Perturbative gravity in the causal approach

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Received 16 March 2009, in final form 14 October 2009
Published 15 December 2009
Online at stacks.iop.org/CQG/27/015013

Abstract
Quantum theory of the gravitation in the causal approach is studied up to the second order of perturbation theory in the causal approach. We emphasize the use of cohomology methods in this framework. After describing in detail the mathematical structure of the cohomology method we apply it in three different situations: (a) the determination of the most general expression of the interaction Lagrangian; (b) the proof of gauge invariance in the second order of perturbation theory for the pure gravity system—massless and massive; (c) the investigation of the arbitrariness of the second-order chronological products compatible with renormalization principles and gauge invariance (i.e. the renormalization problem in the second order of perturbation theory). In case (a) we investigate pure gravity systems and the interaction of massless gravity with matter (described by scalars and spinors) and massless Yang–Mills fields. We obtain a difference with respect to the classical field theory due to the fact that in quantum field theory one cannot enforce the divergenceless property on the vector potential and this spoils the divergenceless property of the usual energy–momentum tensor. To correct this one needs a supplementary ghost term in the interaction Lagrangian. In all three case, the computations are more simple than by the usual methods.

PACS numbers: 04.06.m, 11.10.Gh

1. Introduction

The general framework of perturbation theory consists in the construction of the chronological products such that Bogoliubov axioms are verified [2, 6, 5, 10]; for every set of Wick monomials $W_1(x_1), \ldots, W_n(x_n)$ acting in some Fock space $\mathcal{H}$ generated by the free fields of the model $\mathcal{H}$ one associates the operator $T_\mathcal{H}(W_1(x_1), \ldots, W_n(x_n))$; all these expressions are in fact distribution-valued operators called chronological products. Sometimes it is convenient to use another notation: $T(W_1(x_1), \ldots, W_n(x_n))$. The construction of the chronological products
can be done recursively according to Epstein–Glaser prescription [6, 7] (which reduces the induction procedure to a distribution splitting of some distributions with causal support) or according to Stora prescription [15] (which reduces the renormalization procedure to the process of extension of distributions). These products are not uniquely defined, but there are some natural limitations on the arbitrariness. If the arbitrariness does not grow with \( n \), we have a renormalizable theory. An equivalent point of view uses retarded products [18].

Gauge theories describe particles of higher spin. Usually such theories are not renormalizable. However, one can save renormalizability using ghost fields. Such theories are defined in a Fock space \( \mathcal{H} \) with indefinite metric, generated by physical and unphysical fields (called ghost fields). One selects the physical states assuming the existence of an operator \( Q \) called the gauge charge which verifies \( Q^2 = 0 \) and such that the physical Hilbert space is by definition \( \mathcal{H}_{\text{phys}} \equiv \ker(Q)/\im(Q) \). The space \( \mathcal{H} \) is endowed with a grading (usually called the ghost number) and by construction the gauge charge is raising the ghost number of a state. Moreover, the space of Wick monomials in \( \mathcal{H} \) is also endowed with a grading which follows by assigning a ghost number to every one of the free fields generating \( \mathcal{H} \). The graded commutator \( d_0 \) of the gauge charge with any operator \( A \) of fixed ghost number

\[
d_0 A = [Q, A]
\]

is raising the ghost number by a unit. It means that \( d_0 \) is a co-chain operator in the space of Wick polynomials. From now on \([\cdot, \cdot]\) denotes the graded commutator.

A gauge theory assumes also that there exists a Wick polynomial of null ghost number \( T(x) \) called the interaction Lagrangian such that

\[
[Q, T(x)] = i \partial_\mu T^\mu(x)
\]  

(1.2)

for some other Wick polynomials \( T^\mu \). This relation means that the expression \( T \) leaves invariant the physical states, at least in the adiabatic limit. Indeed, let us abandon for the moment the formal notations of distribution theory and use test functions, i.e. we integrate the preceding relation with a test function \( f \) (this is the spacetime-dependent coupling constant of Bogoliubov):

\[
[Q, T(f)] = -i T^\mu(\partial_\mu f).
\]

(1.3)

As we make the test function \( f \) flatter and flatter, the right-hand side becomes smaller and smaller and we obtain

\[
T(f) \mathcal{H}_{\text{phys}} \subset \mathcal{H}_{\text{phys}}
\]

(1.4)

up to terms which can be made as small as desired; condition (1.2) is equivalent to the usual condition of (free) current conservation. Unfortunately one cannot perform this (adiabatic) limit in a rigorous way so one has to postulate these relations as axioms for gauge models in the causal approach. One must generalize these relations for higher orders of the perturbation theory in a natural way and work only with the chronological products which are free of infra-red divergences in the most general case. The outcome is very rewarding: one obtains in a unique way the structure of many gauge models in lower orders of the perturbation theory. One can prove gauge invariance and study the arbitrariness of the chronological products. The computations are quite long and the main purpose of this paper is to prove that such type of computations can be done more simple and compact if we use a convenient cohomology formalism.

In all known models one finds out that there exists a generalization of (1.2): we have a chain of Wick polynomials \( T^\mu, T^{\mu\nu}, T^{\mu\nu\rho}, \ldots \) such that

\[
[Q, T] = i \partial_\mu T^\mu,
[Q, T^\mu] = i \partial_\nu T^{\mu\nu},
[Q, T^{\mu\nu}] = i \partial_\rho T^{\mu\nu\rho}, \ldots
\]

(1.5)
In all cases $T^\mu_{\nu}, T^\mu_{\nu\rho}, \ldots$ are completely antisymmetric in all indices; it follows that the chain of relations stops at step 4 (if we work in four dimensions). We can also use a compact notation $T^I$ where $I$ is a collection of indices $I = [v_1, \ldots, v_p]$ ($p = 0, 1, \ldots$) and the brackets emphasize the complete antisymmetry in these indices. All these polynomials have the same canonical dimension

$$\omega(T^I) = \omega_0, \quad \forall I,$$

and because the ghost number of $T \equiv T^\emptyset$ is supposed null, then we also have

$$gh(T^I) = |I|.$$ (1.6)

One can write compactly relations (1.5) as follows:

$$d_Q T^I = i\partial^\mu T^{I\mu}.$$ (1.7)

For all known models the equations (1.5) can stop earlier: for instance in the case of gravity $T^\mu_{\nu\rho\sigma} = 0$.

Now we can construct the chronological products

$$T^{l_1,\ldots,l_n}(x_1,\ldots,x_n) \equiv T(T^{l_1}(x_1),\ldots,T^{l_n}(x_n))$$

according to the recursive procedure. We say that the theory is gauge invariant in all orders of the perturbation theory if the following set of identities generalizing (1.8):

$$d_Q T^{l_1,\ldots,l_n} = \sum_{l=1}^n (-1)^{l_j} \frac{\partial}{\partial x_{l_j}} T^{l_1,\ldots,l_{j-1},l_{j+1},\ldots,l_n}$$

are true for all $n \in \mathbb{N}$ and all $l_1,\ldots,l_n$. Here we have defined

$$S_l \equiv \sum_{j=1}^{l-1} |I|_j$$

(see also [4]). In particular, the case $l_1 = \cdots = l_n = \emptyset$ is sufficient for the gauge invariance of the scattering matrix, at least in the adiabatic limit.

Such identities can be usually broken by anomalies, i.e. expressions of type $A^{l_1,\ldots,l_n}$ which are quasi-local and might appear on the right-hand side of relation (1.9). These expressions verify some consistency conditions—the so-called Wess–Zumino equations. One can use these equations in the attempt to eliminate the anomalies by redefining the chronological products. All these operations can be proved to be of cohomological nature and naturally lead to descent equations of the same type as (1.8) but for a different ghost number and canonical dimension.

If one can choose the chronological products such that gauge invariance is true, then there is still some freedom left for redefining them. To be able to decide if the theory is renormalizable one needs the general form of such arbitrariness. Again, one can reduce the study of the arbitrariness to descent equations of the type as (1.8).

Such type of cohomology problems have been extensively studied in the more popular approach to quantum gauge theory based on functional methods (following from some path integration methods). In this setting the co-chain operator is nonlinear and makes sense only for classical field theories. In contrast, in the causal approach the co-chain operator is linear so the cohomology problem makes sense directly in the Hilbert space of the model. For technical reasons one needs however a classical field theory machinery to analyze the descent equations more easily.

In this paper we want to give a general description of these methods and we will apply them to gravitation models. We consider the case of massless and massive gravity. Summarizing
the preceding discussion, we apply the cohomological methods in three different cases: (a) the determination of the most general expression of the interaction Lagrangian; (b) the proof of gauge invariance in the second order of perturbation theory for the pure gravity system—massless and massive; (c) the investigation of the arbitrariness of the second-order chronological products compatible with renormalization principles and gauge invariance (i.e. the renormalization problem in the second order of perturbation theory). It is well known that quantum gravity is not renormalizable (see for instance the review paper [1]). We only emphasize that the arbitrariness of the chronological products (i.e. the list of possible counterterms) is greatly restricted by gauge invariance and it is worthwhile to determine the general form in higher orders of perturbation theory also.

In the next section we recall the axioms verified by the chronological products (for simplicity we give them only for the second order of perturbation theory) and consider the particular case of gauge models. In section 3 we give some general results about the structure of the anomalies and reduce the proof of (1.9) to descent equations. We will use a convenient geometric setting for our problem presented in [11]. In section 4 we determine the cohomology of the operator \( d_Q \) for gravity models. Using this cohomology and the algebraic Poincaré lemma we can solve the descent equations in various ghost numbers in section 5. In section 6 we use these methods to prove the gauge invariance of the pure gravity model in the second order of perturbation theory. We also investigate the arbitrariness of the chronological products in this order of perturbation theory and we find out that it is highly restricted: in fact for the pure gravity model one does not have new counterterms in the second order of perturbation theory; of course this property will not survive in higher orders of perturbation theory. In section 7 we determine the interaction between massless gravity, matter and Yang–Mills fields; we consider here only massless Yang–Mills fields and we will treat the massive case elsewhere. An interesting fact appears in this case due to the fact that, as it is well known, one cannot enforce the property

\[
\partial_\mu v^\mu = 0
\]

for quantum massless fields. This means that the usual expression for the energy–momentum tensor \( T^{\mu\nu} \) will not be divergenceless in the quantum context and this spoils the gauge invariance property of the interaction. Fortunately one can correct this adding a new ghost term in the interaction Lagrangian.

In [16] one can find similar results for massless gravity and its interaction with a scalar field, but the cohomological methods are not used for the proofs.

We note that the renormalizability of quantum gravity has attracted a lot of attention and we mention [8, 14]. Here we concentrate only on some technical procedures of cohomological nature which can be used to simplify the understanding of the lower order of perturbation theory.

2. General gauge theories

We give here the essential ingredients of perturbation theory. For simplicity we emphasize the second order of the perturbation theory.

2.1. Bogoliubov axioms

Suppose that we have a Fock space \( \mathcal{H} \) generated by some set of free fields and consider a set of Wick monomials \( W_j, j = 1, \ldots, n \), acting in this Hilbert space. The chronological products \( T(W_1(x_1), \ldots, W_n(x_n)) = T_{W_1,\ldots,W_n}(x_1, \ldots, x_n), n = 1, 2, \ldots \), are some operator-valued distributions acting in the Fock space and verifying a set of axioms (named Bogoliubov
axioms) explained in detail in [11]. For the case of gravity we will concentrate in this paper on the second order of perturbation theory, so we give the the axioms only for the cases $n = 1, 2$. We postulate the ‘initial condition’

$$T_W(x) = W(x)$$  (2.1)

and we give the axioms for the chronological products $T_{W_1, W_2}(x_1, x_2)$; for the particular case $W_1 = W_2 = T$ one denotes $T_{W_1, W_2}(x_1, x_2) = T_2(x_1, x_2)$ and these axioms guarantee that the scattering matrix

$$S(g) = I + \int_{\mathbb{R}^8} dx \ g(x) T(x) + \int_{\mathbb{R}^{16}} dx_1 dx_2 \ g(x_1) g(x_2) T_2(x_1, x_2) + \cdots$$  (2.2)

is Poincaré covariant, unitary and causal. In general we require the following.

- **Skew symmetry:**

  $$T_{W_1, W_2}(x_1, x_2) = (-1)^{f_1 f_2} T_{W_2, W_1}(x_2, x_1)$$  (2.3)

  where $f_i$ is the number of Fermi fields appearing in the Wick monomial $W_i$.

- **Poincaré invariance:** we have a natural action of the Poincaré group in the space of Wick monomials and we impose that for all $(a, A) \in inSL(2, \mathbb{C})$ we have

  $$U_{a,A} T_{W_1, W_2}(x_1, x_2) U_{a,A}^{-1} = T_{A W_1, A W_2}(A \cdot x_1 + a, A \cdot x_2 + a);$$  (2.4)

  sometimes it is possible to supplement this axiom by other invariance properties: space and/or time inversion, charge conjugation invariance, global symmetry invariance with respect to some internal symmetry group, supersymmetry, etc.

- **Causality:** if $x_1$ succeeds causally $x_2$ (which one denotes $x_1 \geq x_2$), then we have

  $$T_{W_1, W_2}(x_1, x_2) = T_{W_1}(x_1) T_{W_2}(x_2) = W_1(x_1) W_2(x_2);$$  (2.5)

- **Unitarity:**

  $$T_{W_1, W_2}(x_1, x_2) \dagger = -T_{W_2, W_1}(x_1, x_2) + W_1(x_1) W_2(x_2) + W_2(x_2) W_1(x_1).$$  (2.6)

  It can be proved that this system of axioms can be supplemented with

  $$T_{W_1, W_2}(x_1, x_2) = \sum \epsilon(\Omega, \ W_1, W_2(x_1, x_2) \Omega) \ W_1'(x_1), \ W_2''(x_2);$$  (2.7)

  where $W_1'$ and $W_2''$ are Wick submonomials of $W_i$ such that $W_i := W_1' W_2''$; and the sign $\epsilon$ takes care of the permutation of the Fermi fields. This is called the **Wick expansion property**.

