POSTORDER REARRANGEMENT OPERATORS

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Abstract. We investigate the rearrangement of the Haar system induced by the postorder on the set of dyadic intervals in $[0, 1]$ with length greater than or equal to $2^{-N}$. By means of operator norms on $\text{BMO}_N$ we prove that the postorder has maximal distance to the usual lexicographic order.

1. Introduction

Let $\mathcal{D}_N$ be the set of dyadic intervals in $[0, 1]$ with length greater than or equal to $2^{-N}$. Let $\tau$ be any bijective map on $\mathcal{D}_N$ and $(h_I)_{I \in \mathcal{D}_N}$ the $L^\infty$-normalised Haar system. On the space $\text{BMO}_N$ we consider rearrangements of the Haar system induced by the map $\tau$

$$T_\tau: h_I \to h_{\tau(I)}.$$

In recent years boundedness criteria and extrapolation properties for rearrangement operators that rearrange the Haar system have been studied in detail. See, [Sem78, SS81, Sch90, M"ul97, GMP05, GM09, M"ul12, KM13].

In the present work we complement the cited papers by investigating in detail one particular rearrangement and its extremal nature. We introduce the postorder, $\preceq$, on the set of dyadic intervals $\mathcal{D}_N$.

Definition. Let $I, J \in \mathcal{D}_N$. We say $I \preceq J$ if either $I$ and $J$ are disjoint and $I$ is to the left of $J$, or $I$ is contained in $J$.

This specific order defines a bijective map $\tau_N$ on the set $\mathcal{D}_N$, called the postorder rearrangement, that maps the $n^{th}$ interval in postorder onto the $n^{th}$ interval in lexicographic order. Its inverse is denoted by $\sigma_N$.

We show that the postorder has maximal distance to the usual lexicographic order on $\mathcal{D}_N$. We quantify the distance by the product of operator norms

$$\|T_\tau : \text{BMO}_N \to \text{BMO}_N\| \|T_\sigma : \text{BMO}_N \to \text{BMO}_N\|.$$

Particularly, we prove that within a factor of $\sqrt{2}$, on $\text{BMO}_N$, both the operator $T_\tau$ and its inverse $T_\sigma$ reach maximal norm. We denote

$$R^N(\text{BMO}_N) = \sup \{ \|T_\tau : \text{BMO}_N \to \text{BMO}_N\| : \tau : \mathcal{D}_N \to \mathcal{D}_N \text{ bijective} \}.$$

Our main result is

Theorem. For $T = T_\tau$ and $T = T_\sigma$ we have

$$\frac{1}{\sqrt{2}} R^N(\text{BMO}_N) \leq \|T\|_{\text{BMO}_N} \leq R^N(\text{BMO}_N).$$

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This continues the previous study of [MS97], who determine from a different perspective the extremal nature of the postorder and the induced rearrangement. P.F.X. Müller and G. Schechtman show that any block basis of the Haar system \((h_I)_{I \in D_N}\) with respect to the postorder, \(\preceq\), spans spaces that are well isomorphic to \(\ell^p_k\), \(1 < p \neq 2 < \infty\). On the other hand it is easy to find block bases of the Haar system with respect to the lexicographic order (the Rademacher functions) whose span is well isomorphic to \(\ell^2_k\).

The postorder has its origin in computer sciences (see e.g. [BP05, Kn05]). In computer sciences, especially in the design and analysis of algorithms, dyadic trees are commonly used data structures, which enable efficient access to data. Tree traversal algorithms, which systematically walk through a tree and visit each node exactly once, enhance this efficient access. These algorithms define a specific order on the nodes of a tree. This makes it possible to talk about the node following or preceding a given one. The postorder tree traversal visits the left child, then the right child and then the node itself. Considering the dyadic tree structure of \(D_N\) this traversal induces exactly the postorder, \(\preceq\), on \(D_N\). The postorder tree traversal is for example used in the mergesort algorithm, invented by von Neumann in 1945. A more basic application is deallocating memory of all nodes of a tree, i.e. deleting a tree. In calculator programs the postorder tree traversal is used to evaluate postfix notation.

The Mallat algorithm for discrete wavelet transform (DWT) (see [Mal89, Mey93]) determines the wavelet coefficients of a given discrete signal in a specific order which works its way up from the finest level to the coarsest. In case of the Haar transform this order is exactly our postorder, \(\preceq\), cf. figure 1. We discuss the discrete Haar wavelet transform (see e.g. [Wal08]) now in detail. Let \(N \in \mathbb{N}_0\). Suppose a discrete signal on \([0,1]\) is given by the sequence \(c^N = (c^N_0, \ldots, c^N_2^{N-1})\). We process the signal by decomposing it into its trend (approximating coefficients) \(c^{N-1}\) and its fluctuation (detail coefficients) \(d^{N-1}\):

\[
c_k^{N-1} = \frac{1}{\sqrt{2}} (c_{2k}^N + c_{2k+1}^N) \quad \text{and} \quad d_k^{N-1} = \frac{1}{\sqrt{2}} (c_{2k}^N - c_{2k+1}^N).
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{calibration.pdf}
\caption{Calculation of the trend \((c^j_k)\) and the fluctuation \((d^j_k)\) for \(0 \leq j \leq 3\).}
\end{figure}
The trend and the fluctuation are two subsignals of $c^N$ with half of its length. The signal $c^{N-1}$ is again decomposed into its trend $c^{N-2}$ and its fluctuation $d^{N-2}$, which are again subsignals of $c^{N-1}$ with half of its length. Successively we compute from $c^j$ the trends $c^{j-1}$ and the fluctuations $d^{j-1}$. Finally, after $N$ steps, we have decomposed the signal $c^N$ into the coarsest information $c_0^0$ and the detail coefficients $(d_j^0)_{j=0}^{N-1}$, where $d_j^0 = (d_{j0}^0, \ldots, d_{j2^j-1}^0)$.

2. Preliminaries

Throughout this paper we will denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of non-negative integers.

Unless stated otherwise: $\ell, k, N \in \mathbb{N}_0$ such that $0 \leq \ell \leq N$ and $0 \leq k \leq 2^{\ell-1}$.

2.1. Floor and ceiling function. The floor function $\lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$ and the ceiling function $\lceil \cdot \rceil : \mathbb{R} \rightarrow \mathbb{Z}$ are defined as follows:

$$\lfloor x \rfloor = \max \{ z \in \mathbb{Z} : z \leq x \}, \quad \lceil x \rceil = \min \{ z \in \mathbb{Z} : z \geq x \}.$$ 

2.2. Dyadic intervals and trees. Dyadic intervals. An interval $I \subseteq [0, 1]$ is called a dyadic interval, if there exist non-negative integers $\ell$ and $k$ with $0 \leq k \leq 2^{\ell-1}$ such that

$$I = I_{\ell,k} = \left[ k/2^\ell, k+1/2^\ell \right].$$

The length of a dyadic interval $I_{\ell,k}$ is given by $|I_{\ell,k}| = 2^{-\ell}$. In the following we consider for fixed $N \in \mathbb{N}_0$ the set of dyadic intervals with length greater than or equal to $2^{-N}$ given by

$$D_N = \{ I_{\ell,k} : 0 \leq \ell \leq N, 0 \leq k \leq 2^{\ell-1} \}.$$ 

Carleson constant. Let $C \subseteq D_N$. We define the Carleson constant of $C$ as follows

$$[C] = \sup_{I \in C} \frac{1}{|I|} \sum_{J \subseteq I, J \in C} |J|.$$ 

If $C$ is non-empty, then $[C] \geq 1$, otherwise $[C] = 0$.

Dyadic trees. See [BP05, Knu05]. A dyadic tree $T$ consists of a set of nodes that is either empty or has the following properties:

1. One of the nodes, say $R$, is designated the root node.
2. The remaining nodes (if any) are partitioned into two disjoint subsets, called the left subtree and the right subtree, respectively, each of which is a dyadic tree.

The definition yields that every node of a tree is the root of some subtree contained in the tree $T$. The root of the left resp. the right subtree described in property (2) is called the left child resp. the right child of the root $R$. Conversely, the root $R$ is called the parent of the left (resp. right) child. We use the terminology of family trees: parent, children, descendant, etc. The nodes of a dyadic tree $T$ can be partitioned into disjoint sets, called levels, depending on the length $\ell$ of the unique path from a node to the root $R$. The root $R$ is at level 0. The lowest level of $T$ is the set of nodes, whose unique path from the node to the root $R$ has maximal length within the tree $T$. The depth of $T$ is the number of levels in $T$. 
that do not contain the root \( R \). A dyadic tree \( T \) is complete, if every node in \( T \) has exactly two children, except the nodes in the lowermost level, which have exactly zero children, cf. figure 2. In the following we consider complete dyadic trees of depth \( N, N \in \mathbb{N}_0 \). The number of nodes in each level \( \ell, 0 \leq \ell \leq N \), is given by \( 2^\ell \) and the total number of nodes in a complete dyadic tree of depth \( N \) is given by \( 2^N + 1 - 1 \).

