NEAR PERFECT MATCHINGS IN $k$-UNIFORM HYPERGRAPHS II

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Abstract. Suppose $k \nmid n$ and $H$ is an $n$-vertex $k$-uniform hypergraph. A near perfect matching in $H$ is a matching of size $\lfloor n/k \rfloor$. We give a divisibility barrier construction that prevents the existence of near perfect matchings in $H$. This generalizes the divisibility barrier for perfect matchings. We give a conjecture on the minimum $d$-degree threshold forcing a (near) perfect matching in $H$, which generalizes a well-known conjecture on perfect matchings. We also verify our conjecture for various cases. Our proof makes use of the lattice-based absorbing method that the author used recently to solve two other problems on matching and tilings for hypergraphs.

1. Introduction

Given $k \geq 2$, a $k$-uniform hypergraph (in short, $k$-graph) consists of a vertex set $V(H)$ and an edge set $E(H) \subseteq \binom{V(H)}{k}$, where every edge is a $k$-element subset of $V(H)$. A matching in $H$ is a collection of vertex-disjoint edges of $H$. A perfect matching $M$ in $H$ is a matching that covers all vertices of $H$. Clearly, a perfect matching in $H$ exists only if $k$ divides $|V(H)|$. When $k$ does not divide $n = |V(H)|$, we call a matching $M$ in $H$ a near perfect matching if $|M| = \lfloor n/k \rfloor$.

Given a $k$-graph $H$ with a set $S$ of $d$ vertices (where $1 \leq d \leq k - 1$) we define $\deg_H(S)$ to be the number of edges containing $S$ (the subscript $H$ is omitted if it is clear from the context). The minimum $d$-degree $\delta_d(H)$ of $H$ is the minimum of $\deg_H(S)$ over all $d$-vertex sets $S$ in $H$. We refer to $\delta_{k-1}(H)$ as the minimum codegree of $H$.

1.1. Matchings in hypergraphs via degree conditions. For integers $n, k, d, s$ such that $1 \leq d \leq k - 1$ and $0 \leq s \leq n/k$, let $m^*_d(k, n)$ denote the smallest integer $m$ such that $\delta_d(H) \geq m$ forces the existence of a matching in $H$ of size $s$ for any $k$-graph $H$ on $n$ vertices. Throughout this note, $o(1)$ represent a function of $n$ that tends to $0$ when $n$ goes to infinity. The following conjecture \cite{14,15} has received much attention in the last few years: codegree \cite{14,23,24}, and for $1 \leq d \leq k - 2$, approximate $d$-degree \cite{14,19,20}, exact $d$-degree \cite{3,7,12,13,17,25,26,27} (also see surveys \cite{21,28}).

Conjecture 1.1. For $1 \leq d \leq k - 1$ and $k \mid n$,

$$m^*_{d/n}(k, n) = \left( \max \left\{ \frac{1}{2}, 1 - \left( 1 - \frac{1}{k} \right)^{k-d} \right\} + o(1) \right) \binom{n-d}{k-d}.$$

We remark that the quantities in the conjecture come from the so-called divisibility barrier and the space barrier.

Construction 1.2 (Space Barrier). Let $V$ be a set of size $n$ and fix $S \subseteq V$ with $|S| = s < n/k$. Let $H(s)$ be the $k$-graph on $V$ whose edges are all $k$-sets that intersect $S$. It is easy to see that the size of a maximum matching in $H(s)$ is $s < n/k$ and $\delta_d(H(s)) = \left( \binom{n-d}{k-d} - \binom{n-d-s}{k-d} \right) = (1 - (1 - s/n)^{k-d} + o(1)) \binom{n-d}{k-d}$.

Moreover, the maximal value of $\delta_d(H(s))$ is attained by $s = \lfloor n/k \rfloor - 1$, which gives the second term in Conjecture 1.1

Construction 1.3 (Divisibility Barrier with two parts, \cite{23}). Fix integers $j, n$ such that $j \in \{0, 1\}$ and $k \mid n$. Let $V$ be a set of size $n$ with a partition $V_1 \cup V_2$ such that $|V_1| \neq jn/k \mod 2$. Let $H^j$ be the $k$-graph on $V$ whose edges are all $k$-sets $e$ such that $|e \cap V_i| \equiv j \mod 2$.

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To see why $H^j$ does not have a perfect matching, for any matching $M$ in $H^j$, by definition, $|V(M) \cap V_1| \equiv jn/k \mod 2$. This means $V(M) \cap V_1 \neq V_1$ and thus $M$ is not perfect. It is not hard to see that $\delta_d(H^j) \leq (\frac{1}{2} + o(1))(\frac{n-d}{k-d})$ and the equality is attained when $|V_1| \approx |V_2| \approx n/2$. Unfortunately, we do not know which exact values of $|V_1|$ and $|V_2|$ maximize the value of $\delta_d(H^j)$ — this forms a challenging question, see [25] Section 1 for a discussion.

The general cases when $s < n/k$ have also been studied. Note that in the general case we do not require that $k \mid n$. By a simple greedy algorithm, Rödl, Ruciński and Szemerédi [24] proved that $m^*_{k-1}(k, n) = s$ for all $s \leq |n/k| \cdot (k - 2)$. Recently, the author [6] extended this to all $s < n/k$, verifying a conjecture of Rödl, Ruciński and Szemerédi. Note that this is the best one can do, because when $k \mid n$, $m^*_{k-1}(k, n) \approx n/2$ by the main result of [24]. Moreover, Kühn, Osthus and Treglown [17] determined $m^*_1(3, n)$ for all $s \leq n/k$. Bollobás, Daykin and Erdős [2] determined $m^*_1(k, n)$ for $s < n/2k^3 - 1$. Recently, Kühn, Osthus and Townsend [16] determined $m^*_d(k, n)$ asymptotically for $1 \leq d \leq k - 2$ and $s \leq \min\{n/(2d-k), (1-o(1))n/k\}$.

1.2. Near perfect matchings. In this paper we are interested in the case $s = [n/k]$. We write $m^*_d(k, n) := m^*_{\lfloor n/k \rfloor}(k, n)$, namely, the minimum $d$-degree threshold forcing the existence of a (near) perfect matching.

To state our results, we need to define the threshold for almost perfect matchings. For integers $1 \leq d \leq k - 1$, let

$$c^*_d(k) := \lim_{\alpha \to 0, n \to \infty} \left( \frac{m^*_{\lfloor n/k \rfloor}(k, n)}{\binom{n-d}{k-d}} \right).$$

The following theorem follows from [4] Lemma 2.4 and the definition of $c^*_d(k)$.

**Theorem 1.4.** [4] For $1 \leq d \leq k - 1$ and $k \mid n$, we have

$$m^*_{\lfloor n/k \rfloor}(k, n) = \left( \max \left\{ 1/2, c^*_d(k) \right\} + o(1) \right) \left( \frac{n-d}{k-d} \right).$$

Let $\ell \equiv n \mod k$. For any $k$-graph $H$ on $n$ vertices with $\delta_d(H) \geq (\max \left\{ 1/2, c^*_d(k) \right\} + o(1)) \left( \frac{n-d}{k-d} \right)$, we delete arbitrary $\ell$ vertices of $V(H)$ and thus $\delta_d(H)$ decreases at most $\ell \left( \frac{n-d}{k-d} - 1 \right) = o \left( \left( \frac{n-d}{k-d} \right) \right)$. Then we can find a perfect matching by applying Theorem 1.4 and get a near perfect matching of $H$. This gives the following corollary. The lower bound is by the definition of $c^*_d(k)$.

**Corollary 1.5.** For $1 \leq d \leq k - 1$, we have

$$(c^*_d(k) + o(1)) \left( \frac{n-d}{k-d} \right) \leq m^*_{\lfloor n/k \rfloor}(k, n) \leq \left( \max \left\{ 1/2, c^*_d(k) \right\} + o(1) \right) \left( \frac{n-d}{k-d} \right).$$

Corollary 1.5 implies that if $c^*_d(k) \geq 1/2$, then $m^*_{\lfloor n/k \rfloor}(k, n) = (c^*_d(k) + o(1)) \left( \frac{n-d}{k-d} \right)$, which 'coincides' the value of $m^*_{\lfloor n/k \rfloor}(k, n)$. So it is interesting to study $m^*_{\lfloor n/k \rfloor}(k, n)$ when $c^*_d(k) < 1/2$. However, the value of $c^*_d(k)$ for $d < k/2$ is still wide open. Indeed, Construction 1.2 shows that $c^*_d(k) \geq 1 - (1 - 1/k)^{k-d}$. The authors of [16] conjectured that $c^*_d(k) = 1 - (1 - 1/k)^{k-d}$ and verified the cases when $d \geq k/2$. Also, it is shown by the result of [1] that $c^*_d(k) = 1 - (1 - 1/k)^{k-d}$ for $1 \leq k - d \leq 4$.

**Theorem 1.6.** [1] [16] For positive integers $k, d$ such that $k \geq 3$ and $\min\{k/2, k-4\} \leq d \leq k - 1$, we have

$$c^*_d(k) = 1 - \left( \frac{1}{k} \right)^{k-d}.$$ 

When $k \notdivides n$, the knowledge of $m^*_{\lfloor n/k \rfloor}(k, n)$ is very limited. The aforementioned results [6] [17] included $m^*_{k-1}(k, n) = \lfloor n/k \rfloor$ and $m^*_1(3, n) = \left( \frac{n-1}{2} \right) - \left( \frac{2n/3}{2} \right) + 1 = (5/9 + o(1)) \left( \frac{n-1}{2} \right)$. In this paper we give new upper and lower bounds on $m^*_{\lfloor n/k \rfloor}(k, n)$ for $k \geq 4$ and $1 \leq d \leq k - 2$.

