Time dependent models for the interaction of energetic particles in the ISM

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Abstract. We present a general model allowing one to calculate the distribution function of energetic particles (EPs) in the interstellar medium (ISM), and hence any relevant nuclear reaction rate, for any given time-dependent injection function, as well as in situations when the propagation coefficients themselves are non stationary. We review and provide physical interpretation of general formal solutions of a propagation equation taking into account energy losses, nuclear destruction and escape. Both one-zone and extended models are investigated. Our main goal is to provide standard and general tools for subsequent use in nuclear astrophysics, notably to calculate the gamma-ray emission and spallation rates associated with energetic events and specific acceleration mechanisms. Considering the stationary limit of the model, we show that only time-dependent calculations can probe the density dependence of the physical processes under study, while steady-state models can only give information about density-independent processes.

Key words: Energetic particles: theory – Nuclear reactions – ISM: cosmic rays

1. Introduction

The energetic particles (EPs) present in the Galaxy can be probed through their electromagnetic and nuclear interactions within the interstellar medium (ISM). These interactions can lead to different types of radiation, lines (e.g. from nuclear de-excitation or pair annihilation) or continuum (e.g. synchrotron, bremsstrahlung, inverse Compton or \(\pi^0\) disintegration), as well as to the synthesis of secondary nuclei such as Li, Be and B (LiBeB), whose astrophysical significance is well established (e.g. Reeves et al. 1970; Meneguzzi et al. 1971; Fields et al. 1994; Reeves 1994; Ramaty et al. 1996, Vangioni-Flam et al. 1996).

Basically, the calculation of spallation and gamma-ray production rates at any place in the Galaxy requires the knowledge of i) the cross sections for relevant physical processes and ii) the distribution function of all the interacting EP species, i.e. their energy spectrum and number density. In this paper, we indicate how to calculate these distribution functions in some general situations where an exact formal solution exists. The different reaction rates can then be obtained by merely integrating the corresponding cross sections over the EP spectral densities, given the local ISM (target) density and chemical composition. The knowledge of the local magnetic field and radiation background further allows one to compute the synchrotron and inverse Compton continuum emission in much the same way.

In most previous nuclear astrophysics calculations, a steady-state assumption has been used, in which the steady injection of EPs in the region under study (generally as the result of some particle acceleration process) is counterbalanced by a drift in energy space due mainly to Coulombian energy losses. The resulting EP distribution functions are thus independent of time, as are consequently all the reaction rates. In this paper, we investigate the case when i) the injection of EPs and ii) the conditions of their propagation in the surrounding medium are functions of time. The former type of time dependence arises naturally when the acceleration conditions in a specific region are changing, for instance when a supernova explodes, allowing for acceleration locally in space and time. As for the latter, it can happen when the density and/or composition of the ambient medium varies, as for example in an expanding medium such as the interior of a supernova remnant (Parizot & Drury 1999a,b).

In general, a steady-state model is sufficient provided that the time scale of the changes in the injection and/or propagation conditions is larger than the time needed for the EP distribution function to reach its equilibrium, that is, basically, the time scale of energy losses or escape out of the region considered. In this case, the history of EP-ISM interactions is fairly well reproduced by juxtaposing different phases in which the steady-state approximation holds. Such a strategy has been used extensively for instance to describe the Galactic evolution of light element abundances, as induced by GCR interactions with the ISM (most recently, Fields & Olive 1999; Vangioni-Flam et al. 1999). Indeed, the spallation rates at time \(t\) depend on the current ambient metallicity and the current flux of GCR,
which are both functions of time, but with time scales presumably larger than the GCR confinement time. There are some situations, however, where the steady-state approximation does not hold (Parizot et al. 1997a,b,c; Parizot & Drury 1999a,b), and a time dependent model is needed.

It has to be noted that formal solutions of the propagation equation have been known for several decades (e.g. Syrovatskii, 1959; Jones, 1970; Reames, 1974; Ramaty, 1974) and used extensively in the cosmic-ray transport theory, with additional features and complications, such as convection, re-acceleration, spatially dependent diffusion, etc. (Lerche and Schlickeiser 1981, 1988; Schlickeiser 1986; Wang and Schlickeiser, 1987). However, the case when the energy losses and/or the diffusion coefficient are explicitly depending on time, and not only on space, has not yet been considered, to the best of our knowledge, while it is relevant to many applications in nuclear astrophysics (e.g. Parizot & Drury 1999a,b). Moreover, most of the situations of interest for spallative nucleosynthesis and gamma-ray line astronomy do not require the refinements mentioned, so we shall try to summarize and present in a clear and unified way the only results which one needs to develop a general time-dependent code for nuclear astrophysics calculations. In particular, we shall emphasize the simple and intuitive physical meaning of the solutions, describing with special care the more original features, especially useful for numerical implementation, such as the temporal connection of solutions obtained in different phases of the EP propagation.

