Reshetikhin’s Formula for the Jones Polynomial of a Link: Feynman Diagrams and Milnor’s Linking Numbers.

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Abstract

We use Feynman diagrams to prove a formula for the Jones polynomial of a link derived recently by N. Reshetikhin. This formula presents the colored Jones polynomial as an integral over the coadjoint orbits corresponding to the representations assigned to the link components. The large $k$ limit of the integral can be calculated with the help of the stationary phase approximation. The Feynman rules allow us to express the phase in terms of integrals over the manifold and the link components. Its stationary points correspond to flat connections in the link complement. We conjecture a relation between the dominant part of the phase and Milnor’s linking numbers. We check it explicitly for the triple and quartic numbers by comparing their expression through the Massey product with Feynman diagram integrals.

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1 Introduction

Let $\mathcal{L}$ be an $n$-component link in a 3-dimensional manifold $M$. E. Witten presented in [1] the Jones polynomial of $\mathcal{L}$ as a path integral over the gauge equivalence classes of $SU(2)$ connection $A_\mu$ on $M$:

$$Z_{\alpha_1, \ldots, \alpha_n} (M, \mathcal{L}; k) = \int [DA_\mu] \exp \left( \frac{i}{\hbar} S_{CS} \right) \prod_{j=1}^n \text{Tr}_{\alpha_j} \text{Pexp} \left( \oint_{L_j} A_\mu dx^\mu \right),$$

(1.1)

here $S_{CS}$ is the Chern-Simons action

$$S_{CS} = \frac{1}{2} \text{Tr} \epsilon^{\mu
u\rho} \int_M d^3 x \left( A_\mu \partial_\nu A_\rho - \frac{2}{3} A_\mu A_\nu A_\rho \right),$$

(1.2)

$h$ is a “Planck’s constant”:

$$\hbar = \frac{2\pi}{k}, \quad k \in \mathbb{Z},$$

(1.3)

the trace $\text{Tr}$ in eq. (1.2) is taken in the fundamental (2-dimensional) representation and $\text{Tr}_{\alpha_j} \text{Pexp} \left( \oint_{L_j} A_\mu dx^\mu \right)$ are the traces of holonomies along the link components $\mathcal{L}_j$ taken in the $\alpha_j$-dimensional representations.

The path integral (1.1) can be calculated in the stationary phase approximation in the limit of large $k$. The stationary points of the Chern-Simons action (1.2) are flat connections and Witten’s invariant is presented as a sum over connected pieces $\mathcal{M}_c$ of their moduli space $\mathcal{M}$:

$$Z_{\alpha_1, \ldots, \alpha_n} (M, \mathcal{L}; k) = \sum_{\mathcal{M}_c} Z^{(\mathcal{M}_c)}_{\alpha_1, \ldots, \alpha_n} (M, \mathcal{L}; k),$$

$$Z^{(\mathcal{M}_c)}_{\alpha_1, \ldots, \alpha_n} (M, \mathcal{L}; k) = \exp \frac{i}{\hbar} \left( S^{(c)}_{CS} + \sum_{n=1}^{\infty} S^{(c)}_n \hbar^n \right),$$

(1.4)

here $S_{CS}$ is a Chern-Simons action of flat connections of $\mathcal{M}_c$ and $S^{(c)}_n$ are the quantum $n$-loop corrections to the contribution of $\mathcal{M}_c$.

Suppose that $M$ is a rational homology sphere (RHS). Then the trivial connection is an isolated point in the moduli space of flat connections. Let $\mathcal{L}$ have only one component, so that it is a knot $\mathcal{K}$. In our previous paper [2] we gave a “path integral” proof of the following conjecture which P. Melvin and H. Morton formulated in [4] for the case of $M = S^3$:
**Proposition 1.1** The trivial connection contribution to the Jones polynomial of a knot \( K \) in a RHS \( M \) can be expressed as

\[
Z_{(\alpha)}^{(tr)}(M, K; k) = Z^{(tr)}(M; k) \exp \left( \frac{\alpha^2 - 1}{2K} \right) \alpha J(\alpha, K),
\]

(1.5)

here \( \nu \) is a self-linking number of \( K \) and \( J(\alpha, K) \) is a function that has the following expansion in \( K^{-1} \) series:

\[
J(\alpha, K) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} D_{m,n} \alpha^m K^{-n}.
\]

(1.6)

The dominant part of this expansion is related to the Alexander polynomial of \( K \):

\[
\pi a \sum_{n=0}^{\infty} D_{n,n} \alpha^n = \left[ \text{ord} H_1(M, \mathbb{Z}) \right] \frac{\sin \left( \frac{\pi a}{m_2d} \right)}{\Delta_A(M, K; e^{2\pi i m_2/\pi})},
\]

(1.7)

the integer numbers \( m_2 \) and \( d \) are defined in [2], \( m_2 = d = 1 \) if \( M = S^3 \).

The virtue of eq. (1.5) is that it assembles the dominant part of the \( 1/K \) expansion of the Jones polynomial \( Z_{(\alpha)}^{(tr)}(M, K; k) \) into the exponential and puts a restriction on the power of \( \alpha \) in the preexponential factor. This enabled us in [2] to use a stationary phase approximation in the Witten-Reshetikhin-Turaev surgery formula in order to derive a knot surgery formula for the loop corrections \( S_n^{(tr)} \). A similar formula for the Jones polynomial of a link is required in order to generalize this result to link surgery. However the arguments of [2] which led to eq. (1.5) can not be extended directly to links, because there may exist irreducible flat connections in the link complement with arbitrarily small holonomies along the meridians of link components.

A generalization of eq. (1.5) for links was derived recently by N. Reshetikhin [5]. He observed that if the dimensions \( \alpha_i \) in eq. (1.1) are big enough, then the representation spaces can be treated classically: the matrix elements of Lie algebra generators in \( \alpha_j \)-dimensional representation can be substituted by functions on the coadjoint orbit of radius \( \alpha_j \) and a trace over the representation can be substituted by an integral over that orbit.

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2I am indebted to N. Reshetikhin for sharing the results of his unpublished research.
Proposition 1.2 Let $\mathcal{L}$ be an $n$-component link in a RHS $M$. Then the trivial connection contribution to its Jones polynomial can be expressed as a multiple integral over the SU(2) coadjoint orbits:

$$Z_{\alpha_1, \ldots, \alpha_n}^{(tr)}(M, \mathcal{L}, k) = Z^{(tr)}(M; k) \int_{|\vec{a}_j| = \frac{\alpha_j}{K}} \prod_{j=1}^n \left( \frac{K d^2 \vec{a}_j}{4\pi |\vec{a}_j|} \right) \exp \left( \frac{i\pi K}{2} \sum_{m=2}^{\infty} L_m(\vec{a}_1, \ldots, \vec{a}_n) \right) \prod_{j=1}^n \left( K^4 \pi d^2 \vec{a}_j \right) \exp \left( \sum_{l,m=0 \atop l+m \neq 0}^{\infty} K^{-m} P_{l,m}(\vec{a}_1, \ldots, \vec{a}_n) \right),$$

(1.8)

where $\vec{a}_j$ are 3-dimensional vectors with fixed length

$$|\vec{a}_j| = \frac{\alpha_j}{K}$$

(1.9)

and $L_m(\vec{a}_1, \ldots, \vec{a}_n)$, $P_{l,m}(\vec{a}_1, \ldots, \vec{a}_n)$ are homogeneous invariant (under SO(3) rotations) polynomials of degree $M$. In particular,

$$L_2(\vec{a}_1, \ldots, \vec{a}_n) = \sum_{i,j=1}^n l_{ij} \vec{a}_i \cdot \vec{a}_j,$$

(1.10)

$l_{ij}$ is the linking number of the link components $\mathcal{L}_i$ and $\mathcal{L}_j$.

An example of this formula for a torus link is derived in Appendix 1 of [3].

In Section 2 we find a set of Feynman rules to calculate the trivial connection contribution to the Jones polynomial $Z_{\alpha_1, \ldots, \alpha_n}^{(tr)}(M, \mathcal{L}, k)$. This enables us to prove the Proposition 1.2 and to derive a property (Proposition 2.3) of the polynomials $L_m(\vec{a}_1, \ldots, \vec{a}_n)$ that we will use in [3] in order to relate the r.h.s. of eq. (1.8) to the multivariable Alexander polynomial. In Section 3 we formulate a conjecture that the coefficients of the polynomials $L_m(\vec{a}_1, \ldots, \vec{a}_n)$ are related to Milnor’s linking numbers in a way that generalizes eq. (1.10), and present some circumstantial evidence that supports it. In Section 4 we study the properties of flat connections in the link complement in order to further support our conjecture. In Sections 5 and 6 we use Feynman diagrams to calculate the coefficients of the polynomials $L_m$ for $m = 3, 4$. We demonstrate explicitly that they are indeed proportional to Milnor’s triple and quartic linking numbers.

I am thankful to A. Vaintrob and O. Viro for teaching me about these linking numbers.
We will use the following notations throughout this paper: an element \( v \) of the Lie algebra \( su(2) \) can be presented as \( v = i\sigma_a v^a \), here \( \sigma_a \) are \( 2 \times 2 \) Pauli matrices which form an orthogonal basis in the fundamental representation of \( SU(2) \): 
\[
[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c, \quad \text{Tr}\sigma_a\sigma_b = 2\delta_{ab}.
\]
Three components \( v^a, 1 \leq a \leq 3 \) form a 3-dimensional vector \( \vec{v} \). Depending on our needs, we will use \( v, v^a \) or \( \vec{v} \) to denote the same object.

2 Feynman Diagrams

We are going to prove Reshetikhin’s formula (1.8) for the Jones polynomial of a link by using Feynman diagrams in order to calculate the path integral (1.1). An introduction into Feynman diagram techniques can be found in any textbook on quantum field theory. Feynman rules of the Chern-Simons theory (1.2) are described, e.g. in [6]. We also especially recommend [7] for a nice simple introduction into this subject.

Our goal is to “take a logarithm” of the sum of all Feynman diagrams contributing to 
\[
Z_{\alpha_1,\ldots,\alpha_n}^{(tr)}(M, L, k).
\]
This would be easy if the link \( L \) had no components: the logarithm would be equal to the sum of all connected Feynman diagrams. Therefore we have to find the analog of “connectivity” for the diagrams with the end-points on the link components. This will be the connectivity in quantum theory on the coadjoint orbit which describes the \( \alpha_j \)-dimensional representation of \( SU(2) \) assigned to a link component \( L_j \). The key to defining the connectivity is the Campbell-Hausdorf formula\(^4\) which shows how to take a logarithm of the product of two noncommuting exponentials.

Let \( \mathcal{K} \) be a knot in a 3-dimensional manifold \( M \). Let \( A(t) \) be a restriction of an \( SU(2) \) connection \( A_\mu \) onto \( \mathcal{K} \):
\[
A(t) = A_\mu(x(t)) \frac{dx^\mu}{dt},
\]
(2.1)

\(^4\)I am indebted to N. Reshetikhin for pointing to the relevance of the Campbell-Hausdorf formula for his derivation of eq. (1.8).
here $0 \leq t \leq 1$ is a parametrization of $\mathcal{K}$. The trace

$$\text{Tr}_\alpha \text{Exp} \left( \int_\mathcal{K} A_\mu dx^\mu \right) = \text{Tr}_\alpha \text{Exp} \left( \int_0^1 A(t) dt \right)$$

(2.2)
can be considered as a partition function of the quantum theory on the coadjoint orbit of $SU(2)$, $A(t)$ playing the role of an external classical field. We will try to put the trace (2.2) in an exponential form similar to that of eq. (1.4).

