Rotating Black Hole Solutions with Axion Dilaton and Two Vector Fields and Solutions with Metric and Fields of the Same Form

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Abstract

We present two rotating black hole solutions with axion $\xi$, dilaton $\phi$ and two $U(1)$ vector fields. Starting from a non-rotating metric with three arbitrary parameters, which we have found previously, and applying the "Newman-Janis complex coordinate trick" we get a rotating metric $g_{\mu\nu}$ with four arbitrary parameters namely the mass $M$, the rotation parameter $a$ and the charges electric $Q_E$ and magnetic $Q_M$. Then we find a solution of the equations of motion having this $g_{\mu\nu}$ as metric. Our solution is asymptotically flat and has angular momentum $J = Ma$, gyromagnetic ratio $g = 2$, two horizons, the singularities of the solution of Kerr, axion and dilaton singular only when $r = a \cos \theta = 0$ etc. By applying to our solution the $S$-duality transformation we get a new solution, whose axion, dilaton and vector fields have one more parameter. The metrics, the vector fields and the quantity $\lambda = \xi + ie^{-2\phi}$ of our solutions and the solution of : Sen for $Q_E$, Sen for $Q_E$ and $Q_M$, Kerr-Newman for $Q_E$ and $Q_M$, Kerr, Ref. [7], Shapere, Trivedi and Wilczek (STW), Gibbons-Maeda-Garfinkle-Horowitz-Strominger (GM-GHS), Reissner-Nordström, Schwarzschild are the same function of $a$, and two functions $\rho^2 = r(r + b) + a^2 \cos^2 \theta$ and $\Delta = \rho^2 - 2Mr + c$, of $a$, $b$ and two functions for each vector field, and of $a$, $b$ and $d$ respectively, where $a$, $b$, $c$ and $d$ are constants. From our solutions several known solutions can be obtained for certain values of their parameters. It is shown that our two solutions satisfy the weak the dominant and the strong energy conditions outside and on the outer horizon and that all solutions with a metric of our form, whose parameters satisfy some relations satisfy also these energy conditions outside and on the outer horizon. This happens to all solutions given in the Appendix. Mass formulae for our solutions

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and for all solutions which are mentioned in the paper are given. One mass formula for each solution is of Smarr’s type and another a differential mass formula. Many solutions with metric, vector fields and λ of the same functional form, which include most physically interested and well known solutions, are listed in an Appendix.

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1 Introduction

It is well known that the solution of Kerr [1] can be ”derived” from the solution of Schwarzshild by means of the ”Newman and Janis complex coordinate trick” [2]. Later the Kerr-Newman metric with electric charge only was derived from the Reissner-Nordström metric by the same trick and subsequently a vector field was found, which together with the metric gives the Kerr-Newnum solution of the Einstein-Maxwell equations [3]. Recently it was shown that the metric of the axion-dilaton black hole solution of Sen with electric charge only [4] can be derived from the metric of the Gibbons-Maeda-Garfinkle-Horowitz-Strominger (GM-GHS) solution [5] by the Newman and Janis method [6]. The GM-GHS metric depends on two arbitrary parameters, while the metric of Sen on three such parameters. The solution of Sen satisfies the field equations of the Einstein-Maxwell-axion-dilaton gravity in four dimensions, which are obtained from the action

\[ S = \int d^4x \sqrt{-g} \left\{ R - 2\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \epsilon^{\mu \nu \xi \eta} \partial_\mu \xi \partial_\nu \eta - e^{-2\phi} F_{\mu \nu} F^{\mu \nu} - \xi F_{\mu \nu} F^{\mu \nu} \right\} \quad (1) \]

where \( R \) is the Ricci scalar, \( \xi \), \( \phi \) and \( F_{\mu \nu} \) are the axion the dilaton and a \( U(1) \) vector field respectively.

Non-rotating black hole solutions were found recently, whose metric (for \( c = 0 \), where \( c \) is a parameter appearing in the solutions) depends on three arbitrary parameters [7]-[9]. One may ask if the rotating metric with four parameters we get if we apply to this metric the Newman-Janis trick can be the metric of a rotating black hole solution of the equations of motion coming from the action (1). We tried to do that but we failed. Subsequently we proved that this can be done if we add to the Lagrangian of the action (1) a second \( U(1) \) vector field. Actions of the above type with more than one vector fields are frequently used, and it is argued that ”the presence of only one vector field is insufficient to generate all interesting metrics” [10].
In Section 2 we shall apply to our metric the Newman-Janis method and we shall derive a rotating black hole metric with four parameters, which depends on the rotation parameter $a$ and two functions $\Delta$ and $\rho^2$. This metric and the metrics of all solutions, which are mentioned in this paper are the same function of $a$, $\Delta$ and $\rho^2$.

In Section 3 starting from an action of the type (1) but with two vector fields $F_{\mu\nu}$ and $F'_{\mu\nu}$ we obtain the equations of motion. Also we take $F_{r\theta} = F'_{r\theta} = F'_{t\phi} = 0$ and we express the rest four non-vanishing components of the field $F_{\mu\nu}$ in terms of two functions $\zeta$ and $\eta$ and the four non-vanishing components of the field $F'_{\mu\nu}$ in terms of two functions $\zeta'$ and $\eta'$ by the same ansatz. The vector fields of all solutions which are mentioned in this paper can be expressed in terms of two functions by the same ansatz.

In Section 4 we calculate the components of the Ricci tensor $R_{\mu\nu}$ using the metric we have found and we show that the metric equations can be reduced to three relatively simple equations.

In Section 5 we show that the equations of motion of the vector fields can be solved exactly and we express the functions $\zeta$ and $\eta$ of the first vector field in terms of two arbitrary functions of their arguments $G(r)$ and $H(y)$ and the functions $\zeta'$ and $\eta'$ of the second vector field again in terms of two arbitrary functions of their arguments $G'(r)$ and $H'(y)$.

The solution of the complex axion-dilaton equation of motion is the most difficult part of the work. This is done in Section 6, in which $\xi$, $e^{-2\phi}$, $\zeta$, $\eta$, $\zeta'$ and $\eta'$ are determined.

In Section 7 some physical characteristics of the solution are determined. The mass, the angular momentum, the vector potentials, the magnetic dipole moment and the gyromagnetic ratio of the solution are calculated. The two infinite red shift surfaces are obtained and it is shown that the solution has two event horizons and a ring singularity, as the solution of Kerr [1], [11]. Also the area of the outer horizon, the surface gravity, which is proportional to the Hawking temperature, and the angular velocity are determined.

In Section 8 it is shown that for certain values of the four arbitrary parameters of our solution a number of known solutions or metrics of known solutions are obtained as special cases. These are the solution of Sen for electric charge only [4], the metric of the Kerr-Newman solution for equal electric and magnetic charges [12], the solution of Kerr [1], a metric we have found previously [7], [9], the GM-GHS solution [5], the metric of the Reissner-Nordström solution for equal electric and magnetic charges and the Schwarzschild solution.
In Section 9 by applying to the solution we have found the S-duality transformation we obtain a “new solution”, which has the same metric \( g_{\mu\nu} \) but its axion, dilaton and vector fields contain one more arbitrary parameter. Also it is pointed out that the quantity \( \lambda = \xi + ie^{-2\phi} \) of both solutions we have found takes the form of Eq (175); and it is expressed in terms of two real parameters \( a \) (it is \( y = a\cos\theta \)) and \( b \) and one real or complex parameter \( d \). A relation of the same form holds for all solutions with \( \phi \neq 0 \), which are mentioned in this paper.

In Section 10 we show that the solution of Sen with both charges electric and magnetic \[13\] and the solution of Shapere, Trivedi and Wilczek (STW) \[14\] are special cases of the “new solution”.

In Section 11 we show that our two solutions and also all solutions, whose metric is of the form considered in this paper and its parameters satisfy some relations, satisfy the weak the dominant and the strong energy conditions outside and on the outer horizon. All solutions given in the Appendix of the paper satisfy all energy conditions outside and on the outer horizon.

In section 12 we calculate a mass formula for the solutions with metric of the form of Eqs (19), (25) and (26) and angular momentum \( J = Ma \), where \( a \) is a non-zero or zero constant. The formula is homogeneous in its arguments. Applying to this expression Euler’s theorem on homogeneous functions we find a mass formula of Smarr’s type \[15\]. Also taking its differential we find a differential mass formula. The mass formulae of our two solutions and all other solutions of the Appendix with metric of this type are given. Also for solutions with metric of the form of Eqs (19) and (27) and angular momentum \( J = Ma \) a mass formula is obtained, which in the case of the Kerr-Newman solution is homogeneous in its arguments. From this expression mass formulae for the Kerr-Newman solution with arbitrary \( Q_E \) and \( Q_M \) are calculated, whose special cases are the mass formulae of the solution of Kerr and the mass formulae of the solution of Reissner-Nordström.

In the Appendix we make a list of physically interesting solutions, whose metric and fields are of the general form considered in the paper. More specifically for all solutions of the Appendix the metric \( (g_{\mu\nu}) \) is the same function of the rotation parameter \( a \) and two functions \( \rho^2 = r(r+b)+a^2\cos^2\theta \) and \( \Delta = \rho^2 - 2Mr + c \) with \( M, b \) and \( c \) constants, the vector field (or each of their vector fields) is expressed in the same way in terms of \( a, b \) and two functions and the quantity \( \lambda = \xi + e^{-2\phi} \) is the same function of \( a, b \) and \( d \), where \( d \) is a real or complex constant.
2 The Metric

In a model with the action

\[ S = \int d^4x \sqrt{-g} \{ R - 2 \partial_\mu \phi \partial^\mu \phi - e^{-2\phi} F_{\mu \nu} F^{\mu \nu} \} \] (2)

where \( R \) is the Ricci scalar and \( \phi \) and \( F_{\mu \nu} \) are the dilaton and a \( U(1) \) vector field respectively, we found the solution (Eqs (54)-(57) of Ref. \[9\] for \( \psi = 2\phi, \alpha = b \) and \( \psi_0 = 0 \))

\[ ds^2 = -\frac{(r + A)(r + B)}{r(r + b)} dt^2 + \frac{r(r + b)}{(r + A)(r + B)} dr^2 + r(r + b)(d\theta^2 + \sin^2 \theta d\phi^2) \] (3)

\[ e^{2\phi} = 1 + \frac{b}{r}, \quad F = \frac{\sqrt{AB}}{2r^2} dr \wedge dt + \frac{\sqrt{(b - A)(b - B)}}{\sqrt{2}} \sin \theta d\theta \wedge d\phi \] (4)

and in a model with the action

\[ S = \int d^4x \sqrt{-g} \{ R - 2 \partial_\mu \phi \partial^\mu \phi - (g_1 e^{2\phi} + g_2 e^{-2\phi}) \} \] (5)

we found a solution (Eqs (2) and (12)-(14) of Ref \[7\] for \( \psi = 2\phi, \alpha = b \) and \( \psi_0 = 0 \)) with the same \( ds^2 \) and

\[ e^{-2\phi} = 1 + \frac{b}{r}, \quad F = Q dr \wedge dt, \quad g_1 = \frac{AB}{2Q^2}, \quad g_2 = \frac{(b - A)(b - B)}{2Q^2} \] (6)

where \( A, B \) and \( b \) are arbitrary real constants. We shall apply the method of Newman and Janis \[2, 11\] to the metric of expression (3).

Following this method we replace \( dt \) in (3) by \( du \), where

\[ dt = du + \frac{r(r + b)}{(r + A)(r + B)} dr \] (7)

Then Eq. (3) becomes

\[ ds^2 = - \left( 1 + \frac{A}{r} \right) \left( 1 + \frac{B}{r} \right) du^2 - 2 du dr + r^2 (1 + \frac{b}{r}) (d\theta^2 + \sin^2 \theta d\phi^2) \] (8)

If \( g^{\mu \nu} \) is the inverse of the metric of the line element (8) and introduce the vectors

\[ l^\mu = \delta_1^\mu, \quad n^\mu = \delta_0^\mu - \frac{(1 + \frac{A}{r})(1 + \frac{B}{r})}{2(1 + \frac{b}{r})} \delta_1^\mu, \quad m^\mu = \frac{1}{r \sqrt{2(1 + \frac{b}{r})}} \left( \delta_2^\mu + \frac{i}{\sin \theta} \delta_3^\mu \right) \] (9)
we get
\[ g'^{\mu \nu} = l'^{\mu} n'^{\nu} - l'^{\nu} n'^{\mu} + m'^{\mu} \tilde{m}'^{\nu} + m'^{\nu} \tilde{m}'^{\mu} \] (10)

We allow now \( r \) to become complex and keeping the same symbols for the vectors we write
\[ l^{\mu} = \delta^{\mu}_{1}, \quad n^{\mu} = \delta^{\mu}_{0} - \frac{1 + \frac{A'B}{2} \left( \frac{1}{r} + \frac{1}{r'} \right) + \frac{AB}{r}}{2(1 + \frac{b}{2} \left( \frac{1}{r} + \frac{1}{r'} \right))} \delta^{\mu}_{1}, \quad m^{\mu} = \frac{1}{r \sqrt{2(1 + \frac{b}{2} \left( \frac{1}{r} + \frac{1}{r'} \right))}} \left( \delta^{\mu}_{2} \right) \]
\[ + \left( \frac{i}{\sin \theta} \delta^{\mu}_{3} \right) \] (11)

and subsequently we apply to them the complex coordinate transformation
\[ r \rightarrow r' = r + ia \cos \theta, \quad u \rightarrow u' = u - ia \cos \theta \] (12)

This transformation gives the \( l'^{\mu}, n'^{\mu} \) and \( m'^{\mu} \) vectors
\[ l'^{\mu} = \delta^{\mu}_{1}, \quad n'^{\mu} = \delta^{\mu}_{0} - \frac{\Delta'_{1}}{2 \rho'^{2}} \delta^{\mu}_{1}, \quad m'^{\mu} = \frac{1}{\sqrt{2 \rho'^{2}}} \left\{ ia \sin \theta (\delta^{\mu}_{0} - \delta^{\mu}_{1}) + \delta^{\mu}_{2} + \frac{i}{\sin \theta} \delta^{\mu}_{3} \right\} \] (13)

where
\[ \rho'^{2} = r'(r' + b) + a^{2} \cos^{2} \theta, \quad \Delta' = (r' + A)(r' + B) + a^{2} \cos^{2} \theta \] (14)

For the metric \( g'^{\mu \nu} \) defined by
\[ g'^{\mu \nu} = -l'^{\mu} n'^{\nu} - l'^{\nu} n'^{\mu} + m'^{\mu} \tilde{m}'^{\nu} + m'^{\nu} \tilde{m}'^{\mu} \] (15)
we get
\[ g'^{\mu \nu} = \frac{1}{\rho'^{2}} \left( \begin{array}{cccc} a^{2} \sin^{2} \theta & -\rho'^{2} - a^{2} \sin^{2} \theta & 0 & a \\ -\rho'^{2} - a^{2} \sin^{2} \theta & \Delta' + a^{2} \sin^{2} \theta & 0 & -a \\ 0 & 0 & 1 & 0 \\ a & -a & 0 & \frac{1}{\sin^{2} \theta} \end{array} \right) \] (16)

from which we obtain its inverse
\[ g'^{\mu \nu} = \left( \begin{array}{cccc} -\frac{\Delta'}{\rho'^{2}} & -1 & 0 & \frac{a \sin^{2} \theta (\Delta' - \rho'^{2})}{\rho'^{2} \sin^{2} \theta} \\ -1 & 0 & 0 & \frac{a \sin^{2} \theta}{\rho'^{2}} \\ 0 & 0 & \rho'^{2} & 0 \\ \frac{a \sin^{2} \theta (\Delta' - \rho'^{2})}{\rho'^{2} \sin^{2} \theta} & a \sin^{2} \theta & 0 & \sin^{2} \theta (\rho'^{2} + 2a^{2} \sin^{2} \theta) - \frac{a^{2} \Delta' \sin^{4} \theta}{\rho'^{2}} \end{array} \right) \] (17)
If in the $ds^2$ coming from the above $g'_{\mu\nu}$ we make the transformation

$$

du' = dt' - \rho^2 + a^2 \sin^2 \theta \frac{dr'}{\Delta' + a^2 \sin^2 \theta}, \quad d\phi' = a \frac{dr'}{\Delta' + a^2 \sin^2 \theta}
$$