2. Basic physical ingredients

The general structure of a spallation or gamma-ray production calculation can be divided into three stages: acceleration (production of the EPs), propagation (transport in the surrounding medium) and interaction (with the surrounding matter or radiation field). These three logical stages are not necessarily separated in time, as (re-)acceleration may occur while the EPs are propagating, and interactions with the ambient medium just cannot be avoided at any stage of the process. In many situations, however, the acceleration time scale of the particles is much shorter than their interaction and energy loss time scales, so that acceleration can actually be treated simply as injection (of EPs). This means that the acceleration itself is not affected by the conditions of propagation, and therefore the two stages are disconnected. In this case, we are solely concerned with the future of the EPs once they have been accelerated (or advected from a distant acceleration site) and injected into the region of space under study.

Accordingly, the first physical ingredient that we need to know is the injection function, \( q_i(E, r, t) \), defined as the number of particles of species \( i \) introduced (injected) at energy \( E \), location \( r \) and time \( t \), per unit energy, volume and time (in \( \text{MeV/n}^{-1}\text{cm}^{-3}\text{s}^{-1} \)). This injection function can be either postulated, in order to reproduce specific astronomical observations, or calculated as the output of some known acceleration mechanism.

The treatment of the two subsequent stages, propagation and interaction, is also made easier by artificially separating them in time, which can be legitimated in most cases as discussed below. In any case, the influence of the propagation of the EPs on their distribution function can be derived from the knowledge of: i) spatial diffusion properties, ii) drift properties in the energy space, iii) the local production rate (as secondary nuclei) and iv) the 'catastrophic' loss rate. By catastrophic loss we mean the outright disappearance of the particle, either by escape out of the region under study, destruction in a nuclear reaction (in-flight spallation), or radio-active decay in the case of unstable particles. This is mathematically described by the total loss time, \( \tau_{\text{esc}} \), which depends on the nuclear species considered, \( i \), and is \textit{a priori} a function of energy, location and time. It includes the escape, nuclear destruction and decay times according to:

\[
\frac{1}{\tau_{\text{esc}}(E, r, t)} = \frac{1}{\tau_{\text{tot}}(E, r, t)} + \frac{1}{\tau_{\text{D}}(E, r, t)} + \frac{1}{\gamma(E)\tau_{\text{dec}}},
\]

where \( \gamma(E) \) is the Lorentz factor of the particle.

Strictly speaking, the escape time, \( \tau_{\text{esc}} \), is relevant only to one-zone models, when the \( r \) variable is meaningless. Indeed, in spatially extended models, the escape of particles out of the region under study is taken automatically into account through diffusion, and should not be included in \( \tau_{\text{esc}} \).

The nuclear destruction time, \( \tau_{\text{D}} \), is obtained from the total inelastic cross sections \( \sigma_{i,j} \) for a projectile \( i \) in a target of species \( j \), as:

\[
\frac{1}{\tau_i(D)(E, r, t)} = (\sum_{j} \sigma_{i,j}(E)n_j(r, t)|v(E)|,
\]

where \( v(E) \) is the velocity of the particle and \( n_j(r, t) \) is the local number density of the target nuclei of species \( j \), at time \( t \).

Similarly, the production rate of the EP species \( k \) as secondary nuclei, through all the nuclear reactions \( i+j \rightarrow k \) with cross section \( \sigma_{i,j,k} \), is given by:

\[
q_k'(E, r, t) = \sum_{i,j} \int_{0}^{\infty} \text{d}E' n_i(E', r, t)n_j(r, t)\sigma_{i,j,k}(E')v_i(E'),
\]

where \( n_i(E, r, t) \) is the distribution function of energetic nuclei of species \( i \), i.e. their number per unit volume and energy (in \( \text{MeV/n}^{-1}\text{cm}^{-3} \)). Although \( q_k'(E, r, t) \) may be thought of as part of the injection function \( q_i(E, r, t) \), it should be noted that it is actually the part of the injection rate which depends on the EP content of the ISM. This makes \( q_k'(E, r, t) \) mathematically more difficult to implement than
the \textit{primary} injection function, \(q_i\), which is the same whatever the local and current distribution functions \(n_i\) may be.

Apart from being created (synthesized) and removed by catastrophic losses, EPs are subject to energy losses which modify the shape of their distribution function. Whether coulombian, synchrotron, inverse Compton, adiabatic or of any kind, these energy losses can generally be expressed by means of an \textit{energy loss function}, \(dE/dt = E_i(E, r, t)\), giving the energy loss rate \((E_i < 0)\) of nuclei \(i\) (in \(\text{(MeV/\text{n})s}^{-1}\)) as a function of energy, location and time. Note that the energy loss function can also include positive terms corresponding to (non diffusive) steady acceleration.

