Research Article

Map Ideal of Type the Domain of \( r\)-Cesàro Matrix in the Variable Exponent \( \ell_{t(.)} \) and Its Eigenvalue Distributions

Awad A. Bakery and OM Kalthum S. K. Mohamed

1College of Science and Arts at Khulis, Department of Mathematics, University of Jeddah, Jeddah, Saudi Arabia
2Department of Mathematics, Faculty of Science, Ain Shams University, P.O. Box 1156, Cairo, 11566 Abbassia, Egypt
3Department of Mathematics, Academy of Engineering and Medical Sciences, Khartoum, Sudan

Correspondence should be addressed to OM Kalthum S. K. Mohamed; om_kalsoom2020@yahoo.com

Received 10 February 2021; Revised 2 April 2021; Accepted 19 April 2021; Published 28 April 2021

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In this article, we define a new sequence space generated by the domain of \( r\)-Cesàro matrix in Nakano sequence space. Some geometric and topological properties of this sequence space, the multiplication maps defined on it, and the eigenvalue distributions of map ideal with \( s\)-numbers that belong to this sequence space have been examined.

1. Introduction

The vector spaces \( \ell_{t(.)} \) are contained in the variable exponent spaces \( L_{t(.)}\). Regarding the 2nd half of the twentieth century, it used to be fulfilled that these variable exponent spaces constituted the proper framework for the mathematical components of numerous issues for which the classical Lebesgue spaces have been inadequate. The relevancy of these spaces and their homes made them a famous and environment friendly device in the remedy of a range of situations; these days, the region of \( L_{t(.)}(\Omega) \) spaces is a prolific subject of lookup with ramifications achieving into very numerous mathematical specialties \([1]\). Learning about the variable exponent Lebesgue spaces \( L_{t(.)} \) obtained in addition impetus from the mathematical description of the hydrodynamics of non-Newtonian fluids \([2, 3]\). Applications of non-Newtonian fluids additionally known as electrorheological vary from their use in army science to civil engineering and orthopedics. By \( \ell^N, \ell_\infty, \ell_p, \text{ and } c_0 \), we suggest the spaces of each, bounded, \( r\)-absolutely summable and null sequences of complex numbers \( N = \{0, 1, 2, \ldots \} \). We evidence the space of all, finite rank, approximable and compact bounded linear maps from a Banach space \( \mathcal{P} \) into a Banach space \( \mathcal{Q} \) by \( \mathcal{B}(\mathcal{P}, \mathcal{Q}) \), \( \mathcal{F}(\mathcal{P}, \mathcal{Q}) \), \( \mathcal{A}(\mathcal{P}, \mathcal{Q}) \), and \( \mathcal{K}(\mathcal{P}, \mathcal{Q}) \), and if \( \mathcal{P} = \mathcal{Q} \), we mark \( \mathcal{B}(\mathcal{P}), \mathcal{F}(\mathcal{P}), \mathcal{A}(\mathcal{P}), \text{ and } \mathcal{K}(\mathcal{P}) \), respectively (see \([4, 5]\)). The ideal of all, finite rank, approximable and compact maps is denoted by \( \mathcal{B}, \mathcal{F}, \mathcal{A}, \text{ and } \mathcal{K} \). We designate \( e_l = (0, 0, \cdots, 1, 0, 0, \cdots) \), as 1 presents at the \( l\)-th coordinate, with \( l \in N \).

Lemma 1 \([5]\). Pick up \( U \in \mathcal{B}(\mathcal{P}, \mathcal{Q}) \). Assume \( U \notin \mathcal{A}(\mathcal{P}, \mathcal{Q}) \); then, there are maps \( X \in \mathcal{B}(\mathcal{P}) \) and \( Y \in \mathcal{B}(\mathcal{Q}) \) so that \( YUXe_l = e_o \) for every \( l \in N \).

Definition 2 \([5]\). A Banach space \( \mathcal{Y} \) is named simple if the algebra \( \mathcal{B}(\mathcal{Y}) \) includes one and only one nontrivial closed ideal.

Theorem 3 \([5]\). Let \( \mathcal{Y} \) be an infinite dimensional Banach space; then,

\[
\mathcal{F}(\mathcal{Y})\mathcal{B}(\mathcal{Y})\mathcal{A}(\mathcal{Y})\mathcal{K}(\mathcal{Y})\mathcal{B}(\mathcal{Y}).
\]

Definition 4 \([6]\). A map \( U \in \mathcal{B}(\mathcal{Y}) \) is entitled Fredholm if \( \dim(\text{Range}(U))^c < \infty, \dim(\ker(U))^c = \infty, \) and \( \text{Range}(U) \) is closed, where \( \text{Range}(U)^c \) mentions the complement of \( \text{Range}(U) \).
**Definition 5** [7]. A subclass \( \mathcal{W} \) of \( B \) is named a map ideal if every component \( \mathcal{W}(\mathcal{P}, \mathcal{Q}) = \mathcal{W} \cap B(\mathcal{P}, \mathcal{Q}) \) executes the next setup:

(i) \( I \in \mathcal{W} \), if \( \Omega \) illustrates Banach space of one dimension

(ii) \( \mathcal{W}(\mathcal{P}, \mathcal{Q}) \) is a linear space on \( \mathcal{Q} \)

(iii) Suppose \( X \in B(\mathcal{P}, \mathcal{Q}), Y \in \mathcal{W}(\mathcal{P}, \mathcal{Q}), \) and \( Z \in B(\mathcal{Q}, \mathcal{Q}), \) then \( ZYX \in \mathcal{W}(\mathcal{P}, \mathcal{Q}), \) where \( \mathcal{P} \) and \( \mathcal{Q} \) are normed spaces

Faried and Bakery [8] made current the notion of prequasi ideal which is added established than the quasi ideal.

**Definition 6.** A function \( \Psi : \mathcal{W} \rightarrow [0, \infty) \) is named a prequasi norm on the map ideal \( \mathcal{W} \) if the next setting encompasses the following:

1. For each \( X \in \mathcal{W}(\mathcal{P}, \mathcal{Q}), \) \( \Psi(X) \geq 0 \) and \( \Psi(X) = 0 \iff X = 0 \)
2. We have \( E_0 \geq 1 \) so as to \( \Psi(\kappa X) \leq E_0|\kappa|\Psi(X) \), for all \( X \in \mathcal{W}(\mathcal{P}, \mathcal{Q}) \) and \( \kappa \in \mathcal{Q} \)
3. We have \( G_0 \geq 1 \) for \( \Psi(Z_1 + Z_2) \leq G_0[\Psi(Z_1) + \Psi(Z_2)] \), for all \( Z_1, Z_2 \in \mathcal{W}(\mathcal{P}, \mathcal{Q}) \)
4. We have \( D_0 \geq 1 \) if \( X \in B(\mathcal{P}, \mathcal{Q}), Y \in \mathcal{W}(\mathcal{P}, \mathcal{Q}), \) and \( Z \in B(\mathcal{Q}, \mathcal{Q}), \) then \( \Psi(ZYX) \leq D_0||Z||\Psi(Y)||X|| \)

**Theorem 7** [8]. \( \Psi \) is a prequasi norm on the map ideal \( \mathcal{W} \), whenever \( \Psi \) is a quasinorm on the map ideal \( \mathcal{W} \).

**Definition 8** [9]. An \( s \)-number function is a map detailed on \( B(\mathcal{P}, \mathcal{Q}) \) which sort to every map \( X \in B(\mathcal{P}, \mathcal{Q}) \) a nonnegative scalar sequence \( (s_a(X))_{a=0}^{\infty} \) overbearing that the next setting encompasses the following:

\[
\begin{align*}
B_{a}^p &= \{ B_{a}^p (\mathcal{P}, \mathcal{Q}) : \mathcal{P} \text{ and } \mathcal{Q} \text{ are Banach Spaces}\}, \quad \text{where } B_{a}^p (\mathcal{P}, \mathcal{Q}) = \{ X \in B(\mathcal{P}, \mathcal{Q}) : (s_a(X))_{a=0}^{\infty} \in \Psi^p \}, \\
B_{b}^a &= \{ B_{b}^a (\mathcal{P}, \mathcal{Q}) : \mathcal{P} \text{ and } \mathcal{Q} \text{ are Banach Spaces}\}, \quad \text{where } B_{b}^a (\mathcal{P}, \mathcal{Q}) = \{ X \in B(\mathcal{P}, \mathcal{Q}) : (s_a(X))_{a=0}^{\infty} \in \Psi^a \}, \\
B_{d}^e &= \{ B_{d}^e (\mathcal{P}, \mathcal{Q}) : \mathcal{P} \text{ and } \mathcal{Q} \text{ are Banach Spaces}\}, \quad \text{where } B_{d}^e (\mathcal{P}, \mathcal{Q}) = \{ X \in B(\mathcal{P}, \mathcal{Q}) : (d_a(X))_{a=0}^{\infty} \in \Psi^e \}
\end{align*}
\]