(18)

the only non-diagonal term which remains is $dt' d\phi'$ and we get if we drop the primes

$$

g_{\mu\nu} = \begin{pmatrix}
-\frac{\Delta}{\rho^2} & 0 & 0 & \frac{a(\Delta-\rho^2) \sin^2 \theta}{\rho^2} \\
0 & \frac{\rho^2}{\Delta + a^2 \sin^2 \theta} & 0 & 0 \\
0 & 0 & \rho^2 & 0 \\
\frac{a(\Delta-\rho^2) \sin^2 \theta}{\rho^2} & 0 & 0 & \frac{\sin^2 \theta(\rho^2 + 2a^2 \sin^2 \theta) - \frac{a^2 \Delta \sin^2 \theta}{\rho^2}}
\end{pmatrix}
$$

(19)

where

$$

\rho^2 = r(r + b) + a^2 \cos^2 \theta, \quad \Delta = (r + A)(r + B) + a^2 \cos^2 \theta
$$

(20)

The $g_{\mu\nu}$ of expression (19) gives

$$

|g_{\mu\nu}| = -\rho^4 \sin^2 \theta
$$

(21)

$$

ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\frac{\Delta}{\rho^2} dt^2 + \frac{\rho^2}{\Delta + a^2 \sin^2 \theta} dr^2 + \rho^2 d\theta^2 + \frac{2a(\Delta-\rho^2) \sin^2 \theta}{\rho^2} dt d\phi
$$

$$

+ \{\sin^2 \theta(\rho^2 + 2a^2 \sin^2 \theta) - \frac{a^2 \Delta \sin^2 \theta}{\rho^2}\} d\phi^2
$$

(22)

Also from the $g_{\mu\nu}$ of expression (19) we get for any $\Delta$ and $\rho^2$

$$

g^{\mu\nu} = \begin{pmatrix}
\frac{\rho^2 + 2a^2 \sin^2 \theta - a^2 \Delta \sin^2 \theta}{\Delta + a^2 \sin^2 \theta} & 0 & 0 & \frac{a(\Delta-\rho^2)}{\rho^2(\Delta + a^2 \sin^2 \theta)} \\
0 & \frac{\Delta + a^2 \sin^2 \theta}{\rho^2} & 0 & 0 \\
0 & 0 & \frac{1}{\rho^2} & 0 \\
\frac{a(\Delta-\rho^2)}{\rho^2(\Delta + a^2 \sin^2 \theta) \sin^2 \theta} & 0 & 0 & \frac{\Delta}{\rho^2(\Delta + a^2 \sin^2 \theta) \sin^2 \theta}
\end{pmatrix}
$$

(23)

If we introduce the notation

$$

(b - A)(b - B) = 2Q_E^2, \quad AB = 2Q_M^2, \quad b - A - B = 2M
$$

(24)

we get [16]

$$

b = \frac{Q_E^2 - Q_M^2}{M}, \quad \rho^2 = r(r + \frac{Q_E^2 - Q_M^2}{M}) + a^2 \cos^2 \theta
$$

(25)
\[ \Delta = r(r + \frac{Q_E^2 - Q_M^2}{M}) - 2Mr + 2Q_M^2 + a^2 \cos^2 \theta = \rho^2 - 2Mr + 2Q_M^2 \quad (26) \]

We shall find two solutions whose metric is given by Eqs (19), (25) and (26). This metric has four arbitrary parameters namely \( M, a, Q_E \) and \( Q_M \).

Generally all solutions we shall consider in this paper have a metric \( g_{\mu \nu} \) of the form of Eq (19). This metric is a function of the rotation parameter \( a \) and of \( \rho^2 \) and \( \Delta \), which are given by the relations

\[ \rho^2 = r(r + b) + a^2 \cos^2 \theta, \quad \Delta = r(r + b) - 2Mr + c + a^2 \cos^2 \theta \quad (27) \]

where \( M, b \) and \( c \) are constants. It is obvious that a metric of the form of Eqs (19), (25) and (26) is also of the form of Eqs (19) and (27). However there are solutions with \( \rho^2 \) and \( \Delta \) of the form of Eq (27) but not of the form of Eqs (25) and (26), that is with \( b \neq \frac{Q_E^2 - Q_M^2}{M} \) and or \( c \neq 2Q_M^2 \). Such are the Kerr-Newman solution and the Reissner-Nordström solution for electric charge \( Q_E \) and magnetic charge \( Q_M \) with \( Q_E \neq Q_M \). In these two solutions we have \( b = 0 \) and \( c = Q_E^2 + Q_M^2 \).

### 3 Model with two Vector Fields

Consider a model which has besides gravity an axion field \( \xi \), a dilaton field \( \phi \), two vector fields \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( F'_{\mu \nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu \) and action

\[ S = \int d^4x \sqrt{-g} \left\{ R - 2\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} e^{4\phi} \partial_\mu \xi \partial^\mu \xi - e^{-2\phi} (F_{\mu \nu} F^{\mu \nu} + F'_{\mu \nu} F'^{\mu \nu}) \right\} \]

\[ - \xi (F_{\mu \nu} F^{\mu \nu} + F'_{\mu \nu} F'^{\mu \nu}) \} \quad (28) \]

where

\[ F^{\ast \mu \nu} = \frac{1}{2\sqrt{-g}} \epsilon^{\mu \nu \sigma \tau} F_{\sigma \tau}, \quad F'^{\ast \mu \nu} = \frac{1}{2\sqrt{-g}} \epsilon^{\mu \nu \sigma \tau} F'_{\sigma \tau}, \quad \epsilon^{0123} = +1 \quad (29) \]

Also we define \( \lambda, F_{\pm \mu \nu} \) and \( F'^{\pm \mu \nu} \) by

\[ \lambda = \xi + ie^{-2\phi}, \quad F_{\pm \mu \nu} = F_{\mu \nu} \pm i F^{\ast \mu \nu}, \quad F'^{\pm \mu \nu} = F'_{\mu \nu} \pm i F'^{\ast \mu \nu} \quad \text{(30)} \]

Then the action can be written in the form

\[ S = \int d^4x \sqrt{-g} \left\{ R - \frac{1}{2} e^{4\phi} \partial_\mu \lambda \partial^\mu \lambda + \frac{i}{4} (\lambda F_+^2 - \bar{\lambda} F_-^2) + \frac{i}{4} (\lambda F'_+^2 - \bar{\lambda} F'_-^2) \right\} \quad (31) \]

8
and we get the equations of motion

\[ R_{\mu \nu} = \frac{1}{4} e^{4\phi} (\partial_{\mu} \bar{\lambda} \partial_{\nu} \lambda + \partial_{\nu} \bar{\lambda} \partial_{\mu} \lambda) + 2 e^{-2\phi} (F_{\mu \rho} F_{\nu}^{\rho} - \frac{g_{\mu \nu}}{4} F^{2} + 2 e^{-2\phi} (F_{\mu \rho} F_{\nu}^{\rho} - \frac{g_{\mu \nu}}{4} F^{2}) \]

\[ \nabla_{\mu} (\lambda F_{\mu \nu}^{\mu} - \bar{\lambda} F_{\mu \nu}^{\nu}) = 0, \quad \nabla_{\mu} (\lambda F_{\mu \nu}^{\nu} - \bar{\lambda} F_{\mu \nu}^{\mu}) = 0 \]

\[ e^{4\phi} \nabla_{\mu} \bar{\lambda} \partial^{\mu} \lambda + ie^{6\phi} \partial_{\mu} \lambda \partial^{\mu} \lambda - \frac{i}{2} (F_{\nu}^{2} + F_{\nu}^{2}) = 0 \]

(32)

To find a solution of the above equations with the metric of Eqs (19), (25) and (26) we shall make the following ansatz for the fields

\[ F_{\mu \nu} = \zeta, \quad F_{\mu \nu} = -a \sin^{2} \theta \zeta, \quad F_{\theta \phi} = \{r(r+b) + a^{2}\} \sin \theta \eta \]

\[ F_{\phi \theta} = F_{\phi \theta} = 0 \]

(35)

\[ F'_{\mu \nu} = \zeta', \quad F'_{\mu \nu} = -a \sin^{2} \theta \zeta', \quad F'_{\theta \phi} = -a \sin \theta \eta', \quad F'_{\phi \theta} = \{r(r+b) + a^{2}\} \sin \theta \eta' \]

\[ F'_{\phi \theta} = F'_{\phi \theta} = 0 \]

(36)

The previous expressions give for the metric of Eqs (19), (25) and (26)

\[ F^{2} = 2(\eta^{2} - \zeta^{2}), \quad FF^{*} = -4 \zeta \eta, \quad F'^{2} = 2(\eta'^{2} - \zeta'^{2}), \quad F'^{*}F^{*} = -4 \zeta' \eta' \]

(37)

Also we have \( F'^{*} = -F^{2} \) and \( F'^{*} = -F'^{2} \) Generally the vector fields \( F_{\mu \nu} \) of all solutions we shall consider in this paper are of the above form, which means that each \( F_{\mu \nu} \) is expressed in terms of the rotation parameter \( a \), a constant \( b \) and two functions.

4 Reduction of the Metric Equations

For the metric of Eqs (19), (25) and (26) we have the following non zero components of the Ricci tensor 17

\[ R_{tt} = \frac{(\Delta + 2a^{2} \sin^{2} \theta) \kappa}{\rho^{8}}, \quad R_{rr} = \frac{\rho^{2}(b^{2} - 2(Q_{E}^{2} + Q_{M}^{2})) + a^{2}b^{2} \sin^{2} \theta}{2(\Delta + a^{2} \sin^{2} \theta) \rho^{4}} \]

(38)

\[ R_{\theta \theta} = \frac{2K + a^{2}b^{2} \sin^{2} \theta}{2\rho^{4}}, \quad R_{\phi \phi} = -\frac{a \sin^{2} \theta(\Delta + \rho^{2} + a^{2} \sin^{2} \theta) \kappa}{\rho^{8}} \]

(39)

\[ R_{\phi \theta} = \frac{\sin^{2} \theta \{(\rho^{2} + a^{2} \sin^{2} \theta)^{2} + (\Delta + a^{2} \sin^{2} \theta) a^{2} \sin^{2} \theta\} \kappa}{\rho^{8}} \]

(40)
where
\[ K = Q_E (r^2 + a^2 \cos^2 \theta) + Q_M \{(r + b)^2 + a^2 \cos^2 \theta\} \quad (41) \]

Also if the components of \( F_{\mu\nu} \) are given by Eqs (35) we get
\[ F_{t\mu} F^\mu_t - g_{tt} F^2 = \frac{(\Delta + 2a^2 \sin^2 \theta)(\zeta^2 + \eta^2)}{2} \quad (42) \]
\[ F_{r\mu} F^\mu_r - g_{rr} F^2 = -\frac{\rho^2(\zeta^2 + \eta^2)}{2(\Delta + a^2 \sin^2 \theta)} \quad (43) \]
\[ F_{t\mu} F^\mu_o - g_{oo} F^2 = -\frac{a \sin^2 \theta(\Delta + \rho^2 + 2a^2 \sin^2 \theta)(\zeta^2 + \eta^2)}{2\rho^2} \quad (44) \]
\[ F_{\phi\mu} F^\mu_{\phi} - g_{\phi\phi} F^2 = \frac{\sin^2 \theta(a^2 \Delta \sin^2 \theta + (\rho^2 + a^2 \sin^2 \theta)^2 + a^4 \sin^4 \theta)(\zeta^2 + \eta^2)}{2\rho^2} \quad (45) \]
\[ F_{\mu\sigma} F^\sigma_{\nu} - g_{\mu\nu} F^2 = 0 \quad \text{for} \ (\mu, \nu) = (r, t), (t, r), (t, \theta), (\phi, r), (\phi, \theta) \quad (46) \]

The expressions \( F'_{\mu\sigma} F'^{\sigma}_{\nu} - g_{\mu\nu} F'^2 \) are obtained from the above expressions if we replace \( \zeta^2 + \eta^2 \) by \( \zeta'^2 + \eta'^2 \). Also in the paper we shall assume that \( \lambda, F_{\mu\nu} \) and \( F'_{\mu\nu} \) do not depend on \( t \) and \( \phi \) and we shall define \( y \) by the relation
\[ y = a \cos \theta \quad (47) \]

Then Eq. (32) gives
\[ \zeta^2 + \eta^2 + \zeta'^2 + \eta'^2 = \frac{Ke^{2\phi}}{\rho^6} \quad \text{for} \ (\mu, \nu) = (t, t), (\phi, \phi), (t, \phi) \quad (48) \]
\[ \partial_r \lambda \partial_r \bar{\lambda} = \frac{b^2 e^{-4\phi}}{\rho^4} \quad \text{for} \ (\mu, \nu) = (r, r) \quad (49) \]
\[ \partial_y \lambda \partial_y \bar{\lambda} = \frac{b^2 e^{-4\phi}}{\rho^4} \quad \text{for} \ (\mu, \nu) = (\theta, \theta) \quad (50) \]
\[ \partial_r \lambda \partial_y \bar{\lambda} + \partial_y \lambda \partial_r \bar{\lambda} = 0 \quad \text{for} \ (\mu, \nu) = (r, \theta) \quad (51) \]

while it is identically satisfied for \( (\mu, \nu) = (t, r), (t, \theta), (\phi, r) \) and \( (\phi, \theta) \). To derive Eqs (49) and (50) we used Eq. (48). Multiplying Eqs (49) and (50) by parts and using Eq. (51) we get
\[ \partial_r \lambda \partial_r \bar{\lambda} \partial_y \lambda \partial_y \bar{\lambda} = \frac{b^4 e^{-8\phi}}{\rho^8} = -(\partial_y \lambda)^2 (\partial_r \bar{\lambda})^2 \quad (52) \]
From Eqs (49) and (52) we get
\[ \partial_y \lambda \partial_r \bar{\lambda} = \pm i \frac{b^2 e^{-4\phi}}{\rho^4} = \pm i (\partial_r \lambda)(\partial_r \bar{\lambda}) \] (53)

Therefore we get
\[ \partial_y \lambda = \pm i \partial_r \lambda \] (54)

whose solution is \( \lambda = \lambda(r + iy) \) for the + sign and \( \lambda = \lambda(r - iy) \) for the - sign. We shall take
\[ \lambda = \lambda(r + iy) \] (55)

where \( \lambda(r + iy) \) is an arbitrary function of its arguments. If relations (49) and (55) hold Eqs (50) and (51) are satisfied. Therefore if Eqs (48), (49) and (55) hold Eq. (32) is satisfied for all values of \( \mu \) and \( \nu \). In other words all metric equations are reduced to Eqs (48), (49) and (55), where \( y \) is given by Eq. (47).

