Arbitrary Finite Dimensional Generalization of Quantum Coin-Flipping Game

Fasheng Cao*
Department of Cognitive Science Guizhou Minzu University Guiyang, China

*Corresponding author: caofasheng@163.com

Abstract. Two players of coin-flipping game randomly choose one of two strategies, flipping over or not. Meyer quantized states of the game to be described instead by density matrices of the two-dimensional Hilbert space. In order to obtain and unify generation of the two players coin-flipping game with more than two strategies, we renew the need for equivalently describing Meyer’s PQ coin-flipping game by representing player's strategies as elements of permutation groups. Expanding the dimension of quantum vector to arbitrary N-dimension in the coin-flipping game model, we generalize Meyer's PQ coin-flipping game to PQ arbitrary finite strategies game. Owing to the quantum Fourier transform, quantum contrivance of Q has a winning strategy in PQ coin-flipping game with arbitrary finite strategies. Even though the player Q has a small probability of winning his opponent in the classical PQ coin-flipping game with arbitrary finite strategies, he can win his opponent who uses classical mixed strategies by his quantum strategies. Furthermore, we realize the winning quantum strategies of quantum player Q of PQ coin-flipping game with arbitrary finite strategies in macro world.

1. Introduction
Game theory mainly studies the interaction between the formulaic incentive structures, which is a mathematical theory and formal method to study the phenomenon of struggle or competition. It is a theory to provide decision-making schemes for the players. Originally developed in 1940s, it can solve many economic problems, social sciences and biological [10,11,14]. Quantum game is a kind of quantum information theory. Compared with the classical game, the advantage of quantum game is that even if the quantum player can not get any information about his/her opponent's behavior, his/her can get more benefits through quantum entanglement. When the player's state is entangled, the operation of a player's own state will affect the other players’ states, so the player's return may be different from that in the classic game.

Since Mayer put forward the first quantitative game by combining game theory with quantum computing, people have been very interested in what would happen when a classical game is extended into the quantum domain.

The characteristics of quantum strategy two-player zero-sum finite strategies game have been proved that one player always wins if he adopts quantum strategy. [8] The essence of entangled states is that they cannot be prepared with local physical devices. But the quantum game can be even more strange, because it is possible in this case to make the choices of the two players "entangled". On the other hand,
quantum entanglement, being closely related to the quantum non-locality in quantum computation theory, which plays an important role in quantum games. [1,2,3,12]

With Du experimentally realizing the quantum prisoner’s dilemma game on nuclear magnetic resonance quantum computer, many researchers have verified that the quantum game matches the theoretical income to different degrees through experiments. [2,9,13,16,17] Quantum coins are applied to quantum numerical integration algorithm. [16]

In this paper we shall generalize Meyer’s PQ coin-flipping game to PQ arbitrary finite strategies game. Moreover, we shall demonstrate that Q has winning strategy in PQ coin-flipping game with arbitrary finite strategies. Even though the player Q has a small probability of winning his opponent in the classical PQ coin-flipping game with arbitrary finite strategies, he can win his opponent who uses classical mixed strategies by his quantum strategies. Finally, we shall realize the winning quantum strategies of quantum player Q of PQ coin-flipping game with arbitrary finite strategies in macro world.

2. Equivalent description of coin-flipping game

The rules of coin-flipping game are as follows. Player P puts a coin in a black box and keeps its head up, after player Q operates. They operate twice in turn. They can choose whether to toss a coin or not, but they can’t see coin’s states. If the coin keeps its head up, the second player wins, otherwise first player wins. Now Q's quantum strategy is to put a superposition of coin’s head up and down. Since player P only performs the up and down exchange operation, he obviously cannot change the superposition, so Q can win the game by turning the coin back to its original state. [8]

In order to describe equivalently this coin-flipping game, we give an outline of permutation group. More specific permutation groups refer to [5,6,7].

If \( X \) is an \( n \)-element set, a permutation of \( X \) is a bijection correspondence \( f : X \rightarrow X \). Given any two permutations \( f, g \), a new one can be defined as their synthesis, which can be composed to give the permutation \( f \circ g : X \rightarrow X \), \( f \circ g(x) = g(f(x)) \). The set of all permutations of \( X \) forms a group, whose operation is the synthesis of functions, which is denoted \( \text{Sym} (X) \), or \( S_n \) if the specific set is inessential.

The coin-flipping game can be equivalently described as follows.

