The logarithmic derivative for point processes with equivalent Palm measures

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Abstract

The logarithmic derivative of a point process plays a key rôle in the general approach, due to the third author, to constructing diffusions preserving a given point process. In this paper we explicitly compute the logarithmic derivative for determinantal processes on $\mathbb{R}$ with integrable kernels, a large class that includes all the classical processes of random matrix theory as well as processes associated with de Branges spaces. The argument uses the quasi-invariance of our processes established by the first author.

1 Introduction

Let $\mathbb{P}$ be a point process on $\mathbb{R}^d$, or, in other words, a Borel probability measure on the space of locally finite configurations $\text{Conf}(\mathbb{R}^d)$. It is a natural question whether one can construct a diffusion $\xi(t) = (\xi^1(t), \xi^2(t), \ldots, \xi^i(t), \ldots)$ on the space $(\mathbb{R}^d)^\mathbb{N}$ such that the configuration $X(t) = \{\xi^1(t), \xi^2(t), \ldots, \xi^i(t), \ldots\}$ is almost surely locally finite for every $t \in \mathbb{R}^+$, and the process $X(t)$, considered as a process on the space $\text{Conf}(\mathbb{R}^d)$, preserves the measure $\mathbb{P}$. For example, if $\mathbb{P}$ is the standard Poisson point process on $\mathbb{R}^d$, then $\xi^i(t)$ are independent Brownian motions. In the series of papers [6, 9–15] the third author with collaborators developed a general approach to constructing the process $\xi$. The key step is the computation of the logarithmic derivative $d^\mathbb{P}$ of the measure $\mathbb{P}$, $d^\mathbb{P} : \mathbb{R}^d \times \text{Conf}(\mathbb{R}^d) \mapsto \mathbb{R}^d$, introduced by the third author in [10]. The process $\xi$ is then a solution of the infinite-dimensional stochastic differential equation

$$\xi^i(t) = \xi^i(0) + B^i(t) + \frac{1}{2} \int_0^t d^\mathbb{P}(\xi^i(u), X_i(\xi(u)))du, \quad i \in \mathbb{N},$$

where the configuration $X_i$ is defined by the formula $X_i(\xi(u)) := \{\xi^j(u)\}_{j \neq i}$ and $B^i$ are independent Brownian motions. In [6, 10, 15] logarithmic derivatives were calculated for
determinantal processes arising in random matrix theory: sine\textsubscript{2}, Airy\textsubscript{2}, Bessel\textsubscript{2} and the Ginibre point processes. The computation was based on finite particle approximation and had to be adapted for each determinantal process separately.

Theorem 2.3, the main result of this paper, establishes existence and gives an explicit formula for the logarithmic derivative for determinantal point processes on \( \mathbb{R} \) with integrable kernels studied in [2], a class that includes, in particular, determinantal processes mentioned above and those corresponding to de Branges spaces [4].

There are other methods to constructing infinite-dimensional diffusions. In particular, in [7, 8], using extended determinantal kernels, Katori and Tanemura constructed diffusions reversible with respect to the sine\textsubscript{2}, Airy\textsubscript{2}, and Bessel\textsubscript{2} point processes. A different approach to studying the diffusion preserving the sine\textsubscript{2} process is due to L.-C. Tsai [19].

In [1], Borodin and Olshanski gave a construction of infinite-dimensional diffusions as scaling limits of random walks on partitions.

To explain our results in more details we first give an informal definition of the logarithmic derivative. Consider a point process \( P \) on \( \mathbb{R}^d \) which admits a differentiable first correlation function \( \rho_1: \mathbb{R}^d \to \mathbb{R} \). Denote by \( P_a \) the reduced Palm measure conditioned at the point \( a \in \mathbb{R}^d \) and define the reduced Campbell measure \( C_P \) as a Borel sigma-finite measure on the space \( \mathbb{R}^d \times \text{Conf}(\mathbb{R}^d) \) given by

\[
d_C_P(a, X) = \rho_1(a) dP^a(X) da.
\]

Then, informally, the logarithmic derivative \( d_P \) is defined as a gradient of the logarithm of \( C_P \),

\[
d_P(a, X) = \nabla_a (\ln \rho_1(a) + \ln P^a(X)),
\]

see Definition 2.1. The main problem when proving the existence of the logarithmic derivative is to give a sense to the term \( \nabla_a \ln P^a(X) \).

Our first result is Proposition 2.2, where we find the connection between equivalence of the Palm measures conditioned at different points and the existence of the logarithmic derivative. More specifically, we consider a point process \( P \) on \( \mathbb{R}^d \) as above and assume that for any \( a, b \in \mathbb{R}^d \) the reduced Palm measures \( P^a \) and \( P^b \) are equivalent,

\[
dP^b(X) = R_{b,a}(X) dP^a(X),
\]

the Radon-Nikodym derivative \( R_{b,a}(X) \) is continuous with respect to \( b \) in \( L^1(\mathbb{P}^a, \text{Conf}(\mathbb{R}^d)) \) and the derivative \( \nabla_b R_{b,a} \) exists in appropriate sense. We then prove that the logarithmic derivative \( d_P \) exists and the formula (1.1) is valid with

\[
\nabla_a \ln P^a := \nabla_b \big|_{b=a} R_{b,a}.
\]

Our second and main result is the mentioned above Theorem 2.3. To establish it, it suffices to check that assumptions of Proposition 2.2 are satisfied for the considered class of determinantal point processes. To show this, we use the results of the paper [2], where the first author proved that for this class of determinantal processes the reduced Palm measures are equivalent and the Radon-Nikodym derivative has the form

\[
R_{b,a} = \lim_{R \to \infty, \delta \to 0} R_{b,a}^{R,\delta} \quad \text{where} \quad R_{b,a}^{R,\delta} = R_{b,a}^{X \in \text{Conf}(\mathbb{R}^d): |x| < R, [x-a, |x-b|/\delta]} \left( \frac{x-b}{x-a} \right)^2,
\]
and \( C_{b,a}^{R,\delta} \) are some normalizing constants. While the continuity in \( b \) of \( R_{b,a} \) follows immediately from the results of [2], the proof of its differentiability requires some efforts. To get it, we approximate the Radon-Nikodym derivative \( R_{b,a} \) by the function \( R_{b,a}^{R,\delta} \), compute the derivative of the latter, and then pass to the limit \( R \to \infty, \delta \to 0 \) using the techniques of normalized additive and multiplicative functionals developed in [2], which we outline in the appendix. Finally, we find

\[
\nabla b |_{b=a} \mathcal{R}_{b,a} = \lim_{R \to \infty, \delta \to 0} (S_{a}^{R,\delta} - \mathbb{E}^{a} S_{a}^{R,\delta}),
\]

where \( S_{a}^{R,\delta} = \sum_{x \in X : |x| < R, |x-a| > \delta} \frac{2}{a-x} \) and \( \mathbb{E}^{a} \) stands for the expectation with respect to the reduced Palm measure \( \mathbb{P}^{a} \).

