FUNCTIONAL EQUATION AND ZEROS ON THE CRITICAL LINE OF THE QUADRILATERAL ZETA FUNCTION

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Abstract. For $0 < a \leq 1/2$, we define the quadrilateral zeta function $Q(s, a)$ using the Hurwitz and periodic zeta functions and show that $Q(s, a)$ satisfies Riemann’s functional equation studied by Hamburger, Hecke and Knopp. Moreover, we prove that for any $0 < a \leq 1/2$, there exist positive constants $A(a)$ and $T_0(a)$ such that the number of zeros of the quadrilateral zeta function $Q(s, a)$ on the line segment from $1/2$ to $1/2 + iT$ is greater than $A(a)T$ whenever $T \geq T_0(a)$.

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

1.1. Main results. For $0 < a \leq 1$, define the Hurwitz zeta function $\zeta(s, a)$ by

$$
\zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}, \quad \sigma > 1,
$$

and the periodic zeta function $\text{Li}(s, a)$ by

$$
\text{Li}(s, a) := \sum_{n=1}^{\infty} \frac{e^{2\pi i na}}{n^s}, \quad \sigma > 1.
$$

The Dirichlet series of $\zeta(s, a)$ and $\text{Li}(s, a)$ converge absolutely in the half-plane $\sigma > 1$ and uniformly in each compact subset of this half-plane. Moreover, the Hurwitz zeta function has analytic continuation to $\mathbb{C}$ except $s = 1$, where there is a simple pole with residue

2010 Mathematics Subject Classification. Primary 11M35, Secondary 11M26.

Key words and phrases. Functional equation, converse theorems, Quadrilateral zeta function, Zeros on the critical line.
For the whole complex plane except the equation of the quadrilateral zeta function \( Q(s, a) \). Furthermore, the function \( \text{Li}(s, a) \) with \( 0 < a < 1 \) is analytically continuable to the whole complex plane (e.g., [15, Chapter 2.2]). We clearly have \( \zeta(s, 1) = \text{Li}(s, 1) = \zeta(s) \), where \( \zeta(s) \) is the Riemann zeta function.

For \( 0 < a \leq 1/2 \), we define the quadrilateral zeta function \( Q(s, a) \) as

\[
2Q(s, a) := \zeta(s, a) + \zeta(s, 1 - a) + \text{Li}(s, a) + \text{Li}(s, 1 - a). \tag{1.1}
\]

Based on the facts mentioned above, the function \( Q(s, a) \) can be continued analytically to the whole complex plane except \( s = 1 \). The first main theorem is the following functional equation of the quadrilateral zeta function \( Q(s, a) \).

**Theorem 1.1.** For \( 0 < a \leq 1/2 \), it holds that

\[
Q(1 - s, a) = \frac{2\Gamma(s)}{(2\pi)^s} \cos\left(\frac{\pi s}{2}\right)Q(s, a). \tag{1.2}
\]

Moreover, we show the following, which implies that \( Q(s, a) \) has infinitely many zeros on the critical line \( \sigma = 1/2 \).

**Theorem 1.2.** For any \( 0 < a \leq 1/2 \), there exist positive constants \( A(a) \) and \( T_0(a) \) such that the number of zeros of \( Q(s, a) \) on the line segment from \( 1/2 \) to \( 1/2 + iT \) is greater than \( A(a)T \) whenever \( T \geq T_0(a) \).

We share some remarks on the functional equation and zeros on the critical line of zeta functions in the next three subsections. Note that the quadrilateral zeta function \( Q(s, a) \) also has the following remarkable properties. From [17, (2.4)], it holds that

\[
Q(0, a) = -1/2 = \zeta(0) \quad \text{for all} \quad 0 < a \leq 1/2.
\]

For \( n \in \mathbb{N} \), it is shown in [18, Corollary 3.7] that \( Q(-n, a) \) and \( \pi^{-2n}Q(2n, a) \) are rational functions of \( e^{2\pi ia} \) with rational coefficients. By (1.2), the function \( Q(s, a) \) has simple zeros at the negative even integers. Furthermore, it is proved in [17, Theorem 1.1] that there exists \( a_0 = 0.1183751396 \ldots \) such that

1. \( Q(\sigma, a_0) \) has a unique double real zero at \( \sigma = 1/2 \) when \( \sigma \in (0, 1) \),
2. for any \( a \in (a_0, 1/2] \), the function \( Q(\sigma, a) \) has no real zero in \( \sigma \in (0, 1) \),
3. for any \( a \in (0, a_0) \), \( Q(\sigma, a) \) has at least two real zeros in \( \sigma \in (0, 1) \).

In addition, for \( 0 < a \leq 1/2 \), it is shown in [17, Proposition 1.5] that

\[
N(T, Q(s, a)) = \frac{T}{\pi} \log T - \frac{T}{\pi} \log(2\pi a^2) + O_\alpha(\log T),
\]

where \( N(T, F) \) is the number of non-real zeros of a function \( F(s) \) with \( |3(s)| < T \) when \( T \) is sufficiently large. Moreover, we prove in [17, Proposition 1.4] that \( Q(s, a) \) has infinitely many complex zeros in the region of absolute convergence and the critical strip when \( a \in \mathbb{Q} \cap (0, 1/2) \setminus \{1/6, 1/4, 1/3\} \).

**1.2. Zeros of zeta functions on the critical line.** The famous Riemann hypothesis asserts that the real part of every non-real zero of the Riemann zeta function is \( 1/2 \). The study to establish the lower bound for the number of zeros of \( \zeta(s) \) on the critical line \( \sigma = 1/2 \) has long history. Denote by \( N_{\text{Ri}}(T) \) the number of zeros \( \rho = \beta + i\gamma \) of the Riemann zeta function \( \zeta(s) \) with \( \beta = 1/2 \) and \( 0 < \gamma \leq T \). In 1914, Hardy proved that...
Let \( N_{\text{Ri}}(T) \to \infty \) if \( T \to \infty \). Later, Hardy and Littlewood [9] showed the following (see also [5, Chapter 11.2] and [22, Chapter 10.7]):

**Theorem A** (Hardy and Littlewood [9, Theorem A]). There are constants \( A > 0 \) and \( T_0 > 0 \) such that \( N_{\text{Ri}}(T) \geq AT \) whenever \( T > T_0 \).

In 1942, Selberg proved that there exists \( A > 0 \) such that

\[
N_{\text{Ri}}(T) \geq AT \log T.
\]

Note that the numerical value of the constant \( A \) in Selberg’s theorem was very small. However, Levinson [16] greatly improved Selberg’s result and showed that \( A \geq 1/3 \). Furthermore, Conrey [4] proved that \( A \geq 0.4088 \). The current (June 2021) best result, which was proved by Kühn, Robles, and Zeindler [13], for the lower bound of \( A \) is

\[
A \geq 0.410725.
\]

It is well-known that the Riemann zeta function \( \zeta(s) \) does not vanish in the region of absolute convergence by the Euler product. Next, we review some facts about the zeros on the vertical line \( \sigma = 1/2 \) of the Epstein and Hurwitz zeta functions, which have complex zeros in the half-plane \( \sigma > 1/2 \) (e.g., [11, Chapter 7.4.3] and [15, Chapter 8.4]).

