ABSOLUTE ALGEBRAS, CONTRAMODULES, AND DUALITY SQUARES

VICTOR ROCA I LUCIO

ABSTRACT. Absolute algebras are a new type of algebraic structures, endowed with a meaningful notion of infinite sums of operations without supposing any underlying topology. Opposite to the usual definition of operadic calculus, they are defined as algebras over cooperads. The goal of this article is to develop this new theory. First, we relate the homotopy theory of absolute algebras to the homotopy theory of usual algebras via a duality square. It intertwines Bar-Cobar adjunctions with linear duality adjunctions. In particular, we show that linear duality functors between types of coalgebras and types of algebras are Quillen functors and that they induce equivalences between objects with finiteness conditions on their homology. We embed the theory of contramodules as a particular case of the theory of absolute algebras. We study in detail the case of absolute associative algebras and absolute Lie algebras. In [CPRNW23], the authors showed that two nilpotent Lie algebras whose universal enveloping algebras are isomorphic as associative algebras must be isomorphic. We generalize their theorem to any absolute Lie algebras and any minimal absolute $L_{\infty}$-algebras, of which nilpotent Lie algebras and minimal nilpotent $L_{\infty}$-algebras are a particular cases.

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INTRODUCTION

Let $A$ be a vector space. The data of an associative algebra structure amounts to the data of a “multiplication table” of elements of $A$: given two elements, it states which element in $A$ is their product. One can compile this “multiplication table” into a single morphism

\[
\gamma_A : \bigoplus_{n \geq 1} A^\otimes n \to A
\]

which assigns to any $n$-tuple $(a_1, \ldots, a_n)$ the value of their product $\gamma_A(a_1, \ldots, a_n)$. The conditions that an associative algebra structure has to satisfy are encoded by the fact that $\gamma_A$ defines a structure of an algebra over a monad. But in this classical algebraic framework, infinite collections of elements do not have an assigned value by this “multiplication table”. In order to given a value to expressions like $\sum_{n \geq 1} a_{1,n} \otimes \cdots \otimes a_{n,n}$, the classical approach used so far

\begin{flushleft}
Date: September 26, 2023.
2020 Mathematics Subject Classification. Primary 18N40, Secondary 18M70.
Key words and phrases. Absolute algebras, contramodules, operads, universal enveloping algebras, homotopical algebra.

The author was partially supported by the project ANR-20-CE40-0016 HighAGT funded by the Agence Nationale pour la Recherche.
\end{flushleft}
has been to add the data of a topology on $A$, so that the values assigned to any partial sum converge to a well-defined element in $A$. This solution has several issues, specially when one tries to mix homotopical algebra and topology. For instance, neither the category of topological abelian groups nor the category of topological modules over a topological ring are abelian. This type of problems is the main motivation for the recent approaches of [CS19] and [BH19].

Absolute algebras are a new type of algebraic structures where infinite sums of operations have well-defined images by definition. For instance, the data of an absolute associative algebra structure on a vector space $A$ amounts to the data of a “transfinite multiplication table”, where any series of elements in $A$ is assigned a value in $A$. This information compiles into a single structural map

\[
\begin{align*}
\gamma_A : & \prod_{n \geq 1} A^\otimes n \\
\sum_{n \geq 1} a_{i,n} \otimes \cdots \otimes a_{n,n} & \mapsto \gamma_A \left( \sum_{n \geq 1} a_{i,n} \otimes \cdots \otimes a_{n,n} \right).
\end{align*}
\]

The bright point is that this type of structures is defined as an algebra over a monad, hence the category of absolute associative algebras enjoys many desirable properties. For example, it is complete and cocomplete. Any absolute associative algebra structure gives an associative algebra structure in the classical sense, simply by restricting the structural map to finite sums. There are many "absolute analogues" of standard types of algebraic structures: absolute Lie algebras, absolute $A_\infty$-algebra, absolute $L_\infty$-algebras, etc. This stems from the fact that this type of structures are defined as algebras over a cooperad.

Recall that most of classical types of algebraic structures can be successfully encoded by an operad. See [LV12] for instance. Thus, given an operad $P$, we get a category of dg $P$-algebras. The operadic calculus provides us with a powerful tool to study the homotopy category of these dg $P$-algebras. The key element is the Koszul duality for operads, that allows one to construct a Koszul dual conilpotent cooperad $P^!$. This data allows us to construct a Bar-Cobar adjunction

\[
\begin{array}{c}
dg P\text{-alg} \\
\downarrow \Omega \\
dg P^!\text{-coalg}
\end{array}
\xleftarrow{\Omega} \xrightarrow{\text{B}}
\]

between the category of dg $P$-algebras and the category of dg $P^!$-coalgebras. Contrary to popular belief, any coalgebra over a cooperad is, by definition, conilpotent. Hence we omit this adjective when possible. Using this adjunction, one can transfer the model structure where weak-equivalences are given by quasi-isomorphism onto the category of conilpotent $P^!$-coalgebras and obtain a Quillen equivalence. This approach, first developed by V. Hinich for dg Lie algebras in [Hin01] and by K. Lefèvre-Hasegawa in [LH03] for dg associative algebras, and was later extended to all Koszul operads by B. Vallette in [Val20]. Conilpotent coalgebraic types of structures are given by decomposition maps, which assign finite sums to any element. Let $C$ be a vector space. For example, the data of a non-counital conilpotent coassociative coalgebra structure on $C$ is equivalent to the data of a "decomposition table", compiled into a map

\[
\begin{align*}
\Delta_C : & C \\
& \mapsto \bigoplus_{n \geq 1} C^\otimes n \\
c & \mapsto \Delta_C(c) = \sum_{i \in I} c^{(i)}_1 \otimes \cdots \otimes c^{(i)}_n,
\end{align*}
\]

where $I$ is a finite set. It is conilpotent precisely because this sum is finite, and therefore $\Delta_C$ lands on the direct sum instead of the product. But most coassociative coalgebras that appear in nature are not conilpotent, e.g: consider $k$ with its diagonal map.
In the same way as algebras are "the Koszul dual notion" to conilpotent coalgebras, since they both are "finitary types of structures", absolute types of algebras are the "Koszul dual notion" to non-conilpotent coalgebras. The infinite decompositions of elements in these non-conilpotent coalgebras are reflected in the "transfinite multiplication tables" of absolute algebras. The notion of an algebra over a cooperad was introduced in an abstract context by B. Le Grignou and D. Lejay in [GL22]. Their goal was to study the homotopy theory of non necessarily conilpotent coalgebras. Since these types of structures are encoded as coalgebras over an operad, the reasonable thing to do was to look at what an algebra over a cooperad looks like. Starting from a cooperad, one can construct a monad by considering a dual version of the Schur functor. Algebras over a cooperad are defined as algebras over its associated monad. Given an operad $\mathcal{P}$ and its Koszul dual cooperad $\mathcal{P}^!$, the authors of loc.cit construct a complete Bar-Cobar adjunction

$$\text{dg } \mathcal{P}\text{-coalg} \xleftarrow{\hat{\Omega}} \text{dg } \mathcal{P}^!\text{-alg},$$

and show that in some cases, the homotopy theory of these non necessarily conilpotent coalgebra can be recovered from the homotopy theory of the dual absolute algebras with a transferred model structure along this complete Bar-Cobar adjunction.

**Main results.** The goal of this article is to develop the theory of algebras over cooperads, which we call **absolute algebras**. Our first result answers a very natural question: how do these two aforementioned Bar-Cobar constructions relate to each other? We show that there are duality functors that intertwine both of these adjunctions in a duality square of commuting adjunctions.

**Theorem A** (Duality square, Theorem 2.16). There exists a square of adjunctions

$$\begin{array}{ccc}
\text{dg } \mathcal{P}\text{-alg}^\text{op} & \xleftarrow{\hat{\Omega}} & \text{dg } \mathcal{P}^!\text{-coalg}^\text{op} \\
\downarrow & & \downarrow \\
\text{dg } \mathcal{P}\text{-coalg} & \xleftarrow{\hat{\Omega}} & \text{dg } \mathcal{P}^!\text{-alg}, \\
\end{array}$$

which commutes in the following sense: right adjoints going from the top right corner to the bottom left corner are naturally isomorphic.

The technical part of the above results is the construction of the linear duality adjunctions. In particular, the functor $(-)^\psi$ is a generalization of the Sweedler dual functor constructed in [Swe69]. The above theorem admits a much more general formulation: it also holds for any curved twisting morphism in the sense of [RiL22a]. Conceptually, it shows that absolute algebras appear every time one considers the linear dual of a conilpotent coalgebra. For example, taking the linear dual of the Bar construction, which occurs in many instances in the literature: it is a key ingredient in J. Lurie’s proof of that formal moduli problems are encoded by dg Lie algebras [Lur11] and in the subsequent generalization of his result by D. Calaque, R. Campos and J. Nuiten in [CCN22].

This duality square can in favorable cases be made compatible with the respective model structures of each of these categories. Then, this square becomes a square of Quillen adjunctions. This allows us to have, for the first time, a homotopical understanding of the linear duality functor $(-)^\psi$ between types of coalgebras and types of algebras over a cofibrant dg operad $\mathcal{P}$. 
Theorem B (Theorem 2.25). Let \( \mathcal{P} \) be a cofibrant dg operad. There is an equivalence of \( \infty \)-categories

\[
\text{dg } \mathcal{P}\text{-coalg}^{f.d., \pm} \left[ \mathbb{Q}, \text{iso}^{-1} \right] \xrightarrow{(-)^*} \text{dg } \mathcal{P}\text{-alg}^{f.d., \pm} \left[ \mathbb{Q}, \text{iso}^{-1} \right]^{\text{op}},
\]

between the \( \infty \)-category of dg \( \mathcal{P} \)-algebras with degree-wise finite dimensional bounded below (resp. bounded above) homology and the \( \infty \)-category of \( \mathcal{P} \)-coalgebras with degree-wise finite dimensional bounded above (resp. bounded below) homology.

The above two theorems play a seminal role in [RiL22b], where we develop the integration theory of curved absolute \( \mathcal{L}_\infty \)-algebras. There, they allow us to prove that curved absolute \( \mathcal{L}_\infty \)-algebras provide us with rational models, which are dual to Sullivan’s models, and to develop our approach to derived algebraic geometry using curved absolute \( \mathcal{L}_\infty \)-algebras.

We then treat examples and develop the theory in those particular cases of interest. The first example is that of contramodules. Contramodules over coassociative coalgebras where first introduced by S. Eilenberg and J. C. Moore in [EM65] but later somewhat forgotten until they were extensively studied by L. Positselski, see for instance [Pos21]. Since a coassociative coalgebra is a cooperad concentrated in arity one, we show that contramodules are a particular example of absolute algebras. Engulfing this theory provides us with illuminating examples and counterexamples that shed a light into what is to be expected of this new type of algebraic structures. After, we treat in extenso the cases of dg absolute associative algebras and dg absolute Lie algebras. We show that nilpotent associative algebras are a particular examples of absolute associative algebras, and describe various new constructions that can be performed on this category.

Finally, we apply this new framework to Lie theory. In [CPRNW23], the authors proved the following theorem: two nilpotent Lie algebras are isomorphic as Lie algebras if and only if their universal enveloping algebras are isomorphic as associative algebras. For proving this statement, they compared the deformation complexes of \( \mathcal{C}_\infty \) and \( A_{\infty} \)-coalgebras. In this last section, we show how to reinterpret this result on deformation complexes in the more general context of absolute Lie algebras and their universal enveloping absolute algebras. By doing so, we were able to generalize their theorem as follows.

Theorem C (Theorem 4.9 and Theorem 4.14). Let \( k \) be a field of characteristic zero and let \( g \) and \( h \) be two complete graded absolute Lie algebras (resp. two complete minimal absolute \( \mathcal{L}_\infty \)-algebras). They are isomorphic as complete graded absolute Lie algebras (resp. as complete minimal absolute \( \mathcal{L}_\infty \)-algebras) if and only if their universal enveloping absolute algebras (resp. absolute \( A_{\infty} \)-algebras) are isomorphic.

The first thing to mention on this result is that is follows from a more general statement concerning dg absolute Lie algebras and absolute \( \mathcal{L}_\infty \)-algebras. Indeed, we show that two of them are linked by a zig-zag of weak-equivalences if and only if their universal enveloping constructions are. Since these weak equivalences are in particular quasi-isomorphisms, the above result follows when the differential is zero. The second thing is that class of complete graded absolute Lie algebras includes nilpotent graded Lie algebras without degree restrictions. In this particular case, we compute explicitly the universal enveloping absolute algebra, which is given by the completed tensor algebra modulo the standard relation. Analogously, nilpotent \( \mathcal{L}_\infty \)-algebras in the sense of [Get09] are examples of absolute \( \mathcal{L}_\infty \)-algebras. Therefore the above theorem hold for minimal nilpotent \( \mathcal{L}_\infty \)-algebras as well.

Acknowledgments. I would like to thank my former PhD. advisor Bruno Vallette for the numerous discussion we had and for his careful readings of this paper. I would also like to thank Damien Calaque, Geoffroy Horel, Brice Le Grignou, Johan Leray, Joost Nuiten, L. Positseski, and Friedrich Wagemann for interesting discussions. I would like to acknowledge the warm hospitality of Svenska KullagerFabriken and its employees in Göteborg, which provided great working conditions in order to finish this paper. This paper was written during my PhD. thesis.
at the Université Sorbonne Paris Nord, I would like to thank its great mathematical community.

**Conventions.** Let $\mathbb{k}$ be a ground field of characteristic 0. The ground category is the symmetric monoidal category $(\text{pdg-mod}, \otimes, \mathbb{k})$ of pre-differential graded $(\text{pdg})$ $\mathbb{k}$-modules: these are graded modules $V$ endowed with a degree $-1$ endomorphism $d_V$. We work with the homological degree conventions. The tensor product $\otimes$ of pdg modules given by the graded module 

$$(A \otimes B)_n := \bigoplus_{p+q=n} A_p \otimes B_q ,$$

together with the pre-differential

$$d_{A \otimes B}(a \otimes b) := d_A(a) \otimes b + (-1)^{|a|} a \otimes d_B(b) .$$

The isomorphism $\tau_{A,B} : A \otimes B \to B \otimes A$ is given by the Koszul sign rule $\tau(a \otimes b) := (-1)^{|a||b|} b \otimes a$ on homogeneous elements. The suspension of a $V$ is denoted by $sV$, given by $(sV)_p := V_{p-1}$. A **differential graded (dg) module** is pre-differential graded modules $V$ such that $d^2_V = 0$. The category of dg modules is a full subcategory of pdg modules. A $(p)dg$ module $V$ is degree-wise finite dimension if every $V_n$ is a finite dimensional $\mathbb{k}$-module. It is bounded above if there exists an $m \in \mathbb{Z}$ such that $V_n = 0$ for $n \geq m$ and it is bounded below if there exists an $m \in \mathbb{Z}$ such that $V_n = 0$ for $n \leq m$.

A pdg $S$-module $M$ is a collection $(M(n))_{n \in \mathbb{N}}$ of pdg $\mathbb{k}[S_n]$-modules. This category is denoted by pdg $S$-mod. We denote $(\text{pdg } S\text{-mod}, \circ, 1)$ the monoidal category of pre-differential graded $S$-modules endowed with the composition product $\circ$. Again, a dg $S$-module is a pdg $S$-module whose pre-differential squares to zero; they form a full subcategory of pdg $S$-modules. We refer to [LV12] for most of the notations, and to [RiL22c] for a more details explanation of this framework and of the recollections.

1. **Recollections**

The goal of this section is to briefly recall the recent developments in operadic calculus made in [GL22]. Their main idea is encode *non-necessarily conilpotent* types of coalgebras using operads, and use cooperads to encode their "Koszul dual structure". These are the types of structure that we will call "absolute algebras" in the present article. For a more thorough exposition of this theory, see also [RiL22c].

1.1. **Curved standard operadic calculus.** First let us recall the classical side of the operadic calculus in the curved setting, where operads are used to encode algebras and cooperads are used to encode conilpotent coalgebras. We do so in the curved setting of [Gri19], which is an extension of the classical operadic calculus of [GK94]. See also [LV12] for these results.