5 Solution of the Vector Field Equations

We have
\[ \lambda F_+^{\mu\nu} - \bar{\lambda} F_-^{\mu\nu} = 2i(e^{-2\phi} F^{\mu\nu} + \xi F^* F^{\mu\nu}) \] (56)

and the first of Eqs (33) becomes
\[ \partial_\mu(\sqrt{-g}(e^{-2\phi} F^{\mu\nu} + \xi F^* F^{\mu\nu})) = 0 \] (57)

Then since \( \lambda, F^{\mu\nu} \) and \( g \) are not functions of \( t \) and \( \phi \) while \( F_{\tau\phi} = F_{t\phi} = 0 \), the above relation is satisfied for \( \nu = r \) and \( \nu = \theta \). Eq. (57) for the metric of Eqs (19), (25) and (26) gives

for \( \nu = t \) : \( \partial_\tau \{(\rho^2 + a^2 \sin^2 \theta)(e^{-2\phi} \zeta + \xi \eta)\} + \partial_y \{a^2 \sin^2 \theta(\rho^2 e^{-2\phi} \eta - \xi \zeta)\} = 0 \) (58)

and for \( \nu = \phi \) : \( \partial_\tau (e^{-2\phi} \zeta + \xi \eta) + \partial_y (e^{-2\phi} \eta - \xi \zeta) = 0 \) (59)

Multiplying Eq. (59) by \( a^2 \) and subtracting from Eq. (58) we get
\[ \partial_\tau \{r(r + b) (e^{-2\phi} \zeta + \xi \eta)\} - \partial_y \{y^2 (e^{-2\phi} \eta - \xi \zeta)\} = 0 \] (60)

The general solution of Eq. (59) is
\[ e^{-2\phi} \zeta + \xi \eta = \partial_y P(r, y), \quad e^{-2\phi} \eta - \xi \zeta = -\partial_\tau P(r, y) \] (61)
where $P(r, y)$ is an arbitrary function of its arguments. Then substituting the expressions (61) in (60) we get
\[ \partial_r \partial_y \{ (r + b) + y^2 \} P = 0 \] (62)
Therefore we get
\[ P = \frac{G(r) + H(y)}{r(r + b) + y^2} = \frac{G(r) + H(y)}{\rho^2} \] (63)
where $G(r)$ and $H(y)$ are arbitrary functions of their arguments and Eq. (25) was used. If we solve Eqs (61) for $\zeta$ and $\eta$ we get
\[ \zeta = (\lambda \bar{\lambda})^{-1}(\xi \partial_r P + e^{-2\phi} \partial_y P), \quad \eta = (\lambda \bar{\lambda})^{-1}(\xi \partial_y P - e^{-2\phi} \partial_r P) \] (64)
Proceeding in an analogous fashion we can solve the second of Eqs (33) and find $P'$, $\zeta'$ and $\eta'$. We get
\[ P' = \frac{G'(r) + H'(y)}{\rho^2} \] (65)
\[ \zeta' = (\lambda \bar{\lambda})^{-1}(\xi \partial_r P' + e^{-2\phi} \partial_y P'), \quad \eta' = (\lambda \bar{\lambda})^{-1}(\xi \partial_y P' - e^{-2\phi} \partial_r P') \] (66)
with $G'(r)$ and $H'(y)$ arbitrary functions of their arguments. From Eqs (64) and (66) we get
\[ \zeta^2 + \eta^2 + \zeta'^2 + \eta'^2 = \frac{1}{\lambda \bar{\lambda}}((\partial_r P)^2 + (\partial_y P)^2 + (\partial_r P')^2 + (\partial_y P')^2) \] (67)
and Eq. (48) becomes
\[ \frac{K e^{2\phi} \lambda \bar{\lambda}}{\rho^6} = (\partial_r P)^2 + (\partial_y P)^2 + (\partial_r P')^2 + (\partial_y P')^2 \] (68)

6 Solution of the Axion-Dilaton Equation

We shall solve now the axion-dilaton Eq. (34). From Eqs (30), (37), (64) and (66) we get
\[ F^2 = 2(F^2 - iFF^*) = -4(\zeta - i\eta)^2 = -4\lambda^{-2}(\partial_r P - i\partial_y P)^2 \] (69)
\[ F^2 = 2(F'^2 - iF''F'^*) = -4(\zeta' - i\eta')^2 = -4\lambda^{-2}(\partial_r P' - i\partial_y P')^2 \] (70)
and Eq. (34) becomes
\[
\nabla_\mu \partial^\mu \lambda + ie^{2\phi} \partial_\mu \lambda \partial^\mu \lambda + 2ie^{-4\phi} \bar{\lambda}^{-2} \left\{ (\partial_\mu P - i\partial_y P)^2 + (\partial_\mu P' - i\partial_y P')^2 \right\} = 0 \tag{71}
\]
Since for the metric of Eqs (19), (25) and (26) and for \( \lambda = \lambda(r + iy) \) we get
\[
\nabla_\mu \partial^\mu \lambda = \frac{\Delta}{\rho^2} (\partial_\mu \lambda)^2 \tag{72}
\]
Eq. (71) becomes
\[
\Delta \partial_\mu \partial_{\bar{\mu}} \lambda + \left\{ 2(r - iy) + b + \frac{2}{b}(Q_M^2 - Q_E^2) \right\} \partial_\mu \lambda + i\Delta e^{2\phi} (\partial_\mu \lambda)^2
\]
\[
+ 2i\rho^2 e^{-4\phi} \bar{\lambda}^{-2} \left\{ (\partial_\mu P - i\partial_y P)^2 + (\partial_\mu P' - i\partial_y P')^2 \right\} = 0 \tag{74}
\]
To find a solution of Eq. (74) we multiply this equation by \( \partial_\mu \bar{\lambda} \), then we multiply its complex conjugate by \( \partial_{\bar{\mu}} \lambda \) and subsequently we sum the resulting expressions by parts. We get \( \Lambda \partial_\mu \bar{\lambda} + \bar{\Lambda} \partial_{\bar{\mu}} \lambda = 0 \) or if we use Eq. (49)
\[
2ib\{Q_M^2((r + b)^2 - y^2) - Q_E^2(r^2 - y^2)\} = \rho^8\left\{ (\partial_\mu P - i\partial_y P)^2 \partial_\mu (\bar{\lambda}^{-1})
\right.
\]
\[
- (\partial_\mu P + i\partial_y P)^2 \partial_\mu (\lambda^{-1}) \left\} + \rho^8\left\{ (\partial_\mu P' - i\partial_y P')^2 \partial_\mu (\bar{\lambda}^{-1}) - (\partial_\mu P' + i\partial_y P')^2 \partial_\mu (\lambda^{-1}) \right\} \right. \tag{75}
\]
Eq.(75) is satisfied if \( \lambda, P \) and \( P' \) satisfy the relations
\[
- 2ibQ_M^2(r^2 - y^2) = \rho^8\left\{ (\partial_\mu P - i\partial_y P)^2 \partial_\mu (\bar{\lambda}^{-1}) - (\partial_\mu P + i\partial_y P)^2 \partial_\mu (\lambda^{-1}) \right\} \tag{76}
\]
\[
2ibQ_M^2((r + b)^2 - y^2) = \rho^8\left\{ (\partial_\mu P' - i\partial_y P')^2 \partial_\mu (\bar{\lambda}^{-1}) - (\partial_\mu P' + i\partial_y P')^2 \partial_\mu (\lambda^{-1}) \right\} \tag{77}
\]
Of course \( \lambda, P \) and \( P' \) must be of the form given by Eqs (55), (63) and (65) respectively and they must be a solution of Eq. (74). All these requirements are satisfied if
\[
\lambda = \frac{i(r + iy + b)}{r + iy} \tag{78}
\]
\[
P = \frac{Q_E y}{\rho^2}, \quad P' = \frac{Q_M (r + b)}{\rho^2} \tag{79}
\]
From Eq. (78) we get
\[ \xi = \frac{by}{r^2 + y^2}, \quad e^{-2\phi} = \frac{\rho^2}{r^2 + y^2} \] (80)
and Eq. (49) is satisfied. Also from Eqs (64), (66) and (78)-(80) we get
\[ \zeta = \frac{Q_E(r^2 - y^2)}{\rho^4}, \quad \eta = \frac{2Q_Ery}{\rho^4}, \quad \zeta' = -\frac{Q_My(2r + b)}{\rho^4}, \quad \eta' = -\frac{Q_M\{r(r + b) - y^2\}}{\rho^4} \] (81)
and Eq. (48) is satisfied, which means that all equations are satisfied.

In summary our solution has a metric given by Eqs (19), (25) and (26), \( \lambda, \xi \) and \( e^{-2\phi} \) given by Eqs (47), (78) and (80) or by the relations
\[ \lambda = \frac{i(r + ia \cos \theta + b)}{r + ia \cos \theta}, \quad \xi = \frac{ab \cos \theta}{r^2 + a^2 \cos^2 \theta}, \quad e^{-2\phi} = \frac{\rho^2}{r^2 + a^2 \cos^2 \theta} \] (82)
and \( F_{\mu\nu} \) and \( F'_{\mu\nu} \) given by Eqs (35) and (36) with \( \zeta, \eta, \zeta' \) and \( \eta' \) given by Eqs. (47) and (81) or by the relations
\[ \zeta = \frac{Q_E(r^2 - a^2 \cos^2 \theta)}{\rho^4}, \quad \eta = \frac{2Q_Ear \cos \theta}{\rho^4}, \quad \zeta' = -\frac{Q_Ma \cos \theta(2r + b)}{\rho^4} \]
\[ \eta' = -\frac{Q_M\{r(r + b) - a^2 \cos^2 \theta\}}{\rho^4} \] (83)

The solution we have found has \( \xi \) and \( \phi \) fields with zero asymptotic values. However we can easily construct a solution with arbitrary asymptotic values of \( \xi \) and \( \phi \). Indeed we find using Eqs (32)- (34) that the expressions
\[ g_{\mu\nu} = g_{\mu\nu}, \quad \bar{\xi} = e^{-2\phi_\infty} \xi + \xi_\infty, \quad e^{-2\phi} = e^{-2\phi_\infty} e^{-2\phi}, \quad F_{\mu\nu} = e^{\phi_\infty} F_{\mu\nu}, \quad F'_{\mu\nu} = e^{\phi_\infty} F'_{\mu\nu} \] (84)
where \( g_{\mu\nu}, \xi, e^{-2\phi}, F_{\mu\nu}, \) and \( F'_{\mu\nu} \) are given by Eqs (19), (25), (26), (35), (36), (82) and (83) and \( \xi_\infty \) and \( \phi_\infty \) are arbitrary real constants, give a solution with arbitrary asymptotic values \( \xi_\infty \) and \( \phi_\infty \) of \( \xi \) and \( \phi \).

7 Physical Properties of the Solution

We shall describe now some physical properties of our solution. We get for large \( r \)
\[ g_{tt} = -\frac{\Delta}{\rho^2} \sim -(1 - \frac{2M}{r}) + O(\frac{1}{r^2}) \] (85)
Therefore the solution is asymptotically flat and the parameter $M$ is its mass. The $F_{\mu\nu}$ of the first vector field can be obtained from the vector potential

$$A_t = -\frac{Q_E r}{\rho^2}, \quad A_\phi = \frac{Q_E a r \sin^2 \theta}{\rho^2}, \quad A_r = A_\theta = 0$$

(86)

and the $F'_{\mu\nu}$ of the second vector field from the vector potential

$$A'_t = \frac{Q_M a \cos \theta}{\rho^2}, \quad A'_\phi = -\frac{Q_M (\rho^2 + a^2 \sin^2 \theta) \cos \theta}{\rho^2}, \quad A'_r = A'_\theta = 0$$

(87)

Since asymptotically

$$F_{rt} = \frac{Q_E (r^2 - a^2 \cos^2 \theta)}{\rho^4} \sim \frac{Q_E}{r^2} + O\left(\frac{1}{r^3}\right)$$

$$F'_{\theta\phi} = \frac{Q_M \{r(r + b) + a^2\} \{r(r + b) - a^2 \cos^2 \theta\} \sin \theta}{\rho^4} \sim Q_M \sin \theta + O\left(\frac{1}{r}\right)$$

(88)

$Q_E$ is an electric charge and $Q_M$ is a magnetic charge.

The angular momentum $J$ and the magnetic moment $\mu$ of a solution are obtained from the asymptotic form of $g_{t\phi}$ and $A_\phi$ respectively by the relations

$$g_{t\phi} \sim -\frac{2J}{r} \sin^2 \theta + O\left(\frac{1}{r^2}\right), \quad A_\phi \sim \frac{\mu \sin^2 \theta}{r} + O\left(\frac{1}{r^2}\right)$$

(89)

Since for our solution we have asymptotically

$$g_{t\phi} = \frac{a(\Delta - \rho^2) \sin^2 \theta}{\rho^2} \sim -\frac{2Ma \sin^2 \theta}{r} + O\left(\frac{1}{r^2}\right)$$

(90)

$$A_\phi = \frac{Q_E a r \sin^2 \theta}{\rho^2} \sim \frac{Q_E a \sin^2 \theta}{r} + O\left(\frac{1}{r^2}\right)$$

(91)

the angular momentum $J$ and the magnetic moment $\mu$ of our solution are given by the relations

$$J = Ma, \quad \mu = Q_E a$$

(92)

If we define the gyromagnetic ratio $g$ by the relation $\mu = g \frac{Q_E J}{2M}$ we get from Eqs (92)

$$g = 2$$

(93)
The infinite red shift surfaces occur when \[ g_{tt} = \frac{-\Delta}{\rho^2} = 0 \] (94)

From Eqs (26) and (94) we find that we have two infinite red shift surfaces \( r_S^\pm \) where

\[
r_S^\pm = \frac{1}{2M} \{2M^2 - Q_E^2 + Q_M^2 \pm (4M^4 - 4M^2(Q_E^2 + Q_M^2) + (Q_E^2 - Q_M^2)^2 - 4J^2 \cos^2 \theta)^{\frac{1}{2}} \} \]

(95)

Therefore the infinite red shift surfaces are closed axially symmetric surfaces.

