Bounding slopes of \( p \)-adic modular forms

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Abstract

Let \( p \) be prime, \( N \) be a positive integer prime to \( p \), and \( k \) be an integer. Let \( P_k(t) \) be the characteristic series for Atkin’s \( U \) operator as an endomorphism of \( p \)-adic overconvergent modular forms of tame level \( N \) and weight \( k \). Motivated by conjectures of Gouvêa and Mazur, we strengthen a congruence in [W] between coefficients of \( P_k \) and \( P_{k'} \) for \( k' \) \( p \)-adically close to \( k \). For \( p - 1 \mid 12 \), \( N = 1 \), \( k = 0 \), we compute a matrix for \( U \) whose entries are coefficients in the power series of a rational function of two variables. We apply this computation to show for \( p = 3 \) a parabola below the Newton polygon \( N_0 \) of \( P_0 \), which coincides with \( N_0 \) infinitely often. As a consequence, we find a polygonal curve above \( N_0 \). This tightest bound on \( N_0 \) yields the strongest congruences between coefficients of \( P_0 \) and \( P_k \) for \( k \) of large \( 3 \)-adic valuation.

1 Overview and background

Let \( p \) be a prime number, \( N \) be a positive integer relatively prime to \( p \), and \( k \) be an integer. Let \( B \) be a \( p \)-adic ring between \( \mathbb{Z}_p \) and \( \mathbb{O}_p \), the ring of integers in \( \mathbb{C}_p \). Denote by \( M_k(N, B) \) the \( p \)-adic overconvergent modular forms of tame level \( N \) and weight \( k \) and by \( S_k(N, B) \) the subspace of overconvergent cusp forms.

For every weight \( k \), Atkin’s \( U \) operator is an endomorphism of \( M_k(N, B) \) stabilizing \( S_k(N, B) \). Denote by \( U^{(k)} \) the restriction of \( U \) to \( M_k(N, B) \) and by \( U_{(k)} \) the restriction to \( S_k(N, B) \). These are compact operators, so the characteristic series

\[
P_k(t) = \det(1 - tU_{(k)}), \quad Q_k(t) = \det(1 - tU^{(k)})
\]

exist.

Let \( a_m(P_k) \) be the coefficient of \( t^m \) in \( P_k(t) \). As a function on a suitably defined space of weights \( k \), \( a_m(P_k) \) is a rigid analytic function of \( k \).

Wan [W], and Buzzard [B] construct \( \tilde{N}(m) \), which grows as \( O(m^2) \) and depends on \( p \) and \( N \) and not on \( k \) such that \( v_p(a_m(P_k)) > \tilde{N}(m) \).

Gouvêa and Mazur [GM] note, in an earlier work, the existence of \( \mathfrak{N}(m) \) and show, for prime \( p \geq 5 \), integer \( l \) and positive integer \( n \),

\[
v_p(a_m(P_k) - a_m(P_{k+lp^{p}(p-l)}) \geq n + 1.
\]
Following a remark in [Ka], the result in Equation (1) extends to $p = 2, 3$.

In section 2, we show

$$v_p(a_m(P_k) - a_m(P_{k+lp^n(p-1)})) \geq \hat{N}(m - 2) + n + 1.$$  \hspace{1cm} (2)

In section 3, for each $p = 2, 3, 5, 7, 13, N = 1$, we construct a matrix $M$ for $U(0)$ with respect to an explicit basis. We show, for $M_{ij}$ the entries of $M$,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} M_{ij} x^i y^j$$

is the power series expansion of a rational function of two variables.

In section 4, we show for $p = 3$,

$$v_p(a_m(Q_0)) \geq 3 \left( \frac{m}{2} \right) + 2m,$$

with equality if and only if there is positive integer $j$ such that $m = (3^j - 1)/2$. The secant segments joining these vertices of the Newton polygon $N'_0$ of $Q_0$ form a polygon curve above $N'_0$. We find evidence in support of a conjecture in [G] on the distribution of slopes of classical modular forms.

1.1 Motivating conjectures

The zeros of $P_k(t)$ are reciprocals of $U^{(k)}$ eigenvalues. For rational number $\alpha$, let $d(k, \alpha)$ denote the number of $U^{(k)}$ eigenvalues with $p$-adic valuation $\alpha$.

**Conjecture 1.1 (Gouvêa-Mazur)** Let $k, l$ be integers, $n$ be a positive integer, and $\alpha < n$. Then $d(k, \alpha) = d(k + lp^n(p - 1), \alpha)$.

Wan [W] uses Equation (1) and the construction of $\hat{N}(m)$ to compute a quadratic concave up function $f_{Wan}(n)$ such that the conclusion of Conjecture 1.1 holds for $\alpha < f_{Wan}(n)$.

The stronger congruence in Equation (2) together with the method of [W] shows there is quadratic $f(n)$ with quadratic term smaller than that of $f_{Wan}(n)$ such that the conclusion of Conjecture 1.1 holds for $\alpha < f(n)$

**Conjecture 1.2 (Gouvêa)** Let $R_k$ be the multiset of slopes with multiplicity of classical $p$-oldforms in $M_k(N, \mathbb{Z}_p)$. The probability that an element of $R_k$ chosen with uniform distribution is in the interval $\left( \frac{k-1}{p+1}, \frac{p(k-1)}{p+1} \right)$ diminishes to zero as $k$ increases without bound.
1.2 Spaces of overconvergent modular forms

For \( p \geq 5 \), let \( E_{p-1} \) be the level one Eisenstein series. Let \( M_k(N, B) \) be the classical weight \( k \) level \( N \) modular forms with coefficients in \( B \).

