THE UNREASONABLE EFFECTIVENESS OF
QUANTUM FIELD THEORY

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Quantum field theory offers physicists a tremendously wide range of application; it is both a language with which a vast variety of physical processes can be discussed and also it provides a model for fundamental physics, the so-called “standard-model,” which thus far has passed every experimental test. No other framework exists in which one can calculate so many phenomena with such ease and accuracy. Nevertheless, today some physicists have doubts about quantum field theory, and here I want to examine these reservations. So let me first review the successes.

Field theory has been applied over a remarkably broad energy range and whenever detailed calculations are feasible and justified, numerical agreement with experiment extends to many significant figures. Arising from a mathematical account of the propagation of fluids (both “ponderable” and “imponderable”), field theory emerged over a hundred years ago in the description within classical physics of electromagnetism and gravity [1]. Thus its first use was at macroscopic energies and distances, with notable successes in explaining pre-existing data (relationship between electricity and magnetism, planetary perihelion precession) and predicting new effects (electromagnetic waves, gravitational bending of light). Schrödinger’s wave mechanics became a bridge between classical and quantum field theory: the quantum mechanical wave function is also a local field, which when “second” quantized gives rise to a true quantum field theory, albeit a non-relativistic one. This theory for atomic and chemical processes works phenomenally well at electron-volt energy scales or at distances of $O(10^{-5}\text{cm})$. Its predictions, which do not include relativistic and radiative effects, are completely verified by experiment. For example the ground state energy of helium is computed to seven significant figures; the experimental value is determined with six figure accuracy; disagreement, which is seen in the last place, disappears once radiative and relativistic corrections are included. Precisely, in order to incorporate relativity and radiation, quantum field theory of electromagnetism was developed, and successfully applied before World War II to absorption or emission of real photons. Calculation of virtual photon processes followed the war, after renormalization theory succeeded in hiding the infinities that appear in the formalism. [2] Here accuracy of calculation is achieved at the level of one part in $10^8$ (as in the magnetic moment of the electron, whose measured value agrees completely with theory),
where distances of $O(10^{-13}\text{cm})$ are being probed. Further development, culminating with the standard particle physics model, followed after further infinities that afflict theories with massive vector mesons were tamed. Indeed the masses of the vector mesons mediating weak interactions (by now unified with electromagnetism) were accurately predicted, at a scale of 100 $\text{GeV}$.

I have summarized briefly quantum field theoretic successes within elementary particle physics. It is also necessary to mention the equally impressive analyses in condensed matter physics, where many fascinating effects (spontaneous symmetry breaking, both in the Goldstone-Nambu and Anderson-Higgs modes, quantum solitons, fractional charge, etc.) are described in a field theoretic language, which then also informs elementary particle theory, providing crucial mechanisms used in the latter’s model building. This exchange of ideas demonstrates vividly the vitality and flexibility of field theory. Finally we note that quantum field theory has been extrapolated from its terrestrial origins to cosmic scales of distance and energy, where it fuels “inflation” – a speculative, but completely physical analysis of early universe cosmology, which also appears to be consistent with available data.

With this record of accomplishment, why are there doubts about quantum field theory, and why is there vigorous movement to replace it with string theory, to take the most recent instance of a proposed alternative? Several reasons are discerned. Firstly, no model is complete – for example the standard particle physics model requires *ad hoc* inputs, and does not encompass gravitational interactions. Also, intermittently, calculational difficulties are encountered, and this is discouraging. But these shortcomings would not undermine faith in the ultimate efficacy of quantum field theory were it not for the weightiest obstacle: the occurrence of divergences when the formalism is developed into a computation of physical processes.

Quantum field theoretic divergences arise in several ways. First of all, there is the lack of convergence of the perturbation series, which at best is an asymptotic series. This phenomenon, already seen in quantum mechanical examples like the anharmonic oscillator, is a shortcoming of an approximation method and I shall not consider it further.

