The Nullstellensatz for supersymmetric polynomials.

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Abstract

In this paper we prove a Nullstellensatz for supersymmetric polynomials. This gives a bijection between radical ideals and superalgebraic sets. These are algebraic sets which are invariant under the Weyl groupoid of Sergeev and Veselov, [SV11]. Note that the algebra of supersymmetric polynomials is not Noetherian, so the usual Nullstellensatz does not apply. However it does satisfy the ascending chain condition on radical ideals and this allows for the decomposition of superalgebraic sets into irreducible components. Analogous results hold for the ring of Laurent supersymmetric polynomials.

As an application, we give a proof of conjecture 13.5.1 from [Mus12]. This concerns the maximal ideals in the enveloping algebra of the general linear and orthosymplectic Lie superalgebras. The center is closely related to the algebra of supersymmetric polynomials and the result can be thought of as an analog of the weak Nullstellensatz.

1 Introduction

The interest in supersymmetric polynomials comes from several sources. First they often satisfy analogs of combinatorial properties of symmetric polynomials. For example there is an analog of the Jacobi-Trudi identity for Schur polynomials [PT92], and super-Schur polynomials form a Z-basis for the Z-algebra of supersymmetric polynomials. Secondly the center Z(g) of the enveloping algebra U(g) is isomorphic to an algebra of supersymmetric polynomials, when g is the Lie superalgebra gl(m, n) or the orthosymplectic Lie superalgebra osp(2m + 1, 2n). For the Lie superalgebra osp(2m, 2n) a slight modification of this statement is necessary. Lastly it is shown in [SV11] that if g = gl(m/n), then a natural quotient of the Grothendieck group of finite dimensional g-modules is isomorphic to a ring of Laurent supersymmetric polynomials. Suprisingly though, the rings of (Laurent) supersymmetric polynomials have received little interest from a geometric point of view, perhaps because they are

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not Noetherian. Yet as we show the spectra of these rings have some very pleasant geometric properties. In fact they are quotient spaces, except that the quotient arises from the action of a groupoid rather than a group.

2 General results.

We begin with some general results. These will be applied to both supersymmetric polynomials and Laurent supersymmetric polynomials. Unless otherwise stated all rings are commutative. If $I$ is an ideal of $A$, let

$$\text{rad } (I) = \{ f \in A | f^n \in I, \text{ for some } n > 0 \}.$$ 

If $I = \text{rad } (I)$ we say $I$ is a radical ideal.

Lemma 2.1. Suppose $A$ is a subring of $B$, $T \in B$ and $TB \subseteq A$. Let $P$ be an ideal of $A$ with $T \in P$ and suppose $P$ is a radical ideal, then $TB \subseteq P$.

Proof. This holds since $(TB)^2 = T^2B \subseteq AT \subseteq AP = P$. \hfill $\square$

Example 2.2. The result may be false if $P$ is not a radical ideal. Let $A = k + Tk[S,T], B = k[S,T]$. If $P = kT + T^2k[S,T]$, then $T \in P$ but $TB$ is not contained in $P$.

We keep the assumptions of Lemma 2.1 and suppose that $T$ is a non-zero divisor in $B$. Since $TB \subseteq A \subseteq B$, we have $A_T = B_T$. If $L$ is an ideal of $A$ define the extension of $L$ to $B_T$ to be $L^e = L_T$, the localization of $L$ with respect to the powers of $T$. If $M$ is an ideal of $A_T$, set $M^c = M \cap A$, the contraction of $M$ to $A$. By [GW04] Theorem 10.15, contraction and extension provide inverse lattice isomorphisms between the lattice ideals of $A_T$ and the lattice of ideals $C$ of $A$ such that $A/C$ is $T$-torsionfree. Another result we need is that the contraction of a prime ideal is prime. Indeed if $R$ is a subring of the commutative ring $S$ and $P$ is prime in $R$, then $R/R \cap P$ embeds in $S/P$ which is a domain, so $R \cap P$ is prime. We mention this rather obvious fact only because it fails for non-commutative non-Noetherian rings [GW04] Exercise 10.M. It is also easy to check that if $P$ is prime in $R$, then $P_T$ is prime in $R_T$. Thus extension and contraction give a bijection

$$\{ P \in \text{Spec } A | T \notin P \} \leftrightarrow \{ \text{Spec } A_T \}.$$ 

(2.1)

Note that if $P$ is maximal, then by exactness of localization [Rot09] Corollary 4.81, $P_T$ is also maximal. From now on we identify the two sets on either side of (2.1).

Corollary 2.3. If $m$ is a maximal ideal of $A$ and $T \notin m$, the maximal ideal $M$ of $B$ given by $M = mT \cap B$ satisfies $m = M \cap A$. Also $mB \neq B$.

Proof. This first statement follows from Equation (2.1), and the second is an immediate consequence. \hfill $\square$
Theorem 2.4. Let $\phi : A \rightarrow A/BT = C$ be the natural map. Then we have a disjoint union

$$\text{Spec } A = \text{Spec } B_T \cup \phi^{-1}(\text{Spec } C).$$

(2.2)

Proof. As noted above we have $A_T = B_T$. For a prime ideal $P$ of $A$ there are two possibilities. If $T \notin P$, then $P_T$ is a prime ideal of $A_T = B_T$ such that $P_T \cap A = P$. If $T \in P$, then $P$ is the inverse image of the prime ideal of $P/BT$ under $\phi$. \qed

Remarks 2.5. (a) Clearly $\text{Spec } B_T$ is open and $\phi^{-1}(\text{Spec } C)$ is closed in $\text{Spec } A$.

In our applications $B$ will be a finitely generated algebra over $\mathbb{Z}$ or over a field, so $B_T$ will be a Jacobson ring, \cite{Eis95} Theorem 4.19, and then $\text{Spec } B_T$ will be locally closed.

(b) Theorem 2.4 actually holds for noncommutative rings provided $T$ is a normal element, that is $TB_T = BT$, which is not a zero divisor.

Equation (2.2) does not hold if Spec is replaced everywhere by Rad, the set of radical ideals of a ring. Just consider the intersection $P \cap Q$ of two primes lying in different components on the right side of Equation (2.2) to see this. To understand the relationship between prime and radical ideals, it would help to know that every radical ideal is a finite intersection of prime ideals. By \cite{Kap74} Theorem 87, this holds if the ring satisfies the ascending chain condition on radical ideals. We abbreviate this condition to ACCR. We are now ready for the following result.

Lemma 2.6. In the situation of Theorem 2.4, suppose that the rings $B_T$ and $C$ satisfy ACCR. Then so does $A$.