**Theorem 1.7.** For $k \geq 4$, $1 \leq d \leq k - 2$ and $k \notdivides n$, we have

$$\left( \max \{ g(k, d, \ell), c^*_d(k) \} + o(1) \right) \left( \frac{n-d}{k-d} \right) \leq m^*_{\lfloor n/k \rfloor}(k, n) \leq \left( \max \{ g(k, d, 1), c^*_d(k) \} + o(1) \right) \left( \frac{n-d}{k-d} \right).$$

The term $g(k, d, \ell)$ will be defined formally later. Our second result concerns the case $d = k - 2$ and $\ell \in \{2, \ldots, k-1\}$. Note that the approximate version of $k = 3$ case was obtained in [3] and the exact vertex degree condition in [17].
Theorem 1.8. For integers $k, \ell, n$ such that $k \geq 4$ and $n \equiv \ell \mod k$ for some $\ell \in \{2, \ldots, k-1\}$, we have

$$n'_{k-2}(k, n) = \left(1 - \left\lfloor \frac{1}{k} \right\rfloor \right)^2 + o(1) \left(\frac{n-k+2}{2} \right) \left(\frac{2k-1}{k^2} + o(1) \right) \left(\frac{n-k+2}{2} \right).$$

By the previous results [3, 19], it seems that $n'_{\ell}(k,n)$ is determined only by the space barrier. However, we give the following divisibility barrier construction which generalizes Construction 1.3, showing that this is not always the case when $d \leq k-2$.

Construction 1.9. Fix integers $\ell, j, n$ and a real number $0 \leq x \leq 1$ such that $0 \leq \ell \leq k-1$, $j \in \{0, 1, \ldots, \ell+1\}$, $n \equiv \ell \mod k$ and $x n = \left\lfloor \frac{n}{k} \right\rfloor \ell + \ell + 1 \mod \ell + 2$. Let $V$ be a set of size $n$ with a partition $V_1 \cup V_2$ such that $|V_1| = xn$. Let $H^j_\ell(x)$ be the $k$-graph on $V$ whose edges are all $k$-sets $e$ such that $|e \cap V_1| = j \mod \ell + 2$.

Let $M$ be a near perfect matching in $H^j_\ell(x)$ and thus $M$ leaves a set $U$ of exactly $\ell$ vertices uncovered. Suppose $|U \cap V_1| = i'$ for some $0 \leq i' \leq \ell$, then we have $|V(M) \cap V_1| = |V_1| - i' \equiv \left\lfloor \frac{n}{k} \right\rfloor j + \ell + 1 - i' \mod \ell + 2$. On the other hand, because $|e \cap V_1| \equiv j \mod \ell + 2$ for every edge $e \in M$, we have $|V(M) \cap V_1| \equiv \left\lfloor \frac{n}{k} \right\rfloor j \mod \ell + 2$. This is a contradiction because $1 \leq \ell + 1 - i' \leq \ell + 1$, and thus $H^j_\ell(x)$ contains no near perfect matching.

Similar as in Construction 1.3, it is not clear which exact values of $|V_1|, |V_2|$ maximize $\delta_d(H^j_\ell(x))$ in Construction 1.9. Indeed, it seems even harder than its special case Construction 1.3 (when $\ell = 0$) – the maximum of $\delta_d(H^j_\ell(x))$ is not always achieved by the (almost) balanced bipartition, i.e., $x \approx 1/2$ (see Section 2). For this reason we introduce the variable $x$ in Construction 1.9 and focus on the major term by letting $|V_1| = xn$. For $1 \leq d \leq k-1$ and $0 \leq \ell \leq k-1$, we define

$$g(k, d, \ell) := \lim_{n \to \infty} \left(\max_{0<j \leq \ell+1} \max_{0 \leq x \leq 1} \delta_d(H^j_\ell(x))/\left(\frac{n-d}{k-d}\right)\right).$$

We make the following conjecture. As mentioned in the previous section, $g(k, d, 0) = 1/2$, so Conjecture 1.10 is a special case (when $\ell = 0$) of Conjecture 1.10.

Conjecture 1.10. For all $1 \leq d \leq k-1$ and $n \equiv \ell \mod k$ for some $0 \leq \ell \leq k-1$,

$$(1.1) \quad m'_d(k, n) = \max \left\{ g(k, d, \ell), 1 - \left(1 - \frac{1}{k} \right)^{k-d} \right\} + o(1) \left(\frac{n-d}{k-d}\right).$$

Regarding the quantitative values of $g(k, d, \ell)$, we will show that $g(k, d, 1)$ can be determined explicitly by a simple optimization (see Section 2) and we have the following bounds on $g(k, d, \ell)$.

Proposition 1.11. Let $1 \leq d \leq k-1$ and $0 \leq \ell \leq k-1$. Then

(i) When $d \leq \ell + 1$, $g(k, d, \ell) \leq 1/(d+1)$ and when $d \geq \ell + 1$, $g(k, d, \ell) \leq 1/(\ell + 2)$.

(ii) $\left[ \frac{k-d}{3} \right] 2^{k-d} \leq g(k, d, 1) < \frac{1}{3}$ for $d > 1$.

So when $d > 1$ and $k - d$ tends to infinity, $g(k, d, 1)$ tends to $1/3$. We believe that this is true in general, that is, when $k - d$ tends to infinity, $d \geq \ell$ and $\ell$ stays as a constant, $g(k, d, \ell)$ tends to $1/(\ell + 2)$. Note that for $d = k-1$, Proposition 1.11(ii) says that $0 \leq g(k, k-1, 1) < 1/3$. In fact we have $g(k, k-1, \ell) = 0$ for all $1 \leq \ell \leq k-1$. This is shown in the following proposition.

Proposition 1.12. It holds that $g(k, d, \ell) = 0$ for $d \geq \max\{k - \ell, \lfloor k/2 \rfloor + 1\}$.

This explains the result in [6] – when $d = k-1$, we have $d \geq \max\{k - \ell, \lfloor k/2 \rfloor + 1\}$ unless $\ell = 0$ (so Conjecture 1.10 is true for $d = k-1$). We obtain following remarks by Propositions 1.11 and 1.12.

(1) Note that $1 - (1 - 1/k)^{k-d} \geq g(k, d, 1)$ if $d \leq (1 - \ln(3/2))k \approx 0.59k$. Indeed, since $(1 - 1/k)^{k} < 1/e$, we have

$$(1 - 1/k)^{k-d} < \left(\frac{1}{e}\right)^{\frac{k-d}{k}} \leq \left(\frac{1}{e}\right)^{\ln(3/2)} = \frac{2}{3}.$$

So $1 - (1 - 1/k)^{k-d} \geq 1/3 > g(k, d, 1)$ for $d \geq 2$ and $1 - (1 - 1/k)^{k-1} > 1/2 > g(k, 1, 1)$ by Proposition 1.11.
Notations. Throughout this paper, we prove Theorem 1.7 by the lattice-based absorbing method. We show Theorem 1.8 in Section 4.

1.3. Lattice-based absorbing method. We use the lattice-based absorbing method in the proof of Theorem 1.7. The absorbing technique initiated by Rödl, Ruciński and Szemerédi [22] has been shown to be efficient on finding spanning structures in graphs and hypergraphs. Roughly speaking, the goal is to build the absorbing set, which is a small subset of vertices and can be used to ‘absorb’ another small set of arbitrary vertices. For finding perfect matchings, the so-called Strong Absorbing Lemma [3, Lemma 2.4] says that when the minimum degree condition guarantees that every two vertices are reachable, we can find the absorbing set in the hypergraph.

So the question is: What can we do when the minimum degree condition does not guarantee that every two vertices are reachable? Keevash and Mycroft [11] studied this case. They showed that for any $k$-graph $H$ with a bounded minimum codegree which is not close to the space barrier, there exists a partition $\mathcal{P}$ of $V(H)$ such that a nice structure appears in the lattice generated by the edge-vectors on $\mathcal{P}$. Then they showed that $H$ contains a perfect matching if the lattice satisfies certain condition. Their proof used the hypergraph regularity method, rather than the absorbing method. Inspired by their work, we noticed the relation of reachability and the lattice structure and developed the lattice-based absorbing method. Roughly speaking, the reachability information provides us a partition $\mathcal{P}$, for which we can find an absorbing set that works (although not in the usual sense) under the lattice structure. This method was first used in [5] for solving a problem of Karpiński, Ruciński and Szymańska [9], for which an asymptotic result was obtained by Keevash, Knox and Mycroft [10]. Another advantage of the method is that the lattice language allows us to use some basic knowledge in group theory, such as, subgroups and cosets (see Proposition 3.10).

The rest of this paper is organized as follows. We show Propositions 1.11 and 1.12 in Section 2. In Section 3, we prove Theorem 1.8 by the lattice-based absorbing method. We show Theorem 1.8 in Section 4.

Notations. Throughout this paper, $x \ll y$ means that for any $y \geq 0$ there exists $x_0 \geq 0$ such that for any $x \leq x_0$ the following statement holds. Hierarchy of more constants are defined similarly. For simplicity, for a set $S$ and an element $u$, we write $S \cup u$ instead of $S \cup \{u\}$. We use boldface letters to represent vectors, for example, $u, v, i$.

