Two-dimensional QCD, instanton contributions and the perturbative Wu-Mandelstam-Leibbrandt prescription

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Abstract

The exact Wilson loop expression for the pure Yang-Mills $U(N)$ theory on a sphere $S^2$ of radius $R$ exhibits, in the decompactification limit $R \to \infty$, the expected pure area exponentiation. This behaviour can be understood as due to the sum over all instanton sectors. If only the zero instanton sector is considered, in the decompactification limit one exactly recovers the sum of the perturbative series in which the light-cone gauge Yang-Mills propagator is prescribed according to Wu-Mandelstam-Leibbrandt. When instantons are disregarded, no pure area exponentiation occurs, the string tension is different and, in the large-$N$ limit, confinement is lost.

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I. INTRODUCTION

Quantum Yang-Mills theories on compact two-dimensional surfaces have been extensively studied in the past years: in spite of their seeming triviality, many interesting (and non-trivial) results were obtained exploiting their non-perturbative solvability. In particular a string picture, in the large-$N$ limit, was derived in [1], while partition function [2], Wilson loops [3] and field-strength correlators [4] were computed exactly on arbitrary genus.

A further intriguing aspect is the appearance, on genus zero and in the limit of large $N$, of a third order phase transition at a critical value of $g^2NA$ [5] ($g^2$ being the Yang-Mills coupling constant and $A$ the area of the sphere): a strong coupling phase, where a pure area exponentiation for Wilson loops dominates in the large-$A$ limit, is distinguished from a weak coupling phase with no confining behaviour [6]. A clear physical picture of this phenomenon was presented in [7], showing, by explicit computations, that instanton contributions are suppressed in the second case, while playing a prominent role in driving the theory in the strong (confining) phase. The relevance of topologically non-trivial configurations in connection with the Douglas-Kazakov phase transition was also noticed some time before in ref. [8].

At the first sight this could appear as a paradox: confinement in $QCD_2$ is often regarded as a perturbative feature. As a matter of fact the area exponentiation is simply obtained (for any $N$) by summing on the plane the perturbative series, with the t’Hooft-CPV prescription for the gluon exchange potential [7]. Alternatively, from the results of [2,3], one can easily realize that the very same result is obtained for any value of $N$ in the decompactification limit when all instanton sectors are taken into account.

On the other hand an exact resummation of the perturbative series has also been recently done [10] adopting instead the Wu-Mandelstam-Leibbrandt (WML) [11] prescription for the gluon propagator, generalizing to all orders the $O(g^4)$ computation of [12,13]; it leads, as firstly noticed in [12], to a result different from a pure area-law exponentiation (which would be expected from the area-preserving diffeo-invariance of the theory plus positivity
arguments), and, in particular, predicting a different value for the string tension.

Dramatically, the large-$N$ limit exhibits a non-confining behaviour, while one easily realizes that, on the plane, the theory should be in the strong coupling phase (the plane being thought as decompactification of a large sphere).

In this letter we discuss the reasons of these discrepancies, and show how the WML computations presented in [10,12,13] are indeed perturbatively correct, in the sense that what is missing exactly represents the instanton contribution. This contribution can be eventually recovered by expanding the functional integral as a sum over a class of topologically charged field configurations. In so doing the usual expected area-law behaviour is reproduced.

II. THE INSTANTON EXPANSION

Our starting point are the well-known expressions [4] of the exact partition function and of a non self-intersecting Wilson loop for a pure $U(N)$ Yang-Mills theory on a sphere $S^2$ with area $A$

$$Z(A) = \sum_R (d_R)^2 \exp \left[ -\frac{g^2 A}{2} C_2(R) \right],$$

$$W(A_1, A_2) = \frac{1}{Z_N} \sum_{R,S} d_R d_S \exp \left[ -\frac{g^2 A_1}{2} C_2(R) - \frac{g^2 A_2}{2} C_2(S) \right] \int dU \text{Tr}[U] \chi_R(U) \chi_S^\dagger(U),$$

$d_R(S)$ being the dimension of the irreducible representation $R(S)$ of $U(N)$; $C_2(R)$ ($C_2(S)$) is the quadratic Casimir, $A_1 + A_2 = A$ are the areas singled out by the loop, the integral in (2) is over the $U(N)$ group manifold while $\chi_{R(S)}$ is the character of the group element $U$ in the $R(S)$ representation.

Eqs. (1), (2) can be easily deduced from the solution of Yang-Mills theory on the cylinder, using the fact that the hamiltonian evolution is governed by the laplacian on $U(N)$: we call eqs. (1),(2) the heat-kernel representations of $Z(A)$ and $W(A_1, A_2)$, respectively.