The ideals and multiplication mappings possess extensive grazing of mathematics in functional analysis, namely, in the theory of fixed point, eigenvalue distributions theorem, and geometric structure of Banach spaces. A few of map ideals in the class of Banach spaces or Hilbert spaces are evident by inconsistent scalar sequence spaces. For representative the ideal of compact maps is evident by the space \( c_0 \) and \( d_1(X), \) for \( X \in B(\mathcal{P}, \mathcal{Q}) \). Pietsch [5] approved the quasi-ideals \( B_{c}^b, \) for \( 0 < b < \infty \). He investigated that the ideals of nuclear maps and of Hilbert Schmidt maps between Hilbert spaces are explored by \( \xi_1 \) and \( \xi_2 \), respectively. He examined that \( \Xi(\xi_b) \) are dense in \( B(\xi_b) \), and the algebra \( B(\xi_b) \), where \( 1 \leq b < \infty \), constructed simple Banach space. Pietsch [10] approved that \( B_{c}^b \), for \( 0 < b < \infty \), is small. Makarov and Faried [11] examined that for each infinite dimensional Banach space \( \mathcal{P}, \mathcal{Q} \) and \( r > b > 0 \), then \( B_{c}^b(\mathcal{P}, \mathcal{Q}) \) is \( B(\mathcal{P}, \mathcal{Q}) \). Yaying et al. [12] defined and examined the sequence space, \( \chi_{t}^r \), whose \( r \)-Cesàro matrix is in \( \xi_t \), with \( r \in (0, 1] \) and \( 1 \leq t \).
respectively, and examined some topological properties of bounded, regularly convergent, and absolutely convergent series. Basarır and Kara investigated the compact mappings on some Euler $B(m)$-difference sequence spaces [13], some difference sequence spaces of weighted means [14], the Riesz $B(m)$-difference sequence space [15], the $B$-difference sequence space derived by weighted mean [16], and the $m^{th}$ order difference sequence space of generalized weighted mean [17]. Mursaleen and Noman [18, 19] introduced the compact mappings on some difference sequence spaces. The multiplication maps on Cesàro sequence spaces with the Luxemburg norm were studied by Komal et al. [20]. Ilkhan et al. [21] examined the multiplication maps on Cesàro second-order function spaces. Recently, many authors in the literature have considered some nonabsolute-type sequence spaces and introduced recent high-quality papers, for example, Mursaleen and Noman [22] defined the sequence space $\ell_{p}^{0}$ and $\ell_{p}^{\infty}$ of nonabsolute type and proved that the spaces $\ell_{p}^{0}$ and $\ell_{p}^{\infty}$ are linearly isomorphic for $0 < p \leq \infty$. $\ell_{p}^{0}$ is a $p$-normed space and a BK-space in the cases for $0 < p < 1$ and $1 \leq p \leq \infty$, and formed the basis for the space $\ell_{p}^{0}$ for $1 \leq p < \infty$. In [23], they examined the $\alpha$, $\beta$, and $\gamma$-duals of $\ell_{p}^{0}$ and $\ell_{p}^{\infty}$ of nonabsolute type, for $1 \leq p < \infty$. They characterized some related matrix classes and derived the characterizations of some other classes by means of a given basic lemma. On Cesàro summable sequences, Mursaleen and Başar [24] defined some spaces of double sequences whose Cesàro transforms are bounded, convergent in Pringsheim’s sense, null in Pringsheim’s sense, both convergent in Pringsheim’s sense, and bounded, regularly convergent, and absolutely $q$-summable, respectively, and examined some topological properties of those sequence spaces. The addiction inequality will be run down in the development [25]. If $r_{a} \geq 1$ and $x_{a}, z_{a} \in \mathbb{R}$, with $a \in N$, and $h = \sup a_{r}$, then

$$
|x_{a} + z_{a}|^{r} \leq 2^{h-1}(|x_{a}|^{r} + |z_{a}|^{r}).
$$

Suppose $r \in (0, 1)$, $(t_{i}) \in R^{N}$, where $R^{N}$ is the space of all sequences of positive reals, and $t_{j} \geq 1$, with $j \in N$, we define a new sequence space generated by the $r$-Cesàro matrix in Nakano sequence space as

$$
(\text{ces}_{r}^{(t)})_{u} = \{ f = (f_{k}) \in \mathbb{N}^{N} : u(\rho f) < \infty, \text{for some } \rho > 0 \},
$$

where $u(f) = \sum_{l=0}^{\infty}(|\sum_{a=0}^{l} r^{t_{a}} f_{a}|/|l + 1|)^{r}$ and

$$
[l]_{r} = \begin{cases} 
1 - r', & r \neq 1, \\
1 - r, & r = 1.
\end{cases}
$$

In case $(t_{i}) \in \ell_{\infty}$, we have

$$
(\text{ces}_{r}^{(t)})_{u} = \{ f = (f_{k}) \in \mathbb{N}^{N} : u(\rho f) < \infty, \text{for any } \rho > 0 \}. \quad (8)
$$

Remark 10.

1. When $r = 1$ and $t_{i} = t$, with $l \in N$, then $\text{ces}_{r}^{(t)}$ is compressed to $\text{ces}_{r}$, introduced and studied by Ng and Lee [26]. Different types of Cesàro summable sequence spaces of nonabsolute type have been studied by many authors [27–31].

2. If $t_{i} = t$, with $l \in N$, $\text{ces}_{r}^{(t)}$ is truncated to $\chi_{r}$ studied by Yaying et al. [12].

The goal of this paper is efficient like so in Section 2 we offer the sufficient setting on any linear space of sequences $Y'$, and we mark it a private sequence space $(p\mathfrak{f})$, so as to the class $B_{p}$ constructs a map ideal. We apply this theorem on $\text{ces}_{r}^{(t)}$. We define a subclass of the $p\mathfrak{f}$ which we will call a premodular $p\mathfrak{f}$ under the functional $\upsilon : Y' \rightarrow [0, \infty)$. We explain the sufficient conditions on $\text{ces}_{r}^{(t)}$ with definite functional $\upsilon$ to become premodular $p\mathfrak{f}$. Which implies that $\text{ces}_{r}^{(t)}$ is a pre quasi normed $p\mathfrak{f}$. In Section 3, we define a multiplication map on the pre quasi normed $p\mathfrak{f}$, $(\text{ces}_{r}^{(t)})_{u}$, and give the necessity and sufficient setup on this sequence space such that the multiplication map is bounded, approximable, invertible, Fredholm, and closed range. In Section 4, firstly, we introduce the sufficient settings (not necessary) on the premodular $p\mathfrak{f}((\text{ces}_{r}^{(t)})_{u})$ so that $F$ is dense in $B_{(\text{ces}_{r}^{(t)})_{u}}$. This explains a negative answer of Rhoades [32] open problem about the linearity of $s$-type $(\text{ces}_{r}^{(t)})_{u}$ spaces. Secondly, we introduce the conditions on $((\text{ces}_{r}^{(t)})_{u})_{s}$ so that the components of pre quasi ideal $B_{(\text{ces}_{r}^{(t)})_{s}}$ are complete and closed. Thirdly, we investigate the sufficient conditions on $((\text{ces}_{r}^{(t)})_{u})_{s}$ so as to the class of all bounded linear maps which sequence of eigenvalues in $(\text{ces}_{r}^{(t)})_{u}$ equals $B_{(\text{ces}_{r}^{(t)})_{s}}$.

2. Linear Problem

In this section, we offer the enough setting on any linear space of sequences $Y'$, and we mark it private sequence space $(p\mathfrak{f})$, so as the class $B_{\mathfrak{f}_{p}}$ creates a map ideal. We apply this setting on $\text{ces}_{r}^{(t)}$. We define a subclass of $p\mathfrak{f}$ under the functional $\upsilon : Y' \rightarrow [0, \infty)$, which we will call a premodular $p\mathfrak{f}$. We explain the enough setup on $\text{ces}_{r}^{(t)}$ with definite
functional $v$ to become premodular $p\mathfrak{s}\mathfrak{e}\mathfrak{s}$, which implies that $\text{ces}^{(t)}_0$ is a prequasi normed $p\mathfrak{s}\mathfrak{e}\mathfrak{s}$.

Definition 11. The linear space of sequences $\mathcal{Y}$ is named a $p\mathfrak{s}\mathfrak{e}\mathfrak{s}$, if it satisfies the following:

1. $e_b \in \mathcal{Y}$, with $b \in N$

2. $\mathcal{Y}$ is solid, i.e., for $f = (f_b) \in \mathcal{C}^N$, $|g| = (|g_b|) \in \mathcal{Y}$, and $|f_b| \leq |g_b|$, over $b \in N$, then $|f| \in \mathcal{Y}$

3. $(|f_{[2b]}|)_{b=0}^{\infty} \in \mathcal{Y}$, while $[b/2]$ illustrates the integral part of $b/2$, if $(|f_{[b]}|)_{b=0}^{\infty} \in \mathcal{Y}$

Theorem 12. If the linear sequence space $\mathcal{Y}$ is $p\mathfrak{s}\mathfrak{e}\mathfrak{s}$, then $\mathbb{B}_{\mathcal{Y}}$ is a map ideal.