2. Proof of Propositions 1.11 and 1.12

Proof of Proposition 1.11 Fix $0 \leq t \leq d$ and consider any $d$-set $S_t$ containing exactly $t$ vertices in $V_1$. By the definition of $H^t_1(x)$, we have

$$\delta_d(H^t_1(x)) = \min\{\deg_{H^t_1(x)}(S_0), \ldots, \deg_{H^t_1(x)}(S_d)\}.$$ (2.1)

Note that the neighbors of $S_t$ are all $(k-d)$-sets with $j'$ vertices in $V_1$ such that $j' \equiv j-t \mod \ell + 2$ and $0 \leq j' \leq k-d$. Thus,

$$\deg_{H^t_1(x)}(S_t) = \sum_{j' \equiv j-t \mod \ell + 2, 0 \leq j' \leq k-d} \left(\frac{|V_1| - t}{|V_2| - (d - t)}\right)^{k - d - j'}.$$ (2.2)

In fact, their result is on the degree sequence on $k$-complexes, which is more general. Here we state their work in the codegree case on $k$-graphs.
Let \( p = \min\{\ell + 1, d\} \) and note that \( \deg_{H^1(x)}(S_0) + \cdots + \deg_{H^1(x)}(S_p) \leq (1 + o(1))(n^{-d}). \) So \( \delta_d(H^1(x)) \leq (\frac{1}{p+1} + o(1))(n^{-d}). \) This implies that when \( d \leq \ell + 1, g(k, d, \ell) \leq 1/(d+1) \) and when \( d \geq \ell + 1, g(k, d, \ell) \leq 1/(\ell+2) \), proving part (i).

Now consider \( \ell = 1 \) and use \( |V_1| = xn \) and \( |V_2| = (1-x)n \). Then (2.2) implies that

\[
\deg_{H^1(x)}(S_t) = \sum_{j} x^{j'} (1-x)^{k-d-j'} n^{k-d} + o(n^{-d})
\]

\[
= \left( \sum_{j} \left( \begin{array}{c} k-d \\ j' \end{array} \right) x^{j'} (1-x)^{k-d-j'} + o(1) \right) \left( \begin{array}{c} n-d \\ k-d \end{array} \right),
\]

where the sums are on all \( j' \equiv j - t \mod 3 \) and \( 0 \leq j' \leq k - d \). Define three functions \( h_i(k, d, x) = \sum_{j'=i \mod 3, 0 \leq j' \leq k - d} (k-d)/(j') x^{j'} (1-x)^{k-d-j'} \) for \( i = 0, 1, 2 \). Clearly these functions do not depend on \( j \) and

\[
g(k, d, 1) = \max_{x \in [0,1]} \min_{i=0,1,2} \{ h_i(k, d, x) \}.
\]

This implies that, for any \( x \in [0,1] \) and \( \gamma > 0 \), there exists \( n_0 \) such that for any \( n \geq n_0 \), we have

\[
\min_{i=0,1,2} \left\{ \sum_{j'=i \mod 3, 0 \leq j' \leq k - d} (x^n/j) (1-x)^{k-d-j} \right\} \leq (g(k, d, 1) + \gamma/2) \left( \begin{array}{c} n-d \\ k-d \end{array} \right).
\]

To see (ii), let \( \omega_1 \) be one of the nontrivial cubic roots of 1. We consider the following polynomial \((x\omega_1 + 1 - x)^{k-d}. \) It is easy to see that we can write the polynomial in the following form

\[
(x\omega_1 + 1 - x)^{k-d} = h_0(k, d, x) + h_1(k, d, x)\omega_1 + h_2(k, d, x)\omega_1^2.
\]

First, since \( \sum_{i=0,1,2} h_i(k, d, x) = 1 \), we have \( g(k, d, 1) \leq 1/3. \) Moreover, if \( g(k, d, 1) = 1/3 \), then let \( x_0 \) be the value of \( x \) that achieves this, i.e., \( h_i(k, d, x_0) = 1/3 \) for \( i = 0, 1, 2 \). Putting \( x_0 \in \mathbb{R} \) and by \( 1 + \omega_1 + \omega_1^2 = 0 \), we get \((x_0\omega_1 + 1 - x_0)^{k-d} = 0. \) This implies that \( x_0 = 1/(1-\omega_1) \notin \mathbb{R}, \) a contradiction.

To see the lower bound, set \( x = 1/2 \) in (2.5) and we get

\[
\frac{1+\omega_1}{2^{k-d}} = h_0(k, d, 1/2) + h_1(k, d, 1/2)\omega_1 + h_2(k, d, 1/2)\omega_1^2.
\]

For \( i = 0, 1, 2, \) let \( C_i = 2^{k-d}h_i(k, d, 1/2) \). Note that \( C_0, C_1 \) and \( C_2 \) are integers such that \( C_0 + C_1 + C_2 \geq 2^{k-d}. \) On the other hand, note that \( (1+\omega_1)^{k-d} = (-1)^{k-d}\omega_1^{2(k-d)} \), which could be \( \pm 1 \), \( \pm \omega_1 \), or \( \pm \omega_1^2 \), depending on the value of \( k - d \). If \( (1+\omega_1)^{k-d} = 1 \) then by \( 1 + \omega_1 + \omega_1^2 = 0 \), we know that \( C_0 - 1 = C_1 = C_2 \). This implies that \( C_0 = \lceil \frac{2^{k-d}}{3} \rceil \) and \( C_1 = C_2 = \lfloor \frac{2^{k-d}}{3} \rfloor \). Other cases are similar and it is easy to see that in all cases, \( \min\{C_0, C_1, C_2\} = \lfloor \frac{2^{k-d}}{3} \rfloor \). Thus, by definition, we have \( g(k, d, 1) \geq \min_{i=0,1,2} \{ h_i(k, d, 1/2) \} = \lfloor \frac{2^{k-d}}{3} \rfloor 2^{k-d}. \) This concludes part (ii).

Here we remark that for fixed \( k \) and \( d \), the value of \( g(k, d, 1) \) can be determined simply. For example, let \( k = 6 \) and \( d = 3 \) and consider \( H^1(x) \), i.e., all \( 6 \)-sets \( e \) such that \( |e \cap V_1| = 0, 3 \) or \( 6 \). By (2.3), we have

\[
g(6, 3, 1) = \max_{x \in [0,1]} \min \{ x^3 + (1-x)^3, 3x^2(1-x), 3x(1-x)^2 \}.
\]

The answer is about \( 0.283 \) which is achieved by \( x \approx 0.605. \)

**Proof of Proposition 11.12** Given any \( k \)-set \( S \) in \( V(H^1(x)) \), clearly \( |S \cap V_1| \in [0, k]. \) Recall that \( S \in E(H^1(x)) \) if and only if \( |S \cap V_1| \equiv j \mod \ell + 2 \). Consider the longest interval \( I_j = [a, b] \subseteq [0, k] \) such that for any \( i \in I_j, i \not\equiv j \mod \ell + 2 \). If there are two values of \( i \in [0, k] \) such that \( i \equiv j \mod \ell + 2 \), then \( |I_j| = \ell + 1 \). Otherwise, there is only one value of \( i \in [0, k] \) such that \( i \equiv j \mod \ell + 2 \), which cuts the whole interval \([0, k]\) into two pieces, and thus \( |I_j| = \max\{\ell, k - j\} \geq [k/2] \). Moreover, \( |I_{[k/2]}| = \min\{\ell + 1, [k/2]\} \). Altogether \( \min_{0 \leq j \leq \ell + 1} |I_j| = \min\{\ell + 1, [k/2]\} \). Note that the \((a + k - b)\)-sets with \( a \) vertices in \( V_1 \) and \( k - b \) vertices
in $V_2$ have degree zero in $H^j_i(x)$, because all $k$-sets $S$ containing such a set satisfy that $|S \cap V_1| \in I_j$. This means $\delta_d(H^j_i(x)) = 0$ for $d \geq a + k - b = k + 1 - |I_j|$. By definition, this implies that $g(k, d, \ell) = 0$ when $d \geq \max_{0 \leq j \leq \ell+1} k + 1 - |I_j| = k + 1 - \min_{0 \leq j \leq \ell+1} |I_j| = k + 1 - \min\{\ell + 1, [k/2]\} = \max\{k - \ell, [k/2] + 1\}$. □

3. PROOF OF THEOREM

For a $k$-graph $H$, we shall first identify a partition $\mathcal{P}$ of $V(H)$ and then study the so-called robust edge-lattice with respect to this partition.

3.1. A partition of the vertex set. We start with some definitions. We say that two vertices $u$ and $v$ are $(\beta, i)$-reachable in $H$ if there are at least $\beta n^{ik-1}$ $(ik - 1)$-sets $S \subseteq V(H)$ such that both $H[S \cup u]$ and $H[S \cup v]$ have perfect matchings. We say a vertex set $U$ is $(\beta, i)$-closed in $H$ if any two vertices $u, v \in U$ are $(\beta, i)$-reachable in $H$. For any $v \in V(H)$, let $\tilde{N}_\beta(v)$ be the set of vertices in $V(H)$ that are $(\beta, i)$-reachable to $v$.

We first show a lower bound on $|\tilde{N}_\beta(v)|$ for all but a small fraction of vertices $v \in V(H)$. Note that we cannot guarantee that the conclusion of Lemma 3.1 holds for all vertices in $V(H)$. Similar proof tricks are used in [8].

