Hyperbolic Fibonacci Sequence

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Abstract

In this paper, we investigate the hyperbolic Fibonacci sequence and the hyperbolic Fibonacci numbers. Furthermore, we give recurrence relations, the golden ratio and Binet’s formula for the hyperbolic Fibonacci sequence and Lorentzian inner product, cross product and mixed product for the hyperbolic Fibonacci vectors.

1. Introduction

For the Fibonacci sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, . . . , Fn, . . .

defined by the recurrence relation

\[ F_n = F_{n-1} + F_{n-2}, \quad (n \geq 3), \]

with \( F_1 = F_2 = 1 \), it is well known that the n-th term of the Fibonacci sequence \( (F_n) \) [1]-[3]. Some recent generalizations have produced a variety of new and extended results.[4]-[8].

Hyperbolic numbers have applications in different areas of mathematics and theoretical physics. In particular, they are related to the Lorentz-Minkowski (Space-time) geometry in the plane as well as complex numbers are to Euclidean one [9]. The work on the function theory for hyperbolic numbers can be found in [10]-[15]. The set of hyperbolic numbers \( \mathbb{H} \) can be described in the form as

\[ \mathbb{H} = \{ z = x + hy \mid h \notin \mathbb{R}, h^2 = 1, \ x, y \in \mathbb{R} \}. \]  

Addition, substraction and multiplication of any two hyperbolic numbers \( z_1 \) and \( z_2 \) are defined by

\[ z_1 \pm z_2 = (x_1 \pm hy_1) \pm (x_2 \pm hy_2) = (x_1 \pm x_2) \pm h(y_1 \pm y_2), \]

\[ z_1 \times z_2 = (x_1 + hy_1) \times (x_2 + hy_2) = x_1 x_2 + y_1 y_2 + h(x_1 y_2 + y_1 x_2). \]

On the other hand, the division of two hyperbolic numbers are given by

\[ \frac{z_1}{z_2} = \frac{x_1 + hy_1}{x_2 + hy_2}, \quad \frac{x_1 x_2 + y_1 y_2}{x_2^2 - y_2^2} + h \left( \frac{x_1 y_2 + y_1 x_2}{x_2^2 - y_2^2} \right). \]

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If \( x^2 - \frac{1}{2} \neq 0 \), then the division \( \frac{2}{x} \) is possible. Therefore, the hyperbolic number system is a non-division algebra. The hyperbolic conjugation of \( z = x + hy \) is defined by
\[
\tilde{z} = z^\dagger = x - hy, \quad \mathcal{I} = z.
\]
For any \( z_1, z_2 \) hyperbolic numbers, can be written as follows:
\[
\begin{align*}
\tilde{z}_1 + z_2 &= \tilde{z}_1 + \tilde{z}_2, \\
\tilde{z}_1 \cdot z_2 &= \tilde{z}_1 \cdot \tilde{z}_2, \\
\|\tilde{z}\|^2 &= z \cdot \tilde{z} = x^2 - y^2.
\end{align*}
\]
where \( z \) is time-like if \( \|\tilde{z}\|^2 > 0 \), light-like if \( \|\tilde{z}\|^2 = 0 \) and space-like if \( \|\tilde{z}\|^2 < 0 \). The ring of hyperbolic numbers has zero-divisors. Moreover, these zero-divisors are also idempotent elements \( \{e, e^\dagger\} \) for hyperbolic numbers, given by
\[
e = \frac{1 + h}{2}, \quad e^\dagger = \frac{1 - h}{2}.
\]
where \( e e^\dagger = 0, \ e^\dagger = e, \ (e^\dagger)^2 = e^\dagger, \ e + e^\dagger = 1 \) and \( e - e^\dagger = k \). Then, each hyperbolic number \( z \) can be written as follows:
\[
z = x + hy = (x + y)e + (x - y)e^\dagger = z_1 + z_2 e^\dagger.
\]
These numbers are also called double, split, perplex, Lorentz and duplex numbers [12].