Proof. Let $R_1 \subseteq R_2 \subseteq \ldots$ be an ascending chain of radical ideals in $A$. Then $R_1^T \subseteq R_2^T \ldots$ is an ascending chain of radical ideals in $A_T = B_T$, so by assumption, there is an $m$ such that $R_i^T = R_k^T$ for all $i \geq m$. Write $R_i^T = p_1 \cap p_2 \cap \ldots \cap p_r$ for some prime ideals of $B_T$. We can assume that this intersection is irredundant, and all the $p_k$ are minimal over $R_i^T$. Then if $P_k = p_k \cap A$ and $i \geq m$, the $P_k$ are exactly the minimal primes over $R_i^T$ that do not contain $T$. By \cite{Eis95} Corollary 2.12 every radical ideal in a commutative ring is the intersection of a (possibly infinite number) of prime ideals. So for $i \geq m$, write $R_i$ as an intersection of prime ideals in $A$. Say

$$R_i = P_1 \cap \ldots \cap P_r \cap \bigcap_{j \in \Lambda_i} Q_j,$$

where the $Q_j$ are prime ideals that contain $TB$. Thus $D_i = \bigcap_{j \in \Lambda_i} Q_j$ is a radical ideal in $A$ containing $TB$, and if $D_i = D_i/TB$, then $D_1 \subseteq D_2 \subseteq \ldots$ is an ascending chain of radical ideals of $C$. Therefore there is an $n \geq m$, such that $D_i = D_n$ for all $i > n$. Hence $R_i = R^n$ for all $i > n$. \qed

3 Supersymmetric Polynomials.

We use the notation of \cite{Mus12}. Fix nonnegative integers $m, n$. Let $\mathcal{X}_m = (x_1, \ldots, x_m)$, $\mathcal{Y}_n = (y_1, \ldots, y_n)$ be two sets of indeterminates and $L$ a commutative ring. The sym-
metric group $S_m$ acts on the polynomial ring $L[X_m] = L[x_1, \ldots, x_m]$ by the rule
\[ w(x_i) = x_{w(i)}. \]
Similarly $W_{m,n} = S_m \times S_n$ acts on $L[X_m, Y_n] = L[x_1, \ldots, x_m, y_1, \ldots, y_n]$. For $f \in L[X_m, Y_n]$ and $t \in L$, we write $f(x_1 = t, y_1 = -t)$ for the polynomial obtained by substituting $x_1 = -y_1 = t$ in $f$.

A $W_{m,n}$-invariant polynomial $f$ is \textit{supersymmetric} if $f(x_1 = t, y_1 = -t)$ is a polynomial which is independent of $t$. Equivalently by [Mus12] Lemma 12.1.1,\[
\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial y_1} \in (x_1 + y_1). \tag{3.1}
\]

For example if $r \geq 1$ the power sum $p_{m,n}^{(r)}$ defined by\[
p_{m,n}^{(r)} = (x_1^r + \ldots + x_m^r) + (-1)^{r-1}(y_1^r + \ldots + y_n^r) \tag{3.2}
\]
is supersymmetric. The set of all supersymmetric polynomials in with coefficients in $L$ is called the \textit{$L$-algebra of supersymmetric polynomials in $X_m, Y_n$}. We denote this algebra by $I_L(x_1, \ldots, x_m | y_1, \ldots, y_n)$ or $I_L[X_m, Y_n]$.

The ring $I_L[X_m, Y_n]$ has a $\mathbb{Z}$-basis consisting of certain super Schur polynomials. Chapter 12 of [Mus12] gives three equivalent definitions of these polynomials. For our purposes it is most convenient to use the definition of super Schur polynomials via symmetrization operators. We denote these polynomials by $F_\lambda(X_m/Y_n)$. In order to define them we need some combinatorics.

First, denote the Vandermonde determinant by
\[
\Delta(X_m) = \prod_{1 \leq i < j \leq m} (x_i - x_j).
\]
The \textit{Young diagram} of the partition $\lambda$ is the set of points
\[
D_\lambda = \{(i, j) \in \mathbb{N}^2 | 1 \leq j \leq \lambda_i \}.
\]
We represent this diagram by a set of boxes in the fourth quadrant, with the first index $i$ corresponding to rows and the second $j$ to columns. Next the $(m, n)$-\textit{hook} is the set
\[
\{(i, j) \in \mathbb{N}^2 | i \leq m \text{ or } j \leq n \},
\]
and we say that a partition $\lambda$ is \textit{contained in the $(m, n)$-hook} if $D_\lambda$ is contained in this subset, or equivalently if $\lambda_{m+1} \leq n$. Denote by $\mathcal{H}(m, n)$ the set of partitions $\lambda$ contained in the $(m, n)$-hook. Let $D^{m,n}_\lambda$ be the following subset of $D_\lambda$
\[
D^{m,n}_\lambda = \{(i, j) | i \leq m, j \leq n, j \leq \lambda_i \}.
\]
Suppose $\lambda$ is contained in the $(m,n)$-hook. The part of $D_\lambda$ outside $D_{\lambda}^{m,n}$ is determined by two partitions $\mu, \nu$ defined by

$$
\mu_i = \max\{0, \lambda_i - n\}, \quad \nu_j = \max\{0, \lambda'_j - m\},
$$

where $\lambda'$ is the transpose of the partition $\lambda$. The part of $D_\lambda$ to the right of the line $j = n$ is a translate of the diagram $D_\mu$, while the part of $D_\lambda$ below the line $i = m$ is a translate of $D_{\nu'}$.

Now define

$$
g_\lambda(X_m, Y_n) = \prod_{i=1}^{m} x_i^{\mu_i + m - i} \prod_{i=1}^{n} y_i^{\nu_i + n - i} \prod_{(i,j) \in D_{\lambda}^{m,n}} (x_i + y_j),
$$

and

$$
F_\lambda(X_m/Y_n) = \sum_{w \in W_{m,n}} w \left[ \frac{g_\lambda(X_m, Y_n)}{\Delta(X_m)\Delta(Y_n)} \right]. \tag{3.3}
$$

If $\lambda$ is not contained in the $(m,n)$-hook, set $F_\lambda(X_m/Y_n) = 0$.

Assume $m > 0$ and $n > 0$ and set

$$
\mathcal{H}_{m,n}^0 = \mathcal{H}(m,n) \setminus (\mathcal{H}(m,n - 1) \cup \mathcal{H}(m - 1, n))
$$

Note that a partition $\lambda$ in $\mathcal{H}(m,n)$ belongs to $\mathcal{H}_{m,n}^0$ if and only if $(m,n) \in D_\lambda$.

Define

$$
T = T(X_m, Y_n) = \prod_{i=1}^{m} \prod_{j=1}^{n} (x_i + y_j). \tag{3.4}
$$

If $\lambda \in \mathcal{H}_{m,n}^0$, we have

$$
g_\lambda(X_m, Y_n) = T \prod_{i=1}^{m} x_i^{\mu_i + m - i} \prod_{i=1}^{n} y_i^{\nu_i + n - i},
$$

and since $T$ is $W$-invariant, (3.3) becomes

$$
F_\lambda(X_m/Y_n) = T \sum_{w \in W_{m,n}} w \left[ \frac{\Pi_{i=1}^{m} x_i^{\mu_i + m - i} \Pi_{i=1}^{n} y_i^{\nu_i + n - i}}{\Delta(X_m)\Delta(Y_n)} \right]
= TS_\mu(X_m)S_\nu(Y_n), \tag{3.5}
$$

where $S_\mu, S_\nu$ are the usual Schur polynomials.

We summarize the main properties of the super Schur polynomials $F_\lambda(X_m/Y_n)$.

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1There is an erratum in this definition on page 248 of [Mus12]. For a complete list of current known errata see [https://www.ams.org/publications/authors/books/postpub/gsm-131](https://www.ams.org/publications/authors/books/postpub/gsm-131)
Proposition 3.1. Assume $\lambda$ is contained in the $(m,n)$-hook. Then we have

(a) The polynomials $F_{\lambda}(X_m/Y_n)$ are supersymmetric.

(b) The $\mathbb{Z}$-algebra of supersymmetric polynomials in $X_m$ and $Y_n$ has a $\mathbb{Z}$-basis consisting of the $F_{\lambda}(X_m/Y_n)$ as $\lambda$ ranges over partitions contained in the $(m,n)$-hook.