More disturbing are the infinities that are present in every perturbative term, beyond
the first. These divergences occur after integrating or summing over intermediate states – a necessary calculational step in every non-trivial perturbative order. When this integration/summation is expressed in terms of an energy variable, an infinity can arise either from the infrared – low energy – and/or from the ultraviolet – high energy – domains.

The former, infrared infinity afflicts theories with massless fields and is a consequence of various idealizations for the physical situation: taking the region of space-time, which one is studying, to be infinite and supposing that massless particles can be detected with infinitely precise energy-momentum resolution are physically unattainable goals and lead in consequent calculations to the afore-mentioned infrared divergences. In quantum electrodynamics one can show that physically realizable experimental situations are described within the theory by infrared-finite quantities. Admittedly, thus far we have not understood completely the infrared structure in the non-Abelian generalization of quantum electrodynamics – this generalization is an ingredient of the standard model – but we believe that no physical instabilities lurk there either. So the consensus is that infrared divergences, do not arise from any intrinsic defect of the theory, but rather from illegitimate attempts at forcing the theory to address unphysical questions.

Finally, we must confront the high-energy, ultraviolet infinities. These do appear to be intrinsic to quantum field theory, and no physical consideration can circumvent them: unlike the infrared divergences, ultraviolet ones cannot be excused away. But they can be “renormalized.” This procedure allows sidestepping or hiding the infinities and succeeds in unambiguously extracting numerical predictions from the standard model and from other “physical” quantum field theories, with the exception of Einstein’s gravity theory – general relativity – which thus far remains “non-renormalizable.”

The apparently necessary presence of ultraviolet infinities has dismayed many who remain unimpressed by the pragmatism of renormalization: Dirac and Schwinger, who count among the creators of quantum field theory and renormalization theory, respectively, ultimately rejected their constructs because of the infinities. But even among those who accept renormalization, there is disagreement about its ultimate efficacy at well-defining a theory. Some argue that sense can be made only of “asymptotically free” renormalizable field theories (in
these theories the interaction strength decreases with increasing energy). On the contrary, it is claimed that asymptotically non-free models, like electrodynamics and $\phi^4$-theory, do not define quantum theories, even though they are renormalizable – it is said “they do not exist.” Yet electrodynamics is the most precisely verified quantum field theory, while the $\phi^4$-model is a necessary component of the standard model, which thus far has met no experimental contradiction.

The ultraviolet infinities appear as a consequence of space-time localization of interactions, which occur at a point, rather than spread over a region. (Sometimes it is claimed that field theoretic infinities arise from the unhappy union of quantum theory with special relativity. But this does not describe all cases – later I shall discuss a non-relativistic, ultraviolet divergent and renormalizable field theory.) Therefore choosing models with non-local interactions provides a way for avoiding ultraviolet infinities. The first to take this route was Heisenberg, but his model was not phenomenologically viable. These days in string theory non-locality is built-in at the start, so that all quantum effects – including gravitational ones – are ultraviolet finite, but this has been achieved at the expense of calculability: unlike ultravioletly divergent local quantum field theory, finite string theory has not yielded even an approximate calculation of any physical process.

My goal in this talk is to persuade you that the divergences of quantum field theory must not be viewed as unmitigated defects; on the contrary, they convey crucially important information about the physical situation, without which most of our theories would not be physically acceptable. The stage where such considerations play a role is that of symmetry, symmetry breaking and conserved quantum numbers, so next I have to explain these ideas.

Physicists are mostly agreed that ultimate laws of Nature enjoy a high degree of symmetry. By this I mean that the formulation of these laws, be it in mathematical terms or perhaps in other accurate descriptions, is unchanged when various transformations are performed. Presence of symmetry implies absence of complicated and irrelevant structure, and our conviction that this is fundamentally true reflects an ancient aesthetic prejudice – physicists are happy in the belief that Nature in its fundamental workings in essentially simple. Moreover, there are practical consequences of the simplicity entailed by symmetry: it is
easier to understand the predictions of physical laws. For example, working out the details of very-many-body motion is beyond the reach of actual calculations, even with the help of computers. But taking into account the symmetries that are present allows understanding at least some aspects of the motion, and charting regularities within it.

