A cosmic shadow on CSL

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The Continuous Spontaneous Localisation (CSL) model solves the measurement problem of standard quantum mechanics, by coupling the mass density of a quantum system to a white-noise field. Since the mass density is not uniquely defined in general relativity, this model is ambiguous when applied to cosmology. We however show that some well-motivated choices of the density contrast already make current measurements of the cosmic microwave background incompatible with other laboratory experiments.

Addressing the measurement (or macro-objectification) problem is a central issue in quantum mechanics, and three classes of solutions have been put forward [1]. One can either (1) leave quantum theory unmodified and consider different interpretations (e.g. Copenhagen, many worlds, QBism, etc.); (2) extend the mathematical framework and introduce additional degrees of freedom (e.g. de Broglie-Bohm); or (3) consider that quantum theory is an approximation of a more general framework and that, outside its domain of validity, it differs from the standard formulation. Dynamical collapse models [1–5] follow this last reasoning and introduce a non-linear and stochastic modification to the Schrödinger equation. Remarkably, the structure of this modification is essentially unique. Through an embedded amplification mechanism, this allows microscopic systems to be described by the standard rules of quantum mechanics, while preventing macroscopic systems from being in a superposition of macroscopically distinct configurations. It also allows the Born rule to be derived rather than postulated. Because they lead to predictions that are different from that of conventional quantum mechanics, dynamical collapse models are falsifiable contrary to the other options mentioned before (except de Broglie-Bohm theory in the out-of-equilibrium regime [6, 7]).

Different versions of dynamical collapse theories correspond to different choices for the collapse operator (energy, momentum, spin, position), the nature of the stochastic noise (white or non-white) and whether dissipative effects are included or not. Only a collapse operator related to position can ensure proper localisation in space, and three iconic theories have been proposed: (1) the Ghirardi-Rimini-Weber (GRW) model, which is historically the first one but is not formulated in terms of a continuous stochastic differential equation, (2) Quantum Mechanics with Universal Position Localisation (QMUPL), where the collapse operator is position but where the stochastic noise depends on time only, and (3) the Continuous Spontaneous Localisation (CSL) model [4], where the stochastic noise depends on time and space and where the collapse operator is the mass density. This version is the most refined of all three, and features the modified Schrödinger equation

\[ d\Psi = \left\{-i\hat{H}dt + \frac{\sqrt{\gamma}}{m_0} \int d\mathbf{x}_p \left[ \hat{\rho}_{sm}(\mathbf{x}_p) - \langle \hat{\rho}_{sm}(\mathbf{x}_p) \rangle \right] dW_t(\mathbf{x}_p) - \frac{\gamma}{2m_0^2} \int d\mathbf{x}_p \left[ \hat{\rho}_{sm}(\mathbf{x}_p) - \langle \hat{\rho}_{sm}(\mathbf{x}_p) \rangle \right]^2 dt \right\} |\Psi\rangle, \quad (1) \]

where \( \hat{H} \) is the standard Hamiltonian of the system, \( \langle A \rangle \equiv \langle |\Psi\rangle |A|\Psi\rangle \), \( \gamma \) is the first free parameter of the theory, \( m_0 \) is a reference mass (usually the mass of a nucleon), \( W_t(\mathbf{x}_p) \) is an ensemble of independent Wiener processes (one for each point in space), and \( \hat{\rho}_{sm} \) is the smeared mass density operator

\[ \hat{\rho}_{sm}(\mathbf{x}_p) = \frac{1}{(2\pi)^{3/2} r_c^3} \int d\mathbf{y} \hat{\rho}(\mathbf{x}_p + \mathbf{y}) e^{-\frac{|\mathbf{y}|^2}{2r_c^2}}, \quad (2) \]

where \( r_c \) is the second free parameter of the theory. The two parameters \( \gamma \) and \( r_c \) have been constrained in various laboratory experiments. The strongest bounds so far come from X-ray spontaneous emission [8], force noise measurements on ultracold cantilevers [9], and gravitational-wave interferometers [10]. These constraints leave the region of parameter space around \( r_c \sim 10^{-8} \sim 10^{-4}\text{m} \) and \( \lambda \sim 10^{-18} \sim 10^{-10}\text{s}^{-1} \) viable, where \( \lambda \equiv \gamma/(8\pi^{3/2}r_c^3) \), corresponding to the white region in Fig. 3.

Dynamical collapse models can also be constrained in a cosmological context [11–15]. Indeed, the typical physical scales involved in cosmology are many orders of magnitude different from those encountered in the lab and this may lead to competitive constraints (in the early universe, energy scales can be as high as \( \sim 10^{15}\text{GeV} \), corresponding to densities of \( \sim 10^{80} \text{g} \times \text{cm}^{-3} \)). Moreover, one can argue that the quantum measurement problem is even more acute in cosmology than in the lab [16], due to the difficulties in introducing an “observer” as in the standard Copenhagen interpretation [17]. Since the quantum state of cosmological perturbations, \( |\Psi_{2\text{sq}}\rangle \), is not an eigenstate of the Cosmic Microwave Background...
(CMB) temperature anisotropies, how the process

$$|\Psi_{2\text{seq}}\rangle = \sum_{k,\eta} c_n|\hat{n}_k\rangle \rightarrow |\hat{\text{Planck}}\rangle$$

occurred is unclear. This makes the early universe a perfect arena to test CSL.

The leading paradigm to describe this epoch is cosmic inflation [18–22], which was introduced in order to solve the puzzles of the standard hot big-bang phase. Inflation is believed to have been driven by a scalar field $\phi$, named the “inflaton”, the physical nature of which is still unknown although detailed constraints on the shape of its potential now exist [23–31]. Inflation also provides a convincing mechanism for structure formation according to which galaxies and CMB anisotropies are nothing but quantum vacuum fluctuations amplified by gravitational instability and stretched to astrophysical scales [32]. This mechanism fits very well the high-accuracy astrophysical data now at our disposal, in particular the CMB temperature and polarisation anisotropies [33, 34].

The universe is well described by a flat, homogeneous and isotropic metric of the Friedmann-Lemaître-Robertson-Walker (FLRW) type, $\mathrm{d}s^2 = -a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$, where $x^i$ is the comoving spatial coordinate, $t$ refers to cosmic time, and $a(t)$ is the scale factor which depends on time only. During inflation, the expansion is accelerated, $\ddot{a} > 0$, and the Hubble parameter $H = \dot{a}/a$ (where a dot denotes derivation with respect to time) is almost constant, see Fig. 1.

To describe the small quantum fluctuations living on top of this FLRW background, the metric and inflaton fields are expanded according to $g_{\mu\nu} = g_{\mu\nu}^{\text{FLRW}}(t) + \delta g_{\mu\nu}(t, x)$ and $\phi = \phi^{\text{FLRW}}(t) + \delta \phi(t, x)$ with $|\delta g_{\mu\nu}/g_{\mu\nu}^{\text{FLRW}}| \ll 1$ and $|\delta \phi/g^{\text{FLRW}}| \ll 1$. This gives rise to two types of perturbations, scalars and tensors. Tensors correspond to primordial gravitational waves and have not yet been detected, the tensor-to-scalar ratio $r$ being $r \lesssim 0.064$ [34]. Then, scalar perturbations can be described with a single gauge-invariant degree of freedom, the so-called curvature perturbation $\zeta(t, x)$ [32, 35], which can be directly related to temperature anisotropies. Expanding the action of the system (namely the Einstein-Hilbert action plus the action of a scalar field) up to second order in the perturbations leads to the Hamiltonian of the perturbations, $H = \int_{\mathbb{R}^4} \mathrm{d}^4k \left[ \hat{p}_k^2 + \omega^2(k, \eta) \hat{v}_k^2 \right]$, where $\hat{v}_k \equiv \hat{z}_k$ is the Mukhanov-Sasaki variable. One has introduced $z \equiv \sqrt{2\pi} M_\text{Pl}/c_s$, where $c_s$ is the speed of sound ($c_s = 1$ for a scalar field) and $\epsilon_1 \equiv -H/\dot{H}$ is the first Hubble-flow parameter [36, 37]. In the above expressions, the curvature perturbation has been Fourier transformed, $\zeta(n, x) = (2\pi)^{-3/2} \int \mathrm{d}^3k \, \hat{z}_k(n) e^{ik\cdot x}$, as appropriate for a linear theory where the modes evolve independently. The conjugate momentum is $\hat{p}_k \equiv \hat{v}_k^*$, where a prime denotes derivation with respect to the conformal time $\eta$ defined via $\mathrm{d}t = a\mathrm{d}\eta$. Each mode behaves as a parametric oscillator, $\hat{v}_k'' + \omega^2(k, \eta) \hat{v}_k = 0$, with a time-dependent frequency \( \omega^2(k, \eta) = \hat{c}_s^2 k^2 - z''/z \) that involves the background dynamics. This phenomenon, described by the interaction between a quantum field (here the cosmological perturbations) and a time-dependent classical source (here the background spacetime), leads to parametric amplification and can be found in many other branches of Physics (e.g. the Schwinger effect [38], the dynamical Casimir effect [39], Unruh [40] and Hawking [41] effects, etc.).

