ON $q$-DE RHAM COHOMOLOGY VIA $\Lambda$-RINGS

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Abstract. We show that Aomoto’s $q$-deformation of de Rham cohomology arises as a natural cohomology theory for $\Lambda$-rings. Moreover, Scholze’s $(q-1)$-adic completion of $q$-de Rham cohomology depends only on the Adams operations at each residue characteristic. This gives a fully functorial cohomology theory, including a lift of the Cartier isomorphism, for smooth formal schemes in mixed characteristic equipped with a suitable lift of Frobenius. If we attach $p$-power roots of $q$, the resulting theory is independent even of these lifts of Frobenius, refining a comparison by Bhatt, Morrow and Scholze.

Introduction

The $q$-de Rham cohomology of a polynomial ring is a $\mathbb{Z}[q]$-linear complex given by replacing the usual derivative with the Jackson $q$-derivative $\nabla_q(x^n) = [n]_q x^{n-1} dx$, where $[n]_q$ is Gauss’ $q$-analogue $\frac{q^n - 1}{q-1}$ of the integer $n$. In [Sch2], Scholze discussed the $(q-1)$-adic completion of this theory for smooth rings, explaining relations to $p$-adic Hodge theory and singular cohomology, and conjecturing that it is independent of co-ordinates.

We show that $q$-de Rham cohomology naturally arises as a functorial invariant of $\Lambda$-rings (Theorem 1.11), and that its $(q-1)$-adic completion depends only on a $\Lambda_P$-ring structure (Theorem 2.7), for $P$ the set of residue characteristics; a $\Lambda_P$-ring has a lift of Frobenius for each $p \in P$. This recovers the known equivalence between de Rham cohomology and complete $q$-de Rham cohomology over the rationals, while giving no really new functoriality statements for smooth schemes over $\mathbb{Z}$. However, in mixed characteristic, it means that complete $q$-de Rham cohomology depends only on a lift $\Psi_P$ of absolute Frobenius locally generated by co-ordinates with $\Psi_P(x_i) = x_i^p$. Given such data, we construct (Proposition 2.8) a quasi-isomorphism between Hodge cohomology and $q$-de Rham cohomology modulo $[p]_q$, extending the local lift of the Cartier isomorphism in [Sch2] Proposition 3.4).

Taking the Frobenius stabilisation of the complete $q$-de Rham complex of $A$ yields a complex resembling the de Rham–Witt complex. We show (Theorem 3.10) that up to $(q^{1/p^\infty} - 1)$-torsion, the $p$-adic completion of this complex depends only on the $p$-adic completion of $A[\zeta_{p^\infty}]$ (where $\zeta_n$ denotes a primitive $n$th root of unity), with no requirement for a lift of Frobenius or a choice of co-ordinates. The main idea is to show that the stabilised $q$-de Rham complex is in a sense given by applying Fontaine’s period ring construction $A_{\text{inf}}$ to the best possible perfectoid approximation to $A[\zeta_{p^\infty}]$. As a consequence, this shows (Corollary 3.11) that after attaching all $p$-power roots of $q$, $q$-de Rham cohomology in mixed characteristic is independent of choices, which was already known after base change to a period ring, via the comparisons of [BMS] between $q$-de Rham cohomology their theory $A_{\Omega^\bullet}$.

We expect that the dependence of these cohomology theories either on Adams operations at the residue characteristics (for de Rham) or on $p$-power roots of $q$ (for variants of de Rham–Witt) is unavoidable, and that the conjectures of [Sch2] might thus be slightly
optimistic. Some of the strongest evidence for the conjectures is provided by the lifts of the Cartier isomorphisms, which rely on a choice of Frobenius. On the other hand, the comparison theorems of [BMS] can be seen as a manifestation of \(q\)-de Rham–Witt complexes; although they do not require a lift of Frobenius, they involve all \(p\)-power roots of \(q\).

The essence of our construction of \(q\)-de Rham cohomology of \(A\) over \(R\) is to set \(q\) to be an element of rank 1 for the \(\Lambda\)-ring structure, and to look at flat \(\Lambda\)-rings \(B\) over \(R[q]\) equipped with morphisms \(A \to B/(q - 1)\) of \(\Lambda\)-rings over \(R\). If these seem unfamiliar, reassurance should be provided by the observation that \((q - 1)B\) carries \(q\)-analogues of divided power operations (Remark 1.4).

For the variants of de Rham–Witt cohomology in \(\S 3\), the key to giving a characterisation independent of lifts of Frobenius is the factorisation of the tilting equivalence for perfectoid algebra via a category of \(\Lambda_p\)-rings.

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1. Comparisons for \(\Lambda\)-rings

We will follow standard notational conventions for \(\Lambda\)-rings. These are commutative rings equipped with operations \(\lambda^i\) resembling alternating powers, in particular satisfying

\[
\lambda^k(a + b) = \sum_{i=0}^k \lambda^i(a)\lambda^{k-i}(b),
\]

with \(\lambda^0(a) = 1\) and \(\lambda^1(a) = a\). For background, see [Bor] and references therein. The \(\Lambda\)-rings we encounter are all torsion-free, in which case the \(\Lambda\)-ring structure is equivalent to giving ring endomorphisms \(\Psi^n\) for \(n \in \mathbb{Z}_{>0}\) with \(\Psi^{mn} = \Psi^m \circ \Psi^n\) and \(\Psi^p(x) \equiv x^p \mod p\) for all primes \(p\). If we write \(\lambda_t(f) := \sum_{i \geq 0} \lambda^i(f)t^i\) and \(\Psi_t(f) := \sum_{n \geq 1} \Psi^n(f)t^n\), then the families of operations are related by the formula

\[
\Psi_t = -i\frac{d\log \lambda^{-1}}{dt}.
\]

We refer to elements \(x\) with \(\lambda^i(x) = 0\) for all \(i > 1\) (or equivalently \(\Psi^n(x) = x^n\) for all \(n\)) as elements of rank 1.
1.1. The $\Lambda$-ring $\mathbb{Z}[q]$.

**Definition 1.1.** Define $\mathbb{Z}[q]$ to be the $\Lambda$-ring with operations determined by setting $q$ to be of rank 1.

We now consider the $q$-analogues $[n]_q := \frac{q^n - 1}{q - 1} \in \mathbb{Z}[q]$ of the integers, with $[n]_q! = [n]_q[n-1]_q \ldots [1]_q$, and $\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!}$.

**Remark 1.2.** To see the importance of regarding $\mathbb{Z}[q]$ as a $\Lambda$-ring observe that the binomial expressions

$$\lambda^k(n) = \binom{n}{k}, \quad \lambda^k(-n) = (-1)^k \binom{n+k-1}{k}$$

have as $q$-analogues the Gaussian binomial theorems

$$\lambda^k([n]_q) = q^{k(k-1)/2} \binom{n}{k}_q, \quad \lambda^k([-n]_q) = (-1)^k \binom{n+k-1}{k}_q,$$

as well as Adams operations

$$\Psi^i([n]_q) = [n]_{q^i}.$$

For any torsion-free $\Lambda$-ring, localisation at a set of elements closed under the Adams operations always yields another $\Lambda$-ring, since $\Psi^i(a^{-1}) - a^p = (\Psi^i(a)a^p)^{-1}(a^p - \Psi^i(a))$ is divisible by $p$.

**Lemma 1.3.** For the $\Lambda$-ring structure on $\mathbb{Z}[x, y]$ with $x, y$ of rank 1, the elements

$$\lambda^n\left(\frac{y-x}{q-1}\right) \in \mathbb{Z}[q, \{(q^n - 1)^{-1}\}_{n \geq 1}, x, y]$$

are given by

$$\lambda^k\left(\frac{y-x}{q-1}\right) = \frac{(y - x)(y - qx) \ldots (y - q^{k-1}x)}{(q - 1)^k [k]_q!} = \sum_{j=0}^{k} q^{j(j-1)/2} (-x)^j y^{k-j} [j]_q! [k-j]_q!.$$

**Proof.** The second expression comes from multiplying out the Gaussian binomial expansions. The easiest way to prove the first is to observe that $\lambda^k\left(\frac{y-x}{q-1}\right)$ must be a homogeneous polynomial of degree $k$ in $x, y$, with coefficients in the integral domain $\mathbb{Z}[q, \{(q^n - 1)^{-1}\}_{n \geq 1}]$, and to note that

$$\lambda^k\left(\frac{y-x}{q-1}\right) = \lambda^k([n]_q x) = q^{k(k-1)/2} \binom{n}{k}_q x^k.$$

Thus $\lambda^k\left(\frac{y-x}{q-1}\right)$ agrees with the homogeneous polynomial above for infinitely many values of $\frac{y}{x}$, so must be equal to it. \qed

**Remark 1.4.** Note that as $q \to 1$, Lemma 1.3 gives $(q - 1)^k \lambda^k\left(\frac{y-x}{q-1}\right) \to \frac{\exp(at) - 1}{\exp(at) - 1}$. Indeed, for any rank 1 element $x$ in a $\Lambda$-ring we have $\lambda_{(q-1)t}\left(\frac{a}{q-1}\right) = e_q(xt)$, the $q$-exponential, with multiplicative and universality then implying that $\lambda_{(q-1)t}\left(\frac{a}{q-1}\right)$ is a $q$-deformation of $\exp(at)$ for all $a$. Thus $(q - 1)^k \lambda^k\left(\frac{y-x}{q-1}\right)$ is a $q$-analogue of the $k$th divided power $(a^k/k!)$. An explicit expression comes recursively from the formula

$$[k]_q(q - 1)\lambda^k\left(\frac{y-x}{q-1}\right) = \sum_{i>0} \lambda^i(a) \lambda^{k-i}\left(\frac{a}{q-1}\right).$$
Lemma 1.5. For elements $x, y$ of rank 1, the $\Lambda$-subring of $\mathbb{Z}[q, \{(q^n - 1)^{-1}\}_{n \geq 1}, x, y]$ generated by $q, x, y, \frac{y - x}{q - 1}$ has basis $\lambda^i\left(\frac{y - x}{q - 1}\right)$ as a $\mathbb{Z}[q, x]$-module.

