Universal scaling of conserved charge in the stochastic diffusion dynamics

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In this paper, we explore the Kibble-Zurek scaling of the conserved charge, using the stochastic diffusion dynamics. After determining the characteristic scales \(\tau_{KZ}\) and \(l_{KZ}\) and properly rescaling the traditional correlation function and cumulant, we construct universal functions for both the two-point correlation function \(C(y_1 - y_2; \tau)\) and second-order cumulant \(K(\Delta y, \tau)\) of the conserved charge in the critical regime, which are insensitive to the initial temperature and a parameter in the mapping between 3D Ising model and the hot QCD system near the critical point.

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I. INTRODUCTION

One of the main goals of the Beam Energy Scan (BES) program [1–5] at Relativistic Heavy Ion Collider (RHIC) is to probe the phase structure of Quantum Chromodynamics (QCD) matter and to search the critical point [6–15], the endpoint of the 1-st order phase transition boundary of the QCD phase diagram [7–9, 16–22]. At the critical point, the thermal medium is strongly correlated with degree fluctuations of various variables [6–9]. It was also found that the skewness \(\Delta\) and kurtosis \(\kappa\) of the net protons diverge with the correlation length by \(\xi^{4.5}\) and \(\eta\), respectively [23]. In BES experiment, event-by-event multiplicity fluctuations of net protons and net charges have been systematically measured at different collision energies [24–28]. It was found that the kurtosis \(\kappa\) of net protons, presents a non-monotonic behavior and largely deviates from the Poisson baseline at lower collision energies, indicating the potential of discovery the critical point [28, 29].

Recently, it was realized that the non-equilibrium effects are significant for an expanding medium near the critical point [30–46]. In particular, the critical slowing down effects largely influence the non-equilibrium fluctuations, which even reverse the signs of skewness \(\Delta\) and kurtosis \(\kappa\) compared with the equilibrium values [37, 38]. It was also argued that the soft mode of the critical point is a diffusive mode, which is a combination of the order parameter field and the conserved quantities [32]. Recently, diffusion dynamics near the critical point have been developed [42, 45], which showed that the second order cumulant of the conserved charge presents non-monotonic behavior with the change of the rapidity window [42].

For the dynamical model calculations near the critical point, the non-equilibrium fluctuations are non-universal, which depend on various free parameters, such as the relaxation time and the mapping from the 3D Ising model to the hot QCD medium, etc. On the other hand, within the framework of Kibble-Zurek Mechanism (KZM), one can construct some universal variables near the critical point that are insensitive to some non-universal factors [41, 46–51]. The key point of the KZM is that, due to the critical slowing down effects, the systems inevitably get out-of-equilibrium near the critical point, after which these “frozen” systems have correlated regions with characteristic scales, leading to various universal variables. The KZM was first introduced by Kibble in cosmology [52] and then extended by Zurek to the condensed matter physics [53]. In relativistic heavy ion collisions, the KZM was first studied in Ref. [41], which constructed universal functions of the order parameter field that are insensitive to the relaxation time and the evolving trajectory of the system. In Ref. [51], we have investigated the Kibble-Zurek scaling for both the order parameter field and the multiplicity fluctuations of net protons, using the Langevin dynamics of model A. We found that, compared with the original fluctuations of net protons, the oscillating behavior of the constructed approximately universal functions are strongly suppressed.

In this paper, we investigate the critical universal scaling of the conserved charge within the framework of model B. As mentioned above, the soft mode near the QCD critical point is a diffusive mode, which is a linear combination of the order parameter field and the conserved quantities. Moreover, the conserved quantities directly related to the possible experimental observable. Comparing with our early work [51] which only considers the non-conserved order parameter field, this paper explores the possible universal scaling for fluctuation of conserved charge using the stochastic diffusion equation (SDE). We will demonstrate that the constructed universal functions for the two-point correlation function and the second-order cumulant of conserved charge are insensitive to the non-universal factors of two cases 1) the evolving hot medium with different strength of critical component \(c_s\), a parameter in the mapping from 3D Ising to QCD critical point; 2) the evolving system with different initial temperature \(T_0\). Note that, in this paper, we restrict our attention to the possibility of constructing the universal functions for the diffusion dynamics near the critical point, which we only focus on the 1+1-dimensional system with the Bjorken approximation.

