2D PROBLEMS IN GROUPS

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Abstract. We investigate a conjecture about stabilisation of deficiency in finite index subgroups and relate it to the D2 Problem of C.T.C. Wall and the Relation Gap problem. We verify the pro-p version of the conjecture, as well as its higher dimensional abstract analogues.

Given a finitely presented group $G$, the deficiency $\delta(G)$ of $G$ is defined as the maximum of $|X| - |R|$ over all presentations $G = \langle X \mid R \rangle$. We related deficiency of a group with 2-dimensionality in [8] and proposed the following conjecture.

2D Conjecture ([8]). Let $G$ be a residually finite finitely presented group such that $\delta(H) - 1 = [G : H](\delta(G) - 1)$ for every subgroup $H$ of finite index in $G$. Then $G$ has a finite 2-dimensional classifying space $K(G, 1)$.

In this paper, we relate the above conjecture with two well-known problems in topological group theory: Wall’s D2 problem and the Relation Gap problem. The main purpose of the paper is to explain the implications affirmative D2 problem $\Rightarrow$ no relation gap $\Rightarrow$ 2D conjecture

1. Background

Let $G$ be a finitely presented group. Set $d(G)$ to be the cardinality of a minimal generating set of $G$.

We denote by $b_1(G) = \dim_\mathbb{Q} H_1(G, \mathbb{Q})$ and note that $\delta(G) \leq b_1(G) \leq d(G)$. Starting with a presentation $\langle X \mid R \rangle$ for $G$, one obtains a Schreier presentation for $H$ with $[G : H](|X| - 1) + 1$ generators and $[G : H]|R|$ relations showing that

$$\delta(H) - 1 \geq [G : H](\delta(G) - 1).$$

We are interested in the situation when the above inequality is in fact equality for every finite index subgroup $H$ of $G$.

We next introduce the invariant $\mu_n(G)$ of Swan [13]. Let $n \in \mathbb{N}$. A partial free resolution of $\mathbb{Z}$ of length $n$ is an exact sequence

$$\mathcal{F} : (\mathbb{Z}G)^{f_n} \to (\mathbb{Z}G)^{f_{n-1}} \to \cdots \to (\mathbb{Z}G)^{f_0} \to \mathbb{Z} \to 0$$

and we define $\mu_n(\mathcal{F}) = \sum_{i=0}^{n}(-1)^{n-i} f_i$.

Recall the well-known Morse inequalities.
Proposition 1. Let $n \in \mathbb{N}$ and $F$ be a partial free resolution (1) as above. Then

$$\sum_{i=0}^{n} (-1)^{n-i}b_i(G) \leq \mu_n(F).$$

R. Swan [13] defined the following invariant while studying free resolutions of modules of finite groups.

Definition 2. Let $n \in \mathbb{N}$. The invariant $\mu_n(G)$ is defined as the minimum of $\mu_n(F)$ as $F$ ranges over all partial free resolutions $F$ of $Z$.

Given a presentation of $G$ with $e_1$ generators and $e_2$ relations one has the partial free resolution

$$(ZG)^{e_2} \xrightarrow{\partial_2} (ZG)^{e_1} \xrightarrow{\partial_1} ZG \xrightarrow{\partial_0} Z \to 0$$

arising as the cellular chain complex of the universal cover of the presentation complex of $G$. By taking a presentation which realizes the deficiency of $G$ we obtain $\mu_2(G) \leq 1 - \delta(G)$. The case $n = 2$ of the Morse inequalities applied to (2), together with $b_0(G) = 1$ gives the well-known inequality $\delta(G) \leq b_1(G) - b_2(G)$.

1.1. Groups with two dimensional classifying spaces. The deficiency is easy to compute for groups which have finite two-dimensional classifying spaces. Examples of such groups are surface groups or more generally, torsion-free one relator groups and direct products of two free groups.

Lemma 3. If a group $G$ has a finite two-dimensional space $K(G, 1)$, then $\delta(G) = 1 - \chi(G)$ and consequently, $\delta(H) - 1 = |G : H|/(\delta(G) - 1)$ for every subgroup $H$ of finite index in $G$.