**Proposition 1.3 (Katz)** For \( p \geq 5 \) and any \( f \in M_k(N, B) \), there are \( b_j \in M_{k+j(p-1)}(N, B) \) for \( j \geq 0 \) and \( r \in \mathcal{O}_p \) of positive valuation such that

\[
f = b_0 + \sum_{j=1}^{\infty} r^j b_j / E_{p-1}^j,
\]

(3)

There is a distinguished choice of \( b_j \) after choosing \( r \) and direct sum decompositions

\[ M_{k+j(p-1)} = E_{p-1} \cdot M_{k+(j-1)(p-1)} \oplus W_{k+j(p-1)}, \]

such that \( b_j \in W_{k+j(p-1)}(N, B) \) for \( j > 0 \).

See [Ka], Propositions 2.6.1 and 2.8.1. The parameter \( r \) is the growth condition and \( v_p(r) \) is bounded above by the given \( f \).

Let \( \mathcal{M}_k(N, B, r) \) be the space of modular forms with growth condition \( r \). The space \( \mathcal{M}_k(N, B) \) is \( \bigcup_{v_p(r) > 0} \mathcal{M}_k(N, B, r) \).

**Remark 1.3.1** For \( p = 3 \), \( N > 2 \) and prime to 3, Theorem 1.7.1 of [Ka] shows there is a level \( N \) lift of the characteristic 3 Hasse invariant, so an analogous expansion result holds. Proposition 2.8.2 of loc. cit. shows the expansion result for \( N = 2 \).

**Proposition 1.4** Suppose \( p = 2 \) or 3 and \( N \) relatively prime to \( p \). For any \( f \in \mathcal{M}_k(N, B) \) there are \( b_j \in M_{k+4j}(N, B) \) and \( r \in \mathcal{O}_p \) of positive valuation such that

\[
f = b_0 + \sum_{j=1}^{\infty} r^{4j/(p-1)} b_j / E_4^j,
\]

There is a distinguished choice of \( b_j \) after choosing \( r \) and direct sum decompositions

\[ M_{k+4j} = E_4 \cdot M_{k+4(j-1)} \oplus W_{k+4j}(N, B), \]

such that \( b_j \in W_{k+4j}(N, B) \) for \( j > 0 \).

**Proof.** We follow the remark at the end of Subsection 2.1 of loc. cit.. Let \( B \) be the fourth power of the Hasse invariant \( A \) for \( p = 2 \) and the square of \( A \) for \( p = 3 \). In either case, \( B \) is a weight 4 level 1 modular form defined over \( \mathbf{F}_p \). A version of Deligne’s congruence holds: \( B \equiv E_4 \mod 2^4 \) and \( B \equiv E_4 \mod 3 \).

For \( N > 2 \), and relatively prime to \( p \), the functor “isomorphism classes of elliptic curves with level \( N \) structure” is representable by a scheme which is smooth over \( \mathbf{Z}[1/N] \) and the formation of modular forms commutes with base change to a ring in which \( p \) is topologically nilpotent. So we repeat the construction of \( p \)-adic modular forms for \( p = 2, 3 \) and Katz expansions with powers of \( r^{4/(p-1)} E_4^{-1} \).
For $p = 2, 3$ (and 5), and $N = 1$, Section 1.4 of \[Se2\] states weight zero forms have expansions in powers of $\Delta E_4^{-3}$ where $\Delta$ is the weight 12 level 1 cusp form. Coleman\[C2\] shows

$$E_k \cdot \mathcal{M}_0(N, B) = \mathcal{M}_k(N, B).$$

$M_k(N, B)$ is a free $B$ module, so $M_k(N, B) = E_4 \cdot M_{k-4}(N, B) \oplus W_k(N, B)$ for some $W_k(N, B) \subset M_k(N, B)$. \hfill $\square$

**Theorem 1.5 (Coleman)** Let $k_1, k_2$ be weights. Let $G(q) \in M_{k_1-k_2}(N, B)$. Let $\Xi$ be the operator multiplication by $G(q)/G(q^p)$. If $1/G \in M_{k_2-k_1}(N, B)$ then $U^{(k_1)}$ is similar to $U^{(k_2)} \Xi$.

**Remark 1.5.1** The Eisenstein series satisfy the hypothesis of Theorem 1.5.

### 1.3 Notations for matrices and Newton Polygons

Let $M$ be a matrix over a ring, possibly of infinite rank. Let $n$ be a nonnegative integer. Let $s = (s_1, s_2, s_3, \ldots, s_n)$ be a sequence of $n$ distinct natural numbers.

The $n \times n$ diagonal major of $M$ associated to $s$ is the $n \times n$ matrix $A$ whose entry $A_{ij}$ is $M_{s_i,s_j}$.

A selection of a $M$ associated to $s$ and degree $n$ permutation $\pi$ is a sequence of $n$ elements, $(M_{s_{\pi(1)},s_{\pi(2)}}, \ldots, M_{s_{\pi(n)},s_{\pi(n)}})$.

