The informal semantics of Answer Set Programming: A Tarskian perspective

Marc Denecker
Department of Computer Science, KU Leuven, 3001 Leuven, Belgium
(e-mail: marc.denecker@cs.kuleuven.be)

Yuliya Lierler
Department of Computer Science, University of Nebraska at Omaha, Omaha, NE 68182, USA
(e-mail: ylierler@unomaha.edu)

Miroslaw Truszczynski
Department of Computer Science, University of Kentucky, Lexington, KY 40506-0633, USA
(e-mail: mirek@cs.uky.edu)

Joost Vennekens
Department of Computer Science, KU Leuven, Campus De Nayer, 2860 Sint-Katelijne-Waver, Belgium
(e-mail: joost.vennekens@cs.kuleuven.be)

submitted 1 January 2003; revised 1 January 2003; accepted 1 January 2003

Abstract

In Knowledge Representation, it is crucial that knowledge engineers have a good understanding of the formal expressions that they write. What formal expressions state intuitively about the domain of discourse is studied in the theory of the informal semantics of a logic. In this paper we study the informal semantics of Answer Set Programming. The roots of answer set programming lie in the language of Extended Logic Programming, which was introduced initially as an epistemic logic for default and autoepistemic reasoning. In 1999, the seminal papers on answer set programming proposed to use this logic for a different purpose, namely, to model and solve search problems. Currently, the language is used primarily in this new role. However, the original epistemic intuitions lose their explanatory relevance in this new context. How answer set programs are connected to the specifications of problems they model is more easily explained in a classical Tarskian semantics, in which models correspond to possible worlds, rather than to belief states of an epistemic agent. In this paper, we develop a new theory of the informal semantics of answer set programming, which is formulated in the Tarskian setting and based on Frege’s compositionality principle. It differs substantially from the earlier epistemic theory of informal semantics, providing a different view on the meaning of the connectives in answer set programming and on its relation to other logics, in particular classical logic.

KEYWORDS: informal semantics, knowledge representation, answer-set programming

1 Introduction

I am not here in the happy position of a mineralogist who shows his audience a rock-crystal: I cannot put a thought in the hands of my readers with the request that they should examine it from all sides. Something in itself not perceptible by sense, the thought is presented to the reader—and I must be content with that—wrapped up in a perceptible linguistic form.

Gottlob Frege, Der Gedanke
In knowledge representation, a human expert expresses informal propositions about the domain of discourse by a formal expression in some logic $\mathcal{L}$. The latter is formulated in a vocabulary for which the expert has an intended interpretation $\mathcal{I}$ specifying the meaning of the vocabulary symbols in the domain. It is essential that the human expert understands which informal propositions about the problem domain are expressed by formal expressions of $\mathcal{L}$. The fundamental task of the informal semantics of a logic $\mathcal{L}$, sometimes called the declarative reading or the intuitive interpretation of $\mathcal{L}$, is to provide this understanding by explaining formal expressions of $\mathcal{L}$ (or of a fragment of $\mathcal{L}$) as precise informal propositions about the domain of discourse.

In Frege’s terms, an informal proposition is a thought in the mind of the expert. It is not a tangible object and is not perceptible by others. To be observed, studied and used, it must be presented in linguistic form. In this paper, we will therefore assume that the domain expert’s intended interpretation $\mathcal{I}$ maps each vocabulary symbol to some natural language statement that represents the imperceptible “thought” that this symbol is supposed to represent. Likewise, the informal semantics of a logic $\mathcal{L}$ is a mapping that, given such an intended interpretation $\mathcal{I}$, assigns to a formal expression $\varphi$ of $\mathcal{L}$ a natural language reading $\mathcal{I}(\varphi)$ that captures the meaning of $\varphi$ in $\mathcal{I}$.

We already hint here that natural language is to play a key role in any study of informal semantics. Such use of natural language may be controversial. For instance, Barwise and Cooper (1981) say: “To most logicians (like the first author) trained in model-theoretic semantics, natural language was an anathema, impossibly vague and incoherent.” Upon closer inspection, however, the situation is not quite so dire. Indeed, Barwise and Cooper (1981) go on to say that: “To us, the revolutionary idea [...] is the claim that natural language is not impossibly incoherent [...], but that large portions of its semantics can be treated by combining known tools from logic, tools like functions of finite type, the $\lambda$-calculus, generalized quantifiers, tense and modal logic, and all the rest.” In this article, we subscribe to this more optimistic view on natural language. While it is certainly possible to create vague, ambiguous or meaningless natural language statements, we believe that a careful use of suitable parts of natural language can avoid such problems. Indeed, much of science and mathematics throughout the centuries has been developed by means of a clear and precise use of natural language. It is this same clarity and precision that we want to achieve in our study of informal semantics.

The main goal of this paper is to study the informal semantics of answer set programming (ASP) — a broadly used logic-based knowledge representation formalism (Marek and Truszczynski 1999; Niemelä 1999; Brewka et al. 2011; Gebser et al. 2012). ASP has its roots in extended logic programming (ELP) proposed by Gelfond and Lifschitz (1988; 1991). As part of the development of ELP, Gelfond and Lifschitz presented an informal semantics $\mathcal{I}_{ELP}$ for extended logic programs based on epistemic notions of default and autoepistemic reasoning. According to their proposal, an extended logic program expresses the knowledge of a rational introspective agent, where a stable model (or an answer set) represents a possible state of belief of the agent by enumerating all literals that are believed in that state. The Gelfond-Lifschitz informal semantics is attuned to applications in epistemic domains. However, it is not well aligned with others.

A decade after ELP was conceived, researchers realized that, in addition to modeling applications requiring autoepistemic reasoning, the language can be used for modeling and solving combinatorial search and optimization problems (Marek and Truszczynski 1999; Niemelä 1999). The term answer set programming was proposed shortly thereafter by Lifschitz (1999; 2002) to be synonymous with the practice of using extended logic programs for this type of applications. Since then, ASP has gained much attention and evolved into a computational knowledge repre-
The informal semantics of Answer Set Programming: A Tarskian perspective

sentation paradigm capable of solving search problems of practical significance (Brewka et al., 2011). Particularly influential was the emergence of a methodology to streamline the task of programming in this paradigm. It consists of arranging program rules in three groups: one to generate the search space, one to define auxiliary concepts, and one to test (impose) constraints. Lifschitz (2002) coined the term generate-define-test (GDT) to describe it. Programs obtained by following the GDT methodology, or GDT programs, for short, form the overwhelming majority of programs arising in search and optimization applications.

However, for GDT programs, the epistemic informal semantics is inappropriate and ineffective in its role. To illustrate this point, consider the graph coloring problem. One of the conditions of the problem is the following informal proposition:

“each node has a color”.

In the language of ELP of 1999, this condition can be expressed by the rule

\[ \text{Aux} \leftarrow \text{not} \, \text{Aux}, \, \text{Node}(x), \, \text{not} \, \text{Colored}(x). \]  

(2)

The reading that the \( \mathcal{ELP} \) informal semantics provides for rule (2) is: “for every \( x \), Aux holds if the agent does not know Aux and \( x \) is a node and the agent does not know that \( x \) has a color.”

There is an obvious mismatch between this sentence and the simple (objective, non-epistemic) proposition (1) that rule (2) intends to express. In other words, in this example, the explanatory power of the epistemic informal semantics diminishes. It fails to provide a direct, explicit link between the formal expression on the one side, and the property of objects in the domain of discourse it is intended to represent, on the other.

Modern ASP dialects typically provide a more elegant notation for writing down constraints, such as:

\[ \leftarrow \text{Node}(x), \, \text{not} \, \text{Colored}(x). \]  

(3)

However, in itself this does not address or fix the mismatch. Moreover, as we discuss further on in this paper, it is often surprisingly difficult to extend the Gelfond-Lifschitz epistemic informal semantics to cover the new language constructs of modern ASP dialects.

At the root of the mismatch lies the reflective epistemic agent. A key aspect of the original applications for ELP was the presence of such an agent in the domain of discourse; typically it was a knowledge base that reflects on its own content. Such an agent is absent in the graph coloring problem and in typical problems that are currently solved using ASP. For example, there are no benchmarks in the series of ASP competitions (Gebser et al., 2007b; Denecker et al., 2009; Calimeri et al., 2011; Alviano et al., 2013a) that mention or require an epistemic introspective agent.

In this paper, we present a new theory \( \mathcal{OB}_T \) of the informal semantics for ASP. We call it Tarskian because it interprets an answer set of an ASP program in the same way as a model of a first-order logic (FO) theory is interpreted—namely, as an abstraction of a possible state of affairs of the application domain, and not epistemically as a state of beliefs of an agent. We define this theory for the pragmatically important class of GDT programs and their subexpressions. Our informal semantics explains the formal semantics of ASP under the Tarskian view of answer sets. It offers an explanation of the meaning of connectives, including “non-classical” ones, and it satisfies Frege’s compositionality principle. Under the new semantics, the mismatch between the information the user encapsulated in the program and the intended reading of the program disappears. For example, it maps the constraint (3) to the informal proposition (1). It is worth
noting that the epistemic semantics $\mathcal{GL}_I$ reflects the fact that ASP’s roots are in the domain of commonsense reasoning. By contrast, the informal semantics that we introduce here uses the kind of natural language constructs common in mathematical texts.

A major issue in building an informal semantics for ASP concerns the structure of programs. Formally, programs are “flat” collections of rules. However, to a human expert, GDT programs have a rich internal structure. To build our informal semantics $\mathcal{IP}$, we develop an “intermediate” logic ASP-FO that is directly inspired by the GDT methodology, in which the (hidden) internal structure of GDT programs is made explicit. This structure supports the use of the compositionality principle when defining the informal semantics for the logic ASP-FO. We show that by exploiting splitting results for ASP (Ferraris et al., 2011), programs constructed following the GDT methodology can be embedded in the logic ASP-FO. Thanks to the embedding, our discussion applies to the fragment of ASP consisting of GDT programs and establishes an informal semantics for this class of programs.

The paper is organized as follows. We start by reviewing theories of informal semantics for two logics: the one of first-order logic (Section 2), which provides guidance for our effort, and the Gelfond-Lifschitz theory of informal semantics for ELP (Section 3), with which we contrast our proposal. We then discuss the class of GDT programs, the focus of our work (Section 4), and present the logic ASP-FO as a way of making the internal structure of GDT programs explicit (Section 5). Section 6 then presents the main contribution of this paper: the informal semantics $\mathcal{IP}$. Section 7 presents a number of formal results in support of this information semantics. We finish with a discussion of related work (Section 8) and some conclusions (Section 9).

2 The formal and informal semantics of first-order logic

In this section, we introduce classical first order logic (FO), with special attention to its informal semantics. This serves two purposes. First, it is meant as an introduction of the interplay between formal and informal semantics. Much of that will be reused for the logic ASP-FO that we define later in this paper. Second, ASP-FO is a proper superset of FO. Hence, its formal and informal semantics will extend that of FO.

**Formal syntax of FO** We assume an infinite supply of non-logical symbols: predicate and function symbols, each with a non-negative integer arity. Predicate and function symbols of arity 0 are called propositional and object symbols, respectively. A vocabulary $\Sigma$ is a set of non-logical symbols.

A term $t$ is an object symbol or a compound expression $f(t_1, \ldots, t_n)$, where $f$ is an $n$-ary function symbol and the $t_i$’s are terms. An atom is an expression $P(t_1, \ldots, t_n)$, where $P$ is an $n$-ary predicate symbol and the $t_i$’s are terms (in particular, propositional symbols are atoms). A formula is then inductively defined as follows:

- Each atom is a formula;
- If $t_1$, $t_2$ are terms, then $t_1 = t_2$ is a formula;
- If $\varphi_1$ and $\varphi_2$ are formulas, then so are $\varphi_1 \land \varphi_2$, $\varphi_1 \lor \varphi_2$, $\varphi_1 \Rightarrow \varphi_2$ and $\varphi_1 \Leftrightarrow \varphi_2$;
- If $\varphi$ is a formula, then so are $\neg \varphi$, $\exists x \varphi$ and $\forall x \varphi$, where $x$ is an object symbol, here called a variable.

An occurrence of a symbol $\tau$ in a formula $\varphi$ is *bound* if it is within a subformula of the form
\[\exists \tau \, \psi \text{ or } \forall \tau \, \psi.\] Otherwise, the occurrence is free. In FO, the only symbols that occur bound are object symbols.

Given a vocabulary \(\Sigma\), we call a formula \(\varphi\) a sentence over \(\Sigma\) if all symbols with free occurrences in \(\varphi\) belong to \(\Sigma\). The set of sentences over \(\Sigma\) is denoted \(\mathbb{L}_{\Sigma}\). A theory is a finite set of formulas. A theory over \(\Sigma\) is a finite set of sentences over \(\Sigma\).

**Example 1**
The running application in this section is graph coloring. We choose the vocabulary \(\Sigma_{\text{col}}\) that consists of unary predicates \(\text{Node}\) and \(\text{Colour}\); a binary predicate \(\text{Edge}\); and a unary function \(\text{ColourOf}\) intended to be the mapping from nodes to colors. We define \(T_{\text{col}}\) to be the following sentence over \(\Sigma_{\text{col}}\):

\[\forall x \forall y \, (\text{Edge}(x, y) \Rightarrow \neg \text{ColourOf}(x) = \text{ColourOf}(y)).\]

**Formal semantics of FO** The basic semantic objects of FO are structures for a given vocabulary.

**Definition 1**
Let \(\Sigma\) be a vocabulary. A \(\Sigma\)-structure \(\mathfrak{A}\) consists of (i) a non-empty set \(\text{dom}(\mathfrak{A})\), called the domain of \(\mathfrak{A}\), and (ii) an interpretation function \((\cdot)^{\mathfrak{A}}\) that assigns an appropriate value \(\tau^{\mathfrak{A}}\) to each symbol \(\tau\) of \(\Sigma\):

- The value \(\tau^{\mathfrak{A}}\) of an \(n\)-ary function symbol \(\tau\) is an \(n\)-ary total function over \(\text{dom}(\mathfrak{A})\).
- The value \(\tau^{\mathfrak{A}}\) of an \(n\)-ary predicate symbol \(\tau\) is an \(n\)-ary relation over \(\text{dom}(\mathfrak{A})\).

We call \(\tau^{\mathfrak{A}}\) the interpretation or value of \(\tau\) in \(\mathfrak{A}\). Given a \(\Sigma\)-structure \(\mathfrak{A}\), we denote its vocabulary \(\Sigma\) by \(\Sigma_{\mathfrak{A}}\). We say that \(\mathfrak{A}\) interprets a vocabulary \(\Sigma'\) if it interprets each symbol in \(\Sigma'\) (that is, if \(\Sigma' \subseteq \Sigma_{\mathfrak{A}}\)).

For a given vocabulary \(\Sigma\), we denote by \(\mathcal{S}_{\Sigma}\) the class of all \(\Sigma\)-structures. For a \(\Sigma\)-structure \(\mathfrak{A}\), we define the projection of \(\mathfrak{A}\) on \(\Sigma' \subseteq \Sigma\), written \(\mathfrak{A}|_{\Sigma'}\), to be the \(\Sigma'\)-structure with the same domain as \(\mathfrak{A}\) and the interpretation function \((\cdot)^{\mathfrak{A}}\) of \(\mathfrak{A}\) restricted to \(\Sigma'\). We call \(\mathfrak{A}\) an expansion of \(\mathfrak{A}'\) if \(\mathfrak{A}' = \mathfrak{A}|_{\Sigma_{\mathfrak{A}'}}\).

**Example 2**
For the vocabulary \(\Sigma_{\text{col}}\), consider, for instance, the structure \(\mathfrak{A}_{\text{col}}\) defined as follows:

- \(\text{dom}(\mathfrak{A}_{\text{col}}) = \{n_1, n_2, n_3, c_1, c_2\}\)
- \(\text{Node}^{\mathfrak{A}_{\text{col}}} = \{n_1, n_2, n_3\}\),
- \(\text{Colour}^{\mathfrak{A}_{\text{col}}} = \{c_1, c_2\}\),
- \(\text{Edge}^{\mathfrak{A}_{\text{col}}} = \{(n_1, n_2), (n_2, n_3)\}\) and
- \(\text{ColourOf}^{\mathfrak{A}_{\text{col}}} = \{n_1 \mapsto c_1, n_2 \mapsto c_2, n_3 \mapsto c_1, c_1 \mapsto c_1, c_1 \mapsto c_2\}\).

Note that, in FO, a structure must interpret each function symbol by a total function on its domain. Therefore, while we intend \(\text{ColourOf}\) to be a function that maps nodes to colors, it must

1 This definition correctly handles the case of 0-ary function and predicate symbols. Functions from \(\text{dom}(\mathfrak{A})^0\), i.e., the empty tuple (), into \(\text{dom}(\mathfrak{A})\) can be viewed as elements from \(\text{dom}(\mathfrak{A})\), yielding a standard interpretation of 0-ary function symbols (which are often called constants); and each 0-ary predicate symbol is represented by one of exactly two 0-ary relations over \(\text{dom}(\mathfrak{A})\): () or \(\{(\)\}\), i.e., false or true.
also map each of the colors to some value. This could be avoided by, for instance, allowing partial functions, or by using a typed variant of FO. For simplicity, however, we stay with classical definitions of FO.

For a structure $\mathfrak{A}$, a 0-ary function symbol $\tau$ and a corresponding value $v \in \text{dom}(\mathfrak{A})$, we denote by $\mathfrak{A}[\tau : v]$ the structure identical to $\mathfrak{A}$ except that $\tau^{\mathfrak{A}}[\tau] = v$.

**Definition 2**

We extend the interpretation function $(\cdot)^{\mathfrak{A}}$ of structure $\mathfrak{A}$ to all compound terms over $\Sigma_{\mathfrak{A}}$ by the inductive rule

$- f(t_1, \ldots, t_n)^{\mathfrak{A}} = f^{\mathfrak{A}}(t_1^{\mathfrak{A}}, \ldots, t_n^{\mathfrak{A}})$.

We further extend this function to tuples $t$ of terms over $\Sigma_{\mathfrak{A}}$ by defining

$- t^{\mathfrak{A}} = (t_1^{\mathfrak{A}}, \ldots, t_n^{\mathfrak{A}})$, where $t = (t_1, \ldots, t_n)$.

The value/interpretation $t^{\mathfrak{A}}$ is a well-defined object in the domain of $\mathfrak{A}$ provided $t$ is a term and $\mathfrak{A}$ interprets all function and object symbols in $t$.

Next, we define the truth relation, or the satisfaction relation, between structures and formulas.

**Definition 3 (Satisfaction relation $\mathfrak{A} \models \varphi$)**

Let $\varphi$ be an FO formula and $\mathfrak{A}$ a structure interpreting all free symbols of $\varphi$. We define $\mathfrak{A} \models \varphi$ by induction on the structure of $\varphi$:

- $\mathfrak{A} \models P(t)$, where $P$ is a predicate symbol, if $t^{\mathfrak{A}} \in P^{\mathfrak{A}}$;
- $\mathfrak{A} \models \psi \land \varphi$ if $\mathfrak{A} \models \psi$ and $\mathfrak{A} \models \varphi$;
- $\mathfrak{A} \models \psi \lor \varphi$ if $\mathfrak{A} \models \psi$ or $\mathfrak{A} \models \varphi$ (or both);
- $\mathfrak{A} \models \lnot \psi$ if $\mathfrak{A} \not\models \psi$; i.e., if it is not the case that $\mathfrak{A} \models \psi$;
- $\mathfrak{A} \models \exists x \psi$ if for some $d \in \text{dom}(\mathfrak{A})$, $\mathfrak{A}[x : d] \models \psi$;
- $\mathfrak{A} \models \forall x \psi$ if for each $d \in \text{dom}(\mathfrak{A})$, $\mathfrak{A}[x : d] \models \psi$.

When $\mathfrak{A} \models \varphi$, we say that $\varphi$ is true in $\mathfrak{A}$, or that $\mathfrak{A}$ satisfies $\varphi$, or that $\mathfrak{A}$ is a model of $\varphi$.

The satisfaction relation is easily extended from formulas to theories.

**Definition 4**

Let $T$ be an FO theory over $\Sigma$. A $\Sigma$-structure $\mathfrak{A}$ is a model of $T$ (or satisfies $T$), denoted $\mathfrak{A} \models T$, if $\mathfrak{A} \models \varphi$ for each $\varphi \in T$.

The satisfaction relation induces definitions of several other fundamental semantic concepts.

