Variational Heteroscedastic Volatility Model

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We propose Variational Heteroscedastic Volatility Model (VHVM) - an end-to-end neural network architecture capable of modelling heteroscedastic behaviour in multivariate financial time series. VHVM leverages recent advances in several areas of deep learning, namely sequential modelling and representation learning, to model complex temporal dynamics between different asset returns. At its core, VHVM consists of a variational autoencoder to capture relationships between assets, and a recurrent neural network to model the time-evolution of these dependencies. The outputs of VHVM are time-varying conditional volatilities in the form of covariance matrices. We demonstrate the effectiveness of VHVM against existing methods such as Generalised AutoRegressive Conditional Heteroscedasticity (GARCH) and Stochastic Volatility (SV) models on a wide range of multivariate foreign currency (FX) datasets.

Keywords: Multivariate heteroscedasticity; Recurrent neural networks; Variational inference; Volatility forecasting

JEL Classification: C32, C45, C53,

1. Introduction

Financial time series is known to exhibit heteroscedastic behaviour - time-varying conditional volatility (Engle and Patton 2007, Poon and Granger 2003). Being able to model and predict this behaviour is of practical importance to professionals in finance for the purpose of risk management (Christoffersen and Diebold 2000, Long et al. 2020), derivative pricing (J.Duan 1995), and portfolio optimisation (Ranković et al. 2016, Escobar-Anel et al. 2022).

It is well documented in literature that (conditional) volatility is forecastable on hourly or daily frequencies (Christoffersen and Diebold 2000). On a univariate level, this involves predicting time-varying variances of asset returns; this predictability can be attributed to the so-called volatility clustering phenomenon: large (small) changes in asset price are often followed by further large(small) changes (Engle and Patton 2007, Fama 1965, Schwert 1989). On a multivariate level (involving a portfolio of assets), volatility forecasting involves estimating conditional covariances between asset pairs in addition to conditional variances of the assets. One could argue that some of this predictability comes from the so-called spillover effect: the transfer of shock between different financial markets (Jebran et al. 2017, Hassan and Malik 2007, Du et al. 2011). An effective multivariate volatility model therefore needs to capture both intra and inter-time series dynamics.

Traditional volatility forecasting models can be divided into two main categories: Generalised AutoRegressive Conditional Hertoscedasitcity (GARCH) (Bollerslev 1986, Nelson 1991, Glosten...
et al. 1993) and Stochastic Volatility (SV) (Jacquier et al. 1994, Chan and Grant 2016) models.

The two competing classes of models rely on different underlying assumptions (Luo et al. 2018). GARCH models describe a deterministic relationship between future conditional volatility and past conditional volatility and squared returns. SV models assume that conditional volatility follows a latent autoregressive process. Although there is no general consensus that GARCH is always superior to SV (or vice versa), there is some evidence that SV models are more flexible in modelling the characteristics of asset returns (Chan and Grant 2016, Shapovalova 2021). Nonetheless, the popularity of GARCH models seems to surpass that of SV models due to several reasons. Firstly, GARCH models are easier to fit than SV models. The parameters of GARCH are obtained using maximum likelihood estimation, whereas for SV models one needs to obtain samples from an intractible posterior distribution using methods such as Markov chain Monte Carlo (MCMC), which works well when the number of parameters is small, however the convergence can be slow in larger models (Shapovalova 2021). Secondly, there is an abundance of open-source software (packages) for GARCH models, such as fGarch (Wuertz et al. 2017) and rugarch (Galanos and Kley 2022) in the programming language R, and arch (Sheppard et al. 2022) in Python. For SV models however, there was no go-to package for model estimation until the release of stochvol and factorstochvol (Hosszejni and Kastner 2021) in R.

It is worth mentioning that GARCH models too suffer from the curse of dimensionality. For a portfolio of \( n \) assets, the computational complexity of GARCH models scales with \( O(n^5) \), which makes it impossible fit to beyond a portfolio of roughly 5 assets (Wu et al. 2013).

In recent years, the application of deep learning models for time series forecasting has achieved state of the art performances in many domains (Bandara et al. 2019, Böse et al. 2017, Li et al. 2018). Since our observations consist of multiple asset returns time series, a natural research direction would be to investigate whether deep learning models can capture complex dependencies between different assets across time. There are two main obstacles in this task. Firstly, the conditional volatility (covariance matrix) is a latent variable and must be inferred using observational data (Luo et al. 2018). Secondly, for a matrix to be a valid covariance matrix, it must be symmetric and positive definite (Engle and Kroner 1995). How to impose these constraints on a neural network such that its outputs are valid covariance matrices is a challenging task.