  We can also include in the induction hypothesis a limitation on the order of singularity of the vacuum averages of the chronological products associated with arbitrary Wick monomials $W_1, W_2$; explicitly,

  $$\omega(\Omega, T_{W_1, W_2}(x_1, x_2) \Omega)) \leq \omega(W_1) + \omega(W_2) - 1$$  (2.8)

  where by $\omega(\cdots)$ we mean the order of singularity of the (numerical) distribution $\cdots$ and by $\omega(W)$ we mean the canonical dimension of the Wick monomial $W$; in particular this means that we have

  $$T_{W_1, W_2}(x_1, x_2) = \sum g t_g(x_1 - x_2) W_g(x_1, x_2)$$  (2.9)

  where $W_g$ are Wick polynomials of fixed canonical dimension, $t_g$ are distributions in one variable with the order of singularity bounded by the power counting theorem [6]:

  $$\omega(t_g) + \omega(W_g) \leq \omega(W_1) + \omega(W_2) - 1$$  (2.10)

  and the sum over $g$ is essentially a sum over Feynman graphs. We indicate briefly the simplest way to obtain the chronological products [6]. We compute the commutator $[W_1(x_1), W_2(x_2)]$.
and we first consider only the contributions coming from three graphs; we end up with an expression of the form

$$[W_1(x_1), W_2(x_2)]_{\text{tree}} = \sum \frac{\partial}{\partial x^{\mu_1}} \cdots \frac{\partial}{\partial x^{\mu_k}} D_m(x_1 - x_2) W_{\mu_1 \cdots \mu_k}(x_1, x_2)$$

(2.11)

where \( D_m(x_1 - x_2) \) is the Pauli–Villars causal distribution of mass \( m \), \( W_{\mu_1 \cdots \mu_k}(x_1, x_2) \) are Wick polynomials. Now one defines

$$T_{W_1, W_2}(x_1, x_2)_{\text{tree}} = \sum \frac{\partial}{\partial x^{\mu_1}} \cdots \frac{\partial}{\partial x^{\mu_k}} D_m(x_1 - x_2) W_{\mu_1 \cdots \mu_k}(x_1, x_2)$$

(2.12)

obtained from the previous one by replacing the causal distributions by the corresponding Feynman propagators. A similar procedure works for loop graphs also; only the procedure of obtaining the Feynman propagators from the corresponding causal distributions is more complicated (however, as for the tree contributions, is based on a standard procedure of distribution splitting). The resulting chronological products do verify all the axioms.

Up to now, we have defined the chronological products only for Wick monomials \( W_1, W_2 \), but we can extend the definition for Wick polynomials by linearity.

One can modify the chronological products without destroying the basic property of causality iff one can make

$$T_{W_1, W_2}(x_1, x_2) \rightarrow T_{W_1, W_2}(x_1, x_2) + R_{W_1, W_2}(x_1, x_2)$$

(2.13)

where \( R \) are quasi-local expressions; by a quasi-local expression we mean in this case an expression of the form

$$R_{W_1, W_2}(x_1, x_2) = \sum g \left( P_g(\partial) \delta(x_1 - x_2) \right) W_g(x_1, x_2)$$

(2.14)

with \( P_g \) monomials in the partial derivatives and \( W_g \) are Wick polynomials. Because of the delta function, we can consider that \( P_g \) is a monomial only in the derivatives with respect to, say, \( x_2 \). If we want to preserve (2.8) we impose the restriction

$$\deg(P_g) + \omega(W_g) \leq \omega(W_1) + \omega(W_2) - 4$$

(2.15)

and some other restrictions are following from the preservation of Lorentz covariance and unitarity.

The redefinitions of type (2.13) are the so-called finite renormalizations. Let us note that in higher orders of perturbation theory this arbitrariness, described by the number of independent coefficients of the polynomials \( P_g \), can grow with \( n \) and in this case the theory is called non-renormalizable. This can happen if some of the Wick monomials \( W_j, j = 1, \ldots, n \), have canonical dimension greater than 4. This seems to be the case for quantum gravity.

It is not hard to prove that any finite renormalization can be rewritten in the form

$$R(x_1, x_2) = \delta(x_1 - x_2) W(x_1) + \frac{\partial}{\partial x^\mu} \delta(x_1 - x_2) W^\mu(x_1)$$

(2.16)

where the expressions \( W, W^\mu \) are Wick polynomials in one variable. But it is clear that the second term in the above expression is null in the adiabatic limit so we can postulate that these types of finite renormalizations are trivial. This means that we can admit that the finite renormalizations have a much simpler form, namely

$$R(x_1, x_2) = \delta(X) W(x_1)$$

(2.17)

where the Wick polynomial \( W \) is constrained by

$$\omega(W) \leq \omega(W_1) + \omega(W_2) - 4.$$  (2.18)
2.2. Gauge theories and anomalies

From now on we work in the four-dimensional Minkowski space and we have the Wick polynomials \( T^I \) such that the descent equations (1.8) are true and we also have

\[
T^I(x_1)T^J(x_2) = (-1)^{|I||J|} T^J(x_2)T^I(x_1), \quad \forall \ x_1 \sim x_2, \tag{2.19}
\]

i.e. for \( x_1 - x_2 \) space-like these expressions causally commute in the graded sense.

Equations (1.8) are called a relative cohomology problem. The co-boundaries for this problem are of the type

\[
T^I = d_Q B^I + i\partial_\mu B^{I\mu}. \tag{2.20}
\]

In the second order of perturbation theory we construct the associated chronological products

\[
T^{I_1,I_2}(x_1, x_2) = T_{T^{I_1},T^{I_2}}(x_1, x_2).
\]

We will impose the graded symmetry property:

\[
T^{I_1,I_2}(x_1, x_2) = (-1)^{|I_1||I_2|} T^{I_2,I_1}(x_2, x_1). \tag{2.21}
\]

We also have

\[
gh(T^{I_1,I_2}) = |I_1| + |I_2|. \tag{2.22}
\]

In the case of a gauge theory the set of \textit{trivial} finite renormalizations is larger; we can also include co-boundaries because they induce the null operator on the physical space:

\[
R^{I_1,I_2}(x_1, x_2) = d_Q B^{I_1,I_2}(x_1) + i \frac{\partial}{\partial x_2^\mu} \delta(x_1 - x_2) B^{I_1,I_2\mu}(x_1). \tag{2.23}
\]

One can write the gauge invariance condition (1.9) in a more compact form [11] but for \( n = 2 \) this will not be necessary.

We now determine the obstructions for the gauge invariance relations (1.9). These relations are true for \( n = 1 \) according to (1.8). Then one can prove that in order \( n = 2 \) we must have

\[
d_Q T^{I_1,I_2} = i \frac{\partial}{\partial x_1^\mu} T^{I_1,I_2\mu} + i(-1)^{|I_1|} \frac{\partial}{\partial x_2^\mu} T^{I_1,I_2\mu} + A^{I_1,I_2}(x_1, x_2) \tag{2.24}
\]

where the expressions \( A^{I_1,I_2}(x_1, x_2) \) are quasi-local operators:

\[
A^{I_1,I_2}(x_1, x_2) = \sum_k \left[ \frac{\partial}{\partial x_2^\mu_1} \cdots \frac{\partial}{\partial x_2^\mu_k} \delta(x_2 - x_1) \right] W^{I_1,I_2;\{\mu_1,\cdots,\mu_k\}}(x_1) \tag{2.25}
\]

and are called \textit{anomalies}. In this expression the Wick polynomials \( W^{I_1,\cdots,I_n;\{\mu_1,\cdots,\mu_k\}} \) are uniquely defined. From (2.10) we have

\[
\omega(W^{I_1,I_2;\{\mu_1,\cdots,\mu_k\}}) \leq 2\omega_0 - 3 - k \tag{2.26}
\]

where we remind that \( \omega_0 \equiv \omega(T) \); this gives a bound on \( k \) in the previous sum. It is clear that we have from (2.21) a similar symmetry for the anomalies: namely we have

\[
A^{I_1,I_2}(x_1, x_2) = (-1)^{|I_1||I_2|} A^{I_2,I_1}(x_2, x_1) \tag{2.27}
\]

and we also have

\[
gh(A^{I_1,I_2}) = |I_1| + |I_2| + 1 \tag{2.28}
\]

and

\[
A^{I_1,I_2} = 0 \quad \iff \quad |I_1| + |I_2| > 2\omega_0 - 4. \tag{2.29}
\]
We also have some consistency conditions verified by the anomalies. If one applies the operator \( dQ \) to (2.24) one obtains the so-called Wess–Zumino consistency conditions for the cases \( n = 2 \):

\[
d_Q A^{I_1, I_2} = -i \frac{\partial}{\partial x_1} A^{I_1, I_2} - i(-1)^{|I_1|} \frac{\partial}{\partial x_2} A^{I_1, I_2}.
\] (2.30)

Let us note that we can suppose, as for the finite renormalizations (see (2.17)), that all anomalies which are total divergences are trivial because they spoil gauge invariance by terms which can be made as small as one wishes (in the adiabatic limit), i.e. we can take the form

\[
A^{I_1, I_2}(x_1, x_2) = \delta(x_1 - x_2) W^{I_1, I_2}(x_1).
\] (2.31)

In the case of quantum gravity it is not necessary to postulate this relation: one can prove it if one makes convenient finite renormalizations! For Yang–Mills models one can prove even more: such type of relations can be implemented in an arbitrary order of perturbation theory.

Suppose now that we have fixed the gauge invariance (1.9) (for \( n = 2 \)) and we investigate the renormalizability issue, i.e. we make the redefinitions

\[
T^{I_1, I_2} \rightarrow T^{I_1, I_2} + R^{I_1, I_2} \tag{2.32}
\]

where \( R \) are quasi-local expressions. As before we have

\[
R^{I_1, I_2}(x_1, x_2) = (-1)^{|I_1||I_2|} R^{I_2, I_1}(x_2, x_1) \tag{2.33}
\]

We also have

\[
gh(R^{I_1, I_2}) = |I_1| + |I_2| \tag{2.34}
\]

and

\[
R^{I_1, I_2} = 0 \quad \text{iff} \quad |I_1| + |I_2| > 2\omega_0 - 4. \tag{2.35}
\]

If we want to preserve (1.9) it is clear that the quasi-local operators \( R^{I_1, I_2} \) should also verify

\[
d_Q R^{I_1, I_2} = i \frac{\partial}{\partial x_1} R^{I_1, I_2} - i(-1)^{|I_1|} \frac{\partial}{\partial x_2} R^{I_1, I_2} \tag{2.36}
\]

i.e. equations of type (2.30). In this case we note that we have more structure; according to the previous discussion we can impose the structure (2.17):

\[
R^{I_1, I_2}(x_1, x_2) = \delta(x_1 - x_2) W^{I_1, I_2}(x_1) \tag{2.37}
\]

and we obviously have

\[
gh(W^{I_1, I_2}) = |I_1| + |I_2| \tag{2.38}
\]

and

\[
W^{I_1, I_2} = 0 \quad \text{iff} \quad |I_1| + |I_2| > 2\omega_0 - 4. \tag{2.39}
\]

From (2.36) we obtain after some computations that there are Wick polynomials \( R^I \) such that

\[
W^{I_1, I_2} = (-1)^{|I_1||I_2|} R^{I_1, I_2}. \tag{2.40}
\]

Moreover, we have

\[
gh(R^I) = |I| \tag{2.41}
\]

and

\[
R^I = 0 \quad \text{iff} \quad |I| > 2\omega_0 - 4. \tag{2.42}
\]

Finally, the following descent equations are true:

\[
d_Q R^I = i\partial_\mu R^{I, \mu} \tag{2.43}
\]

and we have obtained another relative cohomology problem similar to the one from the introduction.
3. Wess–Zumino consistency conditions

In this section we consider a particular form of (2.24) and (2.30) namely the case when all polynomials $T^I$ have canonical dimension $\omega_0 = 5$ and $T^{\mu\nu\sigma\rho} = 0$. In this case (2.29) becomes

$$A^{I_1;I_2}(X) = 0 \quad \text{iff} \quad |I_1| + |I_2| > 6. \quad (3.1)$$

It is convenient to define

$$A_1 \equiv A^{\beta;\beta}, \quad A_2 \equiv A^{[\mu;\mu]}, \quad A_3 \equiv A^{[\nu;\nu]}, \quad A_4 \equiv A^{[\nu;\nu]}, \quad A_5 \equiv A^{[\mu;\mu];\rho}, \quad A_6 \equiv A^{[\mu;\nu];[\rho;\sigma]}, \quad A_7 \equiv A^{[\mu;\nu];[\rho;\sigma]}, \quad A_8 \equiv A^{[\mu;\nu];[\sigma;\lambda]}, \quad A_9 \equiv A^{[\mu;\nu];[\sigma;\lambda;\omega]} \quad (3.2)$$

where we have emphasized the antisymmetry properties with brackets. We have from (2.24) the following anomalous gauge equations:

$$d\rho T(T(x_1), T(x_2)) = i\frac{\partial}{\partial x_1^\rho} T(T^{\mu}(x_1), T(x_2)) + i\frac{\partial}{\partial x_2^\rho} T(T(x_1), T^{\mu}(x_2)) + A_1(x_1, x_2) \quad (3.3)$$

$$d\rho T(T^{\mu}(x_1), T(x_2)) = i\frac{\partial}{\partial x_1^\rho} T(T^{\mu}(x_1), T(x_2)) + A_2(x_1, x_2) \quad (3.4)$$

$$d\rho T(T^{\mu}(x_1), T^{\nu}(x_2)) = i\frac{\partial}{\partial x_1^\rho} T(T^{\mu}(x_1), T^{\nu}(x_2)) + A_3(x_1, x_2) \quad (3.5)$$

$$d\rho T(T^{\mu}(x_1), T^{\nu}(x_2)) = i\frac{\partial}{\partial x_1^\rho} T(T^{\nu}(x_1), T^{\nu}(x_2)) - i\frac{\partial}{\partial x_2^\rho} T(T^{\mu}(x_1), T^{\nu}(x_2)) + A_4(x_1, x_2) \quad (3.6)$$

$$d\rho T(T^{\mu}(x_1), T^{\nu}(x_2)) = i\frac{\partial}{\partial x_1^\rho} T(T^{\mu}(x_1), T^{\nu}(x_2)) + A_5(x_1, x_2) \quad (3.7)$$

$$d\rho T(T^{\mu}(x_1), T^{\nu}(x_2)) = i\frac{\partial}{\partial x_1^\rho} T(T^{\nu}(x_1), T^{\nu}(x_2)) + A_6(x_1, x_2) \quad (3.8)$$

$$d\rho T(T^{\mu}(x_1), T^{\nu}(x_2)) = -i\frac{\partial}{\partial x_2^\rho} T(T^{\mu}(x_1), T^{\nu}(x_2)) + A_7(x_1, x_2) \quad (3.9)$$

$$d\rho T(T^{\mu}(x_1), T^{\nu}(x_2)) = -i\frac{\partial}{\partial x_2^\rho} T(T^{\mu}(x_1), T^{\nu}(x_2)) + A_8(x_1, x_2) \quad (3.10)$$

$$d\rho T(T^{\mu}(x_1), T^{\nu}(x_2)) = -i\frac{\partial}{\partial x_2^\rho} T(T^{\mu}(x_1), T^{\nu}(x_2)) + A_9(x_1, x_2) \quad (3.11)$$

$$d\rho T(T^{\mu}(x_1), T^{\nu}(x_2)) = 0. \quad (3.12)$$

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From (2.27) we get the following symmetry properties:

\[ A_1(x_1, x_2) = A_1(x_2, x_1) \] (3.13)

and we also have

\[ A_4^{\mu\nu}(x_1, x_2) = -A_4^{\mu\nu}(x_2, x_1), \] (3.14)

\[ A_6^{[\mu\nu];[\rho\sigma]}(x_1, x_2) = A_6^{[\rho\sigma];[\mu\nu]}(x_2, x_1), \] (3.15)

and

\[ A_{10}^{[\mu\nu];[\sigma\lambda\rho]}(x_1, x_2) = -A_{10}^{[\sigma\lambda\rho];[\mu\nu]}(x_2, x_1). \] (3.16)