Quantisation of parametric oscillators yields squeezed states, which are Gaussian states. Solving the Schrödinger equation with the above Hamiltonian leads to $\hat{\Psi}[v] = \prod_{k,\eta} \mathcal{W}_k(v_k \hat{v}_k)$, where $s = \text{R, I}$ labels the real and imaginary parts of $v_k$, with $\mathcal{W}_k(v_k \hat{v}_k) = N_k e^{-\Omega_k(v_k)^2}$, $|N_k| = (2\Re \Omega_k/\pi)^{1/4}$ and $\Omega_k$ obeying the equation $\Omega_k'' = -2\Omega_k^2 + i\omega^2(k, \eta)/2$. In the standard approach, $\langle \dot{\hat{v}}_k \rangle = 0$ and one needs to assume the existence of a specific process (3) that led to a particular realisation corresponding to our universe (this is the macro-objectification problem mentioned above). The dispersion of the different realisations is characterised by the two-point correlation function $\langle \zeta_k^2 \rangle = \int P_\zeta \, \mathrm{d} \ln k$ where $P_\zeta = k^3 |\zeta_k|^2/(2\pi^2)$ is the power spectrum, which is pre-
dicted to be of the form $A_k k^{n_s-1}$ where $n_s$ should be close to one. The recent Planck data (identifying spatial and ensemble averages) have confirmed this result with $\ln(10^{10} A_s) = 3.044\pm0.014$ and $n_s = 0.9649\pm0.0042$ [34].

If quantum theory is described by CSL rather than by the standard framework, the behaviour of the cosmological perturbations is modified according to Eq. (1). In that case, the mass density is given by $\rho = \tilde{\rho} + \delta \rho$, where $\tilde{\rho}$ is the homogeneous component of the energy density satisfying the Friedmann equation $\ddot{\rho} = 3 M_p^2 H^2$, $M_p$ is the reduced Planck mass, and $\delta \rho$ the density fluctuation.

In General Relativity (GR) however, there is no unique definition of the density contrast $\delta \rho/\tilde{\rho}$. While all possible choices coincide on sub-Hubble scales where observations are performed, they can differ on super-Hubble scales. This introduces a fundamental ambiguity when defining CSL in cosmology: each choice for the density contrast leads to a different CSL theory. A physically well-motivated choice, which we adopt in this paper, consists in measuring the energy density relative to the hypersurface which is as close as possible to a “Newtonian” time slicing (denoted $\delta \rho$ in Ref. [42]). This leads to $\delta \rho/\tilde{\rho} = \epsilon_1 (1 + \epsilon_1 a^2 H^2 \partial^2 / 3a H)$ if the universe is dominated by a scalar field. Our aim is certainly not to argue in favour of that specific choice but rather to illustrate that astrophysical data are already accurate enough to constrain some well-justified CSL theories. This, however, does not preclude the existence of other density contrasts (e.g. $\delta \rho$ of Ref. [12] or a GR-generalised version of the proposal made in Ref. [43]) for which the corresponding CSL theory remains compatible.

From the previous considerations, Eq. (1) can be written in Fourier space as a set of independent CSL equations for the real and imaginary parts of each Fourier mode, in which the smeared mass density operator reads $\delta \rho_{\text{sm}} (k) = \alpha_k \hat{v}_k^s + \beta_k \hat{\rho}_k^s$ with

$$\alpha_k \equiv \frac{M_p^2 H^2 \epsilon_1}{a} e^{-\frac{s^2 z}{2 \alpha^2}} \left( 4 + \epsilon_2 - 3 \left( \frac{a H}{k} \right)^2 \epsilon_1 (1 + \epsilon_2) \right),$$

and

$$\beta_k \equiv \frac{M_p^2 H \epsilon_1}{a z} e^{-\frac{s^2 z}{2 \alpha^2}} \left( 3 \epsilon_1 \left( \frac{a H}{k} \right)^2 - 1 \right),$$

where $\epsilon_2 \equiv d \ln \epsilon_1 / d \ln a$ denotes the second Hubble-flow parameter. Due to the presence of the exponential term, the effect of the CSL terms is triggered only once the mode $k$ under consideration crosses out the scale $r_c$, i.e. when its physical wavelength is larger than $r_c$, $k/a < r_c^{-1}$. Depending on the value of $r_c$, this can happen either during inflation or subsequently, see Fig. 1 (cases labeled $r_c$ and $r'_c$, respectively). Physically, it is clear that the CSL terms cannot “localize” a mode if its “size” (its wavelength) is smaller than the localization scale $r_c$. This also means that, at early time, when $k/a < r_c^{-1}$, the standard theory applies, which implies that one of the great advantages of inflation, namely the possibility to choose well-defined initial conditions in the Minkowski limit (the so-called Bunch-Davies vacuum state [44]), is preserved.

We are now in a position to solve Eq. (1). The most general stochastic Gaussian wavefunction can be written as

$$\Psi_k (\hat{v}_k) = |N_k (\eta)| \exp \left\{ - \Re \Omega_k (\eta) [\hat{v}_k - \bar{v}_k (\eta)]^2 + i \sigma_k (\eta) \hat{v}_k - i 3 \Im \Omega_k (\eta) (\hat{v}_k^*)^2 \right\},$$

(6)

where the free functions $\Omega_k, \sigma_k, \sigma_k^* \text{ and } \Gamma_0$ are (a priori) stochastic quantities. This wavepacket is centred around $\langle \hat{v}_k \rangle = \bar{v}_k$ with a variance $\langle (\hat{v}_k^2 - \bar{v}_k^2)^2 \rangle = (4 \Re \Omega_k)^{-1}$. The collapse of the wavefunction happens if the width of $\Psi (\hat{v}_k)$ is much smaller than the typical dispersion of its mean, i.e.

$$R \equiv \frac{\mathbb{E} \left[ (\hat{v}_k^2 - \bar{v}_k^2)^2 \right]}{\mathbb{E} (\hat{v}_k^2)} \ll 1$$

(7)

where $\mathbb{E}$ denotes the stochastic average. In fact, if the collapse occurs according to the Born rule, then $\mathbb{E} \left[ (\hat{v}_k^2)^2 \right] = \langle \hat{v}_k^2 \rangle_{\gamma=0} = (4 \Re \Omega_k |_{\gamma=0})^{-1}$, and $R$ can also be defined as $R = \mathbb{E} \left[ \left( \langle \hat{v}_k^2 \rangle - \langle \hat{v}_k \rangle^2 \right) / \langle \hat{v}_k^2 \rangle_{\gamma=0} \right]$. When the wavefunction has collapsed, its realisations are described by $\hat{v}_k^s$. The power spectrum of the Mukhanov-Sasaki variable (or of curvature perturbation) is thus given by the dispersion of that quantity,

$$P_v (k) = \frac{k^3}{2 \pi^2} \left\{ \mathbb{E} (\hat{v}_k^2) - \mathbb{E} (\hat{v}_k^2) \right\}.$$ (8)

