Existence of Coupled Best Proximity Points of \( p \)-Cyclic Contractions

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Abstract: We generalize the notion of coupled fixed (or best proximity) points for cyclic ordered pairs of maps to \( p \)-cyclic ordered pairs of maps. We find sufficient conditions for the existence and uniqueness of the coupled fixed (or best proximity) points. We illustrate the results with an example that covers a wide class of maps.

Keywords: coupled fixed points; cyclic maps; uniformly convex Banach space; error estimate

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1. Introduction

Banach’s fixed point theorem has proven to be a powerful tool in pure and applied mathematics. Coupled fixed points were initiated in [1] more than 30 year ago. It turns out that the last 10 years there is a great interest on coupled fixed points, both in fundamental results and their applications [2–5]. We would like to mention a new kind of applications in the theory of equilibrium in duopoly markets [6,7].

A notion that generalizes fixed point results for non-self maps is that of cyclic maps [8] i.e., \( T : A \to B, T : B \to A \). Since a cyclic map \( T \) does not necessarily have a fixed point, one can alter the problem \( x = Tx \) to a problem to find an element \( x \) which is in some sense closest to \( Tx \). Best proximity points were introduced for cyclic maps in [9] (\( x \) is called a best proximity points of \( T \) in \( A \) if \( \|x - Tx\| = \text{dist}(A, B) = \inf\{\|a - b\|: a \in A, b \in B\} \) and they are relevant in this perspective. The notion of best proximity points [9] actually generalizes the notion of cyclic maps from [8], as far as if \( A \cap B \neq \emptyset \), then any best proximity point is a fixed point, too. It turns out that best proximity points are interesting not only as a pure mathematical results, but also as a possibility for a new approach in solving of different types of problems [2–7].

We would like to mention just a few very recent results about coupled best proximity points, that can be applied in solving of different types of problems. The authors have investigated a generalization of GKT cyclic \( \Phi \)-contraction mapping in [10] and a non trivial application for solving of initial value problem is presented. The existence of coupled best proximity point for a class of cyclic (or noncyclic) condensing operators are studied in [11] an the main result applied for finding of an optimal solution for a system of differential equations. A new class of mappings called fuzzy proximally compatible mappings are considered in [12], where coupled best proximity point results are obtained and further applied in finding the fuzzy distance between two subsets of a fuzzy metric space.
Unfortunately all of the mentions above results are for $2$-cyclic maps. It is not easy to generalize the results about $2$-cyclic maps to $p$-cyclic maps. The first breakthrough was obtained in [13], where authors succeed to show that for wide classes of maps the distances between the successive sets are equal. The technique from [13] was later widely used [14–17].

We have tried to unify the techniques from [1,13] to get results for the existence and uniqueness of coupled fixed (or best proximity points) for $p$-cyclic maps. The first results related to finding the error estimate for best proximity points is made in [18]. In [19], results for the existence and uniqueness of coupled best proximity points are obtained, as well as an error estimate is obtained. In this article, $p$-cyclic operators are considered, and the results obtained include as a special case the results obtained in [13,19].

2. Preliminaries

We will summarize the notions and the results that we will need.

If $A$ and $B$ are nonempty subsets of the metric space $(X,d)$, then a distance between the sets $A$ and $B$ will be the number $\text{dist}(A,B) = \inf \{d(x,y): x \in A, y \in B\}$.

Let $\{A_i\}_{i=1}^p$ be nonempty subsets of $X$. Just to simplify some of the formulas we will assume the convention that $A_{k+p+i} = A_i$ for $i = 1, 2, \ldots, p$ and $k \in \mathbb{N}$.

Following [13], if $\{A_i\}_{i=1}^p$ be nonempty subsets of a metric space $(X,d)$, then the map $T: \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ is called a $p$-cyclic map if it is satisfied that $T(A_i) \subseteq T(A_{i+1})$ for every $i = 1, 2, \ldots, p$. A point $\xi \in A_i$ is called a best proximity point of the cyclic map $T$ in $A_i$ if $d(\xi, T\xi) = \text{dist}(A_i, A_{i+1})$.

The next two lemmas are fundamental to the best proximity points theory.

**Lemma 1.** ([9]) Let $A$ be a nonempty closed, convex subset, and $B$ be a nonempty closed subset of a uniformly convex Banach space $X, \|\cdot\|$). Let $\{x_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ be two sequences in $A$ and $\{y_n\}_{n=1}^\infty$ be a sequence in $B$ so that:

1. $\lim_{n \to \infty} \|x_n - y_n\| = \text{dist}(A,B)$;
2. $\lim_{n \to \infty} \|z_n - y_n\| = \text{dist}(A,B)$;
then $\lim_{n \to \infty} \|x_n - z_n\| = 0$.

**Lemma 2.** ([9]) Let $A$ be a nonempty closed, convex subset, and $B$ be a nonempty closed subset of a uniformly convex Banach space $(X, \|\cdot\|)$. Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be sequences in $A$ and $\{y_n\}_{n=1}^\infty$ be a sequence in $B$ satisfying:

1. $\lim_{n \to \infty} \|z_n - y_n\| = \text{dist}(A,B)$;
2. for every $\varepsilon > 0$ there is a number $N_0 \in \mathbb{N}$, such that for any $m > n \geq N_0$, $\|x_n - y_n\| \leq \text{dist}(A,B) + \varepsilon$,
then for every $\varepsilon > 0$, there is a number $N_1 \in \mathbb{N}$, so that for all $m > n \geq N_1$, holds the inequality $\|x_n - z_n\| \leq \varepsilon$.

The geometric structure of the underlying space $X$ plays a key role. When we consider the Banach space $(X, \|\cdot\|)$ we will always assume that the distance between the elements is generated by the norm $\|\cdot\|$ i.e., $d(x,y) = \|x - y\|$. 

**Definition 1.** ([20]) Let $(X, \|\cdot\|)$ be a Banach space. For every $\varepsilon \in (0,2]$ we define the modulus of convexity of $\|\cdot\|$ by

$$\delta_{\|\cdot\|}(\varepsilon) = \inf \left\{ 1 - \frac{x + y}{2} : x, y \in B_X, \|x - y\| \geq \varepsilon \right\}.$$ 

The norm is called uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. The space $(X, \|\cdot\|)$ is then called a uniformly convex space.
For any uniformly convex Banach space $X$ there holds the inequality [9]

$$\left\| \frac{x + y}{2} - z \right\| \leq \left( 1 - \delta_X \left( \frac{r}{R} \right) \right) R$$

(1)

for any $x, y, z \in X$, such that $\|x - z\| \leq R$, $\|y - z\| \leq R$ and $\|x - y\| \geq r$, provided that $R, r$ be real numbers and $R > 0, r \in [0, 2R]$.

For any uniformly convex Banach space $(X, \| \cdot \|)$ its modulus of convexity $\delta_X$ is strictly increasing function and thus its inverse function $\delta^{-1}$ exists. If there are constants $C > 0$ and $q > 0$, so that the inequality $\delta_{\| \cdot \|} (\epsilon) \geq C \epsilon^q$ holds for any $\epsilon \in (0, 2]$ we say that the modulus of convexity is of power type $q$ with a constant $C$.

