On the Integrable Chaplygin Type Hydrodynamic Systems and Their Geometric Structure

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Received: 6 February 2020; Accepted: 28 March 2020; Published: 1 May 2020

Abstract: A class of spatially one-dimensional completely integrable Chaplygin hydrodynamic systems was studied within framework of Lie-algebraic approach. The Chaplygin hydrodynamic systems were considered as differential systems on the torus. It has been shown that the geometric structure of the systems under analysis has strong relationship with diffeomorphism group orbits on them. It has allowed to find a new infinite hierarchy of integrable Chaplygin like hydrodynamic systems.

Keywords: Lax-Sato compatibility equations; Chaplygin type hydrodynamical equations; Casimir invariants; torus diffeomorphisms; loop Lie algebra; Lie-Poisson structure

MSC: 17B68; 17B80; 35Q53; 35G25; 35N10; 37K35; 58J70; 58J72; 34A34; 37K05; 37K10

1. Introduction

The discovery of the inverse scattering method gave rise to the active development of research in the field of integrable models or solvable nonlinear partial differential equations [1–3]. The models can be called ubiquitous as they manifest themselves in many areas of physics including plasma physics, solid state physics, nonlinear optics, hydrodynamics and other both theoretical and applied research fields. Besides, they are also related to various branches of mathematics and imply beautiful structures. The following research mainly focuses on geometrical structures of the Chaplygin hydrodynamical system. These structures arise during analysis of the exact solutions and other mathematical properties of the system.

The Introduction, which is Section 1 of the present article, is followed by four sections and Conclusion. Section 2 provides the basics of the differential-geometric approach to the study of the integrable dispersionless dynamical systems, which draws upon Lie groups and algebras on the torus. Section 3 analyses the geometric structures of the one-dimensional completely integrable Chaplygin hydrodynamic system. It has been shown, that this system can be described as a system given on the torus with the corresponding group orbit structure. It allows us to find the seed differential form and the corresponding infinite hierarchy of commuting to each other Hamiltonian systems and Lax-Sato vector fields. Section 4 describes a new infinite hierarchy of the integrable Chaplygin hydrodynamic system, which is generated by the new seed element which is connected with the one found in the Section 3. The recent research [4] has revealed that these dynamical systems have strong connections with the equations of the Monge type. The geometric properties and the corresponding geometric structures were studied within the framework of the general differential systems on jet-manifolds theory, which used the embedding properties of the Grassman manifold [5]. Thus, the relations between different geometric approaches to description of the completely integrable dispersionless differential systems present a new and interesting research problem, which needs in-depth study.
2. Lie-algebraic Approach to the Vector Fields on the Torus

Let us consider the loop diffeomorphisms group \( \tilde{G} := \text{Diff}(\mathbb{T}^n) = \{ C^1 \subset S^1 \rightarrow G := \text{Diff}(\mathbb{T}^n) \} \) [6], and its subgroups \( \text{Diff}_\pm(\mathbb{T}^n) \) of holomorphically extendible elements to the interior \( \mathbb{D}_+ \subset \mathbb{C} \) and to the exterior \( \mathbb{D}_- \subset \mathbb{C} \) regions of the unit centrally located disk \( \mathbb{D}^1 \subset \mathbb{C}^1 \), respectively, and such that for any \( \tilde{g}(\lambda) \in \text{Diff}_-(\mathbb{T}^n) \), \( \lambda \in \mathbb{D}_- \), there exists a limit \( \lim_{\lambda \rightarrow \infty} \tilde{g}(\lambda) = \text{Id} \in \text{Diff}(\mathbb{T}^n) \).