We start with a simple particular case of the Campbell-Hausdorf formula:

$$e^v e^w = \exp \left[ v + \sum_{n=0}^{\infty} \frac{(-1)^n B_n}{n!} (\text{ad}_v)^n w + \mathcal{O}(w^2) \right],$$

(2.3)
here $v$ and $w$ are elements of a Lie algebra (say, $SU(2)$), $\text{ad}_v w = [v, w]$ and $B_n$ are Bernoulli numbers.

According to the Campbell-Hausdorf formula,

$$e^{v_1} \ldots e^{v_n} = \exp[C(v_1, \ldots, v_n)],$$

(2.4)
here $C(v_1, \ldots, v_n)$ is a Lie algebra valued infinite polynomial in commutators of $v_i$. We denote as $C_n(v_1, \ldots, v_n)$ a part of $C(v_1, \ldots, v_n)$ containing the $n$th order monomials which are linear in all $v_i$:

$$C_n^a(v_1, \ldots, v_n) = C_{a_1, \ldots, a_n}^{a_1} v_1^{a_1} \ldots v_n^{a_n}.$$  

(2.5)
A simple corollary of eq. (2.3) is the following

**Proposition 2.1** The coefficient $C_{a_1, \ldots, a_n}^a$ is a sum of tensors coming from the diagrams like those of Figs. 1-4 according to the rule: a dashed line with $m$ vertices which is depicted in Fig. 1, goes from the upper right to the lower left and produces a factor

$$(-1)^m \frac{B_m}{m!} \sum_{S \in S_m} f_{b(a_1, \ldots, a_m)}^a,$$

(2.6)
here $S_m$ is a group of permutations of numbers $1, \ldots, m$ and the tensor $f_{b(a_1, \ldots, a_m)}^a$ is a sum of products of structure constants of the group $SU(2)$:

$$f_{b(a_1, \ldots, a_m)}^a = (-2)^m \epsilon_{a_1 b_1} \epsilon_{a_2 b_2} \cdots \epsilon_{a_m b_{m-1} a}.$$  

(2.7)
the sum over repeated indices is, of course, implied in eq. (2.7). A contribution of the whole diagram is a sum of the factors (2.6) of its dashed lines over the intermediate indices.

More generally, if we pick \( m \) elements \( v_{i_1}, \ldots, v_{i_m}, \ 1 \leq i_1 < \ldots < i_m \leq n \), then the \( m \)th order homogeneous part of \( C(v_1, \ldots, v_n) \) which is linear in them, is equal to

\[
C_{a_{i_1}, \ldots, a_{i_m}}^{a_{i_1}} v_{i_1}^{a_{i_1}} \cdots v_{i_m}^{a_{i_m}}. \tag{2.8}
\]

**Examples**

The diagram for \( n = 2 \) is drawn in Fig. 2:

\[
i\sigma_a C_{a_1, a_2}^{a_1} v_{a_1}^{a_1} v_{a_2}^{a_2} = \frac{1}{2} [v_1, v_2]. \tag{2.9}
\]

The diagrams for \( n = 3 \) are drawn in Fig. 3:

\[
i\sigma_a C_{a_1, a_2, a_3}^{a_1} v_1^{a_1} v_2^{a_2} v_3^{a_3} = \frac{1}{4} ([v_1, v_2], v_3) + \frac{1}{12} ([v_1, [v_2, v_3]] + [v_2, [v_1, v_3]])
\]

\[
= \frac{1}{6} ([v_1, v_2], v_3) + [v_1, [v_2, v_3]]). \tag{2.10}
\]

The diagrams for \( n = 4 \) are drawn in Fig. 4:

\[
i\sigma_a C_{a_1, a_2, a_3, a_4}^{a_1} v_1^{a_1} v_2^{a_2} v_3^{a_3} v_4^{a_4} = \frac{1}{180} ([v_1, [v_2, [v_3, v_4]]] + [v_1, [v_3, [v_2, v_4]]] + [v_2, [v_1, [v_3, v_4]]] + [v_3, [v_1, [v_2, v_4]]] + [v_3, [v_2, [v_1, v_4]]])
\]

\[
+ \frac{1}{24} ([v_1, v_2], [v_3, [v_2, v_4]]] + [v_2, [v_1, v_3], [v_2, v_4]]] + [v_3, [v_2, [v_1, v_4]]])
\]

\[
+ \frac{1}{24} ([v_1, v_3], [v_2, [v_1, v_4]]] + [v_2, [v_3, v_4], [v_1, v_4]]] + [v_3, [v_1, [v_2, v_4]]])
\]

\[
+ \frac{1}{24} ([v_2, v_3], [v_1, v_4]]] + [v_1, [v_2, v_3], [v_2, v_4]]] + [v_3, [v_1, [v_2, v_4]])
\]

\[
+ \frac{1}{24} ([v_2, v_3], [v_1, v_4]]] + [v_1, [v_2, v_3], [v_2, v_4]]] + [v_3, [v_1, [v_2, v_4]])
\]

\[
+ \frac{1}{8} ([v_1, v_2], v_3, [v_2, v_4]]]. \tag{2.11}
\]

The Proposition 2.1 allows us to “take a logarithm” of the parallel transport operator \( P\exp \left( \int_0^1 A(t)dt \right) \). We introduce an “iterated” integral (see also [12]):

\[
\int_0^1 dt_1 \cdots dt_n \{ A(t_1), \ldots, A(t_n) \} = \int_{0 \leq t_n \leq \ldots \leq t_1 \leq 1} A(t_1) \cdots A(t_n). \tag{2.12}
\]

Note that the iterated integral depends on a choice of the zero point of \( t \) parametrization of \( \mathcal{K} \).
Proposition 2.2 The holonomy operator \( P \exp \left( \int_0^1 A(t) dt \right) \) can be presented as an exponential of an infinite sum of iterated integrals:

\[
P \exp \left( \int_0^1 A(t) dt \right) = \exp \left[ i \sigma_a \sum_{n=1}^{\infty} C_{a_1, \ldots, a_n}^a \int_0^1 dt_1 \cdots dt_n \{ A^{a_1}(t_1), \ldots, A^{a_n}(t_n) \} \right]. \tag{2.13}
\]

Let us first present a simple “physical” proof of eq. (2.10). We split the interval \( 0 \leq t \leq 1 \) into many small intervals \( \Delta t_i \) with middle points \( t_i \) so that

\[
P \exp \left( \int_0^1 A(t) dt \right) = e^{A(t_1) \Delta t_1} e^{A(t_2) \Delta t_2} \cdots e^{A(t_n) \Delta t_n}. \tag{2.14}
\]

Then we apply the Campbell-Hausdorf formula to the r.h.s. of eq. (2.14) retaining only those terms of \( C(A(t_1) \Delta t_1, \ldots, A(t_n) \Delta t_n) \) which are at most linear in any particular \( A(t_i) \). According to the Proposition 2.1, such terms are given by eq. (2.8) with \( v_{i_m} \) substituted by \( A(t_{i_m}) \). It is easy to see that a sum of all polynomials of a given order converges to the iterated integral of eq. (2.13).

To prove eq. (2.13) more rigorously we may use the following presentation of the holonomy operator:

\[
P \exp \left( \int_0^1 A(t) dt \right) = \sum_{n=0}^{\infty} \int_0^1 dt_1 \cdots dt_n \{ A(t_1), \ldots, A(t_n) \}
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{s \in S_n} \int_0^1 dt_1, \ldots, dt_n \{ A(t_{s(1)}, \ldots, A(t_{s(n)}) \}
\]

\[
\overset{\text{def.}}{=} \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 dt_1, \ldots, dt_n P[A(t_1, \ldots, A(t_n)], \tag{2.15}
\]

here \( P[A(t_1, \ldots, A(t_n)] \) is a path-ordered product, i.e. the Lie algebra valued forms \( A(t_i) \) are multiplied in the order of the values of their arguments \( t_i \): for the largest \( t_i \), \( A(t_i) \) stands to the left and so on.

The Proposition 2.1 implies the following formula for the product of \( n \) Lie algebra elements \( v_i \):

\[
v_1 \cdots v_n = \partial \varepsilon_1 \cdots \partial \varepsilon_n \sum_{m=1}^{n} \frac{1}{m!} \left( i \sigma_a \sum_{l=1}^{m} \sum_{s \in S_{n}^l} C_{a_1, \ldots, a_l}^{a} v_{s(1)}^{a_1} \cdots v_{s(l)}^{a_l} \varepsilon_{s(1)} \cdots \varepsilon_{s(l)} \right) \bigg|_{\varepsilon_1 = \cdots = \varepsilon_n = 0} \tag{2.16}
\]

here \( S_n^l \) is a set of all injections of \( l \) numbers \( 1, \ldots, l \) into \( n \) numbers \( 1, \ldots, n \) which keeps the order. A more explicit version of this formula requires a splitting of \( n \) elements \( v_i \) into
sets, with \( n_i \) elements in each set, \( m \) being an arbitrary integer number: \( 1 \leq m \leq n \). Denote by \( (s_1, \ldots, s_m) \) an injection of the union of \( m \) sets \( 1, \ldots, n_i \) into the set \( 1, \ldots, n \) which preserves the order within each set \( 1, \ldots, n_i \): \( s_i(i) > s_i(j) \) if \( i > j \). Consider now a symmetrized product

\[
D = \frac{1}{m! \prod_{l=1}^{m} (\# l)!} \sum_{s \in S_m} D_{s(1)} \ldots D_{s(m)},
\]

(2.17)

here

\[
D_i = i \sigma_a C_{a_1, \ldots, a_{n_i}} v_{s_i(1)}^{a_1} \ldots v_{s_i(n_i)}^{a_{n_i}}
\]

(2.18)

and \( \# l \) is the number of indices \( i \) for which \( n_i = l \). The r.h.s. of eq. (2.16) is equal to the sum of all such \( D \) taken over all numbers \( m \), all possible splittings and all injections \( (s_1, \ldots, s_m) \).

Let us apply this presentation to the product \( A(t_{s(1)}) \ldots A(t_{s(n_i)}) \) appearing in the second line of eq. (2.15). Suppose that we permute some of \( A(t_{s(i)}) \). A term \( D \) coming from a particular injection \( (s_1, \ldots, s_m) \) still remains if this permutation does not change the order within the \( n_i \)-element sets into which the elements \( A(t_{s(i)}) \) are split by the injection. Therefore we can combine the integrals over \( 0 \leq A(t_{s(1)}) \leq \ldots \leq A(t_{s(i)}) \leq 1 \) which come with a particular term \( D \) into one integral over the regions \( 0 \leq A(t_{s(s_i(n_i))}) \leq \ldots \leq A(t_{s(s_i(1))}) \leq 1 \) for \( 1 \leq i \leq m \). This operation leaves a subgroup of \( S \) which permutes the images \( s_i(1), \ldots, s_i(n_i) \) for a given \( i \). Its composition with \( S_n^i \) creates a redundant group \( S_n \). The sum over its elements can be removed by relabelling the integration variables and adding an extra factor \( n! \) which cancels the same factor in the denominator of eq. (2.15). It is not hard to see that what remains is the \( m \)th order term in the expansion of the exponential in eq. (2.13). This completes the proof of the Proposition 2.2.