The event horizons are the surfaces [19]

\[
\Delta + a^2 \sin^2 \theta = 0 \]

(96)

Taking into account Eqs (25), (26) and (96) we find that we have two horizon surfaces \( r_H^\pm \)

\[
r_H^\pm = M - \frac{b}{2} \pm \sqrt{(M - \frac{b}{2})^2 - (2Q_M^2 + a^2)}
\]

\[
= \frac{1}{2M} \{2M^2 - Q_E^2 + Q_M^2 \pm (4M^4 - 4M^2(Q_E^2 + Q_M^2) + (Q_E^2 - Q_M^2)^2 - 4J^2)^{\frac{1}{2}} \}
\]

(97)

The above expressions are obtained from the expressions \( r_S^\pm \) for \( \theta = 0 \) and \( \theta = \pi \). The outer and the inner horizons are closed surfaces, are contained within the corresponding infinite red shift surfaces and coincide with them at \( \theta = 0 \) and \( \theta = \pi \). If we compute the Ricci scalar \( R \) and the curvature scalar \( R^2 = R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} \) which are obtained from our metric given by Eqs [19], (25) and (26) we get [17]

\[
R = \frac{b^2(\Delta + 2a^2 \sin^2 \theta)}{2\rho^6}
\]

(98)

\[
R^2 = \frac{Y(r, M, Q_E^2, Q_M^2)}{4\rho^{12}}
\]

(99)

where \( Y(r, M, Q_E^2, Q_M^2) \) is a complicated polynomial of its arguments. This means that we have irremovable singularities only at

\[
\rho^2 = r(r + \frac{Q_E^2 - Q_M^2}{M}) + a^2 \cos^2 \theta = 0
\]

(100)

16
Since we shall assume as previously [9] that
\[ b = \frac{Q_E^2 - Q_M^2}{M} > 0 \] Eq. (100) for \( a \neq 0 \) is satisfied if
\[ r = \cos \theta = 0 \] (101)
Therefore for \( a \neq 0 \) we have a ring singularity, as in the case of the metric of Kerr [11]. Also from Eqs (47) and (80) we find that for \( a \neq 0 \) the axion and the dilaton are singular only when \( r = \cos \theta = 0 \). When \( a = 0 \) from Eqs (47), (80) and (98)-(99) we find that the axion field vanish, the dilaton field becomes singular only at \( r = 0 \) and the metric has irremovable singularity only at \( r = 0 \) [9]. From Eq. (97) we find that the horizons disappear when
\[ 4J^2 > 4M^4 - 4M^2(Q_E^2 + Q_M^2) + (Q_E^2 - Q_M^2)^2 \] and therefore the extremal limit corresponds to
\[ 4J^2 \rightarrow 4M^4 - 4M^2(Q_E^2 + Q_M^2) + (Q_E^2 - Q_M^2)^2 \] (102)
To find the area \( N \) of the outer horizon we consider the line element \( ds^2 \) of Eq. (22) for \( dt = dr = 0 \) and calculate its determinant \( g' \) at \( r = r^+ \). Since
\[ g' = \{r^+(r^+ + b) + a^2\}^2 \sin^2 \theta = \{r^+(r^+ + \frac{Q_E^2 - Q_M^2}{M}) + a^2\}^2 \sin^2 \theta \] (103)
we get
\[ N = \int_0^\pi d\theta \int_0^{2\pi} d\phi \sqrt{g'} = 4\pi \{r^+(r^+ + b) + a^2\} \]
\[ = 4\pi \{2M^2 - Q_E^2 - Q_M^2 + (4M^4 - 4M^2(Q_E^2 + Q_M^2) + (Q_E^2 - Q_M^2)^2 - 4J^2)^{\frac{1}{2}} \} \] (104)
The surface gravity \( k \) is calculated at the outer horizon \( r^+ \) by the relation
\[ k = \lim_{r \rightarrow r^+} \sqrt{g'} \partial_r \sqrt{-g_{tt}}|_{\theta=0} \] (105)
We get
\[ k = \frac{r^+ - r^-}{2\{r^+(r^+ + b) + a^2\}} \]
\[ = \frac{(4M^4 - 4M^2(Q_E^2 + Q_M^2) + (Q_E^2 - Q_M^2)^2 - 4J^2)^{\frac{1}{2}}}{2M\{2M^2 - Q_E^2 - Q_M^2 + (4M^4 - 4M^2(Q_E^2 + Q_M^2) + (Q_E^2 - Q_M^2)^2 - 4J^2)^{\frac{1}{2}}\}} \]
\[ = 2\pi T_H \] (106)
where \( T_H \) is the Hawking temperature. In the extremal limit we have \( k = \frac{1}{4M} \) if \( J = Q_M = 0 \) or \( J = Q_E = 0 \) and \( k = 0 \) in all other cases. The angular velocity \( \Omega \) at the horizon is calculated from the relation [4]
\[ g_{tt} + 2g_{t\phi} \Omega + g_{\phi\phi} = 0 \] (107)
Therefore
\[\Omega = -\frac{g_{\phi\phi}}{g_{\phi\phi}} \bigg|_{r=r^H} = \frac{a}{r^H(r^H + b) + a^2} \]
\[= \frac{J}{M\{2M^2 - Q_E^2 - Q_M^2 + (4M^4 - 4M^2(Q_E^2 + Q_M^2) + (Q_E^2 - Q_M^2)^2 - 4J^2)^{\frac{1}{2}}\}} \quad (108)\]

Expressions (97), (104), (106) and (108) for \(Q_E = Q\) and \(Q_M = 0\) are identical with the corresponding expressions of Sen [4].

8 Special Cases I

The solution we have found has the four parameters \(M, a, Q_E\) and \(Q_M\), which can take arbitrary values. All these parameters appear in the metric and for certain values of them a number of known solutions or metrics of known solutions can be obtained. This will be examined in this section.

(1) Solution of Sen with electric charge \(Q_E\) only [4]

If we put in our solution
\[Q_M = 0 \quad (109)\]
we get from Eqs (25), (26) and (47)
\[b = \frac{Q_E^2}{M}, \quad \rho^2 = r(r + \frac{Q_E^2}{M}) + y^2, \quad \Delta = r(r + \frac{Q_E^2}{M}) - 2Mr + y^2 \quad (110)\]
and from Eqs (78), (80) and (81)
\[\lambda = \frac{i(r + iy + b)}{r + iy}, \quad \xi = \frac{by}{r^2 + y^2}, \quad e^{-2\phi} = \frac{\rho^2}{r^2 + y^2} \quad (111)\]
\[\zeta = \frac{Q_E(r^2 - y^2)}{\rho^4}, \quad \eta = \frac{2Q_Ery}{\rho^4} \quad (112)\]
\[\zeta' = \eta' = 0 \quad (113)\]
Eqs (110)-(112) are identical with Eqs (288)-(290) since \(b\) is given by Eq. (287). Therefore for \(Q_M = 0\) our solution becomes the solution of Sen with electric charge only [4].
(2) Metric of Kerr-Newnman solution with electric charge \( Q_E \) and equal magnetic charge \( Q_M \) \[12\]

If in our solution we put
\[
Q_E = Q_M
\] (114)
we get from Eq. (25) \( b = 0 \) from Eqs (25), (26) and (47)
\[
\rho^2 = r^2 + y^2, \quad \Delta = r^2 - 2Mr + y^2 + 2Q_M^2 = r^2 - 2Mr + y^2 + Q_E^2 + Q_M^2
\] (115)
and from Eqs (13), (80) and (81)
\[
\xi = \phi = 0
\] (116)
\[
\zeta = \frac{Q_E(r^2 - y^2)}{\rho^4}, \quad \eta = \frac{2Q_Ery}{\rho^4}
\] (117)
\[
\zeta' = \frac{-2Q_Mry}{\rho^4}, \quad \eta' = \frac{Q_M(r^2 - y^2)}{\rho^4}
\] (118)
Eqs (115) are identical with Eqs (300), which means that our metric becomes the metric of the Kerr-Newman solution \[12\]. Also the \( \zeta \) and \( \eta \) of Eqs (301) are the sums \( \zeta + \zeta' \) and \( \eta + \eta' \), which are obtained from Eqs (117) and (118).

(3) Solution of Kerr \[1\]

If
\[
Q_E = Q_M = 0
\] (119)
Eq. (25) gives \( b = 0 \) and we get from Eqs (25), (26) and (47)
\[
\rho^2 = r^2 + y^2, \quad \Delta = r^2 - 2Mr + y^2
\] (120)
and from Eqs (13), (80) and (81)
\[
\xi = \phi = \zeta = \eta = \zeta' = \eta' = 0
\] (121)
Eqs (120) and (304) are identical, which means that for \( Q_E = Q_M = 0 \) our solution becomes the solution of Kerr \[1\].

(4) Metric and the axion and the dilaton field of the solution given by Eqs (3) and (4) of this paper (Class. Quant. Grav. 23, 7591 (2006) Eqs (54)-(57) for \( \psi_0 = 0, AB = 2Q_E^2, (\alpha - A)(\alpha - B) = 2Q_M^2, 2M = \alpha - A - B, \psi = 2\phi \) and \( \alpha = b \))
If we introduce new parameters $M$, $Q_E$ and $Q_M$ by the relations

\[(b - A)(b - B) = 2Q_M^2, \quad AB = 2Q_E^2, \quad b - A - B = 2M\] (122)

the solution takes the form given in the Appendix namely its metric is given by Eqs (19) and (307), its fields axion and dilaton are given by Eq. (308) and its vector field is given by Eqs (35) and (309). We can show that the metric, the axion field and the dilaton field of this solution are obtained from the corresponding quantities of our solution for $a = 0$ (123)

Indeed if we put $a = 0$ in our solution we get from Eqs (25), (26), (80) and (81)

\[b = \frac{Q_E^2 - Q_M^2}{M}, \quad \rho^2 = r(r + b), \quad \Delta = r(r + b) - 2Mr + 2Q_M^2\] (124)

\[\xi = 0, \quad e^{-2\phi} = \frac{r + b}{r}\] (125)

\[\zeta = \frac{Q_E}{(r + b)^2}, \quad \eta = \zeta' = 0, \quad \eta' = \frac{Q_M}{r(r + b)}\] (126)

If we introduce new variables $r'$ and $b'$ by the relations

\[r = r' + b', \quad b = -b'\] (127)

Eqs (124)-(126) become

\[b' = \frac{Q_M^2 - Q_E^2}{M}, \quad \rho^2 = r'(r' + b'), \quad \Delta = r'(r' + b') - 2Mr' + 2Q_E^2\] (128)

\[\xi = 0, \quad e^{2\phi} = \frac{r' + b'}{r'}\] (129)

\[\zeta = \frac{Q_E}{r'^2}, \quad \eta = \zeta' = 0, \quad \eta' = \frac{Q_M}{r'(r' + b')}\] (130)

If in (128)-(130) we replace $r'$ by $r$ and $b'$ by $b$ we see that Eqs (128) and (307) are identical and the same think happens to Eqs Eqs (129) and (308). Therefore the metric and the axion and the dilaton fields of the solution given by Eqs (3) and (4) [9] are obtained from the metric and the axion and
the dilaton fields of our solution for \( a = 0 \). Also the solution given by Eqs \( \text{[3]} \) and \( \text{[4]} \) has a vector field with electric charge \( Q_E \) and magnetic charge \( Q_M \), while our solution has two vector fields one with electric charge \( Q_E \) and another with magnetic charge \( Q_M \).

(5) The GM-GHS solution [5]

If we put in our metric
\[
a = Q_E = 0
\]
we get from Eqs \( \text{(25)} \) and \( \text{(26)} \)
\[
b = -\frac{Q_M^2}{M}, \quad \rho^2 = r\left( r - \frac{Q_M^2}{M} \right)
\]
\[
\Delta = r\left( r - \frac{Q_M^2}{M} \right) - 2Mr + 2Q_M^2 = (r - 2M)(r - \frac{Q_M^2}{M})
\]
and from Eqs \( \text{(80)} \) and \( \text{(81)} \)
\[
\xi = 0, \quad e^{-2\phi} = 1 + \frac{b}{r}, \quad \zeta = \eta = \zeta' = 0, \quad \eta' = \frac{Q_M}{\rho^2}
\]
From Eqs \( \text{(132)} \), \( \text{(133)} \), \( \text{(325)} \) and \( \text{(326)} \) we find that for \( a = Q_E = 0 \) one of the vector fields of our solution disappears and our solution becomes the GM-GHS solution [5].

(6) Metric of the Reissner-Nordström solution for electric charge \( Q_E \) and equal magnetic charge \( Q_M \)

If we put in our solution
\[
a = 0 \quad \text{and} \quad Q_E = Q_M
\]
which implies according to Eq. \( \text{(25)} \) that \( b = 0 \), we get from Eqs \( \text{(25)} \) and \( \text{(26)} \)
\[
\rho^2 = r^2, \quad \Delta = r^2 - 2Mr + 2Q_M^2 = r^2 - 2Mr + Q_E^2 + Q_M^2
\]
and from Eqs \( \text{(80)} \) and \( \text{(81)} \)
\[
\xi = \phi = 0, \quad \zeta = \frac{Q_E}{r^2}, \quad \eta = \zeta' = 0, \quad \eta' = \frac{Q_M}{r^2}
\]
Eqs (133) and (330) are identical. Therefore the metric of the Reissner-Nordström solution for $Q_E = Q_M$ is obtained from the metric of our solution for $a = 0$ and $Q_E = Q_M$. Also both solutions have electric charge $Q_E$ and magnetic charge $Q_M$. However in the Reissner-Nordström solution the charges $Q_E$ and $Q_M$ belong to the same vector field, while in our case $Q_E$ belongs to one vector field and $Q_M$ to another field.

(7) The Schwarzschild solution

If we put in our solution

$$a = Q_E = Q_M = 0 \quad (137)$$

from Eq. (25) we get $b = 0$ and Eqs (25), (26), (80) and (81) give

$$\rho^2 = r^2, \quad \Delta = r^2 - 2Mr \quad (138)$$

$$\xi = \phi = \zeta = \eta = \zeta' = \eta' = 0 \quad (139)$$

Eqs (138) and (335) are identical. Therefore if we put $a = Q_E = Q_M = 0$ in our solution we get the solution of Schwarzschild.

9 “New Solution”

It is well known that the equations of motion coming from the action (28) with $F'_{\mu\nu} = 0$ are invariant under the $SL(2, R)$ group of transformations [14]

$$\lambda \rightarrow \lambda' = \frac{\beta \lambda + \gamma}{\delta \lambda + \epsilon}, \quad \beta \epsilon - \gamma \delta = 1 \quad (140)$$

$$F'_{\mu\nu} \rightarrow - (\delta \lambda + \epsilon) F'_{\mu\nu}, \quad F'_{\mu\nu} \rightarrow - (\delta \bar{\lambda} + \epsilon) F'_{\mu\nu} \quad (141)$$

This invariance known as $S$-duality also holds in the case we have an action of the form (28) with multiple vector fields provided that each vector field transforms as above [20]. We get

$$\lambda' = \frac{\beta \delta - 1}{\delta (\delta \lambda + \epsilon)} \quad (142)$$

The real constant $\beta \delta^{-1}$ makes a shift of the axion field. Such a shift, which is allowed by the equations of motion, will be omitted in which case we get for the $\lambda$ of Eq (78)

$$\lambda' = \frac{1}{\delta (\delta \lambda + \epsilon)} = - \frac{r + iy}{\delta \{ \epsilon r - \delta y + i(\delta (r + b) + \epsilon y) \}} = \xi' + i e^{-2\phi'} \quad (143)$$
where \( \lambda' \) and \( \phi' \) are the new axion and dilaton fields respectively. But for this \( \lambda' \) we get asymptotically

\[
\lambda' \sim -\frac{1}{\delta(\epsilon + i\delta)} = -\frac{\epsilon}{\delta(\delta^2 + \epsilon^2)} + \frac{i}{\delta^2 + \epsilon^2}
\]

(144)

which means that the asymptotic values of \( \xi' \) and \( \phi' \) are not zero. To get a solution in which the axion and the dilaton have zero asymptotic values we have according to Eq (84) to make the replacements

\[
\lambda' \rightarrow \lambda'' = \{\lambda' + \frac{\epsilon}{\delta(\delta^2 + \epsilon^2)}\}(\delta^2 + \epsilon^2)
\]

(145)

\[-(\delta\lambda + \epsilon)F^\mu\nu_+ \rightarrow f^\mu\nu_+ = \frac{-\delta\lambda + \epsilon}{\sqrt{\delta^2 + \epsilon^2}}F^\mu\nu_+ \quad -(\delta\lambda + \epsilon)F^\mu\nu_- \rightarrow f^\mu\nu_- = \frac{-\delta\lambda + \epsilon}{\sqrt{\delta^2 + \epsilon^2}}F^\mu\nu_-
\]

(146)

If we define \( \delta' \) and \( \epsilon' \) by the relations

\[
\delta' = \frac{\delta}{\sqrt{\delta^2 + \epsilon^2}}, \quad \epsilon' = \frac{\epsilon}{\sqrt{\delta^2 + \epsilon^2}}
\]

(147)

we get from Eqs (143) and (145)

\[
\lambda'' = \frac{i\{(\epsilon' + i\delta')(r + iy) + \epsilon'b\}}{\epsilon' + i\delta'}(r + iy) + i\delta'b = \xi'' + ie^{-2\phi''}
\]

(148)

where

\[
\xi'' = \frac{b\{\delta'\epsilon'(2r + b) + (\epsilon'^2 - \delta'^2)\theta\}}{\delta'^2\{(r + b)^2 + y^2\} + \epsilon'^2(r^2 + y^2) + 2\delta'\epsilon'b\theta}
\]

(149)

\[
e^{-2\phi''} = \frac{\rho^2}{\delta'^2\{(r + b)^2 + y^2\} + \epsilon'^2(r^2 + y^2) + 2\delta'\epsilon'b\theta}
\]

(150)