Proof. Assume the linear sequence space $\mathcal{Y}$ is $p\mathfrak{s}\mathfrak{e}\mathfrak{s}$.

(i) Suppose $X \in \mathbb{L}(\mathcal{P}, \mathcal{Q})$ and rank $(X) = b$, with $b \in N$. As $e_b \in \mathcal{Y}$, with $b \in N$ and by the linearity of $\mathcal{Y}$, one has $(s_b(X))_{b=0}^{\infty} = \sum_{b=0}^{b-1} s_b(X) e_b \in \mathcal{Y}$. Therefore, $X \in \mathbb{B}_{\mathcal{Y}}(\mathcal{P}, \mathcal{Q})$, this gives $\mathbb{F}(\mathcal{P}, \mathcal{Q}) \subseteq \mathbb{B}_{\mathcal{Y}}(\mathcal{P}, \mathcal{Q})$

(ii) Presume $X_1, X_2 \in \mathbb{B}_{\mathcal{Y}}(\mathcal{P}, \mathcal{Q})$ and $\rho_1, \rho_2 \in \mathbb{C}$, then $(s_b(X_1))_{b=0}^{\infty} \in \mathcal{Y}$ and $(s_b(X_2))_{b=0}^{\infty} \in \mathcal{Y}$. As $b \geq 2[b/2]$, compared to the definition of $s$-numbers and $s_b(X)$ is a decreasing sequence, we get $s_b(\rho_1 X_1 + \rho_2 X_2) \leq s_{[b/2]}(\rho_1 X_1 + \rho_2 X_2) \leq s_{[b/2]}(\rho_1 X_1) + s_{[b/2]}(\rho_2 X_2) = |\rho_1| s_{[b/2]}(X_1) + |\rho_2| s_{[b/2]}(X_2)$, with $b \in N$. By using the linearity of $\mathcal{Y}$, conditions (24) and (25), one can see $(s_b(\rho_1 X_1 + \rho_2 X_2))_{b=0}^{\infty} \in \mathcal{Y}$, so $\rho_1 X_1 + \rho_2 X_2 \in \mathbb{B}_{\mathcal{Y}}(\mathcal{P}, \mathcal{Q})$

(iii) Let $X \in \mathbb{B}(\mathcal{P}_0, \mathcal{P}), Y \in \mathbb{B}_{\mathcal{Y}}(\mathcal{P}, \mathcal{Q})$, and $Z \in \mathbb{B}(\mathcal{Q}, \mathcal{Q}_0)$, one has $(s_b(Y))_{b=0}^{\infty} \in \mathcal{Y}$. As $s_b(YX) \leq ||Z||s_b(Y)||X||$. By using the linearity of $\mathcal{Y}$ and condition (24), we have $(s_b(YXZ))_{b=0}^{\infty} \in \mathcal{Y}$, then $YXZ \in \mathbb{B}_{\mathcal{Y}}(\mathcal{P}_0, \mathcal{Q}_0)$

Here and after, we will denote the space of all increasing sequences of real numbers by $\mathfrak{N}$.

Theorem 13. $\text{ces}^{(t)}_0$ is $p\mathfrak{s}\mathfrak{e}\mathfrak{s}$, if $(t_1) \in \mathfrak{N} \cap \ell_{\infty}$ with $t_0 > 1$.

Proof.

(i) Assume $f, g \in \text{ces}^{(t)}_0$. As $(t_i) \in \ell_{\infty}$, one has

$$
\sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{t_i} r^i f_i + r^i g_i}{[l+1]_r} \right)^t_i \\
\leq 2^{b-1} \left( \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{t_i} r^i f_i}{[l+1]_r} \right)^t_i \right) + \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{t_i} r^i g_i}{[l+1]_r} \right)^t_i \\
< \infty,
$$

so $f + g \in \text{ces}^{(t)}_0$. (1.ii) Suppose $\rho \in \mathbb{C}$, $f \in \text{ces}^{(t)}_0$, and as $(t_i) \in \ell_{\infty}$, we obtain

$$
\sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{t_i} r^i \rho f_i}{[l+1]_r} \right)^t_i \leq \sup_{i \geq 0} \rho^i \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{t_i} r^i f_i}{[l+1]_r} \right)^t_i < \infty.
$$

Hence, $\rho f \in \text{ces}^{(t)}_0$. To relative to (1-i) and (1-ii), we have $\text{ces}^{(t)}_0$ as a linear space.

Also as $(t_i) \in \mathfrak{N} \cap \ell_{\infty}$ with $t_0 > 1$, one has

$$
\sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{t_i} r^i (f_i)^2}{[l+1]_r} \right)^{t_i} \leq \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{t_i} r^i g_i}{[l+1]_r} \right)^{t_i} < \infty.
$$

Hence, $|f| \in \text{ces}^{(t)}_0$.

(ii) Assume $([f_i]) \in \text{ces}^{(t)}_0$, where $(t_i) \in \mathfrak{N} \cap \ell_{\infty}$, we get

$$
\sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{t_i} r^i |f_i|}{[l+1]_r} \right)^{t_i} = \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{t_i} r^i |f_i|}{[l+1]_r} \right)^{t_i} + \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{t_i} r^{2i} |f_i|}{[l+1]_r} \right)^{t_i} \\
\leq \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{t_i} r^{2i} |f_i|}{[l+1]_r} \right)^{t_i} + \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{t_i} r^{2i+1} |f_i|}{[l+1]_r} \right)^{t_i} \\
\leq 2^{b-1} \left( \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{t_i} r^{2i} f_i}{[l+1]_r} \right)^t_i \right) + \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{t_i} r^{2i+1} f_i}{[l+1]_r} \right)^t_i \\
< \infty,
$$

so $([f_{[2b]}]) \in \text{ces}^{(t)}_0$.

By using Theorem 12, we can get the next theorem.
Theorem 14. Pick up \((t_1) \in \mathfrak{T} \cap \ell_\infty\) with \(t_0 > 1\), then \(B^r_{t_0}\) is a map ideal.

Definition 15. A subclass of \(\mathfrak{p}\mathfrak{s}\mathfrak{s}\) is named a premodular \(\mathfrak{p}\mathfrak{s}\mathfrak{s}\), if there is a map \(v: \mathcal{Y} \rightarrow [0, \infty)\) with the following settings:

(i) When \(f \in \mathcal{Y}\), \(f = \theta \Leftrightarrow v(|f|) = 0\), with \(v(f) \geq 0\), where \(\theta\) is the zero element of \(\mathcal{Y}\).

(ii) If \(f \in \mathcal{Y}\) and \(f \in \mathcal{C}\), we have \(E_0 \geq 1\) with \(v(\rho f) \leq |\rho| E_0 v(f)\).

(iii) \(v(f + g) \leq G_0(v(f) + v(g))\) includes for some \(G_0 \geq 1\), with \(f, g \in \mathcal{Y}\).

(iv) For \(b \in N\), \(|f_b| \leq |g_b|\), we get \(v(|f_b|) \leq v(|g_b|)\).

(v) The inequality, \(v(|f_{b0}|) \leq v(|f_{b0}|) \leq D_0 v(|f_{b0}|)\) includes for \(D_0 \geq 1\).

(vi) If \(\mathcal{F}\) denotes the space of all sequences with finite nonzero coordinates, then \(\mathcal{F} = \mathcal{Y}_{\theta}\).

(vii) We have \(\omega > 0\) so that \(v(\rho, 0, 0, 0, \cdots) \geq \omega |\rho| v(1, 0, 0, 0, \cdots), \) with \(\rho \in \mathcal{C}\).

Definition 16. The \(\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathcal{Y}_{\theta}\), is named a prequasi normed \(\mathfrak{p}\mathfrak{s}\mathfrak{s}\), if \(v\) supports the points (i)-(iii) of Definition 15. If \(\mathcal{Y}\) is complete equipped with \(v\), then \(\mathcal{Y}_{\theta}\) is named a prequasi Banach \(\mathfrak{p}\mathfrak{s}\mathfrak{s}\).

Theorem 17. A prequasi normed \(\mathfrak{p}\mathfrak{s}\mathfrak{s}\mathcal{Y}_{\theta}\), whenever it is premodular \(\mathfrak{p}\mathfrak{s}\mathfrak{s}\).

Theorem 18. \((\text{ces}_r^{(t)})\), is a premodular \(\mathfrak{p}\mathfrak{s}\mathfrak{s}\), if \((t_1) \in \mathfrak{T} \cap \ell_\infty\) with \(t_0 > 1\).

Proof.

(i) Easily, \(v(f) \geq 0\) and \(v(|f|) = 0 \Leftrightarrow f = \theta\)

(ii) We have \(E_0 = \max \{1, \sup \{v(\rho f) \mid \rho \in \mathcal{C}\} \} \geq 1\) with \(v(\rho f) \leq E_0 v(f)\), for every \(f \in \text{ces}_r^{(t)}\) and \(\rho \in \mathcal{C}\).

(iii) One has \(v(f + g) \leq 2^{h-1} (v(f) + v(g))\), for each \(f, g \in \text{ces}_r^{(t)}\).

