Abstract

We discuss an infinite–dimensional kählerian manifold associated with the area–preserving diffeomorphisms on two–dimensional torus, and, correspondingly, with a continuous limit of the $A_r$–Toda system. In particular, a continuous limit of the $A_r$–Grassmannians and a related Plücker type formula are introduced as relevant notions for $W_\infty$–geometry of the self–dual Einstein space with the rotational Killing vector.
1. A remarkable correspondence between $W$–geometry of the two–dimensional $A_r$–Toda system and the Plücker embedding has been recently discovered in [1], where it is shown that the Kähler potentials of the pseudo–metrics induced on the corresponding $W$–surfaces coincide with the $A_r$–Toda fields. Note that this fact takes place also for the $W$–surfaces associated with an arbitrary finite–dimensional simple Lie algebra $G$ endowed with the canonical gradation, and the corresponding $G$–Toda fields. In turn this relation gives an independent proof of the infinitesimal Plücker type formula for the curvatures of the pseudo–metrics in terms of the Cartan matrix $k$ of $G \equiv G(k)$.

It seems reasonable to study such a relation for $W_\infty$–geometry, i.e., in a continuous limit [2], [3], [4]

$$\frac{\partial^2 x(z_+, z_-; \tau)}{\partial z_+ \partial z_-} = \exp \frac{\partial^2 x(z_+, z_-; \tau)}{\partial \tau^2} = \exp \frac{\partial^2 x(z_+, z_-; \tau)}{\partial \tau^2},$$

(1)

of the $A_r$–Toda system describing, in particular, the self–dual Einstein space with the rotational Killing vector [10], and associated with the area–preserving diffeomorphisms $S_0 \text{Diff} T^2$ on two–dimensional torus $T^2$; and even for more general dynamical systems like those associated with a continuum Lie algebra with the Cartan operator $\mathcal{K}$.

A limiting continuous procedure à la Volterra method proved itself in a good light as a hint in constructing the continuum Lie algebras. Now, natural step is to study, using similar reasonings, the infinite–dimensional kählerian manifold and $W_\infty$–geometry associated with system (1), and endowed with the structure of the $A_\infty$ algebra. Note, that there is a number of papers, see e.g., [13] and references therein, where infinite–dimensional Kähler geometry and Grassmannians associated with the group of smooth based loops on a connected compact Lie group were investigated. However, there the arising (flag) manifolds behave in many respects like finite–dimensional ones, while those under consideration in the present paper deal with the notions and objects of a novel nature which, as far as we know, have not been treated before. To be honest, let us mention from the very beginning that in our paper the word “manifold ” for the continuous case should be understood in a conventional sense, since our consideration here is rather formal yet.

To proceed with our program, first give briefly some information about a continuum formulation of the algebras in question.

2. The algebra $A_\infty$ is isomorphic to the algebra $S_0 \text{Diff} T^2$ which in turn, as the continuum Lie algebra $\mathcal{G}(E; -\frac{d^2}{d\tau^2}; \text{id})$ with the Cartan operator $-d^2/d\tau^2$, is isomorphic, in accordance with [11], to the Poisson bracket Lie algebra on $T^2$. The last one, considered as $\mathbb{Z}$–graded continuum algebra $\mathcal{G}(E; -i\frac{d}{d\tau}; -i\frac{d}{d\tau}) = \bigoplus_{m \in \mathbb{Z}} \mathcal{G}_m$ is defined by the commutation relations

$$[X_m(\varphi), X_n(\psi)] = iX_{m+n}(n\varphi'\psi - m\varphi\psi'),$$

(2)
Here \( X_m(\phi) = \int d\tau X_m(\tau) \phi(\tau) \) are the elements of the subspaces \( \mathcal{G}_m \) parametrized by the functions \( \phi(\tau) \) belonging to the algebra \( E \) of trigonometrical polynomials on a circle; \( \varphi' \equiv \frac{d \varphi}{d\tau} \). This algebra is of constant (in functional sense) growth since \( \mathcal{G}_n \simeq \mathcal{G}_1 \simeq E \); its Cartan subalgebra \( \varphi \simeq \mathcal{G}_0 \) is infinite-dimensional, the roots are \( n\delta'(\tau) \). Let \( \varphi^* \) be an algebra dual to \( \varphi \), let \( V \) be a \( \mathcal{G} \)-module and \( \lambda \in \varphi^* \). Denote by \( V_\lambda \) a set of vectors \( v \in V \) satisfying \( X_\lambda(\varphi)v = \lambda(\varphi)v \) for all \( \varphi \in E \). Moreover, it can be shown by an appropriate limit procedure (starting from \( A_r \) and using the aforementioned isomorphism \( \mathcal{G}(E; -i\frac{d}{d\tau}; -i\frac{d^2}{d\tau^2}) \simeq \mathcal{G}(E; -\frac{d^2}{d\tau^2}; \text{id}) \simeq A_\infty \)) that there exists a nonzero vector \( \tilde{v} \in V \) such that \( \mathcal{G}_m(\tilde{v}) = 0 \) for \( m > 0 \) and \( U(\mathcal{G})(\tilde{v}) = V \). Here \( U(\mathcal{G}) \) is the universal enveloping algebra for \( \mathcal{G} \). By analogy with the usual ("discrete") case this \( \mathcal{G} \)-module \( V \) is called the highest weight module, and \( \tilde{v} \) the highest weight vector. A symmetrical bilinear invariant form on the local part \( \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{+1} \) of the algebras in question is defined as follows