**The complete dyadic tree \( D_N \).** The set \( D_N \), given by equation (2.1), has a natural dyadic tree structure, cf. figure 2. The root of the complete dyadic tree \( D_N \) is the

![Diagram of complete dyadic tree](image)

**Figure 2.** The dyadic tree structure of \( D_4 \).

The dyadic interval \( I_{0,0} \). The depth of \( D_N \) is equal to \( N \). For an interval \( I_{\ell,k} \in D_N \) the index \( \ell \) denotes its level within the tree and \( k \) its position within the level. The left resp. the right child of an interval \( I_{\ell,k} \in D_N \) is given by

\[
I_{\ell+1,2k} = \left[ \frac{2k}{2^{\ell+1}}, \frac{2k+1}{2^{\ell+1}} \right] \quad \text{resp.} \quad I_{\ell+1,2k+1} = \left[ \frac{2k+1}{2^{\ell+1}}, \frac{2(k+1)}{2^{\ell+1}} \right] .
\]

**Dyadic subtrees.** Let \( I_{\ell,k} \in D_N \). We denote by \( T_{\ell,k}^N \) the complete dyadic subtree of \( D_N \) with root \( I_{\ell,k} \) and depth \( N - \ell \). Note that \( T_{\ell,k}^N = \{ I \in D_N : I \subseteq I_{\ell,k} \} \). Therefore, we get from (2.2), the Carleson constant

\[
|T_{\ell,k}^N| = \frac{1}{|I_{\ell,k}|} \sum_{I \in T_{\ell,k}^N} |I| = N - \ell + 1 .
\]

**2.3. The order on \( D_N \).** See [MS97], [BP05] and [Knob05]. The postorder \( \preceq \) on \( D_N \) is defined as follows.

**Definition 2.1.** Let \( I, J \in D_N \). We say \( I \preceq J \) if either \( I \) and \( J \) are disjoint and \( I \) is to the left of \( J \), or \( I \) is contained in \( J \).

In terms of the dyadic tree structure of \( D_N \) the postorder is defined as follows: children are always smaller than their parent, the left child is always smaller than the right child and smaller than the descendants of the right child, cf. figure 5.

The natural order on the set \( D_N \) is the lexicographic order, \( \leq_l \), on the set \( \{(\ell,k)\} \). The postorder on \( D_N \), in contrast to the lexicographic order depends on the depth
The postorder works its way up from the leftmost node in the lowermost level to the root of the dyadic tree $D_N$. Therefore, it is clear from the definition that the root of the dyadic tree $D_N$ has postorder ordinal number $2^{N+1} - 1$, which is the total number of nodes contained in the tree $D_N$.

Observe that $I_{1,0}$ is the left child and $I_{1,1}$ is the right child of the root $I_{0,0}$. Hence, the complete dyadic subtree $T_{1,0}^N$ resp. $T_{1,1}^N$ of $D_N$ is the left resp. right subtree of the root $I_{0,0}$. The definition of the postorder yields that the left subtree contains the ordinal numbers $1, \ldots, 2^N - 1$ and the right subtree the ordinal numbers $2^N, \ldots, 2^{N+1} - 2$.

2.4. The order intervals. Let $J_1, J_2 \in D_N$. An order interval with respect to the postorder, $\preceq$, is given by

$$(2.5) B_N(J_1, J_2) = \{ I \in D_N : J_1 \preceq I \preceq J_2 \},$$

and with respect to the lexicographic order, $\leq_l$, by

$$(2.6) E(J_1, J_2) = \{ I \in D_N : J_1 \leq_l I \leq_l J_2 \}.$$  

The following definition and proposition is taken from [MS97] and describes order intervals with respect to the postorder, $\preceq$.

**Definition 2.2.** Let $I, J \in D_N$ with $I \subseteq J$.

1. The cone $C = C(I, J)$ of dyadic intervals between $I$ and $J$ is the unique collection of dyadic intervals $C = \{C_1, \ldots, C_n\}$, where $n = \log_2 \frac{|J|}{|I|} + 1$, satisfying $C_1 = I$, $C_n = J$, $|C_i| = \frac{1}{2}|C_{i+1}|$ and $C_i \subseteq C_{i+1}$ for $1 \leq i \leq n - 1$.

2. The right fill-up of the cone $C$ is the collection of dyadic intervals $R = R(I, J) = \bigcup_{i=1}^{n-1} U_{i+1}$, where $U_{i+1} = \{ U \in D_N : U \subseteq C_{i+1} \setminus C_i \}$, if $C_i$ is the left half of $C_{i+1}$ and $C_{i+1} = U_{i+1} \cup C_i$.

**Figure 3.** Cone and right fill-up given by the dyadic intervals $I = I_{4,4}$ and $J = I_{1,0}$ in $D_4$.

**Proposition 2.3.** Let $J_1, J_2 \in D_N$ and $J_1 \preceq J_2$. For the postorder order interval $B_N(J_1, J_2)$ there exists a unique collection $L = \{ L_1, \ldots, L_m \}$ of pairwise disjoint dyadic intervals satisfying

1. $|L_i| < |L_{i-1}|$, if $2 \leq i \leq m - 1$;
(2) \(|L_m| \leq |L_{m-1}|\), if \(m \geq 2\);
(3) \(L_{i+1}\) lies to the right of \(L_i\) and the closures \(\overline{L_i}\) and \(\overline{L_{i+1}}\) intersect in exactly one point, the left endpoint of \(\overline{L_{i+1}}\);
(4) \(J_1 \subseteq L_1, J_2 = L_m\) and

\[ B^N(J_1, J_2) = \mathcal{C}(J_1, L_1) \cup \mathcal{R}(J_1, L_1) \cup_{i=2}^m \mathcal{M}_i, \]

where \(\mathcal{M}_i = \{I \in \mathcal{D}_N : I \subseteq L_i\}\).

**Remark 2.4.** Note that the intervals \((L_i)_{i=1}^m\) are the maximal (with respect to inclusion) dyadic intervals in the postorder order interval \(B^N(J_1, J_2)\).

2.5. The spaces.

**Haar system and Haar support.** We define the \(L^\infty\)-normalised Haar system \((h_I)_{I \in \mathcal{D}_N}\) as follows:

\[
h_I = \begin{cases} 
1 & \text{on the left half of } I, \\
-1 & \text{on the right half of } I, \\
0 & \text{otherwise}.
\end{cases}
\]

Let \((x_I)_{I \in \mathcal{D}_N}\) be a real sequence and let \(f = \sum_{I \in \mathcal{D}_N} x_I h_I\). The Haar support of \(f\) is the set \(\{I \in \mathcal{D}_N : x_I \neq 0\}\). The Haar support of \(f\) is contained in a non-empty collection of dyadic intervals \(\mathcal{C} \subseteq \mathcal{D}_N\) if and only if \(f = \sum_{I \in \mathcal{C}} x_I h_I\). We denote by \(\mathcal{M}(\mathcal{C})\) the space of all functions \(f\) that have Haar support in a non-empty collection \(\mathcal{C} \subseteq \mathcal{D}_N\).

**Dyadic BMO and the dyadic Hardy spaces \(H^p\).** We define here the known spaces \(BMO_N\) and \(H^p_N\), for fixed \(N \in \mathbb{N}_0\), (see e.g. [Mü88]). Let \((x_I)_{I \in \mathcal{D}_N}\) be a real sequence and \(f = \sum_{I \in \mathcal{D}_N} x_I h_I\). We define

\[
\|f\|_{BMO} = \sup_{I \in \mathcal{D}_N} \left( \frac{1}{|I|} \sum_{J \subseteq I} |x_J|^2 |J| \right)^{\frac{1}{2}}
\]
and
\[ \|f\|_{H^p} = \|S(f)\|_{L^p([0,1])}, \quad \text{for } 0 < p < \infty, \]
where \(S(f)\) is the square function of \(f\) defined by
\[ S(f)(t) = \left( \sum_{I \in D_N} |x_I|^2 1_I(t) \right)^{\frac{1}{2}}, \quad t \in [0,1]. \]

Then we define the spaces \(\text{BMO}_N\) and \(H^p_N, 0 < p < \infty\), as follows
\[ \text{BMO}_N = \left( \text{span} \{ h_I : I \in D_N \}, \|\cdot\|_{\text{BMO}} \right), \]
and
\[ H^p_N = \left( \text{span} \{ h_I : I \in D_N \}, \|\cdot\|_{H^p} \right). \]

As stated previously, \(N \in \mathbb{N}_0\) is fixed throughout this paper. Hence, we use the abbreviations
\[ (2.8) \quad \text{BMO} := \text{BMO}_N \quad \text{and} \quad H^p := H^p_N. \]

Paley’s theorem ([Pal32], see also [M"ul05]) asserts that for all \(1 < p < \infty\) there exists a constant \(A_p\) such that for all \(f \in L^p([0,1])\) given by \(f = \sum_{I \in D_N} x_I h_I\)
\[ \frac{1}{A_p} \|f\|_{L^p} \leq \|S(f)\|_{L^p} \leq A_p \|f\|_{L^p}. \]

This theorem identifies \(H^p\) as the dual space of \(H^q\), where \(\frac{1}{p} + \frac{1}{q} = 1\) and \(1 < p < \infty\).