A presentation of the holonomy operator as an exponential enables us to use the Weyl character formula for the calculation of its trace. For an element \( v \) of the Lie algebra \( su(2) \)

\[
\text{Tr}_\alpha e^v = \frac{\sin(\alpha |\vec{v}|)}{\sin |\vec{v}|} = \frac{|\vec{v}|}{\sin |\vec{v}|} \int_{|\vec{a}|=\alpha} \frac{d^2 \vec{a}}{4\pi |\vec{a}|} e^{i\vec{a} \cdot \vec{v}}.
\]

(2.19)

Combining this equation with eq. (2.13) we conclude that

\[
\text{Tr}_\alpha \text{Pexp} \left( \oint_K A_\mu dx^\mu \right) = \frac{\sin(\alpha |\vec{v}|)}{\sin |\vec{v}|} = \frac{|\vec{v}|}{\sin |\vec{v}|} \int_{|\vec{a}|=\alpha} \frac{K d^2 \vec{a}}{4\pi |\vec{a}|} e^{iK \vec{a} \cdot \vec{v}}.
\]

(2.20)
where\[ v^a = \sum_{n=1}^{\infty} C^a_{\alpha_1,\ldots,\alpha_n} \int_0^1 dt_1 \cdots dt_n \{A^{a_1}(t_1), \ldots, A^{a_n}(t_n)\} \tag{2.21} \]

This is the exponential presentation of the trace of holonomy that we were looking for.

We can apply the formula (2.20) to the holonomies along the link components of eq. (1.1):

\[
Z_{\alpha_1,\ldots,\alpha_n}(M, \mathcal{L}; k) = \int_{\bar{a}_j = \frac{\alpha}{2\pi}} d^{\mathcal{L}} a \prod_{j=1}^n \left( \frac{K d^2 \bar{a}_j}{4\pi |\bar{a}_j|} \right) \int [\mathcal{D}A] \left( \prod_{j=1}^n \frac{|\bar{v}_j|}{\sin |\bar{v}_j|} \right) \tag{2.22} \]

\[
\times \exp \left[ -\frac{ik}{\pi} e^{\mu \nu \rho} \int_M d^3 x \left( \frac{1}{2} \bar{A}_\mu \cdot \partial_\nu \bar{A}_\rho + \frac{1}{3} \bar{A}_\mu \cdot (\bar{A}_\nu \times \bar{A}_\rho) \right) \right. \]

\[+ i K \sum_{j=1}^n \bar{a}_j \cdot \bar{v}_j \left. \right] \]

here

\[ v^a_j = \sum_{n=1}^{\infty} C^a_{\alpha_1,\ldots,\alpha_n} \int_0^1 dt_1^{(j)} \cdots dt_n^{(j)} \{A^{(j)}(t_1^{(j)}), \ldots, A^{(j)}(t_n^{(j)})\}, \tag{2.23} \]

and \( t^{(j)} \) is a parametrization of a link component \( \mathcal{L}_j \). We put \( K \) instead of \( k \) as a factor in front of the integral in the exponent of eq. (2.22) in order to be able to ignore the 1-loop corrections to the propagator of the original Chern-Simons theory (1.1) (see e.g. [7] and references therein). The formula (2.22) indicates that the terms \( \bar{a}_j \cdot \bar{v}_j \) may be considered as extra vertices in the quantum Chern-Simons theory. In other words, Feynman rules for the quantum theory (2.22) include a propagator (i.e., a Green’s function)

\[ \langle A^a_\mu(x_1)A^b_\nu(x_2) \rangle = -\frac{i\pi}{K} \delta^{ab} \Omega_{\mu\nu}(x_1, x_2), \tag{2.24} \]

a usual cubic vertex

\[ V^3 = -\frac{iK}{3\pi} e^{\mu \nu \rho} \int_M d^3 x \bar{A}_\mu \cdot (\bar{A}_\nu \times \bar{A}_\rho) \tag{2.25} \]

and extra vertices coming from the holonomies along the link components

\[ V^{(j)}_n = i K \bar{a}_j \cdot \bar{C}^a_{\alpha_1,\ldots,\alpha_n} \int_0^1 dt_1^{(j)} \cdots dt_n^{(j)} \{A^{(j)}(t_1^{(j)}), \ldots, A^{(j)}(t_n^{(j)})\}. \tag{2.26} \]

In particular,

\[ V_1^{(j)} = i K \bar{a}_j \cdot \left( \int_0^1 dt^{(j)} \bar{A}(t^{(j)}) \right), \tag{2.27} \]

\[ V_2^{(j)} = -i K \bar{a}_j \cdot \left( \int_0^1 dt_1^{(j)} dt_2^{(j)} \{\bar{A}(t_1^{(j)}) \times \bar{A}(t_2^{(j)})\} \right), \tag{2.28} \]

\[ V_3^{(j)} = \frac{2}{3} i K \bar{a}_j \cdot \left[ \int_0^1 dt_1^{(j)} dt_2^{(j)} dt_3^{(j)} \left( \{\bar{A}(t_1^{(j)}) \times \bar{A}(t_2^{(j)})\} \times \bar{A}(t_3^{(j)}) \right) \right. \]

\[ + \{\bar{A}(t_1^{(j)}) \times (\bar{A}(t_2^{(j)}) \times \bar{A}(t_3^{(j)}))\}], \tag{2.29} \]
a_j \cdot \{ \vec{A}(t_1^{(j)}) \times \vec{A}(t_2^{(j)}) \} = \epsilon_{abc} a_j^a \{ A^b(t_1^{(j)}), A^c(t_2^{(j)}) \}.

(2.30)

A symmetric bilocal (1,1)-form (i.e., a 1-form in both variables $x$, $y$) $\Omega_{\mu\nu}(x, y)$ of eq. (2.24) is a Green’s function of the operator $\epsilon^{\mu\nu\rho} \partial_\nu$ (i.e., of a differential $d$). It should satisfy an equation

$$d_\mu \Omega(x, y) = \delta^{(3)}(y - x) + d_x \hat{\Omega}(x, y),$$

(2.31)

here $\delta^{(3)}(y - x)$ is a 3-form $\delta$-function while $\hat{\Omega}(x, y)$ is a (0,2)-form, i.e. a 2-form in $y$ and a 0-form in $x$. If $M = S^3$ and $S^3$ is presented as $\mathbb{R}^3$ with an infinite point, then

$$\Omega_{\mu\nu}(x, y) = \frac{1}{4\pi} \epsilon_{\mu\nu\rho} \frac{y^\rho - x^\rho}{|y - x|^3}.$$  

(2.32)

For more information on $\Omega(x, y)$ see, e.g. [7] and references therein.

There are some extra vertices coming from the expansion of the prefactors $\frac{|\vec{v}_j|}{\sin |\vec{v}_j|}$ into the powers of $\vec{v}_j^2$. These vertices do not assemble into an exponential. According to the standard combinatorics of Feynman diagrams, the trivial connection contribution to the path integral over $[DA_\mu]$ in eq. (2.22) can be presented as a product of two factors:

$$Z^{(tr)}_{\alpha_1, \ldots, \alpha_n}(M, \mathcal{L}, k) = (1 + G(\vec{a}_1, \ldots, \vec{a}_n; K)) \exp \left[ \sum_{l=0}^{\infty} K^{1-l} L^{(l)}(\vec{a}_1, \ldots, \vec{a}_n) \right],$$

(2.33)

here $G(\vec{a}_1, \ldots, \vec{a}_n; K)$ is a sum of all Feynman diagrams containing at least one vertex coming from $\frac{|\vec{v}_j|}{\sin |\vec{v}_j|}$ and $L^{(l)}(\vec{a}_1, \ldots, \vec{a}_n)$ is a sum of all connected $l$-loop Feynman diagrams which contain only the vertices (2.25) and (2.26). Each of these vertices carries a factor $K$ while the propagator (2.24) is of order $K^{-1}$. As a result, $l$-loop diagrams have a factor $K^{1-l}$ which we made explicit in the exponent of eq. (2.33). The vertices coming from $\frac{|\vec{v}_j|}{\sin |\vec{v}_j|}$ do not carry the factor $K$, therefore the diagrams that contain such vertices have only zero or negative powers of $K$. Thus, combining the Taylor series expansions

$$L^{(1)}(\vec{a}_1, \ldots, \vec{a}_n) = \sum_{m=2}^{\infty} L_m(\vec{a}_1, \ldots, \vec{a}_n),$$

(2.34)
\[ 1 + G(\vec{a}_1, \ldots, \vec{a}_n; K) \exp \left[ \sum_{l=1}^{\infty} K^{1-l} L^{(l)}(\vec{a}_1, \ldots, \vec{a}_n) \right] \] (2.35)

\[ = 1 + \sum_{l,m=0}^{\infty} K^{-m} P_{m,l}(\vec{a}_1, \ldots, \vec{a}_n), \]

where \( L_m(\vec{a}_1, \ldots, \vec{a}_n) \) and \( P_{l,m}(\vec{a}_1, \ldots, \vec{a}_n) \) are invariant homogeneous polynomials of order \( m \), with eqs. (2.33) and (2.22) we arrive at Reshetikhin’s formula (1.8).

A quadratic polynomial \( L_2 \) comes from the Feynman diagram containing only one propagator (2.24) both endpoints of which are attached to link components. Since

\[ \oint_{L_i} dx_i \oint_{L_j} dx_j \Omega(x_i, x_j) = l_{ij}, \] (2.36)

\( l_{ij} \) being the gaussian linking number, we conclude that

\[ L_2(\vec{a}_1, \ldots, \vec{a}_n) = \sum_{i,j=1}^{n} l_{ij} \vec{a}_i \cdot \vec{a}_j. \] (2.37)

This completes the proof of the Proposition 1.2.

We will also need in the future the polynomials

\[ L_3(\vec{a}_1, \ldots, \vec{a}_n) = \sum_{i,j,k=1}^{n} l^{(3)}_{ijk} \vec{a}_i \cdot (\vec{a}_j \times \vec{a}_k), \] (2.38)

\[ L_4(\vec{a}_1, \ldots, \vec{a}_n) = \sum_{i,j,k,l=1}^{n} l^{(4)}_{ijkl} (\vec{a}_i \times \vec{a}_j) \cdot (\vec{a}_k \times \vec{a}_l). \] (2.39)

It will become clear from the study of Feynman diagrams in Sections 3 and 4 why we use these particular group weight structures.

Consider now the group weights associated to connected tree level diagrams contributing to the polynomials \( L_m(\vec{a}_1, \ldots, \vec{a}_n) \). These diagrams are combinations of tree diagrams of the original Chern-Simons theory (1.2) sewn by the new vertices (2.26) which are themselves tree diagrams (see Figs. 1–4). In both types of tree diagrams the segments are \( \delta \)-symbols \( \delta_{ab} \) and the elementary cubic vertices are proportional to the group structure constants \( \epsilon_{abc} \).

We may associate an invariant monomial \( F_m(\vec{b}_1, \ldots, \vec{b}_m) \) which is linear in all vectors \( \vec{b}_j, 1 \leq j \leq m \) to a combined group weight coming from a Feynman diagram with \( m \)
vertices (2.26) by placing vectors \(\vec{b}_j\) at the bottom of the tree diagrams associated with its vertices (2.26). Then in order to get the actual group weight of the Feynman diagram that would contribute to the polynomial \(L_m(\vec{a}_1, \ldots, \vec{a}_n)\), we should substitute \(n\) vectors \(\vec{a}_j\) for \(m\) vectors \(\vec{b}_j\) depending on which vertex (2.26) comes from which link component \(L_j\).

Since every tree diagram with more than three external legs contains at least two \(Y\)-shaped configurations with two vectors \(\vec{b}\) attached to each of them (see Fig. 5) and since the cubic vertices produce antisymmetric tensors \(\epsilon_{abc}\), we conclude that a polynomial \(F_m(\vec{b}_1, \ldots, \vec{b}_m)\), \(m \geq 4\) is zero if at least \(m - 1\) of \(m\) vectors \(\vec{b}_j\) are parallel. It is obvious that the same is true for \(m = 3\).

**Proposition 2.3** The polynomials \(L_m(\vec{a}_1, \ldots, \vec{a}_n)\) are produced from invariant homogeneous polynomials \(F_m(\vec{b}_1, \ldots, \vec{b}_m)\) of order \(m\) by substituting \(n\) vectors \(\vec{a}_j\) in place of \(m\) vectors \(\vec{b}_j\). The polynomials \(F_m(\vec{b}_1, \ldots, \vec{b}_m), m \geq 3\) are equal to zero if at least \(m - 1\) of \(m\) vectors \(\vec{b}_j\) are parallel.