(c) The polynomials $F_{\lambda}(X_m/Y_n)$ with $\lambda \in \mathcal{H}_{m,n}^0$ form a basis for the $\mathbb{Z}$-module of supersymmetric polynomials in $X_m, Y_n$ for which the substitution $x_m = y_n = 0$ yields the zero polynomial.

(d) Specializing $y_{n+1} = 0$ in the polynomial $F_{\lambda}(X_m/Y_{n+1})$, or $x_{m+1} = 0$ in the polynomial $F_{\lambda}(X_{m+1}/Y_n)$ yields the polynomial $F_{\lambda}(X_m/Y_n)$.

Proof. See [Mus12] Lemmas 12.2.5, 12.2.6 and Proposition 12.2.7.

Corollary 3.2. The homomorphism

$$\phi : I_\mathbb{Z}[X_m,Y_n] \rightarrow I_\mathbb{Z}[X_{m-1},Y_{n-1}]$$

defined by the substitution $x_m = y_n = 0$ is surjective and has kernel equal to $T_\mathbb{Z}[X_m,Y_n]^{W_{m,n}}$.

Proof. By (d) $\phi(F_{\lambda}(X_m/Y_n)) = F_{\lambda}(X_{m-1}/Y_{n-1})$, if $\lambda$ is in the $(m-1, n-1)$ hook. It follows from (b) that $\phi$ is surjective. The second statement follows from (c) and (3.5).

4 Supersymmetric Laurent polynomials and Grothendieck rings of basic classical Lie superalgebras.

We recall some results of Sergeev, [Ser19], changing some of the notation slightly for convenience. The changes are noted in the footnotes. First define the algebra of Laurent symmetric polynomials to be

$$\Lambda_m = \mathbb{Z}[x_1^{\pm1}, \ldots, x_m^{\pm1}]^{S_m}.$$ 

Let $\lambda_1, \ldots, \lambda_m$ be a non-increasing sequence of integers. We define the Euler character $E_{\lambda}(x)$ by means of the following formula

$$E_{\lambda}(x)\Delta_m(x) = \{x_1^{\lambda_1+m-1} \ldots x_m^{\lambda_m}\},$$

where $\Delta_m(x) = \prod_{i<j}(x_i - x_j)$ and the brackets $\{\}$ mean alternation over the group $S_m$,

$${f} = \sum_{\sigma \in S_m} \text{sign}(\sigma)\sigma(f)).$$

The following result is [Ser19] Theorem 2.3.
Theorem 4.1. The \( E_\lambda(x) \) with \( \lambda_1, \ldots, \lambda_m \) a non-increasing sequence of integers form a \( \mathbb{Z} \)-basis of the ring \( \Lambda_m \).

Next the ring

\[
\Lambda_{m,n} = \{ f \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}, y_1^{\pm 1}, \ldots, y_n^{\pm 1}] S_m \times S_n \mid x_i \frac{\partial f}{\partial x_i} + y_j \frac{\partial f}{\partial y_j} \in (x_i - y_j) \}
\]

will be called the ring of Laurent supersymmetric polynomials. Instead of (3.1) we have

\[
x_1 \frac{\partial f}{\partial x_1} + y_1 \frac{\partial f}{\partial y_1} \in (x_1 - y_1).
\]

This accounts for the sign differences in the definitions of the maps (3.6) and (4.1).

If \( mn > 0 \) then by Lemma 10.1 in the Appendix, we have the homomorphism

\[
\phi : \Lambda_{m,n} \rightarrow \Lambda_{m-1,n-1},
\]

defined by setting \( x_m \) equal to \( y_n \). Denote by \( Q(m,n) \) the set of pairs of sequences of non-increasing integers \( (\lambda, \mu) \in \mathbb{Z}^m \times \mathbb{Z}^n \). For \( (\lambda, \mu) \in Q(m,n) \) set

\[
K_{\lambda,\mu} = \prod_{j=1}^n \prod_{i=1}^m \left( 1 - \frac{y_j}{x_i} \right) E_\lambda(x_1, \ldots, x_m) E_\mu(y_1, \ldots, y_n),
\]

A \( \mathbb{Z} \)-basis for \( \Lambda_{m,n} \) is given in [Ser19], Theorem 5.6. From the proof we have the following important statements.

Theorem 4.2. The map \( \phi \) is onto, and the \( K_{\lambda,\mu} \) with \( (\lambda, \mu) \in Q(m,n) \) form a \( \mathbb{Z} \)-basis for \( \ker \phi \).

Now set

\[
T = K_{0,0} = \prod_{j=1}^n \prod_{i=1}^m \left( 1 - \frac{y_j}{x_i} \right),
\]

\[
A = \Lambda_{m,n}, \quad A' = \Lambda_{m-1,n-1} \quad \text{and} \quad B = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}, y_1^{\pm 1}, \ldots, y_n^{\pm 1}] S_m \times S_n = \Lambda_m \times \Lambda_n.
\]

Then from Theorem 4.1 and (4.2), \( TB \subset A \subset B \) and hence \( A_T = B_T \). Also by Theorem 4.2 \( \ker \phi = TB \). Therefore by Equation (2.2) we have a disjoint union

\[
\Spec A = \Spec B_T \cup \phi^{-1}(\Spec A').
\]

Part of the interest in the ring \( \Lambda_{m,n} \) comes from the following result of Sergeev and Veselov, [SV11]. Let \( K(\mathfrak{g}) \) be the quotient of the Grothendieck ring of finite dimensional \( \mathbb{Z}_2 \)-graded representations of the Lie superalgebra \( \mathfrak{g} = \mathfrak{gl}(m|n) \) by the ideal generated by all \( [M] + [\Pi M] \) where \( \Pi \) is the parity change functor, and \( [M] \) is the class of the module \( M \) in the Grothendieck ring. Then the supercharacter yields

\(^2\)In [Ser19] \( \Lambda_{m,n} \) is denoted \( \Lambda_{m,n}^+ \), but we want to add further superscripts later. A similar remark applies to the ring \( \Lambda_m \) defined above.

\(^3\)In [Ser19] there is a more general definition valid for pairs of sequences of non-increasing integers \( (\lambda, \mu) \) which are not in \( \mathbb{Z}^m \times \mathbb{Z}^n \).
an isomorphism from $K(g)$ to $\Lambda_{m,n}$. One of the key techniques of [Ser19] is the map $\phi : \Lambda_{m,n} \to \Lambda_{m-1,n-1}$ from (4.1). It was shown by Hoyt and Reif, [HR18] that this evaluation homomorphism has a natural interpretation using the Duflo-Serganova functor. This map as well as its analog (3.6) for supersymmetric polynomials is also one of the main technical tools of this paper.

5 Prime Ideals in Rings of Supersymmetric Polynomials.

Now we apply our results simultaneously to rings of supersymmetric polynomials and rings of supersymmetric Laurent polynomials. However some definitions still need to be made separately. For this purpose we refer to case S or case L when dealing with supersymmetric polynomials or supersymmetric Laurent polynomials respectively. Set $W = W_{m,n}$ and $W' = W_{m-1,n-1}$. Then in case S, let $A = A_{m,n}$ be the ring of supersymmetric polynomials, and $B = \mathbb{Z}[X_m, Y_n]^W$. If $m \leq 0$ or $n \leq 0$, set $A = \mathbb{Z}$. In addition set $A' = A_{m-1,n-1}$ and $B' = \mathbb{Z}[X_{m-1}, Y_{n-1}]^{W'}$. The map $\phi$ is defined in Equation (3.6) and the element $T$ is as in Equation (3.4).