To tackle the first challenge, we adopt a recent trend that combines a variational autoencoder (VAE) and a recurrent neural network (RNN) (a VRNN) to allow efficient structured inference over a sequence of continuous latent random variables (Chung et al. 2015, Bayer and Osendorfer 2014, Krishnan et al. 2017, Fabius and van Amersfoort 2015, Fraccaro et al. 2016, Karl et al. 2017). The use of a VAE (and hence variational inference) translates posterior approximation into an optimisation task which can be solved using a neural network trained with stochastic gradient descent. The use of an RNN allows information from previous steps to be used in the modelling and forecasting of the latent variable in future steps. In fact, this promising framework has been explored in Yin and Barucca (2022) and Luo et al. (2018). In Yin and Barucca (2022), the authors proposed a neural network adaptation of the GARCH model which showed improved performance over traditional GARCH models. However, this model also suffers from the curse of dimensionality as it is still a GARCH model by design. Since the focus of this paper is on multivariate volatility, not being able to scale up to higher dimensions (beyond at least 5) is a big limitation. The authors in Luo et al. (2018) proposed a purely data driven approach to volatility forecasting under the VRNN framework which we used as a baseline model in our results section.

To tackle the second challenge, one possible approach is to combine traditional econometrics models with deep learning models. Traditional econometrics models have well understood statistical properties and neural networks can be used to enhance the predictive power of the model. In Yin and Barucca (2022), the authors use a neural network to parameterise the time-varying coefficients of the BEKK(1,1) model, which is a multivariate GARCH model proposed by Engle and Kroner (1995). The BEKK model produces symmetric and positive definite covariance matrices by design and therefore no other constraints need to be applied to the neural network output. As mentioned previously however, many econometric models suffer from curse of dimensionality. Instead, we
follow the approach in Dorta et al. (2018) and design our neural network such that it outputs the Cholesky decomposition of the precision matrix (inverse of the covariance matrix) at every time step. We will show later on how this guarantees symmetry and positive definiteness. (Engle and Kroner 1995)

Our main contribution is therefore an end-to-end neural network architecture capable of forecasting valid covariance matrices. We show in the results section that VHVM consistently outperforms existing GARCH and SV baselines on a range of multivariate FX portfolios.

The rest of the paper is structured as follows: in section 2 we present a summary of the field of volatility forecasting and introduce some of the popular models; also discuss how they are related to our proposed model. In section 3 we formally introduce VHVM: its generative and inference components, and model training and forecasting. In section 4 we outline the experiments and data used to show the effectiveness of VHVM against existing baselines from both traditional econometrics and the field of deep learning.

2. Related Work

For a financial asset with price $S_t$ at time $t$, its log returns are computed as $r_t = \log(S_t) - \log(S_{t-1})$. The returns process $r_t$ can be assumed to have a conditional mean $E[r_t | I_{t-1}] = 0$ and a conditional variance $E[r_t^2 | I_{t-1}] = \sigma_t^2$, in other words $r_t | I_{t-1} \sim N(0, \sigma_t^2)$. The information set $I_t$ describes all relevant available at time $t$: $I_t = \{r_{1:t}, \sigma_{1:t}^2\}$. The time variation of conditional variance $\sigma$ is known as heteroscedasticity and the aim of volatility forecasting is to model this behaviour (Engle and Kroner 1995).

2.1. Generalise Autoregressive Conditional Heteroscedasticity

ARCH (Engle 1982) and GARCH (Bollerslev 1986) models have been dominating the field of volatility forecasting since the late 1900s due to their simple model form, explainability, and ease of estimation. Many GARCH model variants have since been proposed to account for well-known stylised facts about financial time series such as volatility clustering and leverage effect (Engle and Patton 2007). The EGARCH (Nelson 1991) and GJR-GARCH (Glosten et al. 1993) models for example, were designed to specifically accommodate the leverage effect. The most general GARCH($p,q$) model proposed by Bollerslev (1986) describes a deterministic relationship between future conditional volatility, past conditional volatility and squared returns:

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i r_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \quad (1)$$

where $p$ and $q$ are lag orders of the ARCH and GARCH terms, under which the returns process $r_t$ has an unconditional mean $E[r_t] = 0$ and unconditional variance $E[r_t^2] = \frac{\omega}{1 - \alpha - \beta}$.

In total, there are many hundreds of GARCH model variants, however there exists little consensus on when to use which GARCH models as their performances tend to vary with the nature and behaviour of the time series being modelled, for example, the leverage effect is frequently observed in stock returns but rarely seen in foreign exchange currency returns (Engle and Patton 2007). In Hansen and Lunde (2005) the authors compared the performances of 330 GARCH variants on Deutsche Mark-US Dollar exchange rates and IBM returns and found that the GARCH(1,1) was not outperformed by any other model in the foreign exchange analysis. In the IBM stock returns analysis however, the authors found that GARCH(1,1) was inferior to models that explicitly accounted for the leverage effect. It has thus become common practice to explore various GARCH variants for the same task (Chan and Grant 2016, Chu et al. 2017, Malik 2005).