**Definition 1.1** (dg operad). An *dg operad* $\mathcal{P}$ amounts to the data $(\mathcal{P}, \gamma, \eta, d_{\mathcal{P}})$ of a monoid in $(\text{dg } S\text{-mod}, \circ, 1)$.

**Definition 1.2** (augmented dg operad). An *augmented dg operad* $\mathcal{P}$ amounts to the data of a dg operad $(\mathcal{P}, \gamma, \eta)$ equipped with a morphism of dg operads $\nu : \mathcal{P} \to I$ such that $\nu \circ \eta = \text{id}$.

If $\mathcal{P}$ is not augmented, then one has to consider curved cooperads on the other side of the Koszul duality. Curved cooperads are particular examples of pdg cooperads, endowed with a curvature that controls how far is the pre-differential to be a differential.

**Definition 1.3** (pdg cooperad). A *pdg cooperad* $\mathcal{C}$ amounts to the data $(\mathcal{C}, \Delta, \epsilon, d_{\mathcal{C}})$ of a comonoid in the monoidal category $(\text{pdg } S\text{-mod}, \circ, 1)$.

**Definition 1.4** (coaugmented pdg cooperad). A *coaugmented pdg cooperad* $\mathcal{C}$ amounts to the data of a pdg cooperad $(\mathcal{C}, \Delta, \epsilon)$ equipped together with a morphism of pdg cooperads $\mu : I \to \mathcal{C}$ such that $\epsilon \circ \mu = \text{id}$.
**Definition 1.5 (curved cooperad).** A curved cooperad \( \mathcal{C} \) amounts to the data \((\mathcal{C}, \Delta, \epsilon, d_{\mathcal{C}}, \Theta_{\mathcal{C}})\) of a pdg cooperad \((\mathcal{C}, \Delta, \epsilon, d_{\mathcal{C}})\) and a morphism of pdg \(S\)-modules \(\Theta_{\mathcal{C}} : (\mathcal{C}, d_{\mathcal{C}}) \to (I, 0)\) of degree \(-2\) called the \emph{curvature}, such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\Delta_{(1)}} & \mathcal{C} \circ_{[1]} \mathcal{C} & \xrightarrow{id \circ_{[1]} \Theta_{\mathcal{C}} - (\Theta_{\mathcal{C}} \circ_{[1]} id)} & (\mathcal{C} \circ_{[1]} I) \oplus (I \circ_{[1]} \mathcal{C}) \\
& & & \xrightarrow{\mathcal{C}} & \mathcal{C} \\
& & & \mathcal{C} & \xrightarrow{d_{\mathcal{C}}} & \mathcal{C}
\end{array}
\]

where \(\circ\) is given by \(+(\mu, \nu) := \mu + \nu\).

The coradical filtration of a coaugmented cooperad (possibly pdg or curved) is the increasing filtration defined using iterations of the partial decompositions. Since it is coaugmented, one can remove trivial decompositions from the partial decomposition maps. An element is in the \(\omega\)-term of this filtration if its image by any possible composition of \(\omega\) (non-trivial) partial decomposition maps is zero. A cooperad is said to be conilpotent if any arbitrary iteration of these (non-trivial) partial decompositions maps of the cooperad ends up being trivial, or equivalently, if the coradical filtration is exhaustive. For a precise definition, see [RiL22a, Section 1.3] or [RiL22c, Chapter 1, Section 4].

To any dg (resp. pdg) \(S\)-module one can associate an endofunctor in the category of dg (resp. pdg) modules via the Schur realization functor:

\[
\mathcal{R} : (p)dg \text{-} S\text{-mod} \longrightarrow \text{End}((p)dg \text{-} mod)
\]

\[
M \longrightarrow \mathcal{R}(M)(-) \cong \bigoplus_{n \geq 0} M(n) \otimes S_n(-)^{\otimes n}.
\]

The realization functor \(\mathcal{R}(-)\) is strong monoidal. Thus, for any dg operad \(\mathcal{P}\) its Schur functor \(\mathcal{R}(\mathcal{P})\) is a monad in dg modules and for any pdg cooperad \(\mathcal{C}\) its Schur functor \(\mathcal{R}(\mathcal{C})\) is a comonad in pdg modules.

**Definition 1.6 (dg \(\mathcal{P}\)-algebra).** Let \(\mathcal{P}\) be an operad. A dg \(\mathcal{P}\)-algebra \(B\) amounts to the data \((B, \gamma_B, d_B)\) of an algebra over the monad \(\mathcal{R}(\mathcal{P})\).

**Definition 1.7 (pdg \(\mathcal{C}\)-coalgebra).** Let \(\mathcal{C}\) be a pdg cooperad. A pdg \(\mathcal{C}\)-coalgebra \(D\) amounts to the data \((D, \Delta_D, d_D)\) of a coalgebra over the comonad \(\mathcal{R}(\mathcal{C})\).

**Definition 1.8 (curved \(\mathcal{C}\)-coalgebra).** Let \(\mathcal{C}\) be a curved cooperad. A pdg coalgebra \(D\) is said to be curved if the following diagram commutes:

\[
D \xrightarrow{\Delta_D} \mathcal{R}(\mathcal{C})(D) \\
\downarrow d_D \\
D \cong \mathcal{R}(I)(D).
\]

The Koszul duality between dg operads and curved cooperads allows to construct a first Bar-Cobar adjunction between algebras over a dg operad and curved coalgebras over a curved cooperad. For the definition of a curved twisting morphism between a curved cooperad and a dg operad, see [Gri19, Section 4] or [RiL22a, Section 6].

**Theorem 1.9 (Bar-Cobar adjunction).** Let \(\alpha : \mathcal{C} \to \mathcal{P}\) be a curved twisting morphism. There is a Bar-Cobar adjunction relative to \(\alpha\):

\[
\text{curv} \mathcal{C}\text{-coalg} \quad \downarrow \quad \text{dg} \mathcal{P}\text{-alg},
\]

\[
\Omega_{\alpha} \quad \downarrow \quad \text{B}_{\alpha}
\]

\[
\text{curv} \mathcal{C}\text{-coalg} \quad \downarrow \quad \text{dg} \mathcal{P}\text{-alg},
\]
There is always a transferred model category structure on the category of dg $\mathcal{P}$-algebras where fibrations are given by epimorphisms and weak-equivalences are given by quasi-isomorphisms. Using the above adjunction, it can be transferred onto the category of curved $\mathcal{C}$-coalgebras. Under certain hypothesis on $\alpha$, this Quillen adjunction becomes a Quillen equivalence.

1.2. Curved algebras over curved cooperads. There is a dual Schur realization functor which is given by

$$\hat{\mathcal{F}}^c : (p)\text{dg } S\text{-mod} \to \text{End}((p)\text{dg mod})$$

$$\hat{\mathcal{F}}^c(M)(-):=\prod_{n\geq 0} \text{Hom}_{S_n}(M(n), (-)^{\otimes n}).$$

Lemma 1.11 ([GL22, Corollary 3.4]). The dual Schur realization functor $\hat{\mathcal{F}}^c(-)$ can be endowed with a lax monoidal structure, that is, there exists a natural transformation $\varphi_{M,N} : \hat{\mathcal{F}}^c(M) \circ \hat{\mathcal{F}}^c(N) \to \hat{\mathcal{F}}^c(M \circ N).$

which satisfies associativity and unitality compatibility conditions with respect to the monoidal structures. Furthermore, $\varphi_{M,N}$ is a degree-wise monomorphism for all pdg $S$-modules $M, N$.

Remark 1.12. The construction of this monomorphism is explained in [RL22c, Chapter 1, Section 5].

This allows us to construct from any pdg cooperad a monad in the category of pdg modules.

Definition 1.13 (pdg $\mathcal{C}$-algebra). A pdg $\mathcal{C}$-algebra $A$ amounts to the data $(A, \gamma_A, d_A)$ of an algebra over the monad $\hat{\mathcal{F}}^c(\mathcal{C})$. This data is equivalent to the data of a map

$$\gamma_A : \prod_{n\geq 0} \text{Hom}_{S_n}(\mathcal{C}(n), A^{\otimes n}) \to A,$$

such that the following diagram commutes:

$$\hat{\mathcal{F}}^c(\mathcal{C}) \circ \hat{\mathcal{F}}^c(\mathcal{C})(A) \xrightarrow{\varphi_{\mathcal{C}, \mathcal{C}}(A)} \hat{\mathcal{F}}^c(\mathcal{C} \circ \mathcal{C})(A) \xrightarrow{\hat{\mathcal{F}}^c(\Delta)} \hat{\mathcal{F}}^c(\mathcal{C})(A).$$

Remark 1.14. The notion of an algebra over a cooperad defines a new type of algebraic structures. The reason is that the structural map

$$\gamma_A : \prod_{n\geq 0} \text{Hom}_{S_n}(\mathcal{C}(n), A^{\otimes n}) \to A$$

gives, for any infinite series of operations in $\mathcal{C}$ labeled with elements of $A$, a well-defined image in $A$. Thus algebras over a cooperad are endowed with infinite summation of structural operations by definition, without presupposing any underlying topology.

The notion of an algebra over a cooperad admits a further description in the case where the cooperad is conilpotent. Let $\mathcal{C}$ be a conilpotent pdg cooperad. Each term of the coradical filtration $\mathcal{R}_\omega \mathcal{C}$ defines a pdg sub-cooperad, and there is a short exact sequence of pdg $S$-modules

$$0 \longrightarrow \mathcal{R}_\omega \mathcal{C} \xrightarrow{\iota_\omega} \mathcal{C} \xrightarrow{\pi_\omega} \mathcal{C}/\mathcal{R}_\omega \mathcal{C} \longrightarrow 0.$$
Definition 1.15 (Canonical filtration on a pdg $C$-algebra). Let $C$ be a conilpotent pdg cooperad and let $A$ be a pdg $C$-algebra. The canonical filtration of $A$ is the decreasing filtration of given by

$$W_\omega A := \text{Im} \left( \gamma_A \circ \hat{S}^C(\pi_\omega)(\text{id}_A) : \hat{S}^C(C/\mathcal{R}_\omega C)(A) \to A \right)$$

where $\mathcal{R}_\omega C$ denotes the $\omega$-th term of the coradical filtration, for all $\omega \geq 0$. Notice that we have

$$A = W_0 A \supseteq W_1 A \supseteq W_2 A \supseteq \cdots \supseteq W_\omega A \supseteq \cdots$$

Definition 1.16 (Completion of a pdg $C$-algebra). Let $A$ be a pdg $C$-algebra. Its completion is given by

$$\hat{A} := \lim_{\omega} A/W_\omega A,$$

where the limit is taken in the category of pdg $C$-algebras.

It comes equipped with a canonical morphism of pdg $C$-algebras $\varphi_A : A \to \hat{A}$.

Proposition 1.17 ([GL22, Proposition 4.24]). Let $A$ be a pdg $C$-algebra. The canonical morphism $\varphi_A : A \to \hat{A}$ is an epimorphism.

Remark 1.18. Conceptually, this comes from the fact that any pdg $C$-algebra already carries a meaningful notion of infinite summation. Thus, ‘nothing needs to be added’ when one applies the completion functor. On the other hand, the topology induced by the canonical filtration of a pdg $C$-algebra might not be Hausdorff. Meaning that the canonical morphism $\varphi_B$ might not be a monomorphism. This completion functor should be considered a sifting functor.

Definition 1.19 (Complete pdg $C$-algebra). The pdg $C$-algebra $A$ is said to be complete if the morphism $\varphi_A$ is an isomorphism.

Example 1.20. Any free pdg $C$-algebra is complete.

Proposition 1.21. Let $C$ be a conilpotent pdg cooperad. The category of complete pdg $C$-algebras forms a reflective subcategory of the category of pdg $C$-algebras, where the reflector is given by the completion functor.

Remark 1.22. Contrary to $C$-coalgebras, which are always conilpotent, there are examples of $C$-algebras which are not complete. See the counterexample given in [GL22, Section 4.5].

Definition 1.23 (Curved $C$-algebra). Let $C$ be a curved cooperad and let $A$ be a pdg $C$-algebra. It is a curved $C$-algebra if the following diagram commutes:

$$A \cong \hat{S}^C[I](A) \xrightarrow{\hat{S}^C(\Theta_C)(\text{id})} \hat{S}^C[C](A) \xrightarrow{\gamma_A} A$$

The category of curved $C$-algebras is a full sub-category of the category of pdg $C$-algebras, and the inclusion functor admits a reflector.

Proposition 1.24 ([GL22, Theorem 7.5]). Let $C$ be a curved cooperad. The category of curved $C$-algebras is a reflective sub-category of the category of pdg $C$-algebras. It is thus presentable.
1.3. **Coalgebras over operads.** Even though that, for an operad \( \mathcal{P} \), its dual Schur functor \( \hat{\mathcal{F}}^\mathcal{P}(\cdot) \) fails to be a comonad, one can still define a notion of a coalgebra over an operad.

**Definition 1.25** (dg \( \mathcal{P} \)-coalgebra). A dg \( \mathcal{P} \)-coalgebra \( C \) amounts to the data \((C, \Delta_C, d_C)\) of a dg module \((C, d_C)\) endowed with a structural map 
\[ \Delta_C : C \to \prod_{n \geq 0} \text{Hom}_{\mathcal{S}}(\mathcal{P}(n), C^\otimes n), \]
such that the following diagram commutes
\[
\begin{array}{ccc}
C & \xrightarrow{\Delta_C} & \hat{\mathcal{F}}^\mathcal{P}(C) \\
\downarrow \Delta_C & & \downarrow \hat{\mathcal{F}}(\gamma^{\mathcal{P}}) \circ \hat{\mathcal{F}}(\mathcal{P})(C) \\
\hat{\mathcal{F}}^\mathcal{P}(C) & \xrightarrow{\hat{\mathcal{F}}(\gamma^{\mathcal{P}})} & \hat{\mathcal{F}}^\mathcal{P}(\mathcal{P} \circ \mathcal{P})(C).
\end{array}
\]

**Remark 1.26.** The data of a \( \mathcal{P} \)-coalgebra \( C \) is equivalent to the data of a morphism of dg operads \( \mathcal{P} \to \text{Coend}_C \), where \( \text{Coend}_D \) stands for the coendomorphisms operad of \( C \), given by the dg \( \mathcal{S} \)-module
\[ \text{Coend}_D(n) := \text{Hom}(C, C^\otimes n), \]
where the operad structure is given by the standard composition of morphisms.

**Theorem 1.27** ([Ane14, Theorem 2.7.11]). Let \( \mathcal{P} \) be a dg operad. The category of dg \( \mathcal{P} \)-coalgebras is comonadic. There exists a comonad \((\mathcal{C}(\mathcal{P}), \omega, \xi)\) in the category of dg modules such that the category of \( \mathcal{C}(\mathcal{P}) \)-coalgebras is equivalent to the category of dg \( \mathcal{P} \)-coalgebras.

In particular, this entails the existence of a cofree dg \( \mathcal{P} \)-coalgebra. While in the general setting of [Ane14], the construction of the comonad \( \mathcal{C}(\mathcal{P}) \) is given by an infinite recursion, the construction of \( \mathcal{C}(\mathcal{P}) \) in the category of dg modules stops at the first step.

**Theorem 1.28** ([Ane14, Theorem 3.3.1]). The endofunctor \( \mathcal{C}(\mathcal{P}) \) is given by the pullback
\[
\begin{array}{ccc}
\mathcal{C}(\mathcal{P}) & \xrightarrow{p_2} & \hat{\mathcal{F}}^\mathcal{P}(\mathcal{P}) \circ \hat{\mathcal{F}}^\mathcal{P}(\mathcal{P}) \\
p_1 \downarrow & & \downarrow \hat{\mathcal{F}}(\gamma^{\mathcal{P}}) \\
\hat{\mathcal{F}}^\mathcal{P}(\mathcal{P}) & \xrightarrow{\hat{\mathcal{F}}(\gamma)} & \hat{\mathcal{F}}^\mathcal{P}(\mathcal{P} \circ \mathcal{P})
\end{array}
\]
in the category of endofunctors. Notice that \( p_1 \) is a degree-wise monomorphism since \( \varphi_{\mathcal{P},\mathcal{P}}(V) \) is a degree-wise monomorphism. The structural map of the comonad
\[ \omega : \mathcal{C}(\mathcal{P}) \to \mathcal{C}(\mathcal{P}) \circ \mathcal{C}(\mathcal{P}) \]
is induced by the map \( p_2 \) in the previous pullback. The counit of the comonad \( \mathcal{C}(\mathcal{P}) \) is given by
\[ \xi : \mathcal{C}(\mathcal{P}) \xrightarrow{p_1} \hat{\mathcal{F}}^\mathcal{P}(\mathcal{P}) \xrightarrow{\hat{\mathcal{F}}(\eta)} V. \]

**Remark 1.29.** Let \( \mathcal{P} \) be a dg operad. For any dg module \( V \), the cofree dg \( \mathcal{P} \)-coalgebra on \( V \) is given by \( \mathcal{C}(\mathcal{P})(V) \).