Symmetries bring with them conservation laws – an association that is precisely formulated by Noether’s theorem. Thus time-translation symmetry, which states that physical laws do not change as time passes, ensures energy conservation; space-translation symmetry – the statement that physical laws take the same form at different spatial locations – ensures momentum conservation. For another example, we note that quantal description makes use of complex numbers (involving $\sqrt{-1}$). But physical quantities are real, so complex phases can be changed at will, without affecting physical content. This invariance against phase redefinition, called gauge symmetry, leads to charge conservation. The above exemplify a general fact: symmetries are linked to constants of motion. Identifying such constants, on the one hand, satisfies our urge to find regularity and permanence in natural phenomena, and on the other hand, we are provided with a useful index for ordering physical data.

However, in spite of our preference that descriptions of Nature be enhanced by a large amount of symmetry and characterized by many conservation laws, it must be recognized that actual physical phenomena rarely exhibit overwhelming regularity. Therefore, at the very same time that we construct a physical theory with intrinsic symmetry, we must find a way to break the symmetry in physical consequences of the model. Progress in physics can be frequently seen as the resolution of this tension.

In classical physics, the principal mechanism for symmetry breaking, realized already within Newtonian mechanics, is through boundary and initial conditions on dynamical equations of motion – for example radially symmetric dynamics for planetary motion allows radially non-symmetric, non-circular orbits with appropriate initial conditions. But this mode of symmetry breaking still permits symmetric configurations - circular orbits, which are rotationally symmetric, are allowed. In quantum mechanics, which anyway does not need initial conditions to make physical predictions, we must find mechanisms that prohibit symmetric configurations altogether.
In the simplest, most direct approach to symmetry breaking, we suppose that in fact
dynamical laws are not symmetric, but that the asymmetric effects are “small” and can
be ignored “in first approximation.” The breaking of rotational symmetry in atoms by an
external electromagnetic field or of isospin symmetry by the small electromagnetic interaction
are familiar examples. However, this explicit breaking of symmetry is without fundamental
interest for the exact and complete theory; we need more intrinsic mechanisms that work for
theories that actually are symmetric.

A more subtle idea is **spontaneous symmetry breaking**, where the dynamical laws are
symmetric, but only asymmetric configurations are actually realized, because the symmetric
ones are energetically unstable. This mechanism, urged upon us by Heisenberg, Anderson,
Nambu and Goldstone, is readily illustrated by the potential energy profile possessing left-
right symmetry and depicted in the Figure. The left-right symmetric value at the origin is
a point of unstable equilibrium; stable equilibrium is attained at the reflection unsymmetric
points ±a. Once the system settles in one or the other location, left-right parity is absent.
One says that the symmetry of the equations of motion is “spontaneously” broken by the
stable solution.

![Potential energy profile](image)

**Left-right symmetric particle energy or field theoretic energy density.** The symmetric point at 0 is
energetically unstable. Stable configurations are at ±a, and a quantum mechanical particle can
tunnel between them. In field theory, the energy barrier is infinite, tunneling is surpressed, the
system settles into state +a or −a and left-right symmetry is spontaneously broken.
But here we come to the first instance where infinities play a crucial role. The above discussion of the asymmetric solution is appropriate to a classical physics description, where a physical state minimizes energy and is uniquely realized by one or the other configuration at \( \pm a \). However, quantum mechanically a physical state can comprise a superposition of classical states, where the necessity of superposing arises from quantum mechanical tunneling, which allows mixing between classical configurations. Therefore if the profile in the Figure describes potential energy of a single quantum particle as a function of particle position, the barrier between the two minima carries finite energy. The particle can then tunnel between the two configurations \( \pm a \), and the lowest quantum state is a superposition, which in the end respects the left-right symmetry. Spontaneous symmetry breaking does not occur in quantum particle mechanics. However, in a field theory, the graph in the Figure describes spatial energy density as a function of the field, and the total energy barrier is the finite amount seen in the Figure, multiplied by the infinite spatial volume in which the field theory is defined. Therefore the total energy barrier is infinite, and tunneling is impossible. Thus spontaneous symmetry breaking can occur in quantum field theory, and Weinberg as well as Salam employed this mechanism for breaking unwanted symmetries in the standard model. But we see that this crucial ingredient of our present-day theory for fundamental processes is available to us precisely because of the infinite volume of space, which also is responsible for infrared divergences!

