Intersections between the norm-trace curve and some low degree curves

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Abstract

In this paper we analyze the intersection between the norm-trace curve over \( \mathbb{F}_{q^3} \) and the curves of the form \( y = ax^3 + bx^2 + cx + d \), giving a complete characterization of the intersection between the curve and the parabolas, as well as sharp bounds for the other cases. This information is used for the determination of the weight distribution of some one-point AG codes constructed on the curve.

Keywords: Norm-trace curve - AG Code - Weight distribution

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1 Introduction

Algebraic-Geometric (AG codes for short) codes form an important class of error correcting codes; see [9,10,24].

Let \( \mathcal{X} \) be an algebraic curve defined over the finite field \( \mathbb{F}_q \) of order \( q \). The parameters of the AG codes associated with \( \mathcal{X} \) strictly depend on some properties of the underlying curve \( \mathcal{X} \). In general, curves with many \( \mathbb{F}_q \)-rational places with respect to their genus give rise to AG codes with good parameters. For this reason maximal curves, that is, curves attaining the Hasse-Weil upper bound, have been widely investigated in the literature: for example the Hermitian curve and its quotients; see for instance [23,25–27].

The determination of the intersection of a given curve and low degree curves is often useful for the determination of informations of the algebraic-geometric codes arising from the curve; see [1,2,6,17,18].

The norm-trace curve is a natural generalization of the Hermitian curve to any extension field \( \mathbb{F}_{q^r} \). It has been widely studied for coding theoretical purposes; see [1,8,13,19,21].
In this paper, we focus our attention on the intersection between the norm-trace curves and the curves of the form $y = ax^3 + bx^2 + cx + d$ over $\mathbb{F}_{q^r}$, arriving at complete characterization of the intersection between the curve and the parabolas as well as sharp bounds for the other cases. To do so, we employ techniques coming from the properties of irreducible cubic surfaces over finite fields. Then we partially deduce the weight distribution of the corresponding one-point codes.

2 Preliminary Results

Let $q$ be a power of a prime and consider $\mathbb{F}_q$, the finite field with $q$ elements. Let $C \subset (\mathbb{F}_q)^n$ be a linear subspace, then $C$ is a linear code and we will indicate, as usual, with $[n,k,d]$ its parameters, where $d$ is its Hamming minimum distance.

2.1 The norm-trace curve

The norm-trace curve $X$ is the plane curve defined over $\mathbb{F}_q$ by the affine equation

$$
\frac{x^{q^r-1}}{q-1} = y^{q^r-1} + y^{q^r-2} + \cdots + y^q + y.
$$

The norm $N_{\mathbb{F}_q}$ and the trace $T_{\mathbb{F}_q}$ are two well-known functions from $\mathbb{F}_q$ to $\mathbb{F}_q$ such that

$$
N_{\mathbb{F}_q}(x) = x^{q^r-1} = x^{q^r-1} + q^r-2 + \cdots + q+1
$$

and

$$
T_{\mathbb{F}_q}(x) = x^{q^r-1} + x^{q^r-2} + \cdots + x^q + x.
$$

When $q$ and $r$ are understood, we will write $N = N_{\mathbb{F}_q}$ and $T = T_{\mathbb{F}_q}$.

The equation $\frac{x^{q^r-1}}{q-1} = y^{q^r-1} + y^{q^r-2} + \cdots + y^q + y$ has precisely $q^{2r-1}$ solutions in $\mathbb{A}^2(\mathbb{F}_q)$, so the curve $X$ has $q^{2r-1} + 1$ rational points: $q^{2r-1}$ of them are affine points plus a single point at the infinity $P_\infty$.

If $r = 2$ $X$ coincides with the Hermitian curve and if $r \geq 3$ $X$ is singular in $P_\infty$. Moreover it is known that its Weierstrass semigroup in $P_\infty$ is generated by $\left\{q^r-1, \frac{q^r-1}{q-1}\right\}$.

Our main aim is the study of the intersection between $X$ and the cubics of the form $y = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{F}_{q^r}$. In particular we focus on the intersections between $X$ and parabolas. The case $r = 2$ is completely investigated in [7,17], so we deal with the more difficult case $r \geq 3$. 

2
2.2 Algebraic-Geometric Codes

In this section we introduce some basics notions on AG codes. For a detailed introduction we refer to [24].

Let $\mathcal{X}$ be a projective curve over the finite field $F_q$, consider the rational function field $F_q(\mathcal{X})$ and the set $\mathcal{X}(F_q) = \{P_1, \ldots, P_N\}$ given by the $F_q$-rational places of $\mathcal{X}$. Given an $F_q$-rational divisor $D = \sum_{i=1}^n m_i P_i$, where $n < N$, the Riemann-Roch space associated to $D$ on $\mathcal{X}$ is the vector space $\mathcal{L}(D)$ over $F_q$ defined as

$$\mathcal{L}(D) = \{f \in F_q(\mathcal{X}) \mid (f) + D \geq 0\} \cup \{0\}.$$ 

It is known that $\mathcal{L}(D)$ is a finite dimensional $F_q$-vector space and the exact dimension can be computed using the Riemann-Roch theorem. We write $\ell(D) = \dim_{F_q} \mathcal{L}(D)$.

Consider now the divisor $D = \sum_{P \in S} P$, $S = \{P_1, \ldots, P_n\} \subseteq \mathcal{X}(F_q)$, where all the $P$'s have weight one. Let $G$ be another $F_q$-rational divisor such that $\text{supp}(G) \cap \text{supp}(D) = \emptyset$. Consider the evaluation map

$$\text{ev} : \mathcal{L}(G) \to (F_q)^n \quad \text{ev}(f) = (f(P_1), \ldots, f(P_n)).$$

This map is $F_q$-linear and it is injective if $n > \deg(G)$.

The AG-code $C_{\mathcal{L}}(D,G)$ associated with the divisors $D$ and $G$ is then defined as $\text{ev}(\mathcal{L}(G))$. It is well known that $\ell(G) > \ell(G - D)$ and that $C_{\mathcal{L}}(D,G)$ is an $[n, \ell(G) - \ell(G - D), d]_q$ code, where $d \geq d^* = n - \deg(G)$, with $d^*$ is the so called designed minimum distance of the code.

