Thermodynamics of black holes in the deformed Hořava-Lifshitz gravity

Yun Soo Myung

Institute of Basic Science and School of Computer Aided Science
Inje University, Gimhae 621-749, Korea

Abstract

We study thermodynamics of black holes in the deformed Hořava-Lifshitz gravity with coupling constant $\lambda$. For $\lambda = 1$, the black hole behaves the Reissner-Norström black hole. Hence, this is different from the Schwarzschild black hole of Einstein gravity. A connection to the generalized uncertainty principle is explored to understand the Hořava-Lifshitz black holes.

\textsuperscript{1}e-mail address: ysmyung@inje.ac.kr
1 Introduction

Recently Hořava has proposed a renormalizable theory of gravity at a Lifshitz point\cite{1}, which may be regarded as a UV complete candidate for general relativity. Very recently, the Hořava-Lifshitz gravity theory has been intensively investigated in \cite{2,3,4,5,6,7,8,9,10,11} and its cosmological applications have been studied in \cite{12,13,14,15,16,17}.

Introducing the ADM formalism where the metric is parameterized \cite{18}
\begin{equation}
    ds^2_{ADM} = -N^2dt^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt),
\end{equation}
the Einstein-Hilbert action can be expressed as
\begin{equation}
    S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{gN} \left( K_{ij}K^{ij} - K^2 + R - 2\Lambda \right),
\end{equation}
where $G$ is Newton’s constant and extrinsic curvature $K_{ij}$ takes the form
\begin{equation}
    K_{ij} = \frac{1}{2N} \left( g_{ij} - \nabla_i N_j - \nabla_j N_i \right).
\end{equation}
Here, a dot denotes a derivative with respect to $t$.

We would like to mention that the IR vacuum of this theory is anti de Sitter (AdS) spacetimes. Hence, it is interesting to take a limit of the theory, which may lead to a Minkowski vacuum in the IR sector. To this end, one may modify the theory by introducing “$\mu^4R$” and then, take the $\Lambda_W \to 0$ limit. This does not alter the UV properties of the theory, but it changes the IR properties. That is, there exists a Minkowski vacuum, instead of an AdS vacuum. Then, one finds the black hole solution, deviating from the Schwarzschild solution.

A deformed action of the non-relativistic renormalizable gravitational theory is given by \cite{19}
\begin{equation}
    S_{HL} = \int dt d^3x \left( \mathcal{L}_0 + \tilde{\mathcal{L}}_1 \right),
\end{equation}
\begin{equation}
    \mathcal{L}_0 = \sqrt{gN} \left\{ \frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2(\Lambda_W R - 3\Lambda_W ^2)}{8(1 - 3\lambda)} \right\},
\end{equation}
\begin{equation}
    \tilde{\mathcal{L}}_1 = \sqrt{gN} \left\{ \frac{\kappa^2 \mu^2(1 - 4\lambda)}{32(1 - 3\lambda)} R^2 - \frac{\kappa^2}{2\mu^2} \left( C_{ij} - \frac{\mu^2}{2} R_{ij} \right) \left( C^{ij} - \frac{\mu^2}{2} R^{ij} \right) + \mu^4 R \right\}
\end{equation}
where $C_{ij}$ is the Cotton tensor
\begin{equation}
    C^{ij} = \epsilon^{ik\ell} \nabla_k \left( R^\ell_j - \frac{1}{4} R^{\ell} \delta^j_\ell \right) = \epsilon^{ik\ell} \nabla_k R^\ell_j - \frac{1}{4} \epsilon^{ikj} \partial_\ell R. \tag{6}
\end{equation}
Comparing $\mathcal{L}_0$ with Eq.(2) of general relativity, the speed of light, Newton’s constant and the cosmological constant are given by

\[
c = \frac{\kappa^2 \mu}{4} \sqrt{\frac{\Lambda_W}{1 - 3\lambda}}, \quad G = \frac{\kappa^2}{32\pi} c, \quad \Lambda = \frac{2}{3} \Lambda_W. \tag{7}
\]

The equations of motion were derived in [20] and [21], but we do not write them due to the length.