But infrared problems are not so significant, so let me focus on the ultraviolet infinities. These are important for a further, even more subtle mode of symmetry breaking, which also is crucial for the phenomenological success of our theories. This mode of symmetry breaking is called anomalous or quantum mechanical, and in order to explain it, let me begin by reminding that the quantum revolution did not erase our reliance on the earlier, classical physics. Indeed, when proposing a theory, we begin with classical concepts and construct models according to the rules of classical, pre-quantum physics. We know, however, such classical reasoning is not in accord with quantum reality. Therefore, the classical model is reanalyzed by the rules of quantum physics, which comprise the true laws of Nature. This two-step procedure is called quantization.
Differences between the physical pictures drawn by a classical description and a quantum description are of course profound. To mention the most dramatic, we recall that dynamical quantities are described in quantum mechanics by operators, which need not commute. Nevertheless, one expects that some universal concepts transcend the classical/quantal dichotomy, and enjoy rather the same role in quantum physics as in classical physics.

For a long time it was believed that symmetries and conservation laws of a theory are not affected by the transition from classical to quantum rules. For example if a model possesses translation and gauge invariance on the classical level, and consequently energy/momentum and charge are conserved classically, it was believed that after quantization the quantum model is still translation and gauge invariant so that the energy/momentum and charge operators are conserved within quantum mechanics, that is, they commute with the quantum Hamiltonian operator. But now we know that in general this need not be so. Upon quantization, some symmetries of classical physics may disappear when the quantum theory is properly defined in the presence of its infinities. Such tenuous symmetries are said to be anomalously broken; although present classically, they are absent from the quantum version of the theory, unless the model is carefully arranged to avoid this effect.

The nomenclature is misleading. At its discovery, the phenomenon was unexpected and dubbed “anomalous.” By now the surprise has worn off, and the better name today is “quantum mechanical” symmetry breaking.

Anomalously or quantum mechanically broken symmetries play several and crucial roles in our present-day physical theories. In some instances they save a model from possessing too much symmetry, which would not be in accord with experiment. In other instances, the desire to preserve a symmetry in the quantum theory places strong constraints on model building and gives experimentally verifiable predictions; more about this later.

Now I shall describe two specific examples of the anomaly phenomenon. Consider first massless fermions moving in an electromagnetic field background. Massive, spin-\(\frac{1}{2}\) fermions possess two spin states – up and down – but massless fermions can exist with only one spin state, called a helicity state, in which spin is projected along the direction of motion. So the massless fermions with which we are here concerned carry only one helicity and these
are an ingredient in present-day theories of quarks and leptons. Moreover, they also arise in condensed matter physics, not because one is dealing with massless, single-helicity particles, but because a well-formulated approximation to various many-body Hamiltonians can result in a first order matrix equation that is identical to the equation for single-helicity massless fermions, \textit{i.e.}, a massless Dirac-Weyl equation for a spinor $\Psi$.

If we view the spinor field $\Psi$ as an ordinary mathematical function, we recognize that it possesses a complex phase, which can be altered without changing the physical content of the equation that $\Psi$ obeys. We expect therefore that this instance of gauge invariance implies charge conservation. However, in a quantum field theory $\Psi$ is a quantized field operator, and one finds that in fact the charge operator $Q$ is not conserved; rather

$$\frac{dQ}{dt} = \frac{i}{\hbar}[H, Q] \propto \int_{\text{volume}} \mathbf{E} \cdot \mathbf{B}$$

(1)

where $\mathbf{E}$ and $\mathbf{B}$ are the background electric and magnetic fields in which our massless fermion is moving; gauge invariance is lost!