Fefferman’s inequality ([FS72], see also [M"ul05])
\[ (2.9) \quad \left| \int fh \right| \leq 2\sqrt{2} \|f\|_{H^1} \|h\|_{\text{BMO}}, \]
and a theorem to the effect that every continuous linear functional \(L : H^1 \rightarrow \mathbb{R}\) is necessarily of the form \(L(f) = \int f\varphi\) with \(\|\varphi\|_{\text{BMO}} \leq \|L\|\) identify \(\text{BMO}\) as dual space of \(H^1\), (see [FS72], [Gar73], [M"ul05]).

2.6. The operators.

Rearrangements of the Haar system. Let \(\tau\) be a bijective map defined on the set \(D_N\). On \(\text{BMO}\) we study rearrangements of the \(L^\infty\)-normalised Haar system \((h_I)_{I \in D_N}\) given by the rearrangement operator
\[ T_\tau : h_I \mapsto h_{\tau(I)}, \]
and on \(H^p, 0 < p < \infty\), rearrangements of the \(L^p\)-normalised Haar system given by the rearrangement operator
\[ T_{\tau,p} : \frac{h_I}{|I|^\frac{1}{p}} \mapsto \frac{h_{\tau(I)}}{|\tau(I)|^\frac{1}{p}}. \]

A standard argument (given below) yields the following norm estimates for rearrangement operators on \(\text{BMO}\)
\[ (2.10) \quad \sup_{C \subseteq D_N, \text{non-empty}} \left[ \frac{\|\tau_N(C)\|^{\frac{1}{2}}}{|C|^{\frac{1}{2}}} \right] \leq \|T_\tau\|_{\text{BMO}} \leq (N+1)^\frac{1}{2}. \]

Note that the lower bound in (2.10) is always greater than or equal to one.
Let \( x = \sum_{I \in \mathcal{D}_N} x_I h_I \). Then
\[
\|T_\tau x\|_{\text{BMO}}^2 = \sup_{I \in \mathcal{D}_N} \frac{1}{|I|} \sum_{J \subseteq I} |x_{\tau^{-1}(J)}|^2 |J| \leq \sup_{I \in \mathcal{D}_N} |x_I|^2 \|D_N\| \leq \|x\|_{\text{BMO}}^2 \|D_N\|.
\]
Definition (2.2) yields \( \|D_N\| = N + 1 \). Let \( C \subseteq \mathcal{D}_N \) be any non-empty collection of dyadic intervals. Let \( x = \sum_{I \in C} x_I h_I \). Then
\[
\|x\|_{\text{BMO}} = [C]^{\frac{1}{2}} \quad \text{and} \quad \|T_\tau x\|_{\text{BMO}} = \|\tau(C)\|^{\frac{1}{2}}.
\]
Let \( x = \sum_{I \in C} x_I h_I \) for some non-empty collection of dyadic intervals \( C \subseteq \mathcal{D}_N \). The above argument provides the following rough upper bound
\[
(2.11) \quad \|T_\tau x\|_{\text{BMO}} \leq \|x\|_{\text{BMO}} \|\tau(C)\|^{\frac{1}{2}}.
\]

The adjoint operator of a rearrangement operator is again a rearrangement operator induced by the inverse rearrangement. By the duality of \( H^1 \) and \( \text{BMO} \) we have that the operator \( T_\tau \) on \( \text{BMO} \) is the adjoint operator of \( T_{\tau^{-1},1} \) on \( H^1 \) with
\[
(2.12) \quad \frac{1}{C_F} \|T_\tau\|_{\text{BMO}} \leq \|T_{\tau^{-1},1}\|_{H^1} \leq C_F \|T_\tau\|_{\text{BMO}},
\]
where \( C_F = 2\sqrt{2} \) is the constant appearing in Fefferman’s inequality (2.9).

**Interpolation and extrapolation of rearrangement operators.** See [GMP05, M"ul05].

The following interpolation resp. extrapolation theorem provides a tool that enables one to deduce norm estimates for the rearrangement operators \( T_{\tau,p} \) on \( H^p \) for every \( 0 < p < 2 \) from norm estimates of some rearrangement operator \( T_{\tau,p_0} \) on \( H^{p_0} \), \( 0 < p_0 < 2 \). The left-hand side inequality corresponds to an extrapolation based on Pisier’s extrapolation norm (see [GMP05]). The right-hand side inequality is obtained by a standard interpolation argument. Note that \( \|T_{\tau,2}\|_{H^2} = 1 \).

**Theorem 2.5.** For all \( 0 < s < r < 2 \) there exists a constant \( c_{r,s} \) such that
\[
\frac{1}{c_{r,s}} \|T_{\tau,s}\|_{H^r}^{\frac{r}{r-s}} \leq \|T_{\tau,r}\|_{H^r}^{\frac{r}{r-s}} \leq c_{r,s} \|T_{\tau,s}\|_{H^r}^{\frac{r}{r-s}}.
\]

The duality of \( H^p \) and \( H^q \), \( 1 < q < 2 \), \( \frac{1}{p} + \frac{1}{q} = 1 \), gives the following corollary to Theorem 2.5. Recall that the adjoint rearrangement operator on \( H^p \) coincides with the inverse rearrangement operator on \( H^q \).

**Corollary 2.6.** For all \( 2 < p < \infty \) there exists a constant \( c_p \) such that
\[
\frac{1}{c_p} \|T_{\tau,p}\|_{H^p} \leq \|T_{\tau^{-1},1}\|_{H^1}^{1-\frac{1}{q}} \leq c_p \|T_{\tau,p}\|_{H^p}.
\]

**Remark 2.7.** Observe that by the above theorem and corollary rearrangement operators \( T_{\tau,p} \) on \( H^p \), \( 0 < p < \infty \), induced by any bijective map \( \tau \) acting on \( \mathcal{D}_N \), have the norm estimate
\[
\|T_{\tau,p}\|_{H^p} \leq c_p (N + 1)^{\frac{1}{2} - \frac{1}{p}}.
\]
3. The main theorem

Let $\tau_N$ be the bijective map on the dyadic intervals that associates to the $n^{th}$ interval in postorder the $n^{th}$ interval in lexicographic order, cf. figure 5. This rearrangement is called *postorder rearrangement*. Its inverse, which associates to the $n^{th}$ interval in lexicographic order the $n^{th}$ interval in postorder, is denoted by $\sigma_N$.

The rearrangements $\tau_N$ and $\sigma_N$ induce rearrangement operators on BMO and on the $H^p$-spaces. On BMO we consider the rearrangement operators

$$T_{\tau_N} : h_I \mapsto h_{\tau_N(I)} \quad \text{and} \quad T_{\sigma_N} : h_I \mapsto h_{\sigma_N(I)}$$

and obtain the following norm estimates for these rearrangement operators applied to functions with Haar support in the sets $T_{\ell,0}^N = \{ I \in D_N : I \subseteq I_{\ell,0} \}$ and $D_{N-\ell}$.

Recall that $\mathcal{M}(T_{\ell,0}^N) = \text{span} \{ h_I : I \in T_{\ell,0}^N \}$ and $\mathcal{M}(D_{N-\ell}) = \text{span} \{ h_I : I \in D_{N-\ell} \}$.
Theorem 3.1. Let \( N \in \mathbb{N}_0 \) and \( 0 \leq \ell \leq N \). Let \( T = T_{\tau_N}|_{\mathcal{M}(T_{\ell,0}^N)} \) or \( T = T_{\sigma_N}|_{\mathcal{M}(D_{N-\ell})} \). Then

\[
\frac{1}{\sqrt{2}}(N - \ell + 1)^{\frac{1}{2}} \leq \|T\|_{\text{BMO}} \leq (N - \ell + 1)^{\frac{1}{2}}.
\]

This theorem in combination with the general upper bound in (2.10) reveals the extremal nature of the rearrangements \( \tau_N \) and \( \sigma_N \) in the sense that for \( T = T_{\tau_N} \) resp. \( T = T_{\sigma_N} \) we have

\[
\frac{1}{\sqrt{2}}R^N(\text{BMO}) \leq \|T\|_{\text{BMO}} \leq R^N(\text{BMO}),
\]

where \( R^N(\text{BMO}) = \sup \{||T_\tau : \text{BMO} \rightarrow \text{BMO}|| : \tau : D_N \rightarrow D_N \text{ bijective}\} \).