**Remark 1.30.** The subspace \( \mathcal{C}(\mathcal{P})(V) \) of \( \hat{\mathcal{F}}^\mathcal{P}(\mathcal{P})(V) \) admits an explicit description in terms of representative functions. See [Ane14, Section 3.1] or [BL85] for the original reference about representative functions in the case of coassociative and cocommutative coalgebras.
1.4. Complete Bar-Cobar adjunctions and model structures. In this section, we reexplain the constructions made in [GL22, Section 8 to 11]. Beware that we use different notations that those used in loc.cit. First, we fix \( P \) a dg operad, \( \mathcal{C} \) a conilpotent curved cooperad, and \( \alpha : \mathcal{C} \to P \) a curved twisting morphism.

**Notation.** Let \( f : X \to Y \) be a map of degree 0 and \( g : X \to Y \) be a map of degree \( p \) between graded modules \( X, Y \). We denote

\[
\text{III}_n(f, g) := \sum_{i=0}^{n} f^{\otimes i-1} \otimes g \otimes f^{\otimes n-i} : X^{\otimes n} \to Y^{\otimes n}
\]

the resulting \( S_n \)-equivariant map of degree \( p \). Let \( M \) be an graded \( S \)-module. It induces a map of degree \( p \)

\[
\prod_{n \geq 0} \text{Hom}_{S_n}(M(n), X^{\otimes n}) \to \prod_{n \geq 0} \text{Hom}_{S_n}(M(n), Y^{\otimes n})
\]

by applying \( \text{Hom}(id, \text{III}_n(f, g)) \) at each arity. By a slight abuse of notation, this map will be denoted by \( \hat{\mathcal{C}}(\text{id})[\text{III}(f, g)] \).

**Definition 1.31** (Complete Bar construction relative to \( \alpha \)). Let \((A, \gamma_A, d_A)\) be a curved \( \mathcal{C}\)-algebra. The complete Bar construction relative to \( \alpha \) of \( A \) is given by

\[
\hat{B}_{\alpha}A := (\mathcal{C}(P)(A), d_{\text{bar}} := d_1 + d_2),
\]

where \( \mathcal{C}(P)(A) \) denotes the cofree graded \( P \)-coalgebra generated by \( A \). The differential \( d_{\text{bar}} \) is given by the sum of two terms \( d_1 \) and \( d_2 \). The term \( d_1 \) is given by

\[
d_1 = \mathcal{C}(d_P)(\text{id}) + \mathcal{C}(\text{id})(\text{III}(\text{id}, d_A))
\]

The term \( d_2 \) is given by the unique coderivation extending

\[
\mathcal{C}(P)(A) \xrightarrow{p_{1}(A)} \hat{\mathcal{C}}(\mathcal{C})(A) \xrightarrow{\hat{\mathcal{C}}(\alpha)\text{id}} \hat{\mathcal{C}}(\mathcal{C})(A) \xrightarrow{\gamma_A} A.
\]

**Proposition 1.32.** For any curved \( \mathcal{C} \)-algebra \( A \), the complete Bar construction \( \hat{B}_{\alpha}A \) forms a dg \( P \)-coalgebra, and it defines a functor

\[
\hat{B}_{\alpha} : \text{curv} \mathcal{C}-\text{alg} \to \text{dg} \mathcal{P}\text{-coalg}.
\]

**Definition 1.33** (Complete Cobar construction relative to \( \alpha \)). Let \((C, \delta_C, d_C)\) be a dg \( P \)-coalgebra. The complete Cobar construction relative to \( \alpha \) of \( C \) is given by

\[
\hat{\Omega}_{\alpha}C := (\hat{\mathcal{C}}(\mathcal{C})(C), d_{\text{cobar}} := d_1 - d_2),
\]

where \( \hat{\mathcal{C}}(\mathcal{C})(C) \) denotes the free complete pdg \( \mathcal{C} \)-coalgebra generated by \( C \). The differential \( d_{\text{cobar}} \) is given by the difference of two terms \( d_1 \) and \( d_2 \). The term \( d_1 \) is given by

\[
d_1 = -\hat{\mathcal{C}}(d_C)(\text{id}) + \hat{\mathcal{C}}(\text{id})(\text{III}(\text{id}, d_C))
\]

The term \( d_2 \) is given by the unique derivation extending

\[
C \xrightarrow{\Delta_C} \hat{\mathcal{C}}(P)(C) \xrightarrow{\hat{\mathcal{C}}(\alpha)\text{id}} \hat{\mathcal{C}}(\mathcal{C})(C).
\]

**Proposition 1.34.** For any dg \( P \)-coalgebra \( D \), the complete Cobar construction \( \hat{\Omega}_{\alpha}D \) forms a complete curved \( \mathcal{C} \)-algebra, and it defines a functor

\[
\hat{\Omega}_{\alpha} : \text{dg} \mathcal{P}\text{-coalg} \to \text{curv} \mathcal{C}\text{-alg}^{\text{comp}}.
\]

**Definition 1.35** (Curved twisting morphism relative to \( \alpha \)). Let \((C, \delta_C, d_C)\) be a dg \( \mathcal{P} \)-coalgebra and let \((A, \gamma_A, d_A)\) be a curved \( \mathcal{C} \)-algebra. A graded morphism

\[
\nu : C \to A
\]
is said to be a \textit{curved twisting morphism relative to $\alpha$} if it satisfies the following equation

$$\gamma_A \cdot \hat{\mathcal{C}}(\alpha)(\nu) \cdot \Delta_C + \partial(\nu) = 0.$$  

The set of curved twisting morphisms relative to $\alpha$ are denoted by $\text{Tw}^\alpha(C,A)$.

\textbf{Remark 1.36.} The set of curved twisting morphisms relative to $\alpha$ can be encoded as the set of Maurer–Cartan elements in a convolution curved absolute $\mathcal{L}_\infty$-algebra. For more on this, see [RiL22b, Section 4].

\textbf{Proposition 1.37 ([GL22, Section 8])}. \textit{There are bijections}

$$\text{Hom}_{\text{curv} \mathcal{C}-\text{alg} \text{comp}} \left( \hat{\Omega}_\alpha C, A \right) \cong \text{Tw}^\alpha(C,A) \cong \text{Hom}_{\text{dg} \mathcal{P}-\text{coalg}} \left( C, \hat{B}_\alpha A \right),$$

\textit{which are natural in $C$ and $A$.}

Therefore we get an adjunction

$$\begin{array}{c}
dg \mathcal{P}-\text{coalg} \quad \Downarrow \quad \text{curv} \mathcal{C}-\text{alg} \text{comp}
\end{array}$$

between the category of dg $\mathcal{P}$-coalgebras and the category of complete curved $\mathcal{C}$-algebras. We call this adjunction the \textit{complete Bar-Cobar adjunction} relative to $\alpha$.

In some case, one can promote this adjunction into a Quillen adjunction. But first, one needs to put a model category structure on the category of dg $\mathcal{P}$-coalgebras.

\textbf{Theorem 1.38 ([GL22, Section 9])}. \textit{Let $\mathcal{P}$ be a cofibrant dg operad. There is a model category structure on the category of dg $\mathcal{P}$-coalgebras, left-transferred along the cofree-forgetful adjunction}

$$\begin{array}{c}
dg\text{-mod} \quad \Downarrow \quad dg \mathcal{P}-\text{coalg}
\end{array}$$

\textit{where}

(1) \text{the class of weak equivalences is given by quasi-isomorphisms,}

(2) \text{the class of cofibrations is given by degree-wise monomorphisms,}

(3) \text{the class of fibrations is given by right lifting property with respect to acyclic cofibrations.}

\textbf{Remark 1.39.} The assumption that $\mathcal{P}$ is a cofibrant dg operad is mandatory. Indeed, one can show that the category of dg uCom-coalgebras does not admit a model structure where weak-equivalences are given by quasi-isomorphisms and where cofibrations are given by degree-wise monomorphisms, where uCom encodes counital cocommutative coalgebras.

Model category structures on coalgebras over dg operads behave well with respect to quasi-isomorphisms at the operadic level.

\textbf{Theorem 1.40 ([GL22, Section 9])}. \textit{Let $f : \mathcal{P} \to \mathcal{Q}$ be a quasi-isomorphism of cofibrant dg operads. The induced adjunction}

$$\begin{array}{c}
dg \mathcal{P}-\text{coalg} \quad \Downarrow \quad dg \mathcal{Q}-\text{coalg}
\end{array}$$

\textit{is a Quillen equivalence.}

When this model category structure exists, it can be transferred along the complete Bar-Cobar adjunction relative to $\alpha$.

\textbf{Theorem 1.41 ([GL22, Section 10])}. \textit{Let $\mathcal{P}$ be a cofibrant dg operad. There is a model structure on the category of complete $\mathcal{C}$-algebras right-transferred along the complete Bar-Cobar adjunction}

$$\begin{array}{c}
dg \mathcal{P}-\text{coalg} \quad \Downarrow \quad \text{curv} \mathcal{C}-\text{alg} \text{comp}
\end{array}$$
where

(1) the class of weak equivalences is given by morphisms $f$ such that $\hat{B}_\alpha(f)$ is a quasi-isomorphism,

(2) the class of fibrations is given by degree-wise epimorphisms,

(3) and the class of cofibrations is given by left lifting property with respect to acyclic fibrations.

In the case where $\mathcal{P}$ is the Cobar construction of $\mathcal{C}$ in the sense of [Gri21], this adjunction can be promoted to a Quillen equivalence.

**Theorem 1.42** ([GL22, Section 11]). The complete Bar-Cobar adjunction relative to the universal curved twisting morphism $\iota : \mathcal{C} \to \Omega \mathcal{C}$

$$
\begin{array}{c}
\mathsf{dg} \Omega \mathcal{C}\mathsf{-coalg} \\
\downarrow \hat{\Omega}_\iota \\
\mathsf{curv} \mathcal{C}\mathsf{-alg}^{\mathsf{comp}}
\end{array}
$$

is a Quillen equivalence.

**Remark 1.43.** The model category structure on the category of complete curved $\mathcal{C}$-algebras transferred using the complete Bar-Cobar adjunction relative to the curved twisting morphism $\iota : \mathcal{C} \to \Omega \mathcal{C}$ is called the **canonical model structure**.

Furthermore, Bar-Cobar adjunctions are "functorial" in the following sense:

**Proposition 1.44** ([GL22, Lemma 9.8]). Let $\mathcal{P}$ and $\mathcal{Q}$ be two cofibrant dg operads, and let $\mathcal{C}$ and $\mathcal{D}$ be conilpotent curved cooperads and

(1) let $\alpha : \mathcal{C} \to \mathcal{P}$ and $\beta : \mathcal{D} \to \Omega$ be two curved twisting morphisms,

(2) let $f : \mathcal{P} \to \mathcal{Q}$ be a morphism of dg operads and let $g : \mathcal{C} \to \mathcal{D}$ be a morphism of conilpotent curved cooperads,

such that the following diagram commutes

$$
\begin{array}{c}
\mathcal{C} \\
g \\
\mathcal{D} \\
\mathcal{P} \\
\mathcal{Q}
\end{array}
\xrightarrow{\alpha} 
\xrightarrow{f} 
\xleftarrow{\beta} 
\xrightarrow{\beta}
$$

The following square

$$
\begin{array}{c}
\mathsf{dg} \mathcal{P}\mathsf{-coalg} \\
\downarrow \hat{\Omega}_\alpha \\
\mathsf{dg} \mathcal{Q}\mathsf{-coalg} \\
\downarrow \hat{\Omega}_\beta \\
\mathsf{curv} \mathcal{C}\mathsf{-alg}^{\mathsf{comp}} \\
\downarrow \hat{B}_\alpha \\
\mathsf{curv} \mathcal{D}\mathsf{-alg}^{\mathsf{comp}} \\
\downarrow \hat{B}_\beta \\
\mathsf{Res}_f \\
\mathsf{Coind}_f
\end{array}
$$

of Quillen adjunctions commutes.

**Remark 1.45.** Any morphism $f : \mathcal{P} \to \mathcal{Q}$ induces a morphism of comonads $\mathcal{C}(f) : \mathcal{C}(\mathcal{Q}) \to \mathcal{C}(\mathcal{P})$, which in turn produces an adjunction $\text{Res}_f \dashv \text{Coind}_f$ between dg $\mathcal{P}$-coalgebras and dg $\mathcal{Q}$-coalgebras.

Any morphism $g : \mathcal{C} \to \mathcal{D}$ induces a morphism of monads $\mathcal{F}_c(f) : \mathcal{F}_c(\mathcal{D}) \to \mathcal{F}_c(\mathcal{C})$, which in turn produces an adjunction $\text{Ind}_g \dashv \text{Res}_g$ between pdg $\mathcal{C}$-algebras and pdg $\mathcal{D}$-algebras.

One can check that this adjunction can be restricted to an adjunction between complete curved $\mathcal{C}$-algebras and complete curved $\mathcal{D}$-algebras.
2. Duality squares

Let us fix a dg operad \( P \) and a conilpotent curved cooperad \( C \), together with a curved twisting morphism \( \alpha : C \rightarrow P \). The goal of this section is to construct two duality adjunctions that interrelate the "classical" Bar-\( \text{Cobar} \) constructions relative to \( \alpha \) with the complete Bar-\( \text{Cobar} \) constructions relative to \( \alpha \). This will allow us to understand the linear duality functor from a homotopy theoretical point of view.

Notation. We use \( B \) for a generic dg \( P \)-algebras, \( C \) for a generic dg \( P \)-coalgebras, \( A \) for a generic curved \( C \)-algebras and \( D \) for a generic curved \( C \)-coalgebras.

2.1. Sweedler functor. The linear dual of a coalgebra over a given operad is naturally an algebra over the same operad.

**Lemma 2.1.** The linear dual defines a functor

\[
\begin{array}{ccc}
\text{(dg } P\text{-coalg)}^\text{op} & \xrightarrow{(-)^*} & \text{dg } P\text{-alg} \\
\end{array}
\]

**Proof.** Let \((C, \Delta_C, d_C)\) be a dg \( P \)-coalgebra. For any an element \( \mu \) in \( P(n) \), the coalgebra structure produces a decomposition map

\[
\Delta_\mu : C \rightarrow C^\otimes n.
\]

By applying linear duality we get a map

\[
\gamma_\mu^* : (C^*)^\otimes n \xrightarrow{i} (C^\otimes n)^* \xrightarrow{(\Delta_\mu)^*} C^*.
\]

One can check that

\[
\begin{align*}
\mathcal{P} & \xrightarrow{\gamma_C^*} \text{End}_{C^*} \\
\mu & \xrightarrow{\gamma_\mu^*}
\end{align*}
\]

is indeed a morphism of dg operads. Therefore the linear dual of a dg \( P \)-coalgebra has a canonical dg \( P \)-algebra structure, and one checks that this association is indeed functorial with respect to morphisms of dg \( P \)-coalgebras since the morphism \( i : (C^*)^\otimes n \rightarrow (C^\otimes n)^* \) is a natural transformation.

**Proposition 2.2.** The linear duality functor

\[
\begin{array}{ccc}
\text{(dg } P\text{-coalg)}^\text{op} & \xrightarrow{(-)^*} & \text{dg } P\text{-alg} \\
\end{array}
\]

admits a left adjoint.