The Wess–Zumino consistency conditions in this case are

\[
d_Q A_1(x_1, x_2) = -i \frac{\partial}{\partial x_1^\mu} A_2^\mu(x_1, x_2) - i \frac{\partial}{\partial x_2^\mu} A_2^\mu(x_2, x_1); \\
d_Q A_2^\mu(x_1, x_2) = -i \frac{\partial}{\partial x_1^\mu} A_3^{[\mu\nu]}(x_1, x_2) + i \frac{\partial}{\partial x_2^\mu} A_3^{[\mu\nu]}(x_1, x_2); \\
d_Q A_3^{[\mu\nu]}(x_1, x_2) = -i \frac{\partial}{\partial x_1^\mu} A_7^{[\mu\nu];\rho}(x_1, x_2) - i \frac{\partial}{\partial x_2^\mu} A_7^{[\mu\nu];\rho}(x_1, x_2); \\
d_Q A_4^{\mu\nu;\rho}(x_1, x_2) = -i \frac{\partial}{\partial x_1^\mu} A_5^{[\mu\nu;\rho]}(x_1, x_2) + i \frac{\partial}{\partial x_2^\mu} A_5^{[\mu\nu;\rho]}(x_1, x_2); \\
d_Q A_5^{[\mu\nu;\rho]}(x_1, x_2) = -i \frac{\partial}{\partial x_1^\mu} A_8^{[\mu\nu;\rho];\sigma}(x_1, x_2) - i \frac{\partial}{\partial x_2^\mu} A_8^{[\mu\nu;\rho];\sigma}(x_1, x_2); \\
d_Q A_6^{[\mu\nu;\rho];\sigma}(x_1, x_2) = i \frac{\partial}{\partial x_1^\mu} A_9^{[\mu\nu;\rho];[\sigma\lambda]}(x_1, x_2); \\
d_Q A_7^{[\mu\nu;\rho];[\sigma\lambda]}(x_1, x_2) = i \frac{\partial}{\partial x_1^\mu} A_{10}^{[\mu\nu;\rho];[\sigma\lambda\rho]}(x_1, x_2); \\
d_Q A_{10}^{[\mu\nu;\rho];[\sigma\lambda\rho]}(x_1, x_2) = 0. \] (3.26)

We suppose from now on that we work in a four-dimensional Minkowski spacetime and we have the following result:

**Theorem 3.1.** One can redefine the chronological products such that

\[
A_1(x_1, x_2) = \delta(x_1 - x_2) W(x_1), \\
A_3^{[\mu\nu]}(x_1, x_2) = \delta(x_1 - x_2) W^{[\mu\nu]}(x_1), \\
A_5^{[\mu\nu;\rho]}(x_1, x_2) = \delta(x_1 - x_2) W^{[\mu\nu;\rho]}(x_1), \\
A_7^{[\mu\nu;[\rho\sigma]]}(x_1, x_2) = -\delta(x_1 - x_2) W^{[\mu\nu;[\rho\sigma]]}(x_1),
\]

and \( A_j = 0, \) \( j = 6, 8, 9, 10. \) Moreover, one has the following descent equations:

\[
d_Q W = -i \partial_{\mu} W^{\mu}, \quad d_Q W^{\mu} = i \partial_{\mu} W^{[\mu\nu]}, \\
d_Q W^{[\mu\nu]} = -i \partial_{\mu} W^{[\mu\nu]}, \quad d_Q W^{[\mu\nu;\rho]} = 0.
\] (3.28)
The expressions $W$, $W^{\mu}$ and $W^{[\mu\nu]}$ are relative co-cycles and are determined up to relative co-boundaries. The expression $W^{[\mu\nu\rho]}$ is a co-cycle and it is determined up to a co-boundary.

The proof relies on the symmetry properties and the Wess–Zumino equations of consistency which are enough to obtain the result from the statement. The computations are straightforward and similar to those from [11] so we skip them. As we can see one can simplify considerably the form of the anomalies in the second order of the perturbation theory if one makes convenient redefinitions of the chronological products. Moreover, the result is of purely cohomological nature, i.e. we did not use the explicit form of the expressions $T$, $T^{\mu}$, $T^{[\mu\nu]}$, $T^{[\mu\nu\rho]}$. It will be a remarkable fact to extend the preceding result for arbitrary order of the perturbation theory. Relations (3.28) from the statement lead naturally to some descent equations.

4. The cohomology of the gauge charge operator

We consider the vector space $\mathcal{H}$ of Fock type generated (in the sense of Borchers theorem) by the symmetric tensor field $h_{\mu\nu}$ (with Bose statistics) and the vector fields $u^\rho$, $\tilde{u}^\sigma$ (with Fermi statistics). The Fermi fields are usually called ghost fields. We suppose that all these (quantum) fields are of null mass. Let $\Omega$ be the vacuum state in $\mathcal{H}$. In this vector space we can define a sesquilinear form $\langle \cdot, \cdot \rangle$ in the following way: the (non-zero) 2-point functions are by definition:

$$\langle \Omega, h_{\mu\nu}(x_1) h_{\rho\sigma}(x_2) \Omega \rangle = - \frac{i}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \eta_{\mu\nu} \eta_{\rho\sigma}) D_0^{(+)0}(x_1 - x_2),$$

$$\langle \Omega, u_\mu(x_1) \tilde{u}_\sigma(x_2) \Omega \rangle = i \eta_{\mu\sigma} D_0^{(+)0}(x_1 - x_2),$$

$$\langle \Omega, \tilde{u}_\mu(x_1) u_\nu(x_2) \Omega \rangle = - i \eta_{\mu\nu} D_0^{(+)0}(x_1 - x_2)$$

(4.1)

and the $n$-point functions are generated according to the Wick theorem. Here $\eta_{\mu\nu}$ is the Minkowski metrics (with diagonal $1, -1, -1, -1$) and $D_0^{(+)0}$ is the positive frequency part of the Pauli–Jordan distribution $D_0$ of null mass. To extend the sesquilinear form to $\mathcal{H}$ we define the conjugation by

$$h^\dagger_{\mu\nu} = h_{\mu\nu}, \quad u^\dagger_\rho = u_\rho, \quad \tilde{u}^\dagger_\sigma = - \tilde{u}_\sigma.$$  (4.2)

Now we can define in $\mathcal{H}$ the operator $Q$ according to the following formulas:

$$[Q, h_{\mu\nu}] = - \frac{i}{2} (\partial_\mu u_\nu + \partial_\nu u_\mu - \eta_{\mu\nu} \partial_\rho u^\rho), \quad [Q, u_\mu] = 0,$$

$$[Q, \tilde{u}_\mu] = i \partial^\mu h_{\mu\nu} \quad Q \Omega = 0$$

(4.3)

where by $[\cdot, \cdot]$ we mean the graded commutator. One can prove that $Q$ is well defined. Indeed, we have the causal commutation relations

$$[h_{\mu\nu}(x_1), h_{\rho\sigma}(x_2)] = - \frac{i}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \eta_{\mu\nu} \eta_{\rho\sigma}) D_0(x_1 - x_2) \cdot I,$$

$$[u_\mu(x_1), \tilde{u}_\sigma(x_2)] = i \eta_{\mu\sigma} D_0(x_1 - x_2) \cdot I$$

(4.4)

and the other commutators are null. The operator $Q$ should leave invariant these relations, in particular

$$[Q, [h_{\mu\nu}(x_1), \tilde{u}_\sigma(x_2)]] + \text{cyclic permutations} = 0$$

(4.5)

which is true according to (4.3). It is useful to introduce a grading in $\mathcal{H}$ as follows: every state which is generated by an even (odd) number of ghost fields and an arbitrary number of vector fields is even (resp. odd). We denote by $|f|$ the ghost number of the state $f$. We note
that the operator $Q$ raises the ghost number of a state (of fixed ghost number) by a unit. The usefulness of this construction follows from

**Theorem 4.1.** The operator $Q$ verifies $Q^2 = 0$. The factor space $\text{Ker}(Q)/\text{Im}(Q)$ is isomorphic to the Fock space of particles of zero mass and helicity 2 (gravitons).

**Proof.**

(i) The fact that $Q$ squares to zero follows easily from (4.3): the operator $Q^2 = 0$ commutes with all field operators and gives zero when acting on the vacuum.

(ii) The generic form of a state $\Psi \in \mathcal{H}^{(1)} \subset \mathcal{H}$ from the one-particle Hilbert subspace is

$$\Psi = \left[ \int f_{\mu
u}(x) h_{\mu
u}(x) + \int g_\mu^{(1)}(x) u_\mu(x) + \int g_\mu^{(2)}(x) \bar{u}_\mu(x) \right] \Omega \quad (4.6)$$

with test functions $f_{\mu
u}$, $g_\mu^{(1)}$, $g_\mu^{(2)}$ verifying the wave equation; we can also suppose that $f_{\mu
u}$ is symmetric. The relation $\Psi \in \text{Ker}(Q)$ i.e. $Q\Psi = 0$ leads to $\partial^\nu f_{\mu\nu} = \frac{1}{2} \partial_\mu f$ (where $f = \eta^{\mu\nu} f_{\mu\nu}$ is the trace of $f_{\mu\nu}$ and $g_\mu^{(2)} = 0$) i.e. the generic element $\Psi \in \mathcal{H}^{(1)} \cap \text{Ker}(Q)$ is

$$\Psi = \left[ \int f_{\mu
u}(x) h_{\mu
u}(x) + \int g_\mu(x) u_\mu(x) \right] \Omega \quad (4.7)$$

with $g_\mu$ arbitrary and $f_{\mu\nu}$ constrained by the transversality condition $\partial^\nu f_{\mu\nu} = \frac{1}{2} \partial_\mu f$, so the elements of $\mathcal{H}^{(1)} \cap \text{Ker}(Q)$ are in one-to-one correspondence with couples of test functions $[f_{\mu\nu}, g_\mu]$ with the transversality condition on the first entry. Now, a generic element $\Psi' \in \mathcal{H}^{(1)} \cap \text{Ran}(Q)$ has the form

$$\Psi' = Q\Phi = \left[ -\frac{1}{2} \left( \partial_\mu g'_\mu + \partial_\nu g'_\mu \right)(x) h_{\mu\nu}(x) + \int \left( \partial^\nu g_{\mu\nu} - \frac{1}{2} \partial_\mu g' \right)(x) u(x) \right] \Omega \quad (4.8)$$

with $g' = \eta^{\mu\nu} g'_{\mu\nu}$ so if $\Psi \in \mathcal{H}^{(1)} \cap \text{Ker}(Q)$ is indexed by the couple $[f_{\mu\nu}, g_\mu]$, then $\Psi + \Psi'$ is indexed by the couple $[f_{\mu\nu} = \frac{1}{2} (\partial_\mu g'_\mu + \partial_\nu g'_\mu), g_\mu + \left( \partial^\nu g_{\mu\nu} - \frac{1}{2} \partial_\mu g' \right)]$. If we take $g'_\mu$ conveniently we can make $g_\mu = 0$ and if we take $g'_\mu$ convenient we can make $f = 0$; in this case the transversality condition becomes $\partial^\nu f_{\mu\nu} = 0$. It follows that the equivalence classes from $(\mathcal{H}^{(1)} \cap \text{Ker}(Q))/(\mathcal{H}^{(1)} \cap \text{Ran}(Q))$ are indexed by wavefunctions $f_{\mu\nu}$ verifying the conditions of transversality and tracelessness $\partial^\nu f_{\mu\nu} = 0$. It remains to prove that the sesquilinear form $\langle \cdot, \cdot \rangle$ induces a positively defined form on $(\mathcal{H}^{(1)} \cap \text{Ker}(Q))/(\mathcal{H}^{(1)} \cap \text{Ran}(Q))$ and we have obtained the usual one-particle Hilbert space for the graviton (i.e. a particle of zero mass and helicity 2).

(iii) The extension of this argument to the nth-particle space is done as in [11] using the Künneth formula [3].

Now we have the physical justification for solving another cohomology problem namely to determine the cohomology of the operator $d_Q = [Q, \cdot]$ induced by $Q$ in the space of Wick polynomials. To solve this problem it is convenient to use the same geometric formalism [9] used in [11]. We consider that the (classical) fields are $h_{\mu\nu}, u_\mu, \bar{u}_\mu$ of null mass and we consider the set $\mathcal{P}$ of polynomials in these fields and their formal derivatives (in the sense of jet bundle theory). The formal derivative operators $d_\mu$ are given by

$$d_\mu y^\alpha_{\nu_1 \cdots \nu_k} \equiv y^\alpha_{\mu \nu_1 \cdots \nu_k} \quad (4.9)$$
where \( y^\alpha \) are the basic variables \( y^\alpha = (h_{\mu\nu}, u_\rho, \tilde{u}_\sigma) \) and \( y^\alpha \) are the jet bundle coordinates (see [11] for details). We note that on \( \mathcal{P} \) we have a natural grading. We introduce by convenience the notation

\[
B_\mu \equiv d^\nu h_{\mu\nu}
\]

and define the graded derivation \( d_Q \) on \( \mathcal{P} \) according to

\[
d_Q h_{\mu\nu} = -\frac{i}{2} (d_\mu u_\nu + d_\nu u_\mu - \eta_{\mu\nu} d_\rho u_\rho),
\]

\[
d_Q u_\mu = 0,
\]

\[
d_Q \tilde{u}_\mu = iB_\mu,\]

\[
[d_Q, d_\mu] = 0.
\]

Then one can easily prove that \( d^2_Q = 0 \) and the cohomology of this operator is isomorphic to the cohomology of the preceding operator (denoted also by \( d_Q \)) and acting in the space of Wick polynomials. The operator \( d_Q \) raises the grading and the canonical dimension by a unit.

To determine the cohomology of \( d_Q \) it is convenient to introduce some notations: first

\[
h \equiv \eta^{\mu\nu} h_{\mu\nu}
\]

\[
\hat{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h
\]

and then we define the Christoffel symbols according to

\[
\Gamma_{\mu;\nu;\rho} \equiv d_\rho \hat{h}_{\mu\nu} + d_\nu \hat{h}_{\mu\rho} - d_\mu \hat{h}_{\nu\rho}.
\]

We observe that

\[
d_Q \Gamma_{\mu;\nu;\rho} = -id_\rho d_\mu u_\nu
\]

and we can express the first-order derivatives through the Christoffel symbols

\[
d_\rho \hat{h}_{\mu\nu} = \frac{1}{2} (\Gamma_{\mu;\nu;\rho} + \Gamma_{\nu;\rho;\mu}).
\]

The expression

\[
R_{\mu\nu;\rho;\sigma} \equiv d_\rho \Gamma_{\mu;\nu;\sigma} - (\rho \leftrightarrow \sigma)
\]

is called the \textit{Riemann tensor}; we can easily prove

\[
R_{\mu\nu;\rho;\sigma} = -R_{\rho\mu;\nu;\sigma} = -R_{\mu\nu;\sigma;\rho} = R_{\rho\sigma;\mu\nu},
\]

\[
d_Q R_{\mu\nu;\rho;\sigma} = 0,
\]

\[
R_{\mu\nu;\rho;\sigma} + R_{\mu\rho;\nu;\sigma} + R_{\nu\sigma;\mu;\rho} = 0,
\]

\[
d_\rho R_{\mu\nu;\rho;\sigma} + d_\nu R_{\mu\rho;\nu;\sigma} + d_\sigma R_{\mu\nu;\sigma;\rho} = 0
\]

and the last two relations are called \textit{Bianchi identities}.