The last physical ingredient that we need to know is the diffusion coefficient \(D_i(E, r, t)\). Together with \(q_i(E, r, t)\), \(\tau_{\text{tot}}^i(E, r, t)\) and \(E_i(E, r, t)\), it allows one to calculate the distribution function of each EP species, \(n_i(E, r, t)\), at least in principle. The rate of any given reaction at any place and time is then obtained by integrating the relevant cross section over \(n_i(E, r, t)\) in the energy space, just as in Eq. (3), where the index \(k\) may represent any secondary nucleus or photon species (e.g. from a given gamma-ray line) in which we are interested.

The above ingredients being assumed known, we can now write down the propagation equation and its solution in a few general cases.

3. One-zone models with stationary propagation conditions

In a number of cases, it is useless to study the spatial diffusion of the EPs, not because it is negligible, but because we are interested in the gamma-ray emission or secondary nuclei production integrated over a sufficiently large volume to be thought of as a box without internal structure. A ‘one-zone model’ can then be used, in which the space coordinates are purely and simply dropped, and the EP distribution function, \(n_i(E, r, t)\), is replaced by its integral over the volume, namely the spectral density, \(N_i(E, t)\), defined as the differential number of EPs at energy \(E\) and time \(t\) (in \(\text{MeV/\text{n}}\)) in the whole box. For example, such models are appropriate for the evaluation of reaction rates in a homogeneous medium, or more generally in an instrumentally unresolved region of space.

In a one-zone model, we have to solve the EP propagation equation in the following form:

\[
\frac{\partial}{\partial t} N_i(E, t) + \frac{\partial}{\partial E} \left(E_i(E)N_i(E, t)\right) = Q_i(E, t) + Q'_i(E, t) - \frac{N_i(E, t)}{\tau_{\text{tot}}^i}, \tag{4}
\]

which merely expresses that the rate of change of the spectral density is equal to what is injected minus what is lost. \(Q_i(E, t)\) and \(Q'_i(E, t)\) are the integrals of \(q_i(E, r, t)\) and \(q'_i(E, r, t)\) (i.e. Eq. 3) over the volume of the box, and the second term in the left hand side describes the drift in the energy space, due to energy losses.

Let us now distinguish between primary EPs, for which \(Q_i(E, t) \gg Q'_i(E, t)\), and secondary EPs, for which \(Q_i(E, t) \ll Q'_i(E, t)\). Primary EPs are typically protons, alpha particles and abundant materials such as C, N, O, Mg, Si, Fe, etc., whose production rates by spallation are usually much smaller than their injection rate from the acceleration of ambient materials. Energetic electrons are also mainly primary EPs. On the contrary, nuclei such as Li, Be and B, as most of the usual spallation products, are secondary EPs. This is also the case of positrons and anti-protons, both of which may be important tracers of cosmic-ray propagation. Such a distinction simplifies the calculation both conceptually and mathematically. Indeed, we shall first solve the propagation equation for the primary nuclei, with the term \(Q'_i(E, t)\) dropped, and then use the spectral densities obtained in this ways to calculate the production rates of secondary nuclei. For EP species with \(Q_i(E, t)\) and \(Q'_i(E, t)\) of the same order of magnitude, one has to solve first the propagation equation for the parent nuclei, deduce \(Q'_i(E, t)\) and then solve Eq. (4) again, in a (hopefully quickly convergent) iterative way.

The formal solution of Eq. (4) reads:

\[
N_i(E, t) = \frac{1}{|E_i(E)|} \int_{E}^{+\infty} Q_i(E_0, t - \tau_i(E_0, E)) \times \exp \left(- \int_{E_0}^{E} \frac{dE'}{E_i(E')} \tau_{\text{tot}, i}(E') \right) dE_0, \tag{5}
\]

where the function \(\tau_i(E_0, E)\) has an important physical meaning. It is defined for each EP species \(i\) as:

\[
\tau_i(E_0, E) = \int_{E_0}^{E} \frac{dE'}{E_i(E')}. \tag{6}
\]

so it has the dimensions of a time, and describes the drift in energy of the EPs, due to the various energy loss mechanis-
otherwise corrects the contribution of the EPs injected at energies $E_0 \geq E$ by weighting it by the probability of survival during the time needed to slow down from energy $E_0$ to energy $E$, that is during the drift time $\tau_i(E_0, E)$. This survival probability is $P_i = \exp(-\langle \tau_i / \tau_{\text{tot}} \rangle)$, where the average accounts for the energy dependence of catastrophic losses, $\tau_{\text{tot}}(E')$, which are indeed varying during the energy drift. The time spent by the particles between energies $E'$ and $E' + dE'$ being $dt(E') = dE' / \dot{E}_i(E')$, one gets:

$$\left\langle \frac{\tau_i}{\tau_{\text{tot}}} \right\rangle = \int_{E_0}^E \frac{dE'}{\dot{E}_i(E') \tau_{\text{tot}, 1}(E')}.$$  

which justifies the form of the exponential factor in Eq. (5).