3 Intersections between $\mathcal{X}$ and a curve $y = A(x)$ of degree $h$

Our aim is to find out the intersection over $F_{q^3}$ of $\mathcal{X}$ with the curve defined by the polynomial $y = A(x)$ of degree $h$, so $A(x) = A_h x^h + \cdots + A_0$, where $A_h \neq 0$ and $A_i \in F_{q^r}$. More precisely, given two curves $\mathcal{X}$ and $\mathcal{Y}$ lying in the affine space $A^2(F_{q^r})$ we call planar intersection (or simply intersection) the number of points in $A^2(F_{q^r})$ that lie in both curves, disregarding multiplicity.

Substituting $y = A(x)$ in the equation of the norm-trace curve, we get, by the linearity of $T$,

$$N(x) = T(A_h x^h) + \cdots + T(A_1 x) + T(A_0).$$

Given a linear basis $B = \{w_0, \ldots, w_{r-1}\}$ of $F_{q^r}$ with respect to $F_q$, we know that there is a vector space isomorphism $\Phi_B : (F_q)^r \to F_{q^r}$ such that $\Phi_B(s_0, \ldots, s_{r-1}) = \sum_{i=0}^{r-1} s_i w_i$. If we consider the maps $N, T : F_{q^r} \to F_q$, we can interpret them from $(F_q)^r$ to $F_q$ in this way

$$\tilde{N} : (F_q)^r \to F_q \quad \tilde{T} : (F_q)^r \to F_q$$

$$\tilde{N} = N \circ \Phi_B \quad \tilde{T} = T \circ \Phi_B.$$
and call $T_i := T(A_i x^i)$ and $\tilde{T}_i := T_i \circ \Phi_B$, $1 \leq i \leq h$. From now on, we will take as $B$ a normal basis, i.e. a basis $B = \{ \alpha, \alpha^q, \ldots, \alpha^{q^{r-1}} \}$. We know that such a basis exists, see [13, Theorem 2.35].

A simple manipulation shows that $\tilde{N}$ and $\tilde{T}_i$ are homogeneous polynomials of degree respectively $r$ and $i$ in $\mathbb{F}_q[x_0, \ldots, x_{r-1}]$. Therefore

$$\tilde{N}(x_0, \ldots, x_{r-1}) = \tilde{T}_h(x_0, \ldots, x_{r-1}) + \cdots + \tilde{T}_1(x_0, \ldots, x_{r-1}) + D$$

(2)

which is the equation of a hypersurface of $\mathbb{A}^r(\mathbb{F}_q)$, where $D = T(A_0)$. Notice that the LHS has degree $r$, while the RHS has degree $h$.

4 Case $r = 3$ and $h = 2$

We are interested in this case to find the number of possible intersections between the norm-trace curve and the parabolas. By parabola we mean a curve $y = Ax^2 + Bx + C$, $A, B, C \in \mathbb{F}_{q^3}$ and $A \neq 0$.

These numbers help to determine some weights for the corresponding AG code, see Section 6. From now on, $B = \{ \alpha, \alpha^q, \alpha^{q^2} \}$.

Specializing to $y = Ax^2 + Bx + C$, equation (2) becomes

$$\tilde{N}(x_0, x_1, x_2) = \tilde{T}_2(x_0, x_1, x_2) + \tilde{T}_1(x_0, x_1, x_2) + D$$

(3)

The map $\Phi_B^{-1} : \mathbb{F}_{q^3} \to (\mathbb{F}_q)^3$ induces a correspondence between $\mathbb{F}_q[x_0, x_1, x_2]$ and $\mathbb{F}_{q^3}[x]$ such that we can substitute $x$ with $x_0 \alpha + x_1 \alpha^q + x_2 \alpha^{q^2}$ and $x^2$ with

$$x_0^2 \alpha^2 + x_1^2 \alpha^{2q} + x_2^2 \alpha^{2q^2} + 2x_0 x_1 \alpha^{q+1} + 2x_0 x_2 \alpha^{q^2+1} + 2x_1 x_2 \alpha^{q^2+q}$$

and the following results holds.

**Remark 4.1.** Chosen two elements $A = A_0 \alpha + A_1 \alpha^q + A_2 \alpha^{q^2}$ and $B = B_0 \alpha + B_1 \alpha^q + B_2 \alpha^{q^2}$ then

$$T(AB) = A_0 B_0 + A_1 B_1 + A_2 B_2$$

**Proof.**

$$T(AB) = T((A_0 \alpha + A_1 \alpha^q + A_2 \alpha^{q^2})(B_0 \alpha + B_1 \alpha^q + B_2 \alpha^{q^2}))$$

$$= T(A_0 B_0 \alpha^2 + A_0 B_1 \alpha^{q+1} + A_0 B_2 \alpha^{q^2+1} + A_1 B_0 \alpha^{q+1} + A_1 B_1 \alpha^{q^2+1} + A_1 B_2 \alpha^{q^2+q} + A_2 B_0 \alpha^{q^2+q} + A_2 B_2 \alpha^{2q^2})$$

$$= A_0 B_0 + A_1 B_1 + A_2 B_2$$

$\square$
Using these relations we want to write down the explicit equation of the surface \( S \).

\[
\tilde{T}_1 = B(x_0 \alpha + x_1 \alpha^q + x_2 \alpha^{q^2}) + B^q (x_0 \alpha^q + x_1 \alpha^{q^2} + x_2 \alpha) + B^{q^2} (x_0 \alpha^{q^2} + x_1 \alpha + x_2 \alpha^q) \\
= x_0 T(\alpha B) + x_1 T(\alpha B^q) + x_2 T(\alpha B^{q^2})
\]

\[
\tilde{T}_2 = A(x_0 \alpha + x_1 \alpha^q + x_2 \alpha^{q^2})^2 + A^q (x_0 \alpha^q + x_1 \alpha^{q^2} + x_2 \alpha)^2 + A^{q^2} (x_0 \alpha^{q^2} + x_1 \alpha + x_2 \alpha^q)^2 \\
= x_0^2 T(A\alpha^2) + x_1^2 T(A\alpha^{2q}) + x_2^2 T(A\alpha^{2q^2}) + 2x_0x_1 T(A\alpha^{q+1}) + 2x_0x_2 T(A\alpha^{q^2+1}) + 2x_1x_2 T(A\alpha^{q^2+q})
\]

\[
\tilde{N} = (x_0 \alpha q^2 + x_1 \alpha + x_2 \alpha q)(x_0 \alpha + x_1 \alpha^q + x_2 \alpha)(x_0 \alpha + x_1 \alpha + x_2 \alpha^q) \\
= (x_0^3 + x_1^3 + x_2^3)N(\alpha) + (x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_0)T(\alpha^{q+2}) + (x_0^2 x_2 + x_1^2 x_0 + x_2^2 x_1)T(\alpha^{2q+1}) \\
+ x_0 x_1 x_2 (3N(\alpha) + T(\alpha^3))
\]