2. Hyperbolic Fibonacci sequence

The hyperbolic Fibonacci sequence defined by
\[
\tilde{F}_n = F_n + F_{n+1}, \quad (h^2 = 1)
\]
with \( \tilde{F}_1 = 1 + h, \ \tilde{F}_2 = 1 + 2h \) where \( h^2 = 1 \). That is, the hyperbolic Fibonacci sequence \( \tilde{F}_n \) is
\[
h, 1 + h, 1 + 2h, 2 + 3h, 3 + 5h, \ldots, (1 + h)F_n + hF_{n-1}, \ldots
\]
Using the equations (2.1) and (2.2), it was obtained
\[
\begin{align*}
\tilde{F}_{n+1} &= (1 + h)F_{n+1} + hF_n, \\
\tilde{F}_{n+2} &= (1 + 2h)F_{n+1} + (1 + h)F_n, \\
\tilde{F}_{n+3} &= (2 + 5h)F_{n+1} + (1 + 2h)F_n, \\
& \vdots \\
\tilde{F}_{n+r} &= (F_n + hF_{n+1})F_{r+1} + (F_{n-1} + hF_n)F_r, \\
& \vdots
\end{align*}
\]
For the hyperbolic Fibonacci sequence, it was obtained the following properties:
\[
\begin{align*}
F_{n+1}^2 + F_{n+2}^2 &= 2\tilde{F}_{n+1} + F_{2n+2}, \\
F_{n+1}^2 - F_{n-1}^2 &= 2F_{2n} + F_{2n+1}, \\
F_{n+r} &= F_{n}F_{r+1} + F_{n-1}F_{r}, \quad (n \geq 3) \\
F_{n+1}F_{n+1} - F_{n}^2 &= h(-1)^n, \\
F_{n+r}F_{n+r} - F_{n}^2 &= h(-1)^{n-r+1}F_n^2, \\
F_{n} + hF_{n+1} &= F_{2n+1}, \\
F_{n}F_{n+1} + F_{n+1}F_{n+1} &= 2\tilde{F}_{n+m+1} + F_{n+m+2}, \\
F_{n}F_{m+1} - F_{n+1}F_{m} &= h(-1)^mF_{n-m}, \\
\tilde{F}_{n+m} - (-1)^{m+1}\tilde{F}_{n+1} &= L_n = F_{n+1} + (-1)^lF_{r-1}.
\end{align*}
\]

**Theorem 2.1.** If \( \tilde{F}_n \) is the hyperbolic Fibonacci number, then
\[
\lim_{n \to \infty} \frac{\tilde{F}_{n+1}}{\tilde{F}_n} = \frac{\alpha^2}{\alpha - 1}
\]
where \( \alpha = \left(1 + \sqrt{5}\right)/2 = 1.618033 \ldots \) is the golden ratio.

**Proof.** We have for the Fibonacci number \( F_n \),
\[
\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \alpha
\]
where \( \alpha = \left(1 + \sqrt{5}\right)/2 = 1.618033 \ldots \) is the golden ratio [3].
Then, using this limit value for the hyperbolic Fibonacci number \( \bar{F}_n \), we obtain
\[
\lim_{n \to \infty} \frac{\bar{F}_{n+1}}{\bar{F}_n} = \lim_{n \to \infty} \frac{\bar{F}_{n+1} + h \bar{F}_{n+2}}{\bar{F}_n + h \bar{F}_{n+1}} = \lim_{n \to \infty} \frac{(\bar{F}_{n+1} + h \bar{F}_{n+2})(\bar{F}_n - h \bar{F}_{n+1})}{\bar{F}_n^2 - \bar{F}_{n+1}^2} \\
= \lim_{n \to \infty} \frac{\bar{F}_n + h (\bar{F}_{n+1} - \bar{F}_n + h \bar{F}_{n+2} - h^2 \bar{F}_n)}{(\bar{F}_n - \bar{F}_{n+1})^2} \\
= \lim_{n \to \infty} \frac{-F_n^2}{F_n - F_{n+1}} + h \lim_{n \to \infty} \frac{(-1)^{n+1}}{F_n - F_{n+1}} \\
= -\frac{\alpha^2}{1-\alpha} + 0 \\
= \frac{\alpha^2}{\alpha - 1}
\]
where the identities \( F_{n+2} = F_n + F_{n+1} \) and \( F_{n+1}F_{n-1} - F_n^2 = (-1)^n \) are used.

\[\square\]

**Theorem 2.2.** The Binet formula \(^1\) for the hyperbolic Fibonacci sequence is as follows;
\[
\bar{F}_n = \frac{1}{\alpha - \beta} (\alpha^n - \beta^n).
\]