Proof. The $\Lambda$-subring clearly contains the $\mathbb{Z}[q, x]$-module $M$ generated by the elements $\lambda^i\left(\frac{y - x}{q - 1}\right)$, which are also clearly linearly independent. Since $\mathbb{Z}[x, q]$ is a $\Lambda$-ring, it suffices to show that $M$ is closed under multiplication.

By Lemma 1.3 we know that

$$\lambda^i\left(\frac{y - x}{q - 1}\right) \lambda^j\left(\frac{y - x}{q - 1}\right) = \binom{i + j}{i}_q \lambda^{i+j}\left(\frac{y - x}{q - 1}\right).$$

We can rewrite $\frac{y - x}{q - 1} = \frac{x - y}{q - 1} - [i]_q x$, so $\lambda^i\left(\frac{y - x}{q - 1}\right) - \lambda^j\left(\frac{y - x}{q - 1}\right)$ lies in the $\mathbb{Z}[q, x]$-module spanned by $\lambda^m\left(\frac{y - x}{q - 1}\right)$ for $m < j$. By induction on $j$, it thus follows that

$$\lambda^i\left(\frac{y - x}{q - 1}\right) \lambda^j\left(\frac{y - x}{q - 1}\right) - \lambda^i\left(\frac{y - x}{q - 1}\right) \lambda^j\left(\frac{y - x}{q - 1}\right) \in M,$$

so the binomial expression above implies $\lambda^i\left(\frac{y - x}{q - 1}\right) \lambda^j\left(\frac{y - x}{q - 1}\right) \in M$. \qed

1.2. $q$-cohomology of $\Lambda$-rings.

Definition 1.6. Given a $\Lambda$-ring $R$, say that $A$ is a $\Lambda$-ring over $R$ if it is a $\Lambda$-ring equipped with a morphism $R \to A$ of $\Lambda$-rings. We say that $A$ is a flat $\Lambda$-ring over $R$ if $A$ is flat as a module over the commutative ring underlying $R$.

Definition 1.7. Given a morphism $R \to A$ of $\Lambda$-rings, we define the category $\text{Strat}^q_{A/R}$ to consist of flat $\Lambda$-rings $B$ over $R[q]$ equipped with a compatible morphism $A \to B/(q - 1)$, such that the map $A \to B/(q - 1)$ admits a lift to $B$; a choice of lift is not taken to be part of the data, so need not be preserved by morphisms.

More concisely, $\text{Strat}^q_{A/R}$ is the Grothendieck construction of the functor

$$(\text{Spec} A)^q_{\text{strat}} : B \mapsto \text{Im}(\text{Hom}_{\Lambda, R}(A, B) \to \text{Hom}_{\Lambda, R}(A, B/(q - 1)))$$

on the category $fA(R[q])$ of flat $\Lambda$-rings over $R[q]$.

Definition 1.8. Given a flat morphism $R \to A$ of $\Lambda$-rings, define $qDR(A/R)$ to be the cochain complex of $R[q]$-modules given by taking the homotopy limit of the functor

$$\text{Strat}^q_{A/R} \to \text{Ch}(R[q])$$

$$B \mapsto B.$$

Equivalently, can we follow the approach of [Gro, Sim] towards the de Rham stack by regarding $qDR(A/R)$ as the quasi-coherent cohomology complex of $(\text{Spec} A)^q_{\text{strat}}$. Writing $\Omega: fA(R[q]) \to \text{Mod}_{R[q]}$ for the forgetful functor to the category of $R[q]$-modules, and $[fA(R[q]), \text{Set}]$ for the category of set-valued functors on $fA(R[q])$, we have

$$qDR(A/R) = R\text{Hom}_{fA(R[q]), \text{Set}}((\text{Spec} A)^q_{\text{strat}}, \Omega),$$

coming from the right-derived functor of the functor $\text{Hom}_{fA(R[q]), \text{Set}}((\text{Spec} A)^q_{\text{strat}}, -)$ of natural transformations with source $(\text{Spec} A)^q_{\text{strat}}$.

Definition 1.9. Given a polynomial ring $\mathbb{R}[x]$, recall from [Sch2] that the $q$-de Rham (or Aomoto–Jackson) cohomology $q\Omega^{\bullet}_{\mathbb{R}[x]/\mathbb{R}[q]}$ is given by the complex

$$\mathbb{R}[x][q] \xrightarrow{\nabla_q} \mathbb{R}[x][q] dx,$$

where

$$\nabla_q(f) = \left. \frac{f(qx) - f(x)}{x(q - 1)} \right| dx,$$
so \( \nabla_q(x^n) = [n]_q x^{n-1}dx. \)

Given a polynomial ring \( R[x_1, \ldots, x_d] \), the \( q \)-de Rham complex \( q\Omega^*_{R[x_1, \ldots, x_d]/R} \) is then set to be

\[
q\Omega^*_{R[x_1]/R} \otimes R[q] q\Omega^*_{R[x_2]/R} \otimes R[q] \cdots \otimes R[q] q\Omega^*_{R[x_d]/R},
\]

so takes the form

\[
R[x_1, \ldots, x_d][q] \xrightarrow{\nabla_q} \Omega^1_{R[x_1, \ldots, x_d]/R[q]} \xrightarrow{\nabla_q} \cdots \xrightarrow{\nabla_q} \Omega^d_{R[x_1, \ldots, x_d]/R[q]}.
\]

**Proposition 1.10.** If \( R \) is a \( \Lambda \)-ring and \( x \) of rank 1, then \( q\text{DR}(R[x]/R) \) can be calculated by a cosimplicial module \( U^* \) given by setting \( U^n \) to be the \( \Lambda \)-subring

\[
U^n \subset R[q, \{(q^n - 1)^{-1}\}_{m \geq 1}, x_0, \ldots, x_n]
\]

generated by \( q \) and the elements \( x_i \) and \( \frac{x_i - x_j}{q-1} \).

**Proof.** For \( X = \text{Spec } R[x] \), the set-valued functor \( X^q_{\text{strat}} \) is not representable, but it can be resolved by the simplicial functor \( X^q_{\text{strat}} \) given by taking the Čech nerve of \( \text{Hom}_{\Lambda, R}(A, B) \to \text{Hom}_{\Lambda, R}(A, B/(q-1)) \), so

\[
(X^q_{\text{strat}})_n(B) := \text{Hom}_{\Lambda, R}(A, B) \times_{\text{Hom}_{\Lambda, R}(A, B/(q-1)) \cdots \times_{\text{Hom}_{\Lambda, R}(A, B/(q-1))}} \text{Hom}_{\Lambda, R}(A, B)
\]

\[
= \text{Hom}_{\Lambda, R}(A, B \times_B/(q-1) \cdots \times_B/(q-1) B).
\]

Observe that any element of \((X^q_{\text{strat}})_n(B)\) gives rise to a morphism \( f: R[q, x_0, \ldots, x_n] \to B \) of \( \Lambda \)-rings over \( R[q] \), with the image of \( x_i - x_j \) divisible by \( (q-1) \). Flatness of \( B \) then gives a unique element \( f(x_i - x_j)/(q-1) \in B \), so we have a map \( f \) to \( B \) from the free \( \Lambda \)-ring \( L \) over \( R[q, x_0, \ldots, x_n] \) generated by elements \( z_{ij} \) with \( (q-1)z_{ij} = x_i - x_j \).