Obviously, the lower bound in (3.1) is the important one for this result and the statement of Theorem 3.1. The upper bound in (3.1) is the trivial one that originates from the depth (in the sense of dyadic trees) of the sets \( D_{N-\ell} \) resp. \( T_{\ell,0}^N \).

Theorem 3.2 provides a tool that enables one to gain insight into the rearrangement operators \( T_{\sigma_N} \) applied to spaces of functions with Haar support in a lexicographic order interval. In Theorem 3.1 we have already seen that on the lexicographic order interval \( D \) one can read off the upper bound from the tree representation of \( N \), cf. figure 4.

Theorem 3.2. Let \( N \in \mathbb{N}_0 \). Let \( \mathcal{E} = \mathcal{E}(E_1, E_2) \) be the lexicographic order interval given by the dyadic intervals \( E_1, E_2 \in D \) with \( E_1 \leq E_2 \). Then

\[
\|T_{\sigma_N}|_{\mathcal{M}(\mathcal{E})}\|_{\text{BMO}}^2 \leq N - \log_2 \frac{1}{|I_1|} + 2,
\]

where \( L_1 \) is the maximal (with respect to inclusion) dyadic interval in the postorder order interval \( B^N(\sigma_N(E_1), \sigma_N(E_2)) \) that contains the left endpoint \( \sigma_N(E_1) \).

Lexicographic order intervals \( \mathcal{E}(E_1, E_2) \) with large Carleson constant are given by endpoints \( E_1, E_2 \) which satisfy the property that \( \log_2 \frac{1}{|I_1|} \) is much smaller than \( \log_2 \frac{1}{|E_2|} \). The upper bound in Theorem 3.2 depends for these order intervals only on the right endpoint \( E_2 \). Particularly, the upper bound is given by

\[
\|T_{\sigma_N}|_{\mathcal{M}(\mathcal{E})}\|_{\text{BMO}}^2 \leq \log_2 \frac{1}{|E_2|} + 2.
\]

The duality relation of \( H^1 \) and BMO, in particular the norm equivalence in equation (2.12), and the interpolation resp. extrapolation procedure in Theorem 2.3 and Corollary 2.4 give equivalent norm estimates as in Theorem 3.1 for the rearrangement operators on \( H^p \), \( 0 < p < \infty \), given by

\[
T_{\tau_N,p} : \frac{h_I}{|I|^{\frac{1}{p}}} \mapsto \frac{h_{\tau_N(I)}}{|\tau_N(I)|^{\frac{1}{p}}} \text{ resp. } T_{\sigma_N,p} : \frac{h_I}{|I|^{\frac{1}{p}}} \mapsto \frac{h_{\sigma_N(I)}}{|\sigma_N(I)|^{\frac{1}{p}}}.
\]
Corollary 3.3. For all $0 < p < \infty$ there exists a constant $C_p$ such that for all $N \in \mathbb{N}_0$, $0 \leq \ell \leq N$ and $T = T_{\tau_N,p}\big|_{M(T^{N}_{\ell,0})}$ or $T = T_{\sigma_N,p}\big|_{M(D_{N-1})}$ the following holds

$$2^{-\frac{1}{p} - \frac{1}{2}} \frac{(N - \ell + 1)^{\frac{1}{p} - \frac{1}{2}}}{C_p} \leq \| T \|_{H^p} \leq C_p (N - \ell + 1)^{\frac{1}{p} - \frac{1}{2}}.$$  

Remark 3.4. By the convexification method ([LIT79, CTS80], see also [MP] for the concrete specialisation to Hardy spaces) one obtains the same result as in Corollary 3.3 for the more general Triebel Lizorkin spaces.

Corollary 3.3 gives, considering the general upper bound in Remark 2.7, the same extremality statement for the rearrangement operators $T = T_{\tau_N,p}$ or $T = T_{\sigma_N,p}$ on the spaces $H^p$, $0 < p < \infty$. For all $0 < p < \infty$ there exists a constant $B_p$ such that

$$2^{-\frac{1}{p} - \frac{1}{2}} B_p R^N(H^p) \leq \| T \|_{H^p} \leq R^N(H^p),$$

where $R^N(H^p) = \sup \{ \| T_\tau : H^p \to H^p \| : \tau : \mathcal{D}_N \to \mathcal{D}_N \text{ bijective} \}$.

4. Proof of Theorem 3.1

4.1. Parameters associated with the postorder rearrangement. For the proof of the main theorem we need formulae that describe the map $\tau_N$ precisely. Recall that $\tau_N$ maps the $n$th dyadic interval in postorder onto the $n$th dyadic interval in lexicographic order. First of all we give formulae that describe the assignment of postorder ordinal numbers and lexicographic ordinal numbers to dyadic intervals in $\mathcal{D}_N$. We denote by $a^{\ell}(k)$ the postorder ordinal number and by $b^{\ell}(k)$ the lexicographic ordinal number of the dyadic interval $I^{\ell,k} \in \mathcal{D}_N$.

The assignment rule for a lexicographic ordinal number to a dyadic interval $I^{\ell,k}$ is given by

$$b^{\ell}(k) = \left(\sum_{i=0}^{\ell-1} 2^i\right) + k + 1 = 2^\ell + k.$$  

We can determine from the ordinal number $b^{\ell}(k)$ the level $\ell$ and the position $k$ of the associated interval $I^{\ell,k}$:

$$\ell = \lfloor \log_2 b^{\ell}(k) \rfloor \quad \text{and} \quad k = b^{\ell}(k) - 2^\ell.$$  

The assignment rule for postorder ordinal numbers to the dyadic intervals is more difficult than in the lexicographic case. Let $j \in \mathbb{N}$ with dyadic expansion $j = \sum \epsilon_i 2^i$. We define $m(j) = \min \{ i \in \mathbb{N} : \epsilon_i \neq 0 \}$.

Lemma 4.1. Let $N \in \mathbb{N}_0$, $0 \leq \ell \leq N$ and $0 \leq k \leq 2^\ell - 1$. The postorder ordinal number of the dyadic interval $I^{\ell,k} \in \mathcal{D}_N$ is given by

$$a^{\ell}(k) = (k + 1) (2^{N-\ell+1} - 1) + \sum_{j=1}^{k} m(j).$$

Proof. Let $1 \leq j \leq 2^\ell - 1$ and let

$$t^{\ell}(j) = a^{\ell}(j) - a^{\ell}(j-1) - 1,$$
where $a^\ell(j-1)$ and $a^\ell(j)$ are the postorder ordinal numbers of two successive dyadic intervals in level $\ell$. This gives the recursive formula $a^\ell(j) = a^\ell(j-1) + t^\ell(j) + 1$ and thereby the assignment rule for the postorder ordinal number:

\begin{equation}
(4.4) \quad a^\ell(k) = a^\ell(0) + k + \sum_{j=1}^{k} t^\ell(j).
\end{equation}

The definition of the postorder and the dyadic tree structure of $\mathcal{D}_N$ yield

\begin{equation}
(4.5) \quad a^\ell(0) = 2^{N-\ell+1} - 1, \quad \text{for all } 0 \leq \ell \leq N.
\end{equation}

In the following we determine a formula for $t^\ell(j)$, $1 \leq j \leq 2^{\ell} - 1$. To this end, we give formulae that associate the postorder ordinal number of a dyadic interval with the postorder ordinal number of its parent. We consider the dyadic interval $I_{\ell,k} \in \mathcal{D}_N$ with the postorder ordinal number $a^\ell(k)$ and its children $I_{\ell+1,2k}$ and $I_{\ell+1,2k+1}$ with the postorder ordinal numbers $a^{\ell+1}(2k)$ and $a^{\ell+1}(2k+1)$. By the definition of the postorder we have $a^{\ell+1}(2k) < a^{\ell+1}(2k+1) < a^\ell(k)$. Furthermore, $a^{\ell+1}(2k)$ is smaller and $a^{\ell+1}(2k+1)$ is greater than the ordinal numbers of the descendants of $I_{\ell+1,2k+1}$. The number of descendants of $I_{\ell+1,2k+1}$ is $2^{N-\ell-2}$. Hence, the definition of the postorder yields the following recursions:

\begin{equation}
(4.6) \quad a^\ell(k) = a^{\ell+1}(2k+1) + 1 \quad \text{and} \quad a^\ell(k) = a^{\ell+1}(2k) + 2^{N-\ell},
\end{equation}

where $0 \leq \ell \leq N$ and $0 \leq k \leq 2^\ell - 1$. Induction shows that for $1 \leq i < \ell$

\begin{equation}
(4.7) \quad a^{\ell-i}(s-1) = a^{\ell}(2^is-1) + i \quad \text{and} \quad a^{\ell-i}(s) = a^{\ell}(2^is) + 2^{N-\ell+1}(2^i-1),
\end{equation}

where $1 \leq s \leq 2^{\ell-i} - 1$.