**Proof.** If we use definition of the hyperbolic Fibonacci sequence and substitute first equation in footnote, then we get
\[
\bar{F}_n = F_n + h F_{n+1} \\
= (\alpha^n - \beta^n) + h (\alpha^{n+1} - \beta^{n+1}) \\
= \alpha^n (1 + h \alpha) - \beta^n (1 + h \beta) \\
= \frac{\alpha^n}{\alpha - \beta} (\alpha \bar{x}) \\
\]
where \( \bar{x} = 1 + h \alpha \) and \( \beta = 1 + h \beta \).

\[\square\]

### 3. Hyperbolic Fibonacci vectors

Let \( \bar{z}_1 = (x_1, x_2, x_3) \) and \( \bar{z}_2 = (y_1, y_2, y_3) \) be vectors in \( \mathbb{R}^3 \). The Lorentzian inner product of \( \bar{z}_1 \) and \( \bar{z}_2 \) is defined as \([16]\)
\[
\bar{z}_1 \cdot \bar{z}_2 = (\bar{z}_1, \bar{z}_2)_L = x_1 y_1 + x_2 y_2 - x_3 y_3.
\]
This space denote by \( \mathbb{L}^2, 1 \) or **Lorentz 3-space** \( \mathbb{L}^3 \).

A hyperbolic Fibonacci vector is defined by
\[
\bar{F}_n = (\bar{F}_n, \bar{F}_{n+1}, \bar{F}_{n+2})
\]
Also, from equations (2.1) and (2.3) it can be expressed as
\[
\bar{F}_n = \bar{F}_n + h \bar{F}_{n+1}
\]
where \( \bar{F}_n = (F_n, F_{n+1}, F_{n+2}) \) and \( \bar{F}_{n+1} = (F_{n+1}, F_{n+2}, F_{n+3}) \) are the hyperbolic Fibonacci vectors.

The product of the hyperbolic Fibonacci vector \( \bar{F}_n \) and the scalar \( \lambda \in \mathbb{R} \) is given by
\[
\lambda \bar{F}_n = \lambda \bar{F}_n + h \lambda \bar{F}_{n+1}
\]
and \( \bar{F}_n \) and \( \bar{F}_m \) are equal if and only if
\[
F_n = F_m \\
F_{n+1} = F_{m+1} \\
F_{n+2} = F_{m+2}
\]

\(^1\) Binet formula is the explicit formula to obtain the \( n \)-th Fibonacci and Lucas numbers. It is well known that for the Fibonacci and Lucas numbers, Binet formulas are
\[
F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}
\]
and
\[
L_n = \alpha^n + \beta^n
\]
respectively, where \( \alpha + \beta = 1, \alpha - \beta = \sqrt{5} \), \( \alpha \beta = -1 \) and \( \alpha = (1 + \sqrt{5})/2, \beta = (1 - \sqrt{5})/2 \). [7],[8].
Some examples of the hyperbolic Fibonacci vectors can be given easily as;
\[
\vec{F}_1 = (F_1, F_2, F_3) = (F_1, F_2, F_3) + h(F_2, F_3, F_4) = (1 + h, 1 + 2h, 2 + 3h)
\]
\[
\vec{F}_2 = (F_2, F_3, F_4) = (F_2, F_3, F_4) + h(F_3, F_4, F_5) = (1 + 2h, 2 + 3h, 3 + 5h)
\]

**Theorem 3.1.** Let \( \vec{F}_n \) and \( \vec{F}_m \) be two hyperbolic Fibonacci vectors. The Lorentzian inner product of \( \vec{F}_n \) and \( \vec{F}_m \) is given by
\[
\left< \vec{F}_n, \vec{F}_m \right> = (F_{n+m+1} - F_{n+m+2}) + h(3F_{n+m+2} + 2F_{n+m+3} - F_{n+1}F_{m+1}).
\] (3.1)

**Proof.** The Lorentzian inner product of \( \vec{F}_n = (\vec{F}_n, \vec{F}_{n+1}, \vec{F}_{n+2}) \) and \( \vec{F}_m = (\vec{F}_m, \vec{F}_{m+1}, \vec{F}_{m+2}) \) defined by
\[
\left< \vec{F}_n, \vec{F}_m \right> = \vec{F}_n \vec{F}_m + \vec{F}_{n+1} \vec{F}_{m+1} - \vec{F}_{n+2} \vec{F}_{m+2}
\]
where \( \vec{F}_n = (F_n, F_{n+1}, F_{n+2}) \) is the hyperbolic Fibonacci vector. Also, the equations (1.1), (1.2) and (1.3), we obtain
\[
\left< \vec{F}_n, \vec{F}_m \right> = F_n F_m + F_{n+1} F_{m+1} - F_{n+2} F_{m+2}
\] (3.2)
\[
\left< \vec{F}_{n+1}, \vec{F}_{m+1} \right> = F_{n+1} F_{m+1} + F_{n+2} F_{m+2} - F_{n+3} F_{m+3}
\] (3.3)
\[
\left< \vec{F}_n, \vec{F}_{m+1} \right> = F_n F_{m+1} + F_{n+1} F_{m+2} - F_{n+2} F_m
\] (3.4)
and
\[
\left< \vec{F}_{n+1}, \vec{F}_m \right> = F_{n+1} F_m + F_{n+2} F_{m+1} - F_{n+3} F_m
\] (3.5)