The above quantity can also be rewritten as $P_v (k) = k^3 \mathbb{E} \mathbb{E} (\hat{v}_k^2) - \mathbb{E} \left[ (\hat{v}_k^2 - \hat{v}_k^s)^2 \right] / (2 \pi^2)$. In order to calculate the quantities (7) and (8), one can insert the stochastic wavefunction (6) into Eq. (1) and solve the obtained stochastic differential equations. One obtains that $\Omega_k$ decouples from the other free functions and obeys $\dot{\Omega}_k = 4 \gamma a^4 \alpha_k \beta_k \Omega_k / m_0^2 - 2 (i + 2 \gamma a^4 \beta_k^2 / m_0^2) \Omega_k^2 + \gamma a^4 \alpha_k^2 \beta_k^2 / m_0^2 + i \omega^2 (k, \eta) / 2$. This equation is non-stochastic, as in the standard case, but contains new terms proportional to $\gamma$. Since it is non-stochastic, $\mathbb{E} \left[ (\hat{v}_k^2 - \bar{v}_k^2)^2 \right] = (4 \Re \Omega_k)$ and this implies that $R = \Re \Omega_k |_{\gamma=0} / \Re \Omega_k$.

In order to obtain the spectrum (8), $\mathbb{E} \left[ (\hat{v}_k^2)^2 \right]$ remains to be determined. This is done by noticing that Eq. (1) can be cast into a Lindblad equation [45] for the averaged density matrix $\hat{\rho} = \mathbb{E} (|\Psi \rangle \langle \Psi |)$. From this Lindblad equation, one can derive a third-order differential equation for $\mathbb{E} \left[ (\hat{v}_k^2) \right]$ that can be solved exactly. Combining the above mentioned results, one obtains

$$P_v (k) \simeq \frac{k^3}{2 \pi^2} \frac{1}{4 \Re \Omega_k |_{\gamma=0}} \left[ 1 + \frac{3 \gamma}{2 m_0^2} \epsilon_1 \bar{\rho}_{\text{int}} \left( \frac{k}{a H} \right)^{-1} \right.$$  

$$\left. - \frac{\Re \Omega_k |_{\gamma=0}}{\Re \Omega_k} \right],$$

(9)
where $\hat{\rho}_{\text{inf}} = 3H^2_{\text{inf}}M^2_{\text{pl}}$ is the energy density during inflation. Depending on the value of $\gamma$, different results can be obtained, that are sketched in Fig. 2. If $\gamma = 0$, the state remains homogeneous and isotropic, and the spectrum vanishes. Then, when $\gamma$ increases above a certain threshold, collapse occurs ($R \ll 1$) so the third term in Eq. (9) can be neglected. Provided the second term remains also negligible, the Born rule is thus recovered, and a scale-invariant power spectrum is obtained, in agreement with observations. Finally, when $\gamma$ continues to increase so as to make the second term large, the power spectrum is no longer frozen on large scales and acquires a spectral index $n_0 = 0$, which is excluded by CMB observations.

The amplitude of the correction to the power spectrum is proportional to the energy density during inflation measured in units of the reference mass, which is clearly huge and illustrates the potential of cosmology to test the quantum theory, given that its characteristic scales differ by orders of magnitude from those in the lab. The correction is also slow-roll suppressed because of the relation between $\delta\rho/\rho$ and $\zeta$ [since only the perturbations are quantized, the classical part $\hat{\rho}$ cancels out in Eq. (1)]. This suppression, however, is not sufficient to compensate for the hugeness of $\hat{\rho}_{\text{inf}}$. 

In the standard situation, since the power spectrum of $\zeta$ is frozen on large scales, its value at the end of inflation is what we observe on the CMB last scattering surface and the calculation can be stopped here. In the CSL theory however, this may no longer be true, hence one needs to extend the present analysis to the radiation era that follows inflation. During this epoch, the quantities $\alpha_k$ and $\beta_k$ read

$$\alpha_k \equiv \frac{24M^2_{\text{pl}}H^2}{z} e^{-\frac{\kappa^2 z^2}{2\pi^2}} \left[ 3 \left( \frac{aH}{k} \right)^2 - 1 \right], \quad (10)$$

$$\beta_k \equiv \frac{12M^2_{\text{pl}}H^2}{a z} e^{-\frac{\kappa^2 z^2}{2\pi^2}} \left[ 1 - 6 \left( \frac{aH}{k} \right)^2 \right]. \quad (11)$$

The power spectrum of the Mukhanov-Sasaki variable can then be determined using the same techniques as before, and one obtains

$$P_v(k) = \frac{k^3}{2\pi^2} \frac{1}{4\Re \Omega_k |_{\gamma = 0}} \left[ 1 - \frac{448}{3} \frac{\gamma}{m_{\text{pl}}^2} \frac{\hat{\rho}_{\text{end}} \epsilon_1}{aH_{\text{end}}} \left( \frac{k}{aH_{\text{end}}} \right)^{-1} - \frac{\Re \Omega_k |_{\gamma = 0}}{\Re \Omega_k} \right], \quad (12)$$

where $\hat{\rho}_{\text{end}}$ is the energy density at the end of inflation. Comparing with Eq. (9), one can see that the power spectrum indeed evolves during the transition between inflation and the radiation era, but quickly settles to a constant value, which is therefore the power spectrum probed by CMB experiments. The CSL terms introduce a correction with a spectral index $n_0 = 0$. One can also determine the collapse criterion $R = 1152\gamma \hat{\rho}_{\text{end}}(-k\eta_{\text{end}})^{-7}/m_{\text{pl}}^2$.

So far, we have assumed that the scale $r_c$ was crossed
out during inflation. Let us now examine the situation where \( r_c \) is crossed out during the radiation era. In that case, prior to crossing and in particular during the entire inflationary phase, the standard results remain valid. After crossing, the CSL terms become important and, using again the same techniques, one obtains

\[
P_v(k) = \frac{k^3}{2 \pi^2} \frac{1}{\Delta \text{Re} \Omega_k} \left[ 1 + \frac{35408}{429 \rho_{\text{end}}} \right] \left( \frac{r_c}{H_\text{end}} \right)^{-9} \left( \frac{k}{a H_\text{end}} \right)^{-10} \left( \frac{\text{Re} \Omega_k}{\text{Re} \Omega_k} \right)_{\gamma=0} \left[ 1 + \frac{35408}{429 \rho_{\text{end}}} \right]^{-1} \left( \frac{k}{a H_\text{end}} \right)^{-10} \left( \frac{\text{Re} \Omega_k}{\text{Re} \Omega_k} \right)_{\gamma=0}.
\]

As before, the spectrum is frozen out on super-Hubble scales, but the CSL correction now has spectral index \( n_S = -9 \). The collapse criterion is given by \( R = 72644/(11 m_0^2) \rho_{\text{end}} (k \eta_{\text{end}})^{-14} (H_{\text{end}} r_c)^{-7} \).

Since the CSL corrections are strongly scale dependent, they are ruled out by CMB measurements. Therefore, that version of CSL is now ruled out. As stressed above, other choices for the density contrast could be made, and it is clear that one can find some for which the CSL model remains valid. This implies that when derived from a more fundamental theory, the CSL model should come with a prescription for the density contrast, that crucially conditions the cosmological constraints. Our results illustrate that some a priori well-motivated prescriptions are already impossible.

Further subtleties could also arise if the CSL model was formulated in a field-theoretic manner [4, 46, 47] (which is in principle required in the present context – although at linear order all Fourier modes decouple and can be treated quantum-mechanically), where parameter values may e.g. run with the energy scale at which the experiment is performed. Other approaches, e.g. Diósi-Penrose model [3, 48] where gravity is responsible for the collapse or scenarios where dissipative effects are taken into account [49], could also lead to different results.

Despite these uncertainties, the fact that astrophysical data can constrain CSL highlights the usefulness of early universe observations to discuss foundational issues in quantum mechanics.

