APPLICATION OF PRESERVATION TECHNOLOGY FOR LIFETIME DEPENDENT PRODUCTS IN AN INTEGRATED PRODUCTION SYSTEM

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ABSTRACT. It is important to adopt precisely the optimum level of preservation technology for deteriorating products, as with every passing day, a larger number of items deteriorate and cause an economic loss. For earning more profit, industries have a tendency to add more preservatives for long lifetime of products. However, realizing the health issues, there is a boundary that no manufacturer can add huge amount of preservatives for infinite lifetime of products. The correlation between the long lifetime along with the price of the product is introduced in this model to show the benefit of the optimum level of investment in preservation technology. To maintain the environmental sustainability, the deteriorated items, which can no longer be preserved by adding preservatives anywhere, are disposed with proper protection. The objective of the study is to obtain profit to show the application through a non-linear mathematical. The model is solved through Kuhn-Tucker and an algorithm. Robustness of the model is verified through numerical experiments and sensitivity analysis. Some comparative analyses are provided, which support the adoption of preservation technology for deteriorating products. Numerical studies proved that the profit increases significantly with the application of proposed preservation technology. Some important managerial insights are provided to help the decision makers while implementing the proposed model in real-world situations.

1. Introduction. Most of the food products, for examples bakery items, fruits, and vegetables start deteriorating as soon as they are produced. Deterioration spoils the quality of the food for human use. Either the quality of the spoiled food is reduced or it become perishable after a certain period of time. Deterioration can occur due to attacks on harvested food items by enzymes, oxidation, and microorganisms, which include, bacterial, mold, yeast, moisture, temperature, and chemical reactions. The deteriorated product can be analyzed on the basis of its appearance, which may not be fresh, change of color and texture, undesirable odor and taste. Such deteriorated food can cause foodborne illness. For cleaner production of such

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products, preservation technology becomes effective by avoiding/reducing the deterioration.

Some of deteriorating products, when close to the expiration date, lose their freshness and therefore, customers do not buy them, resulting in economic loss [39]. Such products include fresh fruits, vegetables, and other food products [33]. There are several reasons that deterioration may occur, which can cause variations in the physical/chemical composition of products. Improved storage conditions and the addition of preservatives to some products can reduce the rate of deterioration to a minimum level [23]. As time passes, inventory items deteriorate at a higher rate. To retain the freshness and usefulness of the products for longer periods, different types of inventory conditions are applied. These conditions include temperature, relative humidity, air velocity, atmospheric composition, and sanitation procedures. Deterioration implies that the product is not fit for use for a particular purpose. For example, fruits are not viewed as being fit for eating if they lose their appearance (fresh look, crispness or juiciness), and nutrition can deteriorate due to transportation or water loss [21].

Preservation technology refers to the methods, which can totally prevent, delay, or otherwise reduce the spoilage of deteriorating products. Several methods are being used to avoid food deterioration, purpose of which is to avoid exposure of the food items with bacterial, fungi, yeast, and other microorganisms as well as to slow down the oxidation, which causes rancidity. Addition of preservatives can significantly increase the life of some products. Several juices and liquids are added with preservatives to increase their shelf life. Refrigeration is vastly used to preserve food items for longer time, though there is a limit of increasing shelf life of the products. Canning is used to preserve food items for very long time. This technique uses vacuum packing to keep oxygen out in order to avoid food oxidation and spoilage. Lactic acid fermentation is also adopted for food preservation. Other methods may include, drying, cooling, freezing, boiling, heating, salting, sugaring, smoking, pickling, jellying, jugging, pasteurization, vacuum packing, irradiation, pulsed electric field electroporation, atmospheric modification, non-thermal plasma, and high pressure food preservation.

Customers considering buying such deteriorating products are conscious of the expiration date [39, 33]. If products are very close to their expiration date, they are still kept for sale, but at a lower price, which affects the revenue and causes economic loss [41, 9, 35]. Some of these products pass their expiration time and are treated as waste. According to an estimate [32], 50% of such waste is inappropriately disposed, resulting in environmental pollution. The health of human beings and other living organisms is largely affected by illegal and inappropriate waste disposals [1]. Due to urbanization of modern world, governments have now realized the importance of pollution reduction and they have legislated to control the disposing habits of the consumers. For example, government of Germany legislated to pay find for food waste. Similarly, government of South Korea has started to charge consumers for wasting the food products. By such legislations, they have enforced the consumers/retailers/manufacturers to dispose their waste in an appropriate manner. They created disposal centers, which dispose the industrial and domestic waste on a prescribed disposal cost.

Price of a product may vary time to time. Basic products of a company are sold generally at some constant prices, while the seasonal and short lifetime products are usually sold at some variable prices. Seasonal products, at the end of season,
are sold at lower price to prevent inventory costs. For examples winter and summer clothing, ski season products, and Easter chocolates etc. are offered at higher price during the season and are sold at very low price when the season is over. Similarly, short lifetime products are offered at low price when they are close to the time of expiration in order to maximize sales before those products expire. Fresh juices, milk, fruits, and vegetables are sold at lower prices when they are close to their time of expiration. Several researchers modelled variable selling-price in their studies. Among pioneer attempts, pricing models have been discussed by Whitin [38], Mills [17], Karlin and Carr [15]. Chew et al. [7] discussed several inventory models for lifetime-dependent perishable products assuming maximum lifetime dependent selling-price.

Considering the several types of variable selling-price, as suggested by Chew et al. [7], this study considers that the selling-price of a product increases with the time before it expires. One can propose two components of selling-price, the constant and the variable component. The variable component of the selling-price depends on maximum lifetime of the product. This way, the preservation efforts payoff economically, which are made to extend the lifetime of a product.

Due to the legislations regarding environmental protection by many governments, corporations are now more careful about reducing their waste. Preservation technology helps to reduce the rate of deterioration and hence the product is sustained for longer periods of time without expiration [13]. Use of this technology also affects the selling-price, because a product with a longer expiration time can be sold at a higher price. For example, milk when processed, is preserved for a longer time and is sold at a higher price. Thus, preservation technology positively affects the price.

This study proposes a production and inventory system for short lifetime deteriorating products, in which such products are preserved for longer periods by the application of preservation technology. Further, products that have deteriorated are properly disposed, thus protecting the environment. The components of the products selling-price vary directly with the lifetime of the product. Structure of the paper follows: Section 2 presents comprehensive literature review. Section 3 includes problem definition, notation/abbreviations, and assumptions. Section 4 contains mathematical model formulation and its solution methodology. Section 5 exhibits numerical experiments, comparative analysis, and sensitivity analysis. Some managerial insights are provided in Section 6. Conclusions and future directions are given in Section 7.

2. Literature review. Many researchers have contributed to the literature regarding the effects of deterioration on inventory and production models. Researchers have considered different types of models and developed mathematical expressions for the rate of deterioration. Earlier attempts of modeling an inventory system with deterioration include the research of Ghare and Schrader [11]. Later, these models were updated by Sachan [22] by incorporating shortages in deteriorating inventory system with constant rate of deterioration. Chang and Dye [4] added partial backlogging in an inventory model with constant rate of deterioration. Skouri et al. [31] extended the research by adding time-varying rate of deterioration. Practically, rate of deterioration increases with time. This idea was modelled by Sarkar [24], where he proposed a time-varying and maximum lifetime-dependent rate of deterioration. Later, Shah et al. [30] included the assumption that a product starts deteriorating after a specific interval of time. Sarkar and Sarkar [28] modelled the rate of deterioration that is linearly proportional to the time. Chen and Teng [5] considered
the expiration time to model rate of deterioration in an inventory model. Effect of temperature on rate of deterioration in an inventory model was investigated by Qin et al. [21]. They proposed a rate of deterioration that varies linearly with time and is exponentially proportional to the temperature.