The paper is organized as follows. In Section 2 we formulate our main results, Proposition 2.2 and Theorem 2.3. Section 3 is devoted to the proofs. In Appendix A we recall some results of [2] needed in the proof of Theorem 2.3.

## 2 Formulation of the main results

### 2.1 Configurations, point processes, Palm distributions

Consider the space of locally finite configurations

\[
\text{Conf}(\mathbb{R}^{d}) := \{ X \subset \mathbb{R}^{d} \mid \text{X does not have limit points in } \mathbb{R}^{d} \}.
\]

A Borel probability measure \( \mathbb{P} \) on \( \text{Conf}(\mathbb{R}^{d}) \) is called a point process. Take a bounded Borel set \( B \in \mathcal{B}(\mathbb{R}^{d}) \) and consider a function \( \#_{B} : \text{Conf}(\mathbb{R}^{d}) \to \mathbb{N} \cup \{0\} \) such that \( \#_{B}(X) \) is equal to the cardinality of the set \( B \cap X \). Assume that the process \( \mathbb{P} \) admits the first correlation function \( \rho_{1} \), that is for any bounded \( B \in \mathcal{B}(\mathbb{R}^{d}) \) the function \( \#_{B} \) is integrable with respect to the measure \( \mathbb{P} \) and there exists a function \( \rho_{1} \in L^{1}_{\text{loc}}(\mathbb{R}^{d}) \) satisfying

\[
\int_{B} \rho_{1}(x) \, dx = \int_{\text{Conf}(\mathbb{R}^{d})} \#_{B}(X) \, d\mathbb{P}(X), \quad \forall B \in \mathcal{B}(\mathbb{R}^{d}), \quad B \text{ is bounded}.
\]

Define the first correlation measure \( \hat{\rho}_{1} \) as \( \hat{\rho}_{1}(B) = \int_{B} \rho_{1}(x) \, dx \).

The Campbell measure \( \hat{C}_{\mathbb{P}} \) is a sigma-finite Borel measure on \( \mathbb{R}^{d} \times \text{Conf}(\mathbb{R}^{d}) \) defined as

\[
\hat{C}_{\mathbb{P}}(B, \mathcal{X}) = \int_{\mathcal{X}} \#_{B}(X) \, d\mathbb{P}(X), \quad \forall B \in \mathcal{B}(\mathbb{R}^{d}), \quad \mathcal{X} \in \mathcal{B}(\text{Conf}(\mathbb{R}^{d})),
\]

where \( \mathcal{B}(\text{Conf}(\mathbb{R}^{d})) \) stands for the Borel sigma-algebra on \( \text{Conf}(\mathbb{R}^{d}) \). Fix a Borel set \( \mathcal{X} \subset \text{Conf}(\mathbb{R}^{d}) \) and consider a sigma-finite measure \( C_{\mathbb{P}}^{\mathcal{X}} \) on \( \mathbb{R}^{d} \) given by the formula

\[
C_{\mathbb{P}}^{\mathcal{X}}(B) = \hat{C}_{\mathbb{P}}(B, \mathcal{X}), \quad \forall B \in \mathcal{B}(\mathbb{R}^{d}).
\]

By definition, for any \( \mathcal{X} \in \mathcal{B}(\text{Conf}(\mathbb{R}^{d})) \) the measure \( C_{\mathbb{P}}^{\mathcal{X}} \) is absolutely continuous with respect to \( \hat{\rho}_{1} \). Then the Palm measure \( \mathbb{P}^{a} \), defined for \( \hat{\rho}_{1} \)-almost every \( a \in \mathbb{R}^{d} \), is a measure
on $\text{Conf}(\mathbb{R}^d)$ given by the relation

$$\hat{\mathbb{P}}^a(\mathcal{A}) = \frac{dC_\rho}{d\rho_1}(a).$$

Equivalently, the Palm measure $\hat{\mathbb{P}}^a$ is the canonical conditional measure of the Campbell measure $\hat{C}_\rho$ with respect to the measurable partition of the space $\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)$ into subsets of the form $\{a\} \times \text{Conf}(\mathbb{R}^d)$, $a \in \mathbb{R}^d$. Thus, we can write

$$d\hat{C}_\rho(a, X) = d\hat{\mathbb{P}}^a(X)\rho_1(a) \, da.$$  

By definition, the Palm measure $\hat{\mathbb{P}}^a$ is supported on the subset of configurations containing a particle at the position $a$. The reduced Palm measure $\mathbb{P}^a$ is defined as the push-forward of the Palm measure $\hat{\mathbb{P}}^a$ under the map $X \mapsto X \setminus \{a\}$ erasing the particle $a$ from the configuration $X$. We then define the reduced Campbell measure $C_\rho$ as

$$dC_\rho(a, X) = d\hat{\mathbb{P}}^a(X)\rho_1(a) \, da. \quad (2.1)$$

Note that, in difference with the notions of the (reduced) Palm measure and the Campbell measure, this definition is not standard. Writing it in a more formal way, we obtain

$$C_\rho(B, \mathcal{A}) = \int_B \int_{\mathcal{A}} d\hat{\mathbb{P}}^a(X)\rho_1(a) \, da \quad \forall B \in \mathcal{B}(\mathbb{R}^d), \quad \mathcal{A} \in \mathcal{B}(\text{Conf}(\mathbb{R}^d)).$$

For more details see e.g. [2] and [5].

### 2.2 Definition of the logarithmic derivative

A function $\varphi : \text{Conf}(\mathbb{R}^d) \to \mathbb{R}$ is called local if there exists a compact set $K \subset \mathbb{R}^d$ such that $\varphi(X) \equiv \varphi(X \cap K)$. For a local function $\varphi$ we define symmetric functions $\varphi_n : \mathbb{R}^{nd} \to \mathbb{R}$, $n \geq 1$, by the relation

$$\varphi_n(x_1, \ldots, x_n) = \varphi(\{x_1, \ldots, x_n\}).$$

We say that a local function $\varphi$ is smooth if the functions $\varphi_n$ are smooth for all $n \in \mathbb{N}$. We denote by $D_0$ the space of all bounded local smooth functions on $\text{Conf}(\mathbb{R}^d)$.

Denote by $B_R$ a ball in $\mathbb{R}^d$ of radius $R$. Let $L^1_{\text{loc}}(\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d), C_\rho)$ be the space of vector-functions $f : \mathbb{R}^d \times \text{Conf}(\mathbb{R}^d) \to \mathbb{R}^d$ satisfying $f \in L^1(B_R \times \text{Conf}(\mathbb{R}^d), C_\rho)$, for all $R > 0$.

Denote by $C^\infty_0$ the space of smooth real-valued functions on $\mathbb{R}^d$ which have compact supports. We say that a function $\varphi : \mathbb{R}^d \times \text{Conf}(\mathbb{R}^d) \to \mathbb{R}$ belongs to the space $C^\infty_0D_0$ if $\varphi = \varphi_1\varphi_2$, where $\varphi_1 \in C^\infty_0(\mathbb{R}^d)$ and $\varphi_2 \in D_0$.