One way to understand this breaking of symmetry is to observe that our model deals with \textbf{massless} fermions and conservation of charge for \textbf{single-helicity} fermions makes sense only if there are no fermion masses. But quantum field theory is beset by its ultraviolet infinities that must be controlled in order to do a computation. This is accomplished by regularization and renormalization, which introduces mass scales for the fermions, and we see that the symmetry is anomalously broken by the ultraviolet infinities of the theory.

The phase-invariance of single-helicity fermions is called \textbf{chiral symmetry} and chiral symmetry has many important roles in the standard model, which involves many kinds of fermion fields, corresponding to the various quarks and leptons. In those channels where a gauge vector meson couples to the fermions, chiral symmetry must be maintained to ensure gauge invariance. Consequently fermion content must be carefully adjusted so that the anomaly disappears. This is achieved, because the proportionality constant in the failed conservation law (1) involves a sum over all the fermion charges, $\sum_n q_n$, so if that quantity vanishes the anomaly is absent. In the standard model the sum indeed vanishes, separately for each of the three fermion families. For a single family this works out as follows:
three quarks \( q_n = \frac{2}{3} \Rightarrow 2 \)

three quarks \( q_n = \frac{-1}{3} \Rightarrow -1 \)

one lepton \( q_n = -1 \Rightarrow -1 \)

one lepton \( q_n = 0 \Rightarrow 0 \)

\[ \sum_n q_n = 0 \]

In channels to which no gauge vector meson couples, there is no requirement that the anomaly vanish, and this is fortunate: a theoretical analysis shows that gauge invariance in the up-down quark channel prohibits the two-photon decay of the neutral pion (which is composed of up and down quarks). But the decay does occur with the invariant decay amplitude of \((.025 \pm .001)(GeV)^{-1}\). Before anomalous symmetry breaking was understood, this decay could not be fitted into the standard model, which seemed to possess the decay-forbidding chiral symmetry. Once it was realized that the relevant chiral symmetry is anomalously broken, this obstacle to phenomenological viability of the standard model was removed. Indeed since the anomaly is completely known, the decay amplitude can be completely calculated (in the approximation that the pion is massless) and one finds \((.025)(GeV)^{-1}\), in excellent agreement with experiment.

We must conclude that Nature knows about and makes use of the anomaly mechanism: fermions are arranged into gauge-anomaly-free representations, and the requirement that anomalies disappear “explains” the charges of elementary fermions; the pion decays into two photons because of an anomaly in an ungauged channel. It seems therefore that in local quantum field theory these phenomenologically desirable results are facilitated by ultraviolet divergences, which give rise to symmetry anomalies.

The observation that infinities of quantum field theory lead to anomalous symmetry breaking allows comprehending a second example of quantum mechanical breaking of yet another symmetry – that of scale invariance. Like the space-time translations mentioned earlier, which lead to energy-momentum conservation, scale transformations also act on space-time coordinates, but in a different manner: they dilate the coordinates, thereby changing the units of space and time measurements. Such transformations will be symmetry operations...
in models that possess no fundamental parameters with time or space dimensionality, and therefore do not contain an absolute scale for units of space and time. Our quantum chromodynamical model (QCD) for quarks is free of such dimensional parameters, and it would appear that this theory is scale invariant – but Nature certainly is not! The observed variety of different objects with different sizes and masses exhibits many different and inequivalent scales. Thus if scale symmetry of the classical field theory, which underlies the quantum field theory of QCD, were to survive quantization, experiment would have grossly contradicted the model, which therefore would have to be rejected. Fortunately, scale symmetry is quantum mechanically broken, owing to the scales that are introduced in the regularization and renormalization of ultraviolet singularities. Once again a quantum field theoretic pathology has a physical effect, a beneficial one: an unwanted symmetry is anomalously broken, and removed from the theory.

Another application of anomalously broken scale invariance, especially as realized in the renormalization group program, concerns high energy behavior in particle physics and critical phenomena in condensed matter physics. A scale-invariant quantum theory could not describe the rich variety of observed effects, so it is fortunate that the symmetry is quantum mechanically broken.