Therefore (3) becomes

\[
0 = - (x_0^3 + x_1^3 + x_2^3)N(\alpha) - (x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_0)T(\alpha^{q+2}) - (x_0^2 x_2 + x_1^2 x_0 + x_2^2 x_1)T(\alpha^{2q+1}) \\
- x_0 x_1 x_2 (3N(\alpha) - T(\alpha^3)) + x_0^2 T(A\alpha^2) + x_1^2 T(A\alpha^{2q}) + x_2^2 T(A\alpha^{2q^2}) + 2x_0 x_1 T(A\alpha^{q+1}) \\
+ 2x_0 x_2 T(A\alpha^{q^2+1}) + 2x_1 x_2 T(A\alpha^{q^2+q}) + x_0 T(\alpha B) + x_1 T(\alpha B^q) + x_2 T(\alpha B^{q^2}) + D
\] (4)

and we call \( S_1 = S_1(\mathbb{F}_q) \) the surface having this equation, which is clearly defined over \( \mathbb{F}_q \). Here and in the following, if \( X \) is a surface or a curve, we denote with \( X(\mathbb{F}_q) \) only its affine \( \mathbb{F}_q \)-rational points.

**Remark 4.2.** By construction the \( \mathbb{F}_q \)-rational points of \( S_1 \), i.e. the points in \( S_1(\mathbb{F}_q) \), correspond to the intersections in \( A^2(\mathbb{F}_q) \) between the norm-trace curve and the parabola \( y = Ax^2 + Bx + C \).

Equation (4) can be also written as

\[
0 = - (x_0 \alpha + x_1 \alpha^q + x_2 \alpha^{q^2})(x_0 \alpha^q + x_1 \alpha^{q^2} + x_2 \alpha)(x_0 \alpha^{q^2} + x_1 \alpha + x_2 \alpha^q) + A(x_0 \alpha + x_1 \alpha^q + x_2 \alpha^{q^2})^2 \\
+ A^q (x_0 \alpha^q + x_1 \alpha^{q^2} + x_2 \alpha)^2 + A^{q^2} (x_0 \alpha^{q^2} + x_1 \alpha + x_2 \alpha^q)^2 + B(x_0 \alpha + x_1 \alpha^q + x_2 \alpha^{q^2}) \\
+ B^q (x_0 \alpha^q + x_1 \alpha^{q^2} + x_2 \alpha) + B^{q^2} (x_0 \alpha^{q^2} + x_1 \alpha + x_2 \alpha^q) + D.
\]
If we apply the following linear change of coordinates in \( \text{GL}(3, \mathbb{F}_q) \)

\[
\begin{align*}
X_0 &= x_0 \alpha + x_1 \alpha^q + x_2 \alpha^{q^2} \\
X_1 &= x_0 \alpha^q + x_1 \alpha^{q^2} + x_2 \alpha \\
X_2 &= x_0 \alpha^{q^2} + x_1 \alpha + x_2 \alpha^q
\end{align*}
\]

we obtain a new surface \( S_2 = S_2(\mathbb{F}_q) \), defined over \( \mathbb{F}_q^3 \), with equation

\[
X_0 X_1 X_2 = A X_0^2 + A^q X_1^2 + A^{q^2} X_2^2 + B X_0 + B^q X_1 + B^{q^2} X_2 + D. \quad (5)
\]

Note that this change of coordinates is bijective since its associated matrix is

\[
M = \begin{pmatrix}
\alpha & \alpha^q & \alpha^{q^2} \\
\alpha^q & \alpha^{q^2} & \alpha \\
\alpha^{q^2} & \alpha & \alpha^q
\end{pmatrix}
\]

which is a Moore matrix, and its determinant is different from zero since we are dealing with three linearly independent elements, see \([15, \text{Corollary 2.38}]\).

**Remark 4.3.** Clearly, all the \( \mathbb{F}_q \)-rational points of \( S_1 \) are mapped to all \( \mathbb{F}_{q^3} \)-rational points of \( S_2 \) of the form \((\beta, \beta^q, \beta^{q^2})\), \( \beta \in \mathbb{F}_{q^3} \). Similarly, \( \mathbb{F}_q \)-rational lines contained in \( S_1 \) are mapped to \( \mathbb{F}_{q^3} \)-rational lines contained in \( S_2 \) having direction \((\beta, \beta^q, \beta^{q^2})\), \( \beta \in \mathbb{F}_{q^3} \). The affinity preserves the absolutely irreducible components of the surfaces and the singularities, since it is in \( \text{GL}(3, \mathbb{F}_q) \).

**Proposition 4.4.** \( S_1 \) is an absolutely irreducible cubic surface.

**Proof.** By Remark 4.3 it is sufficient to prove that \( S_2 \) is absolutely irreducible. We proceed by contradiction. If \( S_2 \) is reducible, since its degree is three, then it must contain a plane. In this case we would have

\[
(h_0 X_0^2 + h_1 X_1^2 + h_2 X_2^2 + h_3 X_0 X_1 + h_4 X_0 X_2 + h_5 X_1 X_2 + h_6 X_0 + h_7 X_1 + h_8 X_2 + h_9)(k_0 X_0 + k_1 X_1 + k_2 X_2 + k_3) = X_0 X_1 X_2 - AX_0^2 - A^q X_1^2 - A^{q^2} X_2^2 - BX_0 - B^q X_1 - B^{q^2} X_2 - D
\]

where \( h_i, k_j \in \mathbb{F}_q \), \( i \in \{0, \ldots, 9\} \) and \( j \in \{0, \ldots, 3\} \), and at least one among \( h_0, \ldots, h_5 \) nonzero and at least one of \( k_0, k_1, k_2 \) nonzero. Applying the identity principle for polynomials to the third degree terms, we obtain
\[
\begin{align*}
\begin{cases}
h_0k_0 = h_1k_1 = h_2k_2 = 0 \\
h_0k_1 + h_3k_0 = h_0k_2 + h_4k_0 = 0 \\
h_1k_0 + h_3k_1 = h_1k_2 + h_5k_1 = 0 \\
h_2k_0 + h_4k_2 = h_2k_1 + h_5k_2 = 0 \\
h_3k_2 + h_4k_1 + h_5k_0 = 0
\end{cases}
\end{align*}
\]  

(7)

There are three possibilities (up to a permutation of the indices):

1. \(k_0 \neq 0\) and \(k_1 = k_2 = 0\). We have that \(h_0 = 0\) and (7) becomes

\[
\begin{align*}
\begin{cases}
h_3k_0 = h_4k_0 = 0 \\
h_1k_0 = 0 \\
h_2k_0 = 0 \\
h_5k_0 = 1
\end{cases}
\end{align*}
\]

which means \(h_1 = h_2 = h_3 = h_4 = 0\) and \(h_5 \neq 0\). At this point (6) becomes

\[
(h_5X_1X_2 + h_6X_0 + h_7X_1 + h_8X_2 + h_9)(k_0X_0 + k_3) = X_0X_1X_2 - AX_0^2 - A^\theta X_0^2 - A^\theta X_1^2 - BX_0 - B^\theta X_1 - B^\theta X_2 - D
\]

and since \(A^\theta \neq 0\) and the LHS does not contain any \(X_1^2\) term we have that this cannot happen.

