Automorphisms of tiled orders

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Abstract

Let $\Lambda$ be a tiled $R$-order. We give a description of $\text{Aut}_R(\Lambda)$ as the semidirect product of $\text{Inn}(\Lambda)$ and a certain subgroup of $\text{Aut}(Q(\Lambda))$, where $Q(\Lambda)$ is the link graph of $\Lambda$. Additionally, we give criteria for determining when an element of $\text{Aut}(Q(\Lambda))$ belongs to this subgroup in terms of the exponent matrix for $\Lambda$.

Key words: tiled order, link graph, automorphism

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1 Introduction

Let $R$ be a complete discrete valuation ring (DVR) with quotient field $K$ and maximal ideal $P = \pi R$. Recall that an $R$-order $\Lambda$ in $M_n(K)$ is called tiled in case it has a complete set of $n$ orthogonal idempotents $\{e_1, \ldots, e_n\}$. Without loss of generality, we may assume that $e_i = e_{ii}$ for all $i$, where $\{e_{ij}\}$ are the standard matrix units in $M_n(K)$. It is then possible to write $\Lambda = (P^\alpha_{ij})_n = (\pi^\alpha_{ij}R)_n$, where the $\alpha_{ij}$ are non-negative integers, $\alpha_{ii} = 0$ for all $i$, and $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ for all $i$, $j$ and $k$. [5, p. 313]. If $\Lambda$ is such a tiled order, then the integral matrix $(\alpha_{ij})$ is called the exponent matrix for $\Lambda$. It is clear that $\Lambda$ is completely determined by its exponent matrix.

In this paper, we study the structure of $\text{Aut}_R(\Lambda)$, the group of $R$-automorphisms of $\Lambda$, in the case where $\Lambda$ is tiled. We give a description of $\text{Aut}_R(\Lambda)$ as the
semidirect product of $\text{Inn}(\Lambda)$ and a certain subgroup of the automorphism group of the link graph $Q(\Lambda)$ of $\Lambda$. (The precise definition is recalled in the next section; for now we note only that $Q(\Lambda)$ is a quiver on $n$ vertices.) This subgroup, which we denote by $\mathcal{O}_\Lambda$, is called the group of liftable automorphisms of $Q(\Lambda)$. As the name suggests, these automorphisms are precisely those which are induced by automorphisms of the order $\Lambda$.

In general, $\mathcal{O}_\Lambda$ will be a proper subgroup of $\text{Aut}(Q(\Lambda))$; we develop necessary and sufficient conditions for a given automorphism to be liftable in terms of the matrix of exponents $(\alpha_{ij})$ of $\Lambda$. Additionally, we show that if $\Lambda$ is basic and $\alpha_{ij} \in \{0, 1\}$ for all $i, j$, then $\mathcal{O}_\Lambda = \text{Aut}(Q(\Lambda))$. Finally, using these liftable automorphisms, we give an example to show that the crossed product $\Lambda \ast \mathcal{O}_\Lambda$ need not be a prime ring. This is in contrast to the hereditary case [4].

2 The link graph of a tiled order

Let $\Lambda = (P^{\alpha_{ij}})_n$ be a tiled order as above. In [8, section 2], Wiedemann and Roggenkamp construct a quiver $Q(\Lambda)$ associated to $\Lambda$, as follows. Let $P_i = \Lambda e_i$ denote the $i$-th column of $\Lambda$. Then $Q(\Lambda)$ has $n$ vertices, and there is an arrow from $i$ to $j$ if and only if $P_i$ is a summand of the projective cover of $\text{rad}(P_j)$. One assigns a value $v$ to the arrows by setting $v(i, j) = \beta$, where $\text{Hom}_\Lambda(P_i, P_j) = P^\beta$. The corresponding valued quiver will be denoted $Q^v(\Lambda)$. In general, distinct orders $\Lambda$ and $\Lambda'$ can have identical quivers: $Q(\Lambda) = Q(\Lambda')$ (see for example [2, Example 10]). However, it is proven in [8, Theorem 1] that $\Lambda$ is uniquely determined by $Q^v(\Lambda)$.