The corresponding Lie subalgebras \( \mathfrak{g}_\pm = \text{Diff}_\pm(\mathbb{T}^n) \simeq \text{Vect}(\mathbb{T}^n) \) of the subgroups \( \text{Diff}_\pm(\mathbb{T}^n) \) are the holomorphic vector fields on \( \mathbb{T}^n \) in regions \( \mathbb{D}_+ \subset \mathbb{C}^1 \), respectively, the split Lie algebra \( \mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_- \), and for any \( \tilde{a}(\lambda) \in \text{Diff}_-(\mathbb{T}^n) \) we have \( \lim_{\lambda \rightarrow \infty} \tilde{a}(\lambda) = 0 \). The adjoint space \( \mathfrak{g}^* := \mathfrak{g}_+^* \oplus \mathfrak{g}_-^* \), where the space \( \mathfrak{g}_+^* \subset \Gamma(\mathbb{T}_C^\mathbb{C}; T^* \mathbb{T}^n) \) consists, respectively, of the differential forms on the torus \( \mathbb{T}^n \), accordingly holomorphic on the set \( \mathbb{D}_+ \subset \mathbb{C}^1 \), and the adjoint space \( \mathfrak{g}_-^* \subset \Gamma(\mathbb{T}_C^\mathbb{C}; T^* \mathbb{T}^n) \) consists, respectively, of the differential forms on the torus \( \mathbb{T}^n \), respectively holomorphic on the set \( \mathbb{D}_- \).

Under this construction the space \( \mathfrak{g}_+^* \) is dual to \( \mathfrak{g}_- \) and \( \mathfrak{g}_-^* \) is dual to \( \mathfrak{g}_+ \) with respect to the following nondegenerate convolution form on the product \( \mathfrak{g}^* \times \mathfrak{g} \)

\[
(\tilde{l}|\tilde{a}) := \text{res}_\lambda \int_{\mathbb{T}^n} (l, a) \, dx
\]

for any vector field \( \tilde{a} := \langle a(x), \partial_x \rangle \in \mathfrak{g} \) and differential form \( \tilde{l} := \langle l(x), dx \rangle \in \mathfrak{g}^* \) on \( \mathbb{T}^n \), depending on the coordinate \( x := (\lambda; x) \in \mathbb{T}^n \), where \( \langle \cdot, \cdot \rangle \) is the scalar product on the Euclidean space \( \mathbb{R}^{n+1} \) and \( \frac{\partial}{\partial x} := \left( \frac{\partial}{\partial \lambda}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right)^T \) is the gradient vector. We can define set \( I(\mathfrak{g}^*) \) of smooth functionals \( h : \mathfrak{g}^* \rightarrow \mathbb{R} \) on the adjoint space \( \mathfrak{g}^* \), which are Casimir invariant at a seed element \( \tilde{l} \in \mathfrak{g}^* \) via the co-adjoint Lie algebra \( \mathfrak{g} \) action

\[
ad_{\nabla h(l)}\tilde{l} = 0.
\]

Then a wide class of multi-dimensional dispersionless completely integrable commuting to each other Hamiltonian systems can be generated [7–9] within the framework of Adler-Kostant-Symes scheme [3,10–12] via

\[
\frac{dl}{dl} = -ad_{\nabla h(l)}l,
\]

for all \( h \in I(\mathfrak{g}^*) \), \( \nabla h(l) := \nabla h_+ (l) \oplus \nabla h_- (l) \in \mathfrak{g}_+ \oplus \mathfrak{g}_- \), on the corresponding functional manifolds. The flows Equation (3) can also be represented on the functional space \( C^2(\mathbb{T}^n; \mathbb{C}) \) as a commuting system of Lax-Sato type [7] vector field equations.

The loop Lie algebra \( \mathfrak{g} \), which was defined before, is used to provide thorough description of the Lax-Sato type compatibly systems. The elements of the Lie algebra are presented as \( a(x; \lambda) := \langle a(x; \lambda), \partial_x \rangle = \sum_{j=1}^{n} a_j(x; \lambda) \partial_{x_j} + a_0(x; \lambda) \frac{\partial}{\partial \lambda} \in \mathfrak{g} \) for some holomorphic vectors \( a(x; \lambda) \in \mathbb{E} \times \mathbb{E}^n \) in \( \lambda \in \mathbb{D}_\pm \) for all \( x \in \mathbb{T}^n \), where \( \frac{\partial}{\partial \lambda} \) is the generalized Euclidean vector gradient defined before. The Lie algebra \( \mathfrak{g} \) allows the natural split into the direct sum of two subalgebras:

\[
\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-.
\]