For the realistic universal observables that might be associated with experimental measurements, one needs to at least numerically simulate the 3+1 dimensional diffusion dynamics and consider the higher-order cumulants of fluctuations. This requires high statistical runs and a large amount of computing.
resources, which we would like to leave it to future study.

The paper is organized as follows: Sec. II briefly reviews the dynamics of conserved charge near the critical point based on the stochastic diffusion equation. In Sec. III, we construct the universal functions for two-point correlation function and the second-order cumulant. Sec. IV presents and discusses the main results of the constructed universal functions. Sec. V summarizes and concludes this paper.

II. DYNAMICS OF CONSERVED CHARGE

A. Stochastic diffusion equation

For a dynamical model near the critical point, the slow modes are the relevant and essential modes, which largely influence the critical behavior of the evolving system. According to the classification of Ref. [54], the critical dynamics of non-conserved and conserved order parameter field belong to model A and B, respectively. While, model H describes a system with a conserved order parameter field, conserved transverse momentum density, and nonzero Poisson bracket between the two. In general, it is believed that the dynamical system near QCD critical point lies in model H [32, 55–57]. However, the related analysis or numerical implementation of model H is complicated, which have not been fully developed. For simplicity, our previous work [51] only focused on the dynamics and universal scaling of the non-conserved order parameter field within the framework of model A. Recently, Ref. [42] has developed the stochastic diffusion dynamics of the conserved charge for model B, which demonstrated that the two-point correlation function and cumulant behave non-monotonically with the change of the rapidity interval and window, respectively. In this paper, we will explore the universal behavior of the conserved charge based on the stochastic diffusion equation described in Ref. [42].

For simplicity, we focus on 1+1-dimensional evolution of the conserved charge density $n(y, \tau)$ with the proper time $\tau = \sqrt{\tau^2 - z^2}$ and the spacetime rapidity $y = \tanh^{-1}(z/\tau)$ for a boost-invariant Bjorken system. The related stochastic diffusion equation is [42]:

$$\frac{\partial}{\partial \tau} n(y, \tau) = D_y(\tau) \frac{\partial^2}{\partial y^2} n(y, \tau) + \frac{\partial}{\partial y} \zeta(y, \tau) \quad (1)$$

Here $\delta n(y, \tau) = n(y, \tau) - \langle n(y, \tau) \rangle$, and $(\cdots)$ denotes the event average. The diffusion coefficient $D_y(\tau)$ is related to the Cartesian one $D_{x}(\tau)$ with $D_{y}(\tau) = D_{x}(\tau) \tau^{-2}$. The noise $\zeta(y, \tau)$ satisfies the fluctuation-dissipation theorem:

$$\langle \zeta(y, \tau) \rangle = 0,$$
$$\langle \zeta(y_1, \tau_1) \zeta(y_2, \tau_2) \rangle = 2 \chi_{x}(\tau) D_{x}(\tau) \delta(y_1 - y_2) \delta(\tau_1 - \tau_2), \quad (2)$$

where $\chi_{x}(\tau)$ is the susceptibility of the conserved charge per unit rapidity, related to the Cartesian one $\chi_{C}(\tau)$ with $\chi_{x}(\tau) / \tau = \chi_{C}(\tau)$. For notational convenience, the subscripts of the diffusion coefficient and susceptibility in the following part of this paper are dropped, which are denoted as $D(\tau) = D_{x}(\tau)$ and $\chi(\tau) = \chi_{x}(\tau)$, respectively.