(iv) Definitely, from the proof part (24) of Theorem 13

(v) Indeed, from the proof part (25) of Theorem 13, \(D_0 \geq 2^{h-1} + 2^{h-1} + 2^h \geq 1\).

(vi) Obviously, \(\mathcal{F} = \text{ces}_r^{(t)}\)

(vii) We have \(0 \leq \omega \leq \sup \{v(\rho f) \mid \rho \in \mathcal{C}\} \geq \omega |\rho| v(1, 0, 0, 0, \cdots), \) for each \(\rho \neq 0\) and \(\omega > 0\), if \(\rho = 0\).

By following Theorems 17 and 18, we determine the next theorem.

Theorem 19. The space \(\mathcal{Y}_{\theta}\) is a prequasi normed \(\mathfrak{p}\mathfrak{s}\mathfrak{s}\), if \((t_1) \in \mathfrak{T} \cap \ell_\infty\) with \(t_0 > 1\).

3. Multiplication Maps on \((\text{ces}_r^{(t)})\)

In this section, we define a multiplication map on the prequasi normed \(\mathfrak{p}\mathfrak{s}\mathfrak{s}\) and investigate the necessity and sufficient setup on \((\text{ces}_r^{(t)})\), so as the multiplication map is bounded, invertible, approximable, Fredholm, and closed range map.

Definition 20. If \(\omega = (\omega_k) \in \mathfrak{C}^N\) and \(\mathcal{Y}_{\theta}\) is a prequasi normed \(\mathfrak{p}\mathfrak{s}\mathfrak{s}\), the map \(H_\omega: \mathcal{Y}_{\theta} \rightarrow \mathcal{Y}_{\theta}\) is called a multiplication map on \(\mathcal{Y}_{\theta}\), when \(H_\omega f = (\omega_k f_k) \in \mathcal{Y}_{\theta}\), with \(f \in \mathcal{Y}_{\theta}\). The multiplication map is named created by \(\omega\), if \(H_\omega \in \mathfrak{B}(\mathcal{Y}_{\theta})\).

Theorem 21. Pick up \(\omega \in \mathfrak{C}^N\) and \((t_1) \in \mathfrak{T} \cap \ell_\infty\) with \(t_0 > 1\), then \(\omega \in \ell_\infty\), if and only if \(H_\omega \in \mathfrak{B}(\text{ces}_r^{(t)})\).

Proof. Suppose the settings are confirmed. Let \(\omega \in \ell_\infty\). Hence, there is \(\omega > 0\) so as to \(|\omega_k| \leq \nu, \) with \(b \in N\). For \(f \in \text{ces}_r^{(t)}\), one has

\[
v(H_\omega f) = v(\omega f) = \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{l} r^i \omega_i f_r}{l + 1} \right) = \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{l} v(\rho f)}{l + 1} \right) \leq \sup \nu \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{l} v(\rho f)}{l + 1} \right) = \sup \nu \sum_{i=0}^{\infty} v(f).
\]

Therefore, \(H_\omega \in \mathfrak{B}(\text{ces}_r^{(t)}),\)

On the contrary, let \(H_\omega \in \mathfrak{B}(\text{ces}_r^{(t)}),\) and \(\omega \notin \ell_\infty\). Hence, for all \(b \in N, \) there are \(x_0 \in N\) so as \(\omega_{x_0} > b\). We have

\[
v(H_\omega e_{x_0}) = v(\omega e_{x_0}) = \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{l} r^i \omega_i (e_{x_0})}{l + 1} \right) = \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{l} r^i \omega_i (e_{x_0})}{l + 1} \right) \geq \sum_{i=0}^{\infty} \left( \frac{b^i r^i \omega_i (e_{x_0})}{l + 1} \right) > b^i v(e_{x_0}).
\]

Hence, \(H_\omega \notin \mathfrak{B}(\text{ces}_r^{(t)}),\) So \(\omega \notin \ell_\infty\).
Theorem 22. Assume \( \omega \in \mathcal{C}^\omega \) and \((ces_r^{(t)})_\omega\) be a prequasi normed \( \mathbb{F} \). Then, \( \omega_b = \varrho \), for every \( b \in \mathbb{N} \) and \( g \in \mathcal{C} \) with \( |g| = 1 \), if and only if \( H_{\omega} \) is an isometry.

Proof. Let the sufficient condition be verified. One has

\[
\nu(H_{\omega}f) = \nu(\omega f)
\]

\[
= \sum_{l=0}^{\infty} \left( \frac{\sum_{l=0}^{\infty} r^l \omega_k(e_{b_l})}{[l+1]} \right)_{l_1}
\]

\[
= \sum_{l=0}^{\infty} \left( \frac{\sum_{l=0}^{\infty} |g|^l \omega_k(e_{b_l})}{[l+1]} \right)_{l_1}
\]

\[
= \nu(f),
\]

with \( f \in (ces_r^{(t)})_\omega \). So \( H_{\omega} \) is an isometry.

Let the necessity condition be satisfied and \( |\omega_b| < 1 \), for some \( b = b_0 \). We get

\[
\nu(H_{\omega}e_{b_0}) = \nu(\omega e_{b_0})
\]

\[
= \sum_{l=0}^{\infty} \left( \frac{\sum_{l=0}^{\infty} r^l \omega_k(e_{b_l})}{[l+1]} \right)_{l_1}
\]

\[
= \sum_{l=0}^{\infty} \left( \frac{\sum_{l=0}^{\infty} |g|^l \omega_k(e_{b_l})}{[l+1]} \right)_{l_1}
\]

\[
< \sum_{l=0}^{\infty} \left( \frac{r^l \omega_{b_0}}{[l+1]} \right)_{l_1}
\]

\[
= \nu(e_{b_0}).
\]

Also, when \( |\omega_{b_0}| > 1 \), it is easy to show that \( \nu(H_{\omega}e_{b_0}) > \nu(e_{b_0}) \), which is an inconsistency for the two cases. Therefore, \( |\omega_b| = 1 \), for all \( b \in \mathbb{N} \).

By \( \mathbb{F} \), we will denote the space of all elements with finite number of elements.

Theorem 23. Raise up \( \omega \in \mathcal{C}^\omega \) and \((t_i) \in \mathcal{A} \cap \ell_{\infty} \) with \( t_0 > 1 \).

Then, \( H_{\omega} \in \mathcal{A}((ces_r^{(t)})_\omega) \), if and only if \( (\omega_b)_{b=0}^{\infty} \in c_0 \).

Proof. Let \( H_{\omega} \in \mathcal{A}((ces_r^{(t)})_\omega) \), so \( H_{\omega} \in \mathcal{A}((ces_r^{(t)})_\omega) \). Suppose \( \lim_{b \to \infty} |\omega_b| = 0 \). Therefore, we have \( \rho > 0 \) so as the set \( K_{\rho} = \{ b \in \mathbb{N} : |\omega_b| \geq \rho \} \neq \emptyset \), if \( \{ \alpha_n \}_{n \in \mathbb{N}} \in K_{\rho} \). Hence, \( \{ e_{a_n} : a_n \in K_{\rho} \} \in \ell_{\infty} \) is an infinite set in \((ces_r^{(t)})_\omega\). Since \( \alpha_n, \beta_n \in K_{\rho} \). Therefore, \( \{ e_{a_n} : a_n \in K_{\rho} \} \in \ell_{\infty} \), which cannot have a convergent subsequence under \( H_{\omega} \). Hence, \( H_{\omega} \notin \mathcal{A}((ces_r^{(t)})_\omega) \), which implies \( H_{\omega} \notin \mathcal{A}((ces_r^{(t)})_\omega) \); this gives an inconsistency. So \( \lim_{b \to \infty} |\omega_b| = 0 \). On the other hand, let \( \lim_{b \to \infty} |\omega_b| = 0 \). Hence, for each \( \rho > 0 \), one has \( K_{\rho} = \{ b \in \mathbb{N} : |\omega_b| \geq \rho \} \in \mathbb{F} \). Hence, for each \( \rho > 0 \), we have \( \dim ((ces_r^{(t)})_\omega)_{K_{\rho}} = \dim (\mathcal{C}^\omega) < \infty \). So \( H_{\omega} \in \mathcal{F}((ces_r^{(t)})_\omega) \). Define \( \omega_a \in \mathcal{C}^\omega \), for all \( a \in \mathbb{N} \), by

\[
(\omega_a)_b = \begin{cases} 
\omega_b, & b \in K_{1/a+1}, \\
0, & \text{otherwise.}
\end{cases}
\]

It is clear that \( H_{\omega_a} \in \mathcal{F}((ces_r^{(t)})_\omega)_{K_{1/a+1}} \) as \( \dim (((ces_r^{(t)})_\omega)_{K_{1/a+1}}) < \infty \), for all \( a \in \mathbb{N} \). From \((t_i) \in \mathcal{A} \cap \ell_{\infty} \) with \( t_0 > 1 \), one can see

\[
u((H_{\omega} - H_{\omega_a})f) = \nu((\omega_b - (\omega_a)_b)f_{b=0}^{\omega})
\]

\[
= \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{\infty} r^i (\omega_b - (\omega_a)_b)f_{b=0}^{\omega}}{[l+1]} \right)_{l_1}
\]

\[
= \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{\infty} r^i (\omega_b - (\omega_a)_b)f_{b=0}^{\omega}}{[l+1]} \right)_{l_1}
\]

\[
= \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{\infty} r^i (\omega_b - (\omega_a)_b)f_{b=0}^{\omega}}{[l+1]} \right)_{l_1}
\]

\[
= \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{\infty} r^i (\omega_b - (\omega_a)_b)f_{b=0}^{\omega}}{[l+1]} \right)_{l_1}
\]

\[
< \frac{1}{(a+1)^{t_0}} \sum_{i=0}^{\infty} \left( \frac{\sum_{i=0}^{\infty} r^i f_{b=0}^{\omega}}{[l+1]} \right)_{l_1}
\]

\[
= \frac{1}{(a+1)^{t_0}} \nu(f).
\]

Hence, \( \|H_{\omega} - H_{\omega_a}\| \leq 1/(a+1)^{t_0} \). This gives that \( H_{\omega} \) is a limit of finite rank maps. Therefore, \( H_{\omega} \in \mathcal{A}((ces_r^{(t)})_\omega) \).