Next we consider, as in the case of the Yang–Mills fields, more convenient variables:
(i) first one can express the derivatives of the Christoffel symbols in terms of the completely symmetric derivatives

\[
\Gamma_{\mu,\rho_1,...,\rho_n} \equiv S_{\rho_1,...,\rho_n} (d_{\rho_1} \cdots d_{\rho_n} \Gamma_{\mu,\rho_1,\rho_2})
\]

and derivatives of the Riemann tensor; (ii) next, one expresses the variables \( \Gamma_{\mu,\rho_1,...,\rho_n} \) in terms of the expressions \( \Gamma^{(0)}_{\mu,\rho_1,...,\rho_n} \) (which is, by definition, the traceless part in \( \rho_1, \ldots, \rho_n \)) and \( B_{\mu,\rho_1,...,\rho_{n-1}} \); (iii) finally one expresses the derivatives of the Riemann tensor \( d_{\lambda_1} \cdots d_{\lambda_n} R_{\mu\nu;\rho;\sigma} \) in terms of the traceless part in all indices \( R^{(0)}_{\mu\nu;\rho;\sigma,\lambda_1,...,\lambda_n} \) and \( B_{\mu,\rho_1,...,\rho_{n+1}} \),

We will use the K"uneth theorem:

**Theorem 4.2.** Let \( \mathcal{P} \) be a graded space of polynomials and \( d \) an operator verifying \( d^2 = 0 \) and raising the grading by a unit. Let us suppose that \( \mathcal{P} \) is generated by two subspaces \( \mathcal{P}_1, \mathcal{P}_2 \) such that \( \mathcal{P}_1 \cap \mathcal{P}_2 = \{0\} \) and \( d \mathcal{P}_j \subset \mathcal{P}_j, j = 1, 2 \). We define by \( d_j \) the restriction of \( d \) to \( \mathcal{P}_j \). Let us consider the cohomology of \( d \) at the space \( \mathcal{P} \). Then

\[
H^* (\mathcal{P}) = \bigoplus_{j=1}^2 H^* (\mathcal{P}_j).
\]
$P_j$. Then there exists the canonical isomorphism $H(d) \cong H(d_1) \times H(d_2)$ of the associated cohomology spaces. (See [3]).

Now we can give a generic description for the co-cycles of $d_Q$; we denote by $Z_Q$ and $B_Q$ the co-cycles and the co-boundaries of this operator. First we define

$$u_{[\mu\nu]} = \frac{1}{2}(d_\mu u_\nu - d_\nu u_\mu)$$

and

$$u_{\mu\nu} = \frac{1}{2}(d_\mu u_\nu + d_\nu u_\mu)$$

such that we have

$$d_\mu u_\nu = u_{[\mu\nu]} + u_{\mu\nu}.$$  

(4.19)

Now we have

**Theorem 4.3.** Let $p \in Z_Q$. Then $p$ is cohomologous to a polynomial in $u_\mu, u_{\mu\nu}$ and $R(0)_{\mu\nu;\rho\sigma;\lambda_1,\ldots,\lambda_n}$.

**Proof.** The idea is to define conveniently two subspaces $P_1, P_2$ and apply the Künneth theorem. We will take $P_1 = P_0$ from the statement and $P_2$ the subspace generated by the variables $B_{\mu;\nu_1,\ldots,\nu_n} (n \geq 0), \Gamma_{\mu;\nu_1,\ldots,\nu_n}^{(0)} (n \geq 2), u_{\mu;\nu_1,\ldots,\nu_n} (n \geq 0), u_{[\mu;\nu_1,\ldots,\nu_n]} (n \geq 2), u_{\mu\nu}$ and $\hat{h}_{\mu\nu}$. We have

$$d_Q P_1 = \{0\}$$

and

$$d_Q u_{[\mu;\nu_1,\ldots,\nu_n]} = 0, \quad d_Q u_{\mu;\nu_1,\ldots,\nu_n} = 0 (n \geq 2)$$

$$d_Q \Gamma_{\mu;\nu_1,\ldots,\nu_n}^{(0)} = -iu_{\mu;\nu_1,\ldots,\nu_n} (n \geq 2)$$

$$d_Q B_{\mu;\nu_1,\ldots,\nu_n} = \Gamma_{\mu;\nu_1,\ldots,\nu_n}^{(0)} (n \geq 2)$$

$$d_Q \hat{h}_{\mu\nu} = -iu_{\mu\nu}$$

(4.21)

so we meet the conditions of Künneth theorem. Let us define in $P_2$ the graded derivation $\mathfrak{h}$ by

$$\mathfrak{h} u_{[\mu;\nu_1,\ldots,\nu_n]} = i\hat{h}_{\mu\nu}$$

$$\mathfrak{h} u_{\mu;\nu_1,\ldots,\nu_n} = i\Gamma_{\mu;\nu_1,\ldots,\nu_n}^{(0)} (n \geq 2)$$

$$\mathfrak{h} B_{\mu;\nu_1,\ldots,\nu_n} = -i\hat{h}_{\mu;\nu_1,\ldots,\nu_n} (n \geq 0)$$

(4.22)

and zero on the other variables from $P_2$. It is easy to prove that $\mathfrak{h}$ is well defined: the condition of tracelessness is essential to avoid conflict with the equations of motion. Then one can prove that

$$[d_Q, \mathfrak{h}] = Id$$

(4.23)

on polynomials of degree 1 in the fields and because the left-hand side is a derivation operator, we have

$$[d_Q, \mathfrak{h}] = n \cdot Id$$

(4.24)

on polynomials of degree $n$ in the fields. It means that $\mathfrak{h}$ is a homotopy for $d_Q$ restricted to $P_2$ so the corresponding cohomology is trivial: indeed, if $p \in P_2$ is a co-cycle of degree $n$ in the fields, then it is a co-boundary $p = \frac{n}{2}d_Q \mathfrak{h} p$.

According to the Künneth formula if $p$ is an arbitrary co-cycle from $P$, it can be replaced by a cohomologous polynomial from $P_0$ and this proves the theorem. 

**Remark 4.4.** There is an important difference with respect to the Yang–Mills case, namely the space $P_0$ is not isomorphic to the cohomology group $H_Q$ and this follows from the fact that
We provide an example of an expression belonging to this intersection. We start with the expression
\[ B_{\mu\nu\rho\sigma\lambda} \equiv u_\mu u_\nu u_\rho u_\sigma u_\lambda; \]  
(4.25)
because of the complete antisymmetry, we have in fact
\[ B_{\mu\nu\rho\sigma\lambda} = 0. \]  
(4.26)
On the other hand, we have
\[ d_\sigma B_{\mu\nu\rho\sigma\lambda} = p_{\mu\nu\rho\sigma} + dQ(\cdots) \]  
(4.27)
where
\[ p_{\mu\nu\rho\sigma} \equiv u_\lambda (u_\mu u_\nu u_\rho u_\sigma u_\lambda + u_\nu u_\rho u_\sigma u_\mu u_\lambda + u_\rho u_\sigma u_\mu u_\nu u_\lambda + u_\sigma u_\mu u_\nu u_\rho u_\lambda). \]  
(4.28)
so we have \( p_{\mu\nu\rho\sigma} \in P_0 \cap BQ. \)

We repeat the whole argument for the case of massive graviton, i.e. particles of spin 1 and positive mass.

We consider a vector space \( \mathcal{H} \) of Fock type generated (in the sense of Borchers theorem) by the tensor field \( h_{\mu\nu} \), the vector field \( v_\mu \) (with Bose statistics) and the vector fields \( u_\mu, \bar{u}_\mu \) (with Fermi statistics). We suppose that all these (quantum) fields are of mass \( m > 0 \). In this vector space we can define a sesquilinear form \( \langle \cdot , \cdot \rangle \) in the following way: the (non-zero) 2-point functions are by definition
\[ \langle \Omega, h_{\mu\nu}(x_1) h_{\rho\sigma}(x_2) \Omega \rangle = -\frac{i}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\nu\rho} \eta_{\mu\sigma} - \eta_{\mu\sigma} \eta_{\nu\rho}) D_m^{(s)}(x_1 - x_2), \]  
(4.29)
\[ \langle \Omega, u_\mu(x_1) \bar{u}_\sigma(x_2) \Omega \rangle = i \eta_{\mu\sigma} D_m^{(s)}(x_1 - x_2), \]  
\[ \langle \Omega, v_\mu(x_1) v_\mu(x_2) \Omega \rangle = i (\partial^\nu h_{\mu\nu} + m v_\mu), \]  
(4.30)
and the \( n \)-point functions are generated according to the Wick theorem. Here \( D_m^{(s)} \) is the positive frequency part of the Pauli–Jordan distribution \( D_m \) of mass \( m \). To extend the sesquilinear form to \( \mathcal{H} \) we define the conjugation by
\[ h_{\mu\nu}^\dagger = h_{\nu\mu}, \quad u_\mu^\dagger = u_\mu, \quad \bar{u}_\sigma^\dagger = -\bar{u}_\sigma, \quad v_\mu^\dagger = v_\mu. \]  
(4.31)
Now we can define in \( \mathcal{H} \) the operator \( Q \) according to the following formulas:
\[ \left[ Q, h_{\mu\nu} \right] = -\frac{i}{2} (\partial_\mu u_\nu + \partial_\nu u_\mu - \eta_{\mu\nu} \partial_\rho u_\rho), \]  
\[ \left[ Q, u_\mu \right] = 0, \]  
\[ \left[ Q, \bar{u}_\mu \right] = i (\partial^\nu h_{\mu\nu} + m v_\mu), \]  
\[ Q\Omega = 0. \]  
(4.32)
One can prove that \( Q \) is well defined. Indeed, we have the causal commutation relations
\[ \left[ h_{\mu\nu}(x_1), h_{\rho\sigma}(x_2) \right] = -\frac{i}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\nu\rho} \eta_{\mu\sigma} - \eta_{\mu\sigma} \eta_{\nu\rho}) D_m(x_1 - x_2) \cdot I, \]  
\[ \left[ u_\mu(x_1), \bar{u}_\sigma(x_2) \right] = i \eta_{\mu\sigma} D_m(x_1 - x_2) \cdot I, \]  
\[ \left[ v_\mu(x_1), v_\mu(x_2) \right] = i \eta_{\mu\sigma} D_m(x_1 - x_2) \cdot I, \]  
and the other commutators are null. The operator \( Q \) should leave invariant these relations, in particular
\[ \left[ Q, [h_{\mu\nu}(x_1), \bar{u}_\sigma(x_2)] \right] + \text{cyclic permutations} = 0, \]  
\[ \left[ Q, [v_\mu(x_1), \bar{u}_\sigma(x_2)] \right] + \text{cyclic permutations} = 0. \]  
(4.33)
We have a result similar to the first theorem of this section:

**Theorem 4.5.** The operator $Q$ verifies $Q^2 = 0$. The factor space $\text{Ker}(Q)/\text{Im}(Q)$ is isomorphic to the Fock space of particles of mass $m$ and spin 2 (massive gravitons).

**Proof.**

(i) The fact that $Q$ squares to zero follows easily from (4.31).

(ii) The generic form of a state $\Psi \in \mathcal{H}^{(1)} \subset \mathcal{H}$ from the one-particle Hilbert subspace is

$$
\Psi = \left[ \int f_{\mu \nu}(x) h^{\mu \nu}(x) + \int g^{(1)}_{\mu}(x) u^{\mu}(x) + \int g^{(2)}_{\mu}(x) \bar{u}^{\mu}(x) + \int h_{\mu}(x) v^{\mu}(x) \right] \Omega
$$

(4.34)

with test functions $f_{\mu \nu}, g^{(1)}_{\mu}, g^{(2)}_{\mu}, h_{\mu}$ verifying the wave equation; we can also suppose that $f_{\mu \nu}$ is symmetric. The condition $\Psi \in \text{Ker}(Q)$ i.e. $Q\Psi = 0$ leads to $h_{\mu} = \frac{2}{m} (\partial^{\nu} f_{\nu \mu} - \frac{1}{2} \partial_{\mu} f)$ (where $f = \eta^{\mu \nu} f_{\mu \nu}$ is the trace of $f_{\mu \nu}$) and $g^{(2)}_{\mu} = 0$, i.e. the generic element $\Psi \in \text{Ker}(Q)$ is

$$
\Psi = \left[ \int f_{\mu \nu}(x) h^{\mu \nu}(x) + \int g_{\mu}(x) u^{\mu}(x) + \frac{2}{m} \int \left( \partial^{\nu} f_{\mu \nu} - \frac{1}{2} \partial_{\mu} f \right)(x) v^{\mu}(x) \right] \Omega
$$

(4.35)

with $g_{\mu}$ and $f_{\mu \nu}$ arbitrary so $\Psi \in \mathcal{H}^{(1)} \cap \text{Ker}(Q)$ is indexed by couples of test functions $[f_{\mu \nu}, g_{\mu}]$. Now, a generic element $\Psi' \in \mathcal{H}^{(1)} \cap \text{Ran}(Q)$ has the form

$$
\Psi' = Q \Phi = \left[ -\frac{1}{2} \int (\partial_{\mu} g'_{\nu} + \partial_{\nu} g'_{\mu})(x) h^{\mu \nu}(x) \\
+ \int \left( \partial^{\nu} g'_{\mu \nu} - \frac{1}{2} \partial_{\mu} g' - \frac{m}{2} h'_{\mu} \right)(x) u^{\mu}(x) \right] \Omega
$$

(4.36)

with $g' = \eta^{\mu \nu} g'_{\mu \nu}$ so if $\Psi \in \mathcal{H}^{(1)} \cap \text{Ker}(Q)$ is indexed by the couple $[f_{\mu \nu}, g_{\mu}]$, then $\Psi + \Psi'$ is indexed by the couple $[f_{\mu \nu} - \frac{1}{2} (\partial_{\mu} g'_{\nu} + \partial_{\nu} g'_{\mu}), g_{\mu} + (\partial^{\nu} g'_{\mu \nu} - \frac{1}{2} \partial_{\mu} g' - \frac{m}{2} h'_{\mu})]$. If we take $h'_{\mu}$ conveniently we can make $g_{\mu} = 0$ and if we take $g'_{\mu \nu}$ convenient we can make $\partial^{\nu} f_{\mu \nu} = 0$. We still have the freedom to change $f_{\mu \nu} \rightarrow f_{\mu \nu} - \frac{1}{2} (\partial_{\mu} g'_{\nu} + \partial_{\nu} g'_{\mu})$ with transverse functions $\partial^{\nu} g'_{\mu \nu} = 0$ without affecting the property $\partial^{\nu} f_{\mu \nu} = 0$. It remains to prove that the sesquilinear form $(\cdot, \cdot)$ induces a positively defined form on $\mathcal{H}^{(1)} \cap \text{Ker}(Q)$ and we have obtained a direct sum of the one-particle Hilbert space for the graviton of mass $m$ (i.e. a particle of mass $m$ and helicity 2) and a scalar particle of the same mass $m$.