There are two same size square cards, which are respectively coloured with two colors, 1 and 2. The card coloured with colour \( i \) \( (i = 1,2) \) is called number \( i \) card. The two cards are arranged in a line. Each arrangement corresponds to a state. So, there are two states represented by two arrangements, \((1,2)\) and \((2,1)\). We respectively map two states of the coin, head up and head down, to the states of two cards, \((1,2)\) and \((2,1)\). We respectively correspond two strategies, flipping over or not, to two permutations, \( f_1 \) or \( f_2 \), which is the identity on \( \text{Sym} \{1,2\} \).

The game is equivalently described as coin-flipping game of two-dimension, where player P places two same size square cards, which are respectively coloured with two colors, 1 and 2, into a black box in numerical order, after player Q operates. They operate twice in turn. They can choose to permutate, or not, the cards in a parallel translational manner. But they are not able to see cards’ order. If the cards end up being in numerical order, the second player wins, otherwise first player wins.

The Q's winning quantum strategy is to put a superposition of coin’s head up and down. Since player P only performs the up and down exchange operation, he obviously cannot change the superposition, so Q can win the game by turning the coin back to its original state.

3. N-dimensional generalization of coin-flipping game

By means of the equivalent description of coin-flipping game, we can obtain the arbitrary \( N \) -dimensional generalization of coin-flipping game with multi-strategies, which is called \( N \) -cards-permutating game with multi-strategies. Naturally, Let \( X \) be the set of \( \{1,2...,N\} \), there are \( N \) same size square cards, which are respectively coloured with \( N \) colors, 1, 2,...and \( N \). Strategies are permutations of these cards in a translational manner.
Two players PQ permutating $N$-cards game with $N$-strategies are described as follows: P places $N$ same size square cards, which are respectively coloured with $N$ colors, into a black box in numerical order, after player Q operates. They operate twice in turn. They can choose a strategy from strategy set to permutate these cards in a parallel translational manner. But they are not able to see cards' order. If the cards end up being in numerical order, the second player wins, otherwise first player wins.

Next, we demonstrate that Q has winning strategy in PQ arbitrary $N$-strategies permutating $N$-cards game. Even though the player Q has a small probability to win his opponent in this classical game, he can win his opponent who uses classical mixed strategies by his quantum strategies.

Player Q change the initial basis state into a superposition, which is invariant on any mixed strategies of player P. Then player Q change it back to the initial basis state. Also, we can use Fig. 1 to visualize the above idea.

**Fig 1.** Quantum player Q realizing his winning quantum strategies in macro world

It is easy to calculate that Q's chance of winning is $\frac{1}{N}$. But if the rules of the game allow the q-quantum strategy, the situation is quite different. Let's see how this is in more detail. To understand how Q to do, let us reanalyze PQ permutating $N$-cards game with $N$-strategies in its broad form of mobile sequence. Traditionally, we can describe the game as a tree, and use the state to represent the vertex of the tree. If the vertex state changes to another vertex state after the operation, then a directed edge is connected between the two vertices. The outgoing edge of each vertex corresponds to the next possible action. For our purposes, it is more useful to study this tree, which is obtained by identifying the vertices with the same game state and the same number of previous moves. Therefore, we show that the game tree of PQ coin-flipping is not a binary tree with an additional height of 3, but a directed graph described by Fig. 2.

**Fig 2.** Quantum player Q realizing his winning quantum strategies in macro world
The vertices are labeled \( s_1, s_2, \ldots, s_N \) in accordance with the state of \( N \) cards, and each \( N \times N \) unitary matrix represents a permutation of the cards in a parallel translational manner. Now it naturally define an \( N \) -dimensional vector space \( V \) attached with basis \( \{|1\rangle, |2\rangle, \ldots, |N\rangle\} \), where

\[
|1\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},
|2\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix},
\ldots,
|N\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.
\] (1)

And use the sequence of \( N \times N \) matrix to represent player strategy. That is, the matrices

\[
F_i = \left( \begin{array}{c|c|c|c|} f_i(1) & f_i(2) & \cdots & f_i(N) \end{array} \right)
\] (2)

respectively correspond to the permutation \( f_i \) on \( N \) square cards — \( f_i \) is defined by \( f_i(j) = i \oplus j \), where \( \oplus \) is denoted additions modulo \( N \) — for \( i = 1, 2, \ldots, N \).