An extensive study of the Geometry of Banach spaces can be found in [21–23].

3. Auxiliary Results

The iterated sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ (defined in ([1] in the statement of Theorem 1 for coupled fixed points and in [24] in the statement of Lemma 3.8 for coupled best proximity points) will play a crucial role in the proofs of the results, as far as the ordered pair $(x, y)$ of coupled fixed (or best proximity) points is obtained as its limit.

**Definition 2.** ([1, 24]) Let $\{A_i\}_{i=1}^{p}$ be nonempty subsets of a metric space $X$ and $T : \bigcup_{i=1}^{p} A_i \times A_i \to A_{i+1}$. For any $(x_0, y_0) \in A_1 \times A_1$ the sequence $\{(x_n, y_n)\}_{\text{even}}$ is defined inductively by $(x_1, y_1) = (T(x_0, y_0), T(y_0, x_0))$ and if $(x_n, y_n)$ has been already defined then $(x_{n+1}, y_{n+1}) = (T(x_n, y_n), T(y_n, x_n))$.

When we consider a sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ we will always assume that it is the iterated sequence defined in Definition 2. Sometimes we will consider a subsequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ of $\{(x_n, y_n)\}_{n=0}^{\infty}$.

The notion of a coupled best proximity point for cyclic maps was defined in [24] and the notion of best proximity point for $p$-cyclic maps was introduced in [13]. We will combine both definitions to define a coupled best proximity point for a $p$-cyclic maps.

**Definition 3.** Let $A_i, i = 1, 2, \ldots, p$ be nonempty subsets of a metric space $(X, d)$ and $T : A_i \times A_i \to A_{i+1}$ for $i = 1, 2, \ldots, p$. A point $(x, y) \in A_1 \times A_1$ is said to be a best proximity point of $T$ in $A_1 \times A_1$, if $d(x, T(x, y)) = d(y, T(y, x)) = d(A_i, A_{i+1})$.

Following [13] we will define a $p$-cyclic contractive condition for $T : \bigcup_{i=1}^{p} A_i \times A_i \to A_{i+1}$.

**Definition 4.** Let $\{A_i\}_{i=1}^{p}$ be nonempty subsets of a metric space $(X, d)$. The map $T$ is called $p$-cyclic contraction, if it satisfies the following condition:

- $T : A_i \times A_i \to A_{i+1}$
- There exists $\alpha, \beta \geq 0$, $\alpha + \beta \in (0, 1)$, such that the inequality
  $$d(T(x, y), T(u, v)) \leq \alpha d(x, u) + \beta d(y, v) + (1 - (\alpha + \beta))d(A_i, A_{i+1})$$

holds for every $(x, y) \in A_i \times A_i, (u, v) \in A_{i+1} \times A_{i+1}, 1 \leq i \leq p$.

**Lemma 3.** Let $\{A_i\}_{i=1}^{p}$ be nonempty subsets of a metric space $(X, d)$ and $T$ be a $p$-cyclic contraction map. Then $\text{dist}(A_i, A_{i+1}) = \text{dist}(A_{i+1}, A_{i+2})$ for $i = 1, 2, \ldots, p$.

**Proof.** Let us put $d_{i+1} = \text{dist}(A_{p-i}, A_{p-1-i})$ for $i = 0, 1, \ldots, p - 1$ (where we use the convention $d_p = \text{dist}(A_1, A_0) = \text{dist}(A_1, A_p)$.
Let us suppose the contrary, that there are two indexes \( k, j \in \{1, 2, \ldots, p\} \), such that \( \max_{i \in \{1, 2, \ldots, p\}} \{d_i\} = d_k > d_j \). Without loss of generality we may assume, that \( k = p \). There exists \( s \in (0, 1) \), such that

\[
d_j = (1 - s)d_p. \tag{3}
\]

Let \( (x_0, y_0) \in A_p \), then \( x_{pn+m}, y_{pn+m} \in A_m, x_{pn}, y_{pn} \in A_p, x_{pn+1}, y_{pn+1} \in A_1 \) and from (2) we get

\[
d(x_{np+1}, x_{np}) = d(T(x_{np}, y_{np}), T(x_{np-1}, y_{np-1})) \leq \alpha d(x_{np}, x_{np-1}) + \beta d(y_{np}, y_{np-1} + (1 - \alpha - \beta)d_1) \tag{4}
\]

and

\[
d(y_{np+1}, y_{np}) = d(T(y_{np}, x_{np}), T(y_{np-1}, x_{np-1})) \leq \alpha d(y_{np}, y_{np-1}) + \beta d(x_{np}, x_{np-1} + (1 - \alpha - \beta)d_1). \tag{5}
\]

Let us, for what follows, to use the notation \( \gamma = \alpha + \beta \). From (4) and (5) we can write the chain of inequalities

\[
S_1 = d(x_{np+1}, x_{np}) + d(y_{np+1}, y_{np}) \leq \gamma d(x_{np}, x_{np-1}) + d(y_{np}, y_{np-1}) + 2(1 - \gamma)d_1
\]

\[
\leq \gamma [\gamma d(x_{np-1}, x_{np-2}) + d(y_{np-1}, x_{np-2}) + 2(1 - \gamma)d_2] + 2(1 - \gamma)d_1
\]

\[
= \gamma^2 (d(x_{np-1}, x_{np-2}) + d(y_{np-1}, x_{np-2}) + 2(1 - \gamma)(d_2 + d_1))
\]

\[
\leq \gamma^3 (d(x_{np-2}, x_{np-3}) + d(y_{np-2}, x_{np-3}) + 2(1 - \gamma)(\gamma^2 d_3 + \gamma d_2 + d_1))
\]

\[
\leq \gamma^p (d(x_{n(p-1)+1}, x_{n(p-1)}) + d(y_{n(p-1)+1}, x_{n(p-1)})) + 2(1 - \gamma) \sum_{i=0}^{p-1} \gamma^i d_{i+1}
\]

\[
\leq \gamma^p (d(x_1, x_0) + d(y_1, x_0)) + 2(1 - 2\alpha) \sum_{i=0}^{p-1} \gamma^i \sum_{k=0}^{n-1} \gamma^k p d_{i+1}
\]

and thus we get

\[
\sum_{i=0}^{p-1} \gamma^i \sum_{k=0}^{n-1} \gamma^k p d_{i+1} = \sum_{j=0}^{p-1} \sum_{k=0}^{n-1} \gamma^{k+i} p d_{j+1} \leq \frac{1}{1 - \gamma^p} \sum_{k=0}^{n-1} \gamma^k p d_{k+1}.
\]