Besides inventory model, in literature, deterioration of products has also been considered in production and supply chain models. A production model for deteriorating products was investigated by Wee and Widyadana [37] proposing reworking and stochastic preventive maintenance. A random rate of deterioration in a production system was modeled by Sarkar [25], where he proposed and solved several models by considering rate of deterioration as uniform, triangular or beta distribution. An uncertain rate of deterioration in a production model was considered by Priyan and Uthayakumar [19]. Assuming that the whole product expires when the rate of deterioration reaches 100%, Chen and Teng [6] proposed a supply chain system of deteriorating products with a time varying rate of deterioration. Oral et al. [18] discussed an economic order quantity model for deteriorating products with selling-price-and displayed-stock dependent demand. Feng et al. [10] suggested an inventory model, where demand of short-life deteriorating products is a function of its selling-price, displayed stocks and expiration date. Li et al. [16] considered a stochastic inventory system under deterioration with stock-dependent demand and variable backlogging rate to develop dynamic pricing and ordering policies.

Product deterioration is controlled by using preservation technology. Additives/preservatives are added during product manufacturing, which slows down different chemical reactions that can cause deterioration, and proper conditions are applied when the products are stored as inventory. Many researchers have studied the application of preservation technology during the last decade. A direct and linear decrease in the rate of deterioration with the application of preservation technology was proposed by Hsu et al. [13]. They optimized the total profit on optimal values of preservation technology cost, replenishment cycle, shortage period, and ordering quantity. He and Huang [12] proposed that the rate of deterioration of seasonal deteriorating products is negatively exponentially proportional to the investment in preservation technology. Dye [8] proposed that the rate of deterioration can be reduced using preservation technology and that the reduced rate is a linear function of preservation technology investment. Yang et al. [40] proposed that reduction in the rate of deterioration is a linear function of investment in preservation technology. Shah et al. [29] assumed that the reduced rate of deterioration is a continuous increasing function of investment in preservation technology. Tsao [34] suggested that cost of preservation technology increases as a quadratic function of preservation efforts, which linearly reduces the rate of deterioration.

Authors’ contribution to the literature is exhibited in Table 1. After reviewing the literature on deterioration and preservation technology, this research proposes an integrated production model for short lifetime deteriorating products, which are preserved for longer time by the application of preservation technology. This research uses a time-varying rate of deterioration that depends on the maximum lifetime of the product, as proposed by [24, 5, 29], and applies preservation technology to increase the maximum lifetime of the product, thus reducing the rate of deterioration. The selling-prices of these products are sensitive to the maximum product’s lifetime. The variable component of the product selling-price varies directly with the product’s lifetime. In order to achieve environmental sustainability and comply with government legislations, the deteriorated products are disposed of
in an appropriate manner. The objective of this research is to obtain best strategy for the investment in preservation technology and production-inventory cycles such that the total profit per unit time is maximized.

| Reference paper              | Deterioration type | Deterioration formulation | Preservation technology | MLD selling-price |
|-----------------------------|--------------------|---------------------------|-------------------------|-------------------|
| Hsu et al. [13]             | constant           | –                         | √                       | –                 |
| Sarkar [24]                 | Time-varying       | MLD                       | –                       | –                 |
| Sarkar and Sarkar [28]      | Time-varying       | Linear                    | –                       | –                 |
| Dye [8]                     | Time-varying       | Linear                    | √                       | –                 |
| Qin et al. [21]             | Time-and temperature varying | Exponential | –                       | –                 |
| Wee and Widyadana [37]      | Constant           | –                         | –                       | –                 |
| Chew et al. [7]             | –                  | –                         | –                       | √                 |
| Sarkar [25]                 | Random             | Uniform, triangular       | –                       | –                 |
| Wang et al. [36]            | Time-varying       | MLD                       | –                       | –                 |
| Priyan and Uthayakumar [19] | Fuzzy              | Triangular                | –                       | –                 |
| Shah et al. [29]            | Time-varying       | MLD                       | √                       | –                 |
| Tsao [34]                   | Constant           | –                         | √                       | –                 |
| This paper                  | Time-varying       | MLD                       | √                       | √                 |

(−) indicates the mentioned assumptions are not applicable or not considered by the authors and MLD means “Maximum Lifetime-Dependent”.

3. Problem definition, notation, and assumptions. This section provides comprehensive problem definition through illustration of process flow. Abbreviations and notation to elaborate the mathematical model are provided. The necessary assumptions required to complement the mathematical model are quoted with proper references.

3.1. Problem definition. From supplier to consumer, the process flow is described in Figure (1). Material is purchased and is sent to the production facility, where the product is produced. The finished products are stored as inventory, which is used to fulfill customers demand. Inventory items start deteriorating at a specific rate and deteriorated items are forwarded to the disposal center. The rate of deterioration varies directly with time and inversely with the maximum products lifetime. Preservation technology is applied to improve products lifetime and reduce the rate of deterioration. The system costs include manufacturing, material, inventory holding, disposal, and setup cost, whereas the hidden cost consists of the loss due to deteriorated items. Production-inventory planning and preservation technology are very important in a system, where the rate of deterioration is a function of time. This study proposes a non-linear mathematical model that calculates the optimal solution for the production-inventory schedule and investment in preservation technology that maximizes the profit.

Sarkar [24] proposed the idea of a time-varying, maximum lifetime-dependent rate of deterioration. He suggested that the rate of deterioration is calculated as $\theta \left[ \frac{1}{1+L-t} \right]$. At any time, say $t = 1 \leq L$, the effect of the maximum products lifetime ($L$) on the rate of deterioration $\theta$ is calculated and illustrated graphically in Figure
From this observation, it is inferred that an increase in the maximum lifetime of the product decreases the rate of deterioration. The maximum products lifetime can be increased via the application of preservation technology, and thus the rate of a products deterioration can be decreased to a minimum level. This research deals with such kinds of deteriorating products, which are preserved for a longer time by reducing the rate of deterioration by increasing their maximum lifetime via the application of preservation technology.

3.2. **Notation.** This subsection gives notation for the mathematical model. It includes variables and parameters which are defined in the following sub-subsections.

3.2.1. **Decision variables.**
- $T$: length of planning horizon (years)
- $t_1$: production run time (years)
3.2.2. **Parameters.**

- $C_p$ cost of preservation technology per unit ($/unit)$
- $C_s$ setup cost per setup ($/setup$)
- $C_{mt}$ material purchasing cost per unit ($/unit$)
- $C_m$ manufacturing cost per unit ($/unit$)
- $h$ inventory holding cost per unit per year ($/unit/year$)
- $C_d$ disposal cost per unit for deteriorated items ($/unit disposed$)
- $P$ selling-price per unit ($/unit$)
- $\varepsilon$ fixed selling-price per unit ($/unit$)
- $\nu$ variable selling-price per unit ($/unit$)
- $R$ revenue per year ($/unit/time$)
- $\pi$ profit per year ($/unit/time$)
- $R_D$ products demand at time $t$ (units/year)
- $R_m$ production rate at time $t$ (units/year)
- $\theta$ rate of deterioration (in years)
- $L$ maximum lifetime of product (years)
- $M$ maximum investment for preservation ($/unit$)
- $I_1$ on hand inventory at time $t$, $0 \leq t \leq t_1$ (units)
- $I_2$ on hand inventory at time $t$, $t_1 \leq t \leq T$ (units)
- $I$ inventory per cycle (units)
- $D$ demand per cycle (units/cycle)
- $N_m$ production per cycle (units/cycle)
- $N_d$ deteriorated items per cycle (units/cycle)
- $\alpha$ fraction by which the lifetime of the product will increase by applying preservation technology
- $\gamma, \delta$ proportionality constants

3.3. **Assumptions.** The following assumptions are considered to develop the model.

