Curved $L_\infty$-algebras and lifts of torsors.

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Abstract

Consider an extension of finite dimensional nilpotent Lie algebras 
$0 \to \mathfrak{h} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$ (over a field $k$ of characteristic zero) corresponding 
to an extension of unipotent algebraic groups $1 \to H \to \tilde{G} \to G \to 1$. For a $G$-torsor $P$ on an algebraic variety $X$ over $k$, we study the 
problem of lifting $P$ to a $\tilde{G}$-torsor $\tilde{P}$. Fixing a trivialization of $P$ on open subsets of an affine cover, we give the Cech complex of $\mathfrak{h}$-valued functions the structure of a curved $L_\infty$-algebra and define a curved 
version of the Deligne-Getzler groupoid. We show that this groupoid 
is isomorphic the groupoid of cocycle level $\tilde{G}$-lifts of $P$.

1 Introduction.

The purpose of this article is to give a curved version of a result due to 
Hinich, cf. [8] on descent of Deligne groupoids, or rather its variant proved 
by Fiorenza-Manetti-Martinengo in [3]. We will describe a specific setting of 
interest. Suppose we are given a scheme over a field $k$ of characteristic zero, 
an open affine covering $U = \{U_i\}$ and a finite dimensional Lie algebra $\mathfrak{g}$ with 
the corresponding unipotent algebraic group $G$. Then any $G$-torsor $P \to G$ 
in either Zariski or fppf topology trivializes on each open subset $U_i$. For 
$G = \mathbb{G}_a$ this follows from the vanishing of cohomology of the structure sheaf 
on each $U_i$ and in general follows by using a filtration on $G$ with successive 
quotients isomorphic to $\mathbb{G}_a$. Therefore, given a fixed trivialization of $P$ 
on each $U_i$, all information about $P$ can be recovered from the transition 
functions $\Phi_{ij} : U_i \cap U_j \to G$ subject to the cocycle condition 
$$\Phi_{ik} = \Phi_{ij} \Phi_{jk}$$
on triple intersections. As usual, a change of trivialization is described by 
regular maps $\Sigma_i : U_i \to G$ which send $\{\Phi\}_{ij}$ to an equivalent cocycle 
$$\hat{\Phi}_{ij} = \Sigma_i^{-1} \Phi_{ij} \Sigma_j.$$
On the other hand, the Čech complex $\mathcal{L}(\mathfrak{g})$ of the sheaf of $\mathfrak{g}$-valued functions, with respect to $\mathcal{U}$, does not inherit the Lie algebra structure from $\mathfrak{g}$ but it can be given the structure of an $L_\infty$ algebra. With appropriate choices, the cocycle condition above can be identified with the Maurer-Cartan equation of $\mathcal{L}(\mathfrak{g})$ and equivalences of cocycles with arrows in the Deligne-Getzler groupoid.

In more detail: on one hand, the $L_\infty$ structure on the Čech complex depends on the choice of a simplicial Dupont homotopy, see [7]. On the other hand, to convert the group valued cocycle condition into a statement involving the Lie algebra one must describe the group product in $G$ via the bracket in $\mathfrak{g}$. This is done by the Campbell-Baker-Hausdorff formula, which can be modified to a family of formulas using the Jacobi identity on the bracket of $\mathfrak{g}$. As it turns out, a choice of Dupont homotopy also fixes a CBH type formula in this family. This can be seen from the work of Getzler, cf. [7] who has related the CBH formula to a horn lifting condition in a Kan groupoid. The two choices (of the $L_\infty$ structure on Čech complex and the description of group product in terms of brackets) turn out to be coordinated so that the cocycle condition can be matched to the Maurer-Cartan equation. See [3] for a more general statement.

We are interested in the problem of lifting a torsor across a Lie algebra extension. Thus, we consider an extension of nilpotent Lie algebras

$$0 \to \mathfrak{h} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

corresponding to the extension of unipotent Lie groups $H \to \tilde{G} \to G$, and study the problem of lifting $P$ to a $\tilde{G}$ torsor $\tilde{P}$.

This kind of problem appears in a range of geometric situations, such as infinitesimal extension of a vector bundle or a torsor from a smooth subvariety $X$ to a smooth ambient variety $Y$, or finding a deformation quantization of a vector bundle on $X$ as a module over a fixed Zariski sheaf $\mathcal{O}_h$ of quantized functions on $X$. For full power appropriate to such applications one should rather work with groups and Lie algebras which are infinite-dimensional proalgebraic groups with the finite dimensional reductive Levi part. In this paper, however, we restrict to the simple finite dimensional unipotent/nilpotent setting, just observing that the case of the nontrivial reductive part $G_L$ would be resolved by working with $G_L$-invariant objects on appropriate $G_L$-torsor over $X$. See e.g. Section 5.1 in [11] for an example.

To formulate our result we also choose a splitting $\tilde{\mathfrak{g}} \simeq \mathfrak{g} \oplus \mathfrak{h}$ as vector spaces. Since each Lie algebra can be identified with its group as an algebraic
variety, we also get a splitting of algebraic varieties $\tilde{G} \simeq G \times H$ (the group operation is compatible with the embedding of $H$ and the projection to $G$).

**Theorem 1.1.** For a fixed cocycle defining $P$ and the choices made above, there exists a curved complete $L_\infty$ algebra structure on the Čech complex $L(\mathfrak{h})$ of $\mathfrak{h}$-valued functions, such that its curved Maurer-Cartan elements are in bijective correspondence with $\tilde{G}$-valued lifts of the cocycle defining $P$, and homotopy equivalences between different solutions are in bijective correspondence with $H$-valued changes of trivialization.

We note that curved Maurer-Cartan solutions behave most reasonably in a complete filtered setting, and while we were able to define the Deligne-Getzler groupoid in our particular situation, we are not aware of any general appropriate construction that implies our case.

The structure of this paper is as follows. In Section 2 we recall definitions and fix notation related to $L_\infty$ algebras, Maurer-Cartan solutions and their equivalence, and formulate results related to homotopy transfer of structure under a contraction. In Section 3 we discuss the construction of an $\infty$-groupoid associated with an $L_\infty$ algebra (which we assume to be non-negatively graded). We also recall the relationship between CBH and horn filling. Section 4 discusses the general theorem of Hinich on descent of Deligne groupoids (for a semicosimplicial DGLA) and its more specific version for semicosimplicial Lie algebras, due to Fiorenza-Manetti-Martinengo. We explain how the latter result relates to deformation of $G$-torsors. Finally, in Section 5-7 we extend it to the curved setting and compare curved Maurer-Cartan equations to lifts of cocycles across extensions. Definitions and results related to curved $L_\infty$ algebras are collected in Section 8 (appendix).

2 Complete $L_\infty$ algebras and Maurer-Cartan equations.

2.1 $L_\infty$ Algebras and $L_\infty$ morphisms

We follow the notation used in [2].

**Definition 2.1.** An $L_\infty$ structure on a graded vector space $V$ is a codifferential $Q$ of degree 1 on the (reduced) symmetric coalgebra $S(V[1]) = \bigoplus_{i \geq 1} V[1]^{\otimes i}$. The pair $(V, Q)$ is called an $L_\infty$ algebra.

Note that the codifferential $Q$ is determined by $Q^1 : S(V[1]) \to V$, see Corollary VIII.34 in [9]. If we break the map apart, we will get maps.
Definition 2.2. An $L_\infty$ morphism of $L_\infty$ algebras $F : (V,Q) \to (W,R)$ is a morphism of dg-coalgebras $G : S(V[1]) \to S(W[1])$ which is given by a family of degree zero maps $g_i = G_1^i : V[1]^{\otimes i} \to W[1]$, $i \geq 1$, such that $G : S(V[1]) \to S(W[1])$ commutes with $Q$ and $R$. $F$ is a strict morphism if $g_i = 0$ for $i \geq 2$. $F$ is an $L_\infty$ quasi-isomorphism if $g_1 : (V[1],q_1) \to (W[1],r_1)$ is a quasi-isomorphism of underlying complexes.

In this paper, we work with algebras that have a complete filtration. Completeness is not needed for the homotopy transfer of structure theorem, but it is essential in the proof of the formal Kuranishi theorem.

Definition 2.3. A complete graded space is a graded space $V$ equipped with a descending filtration $F\cdot V$, $V = F^1 V \supset \cdots \supset F^p V \supset \cdots$, such that $V$ is complete in the induced topology, i.e., the natural $V \to \varprojlim V/F\cdot V$ is an isomorphism of graded spaces. Given complete graded spaces $(W,F\cdot W)$ and $(V,F\cdot V)$, a continuous map of graded spaces is a map $g : W \to V$ such that $g(F^p W) \subset F^p V$ for all $p \geq 1$. A complete dg space is a complete graded space $(V,F\cdot V)$ with a continuous differential $d$.

Definition 2.4. A complete $L_\infty$ algebra is a complete graded space $(V,F\cdot V)$ with an $L_\infty$ structure $Q$ on $V$ such that $q_i$’s are continuous, i.e. $q_i(F^{p_1} V[1] \otimes \cdots \otimes F^{p_i} V[1]) \subset F^{p_1+\cdots+p_i} V[1]$, for all $i,p_1,\ldots,p_i \geq 1$. A continuous $L_\infty$ morphism $G : (W,F\cdot W,R) \to (V,F\cdot V,Q)$ between complete $L_\infty$ algebras is an $L_\infty$ morphism $G : (W, R) \to (V, Q)$ such that $g_i$’s are continuous, i.e. $g_i(F^{p_1} W[1] \otimes \cdots \otimes F^{p_i} W[1]) \subset F^{p_1+\cdots+p_i} V[1]$, for all $i,p_1,\ldots,p_i \geq 1$.

Remark. From now on we assume completeness of algebras and morphisms, unless otherwise specified.

Definition 2.5. The curvature of a complete $L_\infty$ algebra $(V,F\cdot V,Q)$ is the map

$$x \mapsto R_V(x) = \sum_{i \geq 1} \frac{1}{i!} q_i(x \otimes \cdots \otimes x) : V^1 \to V^2$$
Note that the infinite sum above converges because \((V, F^\bullet V, Q)\) is complete.