There is an alternative description of $Q(\Lambda)$ as the link graph of the maximal ideals of $\Lambda$, which we now describe. Note that $\Lambda$ has exactly $n$ distinct maximal two-sided ideals $M_1, \ldots, M_n$, where $M_k$ is obtained from $\Lambda$ by replacing the $R$ in the $(k, k)$-position with a $P$. That is, $M_k = (P^{\beta_{ij}})_n$, where $\beta_{ij} = \alpha_{ij}$ for $(i, j) \neq (k, k)$, and $\beta_{kk} = 1$. The link graph is defined as the quiver with $n$ vertices, with an arrow from $i$ to $j$ if and only if $M_j M_i \neq M_j \cap M_i$ [7]. Then, [1, Proposition 1.2] shows that the link graph of $\Lambda$ coincides with $Q(\Lambda)$.

Lemma 1 There is a group homomorphism $\Phi : \text{Aut}_R(\Lambda) \rightarrow \text{Aut}(Q(\Lambda))$, whose kernel contains $\text{Inn}(\Lambda)$.

proof. An automorphism of $\Lambda$ permutes the maximal ideals $M_1, \ldots, M_n$ of $\Lambda$, and thereby gives rise to a permutation of the vertices of $Q(\Lambda)$. Let us denote the corresponding permutation of $\{1, \ldots, n\}$ as $\sigma$. Then, we have that there is an arrow from $\sigma(i)$ to $\sigma(j)$ in $Q(\Lambda)$ if and only if $M_{\sigma(i)} M_{\sigma(j)} \neq M_{\sigma(j)} \cap M_{\sigma(i)}$, and if and only if $M_j M_i \neq M_j \cap M_i$, if and only if there is an arrow
from $i$ to $j$ in $Q(\Lambda)$. Thus, $\sigma$ gives an automorphism of $Q(\Lambda)$. Finally, if $\varphi$ is an inner automorphism of $\Lambda$, then $\varphi$ fixes each of the maximal ideals $M_1, \ldots, M_n$. Thus $\varphi$ induces the identity on $Q(\Lambda)$. \hfill \Box

We shall see below that in fact $\text{Inn}(\Lambda) = \ker \Phi$. The reason that we work with link graph $Q(\Lambda)$ instead of the valued quiver $Q^v(\Lambda)$ is that the corresponding result fails for valued quivers: There are tiled orders $\Lambda$ for which $\varphi \in \text{Aut}(\Lambda)$ does not induce an automorphism of $Q^v(\Lambda)$. The following example illustrates this.

**Example 2** Consider the order

$$\Lambda = \begin{pmatrix} R & P^2 & P^4 \\ P^3 & R & P^4 \\ P & P & R \end{pmatrix}.$$ 

One computes that the valued quiver $Q^v(\Lambda)$ is

![Valued Quiver Example](image)

Now, a direct verification shows that conjugation by $\left( \begin{smallmatrix} 0 & \pi & 0 \\ 0 & 0 & \pi^3 \\ 1 & 0 & 0 \end{smallmatrix} \right)$ is an automorphism of $\Lambda$, and the induced permutation on the vertices of $Q(\Lambda)$ is $\sigma = (123)$. Since $v(1, 2) = 2$ and $v(\sigma(1), \sigma(2)) = v(2, 3) = 4$, we see that $\sigma$ does not induce an automorphism of $Q^v(\Lambda)$. \hfill \Box

One may at first hope that every automorphism of $Q(\Lambda)$ is induced by an automorphism of $\text{Aut}_R(\Lambda)$ as above; i.e. that $\Phi$ is surjective. Unfortunately, this fails to be the case. Indeed, we shall develop an explicit criterion for determining whether or nor an element of $Q(\Lambda)$ is in $\text{im} \Phi$; we shall call such automorphisms *liftable*.

