ANISOTROPIC FRACTIONAL GAGLIARDO-NIRENBERG, WEIGHTED CAFFARELLI-KOHN-NIRENBERG AND LYAPUNOV-TYPE INEQUALITIES, AND APPLICATIONS TO RIESZ POTENTIALS AND \( p \)-SUB-LAPLACIAN SYSTEMS

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Abstract. In this paper we prove the fractional Gagliardo-Nirenberg inequality on homogeneous Lie groups. Also, we establish weighted fractional Caffarelli-Kohn-Nirenberg inequality and Lyapunov-type inequality for the Riesz potential on homogeneous Lie groups. The obtained Lyapunov inequality for the Riesz potential is new already in the classical setting of \( \mathbb{R}^N \). As an application, we give two-sided estimate for the first eigenvalue of the Riesz potential. Also, we obtain Lyapunov inequality for the system of the fractional \( p \)-sub-Laplacian equations and give an application to estimate its eigenvalues.

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1. Introduction

1.1. Fractional Gagliardo-Nirenberg inequality. In the works of E. Gagliardo [9] and L. Nirenberg [14] (independently), they obtained the following (interpolation) inequality

\[
\|u\|^p_{L^p(\mathbb{R}^N)} \leq C \|\nabla u\|^{N(p-2)/2}_{L^2(\mathbb{R}^N)} \|u\|^{(2p-N(p-2))/2}_{L^2(\mathbb{R}^N)}, \quad u \in H^1(\mathbb{R}^N),
\]

(1.1)

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where
\[
\begin{cases}
2 \leq p \leq \infty \text{ for } N = 2, \\
2 \leq p \leq \frac{2N}{N-2} \text{ for } N > 2.
\end{cases}
\]

The Gagliardo-Nirenberg inequality on the Heisenberg group $\mathbb{H}^n$ has the following form
\[
\|u\|_{L^p(\mathbb{H}^n)}^p \leq C \|\nabla_{\mathbb{H}} u\|_{L^2(\mathbb{H}^n)}^{Q(p-2)/2} \|u\|_{L^2(\mathbb{H}^n)}^{(2p-Q(p-2))/2},
\]
where $\nabla_{\mathbb{H}}$ is a horizontal gradient and $Q$ is a homogeneous dimension of $\mathbb{H}^n$. In [3], the authors established the best constant for the sub-elliptic Gagliardo-Nirenberg inequality (1.2). Consequently, in [20] the best constants in Gagliardo-Nirenberg and Sobolev inequalities were also found for general hypoelliptic (Rockland operators) on general graded Lie groups.

In [15] the authors obtained a fractional version of the Gagliardo-Nirenberg inequality in the following form:
\[
\|u\|_{L^\tau(\mathbb{R}^N)} \leq C [u]_{s,p}^{a} \|u\|_{L^\tau(\mathbb{R}^N)}^{1-a}, \quad \forall u \in C_c^\infty(\mathbb{R}^N),
\]
where $[u]_{s,p}$ is Gagliardo’s seminorm defined by
\[
[u]_{s,p} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dxdy,
\]
for $N \geq 1$, $s \in (0, 1)$, $p > 1$, $\alpha \geq 1$, $\tau > 0$, and $a \in (0, 1]$ is such that
\[
\frac{1}{\tau} = a \left( \frac{1}{p} - \frac{s}{N} \right) + \frac{1-a}{\alpha}.
\]

In this paper we formulate the fractional Gagliardo-Nirenberg inequality on the homogeneous Lie groups. To the best of our knowledge, in this direction systematic studies on the homogeneous Lie groups started by the paper [18] in which homogeneous group versions of Hardy and Rellich inequalities were proved as consequences of universal identities.

1.2. Fractional Caffarelli-Kohn-Nirenberg inequality. In their fundamental work [2], L. Caffarelli, R. Kohn and L. Nirenberg established:

**Theorem 1.1.** Let $N \geq 1$, and let $l_1$, $l_2$, $l_3$, $a$, $b$, $d$, $\delta \in \mathbb{R}$ be such that $l_1, l_2 \geq 1$, $l_3 > 0$, $0 \leq \delta \leq 1$, and
\[
\frac{1}{l_1} + \frac{a}{N}, \quad \frac{1}{l_2} + \frac{b}{N}, \quad \frac{1}{l_3} + \frac{\delta d + (1-\delta)b}{N} > 0.
\]

Then,
\[
|||x|^{5d+(1-\delta)b} u|||_{L^3(\mathbb{R}^N)} \leq C |||x|^a \nabla u|||_{L^1(\mathbb{R}^N)}^{\delta} |||x|^b u|||_{L^2(\mathbb{R}^N)}^{1-\delta}, \quad u \in C_c^\infty(\mathbb{R}^N),
\]
if and only if
\[
\frac{1}{l_3} + \frac{\delta d + (1-\delta)b}{N} = \delta \left( \frac{1}{l_1} + \frac{a-1}{N} \right) + (1-\delta) \left( \frac{1}{l_2} + \frac{b}{N} \right),
\]
\[
a - d \geq 0, \quad \text{if } \delta > 0,
\]
\[
a - d \leq 1, \quad \text{if } \delta > 0 \text{ and } \frac{1}{l_3} + \frac{\delta d + (1-\delta)b}{N} = \frac{1}{l_1} + \frac{a-1}{N},
\]
(1.6)
where $C$ is a positive constant independent of $u$.

In [15] the authors proved the fractional analogues of the Caffarelli-Kohn-Nirenberg inequality in weighted fractional Sobolev spaces. Also, in [11] a fractional Caffarelli-Kohn-Nirenberg inequality for an admissible weight in $\mathbb{R}^N$ was obtained.

Recently many different versions of Caffarelli-Kohn-Nirenberg inequalities have been obtained, namely, in [24] on the Heisenberg groups, in [22] and [23] on stratified groups, in [19] and [21] on (general) homogeneous Lie groups. One of the aims of this paper is to prove the fractional weighted Caffarelli-Kohn-Nirenberg inequality on the homogeneous Lie groups.

1.3. Fractional Lyapunov-type inequality. Historically, in Lyapunov’s work [13] for the following one-dimensional homogeneous Dirichlet boundary value problem (for the second order ODE)

\[
\begin{cases}
u''(x) + \omega(x)u(x) = 0, & x \in (a, b), \\
u(a) = u(b) = 0,
\end{cases}
\]  

(1.7)

it was proved that if $u$ is a non-trivial solution of (1.7) and $\omega(x)$ is a real-valued and continuous function on $[a, b]$, then necessarily

\[
\int_a^b |\omega(x)| \, dx > \frac{4}{b-a}.
\]  

(1.8)

Nowadays, there are many extensions of Lyapunov’s inequality. In [5] the author obtains Lyapunov’s inequality for the one-dimensional Dirichlet $p$-Laplacian

\[
\begin{cases}
(|u'(x)|^{p-2}u'(x))' + \omega(x)u(x) = 0, & x \in (a, b), \\
u(a) = u(b) = 0,
\end{cases}
\]  

(1.9)

where $\omega(x) \in L^1(a, b)$, so necessarily

\[
\int_a^b |\omega(x)| \, dx > \frac{2^p}{(b-a)^{p-1}}, \quad 1 < p < \infty.
\]  

(1.10)

Obviously, taking $p = 2$ in (1.10), we recover the classical Lyapunov inequality (1.8).

In [10] the authors obtained interesting results concerning Lyapunov inequalities for the multi-dimensional fractional $p$-Laplacian $(-\Delta_p)^s$, $1 < p < \infty$, $s \in (0, 1)$, with a homogeneous Dirichlet boundary condition, that is,

\[
\begin{cases}
(-\Delta_p)^s u = \omega(x)|u|^{p-2}u, & x \in \Omega, \\
u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega,
\end{cases}
\]  

(1.11)

where $\Omega \subset \mathbb{R}^N$ is an open set, $1 < p < \infty$, and $s \in (0, 1)$. Let us recall the following result of [10].

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^N$ be an open set, and let $\omega \in L^\theta(\Omega)$ with $1 < \frac{N}{sp} < \theta < \infty$, be a non-negative weight. Suppose that problem (1.11) has a non-trivial weak solution $u \in W_0^{s,p}(\Omega)$. Then

\[
\left(\int_\Omega \omega^\theta(x) \, dx\right)^{\frac{1}{\theta}} > \frac{C}{r_{\Omega}^{sp-s}}.
\]  

(1.12)
where $C > 0$ is a universal constant and $r_\Omega$ is the inner radius of $\Omega$.

In [4], the authors considered a system of ODE for $p$ and $q$-Laplacian on the interval $(a, b)$ with the homogeneous Dirichlet condition in the following form:

\[
\begin{aligned}
&\{-|u'(x)|^{p-2}u'(x)' = f(x)|u(x)|^{\alpha-2}u(x)|v(x)|^\beta, \\
&\{-|v'(x)|^{q-2}v'(x)' = g(x)|u(x)|^{\alpha}|v(x)|^{\beta-2}v(x),
\end{aligned}
\tag{1.13}
\]
on the interval $(a, b)$, with
\[
u(a) = u(b) = v(a) = v(b) = 0,
\tag{1.14}
\]
where $f, g \in L^1(a, b)$, $f, g \geq 0$, $p, q > 1$, $\alpha, \beta \geq 0$ and
\[
\frac{\alpha}{p} + \frac{\beta}{q} = 1.
\]

Then we have Lyapunov-type inequality for system (1.13) with homogeneous Dirichlet condition (1.14):
\[
2^{\alpha+\beta} \leq (b-a)^{\frac{\alpha}{p} + \frac{\beta}{q}} \left( \int_a^b f(x)dx \right)^{\frac{\alpha}{p}} \left( \int_a^b g(x)dx \right)^{\frac{\beta}{q}},
\tag{1.15}
\]
where $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$. In [11], the authors obtained the Lyapunov-type inequality for a fractional $p$-Laplacian system in an open bounded subset $\Omega \subset \mathbb{R}^N$ with homogeneous Dirichlet conditions. One of our goals in this paper is to extend the Lyapunov-type inequality for the Riesz potential and for the fractional $p$-sub-Laplacian system on the homogeneous Lie groups. These results are given in Theorem 5.1 and 5.7. Also, we give applications of the Lyapunov-type inequality for the Riesz potential and for fractional $p$-sub-Laplacian system on the homogeneous Lie groups.