2 Thermodynamics of Hořava-Lifshitz black holes

In this section, we investigate the black hole solution in the limit of $\Lambda_W \to 0$ and its thermodynamic properties. For this purpose, considering $N^i = 0$, the spherically symmetric solutions could be obtained with the metric ansatz [20, 22, 23, 24, 25, 26, 27, 28]

\[
ds_{SS}^2 = -N^2(r) \, dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2).
\tag{8}
\]

Considering the minimal theory, we obtain the Schwarzschild black hole whose metric function is given by

\[
f_S = 1 - \frac{2m}{r}. \tag{9}
\]

In order to obtain the solution, let us substitute the metric ansatz [8] into the action, and then vary the functions $N$ and $f$. This is possible because the metric ansatz shows all the allowed singlets which are compatible with the $SO(3)$ action on the $S^2$. The reduced Lagrangian is given by

\[
\tilde{\mathcal{L}}_1 = \frac{\kappa^2 \mu^2 N}{8(1 - 3\lambda)\sqrt{f}} \left( \frac{\lambda - 1}{2} f'^2 - \frac{2\lambda(f - 1)}{r} f' + \frac{(2\lambda - 1)(f - 1)^2}{r^2} - 2\omega(1 - f - rf') \right), \tag{10}
\]

with $\omega = 8\mu^2(3\lambda - 1)/\kappa^2$. For $\lambda = 1$ ($\omega = 16\mu^2/\kappa^2$), we have a solution where $f$ and $N$ are determined to be

\[
N^2 = f = 1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega m)} \tag{11}
\]

where $m$ is an integration constant related to the mass.

As is shown in Fig. 1, from the condition of $f(r_\pm) = 0$, the outer (inner) horizons are given by

\[
r_\pm = m \left[ 1 \pm \sqrt{1 - \frac{1}{2\omega m^2}} \right]. \tag{12}
\]

In order to have a black hole solution, it requires that

\[
m^2 \geq \frac{1}{2\omega}. \tag{13}
\]
Figure 1: Graphs of two horizons \((r_{\pm})\) and mass \(m\) with \(\omega = 1\). Left graph: the upper solid curve represents the outer horizon \(r_+(m, \omega)\) and the lower shows the inner horizon \(r_-(m, \omega)\). The dashed line denotes the Schwarzschild black hole. Right graph: \(m(r_{\pm}, \omega)\) with \(\omega = 1\). For \(r_- < r_e = 0.71\), the solid curve represents \(m(r_-, \omega)\) while for \(r_+ > r_e\), it represents \(m(r_+, \omega)\). The dashed line denotes the Schwarzschild black hole.

Figure 2: Extremal mass graph. The curve denotes the extremal black hole \(m_e(\omega)\). The naked singularity regions are area below the curve while the non-extremal black hole area are area above the curve.

Further, the degenerate horizon

\[
r_e = m_e = \frac{1}{\sqrt{2\omega}}
\]

appears when

\[
m^2 = m_e^2.
\]

A graph of \(m_e(\omega)\) is shown in Fig. 2, splitting the whole region into the non-extremal black holes and the naked singularity regions. We do not have such a curve in the Schwarzschild black hole because the Schwarzschild black hole has a single horizon only.
Figure 3: Temperature and heat capacity graphs. Left graphs: $T(r_+, \omega)$ with $\omega = 1$. We observe that $T(r_e, \omega) = 0$ at $r_e = 0.71$ while $T = T_m$ at $r_m = 1.64$. The dashed curve denotes the temperature $T_S$ of the Schwarzschild black hole. Right graph: $C(r_+, \omega)$ shows a blow-up point at $x_m = 1.64$, dividing it into $C > 0$ and $C < 0$. Note that $C(r_e, \omega) = 0$. The dashed curve denotes the heat capacity $C_S$ of the Schwarzschild black hole.

The mass function is defined as

$$m(r_\pm, \omega) = \frac{1 + 2\omega r^2_\pm}{4\omega r_\pm}.$$ \hspace{1cm} (16)

The mass is depicted in Fig. 1. For large $r_+$, it reduces to the mass of Schwarzschild black hole ($m_S = r_+/2$).

The temperature is defined by

$$T = \frac{f'}{4\pi} \bigg|_{r=r_+} = \frac{2\omega r^2_+ - 1}{8\pi(\omega r^2_+ + r_+)}.$$ \hspace{1cm} (17)

while for large $r_+$, it leads to the Schwarzschild case

$$T_S = \frac{1}{4\pi r_+}.$$ \hspace{1cm} (18)

We confirm that $T = 0$ for the extremal black hole at $r_e = 1/\sqrt{2\omega}$. As is depicted in Fig. 3, we observe that the maximum point $r_+ = r_m$ appears as

$$x_m = \frac{\sqrt{5\sqrt{33}}}{2\sqrt{\omega}}$$ \hspace{1cm} (19)

which satisfies $dT/dr_+ = 0$. Hence, for $\lambda = 1$, the solution interpolates between AdS$_2 \times$S$^2$ in the near-horizon geometry of extremal black hole and M$_4$ (asymptotically flat spacetimes).