In case L, we set $A = \Lambda_{m,n}$ be the ring of supersymmetric Laurent polynomials, and $B = \Lambda_m \times \Lambda_n$. If $m \leq 0$ or $n \leq 0$, set $A = \Lambda_{m,n} = \mathbb{Z}$. In addition set $A' = \Lambda_{m-1,n-1}$ and $B' = \Lambda_{m-1} \times \Lambda_{n-1}$. The map $\phi$ is defined in Equation (4.1) and the element $T$ in Equation (4.3). In both cases we have a commutative diagram.

$$\begin{array}{ccc}
A & \xrightarrow{\phi} & A' \\
\downarrow \Phi & & \downarrow \Phi' \\
B & \xrightarrow{\phi} & B'
\end{array}$$

The map $\Phi$ is given by evaluation of a polynomial at $x_m = y_n = 0$. The two vertical maps are the obvious inclusions.

Remark 5.1. We make a remark about extension of scalars. For $\mathbb{Z}$-algebras $C, K$, set $C^K = C \otimes_\mathbb{Z} K$. If $K$ is a field of characteristic zero, then $I_\mathbb{Z}[X_m, Y_n]^K = I_K[X_m, Y_n]$. This is shown by first extending scalars to $\mathbb{Q}$ by localization, then to $K$ using a $\mathbb{Q}$-basis for the field extension $\mathbb{Q} \subseteq K$. Furthermore we can extend scalars in the above diagram replacing each algebra $C$ by $C^K$, and the maps $\phi, \Phi$ by $\phi_K = \phi \otimes 1, \Phi_K = \Phi \otimes 1$. This also works when $K = L[T]$ where $L$ is a field of characteristic zero. However it is not clear how to lift a supersymmetric polynomial with coefficients in $\mathbb{F}_p$ to the integers. The polynomial $x^2 + y^2$ is not supersymmetric, but mod 2 it is equal to $p^{(2)}_{1,1}$. Similar remarks apply to extension of scalars for $\Lambda_{m,n}$.

For the remainder of this section $K$ will denote either the ring of integers, or a field of characteristic zero.

Corollary 5.2. If $\phi_K : A^K = \Lambda^K_{m,n} \to \Lambda^K_{m-1,n-1}$ is the map induced by Equation
or, Equation (4.1), we have a disjoint union

$$\text{Spec } \mathcal{A}_{m,n}^K = \text{Spec } B^K \cup \phi^{-1}_K(\text{Spec } \mathcal{A}_{m-1,n-1}^K).$$

(5.1)

**Proof.** Note that $TB^K \subset \mathcal{A}^K \subset B^K$. Also we have $\text{Ker } \phi_K = TB^K$. If $K = \mathbb{Z}$, this follows from Corollary 3.2. For a field argue as in the remark. Hence the result follows from Equation (2.2). □

To prove the Nullstellensatz we need more information on maximal ideals.

**Proposition 5.3.** If $m \in \text{Max } \mathcal{A}_{m,n}^K$, then $mB^K$ is a proper ideal of $B^K$.

**Proof.** If $T \notin m$ this follows from Corollary 2.3. If $T \in m$, we use the above commutative diagram. Here $\phi_K(m)$ is a maximal ideal of $\mathcal{A}_{m-1,n-1}^K$, hence by induction $J = \phi_K(m)B^K_{m-1,n-1}$ is a proper ideal of $B^K_{m-1,n-1}$. Since $\Phi_K(mB^K) = J$, it follows that $mB^K$ is a proper ideal of $B^K$. □

**Theorem 5.4.** If $m$ is a maximal ideal of $\mathcal{A}^K$, then there is a maximal ideal $m$ of $\mathcal{B}^K$ such that $m = M \cap \mathcal{A}^K$.

**Proof.** If $T \notin m$ this follows from Corollary 2.3 extending scalars if necessary. If $T \in m$, the result holds by taking $M$ to be any maximal ideal containing $mB^K$. □

**Remark 5.5.** The ideal $M$ is not unique, but in case $S$, if $k$ is an algebraically closed field the fibers of the map

$$\text{Max } B^k_{m,n} \rightarrow \text{Max } \mathcal{A}^k_{m,n}$$

given by $M \rightarrow M \cap \mathcal{A}^k_{m,n}$ are known, [Mus12] Theorem 13.5.4.

**Theorem 5.6.** In both cases $S$ and $L$, the algebras $\mathcal{A}_{m,n}^K$ satisfy ACCR for $K = \mathbb{Z}$ or any field of characteristic zero. Hence any radical ideal $I$ is a finite intersection of prime ideals. These prime ideals can be taken to be the prime ideals minimal over $I$.

**Proof.** This follows from Lemma 2.6 extending scalars as necessary (compare also Equation (5.1)), once we observe that $B^K_2$ is Noetherian so has ACCR, and that $\mathcal{A}_{m-1,n-1}^K$ has ACCR by induction. □

6 The Strong Nullstellensatz.

First we prove the analog of the weak Nullstellensatz. Let $k$ be an algebraically closed field of characteristic zero. Then set $\mathcal{A} = \mathcal{A}_{m,n}^k$, $\mathcal{B} = \mathcal{B}_{m,n}^k$ and $k^{m|n} = k^m \times k^n$ in case $S$. In case $L$, we set $\mathcal{A} = \Lambda_{m,n}^k$, $\mathcal{B} = \Lambda_m^k \times \Lambda_n^k$ and $k^{m|n} = (k^*)^m \times (k^*)^n$.

**Theorem 6.1.** If $m$ is a maximal ideal of $\mathcal{A}$, there exists $\lambda \in k^{m|n}$ in case $S$, or $\lambda \in k^{m|n}$ in case $L$ such that $m = \{ f \in \mathcal{A} | f(\lambda) = 0 \}$. 

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Proof. This follows from Theorem 5.4. Note that maximal ideals in $B$ correspond to $W$-orbits in $k^{m|n}$ or $k^{*m|n}$.

If $I$ is a subset of $A$, let $\mathcal{V}(I) = \{ x \in k^{m|n} | f(x) = 0 \text{ for all } f \in I \}$ in case S, and $\mathcal{V}(I) = \{ x \in k^{*m|n} | f(x) = 0 \text{ for all } f \in I \}$, in case L. Such a set is called an superalgebraic set. If instead $I$ is a subset of $B$, we say that $\mathcal{V}(I)$ is an algebraic set. Thus any superalgebraic set is algebraic. In addition if $V$ is a subset of $k^{m|n}$ in case S, or $k^{*m|n}$ in case L, set

$$\mathcal{I}_A(V) = \{ f \in A | f(x) = 0 \text{ for all } x \in V \}.$$ 

We will also need

$$\mathcal{I}(V) = \{ f \in B | f(x) = 0 \text{ for all } x \in V \}.$$ 

Theorem 6.2. The maps $\mathcal{I}_A$ and $\mathcal{V}$ are inverse bijections between the set of superalgebraic sets in $k^{m|n}$ (case S) or $k^{*m|n}$ (case L), and the set of radical ideals in $A$. Both maps are order reversing.