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SUPPLEMENTARY MATERIAL

THE CSL MASTER EQUATIONS

The CSL equation is given by (see, for instance, Eq. (4) of Ref. [49])

\[
|d\hat{\Psi}(x_p,t)\rangle = \left\{-i\hat{H}dt + \frac{\sqrt{\gamma}}{m_0} \int dx_p \left[ \hat{C}(x_p) - \langle \hat{C}(x_p) \rangle \right] dW_t(x_p) \right.
\]

\[
- \frac{\gamma}{2m_0^2} \int dx_p \left[ \hat{C}(x_p) - \langle \hat{C}(x_p) \rangle \right]^2 dt \right\} |\Psi(x_p,t)\rangle ,
\]

where \( \gamma \) is a free parameter, \( m_0 \) a reference mass (usually the mass of a nucleon), \( \hat{H} \) the Hamiltonian of the system, \( \hat{C} \) the collapse operator and \( W_t(x_p) \) is an ensemble of independent Wiener processes satisfying \( E[dW_t(x_p)dW_t(x'_p)] = \delta(x_p - x'_p)\delta(t - t')dt^2 \). This equation is written in physical coordinates \( x_p \). However, in cosmology, it is more convenient to work in terms of comoving coordinates defined by \( x_p = ax \), where \( a \) is the time-dependent scale factor and describes how the size of the universe evolves with time. Comoving coordinates are coordinates for which the motion related to the expansion of the universe is subtracted out. In terms of these coordinates, the CSL equation reads

\[
|d\hat{\Psi}(x,t)\rangle = \left\{-i\hat{H}dt + \frac{1}{m_0} \sqrt{\gamma} \int dx a^3 \left[ \hat{C}(x) - \langle \hat{C}(x) \rangle \right] dW_t(x) \right.
\]

\[
- \frac{\gamma}{2m_0^2} \int dx a^3 \left[ \hat{C}(x) - \langle \hat{C}(x) \rangle \right]^2 dt \right\} |\Psi(x,t)\rangle ,
\]

with \( dW_t(x_p) = a^{-3/2}dW_t(x) \) and \( E[dW_t(x)dW_t(x')] = \delta(x - x')\delta(t - t')dt^2 \), this last result coming from the fact that \( E[dW_t(x_p)dW_t(x'_p)] = \delta(ax - ax')\delta(t - t')dt^2 = a^{-3}\delta(x - x')\delta(t - t')dt^2 \).

In the CSL theory, the collapse operator is taken to be the energy density. Moreover, in cosmological perturbations theory, one writes \( \hat{\rho} = \rho + \delta\rho \), where \( \rho \) is the background energy density, and only the fluctuating part is quantised. As a consequence, the classical background part does not contribute to the CSL equation since \( \hat{C}(x) - \langle \hat{C}(x) \rangle = \hat{\rho} + \delta\rho - \langle \hat{\rho} + \delta\rho \rangle = \delta\rho - \langle \delta\rho \rangle \). The collapse operator also needs to be coarse-grained over the distance \( r_c \), where \( r_c \) is the other free parameter in the model. One therefore introduces the Gaussian coarse-graining procedure

\[
f_{cg}(x) = \left( \frac{a}{r_c} \right)^3 \frac{1}{(2\pi)^{3/2}} \int dy f(x + y) e^{-\frac{|x - y|^2}{2r_c^2}} .
\]

This implies that the collapse operator used in the CSL equation reads

\[
\hat{C}(x) = \hat{\rho} \left|_{cg} \right. (x) = 3M_p^2 \frac{\mathcal{H}^2}{a^2} \left. \frac{\delta\rho}{\rho} \right|_{cg} (x) ,
\]

where we have used the Friedmann equation relating \( \mathcal{H} = a'/a \) to \( \hat{\rho} \).

In cosmology, perturbation theory is usually formulated in Fourier space. In the CSL context, this leads to one CSL equation for each mode, namely

\[
|d\hat{\Psi}_k(t)\rangle = \left\{-i\hat{H}_kdt + \frac{\sqrt{\gamma a^3}}{m_0} \left[ \hat{C}^s(k) - \langle \hat{C}^s(k) \rangle \right] dW_t^s(k) \right.
\]

\[
- \frac{\gamma a^3}{2m_0^2} \left[ \hat{C}^s(k) - \langle \hat{C}^s(k) \rangle \right]^2 dt \right\} |\Psi_k(t)\rangle ,
\]

the index \( s \) designating the real and imaginary parts, \( s = R, I \). The correlation functions of the noise in Fourier space are given by

\[
E[dW_t^R(k) dW_t^R(k')] = E[dW_t^I(k) dW_t^I(k')] = \delta(k - k')\delta(t - t')dt^2 , \quad E[dW_t^R(k) dW_t^I(k')] = 0 ,
\]

and the Fourier transform of the collapse operator reads

\[
\hat{C}(k) = 3M_p^2 \frac{\mathcal{H}^2}{a^2} e^{-\frac{k^2}{2r_c^2}} \left. \frac{\delta\rho}{\rho} \right| (k) .
\]
The CSL equation can also be cast into a Lindblad equation, see for instance Eq. (21) of Ref. [49], which takes the form
\[
\frac{d\hat{\rho}}{dt} = -i \left[ \hat{H}, \hat{\rho} \right] - \frac{\gamma}{2m_0} \int dx a^3 \left[ \hat{C}(x), \left[ \hat{C}(x), \hat{\rho} \right] \right]
\]  
for the mean density matrix $\hat{\rho} = \mathbb{E}(|\Psi\rangle\langle\Psi|)$. In Fourier space, this gives rise to one equation per Fourier mode, which can be written as
\[
\frac{d\hat{\rho}_k}{dt} = -i \left[ \hat{H}_k, \hat{\rho}_k \right] - \frac{\gamma}{2m_0} a^3 \left[ \hat{C}^s(k), \left[ \hat{C}^s(k), \hat{\rho}_k \right] \right].
\]

**SOLVING THE LINDBLAD EQUATION**

The stochastic mean of the quantum expectation value of some observable $\hat{O}_k^s$ is given by $\mathbb{E}\left( \langle \hat{O}_k^s \rangle \right) = \text{Tr} \left( \hat{\rho}_k^s \hat{O}_k^s \right)$, where $\hat{\rho}_k^s$ obeys Eq. (22). Differentiating this expression with respect to conformal time (we recall that conformal time $\eta$ is related to cosmic time $t$ by $d\eta = \omega dt$) and making use of Eq. (22), one obtains
\[
\frac{d}{d\eta} \mathbb{E}\left( \langle \hat{O}_k^s \rangle \right) = \mathbb{E}\left( \langle \frac{\partial}{\partial \eta} \hat{O}_k^s \rangle \right) - i \left[ \langle \hat{O}_k^s, \hat{H}_k^s \rangle \right] - \frac{\gamma a^4}{2m_0} \left[ \langle \hat{O}_k^s, \hat{C}_k^s \rangle, \hat{C}_k^s \right].
\]