Theorem 24. Pick up \( \omega \in \mathcal{C}^\omega \) and \((t_i) \in \mathcal{A} \cap \ell_{\infty} \) with \( t_0 > 1 \).

Then, \( H_{\omega} \in \mathcal{A}((ces_r^{(t)})_\omega) \), if and only if \( (\omega_b)_{b=0}^{\infty} \in c_0 \).

Proof. Obviously, since \( \mathcal{A}((ces_r^{(t)})_\omega) \) is \( \mathcal{A}((ces_r^{(t)})_\omega) \).
Corollary 25. If \((t_i) \in \mathfrak{T} \cap \ell_{\infty} \) with \(t_0 > 1\), then \(\mathcal{K}((\mathit{ces}_r^{(1)})_v) \neq \emptyset \mathfrak{B}((\mathit{ces}_r^{(1)})_v)\).

Proof. As \(\omega = (1, 1, \cdots)\) is created the multiplication map \(I\) on \((\mathit{ces}_r^{(1)})_\omega\), then \(\alpha > 0\) and \(\eta > 0\) so as \(\alpha < 0\) \(\omega\) \(\eta\), with \(\omega \neq (\ker (\omega))\), if and only if \(\text{Range}(H_\omega)\) is closed.

Proof. Assume \(\alpha > 0\) so as \(|\omega_0| \geq \rho_0\), with \(b \neq (\ker (\omega))\). To show that \(\text{Range}(H_\omega)\) is closed, if \(g\) is a unit point of \(\text{Range}(H_\omega)\), we have \(H_\omega f \in (\mathit{ces}_r^{(1)})_\omega\) with \(b \neq N\) so that \(\lim_{b \to \infty} H_\omega f = g\). Obviously, the sequence \(H_\omega f b\) is a Cauchy sequence. As \((t_i) \in \mathfrak{T} \cap \ell_{\infty}\), with \(t_0 > 1\), one has

\[
v(H_\omega f b - H_\omega f b) = \sum_{n=0}^{\infty} \left( \frac{\sum_{k=0}^{\infty} \rho(k) (\mathit{ces}_r^{(1)})_\omega} {t+1} \right)\]

\[
\geq 0 \sum_{n=0}^{\infty} \left( \frac{\sum_{k=0}^{\infty} \rho(k) (\mathit{ces}_r^{(1)})_\omega} {t+1} \right)\]

\[
\geq \sup_{t \to \infty} t v(u_n - u_b),
\]

where

\[
(u_n)_k = \begin{cases} (f_a)_k, & k \in (\ker (\omega))\, \text{c}, \\ 0, & k \notin (\ker (\omega))\, \text{c}. \end{cases}
\]

This implies that \((u_n)_k\) is a Cauchy sequence in \((\mathit{ces}_r^{(1)})_\omega\). As \((\mathit{ces}_r^{(1)})_\omega\) is complete, there is \(f \in (\mathit{ces}_r^{(1)})_\omega\) so that \(\lim_{b \to \infty} H_\omega f = f\). Since \(H_\omega \in \mathfrak{B}((\mathit{ces}_r^{(1)})_\omega)\), we have \(\lim_{b \to \infty} H_\omega f = f\). But \(\lim_{b \to \infty} H_\omega f = \lim_{b \to \infty} H_\omega f = g\). So \(H_\omega f = g\). Hence, \(g \neq \text{Range}(H_\omega)\). Therefore, \(\text{Range}(H_\omega)\) is closed. Next, suppose the necessity setup is satisfied. So there is \(\rho > 0\) so as \(v(H_\omega f) \geq \rho v(f)\), with \(f \neq (\mathit{ces}_r^{(1)})_\omega (\ker (\omega))\). If \(K = \{b \neq (\ker (\omega))\} : |\omega_0| < \rho\} \neq \emptyset\), then for \(a_0 \in K\), one has

\[
v(H_\omega e_{a_0}) = v\left(\left(\mathit{ces}_r^{(1)}_v \mathit{ces}_r^{(1)}_v\right)_{e_{a_0}}\right)\]

\[
= \sum_{i \geq 0} \left( \frac{\sum_{k=0}^{\infty} \rho(k) (\mathit{ces}_r^{(1)}_v (\mathit{ces}_r^{(1)}_v)_\omega)_{e_{a_0}}} {t+1} \right)\]

\[
\leq \sup_{t \to \infty} t v(e_{a_0})\]

This gives an inconsistency. Therefore, \(K = \phi\), we have \(|\omega_0| \geq \rho\), with \(b \neq (\ker (\omega))\). This proves the theorem.

Theorem 26. \(H_\omega \in \mathfrak{B}((\mathit{ces}_r^{(1)})_\omega)\), then there are \(\alpha > 0\) and \(\eta > 0\) so as \(\alpha < 0\) \(\omega\) \(\eta\), with \(b \in \mathfrak{B}((\mathit{ces}_r^{(1)})_\omega), \) and \(I \in \mathfrak{B}((\mathit{ces}_r^{(1)})_\omega), \)

\[
v(H_\omega e_{a_0}) = v\left(\left(\mathit{ces}_r^{(1)}_v \mathit{ces}_r^{(1)}_v\right)_{e_{a_0}}\right)\]

\[
= \sum_{i \geq 0} \left( \frac{\sum_{k=0}^{\infty} \rho(k) (\mathit{ces}_r^{(1)}_v (\mathit{ces}_r^{(1)}_v)_\omega)_{e_{a_0}}} {t+1} \right)\]

This gives an inconsistency. Therefore, \(K = \phi\), we have \(|\omega_0| \geq \rho\), with \(b \neq (\ker (\omega))\). This proves the theorem.

Theorem 27. Pick up \(\omega \in \mathfrak{B}^\infty\) and \((\mathit{ces}_r^{(1)})_v\) be a prequasi Banach \(\mathfrak{B}^\infty\). There, then \(\alpha > 0\) and \(\eta > 0\) so as \(\alpha < 0\) \(\omega\) \(\eta\), with \(b \in \mathfrak{B}((\mathit{ces}_r^{(1)})_\omega), \) and \(I \in \mathfrak{B}((\mathit{ces}_r^{(1)})_\omega), \)

\[
v(H_\omega e_{a_0}) = v\left(\left(\mathit{ces}_r^{(1)}_v \mathit{ces}_r^{(1)}_v\right)_{e_{a_0}}\right)\]

\[
= \sum_{i \geq 0} \left( \frac{\sum_{k=0}^{\infty} \rho(k) (\mathit{ces}_r^{(1)}_v (\mathit{ces}_r^{(1)}_v)_\omega)_{e_{a_0}}} {t+1} \right)\]

This gives an inconsistency. Therefore, \(K = \phi\), we have \(|\omega_0| \geq \rho\), with \(b \neq (\ker (\omega))\). This proves the theorem.

Theorem 28. Raise up \((\mathit{ces}_r^{(1)})_v\) to be a prequasi Banach \(\mathfrak{B}^\infty\) and \(H_\omega \in \mathfrak{B}((\mathit{ces}_r^{(1)})_\omega)\). Then, \(H_\omega\) is a Fredholm map, if and only if (i) \(\ker (\omega)\) \(\mathfrak{B}^\infty\) and (ii) \(|\omega_0| \geq \rho\), with \(b \neq (\ker (\omega))\).

Proof. Assume the sufficient condition is satisfied. Let \(\ker (\omega)\) \(\mathfrak{B}^\infty\) be infinite; hence, \(e_0 \in \ker (H_\omega)\), with \(b \neq (\ker (\omega))\). Since \(e_0\) s are linearly independent, this gives dim \((\ker (H_\omega)\) ) = \(\infty\); this implies an inconsistency. Hence, \(\ker (\omega)\) \(\mathfrak{B}^\infty\) must be finite. The condition (ii) comes from Theorem 26. Next, let the setup (i) and (ii) be confirmed. From Theorem 26, the setup (ii) implies that \(\text{Range}(H_\omega)\) is closed. The setting (i) gives that \(\dim (\ker (H_\omega)) < \infty\) and \(\dim ((\text{Range}(H_\omega)) < \infty\). This implies that \(H_\omega\) is Fredholm.