(iii) The extension of this argument to the $n$th-particle space is done as in [11] using the K"unneth formula [3].

□

Now we determine the cohomology of the operator $d_Q = [Q, \cdot]$ induced by $Q$ in the space of Wick polynomials. As before, it is convenient to use the formalism from the preceding section. We consider that the (classical) fields $y^{\alpha}$ are $h_{\mu \nu}, u_{\mu}, \bar{u}_{\mu}, v_{\mu}$ of mass $m$ and we consider the set $\mathcal{P}$ of polynomials in these fields and their derivatives. We introduce by convenience the notation:

$$
C_{\mu} \equiv d^{\nu} h_{\mu \nu} + m v_{\mu}
$$

(4.37)
and define the graded derivation $d_Q$ on $P$ according to
\[
d_Q h_{\mu \nu} = -\frac{1}{2} (d_{\mu} u_{\nu} + d_{\nu} u_{\mu} - \eta_{\mu \nu} d_\rho u^\rho),
\]
\[
d_Q u_{\mu} = 0, \quad d_Q h_{\mu \nu} = i C_{\mu}, \quad d_Q v_{\mu} = -\frac{i m}{2} u_{\mu}, \quad (4.38)
\]
Then one can prove that $d_Q^2 = 0$ and the cohomology of this operator is isomorphic to the cohomology of the preceding operator (denoted also by $d_Q$) and acting in the space of Wick monomials. To determine the cohomology of $d_Q$ it is convenient to introduce the Riemann tensor $R_{\mu \nu; \rho \sigma}$ as before and also
\[
\phi_{\mu \nu} \equiv -d_\mu v_\nu - d_\nu v_\mu + \eta_{\mu \nu} d_\rho v^\rho + mh_{\mu \nu}
\]
and observe that we also have
\[
d_Q \phi_{\mu \nu} = 0. \quad (4.40)
\]
Then we construct new variables as in the massless case: (i) we express the variables $v_{(\mu \nu)} \equiv \frac{1}{n!} (d_{\mu} v_{\nu} + d_{\nu} v_{\mu})$, and $d_{\mu} \cdots d_{\nu} v_{\rho}(n \geq 2)$ through $\phi_{\mu \nu}$, $h_{\mu \nu}$ and their derivatives; (ii) next, we express the derivatives $\phi_{\mu \nu; \rho \sigma}$ through the traceless parts $\phi_{\mu \nu; \rho \sigma}^{(0)}$, $\phi_{\mu \nu; \rho \sigma}^{(1)}$, and $C_{\mu \nu; \rho \sigma}$, (iii) Finally we express the variables: $\Gamma_{\mu \nu; \mu \nu}$ and $d_\lambda \cdots d_\lambda R_{\mu \nu; \rho \sigma}$ in terms of the traceless parts $\Gamma_{\mu \nu; \mu \nu}^{(0)}$, $R_{\mu \nu; \rho \sigma}^{(0)}$, $\phi_{\mu \nu; \rho \sigma}^{(0)}$, $\phi_{\mu \nu; \rho \sigma}^{(1)} C_{\mu \nu; \mu \nu}$ and $h_{\mu \nu}$, $v_{(\mu \nu)}$. Now we can describe the cohomology of the operator $d_Q$ in the massive case.

**Theorem 4.6.** Let $p \in Z_Q$. Then $p$ is cohomologous to a polynomial in the traceless variables: $R_{\mu \nu; \rho \sigma}^{(0)}$, $\phi_{\mu \nu; \rho \sigma}^{(0)}$, $\phi_{\mu \nu; \rho \sigma}^{(1)}$.  

**Proof.** Similar to the proof of theorem 4.3. We take $P_1 = P_0$ as in the statement of the theorem and $P_2$ generated by the other variables. The graded derivation $\mathfrak{h}$ is defined in this case by
\[
\mathfrak{h} u_{\mu} = \frac{2i}{m} v_{\mu}, \quad \mathfrak{h} u_{(\mu \nu)} = i \mathfrak{h} v_{(\mu \nu)}, \quad \mathfrak{h} u_{(\mu \nu)} = \frac{2i}{m} v_{(\mu \nu)}, \quad (4.41)
\]
and it follows that $\mathfrak{h}$ is a homotopy for $d_Q$ restricted to $P_2$ so the corresponding cohomology is trivial.

According to the Künneth formula if $p$ is an arbitrary co-cycle from $P$, it can be replaced by a cohomologous polynomial from $P_0$ and this proves the theorem. \[\square\]

We note that in the case of null mass the operator $d_Q$ raises the canonical dimension by one unit and this fact is not true no longer in the massive case. We are lead to another cohomology group. Let us take as the space of co-chains the space $P^{(n)}$ of polynomials of canonical dimension $m \leq n$; then $Z_Q^{(n)} \subset P^{(n)}$ and $B_Q^{(n)} \equiv d_Q P^{(n-1)}$ are the co-cycles and the co-boundaries respectively. It is possible that a polynomial is a co-boundary as an element of $P$ but not as an element of $P^{(n)}$. The situation is described by the following generalization of the preceding theorem.

**Theorem 4.7.** Let $p \in Z_Q^{(n)}$. Then $p$ is cohomologous to a polynomial of the form $p_1 + d_Q p_2$ where $p_1 \in P_0$ and $p_2 \in P^{(n)}$.

We will call the co-cycles of type $p_1$ (resp. $d_Q p_2$) primary (resp. secondary).
5. The relative cohomology of the operator \( d_Q \)

A polynomial \( p \in \mathcal{P} \) verifying the relation 
\[ d_Q p = id_{\mu}p^\mu \]  
(5.1)

for some polynomials \( p^\mu \) is called a relative co-cycle for \( d_Q \). The expressions of the type 
\[ p = d_Q b + id_{\mu}b^\mu \quad (b, b^\mu \in \mathcal{P}) \]  
(5.2)

are relative co-cycles and are called relative co-boundaries. We denote by \( Z^\text{rel}_Q, B^\text{rel}_Q \) and \( H^\text{rel}_Q \) the corresponding cohomological spaces. In (5.1) the expressions \( p_\mu \) are not unique. It is possible to choose them as Lorentz covariant.

Now we consider the framework and notations of the preceding section in the case \( m = 0 \). Then we have the following result which describes the most general form of the self-interaction of the gravitons. Summation over the dummy indices is used everywhere.

**Theorem 5.1.** Let \( T \) be a relative co-cycle for \( d_Q \) which is trilinear in the fields and is of canonical dimension \( \omega(T) \leq 5 \) and ghost number \( gh(T) = 0 \). Then

(i) \( T \) is (relatively) cohomologous to a non-trivial co-cycle of the form
\[ t = \kappa(2h_{\mu\nu}d^\mu h^\nu + 4h_{\mu\nu}d^\mu d^\nu + 2h_{\mu\nu}d^\mu d^\nu - 4h_{\mu\nu}d^\mu d^\nu - 4h_{\mu\nu}d^\mu d^\nu + 4d^\mu u^\nu d^\mu u^\nu) \]  
(5.3)

where \( \kappa \in \mathbb{R} \).

(ii) The relation
\[ d_Q t = id_{\mu}t^\mu \]  
(5.4)

is verified by
\[ t^\mu = \kappa(-2u^\nu d_{\mu}d^\nu h^\mu + 2u^\nu d_{\mu}d^\nu h^\mu + 2u^\nu d_{\mu}d^\nu h^\mu + 2u^\nu d_{\mu}d^\nu h^\mu) \]  
(5.5)

(iii) The relation
\[ d_Q t^{\mu\nu} = id_{\nu}t^{\mu\nu} \]  
(5.6)

is verified by
\[ t^{\mu\nu} = \kappa[2(-u^\mu d_{\nu}d^\mu h^\nu + u^\mu d_{\nu}d^\mu h^\nu + u^\mu d_{\nu}d^\mu h^\nu + u^\mu d_{\nu}d^\mu h^\nu) - (\nu \leftrightarrow \mu) + 4d^\mu u^\nu d^\mu u^\nu]. \]  
(5.7)

(iv) The relation
\[ d_Q t^{\mu\nu\rho} = id_{\rho}t^{\mu\nu\rho} \]  
(5.8)

is verified by
\[ t^{\mu\nu\rho} = \kappa[2d_{\mu}d^\nu u^{\rho\mu} - u^{\mu\rho}d_{\nu}u^{\rho\nu} + u^{\mu\rho}d_{\nu}u^{\rho\nu} + u^{\mu\rho}d_{\nu}u^{\rho\nu}] + \text{circular perm.} \]  
(5.9)

and we have \( d_Q t^{\mu\nu\rho} = 0 \).

(v) The co-cycles \( t, t^{\mu}, t^{\mu\nu} \) and \( t^{\mu\nu\rho} \) are non-trivial and invariant with respect to parity.

**Proof.**

(i) By hypothesis we have
\[ d_Q T = id_{\mu}T^\mu. \]  
(5.10)

If we apply \( d_Q \) we obtain \( d_{\mu}d_Q T^\mu = 0 \) so with the Poincaré lemma there must exist the polynomials \( T^{\mu\nu} \) antisymmetric in \( \mu, \nu \) such that
\[ d_Q T^\mu = id_{\mu}T^{\mu\nu}. \]  
(5.11)
Continuing in the same way we find $T_{\mu\nu\rho}$ which is completely antisymmetric and we also have

$$d_Q T_{\mu\nu} = i d_\rho T_{\mu\nu}. \quad (5.9)$$

According to a theorem proved in [11] we can choose the expressions $T^i$ to be Lorentz covariant; we also have

$$g h(T^I) = |I|. \quad (5.10)$$

Because $T_{\mu\nu}^{\nu}$ is trilinear in the fields and has ghost number 3, it verifies

$$d_Q T_{\mu\nu}^{\nu} = 0. \quad (5.11)$$

We apply theorem 4.3 and obtain

$$T_{\mu\nu}^{\nu} = d_Q B_{\alpha\beta}^{\mu\nu} + T_{\mu\nu}^{\nu} \quad (5.12)$$

where $T_{\mu\nu}^{\nu} \in T^{(S)}_1$ and we can choose the expressions $B_{\alpha\beta}^{\mu\nu}$ and $T_{\mu\nu}^{\nu}$ completely antisymmetric. The generic form of $T_{\mu\nu}^{\nu}$ is

$$T_{\mu\nu}^{\nu} = a_0 u_\mu u_\nu u_\rho + a_1 (u_\mu u_\alpha u_\nu + u_\mu u_\nu u_\alpha + u_\nu u_\alpha u_\mu) + a_2 u_\lambda (u_\mu u_\nu u_\lambda + u_\nu u_\mu u_\lambda + u_\mu u_\lambda u_\nu) + a_3 e^{\alpha\beta\rho\nu} u_\alpha u_\beta. \quad (5.13)$$

We substitute the expression $T_{\mu\nu}^{\nu}$ in relation $(5.9)$ and obtain

$$d_Q (T_{\mu\nu}^{\nu} - i d_\rho B_{\alpha\beta}^{\nu}) = i d_\rho T_{\mu\nu}^{\nu}. \quad (5.14)$$

The right-hand side must be a co-boundary. If we compute the divergence $d_\rho T_{\mu\nu}^{\nu}$ and impose that it is a co-boundary, we obtain $a_0 = a' = 0$ and no constraints on $a_j (j = 1, 2)$ so apparently we have two possible solutions, namely the corresponding polynomials $T_{\mu\nu}^{\nu} (j = 1, 2)$ from the expression of $T_{\mu\nu}^{\nu}$. However, let us define

$$b_{\alpha\beta\mu\nu} = A u_\mu u_\nu u_\alpha u_\beta$$

where $A$ performs antisymmetrization in all indices. Then it is not hard to obtain that

$$d_\sigma b_{\alpha\beta\mu\nu} = -\frac{1}{2} T_{\mu\nu}^{\nu} - \frac{1}{2} T_{\nu\mu}^{\nu} + d_Q b_{\mu\nu}$$

where we can choose the expression $b_{\mu\nu}$ completely antisymmetric. It follows that if we modify conveniently the expressions $B_{\alpha\beta\mu\nu}$ and $B^{\mu\nu}$, we make $a_1 \rightarrow a_1 + 2c$, $a_2 \rightarrow a_2 + c$ with $c$ arbitrary. In particular we can arrange such that $a_1 = a_2 = 2\kappa$ (this is the choice made in [13]). In this case one can prove rather easily that $T_{\mu\nu}^{\nu} = t_{\mu\nu} + d_Q (\cdots)$ where $t_{\mu\nu}$ is the expression from the statement. It follows that one can exhibit $T_{\mu\nu}^{\nu}$ in the following form:

$$T_{\mu\nu}^{\nu} = t^{\mu\nu} + d_Q B_{\alpha\beta\mu\nu} + i d_\rho B_{\alpha\beta\mu\nu}. \quad (5.15)$$

Now one proves by direct computation that

$$d_\rho t_{\mu\nu} = -i d_Q t_{\mu\nu} \quad (5.16)$$

where $t_{\mu\nu}$ is the expression from the statement so we obtain

$$d_Q (T_{\mu\nu}^{\nu} - t^{\mu\nu} - i d_\rho B_{\alpha\beta\mu\nu}) = 0. \quad (5.17)$$
(ii) We use again theorem 4.3 and obtain

\[ T^{\mu \nu} = t^{\mu \nu} + d_Q B^{\mu \nu} + id_\mu B^{\nu \rho} + T_0^{\mu \nu} \]

(5.18)

where \( T_0^{\mu \nu} \in T_0^{(5)} \) and we can choose the expressions \( B^{\mu \nu} \) and \( T_0^{\mu \nu} \) antisymmetric. The generic form of the expression \( T_0^{\mu \nu} \) is

\[ T_0^{\mu \nu} = b_\mu B^{\nu \rho} + b'_\rho B^{\mu \nu \rho} + T_0^{\mu \nu} \]

(5.19)

the monomials \( u_\mu B^{\nu \rho} \) and \( \epsilon^{\mu \nu \rho \sigma} u_\alpha B^{\rho \sigma} \) can be eliminated if we use the following consequence of the Bianchi identity:

\[ R^{(0)}_{\mu \nu \rho \sigma} + R^{(0)}_{\mu \rho \nu \sigma} + R^{(0)}_{\mu \sigma \nu \rho} = 0 \]

and redefine the expression \( B^{\mu \nu} \). We substitute the expression of \( T^{\mu \nu} \) in (5.8) and get

\[ d_Q(T^{\mu} - i d_\mu B^{\mu \nu}) = id_\mu (T_0^{\mu \nu} + t^{\mu \nu}). \]

(5.21)

But one proves by direct computation that we have

\[ d_\rho t^{\mu \nu} = -id_Q t^{\mu \nu} \]

(5.22)

where \( t^{\mu} \) is the expression from the statement so the preceding relation becomes

\[ d_Q(T^{\mu} - t^{\mu} - id_\mu B^{\mu \nu}) = id_\mu T_0^{\mu \nu}. \]

(5.23)

The right-hand side must be a co-boundary and one easily obtains that \( b = b' = 0 \) so we have

\[ d_Q(T^{\mu} - t^{\mu} - id_\mu B^{\mu \nu}) = 0. \]

(5.24)

(iii) Now it is again the time when we use theorem 4.3 and obtain

\[ T^{\mu} = t^{\mu} + d_Q B^{\mu \nu} + id_\mu B^{\nu \rho} + T_0^{\mu} \]

(5.25)

where \( T_0^{\mu} \in T_0^{(5)} \). But there is no such expression, i.e. \( T_0^{\mu} = 0 \) and we have

\[ T^{\mu} = t^{\mu} + d_Q B^{\mu \nu} + id_\mu B^{\nu \rho}. \]

(5.26)

Now we get from (5.7)

\[ d_Q(T - id_\mu B^{\mu}) = id_\mu t^{\mu}. \]

(5.27)

But we obtain by direct computation that we have

\[ d_\mu t^{\mu} = -id_Q t^{\mu} \]

(5.28)

where \( t^{\mu} \) is the expression from the statement so the preceding relation becomes

\[ d_Q(T - t - id_\mu B^{\mu}) = 0 \]

(5.29)

so a last use of theorem 4.3 gives

\[ T = t + d_Q B + id_\mu B^{\mu} + T_0 \]

(5.30)

where \( T_0 \in T_0^{(5)} \). But there is no such expression, i.e. \( T_0 = 0 \) and we have

\[ T = t + d_Q B + id_\mu B^{\mu}, \]

(5.31)

i.e. we have obtained the first four assertions from the statement.