Then from equation (3.2), (3.3), (3.4) and (3.5), we have the equation (3.1). □

**Special Case-1:** For the Lorentzian inner product of the hyperbolic Fibonacci vectors \( \vec{F}_n \) and \( \vec{F}_{n+1} \), we get
\[
\left< \vec{F}_n, \vec{F}_{n+1} \right> = \vec{F}_n \vec{F}_{n+1} + \vec{F}_{n+1} \vec{F}_{n+2} - \vec{F}_{n+2} \vec{F}_n
\]
\[
= \left< \vec{F}_n, \vec{F}_{n+1} \right> + \left< \vec{F}_{n+1}, \vec{F}_{n+2} \right> + h\left(\left< \vec{F}_n, \vec{F}_{n+2} \right> + \left< \vec{F}_{n+1}, \vec{F}_{n+1} \right> \right)
\]
\[
= (F_{2n+1} - F_{2n+2}) + h(2F_{2n+3} + F_{2n+5} - F_{n+2}F_{n+3})
\]
and
\[
\left< \vec{F}_n, \vec{F}_n \right> = F_n^2 + F_{n+1}^2 - F_{n+2}^2
\]
\[
= \left< \vec{F}_n, \vec{F}_n \right> + \left< \vec{F}_{n+1}, \vec{F}_{n+1} \right> + h\left< \vec{F}_n, \vec{F}_{n+1} \right>
\]
\[
= (F_{2n+1} - F_{2n+4}) + h(2F_{2n+2} - F_{n+2}F_{n+3}).
\]

Then for the Lorentzian inner product of the hyperbolic vector \( \vec{F} \), we have, using identities of the Fibonacci numbers
\[
F_n^2 - F_{n-1}^2 = F_{2n+1}
\]
\[
F_n^2 - F_{n+1}^2 = F_{2n+2}
\]
\[
F_n F_m + F_{n+1} F_{m+1} = F_{n+m+1}
\]

(see, [11]), we have
\[
\| \vec{F}_n \|^2 = \left< \vec{F}_n, \vec{F}_n \right> = F_n^2 + F_{n+1}^2 - F_{n+2}^2
\]
\[
= (F_{2n+1} - F_{2n+2}) + h(F_{2n+2} - F_{n+2}F_{n+3}).
\]

\(^2\)Lorentzian inner product of hyperbolic number as follows:
\[
\left< \vec{F}, \vec{F} \right>_L = x_1^2 + x_2^2 - x_3^2, \quad [11].
\]
The Lorentzian cross product \([16],[17]\) of the vectors \(\vec{z}_1^L\) and \(\vec{z}_2^L\) in \(\mathbb{L}^3\) is

\[
\vec{z}_1^L \times_L \vec{z}_2^L = \begin{vmatrix}
- i & - j & k \\
 x_1 & x_2 & x_3 \\
 y_1 & y_2 & y_3
\end{vmatrix} = - i (x_2 y_3 - x_3 y_2) + j (x_1 y_3 - x_3 y_1) + k (x_1 y_2 - x_2 y_1).
\]

**Theorem 3.2.** Let \(\vec{F}_n\) and \(\vec{F}_m\) be two hyperbolic Fibonacci vectors. The Lorentzian cross product of \(\vec{F}_n\) and \(\vec{F}_m\) is given by

\[
\vec{F}_n \times_L \vec{F}_m = h (-1)^m F_{n-m} (i + j + k).
\]  \hspace{1cm} (3.6)

**Proof.** The Lorentzian cross product of \(\vec{F}_n = \vec{F}_n + h \vec{F}_{n+1}\) and \(\vec{F}_m = \vec{F}_m + h \vec{F}_{m+1}\) defined by

\[
\vec{F}_n \times \vec{F}_m = (\vec{F}_n \times \vec{F}_m) + (\vec{F}_{n+1} \times \vec{F}_{m+1}) + h (\vec{F}_n \times \vec{F}_{m+1} + \vec{F}_{n+1} \times \vec{F}_m)
\]

where \(\vec{F}_n\) is the hyperbolic Fibonacci vector and \(\vec{F}_n \times \vec{F}_m\) is the Lorentzian cross product for the hyperbolic Fibonacci vectors \(\vec{F}_n\) and \(\vec{F}_m\).