Proof. The key point is that $\mathcal{I}_A(\mathcal{V}(I)) \subseteq \text{rad}(I)$. We adapt a well-known argument of Rabinowitsch see [Ful89] Chapter 1. Suppose $G \in \mathcal{I}_A(\mathcal{V}(I))$, and let $J$ be the ideal of $A_{m,n} \otimes k[T]$ generated by $I$ and $TG - 1$. Then $\mathcal{V}(J)$ is empty, since $G$ vanishes whenever all polynomials in $I$ vanish. Therefore by the weak Nullstellensatz, $1 \in J$, and we can write

$$1 = \sum_{i=1}^r A_i F_i + B(TG - 1)$$

where the $F_i$ are in $I$ and $B, A_i \in A_{m,n} \otimes k[T]$. The result follows by multiplying by a large power of $G$. 

Proposition 6.3. The maps $\mathcal{V}$, and $\mathcal{I}_A$ satisfy the following properties. Suppose that $E_\lambda, V_\lambda$ are subsets of $A$, and $k^{m|n}$ (or $k^{*m|n}$) respectively and that $a, b$ are ideals of $A$.

(a) $\mathcal{V}(\bigcup_{\lambda \in A} E_\lambda) = \bigcap_{\lambda \in A} \mathcal{V}(E_\lambda)$.

(b) $\mathcal{V}(a \cap b) = \mathcal{V}(ab) = \mathcal{V}(a) \cup \mathcal{V}(b)$.

(c) $\mathcal{I}_A(\bigcup_{\lambda \in A} V_\lambda) = \bigcap_{\lambda \in A} \mathcal{I}_A(V_\lambda)$.

Proof. Left to the reader.

7 Superalgebraic sets.

To give more meaning to Theorem 6.2 we need to know what superalgebraic sets look like. We show they are just the algebraic sets which are invariant under the Weyl groupoid of Sergeev and Veselov, [SV77]. First we review their work.
7.1 Definition of the Weyl Groupoid.

In [SV11] Sergeev and Veselov associated a certain groupoid \( \mathcal{W} = \mathcal{W}(R) \), which they call Weyl groupoid, to any generalized root system \( R \subset V \) in the sense of Serganova [Ser96]. A groupoid \( \mathcal{G} \) can be defined as a small category with all morphisms invertible. We denote the set of objects by \( \mathcal{B} \) which we call the base. As in [SV11] we use the same notation \( \mathcal{G} \) for the set of morphisms as for the groupoid itself.

To define the Weyl groupoid we need a preliminary construction, namely the semi-direct product groupoid \( \Gamma \rtimes \mathcal{G} \). Let \( \mathcal{G} \) be a groupoid and \( \Gamma \) a group acting on \( \mathcal{G} \) by automorphisms of the corresponding category. In particular, \( \Gamma \) acts on the base \( \mathcal{B} \) of \( \mathcal{G} \). Then the semi-direct product groupoid \( \Gamma \rtimes \mathcal{G} \) has the same base \( \mathcal{B} \), and the morphisms from \( x \) to \( y \) are pairs \((\gamma, f)\), with \( \gamma \in \Gamma \), \( f \in \mathcal{G} \) such that \( f : \gamma x \to y \).

Note that if \((\delta, g)\) is a second morphism with \( g : \delta y \to z \), then we also have morphisms \( \delta(f) : \delta \gamma x \to \delta y \), and \( g \circ \delta(f) : \delta \gamma x \to z \). Hence we can define composition of morphisms as follows:

\[
(\delta, g) \circ (\gamma, f) = (\delta \gamma, g \circ \delta(f)).
\]

Now we can define the Weyl groupoid \( \mathcal{W}(R) \) corresponding to the generalized root system \( R \). Recall that the reflections with respect to the non-isotropic roots generate a finite group denoted \( W_0 \). First consider the following groupoid \( \mathcal{T}_{iso} \) with base \( R_{iso} \), the set of all the isotropic roots in \( R \). The set of morphisms \( \alpha \to \beta \) is non-empty if and only if \( \beta = \pm \alpha \) in which case it consists of just one element. Denote the corresponding morphism \( \alpha \to -\alpha \) by \( \tau_\alpha, \alpha \in R_{iso} \). The group \( W_0 \) acts on \( \mathcal{T}_{iso} \) in a natural way: \( \alpha \to w(\alpha), \tau_\alpha \to \tau_{w(\alpha)} \). Define the Weyl groupoid

\[
\mathcal{W}(R) = W_0 \coprod W_0 \rtimes \mathcal{T}_{iso}
\]

to be the disjoint union of the group \( W_0 \) considered as a groupoid with a single point base \([W_0]\) and the semi-direct product groupoid \( W_0 \rtimes \mathcal{T}_{iso} \) with base \( R_{iso} \). Observe that the disjoint union is a well defined operation on the groupoids.

7.2 Action of \( \mathcal{W}(R) \) on the ambient space \( V \).

For any set \( X \) consider the following groupoid \( \mathcal{G}(X) \), with base consisting of all possible subsets \( Y \subset X \) and morphisms are all possible bijections between them. By an action of a groupoid \( \mathcal{G} \) on a set \( X \) we will mean a natural transformation between the categories \( \mathcal{G} \) and \( \mathcal{G}(X) \). If \( X \) is a vector space, the affine groupoid \( \mathcal{A}(X) \) has base all affine subspaces, and morphisms all affine bijections. Then an affine action of \( \mathcal{G} \) on \( X \) is natural transformation from \( \mathcal{G} \) to \( \mathcal{A}(X) \).

Returning to our generalized root system \( R \subset V \), let \( X = V \) and define the following affine action of the Weyl groupoid \( \mathcal{W}(R) \) on \( V \). The base point \([W_0]\) maps to the whole space \( V \), meaning that an element \( w \in W_0 \) acts on any point of \( V \) in the natural way. The base element corresponding to an isotropic root \( \alpha \) maps to the
hyperplane $\Pi_\alpha$ defined by the equation $(\alpha, x) = 0$. The element $\tau_\alpha$ acts as the shift

$$\tau_\alpha(x) = x + \alpha, \; x \in \Pi_\alpha.$$  

Note that since $\alpha$ is isotropic, $x + \alpha$ also belongs to $\Pi_\alpha$.

Let $V = \mathfrak{h}^*$ be the dual space to a Cartan subalgebra $\mathfrak{h}$ of a basic classical Lie superalgebra $\mathfrak{g}$ with generalized root system $R$. In the case of $\mathfrak{psl}(n|n)$ we consider $\mathfrak{gl}(n|n)$ instead. Using the invariant bilinear form we can identify $V$ and $V^* = \mathfrak{h}$ and consider the elements of the group ring $\mathbb{Z}[\mathfrak{h}^*]$ as functions on $V$. Similarly the symmetric algebra $S(\mathfrak{h})$ consists of functions on $\mathfrak{h}^*$. A function $f$ on $V$ is invariant under the action of the groupoid $\mathcal{W}$ if for any $g \in \mathcal{W}$ we have $f(g(x)) = f(x)$ for all $x$ in the domain of definition of the morphism $g$.