For one-point correlators, $\hat{O}_k^s = v_k^s$ and $\hat{O}_k^s = p_k^s$, this gives rise to
\[
\frac{d}{d\eta} \langle \hat{v}_k^s \rangle = \langle \hat{p}_k^s \rangle, \quad \frac{d}{d\eta} \langle \hat{p}_k^s \rangle = -\omega^2(k, \eta) \langle \hat{v}_k^s \rangle,
\]
which is nothing but the Ehrenfest theorem. For two-point correlators, denoting $P_{vv}(k) = \langle \hat{v}_k^s \hat{v}_k^s \rangle$, $P_{pp}(k) = \langle \hat{p}_k^s \hat{p}_k^s \rangle$, $P_{vp}(k) = \langle \hat{v}_k^s \hat{p}_k^s \rangle$, and $P_{vp}(k) = \langle \hat{p}_k^s \hat{v}_k^s \rangle$, one obtains
\[
\frac{dP_{vv}(k)}{d\eta} = P_{vp}(k) + P_{vp}(k) + \frac{\gamma}{m_0} a^4 \beta_k^2,
\]
\[
\frac{dP_{pp}(k)}{d\eta} = 2P_{pp}(k) - 2\omega^2(k, \eta)P_{vv}(k) - 2a^4 \frac{\gamma}{m_0} \alpha_k \beta_k,
\]
\[
\frac{dP_{vp}(k)}{d\eta} = -\omega^2(k, \eta) \left[ P_{vp}(k) + P_{vp}(k) \right] + a^4 \frac{\gamma}{m_0} \alpha_k^2
\]
where the coefficients $\alpha_k$ and $\beta_k$ have been defined in the main text, see Eqs. (4)-(5) and (10)-(11). These equations can be combined into a single third-order equation for $P_{vv}$ only, which reads
\[
\frac{d^3P_{vv}}{d\eta^3} + 4\omega^2(k, \eta) \frac{dP_{vv}}{d\eta} + 4\omega \frac{d\omega}{d\eta} P_{vv} = S,
\]
where $S$ is the source function given by
\[
S = \frac{\gamma}{m_0} \left[ 2a^4 (\alpha_k^2 + \omega^2 \beta_k^2) - 2 \left( a^4 \alpha_k \beta_k \right) + (a^4 \beta_k)^2 \right].
\]
As we will show below, this source function encodes both the modifications to the power spectrum and the collapsing time. Let us note that it is invariant under phase-space canonical transforms, so the results derived hereafter would be the same if other canonical variables than $v_k$ and $p_k$ were used.

As shown in Ref. [50], Eq. (28) can be solved by introducing the Green function of the free theory,
\[
G(\eta, \bar{\eta}) = \frac{1}{W} \left[ g_k^{0*}(\eta)g_k^{0}(\eta) - g_k^{0*}(\eta)g_k^{0*}(\eta) \right] \Theta(\eta - \bar{\eta}),
\]
where $g_k^0$ is a solution of the Mukhanov-Sasaki equation, $(g_k^0)^{\prime\prime} + \omega^2(k, \eta)g_k^0 = 0$, and where $W = g_k^{0*}g_k^{0*} - g_k^{0}g_k^{0*}$ is its Wronskian. By construction, given the mode equation obeyed by $g_k^0$, it is a constant. Then, the solution to Eq. (28) reads
\[
P_{vv}(k) = g_k^0(\eta)g_k^{0*}(\eta) + \frac{1}{2} \int_{-\infty}^{\eta} S(\bar{\eta}) G^2(\eta, \bar{\eta}) d\bar{\eta}.
\]
Inflation

During inflation $a \simeq -1/(H\eta)$, and at leading order in the Hubble-flow parameters, Eq. (29) gives rise to

$$S_{\text{inf}} \simeq \frac{\gamma}{m_0^2} \epsilon_1 H^2 M_{\text{Pl}}^2 k^2 e^{-(r_c/\lambda)^2} \left( \frac{\ell_H}{\lambda} \right)^6 \left[ 126 \epsilon_1^2 - 75 \ell_1 \left( \frac{\ell_H}{\lambda} \right)^2 + 81 \ell_1^2 \left( \frac{\ell_H}{\lambda} \right)^2 + 18 \left( \frac{\ell_H}{\lambda} \right)^4 - 48 \epsilon_1 \left( \frac{\ell_H}{\lambda} \right)^2 \left( \frac{r_c}{\lambda} \right)^2 \right] + 18 \epsilon_1^2 \left( \frac{r_c}{\lambda} \right)^4 + \left( \frac{\ell_H}{\lambda} \right)^6 + 7 \left( \frac{\ell_H}{\lambda} \right)^4 \left( \frac{r_c}{\lambda} \right)^2 - 12 \left( \frac{\ell_H}{\lambda} \right)^2 \left( \frac{r_c}{\lambda} \right)^4 + 2 \left( \frac{\ell_H}{\lambda} \right)^4 \left( \frac{r_c}{\lambda} \right)^4 \right], \quad (32)$$

where $\ell_H = H^{-1}$ is the Hubble radius and $\lambda = \alpha(\eta)/k$ the wavelength of the Fourier mode with comoving wavenumber $k$. The quantity $\ell_1/\lambda$ can also be written as $\ell_H/\lambda = k/(aH) = -k\eta$. We see that the amplitude of the source is controlled by the energy density during inflation, $\rho_{\text{inf}} = 3H^2 M_{\text{Pl}}^2$, and by the first Hubble-flow parameter $\epsilon_1$ (at next-to-leading order in slow roll, higher-order Hubble flow parameters would appear). The limits we are interested in are $\ell_H/\lambda \ll 1$ (super Hubble limit) and $r_c/\lambda \ll 1$ (otherwise the exponential term turns the source off, see the discussion in the main text). In this regime, the dominant term is the first one, proportional to $126 \epsilon_1^2$ (although it is slow-roll suppressed).

Normalising the mode function in the Bunch-Davies vacuum, at leading order in slow roll one has

$$g_k(\eta) = e^{ik\eta} \sqrt{2k} \left( 1 + \frac{i}{k\eta} \right), \quad (33)$$

from which Eq. (30) gives

$$G_{\text{inf}}(\eta, \bar{\eta}) = \frac{(1 + k^2 \eta'^2)}{(k^2 \eta'^2)} \sin [k(\eta - \eta')] - k(\eta - \eta') \cos [k(\eta - \eta')] \Theta(\eta - \bar{\eta}) \approx \frac{\eta^3 - \bar{\eta}^3}{3\eta \bar{\eta}} \Theta(\eta - \bar{\eta}). \quad (34)$$

The second expression is valid in the super-Hubble limits $-k\eta \to 0$ (since the power spectrum is computed on super-Hubble scales) and $-k\eta' \to 0$ (since we assume $Hr_c \gg 1$, so any mode is super Hubble when it crosses out $r_c$). Plugging Eqs. (32) and (34) into Eq. (31), one obtains at leading order

$$P_{\text{ev}}(k) \simeq |v_k|^2_{\text{standard}} + \frac{9\gamma}{2m_0^2 k} H^2 M_{\text{Pl}}^2 \epsilon_1^2 \left( \frac{k}{aH} \right)^{-3} = |v_k|^2_{\text{standard}} \left[ 1 + \frac{9\gamma}{2m_0^2} H^2 M_{\text{Pl}}^2 \epsilon_1^2 \left( \frac{k}{aH} \right)^{-1} \right], \quad (35)$$

where $|v_k|^2_{\text{standard}} = |g_k|^2$, which is the result used in the main text.

Radiation-dominated epoch

Let us now study what happens during the radiation dominated era. In that case the scale factor is given by $a(\eta) = a_0 (\eta - \eta_r)$ and, as a consequence, $H(\eta) = a'/a = (\eta - \eta_r)^{-1}$. Requiring the scale factor and its derivative (or, equivalently, the Hubble parameter) to be continuous, which is equivalent to the continuity of the first and second fundamental forms, gives $\eta_r = 2\eta_{\text{end}}$ and $a_0 = 1/(H_{\text{end}}\eta_{\text{end}}^2)$.