To demonstrate our techniques we consider the Riesz potential in the Abelian case $(\mathbb{R}^N, +)$ and give two side estimates of the first eigenvalue of the Riesz potential in the Abelian case $(\mathbb{R}^N, +)$.

Summarising our main results of the present paper, we prove the following facts:

- An analogue of the fractional Gagliardo-Nirenberg inequality on the homogeneous group $\mathbb{G}$;
- An analogue of the fractional weighted Caffarelli-Kohn-Nirenberg inequality on $\mathbb{G}$;
- An analogue of the Lyapunov-type inequality for the Riesz potential on $\mathbb{G}$;
- An analogue of the Lyapunov-type inequality for the fractional $p$-sub-Laplacian system on $\mathbb{G}$.

The paper is organised as follows. First we give some basic discussions on fractional Sobolev spaces and related facts on homogeneous Lie groups, then in Section 3 we present the fractional Gagliardo-Nirenberg inequality on $\mathbb{G}$. The fractional weighted Caffarelli-Kohn-Nirenberg inequality on $\mathbb{G}$ is proved in Section 4. In Section 5 we discuss analogues of the Lyapunov-type inequalities for the Riesz potential and fractional $p$-sub-Laplacian system on $\mathbb{G}$. 
2. Preliminaries

We recall that a Lie group \((\mathbb{R}^n) G\) with the dilation
\[ D_\lambda(x) := (\lambda^{\nu_1} x_1, \ldots, \lambda^{\nu_n} x_n), \quad \nu_1, \ldots, \nu_n > 0, \quad D_\lambda: \mathbb{R}^n \to \mathbb{R}^n, \]
which is an automorphism of the group \(G\) for each \(\lambda > 0\), is called a \textit{homogeneous (Lie) group}. In this paper, for simplicity, we use the notation \(\lambda x\) instead of the dilation \(D_\lambda(x)\). The homogeneous dimension of the homogeneous group \(G\) is denoted by
\[ Q := \nu_1 + \ldots + \nu_n. \]

A homogeneous quasi-norm on \(G\) is a continuous non-negative function
\[ G \ni x \mapsto q(x) \in [0, \infty), \quad (2.1) \]
with the properties
\begin{enumerate}
  \item[i)] \(q(x) = q(x^{-1})\) for all \(x \in G\),
  \item[ii)] \(q(\lambda x) = \lambda q(x)\) for all \(x \in G\) and \(\lambda > 0\),
  \item[iii)] \(q(x) = 0\) iff \(x = 0\).
\end{enumerate}
Moreover, the following polarisation formula on homogeneous Lie groups will be used in our proofs: there is a (unique) positive Borel measure \(\sigma\) on the unit quasi-sphere \(\omega_Q := \{x \in G : q(x) = 1\}\), so that for every \(f \in L^1(G)\) we have
\[ \int_G f(x)dx = \int_0^\infty \int_{\omega_Q} f(ry)r^{Q-1}d\sigma(y)dr. \quad (2.2) \]

We refer to [7] for the original appearance of such groups, and to [6] for a recent comprehensive treatment. Let \(p > 1\), \(s \in (0, 1)\), and let \(G\) be a homogeneous Lie group of homogeneous dimension \(Q\). For a measurable function \(u: G \to \mathbb{R}\) we define the Gagliardo quasi-seminorm by
\[ [u]_{s,p,q} = \left( \int_G \int_G \frac{|u(x) - u(y)|^p}{q^{Q+sp}(y^{-1} \circ x)}dxdy \right)^{1/p}. \quad (2.3) \]

Now we recall the definition of the fractional Sobolev spaces on homogeneous Lie groups denoted by \(W^{s,p,q}(G)\). For \(p \geq 1\) and \(s \in (0, 1)\), the functional space
\[ W^{s,p,q}(G) = \{u \in L^p(G) : u \text{ is measurable}, [u]_{s,p,q} < +\infty\}, \quad (2.4) \]
is called the fractional Sobolev space on \(G\).

Similarly, if \(\Omega \subset G\) is a Haar measurable set, we define the Sobolev space
\[ W^{s,p,q}(\Omega) = \{u \in L^p(\Omega) : u \text{ is measurable}, [u]_{s,p,q,\Omega} < +\infty\}. \quad (2.5) \]

Now we recall the definition of the weighted fractional Sobolev space on the homogeneous Lie groups denoted by
\[ W^{s,p,\beta,q}(G) = \{u \in L^p(G) : u \text{ is measurable}, [u]_{s,p,\beta,q} < +\infty\}, \quad (2.6) \]
where $\beta_1, \beta_2 \in \mathbb{R}$ with $\beta = \beta_1 + \beta_2$ and it depends on $\beta_1$ and $\beta_2$.

As above, for a Haar measurable set $\Omega \subset \mathbb{G}$, $p \geq 1$, $s \in (0, 1)$ and $\beta_1, \beta_2 \in \mathbb{R}$ with $\beta = \beta_1 + \beta_2$, we define the weighted fractional Sobolev space

$$W^{s, p, \beta, q}(\Omega) = \{u \in L^p(\Omega) : u \text{ is measurable,}$$
$$\quad \quad \quad [u]_{s, p, \beta, q, \Omega} = \left( \int_{\Omega} \int_{\Omega} \frac{q^{1+p}(x)q^{2p}(y)|u(x) - u(y)|^{p}}{q^{Q+p}(y^{-1} \circ x)} dxdy \right)^{\frac{1}{p}} < +\infty \}.$$ 

(2.7)

Obviously, taking $\beta = \beta_1 = \beta_2 = 0$ in (2.7), we recover (2.5).

The mean of a function $u$ is defined by

$$u_\Omega = \int_{\Omega} u dx = \frac{1}{|\Omega|} \int_{\Omega} u dx, \quad u \in L^1(\Omega),$$ 

(2.8)

where $|\Omega|$ is the Haar measure of $\Omega \subset \mathbb{G}$.

We will also use the decomposition of $\mathbb{G}$ into quasi-annuli $A_{k,q}$ defined by

$$A_{k,q} := \{x \in \mathbb{G} : 2^k \leq q(x) < 2^{k+1}\},$$ 

(2.9)

where $q(x)$ is a quasi-norm on $\mathbb{G}$.

3. FRACTIONAL GALIARDO-NIRENBERG INEQUALITY ON $\mathbb{G}$

In this section we prove an analogue of the fractional Gagliardo-Nirenberg inequality on the homogeneous Lie groups. To prove Gagliardo-Nirenberg’s inequality we need some preliminary results from [12], a version of a fractional Sobolev inequality on the homogeneous Lie groups.

From now on, unless specified otherwise, $\mathbb{G}$ will be a homogeneous group of homogeneous dimension $Q$.

**Theorem 3.1** ([12], Fractional Sobolev inequality). Let $p > 1$, $s \in (0, 1)$, $Q > sp$, and let $q(\cdot)$ be a quasi-norm on $\mathbb{G}$. For any measurable and compactly supported function $u : \mathbb{G} \to \mathbb{R}$ there exists a positive constant $C = C(Q, p, s, q) > 0$ such that

$$||u||_{L^p(\mathbb{G})}^p \leq C [u]_{s, p, q}^p,$$

(3.1)

where $p^* = p^*(Q, s) = \frac{Qp}{Q-sp}$.

**Theorem 3.2.** Assume that $Q \geq 2$, $s \in (0, 1)$, $p > 1$, $\alpha \geq 1$, $\tau > 0$, $a \in (0, 1]$, $Q > sp$ and

$$\frac{1}{\tau} = a \left(\frac{1}{p} - \frac{s}{Q}\right) + \frac{1 - a}{\alpha}.$$ 

Then,

$$||u||_{L^\tau(\mathbb{G})} \leq C [u]_{s,p,q}^a ||u||_{L^\alpha(\mathbb{G})}^{1-a}, \quad \forall \ u \in C^1_c(\mathbb{G}),$$ 

(3.2)

where $C = C(s, p, Q, a, \alpha) > 0$.

**Proof of Theorem 3.2** By using the Hölder inequality, for every $\frac{1}{\tau} = a \left(\frac{1}{p} - \frac{s}{Q}\right) + \frac{1 - a}{\alpha}$ we get

$$||u||_{L^\tau(\mathbb{G})} = \int_{\mathbb{G}} ||u||^\tau dx = \int_{\mathbb{G}} ||u||^{a\tau} ||u||^{(1-a)\tau} dx \leq \||u||_{L^p(\mathbb{G})}^{a\tau} ||u||_{L^\alpha(\mathbb{G})}^{(1-a)\tau},$$ 

(3.3)
where \( p^* = \frac{Qp}{Q - sp} \). From (3.3), by using the fractional Sobolev inequality (Theorem 3.1), we obtain

\[
\|u\|_{L^\tau(G)} \leq \|u\|_{L^\tau(G)}^{(1-a)\tau} \leq C[u]_{s,p,q}^a \|u\|_{L^\alpha(G)}^{1-a},
\]

that is,

\[
\|u\|_{L^\tau(G)} \leq C[u]_{s,p,q}^a \|u\|_{L^\alpha(G)}^{1-a}, \tag{3.4}
\]

where \( C \) is a positive constant independent of \( u \). Theorem 3.2 is proved. \( \square \)

Remark 3.3. In the Abelian case \((\mathbb{R}^N, +)\) with the standard Euclidean distance instead of the quasi-norm, from Theorem 3.2 we get the fractional Gagliardo-Nirenberg inequality which was proved in [15].

4. Weighted fractional Caffarelli-Kohn-Nirenberg inequality on \( G \)

In this section we prove the weighted fractional Caffarelli-Kohn-Nirenberg inequality on the homogeneous Lie groups.

