T-ADIC EXPONENTIAL SUMS OVER AFFINOIDS

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Abstract. We introduce and develop $(\pi, p)$-adic Dwork theory for $L$-functions of exponential sums associated to one-variable rational functions, interpolating $p^k$-order exponential sums over affinoids. Namely, we prove a generalization of the Dwork-Monsky-Reich trace formula and apply it to establish an analytic continuation of the $C$-function $C_f(s, \pi)$. We compute the lower $(\pi, p)$-adic bound, the Hodge polygon, for this $C$-function. Along the way, we also show why a strictly $\pi$-adic theory will not work in this case.

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1. Introduction

Let $p$ be a prime and $q = p^a$, some integer $a \geq 1$. Fix $\ell \geq 1$ distinct elements $P_1, \ldots, P_\ell \in \mathbb{F}_q \cup \{\infty\}$. Without loss of generality, take $P_1 = \infty$ and $P_2 = 0$, assuming $\ell \geq 2$ for the rest of the paper. For $x \in \mathbb{F}_q$, denote by $\hat{x}$ the Teichmüller lift of $x$ in $\mathbb{Z}_q$.

Let $E(x)$ be the Artin-Hasse exponential series, $T$ a formal variable and $\pi$ such that $E(\pi) = 1 + T$. To $f(x) = \sum_{j=1}^\ell \sum_{i=1}^{d_j} \frac{a_{i,j}}{(x-P_j)^i} \in \mathbb{Z}_q\left[\frac{1}{x-P_1}, \ldots, \frac{1}{x-P_\ell}\right]$, $a_{d,j} \neq 0$. 

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we associate a \( \pi \)-adic exponential sum$^1$:

\[
S_f(k, \pi) = \sum_{x \in \mathbb{F}_q^\times, x \neq \hat{P}_1, \ldots, \hat{P}_\ell} (E(\pi))^{\text{Tr}_{\mathbb{Q}_p}/Q_p(f(x))},
\]

and we say the characteristic function, or \( C \)-function, attached to this exponential sum is

\[
C_f(s, \pi) = \exp\left(\infty \sum_{k=1}^{\infty} -\left(q^k - 1\right)^{-1} S_f(k, \pi) \frac{s^k}{k}\right).
\]

When \( T = \zeta_p - 1 \), \( \zeta_p \) a primitive \( p \)th root of unity, (1) becomes the exponential sum over a one-dimensional affinoid studied by Robba in [12] and Zhu in [15]. Oppositely, letting \( \zeta_p^m \) be \( p^m \)th roots of unity and \( T = \zeta_p^m - 1 \) yields exponential sums of \( p^m \)-order over one-dimensional affinoids. In the classical case, these \( p^m \)-order exponential sums were studied by Liu and Wei in [10].

The purpose of \( \pi \)-adic (and \( (\pi, p) \)-adic) theory is to interpolate all of these exponential sums in a single \( C \)-function. Whenever we set \( \pi \) to be a value in \( c \in \mathbb{C}_p \), we say we specialize at \( \pi = c \).

When \( f(x) \) has one or two poles, Liu and Wan ([9]) built a \( T \)-adic Dwork theory and computed, among other things, a Hodge polygon for this \( C \)-function. In this paper, we extend their results to the case when \( \ell \geq 3 \) by generalizing the affinoid Dwork theory used earlier by Zhu in [15]. The bulk of our work is lifting this Dwork theory to the \( \pi \)-adic case. That is, we construct a Banach module \( Z^\pi \) and a completely continuous operator \( \alpha_a \) on \( Z^\pi \) such that

\[
C_f(s, \pi) = \det(1 - \alpha_a s).
\]

Unlike Liu and Wan’s case, however, a purely \( T \)-adic theory is not precise enough. When \( \ell \geq 3 \), the \( \alpha_a \) operator is not \( \pi \)-adically completely continuous and we cannot apply Dwork theory (see Corollary 6.10). To resolve this, we utilize the \( (\pi, p) \)-adic norm, used for the same reason by Li in [6], to produce sharper estimates and make \( \alpha_a \) completely continuous.

Our main result, the computation of the \( (\pi, p) \)-adic Hodge polygon, is as follows: For \( k = 1, \ldots, \ell \), let \( \text{HP}^c_k \) be the Newton polygon with vertices

\[
\{(n, \frac{a(p-1)n(n-1)c}{2d_k})\}_{n \geq 0},
\]

where \( c \) is a real number with \( 0 < c \leq \frac{1}{p-1} \).

We define the \( (\pi^{1/c}, p) \)-adic Hodge polygon, \( \text{HP}^c \), to be the concatenation of \( \text{HP}^c_1, \ldots, \text{HP}^c_\ell \).

**Theorem 1.1.** The \( (\pi^{1/c}, p) \)-adic Newton polygon of \( C_f(s, \pi) \) lies above \( \text{HP}^c \).

As an example, consider the case where \( \pi_1 \) is a root of \( \log(E(x)) \) with \( \text{ord}_p \pi_1 = 1/(p-1) \). After specializing at \( \pi = \pi_1 \), Theorem 1.1 implies, taking \( c = \frac{1}{p-1} \), that the corresponding Hodge polygon is nothing but the concatenation of:

\[
\{(n, \frac{a_n(n-1)}{2d_k})\}_{n \geq 0},
\]

---

$^1$The literature ([9], [8], [6], etc.) generally deals with \( T \)-adic exponential sums, but for convenience, we will do things \( \pi \)-adically. There is no difference and our results can be stated either way.
over \( k = 1, \ldots, \ell \), and this is exactly the same Hodge bound obtained in \([15]\).

Our construction of a \((\pi, p)\)-adic theory opens up many avenues of future development. Liu, Liu and Niu in \([8]\), for instance, compute the generic Newton polygon for the classical \(T\)-adic \(C\)-function, and there is a natural question as to whether their results can be extended to the affinoid case. Similarly, Ren, Wan, Xiao and Yu in \(([14])\) considered exponential sums over higher rank Artin-Schreier-Witt towers and Liu and Liu in \([7]\) studied twisted \(T\)-adic exponential sums. Extending both of these results to the affinoid case might be interesting.

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2. Preliminaries

We will need some results about Tate and Banach algebras. For a more comprehensive review, see \([3]\), \([2]\) and \([4]\).

2.1. Tate Algebras. Let \((A, | \cdot |)\) be an ultrametrically normed ring. Define the Tate algebra over \(A\) to be

\[
A(X_1, \ldots, X_n) = \left\{ \sum_{i_1, \ldots, i_n \in \mathbb{Z}_{\geq 0}} a_{i_1, \ldots, i_n} X_1^{i_1} \cdots X_n^{i_n} \in A[[X_1, \ldots, X_n]] : |a_{i_1, \ldots, i_n}| \to 0 \right\}
\]

as \(i_1 + \cdots + i_n \to \infty\),

and equip \(A(X_1, \ldots, X_n)\) with the gauss norm \(| \cdot |_{\text{gauss}}\)

\[
| \sum_{i_1, \ldots, i_n \in \mathbb{Z}_{\geq 0}} a_{i_1, \ldots, i_n} X_1^{i_1} \cdots X_n^{i_n} |_{\text{gauss}} = \sup_{i_1, \ldots, i_n} |a_{i_1, \ldots, i_n}|.
\]

2.2. Banach algebras and modules. Let \(A\) be a complete unital commutative ring separated with respect to a non-trivial ultrametric norm \(| \cdot |\) such that

(1) \(|1| = 1\)
(2) \(|a + b| \leq \max\{|a|, |b|\}\)
(3) \(|ab| \leq |a||b|\)
(4) \(|a| = 0\) if and only if \(a = 0\),

for all \(a, b \in A\).

We call \(A\) a Banach algebra. Moreover, if \(E\) is an ultrametrically normed complete module over \(A\) such that \(|ae| \leq |a||e|\) for \(a \in A\) and \(e \in E\), we say \(E\) is a Banach module over \(A\). A Banach module \(E\) over \(A\) has an orthonormal basis \(\{e_i\}_{i \in I} \subset E\) if for each \(x \in E\) we can write uniquely \(x = \sum_{i \in I} a_i e_i\) for \(a_i \in A\) with \(|a_i| \to 0\) as \(i \to \infty\).