The **Maurer Cartan set** of a complete \(L_\infty\) algebra \((V, F^\bullet V, Q)\) is the set

\[
\text{MC}(V) := \{ x \in V^1 \text{ s.t. } R_V(x) = 0 \}.
\]

A continuous \(L_\infty\) morphism \(G\) induces a map, see e.g. [2]:

\[
G_* : \text{MC}(W) \to \text{MC}(V), \quad x \mapsto G_*(x) = \sum_{i \geq 1} \frac{1}{i!} g_i(x \odot \cdots \odot x)
\]

Gauge equivalence used with a Maurer-Cartan solutions of a positively graded DGLAs is not defined on Maurer Cartan solutions of an \(L_\infty\) algebra \(L\) as \(L^0\) is not a Lie algebra in general. Instead, we will use the following equivalence for Maurer Cartan solutions, given in terms of the induced \(L_\infty\) algebra \(V \otimes_k k[s, ds]\) where \(k[s, ds]\) is the graded commutative algebra on a degree 0 variable \(s\), and a degree 1 variable \(ds\) subject to \(d(s) = ds\):

**Definition 2.6.** Two Maurer Cartan solutions \(a, a' \in \text{MC}(V)\) are **(homotopy) equivalent** if there exist \(z \in \text{MC}(V \otimes_k k[s, ds])\) such that

\[
z|_{s=0} = a, \quad z|_{s=1} = a'
\]

where the evaluation map is given by \(\text{Eval}_{s=s_0} : V \otimes_k k[s, ds] \to V\)

\[
\text{Eval}_{s=s_0}(x(s) + y(s)ds) = x(s_0)
\]

### 2.2 Homotopy Transfer and Formal Kuranishi Theorem

In this section we will review the homotopy transfer of structure theorem and observe how the Maurer Cartan set behaves under homotopy transfer.

**Definition 2.7.** A **complete contraction**

\[
\begin{array}{ccc}
W & \xrightarrow{f} & V \\
\xleftarrow{g} & & \xleftarrow{\kappa}
\end{array}
\]

is a complete dg space \((V, F^\bullet V, d_V)\) and a dg space \((W, d_W)\), together with dg morphisms \(f : (W, d_W) \to (V, d_V)\), \(g : (V, d_V) \to (W, d_W)\) and a contracting (degree minus one) homotopy \(K : V \to V\), such that

- \(gf = \text{id}_W\) and \(Kd_V + d_VK = fg - \text{id}_V\)
- \(K\) satisfies the side conditions \(Kf = K^2 = gK = 0\)
• $K$ and $fg$ are continuous with respect to the filtration $F^*V$ on $V$.

Then $W$ is equipped with the induced filtration $F^pW = f^{-1}(F^pV)$ such that $f, g$ are continuous morphisms.

We now state the homotopy transfer of structure theorem, see e.g. [1]. It can be used for constructing $L_\infty$ structures on a dg vector space.

Theorem 2.8. Given a complete contraction $W[1] \xleftarrow{f_1} V[1] \xrightarrow{g_1} K$ and a complete $L_\infty$ algebra structure $Q$ on $(V, F^*V)$ with linear part $q_1 = d_{V[1]}$, there is an induced complete $L_\infty$ algebra structure $R$ on $(W, F^*W)$ with linear part $r_1 = d_{W[1]}$, together with continuous $L_\infty$ morphisms $f : (W, R) \rightarrow (V, Q)$, $g : (V, Q) \rightarrow (W, R)$ with linear parts $f_1, g_1$ respectively. Denoting by $f_i^k$ the composition $W[1] \otimes^i \xrightarrow{f_i} S(W[1]) \xrightarrow{f} S(V[1]) \rightarrow V[1] \otimes^k$, $f$ and $R$ are determined recursively by

$$f_i = \sum_{k=2}^{i} K q_k f_i^k \quad r_i = \sum_{k=2}^{i} g_1 q_k f_i^k \quad \text{for } i \geq 2.$$ 

We denote by $K_1^\Sigma : V[1] \otimes^i \rightarrow V[1] \otimes^i$ the degree minus one map defined by

$$K_1^\Sigma(v_1 \otimes \cdots \otimes v_i) = \frac{1}{i!} \sum_{\sigma \in S_i, j=0, \ldots, i} \pm f_1 g_1(v_{\sigma(1)}) \otimes \cdots \otimes f_1 g_1(v_{\sigma(j-1)}) \otimes K(v_{\sigma(j)}) \otimes v_{\sigma(j+1)} \otimes \cdots \otimes v_{\sigma(i)},$$

where $\pm$ is the appropriate Koszul sign (taking into account that $|K| = -1$). Denoting by $Q_i^k$ the composition $V[1] \otimes^i \xrightarrow{Q_i} S(V[1]) \xrightarrow{Q} S(V[1]) \rightarrow V[1] \otimes^k$, the $L_\infty$ morphism $g$ is determined recursively by

$$g_i = \sum_{k=1}^{i-1} g_k Q_i^k K_1^\Sigma \quad \text{for } i \geq 2.$$ 

The following theorem is essentially due to Getzler [7] explains what happens to the Maurer Cartan set under the homotopy transfer. It is stated in this form by Bandiera in [1], where it is called the formal Kuranishi theorem.

Theorem 2.9. Under the hypothesis of the homotopy transfer of structure theorem, the correspondence

$$\rho : MC(V) \rightarrow MC(W) \times K(V^1) : x \rightarrow (MC(g)(x), K(x))$$
is bijective. The inverse \( \rho^{-1} \) admits the following recursive construction: for \( y \in MC(W) \) and \( K(v) \in K(V^1) \), define \( x_n \in V^1 \), \( n \geq 0 \), by

\[
x_0 = 0 \quad \text{and} \quad x_{n+1} = f_1(y) - q_1 K(v) + \sum_{i \geq 2} \frac{1}{i!} (K q_i - f_1 g_i) (x_n \circ i).
\]

This sequence converges (with respect to the complete topology on \( V \)) to a well defined \( x \in V^1 \), and we have \( \rho^{-1}(y, K(v)) = x \). Finally, \( \rho^{-1}(-,0) = MC(f) : MC(W) \to MC(V) \) is a bijection \( MC(W) \to \ker K \cap MC(V) \), whose inverse is the restriction of \( g_1 \).

### 3 Deligne-Getzler \( \infty \)-Groupoid

For a nilpotent Lie algebra \( g \) the exponential map gives a bijection between \( g \) and its unipotent group \( G \) (and the Baker-Campbell-Hausdorff gives the group product in terms of brackets on \( g \)). In the \( L_\infty \) case, \( L^0 \) is not a Lie algebra as the bracket does not satisfy the Jacobi identity. We need an object that generalizes the Lie group \( G \). It turns out that the natural object to consider will be \( \infty \)-groupoids, which are simplicial sets satisfying some additional conditions. In [2], Getzler explains how to integrate a nilpotent \( L_\infty \) algebra to an \( \infty \)-groupoid, which generalizes the way a nilpotent Lie algebra integrates to its exponential group. General Baker-Campbell-Hausdorff product can then be seen as an arrow filing a horn of a Kan complex. A first model for such \( \infty \)-groupoid, \( MC_\infty(L) := MC(\Omega^*(\Delta_\bullet; L)) \), was introduced by Sullivan and studied in depth by Hinich [3]. The problem with \( MC_\infty(L) \) is that it is quite large, e.g. larger than the Cech complex in the geometric setting (and it is not a \( \infty \)-groupoid in a strict sense [2]). When \( g \) is a nilpotent Lie algebra, the nerve \( N(e^g) \) is only a deformation retraction of \( MC_\infty(g) \). Getzler introduced a smaller model \( \gamma_\bullet \) homotopy equivalent to \( MC_\infty(L) \) as a Kan complex. Bandiera rewrote \( \gamma_\bullet \) as \( Del_\infty(L) := MC(C^*(\Delta_\bullet; L)) \) using the formal Kuranishi theorem [1], which is the notation we are going to use.

#### 3.1 Groupoids and \( \infty \)-groupoids

We recall the standard simplicial definitons. Let \( \Delta \) be the simplex category with ordinals \( [n] = (0 < 1 < \cdots < n) \) as objects and non decreasing maps as morphisms. It is generated by the injective face maps \( d_k : [n-1] \to [n] \), \( 0 \leq k \leq n \), (the image of \( d_k \) does not contain \( k \)); and the surjective degeneracy maps \( s_k : [n] \to [n-1] \), \( 0 \leq k \leq n-1 \) (the value \( k \) is repeated twice for \( s_k \)).
**Definition 3.1.** A simplicial set $X$ is a contravariant functor from $\Delta$ to the category of sets. This gives us a sequence of sets $X_n = X([n])$ indexed by the natural numbers $n \in \{0, 1, 2, \ldots \}$, and the maps

\[
\delta_k = X(d_k) : X_n \to X_{n-1}, \quad 0 \leq k \leq n
\]

\[
\sigma_k = X(s_k) : X_{n-1} \to X_n, \quad 0 \leq k \leq n
\]

satisfying the simplicial identities, cf. [14].

In other words, $X$ is a functor $\Delta^{op} \to \text{Sets}$. Similarly, we can define simplicial objects in any category.

**Definition 3.2.** Let $\Delta_n = \Delta(\cdot, [n]) \in \text{SSet}$ be the standard simplicial $n$-simplex in SSet. For $0 \leq i \leq n$, let $\Lambda^i_n \subset \Delta_n$ be simplicial set corresponding to the union of the faces $d_k[\Delta_n-1] \subset \Delta_n$, $k \neq i$. An $n$-horn in $X$ is a simplicial map from $\Lambda^i_n$ to $X$. A simplicial object $X$ satisfies the Kan condition if any morphism of $n$-horn can be extended to a simplicial morphism $\Delta_n \to X$. Such $X$ is called a Kan complex.

We will now introduce the nerve functor, which associate each groupoid (group) a corresponding simplicial set.

**Definition 3.3.** Given a groupoid $G$ (a category where all morphisms are invertible), the nerve $N(G)$ of $G$ is a simplicial set whose 0-simplices are objects of $G$, 1-simplices morphisms of $G$, and $n$-simplices are $n$-tuples of composable morphisms of $G$, i.e.

$$x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n$$

where $x_i$ is an object in $G$ and the $f_i : x_{i-1} \to x_i$ is a morphism from $x_{i-1}$ to $x_i$. The face maps $d_i : N(G)_k \to N(G)_{k-1}$ are given by composition of morphisms at the $i$-th object. The degeneracy maps $s_i : N(G)_k \to N(G)_{k+1}$ are given by inserting identity morphism at the object $x_i$.

**Proposition 3.4.** [2] Given a groupoid $G$, the nerve $N(G)$ is a Kan complex.

Thus Kan complexes give us a generalization for groupoids (groups). We will further narrow down to simplicial sets called $\infty$-groupoids, which Kan complexes with an additional class of thin elements such that every horn has a unique thin filler, see Definition 2.5 in [3].
Definition 3.5. Two parallel 1-simplices $f$ and $g$ of a Kan complex $X$ are homotopic if and only if there exist a 2-simplex in $X$ of either of the following form

\[
\begin{array}{c}
\text{X}_1 \\
\downarrow^f \\
\text{X}_0 \\
\downarrow^g \\
\text{X}_1
\end{array}
\quad \quad
\begin{array}{c}
\text{X}_1 \\
\downarrow^f \\
\text{X}_0 \\
\downarrow^g \\
\text{X}_1
\end{array}
\]

This defines an equivalence relation on the 1-simplices of $X$ [13].