Theorem 4.1. Assume that \( Q \geq 2, s \in (0,1), p > 1, \alpha \geq 1, \tau > 0, a \in (0,1], \beta_1, \beta_2, \beta, \mu, \gamma \in \mathbb{R}, \beta_1 + \beta_2 = \beta \) and

\[
\frac{1}{\tau} + \frac{\gamma}{Q} = a \left( \frac{1}{p} + \frac{\beta - s}{Q} \right) + (1 - a) \left( \frac{1}{\alpha} + \frac{\mu}{Q} \right). \tag{4.1}
\]

Assume in addition that, \( 0 \leq \beta - \sigma \) with \( \gamma = a\sigma + (1 - a)\mu \), and

\[
\beta - \sigma \leq s \text{ only if } \frac{1}{\tau} + \frac{\gamma}{Q} = \frac{1}{p} + \frac{\beta - s}{Q}. \tag{4.2}
\]

Then for \( u \in C^1_c(G) \) we have

\[
\| q^\gamma(x)u \|_{L^\tau(G)} \leq C[u]_{s,p,\beta,q}^a \| q^\mu(x)u \|_{L^\alpha(G)}^{1-a}, \tag{4.3}
\]

when \( \frac{1}{\tau} + \frac{\gamma}{Q} > 0 \), and for \( u \in C^1_c(G \setminus \{e\}) \) we have

\[
\| q^\gamma(x)u \|_{L^\tau(G)} \leq C[u]_{s,p,\beta,q}^a \| q^\mu(x)u \|_{L^\alpha(G)}^{1-a}, \tag{4.4}
\]

when \( \frac{1}{\tau} + \frac{\gamma}{Q} < 0 \). Here \( e \) is the identity element of \( G \).

Remark 4.2. In the Abelian case \((\mathbb{R}^N, +)\) with the standard Euclidean distance instead of quasi-norm in Theorem 4.1, we get the (Euclidean) fractional Caffarelli-Kohn-Nirenberg inequality (see, e.g. [15], Theorem 1.1).

To prove the fractional weighted Caffarelli-Kohn-Nirenberg inequality on \( G \) we will use Theorem 3.2 in the proof of the following lemma.

Lemma 4.3. Assume that \( Q \geq 2, s \in (0,1), p > 1, \alpha \geq 1, \tau > 0, a \in (0,1] \) and

\[
\frac{1}{\tau} \geq a \left( \frac{1}{p} - \frac{s}{Q} \right) + \frac{1-a}{\alpha}.
\]

Let \( \lambda > 0 \) and \( 0 < r < R \) and set

\[
\Omega = \{ x \in G : \lambda r < q(x) < \lambda R \}.
\]
Then, for every \( u \in C^1(\Omega) \), we have
\[
\left( \int_{\Omega} |u - u_{\Omega}|^{\tau} \, dx \right)^{\frac{1}{\tau}} \leq C_{r,R} \lambda \frac{a(s_\Omega - Q)}{p} \left[ \int_{\Omega} |u|^{a} \, dx \right]^{\frac{1}{a}} , \tag{4.5}
\]
where \( C_{r,R} \) is a positive constant independent of \( u \) and \( \lambda \).

**Proof of Lemma 4.3.** Without loss of generality, we assume that \( 0 < s' \leq s \) and \( \tau' \geq \tau \) are such that
\[
\frac{1}{\tau'} = a \left( \frac{1}{p} - \frac{s'}{Q} \right) + \frac{1-a}{\alpha} ,
\]
and \( \lambda = 1 \), then let \( \Omega_1 \) be
\[
\Omega_1 = \{ x \in G : r < q(x) < R \} .
\]

By using Theorem 3.2, Jensen’s inequality and \([u]_{s',p,q,\Omega} \leq C[u]_{s,p,q,\Omega}\), we get
\[
\left( \int_{\Omega_1} |u - u_{\Omega_1}|^{\tau} \, dx \right)^{\frac{1}{\tau}} = \frac{1}{|\Omega_1|^{\frac{\tau}{p}}} \| u - u_{\Omega_1} \|_{\tau} \leq C_{r,R} \| u - u_{\Omega_1} \|_{L^\tau(\Omega_1)}
\]
\[
\leq C_{r,R} [u - u_{\Omega_1}]^{a}_{s',p,q,\Omega_1} \| u \|_{L^{a}(\Omega_1)}^{1-a}
\]
\[
\leq C_{r,R} \left( \int_{\Omega_1} \int_{\Omega_1} \frac{|u(x) - u_{\Omega_1} - u(y) + u_{\Omega_1}|^{p}}{q^{Q + sp}(y^{-1} \circ x)} \, dxdy \right)^{\frac{a}{p}} \| u \|_{L^{a}(\Omega_1)}^{1-a}
\]
\[
\leq C_{r,R} [u]^{a}_{s,p,q,\Omega_1} \| u \|_{L^{a}(\Omega_1)}^{1-a} \leq C_{r,R} [u]^{a}_{s,p,q,\Omega_1} \left( \int_{\Omega_1} |u|^{a} \, dx \right)^{\frac{1-a}{a}} , \tag{4.6}
\]
where \( C_{r,R} > 0 \). Let us set \( u(\lambda x) \) instead of \( u(x) \), then
\[
\left( \int_{\Omega_1} |u(\lambda x) - \int_{\Omega_1} u(\lambda x) \, dx|^{\tau} \, dx \right)^{\frac{1}{\tau}} \leq C_{r,R} \left( \int_{\Omega_1} \int_{\Omega_1} \frac{|u(\lambda x) - u(\lambda y)|^{p}}{q^{Q + sp}(y^{-1} \circ x)} \, dxdy \right)^{\frac{a}{p}}
\]
\[
\times \left( \frac{1}{|\Omega_1|} \int_{\Omega_1} |u(\lambda x)|^{a} \, dx \right)^{\frac{1-a}{a}} . \tag{4.7}
\]
Thus, we compute

\[
\left( \frac{1}{|\Omega|} \int_{\Omega} |u(x) - \int_{\Omega} u(x) dx|^p dx \right)^{\frac{1}{p}} = \left( \frac{1}{|\Omega|} \int_{\Omega} |u(x) - \frac{1}{|\Omega|} \int_{\Omega} u(x) dx|^p dx \right)^{\frac{1}{p}} \\
= \left( \frac{1}{|\Omega|} \int_{\Omega} |u(\lambda y) - \frac{1}{|\Omega|} \int_{\Omega} u(\lambda y) d(\lambda y)|^p d(\lambda y) \right)^{\frac{1}{p}} \\
= \left( \frac{1}{|\Omega|} \int_{\Omega} \lambda^Q |u(\lambda y) - \frac{\lambda^Q}{\lambda^Q |\Omega|} \int_{\Omega} u(\lambda y) dy|^p dy \right)^{\frac{1}{p}} \\
= \left( \frac{1}{|\Omega|} \int_{\Omega} |u(\lambda y) - \frac{1}{|\Omega|} \int_{\Omega} u(\lambda y) dy|^p dy \right)^{\frac{1}{p}} \\
\leq C_{r,R} \left( \int_{\Omega} \int_{\Omega} \frac{|u(\lambda y) - u(\lambda x)|^p}{q^{Q+sp(y^{-1} \circ x)}} dxdy \right)^{\frac{1}{q'}} \left( \frac{1}{|\Omega|} \int_{\Omega} |u(\lambda y)|^\alpha dy \right)^{\frac{1}{\alpha}} \\
= C_{r,R} \left( \int_{\Omega} \int_{\Omega} \frac{\lambda^{sp} |u(\lambda y) - u(\lambda x)|^p}{q^{Q+sp(y^{-1} \circ x)}} dxdy \right)^{\frac{1}{q'}} \left( \frac{1}{|\Omega|} \int_{\Omega} |u(\lambda y)|^\alpha dy \right)^{\frac{1}{\alpha}} \\
= C_{r,R} \left( \int_{\Omega} \int_{\Omega} \frac{\lambda^{sp} |u(x) - u(y)|^p}{q^{Q+sp(y^{-1} \circ x)}} dxdy \right)^{\frac{1}{q'}} \left( \frac{1}{|\Omega|} \int_{\Omega} |u(x)|^\alpha dx \right)^{\frac{1}{\alpha}} \\
= C_{r,R} \lambda^{a(s)p} |u|^{a}_{s,p,q,\Omega} \left( \frac{1}{|\Omega|} \int_{\Omega} |u(x)|^\alpha dx \right)^{\frac{1}{\alpha}}. \tag{4.8}
\]

The proof of Lemma 4.3 is complete. \qed

**Proof of Theorem 4.3** First let us consider the case (4.2), that is, \( \beta - \sigma \leq s \) and \( \frac{1}{\tau} + \frac{\gamma}{Q} = \frac{1}{p} + \frac{\beta - s}{Q} \). By using Lemma 4.3 with \( \lambda = 2^k \), \( r = 1 \), \( R = 2 \) and \( \Omega = A_{k,q} \), we get

\[
\left( \int_{A_{k,q}} |u - u_{A_{k,q}}|^\gamma dx \right)^{\frac{1}{\gamma}} \leq C 2^{a(s)p} \left[ u \right]^{a}_{s,p,q,A_{k,q}} \left( \int_{A_{k,q}} |u|^\alpha dx \right)^{\frac{1}{\alpha}}, \tag{4.9}
\]

where \( A_{k,q} \) is defined in (2.9) and \( k \in \mathbb{Z} \). Now by using (4.9) we obtain

\[
\int_{A_{k,q}} |u|^\gamma dx = \int_{A_{k,q}} |u - u_{A_{k,q}} + u_{A_{k,q}}|^\gamma dx \leq C \left( \int_{A_{k,q}} |u_{A_{k,q}}|^\gamma dx + \int_{A_{k,q}} |u - u_{A_{k,q}}|^\gamma dx \right)
\]
\[ \begin{align*}
&= C \left( \int_{A_{k,q}} |u_{A_{k,q}}|^\tau dx + \frac{|A_{k,q}|}{|A_{k,q}|} \int_{A_{k,q}} |u - u_{A_{k,q}}|^\tau dx \right) \\
&= C \left( |A_{k,q}| |u_{A_{k,q}}|^\tau + |A_{k,q}| \int_{A_{k,q}} |u - u_{A_{k,q}}|^\tau dx \right) \\
\leq& C \left( |A_{k,q}| |u_{A_{k,q}}|^\tau + 2 \frac{a^k(xpQQ)^\tau}{p} |A_{k,q}| \|u\|_{1,s,p,q,A_{k,q}} \left( \frac{1}{|A_{k,q}|} \int_{A_{k,q}} |u|^{\alpha} dx \right)^{(1-a)^\tau} \right) \\
\leq& C \left( 2^{Qk} |u_{A_{k,q}}|^\tau + 2 \frac{a^k(xpQQ)^\tau}{p} 2^{kQ} 2^{-Q(1-a)^\tau} \|u\|_{1,s,p,q,A_{k,q}} \|u\|_{L^\alpha(A_{k,q})}^{(1-a)^\tau} \right). \tag{4.10} \end{align*} \]