On the other hands, as first noted by Witten [14], it is possible to represent $Z(A)$ (and consequently $W(A_1, A_2)$) as a sum over instable instantons, where each instanton
contribution is associated to a finite, but not trivial, perturbative expansion. The easiest
way to see it, is to perform a Poisson resummation in eqs. (1),(2).

To this purpose we write them explicitly for \(N > 1\) in the form

\[
Z(A) = \frac{1}{N!} \exp \left[ -\frac{g^2 A}{24} N(N^2 - 1) \right] \sum_{m_i = -\infty}^{+\infty} \Delta^2(m_1, ..., m_N) \exp \left[ -\frac{g^2 A}{2} \sum_{i=1}^{N} (m_i - \frac{N - 1}{2})^2 \right],
\]

\[
W(A_1, A_2) = \frac{1}{ZN} \exp \left[ -\frac{g^2 A}{24} N(N^2 - 1) \right] \frac{1}{N!} \sum_{k=1}^{N} \sum_{m_i = -\infty}^{+\infty} \Delta(m_1, ..., m_N) \times \Delta(m_1 + \delta_{1,k}, ..., m_N + \delta_{N,k}) \exp \left[ -\frac{g^2 A_1}{2} \sum_{i=1}^{N} (m_i - \frac{N - 1}{2})^2 - \frac{g^2 A_2}{2} \sum_{i=1}^{N} (m_i - \frac{N - 1}{2} + \delta_{i,k})^2 \right].
\]

We have described the generic irreducible representation by means of the set of integers \(m_i = (m_1, ..., m_N)\), related to the Young tableaux, in terms of which we get

\[
C_2(R) = \frac{N}{12} (N^2 - 1) + \sum_{i=1}^{N} (m_i - \frac{N - 1}{2})^2,
\]

\[
d_R = \Delta(m_1, ..., m_N).
\]

\(\Delta\) is the Van der Monde determinant and the integration in eq.(2) has been performed explicitly, using the well-known formula for the characters in terms of the set \(m_i\).

The instanton representation of \(Z(A)\) and of \(W(A_1, A_2)\) is now simply obtained \footnote{3,16} by performing a Poisson resummation over \(m_i\)

\[
\sum_{m_i = -\infty}^{+\infty} F(m_1, ..., m_N) = \sum_{n_i = -\infty}^{+\infty} \tilde{F}(n_1, ..., n_N),
\]

\[
\tilde{F}(n_1, ..., n_N) = \int_{-\infty}^{+\infty} dz_1...dz_N \exp \left[ 2\pi i (z_1 n_1 + ... + z_N n_N) \right] F(z_1, ..., z_N)
\]

for the eqs.(3,4).

We have carefully repeated the original computations of ref. \footnote{7}, paying particular attention to the numerical factors and to the area dependences; as a matter of fact, at variance with \footnote{7}, where interest was focussed on the large-\(N\) limit, we are mainly concerned with decompactification (large \(A\)) and with a comparison with the results of ref. \footnote{10} for any value of \(N\). We have obtained
\[
Z(A) = C(g^2 A, N) \sum_{n_i = -\infty}^{+\infty} \exp[-S_{\text{inst}}(n_i)] Z(n_1, ..., n_N),
\]
\[
W(A_1, A_2) = \frac{1}{Z N} C(g^2 A, N) \exp\left[-g^2 \frac{A_1 A_2}{2 A}\right] \sum_{n_i = -\infty}^{+\infty} \exp[-S_{\text{inst}}(n_i)]
\times \sum_{k=1}^{N} \exp\left[-2\pi i n_k \frac{A_2}{A}\right] W_k(n_1, ..., n_N),
\] (7)

where
\[
C(g^2 A, N) = (i)^{N(N-1)} \frac{(g^2 A)^{-N^2/2}}{N!} \exp\left[-\frac{g^2 A}{24} N(N^2 - 1)\right]
\]
\[
S_{\text{inst}}(n_i) = \frac{2\pi^2}{g^2 A} \sum_{i=1}^{N} n_i^2,
\] (8)

and
\[
Z(n_1, ..., n_N) = \exp(i\pi (N - 1) \sum_{i=1}^{N} n_i) \int_{-\infty}^{+\infty} dz_1...dz_N \exp\left[-\frac{1}{2} \sum_{i=1}^{N} z_i^2\right]
\times \prod_{i<j}^{N} \left(\frac{4\pi^2}{g^2 A} (n_i - n_j)^2 - (z_i - z_j)^2\right).
\]
\[
W_k(n_1, ..., n_N) = \exp(i\pi (N - 1) \sum_{i=1}^{N} n_i) \int_{-\infty}^{+\infty} dz_1...dz_N \exp\left[-\frac{1}{2} \sum_{i=1}^{N} z_i^2\right] \times
\prod_{i<j}^{N} \left[\left(\frac{2\pi}{g^2 A} (n_i - n_j) + \frac{g^2 A_2 - A_1}{2 \sqrt{g^2 A}} (\delta_{i,k} - \delta_{j,k})\right)^2 - ((z_i - z_j) + i \frac{\sqrt{g^2 A}}{2} (\delta_{i,k} - \delta_{j,k}))^2\right].
\] (9)