The left adjoint to the nerve functor, $N : \text{Grpd} \to \text{Kan}$, which takes Kan complexes back to groupoids is called the fundamental groupoid functor.

Definition 3.6. Given a Kan complex $X$, the fundamental groupoid, $\pi_{\leq 1}X$, is the groupoid with the following properties:

- the set of objects are 0-simplices in $X$
- the morphisms are homotopy classes of 1-simplices in $X$
- the identity morphism of $x \in X_0$ is represented by the degenerate 1-simplex $s_0(x)$
- a composition relation $h = g \circ f$ in $\pi_{\leq 1}X$ if and only if for any choices of 1-simplices representing these morphisms, there exist a 2-simplex in $X$ with boundary

\[
\begin{array}{c}
x_1 \\
\downarrow^f \\
x_0 \\
\downarrow^h \\
x_2
\end{array}
\]

The fundamental groupoid of a Kan complex mimics the fundamental groupoid of a topological space. The following proposition tells us that the $\pi_{\leq 1}$ functor preserves the homotopy relation.
Proposition 3.7. If $X$ and $Y$ are homotopy equivalent Kan complexes, then $\pi_{\leq 1}X$ and $\pi_{\leq 1}Y$ are equivalent as groupoids. If $X$ and $Y$ are homotopy equivalent Kan complexes and that $\pi_{\leq 1}X$ and $\pi_{\leq 1}Y$ have the same set of objects, then $\pi_{\leq 1}X$ and $\pi_{\leq 1}Y$ are isomorphic groupoids.

Proof. See [12] for the proof of the first part. For the second, observe that equivalent groupoids with the same set of objects are isomorphic. \qed

3.2 Deligne-Getzler $\infty$-groupoids

In this subsection we first introduce two important complexes and from them construct the Deligne-Getzler $\infty$-groupoid that gives us the general Baker-Campbell-Hausdorff product of a DGLA or an $L_\infty$ algebra.

Definition 3.8. For $n \geq 0$, the differential graded commutative algebra of polynomial differential forms on the $n$-simplex $\Delta_n$ is:

$$\Omega^*_n = \frac{k[t_0, \ldots, t_n, dt_0, \ldots, dt_n]}{(\sum t_i - 1, \sum dt_i)}.$$ 

where the differential is induced by the usual differential for differential forms that sends $t_i \mapsto dt_i$. Notice that $\Omega^*_n$ has a natural structure of simplicial dg commutative algebra. The face map $\partial_i$ annihilates $t_i$ and $dt_i$ and the degeneracy map $s_i$ sends $t_i, dt_i$ to $t_i + t_i + 1$ and $dt_i + dt_i + 1$, respectively.

Given a simplicial set $X$, the space of polynomial 1-forms on $X$ is $\Omega^1(X) := \operatorname{SSet}(X, \Omega^1)$, i.e. the simplicial set morphisms from $X$ to $\Omega^1$, and $\Omega^*(X) := \oplus_{l \geq 0} \Omega^l(X)$. In particular, when $X$ is $\Delta_*$, we have $\Omega^*(\Delta_*) = \Omega^*$.

Given a dg vector space $L$, set $\Omega^*(X; L) = \Omega^*(X) \otimes L$ the complex of polynomial differential forms on $X$ with coefficients in $L$.

Definition 3.9. The complex of non-degenerate simplicial $k$-cochains on $X$ is $C^*(X) := C^*(X; k) = \oplus_{l \geq 0} C^l(X)$ where $C^l(X)$ is the space of $k$-valued $l$-cochains $\alpha : X_l \to k; \sigma \mapsto \alpha_\sigma$ on $X$ vanishing on degenerate simplices. The differential is given by

$$d\alpha(\sigma) = (d\alpha)_\sigma = \sum_{i=0}^{k+1} (-1)^i \alpha_{\partial_i \sigma}$$

where $\partial_i : X_{k+1} \to X_k, i = 0, \ldots, k+1$, are the face maps of $X$.

Given a dg vector space $L$, set $C^*(X; L) = C^*(X) \otimes L$ the complex of non-degenerate simplicial cochains on $X$ with coefficients in $L$. 

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Note that $\Omega^*(X; L)$ inherits an algebraic structure of $L$ (such as Lie or graded commutative) as $\Omega^*(X)$ is graded commutative. The complex $\Omega^*(X; L)$ does not inherit a complete structure in general, but we can replace $\Omega^*(X; L)$ with its completion $\hat{\Omega}^*(X; L) := \varprojlim \Omega^*(X; L/F^p L)$ which will also have the same algebraic structure as $L$.

On the other hand, for a complete $L$ the complex $C^*(X; L)$ is complete with respect to the filtration $F^p C^*(X; L) = C^*(X; F^p L)$, but in general $C^*(X; L)$ does not inherit an algebraic structure of $L$. But a standard contraction from $\Omega^*(X; L)$ to $C^*(X; L)$ (and thus from $\hat{\Omega}^*(X; L)$ to $C^*(X; L)$) can be used to transfer a homotopy version of structure (Lie, graded commutative, etc.) to $C^*(X; L)$ using the structure on $\hat{\Omega}^*(X; L)$.

**Theorem 3.10** (Getzler, [7]). There is a standard contraction from $\hat{\Omega}^*(X; L)$ to $C^*(X; L)$ given by integrating forms over simplices in one direction, inclusion of Whitney’s elementary forms in the other direction, and Dupont homotopy as the contracting homotopy.

In particular when $L$ is a complete $L_\infty$ algebra, so is $\hat{\Omega}^*(X; L)$, and by homotopy transfer $C^*(X; L)$ is a complete $L_\infty$ algebra. We now define the Deligne-Getzler $\infty$-groupoid of a complete DGLA ($L_\infty$ algebra) $L$. Denote

$$\Delta_\bullet : \Delta_0 \longrightarrow \Delta_1 \longrightarrow \Delta_2 \longrightarrow \cdots$$

the standard cosimplicial simplex in $\text{SSet}^\Delta$.

**Definition 3.11.** Given a complete $L_\infty$ algebra $L$, the **Deligne-Getzler** $\infty$-groupoid of $L$ is the simplicial set $\text{Del}_\infty(L)_n := \text{MC}(C^*(\Delta_n; L))$ of Maurer-Cartan cochains with coefficients in $L$, its face an degeneracy maps induced from $\Delta_\bullet$. Further $\text{MC}_\infty(L)$ is the simplicial set with $\text{MC}_\infty(L)_n := \text{MC}(\Omega^*(\Delta_n; L))$ with face and degeneracy maps induced from $\Omega_\bullet$.

The properties of these two simplicial sets are summarized below

**Proposition 3.12.** The following hold:

(a) $\text{MC}_\infty(L)$ and $\text{Del}_\infty(L)$ are homotopy equivalent $\infty$-groupoids.

(b) $\pi_{\leq 1} \text{MC}_\infty(L)$ is isomorphic to $\pi_{\leq 1} \text{Del}_\infty(L)$ as groupoids.

(c) For a non negatively graded nilpotent $L_\infty$ algebra $L$, $\text{Del}_\infty(L)$ is isomorphic to the nerve of $\text{Del}(L) := \pi_{\leq 1} \text{Del}_\infty(L)^{op} \simeq \pi_{\leq 1} \text{MC}_\infty(L)^{op}$. Further, in this case morphisms in $\pi_{\leq 1} \text{MC}_\infty(L)$, are in bijection with 1-simplices of $\text{Del}_\infty(L)$.

**Proof.** For part (a), the Kan property for $\text{MC}_\infty$ is proved in [8], while the Kan property for $\text{Del}_\infty$ and homotopy equivalence in Corollary 5.9 of [7].
(the latter paper actually deals with an isomorphic simplicial set denoted there by $\gamma_\bullet(L)$). Morphism on $\pi_{\leq 1}$ in part (b) follows by Proposition 3.7. In part (c), isomorphism with the nerve is a particular case of Theorem 5.4 of [7]. For statement about morphisms, note that under the assumption on grading, morphisms in $\text{Del}^{\text{op}}(L)$ are given by 1-simplices of $\text{Del}_\infty(L)$ as $\text{Del}_\infty(L) = N(\text{Del}^{\text{op}}(L))$.

### 3.3 Baker-Campbell-Hausdorff and Horn Filling

In this section, we assume that an nilpotent $L_\infty$ algebra $L$ is concentrated in non-negative degrees and, following [2], we relate horn filling in $\text{Del}_\infty(L)$ with Baker-Campbell-Hausdorff product on $L^0$. We will start by looking at a complete contraction given by Bandiera in [1]. Let $L$ be a complete $L_\infty$ algebra. For $i = 0, \ldots, n$, we define a homotopy $h^i : C^*(\Delta_n; L) \to C^{*-1}(\Delta_n; L)$ by writing for $0 \leq i_0 < \cdots < i_k$:

$$h^i(\alpha)_{i_0 \cdots i_k} = \begin{cases} 0 & \text{if } i \in \{i_0, \cdots, i_k\} \\ (-1)^j \alpha_{i_0 \cdots i_{j-1} i_j \cdots i_k} & \text{if } i_{j-1} < i < i_j \end{cases}$$

where $\beta_{i_0 \cdots i_k} \in L^{i-k}$, $0 \leq i_0 < \cdots < i_k \leq n$ is the evaluation of $\beta \in C^i(\Delta_n; L)$ on the $k$-simplex of $\Delta_n$ spanned by the vertices $i_0, \ldots, i_k$. Denote $e_i : \Delta_0 \to \Delta_n$ the inclusion of the $i$-th vertex and by $\pi : \Delta_n \to \Delta_0$ the final morphism. The above $h^i$ gives a homotopy on the complete contraction

$$L = C^*(\Delta_0; L) \xrightarrow{\pi^*} C^*(\Delta_n; L) \xleftarrow{h^i} c_i^* \alpha$$

If $\partial_i : \Delta_{n-1} \to \Delta_n$ is the inclusion of the $i$-th face, then $\partial_i^*$ sends $h^i(C^1(\Delta_n; L))$ isomorphically to $C^0(\Delta_{n-1}; L)$. The formal Kuranishi theorem by with $W = L$, $V = C^*(\Delta_n; L)$ and $K = h^i$, gives the following proposition [1]:

**Proposition 3.13** ([2]). For all $i = 0, \ldots, n$, the correspondence

$$\rho^i : \text{Del}_\infty(L)_n \to \text{MC}(L) \times h^i(C^1(\Delta_n; L)) : \alpha \to (e_i^*(\alpha), h^i(\alpha))$$

is bijective.