1. A single-type of deteriorating product is considered for single period of a system, where products deteriorate at increasing rate with time. Preservation technology is used to reduce the rate of deterioration and thus preserves the product for longer periods.

2. The rate of deterioration depends on the maximum lifetime of the product and is calculated as $\theta = \frac{1}{1+t/L}$, where $L$ is the maximum lifetime of the product [24]. When $t \to 0$, the deterioration rate is minimum. The rate of deterioration keeps increasing with time and it becomes maximum when time approaches maximum lifetime of the product i.e. when $t \to L$, then $\theta = 1$ and the whole product is deteriorated.

3. Short-lifetime products are sold at higher price, when they have longer expiration time. Therefore, this research assumes that the variable component of the products selling-price is a function of its maximum lifetime $\nu = \delta L$ i.e., a product with a longer time to expire is sold at a higher price compared to the same product with a shorter expiration time. The value of $\delta$ is the sensitivity of a products price to its lifetime. More the value of $\delta$, more will the price be sensitive to its lifetime. The similar model for lifetime dependent selling-price has been introduced by Chew et al. [7] for the perishable products being supplied in the airline industry.
4. Preservation technology is used to improve lifetime of the product. The effect of investment in preservation technology on products maximum lifetime is assumed to be linear. Mathematically, improved lifetime of the product is calculated as \( \hat{L} = (1 + \gamma C_p) L \). As the investment in preservation technology is increased, maximum lifetime of the product is increased and it reduces the rate of deterioration according to assumption (2).

5. Due to several factors, some products’ demand increases with time. Considering this fact, the rate of customer demand is assumed to be time-dependent \( (R_D = a + bt) \) as proposed by Chakrabarty et al. [3].

6. Products are produced at a rate, which is related to their demand. Therefore, the rate of production is assumed to be demand-dependent \( (R_m=kR_D=k(a + bt)) \), as proposed by Qin and Liu [20]. During the production cycle, the production rate is higher than the demand, i.e., \( k > 1 \).

7. Food products, once expired, cannot be repaired to recover for the same consumption. Therefore, deteriorated items are considered non-repairable, thus removed from inventory and disposed.

4. **Mathematical model.** This section explains the suggested problem comprehensively and develops a mathematical model to represent the proposed system. Objective of this study is defined through the mathematical model and constraints. Further, this section provides comprehensive solution technique to solve the proposed model.

4.1. **Model development.** At any time \( t \), the rate of deterioration varies inversely with the maximum lifetime of the product. Thus, it is inferred that increases in the products maximum lifetime decrease the rate of deterioration. The maximum lifetime increases with the application of preservation technology. An investment \( C_p \) in preservation technology increases the products maximum lifetime from \( L \) to \( \hat{L} = L + \alpha L \), and thus reduces the rate of deterioration from \( \theta \) to \( \hat{\theta} \), where \( \alpha \) is the fraction by which lifetime of the product increases by applying preservation technology. The increase in lifetime depends on the quality of preservation technology, which can be improved by investing more in the preservation technology. Hence, \( \alpha \) is a function of \( C_p \). Mathematically, \( \alpha = \gamma C_p \) and thus \( \hat{L} = L + \alpha L = (1 + \alpha)L = (1 + \gamma C_p)L \). By applying preservation technology, a modified rate of deterioration is calculated in following equation:

\[
\hat{\theta} = \frac{1}{1 + (1 + \gamma C_p)L - t}
\]

The value of \( \gamma \) is the degree of effectiveness of preservation cost on the lifetime of the products. More the value of \( \gamma \), more effective will be the preservation technology for the product. The value of \( \gamma \) depends on right choice of preservation conditions/preservatives for a specific product. For example, investment for cold storage conditions for ice cream is more effective than the same amount of investment for relative humidity and aeration. In contrast, for vegetables, the investment for cold storage conditions is not as effective as the same amount of investment for relative humidity and proper aeration. Thus, the right choice of preservation conditions is more effective to improve lifetime of the products.
Demand and production rate are time-dependent [3], and production is a function of demand, as expressed below:

\[
R_D = a + bt, \\
R_m = kR_D = k(a + bt).
\]

The demand during one cycle is given by

\[
D = \int_0^T R_D dt = aT + \frac{bT^2}{2}.
\]

Quantity produced during the production cycle fulfills customer demand and maintains product deterioration for the whole production-inventory cycle. During the interval \((0, t_1)\), at any time \(t\), \(R_m\) product items are produced and added to inventory while \(R_D\) items are demanded and \(\hat{\theta}I(t)\) items are deteriorated, which are taken from inventory. The level of product inventory increases during the interval \((0, t_1)\), due to the rate of production being higher than the cumulative rate of demand and deterioration. Similarly during the interval \((t_1, T)\), there is no production and inventory is depleted by the number of items demanded and deteriorated. The behavior of the inventory during the complete cycle is depicted in Figure (4).

The governing differential equations of current inventory are expressed below:

\[
\frac{dI_1(t)}{dt} = R_m - R_D - \hat{\theta}I_1 = 0, \quad 0 \leq t \leq t_1, \\
\frac{dI_2(t)}{dt} = -R_D - \hat{\theta}I_2, \quad t_1 \leq t \leq T
\]

The above differential equations are solved using inventory conditions given below.

\[\text{Figure 3. Inventory behavior during one cycle}\]
to calculate on-hand inventory at any time $t$ as expressed in the following equations:

$$I_1(t) = 0 \text{ at } t = 0,$$
$$I_2(t) = 0 \text{ at } t = T,$$
$$I_1(t) = (k - 1)(1 + \bar{L} - t) \left[ (a + b + b\bar{L}) \ln \left( \frac{1 + \bar{L}}{1 + \bar{L} - t} \right) \right], 0 \leq t \leq t_1$$
$$I_2(t) = (1 + \bar{L} - t) \left[ (a + b + b\bar{L}) \ln \left( \frac{1 + \bar{L} - t}{1 + \bar{L} - T} \right) \right], t_1 \leq t \leq T$$

The system’s total cost is incurred by the material purchasing cost, manufacturing cost, preservation technology cost, disposal cost, inventory holding cost, and setup cost. These costs and the total cost are calculated as follows.

a. Setup cost

This cost includes the expenses to buy apparatus for manufacturing and setup the facility for production and inventory. The setup cost is considered per cycle and is given below:

$$\text{Setup cost} = C_s$$

b. Material cost and production cost

Material is purchased from material supplier and is used to produce the required quantity of the product. The quantity of the material is decided based on production quantity. Therefore, it is important to determine the number of items produced during one cycle as is calculated in Equation (1).

$$N_m = \int_0^{t_1} R_m \, dt = k \left( at_1 + \frac{bt_1^2}{2} \right)$$

Several operations are accomplished to produce single unit of the product and cost incurred at each operation is used to get production cost per unit. Using the number of items produced per cycle, cost of material per unit item and cost of production per unit item, total cost of material and production per cycle is calculated in the following equation:

$$C_{mt}N_m = (C_{mt} + C_m)k \left( at_1 + \frac{bt_1^2}{2} \right)$$

where $N_m$ is calculated in Equation (1).