**Definition 5 (Derived semantic relations)**

A theory (or formula) $T$ entails a formula $\varphi$, denoted $T \models \varphi$, if $\varphi$ is satisfied in every structure $\mathfrak{A}$ that interprets all symbols with free occurrences in $T$ and $\varphi$, and satisfies $T$. A formula $\varphi$ is valid (denoted $\models \varphi$) if it is satisfied in every structure that interprets its free symbols. A formula or theory is satisfiable if it is satisfied in at least one structure. A formula $\varphi_1$ is logically equivalent to $\varphi_2$ (denoted $\varphi_1 \equiv \varphi_2$) if for every structure $\mathfrak{A}$ that interprets the free symbols of both formulas, $\mathfrak{A} \models \varphi_1$ if and only if $\mathfrak{A} \models \varphi_2$. 
The informal semantics of FO

The theory of informal semantics of FO is a coherent system of interpretations of its formal syntax and semantics that explains formulas as objective propositions about the application domain, structures as “states of affairs” of the application domain, and the satisfaction relation as a truth relation between states of affairs and propositions. It also induces informal semantics for the derived semantical relations. We will denote this informal semantics by $\mathcal{FO}$, and the three components of which it consists (i.e., the interpretation of formulas, the interpretation of structures, and the interpretation of semantic relations such as satisfaction) by $\mathcal{FO}^L$, $\mathcal{FO}^S$ and $\mathcal{FO}^{\equiv}$, respectively.

The informal semantics of a formula $\varphi$ is the information that is represented by $\varphi$ about the problem domain. It is essentially a thought. Following the quote by Frege at the beginning of this article, we make these thoughts tangible by giving them a linguistic form. In other words, the first component of the theory of informal semantics of FO consists of a mapping of FO formulas to natural language statements.

The informal semantics of a formula $\varphi$ depends on a parameter — the meaning that we give to the symbols of vocabulary $\Sigma$ of $\varphi$ in the application domain. This is captured by the intended interpretation $\mathcal{I}$ of the vocabulary $\Sigma$. To state the informal semantics of a formula over $\Sigma$ in linguistic form, we specify $\mathcal{I}$ as an assignment of natural language expressions to the symbols of $\Sigma$. For an $n$-ary function $f/n$, $\mathcal{I}(f/n)$ (or $\mathcal{I}(f)$, if the arity is clear or immaterial) is a parameterized noun phrase that specifies the value of the function in the application domain in terms of its $n$ arguments. Similarly, for an $n$-ary predicate $p/n$, $\mathcal{I}(p/n)$ is a parameterized sentence describing the relation between $n$ arguments of $p$. In either case, the $i$th argument is denoted as $\#_i$.

Example 3

In the running example, the intended interpretation $\mathcal{I}_{col}$ of the vocabulary $\Sigma_{col}$ can be expressed in linguistic form as parameterized declarative sentence and parameterized noun phrases:

- $\mathcal{I}_{col}(\text{Node}/1) =$ “$\#_1$ is a node”;
- $\mathcal{I}_{col}(\text{Edge}/2) =$ “there is an edge from $\#_1$ to $\#_2$”;
- $\mathcal{I}_{col}(\text{ColourOf}/1) =$ “the color of $\#_1$”;

Given an intended interpretation $\mathcal{I}$ for a vocabulary $\Sigma$, the informal semantics $\mathcal{FO}^L_?$ of FO terms and formulas over $\Sigma$ is now the inductively defined mapping from formal expressions to natural language expressions specified in Table[1] [1].

A special case in Table[1] is its final row. It gives the meaning of the implicit composition operator of FO, i.e., the operator that forms a single theory out of a number of sentences. The informal semantics of this operator is simply that of the standard (monotone) conjunction.

Example 4

For the theory $T_{col}$ defined in Example[1], $\mathcal{FO}^L_{T_{col}}(T_{col})$ results in the following statement:

For all $x$ in the universe of discourse, for all $y$ in the universe of discourse, if there is an edge from $x$ to $y$, then it is not the case that the color of $x$ and the color of $y$ are the same.

In other words, $T_{col}$ states that adjacent nodes are of different color.
Table 1. The informal semantics of FO formulas.

| \( \Phi \) | \( \mathcal{FOL}_I^L(\Phi) \) |
|---|---|
| \( x \) | \( x \) (where \( x \) is a variable) |
| \( f(t_1, \ldots, t_n) \) | \( \mathcal{I}(f)(\mathcal{FOL}_I^L(t_1), \ldots, \mathcal{FOL}_I^L(t_n)) \) (i.e., the noun phrase \( \mathcal{I}(f) \) with its parameters instantiated to \( \mathcal{FOL}_I^L(t_1), \ldots, \mathcal{FOL}_I^L(t_n) \)) |
| \( P(t_1, \ldots, t_n) \) | \( \mathcal{I}(P)(\mathcal{FOL}_I^L(t_1), \ldots, \mathcal{FOL}_I^L(t_n)) \) (i.e., the declarative sentence \( \mathcal{I}(P) \) with its parameters instantiated to \( \mathcal{FOL}_I^L(t_1), \ldots, \mathcal{FOL}_I^L(t_n) \)) |
| \( \varphi \lor \psi \) | \( \mathcal{FOL}_I^L(\varphi) \) or \( \mathcal{FOL}_I^L(\psi) \) (or both) |
| \( \varphi \land \psi \) | \( \mathcal{FOL}_I^L(\varphi) \) and \( \mathcal{FOL}_I^L(\psi) \) |
| \( \neg \varphi \) | it is not the case that \( \mathcal{FOL}_I^L(\varphi) \) (i.e., \( \mathcal{FOL}_I^L(\varphi) \) is false) |
| \( \varphi \Rightarrow \psi \) | if \( \mathcal{FOL}_I^L(\varphi) \) then \( \mathcal{FOL}_I^L(\psi) \) (in the sense of material implication) |
| \( \exists x \varphi \) | there exists an \( x \) in the universe of discourse such that \( \mathcal{FOL}_I^L(\varphi) \) |
| \( \forall x \varphi \) | for all \( x \) in the universe of discourse, \( \mathcal{FOL}_I^L(\varphi) \) |
| \( t_1 = t_2 \) | \( \mathcal{FOL}_I^L(t_1) \) and \( \mathcal{FOL}_I^L(t_2) \) are the same (i.e., they represent the same elements of the universe of discourse) |
| \( T = \{ \varphi_1, \ldots, \varphi_n \} \) | \( \mathcal{FOL}_I^L(\varphi_1) \) and \( \ldots \) and \( \mathcal{FOL}_I^L(\varphi_n) \) |

**Example 5**

Let us consider an alternative application domain for the FO language considered above, in which we have humans, each of some age, and each possibly with some siblings. We now may have the following intended interpretation \( \mathcal{I}_{\text{sib}} \) of the vocabulary \( \Sigma_{\text{col}} \):

- \( \mathcal{I}_{\text{sib}}(\text{Node}/1) = "\#_1\text{ is a human}"; \n- \( \mathcal{I}_{\text{sib}}(\text{Edge}/2) = "\#_1\text{ and }\#_2\text{ are siblings}"; \n- \( \mathcal{I}_{\text{sib}}(\text{ColorOf}/1) = "\text{the age of }\#_1". \n
For theory \( T_{\text{col}} \), \( \mathcal{FOL}_{I_{\text{sib}}}(T_{\text{col}}) \) yields this statement:

For all \( x \) in the universe of discourse, for all \( y \) in the universe of discourse, if \( x \) and \( y \) are siblings, then it is not the case that the age of \( x \) and the age of \( y \) are the same.
In other words, in the "sibling" application domain, the theory $T_{col}$ states that siblings have different age.

Informal semantics for FO’s semantical concepts  In addition to explaining the informal meaning of syntactical expressions, the informal semantics of FO also offers explanations of FO’s formal semantical objects: structures, the satisfaction relation, and the derived concepts of entailment, satisfiability, and validity.

The basic informal notion behind these concepts is that of a state of affairs. States of affairs differ in the objects that exists in the universe of discourse, or in the relationships and functions amongst these objects. The application domain is in one of many potential states of affairs. In a state of affairs, a proposition of the application domain is either true or false.

The intended interpretation $\mathcal{I}$ in general does not fix the state of affairs. Rather, it determines an abstraction function from states of affairs to $\Sigma$-structures.

Example 6
Under the intended interpretation $\mathcal{I}_{col}$ for the vocabulary $\Sigma_{col}$, the $\Sigma_{col}$-structure $\mathfrak{A}_{col}$ of Example 2 represents any state of affairs with five elements in the universe of discourse: three nodes abstracted as $n_1, n_2, n_3$ with edges corresponding to $(n_1, n_2)$ and $(n_2, n_3)$ and two colors represented by $c_1, c_2$; finally, a coloring mapping that associates colors to all elements (nodes and colors).

In the sequel, we denote the class of states of affairs that abstract under $\mathcal{I}$ to structure $\mathfrak{A}$ as $\mathcal{FO}_I^S(\mathfrak{A})$. We call this the informal semantics of the structure $\mathfrak{A}$ under $\mathcal{I}$. Table 2 expresses the meaning of structures as explained above.

Different intended interpretations $\mathcal{I}$ give rise to different abstractions.

Example 7
Under the alternative intended interpretation $\mathcal{I}_{sib}$ of Example 5, $\mathfrak{A}_{col}$ represents any state of affairs with three persons and two ages, where the sibling relation consists of pairs corresponding to $(n_1, n_2)$ and $(n_2, n_3)$ and where persons $n_1, n_3$ have the same age different from that of $n_2$. However, no possible state of affairs under this intended interpretation abstracts into $\mathfrak{A}_{col}$, since the sibling relation amongst a group of persons is always an equivalence relation while $\text{Edge}^{\mathfrak{A}_{col}}$ is not. Stated differently, $\mathcal{FO}_I^S(\mathfrak{A}_{col})$ contains only impossible states of affairs.

The informal semantics of the satisfaction relation $\models$ between structures and sentences is the relation between states of affairs and true propositions in it. That is, $\mathfrak{A} \models \varphi$ is interpreted as “(the proposition represented by) $\varphi$ is true in the state of affairs (corresponding to) $\mathfrak{A}$.” Table 3 summarizes this observation.

We thus specified for each vocabulary $\Sigma$ and for every intended interpretation $\mathcal{I}$ for $\Sigma$ a triple $\mathcal{FO}_I = (\mathcal{FO}_I^L, \mathcal{FO}_I^F, \mathcal{FO}_I^S)$ that explains the informal semantics of formulas, structures and the satisfaction relation. We call this the (standard) theory of informal semantics of FO.

---

2 More accurately, it determines an abstraction function from states of affairs to classes of isomorphic $\Sigma$-structures.
Table 2. The informal semantics of FO structures.

| \( \mathfrak{A} \) | a state of affairs \( S \in \mathcal{F}O^\lambda_2(\mathfrak{A}) \) that has abstraction \( \mathfrak{A} \) |
|-----------------|----------------------------------------------------------------------------------------------------------|
| \( \text{dom}(\mathfrak{A}) \) | the set of elements in the universe of discourse of \( S \) |
| \( P^\mathfrak{A} \) | the property of \( S \) described by the declarative sentence \( I(P) \) |
| \( f^\mathfrak{A} \) | the function in \( S \) described by the noun phrase \( I(f) \) |

Table 3. The informal semantics of the satisfaction relation of FO.

| \( \models \) | \( \mathcal{F}O^\lambda_2 \) |
|-----------------|----------------------------------------------------------------------------------------------------------|
| \( \mathfrak{A} \models T \) | The property \( \mathcal{F}O^\lambda_2(T) \) holds in the states of affairs \( \mathcal{F}O^\lambda_2(\mathfrak{A}) \) |

Informal semantics of derived semantical relations The standard theory of informal semantics of FO also induces the informal meaning for the derived semantical concepts of entailment, equivalence, validity, satisfiability, and equivalence in a way that reflects our understanding of these concepts in mathematics and formal science.

For instance, under the standard informal semantics of FO, a formal expression \( \psi \models \varphi \) (entailment), becomes the statement that the informal semantics of \( \varphi \) (a statement that expresses a property of the application domain) is true in every state of affairs for that application domain, in which the informal semantics of \( \psi \) (another statement that expresses a property of the application domain) is true (which, in case the informal semantics of \( \psi \) and \( \varphi \) are mathematical propositions, means that the first proposition, the one corresponding to \( \psi \), mathematically entails the second proposition, the one corresponding to \( \varphi \)). Similarly, validity of \( \varphi \) means that the informal semantics of \( \varphi \) is true in every state of affairs, and satisfiability of \( \varphi \) means that the informal semantics of \( \varphi \) is true in at least one state of affairs.

Precision of the informal semantics The informal semantics \( \mathcal{F}O^\lambda_2(\varphi) \) of a sentence of theory \( T \) under intended interpretation \( I \) is a syntactically correct statement in natural language, but is this statement a sensible and precise statement about the application domain? This is a concern given the vagueness, ambiguity and the lack of coherence that is so often ascribed to natural language (Barwise and Cooper, 1981). We now discuss this.

First, when the intended interpretation \( I \) of \( \Sigma \) is vague or ambiguous, then indeed, the informal semantics of FO sentences over \( \Sigma \) will be vague or ambiguous. E.g., if we interpret \( \text{Edge}^\lambda \) as "\#_1 is rather similarly looking as \#_2", then surely the informal semantics of sentences containing this predicate will be vague. FO is not designed to express information given in terms of vague
The informal semantics of Answer Set Programming: A Tarskian perspective

concepts. In FO, as in many other languages, it is the user’s responsibility to design a vocabulary with a clear and precise intended interpretation.

A second potential source of ambiguity lies in the use of natural language connectives such as “and”, “or”, “if…then…”, etc. in Table 1. Several of these connectives are overloaded, in the sense that they may mean different things in different contexts. However, this does not necessarily lead to ambiguity or vagueness, since human readers are skilled in using context to disambiguate overloaded concepts. Let us consider several potential ambiguities.

In natural language, the connective “and” corresponds not only to logical conjunction, but also to temporal consecutive conjunction exemplified in I woke up and brushed my teeth. However, the temporal interpretation arises only in a temporal context. Table 1 is not stated within a temporal context, so it is logical conjunction that is intended and this intended meaning is inherited in every occurrence of $\land$ in every FO sentence. Similarly, the word “or” is used to denote both inclusive and exclusive disjunction. In mathematical texts its accepted meaning is that of inclusive disjunction. The rule for “or” in Table 1 explicitly adds “or both”, to remove any possibility for ambiguity.

The conditional “if . . . then . . . ” is famously ambiguous. It can mean many different things in varying contexts (Dancygier and Sweetser, 2005). Therefore, any choice for the formal and informal semantics of the implication symbol can only cover part of the use of the conditional in natural language. In FO, the informal semantics of $\Rightarrow$ was chosen to be the material implication, which interprets “if A then B” as “not A or B”. It has the benefit of being simple and clear, and it is likely the conditional that we need and use most frequently in mathematical text.

To summarize, the natural language connectives and quantifiers in Table 1 are precise and clear. This precision is inherited by the informal semantics $\mathcal{FOL}_I$ of FO sentences. Consequently, under the assumption that the intended interpretation $\mathcal{I}$ is also clear, the natural language statements produced by $\mathcal{FOL}_I$ are as clear and unambiguous as mathematical text.

Informal semantics as the “empirical” component of logic Formal sentences of a knowledge representation logic are used to specify information about an application domain. The role of the informal semantics theory of a knowledge representation logic is to provide a principled account of which information about an application domain is expressed by formal sentences of the logic.

In other formal empirical sciences, we find theories with a similar role. A physics theory (e.g., quantum mechanics) not only consists of mathematical equations but, equally important, also of a theory that describes, often with extreme precision, how the mathematical symbols used in these equations are linked to observable phenomena and measurable quantities in reality. This second part of the theory therefore plays a role similar to that of the informal semantics theory of a knowledge representation logic.

A physics theory is a precise, falsifiable hypothesis about the reality it models. Such a theory can never be proven. But it potentially can be experimentally falsified by computing a mathematical prediction from the formal theory, and verifying if the measured phenomena match the predictions. If not, the experiment refutes the theory. Otherwise, it corroborates (supports) the theory. Availability of a large and diverse body of corroborating experiments, in the absence of experimental refutation, increases our confidence in the theory.

Likewise, a formal logic with a theory of informal semantics is a precise, falsifiable hypothesis

---

3 This suggests adding symbols to the logic to express other conditionals, which is something we we will do later in this paper).
Alternative theories of informal semantics of FO. That the informal and formal semantics correspond so well to each other is not an accident. The key to this is the tight correspondence between the natural language expressions occurring in Table 1 and the statements used in the bodies of rules of Definition 3, for instance, between the informal and formal semantics of \( \forall \) and \( \exists \left( \text{ColorOf} \right) \). To illustrate, let us consider a set of interpretations in which both the informal and formal semantics are in mismatch. E.g., for the propositional vocabulary \( \Sigma = \{ P, Q \} \), it now holds that \( \{ P, Q \} \models P \vee Q \) while under any intended interpretation of \( P, Q \), the statement “\( \exists x ( P(x) \) or \( \exists x ( Q(x) \) (or both)\)” is true in the state of affairs \( \mathcal{F}_T(\{ P, Q \}) \).

However, while the formal semantics constrains the informal semantics, it does not uniquely determine it. The informal semantics of formulas arises from a coherent set of interpretations of language constructs and of the semantic primitives: structures and \( \models \). By carefully changing this system, we may obtain an informal semantics that still matches with the formal semantics although it assigns a very different meaning to formulas. Let us illustrate this.

In the proposed new informal semantics, we reinterpret all connectives such that the informal semantics of any sentence is exactly the negation of its standard informal semantics, and we reinterpret \( \models \) to mean that the informal semantics of \( \varphi \) is false in states of affairs represented by \( \mathcal{A} \). These two negations then compensate for each other, leading to a theory of informal semantics that still matches with the formal semantics, even though it assigns the negation of the standard informal semantics to each FO sentence! To be precise, the alternative informal semantics theory is given by \( \mathcal{F}_T(\Sigma_T, \mathcal{F}^S_T, \mathcal{F}_T(\Sigma_T)) \), where \( \mathcal{F}_T(\Sigma_T) = \mathcal{F}_T(\Sigma_T) \) that is, the informal semantics of structures as states of affairs is as in the standard informal semantics of FO, and the interpretations of the two non-standard components \( \mathcal{F}_T(\Sigma_T) \) and \( \mathcal{F}_T(\Sigma_T) \) are defined in Table 1. To illustrate, let us consider the formula \( \varphi_{col} \):

\[
\exists x \exists y \left( \text{Edge}(x, y) \land \text{ColorOf}(x) = \text{ColorOf}(y) \right).
\]

The non-standard informal semantics \( \mathcal{F}_T(\Sigma_T) \) interprets the formula \( \varphi_{col} \) as:
For all \( x \) in the universe of discourse, for all \( y \) in the universe of discourse it is not the case that there is an edge from \( x \) to \( y \), or the color of \( x \) and the color of \( y \) are not the same.

Stated differently, \( \mathcal{F}_{T}^{L} \) says that adjacent nodes are of different color. Note that this statement is the negation of the standard informal semantics of this formula and that it has the same meaning as the one produced by the standard informal semantics \( \mathcal{F}_{T}^{L} \), for the (different) formula \( T_{\text{col}} \). Since in the new informal semantics, structures are still interpreted in the same way as before, it follows that \( \mathcal{F}_{T_{\text{col}}}^{L}(\bar{\phi}_{\text{col}}) \) is satisfied in the state-of-affairs \( \mathcal{F}_{T_{\text{col}}}^{S}(\mathcal{A}_{\text{col}}) \). On the formal side, nothing has changed and it is the case that \( \mathcal{A}_{\text{col}} \models \bar{\phi}_{\text{col}} \), i.e., the formula is not formally satisfied in the structure. But under the new informal semantics, the relation \( \models \) is now interpreted as non-satisfaction and hence, \( \mathcal{A}_{\text{col}} \not\models \bar{\phi}_{\text{col}} \) is to be interpreted as the fact that \( \mathcal{F}_{T_{\text{col}}}^{L}(\bar{\phi}_{\text{col}}) \) is (not non-)satisfied in \( \mathcal{F}_{T_{\text{col}}}^{S}(\mathcal{A}_{\text{col}}) \). Which is true!

Thus, even though the formal semantics of a logic strongly constrains its informal semantics, there may nevertheless remain different informal semantics that correspond to the logic’s formal semantics. In the case of FO, an informal semantics such as \( \mathcal{F}_{T}^{L} \) is counterintuitive and of no practical use.

As many logics reuse the connectives of FO, we will use the following terminology. We will say that a connective is classical in a logic under some informal semantics \( \mathcal{I} \) if \( \mathcal{I} \) interprets it in the same way as \( \mathcal{F}_{T}^{L} \) does (i.e., by the same natural language phrase). For instance, in the non-standard informal semantics \( \mathcal{F}_{T}^{L} \) for FO, the negation connective \( \neg \) is classical, whereas the conjunction connective \( \land \) is not.

In the rest of this paper, we often omit the superscript from the notation \( \mathcal{I}^{L}, \mathcal{I}^{S}, \mathcal{I}^{=} \) when it is clear which of the three components of a theory of informal semantics \( \mathcal{I} \) is intended.

3 The original formal and informal semantics of extended logic programs

In this section, we recall the standard formal and informal semantics of extended logic programs as it appeared in Gelfond and Lifschitz (1988; 1991). An alternative review of the informal semantics for this logic can be found in Gelfond and Kahl (2014, Section 2.2.1). For simplicity, here we only consider the propositional case. Logic programs with variables are interpreted by the means of their so-called grounding that transforms them into propositional programs.