Since \( B \) is flat, it embeds in \( B[(q^n - 1)^{-1}]_{m \geq 1} \) (the only hypothesis we really need) implying that the image of \( f \) factors through the image \( U^n \) of \( L \) in \( R[q, \{(q^n - 1)^{-1}\}_{m \geq 1}, x_0, \ldots, x_n] \). To see that \((X^q_{\text{strat}})_n \) is represented by \( U^n \), we only now need to check that \( U^n \) is itself flat over \( R[q] \), which follows because the argument of Lemma 1.4 gives a basis

\[
x_0^{\lambda_0^{\ell_0}}(\frac{x_1 - x_0}{q-1}) \cdots \lambda_n(\frac{x_n - x_{n-1}}{q-1})
\]

for \( U^n \) over \( R[q] \). We therefore have \( q\text{DR}(R[x]/R) \simeq U^* \). \( \Box \)

**Theorem 1.11.** If \( R \) is a \( \Lambda \)-ring, and the polynomial ring \( R[x_1, \ldots, x_d] \) is given the \( \Lambda \)-ring structure for which the elements \( x_i \) are of rank 1, then there are \( R[q] \)-linear zigzags of quasi-isomorphisms

\[
q\text{DR}(R[x_1, \ldots, x_n]/R) \simeq (\Omega^*_{R[x_1, \ldots, x_n]/R[q]}, (q-1)\nabla_q)
\]

\[
\text{L}_{n(q-1)}q\text{DR}(R[x_1, \ldots, x_n]/R) \simeq q\Omega^*_{R[x_1, \ldots, x_n]/R},
\]

where \( \text{L}_{n(q-1)} \) denotes derived décalage with respect to the \( (q-1) \)-adic filtration.

**Proof.** It suffices to prove the first statement, the second following immediately by décalage. Since \((\text{Spec } A \otimes R A')^q_{\text{strat}}(B) = (\text{Spec } A)^q_{\text{strat}}(B) \times (\text{Spec } A')^q_{\text{strat}}(B)\), and coproducts of flat \( \Lambda \)-rings over \( R[q] \) is given by \( \otimes_{R[q]} \), we have \( q\text{DR}((A \otimes R A')/R) \simeq q\text{DR}(A/R) \otimes_{R[q]} q\text{DR}(A'/R) \), so we may reduce to the case \( A = R[x] \).
Proposition 1.10 gives $q\text{DR}(R[x]/R) \simeq U^\bullet$, and in order to compare this with $q$-de Rham cohomology, we now consider the cochain complexes $\tilde{Ω}^\bullet(U^n)$ given by
\[ U^n \xrightarrow{(q-1)\nabla_q} \bigoplus_i U^ndx_i \xrightarrow{(q-1)\nabla_q} \bigoplus_{i<j} U^ndx_i \wedge dx_j \xrightarrow{(q-1)\nabla_q} \ldots. \]

In order to see that this differential is well-defined, observe that
\[
(q-1)\nabla_q x^k \left( \frac{y-x}{q-1} \right) = \frac{1}{y(x)} \left( \frac{y-x}{q-1} \right)\frac{1}{y(q-1)} dy = \frac{x^k \left( \frac{y-x}{q-1} \right)}{y(q-1)} dy = \frac{1}{y(q-1)} dx.
\]

The first calculation also shows that the inclusion $\tilde{Ω}^\bullet(U^{n-1}) \hookrightarrow \tilde{Ω}^\bullet(U^n)$ is a quasi-isomorphism, since
\[
(q-1)\nabla_{x_n} f(x_0, \ldots, x_{n-1}) \frac{x_k}{x_n} \frac{x_n-x_{n-1}}{q-1} = \frac{1}{x_n} \left( \frac{x_n-x_{n-1}}{q-1} \right) \frac{1}{x_n \cdot (q-1)} dx_n.
\]

By induction on $n$ we deduce that the inclusion $\tilde{Ω}^\bullet(U^0) \hookrightarrow \tilde{Ω}^\bullet(U^n)$, and hence the retraction of it given by the diagonal, is a quasi-isomorphism.

Now the complexes $\tilde{Ω}^i(U^\bullet)$ are all acyclic for $i > 0$, consisting of cosimplicial tensor products of $U^\bullet$ with cosimplicial symmetric powers of the acyclic complex given by $\mathbb{Z}dx_0 \oplus \cdots \oplus \mathbb{Z}dx_n$ in level $n$. We therefore have quasi-isomorphisms
\[
U^\bullet \leftarrow \text{Tot } \tilde{Ω}^\bullet(U^\bullet) \rightarrow \tilde{Ω}^\bullet(U^0)
\]
of flat cochain complexes over $R[q]$, so
\[
q\text{DR}(R[x]/R) \simeq \tilde{Ω}^\bullet(R[x]),
\]
and we just observe that $\eta(q-1)\tilde{Ω}^\bullet(R[x]) = (\Omega^\bullet_{R[x]/R[q]}(q-1)\nabla_q).
\]

Remark 1.12. Note that Theorem 1.11 implies that $q\Omega^\bullet_{R[x_1, \ldots, x_n]/R}$ naturally underlies the décalage of a cosimplicial A-ring over $R[q]$. Even the underlying cosimplicial commutative ring structure carries more information than an $E_\infty$-structure when $Q \not\subseteq R$.

Remark 1.13. The complex $(\Omega^\bullet_{R[x_1, \ldots, x_n]/R}, (q-1)\nabla_q)$ is a more fundamental object than its décalage $q\Omega^\bullet_{R[x_1, \ldots, x_n]/R}$, since it has a vestigial memory of the Hodge filtration.

There might be a natural formulation of the theorem not involving décalage, in terms of a $q$-analogue of the crystalline site for a Λ-ring $A$ over $R$, regarded as an $R[q]$-algebra via $R = R[q]/(q-1)$. Following Remark 1.12, this would involve extensions $B \rightarrow A$ of Λ-rings over $R[q]$ equipped with $q$-analogues of divided power operations on the augmentation ideals $I$, looking like $x \mapsto (q-1)^k \lambda^k(\frac{x}{q-1})$.

1.3. Completed $q$-cohomology.

Definition 1.14. Given a morphism $R \rightarrow A$ of Λ-rings, we define the category $\text{Strat}_{A/R} \subset \text{Strat}_{A/R}$ to consist of those objects which are $(q-1)$-adically complete.
Definition 1.15. Given a flat morphism $R \to A$ of $\Lambda$-rings, define $q\hat{\text{DR}}(A/R)$ to be the cochain complex of $R[[q-1]]$-modules given by taking the homotopy limit of the functor

$$\text{Strat}^q_{A/R} \to \text{Ch}(R[[q]])$$

$$B \mapsto B.$$

The following is immediate:

Lemma 1.16. Given a flat morphism $R \to A$ of $\Lambda$-rings, the complex $q\hat{\text{DR}}(A/R)$ is the derived $(q-1)$-adic completion of $q\text{DR}(A/R)$.

Definition 1.17. As in [Sch2], given a formally étale map $\square: R[x_1, \ldots, x_d] \to A$, define $q\hat{\Omega}^\bullet_{A/R, \square}$ to be the complex

$$A[[q-1]] \xrightarrow{\nabla_q} \Omega^1_{A/R}[q-1] \xrightarrow{\nabla_q} \cdots \xrightarrow{\nabla_q} \Omega^n_{A/R}[q-1],$$

where $\nabla_q$ is defined as follows. First note that the ring endomorphisms $\gamma_i$ of $R[x_1, \ldots, x_d][q-1]$ given by $\gamma_i(x_j) = q^{v_i}x_j$ extend uniquely to endomorphisms of $A[[q-1]]$ which are the identity modulo $q-1$, then set

$$\nabla_q(f) := \sum_i \gamma_i(f) - f \frac{dx_i}{(q-1)x_i}.$$

Note that $q\hat{\Omega}^\bullet_{R[x_1, \ldots, x_d]/R}$ is just the $(q-1)$-adic completion of $q\hat{\Omega}^\bullet_{R[x_1, \ldots, x_d]/R}$.

Proposition 1.18. If $R$ is a flat $\Lambda$-ring over $\mathbb{Z}$ and $\square: R[x_1, \ldots, x_d] \to A$ is a formally étale map of $\Lambda$-rings, the elements $x_i$ having rank 1, then there are zigzags of $R[[q]]$-linear quasi-isomorphisms

$$q\hat{\text{DR}}(A/R) \simeq (\Omega^\bullet_{A/R}[q-1], (q-1)\nabla_q), \quad \text{L}(\eta_{q-1})q\hat{\text{DR}}(A/R) \simeq q\hat{\Omega}^\bullet_{A/R, \square}.$$

The induced quasi-isomorphisms

$$q\hat{\text{DR}}(A/R) \otimes^L_{R[[q-1]]} R \simeq (\Omega^\bullet_{A/R}, 0), \quad (\text{L}(\eta_{q-1})q\hat{\text{DR}}(A/R)) \otimes^L_{R[[q-1]]} R \simeq \Omega^\bullet_{A/R}$$

are independent of the choice of framing.

Proof. This is much the same as the proof of Theorem 1.11. The complex $q\hat{\text{DR}}(A/R)$ can be realised as a cosimplicial $\Lambda$-ring $U$, with $U^n$ the $(q-1)$-adically complete $\Lambda$-subring of $A^{\otimes(n+1)}[[q-1]]((q^n-1)^{-1})_{m \geq 1}$ generated by $A^{\otimes(n+1)}[[q-1]]$ and $(q-1)^{-1}\ker(A^{\otimes(n+1)} \to A)[[q-1]]$.