There exists \( N \in \mathbb{N} \), so that for any \( n \geq N \) there holds the inequality

\[
\gamma^p (d(x_1, x_0) + d(y_1, y_0)) \leq \frac{s}{2} \frac{(1 - \gamma)^j}{1 - \gamma^p} d_p,
\]

where \( j \) and \( s \) are the index and the constant from (3), respectively. Therefore using the assumption that \( d_j = (1 - s)d_p = d_p - sd_p \) and that for any \( k \neq p \) there holds \( d_k \leq d_p \) we get

\[
2d_p \leq d(x_{np+1}, x_{np}) + d(y_{np+1}, y_{np}) \leq \gamma^p (d(x_1, x_0) + d(y_1, y_0)) + \frac{2(1 - \gamma)}{1 - \gamma^p} \sum_{k=0}^{p-1} \gamma^k p d_{k+1}
\]

\[
\leq \gamma^p (d(x_1, x_0) + d(y_1, y_0)) - s \frac{(1 - \gamma)^j}{1 - \gamma^p} d_p + \frac{2(1 - \gamma)}{1 - \gamma^p} \sum_{k=0}^{p-1} \gamma^k p d_{p} \tag{7}
\]

\[
= \gamma^p (d(x_1, x_0) + d(y_1, y_0)) - s \frac{(1 - \gamma)^j}{1 - \gamma^p} d_p + 2d_p
\]

\[
< \frac{s}{2} \frac{(1 - \gamma)^j}{1 - \gamma^p} d_p + 2d_p < 2d_p.
\]
which is a contradiction and consequently the assumption that there exists $j$ so that $d_j < \max\{d_i : i = 1, 2, \ldots, p\}$ could not hold.

We have just proven in Lemma 3 that for maps, which satisfy Definition 4, there holds $\text{dist}(A_1, A_2) = \text{dist}(A_2, A_3) = \cdots = \text{dist}(A_{p-1}, A_p) = \text{dist}(A_p, A_1)$ and thus we can denote in the rest of the article the distance between the consecutive sets by $d = \text{dist}(A_i, A_{i+1}), i = 1, 2, \ldots, p$.

An easier to apply inequality, which is a consequence from (2) is the inequality

$$d(T(x, y), T(u, v)) + d(T(y, x), T(v, u)) - 2d \leq \gamma(d(x, u) + d(y, v) - 2d)$$

for every $(x, y) \in A_i \times A_i, (u, v) \in A_{i+1} \times A_{i+1}, 1 \leq i \leq p$.

**Lemma 4.** Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a metric space $(X, d)$ and $T$ be a $p$-cyclic contraction. Then for every $(x_0, y_0) \in A_1 \times A_i$ there hold $\lim_{n \to \infty} d(x_{pn}, x_{pn+1}) = d$, $\lim_{n \to \infty} d(y_{pn}, y_{pn+1}) = d$, $\lim_{n \to \infty} d(x_{pn+p}, x_{pn+1}) = d$ and $\lim_{n \to \infty} d(y_{pn+p}, y_{pn+1}) = d$.

**Proof.** By Lemma 3 we have that $d(A_i, A_{i+1}) = d(A_{i+1}, A_{i+2})$ for $i = 1, 2, \ldots, p - 1$. Let us put $(A_1, A_2) = d$. Therefore there holds the chain of inequalities

$$0 \leq d(x_{pn+1}, x_{pn}) + d(y_{pn+1}, y_{pn}) - 2d \leq \gamma(d(x_{pn}, y_{pn-1}) + d(y_{pn}, y_{pn-1}) - 2d) \leq \gamma^2(d(x_{pn-1}, x_{pn-2}) + d(y_{pn-1}, y_{pn-2}) - 2d) \leq \gamma^n(d(x_1, x_0) + d(y_1, y_0) - 2d).$$

Consequently after taking a limit in (9) when $n \to \infty$ we get $\lim_{n \to \infty} d(x_{pn}, x_{pn-1}) + d(y_{pn}, y_{pn-1})) = 2d$. From the inequalities $d \leq d(x_{pn}, x_{pn-1})$ and $d \leq d(y_{pn}, y_{pn-1})$ it follows that $\lim_{n \to \infty} d(x_{pn}, x_{pn-1}) = \lim_{n \to \infty} d(y_{pn}, y_{pn-1}) = d$.

The proofs of the other two (actually four, because of $\pm$) limits can be done in a similar fashion.

**Lemma 5.** If $(X, \| \cdot \|)$ be a uniformly convex Banach space, $\{A_i\}_{i=1}^p$ be nonempty and convex subsets of $X$. $T$ be a $p$-cyclic contraction. Then for every $(x_0, y_0) \in A_1 \times A_i$ there hold $\lim_{n \to \infty} \|x_{pn} - x_{pn+p}\| = 0$, $\lim_{n \to \infty} \|y_{pn} - y_{pn+p}\| = 0$, $\lim_{n \to \infty} \|x_{pn+1} - x_{pn+p+1}\| = 0$ and $\lim_{n \to \infty} \|y_{pn+1} - y_{pn+p+1}\| = 0$.

**Proof.** By Lemma 4 we have that $\lim_{n \to \infty} \|x_{pn} - x_{pn+1}\| = \lim_{n \to \infty} \|y_{pn+p} - y_{pn+1}\| = d$.

According to Lemma 3 it follows that $\lim_{n \to \infty} \|x_{pn} - x_{pn+p}\| = 0$.

**Lemma 6.** Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a metric space $(X, d)$ and $T$ be a $p$-cyclic contraction. Let $(x_0, y_0) \in A_1 \times A_i$ and the sequence $\{(x_{pn}, y_{pn})\}_{n=0}^\infty$ has a convergent (say to $(\xi, \eta)$) in $A_1 \times A_i$, then $(\xi, \eta)$ is a best proximity point of $T$ in $A_1 \times A_i$.

**Proof.** By the inequality $d \leq d(x_{pn-1}, \xi) \leq d(x_{pn-1}, x_{pn}) + d(x_{pn}, \xi)$, the assumption that $\lim_{j \to \infty} d(x_{pn_j}, \xi) = 0$ and $\lim_{j \to \infty} d(x_{pn_j}, x_{pn_j}) = d$ we get $\lim_{j \to \infty} d(x_{pn_j-1}, \xi) = d$. 


By similar arguments it follows that \( \lim_{j \to \infty} d(y_{pnj-1}, \eta) = d \). Using the continuity of the metric function \( d(\cdot, \cdot) \) and Lemma 4, we can write the chain of inequalities

\[
0 < d(\xi, T(\xi, \eta)) + d(T(\eta, \xi), \eta) - 2d = \lim_{j \to \infty} (d(x_{pnj}, T(\xi, \eta)) + d(T(\xi, \xi), y_{pnj})) - 2d = \lim_{j \to \infty} (d(T(x_{pnj-1}, y_{pnj-1}), T(\xi, \eta)) + d(T(\eta, \xi), T(y_{pnj-1}, x_{pnj-1})) - 2d) \leq \gamma \lim_{j \to \infty} (d(x_{pnj-1, \xi}) + d(y_{pnj-1, \eta}) - 2d) = \gamma \left( \lim_{j \to \infty} (d(x_{pnj-1, x_{pnj}}) + d(y_{pnj-1, y_{pnj}})) - 2d \right) = 0.
\]

Consequently \( d(\xi, T(\xi, \eta)) + d(T(\eta, \xi), \eta) = 2d \) and from the inequalities \( d(\xi, T(\xi, \eta)) \geq d \) and \( d(T(\eta, \xi), \eta) \geq d \) it follows that \( d(\xi, T(\xi, \eta)) = d(\eta, \xi) = d \).