This proposition will play an important role in extracting the multivariable Alexander polynomial from the r.h.s. of eq. (1.8) in [3].

Suppose for a moment that the link \(\mathcal{L}\) has only one component, i.e. it is in fact a knot \(\mathcal{K}\). An immediate consequence of the Proposition 2.3 is that only a quadratic term survives in the exponent of the Reshetikhin’s formula (1.8). As a result, the whole formula is reduced to eq. (1.5). Thus we produced yet another proof of the first part of the Melvin-Morton conjecture 1.1.

### 3 The Exponent of Reshetikhin’s Formula

In the remainder of this paper we are going to study more closely the structure of the exponent of Reshetikhin’s formula (1.8).

**Conjecture 3.1** If \(L_l(\vec{a}_1, \ldots, \vec{a}_n) = 0\) for all \(l < m\), then the coefficients of the polynomial \(L_m(\vec{a}_1, \ldots, \vec{a}_n)\) are proportional to the \(m\)th order Milnor’s linking numbers \(l^{(\mu)}_{i_1, \ldots, i_m}\) of the link.
Here $\mathbf{L} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is a $3$-dimensional vector formed by Pauli matrices.

We use the scalar product $\mathbf{\sigma} \cdot \mathbf{a}_i$ instead of simply $a_i$ in order to stress that we are dealing with the $su(2)$ Lie algebra element in the fundamental representation. Note that $l_{ij}^{(\mu)} = l_{ij}$, so eq. (3.1) for $m = 2$ is consistent with eq. (1.10).

It is known that Milnor’s link invariants of the higher order are not well defined (at least, in $\mathbb{Z}$) if lower order invariants are non-zero. The same property is shared by the polynomials $L_m$. Their coefficients can not be restored unambiguously from the value of the partition function $Z_{\alpha_1,\ldots,\alpha_n}^{(tr)}(M, \mathcal{L}, k)$ because there are changes in integration variables $\mathbf{a}_j$ which keep the form of eq. (1.8) but still alter the coefficients of higher order polynomials $L_m$ if the lower order polynomials are non-zero. Suppose for example that we substitute a vector $\mathbf{a}_j$ by a vector $\mathbf{a}_j'$ obtained by rotating $\mathbf{a}_j$ around another vector $\mathbf{a}_k$ by an angle $\phi_{j,k}:

$$
\mathbf{a}_j' = \sum_{m=0}^{\infty} \frac{\phi_{j,k}^m}{m!} \left[ \mathbf{a}_k \times \mathbf{a}_k \times \ldots \mathbf{a}_k \times \mathbf{a}_j \right] \ldots.
$$

(3.2)

A substitution of eq. (3.2) into the quadratic polynomial $L_2$ of eq. (2.37) generates, among others, new cubic terms, so that

$$
l_{ijk}^{(3)} = l_{ijk}^{(3)} + \lambda_{ij} \phi_{j,k}, \quad 1 \leq i \leq n, \quad i \neq j, k.
$$

(3.3)

To put it differently, we need to know the coefficients $l_{ijk}^{(3)}$ only up to these transformations. In fact, as we will see in Section 5, the Feynman rules of Section 2 predict these coefficients only up to this transformation due to the dependence of the vertices (2.26) on the choice of a zero point in $t^{(j)}$ parametrization of the link component $\mathcal{L}_j$. However this ambiguity disappears if the gaussian linking numbers $l_{ij}$ are equal to zero. Then the coefficients $l_{ijk}^{(3)}$ are well defined and turn out to be proportional to the triple Milnor’s invariants.

There are many other possible rotations of the integration variables $\mathbf{a}_j$. For example, a vector $\mathbf{a}_j$ can be rotated around the vector $\mathbf{a}_i \times \mathbf{a}_j$. This transformation will change the
coefficient $l^{(4)}_{ij,ij}$ by an amount proportional to $l_{ij}$. It will also cause a change in the coefficient $p_{ij}$ of the preexponential polynomial

$$P_{0,2}(\vec{a}_1, \ldots, \vec{a}_n) = \sum_{i,j=1}^{n} p_{ij} \vec{a}_i \cdot \vec{a}_j,$$  \hspace{1cm} (3.4)

due to a nontrivial jacobian factor.

Let us now consider some circumstantial evidence in support of the Conjecture 3.1. Milnor’s linking numbers are invariant under tying a small knot on a link component. Let $\mathcal{L}$ be an $n$-component knot in a RHS $M$ and $\mathcal{K}$ be a knot in $S^3$. We cut an infinitely small 3-dimensional ball $B^3$, whose center belongs to $\mathcal{L}_1$, out of $M$ and we cut another infinitely small ball $\tilde{B}^3$, whose center belongs to $\mathcal{K}$, out of $S^3$. Then we glue the boundaries $\partial (M \setminus B^3)$ and $\partial (S^3 \setminus \tilde{B}^3)$, thus producing a new link $\mathcal{L}'$ in $M$. In other words, we “tie” a small knot $\mathcal{K}$ on the component $\mathcal{L}_1$ of $\mathcal{L}$. Milnor’s invariants of $\mathcal{L}$ and $\mathcal{L}'$ coincide.

Let us see what happens to the exponent of eq. (1.8). According to [1],

$$Z_{\alpha_1,\ldots,\alpha_n}(M, \mathcal{L}; k) = \frac{\sqrt{K}}{2 \sin \left( \frac{\pi K}{4} \right)} \int_{|\vec{a}_1| = \frac{\pi K}{4}} \prod_{j=1}^{n} \left( \frac{K}{2 \pi |\vec{a}_j|} \right) \exp \left[ \frac{i \pi K}{2} \left( \nu \vec{a}_1^2 + \sum_{m=2}^{\infty} L_m(\vec{a}_1, \ldots, \vec{a}_n) \right) \right]$$

Combining the formula (1.8) for $Z_{\alpha_1,\ldots,\alpha_n}(M, \mathcal{L}; k)$ with the formula (1.5) for $Z_{\alpha_1}(S^3, \mathcal{K}; k)$ we can easily derive the formula (1.8) for $Z_{\alpha_1,\ldots,\alpha_n}(M, \mathcal{L}'; k)$:

$$Z_{\alpha_1,\ldots,\alpha_n}(M, \mathcal{L}'; k) = Z_{\alpha_1,\ldots,\alpha_n}(M; k) \int_{|\vec{a}_1| = \frac{\pi K}{4}} \prod_{j=1}^{n} \left( \frac{K}{2 \pi |\vec{a}_j|} \right) \exp \left[ \frac{i \pi K}{2} \left( \nu \vec{a}_1^2 + \sum_{m=2}^{\infty} L_m(\vec{a}_1, \ldots, \vec{a}_n) \right) \right]$$

As we see, the exponent remains the same except for the trivial change of framing of $\mathcal{L}_1$: $l'_{11} = l_{11} + \nu$. This provides a confirmation for our conjecture that the coefficients of the polynomials $L_m(\vec{a}_1, \ldots, \vec{a}_n)$ are proportional to Milnor’s linking numbers.
4 Flat Connections in the Link Complement

The strongest evidence in support of the Conjecture 3.1 is provided by the relation between the large $K$ asymptotics of the Jones polynomial of a link and the flat connections in the link complement. In this section we will assume for simplicity that our manifold $M$ is a 3-dimensional sphere $S^3$. Consider a large $K$ limit of the Jones polynomial (1.1) when the ratios (1.9) are kept constant. Then (see [2] and references therein) the invariant $Z_{\alpha_1,...,\alpha_n}(S^3,\mathcal{L};k)$ can be expressed as a path integral over the connections in the link complement:

$$Z_{\alpha_1,...,\alpha_n}(S^3,\mathcal{L};k) = \int [DA_{\mu}] \exp \left( \frac{i}{\hbar} S'_{CS}[A_{\mu}] \right) ,$$

$$S'_{CS}[A_{\mu}] = \frac{1}{2} \text{Tr} e^{i\mu\rho} \int_{S^3} \sum_{j=1}^{n} \text{Tub}(L_j) d^3x \left( A_{\mu} \partial_{\nu} A_{\rho} - \frac{2}{3} A_{\mu} A_{\nu} A_{\rho} \right) - \frac{1}{2} \sum_{j=1}^{n} \text{Tr} \left[ \left( \oint_{C_{1}^{(j)}} A_{\mu} dx^\mu \right) \left( \oint_{C_{2}^{(j)}} A_{\mu} dx^\mu \right) \right] .$$

In this formula $C_{1}^{(j)}$ and $C_{2}^{(j)}$ are two basic cycles on the boundary of the tubular neighborhood $\text{Tub}(\mathcal{L}_j)$. The cycle $C_{1}^{(j)}$ is a meridian of $\mathcal{L}_j$, it can be contracted through $\text{Tub}(\mathcal{L}_j)$. A cycle $C_{2}^{(j)}$ is a parallel, it has a unit intersection number with $C_{1}^{(j)}$ and it is defined only modulo $C_{1}^{(j)}$. A self-linking number $l_{jj}$ is by definition a linking number between $\mathcal{L}_j$ and $C_{2}^{(j)}$. The path integral (4.1) goes over the gauge equivalence classes of connections $A_{\mu}$ which satisfy the boundary conditions

$$\text{Pexp} \left( \oint_{C_{1}^{(j)}} A_{\mu} dx^\mu \right) = \exp \left( \frac{i\pi}{K} \sigma_3 \alpha_j \right) \equiv \exp(i\pi \vec{\sigma} \cdot \vec{a}_j) \quad \text{up to a conjugation. (4.3)}$$

The large $K$ limit of the path integral (4.1) can be found with the help of the stationary phase approximation. The invariant $Z_{\alpha_1,...,\alpha_n}(S^3,\mathcal{L};k)$ will be presented in the form (1.4), however this time the sum will go over the flat connections in the link complement $S^3 \setminus \sum_{j=1}^{n} \text{Tub}(\mathcal{L}_j)$ satisfying the boundary conditions (4.3). On the other hand, the same large $K$ asymptotics of $Z_{\alpha_1,...,\alpha_n}^{(tr)}(S^3,\mathcal{L};k)$ can be found by applying the stationary phase approximation to the finite dimensional integral over the vectors $\vec{a}_j$ in Reshetikhin’s formula (1.8). The invariant
will be presented as a sum over the conditional stationary points of the phase
\[ \sum_{m=1}^{\infty} L_m(\vec{a}_1, \ldots, \vec{a}_n), \] (4.4)

the conditions being eqs. (1.9). Therefore we conjecture that there is a one-to-one correspondence between the flat connections in the link complement, which are close to the trivial connection, and the stationary phase points of the phase (4.4), so that their contributions to \( Z^{(\text{tr})}_{\alpha_1, \ldots, \alpha_n}(S^3, \mathcal{L}; k) \) are equal.