**Proof.** Consider the following square of functors

\[
\begin{array}{ccc}
\text{(dg } P\text{-coalg)}^\text{op} & \xrightarrow{(-)^*} & \text{dg } P\text{-alg} \\
\mathcal{S}(\mathcal{P})(-) & \xrightarrow{(-)^*} & \text{U} \\
\text{dg mod}^\text{op} & \xrightarrow{T} & \text{dg mod} \\
\end{array}
\]

where \( \mathcal{S}(\mathcal{P})(-) \) is the free dg \( P \)-algebra functor and where \( \mathcal{G}(\mathcal{P})(-) \) is the cofree dg \( P \)-coalgebra functor given by Theorem 1.28. The left hand side adjunction is monadic, since it is the opposite of a comonadic adjunction. All categories involved are complete and cocomplete. We also have that \((-)^* \cdot U^\text{op} \cong U \cdot (-)^* \). Thus we can apply the Adjoint Lifting Theorem \([Joh75, \text{Theorem 2}]\), which concludes the proof.
**Definition 2.3** (Sweedler dual). The *Sweedler duality functor*

\[
\begin{array}{c}
dg P\text{-alg} \\
\xrightarrow{(-)^*} \\
(dg P\text{-coalg})^{\text{op}}
\end{array}
\]

is defined as the functor left adjoint of the linear dual functor.

**Remark 2.4.** The proof of the Adjoint Lifting Theorem [Joh75, Theorem 2] gives an explicit construction of this left adjoint. Let \((B, \gamma_B, d_B)\) be a dg \(P\)-algebra. The Sweedler dual dg \(P\)-coalgebra \(B^o\) is given by the following equalizer:

\[
\text{Eq}
\left(\begin{array}{c}
\mathcal{C}(P)(B^*) \\
\xrightarrow{\rho} \\
\mathcal{C}(P)((\mathcal{C}(P)(B))^*)
\end{array}\right),
\]

where \(\rho\) is an arrow constructed using the comonadic structure of \(\mathcal{C}(P)\) and the canonical inclusion of a dg module into its double linear dual.

**Proposition 2.5.** There is a natural monomorphism \(\epsilon : U^{\text{op}} \cdot (-)^o \hookrightarrow (-)^* \cdot U\), which implies that the Sweedler dual is a sub-dg module of the linear dual functor.

**Proof.** Let \(V\) be a dg module, there is a monomorphism \(\epsilon_V : (\mathcal{C}(P)(V))^o \cong \mathcal{C}(P)(V^*) \xrightarrow{(-)^o V} \mathcal{C}(P)((\mathcal{C}(P)(V))^*) \xrightarrow{(-)^* V} (\mathcal{C}(P)(V))^*\), where the monomorphism \(p_1\) is given by Theorem 1.28, hence the proposition is true on free dg \(P\)-algebras. Any dg \(P\)-algebra can be written as a split coequalizer of free dg \(P\)-algebras. Both \(U^{\text{op}} \cdot (-)^o\) and \((-)^* \cdot U\) send these split coequalizers to split equalizers of dg modules, therefore the monomorphism \(\epsilon\) extends to all dg \(P\)-algebras.

**Remark 2.6** (Beck–Chevalley condition). Let \(B\) be a dg \(P\)-algebra which is of total finite dimension. Then the map \(\epsilon_B : U^{\text{op}} \cdot B^o \cong B^* \cdot U\), is an isomorphism of dg modules. Furthermore, there is a canonical structure of dg \(P\)-coalgebra on the linear dual \(B^*\). Therefore the adjunction \((-)^o \dashv (-)^*\) restricts to an anti-equivalence of categories between the category of total finite dimensional dg \(P\)-algebra and the category of total finite dimensional dg \(P\)-coalgebras. More generally, it suffices that \(B\) is degree-wise finite dimensional and bounded above or bounded below.

**Example 2.7.** Consider the operad \(Ass\) which encodes dg associative algebras as its algebras and dg coassociative coalgebras as its coalgebras. Then the adjunction

\[
\begin{array}{c}
dg Ass\text{-alg} \\
\xleftarrow{(-)^o} \\
(dg Ass\text{-coalg})^{\text{op}}
\end{array}
\]

coincides with the original Sweedler adjunction constructed in [Swe69].

**2.2. Topological dual functor.** Let’s turn to the other side of the Koszul duality, where \(\mathcal{C}\) is a conilpotent curved cooperad.

**Lemma 2.8.** The linear duality defines a functor

\[
(dg \mathcal{C}\text{-coalg})^{\text{op}} \xrightarrow{(-)^*} pdg \mathcal{C}\text{-alg}^{\text{comp}}
\]

from the category of pdg \(\mathcal{C}\)-coalgebras to the category of complete pdg \(\mathcal{C}\)-algebras.
Proof. Let \((D, \Delta_D, d_D)\) be a pdg \(C\)-coalgebra, where
\[
\Delta_D : D \longrightarrow \bigoplus_{n \geq 0} C(n) \otimes_{S_n} D^n
\]
is the structural morphism. By applying the linear duality, we get a map
\[
\gamma_{D^*} : \prod_{n \geq 0} \text{Hom}_{S_n}(C(n), (D^*)^n) \longrightarrow \prod_{n \geq 0} \text{Hom}_{S_n}(C(n), (D^\otimes n)^*) \xrightarrow{(\Delta_D)^*} D^*.
\]
One can check that it defines a pdg \(C\)-algebra structure on \(D^*\). Furthermore, let
\[
F_\omega D := \text{Ker} \left( \Delta_D^* : D \longrightarrow \bigoplus_{n \geq 0} C/\mathcal{R}_\omega C(n) \otimes_{S_n} D^n \right)
\]
be the canonical coradical filtration on \(D\) induced by the coradical filtration on \(C\). Since \(C\) is conilpotent, the coradical filtration of any dg \(C\)-coalgebra is exhaustive, therefore
\[
D \cong \colim_\omega F_\omega D,
\]
which in turn implies that
\[
D^* \cong \lim_\omega (F_\omega D)^*.
\]
One can check that \((F_\omega D)^* \cong D^*/W_\omega D^*\), therefore the image of the linear duality functor \((-)^*\) lies in the sub-category of complete pdg \(C\)-coalgebras.

Example 2.9. Lemma 2.8 provides us with a wealth of examples of absolute algebras. Indeed, it shows that every time one takes the linear dual of some type of conilpotent coalgebras, one gets some type of absolute algebras.

For instance, if one considers the linear dual of the global sections \(\mathcal{O}_G(G)^*\), where \(\mathcal{O}_G\) is the structural sheaf of a (possibly pro)-unipotent algebraic group \(G\), then one gets an absolute associative algebra with a compatible coalgebraic structure.

Proposition 2.10. The linear duality functor
\[
\text{(pdg } C\text{-coalg})^{\text{op}} \xrightarrow{(-)^*} \text{pdg } C\text{-alg}^{\text{comp}}
\]
admits a left adjoint.

Proof. We consider the following square of functors
\[
\begin{array}{ccc}
\text{pdg } C\text{-coalg}^{\text{op}} & \xrightarrow{(-)^*} & \text{pdg } C\text{-alg}^{\text{comp}} \\
\mathcal{F}(C)(-)^{\text{op}} & \xrightarrow{\mathcal{F}^2(C)(-)} & \text{pdg mod}^{\text{op}} \\
\text{pdg mod} & \xrightarrow{(-)^*} & \text{pdg mod}^{\text{op}}
\end{array}
\]
where \(\mathcal{F}(C)(-)\) is the cofree pdg \(C\)-coalgebra functor and where \(\mathcal{F}^2(C)(-)\) is the free pdg \(C\)-algebra functor, which is always complete. Again, vertical adjunctions are monadic and it is clear that \((-)^* \cdot U^{\text{op}} \cong U \cdot (-)^*\). Thus we can apply the Adjoint Lifting Theorem [Joh75, Theorem 2], which concludes the proof.

Definition 2.11 (Topological dual functor). The topological dual functor
\[
\text{pdg } C\text{-alg}^{\text{comp}} \xrightarrow{(-)^\wedge} \text{(pdg } C\text{-coalg})^{\text{op}}
\]
is defined as the functor left adjoint to the linear dual functor.

Remark 2.12. Given a complete pdg $\mathcal{C}$-algebra $(A, \gamma_A, d_A)$, its topological dual $A^\vee$ is given by the following equalizer:

$$\text{Eq} \left( \mathcal{S}(\mathcal{C})(A^\ast) \xrightarrow{\rho} \mathcal{S}(\mathcal{C}) \left( \left( \mathcal{S}(\mathcal{C})(A) \right)^\ast \right) \right),$$

where $\rho$ is an arrow constructed using the comonadic structure of $\mathcal{S}(\mathcal{C})(-)$ and the canonical inclusion of a pdg module into its double linear dual.

Proposition 2.13. There is a natural monomorphism $\epsilon : U^\text{op} \cdot (-)^\vee \hookrightarrow (-)^\ast \cdot U$, which implies that the topological dual is a sub-pdg module of the linear dual functor.

Proof. If the proposition is true on free complete pdg $\mathcal{C}$-algebras, then it is true for any pdg $\mathcal{C}$-algebra. Indeed, any pdg $\mathcal{C}$-algebra can be written as a split coequalizer of free pdg $\mathcal{C}$-algebras. Both composites $U^\text{op} \cdot (-)^\vee$ and $(-)^\ast \cdot U$ send these split coequalizers to split equalizers of pdg modules, therefore the monomorphism $\epsilon$ extends to all pdg $\mathcal{C}$-algebras.

Therefore it amounts to show that for any pdg module $V$, the following map is a monomorphism

$$\epsilon_V : \left( \mathcal{T}(\mathcal{C})(V) \right)^\vee \cong \mathcal{S}(\mathcal{C})(V^\ast) \longrightarrow \left( \mathcal{T}(\mathcal{C})(V) \right)^\ast .$$

The above map is given by the following composite

$$\mathcal{S}(\mathcal{C})(V^\ast)$$
$$\downarrow \eta_{\mathcal{S}(\mathcal{C})(V^\ast)}$$
$$\left( \text{Hom} \left( \bigoplus_{n \geq 0} \mathcal{C}(n) \otimes_{S_n} (V^\ast)^\otimes n, \mathbb{k} \right) \right)^\ast$$
$$\equiv$$
$$\left( \prod_{n \geq 0} \text{Hom}_{S_n} \left( \mathcal{C}(n), (V^\ast)^\otimes n \right) \right)^\ast$$
$$\equiv$$
$$\left( \prod_{n \geq 0} \text{Hom}_{S_n} \left( \mathcal{C}(n), V^\otimes n \right) \right)^\ast$$
$$\equiv$$
$$\left( \mathcal{T}(\mathcal{C})(V) \right)^\ast,$$

where $\eta$ is the inclusion of a pdg module into its double dual. When $V$ is of total finite dimension (or degree-wise finite dimensional and bounded above or bounded below), $\epsilon_V$ coincides with $\eta$ and therefore is clearly a monomorphism. Let us show this without any finiteness condition.

First, we suppose that the conilpotent curved cooperad $\mathcal{C}$ is free as a graded $S$-module, meaning there is an isomorphism $\mathcal{C}(n) \cong \mathcal{C}^0(n) \otimes \mathbb{k}[S_n]$ of graded $S_n$-modules for all $n \geq 0$. 
Let us choose a basis of \( \{ c_n^\alpha \}_{\alpha \in I(n)} \) of \( \mathcal{C}^g(n) \) as a graded \( \mathbb{R} \)-module and let us choose a basis \( \{ f^\beta \}_{\beta \in J} \) for \( V^\ast \) as a graded \( k \)-module. The morphism \( \epsilon_V \) is determined by sending

\[
\epsilon_V : \bigoplus_{n \geq 0} \mathcal{C}^g(n) \otimes (V^\ast)^{\otimes n} \rightarrow \left( \prod_{n \geq 0} \operatorname{Hom}(\mathcal{C}^g(n), V^{\otimes n}) \right)^{\ast}
\]

\[
c_n^\alpha \otimes (f_1^{b_1} \otimes \cdots \otimes f_n^{b_n}) \rightarrow \left[ \epsilon_{V,(c_n^\alpha,f_1^{b_1},\ldots,f_n^{b_n})} : \sum_{n \geq 0} g_n \mapsto (f_1^{b_1} \otimes \cdots \otimes f_n^{b_n})(g_n(c_n^\alpha)) \right],
\]

where \( g_n \) is an element in \( \operatorname{Hom}(\mathcal{C}^g(n), V^{\otimes n}) \). Let us show that the kernel of \( \epsilon_V \) is trivial and therefore that it is injective. Let us consider a non-trivial element

\[
e = \sum_{n \geq 0} \sum_{\alpha \in I(n)} \sum_{(\beta_1,\ldots,\beta_n) \in J^n} \lambda(\alpha,\beta_1,\ldots,\beta_n)c_n^\alpha \otimes s_n(f_1^{\beta_1} \otimes \cdots \otimes f_n^{\beta_n}).
\]

Since \( e \) is non-zero, there exists a some \( n \geq 0 \) and \( (a,b_1,\ldots,b_n) \in I(n) \times J^n \) such that \( \lambda(a,b_1,\ldots,b_n) \neq 0 \). Since \( f_i^{b_i} \) are non-zero, there exists an element \( v_1 \) in \( V \) such that \( f_i^{b_i}(v_1) = 1 \), for all \( 1 \leq i \leq n \). Let us consider the morphism of graded \( k \)-modules given by

\[
\mathbb{1}_{(c_n^\alpha,v_1,\ldots,v_n)} : \mathcal{C}^g(n) \rightarrow V^{\otimes n}
\]

\[
c_n^\alpha \mapsto (v_1 \otimes \cdots \otimes v_n),
\]

and which sends any \( c_n^\alpha \) to zero if \( \alpha \neq a \). Then

\[
\epsilon_V(e)(\mathbb{1}_{(c_n^\alpha,v_1,\ldots,v_n)}) = \epsilon_{V,(c_n^\alpha,f_1^{b_1},\ldots,f_n^{b_n})}(\mathbb{1}_{(c_n^\alpha,v_1,\ldots,v_n)}) = (f_1^{b_1} \otimes \cdots \otimes f_n^{b_n})(v_1 \otimes \cdots \otimes v_n) = 1,
\]

therefore \( \epsilon_V(e) \neq 0 \) and the morphism \( \epsilon_V \) is injective.

Now let \( \mathcal{C} \) be any conilpotent curved cooperad. Over a characteristic zero field, there exists a free graded \( S \)-module \( M \) together with a monomorphism

\[
\phi_{\mathcal{C}} : \mathcal{C} \rightarrow M,
\]

since every graded \( S \)-module is arity-wise projective. The following square

\[
\mathcal{J}(\mathcal{C})(V^\ast) \rightarrow \left( \mathcal{J}^\mathcal{C}(\mathcal{C})(V) \right)^{\ast}
\]

\[
\mathcal{J}(\phi_{\mathcal{C}})(\operatorname{id}) \downarrow \quad \quad \quad \quad \quad \quad \downarrow \left( \mathcal{J}^\mathcal{C}(\phi_{\mathcal{C}})(V) \right)^{\ast}
\]

\[
\mathcal{J}(M)(V^\ast) \rightarrow \left( \mathcal{J}^\mathcal{C}(M)(V) \right)^{\ast}
\]

is commutative, therefore the top horizontal map is also a monomorphism, which concludes the proof.

\[\Box\]

**Proposition 2.14.** The adjunction

\[
\operatorname{pdg} \mathcal{C}\text{-alg}^{\text{comp}} \leftrightarrow \left( \operatorname{pdg} \mathcal{C}\text{-coalg} \right)^{\text{op}},
\]

restricts to an adjunction

\[
\operatorname{curv} \mathcal{C}\text{-alg}^{\text{comp}} \leftrightarrow \left( \operatorname{curv} \mathcal{C}\text{-coalg} \right)^{\text{op}}.
\]
Proof. Denote $\Theta_C : C \rightarrow I$ the curvature of $C$. Let $(D, \Delta_D, d_D)$ be a pdg $C$-coalgebra. Recall that it is curved if the following diagram commutes

$$
\begin{array}{ccc}
D & \xrightarrow{\Delta_D} & \mathcal{J}(C)(D) \\
& \downarrow{d_D^2} & \downarrow{(\Theta_C)(\text{id})} \\
D & \cong & \mathcal{J}(I)(D) \\
\end{array}
$$

Therefore we have

$$
\begin{array}{ccc}
\mathcal{F}^C(C)(D^*) & \xrightarrow{(\mathcal{J}(C)(D))^*} & (\mathcal{J}(C)(D))^* \\
& \downarrow{(\Delta_D)^*} & \downarrow{(d_D^2)^*} \\
& \mathcal{F}^C(I)(D^*) & \cong (\mathcal{J}(I)(D))^* \cong D^* \\
\end{array}
$$

The left square commutes by naturality of the inclusion. The right triangle commutes since it is the image of a commutative triangle by the functor $(-)^*$. Thus the big diagram commutes. Finally, one can observe that $(d_D^2)^* = -d_D^2$. Therefore $D^*$ is indeed a complete curved $C$-algebra.