Then, from (4.10) we get

\[ \begin{align*}
& \int_{A_{k,q}} q^{\tau}(x)|u|^\tau dx \leq 2^{(k+1)\tau} \int_{A_{k,q}} |u|^\tau dx \leq C 2^{(Q+\gamma)^k} |u_{A_{k,q}}|^\tau \\
& + C 2^{(\gamma+Q+\frac{a^k(xpQQ)^\tau}{p})} 2^{kQ} 2^{-Q(1-a)^\tau} \|u\|_{L^\alpha(A_{k,q})} \|u\|_{1,s,p,q,A_{k,q}} \left( \frac{1}{|A_{k,q}|} \int_{A_{k,q}} |u|^{\alpha} dx \right)^{(1-a)^\tau} \\
& + C 2^{(\gamma+Q+\frac{a^k(xpQQ)^\tau}{p})} \frac{Q(1-a)^\tau}{\alpha} a^{\beta\tau - \mu(1-a)} k \left( \int_{A_{k,q}} \int_{A_{k,q}} \frac{q^{pQ}(x)q^{pQ}(y) |u(x) - u(y)|^p}{q^{2pQ}(y-1 \circ x)} dx dy \right)^{\frac{\alpha}{\tau}} \\
& \times \left( \int_{A_{k,q}} \frac{2^{kQ}}{2^{kQ}} |u(x)|^{\alpha} dx \right)^{(1-a)^\tau} \\
& \leq C 2^{(Q+\gamma)^k} |u_{A_{k,q}}|^\tau \\
& + C 2^{(\gamma+Q+\frac{a^k(xpQQ)^\tau}{p})} \frac{Q(1-a)^\tau}{\alpha} a^{\beta\tau - \mu(1-a)} k \left( \int_{A_{k,q}} \int_{A_{k,q}} \frac{q^{pQ}(x)q^{pQ}(y) |u(x) - u(y)|^p}{q^{2pQ}(y-1 \circ x)} dx dy \right)^{\frac{\alpha}{\tau}} \\
& \times \left( \int_{A_{k,q}} q^{\mu}(x)|u(x)|^{\alpha} dx \right)^{(1-a)^\tau} \\
& \leq C 2^{(Q+\gamma)^k} |u_{A_{k,q}}|^\tau \\
& + C 2^{(\gamma+Q+\frac{a^k(xpQQ)^\tau}{p})} \frac{Q(1-a)^\tau}{\alpha} a^{\beta\tau - \mu(1-a)} k \|u\|_{1,s,p,q,A_{k,q}} \|q^{\mu}(x)u\|_{L^\alpha(A_{k,q})}^{(1-a)^\tau}. \tag{4.11} \end{align*} \]

Here by (4.11), we have

\[ \begin{align*}
\gamma + Q + \frac{a(sp - Q)^\tau}{p} - \frac{Q(1-a)^\tau}{\alpha} - a^\beta\tau - \mu(1-a) \\
& = Q\left( \frac{\gamma}{Q} + 1 + \frac{a(sp - Q)}{Qp} - \frac{(1-a)}{\alpha} - a^\beta\tau - \mu(1-a) \right) \\
& = Q\left( a \left( \frac{1}{Q} + \frac{\beta - s}{Q} \right) + (1-a) \left( \frac{1}{\alpha} + \frac{\mu}{Q} \right) + \frac{a^{sp - Q}}{Qp} - \frac{(1-a)}{\alpha} - a^\beta\tau - \mu(1-a) \right) \\
& = 0. \tag{4.12} \end{align*} \]

Thus, we obtain

\[ \begin{align*}
\int_{A_{k,q}} q^{\gamma}(x)|u|^\tau dx \leq C 2^{(\gamma+Q)^k} |u_{A_{k,q}}|^\tau + C |u|^{\alpha} \|q^{\mu}(x)u\|_{L^\alpha(A_{k,q})}^{(1-a)^\tau}. \tag{4.13} \end{align*} \]
and by summing over \( k \) from \( m \) to \( n \), we get

\[
\int_{\bigcup_{k=m}^{n} A_{k,q}} q^{\gamma \tau}(x) |u|^\tau \, dx = \int_{\{2^m < q(x) < 2^{m+1}\}} q^{\gamma \tau}(x) |u|^\tau \, dx \leq C \sum_{k=m}^{n} 2^{(\gamma \tau + Q)k} |u_{A_{k,q}}|^\tau \\
+ C \sum_{k=m}^{n} [u]_{s,p,\beta,q,A_{k,q}} \|q^{\mu}(x) u\|_{L^\infty(A_{k,q})}^{(1-a)\tau}, \quad (4.14)
\]

where \( k, m, n \in \mathbb{Z} \) and \( m \leq n - 2 \).

To prove \( 4.3 \) let us choose \( n \) such that

\[
\text{supp} \, u \subset B_{2^n}, \quad (4.15)
\]

where \( B_{2^n} \) is a quasi-ball of \( G \) with the radius \( 2^n \).

The following known inequality will be used in the proof.

**Lemma 4.4** (Lemma 2.2, [16]). Let \( \xi > 1 \) and \( \eta > 1 \). Then exists a positive constant \( C \) depending \( \xi \) and \( \eta \) such that \( 1 < \zeta < \xi \),

\[
(|a| + |b|)^\eta \leq \zeta |a|^\eta + \frac{C}{(\zeta - 1)^{\eta-1}} |b|^\eta, \quad \forall \, a, b \in \mathbb{R}. \quad (4.16)
\]

Let us consider the following integral

\[
f_{A_{k+1,q} \cup A_{k,q}} |u - f_{A_{k+1,q} \cup A_{k,q}} u|^\tau \, dx \\
= \frac{1}{|A_{k+1,q}| + |A_{k,q}|} \int_{A_{k+1,q} \cup A_{k,q}} |u - f_{A_{k+1,q} \cup A_{k,q}} u|^\tau \, dx \\
= \frac{1}{|A_{k+1,q}| + |A_{k,q}|} \left( \int_{A_{k+1,q}} |u - f_{A_{k+1,q} \cup A_{k,q}} u|^\tau \, dx + \int_{A_{k,q}} |u - f_{A_{k+1,q} \cup A_{k,q}} u|^\tau \, dx \right).
\]
On the other hand, a direct calculation gives

\[
\begin{align*}
&\frac{1}{|A_{k+1,q}| + |A_{k,q}|} \left( \int_{A_{k+1,q}} \left| u - \int_{A_{k+1,q} \cup A_{k,q}} u \right|^\tau dx \right) \\
&= \frac{1}{|A_{k+1,q}| + |A_{k,q}|} \left( \int_{A_{k+1,q}} \left| u - \int_{A_{k+1,q}} u \right|^\tau dx \right) \\
& \geq \frac{1}{|A_{k+1,q}| + |A_{k,q}|} \left( \int_{A_{k+1,q}} \left( u - \int_{A_{k+1,q}} u \right) dx \right)^\tau \\
&= \frac{1}{|A_{k+1,q}| + |A_{k,q}|} \left( \int_{A_{k,q}} u dx - \int_{A_{k+1,q}} u dx \right)^\tau \\
&= \frac{1}{(|A_{k+1,q}| + |A_{k,q}|)^2} \left( |A_{k+1,q}| \int_{A_{k,q}} u dx - |A_{k,q}| \int_{A_{k+1,q}} u dx \right)^\tau \\
&= \frac{|A_{k+1,q}| |A_{k,q}|}{(|A_{k+1,q}| + |A_{k,q}|)^2} \left( \frac{1}{|A_{k,q}|} \int_{A_{k,q}} u dx - \frac{1}{|A_{k+1,q}|} \int_{A_{k+1,q}} u dx \right)^\tau \\
&= \frac{|A_{k+1,q}| |A_{k,q}|}{(|A_{k+1,q}| + |A_{k,q}|)^2} \left( |u_{A_{k+1,q}} - u_{A_{k,q}}| \right)^\tau \geq C \frac{2^{Qk_2Q(k-1)}}{2^{Qk_2Q(k-1)}} |u_{A_{k+1,q}} - u_{A_{k,q}}|\tau \\
& \geq C \frac{2^{Qk_2Q(k-1)}}{2^{Qk_2Q(k-1)}} |u_{A_{k+1,q}} - u_{A_{k,q}}|\tau \geq C |u_{A_{k+1,q}} - u_{A_{k,q}}|\tau. \quad (4.17)
\end{align*}
\]

From (4.17) and Lemma 4.3 we obtain

\[
|u_{A_{k+1,q}} - u_{A_{k,q}}|\tau \leq C \int_{A_{k+1,q} \cup A_{k,q}} \left| u - \int_{A_{k+1,q} \cup A_{k,q}} u \right|^\tau dx
\]

\[
\leq C2^{a(2Q-k)Q} \left[ u \right]_{s,p,A_{k+1,q} \cup A_{k,q}}^{\tau} \left( \int_{A_{k+1,q} \cup A_{k,q}} |u|^a dx \right)^{(1-a)\tau}. \quad (4.18)
\]

By using this fact, taking \( \tau = 1 \) we have

\[
|u_{A_{k,q}}| \leq |u_{A_{k+1,q}} - u_{A_{k,q}}| + |u_{A_{k+1,q}}|
\]

\[
\leq |u_{A_{k+1,q}}| + C2^{a(2Q-k)Q} \left[ u \right]_{s,p,A_{k+1,q} \cup A_{k,q}}^{a} \left( \int_{A_{k+1,q} \cup A_{k,q}} |u|^a dx \right)^{(1-a)\tau}, \quad (4.19)
\]
and by using Lemma 4.4 with \( \eta = \tau, \zeta = 2^{\gamma \tau + Q} c \), where \( c = \frac{2}{1 + 2^{\gamma \tau + Q}} < 1 \), since \( \gamma \tau + Q > 0 \), we have

\[
2^{(\gamma + Q)k} |u_{A_{k,q}}|^\tau \leq c2^{(k+1)(\gamma + Q)} |u_{A_{k+1,q}}|^\tau + C[u]^{\tau}_{s,p,\beta,q,A_{k+1,q} \cup A_{k,q}} \|q^\mu(x)u\|_{L^n(A_{k+1,q} \cup A_{k,q})}^{(1-\alpha)\tau}.
\]