Now, we calculate the cross products \(\vec{F}_n \times \vec{F}_m\), \(\vec{F}_{n+1} \times \vec{F}_{m+1}\), \(\vec{F}_n \times \vec{F}_{m+1}\) and \(\vec{F}_{n+1} \times \vec{F}_m\): Using the property \(F_n F_{m+1} - F_{m+1} F_n = (-1)^m F_{m-m}\), we get

\[
\vec{F}_n \times \vec{F}_m = (-1)^m F_{n-m} (i + j + k)
\]  \hspace{1cm} (3.7)

\[
\vec{F}_{n+1} \times \vec{F}_{m+1} = (-1)^{m+1} F_{n-m} (i + j + k)
\]  \hspace{1cm} (3.8)

\[
\vec{F}_n \times \vec{F}_{m+1} = (-1)^{m+1} F_{n-m} (i + j + k)
\]  \hspace{1cm} (3.9)

and

\[
\vec{F}_{n+1} \times \vec{F}_m = (-1)^m F_{n-m+1} (i + j + k).
\]  \hspace{1cm} (3.10)

Then from the equations (3.7), (3.8), (3.9) and (3.10), we obtain the equation (3.6).

**Theorem 3.3.** Let \(\vec{F}_n, \vec{F}_m\) and \(\vec{F}_\ell\) be the hyperbolic Fibonacci vectors. The Lorentzian mixed product of these vectors is

\[
\langle \vec{F}_n \times_L \vec{F}_m, \vec{F}_\ell \rangle = 0.
\]  \hspace{1cm} (3.11)

**Proof.** Using the properties

\[
\vec{F}_n \times \vec{F}_m = (\vec{F}_n \times \vec{F}_m) + (\vec{F}_{n+1} \times \vec{F}_{m+1}) + h (\vec{F}_n \times \vec{F}_{m+1} + \vec{F}_{n+1} \times \vec{F}_m)
\]

and

\[
\vec{F}_\ell = \vec{F}_\ell + h \vec{F}_{\ell+1}
\]

we can write,

\[
\langle \vec{F}_n \times L \vec{F}_m, \vec{F}_\ell \rangle = \langle \vec{F}_n \times \vec{F}_m, \vec{F}_\ell \rangle + \langle \vec{F}_{n+1} \times \vec{F}_{m+1}, \vec{F}_\ell \rangle + h \langle \vec{F}_n \times \vec{F}_{m+1}, \vec{F}_{\ell+1} \rangle + h \langle \vec{F}_{n+1} \times \vec{F}_m, \vec{F}_{\ell+1} \rangle + h \langle \vec{F}_n \times \vec{F}_m, \vec{F}_{\ell+1} \rangle + h \langle \vec{F}_{n+1} \times \vec{F}_{m+1}, \vec{F}_{\ell+1} \rangle.
\]

Then from equations (3.7), (3.8), (3.9) and (3.10) and by using the Lorentzian inner product definition of the hyperbolic number, we obtain

\[
\langle (i + j + k), \vec{F}_\ell \rangle = F_\ell + F_{\ell+1} - F_{\ell+2} = 0,
\]

\[
\langle (i + j + k), \vec{F}_{\ell+1} \rangle = F_{\ell+1} + F_{\ell+2} - F_{\ell+3} = 0.
\]

Thus, we have the equation (3.11).
4. Conclusion

The hyperbolic Fibonacci sequence defined by
\[
\tilde{F}_n = F_n + hF_{n+1}, \quad (h^2 = 1),
\]
with \(\tilde{F}_1 = 1 + h, \tilde{F}_2 = 1 + 2h\) where \(h^2 = 1\).

In addition, limit for the hyperbolic Fibonacci sequence and Binet’s formula for the hyperbolic Fibonacci sequence is given. Furthermore, vectors and the Lorentzian inner product, cross product and mixed product of these vectors are given.

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