Let us make this relation more precise. Consider a 1-parametric family of flat connections \( A_\mu(x, \tau), \tau \geq 0 \) in the link complement, which starts at the trivial connection: \( A_\mu(x, 0) = 0 \). The holonomies of a flat connection define a homomorphism from the fundamental group of the link \( \pi_1(\mathcal{L}) \) into \( SU(2) \). In Wirtinger’s presentation of \( \pi_1(\mathcal{L}) \) the link is projected onto a plane and the fundamental group is generated by the meridians \( C_{j,i} \) of the pieces into which the link components \( \mathcal{L}_j \) are split by overcrossings. If two pieces \( \mathcal{L}_{j,i} \) and \( \mathcal{L}_{j,i+1} \) of a link component \( \mathcal{L}_j \) are joined at the overcrossing of \( \mathcal{L}_j \) by a piece \( \mathcal{L}_{k,l} \) of \( \mathcal{L}_k \), then the elements \( C_{j,i}, C_{j,i+1}, C_{k,l} \in \pi_1(\mathcal{L}) \) satisfy the equation
\[ C_{j,i+1} = C_{k,l}^{\pm 1} C_{j,i} C_{k,l}^{\pm 1}, \] (4.5)
the signs in the exponents depend on the signature of the overcrossing. These relations imply that for a given \( j \) the holonomies of \( A_\mu(x, \tau) \) along the meridians \( C_{j,i} \) are all equal to the leading order in \( \tau \):
\[ \text{Pexp} \left( \oint_{C_{j,i}} A_\mu(x, \tau) dx^\mu \right) = \exp \left[ i\pi \tau \vec{\sigma} \cdot \vec{a}_j \right] + O(\tau^2), \] (4.6)
here \( \vec{a}_j \) are the vectors indicating the directions in the Lie algebra \( su(2) \) in which the trivial connection of \( \tau = 0 \) in deformed as \( \tau \) grows.

Consider a knot \( \mathcal{K} \) in the link complement \( S^3 \setminus \sum_{j=1}^{n} \text{Tub}(\mathcal{L}_j) \), we denote as \( l_{0j} \) the linking numbers of \( \mathcal{K} \) and \( \mathcal{L}_j \). It is not hard to see that to the leading order in \( \tau \)
\[ \text{Pexp} \left( \oint_{\mathcal{K}} A_\mu(x, \tau) dx^\mu \right) = \exp \left[ i\pi \tau \vec{\sigma} \cdot \left( \sum_{j=1}^{n} l_{0j} \vec{a}_j \right) \right] + O(\tau^2), \] (4.7)
so that if we attach a $\beta$-dimensional representations to $\mathcal{K}$, then

$$\text{Tr}_\beta \text{Pexp} \left( \int_{\mathcal{K}} A_\mu(x, \tau) dx^\mu \right) = \beta \left[ 1 - \tau^2 \frac{\pi^2}{6}(\beta^2 - 1) \left( \sum_{j=1}^{n} l_{0j} \vec{a}'_j \right)^2 \right] + O(\tau^3). \quad (4.8)$$

Consider now a family of conditional stationary points of the phase (4.4):

$$\vec{a}^{(\text{st})}_j(\tau) = \vec{a}_j'' + O(\tau^2). \quad (4.9)$$

Let us add the knot $\mathcal{K}$ as a 0th component to the link $\mathcal{L}$ (i.e., multiply the r.h.s. of eq. (1.1) by $\text{Tr}_\beta \text{Pexp} (f_{\mathcal{K}} A_\mu dx^\mu)$) and see what happens to the contribution of the stationary phase points (4.9). The new exponent of the formula (1.8) should include the terms containing the vector $\vec{b}$ corresponding to $\mathcal{K}$: $|\vec{b}| = \beta/K$. We assume that $\beta \sim 1$ as $K \to \infty$. Then, to the leading order in $\tau, K$ and $\beta$, we should account only for the bilinear term $\vec{b} \cdot \left( \sum_{j=1}^{n} l_{0j} \vec{a}_j \right)$ in the new exponent. As a result, the contribution of the stationary phase point (4.9) is multiplied by the factor

$$\int_{|\vec{b}|=\frac{K}{\pi}} \frac{K}{4\pi} d^2 \vec{b} \exp \left[ i \pi \tau K \vec{b} \cdot \left( \sum_{j=1}^{n} l_{0j} \vec{a}'_j \right) \right] = \beta \left[ 1 - \frac{1}{6} \pi^2 \tau^2 \beta^2 \left( \sum_{j=1}^{n} l_{0j} \vec{a}'_j \right)^2 \right] + O(\tau^3). \quad (4.10)$$

This factor should be interpreted as the trace of the holonomy of the flat connection corresponding to the stationary point (4.9). Comparing eqs. (4.10) and (4.8) we conclude that

$$\vec{a}_j' = \vec{a}_j'' \quad (4.11)$$

for the family of the flat connections $A_\mu(x, \tau)$ that corresponds to the family of the stationary phase points (4.9).

Milnor’s linking numbers $l_{i_1, \ldots, i_m}^{(\mu)}$ allow us to formulate the necessary conditions that the vectors $\vec{a}_j'$ have to satisfy so that the flat connections $A_\mu(x, \tau)$ with the holonomies (4.6) exist. Let us briefly review the algebraic definition of these numbers. Consider a parallel $\tilde{C}_j$, by definition it is an element of $\pi_1(\mathcal{L})$ which is homologically equivalent to the parallel $C_2^{(j)}$ and also commutes with the meridian $C_{j,1}$. The element $\tilde{C}_j$ can be expressed as a product of powers of the meridians $C_{k,1}$ since they generate the whole group $\pi_1(\mathcal{L})$. Milnor showed [8] that the parallel $\tilde{C}_j$ can be expressed only in terms of the meridians $C_{k,1}$ (one meridian
per link component) modulo the elements of the lower central subgroup \( \pi_1^{(q)}(L) \) of \( \pi_1(L) \) \((\pi_1^{(1)}(L) = \pi_1(L), \pi_1^{(n+1)}(L) = [\pi_1^{(n)}, \pi_1(L)]))\) for any arbitrarily big value of \( q \in \mathbb{Z} \). Suppose that the first non-zero Milnor’s numbers appear at order \( m \). Then we choose \( q > m \). It is easy to see that for the family of flat connections (4.6) the holonomy along the elements of \( \pi_1^{(q)}(L) \) is equal to 1 up to the order \( \tau^q \). We are interested in the holonomy along \( \tilde{C}_j \) up the order \( \tau^{m-1} \), so we can neglect the elements of \( \pi_1^{(q)}(L) \) and use the expression for \( \tilde{C}_j \) in terms of \( C_{k,1} \) modulo \( \pi_1^{(q)}(L) \). Then, according to eqs. (4.6) and Milnor’s definition of \( l^{(\mu)}_{i_1,\ldots,i_m} \) as the coefficients of the Magnus expansion of this expression,

\[
\exp \left( \oint_{\tilde{C}_j} A_\mu(x, \tau) dx^\mu \right) = \exp \left( \sum_{1 \leq i_1, \ldots, i_{m-1} \leq n} \frac{l^{(\mu)}_{i_1,\ldots,i_{m-1},j}}{l^{(\mu)}_{i_1,\ldots,i_{m-1},j}} (\vec{\sigma} \cdot \vec{a}_{i_1}^\tau) \cdots (\vec{\sigma} \cdot \vec{a}_{i_{m-1}}^\tau) \right) + \mathcal{O}(\tau^m)
\]

(we actually used the relation \( l_{i_1} = \exp[i \pi \tau \vec{\sigma} \cdot \vec{a}_i] \) coming from eq. (4.6) rather than \( C_{i,1} = 1 + X_i, X_j \) being an indeterminate, which is a standard form of the Magnus expansion, see also [10]). Since the parallel \( \tilde{C}_j \) commutes with the meridian \( C_{j,1} \), we come to the following conclusion:

**Proposition 4.1** If the 1-parametric family of flat connections \( A_\mu(x, \tau) \) defined by eq. (4.6) exists in the link complement, then the vectors \( \vec{a}_j \) satisfy the condition

\[
\left[ \vec{\sigma} \cdot \vec{a}_j^\tau, \sum_{1 \leq i_1, \ldots, i_{m-1} \leq n} l^{(\mu)}_{i_1,\ldots,i_{m-1},j} (\vec{\sigma} \cdot \vec{a}_{i_1}^\tau) \cdots (\vec{\sigma} \cdot \vec{a}_{i_{m-1}}^\tau) \right] = 0, \quad 1 \leq j \leq n.
\]

(4.13)

Let us compare eq. (4.13) with the stationary phase condition for the integral of eq. (1.8). Suppose that \( L_l(\vec{a}_1, \ldots, \vec{a}_n) = 0 \) for \( l < m \). Then eq. (4.9) presents a family of conditional stationary points of the phase (4.4) if

\[
\frac{\partial L_m(\vec{a}_1', \ldots, \vec{a}_n')}{\partial \vec{a}_j'} \times \vec{a}_j' = 0.
\]

(4.14)

Then it follows from the invariance of Milnor’s linking numbers \( l^{(\mu)}_{i_1,\ldots,i_m} \) under a cyclic permutation of the indices that

**Proposition 4.2** The identification (4.11) of the flat connections (4.6) with conditional stationary points (4.9) of the phase (4.4) is consistent with the conjectured expression (3.1) for the polynomials \( L_m(\vec{a}_1, \ldots, \vec{a}_n) \).
This proposition supports the Conjecture 3.1 but it does not help us to fix the coefficient in front the trace in eq. (3.1). This can be done by comparing the dominant part of the stationary phase of eq. (1.8)

\[ \frac{i\pi K}{2} \tau^m L_m(\vec{a}_1', \ldots, \vec{a}_n') \]  

(4.15)

with the Chern-Simons action (4.2) of the flat connection \( A_\mu(x, \tau) \). The action \( S'_{CS} \) has the following property (see, e.g. [9]): if the connections \( A_\mu \) and \( A_\mu + \delta A_\mu \) are both flat, then

\[ S'_{CS}[A_\mu + \delta A_\mu] - S'_{CS}[A_\mu] = -\sum_{j=1}^n \text{Tr} \left[ \left( \oint_{C_1^{(j)}} \delta A_\mu dx^\mu \right) \left( \oint_{C_2^{(j)}} A_\mu dx^\mu \right) \right] \]  

(4.16)

(this is a general property of a classical action: \( \delta \left( \int_{t_1}^{t_2} [p\dot{q} - H(p, q)] dt \right) = p \delta q|_{t_1}^{t_2} \)). On the other hand, eq. (4.12) implies that

\[ \oint_{C_2^{(j)}} A_\mu(x, \tau) dx^\mu = (i\pi \tau)^m \sum_{1 \leq i_1, \ldots, i_{m-1} \leq n} l_{i_1, \ldots, i_{m-1}, j}^{(\mu)} (\vec{\sigma} \cdot \vec{a}_{i_1}') \cdots (\vec{\sigma} \cdot \vec{a}_{i_{m-1}}') + \mathcal{O}(\tau^m). \]  

(4.17)

Combining eqs. (4.16) and (4.17) we conclude that

\[ \frac{dS'_{CS}[A_\mu(x, \tau)]}{d\tau} = -(i\pi)^m \tau^{m-1} \text{Tr} \sum_{1 \leq i_1, \ldots, i_m \leq n} l_{i_1, \ldots, i_m}^{(\mu)} (\vec{\sigma} \cdot \vec{a}_{i_1}') \cdots (\vec{\sigma} \cdot \vec{a}_{i_m}'). \]  

(4.18)

After integrating this equation over \( \tau \) we arrive at eq. (3.1).

5 A Triple Milnor’s Linking Number

Milnor’s invariants \( l_{i_1, \ldots, i_m}^{(\mu)} \) can be expressed as integrals of differential forms constructed with the help of the Massey product (see [13], [14] and references therein, a simple introduction into this subject together with the relevant formulas can be found in [15]). We are going to check the Conjecture 3.1 for the polynomials \( L_3 \) and \( L_4 \) by comparing the Feynman diagram formulas for their coefficients with these expressions.