It enables the introduction of the classical \( \mathcal{R} \)-structure on \( \mathfrak{g} \) for any \( \tilde{a}, \tilde{b} \in \mathfrak{g} \):

\[
[\tilde{a}, \tilde{b}]_\mathcal{R} := [\mathcal{R}\tilde{a}, \tilde{b}] + [\tilde{a}, \mathcal{R}\tilde{b}],
\]

where

\[
\mathcal{R} := (P_+ - P_-)/2
\]

and

\[
P_\pm \mathfrak{g} := \mathfrak{g}_\pm \subset \mathfrak{g}.
\]
Being adjoint to the Lie algebra $\tilde{G}$ of vector fields on $\mathbb{T}^n$ the space $\tilde{G}^* \simeq \tilde{A}^1(\mathbb{T}^n)$ can be identified with $\tilde{G}$ with regard to the metric Equation (1). Thus, it is possible to determine two Lie–Poisson brackets

$$\{f, g\} := (\tilde{l}, [\nabla f(\tilde{l}), \nabla g(\tilde{l})])$$

(8)

and

$$\{f, g\}_R := (\tilde{l}, [\nabla f(\tilde{l}), \nabla g(\tilde{l})]_R),$$

(9)

for arbitrary $f, g \in \text{D}(\tilde{G}^*)$, with the gradient elements $\nabla f(\tilde{l})$ and $\nabla g(\tilde{l}) \in \tilde{G}$ being calculated with respect to the metric Equation (1) at any seed element $\tilde{l} \in \tilde{G}^*$.

Let us define the generators of the Hamiltonian vector field for some special integers $p_y, p_t \in \mathbb{Z}_+$ in the following way

$$\nabla h^{(y)}_+(l) := (\nabla \gamma^{(p_y)}(l))|_+, \quad \nabla h^{(t)}_+(l) := (\nabla \gamma^{(p_t)}(l))|_+, \quad \nabla \gamma := \nabla \gamma^{(p)} \sim \lambda^p \sum_{j \in \mathbb{Z}_+} \nabla \gamma_j^{(p)} \lambda^{-j}. \quad \text{(10)}$$

(10)

The condition Equation (10) can be rewritten as

$$l \left( \frac{\partial}{\partial x} \nabla \gamma(l) \right) + \left( \nabla \gamma(l), \frac{\partial}{\partial x} \right) l + \left( l, \left( \frac{\partial}{\partial x} \nabla \gamma(l) \right) \right) = 0 \quad \text{(11)}$$

(11)

for the Casimir function $\gamma \in \text{D}(\tilde{G}^*)$ and should be solved analytically. When the seed element $\tilde{l} \in \tilde{G}^*$ is singular when $|\lambda| \to \infty$, the following asymptotic expansion is used for some respectively chosen $p \in \mathbb{Z}_+$

$$\nabla \gamma := \nabla \gamma^{(p)} \sim \lambda^p \sum_{j \in \mathbb{Z}_+} \nabla \gamma_j^{(p)} \lambda^{-j}. \quad \text{(12)}$$

(12)

The expansion Equation (13) should be used in Equation (12), which allows to solve the Equation (12) recurrently.

Let us define the generators of the Hamiltonian vector field for some special integers $p_y, p_t \in \mathbb{Z}_+$ in the following way

$$\nabla h^{(y)}_+(l) := (\nabla \gamma^{(p_y)}(l))|_+, \quad \nabla h^{(t)}_+(l) := (\nabla \gamma^{(p_t)}(l))|_+, \quad \nabla \gamma := \nabla \gamma^{(p)} \sim \lambda^p \sum_{j \in \mathbb{Z}_+} \nabla \gamma_j^{(p)} \lambda^{-j}. \quad \text{(13)}$$

(13)

for the Casimir invariants $h^{(y)}, h^{(t)} \in \text{I}(\tilde{G}^*)$ defined above. These invariants generate two commuting flows:

$$\frac{dl}{dt} = - \left( \frac{\partial}{\partial x}, \nabla h^{(y)}_+(l) \right)_l - \left( l, \left( \frac{\partial}{\partial x} \nabla h^{(t)}_+(l) \right) \right), \quad \text{(14)}$$

(14)

$$\frac{dl}{dy} = - \left( \frac{\partial}{\partial x}, \nabla h^{(y)}_+(l) \right)_l - \left( l, \left( \frac{\partial}{\partial x} \nabla h^{(t)}_+(l) \right) \right), \quad \text{(15)}$$