4. Prequasi Ideal

In this section, firstly, we introduce the sufficient setting (not necessary) on \((\mathit{ces}_r^{(1)})_v\) such that \(F\) is dense in \(\mathfrak{B}^\infty((\mathit{ces}_r^{(1)})_v)\). This investigates a negative answer of Rhoades [32] open problem about the linearity of \(s\)-type \((\mathit{ces}_r^{(1)})_v\) spaces. Secondly, for which conditions on \((\mathit{ces}_r^{(1)})_v\) are \(\mathfrak{B}^\infty((\mathit{ces}_r^{(1)})_v)\) complete and closed? Thirdly, we give the sufficient setup on \((\mathit{ces}_r^{(1)})_v\) such
that $B^*(ces^{(t)})_{l}$ is strictly contained for different powers. We explain the settings in order that $B^*(ces^{(t)})_{l}$ is minimum.

Fourthly, we explain the conditions so that the Banach pre quasi ideal $B^*(ces^{(t)})_{l}$ is simple. Fifthly, we give the sufficient conditions on $(ces^{(t)})_{l}$ such that the space of all bounded linear maps which sequence of eigenvalues in $(ces^{(t)})_{l}$ equals $B^*(ces^{(t)})_{l}$.

4.1. Finite Rank Pre quasi Ideal

**Theorem 29.** $B^*(ces^{(t)})_{l}((\mathcal{P}, \mathcal{Q}) \subseteq F(\mathcal{P}, \mathcal{Q})$, whenever $(t) \in \mathfrak{S} \cap \ell_{\infty}$ with $t_{0} > 1$. But the converse is not necessarily true.

Proof. To show that $F(\mathcal{P}, \mathcal{Q}) \subseteq B^*(ces^{(t)})_{l}((\mathcal{P}, \mathcal{Q})$, as $e_{l} \in (ces^{(t)})_{l}$, with $l \in \mathbb{N}$ and $(ces^{(t)})_{l}$ is a linear space, let $Z \in F(\mathcal{P}, \mathcal{Q})$, one has $(s_{l}(Z))_{l=0}^{\infty} \in \mathcal{F}$. To show that $B^*(ces^{(t)})_{l}((\mathcal{P}, \mathcal{Q}) \subseteq F(\mathcal{P}, \mathcal{Q})$, as $t_{0} > 1$ and $(t) \in \mathfrak{S} \cap \ell_{\infty}$, one can see $\sum_{l=0}^{\infty}(1/[l + 1])^{t_{l}} < \infty$. For $Z \in B^*(ces^{(t)})_{l}((\mathcal{P}, \mathcal{Q})$, we have $(s_{l}(Z))_{l=0}^{\infty} \in (ces^{(t)})_{l}$. As $v\nu(s_{l}(Z))_{l=0}^{\infty} < \infty$, suppose $\rho \in (0, 1)$, then there is $l_{0} \in \mathbb{N} - \{0\}$ with $v\nu(s_{l}(Z))_{l_{0}}^{\infty} < \rho/2^{b+3}d$, for some $d \geq 1$, where $d = \max \{1, \sum_{l=0}^{\infty}(1/[l + 1])^{t_{l}}\}$. As $s_{l}(Z)$ is decreasing, one has

\[
\sum_{l=l_{0}+1}^{2l} \left( \frac{\sum_{j=0}^{t_{l}}r_{j}^{2}e_{j}(Z)}{[l + 1]_{r}} \right)^{t_{l}} \leq \sum_{l=0}^{\infty} \left( \frac{\sum_{j=0}^{t_{l}}r_{j}^{2}e_{j}(Z)}{[l + 1]_{r}} \right)^{t_{l}} \leq \sum_{l=0}^{\infty} \left( \frac{\sum_{j=0}^{t_{l}}r_{j}^{2}e_{j}(Z)}{[l + 1]_{r}} \right)^{t_{l}} < \frac{\rho}{2^{b+3}d}.
\]

Therefore, there is $Y \in F_{2l_{0}}(\mathcal{P}, \mathcal{Q})$ so that rank$(Y) \leq 2l_{0}$ and

\[
\sum_{l=2l_{0}+1}^{3l_{0}} \left( \frac{\sum_{j=0}^{t_{l}}r_{j}^{2}|Z - Y|}{[l + 1]_{r}} \right)^{t_{l}} \leq \sum_{l=0}^{3l_{0}} \left( \frac{\sum_{j=0}^{t_{l}}r_{j}^{2}|Z - Y|}{[l + 1]_{r}} \right)^{t_{l}} < \frac{\rho}{2^{b+3}d}.
\]

since $(t) \in \mathfrak{S} \cap \ell_{\infty}$, we have

\[
\sup_{l_{0}} \left( \sum_{l=0}^{l_{0}} r_{l}^{2}||Z - Y|| \right)^{t_{l}} < \frac{\rho}{2^{b+3}d}.
\]

Therefore, one has

\[
\sum_{l=0}^{l_{0}} \left( \frac{\sum_{j=0}^{t_{l}}r_{j}^{2}|Z - Y|}{[l + 1]_{r}} \right)^{t_{l}} < \frac{\rho}{2^{b+3}d}.
\]

As $(t) \in \mathfrak{S} \cap \ell_{\infty}$, $t_{0} > 1$ and by using inequalities (5) and (24)–(27), one has

\[
d(Z, Y) = \nu(s_{l}(Z - Y))_{l=0}^{\infty}
\]

\[
= \sum_{l=0}^{l_{0}} \left( \frac{\sum_{j=0}^{t_{l}}r_{j}^{2}e_{j}(Z - Y)}{[l + 1]_{r}} \right)^{t_{l}} + \sum_{l=l_{0}+1}^{\infty} \left( \frac{\sum_{j=0}^{t_{l}}r_{j}^{2}e_{j}(Z - Y)}{[l + 1]_{r}} \right)^{t_{l}} \leq \sum_{l=0}^{l_{0}} \left( \frac{\sum_{j=0}^{t_{l}}r_{j}^{2}|Z - Y|}{[l + 1]_{r}} \right)^{t_{l}} + \sum_{l=l_{0}+1}^{\infty} \left( \frac{\sum_{j=0}^{t_{l}}r_{j}^{2}|Z - Y|}{[l + 1]_{r}} \right)^{t_{l}} \leq \sum_{l=0}^{l_{0}} \left( \frac{\sum_{j=0}^{t_{l}}r_{j}^{2}|Z - Y|}{[l + 1]_{r}} \right)^{t_{l}} + \sum_{l=l_{0}+1}^{\infty} \left( \frac{\sum_{j=0}^{t_{l}}r_{j}^{2}|Z - Y|}{[l + 1]_{r}} \right)^{t_{l}} \leq \sum_{l=0}^{l_{0}} \left( \frac{\sum_{j=0}^{t_{l}}r_{j}^{2}|Z - Y|}{[l + 1]_{r}} \right)^{t_{l}} + \sum_{l=0}^{\infty} \left( \frac{\sum_{j=0}^{t_{l}}r_{j}^{2}|Z - Y|}{[l + 1]_{r}} \right)^{t_{l}} < \rho.
\]

(28)

 Conversely, we give a counterexample as $I_{4} \in B^*(ces^{(t)})_{l}((\mathcal{P}, \mathcal{Q}) \subseteq F(\mathcal{P}, \mathcal{Q})$, but $0 < r < 1$ is not verified. This confirms the proof.

4.2. Banach and Closed Pre quasi Ideal

**Theorem 30.** The function $\Psi$ is a pre quasi norm on $B^*(\mathcal{F})_{l}$, where $\Psi(Z) = v\nu(s_{l}(Z))_{l=0}^{\infty}$ for all $Z \in B^*(\mathcal{F})_{l}((\mathcal{P}, \mathcal{Q})$, if $(\mathcal{F})_{l}$ is a premodular $\Psi$.

Proof. Let $(\mathcal{F})_{l}$ be a premodular $\Psi$; hence, $\Psi$ satisfies the following conditions:

(1) For all $X \in B^*(\mathcal{F})_{l}((\mathcal{P}, \mathcal{Q})$, $\Psi(X) = v\nu(s_{l}(X))_{l=0}^{\infty} \geq 0$ and $\Psi(X) = v\nu(s_{l}(X))_{l=0}^{\infty} = 0$, if and only if $s_{l}(X) = 0$, for all $b \in \mathbb{N}$, if and only if $X = 0$

(2) There is $E \geq 1$ such that $\Psi(pX) = v\nu(s_{l}(pX))_{l=0}^{\infty} \leq E\Psi(X)$, for all $X \in B^*(\mathcal{F})_{l}((\mathcal{P}, \mathcal{Q})$ and $p \in \mathbb{C}$

(3) There is $D \geq 1$ such that for all $X_{1}, X_{2} \in B^*(\mathcal{F})_{l}((\mathcal{P}, \mathcal{Q})$, we have

\[
\Psi(X_{1} + X_{2}) = v\nu(s_{l}(X_{1} + X_{2}))_{l=0}^{\infty} \leq G_{0}\Psi(X_{1}) + v\nu(s_{l}(X_{2}))_{l=0}^{\infty} \leq G_{0}D\Psi(X_{1}) + v\nu(s_{l}(X_{2}))_{l=0}^{\infty} \leq D[\Psi(X_{1}) + \Psi(X_{2})]_{l=0}^{\infty}.
\]

(29)
(4) There is $\rho \geq 1$ such that if $X \in B(\mathcal{P}, \mathcal{Q})$, $Y \in B(\mathcal{Q}, \mathcal{R})$, and $Z \in B(\mathcal{R}, \mathcal{C})$, then $\Psi(ZYX) = \nu(s_b(ZYX))^{(v)} = \nu(\|X\|\|Z\|s_b(Y))^{(v)} \leq \rho||X||\|\Psi(Y)||$.