**Definition 2.1.** Let $\mathbb{P}$ be a point process on $\mathbb{R}^d$ that admits the first correlation function. A function $d\mathbb{P} \in L^1_{\text{loc}}(\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d), C_\rho)$ is called the logarithmic derivative of $\mathbb{P}$ if for any observable $\varphi \in C^\infty_0D_0$ we have

$$\int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} \nabla_a \varphi(a, X) \, dC_\rho(a, X) = -\int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} d\mathbb{P}(a, X) \varphi(a, X) \, dC_\rho(a, X).$$

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2.3 Logarithmic derivative of a point process with equivalent Palm measures

Consider a point process \( P \) on \( \mathbb{R}^d \) that admits the first correlation function \( \rho_1 \); recall that we denote by \( \hat{\rho}_1 \) the first correlation measure,

\[
\hat{\rho}_1(da) = \rho_1(a) \, da.
\]

In this subsection we give a general scheme for the computation of the logarithmic derivative \( dP \) under the following assumption.

**Assumption 1.**

1. The first correlation function \( \rho_1 \) is \( C^1 \)-smooth.
2. For \( \hat{\rho}_1 \)-almost all \( a, b \in \mathbb{R}^d \) the reduced Palm measures \( P^a \) and \( P^b \) are equivalent.

Denote by \( R_{b,a} \) their Radon-Nikodym derivative, so that

\[
dP^b(X) = R_{b,a}(X) \, dP^a(X).
\]

3. For \( \hat{\rho}_1 \)-almost all \( a \in \mathbb{R}^d \) we have \( R_{b,a} \to 1 \) as \( b \to a \) in \( L^1(\mathbb{P}^a, \text{Conf}(\mathbb{R}^d)) \).

For a function \( \varphi \in C_0^\infty(\mathbb{R}^d) \mathcal{D}_0 \) we define the function \( f_\varphi : \mathbb{R}^d \to \mathbb{R} \) as

\[
f_\varphi(\varepsilon) := \int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} R_{a+\varepsilon,a}(X) \varphi(a, X) \, dC_P(a, X).
\]

4. For any \( \varphi \in C_0^\infty(\mathbb{R}^d) \mathcal{D}_0 \) the function \( f_\varphi \) admits partial derivatives in \( \varepsilon \) at the point \( \varepsilon = 0 \). There exist functions \( \partial_i R : \mathbb{R}^d \times \text{Conf}(\mathbb{R}^d) \to \mathbb{R} \) such that for any \( \varphi \) as above and any \( 1 \leq i \leq d \) we have

\[
\partial_\varepsilon f_\varphi(0) = \int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} \partial_i R(a, X) \varphi(a, X) \, dC_P(a, X).
\]

Set

\[
\nabla R := (\partial_1 R, \ldots, \partial_d R).
\]

**Proposition 2.2.** Let \( \mathbb{P} \) be a point process on \( \mathbb{R}^d \) satisfying Assumption 1. Then for \( \hat{\rho}_1 \)-almost all \( a \in \mathbb{R} \) its logarithmic derivative \( dP \) exists and has the form

\[
dP(a, X) = \nabla_a \ln \rho_1(a) + \nabla R(a, X).
\]  

(2.2)

Note that there is no need to define the logarithmic derivative at the points \( a \in \mathbb{R}^d \) where \( \rho_1(a) = 0 \) since the measure \( \hat{\rho}_1 \) of the set of such \( a \) is zero. Proof of Proposition 2.2 is given in Section 3.1. It is based on the differentiation by parts, that is why we crucially need the absolute continuity of the measure \( \hat{\rho}_1 \) and the differentiability of its density, which is the first correlation function \( \rho_1 \).
2.4 Logarithmic derivative of a determinantal process on \( \mathbb{R} \) with an integrable kernel

In this section we construct the logarithmic derivative for a class of determinantal processes on \( \mathbb{R} \). A point process \( P \) on Conf(\( \mathbb{R} \)) is called determinantal if there exists a locally trace class operator \( P : L^2(\mathbb{R}, dx) \mapsto L^2(\mathbb{R}, dx) \) such that for any bounded measurable function \( h \), for which the support \( \text{supp}(h - 1) =: D \) is a compact set, we have

\[
\mathbb{E} \left( \prod_{x \in X} h(x) \right) = \det (1 + (h - 1)P \mathbb{1}_D).
\]

Here the expectation is taken with respect to the measure \( P \), \( \det \) stands for the Fredholm determinant and \( \mathbb{1}_D \) denotes the indicator function of the set \( D \). See for details [16, 18]. Since the operator \( P \) is locally trace class, it admits a kernel which we denote by \( \Pi \). We impose the following restrictions for \( P \) and \( \Pi \).

Assumption 2.

1. The operator \( P \) is an orthogonal projection onto a closed subspace \( L \subset L^2(\mathbb{R}, dx) \).
2. For \( \hat{\rho}_1 \)-almost all \( a \in \mathbb{R} \), given any function \( \varphi \in L \) satisfying \( \varphi(a) = 0 \), we have \((x - a)^{-1}\varphi \in L\).
3. The kernel \( \Pi \) is \( C^2 \)-smooth on \( \mathbb{R}^2 \).
4. We have \( \int \frac{\Pi(x, x)}{1 + x^2} dx < \infty \).

Also, note that for any \( a \in \mathbb{R} \) the function \( \Pi(a, \cdot) \) belongs to \( L^2(\mathbb{R}, dx) \).

Take \( a \in \mathbb{R} \), \( R \gg 1 \) and \( \delta \ll 1 \), and consider the additive functional

\[
S_{a}^{R,\delta} : \text{Conf}(\mathbb{R}) \mapsto \mathbb{R}, \quad S_{a}^{R,\delta}(X) = \sum_{x \in X : |x| < R, |x - a| > \delta} \frac{2}{a - x}. \tag{2.3}
\]

The additive functional \( S_{a}^{R,\delta} \) may diverge as \( R \to \infty \). To overcome this difficulty we define the normalized additive functional

\[
\overline{S}_{a}^{R,\delta} := S_{a}^{R,\delta} - \mathbb{E}^a S_{a}^{R,\delta},
\]

where \( \mathbb{E}^a \) stands for the expectation with respect to the reduced Palm measure \( \mathbb{P}^a \). Results obtained in [2] imply that, under Assumption 2, for \( \hat{\rho}_1 \)-almost all \( a \in \mathbb{R} \) there exists a function \( \overline{S}_{a} : \text{Conf}(\mathbb{R}) \mapsto \mathbb{R} \), such that

\[
\overline{S}_{a}^{R,\delta} \to \overline{S}_{a} \quad \text{as} \quad R \to \infty, \delta \to 0 \quad \text{in} \quad L^2(\text{Conf}(\mathbb{R}), \mathbb{P}^a). \tag{2.4}
\]

Moreover, the convergence (2.4) holds uniformly in \( a \) as \( a \in \mathbb{R} \) ranges in a compact set. The required theory from [2] is recalled in Appendix A.1 and the convergence (2.4) is established in Corollary A.2.