Finally, the heat capacity defined by $C = \left(\frac{dm}{dT}\right)_\lambda$ takes the form

$$C(r_+, \omega) = -\frac{2\pi}{\omega} \frac{(\omega r^2_+ + 1)^2(2\omega r^2_+ - 1)}{2\omega^2 r^4_+ - 5\omega r^2_+ - 1}.$$ \hspace{1cm} (20)
An isolated black hole like Schwarzschild black hole is never in thermal equilibrium because it decays by the Hawking radiation. It could be represented by its negative heat capacity. However, the Schwarzschild black hole can be in thermal equilibrium in anti-de Sitter spacetimes (AdS\(_4\)) because the AdS\(_4\) plays the role of a box. In this case, we have small black hole with negative heat capacity and large black hole with positive heat capacity. Hence one should consider not just the black hole itself, but also its surroundings of the heat reservoir. Actually, the thermodynamic stability argument based on the sign of heat capacity is closely related to the stability argument based on the sign of eigenfrequency \(w^2\) in the perturbation theory [29]. Here we have the correspondence between two stabilities:

| stability | Thermodynamic stability | Classical stability |
|-----------|------------------------|---------------------|
| stability | \(C > 0\)              | \(w=\text{real} (w^2 > 0)\) |
| instability | \(C < 0\)             | \(w=\text{imaginary} (w^2 < 0)\) |

As is shown in Fig. 3, the heat capacity blows up at \(r_+ = r_m\). Hence, small black holes of \(r_+ < r_m\) are stable because their heat capacities are positive, while large black hole of \(r_+ > r_m\) are unstable because their heat capacities are negative, like the heat capacity \(C_S = -2\pi r_m^2\) of Schwarzschild black hole. In this case, we have positive heat capacities for small black holes even in asymptotically flat spacetime because of the presence of \(\omega\), like the charge \(Q\) in the Reissner-Nordström black hole.

### 3 GUP and Hořava-Lifshitz gravity

The solution in Eq.(12) reminds us the Schwarzschild black hole modified with the generalized uncertainty principle (GUP) [30]. Hence, we make a connection between the GUP and the Hořava-Lifshitz gravity as a UV complete candidate for general relativity. A commutation relation of

\[
[x, p] = i\hbar(1 + \beta^2 p^2)
\]

leads to the generalized uncertainty relation

\[
\Delta x \Delta p \geq \hbar \left[ 1 + \alpha^2 l_p^2 \left( \frac{\Delta p}{\hbar} \right)^4 \right]
\]

with \(l_p = \sqrt{G\hbar/c^3}\) the Planck length. Here a parameter \(\alpha = \hbar\sqrt{\beta}/l_p\) is introduced to indicate the GUP effect. The Planck mass is given by \(m_p = \sqrt{\hbar c/G}\). The above implies a lower bound on the length scale

\[
\Delta x \geq 2\alpha l_p,
\]
which means that the Planck length plays the role of a fundamental scale. Also, the GUP may be used to derive the modified black hole temperature by introducing $\Delta p$ as the energy (temperature) of radiated photons. The momentum uncertainty for radiated photons can be found to be

$$\frac{\Delta x}{2\alpha^2 l_p^2} \left[ 1 - \sqrt{1 - \frac{4\alpha^2 l_p^2}{(\Delta x)^2}} \right] \leq \frac{\Delta p}{\hbar} \leq \frac{\Delta x}{2\alpha^2 l_p^2} \left[ 1 + \sqrt{1 - \frac{4\alpha^2 l_p^2}{(\Delta x)^2}} \right]. \tag{24}$$

The left inequality provides small corrections to the Heisenberg’s uncertainty principle for $\Delta x \gg \alpha l_p$ as $\Delta p \geq \hbar/\Delta x + \hbar\alpha^2 l_p^2/(\Delta x)^3 + \cdots$ [31]. On the other hand, the right inequality implies that $\Delta p$ cannot be arbitrarily large in order that the GUP in (22) makes sense.

For simplicity we use the Planck units of $c = \hbar = G = k_B = 1$ which imply that $l_p = m_p = 1$ and $\beta = \alpha^2$. Considering the GUP effect on the near-horizon and $\Delta x = r_+ = 2m$ from Eq.(9)\footnote{By modelling a black hole as a black box with linear size $r_+$, the uncertainty in the position of an emitted particle by the Hawking radiation may take $\Delta x \sim r_+$ with $r_+$ the radius of event horizon. In the absence of the GUP effect, the horizon radius is given by $r_+ = 2m$. In the presence of the GUP effect, we do not know a precise form of the horizon radius $r_+ = r_+(m, \beta)$ unless a GUP-corrected metric is known. However, we assume that $\Delta x \sim r_+$ would be valid even with the GUP effect.}, the relation (24) reduces to

$$m \left[ 1 - \sqrt{1 - \frac{\beta}{m^2}} \right] \leq \beta \Delta p \leq m \left[ 1 + \sqrt{1 - \frac{\beta}{m^2}} \right]. \tag{25}$$

Replacing $\beta$ with $1/2\omega$, the above leads to a relation

$$r_- \leq \frac{\Delta p}{2\omega} \leq r_+. \tag{26}$$

Here we wish to mention that a replacement of $\beta \rightarrow 1/2\omega$ was done because both sides of Eq.(25) have mathematically the same form as Eq.(12). It seems that Eq.(26) indicates a connection between quantum and classical properties of black holes in the deformed Hořava-Lifshitz gravity.