4. One-zone models with time-dependent conditions of propagation

In the previous section we assumed that the energy loss rate, $\dot{E}_i(E)$, depended on the EP energy, but not on time. Now let us consider the case of ionization losses in a medium of density $\rho_0$, whose chemical composition is given by the relative abundance (by number) of each nuclear species, $j$: $\alpha_j$. Let $dE / dx |_j$ be the particle energy loss per unit grammage passed through in a medium of pure $j$ nuclei (expressed in g/cm$^2$). Then the energy loss rate of particles $i$ in the ISM is given by:

$$\dot{E}_i(E) = \frac{dE}{dx} |_j \times \alpha_j \rho_0 v.$$  

As a consequence, the EP energy loss rate cannot be considered constant whenever the density or composition of the propagation medium are functions of time. Now such situations are not unusual in astrophysics. One has to think of the medium surrounding an active massive star: it is composed of successive layers of different density and composition, resulting from successive phases of stellar wind (main sequence, red supergiant, Wolf-Rayet of type N, C, etc.). Such a situation has been considered by Parizot et al. (1997a,b,c). Likewise, EPs interacting in the interior of a supernova remnant will experience variations in the ambient density and composition (due to the expansion of the remnant and the dilution of the ejecta by the swept-up material). This situation is addressed in detail in Parizot & Drury (1999a,b).

To deal with this non-stationarity of the conditions of propagation, we divide the process into successive phases during which the density and composition of the target can legitimately be considered constant. The solution of the propagation equation restricted to any individual phase is obtained in exactly the same way as above (Eq. (5)), and the only technical difficulty lies in the proper connection between the successive stationary solutions. To see how this works, let us consider the situation of Fig. 6.

We are interested in the EP distribution function at a succession of times $t$, during three distinct phases following the onset of the EP injection process, assumed to occur at $t = 0$ (by definition, the ambient density and composition are constant during each of these phases). The set of these distribution functions would typically allow one to calculate observables such as the gamma-ray line emission rate as a function of time. Let us first calculate the distribution function during phase 1, i.e for $0 \leq t \leq t_1$, where $t_1$ marks the end of the first phase. Since the conditions of propagation are constant during each phase, the EP distribution function at any time before $t_1$ is readily given by Eq. (6). Dropping the species index, $i$, for convenience and noting $\dot{E}_1(E)$ and $\tau_1(E_0, E)$ the energy loss function and the energy drift function during phase 1, we get:

$$N(E, t) = \frac{1}{|E_1(E)|} \int_0^\infty dE' Q(E', t - \tau_1(E', E)) P_1(E', E),$$  

where $P_1(E', E) = \exp(-\langle \tau_i / \tau_{\text{tot}, 1} \rangle)$ is the probability (calculated under the conditions of phase 1) for an EP injected at energy $E'$ to reach energy $E$ without escaping or being altered by nuclear reactions (see Eq. (6)).

Note that since the injection of EPs began at time $t = 0$, the injection rate is null at negative times, so that $Q(E', t - \tau_1(E', E))$, considered as a function of the injection energy, $E'$, vanishes for all energies greater than a maximum value, $E_0$, defined by:

$$\tau_1(E_0, E) = t.$$  

Indeed, any particle injected at a higher energy would need a time greater than $t$ to slow down to energy $E$, and consequently cannot contribute to $N(E, t)$. The upper limit in the integral of Eq. (9) can thus be replaced by the energy $E_0(E, t)$, implicitly defined in Eq. (10). In order to shorten the writing of equations, we define the retarded injection function, $f_1(E', E)$, as:

$$f_1(E', E) = Q(E', t - \tau_1(E', E)) \times P_1(E', E),$$  

whose physical meaning is straightforward, so that Eq. (9) can be re-written as:

$$N(E, t) = \frac{1}{|E_1(E)|} \int_E^{E_0(E, t)} f_1(E', E) dE'.$$  

Let us now calculate the EP spectral density, $N(E, t)$, during phase 2, i.e. at times $t_1 \leq t \leq t_2$. One can still define a limiting energy, $E_1(E, t)$, such that the particles having this energy at time $t_1$ (end of phase 1, beginning of phase 2) have exactly energy $E$ at time $t$. $E_1(E, t)$ is defined similarly to $E_0$ in Eq. (11), except that the energy drift function, $\tau$, now corresponds to the specific propagation conditions prevailing during phase 2:

$$\tau_2(E_1, E) = t - t_1.$$  

As above, any particle injected at an energy $E' \geq E_1$ cannot contribute to the spectral density $N(E,t)$ unless it has been injected earlier than $t_1$, i.e. during phase 1.