The $n \times n$ diagonal minor of $M$ associated to $s$ is the determinant of the $n \times n$ diagonal major of $M$ associated to $s$.

The upper $n \times n$ diagonal major of $M$ is the diagonal major associated to the sequence $(1, 2, 3, \ldots, n)$.

The diagonal matrix $D = \text{diag}(d_i : i \geq 1)$ is the matrix with entries $D_{ii} = d_i$ and zero elsewhere.

The Newton polygon of power series $P(t)$ is the function $N(m)$ which is the lower convex hull of the set $(m, v_p(a_m(P)))$, defined for real $m \geq 0$.

A vertex of the Newton polygon $N(m)$ is a point $(m, N(m))$ such that $N(m) = v_p(a_m(P))$.

A side of a Newton polygon $N(m)$ is a line segment whose endpoints are vertices.

The slopes of a Newton polygon are the slopes of its sides.

The multiplicity of a slope is the difference of the first coordinates of its endpoints.

We denote by $N_k(m)$ the Newton polygon of $P_k$, and by $N_k(m)$ a function such that $N_k(m) \geq N_k(m)$. We indicate by $N_k(m)$ a function such that for all weights $k$, $N_k(m) \geq N_k(m)$.

We denote by $N_k'(m)$ the Newton polygon of $Q_k$, and by $N_k'(m)$ a function such that $N_k'(m) \geq N_k'(m)$.

We state as Lemma \[K2\] that if $p - 1 \mid 12$ and $N = 1$, then $P_k(t) = (1-t)Q_k(t)$. For these cases, $N_k(m) = N_k(m + 1)$. 

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2 Comparing Newton polygons for $U$ in different weights

Retain $p, N, k$ as before, and let $l$ be an integer and $n$ be a positive integer. For $p = 2$, we require $n \geq 2$. Let $k' = k + l(p - 1)p^n$. At the end of the section, we show there is a quadratic $\tilde{N}(m)$ such that

$$v_p(a_m(P_k) - a_m(P_{k'})) \geq \tilde{N}(m - 2) + n + 1.$$  

We now describe only the case $p > 3$ for clarity. Section 1.2 reviews the differences for $p = 2, 3$ from the case $p > 3$.

Let $r = p^{1/(p+1)}$. Choose a basis $\{b_{0,s}\}$ for the module $M_k(N, B)$. For $i > 0$, choose a basis $b_{i,s}$ for the module $W_{k+i(p-1)}(N, B)$.

Let $e_{i,s} = r^i E_{p-1}^{-i} b_{i,s}$. Let $M$ be the matrix for $U^{(k)}$ with respect to the basis $\{e_{i,s}\}$.

Let $N_k(m)$ be the Newton polygon of $P_k(t)$.

Lemma 2.1 of [W] includes

Lemma 2.2 (Wan) Let $k$ be a weight. If $d_v \leq m < d_{v+1}$ for some $v \geq 0$, then

$$N_k(m) \geq \frac{p - 1}{p + 1} \left( \sum_{u=0}^{v} um_a + (v + 1)(m - d_v) \right) - m.\tag{4}$$

Definition 2.1 Let $\tilde{N}_k(m)$ be the right side of Equation (4).

The $m_a$ have an upper bound, depending on $p$ and $N$, so $\tilde{N}_k(m)$ grows quadratically. Wan shows $N_k(m) = \tilde{N}_k(m)$ when both are less than $n + 1$.

Lemma 2.3 Let $A$ be the matrix for $U^{(k)}$ with respect to basis $e_{i,s} = r^{-i} e_{i,s}$. Then

$$v_p(A_{i,s}^{u,v}) \geq \frac{(up - i)}{(p + 1)}$$

and also at least zero.

Proof. $U^{(k)}$ stabilizes $M_k(N, B, 1)$, as shown in [GM].  

Proposition 2.4 $E_{p-1}^n(q)/E_{p-1}^n(q^p) \in 1 + p^{n+1} M_0(1, \mathbb{Z}_p, 1)$.

Proof. In weight zero, the only $e_{i,s}$ not 0 at the cusp $\infty$ is the constant function 1. The $q$-expansion of $(E_{p-1} - 1)/p$ is in $q\mathbb{Z}[[q]]$.

Theorem 2.5 For $k, k'$ as above, $v_p(a_m(P_k) - a_m(P_{k'})) \geq \tilde{N}_k(m - 2) + n + 1$.  

PROOF. Let $C$ be the matrix with respect to the basis $\epsilon_{i,s}$ for multiplication by $E_p^n(q)/E_p^{n-1}(q^p)$ considered as an operator on $M_k(N,B,r)$.

Let $M^{(k')} = MC$. By Theorem 1.5, $M^{(k')}$ is a matrix for an operator similar to $U^{(k')}$ on $M_{k'}(N,B,r)$ and $M^{(k')}$ acts on $M_k(N,B,r)$.

By Proposition 2.4, the matrix $C^{-1}$ is a matrix with entries in $p^{n+1}B$, so $M - M^{(k')}$ has entries in $p^{n+1}B$.