In other words, if we fix a Maurer Cartan element in $L$ and a cochain in $h^i(C^1(\Delta_n; L))$, we can recover the unique cochain in $\text{Del}_\infty(L)_n$ by the recursive formula of the formal Kuranishi theorem.

This allows us to recover (slightly generalized) gauge action of $L^0$ on Maurer-Cartan solutions. Indeed, take $x \in \text{MC}(L) \subset L_1$ and $a \in L^0$ and
consider the pair \((x, \alpha)\) where \(\alpha \in h^0(C^1(\Delta_1; L) \subset C^0(\Delta_0; L)\) takes the value \(a\) on the unique non-degenerate 0-simplex of \(\Delta_0\). Then \(z = (\rho^0)^{-1}(x, \alpha) \in Del_\infty(L_1)\) and \(\partial_0^0(z)\) is the Maurer-Cartan element that we denote by \(a \cdot x \in L^1\). By Section 5.2 in [2] this gives the usual gauge action of \(L^0\) on Maurer-Cartan elements when \(L\) is a DGLA.

Now take any \(x \in MC(L)\) (say \(x = 0\)) and \(a, b \in L^0\). We want to get the Baker-Campbell-Hausdorff product of \(a\) and \(b\) through the horn filling in \(Del_\infty(L)_2\). Consider the following 2-horn: Put \(x\) on the \([1]\) vertex, \(a\) on the \([01]\) edge, \(b\) on the \([12]\) edge, and \(0\) on \([012]\)

Now consider the 1-cochain \(\alpha'\) in \(h^1(C^1(\Delta_2; L)) \subset C^0(\Delta_2, L)\) with the only nonzero values on non-degenerate simplices in \(\Delta_2\) given by

\[
\alpha'([0]) = a, \quad \alpha'([2]) = b.
\]

We can then apply the recursive formula from the formal Kuranishi theorem and get an unique cochain \(\alpha \in MC(C^*(\Delta_2; L)) = Del_\infty(L)_2\). The Baker-Campbell-Hausdorff product \(\rho^2_x(-)\) between the morphism \(a\) and \(b\) is then defined by evaluating \(\alpha\) on the face \(\partial_1 \Delta_2\) opposite to the vertex \([1]\).

Then \(\rho^2_x(a, b) = a \ast b\), the usual Baker-Campbell-Hausdorff product, by Proposition 5.2.36 in [2]. In general we have higher general Baker-Campbell-Hausdorff products obtained by filling in higher dimensional horns but in this paper we only deal with the products between elements in \(L^0\).
4 Descent of Deligne groupoids

Given a nilpotent Lie algebra $\mathfrak{g}$ with the unipotent group $G = \exp(\mathfrak{g})$ and a $G$ torsor $P$ over $X$, the result of Hinich in \cite{Hinich} allows us to use combinatorial tools to study formal deformations of $P$. One can either work with the Thom-Whitney complex constructed from an affine covering of $X$, or a quasi-isomorphic $L_\infty$ algebra constructed on the underlying Cech complex.

This is a special case of the $L_\infty$ structure for semicosimplicial Lie algebras (in degree 0) is studied by Fiorenza, Manetti, and Martinengo. They show in \cite{FM} that the solutions to the deformation equation (i.e. cocycle condition on transition functions) are exactly the Maurer Cartan solutions of the $L_\infty$ Cech complex and the equivalences of deformations are exactly the equivalence of Maurer Cartan solutions. We use Bandiera’s reformulation of the result in \cite{FM} using Deligne-Getzler $\infty$-groupoids.

4.1 Semicosimplicial DGLAs, totalization and homotopy limit

Definition 4.1. A semicosimplicial differential graded Lie algebra $L_\bullet$ is a covariant functor $\Delta^\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow 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The resulting object is a DGLA, with a simpler Maurer Cartan equation, but the complex is quite large and harder to interpret geometrically. Bandiera introduced a smaller version of the homotopy limit with an $L_{\infty}$ structure through homotopy transfer. In the case of interest to us, it is isomorphic to the Cech complex as a DG vector space.

**Definition 4.3.** Given a semicosimplcial complete DGLA $L_\bullet \in \overline{\text{DGLA}}^\Delta$, its **Thom-Whitney complex** is the complete DGLA

$$\text{Tot}_{\text{TW}}(L_\bullet) = \left\{ (\alpha_0, \ldots, \alpha_n, \ldots) \in \prod_{n \geq 0} \hat{\Omega}^*(\Delta_n; L_n) \text{ s.t. } \partial^j_\bullet(\alpha_{n-1}) = \delta^*_n(\alpha_n) \right\}$$

where the morphism $\partial^j_\bullet : \hat{\Omega}^*(\Delta_{n-1}; L_{n-1}) \to \hat{\Omega}^*(\Delta_{n-1}; L_n)$ is the push-forward by the $j$-th cofaces of $L_\bullet$ and $\delta^*_n : \hat{\Omega}^*(\Delta_n; L_n) \to \hat{\Omega}^*(\Delta_{n-1}; L_n)$ is the pull back by the $j$-th coface of $\Delta$. The Thom-Whitney complex inherits a DGLA structure from $L_\bullet$ since the product on differential forms is graded commutative.

**Definition 4.4.** Given a semicosimplcial complete DGLA $L_\bullet \in \overline{\text{DGLA}}^\Delta$, its **totalization** $\text{Tot}(L_\bullet)$ is the complete $L_{\infty}$ algebra

$$\text{Tot}(L_\bullet) = \left\{ (\alpha_0, \ldots, \alpha_n, \ldots) \in \prod_{n \geq 0} C^*(\Delta_n; L_n) \text{ s.t. } \partial^j_\bullet(\alpha_{n-1}) = \delta^*_n(\alpha_n) \right\}$$

where the morphism $\partial^j_\bullet : C^*(\Delta_{n-1}; L_{n-1}) \to C^*(\Delta_{n-1}; L_n)$ is the push-forward by the $j$-th cofaces of $L_\bullet$ and $\delta^*_n : C^*(\Delta_n; L_n) \to C^*(\Delta_{n-1}; L_n)$ is the pull back by the $j$-th coface of $\Delta$. The $L_{\infty}$-structure on $\text{Tot}(L_\bullet)$ is induced from the DGLA structure of $\text{Tot}_{\text{TW}}(L_\bullet)$ via the contraction induced by the contraction of differential forms $\Delta_n$ onto cochains on $\Delta_n$ (with a fixed choice of Dupont homotopy, cf. [2]).

**Proposition 4.5.** In $L_\bullet$ is a semicosimplcial Lie algebra (i.e. all $L_k$ are in homological degree zero) then underlying dg vector space of $\text{Tot}(L_\bullet)$ is the complex $L_0 \to L_1 \to L_2 \ldots$ with the differential $\sum_j (-1)^j \partial_{j,q}$.

**Proof.** A degree $m$ element $\alpha$ in $\text{Tot}(L_\bullet)$ is of the form $\alpha = (\alpha_0, \ldots, \alpha_n, \ldots)$, where $\alpha_n$ is in $C^m(\Delta_n; L_n)$ (with additional compatibility conditions on $\alpha_n$). Since $L_m$ is in homological degree zero, $\alpha_m$ only takes nonzero value on (nondegenerate) $m$-simplicies of $\Delta_n$. In particular $\alpha_n = 0$ for $n < m$. 

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For $n > m$, by definition of $\text{Tot}$ evaluation of $\alpha_n$ on $m$-simplices of $\Delta_n$ is given by restricting to an $(n - 1)$ dimensional face containing and $m$-simplex and then computing the value by applying a cosimplicial map to the value of $\alpha_{n-1}$. Thus, a degree $m$ cochain $\alpha$ is uniquely determined by $\alpha_m$.

An explicit check also shows that this correspondence is compatible with differentials: if $l$ is an element in $L_k$, then $d(l) = \sum_{j=0}^{k+1} (-1)^j \partial_{j,k+1}(l) \in L_{k+1}$. Now consider $\alpha$ an element of degree $k$ in $\text{Tot}(L\bullet)$ whose value on the $k$-simplices in $\Delta_k$ is $l$. $\alpha$ is of the form $\alpha = (0, \ldots, 0, \alpha_k, \alpha_{k+1}, \ldots)$ and $d(\alpha) = (\delta^*(0), \ldots, \delta^*(0), \delta^*(\alpha_k), \delta^*(\alpha_{k+1}), \ldots)$.

Note that $\delta^*(\alpha_i) = \partial_\ast(\alpha_{i-1})$ by the construction of $\text{Tot}$ and $\alpha_{k-1}$ is the 0 cochain, so $\delta^*(\alpha_k) = 0$. By the same reasoning we have $\delta^*(\alpha_{k+1}) = \partial_\ast(\alpha_k)$, whose evaluation at the $k+1$ simplex in $\Delta_{k+1}$ is $\sum_{j=0}^{k+1} (-1)^j \partial_j(l) = d(l)$, so we have $d(\alpha) = (0, \ldots, 0, 0, \delta^*(\alpha_{k+1}) = d(l), \ldots)$. Thus by our previous discussion, $d(\alpha)$ must be a degree $k+1$ element in $\text{Tot}(L\bullet)$ which under our bijection will precisely be $d(l)$ in $\bigoplus_n L_n[-n]$.

**Notation.** In the geometric situation with the sheaf of Lie algebras $\mathfrak{g}$ and an open covering $\mathcal{U}$ we will denote $\text{Tot}(\mathfrak{g}(\mathcal{U})\bullet)$ by $\mathcal{L}(\mathfrak{g})$ and $\text{Tot}_{\mathcal{W}}(\mathfrak{g}(\mathcal{U})\bullet)$ by $\hat{\mathcal{L}}(\mathfrak{g})$. We will also use $[-,-]_\mathfrak{g}$ to denote $[-,-]_{\text{Tot}_{\mathcal{W}}(\mathfrak{g})}$. By the previous result, $\mathcal{L}(\mathfrak{g})$ is an $\mathbb{L}_\infty$ algebra for which the underlying vector space is just the Cech complex of $\mathfrak{g}$ with respect to $\mathcal{U}$. It also depends on the choice of Dupont homotopy but with suppress both dependences (on the covering and on the homotopy) from notation, assuming that both are fixed.

### 4.2 Theorems on Descent of Deligne Groupoids

Now we state Hinich’s theorem on descent of Deligne groupoid. We will also sketch a proof of Fiorenza, Manetti, and Martinengo’s result that for a semicosimplicial DGLA in degree 0, we get an isomorphism of groupoids.