(15)

where $y, t \in \mathbb{R}$ serve as the corresponding evolution parameters. The flows Equations (14) and (15) commute to each other owing to the commutations of $h^{(l)}(l), h^{(y)}(l) \in \text{I}(\tilde{G}^*)$ with respect to the bracket Equation (9). As a result the Lax-Sato compatibility condition

$$\frac{\partial}{\partial y} \nabla h^{(l)}_+(l) - \frac{\partial}{\partial t} \nabla h^{(y)}_+(l) = \left[ \nabla h^{(l)}_+(l), \nabla h^{(y)}_+(l) \right]$$

(16)

(16)
is satisfied for the generators of the Hamiltonian vector fields

\[ \nabla h^{(t)}(\bar{l}) := \left\langle \nabla h^{(t)}(l), \frac{\partial}{\partial x} \right\rangle , \]

(19)

\[ \nabla h^{(y)}(\bar{l}) := \left\langle \nabla h^{(y)}(l), \frac{\partial}{\partial x} \right\rangle , \]

(20)

for all parameters \( t, y \in \mathbb{R} \). The Equation (18) is equivalent to the compatibility condition of two linear equations

\[ \left( \frac{\partial}{\partial t} + \nabla h^{(t)}(\bar{l}) \right) \psi = 0, \]

(21)

\[ \left( \frac{\partial}{\partial y} + \nabla h^{(y)}(\bar{l}) \right) \psi = 0 \]

(22)

for all \( y, t \in \mathbb{R} \), any \( \lambda \in \mathbb{C} \) and some function \( \psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^n; \mathbb{C}) \). Thus, the following proposition holds.

**Proposition 1.** Let us be given the loop Lie algebra \( \tilde{G} \), metric \((\cdot | \cdot)\) and co-adjoint action on the \( \tilde{G}^* \). Then the seed vector field \( \tilde{l} \in \tilde{G}^* \) generates the dynamical systems

\[ \frac{\partial \tilde{l}}{\partial t} = -ad^*_{\nabla h^{(t)}(\bar{l})} \tilde{l}, \]

(23)

\[ \frac{\partial \tilde{l}}{\partial y} = -ad^*_{\nabla h^{(y)}(\bar{l})} \tilde{l} \]

(24)

which are the commuting Hamiltonian flows for all \( t, y \in \mathbb{R} \), and where \( h^{(t)}, h^{(y)} \in I(\tilde{G}^*) \) are the Casimir functions. The vector fields representations Equations (21) and (22) are the consequence of the compatibility condition Equation (18) of the flows Equations (23) and (24).

**Remark 1.** In case when chosen seed element \( \tilde{l} \in \tilde{G}^* \) is singular when \( |\lambda| \to 0 \) the expression Equation (13) should be taken as

\[ \nabla \gamma^{(p)}(l) \sim \lambda^{-p} \sum_{j \in \mathbb{Z}_+} \nabla \gamma^{(j)}(l) \lambda^j \]

(25)

where \( p \in \mathbb{Z}_+ \) are correspondingly chosen integers. Then the reduced gradients of the Casimir function should be represented by the generators

\[ \nabla h^{(y)}(l) := (\lambda^{-p_y} \nabla \gamma^{(p_y)}(l))_-, \]

(26)

\[ \nabla h^{(t)}(l) := (\lambda^{-p_t} \nabla \gamma^{(p_t)}(l))_- \]

for some chosen positive integers \( p_y, p_t \in \mathbb{Z}_+ \). Consequently, the Hamiltonian flows in this case have the following representation

\[ \frac{\partial \tilde{l}}{\partial t} = ad^*_{\nabla h^{(t)}(l)} \tilde{l}, \]

(27)

\[ \frac{\partial \tilde{l}}{\partial y} = ad^*_{\nabla h^{(y)}(l)} \tilde{l} \]

(28)

It is possible to consider a wider class of the integrable dispersionless equations, which can be considered as compatible Hamiltonian flows with respect to the semidirect product \( \tilde{G} \ltimes \tilde{G}^* \) of the holomorphic loop Lie algebra \( \tilde{G} \) of vector fields on the torus \( \mathbb{T}^n \) and its regular co-adjoint space \( \tilde{G}^* \), with additional corresponding cocycles in the frame of the Ovsienko’s scheme \([8,9]\).