For a bounded Banach module operator \(\phi : B \to C\), we write the standard operator norm \(\| \cdot \|_{\text{op}}\)

\[
\| \phi \|_{\text{op}} = \sup_{b \in B, |b| = 1} |\phi(b)|.
\]

If \(\{e_i\}_{i \in I}\) is an orthonormal bases for \(B\), then an endomorphism of \(B\), \(\phi\), is completely continuous if

\[
\lim_{i \to \infty} \sup_{j \in I} |b_{ij}| = 0,
\]

where \(\phi(e_i) = \sum_{j \in I} b_{ij} e_j\).
3. \( p \)-adic Spaces

Once and for all, fix \( 0 < r < 1 \) and \( R \in \mathbb{C}_p \) with \( |R|_p = r \) and let \( s \) be a \( p \)-power. Define \( H_{r,s} = \mathbb{C}_p \langle \frac{R}{x - \hat{P}_1}, \ldots, \frac{R}{x - \hat{P}_\ell} \rangle \) to be the Tate algebra of rigid analytic functions over an affinoid \( A_{r,s} = \{ x \in \mathbb{C}_p : |x|_p \leq 1/r, |x - \hat{P}_j|_p \geq r \text{ for } 2 \leq j \leq \ell \} \) with supremum norm \( \| \cdot \|_{r,s} \):

\[
\| \xi \|_{r,s} = \sup_{x \in A_{r,s}} |\xi(x)|_p.
\]

**Remark.** Let \( A \) be any algebraically closed and ultrametrically normed field and consider the Tate algebra \( A \langle X_1, \ldots, X_n \rangle \). It is well known that if \( Z = \{ (x_1, \ldots, x_n) \in A^n : |x_i|_p \leq 1 \} \), then for \( f \in A(X_1, \ldots, X_n) \),

\[
\sup_{(x_1, \ldots, x_n) \in Z} |f(x_1, \ldots, x_n)| = |f|_{\text{gauss}}.
\]

However in the above, when \( X_i = \frac{R}{x - \hat{P}_j} \), we see that \( (X_1, \ldots, X_n) \in Z \) if and only if, \( i \neq 1, |\frac{R}{x - \hat{P}_j}|_p \leq 1 \), which implies \( |x - \hat{P}_j|_p \geq r \), and for \( i = 1, |Rx|_p \leq 1 \), which yields \( |x|_p \leq 1/r \). Hence \( Z = A_{r,s} \) and \( | \cdot |_{\text{gauss}} = \| \|_{r,s} \) on \( H_{r,s} \).

\( H_{r,s} \) has two important orthonormal bases that we will utilize.

**Proposition 3.1.** The set

\[
\left\{ \left( \frac{R}{x - \hat{P}_j} \right)^i \right\}_{1 \leq j \leq \ell, 0 \leq i}
\]

forms an orthonormal basis for \( H_{r,s} \) over \( \mathbb{C}_p \). (When convenient, we will use the notation \( B_{ij} = \frac{1}{(x - \hat{P}_j)^i} \).)

**Proof.** See Lemma 2.1 and the comment following its proof on p.1535 in [15]. \( \square \)

**Proposition 3.2.** Let \( v(x) = (x - \hat{P}_1) \cdots (x - \hat{P}_\ell) \). The set

\[
\left\{ \frac{x^i}{R^{j} - j^i v^j} \right\}_{i \geq 0, (i, \ell) = 1, j \geq 0}
\]

forms an orthonormal basis for \( H_{r,s} \) over \( \mathbb{C}_p \).

**Proof.** See Theorem 2 and the remark following it in [11]. \( \square \)

4. \( (\pi, p) \)-adic Spaces

Let \( \pi \) be a formal variable.

**Definition 4.1.** For \( f(\pi) = \sum_{i=0}^{\infty} b_i \pi^i \in \mathbb{Z}_q[[\pi]] \), define the \( (\pi, p) \)-norm on \( \mathbb{Z}_q[[\pi]] \)

\[
|\cdot|_{\pi,p} : \sum_{i=0}^{\infty} b_i \pi^i |_{\pi,p} = \max_i |b_i|_p p^{-i}.
\]

**Lemma 4.2.** \( | \cdot |_{\pi,p} \) is a complete multiplicative norm on \( \mathbb{Z}_q[[\pi]] \).
Proof. Let \( f(x) = \sum_{i=0}^{\infty} b_i \pi^i \), \( g(x) = \sum_{i=0}^{\infty} c_i \pi^i \in \mathbb{Z}_q[[\pi]] \). The only nontrivial thing to prove is \( | \cdot |_{\pi, p} \) is a norm has \( | f + g |_{\pi, p} \leq \max (|f|_{\pi, p}, |g|_{\pi, p}) \). Then:

\[
|f + g|_{\pi, p} = \max_i (|b_i + c_i|_p p^{-i}) \leq \max_i (\max_i (|b_i|_p, |c_i|_p) p^{-i})
\]

\[
= \max_i (\max_i |b_i|_p p^{-i}, \max_i |c_i|_p p^{-i}) = \max (|f|_{\pi, p}, |g|_{\pi, p}).
\]

To see that \( \mathbb{Z}_q[[\pi]] \) is complete with respect to this norm, observe that \( | \cdot |_{\pi, p} \) is just the norm induced by the \((\pi, p)\)-topology on \( \mathbb{Z}_q[[\pi]] \), and

\[
\lim_i \mathbb{Z}_q[[\pi]]/(\pi, p)^i \cong \mathbb{Z}_q[[\pi]].
\]

One direction of the inequality to show \( | \cdot |_{\pi, p} \) is multiplicative is clear:

\[
|fg|_{\pi, p} = \max_i \left| \sum_{j,k \geq 0} b_j c_k |p| p^{-i} \right| \leq \max (\max_j (|b_j|_p p^{-j} \cdot |c_k|_p p^{-k})) \leq |f|_{\pi, p} |g|_{\pi, p}.
\]

For the opposite inequality, let \( i_0 \) and \( j_0 \) be the minimal integers such that \( |f|_{\pi, p} = |b_{i_0}|_p p^{-i_0} \) and \( |g|_{\pi, p} = |c_{j_0}|_p p^{-j_0} \). If we write \( fg = \sum_{i=0}^{\infty} a_i \pi^i \), then

\[
|a_{i_0 + j_0}|_p = |b_{i_0} c_{j_0} + \sum_{i+j=i_0+j_0, i \neq i_0, j \neq j_0} b_i c_j|_p.
\]

Take some \( i, j, i \neq i_0 \) and \( j \neq j_0 \), with \( i + j = i_0 + j_0 \) so that either \( i < i_0 \) and \( j > j_0 \) or \( j < j_0 \) and \( i > i_0 \). In either case, by the minimality of \( i_0 \) and \( j_0 \), \( |b_i|_p c_j|_p < |b_{i_0}|_p c_{j_0}|_p \), and so \( |a_{i+0+j_0}| = |b_{i_0} c_{j_0}|_p \). Hence:

\[
|fg|_{\pi, p} = \max_i |a_i|_p p^{-i} \geq |a_{i_0 + j_0}| p^{-(i_0 + j_0)} = |f|_{\pi, p} |g|_{\pi, p}.
\]

Because both \( \mathbb{Z}_q[[\pi]] \) and \( \mathcal{H}_{r,s} \) are Banach modules over \( \mathbb{Z}_q \), we can consider the following completed tensor product of \( \mathbb{Z}_q \)-Banach modules (again see [4], p.424):

**Definition 4.3.** Define a module

\[ \mathcal{H}^\pi_{r,s} = \mathbb{Z}_q[[\pi]] \hat{\otimes}_{\mathbb{Z}_q} \mathcal{H}_{r,s} \]

equipped with the norm coming from the completed tensor product \( \| \cdot \|_{r,s} \)

\[
\|n\|_{r,s} = \inf_i \sup b_i(\pi) \|_{\pi, p} \|_{r,s},
\]

where the infimum is taken over all representations of \( n = \sum_i b_i(\pi) \otimes \xi_i \), with \( |b_i(\pi)|_{\pi, p} \|_{r,s} \to 0 \) as \( i \to \infty \).

Note that for the sake of notation when referring to simple tensors in \( \mathcal{H}^\pi_{r,s} \), we will just write \( a \otimes b \) rather than \( a \hat{\otimes} b \).