Preliminaries

We start by introducing some useful notations. Let \( \mathcal{L} \) be an \( n \)-component link in a RHS \( M \). Suppose that we cut out a tubular neighborhood Tub(\( \mathcal{L}_j \)) from the manifold \( M \)
and then glue it back after switching its meridian $C_1^{(j)}$ and its parallel $C_2^{(j)}$. We call such procedure an $S$-surgery on $L_j$. With a slight abuse of notations, we denote as $\text{Tub}'(L_j)$ the tubular neighborhood when it is glued back as a result of $S$-surgery. We denote as $M_{t_1 \ldots t_m \bar{j}_1 \ldots \bar{j}_l}$ the manifold constructed from $M$ by removing the tubular neighborhoods $\text{Tub}(L_{i_1}), \ldots, \text{Tub}(L_{i_m})$ (we will also assume these neighborhoods to be infinitely thin) and performing $S$-surgeries on the link components $L_{j_1}, \ldots, L_{j_l}$.

We denote as $\omega_j$ a closed 1-form defined in $M_j$ by a condition

$$\oint_{C_1^{(j)}} \omega_j = 1$$

(5.1)

This form can be expressed with the help of the Green’s function (2.24):

$$\omega_j(\cdot) = \oint_{L_j} \Omega(t^{(j)}; \cdot),$$

(5.2)

here $t^{(j)}$ is a parametrization of $L_j$ and we slightly abused the notations by using $t^{(j)}$ instead of $x(t^{(j)})$ as the argument of $\Omega$. The linking numbers $l_{ij}$ can be expressed with the help of the forms $\omega_j$:

$$l_{ij} = \oint_{L_i} \omega_j = \oint_{L_j} \omega_i = \oint_{L_i} \oint_{L_j} \Omega(t^{(i)}, t^{(j)}).$$

(5.3)

Another useful property of $\omega_j$ is that if $\omega$ is a smooth 1-form in an infinitely thin tubular neighborhood $\text{Tub}(L_j)$, then

$$\int_{\partial \text{Tub}(L_j)} \omega_j \wedge \omega = \oint_{L_j} \omega.$$

(5.4)

The following object appears naturally in the formulas for Milnor’s invariants. Let $\omega_1, \omega_2$ be two 1-forms defined on a knot $K$ parametrized by $0 \leq t \leq 1$. An “iterated commutator” $[\omega_1, \omega_2]$ is a bilocal $(1,1)$-form

$$[\omega_1, \omega_2](t_1, t_2) = \text{sign}(t_1 - t_2) \omega_1(t_1) \omega_2(t_2).$$

(5.5)

If both forms $\omega_1$ and $\omega_2$ are multilocal, then in our notations

$$\int_K [\omega_1, \omega_2] \equiv \int_{\text{min}(t_1, \ldots, t_m) \geq \text{max}(t_{m+1}, \ldots, t_{m+n})} \prod_{j=1}^{m+n} dt_j \omega_1(t_1, \ldots, t_m) \omega_2(t_{m+1}, \ldots, t_{m+n})$$

$$- \int_{\text{min}(t_{m+1}, \ldots, t_{m+n}) \geq \text{max}(t_1, \ldots, t_m)} \prod_{j=1}^{m+n} dt_j \omega_1(t_1, \ldots, t_m) \omega_2(t_{m+1}, \ldots, t_{m+n}).$$

(5.6)
Obviously, the definition of the iterated commutator (5.5) depends on the choice of the zero-point of $t$ parametrization. If this zero-point is shifted by $\Delta t$, then

$$\lim_{\Delta t \to 0} \frac{\Delta f_K[\omega_1, \omega_2]}{\Delta t} = 2\omega_1(0) \oint_K \omega_2 - 2\omega_2(0) \oint_K \omega_1.$$  

(5.7)

Therefore if both integrals $\oint_K \omega_{1,2}$ are equal to zero, then the integral $f_K[\omega_1, \omega_2]$ is well defined.

The iterated commutator appears in our calculations due to the following

**Proposition 5.1** Let $K$ be a knot in a manifold $M$, $C_{1,2}$ being the meridian and parallel on $\partial \text{Tub}(K)$. Let $M'$ be a manifold constructed by $S$-surgery on $K$. Let $\omega$ be a closed form in $M \setminus \text{Tub}(K)$ satisfying a condition

$$\oint_{C_1} \omega = 1.$$  

(5.8)

Let $\omega_1, \omega_2$ be two closed 1-forms in $M$ and suppose that

$$\oint_{C_2} \omega = \oint_K \omega_1 = \oint_K \omega_2 = 0.$$  

(5.9)

Then the forms $\omega_1, \omega_2$ and $\omega$ can be extended into $M'$. If the tubular neighborhood $\text{Tub}(K)$ is infinitely thin then

$$\int_{\text{Tub}'(K)} \omega_1 \wedge \omega_2 \wedge \omega = \frac{1}{2} \oint_K [\omega_1, \omega_2].$$  

(5.10)

To prove this proposition we introduce the functions $f_1$ and $f'_1$ such that

$$df_1 = \omega_1 \quad \text{inside } \text{Tub}(K),$$  

(5.11)

$$df'_1 = \omega_1 \quad \text{inside } \text{Tub}'(K)$$  

(5.12)

and they coincide on the common boundary:

$$f_1|_{\partial \text{Tub}(K)} = f'_1|_{\partial \text{Tub}'(K)}.$$  

(5.13)

Then

$$\int_{\text{Tub}'(K)} \omega_1 \wedge \omega_2 \wedge \omega = \int_{\text{Tub}'(K)} df'_1 \wedge \omega_2 \wedge \omega = \int_{\partial \text{Tub}(K)} f_1 \omega_2 \wedge \omega = -\oint_K f_1 \omega_2.$$  

(5.14)
and
\[ \int_{K} [\omega_1, \omega_2] = \int_0^1 dt_2 \omega_2(t_2) \left[ \int_{t_2}^1 dt_1 \omega_1(t_1) - \int_0^{t_2} dt_1 \omega_1(t_1) \right] = -2 \oint_{K} f_1 \omega_2. \quad (5.15) \]
This proves the proposition.

**Milnor’s Invariant**

Suppose that the following linking numbers are all equal to zero:
\[ l_{ij} = 0, \quad 1 \leq i, j \leq 3 \quad (5.16) \]
(a condition \( l_{jj} = 0, 1 \leq j \leq 3 \) is not necessary to define the triple Milnor’s invariants but it will simplify our formulas). These conditions allow us to extend the forms \( \omega_1, \omega_2, \omega_3 \) into the manifold \( M_{123} \). The triple Milnor’s linking number of the link components \( \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \) is equal to the intersection number
\[ l^{(\mu)}_{123} = \int_{M_{123}} \omega_1 \wedge \omega_2 \wedge \omega_3. \quad (5.17) \]
The r.h.s. of this formula can be expressed as an integral in \( M \). Indeed, we may split the integral (5.17):
\[ l^{(\mu)}_{123} = \int_{M_{123}} \omega_1 \wedge \omega_2 \wedge \omega_3 + \sum_{j=1}^3 \int_{\text{Tub}(\mathcal{L}_j)} \omega_1 \wedge \omega_2 \wedge \omega_3. \quad (5.18) \]
If the tubular neighborhoods \( \text{Tub}(\mathcal{L}_j) \) are infinitely thin then the first integral in the r.h.s. of eq. (5.18) is equal to
\[ \int_{M_{123}} \omega_1 \wedge \omega_2 \wedge \omega_3 = Y^{(6)}_{123} \equiv \int_{M} \omega_1 \wedge \omega_2 \wedge \omega_3, \quad (5.19) \]
while the integrals in the sum can be transformed with the help of the Proposition [5.1]:
\[ \int_{\text{Tub}(\mathcal{L}_k)} \omega_i \wedge \omega_j \wedge \omega_k = X^{(7)}_{ij,k} \equiv \frac{1}{2} \int_{\mathcal{L}_k} [\omega_i, \omega_j], \quad i \neq j \neq k. \quad (5.20) \]
Therefore
\[ l^{(\mu)}_{123} = Y^{(6)}_{123} + X^{(7)}_{12,3} + X^{(7)}_{31,2} + X^{(7)}_{23,1}. \quad (5.21) \]
Now we will use the Feynman rules derived in Section 2 in order to calculate the cubic coefficient $l_{123}^{(3)}$. The Feynman diagram contributions to $l_{123}^{(3)}$ are depicted in Figs. 6 and 7 up to the permutations. The diagram of Fig. 6 contains three propagators (2.24), one cubic vertex (2.25) and tree vertices (2.27). Its contribution to the exponent of eq. (1.8) is equal to

$$D_{123}^{(6)} = -2i\pi^2 K\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) \int_M d^3y \epsilon^{\nu_1\nu_2\nu_3} \times \oint_{L_1} dx_1^{\mu_1} \Omega_{\mu_1\nu_1}(x_1, y) \oint_{L_2} dx_2^{\mu_2} \Omega_{\mu_2\nu_2}(x_2, y) \oint_{L_3} dx_3^{\mu_3} \Omega_{\mu_3\nu_3}(x_3, y),$$

or, in view of eq. (5.2),

$$D_{123}^{(6)} = -2i\pi^2 K\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) Y_{123}^{(6)}. \quad (5.22)$$

A diagram of Fig. 7 consists of two propagators (2.24), two vertices (2.27) and one vertex (2.28). Its contribution is equal to

$$D_{23,1}^{(7)} = -i\pi^2 K\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) \int_{L_1} \left[ \oint_{L_2} dx_2 \Omega(x_2, \cdot), \oint_{L_3} dx_3 \Omega(x_3, \cdot) \right], \quad (5.23)$$

or, in view of eq. (5.2)

$$D_{23,1}^{(7)} = -i\pi^2 K\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) X_{23,1}^{(7)}. \quad (5.24)$$

A corresponding cubic term in the exponent of eq. (1.8) is

$$3i\pi K l_{123}^{(3)} \vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) \quad (5.25)$$

so that

$$l_{123}^{(3)} = -\frac{2}{3} \pi \left( Y_{123}^{(6)} + X_{123}^{(7)} + X_{31,2}^{(7)} + X_{23,1}^{(7)} \right). \quad (5.26)$$

Comparing this expression with the formula (5.21) for the triple linking number we see that
Proposition 5.2 If the linking numbers \( l_{ii}, l_{jj}, l_{kk}, l_{ij}, l_{jk}, l_{ik} \) are all equal to zero, then the cubic coefficient \( l^{(3)}_{ijk} \) in the exponent of Reshetikhin’s formula (1.8) is proportional to the triple Milnor’s linking number \( l^{(\mu)}_{ijk} \):

\[
l^{(\mu)}_{ijk} = -\frac{3}{2\pi} l^{(3)}_{ijk}. \tag{5.28}
\]

The obvious symmetry

\[
l^{(3)}_{ijk} = -l^{(3)}_{jik} \tag{5.29}
\]

leads to the following relation:

\[
\sum_{1 \leq i,j,k \leq n} l^{(3)}_{ijk} \text{Tr}(\bar{\sigma} \cdot \bar{a}_i)(\bar{\sigma} \cdot \bar{a}_j)(\bar{\sigma} \cdot \bar{a}_k) = 2i \sum_{1 \leq i,j,k \leq n} l^{(3)}_{ijk} (\bar{a}_i \times \bar{a}_j) \cdot \bar{a}_k. \tag{5.30}
\]

After substituting eq. (5.28) into eq. (3.1) and using eq. (5.30) we recover eq. (2.38). This proves the Conjecture 3.1 for \( m = 3 \).

Note that the contribution (5.25) of the Feynman diagram of Fig. 7 is proportional to the integral \( X_{23,1}^{(7)} \) of eq. (5.20) whose value depends on the choice of the zero-point it the parametrization \( t_1 \) of the link component \( L_1 \). However it follows from eq. (5.7) that this ambiguity can be compensated by the change of integration variables (3.2).