Uniqueness of lifts with respect to the formally étale framing ensures that the endomorphisms $\gamma_i$ commute with the Adams operations, so are $\Lambda$-ring endomorphisms of $R$. Since the formal completion of $A \otimes_R A \to A$ is just the $\Lambda$-ring

$$A[[x_1 - y_1], (x_2 - y_2), \ldots, (x_d - y_d)],$$

the calculations of Theorem 1.11 now adapt to give quasi-isomorphisms

$$(\Omega^\bullet_{A/R}[q], (q-1)\nabla_q) \leftarrow \text{Tot}^\bullet(U^n) \to U^\bullet,$$

where $\text{Tot}^\bullet(U^n)$ is the $(q-1)$-adic completion of $(U^n \otimes_{A^{\otimes(n+1)}} (\Omega^\bullet_{A/R})^{\otimes(n+1)}, (q-1)\nabla_q))$. Reduction of this or its décalage modulo $(q-1)$ replaces $\nabla_q$ with $d$, removing the dependence on co-ordinates. \qed
Remark 1.19. As in [Sch2, Definition 7.3], there is a notion of $q$-connections on projective $A[q-1]$-modules $M$. Adapting the ideas of Proposition 1.18 these will be equivalent to projective modules over $X^q_{\text{strat}}$, so flat Cartesian $q\text{DR}(A/R)$-modules $N$ with $N \otimes_{q\text{DR}(A/R)} A[q-1] = M$, together with a condition that the $(q\text{DR}(A/R)/(q-1))$-module $N/(q-1)$ is just given by pullback of the $A$-module $M/(q-1)$.

Via Lemma [La] these data are equivalent to specifying an operator $\partial^1: M \to \bigoplus_{k_1,\ldots,k_d} M \lambda^{k_1}(\frac{y_1^1-x_1^1}{q-1}) \cdots \lambda^{k_d}(\frac{y_d^d-x_d^d}{q-1})$ satisfying a cocycle condition and congruent to the identity modulo $(q-1)$. Such operators then arise from $q$-connections $(\nabla_1,q,\ldots,\nabla_d,q)$ as $q$-Taylor series

$$\partial^1(f) := \sum_{k_1,\ldots,k_d} (q-1)\sum_{k_i}(\nabla_{1,q}^{k_1} \cdots \nabla_{d,q}^{k_d})(f) \lambda^{k_1}(\frac{y_1^1-x_1^1}{q-1}) \cdots \lambda^{k_d}(\frac{y_d^d-x_d^d}{q-1}).$$

2. Comparisons for $\Lambda_P$-rings

Since very few étale maps $R[x_1,\ldots,x_d] \to A$ give rise to $\Lambda$-ring structures on $A$, Proposition [La] is fairly limited in its scope for applications. We now show how the construction of $q\text{DR}$ and the comparison quasi-isomorphism survive when we weaken the $\Lambda$-ring structure by discarding Adams operations at invertible primes.

2.1. $q$-cohomology for $\Lambda_P$-rings. Our earlier constructions for $\Lambda$-rings all carry over to $\Lambda_P$-rings, as follows.

Definition 2.1. Given a set $P$ of primes, we define a $\Lambda_P$-ring $A$ to be a $\Lambda_{\mathbb{Z},P}$-ring in the sense of [Bot]. This means that it is a coalgebra in commutative rings for the comonad given by the functor $W(P)$ of $P$-typical Witt vectors. When a commutative ring $A$ is flat over $\mathbb{Z}$, giving a $\Lambda_P$-ring structure on $A$ is equivalent to giving commuting Adams operations $\Psi^p$ for all $p \in P$, with $\Psi^p(a) \equiv a^p \mod p$ for all $a$.

Thus when $P$ is the set of all primes, a $\Lambda_P$-ring is just a $\Lambda$-ring; a $\Lambda_0$-ring is just a commutative ring; for a single prime $p$, we write $\Lambda_p := \Lambda_{\{p\}}$, and note that a $\Lambda_p$-ring is a $\delta$-ring in the sense of [Joy].

Definition 2.2. Given a $\Lambda_P$-ring $R$, say that $A$ is a $\Lambda_P$-ring over $R$ if it is a $\Lambda_P$-ring equipped with a morphism $R \to A$ of $\Lambda_P$-rings. We say that $A$ is a flat $\Lambda_P$-ring over $R$ if $A$ is flat as a module over the commutative ring underlying $R$.

Definition 2.3. Given a morphism $R \to A$ of $\Lambda_P$-rings, we define the category $\text{Strat}^q_{A/R}$ to consist of flat $\Lambda_P$-rings $B$ over $R[q]$ equipped with a compatible morphism $A \to B/(q-1)$, such that the map $A \to B/(q-1)$ admits a lift to $B$. We define the category $\text{Strat}^q_{A/R} \subset \text{Strat}^q_{A/R}$ to consist of those objects which are $(q-1)$-adically complete.

More concisely, $\text{Strat}^q_{A/R}$ (resp. $\text{Strat}^q_{A/R}$) is the Grothendieck construction of the functor

$$(\text{Spec} A^q_{\text{strat}})_R: B \mapsto \text{Im} \left( \text{Hom}_{\Lambda_P,R}(A,B) \to \text{Hom}_{\Lambda_P,R}(A,B/(q-1)) \right)$$

of the category of flat $\Lambda_P$-rings (resp. $(q-1)$-adically complete flat $\Lambda_P$-rings) over $R[q]$.
**Definition 2.4.** Given a flat morphism $R \to A$ of $\Lambda_P$-rings, define $q\operatorname{DR}_P(A/R)$ to be the cochain complex of $R[q]$-modules given by taking the homotopy limit of the functor

$$\operatorname{Strat}^q_{A/R} \to \operatorname{Ch}(R[q])$$

$$B \mapsto B.$$ 

Define $q\operatorname{DR}_P(A/R)$ to be the cochain complex of $R[q - 1]$-modules given by the corresponding homotopy limit over $\operatorname{Strat}^q_{A/R}$. 

Thus when $P$ is the set of all primes, we have $q\operatorname{DR}_P(A/R) = q\operatorname{DR}(A/R)$. At the other extreme, for $A$ smooth, $q\operatorname{DR}_Q(A/R)$ is the Rees construction of the Hodge filtration on the infinitesimal cohomology complex of $A$ over $R$, with formal variable $(q - 1)$. In more detail, there is a decreasing filtration $F$ of $\mathcal{O}_{\text{inf}}$ given by powers of the augmentation ideal $\mathcal{O}_{\text{inf}} \to \mathcal{O}_{\text{zar}}$, and $q\operatorname{DR}_Q(A/R) \simeq \bigoplus_{v \in \mathbb{Z}} (q - 1)^{-v} \Gamma(\text{Spec } A, F'_{\mathcal{O}_{\text{inf}}})(q - 1)^{-v}$.

**Lemma 2.5.** For a set $P$ of primes, the forgetful functor from $\Lambda$-rings to $\Lambda_P$-rings has a right adjoint $W^{(\mathbb{F}_P)}$. There is a canonical ghost component morphism

$$W^{(\mathbb{F}_P)}(B) \to \prod_{n \in \mathbb{N}, n \neq 1, \forall \nu \in P} B,$$

which is an isomorphism when $P$ contains all the residue characteristics of $B$.

**Proof.** Existence of a right adjoint follows from the comonadic definitions of $\Lambda$-rings and $\Lambda_P$-rings. The ghost component morphism is given by taking the Adams operations $\Psi^n$ given by the $\Lambda$-ring structure on $W^{(\mathbb{F}_P)}(B)$, followed by projection to $B$. When $P$ contains all the residue characteristics of $B$, a $\Lambda$-ring structure is the same as a $\Lambda_P$-ring structure with compatible commuting Adams operations for all primes not in $P$, leading to the description above. \qed

Note that the big Witt vector functor $W$ on commutative rings thus factorises as $W = W^{(\mathbb{F}_P)} \circ W^{(P)}$, for $W^{(P)}$ the $P$-typical Witt vectors.

**Proposition 2.6.** Given a morphism $R \to A$ of $\Lambda$-rings, and a set $P$ of primes, there are natural maps

$$q\operatorname{DR}_P(A/R) \to q\operatorname{DR}(A/R), \quad q\operatorname{DR}_P(A/R) \to q\operatorname{DR}(A/R),$$

and the latter map is a quasi-isomorphism when $P$ contains all the residue characteristics of $A$.