For a portfolio of assets, models tend to be multivariate generalisations of the univariate GARCH
model. In addition to modelling conditional variances for each asset, one also needs to model time varying covariances between different asset pairs. The output for a multivariate GARCH model is a time-varying covariance matrix which describes the instantaneous intra and inter-asset relationships. Notable examples of multivariate GARCH models include the VEC model (Bollerslev et al. 1988), the BEKK model (Engle and Kroner 1995), the GO-GARCH model (Van Der Weide 2002), and the DCC-GARCH model (Christodoulakis and Satchell 2002, Tse and Tsui 2002, Engle 2002). For our analysis, we used the DCC-GARCH (dynamic conditional correlation) model as a multivariate GARCH baseline. The key difference between DCC-GARCH and BEKK (a popular multivariate GARCH) is that BEKK assumes constant conditional correlation between assets, i.e. the change in the covariance between two assets with time is due to the changes in the two variances (but the conditional correlation is constant) (Huang et al. 2010). The constant conditional correlation (CCC) assumption is rather crude since during different market regimes one would expect the correlation between assets to vary. The DCC-GARCH is a generalisation of a CCC-GARCH that accounts for dynamic correlation. During the estimation procedure, various univariate GARCH models are fit for the assets, followed by estimations of the parameters for conditional correlation.

When fitting a multivariate GARCH model under the assumption of normal innovations \( r_t \sim \mathcal{N}(0, \Sigma_t) \), we seek to maximise the multivariate normal log likelihood function (Bauwens et al. 2006):

\[
\mathcal{L}(\theta) = -\frac{1}{2} \sum_{t=1}^{T} (\log |\Sigma_t| + r_t^T \Sigma_t^{-1} r_t),
\]

which becomes computationally expensive in higher dimensions since we are required (for a portfolio of \( n \) assets) to invert an \( n \times n \) covariance matrix \( \Sigma_t \) for every time step. One solution to alleviate this burden is to directly work with the precision matrix (inverse covariance matrix) instead: \( P = \Sigma_t^{-1} \). Setting a neural network output to be a precision matrix rather than a covariance matrix allows us to compute the log likelihood straightaway; hence bypassing the expensive matrix inversion step during model training. When the actual covariance matrix is required during the testing phase, one could simply invert the precision matrix to obtain the covariance matrix (Luo et al. 2018, Dorta et al. 2018).

### 2.2. Stochastic Volatility

Stochastic Volatility is an alternative class of models that rely on assumption that the log conditional variance follows a non-deterministic autoregressive AR(\( p \)) (usually \( p = 1 \)) process (Shapovalova 2021):

\[
\ln \sigma^2_{t+1} = \mu + \phi \ln \sigma^2_t + \sigma_\eta \eta_{t+1},
\]

where \( \eta_t \sim \mathcal{N}(0, 1) \) describes the innovation of the log variance process. For the rest of the section we refer the log volatility \( \ln \sigma^2_t \) as \( h_t \) such that \( r_t = \exp(h_t/2) \epsilon_t \) where \( \epsilon_t \sim \mathcal{N}(0, 1) \).

In a multivariate setting, we seek to simultaneously model the volatility movements of a group of assets (Platanioti et al. 2005). Related movements between different asset classes, financial markets or exchange rates are often observed due to them being influenced by common unobserved drivers (or factors) (Aydemir 1998). Diebold and Nerlove (1989) investigated the behaviour of seven dollar exchange rates for a period of 12 years and found that the seven series showed similarities in volatility behaviour in response to actions taken by the US government such as intervention efforts. Since stochastic volatility models are defined in terms of the log volatility process, it is harder to generalise a univariate model to its multivariate counterpart than a GARCH model (Platanioti et al. 2005). Related movements between different asset classes, financial markets or exchange rates are often observed due to them being influenced by common unobserved drivers (or factors) (Aydemir 1998). Diebold and Nerlove (1989) investigated the behaviour of seven dollar exchange rates for a period of 12 years and found that the seven series showed similarities in volatility behaviour in response to actions taken by the US government such as intervention efforts.
In this paper we take as baseline the factor model independently proposed by Pitt and Shephard (1999) and Aguilar and West (2000). An open-source package (factorstochvol) was developed in the programming language R by Hosszejni and Kastner (2021) which we used to run the baseline SV model in our analysis. The factor volatility model (Hosszejni and Kastner 2021) for a portfolio of \( n \) assets assumes \( m \) latent common factors where \( m < n \). We have that:

\[
\begin{align*}
    \mathbf{r}_t | \mathbf{f}_t & \sim \mathcal{N}(\mathbf{\Lambda f}_t, \bar{\Sigma}_t), \\
    \mathbf{f}_t & \sim \mathcal{N}(0, \hat{\Sigma}_t),
\end{align*}
\]

where \( \mathbf{f}_t = (f_{t1}, \ldots, f_{tm})^T \) is the vector of \( m \) factors, \( \mathbf{\Lambda} \in \mathbb{R}^{n \times m} \) is a matrix of factor loadings. The covariance matrices \( \bar{\Sigma}_t \) and \( \hat{\Sigma}_t \) are diagonal and are defined as:

\[
\begin{align*}
    \bar{\Sigma}_t &= \text{diag}(\exp(\bar{h}_{t1}), \ldots, \exp(\bar{h}_{tn})), \\
    \hat{\Sigma}_t &= \text{diag}(\exp(\hat{h}_{t1}), \ldots, \exp(\hat{h}_{tm})),
\end{align*}
\]

where \( \bar{h} \) and \( \hat{h} \) are the log variances of the \( n \) assets and \( m \) latent factors defined as follows (the AR(1) process given in (3)):

\[
\begin{align*}
    \bar{h}_{ti} & \sim \mathcal{N}(\bar{\mu}_i + \bar{\phi}_i(\bar{h}_{t-1,i}), \bar{\sigma}_i^2), i = 1, \ldots, n, \\
    \hat{h}_{tj} & \sim \mathcal{N}(\hat{\mu}_j + \hat{\phi}_j(\hat{h}_{t-1,j}), \hat{\sigma}_j^2), j = 1, \ldots, m.
\end{align*}
\]

Given the above, the multivariate returns process follows a 0 mean multivariate normal distribution with \( \mathbf{r}_t \sim \mathcal{N}(0, \Sigma_t) \), where \( \Sigma_t = \mathbf{\Lambda} \hat{\Sigma}_t \mathbf{\Lambda}^T + \bar{\Sigma}_t \). We see that the factor volatility model in (4) is by nature a state space model with a random walk latent transition process and a linear emission process \( \mathbf{r}_t | \mathbf{f}_t \). We will show later on that our end-to-end neural network architecture follows the same theoretical framework: a neural network (an RNN) that models the non-linear latent transition process \( \mathbf{f}_t | \mathbf{f}_{t-1} \), a neural network (VAE) that infers the latent factors from observational data \( \mathbf{f}_t | \mathbf{r}_{1:t} \), and a neural work (multilayer perceptron (MLP)) that parameterises the emission distribution \( \mathbf{r}_t | \mathbf{f}_t \).

### 2.3. Amortised Variational Inference

For a state space model with latent variable \( \mathbf{z}_t \) and observations \( \mathbf{r}_t \), we are interested in the posterior distribution \( P(\mathbf{z}_t | \mathbf{r}_{1:t}) \) - also known as the filtering distribution in time series literature. Note that in the previous section we denoted the latent factors of a SV model as \( \mathbf{f}_t \) as this is a common choice of notation in SV literature. From this section onwards we will use \( \mathbf{z}_t \) instead of \( \mathbf{f}_t \) since \( \mathbf{z}_t \) is more commonly used to represent latent variables in machine learning literature.

The variational autoencoder (VAE) (Kingma and Welling 2014) is a neural network architecture trained with stochastic gradient descent. VAE consists of an encoder neural network that parameterises the posterior distribution \( P(\mathbf{z} | \mathbf{r}) \), and a decoder neural network that parameterises the emission distribution \( P(\mathbf{r} | \mathbf{z}) \). Here we have omitted the subscript \( t \) since VAEs were traditionally designed to work in a static setting and has been used extensively as generative models in computer vision (Dorta et al. 2018). Our aim is maximise the marginal log likelihood \( \log P(\theta | \mathbf{r}) \) where the latent variable \( \mathbf{z} \) has been integrated out and \( \theta \) represents the model parameters we are optimising over. This task however involves an intractable integral.
\[ \log P_{\theta}(r) = \log \int P_{\theta}(r|z)P_{\theta}(z) \, dz. \] (7)

With variational inference [Blei et al. 2017], we use a much simpler distribution \( q_{\phi}(z|r) \) to approximate the actual intractable posterior \( P_{\theta}(z|r) \). We can express the marginal log likelihood in terms of an lower bound - evidence lower bound (or ELBO) - and the Kullback Leibler divergence between our variational approximation \( q_{\phi}(z|r) \) and actual posterior \( P_{\theta}(z|r) \):

\[ \log P_{\theta}(r) = ELBO(\theta, \phi) + KL(q_{\phi}(z|r) || P_{\theta}(z)); \] (8)

since \( \log P_{\theta}(r) \) depends only on \( \theta \), minimising \( KL(q_{\phi}(z|r) || P_{\theta}(z)) \) with respect to \( \phi \) is equivalent to maximising \( ELBO(\theta, \phi) \) w.r.t. \( \phi \). Maximising \( ELBO(\theta, \phi) \) w.r.t. \( \theta \) corresponds to maximising \( \log P_{\theta}(r) \).