2. \(k_0, k_1 \neq 0\) and \(k_2 = 0\). We have that \(h_1 = h_2 = 0\) and (7) becomes

\[
\begin{align*}
\begin{cases}
h_3k_0 = h_4k_0 = 0 \\
h_3k_1 = h_5k_1 = 0 \\
h_2k_0 = h_2k_1 = 0 \\
h_4k_1 + h_5k_0 = 1
\end{cases}
\end{align*}
\]

which means \(h_2 = h_3 = h_4 = h_5 = 0\) and this is a contradiction;

3. \(k_0, k_1, k_2 \neq 0\). We have that \(h_0 = h_1 = h_2 = 0\) and (7) becomes

\[
\begin{align*}
\begin{cases}
h_3k_0 = h_4k_0 = 0 \\
h_3k_1 = h_5k_1 = 0 \\
h_4k_2 = h_5k_2 = 0 \\
h_3k_2 + h_4k_1 + h_5k_0 = 1
\end{cases}
\end{align*}
\]

\]
which means \( h_2 = h_3 = h_4 = h_5 = 0 \) and this is a contradiction.

What we want to do now is to estimate the number of \( \mathbb{F}_q \)-rational points of \( S_1 \). Since they correspond to the intersections between \( X \) and \( y = Ax^2 + Bx + C \), by applying the Bézout theorem we get that

\[
|S_1(\mathbb{F}_q)| \leq 2(q^2 + q + 1).
\]

This bound can be improved, as we will see. Using the fact that the surface is irreducible, we can apply the well-known Lang-Weil bound.

**Theorem 4.5** ([14]). Given nonnegative integers \( n, d \) and \( r \), with \( d > 0 \), there is a positive constant \( A(n, d, r) \) such that for every finite field \( \mathbb{F}_q \), and every irreducible subvariety \( X \subseteq \mathbb{P}^n(\mathbb{F}_q) \) of dimension \( r \) and degree \( d \), we have

\[
||X(\mathbb{F}_q)| - q^r| \leq (d - 1)(d - 2)q^{r - \frac{1}{2}} + A(n, d, r)q^{r - 1}
\]

**Corollary 4.6.** The number of \( \mathbb{F}_q \)-rational points on the surface \( S_1(\mathbb{F}_q) \) is limited by

\[
q^2 + 2q^{\frac{3}{2}} + A(3, 3, 2)q.
\]

This bound is better than Bézout’s, and other theoretical estimates are known (see [3]), but we want to improve the estimation and arrive at a bound in the form

\[
S_1(\mathbb{F}_q) \leq q^2 + \eta q + \mu
\]

where \( \mu < q \) and \( \eta \) is upper bounded by a constant (independent from \( q \) and \( \mu \)). Experimentally we found the following

**Fact 4.7.** For \( q \in \{2, \ldots, 29\} \) it is \( |\eta| \leq 2 \) and \( \mu = 1 \).

**Conjecture 4.8.** \( |\eta| \leq 2 \) and \( \mu = 1 \) for all \( q \).

Let us recall some previous results

**Theorem 4.9** (Theorem 27.1, [16]). Let \( S \) be a cubic surface over \( \mathbb{F}_q \). If \( S \) is birationally trivial then

\[
|S(\mathbb{F}_q)| \equiv 1 \mod q.
\]
In the case in which $S_1$ is smooth we also know the possible values for $|S_1(\mathbb{F}_q)|$.

**Theorem 4.10** (Theorem 23.1, [16]). Let $S$ be a smooth irreducible cubic surface over $\mathbb{F}_q$, then the number of points of $S(\mathbb{F}_q)$ is exactly

$$|S(\mathbb{F}_q)| = q^2 + \eta q + 1$$

where $\eta \in \{-2, -1, 0, 1, 2, 3, 4, 5, 7\}$.

Theorem (4.10) suggests us to consider separately the case in which $S_1$ is smooth from the case in which it is singular, indeed (4.10) gives a good bound for the smooth case.

## 5 Preliminaries on the singular case

From now on we investigate when $S_1$ is singular. We start with observing that the possible singular points can only be isolated double points, since $S_1$ is a cubic irreducible surface. In this context the following result is very helpful.

**Theorem 5.1** ([5]). Let $S \subset \mathbb{P}^3(\mathbb{K})$ be a singular irreducible cubic surface defined on the field $\mathbb{K}$. Let $\bar{S} = S(\bar{\mathbb{K}})$ be the surface defined by $S$ over $\bar{\mathbb{K}}$, the algebraic closure of $\mathbb{K}$. Let $\delta$ be the number of isolated double points of $\bar{S}$. Then $\delta \leq 4$ and $S$ is birationally equivalent (over $\mathbb{K}$) to

(i) $\mathbb{P}^2(\mathbb{K})$ if $\delta = 1, 4$;

(ii) a smooth Del Pezzo surface of degree 4 if $\delta = 2$;

(iii) a smooth Del Pezzo surface of degree 6 if $\delta = 3$.

Recall that a smooth Del Pezzo surface is a smooth projective surface $V$ whose anticanonical class is ample. Many arithmetic properties of these surfaces were investigated by Manin; see [16].

What we want to do now is to find a bound in the desired form for the four possible cases of singularities ($\delta = 1, 2, 3, 4$).