Lemma 3.1. Given $\delta, \epsilon > 0$ such that $\delta \geq 3\epsilon$, there exists $\alpha > 0$ such that the following holds for sufficiently large $n$. Let $H = (V, E)$ be an $n$-vertex $k$-graph with $\delta_1(H) \geq \delta \left(\frac{n}{k-1}\right)$, there exists a set $V'_0 \subseteq V$ of size at most $\alpha n$ such that for any $v \in V \setminus V'_0$, $|\tilde{N}_\alpha(v)| \geq \frac{3\epsilon^2}{2} n$.

Proof. If an edge $e \in E$ contains a $(k - 1)$-set $S \subseteq \left(\frac{\epsilon n}{k-1}\right)$ with $\deg_H(S) \leq \epsilon^2 n$, then it is called weak, otherwise called strong. Note that by the definition, the number of weak edges in $H$ is at most $\left(\frac{n}{k-1}\right)^2 n$. Let $H'$ be the subhypergraph of $H$ induced on strong edges. Let

$$V'_0 = \left\{ v \in V : v \text{ is contained in at least }\epsilon \left(\frac{n}{k-1}\right) \text{ weak edges} \right\}.$$ 

So $|V'_0| \leq \alpha n$ – otherwise there are more than $\alpha n \cdot \epsilon \left(\frac{n}{k-1}\right) / k = \left(\frac{n}{k-1}\right)^2 n$ weak edges in $H$, a contradiction.

Let $\alpha = \frac{\delta^2}{3 \epsilon^2}$. For any $x \in V \setminus V'_0$, since $x$ is contained in at most $\epsilon \left(\frac{n}{k-1}\right)$ weak edges, we have

$$\deg_{H'}(x) \geq \deg_H(x) - \epsilon \left(\frac{n}{k-1}\right) \geq \delta_1(H) - \epsilon \left(\frac{n}{k-1}\right) \geq \frac{\delta}{2} \left(\frac{n-1}{k-1}\right).$$

To see $|\tilde{N}_\alpha(x)| \geq \frac{3\epsilon^2}{4} n$, let

$$D = \left\{ v \in V : |N_{H'}(v) \cap N_{H'}(x)| \geq \frac{\delta \epsilon^2}{8} \left(\frac{n-1}{k-1}\right) \right\}.$$ 

By definition, two vertices $x, v \in V$ are $(\alpha, 1)$-reachable in $H'$ (so in $H$) if $|N_{H'}(v) \cap N_{H'}(x)| \geq \delta \epsilon^2 \left(\frac{n-1}{k-1}\right)/8 > \alpha n^{k-1}$. Therefore $D \subseteq \tilde{N}_\alpha(x)$. Let $t$ be the number of pairs $(S, u)$ such that $S \subseteq N_{H'}(x)$ and $u \in N_{H'}(S)$. Since all edges of $H'$ are strong, we have $t \geq \deg_{H'}(x) \cdot \epsilon^2 n$. By counting, we have

$$\deg_{H'}(x) \epsilon^2 n \leq t \leq n \cdot \frac{\delta \epsilon^2}{8} \left(\frac{n-1}{k-1}\right) + |D| \cdot \deg_{H'}(x),$$

which implies $|D| \geq \epsilon^2 n - \delta \epsilon^2 n^{k-1}/(8 \deg_{H'}(x))$. By (3.1), $|\tilde{N}_\alpha(x)| \geq |D| \geq \frac{3\epsilon^2}{4} n$ and we are done. □

We will use the following simple result from [18] here and in the next subsection.

Proposition 3.2. [18] Proposition 2.1 For $\epsilon, \beta > 0$ and integer $i \geq 1$, there exists $\beta_0 > 0$ and an integer $n_0$ satisfying the following. Suppose $H$ is a $k$-graph of order $n \geq n_0$ and there exists a vertex $x \in V(H)$ with $|\tilde{N}_{\beta}(x)| \geq \epsilon^2 n/2$. Then for all $0 < \beta' \leq \beta_0$, $\tilde{N}_{\beta'}(x) \subseteq \tilde{N}_{\beta}(x)$.

The main tool to identify the partition of $V(H)$ is the following lemma, which is a variant of [5] Lemma 3.8 (see also [8] Lemma 3.8). Since there may be some vertices $v$ such that $|\tilde{N}_\beta(v)|$ is small, we cannot get a perfect partition as in [5] Lemma 3.8 – $\mathcal{P}$ will contain a trash set $V_0$. 

6
Lemma 3.3. Given $0 < \epsilon \ll \delta$, there exists $\beta > 0$ satisfying the following. Let $H = (V, E)$ be an $n$-vertex k-graph such that $\delta_1 (H) \geq (\delta + k^2 \epsilon)\binom{n-1}{k-1}$. Then there is a partition $\mathcal{P}$ of $V(H)$ into $V_0, V_1, \ldots, V_r$ with $r \leq \lceil 1/\delta \rceil$ such that $|V_0| \leq \sqrt{en}$ and for any $i \in [r]$, $|V_i| \geq \epsilon^2 n$ and $V_i$ is $(\beta, 2^{(1/\delta)-1})$-closed in $H$.

Proof. We first apply Lemma 3.1 with $\delta, \epsilon$ and get $\alpha > 0$. Let $c = \lceil 1/\delta \rceil$ (then $(c+1)\delta - 1 > 0$). We choose constants satisfying the following hierarchy

$$1/n \ll \beta = \beta_{c-1} \ll \beta_{c-2} \ll \cdots \ll \beta_1 \ll \beta_0 \ll \epsilon, (c+1)\delta - 1.$$

Throughout this proof, given $v \in V(H)$ and $0 \leq i \leq c-1$, we write $\tilde{N}_i (v)$ as $\tilde{N}_i (v)$ for short. We also say $2^i$-reachable (or $2^i$-closed) for $(\beta_i, 2^i)$-reachable (or $(\beta_i, 2^i)$)-closed. We first apply Lemma 3.1 on $H$ and get a vertex set $V_0'$ of size at most $kcn$ such that for any $v \in V \setminus V_0'$, we have $|\tilde{N}_0 (v)| \geq |\tilde{N}_c(v)| \geq \frac{3}{4} \epsilon^2 n$. By Proposition 3.2 and the choice of $\beta_i$'s, we may assume that $\tilde{N}_i (v) \subseteq \tilde{N}_{i+1} (v)$ for all $0 \leq i < c-1$ and all $v \in V(H')$. Hence, if $W \subseteq V(H)$ is $2^i$-closed in $H$ for some $i \leq c-1$, then $W$ is $2^{i-1}$-closed. Let $n' = |V \setminus V_0'|$ and $H' = H[V \setminus V_0']$. Note that

$$\delta_1 (H') \geq \delta_1 (H) - kcn \left\lfloor \frac{n-2}{k-2} \right\rfloor \geq \delta \left\lfloor \frac{n-1}{k-1} \right\rfloor.$$

Recall that two vertices $u$ and $v$ are $1$-reachable in $H$ if $|N_H (u) \cap N_H (v)| \geq \beta_0 n^{k-1}$. We first note that any set of $c+1$ vertices in $V(H')$ contains two vertices that are $1$-reachable to each other because $\delta_1 (H') \geq \delta \left\lfloor \frac{n-1}{k-1} \right\rfloor$ and $(c+1)\delta - 1 \geq \left\lfloor \frac{c+1}{2} \right\rfloor \beta_0$. Also we can assume that there are two vertices that are not $2^{c-1}$-reachable to each other, as otherwise $V(H')$ is $2^{c-1}$-closed and we get a partition $\mathcal{P}_0 = \{V_0', V(H')\}$.

Let $d$ be the largest integer such that there exist $v_1, \ldots, v_d \in V(H')$ such that no pair of them are $2^{c+1-d}$-reachable to each other. Note that $d$ exists by our assumption and $2 \leq d \leq c = \lceil 1/\delta \rceil$ by our observation. Fix such $v_1, \ldots, v_d \in V(H')$, by Proposition 3.2 we may assume that any two of them are not $2^{c-d}$-reachable to each other. Consider $\tilde{N}_{c-d} (v_i)$ for all $i \in [d]$. Then we have the following facts.

(i) Any $v \in V(H) \setminus \{v_1, \ldots, v_d\}$ must be in $\tilde{N}_{c-d} (v_i)$ for some $i \in [d]$, as otherwise $v, v_1, \ldots, v_d$ contradicts the definition of $d$.

(ii) $|\tilde{N}_{c-d} (v_i) \cap \tilde{N}_{c-d} (v_j)| < \epsilon n$ because $v_i, v_j$ are not $2^{c+1-d}$-reachable to each other. Indeed, otherwise we get at least

$$\epsilon n (\beta_{c-d} n^{2^{c-d} - 1} - n^{2^{c-d} - 2}) (\beta_{c-d} n^{2^{c-d} - 1} - 2^{c-d} \epsilon n^{2^{c-d} - 2}) \geq \beta_{c+1-d} n^{2^{c+1-d} - 1},$$

reachable $(2^{c+1-d} - 1)$-sets for $v_i, v_j$, which means that they are $2^{c+1-d}$-reachable, a contradiction.

For $i \in [d]$, let $U_i = \overline{(\tilde{N}_{c-d} (v_i) \cup \{v_i\}) \setminus \bigcup_{j \in [d] \setminus \{i\}} \tilde{N}_{c-d} (v_j)}$. Note that for $i \in [d]$, $U_i$ is $2^{c-d}$-closed. Indeed, if there exist $u_1, u_2 \in U_i$ that are not $2^{c-d}$-reachable to each other, then $\overline{(u_1, u_2) \cup \{v_i\}}$ contradicts the definition of $d$.