These formulae have a nice interpretation in terms of instantons. Indeed, on \(S^2\), there are non trivial solutions of the Yang-Mills equation, labelled by the set of integers
\[
A_\mu(x) = \begin{pmatrix}
n_1 A_\mu^0(x) & 0 & \ldots & 0 \\
0 & n_2 A_\mu^0(x) & \ldots & 0 \\
0 & \ldots & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & n_N A_\mu^0(x)
\end{pmatrix}
\] (10)

where \(A_\mu^0(x) = A_\mu^0(\theta, \phi)\) is the Dirac monopole potential,
\[
A_\theta^0(\theta, \phi) = 0, \quad A_\phi^0(\theta, \phi) = \frac{1 - \cos \theta}{2},
\]
\( \theta \) and \( \phi \) being the polar (spherical) coordinates on \( S^2 \).

The integer nature of the coefficients is of course a consequence of Dirac quantization condition or (more mathematically) of the fact that the original \( U(N) \)-bundle has been reduced to a non-trivial \( N \)-torus bundle (see [14,17] for details). In the light of the above considerations, eqs.\( (7) \) can be interpreted as follows: \( S_{\text{inst}}(n_i) \) represents the classical action evaluated on the non-trivial solutions \( (10) \), the exponential factor inside the sums is their contributions to the Wilson loop, while \( Z(n_i) \) and \( W_k(n_i) \) are the quantum corrections, as anticipated by Witten using the Duistermaat-Heckman theorem; a direct path-integral evaluation is presented in [17].

¿From the above representations it is rather clear why the decompactification limit \( A \to \infty \) should not be performed too early. Indeed on the plane it is not easy to distinguish fluctuations around the instanton solutions from Gaussian fluctuations around the trivial field configuration, since \( S_{\text{inst}}(n_i) \) goes to zero for any finite set \( n_i \) when \( A \to \infty \). For finite \( A \) and finite \( n_i \) instead, in the limit \( g \to 0 \), only the zero instanton sector can survive in the Wilson loop expression (notice that the power-like singularity \( (g^2)^{-N^2/2} \) in the coefficient \( C(g^2 A, N) \) exactly cancels in the normalization). In this limit each instanton contribution is \( \mathcal{O}(\exp(\frac{-1}{g^2})) \); therefore instantons become crucial only when they are completely resummed.

On the other hand the zero instanton contribution should be obtainable in principle by means of perturbative calculations.

In the following we compute from eqs.\( (7) \) the exact expression on the sphere \( S^2 \) of the zero instanton contribution to the Wilson loop, obviously normalized to zero instanton partition function.

### III. RELATION WITH PERTURBATION THEORY

We write eq.\( (7) \) for the zero instanton sector \( n_i = 0 \). Thanks to its symmetry, we can always choose \( k = 1 \) and the equation becomes
\[ W_0 = (2\pi)^{-N} \prod_{n=0}^{N} \frac{1}{n!} \exp \left[ -g^2 \frac{A_1 A_2}{2A} \right] \int_{-\infty}^{+\infty} dz_1 \ldots dz_N \exp \left[ -\frac{1}{2} \sum_{i=1}^{N} z_i^2 \right] \]
\[ \times \prod_{j=2}^{N} \left[ (z_1 - z_j)^2 + i \sqrt{g^2 A(z_1 - z_j) - g^2 A_1 A_2} \Delta^2(z_2, \ldots, z_N) \right]. \tag{11} \]

We introduce the two roots of the quadratic expression in the integrand \( z_\pm = z_1 + i\alpha \pm i\beta \) with \( \alpha = \frac{\sqrt{g^2 A^2}}{2} \) and \( \beta = \frac{\sqrt{g^2 (A_1 - A_2)}}{2\sqrt{A}} \). The previous equation then becomes

\[ W_0 = (2\pi)^{-N} \prod_{n=0}^{N} \frac{1}{n!} \exp \left[ -g^2 \frac{A_1 A_2}{2A} \right] \int_{-\infty}^{+\infty} dz_1 \ldots dz_N \exp \left[ -\frac{1}{2} \sum_{i=1}^{N} z_i^2 \right] \]
\[ \times \Delta(z_+, z_2, \ldots, z_N) \Delta(z_-, z_2, \ldots, z_N). \tag{12} \]