A different perspective on the anomaly phenomenon comes from the path integral formulation of quantum theory, where one integrates over classical paths the phase exponential of the classical action.

\[
\text{Quantum Mechanics } \leftrightarrow \int_{\text{paths}} e^{i/\hbar \text{classical action}} \quad (2)
\]

When the classical action possess a symmetry, the quantum theory will respect that symmetry if the measure on paths is unchanged by the relevant transformation. In the known examples (chiral symmetry, scale symmetry) anomalies arise precisely because the measure fails to be invariant and this failure is once again related to infinities: the measure is an infinite product of measure elements for each point in the space-time where the quantum (field) theory is defined; regulating this infinite product destroys its apparent invariance.

Yet another approach to chiral anomalies, which arise in (massless) fermion theories,
makes reference to the first instance of regularization/renormalization, used by Dirac to remove the negative-energy solutions to his equation. Recall that to define a quantum field theory of fermions, it is necessary to fill the negative-energy sea and to renormalize the infinite mass and charge of the filled states to zero. In modern formulations this is achieved by “normal ordering” but for our purposes it is better to remain with the more explicit procedure of subtracting the infinities, i.e. renormalizing them.

It can then be shown that in the presence of a gauge field, the distinction between “empty” positive-energy states and “filled” negative-energy states can not be drawn in a gauge invariant manner, for massless, single-helicity fermions. Within this framework, the chiral anomaly comes from the gauge non-invariance of the infinite negative-energy sea. Since anomalies have physical consequences, we must assign physical reality to this infinite negative-energy sea.

Actually, in condensed matter physics, where a Dirac-type equation governs electrons, owing to a linearization of dynamical equations near the Fermi surface, the negative-energy states do have physical reality: they correspond to filled, bound states, while the positive energy states describe electrons in the conduction band. Consequently, chiral anomalies also have a role in condensed matter physics, when the system is idealized so that the negative-energy sea is taken to be infinite.

In this condensed matter context another curious, physically realized, phenomenon has been identified. When the charge of the filled negative states is renormalized to zero, one is subtracting an infinite quantity and rules have to be agreed upon so that no ambiguities arise when infinite quantities are manipulated. With this agreed-upon subtraction procedure, the charge of the vacuum is zero, and filled states of positive energy carry integer units of charge. Into the system one can insert a soliton – a localized structure that distinguishes between different domains of the condensed matter. In the presence of such a soliton, one needs to recalculate charges using the agreed-upon rules for handling infinities and one finds, surprisingly, a non-integer result, typically half-integer: the negative-energy sea is distorted by the soliton to yield a half a unit of charge. The existence of fractionally charged states in the presence of solitons has been experimentally identified in polyacetylene. We thus
have another example of a physical effect emerging from infinities of quantum field theory.

Let me conclude my qualitative discussion of anomalies with an explicit example from quantum mechanics, whose wave functions provide a link between particle and field theoretic dynamics. My example also dispels any suspicion that ultraviolet divergences and the consequent anomalies are tied to the complexities of relativistic quantum field theory: the non-relativistic example shows that locality is what matters.

Recall first the basic dynamical equation of quantum mechanics: the time independent Schrödinger equation for a particle of mass \( m \) moving in a potential \( V(\mathbf{r}) \) with energy \( E \).

\[
\left( -\nabla^2 + \frac{2m}{\hbar^2} V(\mathbf{r}) \right) \psi(\mathbf{r}) = \frac{2m}{\hbar^2} E \psi(\mathbf{r}) .
\]  (3)

In its most important physical applications, this equation is taken in three spatial dimensions and \( V(\mathbf{r}) \) is proportional to \( 1/r \) for the Coulomb force relevant in atoms. Here we want to take a different model with potential that is proportional to the inverse square, so that the Schrödinger equation is presented as

\[
\left( -\nabla^2 + \lambda \frac{r}{r^2} \right) \psi(\mathbf{r}) = k^2 \psi(\mathbf{r}) , \quad k^2 \equiv \frac{2m}{\hbar^2} E .
\]  (4)