Let \( B(x, y) = ax^2 + bxy + cy^2 \) be a positive definite integral binary quadratic form, and denote by \( r_B(n) \) the number of solutions of the equation \( B(x, y) = n \) in integers \( x \) and \( y \). Then, the Epstein zeta function for the form \( B \) is defined by the ordinary Dirichlet series

\[
\zeta_B(s) := \sum_{(x,y) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{B(x, y)^s} = \sum_{n=1}^{\infty} \frac{r_B(n)}{n^s}
\]

for \( \sigma > 1 \). It is widely known that the function \( \zeta_B(s) \) admits analytic continuation into the entire complex plane except for a simple pole at \( s = 1 \) with residue \( 2\pi \Delta^{-1} \), where \( \Delta := \sqrt{4ac - b^2} \) (e.g., [6, Section 1]). Moreover, the function \( \zeta_B(s) \) satisfies the functional equation

\[
\left( \frac{\Delta}{2\pi} \right)^s \Gamma(s) \zeta_B(s) = \left( \frac{\Delta}{2\pi} \right)^{1-s} \Gamma(1-s) \zeta_B(1-s).
\]

Denote by \( N_{\text{Ep}}(T) \) the number of zeros of the Epstein zeta function \( \zeta_B(s) \) on the critical line and whose imaginary part is smaller than \( T > 0 \). In 1935, Potter and Titchmarsh [19] showed that \( N_{\text{Ep}}(T) \gg T^{1/2-\varepsilon} \). Subsequently, Sankaranarayanan [20] obtained \( N_{\text{Ep}}(T) \gg T^{1/2}/\log T \), and Jutila and Srinivas [14] proved that \( N_{\text{Ep}}(T) \gg T^{5/11-\varepsilon} \). As the current (June, 2021) best result, Baier, Srinivas, and Sangale [2] showed that

\[
N_{\text{Ep}}(T) \gg T^{4/7-\varepsilon}.
\]

A key to the proof of the estimation \( N_{\text{Ep}}(T) \gg T^{4/7-\varepsilon} \) shown in [2] is the first power mean of an ordinary Dirichlet series \( \sum_{n=1}^{\infty} b_n n^{-s} \) with \( b_n \in \mathbb{C} \) satisfying certain conditions cannot be too small. By zeros on the critical line and the functional equation of the Epstein zeta function \( \zeta_B(s) \) mentioned above, the quadrilateral zeta function \( Q(s, a) \) has many analytical properties in common with the Epstein zeta function (and the Riemann zeta function). It should be mentioned that the gamma factor of \( Q(s, a) \) does not depend on the parameter \( 0 < a < 1/2 \) from Theorem 1.1, but the gamma factor of \( \zeta_B(s) \) depends on the discriminant \( \Delta \) of the positive definite integral binary quadratic form \( B(x, y) \). Furthermore, we can see that the lower bound for the zeros of \( Q(s, a) \) on the critical line is better than that of \( \zeta_B(s) \) by virtue of Theorem 1.2 at present.
For the Hurwitz zeta function $\zeta(s, a)$ with $a = 1/3, 2/3, 1/4, 3/4, 1/6, or 5/6$, Gonek [7] showed that there exists a constant $0 < c < 1$ such that the number of zeros (including multiplicities) of $\zeta(s, a)$ on the segment $(1/2, 1/2 + iT)$ is $\leq (c + o(1))(T/2\pi)\log T$ as $T$ tends to infinity. Moreover, he concluded with the following conjecture:

Conjecture (Gonek [7]). If $0 < a < 1$ is rational and $a \neq 1/2$, then the Hurwitz zeta function $\zeta(s, a)$ has $\ll T$ zeros on the segment $(1/2, 1/2 + iT)$.

Based on this conjecture and the facts above, we can guess that proving the existence of $\gg T$ zeros on the line segment $(1/2, 1/2 + iT)$ of the Hurwitz or Epstein zeta functions is difficult because these zeta functions have no Euler product in general. However, we show that the quadrilateral zeta function $Q(s, a)$ has $\gg T$ zeros on the line segment $(1/2, 1/2 + iT)$ even though $Q(s, a)$ cannot be written as an Euler product (see (1.1)).

Remark. The quadrilateral zeta function $Q(s, a)$ with $a \in \mathbb{Q}$ can be essentially expressed as a linear combination of Euler products from (1.6). Hence, under the GRH and some assumptions on well-spacing of zeros for Dirichlet $L$-functions, we could show that $Q(s, a)$ with $a \in \mathbb{Q} \cap (0, 1/2) \setminus \{1/3, 1/4, 1/6\}$ have $100$% of zeros on the line $\sigma = 1/2$ if we could replace the function $\sum_{j=1}^{N} b_j L(s, \chi_j)$, where $b_j \in \mathbb{R} \setminus \{0\}$, in Bombieri and Hejhal [3, Theorem A] by the function $\sum_{j=1}^{N} (\beta_{1j} + \beta_{2j}q^a)L(s, \chi_j)$, where $\beta_{1j}, \beta_{2j} \in \mathbb{C} \setminus \{0\}$ and $q$ is a natural number. However, it seems to be extremely difficult for us to relax their assumptions in [3, Theorem A] as above (even when $\beta_{2j} = 0$). It is worth noting that $Q(s, a)$ with $a \in \mathbb{R} \setminus \mathbb{Q}$ can be expressed as neither an ordinary Dirichlet series nor a linear combination of Euler products. Despite these facts, we can prove Theorem 1.2 by modifying the proof of Hardy and Littlewood’s classical ideas in [9, Sections 2, 3, and 4] (see also [5, Chapter 11.2] and [22, Chapter 10.7]).

1.3. Hamburger’s, Hecke’s and Knopp’s Theorems. It is widely known that $\zeta(s)$ satisfies the functional equation

$$
\zeta(1-s) = \frac{2\Gamma(s)}{(2\pi)^s} \cos\left(\frac{\pi s}{2}\right) \zeta(s).
$$

(1.3)

As the first converse theorem for the Riemann zeta function $\zeta(s)$, Hamburger [8] proved that $\zeta(s)$ is characterized by the functional equation (1.3) (see also Siegel [21] and Titchmarsh [22, Chapter 2.13]).

Theorem B (Hamburger [8, Satz 1]). Let $G(s)$ be an entire function of finite order, $P(s)$ be a polynomial, and suppose that

$$
f(s) := \frac{G(s)}{P(s)} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad a(n) \in \mathbb{C}, \quad (H1)
$$

the series being absolutely convergent for $\sigma > 1$. Assume that

$$
\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)f(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)f(1-s).
$$

(H2)

Then, we have $f(s) = C\zeta(s)$, where $C$ is a constant.

Hecke [10, Section 1] showed that Hamburger’s Theorem can be rewritten as: the following three conditions characterize $\zeta(2s)$ up to a constant factor.
- (1) - The function $\phi(s)$ is meromorphic, and $P(s)\phi(s)$ is an entire function of finite genus with a suitable polynomial $P(s)$.

- (2) - The function $R(s) = \pi^{-s}\Gamma(s)\phi(s)$ satisfies the functional equation $R(s) = R(1/2 - s)$.

- (3a) - Both functions $\phi(s)$ and $\phi(s/2)$ can be expanded in a Dirichlet series that converges in a half-plane.

Moreover, Hecke [10] proved that the expressibility of $\phi(s/2)$ as a Dirichlet series in -(3a)- can be replaced by the following restriction on the poles of $\phi(s)$. More precisely, he showed that $\zeta(2s)$ (up to a constant factor) is uniquely determined by -(1)-, -(2)-, and -(3b)-. The function $\phi(s)$ can be expanded in a Dirichlet series that converges somewhere and the only pole allowed for $\phi(s/2)$ is $s = 1$.

It is quite natural to relax the conditions introduced by Hecke. Knopp [12] showed the following, which implies that there are infinitely many linearly independent solutions if we drop the pole condition -(3b)- above by using the Riemann-Hecke correspondence between ordinary Dirichlet series with functional equations and modular forms or the generalized Poincaré series.

**Theorem C** (Knopp [12, Theorem 1]). Let $\sigma_0 \geq 1/4$ and $A(\sigma_0)$ be the space of all rational functions $A(s)$ with poles restricted to the vertical strip $1/2 - \sigma_0 \leq \Re(s) \leq \sigma_0$ and satisfying $A(1/2 - s) = A(s)$. Let $A_H(\sigma_0)$ be the subspace of $A$ in $A(\sigma_0)$ such that $R(s) - A(s)$ is entire for some $R(s) = \pi^{-s}\Gamma(s)\phi(s)$ satisfying -(1)-, -(2)-, and -(3) - The function $\phi(s)$ can be expanded in a Dirichlet series that converges somewhere. Then with $N \geq n > [\sigma_0/2 + 3/8] + 2$ and $A_1, \ldots, A_n \in A(\sigma_0)$, some nontrivial linear combination of the $A_j$ is in $A_H(\sigma_0)$.

