the groundstate are to be considered since the other terms are diminished by $e^{-\beta \varepsilon_{\text{gap}} L}$. (If the bandgap between two bands has vanished we will consider them to be a single band.) Within the same band the energy difference, $\Delta \varepsilon$, between two successive states goes to zero as $Ma \rightarrow \infty$ so we need to keep all terms.

The presence of the $c_n$ and $\tilde{c}_m$ factors in Eq. (G2) informs us that the initial conditions are important. For example, if the polymer chain is pinned at its two ends, its average tilt cannot change much as system parameters are varied. In the molecular dynamics simulations, we allow for free unbiased wandering of the polymer ends. Therefore, $c_n$ and $\tilde{c}_m$ are random variables which we can ensemble-average over such that the initial conditions become unimportant.

Now, it is known that for $k$ near $k_b = \pm \frac{\pi}{a} \pm iF \beta \theta_c$ the single-body energy goes as $\varepsilon(k) \approx \varepsilon(k_b) + R e^{i\frac{\pi}{2}} (k - k_b)^{1/2}$ where $R$ is a real constant. The square-root behaviour comes from the fact that the energy bands meet at a branch point of order one. For the neighbourhood of the branch point, $\theta \sim \theta_c$ and $k = \frac{\pi}{a} - \delta k$, we have to first order, $\varepsilon(k) = \frac{\pi}{a} + \delta k + iF \beta \theta \approx \varepsilon(k_b) +$
Re^{\frac{2}{1+\sigma}}(F\beta a|\theta - \theta_c|)^{1/2}$ where $\sigma$ is the sign of $\theta - \theta_c$. For $\theta < \theta_c$ we have $\text{Im}(\varepsilon(k)) \approx \text{Im}(\varepsilon(k_0))$ which has no $\theta$ dependence but for $\theta > \theta_c$ we get $\text{Im}(\varepsilon(k)) \approx \text{Im}(\varepsilon(k_0)) + R(F\beta a|\theta - \theta_c|)^{1/2}$.

For commensurate filling at $\theta \geq \theta_c$, the ground state term is again zero since its real-valued. The next significant terms are the first excited states which have ‘particles’ and ‘holes’ near the branch point. Such terms then contain the above factors of $\text{Im}(\varepsilon(k))$ which have the $\phi \sim |\theta - \theta_c|^{1/2}$ dependence. Higher excited states are diminished due to the factor of $e^{-\beta\varepsilon(k)L}$.

For incommensurate filling at $\theta \sim \theta_c$, the ‘particles’ and ‘holes’ of the first excited states are away from the branch point. We expect such single-body energy values away from the branch point to have a linear in $\theta$ dependence recovering Eq. (G2). Indeed, for a system without a periodic potential ($V_0 = 0$) we have $\varepsilon_n = \frac{k^2}{2m} + i\frac{\theta k_n}{\pi}$.

**Appendix H: Rotated frame, adiabatic approximation, and quantized transport**

In the rotated frame we get Eq. (5) as the time-dependent quantum Hamiltonian which leads to the following differential equation for each polymer chain,

$$\frac{\partial \Psi(x, \tau)}{\partial \tau} = \left(\frac{1}{2F\beta} \frac{\partial^2}{\partial x^2} - \beta V(x + \theta \tau)\right)\Psi(x, \tau). \quad (H1)$$

This Floquet partial differential equation (73) can also be derived by changing variables from $(x, \tau)$ to $(x', \tau') = (x + \theta \tau, \tau)$ in Eq. (B5).

The Hamiltonian varies slowly in time which suggests the use of the quantum adiabatic theorem. However, the conventional Hermitian derivation relies on the vanishing of both $\text{Re}(\varepsilon(\tau))$, the real part of the instantaneous eigen-energy, and $\text{Im}(i\Psi^*_n|\partial_\tau \Psi_n)$, the imaginary part of the Berry connection (74 73). (Here, $L(R)$ denotes the left(right) eigenstates of the Hamiltonian.) Neither of these conditions are guaranteed for a generic non-Hermitian Hamiltonian even if it is $\mathcal{PT}$ symmetric 70 as is the case in Eq. (5).