Mixed operations are convex linear combination on the set \( \{F_i | i = 1, 2, \ldots, N\} \), which acts as an \( N \times N \) stochastic matrix:

\[
U_P = \begin{pmatrix} p_1 & p_2 & p_3 & \cdots & p_N \\ p_N & p_1 & p_2 & \cdots & p_{N-1} \\ p_{N-1} & p_N & p_1 & \cdots & p_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_2 & p_3 & p_4 & \cdots & p_1 \end{pmatrix}
\] (3)

where \( \sum p_i = 1 \). If the game player permutates \( N \) square cards with probability \( p_i \in (0,1) \).

An ordered group of mixed operations turns the state of the \( N \) -cards into a linear combination \( \sum p_i |i\rangle \), which indicates that if the box is opened the \( N \) -cards will be in a state of probability \( p_N \). In the permutation group \( f_i \) is the identity on \( \text{Sym} \{\{1,2,\ldots,N\}\} \), the composition rule satisfies that \( f_i(j) = i \oplus j \).

Q, however, is adopting quantum strategy, that is, a single sequence, rather than a random sequence, acts on the matrices of square card. His winning strategy in PQ permutating \( N \) -cards game is \( U_QU_Q \), where \( U_Q \) is a quantum Fourier transform

\[
\frac{1}{\sqrt{N}} \begin{pmatrix} \omega_i^1 & \omega_i^2 & \omega_i^3 & \cdots & \omega_i^N \\ \omega_i^2 & \omega_i^3 & \omega_i^4 & \cdots & \omega_i^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_i^{N-1} & \omega_i^{N-2} & \omega_i^{N-3} & \cdots & \omega_i^1 \end{pmatrix}
\] (4)

It is easy that \( \{F_i | i = 1, 2, \ldots, N\} \), \( U_P \), and \( U_Q \) are unitary matrices.

The density operator of a quantum pure state \( \rho \) is the projection matrix and the quantum mixed state’s density operator is the corresponding convex combination of sequence of pure quantum state
density matrices. Unitary operations act on quantum state density matrices. The $N$-cards start in the state $\rho_0 = |N\rangle\langle N|$ and Q's first action, $U_Q$, can get the state:

$$\rho_1 = U_Q\rho_0 U_Q^\dagger = \frac{1}{N} \begin{pmatrix}
1 & 1 & 1 & \cdots & 1
\end{pmatrix}$$

(5)

An ordered group of mixed actions turns the state $\rho_1$ of the $N$-cards into a linear combination $\sum p_i F_i \cdot \rho_2 = \sum p_i F_i \rho_1 F_i^\dagger = \rho_1$. Q’s second action, $U_Q$, puts it into the initial state, $\rho_3 = U_Q\rho_2 U_Q^\dagger = \rho_0$, namely the initial state. So, Q can win his opponent P who uses classical mixed strategies by his quantum strategies.

It is an accurate observation that the same consequences is yielded in PQ permutating $N$-cards game with arbitrary finite strategies.

**Remark** In two players game where they take turns (Q, then P, then Q), if player Q has a strategy to move the initial basis state into a state, which is a common fixed point of mixed strategies of player P. Then player Q uses the strategies to manipulation into making a lot of payoffs.

**Fig 3.** Quantum player Q realizing his winning quantum strategies in macro world
If both game players implement quantum strategies, we are concerned about what will happen. Meyer's theorem gave answer that there is often a (mixed quantum, mixed quantum) equilibrium in two-player zero-sum finite strategies game [8].

4. Realizing quantum strategies of general coin-flipping game in macro world description

In classical PQ permutating $N$-cards game with multi-strategies, an arrangement $f_i (i = 1, 2, ..., N)$ on the set \{1, 2, ..., $N$\} is a basis state.

Also, we can use Fig.3 to visualize our idea of quantum player Q realizing his winning quantum strategies in macro world. P places $N$ same size square cards, which are respectively coloured by $N$ colors, into a black box in numerical order.

Quantum player Q realizing his winning quantum strategies in macro world are composed of two procedures, disassembly ($Q_1$) and assembly ($Q_2$). The first procedure $Q_1$ is described as follows.

He divides each square card into $N$ equal rectangle strips, and takes out $N$ rectangle strips whose colors are pairwise various. Then splicing the $N$ rectangle strips into a square card in numerical order, he gets $N$ new same square cards which are all colored by $N$ colors in numerical order. Finally, arranging $N$ same square cards, he gets a superposition state which is invariant on any mixed strategy of player P.

The other, $Q_2$, is described as follows. He disassembles the cards, which are spliced in the first procedure, and pieces together $N$ same color segments into the $N$ original cards. Finally, arranging the $N$ original cards in numerical order, he obtains the initial state, which enabled him to win the game.