**Proposition 4.4.** For \( g, h \in \mathcal{H}^\pi_{r,s} \), \( \|gh\|_{r,s} \leq \|g\|_{r,s} \|h\|_{r,s} \).

Proof. For \( g, h \in \mathcal{H}^\pi_{r,s} \) with arbitrary representations \( g = \sum_i b_i \otimes g_i \) and \( h = \sum_i c_i \otimes h_i \),

\[
gh = (\sum_i b_i \otimes g_i)(\sum_j c_j \otimes h_j) = \sum_{i,j} b_i c_j \otimes g_i h_j.
\]
Hence by Lemma 4.2 and the fact that the norm on the Tate algebra is multiplicative,
\[
\|gh\|_{r,s} = \inf_{g=\sum_i b_i \otimes g_i} \sup_i \|e_i\|_{\pi,p} \|\xi_i\|_{r,s} \leq \inf_{g=\sum_i b_i \otimes g_i} \sup_{i,j} \|b_i c_j\|_{\pi,p} \|g_i h_j\|_{r,s}
\]
\[
\leq \inf_{g=\sum_i b_i \otimes g_i} \sup_{i,j} (\|b_i\|_{\pi,p} \|h_j\|_{r,s}) (\|c_j\|_{\pi,p} \|g_i\|_{r,s}) = \|g\|_{r,s} \|h\|_{r,s}.
\]

\[\square\]

Let \( \mathbb{C} = \mathbb{Z}_q[[\pi]] \hat{\otimes}_{\mathbb{Z}_q} \mathbb{C}_p \) and define a \( \mathbb{C} \)-module structure on \( \mathcal{H}^\pi_{r,s} \) in the following way: for a tensor \( b \otimes \xi \in \mathcal{H}^\pi_{r,s} \) and a tensor \( b' \otimes \xi' \) in \( \mathbb{C} \),
\[
(b \otimes \xi)(b' \otimes \xi') = bb' \otimes \xi \xi',
\]
and extend linearly. The \( \mathbb{Z}_q \)-Banach module \( \mathbb{C} \) also has an induced tensor product norm defined similarly to the above. Abusing notation, we will write it as \( \| \cdot \|_{\pi,p} \).

**Proposition 4.5.** \( \mathcal{H}^\pi_{r,s} \) is a \( \mathbb{C} \)-Banach module and if \( \{e_i\}_{i \in I} \) is an orthonormal basis for \( \mathcal{H}_{r,s} \) over \( \mathbb{C}_p \) then \( \{1 \otimes e_i\}_{i \in I} \) is an orthonormal basis for \( \mathcal{H}^\pi_{r,s} \) over \( \mathbb{C} \).

**Proof.** The first statement is clear; see Section 3.1.1 in [3] to prove that this multiplication is well-defined.

For the second statement, by Proposition 3 in Appendix B of [2] and a basic identity about completed tensor products, there is an isomorphism of \( \mathbb{Z}_q \)-Banach modules:
\[
\mathcal{C} \otimes_{\mathbb{C}_p} \mathcal{H}_{r,s} \cong (\mathbb{Z}_q[[\pi]] \hat{\otimes}_{\mathbb{Z}_q} \mathbb{C}_p) \otimes_{\mathbb{C}_p} \mathcal{H}_{r,s} \cong \mathbb{Z}_q[[\pi]] \otimes_{\mathbb{Z}_q} (\mathbb{C}_p \otimes_{\mathbb{C}_p} \mathcal{H}_{r,s})
\]
\[
\cong \mathbb{Z}_q[[\pi]] \hat{\otimes}_{\mathbb{Z}_q} \mathcal{H}_{r,s}.
\]

So by Proposition A1.3 in [4], \( \{1 \otimes 1 \otimes e_i\}_{i \in I} \) is an orthonormal basis for \( \mathcal{C} \otimes_{\mathbb{C}_p} \mathcal{H}_{r,s} \) over \( \mathbb{C} \), which implies that \( \{1 \otimes e_i\}_{i \in I} \) is an orthonormal basis for \( \mathbb{Z}_q[[\pi]] \hat{\otimes}_{\mathbb{Z}_q} \mathcal{H}_{r,s} \) over \( \mathbb{C} \).

\[\square\]

Let \( (\mathcal{H}_{r,s})_j = \mathbb{C}_p(\mathbb{Z}_q[[\pi]]_{\pi,p}/\mathbb{Z}_q[[\pi]]_{\pi,p})_j \) and define \( (\mathcal{H}^\pi_{r,s})_j = \mathbb{Z}_q[[\pi]] \hat{\otimes}_{\mathbb{Z}_q} (\mathcal{H}_{r,s})_j \). For each \( j \), let \( \| \cdot \|_j \) be the norm coming from the tensor product in \( (\mathcal{H}_{r,s})_j \).

**Proposition 4.6** (Mittag-Leffler). There is a decomposition of \( \mathbb{Z}_q[[\pi]] \)-Banach modules
\[
\mathcal{H}^\pi_{r,s} \cong \bigoplus_{j=1}^\ell (\mathcal{H}^\pi_{r,s})_j,
\]
Moreover, if for \( \xi \in \mathcal{H}^\pi_{r,s} \) we write \( \xi = \sum_{j=1}^\ell \xi_j \in \bigoplus_{j=1}^\ell (\mathcal{H}^\pi_{r,s})_j \), then \( \|\xi\|_{r,s} = \max_{1 \leq j \leq \ell} \|\xi_j\|_j \).

**Proof.** By Proposition 6 in section 2.1.7 of [3],
\[
\mathcal{H}^\pi_{r,s} \cong \mathbb{Z}_q[[\pi]] \hat{\otimes}_{\mathbb{Z}_q} \mathcal{H}_{r,s} \cong \mathbb{Z}_q[[\pi]] \hat{\otimes}_{\mathbb{Z}_q} (\mathcal{H}_{r,s})_j = \bigoplus_{j=1}^\ell (\mathbb{Z}_q[[\pi]] \hat{\otimes}_{\mathbb{Z}_q} (\mathcal{H}_{r,s})_j).
\]

The norm relationship follows from Proposition 4.5.\[\square\]
4.1. The Submodule $Z^\pi$. For the purposes of our Dwork theory, it will suffice to work in an integral submodule $Z^\pi$ of $\mathcal{H}^\pi_{1,1}$.

**Definition 4.7.** Consider the $\mathbb{Z}_p$ and $\mathbb{Z}_q$-Banach modules:

$$O_1 = \mathbb{Z}_p[[\pi]] \hat{\otimes} \mathbb{Z}_p \mathbb{Z}_p$$

and define $Z^\pi$ to be the submodule of $\mathcal{H}^\pi_{1,1}$ generated by tensors of the form $1 \otimes B_{ij}$ with coefficients in $O_a$.

By Proposition 4.5, every $\xi \in Z^\pi \subset H^\pi_{1,1}$ can be uniquely represented as a sum:

$$\xi = \sum_{1 \leq j \leq \ell \atop i \geq 0} c_{ij} (1 \otimes B_{ij}),$$

with $c_{ij} \in O_a$. Or, via Proposition 3.2 and Proposition 4.5, each $\xi \in Z^\pi$ can be uniquely represented as

$$\xi = \sum_{1 \leq j \leq \ell \atop i \geq 0} e_{ij} (1 \otimes x^i),$$

again with $e_{ij} \in O_a$.

If $\text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p) = \{\tau\}$, $O_a$ can be endowed with a natural $\tau$ action,

$$\tau(b(\pi) \otimes r) \mapsto \tau(b(\pi)) \otimes \tau(r),$$

with the action of $\tau$ on $\mathbb{Z}_q[[\pi]]$ defined coefficient-wise acting as the identity on $\pi$. Furthermore, letting $\tau$ act as the identity on $x$, we get a $\tau$ action on $Z^\pi$:

$$\sum_{1 \leq j \leq \ell \atop i \geq 0} c_{ij} (1 \otimes \frac{1}{(x - P_j)^i}) \mapsto \sum_{1 \leq j \leq \ell \atop i \geq 0} \tau(c_{ij}) (1 \otimes \frac{1}{(x - \tau(P_j))^i}).$$

(Note that this $\tau$ action is essentially the same action as $\tau_*$ from [15].)

We also will need to define two handy maps associated to $Z^\pi$.