Clearly the singular points on $S_2$ correspond to the solutions of

$$
\begin{align*}
X_0X_1X_2 &= AX_0^2 + A^qX_1^2 + A^{q^2}X_2^2 + BX_0 + B^qX_1 + B^{q^2}X_2 + D \\
X_1X_2 &= 2AX_0 + B \\
X_0X_2 &= 2A^qX_1 + B^q \\
X_0X_1 &= 2A^{q^2}X_2 + B^{q^2}
\end{align*}
$$

(8)
Remark 5.2. Since $S_1$ is defined over $\mathbb{F}_q$ if $P \in S_1(\mathbb{F}_q)$ is a singular point then its conjugates with respect to the Frobenius automorphism are singular.

Remark 5.3. Notice also if a singular point of $S_2$ is $\mathbb{F}_q^6$-rational the corresponding singularity of $S_1$ will be $\mathbb{F}_q^2$-rational.

Before delving into the classification of the four cases arising from different values of $\delta$, we need to examine separately the case $B = 0$, which turns out to be special.

5.1 Case $B = 0$

In this case the singularities of the surface correspond to the solutions of

$$\begin{cases} 
X_0X_1X_2 = AX_0^2 + A^qX_1^2 + A^{q^2}X_2^2 + D \\
X_1X_2 = 2AX_0 \\
X_0X_2 = 2A^qX_1 \\
X_0X_1 = 2A^{q^2}X_1 
\end{cases} \quad (9)$$

A direct computation leads to the fact that if $(\bar{x}_0, \bar{x}_1, \bar{x}_2) \neq (0, 0, 0)$ is a singular point, then each $\bar{x}_i$ is different from zero.

Proposition 5.4. The possible singular case for $B = 0$ are

(i) $P = (0, 0, 0)$ is the only singular point, this happens if and only if $D = 0$;

(ii) $q$ is odd and $\delta = 4$, this happens if and only if $-\frac{D}{A}$ is a square.

Proof. Direct computations show that (i) comes from Equation [9], so we are left with the case $q$ odd and $(0, 0, 0)$ not singular. Substituting the derivatives into the equation that defines the surface we get

$$\begin{cases} 
-AX_0^2 + A^qX_1^2 + A^{q^2}X_2^2 + D = 0 \\
AX_0^2 - A^qX_1^2 + A^{q^2}X_2^2 + D = 0 \\
AX_0^2 + A^qX_1^2 - A^{q^2}X_2^2 + D = 0.
\end{cases}$$

Summing pairwise the equations gives us

$$\begin{cases} 
2AX_0^2 + 2D = 0 \\
2A^qX_1^2 + 2D = 0 \\
2A^{q^2}X_2^2 + 2D = 0
\end{cases}$$
and, since $q$ is odd and $A \neq 0$, we can deduce the system

$$\begin{cases} X_0^2 = -\frac{D}{A} \\ X_1^2 = -\frac{D}{A^q} \\ X_2^2 = -\frac{D}{A^{q^2}} \end{cases}$$ (10)

The fact that $\beta \in \mathbb{F}_q$ is a square if and only if $\beta^q$ is a square implies that all the equations of (10) are solvable if and only if the first one is. Therefore (10) is solvable if and only if $-\frac{D}{A}$ is a square.

Suppose that $-\frac{D}{A}$ is a square, $-\frac{D}{A} = \gamma^2$. From the equation of the surface it follows that $S_2$ has four singularities of the form $(\gamma, \gamma^q, \gamma^{q^2}), (\gamma, -\gamma^q, -\gamma^{q^2}), (-\gamma, \gamma^q, -\gamma^{q^2}), (-\gamma, -\gamma^q, \gamma^{q^2})$.

**Remark 5.5.** Notice that in case $D \neq 0$ and $-\frac{D}{A}$ square, the four singular points cannot be all conjugates with respect to the Frobenius automorphism.

### 5.2 One singular point

From now on we can consider $B \neq 0$. From Remark 5.2 if $S_1$ has one singular (double) point $P$ then $P$ has to be $\mathbb{F}_q$-rational, otherwise also its conjugate should be singular. Consider now the sheaf of $\mathbb{F}_q$-rational lines passing through $P$: each line, not contained in $S_1(\mathbb{F}_q)$, can intersect $S_1(\mathbb{F}_q)$ in at most one more point since $P$ is a double point and $S_1$ has degree three. So the number of $\mathbb{F}_q$-rational points of $S_1$ is given by

$$|S_1(\mathbb{F}_q)| \leq (q^2 + 1) + h(q - 1) = q^2 + hq + 1 - h$$

where $h$ is the number of lines contained in $S_1$ and passing through $P$.

**Proposition 5.6.** With the same notation as before we have $h = 0$.

**Proof.** We want to give a bound for the maximal number of ($\mathbb{F}_q$-rational) lines fully contained in $S_1$ and passing through $P$. For simplicity we proceed with the computations on $S_2$, since the number of these lines will be the same. Suppose the corresponding singular point $Q$ on $S_2$ has coordinates $(a, a^q, a^{q^2})$. Then, since it is the only singular point, we have that $Q$ is the only point that satisfies (8). Consider now the sheaf of lines passing through $Q$, which has parametric equation, for $b \neq 0$

$$\begin{cases} X_0 = bt + a \\ X_1 = b^q t + a^q \\ X_2 = b^{q^2} t + a^{q^2} \end{cases}$$
and after doing the substitution we get that, if the line is contained into \( S_2 \),
\[
p_3 t^3 + p_2 t^2 + p_1 t
\]
has to be the zero polynomial in \( \mathbb{F}_q[t] \), where
\[
p_3 := b^{q^2+q+1}
p_2 := -Ab^2 - (Ab^2)^q - (Ab^2)^{q^2} + b^{q+1}a^{q^2} + b^{q^2+1}a^{q^2} + b^{q^2+q}a
\]
\[
p_1 := -2Aab - 2(Aab)^q - 2(Aab)^{q^2} - Bb - (Bb)^q - (Bb)^{q^2} + ba^{q^2+q} + b^a a^{q^2+1} + b^{q^2}a^{q^2+1}
\]
From the fact that \( p_3 = 0 \) we have that \( N(b) \) has to be equal to zero, but this means that \( b \) is equal to zero, which is a contradiction.

Putting together the previous observations we have the following result.

**Proposition 5.7.** If \( S_1 \) has one singular \( \mathbb{F}_q \)-rational point then
\[
|S_1(\mathbb{F}_q)| \leq q^2 + 1. \tag{11}
\]

### 5.3 Two singular points

Call \( P_1 \) and \( P_2 \) the two singular points of \( S_1 \), from Remark 5.2 there are two possibilities:

(i) \( P_1 \) and \( P_2 \) are \( \mathbb{F}_q \)-rational;

(ii) \( P_1 \) and \( P_2 \) are \( \mathbb{F}_q^2 \)-rational and conjugates.