For an arbitrary chosen \((z, v) \in A_1 \times A_j \), let us denote \( T^2(z, v) = T(T(z, v), T(v, z)) \) and \( T^2(v, z) = T(T(v, z), T(z, v)) \) and if we have already defined \((T^{p-1}(z, v), T^{p-1}(v, z))\), then put

\[
T^p(z, v) = T(T^{p-1}(z, v), T^{p-1}(v, z)) \quad \text{and} \quad T^p(v, z) = T(T^{p-1}(v, z), T^{p-1}(z, v)).
\]

**Lemma 7.** Let \( \{A_i\}_{i=1}^p \) be nonempty closed subsets of a metric space \((X, d)\) and \( T \) be a \( p \)-cyclic contraction. If there exists a coupled best proximity point \((z, v) \in A_1 \times A_j \), then \((T^p(z, v), T^p(v, z))\) is a coupled best proximity point of \( T \) in \( A_{i+1} \times A_{i+1} \). If \((z, v) \) is a limit of the sequence \( \{(x_{pn}, y_{pn})\}_{n=0}^{\infty} \), then the ordered pair \((z, v)\) is a \( p \)-periodic point of \( T \), i.e., \( z = T^p(z, v) \) and \( v = T^p(v, z) \) for \( n \in \mathbb{N} \) and any sequence \( \{(x_{pn}, y_{pn})\}_{n=0}^{\infty} \) converges to \((z, v)\).

**Proof.** Let \((z, v)\) be any ordered pair, which is a coupled best proximity points of \( T \) in \( A_1 \times A_1 \). From the inequality

\[
S_2 = ||T(z, v) - T^2(z, v)|| + ||T(v, z) - T^2(v, z)|| - 2d \
\leq \gamma(||z - T(z, v)|| + ||v - T(v, z)|| - 2d) = 0
\]

it follows that \((T(z, v), T(v, z))\) is an ordered pair, which is a coupled best proximity points of \( T \) in \( A_{i+1} \times A_{i+1} \). From

\[
S_3 = ||T^2(z, v) - T^3(z, v)|| + ||T^2(v, z) - T^3(v, z)|| - 2d \
\leq \gamma^2(||z - T(z, v)|| + ||v - T(v, z)|| - 2d) = 0
\]

it follows that \((T^2(z, v), T^2(v, z))\) is a coupled best proximity points of \( T \) in \( A_{i+2} \times A_{i+2} \). By induction we can prove that \((T^n(z, v), T^n(v, z))\) is a coupled best coincidence points of \( T \) in \( A_{i+n} \times A_{i+n} \).

Therefore we have

\[
0 \leq ||z - T^{p+1}(z, v)|| + ||v - T^{p+1}(v, z)|| - 2d = \lim_{n \to \infty} \left(||z_{pn} - T^{p+1}(z, v)|| + ||v_{pn} - T^{p+1}(v, z)|| - 2d\right) = \lim_{n \to \infty} \left(||T(z_{pn-1}, v_{pn-1}) - T^{p+1}(z, v)|| + ||T(v_{pn-1}, z_{pn-1}) - T^{p+1}(v, z)|| - 2d\right) \leq \gamma^p \lim_{n \to \infty} \left(||z_{p(n-1)} - T(z, v)|| + ||v_{p(n-1)} - T(v, z)|| - 2d\right) = \gamma^p \left(||z - T(z, v)|| + ||v - T(v, z)|| - 2d\right) = 0.
\]

Thus \( ||z - T^{p+1}(z, v)|| = ||v - T^{p+1}(v, z)|| = d \). From \( ||z - T(z, v)|| = ||v - T(v, z)|| = d \) and Lemma 2 it follows that \( T(z, v) = T^{p+1}(z, v) \) and \( T(v, z) = T^{p+1}(v, z) \). From

\[
||z - T(z, v)|| + ||v - T(v, z)|| - 2d = 0,
\]

we have that \( d(z, v) = d(v, z) \).
Let $T$ be a priori estimate that the inequality

$$\|T(z, v) - T(v, z)\| + \|T(v, z) - T(v, z)\| - 2d$$

From (10) we get that $z = T^p(z, v)$ and $v = T^p(v, z)$. Now, by a similar calculations we can obtain that $T^{2p}(z, v) = T^p(z, v) = z$, $T^{2p}(v, z) = T^p(v, z) = v$ and by induction, that $T^{np}(z, v) = z$ and $T^{np}(v, z) = v$.

Let there exists $(\xi, \eta) \in A_1 \times A_1$, which is a coupled best proximity points of $T$ in $A_1 \times A_1$, i.e., $\|\xi - T(\xi, \eta)\| = \|\eta - T(\eta, \xi)\| = d$, that is different from $(z, v)$ and obtained as a limit of a sequence $\{(\xi_n, \eta_n)\}_{n=0}^{\infty}$. Using the continuity of the norm function, the equality $T(z, v) = T^{p+1}(z, v), T(v, z) = T^{p+1}(v, z)$ we get the inequality

$$\|\xi - T(z, v)\| + \|\eta - T(v, z)\| - 2d < \|\xi - T(z, v)\| + \|\eta - T(v, z)\| - 2d = 0.$$

Consequently $\|\xi - T(z, v)\| + \|\eta - T(v, z)\| = 2d$. Therefore $\|\xi - T(z, v)\| = \|\eta - T(v, z)\| = d$ and from $\|z - T(z, v)\| = \|v - T(v, z)\| = d$ and by Lemma 2 it follows that $(z, v) = (\xi, \eta)$. □

**4. Main Results**

**Theorem 1.** Let $\{A_i\}_{i=1}^p$ be nonempty, closed and convex subsets of a complete metric space $(X, d)$. Let $T : \bigcup_{i=1}^p A_i \times A_i \rightarrow A_i \times A_i$ be a $p$-cyclic map, so that exist $\alpha, \beta \geq 0$, $\alpha + \beta \in (0, 1)$, such that the inequality

$$d(T(x, y), T(u, v)) \leq \alpha d(x, u) + \beta d(y, v)$$

holds for every $(x, y) \in A_1 \times A_1, (u, v) \in A_{i+1} \times A_{i+1}, 1 \leq i \leq p$.

Then there exists an order pair $(z, v) \in \cap_{i=1}^p (A_i \times A_i)$, such that, if $(x_0, y_0) \in A_1 \times A_1$ be an arbitrary point of $A_1 \times A_1$, the sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ converges to $(z, v)$ and the order pair $(z, v)$ is a unique coupled fixed point of $T$. Moreover, there hold

- the a priori estimate $\max \{\rho(x_n, z), \rho(y_n, v)\} \leq \frac{\gamma^n}{1 - \gamma} (\rho(x_1, x_0) + \rho(y_1, y_0))$
- the a posteriori estimate $\max \{\rho(x_n, z), \rho(y_n, v)\} \leq \frac{\gamma^n}{1 - \gamma} (\rho(x_{n-1}, x_n) + \rho(y_{n-1}, y_n))$
- the rate of convergence $\rho(x_n, z) + \rho(y_n, v) \leq \gamma (\rho(x_{n-1}, z) + \rho(y_{n-1}, v))$

where $\gamma = \alpha + \beta$.