**Proof.** We have functors

$$(\text{Spec } A)^q_{\text{strat}} \circ W^{(\mathbb{F}_P)} : B \mapsto \text{Im}(\operatorname{Hom}_{A,R}(A, W^{(\mathbb{F}_P)}B) \to \operatorname{Hom}_{A,R}(A, (W^{(\mathbb{F}_P)}B)/(q - 1)))$$

$$(\text{Spec } A)^q_{\text{strat}}' : B \mapsto \text{Im}(\operatorname{Hom}_{A,P,R}(A, B) \to \operatorname{Hom}_{A,P,R}(A, B/(q - 1)))$$

on the category of flat $\Lambda_P$-rings over $R[q]$. There is an obvious map

$$(W^{(\mathbb{F}_P)}B)/(q - 1) \to W^{(\mathbb{F}_P)}(B/(q - 1)),$$

and hence a natural transformation $(\text{Spec } A)^q_{\text{strat}} \circ W^{(\mathbb{F}_P)} \to (\text{Spec } A)^q_{\text{strat}}'$, which induces the morphism $q\operatorname{DR}_P(A/R) \to q\operatorname{DR}(A/R)$ on cohomology.
When $P$ contains all the residue characteristics of $A$, the map $(W(\xi P)B)/(q - 1) \to W(\xi P)(B/(q - 1))$ is just
\[ \prod_{n \in \mathbb{N}} B/(q^n - 1) \to \prod_{n \in \mathbb{N}} B/(q - 1), \]
since the morphism $R[q] \to W(\xi P)B$ is given by Adams operations, with $\Psi^n(q - 1) = q^n - 1$.

We have $(q^n - 1) = (q - 1)[n]_q$, and $[n]_q$ is a unit in $\mathbb{Z}[[1/n]]$, hence a unit in $B$ when $n$ is coprime to the residue characteristics. Thus the map $(W(\xi P)B)/(q - 1) \to W(\xi P)(B/(q - 1))$ gives an isomorphism whenever $B$ is $(q - 1)$-adically complete and admits a map from $A$, so the transformation $(\text{Spec } A)^q_{\text{strat}} \circ W(\xi P) \to (\text{Spec } A)^q_{\text{strat}}$ is a natural isomorphism on the category of flat $(q - 1)$-adically complete $\Lambda_P$-rings over $R[q]$, and so $q\Omega_{R,P}(A/R) \to q\Omega_{R}(A/R)$. \qed

Over $\mathbb{Z}[\frac{1}{\mathfrak{p}}]$, every $\Lambda_P$-ring can be canonically made into a $\Lambda$-ring, by setting all the additional Adams operations to be the identity. However, this observation is of limited use in establishing functoriality of $\xi$-de Rham cohomology, because the resulting $\Lambda$-ring structure will not satisfy the conditions of Proposition 1.18. We now give a more general result which does allow for meaningful comparisons.

**Theorem 2.7.** If $R$ is a flat $\Lambda_P$-ring over $\mathbb{Z}$ and $\square$: $R[x_1, \ldots, x_d] \to A$ is a formally étale map of $\Lambda_P$-rings, the elements $x_i$ having rank 1, then there is a zigzag of $R[q]$-linear quasi-isomorphisms
\[ L\eta(q-1)\hat{q}\Omega_{R,P}(A/R) \simeq q\Omega_{\hat{A}/\hat{R},\square} \]
whenever $P$ contains all the residue characteristics of $A$.

**Proof.** The key observation to make is that formally étale maps have a unique lifting property with respect to nilpotent extensions of flat $\Lambda_P$-rings, because the Adams operations must also lift uniquely. In particular, this means that the operations $\gamma_i$ featuring in the definition of $q$-de Rham cohomology are necessarily endomorphisms of $A$ as a $\Lambda$-ring.

Similarly to Proposition 1.18 $\hat{q}\Omega_{R,P}(A/R)$ is calculated using a cosimplicial $\Lambda_P$-ring given in level $n$ by the $(q - 1)$-adic completion $\hat{U}_{P,A}$ of the $\Lambda_P$-ring over $R[q]$ generated by $A^{\oplus n(n+1)}[q]$ and $(q - 1)^{-1}\ker(A^{\oplus n(n+1)}[q] \to A)[q]$. The observation above shows that $\hat{U}_{P,A} \cong \hat{U}_{P,R[x_1, \ldots, x_d]} \otimes_{R[x_1, \ldots, x_d]} A$, changing base along $\square$ applied to the first factor.

As in Proposition 2.6 $\hat{U}_{P,R[x_1, \ldots, x_d]}$ is just the $(q - 1)$-adic completion of the complex $U^*$ from Proposition 1.10. Further application of the key observation above then allows us to adapt the constructions of Theorems 1.11 giving the desired quasi-isomorphisms. \qed

### 2.2. Cartier isomorphisms in mixed characteristic.

The only setting in which Theorem 2.7 leads to results close to the conjectures of Scholze is when $R = W(p)(k)$, the $p$-typical Witt vectors of a field of characteristic $p$, and $A = \lim_{\leftarrow n} A_n$ is a formal deformation of a smooth $k$-algebra $A_0$. Then any formally étale morphism $W(p)(k)[x_1, \ldots, x_d] \to A$ gives rise to a unique compatible lift $\Psi$ of absolute Frobenius on $A$ with $\Psi(x_i) = x_i^p$, so gives $A$ the structure of a topological $\Lambda_p$-ring. The framing still affects the choice of $\Lambda_p$-ring structure, but at least such a structure is
guaranteed to exist, giving rise to a complex $q \text{DR}_p(A/R)^{\wedge_p} := \varprojlim_n q \text{DR}_p(A/R) \otimes_R R_n$
depending only on the choice of $\Psi$, where $R_n = W'_n(k)$.

Our constructions now allow us to globalise the quasi-isomorphism

$$(q'\Omega_{A/R,\square}^{\wedge_p})^{/[p]}_q \simeq (\Omega_{A/R}^*)^{\wedge_p}[q-1]/[p]_q$$

of [Sch2 Proposition 3.4], where $\Omega_{A/R}^*$ denotes the complex $A \to A^1_B \to \Omega_{A/R}^2 \to \ldots$.

**Proposition 2.8.** Take a smooth formal scheme $\mathfrak{X}$ over $R = W^{(p)}(k)$ equipped with a lift $\Psi$ of Frobenius which étale locally admits co-ordinates $\{x_i\}$ as above with $\Psi(x_i) = x_i^p$.

Then there is a global quasi-isomorphism

$$C^{-1}_q: (\Omega_{\mathfrak{X}/R}^*)^{\wedge_p}[q-1]/[p]_q \to (\mathcal{L}n_{(q-1)}q \text{DR}_p(\mathcal{O}_{\mathfrak{X}/R}))^{\wedge_p}/[p]_q$$
in the derived category of étale sheaves on $\mathfrak{X}$.

**Proof.** Functoriality of the construction $q \text{DR}_p$ for rings with Frobenius lifts gives us a sheaf $q \text{DR}_p(\mathcal{O}_{\mathfrak{X}/R})^{\wedge_p}$ on $\mathfrak{X}$. We then have maps

$$\Psi^p: q \text{DR}_p(\mathcal{O}_{\mathfrak{X}/R})^{\wedge_p} \to q \text{DR}_p(\mathcal{O}_{\mathfrak{X}/R})^{\wedge_p}$$

and thus, denoting good truncation by $\tau$,

$$(q-1)^i\Psi^p: \tau^{\leq i}(q \text{DR}_p(\mathcal{O}_{\mathfrak{X}/R})^{\wedge_p}/(q-1)) \to (\mathcal{L}n_{(q-1)}q \text{DR}_p(\mathcal{O}_{\mathfrak{X}/R})^{\wedge_p})/[p]_q;$$

the left-hand side is quasi-isomorphic to $\bigoplus_{j \leq i} (\Omega_{\mathfrak{X}/R}^j)^{\wedge_p}[-j]$ by Proposition 1.18.

Extending the construction $R[q]$-linearly and restricting to top summands therefore gives us the global map $C^{-1}_q$. For a local choice of framing, the map $\Psi^p$ necessarily corresponds via Theorem 2.7 to the chain map $adx^i \mapsto \Psi^p(a)x^i(x-1)dx^i$ on the complex $(\Omega_{A/R}^*[q-1],(q-1)(-1)\nabla_q)$. This gives equivalences

$$(q-1)^i\Psi^p \simeq \sum_{j \leq i} (q-1)^{-j}(\tilde{C}^{-1})^j$$

for Scholze’s locally defined lifts $\langle \tilde{C}^{-1} \rangle: (\Omega_{A/R}^j)^{\wedge_p}[-j] \to (q'\Omega_{A/R,\square}^{\wedge_p})^{/[p]}_q$ of the Cartier quasi-isomorphism. The local calculation of [Sch2 Proposition 3.4] then ensures that $C^{-1}_q$ is a quasi-isomorphism. \square

**3. Functoriality via analogues of de Rham–Witt cohomology**

In order to obtain a cohomology theory for smooth commutative rings rather than for $\Lambda_p$-rings, we now consider $q$-analogues of de Rham–Witt cohomology. Our starting point is to observe that if we allow roots of $q$, we can extend the Jackson differential to fractional powers of $x$ by the formula

$$\nabla_q(x^{m/n}) = \frac{q^{m/n} - 1}{q - 1} x^{m/n} d \log x,$$

so terms such as $[n]_{q^{1/n}} x^{m/n}$ have integral derivative, where $[n]_{q^{1/n}} = \frac{q^{-1}}{q^{1/n} - 1}$. 