Theorem 31. Pick up $(t_j) \in X \cap \ell_\infty$ with $t_0 > 1$, then $(B'_{(ces)^{(v)}}, \Psi)$ is a prequasi Banach ideal, where $\Psi(X) = \nu(s_b(X))^{(v)}$.

Proof. As $(ces)^{(v)}$ is a premodular $\mathfrak{ps}$, hence from Theorem 30, $\Psi$ is a prequasi norm on $B'_{(ces)^{(v)}}$. Suppose $(X_b)_{b \in N}$ is a Cauchy sequence in $B'_{(ces)^{(v)}}(\mathcal{P}, \mathcal{Q})$. As $B(\mathcal{P}, \mathcal{Q}) \supseteq B'_{(ces)^{(v)}}(\mathcal{P}, \mathcal{Q})$, one has

$$\Psi(X_a - X_b) = \sum_{l=0}^{\infty} \left( \frac{\sum_{i=0}^{l} s_{r} s_{r} (X_a - X_b)}{l+1} \right)^{v} \leq 2^{b_1 - 1} \sum_{l=0}^{\infty} \left( \frac{\sum_{i=0}^{l} s_{r} s_{r} (X_a - X_b)}{l+1} \right)^{v} + 2^{b_1 - 1} D_0 \sum_{l=0}^{\infty} \left( \frac{\sum_{i=0}^{l} s_{r} s_{r} (X_a - X_b)}{l+1} \right)^{v} < \infty.$$ 

Hence, $(s_b(X))_{b \in N}$ is a convergent sequence in $B(\mathcal{P}, \mathcal{Q})$. Since $B(\mathcal{P}, \mathcal{Q})$ is a Banach space, there is $X \in B(\mathcal{P}, \mathcal{Q})$ with $\lim_{b \to \infty} \|X_b - X\| = 0$. Since $(s_b(X))_{b \in N}^{(v)}$ for all $b \in N$, from Definition 15 parts (ii), (iii), and (v), one can see

$$\Psi(X) = \sum_{l=0}^{\infty} \left( \frac{\sum_{i=0}^{l} s_{r} s_{r} (X)}{l+1} \right)^{v} \leq 2^{b_1 - 1} \sum_{l=0}^{\infty} \left( \frac{\sum_{i=0}^{l} s_{r} s_{r} (X_a - X_b)}{l+1} \right)^{v} + 2^{b_1 - 1} \sum_{l=0}^{\infty} \left( \frac{\sum_{i=0}^{l} s_{r} s_{r} (X_a - X_b)}{l+1} \right)^{v} \leq 2^{b_1 - 1} D_0 \sum_{l=0}^{\infty} \left( \frac{\sum_{i=0}^{l} s_{r} s_{r} (X_a - X_b)}{l+1} \right)^{v} < \infty.$$ 

We get $(s_b(X))_{b \in N}^{(v)} \in (ces)^{(v)}$, so $X \in B'_{(ces)^{(v)}}(\mathcal{P}, \mathcal{Q})$.

4.3. Minimum Prequasi Ideal

Theorem 33. For any infinite dimensional Banach spaces $\mathcal{P}$, $\mathcal{Q}$ and $(t_i^{(1)}) \in \ell_\infty$, $(t_j^{(2)}) \in \ell_\infty$, with $1 < t_i^{(1)} < t_j^{(2)}$, for all $l \in N$, then

$$B'_{ces^{(v)}}(\mathcal{P}, \mathcal{Q}) \cup B'_{ces^+(v)}(\mathcal{P}, \mathcal{Q}).$$

Proof. Suppose $Z \in B'_{ces^+^{(v)}}(\mathcal{P}, \mathcal{Q})$, then $(s(Z)) \in (ces^+(v))^{(v)}$. One has

$$\sum_{i=0}^{\infty} \left( \frac{1}{l+1} \sum_{z=0}^{l} r^z s_{r} (Z) \right)^{v} < \sum_{i=0}^{\infty} \left( \frac{1}{l+1} \sum_{z=0}^{l} r^z s_{r} (Z) \right)^{v} < \infty.$$ 

Then, $Z \in B'_{ces^+^{(v)}}(\mathcal{P}, \mathcal{Q})$. After, if we choose $(s(Z))_{b \in N}^{(v)}$ so as

$$\sum_{i=0}^{\infty} \left( \frac{1}{l+1} \sum_{z=0}^{l} r^z s_{r} (Z) \right)^{v} < \infty$$

we have $Z \in B(\mathcal{P}, \mathcal{Q})$ such that
\[
\sum_{l=0}^{\infty} \left( \frac{1}{l+1} \sum_{r=0}^{l} r^2 s_r(Z) \right) t^{(i)}_{l} = \sum_{l=0}^{\infty} \frac{1}{l+1} = \infty,
\]
\[
\sum_{l=0}^{\infty} \left( \frac{1}{l+1} \sum_{r=0}^{l} r^2 s_r(Z) \right) t^{(2)}_{l} = \sum_{l=0}^{\infty} \left( \frac{1}{l+1} \right) t^{(2)}_{l} < \infty.
\] (36)

So \( X \notin \mathbb{B}^s_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q}) \) and \( X \in \mathbb{B}^s_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q}) \). Clearly, \( \mathbb{B}^s_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q}) \subset \mathbb{B}(\mathcal{P}, \mathcal{Q}) \). Next, if we take \( (s_l(X))_{\eta}^{\infty} \) such that \( \sum_{l=0}^{\infty} r^2 s_r(Z) = \frac{l+1}{\sqrt{l+1}} \). We have \( X \in \mathbb{B} (\mathcal{P}, \mathcal{Q}) \) so that \( X \notin \mathbb{B}^s_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q}) \). This confirms the proof.

**Theorem 34.** Pick up any infinite dimensional Banach spaces \( \mathcal{P}, \mathcal{Q} \) and \( (t_1) \in \mathfrak{F} \cap \ell_{\infty} \) with \( t_0 > 1 \); hence, \( \mathbb{B}^a_{\{\langle\rangle\}} \) is minimum.

**Proof.** Assume the setup is confirmed. So \( (\mathbb{B}^a_{\{\langle\rangle\}}, \Psi) \), where

\[\Psi(Z) = \sum_{l=0}^{\infty} \left( \frac{1}{l+1} \sum_{r=0}^{l} r^2 \alpha_r(Z) \right)^t,\]

is a quasi Banach ideal. Let \( \mathbb{B}^a_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q}) = \mathbb{B} (\mathcal{P}, \mathcal{Q}) \); hence, there is \( \eta > 0 \) so as \( \Psi(Z) \leq \eta \|Z\| \), for each \( Z \in \mathbb{B}(\mathcal{P}, \mathcal{Q}) \). Then, by Dvoretzky’s theorem [33] with \( b \in N \), one has quotient spaces \( \mathcal{P}/Y_b \) and subspaces \( M_b \) of \( \mathcal{Q} \) which can be mapped onto \( \ell_2^b \) by isomorphisms \( V_b \) and \( X_b \) with \( \|V_b\| \|V_b^{-1}\| \leq 2 \) and \( \|X_b\| \|X_b^{-1}\| \leq 2 \). If \( I_b \) is the identity map on \( \ell_2^b \), \( T_b \) is the quotient map from \( \mathcal{P} \) onto \( \mathcal{P}/Y_b \) and \( I_b \) is the natural embedding map from \( M_b \) into \( \mathcal{Q} \). Assume \( m_2 \) be the Bernstein numbers [34]; hence,

\[
1 = m_2(I_b) = m_2(X_b X_b^{-1} I_b V_b V_b^{-1}) \\
\leq \|X_b\| m_2(X_b X_b^{-1} I_b V_b V_b^{-1}) \|V_b^{-1}\| \\
= \|X_b\| m_2(J_b X_b^{-1} I_b V_b V_b^{-1}) \|V_b^{-1}\| \\
= \|X_b\| d_2(J_b X_b^{-1} I_b V_b V_b^{-1}) \|V_b^{-1}\| \\
\leq \|X_b\| \alpha_2(J_b X_b^{-1} I_b V_b V_b^{-1}) \|V_b^{-1}\|,
\] (37)

for \( 0 \leq l \leq b \). We have

\[
\sum_{l=0}^{1} r^2 \leq \sum_{l=0}^{1} \|X_b\| \|r^2 \alpha_2(J_b X_b^{-1} I_b V_b V_b^{-1}) \|V_b^{-1}\| \|V_b^{-1}\| \
\leq (\|X_b\| \|V_b^{-1}\|) t^{(i)}_{l} \left( \sum_{l=0}^{b} \frac{r^2 \alpha_2(J_b X_b^{-1} I_b V_b V_b^{-1})}{l+1} \right)^t.
\] (38)