A literal is either an atom \( A \) or an expression \( \neg A \), where \( A \) is an atom. An extended logic programming rule is an expression of the form

\[
A \leftarrow B_{1}, \ldots, B_{n}, \text{not } C_{1}, \ldots, \text{not } C_{m}, \tag{4}
\]

where \( A, B_{i}, \) and \( C_{j} \) are propositional literals. If all the literals \( A, B_{i} \) and \( C_{i} \) are atoms (i.e., symbol \( \neg \) does not appear in the rule), then such a rule is called normal. The literal \( A \) is the head of the rule and expression \( B_{1}, \ldots, B_{n}, \text{not } C_{1}, \ldots, \text{not } C_{m} \) is its body. We may abbreviate this rule as \( \text{Head } \leftarrow \text{Body} \). An extended logic program is a finite set of such rules. We denote the set of all programs in a given vocabulary \( \Sigma \) by \( \mathbb{P}_{\Sigma} \).

A consistent set of propositional literals is a set that does not contain both \( A \) and its complement \( \neg A \) for any atom \( A \). A believed literal set \( X \) is a consistent set of propositional literals. We denote the set of all believed literal sets for a vocabulary \( \Sigma \) by \( \mathbb{B}_{\Sigma} \). A believed literal set \( X \) satisfies a rule \( r \) of the form (4) if \( A \) belongs to \( X \) or there exists an \( i \in \{1, \ldots, n\} \) such that \( B_{i} \not\in X \) or a \( j \in \{1, \ldots, m\} \) such that \( C_{j} \in X \). A believed literal set is a model of a program \( P \) if it satisfies all rules \( r \in P \).
Table 4. A non-standard informal semantics of FO. The informal semantics of terms is as in Table 1 and the informal semantics of structures as in Table 2.

| \( \Phi \) | \( \tilde{\mathcal{F}}\mathcal{O}_{I}^{L}(\Phi) \) |
|----------------|----------------------------------|
| \( P(t) \) | it is not the case that \( I(P)\langle \tilde{\mathcal{F}}\mathcal{O}_{I}^{L}(t_{1}), \ldots, \tilde{\mathcal{F}}\mathcal{O}_{I}^{L}(t_{n}) \rangle \) |
| \( \varphi \lor \psi \) | \( \tilde{\mathcal{F}}\mathcal{O}_{I}^{L}(\varphi) \) and \( \tilde{\mathcal{F}}\mathcal{O}_{I}^{L}(\psi) \) |
| \( \varphi \land \psi \) | \( \tilde{\mathcal{F}}\mathcal{O}_{I}^{L}(\varphi) \) or \( \tilde{\mathcal{F}}\mathcal{O}_{I}^{L}(\psi) \) |
| \( \neg \varphi \) | it is not the case that \( \tilde{\mathcal{F}}\mathcal{O}_{I}^{L}(\varphi) \) |
| \( \exists x \varphi \) | for all \( x \) in the universe of discourse, \( \tilde{\mathcal{F}}\mathcal{O}_{I}^{L}(\varphi) \) |
| \( \forall x \varphi \) | there exists an \( x \) in the universe of discourse such that \( \tilde{\mathcal{F}}\mathcal{O}_{I}^{L}(\varphi) \) |
| \( t_{1} = t_{2} \) | \( \tilde{\mathcal{F}}\mathcal{O}_{I}^{L}(t_{1}) \) and \( \tilde{\mathcal{F}}\mathcal{O}_{I}^{L}(t_{2}) \) are not the same |
| \( T = \{ \varphi_{1}, \ldots, \varphi_{n} \} \) | \( \tilde{\mathcal{F}}\mathcal{O}_{I}^{L}(\varphi_{1}) \) or \( \ldots \) or \( \tilde{\mathcal{F}}\mathcal{O}_{I}^{L}(\varphi_{n}) \) |

\[ \models \tilde{\mathcal{F}}\mathcal{O}^{m} \]
\[ A \models T \quad \text{The property } \tilde{\mathcal{F}}\mathcal{O}_{I}^{L}(T) \text{ does not hold in the state-of-affairs } \tilde{\mathcal{F}}\mathcal{O}_{I}^{L}(A) \]

For a rule \( r \) of the form \( (4) \) and a believed literal set \( X \), the reduct \( r^{X} \) is defined whenever there is no literal \( C_{j} \) for \( j \in \{ 1, \ldots, m \} \) such that \( C_{j} \in X \). If the reduct \( r^{X} \) is defined, then it is the rule

\[ A \leftarrow B_{1}, \ldots, B_{n} \]

The reduct \( P^{X} \) of the program \( P \) consists of the rules \( r^{X} \) for all \( r \in P \), for which the reduct is defined. A believed literal set \( X \) is a stable model or answer set of \( P \), denoted \( X \models_{st} P \), if it is a \( \subseteq \)-least model of \( P^{X} \).

The formal logic of extended logic programs consists of the triple \( (\mathcal{L}_{E}, \mathcal{B}_{\Sigma}, \models_{st}) \). Gelfond and Lifschitz [1988, 1991] described an informal semantics for such programs based on epistemic notions of default and autoepistemic reasoning. Just as with \( \tilde{\mathcal{F}}\mathcal{O}_{I} \), the informal semantics \( \mathcal{G}\mathcal{L}_{I} \) arises from a system of interpretations of the formal syntactical and semantical concepts. We now recall this informal semantics, which we name \( \mathcal{G}\mathcal{L}_{I} \). We denote its three components by \( \mathcal{G}\mathcal{L}_{I}^{T}, \mathcal{G}\mathcal{L}_{I}^{S} \) and \( \mathcal{G}\mathcal{L}_{I}^{r} \).

One of the key aspects of \( \mathcal{G}\mathcal{L}_{I} \) is that it views a believed literal set \( X \) as an abstraction of
a belief state of some agent. An agent in some belief state considers certain states of affairs as possible and the others as impossible. The corresponding believed literal set $X$ is the set of all literals $L$ that the agent believes in, that is, those that are true in all states of affairs that agent regards as possible. Importantly, it is not the case that a literal $L$ that does not belong to $X$ is believed to be false by the agent. Rather, it is not believed by the agent: the literal is false in some states of affairs the agent holds possible, and it may be true in others. In fact, $L$ is true in at least one of the agents possible state of affairs unless the complement of $L$ belongs to $X$ (unless the agent believes the complement of $L$). Thus, the informal semantics $\mathcal{GL}_I$ explains the meaning of programs in terms of what literals an agent with incomplete knowledge of the application domain might believe in.

Example 9

Consider the believed literal set

$$X = \{\text{student}(\text{mary}), \text{male}(\text{john})\}$$

under the obvious intended interpretation $I$ for the propositional atoms. This $X$ is the abstraction of any belief state in which the agent both believes that Mary is a student and that John is male, and does not believe that John is a student or that Mary is male. One such belief state is the state $B_0$ in which the agent considers the following states of affairs as possible:

1. John is the only male in the domain of discourse; Mary is the only student.
2. John and Mary are both male students.
3. John and Mary are both male; Mary is the only student.
4. John is the only male; John and Mary are both students

Another belief state corresponding to $X$ is the state $B_1$ in which the agent considers the states of affairs $2, 4$ of $B_0$ as possible. In both, Mary is a student and John a male in all possible states of affairs. John is student in worlds $2, 4$; Mary is male in worlds $2, 3$. Hence, literals $\neg \text{student}(\text{john})$ and $\neg \text{male}(\text{mary})$ are not believed.

Although both belief states abstract to the same believed literal set, they are different. When compared to $B_0$, the belief state $B_1$ contains the additional belief that either John is a student or Mary is male. Since this additional belief is not atomic, a formal believed literal set cannot distinguish belief states in which it holds (e.g., $B_1$) from belief states in which it does not (e.g., $B_0$). In this way, different informal belief states are still abstracted to the same formal believed literal set. This shows that believed literal sets are a rather coarse way of abstracting belief states, compared to, e.g., believed formula sets or Kripke structures.

We denote the class of informal belief states that are abstracted to a given formal believed literal set $X$ under an intended interpretation $I$ as $\mathcal{GL}_I^S(X)$. Table 5 summarizes this abstraction function.

Table 6 shows the Gelfond-Lifschitz informal semantics $\mathcal{GL}_I^E$ of programs. As in the informal semantics for FO, each atom $A$ has an intended interpretation $I(A)$ which is represented linguistically as a noun phrase about the application domain. The intended interpretation $I(\neg A)$ is “it is not the case that $I(A)$”. As is clear from this table, under $\mathcal{GL}_I^E$, extended logic programs have both classical and non-classical connectives. On the one hand, the comma operator is classical conjunction and the rule operator $\leftarrow$ is classical implication. On the other hand, the implicit composition operator (constructing a program out of individual rules) is non-classical, because
Table 5. The Gelfond-Lifschitz informal semantics of belief sets.

| A belief set \( X \) | A belief state \( B \in \mathcal{GL}_I(X) \) that has abstraction \( X \) |
|----------------------|----------------------------------------------------------------------------------|
| \( A \in X \) for atom \( A \) | \( B \) has the belief that \( \mathcal{I}(A) \) is true; i.e., \( \mathcal{I}(A) \) is true in all states of affairs possible in \( B \) |
| \( \neg A \in X \) for atom \( A \) | \( B \) has the belief that \( \mathcal{I}(A) \) is false; i.e., \( \mathcal{I}(A) \) is false in all states of affairs possible in \( B \) |
| \( A \notin X \) for atom \( A \) | \( B \) does not have the belief that \( \mathcal{I}(A) \) is true; i.e., \( \mathcal{I}(A) \) is false in some state of affairs possible in \( B \) |
| \( \neg A \notin X \) for atom \( A \) | \( B \) does not have the belief that \( \mathcal{I}(A) \) is false; i.e., \( \mathcal{I}(A) \) is true in some state of affairs possible in \( B \) |

Table 6. The Gelfond-Lifschitz (1988; 1991) informal semantics for ASP formulas.

| \( \Phi \) | \( \mathcal{GL}_I(\Phi) \) |
|----------|-------------------|
| propositional atom \( A \) | \( \mathcal{I}(A) \) |
| propositional literal \( \neg A \) | it is not the case that \( \mathcal{I}(A) \) |
| expression of the form \( \text{not} \ C \) | the agent does not know that \( \mathcal{GL}_I(C) \) |
| expression of the form \( \Phi_1, \Phi_2 \) | \( \mathcal{GL}_I(\Phi_1) \) and \( \mathcal{GL}_I(\Phi_2) \) |
| rule \( \text{Head} \leftarrow \text{Body} \) | if \( \mathcal{GL}_I(\text{Body}) \) then \( \mathcal{GL}_I(\text{Head}) \) (in the sense of material implication) |
| program \( P = \{r_1, \ldots, r_n\} \) | All the agent knows is: |
| | \( \mathcal{GL}_I(r_1) \) and |
| | \( \ldots \) |
| | \( \mathcal{GL}_I(r_n) \) |

it performs a closure operation: the agent knows only what is explicitly stated. Of the two negation operators, symbol \( \neg \) is classical negation, whereas \( \text{not} \) is a non-classical negation, which is called default negation.

The final component of the GL theory of informal semantics is \( \mathcal{GL}_I(=^*) \), which explains what it means for a set of literals \( X \) to be a stable model of a program \( P \). As can be seen in Table 7, this means that, given that \( \mathcal{GL}_I(P) \) represents precisely the knowledge of the agent, \( X \) could be the set of literals the agent believes.
The informal semantics of Answer Set Programming: A Tarskian perspective

Table 7. The Gelfond-Lifschitz [1988, 1991] informal semantics for the ASP satisfaction relation.

| |=_{st} | \text{GL}_{L}^{\leq_{st}} |
|---|---|
| X |=_{st} P | Given that all the agent knows is \text{GL}_{L}^{\leq_{st}}(P), X could be the set of literals the agent believes |

To illustrate this informal semantics, let us consider Gelfond and Lifschitz’ well-known interview example.

**Example 10**

Whether students of a certain school are eligible for a scholarship depends on their GPA and on their minority status. The school has an incomplete database about candidate students. Students for which the school has insufficient information to decide eligibility should be invited for an interview. The following ELP program expresses the school’s knowledge.

\[
\begin{align*}
\text{Eligible}(x) & \leftarrow \text{HighGPA}(x). \\
\text{Eligible}(x) & \leftarrow \text{FairGPA}(x), \text{Minority}(x). \\
\neg\text{Eligible}(x) & \leftarrow \neg\text{FairGPA}(x), \neg\text{HighGPA}(x). \\
\text{Interview}(x) & \leftarrow \neg\text{Eligible}(x), \neg\text{Eligible}(x) \\
\text{Minority}(\text{brit}). \\
\text{HighGPA}(\text{mary}). \\
\neg\text{Minority}(\text{david}). \\
\text{FairGPA}(\text{david}).
\end{align*}
\]

The three rules for *Eligible* specify a partial policy for eligibility: they determine the eligibility for all students except for non-minority students with fair GPA. In this sense, under \text{GL}_{L}^{\leq_{st}}, this program does not actually define when a student is eligible. The next rule is epistemic. It expresses that the school interviews a person whose eligibility is unknown. The remaining rules specify partial data on students Mary, Brit and David. In particular, *FairGPA*(brit) and even *FairGPA*(mary) are unknown.

For Mary, the first rule applies and the school knows that she is eligible. The epistemic fourth rule will therefore not conclude that she should be interviewed. Incidentally, nor is it implied that she will not be interviewed; the school (formally, the program) simply does not know. This follows from the informal semantics of the implicit composition operator: “all the agent knows is...”. For Brit and David, their eligibility is unknown. However, the reasons are different: for Brit because of lack of data, for David because the policy does not specify it. Therefore, both will be interviewed. The unique answer set extends the student data with the following literals:

\[
\begin{align*}
\text{Eligible}(\text{mary}), \text{Interview}(\text{brit}), \text{Interview}(\text{david}).
\end{align*}
\]

The crucial property of this example is that whether a student should be interviewed does not only depend on properties of the student alone, but also on the agent’s knowledge about this student. In other words, it is perfectly possible that the same student should be interviewed
when applying to school \( A \) but not when applying to school \( B \), even when the eligibility criteria used by the two schools are precisely the same. Indeed, this can happen if school \( B \) has more information on record about the student in question. Because of this property, a logic with a subjective informal semantics is required here.

It is illustrative to compare this example to, for instance, graph coloring. Whether a graph is colorable given a certain number of colors is purely a property of this graph itself. It does not depend on anyone’s knowledge about this graph, and it does not depend on which agent is doing the coloring. In other words, graph coloring lacks the subjective, epistemic component of the interview example. Consequently, as we illustrated in Section 1 applying \( \mathcal{GL}_I \) to the rules of a typical GDT graph coloring program produces a misleading result. It refers to the knowledge of an agent, when in fact there exists no agent whose knowledge is relevant to the problem.

The interview example, with its inherent epistemic component, is a clear case where the informal semantics \( \mathcal{GL}_I \) applies. Similar informal semantics have also been developed for other formalisms for modeling reasoning with incomplete information, including default logic (Reiter, 1980) and autoepistemic logic (Moore, 1985), and much of our discussion above extends to such formalisms, too. They all have a role to play but it is important to be aware of the scope of their applicability.

To the best of our knowledge, \( \mathcal{GL}_I \) is the only informal semantics developed in the literature for the language of extended logic programs. Answer set programming adopted it, when it adopted the language of extended logic programming. However, the informal semantics \( \mathcal{GL}_I \) is not suitable for typical answer set programing applications. Moreover, over time, answer set programming has developed a richer language with features not found in the original papers by Gelfond and Lifschitz, including choice rules, aggregates and weight constraints. If is often non-trivial to extend \( \mathcal{GL}_I \) to these richer languages, as illustrated in the next paragraph.

Extending \( \mathcal{GL}_I \) to modern ASP is difficult As a simple example of a modern ASP language feature, we consider the constraint (3) used to model the property that nodes should be colored. As stated in the introduction, \( \mathcal{GL}_I \) provides the following informal semantics for this rule: “for every \( x \), Aux holds if the agent does not know Aux and \( x \) is a node and the agent does not know that \( x \) has a color.”

Can a simpler informal semantics (that avoids the predicate Aux) be given for constraint (3) directly? This question is not easily answered. Starting from the way in which Table 6 interprets atoms and the operator not, one plausible candidate for such an informal semantics is the sentence: “for every \( x \), it is not the case that \( x \) is a node and the agent does not know that \( x \) has a color’.

However, upon closer analysis, this sentence turns out not to match with the formal semantics. This can be seen as follows. Let us suppose for simplicity that the Herbrand universe consists of only one constant \( a \). The proposed informal semantics boils down then to:

\( (*) \) “It is not the case that \( a \) is a node and the agent does not know that \( a \) has a color.”

Consider now the ASP program \( P \) that consists only of the constraint (3). This program has a unique answer set \( \emptyset \), the empty believed literal set. Let \( B \) be any (informal) belief state that satisfies (\( * \)). In order for our proposed sentence (\( * \)) to be a correct informal semantics for (3), this belief state \( B \) must abstract to the unique answer set \( \emptyset \) of \( P \). If so, then in \( B \) it is not known that \( a \) is colored (because \( \text{Colored}(a) \not\in \emptyset \)). It follows that \( B \) must satisfy the property “it is not the case that \( a \) is a node.” In other words, \( B \) cannot consider as possible any state of affairs in which
a is a node. Or, to put it in even simpler terms, the belief state $B$ must contain the belief that $a$ is not a node. However, since $B$ abstracts to $\emptyset$, it must then be the case that $\neg \text{Node}(a) \in \emptyset$, which is clearly false. Therefore, our proposed informal semantics of (3) is incorrect.

It is not the purpose of this article to present an epistemic informal semantics for modern ASP languages. We bring up this issue to point out that doing so is often more problematic than one might expect. This is not only the case for constraints, but also for choice rules, whose informal epistemic semantics has been the topic of lively debates in the ASP community\(^4\). For the sake of completeness, we conclude this discussion by mentioning that a correct informal epistemic reading of (3) is:

\[
(\ast\ast) \text{ "for every } x \text{, it is not the case that the agent knows that } x \text{ is a node and the agent does not know that } x \text{ has a color."}
\]

Writing constraints such as (3) is common practice among ASP programmers. At the same time, the fact that (*) is not a correct informal semantics for them while (\ast\ast) is, appears to be far from common knowledge. This illustrates the pitfalls of using the Gelfond and Lifschitz’ epistemic informal semantics with the GDT programming methodology.

Our goal in this paper is to develop an alternative informal semantics $\mathcal{O}^B_T$ for answer set programming, which, unlike $\mathcal{G}^C_T$, is objective, that is, not epistemic, and extends to new features of the language that ASP now takes for granted. The rationale behind this effort is our conviction that such an objective informal semantics will be better aligned with typical GDT programs. Before we introduce it, we first review the GDT paradigm.

4 Generate-Define-Test methodology

The generate-define-test (GDT) methodology (Lifschitz, 2002) was proposed as a principled way to encode search problems in ASP. Over the years, it became the de facto standard in the field. The GDT methodology yields programs that consist of three parts: generate, define and test.

The role of generate is to generate the search space. In modern ASP languages this task is often accomplished by a set of choice rules:

\[
\{A\} \leftarrow B_1, \ldots, B_n, \text{not } C_1, \ldots, \text{not } C_m,
\]

where $A$, $B_i$, and $C_j$ are atoms (possibly, non propositional). Intuitively, rule (5) states that the atom in the head can take any of the values true and false, if the condition expressed by the body of the rule holds. We call the predicate symbol of $A$ generated.

The define part is a set of definitions of input and auxiliary predicates. Each definition is encoded by a group of normal rules (i.e., rules of the form (4) in which the expressions $A$, $B_i$ and $C_j$ are all atoms). In such a rule, the predicate symbol of the head $A$ is the one being defined. Input predicates are defined by exhaustive enumeration of their extensions. That is, their definitions consist of sets of facts, i.e., rules of form (4) that have empty bodies and contain no variables. For facts, the symbol $\leftarrow$ is often omitted from the notation. Auxiliary predicates are defined by sets of rules (4) that specify these symbols by describing how to derive their extensions from the extensions of the generated and input symbols.

\[^4\] This is illustrated by the Texas Action Group discussion on the matter [http://www.cs.utexas.edu/users/vl/tag/choice_discussion].
Finally, the test part eliminates some generated candidate answer sets. It consists of constraint rules:

\[ \leftarrow B_1, \ldots, B_n, \text{not } C_1, \ldots, \text{not } C_m, \]

where \( B_i \) and \( C_j \) are atoms. Each constraint rule eliminates those candidate answer sets, in which all \( B_i \) are true and all \( C_j \) are false.