Using Kostant-Souriau-Symes scheme it allows to construct the Hamiltonian flows on the Lie algebra...
$\mathcal{G} \times \mathcal{G}^*$, which can be identified with its adjoint space $(\mathcal{G} \times \mathcal{G}^*)^*$. This interesting problem will be considered in the forthcoming paper.

3. The Structure of the Group Orbit of the Chaplygin Hydrodynamical System

In this section we apply the scheme, which was developed above, to the Chaplygin hydrodynamic system, which is also known as the Chaplygin gas equation [13–15] and widely used in the cosmological models of universe

\begin{equation}
    u_t = -uu_x - kv_x v^{-3}, \\
    v_t = -(uv)_x.
\end{equation}

Here $k \in \mathbb{R}$ is a constant parameter, $2\pi$-periodic dynamical variables $(u,v) \in M \subset C^\infty(\mathbb{R}/2\pi \mathbb{Z};\mathbb{R}^2)$ are given on the functional manifold $M$, and $t \in \mathbb{R}$ is the evolution parameter. Let us take the loop Lie algebra $\hat{\mathcal{G}} := \text{diff}(\mathbb{T}^1)$, which is defined on the one-dimensional torus $\mathbb{T}^1$ and a seed element $\hat{l} \in \hat{\mathcal{G}}^*$ in the following form:

\begin{equation}
\hat{l} = \left[ \left( \frac{1}{8} \lambda_x + uu_x \right) \lambda + \frac{1}{2} u_x \lambda^3 \right] dx + \left[ \frac{3}{8} \left( \alpha + 4u^2 \right) + \frac{5}{2} u \lambda^2 + \lambda^4 \right] d\lambda,
\end{equation}

where for simplicity we introduced $\alpha := kv^{-2} + u^2$. Let us define gradients of some Casimir functionals $h^{(g)}, h^{(t)}$ and $h^{(s)} \in \mathcal{I}(\hat{\mathcal{G}}^*)$:

\begin{equation}
\begin{align*}
\nabla h^{(g)}(l) &:= \nabla h^{(2)}(l), \\
\nabla h^{(t)}(l) &:= \nabla h^{(4)}(l), \\
\nabla h^{(s)}(l) &:= \nabla h^{(6)}(l),
\end{align*}
\end{equation}

where

\begin{equation}
\begin{align*}
\nabla h^{(2)}(l) &= \left( \begin{array}{c} -2 \\ 0 \end{array} \right) \lambda^2 + \left( \begin{array}{c} 0 \\ u_x \end{array} \right) \lambda^1 + \left( \begin{array}{c} u \\ 0 \end{array} \right) \lambda^0 + O(\lambda^{-1}) \\
\nabla h^{(4)}(l) &= \left( \begin{array}{c} -8 \\ 0 \end{array} \right) \lambda^4 + \left( \begin{array}{c} 0 \\ 4u_x \end{array} \right) \lambda^3 + \left( \begin{array}{c} -4u \\ 0 \end{array} \right) \lambda^2 \\
&+ \left( \begin{array}{c} 0 \\ \alpha_x \end{array} \right) \lambda^1 + \left( \begin{array}{c} \alpha \\ 0 \end{array} \right) \lambda^0 + O(\lambda^{-1})
\end{align*}
\end{equation}

and

\begin{equation}
\begin{align*}
\nabla h^{(6)}(l) &= \left( \begin{array}{c} -16 \\ 0 \end{array} \right) \lambda^6 + \left( \begin{array}{c} 0 \\ uu_x \end{array} \right) \lambda^5 + \left( \begin{array}{c} -3u \\ 0 \end{array} \right) \lambda^4 \\
&+ \left( \begin{array}{c} 0 \\ \alpha_x / 4 + uu_x \end{array} \right) \lambda^3 + \left( \begin{array}{c} -\alpha / 4 - 1/2u^2 \\ 0 \end{array} \right) \lambda^2 \\
&+ \left( \begin{array}{c} 0 \\ -(u\alpha)_x / 8 \end{array} \right) \lambda^1 + \left( \begin{array}{c} u\alpha / 8 \\ 0 \end{array} \right) \lambda^0 + O(\lambda^{-1}).
\end{align*}
\end{equation}

and calculate asymptotic expansions when $\lambda \to \infty$. Then the Lax-Sato vector field generators will be equal to the expressions