Now we can determine an explicit formula for $t^\ell(j)$. If $j$ is odd, then the formulae in (4.6) yield $a^{\ell-1} \left(\frac{j-1}{2}\right) = a^\ell(j)+1$ and $a^{\ell-1} \left(\frac{j+1}{2}\right) = a^\ell(j-1)+2^{N-\ell+1}$. Therefore, by equation (4.3)

\begin{equation}
(4.8) \quad t^\ell(j) = 2^{N-\ell+1} - 2, \quad \text{if } j \text{ is odd}.
\end{equation}

If $j$ is even, then there exists an integer $i$, $1 \leq i < \ell$, given by $i = m(j)$, and an odd integer $s$, $1 \leq s \leq 2^{\ell-i} - 1$ such that $j = 2^is$. Equation (4.3) and the formulae in (4.7) yield

\begin{equation}
(4.9) \quad t^\ell(j) = m^\ell(j) + 2^{N-\ell+1} - 2.
\end{equation}

Note that $m(j) = 0$, if $j$ is odd. Putting this into equation (4.4) yields the statement.

Given the ordinal numbers of a dyadic interval with respect to both the postorder and the lexicographic order on $\mathcal{D}_N$ we can describe the postorder rearrangement $\tau_N$ as follows. Let $I_{L,K}$ is an ordinal number of the dyadic interval $I_{L,K} \in \mathcal{D}_N$. The
postorder rearrangement $\tau_N$ is then the bijective map on $D_N$ that maps the dyadic interval $I_{l,k}$ onto the dyadic interval $I_{l,K}$, cf. figure 5.

In the following section we describe the determination of $L$ and $K$ such that $a^{\ell}(k) = 2^L + K$. In the following we use the notation

$$\text{Level}(a^{\ell}(k)) = L \quad \text{and} \quad \text{Pos}(a^{\ell}(k)) = K.$$ 

According to (4.11) we have

$$\text{Level}(a^{\ell}(k)) = \lfloor \log_2 (a^{\ell}(k)) \rfloor \quad \text{and} \quad \text{Pos}(a^{\ell}(k)) = a^{\ell}(k) - 2^\text{Level}(a^{\ell}(k)).$$ 

The following two Lemmata give formulae for $\text{Level}(a^{\ell}(k))$ and $\text{Pos}(a^{\ell}(k))$, which do not involve the postorder ordinal number $a^{\ell}(k)$ but only the level $\ell$ and the position $k$ of the corresponding dyadic interval $I_{l,k}$.

**Lemma 4.2.** Let $N \in \mathbb{N}_0$ and $0 \leq \ell \leq N$. For all $0 \leq k \leq 2^\ell - 1$ we have

$$\text{Level}(a^{\ell}(k)) = \lfloor \log_2 (k + 1) \rfloor + N - \ell.$$ 

**Proof.** The definition of the postorder yields $a^{\ell}(0) = 2^{N-\ell+1} - 1$. Therefore, by equation (4.10) we have $\text{Level}(a^{\ell}(0)) = N - \ell$. Now we show that for all $1 \leq s \leq \ell$

$$\text{Level}(a^{\ell}(2^{s-1})) = \cdots = \text{Level}(a^{\ell}(2^s - 1)) = s + N - \ell.$$ 

By Lemma 4.1 we have

$$a^{\ell}(2^{s-1}) = 2^{N-\ell+s} + 2^{N-\ell+1} - 2^{s-1} - 1 + \sum_{j=1}^{2^{s-1}-1} m(j).$$ 

Recall that for $j \in \mathbb{N}$ given by its dyadic expansion $j = \sum \epsilon_i 2^i$ we have $m(j) = \min \{i \in \mathbb{N} : \epsilon_i \neq 0\}$. Hence, $m(j) = 0$ for all odd integers $j$. We can split the sum on the right-hand side of equation (4.13) as follows

$$\sum_{j=1}^{2^{s-1}-1} m(j) = \sum_{j=1}^{2^{s-2}} m(2j) + \sum_{j=1}^{s-2} \sum_{j_1=1}^{j-1} m(2^{j_1} + 2j_2) + \cdots$$

$$\cdots + \sum_{j_1=1}^{s-2} \cdots \sum_{j_{s-2}=1}^{j_{s-3}-1} m(2^{j_1} + \cdots + 2^{j_{s-2}}).$$ 

By definition, $m(2^{j_0}) = j_0$ and $m(2^{j_1} + \cdots + 2^{j_i}) = j_i$ for $2 \leq i \leq s - 2$. Hence,

$$\sum_{j=1}^{2^{s-1}} m(j) = \sum_{j_0=1}^{s-1} j_0 + \sum_{j_1=1}^{s-2} j_2 + \cdots + \sum_{j_{s-2}=1}^{s-2} \sum_{j_{s-3}=1}^{j_{s-2}} j_{s-2}$$

$$= \sum_{k=1}^{s-1} \binom{s-1}{k} = 2^{s-1} - 1.$$ 

Putting this into formula (4.13) we get $a^{\ell}(2^{s-1}) = 2^{N-\ell+1} + 2^{N-\ell+s} - 2$. Lemma 4.1 yields

$$a^{\ell}(2^s - 1) = 2^{N-\ell+s+1} - 2^s - m(2^s) + \sum_{j=1}^{2^s} m(j).$$
Note that $m(2^s) = s$. By equation (4.14) we have $a^\ell(2^s - 1) = 2^{N-\ell+s+1} - s - 1$.

Equation (4.10) yields

$$\text{Level}(a^\ell(2^s-1)) = \lfloor \log_2 (2^{N-\ell+1} + 2^{N-\ell+s} - 2) \rfloor = N - \ell + s,$$

$$\text{Level}(a^\ell(2^s - 1)) = \lfloor \log_2 (2^{N-\ell+1+s} - s - 1) \rfloor = N - \ell + s.$$

Note that the map $k \mapsto \text{Level}(a^\ell(k))$ is monotonically increasing for all $0 \leq k \leq 2^\ell - 1$. Therefore, (4.12) is proven.

Let $0 \leq k \leq 2^\ell - 1$ and $s = \lfloor \log_2 (k + 1) \rfloor$. Then $2^{s-1} \leq k \leq 2^s - 1$ and equation (4.10) yields Level$(a^\ell(k)) = s + N - \ell$. □

The next Lemma describes the determination of $K = \text{Pos}(a^\ell(k))$. As stated previously, Pos$(a^\ell(k))$ depends on $L = \text{Level}(a^\ell(k))$, which was determined in Lemma 4.2. Recall that for $j \in \mathbb{N}$ with dyadic expansion $j = \sum \epsilon_i 2^i$ we have $m(j) = \min \{i \in \mathbb{N} : \epsilon_i \neq 0\}$.

**Lemma 4.3.** Let $N \in \mathbb{N}_0$ and $0 \leq \ell \leq N$. Then $\text{Pos}(a^\ell(0)) = 2^{N-\ell} - 1$ and for all $0 < k \leq 2^\ell - 1$

$$\text{Pos}(a^\ell(k)) = (k + 1) (2^{N-\ell+1} - 1) + 2^{L-N+\ell-1} - 2^L - 1 + \sum_{j=2^{L-N+\ell-1+1}}^k m(j).$$

**Proof.** Recall that $a^\ell(0) = 2^{N-\ell+1} - 1$ and Level$(a^\ell(0)) = N - \ell$. Therefore, by equation (4.10) we have Pos$(a^\ell(0)) = 2^{N-\ell} - 1$.