Had the injection of EPs began at time $t_1$, we could easily write the solution of the propagation equation during phase 2, just as above:

$$N(E,t) = \frac{1}{|E_2(E)|} \int_{E_0(E,t)}^{E_1(E,t)} f_2(E',E)dE', \quad (14)$$

where the retarded injection function is now

$$f_2(E',E) = Q(E',t-\tau_2(E',E)) \times P_2(E',E), \quad (15)$$

and $\tau_2(E',E)$ and $P_2(E',E)$ are the new energy drift function and EP survival probability, acknowledging the new target density and composition (i.e. corresponding to phase 2). But to obtain the actual EP spectral density, we still have to add the contribution of all the EPs injected during phase 1. Now by definition of $E_1(E,t)$, $N(E,t)$ collects at time $t$ all the EPs which had energy $E_1(E,t)$ at the end of phase 1, i.e. at time $t_1$. But their number per unit energy, $N(E_1(E,t), t_1)$, has already been calculated. It is given by Eq. (3) (or its condensed version Eq. (2)):

$$N(E_1(E,t), t_1) = \frac{1}{|E_1(E_1(E,t))|} \times \int_{E_0(E,t)}^{E_1(E_1(E,t))} f_1(E',E_1(E,t))dE', \quad (16)$$

where we recognized that

$$E_0(E_1(E,t), t_1) = E_0(E,t). \quad (17)$$

Before adding this contribution from phase 1 to the spectral density of the EPs at time $t$, we still need to take into account the catastrophic losses of particles since the end of phase 1. This is achieved by multiplying $N(E_1(E,t), t_1)$ by the survival probability (under the conditions of phase 2) from energy $E_1(E,t)$ to energy $E$.

Finally, we must remember that the distribution function $N(E,t)$ gives the differential number of EPs at energy $E$, that is the number of EPs between $E$ and $E + dE$. Accordingly, we must re-scale $N(E_1(E,t), t_1)$ by a factor $|\hat{E}_2(E_1(E,t))|/|\hat{E}_2(E)|$ representing the contraction (or dilation) of the ‘comoving’ energy interval.

Summing up all this ‘book keeping’, we finally write the spectral density of the EPs at any time $t$ during

**Fig. 1.** Diagram corresponding to an EP propagation process divided into three phases, during which the ambient density and chemical composition, and thus the energy loss rate, are assumed constant. Instants $t_1$ and $t_2$ correspond to the changes of phase, and $t$ is the ‘current’ time, at which we want to calculate the EP energy spectrum. The energy axis goes backward in time, because the energy losses produce a continual decrease of the energy of particles. Particles having energy $E$ at time $t$ (whose density, $N(E,t)$, we are looking for), had energy $E_2$ at time $t_2$, $E_1$ at time $t_1$, and $E_0$ at time $t_0 = 0$. These limiting energies of course depend on $E$ and $t$. The diagram can be easily generalized to as much phases as one wishes. The contribution of all phases to the current spectral density must be added, that of phase $i$ being limited to a domain of energy stretching from $E_{i-1}$ to $E_i$ (see text).
phase 2 as :

\[ N(E, t) = \frac{1}{|E_2(E)|} \int_E^{E_1(E, t)} f_2(E', E) dE' + N(E_1(E, t), t_1) P_2(E_1(E, t), E) \frac{|E_2(E_1(E, t))|}{|E_2(E)|} , \]  

\[ (18) \]

or, substituting from Eq. (16) and conveniently abbreviating \( E_i(E, t) \) to \( E_i \),

\[ N(E, t) = \frac{1}{|E_2(E)|} \left[ \int_E^{E_1(E)} f_2(E', E) dE' + \frac{|E_2(E_1)|}{|E_1(E_1)|} P_2(E_1, E) \int_{E_1}^{E_0} f_1(E', E_1) dE' \right] . \]

\[ (19) \]

Note that we could have alternately obtained Eq. (15) by integrating the propagation equation, (4), beginning at time \( t_1 \) and replacing the injection function \( Q(E, t) \) by an effective injection function, \( Q_{\text{eff}}(E, t) \), defined by :

\[ Q_{\text{eff}}(E, t) = Q(E, t) + N(E, t_1) \delta(t - t_1) . \]

\[ (20) \]

Although mathematically equivalent, we feel however that the derivation given above gives a deeper physical insight of the situation.