As such, the GDT methodology identifies different components of programs by assigning them a particular operational task (e.g., "generate search space", "test candidate solution") in the computation of answer sets. Thus, it is about the computation process of answer sets and not about the meaning of expressions in ASP programs. In other words, the GDT methodology does not specify an informal semantics for GDT programs.

We call finite sets of choice, normal, and constraint rules core ASP programs. For these core ASP programs, we will adopt the formal semantics proposed by Ferraris et al. (2011). The main concepts of this semantics are briefly reviewed in the appendix; we omit the details, as they are not relevant for the purposes of this paper.

While syntactically simple, core ASP programs are expressive enough to support the GDT methodology. Moreover, they are a part of almost all ASP dialects, modulo minor syntactic differences. Thus, they form a convenient target language for our study of GDT programs. However, the GDT methodology is not restricted to core ASP. Many applications rely on extensions such as aggregates and weight expressions, or use different means to implement the generate task (pairs of rules \( P \leftarrow \text{not } P^*; P^* \leftarrow \text{not } P \); disjunctions \( P \lor P^* \); or rules \( P \leftarrow \text{not not } P \)). The discussion we present here also applies to programs that contain such expressions. We touch on this subject in Section 8.

To illustrate the GDT methodology, we present a core ASP program that encodes the Hamiltonian cycle problem.

\[
\begin{align*}
\text{generate} & \quad \{ \text{In}(x, y) \} \leftarrow \text{Edge}(x, y). \\
\text{define} & \quad \text{Node}(V). \ldots \text{Node}(W). \quad \text{Edge}(V, V'). \ldots \text{Edge}(W, W'). \\
& \quad T(x, y) \leftarrow \text{In}(x, y), \\
& \quad T(x, y) \leftarrow T(x, z), T(z, y). \\
\text{test} & \quad \leftarrow \text{In}(x, y), \text{In}(x, z), y \neq z. \\
& \quad \leftarrow \text{In}(x, z), \text{In}(y, z), x \neq y. \\
& \quad \leftarrow \text{Node}(x), \text{Node}(y), \text{not } T(x, y).
\end{align*}
\]

The long horizontal lines indicate the partition of the program into the generate, define, and test parts respectively. The short lines inside the define part separate groups of rules defining individual predicates. Inside the test part, they separate individual constraints. Predicate symbol \( \text{In} \) is a generated symbol, symbols \( \text{Node} \) and \( \text{Edge} \) are input symbols, and \( T \) is an auxiliary one. The generate part specifies all subsets \( \text{In} \) of the \( \text{Edge} \) relation as candidate solutions. The relations \( \text{Node} \) and \( \text{Edge} \) represent all vertices and edges of the input graph and are defined by enumeration. The relation \( T \) represents the auxiliary concept of the transitive closure of the relation \( \text{In} \). The test part "weeds out" those candidate answer sets, whose \( \text{In} \) relation does not
correspond to a Hamiltonian cycle. The auxiliary relation $T$ is necessary to state the constraint that a Hamiltonian cycle connects every pair of vertices.

As this example illustrates, a GDT program often has a rich internal structure. For instance, in the example above

- rules can be partitioned into three groups *generate*, *define* and *test*
- *define* contains separate definitions for three predicates *Node*, *Edge* and *T*, and
- *test* consists of three independent constraints.

The fact that the internal structure of programs remains implicit in the standard ASP formalism motivates us to introduce the ASP-FO language in the next section that makes the structure of programs explicit. In ASP-FO, rules in *generate*, *define* and *test* can be further split out in independent modules called G-modules, D-modules and T-modules.

5 The logic ASP-FO

We now turn to our goal of developing an informal semantics for GDT programs. To this end, motivated by the GDT methodology, we propose a logic ASP-FO and develop for it an informal semantics. We design the logic ASP-FO so that each core ASP program can be cast as an ASP-FO program without any essential changes. In this way, the informal semantics for ASP-FO can be used for ASP programs and, as we argue, becomes particularly effective in explaining the meaning of GDT programs.

5.1 Syntax

As in FO, expressions of ASP-FO are built from predicate and function symbols of some vocabulary $\Sigma$. Theories of ASP-FO consist of three types of modules: G-modules, D-modules and T-modules.

A choice rule is an expression of the form:

$$\forall \bar{x} \ (\{ P(\bar{t}) \} \leftarrow \varphi),$$

where $\varphi$ is an FO formula, $P(\bar{t})$ is an atom, and $\bar{x}$ includes all free variables appearing in the rule. We call the expression $\{ P(\bar{t}) \}$ the head of the rule and refer to $P$ as its head predicate. We call $\varphi$ the body of the rule.

**Definition 6 (G-module)**

A G-module is a finite set of choice rules with the same head predicate. Moreover, the head predicate may not appear in the body of the rules.

While many modern ASP solvers allow recursive choice rules, our concept of G-modules is more restrictive. This is in keeping with our view of G-modules as modules that generate the search space from a given problem instance. To generate such a search space, recursion does not seem to be required.

Recursion is allowed and, in fact, necessary in the *define* part. A define rule is an expression of the form

$$\forall \bar{x} \ (P(\bar{t}) \leftarrow \varphi),$$

where $\varphi$ is an FO formula, $P(\bar{t})$ is an atom, and $\bar{x}$ includes all free variables appearing in the rule. The concepts of the head, the head predicate and the body of the rule are defined in an obvious way similarly as above.
Definition 7 (D-module)
A D-module $D$ is a pair $\langle \text{Def}, \Pi \rangle$, where $\text{Def}$ is a finite set of predicates, called defined predicates, and $\Pi$ is a finite set of define rules such that the head predicate of each rule belongs to $\text{Def}$.

For a D-module $D$, we denote the set $\text{Def}$ of its defined predicate symbols by $\text{Def}(D)$. We write $\text{Par}(D)$ for the set of all predicate and function symbols in $\Pi$ other than the defined predicates. We call $\text{Par}(D)$ the set of parameter symbols of $D$.

For a set of define rules $\Pi$, by $\text{hd}(\Pi)$ we denote the set of all predicate symbols appearing in the heads of rules in $\Pi$. Note that if $\Pi$ is the set of define rules of a D-module $D$, then $\text{hd}(\Pi) \subseteq \text{Def}(D)$. If the inclusion is proper, the D-modules $\langle \text{hd}(\Pi), \Pi \rangle$ and $D$ are different; the latter makes all predicates in $\text{Def}(D) \setminus \text{hd}(\Pi)$ universally false. In the following, we use $\Pi$ as a shorthand notation for a D-module $\langle \text{hd}(\Pi), \Pi \rangle$.

Definition 8 (T-module)
A T-module is an FO sentence.

While G-modules and D-modules are sets of expressions, a T-module is not. Since any finite set of FO sentences $\varphi_1, \ldots, \varphi_n$ can be equivalently represented by its conjunction $\varphi_1 \land \cdots \land \varphi_n$, the restriction of T-modules to single formulas does not result in any loss of generality.

Definition 9
An ASP-FO theory is a finite set of G-modules, D-modules, and T-modules.

We say that an ASP-FO module or theory $\Psi$ is over vocabulary $\Sigma$ if every non-logical symbol mentioned in $\Psi$ belongs to $\Sigma$. In case of a D-module $\langle \text{Def}, \Pi \rangle$, also symbols in $\text{Def}$ should belong to $\Sigma$.

We say that a predicate is specified by a G-module or D-module if it is the head predicate of the G-module or a defined predicate of the D-module. Unless stated differently, we assume that no predicate is specified by more than one module in an ASP-FO theory as this suffices to express GDT programs. However, the formal definitions of syntax and semantics of ASP-FO do not require this limitation.

5.2 From Core ASP to ASP-FO Informally

There is an obvious match between language constructs of ASP-FO and those used in ASP to express generate, define and test. Specifically, an ASP choice rule (5) corresponds to an ASP-FO choice rule
$$\forall \bar{x} \ (\{A\} \leftarrow B_1 \land \cdots \land B_n \land \neg C_1 \land \cdots \land \neg C_m),$$
a normal rule (4) corresponds to an ASP-FO define rule
$$\forall \bar{x} \ (A \leftarrow B_1 \land \cdots \land B_n \land \neg C_1 \land \cdots \land \neg C_m),$$
and an ASP constraint (6) corresponds to the T-module given by the FO sentence
$$\forall \bar{x} \ (\neg(\leftarrow (B_1 \land \cdots \land B_n \land \neg C_1 \land \cdots \land \neg C_m))),$$
where in each case, $\bar{x}$ is the set of variables occurring in the ASP expression (5), (4) and (6), respectively. These syntactical translations turn both the constraint operator $\leftarrow$ and the negation-as-failure symbol $\text{not}$ in (6) into the negation symbol $\neg$.

Consider the encoding (7) of the Hamiltonian cycle problem. It can be embedded in ASP-FO
in several ways, including the following encoding that makes explicit the hidden structure of that program:

\[
\begin{align*}
generate & \{ \forall x \forall y(\{ In(x, y) \} \leftarrow Edge(x, y)) \} \\
define & \{ Node(V) \leftarrow \top, \ldots, Node(W) \leftarrow \top \} \{ Edge(V, V') \leftarrow \top, \ldots, Edge(W, W') \leftarrow \top \} \{ \forall x \forall y(T(x, y) \leftarrow In(x, y)) \} \{ \forall x \forall y \forall z(T(x, y) \leftarrow T(x, z) \land T(z, y)) \}
\end{align*}
\] (10)

Merging the three D-modules into one would yield another embedding, with less internal structure.

Any answer set program written in the base formalism of Section 4 has a straightforward syntactic translation to an ASP-FO theory: choice rules are grouped according to their head predicate; normal rules are grouped together in one D-module, and each constraint can be represented as a single T-module (in each case, after rewritings described above). In the case of a D-module we can often reveal its hidden structure by splitting it into several smaller D-modules (as we did in the theory (10)). Section 5.5 provides a detailed formal account on the relation between answer set programs and ASP-FO theories.

5.3 Semantics

We now introduce the formal semantics of ASP-FO. As in FO and in a growing number of ASP semantics, the semantics is based on the standard notion of \(\Sigma\)-structures instead of literal sets. Using the terminology of logic programming and ASP, we call \(\Sigma\)-structures also \(\Sigma\)-interpretations.

A crucial feature of the semantics of ASP-FO is modularity: a structure/interpretation is a model of an ASP-FO theory if it is a model of each of its modules. In other words, an ASP-FO theory can be understood as a standard monotone conjunction of its modules.

**Definition 10**

An interpretation \(\mathcal{A}\) satisfies (is a model of) an ASP-FO theory \(T\), written \(\mathcal{A} \models T\), if \(\mathcal{A}\) satisfies (is a model of) each module in \(T\).

To complete the definition of the semantics of ASP-FO theories, we now define the semantics of individual modules.

**T-modules** The case of T-modules is straightforward. T-modules are FO sentences and we use the classical definition of satisfaction (cf. Definition 3) to specify when a T-module holds in a structure.

**G-modules** The role of choice rules in GDT programs is to “open up” some atoms \(P(\bar{t})\) — to allow them to take any of the values true and false. We take this idea as the basis of our formalization of the semantics of G-modules.

**Definition 11**
An interpretation \( \mathcal{M} \) is a model of a G-module \( \mathcal{G} \) with head predicate \( P \) if for each tuple \( \vec{d} \) of elements in the domain of \( \mathcal{M} \) such that \( \mathcal{M}[\vec{x} : \vec{d}] \models P(\vec{x}) \), there is a choice rule \( \forall \vec{y} \left( \{ P(\vec{t}) \} \leftarrow \varphi \right) \) in \( \mathcal{G} \) and a tuple \( \vec{d}' \) so that \( \mathcal{M}[\vec{x} : \vec{d}, \vec{y} : \vec{d}'] \models \varphi \).

D-modules We define the semantics of D-modules by adapting the stable semantics of definitions introduced by Pelov et al. (2007). That semantics is based on the three-valued immediate consequence operator. It is obtained as a special case from the approximation fixpoint theory (Denecker et al., 2000), which defines stable and well-founded fixpoints for arbitrary lattice operators. The semantics proposed by Pelov et al. (2007) is a generalization of the original Gelfond-Lifschitz formal semantics. In particular, unlike the original semantics it is not restricted to Herbrand interpretations only. Our presentation follows that proposed by Vennekens et al. (2007) and developed further by Denecker et al. (2012).

Definition 12 (Satisfaction by pairs of interpretations)
Let \( \varphi \) be an FO formula, \( \mathfrak{A} \) and \( \mathfrak{B} \) interpretations of all free symbols in \( \varphi \) (including free variables) such that \( \mathfrak{A} \) and \( \mathfrak{B} \) have the same domain and assign the same values to all function symbols. We define the relation \( (\mathfrak{A}, \mathfrak{B}) \models \varphi \) by induction on the structure of \( \varphi \) (for simplicity, we consider only the connectives \( \neg \) and \( \lor \), and the existential quantifier \( \exists \)):

- \( (\mathfrak{A}, \mathfrak{B}) \models P(\vec{t}) \) if \( \mathfrak{A} \models P(\vec{t}) \);
- \( (\mathfrak{A}, \mathfrak{B}) \models \neg \varphi \) if \( \mathfrak{B} \not\models \varphi \);
- \( (\mathfrak{A}, \mathfrak{B}) \models \varphi \lor \psi \) if \( (\mathfrak{A}, \mathfrak{B}) \models \varphi \) or \( (\mathfrak{A}, \mathfrak{B}) \models \psi \);
- \( (\mathfrak{A}, \mathfrak{B}) \models \exists \varphi \) if for some \( d \in \text{dom}(\mathfrak{A}) \), \( (\mathfrak{A}[x : d], \mathfrak{B}[x : d]) \models \psi \).

When \( \varphi \) is a sentence, we define \( \varphi(\mathfrak{A}, \mathfrak{B}) = t \) if \( (\mathfrak{A}, \mathfrak{B}) \models \varphi \) and \( \varphi(\mathfrak{A}, \mathfrak{B}) = f \) otherwise.

This is the standard satisfaction relation, except that positive occurrences of atoms are interpreted in \( \mathfrak{A} \), and negative occurrences in \( \mathfrak{B} \). When \( \mathfrak{A} = \mathfrak{B} \), this relation collapses to the standard satisfaction relation of FO so that if \( \varphi \) is a sentence then \( \varphi^\mathfrak{A} = \varphi(\mathfrak{A}, \mathfrak{A}) \). Also if \( \mathfrak{A} \leq^t \mathfrak{A}' \) and \( \mathfrak{B} \leq^t \mathfrak{B}' \), then \( \varphi(\mathfrak{A}, \mathfrak{B}) \leq^t \varphi(\mathfrak{A}', \mathfrak{B}') \) (and so, in particular, \( \varphi(\cdot, \cdot) \) is monotone in its first and antitone in its second argument). These two properties imply that the satisfaction relation in Definition 12 can be used to approximate the standard truth value, in the sense that if \( \mathfrak{A} \leq^t \mathfrak{A}_u \), then for each sentence \( \varphi \), \( \varphi(\mathfrak{A}, \mathfrak{A}_u) \leq^t \varphi^\mathfrak{A} \leq^t \varphi(\mathfrak{A}, \mathfrak{A}_t) \).

Essentially, this satisfaction relation represents Kleene’s and Belnap’s three- and four-valued truth assignment functions (Feferman, 1984). The connection is based on the bilattice correspondence between the four truth values \( f, t, u, i \) and pairs of lower and upper estimates \( (f, f), (t, t), (f, t), \) and \( (t, f) \), respectively. In this view, three- and four-valued interpretations \( \mathfrak{A} \) correspond to pairs of interpretations \( (\mathfrak{A}_t, \mathfrak{A}_u) \) sharing the domain and the interpretations of function symbols, and the truth value \( \varphi^\mathfrak{A} \in \{ f, t, u, i \} \) corresponds to the pair \( (\varphi(\mathfrak{A}_t, \mathfrak{A}_u), \varphi(\mathfrak{A}_u, \mathfrak{A}_t)) \).

Definition 13
A pair of interpretations \( (\mathfrak{A}, \mathfrak{B}) \) sharing the domain and the interpretations of function symbols satisfies a define rule \( \forall \vec{x} \left( P(\vec{t}) \leftarrow \varphi \right) \) if for each tuple \( \vec{d} \) of domain elements, if \( (\mathfrak{A}[\vec{x} : \vec{d}], \mathfrak{B}[\vec{x} : \vec{d}]) \models \varphi \) then \( \mathfrak{A}[\vec{x} : \vec{d}] \models P(\vec{t}) \).

In the context of Herbrand interpretations, the two definitions reflect the way Gelfond and Lifschitz used the reduct to define stable models of normal logic programs. Let us recall that an extended logic program is normal if it consists of normal program rules only, and let us consider a D-module that corresponds to a normal program \( \Pi \). Let \( \mathcal{M} \) be an Herbrand interpretation. For
a body $\varphi$ of a rule in $\text{grn}(\Pi)$ (the ground instantiation of $\Pi$), we write $\varphi^M$ for its reduced form obtained by replacing the negative literals in $\varphi$ by their evaluation in $M$. Clearly, for every rule $P \leftarrow \varphi$ in $\text{grn}(\Pi)$, an Herbrand interpretation $\mathfrak{A}$ satisfies the reduced body $\varphi^M$ if and only if $(\mathfrak{A}, M) \models \varphi$. Since the reduct $\text{grn}(\Pi)^M$ can essentially be viewed as obtained from $\text{grn}(\Pi)$ by replacing the body of each rule by its reduced form, we have the following property explaining the connection of Definitions 12 and 13 to the concept of a stable model of a normal program.

**Proposition 1 (Denecker et al. (2012))**

For a normal program $\Pi$ and Herbrand interpretations $\mathfrak{A}$ and $M$, the interpretation $\mathfrak{A}$ is a model of the reduct $\text{grn}(\Pi)^M$ if and only if the pair of interpretations $(\mathfrak{A}, M)$ satisfies all rules of $\Pi$ (viewed as define rules). Further, $M$ is a stable model of $\Pi$ if and only if $M$ is the $\leq_{t}$-least Herbrand interpretation $\mathfrak{A}$ such that $(\mathfrak{A}, M)$ satisfies all rules of $\Pi$ (viewed as define rules).

In the setting of D-modules we must account for the parameters (input symbols) that may appear in the rules. For two structures $\mathfrak{A}$ and $B$ that have the same domain and interpret disjoint vocabularies, let $\mathfrak{A} \circ B$ denote the structure that

1. interprets the union of the vocabularies of $\mathfrak{A}$ and $B$,
2. has the same domain as $\mathfrak{A}$ and $B$, and
3. coincides with $\mathfrak{A}$ and $B$ on their respective vocabularies.

Following Gelfond and Lifschitz, to define a parameterized version of the stable-model semantics, we perform minimization with respect to the truth order $\leq_{t}$.

**Definition 14 (Parameterized stable-model semantics)**

For a D-module $D$, an interpretation $M$ of $\text{Def}(D)$ is a stable model of $D$ relative to an interpretation $\mathfrak{A}_p$ of $\text{Par}(D)$ if $M$ is the $\leq_{t}$-least among all interpretations $\mathfrak{A}$ of $\text{Def}(D)$ such that $\mathfrak{A}$ has the same domain as $\mathfrak{A}_p$ and $(\mathfrak{A}_p \circ \mathfrak{A}, \mathfrak{A}_p \circ M)$ satisfies all rules of $D$.

This parameterized stable-model semantics of D-modules extends the original stable-model semantics of normal logic programs in three ways:

1. it is parameterized, that is, it builds stable models on top of a given interpretation of the parameter symbols;
2. it handles FO bodies; and
3. it works for arbitrary (also non-Herbrand) interpretations.

It shares these properties with other recent generalizations of the original ASP formalism, such as Pearce and Valverde (2005), Lee and Meng (2008), Ferraris et al. (2011), Zhang and Zhou (2010), Zhou and Zhang (2011) and Asuncion et al. (2012).

The parameterized stable-model semantics turns a D-module $D$ into a non-deterministic function from interpretations of $\text{Par}(D)$ to interpretations of $\text{Def}(D)$. An interpretation $\mathfrak{A}$ satisfies $D$ if it agrees with the function defined by $D$, i.e., if its interpretation of $\text{Def}(D)$ is one of the possible images under this function of its interpretation of $\text{Par}(D)$.

---

5 It is a simple consequence of the monotonicity of $(\mathfrak{A}, B) \models \varphi$ in $\mathfrak{A}$ that this $\leq_{t}$-least interpretation always exists (Vennekens et al. 2007).
Definition 15 (The semantics of D-modules)

An interpretation \( \mathfrak{A} \) is a model of a D-module \( D \), written \( \mathfrak{A} \models D \), if \( \mathfrak{A}|_{\text{Def}(D)} \) is a stable model of \( D \) relative to \( \mathfrak{A}|_{\text{Par}(D)} \).

We stress that we use the term model here to distinguish the present concept from the stable model relative to an interpretation (Definition 14).