If (i) happens then (11) holds and we can use that bound.

We look for a bound when (ii) happens: call \( r \) the line passing through \( P_1 \) and \( P_2 \), since it fixes the conjugate points then it has to be \( \mathbb{F}_q \)-rational and moreover this line has to be contained in \( S_1(\mathbb{F}_q) \) since the intersection multiplicity of this line is at least 2 in both \( P_1 \) and \( P_2 \) and the surface has degree 3. Now consider the pencil of planes passing through \( r \) and consider the cubic curve \( C \) defined as intersection between any of these planes and \( S_1 \). Clearly \( C \) is reducible and there are two possible situations

1. \( C \) is completely reducible. In this case \( C \) is the product of three lines contained in the surface.

   Call \( s \) and \( s' \) the two lines different from \( r \): \( s \) and \( s' \) cannot be \( \mathbb{F}_q \)-rational since they do not fix the conjugates, so they are \( \mathbb{F}_q^2 \)-rational. From the fact that they are contained in \( S_1 \) and they pass through conjugate points we have that \( s' = s^q \). From this fact we have that the number of \( \mathbb{F}_q \)-rational points on \( C \setminus r \) is 1 and that point is \( s \cap s' \).
2. $\mathcal{C}$ is the product of $r$ and an irreducible conic $\mathcal{D}$ contained in the surface and it contains exactly $q$ points, see [12, Lemma 7.2.3]. In this case the number of $\mathbb{F}_q$ rational points of $\mathcal{D}$ not contained in $r$ is exactly $q - 2$.

From the analysis of the two possible cases, recalling that the maximum number of lines contained in a cubic surface is 27 (see [16, Chapter IV]), the first situation can happen at most in 13 cases, and so we have:

\[
q + (q - 13)(q - 2) + 13 \leq |S_1(\mathbb{F}_q)| \leq q(q - 2) + q
\]

Putting together the previous observations we have the following result.

**Proposition 5.8.** If $S_1$ has two singular $\mathbb{F}_q^2$-rational conjugate points then

\[
q^2 - 14q + 39 \leq |S_1(\mathbb{F}_q)| \leq q^2 - q.
\]
5.4 Three singular points

Call $P_1$, $P_2$ and $P_3$ the singular points of $S_1$, from Remark 5.2 we have the following configurations:

(i) At least one among $P_1$, $P_2$ and $P_3$ is $\mathbb{F}_q$-rational;

(ii) $P_1$, $P_2$ and $P_3$ are $\mathbb{F}_{q^3}$-rational and conjugates.

If (i) happens then (11) holds, so our task now is to find a bound when (ii) happens.

We start with observing that the three points cannot be collinear, which comes directly from the following proposition.

**Proposition 5.9.** Let $C$ be a cubic curve such that it has three double points. Then $C$ is completely reducible and splits in the product of three lines, each passing through a pair of its singular points.

**Proof.** Direct consequence of Bézout’s theorem. □

In order to get an estimation of $|S_1(\mathbb{F}_q)|$ for (ii) we change the model of the surface as the following proposition suggests.

**Proposition 5.10.** Let $S$ be a cubic surface over $\mathbb{P}^3(\mathbb{F}_q)$, considered with projective coordinates $[r_0 : r_1 : r_2 : T]$, and such that it has exactly three conjugates $\mathbb{F}_{q^3}$-rational double points, namely $P_1, P_2$ and $P_3$. Then $S$ is projectively equivalent to the surface having affine equation, for certain $\beta, \gamma \in \mathbb{F}_q$:

$$r_0r_1r_2 + \beta r_0r_1r_2 + \beta^q r_0r_2 + \gamma r_0 + \gamma^q r_1 + \gamma^q r_2 = 0.$$

**Proof.** Up to a change of projective frame we can consider the following situation

- The plane passing through the three points is the plane at infinity $T = 0$ and the triangle of lines through them in that plane is given by $r_0$, $r_1$ and $r_2$;

- $O = (0 : 0 : 0 : 1) \in S$.

From these choices we obtain the following equation for the surface $S$

$$r_0r_1r_2 + T(\alpha_0r_0^2 + \alpha_1r_1^2 + \alpha_2r_2^2 + \beta_0r_0r_1 + \beta_1r_1r_2 + \beta_2r_0r_2) + T^2(\gamma_0r_0 + \gamma_1r_1 + \gamma_2r_2) = 0$$

where $\alpha_i, \beta_i, \gamma_i \in \mathbb{F}_{q^3}$ for $i \in \{0, 1, 2\}$. From the fact that $P_1, P_2$ and $P_3$ are conjugates it follows that $r_0, r_1$ and $r_2$ are conjugates and then we get that $\alpha_1 = \alpha_0^q$, $\alpha_2 = \alpha_0^{q^2}$, $\beta_1 = \beta_0^q$, $\beta_2 = \beta_0^{q^2}$, $\gamma_1 = \gamma_0^q$ and $\gamma_2 = \gamma_0^{q^2}$. Consider now the plane $\pi$ passing through $P_1, P_2$ and $O$. Without loss
of generality, $P_1$ is the singular point satisfying $T = r_1 = r_2 = 0$, then its coordinates will be $P_1 = (p_1, p_2, p_3, 0)$. Consider now the line, namely $s$ passing through $P_1$ and $O$. A general point on that line has coordinates $P_{\lambda, \mu} = (\lambda p_0, \lambda p_1, \lambda p_2, \mu)$. Substituting the coordinates of $P_{\lambda, \mu}$ into the equation of $S$ we obtain

$$0 = \alpha_0 \lambda r_0^2(P_{\lambda, \mu}) + \beta_0 \lambda^2 r_0(P_{\lambda, \mu}) = \lambda(\alpha_0 r_0^2(P_{\lambda, \mu}) + \beta_0 \lambda r_0(P_{\lambda, \mu})).$$

Now since $r_0(P_1) \neq 0$ and we want $(0, \mu)$ as double solution then $\alpha_0 = 0$. Iterating this process the equation of the surface becomes

$$r_0 r_1 r_2 + T(\beta_0 r_0 r_1 + \beta_0^2 r_1 r_2 + \beta_0^2 r_0 r_2) + T^2(\gamma_0 r_0 + \gamma_0^2 r_1 + \gamma_0^2 r_2) = 0.$$

We want to reduce the problem of counting the points in the form $(\alpha, \alpha^q, \alpha^{q^2})$ on the cubic surface to the problem of counting the points in the same form on a certain quadric. To achieve the result we apply the Cremona transform, call

$$z_1 := \frac{1}{r_1}, \quad z_2 := \frac{1}{r_2}, \quad z_3 := \frac{1}{r_3},$$

dividing the equation of the surface by $r_1 r_2 r_3$ we obtain

$$Q : \beta z_3 + \beta^q z_1 + \beta^{q^2} z_2 + \gamma z_2 z_3 + \gamma^q z_1 z_3 + \gamma^{q^2} z_1 z_2 - 1 = 0.$$

Note that if $\gamma = 0$ then $Q$ collapse to a plane.