We first fix some notation. Given $x \in GL_n(K)$, we denote conjugation by $x$ as $t_x$. We shall identify automorphisms of $Q(\Lambda)$ with the induced permutation of the vertices of $Q(\Lambda)$; thus $\sigma$ will simultaneously denote an element of
Lemma 3 Let $\sigma \in S_n$. Given $\sigma \in S_n$, we let $v(\sigma) \in M_n(K)$ denote the corresponding permutation matrix: $v(\sigma)_{ij} = 1$ if $j = \sigma(i)$, 0 otherwise.

**PROOF.** The Skolem-Noether Theorem shows that any automorphism of $\Lambda$ is given by conjugation by $x$ for some $x \in GL_n(K)$. The automorphism $\varphi$ takes the set of orthogonal primitive idempotents $\{e_1, \ldots, e_n\}$ to another set of orthogonal primitive idempotents, say $\{f_1, \ldots, f_n\}$. Since $\Lambda$ is semiperfect, there is a $u \in \Lambda^*$ and $\sigma \in S_n$ such that $\iota_u f_i = e_{\sigma(i)}$ for all $i$ [6, Proposition 3.7.3].

So, the automorphism $\iota_u \varphi$ which is conjugation by some $x \in GL_n(K)$, acts as a permutation $\sigma$ on $\{e_1, \ldots, e_n\}$. If one writes out in explicit matrix form the condition that $x^{-1} e_i x = e_{\sigma(i)}$ for all $i$, then one sees that only the $(i, \sigma(i))$ entry of $x$ is nonzero, for $i = 1, \ldots, n$. We may write the $(i, \sigma(i))$ entry of $x$ as $r_i \pi^{d_i}$, where $r_i \in R^*$ and $d_i \in \mathbb{Z}$. In particular, $x$ factors as $y d v(\sigma)$, where $y = \text{diag}(r_1, \ldots, r_n)$ and $d = \text{diag}(\pi^{d_1}, \ldots, \pi^{d_n})$. Now, since $\iota_u \varphi = \iota_y d v(\sigma)$, we have $\varphi = \iota_u^{-1} y d v(\sigma)$. Since each of $u^{-1}$ and $y$ is in $\Lambda^*$, we see that $\varphi$ has the indicated form. $\square$

**Proposition 4** An automorphism $\sigma$ of $Q(\Lambda)$ is liftable if and only if there exists a diagonal matrix $d = \text{diag}(\pi^{d_1}, \ldots, \pi^{d_n})$ such that $\iota_d v(\sigma) \in \text{Aut}_R(\Lambda)$.

**PROOF.** Suppose that there is an automorphism of $\Lambda$ of the given form. Since conjugation by $d$ fixes the primitive orthogonal idempotents $\{e_1, \ldots, e_n\}$, we see that $\iota_d v(\sigma)(e_i) = e_{\sigma(i)}$ for all $i$. Since the maximal ideal $M_i$ of $\Lambda$ can be characterized as the unique maximal ideal of $\Lambda$ which does not contain $e_i$, this implies that $\iota_d v(\sigma)(M_i) = M_{\sigma(i)}$ for all $i$. Thus $\iota_d v(\sigma)$ induces the permutation $\sigma$ on the vertices of $Q(\Lambda)$, and is necessarily an automorphism of $Q(\Lambda)$. Hence, $\sigma = \Phi(\iota_d v(\sigma))$ is liftable.