Example 11

Let us consider a D-module:

\[
D = \{ \forall x \ (p(x) \leftarrow \neg q(x)) \quad \forall x \ (q(x) \leftarrow \forall y \ r(y, x)). \}
\]

There are no function symbols here, \( r \) is the only parameter of \( D \), and \( p \) and \( q \) are the defined symbols. Each interpretation \( \mathfrak{A}_p \) of the parameter \( r \) determines the corresponding set of stable models relative to \( \mathfrak{A}_p \). Each such stable model is an interpretation of the defined symbols \( p \) and \( q \). Let us investigate how these stable models are to be obtained.

The class of candidates for a stable model relative to a given \( \mathfrak{A}_p \) consists of interpretations \( M \) of the defined symbols \( p, q \) that have the same domain as \( \mathfrak{A}_p \). For each such \( M \), \( \mathfrak{A}_p \circ M \) is an interpretation of all symbols occurring in \( \Pi \), that matches \( \mathfrak{A}_p \) on the parameters and \( M \) on the defined symbols. Let us fix one such \( M \) and consider the set of all interpretations \( \mathfrak{A} \) such that \((\mathfrak{A}_p \circ \mathfrak{A}, \mathfrak{A}_p \circ M)\) satisfies this rule set. In the evaluation of the first rule, \( q \) occurs negatively and so, it is evaluated with respect to \( \mathfrak{A}_p \circ M \). Moreover, since \( q \) is a defined predicate, it is evaluated in \( M \). For the second rule, the parameter \( r \) is evaluated in \( \mathfrak{A}_p \). Thus, the set of all interpretations \( \mathfrak{A} \) such that \((\mathfrak{A}_p \circ \mathfrak{A}, \mathfrak{A}_p \circ M)\) satisfies \( D \) contains each interpretation \( \mathfrak{A} \) such that

- \( q^\mathfrak{A} \supseteq \{ d \in \text{dom}(\mathfrak{A}_p) | \forall d' \in \text{dom}(\mathfrak{A}_p) : (d', d) \in r^\mathfrak{A}_p \} \),
- \( p^\mathfrak{A} \supseteq \text{dom}(\mathfrak{A}_p) \setminus q^\mathfrak{M} \).

According to the definition, \( M \) is a stable model of \( D \) relative to \( \mathfrak{A}_p \), if it is the smallest interpretation of \( p \) and \( q \) to satisfy this condition. This is the case precisely if neither of these two set inclusions is strict, that is, if

- \( q^M = \{ d \in \text{dom}(\mathfrak{A}_p) | \forall d' \in \text{dom}(\mathfrak{A}_p) : (d', d) \in r^\mathfrak{A}_p \} \),
- \( p^M = \text{dom}(\mathfrak{A}_p) \setminus q^M = \{ d \in \text{dom}(\mathfrak{A}_p) | \exists d' \in \text{dom}(\mathfrak{A}_p) : (d', d) \notin r^\mathfrak{A}_p \} \).

This shows that each interpretation \( \mathfrak{A}_p \) of \( r \) determines a unique stable model of \( D \).

Applying Definition 15 to this example, we see that an interpretation \( \mathfrak{A} \) of the vocabulary \( \Sigma = \{ r, p, q \} \) is a model of the D-module \( D \) if \( M = \mathfrak{A}|_{\{p,q\}} \) satisfies the equations on \( p \) and \( q \) obtained from the equations for \( p^M \) and \( q^M \) by substituting \( M|_{\{r\}} \) for \( \mathfrak{A}_p \).

The language design of ASP-FO was guided by our aim to develop an informal semantics for the most basic expressions and connectives of ASP. It is straightforward to extend the language ASP-FO with additional types of modules and language expressions. Section 5.4 introduces one new module called an Herbrand module. In Section 8 we discuss other possible extensions with aggregates, weight expressions and disjunction in the head.

5.4 Herbrand Modules

In applications where the domain of all relevant objects is known, it is common to design a vocabulary \( \Sigma \) such that each domain element is denoted by exactly one ground term. In such
a case, considering Herbrand interpretations is sufficient. ASP is tailored to such applications. Consequently, it typically restricts its semantics to Herbrand interpretations only. The logic ASP-FO is an open domain logic with uninterpreted function symbols. We now introduce an Herbrand module into the language of ASP-FO. Its role is to express the proposition that the domain of discourse is the Herbrand universe. It allows us to restrict the semantics of ASP-FO to Herbrand interpretations and will facilitate the formal embedding of (standard) ASP into ASP-FO.

Definition 16
An Herbrand module over a set $\sigma$ of function symbols is the expression $H(\sigma)$. We say that $M$ is a model of $H(\sigma)$, denoted $M \models H(\sigma)$, if $\text{dom}(M)$ is the set of terms that can be built from $\sigma$, and if for each such term $t$, $t^M = t$.

If $\Sigma_F$ is the set of all function symbols of $\Sigma$, then the models of the Herbrand module $H(\Sigma_F)$ are precisely the Herbrand interpretations of $\Sigma$.

We now extend the definition of an ASP-FO theory as follows: an ASP-FO theory is a finite set of G-modules, D-modules, T-modules, and Herbrand modules.

An Herbrand module $H(\sigma)$ in ASP-FO can be seen as a shorthand for the combination of the domain closure axiom $DCA(\sigma)$ and the FO unique name axioms $UNA(\sigma)$. A combination $DCA(\sigma)$ and $UNA(\sigma)$ can also be expressed in ASP-FO by means of D- and T-modules\footnote{It is a well-known consequence of the compactness theorem for FO that $DCA(\sigma)$ cannot be represented in FO if $\sigma$ contains at least one constant and one function symbol of arity $\geq 1$.}. Deheker (2000) illustrated how the logic FO(ID) captures this combination. The same method is applicable in ASP-FO. The idea is to introduce a new predicate symbol $U/1$ and then the D-module:

\[
\begin{align*}
&\forall x_1 \ldots \forall x_n (U(f_j(x_1, \ldots, x_n)) \leftarrow U(x_1) \land \cdots \land U(x_n)) \\
&\cdots
\end{align*}
\]

that has one such rule as above for every function symbol $f_j/n$ in $\sigma$. In the case of a constant symbol $C$, the rule reduces to $U(C) \leftarrow \top$. This D-module defines $U$ as the set of all ground terms of $\sigma$. The following T-module is added to express that there are no other objects in the domain:

\[
\forall x U(x).
\]

Combining the above D-module and T-module with FO axioms $UNA(\sigma)$ yields an ASP-FO theory whose models are (isomorphic to) the structures satisfying $H(\sigma)$. Thus Herbrand modules are redundant in ASP-FO and serve only as useful and intuitive abbreviations.

5.5 Formal Relation to ASP

We now show that ASP-FO is a conservative extension of the core ASP language so that we can formally relate core ASP programs and ASP-FO theories. For a core ASP program $\Pi$, by $\hat{\Pi}$ we denote the collection of rules obtained by rewriting the rules in $\Pi$ in the syntax of ASP-FO as illustrated in Section 5.2. Further, for a vocabulary $\Sigma$ we write $\Sigma_P$ and $\Sigma_F$ for the sets of predicate and function symbols in $\Sigma$, respectively.

What we are looking for is a connection between an ASP program and an ASP-FO theory that in the case of GDT programs makes their implicit structure explicit. To establish such
a connection, we use the results by Ferraris et al. (2009) on “splitting”. Splitting is a common method for uncovering the structure of programs for their further analyses. For instance, Erdoğan and Lifschitz (2004) use it to prove correctness of GDT programs.

Let $\Pi$ be a core ASP program of the syntactic form considered in Section 4 with rules of the form (5), (4) and (6). We define the positive predicate dependency graph of $\Pi$ as the directed graph that has all predicates of $\Pi$ as its vertices and that has an edge from a predicate $P$ to a predicate $Q$ whenever $P$ appears in the head of a rule that has a non-negated $Q$ in its body. We say that $P$ positively depends on $Q$ if there is a non-zero length path from $P$ to $Q$ in this graph.

**Definition 17**

A partition $\{\Pi_1, \ldots, \Pi_n\}$ of $\Pi$ is a splitting of $\Pi$ if:

1. each $\Pi_i$ that contains a constraint is a singleton;
2. for each predicate $P$, all rules with $P$ in the head belong to the same $\Pi_i$; and
3. if two predicates positively depend on each other, their rules belong to the same module $\Pi_i$.

Splitting tends to decompose a program in components that can be understood independently. For instance, the horizontal lines in the Hamiltonian cycle program (7) identify a splitting in which each component has a simple and natural understanding.

**Definition 18**

A splitting $\{\Pi_1, \ldots, \Pi_n\}$ of $\Pi$ is proper if no module $\Pi_i$ contains choice rules for two different predicates, or both a normal rule and a choice rule. In addition, no head predicate of a choice module may positively depend on itself.

The following proposition is straightforward. We use here a simplified notation, where $\hat{\Pi}$ denotes the D-module $\langle \text{hd}(\Pi), \hat{\Pi} \rangle$.

**Proposition 2**

If $\Pi$ has a proper splitting $\{\Pi_1, \ldots, \Pi_n\}$ then $\{\hat{\Pi}_1, \ldots, \hat{\Pi}_n\}$ is a well-formed ASP-FO theory.

For instance, the splitting of the Hamiltonian cycle program (7) identified by the horizontal lines is a proper splitting, and its translation is the ASP-FO theory (10).

Typically, a program $\Pi$ with a proper splitting $\{\Pi_1, \ldots, \Pi_n\}$ is equivalent to the corresponding ASP-FO theory augmented with the Herbrand module. However, this is not the case for programs that contain predicates which do not appear in the head of any rule. In core ASP, such predicates are universally false. To obtain the same effect in ASP-FO, we add an additional “empty” D-module $(\text{Def}, \{\})$, where $\text{Def}$ is the set of these predicates.

**Theorem 3**

Let $\Pi$ be a core ASP program over a finite vocabulary $\Sigma$ with a proper splitting $\{\Pi_1, \ldots, \Pi_n\}$. Then an interpretation $M$ is an answer set of $\Pi$ if and only if $M$ is a model of the ASP-FO theory $\{\mathcal{H}(\Sigma_F), \hat{\Pi}_1, \ldots, \hat{\Pi}_n, (\text{Def}, \{\})\}$, where $\text{Def} = \Sigma \setminus \text{hd}(\Pi)$.

This theorem essentially follows from the splitting result of Ferraris et al. (2011). We present an argument in the appendix, as it requires technical notation and concepts not related to the main topic of the paper.

Theorem 3 implies that answer sets of the GDT program (7) coincide with Herbrand models of the ASP-FO theory (10). More generally, Theorem 3 states that properly splittable ASP programs can be embedded in ASP-FO while preserving their implicit internal structure. It therefore
implies that such a program can be understood as the *monotone conjunction* of the ASP-FO modules of which it consists. Thus, even though ASP is a nonmonotonic logic, its nonmonotonicity is restricted to individual ASP-FO modules of which it consists. In this way, the theorem paves the way towards an informal semantics of GDT programs by reducing the problem to finding the informal semantics of their generate, define and test modules.

6 Informal semantics of ASP-FO

In this section, we develop the theory of informal semantics $\mathcal{O}_B^I$ for ASP-FO. We use the Hamiltonian cycle ASP-FO theory (10) as a test case. Through the model-preserving embedding of Theorem 3, this informal semantics applies to the original GDT program (7).

The Hamiltonian cycle theory expresses that a graph $In$ is a Hamiltonian cycle of graph $Edge$: a linear cyclic subgraph of $Edge$ that includes all vertices. The intended interpretation $I$ of the predicate symbols of this theory can be specified as follows:

- $I(Node)$: “$\#_1$ is a vertex”
- $I(Edge)$: “there is an edge from $\#_1$ to $\#_2$ in graph $Edge$”
- $I(In)$: “there is an edge from $\#_1$ to $\#_2$ in graph $In$” and
- $I(T)$: “$\#_2$ is reachable from $\#_1$ in graph $In$”.

In addition to the above predicate symbols, the vocabulary of this theory also contains a number of constant symbols $v, w, \ldots$. These are intended to represent the nodes of the graph. We therefore also add the Herbrand module $\mathcal{H}(\Sigma_F)$ to the theory.

The composition operator of ASP-FO theories

Formally, a structure $A$ is a model of an ASP-FO theory if it is a model of each of its modules. In our Tarskian perspective, this means that a world is possible according to a theory if it is possible according to each of its modules. Thus, as for FO, the composition operator that describes how the meaning of a theory depends on the meaning of its elements is simply the standard conjunction: if an ASP-FO theory $T$ consists of modules $\Psi_1, \ldots, \Psi_n$, then $\mathcal{O}_B^I(T)$ is the conjunction of the statements $\mathcal{O}_B^I(\Psi_i), \ldots, \mathcal{O}_B^I(\Psi_n)$. Therefore, adding a new module to an ASP-FO theory is a monotone operation, in the same way that adding an additional formula to an FO theory is.

Theorem 3 shows that a GDT program (to be precise, a core ASP program with a proper splitting) can be viewed as a monotone conjunction of its components. Thus, the nonmonotonicity of an ASP program in the GDT style is confined to individual components. Indeed, we will see that the informal composition operators that construct G-modules and D-modules from individual rules are not monotone. That is, the meaning of G-modules and D-modules cannot be understood as a simple conjunction of the meanings of their rules.

Informal semantics of T-modules (FO sentences)

Formally, a T-module $T$ consists of FO sentences under their classical semantics. Therefore, we set $\mathcal{O}_B^I(T) = \mathcal{F}^I(T)$. In the case of theory (10), this yields the following readings for its T-modules. The T-module

\[ \forall x \forall y \forall z (In(x, y) \land In(x, z) \land y \neq z) \]
states that for all \( x \), for all \( y \) and for all \( z \), it is not the case that there are edges from \( x \) to \( y \) and from \( x \) to \( z \) in graph \( I^n \) and that \( y \) and \( z \) are not the same. This can be restated as: each domain element is the start of at most one edge in the graph \( I^n \).

The T-module

\[
\forall x \forall y \forall z \neg (I^n(x, z) \land I^n(y, z) \land x \neq y)
\]

has a similar reading, which can be equivalently stated as: each domain element is the end of most one edge in the graph \( I^n \).

The T-module

\[
\forall x \forall y (\neg (\text{Node}(x) \land \text{Node}(y) \land \neg T(x, y))
\]

says that for every \( x \) and for every \( y \), it is not the case that \( x \) and \( y \) are nodes, and \( y \) is not reachable from \( x \) in graph \( I^n \). This can be equivalently stated as: every node is reachable from every other node in graph \( I^n \).

The three propositions above are precisely the properties that the graph \( I^n \) should satisfy to be a Hamiltonian cycle of the graph \( E^n \). They therefore represent precisely what the ASP programmer intended to encode.

**Informal semantics of G-modules (choice rules)**

Choice rules in ASP are often explained in a computational way, as “generators of the search space.” In this section, we develop a declarative interpretation for choice rules in ASP-FO.

We start by rewriting G-modules using a process similar to predicate completion (Clark, 1978). First, every choice rule (8) in a G-module is rewritten as

\[
\forall \bar{y} \left( \{ P(\bar{y}) \} \leftarrow \exists \bar{x} (\bar{y} = \bar{t} \land \varphi) \right)
\]

Second, all resulting choice rules, say,

\[
\forall \bar{x} \left( \{ P(\bar{x}) \} \leftarrow \varphi_i \right)
\]

for \( i = 1, \ldots, n \)

are combined into a single one:

\[
\forall \bar{x} \left( \{ P(\bar{x}) \} \leftarrow \varphi_1 \lor \cdots \lor \varphi_n \right)
\]

We denote the result of this rewriting of a G-module \( G \) to a singleton G-module by \( S(G) \). It is evident that the rewriting preserves models.

**Theorem 4**

Every G-module \( G \) is equivalent to the singleton G-module \( S(G) \).

This result is important because singleton G-modules have a simple representation as FO sentences.

**Theorem 5**

An interpretation \( M \) satisfies a singleton G-module \( \{ \forall \bar{x} \left( \{ P(\bar{x}) \} \leftarrow \varphi \} \) if and only if \( M \) satisfies FO sentence \( \forall \bar{x} (P(\bar{x}) \Rightarrow \varphi) \).

**Proof.** By Definition \ref{def:fo} \( M \) satisfies \( \{ \forall \bar{x} \left( \{ P(\bar{x}) \} \leftarrow \varphi \} \) if and only if for each variable assignment \( \theta \) such that \( M, \theta \models P(\bar{x}) \) it holds that \( M, \theta \models \varphi \). This is precisely the condition for \( M \) to satisfy \( \forall \bar{x} (P(\bar{x}) \Rightarrow \varphi) \). QED

For instance, the singleton G-module

\[
\{ \forall x \forall y (\{ I^n(x, y) \} \leftarrow E^n(x, y) \})
\]
of the ASP-FO theory \(^{(10)}\) corresponds to the FO sentence
\[
\forall x \forall y (\text{In}(x, y) \Rightarrow \text{Edge}(x, y)).
\] (12)

We call the result of first rewriting a G-module \(G\) to a singleton G-module and then translating the latter to FO the completion of \(G\). We denote it by \(\text{Gcompl}(G)\). The following consequence of Theorem\(^{[4]}\) provides the key property of \(\text{Gcompl}(G)\).

**Corollary 6**
Every G-module \(G\) is equivalent to the FO sentence \(\text{Gcompl}(G)\).

This corollary demonstrates that G-modules can be simulated by T-modules. It follows that ASP-FO theories can be seen as consisting of FO sentences and D-modules.

A G-module \(G\) for a predicate \(P\) is a set of choice rules
\[
\{ \{ \text{P}(\bar{t}_1) \} \leftarrow \phi_1, \ldots, \{ \text{P}(\bar{t}_n) \} \leftarrow \phi_n \}.
\] (13)

By Corollary\(^{[6]}\) such \(G\) is equivalent to \(\text{Gcompl}(G)\):
\[
\forall \bar{y} (\text{P}(\bar{y}) \Rightarrow \exists \bar{x}_1 (\bar{y} = \bar{t}_1 \land \phi_1) \lor \cdots \lor \exists \bar{x}_n (\bar{y} = \bar{t}_n \land \phi_n)).
\]

Thus, given some intended interpretation \(I\) for a vocabulary \(\Sigma\), and a G-module \(G\) over \(\Sigma\) of the form \((13)\), the informal semantics \(\text{OB}_I(G)\) must be equivalent to the informal semantics \(\text{FO}_I(\text{Gcompl}(G))\). With this in mind, we define \(\text{OB}_I(G)\) by restating \(\text{FO}_I(\text{Gcompl}(G))\) as follows:

**In general,** for each \(\bar{x}\), \(\text{P}^I[\bar{x}]\) is false. However, there are exceptions as expressed by the following rules:

- If \(\text{FO}_I(\phi_1)\), then it might be that \(\text{FO}_I(\text{P}(\bar{t}_1))\).
- \(\cdots\)
- If \(\text{FO}_I(\phi_n)\), then it might be that \(\text{FO}_I(\text{P}(\bar{t}_n))\).
- There are no other exceptions.

This definition implicitly specifies the meaning of the logical connectives occurring in choice rule bodies, the rule operator \(\leftarrow\), and the composition operator that forms a G-module out of its rules. We now make this meaning explicit.

First, in the translation of G-modules to FO sentences, choice rule bodies \(\phi_i\) are treated as “black boxes,” that are simply copied and pasted into FO expressions. This shows that choice rule bodies in ASP-FO not only look like, but in fact are FO expressions, with all FO logical symbols retaining their informal semantics. In particular, this illustrates that the negation operator in the bodies of choice rules of an ASP-FO G-module is just classical negation.

Second, to explicate the informal semantics of the rule operator and the composition operator of G-modules, we note that the informal reading \(\text{OB}_I(G)\) interprets a G-module as a local closed world assumption (LCWA) on predicate \(P\) (local refers to the scope of the assumption, which is restricted to \(P\)), but provides an exception mechanism to relax it. Each rule of a G-module expresses an exception to the LCWA and reinstalls uncertainty, the open world assumption (OWA), on the head atom. For instance, the G-module
\[
\{ \{ \text{In}(x, y) \} \leftarrow \text{Edge}(x, y) \}
\]
states that “The Hamiltonian path (In) is empty except that if \((x, y)\) is an edge of the graph \(G\), then \((x, y)\) might belong to it.” We note that this yields an informal but precise linguistic reading of \(G\) and of rules in \(G\), a reading that is indeed equivalent to the informal semantics of \(G\)’s translation into FO which states that \(\text{In}\) is a subgraph of \(\text{Edge}\).
It can be seen in $\mathcal{O}\mathcal{B}_\mathcal{I}(\mathcal{G})$ that the rule operator in G-modules has unusual semantic properties. Each rule of a G-module is a conditional “if . . . then $P(\overline{t})$ might be true”. Its “conclusion” removes information (namely that $P(\overline{t})$ is false) rather than adding some. This is unlike any other conditional or expression in logic that we are aware of.