By summing over \( k \) from \( m \) to \( n \) and by using (4.15) we have

\[
\sum_{k=m}^{n} 2^{(\gamma + Q)k} |u_{A_{k,q}}|^\tau \leq \sum_{k=m}^{n} c2^{(k+1)(\gamma + Q)} |u_{A_{k+1,q}}|^\tau + C \sum_{k=m}^{n} [u]^{\tau}_{s,p,\beta,q,A_{k+1,q} \cup A_{k,q}} \|q^\mu(x)u\|_{L^n(A_{k+1,q} \cup A_{k,q})}^{(1-\alpha)\tau}.
\]

By using (4.20), we compute

\[
(1 - c) \sum_{k=m}^{n} 2^{(\gamma + Q)k} |u_{A_{k,q}}|^\tau \leq 2^{(\gamma + Q)m} |u_{A_{m,q}}|^\tau + (1 - c) \sum_{k=m+1}^{n} 2^{(\gamma + Q)k} |u_{A_{k,q}}|^\tau
\]

\[
\leq C \sum_{k=m}^{n} [u]^{\tau}_{s,p,\beta,q,A_{k+1,q} \cup A_{k,q}} \|q^\mu(x)u\|_{L^n(A_{k+1,q} \cup A_{k,q})}^{(1-\alpha)\tau}.
\]

This yields

\[
\sum_{k=m}^{n} 2^{(\gamma + Q)k} |u_{A_{k,q}}|^\tau \leq C \sum_{k=m}^{n} [u]^{\tau}_{s,p,\beta,q,A_{k+1,q} \cup A_{k,q}} \|q^\mu(x)u\|_{L^n(A_{k+1,q} \cup A_{k,q})}^{(1-\alpha)\tau}.
\]

From (4.14) and (4.22), we have

\[
\int_{\{2^m < q(x) < 2^{n+1}\}} q^{\gamma \tau}(x) |u|^\tau dx \leq C \sum_{k=m}^{n} [u]^{\tau}_{s,p,\beta,q,A_{k+1,q} \cup A_{k,q}} \|q^\mu(x)u\|_{L^n(A_{k+1,q} \cup A_{k,q})}^{(1-\alpha)\tau}.
\]

Let \( s, t \geq 0 \) be such that \( s + t \geq 1 \). Then for any \( x_k, y_k \geq 0 \), we have

\[
\sum_{k=m}^{n} x_k^s y_k^t \leq \left( \sum_{k=m}^{n} x_k \right)^s \left( \sum_{k=m}^{n} y_k \right)^t.
\]

By using this inequality in (4.23) with \( s = \frac{\alpha}{p}, t = \frac{(1-\alpha)\tau}{\alpha} : \frac{\alpha}{p} + \frac{1-\alpha}{\alpha} \geq \frac{1}{\gamma} \) and \( s \geq \beta - \sigma \), we obtain

\[
\int_{\{q(x) > 2^m\}} q^{\gamma \tau}(x) |u|^\tau dx \leq C[u]^{\tau}_{s,p,\beta,q,\cup_{k=m}^{\infty} A_{k,q}} \|q^\mu(x)u\|_{L^n(A_{k+1,q} \cup A_{k,q})}^{(1-\alpha)\tau}.
\]

Inequality (4.3) is proved.

Let us prove (4.4). The strategy of the proof is similar to the previous case. Choose \( m \) such that

\[
supp u \cap B_{2^m} = \emptyset.
\]

From Lemma 4.3 we have

\[
|u_{A_{k+1,q}} - u_{A_{k,q}}|^\tau \leq C' \frac{a_k(x_{p-Q})}{p} [u]^{\tau}_{s,p,\beta,q,A_{k+1,q} \cup A_{k,q}} \left( \int_{A_{k+1,q} \cup A_{k,q}} |u|^\alpha dx \right)^{(1-\alpha)\tau}.\]
By Lemma 4.4 and choosing \( c = \frac{1+2q+Q}{2} < 1 \), since \( \gamma \tau + Q < 0 \), we have
\[
2^{(\gamma + Q)(k + 1)} |u_{A_{k+1,q}}|^\tau \leq c 2^{k(\gamma + Q)} |u_{A_{k,q}}|^\tau + C |u_{A_{k,q}}|^\tau + C |u_{A_{k,q}}|^\tau \|q^\mu(x)u\|^{(1-a)\tau}_{L^\infty(A_{k+1,q} \cup A_{k,q})},
\]
and by summing over \( k \) from \( m \) to \( n \) and by using (4.26) we obtain
\[
\sum_{k=m}^{n} 2^{(\gamma + Q)k} |u_{A_{k,q}}|^\tau \leq C \sum_{k=m-1}^{n-1} |u_{s,p,\beta,q,A_{k+1,q} \cup A_{k,q}}|^\tau \|q^\mu(x)u\|^{(1-a)\tau}_{L^\infty(A_{k+1,q} \cup A_{k,q})}. \tag{4.27}
\]
From (4.14) and (4.27), we establish that
\[
\int_{\{2^m < q(x) < 2^{m+1}\}} q^\gamma(x) |u|^\tau dx \leq C \sum_{k=m-1}^{n-1} |u_{s,p,\beta,q,A_{k+1,q} \cup A_{k,q}}|^\tau \|q^\mu(x)u\|^{(1-a)\tau}_{L^\infty(A_{k+1,q} \cup A_{k,q})}. \tag{4.28}
\]
Now by using (4.24) we get
\[
\int_{\{q(x) < 2^{m+1}\}} q^\gamma(x) |u|^\tau dx \leq C |u_{s,p,\beta,q,\cup_{k=-\infty}^m A_{k,q}}|^\tau \|q^\mu(x)u\|^{(1-a)\tau}_{L^\infty(\cup_{k=-\infty}^m A_{k,q})}. \tag{4.29}
\]
The proof of the case \( s \geq \beta - \sigma \) is complete.

Let us prove the case of \( \beta - \sigma > s \). Without loss of generality, we assume that
\[
|u|_{s,p,\beta,q} = \|u\|_{L^\infty(\mathbb{R})} = 1, \tag{4.30}
\]
where
\[
\frac{1}{p} + \frac{\beta - s}{Q} \neq \frac{1}{\alpha} + \frac{\mu}{Q}. \tag{4.31}
\]
We also assume that \( a_1 > 0, \) \( 1 > a_2 \) and \( \tau_1, \tau_2 > 0 \) with
\[
\frac{1}{\tau_1} = \frac{a_2}{p} + \frac{1 - a_2}{\alpha}, \tag{4.32}
\]
and
\[
\text{if } \frac{a}{p} + \frac{1 - a}{\alpha} - \frac{as}{Q} > 0, \text{ then } \frac{1}{\tau_1} = \frac{a_1}{p} + \frac{1 - a_1}{\alpha} - \frac{a_1s}{Q}, \tag{4.33}
\]
\[
\text{if } \frac{a}{p} + \frac{1 - a}{\alpha} - \frac{as}{Q} \leq 0, \text{ then } \frac{1}{\tau_1} > \frac{1}{\tau_1} = \frac{a_1}{p} + \frac{1 - a_1}{\alpha} - \frac{a_1s}{Q}. \tag{4.34}
\]
Taking \( \gamma_1 = a_1(1 - a_1)\mu \) and \( \gamma_2 = a_2(\beta - s) + (1 - a_2)\mu \), we obtain
\[
\frac{1}{\tau_1} + \frac{\gamma_1}{Q} \geq a_1 \left( \frac{1}{p} + \frac{\beta - s}{Q} \right) + (1 - a_1) \left( \frac{1}{\alpha} + \frac{\mu}{Q} \right) \tag{4.35}
\]
and
\[
\frac{1}{\tau_2} + \frac{\gamma_2}{Q} = a_2 \left( \frac{1}{p} + \frac{\beta - s}{Q} \right) + (1 - a_2) \left( \frac{1}{\alpha} + \frac{\mu}{Q} \right). \tag{4.36}
\]
Let \( a_1 \) and \( a_2 \) be such that
\[
|a - a_1| \text{ and } |a - a_2| \text{ are small enough,} \tag{4.37}
\]
\[
a_2 < a < a_1, \text{ if } \frac{1}{p} + \frac{\beta - s}{Q} > \frac{1}{\alpha} + \frac{\mu}{Q}, \tag{4.38}
\]
\[
a_1 < a < a_2, \text{ if } \frac{1}{p} + \frac{\beta - s}{Q} < \frac{1}{\alpha} + \frac{\mu}{Q}. \tag{4.39}
\]
By using (4.35)–(4.37) in (4.33), (4.34) and (4.11), we establish
\[
\frac{1}{\tau_1} + \frac{\gamma_1}{Q} > \frac{1}{\tau} + \frac{\gamma}{Q} > \frac{1}{\tau_2} + \frac{\gamma_2}{Q} > 0. \tag{4.38}
\]
From (4.32) in the case \(\frac{s}{p} + \frac{\beta}{\alpha} - \frac{a}{Q} > 0\) with \(a > 0, \beta - \sigma > s\) and (4.35), we get
\[
\frac{1}{\tau} - \frac{1}{\tau_1} = (a - a_1) \left(\frac{1}{p} - \frac{s}{Q} - \frac{1}{\alpha}\right) + \frac{a}{Q} (\beta - \sigma) > 0, \tag{4.39}
\]
and
\[
\frac{1}{\tau} - \frac{1}{\tau_2} = (a - a_2) \left(\frac{1}{p} - \frac{1}{\alpha}\right) + \frac{a}{Q} (\beta - \sigma - s) > 0. \tag{4.40}
\]
From (4.32), (4.39) and (4.40), we have
\[
\tau_1 > \tau, ~ \tau_2 > \tau.
\]
Thus, using this, (4.35) and Hölder’s inequality, we obtain
\[
\|q^\gamma(x)u\|_{L^\tau(G; B_1)} \leq C\|q^{\gamma_1}(x)u\|_{L^{\tau_1}(G)}, \tag{4.41}
\]
and
\[
\|q^\gamma(x)u\|_{L^\tau(B_1)} \leq C\|q^{\gamma_2}(x)u\|_{L^{\tau_2}(G)}, \tag{4.42}
\]
where \(B_1\) is the unit quasi-ball. By using the previous case, we establish
\[
\|q^{\gamma_1}(x)u\|_{L^{\tau_1}(G)} \leq C [u]_{s,p,\beta,q}^{a_1}\|q^\mu(x)u\|_{L^\sigma(G)}^{1-a_1} \leq C, \tag{4.43}
\]
and
\[
\|q^{\gamma_2}(x)u\|_{L^{\tau_2}(G)} \leq C [u]_{s,p,\beta,q}^{a_2}\|q^\mu(x)u\|_{L^\sigma(G)}^{1-a_2} \leq C. \tag{4.44}
\]
The proof of Theorem 4.1 is complete. \(\square\)

**Remark 4.5.** By taking in (4.4) \(a = 1, \tau = p, \beta_1 = \beta_2 = 0, \) and \(\gamma = -s,\) we get an analogue of the fractional Hardy inequality on homogeneous Lie groups (Theorem 2.9, [12]).