In this model, transforming the length scale is a symmetry: because the Laplacian scales as \( r^{-2} \), \( \lambda \) is dimensionless and in (4) there is no intrinsic unit of length. A consequence of scale invariance is that the scattering phase shifts and the \( S \) matrix, which in general depend on energy, \( i.e. \) on \( k \), are energy-independent in scale invariant models, and indeed when the above Schrödinger equation is solved, one verifies this prediction of the symmetry by finding an energy-independent \( S \) matrix. Thus scale invariance is maintained in this example – there are no surprises.

Let us now look to a similar model, but in two dimensions with a \( \delta \)-function potential, which localizes the interaction at a point.

\[
\left( -\nabla^2 + \lambda \delta^2(\mathbf{r}) \right) \psi(\mathbf{r}) = k^2 \psi(\mathbf{r}) .
\]  (5)

Since in two dimensions the two-dimensional \( \delta \)-function scales as \( 1/r^2 \), the above model also appears scale invariant; \( \lambda \) is dimensionless. But in spite of the simplicity of the local contact
interaction, the Schrödinger equation suffers a short-distance, ultraviolet singularity at \( r = 0 \), which must be renormalized. Here is not the place for a detailed analysis, but the result is that only the \( s \)-wave possesses a non-vanishing phase shift \( \delta_0 \), which shows a logarithmic dependence on energy.

\[
\text{ctn} \, \delta_0 = \frac{2}{\pi} \ln kR + \frac{1}{\lambda}
\]

(6)

\( R \) is a scale that arises in the renormalization, and scale symmetry is decisively and quantum mechanically broken. Moreover, the scattering is non-trivial solely as a consequence of broken scale invariance. It is easily verified that the two-dimensional \( \delta \)-function in classical theory, where it is scale invariant, produces no scattering.

Furthermore the \( \delta \)-function model may be second quantized, by promoting the wavefunction to a field operator \( \hat{\psi} \), and positing a field theoretic Hamiltonian density operator of the form

\[
\mathcal{H} = \frac{\hbar^2}{2m} \nabla \hat{\psi}^* \cdot \nabla \hat{\psi} + \frac{\lambda}{2} (\hat{\psi}^* \hat{\psi})^2
\]

(7)

The physics of a second-quantized non-relativistic field theory is the same as that of many body quantum particle mechanics, and the two-body problem (with center-of-mass motion removed) is governed by the above-mentioned Schrödinger equation: the \( \delta \)-function interaction is encoded in the \( \frac{\lambda}{2} (\hat{\psi}^* \hat{\psi})^2 \) interaction term, which is local.

Analysis of the field theory confirms its apparent scale invariance, but the local interaction produces ultraviolet divergences, that must be regulated and renormalized, thereby effecting an anomalous, quantum mechanical breaking of the scale symmetry.

The above list of examples persuades me that the infinities of local quantum field theory – be they ultraviolet divergences, or an infinite functional measure, or the infinite negative-energy sea – are not merely undesirable blemishes on the theory, which should be hidden or – even better – not present in the first place. On the contrary the successes of various field theories in describing physical phenomena depend on the occurrence of these infinities. One cannot escape the conclusion that Nature makes use of anomalous symmetry breaking, which occurs in local field theory owing to underlying infinities in the mathematical description.
The title of my talk expresses the surprise at this state of affairs: surely it is unreasonable that some of the effectiveness of quantum field theory derives from its infinities. Of course my title is also a riff on Wigner’s well-known aphorism about the unreasonable effectiveness of mathematics in physics.[10] We can understand the effectiveness of mathematics: it is the language of physics and any language is effective in expressing the ideas of its subject. Field theory, in my opinion, is also a language, a more specialized language that we have invented for describing fundamental systems with many degrees of freedom. It follows that all relevant phenomena will be expressible in the chosen language, but it may be that some features can be expressed only awkwardly. Thus for me chiral and scale symmetry breaking are completely natural effects, but their description in our present language – quantum field theory – is awkward and leads us to extreme formulations, which make use of infinities. One hopes that there is a more felicitous description, in an as yet undiscovered language. It is striking that anomalies afflict precisely those symmetries that depend on absence of mass: chiral symmetry, scale symmetry. Perhaps when we have a natural language for anomalous symmetry breaking we shall also be able to speak in a comprehensible way about mass, which today remains a mystery.