According to the theorems by Hamburger, Hecke, and Knopp, we can see that the conditions to characterize $\zeta(s)$ introduced by Hamburger or Hecke are so polished that a slight weakening of their conditions leads to infinitely many counterexamples, as mentioned by Knopp. Note that Knopp’s theorem does not provide any explicit representation for the coefficients of $a(n)$ of the Dirichlet series satisfying condition -(3)-. However, as analogues or improvements to Knopp’s Theorem, in the next subsection, we show that the zeta function $Q(s, a)$ defined explicitly in Section 1.1 fulfills the assumption -(2)- and some modified conditions of -(1)- and -(3a)- or -(3b)-.

### 1.4. Variations of Knopp’s Theorem

Now, we consider some variations of Knopp’s Theorems, namely, we properly modify conditions -(1)-, -(3a)-, and -(3b)- introduced by Hecke and prove that $Q(s, a)$ fulfills the reshaped conditions.

First, we have the following immediately from Theorem 1.1.

**Corollary 1.3.** The function $Q(2s, a)$ satisfies -(1)-, -(2)- and -(3a') - Both functions $\phi(s)$ and $\phi(s/2)$ can be expanded in a general Dirichlet series that converges in a half-plane.

Next, let $\varphi$ be the Euler totient function and $\chi$ be a primitive Dirichlet character of the conductor of $q$. Let $L(s, \chi) := \sum_{n=1}^{\infty} \chi(n)n^{-s}$ be the Dirichlet $L$-function. Then, for $0 < r < q$, where $q$ and $r$ are relatively prime integers, we have

$$
\zeta(s, r/q) = \sum_{n=0}^{\infty} \frac{1}{(n+r/q)^s} = \sum_{n=0}^{\infty} \frac{q^s}{(r+qn)^s} = q^s \sum_{\chi \mod q} \chi(r)L(s, \chi).
$$

(1.4)
In addition, let $G(r, \chi)$ denote the (generalized) Gauss sum $G(r, \chi) := \sum_{n=1}^{q} \chi(n)e^{2\pi i rn/q}$ associated with a Dirichlet character $\chi$. Then we have

$$\text{Li}(s, r/q) = q^{-s} \sum_{n=1}^{q} e^{2\pi i rn/q} \zeta(s, n/q) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} G(r, \chi)L(s, \chi).$$

Hence, from (1.4) and (1.5), it holds that

$$Q(s, r/q) = \frac{1}{2\varphi(q)} \sum_{\chi \bmod q} (1 + \chi(-1)) (\chi(r)q^s + G(r, \chi))L(s, \chi).$$

Therefore, we have the following from the functional equation (1.2).

**Corollary 1.4.** The function $Q(2s, r/q)$ satisfies -(1)-, -(2)-, and -(3b')- There exists a positive integer $q$ such that $q^{-2s}\phi(s)$ can be expanded in a Dirichlet series that converges somewhere, and the only pole allowed for $\phi(s/2)$ is $s = 1$.

For $q \in \mathbb{N}$, put $H(s, q) := (q^s + q^{1-s})^{-1}$. Then, we can see that $H(s, q) = H(1-s, q)$, and $q^s H(s, q)$ is written as an ordinary Dirichlet series by

$$q^s H(s, q) = \frac{q^s}{q^s + q^{1-s}} = \frac{1}{1 + q^{1-2s}} = \sum_{k=0}^{\infty} \frac{(-q)^k}{q^{2ks}}, \quad \sigma > 1/2.$$

From (1.6), the function $q^{-s}Q(s, r/q)$ can be expressed as an ordinary Dirichlet series. Therefore, we can see that

$$H(s, q)Q(s, r/q) = q^s H(s, q) \cdot q^{-s}Q(s, r/q)$$

is also written by an ordinary Dirichlet series. Moreover, the function

$$(1 - q^{1-s})(1 + q^{1-2s})H(s, q)Q(s, r/q)$$

is entire. Hence, we have the following from Theorem 1.1.

**Corollary 1.5.** The function $H(2s, q)Q(2s, r/q)$ satisfies (2), (3a), and -(1')- The function $\phi(s)$ is meromorphic and $D(s)\phi(s)$ is an entire function of finite genus with a suitable Dirichlet polynomial $D(s)$.

Corollaries 1.3, 1.4 and 1.5 can be regarded as analogues of Theorem C. Note that the functions appearing in Knopp’s theorem have poles in the strip $1/2 - \sigma_0 \leq \Re(s) \leq \sigma_0$, where $\sigma_0 \geq 1/4$, from the condition -(3)-, but the zeta function $Q(2s, a)$ has only one pole at $s = 1/2$ (see condition -(3b')- due to Hecke). Furthermore, the function $Q(2s, a)$ also fulfills the following splendid property by Theorem 1.2:

-(0)- The function $\phi(s)$ has infinitely many zeros on the line $\sigma = 1/4$.

Because Knopp did not explicitly provide solutions composed of zeta or $L$-functions that satisfy the conditions -(1)-, -(2)-, and -(3)-, we cannot see that the zeta or $L$-functions in his theorem fulfill any other noteworthy property. However, we show that the zeta function $Q(s, a)$ defined by (1.1) satisfies the functional equation (H2) in Theorem 1.1, or fulfills the condition -(2)- and some variations of -(1)- and -(3a)-, -(3b')-, or -(3)- in Corollaries 1.3, 1.4, and 1.5. Furthermore, we prove that the zeta function $Q(s, a)$ has $\gg T$ zeros on the segment $(1/2, 1/2 + iT)$ in Theorem 1.2.
2. Proofs

2.1. Functional equation and integral representation. First, we prove Theorem 1.1, namely, the functional equation (1.2)

Proof of Theorem 1.1. For simplicity, we put

\[ Z(s, a) := \zeta(s, a) + \zeta(s, 1 - a), \quad P(s, a) := \text{Li}(s, a) + \text{Li}(s, 1 - a). \]

Note that one has \(2Q(s, a) = Z(s, a) + P(s, a)\). For \(\sigma > 1\), it is known that

\[ \zeta(1 - s, a) = 2\Gamma(s) \left\{ \cos \left( \frac{\pi s}{2} \right) \sum_{n=1}^{\infty} \frac{\cos 2\pi na}{n^s} + \sin \left( \frac{\pi s}{2} \right) \sum_{n=1}^{\infty} \frac{\sin 2\pi na}{n^s} \right\} \]

(see for example [15, Theorem 2.3.1]). Thus, we have

\[ Z(1 - s, a) = 4\Gamma(s) \cos \left( \frac{\pi s}{2} \right) \sum_{n=1}^{\infty} \frac{\cos 2\pi na}{n^s} = 2\Gamma(s) \cos \left( \frac{\pi s}{2} \right) P(s, a). \]

In addition, for \(\sigma < 0\), it holds that

\[ \text{Li}(1 - s, a) = \frac{\Gamma(s)}{(2\pi)^s} \left( e^{\pi is/2} \zeta(s, a) + e^{-\pi is/2} \zeta(s, 1 - a) \right), \quad 0 < a < 1 \]

from [1, Exercises 12.2 and 12.3]. Hence, one has

\[ P(1 - s, a) = \frac{2\Gamma(s)}{(2\pi)^s} \left( \cos \left( \frac{\pi s}{2} \right) \zeta(s, a) + \cos \left( \frac{\pi s}{2} \right) \zeta(s, 1 - a) \right) = \frac{2\Gamma(s)}{(2\pi)^s} \cos \left( \frac{\pi s}{2} \right) Z(s, a). \]

We note that the functions \(\zeta(s, a)\) and \(Z(s, a)\) are regular for all \(s \in \mathbb{C}\) except \(s = 1\) (see [1, Chapter 12]) and the functions \(\text{Li}(s, a)\) and \(P(s, a)\) with \(0 < a < 1\) are analytically continuable to an entire function (see [15, Chapter 2.2]). Hence, the function \(Q(s, a)\) is regular for all \(s \in \mathbb{C}\) except \(s = 1\), where there is a simple pole with residue 1. Therefore, by using the identity theorem and the functional equations of \(Z(s, a)\) and \(P(s, a)\) above, we have

\[ 2Q(1 - s, a) = Z(1 - s, a) + P(1 - s, a) \]

\[ = \frac{2\Gamma(s)}{(2\pi)^s} \cos \left( \frac{\pi s}{2} \right) (P(s, a) + Z(s, a)) = \frac{2\Gamma(s)}{(2\pi)^s} \cos \left( \frac{\pi s}{2} \right) 2Q(s, a) \]

which implies the functional equation of \(Q(s, a)\). \(\square\)

Next, we show the following integral representation of \(\pi^{-s/2} \Gamma(s/2)Q(s, a)\).