c. Inventory holding cost

The storage of the produced items needs proper conditions and the investment to maintain these conditions is termed as inventory holding cost. In the context of presented food items problem, inventory conditions are important to keep the products fresh and nutritious. Important inventory conditions include temperature, relative humidity, light, and air circulations. Using the expressions for $I_1(t)$ and $I_2(t)$, as defined above, the total inventory carried during one cycle is calculated and demonstrated in the following equation:
\[ I = \int_0^{t_1} I_1(t)\,dt + \int_{t_1}^{T} I_2(t)\,dt \]
\[ = \frac{1}{12} \left[ 6(a + b + b\hat{L}) \left( (1 + \hat{L})^2 \left( \ln(1 + \hat{L} - t_1) - \ln(1 + \hat{L} - T) \right) \right) \right. \]
\[- t_1(2 + 2\hat{L} - t_1)\ln \frac{1 + \hat{L} - t_1}{1 + \hat{L} - T} + (T - t_1) \left( (3a + 3b\hat{L})(T + t_1 - 2\hat{L} - 2) \right) \]
\[- b \left( 6 + 4t_1^2 + 3T - 2T^2 + 6\hat{L}(1 + L) - t_1(9 + 2T + 6\hat{L}) \right) \]
\[ + \left( k - 1 \right) \left( - (1 + \hat{L})^2 \left( 3a + 3b\hat{L} + b(5 + 2\hat{L}) \right) \right) \]
\[ + \left( 1 + \hat{L} - t_1 \right) \left( 3a + 3b\hat{L} + b(5 + 4t_1 + 2\hat{L}) + 6(a + b + b\hat{L}) \ln \frac{1 + \hat{L}}{1 + \hat{L} - t_1} \right) \]

Thus, the inventory holding cost per cycle is calculated below:

\[ hI = \frac{h}{12} \left[ 6(a + b + b\hat{L}) \left( (1 + \hat{L})^2 \left( \ln(1 + \hat{L} - t_1) - \ln(1 + \hat{L} - T) \right) \right) \right. \]
\[- t_1(2 + 2\hat{L} - t_1)\ln \frac{1 + \hat{L} - t_1}{1 + \hat{L} - T} + (T - t_1) \left( (3a + 3b\hat{L})(T + t_1 - 2\hat{L} - 2) \right) \]
\[- b \left( 6 + 4t_1^2 + 3T - 2T^2 + 6\hat{L}(1 + L) - t_1(9 + 2T + 6\hat{L}) \right) \]
\[ + \left( k - 1 \right) \left( - (1 + \hat{L})^2 \left( 3a + 3b\hat{L} + b(5 + 2\hat{L}) \right) \right) \]
\[ + \left( 1 + \hat{L} - t_1 \right) \left( 3a + 3b\hat{L} + b(5 + 4t_1 + 2\hat{L}) + 6(a + b + b\hat{L}) \ln \frac{1 + \hat{L}}{1 + \hat{L} - t_1} \right) \]

**d. Preservation technology cost**

Preservation conditions are maintained to preserve the products for longer time during their storage to increase the lifetime. The cost invested for this purpose is determined as preservation technology cost and it is calculated per unit per unit time of their storage. The cost of preservation technology for one cycle is calculated below:

\[ C_pI = \frac{C_p}{12} \left[ 6(a + b + b\hat{L}) \left( (1 + \hat{L})^2 \left( \ln(1 + \hat{L} - t_1) - \ln(1 + \hat{L} - T) \right) \right) \right. \]
\[- t_1(2 + 2\hat{L} - t_1)\ln \frac{1 + \hat{L} - t_1}{1 + \hat{L} - T} + (T - t_1) \left( (3a + 3b\hat{L})(T + t_1 - 2\hat{L} - 2) \right) \]
\[-b \left( 6 + 4t_1^2 + 3T - 2T^2 + 6\hat{L}(1 + L) - t_1(9 + 2T + 6\hat{L}) \right) \]
\[+ \ (k - 1) \left( - (1 + \hat{L})^2 \left( 3a + 3b\hat{L} + b(5 + 2\hat{L}) \right) \right. \]
\[+ \ (1 + \hat{L} - t_1) \left( 3a + 3b\hat{L} + b(5 + 4t_1 + 2\hat{L}) + 6(a + b + b\hat{L}) \ln \frac{1 + \hat{L}}{1 + L - t_1} \right) \]

where \(I\) is defined during inventory calculations.

d. Disposal cost

Despite of applying preservation technology, some of the products are deteriorated and are removed from other good products. These deteriorated items are sent to a disposal center. The number of items deteriorated during one cycle are calculated and expressed below:

\[N_d = \int_0^{t_1} \theta(t)I_1(t)dt + \int_{t_1}^{T} \theta(t)I_2(t)dt\]
\[= \frac{1}{2} \left[ t_1 \left( 2k(a + b + b\hat{L}) + bt_1(k - 2) \right) - 2T(a + b + b\hat{L} - bt_1) - bT^2 \right. \]
\[+ \ 2(a + b + b\hat{L}) \left( (1 - k)(1 + \hat{L} - t_1) \ln \frac{1 + \hat{L}}{1 + L - t_1} + (1 + \hat{L}) \ln(1 + \hat{L} - t_1) \right. \]
\[\left. - \ t_1 \ln \frac{1 + \hat{L} - t_1}{1 + L - T} - (1 + \hat{L}) \ln(1 + \hat{L} - T) \right) \]

These deteriorated items are disposed through proper processing that does not harm the environment. For this purpose, to comply with the legislative requirements, government approved disposal centers are established, which dispose such waste products for a specific cost. The cost of disposal per cycle is calculated by using per unit disposal cost and the total number of items deteriorated during one cycle. The disposal cost per cycle is computed as follows:

\[C_d N_d = \frac{C_d}{2} \left[ t_1 \left( 2k(a + b + b\hat{L}) + bt_1(k - 2) \right) - 2T(a + b + b\hat{L} - bt_1) - bT^2 \right. \]
\[+ \ 2(a + b + b\hat{L}) \left( (1 - k)(1 + \hat{L} - t_1) \ln \frac{1 + \hat{L}}{1 + L - t_1} + (1 + \hat{L}) \ln(1 + \hat{L} - t_1) \right. \]
\[\left. - \ t_1 \ln \frac{1 + \hat{L} - t_1}{1 + L - T} - (1 + \hat{L}) \ln(1 + \hat{L} - T) \right) \]

In some scenario disposal cost can be negative. For example, in several European countries, companies and NGOs have come up in recent years that collect expired food products at companies and either distribute them among poor people free, or that buy these products and use them to produce energy from organic material. In these two examples, the disposal cost would be negative or zero. Such specific
scenarios can be considered to model a reverse logistics system, where waste food items can be used to fulfil other purposes, which could earn more profit.

**Total cost of the production system per cycle**

The total cost is calculated by summing up all costs explained above. Total cost per unit time of the proposed system is expressed in the equation below:

$$
\text{Total cost (TC)} = \frac{1}{T} (C_s + C_{m1}N_m + C_mN_m + hI + C_pI + C_dN_d)
$$

**Selling-price and revenue per cycle**

This research proposes two components of the selling-price for short-life products. The fixed base price $\varepsilon$ and the variable price $\nu$ which depends on the maximum lifetime of the product and is expressed as $\nu = \delta L$. The selling-price of a single item is calculated below:

$$
\text{Product’s selling price} = \varepsilon + \nu = \varepsilon + \delta L
$$

Products having a longer time to expire are sold at a higher price. The variable component of selling-price varies with the variation in maximum lifetime of the product. As the variation in products lifetime is related to the investment in preservation technology, it is therefore concluded that the variable selling-price increases with an increase in the preservation technology investment. Here, it is important to consider that increase in the products lifetime and selling-price are limited to a specific level. This study considers a limited investment in preservation technology that is bounded by the value $M \leq [12, 8]$. This research can be further continued by considering how the products lifetime $\hat{L}$ and variable selling-price $\nu$ are affected by varying the investment in preservation technology $C_p$. The revenue per unit time for the preserved products is expressed in the equation below:

$$
\text{Revenue (R)} = \frac{1}{T} \left( \varepsilon + \delta(1 + \gamma C_p)L \left( aT + \frac{bT^2}{2} \right) \right)
$$

$$
= \left( \varepsilon + \delta(1 + \gamma C_p)L \right) \left( a + \frac{bT}{2} \right)
$$

$$(2)$$

where $\delta$ and $\gamma$ are proportionality constants that depend on the system properties.