(iv) We prove now that \( t \) from the statement is not a trivial (relative) co-cycle. Indeed, if this would be true i.e. \( t = d_Q B + id_\mu B^{\mu} \), then we get \( d_\mu (t^{\mu} - d_Q B^{\mu}) = 0 \) so with the Poincaré lemma we have \( t^{\mu} = d_Q B^{\mu} + id_\mu B^{\mu \nu} \). In the same way we obtain from here: \( t^{\mu \nu} = d_Q B^{\mu \nu} + id_\mu B^{\mu \nu \rho} \) and \( t^{\mu \nu \rho} = d_Q B^{\mu \nu \rho} \). This relation contradicts the fact that \( t^{\mu \nu \rho} \) is a non-trivial co-cycle for \( d_Q \) as it follows from theorem 4.3. The invariance with respect to parity invariance is obvious. □
If $T$ is bilinear in the fields, we cannot use the Poincaré lemma but we can make a direct analysis. The result is easy to obtain and is omitted.

Now we extend this result to the case $m > 0$.

**Theorem 5.2.** Let $T_m$ be a relative co-cycle for $d_Q$ which is trilinear in the fields and is of canonical dimension $\omega(T_m) \leq 5$ and ghost number $gh(T_m) = 0$. Then (i) $T_m$ is (relatively) cohomologous to a non-trivial co-cycle of the form

$$t_m = t + \kappa \left[ m^2 \left( \frac{4}{3} h^\mu\nu h_{\nu\rho} h_{\mu\rho} - h^\mu\nu h_{\mu\nu} + \frac{1}{6} h^3 \right) + 4m u_\rho d^\rho v^\lambda u_\lambda - 4d^\rho v^\sigma d^\lambda v_{\rho\lambda} \right]$$

(5.32)

where $t$ is the expression from the preceding theorem.

(ii) The relation $d_Q t_m = id_\mu T^\mu_m$ is verified by

$$t^\mu_m = t^\mu + \kappa \left[ 4u_\rho d^\rho v_\lambda v^\lambda - 2u^\mu d^\rho v_\rho h^\lambda - u^\mu d^\rho v_\rho h^\lambda - m^2 u^\mu (h^\rho\sigma h_{\rho\sigma} - \frac{1}{2} h^2) \right]$$

(5.33)

where $t_\mu$ is the expression from the preceding theorem.

(iii) The relation $d_Q t^\mu_\nu m = id_\rho T^\rho_\mu_\nu$ is verified by

$$t^\mu_\nu m = t^\mu_\nu + 2\kappa m (v^\mu u^\nu - v^\nu u^\mu) d^\rho u_\rho$$

(5.34)

where $t^\mu_\nu$ is the expression from the preceding theorem.

(iv) The relation $d_Q t^\mu_\nu_\rho m = id_\sigma T^\sigma_\mu_\nu_\rho$ is verified by

$$t^\mu_\nu_\rho m = t^\mu_\nu_\rho + 2\kappa m^2 u^\mu u^\nu u^\rho$$

(5.35)

where $t^\mu_\nu_\rho$ is the expression from the preceding theorem. We also have $d_Q t^\mu_\nu_\rho = 0$. (v) The co-cycles $t_m, t^\mu_m, t^\mu_\nu_m$ and $t^\mu_\nu_\rho_m$ are non-trivial, parity invariant and have smooth limit for $m \searrow 0$.

**Proof.**

(i) As in the preceding theorem we can prove that we must have

$$d_Q T_m = id_\mu T^\mu_m$$

(5.36)

and

$$d_Q T^\mu_m = id_\rho T^\mu_\rho_m, \quad d_Q T^\mu_\nu_m = id_\rho T^\mu_\nu_\rho_m$$

(5.37)

and

$$d_Q T^\mu_\nu_\rho_m = 0.$$  

According to a theorem proved in [11] we can choose the expressions $T^I_m$ to be Lorentz covariant; we also have

$$gh(T^I_m) = |I|.$$  

(5.38)

Because we have $d_Q T^\mu_\nu_\rho_m = 0$, we have from theorem 4.7

$$T^\mu_\nu_\rho_m = d_Q B^\mu_\nu_\rho + T^\mu_\nu_\rho_0, \quad T^\mu_\nu_\rho_0 = 0$$

(5.39)

where $T^\mu_\nu_\rho_0 \in T_0^{(5)}$ and we can choose the expressions $B^\mu_\nu_\rho$ and $T^\mu_\nu_\rho_0$ completely antisymmetric. The generic form of $T^\mu_\nu_\rho_0$ is

$$T^\mu_\nu_\rho_0 = T^\mu_\nu_0$$

(5.40)
where \( T^\mu_{\nu\rho} \) is expression (5.13) from the massless case with \( a = 0 \) (the corresponding term is a secondary co-cycles). We substitute the expression \( T^\mu_{\nu\rho} \) into the first relation (5.37) and obtain

\[
d Q(T^\mu_{\nu\rho} - i d_\rho B^\mu_{\nu\rho}) = i d_\rho T^\mu_{0\nu,0}.
\]  

(5.41)

The right-hand side must be a co-boundary. If we compute the divergence \( d_\rho T^\mu_{0\nu,0} \) and impose that it is a co-boundary, we obtain immediately \( a' = 0 \) (the corresponding term is a secondary co-cycles). We substitute the expression \( T^\mu_{\nu\rho} \) into the first relation (5.37) and obtain

\[
d Q(T^\mu_{\nu\rho} - i d_\rho B^\mu_{\nu\rho}) = i d_\rho T^\mu_{0\nu,0}.
\]  

(5.42)

Now one proves by direct computation that

\[
d_\rho t^\mu_{\nu\rho} = -i d_\rho t^\mu_{0\nu}
\]  

(5.43)

where \( t^\mu_{0\nu} \) is the expression from the statement. We substitute this expression in the second relation (5.37) and obtain

\[
d Q(T^\mu_{\nu\rho} - t^\mu_{\nu\rho} - i d_\rho B^\mu_{\nu\rho}) = 0.
\]  

(5.44)

(ii) We use again theorem 4.36 and obtain the generic form of the expression

\[
T^\mu_{\nu\rho} = t^\mu_{\nu\rho} + d Q B^\mu_{\nu\rho} + i d_\rho B^\mu_{\nu\rho} = T^\mu_{0\nu,0}
\]  

(5.45)

where \( T^\mu_{0\nu,0} \in P^0(\mathcal{V}) \) and we can choose the expressions \( B^\mu_{\nu\rho} \) and \( T^\mu_{0\nu,0} \) antisymmetric. The generic form of the expression \( T^\mu_{0\nu,0} \) is the same as in the massless case \( T^\mu_{0\nu,0} = T^\mu_{0\nu} \) so we obtain from the first relation in (5.37):

\[
d Q(T^\mu_{\nu\rho} - i d_\rho B^\mu_{\nu\rho}) = i d_\rho T^\mu_{0\nu,0}.
\]  

(5.46)

But one proves by direct computation that we have

\[
d_\rho t^\mu_{\nu\rho} = -i d_\rho t^\mu_{0\nu}
\]  

(5.47)

where \( t^\mu_{0\nu} \) is the expression from the statement so the preceding relation becomes

\[
d Q(T^\mu_{\nu\rho} - t^\mu_{\nu\rho} - i d_\rho B^\mu_{\nu\rho}) = i d_\rho T^\mu_{0\nu,0}.
\]  

(5.48)

The right-hand side must be a co-boundary and one obtains as in the massless case \( T^\mu_{0\nu,0} = 0 \) so we have

\[
T^\mu_{\nu\rho} = t^\mu_{\nu\rho} + d Q B^\mu_{\nu\rho} + i d_\rho B^\mu_{\nu\rho}
\]  

(5.49)

and

\[
d Q(T^\mu_{\nu\rho} - t^\mu_{\nu\rho} - i d_\rho B^\mu_{\nu\rho}) = 0.
\]  

(5.50)

(iii) Now it is again the time when we use theorem 4.36 and obtain

\[
T^\mu_{\nu\rho} = t^\mu_{\nu\rho} + d Q B^\mu_{\nu\rho} + i d_\rho B^\mu_{\nu\rho} + T^\mu_{0\nu,0}
\]  

(5.51)

where \( T^\mu_{0\nu,0} \in P^0(\mathcal{V}) \). The generic form of such an expression is

\[
T^\mu_{0\nu,0} = d_1 u^\mu \phi^2 + d_2 u^\mu \phi \phi_\rho^\sigma + d_3 u^\mu \phi^\rho \phi^\sigma + d_4 u^\mu \phi^\rho \phi^\sigma \phi_\rho^\sigma
\]  

(5.52)

and we have from relation (5.36)

\[
d Q(T^\mu_{\nu\rho} - i d_\rho B^\mu_{\nu\rho}) = i(d_1 t^\mu_{\nu\rho} + d_2 T^\mu_{0\nu,0})
\]  

(5.53)
so the right-hand side must be a co-boundary. But one proves by direct computation that
\[ d_\nu t^\mu_m = -i d_\rho t^\mu_m \] (5.54)
where \( t_m \) is the expression from the statement so the preceding relation becomes
\[ d_\rho (T_m - t_m - i d_\nu B^\mu) = i d_\nu T^\mu_0,m \] (5.55)
so the expression \( d_\mu T^\mu_0,m \) must be a co-boundary. By direct computation we obtain from this condition \( d_j = 0 (j = 1, \ldots, 4) \), i.e. \( T^\mu_0,m = 0 \). It follows that
\[ T^\mu_0,m = t^\mu_m + d_\rho B^\mu + i d_\nu B^\mu \nu \] (5.56)
and
\[ d_\rho (T_m - t_m - i d_\nu B^\mu) = 0 \] (5.57)
so a last use of theorem 4.36 gives
\[ T_m = t_m + d_\rho B + i d_\nu B^\mu \nu \] (5.58)
where \( T_{0,m} \in T^{(5)}_0 \). But there is no such expression, i.e. \( T_{0,m} = 0 \) and we have
\[ T_m = t_m + d_\rho B + i d_\nu B^\mu \] (5.59)
i.e. we have obtained the first four assertions from the statement.
(iv) We prove now that \( t_m \) from the statement is not a trivial (relative) co-cycle as in the massless case. Parity invariance and the existence of a smooth limit \( m \searrow 0 \) are obvious. \( \square \)

For the bilinear we can make a direct analysis as in the massless case.

6. Gauge invariance and renormalization in the second order of perturbation theory

In the same way one can analyze the descent equations (3.28) and study the form of the anomalies in the second order of perturbation theory.

**Theorem 6.1.** In the massless case the second-order chronological products can be chosen such that the expression \( W^\delta \) from theorem 3.1 is
\[ W = 0, \quad W^\mu = 0, \quad W^\mu \nu = 0 \] (6.1)

and
\[ W^{\mu \nu \rho} = d_\nu B^{\mu \nu \rho} \] (6.2)
with the expression \( B^{\mu \nu \rho} \) completely antisymmetric and constrained by \( \omega(B^{\mu \nu \rho}) \leq 6 \) and \( gh(B^{\mu \nu \rho}) = 4 \).

**Proof.** We will need relations (3.28) in which we prefer to change some signs \( W^\mu \rightarrow -W^\mu, W^{\mu \nu} \rightarrow -W^{\mu \nu} \), i.e.
\[ d_\rho W = i \partial_\rho W^\mu, \quad d_\rho W^\mu = i \partial_\rho W^{\mu \nu}, \quad d_\rho W^{\mu \nu} = i \partial_\rho W^{\mu \nu}, \quad d_\rho W^{\mu \nu \rho} = 0, \] (6.3)
and we also have from (2.26) the bound \( \omega(W^I) \leq 7 \). Moreover, the parity invariance obtained in theorem 4.3 can be used to prove that the polynomials \( W^\delta \) are also parity invariant.