### 7.3 Description of Superalgebraic sets.

When dealing with a prime ideal $P$ of $\mathcal{A}$ or $\mathcal{B}$ with $T \notin P$, it is helpful to introduce $k^{m,n} \times k$ or $k^{m,n} \times k$ using $z$ as the coordinate for the last copy of $k$. Then for $R = \mathcal{A}$ or $\mathcal{B}$ we have $R_T \cong R[z]/(Tz - 1)$. Under this isomorphism $P_T$ corresponds to $P[z]/(Tz - 1)$. A point $y \in k^{m,n}$ with $T(y) \neq 0$ corresponds to the point $(y, T(y)^{-1}) \in k^{m,n} \times k$. This allows us to identify

$$\mathcal{V}(P_T) = \{ y \in k^{m,n} | f(y) = 0 \text{ for all } f \in P, T(y) \neq 0 \}$$

with

$$\{ y \in k^{m,n} \times k | f(y) = 0 \text{ for all } f \in P[z], (Tz - 1)(y) \neq 0 \}.$$  

Note that $T$ is fixed by every element of $W = W_{m,n}$, so defining $wz = z$ for all $w \in W$ makes the map

$$y \mapsto (y, T(y)^{-1})$$

when $T(y) \neq 0$, $W$-equivariant. Also $T(\Pi_\alpha) = 0$ for all $\alpha \in R_{iso}$, the action of $W_0 \ltimes \mathfrak{T}_{iso}$ on

$$\mathcal{V}_T = \{ y \in k^{m,n} | T(y) \neq 0 \}$$

is trivial (that is no element of $\mathcal{V}_T$ is in the domain of definition of any morphism in $W_0 \ltimes \mathfrak{T}_{iso}$). Thus defining the action of $W_0 \ltimes \mathfrak{T}_{iso}$ on $k^{m,n} \times k$ to be trivial, actually makes the map in (7.1) $\mathcal{W}(R)$-equivariant.

**Theorem 7.1.** The superalgebraic sets are exactly the algebraic sets that are invariant under the Weyl groupoid $\mathcal{W}$.

**Proof.** There are two things to check. If $I$ is an ideal of $\mathcal{A}_{m,n}$ then $\mathcal{V}(I)$ is invariant under $\mathcal{W}$. This holds because elements of $I$ are supersymmetric, so $\mathcal{V}(I)$ is a union of orbits. Conversely, suppose that $V$ is an algebraic set in the usual sense, which is invariant under $\mathcal{W}$. We want to show that $V$ is superalgebraic. Use the notation before Theorem 6.2. Set

$$V' = \{ v \in V | T(v) \neq 0 \}, \text{ and } V^c := V \backslash V'$$

from this point on substitute $k^{m,n}$ for $k^{m,n}$ if necessary.
(a) Suppose that \( P_1, \ldots, P_r \) are the prime ideals of \( \mathcal{B} \) minimal over \( \mathcal{I}(V) \) such that \( T \notin P_i \). Then set \( Q_i = P_i \cap \mathcal{A} \), a prime ideal of \( \mathcal{A} \). Using the fact that \( T \notin Q_i \), it is easy to see that \( P_i = Q_i \mathcal{B} \). Therefore \( \mathcal{V}(P_i) = \mathcal{V}(Q_i) \). Using the remarks preceding the proof this shows that \( V' \) is superalgebraic.

(b) Now set \( J = \mathcal{I}(V) + T \mathcal{B} \). Then \( \mathcal{V}(J) = V^c \). This corresponds to an algebraic sets in \( \text{Spec} \, \mathcal{B}' \). Moreover \( V^c \) is \( \mathcal{M} \)-invariant so is superalgebraic by induction.

It follows from Proposition 6.3 that \( V \) is superalgebraic. \( \square \)

7.4 Prime ideals and irreducible components.

We say that a superalgebraic set is irreducible if it cannot be written as the union of two proper superalgebraic subsets.

**Proposition 7.2.** If \( I \) is a radical ideal and \( V = \mathcal{V}(I) \) is the corresponding superalgebraic set, then \( I \) is prime if and only if \( V \) is irreducible.

**Proof.** The proof is completely analogous to the classical case \([\text{Ful} \, 89]\) Proposition 1, page 7. \( \square \)

In general if \( I \) is a radical ideal of \( \mathcal{A} \), then using Theorem 5.6, we can write \( I \) uniquely in the form \( I = P_1 \cap \ldots \cap P_r \), where the \( P_i \) are the prime ideals of \( \mathcal{A} \) which are minimal over \( I \). In this situation we call the superalgebraic sets \( \mathcal{V}(P_i) \) the irreducible components of \( \mathcal{V}(I) \). By Proposition 7.2 they are irreducible.

**Corollary 7.3.** Every superalgebraic set is uniquely a finite union of irreducible components.

**Proof.** This follows from the Nullstellensatz and Proposition 6.3 (b). \( \square \)

8 The degree of atypicality of a prime ideal.

We can iterate the process described in Equations (3.6), (4.1) and Theorem 2.4. Using the definitions of \( \mathcal{A}_{m,n} \) and \( \mathcal{B}_{m,n} \) given near the start of section \( 5 \) set \( \mathcal{A}_i = \lambda \mathcal{A}_{m-i,n-i} \), and \( \mathcal{B}_i = \mathcal{B}_{m-i,n-i} \). Then using the \( r + 1 \) pairs

\[ \{x_m, y_n\}, \ldots, \{x_{m-i}, y_{n-r}\}, \]

we obtain surjective homomorphisms

\[ \mathcal{A}_0 \overset{\phi_1}{\longrightarrow} \mathcal{A}_1 \overset{\phi_2}{\longrightarrow} \ldots \overset{\phi_r}{\longrightarrow} \mathcal{A}_r \] (8.1)

Set \( \psi_i = \phi_i \ldots \phi_2 \phi_1 \), and in case \( S \) set

\[ T_i = \prod_{k=1}^{m-i} \prod_{j=1}^{n-i} (x_k + y_j) \in \mathcal{A}_i, \]

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\mathcal{B}_i = \mathbb{Z}[\mathcal{X}_{m-i}, \mathcal{Y}_{n-i}]_{W_i}^W$, and \( \hat{\mathcal{B}}_i = \mathbb{Z}[\mathcal{X}_{m-i}, \mathcal{Y}_{n-i}]_{T_i}^W \), where \( W_i \) is a direct product of symmetric groups \( W_i = S_{m-i} \times S_{n-i} \).

In case L, set \( \mathcal{A}_i = \Lambda_{m-i,n-i} \), \( \mathcal{B}_i = \Lambda_{m-i} \times \Lambda_{n-i} \),

\[
T_i = \prod_{j=1}^{n-i} \prod_{i=1}^{m-i} \left( 1 - \frac{y_j}{x_i} \right) \in \mathcal{A}_i,
\]

and \( \hat{\mathcal{B}}_i = (\Lambda_{m-i} \times \Lambda_{n-i})_{T_i} \). Note that \( T_i \hat{\mathcal{B}}_i \subset \mathcal{A}_i \subset \mathcal{B}_i \). Repeating our earlier arguments leads to a disjoint union of locally closed sets

\[
\text{Spec} \mathcal{A} = \text{Spec} B_T \cup \psi_1^{-1}(\text{Spec} \hat{\mathcal{B}}_1) \cup \ldots \cup \psi_r^{-1}(\text{Spec} \hat{\mathcal{B}}_r-1) \cup \psi_r^{-1}(\text{Spec} A_r).
\]

(8.2)

Now suppose \( p = \psi_r^{-1}(P) \), where \( P \in \text{Spec} A_r \) and that \( T_r \notin P \). In this case \( P \) corresponds by localization to a prime ideal of \( \hat{\mathcal{B}}_r \), and we say that \( p \) has degree of atypicality \( r \). In the case that \( P = M \) is a maximal ideal in \( A_r \), \( M \) corresponds to a maximal ideal \( M_{T_r} \) of \( \hat{\mathcal{B}}_r \), and so \( \mathcal{V}(M) \) consists of a single \( W_r \) orbit of points in \( \text{Spec} A_r \). This shows that \( p = \psi_r^{-1}(P) \) has degree of atypicality \( r \) in the usual sense. Since \( \hat{\mathcal{B}}_r \) is a Jacobson ring, we have the following.