The difference $a_m(P_k) - a_m(P_{k'})$ is equal to

$$\text{tr } M - \text{tr } M^{(k')}.$$ 

These traces are the sums of all the different $m \times m$ diagonal minors of $M$ and $M^{(k')}$, so the difference contains terms (up to sign)

$$\prod_{i=1}^{m} M^{(k')}_{s_i,s_{\pi(i)}} - \prod_{i=1}^{m} M_{s_i,s_{\pi(i)}},$$

where $s$ is a sequence of $m$ integers, $\pi$ is a permutation of degree $m$.

Let

$$Z = \prod_{i=1}^{m} (z_i + w_i) - \prod_{i=1}^{m} (z_i),$$

where $z_i \in B$ and $w_i \in p^{n+1}B$, be instance of equation (5).

The sequence $(z_1, z_2, \ldots, z_m)$ is a selection of $M$. By Lemma 2.3, the product of any $m - j$ of them has valuation at least $\hat{N}_k (m - 2j)$. The product of any $j$ of the $w_i$ has valuation at least $j(n+1)$.

Rewrite (6) as

$$Z = \sum_{\emptyset \neq s \subseteq \{1,2,\ldots,m\}} \prod_{i \in s} w_i \prod_{i \notin s} z_i.$$ 

(7)

For any subset $s$ of size $j$,

$$v_p(\prod_{i \in s} w_i \prod_{i \notin s} z_i) \geq \hat{N}_k (m - 2j) + j(n+1).$$

The set $s$ is nonempty, so,

$$v_p(Z) \geq \hat{N}_k (m - 2) + n + 1,$$

for every instance of Equation (6).

\[ \square \]

Corollary 2.5.1 There is a quadratic $\hat{N}(m)$ independent of $k$ such that the conclusion of Theorem 2.5 holds.

PROOF. Given $p,N$, Wan shows there are finitely many different $\hat{N}_k(m)$. Let $\hat{N}(m)$ be the infimum of them. \[ \square \]
3 Computing tame level 1 $U$ for $p \in \{2, 3, 5, 7, 13\}$

Let $p$ be a prime such that $X_0(p)$ has genus 0, that is, $p \in \{2, 3, 5, 7, 13\}$ and $N = 1$. We show how to compute $U(0)$ with respect to an explicit basis.

The curve $X_0(p)$ has a uniformizer

$$d_p = p^{-1} \sqrt[3]{\Delta(q^p)/\Delta(q)}$$

with simple zero at the cusp $\infty$, pole at the cusp 0, and leading $q$ expansion coefficient 1.

Let $\pi: X_0(p) \to X_0(1)$ be the map which ignores level $p$ structure. Let $\hat{j} = \pi^*(j)$. The map $\pi$ is ramified above $j = 0, 1728, \infty$ only.

Proposition 3.1 There is a degree $p + 1$ polynomial $H_p$ over $\mathbb{Z}$ with constant term 1 such that

$$d_p \hat{j} = H_p(d_p).$$

Proof. The map $\pi$ has degree $p + 1$. The product $d_p \hat{j}$ has a pole only at the cusp 0. Hence, there is a polynomial $H_p$ satisfying the proposition.

$H_p$ has integer coefficients, because the $q$-expansion of $d_p \hat{j}$ at $\infty$ is in $1 + q\mathbb{Z}[[q]]$. \square

Remark 3.1.1 The ramification degrees of $\pi$ over $j = 0$ are 1 and 3, yielding roots of multiplicity 1 or 3 of $H_p(d_p)$. Points over $j = 1728$ are roots of multiplicity 1 or 2 of $H_p(d_p) - 1728d_p$. We calculate $H_p$ by equating $q$-expansions.

Lemma 3.2 $P_k(t) = (1 - t)Q_k(t)$

Proof. $X_0(p)$ has genus 0, so the only weight zero noncuspidal eigenforms are constants and the eigenvalue is 1. By a theorem of [H], or as a consequence of Theorem 1.5, in every weight $k$, $d(k, 0) = 1$ and a slope zero eigenform is noncuspidal. \square

Let $t_2 = 4$, $t_3 = 3$. For $p \geq 5$, let $t_p = 1$.

Let $c_2 = 0$, $c_3 = 1728$, $c_5 = 0$, $c_7 = 1728$, and $c_{13} = 432000/691$.

Let $e = 12/(p^2 - 1)$.

Lemma 3.3 The Newton polygon of $H_p(d_p) - c_p d_p$, as a polynomial in $d_p$, has a single side of slope $ep$.

Lemma 3.4 The weight 12 power of $E_{t_p(p-1)}$ is $(j - c_p)\Delta$.

The lemmas are direct computations.

Proposition 3.5 Let $r < p/(p + 1)$. The disc $D = \{z: z \in X_0(1), v_p(E_{t_p(p-1)}(z)) < t_pr\}$ is isomorphic to $\{z: z \in X_0(p), v_p(d_p(z)) > -er(p+1)\}$. 


Proof. When \( z \in X_0(1) \) is a point of supersingular reduction, \( \Delta(z) \) is a unit. At a point of ordinary reduction, \( E_{p(p-1)}(z) \) is a unit and \( v_p(\Delta(z)) \geq 0 \). By Lemma 3.3, \( D = \{ z : v_p(j(z) - c_p) < er(p + 1) \} \).