Using the coefficients $\alpha_k$ and $\beta_k$ given in Eqs. (10) and (11), the source function (29) reads

$$S_{\text{rad}} = \frac{\gamma}{m_0^2} H^2_{\text{end}} M_{\text{Pl}}^2 k^2 e^{-(r_c/\lambda)^2} \left( \frac{a_{\text{end}}}{a} \right)^4 \left( \frac{\ell_H}{\lambda} \right)^6 \left[ 3024 - 414 \left( \frac{\ell_H}{\lambda} \right)^2 + \left( \frac{\ell_H}{\lambda} \right)^6 - 1836 \left( \frac{a_{\text{end}}}{a} \right)^2 \left( \frac{r_c}{\lambda} \right)^2 \right] + 216 \left( \frac{a_{\text{end}}}{a} \right)^4 \left( \frac{r_c}{\lambda} \right)^4_{\text{end}} - 72 \left( \frac{a_{\text{end}}}{a} \right)^4 \left( \frac{\ell_H}{\lambda} \right)^2 \left( \frac{r_c}{\lambda} \right)^2_{\text{end}} + 432 \left( \frac{a_{\text{end}}}{a} \right)^2 \left( \frac{\ell_H}{\lambda} \right)^4 \left( \frac{r_c}{\lambda} \right)^2_{\text{end}} + 6 \left( \frac{a_{\text{end}}}{a} \right)^2 \left( \frac{\ell_H}{\lambda} \right)^2 \left( \frac{r_c}{\lambda} \right)^4_{\text{end}} - 21 \left( \frac{a_{\text{end}}}{a} \right)^2 \left( \frac{\ell_H}{\lambda} \right)^4 \left( \frac{r_c}{\lambda} \right)^2_{\text{end}} \right]. \quad (36)$$

Its form is similar to that of the source during inflation, see Eq. (32), although the amplitude is now proportional to the energy density at the end of inflation, $\rho_{\text{end}} = 3H^2_{\text{end}} M_{\text{Pl}}^2$, and is no longer slow-roll suppressed as is expected in the radiation-dominated era. The coefficients of the expansion depend on $(r_c/\lambda)_{\text{end}}$, the ratio between the CSL scale and the mode wavelength evaluated at the end of inflation. This dependence on quantities evaluated at the end of inflation comes from the matching procedure.
At the perturbative level, the Mukhanov-Sasaki variable now obeys $g_{kk}'' + \left( c_{g}^{2}k^{2} - \frac{z''}{z} \right) g_{k} = 0$ with $c_{g}^{2} = 1/3$ and $z = aM_{Pl}^{2}/c_{s} = 2\sqrt{3a}M_{Pl}$. The solution reads

$$g_{k}^{0}(\eta) = A_{k}e^{-ik\frac{\eta - \eta_{end}}{c_{s}}} + B_{k}e^{ik\frac{\eta - \eta_{end}}{c_{s}}}.$$  \hfill (37)

On super-Hubble scales, continuity of the first and second fundamental forms is equivalent to the continuity of $\zeta$ and $\zeta'$. This implies

$$A_{k} = \frac{e^{ik\eta_{end}(1-\frac{\eta}{\lambda})}}{2\sqrt{2k}} \left[ 1 + \frac{i}{k\eta_{end}} - \sqrt{3} - \frac{i\sqrt{3}}{k\eta_{end}} + \sqrt{3} \left( \frac{k\eta_{end}}{k_{\eta_{end}}} \right)^{2} \right] \sqrt{6} \epsilon_{1}$$ \hfill (38)

$$B_{k} = \frac{e^{ik\eta_{end}(1+\frac{\eta}{\lambda})}}{2\sqrt{2k}} \left[ 1 + \frac{i}{k\eta_{end}} + \sqrt{3} + \frac{i\sqrt{3}}{k\eta_{end}} - \sqrt{3} \left( \frac{k\eta_{end}}{k_{\eta_{end}}} \right)^{2} \right] \sqrt{6} \epsilon_{1}.$$ \hfill (39)

As a consequence, during the radiation-dominated era, the Mukhanov-Sasaki variable takes the following form (at leading order in the super-Hubble limit)

$$g_{k}^{0}(\eta) = \frac{i\sqrt{3}}{\sqrt{k\epsilon_{1}(k_{\eta_{end}})^{2}}} \left\{ (k\eta_{end}) \cos \left[ \frac{k}{\sqrt{3}} (\eta - \eta_{end}) \right] - \sqrt{3} \sin \left[ \frac{k}{\sqrt{3}} (\eta - \eta_{end}) \right] \right\}.$$ \hfill (40)

Plugging this expression into Eq. (30), one obtains

$$G_{\text{rad}}(\eta, \bar{\eta}) = \frac{\sqrt{3}}{k} \sin \left[ \frac{k}{\sqrt{3}} (\eta - \bar{\eta}) \right] \Theta (\eta - \bar{\eta}) \simeq (\eta - \bar{\eta}) \Theta (\eta - \bar{\eta}).$$ \hfill (41)

At this stage, one must distinguish between two situations: either the Fourier mode under consideration crosses out the scale $r_{c}$ during inflation or during the radiation-dominated era.

**Case where the mode crosses out $r_{c}$ during inflation**

In the standard situation, the power spectrum of $\zeta$ computed at the end of inflation is frozen on super Hubble scales and can be directly propagated to the last scattering surface. Here, however, a priori, the power spectrum continues to evolve during the radiation-dominated era even on large scales.

The integral appearing in Eq. (31) can be split in two parts: one for which $-\infty < \bar{\eta} < \eta_{end}$, which was already calculated above during inflation, and one for which $\eta_{end} < \bar{\eta} < \eta$ that we now calculate. If the scale $r_{c}$ is crossed out during inflation, then $(r_{c}/\lambda)_{end} \ll 1$ and all the terms in the source but the one proportional to 3024 can be ignored. At leading order in $\ell_{H}/\lambda_{end}$ and $r_{c}/\lambda_{end}$, one obtains that, after a few $e$-folds, the power spectrum freezes to

$$P_{\text{rad}}(k) = |v_{k}|^{2} \text{standard} \left[ 1 + 448 \frac{\gamma}{m_{0}^{2}} H_{end}^{2} M_{Pl}^{2} \left( \frac{k}{aH} \right)_{end}^{-1} \right].$$ \hfill (42)

**Case where the mode crosses out $r_{c}$ during the radiation-dominated era**

The mode crosses out $r_{c}$ when $a_{\text{cross}} = kr_{c}$, i.e. at $\eta_{\text{cross}} = \eta_{c} + k_{\text{cross}}^{2} H_{end} r_{c}$, which implies that $(a_{end}/a_{\text{cross}})(r_{c}/\lambda)_{end} = 1$. As a consequence, in the source (36), the terms proportional to 3024, $-1836$ and $216$ are of the same order of magnitude initially, while the others are negligible since suppressed by powers of $\ell_{H}/\lambda$ and can be safely neglected. This gives rise to

$$P_{\text{rad}}(k) = |v_{k}|^{2} \text{standard} \left[ 1 + 35408 \frac{\gamma}{m_{0}^{2}} H_{end}^{2} M_{Pl}^{2} \left( \frac{r_{c}}{\ell_{H}} \right)_{end}^{-9} \left( \frac{k}{aH} \right)_{end}^{-10} \right].$$ \hfill (43)
SOLVING THE CSL EQUATION

The CSL equation (14) admits Gaussian solutions [as revealed e.g. from the fact that its Lindblad counterpart (22) is linear mode by mode]. Therefore, since the initial vacuum state, the Bunch-Davies state, is Gaussian, it remains so at any time and the stochastic wave function can be written as

$$\Psi_k^s(\eta, v_k^s) = |N_k(\eta)| \exp \left\{ -\Re \Omega_k(\eta)[v_k^s - \bar{v}_k^s(\eta)]^2 + i\sigma_k^s(\eta) + i\chi_k^s(\eta)v_k^s - i3m\Omega_k(\eta)(v_k^s)^2 \right\},$$

where, for the state to be normalised, one has

$$|N_k| = \left( \frac{2\Re \Omega_k}{\pi} \right)^{1/4}. \tag{45}$$

In the standard picture, the quantum state evolves into a two-mode strongly squeezed state. Here, one has $\langle \hat{v}_k^s \rangle = \bar{v}_k^s$ and $\langle \hat{p}_k^s \rangle = -i(\partial/\partial v_k^s) = \chi_k^s - 23m\Omega_k\bar{v}_k^s$, giving rise to $\langle \hat{C}^s(k) \rangle = (\alpha_k - 23m\Omega_k\beta_k)\bar{v}_k^s + \beta_k\chi_k^s$.