Let $(A, \gamma_A, d_A)$ be a complete curved $C$-coalgebra. Consider the following diagram

$$
\begin{array}{ccc}
(\mathcal{F}^C(C)(A))^* & \xrightarrow{(\gamma_A)^*} & A^* \\
& \downarrow{(\mathcal{J}(\Theta_C)(\text{id}))^*} & \downarrow{(-d_A^2)^*} \\
& (\mathcal{F}^C(I)(A))^* \cong A^* & \\
\mathcal{J}(C)(A^\vee) & \xrightarrow{\Delta_A^\vee} & A^\vee \\
& \downarrow{\mathcal{J}(\Theta_C)(\text{id}_{A^\vee})} & \downarrow{d_A^2} \\
\mathcal{J}(I)(A^\vee) & \cong A^\vee, & \\
\end{array}
$$

in the category of pdg modules, where the vertical arrows are given by Proposition 2.13. The top triangle commutes since it is the image of a commuting triangle via the functor $(-)^*$. Each of the vertical faces commutes as well, where $d_A^2$ is simply given by $(d_A)^\vee \circ (d_A)^\vee$. Since every vertical map is a monomorphism, the bottom triangle also commutes and thus $A^\vee$ is indeed a curved $C$-coalgebra. 

Remark 2.15. As far as we know, the category of curved $C$-coalgebras might not be comonadic. Thus in order to prove Proposition 2.14, one cannot use the same arguments as in the proof of Proposition 2.10.

2.3. The algebraic duality square. Using the two duality adjunctions constructed so far, one can interrelate the “standard” Bar-Cobar adjunction with the complete Bar-Cobar adjunction in a square of commuting adjunctions.
Theorem 2.16 (Duality square). The square of adjunctions

\[
\begin{array}{ccc}
\text{(dg P-alg)}^{op} & \xrightarrow{B^{op}_\alpha} & \text{curv C-coalg}^{op} \\
\downarrow_{\Omega^{op}_\alpha} & & \downarrow \\
\text{dg P-coalg} & \xleftarrow{\hat{\Omega}_\alpha} & \text{curv C-alg}^{comp}
\end{array}
\]

commutes in the following sense: right adjoints going from the top right to the bottom left are naturally isomorphic.

Proof. Let \( V \) be a graded module. There is an isomorphism

\[
(\mathcal{S}(\mathcal{P})(V))^\circ \cong \mathcal{C}(\mathcal{P})(V^*)
\]

of graded \( \mathcal{P} \)-coalgebras. Let \((D, \Delta_D, d_D)\) be a curved \( \mathcal{C} \)-coalgebra, one can check by direct inspection that the above isomorphism commutes with the differential and therefore gives an isomorphism

\[
(\Omega_{\alpha}D)^\circ \cong \hat{B}_{\alpha}(D^*)
\]

of dg \( \mathcal{P} \)-coalgebras. This isomorphism is natural in \( D \). \( \square \)

Remark 2.17. Let \( C \) be a dg \( \mathcal{P} \)-coalgebra. Then we have an isomorphism

\[
B_{\alpha}C^* \cong (\hat{\Omega}_{\alpha}C)^{\vee}
\]

of dg \( \mathcal{C} \)-coalgebras which is natural in \( C \), which is given by the mate of the isomorphism constructed in the proof of Theorem 2.16.

Proposition 2.18. There is a natural monomorphism

\[
\zeta : \hat{\Omega}_{\alpha} \cdot (-)^\circ \rightarrow (-)^* \cdot B_{\alpha}
\]

of complete curved \( \mathcal{C} \)-algebras.

Proof. This monomorphism is build using the monomorphisms constructed in Proposition 2.5 and Proposition 2.13. \( \square \)

Proposition 2.19. Let \( B \) be a dg \( \mathcal{P} \)-algebra degree-wise finite dimensional and bounded above or bounded below. There is an isomorphism

\[
\hat{\Omega}_{\alpha}B^* \cong (B_{\alpha}B)^*
\]

of complete curved \( \mathcal{C} \)-algebras.

Proof. Let \( V \) be a graded module degree-wise finite dimensional and bounded above or bounded below. There is an isomorphism

\[
(\mathcal{S}(\mathcal{C})(V))^* \cong \mathcal{S}(\mathcal{C})(V^*)
\]

of complete graded \( \mathcal{C} \)-algebras. Now let \( B \) be a dg \( \mathcal{P} \)-algebra degree-wise finite dimensional and bounded above or bounded below, one can check that the above isomorphism commutes with the respective pre-differentials. \( \square \)

Remark 2.20. The subcategory of dg \( \mathcal{P} \)-algebras which satisfy the Beck-Chevalley condition with respect to the duality square of Theorem 2.16 contains the subcategory of degree-wise finite dimensional and bounded above or bounded below dg \( \mathcal{P} \)-algebras.
2.4. **Homotopical duality square.** The duality square of Theorem 2.16 behaves well with respect to model structures. We now restrict to the case where \( \mathcal{P} = \Omega \mathcal{C} \) and where the curved twisting morphism considered is the canonical curved twisting morphism \( \iota : \mathcal{C} \to \Omega \mathcal{C} \), in order to ensure the existence of a model category structure on the category of dg \( \mathcal{P} \)-coalgebras.

**Lemma 2.21.** The adjunction
\[
\begin{array}{ccc}
\text{dg } \Omega \mathcal{C}\text{-coalg} & \xleftarrow{(-)^*} & \text{dg } \Omega \mathcal{C}\text{-alg}^\text{op} \\
\xrightarrow{(-)^*} & & \xrightarrow{(-)^*} \\
\Omega_\mathcal{P}^\text{op} & \xlongleftarrow{\Omega_\mathcal{P}} & \text{curv } \mathcal{C}\text{-coalg}^\text{op} \\
\text{B}_\mathcal{P}^\text{op} & \xlongleftarrow{\Omega_\mathcal{P}} & \text{curv } \mathcal{C}\text{-alg}^\text{comp}
\end{array}
\]
is a Quillen adjunction.

**Proof.** The left adjoint \(( - )^*\) sends degree-wise monomorphisms to degree-wise epimorphisms. Thus it preserves cofibrations. It also preserves quasi-isomorphisms. Therefore we have a Quillen adjunction. \(\square\)

**Theorem 2.22** (Homotopical properties of the duality square). All the adjunctions in the square
\[
\begin{array}{ccc}
\text{dg } \Omega \mathcal{C}\text{-alg}^\text{op} & \xleftarrow{(-)^*} & \text{curv } \mathcal{C}\text{-coalg}^\text{op} \\
\xrightarrow{(-)^*} & & \xrightarrow{(-)^*} \\
\text{B}_\mathcal{P} & \xlongleftarrow{\Omega_\mathcal{P}} & \text{curv } \mathcal{C}\text{-alg}^\text{comp} \\
\text{B}_\mathcal{P} & \xlongleftarrow{\Omega_\mathcal{P}} & \text{curv } \mathcal{C}\text{-alg}^\text{comp}
\end{array}
\]
are Quillen adjunctions.

**Proof.** The only thing left to check is that the adjunction
\[
\begin{array}{ccc}
\text{curv } \mathcal{C}\text{-alg}^\text{comp} & \xrightarrow{(-)^*} & \text{curv } \mathcal{C}\text{-coalg}^\text{op} \\
\xleftarrow{(-)^*} & & \xleftarrow{(-)^*} \\
\text{B}_\mathcal{P} & \xlongleftarrow{\Omega_\mathcal{P}} & \text{curv } \mathcal{C}\text{-alg}^\text{comp} \\
\text{B}_\mathcal{P} & \xlongleftarrow{\Omega_\mathcal{P}} & \text{curv } \mathcal{C}\text{-alg}^\text{comp}
\end{array}
\]
is indeed a Quillen adjunction, where the model structure considered on curved \( \mathcal{C} \)-coalgebras is obtained by transfer along the adjunction \( \Omega_\mathcal{C} \dashv \text{B}_\mathcal{C} \) and where the model structure considered on the category of complete curved \( \mathcal{C} \)-algebras is obtained by transfer along the adjunction \( \hat{\Omega}_\mathcal{C} \dashv \hat{\text{B}}_\mathcal{C} \).

Let us check that \(( - )^*\) is a right Quillen functor. It sends monomorphisms to epimorphisms, thus preserves fibrations. Since every curved \( \mathcal{C} \)-coalgebra is cofibrant (fibrant in the opposite category), we are left to show that \(( - )^*\) preserves weak equivalences of curved \( \mathcal{C} \)-coalgebras. Let
\[
f : D_1 \xRightarrow{\sim} D_2
\]
be a weak equivalence of curved \( \mathcal{C} \)-coalgebras, that is,
\[
\Omega_\mathcal{C}(f) : \Omega_\mathcal{C}D_1 \Rightarrow \Omega_\mathcal{C}D_2
\]
is a quasi-isomorphism of dg \( \Omega \mathcal{C} \)-algebras, by Lemma 2.21, we know that the Sweedler dual functor \(( - )^\vee\) is a right Quillen functor. Therefore it preserves weak-equivalences between fibrant objects (i.e: quasi-isomorphisms between cofibrant dg \( \Omega \mathcal{C} \)-algebras). Thus
\[
(\Omega_\mathcal{C}(f))^\vee : (\Omega_\mathcal{C}D_1)^\vee \Rightarrow (\Omega_\mathcal{C}D_2)^\vee
\]
is a quasi-isomorphism of dg \( \Omega \mathcal{C} \)-coalgebras. Using the commutativity of the square, we get that
\[
\hat{\text{B}}_\mathcal{C}(f^\ast) : \hat{\text{B}}\mathcal{C}D_1^\ast \Rightarrow \hat{\text{B}}\mathcal{C}D_2^\ast
\]
is also a quasi-isomorphism of dg \( \Omega \mathcal{C} \)-coalgebras. Therefore \( f^\ast : D_1^\ast \to D_2^\ast \) is a weak equivalence in the model category of complete curved \( \mathcal{C} \)-algebras. \(\square\)
Remark 2.23. There are many examples where one ends up considering the linear dual of a Bar construction \((B_\alpha(-))^\ast\). The above square shows that this construction lands naturally in a category of absolute algebras.

For instance, taking the linear dual of the Bar construction is crucial in the study of formal moduli problems. J. Lurie’s original approach [Lur11] involves considering the linear dual of the Chevalley-Eilenberg complex. The linear dual of a Bar construction also plays a crucial role in the approach of D. Calaque, R. Campos and J. Nuiten [CCN22] to formal moduli problems over algebras over a dg operad. Using the above duality square to study formal moduli problems will be the subject of a future work.

The above duality square allows us to state a precise comparison theorem between the homotopy theory of dg algebras over a cofibrant operad and the homotopy theory of dg coalgebras over the same cofibrant operad.

Lemma 2.24. Let \(B\) be a dg \(\Omega\mathcal{C}\)-algebra whose homology is degree-wise finite dimensional and bounded above or bounded below. Then the derived unit of adjunction

\[
\mathcal{R}(\eta_B) : B \rightarrow (\mathcal{R}(B^\circ))^\ast
\]

is a quasi-isomorphism. Therefore the functor \((-)^\circ\) is homotopically fully faithful on the full sub-\(\infty\)-category of degree-wise finite dimensional bounded above dg \(\Omega\mathcal{C}\)-algebras and on the full sub-\(\infty\)-category of degree-wise finite dimensional bounded below dg \(\Omega\mathcal{C}\)-algebras.

Proof. Let \(B\) be a dg \(\Omega\mathcal{C}\)-algebra such that its homology is degree-wise finite dimensional and bounded above. The bounded below case is analogous. Since \(k\) is a field of characteristic zero, using the Homotopy Transfer Theorem, one can replace \(B\) by its homology \(H_* (B)\) in order to compute this derived unit of adjunction. Indeed, there is a direct quasi-isomorphism

\[
\Omega(i_\ast) : \Omega_i B_\ast H_\ast (B) \rightarrow \Omega_i B_i B
\]

of dg \(\Omega\mathcal{C}\)-algebras between their cofibrant resolutions given by the Bar-Cobar adjunction relative to \(i\). Here \(i_\ast : B_\ast H_\ast (B) \rightarrow B_i B\) is the \(\infty\)-quasi-isomorphism given by the choice of a homotopy contraction between \(B\) and \(H_\ast (B)\). Therefore we obtain the following commutative square

\[
\begin{array}{ccc}
\Omega_i B_\ast H_\ast (B) & \xrightarrow{\Omega(i_\ast)} & \Omega_i B_i B \\
\downarrow \mathcal{R}(\eta_{H_\ast (B)}) & & \downarrow \mathcal{R}(\eta_B) \\
((\Omega_i B_\ast H_\ast (B))^\circ)^\ast & \xrightarrow{((\Omega(i_\ast))^\circ)^\ast} & ((\Omega_i B_i B)^\circ)^\ast,
\end{array}
\]

Using 2 out of 3, we conclude that \(\mathcal{R}(\eta_B)\) is a quasi-isomorphism of dg \(\Omega\mathcal{C}\)-algebras if and only if \(\mathcal{R}(\eta_{H_\ast (B)})\) is. We compute that

\[
(\Omega_i B_\ast H_\ast (B))^\circ \cong \widehat{B}_i (B_\ast H_\ast (B))^\ast \cong \widehat{B}_i \Omega_i (H_\ast (B))^\ast,
\]

first using the commutativity of the square of Theorem 2.22 and then by applying Proposition 2.19 since \(H_\ast (B)\) is degree-wise finite dimensional and bounded above. And the unit of the complete Bar-Cobar adjunction \(\Omega_i \dashv B_i\)

\[
\eta_{(H_\ast (B))^\ast} : (H_\ast (B))^\ast \rightarrow \widehat{B}_i \Omega_i (H_\ast (B))^\ast
\]

is a quasi-isomorphism of dg \(\Omega\mathcal{C}\)-coalgebras since this adjunction is a Quillen equivalence. We can apply the linear dual functor to this quasi-isomorphism, resulting in

\[
H_\ast (B) \cong (H_\ast (B))^\ast \rightarrow (\widehat{B}_i \Omega_i (H_\ast (B))^\ast)^\ast = (\mathcal{R}(H_\ast (B)^\circ))^\ast,
\]
using again that \( H_*(-(B)) \) is of degree-wise finite dimensional and bounded above. Using 2 out of 3, we conclude that

\[
\mathbb{R}(\eta_{H_*(-(B))}) : H_*(B) \rightarrow (\mathbb{R}(H_*(B)))^*
\]

must also be a quasi-isomorphism of dg \( \Omega \mathcal{C} \)-algebras.

**Theorem 2.25.** There is an equivalence of \( \infty \)-categories

\[
dg \Omega \mathcal{C} \text{-coalg}^{f.d.\pm} \xrightarrow{\sim} \text{dg} \Omega \mathcal{C} \text{-alg}^{f.d.\pm},
\]

between the \( \infty \)-category of \( \text{dg} \Omega \mathcal{C} \)-algebras with degree-wise finite dimensional and bounded below (resp. bounded above) homology and the \( \infty \)-category of \( \Omega \mathcal{C} \)-coalgebras with degree-wise finite dimensional and bounded above (resp. bounded below) homology.