Lemma 3.3 shows the relation \((j - c_p)d_p = H_p - c_p d_p\) is uniquely invertible for \( d_p \) such that \( v_p(d_p(z)) > -er(p + 1) \), establishing the isomorphism. \( \square \)

Corollary 3.5.1 \( S_0(1, Z_p) \subset d_p Z_p[[d_p]]. U_{(0)} \) acts as a matrix \( M \) on a basis of powers of \( d_p \).

Let \( \mathcal{W} \) be the rigid subspace of \( X_0(p) \) where \( v_p(\pi^*(E_{tp(p-1)})) < tp/(p + 1) \). The section \( s \) of \( \pi \) over \( \pi(W) \) such that for elliptic curve \( E \), \( s(E) \) is the pair \((E, C)\) for \( C \) the canonical order \( p \) subgroup of \( E \) is an isomorphism.

Let \( V \) be the pullback of \( \phi \), the Deligne-Tate lift of Frobenius on \( X_0(1)/F_p \). Let \( w_p \) be the Atkin-Lehner involution on \( X_0(p) \).

Lemma 3.6 For points of \( \mathcal{W} \),

\[
V(j) \circ \pi = j \circ w_p.
\]

Proof. The Atkin-Lehner involution acts as

\[
w_p: (E, C) \to (E/C, E[p]/C).
\]

\( E \) has a canonical subgroup of order \( p \), and

\[
V: E \to E/\ker \phi^*.
\]

coincides with \( s^* \circ w_p^* \circ \pi^* \).

We identify \( \mathcal{W} \) with \( \pi(W) \) via section \( s \). \( \square \)

Proposition 3.7 For points of \( \mathcal{W} \),

\[
H_p(p^{12/(1-p)}/d_p)V(d_p) - p^{12/(1-p)}H_p(V(d_p))/d_p = 0.
\] (8)

Proof. The modular equation

\[
H_p(w_p^*(d_p))V(d_p) = H_p(V(d_p))w_p^*(d_p)
\]

holds on \( \mathcal{W} \) and \( w_p(d_p) = (p^{12/(1-p)}/d_p) \). \( \square \)

Theorem 3.8 There is an algebraic function \( I_p(y, x) \) and a matrix \( M \) for \( U_{(0)} \) with respect to the basis \( d_p^n \) such that entries \( M_{ij} \) satisfy a generating function equation

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} M_{ij} x^i y^j = \frac{y}{p} \frac{d}{dy} \log I_p(x, y). \tag{9}
\]
Proof. Clear denominators and factor $V(d_p) - w_p^*(d_p)$ from Equation (8) to determine an algebraic relation
\[ d_p^N I_p(V(d_p), 1/d_p) \]
between $d_p$ and $V(d_p)$, of degree $p$ in $d_p$. The inverse of $V$ applied to coefficient of $d_p^{p-1}$ is $\text{tr} V(d_p) = pH(d_p)$.

The values of $U(d_p^n)$ for $n = 0$ to $p - 1$ and the coefficients of $I_p$ determine a recurrence for $U(d_p^n)$ for $n \geq p$. □

Remark 3.8.1 The $I_p(x,y)$ for $p = 2, 3, 5, 7, 13$ are

\[
\begin{align*}
I_2 &= 1 - (2^{12}x^2 + 3 \cdot 2^4)x - xy^2, \\
I_3 &= 1 - (3^{12}x^3 + 4 \cdot 3^5x^2 + 10 \cdot 3^3x)y - (3^{6}x^2 + 4 \cdot 3^2x)y^2 - xy^3, \\
I_5 &= 1 - (5^{12}x^5 + 6 \cdot 5^{10}x^4 + 63 \cdot 5^7x^3 + 52 \cdot 5^5x^2 + 63 \cdot 5^2x)y \\
&\quad - (5^{6}x^4 + 6 \cdot 5^3x^2 + 52 \cdot 5^2x)y^2 \\
&\quad - (5^{5}x^3 + 6 \cdot 5^4x^2 + 63 \cdot 5x)y^3 - (5^{3}x^2 + 6 \cdot 5)x)y^4 - xy^5, \\
I_7 &= 1 - (7^{12}x^7 + 4 \cdot 7^{11}x^6 + 46 \cdot 7^9x^5 + 272 \cdot 7^7x^4 + \\
&\quad 845 \cdot 7^5x^3 + 176 \cdot 7^2x^2 + 82 \cdot 7x)y - \ldots - xy^7, \\
I_{13} &= 1 - (13^{12}x^{13} + 2 \cdot 13^{12}x^{12} + 25 \cdot 13^{11}x^{11} + 196 \cdot 13^{10}x^{10} + \\
&\quad 1064 \cdot 13^9x^9 + 4180 \cdot 13^8x^8 + 12086 \cdot 13^7x^7 + \\
&\quad 25660 \cdot 13^6x^6 + 39182 \cdot 13^5x^5 + 41140 \cdot 13^4x^4 + \\
&\quad 27272 \cdot 13^3x^3 + 9604 \cdot 13^2x^2 + 1165 \cdot 13)x)y - \ldots - xy^{13}.
\end{align*}
\]

Proposition 3.9 The $p$-adic valuation of $M_{ij}$ is at least $\epsilon(p \cdot i - j) - 1$. There is a parabola $N(m)$ with quadratic coefficient $6/(p+1)$ such that $N_0(m) \geq N(m)$.