After solving the SDE (1), one could obtain the correlation function

$$C(y_1, y_2; \tau) \equiv \langle \delta n(y_1, \tau) \delta n(y_2, \tau) \rangle = \chi(\tau) \delta(y_1 - y_2) - \int_{\tau_0}^{\tau} d\tau' \chi'(\tau')G(y_1 - y_2; 2d(\tau', \tau)), \quad (3)$$

where $\chi'(\tau) = d\chi(\tau)/d\tau$. Here the normalized Gauss distribution is:

$$G(\tilde{y}; d) = \frac{1}{\sqrt{\pi d}} e^{-\tilde{y}^2/d^2}, \quad (4)$$

and

$$d(\tau_1, \tau_2) = \left(2 \int_{\tau_1}^{\tau_2} d\tau' D(\tau') \right)^{1/2} \quad (5)$$

represents the diffusion “length” in rapidity space from $\tau_1$ to $\tau_2$ with $\tau_1 \leq \tau_2$.

The amount of the charge deposed within a finite rapidity window $\Delta y$ at mid-rapidity and at a proper time $\tau$ can be calculated as:

$$Q_{\Delta y}(\tau) \equiv \int_{-\Delta y/2}^{\Delta y/2} dy n(y, \tau). \quad (6)$$

Correspondingly, the second-order cumulant of $Q_{\Delta y}(\tau)$ takes the following form:

$$K(\Delta y, \tau) \equiv \langle \delta Q_{\Delta y}(\tau)^2 \rangle / \Delta y$$
$$= \frac{1}{\Delta y} \int_{-\Delta y/2}^{\Delta y/2} dy_1 dy_2 \langle \delta n(y_1, \tau) \delta n(y_2, \tau) \rangle$$
$$= \chi(\tau) - \int_{\tau_0}^{\tau} d\tau' \chi'(\tau') F \left( \frac{\Delta y}{2d(\tau', \tau)}, \right), \quad (7)$$

where

$$F(X) \equiv \frac{2}{\sqrt{\pi}} \int_{0}^{X} dz \left(1 - \frac{z}{X} \right) e^{-z^2}. \quad (8)$$

For the detailed derivation, please refer to Appendix. A.

Note that the SDE (1) used in the present study only considers the two-point interaction and neglect the higher order contributions. The advantage of such simplification is that it can be analytically solved, as shown in Eqs. (3) and (7). It is adequate for our first attempt to study the Kibble-Zurek scaling for the two-point correlations in the diffusion dynamics without further considering the higher order cumulants.

B. Parametrizing the susceptibility $\chi(\tau)$ and diffusion coefficient $D(\tau)$

Both the correlation function (3) and the cumulant (7) depend on the susceptibility $\chi(\tau)$ and diffusion coefficient $D(\tau)$, which needs some additional parametrizations. In general, the susceptibility $\chi$ and diffusion coefficient $D$ include both the
singular parts $\chi^{sr}$, $D_c^{sr}$ and the regular parts $\chi^{cr}$, $D_c^{cr}$, respectively [42]. As the system evolves near the critical point, the singular contributions become dominant. We thus neglect the regular parts to simplify the following study of the Kibble-Zurek scaling. The susceptibility $\chi(\tau)$ and diffusion coefficient $D(\tau)$ with only the singular parts are then written as:

$$\chi(\tau) = \chi^{sr}(\tau),$$

$$D(\tau) = D_c^{sr}/\tau^2. \tag{9}$$

Here, we construct the singular part $\chi^{sr}$ and $D_c^{sr}$ through a mapping between the hot QCD matter and the 3D Ising model. In the linear parametric model [58, 59], the magnetization of the 3D Ising systems is parameterized with two variables $R$ and $\theta$:

$$M(R, \theta) = m_0 R^{1/3} \theta, \tag{11}$$

where the reduced temperature $r$ and the dimensionless magnetic field $H$ are expressed as:

$$r(R, \theta) = R(1 - \theta^2), \tag{12}$$

$$H(R, \theta) = h_0 R^{5/3}(3 - 2\theta^2), \tag{13}$$

Here, we have adopted the values of the Ising critical exponents [60], and the normalization constants $m_0$ and $h_0$ are fixed by the conditions $M(r = -1, H = 0^*) = 1$ and $M(r = 0, H = 1) = 1$. From Eq. (11), one could calculate the susceptibility of the 3D Ising model:

$$\chi_M(r, H) = \frac{\partial M(r, H)}{\partial H} \bigg|_r = \frac{m_0}{h_0} \frac{1}{R^{1/3}(3 + 2\theta^2)}. \tag{14}$$