Proposition 2.1. Define \(G_a(u)\) by

\[ G_a(u) := \sum_{n \in \mathbb{Z}} \left( \exp(-\pi u^2 (n + a)^2) + \exp(-\pi u^2 n^2 + i2\pi na) \right). \]

Then, for \(0 < \Re(s) < 1\), we have

\[ \pi^{-s/2} \Gamma\left( \frac{s}{2} \right) Q(s, a) = \int_0^{\infty} u^{-s} \left( G_a(u) - \frac{1}{u} \right) du. \]
Proof. For \( \Re(s) > 1 \), we have
\[
2 \int_0^\infty u^{s-1} \sum_{n \in \mathbb{Z}} \exp(-\pi u^2 (n + a)^2) \, du
\]
\[
= 2 \sum_{n=0}^\infty \int_0^\infty u^{s-1} e^{-\pi u^2(n+a)^2} \, du + 2 \sum_{n=0}^\infty \int_0^\infty u^{s-1} e^{-\pi u^2(n+1-a)^2} \, du.
\]

The first infinite series can be written as
\[
\sum_{n=0}^\infty \int_0^\infty e^{-v} \left( \frac{v/\pi}{(n+a)^2} \right)^{s/2-1} \frac{dv/\pi}{(n+a)^2} = \pi^{-s/2} \Gamma \left( \frac{s}{2} \right) \sum_{n=0}^\infty (n+a)^{-s}.
\]
Hence, we obtain
\[
2 \int_0^\infty u^{s-1} \sum_{n \in \mathbb{Z}} \exp(-\pi u^2 (n + a)^2) \, du = \pi^{-s/2} \Gamma \left( \frac{s}{2} \right) (\zeta(s, a) + \zeta(s, 1 - a)).
\]

Similarly, when \( \Re(s) > 1 \), one has
\[
2 \int_0^\infty u^{s-1} \sum_{n \neq n \in \mathbb{Z}} \exp(-\pi u^2 n^2 + i2\pi na) \, du
\]
\[
= 2 \sum_{n=0}^\infty \int_0^\infty u^{s-1} e^{2\pi in - \pi u^2 n^2} \, du + 2 \sum_{n=0}^\infty \int_0^\infty u^{s-1} e^{2\pi in(1-a) - \pi u^2 n^2} \, du.
\]

The first infinite series can be expressed as
\[
\sum_{n=0}^\infty \int_0^\infty e^{2\pi in} e^{-v} \left( \frac{v/\pi}{n^2} \right)^{s/2-1} \frac{dv/\pi}{n^2} = \pi^{-s/2} \Gamma \left( \frac{s}{2} \right) \sum_{n=0}^\infty \frac{e^{2\pi in}}{n^s}.
\]
Thus, it holds that
\[
2 \int_0^\infty u^{s-1} \sum_{n \neq n \in \mathbb{Z}} \exp(-\pi u^2 n^2 + i2\pi na) \, du = \pi^{-s/2} \Gamma \left( \frac{s}{2} \right) (\text{Li}(s, a) + \text{Li}(s, 1 - a)).
\]

Therefore, when \( \Re(s) > 1 \), we have
\[
\pi^{-s/2} \Gamma \left( \frac{s}{2} \right) Q(s, a) = \int_0^\infty u^{s-1} (G_a(u) - 1) \, du.
\]

For \( a, u > 0 \), it is well-known that (see [11, p. 13, (6)])
\[
\sum_{n \in \mathbb{Z}} \exp(-\pi u^2 (n + a)^2) = \frac{1}{u} \sum_{n \in \mathbb{Z}} \exp(-\pi u^{-2} n^2 + i2\pi na),
\]
\[
\sum_{n \in \mathbb{Z}} \exp(-\pi u^2 n^2 + i2\pi na) = \frac{1}{u} \sum_{n \in \mathbb{Z}} \exp(-\pi u^{-2} (n + a)^2).
\]
Hence, we easily obtain
\[
G_a(u) = u^{-1} G_a(u^{-1}), \quad u > 0.
\] (2.1)
By using the equation above and changing the variable \( u \to v^{-1} \), we have
\[
\pi^{-s/2} \Gamma \left( \frac{s}{2} \right) Q(s, a) = \int_0^\infty v^{-s} \left( G_a(v) - \frac{1}{v} \right) \, dv.
when $\Re(s) > 1$. Hence, we have
\[
\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) Q(s, a) = \int_0^1 v^{-s} \left(G_a(v) - \frac{1}{v}\right) dv + \int_1^\infty v^{-s} \left(G_a(v) - \frac{1}{v}\right) dv
\]
\[
= \int_0^1 v^{-s} \left(G_a(v) - \frac{1}{v}\right) dv + \int_1^\infty v^{-s} \left(G_a(v) - 1 - \frac{1}{v}\right) dv + \frac{1}{s - 1} \tag{2.2}
\]
Let $a_* := \min\{a, 1 - a\}$. From the definition of $G_a(u)$ and (2.1), it holds that
\[
G_a(u) = \frac{1}{u} \left(1 + O\left(\exp(-a_+^2 u^{-2})\right)\right), \quad u \to 0.
\]
Let $\Re(s) > 0$. By the definition of $G_a(u)$ and the estimation above, we have
\[
\int_0^1 v^{-s} \left(G_a(v) - \frac{1}{v}\right) dv \ll \int_0^1 v^{-\sigma-1} \exp(-a_*^2 v^{-2}) dv < \infty,
\]
\[
\int_1^\infty v^{-s} \left(G_a(v) - 1 - \frac{1}{v}\right) dv \ll \int_1^\infty v^{-\sigma-1} dv + \int_1^\infty \left(G_a(v) - 1\right) dv
\]
\[
\ll \int_1^\infty \sum_{n=1}^\infty \exp(-\pi vn^2) dv < \infty.
\]
Hence, both integrals in (2.2) converge when $\Re(s) > 0$. Clearly, one has
\[
\frac{1}{s - 1} = -\int_0^1 v^{-s} dv
\]
for $\Re(s) < 1$. Therefore, we obtain the integral representation in Proposition 2.1 \(\square\)

2.2. **Lemmas.** We contrast the integrals $J_a(t)$ and $I_a(t)$ given by
\[
J_a(t) := \frac{1}{2\pi} \int_{t-k}^{t+k} \pi^{-1/4 - iu/4} \Gamma\left(\frac{1 + 2iu}{4}\right) Q(1/2 + iu, a) e^{\pi u/4} e^{-u\delta/2} du
\]
\[
I_a(t) := \frac{e^{i\pi/4} e^{-i\delta/4}}{2\pi} \int_{t-k}^{t+k} \pi^{-1/4 - iu/4} \Gamma\left(\frac{1 + 2iu}{4}\right) Q(1/2 + iu, a) e^{\pi u/4} e^{-u\delta/2} du
\]
to show Theorem 1.2. This idea is parallel to the proof in [5, Chapter 11.2] or [22, Chapter 10.7]. Note that $|J_a(t)| > |I_a(t)|$ if the interval of integration of $I_a(t)$ contains roots of $Q(s, a) = 0$ on $\Re(s) = 1/2$ and $|J_a(t)| = |I_a(t)|$ otherwise. The key to the proof of Theorem 1.2 is to estimate the total length of all the integral intervals satisfying $|J_a(t)| > |I_a(t)|$. For this purpose, we present some lemmas.