**Proof.** Let $(x_0, y_0) \in \bigcup_{i=1}^p (A_i \times A_i)$ be arbitrary chosen. Let us consider the iterated sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$. Then there hold the inequalities

$$d(x_{n+1}, x_n) = d(T(x_n, y_n), T(x_{n-1}, y_{n-1})) \leq \alpha d(x_n, x_{n-1}) + \beta d(y_n, y_{n-1})$$

and

$$d(y_{n+1}, y_n) = d(T(y_n, x_n), T(y_{n-1}, x_{n-1})) \leq \alpha d(y_n, y_{n-1}) + \beta d(x_n, x_{n-1}).$$

After summing up the above two inequalities we get

$$d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \leq \gamma (\rho(x_n, x_{n-1}) + \rho(y_n, y_{n-1})).$$

From (11) we get that there holds true

$$\max \{d(x_{n+1}, x_n), d(y_{n+1}, y_n)\} \leq \frac{\gamma^n}{1 - \gamma} (\rho(x_1, x_0) + \rho(y_1, y_0)).$$
Thus
\[
d(x_n, x_{n+p}) \leq \sum_{k=0}^{p-1} d(x_{n+k}, x_{n+k+1}) \leq \sum_{k=0}^{p-1} \gamma^{n+k}(d(x_1, x_0) + d(y_1, y_0))
\]
(12)

Therefore \( \{ x_n \}_{n=1}^{\infty} \) is a Cauchy sequence in \( \bigcup_{i=1}^{p} A_i \). From the assumption that \( A_i \) are closed subsets of the complete metric space \( (X, d) \) it follows that \( \{ x_n \}_{n=1}^{\infty} \) is convergent to some point \( z \in \bigcup_{i=1}^{p} A_i \). The sequence \( \{ x_n \}_{n=1}^{\infty} \) is an iterated sequence defined by the \( p \)-cyclic map \( T \) and thus it has infinite number of terms that belong to each \( A_i, i = 1, 2, \ldots, p \). Consequently \( z \in \cap_{i=1}^{p} A_i \).

By literary the same arguments we get that \( \lim_{n \to \infty} y_n = v \in \cap_{i=1}^{p} A_i \).

We will show that \( (z, v) \) is a coupled fixed point of \( T \). Indeed from

\[
S_6 = \quad \quad d(z, T(z, v)) + d(v, T(v, z))
\]
\[
= \quad \quad \lim_{n \to \infty} d(x_n, T(z, v)) + d(y_n, T(v, z))
\]
\[
= \quad \quad \lim_{n \to \infty} d(T(x_{n-1}, y_{n-1}), T(z, v)) + d(T(y_{n-1}, x_{n-1}), T(v, z))
\]
\[
\leq \quad \quad \gamma \lim_{n \to \infty} d(x_{n-1}, z) + d(y_{n-1}, v) = 0
\]

it follows that \( d(z, T(z, v)) = d(v, T(v, z)) = 0 \), i.e., \( (z, v) \) is a coupled fixed point of \( T \).

We will prove that \( (z, v) \) is a unique coupled fixed point by assuming the contrary. Let \( (x, y) \) be a coupled fixed point of \( T \), different from \( (z, v) \). If \( x \in A_i \), then by the definition of a coupled fixed point it follows that \( y \in A_i \), too. From the assumption that \( T \) is a \( p \)-cyclic map it follows that \( (x, y) = (T(x, y), T(y, x)) \in A_{i+1} \times A_{i+1} \) and therefore \( (x, y) \in \cap_{i=1}^{p} (A_i \times A_i) \). From the inequality

\[
d(x, z) + d(y, v) = d(T(x, y), T(z, v)) + d(T(y, x), T(v, z)) \leq \gamma (d(x, z) + d(y, v))
\]

and the assumption that \( \gamma \in (0, 1) \) it follows that \( d(x, z) = d(y, v) = 0 \), i.e., the coupled fixed point \( (z, v) \) of \( T \) is unique.

After taking a limit in (12) we get

\[
d(x_n, z) = \lim_{p \to \infty} d(x_n, x_{n+p}) \leq \frac{\gamma^n}{1-\gamma} (d(x_1, x_0) + d(y_1, y_0)).
\]

and

\[
d(y_n, z) = \lim_{p \to \infty} d(y_n, y_{n+p}) \leq \frac{\gamma^n}{1-\gamma} (d(x_1, x_0) + d(y_1, y_0)).
\]

Consequently there holds the a priori estimate

\[
\max \{ d(x_n, z), d(y_n, z) \} \leq \frac{\gamma^n}{1-\gamma} (d(x_1, x_0) + d(y_1, y_0)).
\]

From the chain of inequalities

\[
d(x_n, x_{n+p}) \leq \sum_{k=0}^{p-1} d(x_{n+k}, x_{n+k+1}) \leq \sum_{k=1}^{p} \gamma^k (d(x_{n-1}, x_n) + d(y_{n-1}, y_n))
\]
\[
= \quad \quad \frac{\gamma}{1-\gamma} (d(x_{n-1}, x_n) + d(y_{n-1}, y_n)).
\]

After taking a limit, when \( p \to \infty \), in the above inequality, we get

\[
d(x_n, z) = \lim_{p \to \infty} d(x_n, x_{n+p}) \leq \frac{\gamma}{1-\gamma} (d(x_{n-1}, x_n) + d(y_{n-1}, y_n)).
\]
Consequently, after using the same arguments for \( d(y_n, v) \), there holds the a posteriori estimate

\[
\max \{d(x_n, z), d(y_n, v)\} \leq \frac{\gamma}{1 - \gamma} (d(x_{n-1}, x_n) + d(y_{n-1}, y_n)).
\]

From the inequality

\[
d(x_n, z) + d(y, v) = d(T(x_n, y_{n-1}), T(z, v)) + d(T(y_{n-1}, x_n), T(v, z)) \leq \gamma(d(x_{n-1}, z) + d(y_{n-1}, v))
\]

we get the estimate the rate of convergence. \( \square \)

We will use the notations \( P_{n,m} = \|x_n - x_m\| + \|y_n - y_m\| \) and \( W_{n,m} = \|x_n - x_m\| + \|y_n - y_m\| - 2d \), where \( \{x_n\}_{n=0}^{\infty} \) and \( \{y_n\}_{n=0}^{\infty} \) be the sequences from Definition 2, when the text field is too short.

We have proven in Lemma 3, that for any \( p \)-cyclic contraction the distances between the consecutive sets are equal. Therefore in the next theorem we will denote \( d = \text{dist}(A_i, A_{i+1}) \), \( i = 1, 2, \ldots, p \).