**Remark 4.6.** In the Abelian case \((\mathbb{R}^N, +)\) with the standard Euclidean distance instead of the quasi-norm and by taking in (4.4) \(a = 1, \tau = p, \beta_1 = \beta_2 = 0, \) and \(\gamma = -s,\) we get the fractional Hardy inequality (Theorem 1.1, [8]).

Now we consider the critical case \(\frac{1}{\tau} + \frac{\gamma}{Q} = 0.\)

**Theorem 4.7.** Assume that \(Q \geq 2, s \in (0,1), p > 1, \alpha \geq 1, \tau > 1, a \in (0,1], \beta_1, \beta_2, \beta, \mu, \gamma \in \mathbb{R}, \beta_1 + \beta_2 = \beta;\)
\[
\frac{1}{\tau} + \frac{\gamma}{Q} = a \left(\frac{1}{p} + \frac{\beta - s}{Q}\right) + (1 - a) \left(\frac{1}{\alpha} + \frac{\mu}{Q}\right). \tag{4.45}
\]
Assume in addition that, \(0 \leq \beta - \sigma \leq s\) with \(\gamma = a\sigma + (1 - a)\mu.\)
If \(\frac{1}{\tau} + \frac{\gamma}{Q} = 0\) and supp \(u \subset B_\tau,\) then, we have
\[
\left\|\frac{q^\gamma(x)u}{\ln \frac{2r}{q(x)}}\right\|_{L^\tau(G)} \leq C[u]_{s,p,\beta,q}^a\|q^\mu(x)u\|_{L^\sigma(G)}^{1-a}, ~ u \in C^1_c(G), \tag{4.46}
\]
where \(B_R = \{x \in G : q(x) < R\}\) is the quasi-ball and \(0 < r < R.\)
Proof of Theorem 4.7. The proof is similar to the proof of Theorem 4.1, summing over $k$ from $m$ to $n$ and fixing $\varepsilon > 0$, we have

$$
\int_{\{q(x) > 2^m\}} \frac{q^{\g(T)(x)}}{\ln^{1+\varepsilon}\left(\frac{2R}{q(x)}\right)} |u|^\tau \, dx \leq C \sum_{k=m}^{n} \frac{1}{(n+1-k)^{1+\varepsilon}} |u|_{A_{k,q}}^\tau + C \sum_{k=m}^{n} [u]_{s,p,\beta,q,A_{k,q}}^{\alpha} \|q^\mu(x)u\|_{L^\alpha(A_{k,q})}^{(1-\alpha)^\tau}. \tag{4.47}
$$

From Lemma 4.3, we have

$$
|u_{A_{k+1,q}} - u_{A_{k,q}}| \leq C 2^{\frac{\alpha(k-Q)}{\alpha}} [u]^a_{s,p,q,A_{k+1,q} \cup A_{k,q}} \left( \int_{A_{k+1,q} \cup A_{k,q}} |u|^{\alpha} \, dx \right)^{\frac{1-\alpha}{\alpha}}.
$$

By using Lemma 4.4 with $\zeta = \frac{(n+1-k)^\varepsilon}{(n+\frac{3}{2}-k)^\varepsilon}$ we get

$$
\frac{|u_{A_{k+1,q}}^\tau}{(n+1-k)^{\tau}} \leq \frac{|u_{A_{k,q}}^\tau|}{(n+\frac{3}{2}-k)^{\tau}} + C(n+1-k)^{-1-\varepsilon} [u]_{s,p,\beta,q,A_{k+1,q} \cup A_{k,q}}^{\alpha} \|q^\mu(x)u\|_{L^\alpha(A_{k+1,q} \cup A_{k,q})}^{(1-\alpha)^\tau}. \tag{4.48}
$$

For $\varepsilon > 0$ and $n \geq k$, we have

$$
\frac{1}{(n-k+1)^{\varepsilon}} - \frac{1}{(n-k+\frac{3}{2})^{\varepsilon}} \sim \frac{1}{(n-k+1)^{1+\varepsilon}}. \tag{4.49}
$$

By using this fact, (4.48), (4.49) and $\varepsilon = \tau - 1$, we obtain

$$
\sum_{k=m}^{n} \frac{|u_{A_{k,q}}^\tau|}{(n+1-k)^{\tau}} \leq C \sum_{k=m}^{n} [u]_{s,p,\beta,q,A_{k+1,q} \cup A_{k,q}}^{\alpha} \|q^\mu(x)u\|_{L^\alpha(A_{k+1,q} \cup A_{k,q})}^{(1-\alpha)^\tau}. \tag{4.50}
$$

From (4.47) and (4.50), we establish

$$
\int_{\{q(x) > 2^m\}} \frac{q^{\g(T)(x)}}{\ln^{1+\varepsilon}\left(\frac{2R}{q(x)}\right)} |u|^\tau \, dx \leq C \sum_{k=m}^{n} [u]_{s,p,\beta,q,A_{k+1,q} \cup A_{k,q}}^{\alpha} \|q^\mu(x)u\|_{L^\alpha(A_{k+1,q} \cup A_{k,q})}^{(1-\alpha)^\tau}. \tag{4.51}
$$

By using (4.24) with (4.45) and $0 \leq \beta - \sigma \leq s$, where $s = \frac{\tau a}{p}$, $t = \frac{(1-\alpha)^\tau}{\alpha}$, we have $s + t \geq 1$ and we arrive at

$$
\int_{\{q(x) > 2^m\}} \frac{q^{\g(T)(x)}}{\ln^{1+\varepsilon}\left(\frac{2R}{q(x)}\right)} |u|^\tau \, dx \leq C \sum_{k=m}^{n} [u]_{s,p,\beta,q,A_{k,q}}^{\alpha} \|q^\mu(x)u\|_{L^\alpha(A_{k,q})}^{(1-\alpha)^\tau}. \tag{4.52}
$$

Theorem 4.7 is proved. \qed

5. Lyapunov-type inequalities for the fractional operators on $\mathbb{G}$

In this section we prove the Lyapunov-type inequality for the Riesz potential and for the fractional $p$-sub-Laplacian system on homogeneous Lie groups. Note that the Lyapunov-type inequality for the Riesz operator is new even in the Abelian case $(\mathbb{R}^N, +)$. Also, we give applications of the Lyapunov-type inequality, more precisely, we give two side estimates for the first eigenvalue of the Riesz potential of the fractional $p$-sub-Laplacian system.
Let us consider the Riesz potential on a Haar measurable set $\Omega \subset \mathbb{G}$ that can be defined by the formula
\[
\mathcal{R}u(x) = \int_{\Omega} \frac{u(y)}{q^{2s}(y^{-1} \circ x)} dy, \quad 0 < 2s < Q. \tag{5.1}
\]
The (weighted) Riesz potential can be also defined by
\[
\mathcal{R}(\omega u)(x) = \int_{\Omega} \frac{\omega(y)u(y)}{q^{2s}(y^{-1} \circ x)} dy, \quad 0 < 2s < Q. \tag{5.2}
\]

**Theorem 5.1.** Let $\Omega \subset \mathbb{G}$ be a Haar measurable set and let $Q \geq 2 > 2s > 0$ and let $1 < p < 2$. Assume that $\omega \in L^{\frac{p}{\theta}}(\Omega), \frac{1}{q^{2s}(y^{-1} \circ x)} \in L^{\frac{p}{\theta}}(\Omega \times \Omega)$ and $C_0 = \left\| \int_{q^{2s}(y^{-1} \circ x)} \right\|_{L^{\frac{p}{\theta}}(\Omega \times \Omega)}$. Let $u \in L^{\frac{p}{\theta}}(\Omega), u \neq 0$, satisfy
\[
\mathcal{R}(\omega u)(x) = \int_{\Omega} \frac{\omega(y)u(y)}{q^{2s}(y^{-1} \circ x)} dy = u(x), \text{ for a.e. } x \in \Omega. \tag{5.3}
\]
Then
\[
\|\omega\|_{L^{\frac{p}{\theta}}(\Omega)} \geq \frac{1}{C_0}. \tag{5.4}
\]

**Proof of Theorem 5.1.** In (5.3), by using Hölder’s inequality for $p, \theta > 1$ with $\frac{1}{p} + \frac{1}{\theta} = 1$ and $\frac{1}{q} + \frac{1}{p'} = 1$, we have
\[
|u(x)| = \left| \int_{\Omega} \frac{\omega(y)u(y)}{q^{2s}(y^{-1} \circ x)} dy \right| \leq \left( \int_{\Omega} |\omega(y)|u(y)| dy \right)^{\frac{1}{p'}} \left( \int_{\Omega} \left| \frac{1}{q^{2s}(y^{-1} \circ x)} \right|^{\frac{1}{p'}} dy \right)^{\frac{1}{p'}} \tag{5.5}
\]
\[
\leq \left( \int_{\Omega} |\omega(y)|\theta dy \right)^{\frac{1}{p'}} \left( \int_{\Omega} |u(y)|\theta dy \right)^{\frac{1}{p'}} \left( \int_{\Omega} \left| \frac{1}{q^{2s}(y^{-1} \circ x)} \right|^{\frac{1}{p'}} dy \right)^{\frac{1}{p'}} \tag{5.5}
\]
\[
= \|\omega\|_{L^{\frac{p}{\theta}}(\Omega)}\|u\|_{L^{\frac{p}{\theta}}(\Omega)} \left( \int_{\Omega} \left| \frac{1}{q^{2s}(y^{-1} \circ x)} \right|^{\frac{p}{p'}} dy \right)^{\frac{1}{p}}. \tag{5.5}
\]
Let $p'$ be such that $p' = p\theta'$ and then $\theta = \frac{1}{2-p'}$. Thus, we get
\[
|u(x)| \leq \|\omega\|_{L^{\frac{p}{\theta}}(\Omega)}\|u\|_{L^{\frac{p}{\theta}}(\Omega)} \left( \int_{\Omega} \left| \frac{1}{q^{2s}(y^{-1} \circ x)} \right|^{\frac{p}{p'}} dy \right)^{\frac{1}{p}}. \tag{5.6}
\]
From (5.6) we calculate
\[
\|u\|_{L^{\frac{p}{\theta}}(\Omega)} \leq \|\omega\|_{L^{\frac{p}{\theta}}(\Omega)}\|u\|_{L^{\frac{p}{\theta}}(\Omega)} \left( \int_{\Omega} \int_{\Omega} \left| \frac{1}{q^{2s}(y^{-1} \circ x)} \right|^{\frac{p}{p'}} dx dy \right)^{\frac{1}{p}} \tag{5.7}
\]
\[
= C_0\|\omega\|_{L^{\frac{p}{\theta}}(\Omega)}\|u\|_{L^{\frac{p}{\theta}}(\Omega)}. \tag{5.7}
\]
Finally, since $u \neq 0$, this implies
\[
\|\omega\|_{L^{\frac{p}{\theta}}(\Omega)} \geq \frac{1}{C_0}. \tag{5.8}
\]
Theorem 5.1 is proved. \qed
Let us consider the following spectral problem for the Riesz potential:
\[
\mathcal{R}u(x) = \int_\Omega \frac{u(y)}{|x - y|^{N-2s}} dy = \lambda u(x), \quad x \in \Omega, \quad 0 < 2s < Q.
\] (5.9)