\[
\text{tr } (X_i(f)X_j(g)) = \delta_{i+j,0}(f,g), \quad i, j = 0, \pm 1;
\]

where \( (f, g) = \int d\tau \frac{df(\tau)}{d\tau} \frac{dg(\tau)}{d\tau} \) for

\[
g(E; -i\frac{d}{d\tau}; -i\frac{d^2}{d\tau^2}) \quad \text{and } \quad g(E; -\frac{d^2}{d\tau^2}; \text{id}).
\]

In what follows a continuous version (for \( A_\infty \)) of the highest weight vectors of the fundamental representations of \( A_r \) is denoted by \( |\tau > \), for which

\[
X_0(\phi)|\tau > = \phi(\tau)|\tau >; \quad X_m(\phi)|\tau > = 0 \text{ for } m > 1;
\]

and \( X_{-1}(\tilde{\tau})|\tau > = 0 \) for \( \tilde{\tau} \neq \tau \). (3)

3. A relevant object for the description of the \( A_r -W \) - geometry of \( \mathbf{C}^r \) - target manifolds with a positive curvature form is the Plücker embeddings of the Grassmannians \( \mathcal{G}r(N|p) \) in the projective spaces, \( \mathcal{G}r(N|p) \Rightarrow \mathbf{C}P^{(N)}_{p-1} \). Here inhomogeneous coordinates \( Z_{ij} = \det \mathcal{R}_{a-i+j}/\det \mathcal{R}_a \) of this map are the determinants of all the \( p \times p \) minors \( \mathcal{R}_a \) of the element \( \mathcal{R} \) of \( \mathcal{G}r(N|p) \) in its matrix realization. A natural continuous version of the Plücker representation for the pseudo–metric in terms of the norms \( |\mathcal{R}_p| = \exp(-Q_p) \), and the curvature form \( R_p \) of the pseudo–metric \( ds^2 \) (infinitesimal Plücker formula),

\[
ds^2_p = \frac{i}{2} \partial \bar{\partial} \log |\mathcal{R}_p|^2 = \frac{i}{2} \prod_q e^{2k_qQ_q}dZ \wedge d\bar{Z}; \quad R_p = \sum_q k_q ds^2_q,
\]

can be written as

\[
ds^2(\tau) = \frac{i}{2} \partial \bar{\partial} Q(\tau) = \frac{i}{2} e^{2\bar{Q}(\tau)/\partial \tau^2} dZ \wedge d\bar{Z}; \quad R(\tau) = \frac{\partial^2}{\partial \tau^2} ds^2(\tau)
\]

Of course, at this stage our arguments are very formal since the objects which we have introduced are not yet specified enough, and we need to provide some parametrization for them to clarify their group–algebraic meaning.

For this goal let us recall that for \( A_r -W \) - geometry \([\mathbb{I}]\), the trivial \( \mathbf{C}^N \) - target manifolds are simply \( \mathbf{C}^N \) Riemannian manifolds with \( 2N \) real dimensions and with the homogeneous (Euclidean) coordinates \( Y^A \) and \( \bar{Y}^A \), \( 0 \leq A, \bar{A} \leq N - 1 \), such that the linear element \( ds^2 = \sum_A dY^A d\bar{Y}^A \). The corresponding \( \mathbf{C}^N-W \) - surface
is a two-dimensional manifold with a chiral embedding into $\mathbb{C}^N$, which is defined by the independent functions $Y^A(z_-)$ and $\bar{Y}^\dot{A}(z_+)$, $0 \leq A, \dot{A} \leq N - 1$ with $N$ being the dimension of the corresponding representation of $A_r$. Here an explicit parametrization of the cosets that are kählerian manifolds associated with the fundamental representations of $A_r$ with the highest weight states $|\lambda_i >$, $1 \leq i \leq r$, is based on the following scheme. Denote by $S_i \subset A_r$ the stability group of $|\lambda_i >$. Then the Grassmannian $Gr(N|i) \equiv A_r/S_i$ is parametrized by the formula

$$\exp(\sum Y^A F_A)|\lambda_i >, 1 \leq A \leq \dim A_r/S_i, F_A \in A_r/S_i,$$

and $\bar{|}\lambda_i >$. The exponential of the corresponding Kähler potential is the highest weight matrix element

$$\exp K_i = <\lambda_i|\exp(-\sum A Y^A F_A')\exp(\sum A Y^A F_A)|\lambda_i >.$$