Conversely, let $\sigma$ be liftable. Then there is an automorphism $\varphi$ with $\Phi(\varphi) = \sigma$. By Lemma 3, we can write $\varphi = \iota_u d v(\tau)$, where $u \in \Lambda^*$, $d = \text{diag}(\pi^{d_1}, \ldots, \pi^{d_n})$, and $\tau \in S_n$. Now, $\Phi(\varphi) = \Phi(\iota_d v(\tau))$ (since $\Phi(\iota_u) = 0$) and, as in the previous paragraph, $\iota_d v(\tau)(M_i) = M_{\tau(i)}$ for all $i$. It follows that $\tau = \sigma$, so that $\iota_d v(\sigma) \in \text{Aut}_R(\Lambda)$. $\square$
3 Main results

We are now in a position to state and prove the main theorem of this paper.

**Theorem 5** Let $\Lambda$ be a tiled order in $M_n(K)$ with exponent matrix $(\alpha_{ij})$ and link graph $Q(\Lambda)$.

(a) $\sigma \in \text{Aut}(Q(\Lambda))$ is liftable if and only if the linear system

$$x_i - x_j = \alpha_{ij} - \alpha_{\sigma(i)\sigma(j)}, \quad i < j \quad (1)$$

has a solution $x = (x_1, \ldots, x_n)$ in integers $x_i$.

(b) If $\sigma$ is liftable, then $\iota_{d(x)v(\sigma)} \in \text{Aut}_R(\Lambda)$, where $d(x) = \text{diag}(\pi^{x_1}, \ldots, \pi^{x_n})$.

(c) If $\varphi \in \text{Aut}_R(\Lambda)$ is written as $\iota_{udv(\sigma)}$ with $d = \text{diag}(\pi^{d_1}, \ldots, \pi^{d_n})$ as in Lemma 3, then $x = (d_1, \ldots, d_n)$ is a solution to (1).

(d) Let $\mathcal{O}_\Lambda = \{\iota_{d(x)v(\sigma)} : \sigma \text{ is liftable}\}$. Then $\mathcal{O}_\Lambda$ is a subgroup of $\text{Aut}_R(\Lambda)$, and $\text{Aut}_R(\Lambda) = \text{Inn}(\Lambda) \rtimes \mathcal{O}_\Lambda$.

**PROOF.** (a) Suppose that $\sigma$ is liftable. By Proposition 4, $\sigma$ must lift to an automorphism of the form $\iota_{d(x)v(\sigma)}$ for some diagonal matrix $d = \text{diag}(\pi^{d_1}, \ldots, \pi^{d_n})$. Now, the exponent matrix for the order $d^{-1}\Lambda d$ is $(\alpha_{ij} - d_i + d_j)$; this follows because left multiplication by $d^{-1}$ subtracts $d_i$ from the $i$-th row of the exponent matrix, while right multiplication by $d$ adds $d_j$ to the $j$-th column. Conjugating $d^{-1}\Lambda d$ by $v(\sigma)$ has the effect of applying $\sigma$ to the indices in the exponent matrix; that is, the exponent matrix for $v(\sigma)^{-1}d^{-1}\Lambda dv(\sigma)$ is $(\beta_{ij}) = (\alpha_{\sigma(i)\sigma(j)} - d_{\sigma(i)} + d_{\sigma(j)})$. Since $\iota_{d(x)v(\sigma)}$ is an automorphism of $\Lambda$, $\beta_{ij} = \alpha_{ij}$ for all $i$ and $j$; i.e.

$$\alpha_{ij} = \alpha_{\sigma(i)\sigma(j)} - d_{\sigma(i)} + d_{\sigma(j)}$$

for all $i$ and $j$. It follows that $x = (d_1, \ldots, d_n)$ is a solution to (1).

Conversely, let $x = (x_1, \ldots, x_n)$ be a solution to (1), and let $d(x) = \text{diag}(\pi^{x_1}, \ldots, \pi^{x_n})$. Then one computes as in the previous paragraph that the exponent matrix for $v(\sigma)^{-1}d(x)^{-1}\Lambda d(x)v(\sigma)$ is $\alpha_{ij}$. This shows that $\sigma$ is liftable.

(b) The computation in the previous paragraph shows that $\iota_{d(x)v(\sigma)}$ is an automorphism of $\Lambda$ whenever $\sigma$ is liftable.