**Lemma 4.8.** There is an $\mathbb{Z}_q$-Banach algebra isomorphism:

$$[\mathfrak{C}] : O_a \rightarrow \mathbb{Z}_q[[\pi]]$$

$$b(\pi) \otimes r \mapsto rb(\pi),$$

and, for $x_0 \in \mathbb{A}_{1,1}$, there is an evaluation map:

$$[\mathfrak{P}_{x_0}] : Z^\pi \rightarrow \mathcal{C}$$

$$\sum_{ij} c_{ij} (1 \otimes \left(\frac{1}{x - P_j}\right)^i) \mapsto \sum_{ij} c_{ij} \left(\frac{1}{x_0 - P_j}\right)^i.$$

**Proof.** Defining the obviously bounded $\mathbb{Z}_q$-algebra homomorphisms

$$\phi_1 : \mathbb{Z}_q[[\pi]] \rightarrow \mathbb{Z}_q[[\pi]] : b(\pi) \mapsto b(\pi)$$

$$\phi_2 : \mathbb{Z}_q \rightarrow \mathbb{Z}_q[[\pi]] : a \mapsto a,$$

by Proposition 2 in 3.1.1 of [3], there is a unique bounded $\mathbb{Z}_q$-algebra homomorphism

$$\psi : \mathbb{Z}_q[[\pi]] \hat{\otimes} \mathbb{Z}_q \rightarrow \mathbb{Z}_q[[\pi]].$$

If $a(\pi) \otimes b \in \mathbb{Z}_q[[\pi]] \hat{\otimes} \mathbb{Z}_q$, it’s easy to see that
\[ a(\pi) \otimes b = ba(\pi) \otimes 1, \text{ and so by the induced action of } \phi_1 \text{ and } \phi_2 \text{ through } \psi, \]
\[ \psi'(a(\pi) \otimes b) = ba(\pi). \]
Hence if we define
\[ \psi' : \mathbb{Z}_q[[\pi]] \to \mathbb{Z}_q[[\pi]] \otimes_{\mathbb{Z}_q} \mathbb{Z}_q \]
\[ a(\pi) \mapsto a(\pi) \otimes 1, \]
one sees that \( \psi \circ \psi' \) is the identity and thus \( \psi \) is a bijection.

The only thing left is to check is that \( \rho_{x_0} \) is well-defined. If \( x_0 \in \mathbb{A}_{1,1} \), then
\[ |x_0 - \hat{P}_j|_p \geq 1 \text{ and so } \left| \left( \frac{1}{\pi - \hat{P}_j} \right)^i \right|_p \leq 1. \]
Hence \( |c_{ij} \left( \frac{1}{\pi - \hat{P}_j} \right)^i|_{\pi,p} \to 0 \text{ as } i,j \to \infty \)
since \( |c_{ij}|_{\pi,p} \to 0 \text{ as } i,j \to \infty \), and the claim follows. Observe that if \( x_0 \in \mathbb{Z}_q \), then
\[ \rho_{x_0} : \mathcal{Z} \to \mathcal{O}_a \text{ and } \iota \circ \rho_{x_0} : \mathcal{Z} \to \mathbb{Z}_q[[\pi]]. \]

\[ \square \]

We will also need a twisting of \( \mathcal{Z}, \mathcal{Z}^{\infty} \), which is defined to be the submodule of elements of the form
\[ (4) \quad \xi = \sum_{ij} c_{ij}(1 \otimes \frac{1}{(x - \hat{P}^p_j)^i}), \]
with \( c_{ij} \in \mathcal{O}_a \). We will write \( \mathcal{B}_{ij} = \frac{1}{(x - \hat{P}^p_j)^i} \).

5. A Trace Formula

In this section we develop key trace formulas that will form the foundation for our corresponding Dwork theory. We will work towards proving the following theorem:

**Theorem 5.1.** Let \( k \geq 1 \) and \( g \in \mathcal{Z} \) with \( U^a \circ g \) completely continuous. Then
\[ \text{Tr}((U^a \circ g)^k|\mathcal{Z}) = (q^k - 1)^{-1} \sum_{x_0 \in \mathbb{F}_q^\infty, x_0 \neq \hat{P}_1, \ldots, \hat{P}_e} \rho_{x_0} \circ (g(x) \cdots g(x^{k-1})), \]
where \( U \) is defined below.

5.1. The \( U_p \) Operator. Let \( U_p \) be the operator on \( \mathcal{H}_{r,s} \) from [15], namely:
\[ \mathcal{U}_p : \mathcal{H}_{r,s} \to \mathcal{H}_{r,s,p} \]
\[ \xi(x) \mapsto \frac{1}{p} \sum_{\xi(z) = x} \xi(z). \]

We can extend the \( \mathbb{C}_p \)-linear operator \( U_p \) to a \( \mathcal{C} \)-linear operator on \( \mathcal{H}_{r,s}^\pi \):

**Definition 5.2.** Let \( U \) be the \( \mathcal{C} \)-linear operator given by
\[ \mathcal{U} : \mathcal{H}_{r,s}^\pi \to \mathcal{H}_{r,s,p} \]
\[ b \otimes \xi \mapsto b \otimes U_p(\xi), \]
and extended linearly.

**Proposition 5.3.** The operator \( U_p \) has the following properties:

1. For \( \xi \) and \( g \), \( U(\xi(x^g)g(x)) = \xi(x)U(g(x)) \).
2. Let \( h(x) = \sum_{i=-\infty}^{\infty} h_i x^i \in \mathbb{C}_p[[x,x^{-1}]] \). Then \( U_p h = \sum_{i=-\infty}^{\infty} h_i x^i \).

**Proof.** The first result is trivial and the second is well known, see [12], p.238. \( \square \)
To prove the trace formula we’ll need to understand exactly how \( U \) acts on the \( B_{ij} \):

**Lemma 5.4.** Let \( x \in k_{r,1} \) and \( B_{ij}^{\pi,\tau} = \frac{1}{(x-P_j)^i} \). Then

\[
UB_{ij}^{\pi} = \sum_{n=[i/p]}^i (U_{(i,j),n} \otimes \hat{P}_j^{np-i})B_{nj}^{\pi,\tau},
\]

with \( U_{(i,j),n} \in \mathbb{Z}_p \). For \( j = 1, 2 \), \( U_{(i,j),n} = 0 \) unless \( n = i/p \), in which case \( U_{(i,j),i/p} = 1 \). When \( j \geq 3 \), \( U_{(i,j),i/p} \in \mathbb{Z}_p^\times \) and \( \text{ord}_p U_{(i,j),n} \geq \frac{np-i}{p-1} - 1 \).

**Proof.** Apply Lemma 3.1 from [15]. See also section 5.3 in [5]. □

Hence \( U \) maps \( \mathbb{Z}^{\pi} \) to \( \mathbb{Z}^{\pi,\tau} \), implying that \( U^a \) maps \( \mathbb{Z}^{\pi} \) to \( \mathbb{Z}^{\pi,\tau} \), i.e. \( U^a \) is an endomorphism of \( \mathbb{Z}^{\pi} \).

Let us finish this subsection by proving that \( U^a \) is not only an endomorphism of \( \mathbb{Z}^{\pi} \), but that it’s a continuous endomorphism.

**Proposition 5.5.** Let \( h \in \mathbb{Z}^{\pi} \). Then \( U^a \circ h \) is a continuous linear operator, \( h \) acting by multiplication, of norm \( \leq q \|h\|_{r,s} \).

**Proof.** We’ll first prove that \( U \) is a continuous linear operator of norm less than or equal to \( p \). Unless noted, all of the following suprema are taken over \( g \in H_{r,s}^{\pi} \), \( \|g\|_{r,s} = 1 \), and we write \( g = \sum_{i,j} c_{ij} (1 \otimes B_{ij}) \). Because

\[
\|U\|_{op} = \sup \|U \circ g\|_{r,s} = \sup \| \sum_{i,j} c_{ij} (1 \otimes U_p \circ B_{ij}(x)) \|_{r,s}
\]

\[
\leq \sup \|c_{ij}\|_{\pi,p} (\sup \|U_p \circ B_{ij}(x)\|_{r,s}) = \sup \|g\|_{r,s} \|U_p \circ B_{ij}(x)\|_{op} \leq p,
\]

by Proposition 6 in [11], and so \( U \) is continuous.

We conclude:

\[
\|U^a \circ h\|_{op} = \sup \|U^a(hg)\| \leq \sup \|U^a\|_{op} \|hg\|_{r,s} = q \|h\|_{r,s}.
\]

5.2. Building the Trace Formula. This subsection contains the proof of our desired trace formula. The first step is to develop an analogue trace formula on a polynomial submodule, \( P^{\pi} \). Using a limiting process, we can then lift this formula to \( \mathbb{Z}^{\pi} \), and this consequently yields Theorem 5.1.