**Proof.** By Lemma 2.24, we know that the Sweedler duality functor \((-)^0\) is homotopically fully faithful restricted to \( \text{dg} \Omega \mathcal{C} \)-algebras with degree-wise finite dimensional and bounded below (resp. bounded above) homology. Using the Homotopy Transfer Theorem for \( \text{dg} \Omega \mathcal{C} \)-coalgebras, one can reproduce the same arguments in order to show that the linear duality functor \( (-)^* \) is homotopically fully faithful restricted to \( \text{dg} \Omega \mathcal{C} \)-coalgebras with degree-wise finite dimensional and bounded above (resp. bounded below) homology. It is immediate to see that the adjunction \((-)^* \dashv (-)^0\) sends \( \text{dg} \Omega \mathcal{C} \)-coalgebras with degree-wise finite dimensional and bounded above (resp. bounded below) homology to \( \text{dg} \Omega \mathcal{C} \)-algebra with degree-wise finite dimensional and bounded below (resp. bounded above) homology and vice-versa. \( \square \)

**Remark 2.26.** Since any cofibrant dg operad \( \mathcal{P} \) is deformation retract of a dg operad of the form \( \Omega \mathcal{C} \), therefore Theorem 2.25 also hold for any cofibrant dg operad.

**Remark 2.27.** The above result plays a major role in the approach to rational homotopy theory and derived algebraic geometry developed in [RiL22b].

### 3. Examples: Contramodules and Absolute Associative Algebras

The goal of this section is to illustrate the new notion of an absolute algebras through various examples. First, we show that the theory of contramodules is a particular case of the theory of absolute algebras. Then, we explore the notion of absolute associative algebras and absolute Lie algebras and explain how well-known constructions are instances of these structures.

#### 3.1. Contramodules

The notion of a contramodule over a dg counital coassociative coalgebra appears naturally as the dual definition of a comodule. This notion was introduced in the seminal work of S. Eilenberg and J. C. Moore in [EM65]. After being almost forgotten for a number of years, this notion reemerged in the work of L. Positseski, see [Pos21] for an extensive account. In this subsection, we show that contramodules appear as algebras over cooperads concentrated in arity one. This allows us to subsume the theory of contramodules, obtaining many examples of interest of "absolute algebras". To the best of our knowledge, the duality square stated in the previous section provides is also a new result in the context of contramodules.

**Definition 3.1 (Contramodule).** Let \((C, \Delta, \epsilon, d_C)\) be a dg counital coassociative coalgebra. A dg \( C \)-contramodule \( M \) is the data \((M, \gamma_M, d_M)\) of a dg module \((M, d_M)\) together with a morphism of dg modules

\[
\gamma_M : \text{Hom}(C, M) \longrightarrow M,
\]
such that the following diagrams commute

\[
\begin{align*}
\text{Hom}(C \otimes C, M) & \xrightarrow{\Delta^*} \text{Hom}(C, M) \\
\text{Hom}(C, \text{Hom}(C, M)) & \xrightarrow{\gamma_M} \text{Hom}(C, M)
\end{align*}
\]

\[
\begin{align*}
M \cong \text{Hom}(k, M) & \xrightarrow{e^*} \text{Hom}(C, M) \\
\text{Hom}(C, M) & \xrightarrow{\gamma_M} M.
\end{align*}
\]

Remark 3.2. If \((C, \Delta, \epsilon, d_C)\) is a dg counital coassociative coalgebra, one can show that the endofunctor

\[\text{Hom}(C, -) : \text{dg mod} \rightarrow \text{dg mod}\]

admits a monad structure such that dg \(C\)-contramodules are algebras over this monad. In fact, it admits \emph{two} monad structures, depending on the choice of \(\sigma\) in the above definition. Indeed, there are two isomorphisms

\[\text{Hom}(C \otimes C, M) \cong \text{Hom}(C, \text{Hom}(C, M)),\]

depending on the choice of either the left or the right factor in \(C \otimes C\). One should speak of \emph{left} or \emph{right} \(C\)-contramodules in each of these cases. We omit these subtleties when possible.

Example 3.3. Let \(C\) be a dg counital coassociative coalgebra and \(V\) be a dg module. The free dg \(C\)-contramodule on \(V\) is given by \(\text{Hom}(C, V)\) with the obvious structure maps.

Example 3.4. Let \(C\) be a dg counital coassociative coalgebra and let \(M\) be a right dg \(C\)-comodule. For any dg module \(V\), the dg module \(\text{Hom}(V, D)\) inherits a canonical left dg \(C\)-contramodule structure.

Proposition 3.5.

1. The data of a dg counital coassociative coalgebra \((C, \Delta, \epsilon, d_C)\) is equivalent to the data of a dg cooperad \((\mathcal{C}, \Delta_C, \epsilon, d_C)\) whose dg \(S\)-module \(\mathcal{C}\) is concentrated in arity one.

2. In the case above, the definition of a dg \(C\)-contramodule is equivalent to the definition of a dg \(\mathcal{C}\)-algebra.

Proof. It is a straightforward computation from the definitions. \qed

This equivalence is a great source of examples for algebras over cooperads. Moreover, it is also a great source of counter-examples.

Example 3.6 (Contramodules over formal power series). Consider the cofree conilpotent coalgebra \(k[t]^c\) cogenerated by a single element \(t\) of degree zero. In this case, one can see that the data of \(k[t]^c\)-contramodule structure on \(M\) is equivalent to the data of a structural map

\[\gamma_M : k[[t]] \otimes M \cong \text{Hom}(k[t]^c, M) \rightarrow M.\]

which satisfies unital and associativity conditions with respect to the algebra of formal power series in one variable \(k[[t]]\). Therefore, for any formal power series

\[\sum_{n \geq 0} m_n \otimes t^n \text{ in } k[[t]] \otimes M,\]

the structural morphism \(\gamma_M\) gives a well-defined image \(\gamma_M \left( \sum_{n \geq 0} m_n \otimes t^n \right)\) in \(M\), without presupposing any topology on \(M\).
Let \( C \) be a dg conilpotent counital coassociative coalgebra and let \( \mathcal{C} \) be its coaugmentation ideal. Any dg \( C \)-contramodule \( M \) admits a canonical decreasing filtration given by
\[
W_\omega M := \text{Im} \left( \gamma^{\geq \omega} M : \text{Hom} \left( \mathcal{C}/\mathcal{R}_\omega \mathcal{C}, M \right) \to M \right),
\]
where \( \mathcal{R}_\omega \mathcal{C} \) denotes the \( \omega \)-stage of the coradical filtration. One says that \( M \) is complete if the canonical morphism of dg \( C \)-contramodules
\[
\varphi_M : M \to \lim_\omega M/W_\omega M
\]
is an isomorphism. Again, this is a particular case of the more general notion of canonical filtration for algebras over conilpotent cooperads given in Definition 1.15.

**Counter-example 3.7.** Consider again the cofree conilpotent coalgebra \( \mathbb{k}[t]^c \) cogenerated by a single element \( t \) of degree zero. There exists a \( \mathbb{k}[t]^c \)-contramodule \( M \) and a family of elements \( \{m_n\}_{n \geq 0} \) in \( M \) such that
\[
\gamma^*_M (m_n \otimes t^n) = 0 \quad \text{for all} \quad n \geq 0 \quad \text{and} \quad \gamma^*_M \left( \sum_{n \geq 0} m_n \otimes t^n \right) \neq 0.
\]
In particular, the topology induced by the canonical filtration on \( M \) is not complete. See [Pos21, Section 1.5] for this construction.

Let \( A \) be a dg unital associative algebra (viewed as a dg operad concentrated in arity one), and let \( C \) be a dg conilpotent counital coassociative algebra (viewed as a conilpotent dg cooperad concentrated in arity one). Let \( \alpha : C \to A \) be a twisting morphism between them. It induces a first adjunction
\[
\begin{array}{ccc}
\text{dg } C\text{-comod}^1 & \xleftarrow{\perp} & \text{dg } A\text{-mod}^1 \\
\alpha \otimes - & \downarrow & \alpha \otimes - \\
C \otimes - & \xrightarrow{\perp} & \text{dg } A\text{-mod}^1
\end{array}
\]
between the category of left dg \( C \)-comodules and the category of right dg \( A \)-modules. Furthermore, the cofree right dg \( A \)-module functor is given by \( \text{Hom}(A, -) \).

**Remark 3.10.** When \( A \) is the group algebra \( \mathbb{k}[G] \) for some group \( G \), the functor \( \text{Hom}(\mathbb{k}[G], -) \) is usually called the coinduced representation.
between the category of right dg C-contramodules and the category of right dg A-modules, where \( \text{Hom}^\alpha(\cdot, \cdot) \) denotes the twisted hom space, see [LV12, Section 2.1] for an account of this twisted hom space. Once again, one can check that this adjunction corresponds to the complete Bar-Cobar adjunction relative to \( \alpha \). When we endow the category of right dg A-module with the injective model structure and the category of right dg C-contramodules with the transferred model structure, this becomes a Quillen adjunction.

The two structures defined above are compatible under the following duality square.

**Proposition 3.11.** The following square diagram

\[
\begin{array}{ccc}
\left( \text{dg A-mod}^1 \right)^{\text{op}} & \xrightarrow{\text{C} \otimes_{\text{A}} \cdot} & \left( \text{dg C-comod}^1 \right)^{\text{op}} \\
\downarrow & & \downarrow \\
\text{dg A-mod}^r & \xleftarrow{\text{Hom}^\alpha(\cdot, \cdot)} & \text{dg C-contra}^r \\
\end{array}
\]

is commutative and is made of Quillen adjunctions.

**Proof.** This is a direct consequence of Theorem 2.22. □

One can view the adjunction

\[
\left( \text{dg C-comod}^1 \right)^{\text{op}} \xleftarrow{\text{Hom}^\alpha(\cdot, \cdot)} \text{dg A-mod}^r \xrightarrow{\text{C} \otimes_{\text{A}} \cdot} \left( \text{dg C-contra}^r \right)
\]

as a contravariant version of the co-contra correspondence constructed of [Pos11]. Indeed, this adjunction identifies finitely generated cofree C-comodules and finitely generated C-contramodules: let \( M \) be a dg module which is degree-wise finite dimensional and bounded above or below, we have that

\[
(C \otimes M)^* \cong \text{Hom}(C, M^*) \quad \text{and} \quad (\text{Hom}(C, M))^\vee \cong C \otimes M.
\]

Furthermore, when both of these categories are endowed with the canonical model structure transferred using the twisting morphism \( \iota : C \rightarrow \Omega C \), then this correspondence becomes a correspondence between the finitely generated fibrant-cofibrant objects of each of these categories. Therefore it gives a contravariant version of the derived co-contra correspondence, see again [Pos11] for more on this.

**Remark 3.12.** The data of a curved cooperad concentrated in arity one is equivalent to the data of a curved counital coassociative coalgebra. This subsection can be generalized \textit{mutatis mutandis} to the curved setting.

### 3.2 Absolute associative algebras and absolute Lie algebras.

In this subsection, we explore the notion of algebras over the conilpotent cooperad \( \text{Ass}^* \), which we call \textbf{absolute associative algebras}. We compare this category with the standard notion of non-unital associative algebras, encoded by the operad \( \text{Ass} \). Furthermore, we also introduce \textbf{absolute Lie algebras}, encoded by the cooperad \( \text{Lie}^* \), and construct the universal enveloping absolute algebra functor. These two examples provide supplementary intuition on the notion of an algebra over a cooperad, in one of its simplest cases. Many of the subsequent results can be generalized, \textit{mutatis mutandis}, to any well-behaved binary cooperad.

**Definition 3.13** (dg absolute associative algebra). A \textbf{dg absolute associative algebra} \((A, \gamma_A, d_A)\) is the data of a \textit{dg Ass}*-algebra.
Let us unravel this definition. Recall that a dg\textsuperscript{Ass}*-algebra \((A, \gamma_A, d_A)\) is the data of a dg module \((A, d_A)\) together with a structural morphism of dg modules
\[
\gamma_A : \prod_{n \geq 0} \text{Hom}_{S_n}(\text{Ass}^*(n), A^\otimes n) \rightarrow A,
\]
which satisfies the axioms of Definition 1.13.

**Lemma 3.14.** There is an isomorphism of dg modules
\[
\prod_{n \geq 0} \text{Hom}_{S_n}(\text{Ass}^*(n), A^\otimes n) \cong \prod_{n \geq 1} A^\otimes n.
\]

**Proof.** The \(S_n\)-module \(\text{Ass}^*(n)\) is given by the regular representation of \(S_n\), for \(n \geq 1\), and by 0, when \(n = 0\). \(\square\)

**Remark 3.15.** This implies that there is a monad structure on the endofunctor
\[
\prod_{n \geq 1} (-)^\otimes n : \text{dg mod} \rightarrow \text{dg mod},
\]
such that a dg absolute associative algebra is the data of an algebra over this monad.

The structural map \(\gamma_A\) of an dg absolute associative algebra \(A\) is equivalently given by a morphism of dg modules
\[
\gamma_A : \prod_{n \geq 1} A^\otimes n \rightarrow A.
\]

It associates to any series
\[
\sum_{n \geq 1} \sum_{i \in I_n} a_1^{(i)} \otimes \cdots \otimes a_n^{(i)},
\]
where the \(a_j\) are elements in \(A\) and where \(I_n\) is a finite set, a well-defined element
\[
\gamma_A \left( \sum_{n \geq 1} \sum_{i \in I_n} a_1^{(i)} \otimes \cdots \otimes a_n^{(i)} \right) \text{ in } A,
\]
without presupposing any topology on the dg module \(A\).

**Remark 3.16.** For a general dg absolute associative algebra \((A, \gamma_A, d_A)\), notice that
\[
\gamma_A \left( \sum_{n \geq 1} \sum_{i \in I_n} a_1^{(i)} \otimes \cdots \otimes a_n^{(i)} \right) \neq \sum_{n \geq 1} \sum_{i \in I_n} \gamma_A \left( a_1^{(i)} \otimes \cdots \otimes a_n^{(i)} \right),
\]
as the latter sum is not even well-defined in \(A\).

**Differential condition:** The condition that \(\gamma_A\) is a morphism of dg modules can be rewritten as
\[
\gamma_A \left( \sum_{n \geq 1} \sum_{i \in I_n} (-1)^j a_1^{(i)} \otimes \cdots \otimes d_A \left( a_j^{(i)} \otimes \cdots \otimes a_n^{(i)} \right) \right) = d_A \left( \sum_{n \geq 1} \sum_{i \in I_n} \gamma_A \left( a_1^{(i)} \otimes \cdots \otimes a_n^{(i)} \right) \right),
\]
for any series in \(\prod_{n \geq 1} A^\otimes n\).

**Associativity condition:** The condition that the structural morphism \(\gamma_A\) defines an algebra over the monad \(\prod_{n \geq 1} (-)^\otimes n\) can be rewritten as
\[
\gamma_A \left( \sum_{k \geq 1} \gamma_A \left( \sum_{i_{j_{1}} \geq 1} \sum_{j_{1} \in i_{1}^{(k)}} a^{(1,j_{1})}_{i_{1}} \otimes \cdots \otimes a^{(1,j_{1})}_{i_{1}} \right) \otimes \cdots \otimes \gamma_A \left( \sum_{i_{k} \geq 1} \sum_{j_{k} \in i_{k}^{(k)}} a^{(k,j_{k})}_{i_{k}} \otimes \cdots \otimes a^{(k,j_{k})}_{i_{k}} \right) \right)
\]

\[
= \gamma_A \left( \sum_{n \geq 1} \sum_{k \geq 1} \sum_{i_{1} \geq 1} \cdots \sum_{i_{k} \in i_{k}^{(k)}} \sum_{j_{1} \in i_{1}^{(1)}}, \cdots, j_{k} \in i_{k}^{(k)} \right) a^{(1,j_{1})}_{i_{1}} \otimes \cdots \otimes a^{(1,j_{1})}_{i_{1}} \otimes \cdots \otimes a^{(k,j_{k})}_{i_{k}} \otimes \cdots \otimes a^{(k,j_{k})}_{i_{k}} .
\]

**Example 3.17.** Let \((V, d_V)\) be a dg module. The free dg absolute associative algebra on \(V\) is given by

\[
\mathcal{F}^\wedge(V) := \prod_{n \geq 1} V^{\otimes n},
\]

that is, the completed non-unital tensor algebra on \(V\). If \(V\) is a \(k\)-vector space of dimension \(n\), this algebra is the algebra of non-commutative formal power series in \(n\) variables without constant terms.