**Lemma 2.2.** Let $a \neq 1/4$, $t$ and $k$ be sufficiently large. Then, we have
\[
\int_{t-k}^{t+k} \left|Q(1/2 + iv, a)\right| dv \geq 2k \cos(2\pi a) - C_1(a) - \frac{C_2(a)k^2}{(t - k)^{1/2}}
\]
\[
- \sum_{a_* = a, 1-a} \sum_{2 \leq n < t/a_*} \left(\frac{\sin(k \log(n + a_*))}{(n + a_*)^{1/2 + it} \log(n + a_*)} + \frac{e^{2\pi i a_* n} \sin(k \log n)}{n^{1/2 + it} \log n}\right), \tag{2.3}
\]
where $C_1(a)$ and $C_2(a)$ are some positive constants depend on $a$. 
Proof. For $a_*=a$ or $1-a$, the following approximate functional equations are shown (see [15, Theorems 3.1.3 and 3.1.2]). Suppose $2\pi \leq |\tau| \leq \pi x$. Then, one has
\[
\zeta(1/2 + i\tau, a_*) = \sum_{n=0}^{x} \frac{1}{(n + a_*)^{1/2 + i\tau}} + O(x^{-1/2}).
\]
Moreover, for $|t| \leq \pi a_*x$, it holds that
\[
\text{Li}(1/2 + i\tau, a_*) = \sum_{n=1}^{x} \frac{e^{2\pi i a_* n}}{n^{1/2 + i\tau}} + O(x^{-1/2}).
\]
Let $a_* := \min\{a, 1-a\}$ again. Then, for $t - k \leq v \leq t + k$, we have
\[
\zeta(1/2 + iv, a_*) = \sum_{0 \leq n < t/a_*} \frac{1}{(n + a_*)^{1/2 - iv}} + E_\zeta(v, a_*) + O(v^{-1/2}),
\]
\[
\text{Li}(1/2 + iv, a_*) = \sum_{1 \leq n < t/a_*} \frac{e^{2\pi i a_* n}}{n^{1/2 + iv}} + E_{\text{Li}}(v, a_*) + O(v^{-1/2}),
\]
where $E_\zeta(v, a_*)$ is defined by
\[
E_\zeta(v, a_*) := \begin{cases} \sum_{t/a_* \leq n < v/a_*} (n + a_*)^{-1/2 - iv} & t \leq v, \\ - \sum_{v/a_* \leq n < t/a_*} (n + a_*)^{-1/2 - iv} & t > v. \end{cases}
\]
The function $E_{\text{Li}}(v, a_*)$ is defined similarly. Because $E_\zeta(v, a_*)$ consists of at most $2k/a_*$ terms each of modulus at most $(n + a_*)^{-1/2} \leq a_*^{-1/2}(t-k)^{-1/2}$, we obtain that for $t - k \leq v \leq t + k$,
\[
\Re\left( \sum_{0 \leq n < t/a_*} \frac{1}{(n + a_*)^{1/2 - iv}} - \zeta(1/2 + iv, a_*) \right) \leq \frac{2k a_*^{-3/2}}{(t-k)^{1/2}} + O(v^{-1/2}) \leq \frac{C_2'(a_*) k}{(t-k)^{1/2}}
\]
for some positive constant $C_2'(a_*)$. Similarly, for some constant $C_2''(a_*) > 0$, it holds that
\[
\Re\left( \sum_{1 \leq n < t/a_*} \frac{e^{2\pi i a_* n}}{n^{1/2 + iv}} - \text{Li}(1/2 + iv, a_*) \right) \leq \frac{2k a_*^{-3/2}}{(t-k)^{1/2}} + O(v^{-1/2}) \leq \frac{C_2''(a_*) k}{(t-k)^{1/2}}.
\]
Assume $0 < a < 1/4$ and put $M(k, a) := 4k \cos(2\pi a) - 2C_1(a)$ and
\[
M(k, t, a) := 4k \cos(2\pi a) - 2C_1(a) - \frac{2C_2(a) k^2}{(t-k)^{1/2}},
\]
where $0 < C_2'(a) + C_2''(a) + C_2'(1-a) + C_2''(1-a) \leq C_2(a)$. For simplicity, we put $\sum_{*,*} := \sum_{a_*=a,1-a} \sum_{2 \leq n < t/a_*}$. Then, we have
\[
2 \int_{t-k}^{t+k} |Q(1/2 + iv, a)| dv \geq 2 \int_{t-k}^{t+k} \Re(Q(1/2 + iv, a)) dv
\]
\[
\geq M(k, a) + \int_{t-k}^{t+k} \Re(\sum_{*,*} \left( \frac{1}{(n + a_*)^{1/2 + iv}} + \frac{e^{2\pi i a_* n}}{n^{1/2 + iv}} \right) - \frac{C_2(a) k}{(t-k)^{1/2}}) dv
\]
\[
= M(k, t, a) + \Re\sum_{*,*} \left( \frac{2 \sin(k \log(n + a_*))}{(n + a_*)^{1/2 + iv} \log(n + a_*)} + \frac{2 e^{2\pi i a_* n} \sin(k \log n)}{n^{1/2 + iv} \log n} \right)
\]
which implies Lemma 2.2 with $0 < a < 1/4$. When $1/4 < a < 1/2$, based on the inequality
\[
\int_{t-k}^{t+k} |Q(1/2 + iv, a)| dv \geq - \int_{t-k}^{t+k} \Re(Q(1/2 + iv, a)) dv,
\]
we obtain Lemma 2.2 with $1/4 < a < 1/2$. □

**Lemma 2.3.** Put $a_* := \min\{a, 1-a\}$ and $a_* = a$ or $1 - a$. Let $A \leq t \leq B$ with $B \geq A \geq 2/a_*$. Then, it holds that
\[
\int_{A}^{B} \left| \sum_{2 \leq n < t/a_*} \frac{\sin(k \log(n + a_*))}{(n + a_*)^{1/2 + it} \log(n + a_*)} \right| dt \leq C_3(a_*) B,
\]
\[
\int_{A}^{B} \left| \sum_{2 \leq n < t/a_*} \frac{e^{2 \pi i n a_*} \sin(k \log n)}{n^{1/2 + it} \log n} \right| dt \leq C_4(a_*) B,
\]
for some positive constants $C_3(a_*)$ and $C_4(a_*)$.

**Proof.** First, we estimate the integral
\[
\int_{A}^{B} \left| \sum_{2 \leq n < t/a_*} \frac{\sin(k \log(n + a_*))}{(n + a_*)^{1/2 + it} \log(n + a_*)} \right|^2 dt =
\]
\[
\int_{A}^{B} \sum_{2 \leq n, m < t/a_*} \frac{\sin(k \log(n + a_*)) \sin(k \log(m + a_*))}{(n + a_*)^{1/2}(m + a_*)^{1/2} \log(n + a_*) \log(m + a_*)} \frac{(m + a_*)^it}{n + a_*} dt.
\]
For the terms with $n = m$, we have
\[
\int_{A}^{B} \sum_{2 \leq n < t/a_*} \frac{\sin^2(k \log(n + a_*))}{(n + a_*) \log(n + a_*)^2} dt \ll \int_{A}^{B} \sum_{2 \leq n < t/a_*} \frac{1}{n \log(n)^2} dt.
\]
It should be noted that the infinite series $\sum_{n=2}^{\infty} n^{-1} (\log n)^{-2}$ converges. Each of the terms with $n \neq m$ is of the form
\[
\frac{\sin(k \log(n + a_*)) \sin(k \log(m + a_*))}{(n + a_*)^{1/2}(m + a_*)^{1/2} \log(n + a_*) \log(m + a_*)} \int_{b}^{B} \frac{(m + a_*)^it}{n + a_*} dt,
\]
where $b := \max(A, a_* m, a_* n)$. Thus, regardless of the value of $b$, its absolute value is bounded above by
\[
\frac{2(n + a_*)^{-1/2}(m + a_*)^{-1/2}}{\log(n + a_*) \log(m + a_*) \log((n + a_*)/(m + a_*))} \ll \frac{n^{-1/2} m^{-1/2}}{\log n \log m | \log(n/m)|}.
\]
It is shown in [5, p. 236] that
\[
\sum_{2 \leq n \neq m \leq B} \frac{n^{-1/2} m^{-1/2}}{\log n \log m | \log(n/m)|} \ll B. \quad (2.4)
\]
Therefore, it holds that
\[
\int_{A}^{B} \left| \sum_{2 \leq n < t/a_*} \frac{\sin(k \log(n + a_*))}{(n + a_*)^{1/2 + it} \log(n + a_*)} \right|^2 dt \ll B.
\]
From the inequality above and the the Cauchy-Schwarz inequality, we have

\[ \int_A^B \left| \sum_{2 \leq n < t/a} \frac{\sin(k \log(n + a_*))}{(n + a_*)^{1/2 + u} \log(n + a_*)} \right| dt \ll (B - A)^{1/2} B^{1/2} \ll B, \]

which implies the first inequality of Lemma 2.3.