Proof. Let $M_{ij}' = p^{e(j-1)}M_{ij}$. The matrix $(M_{ij}')$ is similar to $(M_{ij})$. Theorem [3.8] shows
\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} M_{ij}'x^iy^j = \frac{y}{d_p} \log I_p(p^{-e}x, p^e y). 
\tag{10}
\]

Direct calculation shows $I_p(p^{-e}x, p^e y)$, for the $I_p$ displayed in Remark [3.8.1] is a polynomial in $p^{e(p-1)}x$ and $y$ with integer coefficients. Hence, $v_p(M_{ij}') \geq i \cdot e(p - 1)$. □

3.1 Tame level 1 and $p = 2$ or 3

Emerton [E] calculates the lowest positive slope 2-adic modular forms of every weight. Concise expressions for the $q$-expansions of a few forms facilitate computation.

Serre [S2] observes that for a compact operator $M$ expressed as a matrix on a basis of a Banach space, if $c_i$ is the infimum of the valuations of column $i$ of $M$, then $\text{tr} (\Lambda^n M)$ has valuation at least the sum of the $n$ smallest $c_i$. 

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**Proposition 3.10** For $p = 2$ and even weight $k$, there is an $\mathcal{O}_2$ basis $\{e_n\}_{n \geq 1}$ of $S_k(1, \mathcal{O}_2)$ such that the image of $U(k)$ is a subset of $\bigoplus S^n e_n \mathcal{O}_2$.

**Proof.** This is a rewriting of Proposition 3.21 of [E] in language amenable to the noted observation of Serre. The basis element $e_n$ is $F_k d_2^n$ for a certain weight $k$ form $F$.

Recall $N'_k(m)$ is the Newton polygon of $Q_k(t)$.

**Corollary 3.10.1** $N'_k(m) \geq 3 \binom{m+1}{2}$.

**Lemma 3.11** Suppose $p = 3$. Let $S = \sqrt{3} V(\Delta)$. $S^2$ is in $M_6(1, \mathbb{Z}_p)$ and does not vanish at the cusp $\infty$. The quotient $S/V(S)$ is in $M_0(1, \mathcal{O}_3, 3/2)$ and as a power series in $Z[[d_3]]$, $S/V(S) - 1$ is in the ideal $(9d_3, 27d_3^3)$.

**Proof.** Direct calculation and comparison of $q$ expansions shows $S$ is the Eisenstein series for level 3, weight 3 and character $\tau$, the 3-adic Teichmuller character. $S^2$ is a level 3 weight 6 classical modular form and thus a tame level 1 weight 6 overconvergent modular form.

The curve $X_0(9)$ has genus zero and uniformizer

$$d_9 = \sqrt[3]{V(\Delta)/\Delta}.$$ 

The ramification of the forgetful map to $X_0(3)$ shows

$$d_3 = d_9 + 9d_9^2 + 27d_9^3.$$ 

Reversal of this relation between $d_3$ and $d_9$ and the observation

$$S/V(S) = d_9/d_3$$

shows $S/V(S)$ is in $M_0(1, \mathcal{O}_3, 3/2)$, has constant term 1, and $S/V(S) - 1 \in (9d_3, 27d_3^3) \subset Z[[d_3]]$. 

**Proposition 3.12** For $p = 3$ and even weight $k$ divisible by 3, $N'_k(m) \geq 3 \binom{m}{2}$.

**Proof.** Let $R$ be the multiplication by $(S/V(S))^{k/3}$ operator. Theorem 1.5 shows the composition $U(0) R$ is similar to $U(k)$. Lemma 3.11 shows the conclusions of Proposition 3.9 hold for $U(0) R$. 

[10]
3.2 Further example for $p = 3$, $N = 1$, $k = 0$.

Let $p = 3$, $N = 1$ and 
\[ \hat{N}'_0(m) = \frac{3}{2}m(m-1) + 2m. \]

We work an example of Proposition 3.9.

Lemma 3.13 $N'_0(m) \geq \hat{N}'_0(m)$.

Proof. Recall $e = 3/2$. Equation (10) shows 
\[
3 \sum_{i,j} M'_{ij} x^i y^j = \frac{9(10xy + 8\sqrt{3}x^2y + 3xy^2) + 3^5(4\sqrt{3}x^2y + 2x^2y^2) + 3^8x^3y}{1 - 3^3(10xy + 4\sqrt{3}x^2y + x^3y) - 3^6(4\sqrt{3}x^2y + x^2y^2) - 3^9x^3y}.
\] 

(11)

Following the last step of Proposition 3.9, substitute $\delta = 3^3x$ into the right side of Equation (11) to get 
\[
G(\delta, y) = \frac{10\delta y + 8\sqrt{3}\delta y^2 + 3\delta y^3 + 4\sqrt{3}\delta^2 y + 2\delta^2 y^2 + \delta^3 y}{1 - 10\delta y - 4\sqrt{3}(\delta y^2 + \delta^2 y) - (\delta y^3 + \delta^2 y^2 + \delta^3 y)}.
\]

(12)

The valuation of $M'_{ij}$ is at least $i \cdot e(p-1) - 1 = 3i - 1$. So 
\[ N'_0(m) \geq \sum_{i=1}^{m} 3i - 1 = \hat{N}'_0(m). \]

\[\square\]

4 For $p = 3$, $N = 1$, $\hat{N}'_0$ is a sharp parabola below $N'_0$

Let $p = 3$ and $N = 1$ and 
\[ m_i = \sum_{j=0}^{i-1} 3^j = \frac{3^i - 1}{2}. \]

Theorem 4.1 The set $E = \{m : m \in \mathbb{Z}, N'_0(m) = \hat{N}'_0(m)\}$ is the same as $\{m_i : i \geq 0\}$.