**Theorem 8.1.** If \( p \) is a prime ideal of \( \mathcal{A} \) with degree of atypicality \( r \), then \( p \) is an intersection of maximal ideals of \( \mathcal{A} \) each having degree of atypicality \( r \).

Note also that if \( r = \min(m,n) \), Equation (8.2) gives a natural stratification of \( \text{Max} \ A_{m,n} \) according to the degree of atypicality.

### 9 Applications to \( \text{Z}(\mathfrak{g}) \).

#### 9.1 The Image of the Harish-Chandra map.

In this and the following subsection we apply our results to the center \( \text{Z}(\mathfrak{g}) \) the enveloping algebra \( U(\mathfrak{g}) \). Throughout we work over an algebraically closed field \( k \) of characteristic zero. For a basic classical simple Lie algebra the Harish-Chandra map, due to Gorelik and Kac, gives an isomorphism \( \text{Z}(\mathfrak{g}) \rightarrow I(\mathfrak{h}) \), where \( I(\mathfrak{h}) \) is a certain subalgebra of \( S(\mathfrak{h})_{W}^W \), for further details see [Mus12] Theorem 13.1.1. In the next result we give a description of \( I(\mathfrak{h}) \) when \( \mathfrak{g} = \mathfrak{gl}(m,n) \) or an orthosymplectic Lie superalgebra. Let \( \epsilon_1, \ldots, \epsilon_m, \delta_1, \ldots, \delta_n \) be the usual basis of \( \mathfrak{h}^* \), and \( h_1, \ldots, h_m, h'_1, \ldots, h'_n \) the dual basis for \( \mathfrak{h} \), see [Mus12] Equations (2.2.4), (2.2.5) and (2.3.5).

If \( \mathfrak{g} = \mathfrak{gl}(m,n) \) or \( \mathfrak{g} = \mathfrak{osp}(2m+1,2n) \), \( I(\mathfrak{h}) \) is easily expressed in terms of supersymmetric polynomials. However if \( \mathfrak{g} = \mathfrak{osp}(2m,2n) \) the situation is more complicated. In this case, set

\[
J(\mathfrak{h}) = I_k(h_1^2, \ldots, h_m^2, h'_1^2, \ldots, h'_n^2), \quad T = \prod_{i,j} (h_i^2 - (h'_j)^2), \quad \Phi = (h_1 \ldots h_m) T.
\]
Also consider the group
\[ W' = (\mathbb{Z}_2^m \times S_m) \times (\mathbb{Z}_2^n \times S_n) \]
where \( \times \) denotes a semidirect product. There is an action of \( W' \) on \( S(\mathfrak{h}) = k[h_1, \ldots, h_m, h'_1, \ldots, h'_n] \), where the symmetric groups \( S_m \) and \( S_n \) permute the \( h_i, h'_i \) respectively, and \( \mathbb{Z}_2^m, \mathbb{Z}_2^n \) change their signs. The Weyl group \( W \) of \( g \) is a subgroup of index two in \( W' \). Namely \( W \) consists of all elements of \( W' \) which change an even number of signs of the \( h_i \). Then we have \[ \text{Ser99, Mus12 Theorem 13.4.1.} \]

**Theorem 9.1.** With the above notation

(a) If \( g = \mathfrak{gl}(m, n) \) we have \( I(h) = I_k(h_1, \ldots, h_m; h'_1, \ldots, h'_n) \).

(b) If \( g = \mathfrak{osp}(2m + 1, 2n) \) we have \( I(h) = J(h) \).

(c) If \( g = \mathfrak{osp}(2m, 2n) \), we have
\[
I(h) = J(h) + \Phi S(h)^{W'}. \tag{9.1}
\]

9.2 The case of \( \mathfrak{osp}(2m, 2n) \).

The work on supersymmetric polynomials applies directly to Spec \( I(h) \) when \( g = \mathfrak{gl}(m, n) \) or \( \mathfrak{osp}(2m + 1, 2n) \). Now assume that \( g = \mathfrak{osp}(2m, 2n) \). The first step is to observe that the sum in Equation (9.1) is direct. Let \( \sigma : W' \to \{\pm 1\} \) be the character with Ker \( \sigma = W \). If \( R \) is a ring on which \( W' \) acts, such that \( W \) acts trivially, we have \( R = R_1 \oplus R_{\sigma} \) where \( R_1 = R^{W'} \) and
\[
R_{\sigma} = \{ r \in R | wr = \sigma(w)r \text{ for all } w \in W' \}.
\]

Note that \( R_1 \) is a subring of \( R \) and \( R_{\sigma} \) is an ideal of \( R \). If \( C = S(h)^W \) we have \( C_{\sigma} = h_1 \ldots h_m C_1 \), and likewise for any localization of \( C \) with respect to a \( W' \)-invariant element. If \( R = I(h) \), we have \( I(h)_1 = J(h) \) and \( I(h)_{\sigma} = \Phi S(h)^{W'} \). So the sum in (9.1) is just \( I(h) = I(h)_1 \oplus I(h)_{\sigma} \).

Now we use the projection \( \psi : I(h) \to J(h) \) with kernel \( I(h)_{\sigma} \) to obtain an analog of Equation (2.2). At this point the argument diverges slightly from the proof of Theorem 2.4, so we give the details again. First observe that for any prime ideal \( P \) of \( I(h) \) with \( T \in P \), we have \( I(h)_{\sigma} \subseteq P \). Indeed \( T \) is \( W' \)-invariant, and \( TS(h)^{W'} \subseteq J(h) \subset I(h) \). Hence
\[
(I(h)_{\sigma})^2 = \Phi^2 S(h)^{W'} = T(h_1^2 \ldots h_m^2 S(h)^{W'} \subseteq PI(h) = P. \tag{9.2}
\]

Furthermore
\[
J(h)_T = S(h)^{W'}_T \text{ and } \Phi S(h)^{W'}_T = h_1 \ldots h_m S(h)^{W'}_T,
\]
so \( I(h)_T = S(h)^{W'}_T \). Thus arguments similar to those leading to (2.2) yield a disjoint union
\[
\text{Spec } I(h) = \text{Spec } S(h)^{W'}_T \cup \psi^{-1}(\text{Spec } J(h)).
\]
The analysis of maximal ideals is also slightly different. We use the commutative diagram

\[
\begin{array}{ccc}
I(\mathfrak{h}) & \xrightarrow{\phi} & J(\mathfrak{h}) \\
\downarrow & & \downarrow \\
S(\mathfrak{h})^W & \xrightarrow{\Phi} & S(\mathfrak{h})^{W'}
\end{array}
\]

The map \(\phi\) has kernel \(I(\mathfrak{h})_\sigma\) and \(\Phi\) has kernel \(S(\mathfrak{h})^W_\sigma\). The two vertical maps are the obvious inclusions.

**Proposition 9.2.** If \(m \in \text{Max } I(\mathfrak{h})\), then \(mS(\mathfrak{h})^W\) is a proper ideal of \(S(\mathfrak{h})^W\).