In a variational autoencoder [Kingma and Welling 2014] the ELBO is expressed as:

\[ ELBO(\theta, \phi) = E_{z \sim q_{\phi}(z|r)} [\log P_{\theta}(r|z)] - KL(q_{\phi}(z|r) || P_{\theta}(z)), \] (9)

where \( P_{\theta}(z) \) is the prior distribution over \( z \), which is usually set to an uninformative \( N(0, I) \). We seek to maximise the \( ELBO(\theta, \phi) \) using the encoder neural network \( q_{\phi}(z|r) \) and decoder neural network \( P_{\theta}(z|r) \):

\[ \{\theta^*, \phi^*\} = \arg\max_{\theta, \phi} ELBO(\theta, \phi). \] (10)

We follow a recent trend which adapts a VAE from a static model to a sequential model [Chung et al. 2015, Bayer and Osendorfer 2014, Krishnan et al. 2017, Fabius and van Amersfoort 2015, Fraccaro et al. 2016, Karl et al. 2017]. This usually involves using another neural network to parameterise a learned prior distribution \( P_{\theta}(z_t|z_{t-1}) \) to replace the uninformative prior \( N(0, I) \).

This learned prior describes the latent transition dynamics of the state space model, similar to \( f_t \) of the stochastic volatility model in (4) but conditioned on \( f_{t-1} \) as opposed to a random walk. The decoder neural network parameterises \( P_{\theta}(r_t|z_t) \), which corresponds to the emission mechanism \( r_t|f_t \) in (4). The posterior approximated by the encoder network becomes \( P_{\theta}(z_t|r_{1:t}) \) as opposed to \( P_{\theta}(z|r) \) in a static setting. This requires a sequence to serve as input into an inference model, which is carried by an RNN since the hidden state \( h_t \) is a summary of the sequence \( r_{1:t} \).

The VAE-RNN (or VRNN) framework has achieved state of the art performance in sequence modelling tasks such as video prediction [Franceschi et al. 2020, Denton and Fergus 2018]; hence we also leveraged this framework to design our volatility model.

### 3. Materials and Methods

#### 3.1. Covariance Matrix Parameterisation

A covariance matrix is required to be both symmetric and positive definite [Engle and Kroner 1995]. Under the assumption that the returns time series follows a multivariate normal distribution \( r_t \sim \mathcal{N}(0, \Sigma_t) \), we wish to evaluate the log determinant \( \log |\Sigma_t| \) and Mahalanobis distance \( r_t^T \Sigma_t^{-1} r_t \) from the log likelihood (2). We follow the parameterisation scheme in [Dorta et al. 2018] and perform a Cholesky decomposition on the precision matrix:
\[ \mathbf{P}_t = \Sigma_t^{-1} = L_t L_t^T, \]  

which ensures \( \mathbf{P}_t \) is symmetric by construction. To ensure positive definiteness, we require that the diagonal entries of \( \mathbf{L}_t \) to be strictly positive; this could achieved by applying a Softplus function on top of the neural network output. We see from the log likelihood \( [2] \) that the covariance matrix needs to be inverted before evaluating the Mahalanobis distance; this process is costly at higher dimensions. Working with the precision matrix allows us to bypass the inversion during model training as the Mahalanobis distance is simply \( r_t^T \mathbf{L}_t \mathbf{L}_t^T r_t \). To evaluate the log determinant, we have \( \log |\Sigma_t| = -2 \sum_{i=1}^n \log(l_{ii,t}) \), where \( l_{ii,t} \) is the \( i^{th} \) element in the diagonal of \( \mathbf{L}_t \). Under this scheme, for a portfolio of \( n \) assets, the output of our neural network is simply a vector of size \( \frac{n(n+1)}{2} \) (denoted \( \mathbf{z}_t \) thereon). We convert the vector \( \mathbf{z}_t \) into a lower triangular matrix in a deterministic way using the `torch.tril_indices()` method in PyTorch (e.g. \( f([a, b, c]^T) = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \)). We then apply a Softplus function to the diagonal elements of this matrix (to ensure positive definiteness) and the resulting matrix is the lower Cholesky matrix \( \mathbf{L}_t \). This procedure is carried out at every time step to produce time-varying precision matrices. When the actual covariance matrix is required, for example in the test set to evaluate model performance, matrix \( \mathbf{P}_t \) is inverted to obtain \( \Sigma_t \).

### 3.2. Generative Model

The generative model defines the joint distribution \( P_\theta(\mathbf{r}_{1:T}, \mathbf{L}_{1:T}, \mathbf{z}_{1:T}) \), where \( \mathbf{r}_t \) is the multivariate returns process; \( \mathbf{L}_t \) is the lower Cholesky decomposition of the precision matrix \( \mathbf{P}_t \); vector \( \mathbf{z}_t \) is the neural network output of size \( \frac{n(n+1)}{2} \), which is the latent variable that we try to infer using observational data. We factorise the joint distribution as follows:

\[ P_\theta(\mathbf{r}_{1:T}, \mathbf{L}_{1:T}, \mathbf{z}_{1:T}) = \prod_{t=1}^T P_\theta(\mathbf{r}_t|\mathbf{L}_t)P_\theta(\mathbf{L}_t)|\mathbf{z}_t)P_\theta(\mathbf{z}_t|\mathbf{r}_{1:t-1}), \]  

where \( P_\theta(\mathbf{z}_t|\mathbf{r}_{1:t-1}) \) is a learned prior distribution which describes the transition dynamics of the latent variable \( \mathbf{z}_t \). Information about the sequence \( \mathbf{r}_t|\mathbf{r}_{t-1} \) is carried by an RNN known as the gated recurrent unit (GRU) \( [\text{Cho et al. 2014}] \) with hidden state \( \mathbf{h}_t \) such that:

\[ P_\theta(\mathbf{z}_t|\mathbf{r}_{1:t-1}) = P_\theta(\mathbf{z}_t|\mathbf{h}_{t-1}) = \mathcal{N}(\mu_{z,t}, \Sigma_{z,t}). \]  