In the case of the hot QCD systems, the susceptibility $\chi^{cr}$ for the conserved charge satisfies a similar critical behavior near the critical point:

$$\chi^{cr}(r, H) = c_c \chi_M(r, H) = c_c \frac{m_0}{h_0} \frac{1}{R^{1/3}(3 + 2\theta^2)}, \tag{15}$$

where the dimensionless factor $c_c$ is treated as a free parameter. $\chi^H$ is the susceptibility in the hadronic medium, which can be absorbed by the definitions $C'(y_1 - y_2; \tau) \equiv C(y_1 - y_2; \tau)/\chi^H$ and $K'(\Delta y, \tau) \equiv K(\Delta y, \tau)/\chi^H$. In the following calculations, we will omit the prime to simplify the notation.

Considering that the evolving hot QCD system belongs to model H in the classification of Ref. [54], we scale the diffusion coefficient $D_c$ with the correlation length $\xi$ as: $D_c^{sr} \sim \xi^{-2-\chi_{\xi\eta}}$ with the exponents $\chi_\eta \approx 0.04$ and $\chi_\xi \approx 0.916$ [54]. The correlation length $\xi$ is connected to the susceptibility $\chi^{cr}$ as:

$$\xi = \xi_0 \left(\frac{\chi^{cr}}{\chi^H}\right)^{1/(2-\chi_\eta)} \tag{16},$$

where the constant $d_c = 1$ fm, as used in Ref. [42].

After the above parametrization, the susceptibility $\chi^{cr}(T, \mu)$ and diffusion coefficient $D_c^{cr}(T, \mu)$ as functions of temperature $T$ and chemical potential $\mu$ can be obtained from $\chi^{cr}(r, H)$ and $D_c^{cr}(r, H)$ with the following linear mapping [37, 42]

$$\frac{T - T_c}{\Delta T} = \frac{H}{\Delta H}, \quad \frac{\mu - \mu_c}{\Delta \mu} = -\frac{r}{\Delta r}, \tag{18}$$

where $r$ is treated as a free parameter to simulate the change of $\mu$ [42]. The critical temperature is set to $T_c = 160$ MeV and the width of the critical region is set to $\Delta T/\Delta H = 10$ MeV.

Again, we only focus on an evolving system in 1+1-dimension with Bjorken expansion. We assume the heat bath is evolving along a trajectory with fixed $r$ and the temperature $T$ dropping down with the proper time $\tau$ as [37]:

$$T(\tau) = T_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{c_s^2}{\tau}}, \tag{19}$$

where the speed of sound is taken as: $c_s^2 = 0.15$. The values of the initial time $\tau_0$ and the corresponding temperature $T_0$ will be explained in Sec. IV.

### III. THE KIBBLE-ZUREK SCALING

The correlation (3) and cumulant (7) obtained from solving SDE (1) are non-universal and sensitive to some inputs in the parametrization of $\chi(\tau)$ and $D(\tau)$, such as the strengths of the critical component $c_c$, the initial temperature $T_0$, etc. In Refs. [41] and [51], the universal functions have been constructed within the framework of Kibble-Zurek Mechanism for model A, that involves with the evolving non-conserving order parameter field near the critical point. In this section, we will study the possible universal behavior of the correlation function (3) and cumulant (7) for the evolving conserved charge of model B.

For a dynamical system near the critical point, there are two competitive time scales, the relaxation time $\tau_{rel}$ that describes the time for the system to equilibrate and the quench time $\tau_{quench}$ that characterizes the changing rate of the external potential.