For convenience, let us rewrite the CSL equation (14) in terms of conformal time,

$$\frac{d}{d\eta}\Psi_k^s(\eta) = \left\{ -i\hat{H}_k^s \frac{d\eta}{m_0} + \frac{\sqrt{\gamma}a^4}{m_0} \left[ \hat{C}^s(k) - \langle \hat{C}^s(k) \rangle \right] dW_\eta - \frac{\gamma a^4}{2m_0^2} \left[ \hat{C}^s(k) - \langle \hat{C}^s(k) \rangle \right]^2 d\eta \right\} |\Psi_k^s(\eta)|, \tag{46}$$

where $\hat{H}_k^s = (\hat{p}_k^s)^2/2 + \omega^2(k, \eta)(\hat{v}_k^s)^2/2$ and where the noise $dW_\eta$ is defined by $dW_\eta^r = a^{1/2}dW_\eta^s$ such that

$$E\left[ dW_\eta^r dW_\eta^{s'} \right] = (\delta(k - k')\delta^{ss'}\delta(\eta - \eta')d\eta^2. \tag{47}$$

Making use of the representation $\hat{C}^s(k) = \alpha_k\hat{v}_k^s - \beta_ki\partial/\partial\hat{v}_k^s$, the CSL equation becomes

$$\frac{d}{d\eta}\Psi_k^s(\eta) = \left\{ - \left( \frac{i}{2} \omega^2(k, \eta) + \frac{\gamma a^4}{2m_0^2} \right) (v_k^s)^2 + \left( \frac{i}{2} + \frac{\gamma a^4}{2m_0^2} \right) \frac{\partial^2}{\partial(v_k^s)^2} + i\frac{\gamma a^4}{2m_0^2}\alpha_k\beta_k v_k^s \frac{\partial}{\partial v_k^s} \\
+ \alpha_k \left[ \frac{\sqrt{\gamma} a^4}{m_0^2} dW_\eta + \frac{\gamma a^4}{m_0^2} \left( \alpha_k\bar{v}_k^s - 23m\Omega_k\beta_k\bar{v}_k^s + \beta_k\chi_k^s \right) \right] v_k^s \\
- i\beta_k \left[ \frac{\sqrt{\gamma} a^4}{m_0^2} dW_\eta + \frac{\gamma a^4}{m_0^2} \left( \alpha_k\bar{v}_k^s - 23m\Omega_k\beta_k\bar{v}_k^s + \beta_k\chi_k^s \right) \right] \frac{\partial}{\partial v_k^s} \\
- \frac{\gamma a^4}{m_0^2} \left( \alpha_k\bar{v}_k^s - 23m\Omega_k\beta_k\bar{v}_k^s + \beta_k\chi_k^s \right) dW_\eta \\
\right\} |\Psi_k^s(\eta)|. \tag{48}$$

Plugging Eq. (44) into Eq. (48) and making use of Itô calculus, one can identify terms proportional to $(v_k^s)^2$, $v_k^s$ and 1. This gives rise to the set of differential equations

$$\frac{d\Re \Omega_k}{d\eta} = \gamma \left[ \frac{\gamma a^4}{m_0^2} - \frac{4}{m_0^2} a^4 \frac{\partial}{\partial(\Re \Omega_k)^2} \right] \Re \Omega_k \Re \Omega_k + 4\Re \Omega_k \Im \Omega_k - \frac{4}{m_0^2} a^4 \alpha_k \beta_k \Im \Omega_k, \tag{49}$$

$$\frac{d\Im \Omega_k}{d\eta} = \frac{1}{2} \omega^2(k, \eta) - 2 \left[ (\Re \Omega_k)^2 - (\Im \Omega_k)^2 \right] - 8\gamma \left[ \frac{\gamma a^4}{m_0^2} \Re \Omega_k \Im \Omega_k + \frac{4}{m_0^2} a^4 \alpha_k \beta_k \Re \Omega_k \right], \tag{50}$$

$$\frac{d\ln |N_k(\eta)|}{d\eta} = \frac{1}{4\Re \Omega_k} \frac{d\Re \Omega_k}{d\eta}, \tag{51}$$

$$\frac{d\chi_k^s}{d\eta} = \chi_k^s - 2\eta k \Im \Omega_k + \frac{\sqrt{\gamma} a^2}{2m_0 \Re \Omega_k} (\alpha_k - 2\beta_k \Im \Omega_k) dW_\eta, \tag{52}$$

$$\frac{d\sigma_k^s}{d\eta} = -\Re \Omega_k + 2(\Re \Omega_k)^2 \bar{v}_k^s - \frac{\chi_k^s}{2} + \frac{\gamma a^4}{2m_0^2} \beta_k (\alpha_k - 2\beta_k \Im \Omega_k) (1 - 8\Re \Omega_k \bar{v}_k^s) \\
- 2\frac{\sqrt{\gamma} a^2}{m_0} \beta_k \Re \Omega_k \bar{v}_k^s dW_\eta, \tag{53}$$

$$\frac{d\chi_k^s}{d\eta} = 2\eta k \Im \Omega_k - 4(\Re \Omega_k)^2 \bar{v}_k^s + 7\frac{\gamma a^4}{m_0^2} \beta_k \Re \Omega_k \bar{v}_k^s (\alpha_k - 2\beta_k \Im \Omega_k) + 2\frac{\sqrt{\gamma} a^2}{m_0} \beta_k \Re \Omega_k dW_\eta. \tag{54}$$
Two remarkable properties are to be noticed: $\Omega_k$ decouples from the other parameters of the wavefunction, and its dynamics is not stochastic though modified by the CSL terms. Combining the first two above equations, one can derive an equation for $\Omega_k = \Re \Omega_k + i \Im \Omega_k$, namely
\[
\Omega_k' = -2 \left( i + 2 \frac{\gamma}{m_0^2} a^4 \beta_k^2 \right) \Omega_k^2 + 4 \frac{\gamma}{m_0^2} a^4 \alpha_k \beta_k \Omega_k + \frac{\gamma}{m_0^2} a^4 \alpha_0^2 + \frac{i}{2} \omega^2(k, \eta).
\] (55)
This is a Ricatti equation that can be made linear by introducing the function $g_k(\eta)$ defined by the following expression
\[
\Omega_k = \frac{1}{2} \left( i + 2 \frac{\gamma a^4 \beta_k^2/m_0^2}{1 - 2i \gamma a^4 \beta_k^2/m_0^2} \right) \left( \frac{g_k'}{g_k} - \frac{1}{2} C_1 \right),
\] (56)
and obeying
\[
g_k'' + \left( \frac{1}{2} C_1 - \frac{1}{4} C_1^2 + C_2 \right) g_k = 0.
\] (57)

The coefficients $C_1$ and $C_2$ are given by
\[
C_1 \equiv -2i \frac{\gamma}{m_0^2} \left[ 2a^4 \alpha_k \beta_k - \frac{(a^4 \beta_k^2)'}{1 - 2i \gamma a^4 \beta_k^2/m_0^2} \right], \quad C_2 \equiv \left( 1 - 2i \frac{\gamma}{m_0^2} a^4 \beta_k^2 \right) \left[ \omega^2(k, \eta) - 2i \frac{\gamma}{m_0^2} a^4 \alpha_0^2 \right],
\] (58)
from which it follows that $-C_1'/2 - C_1^2/4 + C_2 = \omega^2(k, \eta) + \Delta \omega^2(k, \eta)$, where $\Delta \omega^2(k, \eta)$ is a function which vanishes when $\gamma = 0$ and can easily be determined from the expressions of $C_1$ and $C_2$. Quite remarkably, one has
\[
\Delta \omega^2(k, \eta) = -i S + O(\gamma^2),
\] (59)
where $S$ is the source function introduced in Eq. (29), and computed in Eqs. (32) and (36) for inflation and radiation respectively. Solving Eq. (57) exactly is difficult but can be done perturbatively in $\gamma$. The perturbed solution can be written as
\[
g_k(\eta) = g_0^0(\eta) + \frac{\gamma}{m_0^2} h_k(\eta) + O(\gamma^2),
\] (60)
where $g_0^0(\eta)$ is the solution of the mode equation for $\gamma = 0$ introduced above. Plugging this expansion into Eq. (57), the function $h_k(\eta)$ obeys
\[
h_k'' + \omega^2(k, \eta) h_k = i \frac{m_0^2 S}{\gamma} g_0^0,
\] (61)
which is solved as
\[
h_k(\eta) = i \int_{-\infty}^{\eta} G(\eta, \eta') \frac{m_0^2 S(\eta')}{\gamma} g_0^0(\eta') d\eta,
\] (62)
where the Green function $G(\eta, \eta)$ has been introduced in Eq. (30). Let us recall that the quantity $m_0^2 S/\gamma$ is of order $O(\gamma^0)$ at leading order. Inserting the expansion (60) into Eq. (56) finally leads to
\[
\Omega_k = \frac{1}{2i} \frac{g_k''}{g_k} \left\{ 1 - \frac{\gamma}{m_0^2} \left( \frac{h_k}{g_k} - \frac{h_0}{g_0} \right) + i \frac{\gamma}{m_0^2} \frac{g_0}{g_k} \left[ 2a^4 \alpha_k \beta_k - (a^4 \beta_k^2)' \right] + 2i \frac{\gamma}{m_0^2} a^4 \beta_k^2 + O(\gamma^2) \right\}.
\] (63)