(c) Suppose $\varphi = \iota_{udv(\sigma)}$. Since $u \in \Lambda^*$, we have that $\iota_{u^{-1}}: \varphi = \iota_{udv(\sigma)} \in \text{Aut}_{R}(\Lambda)$. Now, the computation in the proof of part (a) shows that $x = (d_1, \ldots, d_n)$ is a solution to (1), where $d = \text{diag}(\pi^{d_1}, \ldots, \pi^{d_n})$.

(d) We first show that $\mathcal{O}_\Lambda$ is a subgroup of $\text{Aut}_R(\Lambda)$. Let $\iota_{d(x)v(\sigma)}$ and $\iota_{d(x)v(\tau)}$ be in $\text{Aut}_R(\Lambda)$, where $d = \text{diag}(\pi^{d_1}, \ldots, \pi^{d_n})$ and $\delta = \text{diag}(\pi^{\delta_1}, \ldots, \pi^{\delta_n})$. Then
\[\iota_{dv(\sigma)} \iota_{d\delta v(\tau)} = \iota_{dv(\sigma)\delta v(\tau)}.\] Now \(dv(\sigma)\delta v(\tau) = d\delta^\sigma v(\sigma)\delta v(\tau) = d\delta^\sigma v(\sigma\tau),\) where \(\delta^\sigma = \text{diag}(\pi^{d_\sigma(1)}, \ldots, \pi^{d_\sigma(n)}).\) Since \(\iota_{d\delta^\sigma v(\sigma\tau)}\) induces \(\sigma\tau\) on \(Q(\Lambda),\) we see that \(d\delta^\sigma = d(x)\) for some solution \(x\) of (1), by part (c). It follows that \(\iota_{dv(\sigma)} \iota_{d\delta v(\tau)} \in \mathcal{O}_\Lambda\) and \(\mathcal{O}_\Lambda\) is a subgroup.

Next, we show that \(\mathcal{O}_\Lambda \cap \text{Inn}(\Lambda) = 1.\) If \(\iota_{dv(\sigma)} \in \mathcal{O}_\Lambda \cap \text{Inn}(\Lambda),\) then \(\Phi(\iota_{dv(\sigma)}) = 1\) by Lemma 1. Thus \(\sigma\) must be the identity automorphism of \(Q(\Lambda)\) by the proof of Proposition 4. Now, the system (1) has only the trivial solutions \(x = (x, \ldots, x)\) for some \(x \in \mathbb{Z},\) so that \(d = \text{diag}(\pi^x, \ldots, \pi^x)\) for some \(x \in \mathbb{Z}.\) Thus \(\iota_{dv(\sigma)} = \iota_d = 1.\)

To finish the proof, we need to show that \(\text{Aut}_R(\Lambda) = \text{Inn}(\Lambda)\mathcal{O}_\Lambda.\) Given any \(\varphi \in \text{Aut}_R(\Lambda),\) we can write \(\varphi = \iota_u \iota_{dv(\sigma)}\) with \(\iota_u \in \text{Inn}(\Lambda)\) by Lemma 3. By part (c) \(\iota_{dv(\sigma)} \in \mathcal{O}_\Lambda,\) completing the proof. \(\square\)

**Example 6** We illustrate Theorem 5 for the order

\[
\Lambda = \begin{pmatrix}
R & P^2 & P^4 \\
P^3 & R & P^4 \\
P & P & R
\end{pmatrix}.
\]

One computes that the link graph of \(\Lambda\) is the complete quiver on 3 vertices, so that \(\text{Aut}(Q(\Lambda)) = S_3.\) If we let \(\tau = (12)\) and \(\sigma = (123),\) then \(\tau\) and \(\sigma\) generate \(S_3.\)