At first sight, it is unclear why the GUP effect on the Schwarzschild black hole is related to black holes in the deformed Hořava-Lifshitz gravity. However, it was known that the generalized uncertainty principle provides naturally a UV cutoff to the local quantum field theory as gravity effects [32, 33]. The GUP relation of Eq.(21) has an effect on the density of states in momentum space as

$$\frac{d^3 \vec{p}}{(1 + \beta \vec{p}^2)^3} \tag{27}$$
with an important factor of \(1/(1 + \beta p^2)^3\), which effectively cuts off the integral beyond 
\(p = 1/\sqrt{\beta}\). We wish to mention that this factor may be related to the Cotton-term of 
\(C_{ij}C^{ij}\) in Eq.\((5)\) because the latter contains a six-order derivative.

The right-hand side of Eq.\((21)\) includes a \(\vec{p}\)-dependent term and thus affect the cell size 
in phase space as “being \(\vec{p}\)-dependent”. Making use of the Liouville theorem, one could 
show that the invariant weighted phase space volume under time evolution is given by

\[
\frac{d^3x d^3p}{(1 + \beta p^2)^3}, \tag{28}
\]

where the classical commutation relations corresponding to the quantum commutation relation 
of Eq.\((21)\), namely, \(\{x_i, p_j\} = (1 + \beta p^2)\delta_{ij}, \{p_i, p_j\} = 0, \text{and } \{x_i, x_j\} = 2\beta(p_i x_j - p_j x_i)\), 
are used. Actually, \(\beta(= \alpha^2)\) plays the role of a UV cutoff of the consequent momentum 
integration \[33\].

In order to make a further connection between the GUP and the Hořava-Lifshitz gravity, 
we calculate the GUP-corrected temperature. If the mass of the emitted particle is 
neglected, the uncertainty in the energy of the emitted particle is given by

\[
\Delta E \sim \Delta p. \tag{29}
\]

By considering that \(\Delta E\), which could be identified with the characteristic temperature 
of the Hawking radiation, saturates the left inequality in Eq.\((26)\), one has the Hawking temperature

\[
T_{\text{GUP}} = \frac{r_+}{8\pi \beta} \left[ 1 - \sqrt{1 - \frac{4\beta}{r_+}} \right]. \tag{30}
\]

Here a calibration factor “\(1/4\pi\)” was introduced to have agreements with the usual Hawking 
temperature of the Schwarzschild black hole in the leading term, for a large black hole with 
\(r_+ \gg \beta\):

\[
T_{\text{GUP}} \simeq \frac{1}{4\pi} \left[ \frac{1}{r_+} + \frac{\beta}{r_+^2} + \cdots \right]. \tag{31}
\]

On the other hand, from Eq.\((17)\) we have temperature for large Hořava-Lifshitz black holes

\[
T \simeq \frac{1}{4\pi} \left[ \frac{1}{r_+} - \frac{3}{2\omega r_+^4} + \cdots \right]. \tag{32}
\]

Comparing Eq.\((31)\) with Eq.\((32)\) leads to a difference that the GUP effect (\(\beta\)) with the minimum length \(2\sqrt{\beta}l_p\) always increases the Hawking temperature, while the Hořava-Lifshitz gravity \(\omega\) decreases the Hawking temperature and it arrives at \(T = 0\). Furthermore, we 
could not confirm the replacement of \(\beta \to 1/2\omega\) on the temperature side. Thus, the relation 
of Eq.\((26)\) is still obscure.
We have studied thermodynamics of black holes in the deformed Hořava-Lifshitz gravity. For $\lambda = 1$, the black hole behaves like the Reissner-Nordström black hole because of the extremal point with $m = m_e$, $T = 0$, $C = 0$ and the maximum point as the Davies’ point with $T = T_m$, $C_m = [\infty \to -\infty]^{34}$. Hence, for small black hole, it is quite different from the Schwarzschild black hole of Einstein gravity even though the Schwarzschild black hole is recovered for large $r_+$. Explicitly, there is no such points of $r_+ = r_e, r_m$ in the Schwarzschild black hole. In this work, we did not obtain the entropy of the deformed Hořava-Lifshitz black holes. However, we would like to mention that either the Wald formalism or the entropy function formalism may be applied to the Hořava-Lifshitz action to find the entropy correctly because this action contains higher order curvature terms like $R^2$ and $R_{ij}R^{ij}$.

Finally, we could not confirm a solid connection between the GUP and Hořava-Lifshitz gravity, although we have obtained partial connections between them.

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