**Proposition 5.11.** The quadric surface $Q$ is absolutely irreducible.

**Proof.** If $\gamma = 0$ there is nothing to prove, since $Q$ is a plane. Suppose $\gamma \neq 0$ and that $Q$ splits in the product of two planes $\pi_1$ and $\pi_2$, then

$$\beta z_3 + \beta^q z_1 + \beta^{q^2} z_2 + \gamma z_2 z_3 + \gamma^q z_1 z_3 + \gamma^{q^2} z_1 z_2 - 1 = (a_1 z_1 + a_2 z_2 + a_3 z_3 + a_4)(d_1 z_1 + d_2 z_2 + d_3 z_3 + d_4).$$

From the identity principles of polynomials we get that $a_1 d_1 = a_2 d_2 = a_3 d_3 = 0$ Without loss of generality we can consider $a_1 = a_2 = d_3 = 0$ and then the equation becomes

$$\beta z_3 + \beta^q z_1 + \beta^{q^2} z_2 + \gamma z_2 z_3 + \gamma^q z_1 z_3 + \gamma^{q^2} z_1 z_2 - 1 = (a_3 z_3 + a_4)(d_1 z_1 + d_2 z_2 + d_4)$$

and this cannot happen since in the right hand side of this equality we do not have the term $z_1 z_2$. \qed
We want to count the points on the quadric $Q$ in the form $(\delta, \delta q, \delta q^2)$, where $\delta \in \mathbb{F}_{q^3}$. Writing down $\delta$ on the normal basis $B$ we get $\delta = w_1 \alpha + w_2 \alpha^q + w_3 \alpha^{q^2}$. Taking $w_1, w_2$ and $w_3$ as a set of variables (on $\mathbb{F}_q$) we obtain a $\mathbb{F}_q$-rational quadric surface and its $\mathbb{F}_q$-rational points are in one-to-one correspondence with the searched ones.

$$\beta(w_1 \alpha^{q^2} + w_2 \alpha + w_3 \alpha^q) + \beta q(w_1 \alpha + w_2 \alpha^q + w_3 \alpha^{q^2}) + \beta q^2(w_1 \alpha^q + w_2 \alpha^{q^2} + w_3 \alpha) +$$

$$\gamma(w_1 \alpha + w_2 \alpha^q + w_3 \alpha^{q^2})(w_1 \alpha^q + w_2 \alpha q^2 + w_3 \alpha) + \gamma q(w_1 \alpha^{q^2} + w_2 \alpha + w_3 \alpha^q)(w_1 \alpha^q + w_2 \alpha^{q^2} + w_3 \alpha) +$$

$$\gamma q^2(w_1 \alpha q^2 + w_2 \alpha + w_3 \alpha^q)(w_1 \alpha + w_2 \alpha^q + w_3 \alpha^{q^2}) - 1 = 0.$$  

The points we were looking for of the first surface are in one-to-one correspondence with the $\mathbb{F}_q$-rational points on the quadric surface above. It is widely known (see [11, Section 15.3]) that, in this case

$$|S_1(\mathbb{F}_q)| = q^2 + \eta q + 1, \quad \eta \in \{0, 1, 2\}$$  \hspace{1cm} (13)$$

since the quadric surface $Q$ is irreducible.

5.5 Four singular points

Call $P_1$, $P_2$, $P_3$ and $P_4$ the singular points of $S_1$, applying Remark 5.2 we have the following possibilities:

(i) At least one among $P_1$, $P_2$, $P_3$ and $P_4$ is $\mathbb{F}_q$-rational;

(ii) There are two couples of $\mathbb{F}_{q^2}$-rational and conjugates singular points.

(iii) $P_1$, $P_2$, $P_3$ and $P_4$ are $\mathbb{F}_{q^4}$-rational and conjugates.

If (i) or (ii) hold then we have already found out a good bound before, the last thing we have to do is show that (iii) never holds.

Proposition 5.12. Case (iii) never holds.

Proof. In order to solve this problem we use a multivariate approach, calculating the elimination ideal with respect to all the variables less one. Consider the equations in (8): it is clear that, given $X_1$ and $X_2$, the value of $X_0$ is uniquely determined. For this reason we proceed with eliminating the variables $X_0$ and $X_1$ and we obtain the elimination ideal $I_{x_0,x_1} = \langle p_1, p_2 \rangle$, where
\[ p_1(X_1) = 2X_1^5 A^q + X_1^4 B^q - 16X_1^3 A^{q^2+q+1} - 8X_1^2 A^{q^2+1} B^q - X_1^2 B^{q^2+1} + 32X_1 A^2 q^2 + q + 2 \\
- 2X_1 A B^{2q^2} - 2X_1 A^q B^2 + 16A^{2q^2+2} B^q - 4A^{q^2+1} B^{q^2+1} \\
p_2(X_1) = (X_1^2 - 4A^{q^2+1})(X_1^4 A^q + X_1^3 B^q - 4X_1^2 A^{q^2+q+1} + X_1^2 D - 4X_1 A^{q^2+1} B^q \\
+ x_1 B^{q^2+1} - 4A^{q^2+1} D + AB^{2q^2} + A^{q^2} B^2). \]

On the other hand, if we proceed eliminating the variables \( X_0 \) and \( X_1 \) we get the elimination ideal \( J_{x_0,x_1} = \langle q_1, q_2 \rangle \), where \( q_1 = p_1(X_2)^q \) and \( q_2 = p_2(X_2)^q \). The fact that the two ideals are generated by conjugate polynomials will continue to be true after symmetric annihilation of some of their terms. After further computations using the software MAGMA, which can be completely seen in [3], we get that one of the generators of \( I_{x_1,x_2} \) is a polynomial of degree lower or equal to two, namely \( f(X_1) \), and one of the generators of \( J_{x_0,x_2} \) is \( f(X_2)^q \). From this fact we get that the singularities of \( S_2 \) are at most four and if this value is achieved then they belong (at most) to the field \( \mathbb{F}_{q^2} \), which means that the singularities of \( S_1 \) are at most in the field \( \mathbb{F}_{q^2} \).