**Example 3.18 (Convolution absolute associative algebra).** Let \((D, \Delta, d_D)\) be a non-counital conilpotent dg coassociative coalgebra and let \((B, \mu, d_B)\) denote a non-unital dg associative algebra. It is well-known that the dg module of graded morphisms

\[
\text{hom}(D, B, \partial)
\]

admits a convolution dg associative algebra structure, where the product of \(f : D \rightarrow B\) and \(g : D \rightarrow B\) is given by

\[
f \ast g := \begin{array}{c}
D \xrightarrow{\Delta} D \otimes D \xrightarrow{f \otimes g} B \otimes B \xrightarrow{\mu} B
\end{array}.
\]

This convolution product can be extended into an dg absolute associative algebra structure: let \(\mu^n\) (resp. \(\Delta^n\)) denote the \(n\)-iterated product (resp. coproduct), we define

\[
\gamma_{\text{hom}(D,B)} := \begin{array}{c}
\prod_{n \geq 1} \text{hom}(D, B)^{\otimes n} \xrightarrow{\mu^n} \text{hom}(D, B)
\end{array}.
\]

The latter infinite sum is a well-defined linear map since, for any element \(d\) in \(D\), the sum is finite. Indeed, \(D\) is conilpotent, and therefore the image of \(d\) under iterated coproducts is eventually zero. One can check that \(\gamma_{\text{hom}(D,B)}\) satisfies the axioms of a dg absolute associative algebra.

Convolution algebras appear as an essential part of the Koszul duality for associative algebras defined by S. Priddy in [Pri70] since they encode twisting morphisms. For the same reason, they play a role in defining twisted tensor products as done by E. Brown in [Bro59]. They also appear in Quantum Algebra, see [Kas95].

Let us now compare absolute associative algebras with associative algebras.

**Lemma 3.19.** The inclusion

\[
\iota : \bigoplus_{n \geq 1} (-)^{\otimes n} \rightarrow \prod_{n \geq 1} (-)^{\otimes n}
\]

defines a morphism of monads in the category of dg modules.

**Proof.** It is straightforward to check. \(\square\)
Proposition 3.20. There is an adjunction
\[ \text{dg abs assoc-alg} \xleftrightarrow{\text{Res}} \text{dg assoc-alg}, \]
between the category of dg absolute associative algebras and the category of dg associative algebras, where the left adjoint \( \text{Abs} \) is called the absolute envelope functor.

Proof. This follows from Lemma 3.19. □

Let us describe these functors. Let \((A, \gamma_A, d_A)\) be a dg absolute associative algebra. Its restriction algebra \((\text{Res}(A), \text{Res}(\gamma_A), d_A)\) is given by the dg module \((A, d_A)\) together with the structural map
\[ \text{Res}(\gamma_A) : \bigoplus_{n \geq 1} A \otimes n \xrightarrow{l_A} \prod_{n \geq 1} A \otimes n \xrightarrow{\gamma_A} A, \]
which endows \(\text{Res}(A)\) with a dg associative algebra structure. Let \((B, \gamma_B, d_B)\) be a dg associative algebra, its absolute envelope \(\text{Abs}(B)\) is given by the following coequalizer
\[ \text{Coeq} \left( \prod_{n \geq 1} \left( \bigoplus_{k \geq 1} B \otimes k \right) \otimes n \xrightarrow{\psi_B} \prod_{n \geq 1} B \otimes n \right), \]
in the category of dg modules, where \(\psi_B\) is given by
\[ \prod_{n \geq 1} \left( \bigoplus_{k \geq 1} B \otimes k \right) \otimes n \xrightarrow{\gamma_B} \prod_{n \geq 1} B \otimes n. \]

Completeness: Any dg absolute associative algebra \((A, \gamma_A, d_A)\) comes equipped with a canonical decreasing filtration given by
\[ W_\omega A := \text{Im} \left( \gamma_A^{\geq \omega} : \prod_{n \geq \omega + 1} A \otimes n \rightarrow A \right). \]
This filtration is the same filtration defined in Definition 1.15 since the cooperad \(\text{Ass}^*\) is a binary cooperad. A dg absolute associative algebra \((A, \gamma_A, d_A)\) is therefore said to be complete if the canonical epimorphism
\[ \varphi_A : A \rightarrow \lim_{\omega} A/W_\omega A \]
is an isomorphism of dg absolute associative algebras.

Remark 3.21. Let \((A, \gamma_A, d_A)\) be a complete dg absolute associative algebra, then
\[ \gamma_A \left( \sum_{n \geq 1} \sum_{i \in I_n} a_1^{(i)} \otimes \cdots \otimes a_n^{(i)} \right) = \sum_{n \geq 1} \sum_{i \in I_n} \gamma_A \left( a_1^{(i)} \otimes \cdots \otimes a_n^{(i)} \right), \]
using the completeness of \(A\).

Similarly, when \((B, \gamma_B, d_B)\) is a dg associative algebra, one has a canonical decreasing filtration as well given by
\[ F_\omega B := \text{Im} \left( \gamma_B^{\geq \omega} : \bigoplus_{n \geq \omega + 1} B \otimes n \rightarrow B \right). \]
A dg associative algebra \((B, \gamma_B, d_B)\) is said to be complete if the canonical morphism
\[ \lambda_B : B \rightarrow \lim_{\omega} B/F_\omega B \]
is an isomorphism of dg associative algebras. Let us try to compare these two filtrations.
Definition 3.22 (Nilpotent dg absolute associative algebra). A dg absolute associative algebra \((A, \gamma_A, d_A)\) is said to be nilpotent if there exists \(\omega_0 \geq 1\) such that \(W_{\omega_0}A = 0\). The integer \(\omega_0\) is called the nilpotency degree.

Remark 3.23. A nilpotent dg absolute associative algebra is in particular complete.

Definition 3.24 (Nilpotent dg associative algebra). A dg associative algebra \((B, \gamma_B, d_B)\) is said to be nilpotent if there exists \(\omega_0 \geq 1\) such that \(F_{\omega_0}B = 0\). The integer \(\omega_0\) is called the nilpotency degree.

Remark 3.25. This definition coincides with the standard notion of nilpotency for associative algebras present in the literature.

Proposition 3.26. The data of a nilpotent dg absolute associative algebra is equivalent to the data of a nilpotent dg associative algebra with the same nilpotency degree.

Proof. Let \(\omega_0 \geq 1\). A nilpotent dg absolute associative algebra \((A, \gamma_A, d_A)\) with nilpotency degree \(\omega_0\) amounts to the data of an algebra over the monad \(\prod_{n=1}^{\omega_0+1} (-) \otimes n\). On the other hand, a nilpotent dg associative algebra \((B, \gamma_B, d_B)\) amounts to the data of an algebra over the monad \(\bigoplus_{n=1}^{\omega_0+1} (-) \otimes n\). There is an isomorphism of monads
\[
\prod_{n=1}^{\omega_0+1} (-) \otimes n \cong \bigoplus_{n=1}^{\omega_0+1} (-) \otimes n,
\]
since the product only involves a finite amount of terms. \(\square\)

Remark 3.27. The image of a nilpotent dg associative algebra \((B, \gamma_B, d_B)\) via the absolute envelope functor \(\text{Abs}\) is simply given by \((B, \gamma_B, d_B)\) considered as a dg absolute associative algebra. Indeed, we have a natural isomorphism
\[
\text{Hom}_{\text{dg abs assoc}}(B, -) \cong \text{Hom}_{\text{assoc alg}}(B, \text{Res}(-)),
\]
hence, by Yoneda lemma, \(\text{Abs}(B) \cong B\).

Proposition 3.28. Let \((A, \gamma_A, d_A)\) be a dg absolute associative algebra. There is a monomorphism of dg modules
\[
F_\omega \text{Res}(A) \hookrightarrow W_\omega A.
\]
Therefore, if \(A\) is a complete dg absolute associative algebra, the topology induced by the canonical filtration on \((\text{Res}(A), \text{Res}(\gamma_A), d_A)\) is separated.

Proof. It is straightforward to notice that \(F_\omega \text{Res}(A)\) consists of finite sums of products of arity greater than \(\omega + 1\), hence it is included in \(W_\omega A\). For the second point, notice that if \((A, \gamma_A, d_A)\) is complete, then
\[
\bigcap_{\omega \geq 0} F_\omega \text{Res}(A) \hookrightarrow \bigcap_{\omega \geq 0} W_\omega A = \{0\},
\]
hence the canonical morphism of dg associative algebras
\[
\lambda_{\text{Res}(A)} : \text{Res}(A) \longrightarrow \lim_{\omega} (\text{Res}(A)/F_\omega \text{Res}(A))
\]
is a monomorphism and the topology induced by \(F_\omega \text{Res}(A)\) is separated. \(\square\)

Counter-example 3.29. Suppose \((A, \gamma_A, d_A)\) is a complete dg absolute associative algebra, then \(\text{Res}(A)\) might not be complete as a dg associative algebra. This is for instance the case of the free absolute associative algebra on a dg module \((V, d_V)\); its restriction algebra \(\text{Res}(\prod_{n \geq 1} V^{\otimes n})\) is not complete as a dg associative algebra. Indeed, one can check that the canonical morphism \(\lambda_{\prod_{n \geq 1} V^{\otimes n}}\) is not an epimorphism in this case.
The above results can be generalized *mutatis mutandis* to algebras over a binary cooperad $C$ with homogeneous corelations. Particular examples of these include: $\text{Com}^*$, which encodes dg absolute commutative algebras, and $\text{Lie}^*$, which encodes dg absolute Lie algebras.

**A duality square for dg associative algebras.** Let us describe explicitly a duality square one can construct in order to relate dg associative algebras with dg absolute associative algebras.

In the following, we consider dg associative algebras endowed with the standard model structure where weak-equivalences are given by quasi-isomorphisms and fibrations by epimorphisms. We endow conilpotent dg coassociative coalgebras with the transferred model structure via the Bar-Cobar adjunction constructed by K. Lefèvre-Hasegawa in [LH03]. We endow *non-necessarily conilpotent* dg coassociative coalgebras with the model structure constructed by E. Getzler and P. Goerss in [GG99], where weak-equivalences are given by quasi-isomorphisms and cofibrations by monomorphisms. This latter structure can be obtained directly using the coassociative coalgebra structure on the interval object of dg modules and the theory developed in [HKRS17]. Finally, this structure can be transferred via the complete Bar-Cobar adjunction onto dg absolute associative algebras, as it was shown in [GL22].

**Proposition 3.31.** The square of Quillen adjunctions

$$
\begin{array}{ccc}
(dg \text{ assoc-alg})^{op} & \xleftarrow{\Omega^{op}} & (dg \text{ coassoc-coalg}^{conil})^{op} \\
\downarrow & \downarrow & \downarrow \\
(dg \text{ coassoc-coalg}) & \xleftarrow{\hat{B}} & (dg \text{ abs assoc-alg}^{comp})
\end{array}
$$

commutes in the following sense: right adjoints going from the top right to the bottom left are naturally isomorphic.

**Proof.** The existence of this commuting square of adjunctions follows directly from Theorem 2.16 applied to the twisting morphism $\kappa : \text{Ass}^s \longrightarrow \text{Ass}$.

The fact that these adjunctions are Quillen adjunction can be shown using the same arguments as in the proof of Theorem 2.22 applied to this case. □

**Remark 3.32.** This duality square intertwines the classical Bar construction of [CE56] and the classical Cobar construction of [Ada56], and the Sweedler dual functor constructed in [Swe69] with the theory of dg absolute algebras. The first two have played a seminal role in Algebra and in Algebraic Topology since their introduction. The Sweedler dual functor plays an important role in the theory of Hopf algebras. Notice that it was not known to be a right Quillen functor before. Also notice that this square shows that the linear dual of the classical Bar construction for dg associative algebras is naturally a dg absolute associative algebra.

**Remark 3.33.** It was pointed to us by L. Positselski that the complete Bar construction only depends on the underlying associative algebra structure of the absolute associative algebra, as the twisting morphism $\kappa : s\text{Ass}^s \longrightarrow \text{Ass}^s$ only sees the binary product. It therefore coincides with the Bar construction of M. Anel and A. Joyal of [AJ13]. Although this complete Bar-Cobar construction is not known to be a Quillen equivalence, it was used by A. Guan and A. Lazarev in [GL21] to establish Koszul duality type of equivalences between exotic derived categories of dg $A$ modules and dg $\hat{B}(A)$-comodules.

**Remark 3.34.** An analogue duality square can be constructed by replacing dg (co)associative (co)algebras with $A_\infty$-(co)algebras on the left hand side of the duality square. In this case, the complete Bar-Cobar adjunction is known to be a Quillen equivalence.
Universal enveloping absolute algebra functor. As with operads, a morphism of cooperads induces an adjunction between their respective categories of algebras. This allows us to construct the universal enveloping absolute algebra of a dg absolute Lie algebra.

**Definition 3.35** (dg absolute Lie algebra). A *dg absolute Lie algebra* \((\mathfrak{g}, \gamma_\mathfrak{g}, d_\mathfrak{g})\) is the data of a dg \(\mathcal{L}ie^*\)-algebra.

There is a morphism of operads \(\text{Skew} : \mathcal{L}ie \rightarrow \text{Ass}\) given by the skew-symmetrization of the associative product. By taking the linear dual \(\text{Skew}^* : \text{Ass}^* \rightarrow \mathcal{L}ie^*\), we obtain a morphism of conilpotent cooperads.

**Proposition 3.36.** There is an adjunction

\[
\begin{array}{ccc}
\text{dg abs Lie-alg} & \overset{\text{\(\hat{U}\)}}{\longrightarrow} & \text{dg abs assoc-alg},
\end{array}
\]

between the category of dg absolute Lie algebras and the category of dg absolute associative algebras. The left adjoint functor \(\hat{U}\) is called the universal enveloping absolute algebra functor.

**Proof.** The morphism \(\text{Skew}^* : \text{Ass}^* \rightarrow \mathcal{L}ie^*\) induces a natural transformation

\[
\hat{\mathcal{T}} \circ (\text{Skew}^*) : \hat{\mathcal{T}}(\mathcal{L}ie^*) \rightarrow \hat{\mathcal{T}}(\text{Ass}^*)
\]

between the corresponding monads encoding dg absolute associative algebras and dg absolute Lie algebras. \(\square\)

**Remark 3.37.** Let \((\mathfrak{g}, \gamma_\mathfrak{g}, d_\mathfrak{g})\) be a dg absolute Lie algebra. The universal enveloping absolute algebra \(\hat{\mathcal{U}}(\mathfrak{g})\) of \(\mathfrak{g}\) is given by

\[
\text{Coeq}
\left(\hat{\mathcal{T}}^\wedge \left(\hat{\mathcal{L}}\mathcal{L}ie(\mathfrak{g})\right) \overset{\psi_\mathfrak{g}}{\longrightarrow} \hat{\mathcal{T}}^\wedge (\mathfrak{g})\right),
\]

where \(\hat{\mathcal{L}}\mathcal{L}ie(\mathfrak{g})\) denotes the free completed Lie algebra on \(\mathfrak{g}\) and where \(\psi_\mathfrak{g}\) is given by the composition

\[
\psi_\mathfrak{g} : \hat{\mathcal{T}}^\wedge (\hat{\mathcal{L}}\mathcal{L}ie(\mathfrak{g})) \xrightarrow{\hat{\mathcal{T}}^\wedge (\text{Skew}^*)} \hat{\mathcal{T}}^\wedge (\gamma_\mathfrak{g}) \xrightarrow{\gamma (\mathfrak{g})} \hat{\mathcal{T}}^\wedge (\mathfrak{g})\]

Hence \(\hat{\mathcal{U}}(\mathfrak{g})\) is quotient of the completed tensor algebra \(\hat{\mathcal{T}}^\wedge (\mathfrak{g})\) on \(\mathfrak{g}\) where one not only identifies \(x \otimes y - (-1)^{|x||y|} y \otimes x\) with the Lie bracket \([x, y]\), but this identification has also to be done for all formal power series of brackets of elements of \(\mathfrak{g}\).