(i) From the last relation (6.3) and theorem 4.3 we obtain
\[ W^{\mu \nu \rho} = d_\rho B^{\mu \nu \rho} + W^{\mu \nu \rho}_0 \] (6.4)
with $W^\mu_{\nu \rho} \in \mathcal{P}^7_0$ and we can choose the expressions $B^\mu_{\nu \rho}$ and $W^\mu_{0 \nu \rho}$ completely antisymmetric. The generic form of $W^\mu_{0 \nu \rho}$ is

$$W^\mu_{0 \nu \rho} = a_1 (u^{\mu \lambda} u^\nu u^\rho + u^{\nu \lambda} u^\mu u^\rho + u^{\rho \lambda} u^\mu u^\nu) u_\lambda + a_2 (u^{\mu \nu} u^{\rho \lambda} + u^{\nu \rho} u^{\mu \lambda} + u^{\rho \mu} u^{\nu \lambda}) u_\lambda u_\sigma + a_3 (u^{\mu \nu} u^{\rho \lambda} u_\sigma + u^{\nu \rho} u^{\mu \lambda} u_\sigma + u^{\rho \mu} u^{\nu \lambda} u_\sigma) u_\lambda + a_4 (u^{\mu \nu} u^{\rho \lambda} u_\sigma + u^{\nu \rho} u^{\mu \lambda} u_\sigma + u^{\rho \mu} u^{\nu \lambda} u_\sigma) u_\lambda u_\sigma$$

(6.5)

with $a_j \in \mathbb{R}$ ($j = 1, \ldots, 4$). We denote by $T_j$ ($j = 1, \ldots, 4$) the polynomials multiplied by $a_j$ ($j = 1, \ldots, 4$) respectively. If we define the completely antisymmetric expressions

$$b^1_{\mu \nu \rho} \equiv u^\mu u^\nu u^\rho$$
$$b^2_{\mu \nu \rho} \equiv (u^{\mu \lambda} u^\nu u^\rho + u^{\nu \lambda} u^\mu u^\rho + u^{\rho \lambda} u^\nu u^\mu) u_\lambda + a_2 (u^{\mu \nu} u^{\rho \lambda} u_\sigma + u^{\nu \rho} u^{\mu \lambda} u_\sigma + u^{\rho \mu} u^{\nu \lambda} u_\sigma) u_\lambda + a_3 (u^{\mu \nu} u^{\rho \lambda} u_\sigma + u^{\nu \rho} u^{\mu \lambda} u_\sigma + u^{\rho \mu} u^{\nu \lambda} u_\sigma) u_\lambda$$

(6.6)

then it is not hard to obtain that

$$d_\sigma b^1_{\mu \nu \rho} = -T^1_{\mu \nu \rho} + dQ b^1_{\mu \nu \rho}$$
$$d_\sigma b^2_{\mu \nu \rho} = -T^2_{\mu \nu \rho} + dQ b^2_{\mu \nu \rho}$$
$$d_\sigma b^3_{\mu \nu \rho} = T^3_{\mu \nu \rho} - T^4_{\mu \nu \rho} + dQ b^3_{\mu \nu \rho}$$

(6.7)

where we can choose the expressions $b^j_{\mu \nu \rho}$ ($j = 1, \ldots, 3$) completely antisymmetric. It follows that we can rewrite (6.4) in the form

$$W^\mu_{\nu \rho} = dQ B^\mu_{\nu \rho} + i d_\rho B^\mu_{\nu \rho} + W^\mu_{0 \nu \rho}$$

(6.8)

where in the expression of $W^\mu_{0 \nu \rho}$ we can make $a_1 = a_3 = a_4 = 0$. We substitute the preceding expression in the third relation (6.3) and get

$$dQ (W^\mu_{\nu \rho} - i d_\rho B^\mu_{\nu \rho}) = d_\rho W^\mu_{0 \nu \rho}$$

(6.9)

so the right-hand side must be a co-boundary. From this condition we easily find $a_2 = 0$ so in fact we can take $T^3_{\mu \nu \rho} = 0$. It follows that one can exhibit $W^\mu_{\nu \rho}$ in the following form:

$$W^\mu_{\nu \rho} = dQ B^\mu_{\nu \rho} + i d_\rho B^\mu_{\nu \rho}$$

(6.10)

and we also have

$$dQ (W^\mu_{\nu \rho} - i d_\rho B^\mu_{\nu \rho}) = 0.$$  

(6.11)

(ii) We use again theorem 4.3 and obtain

$$W^\mu = dQ B^\mu + i d_\rho B^\mu + W^\mu_0$$

(6.12)

where $W^\mu_0 \in \mathcal{P}^7_0$ and we can choose the expressions $B^\mu_{\nu \rho}$ and $W^\mu_0$ antisymmetric. The generic form of the expression $W^\mu_0$ is

$$W^\mu_0 = b_1 (u^{\mu \rho} u^\nu - u^{\mu \nu} u^\rho) u_\lambda + b_2 u^{\mu \rho} u^{\nu \lambda} u_\rho + b_3 (R^{0 \mu \nu \rho \lambda}_{\mu \lambda}) u_\alpha u_\beta u_\gamma + b_4 (R^{0 \mu \nu \rho \lambda}_{\mu \lambda}) u_\alpha u_\beta u_\rho + b_5 (R^{0 \mu \nu \rho \lambda}_{\mu \lambda}) u_\alpha u_\beta u_\rho$$

(6.13)

and many other possible expressions can be eliminated, or reduced to these above if we use the Bianchi identity. We denote by $T_j$ ($j = 1, \ldots, 5$) the polynomials multiplied by $b_j$ ($j = 1, \ldots, 5$) respectively. If we define the completely antisymmetric expressions

$$b^1_{\mu \nu \rho} \equiv u^\mu u^\nu u^\rho$$
$$b^2_{\mu \nu \rho} \equiv (R^{0 \mu \nu \rho \lambda}_{\mu \lambda}) u_\alpha u_\beta u_\rho$$

(6.14)
we easily obtain that
\[
d_\rho b^{\mu \nu} = -T_1^{\mu \nu} + dQ b^{\mu \nu}_1
\]
where we can choose the expressions \(b^{\mu \nu}_j\) \((j = 1, \ldots, 3)\) antisymmetric. It follows that if we redefine the expressions \(B^{\mu \nu}\) from (6.12), we can make \(b_1 = b_4 = 0\) in \(W_0^{\mu \nu}\). We substitute the expression of \(W^{\mu \nu}\) in the second relation (6.3) and get
\[
d_Q \left( W^{\mu \nu} - id_\rho B^{\mu \nu} \right) = id_\rho W_0^{\mu \nu}
\]
so the right-hand side must be a co-boundary. One easily obtains that \(b_3 = b_5 = 0\) so we are left with one non-trivial co-cycle corresponding to \(b \equiv b_2:\)
\[
W_0^{\mu \nu} = bu^{\mu \nu} u_{\rho \lambda} u_{\rho \lambda}.
\]
Now one proves immediately that
\[
d_\nu W_0^{\mu \nu} = -id_\rho U^{\mu}
\]
and we have
\[
d_Q \left( W^{\mu \nu} - U^{\mu} - id_\rho B^{\mu \nu} \right) = 0.
\]
(iii) Now it is again the time when we use theorem 4.3 and obtain
\[
W^{\mu \nu} = U^{\mu} + dQ B^{\mu \nu} + id_\rho B^{\mu \nu} + W_0^{\mu \nu}
\]
where \(W_0^{\mu \nu} \in T_0^{(7)}\). The generic form of such an expression is
\[
W_0^{\mu \nu} = c_1 u^{\mu \nu} u_\rho u_\lambda + c_2 R^{(0)\mu \nu; \alpha \beta} u_\alpha u_\beta
\]
we denote by \(T_j\) \((j = 1, 2)\) the polynomials multiplied by \(c_j\) \((j = 1, 2)\). However, let us consider the antisymmetric expressions
\[
b_1^{\mu \nu} \equiv u^{\mu \nu}
b_2^{\mu \nu} \equiv R^{(0)\mu \nu; \alpha \beta} u_\alpha u_\beta
\]
and we have
\[
d_\rho b_1^{\mu \nu} = -T_1^{\mu \nu} + dQ b_1^{\mu \nu}
d_\rho b_2^{\mu \nu} = T_2^{\mu \nu} + dQ b_2^{\mu \nu}
\]
so if we redefine the expressions \(B^{\mu \nu}\) and \(B^{\mu}\) from (6.22), we can make \(c_1 = c_2 = 0\), i.e. \(W_0^{\mu \nu} = 0\). As a consequence, we have
\[
W^{\mu \nu} = U^{\mu} + dQ B^{\mu \nu} + id_\rho B^{\mu \nu}.
\]
Now we get from the first equation (6.3)
\[
d_Q \left( W - id_\rho B^{\mu \nu} \right) = id_\rho U^{\mu}
\]
so the right-hand side must be a co-boundary. One can prove that this is not possible so we must have \(b = 0\), i.e. \(U^{\mu} = 0\). As a consequence, we have
\[
W^{\mu \nu} = dQ B^{\mu \nu} + id_\rho B^{\mu \nu}
\]
and
\[
d_Q \left( W - id_\rho B^{\mu \nu} \right) = 0
\]
so a last use of theorem 4.3 gives

\[ W = d_Q B + i d_\mu B^\mu + W_0 \]  

(6.29)

where \( W_0 \in T_0 \). But there is no such expression, i.e. \( W_0 = 0 \) and we have

\[ W = d_Q B + i d_\mu B^\mu. \]  

(6.30)

Now we use finite renormalizations to eliminate the expressions \( B^I(|I| \leq 3) \) as in the end of theorem 3.1 and end up with the expression from the statement. \( \square \)

Remark 6.2.

(i) One can extend the preceding result to the massive case also. The complications are only of technical nature: more terms can appear in the generic expressions of the expressions \( W_0^I \), but they eventually are eliminated.

(ii) The preceding proof stays true if we do not use parity invariance: as in the preceding remark, more terms can appear in the expressions \( W_0^I \).

(iii) We cannot eliminate \( B^\mu_{\nu\rho\sigma} \) by finite renormalizations.

(iv) In higher orders of perturbation theory some expressions of type \( W_0^I \) will survive the algebraic machinery we have used and new ideas are needed to eliminate them.

We can reduce the gauge invariance problem in the second order to a much simple computation namely of the anomaly \( A_7 \) where the expression \( W^\mu_{\nu\rho} \) does appear. But we have

**Theorem 6.3.** The anomaly \( A_7 \) can be eliminated by finite renormalizations.

**Proof.** We consider the massless case. The standard procedure is to show by direct computation that

\[ [t^\mu_{\nu\rho}(x_1), \ell^\sigma(x_2)] = A^{\mu\nu\rho}(x_1, x_2) \partial^\sigma D_0(x_1 - x_2) + A^{\mu\nu\rho;\alpha}(x_1, x_2) \partial^\sigma \partial_\alpha D_0(x_1 - x_2) + \cdots \]  

(6.31)

where by \( \cdots \) we mean terms for which the index \( \sigma \) does not act on the Pauli–Jordan distribution.

If we transform this distribution in the corresponding Feynman propagator \( D^\mu_{\nu\rho} \), we obtain on the left-hand side of relation (3.9) an anomaly of the form

\[ A_7^{[\mu\nu\rho]}(x_1, x_2) = \delta(x_2 - x_1) A^{\mu\nu\rho}(x_1, x_2) + [\partial_\rho \delta(x_2 - x_1)] A^{\mu\nu;\alpha}(x_1, x_2). \]  

(6.32)

Now we make simple computation to put the preceding expression in the standard form:

\[ A_7^{[\mu\nu\rho]}(x_1, x_2) = \delta(x_2 - x_1) W^{\mu\nu\rho}(x_1) + \partial_\rho \delta(x_2 - x_1) W^{\mu\nu;\alpha}(x_1). \]  

(6.33)

The second term can be eliminated by a finite renormalization of the chronological products \( T(T^{[\mu\nu\rho]}(x_1), T^{\nu}(x_2)) \). Now the only thing to prove is that the first term is a co-boundary:

\[ W^{\mu\nu\rho} = d_Q B^{\mu\nu\rho} \]  

(6.34)

and we can eliminate it by a redefinition of the chronological products \( T(T^{[\mu\nu\rho]}(x_1), T^{\nu}(x_2)) \).

Finally one can prove that in the massive case no new contributions to the anomaly \( A_7 \) do appear. \( \square \)

The preceding result means that we can take \( W^{[\mu\nu\rho]} = 0 \) from the very beginning, i.e. we can start in theorem 6.1 with step (ii) and we obtain gauge invariance in the second order of perturbation theory for quantum gravity.

We now turn to the renormalization problem for the second order of the perturbation theory. We prove that the form of the possible counterterms is greatly restricted by gauge
invariance. By accident, in the massless case, we do not have new counterterms except one leading to a renormalization of the coupling constant. We have the following result:

**Theorem 6.4.** In the massless case the finite renormalizations for the second-order chronological products are of the form

\[
R^I = t' + dQ B^I + i\hat{d}_\mu B^I\mu
\]  

(6.35)

where \(t'\) has the same form as the interaction Lagrangian \(t\) from theorem 5.1 (but with a different overall constant) and can be eliminated by a redefinition of the gravitational coupling \(\kappa\). The rest of the terms can be eliminated by finite renormalizations of the chronological products.

**Proof.** According to (2.43) we have the following descent procedure:

\[
\begin{align*}
    d_Q R &= id_\mu R^\mu, \\
    d_Q R^\mu &= id_\nu R^\mu, \\
    d_Q R^\mu\nu &= id_\rho R^{\mu\nu}, \\
    d_Q R^\mu\nu\rho &= id_\gamma R^{\mu\nu\rho}, \\
    d_Q R^\mu\nu\rho\sigma &= 0,
\end{align*}
\]

(6.36)

and we have the limitations \(\omega(R^I) \leq 6, gh(R^I) = |I|\) and also the expressions \(T^I\) are Lorentz covariant. We consider only the case \(\omega(R^I) = 6\) because the case \(\omega(R^I) \leq 5\) has been covered by theorem 5.1.

From the last relation we find, using theorem 4.3, that

\[
R^{\mu\nu\rho\sigma} = d_Q B^{\mu\nu\rho\sigma} + R_0^{\mu\nu\rho\sigma}
\]

(6.37)

with \(R_0^{\mu\nu\rho\sigma} \in T_0^{(6)}\) and we can choose the expressions \(B^{\mu\nu\rho\sigma}\) and \(R_0^{\mu\nu\rho\sigma}\) completely antisymmetric. The generic form of \(R_0^{\mu\nu\rho\sigma}\) is

\[
R_0^{\mu\nu\rho\sigma} = a_1 (u^\mu u^\nu u^\rho u^\sigma + \ldots) + a_2 (u^\mu u^\nu u^\rho u^\sigma + \ldots) + a' \epsilon^{\mu\nu\rho\sigma} u^\alpha u^\beta u^\gamma u^\delta
\]

(6.38)

where by \(\ldots\) we mean the rest of the terms needed to make the expression completely antisymmetric. We denote by \(R_j (j = 1, 2)\) the polynomials multiplied by \(a_j (j = 1, 2)\) respectively. We define the completely antisymmetric expression

\[
b^{\mu\nu\rho\sigma\lambda} \equiv u^\mu u^\nu u^\rho u^\sigma u^\lambda + \ldots
\]

(6.39)

which must be obviously null. On the other hand, we have

\[
d_j b^{\mu\nu\rho\sigma\lambda} = -R_1^{\mu\nu\rho\sigma\lambda} + 2R_2^{\mu\nu\rho\sigma\lambda} + d_Q b^{\mu\nu\rho\sigma\lambda}
\]

where we can take \(b^{\mu\nu\rho\sigma}\) completely antisymmetric. In other words, we have proved that

\[
R_1^{\mu\nu\rho\sigma} - 2R_2^{\mu\nu\rho\sigma} = d_Q b^{\mu\nu\rho\sigma}
\]

(6.40)

and this relation can be used to make \(a_1 = 0\) in the expression of \(R_0^{\mu\nu\rho\sigma}\). We substitute the expression of \(R_1^{\mu\nu\rho\sigma}\) in the fourth relation (6.36) and get

\[
d_Q (R^{\mu\nu\rho\sigma} - id_\sigma B^{\mu\nu\rho\sigma}) = id_\sigma R_0^{\mu\nu\rho\sigma}
\]

(6.41)

so the right-hand side must be a co-boundary. From this condition we easily find \(a_2 = a' = 0\) so in fact we can take \(R_0^{\mu\nu\rho\sigma} = 0\). It follows that one can exhibit \(R_0^{\mu\nu\rho\sigma}\) in the following form:

\[
R_0^{\mu\nu\rho\sigma} = d_Q B^{\mu\nu\rho\sigma}
\]