Let $U_0 = V(H) \setminus (V_0' \cup U_1 \cup \cdots \cup U_d)$. By (i) and (ii), we have $|U_0| \leq \left( \frac{d}{2} \right) \epsilon n$. We add vertices of $U_0$ and the vertices of $U_i$ with $|U_i| \leq \epsilon^2 n$ to $V_0'$ and denote the resulting set by $V_0''$. Let the resulting partition of $V(H)$ be $V_0, V_1, \ldots, V_r$ for some $r \leq d \leq \lceil 1/\delta \rceil$. By definition we have $|V_i| \geq \epsilon^2 n$ for $i \in [r]$ and $|V_0| \leq |V_0'| + |U_0| + \epsilon^2 n \leq \sqrt{en}$, as $\epsilon \ll \delta$. Moreover, each $V_i$ is $2^{c-d}$-closed, and thus $2^{c-1}$-closed. \hfill \Box

3.2. The robust edge-lattice. We need some definitions from [11]. Fix an integer $r > 0$, let $H$ be a $k$-graph and let $\mathcal{P} = \{V_0, V_1, \ldots, V_r\}$ be a partition of $V(H)$. By Lemma 3.3, $V_0$ does not have the reachability information. So when we work on the edge-lattice, we consider the $r$-dimensional vectors on the parts of $\mathcal{P}$ except $V_0$. Formally, the index vector $\text{iv}(S) \in Z^r$ of a subset $S \subseteq V(H)$ with respect to $\mathcal{P}$ is the vector whose coordinates are the sizes of the intersections of $S$ with each part of $\mathcal{P}$ except $V_0$, i.e., $\text{iv}(S)_{V_i} = |S \cap V_i|$ for $i \in [r]$. We call a vector $i \in Z^r$ an $s$-vector if all its coordinates are nonnegative and their sum equals $s$ and denote the set of all $s$-vectors by $I^s_r$. Given $\mu > 0$, a $k$-vector $v$ is called a $\mu$-robust edge-vector if at least $\mu |V(H)|^k$ edges $e \in E(H)$ satisfy $\text{iv}(e) = v$. Let $I^\mu_r (H) \subseteq I^r_k$ be the set of all $\mu$-robust edge-vectors and let $L^\mu_r (H)$ be the lattice (additive subgroup) generated by the vectors of $I^\mu_r (H)$. For $j \in [r]$, let $u_j \in Z^r$ be the $j$-th unit vector, namely, $u_j$ has 1 on the $j$-th coordinate and 0 on other coordinates. A transferral is the vector $u_i - u_j$ for some $i \neq j$.
Suppose $I$ is a set of $k$-vectors in $\mathbb{Z}^r$ and $J$ is a set of vector in $\mathbb{Z}^r$ such that any $i \in J$ can be written as a linear combination of vectors in $I$, namely,

$$i = \sum_{v \in I} a_v v.$$  

(3.2)

We denote by $C(r, k, I, J)$ as the maximum of $|a_v|$, $v \in I$ over all $i \in J$ and $C(k', k, J) := \max_{r \leq k'} C(r, k, I, J)$ for some integer $k'$.

Given a $k$-graph $H$ with $\delta_1(H) \geq (\delta + k^2\epsilon)(\frac{\log n}{\log \log n})^{-1}$ and let $\mathcal{P} = \{V_0, V_1, \ldots, V_1\}$ be the partition of $V(H)$ output by Lemma 3.4. We pick a constant $0 < \mu \ll \epsilon$ and consider $I^*_P(H)$ and $L^*_P(H)$. Our next result shows that $V_i \cup V_j$ is closed in $H$ if $u_i - u_j \in L^*_P(H)$, which means that, we can merge $V_i$ and $V_j$ and keep the closedness.

**Lemma 3.4.** Let $0 < \{\mu, \beta\} \ll \epsilon \ll 1/\eta_0, 1/k', 1/k$, then there exist $0 < \mu' \ll \beta$ and an integer $t \geq i_0$ such that the following holds. Let

Given two vertex-disjoint reachable $(\beta', t)$-sets. Finally, since $V_1$ and $V_2$ are $(\beta', t)$-closed in $H$, by Proposition 3.2 there exists $\beta''$ such that $V_1$ and $V_2$ are $(\beta'', t)$-closed in $H$ and by the assumption above, we get that $V_1 \cup V_2$ is $(\beta', \beta'')$-closed in $H$, where $\beta = \min(\beta', \beta'')$.

Let $J = \{u_1 - u_2\}$. By our assumption, there are nonnegative integers $p_v, q_v \leq C(k', k, J)$, for each $v \in I^*_P(H)$, such that

$$u_1 - u_2 = \sum_{v \in I^*_P(H)} (p_v - q_v) v \quad \text{i.e.,} \quad \sum_{v \in I^*_P(H)} q_v v + u_1 = \sum_{v \in I^*_P(H)} p_v v + u_2.$$  

(3.3)

By comparing the sums of all the coordinates from two sides of either equation in (3.3), we obtain that

$$\sum_{v \in I^*_P(H)} p_v = \sum_{v \in I^*_P(H)} q_v.$$  

Denote this constant by $C'$ and note that $C' \leq C(k', k, J)|I^*_P(H)| \leq C(k', k, J)(\frac{k + i_0 - 1}{r - 1})$, which is independent of $n$. Since $n$ is large enough, we have $C' \ll n$.

Fix $x_1 \in V_1$ and $x_2 \in V_2$. For each $v \in I^*_P(H)$, we select $p_v + q_v$ disjoint edges with index vector $v$ that do not contain $x_1$ or $x_2$, and form two disjoint matchings $M_P$ and $M^g$, where $M_P$ consists of $p_v$ edges with index vector $v$ for all $v \in I^*_P(H)$ and $M^g$ consists of $q_v$ edges with index vector $v$ for all $v \in I^*_P(H)$. Note that $|V(M_P)| = |V(M^g)| = kC'$. When we select any edge, we need to avoid at most $2kC'$ vertices, which are incident to at most $2kC'n^{k-1} \leq \mu n^{k/2}$ edges, as $n$ is large enough. Therefore, the number of choices for the two matchings is at least $(\mu n^{k/2})^{2kC'}$.

By (3.3), we have $I_P(V(M^g)) + u_1 = I_P(V(M_P)) + u_2$. Fix two vertices $x'_1 \in V(M_P) \cap V_1$ and $x'_2 \in V(M^g) \cap V_2$. We match each of the vertices from $V(M_P) \setminus \{x'_1\}$ with a different vertex from $V(M^g) \setminus \{x'_2\}$ such that two matched vertices are from the same part of $H$ and thus are $(\beta, i_0)$-reachable to each other. We next select a reachable $(i_0k - 1)$-set for each pair of the matched vertices such that all these $kC' - (i_0k - 1)$-sets are vertex disjoint and also disjoint from $V(M_P \cup M^g) \cup \{x_1, x_2\}$. There are at least $\frac{n^{i_0k - 1}}{2}$ choices for each of these $(i_0k - 1)$-sets. Finally, since $x_1$ and $x'_1$ respectively, $x_2$ and $x'_2$ are $(\beta, i_0)$-reachable, we pick two vertex-disjoint reachable $(i_0k - 1)$-sets for them such that these two $(i_0k - 1)$-sets are also disjoint from all existing $(i_0k - 1)$-sets and $V(M_P \cup M^g) \cup \{x_1, x_2\}$. The union of these $kC' + 1$ $(i_0k - 1)$-sets and $V(M_P \cup M^g)$ forms a reachable $(i_0k^2C' + kC' + i_0k - 1)$-set for $x_1$ and $x_2$. There are at least

$$\left(\frac{\mu}{2}\right)^{2kC'} \left(\frac{\beta}{2}\right)^{i_0k - 1} \left(\frac{kC' + 1}{2}\right)^{i_0k^2C' + kC' + i_0k - 1}$$

such reachable sets. Thus, we take $\beta'' = (\frac{\mu}{2})^{2kC'} (\frac{\beta}{2})^{i_0k - 1}$ and $t = i_0kC' + C' + i_0$ and the proof is complete. \qed
3.3. Proof of Theorem 1.7  Fix an integer \( i > 0 \). For a \( k \)-vertex set \( S, \) we say a set \( T \) is an absorbing \( i \)-set for \( S \) if \( |T| = i \) and both \( H[T] \) and \( H[T \cup S] \) contain perfect matchings. We use the absorbing lemma from \([5]\) Lemma 3.4 with some quantitative changes, but it easily follows from the original proof.

**Lemma 3.5.** \([5]\) Suppose \( r \leq k \) and

\[
1/n < \alpha \leq \beta, \mu \leq \{1/k, 1/t\}.
\]

Suppose that \( P_0 = \{V_0, V_1, \ldots, V_r\} \) is a partition of \( V(H) \) such that for \( i \in [r], \) \( V_i \) is \((\beta, t)\)-closed. Then there is a family \( F_{abs} \) of disjoint \( tk^2 \)-sets with size at most \( \beta n \) such that \( H[V(F_{abs})] \) contains a perfect matching and every \( k \)-vertex set \( S \) with \( \Pi_{P_0}(S) \in I^t_{P_0}(H) \) has at least one absorbing \( tk^2 \)-sets in \( F_{abs} \).

Another key step in the proof of Theorem 1.7 is the following proposition. We postpone its proof to the next subsection.