Fix one dyadic interval $I_{\ell,k}$, $k > 0$ with corresponding postorder ordinal number $a^\ell(k)$. Let $L = \text{Level}(a^\ell(k))$. Lemma 4.2 states that Level$(a^\ell(j)) = L$, for all $2^{L-N+\ell-1} \leq j \leq 2^{L-N+\ell} - 1$. Recall from the proof of Lemma 4.1 that $a^\ell(j) = a^\ell(j - 1) + 1 + t^\ell(j)$, where $t^\ell(j) = m(j) + 2^{N-\ell+1} - 2$. Hence, by equation (4.10) we have the following recursive formula

$$\text{Pos}(a^\ell(j)) = \text{Pos}(a^\ell(j - 1)) + 1 + t^\ell(j), \quad 2^{L-N+\ell-1} < j < 2^{L-N+\ell} - 1$$

and therefore,

$$(4.15) \quad \text{Pos}(a^\ell(k)) = \text{Pos}(a^\ell(2^{L-N+\ell-1})) + k - 2^{L-N+\ell-1} + \sum_{j=2^{L-N+\ell-1+1}}^k m(j).$$

Since $t^\ell(j) = m(j) + 2^{N-\ell+1} - 2$, it follows that

$$(4.16) \quad \sum_{j=2^{L-N+\ell-1+1}}^k t^\ell(j) = k (2^{N-\ell+1} - 2) + 2^{L-N+\ell} - 2^L + \sum_{j=2^{L-N+\ell-1+1}}^k m(j).$$

Lemma 4.1 and equation (4.14) yield $a^\ell(2^{L-N+\ell-1}) = 2^L + 2^{N-\ell+1} - 2$ and by equation (4.10) we have

$$(4.17) \quad \text{Pos}(a^\ell(2^{L-N+\ell-1})) = 2^{N-\ell+1} - 2.$$

Putting equation (4.16) and (4.17) into equation (4.15) yields the statement. □
4.2. Dyadic subtrees and their lowermost level in $D_N$. In this section we examine the behaviour of the postorder rearrangement $\tau_N$ on complete dyadic subtrees in $D_N$ given by

$$T_{\ell,k}^N = \{I \in D_N : I \subseteq I_{\ell,k}\}$$

and on their lowermost level in $D_N$ given by

$$E_{\ell,k}^N = \{I \in D_N : I \subseteq I_{\ell,k}, \, |I| = 2^{-N}\}.$$ 

Note that $E_{\ell,k}^N$ is a collection of disjoint dyadic intervals and hence, $|E_{\ell,k}^N| = 1$. We know from (2.4) that $|T_{\ell,k}^N| = N - \ell + 1$. We measure the behaviour of the rearrangement by the Carleson constants $[\tau_N(T_{\ell,k}^N)]$ and $[\tau_N(E_{\ell,k}^N)]$. The following two theorems and the corresponding proofs reveal a remarkable phenomenon of the postorder rearrangement $\tau_N$. A complete dyadic subtree as well as its lowermost level in $D_N$ is mapped under $\tau_N$ onto collections of dyadic intervals with large Carleson constant, if it contains the leftmost interval $I_{N,0}$, cf. Theorem 4.4. Otherwise, it is mapped under $\tau_N$ onto a collection of disjoint dyadic intervals of equal length, cf. Theorem 4.7.

**Theorem 4.4.** Let $N \in \mathbb{N}_0$ and $0 \leq \ell \leq N$. Then

$$\|[\tau_N(T_{\ell,0}^N)]\| = N - \ell + 1 \quad \text{and} \quad \|[\tau_N(E_{\ell,0}^N)]\| \geq \frac{N - \ell + 1}{2}.$$

**Proof.** Recall that $\tau_N$ maps the $n$th interval in postorder onto the $n$th interval in lexicographic order. The definition of the postorder yields that the dyadic intervals in $T_{\ell,0}^N$ have the corresponding postorder ordinal numbers $1, \ldots, 2^{N-\ell+1} - 1$, cf. Section 2.3. These are exactly the lexicographic ordinal numbers of the dyadic intervals in $D_{N-\ell}$. Hence, $\tau_N(T_{\ell,0}^N) = D_{N-\ell}$ and $[\tau_N(T_{\ell,0}^N)] = N - \ell + 1$.

The lowermost level of $T_{\ell,0}^N$ in $D_N$ is given by

$$E_{\ell,0}^N = \{I_{N,r} : 0 \leq r \leq 2^{N-\ell} - 1\}.$$

By the characterisation of the postorder rearrangement $\tau_N$ in Section 4.1 we have

$$\tau_N(E_{\ell,0}^N) = \left\{I_{L,K} : L = \text{Level}(a_N(r)), \, K = \text{Pos}(a_N(r)), \, 0 \leq r \leq 2^{N-\ell} - 1\right\}.$$

Lemma 4.2 and Lemma 4.3 give that $\text{Level}(a_N(0)) = 0$ and $\text{Pos}(a_N(0)) = 0$. Therefore, $I_{0,0} \in \tau_N(E_{\ell,0}^N)$ and by definition (2.2)

$$[\tau_N(E_{\ell,0}^N)] \geq \frac{1}{|I_{0,0}|} \sum_{J \in \tau_N(E_{\ell,0}^N)} |J| = \sum_{J \in \tau_N(E_{\ell,0}^N)} |J|.$$

Note that $\tau_N(E_{\ell,0}^N) \subseteq D_{N-\ell}$. We split the sum on the right hand side into levels and get

$$[\tau_N(E_{\ell,0}^N)] \geq \sum_{m=0}^{N-\ell} 2^{-m} |B(m)|,$$

where $B(m)$ is the set of dyadic intervals in the collection $\tau_N(E_{\ell,0}^N)$ that have length $2^{-m}$. We denote by $A(m)$ the set of postorder ordinal numbers corresponding to $B(m)$. Then $|B(m)| = |A(m)|$ and

$$A(m) = \{a^N(r) : 0 \leq r \leq 2^{N-\ell} - 1, \, \text{Level}(a^N(r)) = m\}.$$
Obviously, \(|A(0)| = 1\). By Lemma 4.2 we have \(|A(m)| = 2^{m-1}\) for all \(1 \leq m \leq N - \ell\).

Hence,

\[
[\tau_N(E_{\ell,0}^N)] \geq 1 + \sum_{m=1}^{N-\ell} 2^{-1} = \frac{N - \ell}{2} + 1 \geq \frac{N - \ell + 1}{2}.
\]

\[\square\]

**Remark 4.5.** Let \(N \in \mathbb{N}_0\). An easy computation shows that for \(N - 1 \leq \ell \leq N\)

\[
[\tau_N(E_{\ell,0}^N)] = 1 + \frac{N - \ell}{2}.
\]

Obviously, for \(0 \leq \ell \leq N - 2\) we have the upper bound

\[
[\tau_N(E_{\ell,0}^N)] \leq N - \ell + 1.
\]

**Conjecture 4.6.** Let \(N \in \mathbb{N}, N \geq 2\) and \(0 \leq \ell \leq N - 2\). The supremum in definition (2.22) for the Carleson constant \([\tau_N(E_{\ell,0}^N)]\) is attained for the interval \(I_{1,0}\).

This gives the following formula

\[
[\tau_N(E_{\ell,0}^N)] = \frac{N - \ell}{2} + \frac{3}{2} - 2^{N-\ell+1}.
\]

Now we consider those dyadic trees in \(D_N\) that are mapped under the postorder rearrangement \(\tau_N\) onto collections of disjoint dyadic intervals.

**Theorem 4.7.** Let \(N \in \mathbb{N}, 0 < \ell \leq N\) and \(0 < k \leq 2^{\ell-1}\). Then

(4.22)

\[
[\tau_N(T_{\ell,k}^N)] = [\tau_N(E_{\ell,k}^N)] = 1.
\]

**Proof.** The complete dyadic subtree \(T_{\ell,k}^N\) is given by

\[
T_{\ell,k}^N = \{I_{m,r} : \ell \leq m \leq N, k2^{m-\ell} \leq r \leq (k+1)2^{m-\ell} - 1\}.
\]

We associate the collection \(T_{\ell,k}^N\) with the set of postorder ordinal numbers

(4.23)

\[
\{a^m(r) : \ell \leq m \leq N, k2^{m-\ell} \leq r \leq (k+1)2^{m-\ell} - 1\}.
\]

By the characterization of the postorder rearrangement \(\tau_N\) in Section 4.1 we have

\[
\tau_N(T_{\ell,k}^N) = \{I_{L,K} : L = \text{Level}(a^m(r)), K = \text{Pos}(a^m(r))\}.
\]

We show that for all \(m, \ell \leq m \leq N\), there exists an integer \(s, 1 \leq s \leq m\), such that for all \(r\) as in (4.23) we have

\[
\text{Level}(a^m(r)) = s + N - \ell.
\]

Let \(\overline{\sigma} = m - \ell + \lceil \log_2 (k + 1) \rceil\) so that \(2^{\overline{\sigma} - 1} \leq k2^{m-\ell}\) and \((k+1)2^{m-\ell} - 1 \leq 2^{\overline{\sigma}} - 1\).