Second, consider the integral

\[ \int_A^B \left| \sum_{2 \leq n < t/a} \frac{e^{2\pi i a_*} \sin(k \log n)}{n^{1/2 + u} \log n} \right|^2 dt = \int_A^B \sum_{2 \leq n, m < t/a} \frac{e^{2\pi i a_*} e^{-2\pi i a_*} \sin(k \log n) \sin(k \log m) \left( \frac{m}{n} \right)^u}{n^{1/2} m^{1/2} \log n \log m} dt. \]

Obviously, for the terms with \( n = m \), we have

\[ \int_A^B \sum_{2 \leq n < t/a} \frac{e^{2\pi i a_*} e^{-2\pi i a_*} \sin(k \log n) \sin(k \log m) \left( \frac{m}{n} \right)^u}{n^{1/2} m^{1/2} \log n \log m} dt \leq \int_A^B \sum_{2 \leq n < t/a} \frac{1}{n^{1/2} m^{1/2} \log n \log m} dt. \]

For each of the terms with \( n \neq m \), we have

\[ \left| \int_b^B \frac{e^{2\pi i a_*} e^{-2\pi i a_*} \sin(k \log n) \sin(k \log m) \left( \frac{m}{n} \right)^u}{n^{1/2} m^{1/2} \log n \log m} dt \right| \ll \frac{n^{-1/2} m^{-1/2}}{\log n \log m \log(n/m)}. \]

Thus, we have the second inequality of Lemma 2.3 from (2.4) and the Cauchy-Schwarz inequality.

\[ \square \]

2.3. Proof of the existence of zeros on the critical line. Let \( k \) be a positive real number, \( x \) be a complex number with \( |x| = 1 \) and \( |\Im(x)| \leq \pi/4 \), and put

\[ I_{x,k}(s, a) := \frac{1}{2\pi i} \int_{s - ik}^{s + ik} \pi^{v/2} \Gamma\left(\frac{v}{2}\right) Q(v, a) x^{v-1} dv. \]  

(2.5)

Then, by Proposition 2.1, the function \( I_{x,k}(s, a) \) can be expressed as

\[ \frac{1}{2\pi i} \int_{s - ik}^{s + ik} \left( \frac{u}{x} \right)^{-v} \left( G_a(u) - 1 - \frac{1}{u} \right) \frac{dudv}{x} = \frac{1}{2\pi i} \int_{s - ik}^{s + ik} \left( \frac{G_a(xw) - 1 - \frac{1}{xw}}{w^v} \right) \frac{dwdv}{w^v}. \]

By Cauchy’s integral theorem and the fact that the function \( G_a(u) - 1 \) approaches zero rapidly as \( u \) tends to infinity along any ray \( u = xw \) in the wedge \( |\Im(x)| \leq \pi/4, w \in \mathbb{R} \), the integral above is equal to

\[ \frac{1}{2\pi i} \int_{s - ik}^{s + ik} \int_0^\infty w^{-v} \left( G_a(xw) - 1 - \frac{1}{xw} \right) dwdv = \int_0^\infty \left( \frac{1}{2\pi i} \int_{s - ik}^{s + ik} w^{-v} dv \right) \left( G_a(xw) - 1 - \frac{1}{xw} \right) dw = \frac{1}{\pi} \int_0^\infty \frac{w^{-s} \sin(k \log w)}{\log w} \left( G_a(xw) - 1 - \frac{1}{xw} \right) dw. \]
This expresses $I_{x,k}(s,a)$ as the transform of an operator and shows, from the Parseval-Plancherel identity (see [5, p. 216, line 7]), that

\[
\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} |I_{x,k}(s,a)|^2 ds = \frac{1}{\pi^2} \int_0^\infty \left| \frac{\sin(k \log w)}{\log w} \right|^2 \left| G_a(xw) - 1 - \frac{1}{xw} \right|^2 dw. \tag{2.6}
\]

Note that under the change of variable $w \to w^{-1}$, the form $dw$ becomes $-dw/w^2$, the factor $\sin(k \log w)/\log w$ is unchanged, and the function $G_a(xw) - 1 - (xw)^{-1}$ becomes

\[
G_a\left(\frac{1}{xw}\right) - 1 - \bar{x}w = \bar{x}w\left(\frac{1}{xw}G_a\left(\frac{1}{xw}\right) - \frac{1}{xw} - 1\right) = \bar{x}w\left(G_a(\bar{x}w) - 1 - \frac{1}{\bar{x}w}\right),
\]

where $x^{-1} = \bar{x}$, from (2.1). Thus, the integral of the right-hand side of (2.6) is equal to twice the integral from 1 to $\infty$. The first step is deriving an upper bound of the integral given by (2.6).

**Lemma 2.4.** Let $I_{x,k}(s,a)$ be defined as in (2.5) with $x = e^{-i\pi/4} e^{i\delta/2}$. Then, there exists a constant $K > 0$ such that given $\varepsilon > 0$ the inequality

\[
\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} |I_{x,k}(s,a)|^2 ds < \frac{Kk + \varepsilon k^2}{\delta^{1/2}}
\]

holds for all sufficiently small $\delta > 0$ (with $k > 0$ being arbitrary).

**Proof.** From the inequality

\[
|\sin ky| \leq \begin{cases} \frac{k}{y} & 0 \leq \pi/k, \\ \frac{1}{\pi/k} & y \geq \pi/k, \end{cases}
\]

the integral of the left-hand side of (2.6) is bounded above by

\[
\frac{2k^2}{\pi^2} \int_1^{e^{\pi/k}} \left| G_a(xw) - 1 - \frac{1}{xw} \right|^2 dw + \frac{2}{\pi^2} \int_{e^{\pi/k}}^\infty \left| G_a(xw) - 1 - \frac{1}{xw} \right|^2 \frac{dw}{(\log w)^2}. \tag{2.7}
\]

According to the inequality $|A + B|^2 \leq 2|A|^2 + 2|B|^2$, where $A, B \in \mathbb{C}$, the first integral in (2.7) is at most

\[
\frac{4k^2}{\pi^2} \int_1^{e^{\pi/k}} \left| G_a(xw) - 1 \right|^2 dw + \frac{4k^2}{\pi^2} \int_1^{e^{\pi/k}} \frac{dw}{w^2}.
\]

Obviously, the second definite integral is $4k^2 \pi^{-2} (1 - e^{-\pi/k})$. From $|A + B|^2 \leq 2|A|^2 + 2|B|^2$ again, the first integral is bounded above by

\[
\frac{8k^2}{\pi^2} \int_1^{e^{\pi/k}} \left| \sum_{n \in \mathbb{Z}} \exp(-\pi x^2 w^2(n + a)^2) \right|^2 dw + \frac{8k^2}{\pi^2} \int_1^{e^{\pi/k}} \left| \sum_{0 \neq n \in \mathbb{Z}} \exp(-\pi x^2 w^2 n^2 + i2\pi na) \right|^2 dw
\]