We recall the Rayleigh quotient for the Riesz potential:
\[
\lambda_1(\Omega) = \sup_{u \neq 0} \frac{\int_\Omega \int_\Omega \frac{u(x)u(y)}{|x - y|^{N-2s}} dxdy}{\|u\|^2_{L^2(\Omega)}},
\] (5.10)
where \(\lambda_1(\Omega)\) is the first eigenvalue of the Riesz potential.

So, a direct consequence of Theorem 5.1 is

**Theorem 5.2.** Let \(\Omega \subset \mathbb{G}\) be a Haar measurable set and \(Q \geq 2 > 2s > 0\) and let \(1 < p < 2\). Assume that \(\omega = \frac{1}{\lambda_1(\Omega)}\), we obtain
\[
\lambda_1(\Omega) \leq C_0 |\Omega|^{\frac{2-p}{p}},
\] (5.11)

where \(C_0 = \left\| \frac{1}{|x - y|^{N-2s}} \right\|_{L^{\frac{p}{p-1}}(\Omega \times \Omega)}\).

**Proof of Theorem 5.2** By using (5.10), Theorem 5.1 and \(\omega = \frac{1}{\lambda_1(\Omega)}\), we obtain
\[
\lambda_1(\Omega) \leq C_0 |\Omega|^{\frac{2-p}{p}}.
\] (5.12)

Theorem 5.2 is proved. \(\square\)

In the Abelian group \((\mathbb{R}^N, +)\) we have the following consequences. To the best of our knowledge, these results seem new (even in this Euclidean case).

Let us consider the Riesz potential on \(\Omega \subset \mathbb{R}^N\):
\[
\mathcal{R}u(x) = \int_\Omega \frac{u(y)}{|x - y|^{N-2s}} dy, \quad 0 < 2s < N,
\] (5.13)
and the weighted Riesz potential
\[
\mathcal{R}(\omega u)(x) = \int_\Omega \frac{\omega(y)u(y)}{|x - y|^{N-2s}} dy, \quad 0 < 2s < N.
\] (5.14)

Then we have following theorem:

**Theorem 5.3.** Let \(\Omega \subset \mathbb{R}^N, \quad N \geq 2\), be a measurable set with \(|\Omega| < \infty, \quad 1 < p < 2\) and let \(N \geq 2 > 2s > 0\). Assume that \(\omega \in L^{\frac{p}{p-1}}(\Omega), \quad \frac{1}{|x - y|^{N-2s}} \in L^{\frac{p}{p-1}}(\Omega \times \Omega)\) and let \(S = \left\| \frac{1}{|x - y|^{N-2s}} \right\|_{L^{\frac{p}{p-1}}(\Omega \times \Omega)}\). Assume that \(u \in L^{\frac{p}{p-1}}(\Omega), \quad u \neq 0\), satisfies
\[
\mathcal{R}(\omega u)(x) = u(x), \quad x \in \Omega.
\]

Then
\[
\|\omega\|_{L^{\frac{p}{p-1}}(\Omega)} \geq \frac{1}{S}.
\] (5.15)

**Proof of Theorem 5.3** In Theorem 5.1 we set \(\mathbb{G} = (\mathbb{R}^N, +)\) and take the standard Euclidean distance instead of the quasi-norm. \(\square\)
Proof of Theorem 5.4.
The proof of \((5.16)\) with 

where \(B\) is a quasi-ball with respect to 

Namely, let us consider the fractional 

Let \(\Omega \subset \mathbb{R}^N, N \geq 2\), be a set with \(|\Omega| < \infty\), \(1 < p < 2\) and \(N \geq 2 > 2s > 0\) and \(1 < p < 2\). Assume that \(\omega \in L^{\frac{1}{q-p}}(\Omega)\), \(\frac{1}{q-p} \in L^{\frac{1}{p-1}}(\Omega \times \Omega)\) and 

Then for the spectral problem \((5.16)\) we have,

where \(B \subset \mathbb{R}^N\) is an open ball, \(\lambda_1(\Omega)\) is the first eigenvalue of the spectral problem \((5.16)\) with \(|\Omega| = |B|\).

Proof of Theorem 5.4. The proof of \(\lambda_1(B) \leq S|B|^{\frac{2-p}{p}}\) is the same as the proof of Theorem 5.2. From [17] we have 

The proof of Theorem 5.4 is complete. \(\square\)

In [12] the authors proved a Lyapunov-type inequality for the fractional \(p\)-sub-Laplacian with the homogeneous Dirichlet condition. Here we establish Lyapunov-type inequality for the fractional \(p\)-sub-Laplacian system for the homogeneous Dirichlet problem. Namely, let us consider the fractional \(p\)-sub-Laplacian system:

with homogeneous Dirichlet conditions

where \(\Omega \subset \mathbb{G}\) is a Haar measurable set, \(\omega_i \in L^1(\Omega), \omega_i \geq 0, s_i \in (0, 1), p_i \in (1, \infty)\) and \((-\Delta_{p,q})^s\) is the fractional \(p\)-sub-Laplacian on \(\mathbb{G}\) defined by

Here \(B_q(x, \delta)\) is a quasi-ball with respect to \(q\), with radius \(\delta\), centred at \(x \in \mathbb{G}\), and \(\alpha_i\) are positive parameters such that

To prove a Lyapunov-type inequality for the system we need some preliminary results from [12], the so-called fractional Hardy inequality on the homogeneous Lie groups.
**Theorem 5.5 (\textbf{[12]}, Fractional Hardy inequality).** For all \( u \in C^\infty_c(\mathbb{G}) \) we have

\[
C \int_\mathbb{G} \left| \frac{u(x)}{q^{ps}(x)} \right|^p dx \leq \int_\mathbb{G} \left| u \right|_{s,p,q}^p,
\]

(5.22)

where \( p \in (1, \infty) \), \( s \in (0, 1) \), and \( C \) is a positive constant.

We denote by \( r_{\Omega,q} \) the inner quasi-radius of \( \Omega \), that is,

\[
r_{\Omega,q} = \max\{ q(x) : x \in \Omega \}. \tag{5.23}
\]

**Definition 5.6.** We say that \((u_1, \ldots, u_n) \in \prod_{i=1}^n W_0^{s_i,p_i}(\Omega) \) is a weak solution of

\[
\begin{align*}
&\int_\mathbb{G} \int_\mathbb{G} |u_i(x) - u_i(y)|^{p_i-2}(u_i(x) - u_i(y))(v_i(x) - v_i(y))
\quad \cdot |y - x|^{-\alpha_i} dx dy \\
&= \int_\Omega \omega_i(x) \left( \prod_{j=1}^{i-1} |u_j(x)|^{\alpha_j} \right) \left( \prod_{j=i+1}^n |u_j(x)|^{\alpha_j} \right) |u_i(x)|^{-2} u_i(x) v_i(x) dx,
\end{align*}
\]

(5.24)

for every \( i = 1, \ldots, n \).

Now we present the following analogue of the Lyapunov-type inequality for the fractional \( p \)-sub-Laplacian system on \( \mathbb{G} \).

**Theorem 5.7.** Let \( s_i \in (0, 1) \) and \( p_i \in (1, \infty) \) be such that \( Q > s_i p_i \) for all \( i = 1, \ldots, n \). Let \( \omega_i \in L^\theta(\Omega) \) be a non-negative weight and assume that

\[
1 < \max_{i=1,\ldots,n} \left\{ \frac{Q}{s_i p_i} \right\} < \theta < \infty.
\]

If \((5.18)-(5.19)\) admits a nontrivial weak solution, then

\[
\prod_{i=1}^n \| \omega_i \|_{L^\theta(\Omega)} \geq C r_{\Omega,q}^{Q - \theta \sum_{j=1}^n s_j \alpha_j},
\]

(5.25)

where \( C > 0 \) is a positive constant.

**Remark 5.8.** In Theorem 5.7, by taking \( n = 1 \) and \( \alpha_1 = p \), we establish the Lyapunov-type inequality for the fractional \( p \)-sub-Laplacian on \( \mathbb{G} \) (see, e.g. [12, Theorem 3.1]).