On the $W$–surface associated with the dynamical system, here – the Abelian $A_r$–Toda system, this matrix element (realizing the highest vector of the $i$-th fundamental representation of $A_r$) is given by the corresponding $\tau$–function, and $-\mathcal{K}_i = Q_i$ coincide \[4\] with the $A_r$–Toda fields, whose general solution is determined, in accordance with \[13\], by $2r$ arbitrary functions of $z_+$ and $z_-$ entering the corresponding tau–function.\[4\]

A realization of the object which seems to be natural to call a continuous version of the Grassmannian, $Gr(E|\tau)$, deals with the homogeneous coordinates $Y(z_-, \tau)$ and $\bar{Y}(z_+, \tau)$ of the manifold which arise as continuous analogues of the euclidean coordinates of the finite–dimensional picture. With account of definition \[11\] of the highest weight state $|\tau >$, the exponential of the Kähler potential $K(\tau)$ for the infinite–dimensional manifold under consideration is related to the continuous Toda field $x(z_+, z_-; \tau)$ satisfying equation \[11\], by the formula

$$\exp K(\tau) = |\mathcal{R}(\tau)| = e^{-x(z_+, z_-; \tau)}$$

$$= e^{K^{-1}\log[\phi_-(z_-, \tau)\phi_+(z_+, \tau)]} <\tau|\mathcal{M}_+^{-1}\mathcal{M}_-|\tau >, \quad \text{(6)}$$

which is a direct continualization of the corresponding $A_r$–Toda system. Here $K \equiv -\frac{d^2}{d\tau^2}$, and the dependence on $z_\pm$ is omitted for brevity. The functions $\mathcal{M}_\pm(z_\pm)$ are defined as the solution (multiplicative integral) of the initial-value problem for the equations

$$\partial_\pm \mathcal{M}_\pm = \mathcal{M}_\pm X_{\pm 1}(\phi_\pm). \quad \text{(7)}$$

The matrix element $<\tau|\mathcal{M}_-^{-1}\mathcal{M}_-|\tau >$ realizes the continuous version of the tau–function depending on two arbitrary functions $\phi_\pm(z_\pm, \tau)$ which determine the general solution \[11, 12, 13\] to equation \[11\]. In terms of the solution to equation \[11\], symbolically expressed by the $\mathcal{Z}_\pm$–ordered exponential, $\mathcal{M}_\pm = \mathcal{Z}_ \pm \exp\int dz_\pm' X_{\pm 1}(\phi_\pm(z_\pm'))$, $z_\pm' \in [c_\pm, z_\pm], c_\pm = \text{const}$, the continuous version of the Grassmannian $Gr(E|\tau)$ on the $W_{\infty}$–surface is written as

$$\mathcal{Z}_- \exp[\int_{c_-} z_- dz_- X_{-1}(e^{K\Phi(z_-')})] \exp[-X_0(\Phi)]|\tau >, \quad \Phi \equiv K^{-1}\log\phi_- \quad \text{(8)}$$

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\[6\] In other words, one does not assume here that $\bar{Y}^\dot{A}$ is the complex conjugate of $Y^A$.

\[7\] For the general solution to $G$–Toda system associated with an arbitrary finite–dimensional simple Lie algebra, see \[13\] and also \[13\].

\[8\] For the solution of the Cauchy (initial value) problem to equation \[11\], see \[16\].

\[9\] A different representation of the tau–function for equation \[11\] has been discussed in \[17\]. For a more general system of such a type, associated with the universal Whitham hierarchy, the solutions determining by an infinite number of arbitrary functions of two variables, $\tau$ and, say $z_+ + z_-$, are given in \[18\].
Finally, in these terms a continuous version $I(\tau)$ of a topological characteristic of the $A_r$–$W$–surface, that is used to be the instanton number in \cite{[1]}, can be written as

$$I(\tau) = \frac{i}{2\pi} \int dz_+dz_- \exp \partial^2 x(z_+, z_-; \tau)/\partial\tau^2$$

under appropriate conditions imposed on the functions $\phi_\pm(z_\pm)$ determining the general solution (9) for system (1).

As we have already mentioned, our construction is quite formal and requires to be comprehended in more detail. The main, but surmountable difficulty here is caused by the fact that the exponential mapping of the algebra $G(E; -\frac{d^2}{d\tau^2}; \text{id}) \simeq A_\infty$ gives a differentiable structure which is weaker than that of a Lie group in the classical sense. Some of these problems were considered and reviewed in the paper of M. Adams, T. Ratiu, and R. Schmid in \cite{[13]}, see also references therein.

Moreover, it will be very interesting to extend the consideration to the case of infinite–dimensional manifolds associated with some other continuum Lie algebras with invertible Cartan operators so to have parametrizations analogous to (6) and (8).

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