Bjorken expansion of the hot medium Eq. (19) introduces the variation of the susceptibility $\chi(\tau)$ and diffusion coefficient $D(\tau)$ with which the quench time can be calculated as:

$$\tau_{quench} = \left|\frac{\xi(\tau)}{\partial \xi(\tau)/\partial \tau}\right|. \tag{20}$$

For a diffusion system near the critical point, the relaxation time of the two-point correlation function takes the form $\tau_{rel} = [2D(\tau)\xi^2]^{-1}$ for a particular mode $q$. For the slow modes with $q \ll \xi^{-1}$, the relaxation time is large compared to $\tau_{quench}$, which leads to these modes out-of-equilibrium as the system evolves near the critical point. For the fast modes with $q \gg \xi^{-1}$, the relaxation times are small, which corresponds to fast enough equilibration even near the critical point. In this work,
we focus on the mode with $q\xi = 1$ and the relaxation time is given by:

$$\tau_{\text{rel}} = \frac{\xi^2}{2D(\tau)}. \quad (21)$$

Note that the relaxation time $\tau_{\text{rel}}$ strongly enhances as the system cools down to the critical point and the quench time $\tau_{\text{quench}}$ continuously decreases. As a consequence, there exists a point $\tau^*$, where the relaxation time equals to quench time, after which the system becomes out-of-equilibrium with the formation of correlated patches. According to the Kibble-Zurek Mechanism, the characteristic time scale $\tau_{KZ}$ and length scale $l_{KZ}$ are determined by $\tau^*$ with [41]:

$$\tau_{KZ} = \tau_{\text{rel}}(\tau^*) = \tau_{\text{quench}}(\tau^*), \quad l_{KZ} = \xi_{\text{rel}}(\tau^*). \quad (22)$$

In Fig. 1, we plot the relaxation time $\tau_{\text{rel}}$ and quench time $\tau_{\text{quench}}$ as functions of $\tau - \tau_c$, where $\tau_c$ is the time when the temperature of the system hits the critical temperature $T_c$. It shows that the relaxation time $\tau_{\text{rel}}$ increases and the quench time $\tau_{\text{quench}}$ decreases as the system approaches to the critical point, and the proper time $\tau^*$ can be determined by Eq. (22).

After obtained the characteristic scales $\tau_{KZ}$, $l_{KZ}$ in Eq. (22), one can construct the universal function with the following redefined variables:

$$\tilde{\tau} \equiv (\tau - \tau_c)/\tau_{KZ}, \quad \tilde{y} \equiv y/l_{KZ}, \quad \tilde{\xi} \equiv \xi/l_{KZ},$$

$$\tilde{D} \equiv D/l_{KZ}^{2+\chi y}, \quad \tilde{\chi} \equiv \chi/l_{KZ}^{2-\chi y}. \quad (23)$$

For example, the rescaled correlation function $\tilde{C}(y_1 - y_2, \tilde{\tau})$ and the rescaled function of cumulant $\tilde{K}(\Delta y/l_{KZ}, \tilde{\tau})$ can be constructed as:

$$C(y_1 - y_2, \tau) = l_{KZ}^{1+\chi y} \left\{ \tilde{C}(\tilde{\tau}) \delta(\tilde{y}_1 - \tilde{y}_2) \right. \right.$$

$$\left. \left. - \int_{\tau_0}^{\tau} d\tilde{\tau} \left\{ \int_{\tau_0}^{\tau} d\tilde{\tau}' \tilde{D} \right\}^{-1/2} \exp \left\{ \frac{1}{2} \int_{\tau_0}^{\tau} d\tilde{\tau}' \tilde{D} \right\} \right\} \right.$$}

$$\equiv l_{KZ}^{1+\chi y} C(y_1 - y_2, \tilde{\tau}), \quad (24)$$

$$K(\Delta y, \tau) = l_{KZ}^{2-\chi y} \left\{ \tilde{K}(\tilde{\tau}) - \int_{\tau_0}^{\tau} d\tilde{\tau} \tilde{D} \right\}^{-1/2} \exp \left\{ \frac{\Delta y/l_{KZ}}{2(2\int_{\tau_0}^{\tau} d\tilde{\tau}' \tilde{D} \tilde{D})^{1/2}} \right\} \right.$$}