Theorem 2.3. Let \( \mathbb{P} \) be a determinantal process on \( \mathbb{R} \) satisfying Assumption 2. Then for \( \hat{\rho}_1 \)-almost all \( a \in \mathbb{R} \) the logarithmic derivative \( d_{\mathbb{P}} \) exists and has the form

\[
d_{\mathbb{P}}(a, X) = \frac{d}{da} \ln \rho_1(a) + \overline{S}_{a}(X).
\]
Theorem 2.3 is proven in Section 3.2. There, using results of [2], we show that Assumption 2 implies Assumption 1 with \( \nabla R = \mathbf{f}_a \). Then Theorem 2.3 follows from Proposition 2.2.

3 Proofs of the main results

3.1 Proof of Proposition 2.2

Take a function \( \varphi \in C_0^\infty D_0 \). We have

\[
I := - \int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} \partial_{\alpha_i} \varphi(a, X) \, dC_\varphi(a, X) = - \int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} \lim_{\varepsilon \to 0} \frac{\varphi(a + \varepsilon_i, X) - \varphi(a, X)}{\varepsilon} \, dC_\varphi(a, X),
\]

where \( \varepsilon_i := \varepsilon e_i \) and \( e_i \) is the \( i \)-th basis vector of \( \mathbb{R}^d \). Using the dominated convergence theorem, we exchange the limit with the integral. The latter can be applied since

\[
\left| \frac{\varphi(a + \varepsilon_i, X) - \varphi(a, X)}{\varepsilon} \right| \leq \sup_{X \in \text{Conf}(\mathbb{R}^d), x \in \mathbb{R}^d} \left| \partial_{\varepsilon_i} \varphi(x, X) \right|
\]

and \( \varphi \in C_0^\infty D_0 \). We get

\[
I = - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} \varphi(a + \varepsilon_i, X) \, dC_\varphi(a, X) - \int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} \varphi(a, X) \, dC_\varphi(a, X) \right)
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} \varphi(a, X) \, dC_\varphi(a + \varepsilon_i, X) - \int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} \varphi(a, X) \, dC_\varphi(a, X) \right),
\]

(3.1)

where in the last line of (3.1) we put \( \varepsilon := -\varepsilon \). Using the definition of the reduced Campbell measure (2.1), we find

\[
I = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} \varphi(a, X) \rho_1(a + \varepsilon_i) \, d\mathbb{P}^{a+\varepsilon_i}(X) \, da 
- \int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} \varphi(a, X) \rho_1(a) \, d\mathbb{P}^a(X) \, da \right)
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} \varphi(a, X) \left( \rho_1(a + \varepsilon_i) - \rho_1(a) \right) \, d\mathbb{P}^{a+\varepsilon_i}(X) \, da 
+ \int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} \varphi(a, X) \rho_1(a) \left( d\mathbb{P}^{a+\varepsilon_i}(X) - d\mathbb{P}^a(X) \right) \, da \right)
= \lim_{\varepsilon \to 0} (I_1^\varepsilon + I_2^\varepsilon).
\]

Using Assumption 1(3), we obtain

\[
\lim_{\varepsilon \to 0} I_1^\varepsilon = \int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} \varphi(a, X) \partial_{\alpha_i} \rho_1(a) \, d\mathbb{P}^a(X) \, da = \int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} \varphi(a, X) \partial_{\alpha_i} \left( \ln \rho_1(a) \right) \, dC_\varphi(a, X).
\]

(3.2)
In view of Assumption 1(2), we have
\[
\lim_{\varepsilon \to 0} I^\varepsilon_2 = \frac{1}{\varepsilon} \left( \int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} \varphi(a, X) \mathcal{R}_{a+\varepsilon, a}(X) \, d\mathbb{P}(a, X) - \int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} \varphi(a, X) \, d\mathbb{P}(a, X) \right).
\]
Then, because of the identity \( \mathcal{R}_{a,a}(X) \equiv 1 \), Assumption 1(4) implies
\[
\lim_{\varepsilon \to 0} I^\varepsilon_2 = \int_{\mathbb{R}^d \times \text{Conf}(\mathbb{R}^d)} \varphi(a, X) \partial_i \mathcal{R}(a, X) \, d\mathbb{P}(a, X).
\]
Combining (3.2) with (3.3) we obtain the desired relation (2.2).

### 3.2 Proof of Theorem 2.3

In view of Proposition 2.2, it suffices to check that Assumption 1 is satisfied with \( \nabla \mathcal{R}(a, X) = S_a(X) \). Item 1 of Assumption 1 immediately follows from item 2 of Assumption 2. The proof of the other items relies on the results obtained in the paper [2]; see also [3]. One of the main tools we use borrowed from the works above is the following lemma. Take \( a, b \in \mathbb{R} \) and consider the normalized multiplicative functional \( \Psi_{R,\delta}^{a,b} : \text{Conf}(\mathbb{R}) \mapsto \mathbb{R} \) given by
\[
\Psi_{R,\delta}^{a,b}(X) := C_{a,b} \prod_{x \in X : |x| < R, |x-a| > \delta} \left( \frac{x-b}{x-a} \right)^2,
\]
where the constant \( C_{a,b}^{R,\delta} \) is specified by the normalization requirement \( \mathbb{E}^a \Psi_{R,\delta}^{a,b} = 1 \). Here and further on we set \( \prod_{x \in \emptyset} f(x) = 1 \), for any function \( f \).

**Lemma 3.1.** 1. Under Assumption 1, there exists \( \alpha > 0 \) and a function \( \Psi_{b,a} : \text{Conf}(\mathbb{R}) \mapsto \mathbb{R} \) satisfying
\[
\Psi_{b,a} \mapsto \Psi_{b,a} \text{ as } R \to \infty, \delta \to 0 \text{ in } L^{1+\alpha}(\text{Conf}(\mathbb{R}), \mathbb{P}^a),
\]
for \( \hat{\rho}_1 \)-almost all \( a \in \mathbb{R} \), uniformly in \( a, b \) which range in compact subsets of \( \mathbb{R} \).

2. For \( \hat{\rho}_1 \)-almost all \( a, b \in \mathbb{R}^d \), the function \( \Psi_{b,a} \) is the Radon-Nikodym derivative of the Palm measure \( \mathbb{P}^b \) with respect to the the Palm measure \( \mathbb{P}^a \), i.e.
\[
d\mathbb{P}^b(X) = \Psi_{b,a}(X) \, d\mathbb{P}^a(X).
\]

**Proof.** Item (1) is established in Corollary A.4(1) and follows from results obtained in [2], which we explain in Appendix A.2. Item (2) is a particular case of Theorem 1.4 (1) from [2]. Note that in [2] the multiplicative functional is defined as the product over the set \( \{ x \in X : |x| < R, |x-a| > \delta \} \), so that in difference with the definition (3.4) the point \( b \) is not isolated. It can be checked directly that this does not affect the proof at all. 