**Definition 5.6.** Let \( P^{\pi} \) be a submodule of \( \mathbb{Z}^{\pi} \) spanned by tensors of the form \( 1 \otimes x^i, i \geq 0 \), over \( \mathbb{O}_a \).

For \( g \in P^{\pi} \) (or \( \mathbb{Z}^{\pi} \)), we say that \( g \) is finite if it can be written as a finite sum:

\[
g = \sum_{j=1}^i \sum_{i=1}^{N_j} c_{ij} (1 \otimes B_{ij}),
\]

where \( N_j < \infty \).

**Proposition 5.7.** Let \( h \in P^{\pi} \) and suppose that \( U^a \circ h \) is completely continuous. Then

\[
\text{Tr}(U^a \circ h|P^{\pi}) = (q - 1)^{-1} \sum_{x_0 \in \mathbb{F}_q^\times} \rho^{x_0} \circ h.
\]
Proof. Write $h = \sum_{i=0}^{\infty} c_i (1 \otimes x^i)$, $c_i \in \mathcal{O}_a$. Applying Proposition 5.3

$$(U^a \circ h)(x) = \sum_{i=0}^{\infty} c_i (1 \otimes U(x^i)) = \sum_{i=0}^{\infty} c_i (1 \otimes x^i).$$

Hence,

$$(U^a h)(1 \otimes x^j) = \sum_{i=0}^{\infty} c_i (1 \otimes x^{i+j}) = \sum_{i=0}^{\infty} c_{q^i-j} (1 \otimes x^i),$$

and so $\text{Tr}(U^a \circ h | \mathcal{P}^\pi) = \sum_{i=0}^{\infty} c_{(q-1)i}$. The elementary fact that

$$\sum_{x_0 \in \mathbb{Z}_p^\times} x^w = \begin{cases} (q-1), & \text{if } (q-1) \mid w \\ 0, & \text{if } (q-1) \nmid w \end{cases}$$

yields the claim. \qed

Recall that in Reich’s basis for $\mathcal{H}_{r,s}$ we used a polynomial $\mathbf{v}(x) = (x - \hat{P}_1) \cdots (x - \hat{P}_l)$. In what follows, we will need a lifting of $\mathbf{v}$, $\mathbf{V}^\pi = 1 \otimes \mathbf{v}$.

**Lemma 5.8.** For $x \in \mathbb{A}_{1,1}$,

$$|(v(x))(q-1)p^b - (v(x)/v(x)p^b)|_p \leq p^{-(b+1)},$$

and consequently, $|(v^\pi(x))(q-1)p^b - (v^\pi(x)/v^\pi(x)p^b)|_{r,s} \leq p^{-(b+1)}$.

**Proof.** See the proof of Theorem 4 in [11] \qed

**Proposition 5.9.** Let $g = \sum_{i,j} c_{ij} (1 \otimes B_{ij}) \in \mathbb{Z}^\pi$ be finite and suppose that $U^a \circ g$ is completely continuous. Then

$$\text{Tr}(U^a \circ g | \mathcal{P}^\pi) = \lim_{b \to \infty} \text{Tr}(U^a \circ g(v^\pi)(q-1)p^b | \mathcal{P}^\pi).$$

**Proof.** Take $b$ to be sufficiently large so that for every $j$, $g(v^\pi)(q-1)p^b \in \mathcal{P}^\pi$ and note that $U^a(\mathcal{P}^\pi) \subseteq \mathcal{P}^\pi$. (Such a $b$ exists since $g$ is finite.) In other words, $U^a \circ g(v^\pi)(q-1)p^b$ is an operator on $\mathcal{P}^\pi$, and we can write

$$(5) \quad U^a \circ g(v^\pi)(q-1)p^b(1 \otimes \frac{x^i}{v^j}) = \sum_{r,s} \gamma_{i,j,r,s}^{(b)} \otimes \frac{x^r}{v^s},$$

for some $\gamma_{i,j,r,s}^{(b)} \in \mathbb{Z}_q$ and $r \geq 0$, $(r, \ell) = 1$ and $j \geq 0$. Similarly, $U^a \circ g$ is an operator on $\mathbb{Z}^\pi$, and so we expand it as

$$(6) \quad U^a \circ g(1 \otimes \frac{x^i}{v^j}) = \sum_{r,s} \gamma_{i,j,r,s} \otimes \frac{x^r}{v^s},$$

again some $\gamma_{i,j,r,s} \in \mathbb{Z}_q$.

Let $m$ be an integer such that $\frac{\min_{ij} |c_{ij}|_{\mathbb{Z}_q}}{p^{b+1}} = p^{m-(b+1)}$. Combining Lemma 5.8 and Proposition 5.3 yields

$$\|U^a \circ g \circ ((v^\pi(x))(q-1)p^b - (v^\pi(x)/v^\pi(x)p^b))\|_p \leq p^{m-(b+1)}.$$
But
\[ U^a \circ g \circ ((v^u(x))^{(q-1)p^b} - (v^u(x^a)/v^u(x))^{p^b}) = U^a \circ g(v^u(x))^{(q-1)p^b} - (v^u(x))^{p^b} \circ U^a \circ g(v^u(x))^{-p^b}, \]
and multiplying by \((v^u(x))^{-p^b}\) yields
\[ \|((v^u(x))^{-p^b} \circ (U^a \circ g(v^u(x))^{(q-1)p^b}) - U^a \circ g(v^u(x))^{-p^b}\|_{op} \leq p^{m-(b+1)}. \]
Substituting the expansions in (5) and (8) into (7) yields
\[ \| \sum_{r,s} \gamma_{i,j,r,s}^{(b)} \otimes \frac{x^r}{v^r + p^s} - \sum_{r,s} \gamma_{i,j,p^b,r,s} \otimes \frac{x^r}{v^s} \|_{r,s} \leq p^{m-(b+1)}. \]
By definition then, (8) implies
\[ |\gamma_{i,j,i,j}^{(b)} - \gamma_{i,j-p^b,i,j-p^b}|_p \leq p^{m-(b+1)}, \]
and so
\[ |\sum_{i,j} \gamma_{i,j}^{(b)} - \sum_{i''=0,j'' \geq p^b} \gamma_{i,j}^{(b)}|_p \leq p^{m-(b+1)}. \]
As \(b \to \infty\) then, the identity follows. \(\square\)

**Theorem 5.10.** Let \(k \geq 1, g \in \mathbb{Z}^\pi\) and suppose that \(U^a \circ g\) is completely continuous. Then
\[ \text{Tr}(U^a \circ g|\mathbb{Z}^\pi) = (q - 1)^{-1} \sum_{x_0 \notin \mathbb{F}_q \cup \mathbb{P}_1 \cup \ldots \cup \mathbb{P}_k} \rho_{x_0} \circ g. \]

**Proof.** First suppose that \(g\) is finite. Applying Proposition 5.9 and Proposition 5.7 yields:
\[ \text{Tr}(U^a \circ g|\mathbb{Z}^\pi) = \lim_{b \to \infty} \text{Tr}(U^a \circ g(v^u)^{(q-1)p^b}|\mathbb{P}^\pi) \]
\[ = \lim_{b \to \infty} (q - 1)^{-1} \sum_{x_0 \in \mathbb{F}_q} \rho_{x_0} \circ (g(v^u)^{(q-1)p^b}). \]
Now, if \(x_0 = \mathbb{P}_j\) for any \(j\), then for large \(b\) it is clear that \(\rho_{x_0} \circ (g(v^u)^{(q-1)p^b}) = 0\). On the other hand, if \(x_0 \neq \mathbb{P}_j\) for all \(j\), observe that since \(x_0\) and \(\mathbb{P}_j\) are Teichmuller lifts, \(|x_0|_p = |\mathbb{P}_j|_p = 1\). By assumption \(x_0 \neq P_j \in \mathbb{F}_q\), so \(x_0 - \mathbb{P}_j \neq 1\) and \(|x_0 - \mathbb{P}_j|_p = 1\) and \(x_0 - \mathbb{P}_j \in \mathbb{Z}^\times_q\). Therefore, by the discussion on p.150 in [13], \(\lim_{b \to \infty} (x_0 - \mathbb{P}_j)^{(q-1)p^b} = 1\), which implies that
\[ \lim_{b \to \infty} \rho_{x_0} \circ (g(v^u)^{(q-1)p^b}) = \rho_{x_0} \circ g. \]
Consequently,
\[ \text{Tr}(U^a \circ g|\mathbb{Z}^\pi) = (q - 1)^{-1} \sum_{x_0 \in \mathbb{F}_q, x_0 \neq \mathbb{P}_1 \cup \ldots \cup \mathbb{P}_k} \rho_{x_0} \circ g. \]
The result for arbitrary \(g\) then follows by taking limits. \(\square\)
The proof of Theorem 5.1 follows similarly. (Apply property (1) from Proposition 5.3 to \((U^a \circ g)^k\) and replace \(a\) with \(ak\) in the above proofs.)