**Total profit of the production system per cycle**

The total profit per unit time of the given system as a function of $t_1$, $T$ and $C_p$, say $\pi(t_1, T, C_p)$, is given by

$$
\pi(t_1, T, C_p) = R - TC = \left( \varepsilon + \delta(1 + \gamma C_p)L \right) \left( a + \frac{bT}{2} \right)
$$

$$
- \frac{1}{T} (C_s + C_{m1}N_m + C_mN_m + hI + C_pI + C_dN_d)
$$

$$
= \psi
$$

See appendix for an evaluation of $\psi$

The aim of this study is to obtain the optimal values for the decision variables that maximize the profit per unit time and is expressed below:

Maximize $\pi(t_1, T, C_p)$
subject to the following constraints:

\[ N_m - N_d \geq D \]  \hspace{1cm} (4)
\[ C_p \leq M, \]  \hspace{1cm} (5)
\[ t_1, T, C_p > 0. \]  \hspace{1cm} (6)

The objective function (3) is the profit earned per unit time. Constraint (4) ensures that the customer demand is satisfied within the planning horizon. Constraint (5) restricts the maximum investment in preservation technology because in a specific system, it is not possible to invest an unlimited amount into preservation technology. Constraint (6) shows the non-negativity of the decision variables.

4.2. Solution methodology. Due to the high degree of non-linearity, a closed form solution of the objective function is not possible. For such objective functions with inequality constraints, Kuhn-Tucker method is an appropriate solution technique [27, 14]. The decision variables are used in logarithmic functions; thus it is not possible to obtain optimal values of the decision variables directly. Therefore, a solution algorithm is designed to calculate the optimal solution. The solution methodology is explained below.

4.2.1. Kuhn-Tucker method of constrained optimization. Lagrange function of the objective function is given below:

\[ L(t_1, T, C_p, \lambda_1, \lambda_2) = \pi + \lambda_1 X_1 + \lambda_2 X_2 \]

where \( \pi \) is defined in Equation (3), \( \lambda_1, \lambda_2 \) are Lagrange multipliers, and \( X_1, X_2 \) are calculated using constraints (4 and 5) as given below:

\[ X_1 = D + N_d - N_m \]
\[ X_2 = C_p - M \]

The necessary Kuhn-Tucker conditions for the optimal point that maximizes the profit value considering the defined constraints are given below:

\[ \frac{\partial L(t_1, T, C_p, \lambda_1, \lambda_2)}{\partial t_1} \geq 0; \quad t_1 \left\{ \frac{\partial L(t_1, T, C_p, \lambda_1, \lambda_2)}{\partial t_1} \right\} = 0 \]
\[ \frac{\partial L(t_1, T, C_p, \lambda_1, \lambda_2)}{\partial T} \geq 0; \quad T \left\{ \frac{\partial L(t_1, T, C_p, \lambda_1, \lambda_2)}{\partial T} \right\} = 0 \]
\[ \frac{\partial L(t_1, T, C_p, \lambda_1, \lambda_2)}{\partial C_p} \geq 0; \quad C_p \left\{ \frac{\partial L(t_1, T, C_p, \lambda_1, \lambda_2)}{\partial C_p} \right\} = 0 \]
\[ \lambda_1 X_1 = 0 \]
\[ \lambda_2 X_2 = 0 \]

Considering the fact that the values of the variables \( t_1, T, \) and \( C_p \) cannot be zero, the above conditions are satisfied when the first derivative of Lagrange function with respect to each decision variable is equal to zero. These calculations are given below.

Differentiation of Lagrange function with respect to \( t_1, \) one can obtain

\[ \frac{\partial L(t_1, T, C_p, \lambda_1, \lambda_2)}{\partial t_1} = \phi_1(t_1, T, C_p, \lambda_1) = 0 \]  \hspace{1cm} (7)
Differentiation of Lagrange function with respect to $T$, one can obtain

$$\frac{\partial L(t_1, T, C_p, \lambda_1, \lambda_2)}{\partial T} = \phi_2(t_1, T, C_p, \lambda_1) = 0$$ (8)

Differentiation of Lagrange function with respect to $C_p$, one can obtain

$$\frac{\partial L(t_1, T, C_p, \lambda_1, \lambda_2)}{\partial C_p} = \phi_3(t_1, T, C_p, \lambda_1, \lambda_2) = 0$$ (9)

$$\lambda_1 X_1 = \lambda_1 \left[ -b \left( t_1^2 + T(1 + \hat{L}) - t_1(T + k(1 + \hat{L})) \right) + \left( a + b(1 + \hat{L}) \right) \left( 1 + \hat{L} \right) \left( \ln(1 + \hat{L} - t_1) - \ln(1 + \hat{L} - T) \right) - (k - 1)(1 + \hat{L} - t_1) \ln \left( 1 + \frac{t_1}{1 + \hat{L} - t_1} \right) - t_1 \ln \left( 1 + \frac{T - t_1}{1 + \hat{L} - T} \right) \right] = 0$$ (10)

where $\phi_1(t_1, T, C_p, \lambda_1)$, $\phi_2(t_1, T, C_p, \lambda_1)$, and $\phi_3(t_1, T, C_p, \lambda_1, \lambda_2)$ are defined in appendix.

$$\lambda_2 X_2 = 0$$

$$\lambda_2(C_p - M) = 0$$ (11)

The values of $\lambda_1$ and $\lambda_2$ are defined in the equations below:

$$\lambda_1 = \frac{\xi_1}{\xi_2}$$ (12)

$$\lambda_2 = \frac{\xi_1 \xi_4}{\xi_2} - \xi_3$$ (13)

[where $\xi_1$, $\xi_2$, $\xi_3$, and $\xi_4$ are defined in Appendix.]

4.2.2. Solution algorithm.

**Step 1.** Give parameters’ values and initial values of $\lambda_1$ and $\lambda_2$ as 0 i.e. $\lambda_1 = 0$, $\lambda_2 = 0$.

**Step 2.** Calculate the values of $t_1$, $T$, $C_p$ using Equations (7–9), which satisfy the Equations (10) and (11).

**Step 3.** Using the values of $t_1$, $T$, $C_p$ from **Step 2**, compute the value of the profit ($\pi$) defined in Equation (2).

**Step 4.** From Equations (12 and 13), using the values of $t_1$, $T$, $C_p$ from Step 2, compute the values of $\lambda_1$, $\lambda_2$.

**Step 5.** Using the values of $\lambda_1$, $\lambda_2$ from **Step 4**, repeat **Steps 1, 2, and 3**, $n$ times until the value of the profit ($\pi$) stops increasing.

**Step 6.** Stop further iterations. The values of $t_1$, $T$, $C_p$ calculated in the final iteration are optimum and ($\pi$) is the maximum value of the profit.

5. Numerical experiments. To exhibit the application of the proposed model, a comprehensive numerical study is performed in this section. Wolfram Mathematica 9 and the fmincon optimization tool of MATLAB (2015) with an Interior Point algorithm are used to find the optimality of the objective function using optimal values of the decision variables.
5.1. **Input parameters for numerical study.** The input parameter values are summarized in Table 2.

| Parameter | Value |
|-----------|-------|
| $C_s$     | $500$/setup |
| $h$       | $0.8$/unit/month |
| $a$       | 1500 units/month |
| $L$       | 4 months |
| $\delta$  | 0.05 |
| $C_{mt}$  | $15$/unit |
| $C_d$     | $0.5$/unit |
| $b$       | 60 units/month |
| $k$       | 3.2 |
| $C_m$     | $10$/unit |
| $\varepsilon$ | $100$/unit |
| $M$       | $10$/unit |
| $\gamma$  | 0.005 |

5.2. **Results and discussions.**

5.2.1. **Example 1.** The optimal solution for the objective function when preservation technology is applied is provided in Table 3.

| Parameter | Value |
|-----------|-------|
| $t^*_1$   | 0.21 month |
| $T^*$     | 0.68 month |
| $C_p^*$   | $1.78$/unit/unit time |
| $\pi^*$   | $115955$/month |

This optimal solution satisfies the model constraints.