$$\equiv l_{KZ}^{2-\chi y} K(\Delta y/l_{KZ}, \tilde{\tau}). \quad (25)$$

The rescaled functions $\tilde{C}(y_1 - y_2, \tilde{\tau})$ and $\tilde{K}(\Delta y/l_{KZ}, \tilde{\tau})$ as functions of the redefined variables $\tilde{y}_1 - \tilde{y}_2$, $\tilde{\tau}$ and $\Delta y/l_{KZ}$ are universal and insensitive on the some free parameters, which will be demonstrated in the next section. Note that the calculated correlation function $C(y_1 - y_2, \tau)$ and cumulant $K(\Delta y, \tau)$ evolving with respect to proper time $\tau$, while the Kibble-Zurek scaling procedure is over the relative time $\tau - \tau_c$ as shown in Eq. (22). Therefore, the above rescaling formulae (24) and (25) valid near the critical point, where the relative time $\tau - \tau_c$ is small.

### IV. RESULTS AND DISCUSSIONS

In this section, we will demonstrate the constructed universal functions Eq. (24) and Eq. (25) are insensitive to the free inputs, the strength of critical component $c_c$ and the initial temperature $T_0$.

Firstly, we numerically calculate the correlation function $C(y_1 - y_2, \tau)$ and cumulant $K(\Delta y, \tau)$ along a particular trajectory with the fixed chemical potential $r = 0.1$. The temperature drops down according to Eq. (19) with the initial temperature $T_0 = 190\text{MeV}$ and the initial time $\tau_0$ is set at: $\tilde{\tau}_0 \equiv (\tau_0 - \tau_c)/\tau_{KZ} = -2.5$.

The left panel of Fig. 2 presents the correlation function $C(y_1 - y_2, \tau)$ as a function of $y_1 - y_2$ with different strength of critical component $c_c = 1, 2, 3$ at a fixed rescaled time $\tilde{\tau} = -0.5$, where the corresponding temperature $T$ is larger than but also close to $T_c$. As shown in Ref. [42], the correlation function as a function of $y_1 - y_2$ has a local minimum at very small $y_1 - y_2$ due to the $\delta(\tilde{y}_1 - \tilde{y}_2)$ contribution in Eq. (3). As expected, the correlation function (3) is sensitive to the strength of the critical component $c_c$. In the right panel of Fig. 2, we investigate the universal behavior of the reconstructed correlation function (24) within the framework of KZM. As shown in Fig. 1, the relaxation time $\tau_{\text{rel}}$ strongly enhances as the system approaches to the critical point and quench time $\tau_{\text{quench}}$ decreases, which results in a point $\tau^*$ where

![Graphical representation of the quench time and relaxation time](image-url)
the relaxation time equals to the quench time. With the obtained characteristic scales \( l_{KZ} \) and \( \tau_{KZ} \) at \( \tau^* \) and the redefined variables (23), we construct the universal correlation function according to Eq. (24). The right panel of Fig. 2 plots the constructed universal correlation function \( \hat{C}(\tilde{y}_1 - \tilde{y}_2, \tilde{\tau}) \) at \( \tilde{\tau} = -0.5 \) with different \( c_c \). Compared with the original correlation function \( C(y_1 - y_2, \tau) \) that is sensitive to critical component \( c_c \), these constructed correlation function \( \hat{C}(\tilde{y}_1 - \tilde{y}_2, \tilde{\tau}) \) perfectly converge into one universal curve.

In Fig. 3, we plot the correlation function \( C(y_1 - y_2, \tau) \) as a function of \( y_1 - y_2 \) at the rescaled time \( \tilde{\tau} = 0.5 \), where the temperature \( T \) is close but below \( T_c \). Different to the local minimum of \( C(y_1 - y_2, \tau) \) as a function of \( y_1 - y_2 \) in Fig. 2 which arises from the \( \delta(y_1 - y_2) \) contribution, the one in the left panel of Fig. 3 is due to the changing sign of \( \chi'(\tau) \) in Eq. (3) when \( T < T_c \), indicating the susceptibility \( \chi(\tau) \) has a maximum with respect to the proper time \( \tau \). Meanwhile, \( C(y_1 - y_2, \tau) \) at \( \tilde{\tau} = 0.5 \) also show sensitivity to the strength of the critical component \( c_c \). After the same scaling procedure as above, the constructed universal correlation function \( \hat{C}(\tilde{y}_1 - \tilde{y}_2, \tilde{\tau}) \) nicely converge into one curve.