Taking into account Eqs (147) Eqs (146) give for the \( f^\mu_\nu \) field and the second vector field \( f'_{\mu\nu} \) [13]-[14]

\[
f^\mu_\nu = -(\delta'\xi + \epsilon')F^\mu_\nu + \delta' e^{-2\phi}F^*_{\mu\nu}
\]

(151)

\[
f'_{\mu\nu} = -(\delta'\xi + \epsilon')F'_\mu_\nu + \delta' e^{-2\phi}F'^*_{\mu\nu}
\]

(152)

From the \( F^\mu_\nu \) field of Eq (35) we can calculate the \( F^*_{\mu\nu} \) field using Eq. (29). We get

\[
F_{\nu t}^* = \eta, \quad F_{r\phi}^* = -a\sin^2\theta\eta, \quad F_{\theta t}^* = a\sin\theta\xi, \quad F_{\phi\theta}^* = -\{r(r + b) + a^2\}\sin\theta\xi
\]

23
\[ F^*_{r\theta} = F^*_{t\phi} = 0 \quad (153) \]

which are obtained from the \( F_{\mu\nu} \) field if we make the substitutions \( \zeta \rightarrow \eta \) and \( \eta \rightarrow -\zeta \). Generally if \( F_{\mu\nu} \) is of the form of Eq. (35) and \( g_{\mu\nu} \) is given by expression (19) with \( \rho^2 \) given by Eq. (25) its dual \( F^*_\mu = \frac{g_{\mu\nu} g_{\rho\psi}}{2 \sqrt{-g}} \epsilon^{\sigma\tau\chi\psi} F_{\chi\psi} \) is obtained from \( F_{\mu\nu} \) by the substitution

\[ \zeta \rightarrow \eta, \quad \eta \rightarrow -\zeta \quad (154) \]

Also from Eq. (36) we get for the second vector field

\[ F^*_{rt} = \eta', \quad F^*_{r\phi} = -a \sin^2 \theta \eta', \quad F^*_{\theta t} = a \sin \theta \zeta', \quad F^*_{\theta\phi} = -\{r(r+b)+a^2\} \sin \theta \zeta' \]

\[ F^*_{rt} = F^*_{t\phi} = 0 \quad (155) \]

The vector fields \( f_{\mu\nu} \) and \( f'_{\mu\nu} \) of the new solution are obtained from Eqs (35), (36), (80), (81) and (151)-(155). We get

\[ f_{rt} = \sigma, \quad f_{r\phi} = -a \sin^2 \theta \sigma, \quad f_{\theta t} = -a \sin \theta \tau, \quad f_{\theta\phi} = \{r(r+b)+a^2\} \sin \theta \tau \]

\[ f_{rt} = f_{t\phi} = 0 \quad (156) \]

\[ f'_{rt} = \sigma', \quad f'_{r\phi} = -a \sin^2 \theta \sigma', \quad f'_{\theta t} = -a \sin \theta \tau', \quad f'_{\theta\phi} = \{r(r+b)+a^2\} \sin \theta \tau' \]

\[ f'_{rt} = f'_{t\phi} = 0 \quad (157) \]

where

\[ \sigma = -\frac{\{\delta'b + \epsilon'(r^2 + y^2)\} \zeta - \delta' \rho^2 \eta}{r^2 + y^2} = -\frac{Q_E}{\rho^4} \{y(2r+b)-\epsilon'(r^2-y^2)\} \quad (158) \]

\[ \tau = -\frac{\{\delta'b + \epsilon'(r^2 + y^2)\} \eta + \delta' \rho^2 \zeta}{r^2 + y^2} = -\frac{Q_E}{\rho^4} \{\delta' \{r(r+b)-y^2\} + 2 \epsilon' r y\} \quad (159) \]

\[ \sigma' = -\frac{\{\delta'b + \epsilon'(r^2 + y^2)\} \zeta' - \delta' \rho^2 \eta'}{r^2 + y^2} = -\frac{Q_M}{\rho^4} \{\delta' \{(r+b)^2-y^2\} + \epsilon'(2r+b)y\} \quad (160) \]

\[ \tau' = -\frac{\{\delta'b + \epsilon'(r^2 + y^2)\} \eta' + \delta' \rho^2 \zeta'}{r^2 + y^2} = -\frac{Q_M}{\rho^4} \{2 \delta' y(r+b) - \epsilon' \{r(r+b)-y^2\}\} \quad (161) \]

For large \( r \) we have

\[ f_{rt} \sim -\frac{Q_E}{r^2} + O\left(\frac{1}{r^3}\right), \quad f_{\theta\phi} \sim -Q_E \delta' \sin \theta + O\left(\frac{1}{r}\right) \quad (162) \]
\[ f'_{rt} \sim \frac{Q_M \delta'}{r^2} + O\left(\frac{1}{r^3}\right), \quad f'_{\theta\phi} \sim -Q_M \epsilon' \sin \theta + O\left(\frac{1}{r}\right) \quad (163) \]

Therefore the electric charge \( q_E \) and the magnetic charge \( q_M \) of the first vector field are

\[ q_E = -Q_E \epsilon', \quad q_M = -Q_E \delta' \quad (164) \]

and the electric charge \( q'_E \) and the magnetic charge \( q'_M \) of the second vector field are

\[ q'_E = Q_M \delta', \quad q'_M = -Q_M \epsilon' \quad (165) \]

The following relations are satisfied

\[ q_E q'_E + q_M q'_M = 0, \quad q_E^2 + q_M^2 = Q^2_E, \quad q'_E^2 + q'_M^2 = Q^2_M \quad (166) \]

where \( Q_E \) and \( Q_M \) are arbitrary. Therefore only three of the charges \( q_E, q_M, q'_E \) and \( q'_M \) can take arbitrary values. Using Eqs (164) and (165) we can express \( \sigma, \tau, \sigma' \) and \( \tau' \) in terms of the charges \( q_E, q_M, q'_E \) and \( q'_M \). We get

\[ \sigma = \frac{q_E (r^2 - y^2) - q_M y (2r + b)}{\rho^4}, \quad \tau = \frac{q_M \{r (r + b) - y^2\} + 2q_E y}{\rho^4} \quad (167) \]

\[ \sigma' = \frac{q'_E ((r + b)^2 - y^2)}{\rho^4} - q'_M y (2r + b), \quad \tau' = \frac{q'_M \{r (r + b) - y^2\} + 2q'_E y (r + b)}{\rho^4} \quad (168) \]

The vector fields \( f_{\mu\nu} \) and \( f'_{\mu\nu} \) can be obtained from the vector potentials \( a_{\mu} \) and \( a'_{\mu} \) respectively where

\[ a_t = \frac{Q_E}{\rho^2} (\epsilon' r - \delta' y), \quad a_{\phi} = \frac{Q_E}{\rho^2} \{ -\epsilon' r a \sin^2 \theta + \delta' \{ r (r + b) + a^2 \} \cos \theta \}, \quad a_r = a_{\theta} = 0 \quad (169) \]

\[ a'_t = -\frac{Q_M}{\rho^2} \{ \epsilon' a \cos \theta + \delta' (r + b) \}, \quad a'_{\phi} = \frac{Q_M}{\rho^2} \{ \epsilon' \{ r (r + b) + a^2 \} \cos \theta + \delta' (r + b) a \sin^2 \theta \}, \quad a'_r = a'_{\theta} = 0 \quad (170) \]

For the “new solution” equations of the type (61) must hold. If we write for the first vector field

\[ e^{-2 \phi''} \sigma + \xi'' \tau = \partial_y S, \quad e^{-2 \phi''} \tau - \xi'' \sigma = -\partial_r S \quad (171) \]

and for the second vector field

\[ e^{-2 \phi''} \sigma' + \xi'' \tau' = \partial_y S', \quad e^{-2 \phi''} \tau' - \xi'' \sigma' = -\partial_r S' \quad (172) \]
and use Eqs (149), (150) and (158)-(161) we get

\[ S = -\frac{Q_E}{\rho^2}(\delta' r + \epsilon' y) \]  

(173)

\[ S' = \frac{Q_M}{\rho^2}\{\delta' y - \epsilon'(r + b)\} \]  

(174)

which are of the form of Eqs (63) and (65) respectively.

Eq. (148) can be written in the form

\[ \lambda'' = \frac{i}{r + iy + (1 - d)b} \]  

(175)

where

\[ d = \frac{i\delta'}{\epsilon' + i\delta'} \]  

(176)

The quantity \( \lambda \) of our first solution given by Eq. (78) is again of the form of Eq. (175) with \( d = 0 \) (177).

All solutions with \( \phi \neq 0 \) which are mentioned in this paper have a \( \lambda \) of the form of Eq. (175) with \( d \) a real or a complex constant. In the Appendix we calculate the constant \( d \) for all solutions with \( \phi \neq 0 \) which are listed there. For the fields \( \xi \) and \( \phi \) coming from an expression of the form of Eq. (175) we get \( \xi_\infty = \phi_\infty = 0 \), where \( \xi_\infty \) and \( \phi_\infty \) are their asymptotic values respectively. Therefore before we check if the \( \lambda \) of a solution can be written in the form of Eq. (175) we have to take out the asymptotic values \( \xi_\infty \) and \( \phi_\infty \), if they are non-vanishing.

### 10 Special cases II

If in Eqs (148) and (158)-(161) we put

\[ \delta' = 0, \quad \epsilon' = -1 \]  

(178)

we get expressions (78) and (81). Therefore our solution is a special case of our “new solution” and it is obtained for \( \delta' = 0 \) and \( \epsilon' = -1 \). This means that all solutions of Section 8, which are special cases of our solution, are also special cases of our “new solution”. In addition we shall show that two more
solution [13],[14] are obtained from our “new solution” for certain values of its parameters.

(1) Solution of Sen with both charges electric and magnetic [13]

If in our “new solution” we put

$$Q_M = 0$$  \hfill (179)

we get from Eqs (157), (160) and (161)

$$f'_{\mu\nu} = 0$$  \hfill (180)

and from Eqs (25) and (26)

$$\rho^2 = r(r + \frac{Q^2_E}{M}) + y^2, \quad \Delta = r(r + \frac{Q^2_E}{M}) - 2Mr + y^2$$  \hfill (181)

Also if in Eqs (181), (148)-(150), (158) and (159) we put

$$Q^2_E = q^2_E + q^2_M, \quad \delta' = -\frac{q_M}{\sqrt{q^2_E + q^2_M}}, \quad \epsilon' = -\frac{q_E}{\sqrt{q^2_E + q^2_M}}$$  \hfill (182)

and in the resulting expressions we replace $q_E$ by $Q_E$ and $q_M$ by $Q_M$ we get Eqs (293)-(297). Therefore the solution of Sen with both charges electric and magnetic [13] is obtained from our “new solution” for $Q_M = 0$.

(2) Solution of Shapere, Trivedi and Wilczek [14]

If in our “new solution” we put

$$a = Q_E = 0$$  \hfill (183)

we get from Eqs (25) and (26)

$$b = -\frac{Q^3_M}{M}, \quad \rho^2 = r(r - \frac{Q^3_M}{M})$$  \hfill (184)

$$\Delta = r(r - \frac{Q^2_M}{M}) - 2Mr + 2Q^2_M = (r - 2M)(r - \frac{Q^2_M}{M})$$  \hfill (185)

from Eqs (156), (158) and (159)

$$f_{\mu\nu} = 0$$  \hfill (186)
from Eq (148)

\[
\lambda'' = \frac{i\{(\epsilon' + i\delta')r + \epsilon'b\}}{(\epsilon' + i\delta')r + i\delta'b}
\]  

(187)

and from Eqs (160) and (161)

\[
\sigma' = \frac{Q_M\delta'}{r^2}, \quad \tau' = -\frac{Q_M\epsilon'}{\rho^2}
\]  

(188)

If we define \(Q'_E\) and \(Q'_M\) by the relations

\[
Q'_E = Q_M\delta', \quad Q'_M = -Q_M\epsilon'
\]  

(189)

which imply according to Eq (147) that

\[
Q'_E^2 + Q'_M^2 = Q_M^2
\]  

(190)

using these relations eliminate \(\delta', \epsilon'\) and \(Q_M\) from the expressions (184), (185), (187) and (188), write \(\zeta\) instead of \(\sigma'\) and \(\eta\) instead of \(\tau'\) and subsequently drop the primes we get the relations

\[
\rho^2 = r(r - \frac{Q_E^2 + Q_M^2}{M})
\]  

(191)

\[
\Delta = r(r - \frac{Q_E^2 + Q_M^2}{M}) - 2Mr + 2(Q_E^2 + Q_M^2) = (r - 2M)(r - \frac{Q_E^2 + Q_M^2}{M})
\]  

(192)

\[
\lambda = \frac{i\{(Q_M - iQ_E)r + Q_Mb\}}{(Q_M - iQ_E)r - iQ_Eb}
\]  

(193)

\[
\zeta = \frac{Q_E}{r^2}, \quad \eta = \frac{Q_M}{\rho^2}
\]  

(194)

Eqs (191)-(194) are identical with Eqs (317)-(319) and (321). Therefore the STW solution [14] is obtained from our “new solution” for \(a = Q_E = 0\).
11 Energy Conditions

Using Einstein’s equations and the expressions (38)-(41) and (98) for the components $R_{\mu\nu}$ of the Ricci tensor and the Ricci scalar $R$ we find the components $T_{\mu\nu}$ of the energy-momentum tensor

$$T_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu} \frac{R}{2}$$  \hspace{1cm} (195)

The eigenvalues $w_\mu$ of the tensor $T^\mu_\nu$ are obtained if we equate to zero the determinant of the eigenvalue equation

$$(T^\mu_\mu - w_\delta^\mu_\mu)u^\mu = 0$$  \hspace{1cm} (196)

namely from the relation

$$|T^\mu_\mu - w_\delta^\mu_\mu| = 0 =$$

$$
\begin{vmatrix}
R_{tt}g^{tt} + R_{t\phi}g^{t\phi} - \frac{R}{2} - w & 0 & 0 & R_{tt}g^{t\phi} + R_{t\phi}g^{t\phi} \\
0 & R_{rr}g^{rr} - \frac{R}{2} - w & 0 & 0 \\
0 & 0 & R_{\theta\theta}g^{\theta\theta} - \frac{R}{2} - w & 0 \\
R_{\phi\phi}g^{\phi\phi} + R_{\phi\phi}g^{t\phi} & 0 & 0 & R_{\phi\phi}g^{\phi\phi} + R_{\phi\phi}g^{t\phi} - \frac{R}{2} - w
\end{vmatrix}
\hspace{1cm} (197)
$$

Using the relations

$$R_{tt}g^{tt} + R_{t\phi}g^{t\phi} + 2R_{t\phi}g^{t\phi} = 0$$  \hspace{1cm} (198)

$$R_{tt}R_{\phi\phi} - (R_{t\phi})^2 = \frac{K^2 \sin^2 \theta (\Delta + a^2 \sin^2 \theta)}{\rho^{12}}$$  \hspace{1cm} (199)

$$g^{tt}g^{\phi\phi} - (g^{t\phi})^2 = -\frac{1}{(\Delta + a^2 \sin^2 \theta) \sin^2 \theta}$$  \hspace{1cm} (200)

which hold for any metric of the form of Eq. (19), we find that the energy density $\mu = -w_0$, where $w_0$ is the eigenvalue which corresponds to the timelike eigenvector and the principal pressures $p_i = w_i \ i = 1, 2, 3$ of our solution are

$$\mu = \frac{K}{\rho^6} + \frac{R}{2}$$  \hspace{1cm} (201)

$$p_1 = \frac{K}{\rho^6} - \frac{R}{2}, \quad p_2 = \frac{K}{\rho^6} - \frac{R}{2} + \frac{a^2 b^2 \sin^2 \theta}{2 \rho^6}, \quad p_3 = -\frac{K}{\rho^6} + \frac{R}{2} - \frac{a^2 b^2 \sin^2 \theta}{2 \rho^6}$$  \hspace{1cm} (202)
with $K$ given by Eq (41).