We first show that \(\sigma\) is liftable. The linear system of equations becomes

\[
\begin{align*}
x_1 - x_2 &= \alpha_{12} - \alpha_{23} = -2 \\
x_1 - x_3 &= \alpha_{13} - \alpha_{21} = 1 \\
x_2 - x_3 &= \alpha_{23} - \alpha_{31} = 3.
\end{align*}
\]

which has the solution \(x_1 = 1 + a, x_2 = 3 + a, x_3 = a.\) Taking \(a = 0,\) we see that conjugation by \(dv(\sigma)\) is an automorphism of \(\Lambda,\) where \(d = \text{diag}(\pi, \pi^3, 1).\)

We next see that \(\tau\) is *not* liftable. The linear system for \(\tau\) is

\[
\begin{align*}
x_1 - x_2 &= \alpha_{12} - \alpha_{21} = -1 \\
x_1 - x_3 &= \alpha_{13} - \alpha_{23} = 0 \\
x_2 - x_3 &= \alpha_{23} - \alpha_{13} = 0.
\end{align*}
\]

One checks easily that this system is inconsistent, so that \(\tau\) is not liftable.

It follows that \(\text{Aut}_R(\Lambda) = \text{Inn}(\Lambda) \rtimes \mathcal{O}_\Lambda,\) where \(\mathcal{O}_\Lambda\) is cyclic of order 3, generated by conjugation by \(dv(\sigma) = \begin{pmatrix} 0 & \pi & 0 \\ 0 & 0 & \pi^3 \\ 1 & 0 & 0 \end{pmatrix}.\) \(\square\)
Remark 7 Suppose that $\Lambda$ is a basic, hereditary $R$-order in $M_n(K)$. Then without loss of generality we may assume that the exponent matrix for $\Lambda$ is given by $\alpha_{ij} = 0$ if and only if $i \geq j$, so that $\Lambda$ has the form

$$
\Lambda = \begin{pmatrix}
R & R & \ldots & R & R \\
P & R & \ldots & R & R \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
P & P & \ldots & P & R
\end{pmatrix}.
$$

The radical of $\Lambda$ is obtained by replacing the $R$'s in the diagonal by $P$'s.

If we let $P_i$ denote the $i$-th column of $\Lambda$, then the description of $\text{rad}(\Lambda)$ shows that $\text{rad}(P_i) = P_{i+1}$, where the indices are taken modulo $n$. Consequently, the link graph of $\Lambda$ is

$$
\begin{array}{ccccccc}
1 & \rightarrow & 2 & \rightarrow & 3 & \ldots & n-1 & \rightarrow & n \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\end{array}
$$

We see immediately that $\text{Aut}(Q(\Lambda))$ is cyclic of order $n$, generated by $\sigma = (12\ldots n)$. The corresponding linear system is $x_i - x_j = 0$ for $i < j < n$, and $x_i - x_n = 1$, which has the solution $x = (0, \ldots, 0, -1)$. It follows that $\sigma$ is liftable, and that $O_\Lambda$ is cyclic of order $n$, generated by conjugation by $\left( t_n^0, \pi^{-1}_0 \right)$, where $I_{n-1}$ is the $(n-1) \times (n-1)$ identity matrix. Thus we may use Theorem 5 to recover [4, Proposition 4.10] $\square$

We call a tiled order $\Lambda$ a $(0,1)$-order if $\alpha_{ij} \in \{0,1\}$ for all $i,j$. Fujita has shown that the link graph is an isomorphism invariant for basic $(0,1)$-orders [2, Theorem]. Using this fact, we show that every automorphism of $Q(\Lambda)$ is liftable for such $\Lambda$.

Theorem 8 Let $\Lambda$ be a basic $(0,1)$-order. Then every $\sigma \in \text{Aut}(Q(\Lambda))$ is liftable. Consequently, $\text{Aut}_R(\Lambda) \cong \text{Inn}(\Lambda) \rtimes \text{Aut}(Q(\Lambda))$.