Hence, for some \( \rho \geq 1 \), one has

\[
b + 1 \leq \rho \|X_b\| \|V_b^{-1}\| \left( \sum_{l=0}^{b} \frac{r^2 \alpha_2(J_b X_b^{-1} I_b V_b V_b^{-1})}{l+1} \right)^t \\
\leq \rho \|X_b\| \|V_b^{-1}\| \|J_b X_b^{-1} I_b V_b V_b^{-1}\| \Rightarrow b + 1 \\
\leq \rho \|X_b\| \|V_b^{-1}\| \|J_b X_b^{-1} I_b V_b V_b^{-1}\| \Rightarrow b + 1 \\
\leq \rho \|X_b\| \|V_b^{-1}\| \|J_b X_b^{-1} I_b V_b V_b^{-1}\| \|V_b T_b\| \\
= \rho \|X_b\| \|V_b^{-1}\| \|X_b^{-1} I_b V_b V_b^{-1}\| \|V_b T_b\| \leq 4 \rho \eta.
\] (39)

We have an inconsistency, as \( b \) is an arbitrary. Then, \( \mathcal{P} \) and \( \mathcal{Q} \) both cannot be infinite dimensional when \( \mathbb{B}^a_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q}) = \mathbb{B}(\mathcal{P}, \mathcal{Q}) \). This completes the proof.

**Theorem 35.** Upon any infinite dimensional Banach spaces \( \mathcal{P}, \mathcal{Q} \) and \( (t_1) \in \mathfrak{F} \cap \ell_{\infty} \) with \( t_0 > 1 \), then \( \mathbb{B}^d_{\{\langle\rangle\}} \) is minimum.

4.4. Simple Banach Prequasi Ideal

**Theorem 36.** Presume \( \mathcal{P}, \mathcal{Q} \) be infinite dimensional Banach spaces. Let \( (t_1^{(1)}) \in \ell_{\infty} \) and \( (t_1^{(2)}) \in \ell_{\infty} \) with \( 1 < t_1^{(1)} < t_1^{(2)} \), with \( l \in N \), then

\[
\mathcal{P} \left( \mathbb{B}^d_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q}), \mathbb{B}^d_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q}) \right) = \mathcal{P} \left( \mathbb{B}^d_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q}), \mathbb{B}^d_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q}) \right)
\]

(40)

**Proof.** For \( X \in \mathbb{B}^d_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q}), \mathbb{B}^d_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q}) \) and \( X \notin \mathcal{P} (\mathbb{B}^d_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q}), \mathbb{B}^d_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q})) \). From Lemma 1, one has \( Y \in \mathbb{B}^d_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q}) \) and \( Z \in \mathbb{B}^d_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q}) \) with \( ZXY = I_b \). Therefore, for each \( b \in N \), we have

\[
\|I_b\|_{\mathbb{B}^d_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q})} = \sum_{l=0}^{\infty} \left( \frac{\sum_{l=0}^{b} \frac{r^2 \alpha_2(J_b X_b^{-1} I_b V_b V_b^{-1})}{l+1} \right)^t \\
\leq \|ZXY\| \|I_b\|_{\mathbb{B}^d_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q})} \\
\leq \sum_{l=0}^{\infty} \left( \frac{\sum_{l=0}^{b} \frac{r^2 \alpha_2(J_b X_b^{-1} I_b V_b V_b^{-1})}{l+1} \right)^t.
\]

This defies Theorem 33. Then, \( X \in \mathcal{P} (\mathbb{B}^d_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q}), \mathbb{B}^d_{\{\langle\rangle\}} (\mathcal{P}, \mathcal{Q})) \), which affects the proof.
Corollary 37. Upon any infinite dimensional Banach spaces $\mathcal{P}$ and $\mathcal{Q}$, if $(t_{i}^{(1)}) \in \ell_{\infty}$ and $(t_{i}^{(2)}) \in \ell_{\infty}$ with $1 < t_{i}^{(1)} < t_{i}^{(2)}$, for every $l \in \mathbb{N}$, then

$$
\mathcal{H} \left( \mathbb{B}_{\ell_{\infty}}^{s} \left( \mathcal{P}, \mathcal{Q} \right) ; \mathcal{P} \text{ and } \mathcal{Q} \text{ are Banach spaces} \right) = \mathcal{H} \left( \mathbb{B}_{\ell_{\infty}}^{s} \left( \mathcal{P}, \mathcal{Q} \right) ; \mathcal{P} \text{ and } \mathcal{Q} \text{ are Banach spaces} \right).
$$

Proof. Easily, as $\mathcal{A} \subset \mathcal{H}$.

Theorem 38. Raise up $(t_{i}) \in \mathbb{N} \cap \ell_{\infty}$ with $t_{0} > 1$, and then,

$$
\mathbb{B}_{\ell_{\infty}}^{s} \left( \mathcal{P}, \mathcal{Q} \right) \text{ be simple.}
$$

Proof. Let the closed ideal $\mathcal{H}(\mathbb{B}_{\ell_{\infty}}^{s} \left( \mathcal{P}, \mathcal{Q} \right))$ include a map $X \notin \mathcal{H}(\mathbb{B}_{\ell_{\infty}}^{s} \left( \mathcal{P}, \mathcal{Q} \right))$. From Lemma 1, one has $Y, Z \in \mathbb{B}(\mathbb{B}_{\ell_{\infty}}^{s} \left( \mathcal{P}, \mathcal{Q} \right))$ with $XZYI_{b} = I_{b}$. This gives that $I_{b} = \mathbb{B}(\mathbb{B}_{\ell_{\infty}}^{s} \left( \mathcal{P}, \mathcal{Q} \right))$. Accordingly, $\mathbb{B}(\mathbb{B}_{\ell_{\infty}}^{s} \left( \mathcal{P}, \mathcal{Q} \right))$ is a simple Banach space.

4.5. Eigenvalues of $s$-Type Mappings

Notation 39.

Theorem 40. Pick up any infinite dimensional Banach spaces $\mathcal{P}$ and $\mathcal{Q}$. Suppose $(t_{i}) \in \mathbb{N} \cap \ell_{\infty}$ with $t_{0} > 1$, then

$$
\left( \mathbb{B}_{\ell_{\infty}}^{s} \left( \mathcal{P}, \mathcal{Q} \right) \right)^{s} \left( \mathcal{P}, \mathcal{Q} \right) = \mathbb{B}_{\ell_{\infty}}^{s} \left( \mathcal{P}, \mathcal{Q} \right).
$$

Proof. Let $X \in \left( \mathbb{B}_{\ell_{\infty}}^{s} \left( \mathcal{P}, \mathcal{Q} \right) \right)^{s} \left( \mathcal{P}, \mathcal{Q} \right)$; hence, $(\rho_{l}(X))_{l=0}^{\infty} \in \left( \mathbb{B}_{\ell_{\infty}}^{s} \left( \mathcal{P}, \mathcal{Q} \right) \right)^{s}$ and $|X - \rho_{l}(X)| > 0$, for all $l \in \mathbb{N}$. We have $X = \rho_{l}(X)I$, with $l \in \mathbb{N}$, so $s_{l}(X) = s_{l}(\rho_{l}(X)I) = |\rho_{l}(X)|$, with $l \in \mathbb{N}$. Therefore, $(s_{l}(X))_{l=0}^{\infty} \in \left( \mathbb{B}_{\ell_{\infty}}^{s} \left( \mathcal{P}, \mathcal{Q} \right) \right)^{s}$.

Secondly, let $X \in \mathbb{B}_{\ell_{\infty}}^{s} \left( \mathcal{P}, \mathcal{Q} \right)$. Therefore, $(s_{l}(X))_{l=0}^{\infty} \in \left( \mathbb{B}_{\ell_{\infty}}^{s} \left( \mathcal{P}, \mathcal{Q} \right) \right)^{s}$. Hence, we have

$$
\sum_{i=0}^{\infty} \frac{t_{l}^{(1)}}{t_{l}^{(2)}} s_{l}(X) \geq \sum_{i=0}^{\infty} |s_{l}(X)|^{t_{l}}.
$$

5. Conclusion

In this article, we explain some topological and geometric structure, of the multiplication maps defined on $(\text{ces}^{(t)}_{\ell_{\infty}})$, of the class $\mathcal{B}_{\ell_{\infty}}^{s}$, and of the class $\mathcal{B}_{\ell_{\infty}}^{s}$, of nonabsolute type.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors’ Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

This work was funded by the University of Jeddah, Saudi Arabia, under grant no. UI-20-078-DR. The authors,
therefore, acknowledge with thanks the university’s technical and financial support.

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