The prior distribution is parameterised by a multilayer perceptron (MLP):

\[ \{\mu_{z,t}, \Sigma_{z,t}\}_{\text{prior}} = MLP_{\text{Gen}}(\mathbf{h}_{t-1}). \]  

\( P_\theta(\mathbf{L}_t|\mathbf{z}_t) \) is a delta distribution centered on the output of the deterministic function: `torch.tril_indices` followed by a Softplus function on the diagonal elements, which converts neural network output vector \( \mathbf{z}_t \) into \( \mathbf{L}_t \). The emission distribution \( P_\theta(\mathbf{r}_t|\mathbf{L}_t) \) (the decoder) describes the 0 mean multivariate normal likelihood given in \( [2] \) since \( P_\theta(\mathbf{r}_t|\mathbf{L}_t) = P_\theta(\mathbf{r}_t|(\mathbf{L}_t \mathbf{L}_t^T)^{-1} = \Sigma_t). \)

A graphical presentation of the generative model is given in Fig 1. We refer to the parameters of the generative model collectively as \( \theta \), and the parameters of the inference model as \( \phi \), which are jointly optimised using stochastic gradient variational Bayes.
There are various ways to design the prior distribution $P_\theta(z_t|I_{t-1})$, where $I_{t-1} = \{r_{1:t-1}, z_{1:t-1}, \Sigma_{1:t-1}\}$ represents all available information up to time $t - 1$. We tested other design schemes such as $P_\theta(z_t|r_{1:t-1}, z_{1:t-1})$ and $P_\theta(z_t|r_{1:t-1}, \Sigma_{1:t-1})$, and found that in general the temporal dynamics of the latent variable could be well predicted using past returns alone; hence we decided on $P_\theta(z_t|r_{1:t-1})$. Choosing the prior this way keeps the number of neural network parameters lower than the other two specifications, which reduces overfitting; also we do not need to evaluate the covariance matrix during training.

![Figure 1. Generative model of VHVM. The generative MLP takes as input $\{r_{1:t-1}\}$ and predicts the next-period latent factor $z_t$, conditioned on which we can obtain an estimate of the covariance matrix $\Sigma_t$.](image)

**3.3. Inference Model**

The inference model defines the joint distribution $q_\phi(L_{1:T}, z_{1:T}|r_{1:T})$ which we factorise as follows:

\[
q_\phi(L_{1:T}, z_{1:T}|r_{1:T}) = \prod_{t=1}^{T} q_\phi(L_t|z_t)q_\phi(z_t|r_{1:t}),
\]

(15)

where the posterior distribution over latent variable $q_\phi(z_t|r_{1:t})$ is parameterised by the encoder (MLP) of the VAE:

\[
\{\mu_{z,t}, \Sigma_{z,t}\}_{post} = MLP_{inf}(h_t);
\]

(16)

this represents the filtering distribution, which is our inference of $z_t$ given the most up-to-date observational data $r_{1:t}$. Since $q_\phi(z_t|r_{1:t})$ is our variational approximation of the actual posterior $P_\theta(z_t|r_{1:t})$, we see from (8) that maximising the ELBO($\theta, \phi$) w.r.t. $\phi$ is equivalent to minimising the KL divergence between the variational posterior and the actual posterior. The deterministic function to obtain $L_t$ given $z_t$ is the same as in the generative model; i.e. $q_\phi(L_t|z_t) = P_\theta(L_t|z_t)$, since this simply torch.tril_indices() followed by Softplus.