For example $\delta(F_n \times F_m) = -(n - 1)(m - 1)$ while the deficiency of a torsion-free one relator group defined on $d$ generators is $d - 2$.

The 2D Conjecture stated in the introduction proposes that the converse of Lemma 3 holds. Note that its 1-dimensional analogue is true as shown by R. Strebel [12] (see also [11, Theorem 7] for a different perspective).

Proposition 4 ([12]). Let $G$ be a finitely generated residually finite group. Then $G$ is a free group if and only if $d(H) - 1 = |G : H|/(d(G) - 1)$ for every subgroup $H$ of finite index in $G$.

Strebel proved Proposition 4 as an answer to a question of Lubotzky and van den Dries [10], who had shown that its analogue does not hold in the class of profinite groups. At the same time Lubotzky [9, Proposition 4.2] proved that the analogue of Proposition 4 is true in the class of pro-$p$ groups. We will return to pro-$p$ groups in section 5 below.

We remark that the 2D conjecture is closely connected with gradients in groups and their $L^2$ cohomology. The following basic result characterizes groups $G$ with two dimensional classifying spaces in terms of their $L^2$ Betti numbers $\beta_i(G)$.
Lemma 5 ([8]). Let \( G \) be an infinite finitely presented group. Then \( \delta(G) - 1 \leq \beta_1(G) - \beta_2(G) \) with equality if and only if \( G \) has a two dimensional classifying space.

In particular any counterexample to the 2D conjecture must be a group \( G \) with deficiency gradient strictly less than \( \beta_1(G) - \beta_2(G) \), see [8] for more details on this connection.

2. Wall’s D2 Problem

Wall’s D2 problem is a generalisation of the Eilenberg Ganea Conjecture and belongs to the class of questions that explore links between homological and geometric dimensions. A finite CW-complex \( X \) is said to be a D2 complex if it has cohomological dimension 2. The D2 Problem for a finitely presented group \( G \) asks if every finite D2 complex with fundamental group \( G \) is homotopy equivalent to a finite 2-complex. If the answer is affirmative we shall say that \( G \) has the D2 property. The problem was proposed by C.T.C. Wall in 1965 [14] and little is known about it except in the case when \( G \) is finite, free or abelian, see [7].

The Eilenberg-Ganea Conjecture asks if every group of cohomological dimension 2 is of geometric dimension 2. Note that a group of cohomological dimension 2 does not necessarily have a finite classifying space, as famously shown by M. Bestvina and N. Brady [2]. However, if one assumes that a group \( G \) of cohomological dimension 2 has a finite classifying space \( X \), then \( X \) is a D2 complex. If in addition \( G \) has the D2 property, then \( X \) is homotopy equivalent to a finite 2-complex. So, \( G \) has geometric dimension two, as predicted by Eilenberg-Ganea.

3. The Relation Gap problem

Suppose that a finitely presented group \( G \) is given by the quotient \( F/N \) where \( F \) is free on the group generators \( X \) and \( N \) is normally generated in \( F \) by the relators \( R \subseteq F \). The action of \( F \) by conjugation on \( N \) induces an action of \( G \) on the abelianisation \( N^{ab} \) of \( N \). This makes \( N^{ab} \) into a \( G \)-module called the relation module of the presentation. Evidently, the \( G \)-module \( N^{ab} \) can be generated by \( |R| \) elements and so the \( G \)-rank of \( N^{ab} \), written \( d_G(N^{ab}) \), satisfies \( d_G(N^{ab}) \leq d_F(N) \), where \( d_F(N) \) is the minimum number of normal generators required for \( N \).

A presentation is said to have a relation gap if \( d_G(N^{ab}) \neq d_F(N) \) and the relation gap problem asks, if there exists a finitely presented group with a relation gap. As with the D2 problem, very little is known about the relation gap problem and most proposed counterexamples are not torsion-free, see [5].

We give a proof to the following.