**Theorem 2.** Let \( \{A_i\}_{i=1}^{p-1} \) be nonempty, closed and convex subsets of a uniformly convex Banach space \( (X, \| \cdot \|) \). Let \( T : \bigcup_{i=1}^{p-1} A_i \times A_i \to A_{i+1} \) be a \( p \)-cyclic contraction. Then there exists a unique ordered pair \( (z_i, v_i) \in A_i \setminus A_i \) \( (1 \leq i \leq p) \), which is a limit of the subsequence \( \{(x_{pn}, y_{pn})\}_{n=0}^{\infty} \subset \{(x_n, y_n)\}_{n=0}^{\infty} \). For any initial guess \( (x_0, y_0) \in A_i \times A_i \) and it is a coupled best proximity point of \( T \) in \( A_i \times A_i \). Moreover, \( (T(z_i, v_i), T(v_i, z_i)) \) is a coupled best proximity point of \( T \) in \( A_i \times A_i \) and \( (z_i, v_i) \) is a \( p \)-periodic point of \( T \).

- If \( d > 0 \) and \((X, \| \cdot \|)\) be with a modulus of convexity of power type \( q \) with a constant \( C \), then there holds the a priori error estimate

\[
\max \{|x_{pn} - z_i|, |y_{pn} - v_i|\} \leq P_{0,1} \sqrt{W_{0,1} \over C d} \left( {\sqrt{d} \over 1 - \sqrt{d}} \right)^{pm},
\]

and the a posteriori error estimate

\[
\max \{|x_{pn} - z_i|, |y_{pn} - v_i|\} \leq P_{pn, pn-1} \sqrt{W_{pn, pn-1} \over C d} \left( {\sqrt{d} \over 1 - \sqrt{d}} \right)^{pm},
\]

where \( \gamma = \alpha + \beta \), \( \alpha \) and \( \beta \) be the constants for Definition 4.

- If \( d = 0 \), then there hold the error estimates of Theorem 1.

If \( p = 2 \), we get as a particular case the results from [19].

**Proof.** If \( \text{dist}(A_i, A_{i+1}) = 0 \) for some \( i \), then by Lemma 3 it follows that \( \text{dist}(A_j, A_{j+1}) = 0 \) for all \( i \). Then the contraction condition induced on \( T \) is equivalent to (10). Thus by Theorem 1, \( T \) has a unique coupled fixed point and the error estimates from Theorem 1 holds.

Let us assume that \( d > 0 \). Let \((x_0, y_0) \in A_i \times A_i \). Then \( x_{np} \in A_i \) and \( x_{np+1} \in A_{i+1} \) for all \( n \). By Lemma 4, \( \lim_{n \to \infty} \|x_{np} - x_{np+1}\| = d \). If, for any arbitrary chosen \( \varepsilon > 0 \), there exists an \( n_0 \in \mathbb{N} \), such that for all \( m > n > n_0 \) the inequality holds

\[
\|x_{pn} - x_{pn+1}\| + \|y_{pn} - y_{pn+1}\| \leq 2d + \varepsilon,
\]

by the inequalities \( \|x_{pn} - x_{pn+1}\| \geq d \) and \( \|y_{pn} - y_{pn+1}\| \geq d \) it follows the inequality

\[
\max \{|x_{pn} - x_{pn+1}|, |y_{pn} - y_{pn+1}|\} \leq d + \varepsilon \]

holds for all \( m > n > n_0 \). Then by Lemma 1, for any \( \varepsilon_1 > 0 \), there exists \( n_1 \in \mathbb{N} \), such that for all \( m > n > n_1 \) the inequality \( \frac{\|x_{pn} - x_{np}\|}{\|y_{pn} - y_{np}\|} \leq \varepsilon_1 \) holds, i.e., \( \{x_{np}\}_{n=1}^{\infty} \) and \( \{y_{np}\}_{n=1}^{\infty} \) are Cauchy sequences and thus converges to some \((z, v) \in A_i \times A_i \). By Lemma 6 \((z, v)\) will be a best proximity point of \( T \) in \( A_i \times A_i \).
Let us assume contrary of (13). Then, there exists an \( \epsilon_0 > 0 \) such that, for every \( k \in \mathbb{N} \), there exists \( m_k > n_k \geq k \) such that,

\[
\|x_{pm_k} - x_{pm_k+1}\| + \|y_{pm_k} - y_{pm_k+1}\| \geq 2d + \epsilon_0.
\]

(14)

Let \( m_k \) be the smallest integer greater than \( n_k \), to satisfy the above inequality. Now

\[
S_7 = 2d + \epsilon_0 \leq \|x_{pm_k} - x_{pm_k+1}\| + \|y_{pm_k} - y_{pm_k+1}\|
\]

\[
\leq \|x_{pm_k} - x_{pm_k-p}\| + \|x_{pm_k-p} - x_{pm_k+1}\| + \|y_{pm_k} - y_{pm_k-p}\| + \|y_{pm_k-p} - y_{pm_k+1}\|.
\]

By Lemma 4 we have \( \lim_{k \to \infty} \|x_{pm_k} - x_{pm_k+p}\| = 0 \) and \( \lim_{k \to \infty} \|y_{pm_k} - y_{pm_k+p}\| = 0 \). Therefore, using the choice of \( m_k \) to be the smallest natural, so that to holds the inequality (14), we get

\[
2d + \epsilon_0 \leq \lim_{k \to \infty} \left( \|x_{pm_k} - x_{pm_k+1}\| + \|y_{pm_k} - y_{pm_k+1}\| \right)
\]

\[
\leq \lim_{k \to \infty} \left( \|x_{pm_k-p} - x_{pm_k+1}\| + \|y_{pm_k-p} - y_{pm_k+1}\| \right) \leq 2d + \epsilon_0.
\]

i.e., \( \lim_{k \to \infty} \|x_{pm_k} - x_{pm_k+1}\| + \lim_{k \to \infty} \|y_{pm_k} - y_{pm_k+1}\| = 2d + \epsilon_0. \)

From the inequality

\[
2d + \epsilon_0 \leq \|x_{pm_k} - x_{pm_k+1}\| + \|y_{pm_k} - y_{pm_k+1}\|
\]

\[
\leq \|x_{pm_k} - x_{pm_k+p}\| + \|x_{pm_k+p} - x_{pm_k+1}\| + \|y_{pm_k} - y_{pm_k+p}\| + \|y_{pm_k+p} - y_{pm_k+1}\|.
\]

by using Lemma 4 we have \( \lim_{k \to \infty} \|x_{pm_k} - x_{pm_k+p}\| = \lim_{k \to \infty} \|y_{pm_k+p} - y_{pm_k+1}\| = 0 \) and thus

\[
\epsilon_0 = \lim_{k \to \infty} \left( \|x_{pm_k} - x_{pm_k+1}\| + \|y_{pm_k} - y_{pm_k+1}\| - 2d \right)
\]

\[
\leq \lim_{k \to \infty} \left( \|x_{pm_k+p} - x_{pm_k+p+1}\| + \|y_{pm_k+p} - y_{pm_k+p+1}\| - 2d \right)
\]

\[
\leq \gamma^p \lim_{k \to \infty} \left( \|x_{pm_k} - x_{pm_k+1}\| + \|y_{pm_k} - y_{pm_k+1}\| - 2d \right)
\]

\[
= \gamma^p \epsilon_0.
\]

That is, \( \epsilon_0 \leq \gamma^p \epsilon_0 \), which is a contradiction, because \( \gamma \in (0, 1) \).