Lemma 3.1(2) implies item 2 of Assumption 1 with \( \mathcal{R}_{b,a} = \Psi_{b,a} \). Because of the bounds \( |x| < R \) and \( |x-a| > \delta \), the functions \( \Psi_{b,a}^{R,\delta}(X) \) are \( \mathbb{P}^a \)-almost surely continuous with respect to \( b \). Then, using the dominated convergence theorem, we see that they are continuous in \( L^1(\text{Conf}(\mathbb{R}), \mathbb{P}^a) \). Then the uniformity in \( b \) of convergence from Lemma 3.1(1)
implies that the functions $\Psi_{b,a}$ also are continuous with respect to $b$ in $L^1(\text{Conf}(\mathbb{R}), \mathbb{P}^a)$. So that item 3 of Assumption 1 is fulfilled as well.

It remains to check that item 4 of Assumption 1 holds with $\nabla R(a, X) = \nabla a(X)$. Due to Lemma 3.1(2), we have

$$f_\varphi(\varepsilon) = \int_{\mathbb{R} \times \text{Conf}(\mathbb{R})} \Psi_{a+\varepsilon,a}(X) \varphi(a, X) \, d\mathbb{C}_p(a, X).$$

We need to show that the function $f_\varphi$ is differentiable at zero and

$$\frac{d}{d\varepsilon} f_\varphi(0) = \int_{\mathbb{R} \times \text{Conf}(\mathbb{R})} \nabla a(X) \varphi(a, X) \, d\mathbb{C}_p(a, X). \quad (3.5)$$

Due to Lemma 3.1(1), we have

$$f_\varphi(\varepsilon) = \lim_{R \to \infty, \delta \to 0} f^{R,\delta}_\varphi(\varepsilon), \quad (3.6)$$

where

$$f^{R,\delta}_\varphi(\varepsilon) = \int_{\mathbb{R} \times \text{Conf}(\mathbb{R})} \Psi^{R,\delta}_{a+\varepsilon,a}(X) \varphi(a, X) \, d\mathbb{C}_p(a, X).$$

**Proposition 3.2.** The function $f^{R,\delta}_\varphi$ is $C^1$-smooth. For any $\varepsilon$ from a small neighbourhood of zero its derivative $\frac{d}{d\varepsilon} f^{R,\delta}_\varphi(\varepsilon)$ converges as $R \to \infty, \delta \to 0$, uniformly in $\varepsilon \ll 1$.

Proof of Proposition 3.2 is given in the next subsection. Jointly with (3.6), Proposition 3.2 implies that the function $f_\varphi$ is differentiable in a small neighbourhood of zero and

$$\frac{d}{d\varepsilon} f_\varphi(\varepsilon) = \lim_{R \to \infty, \delta \to 0} \frac{d}{d\varepsilon} f^{R,\delta}_\varphi(\varepsilon). \quad (3.7)$$

We have

$$\frac{d}{d\varepsilon} f^{R,\delta}_\varphi(\varepsilon) = \int_{\mathbb{R} \times \text{Conf}(\mathbb{R})} \frac{d}{d\varepsilon} \Psi^{R,\delta}_{a+\varepsilon,a}(X) \varphi(a, X) \, d\mathbb{C}_p(a, X).$$

By definition (3.4) of the multiplicative functional $\Psi^{R,\delta}_{a+\varepsilon,a}$, we have

$$\frac{d}{d\varepsilon} \Psi^{R,\delta}_{a+\varepsilon,a} = \frac{d}{d\varepsilon} \exp \left( \ln C^{R,\delta}_{a+\varepsilon,a} + 2 \sum_{x \in X, |x| < R, |x-a|, |x-(a+\varepsilon)| > \delta} \ln \left| \frac{x-(a+\varepsilon)}{x-a} \right| \right) = \Psi^{R,\delta}_{a+\varepsilon,a} \left( \frac{d}{d\varepsilon} \ln C^{R,\delta}_{a+\varepsilon,a} + S^{R,\delta}_{a,a+a+\varepsilon} \right), \quad (3.8)$$

where

$$S^{R,\delta}_{a,b} := \sum_{x \in X, |x| < R, |x-a|, |x-b| > \delta} \frac{2}{b-x}.$$  

Since, by definition, $E^a \Psi^{R,\delta}_{a+\varepsilon,a} = 1$, we have

$$E^a \frac{d}{d\varepsilon} \Psi^{R,\delta}_{a+\varepsilon,a} = \frac{d}{d\varepsilon} E^a \Psi^{R,\delta}_{a+\varepsilon,a} = 0.$$
Then, taking the expectation $\mathbb{E}^a$ of the both sides of (3.8), we get

$$
\frac{d}{d\varepsilon} \ln C_{a+\varepsilon,a}^{R,\delta} = -\mathbb{E}^a \left( \Psi_{a+\varepsilon,a}^{R,\delta} S_{a,a+\varepsilon}^{R,\delta} \right).
$$

(3.9)

Now (3.8) together with (3.9) implies

$$
\frac{d}{d\varepsilon} f^R_{\varphi}(\varepsilon) = \int_{\mathbb{R} \times \text{Conf}(\mathbb{R})} \Psi_{a+\varepsilon,a}^{R,\delta} \left( S_{a,a+\varepsilon}^{R,\delta} - \mathbb{E}^a \Psi_{a+\varepsilon,a}^{R,\delta} S_{a,a+\varepsilon}^{R,\delta} \right) \varphi(a, X) \, dC_{\varphi}(a, X).
$$

(3.10)

Since $\Psi_{a,a}^{R,\delta} = 1$ and $S_{a,a}^{R,\delta} = S_a^{R,\delta}$, where the additive functional $S_a^{R,\delta}$ is defined in (2.3), we obtain

$$
\frac{d}{d\varepsilon} f^R_{\varphi}(0) = \int_{\mathbb{R} \times \text{Conf}(\mathbb{R})} \left( S_a^{R,\delta} - \mathbb{E}^a S_a^{R,\delta} \right) \varphi(a, X) \, dC_{\varphi}(a, X).
$$

(3.11)

Due to (2.4), the function $S_a^{R,\delta} - \mathbb{E}^a S_a^{R,\delta} = \overline{\Psi}_a^{R,\delta}$ converges to $\overline{\Psi}_a$ in $L^2(\text{Conf}(\mathbb{R}), \mathbb{P}^a)$ as $R \to \infty, \delta \to 0$, uniformly in $\hat{\rho}_1$-almost all $a \in X \cap \text{Conf}(\mathbb{R})$ supp $\varphi(\cdot, X)$, since the latter set is compact. Then the right-hand side of (3.11) converges to that of (3.5). In view of (3.7), this concludes the proof of the theorem.

3.3 Proof of Proposition 3.2

The $C^1$-smoothness of the function $f^R_{\varphi}$ is obvious since its derivative has the form (3.10).