**Remark 1.** It can be found in Table 3 that the optimal production run time is 0.21 month and the optimal cycle time is 0.68 month.

**Remark 2.** The maximum profit per month is $115955$. The profit can be further increased above the optimal value of time variables, but for those values, the model constraints are violated and thus the solution becomes infeasible.

**Remark 3.** Optimum value of investment in preservation technology per unit product is $1.78$, which significantly increases the lifetime of the product, the selling-price, and hence contributes to the total profit.

5.2.2. **Example 2.** The optimal solution for the objective function when preservation technology is not applied is provided in Table 4.

| Parameter | Value |
|-----------|-------|
| $t^*_1$   | 0.18 month |
| $T^*$     | 0.54 month |
| $\pi^*$   | $111106$/month |

This optimal solution satisfies the model constraints.

**Remark 1.** It can be found in Table 4 that the optimal production run time is 0.18 month and the optimal cycle time is 0.54 month.

**Remark 2.** The maximum profit per month is $111106$. The profit can be further increased above the optimal value of time variables, but for those values, the model constraints are violated and thus the solution becomes infeasible.

**Remark 3.** There is significant improvement in profit when preservation technology was adopted. Moreover, the cycle time was increased to sell more products with the application of preservation technology. The increase in profit with the application of preservation technology is 4.36 percent, which is illustrated in Figure 4.
Figure 4. Improvement of profit with application of proposed preservation technology

| Considered model       | Rate of deterioration | Percentage improvement by using this model |
|------------------------|-----------------------|------------------------------------------|
| This model             | 0.01                  | Not Applicable                           |
| Sarkar [30]            | 0.22                  | 95.45%                                   |
| Shah et al. [29] Case 1| 0.01                  | Not Applicable                           |
| Shah et al. [29] Case 2| 0.04                  | 75%                                      |
| Shah et al. [29] Case 3| 0.07                  | 85.70%                                   |

5.2.3. Comparative analysis of rates of deterioration with Sarkar [24] and Shah et al. [29]. The comparative analysis shows that the proposed theory for the application of preservation technology reduces the rate of deterioration to the minimum as compared to the other models.

5.3. Sensitivity analysis of the key parameters. Table 5 shows the effects of changing by certain percentage (50%, 25%, +25%, +50%) the specific costs for the optimal values of decision variables and the objective function.

The results presented in Table 9 are summarized as following:

- The optimal value of the profit is highly sensitive to the material cost, and the profit margin can significantly be improved by selecting a material supplier that offers lower material costs.
- The cost of production also affects the profit, improving the production system and reducing the production costs can help increase the systems profit.
- Similarly, strategic decisions regarding the setup cost reduction can enhance the profit by a fair amount.
- The effects of variations in the holding cost and disposal cost on the profit are not significant.
Table 5. Sensitivity analysis

| Parameters | Changes of parameters (in %) | $t_1^*$ (in %) | $T^*$ | $C_p^*$ | $\pi^*$ |
|------------|-----------------------------|----------------|-------|--------|--------|
| $C_s$      | -50%                        | -26.82         | -22.62| -19.26 | +10.72 |
|           | -25%                        | -11.73         | -9.31 | -10.00 | +5.03  |
|           | +25%                        | +9.50          | +7.98 | +8.15  | -8.92  |
|           | +50%                        | +18.44         | +14.86| +15.19 | -8.92  |
| $C_{mt}$   | -50%                        | +50.84         | +38.14| +39.63 | +53.65 |
|           | -25%                        | +19.55         | +15.52| +15.93 | +26.11 |
|           | +25%                        | +13.41         | -11.09| -11.85 | -25.15 |
|           | +50%                        | +23.46         | -19.29| -20.37 | -49.63 |
| $C_m$      | -50%                        | +12.29         | +9.76 | +10.00 | +17.27 |
|           | -25%                        | +5.59          | +4.66 | +4.44  | +8.58  |
|           | +25%                        | +5.03          | -3.99 | -4.44  | -8.47  |
|           | +50%                        | -9.50          | -7.76 | -8.15  | -16.85 |
| $h$        | -50%                        | +11.17         | +9.09 | +9.26  | +3.30  |
|           | -25%                        | +5.03          | +4.21 | +4.07  | +1.60  |
|           | +25%                        | -5.03          | -3.55 | -4.07  | -1.52  |
|           | +50%                        | -8.94          | -6.87 | -7.41  | -2.95  |
| $C_d$      | -50%                        | +0.56          | +0.67 | +0.37  | +0.22  |
|           | -25%                        | +0.00          | +0.22 | +0.00  | +0.10  |
|           | +25%                        | -0.56          | -0.22 | -0.37  | -0.10  |
|           | +50%                        | -0.56          | -0.44 | -0.74  | -0.22  |

- The scheduling variables and investment in preservation technology are highly sensitive to the costs of the material, the manufacturing cost and the setup cost.
- The inventory holding cost affects the optimality of the decision variables and the objective function to some extent.
- Variations in the optimal values of the scheduling variables and the investment in preservation technology are directly related to variation in the setup cost, which are inversely related to variations in the material cost, manufacturing cost and inventory holding cost.

The effect of variations in the system parameters on variations of the objective function and decision variables are illustrated in Figures (5-8).

6. Managerial insights. The proposed model is applicable for the short lifecycle deteriorating products, demand of which is time-varying. Using the model, managers of such products can decide optimum investment in preservation technology, plan the schedule of production, and inventory more precisely, which helps cleaning of production system by avoiding/ reducing deterioration of the products. Some important and relevant managerial insights are explained as following.

**Insight 1**
An important business objective is profit maximization. In the given system, profit can be increased by improving several cost parameters. As concluded from numerical experiments, production and material costs are the pertinent cost parameters, which can increase the profit considerably. There is a trade-off between cost of material and production cost. Better quality raw material is expensive but it reduces the cost of production [26] and vice versa. Therefore, material cost should be reduced by doing better market analysis and selecting a supplier who delivers...
Figure 5. Variation in production time by varying the setup cost, material cost, manufacturing cost, inventory holding cost and disposal cost.

Figure 6. Variation in cycle time by varying the setup cost, material cost, manufacturing cost, inventory holding cost and disposal cost.
Figure 7. Variation in cost of preservation by varying the setup cost, material cost, manufacturing cost, inventory holding cost and disposal cost

Figure 8. Variation in profit by varying the setup cost, material cost, manufacturing cost, inventory holding cost and disposal cost

better quality raw material at reduced cost. Further, they can also negotiate the transportation cost, which is considered part of cost of raw material, with material suppliers to bear the transportation cost. Similarly, production cost can be reduced
by improving various parameters in production system. For example, reducing work-in-process inventory, prompt feedback loops to spot and correct mistakes, and the creation of an agile facility that can quickly switch to creating new products on short notice [2].

**Insight 2**

Reduction in material and production cost not only increases the profit but it also increases the cycle time and production time. While reducing these costs, managers should consider the flexibility of their schedules and capacity of their systems. This research provides the managers with comprehensive calculations by which they can calculate the improvement in profit and can estimate the variation in their schedule. Therefore, they can decide the extent to which these costs can be reduced.