3.1. Motivation.

Definition 3.1. Given a $\Lambda_p$-ring $B$, define $\Psi^{1/p^\infty}B$ to be the smallest $\Lambda_p$-ring containing $B$ on which the Adams operations are automorphisms.

In the case $P = \{p\}$, the $\Lambda_p$-ring $\Psi^{1/p^\infty}B$ is thus the colimit of the diagram

$$B \xrightarrow{\Psi^p} B \xrightarrow{\Psi^p} B \xrightarrow{\Psi^p} \ldots$$

The proof of Theorem 3.2 allows us to replace $\widehat{qDR_p}(A/R)$ with the complex $(\Omega_{A/R}^{*}[q-1], (q-1)\nabla_q)$; under this quasi-isomorphism, the Adams operations on $A$ extend to $\Omega_{A/R}^{*}[q-1]$ by setting $\Psi^n(dx_i) := x_i^{n-1}dx_i$. As an immediate consequence we have:

Lemma 3.2. If $R$ is a flat $\Lambda_p$-ring over $\mathbb{Z}$ with $\Psi^{1/p^\infty}R = R$ and residue characteristic $p$, then $\Psi^{1/p^\infty}qDR_p(R[x]/R) \simeq (\Omega_{R[x]}^{*}[q^{1/p^\infty}]/(q-1)\nabla_q)$, so the décalage $L_{(q-1)}\Psi^{1/p^\infty}qDR_p(R[x]/R)$ is quasi-isomorphic to the $(q-1)$-adic completion of the complex

$$\{a \in R[x^{1/p^\infty}, q^{1/p^\infty}] : \nabla_q a \in R[x^{1/p^\infty}, q^{1/p^\infty}]d\log x\} \xrightarrow{\nabla_q} \{b d\log x \in R[x^{1/p^\infty}, q^{1/p^\infty}]d\log x : b(0, q) = 0\}$$

Thus in level 0 (resp. level 1), $L_{(q-1)}\Psi^{1/p^\infty}qDR_p(R[x]/R)$ is spanned by elements of the form $[p^n]_{q^{1/p^n}}, x^m/p^n$ (resp. $x^m/p^n d\log x$), so setting $q^{1/p^\infty} = 1$ gives a complex whose $p$-adic completion is the $p$-typical de Rham–Witt complex.

Lemma 3.3. Let $R$ and $A$ be flat $p$-adically complete $\Lambda_p$-algebras over $\mathbb{Z}_p$, with $\Psi^{1/p^\infty}R = R$ and, for elements $x_i$ of rank 1, a map $\square : R[x_1, \ldots, x_d]^{\psi_p} \to A$ of $\Lambda_p$-rings which is a flat $p$-adic deformation of an étale map. Then the map

$$L_{(q-1)}\Psi^{1/p^\infty}qDR_p(A/R)^{\psi_p} \to L_{(q-1)}(\Psi^{1/p^\infty}qDR_p(A/R))^{\psi_p}$$

is a quasi-isomorphism.

Proof. The map $\Psi^p : A \otimes_R[x_1, \ldots, x_d] R[x^{1/p}, \ldots, x_d^{1/p}] \to A$ becomes an isomorphism on $p$-adic completion, because $\square$ is flat and we have an isomorphism modulo $p$. Thus $\Psi^{1/p^\infty}A \simeq A[x_1^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]^{\psi_p} := (A \otimes_R[x_1, \ldots, x_d] R[x^{1/p^\infty}, \ldots, x_d^{1/p^\infty}])^{\psi_p}$.

Combined with the calculation of Lemma 3.2 this gives us a quasi-isomorphism between $(\Psi^{1/p^\infty}qDR_p(A/R))^{\psi_p}$ and the $(p, q - 1)$-adic completion of

$$\bigoplus_{\alpha} A[[q-1]]_{x_1}^{\alpha_1} \ldots x_d^{\alpha_d} dx^I,$$

where $\alpha \in p^{-\infty}\mathbb{Z}^d$ such that $0 \leq \alpha_i < 1$ if $i \notin I$ and $-1 < \alpha_i \leq 0$ if $i \in I$.

We then observe that the contributions to the décalage $\eta_{(q-1)}$ from terms with $\alpha \neq 0$ must be acyclic, via a contracting homotopy defined by the restriction to $\eta_{(q-1)}$ of the $q$-integration map

$$f x_1^{a_1} \ldots x_d^{a_d} dx^I \mapsto f x_1^{a_1} \ldots x_d^{a_d} \sum_{i \in I} \pm x_i [\alpha_i]_q^{-1} d x_i^{(I \setminus i)},$$

where $[\frac{m}{p^n}]_q^{-1} = [m]_{q^{1/p^n}}^{-1}$ for $m$ coprime to $p$, noting that $[m]_{q^{1/p^n}}$ is a unit in $\mathbb{Z}_{q^{1/p^n}}^{\psi_p}(p, q - 1)$. $\square$
Remark 3.4. The endomorphism given on $\Psi^{1/P^\infty}_q\hat{\text{DR}}_P(A/R)$ by

$$a \mapsto \Psi^{1/n}([n]_q a) = [n]_{q^{1/n}} \Psi^{1/n}a$$

descends to an endomorphism of $H^0(\Psi^{1/P^\infty}_q\hat{\text{DR}}_P(A/R)/(q-1))$, which we may denote by $V_n$ because it mimics Verschiebung in the sense that $\Psi^n V_n = n \cdot \text{id}$.

For $A$ smooth over $\mathbb{Z}$, we then have

$$H^0(\Psi^{1/P^\infty}_q\hat{\text{DR}}_P(A/\mathbb{Z})/(q-1))/(V_p : p \in P) \cong A[q^{1/P^\infty}]/([p]_{q^{1/p}} : p \in P) \cong A[\zeta_{P^\infty}],$$

for $\zeta_n$ a primitive $n$th root of unity.

By adjunction, this gives an injective map

$$H^0(\Psi^{1/P^\infty}_q\hat{\text{DR}}_P(A/\mathbb{Z})/(q-1)) \hookrightarrow W^P(A[\zeta_{P^\infty}])$$

of $A_P$-rings, which becomes an isomorphism on completing $\Psi^{1/P^\infty}_q\hat{\text{DR}}(A/\mathbb{Z})$ with respect to the system $\{(\mathbb{Z}[q^{1/n}])_{n \in \mathbb{P}}\}$ of ideals, where we write $P^\infty$ for the set of integers whose prime factors are all in $P$. This implies that the cokernel is annihilated by all elements of $(q^{1/P^\infty})_P^{-1}$, so leads us to consider almost mathematics as in [GR].

3.2. Almost isomorphisms. Combined with Lemma 3.3 Remark 3.4 allows us to regard $L_{(q-1)}\Psi^{1/P^\infty}_q\hat{\text{DR}}_P(A/\mathbb{Z})_{/P}$ as being almost a $q^{1/P^\infty}$-analogue of $P$-typical de Rham–Witt cohomology. (From now on, we consider only the case $P = \{p\}$.)

The ideal $(q^{1/P^\infty} - 1)^{(1/p - 1)} = \ker(Z[q^{1/P^\infty}]^{1/(p-1)} \rightarrow Z_p)$ is equal to the $p$-adic completion of its square, since we may write it as the kernel $W^P(m)W^P(F_p[q^{1/P^\infty}]^{1/(p-1)}) \rightarrow W^P(F_p)$, for the idempotent maximal ideal $m = ((q-1)^{1/P^\infty})^{1/(p-1)}$ in $F_p[q^{1/P^\infty}]^{1/(p-1)}$.

If we set $h^{1/P^\infty}$ to be the Teichmüller element $[q^{1/P^\infty} - 1] = \lim_{r \to \infty} (q^{1/r^{nr}} - 1)^{nr} \in Z[q^{1/P^\infty}]^{1/(p-1)}$, then $W^P(m) = (h^{1/P^\infty})^{1/(p-1)}$. Although $W^P(m)/p^n$ is not maximal in $Z[h^{1/P^\infty}]^{1/(p-1)}/p^n$, it is idempotent and flat, so gives a basic setup in the sense of [GR] 2.1.1. We thus regard the pair $(Z[q^{1/P^\infty}]^{1/(p-1)}, W^P(m))$ as an inverse system of basic setups for almost ring theory.

We then follow the terminology and notation of [GR], studying $p$-adically complete $(Z[q^{1/P^\infty}]^{1/(p-1)})^a$-modules (almost $Z[q^{1/P^\infty}]^{1/(p-1)}$-modules) given by localising at almost isomorphisms, the maps whose kernel and cokernel are $W^P(m)$-torsion.