**Theorem 4.6** (Hinich, [8]). For semicosimplicial DGLAs $L\bullet$ concentrated in non negative degrees, the Deligne functor commutes with homotopy limits, i.e., there is a natural equivalence of groupoids

$$\text{Del}(\text{Tot}(L\bullet)) \simeq \text{Tot}(\text{Del}(L\bullet)).$$

$\text{Tot}(\text{Del}(L\bullet))$ (the left hand side) is called the groupoid of descent data on $L\bullet$. In the case where $L\bullet$ is a secosimplicial Lie algebra, its objects are the nonabelian 1-cocycles

$$Z^1(\exp(L_1)) = \{ m \in L_1 | e^{\partial_1(m)} e^{-\partial_1(m)} e^{\partial_2(m)} = 1 \}$$
and its morphisms between two cocycles $m_0$ and $m_1$ are 
\[ \{ a \in L_0 | e^{-\partial_1(a)}e^{m_1}\partial_0(a) = e^{m_0} \} \]

We will give details on the above theorem but rather move to the case when $L_\bullet$ is formed by Lie algebras in degree zero (as it happens for the geometric situation involving an open cover and a sheaf of Lie algebras). Then Fiorenza, Manetti, and Martinengo proved in [3] that instead of just equivalence, we are getting an isomorphism of groupoids, that is, the nonabelian 1-cocycles, as a subset of $\text{Tot}(L_\bullet)$, are the same as solutions of the $L_\infty$ Maurer Cartan equation on $\text{Tot}(L_\bullet)$, and that two nonabelian cocycles are equivalent iff they are equivalent Maurer Cartan elements. We sketch the proof briefly, focusing on the parts which will be needed later for studying torsors.

**Theorem 4.7** (Fiorenza, Manetti, and Martinengo, [3]). For semicosimplicial Lie algebra $L_\bullet$, there is an isomorphism of $\infty$-groupoids
\[ \text{Del}_\infty(\text{Tot}(L_\bullet)) \cong \text{Tot}(\text{Del}_\infty(L_\bullet)). \]

and thus an isomorphism of groupoids
\[ \text{Del}(\text{Tot}(L_\bullet)) \cong \text{Tot}(\text{Del}(L_\bullet)). \]

First, we should define the totalization of the semicosimplicial simplicial set $\text{Del}_\infty(L_\bullet)$ (the right hand side of our isomorphism). The totalization of semicosimplicial simplicial sets is defined the same way as the totalization of semicosimplicial complete $L_\infty$ algebras by simply replacing $C^*(\Delta_i; L_i)$ with $\text{SSet}(\Delta_i, L_i)$. This totalization anjots a universal property similar to the totalization of semicosimplicial DGLA. Using this definition, we have
\[
\text{Tot}(\text{Del}_\infty(L_\bullet)) = \left\{ (\alpha_n) \in \prod_{n \geq 0} \text{SSet}(\Delta_n, \text{Del}_\infty(L_n)) \mid \partial^i_*(\alpha_{n-1}) = \delta^*_j(\alpha_n) \right\}.
\]

Notice that $\text{SSet}(X, Y)_n = \text{SSet}(\Delta_n \times X, Y)$, so
\[
\text{Tot}(\text{Del}_\infty(L_\bullet))_i = \{ (\alpha_n) \in \prod_{n \geq 0} \text{SSet}(\Delta_i \times \Delta_n, \text{Del}_\infty(L_n)) \mid \partial^i_*(\alpha_{n-1}) = \delta^*_j(\alpha_n) \}\]

**Proof.** Since $\text{Tot}(L_\bullet)$ is concentrated in non negative degrees, by Proposition 3.12 $\text{Del}_\infty(\text{Tot}(L_\bullet)) = \mathcal{N}(\text{Del}^{op}(\text{Tot}(L_\bullet)))$. Hence $\text{Del}_\infty(\text{Tot}(L_\bullet))$ is uniquely determined by its 0-simplices, 1-simplices and 2-simplices. On the other hand, since $L_i$’s are all concentrated in degree 0, $\text{Del}_\infty(L_i) =$
$N(\text{Del}^{op}(L_*))$. It is easy to check from definition that $\text{Tot}$ and $N$ commute in this case. Thus, $\text{Del}_\infty(\text{Tot}(L_*))$ is also uniquely determined by the 0,1,2-simplices and same holds for its homotopy limit $\text{Tot}(\text{Del}_\infty(L_*))$.

Thus we will be comparing the 0,1,2-simplices for $\text{Del}_\infty(\text{Tot}(L_*))$ and $\text{Tot}(\text{Del}_\infty(L_*))$. Using results of the previous sections to untangle definitions (we omit the straightforward computational details), we see that on both sides simplices have identical descriptions. We record them for future use.

**0-simplices** of both $\text{Del}_\infty(\text{Tot}(L_*))$ and $\text{Tot}(\text{Del}_\infty(L_*))$ can be identified with $\alpha_1 \in C^1(\Delta_1, L_1) = L_1$ such that $\partial_0^0\alpha_1 \circ (-\partial_1^1\alpha_1) \circ \partial_2^2\alpha_1 = 0$, i.e with nonabelian 1-cocycles in $\text{Tot}(L_*)$ ($\circ$ denotes the CBH product).

**1-simplices** of both $\text{Del}_\infty(\text{Tot}(L_*))$ and $\text{Tot}(\text{Del}_\infty(L_*))$ are in bijection with the set of $\alpha_1, \alpha_1' \in L_1$ and $l_0 \in L_0$ such that the diagram below commutes:

![Diagram](image)

**2-simplices** of $\text{Del}_\infty(\text{Tot}(L_*))$ of the following form:

![Diagram](image)

where $a$ is a 1-cocycle of the $\text{Tot}(L_*)$, $l_0 \cdot a$ and $-l_1 \cdot a$ are the resulting 1-cocycles when $l_0$ and $-l_1$ act on $a$, $l_0, l_1 \in L_0$ and their composition is given by the Baker-Campbell-Hausdorff formula in $L_0$.

We proved an isomorphism of $\infty$-groupoids

$$\text{Del}_\infty(\text{Tot}(L_*)) \cong \text{Tot}(\text{Del}_\infty(L_*))$$

Applying $\pi_{\leq 1}(...)^{op}$ gives $\text{Del}(\text{Tot}(L_*)) \cong \text{Tot}(\text{Del}(L_*))$.  

$\square$
4.3 Unipotent torsors and deformations.

We return to the setting of our Introduction, with an algebraic scheme $X$, its
affine open cover $\mathcal{U} = \{U_i\}$ and a $G$-torsor $P$ give by the transition functions
$\Phi_{ij} : U_i \cap U_j \to G$ viewed as exponents of $\varphi_{ij} : U_i \cap U_j \to \mathfrak{g}$. The usual
cocycle condition $\Phi_{ij} \Phi_{jk} = \Phi_{ik}$ can be understood as a condition imposed
on the element $\varphi = (\varphi_{ij}) \in \mathcal{L}(\mathfrak{g})^1$ in the degree 1 component of the Cech
$L_\infty$ algebra of the sheaf of $\mathfrak{g}$-valued functions on $X$.

Similarly, a change of trivialization of $P$ on each $U_i$ is give by a collection
of regular maps $\Sigma_i : U_i \to G$, which changes the cocycle as follows:

$$\Phi_{ij} \mapsto \Sigma_i^{-1} \Phi_{ij} \Sigma_j$$

Define the groupoid $\text{Tors}(X, P, \mathcal{U})$ of $P$-torsors on $X$ (with respect to the
fixed choice of an affine cover) by viewing cocycles $\Phi_{ij}$ as objects and changes
of trivializations $\Sigma_j$ as morphisms. Composition of morphisms is the obvious
product of $G$ valued cocycles (which can be rephrased in terms of CBH product
on elements $\sigma = (\sigma_j) \in \mathcal{L}(\mathfrak{g})^0$). Comparing this with the descriptions of
0, 1, and 2-simplices in the proof of 4.7 we obtain a

**Corollary 4.8.** There exists an isomorphism the Deligne groupoid of the
Cech $L_\infty$-algebra of the sheaf of $\mathfrak{g}$-valued regular functions, and $\text{Tors}(X, G, \mathcal{U})$.

A version of this statement can be formulated for the problem of deforming
a fixed $G$-torsor $P$ over an Artinian $k$-algebra $A = k \oplus m_A$ (with finite
dimensional $m_A$, for the sake of simplicity). Then we fix a covering $\mathcal{U}$, a
cocycle giving $P$ and reduce the question of constructing a deformation to
the question of finding an element $\phi \in \mathcal{L}(\mathfrak{g}_P \otimes m_A)$ satisfying appropriate
cocycle condition. Here $\mathfrak{g}_P$ is the adjoint sheaf of Lie algebras, associated
to $P$. As our further discussion will be a generalization of this picture, we
leave the detailed discussion until later, just stating here the

**Corollary 4.9.** There is an isomorphism between the Deligne groupoid of
the Cech $L_\infty$ algebra of $\mathfrak{g}_P \otimes m_A$ and the groupoid of deformations of $P$ over
the spectrum of the Artinian algebra $A = k \oplus m_A$.

In this form the statement can be extended to a wider range of groups $G$
as tensoring with $m_A$ automatically creates a sheaf of nilpotent Lie algebras.
5 Lifting $G$-torsors across extensions.

Now we finally move to the problem discussed in the introduction. Suppose we have an extension of nilpotent Lie algebras:

$$0 \to h \to \tilde{g} \to g \to 0$$

corresponding to the extension of unipotent groups $1 \to H \to \tilde{G} \to G \to 1$. For a $G$-torsor $P$ on a scheme $X$ we want to study its different lifts $\tilde{P}$ to a $\tilde{G}$-torsor. To rigidify the problem, fix an open cover of $X$ and a cocycle $\Phi_{ij} : U_i \cap U_j \to G$ defining $P$, and study the groupoid of $\tilde{G}$-valued cocycles $\tilde{\Phi}_{ij} : U_i \cap U_j \to \tilde{G}$ lifting $\Phi_{ij}$. Morphisms between two such lifts are given by change of coordinate maps $\tilde{\Sigma}_j : U_j \to \tilde{G}$ which map to identity in $G$. This means that $\tilde{\Sigma}_j$ take values in the subgroup $H \subset \tilde{G}$. We will show that our choices induce on the Cech complex $\mathcal{L}(h)$ of $h$-valued functions the structure of a curved $L_\infty$-algebra and that the groupoid of lifts of $P$ is isomorphic to the (appropriately defined!) Deligne groupoid of $\mathcal{L}(h)$.