In Fig. 4, we investigate the universal behavior of the cumulant \( K(\Delta y, \tau) \) according to Eq. (25). The system evolves with the same parameters as the two above cases, except for changing the chemical potential to \( r = 0.3 \). The left panel of Fig. 4 shows the time evolution of \( K(\Delta y, \tau) \) with different strengths of critical component \( c_c \), where \( \Delta y \) is fixed at \( \Delta y/l_{KZ} = 4 \). Similar to the two above cases of correlation function, the time evolution of second-order cumulant \( K(\Delta y, \tau) \) strongly depends on \( c_c \). After rescaling \( K(\Delta y, \tau) \) and \( \tau - \tau_c \) with \( l_{KZ}^2 \chi' \) and \( \tau_{KZ} \), the constructed universal cumulant \( \tilde{K}(\Delta y/l_{KZ}, \tilde{\tau}) \) is independent on the strength of the critical component \( c_c \), as expected in Eq. (25).

In the last paragraph of this section, we will show that the
FIG. 4: (Color online) Time evolution of the second-order cumulants $K(\Delta y, \tau)$ for the conserved charge with different strength of critical component $c_c = 1, 2, 3$. Right panel: the corresponding universal function $\tilde{K}(\Delta y/l_{KZ}, \tilde{\tau})$ as a function of rescaled time $\tilde{\tau}$.

FIG. 5: (Color online) Similar to Fig. 4, but evolving the system with different initial temperatures $T_0 = 170, 180, 190$ MeV.

constructed universal function $\tilde{K}(\Delta y/l_{KZ}, \tilde{\tau})$ is also not sensitive to the initial temperature $T_0$. For this case, we evolve the systems with different initial temperature $T_0 = 170, 180, 190$ MeV at a fixed initial rescaled time $\tilde{t}_0 = -2.5$ along a trajectory with fixed chemical potential $r = 0.3$. Again, we assume one dimensional Bjorken expansion and the temperature drops according to Eq. (19). The left panel of Fig. 5 plots the time evolution of the second-order cumulant $K(\Delta y, \tau)$ with $c_c = 3$ and $\Delta y/l_{KZ} = 4$, which shows a significant dependence on the initial temperature $T_0$. After the same rescaling procedure as described above, the universal cumulant $\tilde{K}(\Delta y/l_{KZ}, \tilde{\tau})$ is constructed, which is insensitive to the initial temperature $T_0$ as shown in right panel of Fig. 5.

V. SUMMARY AND OUTLOOK

In this paper, we explored the Kibble-Zurek scaling for the critical fluctuation of the conserved charge within the framework of stochastic diffusion dynamics. By analytically solving the stochastic diffusion equation (1), the time evolution of the two-point correlation function $C(y_1 - y_2, \tau)$ and the second-order cumulant $K(\Delta y, \tau)$ of conserved charge are obtained, which are non-universal in terms of some free inputs in the model calculations, such as the initial temperature $T_0$ and the strengths of the critical components $c_c$ in the mapping between the QCD medium and 3D-Ising model.

With determining the time $\tau^*$ after which the system falls out-of-equilibrium, we calculated the characteristic scales $\tau_{ez}$ and $l_{KZ}$ of the “frozen” system near the critical point. Using these obtained scales and rescaling the traditional two-point
correlation function $C(y_1 - y_2, \tau)$ and cumulant $K(\Delta y, \tau)$, we constructed the universal correlation function $\tilde{C}(\tilde{y}_1 - \tilde{y}_2, \tilde{\tau})$ and cumulant $\tilde{K}(\Delta y/l_{zk}, \tilde{\tau})$ in terms of the rescaled rapidity $\tilde{y}$ and proper time $\tilde{\tau}$, respectively. These rescaled functions are universal in terms of different free parameters. For instance, we have numerically shown that the universal functions $\tilde{C}(\tilde{y}_1 - \tilde{y}_2, \tilde{\tau})$ and $\tilde{K}(\Delta y/l_{zk}, \tilde{\tau})$ nicely converge into one curve which are insensitive to the strength of critical component $\epsilon_c$ and initial temperature $T_0$, respectively.