The obvious functor $(-)^a$ from modules to almost modules has a right adjoint $(-)_a$, given by $N_a := \text{Hom}_{Z[q^{1/P^\infty}]^{1/(p-1)}}(W^P(m), N)$, the module of almost elements. Since the counit $(M_a)^a \rightarrow M$ of the adjunction is an (almost) isomorphism, we may also regard almost modules as a full subcategory of the category of modules, consisting of those $M$ for which the natural map $M \rightarrow (M_a)^a$ is an isomorphism. We can define $p$-adically complete $(Z[q^{1/P^\infty}]^{1/(p-1)})^a$-algebras similarly, forming a full subcategory of $Z[q^{1/P^\infty}]^{1/(p-1)}$-algebras.

**Lemma 3.5.** For any $Z[q - 1]$-module $M$, we may recover the $Z[q^{1/P^\infty}]^{1/(p-1)}$-module $(M \otimes_{Z[q]} Z[q^{1/P^\infty}]^{1/P})$ as the module of almost elements of the associated almost $Z[q^{1/P^\infty}]^{1/(p-1)}$-module.

**Proof.** Since $M \otimes_{Z[q]} Z[q^{1/P^\infty}] = \bigoplus_\alpha M \otimes q^\alpha$ for $\alpha \in p^{-\infty}\mathbb{Z}$ with $0 \leq \alpha < 1$, calculation shows that $(M \otimes_{Z[q]} Z[q^{1/P^\infty}]^{1/P}) \rightarrow (M \otimes_{Z[q]} Z[q^{1/P^\infty}])^a$ must be an isomorphism. □
3.3. Perfectoid algebras. We now relate Scholze’s perfectoid algebras to a class of \( \Lambda_p \)-rings.

**Definition 3.6.** Define a perfectoid \( \Lambda_p \)-ring to be a flat \( p \)-adically complete \( \Lambda_p \)-algebra over \( \mathbb{Z}_p \), on which the Adams operation \( \Psi_p \) is an isomorphism.

For a perfectoid field \( K \) in the sense of [Sch1], there is a tilt \( K^\flat \) (a complete perfect field of characteristic \( p \)). The subring of power-bounded elements is denoted \( K^\diamond \subset K \).

**Lemma 3.7.** Given a perfectoid field \( K \), we have equivalences

\[
\begin{array}{c|c}
\text{perfectoid almost } K^\diamond\text{-algebras} & \mathcal{A}_{\text{inf}}(K^\diamond) \\
\hline
\mathbb{F}_p \otimes \mathbb{Z}_p & W(p) \\
\hline
\text{perfectoid almost } \Lambda_p\text{-rings over } \mathcal{A}_{\text{inf}}(K^\diamond) \\
\hline
\end{array}
\]

of categories, where \( \mathcal{A}_{\text{inf}}(C) := \varprojlim_{q \rightarrow 0} W^{(q)}(C) \).

**Proof.** A perfectoid \( \Lambda_p \)-ring \( B \) is a deformation of the \( \mathbb{F}_p \)-algebra \( B/p \). As in [Sch1 Proposition 5.13], a perfect \( \mathbb{F}_p \)-algebra \( C \) has a unique deformation \( W^{(q)}(C) \) over \( \mathbb{Z}_p \), to which Frobenius must lift uniquely; this gives the bottom pair of equivalences.

We then observe that since \( B := \mathcal{A}_{\text{inf}}(C) \) is a perfectoid \( \Lambda_p \)-ring for any flat \( p \)-adically complete \( \mathbb{Z}_p \)-algebra \( C \), we must have \( B \cong W^{(q)}(B/p) \). Comparing rank 1 elements then gives a monoid isomorphism \( (B/p) \cong \varprojlim_{x \rightarrow x^p} C \), from which it follows that

\[\mathbb{F}_p \otimes \mathbb{Z}_p \mathcal{A}_{\text{inf}}(C) \cong \varprojlim_{q \rightarrow 0} \varprojlim_{C/p} C^\diamond \]

whenever \( C \) is perfectoid. Since tilting gives an equivalence of almost algebras by [Sch1 Theorem 5.2], this completes the proof. \( \square \)

We will only apply Lemma 3.7 to perfectoid almost \( \Lambda_p \)-rings over \( \mathbb{Z}[q^{1/p\infty}]^{\wedge(p-1)} \), in which case it shows that reduction modulo \( [p]_{q^{1/p}} \) (resp. \( p \)) gives an equivalence with perfectoid \( (\mathbb{Z}[q^{1/p\infty}]^{\wedge(p-1)})^\diamond \)-algebras (resp. perfectoid \( (\mathbb{F}_p[q^{1/p\infty}]^{\wedge(q-1)})^\diamond \)-algebras).

3.4. Functoriality of \( q \)-de Rham cohomology. Since \( (\Psi^{1/p\infty}_q \mathcal{D} \mathbb{R}^p_A(A/\mathbb{Z}_p))^\wedge(p) \) is represented by a cosimplicial perfectoid \( \Lambda_p \)-ring over \( \mathbb{Z}[q^{1/p\infty}]^{\wedge(p-1)} \) for any flat \( \Lambda_p \)-ring \( A \) over \( \mathbb{Z}_p \), it corresponds under Lemma 3.7 to a cosimplicial perfectoid \( (\mathbb{Z}[\zeta_p^{1/p\infty}]^{\wedge(p-1)})^\diamond \)-algebra, representing the following functor:

**Lemma 3.8.** For a perfectoid \( (\mathbb{Z}[\zeta_p^{1/p\infty}]^{\wedge(p)})^\diamond \)-algebra \( C \), and a \( \Lambda_p \)-ring \( A \) over \( \mathbb{Z}_p \) with \( X = \text{Spec} A \), there is a canonical isomorphism

\[X^q_p \text{strat}(\varprojlim_{q \rightarrow 0} W^{(p)}(C)_*) \cong \text{Im} \left( \varprojlim_{q \rightarrow 0} X(C_*) \rightarrow X(C_*) \right),\]

for the ring \( C_* \) of almost elements.

**Proof.** By definition, \( X^q_p \text{strat}(\varprojlim_{q \rightarrow 0} W^{(p)}(C)_*) \) is the image of

\[\text{Hom}_{\Lambda_p}(A, \varprojlim_{q \rightarrow 0} W^{(p)}(C)_*) \rightarrow \text{Hom}_{\Lambda_p}(A, (\varprojlim_{q \rightarrow 0} W^{(p)}(C)_*)/(q - 1)).\]
Since right adjoints commute with limits, we may rewrite the first term as $$\lim_{\Psi p} \text{Hom}_{A_p}(A, W^{(p)}(C *)) = \lim_{\Psi p} X(C_*)$$.

Setting $$B := \lim_{\Psi p} W^{(p)}(C)_*$$, observe that because $$[p^n]_{q^{1/p^n}} (q^{1/p^n} - 1) = (q - 1)$$, we have $$\bigcap [p^n]_{q^{1/p^n}} B = (q - 1)B$$, any element on the left defining an almost element of $$(q - 1)B$$, hence a genuine element since $$B = B_*$$ is flat. Then note that since the projection map $$\theta : B \rightarrow C_*$$ has kernel $$([p]_{q^{1/p^n}})$$, the map $$\theta \circ \Psi^{p - 1}$$ has kernel $$([p]_{q^{1/p^n}}$$), and so $$B \rightarrow W^{(p)}C_*$$ has kernel $$\bigcap [p^n]_{q^{1/p^n}} B$$. Thus

$$\text{Hom}_{A_p}(A, \lim_{\Psi p} W^{(p)}(C)_*) / (q - 1) \rightarrow \text{Hom}_{A_p}(A, W^{(p)}(C)_*) = X(C_*)$$.

\[\Box\]

In fact, the tilting equivalence gives $$\lim_{\Psi p} X(C_*) \cong X(C_*^\circ)$$, so the only dependence of $$((\Psi^{1/p^n} \hat{\text{DR}}_p(A/\mathbb{Z}_p))^{\wedge p})^a$$ on the Frobenius lift $$\Psi^p$$ is in determining the image of $$X(C_*^\circ) \rightarrow X(C_*)$$.

Although this map is not surjective, it is almost so in a precise sense, which we now use to establish independence of $$\Psi^p$$, showing that, up to faithfully flat descent, $$\hat{\text{DR}}_p(A/\mathbb{Z}_p)^{\wedge p} / [p]_{q^{1/p^n}}$$ is the best possible perfectoid approximation to $$A[\kappa^{\infty}]^{\wedge p}$$.

**Definition 3.9.** Given a functor $$X$$ from $$(\mathbb{Z}[\kappa^{\infty}]^{\wedge p})^{\wedge p}$$-algebras to sets and a functor $$\mathcal{A}$$ from perfectoid $$(\mathbb{Z}[\kappa^{\infty}]^{\wedge p})^{\wedge p}$$-algebras to abelian groups, we write

$$\text{R} \Gamma_{\text{Pfd}}(X, \mathcal{A}) := \text{R} \text{Hom}_{\text{Pfd}(\mathbb{Z}[\kappa^{\infty}]^{\wedge p})}$$. Set}(X, \mathcal{A})$$,

where Pfd($$S^a$$) denotes the category of perfectoid almost $$S$$-algebras, and $$(C, \text{Set})$$ denotes set-valued functors on $$C$$. When $$X$$ is representable by a $$(\mathbb{Z}[\kappa^{\infty}]^{\wedge p})^{\wedge p}$$-algebra $$C$$, we simply denote this by $$\text{R} \Gamma_{\text{Pfd}}(C, \mathcal{A})$$ — when $$C$$ is perfectoid, this will just be $$\mathcal{A}(C)$$.