Let $x = e^{i\pi/4} e^{-i\delta/2}$ so that $x^2 = \sin \delta + i \cos \delta$. Then, the integrand in the first and second integrals respectively become

\[
\sum_{a_0, a_1} \sum_{n,m=0}^\infty \exp(G_{n,m}(w, \delta)), \quad \sum_{a_0, a_1} \sum_{n,m=1}^\infty \exp(H_{n,m}(w, \delta)), \tag{2.8}
\]
where the sum $\sum_{a_{1}, a_{2}}$ taken over $(a_{1}, a_{2}) = (a, a), (a, 1-a), (1-a, a), (1-a, 1-a)$ and the functions $G_{a_{1}, a_{2}}^{a_{1}, a_{2}}(w, \delta)$ and $G_{a_{1}, a_{2}}^{a_{1}, a_{2}}(w, \delta)$ are defined by

\[
G_{a_{1}, a_{2}}^{a_{1}, a_{2}}(w, \delta) := -\pi ((n + a_{1})^{2} + (m + a_{2})^{2})w^{2} \sin \delta - i\pi ((n + a_{1})^{2} - (m + a_{2})^{2})w^{2} \cos \delta,
\]

\[
H_{a_{1}, a_{2}}^{a_{1}, a_{2}}(w, \delta) := -\pi (n^{2} + m^{2})w^{2} \sin \delta - i\pi (n^{2} - m^{2})w^{2} \cos \delta + i2\pi na_{1} - i2\pi ma_{2},
\]

respectively. We divide each double sum $\sum_{n,m=0}^{\infty}$ and $\sum_{n,m=1}^{\infty}$ into three sums, one in which $n = m$, one in which $n > m$ and one in which $n < m$ because both double sums in (2.8) converge absolutely. If $n = m$, we have

\[
\sum_{n,m=0}^{\infty} \exp(G_{a_{1}, a_{2}}^{a_{1}, a_{2}}(w, \delta)) + \sum_{n=1}^{\infty} \exp(H_{a_{1}, a_{2}}^{a_{1}, a_{2}}(w, \delta)) \leq C_{5}(a) \sum_{n=1}^{\infty} \exp\left(-2\pi n^{2}w^{2} \sin \delta\right)
\]

for some positive constant $C_{5}(a)$. From the inequality

\[
\frac{8k^{2}}{\pi^{2}} \int_{1}^{e^{\pi/k}} \sum_{n=0}^{\infty} \exp\left(-2\pi n^{2}w^{2} \sin \delta\right) dw \leq \frac{k^{2}}{\pi^{2}} \int_{1}^{e^{\pi/k}} \frac{dw}{w(2 \sin \delta)^{1/2}} \leq k\delta^{-1/2},
\]

which is proved in [5, p. 231, line 7 from the bottom], for some positive constant $C_{6}(a)$, it holds that

\[
\frac{8k^{2}}{\pi^{2}} \int_{1}^{e^{\pi/k}} \sum_{a_{1}, a_{2}} \left(\sum_{n=0}^{\infty} \exp(G_{a_{1}, a_{2}}^{a_{1}, a_{2}}(w, \delta)) + \sum_{n=1}^{\infty} \exp(H_{a_{1}, a_{2}}^{a_{1}, a_{2}}(w, \delta))\right) dw
\]

\[
\leq C_{6}(a)k\delta^{-1/2}.
\]

Next, we will estimate the definite integral of the remaining terms $n \neq m$ of (2.8) from 1 to $e^{\pi/k}$. The terms with $m > n$ are the complex conjugates of those with $m < n$, so it suffices to estimate the latter. Consider the integral

\[
\frac{8k^{2}}{\pi^{2}} \int_{1}^{e^{\pi/k}} \exp\left(-\pi ((n + a)^{2} + (m + a)^{2})w^{2} \sin \delta - i\pi ((n + a)^{2} - (m + a)^{2})w^{2} \cos \delta\right) dw
\]

when $n > m$. The real part of the integral is

\[
\frac{8k^{2}}{\pi^{2}} \int_{1}^{e^{\pi/k}} f(w, a) \cos V(w, a) dw,
\]

where

\[
\begin{cases}
 f(w, a) := \exp\left(-\pi ((n + a)^{2} + (m + a)^{2})w^{2} \sin \delta\right), \\
 V(w, a) := \pi ((n + a)^{2} - (m + a)^{2})w^{2} \cos \delta.
\end{cases}
\]

Because $\cos \delta$ is positive for a small $\delta > 0$, $V(w, a)$ is a monotone increasing function with respect to $w$, and this integral can be rewritten in terms of the variable $V$ as

\[
\frac{8k^{2}}{\pi^{2}} \int_{V(1)}^{V(e^{\pi/k})} \frac{f}{V'} \cos V dV,
\]
Moreover, it is shown in [5, p. 233, line 12 from the bottom] that
above is not more than
\[ f \leq \frac{8k^2 2f(1)}{\pi^2 V'(1)} = \frac{16k^2 \exp(-\pi((n + a)^2 + (m + a)^2)w^2 \sin \delta)}{\pi^2 2\pi((n + a)^2 - (m + a)^2) \cos \delta}. \]

A similar estimate can be applied to the imaginary part and to both the real and imaginary parts of the following integral
\[ \frac{8k^2}{\pi^2} \int_1^{e^{\pi/k}} \exp(-\pi(n^2 + m^2)w^2 \sin \delta - i\pi(n^2 - m^2)w^2 \cos \delta + i2\pi na - i2\pi ma) dw. \]

Hence, for some positive constant \( C_7(a) \) and \( a_x, a_y = a \) or \( 1 - a \), we have
\[ \frac{8k^2}{\pi^2} \left| \int_1^{e^{\pi/k}} \exp(G_{n,m}(w, \delta)) dw \right|, \quad \frac{8k^2}{\pi^2} \left| \int_1^{e^{\pi/k}} \exp(H_{n,m}(w, \delta)) dw \right| \leq C_7(a)k^2 \frac{\exp(-\pi(n^2 + m^2) \sin \delta)}{(n^2 - m^2) \cos \delta}. \]

(2.10)

For any \( \varepsilon > 0 \) and for all sufficiently small \( \delta > 0 \), we have
\[ \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} k^2 \exp(-\pi(n^2 + m^2) \sin \delta) \frac{\exp(-n^2 \sin \delta) \log n}{n} \leq \varepsilon k^2 \delta^{-1/2}. \]

from [5, p. 233, line 1]. Therefore, it holds that
\[ \frac{8k^2}{\pi^2} \int_1^{e^{\pi/k}} \sum_{a_x, a_y = a} \sum_{n=1}^{\infty} \left( \sum_{m=0}^{n-1} \exp(G_{n,m}(w, \delta)) + \sum_{m=1}^{n-1} \exp(H_{n,m}(w, \delta)) \right) dw \leq \varepsilon k^2 \delta^{-1/2}. \]

Similar methods prove that the same estimation applies to the second integral in (2.7). We can easily see that
\[ \frac{2}{\pi^2} \int_1^{\infty} (\log w)^2 \left| \frac{1}{|w|} \right|^2 dw \leq \frac{2}{\pi^2} (\log e^{\pi/k})^2 - \frac{\varepsilon k^2}{\pi^2} \delta^{-1/2}. \]

It is shown in [5, p. 233, line 15] that
\[ \int_1^{\infty} (\log w)^{-2} \sum_{n=1}^{\infty} \exp(-2\pi n^2 w^2 \sin \delta) \ll k \delta^{-1/2}. \]

Hence, from (2.9), it holds that
\[ \int_1^{\infty} (\log w)^{-2} \sum_{a_x, a_y = a} \left( \sum_{n=0}^{\infty} \exp(G_{n,n}(w, \delta)) + \sum_{n=1}^{\infty} \exp(H_{n,n}(w, \delta)) \right) dw \ll k \delta^{-1/2}. \]

Moreover, it is shown in [5, p. 233, line 12 from the bottom] that
\[ \int_1^{\infty} \sum_{n=1}^{\infty} \sum_{m<n} \exp\left( -\pi(n^2 + m^2)w^2 \sin \delta - i\pi(n^2 - m^2)w^2 \cos \delta \right) \frac{dw}{(\log w)^2} \ll \sum_{n=1}^{\infty} \sum_{m<n} \frac{\exp(-\pi(n^2 + m^2) \sin \delta)}{(\log e^{\pi/k})^2(n^2 - m^2)} \leq \varepsilon k^2 \delta^{-1/2}. \]
Thus, by the inequality above and modifying the proof of (2.10), we have
\[
\sum_{a_x, a_z} \sum_{n=1}^{\infty} \int_{e^{\pi/k}}^{\infty} \left( \sum_{m=0}^{n-1} \exp(G_{n,m}^{a_x, a_z}(w, \delta)) + \sum_{m=1}^{n-1} \exp(H_{n,m}^{a_x, a_z}(w, \delta)) \right) \frac{dw}{(\log w)^2} \leq \varepsilon k^2 \delta^{-1/2}.
\]