\begin{equation}
\begin{align*}
\nabla h^{(t)}_+(l) &:= (\nabla h^{(2)}(l))_+ = \left( \begin{array}{c} -2 \\ 0 \end{array} \right) \lambda^2 + \left( \begin{array}{c} 0 \\ u_x \end{array} \right) \lambda^1 + \left( \begin{array}{c} u \\ 0 \end{array} \right) \lambda^0, \\
\nabla h^{(g)}_+(l) &:= (\nabla h^{(4)}(l))_+ = \left( \begin{array}{c} -8 \\ 0 \end{array} \right) \lambda^4 + \left( \begin{array}{c} 0 \\ 4u_x \end{array} \right) \lambda^3 \\
&+ \left( \begin{array}{c} 0 \\ \alpha_x \end{array} \right) \lambda^1 + \left( \begin{array}{c} \alpha \\ 0 \end{array} \right) \lambda^0,
\end{align*}
\end{equation}

\begin{equation}
\begin{align*}
\nabla h^{(s)}_+(l) &:= (\nabla h^{(6)}(l))_+ = \left( \begin{array}{c} -16 \\ 0 \end{array} \right) \lambda^6 + \left( \begin{array}{c} 0 \\ uu_x \end{array} \right) \lambda^5 \\
&+ \left( \begin{array}{c} 0 \\ \alpha_x / 4 + uu_x \end{array} \right) \lambda^3 + \left( \begin{array}{c} -\alpha / 4 - 1/2u^2 \\ 0 \end{array} \right) \lambda^2 \\
&+ \left( \begin{array}{c} 0 \\ -(u\alpha)_x / 8 \end{array} \right) \lambda^1 + \left( \begin{array}{c} u\alpha / 8 \\ 0 \end{array} \right) \lambda^0.
\end{align*}
\end{equation}
\[ \nabla h^s(l) := (\nabla h^t(l))|_+ = \left( \begin{array}{c} -2 \\ 0 \\ u_x \end{array} \right) \lambda^6 + \left( \begin{array}{c} 0 \\ u_x \end{array} \right) \lambda^5 + \]
\]
\[ + \left( \begin{array}{c} -3u \\ 0 \\ 0 \end{array} \right) \lambda^4 + \left( \begin{array}{c} 0 \\ 1/4a_x + uu_x \end{array} \right) \lambda^3 + \]
\[ + \left( \begin{array}{c} -1/4a - 1/2u^2 \\ 0 \\ 0 \end{array} \right) \lambda^2 + \left( \begin{array}{c} 0 \\ 0 \\ -1/8(ua)_x \end{array} \right) \lambda^1 + \left( \begin{array}{c} 1/8u \alpha \\ 0 \\ 0 \end{array} \right) \lambda^0, \]
\]
\[ \text{as } \lambda \to \infty. \] Substituting Equations (34), (36) and (38) into Equation (3) we obtain the representation of the evolution flow with respect to the evolution parameter \( t \in \mathbb{R} \):
\[ \frac{\partial \bar{l}}{\partial t} = -ad^*_{\nabla h^t(l)} \bar{l} \sim \begin{cases} u_t = -\frac{1}{2}(u^2 - kv^2)_x, \\ v_t = -((u^2)_{xx}), \end{cases} \]
\[ \text{which coincides with the hydrodynamical system Equation (29), the representation of the evolution flow with respect to the evolution parameter } y \in \mathbb{R}, \]
\[ \frac{\partial \bar{l}}{\partial y} = -ad^*_{\nabla h^y(l)} \bar{l} \sim \begin{cases} u_y = -[u(u^2/3 - kv^2)]_x, \\ v_y = -([u^2 + kv^2]v)_x, \end{cases} \]
\[ \text{and the representation of the evolution flow with respect to the evolution parameter } s \in \mathbb{R}, \]
\[ \frac{\partial \bar{l}}{\partial s} = -ad^*_{\nabla h^s(l)} \bar{l} \sim \begin{cases} u_s = -\frac{1}{3}(-3a^2 + 4u^4)_x \\ v_s = -\frac{1}{3}[(u^2 + kv^2)uv]_x, \end{cases} \]
\[ \text{The flows Equations (39)–(41) are commuting to each other by construction. This can be presented as a set of Lax-Sato type vector fields} \]
\[ \left[ \frac{\partial}{\partial t} + \nabla h^t(l), \frac{\partial}{\partial y} + \nabla h^y(l) \right] = 0, \]
\[ \left[ \frac{\partial}{\partial t} + \nabla h^t(l), \frac{\partial}{\partial s} + \nabla h^s(l) \right] = 0, \]
\[ \left[ \frac{\partial}{\partial s} + \nabla h^s(l), \frac{\partial}{\partial y} + \nabla h^y(l) \right] = 0. \]
\[ \text{which are commuting to each other for all parameters } t, y \text{ and } s \in \mathbb{R} \text{ on the torus } \mathbb{T}^4. \] Consequently, we have obtained three new compatible systems of integrable dispersionless equations.