For a metric of the form of Eqs (19) and (20) the event horizons are the surfaces given by Eq (96), from which we get for the outer horizon $r_+^H$

$$2r_+^H + A + B = \sqrt{(A - B)^2 - 4a^2}$$  \hspace{1cm} (203)

The Ricci scalar $R$ given by Eq (98) becomes on the outer horizon

$$R = \frac{b^2(\Delta + 2a^2b^2 \sin^2 \theta)}{2\rho^2} = \frac{a^2b^2 \sin^2 \theta}{2\rho^2} \geq 0$$  \hspace{1cm} (204)

Also outside the outer horizon where $r = r_+^H + r'$, with $r' > 0$, we have

$$\Delta + a^2 \sin^2 \theta = r'(r' + \sqrt{(A - B)^2 - 4a^2}) > 0$$  \hspace{1cm} (205)

Therefore for all solutions with metric of the form given by Eqs (19) and (20) (obviously Eq. (20) can be replaced by Eqs (24)-(26)) we have outside and on the outer horizon

$$\Delta + a^2 \sin^2 \theta \geq 0 \quad \text{and} \quad R \geq 0$$  \hspace{1cm} (206)

For our solution we have

$$b > 0, \quad A < 0, \quad B < 0$$  \hspace{1cm} (207)

so that we get from Eq (24) the relations

$$2Q^2_E = (b - A)(b - B) > 0, \quad 2Q^2_M = AB > 0$$  \hspace{1cm} (208)

which give according to Eq. (41) the relation

$$K > 0$$  \hspace{1cm} (209)

everywhere. Therefore we get from Eqs (201), (202), (206) and (209) outside and on the outer horizon

$$\mu = \frac{K}{\rho^6} + \frac{R}{2} > 0, \quad \mu + p_1 = \frac{2K}{\rho^6} > 0, \quad \mu + p_2 = \frac{2K}{\rho^6} + \frac{a^2b^2 \sin^2 \theta}{2\rho^6} > 0$$

$$\mu + p_3 = R - \frac{a^2b^2 \sin^2 \theta}{2\rho^6} = \frac{b^2(\Delta + a^2 \sin^2 \theta)}{2\rho^6} \geq 0, \quad \mu - p_1 = R \geq 0$$
\[
\mu - p_2 = R - \frac{a^2 b^2 \sin^2 \theta}{2 \rho^6} = \frac{b^2 (\Delta + a^2 \sin^2 \theta)}{2 \rho^6} \geq 0, \quad \mu - p_3 = \frac{2K}{\rho^6} + \frac{a^2 b^2 \sin^2 \theta}{2 \rho^6} > 0
\]

(210)

and the dominant energy conditions, which are defined by the relations [21]

\[
\mu \geq 0, \quad -\mu \leq p_i \leq \mu, \quad i = 1, 2, 3
\]

(211)

are satisfied. Of course the weak energy conditions, which are defined by the relations [21]

\[
\mu \geq 0, \quad \mu + p_1 \geq 0, \quad i = 1, 2, 3
\]

(212)

are satisfied. Also we have

\[
\mu + p_1 + p_2 + p_3 = \frac{2K}{\rho^6} > 0
\]

(213)

and the strong energy conditions, which are defined by the relations [21]

\[
\mu + p_1 \geq 0, \quad i = 1, 2, 3, \quad \mu + p_1 + p_2 + p_3 \geq 0
\]

(214)

are satisfied. Therefore our solution and also our “new solution”, whose metric is obtained from the metric of our solution if we make the substitutions

\[
Q^2_E \longrightarrow q_E^2 + q_M^2 \equiv Q^2, \quad Q^2_M \longrightarrow q_E^2 + q_M^2 \equiv Q^2
\]

(215)

satisfy the dominant, the weak and the strong energy conditions outside and on the outer horizon.

We observe that all solutions with metric of the form of Eq. (19) with \( \rho^2 \) and \( \Delta \) given by Eq. (20) or by Eqs (24)-(26) satisfy all energy conditions outside and on the outer horizon if the relations \( \Delta + a^2 \sin^2 \theta \geq 0 \), \( R \geq 0 \) and \( K \geq 0 \) are satisfied outside and on the outer horizon, where \( R \) is the Ricci scalar of the metric and \( K \) is defined by Eq. (41). But we proved before that all solutions with metric of the above form satisfy the relation \( \Delta + a^2 \sin^2 \theta \geq 0 \) and \( R \geq 0 \) outside and on the outer horizon. Also we observe that if we define \( Q_E, Q_M \) and \( K \) by the relations (24) and (41) and take

\[
A < 0, \quad B \leq 0, \quad b \geq 0 \quad \text{or} \quad b = B < 0
\]

(216)

we get \( K \geq 0 \) everywhere. Therefore all solutions with metric of the form of Eq. (19) with \( \rho^2 \) and \( \Delta \) given by Eq. (20) or by Eqs (24)-(26), whose parameters \( A, B \) and \( b \) satisfy the relations (216), satisfy the dominant,
the weak and the strong energy conditions outside and on the outer horizon. This happens to all solutions given in the Appendix except the Kerr-Newman solution and the Reissner-Nordström solution if their electric charge \( Q_E \) and magnetic charge \( Q_M \) satisfy the relation \( Q_E \neq Q_M \), for which we cannot reach this conclusion because Eqs (24)-(26) are not satisfied. But we can easily show that these solutions also satisfied all energy conditions. To do that consider a Kerr-Newman solution with charges \( Q'_E \) and \( Q'_M \) for which \( Q'_E \neq Q'_M \) and our solution with charges \( Q_E \) and \( Q_M \) which satisfy the relations

\[
Q_E^2 = Q_M^2 = \frac{Q'_E^2 + Q'_M^2}{2} \tag{217}
\]

Both solutions have the same metric, which means that they have the same components of the Ricci tensor \( R_{\mu\nu} \) and the same Ricci scalar \( R \). Therefore if in \( R_{\mu\nu} \) and \( R \) of Eqs (38)-(41) and (98) we make the substitutions of Eq. (217) we get the \( R_{\mu\nu} \) and \( R \) of the Kerr-Newman solution with \( Q'_E \neq Q'_M \). More specifically we get since \( b = 0 \)

\[
K = (Q'_E^2 + Q'_M^2)(r^2 + a^2 \cos^2 \theta) \quad \text{and} \quad R = 0 \tag{218}
\]

Therefore we get from Eqs (201) and (202)

\[
\mu = p_1 = p_2 = -p_3 = \frac{K}{\rho^6} \tag{219}
\]

Since we have

\[
\mu > 0, \quad \mu + p_1 = \mu + p_2 = \mu - p_3 = \mu + p_1 + p_2 + p_3 = \frac{2K}{\rho^6} > 0
\]

\[
\mu + p_3 = \mu - p_1 = \mu - p_2 = 0 \tag{220}
\]

Eqs (211), (212) and (214) are satisfied everywhere, which means that the Kerr-Newman solution satisfies all energy conditions.

Also since the Reissner-Nordström solution is obtained from the Kerr-Newman solution for \( a = 0 \) for which \( K = (Q_E^2 + Q'_M^2)r^2 > 0 \) Eqs (211), (212) and (214) are satisfied everywhere, which means that the Reissner-Nordström solution for arbitrary \( Q_E \) and \( Q_M \) satisfies all energy conditions.
12 Mass Formulae

In this section we shall consider mass formulae of solutions with metric $g_{\mu\nu}$ of the form of Eqs (19), (25) and (26) and of the form of Eqs (19) and (27) and angular momentum $J = Ma$, where $a$ is a non-zero or a zero constant.

To find the expressions for $M$ and $dM$ assuming that we have a metric of the form of Eqs (19), (25) and (26) and angular momentum $J \neq 0$ we proceed as follows: We get from Eq (97)

$$r_H^+ r_H^- - a^2 = 2Q^2_M$$  \hspace{1cm} (221)

from Eqs (92) and (97) for $a \neq 0$

$$a(r_H^+ + r_H^- + b) = 2Ma = 2J$$  \hspace{1cm} (222)

and from Eq (104)

$$r_H^+(r_H^+ + b) = \frac{N}{4\pi} - a^2$$  \hspace{1cm} (223)

The above system is equivalent to the system formed by Eq (221) and the relations

$$\frac{2Jr_H^+}{a} = \frac{N}{4\pi} + 2Q^2_M$$  \hspace{1cm} (224)

$$\frac{2Jr_H^-}{a} = \frac{4J^2}{a^2} - \frac{N}{4\pi} - 2Q^2_M - \frac{2Jb}{a} = \frac{4J^2}{a^2} - \frac{N}{4\pi} - 2Q^2_E$$  \hspace{1cm} (225)

In writing Eq (225) we used the relations $J = Ma$ and $b = \frac{Q_E^2 - Q_M^2}{M}$. Multiplying Eqs (221) and (223) by parts and using Eq (221) and the relation $J = Ma$ we get the mass formula

$$M = \left\{ \frac{4\pi J^2}{N} + \frac{\pi}{N} \left( \frac{N}{4\pi} + 2Q^2_E \right) \left( \frac{N}{4\pi} + 2Q^2_M \right) \right\}^{\frac{1}{2}}$$  \hspace{1cm} (226)

The above $M$ is a homogeneous function of degree $\frac{1}{2}$ of the variables $x^1 = J$, $x^2 = N$, $x^3 = Q^2_E$ and $x^4 = Q^2_M$. Therefore by applying to it Euler’s theorem on homogeneous functions and by taking the differential of $M$ we get respectively

$$M = 2x^i \frac{\partial M}{\partial x^i} \quad \text{and} \quad dM = \frac{\partial M}{\partial x^i} dx^i$$  \hspace{1cm} (227)

Since we have

$$\frac{\partial M}{\partial J} = \frac{4\pi J}{MN} = \frac{a}{r_H^+(r_H^+ + b) + a^2}$$  \hspace{1cm} (228)
\[
\frac{\partial M}{\partial Q_E} = \frac{1}{4M} \left(1 + \frac{8\pi Q_E^2}{N}\right) = \frac{r_H^+}{2\{r_H^+(r_H^+ + b) + a^2\}} \tag{229}
\]

\[
\frac{\partial M}{\partial Q_M} = \frac{1}{4M} \left(1 + \frac{8\pi Q_M^2}{N}\right) = \frac{r_H^+ + b}{2\{r_H^+(r_H^+ + b) + a^2\}} \tag{230}
\]

\[
\frac{\partial M}{\partial N} = \frac{1}{2M} \left(\frac{4\pi J^2}{N^2} - \frac{4\pi Q_E^2 Q_M^2}{N^2}\right) = \frac{r_H^+ - r_H^+}{16\pi \{r_H^+(r_H^+ + b) + a^2\}} \tag{231}
\]

we find that \( M \) and \( dM \) are given by the relations

\[
M = 2J\Omega + Q_E\phi_E + Q_M\phi_M + \frac{\kappa}{4\pi} N \tag{232}
\]

\[
dM = \Omega dJ + \phi_E dQ_E + \phi_M dQ_M + \frac{\kappa}{8\pi} dN \tag{233}
\]

where the area of the outer horizon \( N \), the angular velocity \( \Omega \) and the surface gravity \( \kappa \) are given by Eqs (104), (108) and (106) respectively, and \( \phi_E \) and \( \phi_M \), which are given by the relations

\[
\phi_E = \frac{Q_E r_H^+}{r_H^+(r_H^+ + b) + a^2}, \quad \phi_M = \frac{Q_M(r_H^+ + b)}{r_H^+(r_H^+ + b) + a^2} \tag{234}
\]

are the electric potential and the magnetic potential on the outer horizon respectively. Eq. (232) is a mass formula of Smarr’s type [15] and Eq. (233) a differential mass formula. Expressions (226), (232) and (233) are the mass formulae of our solution.

According to Eq. (166) the metric \( g_{\mu\nu} \) of our “new solution” is obtained from the metric of our solution if we make the substitutions of Eq. (215). Also its angular momentum is again \( J = Ma \). Since the mass formulae were obtained from the metric and the relation \( J = Ma \) we get for our “new solution” the relations

\[
M = \left\{\frac{4\pi J^2}{N} + \frac{\pi N}{4\pi} + 2Q^2\right\}\left(\frac{N}{4\pi} + 2Q^2\right)^{\frac{1}{2}} \tag{235}
\]

\[
M = 2J\Omega + Q\phi + Q'\phi' + \frac{\kappa}{4\pi} N \tag{236}
\]

\[
dM = \Omega dJ + \phi dQ + \phi' dQ' + \frac{\kappa}{8\pi} dN \tag{237}
\]

34
where the area of the outer horizon $N$ the angular velocity $\Omega$ and the surface gravity $\kappa$ are given by Eqs (104), (108) and (106) respectively, the potentials $\phi$ and $\phi'$ by the relations

$$\phi = \frac{Q r_+^H}{r_+^H(r_+^H + b) + a^2}$$

and

$$\phi' = \frac{Q'(r_+^H + b)}{r_+^H(r_+^H + b) + a^2}$$

(238)

and $b$ by the relation

$$b = \frac{Q^2 - Q'^2}{M}$$

(239)

The solution of Sen with electric charge only is obtained from our solution for $Q_M = 0$. Therefore the mass formulae of this solution of Sen are obtained from the mass formulae of our solution for $Q_M = 0$ that is from the expressions

$$M = \left\{ \frac{4\pi J^2}{N} + \frac{1}{4\pi} \left( N + 2Q_E^2 \right) \right\}^{\frac{1}{2}}$$

(240)

$$M = 2J\Omega + Q_E\phi_E + \frac{\kappa}{4\pi}N$$

(241)

$$dM = \Omega dJ + \phi_E dQ_E + \frac{\kappa}{8\pi}dN$$

(242)

which are obtained from Eqs (226), (232) and (233) respectively for $Q_M = 0$. The quantities $J$, $\Omega$, $N$, $\kappa$ and $\phi_E$ in Eqs (240)-(242) are given by Eqs (92), (108), (104), (106) and (234) respectively and $b = \frac{Q^2}{M^2}$.

Since the solution of Sen with electric charge $q_E$ and magnetic charge $q_M$ is obtained from our “new solution” for $Q' = 0$ the mass formulae of this solution of Sen are obtained if we make the substitutions $Q' = 0$ and $Q^2 = q_E^2 + q_M^2$ in the mass formulae of our “new solution”, which are given by Eqs (235)-(239).

We have proven that Eqs (226), (232) and (233) hold for solutions with a metric $g_{\mu\nu}$ of the form of Eqs (19), (25) and (26) and $J = Ma \neq 0$. By the same method we can show that they also hold for solutions with a metric of the same form and $J = 0$. The solution given by Eqs (3) and (4), which is the same with the solution A7 of the Appendix and which has $J = 0$, is obtained from our solution if we put $a = 0$, make the coordinate transformation of Eq. (127) and drop the primes. Since the coordinate transformation gives the same metric with the charges $Q_E$ and $Q_M$ interchanged the expressions for $M$ and $dM$ of this solution are obtained from Eqs (226), (232) and (233).
if we put \( J = 0 \) and interchange the charges \( Q_E \) and \( Q_M \) and their potentials \( \phi_E \) and \( \phi_M \), that is from the relations

\[
M = \left\{ \frac{\pi}{N} \frac{N}{4\pi} + 2Q_E^2 \left( \frac{N}{4\pi} + 2Q_M^2 \right) \right\}^{\frac{1}{2}} \tag{243}
\]

\[
M = Q_E\phi_E + Q_M\phi_M + \frac{\kappa}{4\pi} N \tag{244}
\]

\[
dM = \phi_E dQ_E + \phi_M dQ_M + \frac{\kappa}{8\pi} dN \tag{245}
\]

In the above expressions the quantity \( b \), the area of the outer horizon \( N \) and the surface gravity \( \kappa \) are given by the relations

\[
b = \frac{Q_M^2 - Q_E^2}{M}, \quad N = 4\pi r_+^H (r_+^H + b), \quad k = \frac{r_+^H - r_-^H}{2r_+^H (r_+^H + b)} \tag{246}
\]

and the electric potential \( \phi_E \) and the magnetic potential \( \phi_M \) by the relations

\[
\phi_E = \frac{Q_E}{r_+^H}, \quad \phi_M = \frac{Q_M}{r_+^H + b} \tag{247}
\]

which are obtained from Eq. (234) if we put \( a = 0 \) and make the interchanges \( \phi_E \to \phi_M \) and \( \phi_M \to \phi_E \).