The estimate of the integral (2.6) is unchanged under \(e^{i\pi/4}e^{-i\delta/2} \rightarrow e^{-i\pi/4}e^{i\delta/2}\) by the complex conjugate of \(I_{x,k}(s, a)\). Hence, Lemma 2.4 is obtained. \(\square\)

Next, we estimate the integral
\[
J_k(t, a) := \frac{1}{2\pi} \int_{t-k}^{t+k} \pi^{-1/4-iu/4} \Gamma\left(\frac{1+2iu}{4}\right) Q(1/2 + iu, a) e^{\pi u/4} e^{-u\delta/2} \, du.
\]
When \(x = e^{-i\pi/4}e^{i\delta/2}\), we have
\[
I_{x,k}(1/2 + it, a) = \frac{1}{2\pi} \int_{t-k}^{t+k} \pi^{-1/4-iu/4} \Gamma\left(\frac{1+2iu}{4}\right) Q(1/2 + iu, a) x^{-1/2} x^i u \, du
\]
\[
= \frac{x^{-1/2}}{2\pi} \int_{t-k}^{t+k} \pi^{-1/4-iu/4} \Gamma\left(\frac{1+2iu}{4}\right) Q(1/2 + iu, a) e^{\pi u/4} e^{-u\delta/2} \, du.
\]

For simplicity, we put
\[
I_a(t) := I_{x,k}(1/2 + it, a), \quad J_a(t) := J_k(t, a).
\]

Then, we have \(J_a(t) \geq |I_a(t)|\) for all \(t \in \mathbb{R}\) and \(J_a(t) = |I_a(t)|\) whenever the interval of integration of \(I_a(t)\) contains no roots of \(Q(s, a) = 0\) on the line \(\mathbb{R}(s) = 1/2\). The basic idea of the proof is to show that in a suitable sense \(J_a(t)\) is much larger than \(|I_a(t)|\) on average. Thus, estimates of \(J_a(t)\) from below are required. Stirling’s formula yields
\[
\Gamma\left(\frac{1+2iu}{4}\right) |Q(1/2 + iu, a)| e^{-\pi u/4} u^{-1/4} |Q(1/2 + iu, a)| e^{\pi u/4}
\]
\[
\gg u^{-1/4} |Q(1/2 + iu, a)|
\]
when \(u\) is sufficiently large. Thus, we have the following:

**Lemma 2.5.** For sufficiently large \(t > 0\), it holds that
\[
J_a(t) \gg (t + k)^{-1/4} e^{-(t+k)\delta/2} \int_{t-k}^{t+k} |Q(1/2 + iu, a)| \, du.
\]

Now, we are in a position to prove the main theorem. Note that the proof below is based on the argument in [5, Chapter 11.2] (see also [22, Chapter 10.7]). When \(a = 1/4\) or \(1/2\), it holds that
\[
Q(s, 1/2) = (2^s + 2^{1-s} - 2) \zeta(s), \quad 2Q(s, 1/4) = (2^s - 2^s + 2^{2-2s} - 2^{1-s}) \zeta(s)
\]
by (1.6) (see also [17, Section 2.2]). Therefore, we suppose \(0 < a < 1/4\) or \(1/4 < a < 1/2\) which implies that \(\cos(2\pi a) \neq 0\) (see Lemma 2.2).

**Proof of Theorem 1.2.** Let \(\nu\) be the number of zeros of \(Q(1/2 + it, a)\) in the interval \(0 \leq t \leq B + k\). And let the line \(\mathbb{R}(s) = 1/2\) be divided into intervals of length \(k\) and for each of the \(\nu\) zeros strike out the interval which contains it and the intervals which adjoin this one. Let \(S\) be the subset of \(\{A \leq t \leq B\}\) consisting of points which do not lie in the stricken intervals. Then, the total length of the intervals of \(S\) is not less than \(B - A - 3\nu k\) because a length of at most \(3k\) was stricken for each zero. Note that \(|I_a(t)| = J_a(t)|\ for
all \( t \in S \). Put \( I := \int_S |I_a(t)|dt \). Then, by Lemma 2.5 and the fact that there are no zeros between \( t - k \) and \( t + k \), we have

\[
I = \int_S J_a(t) \, dt \gg \int_S (B + k)^{-1/4} e^{-(B + k)\delta^2/2} \int_{t-k}^{t+k} |Q(1/2 + iu, a)| \, du \, dt.
\]

From Lemmas 2.2 and 2.3, we have

\[
I \gg (B + k)^{-1/4} e^{-(B + k)\delta^2/2} \int_S \left( 2k \cos |(2\pi a)| - C_1(a) - \frac{C_2(a)k^2}{(t - k)^{1/2}} \right)
\]
\[
- \left| \sum_{a_s = a, 1 - a} \sum_{2 \leq n < t/a_s} \left( \frac{\sin(k \log(n + a_s))}{(n + a_s)^{1/2 + it} \log(n + a_s)} + \frac{e^{2\pi i a_s n} \sin(k \log n)}{n^{1/2 + it} \log n} \right) \right| dt
\]
\[
\gg (B + k)^{-1/4} e^{-(B + k)\delta^2/2} \left( 2k(B - A - 3\nu k) \cos(2\pi a) - C_7(a)B - C_8(a)k^2 B^{1/2} \right)
\]

for some positive constants \( C_7(a) \) and \( C_8(a) \). Let \( (B + k)\delta = 2 \), which can be regarded as a choice of \( B \) given \( \delta \) and \( k \), and let \( B - A = \delta^{-1} \) which can be regarded as a choice of \( A \). Then, the estimation above becomes

\[
I \gg \delta^{1/4} \left( 2k(\delta^{-1} - 3\nu k) \cos(2\pi a) - 2C_7(a)\delta^{-1} - \sqrt{2}C_8(a)k^2 \delta^{-1/2} \right)
\]
\[
\gg K_1 k \delta^{-3/4} - K_2 k^2 \delta^{1/4} \nu - K_3 \delta^{-3/4} - K_4 k^2 \delta^{-1/4},
\]

where \( K_1, K_2, K_3 \) and \( K_4 \) are positive constants (depending on \( a \)). In contrast, from Lemma 2.4, we have

\[
I \leq \int_A^B \frac{1}{A} I_a(t) \, dt \leq \left( \frac{1}{A} \int_A^B |I_a(t)| \, dt \right)^{1/2} \left( \frac{1}{1} \int_A^B |I_a(t)|^2 \, dt \right)^{1/2}
\]
\[
\leq B^{1/2} \left( \int_{1/2+i\infty}^{1/2-i\infty} |I_x, k(s, a)|^2 \, ds \right)^{1/2} = K_5 \frac{(Kk + \varepsilon k^2)^{1/2}}{\delta^{3/4}}.
\]

Therefore, it holds that

\[
K_1 k \delta^{-3/4} - K_2 k^2 \delta^{1/4} \nu - K_3 \delta^{-3/4} - K_4 k^2 \delta^{-1/4} \leq K_5 \delta^{-3/4}(Kk + \varepsilon k^2)^{1/2},
\]

which is equivalent to

\[
\nu \geq \frac{K_1}{K_2} k^{-1} \delta^{-1} - \frac{K_3}{K_2} k^{-2} \delta^{-1} - \frac{K_4}{K_2} k^{-1} \delta^{-1/2} - \frac{K_5}{K_2} k^{-1} \delta^{-1} \left( \frac{K}{k} + \varepsilon \right)^{1/2}.
\]

We can make the coefficient of \( k^{-1} \delta^{-1} \) on the right-hand side positive by choosing \( \varepsilon > 0 \) and \( k^{-1} > 0 \) to be sufficiently small. Hence, with this fixed \( k > 0 \), it has been shown that for all sufficiently small \( \delta > 0 \), the number of roots on the line segment from \( 1/2 \) to \( 1/2 + i2\delta^{-1} \) is greater than \( K_6 \delta^{-1} - K_7 \delta^{-1/2} \) with \( K_6 > 0 \).

**Acknowledgments.** The author would like to thank Professors Kohji Matsumoto and Masatoshi Suzuki for their useful advice. The author would like to also thank the referee for a careful reading of the manuscript and valuable comments and remarks. The author was partially supported by JSPS, grant no. 16K05077.
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