6. \((\pi, p)\)-adic exponential sums

In this section we apply the above analysis to \((\pi, p)\)-adic exponential sums. We describe \(C_f(s, \pi)\) as the determinant of a completely continuous operator and compute estimates that will be fundamental to the computation of the Hodge polygon in Section 7.

Recall that \(E(x) = \sum_{k=0}^{\infty} u^k x^k \in \mathbb{Z}_p[[x]]\) is the Artin-Hasse exponential function and \(\pi \in 1+\mathbb{Q}_p[[x]]\) is such that \(E(\pi) = 1+T\). Let \(f(x) = \sum_{j=1}^\ell \sum_{i=1}^{d_j} a_{ij} \left( \frac{1}{x-P_j^i} \right)^t\), \(a_{ij} \in \mathbb{Z}_q\), and define its associated data:

**Definition 6.1.**

\[
S_f(k, \pi) = \sum_{x \in \mathbb{F}_q^\times, \ x \neq \hat{P}_1, \ldots, \hat{P}_t} E(\pi) ^ {Tr_{\mathbb{Q}_p/F_q}(f(x))} \]

\[
L_f(k, \pi) = \exp\left( \sum_{k=1}^{\infty} S_f(k, \pi) \frac{\pi^k}{k} \right) \]

\[
C_f(k, \pi) = \exp\left( \sum_{k=1}^{\infty} -(q^k - 1)^{-1} S_f(k, \pi) \frac{\pi^k}{k} \right) = \prod_{j=0}^{\infty} L_f(q^j s, \pi). \]

The function \(f\) has the the splitting functions:

**Definition 6.2.**

\[
F_j(x) = \prod_{i=1}^{d_j} E(\pi a_{ij} \otimes B_{ij}) \]

\[
\hat{F}(x) = \prod_{j=1}^{\ell} F_j(x) \]

\[
\hat{F}[a](x) = \prod_{m=0}^{a-1} (\tau^m F)(x^{p^m}). \]

Our main object of study will be the maps \(\alpha_a = U^a \circ \hat{F}[a]\) and \(\alpha_1 = \tau^{-1} \circ U \circ \hat{F}\). Note that \(\alpha_1\) is a \(O_1\)-linear endomorphism of \(Z^\pi\) while \(\alpha_a\) is a \(O_a\)-linear endomorphism of \(Z^\pi\). They are related in the following manner:

**Proposition 6.3.** As \(O_1\)-linear maps, \(\alpha_a = \alpha_1^a\) and \(\det_{O_a}(1-\alpha_a s)^a = \det_{O_1}(1-\alpha_1 s)\).

**Proof.** The proof of this proposition is similar the proof of Lemma 2.9 in [15] (or originally (43) in [1].)

6.1. \((\pi, p)\)-adic Estimates. The following are \((\pi, p)\)-adic liftings of the \(p\)-adic approximations from [15]. Lemma 6.5 and Lemma 6.6 are purely \(\pi\)-adic estimates, and the key computation, Proposition 6.8, blends these two \(\pi\)-adic estimates with the \(p\)-adic nature of the \(U\) operator, Lemma 5.4.
For the sake of notation, we will write our (unweighted) basis as $B^\pi_{ij} = 1 \otimes B_{ij}$ (similarly $B^\pi_{ij} = 1 \otimes B_{ij}$) and define a weighted basis $W^\pi_{ij} = \pi^m \otimes B_{ij}$.

**Definition 6.4.** Let $i \geq 0$ and $0 \leq j, k \leq \ell$ and define

$$U(B^\pi_{ij}) = \sum_{i,j} U(i,j),n B^\pi_{ij} , \quad U(i,j),n \in \mathbb{Z}_p$$

$$F_j(x) = \sum_{n=0}^{\infty} F_{n,j} \otimes B_{nj}, \quad F_{n,j} \in \mathbb{Z}_p[[\pi]]$$

$$(FB^\pi_{ij})_k = \sum_{n=0}^{\infty} (F(i,j),(n,k)) \otimes B_{nk}, F(i,j),(n,k) \in \mathbb{Z}_p[[\pi]].$$

**Lemma 6.5.** The coefficient $F(x) \in \mathbb{Z}^\pi$ and $\text{ord}_\pi F_{n,j} \geq \left\lfloor \frac{n}{d_j} \right\rfloor$ for each $j$. Moreover, if $d_j | n$, equality holds.

**Proof.** By definition,

$$F_j(x) = \prod_{i=0}^{d_j} \left( \sum_{k=0}^{\infty} u_k a_{ij}^k n^k \otimes B_{ij}^k \right) = \sum_{n=0}^{\infty} \left( \sum_{\sum_{k=1}^{d_j} k n_{ij}=n} \prod_{k=1}^{d_j} u_{nk} a_{ij}^{n_k} \pi^{n_k} \right) \otimes B_{nj},$$

and so

$$F_{n,j} = \sum_{\sum_{k=1}^{d_j} k n_{ij}=n} \left( \prod_{k=1}^{d_j} u_{nk} a_{ij}^{n_k} \right) \pi^{\sum_{k=1}^{d_j} n_k}.$$

Taking $n_{d_j} = \left\lfloor \frac{n}{d_j} \right\rfloor$ and $n \mod d_j$ to be either 0 or 1 depending on if $n \mod d_j = 0$ or $n \mod d_j \neq 0$ respectively yields the claim. When $d_j | n$, equality follows from the fact that both $a_{d_j,i}$ and $u_{d_j}$ are nonzero. (The Artin-Hasse coefficient $u_n$ can be expressed as $u_n = h_n / n!$, where $h_n$ is the number of $p$-elements in the permutation group $S_n$. The fact that $u_n \neq 0$ is then immediate.)

**Lemma 6.6.** Fix $i, n \geq 0$ and $1 \leq j, k \leq \ell$. Then:

$$\text{ord}_\pi F_{(i,j),(nk)} \geq \begin{cases} \frac{n-j}{d_k} & \text{if } j = k \\ \frac{n+\ell}{d_k} & \text{if } j \neq 1, k = 1 \\ \frac{n}{d_k} & \text{if } j \neq k, k \neq 1, \end{cases}$$

and equality holds when $d_k | (n-i)$, $d_k | (n+i)$ or $d_k | n$ respectively.

**Proof.** First, observe

$$(11) \quad FB^\pi_{ij} = \left( \sum_{m=0}^{\infty} F_{m,j} \otimes B_{m+i,j} \right) \prod_{v=1}^{\ell} \left( \sum_{n=0}^{\infty} F_{m,v} \otimes B_{m,v} \right),$$

where the only $\pi$-adic terms come from the $F_{m,k}$ and $F_{m,v}$ terms. If we want to compute $(FB^\pi_{ij})_k$, we need to expand each $B_{m,v}, v \neq k$, in terms of $\frac{1}{x-P_k}$. There are several cases to consider:
If $v \geq 2$ and $k \geq 3$, $v \neq k$, to expand $\frac{1}{x - P_v}$ in terms of $\frac{1}{x - P_k}$:

\[
(12) \quad \frac{1}{x - P_v} = \frac{1}{P_k - P_v} \cdot \frac{1}{1 - \frac{x - P_v}{P_k - P_v}} = \sum_{m=0}^{\infty} (-1)^m (P_v - P_k)^{-(m+1)} (x - P_k)^m,
\]

which is analytic on the ball with $|x - P_k|_p < |P_v - P_k|_p = 1$.

If $v \geq 3$ and $k = 1$, use

\[
(13) \quad \frac{1}{x - P_v} = \frac{1}{x}, \quad \frac{1}{1 - \frac{x}{P_v}} = \sum_{m=0}^{\infty} \frac{P_v^m}{x^{m+1}},
\]

which converges on $|x|_p > 1$.