The two procedures can be implemented without seeing cards’ order by two same algorithms $\mathcal{R}$, matrix transposition algorithms [18]. Q corresponds a row $(1, 2, ..., N)$ to an $N \times N$ matrix

$$
\begin{pmatrix}
1 & 2 & \ldots & N \\
1 & 2 & \ldots & N \\
\vdots & \vdots & \ddots & \vdots \\
1 & 2 & \ldots & N
\end{pmatrix}
$$

(6)

then transposes to its transposition matrix

$$
\begin{pmatrix}
1 & 1 & \ldots & 1 \\
2 & 2 & \ldots & 2 \\
\vdots & \vdots & \ddots & \vdots \\
N & N & \ldots & N
\end{pmatrix}
$$

(7)

which is invariant on any column permutation of player P. Q transposes back its transposition matrix, after player P acts column permutation.

Similarly, the player Q realizes the quantum winning strategies in PQ permutating N-cards game with arbitrary finite strategies in macro world, which shows that single quantum bit is not a real quantum system. [4]

5. Conclusion

In summary, we have generalized Meyer's PQ coin-flipping game to PQ arbitrary finite strategies game and have theorised that he can win his opponent who uses classical mixed strategies by his quantum strategies, even though the player Q has a small probability of winning his opponent in the classical PQ coin-flipping game with arbitrary finite strategies. Furthermore, it is essential in this letter that the matrix transposition algorithms give us help to realize the quantum winning strategies of quantum player Q of PQ coin-flipping game with arbitrary finite strategies in macro world.
Acknowledgment
We thank Ju Wang, Xishun Zhao and Shier Ju. We acknowledge that the research was partially supported by NSFC Grant 11661046.

References
[1] J. Du, X Xu, H Li, et al. “Entanglement playing a dominating role in quantum games,” Physics Letters A, 2001, vol.289(1-2), pp.9-15.
[2] J.Du,H.Li,X.Xu, et al. “Experimental Realization of Quantum Games on a Quantum Computer,” Physical Review Letters.2002 vol.88, 137902.
[3] J.Eisert, M. Wilkens and M.Lewenstein, “Quantum games and quantum strategies,” Physical Review Letters.1999 vol. 83, 3077.
[4] S.J. van Enk, Comment: Quantum and Classical Game Strategies, Physical Review Letters.1999 vol.84, 789.
[5] D. W. Farmer, Groups and symmetry: a guide to discovering mathematics. American Mathematical Soc. 1996 pp.68-73.
[6] M. Hamermesh, Group theory and its application to physical problems. Courier Dover Publications, 1989, pp.18-23.
[7] N. Jacobson, Lectures in abstract algebra. Princeton: van Nostrand, 1964, pp.57-87.
[8] D. A. Meyer, “Quantum Strategies,” Physical Review Letters. 1999 vol.82, 1052.
[9] A. Mitra, K. Sivapriya, A. Kumar, et al. “Experimental implementation of a three qubit quantum game with corrupt source using nuclear magnetic resonance quantum information processor,” 2007 vol.187, pp.306-313.
[10] J. von Neumann, O. Morgenstern, Theory of Games and Economic Behaviour, Princeton University Press, 2nd ed., 1947.
[11] M. A. Nowak, K. Sigmund, “Phage-lift for game theory,” Nature 1999 vol.398, pp.367-368.
[12] I. Peterson, “Quantum Games,” Science News 1999 vol.156, pp.334-335.
[13] R. Prevedel, A. Stefanov, P. Walther, et al. “Experimental realization of a quantum game on a one-way quantum computer,” New Journal of Physics, 2007 vol.9, pp.205-217.
[14] E. Rasmusen, Games and Information, Blackwell, Oxford, 1995.
[15] C. Schmid, A. P. Flitney, W. Wieczorek, et al. “Experimental implementation of a four-player quantum game,” New Journal of Physics 2010 vol.12, 063031.
[16] N. H. Shimada, Toshiya Hachisuka. “Flipping Game Development,” October 2019, arxiv.org. 1910.00263v1.
[17] W. Wieczorek, C. Schmid, N. Kiesel, et al. “Experimental observation of an entire family of four-photon entangled states,” Physical Review Letters. 2008 vol.101, 010503.
[18] The matrix transposition algorithms $R$ is described as follow in C language: void transposition(int a[m][n]) {
    int i,j,b[m][n];
    for $(i=1; i<m;i++)$
    for $(j=1; j<n;j++)$
        $b[i][j]$ = a[i][j];
}