If \( a = 0 \) we get from Eqs (19), (25) and (26)

\[
g_{rr} = \rho^2 \Delta = \frac{r(r+b)}{(r-r_+)(r-r_-)} \tag{248}
\]

where \( r_+ > r_- \). Therefore if \( r_- = -b \) the solution has only one horizon at \( r = r_+ \). This is the case of the solution of STW, the solution of GM-GHS and the solution of Schwarzschild. In this case for solutions which are obtained from our solution for some values of its parameters Eqs (226) and (232)-(234) hold, while for solutions which are obtained from our “new solution” Eqs (235)-(239) hold.

The solution of STW is obtained from our “new solution” for \( a = 0 \) and \( Q_E = 0 \) or \( Q = 0 \) in the notation of Eq. (235). If the STW solution has electric charge \( Q_E \) and magnetic charge \( Q_M \) Eqs (235)-(239) hold with \( Q = 0 \) and \( Q^2 = Q_E^2 + Q_M^2 \). These equations give

\[
M = \frac{1}{2} \left\{ \frac{N}{4\pi} + 2(Q_E^2 + Q_M^2) \right\}^{\frac{1}{2}} \tag{249}
\]
\[ M = Q_E \phi_E + Q_M \phi_M + \frac{\kappa}{4\pi} N \]  
\[ dM = \phi_E dQ_E + \phi_M dQ_M + \frac{\kappa}{8\pi} dN \]  
\[ \phi_E = \frac{Q_E}{r_+}, \quad \phi_M = \frac{Q_M}{r_+}, \quad b = -\frac{Q_E^2 + Q_M^2}{M} \]  
and since \( r_- = -b = \frac{Q_E^2 + Q_M^2}{M} \) Eqs (104) and (106) give for the area of the horizon \( N \) and the surface gravity \( \kappa \)

\[ N = 4\pi r_+ \left( r_+ - \frac{Q_E^2 + Q_M^2}{M} \right) \quad \text{and} \quad \kappa = \frac{1}{2r_+} \]  
Eq. (251) is the differential mass formula and Eq. (250) is the Smarr’s type mass formula of the STW solution.

The solution of GM-GHS is obtained from our solution for \( a = Q_E = 0 \). Therefore from Eqs (226) and (232)-(233) we get

\[ M = \frac{1}{2} \left( \frac{N}{4\pi} + 2Q_M^2 \right)^{1/2} \]  
\[ M = Q_M \phi_M + \frac{\kappa}{4\pi} N \]  
\[ \phi_M = \frac{Q_M}{r_+}, \quad \kappa = \frac{1}{2r_+} \]  
where \( r_- = -b = \frac{Q_M^2}{M} \) and

\[ N = 4\pi r_+ \left( r_+ - \frac{Q_M^2}{M} \right), \quad \phi_M = \frac{Q_M}{r_+}, \quad \kappa = \frac{1}{2r_+} \]  

The solution of Schwarzschild is obtained from our solution for \( a = Q_E = Q_M = 0 \). Therefore since \( r_- = 0 \) for this solution Eqs (232)-(233) give the well known expressions

\[ M = \frac{\kappa}{4\pi} N, \quad dM = \frac{\kappa}{8\pi} dN \]  
where \( N = 4\pi r_+^2 \), and \( \kappa = \frac{1}{2r_+} \).

Consider now the case in which the metric \( g_{\mu\nu} \) is of the form of Eq. (19) and \( \Delta \) and \( \rho^2 \) are given by Eq. (27), that is we don’t have necessarily \( b = \frac{Q_E^2 - Q_M^2}{M} \) and \( c = 2Q_M^2 \). If we consider solutions with angular momentum
\( J = Ma \), where \( a \) is a non-zero or zero constant and proceed as before we get instead of Eq. (226) the relation

\[
M = \left\{ \frac{4\pi J^2}{N} + \frac{\pi}{N} \left( \frac{N}{4\pi} + c \right) \left( \frac{N}{4\pi} + c + 2Mb \right) \right\}^{\frac{1}{2}} \tag{259}
\]

If we have a solution we know \( b \) and \( c \) as functions of \( M \) and the charges of the solution. Therefore taking the differential of the above expression we get the differential mass formula of this solution. Also if the function \( M \) of Eq. (259) is a homogeneous function of its arguments we can use Euler’s theorem on homogeneous functions and get the Smarr’s type mass formula of this solution.

As an example consider the Kerr-Newman solution with arbitrary electric charge \( Q_E \) and arbitrary magnetic charge \( Q_M \), which has \( b = 0 \) and \( c = Q_E^2 + Q_M^2 \equiv Q^2 \). Then Eq. (259) for this solution takes the form

\[
M = \left\{ \frac{4\pi J^2}{N} + \frac{\pi}{N} (\frac{N}{4\pi} + Q^2)^2 \right\}^{\frac{1}{2}} \tag{260}
\]

and is a homogeneous function of its arguments \( x^1 = J, x^2 = N \) and \( x^3 = Q^2 \) of degree \( \frac{1}{2} \). Therefore proceeding as before we get the following expressions for the Smarr’s type mass formula \( M \) and the differential mass formula \( dM \)

\[
M = 2J\Omega + \frac{Q^2 r_+}{(r_+^H)^2 + a^2} + \frac{\kappa}{4\pi} N = 2J\Omega + Q_E \phi_E + Q_M \phi_M + \frac{\kappa}{4\pi} N \tag{261}
\]

\[
dM = \Omega dJ + \frac{r_+^H dQ^2}{2 \{(r_+^H)^2 + a^2\}} + \frac{\kappa}{8\pi} dN = \Omega dJ + \phi_E dQ_E + \phi_M dQ_M + \frac{\kappa}{8\pi} dN \tag{262}
\]

where the angular velocity \( \Omega \) and the surface gravity \( \kappa \) are given by Eqs (108) and (106) respectively and the electric potential \( \phi_E \) and the magnetic potential \( \phi_M \) on the outer horizon are given by the relations

\[
\phi_E = \frac{Q_E r_+^H}{(r_+^H)^2 + a^2}, \quad \phi_M = \frac{Q_M r_+^H}{(r_+^H)^2 + a^2} \tag{263}
\]

We obtain the mass formulae of the solution of Kerr from Eqs (260)-(263) for \( Q_E = Q_M = 0 \) and the mass formulae of the solution of Reissner-Nordström for arbitrary electric charge \( Q_E \) and magnetic charge \( Q_M \) from the same equations for \( J = 0 \).
A Appendix

List of Solutions with $g_{\mu\nu}, F_{\mu\nu}$ and $\lambda$ of the Form Presented in the Paper

In this Appendix we shall present the solutions found in this paper and a list of known solutions in order to show that these known solutions have metric, vector field and $\lambda = \xi + ie^{-2\phi}$ of the form of Eqs (19), (35) and (175) respectively and in order to make clear and simple the comparison of our solutions with the known solutions. To make this Appendix easier to use we have repeated in it some formulas already existing in the paper.

All solutions of the Appendix have the metric

$$g_{\mu\nu} = \begin{pmatrix} -\frac{\Delta}{\rho^2} & 0 & 0 & \frac{a(\Delta - \rho^2) \sin^2 \theta}{\rho^2} \\ 0 & \frac{\rho^2}{\Delta + a^2 \sin^2 \theta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ \frac{a(\Delta - \rho^2) \sin^2 \theta}{\rho^2} & 0 & 0 & \sin^2 \theta (\rho^2 + 2a^2 \sin^2 \theta) - \frac{a^2 \Delta \sin^4 \theta}{\rho^2} \end{pmatrix}$$

which is a function of $a$ and two functions $\rho^2$ and $\Delta$. The functions $\rho^2$ and $\Delta$ are of the form

$$\rho^2 = r(r + b) + a^2 \cos^2 \theta \quad (265)$$

$$\Delta = r(r + b) - 2Mr + c + a^2 \cos^2 \theta = \rho^2 - 2Mr + c \quad (266)$$

where $M$ and $a$ are the mass and the rotation parameter of the solution and $b$ and $c$ are constants. From the above metric we get

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\frac{\Delta}{\rho^2} dt^2 + \frac{\rho^2}{\Delta + a^2 \sin^2 \theta} dr^2 + \rho^2 d\theta^2 + \frac{2a(\Delta - \rho^2) \sin^2 \theta}{\rho^2} dtd\phi + \{\sin^2 \theta (\rho^2 + 2a^2 \sin^2 \theta) - \frac{a^2 \Delta \sin^4 \theta}{\rho^2}\} d\phi^2 \quad (267)$$

$$g^{\mu\nu} = \begin{pmatrix} -\frac{\rho^2 + 2a^2 \sin^2 \theta - \frac{a^2 \Delta \sin^2 \theta}{\rho^2}}{\Delta + a^2 \sin^2 \theta} & 0 & 0 & \frac{a(\Delta - \rho^2)}{\rho^2 (\Delta + a^2 \sin^2 \theta)} \\ 0 & \frac{\Delta + a^2 \sin^2 \theta}{\rho^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\rho^2} & 0 \\ \frac{a(\Delta - \rho^2)}{\rho^2 (\Delta + a^2 \sin^2 \theta)} & 0 & 0 & \frac{\Delta}{\rho^2 (\Delta + a^2 \sin^2 \theta) \sin^2 \theta} \end{pmatrix} \quad (268)$$

$$|g_{\mu\nu}| = \frac{1}{|g^{\mu\nu}|} = -\rho^4 \sin^2 \theta \quad (269)$$
All solutions of this Appendix have vector field, or vector fields if they have two such fields, which can be expressed by the relations

\[ F_{rt} = \zeta, \quad F_{r\phi} = -a \sin^2 \theta \zeta, \quad F_{\theta t} = -a \sin \theta \eta, \quad F_{\theta \phi} = \{r(r+b) + a^2\} \sin \theta \eta \]

\[ F_{r\theta} = F_{t\phi} = 0 \]  

(270)
in terms of the rotation parameter \( a \) the constant \( b \) and two functions \( \zeta \) and \( \eta \).

Also if from the axion field \( \xi \) and the dilaton field \( \phi \) with asymptotic values \( \xi_\infty = \phi_\infty = 0 \) we construct the complex quantity \( \lambda = \xi + ie^{-2\phi} \) all solutions of this Appendix with \( \phi \neq 0 \) have a \( \lambda \) of the form

\[ \lambda = \frac{i\{r + iy + (1 - d)b\}}{r + iy + db} \]  

(271)

where \( y \) is define by the relation

\[ y = a \cos \theta \]  

(272)

and \( d \) is a constant real or complex. For each solution we shall give the corresponding action \( S \). The solution we are talking about is a solution of the equations of motion coming from this action. Also we shall indicate if for this solution the rotation parameter \( a \) is vanishing or not (non-rotating or rotating solution) and to simplify the notation we shall leave the parameter \( b \) in \( \lambda, \xi, e^{-2\phi}, \zeta, \eta, \zeta' \) and \( \eta' \).

### A.1 Our solution (Section 6 of the Paper)

Action \( S \)

\[ S = \int d^4x \sqrt{-g} \left\{ R - 2\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} e^{4\phi} \partial_\mu \xi \partial^\mu \xi - e^{-2\phi} (F_{\mu\nu} F^{\mu\nu} + F'_{\mu\nu} F'^{\mu\nu}) ight. \\
- \left. \xi (F_{\mu\nu} F^{*\mu\nu} + F'_{\mu\nu} F'^{*\mu\nu}) \right\}, \quad F^{*\mu\nu} = \frac{1}{2\sqrt{-g}} \epsilon^{\mu\nu\sigma\tau} F_{\sigma\tau} \]  

(273)

Parameters

\[ M, \quad a \neq 0, \quad b = \frac{Q_E^2 - Q_M^2}{M}, \quad c = 2Q_M^2, \quad d = 0, \quad Q_E, \quad Q_M \]  

(274)
Functions $\rho^2$, $\Delta$

$$
\rho^2 = r(r + \frac{Q_E^2 - Q_M^2}{M}) + y^2
$$

$$
\Delta = r(r + \frac{Q_E^2 - Q_M^2}{M}) - 2Mr + 2Q_M^2 + y^2
$$

Fields and field factors

$$
\lambda = \frac{i(r + iy + b)}{r + iy}, \quad \xi = \frac{by}{r^2 + y^2}, \quad e^{-2\phi} = \frac{\rho^2}{r^2 + y^2}
$$

$$
\zeta = \frac{Q_E(r^2 - y^2)}{\rho^4}, \quad \eta = \frac{2QEry}{\rho^4}, \quad \zeta' = -\frac{Q_My(2r + b)}{\rho^4}, \quad \eta' = \frac{Q_M\{r(r + b) - y^2\}}{\rho^4}
$$

Arbitrary parameters

$M$, $a$, $Q_E$, and $Q_M$: arbitrary parameters in the metric and in the solution

A.2 Our “new solution” (Section 9 of the Paper)

Action $S$

$$
S = \int d^4x\sqrt{-g}\{R - 2\partial_{\mu}\phi\partial^{\mu}\phi - \frac{1}{2}e^{4\phi}\partial_{\mu}\xi\partial^{\mu}\xi - e^{-2\phi}(F_{\mu\nu}F^{\mu\nu} + F'_{\mu\nu}F'^{\mu\nu})

- \xi(F_{\mu\nu}F^{\ast\mu\nu} + F'_{\mu\nu}F'^{\ast\mu\nu})\}, \quad F_{\mu\nu} = \frac{1}{2\sqrt{-g}}\epsilon_{\mu\nu\sigma\tau}F_{\sigma\tau}
$$

Parameters

$M$, $a \neq 0$, $b = \frac{Q_E^2 - Q_M^2}{M}$, $c = 2Q_M^2$, $d = \frac{i\delta}{\epsilon + i\delta}$, $\delta$, $\epsilon$,

where $\delta^2 + \epsilon^2 = 1$, $Q_E$, $Q_M$

Functions $\rho^2$, $\Delta$

$$
\rho^2 = r(r + \frac{Q_E^2 - Q_M^2}{M}) + y^2, \quad \Delta = r(r + \frac{Q_E^2 - Q_M^2}{M}) - 2Mr + 2Q_M^2 + y^2
$$

Fields and field factors

$$
\lambda = \frac{i\{(\epsilon + i\delta)(r + iy) + \epsilon b\}}{(\epsilon + i\delta)(r + iy) + i\delta b}
$$
\[ \xi = \frac{b\delta(2r + b) + (e^2 - \delta) y}{\delta^2((r + b)^2 + y^2) + e^2(r^2 + y^2) + 2\delta eby} \]  

\[ e^{-2\phi} = \frac{\rho^2}{\delta^2((r + b)^2 + y^2) + e^2(r^2 + y^2) + 2\delta eby} \]

\[ \zeta = \frac{Q_E}{\rho^4}\{\delta y(2r + b) - \epsilon(r^2 - y^2)\}, \quad \eta = -\frac{Q_E}{\rho^4}\{\delta(r(r + b) - y^2) + 2\epsilon ry\} \]

\[ \zeta' = \frac{Q_M}{\rho^4}\{\delta((r + b)^2 - y^2) + \epsilon(2r + b)y\}, \quad \eta' = \frac{Q_M}{\rho^4}\{2\delta y(r + b) - \epsilon(r(r + b) - y^2)\} \]

Arbitrary parameters

\[ M, \ a, \ Q_E, \ Q_M : \text{arbitrary parameters of the metric} \]

\[ M, \ a, \ Q_E, \ Q_M, \ \delta : \text{arbitrary parameters of the solution} \]

The quantities \( \lambda''', \xi''', \phi''', \sigma, \tau, \sigma', \tau' \) of Section 9 were renamed \( \lambda, \xi, \phi, \zeta, \eta, \zeta' \) and \( \eta' \) here.