4. The Geometric Structure of the Chaplygin Type Hydrodynamical Systems

The seed element Equation (30) generates the gradient expressions Equations (31)–(33) of the Casimir functionals \( h^t, h^y \) and \( h^s \in I(\tilde{G}^*) \). This Section shows that the analytical form of the gradient expressions of the Casimir functionals \( h^t, h^y \) and \( h^s \in I(\tilde{G}^*) \) incorporates the covert geometric structure. The following theorem is true.
Theorem 1. Let the gradient vector field $\nabla h^{(y)}(\vec{I}) := \nabla h^{(4)}(\vec{I}) \in \mathcal{G}$ be coinciding with that given by the expression Equation (32). Then the shifted vector field $\nabla \hat{h}^{(t)}(\vec{I}) := \lambda^{-2} \nabla h^{(y)}(\vec{I}) \in \mathcal{G}$, where

$$
\nabla \hat{h}^{(t)}(\vec{I}) := \lambda^{-2} \nabla h^{(y)}(\vec{I}) = \begin{pmatrix} -8 \\ 0 \\ 0 \\ 0 \end{pmatrix} \lambda \lambda^2 + \begin{pmatrix} 0 \\ 4u_x \end{pmatrix} \lambda \lambda^1 + \begin{pmatrix} -4u \\ 0 \end{pmatrix} \lambda \lambda^0 + \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \lambda \lambda^{-1} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \lambda \lambda^{-2} + O(\lambda^{-3}),
$$

as $\lambda \to \infty$, is related to the Casimir functional $\hat{h}^{(t)}(\vec{I}) \in 1(\mathcal{G}^*)$ with respect to the new seed element

$$
\hat{I} = \left[u_x \lambda^2/2 + (\alpha + u^2)_x/8 \right] dx + (\lambda^3 + u \lambda) d\lambda = d\left[(\alpha + u^2)/8 + u \lambda^2/2 + \lambda^4/4 \right] := d\hat{a},
$$

defined on the torus $T^1$, and generating additional Casimir functionals $\hat{h}^{(y_1)}$ and $\hat{h}^{(s_1)} \in 1(\mathcal{G}^*)$ on the adjoint space $\mathcal{G}^*$, whose gradients are equal to the following expressions:

$$
\nabla \hat{h}^{(y_1)}(\vec{I}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \lambda^3 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} u \\ -u_x \end{pmatrix} \lambda \lambda^1 + \begin{pmatrix} \frac{u}{2} \\ -\frac{u}{2} \end{pmatrix} \lambda \lambda^0 + O(\lambda^{-1}),
$$

$$
\nabla \hat{h}^{(s_1)}(\vec{I}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \lambda^4 + \begin{pmatrix} 1 \\ -\frac{u}{2} \end{pmatrix} \lambda^3 + \begin{pmatrix} \frac{3u}{2} \\ -\frac{u}{2} \end{pmatrix} \lambda \lambda^2 + \begin{pmatrix} \frac{1}{5}(\alpha + 2u^2) \\ -\frac{1}{5} \end{pmatrix} \lambda \lambda^1 + \begin{pmatrix} \frac{1}{5}(\alpha + 4u u_x) \\ -\frac{1}{5} \end{pmatrix} \lambda \lambda^0 + O(\lambda^{-1}),
$$

generating the compatible evolution flows:

$$
u_{t_1} = -\frac{1}{2} u_x, \quad v_{t_1} = \frac{1}{2} u v_x,$$

$$
u_{s_1} = -\frac{1}{8} (u x)_x, \quad v_{s_1} = \frac{1}{8} (v u_x(u^2 - \frac{k}{v^2}),$$

$$
u_{y_1} = 2ax, \quad v_{y_1} = -4uv_x$$

on the functional manifold $M$ with respect to the evolution parameters $t_1, y_1$ and $s_1 \in \mathbb{R}$. Moreover, the following differential form relationship

$$\hat{I} \wedge d\lambda - d(2\hat{I} + 3/4u^2 d\lambda) = 0$$

holds on $T^1$ for all points $(u, v) \in M$.