**Inflation**

We now apply these general considerations to the case of inflation, where the Green function is given by Eq. (34) and the free source function by the expression above that equation. As already mentioned, the first term in the inflationary source $S_{infl}$ given in Eq. (32), i.e. the one proportional to $126c_2^2$, is the dominant one. Keeping only this term in Eq. (61), Eq. (62) leads to the explicit expression of $h_k(\eta)$ which can then be used to calculate the first correction in Eq. (63). The next step consists in calculating the two additional contributions in Eq. (63). Using the expressions of $\alpha_k$ and $\beta_k$ during inflation, see Eqs. (4) and (5), one obtains at leading order in slow roll
\[2a^4 \alpha_k \beta_k - (a^4 \beta_k^2)' \simeq 27H^2 M_{pl}^2 \epsilon_1^3/[(k \eta)^4] \text{ and } 2a^4 \beta_k^2 \simeq 9H^2 M_{pl}^2 \epsilon_1^3/(k \eta)^4.\] Inserting these results into Eq. (63), one finds an exact cancellation, meaning that it is necessary to go to next-to-leading order in slow roll, where the result takes the following form

\[
\Omega_k = \Omega_k|_{\gamma=0} \left[ 1 + i \frac{\gamma}{m_0^2} c^3 \mathcal{O}(\epsilon) \tilde{\rho}_{\text{inf}}(-k \eta)^{-1} + \mathcal{O}(\gamma^2) \right].
\] (64)

Here, \(\mathcal{O}(\epsilon)\) is a linear combination of the Hubble flow parameters. Given that \(\Re \Omega_k|_{\gamma=0} = k(k \eta)^2/2\) and \(\Im \Omega_k|_{\gamma=0} = 1/(2\eta)\), one finally obtains

\[
\Re \Omega_k = \Re \Omega_k|_{\gamma=0} \left[ 1 + \frac{\gamma}{m_0^2} c^3 \mathcal{O}(\epsilon) \tilde{\rho}_{\text{inf}}(-k \eta)^{-7} \right].
\] (65)

We notice that the relative correction to \(\Re \Omega_k\) increases with time, which is what is needed in order for the collapse to occur, \(\Re \Omega_k \gg \Re \Omega_k|_{\gamma=0}\). If one requires the collapse to happen during inflation, a lower bound on the parameter \(\gamma\), defined to be its value such that the relative correction evaluated at \(\eta_{\text{end}}\) is larger than one, can be placed. Of course, this limit depends on the unknown factor \(\mathcal{O}(\epsilon)\). However, as discussed below, the collapse is more efficient during the radiation-dominated era, and the precise value of that quantity plays no role.

### Radiation dominated epoch

During radiation, the Green function is given by Eq. (41) and the free mode function by Eq. (40). Using the expressions of \(\alpha_k\) and \(\beta_k\) during the radiation-dominated era, one also has, for the last two terms in Eq. (63), \(2a^4 \alpha_k \beta_k - (a^4 \beta_k^2)' \simeq 864\eta_{\text{end}} H_{\text{end}}^2 M_{pl}^2 \left[3(\eta - \eta_k)^2 - k^2 \eta_{\text{end}}^2 H_{\text{end}}^2 r_c^2 \right]/\left[k^4(\eta - \eta_k)^8\right]\) and \(2a^4 \beta_k^2 \simeq 864\eta_{\text{end}} H_{\text{end}}^2 M_{pl}^2/[k^4(\eta - \eta_k)^8]\).

**Case where the mode crosses out \(r_c\) during inflation**

As explained above, the first term in Eq. (36) for \(S_{\text{rad}}\) is the dominant one in that case, and at leading order in \(r_c/\lambda_{\text{end}}\), one obtains

\[
\Omega_k \simeq \Omega_k|_{\gamma=0} \left[ 1 + i \frac{\gamma}{m_0^2} 1152 \frac{\tilde{\rho}_{\text{end}}}{k^4(\eta_{\text{end}})^3(\eta - \eta_k)} + \mathcal{O}(\gamma^2) \right].
\] (66)

Given that \(\Re \Omega_k|_{\gamma=0} = (k \eta_{\text{end}})^4/[2k(\eta - \eta_k)]^2\) and \(\Im \Omega_k|_{k} = -[2(\eta - \eta_k)]^{-1}\), we notice that the correction has the same time dependence as \(\Re \Omega_k|_{\gamma=0}\), so its relative value is frozen to

\[
\Re \Omega_k \simeq \Re \Omega_k|_{\gamma=0} \left[ 1 + \frac{\gamma}{m_0^2} 1152 \tilde{\rho}_{\text{end}}(-k \eta_{\text{end}})^{-7} + \mathcal{O}(\gamma^2) \right].
\] (67)

This correction is larger than in Eq. (65), which justifies the statement that the collapse is more efficient in the radiation-dominated epoch. The condition for the collapse, i.e. having a relative correction of order one, is then

\[
\frac{\gamma}{m_0^2} > (1152 \tilde{\rho}_{\text{end}})^{-1}(-k \eta_{\text{end}})^7.
\] (68)

**Case where the mode crosses out \(r_c\) during the radiation-dominated era**

As already discussed, three terms must be kept in the expansion (36) of \(S_{\text{rad}}\), namely the terms proportional to the coefficients 3024, 216 and 1836. This gives rise to

\[
\Omega_k \simeq \Omega_k|_{\gamma=0} \left[ 1 + i \frac{\gamma}{m_0^2} \frac{21792}{11} \frac{H_{\text{end}}^2 M_{pl}^2}{k(-k \eta_{\text{end}})\eta_{\text{end}}}(\eta - \eta_k) - i \frac{\gamma}{m_0^2} \frac{864\eta_{\text{end}}^4 H_{\text{end}}^2 M_{pl}^2}{k^4(\eta - \eta_k)^8} \right]
\]

\[+ \frac{\gamma}{m_0^2} \frac{864\eta_{\text{end}}^4 H_{\text{end}}^2 M_{pl}^2}{k^4(\eta - \eta_k)^8} + \mathcal{O}(\gamma^2) \].
\] (69)
We see that the two last terms are subdominant. In this approximation, the relative correction is again time-independent and given by

$$\Re \Omega_k \simeq \Re \Omega_k|_{\gamma=0} \left[ 1 + \frac{7264}{11} \frac{\gamma}{m_0^2} \rho_{\text{end}} (k \eta_{\text{end}})^{-14} (H_{\text{end}} r_c)^{-7} + O (\gamma^2) \right].$$  (70)

The lower bound on the parameter $\gamma$ can therefore be expressed as

$$\frac{\gamma}{m_0^2} > \left( \frac{7264}{11} \rho_{\text{end}} \right)^{-1} (-k \eta_{\text{end}})^{14} (H_{\text{end}} r_c)^7.$$  (71)