Some evidence for a description of anomalies without the paradox of field theoretic infinities comes from the fact that they have a very natural mathematical expression. For example the $E \cdot B$ in our anomalous non-conservation of chiral charge is an example of a Chern-Pontryagin density, whose integral measures the topological index of gauge field configurations and enters in the Atiyah-Singer index theorem. Also the fractional charge phenomenon, which physicists found in Dirac’s infinite negative energy sea, can alternatively be related to the mathematicians’ Atiyah-Patodi-Singer spectral flow.[11]

The relation between mathematical entities and field theoretical anomalies was realized twenty years ago and has led to a flourishing interaction between physics and mathematics, which today culminates in the string program. However, it seems to me that here mathematical ideas have taken the lead, without advancing our physical understanding – the string program thus far has not illuminated physical questions. In particular I wonder where within the completely finite and non-local dynamics of string theory are we to find the mechanisms
for symmetry breaking that are needed in order to explain the world around us.
1. For an account of the origins of classical field theory, see L.P. Williams, *The Origins of Field Theory* (Random House, New York, NY 1966).

2. For an account of the origins of quantum field theory, specifically quantum electrodynamics, see S.S. Schweber, *QED and the Men Who Made It* (Princeton University Press, Princeton, NJ 1994).

3. For an account of quantum anomalies in conservation laws, see R. Jackiw “Field Theoretic Investigations in Current Algebra” and “Topological Investigation of Quantized Gauge Theories” in S. Treiman, R. Jackiw, B. Zumino and E. Witten, *Current Algebra and Anomalies* (Princeton University Press/World Scientific, Princeton, NJ/Singapore 1985).

4. For a review, see K. Wilson, “The Renormalization Group, Critical Phenomena and the Kondo Problem,” *Rev. Mod. Phys.* 47, 773 (1975).

5. See R. Jackiw, “Effect of Dirac’s Negative Energy Sea on Quantum Numbers,” *Helv. Phys. Acta* 59, 835 (1986).

6. For a selection of applications, see H.B. Nielsen and M. Ninomiya, “The Adler-Bell-Jackiw Anomaly and Weyl Fermions in a Crystal,” *Phys. Lett.* 130B, 389 (1983); I. Krive and A. Rozhavsky, “Evidence for a Chiral Anomaly in Solid State Physics,” *Phys. Lett.* 113A, 313 (1983); M. Stone and F. Gaitan, “Topological Charge and Chiral Anomalies in Fermi Superfluids,” *Ann. Phys.* (NY) 178, 89 (1987).

7. See R. Jackiw and J.R. Schrieffer, “Solitons with Fermion Number 1/2 in Condensed Matter and Relativistic Field Theories,” *Nucl. Phys.* B190 [FS3], 253 (1981) and R. Jackiw, “Fractional Fermions,” *Comments Nucl. Part. Phys.* 13, 15 (1984).

8. See R. Jackiw, “Introducing Scale Symmetry” *Physics Today* 25, No. 1, 23 (1972) and “Delta-Function Potentials in Two- and Three- Dimensional Quantum Mechanics” in *M.A.B. Bég Memorial Volume*, A. Ali and P. Hoodbhoy eds. (World Scientific, Singapore 1991).

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10. E.P. Wigner, “The Unreasonable Effectiveness of Mathematics in the Natural Sciences;” see also his “Symmetry and Conservation Laws” and “Invariance in Physical Theory.” All three articles are reprinted in *The Collected Works of Eugene Paul Wigner*, Vol. VI, Part B, J. Mehra ed. (Springer Verlag, Berlin, 1995).

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