PROOF. Let $\sigma \in \text{Aut}(Q(\Lambda))$, and let $\Gamma = v(\sigma)\Lambda v(\sigma)^{-1}$. Since $\Lambda$ and $\Gamma$ are isomorphic, we have $Q(\Gamma) = Q(\Lambda)$. By [2, section 2], there is a diagonal matrix $d$ such that $\Gamma = d^{-1}\Lambda d$. It follows that

$$
\Lambda = v(\sigma)^{-1}\Gamma v(\sigma) = v(\sigma)^{-1}d^{-1}\Lambda d v(\sigma),
$$

so that $v_{d v(\sigma)} \in \text{Aut}(\Lambda)$. By Proposition 4, $\sigma$ is liftable. $\square$
4 Non-primeness of crossed products

One of the initial motivations of this work was to extend results of [4] on the structure of crossed products over hereditary orders. There it was shown that, if Γ is a prime, hereditary order and $G$ is a subgroup of $O_\Gamma$, then the crossed product $\Gamma \ast G$ was again prime and hereditary. We had originally hoped that this result might generalize to the case of tiled orders as well. However, as the following example illustrates, $\Lambda \ast O_\Lambda$ need not be a prime order, even if $\Lambda$ is prime and $O_\Lambda$ acts transitively on the primitive idempotents of $\Lambda$.

**Example 9** Let $\Lambda = \left( \begin{array}{c|c} R & P \\ \hline P & R \end{array} \right)$, where $R/P$ does not have characteristic 2. Then it is easy to see that $O_\Lambda$ is cyclic of order 2, generated by conjugation by $\left( \begin{array}{rr} 0 & 1 \\ 1 & 0 \end{array} \right)$, which we will denote by $\tau$. We compute directly that $\Lambda \ast O_\Lambda \cong M_2(R) \oplus M_2(R)$ as rings, proving the claim.

We realize $\Lambda \ast O_\Lambda$ as a free $\Lambda$-module of rank 2, with basis $\{1, \overline{\tau}\}$, where multiplication is determined by $e_{11}$ and $e_{22}$ denote the standard matrix units $\left( \begin{array}{rr} 1 & 0 \\ 0 & 0 \end{array} \right)$ and $\left( \begin{array}{rr} 0 & 0 \\ 0 & 1 \end{array} \right)$ respectively, then $\overline{\tau}e_{11} = e_{22}\overline{\tau}$ and $\overline{\tau}e_{22} = e_{11}\overline{\tau}$. If we set $e_{12} = e_{11}\overline{\tau}$ and $e_{21} = e_{22}\overline{\tau}$, then it is straightforward to check that $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ form a set of matrix units in $\Lambda \ast O_\Lambda$. Thus $\Lambda \ast O_\Lambda \cong M_2(S)$, where $S \cong e_1(\Lambda \ast O_\Lambda)e_1$. We finish by showing that $S \cong R \oplus R$ as rings.

We identify $R$ with $e_{11}\Lambda e_{11}$ (that is, with the subset $\left( \begin{array}{c} R \\ 0 \end{array} \right)$ of $\Lambda$), and define a homomorphism $\varphi : R[x] \rightarrow S$ by sending 1 to $e_{11}$ and $x$ to $e_{11}\left( \begin{array}{rr} 0 & 5 \\ 5 & 0 \end{array} \right)\overline{\tau}e_{11}$. One then computes that $x^2$ maps to $\left( \begin{array}{rr} \pi^2 & 4 \\ 4 & 0 \end{array} \right) \in e_{11}\Lambda e_{11}$. It follows that the kernel of $\varphi$ contains $x^2 - \pi^2$; the fact that $S$ and $R[x]/(x^2 - \pi^2)$ both have rank 2 over $R$ shows that $\ker \varphi = (x^2 - \pi^2)$. Since the characteristic of $R/P$ is not 2, $S \cong R[x]/(x^2 - \pi^2) \cong R \oplus R$ as rings. □

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