Theorem 6. A finitely presented group \( G \) with the D2 property does not have a relation gap for presentations realizing \( \delta(G) \).
This may be known to topological group theorists but we have not found it in the literature. There is a result of Dyer [4] Theorem 3.5 with the same statement but with the additional hypothesis $H^3(G, ZG) = 0$.

We need the following.

**Proposition 7** ([3] Proposition 4.3, or [3], Remark 1.3). Let $G$ be a finitely presented group with the D2 property. Then $\mu_2(G) = 1 - \delta(G)$.

For completeness we give a proof of Proposition 7 following [3], based on the following theorem of Wall.

**Theorem 8** ([14], Theorem 4). Let $X$ be a connected CW-complex, $G = \pi_1(X)$ and let $A_*$ be a positive free chain complex equivalent to the cellular chain complex $C_*(X)$ of the universal cover of $X$. Let $K^2$ be a connected CW-complex with fundamental group $G$. There exists another CW complex $Y$ and a homotopy equivalence $h : Y \to X$ such that $Y$ is obtained from $K^2$ by adding 2-cells and 3-cells at the base point to obtain a D2 complex $Y_0$ and then further cells such that $C_*(Y, Y_0)$ is the part of $A_*$ in dimension $\geq 3$.

If the symbol $\alpha_i$ denotes the number of $i$-cells or of generators in degree $i$ then

$$\begin{align*}
\alpha_2(Y_0 - K^2) &= \alpha_2(A) + \alpha_1(K) + \alpha_0(A), \\
\alpha_3(Y^0 - K^2) &= \alpha_2(K) + \alpha_1(A) + \alpha_0(K).
\end{align*}$$

**Proof of Proposition 7**. Let

$$(ZG)^{f_2} \to (ZG)^{f_1} \to (ZG)^{f_0} \to Z \to 0$$

be a partial free resolution of $Z$ with $f_2 - f_1 + f_0 = \mu_2(G)$. Extend this to a free resolution $A_*$ and let $X$ be a CW complex which is a classifying space for $G$. Now $A_*$ is homotopy equivalent to the cellular complex $C_*(X)$ of $X$ and therefore starting with any finite presentation complex $K^2$ for $G$ we can apply Theorem 8 above. In particular there exists a finite 3-dimensional D2 complex $Y_0$ with $\pi_1(Y_0) = G$ and we compute

$$\chi(Y_0) = \sum_{i=0}^{3} (-1)^i \alpha_i(Y_0) = \sum_{i=0}^{2} (-1)^i \alpha_i(A_*) = \mu_2(G).$$

We are assuming that the D2 Problem has positive solution for $G$, therefore $Y_0$ is homotopy equivalent to a finite 2-dimensional complex $L$. We have $G = \pi_1(K) = \pi_1(L)$ and $\chi(L) = \chi(Y_0) = \mu_2(G)$. Hence

$$\delta(G) - 1 \geq \alpha_1(L) - \alpha_0(L) - \alpha_2(L) = -\chi(L) = -\mu_2(G).$$

Therefore $1 - \delta(G) \leq \mu_2(G)$. Since the opposite inequality $\mu_2(G) \leq 1 - \delta(G)$ always holds we have equality.

**Proof of Theorem 6**. Let $G$ be a group with the D2 property. Take a presentation $\langle X \mid R \rangle$ for $G$ with $e_1$ generators and $e_2$ relations such that $e_1 - e_2 = \delta(G)$. We have $G \cong F/N$ where $F$ is a free group of rank $e_1$.
on $X$ and $N$ is the normal closure of the relations $R$. Since $e_1 - e_2$ realises the deficiency of $G$ it follows that $e_2 = d_F(N)$. Let $M = N^{ab}$ be the relation module of this presentation. Recall the chain complex $\{2\}$ above. We have $M \cong \ker \partial_1 = \text{im} \partial_2$. If $M$ has relation gap then $u := d_G(M) < e_2$ and in particular there is a surjection of $\mathbb{Z}G$ modules $f : (\mathbb{Z}G)^u \to \ker \partial_1$. Therefore we can amend the partial resolution above to

$$(\mathbb{Z}G)^u \xrightarrow{f} (\mathbb{Z}G)^{e_1} \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\partial_2} \mathbb{Z} \to 0.$$  