So that, we only need to establish the uniform in $\varepsilon$ convergence of the derivative $\frac{d}{d\varepsilon} f^R_{\varphi}$ as $R \to \infty, \delta \to 0$. Clearly, it suffices to show that the function

$$
J^R_{\varphi}(a, b) := \Psi_{b,b}^{R,\delta} \left( S_{a,b}^{R,\delta} - \mathbb{E}^a \Psi_{b,b}^{R,\delta} S_{a,b}^{R,\delta} \right)
$$

converges as $R \to \infty, \delta \to 0$,

(3.12)

in $L^1(\text{Conf}(\mathbb{R}), \mathbb{P}^a)$ uniformly in $b$ and $\hat{\rho}_1$-almost all $a$, where $a$ ranges in a compact set and $b$ satisfies $|a - b| < \vartheta$, for some fixed $\vartheta < 1$. All convergences below will be uniform in $a, b$ satisfying these restrictions, and we do not mention it any more. Further on we assume $R$ to be sufficiently large and $\delta$ to be sufficiently small where it is needed.

For real Borel functions $f, g$ where $g$ is non-negative we define the additive and multiplicative functionals $S_f, \overline{S}_f, \Psi_g, \overline{\Psi}_g, \Psi_g, \overline{\Psi}_g : \text{Conf}(\mathbb{R}) \mapsto \mathbb{R}$ by the formulas (A.1), (A.2), (A.4) and (A.5). Clearly, they are well-defined if the functions $f$ and $g$ are bounded and the supports supp $f$, supp $(g - 1)$ are compact. However, the normalized functionals $\overline{S}_f, \overline{\Psi}_g$ and $\overline{\Psi}_g$ can be defined for larger classes of functions, see Appendices A.1 and A.2.

Let us define

$$
h^R_{\geq}(x) := \frac{2}{b - x} \mathbb{I}_{\{x < R, |x - a| > \delta, |x - b| \geq 1\}}(x) = \frac{2}{b - x} \mathbb{I}_{\{|x| < R, |x - b| \geq 1\}}(x),
$$

where we have used that $|a - b| \ll 1$, so that the constraint $|x - a| > \delta$ holds automatically. Set also

$$
h^R_{\leq}(x) := \frac{2}{b - x} \mathbb{I}_{\{|x| > R, |x - a| > \delta, \delta < |x - b| < 1\}}(x) = \frac{2}{b - x} \mathbb{I}_{\{|x - a| > \delta, \delta < |x - b| < 1\}}(x).
$$

Then we have

$$
S_{a,b}^{R,\delta} = S_{h^R_{\geq}} + S_{h^R_{\leq}}.
$$
Recall that $E^a\tilde{\Psi}_{b,a}^{R,\delta} = 1$. Then, subtracting in the brackets of (3.12) the term $E^a S_{h_R^\delta}$ and adding the term $E^a\tilde{\Psi}_{b,a}^{R,\delta} E^a S_{h_R^\delta}$, we obtain

$$J^{R,\delta}(a,b) := \tilde{\Psi}_{b,a}^{R,\delta}(S_{h_R^\delta} + S_{h_R^\delta}) - \tilde{\Psi}_{b,a}^{R,\delta}(E^a (\tilde{\Psi}_{b,a}^{R,\delta}(S_{h_R^\delta} + S_{h_R^\delta}))).$$

(3.13)

Now, to establish the convergence (3.12) it suffices to show that the functions $\tilde{\Psi}_{b,a}^{R,\delta} S_{h_R^\delta}$ converge as $R \to \infty$, $\delta \to 0$ in $L^1(\text{Conf}(\mathbb{R}), \mathbb{P}^a)$. Indeed, in view of Lemma 3.1(1), the function $\tilde{\Psi}_{b,a}^{R,\delta}$ converges in $L^1(\text{Conf}(\mathbb{R}), \mathbb{P}^a)$ itself, so that we will see that the second summand from (3.13) converges.

Term $\tilde{\Psi}_{b,a}^{R,\delta} S_{h_R^\delta}$. Let

$$h_>(x) := \frac{2}{b-x}1_{\{|x-b| \geq 1\}}(x).$$

Due to Corollary A.2(2), the additive functional $S_{h_\geq}$ is well-defined and we have the convergence

$$S_{h_R^\delta} \to S_{h_\geq}$$

as $R \to \infty$ in $L^p(\text{Conf}(\mathbb{R}), \mathbb{P}^a)$

(3.14)

with $p = 2$. We claim that it takes place for any $p > 2$ as well. This concludes consideration of the term $\tilde{\Psi}_{b,a}^{R,\delta} S_{h_R^\delta}$ since, using the H"older inequality, from (3.14) joined with Lemma 3.1(1) we obtain

$$\tilde{\Psi}_{b,a}^{R,\delta} S_{h_R^\delta} \to \tilde{\Psi}_{b,a} S_{h_\geq}$$

as $\delta \to 0$, $R \to \infty$ in $L^1(\text{Conf}(\mathbb{R}), \mathbb{P}^a)$.

Denote

$$\Delta^R := h_R^\delta - h_\geq.$$

Due to the Cauchy-Bunyakovsky-Schwarz inequality, we have

$$E^a |S_{h_R^\delta} - S_{h_\geq}|^p \leq E^a |S_{\Delta^R}|^p \leq \sqrt{E^a(S_{\Delta^R})^{2p-2}} \sqrt{E^a(S_{\Delta^R})^2}.$$

Due to the convergence (3.14) with $p = 2$, we have $E^a(S_{\Delta^R})^2 \to 0$ as $R \to \infty$. Thus, it suffices to prove that the expectation $E^a|S_{\Delta^R}|^q$ is bounded uniformly in $R$, for any $q > 0$. We have

$$|S_{\Delta^R}|^q \leq C_q(e^{S_{\Delta^R}} + e^{-S_{\Delta^R}}).$$

(3.15)

Let us write

$$e^{S_{\Delta^R}} = \tilde{\Psi}_{\exp(\Delta^R)}.$$

Due to Corollary A.4(2), we have

$$\tilde{\Psi}_{\exp(\Delta^R)} \to \tilde{\Psi}_1 = 1$$

as $R \to \infty$ in $L^1(\text{Conf}(\mathbb{R}), \mathbb{P}^a)$,

(3.16)

where $\tilde{\Psi}_1$ is the multiplicative functional $\tilde{\Psi}_g$ corresponding to the function $g = 1$. In particular, the $L^1$-norm $E^a e^{S_{\Delta^R}} = E^a \tilde{\Psi}_{\exp(\Delta^R)}$ is bounded uniformly in $R$. Replacing $\Delta^R$ by $-\Delta^R$, the same argument implies that the expectation $E^a e^{-S_{\Delta^R}}$ is also bounded uniformly in $R$. Then, due to (3.15), we see that the expectation $E^a|S_{\Delta^R}|^q$ is bounded uniformly in $R$ as well. So that, we obtain the desired convergence (3.14).
Term $\Psi_{b,a}^{R,\delta} S_{h_\delta}$. Let us factorize

$$\Psi_{b,a}^{R,\delta} S_{h_\delta} = \frac{\tilde{\Psi}_{g_4^R}^{\Psi_{g_2}^{\Psi_{g_3}^{S_{h_\delta}}}}}{\mathbb{E}^{\alpha}(\Psi_{g_4}^{\Psi_{g_2}^{\Psi_{g_3}}})}$$