(6.42)

and we also have

\[
d_Q (R^{\mu\nu\rho\sigma} - id_\sigma B^{\mu\nu\rho\sigma}) = 0.
\]

(6.43)
We use again theorem 4.3 and obtain

\[ R^{\mu \nu \rho} = d_Q B^{\mu \nu \rho} + i d_\sigma B^{\mu \nu \rho \sigma} + R^{\mu \nu \rho}_0 \] (6.44)

where \( R^{\mu \nu \rho}_0 \in \mathcal{P}_0^{(6)} \) and we can choose the expressions \( B^{\mu \nu \rho} \) and \( R^{\mu \nu \rho}_0 \) completely antisymmetric. The generic form of the expression \( R^{\mu \nu \rho}_0 \) is

\[ R^{\mu \nu \rho}_0 = b(u^{\mu} R^{(0) \nu \rho; \alpha \beta} + u^{\nu} R^{(0) \mu \rho; \alpha \beta} + u^{\rho} R^{(0) \mu \nu; \alpha \beta}) u_\alpha u_\beta; \] (6.45)

other possible expressions can be eliminated, or reduced to this above if we use the Bianchi identity. We substitute the expression of \( R^{\mu \nu \rho} \) in the third relation (6.36) and get

\[ d_Q (R^{\mu \nu} - i d_\rho B^{\mu \nu \rho}) = i d_\rho R^{\mu \nu \rho}_0 \] (6.46)

so the right-hand side must be a co-boundary. One easily obtains that \( b = 0 \) so we have \( R^{\mu \nu \rho}_0 = 0 \). This means that

\[ R^{\mu \nu \rho} = d_Q B^{\mu \nu \rho} + i d_\sigma B^{\mu \nu \rho \sigma} \] (6.47)

and we have

\[ d_Q (R^{\mu \nu} - i d_\rho B^{\mu \nu}) = 0. \] (6.48)

(iii) Now it is again the time to use theorem 4.3 and obtain

\[ R^{\mu \nu} = d_Q B^{\mu \nu} + i d_\rho B^{\mu \nu \rho} + R^{\mu \nu}_0 \] (6.49)

where \( R^{\mu \nu}_0 \in \mathcal{P}_0^{(6)} \) and we can take the expressions \( B^{\mu \nu} \) and \( R^{\mu \nu}_0 \) antisymmetric. But there is no such expression \( R^{\mu \nu}_0 \) i.e. we have \( R^{\mu \nu}_0 = 0 \) and it follows that

\[ R^{\mu \nu} = d_Q B^{\mu \nu} + i d_\rho B^{\mu \nu \rho}. \] (6.50)

We substitute this in the second relation (6.36) and we obtain

\[ d_Q (R^{\mu} - i d_\nu B^{\mu \nu}) = 0. \] (6.51)

(iv) Once more we use theorem 4.3 and get

\[ R^{\mu} = d_Q B^{\mu} + i d_\nu B^{\mu \nu} + R^{\mu}_0 \] (6.52)

with \( R^{\mu}_0 \in \mathcal{P}_0^{(6)} \); but there is no such expression, i.e. we have \( R^{\mu}_0 = 0 \) so in fact

\[ R^{\mu} = d_Q B^{\mu} + i d_\nu B^{\mu \nu}. \] (6.53)

We substitute this in the first relation (6.36) and we obtain

\[ d_Q (R - i d_\mu B^{\mu}) = 0. \] (6.54)

(v) A last use of theorem 4.3 gives

\[ R = d_Q B + i d_\mu B^{\mu} + R_0 \] (6.55)

where \( R_0 \in \mathcal{P}_0^{(7)} \). But there is no such expression i.e. \( R_0 = 0 \) and we have

\[ R = d_Q B + i d_\mu B^{\mu}. \] (6.56)

and this finishes the massless case. The massive case brings some new terms in \( R_0 \) which survive and are given in the statement. □

**Remark 6.5.**

(i) Expression (6.40) is another example of a non-trivial element from \( \mathcal{P}_0 \cap B_Q \).

(ii) In higher orders we can have, even in the massless case, expressions \( R_0 \) which cannot be eliminated by the descent procedure. This is consistent with the usual assertion concerning the non-renormalizability of quantum gravity.
7. The interaction of gravity with other quantum fields

In [11] we have given the generic structure of the interaction between a system of Yang–Mills fields (particles of spin 1 and mass \( m \geq 0 \)) with ‘matter’ fields i.e scalar fields of spin 0 and Dirac fields of spin 1/2. In this section we consider the interaction of massless gravitons with massless Yang–Mills fields, scalar fields and Fermi fields.

First we remind the results from [11]. We consider the set of massless fields \( v^\mu_a, u_a, \tilde{u}_a, a = 1, \ldots, r \), where \( v^\mu_a \) are Bose vector fields and \( u_a, \tilde{u}_a \) are Fermi scalar fields (the ghost YM fields).

Then one considers the Hilbert space generated by these fields applied on the vacuum and we define in \( \mathcal{H} \) the operator \( Q \) according to

\[
\begin{align*}
[Q, v^\mu_a] &= i\partial^\mu u_a, \\
[Q, u_a] &= 0, \\
[Q, \tilde{u}_a] &= -i\partial^\mu v^\mu_a.
\end{align*}
\]

(7.1)

and

\[
Q\Omega = 0.
\]

(7.2)

Here \([\cdot, \cdot]\) is the graded commutator. The physical Hilbert space is the the factor space \( \mathcal{H}_{phys} = \text{Ker}(Q)/\text{Im}(Q) \). In [13] we have determined the most general interaction between these fields: such an expression is relatively cohomologous to the expression of the form:

\[
t_{YM} = f_{abc} \left( \frac{1}{2} v^\mu_a v^\nu_b F_{\nu\mu}^c + u^\mu_a v^\nu_b \tilde{u}^c d^\nu \right)
\]

(7.3)

and the constants \( f_{abc} \) are completely antisymmetric. (This is the well-known QCD interaction Lagrangian in the first order of perturbation theory.)

Now we include massless gravitation also. We include in the set of fields generating the Hilbert space \( \mathcal{H} \) the fields \( h_{\mu\nu}, u^\rho, \tilde{u}^\sigma \) the first one being a tensor fields with Bose statistics and the last are vector fields with Fermi statistics. We also extend the definition of the gauge charge \( Q \) given by (7.1) with

\[
[Q, h_{\mu\nu}] = -\frac{1}{2} (\partial_\mu u_\nu + \partial_\nu u_\mu - \eta_{\mu\nu} \partial_\rho u^\rho), \\
[Q, u_\mu] = 0, \\
[Q, \tilde{u}_\mu] = i\partial^\nu h_{\mu\nu},
\]

(7.4)

and we can easily generalize theorem 4.1 and the corresponding result for the Yang–Mills system from [11]: the Fock space describes in this case massless gravitons and and massless spin 1 particles. By definition the ghost number is the sum of the ghost numbers of the YM and gravity sectors. Besides the expression \( t_{YM} \) given above and \( t_{gh} \) determined in theorem 4.3 we need the interaction between the two sets of fields (Yang–Mills and gravitational). The result is described in the following:

**Theorem 7.1.** Suppose that the interaction Lagrangian \( T_{int} \) is restricted by Lorentz covariance, is trilinear in the fields and \( \omega(T_{int}) \leq 5, gh(T_{int}) = 0 \). Then (i) \( T_{int} \) is relatively cohomologous to the expression

\[
t_{int} = f_{ab} \left( 4h_{\mu\nu} F_{\mu\rho}^{ab} F_{\nu\sigma}^{b} - h F_{\mu\nu} F_{\rho\sigma}^{\mu\nu} + 4u^\rho d^\nu \tilde{u}^a F_b^{\mu\nu} \right)
\]

(7.5)

where the constants \( f_{ab} \) are symmetric \( f_{ab} = f_{ba} \).

(ii) The relation \( d_Q t_{int} = i\partial_\mu t_{int} \) is verified by

\[
t_{int}^\mu = f_{ab} \left( u^\mu F_{\rho\sigma}^{ab} F_{\rho\sigma} + 4u^\rho F_{\mu\nu}^{ab} F_{\nu\rho} \right)
\]

(7.6)

and we also have

\[
d_Q t_{int} = 0.
\]

(7.7)

(iii) The constants \( f_{ab} \) are real.
Proof.

(i) By hypothesis we have
\[ d_Q T = id_\mu T \int \]  \hspace{1cm} (7.8)
and the descent procedure leads to
\[ d_Q T = id_\mu T \int \int \]  \hspace{1cm} (7.9)
\[ d_Q T = id_\mu T \int \int \int \]  \hspace{1cm} (7.10)
and can choose the expressions \( T \) to be Lorentz covariant; we also have
\[ gh(T) = |I|, \quad \omega(T) \leq 5. \]  \hspace{1cm} (7.11)

From the last relation we obtain
\[ T = d_Q B + id_\sigma B + T \int \int \]  \hspace{1cm} (7.12)
so the right-hand side must be a co-boundary and a direct computation gives that in fact \( T = 0 \). It follows that
\[ T = d_Q B + id_\sigma B + T \int \int \]  \hspace{1cm} (7.13)
and
\[ d_Q (T - id_\sigma B) = 0. \]  \hspace{1cm} (7.14)

(ii) We obtain from the preceding relation that
\[ T = d_Q B + id_\sigma B + T \int \int \]  \hspace{1cm} (7.15)
where \( T \in P \) and we can choose the expressions \( B \) and \( T \int \) to be antisymmetric. Again we do not give the generic form of the expression \( T \int \), but we give the final result of this standard computation: by conveniently modifying the expressions \( B \) we can arrange such that \( T = 0 \). We substitute the expression of \( T \) in the first relation (7.9) and get:
\[ d_Q (T - id_\sigma B) = 0. \]  \hspace{1cm} (7.16)

(iii) Now it is again the time when we use known results and obtain
\[ T = d_Q B + id_\sigma B + T \int \int \]  \hspace{1cm} (7.17)
where \( T \in P \). Now we get from the first relation (7.9)
\[ d_Q (T - id_\sigma B) = 0. \]  \hspace{1cm} (7.18)
so the right-hand side must be a co-boundary. If one writes the generic form of \( T \), one gets after tedious computations that by modifying the expressions \( B \) one can take
\[ T = t \int \]  \hspace{1cm} (7.19)
with $t^\mu_{\text{int}}$ the expression from the statement of the theorem. Because we have $d_Q t^\mu_{\text{int}} = \text{id}_\mu t^\mu_{\text{int}}$, we get

$$d_Q (T_{\text{int}} - t_{\text{int}} - \text{id}_\mu B^\mu) = 0$$  \hspace{1cm} (7.20)

so known results lead to

$$T_{\text{int}} = t_{\text{int}} + d_Q B + \text{id}_\mu B^\mu + T_{\text{int.0}}$$  \hspace{1cm} (7.21)

where $T_{\text{int.0}} \in \mathcal{F}^{(5)}_0$. But there is no such expression, i.e. $T_{\text{int.0}} = 0$ and we have

$$T_{\text{int}} = t_{\text{int}} + d_Q B + \text{id}_\mu B^\mu$$  \hspace{1cm} (7.22)

which is the final result.

□

**Remark 7.2.**

(i) We note that we have obtained in a natural way the known expression of the energy–momentum tensor $T_{\mu\nu}$ which is, up to a factor, the coefficient of $h^{\mu\nu}$ from the expression $t_{\text{int}}$. However, there is a supplementary ghost term in the first line of formula (7.5). This is due to the fact already explained in the introduction: because we cannot impose in the quantum framework the Maxwell equation

$$\partial_\mu v^\mu_a = 0$$ \hspace{1cm} (7.23)

we cannot impose the divergenceless condition

$$\partial_\mu T^{\mu\nu} = 0$$ \hspace{1cm} (7.24)

and without the extra ghost term in the first line of formula (7.5) we do not have gauge invariance.

We note however that the ghost term from the first line of formula (7.5) gives a null contributions between physical states (described as in theorem 4.1) and this result propagates to all orders of perturbation theory. So it can be neglected in practical computations.

(ii) There are other approaches to the quantization of the massless vector fields in which one can impose the condition $\partial_\mu v^\mu_a = 0$ namely the so-called Coulomb gauge, but the price to pay is the loss of the manifest Lorentz covariance and the appearance of a non-local interaction term so the Epstein–Glaser method cannot be implemented in this approach.

(iii) One can prove that there is no bilinear solution for the interaction.

In the same way one can determine the generic form of the interaction of the massless gravitational field with some scalar fields $\Phi_c$, $c = 1, \ldots, s$, of masses $m_c$ and some Fermi fields $\Psi_A$, $A = 1, \ldots, N$, of masses $M_A$. One can prove that the interaction Lagrangian is relatively cohomologous to the expression

$$t = f'_{cd} \left( h_{\mu\nu} d^\mu \Phi_c d^\nu \Phi_d - \frac{m_c^2 + m_d^2}{4} h_{\mu\nu} \Phi_c \Phi_d \right) + h_{\mu\nu} (d^\mu \bar{\psi} c^\nu \gamma^\nu \gamma^\rho \psi - \bar{\psi} c^\nu \gamma^\nu \gamma^\rho d^\mu \psi)$$  \hspace{1cm} (7.26)

where (i) the expressions $f'_{cd}$ are symmetric, real and commute with the mass matrix of the scalar fields $m_{ab} = \delta_{ab} m_a$; (ii) the matrices $c^\nu$ verify

$$c^\nu M = M c^-\nu$$  \hspace{1cm} (7.27)

where $M$ is the mass matrix of the Dirac fields: $M_{AB} = M_A \delta_{AB}$ and we also have the Hermiticity property $(c^\nu)^\dagger = c^-\nu$.  

8. Conclusions

The cohomological methods presented in a previous paper [11] lead to the simple understanding of quantum gravity in lower orders of perturbation theory. We work only with the chronological products so we completely avoid the infra-red problems. We have analyzed three distinct problems with these methods.

First, the descent technique can be used to give the most general interaction including Yang–Mills fields (massless and massive), matter and massless gravity. Comparison with the usual methods [16, 12] shows that the computations are more simple (however, the computations remain highly non-trivial). In this paper we have considered only massless Yang–Mills fields and the general case will be treated in a forthcoming paper. Further restrictions follow from the cancellation of the anomalies in the second order of the perturbation theory. The analysis can be extended to the third order of perturbation theory and it will also be done elsewhere. One should expect the appearance of the known gravitational anomaly (see for instance [19].)

Second, if we use the consistency Wess–Zumino equations, we can give simple proofs for the gauge invariance in the second order of perturbation theory for the massless and massive pure gravity. And finally, one can determine the general form of possible counterterms compatible with Bogoliubov axioms and gauge invariance. As one expects, in higher orders of perturbation theory the number of counterterms is increasing. However, the cohomological analysis shows that these counterterms can be chosen, essentially, as functions of the Riemann tensor only. This type of results can be useful to a deeper understanding of quantum gravity as a consistent quantum theory.

Acknowledgments

The author wishes to thank Professor G Scharf for the critical reading of the typescript and many valuable suggestions. This paper was partially supported by the CNCSIS Programme, project IDEI 454/2009, cod CNCSIS ID-44.

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