This gives $\mu_2(G) \leq 1 + u - e_2 < 1 - \delta(G)$ contradicting Proposition $[7]$. Therefore presentations of $G$ which realize $\delta(G)$ have no relation gap.

\[ \square \]

4. Relation Gap Problem v.s. 2D Conjecture

**Theorem 9.** If $G$ is a counterexample to the 2D conjecture then there exists a finite index subgroup $H$ of $G$ such that $H$ has a presentation with relation gap.

**Proof.** Suppose that $G$ is a finitely presented group; assume that $X$ is a presentation 2-complex for $G$ realising the deficiency $\delta(G)$. If $X$ is not aspherical, then by Whitehead’s Theorem, $H_2(\tilde{X}) \neq 0$. Let $e_i$ denote the number of $i$-cells in $X$. So $\delta(G) - 1 = e_1 - e_2 - 1$. We have the exact sequence of $G$-modules

$$F : 0 \to H_2(\tilde{X}) \to \mathbb{Z}G^{e_2} \xrightarrow{\partial_2} \mathbb{Z}G^{e_1} \xrightarrow{\partial_1} \mathbb{Z}G \to \mathbb{Z} \to 0$$

where $H_2(\tilde{X}) = \ker \partial_2$. The relation module $R$ associated to $X$ is isomorphic to $\ker \partial_1 = \text{im} \partial_2 \cong \mathbb{Z}G^{e_2}/H_2(\tilde{X})$. Take a non-zero element $\rho$ of $H_2(\tilde{X})$. As an element of $\mathbb{Z}G^{e_2}$, $\rho$ has a representation as a non-zero tuple $(a_1, \ldots, a_{e_2})$, where each $a_i$ is a linear combination in $\mathbb{Z}G$ with support $C_i$ as follows:

$$a_i = \sum_{g \in C_i} a_i^g g$$

Let $C = \cup C_i$; this is a finite collection of elements of $G$. There exists a finite index normal subgroup of $G$, say $H$ such that the elements of $C$ project to distinct cosets in $G/H$. The natural structure of $\mathbb{Z}G$ as a $ZH$-module makes $F$ into the chain complex for the action of $H$ on $\tilde{X}$. Let $E$ be a collection of coset representatives for $H$ in $G$ such that $C \subseteq E$. Consider

$$\mathbb{Z}G^{e_2} = \left( \bigoplus_{g \in E} \mathbb{Z}H.g \right)^{e_2} \cong \mathbb{Z}H^{e_2[G:H]}$$

Let $d$ be the greatest common divisor of the integers $\{a_i^g \mid g \in C_i, \ i = 1, 2, \ldots, e_2\}$. Then $\rho = d\rho'$, where $\rho' \in \mathbb{Z}G^{e_2}$ and all its coefficients are coprime. As $\rho$ is an element of $\ker \partial_2$, $\rho$ is a homomorphism of torsion-free abelian groups, we deduce that $\rho'$ is also an element of $\ker \partial_2$. Therefore, we can assume that $d = 1$. 

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Consider the presentation for $H$ arising from the action of $H$ on $\tilde{X}$: this presentation has $(e_1 - 1)[G : H] + 1$ generators and $e_2[G : H]$ relations. The relation module $R'$ for this presentation of $H$ is the restriction $R'_{\downarrow H}$ of the relation module $R$, wherein $\rho$ represents the zero element. We have assumed that the coefficients of $\rho$ are co-prime and so $\rho$ is a primitive element in the abelian group $(\mathbb{Z}E)^{e_2}$ containing its support in $\mathbb{Z}H^{e_2[G:H]}$. Consequently $R' \cong \mathbb{Z}H^{e_2[G:H]}/H_2(\tilde{X})$ can be generated by fewer than $e_2[G : H]$ elements as an $H$-module. If the above presentation of $H$ has no relation gap then it needs strictly fewer than $e_2[G : H]$ relations and hence $\delta(H) - 1 > [G : H](e_1 - e_2 - 1) = [G : H](\delta(G) - 1)$, contradiction.