**Proof of Theorem 5.7.** For all \( i = 1, \ldots, n \), let us define

\[
\xi_i = \gamma_i p_i + (1 - \gamma_i)p_i^*,
\]

(5.26)

and

\[
\gamma_i = \frac{\theta - \frac{Q}{s_i p_i}}{\theta - 1},
\]

(5.27)

where \( p_i^* = \frac{Q}{Q - s_i p_i} \) is the Sobolev conjugate exponent as in Theorem 5.4. Notice that for all \( i = 1, \ldots, n \) we have \( \gamma_i \in (0, 1) \) and \( \xi_i = p_i \theta' \), where \( \theta' = \frac{\theta}{\theta - 1} \). Then for every \( i \in \{1, \ldots, n\} \) we get

\[
\int_\Omega \left| u_i(x) \right|^{\xi_i} dx \leq \int_\Omega \left| u_i(x) \right|^{\xi_i} dx,
\]

where
and by using Hölder’s inequality with the following exponents \( \nu_i = \frac{1}{\gamma_i} \) and \( \frac{1}{\nu_i} + \frac{1}{\nu'_i} = 1 \), we get

\[
\int_\Omega \frac{|u_i(x)|^{\xi_i}}{q^{\gamma_i,p_i}(x)} \, dx = \int_\Omega \frac{|u_i(x)|^{\gamma_i,p_i}|u_i(x)|^{(1-\gamma_i)p'_i}}{q^{\gamma_i,p_i}(x)} \, dx \\
\leq \left( \int_\Omega \frac{|u_i(x)|^{p_i}}{q^{\gamma_i,p_i}(x)} \, dx \right)^{\gamma_i} \left( \int_\Omega |u_i(x)|^{p'_i} \, dx \right)^{1-\gamma_i}.
\]

(5.28)

On the other hand, from Theorem 3.1, we obtain

\[
\left( \int_\Omega |u_i(x)|^{p_i} \, dx \right)^{1-\gamma_i} \leq C[u_i]_{s_i,p_i,q_i}^{p_i(1-\gamma_i)},
\]

and from Theorem 5.5, we have

\[
\left( \int_\Omega \frac{|u_i(x)|^{p_i}}{q^{\gamma_i,p_i}(x)} \, dx \right)^{\gamma_i} \leq C[u_i]_{s_i,p_i,q_i}^{p_i \gamma_i}.
\]

Thus, from (5.28) and by taking \( u_i(x) = v_i(x) \) in (5.21), we get

\[
\int_\Omega \frac{|u_i(x)|^{\xi_i}}{q^{\gamma_i,p_i}(x)} \, dx \leq C([u_i]_{s_i,p_i,q_i})^{\frac{\xi_i}{\gamma_i}} \leq C([u_i]_{s_i,p_i,q_i})^{\frac{\xi_i}{\gamma_i}}
\]

\[
= C \left( \int_\Omega \omega_i(x) \prod_{j=1}^n |u_j|^{\alpha_j} \, dx \right)^{\frac{\xi_i}{\gamma_i}} = C \left( \int_\Omega \omega_i(x) \prod_{j=1}^n |u_j|^{\alpha_j} \, dx \right)^{\theta'},
\]

for every \( i = 1, \ldots, n \). Therefore, by using Hölder’s inequality with exponents \( \theta \) and \( \theta' \), we obtain

\[
\int_\Omega \frac{|u_i(x)|^{\xi_i}}{q^{\gamma_i,p_i}(x)} \, dx \leq C[\omega_i]_{L^{\theta}(\Omega)} \int_\Omega \prod_{j=1}^n |u_j|^{\alpha_j} \, dx.
\]

By using Hölder’s inequality and (5.21), we get

\[
\int_\Omega \prod_{j=1}^n |u_j(x)|^{\alpha_j} \, dx \leq \prod_{j=1}^n \left( \int_\Omega |u_j|^{\theta j} \, dx \right)^{\frac{\alpha_j}{\theta j}}.
\]

This implies that

\[
\int_\Omega \frac{|u_i(x)|^{\xi_i}}{q^{\gamma_i,p_i}(x)} \, dx \leq C[\omega_i]_{L^{\theta}(\Omega)} \prod_{j=1}^n \left( \int_\Omega |u_j|^{\theta j} \, dx \right)^{\frac{\alpha_j}{\theta j}}.
\]

So we establish

\[
\int_\Omega \frac{|u_i(x)|^{\xi_i}}{r_{i\Omega,q}^{\gamma_i,p_i}} \, dx \leq \int_\Omega \frac{|u_i(x)|^{\xi_i}}{q^{\gamma_i,p_i}(x)} \, dx \\
\leq C[\omega_i]_{L^{\theta}(\Omega)} \prod_{j=1}^n \left( \int_\Omega |u_j|^{\theta j} \, dx \right)^{\frac{\alpha_j}{\theta j}}.
\]

Thus, for every \( e_i > 0 \) we have

\[
\left( \int_\Omega \frac{|u_i(x)|^{\xi_i}}{r_{i\Omega,q}^{\gamma_i,p_i}} \, dx \right)^{e_i} = \frac{1}{r_{i\Omega,q}^{\gamma_i,p_i}} \left( \int_\Omega |u_i(x)|^{\xi_i} \, dx \right)^{e_i},
\]
Consequently, from (5.21) we have the solution of this system

\[ \sum_{j=1}^{n} \frac{1}{\gamma_j s_j p_j e_j} \prod_{i=1}^{n} \left( \int_{\Omega} |u_i(x)|^{p_i} \, dx \right)^{\frac{\alpha_j}{p_i}} \leq C \left( \prod_{i=1}^{n} \left( \int_{\Omega} |u_i(x)|^{p_i} \, dx \right)^{\frac{\alpha_j}{p_i}} \right), \]  

so that

\[ \prod_{i=1}^{n} \left( \int_{\Omega} |u_i(x)|^{p_i} \, dx \right)^{\frac{\alpha_j}{p_i}} \leq C \left( \prod_{i=1}^{n} \left( \int_{\Omega} |u_i(x)|^{p_i} \, dx \right)^{\frac{\alpha_j}{p_i}} \right), \]  

where \( C \) is a positive constant. Then, we choose \( e_i, i = 1, \ldots, n, \) such that

\[ \frac{\alpha_i}{p_i} \sum_{j=1}^{n} e_j - e_i = 0, \quad i = 1, \ldots, n. \]  

Consequently, from (5.21) we have the solution of this system

\[ e_i = \frac{\alpha_i}{p_i}, \quad i = 1, \ldots, n. \]  

From (5.21), (5.27) and (5.30) we arrive at

\[ \prod_{i=1}^{n} \left( \int_{\Omega} |u_i(x)|^{p_i} \, dx \right)^{\frac{\alpha_i}{p_i}} \geq C \prod_{i=1}^{n} \left( \int_{\Omega} |u_i(x)|^{p_i} \, dx \right)^{\frac{\alpha_i}{p_i}}, \]  

Theorem 5.7 is proved. \( \square \)

Now, let us discuss an application of the Lyapunov-type inequality for the fractional \( p \)-sub-Laplacian system on \( \mathcal{G} \). In order to do it we consider the spectral problem for the fractional \( p \)-sub-Laplacian system in the following form:

\[ \begin{cases} (-\Delta_{p_1,q})^{s_1} u_1(x) = \lambda_1 \alpha_1 \varphi(x) |u_1(x)|^{\alpha_1-2} u_1(x) |u_2(x)|^{\alpha_2} \ldots |u_n(x)|^{\alpha_n}, & x \in \Omega, \\ (-\Delta_{p_2,q})^{s_2} u_2(x) = \lambda_2 \alpha_2 \varphi(x) |u_1(x)|^{\alpha_1} |u_2(x)|^{\alpha_2-2} u_2(x) \ldots |u_n(x)|^{\alpha_n}, & x \in \Omega, \\ \vdots \\ (-\Delta_{p_n,q})^{s_n} u_n(x) = \lambda_n \alpha_n \varphi(x) |u_1(x)|^{\alpha_1} |u_2(x)|^{\alpha_2} \ldots |u_n(x)|^{\alpha_n-2} u_n(x), & x \in \Omega, \end{cases} \]  

with

\[ u_i(x) = 0, \quad x \in \mathcal{G} \setminus \Omega, \quad i = 1, \ldots, n, \]  

where \( \Omega \subset \mathcal{G} \) is a Haar measurable set, \( \varphi \in L^1(\Omega), \varphi \geq 0 \) and \( s_i \in (0, 1), \) \( p_i \in (1, \infty), \) \( i = 1, \ldots, n. \)

**Definition 5.9.** We say that \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is an eigenvalue if the problem (5.32) - (5.33) admits at least one nontrivial weak solution \( (u_1, \ldots, u_n) \in \prod_{i=1}^{n} W_{0}^{s_i,p_i}(\Omega). \)
Theorem 5.10. Let \( s_i \in (0, 1) \) and \( p_i \in (1, \infty) \) be such that \( Q > s_i p_i \), for all \( i = 1, \ldots, n \), and

\[
1 < \max_{i=1,\ldots,n} \left\{ \frac{Q}{s_i p_i} \right\} < \infty.
\]

Let \( \varphi \in L^\theta(\Omega) \) with \( \| \varphi \|_{L^\theta(\Omega)} \neq 0 \). Then, we have

\[
\lambda_k \geq \frac{C}{\alpha_k} \left( \prod_{i=1, i \neq k}^n \frac{\alpha_i}{\lambda_i} \right)^{\frac{p_k}{n_k}} \left( \prod_{i=1, i \neq k}^n \frac{1}{\alpha_i} \right)^{\frac{p_k}{n_k}} \int_{\Omega} \varphi^\theta(x) dx,
\]

\( k = 1, \ldots, n \).

where \( C \) is a positive constant and \( k = 1, \ldots, n \).

Proof of Theorem 5.10. In Theorem 5.7 by taking \( \omega_k = \lambda_k \alpha_k \varphi(x) \), \( k = 1, \ldots, n \), we have

\[
\frac{\alpha_k}{\lambda_k} \prod_{i=1, i \neq k}^n (\alpha_i \lambda_i)^{\frac{p_k}{n_k}} \int_{\Omega} \varphi^\theta(x) dx \geq C \frac{Q^\theta \sum_{j=1}^n s_j \alpha_j}{r^\theta_{\Omega,q} \prod_{i=1, i \neq k}^n \alpha_i}.
\]

This implies

\[
\lambda_k \geq \frac{C}{\alpha_k} \left( \prod_{i=1, i \neq k}^n \frac{\alpha_i}{\lambda_i} \right)^{\frac{p_k}{n_k}} \left( \prod_{i=1, i \neq k}^n \frac{1}{\alpha_i} \right)^{\frac{p_k}{n_k}} \int_{\Omega} \varphi^\theta(x) dx.
\]

Finally, we get that

\[
\lambda_k \geq \frac{C}{\alpha_k} \left( \prod_{i=1, i \neq k}^n \frac{\alpha_i}{\lambda_i} \right)^{\frac{p_k}{n_k}} \left( \prod_{i=1, i \neq k}^n \frac{1}{\alpha_i} \right)^{\frac{p_k}{n_k}} \int_{\Omega} \varphi^\theta(x) dx,
\]

\( k = 1, \ldots, n \). \( \square \)

Theorem 5.10 is proved.

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