To summarise, VHVM is consisted of three neural networks: (1) an MLP ($MLP_{Gen}$) for the prior prediction model $P_\theta(z_t|r_{1:t-1})$, also known as the decoder of the VAE which models the transition of the latent variable; (2) an MLP ($MLP_{Inf}$) for the variational posterior $q_\phi(z_t|r_{1:t})$, which is the encoder the VAE; (3) a GRU with hidden states $h_t$ to carry sequential information about the multivariate returns process $\{r_{1:T}\}$ and is shared by the generative and inference models.
3.4. Model Training and Prediction

To perform variational inference we seek to maximise the $ELBO(\theta, \phi)$ w.r.t. $\theta$ and $\phi$ jointly (Kingma and Welling 2014). The expression for the evidence lower bound is given in (17). VHVM is designed to output one step-ahead volatility prediction. When new observations become available, we update the hidden state $h_t$ of the GRU, which serves as the input to the prediction network $MLP_{Gen}$ to predict the next period lower Cholesky matrix.

$$ELBO(\theta, \phi) = \sum_{n=1}^{T} E_{z_t \sim q(\cdot)} [\log P(\mathbf{r}_t | z_t)] - KL(q(\cdot) \| P(z_t | \mathbf{r}_{1:t-1})),$$

(17)

4. Experiments

We test VHVM on foreign exchange data obtained from the Trading Academy website (eatradingacademy.com). We compute daily log returns using data from the period 24/01/2012 to 23/01/2022 (a total of 3653 observations), from which we remove weekend readings where the change in asset price was 0. We constructed various portfolios using our collection of FX series for $n = 5, 10, 20,$ and 50. For model construction, we used a train:validation:test ratio of 80:10:10 and training for 50 epoches.

For model benchmarking, we compared VHVM against three benchmarks: (1) the DCC-GARCH model (Engle 2002), (2) a factor SV model with MCMC sampling (Hosszejni and Kastner 2021), and (3) Neural Stochastic Volatility model (Luo et al. 2018). The three baselines are representative models from the current approaches to volatility forecasting: GARCH models, SV models, and deep learning based models. We have chosen DCC-GARCH due to its ability to model dynamic conditional correlation between assets; we implemented the model in R using the package "rmgarch" (Galanos 2022). For the factor SV model with MCMC sampler (MCMC-SV), we used the recently developed "factorstochvol" package in R (Hosszejni and Kastner 2021).

The Neural Stochastic Volatility model (NSVM) Luo et al. (2018) is perhaps most relevant to our work since it was also designed under the VRNN framework. NSVM uses four recurrent neural networks to model temporal dynamics: one for the observed returns series $\mathbf{r}_t$ and another for the latent factor $\mathbf{z}_t$ in the generative model; similarly for the inference model. In our model however, we attempted to keep the number of model parameters low by using only one RNN but inputting the hidden state at different time steps to perform prediction and inference. Another key difference between NSVM and VHVM is that the output of NSVM is a low-rank approximation of the time-varying covariance matrix, whereas VHVM outputs the full covariance matrix. A low rank approximation may offer faster computations for higher dimensional portfolios, however we
show that VHVM is consistently better in terms of performance.

For model evaluation, we perform one step-ahead covariance matrix forecasting on the test set, and following [Wu et al. (2013)] and [Luo et al. (2018)], use the log likelihood (2) as our performance metric since it describes the likelihood of the observed data falling under our estimated distribution. We have also included the hyperparameters of our model and the baselines in the Appendix for reproduction purposes.

5. Results and Discussion

In Table 1 to 5 we show the performance of VHVM against the three baseline models: NSVM, DCC-GARCH, and MCMC-SV on various 5 dimensional FX portfolios. For every time step we forecast a $5 \times 5$ covariance matrix and in the tables we report the cumulative log likelihood of the test set. We have highlighted in bold the best performing model in terms of log likelihood (higher is better). We observe that VHVM performs best in 17 out of the 20 constructed portfolios. The neural network baseline NSVM however performs best in only one of the portfolios. As previously mentioned, the two key differences between VHVM and NSVM are: (1) VHVM uses a single RNN to carry information about $r_{1:t}$ and the hidden state at different time steps is used for forecasting ($h_{t-1}/h_t$), whereas NSVM uses four separate RNNs to model $z_t$ and $r_t$ in generation and inference; (2) NSVM outputs low rank approximations of the covariance matrix whereas VHVM outputs estimates of the full covariance matrix. We believe the simpler structure (fewer parameters) of our model has helped to reduce overfitting, and parameterising the full covariance matrix is more expressive than a low rank approximation at the expense of computational complexity ($O(n^2)$ vs $O(n)$).

Table 1. Log likelihoods of 5 dimensional Euro-denominated portfolios. The best performing model is highlighted in bold; higher log likelihood is better.

| FX pairs | VHVM (ours) | NSVM | DCC-GARCH | MCMC-SV |
|----------|-------------|------|-----------|---------|
| EURAUD, EURHKD, EURCAD, EURCNY, EURDKK | -1013.489 | -1392.964 | -1134.183 | -1144.132 |
| EURCNY, EURGBP, EURHKD, EURHUF, EURIDR | -1189.944 | -1456.841 | -1235.496 | -1279.841 |
| EURGBP, EURJPY, EURKRW, EURMXN, EURNOK | -1418.791 | -1493.457 | -1506.990 | -1471.722 |
| EURJPY, EURNZD, EURRUB, EURSGD, EURTHUB | -1222.507 | -1331.354 | -1301.942 | -1262.876 |

Table 2. Log likelihoods of 5 dimensional GBP-denominated portfolios. The best performing model is highlighted in bold; higher log likelihood is better.