6 A Quartic Milnor’s Linking Number

Preliminaries

We begin with establishing another formula for the triple Milnor’s linking number \( l^{(\mu)}_{ijk} \) by splitting the integral (5.17) in a different way:

\[
l^{(\mu)}_{ijk} = \int_{M_{\kappa,ij}} \omega_i \wedge \omega_j \wedge \omega_k + \int_{\text{Tub}(C_{\kappa})} \omega_i \wedge \omega_j \wedge \omega_k. \tag{6.1}
\]

All intersection numbers of the closed 2-form \( \omega_i \wedge \omega_j \) in the manifold \( M_{ij} \) are equal to zero, so it is exact:

\[
\omega_i \wedge \omega_j = d\omega_{ij}. \tag{6.2}
\]
Therefore
\[ \int_{M_{k,ij}} \omega_i \wedge \omega_j \wedge \omega_k = \int_{M_{k,ij}} d\omega_{ij} \wedge \omega_k = - \int_{\partial \text{Tub}(L_k)} \omega_{ij} \wedge \omega_k = \oint_{L_k} \omega_{ij} \] (6.3)

and
\[ f^{(ii)}_{ijk} = \oint_{L_k} \left( \omega_{ij} + \frac{1}{2} [\omega_i, \omega_j] \right). \] (6.4)

If we introduce the functions \( f_{i,j}, f'_{i,j} \) satisfying equations
\[ df_{i,j} = \omega_i \quad \text{inside \ Tub}(L_j), \] (6.5)
\[ df'_{i,j} = \omega_i \quad \text{inside \ Tub'}(L_j), \] (6.6)
\[ f_{i,j}|_{\partial \text{Tub}(L_j)} = f'_{i,j}|_{\partial \text{Tub'}(L_j)}, \] (6.7)

then the integrals of the iterated commutators can be presented in the way similar to eq. (5.15):
\[ \int_{L_k} [\omega_i, \omega_j] = -2 \oint_{L_k} f_{i,k} \omega_j = 2 \oint_{L_k} f_{j,k} \omega_i = \oint_{L_k} (f_{j,k} \omega_i - f_{i,k} \omega_j). \] (6.8)

A 1-form \( \omega_{ij} \) is defined by eq. (6.2) in the manifold \( M_{ij} \) only up to an addition of a closed form (that is either \( \omega_i \) or \( \omega_j \)). One possible candidate for \( \omega_{ij} \) can be obtained with the help of the propagator (1,1)-form \( \Omega(x, y) \):
\[ \omega_{ij}(y) = \int_{M_{ij}} d^3 x \, \Omega(x, y) \wedge \omega_i(x) \wedge \omega_j(x) = \frac{1}{2} \int_{L_i} [\Omega(_, y), \omega_j_] + \frac{1}{2} \int_{L_j} [\Omega(_, y), \omega_i] \] (6.9)

Indeed, it follows from eqs. (2.31), (5.4) and (6.8) that
\[ d\omega_{ij}(y) = \omega_i(y) \wedge \omega_j(y) - \int_{\partial M_{ij}} d^2 x \, \bar{\Omega}(x, y) \omega_i(x) \wedge \omega_j(x) \] (6.10)
\[ -\frac{1}{2} \int_{L_i} [d\bar{\Omega}(\_, y), \omega_j_] - \int_0^1 dt^{(i)} \delta(y - x(t^{(i)})) \int_0^{t^{(i)}} dt^{(i)} \omega_j(x(t^{(i)})) \]
\[ + \frac{1}{2} \int_{L_j} [d\bar{\Omega}(\_, y), \omega_i_] + \int_0^1 dt^{(j)} \delta(y - x(t^{(j)})) \int_0^{t^{(j)}} dt^{(j)} \omega_i(x(t^{(j)})) \]
\[ = \omega_i(y) \wedge \omega_j(y) - \int_0^1 dt^{(i)} \delta(y - x(t^{(i)})) \int_0^{t^{(i)}} dt^{(i)} \omega_j(x(t^{(i)})) \]
\[ + \int_0^1 dt^{(j)} \delta(y - x(t^{(j)})) \int_0^{t^{(j)}} dt^{(j)} \omega_i(x(t^{(j)})) \]
The last two terms of this equation are equal to zero when \( y \in M_{ij} \), however they are useful in deriving the formula for contour integrals of \( \omega_{ij} \) by using Stoke’s theorem:

**Proposition 6.1** Let \( C \) be a contour in \( M_{ij} \) which is a boundary of a 2-dimensional orientable surface \( \Sigma \in M \). Let \( \{P_i\} \) and \( \{P_j\} \) be sets of points where \( \Sigma \) intersects \( \mathcal{L}_i \) and \( \mathcal{L}_j \), sign \( (P_i) \) and sign \( (P_j) \) being the signatures of these intersections. Then for \( \omega_{ij} \) given by eq. (6.9)

\[
\oint_C \omega_{ij} = \int_\Sigma \omega_i \wedge \omega_j - \sum_{P_i} \text{sign} (P_i) \int_0^{t(i)} \omega_j (x(t(i))) \int_0^{t(j)} dt_j^{(j)} \omega_j (x(t(j))) = \int_0^{t(i)} dt_i^{(i)} \omega_i (x(t(i))).
\]

Let us choose an infinitely small meridian \( C_1^{(i)}(t^{(i)}) \) near the point \( t^{(i)} \) as the contour \( C \) and a small disk which intersects \( \mathcal{L}_i \) at the point \( t^{(i)} \) as the surface \( \Sigma \). Then the first integral of eq. (6.11) is infinitely small because the singularity of the form \( \omega_i \) near \( \mathcal{L}_i \) is compensated by the jacobian measure factor in polar coordinates.

**Corollary 6.1** If the meridians \( C_1^{(i)}(t^{(i)}) \) and \( C_1^{(j)}(t^{(j)}) \) are infinitely small, then

\[
\oint_{C_1^{(i)}(t^{(i)})} \omega_{ij} = - \int_0^{t^{(i)}} \omega_j, \quad \oint_{C_1^{(j)}(t^{(j)})} \omega_{ij} = - \int_0^{t^{(j)}} \omega_i.
\]

**Milnor’s Invariant**

Suppose that in addition to the gaussian linking numbers, the triple Milnor’s invariants are also equal to zero:

\[
l_{ij} = l_{ijk}^{(u)} = 0, \quad 1 \leq i, j, k \leq 4.
\]

Then the 2-forms \( \omega_i \wedge \omega_j \) are exact in the manifold \( M_{1234} \) because the intersection numbers (5.17) are equal to zero. Therefore we can introduce the 1-forms \( \omega_{ij} \) of eq. (6.2) in
$M_{1234}$. Their values in $M_{1234} \in M_{1234}$ may be given by eq. (6.9). At this point we may use the Massey product (see e.g. [12], [15] and references therein). The following 2-form

$$\omega_{ij,k} = \omega_{ij} \wedge \omega_{3} - \frac{1}{2} \omega_{ki} \wedge \omega_{j} - \frac{1}{2} \omega_{23} \wedge \omega_{i}$$

(6.14)
is closed and the integral

$$l_{ij,kl}^{(M)} = \int_{M_{1234}} \omega_{ij,k} \wedge \omega_{l}$$

(6.15)
is a link invariant which is related to the quartic Milnor’s linking number:

$$l_{ijkl}^{(\mu)} = \frac{2}{3} (l_{ij,kl}^{(M)} - l_{jk,il}^{(M)}).$$

(6.16)

An equivalent but more symmetric formula for $l_{ij,kl}^{(M)}$ can be obtained by integrating by parts:

$$l_{ij,kl}^{(M)} = \int_{M_{1234}} \omega_{ij,kl},$$

(6.17)

$$\omega_{ij,kl} = \frac{1}{2} \left( \omega_{ij} \wedge \omega_{k} \wedge \omega_{l} + \omega_{ki} \wedge \omega_{l} \wedge \omega_{j} - \frac{1}{2} \left( \omega_{ki} \wedge \omega_{j} \wedge \omega_{l} + \omega_{jl} \wedge \omega_{k} \wedge \omega_{i} \right) - \frac{1}{2} \left( \omega_{jk} \wedge \omega_{i} \wedge \omega_{j} + \omega_{il} \wedge \omega_{j} \wedge \omega_{k} \right) \right).$$

(6.18)

We will work directly with the invariants $l_{ij,kl}^{(M)}$ because they have the obvious symmetries:

$$l_{ij,kl}^{(M)} = -l_{ji,kl}^{(M)} = -l_{ij,kl}^{(M)} = l_{kl,ij}^{(M)}.$$ (6.19)

They also satisfy a Jacobi identity:

$$l_{12,34}^{(M)} + l_{31,24}^{(M)} + l_{23,14}^{(M)} = 0.$$ (6.20)

which indicates that the space of quartic invariants of a 4-component link is 2-dimensional.

Our goal is to express the invariants $l_{ij,kl}^{(M)}$ as integrals in $M$. We will work with a particular invariant $l_{12,34}^{(M)}$ in order to simplify our notations. We split the integral (6.17):

$$\int_{M_{1234}} \omega_{12,34} = \int_{M_{1234}} \omega_{12,34} + \sum_{j=1}^{4} \int_{\text{Tub}(L_{j})} \omega_{12,34}.$$ (6.21)

The integral $\int_{M_{1234}} \omega_{12,34}$ becomes $\int_{M} \omega_{12,34}$ in the limit of infinitely thin tubular neighborhoods $\text{Tub}(L_{j})$. For any 3-form $\omega_{ij} \wedge \omega_{k} \wedge \omega_{l}$ we can use eq. (6.9) in order to obtain the formula

$$\int_{M} \omega_{ij} \wedge \omega_{k} \wedge \omega_{l} = Y_{ij,kl}^{(7)} + X_{i,j,kl}^{(10)} - X_{j,i,kl}^{(10)},$$ (6.22)
here

\begin{align*}
Y_{ij,kl}^{(7)} &= \int_M d^3y \left( \int_M d^3x \omega_i(x) \wedge \omega_j(x) \wedge \Omega(x, y) \right) \wedge \omega_k(y) \wedge \omega_l(y), \\
X_{i,j,kl}^{(10)} &= \frac{1}{2} \int_M d^3y \left( \int_{L_j} [\Omega(\cdot, y), \omega_i(\cdot)] \right) \wedge \omega_k(y) \wedge \omega_l(y).
\end{align*}

As a result,

\begin{align*}
\int_{M_{1234}} \omega_{12,34} &= Z_{12,34}^{(0)} - \frac{1}{2} Z_{31,24}^{(0)} - \frac{1}{2} Z_{23,14}^{(0)},
\end{align*}

here

\begin{align*}
Z_{ij,kl}^{(0)} &= Y_{ij,kl}^{(7)} + \frac{1}{2} \left( X_{i,j,kl}^{(10)} - X_{j,i,kl}^{(10)} + X_{k,l,ij}^{(10)} - X_{l,k,ij}^{(10)} \right).
\end{align*}

Next we turn to the integrals over $\text{Tub}'(L_j)$ in eq. (6.21). Consider, for example, $\int_{\text{Tub}'(L_4)} \omega_{12,34}$. After integrating by parts, we can turn this integral onto a sum

\begin{align*}
\int_{\text{Tub}'(L_4)} \omega_{12,34} &= I_1 + I_2, \\
I_1 &= \int_{\text{Tub}'(L_4)} \left( \omega_{12} \wedge \omega_3 - \frac{1}{2} \omega_{31} \wedge \omega_2 - \frac{1}{2} \omega_{23} \wedge \omega_1 \right) \wedge \omega_4, \\
I_2 &= \frac{1}{2} \int_{\partial \text{Tub}(L_4)} \left( \omega_{12} \wedge \omega_{34} - \frac{1}{2} \omega_{31} \wedge \omega_{24} - \frac{1}{2} \omega_{23} \wedge \omega_{14} \right).
\end{align*}