Here we show that the adiabatic theorem can indeed be applied 57 to this non-Hermitian Hamiltonian recovering the well-known result of current quantized to multiples of the Chern number 59. Essentially this is because the instantaneous energies are real and the inner product, and therefore the Berry connection, we define uses the right eigenstates only.

Similar to the derivation of the Hermitian adiabatic theorem we first consider the instantaneous eigenstates given by

$$\left(-\frac{1}{2F\beta^2} \frac{\partial^2}{\partial x^2} + V(x + \theta \tau)\right)\Psi_n(x, \tau) = \varepsilon_n(\tau)\Psi_n(x; \tau). \quad (H2)$$

We can shift the origin by the transformation, $x' = x + \theta \tau$ (note that $\tau$ is held constant by the assumption of instantaneity) to give

$$\left(-\frac{1}{2F\beta^2} \frac{\partial^2}{\partial x'^2} + V(x')\right)\Psi_n(x'; \tau) = \varepsilon_n(\tau)\Psi_n(x'; \tau). \quad (H3)$$

In the mapped quantum system the Hamiltonian corresponding to this equation is $\frac{2}{2m} + V(x)$, the time-independent Hermitian Hamiltonian for a Bloch wave. The instantaneous eigenstate in Eq. (H2) is then given by $\Psi_{nk}(x; \tau) = e^{i(k(x + \theta \tau))}u_{nk}(x + \theta \tau)$. Furthermore, the instantaneous eigenvalue $\varepsilon_{nk}(\tau)$ is real (due to Hermiticity) and time-independent.

We define the inner product $\langle \Psi_{nk}, \Psi_{mq} \rangle$ between two states $\Psi_{nk}(x; \tau)$ and $\Psi_{mq}(x; \tau)$ as

$$\langle \Psi_{nk}, \Psi_{mq} \rangle = \int_0^{Ma} dx \Psi_{nk}(x; \tau)\Psi_{mq}(x; \tau) = \int_0^{Ma + \theta \tau} dx' \Psi^*_{nk}(x')\Psi_{mq}(x') = \delta_{nm}\delta_{kq}. \quad (H4)$$

where the last line follows from the properties of conventional (Hermitian) Bloch states in a space with periodic boundary conditions.

This definition is in contrast to the one commonly used in non-Hermitian literature, $\langle \Psi_{nk}, \Psi_{mq} \rangle = \int dx \Psi_{nk}^L(x; \tau)\Psi_{mq}(x; \tau)$ where the left eigenstate is defined by $\langle \Psi_{nk}^L | H(\theta) | \Psi_{lk}^R \rangle$. (We suppress the $R$ label for the right eigenstate.) This different definition of the inner product leads to the “left-right Berry connection”, $i\langle \Psi_{nk}^L | \partial_\tau \Psi_{lk}^R \rangle$ as opposed to the “right-right Berry connection”, $i\langle \Psi_{nk}^L | \partial_\tau \Psi_{nk}^R \rangle$, which we follow. The computed Chern number is the same regardless of which Berry connection is used 17.

The rest of the derivation closely follows the conventional Hermitian derivation given in Ref. 59 or Ref. 44 in the context of polymers. The tilt of the polymer is proportional to the Chern number which for a sliding potential of the form, $V(x + \alpha \tau)$ is one.

The eventual delocalization when the speed of the sliding potential, $\theta$, reaches $\theta_c$ is still not well understood. One possibility is a breakdown in adiabaticity due to mixing with the higher bands. However, the fact that the transition is so sudden and that in this picture the band gap does not change makes things unclear. In this context, Ref. 61 discusses a system for which the instantaneous eigenvalues are time-independent and for which the breakdown in adiabaticity happens because of hitting a Floquet exceptional point. This is another possibility for the mechanism of the delocalization in this picture.
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