**Proof.** If \(T \notin m\) this follows as in Corollary 2.3. If \(T \in m\) then \(I(\mathfrak{h})_\sigma \subseteq m\), as shown in Equation (9.2), so \(\phi(m)\) is a maximal ideal of \(J(\mathfrak{h})\). Since \(J(\mathfrak{h})\) is an algebra of supersymmetric polynomials, Proposition 5.3 shows that \(L = \phi(m)S(\mathfrak{h})^W\) is a proper ideal of \(S(\mathfrak{h})^W\). Because \(\Phi(mS(\mathfrak{h})^W) = L\), it follows that \(mS(\mathfrak{h})^W\) is a proper ideal of \(S(\mathfrak{h})^W\).

Using this result the weak and strong Nullstellensatz for \(I(\mathfrak{h})\) may be deduced as before. In particular we have the following analog of Theorem 5.4.

**Theorem 9.3.** If \(m\) is a maximal ideal of \(I(\mathfrak{h})\), then there is a maximal ideal \(M\) of \(S(\mathfrak{h})^W\) such that \(m = M \cap I(\mathfrak{h})\).

Now Conjecture 13.5.1 from [Mus12] states that if \(I(\mathfrak{h})\) is as in Theorem 9.1 and \(m\) is any maximal ideal io of \(I(\mathfrak{h})\), then there is a \(\lambda \in \mathfrak{h}^*\) such that

\[
m = \{f \in I(\mathfrak{h}) | f(\lambda) = 0\}. \quad (9.3)
\]

**Proof of the Conjecture.** Since \(I(\mathfrak{h})\) is isomorphic to an algebra of supersymmetric polynomials in cases (a) and (b) of Theorem 9.1 we have \(m = M \cap I(\mathfrak{h})\) for some ideal of \(S(\mathfrak{h})^W\), by Theorem 5.4. By Theorem 9.3 the same holds in case (c). Now the maximal ideal \(M\) in \(S(\mathfrak{h})^W\) corresponds to a \(W\)-orbit in \(\mathfrak{h}^*\), and Equation (9.3) holds for any \(\lambda\) in this orbit. \(\Box\)

### 10 Appendix: Characterizations of Laurent supersymmetric polynomials.

We mention some alternative characterizations of Laurent supersymmetric polynomials with coefficients in a field \(K\). Set

\[
S = K[x_1^{\pm 1}, \ldots, x_m^{\pm 1}, y_1^{\pm 1}, \ldots, y_n^{\pm 1}].
\]

The direct product of symmetric groups \(W = S_m \times S_n\) acts on \(S\) in the obvious way. Obviously if \(f \in S\) is \(W\)-invariant, and

\[
f(x_1 = t, y_1 = t) \text{ is a polynomial which is independent of } t, \quad (10.1)
\]
then \( f(x_i = t, y_j = t) \) is independent of \( t \) for all \( i, j \). We consider the condition (10.1) independently of \( W \)-invariance. Let \( T \) be the algebraic torus \((K^*)^{m+n} \) and let \( \{\epsilon_1, \ldots, \epsilon_m, \delta_1, \ldots, \delta_n\} \) be coordinates on \( T \). Let \( T_i T_j \) be the subtori of \( T \) defined by the equation \( T = \ker \epsilon_1 \delta_1^{-1}, T_1 = \ker \epsilon_1 \cap \ker \delta_1 \), and let \( T_2 = \{ (t, 1, \ldots, 1 | t, 1, \ldots, 1) | t \in K^* \} \). Then we have a direct product \( T = T_1 T_2 \). Next set \( R = K[x_2, \ldots, x_m | y_2, \ldots, y_n ], x = x_1, y = y_1, z_+ = (1 - \frac{x}{y}), \) and \( z_- = (1 - \frac{y}{x}) \).

Note that \( (1 - z_-)^{-1} = (1 - z_+)^{-1} \) and hence \( z_+ = z_+(z_+ - 1)^{-1} \). It follows that \( S = R[x_1, y_1] = R[x_1, z_+, z_-], S z_+ = S z_- \), and

\[
S = R[x_1, (1 - z_+)^{\pm 1}].
\]

(10.2)

If \( \lambda = (x_1, x_2, \ldots, x_m | y_1, y_2, \ldots, y_n), q \neq 0 \in K \) and \( f \in S \), set

\[
f(q)(\lambda) = f(q x_1, x_2, \ldots, x_m | q y_1, y_2, \ldots, y_n).
\]

Then if \( \alpha = (1, 0, \ldots, 0 | 1, 0, \ldots, 0) \), we define the \textit{Laurent directional derivative} \( D_\alpha f \) in the direction of \( \exp(q \alpha) \) by

\[
(D_\alpha f)(\lambda) = \lim_{q \to 1} \frac{f(q \lambda) - f(\lambda)}{q - 1}.
\]

This makes sense since we only have to differentiate Laurent polynomials. Note that the directional derivative \( D_\alpha f \) satisfies \( D_\alpha f = x \partial f / \partial x + y \partial f / \partial y \), where the partial derivatives vanish on \( R \). The result below is the Laurent analog of [Mus12] Lemma 12.1.1.

**Lemma 10.1.** Let \( z = z_+ \). For \( f \in S \) the following conditions are equivalent

(a) \( f \in R + Sz \)

(b) \( f(x = y = t) \) is independent of \( t \neq 0 \)

(c) For \( \lambda \in T, t \in T_2 \), we have \( f(\lambda) = f(t \lambda) \).

(d) \( x \partial f / \partial x + y \partial f / \partial y \in (x - y) \)

**Proof.** (a) \( \Rightarrow \) (b) If \( f \in R \) then \( f(x = y = t) = f \) is independent of \( t \), while if \( f \in Sz \) then \( f(x = y = t) = 0 \).

(a) \( \Rightarrow \) (d) This is similar to the proof of (a) \( \Rightarrow \) (b).

(c) \( \Leftrightarrow \) (d) This holds since \( D_\alpha f \) vanishes on \( T \) if and only if \( f(q \lambda) \) is constant for all \( \lambda \in T \).

(b) \( \Rightarrow \) (a) Using (10.2), we can write \( f \) uniquely as a finite sum

\[
f = \sum_{i,j \in \mathbb{Z}} r_{ij} x^i (1 - z)^j,
\]

(10.3)
with \( r_{i,j} \in R \) for all \( i, j \). Then \( f(x = y = t) = \sum_{i,j \in \mathbb{Z}} r_{i,j} t^i \), so if (b) holds then for all \( i \neq 0 \), we have
\[
 r_{i,0} = -\sum_{j \neq 0} r_{i,j}.
\] (10.4)

Making this substitution in Equation (10.3) shows that \( f \in R + Sz + Sz_+ = R + Sz \).

(c) \( \Rightarrow \) (a) Given \( \lambda \in T \), we can find \( \lambda' \in T_1 \) and \( t' \in T_2 \) such that \( \lambda = t' \lambda' \). Hence (c) is equivalent to \( f(\lambda) = f(t\lambda) \) for all \( t \in T_2 \) and \( \lambda \in T_1 \). Write \( f \) as in (10.3). Then for \( \lambda \in T_1 \) we have \( f(\lambda) = \sum_{i \in \mathbb{Z}, j \in \mathbb{Z}} r_{i,j}(\lambda) \) and \( f(t\lambda) = \sum_{i \in \mathbb{Z}, j \in \mathbb{Z}} r_{i,j}(\lambda)t^i \). Thus if (c) holds then again (10.4) holds (with \( r_{i,j}(\lambda) \) replacing \( r_{i,j} \)) for all non-zero \( i \in \mathbb{Z} \), and (a) follows as before.

\[ \square \]

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