Therefore if $X$ is not aspherical some finite index subgroup of $G$ has a relation gap. \hfill \Box

We note that the argument above gives the following general criterion for freeness of $\mathbb{Z}G$-modules.

**Proposition 10.** Let $G$ be a residually finite group and let $M$ be a finitely generated $\mathbb{Z}G$-module. Assume that $M$ is torsion free as an abelian group and let $f : (\mathbb{Z}G)^r \to M$ be a surjective homomorphism of $\mathbb{Z}G$ modules. Then $f$ is an isomorphism if and only of $d_H(M) = r[G : H]$ for each subgroup $H$ of finite index in $G$.

In particular $M$ is a free module if and only if $d_H(M) = [G : H]d_G(M)$ for each subgroup $H$ of finite index in $G$.

**Proof.** If $f$ is not injective we can find an element $\rho = (a_1, \ldots, a_r) \in \ker f$ with support $C = \cup_{i=1}^r C_i$ and coefficients $a_i^j \in \mathbb{Z}$ defined by $a_i = \sum_{g \in C_i} a_i^j g$. Since $M$ is torsion free we can assume that the greatest common divisor of all integers $a_i^j$ is 1. There is a finite index subgroup $H$ of $G$ such that $C$ projects injectively into $G/H$ and arguing in the same way as in the proof of Theorem 9, we deduce $d_H(M) < r[G : H]$, contradiction. Therefore $f$ is a bijection and $M$ is a free module. \hfill \Box

5. The 2D Conjecture for pro-$p$ groups.

In this section $G$ denotes a finitely presented pro-$p$ group, where we consider presentations in the category of pro-$p$ groups. We keep the notation $\delta(G)$ for the maximum of $|X| - |R|$ over all pro-$p$ presentations $\langle X, R \rangle$ of $G$.

Below we prove the analogue of the 2D conjecture for $G$:

**Theorem 11.** Let $G$ be a finitely presented pro-$p$ group. The following are equivalent:

(i) $\delta(G) - 1 = [G : H](\delta(H) - 1)$ for every open subgroup $H$ of $G$.

(ii) $cd_p(G) \leq 2$.

It will be interesting to find a characterization of the finitely presented profinite groups $G$ for which the condition (i) above holds. Note that already the 1-dimensional situation for profinite groups is quite different. See [10] for examples of profinite groups which satisfy Schreier’s rank-index formula for all open subgroups, but are not projective.
We have the partial free resolution arising from the presentation of \( \delta \) with \( \nu \) normal subgroup \( \langle d \rangle \) of \( \langle F \rangle \) of type \( F \).

\[ \text{Proof.} \]

For pro-p groups \( \delta(G) = \dim_{\mathbb{F}_p} H^1(G) - \dim_{\mathbb{F}_p} H^2(G) \) where we write \( H^i(G) = H^i(G, \mathbb{F}_p) \), see [11] I.4.2 & I.4.3. Hence, if \( cd_p(G) \leq 2 \) then \( \delta(G) - 1 = -\chi(G) \), the pro-p Euler characteristic of \( G \) and therefore (1) holds.

Conversely, suppose that (1) holds and let \( \nu_i = \dim_{\mathbb{F}_p} H^i(G) \) for \( i = 1, 2 \). We have the partial free resolution

\[ \mathbb{F}_p[[G]]^{e_2} \xrightarrow{d_2} \mathbb{F}_p[[G]]^{e_1} \xrightarrow{d_1} \mathbb{F}_p[[G]] \longrightarrow \mathbb{F}_p \longrightarrow 0, \]

arising from the presentation of \( G \) with \( e_1 \) generators and \( e_2 \) relations. We claim that \( J := \ker d_2 \) must be zero. Suppose not. Then we can find an open normal subgroup \( N \) of \( G \) such that the image \( \tilde{J} \) of \( J \) under the reduction \( (\mathbb{F}_p[[G]])^{e_2} \rightarrow (\mathbb{F}_p[G/N])^{e_2} \) is non-zero.