If $v \geq 3$ and $k = 2$, use

\[
(14) \quad \frac{1}{x - P_v} = -\frac{1}{P_v} \cdot \frac{1}{1 - \frac{x}{P_v}} = -\frac{1}{P_v} \sum_{m=0}^{\infty} x^m,
\]

which converges on $|x|_p < |P_v|_p = 1$. If $v = 1$ and $k \geq 3$, just use the trivial expansion $x = (x - P_k) + P_k$. Finally, if $v = 1$ and $k = 2$ (or vice versa), no expansion is necessary.

Let’s start with the case $j = k = 1$:

\[
(15) \quad FB_{i,1}^\pi = \left( \sum_{m=0}^{\infty} F_{m,1} \otimes x^{m+i} \right) \left( \sum_{m=0}^{\infty} F_{m,2} \otimes \frac{1}{x^m} \right) \prod_{v=3}^{\ell} \left( \sum_{m=0}^{\infty} F_{m,v} \otimes \left( \sum_{w=0}^{\infty} \frac{P_v^w}{x^{w+i+1}} \right)^m \right).
\]

Since we only care about the $\pi$-terms, it’s clear that the minimum occurs from the term $F_{n-i,1} \otimes x^n$, and the bound follows from Lemma 6.3. The case for $j = k = 2$ is similar.

Now, let’s look at the case $j = k \geq 3$. For each $v \neq j$, expand $B_{m,v}$ as above. Then $F_{(ij),(nk)}$ is the coefficient of $B_{nk}^\pi$ in (11) after substituting all appropriate expansions. Each expansion has only positive powers of $(x - P_k)$, and so

\[
(16) \quad \text{ord}_x F_{(ij),(nk)} \geq \min_{(n_1, \ldots, n_{\ell})} \text{ord}_x \prod_{v=1}^{\ell} F_{v,n_v},
\]

where the minimum is over all $(n_1, \ldots, n_{\ell}) \in \mathbb{Z}_{\geq 0}^\ell$ such that $n_k - \sum_{v \neq k} n_v = n - i$.

Clearly this occurs when $n_k = n - i$ and $n_v = 0$ for $v \neq k$. The bound follows after applying Lemma 6.3 to 11.

In the case $j \neq 1$, $k = 1$, if $j = 2$,

\[
FB_{i,1} = \left( \sum_{m=0}^{\infty} F_{m,1} \otimes x^m \right) \left( \sum_{m=0}^{\infty} F_{m,2} \otimes \frac{1}{x^m} \right) \prod_{v=3}^{\ell} \left( \sum_{m=0}^{\infty} F_{m,v} \otimes \left( \sum_{w=0}^{\infty} \frac{P_v^w}{x^{w+i+1}} \right)^m \right) \cdot \frac{1}{x^i},
\]

and so again the term contributing to the coefficient of $B_{nk}$ giving smallest $\pi$-adic term is $F_{n+i} \otimes x^{n+i}$. The case $j \geq 3$ is similar.
Finally, there’s the case $j \neq k$, $k \neq 1$. Suppose that $j, k \geq 3$. (The other cases are again similar.) Then the expansion of each $B_{n,v}$ in terms of $k$, including the $B_{ij}$ have only positive powers of $(x - \hat{P}_k)$ and so the minimum occurs simply at $F_{n,k} \otimes x^n$.

Note that in all of the above estimates, if $d_k | (n - i)$, then by Lemma 6.5 the minimum obtained in (16) is unique and sharp and equality holds. \hfill \Box

**Definition 6.7.** Fix $i, n \geq 0$ and $1 \leq j, k \leq \ell$ and recall that for $\xi \in H_{r,n}^\pi$, $(\xi)_j$ is the Laurent expansion at $\hat{P}_j$. We write:

$$(\alpha_1 B_{ij}^\pi)_k = \sum_{n=0}^{\infty} C_{(i,j),(n,k)} \otimes B_{n,k}^\pi, \ C_{(i,j),(n,k)} \in \mathbb{Z}_q[[\pi]]$$

$$(\alpha_1 W_{ij}^\pi)_k = \sum_{n=0}^{\infty} D_{(i,j),(n,k)} \otimes W_{n,k}^\pi, \ D_{(i,j),(n,k)} \in \mathbb{Z}_q[[\pi]].$$

**Proposition 6.8.** Fix $i, n \geq 0$ and $1 \leq j, k \leq \ell$. Then if $k = 1, 2$:

$$\text{ord}_\pi C_{(i,j),(n,k)} \geq \frac{pn - i}{d_k}.$$ 

For $k \geq 3$,

$$\text{ord}_\pi C_{(i,j),(n,k)} \geq \begin{cases} \frac{n-i}{d_k} & \text{if } j = k \\ \frac{n+i}{d_k} & \text{if } j \neq 1, k = 1 \\ \frac{c}{d_k} & \text{if } j \neq k, k \neq 1, \end{cases}$$

and equality holds when $d_k | (n - i)$, $d_k | (n + i)$ or $d_k | n$ respectively. For $k \geq 3$ and any real number $c > 0$, $C_{(i,j),(n,k)}$ also has the following $(\pi^{1/c}, p)$-adic estimates:

$$\text{ord}_{\pi^{1/c}, p} C_{(i,j),(n,k)} \geq \begin{cases} \frac{(n-1)p-(i-1)}{d_k}c & \text{if } d_k \geq c(p-1) \\ \frac{n-i}{d_k}c + n - 1 & \text{if } d_k < c(p-1). \end{cases}$$

**Proof.** We’ll prove the $(\pi, p)$-adic bound, and the $\pi$-adic bounds follow easily. Let $B_{n,k}^{\pi, \tau} = \tau(B_{ij}^\pi)$. Then,

$$\tau \circ \alpha_1 B_{ij}^\pi = (U \circ F) B_{ij}^\pi = U \circ \left( \sum_{k=1}^{\ell} (FB_{ij}^\pi)_k \right) = \sum_{k=1}^{\ell} \sum_{n=0}^{\infty} F_{(i,j),(n,k)} \otimes U(B_{n,k})$$

$$= \sum_{k=1}^{\ell} \sum_{n=0}^{\infty} F_{(i,j),(n,k)} \sum_{m=[n/p]}^{n} (U_{(n,k),m} \otimes \hat{P}_k^{mp-n}) B_{mk}^{\pi, \tau}$$

$$= \sum_{k=1}^{\ell} \sum_{m=0}^{mp} \sum_{n=m}^{\infty} F_{(i,j),(n,k)} (U_{(n,k),m} \otimes \hat{P}_k^{mp-n}) B_{nk}^{\pi, \tau},$$

and so

$$C_{(i,j),(n,k)} = \tau^{-1} \circ \sum_{m=n}^{mp} F_{(i,j),(n,k)} (U_{(n,k),m} \hat{P}_k^{mp-n}).$$

For $k = 1$ and 2, Proposition 5.3 implies that $U_{(n,k),m} = 0$ for $m \neq np$, and combined with Lemma 6.6, this yields the first part of the claim.
For $k \geq 3$, by (17),

$$
\text{ord}_{\pi^{1/p}} C_{ij},(mk) \geq \min_{m \leq n \leq mp} \left( \text{ord}_{\pi^{1/p}} F_{ij},(nk) + \text{ord}_p U_{nk,m} \right).
$$

Let $n_0 = (m-1)p + 1$. By Lemma 5.4 if $n_0 < n \leq mp$, $\text{ord}_p U_{nk,m} = 0$ and so (18) yields $\text{ord}_{\pi^{1/p}} F_{ij},(nk) \geq \frac{m-n}{d_k}c$. On the other hand, if $m \leq n \leq n_0$,

$$
\text{ord}_{\pi^{1/p}} C_{ij},(mk) \geq \min_{m \leq n \leq n_0} \left( \frac{n-i}{d_k} + \frac{mp-n}{p-1} - 1 \right)
$$

(19)

$$
\geq \min_{m \leq n \leq n_0} \left( \frac{-ic + mp}{d_k} + \frac{1}{p-1} \right).
$$

There are now three cases to consider. First, if $\frac{c}{d_k} - \frac{1}{p-1} < 0$, then (19) has minimum at $n = n_0 = (m-1)p + 1$, which yields $\text{ord}_{\pi^{1/p}} C_{ij},(mk) \geq \frac{(m-1)p-(i-1)}{d_k}c$. If $\frac{c}{d_k} - \frac{1}{p-1} \geq 0$, then (19) has minimum at $n = m$, and lower bound $\frac{m-n}{d_k}c+(m-1)$. □

**Theorem 6.9.** Fix $i, n \geq 0$ and $1 \leq j, k \leq \ell$. Using the relation $D_{ij},(nk) = \pi^{i/d_j-n/d_k} C_{ij},(nk)$ and Proposition 6.8 if $k = 1, 2$:

$$
\text{ord}_\pi D_{ij},(nk) \geq \frac{(p-1)n}{d_k}.
$$

For $k \geq 3$,

$$
\text{ord}_\pi D_{ij},(nk) \geq 0
$$

and equality holds when $d_k|(n-i)$ and $j = k$. Furthermore, for a real number $c > 0$,

$$
\text{ord}_{\pi^{1/p}} D_{ij},(nk) \geq \begin{cases} \frac{(n-1)(p-1)c}{d_k} & \text{if } d_k \geq c(p-1) \\ \frac{n-1}{p-1} & \text{if } d_k < c(p-1). \end{cases}
$$

**Corollary 6.10.** Neither $\alpha_1$ nor $\alpha_a$ are $\pi$-adically completely continuous operators, but for $c > 0$, they are both $(\pi^{1/p}, p)$-adically completely continuous operators.