We finish the study of the multiphase case by computing the EP spectral density during phase 3. The calculation leading to Eq. (13) generalizes straightforwardly if we introduce the energy, \( E_2(E, t) \), that a particle of current energy \( E \) (i.e. at time \( t \)) had at the time \( t_2 \) marking the end of phase 2 and the beginning of phase 3. As above, \( E_2(E, t) \) is defined by :

\[ \tau_3(E_2, E) = t - t_2 , \]

\[ (21) \]

where the subscript ‘3’ refers to the conditions of propagation during phase 3.

Recursively adding the contributions of each phase to the spectral density at time \( t \), we obtain :

\[ |\hat{E}_3(E)| N(E, t) = \int_E^{E_2} f_3(E', E) dE' + \frac{|\hat{E}_3(E_2)|}{|E_2(E_2)|} P_3(E_2, E) \int_{E_2}^{E_1} f_2(E', E_2) dE' + \frac{|\hat{E}_2(E_1)|}{|E_1(E_1)|} |\hat{E}_3(E_2)| P_3(E_1, E_2) P_3(E_2, E) \]

\[ \times \int_{E_1}^{E_0} f_1(E', E_1) dE' , \]

\[ (22) \]

with straightforward generalization to any number of phases.

5. Extended models with homogeneous spatial diffusion

Although one-zone models are both powerful and easy to handle, gamma-ray observations with high angular resolution might make it also necessary to use extended models, allowing one to study inhomogeneous astrophysical sites. As an example, one may be interested in situations where energetic particles are accelerated in a small region of space and diffuse in the surrounding medium, exploring sites with various densities and magnetic properties. The EP distribution function, and hence the gamma-ray emission, should thus be different in neighboring regions which can be resolved by the instruments. Moreover, as was discussed in Parizot (1997), the detailed study of spatial diffusion is sometimes indispensable, even in homogeneous media, if the loss of EPs by escape out of the confinement region has to be taken into account.

Let us then write the propagation equation of the EPs in the case of an extended model. We are now interested in the EP distribution function, \( n_i(E, r, t) \), rather than in the integrated spectral density, \( N_i(E, t) \). It satisfies an equation similar to Eq. (4), with new terms describing the spatial transport of particles. In a number of applications, convection can be neglected, because one can work in the referential of the interstellar plasma, or average over large enough regions of space for the plasma to be considered globally at rest. Keeping only the diffusion term, we then have the EP propagation equation in the extended case :

\[ \frac{\partial}{\partial t} n_i(E, r, t) + \frac{\partial}{\partial E} (\hat{E}_i(E) n_i(E, r, t)) = q_i(E, r, t) - n_i(E, r, t) \frac{\tau_{\text{tot}}}{\tau_{\text{tot}}} + \nabla (D(E, r, t) \nabla n_i(E, r, t)) . \]

\[ (23) \]

This equation can of course be solved numerically, but it is much better (be it only regarding computation time) to solve it formally once for all, for any injection function \( q_i(E, r, t) \), as in the one-zone case. To do this, however, we shall assume that the diffusion coefficient does actually not depend on the location, \( r \), and time, \( t \). On the other hand, it can still depend on (and be any function of) the energy, as has to be the case in most realistic astrophysical situations. Note that although the condition \( D(E, r, t) = D(E) \) may be too restrictive in some specific occasions, there will be a way around it in most cases. Indeed the diffusion conditions (gathered mathematically in the diffusion coefficient, \( D \)) do not generally change smoothly, but rather sharply, while passing from one type of environment to another, like from the ISM to a dense cloud or from a superbubble to the ISM. In such cases, one can use a procedure analogous to that of Sect. 4 and solve the propagation equation locally, in regions with homogeneous diffusion, and connect the solutions at the boundaries in much the same way as we did above in the multi-phase case. In practical terms, this amounts to adding to the injection function, \( q_i(E, r, t) \), a term of the form \( \int_S \delta(r - r_0) dS \).
corresponding to the flux of particles across the boundary of each region. This flux 'inherited' from the propagation of EPs in neighboring regions is exactly analogous to the spectral density 'inherited' from the propagation during the earlier phase, as we encountered it in Sect. 1.

The solution of Eq. (23) (with homogeneous diffusion) can be given in a convenient way by introducing a new function, \( \chi_i(E_0, E) \), in the spirit of the energy drift function given by Eq. (11):

\[
\chi_i(E_0, E) = \int_{E_0}^{E} D(E') \frac{dE'}{E(E')}. \tag{24}
\]

Physically, \( \chi_i(E_0, E) \) represents the effective value of 'Dt' when the particle slows down from energy \( E_0 \) (at which it has been injected) to energy \( E \) (which it has at time \( t \)). This is easily seen by noting that \( dE'/E(E') \) represents the time passed by the particle between energies \( E' + dE' \) and \( E' \). The introduction of this 'effective' diffusion parameter is necessary because the energy of the particle decreases with time, while the diffusion coefficient depends on energy. In the case when \( D(E) = D_0 \) is constant, we have in fact \( \chi_i(E_0, E) = D_0 \tau_i(E_0, E) \). Note finally that \( \chi_i(E_0, E) \) has the dimensions of a surface, so we shall call it the diffusion section of the particle (when passing from \( E_0 \) to \( E \)).