| FX pairs | VHVM | NSVM | DCC-GARCH | MCMC-SV |
|----------|------|------|-----------|---------|
| GBPAUD, GBPBGX, GBPBDL, GBPCAD, GBPCHF | -1156.903 | -1364.981 | -1328.515 | -1300.182 |
| GBPCHF, GBPCNY, GBPDKK, GBDKK, GBPIK | -898.588 | -1571.065 | -1047.304 | -1076.312 |
| GBPCNY, GBPTMK, GBPJPY, GBPSAX, GBPKR | -1132.915 | -1404.454 | -1248.320 | -1211.265 |
| GBFRUB, GBPSEK, GBPTRY, GBPJPY, GBPCAD | -1639.355 | -1628.892 | -2969.575 | -2900.870 |

Table 3. Log likelihoods of 5 dimensional USD-denominated portfolios. The best performing model is highlighted in bold; higher log likelihood is better.

| FX pairs | VHVM | NSVM | DCC-GARCH | MCMC-SV |
|----------|------|------|-----------|---------|
| USDAUD, USDBGN, USDCAD, USDCRF, USDCNY | -1309.923 | -1663.190 | -1391.191 | -1331.638 |
| USDCNY, USDEUR, USDDGP, USDKDK, USDNZD | -1416.474 | -1462.547 | -1536.093 | -1414.045 |
| USDEUR, USDHUF, USDKK, USDPJPY, USDNZD | -1237.078 | -1461.547 | -1306.385 | -1293.873 |
| USDGDP, USJDPY, USDKRW, USDMX, USDT | -1609.177 | -1807.927 | -3490.222 | -3129.284 |

To better gauge the relative performances of the four models, we follow [Ismail Fawaz et al. (2019)] and plot a critical difference (CD) diagram showing the average ranking of the four model in Figure 3. Within a CD diagram, two models without a statistically significant difference (s.s.d.) in average ranking are connected with a horizontal line; the absence of such lines in Figure 3 indicates that the four models are s.s.d. in performance across the 20 5 dimensional experiments. According to Figure 3, VHVM has the best overall average ranking (1.25), followed by MCMC-SV(2.2), DCC-GARCH(3), and NSVM(3.55). The fact that MCMC-SV performs slightly better
than DCC-GARCH is in accordance with claims that SV models are more flexible at modelling heteroscedastic behaviour in financial time series [Shapovalova 2021].

Table 4. Log likelihoods of 5 dimensional CNY-denominated portfolios. The best performing model is highlighted in bold; higher log likelihood is better.

| FX pairs                  | VHVM   | NSVM   | DCC-GARCH | MCMC-SV |
|---------------------------|--------|--------|-----------|---------|
| CNYCAD, CNYEUR, CNYGBP, CNYIDR, CNYJPY | -1771.234 | -1447.516 | -1304.421 | -1272.866 |
| CNYKRW, CNYMXN, CNYMYR, CNYRUB, CNYSEK     | -1270.803 | -1346.751 | -1354.709 | -1350.663 |
| CNYGBP, CNYJPY, CNYSEK, CNYSGD, CNYTHB     | -1236.308 | -1427.729 | -1397.665 | -1283.031 |
| CNYEUR, CNYMXN, CNYCAD, CNYUSD, CNYTHB     | -1604.344 | -1592.762 | -1535.586 | -1541.026 |

Table 5. Log likelihoods of 5 dimensional mixed currency portfolios. The best performing model is highlighted in bold; higher log likelihood is better.

| FX pairs                  | VHVM   | NSVM   | DCC-GARCH | MCMC-SV |
|---------------------------|--------|--------|-----------|---------|
| EURAUD, GBPCAD, USDCHF, USDNOK, CNYGBP     | -1354.807 | -1580.274 | -1423.861 | -1363.748 |
| EURHKD, GBPJPY, USDCHF, CNYRUB, CNYCAD    | -1139.711 | -1343.263 | -1290.668 | -1271.675 |
| USDGBP, USDJPY, GBPLSE, CNYSGD, GBPMXN     | -1312.835 | -1379.446 | -1379.766 | -1313.195 |
| CNYEUR, CNYGBP, EURHRD, USDINR, GBPUSD     | -1041.155 | -1247.187 | -1172.433 | -1146.655 |

6. Conclusion

In this paper we propose Variational Heteroscedastic Volatility model (VHVM): an end-to-end neural network architecture capable of forecasting one step-ahead covariance matrices. VHVM outputs the lower Cholesky decomposition of a time-varying conditional precision matrix, which enforces two necessary constraints of a covariance matrix: symmetry and positive definiteness. Furthermore,
by setting the neural network to output the precision matrix, we bypass the computationally expensive matrix conversion step in the evaluation of the multivariate normal log likelihood function. We demonstrated the effectiveness of VHVM against GARCH, SV, and deep learning baseline models and we observed that VHVM consistently outperformed its competitors.

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