**Proposition 3.6.** Given \( \min\{3, k/2\} \leq k - 2 \) or \((k, d) = (5, 2), \) \( 0 < \mu \leq \epsilon < \gamma \) and let \( n \) be sufficiently large. Let \( H \) be an \( n \)-vertex \( k \)-graph with \( \delta(H) \geq (g(k, d, 1) + \gamma)(\frac{n}{k-d}) \) and let \( P = \{V_0, \ldots, V_r\} \) be a partition of \( V(H) \) with \( r \leq 3 \) such that \( |V_0| \leq \sqrt{\epsilon} n \) and for any \( i \in [r], |V_i| \geq \epsilon^2 n, \) and \( L^t_{P_0}(H) \) contains no transferral. Then for any \( U \subseteq V(H) \setminus V_0 \) with \|U\| = k + 1, there exists \( i \in [r] \) such that \( 1_{P}(U) - u_i \in L^t_{P_0}(H) \).

Proof of Theorem 1.7. The lower bound in the theorem is shown by Construction 1.9 and the definition of \( c^*_d(k) \). For the upper bound, note that Proposition 3.6 does not cover the cases when \( d = 1 \) and \( k \geq 4 \) or \( d = 2 \) and \( k \geq 6 \). In fact, in all of these cases, we have that \( c^*_d(k) \geq 1 - (1 - 1/k)^{k-d} > 1/2 \geq g(k, d, 1) \).

Fix \( \gamma > 0 \) and pick \( 0 < \mu_0 < \epsilon_0 < \gamma. \) We first apply Lemma 3.3 with \( \delta = g(k, d, 1) + \gamma/2, \epsilon_0 \) and get \( 0 < \beta_0 \leq \epsilon_0. \) We then apply Lemma 3.3 on \( H \) with \( \delta = g(k, d, 1) + \gamma/2. \) Since \( g(k, d, 1) \geq 1/4, \) we get a partition \( P = \{V_0, V_1', \ldots, V_\ell'\} \) with \( \ell \leq 3 \) such that \( |V_0| \leq \sqrt{\epsilon} n \) and for any \( i \in [\ell], |V_i'| \geq \epsilon^2 n, \) \( V_i' \) is \((3, \epsilon')\)-closed in \( H. \) If \( u_i - u_j \in L^t_{P_0}(H) \) for some \( i, j \in [r], i \neq j, \) then we merge \( V_i \) and \( V_j \) to one part and by Lemma 3.4 \( V_i \cup V_j \) is \((\beta'', \epsilon'')\)-closed for some \( \beta'' > 0 \) and \( \epsilon'' \geq 4. \) We greedily merge the parts until there is no transferral in the \( \mu\)-robust edge-lattice. Let \( P_0 = \{V_0, \ldots, V_r\} \) be the resulting partition for some \( 1 \leq \ell' < 3. \) Note that we have applied Lemma 3.3 at most twice and by Proposition 3.2 we conclude that for each \( i \in [r'], V_i \) is \((\beta', t')\)-closed by the choice of \( \beta'. \) We apply Lemma 3.3 on \( H \) and get \( F_{abs} \) such that \( |V(F_{abs})| \leq tk^2 \beta'^2 n. \)

We build a matching \( M_1 \) in \( H \) as follows. First note that \( |I^t_{P_0}(H)| \leq \binom{k+t'-1}{r-1} \leq \binom{k}{2}. \) For each \( \nu \in I^t_{P_0}(H), \) we greedily pick a matching \( M_{\nu} \) of size \( C\alpha^2 n \) such that \( \Pi_{P_0}(\nu) = \nu \) for every \( \nu \in M_{\nu}. \) Then let \( M_1 \) be the union of all \( M_{\nu} \) for all \( \nu \in I^t_{P_0}(H). \) It is possible to pick \( M_1 \) because there are at least \( \mu n^k \) edges with \( \Pi_{P_0}(\nu) = \nu \in I^t_{P_0}(H). \) Indeed, since \( 1/4 \leq \beta' \leq \mu \leq 1/t, 1/C, \) we have

\[
|V(M_1) \cup V(F_{abs})| \leq k|I^t_{P_0}(H)|C\alpha^2 n + tk^2 \beta'^2 n < \mu n,
\]

which implies that the number of edges intersecting these vertices is less than \( \mu n \) and we are done.

Next we will greedily match the vertices in \( V_0 \setminus V(F_{abs}). \) More precisely, we will find a matching \( M_2 \) that covers all vertices of \( V_0 \setminus V(F_{abs}). \) Note that \( |M_2| \leq |V_0| \leq \sqrt{\epsilon} n. \) When we greedily match a vertex \( v \in V_0 \setminus V(F_{abs}), \) we need to avoid at most \( k|M_2| + |V(M_1) \cup V(F_{abs})| \leq k\sqrt{\epsilon} n + \mu n \leq 2k\sqrt{\epsilon} n \) vertices, and thus at most \( 2k\sqrt{\epsilon} n^{k-1} (k-1) \)-sets. Since \( \delta(H) > \gamma(n-1) > 2k\sqrt{\epsilon} n^{k-1}, \) we can always find a desired edge containing \( v \) and add it to \( M_2. \)

---

\(^2\text{This can be proved by showing that } f(k) = 1 - (1 - 1/k)^{k-d} \text{ is increasing for } d = 1, 2.\)
Let \( V' = V \setminus (V(F_{abs}) \cup V(M_1 \cup M_2)) \) and \( H' = H[V'] \). By previous calculations, we have \(|V(F_{abs}) \cup V(M_1 \cup M_2)| \leq 2k\sqrt{\epsilon n}\) and thus

\[
\delta_\epsilon(H') \geq (c_\epsilon^*(\gamma) + \gamma) \left( \frac{n - d}{k - d} \right) - 2k\sqrt{\epsilon n} \cdot n^{k-d-1} \geq (c_\epsilon^*(\gamma) + \gamma/2) \left( \frac{|V'| - d}{k - d} \right),
\]

as \( \epsilon \ll \gamma \). So by the definition of \( c_\epsilon^*(\gamma) \) and that \(|V'| \geq n/2\) is large enough, we can find a matching \( M_3 \) in \( H' \) which leaves at most \( \alpha^2|V'| \leq \alpha^2n \) vertices uncovered.

Now we absorb the uncovered vertices by \( F_{abs} \). Fix any set \( U \) of \( k+1 \) uncovered vertices, by Proposition 3.6 there exists \( i \in \{v'\} \) such that \( i_{\rho}(U) - u_i \in L_\rho^\mu(H) \). Note that this does not guarantee that we can delete one vertex \( v \) from \( U \) such that \( i_{\rho}(U \setminus \{v\}) \in L_\rho^\mu(H) \), because it is possible that \( U \cap V_i = \emptyset \) for the \( i \) returned by the proposition. By the degree condition, there is a vector \( v \in L_\rho^\mu(H) \) such that \( v_{V_i} > 0 \) and note that \( M_1 \) contains \( Ca^2n \) edges with index vector \( v \). Fix one such edge \( e \) and a vertex \( v \in e \cap V_i \). We delete \( e \) from \( M_1 \) and let \( U' = U \cup (v \setminus \{v\}) \). Clearly, \( i_{\rho}(U') \in L_\rho^\mu(H) \) and \(|U'| = 2k\). Thus, by definition, there exist nonnegative integers \( b_v, c_v \) for all \( v \in L_\rho^\mu(H) \) such that

\[
i_{\rho}(U') = \sum_{v \in L_\rho^\mu(H)} b_vv - \sum_{v \in L_\rho^\mu(H)} c_vv \quad i.e., \quad i_{\rho}(U') + \sum_{v \in L_\rho^\mu(H)} c_vv = \sum_{v \in L_\rho^\mu(H)} b_vv.
\]

By the definition of \( C \), we know that \( b_v, c_v \leq C \). For each \( v \in L_\rho^\mu(H) \), we pick \( c_v \) edges in \( M_1 \) with index vector \( v \). By the equation above, the union of these edges and \( U' \) can be partitioned as a collection of \( k \)-sets, which contains exactly \( b_v \) \( k \)-sets \( F \) with \( i_{\rho}(F) = v \) for each \( v \in L_\rho^\mu(H) \). We repeat the process at most \( \alpha^2n/k \) times until there are exactly \( \ell \) vertices left. Note that for each \( v \in L_\rho^\mu(H) \), our algorithm consumes at most \( (1 + C)a^2n/k < Ca^2n \) edges from \( M_1 \) with index vector \( v \) – this is possible by the definition of \( M_1 \). Moreover, after the process, we get at most \( (2 + |L_\rho^\mu(H)|C)a^2n/k \leq (2 + (k/2)C)a^2n/k < \alpha n \) \( k \)-sets \( S \) with \( i_{\rho}(S) \in L_\rho^\mu(H) \), because \( \alpha \ll 1/k, 1/C \). By the definition of \( F_{abs} \), we can greedily absorb them by \( F_{abs} \) and get a matching \( M_4 \). Thus, we get a near perfect matching of \( H \).

\[ \square \]

3.4. The transferral-free lattices. In this subsection we prove Proposition 3.6. We study the lattice structure \( L_\rho^\mu(H) \) when it contains no transferral.