By Lemma 4.2 it follows that for all \(k2^{m-\ell} \leq r \leq (k+1)2^{m-\ell} - 1\) we have

\[
\text{Level}(a^m(r)) = \overline{\sigma} + N - m = s + N - \ell,
\]

where \(s = \lceil \log_2 (k + 1) \rceil\). The image of \(T_{\ell,k}^N\) is then given by

\[
\tau_N(T_{\ell,k}^N) = \{I_{s+N-\ell,K} : K = \text{Pos}(a^m(r))\}.
\]

\(\tau_N(T_{\ell,k}^N)\) is a collection of disjoint dyadic intervals and therefore, \([\tau_N(T_{\ell,k}^N)] = 1\).

Since \(E_{\ell,k}^N \subseteq T_{\ell,k}^N\), it follows that \(\tau_N(E_{\ell,k}^N) \subseteq \tau_N(T_{\ell,k}^N)\). Therefore, \(\tau_N(E_{\ell,k}^N)\) is also a collection of disjoint dyadic intervals with \([\tau_N(E_{\ell,k}^N)] = 1\).

\[\square\]
Finally, we have all ingredients that we need to prove the statement of Theorem 3.1. For convenience we recall the statement. The rearrangement operators $T = T_{\tau_N}|_{\mathcal{M}(T_{\ell,0})}$ and $T = T_{\tau_N}|_{\mathcal{M}(D_{N^{-\ell}})}$ satisfy the following norm estimates

$$\frac{1}{\sqrt{2}} (N - \ell + 1)^{\frac{3}{2}} \leq \|T\|_{\text{BMO}} \leq (N - \ell + 1)^{\frac{3}{2}}.$$  

Recall that $\mathcal{M}(T_{\ell,0}) = \text{span} \{ h_I : I \in T_{\ell,0} \}$ and $\mathcal{M}(D_{N^{-\ell}}) = \text{span} \{ h_I : I \in D_{N^{-\ell}} \}$. The proof uses the norm estimates for rearrangement operators on BMO given in Section 2.0 and the estimates for Carleson constants given in Theorem 1.3 and Theorem 4.7.

Proof of Theorem 3.1. Let $x \in \mathcal{M}(T_{\ell,0})$. The norm estimate (2.11) and the statement of Theorem 4.4 yield

$$\|T_{\tau_N} x\|_{\text{BMO}}^2 \leq \|\tau_N(T_{\ell,0})\| x\|_{\text{BMO}}^2 \leq (N - \ell + 1) \|x\|_{\text{BMO}}^2.$$  

This gives the upper bound

$$\left\| T_{\tau_N}|_{\mathcal{M}(T_{\ell,0})} \right\|_{\text{BMO}} \leq (N - \ell + 1)^{\frac{3}{2}}.$$  

Equation (2.10) gives the lower bound

$$\left\| T_{\tau_N}|_{\mathcal{M}(T_{\ell,0})} \right\|_{\text{BMO}}^2 \geq \sup_{C \subseteq T_{\ell,0}} \frac{\|\tau_N(C)\|_{\text{BMO}}^{\frac{3}{2}}}{\|C\|^{\frac{3}{2}}}.$$  

We consider the lowermost level $E_{\ell,0}^N$ of the complete dyadic subtree $T_{\ell,0}^N$ in $D_N$. Obviously, $E_{\ell,0}^N \subseteq T_{\ell,0}^N$. We know that $\|E_{\ell,0}^N\| = 1$. Hence, by the statement of Theorem 4.4 we have

$$\left\| T_{\tau_N}|_{\mathcal{M}(T_{\ell,0}^N)} \right\|_{\text{BMO}} \geq \|\tau_N(E_{\ell,0}^N)\| \geq \frac{1}{2} (N - \ell + 1).$$  

Let $x \in \mathcal{M}(D_{N^{-\ell}})$. The norm estimate (2.11) yields

$$\|T_{\sigma_N} x\|_{\text{BMO}}^2 \leq \|\sigma_N(D_{N^{-\ell}})\| x\|_{\text{BMO}}^2.$$  

By the proof of Theorem 4.4 it follows that $\tau_N(T_{\ell,0}^N) = D_{N^{-\ell}}$. Since $\sigma_N = \tau_N^{-1}$, we have $\sigma_N(D_{N^{-\ell}}) = T_{\ell,0}^N$ and $\|\sigma_N(D_{N^{-\ell}})\| = \|T_{\ell,0}^N\| = N - \ell + 1$. Hence,

$$\left\| T_{\sigma_N}|_{\mathcal{M}(D_{N^{-\ell}})} \right\|_{\text{BMO}} \leq (N - \ell + 1)^{\frac{3}{2}}.$$  

Equation (2.10) gives the lower bound

$$\left\| T_{\sigma_N}|_{\mathcal{M}(D_{N^{-\ell}})} \right\|_{\text{BMO}}^2 \geq \sup_{C \subseteq D_{N^{-\ell}}, \text{non-empty}} \frac{\|\sigma_N(C)\|_{\text{BMO}}^{\frac{3}{2}}}{\|C\|^{\frac{3}{2}}}.$$  

Let $\ell < N$. The proof of Theorem 1.7 asserts that $\tau_N(T_{\ell+1,1}^N) \subseteq D_{N^{-\ell}}$ and $\|\tau_N(T_{\ell+1,1}^N)\| = 1$. Hence, we have the lower bound

$$\left\| T_{\tau_N}|_{\mathcal{M}(D_{N^{-\ell}})} \right\|_{\text{BMO}} \geq \|T_{\ell+1,1}^N\| = N - \ell \geq \frac{1}{2} (N - \ell + 1).$$  

The case $\ell = N$ is trivial. \(\square\)
5. Proof of Theorem 3.2

As mentioned in Section 3, the proof of Theorem 3.2 uses a geometric representation of order intervals with respect to the postorder, \( \preceq \). This geometric representation is given in Proposition 2.3 and Definition 2.2 as follows. For every postorder order interval

\[ \mathcal{B}^N(I_1, I_2) = \{ I \in \mathcal{D}_N : I_1 \preceq I \preceq I_2 \}, \]

there exists a collection of maximal intervals \( \mathcal{L} = \{ L_1, \ldots, L_m \} \) such that

\[ \mathcal{B}^N(I_1, I_2) = \mathcal{C}(I_1, L_1) \cup \mathcal{R}(I_1, L_1) \cup \bigcup_{i=2}^m \mathcal{M}_i, \]

where \( \mathcal{C}(I_1, L_1) \) is the cone of dyadic intervals between \( I_1 \) and \( L_1 \), \( \mathcal{R}(I_1, L_1) \) is the right fill-up of the cone and \( \mathcal{M}_i \) is the complete dyadic subtree with root \( L_i \) given by \( \mathcal{M}_i = \{ I \in \mathcal{D}_N : I \subseteq L_i \} \).

For the norm estimate in Theorem 3.2 we need an estimate for the Carleson constant \( \| \mathcal{B}^N(I_1, I_2) \| \). By the geometric representation of \( \mathcal{B}^N \) given above we have that the Carleson constant \( \| \mathcal{B}^N(I_1, I_2) \| \) is related to the Carleson constant of the cone and the right fill-up. Therefore, we start examining the Carleson constant \( \| \mathcal{C}(I, J) \cup \mathcal{R}(I, J) \| \) for two non-disjoint dyadic intervals \( I, J \in \mathcal{D}_N \).

**Theorem 5.1.** Let \( N \in \mathbb{N}_0 \). Let \( I, J \in \mathcal{D}_N \) and \( I \subseteq J \). If \( \mathcal{R}(I, J) \neq \emptyset \), then

\[ N - \log_2 \frac{1}{|I|} + 1 \leq \| \mathcal{C}(I, J) \cup \mathcal{R}(I, J) \| \leq N - \log_2 \frac{1}{|J|} + 2. \]

**Proof.** The definition of the Carleson constant (2.2) yields

\[ \| \mathcal{R}(I, J) \| \leq \| \mathcal{C}(I, J) \cup \mathcal{R}(I, J) \| \leq \| \mathcal{R}(I, J) \| + \| \mathcal{C}(I, J) \|. \]

Recall that the cone \( \mathcal{C}(I, J) \) is a collection of dyadic intervals \( \mathcal{C} = \{ C_1, \ldots, C_n \} \), where \( n = \log_2 \frac{1}{|I|} + 1 \), which satisfies the following properties: \( C_1 = I \), \( C_n = J \), \( |C_i| = \frac{1}{2} |C_{i+1}| \) and \( C_i \subset C_{i+1} \) for \( 1 \leq i \leq n - 1 \). This yields

\[ \| \mathcal{C}(I, J) \| = \sum_{i=1}^{n-1} \frac{1}{|C_i|} \sum_{J \subseteq C_i, J \in \mathcal{C}} |J| = \sup_{i=1}^{n} \frac{1}{|C_i|} \sum_{s=1}^{i} |C_s|. \]

Since \( |C_i| = 2^{i-1} |C_1| \), it follows that \( \| \mathcal{C}(I, J) \| \leq 2 \).