**Insight 3**

For the manufacturers of short-life deteriorating products, it is very important to plan precisely the production and inventory schedules because with every passing hour/day, more number of items deteriorate and the system becomes less profitable. Therefore, more frequent inventory replenishments and short production cycles favor such systems of deteriorating products. This study provides comprehensive solutions to optimize the profit for deteriorating products by wisely deciding the schedules of inventory and production.

**Insight 4**

An important assumption of this research is the lifetime-dependent selling-price. Those products, which have longer expiration time are sold at higher price and vice versa. It is common phenomenon for short-life deteriorating products that such products are put for sale at a lower price that not only sells the products immediately due to customers trend to buy cheaper products which can be used readily but also avoids the disposal cost that would merely be a loss if expiration time has already passed. This is also a new research dimension to estimate effect of lowering the selling-price on demand of the reduced lifetime deteriorating products as is discussed by Feng et al. [10].

7. **Concluding remarks.** The aim of the model was to introduce the correlation between increases of lifetime, by considering the preservation technology, and price of products for cleaning the production system. The proposed model suggested that the lifetimes of products were increased significantly by applying preservation technology and consequently, the rate of deterioration was decreased. The hypothesis that products having longer expiration times were sold at a higher price which contributed to an increase in the profit, was verified through numerical examples that showed a remarkable increase in products lifetime and selling-price when preservation technology was applied. Waste disposal, an important factor in environmental sustainability, proved the effectiveness in this model and deteriorated items were disposed through the proper channels in order to fulfil the legislative requirements for a sustainable system. A sensitivity analysis was presented to check the effects of variations of the system parameters on the optimality of the decision variables and objective function, which directly indicates the usefulness for cleaning of the production system. This research can be further continued to study how investments in preservation technology affects the maximum lifetime of the product to make a sustainable production system. Moreover, it can be studied that how the variable components of the product selling-price are affected by the maximum lifetime of the product. The model can be extended by assuming that temperature affects the rate of deterioration, as proposed by Qin et al. [21]. Another way to extend the
model is by considering shortages and backlogging of items that deteriorate during the planning horizon, as proposed by Sarkar and Sarkar [28]. For a given product after it has expired, it can potentially be converted into multiple other products, which is a more realistic scenario that may be considered for further research.

Appendix A.

\[
\psi = \left( \varepsilon + \delta(1 + \gamma C_P) L \right) \left( a + \frac{b T}{2} \right) - \frac{1}{T} \left[ C_s + (C_{mt} + C_m) k \left( a t_1 + \frac{b t_1^2}{2} \right) \right] - \frac{h + C_P}{12} \left[ 6(a + b + b \hat{L}) \left( 1 + \hat{L} \right)^2 \left( \ln(1 + \hat{L} - t_1) - \ln(1 + \hat{L} - T) \right) \right. \\
- \left. t_1(2 + 2 \hat{L} - t_1) \ln \frac{1 + \hat{L} - t_1}{1 + \hat{L} - T} \right) + (T - t_3) \left( (3a + 3b \hat{L})(T + t_1 - 2 \hat{L} - 2) - b \left( 6 + 4t_1^2 + 3T - 2T^2 + 6 \hat{L}(1 + L) - t_1(9 + 2T + 6 \hat{L}) \right) \right) \\
+ (k - 1) \left( - (1 + \hat{L})^2 \left( 3a + 3b \hat{L} + b(5 + 2 \hat{L}) \right) \right] + (1 + \hat{L} - t_1) \left( 3a + 3b \hat{L} + b(5 + 4t_1 + 2 \hat{L}) + 6(a + b + b \hat{L}) \ln \frac{1 + \hat{L}}{1 + \hat{L} - t_1} \right) \\
+ \frac{C_d}{2} \left[ t_1 \left( 2k(a + b + b \hat{L}) + b t_1(k - 2) \right) - 2T(a + b + b \hat{L} - b t_1) - b T^2 \right] \\
+ 2(a + b + b \hat{L}) \left( (1 - k)(1 + \hat{L} - t_1) \ln \frac{1 + \hat{L}}{1 + \hat{L} - t_1} + (1 + \hat{L}) \ln(1 + \hat{L} - t_1) \right) \\
- t_1 \ln \frac{1 + \hat{L} - t_1}{1 + \hat{L} - T} - (1 + \hat{L}) \ln(1 + \hat{L} - T) \right] \right]
\]

\[
\phi_1 = -\frac{1}{T} \left[ \left( h + C_P \right)(1 + \hat{L} - t_1) + C_d \right] \left\{ b(k t_1 - 2t_1 + T) \right\} \\
+ \left( a + b(1 + \hat{L}) \right) \left( (k - 1) \ln \left( 1 + \frac{t_1}{1 + \hat{L} - t_1} \right) - \ln \left( 1 + \frac{T - t_1}{1 + \hat{L} - T} \right) \right) \right\} \\
+ k(a + bt_1)(C_{mt} + C_m) \right] + \lambda_1 \left[ - ak + b(T - 2t_1) \right] \\
+ (a + b(1 + \hat{L})) \left( (k - 1) \ln \left( 1 + \frac{t_1}{1 + \hat{L} - t_1} \right) - \ln \left( 1 + \frac{T - t_1}{1 + \hat{L} - T} \right) \right) \right] \right]
\[ \phi_2 = \frac{1}{2} b (\varepsilon + \delta L(1 + \gamma C_p)) + \frac{1}{2T(1 + \hat{L} - T)} \left[ (t_1 - T)(a + bT)(h + C_p)(t_1 + T - 2 - 2\hat{L}) + 2C_d \right] + \frac{1}{T^2} \left[ \frac{h + C_p}{12} \left( t_1 - T \right) - 3(a + b\hat{L})(t_1 + T - 2 - 2\hat{L}) + b \left( 6 + 4t_1^2 + 3T - 2T^2 + 6\hat{L}(1 + T) - t_1(9 + 2T + 6\hat{L}) \right) \right] \\
+ (k - 1) \left( (1 + \hat{L})^2(3a + 3b\hat{L} + b(5 + 2\hat{L})) - (1 + \hat{L} - t_1)^2 \right) \\
+ (3a + 3b\hat{L} + b(5 + 4t_1 + 2\hat{L})) + 6(a + b\hat{L}) \ln \frac{1 + \hat{L}}{1 + \hat{L} - t_1} \right) \\
+ (a + b + b\hat{L}) \left( (1 + \hat{L})^2 \ln (1 + \hat{L} - t_1) \right) \\
+ \frac{C_d}{2} \left\{ -bT^2 - 2T(a + b - bt_1 + b\hat{L}) + t_1 \left( bt_1(k - 2) + 2k(a + b + b\hat{L}) \right) \\
+ 2(a + b + b\hat{L}) \left( (k - 1)(t_1 - \hat{L} - 1) \ln \frac{1 + \hat{L} - t_1}{1 + \hat{L} - T} + (1 + \hat{L} - t_1) \ln \frac{1 + \hat{L} - t_1}{1 + \hat{L} - T} \right) \right\} \\
+ k(C_{mt} + C_m) \left( at_1 + \frac{bt_1^2}{2} \right) + C_s \right]\]

\[ \lambda_1 \left[ a + bT + \frac{1}{2} \right] \left\{ -2bT - 2(a + b - bt_1 + b\hat{L}) + 2(a + b + b\hat{L}) \left( \frac{1 + \hat{L} - t_1}{1 + \hat{L} - T} \right) \right\] \\