Fix \( 1 \leq p \leq k-1 \) and any \( p \)-vector \( v \), its neighborhood \( N(v) \) is the set of vectors \( v' \) such that \( v + v' \in L_\rho^\mu(H) \). Note that by definition, the vectors in \( N(v) \) may contain negative coordinates. Moreover, assume \( r = 2 \), we claim that \( N(v) \cap L_{k-p}^\mu = \emptyset \) for any \( 1 \leq p \leq d \) and any \( p \)-vector \( v = (i, p - i) \). Indeed, otherwise, let \( v \) be a \( p \)-vector such that \( N(v) \cap L_{k-p}^\mu = \emptyset \). This implies that the number of edges in \( H[V \setminus V_0] \) with index vector \( i \) such that \( i - v \in L_{k-p}^\mu \) is at most \( |L_{k-p}^\mu| n^{k-p} \leq 2^{k-p} \mu n^k \). Let \( A_v \) be the set of all \( p \)-sets \( S \) with \( i_{\rho}(S) = v \) and thus \( |A_v| = \left( \frac{|V_i|}{|V_i|} \binom{|V_2|}{p} \right) \geq \left( \frac{2^\mu}{2^p} \right) \). By averaging, there is a \( p \)-set \( S \) in \( A_v \) such that

\[
deg_{H}(S) \leq |L_{k-p}^\mu| n^{k-p} |A_v| + |V_0| n^{k-p-1} \leq 2^{k-p} \mu n^k k \left( \binom{k}{p} \right) + \sqrt{\epsilon} n^{k-p} < \gamma \left( \frac{n-p}{k-p} \right),
\]

by \( \mu \ll \epsilon \ll \gamma \). Since \( p \leq d \), this contradicts that \( \delta_\epsilon(H) \geq (g(k, d, 1) + \gamma) \left( \frac{n-d}{k-d} \right) \). Note that a similar argument works for \( r = 3 \), namely, for any \( p \)-vector \( v = (i, i', p - i - i') \) with \( 1 \leq p \leq d \), \( N(v) \cap L_{k-p}^\mu = \emptyset \).

Claim 3.7. Given \( \min\{3, k/2\} \leq d \leq k-2 \) or \( (k, d) = (5, 2) \), \( 0 < \mu \ll \epsilon \ll \gamma \) and let \( n \) be sufficiently large. Let \( H \) and \( P \) be as defined in Proposition 3.6. If \( r = 2 \), then \( (2, -2) \notin L_\rho^\mu(H) \). If \( r = 3 \), then \( (-2, 1, 1), (1, -2, 1), (1, 1, -2) \notin L_\rho^\mu(H) \).

Proof. First assume that \( r = 2 \). Fix \((a_0, b_0) \in L_\rho^\mu(H) \). For the sake of a contradiction, assume that \((2, -2) \notin L_\rho^\mu(H) \). Let \( L_0 \) be the sublattice (subgroup) of \( L_\rho^\mu(H) \) such that \((a, b) \in L_0 \) if \( a + b = 0 \) and let \( L_{k-d} = \{(a, b) \mid a + b = k - d\} \). Let \( t \) be the smallest positive integer such that \((t, -t) \in L_0 \) and it is easy to see that \( L_0 \) is generated by \((t, -t) \). By our assumption, \((1, -1), (2, -2) \notin L_\rho^\mu(H) \), and thus \( t \geq 3 \). Let \( t_0 = \min\{t, k-d+1\} \). It is easy to see that \( L_0 \) partitions \( L_{k-d} \) into \( t_0 \) cosets \( C_0, \ldots, C_{t_0-1} \) such that \( C_i = (k-d-i, i) + L_0 \) for all \( 0 \leq i \leq t_0 - 1 \). For any \( 0 \leq j \leq d \) and \( v_j := (d-j, j) \), we have

\[
N(v_j) = (a_0, b_0) - (d-j, j) + L_0 = (k-d-(b_0-j), b_0-j) + L_0.
\]

This means that \( N(v_j) = C_i \), where \( i_j \equiv b_0-j \mod t_0 \). We split into two cases.

10
Note that if \( N(v_0) \cap N(v_1) \neq \emptyset \), say, \( i \in N(v_0) \cap N(v_1) \), then we have \( i + v_0, i + v_1 \in L_p^\mu(H) \) and thus \( (2, -2) = 2(v_0 - v_1) \in L_p^\mu(H) \), a contradiction. Similarly \( N(v_1) \cap N(v_2) = N(v_0) \cap N(v_2) = \emptyset \) and thus \( N(v_0), N(v_1), N(v_2) \) are pairwise disjoint. Consequently \( \{N(v_0), N(v_1), N(v_2)\} = \{C_0, C_1, C_2\} \). Recall that \( C_i = (k - d - i) + L_0 \) for \( i = 0, 1, 2 \), namely, all \((k - d)\)-vectors with their second coordinates congruent to \( i \) modulo \( 3 \). For \( i = 0, 1, 2 \), let \( B_i \) be the collection of \( d \)-sets with exactly \( j \) vertices in \( V_2 \) such that \( j \equiv i \mod 3 \) and thus \( |B_i| \geq \binom{|V_2|}{d} \binom{|V_1|}{d - j} \geq \left( \frac{c^n}{d} \right) \). By averaging, there exists a \( d \)-set \( S_i \in B_i \) such that

\[
\deg_H(S_i) \leq \sum_{j=0, \text{mod } 3, 0 \leq j \leq k-d} \left( \binom{|V_2|}{j} \binom{|V_1|}{k-d-j} + 2^{k-d} \mu n^k/|B_i| + |V_0|n^{k-d-1} \right)
\]

by \( \mu \ll \epsilon \ll \gamma \). Thus we have

\[
\min_{i=0,1,2} \deg_H(S_i) \leq \min_{i=0,1,2} \left( \sum_{j=0, \text{mod } 3, 0 \leq j \leq k-d} \left( \binom{|V_2|}{j} \binom{|V_1|}{k-d-j} + \frac{\gamma}{2} \binom{n-d}{k-d} \right) \right)
\]

By \([2.4]\), we know that the first term in the right hand side of the inequality above is at most \( (g(k, d, 1) + \gamma/2)(n-d) \), as \( n \) is large enough. Thus, we get \( \delta_d(H) \leq \min_{i=0,1,2} \deg_H(S_i) < (g(k, d, 1) + \gamma)(n-d) \), a contradiction.

**Case 2.** \( t_0 \geq 4 \). Since \( t_0 = \min \{ t, k-d+1 \} \), we have \( t \geq 4 \) and \( d \leq k-3 \). First assume that \( d \geq 3 \). Recall that \( N(v_j) = C_{i_j} \), where \( i_j = b_0 - j \mod t_0 \). Since \( t_0 \geq 4, b_0 - j \) for \( j \in \{0, \ldots, 3\} \) are pairwise distinct. This implies that \( N(v_0), \ldots, N(v_3) \) are four distinct classes. For \( j = 0, \ldots, 3 \), consider the following sums

\[
\sum_{(k-d-i,j) \in C_{i_j}, 0 \leq j \leq k-d} \left( \binom{|V_1|}{k-d-i} \binom{|V_2|}{i_j} \right)
\]

and note that their sum is at most \( \binom{n}{k-d} \). By the pigeonhole principle, there exists \( j' \) such that the \( j'\)-th sum is at most \( \frac{1}{4} \binom{n}{k-d} \). This implies that

\[
\delta_d(H) \leq \frac{1}{4} \binom{n}{k-d} + |I^2_{k-d}| \mu n^k + \binom{|V_1|}{d} + |V_0|n^{k-d-1} < (1/4 + \gamma) \binom{n-d}{k-d},
\]

by \( \mu \ll \epsilon \ll \gamma \). This is a contradiction because \( g(k, d, 1) \geq \frac{2^{k-d}}{3} \geq 1/4 \).

Now we assume \( d = 2 \) and \( d \leq k-3 \), we have \( k = 5 \). Thus, \( k-d = 3, t_0 = 4 \) and for \( i = 0, \ldots, 3 \), \( C_i \cap I^2_3 = \{(3-i, i)\} \). Since \( (1, -1), (2, -2) \notin L_p^\mu(H) \), we know that for \( i = 0, 1, 2, N(v_j) \cap I^2_3 = C_{i_j} \cap I^2_3 = \{(3-i, i)\} \) are distinct. So \( N(v_i) \cap I^2_3 \) equals \( I^2_3 \setminus \{(3, 0)\}, I^2_3 \setminus \{(2, 1)\}, I^2_3 \setminus \{(1, 2)\} \) or \( I^2_3 \setminus \{(0, 3)\} \). If \( \{N(v_i) \cap I^2_3 \}_{i=1,2} = I^2_3 \setminus \{(0, 3)\}, \{I^2_3 \setminus \{(3, 0)\}, \{(2, 1)\}, \{(1, 2)\}\}, \) then

\[
\min \left\{ \binom{|V_1|}{3}, \binom{|V_1|}{2}, \binom{|V_2|}{2} \right\} \leq \min \left\{ \binom{|V_1|}{3} + \binom{|V_2|}{3}, \binom{|V_1|}{2}, \binom{|V_2|}{2} \right\} \leq (g(5, 2, 1) + \gamma/2) \binom{n-2}{3},
\]

by \([2.4]\) and that \( n \) is large enough. By averaging, we get that

\[
\delta_2(H) < (g(5, 2, 1) + \gamma/2) \binom{n-2}{3} + 2^3 \mu n^3/ \left( \binom{c^n}{2} \right) + |V_0|n^2 < (g(5, 2, 1) + \gamma) \binom{n-2}{3},
\]

by \( \mu \ll \epsilon \ll \gamma \), a contradiction. The other three cases are similar.