The two Van der Monde determinants can be expressed in terms of Hermite polynomials and then expanded in the usual way. The integrations over \( z_2, \ldots, z_N \) can be performed, taking the orthogonality condition into account; we get

\[ W_0 = (2\pi)^{-\frac{N}{2}} \prod_{n=0}^{N} \frac{1}{n!} \exp \left[ -g^2 \frac{A_1 A_2}{2A} \right] \prod_{k=2}^{N} (j_k - 1)! z_{j_1 \ldots j_N} z_{j_1 \ldots j_N} \]
\[ \times \int_{-\infty}^{+\infty} dz_1 \exp \left[ -\frac{z_1^2}{2} \right] H_{j_1-1}(z_+) H_{j_1-1}(z_-). \tag{13} \]

Thanks to the relation

\[ \int_{-\infty}^{+\infty} dz_1 \exp \left[ -\frac{z_1^2}{2} \right] H_{j_1-1}(z_+) H_{j_1-1}(z_-) = \sqrt{2\pi(j_1 - 1)!} L_{j_1-1}(\alpha^2 - \beta^2), \tag{14} \]

we finally obtain our main result

\[ W_0 = \frac{1}{N} \exp \left[ -g^2 \frac{A_1 A_2}{2A} \right] L_{N-1}(g^2 \frac{A_1 A_2}{A}). \tag{15} \]

At this point we remark that, in the decompactification limit \( A \to \infty, A_1 \) fixed, the quantity in the equation above exactly coincides, for any value of \( N \), with eq.(11) of ref. [10], which was derived following completely different considerations. We recall indeed that their result was obtained by a full resummation at all orders of the perturbative expansion of the Wilson loop in terms of Yang-Mills propagators in light-cone gauge, endowed with the WML prescription.

Several considerations can now be drawn.
First of all we notice that $W_0$ does not exhibit the usual area-law exponentiation; actually, in the large-$N$ limit, exponentiation (and thereby confinement) is completely lost, as first noticed in [10]. As a matter of fact, from eq. (15), taking the limit $N \to \infty$, we easily get

$$\lim_{N \to \infty} W_0 = \sqrt{\frac{A_1 + A_2}{g^2 A_1 A_2}} J_1 \left( \sqrt{\frac{4g^2 A_1 A_2}{A_1 + A_2}} \right)$$

with $g^2 = g^2 N$. At this stage, however, this is no longer surprising since $W_0$ does not contain any genuine non perturbative contribution, \textit{viz} instantons. If on the sphere $S^2$ we consider the weak coupling phase $g^2 NA < \pi^2$, instanton contributions are suppressed. As a matter of fact, eq. (16) provides the complete Wilson loop expression in the weak coupling phase [5,6]. In turn confinement occurs in the strong coupling phase [7].

For any value of $N$ the pure area law exponentiation follows, after decompactification, from resummation of all instanton sectors, changing completely the zero sector behaviour and, in particular, the value of the string tension.

In the light of the considerations above, there is no contradiction between the use of the WML prescription in the light-cone propagator and the pure area law exponentiation; this prescription is correct but the ensuing perturbative calculation can only provide us with the expression for $W_0$. The paradox of ref. [10] is solved by recognizing that they did not take into account the genuine $O(\exp(-\frac{1}{g^2}))$ non perturbative quantities coming, after decompactification, from the instantons on the sphere.

What might instead be surprising in this context is the fact that, using the instantaneous \textsc{t} Hooft-CPV potential and just resumming at all orders the related perturbative series, one still ends up with the correct pure area exponentiation. It can perhaps be naively understood if the exchange is interpreted as a simple “instantaneous” increasing potential between a $q\bar{q}$ pair, giving rise to hadronic strings in a natural way. This feature is likely to be linked to some peculiar properties of the light-front vacuum (we remind the reader that the light-cone CPV prescription follows from canonical light-front quantization [18]; still we know it is perturbatively unacceptable in higher dimensions [19] and cannot be smoothly continued to any Euclidean formulation).
As a final remark we notice that, for $N = 1$, we always find the pure area exponentiation in the decompactification limit. This can be understood by realizing that $W_k(n) = 1$ in this case and therefrom all instanton sectors provide equal contributions in this limit. Still ’t Hooft and WML prescriptions lead to the same final result in a non trivial way as only planar diagrams contribute in the first case.

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