The right fill-up \( \mathcal{R}(I, J) \) of the cone is the collection of dyadic intervals \( \bigcup_{i=1}^{n-1} \mathcal{U}_{i+1} \), where \( \mathcal{U}_{i+1} = \emptyset \), if \( C_i \) is the right half of \( C_{i+1} \) and \( \mathcal{U}_{i+1} = \{ U \in \mathcal{D}_N : U \subseteq C_{i+1} \setminus C_i \} \), if \( C_i \) is the left half of \( C_{i+1} \). Note that by definition \( \mathcal{U}_i \cap \mathcal{U}_j = \emptyset \) for every \( i \neq j \).

Therefore,

\[ \| \mathcal{R}(I, J) \| = \sup_{i=1}^{n-1} \| \mathcal{U}_{i+1} \|. \]

If \( \mathcal{U}_{i+1} \neq \emptyset \), then \( \mathcal{U}_{i+1} \) is a dyadic subtree of \( \mathcal{D}_N \) with root \( C_{i+1} \setminus C_i \) and depth \( N - \log_2 \frac{1}{|C_{i+1} \setminus C_i|} \). We know that \( |C_{i+1} \setminus C_i| = |C_i| \) and \( |C_i| = 2^{i-1} |I| \). Therefore, by equation (2.4) we have \( \| \mathcal{U}_{i+1} \| = N - \log_2 \frac{2^{i-1} |I|}{|I|} + 1 = N + i - \log_2 \frac{1}{|I|} \). Hence, by equation (5.3)

\[ N + 1 - \log_2 \frac{1}{|I|} \leq \| \mathcal{R}(I, J) \| \leq N + n - 1 - \log_2 \frac{1}{|J|}. \]
Recall that $n = \log_2 \frac{|I|}{|J|} + 1$. This gives the upper bound

$$\| \mathcal{R}(I, J) \| \leq N - \log_2 \frac{1}{|J|}.$$

Summarizing we have \[5.1\].

The statement of Proposition \[2.3\] and the estimates from Theorem \[5.1\] yield the following estimates for the Carleson constant $[\mathcal{B}^N(I_1, I_2)]$.

**Theorem 5.2.** Let $N \in \mathbb{N}_0$ and $\mathcal{B}^N(I_1, I_2)$ the postorder order interval given by $I_1, I_2 \in \mathcal{D}_N$ with $I_1 \subseteq I_2$. Let $L_1$ be the maximal interval in $\mathcal{B}^N(I_1, I_2)$ such that $I_1 \subseteq L_1$. Then

\[
N - \log_2 \frac{1}{|I_1|} + 1 \leq [\mathcal{B}^N(I_1, I_2)] \leq N - \log_2 \frac{1}{|I_1|} + 2.
\]

**Proof.** Let $\mathcal{L} = \{L_1, \ldots, L_m\}$ be the maximal (with respect to inclusion) elements of $\mathcal{B}^N(I_1, I_2)$, as given in Proposition \[5.2\]. Since

$$\mathcal{B}^N(I_1, I_2) = \mathcal{C}(I_1, L_1) \cup \mathcal{R}(I_1, L_1) \cup_{i=2}^m \mathcal{M}_i,$$

where $\mathcal{M}_i = \{I \in \mathcal{D}_N : I \subseteq L_i\}$, and since $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ for all $i \neq j$ and $\mathcal{M}_i \cap (\mathcal{C}(I_1, L_1) \cup \mathcal{R}(I_1, L_1)) = \emptyset$ for all $i$, we have

\[
[\mathcal{B}^N(I_1, I_2)] = \max \{[\mathcal{C}(I_1, L_1) \cup \mathcal{R}(I_1, L_1)]\}, \max_{i=2, \ldots, m} \| \mathcal{M}_i \|.
\]

If $I_1 \subseteq I_2$, then there is only one maximal interval $L_1 = I_2$. Hence, $m = 1$ and $I \in \mathcal{B}^N(I_1, I_2)$ if and only if $I \in \mathcal{C}(I_1, I_2) \cup \mathcal{R}(I_1, I_2)$, cf. proof of Proposition \[5.2\] in [MS97]. Therefore, we have

\[
[\mathcal{B}^N(I_1, I_2)] = \| \mathcal{C}(I_1, I_2) \cup \mathcal{R}(I_1, I_2) \|.
\]

Theorem \[5.1\] yields the statement.

If $I_1 \cap I_2 = \emptyset$, then there exist maximal intervals $\mathcal{L} = \{L_1, \ldots, L_m\}$, $m \geq 2$. For $2 \leq i \leq m$, $\mathcal{M}_i$ is a dyadic subtree of $\mathcal{D}_N$ with root $L_i$ and depth $N - \log_2 \frac{1}{|L_i|}$.

Equation \[2.4\] gives the Carleson constant $\| \mathcal{M}_i \| = N - \log_2 \frac{1}{|L_i|} + 1$. Proposition \[2.3\] yields $|L_m| \leq |L_{m-1}| \cdots < |L_2|$. Hence, $[\mathcal{M}_2] = \max_{i=2, \ldots, m} \| \mathcal{M}_i \|$ and

$$\| \mathcal{B}^N(I_1, I_2) \| = \max \{|[\mathcal{C}(I_1, L_1) \cup \mathcal{R}(I_1, L_1)], [\mathcal{M}_2]|\}.$$

By Theorem \[5.1\] we have the following lower and upper bound.

\[
[\mathcal{B}^N(I_1, I_2)] \geq \max \{N - \log_2 \frac{1}{|I_1|} + 1, N - \log_2 \frac{1}{|L_2|} + 1\}
\]

and

\[
[\mathcal{B}^N(I_1, I_2)] \leq \max \{N - \log_2 \frac{1}{|I_1|} + 2, N - \log_2 \frac{1}{|L_2|} + 1\}.
\]

$|L_2| \leq |L_1|$ by Proposition \[2.3\] Therefore, \[5.8\] yields the upper bound in \[5.4\]. \qed
5.1. **The proof of Theorem 3.2.** Now we have all ingredients for the proof of Theorem 3.2. For convenience we give the statement of the Theorem. We have the following operator norm estimate for the rearrangement operator $T_{\sigma_N}$ acting on lexicographic order intervals $\mathcal{E}(E_1, E_2)$ given by the endpoints $E_1, E_2 \in \mathcal{D}_N$ with $E_1 \leq I \leq E_2$:

$$\|T_{\sigma_N}\|_{\mathcal{M}(\mathcal{E})}^2 \leq N - \log_2 \frac{1}{|L|} + 2,$$

where $L_I$ is the maximal (with respect to inclusion) dyadic interval in the postorder order interval $B^N(\sigma_N(E_1), \sigma_N(E_2))$ that contains the left endpoint $\sigma_N(E_1)$. Recall that $\mathcal{M}(\mathcal{E}) = \text{span}\{I \in \mathcal{D}_N : I \in \mathcal{E}\}$.

**Proof of Theorem 3.2.** Let $x \in \mathcal{M}(\mathcal{E})$. The estimates of rearrangement operators on $\mathcal{BMO}$ in Section 2.6 give the upper bound

$$\|T_{\sigma_N}x\|_{\mathcal{BMO}}^2 \leq \|\mathcal{BMO}(\mathcal{E})\| \|x\|_{\mathcal{BMO}}^2.$$

$\sigma_N$ is the bijective map on $\mathcal{D}_N$ that maps lexicographic order intervals onto postorder order intervals. Hence, for every lexicographic order interval $\mathcal{E} = \mathcal{E}(E_1, E_2)$ there exists a unique postorder order interval $\mathcal{B} = B^N(\sigma_N(E_1), \sigma_N(E_2))$ so that $\mathcal{B} = \mathcal{B}$. Hence, by equation (5.9) and Theorem 5.2 we have

$$\|T_{\sigma_N}x\|_{\mathcal{BMO}}^2 \leq \|\mathcal{B}\| \|x\|_{\mathcal{BMO}} \leq (N - \log_2 \frac{1}{|L|} + 2) \|x\|_{\mathcal{BMO}}^2,$$

where $L_I$ is the maximal interval in $B^N(\sigma_N(E_1), \sigma_N(E_2))$ with $\sigma_N(E_1) \subseteq L_I$. \hfill \Box

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