Proof. It is easy to check that the functional gradient Equation (43) satisfies the relationships

$$ad^*_{\lambda^2 \nabla h^{(y)}(\vec{I})} \hat{I} = 0$$

on the torus $T^1$, determining the new seed element Equation (44), on whose basis one derives the Casimir gradients equalities

$$ad^*_{\nabla h^{(t)}(\vec{I})} \hat{I} = 0,$$

$$ad^*_{\nabla h^{(y_1)}(\vec{I})} \hat{I} = 0$$
for suitably chosen Casimir functionals $\hat{h}^{(t_1)}$ and $\hat{h}^{(y_1)} \in \mathcal{I}(\tilde{G}^*)$. Moreover, it is possible to check that the following differential equalities

$$d\tilde{l} - d(2\hat{a} + 3/4u^2) \wedge d\lambda = 0, \quad \tilde{l} = d\hat{a},$$

(51)

where $\hat{a} := (a + u^2)/8 + u\lambda^2/2 + \lambda^4/4$, hold on the torus $\mathbb{T}^1$, giving rise to the relationship Equation (48). Taking into account that the gradients Equations (43), (45) and (46) are generated by Casimir functionals $\hat{h}^{(t_1)}, \hat{h}^{(y_1)}$ and $\hat{h}^{(s_1)} \in \mathcal{I}(\tilde{G}^*)$ on the adjoint space $\tilde{G}^*$, we infer that the Hamiltonian evolution flows Equation (50) are the new compatible completely integrable Chaplygin type hydrodynamic systems on the functional manifold $M$, which proves the theorem.

As a simple consequence of Theorem 1 we obtain that the commuting Hamiltonian flows

$$\frac{\partial \tilde{l}}{\partial t_1} := -ad^*_{\nabla \hat{h}^{(t_1)}(\tilde{l})},$$

$$\frac{\partial \tilde{l}}{\partial y_1} := -ad^*_{\nabla \hat{h}^{(y_1)}(\tilde{l})},$$

$$\frac{\partial \tilde{l}}{\partial s_1} := -ad^*_{\nabla \hat{h}^{(s_1)}(\tilde{l})},$$

(52)

coinciding with Equation (47) on the functional manifold $M$ and generating, respectively, the corresponding two compatible Lax-Sato vector fields on the torus $\mathbb{T}^1$:

$$\left[ \frac{\partial}{\partial t_1} + \nabla \hat{h}^{(t_1)}(\tilde{l}), \frac{\partial}{\partial s_1} + \nabla \hat{h}^{(s_1)}(\tilde{l}) \right] = 0$$

(53)

for all $t_1, s_1 \in \mathbb{R}$, which are equivalent to a new integrable heavenly type system of dispersionless equations on the manifold $M$. It can be observed that the flows with respect to evolution parameters $t_1$ and $y_1 \in \mathbb{R}$ are scaling equivalent.

5. Conclusions

In the present research the one-dimensional completely integrable Chaplygin dispersionless systems has been studied within the framework of the Lie-algebraic approach to the vector fields on the torus. It has been demonstrated that Chaplygin hydrodynamic system has hidden geometrical structure which is manifested through the existence of the dual seed element which generates the hierarchy of the new evolution systems. The corresponding Casimir functionals proved to be connected with the origin functionals via the affine shifting symmetry.

Funding: This research received no external funding.

Acknowledgments: The author wants to express his deep gratitude to Artur Sergueyev for the unique opportunity of discussion of the findings of the present research at the conference "Dynamics, Geometry and Analysis: 20 years of Mathematical Institute in Opava” held on 8–13 September 2019 in Opava, Czech Republic.

Conflicts of Interest: The author declares no conflict of interest.

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