Note that the free \( \mathbb{F}_p[[G]] \) resolution above is also a partial free resolution of \( \mathbb{F}_p[[N]] \) modules. We apply the functor \( \text{Hom}_N(-, \mathbb{F}_p) \) to the above resolution, using \( \text{Hom}_N(\mathbb{F}_p G, \mathbb{F}_p) \simeq (\mathbb{F}_p[G/N])^* \), where by \( V^* \) we denote the dual of the vector space \( V \) over \( \mathbb{F}_p \). We obtain the chain complex

\[ 0 \leftarrow \tilde{J}^* \leftarrow (\mathbb{F}_p[G/N]^*)^{e_2} \xrightarrow{d_2^*} (\mathbb{F}_p[G/N]^*)^{e_1} \xrightarrow{d_1^*} \mathbb{F}_p[G/N]^* \leftarrow 0. \]

which is exact at \( \tilde{J}^* \) and whose homology group in degree \( i \) is \( H^i(N) \). Therefore

\[ \delta(N) - 1 = \sum_{i=0}^{2} (-1)^{i+1} \dim H^i(N) = \]

\[ = (e_1 - e_2 - 1)[G : N] + \dim \tilde{J}^* > [G : N](\delta(G) - 1), \]

since \( \tilde{J}^* \neq \{0\} \), a contradiction to (i). Therefore \( J = \{0\} \) and \( cd_p(G) \leq 2 \). \( \square \)

6. Higher dimensional analogues

Deficiency can be viewed as one of the partial Euler characteristics, which are defined as follows:

Let \( n \geq 2 \) be an integer and let \( G \) be a group of type \( F_n \). Define \( \nu_n(G) \) to be the minimum of \(-1)^n \chi(X) \) where \( X \) is a finite CW complex of dimension \( n \) such that \( \pi_1(X) = G \) and \( \pi_i(X) = \{0\} \) for \( i = 2, 3, \ldots, n-1 \) (i.e its universal cover \( \tilde{X} \) is \( n-1 \)-connected). Note that \( \nu_2(G) = 1 - \delta(G) \) and for completeness we define \( \nu_1(G) = d(G) - 1 \). From the definition of \( \nu_n \) and \( \mu_n \) we have \( \nu_n(G) \geq \nu_n(G) \) for all \( n \). We note that Theorem [8] above implies

**Proposition 12.** \( \nu_n(G) = \mu_n(G) \) when \( n \geq 3 \).

Here we prove the higher dimensional analogue of the 2D conjecture.

**Theorem 13.** Let \( n > 2 \) be an integer and let \( G \) be a residually finite group of type \( F_n \). Then \( G \) has finite classifying space of dimension \( n \) if and only if \( \nu_n(H) = \nu_n(G)[G : H] \) for every subgroup \( H \) of finite index in \( G \).
Proof. Suppose that $X$ is an $n$-dimensional $K(G,1)$ complex for $G$, then 
$$\nu_n(G) \leq (-1)^n \chi(X)$$ from the definition of $\nu_n(G)$. On the other hand the 
Morse inequalities give 
$$\nu_n(G) \geq \sum_{i=0}^{n} (-1)^{n-i} b_i(G) = (-1)^n \chi(X).$$ 
Therefore $\nu_n(G) = (-1)^n \chi(X)$ and in the same way $\nu_n(H) = (-1)^n \chi(X')$, where 
$X'$ is the cover of $X$ corresponding to $H$. Since $\chi(X') = [G : H] \chi(X)$ the 
equality follows.

For the other direction we could use Proposition \[12\]. Instead we take a 
more elementary approach and argue directly using Proposition \[10\].