To calculate $I_1$ we rewrite it as

\begin{align*}
I_1 &= \int_{\text{Tub}'(L_4)} \left[ \left( \omega_{12} - \frac{1}{2} f'_{1,4} \omega_2 + \frac{1}{2} f'_{2,4} \omega_1 \right) \wedge \omega_3 \\
&\quad - \frac{1}{2} (\omega_{31} + f'_{1,4} \omega_3) \wedge \omega_2 - \frac{1}{2} (\omega_{23} - f'_{2,4} \omega_3) \wedge \omega_1 \right] \wedge \omega_4.
\end{align*}

All three 1-forms

\begin{align*}
\omega_{12} - \frac{1}{2} f'_{1,4} \omega_2 + \frac{1}{2} f'_{2,4} \omega_1, \quad \omega_{31} + f'_{1,4} \omega_3, \quad \omega_{23} - f'_{2,4} \omega_3
\end{align*}

are closed. They also satisfy the condition (5.9) of the Proposition 5.1 for $K = L_4$ because of eqs. (6.4), (6.8) and (6.13). Therefore we can apply the Proposition 5.1 to the r.h.s. of eq. (6.30):

\begin{align*}
I_1 &= \frac{1}{2} \int_{L_4} \left[ \left( \omega_{12} + \frac{1}{2} [\omega_1, \omega_2], \omega_3 \right) - \frac{1}{2} [\omega_{31}, \omega_2] - \frac{1}{2} [\omega_{23}, \omega_1] \right].
\end{align*}
At the same time, Corollary 6.1 allows us to express $I_2$ also as a contour integral. Since the form $\omega_{ij}$ is nonsingular in $\text{Tub}(L)$, then in the limit of infinitely thin tubular neighborhood

$$\int_{\partial\text{Tub}(L)} \omega_{ij} \wedge \omega_{kl} = -\int_0^1 dt^l \omega_{lj}(t^l) \oint_{L(t^l)} \omega_{kl} = -\frac{1}{2} \int_{L_i} [\omega_{ij}, \omega_k], \quad (6.33)$$

so that

$$I_2 = -\frac{1}{4} \int_{L_4} \left( [\omega_{12}, \omega_3] - \frac{1}{2} [\omega_{31}, \omega_2] - \frac{1}{2} [\omega_{23}, \omega_1] \right). \quad (6.34)$$

Combining eqs. (6.27), (6.32) and (6.34) we conclude that

$$\int_{\text{Tub}^\prime(L_i)} \omega_{12,34} = \frac{1}{4} \int_{L_4} \left( [\omega_{12}, \omega_3] - \frac{1}{2} [\omega_{31}, \omega_2] - \frac{1}{2} [\omega_{23}, \omega_1] \right) + \frac{1}{4} \int_{L_4} [[\omega_1, \omega_2], \omega_3], \quad (6.35)$$

or after substituting eq. (6.9) for $\omega_{ij}$,

$$\int_{\text{Tub}^\prime(L_i)} \omega_{ij,kl} = Z_{ij,kl}^{(l)} - \frac{1}{2} Z_{ki,jl}^{(l)} - \frac{1}{2} Z_{jk,il}^{(l)} + \frac{3}{2} X_{ij,k,l}^{(9)}, \quad (6.36)$$

here

$$Z_{ij,kl}^{(l)} = \frac{1}{2} \left( X_{k,i,l,j}^{(10)} - X_{j,i,l,k}^{(11)} + X_{i,j,l,k}^{(11)} \right), \quad (6.37)$$

$$X_{i,j,k,l}^{(11)} = \frac{1}{4} \int_{L_k} \left( \int_{L_j} [\Omega(\cdot, \cdot), \omega_l(\cdot)], \omega_l(\cdot) \right), \quad (6.38)$$

$$X_{ij,k,l}^{(9)} = \frac{1}{6} \int_{L_j} [[\omega_i, \omega_j], \omega_k]. \quad (6.39)$$

It remains now to put together eqs. (6.17), (6.21), (6.25) and (6.36):

$$l_{ij,kl}^{(M)} = Z_{ij,kl} - \frac{1}{2} Z_{ki,jl} + \frac{1}{2} Z_{jk,il} + \frac{3}{2} Z_{ij,k,l}^{'}, \quad (6.40)$$

here

$$Z_{ij,k} = Z_{ij,k}^{(0)} + Z_{ij,k}^{(i)} + Z_{ij,k}^{(j)} + Z_{ij,k}^{(k)} + Z_{ij,k}^{(l)} = Y_{ij,k}^{(7)} + X_{i,j,k}^{(10)} - X_{j,i,l,k}^{(10)} + X_{k,i,l,j}^{(10)} - X_{t,k,i,j}^{(10)}$$

$$- X_{j,i,l,k}^{(11)} + X_{i,j,l,k}^{(11)} - X_{j,i,k,l}^{(11)} + X_{i,j,k,l}^{(11)}, \quad (6.41)$$

$$Z_{ij,k,l} = X_{ij,k,l}^{(9)} - X_{ij,l,k}^{(9)} + X_{kl,i,j}^{(9)} - X_{kl,i,j}^{(9)}. \quad (6.42)$$
The Feynman diagrams that contribute to the quartic term

\[(\vec{a}_i \times \vec{a}_j) \cdot (\vec{a}_k \times \vec{a}_l)\]  
(6.43)

are drawn in Figs. 8–11 up to permutations. Their contributions are easy to calculate with the help of the Feynman rules derived in Section 2:

\[
D^{(8)}_{ij,kl} = 4i\pi^3 K(\vec{a}_i \times \vec{a}_j) \cdot (\vec{a}_k \times \vec{a}_l)Y^{(7)}_{ij,kl},
\]
(6.44)

\[
D^{(9)}_{ij,k,l} = 4i\pi^3 K(\vec{a}_i \times \vec{a}_j) \cdot (\vec{a}_k \times \vec{a}_l)X^{(9)}_{ij,k,l},
\]
(6.45)

\[
D^{(10)}_{i,j,kl} = 4i\pi^3 K(\vec{a}_i \times \vec{a}_j) \cdot (\vec{a}_k \times \vec{a}_l)X^{(10)}_{i,j,kl},
\]
(6.46)

\[
D^{(11)}_{i,j,k,l} = -4i\pi^3 K(\vec{a}_i \times \vec{a}_j) \cdot (\vec{a}_k \times \vec{a}_l)X^{(11)}_{i,j,k,l}.
\]
(6.47)

In eq. (6.45) we kept only the part of the contribution of the diagram of Fig. 9 which is proportional to the quartic term (6.43). The sum of all these diagrams with appropriate permutations should be equal to the quartic term in the exponent of Reshetikhin’s formula (1.8):

\[
4i\pi K l^{(4)}_{ij,kl}(\vec{a}_i \times \vec{a}_j) \cdot (\vec{a}_k \times \vec{a}_l).
\]
(6.48)

Therefore comparing eqs. (6.44)–(6.47) with eqs. (6.41) and (6.42) we conclude that

\[
l^{(4)}_{ij,kl} = \pi^2 (Z_{ij,kl} + Z'_{ij,kl}).
\]
(6.49)

The coefficients \(l^{(4)}_{ij,kl}\) as defined by this equation, have the symmetry properties (6.19), however they do not satisfy eq. (6.20). This can be fixed with the help of the Jacobi identity

\[
(\vec{a}_i \times \vec{a}_j) \cdot (\vec{a}_k \times \vec{a}_l) + (\vec{a}_k \times \vec{a}_i) \cdot (\vec{a}_j \times \vec{a}_l) + (\vec{a}_j \times \vec{a}_k) \cdot (\vec{a}_i \times \vec{a}_l) = 0,
\]
(6.50)

which allows us to add the same quantity to the coefficients \(l^{(3)}_{ijk}, l^{(4)}_{ki,jl}\) and \(l^{(4)}_{jk,il}\) (and the ones obtained by permutations (6.19)) without changing the value of the sum (2.39). Choosing this quantity to be

\[
-\frac{\pi^3}{3} (Z_{ij,kl} + Z_{ki,jl} + Z_{jk,il})
\]
(6.51)
we get a new set of coefficients

\[
l^{(3)}_{ijk} = \pi^2 \left[ \frac{2}{3} Z_{ij,kl} - \frac{1}{3} Z_{ki,jl} - \frac{1}{3} Z_{jk,il} + Z'_{ij,kl} \right],
\]

which satisfy the analog of the Jacobi identity (6.20)

\[
l^{(4)}_{ij,kl} + l^{(4)}_{li,jk} + l^{(4)}_{jl,ik} = 0 \tag{6.53}
\]

in addition to the symmetries

\[
l^{(4)}_{ij,kl} = -l^{(4)}_{ji,kl} = -l^{(4)}_{ij,lk} = l^{(4)}_{kl,ij}. \tag{6.54}
\]

Finally, comparing eqs. (6.52) and eq. (6.40) we arrive at the following

**Proposition 6.2** If all the gaussian linking numbers and triple Milnor’s invariants involving the indices \(i, j, k, l\) are equal to zero, then the 4th order coefficients \(l^{(4)}_{ij,kl}\) of the exponent of Reshetikhin’s formula (1.8) normalized by the symmetries (6.54) and Jacobi identities (6.53) are related to the quartic Milnor’s linking numbers \(l^{(\mu)}_{ij,kl}\):

\[
l^{(\mu)}_{d,kl} = \frac{1}{\pi^2} (l^{(4)}_{ij,kl} - l^{(4)}_{jk,il}) \tag{6.55}
\]

The symmetries (6.54) lead to the equation

\[
\sum_{1 \leq j_1, \ldots, j_4 \leq n} l^{(4)}_{j_1j_2j_3j_4} \text{Tr}(\bar{\sigma} \cdot \vec{a}_{j_1}) \cdots (\bar{\sigma} \cdot \vec{a}_{j_4}) = -2 \sum_{1 \leq j_1, \ldots, j_4 \leq n} l^{(4)}_{j_1j_2j_3j_4} (\vec{a}_{j_1} \times \vec{a}_{j_2}) \cdot (\vec{a}_{j_3} \times \vec{a}_{j_4}) \tag{6.56}
\]

Substituting it together with eq. (6.55) into the r.h.s. of eq. (3.1) we obtain eq. (2.39). Thus we verified the Conjecture \(3.1\) for \(m = 4\).

### 7 Discussion

Reshetikhin’s formula (1.8) seems to be an excellent tool for the study of links. It separates the exponent, which is of order \(K^1\), and the preexponential factor, which is at most of order \(K^0\), for future use in stationary phase approximation. It also incorporates naturally the
Feynman diagrams of links and seems to relate them to Milnor’s invariants. This result may not be so surprising in view of the fact that, as it was demonstrated in [10] and [11], Milnor’s linking numbers are Vassiliev’s invariants of the link. The ambiguities of link Feynman diagrams also seem to match the ambiguities of Milnor’s linking numbers. Both of these ambiguities appear to be related to the freedom of changing the integration variables in Reshetikhin’s integral (see eqs. (3.2) and (3.3)). It is remarkable that Feynman diagrams of the Chern-Simons theory are somehow connected to the Massey product.

The set of Feynman rules that we derived in Section 2 is not unique because of Lie algebra Jacobi identities. In fact, it could possibly be improved in order to incorporate some natural combinatorial symmetries. This may facilitate a proof of Conjecture 3.1 about a relation between the exponent of Reshetikhin’s formula and Milnor’s linking numbers.

The exponent of Reshetikhin’s formula might deserve further study. Its stationary phase points are in one-to-one correspondence with the flat connections in the link complement and the determinant of the second derivatives of the phase is related to their Reidemeister-Ray-Singer torsion (it is proportional to the dominant part of the torsion in the limit of small phases of the holonomies along the meridians of link components). We use this fact in [3] in order to calculate the multivariable Alexander polynomial of the link in the way that generalizes the recent Melvin-Morton conjecture [4].

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