(3.17)

where

$$g_4^R := \left(\left(\frac{x - b}{x - a}\right)^2 - 1\right)\mathbb{I}_{\{x: |x| < R, |x-b| \geq 1\}} + 1$$

and

$$g_2^\delta := \left(1 - \frac{1}{(x-a)^2}\right)\mathbb{I}_{\{|x-a| > \delta, \delta < |x-b| \leq 1\}} + 1, \quad g_3^\delta := \left((x-b)^2 - 1\right)\mathbb{I}_{\{|x-a| > \delta, \delta < |x-b| < 1\}} + 1.$$

Set

$$g_1 := \left(\left(\frac{x - b}{x - a}\right)^2 - 1\right)\mathbb{I}_{\{x: |x-b| \geq 1\}} + 1$$

and

$$g_2 := \left(1 - \frac{1}{(x-a)^2}\right)\mathbb{I}_{\{x: |x-b| < 1\}} + 1, \quad g_3 := \left((x-b)^2 - 1\right)\mathbb{I}_{\{x: |x-b| < 1\}} + 1.$$ 

Corollary A.4(2) states that

$$\tilde{\Psi}_{g_1^R} \to \tilde{\Psi}_{g_1} \text{ as } R \to \infty \text{ in } L^p(\text{Conf}(\mathbb{R}), \mathbb{P}^a),$$

(3.18)

for any $p > 0$, and

$$\tilde{\Psi}_{g_2^\delta} \to \tilde{\Psi}_{g_2} \text{ as } \delta \to 0 \text{ in } L^{1+\alpha}(\text{Conf}(\mathbb{R}), \mathbb{P}^a),$$

(3.19)

for some $\alpha > 0$. Since the functions $g_3^\delta, g_3$ are bounded uniformly in $\delta$ and $(g_3^\delta - 1), (g_3 - 1)$ have compact supports, we obviously have

$$\Psi_{g_3^\delta} \to \Psi_{g_3} \text{ as } \delta \to 0 \text{ in } L^p(\text{Conf}(\mathbb{R}), \mathbb{P}^a),$$

for any $p > 0$. Then the Hölder inequality implies

$$\mathbb{E}^{\alpha}(\tilde{\Psi}_{g_1^R}^{\Psi_{g_2}^{\Psi_{g_3}^{S_{h_\delta}}}}) \to \mathbb{E}^{\alpha}(\tilde{\Psi}_{g_1}^{\tilde{\Psi}_{g_2}^{\Psi_{g_3}}}) \quad \text{as } R \to \infty, \quad \delta \to 0.$$

To prove that the nominator of (3.17) converges, we note that

$$\Psi_{g_4^R}^{S_{h_\delta}} = 2 \sum_{x \in X: |x-a| > \delta, \delta < |x-b| < 1} (b-x) \prod_{y \in X: y \neq x, |y-a| > \delta, \delta < |y-b| < 1} (y-b)^2.$$ 

(3.20)

Clearly, the right-hand side of (3.20) converges as $\delta \to 0$ in $L^p(\text{Conf}(\mathbb{R}), \mathbb{P}^a)$, for any $p > 0$. Together with (3.18)-(3.19), by the Hölder inequality this implies that the numerator of (3.17) converges in $L^1(\text{Conf}(\mathbb{R}), \mathbb{P}^a)$ as $R \to \infty, \delta \to 0$, so that the function $\Psi_{b,a}^{R,\delta} S_{h_\delta}$ also does.

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A  Regularization of additive and multiplicative functionals

In this appendix we consider a determinantal point process $\mathbb{P}$ on $\mathbb{R}$ with the kernel $\Pi$ and assume that $\Pi$ satisfies Assumption 2. We explain results from [2] which we use in this paper and prove some auxiliary convergence results.

A.1  Additive functionals

Let $f : \mathbb{R} \to \mathbb{C}$ be a Borel function. Define the corresponding additive functional

$$S_f : \text{Conf}(\mathbb{R}) \mapsto \mathbb{R}, \quad S_f(X) = \sum_{x \in X} f(x), \quad (A.1)$$

where the series may diverge. If $S_f \in L_1(\text{Conf}(\mathbb{R}), \mathbb{P})$, then we introduce the normalized additive functional

$$\overline{S}_f = S_f - ES_f. \quad (A.2)$$

Now we will show that the normalized additive functional can be defined even when the additive functional itself is not well-defined. Introduce the Hilbert space $V(\Pi)$ of real functions with the norm

$$\|f\|_{V(\Pi)}^2 = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x) - f(y)|^2 |\Pi(x, y)|^2 dxdy.$$  

Here we identify functions which differ by a constant. If a function $f$ is such that $S_f \in L^2(\text{Conf}(\mathbb{R}), \mathbb{P})$, we have

$$E|\overline{S}_f|^2 = \text{Var} S_f = \|f\|_{V(\Pi)}^2.$$  

In particular, this is the case if the function $f$ is bounded and has compact support. Thus, the correspondence $f \mapsto \overline{S}_f$ is an isometric embedding of a dense subset of $V(\Pi)$ into $L_2(\text{Conf}(\mathbb{R}), \mathbb{P})$. It therefore admits a unique isometric extension onto the whole space $H$, and we get

Proposition A.1. There exists a unique linear isometric embedding

$$\overline{S} : V(\Pi) \hookrightarrow L_2(\text{Conf}(\mathbb{R}), \mathbb{P}), \quad \overline{S} : f \mapsto \overline{S}_f$$

such that
1. $\mathbb{E}S_f = 0$ for all $f \in V(\Pi)$;

2. if $S_f \in L_1(\text{Conf}(\mathbb{R}), \mathbb{P})$, then $S_f$ is given by (A.2).

For more details see Proposition 4.1 in [2].

Let $\mathbb{P}^a$ be the reduced Palm measure of the measure $\mathbb{P}$, conditioned at the point $a$.

**Corollary A.2.** 1. For $\hat{\rho}_1$-almost every $a \in \mathbb{R}$ there exists a function $\overline{S}_a : \text{Conf}(\mathbb{R}) \mapsto \mathbb{R}$ such that the convergence (2.4) takes place, uniformly in $a \in \mathbb{R}$ ranging in a compact set.