Second we assume that \( r = 3 \). Indeed, in this case, it suffices to have \( \delta_2(H) \geq (1/4 + \gamma) \binom{n-2}{k-2} \). Consider the set of \( 2 \)-vectors

\[
I^2_3 = \{(2, 0, 0), (0, 2, 0), (0, 0, 2), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}.
\]

Note that \( N((1, 1, 0)) \), \( N((1, 0, 1)) \) and \( N((0, 1, 1)) \) are pairwise disjoint – because \( L_p^\mu(H) \) contains no transfer. Similarly, \( N((2, 0, 0)) \cap N((1, 1, 0)) = \emptyset \) and \( N((2, 0, 0)) \cap N((1, 0, 1)) = \emptyset \) (and similar equations hold...
Proof of Proposition 3.6. Given such a k-graph H and a partition P. The conclusion is trivial if r = 1. So we may assume that r = 2 or 3. We first apply Claim 5.7 and conclude that (2, −2) ∈ L^µ_H (for r = 2) or (−2, 1, 1), (1, −2, 1), (1, 1, −2) ∈ L^µ_H (for r = 3).

If r = 2, then fix any U ⊆ V (H) \ V_0 with i_p (U) = (a, k + 1 − a) for some 0 ≤ a ≤ k + 1 and pick any (a_0, b_0) ∈ P^µ_H (H). Since (2, −2) ∈ L^µ_H (H), then (a_0 + 2i, b_0 − 2i) ∈ L^µ_H (H) for any integer i. Note that a − 1 and a have different parities, so exactly one of (a − 1, k + 1 − a) and (a, k − a) is in L^µ_H (H).

Now assume that r = 3 and consider any U ⊆ V (H) \ V_0 with i_p (U) = (a_1, a_2, a_3) for some nonnegative integers a_1 + a_2 + a_3 = k + 1. Pick any (b_1, b_2, b_3) ∈ P^µ_H (H) and let c_j = a_j − b_j for j ∈ [3]. Note that exactly one of the three consecutive integers c_3 − c_2 − 1, c_3 − c_2 and c_3 − c_2 + 1 is divisible by 3. Thus let i ∈ [3] such that v := (c_1′, c_2′, c_3′) = (c_1, c_2, c_3) − u_i satisfies that c_3′ − c_2′ is divisible by 3. Let m = (c_3′ − c_2′)/3 and m′ = m + c_2′. Note that c_2′ + c_3′ = 0 and it is easy to see that

\[ i_p (U) − u_i = (b_1, b_2, b_3) = v = m′ (2, 1, 1) − m (1, 1, −2) \in L^µ_H (H). \]

Thus, i_p (U) − u_i ∈ L^µ_H (H) and the proof is complete.

4. Proof of Theorem 1.8

We prove Theorem 1.8 in this section. Note that \( 1 − (1 − 1/k)^k = 2k^{-1} \) when \( d = k − 2 \). For \( k = 4, 5, 2k^{-1} > 1/3 > g(k, k, k, 1) \) and thus for this range, the result is covered by Corollary 1.13. For the cases \( k \geq 6 \), we will use the absorbing method adapted from [24].

Given a set S of k + 2 vertices, we call an edge e ∈ E (H) disjoint from S S-absorbing if there are two disjoint edges e_1 and e_2 in E (H) such that |e_1 ∩ S| = k − 2, |e_1 ∩ e| = 2, |e_2 ∩ S| = 4, and |e_2 ∩ e| = 4 − k. Note that this is not the absorbing in the usual sense because e_1 ∪ e_2 misses two vertices of S ∪ e. Let us explain how such absorbing works. Let S be a (k + 2)-set and M be a matching, where V (M) ∩ S = ∅, which contains an S-absorbing edge e. Then M can “absorb” S by replacing e in M by e_1 and e_2 (two vertices of e become uncovered).

Lemma 4.1. For all k ≥ 6, c > 0 there exists \( \beta_0 > 0 \) such that the following holds for all 0 < \( \beta \leq \beta_0 \) and sufficiently large integer n. Let H be an n-vertex k-graph with \( \delta_{k−2} (H) \geq cn^2 \), then there exists a matching \( M' \) in H of size \( |M'| \leq \beta n \) such that for every \( (k + 2) \)-tuple S of vertices of H, the number of S-absorbing edges in \( M' \) is at least \( \beta^2 n \).

Proof. Our proof is adapted from the proofs of [24] Fact 2.2, Fact 2.3. Let \( \beta_0 = c^3/(12k!) \) and \( 0 < \beta \leq \beta_0 \). Let H be an n-vertex k-graph with n sufficiently large and \( \delta_{k−2} (H) \geq cn^2 \). Given any \( (k + 2) \)-set of vertices S, we will show that there are many S-absorbing edges. Let us fix four vertices u_1, ..., u_4 in S and count only those S-absorbing edges e for which the corresponding edge e_2 contains u_1, ..., u_4. We count the ordered k-tuples of distinct vertices \( (v_1, ..., v_k) \) such that e = \{v_1, ..., v_k\} is disjoint from S, \( e_1 \cap e = \{v_{k−3}, v_{k−2}\} \) and \( e_2 = \{v_1, ..., v_{k−4}, u_1, ..., u_4\} \), and divide the result by k!. For each \( j = 1, ..., k−6 \), there are precisely \( n−j−k \) choices of vertex v_j. Having selected \( v_1, ..., v_{k−6} \), each of \( \{v_5, v_6, ..., v_k\}, \{v_{k−3}, v_{k−2}\} \) and \( \{v_{k−1}, v_k\} \) must be a neighbor of an already fixed \( (k + 2) \)-tuple of vertices. Thus, there are at least \( \delta_{k−2} (H) − 2kn \) choices for each pair. Altogether since n is large enough,
there are at least $(n - 2k)^{k-6}(\delta_{k-2}(H) - 2kn)^3 \geq \frac{1}{2}c^3n^k$ choices of the desired ordered $k$-tuples. So there are at least $\frac{1}{2}c^3n^k/k!$ $S$-absorbing edges in $H$.

Now we pick the absorbing matching $M'$. Select a random subset $M$ of $E(H)$, where each edge is chosen independently with probability $p = \beta n^{1-k}$. Then, the expected size of $M$ is at most $\binom{n}{k}p < \beta n^{1-k}$, and the expected number of intersecting pairs of edges in $M$ is at most $n^{2k-1}p^2 = \beta^2 n$. Hence, by Markov's inequality, with probability at least $1 - \frac{1}{2} - 1/k!, |M| \leq \beta n$ and $M$ contains at most $2\beta^2 n$ intersecting pairs of edges. Moreover, for every $(k + 2)$-set of vertices $S$, let $X_S$ be the number of $S$-absorbing edges in $M$. Then we have

$$E(X_S) \geq p \cdot \frac{1}{2}c^3n^k/k! \geq \frac{\beta^3n}{2k!}.$$  

By Chernoff’s bound, with probability $1 - o(1)$, we have that $X_S \geq \frac{1}{4}E(X_S) \geq \frac{\beta^3n}{4k!}$ for all $(k + 2)$-sets $S$ in $H$.

Thus, there is an $M \subseteq E(H)$ satisfying all the properties above. We delete one edge from each intersecting pair of edges and denote the resulting matching by $M'$. So $|M'| \leq \beta n$ and for every $(k + 2)$-set of vertices $S$, $M'$ contains at least $\frac{\beta^3n}{4k!} - 2\beta^2 n \geq 2\beta^2 n$ $S$-absorbing edges, by the definition of $\beta$. □

Proof of Theorem 1.6. Fix $\gamma > 0$. We apply Lemma 4.1 with $c = \frac{\beta \gamma}{\sqrt{\beta}}$ and get $\beta_0$. Let $\beta = \min\{\beta_0, \gamma/(3k)\}$. Let $H$ be an $n$-vertex $k$-graph such that $n \equiv \ell \mod k$ for some $\ell \in \{2, \ldots, k - 1\}$ is sufficiently large and $\delta_{k-2}(H) \geq \frac{2k \ell - 1}{k^2} + \gamma \frac{n}{k^{k+2}}$. We apply Lemma 4.1 and get the absorbing matching $M'$ of size at most $\beta n$ and satisfying the absorbing property.

Let $H' = H[V(H) \setminus V(M')]$ and $n' = |V(H')|$. Note that

$$\delta_{k-2}(H') \geq \left(\frac{2k \ell - 1}{k^2} + \gamma\right)\frac{n-k+2}{2} - k\beta n(n-1) \geq \left(\frac{2k \ell - 1}{k^2} + \gamma/2\right)\frac{n'-k+2}{2}.$$  

Thus, by Theorem 1.6, $H'$ contains a matching $M_1$ that leaves at most $\beta^2 n$ vertices uncovered. Fix any $(k + 2)$-tuple of uncovered vertices $S$, $M'$ contains at least $\beta^2 n$ $S$-absorbing edges. Fix an $S$-absorbing edge $e$, we replace $M'$ by $M_2 := (M' \{e\}) \cup \{e_1, e_2\}$, decreasing the number of uncovered vertices by $k$. Since we have at most $\beta^2 n/k$ iterations, there will always be an available $S$-absorbing edge in $M'$. In the end, we have exactly $\ell$ vertices left uncovered and we are done. □

5. Concluding Remarks

We remark that Proposition 3.6 is the bottleneck in the proof of Theorem 1.7. Indeed, in the current proof, by the assumption $\delta_d(H) \geq (g(k, d, 1) + o(1))\binom{n-d}{k-d} \geq (1/4 + o(1))\binom{n-d}{k-d}$, it suffices to study vertex partitions $P = \{V_0, \ldots, V_r\}$ with $r \leq 3$. So, to improve Theorem 1.7 one has to analyze the vertex partition with more parts and prove a stronger version of Proposition 3.6.

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