\[ \square \]

6 Case \( r = 3 \) and \( h = 3 \)

Consider the case of the intersection over \( \mathbb{F}_{q^3} \) between \( \mathcal{X} \) and the curves \( y = Ax^3 + Bx^2 + Cx + D \), \( A, B, C, D \in \mathbb{F}_{q^3} \) and \( A \neq 0 \). After doing similar computations to those done for the case \( r = 3 \) and \( h = 2 \) we arrive at an equation of a cubic surface \( \tilde{S}_1 = \tilde{S}_1(\mathbb{F}_q) \) defined over \( \mathbb{F}_q \), affinely equivalent to a surface \( \tilde{S}_2 = \tilde{S}_2(\mathbb{F}_q) \) defined over \( \mathbb{F}_{q^3} \), having equation

\[ X_0X_1X_2 = AX_0^3 + A^q X_1^3 + A^{q^2} X_2^3 + BX_0^2 + B^q X_1^2 + B^{q^2} X_2^2 + CX_0 + C^q X_1 + C^{q^2} X_2 + E \]

where \( E = T(D) \). In this more general case \( \tilde{S}_1 \) may be reducible, which can possibly increase the number of \( \mathbb{F}_q \)-rational points of \( \tilde{S}_1 \), but on the other hand the reasonings done for \( r = 3 \) and \( h = 2 \) can be completely extended if \( \tilde{S}_1 \) is irreducible, so we claim the following result.

**Theorem 6.1.** Let \( r = h = 3 \) and consider the \( \mathbb{F}_q \)-rational cubic surface \( \tilde{S}_1 \) associated to the intersections between \( \mathcal{X} \) and \( y = Ax^3 + Bx^2 + Cx + D \). If \( \tilde{S}_1 \) is irreducible then

\[ |\tilde{S}_1| \leq q^2 + 7q + 1. \]
7 AG codes from the Norm-Trace curves

Consider the norm-trace over the field $\mathbb{F}_{q^3}$: since $r = 3$, $\mathcal{X}$ has $N = q^{2r-1} = q^5 \mathbb{F}_{q^3}$-rational points in $\mathbb{A}^2(\mathbb{F}_{q^3})$. We also know that $L_{\mathbb{F}_{q^3}}(2q^2P_{\infty}) = \{ay + bx^2 + cx + d \mid a, b, c, d \in \mathbb{F}_{q^3}\}$. Considering the evaluation map

$$\text{ev} : L_{\mathbb{F}_{q^3}}(2q^2P_{\infty}) \longrightarrow (\mathbb{F}_{q^3})^{q^5}$$

$$f = \tilde{a}y + \tilde{b}x^2 + \tilde{c}x + \tilde{d} \longrightarrow (f(P_1), \ldots, f(P_N))$$

the associated one-point code will be $C_{\mathcal{L}}(D, 2q^2P_{\infty}) = \text{ev}(L_{\mathbb{F}_{q^3}}(2q^2P_{\infty}))$, where the divisor $D$ is the formal sum of all the $q^5$-rational affine points of $\mathcal{X}(\mathbb{F}_{q^3})$. The weight of a codeword associated to the evaluation of a function $f \in L_{\mathbb{F}_{q^3}}(2q^2P_{\infty})$ corresponds to

$$w(\text{ev}(f)) = |\mathcal{X}(\mathbb{F}_{q^3})| - |\mathcal{X}(\mathbb{F}_{q^3}) \cap \{\tilde{a}y + \tilde{b}x^2 + \tilde{c}x + \tilde{d} = 0\}|.$$  

1. If $\tilde{a} = 0$ then we have to study the common zeroes of $\tilde{b}x^2 + \tilde{c}x + \tilde{d}$ and $\mathcal{X}(\mathbb{F}_{q^3})$.
   (a) if $\tilde{b} = \tilde{c} = \tilde{d} = 0$ then $w(\text{ev}(f)) = 0$;
   (b) if $\tilde{b} = \tilde{c} = 0$ and $\tilde{d} \neq 0$ then $w(\text{ev}(f)) = q^5$;
   (c) if $\tilde{b} = 0$ and $\tilde{c} \neq 0$ then $w(\text{ev}(f)) = q^5 - q^2$;
   (d) if $\tilde{c} \neq 0$ and $\tilde{c}^2 - 4\tilde{b}\tilde{d} = 0$ then $w(\text{ev}(f)) = q^5 - q^2$;
   (e) otherwise $w(\text{ev}(f)) = q^5 - 2q^2$.

2. On the other hand, if $\tilde{a} \neq 0$ then we have to study the common zeroes between $\mathcal{X}(\mathbb{F}_{q^3})$ and $\tilde{a}y + \tilde{b}x^2 + \tilde{c}x + \tilde{d}$.
   (a) if $\tilde{b} = \tilde{c} = \tilde{d} = 0$ then $w(\text{ev}(f)) = q^5 - 1$;
   (b) if $\tilde{b} = \tilde{c} = 0$ and $\tilde{d} \neq 0$ then $w(\text{ev}(f)) = q^5 - q^2$;
   (c) if $\tilde{b} = 0$ and $\tilde{c} \neq 0$ then, applying Bézout theorem, we have that $w(\text{ev}(f)) \geq q^5 - (q^2 + q + 1)$;
   (d) otherwise, from what we said previously, $w(\text{ev}(f)) \geq q^5 - (q^2 + 7q + 1)$.

We can summarize our reasonings in the following result.

**Theorem 7.1.** Consider the norm-trace curve $\mathcal{X}$ over the field $\mathbb{F}_{q^3}$, $q \geq 8$, and the AG code $C = C(D, 2q^2P_{\infty})$ arising from $\mathcal{X}$, where $D = \sum_{P \in \mathcal{X}(\mathbb{F}_{q^3}) \setminus P_{\infty}} P$. Let $\{A_w\}_{0 \leq w \leq q^5}$ be the weight distribution of $C$, then the following results hold
(i) \( A_0 = 1; \)

(ii) The minimum distance of \( C \) is \( q^5 - 2q^2; \)

(iii) If \( w > q^5 - 2q^2 \) and \( A_w \neq 0 \) then \( w \geq q^5 - q^2 - 7q - 1; \)

In the cases \( q < 8 \), i.e. \( q = 2, 3, 5, 7 \), the complete \( \{A_w\}_{w \leq q^5} \) can be determined with a computer and we do not report it here.

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