5.1 Lie Algebra Extensions

**Definition 5.1.** Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras. An extension $\mathfrak{g}$ of $\mathfrak{g}$ by $\mathfrak{h}$ is a short exact sequence of the form

$$0 \to \mathfrak{h} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0.$$

**Definition 5.2.** Let $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}'$ be two extensions of $\mathfrak{g}$ by $\mathfrak{h}$. $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}'$ are said to be equivalent if there exits a commutative diagram

$$
\begin{array}{cccccc}
0 & \to & \mathfrak{h} & \to & \tilde{\mathfrak{g}} & \to & \mathfrak{g} & \to & 0 \\
& & \downarrow{\varphi} & & \parallel & & \\
0 & \to & \mathfrak{h} & \to & \tilde{\mathfrak{g}}' & \to & \mathfrak{g} & \to & 0
\end{array}
$$

**Definition 5.3.** A non-abelian 2-cocycle on $\mathfrak{g}$ with values in $\mathfrak{h}$ is a couple $(c, b)$ of linear maps $c : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{h}$ and $b : \mathfrak{g} \to \text{Der}(\mathfrak{h})$ satisfying

$$[b(x), b(y)] - b([x, y]) = \text{ad}(c(x, y))$$

and

$$\sum b(x, c(y, z)) - c(b(x, y), z) = 0$$

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where the sum is over cyclic permutations of $x$, $y$, and $z$. Two non-abelian 2-cocycles are equivalent, $(c, b) \sim (c', b')$ if there exists $\beta : \mathfrak{g} \to \mathfrak{h}$ satisfying 

$$b'_x = b_x + \text{ad}_{\beta(x)}$$

and 

$$c'(x, y) = c(x, y) + b_x(\beta(y)) - b_y(\beta(x)) - \beta([x, y]) + [\beta(x), \beta(y)]$$

Choosing a vector space splitting $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{h}$ compatible with the embedding of $\mathfrak{h}$ and projection to $\mathfrak{g}$, and writing out the bracket of $\tilde{\mathfrak{g}}$ gives such a non-abelian cocycle. For $x \in \mathfrak{g}$ and $y \in \mathfrak{h}$, $b : \mathfrak{g} \to \text{Der}(\mathfrak{h})$ is given by $b(x)(y)$ equal to the projection of $[x, y]_{\tilde{\mathfrak{g}}}$ onto the $\mathfrak{h}$-component, and for $x, x' \in \mathfrak{g}$, $c : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{h}$ is given by the $\mathfrak{h}$ component of $[x, x']_{\tilde{\mathfrak{g}}}$. A direct computation shows that extensions of $\mathfrak{g}$ by $\mathfrak{h}$ are classified by equivalence classes of non-abelian cocycles, which can further be reformulated as equivalence classes of Maurer Cartan solutions for some DGLA, see [4]. For our purpose, it is enough for us to know that the maps $b$ and $c$ fully describe an extension.

5.2 Twisted Cocycles and their Equivalence

We can identify $\tilde{G}$ with the product $G \times H$ as varieties (not groups!) by choosing a vector space splitting of $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{h}$ and thus get an embedding $G \hookrightarrow \tilde{G}$ (of varieties!) by embedding $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ and using the exponential map. Multiplication in $\tilde{G}$ is not the regular multiplication of $G \times H$ and is determined by $c$ and $b$ (via the CBH formula). Our question is then what are the principal $\tilde{G}$-bundles $\tilde{P}$ that extend $P$. The splitting $\tilde{G} = G \times H$ allows us to rewrite the unknown lifted cocycle as 

$$\tilde{\Phi}_{ij} = \Phi_{ij} \Psi_{ij} = \exp(\varphi_{ij}) \exp(\psi_{ij}); \quad \varphi_{ij} \in \Gamma(U_{ij}, \mathfrak{g}), \psi_{ij} \in \Gamma(U_{ij}, \mathfrak{h})$$

The cocycle condition can then be rewritten as (product is taken in $\tilde{G}$) 

$$\exp(\varphi_{ij}) \exp(\psi_{ij}) \exp(\varphi_{jk}) \exp(\psi_{jk}) = \exp(\varphi_{ik}) \exp(\psi_{ik})$$

Comparing with the product of $\{\exp(\varphi_{ab})\}$ in $G$ we get 

$$\exp(\varphi_{ij}) \cdot_{\tilde{G}} \exp(\varphi_{jk}) = (\exp(\varphi_{ij}) \cdot_{G} \exp(\varphi_{jk})) \mathcal{C}(\varphi_{ij}, \varphi_{jk}) = \exp(\varphi_{ik}) \mathcal{C}(\varphi_{ij}, \varphi_{jk})$$

where $\mathcal{C}(\varphi_{ij}, \varphi_{jk})$ is the $H$ component of $\exp(\varphi_{ij}) \cdot_{G} \exp(\varphi_{jk})$. If we rewrite $\exp(\varphi_{ij}) \cdot_{G} \exp(\varphi_{jk})$ in Lie algebra terms using the Baker-Campbell-Hausdorff
formula on \( \tilde{\mathfrak{g}} \), i.e. \( \varphi_{ij} \ast \tilde{g} \varphi_{jk} \), then \( C(\varphi_{ij}, \varphi_{jk}) \) is precisely the exponent of the \( \mathfrak{h} \) component of \( \varphi_{ij} \ast \tilde{g} \varphi_{jk} \). Combining this with the fact that

\[
\exp(-\varphi_{jk}) \exp(\psi_{ij}) \exp(\varphi_{jk}) = \exp\left( \sum_{s=0}^{\infty} (-1)^s \frac{(\text{ad} \varphi_{jk})^s}{s!}(\psi_{ij}) \right),
\]

and denoting

\[
\exp(\psi)^{\varphi} = \exp\left( \sum_{s=0}^{\infty} \frac{(\text{ad} \varphi)^s}{s!}(\psi) \right)
\]

the twisted group cocycles extending \( P \) can then be rewritten as

\[
C(\varphi_{ij}, \varphi_{jk}) \exp(\psi_{ij})^{-\varphi_{jk}} \exp(\psi_{jk}) = \exp(\psi_{ik}).
\]

Passing to a different lift of the same cocycle for \( P \) corresponds to

\[
\tilde{\Phi}_{ij} \mapsto \Sigma_i^{-1} \tilde{\Phi}_{ij} \Sigma_j = \Phi_{ij}(\Phi_{ij}^{-1} \Sigma_i^{-1} \Phi_{ij}) \Psi_{ij} \Sigma_j
\]

where \( \Sigma_a \in \Gamma(U_a, \mathfrak{h}) \). Writing \( \Sigma_a = \exp(\sigma_a) \) and comparing the \( \mathfrak{h} \) components written in Lie algebra terms we get that two twisted cocycles \( \{\exp(\psi_{ij})\} \) and \( \{\exp(\psi'_{ij})\} \) are equivalent iff there exist \( \{\sigma_a \in \Gamma(U_a, \mathfrak{h})\} \) such that

\[
\exp(\psi'_{ij}) = \exp(-\sigma_i)^{-\varphi_{ij}} \exp(\psi_{ij}) \exp(\sigma_j)
\]

on all double overlaps \( U_i \cap U_j \).

Thus, lifting the cocycle of \( P \) is the same as lifting a Maurer-Cartan element \( a \) from \( \mathcal{L}(\mathfrak{g}) \) to \( \mathcal{L}(\tilde{\mathfrak{g}}) \). Lifting a Maurer Cartan solution over a surjection is difficult in general but we use a vector space splitting \( \mathcal{L}(\tilde{\mathfrak{g}}) = \mathcal{L}(\mathfrak{h}) \oplus \mathcal{L}(\mathfrak{g}) \) and to define a curved \( L_\infty \) algebra structure on \( \mathcal{L}(\mathfrak{h}) \). Our goal is to show that the twisted cocycle condition (2) is the same as the curved Maurer Cartan equation on \( \mathcal{L}(\mathfrak{h}) \) and the equivalence of twisted cocycles (3) is the same as the equivalence of curved Maurer Cartan solutions on \( \mathcal{L}(\mathfrak{h}) \).

### 6 Maurer Cartan Solutions for Thom Whitney

#### 6.1 Bijection between Maurer Cartan Solutions

Recall that the Thom-Whitney complex \( \hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \) of \( \tilde{\mathfrak{g}}(\mathcal{U}) \), has the structure of a DGLA. The vector space splitting \( \tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{h} \) induces a splitting of complexes
\( \mathcal{L}(\tilde{g}) = \mathcal{L}(g) \oplus \mathcal{L}(h) \). Now fix a Maurer-Cartan solution \( a \in \mathcal{L}(g)^1 \), and consider \( \alpha \in \mathcal{L}(h)^1 \). Then the Maurer-Cartan equation for \( a + \alpha \)

\[
d(a + \alpha) + \frac{1}{2} [a + \alpha, a + \alpha]_{\tilde{g}} = 0
\]

reduces to

\[
\frac{1}{2} c(a, a) + (d + \text{ad}_a)(\alpha) + \frac{1}{2} [\alpha, \alpha]_h = 0
\]

This can be viewed as a curved Maurer-Cartan equation with respect to the new (curved) differential \( d_h \) on \( \mathcal{L}(h) \) given by the restriction of \( d + \text{ad}_a \).

Setting \( C = \frac{1}{2} c(a, a) \), we obtain by a straightforward computation that \( d_h^2 = [C, -] \). This means that \( (\mathcal{L}(h), C, d_h, [-, -]_h) \) is a curved DGLA (see Appendix for general theory).

For a fixed Maurer-Cartan element \( a \in \mathcal{L}(g) \), let \( \text{MC}_a(\mathcal{L}(\tilde{g})) \) be the set of Maurer Cartan solutions in \( \mathcal{L}(\tilde{g}) \) which have the form \( a + \alpha \) with \( \alpha \in \mathcal{L}(h) \). By the above discussion we have a bijection of this set

\[
\text{MC}_a(\mathcal{L}(\tilde{g})) \cong \text{MC}(\mathcal{L}(h))
\]

with the set \( \text{MC}(\mathcal{L}(h)) \) of curved Maurer Cartan solutions for \( \mathcal{L}(h) \).

### 6.2 Equivalences of Maurer Cartan Solutions

Recall that an equivalence of two Maurer Cartan solutions \( z, z' \in \text{MC}(\mathcal{L}(\tilde{g})) \) (a morphism of the corresponding groupoid) is an element

\[
\tilde{z} \in \text{MC}(\mathcal{L}(\tilde{g}) \otimes k[s, ds]) \quad \text{such that} \quad \tilde{z}|_{s=0} = z, \quad \tilde{z}|_{s=1} = z'.
\]

Recall that \( k[s, ds] \) is the graded commutative algebra of polynomial forms on a line and the tensor product \( \mathcal{L}(\tilde{g}) \otimes k[s, ds] \) has the induced DGLA structure. The evaluation map \( \text{Eval}_{s=s_0} \) is given in Section 2.1.

We will check that an equivalence between Maurer-Cartan solutions \( a + \alpha \) and \( a + \alpha' \), which is constant in the first component, essentially amounts to an equivalence of curved Maurer-Cartan solutions \( a \) and \( a' \).