Hence \( \{x_{p^n}\}_{n=1}^\infty \) and \( \{y_{p^n}\}_{n=1}^\infty \) are Cauchy sequences, converging to some \( (x, y) \in A_i \times A_i \) such that \( \|x - T(x, y)\| = \|T(y, x) - y\| = d. \)

From Lemma 7 it follows that \( (x, y) \), which is a limit of the iterated sequences is unique, for an arbitrary chosen initial guess, \( (T^n(x, y), T^n(y, x)) \) is a coupled best proximity point of \( T \) in \( A_{i+n} \times A_{i+n} \), \( (x, y) \) is a \( p \)-periodic point of \( T \).

It has remained to prove that \( x = y \). It holds

\[
S_8 = \|x - T(y, x)\| + \|y - T(x, y)\| - 2d
\]

\[
= \|T^p(x, y) - T(y, x)\| + \|T^p(y, x) - T(y, x)\| - 2d
\]

\[
\leq \gamma(\|T^{p-1}(x, y) - y\| + \|T^{p-1}(y, x) - x\| - 2d)
\]

\[
= \gamma(\|T^{p-1}(x, y) - T^p(y, x)\| + \|T^{p-1}(y, x) - T^p(x, y)\| - 2d)
\]

\[
\leq \gamma^p(\|x - T(y, x)\| + \|y - T(x, y)\| - 2d).
\]

Consequently \( \|x - T(x, y)\| = \|y - T(x, y)\| = d \). From \( \|y - T(y, x)\| = \|x - T(x, y)\| = d \) and the uniform convexity of \( X \) it follows that \( x = y. \)

From (8) there holds the inequality

\[
\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| - 2d \leq \gamma^k(\|x_{n+1-k} - x_{n-k}\| + \|y_{n+1-k} - y_{n-k}\| - 2d).
\]

Thus we get

\[
\max\{\|x_{n+1} - x_n\|, \|y_{n+1} - y_n\|\} \leq \gamma^k(\|x_{n+1-k} - x_{n-k}\| + \|y_{n+1-k} - y_{n-k}\| - 2d) + d.
\]
There hold the inequalities
\[
\|x_{pn+1} - x_{pn}\| \leq d + \gamma^l(\|x_{pn+1-l} - x_{pn-l}\| + \|y_{pn+1-l} - y_{pn-l}\| - 2d), \\
\|x_{pn+p} - x_{pn+p+1}\| \leq d + \gamma^{l+1}(\|x_{pn+1-l} - x_{pn-l}\| + \|y_{pn+1-l} - y_{pn-l}\| - 2d) \\
\leq d + \gamma^l(\|x_{pn+1-l} - x_{pn-l}\| + \|y_{pn+1-l} - y_{pn-l}\| - 2d)
\]
and
\[
\|x_{pn+p} - x_{pn}\| \leq \|x_{pn+p} - x_{pn+1}\| + \|x_{pn+1} - x_{pn}\| \\
\leq 2\left(d + \gamma^l(\|x_{pn+1-l} - x_{pn-l}\| + \|y_{pn+1-l} - y_{pn-l}\| - 2d)\right).
\]

After a substitution in (1) with \(x = x_{pn}, y = x_{pn+p}, z = x_{pn+1}, R = d + \gamma^l(\|x_{pn+1-l} - x_{pn-l}\| + \|y_{pn+1-l} - y_{pn-l}\| - 2d)\) and \(r = \|x_{pn+p} - x_{pn}\|\) and using the convexity of the set \(A\) we get the chain of inequalities
\[
d \leq \frac{\left\|x_{pn+p} - x_{pn+1}\right\|}{2} \leq \left(1 - \delta_{\|\|}^l\left(\frac{\|x_{pn+p+k} - x_{2n}\|}{W}\right)\right)W,
\]
where we have denoted \(W = d + \gamma^l(\|x_{pn+1-l} - x_{pn-l}\| + \|y_{pn+1-l} - y_{pn-l}\| - 2d)\).

From (15) we obtain the inequality
\[
\delta_{\|\|}^l\left(\frac{\|x_{pn+p+k} - x_{2n}\|}{W}\right) \leq \frac{\gamma^lW_{pn+1-l+k, pn-l+k}(x, y)}{W}.
\]

From the uniform convexity of \(X\) is follows that \(\delta_{\|\|}^l\) is strictly increasing and therefore there exists its inverse function \(\delta_{\|\|}^{-1}\), which is strictly increasing too. From (16) we get
\[
\|x_{pn} - x_{pn+p}\| \leq W\delta_{\|\|}^{-1}\left(\frac{\gamma^lW_{pn+1-l, pn-l}(x, y)}{W}\right).
\]

By the inequality \(\delta_{\|\|}^l(t) \geq Ct^q\) it follows that \(\delta_{\|\|}^{-1}(t) \leq \left(\frac{t}{C}\right)^{1/q}\).

From (17) and the inequalities
\[
d \leq d + \gamma^l(\|x_{pn+1-l} - x_{pn-l}\| + \|y_{pn+1-l} - y_{pn-l}\| - 2d) \\
\leq \|x_{pn+1-l} - x_{pn-l}\| + \|y_{pn+1-l} - y_{pn-l}\|
\]
we obtain
\[
\|x_{pn} - x_{pn+p}\| \leq \left(\frac{\gamma^lW_{pn+1-l, pn-l}}{C(d + \gamma^lW_{pn+1-l, pn-l})}\right)^{1/2} \frac{W_{pn+1-l, pn-l}(x, y)}{Cd}.
\]

There exists a unique pair \((z, v) \in A_1 \times A_1\) such that \(\|z - T(z, v)\| = d\) and \(z\) is a limit of the sequence \(\{x_{pn}\}_{n=1}^{\infty}\) for any \((x, y) \in A_1 \times A_1\).