Suppose that $\nu_n(H) = \nu_n(G)[G : H]$ for every subgroup $H$ of finite index 
in $G$. Let $X$ be the $n$-dimensional CW complex which realises $\nu_n(G)$. Let 
e$_i$ be the number of $i$-dimensional cells of $X$ and let 
$$F_n \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} F_0 \rightarrow \mathbb{Z} \rightarrow 0$$ 
with $F_i = (\mathbb{Z}G)^{e_i}$ be the chain complex of the universal cover $\tilde{X}$. By the 
Hurewicz theorem $\pi_n(X) \simeq H_n(X) = \ker \partial_n$ and thus $X$ is aspherical if and 
only if $\partial_n$ is injective.

Suppose ker $\partial_n \neq \{0\}$ and consider $M = \ker \partial_{n-1} = \text{im}\partial_n$. We apply 
Proposition \[14\] to the $\mathbb{Z}G$-homomorphism $\partial_n : F_n \rightarrow M$, where $F_n = (\mathbb{Z}G)^{e_n}$ to 
deduce that $u := d_H(M) < e_n[G : H]$ for some subgroup $H$.

Choose a set of generators $\alpha_1, \ldots, \alpha_u$ of the $\mathbb{Z}H$-module $M$. Let $Y$ be 
the cover of $X$ with degree $[Y : X] = [G : H]$ and $\pi_1(Y) = H$. Let 
p : $\tilde{X} \rightarrow Y$ be the universal covering map. Denote by $Y_n$ the 
$(n-1)$-skeleton of $Y$ and $\tilde{X}$ respectively and observe that $\pi_{n-1}(Y_n) \simeq 
H_{n-1}(\tilde{X}_n) = \ker \partial_{n-1} = M$ by the Hurewicz theorem. Therefore for each 
i = 1, \ldots, u we can find a cellular map $j_i : S^{n-1} \rightarrow \tilde{X}_n$ representing $\alpha_i$. 
This means that $H_{n-1}(j_i)$ sends the generator of $H_{n-1}(S^{n-1})$ to the element 
$\alpha_i \in H_{n-1}(\tilde{X}_n) = M$.

We now attach $n$-dimensional cells $\sigma_i^n$ to $Y_n$ for $i = 1, \ldots u$ with boundary 
attaching maps 
$$S^{n-1} \xrightarrow{j_i} \tilde{X}_n \xrightarrow{p} Y_n$$ 
and define $Z := Y_n \cup \bigcup_{i=1}^{u} \sigma_i^n$. Note that since $Y_n = Z^n$ we have 
$\pi_i(Z) = \pi_i(Y)$ for $i = 1, \ldots n - 2$. We claim that $\pi_{n-1}(Z) = \{0\}$. It 
is sufficient to prove that $H_{n-1}(\tilde{Z}) = \{0\}$ for the universal cover $\tilde{Z}$ of $Z$. 
Since the $(n-1)$-skeletons of $Z$ and $X$ coincide, the boundary maps $\partial_{n-1}$ on the 
chain complex of $\tilde{Z}$ and $\tilde{X}$ are the same and hence ker $\partial_{n-1} = M$. On the 
other hand the boundary map $\partial_n : (\mathbb{Z}H)^u \rightarrow M$ of degree $n$ of the 
chain complex of $\tilde{Z}$ is surjective since by construction its image contains the 
generators $\alpha_i$. Therefore $H_{n-1}(\tilde{Z}) = \{0\}$ and so $\tilde{Z}$ is $(n-1)$-connected 
as claimed.

Note that $Z$ has $[G : H] e_i$ cells in dimension $i$ for $i = 0, 1, \ldots, n - 1$ and 
u cells in dimension $n$. Since $u < e_n[G : H]$ it follows that 
$$\nu_n(H) \leq (-1)^n \chi(Z) = u + \sum_{i=0}^{n-1} (-1)^{n-i} e_i[G : H] \nu_n(G)[G : H],$$
contradiction. Therefore $H_n(\tilde{X}) = \{0\}$ and $X$ is a finite $K(G,1)$-complex of dimension $n$. □

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