Notice that \( \tilde{z} \in \mathcal{L}(\tilde{g}) \otimes k[s, ds] \) can be written as \( \tilde{a} + \tilde{\alpha} \) where \( \tilde{a} \in \mathcal{L}(g) \otimes k[s, ds] \) and \( \tilde{\alpha} \in \mathcal{L}(h) \otimes k[s, ds] \).

But since we are not changing the trivialization of \( P \), we are only interested in the case where \( \tilde{a} \in \mathcal{L}(g) \otimes k[s, ds] \) is constant and equal to \( a \).

A straightforward check shows that \( (\mathcal{L}(h) \otimes k[s, ds], \tilde{d}_h, [-, -]_h) \) does in fact have a curved DGLA structure and that the curved Maurer Cartan
equation for $\alpha \in \hat{\mathcal{L}}(\mathfrak{h}) \otimes k[s, ds]$ is exactly the same as the Maurer Cartan equation for $a + \alpha \in \hat{\mathcal{L}}(\mathfrak{g}) \otimes k[s, ds]$.

Thus, if $a + \tilde{\alpha}(s, ds)$ gives a homotopy equivalence between Maurer-Cartan solutions $a + \alpha$ and $a + \alpha'$ then $\tilde{\alpha}(s, ds)$ gives a curved homotopy equivalence between curved Maurer-Cartan solutions $\alpha$ and $\alpha'$ in $\hat{\mathcal{L}}(\mathfrak{h})$.

Recall that for an $L_\infty$ algebra $L$ we have a contraction

$$C^*(\Delta_1; L) \xrightarrow{\sim} L \otimes \Omega_1 \xleftarrow{\sim} K$$

where $K$ is the Dupont homotopy, and that application of formal Kuranishi theorem identifies 1-simplices of $\text{Del}(\hat{\mathcal{L}}(\mathfrak{h}))$ with the set

$$\text{MC}(L \otimes \Omega_1, K) = \{ x \in \text{MC}(L \otimes \Omega_1) | K(x) = 0 \}$$

Now define $\text{MC}_a(\hat{\mathcal{L}}(\mathfrak{g}) \otimes \Omega_1, K)$ as the subset of $\text{MC}(\hat{\mathcal{L}}(\mathfrak{g}) \otimes \Omega_1, K)$ where elements are of the form $a + \tilde{\alpha}(s, ds)$. Since $K(a) = 0$, $K(a + \tilde{\alpha}) = 0$ iff $K(\tilde{\alpha}) = 0$. We have a bijection

$$\text{MC}_a(\hat{\mathcal{L}}(\mathfrak{g}) \otimes \Omega_1, K) \cong \text{MC}(\hat{\mathcal{L}}(\mathfrak{h}) \otimes \Omega_1, K).$$

(Where similarly, the right hand side is defined as curved Maurer-Cartan solutions annihilated by $A$ similar argument with $\Omega_2$ shows that compositions of morphisms agree too; i.e. if $(a + \tilde{\alpha}) \circ (a + \tilde{\alpha}') = a + \tilde{\alpha}''$, then $\tilde{\alpha} \circ \tilde{\alpha}' = \tilde{\alpha}''$.

Denote $\text{Del}_a(\hat{\mathcal{L}}(\mathfrak{g}))$ the groupoid with objects $\text{MC}_a(\hat{\mathcal{L}}(\mathfrak{g}))$ and morphisms $\text{MC}_a(\hat{\mathcal{L}}(\mathfrak{g}) \otimes \Omega_1, K)$, $\text{Del}_a(\hat{\mathcal{L}}(\mathfrak{g}))$ is then a subgroupoid (i.e. subcategory) of $\text{Del}(\hat{\mathcal{L}}(\mathfrak{g}))$. The identification of $a + \alpha$ with $\alpha$ clearly gives us a morphism of groupoids, so we have an isomorphism of groupoids

$$\text{Del}_a(\hat{\mathcal{L}}(\mathfrak{g})) \cong \text{Del}(\hat{\mathcal{L}}(\mathfrak{h})).$$

**Remark.** In general if $L$ is a curved $L_\infty$ algebra, $\text{Del}(L)$ might not be well defined. However, in our situation, if we define $\text{Del}(\hat{\mathcal{L}}(\mathfrak{h}))$, by mimicking $\text{Del}$ in the non curved case, as the groupoid whose objects are $\text{MC}(\hat{\mathcal{L}}(\mathfrak{h}))$ and whose (opposite) morphisms $\text{MC}(\hat{\mathcal{L}}(\mathfrak{h}) \otimes \Omega_1, K)$, we will get a groupoid isomorphic to the subgroupoid of $\text{Del}(\hat{\mathcal{L}}(\mathfrak{g}))$. Furthermore, we can define $\text{Del}_\infty(\hat{\mathcal{L}}(\mathfrak{h}))$ as $\mathcal{N}(\text{Del}^{op}(\hat{\mathcal{L}}(\mathfrak{h})))$.

### 7 Extensions of Cocycles and Curved $L_\infty$ Maurer Cartan Solutions

We can now put every piece together by using formal Kuranishi to relate $\text{MC}(\hat{\mathcal{L}}(\mathfrak{g}))$ to the cocycles for extensions of principal $G$-bundles $P$ and relate
MC(\(\hat{\mathcal{L}}(\mathfrak{g})\)) to the Maurer Cartan set of the curved \(L_\infty\) algebra \(\mathcal{L}(\mathfrak{h})\). If \(P\) is trivialized on the elements of an affine cover \(\mathcal{U}\), the transition functions of \(P\) give us a Maurer Cartan element \(a \in MC(\mathcal{L}(\mathfrak{g}))\). By the formal Kuranishi theorem, this lifts to a unique Maurer-Cartan solution of the Thom-Whitney algebra \(\hat{\mathcal{L}}(\mathfrak{g})\) which is annihilated by the Dupont contraction. To simplify the notation, we will denote both elements \(a\).

Our goal is to prove a version of the groupoid isomorphism in the previous section, but replacing Thom-Whitney DGLAs with Cech \(L_\infty\)-algebras. Thus we denote by denote \(MC_a(\mathcal{L}(\mathfrak{g})) \subset \mathcal{L}(\mathfrak{g})\) the set of Maurer Cartan solutions for the form \(a + \alpha\), \(\text{Del}_{a}(\mathcal{L}(\mathfrak{g}))\) the groupoid with objects \(MC_a(\mathcal{L}(\mathfrak{g}))\) and the morphisms being those morphisms in the Deligne groupoid \(\text{Del}(\mathcal{L}(\mathfrak{g}))\) which project on the identity of \(a\) in \(\text{Del}(\mathcal{L}(\mathfrak{g}))\).

**Theorem 7.1.** There is an isomorphism of groupoids

\[
\text{Del}_{a}(\mathcal{L}(\mathfrak{g})) \cong \text{Del}(\mathcal{L}(\mathfrak{h}))
\]

where \(\mathcal{L}(\mathfrak{h})\) has a curved \(L_\infty\) structure obtained by contraction from the curved DGLA \(\hat{\mathcal{L}}(\mathfrak{h})\) whose differential is given by \(d + \text{ad}_a\) and curvature \(\frac{1}{2}c(a,a)\) (see Appendix).

Before we start the proof, recall that \(L_\infty\) structure on \(\mathcal{L}(\mathfrak{f})\) is obtained by homotopy transfer from \(\hat{\mathcal{L}}(\mathfrak{f})\) with homotopy \(\hat{K}\) which is termwise the Dupont homotopy \([7]\). The formal Kuranishi theorem gives us \(MC(\hat{\mathcal{L}}(\mathfrak{f}), \hat{K}) \cong MC(\mathcal{L}(\mathfrak{f}))\) where \(MC(\hat{\mathcal{L}}(\mathfrak{f}), \hat{K}) = \{z \in MC(\hat{\mathcal{L}}(\mathfrak{f}))| \hat{K}(z) = 0\}\).

Similarly, the formal Kuranishi theorem applied to the contraction

\[
\mathcal{L}(\mathfrak{f}) \otimes \Omega^* \leftrightarrow \hat{\mathcal{L}}(\mathfrak{f}) \otimes \Omega^* \leftrightarrow \hat{K} \otimes \text{id}
\]

gives a simplicial isomorphism \(MC(\hat{\mathcal{L}}(\mathfrak{f}) \otimes \Omega^*, \hat{K} \otimes \text{id}) \cong MC(\mathcal{L}(\mathfrak{f}) \otimes \Omega^*)\).

Note that curved homotopy transfer of structure theorem and thus curved formal Kuranishi theorem only apply when we have a correct filtration on our complexes. We will discuss this in more detail as a remark when we apply the curved homotopy transfer of structure theorem in the proof.

**Proof.** Define \(MC_a(\hat{\mathcal{L}}(\mathfrak{g}), \hat{K})\) as the set of Maurer Cartan solutions \(a + \alpha \in MC(\hat{\mathcal{L}}(\mathfrak{g}))\), where \(\alpha \in \hat{\mathcal{L}}(\mathfrak{h})\), such that \(\hat{K}(a + \alpha) = 0\). By the remarks before the proof and the previous section we have

\[
MC_a(\mathcal{L}(\mathfrak{g})) \cong MC_a(\hat{\mathcal{L}}(\mathfrak{g}), \hat{K}) \cong MC(\hat{\mathcal{L}}(\mathfrak{h}), \hat{K}) \cong MC(\mathcal{L}(\mathfrak{h}))
\]
where the last bijection follows by applying the curved formal Kuranishi theorem (see [5], [6] and Section 8.2 below) to the contraction

\[ \overset{\hat{L}(\mathfrak{h})}{\longrightarrow} \overset{L(\mathfrak{h})}{\longrightarrow} \overset{\hat{K}}{\longrightarrow} \]

Thus, the Maurer Cartan solutions are in bijection.