\[ \phi_3 = \gamma \delta L \left( a + \frac{bT}{2} \right) - \frac{1}{12T} \left[ (t_1 - T) \left\{ 6b - 3a(t_1 + T - 2 - 2\hat{L}) \right\} \\
+ b \left( 4t_1^2 + (3 - 2T)T + 3\hat{L}(4 + T) + 6\hat{L}^2 - t_1(9 + 2T + 9\hat{L}) \right) \right] \\
+ (k - 1) \left\{ (1 + \hat{L})^2(3a + 5b(1 + \hat{L})) - (1 + \hat{L} - t_1)^2 \right\} \\
+ \left( 3a + b(5 + 4t_1 + 5\hat{L}) + 6(a + b(1 + \hat{L})) \ln \left( \frac{1 + \frac{t_1}{1 + \hat{L} - t_1}}{1 + \hat{L} - t_1} \right) \right) \right\} \]
\[\begin{align*}
+ & \ 6(a + b(1 + \hat{L})) \left\{ (1 + \hat{L})^2 \left( \ln(1 + \hat{L} - t_1) - \ln(1 + \hat{L} - T) \right) \right. \\
+ & \ t_1(t_1 - 2(1 + \hat{L})) \ln \left( 1 + \frac{T - t_1}{1 + \hat{L} - T} \right) \right\} \\
+ & \ 3\gamma L(h + C_p) \left\{ - 4bt_1^2 + 4akt_1 + 6bkt_1 - bkt_1^2 + 6bkt_1 L + 4aT - 6bT + 8bt_1 T \right. \\
- & \ 6bL - 3bT^2 + 6C_p bL \gamma (kt_1 - T) \\
+ & \ 2(1 + \hat{L})(2a + 3b(1 + \hat{L})) \left( \ln(1 + \hat{L} - t_1) - \ln(1 + \hat{L} - T) \right) \\
- & \ \frac{2a(k - 1)t_1^2}{1 + L} - \frac{2(t_1 - T)^2(a + bT)}{1 + L - T} \\
+ & \ 2 \left( (1 - k)(1 + \hat{L} - t_1)(2a + b(3 - t_1 + 3\hat{L})) \right) \\
\ln \left( 1 + \frac{t_1}{1 + L - t_1} \right) + t_1(-2a + b(t_1 - 4(1 + \hat{L})) \ln \left( 1 + \frac{T - t_1}{1 + L - T} \right) \right) \right\} \\
+ & \ 12\gamma L C_d \left\{ 2b(kt_1 - T) + (a + b(1 + \hat{L})) \left( \ln(1 + \hat{L} - t_1) - \ln(1 + \hat{L} - T) \right) \right. \\
+ & \ (k - 1)(a + b(2 + 2\hat{L} - t_1)) \ln \left( 1 + \frac{t_1}{1 + L - t_1} \right) \\
- & \ bt_1 \ln \left( 1 + \frac{T - t_1}{1 + L - T} \right) + \frac{a(k - 1)t_1}{1 + L} + \frac{(t_1 - T)(a + bT)}{1 + L - T} \right\} \right\} \\
+ & \ \lambda_1 \gamma L \left\{ 2b(kt_1 - T) + (a + b(1 + \hat{L})) \left( \ln(1 + \hat{L} - t_1) - \ln(1 + \hat{L} - T) \right) \right. \\
+ & \ (k - 1)(a + b(2 + 2\hat{L} - t_1)) \ln \left( 1 + \frac{t_1}{1 + L - t_1} \right) \\
- & \ bt_1 \ln \left( 1 + \frac{T - t_1}{1 + L - T} \right) + \frac{a(k - 1)t_1}{1 + L} + \frac{(t_1 - T)(a + bT)}{1 + L - T} \right\} \right] \\
+ & \ \lambda_2 \\
\text{where, } \hat{L} = L + \gamma L C_p \\
\xi_1 & = \ - \frac{1}{T} \left\{ \left( (h + C_p)(1 + \hat{L} - t_1) + C_d \right) \left\{ b(kt_1 - 2t_1 + T) \right. \right. \\
+ & \ (a + b(1 + \hat{L})) \left( (k - 1) \ln \left( 1 + \frac{t_1}{1 + L - t_1} \right) - \ln \left( 1 + \frac{T - t_1}{1 + L - T} \right) \right) \right\} \\
+ & \ k(a + bt_1)(C_{mt} + C_m) \right] 
\end{align*}\]
ξ_2 = \left[ -ak + b(T - 2t_1) + (a + b(1 + \hat{L})) \left( (k - 1) \ln \left( 1 + \frac{t_1}{1 + L - t_1} \right) - \ln \left( 1 + \frac{T - t_1}{1 + L - T} \right) \right) \right]

ξ_3 = -\frac{1}{12T} \left( t_1 - T \right) \left\{ 6b - 3a(t_1 + T - 2 - 2\hat{L}) + b \left( 4t_1^2 + (3 - 2T)T + 3\hat{L}(4 + T) + 6\hat{L}^2 - t_1(9 + 2T + 9\hat{L}) \right) \right\} \\
+ (k - 1) \left\{ (1 + \hat{L})^2(3a + 5b(1 + \hat{L})) - (1 + \hat{L} - t_1)^2 \left( 3a + b(5 + 4t_1 + 5\hat{L}) + 6(a + b(1 + \hat{L})) \ln \left( 1 + \frac{t_1}{1 + L - t_1} \right) \right) \right\} \\
+ 6(a + b(1 + \hat{L})) \left\{ (1 + \hat{L})^2 \left( \ln(1 + \hat{L} - t_1) - \ln(1 + \hat{L} - T) \right) + t_1(t_1 - 2(1 + \hat{L})) \ln \left( 1 + \frac{T - t_1}{1 + L - T} \right) \right\} \\
+ 3\gamma L(h + C_p) \left\{ -4bt_1^2 + 4akt_1 + 6bkt_1 - bkt_1^2 + 6bkt_1L + 4aT - 6bT + 8bt_1T - 6bLT - 3bT^2 + 6C_p bL \gamma (kt_1 - T) + 2(1 + \hat{L})(2a + 3b(1 + \hat{L})) \left( \ln(1 + \hat{L} - t_1) - \ln(1 + \hat{L} - T) \right) \right\} \\
- \frac{2a(k - 1)t_1^2}{1 + L} - \frac{2(t_1 - T)^2(a + bT)}{1 + L - T} \\
+ 2 \left( (1 - k)(1 + \hat{L} - t_1)(2a + b(3 - t_1 + 3\hat{L})) \ln \left( 1 + \frac{t_1}{1 + L - t_1} \right) + t_1(-2a + b(t_1 - 4(1 + \hat{L}))) \ln \left( 1 + \frac{T - t_1}{1 + L - T} \right) \right) \right\} \\
+ 12\gamma LC_d \left\{ 2b(kt_1 - T) + (a + 2b(1 + \hat{L})) \left( \ln(1 + \hat{L} - t_1) - \ln(1 + \hat{L} - T) \right) \right\} \\
+ (k - 1) \left\{ (a + b(2 + 2\hat{L} - t_1)) \ln \left( 1 + \frac{t_1}{1 + L - t_1} \right) \right\} \\
- bt_1 \ln \left( 1 + \frac{T - t_1}{1 + L - T} \right) + \frac{a(k - 1)t_1}{1 + L} + \frac{(t_1 - T)(a + bT)}{1 + L - T} \right\}

ξ_4 = \gamma L \left\{ 2b(kt_1 - T) + (a + 2b(1 + \hat{L})) \left( \ln(1 + \hat{L} - t_1) - \ln(1 + \hat{L} - T) \right) \right\} \\
+ (k - 1) \left\{ (a + b(2 + 2\hat{L} - t_1)) \ln \left( 1 + \frac{t_1}{1 + L - t_1} \right) \right\} \\
- bt_1 \ln \left( 1 + \frac{T - t_1}{1 + L - T} \right) + \frac{a(k - 1)t_1}{1 + L} + \frac{(t_1 - T)(a + bT)}{1 + L - T} \right\
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