Proof. We show for all $m \geq 0$, that $m \in E$ if and only if $(m - 1)/3 \in E$.

The leading coefficient of $P_0$ is 1, so $0 \in E$.

Let $M'$ be the matrix for $U(0)$ with respect to basis $\{3^{3m/2}d^m\}$.

Lemma 3.13 shows $M'_{ij}$ has valuation at least $3i - 1$, so there is a matrix $K$ over \(\mathbb{Z}[\sqrt{3}]\) and diagonal matrix $D = diag(3^{3i-1})$ such that $M' = DK$.

Let $\bar{K} = K \mod \sqrt{3}\mathbb{Z}[\sqrt{3}]$ and let $c_m(\bar{K})$ be its upper $m \times m$ diagonal minor.
Every $m \times m$ diagonal minor of $M'$ has valuation at least $\hat{N}_0'(m)$ and the inequality is strict except for the upper $m \times m$ diagonal minor. So we have reduced the theorem to showing that $m \in E$ if and only if $c_{m}(\bar{K}) \neq 0$.

Call a degree $m$ permutation $\pi$ excellent if the selection of $\bar{K}$ associated to $(1, 2, \ldots, m)$ and $\pi$ is a sequence of nonzero entries of $\bar{K}$.

**Claim 1.** If there is a degree $m$ excellent $\pi$, then $m = m_i$ for some $i$.

We establish Claim 1 by induction. The trivial degree 0 permutation is excellent.

The entries of $K$ satisfy a linear recurrence. Equation (12) with $x$ substituted for $\delta$ is

$$G(x, y) = \frac{10xy + 4\sqrt{3}xy(x + 2y) + xy(x^2 + 2xy + 3y^2)}{1 - xy(10 + 4\sqrt{3}(x + y) + x^2 + xy + y^2)}.$$

The coefficient of $x^iy^j$ is the entry of $K$ in row $i$ and column $j$.

Let $\bar{G}$ be the generating function for entries of $\bar{K}$. $\bar{G}$ is the reduction of $G$ to $\mathbb{F}_3[[x, y]]$.

Let

$$R(i) = (1 + (xy + x^3y + x^2y^2 + xy^3) + (xy + x^3y + x^2y^2 + xy^3)2^i),$$

and

$$\bar{G}_0(x, y) = xy(1 - xy + y^2).$$

Let

$$\bar{G}_j = \bar{G}_0 \cdot \prod_{i=0}^{j-1} R(i)$$

and

$$\bar{C}_j = \prod_{i=j}^{\infty} R(i).$$

For all nonnegative integers $j$, $\bar{C}_j^3 = \bar{C}_{j+1}$ and $\bar{G} = \bar{G}_j \bar{C}_j$.

By direct computation,

$$\bar{G}_1 = (x^{-1}y + 1 - xy^{-1} + y^{-2})\bar{G}_0^3 + xy + x^2y^4 + x^6y^2, \quad (13)$$

and so

$$\bar{G} = (x^{-1}y + 1 - xy^{-1} + y^{-2})\bar{G}_0^3 + (xy + x^2y^4 + x^6y^2)\bar{C}_1. \quad (14)$$

Equation (14) shows the coefficient of $x^iy^j$ in $\bar{G}$ is the same as the coefficient of $x^iy^j$ in $\bar{G}^3$. This coefficient is zero if $i$ is not divisible by 3.

Suppose degree $m$ permutation $\pi$ is excellent. The only unit in row 1 is in column 1, so $\pi(1) = 1$. The functions

$$\sigma(i) = \pi(3i)/3, \quad \sigma'(i) = (\pi(3i - 1) - 1)/3, \quad \sigma''(i) = (\pi(3i + 1) + 1)/3 \quad (15)$$

are excellent degree $\lfloor m/3 \rfloor$ permutations, and $3 | (m - 1)$.

The inductive step is complete.

**Claim 2.** For any $m_i$, there is a unique degree $m_i$ excellent $\pi$. 

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We proceed by induction. The unique degree 0 permutation is excellent.

Equation (14) shows for excellent degree \( m \) permutations \( \sigma, \sigma', \sigma'' \), there is an excellent degree \( m \) permutation \( \pi \), computed by reversing Equations (15).

If there is a unique degree \( (m - 1)/3 \) excellent \( \sigma \), then there is a unique degree \( m \) excellent \( \pi \). Claim 2 is established.