For the arrows (equivalences), we consider two contractions

\[ C^*(\Delta_1; \overset{\hat{L}(\mathfrak{g})}{\longrightarrow} \overset{\hat{L}(\mathfrak{g}) \otimes \Omega_1}{\longrightarrow} \overset{\hat{K}}{\longrightarrow} \]

\[ \overset{\mathcal{L}(\mathfrak{g}) \otimes \Omega_1}{\longrightarrow} \overset{\hat{L}(\mathfrak{g}) \otimes \Omega_1}{\longrightarrow} \overset{\hat{K} \otimes \text{id}}{\longrightarrow} \]

Notice that \( K \) is a contraction that contracts the \( \Omega_1 \) part without changing the coefficients and \( \hat{K} \otimes \text{id} \) contracts the coefficients without changing the \( s, ds \in \Omega_1 = k[s, ds] \). We will denote by \( \text{MC}(\ldots, K, \hat{K} \otimes \text{id}) \) and \( \text{MC}_a(\ldots, K, \hat{K} \otimes \text{id}) \) subsets of Maurer-Cartan solutions annihilated by both homotopies (we apply this to sets arising from \( \hat{g} \) and their curved analogues arising from \( \mathfrak{h} \)). Then

\[ \text{MC}_a(\hat{L}(\mathfrak{g}) \otimes \Omega_1, \hat{K} \otimes \text{id}) \cong \text{MC}_a(\mathcal{L}(\mathfrak{g}) \otimes \Omega_1) \]

and hence

\[ \text{MC}_a(\mathcal{L}(\mathfrak{g}) \otimes \Omega_1, K) \cong \text{MC}_a(\hat{L}(\mathfrak{g}) \otimes \Omega_1, K, \hat{K} \otimes \text{id}). \]

The previous section implies that the set on the right can be identified with \( \text{MC}(\hat{L}(\mathfrak{h}) \otimes \Omega_1, K, \hat{K} \otimes \text{id}) \), which by a similar reasoning applied to \( \mathfrak{h} \) is bijective to \( \text{MC}(\mathcal{L}(\mathfrak{h}) \otimes \Omega_1, K) \). Thus we get a bijection

\[ \text{MC}_a(\mathcal{L}(\mathfrak{g}) \otimes \Omega_1, K) \cong \text{MC}(\mathcal{L}(\mathfrak{h}) \otimes \Omega_1, K). \]

Now suppose \( a + \alpha \in \text{MC}_a(\hat{L}(\mathfrak{g}) \otimes \Omega_1, K) \) gives an equivalence of two Maurer Cartan solutions \( a + \alpha, a + \alpha' \in \text{MC}_a(\mathcal{L}(\mathfrak{g})) \). Because of the bijections \( \text{MC}_a(\mathcal{L}(\mathfrak{h})) \cong \text{MC}(\mathcal{L}(\mathfrak{h})) \) and \( \text{MC}_a(\mathcal{L}(\mathfrak{h}) \otimes \Omega_1, K) \cong \text{MC}(\mathcal{L}(\mathfrak{h}) \otimes \Omega_1, K) \), we can uniquely lift \( a + \alpha, a + \alpha' \), and \( a + \alpha \) in \( \text{MC}(\mathcal{L}(\mathfrak{h})) \) and \( \text{MC}(\mathcal{L}(\mathfrak{h}) \otimes \Omega_1, K) \) respectively. Note that after the lift we still have \( \hat{a}|_{s=0} = \alpha \) and \( \hat{a}|_{s=1} = \alpha' \). The lifts agree because Getzler defines them as solutions to differential equations with initial conditions. Since the bijection between

\[ \text{MC}_a(\hat{L}(\mathfrak{g}) \otimes \Omega_1, K) \cong \text{MC}(\mathcal{L}(\mathfrak{h}) \otimes \Omega_1, K) \]
respects composition as shown in the last section, and the homotopy transfer also respects composition of morphisms are respected, we have an isomorphism of groupoids $\text{Del}_a(\mathcal{L}(\tilde{g})) \cong \text{Del}(\mathcal{L}(h))$.

**Remark.** Note that both totalizations of a semicosimplicial Lie algebra $L_\bullet$, $\text{Tot}_{TW}(L_\bullet)$ and $\text{Tot}(L_\bullet)$, are equipped with the filtrations

$$F^{-1} \subset F^0 \subset F^1 \subset \ldots$$

When $L_\bullet$ arises from a sheaf of Lie algebras and an open cover $F^i$ is given by direct sums of terms coming from sections on $\geq (i + 2)$ overlapping open sets. These filtrations are complete.

The curved $L_\infty$ structure on $\mathcal{L}(h)$, in Getzler’s sense, must then be in $F^1 S^1(\mathcal{L}(h), \mathcal{L}(h))$ as $C$ is an element in the triple intersection, $d$ and $[-,-]$ are degree 1 maps in $\mathcal{L}(h)[1]$. This means that $\mathcal{L}(h)$ is pro-nilpotent and thus we can apply curved homotopy transfer of structure theorem and curved formal Kuranishi theorem. See Appendix for details on curved $L_\infty$ algebras.

**Corollary 7.2.** Suppose we have a principal $G$-bundle $P$ over the base space $X$, $\pi : P \to X$, and $G = \exp(g)$, where $G$ is unipotent and $g$ is nilpotent. Suppose we have a Lie algebra extension $\tilde{g}$ of $g$ by another nilpotent Lie algebra $h$

$$0 \to h \to \tilde{g} \to g \to 0,$$

then extensions of the principal $G$-bundle $P$ are given by the curved Maurer Cartan solutions $\text{MC}(\mathcal{L}(h))$ and the equivalence of extension is precisely the equivalence of curved Maurer Cartan solutions, i.e. the solution for the twisted cocycle condition \((\mathcal{A})\) is in bijection with the curved Maurer Cartan solutions for $\mathcal{L}(h)$ and the twisted equivalence \((\mathcal{B})\) is in bijection with the morphisms between curved Maurer Cartan solution for $\mathcal{L}(h)$ and this bijection respects composition of morphisms (change of trivialization).

**Proof.** Result follows directly from Theorem 7.1 and 4.7.

**Example 7.3.** In the case where the image of $c$ is in the center of $h$, i.e. $c(x, y) \in Z(h) \forall x, y \in g$, we will have an honest action of $g$ (G) on $h$. The extensions of the bundle $P$ are then given by the curved $L_\infty$ Cech complex, $\mathcal{L}(h)$, of the associated bundle $P_h = (P \times h)/G$ (which a bundle of Lie algebras) where its curvature is given by $c$. 

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8 Appendix: curved homotopy transfer

8.1 Definitions

Definition 8.1. Let $L$ be a complete graded vector space; a codifferential $Q$ of degree 1 on the symmetric coalgebra $S(L[1]) = \bigoplus_{n \geq 0} L^\otimes n$ is called a **curved $L_\infty$ structure** on $L$. A curved $L_\infty$ algebra is a complete graded space $(L,F^\bullet L,d)$ together with a curved $L_\infty$ structure $Q$ on $L$.

As in the non curved case, $Q$ is determined by $Q_1: S(L[1]) \to L$. The maps $q_i = Q_i^1: \bigotimes^i L \to L$ give us the (higher) brackets on $L[1]$; $q_0: k \to L$ in particular gives us the **curvature** element. The series of equations (general Jacobi identities) given by $Q^2 = 0$ are different from the non curved case as we have to take into account the $q_0$. In particular, we have

$$q_1^2(x) = q_2(q_0,x), \quad q_1(q_0) = 0 \quad x \in L.$$

Thus, $q_1$ is no longer a differential for $L$ and its cohomology is not well defined. $S(L[1])$ is equipped with a coaugmentation $\eta: k \to S(L[1])$. When $Q\eta = 0$, we have $q_0 = 0$ and we recover non curved $L_\infty$ algebras.

Definition 8.2. A curved $L_\infty$ morphism $F: (L,Q) \to (M,R)$ between curved $L_\infty$ algebras is a morphism $F: S(L[1]) \to S(M[1])$ that commutes with the coproducts, counits, and codifferentials $Q$ and $R$.

Again, $F$ is determined by $F_i^1 = f_i: \bigotimes^i L[1] \to M[1]$ and $F$ can be computed in a fashion similar to the non curved case (but $i \geq 0$).

Definition 8.3. For a curved $L_\infty$ algebra $(L,Q)$

$$\sum_{n=0}^{\infty} \frac{1}{n!} q_n(x, \ldots, x) = 0 \quad x \in L^1.$$ 

is called the **Maurer-Cartan equation**. Its solutions form a (possibly empty!) **Maurer-Cartan set** $MC(L)$ of the curved $L_\infty$ algebra $L$. Two Maurer-Cartan solutions $a, a' \in MC(L)$ are **(homotopy) equivalent** if there exist $z \in MC(L \otimes k[s, ds])$ such that $z|_{s=0} = a, z|_{s=1} = a'$ where the evaluation map $Eval_{s=s_0}: L \otimes k[s, ds] \to L$ is given by

$$Eval_{s=s_0}(x(s) + y(s)ds) = x(s_0)$$
8.2 Curved Homotopy Transfer and Kuranishi Theorem

Here we state the main result of Getzler’s paper [6], the curved version of formal Kuranishi theorem. For complete filtered complexes \( L, M \) let \( S^i(L, M) \) be the set of sequences \((a_0, a_1, \ldots)\) where \( a_0 \in F_1M \) and for \( n \geq 1 \) each \( a_n \) is a filtered graded symmetric \( n \)-linear map from \( L \) to \( M \) of degree \( i \). It carries a filtration with \( F_kS^i \) given by multilinear maps that deepen the filtration at least by \( k \) steps.

**Definition 8.4.** A curved \( \mathcal{L}_\infty \) algebra \((L, \lambda)\) is **pro-nilpotent** if \( \lambda \in F_1S^1(L, L) \).

**Definition 8.5.** Given \( a \in S^i(L, M) \) and \( b \in S^0(K, L) \), define the composition \( a \bullet b \in S^i(K, M) \) by

\[
(a \bullet b)_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \times \\
\times \sum_{k=0}^n \frac{1}{k!} \sum_{n_1 + \cdots + n_k = n} \frac{1}{n_1! \cdots n_k!} a_k(b_{n_1}(x_{\sigma(1)}, \ldots), \ldots, b_{n_k}(\ldots, x_{\sigma(n)}))
\]

For \( \bullet \) to be well defined, we need \( b_0 \in F^1L \). The following theorem was originally shown by Fukaya and stated in the current form by Getzler:

**Theorem 8.6 (Fukaya, [5],[6]).** Consider a complete contraction of filtered complexes \( M \xrightarrow{f} L \xleftarrow{g} h \) with continuous \( f \) and \( g \). Suppose \( L \) is equipped with a pro-nilpotent curved \( \mathcal{L}_\infty \) structure \( \lambda \). Then there is a unique solution in \( S^0(M, L) \) of the fixed-point equation

\[
F = f - h \lambda \bullet F.
\]

Furthermore, \( \mu = g \lambda \bullet F \in S^1(M, M) \) is a curved \( \mathcal{L}_\infty \) structure on \( M \), and \( F \) is a curved \( \mathcal{L}_\infty \) morphism from \((M, F \bullet M, d, \mu)\) to \((L, F \bullet L, \delta, \lambda)\).

Pro-nilpotence is needed to make \( F \mapsto f - h \lambda \bullet F \) a contraction mapping under the metric \( d_c(x, y) = \inf\{c^{-k}|x - y \in F^kL\} \) where \( c \in \mathbb{R}^>1 \).

**Theorem 8.7 (Getzler, [6]).** Under the same setting as in Theorem 8.6, the morphism \( g \) induces a bijection from \( \text{MC}(L, h) \rightarrow \text{MC}(M) \).

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