We can now give the EP distribution function at energy \( E \), location \( r \) and time \( t \), in the case of a homogeneous, though energy dependent diffusion coefficient:

\[
n_i(E, r, t) = \frac{1}{|\mathcal{E}_i(E)|} \times \int_{E}^{\infty} dE' \int dE' f_i(r', E', E) g_i(r', r; E', E), \tag{25}
\]

where \( f_i \) is the 3D generalization of the retarded injection function (Eq. (11)):

\[
f_i(r', E'; E, E) = q_i(E', E', t - \tau_i(E', E)) \times \exp \left( -\int_{E'}^{E} \frac{dE''}{E_i(E'') \tau_{\text{tot}, i}(E'')} \right), \tag{26}
\]

and \( g_i \) is given by:

\[
g_i(r', r; E', E) = \frac{1}{8(\pi\chi(E', E'))^{3/2}} \exp \left( -\frac{|r - r'|^2}{4\chi(E', E')} \right). \tag{27}
\]

This last function proves to have a very simple physical interpretation. To see this, let us consider the simplest equation of diffusion, for identical particles in a homogeneous and isotropic medium, \( \partial N/\partial t - D\Delta N = 0 \), where \( N \) is the number density of particles, \( D \) is the diffusion coefficient, and \( \Delta \) denotes the Laplacian operator. In an infinite medium, if \( N_0 \) particles are initially injected at \( r = r_{\text{in}} \), then we have the well-known solution:

\[
N(r, t) = \frac{N_0}{8(\pi Dt)^{3/2}} \exp \left( -\frac{|r - r_{\text{in}}|^2}{4Dt} \right). \tag{28}
\]

One way to look at this solution, emphasizing the stochasticity of the diffusion process, is to say that if a particle is introduced in \( r_{\text{in}} \) at \( t = 0 \), the density of probability that it be in \( r \) at time \( t \) is \( \mathcal{P}_i(r_{\text{in}}, r) = N(r, t)/N_0 \). Now comparing the expressions for \( \mathcal{P}_i(r_{\text{in}}, r) \) and \( g(r', r; E', E) \) (given by Eq. (27)), while remembering that \( \chi(E', E) \) is the effective 'Dt' during the time it takes the particles to slow down from \( E' \) to \( E \), it is clear that

\[
g_i(r', r; E', E) = \mathcal{P}_i(E', E)(r', r), \tag{29}
\]

or, to say it in words, the function \( g_i(r', r; E', E) \) appearing in Eq. (29) is nothing but the probability for a particle injected at location \( r' \) at energy \( E' \) to be found in \( r \) at energy \( E \).

With the definition \( h_i = f_i \times g_i \), we can now re-write the solution (24) of the propagation equation in the simplest way:

\[
n_i(E, r, t) = \frac{1}{|\mathcal{E}_i(E)|} \int_{E}^{\infty} dE' \int dE' h_i(r', r; E', E), \tag{30}
\]

where \( h_i \) should be thought of as the 'full' retarded injection function, i.e. retarded in both space and time (factors \( g_i \) and \( f_i \), respectively). The physical interpretation of the solution is then obvious: the particles which one finds at energy \( E \), time \( t \) and location \( r \), are the collection of all the particles which have been injected at a higher energy \( E' \), anywhere in the volume considered (integration over \( r' \)), but at a time in the past such that, since their injection, the energy losses have brought their energy down from \( E' \) to exactly \( E \). In addition, each of these contributions must be weighted by the probability that the particle survived during this time (exponential factor in Eq. (28)) and that, during this time again, it diffused from its injection site, \( r' \), to the place now probed, in \( r \) (function \( g_i \), Eq. (27)).

### 6. The stationary limit of the models

As was discussed in the introduction, some astrophysical situations need not be studied in a time-dependent scheme. In steady-state, after a transitory regime lasting about the time-scale of energy losses, the spectral density of the EP species \( i \) reaches its equilibrium value, \( N_i(E) \), which satisfies:

\[
\frac{\partial}{\partial E}(E_i(E)N_i(E)) = Q_i(E) - \frac{N_i(E)}{\tau_{\text{tot}}}. \tag{31}
\]