Also the semantic properties of the composition operator of G-modules are unique. The composition operator underlying G-modules is neither conjunction nor disjunction. It is also not truth functional and not monotone. Adding a rule to a module corresponds to adding a disjunct to the corresponding FO sentence. Hence, the underlying composition operator is anti-monotone: the module becomes weaker with each rule added. This agrees with the role of a choice rule for expressing an exception to the LCWA on $P$. The more rules there are, the more exceptions and hence, the weaker the LCWA.

To recap, G-modules are given a precise informal semantics $\mathcal{O}\mathcal{B}_\mathcal{I}(\mathcal{G})$ as a form of LCWA with an exception mechanism to relax it. Logical connectives in the bodies of choice rules retain their classical meaning. From a logical point of view, the rule operator and the composition operator of G-modules have uncommon semantical properties. Still, $\mathcal{O}\mathcal{B}_\mathcal{I}(\mathcal{G})$ identifies a natural language conditional that explains formal choice rules in a declarative way. For example, when applied to the G-module of ASP-FO theory (10), it yields a correct reading of its G-module.

Informal semantics of D-modules

Humans use definitions to express abstractions of concepts they encounter. These abstractions are necessary for us to understand the world in which we live and function, and to the ability to relay this understanding to others. We communicate these definitions in natural language: already as children, we are trained to compose, understand, and use them effectively. Definitions also appear in the rigorous setting of scientific discourse. In fact, they are the main building blocks of formal science. In scientific and mathematical texts, definitions embody the most precise and objective forms of human knowledge. While definitions in a mathematical and scientific context are typically formulated with more precision than the definitions we use in everyday life, they are still informal, in the sense that they are not written in a formal language. We therefore refer to the unambiguous, precise natural language definitions of concepts we find in everyday life or in science and mathematics as informal definitions.

The stated goal of D-modules of an ASP-FO theory (and the define components of a GDT program) is to define concepts formally. We will now provide D-modules with an informal semantics matching precisely the formal one. The linguistic constructs used by humans to so effectively specify (informal) definitions are natural candidates for that task. We therefore start by reviewing some of these natural language expressions.

While there are no “official” linguistic rules on how to write an informal definition in a mathematical or scientific text, several conventions exist. Simple definitions often take the form of “if” or “if and only if”-statements. More complex cases are inductive (recursive) definitions, which are frequently represented as a set of informal rules, possibly with an induction order. A good example is Definition 3 where the satisfaction relation $\models$ is defined over the subformula induction order. When written according to these linguistic conventions, a definition has a precise and objective meaning to us. Consider an intended interpretation $\mathcal{I}$ for a vocabulary $\Sigma$, a D-module $\mathcal{D} = \langle \text{Def}, \Pi \rangle$ in this vocabulary, with $\text{Def} = \{P_1, \ldots, P_n\}$, and $\Pi = \{\forall \overline{x}_1(A_1 \leftarrow \varphi_1), \ldots, \forall \overline{x}_m(A_m \leftarrow \varphi_m)\}$. Assume the following translation $\mathcal{O}\mathcal{B}_\mathcal{I}(\mathcal{D})$ of $\mathcal{D}$ into natural language.
We define the relations \( P_1, \ldots, P_n \) in terms of \( \text{Par}(D) \) by the following (simultaneous) induction:

\[
\begin{align*}
\forall \mathbf{A} \in D & : \text{OB}_I(\mathbf{A}_1) \quad \text{if} \quad \text{OB}_I(\varphi_1) \\
& \quad \ldots \\
\forall \mathbf{A} \in D & : \text{OB}_I(\mathbf{A}_m) \quad \text{if} \quad \text{OB}_I(\varphi_m) \\
\text{In no other cases, } P_1, \ldots, P_n & \text{ hold.}
\end{align*}
\]

This last clause (“In no other cases...”) is usually left implicit if it is clear that we are giving a definition. The question that we address here is whether this is a suitable informal semantics for D-modules.

The above translation turns a D-module \( D \) into a natural language statement that follows the linguistic conventions used to express inductive definitions. If the D-module is not recursive, the phrase “by the following simultaneous induction” should be dropped; what remains then is a definition by exhaustive enumeration, in which each rule represents one case. This translation again makes use of the natural language connective “if”. As before, however, when this word is encountered in the context of a case of an inductive definition, it has a precise and unambiguous meaning which is clear to any mathematician. We refer to this meaning as the “definitional implication”. We later discuss how this conditional relates to material implication.

We now test the stated informal semantics on the three D-modules of the Hamiltonian cycle theory (10). The first two modules correspond to non-recursive definitions by exhaustive enumeration of elements in the extensions of the Node and Edge relations, respectively. The reading \( \text{OB}_I(D) \) of the remaining D-module is as follows:

We define \( T \) in terms of the graph \( I_n \) by induction:

\[
\begin{align*}
\forall x, y & : \text{OB}_I(x, y) \quad \text{if} \quad \text{there is an edge from } x \text{ to } y \\
\forall x, y, z & : \text{OB}_I(x, y) \quad \text{if} \quad \text{there is an edge from } x \text{ to } y \text{ in } I_n \text{ and } z \text{ is reachable from } x \text{ in } I_n \text{ and } y \text{ is reachable from } z \text{ in } I_n
\end{align*}
\]

This is a standard monotone inductive definition of the transitive closure of graph \( I_n \), which is the intended interpretation of \( T/2 \).

Thus, we now have a proposal for a precise informal semantics \( \text{OB}_I(\cdot) \) for D-modules, and, through the embedding result of Theorem 3, therefore also for define components in GDT programs. The informal semantics we specified reflects the role of these define components in the GDT-methodology. The rest of the section is concerned with the following questions:

(a) For which D-modules \( D \) is \( \text{OB}_I(D) \) a sensible informal definition of the relations in \( \text{Def}(D) \)?

(b) If \( \text{OB}_I(D) \) is a sensible informal definition, are the relations defined by this informal definition indeed the same relations as produced by the parametrized stable semantics of the D-module?

(c) What is the meaning of the logical connectives in such D-modules and, through the embedding of GDT programs, in define components of GDT programs?

In the case of FO, and therefore also of T-modules and G-modules, the correspondence between the informal semantics \( \text{FO}_I(\cdot) \) and the formal semantics of FO is made plausible by the great similarity between Table 1 and Definition 3. In the case of D-modules, however, the situation is more complex. The reason for this is that there is no obvious connection between the way (parametrized) stable models are defined and the way we understand informal inductive definitions.

To address the questions (a) - (c), we will borrow from the work on the logic FO(ID) (Denecker, 2004).
The logic FO(ID) was conceived as a conservative extension of FO with a formal definition construct. Syntactically, FO(ID) corresponds to the fragment of ASP-FO without G-modules. A definition in the logic FO(ID) is a set of rules of exactly the same form as rules in D-modules. There is however a semantic difference: formal definitions in FO(ID) are interpreted under the two-valued parametrized well-founded semantics rather than the stable semantics.

The three questions formulated above for D-modules of the logic ASP-FO arise also for FO(ID)’s definitions. They were investigated in detail by Denecker and Vennekens (2014). The view taken in that study is that an informal inductive or recursive definition defines a set by specifying how to construct it: starting from the empty set, it proceeds by iterated rule application (possibly along some induction order) until the constructed set is saturated under rule application. Denecker and Vennekens formalized this induction process for FO(ID) definitions \( \mathcal{D} \) parameterized by a \( \mathcal{Par}(\mathcal{D}) \)-structure \( \mathfrak{A} \), and compared it with the well-founded model construction. They argued that if the well-founded model is two-valued, the informal semantics \( \mathfrak{OB}_T(\mathcal{D}) \) of such a formal rule set is a sensible inductive definition and proved that all formal induction processes converge to the well-founded model. If the well-founded model is not two-valued, then the informal semantics \( \mathfrak{OB}_T(\mathcal{D}) \) of such a formal rule set is not a sensible informal definition and the induction processes do not converge. This motivated them to call a definition \( \mathcal{D} \) of FO(ID) total in a \( \mathcal{Par}(\mathcal{D}) \)-structure \( \mathfrak{A} \) if the parametrized well-founded model of \( \mathcal{D} \) in \( \mathfrak{A} \) is two-valued. Their punch line is that for definitions \( \mathcal{D} \) that are total in \( \mathfrak{A} \), the informal semantics \( \mathfrak{OB}_T(\mathcal{D}) \) presented above is a sensible informal definition and the relations that it defines are given by the well-founded model of \( \mathcal{D} \) extending \( \mathfrak{A} \).

We now observe that the logics FO(ID) and ASP-FO are tightly related, not only syntactically, but also semantically. As long as we restrict attention to D-modules that have two-valued well-founded models, both logics are identical.

\textbf{Theorem 7 (Pelov et al. (2007))}

Let \( \mathcal{D} \) be a formal definition of FO(ID) or a D-module of ASP-FO, and let \( \mathfrak{A} \) be a \( \mathcal{Par}(\mathcal{D}) \)-structure. If \( \mathcal{D} \) is total in \( \mathfrak{A} \), then the parameterized well-founded model of \( \mathcal{D} \) in \( \mathfrak{A} \) is the unique parametrized stable model of \( \mathcal{D} \) that expands \( \mathfrak{A} \).

Thus, we obtain an answer to question (b) for the logic ASP-FO. Provided that a D-module \( \mathcal{D} \) is total in \( \mathfrak{A} \) (a \( \mathcal{Par}(\mathcal{D}) \)-structure), \( \mathfrak{OB}_T(\mathcal{D}) \) is a correct and precise informal semantics for \( \mathcal{D} \) under the parametrized stable-model semantics.

The totality condition on D-modules (or FO(ID) definitions) addresses the question (a) as it serves as a general semantic criterion for a sensible definition. Broad classes of D-modules (FO(ID) definitions) are total in every context \( \mathfrak{A} \). Others are total only in some contexts \( \mathfrak{A} \). To provide some sense of scope, the classes of non-recursive, definite, stratified and locally stratified logic programs have straightforward generalizations as D-modules in ASP-FO (definitions in FO(ID)). Non-recursive, definite, and stratified normal programs give rise to D-modules that are total in every interpretation of the parameter symbols \( \mathcal{H}(\sigma) \). Locally stratified normal programs give rise to definitions that are total in any Herbrand interpretation, and hence in the context of any theory that contains \( \mathcal{H}(\sigma) \).

Informal semantics of connectives in D-modules in ASP-FO and FO(ID) We now address question (c). Just as in the case of G-modules, the informal semantics \( \mathfrak{OB}_T(\mathcal{D}) \) of D-modules im-
The informal semantics of Answer Set Programming: A Tarskian perspective

35

implicitly determines the informal semantics of the logical connectives occurring in D-module rule bodies and of the rule operator $\leftarrow$. Moreover, it also determines the semantical composition operator that “forms” the meaning of a D-module from its rules. The discussion below is restricted to D-modules $D$ for which $\mathcal{OB}_I(D)$ is a sensible inductive definition.

The translation $\mathcal{OB}_I(D)$ treats a rule body $\varphi$ by simply applying the standard informal semantics of FO to it. All logical symbols in rule bodies therefore retain their classical meaning. In particular, the negation symbol in rule bodies of a D-module, and therefore, the negation as failure symbol in rule bodies of GDT-programs is classical negation. After nearly 40 years of controversy on the nature of negation as failure, this can be called a surprising conclusion.

The rule operator $\leftarrow$ in D-modules represents the sort of conditional that is found in inductive definitions in mathematical text. For example, we can phrase

$$\mathcal{OB}_I(\forall x (\text{Even}(S(x)) \leftarrow \neg \text{Even}(x)))$$

as the conditional “$n + 1$ is even if $n$ is not even”. In the context of a definition, such a rule sounds like a material implication. However, while it indeed entails the material implication (i.e., its head must be true whenever its body is true), it is in fact much stronger than that (in particular, its head may not be arbitrarily true) and is not even a truth functional object. In particular, each rule is an “instruction” in the iterative “recipe” provided by an informal definition to construct the defined relations. This is the third kind of conditional that we encounter in this paper. In other studies of inductive definitions, this kind of conditional has also been called a production (Martin-Löf, 1971).

The remaining question concerns the global informal composition operator of D-modules. In mathematical text, this composition operator and the modular nature of definitions surface most clearly when an existing informal definition is extended with new cases. For instance, the syntax of modal propositional logic may be derived from that of propositional logic by a phrase such as: “We extend the definition of propositional logic with the additional case that if $\varphi$ is a formula, then so is $K\varphi$”. In our terminology, this natural language statement is invoking the informal composition operator of inductive definitions to add an additional rule to an existing definition. Such an additional rule has an impact on the construction process specified by the definition, and therefore also on the relation that is eventually constructed. After the addition, the defined set of formulas becomes strictly larger since more formulas can be constructed. However, the extension has a non-monotonic effect (in the sense used in the area of non-monotonic reasoning). Indeed, before the addition, the definition entailed for each propositional symbol $p$ that $Kp$ was not a formula; after adding the rule, the definition entails that $Kp$ is a formula. This is a revision and neither monotone nor antimonotone.

We observe that from a logical perspective, the rule operator and the global D-module composition have very unusual properties and are truly non-classical. In themselves, they are not truth-functional. Yet, the definitions they form are — a definition expresses a particular logical relation between parameter and defined symbols that can be true or false in structures interpreting these symbols. In summary, these non-standard features do not stop human experts from understanding inductive definitions and the compositional nature of definitions, allowing them e.g., to properly judge how an additional case changes the defined concept.
How much of ASP practice is covered by $\mathcal{OB}_I(-)$?

The transformation to ASP-FO in Theorem 3 is equivalence preserving. Consequently, $\mathcal{OB}_I(-)$ provides a precise informal semantics that captures the content of splittable ASP programs, under the condition that the resulting D-modules are total. Here, we assess how much of ASP practice is covered by these translations.

Theorem 3 applies to core ASP programs $\Pi$ that have a proper splitting. Experience suggests that define components in GDT programs map frequently to classes of D-modules that are known to be total (non-recursive, definite, stratified, locally stratified). For example, in the Hamiltonian cycle program (7), two D-modules are non-recursive and one is negation-free. In an attempt to verify this on a broader scale, we examined benchmark programs of the 2013 ASP system competition (Alviano et al., 2013a). In this experiment, we used an extended version of ASP-FO that supports weight constraints and aggregate expressions which occur in many practical ASP programs. The reading $\mathcal{OB}_I$ and Theorem 3 can be generalized for this formalism (see the next section). A few of the benchmarks such as the strategic company program contain disjunction in the head; to these our theory does not apply. Other benchmarks were properly splittable and could easily be translated into ASP-FO following almost literally the embedding of Theorem 3. In most cases, we could split and apply the trivial syntactic transformations exemplified in transforming (7) to (10). Few places required more than these trivial transformations to express generate parts. Most importantly, we observed that in all our experiments, the D-modules obtained after splitting yielded total definitions. Indeed, the rule sets belonged to one of the aforementioned classes of total D-modules (non-recursive, positive, stratified or locally stratified) and they clearly expressed definitions of the head predicates in terms of the parameter symbols. Thus, $\mathcal{OB}_I(-)$ provided a precise and correct interpretation for the benchmark programs considered.

This observation provides experimental evidence for the claim that the GDT paradigm requires only total D-modules, and that the informal semantics $\mathcal{OB}_I(D)$ therefore suffices to cover GDT practice. A similar observation was made by Erdoğan and Lifschitz (2004), who note that

[w]e expect the rules in the define part of a program to not add or remove potential solutions, but just to extend each of them by adding the defined atoms appropriately.

This is obviously in keeping with our restriction to total D-modules, which have a unique stable model for each interpretation of their parameters. To ensure this property, Erdoğan and Lifschitz (2004) restrict attention to D-modules without negated occurrences of defined atoms—i.e., those that correspond to monotone inductive definitions such as that of transitive closure. Using the results of Denecker and Ternovska (2008) allows us to be more general, by considering also stratified non-monotone inductive definitions such as that of the satisfaction relation.

Table 8 recaps the essence of the new theory $\mathcal{OB}_I$ of informal semantics. Here, rows 9 and 10 give the informal semantics of T- and G-modules, whereas row 11 gives the informal semantics for definitional rules for D-modules. Row 12 specifies that the informal composition operator underlying D-modules is the one underlying inductive definitions. Row 13 gives the implicit composition operator of ASP-FO itself (i.e., it explains what it means to gather a number of modules into a theory). As a comparison to Table 1 shows, the D-module composition operator (row 12) and the rule operator (row 11) are the only non-classical elements.
Table 8. The objective informal semantics for ASP-FO.

| Rule | Formal Expression | Description |
|------|-------------------|-------------|
| 1    | \( f(t_1, \ldots, t_n) \) | \( \mathcal{I}(f(\mathcal{FO}_T(t_1), \ldots, \mathcal{FO}_T(t_n))) \) |
| 2    | \( P(t_1, \ldots, t_n) \) | \( \mathcal{I}(P(\mathcal{FO}_T(t_1), \ldots, \mathcal{FO}_T(t_n))) \) |
| 3    | \( \varphi \lor \psi \) | \( \mathcal{OB}_I(\varphi) \) or \( \mathcal{OB}_I(\psi) \) (or both) |
| 4    | \( \varphi \land \psi \) | \( \mathcal{OB}_I(\varphi) \) and \( \mathcal{OB}_I(\psi) \) |
| 5    | \( \neg \varphi \) | it is not the case that \( \mathcal{OB}_I(\varphi) \) (i.e., \( \mathcal{OB}_I(\varphi) \) is false) |
| 6    | \( \varphi \Rightarrow \psi \) | if \( \mathcal{OB}_I(\varphi) \) then \( \mathcal{OB}_I(\psi) \) (in the sense of material implication) |
| 7    | \( \exists x \varphi \) | there exists an \( x \) in the universe of discourse such that \( \mathcal{OB}_I(\varphi) \) |
| 8    | \( \forall x \varphi \) | for all \( x \) in the universe of discourse, \( \mathcal{OB}_I(\varphi) \) |
| 9    | T-module \( \{ \varphi \} \) | \( \mathcal{FO}_T(\varphi) \) |
| 10   | G-module \( G \) | \( \mathcal{FO}_T(\text{Gcompl}(G)) \) |
| 11   | \( A \leftarrow \varphi \) | if \( \mathcal{OB}_I(\varphi) \) then \( \mathcal{OB}_I(A) \) (in the sense of definitional implication) |
| 12   | D-module \( D = \{ r_1, \ldots, r_n \} \) | The relations \( \mathcal{I}(\text{Def}(D)) \) are defined in terms of \( \mathcal{I}(\text{Par}(D)) \) by the following (simultaneous) induction: |
|      |                  | - \( \mathcal{OB}_I(r_1) \) |
|      |                  | - ... |
|      |                  | - \( \mathcal{OB}_I(r_n) \) |
| 13   | ASP-FO theory \( T = \{ M_1, \ldots, M_n \} \) | \( \mathcal{OB}_T(M_1) \) and ... and \( \mathcal{OB}_T(M_n) \) |

7 ASP-FO as a classical logic

The presented informal semantics \( \mathcal{OB}_T \) is for the most part classical. Thus, we expect ASP-FO to share many properties with FO. If not, then \( \mathcal{OB}_T \) should be easily refutable. In this section, we investigate a number of FO properties in the context of ASP-FO.

The following simple property is a direct consequence of our definitions.

**Proposition 8**
Let \( \Phi \) be an ASP-FO module over vocabulary \( \Sigma \). If \( \mathfrak{A}, \mathfrak{B} \) are two interpretations such that \( \mathfrak{A}|_{\Sigma} = \mathfrak{B}|_{\Sigma} \), then \( \mathfrak{A} \models \Phi \) if and only if \( \mathfrak{B} \models \Phi \).

This result states that, like an FO formula, an ASP-FO module does not impose constraints on symbols that do not appear in it, i.e., any expansion of a model of an ASP-FO module is again a model. In particular, ASP-FO has no implicit global closed world assumption (CWA).

A key property of FO is that substituting a formula \( \varphi \) for a formula \( \psi \) that is equivalent to \( \varphi \) preserves equivalence. We should hope that the same proposition holds in ASP-FO. The following theorem states this property for T-modules and G-modules.

**Theorem 9 (Substitution property for G- and T-modules)**

Let \( \psi, \varphi \) be two equivalent FO formulas. Let \( T' \) be obtained from an ASP-FO theory \( T \) by substituting any number of occurrences of \( \psi \) for \( \varphi \) in T-modules and in the bodies of rules in G-modules. Then \( T \) and \( T' \) have the same models.

Proof. This is a consequence of the substitution property in FO and the fact that T-modules and G-modules are equivalent to FO formulas through the transformation of Corollary 6.

QED

The situation is less straightforward for D-modules, where the formal semantics depends on the concept of the Kleene three-valued truth assignment (Kleene, 1952). As explained in the discussion following Definition 12, Kleene’s three-valued truth assignment can be derived from the notion of the satisfaction relation for pairs of interpretations. Nevertheless, here it is useful to give the explicit definition.