**Example 3.38** (Convolution absolute Lie algebra). Recall the convolution absolute algebra structure on \(\text{hom}(D, B)\) constructed in Example 3.18, where \(D\) is a conilpotent dg coassociative coalgebra and \(B\) a dg associative algebra. Its skew-symmetrization is naturally a dg absolute Lie algebra, which encodes twisting morphisms between \(D\) and \(B\) as its Maurer-Cartan elements.

**Proposition 3.39.** Let \((\mathfrak{g}, \gamma_\mathfrak{g}, d_\mathfrak{g})\) be a nilpotent dg absolute Lie algebra. There is an isomorphism

\[
\hat{\mathcal{U}}(\mathfrak{g}) \cong \frac{\hat{\mathcal{T}}^\wedge (\mathfrak{g})}{(x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y])}
\]

of dg absolute associative algebras.
Proof. The following square of adjunctions

\[
\begin{array}{ccc}
\text{dg assoc-alg} & \xrightarrow{\text{Abs}} & \text{dg abs assoc-alg} \\
\downarrow & & \downarrow \\
\text{dg Lie-alg} & \xrightarrow{\text{Abs}} & \text{dg abs Lie-alg}
\end{array}
\]

commutes, where \(\mathcal{U}\) is the classical universal enveloping algebra functor. Indeed, one easily checks that

\[
\text{Res} \cdot \text{Skew} \cong \text{Skew} \cdot \text{Res}.
\]

This implies that \(\hat{\mathcal{U}}(g) \cong \text{Ab}\left(\frac{\mathcal{F}(g)}{(x \otimes y - (-1)^{|x||y|} y \otimes x - [x,y])}\right) \cong \text{F}^\wedge(\hat{g})\).

□

4. Applications to Absolute Lie Theory

In this section, we give a generalization of one the main results obtained by R. Campos, D. Petersen, D. Robert-Nicoud and F. Wierstra in [CPRNW23], by rephrasing and reinterpreting their constructions with the language of algebras over cooperads, and by using the formalism developed so far. We now consider the dg operads \(\Omega \text{Lie}^*\) and \(\Omega \text{Ass}^*\) which encode, respectively, shifted \(C_\infty\) and shifted \(A_\infty\) algebras (and coalgebras). From now on, the adjective shifted will be implicit.

**Lemma 4.1.** There is a morphism of dg operads

\[
\varphi : \Omega \text{Ass}^* \longrightarrow \Omega \text{Lie}^*,
\]

given by \(\varphi \equiv \Omega(\text{Skew}^*)\).

Proof. It is straightforward from the definition. □

**Proposition 4.2.** The morphism of dg operads \(\varphi : \Omega \text{Ass}^* \longrightarrow \Omega \text{Lie}^*\) induces a Quillen adjunction

\[
\begin{array}{ccc}
\mathcal{C}_\infty\text{-coalg} & \xleftarrow{\text{Res}_\varphi} & \mathcal{A}_\infty\text{-coalg} \\
\downarrow & & \downarrow \\
\mathcal{C}_\infty\text{-coalg} & \xrightarrow{\text{Coind}_\varphi} & \mathcal{A}_\infty\text{-coalg}
\end{array}
\]

between the model category of \(\mathcal{C}_\infty\)-coalgebras and the model category of \(\mathcal{A}_\infty\)-coalgebras.

Proof. Since both dg operads \(\Omega \text{Ass}^*\) and \(\Omega \text{Lie}^*\) are cofibrant, their respective categories of dg coalgebras admit a left-transferred model structure where weak equivalences are given by quasi-isomorphisms and where cofibrations are given by degree-wise monomorphisms, by Theorem 1.38. Both categories are comonadic by Theorem 1.28, and the morphism \(\varphi\) induces a morphism between their respective comonads. Thus it induces an adjunction. Finally, it is straightforward to check that \(\text{Res}_\varphi\) preserves cofibrations and quasi-isomorphisms, since it does not change the underlying chain complex of the \(\mathcal{C}_\infty\)-coalgebra. □

**Theorem 4.3** ([CPRNW23, Theorem 4.27]). Let \(C_1\) and \(C_2\) be two \(\mathcal{C}_\infty\)-coalgebras. There exists a zig-zag of quasi-isomorphisms of \(\mathcal{C}_\infty\)-coalgebras

\[
C_1 \xleftarrow{\cdots} \xrightarrow{\cdots} C_2
\]

if and only if there exists a zig-zag of \(\mathcal{A}_\infty\)-coalgebras

\[
\text{Res}_\varphi(C_1) \xleftarrow{\cdots} \xrightarrow{\cdots} \text{Res}_\varphi(C_2).
\]
We now use the compatibility of the complete Bar-Cobar adjunctions of Proposition 1.44 in order to state a stronger version of [CPRNW23, Theorem B].

**Proposition 4.4.** There is a commuting square

\[
\begin{array}{ccc}
A_\infty\text{-coalg} & \xrightarrow{\hat{\Omega}_i} & \text{dg abs assoc-} \text{alg}^{\text{comp}} \\
\uparrow & & \uparrow \\
\text{Res}_\varphi \swarrow & & \downarrow \text{Coind}_\varphi \\
\mathcal{C}_\infty\text{-coalg} & \xrightarrow{\hat{\Omega}_i} & \text{dg abs Lie-} \text{alg}^{\text{comp}}
\end{array}
\]

of Quillen adjunctions.

**Proof.** It is immediate to check that the square

\[
\begin{array}{ccc}
\text{Ass}^* & \xrightarrow{\iota} & \Omega\text{Ass}^* \\
\downarrow \text{Skew}^* & \Downarrow & \Downarrow \varphi \\
\text{Lie}^* & \xrightarrow{\iota} & \Omega\text{Lie}^*
\end{array}
\]

is commutative. The result follows from Proposition 1.44. \qed

**Theorem 4.5.** Let \( g \) and \( h \) be two dg absolute Lie algebras. There exists a zig-zag of weak equivalences of dg absolute Lie algebras

\[
g \xleftarrow{\cdot} \rightarrow h
\]

if and only if there exists a zig-zag of weak equivalences of dg absolute associative algebras

\[
\hat{\Omega}(g) \xleftarrow{\cdot} \rightarrow \hat{\Omega}(h)
\]

**Proof.** It is a direct consequence of Theorem 4.3, using the fact that the horizontal Quillen adjunctions of Proposition 4.4 are Quillen equivalences. \qed

Let us make more explicit what these weak equivalences of dg absolute associative algebras or dg absolute Lie algebras look like. We state the analogue of [Val20, Proposition 2.5] for the model structure on algebras over a conilpotent dg cooperad transferred along the complete Bar-Cobar adjunction.

**Proposition 4.6.** Let \( f : g \rightarrow h \) be a weak-equivalence of complete dg absolute Lie algebras. It is in particular a quasi-isomorphism.

**Proof.** First notice that the dg operad \( \Omega\text{Lie}^* \) is augmented, that is, there is a morphism of dg operads \( \varepsilon : \Omega\text{Lie}^* \rightarrow I \). Now we consider the following commutative square

\[
\begin{array}{ccc}
\text{Lie}^* & \xrightarrow{\iota} & \Omega\text{Lie}^* \\
\downarrow \text{id}_{\text{Lie}^*} & & \downarrow \varepsilon \\
\text{Lie}^* & \xrightarrow{\varepsilon \circ \iota} & I
\end{array}
\]

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where $\varepsilon \cdot \iota$ is the push forward of the twisting morphism $\iota$ by $\varepsilon$. It induces a commutative square of Quillen adjunctions

\[
\begin{array}{ccc}
\mathcal{C}_\infty\text{-coalg} & \xrightarrow{\hat{\Omega}_\iota} & \text{dg abs Lie-alg}^{\text{comp}} \\
\text{Res}_\iota \downarrow & & \downarrow \text{Id} \\
\text{dg-mod} & \xleftarrow{\hat{\Omega}_{\iota \cdot \iota}} & \text{dg abs Lie-alg}^{\text{comp}} \\
\text{Coind}_\iota \downarrow & & \downarrow \text{Id}
\end{array}
\]

One can easily see that the bottom adjunction is the free-forgetful adjunction. The category of dg modules is equivalent to the category of coalgebras over the trivial operad $I$, thus we can transfer its model category structure along the free-forgetful adjunction using the results of [GL22, Section 10] to obtain a Quillen adjunction, where the model category structure on dg absolute Lie algebras has quasi-isomorphisms as weak-equivalences. Since the identity functor

\[
\text{Id} : (\text{dg abs Lie-alg}^{\text{comp}}, \mathcal{W}) \longrightarrow (\text{dg abs Lie-alg}^{\text{comp}}, \text{Q.iso})
\]

is a right Quillen functor and since every object is fibrant, it sends weak-equivalences of dg absolute Lie algebras to quasi-isomorphisms. Therefore any weak-equivalence of dg absolute Lie algebras is a quasi-isomorphism. \(\square\)

**Remark 4.7.** The above proposition can be generalized *mutatis mutandis* to the case of any twisting morphism $\alpha : \mathcal{P} \longrightarrow \mathcal{C}$ between a reduced cofibrant dg operad $\mathcal{P}$ and a conilpotent dg cooperad $\mathcal{C}$. In particular, this also holds for weak-equivalences of dg absolute associative algebras.

**Remark 4.8.** Model category structures on algebras over a conilpotent dg cooperads behave in an analogous way to what happens with model category structures on coalgebras over conilpotent dg cooperads, as described in [DCH16].

**Theorem 4.5** is stronger than [CPRNW23, Theorem B]. Indeed, the first thing to notice is that the $\text{Abs} \dashv \text{Res}$ adjunction between dg Lie algebras and dg absolute Lie algebras

\[
\begin{array}{ccc}
\text{dg abs Lie-alg} & \xrightarrow{\text{Res}} & \text{dg Lie-alg} \\
\text{Abs} & \xleftarrow{\iota} &
\end{array}
\]

is a Quillen adjunction since $\text{Res}$ preserves fibrations and, by Proposition 4.6, it also preserves weak-equivalences. A dg Lie algebra $\mathfrak{g}$ is *homotopy complete* precisely when the derived unit of adjunction

\[
\mathbb{L}(\eta) : \mathfrak{g} \xrightarrow{\iota} \text{Res} \mathbb{L}\text{Abs}(\mathfrak{g})
\]

is a quasi-isomorphism. See [HH13] for more details on homotopy completeness.

Let $\mathfrak{g}, \mathfrak{h}$ be two cofibrant dg Lie algebras such that $\mathfrak{U}(\mathfrak{g}) \simeq \mathfrak{U}(\mathfrak{h})$. Then $\mathfrak{U}(\mathfrak{g}) \simeq \mathfrak{U}(\mathfrak{h})$, since they are cofibrant. Now, using the commutativity of the square of adjunctions in the proof of Proposition 3.39 together with Theorem 4.5, it follows that $\text{Abs}(\mathfrak{g})$ and $\text{Abs}(\mathfrak{h})$ are weakly-equivalent as dg absolute Lie algebras. So, in particular, when $\mathfrak{g}, \mathfrak{h}$ are also homotopy complete, it follows that they are quasi-isomorphic.

**Theorem 4.9.** Let $\mathfrak{g}$ and $\mathfrak{h}$ be two complete graded absolute Lie algebras. They are isomorphic as complete graded absolute Lie algebras if and only if their universal enveloping absolute algebras are isomorphic.
Proof. This follows from Theorems 4.5 and Proposition 4.6, considering the fact that any weak-equivalence is in particular a quasi-isomorphism, and that any quasi-isomorphism between graded modules with zero differential is an isomorphism. □

Remark 4.10. Any nilpotent graded Lie algebra is a complete graded absolute Lie algebra by the analogue of Proposition 3.26 for absolute Lie algebras. Therefore this result is a generalization of [CPRNW23, Corollary 0.12].

This approach also allows us to generalize the above theorems to the universal enveloping absolute $A_{\infty}$-algebra of an absolute $L_{\infty}$-algebra. We change our shifting conventions, considering this time unshifted $C_{\infty}$ or $A_{\infty}$-coalgebras, and therefore shifting absolute $L_{\infty}$-algebras and absolute $A_{\infty}$-algebras.

Theorem 4.11. Let $g$ and $h$ be two absolute $L_{\infty}$-algebras. There exists a zig-zag of weak equivalences of absolute $L_{\infty}$-algebras

$$g \leftrightarrow \cdot \rightarrow h$$

if and only if there exists a zig-zag of weak equivalences of absolute $A_{\infty}$-algebras

$$\hat{U}_{\infty}(g) \leftrightarrow \cdot \rightarrow \hat{U}_{\infty}(h).$$

Proof. We consider the dg operad $\Omega B\text{Com}$. Since it is a cofibrant resolution for the operad $\text{Com}$, there exists a quasi-isomorphism of dg operads $f : \Omega B\text{Com} \rightarrow \Omega s\text{Lie}^\ast$. Therefore there is a Quillen equivalence between dg $\Omega B\text{Com}$-coalgebras and $C_{\infty}$-coalgebras. Likewise, there is a Quillen equivalence between dg $\Omega B\text{Ass}$-coalgebras and $A_{\infty}$-coalgebras. We consider the adjunction

$$\text{dg } \Omega B\text{Com-coalg} \xleftarrow{\text{Res}_\rho} \downarrow \text{Coind}_\rho \xrightarrow{\text{dg } \Omega B\text{Ass-coalg}}$$

induced by the morphism of dg operads $\rho : \Omega B\text{Ass} \rightarrow \Omega B\text{Com}$. Two dg $\Omega B\text{Com}$-coalgebras $C_1$ and $C_2$ are linked by a zig-zag of quasi-isomorphisms if and only if $\text{Res}_\rho(C_1)$ and $\text{Res}_\rho(C_2)$ are linked by a zig-zag of quasi-isomorphisms of dg $\Omega B\text{Ass}$-coalgebras. Using the commutative square

$$\text{dg } \Omega B\text{Ass-coalg} \xleftarrow{\text{Res}_\rho} \downarrow \text{Coind}_\rho \xrightarrow{\text{abs } A_{\infty}\text{-alg}^{\text{comp}}}$$

and the fact that the horizontal adjunction are Quillen equivalences concludes the proof. □

Proposition 4.12. Let $f : g \rightarrow h$ be a weak equivalence of complete absolute $L_{\infty}$-algebras. It is in particular a quasi-isomorphism.

Proof. The arguments are the same as in the proof of Theorem 4.6 apply in this situation. □

Definition 4.13 (Minimal absolute $L_{\infty}$-algebra). Let $(g, \gamma, d_g)$ be an absolute $L_{\infty}$-algebra. It is minimal if the differential $d_g$ is equal to zero.

Theorem 4.14. Let $g$ and $h$ be two complete minimal absolute $L_{\infty}$-algebras. They are isomorphic as complete minimal absolute $L_{\infty}$-algebras if and only if their universal enveloping absolute $A_{\infty}$-algebras are isomorphic.

Proof. This is a direct corollary of Proposition 4.12 and Theorem 4.11, using the same arguments as in the proof of Theorem 4.9. □
EXAMPLE 4.15 (Arity-wise nilpotent $\mathcal{L}_\infty$-algebras). Nilpotent $\mathcal{L}_\infty$-algebras in the sense of [Get09] are particular examples of absolute $\mathcal{L}_\infty$-algebras. Therefore the above theorem applies to minimal nilpotent $\mathcal{L}_\infty$-algebras without any degree restriction.

REMARK 4.16. The analogues of Theorems 4.5 and 4.11 should also hold when we replace the categories of absolute Lie/$\mathcal{L}_\infty$-algebras by their curved counterparts. Indeed, using [RiL22a, Remark 3.14], one should get analogue statements as in [CPRNW23] concerning the deformation complexes of unital $\mathcal{C}_\infty$-coalgebras and unital $\mathcal{A}_\infty$-coalgebras. Then applying the same formalism is straightforward.

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VICTOR ROCA I LUCIO, ECOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE, EPFL, CH-1015 LAUSANNE, SWITZERLAND
Email address: victor.rocalucio@epfl.ch