At last we would like to point out that this work focuses on the universal scaling of the two-point correlation function and second-order cumulant for the conserved charge based on the stochastic diffusion equation without the higher order coupling (1). At current stage, one can not also expect to connect our constructed universal functions with the experimental data since we used the 1+1-dimensional heat bath with Bjorken approximation to simplify the calculations. On the other hand, there are many natural extensions to this current study. For example, with the higher order contribution added to the stochastic diffusion equation of the conserved charge, one not only study the universal scaling of the two-point correlation function, but also the ones of multi-point correlation functions and related higher-order cumulants. Besides, studying the universal scaling with a more realistic evolving medium are also important for a realistic predictions of the possible observable that might be measured in experiment. These work are complicated, but worthwhile to be investigated in the near future.

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**Appendix A: Derivation for the time evolution of correlation function**

In this appendix, we presents the detail derivation of correlation function (3) from the stochastic diffusion equation (1).

With the Fourier transform

$$n(q, \tau) = \int dy e^{-iqy}n(y, \tau), \quad (A1)$$

SDE (1) in the Fourier space is written as:

$$\frac{\partial}{\partial \tau} \delta n(q, \tau) = -D(\tau)q^2 \delta n(q, \tau) + iq \xi(q, \tau), \quad (A2)$$

and the noise satisfies

$$\langle \xi(q, \tau) \rangle = 0, \quad \langle \xi(q_1, \tau_1) \xi(q_2, \tau_2) \rangle = 4\pi \chi(\tau)D(\tau) \delta(q_1 + q_2)\delta(\tau_1 - \tau_2). \quad (A3)$$

Therefore, one could obtain the time evolution of correlation function in $q$ space:

$$\frac{\partial}{\partial \tau} \langle \delta n(q_1, \tau) \delta n(q_2, \tau) \rangle = -D(\tau)(q_1^2 + q_2^2)\langle \delta n(q_1, \tau) \delta n(q_2, \tau) \rangle + 4\pi q_1 q_2 \chi(\tau)D(\tau)\delta(q_1 + q_2),$$

based on which the relaxation time of the correlation function is obtained as: $\tau_{nl} = [D(\tau)(q_1^2 + q_2^2)]^{-1}$. With the assumption of the locality in the initial fluctuation

$$\langle \delta n(q_1, \tau_0) \delta n(q_2, \tau_0) \rangle = 2\pi \delta(q_1 + q_2)\chi(\tau_0), \quad (A4)$$

the solution of Eq. (A4) is calculated to be

$$\langle \delta n(q_1, \tau) \delta n(q_2, \tau) \rangle = 2\pi \delta(q_1 + q_2)\left[\chi(\tau_0)e^{-q_1^2d(\tau_0, \tau)^2} + 2q_1^2\int_{\tau_0}^{\tau} d\tau' \chi(\tau')D(\tau')e^{-q_1^2d(\tau_0, \tau')^2}\right]. \quad (A5)$$

Then, the correlation function in $y$ space is computed as

$$\langle \delta n(y_1, \tau) \delta n(y_2, \tau) \rangle = \chi(\tau_0)G(y_1 - y_2; 2d(\tau_0, \tau)) + \int_{\tau_0}^{\tau} d\tau' \chi(\tau')\frac{d}{d\tau'}G(y_1 - y_2; 2d(\tau', \tau)) \quad (A6)$$

$$= \chi(\tau)\delta(y_1 - y_2) - \int_{\tau_0}^{\tau} d\tau' \chi(\tau')G(y_1 - y_2; 2d(\tau', \tau)).$$

Meanwhile, the second order cumulant $K(\Delta y, \tau)$ can straightforwardly calculated, as shown in Eq. (7).
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