After a substitution with \(l = pn\) and \(k = 0\) in (18) we get the inequality
\[
\sum_{n=1}^{\infty} (\|x_{pn} - x_{pn+p}\| + \|y_{pn} - y_{pn+p}\|) \leq P_{0,1}(x, y)\sqrt{\frac{W_{0,1}(x, y)}{Cd}} \sum_{n=1}^{\infty} (\psi^q)^{pn} = P_{0,1}(x, y)\sqrt{\frac{W_{0,1}(x, y)}{Cd}} \frac{\psi^q^{pn}}{1 - \psi^{pn}}.
\]
and consequently the series $\sum_{n=1}^{\infty} (x_{pm} - x_{pm+p})$ is absolutely convergent. Thus for any $m \in \mathbb{N}$ there holds $z = x_{pm} - \sum_{n=m}^{\infty} (x_{pm} - x_{pm+p})$ and therefore we get the inequality

$$
\|z - x_{pm}\| \leq \sum_{n=m}^{\infty} \|x_{pm} - x_{pm+p}\| \leq P_{0,1}(x, y) \sqrt{\frac{W_{0,1}(x, y)}{Cd}} \frac{1}{1 - \sqrt[3]{\gamma}}^pm.
$$

The proof for $\|v - y_{pm}\|$ can be done in a similar fashion. After a substitution with $l = 1 + 2i$ in (18) we obtain

$$
\|x_{pm+i} - x_{pm+p+i}\| \leq P_{pm-1,pm}(x, y) \sqrt{\frac{W_{pm-1,pm}(x, y)}{Cd}} (\sqrt[3]{\gamma})^{1+2i}.
\tag{19}
$$

From (19) we get that there holds the inequality

$$
\|x_{pm} - x_{p(n+m)}\| \leq \sum_{i=0}^{m-1} \|x_{pm+i} - x_{pm+p(i+1)}\|
\leq \sum_{i=0}^{m-1} P_{pm-1,pm}(x, y) \sqrt{\frac{W_{pm-1,pm}(x, y)}{Cd}} (\sqrt[3]{\gamma})^{1+pi}
= \left\{\begin{array}{ll}
P_{pm-1,pm}(x, y) & \sqrt{1 + \gamma^{m-1}} (\sqrt[3]{\gamma})^{1+pi} \\
1 & \sqrt{1 - \sqrt[3]{\gamma}}
\end{array}\right.
\tag{20}
$$

and after letting $m \to \infty$ in (20) we obtain the inequality

$$
\|x_{pm} - z\| \leq P_{pm-1,pm}(x, y) \sqrt{\frac{W_{pm-1,pm}(x, y)}{Cd}} \frac{\sqrt[3]{\gamma}}{1 - \sqrt[3]{\gamma}}.
$$

The proof for $\|y_{pm} - v\|$ can be done in a similar fashion. \(\square\)

5. Applications

Let $\varphi, \psi : [1, +\infty) \to [1, +\infty)$ be such that $\max\{\varphi(x), \psi(x)\} \leq x$ for any $x \in [1, +\infty)$. Let us define the function $f(x, y) = \lambda + (1 - \lambda)(\mu\varphi(x) + (1 - \mu)\psi(y))$. Let us consider the system of equations

$$
\left|\begin{array}{l}
x^p + [\lambda + (1 - \lambda)(\mu\varphi(x) + (1 - \mu)\psi(y))]^p = 2 \\
y^p + [\lambda + (1 - \lambda)(\mu\psi(y) + (1 - \mu)\varphi(x))]^p = 2 \\
x - f(f(x, y), f(y, x), f(f(y, x), f(x, y))) = 0 \\
y - f(f(y, x), f(x, y), f(f(x, y), f(y, x))) = 0
\end{array}\right.
\tag{21}
$$

for $x, y \geq 0$ and $\lambda, \mu \in (0, 1)$.

Let $A_1 = \{(x, 0, 0) : x \geq 1\}$, $A_2 = \{(0, x, 0) : x \geq 1\}$, $A_3 = \{(0, 0, x) : x \geq 1\}$ be subsets of $\mathbb{R}^3, \|\cdot\|_p$, $p \in (1, \infty)$. Let us define the map $T$ by $T((x, 0, 0), (y, 0, 0)) = (0, f(x, y), 0)$; $T((0, x, 0), (0, y, 0)) = (0, 0, f(x, y))$; $T((0, 0, x), (0, 0, y)) = (f(x, y), 0, 0)$ for some $\lambda, \mu \in (0, 1)$. It is easy to see that for any $x, y \geq 1$ there holds $f(x, y) \geq 1$ and therefore $T : A_i \times A_i \to A_{i+1}$.
From the inequality, using that \((1 - (1 - \lambda))\mu - (1 - \lambda)(1 - \mu) = \lambda\)

\[
S_9 = \|T((x, 0, 0), (y, 0, 0)) - T((0, u, 0), (0, v, 0))\|_p \\
= \|(0, f(x,y), f(u,v))\|_p = \sqrt{\|f(x,y)\|^p + \|f(u,v)\|^p} \\
\leq \sqrt{\lambda^p + \lambda^p + (1 - \lambda)\sqrt{|\mu\phi(x) + (1 - \mu)\psi(y)|^p + |\mu\phi(u) + (1 - \mu)\psi(v)|^p}} \\
\leq \lambda \sqrt{\lambda^p + (1 - \lambda)\mu \sqrt{|\phi(x)|^p + 1|\psi(y)|^p + (1 - \lambda)(1 - \mu) \sqrt{|\psi(y)|^p + |\psi(v)|^p}} \\
\leq \lambda \sqrt{\lambda^p + (1 - \lambda)\mu \sqrt{|x|^p + |u|^p + (1 - \lambda)(1 - \mu) \sqrt{|y|^p + |v|^p}} \\
= \lambda \text{dist}(A_1, A_2) + (1 - \lambda)\mu \|x - u\|_2 + (1 - \lambda)(1 - \mu) \|y - v\|_p
\]

and

\[
S_{10} = \|T((x, 0, 0), (y, 0, 0)) - T((0, u, 0), (0, v, 0))\|_p \\
= \|T((0, x, 0), (0, y, 0)) - T((0, 0, u), (0, 0, v))\|_p \\
= \|T((0, 0, x), (0, 0, y)) - T((0, 0, 0), (0, 0, 0))\|_p
\]

it follows that \(T\) satisfies the conditions of Theorem 2. Therefore there exist \((z, z)\), which is a coupled best proximity point of \(T\) in \(A_1 \times A_1\) and it is easy to see that \(z = (1, 0, 0)\). Consequently \((z, z)\) is the unique solution of the system of equations

\[
\begin{align*}
\|x - T(x, y)\|_p &= 2 \\
\|y - T(y, x)\|_p &= 2 \\
x - T^3(x, y) &= 0 \\
y - T^3(y, x) &= 0,
\end{align*}
\]

which is the solution of (21).

If we try to solve (21) with the use of some Computer Algebraic System, for example Maple, the software could not find the exact solution even for not too complicated functions \(p = 2, \phi(x) = x^{1/2}, \psi(x) = x\). If we try to solve it numerically, Maple finds that \(x = y = 1\), but could not find that this is a solution for every \(\lambda, \mu \in (0, 1)\) and presents two approximations of \(\lambda\) and \(\mu\).

If we consider the particular case \(p = 3, \phi(x) = \sqrt{x}\) and \(\psi(x) = \sqrt[3]{\log(x) + 1}\), then Maple could not solve (21) even numerically.

6. Discussion

It is interesting whether same conclusions can be made for existence of coupled fixed (or best proximity) points \(p\)-cyclic Meir–Keeler maps [16], Reich Maps \(p\)-cyclic maps [25].

We were not able to prove a uniqueness of the coupled best proximity points, as like as [13,16]. We were able to prove just uniqueness of the best proximity points, if obtained by the sequence of successive iterations, which is not the case of \(2\)-cyclic maps. It will be interesting if this gap can be filled.

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