**Proof.** To see that $\alpha_1$ is not completely continuous $\pi$-adically, see by Theorem 6.9 that if $j = k$ and $d_k|(n-i)$, then $\text{ord}_\pi D_{ij},(nk) = 0$. Hence

$$
\lim_{(n,k) \to \infty} \inf \text{ord}_\pi D_{ij},(n,k) = 0,
$$

and so $\alpha_1$ cannot be completely continuous with respect to $\pi$.

On the other hand, the $(\pi, p)$-adic bound from Theorem 6.9 (without loss of generality, take $k \geq 3$ and $d_k > p-1$) implies that

$$
\text{ord}_{\pi^{1/p}} D_{ij},(n,k) \geq \frac{(p-1)(n-1)}{d_k}c \to \infty \text{ as } n \to \infty.
$$

The complete continuity of $\alpha_a$ follows from the relation $\alpha_a = \alpha_1^a$. □

**6.2. Dwork Theory.**

**Lemma 6.11.** Let $x_0 \in \hat{F}_{q^k}^\times$ such that $x_0 \neq \hat{P}_j$ for all $1 \leq j \leq \ell$. Then:

$$
t \circ \rho_{x_0} \circ \prod_{i=0}^{k-1} \left( F_{a[i]}(x_0^q) = (1 + T)^{T_{q^k/q^a}(f(x_0))} \right).
$$
Proof. Let \( x_0 \in \mathbb{F}_q^\times \) with \( x_0 \neq \hat{P}_j \) for all \( 1 \leq j \leq \ell \). An easy calculation shows that

\[
(1 + T)^{\text{Tr}_{Q_{k}/Q}(f(x_0))} = E(\pi) \sum_{j=1}^{\ell} \sum_{i=1}^{d_j} \sum_{m=0}^{ak_j-1} E(\pi(a_{ij}^m (x_0^m - \hat{P}_j^{p^m})^{-i})).
\]

On the other hand,

\[
\prod_{i=0}^{k-1} F_{[a]}(x_0^q) = \prod_{m=0}^{ak-1} (\tau^m F)(x) = \prod_{j=1}^{\ell} \prod_{i=1}^{d_j} \prod_{m=0}^{ak_j-1} E(\pi(a_{ij}^m \otimes (x_0 - \hat{P}_j^{p^m})^{-i})),
\]

and the identity follows. \( \square \)

Proposition 6.12. For \( k \geq 1 \),

\[
\iota \circ \text{Tr}(\alpha_a^k|Z) = (q - 1)^{-1} S_f(k, \pi).
\]

Proof. Applying Theorem 5.1 to the function \( F_{[a]}(x) = \prod_{m=0}^{ak-1} (\tau^m F)(x^m) \) and using the identity from Lemma 6.11 yields:

\[
\iota \circ \text{Tr}(\alpha_a^k|Z) = (q - 1)^{-1} \sum_{x_0 \in \mathbb{F}_q^\times} \rho_{x_0} \circ (F_{[a]}(x) \cdots F_{[a]}(x^{q-1}))
\]

\[
= (q - 1)^{-1} \sum_{x_0 \in \mathbb{F}_q^\times} (1 + T)^{\text{Tr}_{Q_{k}/Q}(f(x_0))}
\]

\[
= (q - 1)^{-1} S_f(k, \pi).
\]

\( \square \)

Theorem 6.13. We have

\[
C_f(s, \pi) = \iota \circ \det(1 - \alpha_a s|Z).
\]

Proof. By definition and the trace formula in Corollary 6.12

\[
C_f(s, \pi) = \exp\left(-\sum_{k=1}^{\infty} \left(q^k - 1\right)^{-1} S_f(k, \pi) \frac{s^k}{k}\right)
\]

\[
= \exp\left(-\sum_{k=1}^{\infty} \iota \circ \text{Tr}(\alpha_a^k|Z) \frac{s^k}{k}\right)
\]

\[
= \iota \circ \det(1 - \alpha_a s|Z).
\]

\( \square \)

7. The Hodge Bound

We call the lower bound for \( C_f(s, \pi) \) obtained from Theorem 6.9 the Hodge bound. For two Newton polygons \( \text{NP}_1 \) and \( \text{NP}_2 \), let \( \text{NP}_1 \boxplus \text{NP}_2 \) denote the concatenation of the Newton polygons \( \text{NP}_1 \) and \( \text{NP}_2 \), reordering so that the slopes are in increasing order. The Hodge polygon is then given by:
Definition 7.1. For $k = 1, 2$, let $\text{HP}_k^c$ be the Hodge polygon with vertices \\
\[ \{(n, \frac{(p-1)n(n-1)}{2d_k}) \}_{n \geq 0}. \]

For $3 \leq k \leq \ell$, let $\text{HP}_k^c$ be the Hodge polygon with vertices $\{(n, y_n)\}_{n \geq 0}$, where \\
y_n = \begin{cases} \\
\frac{a(p-1)n(n-1)}{2d_k}c & \text{if } d_k \geq c(p-1) \\
\frac{an(n-1)}{2} & \text{if } d_k < c(p-1). \\
\end{cases}

The $(\pi, p)$-adic Hodge polygon, $\text{HP}^c$, is the polygon given by \\
$\bigoplus_{k=1}^{\ell} \text{HP}_k^c$.

Theorem 7.2. The $(\pi^{1/c}, p)$-adic Newton polygon of $C_f(s, \pi)$ lies above $\text{HP}^c$.

Proof. Let $M$ represent the matrix for $\alpha_1$ with respect to the basis $\{W_{ij}^\pi\}$, with the entries of $M$ lying in $\mathcal{O}_a$. Write:
\\
det(1 - Ms) = 1 + \sum_{k=1}^{\infty} C_k s^k \in \mathcal{O}_a[[s]],
\\
so that \\
\begin{equation}
C_k = \sum_{S \subseteq \mathbb{Z}_{\geq 0} \times \{1, \ldots, \ell\}} \sum_{\sigma \in \text{Sym}(S)} \text{sgn } \sigma \prod_{(i,j) \in S} D_{(i,j), \sigma(i,j)}.
\end{equation}

Let $m_i$ be the $i$th slope of $\text{HP}^c$. The smallest $(\pi, p)$-adic valuation that \\
$\prod_{(i,j) \in S} D_{(i,j), \sigma(i,j)}$ can have is $\sum_{i=1}^{k} m_i$, by Theorem 6.9, and so the desired Hodge bound holds for $\det_{\mathcal{O}_a}(1 - \alpha_1 s)$.

However we need to show the Hodge bound holds for $\det_{\mathcal{O}_a}(1 - \alpha_1 s)$, so let $\eta, \ldots, \eta^{n-1}$ be a normal basis for $\mathbb{Z}_q/\mathbb{Z}_p$. Consider the $\mathcal{O}_1$-basis $\eta^i \otimes 1, 0 \leq i \leq \ell$, for $\mathcal{O}^n$. Because $\alpha_1$ is $\tau^{-1}$-linear,
\\
$\alpha_1((\eta^i \otimes 1) \cdot C_{(i',j'),(nk)}) = (\eta^i \otimes 1) \cdot \alpha_1(C_{(i',j'),(nk)})$,
\\
and so the bound follows from Proposition 6.3 and Theorem 6.9. \hfill \Box

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