The following gives a refinement of [BMS Theorem 1.17], addressing some of the questions in [BMS, Remark 1.11]:

**Theorem 3.10.** If $$R$$ is a $$p$$-adically complete $$\Lambda_p$$-ring over $$\mathbb{Z}_p$$, and $$A$$ a formal $$R$$-deformation of a smooth ring over $$(R/p)$$, then the complex

$$\text{R} \Gamma_{\text{Pfd}}((A[\kappa^{\infty}] \otimes_R \Psi^{1/p^n} R)^{\wedge p}, \mathcal{A}_{\text{inf}})$$

of $$\mathbb{Z}[q^{1/p^n}]^{\wedge p}$$-modules is almost quasi-isomorphic to $$(\Psi^{1/p^n} \hat{\text{DR}}_p(A/R))^{\wedge p}$$ for any $$\Lambda_p$$-ring structure on $$A$$ coming from a framing over $$R$$ as in Theorem 2.7.

**Proof.** First observe that $$(\Psi^{1/p^n} \hat{\text{DR}}_p(A/R))^{\wedge p}$$ is the completion of $$\hat{\text{DR}}_p(A/R)$$ with respect to the category of cosimplicial perfectoid almost $$\Lambda_p$$-rings over $$\mathbb{Z}[q^{1/p^n}]^{\wedge p}$$.

Combining the definition of $$\hat{\text{DR}}_p$$ with Lemma 3.7, it then follows that for $$X = \text{Spec} A$$ and $$Y = \text{Spec} R$$, the complex $$(\hat{\text{DR}}_p(A/R))^{\wedge p}$$ is given by the homotopy limit

$$\text{R} \Gamma_{\text{Pfd}}((X^q_\text{strat} \times Y^q_\text{strat} Y) \circ (\mathcal{A}_{\text{inf}})_*, (\mathcal{A}_{\text{inf}})_*)$$.

Writing $$X^\infty(C) := \text{Im}(\lim_{\Psi p} X(C_*) \rightarrow X(C_*)$$), Lemma 3.8 then combines with the description above to give

$$(\hat{\text{DR}}_p(A/R))^{\wedge p} \simeq \text{R} \Gamma_{\text{Pfd}}(X^\infty \times Y^\infty \lim_{\Psi p} Y, (\mathcal{A}_{\text{inf}})_*)$$,

$$\simeq \text{R} \Gamma_{\text{Pfd}}(X^\infty \times Y^\infty \lim_{\Psi p} Y, (\mathcal{A}_{\text{inf}})_*)$$.
We now introduce a Grothendieck topology on the category [Pfd],[\mathbb{Z}][\mathbb{Z}_p]\),\mathbb{Z}^\infty,\mathbb{Z}_p]\), by taking covering morphisms to be those maps \(C \rightarrow C'\) of perfectoid algebras which are almost faithfully flat modulo \(p\). Since \(C^p = \lim_{\Phi} (C/p)\), the functor \(\mathcal{A}_{\text{inf}}\) satisfies descent with respect to these coverings, so the map

\[
R\Gamma_{\text{Pfd}}((X^\infty \times Y \lim_{\mathcal{X}_p}) X_{p}), (\mathcal{A}_{\text{inf}})_*) \rightarrow R\Gamma_{\text{Pfd}}(X^\infty \times Y \lim_{\mathcal{X}_p}) X_{p}, (\mathcal{A}_{\text{inf}})_*)
\]

is a quasi-isomorphism, where \((-)^{\flat}\) denotes sheafification.

In other words, the calculation of \((q^\text{DR}_p(A/R))^\wedge_p\) is not affected if we tweak the definition of \(X^\infty\) by taking the image sheaf instead of the image presheaf. We then have

\[
(X^\infty)^\sharp(C) = \bigcup_{C \rightarrow C'} \text{Im}(X(C_*) \times X(C'_*) \lim_{\mathcal{X}_p} X(C'_*) \rightarrow X(C_*)),
\]

where \(C \rightarrow C'\) runs over all covering morphisms.

Now, \(\lim_{\mathcal{X}_p} X\) is represented by the perfectoid algebra \((\Psi^1/p^\infty A)^\wedge_p\), which is isomorphic to \(A[x_1/p^\infty, \ldots, x_d/p^\infty]^\wedge_p\) as in the proof of Lemma 3.3. This allows us to appeal to Andrè’s results \([\text{And.} \quad \frac{2.5}{\text{Bha.}}\)] as generalised in \([\text{Bha.} \quad \text{Theorem} 2.3]\). For any morphism \(f : A \rightarrow C\), there exists a covering morphism \(C \rightarrow C_1\) such that \(f(x_i)\) has arbitrary \(p\)-power roots in \(C_1\). Setting \(C' := C_1 \otimes C \ldots \otimes C C_d\), this means that the composite \(A \rightarrow C \rightarrow C'\) extends to a map \((\Psi^1/p^\infty A)^\wedge_p \rightarrow C'\), so \(f \in (X^\infty)^\sharp(C)\). We thus have shown that \((X^\infty)^\sharp = X\), giving the required equivalence

\[
((\Psi^1/p^\infty q^\text{DR}_p(A/R))^\wedge_p)_* \simeq R\Gamma_{\text{Pfd}}(X \times Y \lim_{\mathcal{X}_p}) X_{p}, (\mathcal{A}_{\text{inf}})_*).
\]

\[ \square \]

**Corollary 3.11.** If \(R\) is a \(p\)-adically complete \(\Lambda_p\)-ring over \(\mathbb{Z}_p\), and \(A\) a formal \(R\)-deformation of a smooth ring over \((R/p)\), then the \(q\)-de Rham cohomology complex \((q(\Omega^\bullet_{A/R,\Box} \otimes R[q])(\Psi^1/p^\infty R[q^1/p^\infty])^\wedge_p\) is, up to quasi-isomorphism, independent of a choice of co-ordinates \(\square\). It is naturally an invariant of the commutative \(p\)-adically complete \((\Psi^1/p^\infty R)[\mathbb{Q}_p^\wedge]/p^\wedge\)-algebra \(A[\mathbb{Q}_p^\wedge] \otimes R \Psi^1/p^\infty R)^\wedge_p\).

**Proof.** By Theorem 3.10 we know that the complex \(((\Psi^1/p^\infty q^\text{DR}_p(A/R))^\wedge_p)_*\) depends only on \((A[\mathbb{Q}_p^\wedge] \otimes R \Psi^1/p^\infty R)^\wedge_p\). Since

\[
\Psi^1/p^\infty q^\text{DR}_p(A/R) = \Psi^1/p^\infty q^\text{DR}_p((A \otimes_R \Psi^1/p^\infty R)/\Psi^1/p^\infty R),
\]

Theorem 2.7 combines with Lemmas 3.3 and 3.5 to give

\[
(q(\Omega^\bullet_{A/R,\Box} \otimes R[q])(\Psi^1/p^\infty R[q^1/p^\infty])^\wedge_p) \simeq L\eta_{(q-1)}((\Psi^1/p^\infty q^\text{DR}_p(A/R))^\wedge_p)_*,
\]

which completes the proof. \[ \square \]

**Remark 3.12** (Scholze’s conjectures). If we weaken the conjectures of \([\text{Sch2} \quad \text{Conjecture} 3.1]\) (co-ordinate independence of \(q\)-de Rham cohomology over \(\mathbb{Z}\)) via an arithmetic fracture square. Taking more general base rings \(R\) in Corollary 3.11 gives an analogue of \([\text{Sch2} \quad \text{Conjecture} 7.1]\), further weakened by having to invert all Adams operations on \(R\).

The description of Theorem 3.10 is very closely related to the definition of \(A\Omega^\bullet\) in \([\text{BMS} \quad \text{Definition} 1.12]\), giving an analogue of \([\text{Sch2} \quad \text{Conjecture} 4.3]\), and hence the comparison with singular cohomology in \([\text{Sch2} \quad \text{Conjecture} 3.3]\). The operations
described in [Sch2, Conjectures 6.1 and 6.2] correspond to the Adams operations on $q\text{-DR}$ respectively at and away from the residue characteristics. Remark 1.19 provides a category of $q$-connections as described in [Sch2, Conjecture 7.5]; these will correspond via Theorem 1.11 to projective $\mathcal{A}_{\text{inf}}$-modules on the site of integral perfectoid algebras over $A[\zeta_{p^\infty}]^{1/p}$, so are again independent of co-ordinates after base change to $\mathbb{Z}[\eta^{1/p^\infty}]$.

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