Claim 1 shows for \( m \) not equal to any \( m_i \), that \( c_m(K) = 0 \). Claim 2 shows for each \( m_i \), there is a unique selection of the upper \( m_i \times m_i \) diagonal major of \( K \) which contributes a nonzero term to \( c_{m_i}(K) \). Hence, \( c_m(K) \neq 0 \) if and only if there is \( i \) such that \( m = m_i \).

\[ \square \]

**Corollary 4.1.1** Let \( L \) be the secant line such that \( L(m_i) = \tilde{N}_0'(m_i) \) and \( L(m_{i+1}) = \tilde{N}_0'(m_{i+1}) \). If \( m \) is such that \( m_i < m < m_{i+1} \), then

\[ \tilde{N}_0'(m) < N_0'(m) \leq L(m). \]

**Proposition 4.2** Let \( l \) be an integer, \( n \) be a nonnegative integer. Let \( k = 2 \cdot 3^{n+1} \cdot l \).

Let \( s \) be an integer, \( 0 \leq s < 2 \cdot 3^{n-1} \). If \( N_0'(s) = \tilde{N}_0'(s) \), then \( N_k'(s) = \tilde{N}_0'(s) \).

**Proof.** Let \( R = (S/V(S))^{k/3} \). The binomial theorem shows the coefficient of \( d_3^m \) in \( R \) has valuation at least \( [3m/2] + n - v_3(m) \).

Let \( C \) be the matrix for the multiplication by \( R \) operator on \( S_0(1, O_p) \) with respect to the basis \( \{3^{3m/2}d^m\} \).

Let \( M' \) be the matrix for \( U_{(0)} \) with respect to the same basis.

By Theorem 1.5, \( M'C \) is similar to a matrix for \( U_{(k)} \).

For all \( i, j, v_3(M'_{ij}) \geq 3i - 1 \). For \( i > j \) or \( j > 3i \), \( M_{ij}' = 0 \).

For all \( j > 0 \), \( C_{jj} = 1 \). For \( j, m > 0 \), \( v_3(C_{j+m,j}) \geq n - v_3(m) \) and \( C_{j,j+m} = 0 \).

For odd \( m \), including \( m = 3n-1 \), \( v_3(C_{3j+m,j}) \geq \frac{1}{2} \).

For all \( i, v_3(M'_{ij} - (M'C)_{ij}) \geq 3i - 1 \).

For \( i \leq s \), \( v_3(M'_{ij} - (M'C)_{ij}) > 3i - 1 \), because

\[ (M'C)_{ij} = \sum_{k=j}^{3i} M'_{ik}C_{kj}, \]

and \( 3i \leq 3s < 2 \cdot 3^n \).

If \( N_0'(s) = \tilde{N}_0'(s) \) then \( N_k'(s) = N_0'(s) \).

\[ \square \]

**Corollary 4.2.1** Let \( l \) be an integer and \( n \) be a nonnegative integer. Let \( k = 2 \cdot 3^{n+1} \cdot l \).

For integer \( i \), \( 0 \leq i < n - 1 \), there are exactly \( 3^i \) overconvergent \( 3 \)-adic modular forms of weight \( k \) with slope in \( [m_{i+1} + 1, m_{i+2} - 2] \), and these have average slope \( 3^{i+1} - 1 \).

**Proof.** By Proposition 1.2 \( N_k'(m_i) = N_0'(m_i) \) and \( N_k'(m_{i+1}) = N_0'(m_{i+1}) \). There are \( 3^i = m_{i+1} - m_i \) slopes with multiplicity accounted for by the edges joining these vertices of the Newton polygon \( N_k' \). The difference \( N_k'(m_{i+1}) - N_k'(m_i) \) is \( 3^i(3^{i+1} - 1) \).

The average slope is \( 3^{i+1} - 1 \). The minimum of these slopes is at least \( 3m_i + 2 \) and the maximum at most \( 3m_{i+1} - 1 \).

\[ \square \]
Corollary 4.2.2 Let $k$ be an even integer and $i$ be a positive integer. If
\[ v_3(k) \geq \frac{[\hat{N}'_0(m_{i+1}) + \hat{N}'_0(m_i)]}{2} - \hat{N}'_0((m_{i+1} + m_i)/2) + i + 2, \]
then for $m \leq m_{i+1}$, $N'_0(m) = N'_k(m)$.

Proof. The Newton polygons $N'_0$ and $N'_k$ both have vertices $(m_i, \hat{N}'_0(m_i))$ and $(m_{i+1}, \hat{N}'_0(m_{i+1}))$.

By Corollary 4.1.1 and Theorem 2.5, $v_3(a_m(P_k)) = v_3(a_m(P_0))$ for every $m$ between $m_i$ and $m_{i+1}$.

Affirming a pattern noticed by Gouvêa [G],

Corollary 4.2.3 Let $k = 2 \cdot 3^n + 1$. The classical weight $k$ level 3 oldforms have slopes outside $[k/4, 3k/4]$.

Proof. There are $m_n = \frac{k}{12} - \frac{1}{2}$ cuspidal level 1 normalized eigenforms. There are $2m_i + 2$ classical level 3 oldforms, and one pair of these comes from the weight $k$ Eisenstein series. The slopes of the forms in this pair are 0 and $k - 1$.

By Proposition 4.1.2, $N'_k(m_n) = \hat{N}'_0(m_n)$, because $m_n < 2 \cdot 3^{n-1}$.

The slope $N'_k(m_n) - N'_k(m_n - 1)$ is less than
\[ \hat{N}'_0(m_n) - \hat{N}'_0(m_n - 1) = 3m_n - 1 = \frac{k}{4} - 1. \]

The mates of these $m_i$ oldforms have slopes greater than $\frac{3k}{4}$.

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