Integrating this equation directly or, equivalently, taking the stationary limit of the general, time-dependent solution given above (Eq. (30)), one obtains:

\[
N_i(E) = \frac{1}{|\mathcal{E}_i(E)|} \int_{E}^{\infty} Q_i(E)\mathcal{P}_i(E_0, E)dE_0, \tag{32}
\]

where the survival probability \( \mathcal{P}_i \) has the same expression as above.
It is important to note that, in the steady-state approximation, the spectral density of stable energetic particles is inversely proportional to the ambient density, provided that the escape can be neglected, i.e. for low energy particles or in the case of a thick target. Indeed, the Coulombian energy loss rate, $\dot{E}_i(E)$, is always proportional to the ambient density, $n_0$, while the catastrophic loss time, $\tau_{\text{cat}}$, is then equal to the nuclear destruction time, $\tau_{\text{D}}$, which according to Eq. (2) behaves as $n_0^{-1}$. As a result, the survival probability, $P_{\text{s}}$, is independent of $n_0$, and inspection of Eq. (22) shows that $N_i(E)$ scales as $n_0^{-1}$. Now the nuclear reaction rates, due to the interaction of the EPs with the ISM, are obtained by integrating the cross-sections over the spectral density:

$$Q_k = \sum_{i,j} \int_0^{+\infty} n_j N_i(E) \sigma_{ij\rightarrow k}(E) v_i(E) \, dE, \quad (33)$$

where the notations are the same as in Eq. (3).

It is clear, then, that the $n_0^{-1}$ dependence of $N_i$ is exactly compensated by the factor $n_j$, which is just $n_0$ times the relative abundance of nuclei $j$ in the ISM. As a conclusion, all the nuclear reaction rates are independent of density in the steady-state approximation. This is worth emphasizing, because it is one fundamental phenomenological distinction between steady-state and time-dependent models. In essence, steady-state models cannot acknowledge density dependences. This is no longer true for time-dependent models, as we have shown elsewhere on various examples (Parizot et al. 1997a,b,c; Parizot & Drury 1999a,b).

7. Conclusion

In conclusion, we have reviewed the solutions of the differential equation describing the propagation of energetic particles (EPs) in the ISM, in the case of both one-zone and extended models, taking into account homogeneous spatial diffusion. The solutions obtained admit a simple physical interpretation, which we emphasized introducing intermediate functions: the energy drift function, $\tau_i(E_0, E)$, and the diffusion section, $\chi_i(E_0, E)$, which both depend on the nuclear species of the EPs. The first one describes the time it takes a particle to slow down from its injection energy, $E_0$, to its current energy, $E$, because of energy dependent energy losses. As for the latter, it plays the role of an effective ‘diffusion coefficient-times-duration’, taking into account the energy dependence of the diffusion coefficient. We have used these functions to construct retarded injection functions which sum up the energy history of the EPs and allow one to write the solution of the propagation equation in a simple and intelligible way.

We also studied the case when the conditions of propagation vary, which arises when either the composition or the density of the ambient medium are not constant. We have shown how it is possible to use the standard results in this case, dividing the process into as much successive phases as required so that the conditions of propagation can be considered constant during each phase. We outlined the detailed procedure for fitting the phases altogether, relying on, and emphasizing again, the physical content of the mathematics involved. The procedure can be used to solve any type of time-dependent astrophysical situation involving the interaction of energetic particles in the interstellar medium. Indeed, the formal solutions derived here can be easily integrated to obtain the distribution function or spectral density of the EPs, from which the nuclear or electromagnetic reaction rates are deduced directly from the knowledge of the cross sections involved.

Finally, we considered the stationary limit of the models, stressing that the assumptions inherent in steady-state make it impossible in principle to investigate the influence of the ambient density on the processes under study. This is one crucial superiority of the time-dependent models which, for this reason, cannot be avoided in a great number of astrophysical situations. As an example, Parizot et al. (1997a,b,c) have studied the time-dependent gamma-ray line emission induced by the winds of massive stars, and Parizot & Drury (1999a,b) calculated the spallative nucleosynthesis within an expanding supernova remnant, taking the dynamics of the process into account. In each case, the influence of the ambient density has been investigated, and in each case it proved very significant.

We believe that the improvement of the spatial resolution of gamma-ray detectors as well as the accumulation of data on variable gamma-ray sources will make the use of time-dependent models more and more necessary.

In the same spirit, as the Galactic chemical evolution models are getting more and more precise, the question should be raised whether steady-state models are still appropriate in all the situations. In particular, recent studies of the evolution of the spallation products Li, Be and B, have focused mainly on the very early stages of Galactic evolution, when the metallicity of the ISM was less than one thousandth of the solar metallicity. Depending on the models, this corresponds to an age of the order of the confinement or the energy loss time scales. As a consequence, one should be careful in assuming steady-state, because the transitory regime may have not yet totally decreased. In addition, the injection of cosmic rays is usually assumed to follow closely the supernova activity in the Galaxy, but it should be kept in mind that the supernova explosion rate may have changed on a very small time scale in the early Galaxy, which would be one more reason to be careful with steady-state models, and to investigate possible significant time-dependent effects.

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