### A.3 Solution of Sen with electric charge \( Q_E \) only (Phys. Rev. Lett. 69, 1006 (1992))

Action \( S \)

\[ S = \int d^4x \sqrt{-g}\left\{ R - 2\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} e^{4\phi} \partial_\mu \xi \partial^\mu \xi - e^{-2\phi} F_{\mu\nu} F^{\mu\nu} \right\} \]

\[ - \xi F^{*}_{\mu\nu} F^{*\mu\nu} \}, \quad \xi' F^{*}_{\mu\nu} F^{*\mu\nu} = \frac{1}{2\sqrt{-g}} \epsilon^{\mu\nu\sigma\tau} F_{\sigma\tau} \]  

Parameters

\[ M, \ a \neq 0, \ b = \frac{Q_E^2}{M}, \ c = 0, \ d = 0, \ Q_E \]

Functions \( \rho^2, \ \Delta \)

\[ \rho^2 = r(r + \frac{Q_E^2}{M}) + y^2; \quad \Delta = r(r + \frac{Q_E^2}{M}) - 2Mr + y^2 \]

Fields and field factors

\[ \lambda = \frac{i(r + iy + b)}{r + iy}, \quad \xi = \frac{by}{r^2 + y^2}, \quad e^{-2\phi} = \frac{\rho^2}{r^2 + y^2} \]
\[ \zeta = \frac{Q_E(r^2 - y^2)}{\rho^4}, \quad \eta = \frac{2Q_Ery}{\rho^4} \] (290)

Arbitrary parameters
\( M, \ a, \ Q_E \) : arbitrary parameters in the metric and in the solution

A.4 Solution of Sen with electric charge \( Q_E \) and magnetic charge \( Q_M \) (Ref. [13]. The complete solution is not given explicitly)

Action \( S \)
\[ S = \int d^4x \sqrt{-g} \{ R - 2\partial_{\mu}\phi\partial^{\mu}\phi - \frac{1}{2} e^{4\phi} \partial_{\mu}\xi\partial^{\mu}\xi - e^{-2\phi} F_{\mu\nu}F^{\mu\nu} \]
\[ - \xi F_{\mu\nu}F^{*\mu\nu} \}, \quad F^{*\mu\nu} = \frac{1}{2\sqrt{-g}} \epsilon^{\mu\nu\sigma\tau} F_{\sigma\tau} \] (291)

Parameters
\( M, \ a \neq 0, \ b = \frac{Q_E^2 + Q_M^2}{M}, \ c = 0, \ d = \frac{iQ_M}{Q_E + iQ_M}, \ Q_E, \ Q_M \) (292)

Functions \( \rho^2, \ \Delta \)
\[ \rho^2 = r(r + \frac{Q_E^2 + Q_M^2}{M}) + y^2, \quad \Delta = r(r + \frac{Q_E^2 + Q_M^2}{M}) - 2Mr + y^2 \] (293)

Fields and field factors
\[ \lambda = \frac{i\{(Q_E + iQ_M)(r + iy) + Q_Eb\}}{(Q_E + iQ_M)(r + iy) + iQ_Mb} \] (294)
\[ \xi = \frac{b\{Q_EQ_M(2r + b) + (Q_E^2 - Q_M^2)y\}}{Q_M^2\{(r + b)^2 + y^2\} + Q_E^2(r^2 + y^2) + 2Q_EQ_Mby} \] (295)
\[ e^{-2\phi} = \frac{(Q_E^2 + Q_M^2)\rho^2}{Q_M^2\{(r + b)^2 + y^2\} + Q_E^2(r^2 + y^2) + 2Q_EQ_Mby} \] (296)
\[ \zeta = \frac{Q_E(r^2 - y^2) - Q_M(2r + b)y}{\rho^4}, \quad \eta = \frac{Q_M\{r(r + b) - y^2\} + 2Q_EQ_My}{\rho^4} \] (297)

Arbitrary parameters
\( M, \ a, \ Q_E^2 + Q_M^2 \) : arbitrary parameters of the metric
\( M, \ a, \ Q_E, \ Q_M \) : arbitrary parameters of the solution
A.5  Kerr-Newman solution for electric charge $Q_E$ and magnetic charge $Q_M$ (Ref. [12])

Action $S$

$$S = \int d^4x \sqrt{-g} (R - F_{\mu\nu}F^{\mu\nu})$$  \hspace{1cm} (298)

Parameters

$$M, \ a \neq 0, \ b = 0, \ c = Q_E^2 + Q_M^2, \ Q_E, \ Q_M$$  \hspace{1cm} (299)

Functions $\rho^2$, $\Delta$

$$\rho^2 = r^2 + y^2, \ \Delta = r^2 - 2Mr + Q_E^2 + Q_M^2 + y^2$$  \hspace{1cm} (300)

Field factors

$$\zeta = \frac{Q_E(r^2 - y^2) - 2Q_Mry}{\rho^4}, \ \eta = \frac{Q_M(r^2 - y^2) + 2Q_Ery}{\rho^4}$$  \hspace{1cm} (301)

Arbitrary parameters

$M, \ a, \ Q_E^2 + Q_M^2$ : arbitrary parameters of the metric

$M, \ a, \ Q_E, \ Q_M$ : arbitrary parameters of the solution

A.6  Solution of Kerr (Phys. Rev. Lett. 11, 237 (1963))

Action $S$

$$S = \int d^4x \sqrt{-g} R$$  \hspace{1cm} (302)

Parameters

$$M, \ a \neq 0, \ b = 0, \ c = 0$$  \hspace{1cm} (303)

Functions $\rho^2$, $\Delta$

$$\rho^2 = r^2 + y^2, \ \Delta = r^2 - 2Mr + y^2$$  \hspace{1cm} (304)

Arbitrary parameters

$M, \ a, \ :$ arbitrary parameters in the metric and in the solution

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A.7 Solution given by Eqs (3) and (4) of this paper

(Class. Quant. Grav. 23, 7591 (2006) Eqs (54)-(57) for \( \psi_0 = 0 \),
\( AB = 2Q_E^2 \), \((\alpha - A)(\alpha - B) = 2Q_M^2 \), \( 2M = \alpha - A - B \), \( \psi = 2\phi \) and \( \alpha = b \)

Action \( S \)

\[
S = \int d^4x \sqrt{-g} \left\{ R - 2\partial_{\mu} \phi \partial^{\mu} \phi - e^{-2\phi} F_{\mu\nu} F^{\mu\nu} \right\} \tag{305}
\]

Parameters

\[
M, \ a = 0, \ b = \frac{Q_M^2 - Q_E^2}{M}, \ c = 2Q_E^2, \ d = 1, \ Q_E, \ Q_M \tag{306}
\]

Functions \( \rho^2, \Delta \)

\[
\rho^2 = r(r + \frac{Q_M^2 - Q_E^2}{M}), \quad \Delta = r(r + \frac{Q_M^2 - Q_E^2}{M}) - 2Mr + 2Q_E^2 \tag{307}
\]

Fields and field factors

\[
\xi = 0, \quad e^{2\phi} = 1 + \frac{b}{r} \tag{308}
\]

\[
\zeta = \frac{Q_E}{r^2}, \quad \eta = \frac{Q_M}{\rho^2} \tag{309}
\]

Arbitrary parameters

\( M, \ Q_E, \ Q_M \) : arbitrary parameters in the metric and in the solution

Line element

\[
ds^2 = -\left\{1 - \frac{2M}{r} + \frac{2Q_M^2 M}{r(Mr + Q_M^2 - Q_E^2)}\right\} dt^2 + \left\{1 - \frac{2M}{r} + \frac{2Q_M^2 M}{r(Mr + Q_M^2 - Q_E^2)}\right\}^{-1} dr^2
\]

\[
+ r(r + Q_M^2 - Q_E^2)(d\theta^2 + \sin^2 \theta d\phi^2) \tag{310}
\]

The solution takes simpler form in terms of the parameters \( A, B, b \) where
\( 2Q_E^2 = AB, \ 2Q_M^2 = (b - A)(b - B), \ 2M = b - A - B \).

Functions \( \rho^2, \Delta \)

\[
\rho^2 = r(r + b), \quad \Delta = (r + A)(r + B) \tag{311}
\]

Fields and field factors

\[
\xi = 0, \quad e^{2\phi} = 1 + \frac{b}{r} \tag{312}
\]

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\[ \zeta = \frac{\sqrt{AB}}{\sqrt{2}r^2}, \quad \eta = \frac{\sqrt{(b - A)(b - B)}}{\sqrt{2}\rho^2} \] (313)

Arbitrary parameters
\( A < 0, \; B < 0, \; b > 0 \): arbitrary parameters in the metric and in the solution

Line element
\[ ds^2 = -\frac{(r + A)(r + B)}{r(r + b)}dt^2 + \frac{r(r + b)}{(r + A)(r + B)}dr^2 + r(r + b)(d\theta^2 + \sin^2\theta d\phi^2) \] (314)

A.8 Solution of Shapere, Trivedi and Wilczek (Mod. Phys. Lett. A6,2677 (1991))

Action
\[ S = \int d^4x \sqrt{-g} \left\{ R - 2\partial_\mu \phi \partial^{\mu} \phi - \frac{1}{2} e^{4\phi} \partial_\mu \xi \partial^{\mu} \xi - e^{-2\phi} F_{\mu\nu} F^{\mu\nu} \right\} - \xi F_{\mu\nu} F^{*\mu\nu}, \quad F^{*\mu\nu} = \frac{1}{2\sqrt{-g}} e^{\mu\nu\sigma\tau} F_{\sigma\tau} \] (315)

Parameters
\( M, \; a = 0, \; b = -\frac{Q_E^2 + Q_M^2}{M}, \; c = 2(Q_E^2 + Q_M^2), \; d = -\frac{iQ_E}{Q_M - iQ_E}, \; Q_E, \; Q_M \) (316)

Functions \( \rho^2, \; \Delta \)
\[ \rho^2 = r(r - \frac{Q_E^2 + Q_M^2}{M}), \] (317)
\[ \Delta = r(r - \frac{Q_E^2 + Q_M^2}{M}) - 2Mr + 2(Q_E^2 + Q_M^2) = (r - 2M)(r - \frac{Q_E^2 + Q_M^2}{M}) \] (318)

Fields and field factors [14]
\[ \lambda = \frac{i\{(Q_M - iQ_E)r + Q_Mb\}}{(Q_M - iQ_E)r - iQ_Eb} \] (319)
\[ \xi = -\frac{Q_EQ_M b(2r + b)}{Q_E^2(r + b)^2 + Q_M^2 r^2}, \quad e^{-2\phi} = \frac{(Q_E^2 + Q_M^2)\rho^2}{Q_E^2(r + b)^2 + Q_M^2 r^2} \] (320)
\[ \zeta = \frac{Q_E}{r^2}, \quad \eta = \frac{Q_M}{\rho^2} \] (321)
Arbitrary parameters
\( M, \ Q_E + Q_M^2 \) : arbitrary parameters of the metric
\( M, \ Q_E, \ Q_M \) : arbitrary parameters of the solution

Line element
\[
dt^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r(r - \frac{Q_E^2 + Q_M^2}{M})(d\theta^2 + \sin^2 \theta d\phi^2)
\]  
\( (322) \)

A.9 GM-GHS Solution (G. Gibbons, Nucl. Phys. B207, 337 (1982); G. Gibbons and Maeda Nucl. Phys. B298, 741 (1988); D. Garfinkle, G. Horowitz and A. Strominger, Phys. Rev. D43, 3140 (1991) for \( \phi_0 = 0 \); Erratum Phys. Rev. D45, 3888 (1992) )

Action \( S \)
\[
S = \int d^4x \sqrt{-g} \left\{ R - 2\partial_\mu \phi \partial^\mu \phi - e^{-2\phi} F_{\mu\nu} F^{\mu\nu} \right\}
\]  
\( (323) \)

Parameters
\[
M, \ a = 0, \ b = -\frac{Q_M^2}{M}, \ c = 2Q_M^2, \ d = 0, \ Q_M
\]  
\( (324) \)

Functions \( \rho^2, \Delta \)
\[
\rho^2 = r(r - \frac{Q_M^2}{M})
\]
\[
\Delta = r(r - \frac{Q_M^2}{M}) - 2Mr + 2Q_M^2 = (r - 2M)(r - \frac{Q_M^2}{M})
\]  
\( (325) \)

Fields and field factors
\[
\xi = 0, \quad e^{-2\phi} = 1 + \frac{b}{r}, \quad \zeta = 0, \quad \eta = \frac{Q_M}{\rho^2}
\]  
\( (326) \)

Arbitrary parameters
\( M, \ Q_M \) : arbitrary parameters in the metric and in the solution

Line element
\[
dt^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r(r - \frac{Q_M^2}{M})(d\theta^2 + \sin^2 \theta d\phi^2)
\]  
\( (327) \)
A.10  Reissner-Nordström solution for electric charge $Q_E$ and magnetic charge $Q_M$

Action $S$

$$S = \int d^4x \sqrt{-g}(R - F_{\mu\nu}F^{\mu\nu})$$ (328)

Parameters

$$M, \ a = 0, \ b = 0, \ c = Q_E^2 + Q_M^2, \ Q_E, \ Q_M$$ (329)

Functions $\rho^2, \ \Delta$

$$\rho^2 = r^2, \ \Delta = r^2 - 2Mr + Q_E^2 + Q_M^2$$ (330)

Field factors

$$\zeta = \frac{Q_E}{r^2}, \ \eta = \frac{Q_M}{r^2}$$ (331)

Arbitrary parameters

$M, \ Q_E^2 + Q_M^2$: arbitrary parameters of the metric

$M, \ Q_E, \ Q_M$: arbitrary parameters of the solution

Line element

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q_E^2 + Q_M^2}{r^2})dt^2 + (1 - \frac{2M}{r} + \frac{Q_E^2 + Q_M^2}{r^2})^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$ (332)

A.11  Schwarzschild solution

Action $S$

$$S = \int d^4x \sqrt{-g}R$$ (333)

Parameters

$$M, \ a = 0, \ b = 0, \ c = 0$$ (334)

Functions $\rho^2, \ \Delta$

$$\rho^2 = r^2, \ \Delta = r^2 - 2Mr$$ (335)

Line element

$$dt^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$ (336)
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The axion field of this solution and the axion field of Eq. (193) have opposite signs. This sign difference is corrected if we take in our paper $\epsilon_{0123} = -1$. The axion field of the solution A8 of the Appendix has the sign of the axion field of Eq. (193).

To make the notation simpler the function $(\Delta)_{\text{paper}}$ of the present paper is slightly different from the function $(\Delta)_{\text{MTW}}$ of the book of C. W. Misner, K. S. Thorne and J. A. Wheeler, 1973 Gravitation, (W. H. Freeman and Company, San Francisco) and of several other books, which define $\Delta$ as in the book of C. W. Misner et. al. It is $(\Delta)_{\text{MTW}} = (\Delta)_{\text{paper}} + a^2 \sin^2 \theta$.

The computation of the Ricci tensor $R_{\mu\nu}$, the Ricci scalar $R$ and the curvature scalar $R^\mu\nu\sigma\tau R_{\mu\nu\sigma\tau}$ was done with the help of a program given to me by S. Bonanos, whom I thank.

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