Let \( \mathfrak{A} \) be a three-valued structure, i.e., one which interprets each predicate symbol \( P/n \) as a function from \( dom(\mathfrak{A})^n \) to the set of truth values \( \{t, f, u\} \). We order these truth values under the truth order as \( f \leq t \leq u \) and define the complement operator \( f^{-1} = t, t^{-1} = f \) and \( u^{-1} = u \).

Proceeding by induction in a similar way as in Definition 12, we define the truth value \( \varphi^\mathfrak{A},\theta \) of a formula \( \varphi \) with respect to \( \mathfrak{A} \). This definition follows Kleene’s truth tables.

\[
\begin{align*}
- P(t_1, \ldots, t_n)^\mathfrak{A} & \colon= P^\mathfrak{A}(t_1^\mathfrak{A}, \ldots, t_n^\mathfrak{A}), \\
- (\neg \psi)^\mathfrak{A} & \colon= (\psi^\mathfrak{A})^{-1}, \\
- (\psi \land \varphi)^\mathfrak{A} & \colon= Min(\psi^\mathfrak{A}, \varphi^\mathfrak{A}), \\
- (\psi \lor \varphi)^\mathfrak{A} & \colon= Max(\psi^\mathfrak{A}, \varphi^\mathfrak{A}), \\
- (\exists x \psi)^\mathfrak{A} & \colon= Max(\{\psi^\mathfrak{A}[x:d]|d \in D\}), \\
- (\forall x \psi)^\mathfrak{A} & \colon= Min(\{\psi^\mathfrak{A}[x:d]|d \in D\}).
\end{align*}
\]

To link this definition with Definition 12, each three-valued structure \( \mathfrak{A} \) corresponds to a pair \( (\mathfrak{A}_l, \mathfrak{A}_u) \) of two-valued structures. To obtain \( \mathfrak{A}_l \) and \( \mathfrak{A}_u \) from \( \mathfrak{A} \), \( u \) is mapped to \( f \) and to \( t \), respectively. Consequently, \( \mathfrak{A}_l \) represents a lower approximation of the (two-valued) structures represented by \( \mathfrak{A} \), and \( \mathfrak{A}_u \) represents an upper approximation. The relationship between \( \varphi^\mathfrak{A} \) and \( (\varphi(\mathfrak{A}_l, \mathfrak{A}_u), \varphi(\mathfrak{A}_u, \mathfrak{A}_l)) \) is given by the following correspondences: \( t \leftrightarrow (t, t) \), \( f \leftrightarrow (f, f) \), and \( u \leftrightarrow (f, t) \). The tuple \( (t, f) \) does not arise since \( \mathfrak{A}_l \leq_t \mathfrak{A}_u \).

**Definition 19**

We call FO formulas \( \psi, \varphi \) 3-equivalent, denoted \( \psi \equiv_3 \varphi \), if for every three-valued structure \( \mathfrak{A} \) interpreting all symbols of \( \psi, \varphi \), \( \psi^\mathfrak{A} = \varphi^\mathfrak{A} \).
Two 3-equivalent FO formulas are also equivalent since (two-valued) interpretations are a special case of three-valued interpretations. The inverse is not true and some properties of FO, such as the law of excluded middle, do not hold in three-valued logic. For instance, $\top$ (true) and $\phi \lor \neg \phi$, or $\phi$ and $(\phi \land \psi) \lor (\phi \land \neg \psi)$ are not 3-equivalent. However, most standard equivalences are also 3-equivalences:

- $\neg \neg \phi \equiv_3 \phi$ (double negation);
- $\neg (\phi \land \psi) \equiv_3 \neg \phi \lor \neg \psi$ (De Morgan);
- $\neg (\phi \lor \psi) \equiv_3 \neg \phi \land \neg \psi$ (De Morgan);
- $(\phi \land \psi) \lor (\neg \phi \land \neg \psi) \equiv_3 (\neg \phi \lor \psi) \land (\phi \lor \neg \psi)$ (these are two rewritings of $\phi \leftrightarrow \psi$);
- $\neg \exists \vec{x} \phi \equiv_3 \exists \vec{x} \neg \phi$;
- $\neg \forall \vec{x} \phi \equiv_3 \forall \vec{x} \neg \phi$;
- distributivity laws, commutativity and associativity laws, idempotence, etc.

**Theorem 10 (Substitution property for D-modules)**

Let formulas $\psi$ and $\phi$ be 3-equivalent. If an ASP-FO theory $T'$ is obtained from an ASP-FO theory $T$ by substituting any number of occurrences of $\psi$ for $\phi$ in bodies of rules in D-modules, then $T$ and $T'$ have the same models.

Proof. In [Pelov et al., 2007], it was shown that the parametrized stable models of a D-module $D$ can be characterized as a specific kind of fixpoints, called stable fixpoints, of the three-valued immediate consequence operator associated to $D$. Since any substitution of a formula by a 3-equivalent formula preserves the operator, it also preserves its stable models. Consequently, models are preserved, too.

Thus, most standard FO transformations are equivalence preserving D-modules as well. In other words, virtually all standard “laws of thought” apply in ASP-FO: the De Morgan laws, double negation, distributivity, associativity, commutativity, idempotence. This property of ASP-FO has deep practical implications. It means that the programmer has (almost) the same freedom as in FO to express an informal proposition in ASP-FO. It also implies that the correctness of the programmer’s formalization does not depend on subtleties of the formalization that go beyond common understanding.

Here is an example of a standard FO transformation that is not equivalence preserving in the context of a D-module. Since the law of excluded middle does not hold in 3-valued logic, the formulas $\top$ and $P \lor \neg P$ are equivalent but not 3-equivalent. Substituting the second for the first in the body of the rule of the D-module:

$$\{ P \leftarrow \top \}$$

results in the non-equivalent D-module:

$$\{ P \leftarrow P \lor \neg P \}.$$

Indeed, the first module has a unique model $\{P\}$, while the second has no models. Thus, reasoning by cases does not in general preserve equivalence in D-modules. The explanation of this phenomenon lies in the nature of inductive definitions (and not in, e.g., the nature of negation in D-modules). Indeed, (inductive) definitions are sensitive to negated propositions in rule bodies, because such propositions constrain the order in which rules may be applied [Denecker and Vennekens, 2014]. Therefore, rewriting rule bodies while adding such propositions may disturb the rule application process in an irrecoverable way and turn a sensible
definition in a non-sensible one. Denecker and Vennekens (2014) demonstrate that similar phenomenon can be observed in mathematical texts.

While this example shows that reasoning by cases cannot be applied in general, the following theorem illustrates that any equivalence preserving transformation of classical logic, including reasoning by cases, can be applied to D-module rule bodies, provided it is used with care. In particular, the transformation should not destroy the totality of the definition.

Theorem 11 (Second substitution property for D-modules)

Let $\psi \equiv \varphi$ and let $T'$ be obtained from the ASP-FO theory $T$ by substituting occurrences of $\psi$ for $\varphi$ in the bodies of rules in D-modules. If $T$ and $T'$ are both total, then $T$ and $T'$ have the same models.

Proof. Also this theorem was proven by Pelov et al. (2007). In fact, it is a consequence of a more general property (Pelov et al., 2007) that if two D-modules have the same 2-valued immediate consequence operator, then the well-founded models of both may be different but they are not contradicting each other. That is: there are no atoms that are true in one and false in the other. Any application of an equivalence preserving rule on a body of a D-module obviously preserves the 2-valued immediate consequence operator. If both D-modules are total, their well-founded models are 2-valued and hence, identical. These models are also the unique stable models of the two D-modules.

These results essentially show that we are free to apply any equivalence preserving transformation of FO to the rule bodies of a D-module, as long as we are careful not to turn the D-module into a nonsensical inductive definition.

8 Related Work and Discussion

This section discusses the scope of the results in this article and situates them within the ASP literature.

Extending the core ASP language A limitation of the core ASP language studied in this article is that it lacks aggregates or weight constraints. Indeed, such constructs are used in many ASP applications. Pelov et al. (2007) extend FO(ID) with aggregates. That work can be adopted “verbatim” to the case of the logic ASP-FO. Importantly, extending $\mathcal{O}\mathcal{B}_T(\varphi)$ to theories with aggregates is also not problematic. A clear-cut example of a D-module involving induction over aggregates is the following definition specifying that a company $x$ controls a company $y$ if the sum of the shares of $y$ that $x$ owns directly and of the shares of $y$ owned by companies $c$ controlled by $x$ is more than 50%.

This is an example of a monotone inductive definition with recursion over aggregate expressions. Under the intended informal semantics for the $\text{Sum}$ aggregate, $\mathcal{O}\mathcal{B}_T(\cdot)$ produces the following informal reading of this D-module:

The relation “$x$ controls $y$” is defined in terms of the relation “$x$ holds $s$ percent of shares in $y$” by the following induction:
• Consider the sum of the percentages $s$ for companies $c$ such that either $c$ and $x$ are the same and $x$ holds $s$ percent of shares in $y$ or $x$ controls $c$ and $c$ holds $s$ percent of shares in $y$. If this sum is greater then 50, then $x$ controls $y$.

This informal inductive definition provides a precise and meaningful interpretation of the recursion over an aggregate in the D-module above.

**Links to other developments in ASP** [Pearce (1997)] proposed to use the logic of Here and There (HT) as a meta-logic to study ASP semantics. Pearce’s work maps an ASP program to a theory in HT and characterizes its answer sets as a specific subclass of the models of this theory, called equilibrium models. [Pearce and Valverde (2004, 2005)] generalized these ideas to arbitrary first order formulas. Also [Ferraris et al. (2011)] conservatively lifted ASP to the full FO syntax using a form of circumscription – an operator $SM$ defined in second order logic. These characterizations proved to be useful for analyzing and extending ASP semantics. For example, the logic HT was shown to underlie the notion of strong equivalence of programs ([Lifschitz et al. 2001, 2007]), while the use of operator $SM$ provided an elegant alternative to the definition of stable models of logic programs containing choice rules. The relation between these different approaches was investigated in [Lin and Zhou (2011)].

While these characterizations are powerful formal tools (e.g., our Theorem 3 follows from a result proved by Ferraris et al. (2011) about the $SM$ operator), they do not in themselves directly contribute to the understanding of the informal semantics of ASP. For example, neither the informal semantics of HT nor of equilibrium logic has been developed so far. Similar arguments apply to the semantics of logic programs under operator $SM$, where the effect of this operator on the informal semantics of the formulas has not yet been studied.

Several features of ASP-FO also appear in other variants of ASP. As we observed earlier, the intuitive structure of a GDT-program is hidden in an ASP program. Techniques developed in ASP to cope with this include splitting to detect natural components of the program ([Lifschitz and Turner 1994; Janhunen et al. 2009; Ferraris et al. 2009]), and module systems, e.g., in ([Gelfond 2002; Oikarinen and Janhunen 2008; Lierler and Truszczynski 2013]).

Due to its non-Herbrand semantics, ASP-FO imposes neither Domain Closure Axiom nor the Unique Names Axiom. Thus, function symbols that are not constrained by a Herbrand module act as function symbols in classical logic. Recent extensions of ASP with similar features are open domain ASP logics such as those of [Ferraris et al. 2011, Lifschitz et al. 2012] and ASP with functions ([Lin and Wang 2008; Balduccini 2012; Cabal, 2011]). The progression semantics of [Zhang and Zhou (2010)] and [Zhou and Zhang (2011)] also allows non-Herbrand models and makes a distinction between the intensional and extensional predicates of a theory. The ordered completion semantics ([Asuncion et al. 2012]) provides another way to define ASP for non-Herbrand models, which has also been extended to aggregates ([Asuncion et al. 2012]). A detailed comparison with these languages would be interesting but is beyond the scope of this paper.

**Well-founded versus stable semantics** Comparisons between the well-founded and stable semantics have long been the subject of discussion. Our position is that once these semantics are generalized to their parametrized versions and the internal structure of a program (in particular its define components) is identified, then the differences between both semantics disappear for practically relevant programs. They are just different mathematical formalizations for the same informal semantics: sets of clauses that define certain predicates/relations in terms of parameter
symbols. However, while the two semantics are equivalent in the case of sensible (i.e., total) definitions, they are not equivalent in the case of other rule sets. The well-founded semantics identifies non-total definitions by producing a 3-valued model. In the stable semantics, nonsensical definitions are revealed by the absence of stable models or by multiple stable models. However, some programs have a unique stable model, but are not sensible definitions. For instance, the following logic program with three defined symbols $a$, $b$ and $f$ and no parameter symbols

$$\begin{align*}
  f &\leftarrow \neg f \land b \\
  a &\leftarrow \neg b \\
  b &\leftarrow \neg a
\end{align*}$$

has a unique stable model $\{a\}$ but does not represent a sensible definition of $a$, $b$, and $f$.

In practice, if ASP programmers avoid rule sets with cycles over negation, definitions are total and the two semantics coincide. The evolving GDT-programming methodology in ASP discourages cycles over negation as an encoding technique in favour of more direct representations based on the use of choice rules. This means that the debate between both semantics is losing its practical relevance.

**Tools for ASP-FO and FO(ID)** ASP-FO is more than a theoretical mechanism for a semantic study of GDT programs. It is also a viable logic for which efficient tools already exist.

Similarly to FO and FO(ID), ASP-FO is an open domain logic and its models can be infinite. In general, its satisfiability problem is undecidable (and not just co-semidecidable) — this can be proved by adapting the corresponding result concerning the logic FO(ID) \cite{denecker2008}. In many search problems, however, a finite domain is given. That opens a way to practical problem solving. One can apply, e.g., finite model checking, finite satisfiability checking, finite Herbrand model generation or, in case the domain and data are available as an input structure, model expansion \cite{mitchell2005}.

Answer set programming solvers such as Smodels \cite{niemela2000}, DLV \cite{leone2006}, CMODELS \cite{giunchiglia2006}, clasp \cite{gebser2007a}, and WASP \cite{alviano2013} can be viewed as systems supporting a subset of ASP-FO (modulo the rewriting that we proposed). Also, ASP-FO/FO(ID) is formally an extension of Abductive Logic Programming, and hence, abductive reasoners such as the solver $A$-system \cite{kakas2001} can be seen to implement abductive reasoning for a fragment of ASP-FO/FO(ID).

Several systems have been designed with the intention to support extensions or variants of FO with rules. An early finite Herbrand model generator that supported a fragment of ASP-FO/FO(ID) was developed by East and Truszczynski \cite{east2006}. This system supports clausal logic and a set of definite Horn rules under the minimal model semantics; in our terminology this is a negation-free D-module representing a monotone inductive definition. Another solver is the Enfragmo system \cite{aavani2012} that supports model expansion for FO and non-recursive definitions. Both systems support aggregates. Also, Microsoft’s system FORMULA \cite{jackson2013} supports a form of satisfiability checking for a similar logic.

At present, the IDP system \cite{wittocx2008,bruynooghe2015} offers the most complete implementation of ASP-FO as well as FO(ID). IDP is a knowledge base system that provides various forms of inference, including model expansion and Herbrand model generation. From a KR point of view, the system was developed to support essentially the GDT methodology: the representation of assertional knowledge (corresponding to G- and T-modules) and definitional knowledge (D-modules). The language supported by IDP includes FO, D-modules, aggregates,
quantified existential quantifiers, multiple definitions, bounded forms of arithmetic, uninterpreted functions and constructor functions, etc. G-modules are to be “emulated” by FO formulas. A flag can be used to select the parametrized stable or well-founded semantics for rule sets; this switches the system effectively between (extensions of) ASP-FO and FO(ID).

**ASP-FO and FO(ID) in a historical perspective** Originally, logic programming was seen as the Horn fragment of classical logic. This view soon became untenable due to negation as failure. One of the earliest proposed explanations was the view of logic programs as definitions. It underlied the work of Clark (1978) and Chandra and Harel (1982). It was also present in Kowalski’s book (1979). In 1988, Gelfond and Lifschitz proposed the autoepistemic view of logic programs as a synthesis of logic programming and nonmonotonic reasoning. This proposal led to the development of ELP in 1991. Note that nonmonotonic reasoning, one of the roots of ELP, had been developed fifteen years earlier as a reaction against the shortcomings of classical logic for common-sense knowledge representation.

The research direction set out by Chandra and Harel was followed up by Apt et al. (1988) and Van Gelder et al. (1991). Although the link between logic programs and inductive definitions was at the heart of these developments, it was not made explicit. The link was strengthened again by Schlipf (1995) and later fully explicated by Denecker et al. (2001); it led to the logic FO(ID) (Denecker 2000; Denecker and Ternovska 2008). Interestingly, the latter logic grew out of a semantic study of another knowledge representation extension of logic programming: Abductive Logic Programming (Kakas et al., 1992). In ASP-FO and FO(ID), the main relict of logic programs is the D-module which is viewed as a (possibly inductive) definition. Our Tarskian perspective is a proposal to “backtrack” to the early view of logic programs as definitions. Thus, the present paper is a confluence of many research directions. ASP arose at the meeting point of two logic paradigms—nonmonotonic reasoning and logic programming—that in origin were antitheses to classical logic. One of the contributions here is to reconcile ASP with the objective informal semantics of classical logic. This is the crucial step towards a synthesis of these languages, as is achieved in ASP-FO and FO(ID).

**9 Conclusion**

The goal of this paper was to develop a theory of the informal semantics of ASP that:

- explains the ASP practice of GDT programming, and
- matches the informal reading of an ASP expression with the informal proposition that the human programmer has in mind when he writes it.

To conduct our analysis, we presented the formalism ASP-FO, whose modular structure is geared specifically towards the GDT paradigm and in which the internal structure of GDT programs is made explicit. By reinterpreting answer sets as objective possible worlds rather than as belief sets, we obtained an informal semantics for the GDT fragment of ASP that combines modules by means of the standard conjunction, and captures the roles of different modules in GDT-based programs. This allowed us to clarify the nature of the three sorts of conditionals found in ASP, and of the negation symbol in G, D, and T-modules. In addition, the close connection between ASP-FO and FO(ID) assisted us in providing, to the best of our knowledge, the first argument for the correctness of the stable model semantics as a formalization of the concept of an (inductive) definition.
A study of a logic’s informal semantics is an investigation into the foundations of the logic. We showed that explaining ASP from a Tarskian point of view has a deep impact on our view of the ASP language. All together, our study forces us to reconsider the intuitive meaning of ASP’s basic connectives, it redefines ASP’s position in the spectrum of logics and it shows much tighter connections with existing logics including FO and FO(ID).

Acknowledgments

The first and fourth author are supported by Research Foundation-Flanders (FWO-Vlaanderen) and by GOA 2003/08 ”Inductive Knowledge Bases”. The second author was supported by a CRA/NSF 2010 Computing Innovation Fellowship and FRI-2013: Faculty Research International Grant. The third author was supported by the NSF grant IIS-0913459. For sharing their views on topics related to this paper, we would like to thank the following people: Maurice Bruynooghe, Vladimir Lifschitz, Michael Gelfond and Eugenia Ternovska.

References

Aavani, A., Wu, X., Tasharrofi, S., Ternovska, E., Mitchell, D. G., 2012. Enfragmo: A system for modelling and solving search problems with logic. In: Bjørner, N., Voronkov, A. (Eds.), Proceedings of the 18th International Conference on Logic for Programming, Artificial Intelligence, and Reasoning, LPAR 2012. Vol. 7180 of Lecture Notes in Computer Science. Springer, Berlin, pp. 15–22.

Alviano, M., Calimeri, F., Charwat, G., Dao-Tran, M., Dodaro, C., Ianni, G., Krennwallner, T., Kronegger, M., Oetsch, J., Pfandler, A., Pührer, J., Redl, C., Ricca, F., Schneider, P., Schweneger, M., Spendier, L. K., Wallner, J. P., Xiao, G., 2013a. The fourth answer set programming competition: Preliminary report. In: LPNMR. pp. 42–53.

Alviano, M., Dodaro, C., Faber, W., Leone, N., Ricca, F., 2013b. WASP: A native ASP solver based on constraint learning. In: Logic Programming and Nonmonotonic Reasoning, 12th International Conference, LPNMR 2013. pp. 54–66.

Apt, K. R., Blair, H. A., Walker, A., 1988. Towards a theory of declarative knowledge. In: Foundations of Deductive Databases and Logic Programming. Morgan Kaufmann, pp. 89–148.

Asuncion, V., Lin, F., Zhang, Y., Zhou, Y., 2012. Ordered completion for first-order logic programs on finite structures. Artificial Intelligence 177-179.

Balduccini, M., 2012. An answer set solver for non-Herbrand programs: Progress report. In: Dovier, A., Costa, V. S. (Eds.), ICLP (Technical Communications). Vol. 17 of LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, pp. 49–60.

Barwise, J., Cooper, R., 1981. Generalized quantifiers and natural language. Linguistics and Philosophy 4, 159219.

Brewka, G., Eiter, T., Truszczynski, M., 2011. Answer set programming at a glance. Commun. ACM 54 (12), 92–103.

Bruynooghe, M., Blockeel, H., Bogaerts, B., De Cat, B., De Pooter, S., Jansen, J., Labarre, A., Ramon, J., Denecker, M., Verwer, S., Nov. 2015. Predicate logic as a modeling language: Modeling and solving some machine learning and data mining problems with IDP3. Theory and Practice of Logic Programming 15 (6), 783 –817.