2. The convergence (3.14) takes place for $p = 2$, uniformly in $a, b \in \mathbb{R}$ ranging in a compact set.

**Proof.** In [17] it is proven that the reduced Palm measure $\mathbb{P}^a$ coincides for $\hat{\rho}_1$-almost every $a \in \mathbb{R}$ with the determinantal measure associated with the kernel $\Pi^a$, given by

$$\Pi^a(x, y) := \frac{\Pi(x, a) \Pi(a, y)}{\Pi(a, a)} \quad \text{if} \quad \Pi(a, a) \neq 0$$  

(A.3)

and $\Pi^a(x, y) := \Pi(x, y)$ if $\Pi(a, a) = 0$. It can be checked directly that the kernel $\Pi^a$ satisfies Assumption 2. Then item 1 follows from Proposition A.1 applied to the kernel $\Pi^a$. Indeed, we have

$$\overline{S}^R_{\delta} = S^R_{f^a} \quad \text{with} \quad f^R_{\delta}(x) = 2 \frac{a - x}{a - x} \mathbb{1}\{|x| < R, |x - a| > \delta\}.$$

Clearly, $f^R_{\delta} \rightarrow f_a$ in $V(\Pi^a)$ uniformly in $a$ as $R \rightarrow \infty$, $\delta \rightarrow 0$, where

$$f_a(x) := \frac{2}{a - x}.$$

Then Proposition A.1 implies the desired convergence with $\overline{S}_a := \overline{S}^R_{f^a}$. Item 2 can be proven by the same argument. □

### A.2 Multiplicative functionals

For a bounded nonnegative function $g$ with compact support we define the *multiplicative functionals* $\Psi_g, \tilde{\Psi}_g : \text{Conf}(\mathbb{R}) \mapsto \mathbb{R}$ as

$$\Psi_g = \prod_{x \in X} g(x) = e^{S_{\log g}} \quad \text{and} \quad \tilde{\Psi}_g = e^{S_{\log g}}.$$  

(A.4)

Here we set $\Psi_g(X) = \tilde{\Psi}_g(X) = 0$ if there is $x \in X$ such that $g(x) = 0$. In view of Proposition A.1, we can extend the multiplicative functional $\tilde{\Psi}_g$ to functions $g$ satisfying $\|\log g\|_{V(\Pi)} < \infty$. If $\tilde{\Psi}_g \in L^1(\text{Conf}(\mathbb{R}), \mathbb{P})$ we define the normalized multiplicative functional as

$$\Psi_g = \frac{\tilde{\Psi}_g}{\mathbb{E}\tilde{\Psi}_g}.$$  

(A.5)

Fix positive numbers $\alpha > 0, \varepsilon > 0, M > \varepsilon$ and two bounded Borel subsets $B^1, B^2 \in \mathcal{B}(\mathbb{R})$ satisfying

$$\|\mathbb{P}_{B^1 \cup B^2}\| < 1.$$

Denote by $G$ the set of nonnegative Borel functions $g : \mathbb{R} \mapsto \mathbb{R}$ satisfying
1. \( \{ x : g(x) < \varepsilon \} \subset B^1 \);
2. \( \{ x : g(x) > M \} \subset B^2 \);
3. \( \int_{B^2} |g(x)|^{1+\alpha} \Pi(x, x) dx + \int_{R \setminus B^2} |g(x) - 1|^2 \Pi(x, x) dx < \infty \).

We metrize \( \mathcal{G} \) by equipping it with the distance

\[
d_{\mathcal{G}}(g_1, g_2) = \int_{B^2} |g_1(x) - g_2(x)|^{1+\alpha} \Pi(x, x) dx + \int_{R \setminus B^2} |g_1(x) - g_2(x)|^2 \Pi(x, x) dx.
\]

Then \( \mathcal{G} \) becomes a complete separable metric space. Below we formulate Proposition 4.3 from [2].

**Proposition A.3.** For any \( \alpha' : 0 < \alpha' < \alpha \), the correspondences \( g \rightarrow \tilde{\Psi}_g \), \( g \rightarrow \hat{\Psi}_g \) induce continuous mappings from \( \mathcal{G} \) to \( L_{1+\alpha'}(\text{Conf}(\mathbb{R}), \mathbb{P}) \).

**Corollary A.4.** 1. Assertion of Lemma 3.1(1) is satisfied.

2. For \( \hat{\rho}_1 \)-almost all \( a \in \mathbb{R} \), convergences (3.16), (3.18) and (3.19) take place and are uniform in \( a, b \in \mathbb{R} \) as \( a \) range in a compact set and \( b \) satisfies \( |b - a| < \theta \ll 1 \).

**Proof.** Item 1. As we have already explained in the proof of Corollary A.2, the reduced Palm measure \( \mathbb{P}^a \) coincides for \( \hat{\rho}_1 \)-almost all \( a \in \mathbb{R} \) with the determinantal measure associated with the kernel \( \Pi^a \) given by (A.3). Moreover, \( \Pi^a \) satisfies Assumption 2. Indeed, one can check that \( \Pi^a \) is an orthogonal projection kernel onto the space \( L^a \subset L^2(\mathbb{R}, dx) \) defined as \( L^a := \{ \varphi \in L : \varphi(a) = 0 \} \), see Section 2.16 of [2]. Then Assumptions 2(1,2) follow. Assumptions 2(3,4) are obvious. Let

\[
g_{a, b}(x) := \left( \frac{x - b}{x - a} \right)^2 \quad \text{and} \quad g_{a, b}^{R, \delta}(x) := (g_{a, b}(x) - 1) \mathbb{I}_{\{|x| > R, |x-b| > \delta, |x-a| > \delta\}} + 1.
\]

Using that the function \( \Pi^a_{\text{diag}}(x) := \Pi^a(x, x) \) has zero of second order at the point \( x = a \), we find that \( g_{a, b}, g_{a, b}^{R, \delta} \in \mathcal{G} \), for an appropriate choice of the sets \( B^1, B^2 \) and numbers \( \alpha, M \) (independent from \( R, \delta \) and \( a, b \) ), where the space \( \mathcal{G} \) is defined with respect to the kernel \( \Pi^a \). Moreover,

\[
d_{\mathcal{G}}(g_{a, b}^{R, \delta}, g_{a, b}) \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty, \delta \rightarrow 0,
\]

(A.6)

uniformly in \( a, b \) as they range in compact sets. Then Proposition A.3 implies that

\[
\overline{\Psi}^{R, \delta}_{g_{a, b}} \rightarrow \overline{\Psi}_{g_{a, b}} \quad \text{as} \quad R \rightarrow \infty, \delta \rightarrow 0 \quad \text{in} \quad L_{1+\alpha'}(\text{Conf}(\mathbb{R}), \mathbb{P}^a),
\]

for any \( 0 < \alpha' < \alpha \). Since \( \overline{\Psi}^{R, \delta}_{a, b} = \overline{\Psi}^{R, \delta}_{g_{a, b}} \), we get the desired convergence with \( \overline{\Psi}_{a, b} = \overline{\Psi}_{g_{a, b}} \).

Its uniformity in \( a, b \) follows from the uniformity of convergence (A.6) by a direct analysis of the proof of Proposition A.3 (i.e. of Proposition 4.3 from [2]).

**Item 2.** Similarly with the last item, the desired convergences follow by applying Proposition A.3 with the kernel \( \Pi^a \). The convergence (3.18) takes place for arbitrary \( p > 0 \) since we assume \( |a - b| \ll 1 \), so that the functions \( g_{1}^{R} \) and \( g_{1} \) are bounded, and then \( \alpha \) can be chosen arbitrarily large.
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