Cabalar, P., 2011. Functional answer set programming. TPLP 11 (2-3), 203–233.
Calimeri, F., Ianni, G., Ricca, F., Alviano, M., Bria, A., Catalano, G., Cozza, S., Faber, W., Febraro, O., Leone, N., Manna, M., Martello, A., Panetta, C., Perri, S., Reale, K., Santoro, M. C., Sirianni, M., Terracina, G., Veltri, P., 2011. The third answer set programming system competition: Preliminary report of the system competition track. In: J.P. Delgrande, W. F. (Ed.), Proceedings of the International Conference on Logic Programming and Nonmonotonic Reasoning. LPNMR 2011. Vol. 6645 of Lecture Notes in Computer Science. Springer, Berlin, pp. 388–403.

Chandra, A. K., Harel, D., 1982. Structure and complexity of relational queries. J. Comput. Syst. Sci. 25 (1), 99–128.

Clark, K. L., 1978. Negation as failure. In: Gallaire, H., Minker, J. (Eds.), Symposium on Logic and Data Bases. Plenum Press, pp. 293–322.

Dancygier, B., Sweetser, E., 2005. Mental Spaces in Grammar: Conditional Constructions. Vol. 108 of Cambridge Studies in Linguistics. Cambridge University Press.

Denecker, M., 2000. Extending classical logic with inductive definitions. In: Lloyd, J., Dahl, V., Furbach, U., Kerber, M., Lau, K.-K., Palamidessi, C., Pereira, L., Sagiv, Y., Stuckey, P. (Eds.), Proceedings of First International Conference on Computational Logic, CL 2000. Vol. 1861 of Lecture Notes in Computer Science. Springer, Berlin, pp. 703–717.

Denecker, M., Bruynooghe, M., Marek, V. W., 2001. Logic programming revisited: Logic programs as inductive definitions. ACM Transactions on Computational Logic (TOCL) 2 (4), 623–654.

Denecker, M., Bruynooghe, M., Vennekens, J., 2012. Approximation fixpoint theory and the semantics of logic and answers set programs. In: Erdem, E., Lee, J., Lierler, Y., Pearce, D. (Eds.), Correct Reasoning. Vol. 7265 of Lecture Notes in Computer Science. Springer.

Denecker, M., Ternovska, E., 2008. A logic of nonmonotone inductive definitions. ACM Transactions on Computational Logic 9 (2).

Denecker, M., Vennekens, J., Jul. 2014. The well-founded semantics is the principle of inductive definition, revisited. In: International Conference on Principles of Knowledge Representation and Reasoning, Vienna, 20-24 July 2014.

Denecker, M., Vennekens, J., Bond, S., Gebser, M., Truszczyński, M., 2009. The second answer set programming system competition. In: Erdem, E., Lin, F., Schaub, T. (Eds.), Proceedings of the International Conference on Logic Programming and Nonmonotonic Reasoning, LPNMR 2009. Vol. 5753 of Lecture Notes in Computer Science. Springer, Berlin, pp. 637–654.

East, D., Truszczynski, M., 2006. Predicate-calculus-based logics for modeling and solving search problems. ACM Transactions on Computational Logic (TOCL) 7 (1), 38–83.

Erdoğan, S., Lifschitz, V., 2004. Definitions in answer set programming. In: Proceedings of International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR), pp. 114–126.

Feferman, S., 1984. Toward useful type-free theories. Journal of Symbolic Logic 49 (1), 75–111.

Ferraris, P., Lee, J., Lifschitz, V., 2011. Stable models and circumscription. Artificial Intelligence 175, 236–263.

Ferraris, P., Lee, J., Lifschitz, V., Palla, R., 2009. Symmetric splitting in the general theory of
stable models. In: Proceedings of International Joint Conference on Artificial Intelligence, IJCAI 2009, pp. 797–803.

Gebser, M., Kaufmann, B., Neumann, A., Schaub, T., 2007a. Conflict-driven answer set solving. In: Proceedings of 20th International Joint Conference on Artificial Intelligence (IJCAI’07). MIT Press, pp. 386–392.

Gebser, M., Kaufmann, B., Schaub, T., 2012. Conflict-driven answer set solving: From theory to practice. Artif. Intell. 187, 52–89.

Gebser, M., Liu, L., Namasivayam, G., Neumann, A., Schaub, T., Truszczynski, M., 2007b. The first answer set programming system competition. In: Baral, C., Brewka, G., Schlipf, J. (Eds.), Proceedings of the International Conference on Logic Programming and Nonmonotonic Reasoning, LPNMR 2007. Vol. 4483 of Lecture Notes in Computer Science. Springer, Berlin, pp. 3–17.

Gelfond, M., 2002. Representing knowledge in A-Prolog. Lecture Notes in Computer Science 2408, 413–451.

Gelfond, M., Kahl, Y., 2014. Knowledge representation, reasoning, and the design of intelligent agents. Cambridge University Press.

Gelfond, M., Lifschitz, V., 1988. The stable model semantics for logic programming. In: Kowalski, R., Bowen, K. (Eds.), Proceedings of International Logic Programming Conference and Symposium. MIT Press, Cambridge, MA, pp. 1070–1080.

Gelfond, M., Lifschitz, V., 1991. Classical negation in logic programs and disjunctive databases. New Generation Computing 9, 365–385.

Giunchiglia, E., Lierler, Y., Maratea, M., 2006. Answer set programming based on propositional satisfiability. Journal of Automated Reasoning 36, 345–377.

Jackson, E. K., Schulte, W., 2013. FORMULA 2.0: A language for formal specifications. In: Liu, Z., Woodcock, J., Zhu, H. (Eds.), ICTAC Training School on Software Engineering. Vol. 8050 of Lecture Notes in Computer Science. Springer, pp. 156–206.

Janhunen, T., Oikarinen, E., Tompits, H., Woltran, S., 2009. Modularity aspects of disjunctive stable models. Journal of Artificial Intelligence Research 35, 813–857.

Kakas, A., Kowalski, R., Toni, F., 1992. Abductive logic programming. Journal on Logic and Computation 2 (6), 719–770.

Kakas, A. C., Van Nuffelen, B., Denecker, M., 2001. A-system: Problem solving through abduction. In: IJCAI. pp. 591–596.

Kleene, S. C., 1952. Introduction to Metamathematics. Van Nostrand.

Kowalski, R., 1979. Logic for problem solving. Elsevier Science Publishing Co.

Lee, J., Meng, Y., 2008. On loop formulas with variables. In: Proceedings of Eleventh International Conference on Principles of Knowledge Representation and Reasoning (KR), pp. 444–453.

Leone, N., Pfeifer, G., Faber, W., Eiter, T., Gottlob, G., Perri, S., Scarcello, F., 2006. The DLV system for knowledge representation and reasoning. ACM Transactions on Computational Logic 7(3), 499–562.

Lierler, Y., Truszczynski, M., 2013. Modular answer set solving. In: Conference on Artificial Intelligence (AAAI).

URL [http://www.aaai.org/ocs/index.php/WS/AAAIW13/paper/view/7077](http://www.aaai.org/ocs/index.php/WS/AAAIW13/paper/view/7077)

Lifschitz, V., 1999. Answer set planning. In: De Schreye, D. (Ed.), Logic programming, Proceedings of the 1999 International Conference on Logic Programming. MIT Press, pp. 23–37.

Lifschitz, V., 2002. Answer set programming and plan generation. Artificial Intelligence 138, 39–54.
Lifschitz, V., Pearce, D., Valverde, A., 2001. Strongly equivalent logic programs. ACM Transactions on Computational Logic 2, 526–541.

Lifschitz, V., Pearce, D., Valverde, A., 2007. A characterization of strong equivalence for logic programs with variables. In: Proceedings of International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR). pp. 188–200.

Lifschitz, V., Pichotta, K., Yang, F., 2012. Relational theories with null values and non-herbrand stable models. TPLP 12 (4-5), 565–582.

Lifschitz, V., Turner, H., 1994. Splitting a logic program. In: Hentenryck, P. V. (Ed.), Proceedings of the 11th International Conference on Logic Programming, ICLP 1994. MIT Press, Cambridge, MA, pp. 23–37.

Lin, F., Wang, Y., 2008. Answer set programming with functions. In: Brewka, G., Lang, J. (Eds.), KR. AAAI Press, pp. 454–465.

Lin, F., Zhou, Y., 2011. From answer set logic programming to circumscription via logic of GK. Artificial Intelligence 175 (1), 264–277.

Marek, V., Truszczyński, M., 1999. Stable models and an alternative logic programming paradigm. In: Apt, K., Marek, V., Truszczyński, M., Warren, D. (Eds.), The Logic Programming Paradigm: a 25-Year Perspective. Springer, Berlin, pp. 375–398.

Martin-Löf, P., 1971. Hauptsatz for the intuitionistic theory of iterated inductive definitions. In: Fenstad, J. (Ed.), Proceedings of the Second Scandinavian Logic Symposium. Vol. 63 of Studies in Logic and the Foundations of Mathematics. pp. 179–216.

Mitchell, D., Ternovska, E., 2005. A framework for representing and solving NP search problems. In: Proceedings of the 20th National Conference on Artificial Intelligence, AAAI 2005. AAAI Press, pp. 430–435.

Moore, R. C., 1985. Semantical considerations on nonmonotonic logic. Artificial Intelligence 25 (1), 75–94.

Niemelä, I., 1999. Logic programs with stable model semantics as a constraint programming paradigm. Annals of Mathematics and Artificial Intelligence 25, 241–273.

Niemelä, I., Simons, P. 2000. Extending the Smodels system with cardinality and weight constraints. In: Minker, J. (Ed.), Logic-Based Artificial Intelligence. Kluwer, pp. 491–521.

Oikarinen, E., Janhunen, T., 2008. Achieving compositionality of the stable model semantics for Smodels programs. Theory and Practice of Logic Programming 5–6, 717–761.

Pearce, D., 1997. A new logical characterization of stable models and answer sets. In: Dix, J., Pereira, L., Przymusinski, T. (Eds.), Non-Monotonic Extensions of Logic Programming (Lecture Notes in Artificial Intelligence 1216). Springer, pp. 57–70.

Pearce, D., Valverde, A., 2004. Towards a first order equilibrium logic for nonmonotonic reasoning. In: Proceedings of European Conference on Logics in Artificial Intelligence (JELIA). pp. 147–160.

Pearce, D., Valverde, A., 2005. A first order nonmonotonic extension of constructive logic. Studia Logica 80, 323–348.

Pelov, N., Denecker, M., Bruynooghe, M., 2007. Well-founded and stable semantics of logic programs with aggregates. Theory and Practice of Logic Programming 7 (3), 301–353.

Reiter, R., 1980. A logic for default reasoning. Artificial Intelligence 13, 81–132.

Schlipf, J. S., 1995. The expressive powers of the logic programming semantics. J. Comput. Syst. Sci. 51 (1), 64–86.

Truszczynski, M., 2012. Connecting first-order ASP and the logic FO(ID) through reducts. In: Erdem, E., Lee, J., Lierler, Y., Pearce, D. (Eds.), Correct Reasoning - Essays on Logic-Based AI
Appendix

To prove Theorem 3 we rely on results for the extension of ASP to the syntax of classical logic and to arbitrary interpretations presented in [Ferraris et al., 2011]. For an FO sentence \( \Pi \) over a finite vocabulary \( \Sigma \) and a finite set \( p \subseteq \Sigma \) of predicate symbols, [Ferraris et al.] introduced a second-order formula \( SM_p(\Pi) \) and defined a (possibly non-Herbrand) structure \( M \) to be a general answer set of \( \Pi \) relative to \( p \) if \( M \) is a model of \( SM_p(\Pi) \). They defined a general answer set of \( \Pi \) as a general answer set of \( \Pi \) relative to the set of all predicates in \( \Sigma \). We refer to the paper by [Ferraris et al.] for details; the actual definition of the operator \( SM_p \) is not important for our argument and so we omit it.

The core ASP language is embedded in this generalized formalism by a modular transformation. The transformation maps constraints \( \leftarrow L_1, \ldots, L_n \) to the same FO sentences as in ASP-FO: \( \neg \exists \overline{x} (L_1 \land \cdots \land L_n) \). It maps rules \( p(\overline{t}) \leftarrow L_1, \ldots, L_n \) to formulas \( \forall \overline{x} (L_1 \land \cdots \land L_n \Rightarrow p(\overline{t})) \) and it maps choice rules \( \{ p(\overline{t}) \} \leftarrow L_1, \ldots, L_n \) to formulas \( \forall \overline{x} (\neg \neg p(\overline{t}) \land L_1 \land \cdots \land L_n \Rightarrow p(\overline{t})) \).

The mapping of a rule \( r \) is denoted \( \tilde{r} \). The embedding of any set \( \Pi \) of rules is the conjunction of the mapping of its rules and is denoted \( \tilde{\Pi} \).

[Ferraris et al., 2011] defined a structure \( M \) to be a general answer set of a core ASP program \( \Pi \) over a finite vocabulary \( \Sigma \) and \( \{ \Pi_1, \ldots, \Pi_n \} \) be a proper splitting of \( \Pi \). Then an interpretation \( M \) is a general answer set of \( \Pi \) if and only if \( M \) is a model of the ASP-FO theory \( \{ \tilde{\Pi}_1, \ldots, \tilde{\Pi}_n, (Def, \{\}) \} \).

**Theorem 12**

Let \( \Pi \) be a core ASP program over a finite vocabulary \( \Sigma \) and \( \{ \Pi_1, \ldots, \Pi_n \} \) be a proper splitting of \( \Pi \). Then an interpretation \( M \) is a general answer set of \( \Pi \) if and only if \( M \) is a model of the ASP-FO theory \( \{ \tilde{\Pi}_1, \ldots, \tilde{\Pi}_n, (Def, \{\}) \} \).

**Theorem 12** is a generalization of Theorem 3 since it holds for (non-Herbrand) general answer sets.

**Proof.** We will apply the Symmetric Splitting Theorem [Ferraris et al., 2009]. That theorem is stated in the language of arbitrary FO sentences. It applies to finite programs under the rewriting of rules as sentences we discussed above.
By definition, a structure $\mathcal{M}$ is a general answer set of $\Pi$ if and only if it satisfies $SM_{\Sigma_{\Pi}}(\tilde{\Pi})$ or, equivalently, of $SM_{\Sigma_{\Pi}}(\Pi_1 \land \cdots \land \Pi_n)$.

Without loss of generality, we assume that in the splitting $\{\Pi_1, \ldots, \Pi_n\}$, programs $\Pi_1, \ldots, \Pi_i$ consist of choice rules, $\Pi_{i+1}, \ldots, \Pi_j$ of normal program rules, and $\Pi_{j+1}, \ldots, \Pi_n$ of constraints. Since the sentences corresponding to constraints are of the form $\neg \psi$, results of Ferraris et al. (2011) imply that $SM_{\Sigma_{\Pi}}(\tilde{\Pi})$ is equivalent to

$$SM_{\Sigma_{\Pi}}(\Pi_1 \land \cdots \land \Pi_j) \land \Pi_{j+1} \land \cdots \land \Pi_n.$$ 

Next, we observe that $\Pi_1 \land \cdots \land \Pi_j$ can be written as $\Pi_1 \land \cdots \land \Pi_j \land \top$. Moreover, since $\{\Pi_1, \ldots, \Pi_n\}$ is a proper splitting of $\Pi$, the sentences $\Pi_1, \ldots, \Pi_j, \top$ together with the sets $hd(\Pi_1), \ldots, hd(\Pi_j), \Sigma_{\Pi}, \top$ of predicates satisfy the assumptions of the Symmetric Splitting Theorem of Ferraris et al. (2009). Consequently, $SM_{\Sigma_{\Pi}}(\Pi_1 \land \cdots \land \Pi_j)$ is equivalent to

$$SM_{hd}(\Pi_1) \land \cdots \land SM_{hd}(\Pi_j) \land SM_{\Sigma_{\Pi}} \setminus \{hd(\Pi)\}(\top).$$

To deal with choice rules in a program $\Pi_k$, $1 \leq k \leq i$, we use the generalized Clark’s completion Ferraris et al. (2011) defined for the case of FO sentences. To describe it, recall that $\tilde{\Pi}_k$ consists of sentences of the form $\forall \bar{x} \left(\neg \neg p(\bar{t}) \land \varphi \Rightarrow p(\bar{t})\right)$. Each such sentence is first rewritten as $\forall \bar{y} \left(\neg \neg p(\bar{y}) \land \exists \bar{x} \left(\bar{t} = \bar{y} \land \varphi \Rightarrow p(\bar{y})\right)\right)$, where $\bar{y}$ are fresh variables. Next, we combine all formulas obtained in this way into a single one, which has the form

$$\forall \bar{y} \left(\neg \neg p(\bar{y}) \land (\psi_1 \lor \cdots \lor \psi_s) \Rightarrow p(\bar{y}),\right)$$

where $s$ is the number of sentences in $\Pi_k$. The generalized Clark’s completion of the original set $\Pi_k$ is obtained from this sentence by substituting equivalence for implication:

$$\forall \bar{y} \left(\neg \neg p(\bar{y}) \land (\psi_1 \lor \cdots \lor \psi_s) \Leftrightarrow p(\bar{y})\right).$$

One can verify that this sentence is equivalent in FO to the sentence

$$\forall \bar{y} \left(p(\bar{y}) \Rightarrow (\psi_1 \lor \cdots \lor \psi_s)\right).$$

Since the splitting $\{\Pi_1, \ldots, \Pi_n\}$ is proper, all rules of $\Pi_k$ have the same predicate in their heads, and this predicate has no occurrences in the bodies of rules of $\Pi_k$. Consequently, $\Pi_k$ is tight (for a definition of tightness we refer to Ferraris et al. (2011)). By the result on tight programs proved by Ferraris et al. (2011), models of $SM_{hd}(\Pi_k)(\Pi_k)$ and models of the Clark’s completion of $\tilde{\Pi}_k$ coincide.

We now note that the process of completion applied we described in the section on the semantics of the logic ASP-FO, when applied to the $G$-module $\Pi_k$ (that is, formally, the G-module $(hd(\Pi_k), \Pi_k)$), results in an equivalent G-module $\{\forall \bar{y} p(\bar{y}) \leftarrow (\psi_1 \lor \cdots \lor \psi_s)\}$ (cf. Theorem 4 and the discussion that precedes it). By Theorem 5 models of that G-module coincide with models of the sentence $\forall \bar{y} (p(\bar{y}) \Rightarrow (\psi_1 \lor \cdots \psi_s))$. Thus, models of $SM_{hd}(\Pi_k)(\Pi_k)$ and of the G-module $\Pi_k$ are the same.

Next, we observe that programs $\Pi_k$, $i + 1 \leq k \leq j$, are normal. As a consequence of the results by Truszczynski (2012), we obtain that models of $SM_{hd}(\Pi_k)(\Pi_k)$ and of the ASP-FO D-module $\Pi_k$ coincide.

Finally, models of $SM_{\Sigma_{\Pi}}(hd(\Pi))(\top)$ are precisely those interpretations of $\Sigma$ that interpret each predicate symbol in $\Sigma_{\Pi} \setminus \{hd(\Pi)\}$ with the empty relation. Applying the generalized Clark’s com-
pletion to $\top$ (with respect to the vocabulary $\Sigma_P \setminus \text{hd}(\Pi)$) results in the FO sentence

$$\bigwedge_{Q \in \Sigma_P \setminus \text{hd}(\Pi)} \forall \bar{y} \ (Q(\bar{y}) \iff \bot) \quad (14)$$

Note that $\top$ is a tight sentence [Ferraris et al. (2011)] and hence models of $SM_{\Sigma_P \setminus \text{hd}(\Pi)}(\top)$ and (14) coincide. It is easy to see that models of $CWA(\Sigma_P \setminus \text{hd}(\Pi))$ coincide with models of (14).

Gathering all earlier observations together, we obtain that models of $SM_P(\tilde{\Pi})$, that is answer sets of $\Pi$, coincide with models of the ASP-FO theory $\{\Pi_1, \ldots, \Pi_n, CWA(\Sigma_P \setminus \text{hd}(\Pi))\}$. QED

Theorem 3 is a corollary of the above result limited to Herbrand structures.

**Theorem 3** Let $\Pi$ be a core ASP program over a finite vocabulary $\Sigma$ with a proper splitting $\{\Pi_1, \ldots, \Pi_n\}$. Then an interpretation $M$ is an answer set of $\Pi$ if and only if $M$ is a model of the ASP-FO theory $\{H(\Sigma_F), \tilde{\Pi}_1, \ldots, \tilde{\Pi}_n, (Def, \{\})\}$, where $Def = \Sigma_P \setminus \text{hd}(\Pi)$.

Proof: The result follows from Theorem 12 and from the observations that answer sets are Herbrand interpretations and the only interpretations that satisfy the Herbrand module $H(\Sigma_F)$ are Herbrand ones. QED