Random variables as pathwise integrals with respect to fractional Brownian motion✩

Yuliya Mishura*, Georgiy Shevchenko#,∗, Esko Valkeila#

*Kiev National Taras Shevchenko University, Department of Mechanics and Mathematics, Volodomirska 60, 01601 Kiev, Ukraine
#Aalto University, Department of Mathematics and Systems Analysis, P.O. Box 11100, FI-00076 Aalto, Finland

Abstract

We show that a pathwise stochastic integral with respect to fractional Brownian motion with an adapted integrand $g$ can have any prescribed distribution, moreover, we give both necessary and sufficient conditions when random variables can be represented in this form. We also prove that any random variable is a value of such integral in some improper sense. We discuss some applications of these results, in particular, to fractional Black–Scholes model of financial market.

Keywords: fractional Brownian motion, pathwise integral, generalized Lebesgue–Stieltjes integral, arbitrage, replication, divergence integral

2010 MSC: 60G22, 60H05, 60G15, 91G10

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space endowed with a $P$-complete left-continuous filtration $\mathcal{F} = \{\mathcal{F}_t, t \in [0, 1]\}$. On this stochastic basis we consider a fractional Brownian motion $B^H$ with a Hurst parameter $H > \frac{1}{2}$, that is an $\mathcal{F}$-adapted centered Gaussian process with the covariance function

$$
\mathbb{E} \left[ B^H_t B^H_s \right] = \frac{1}{2} \left( s^{2H} + t^{2H} - |s-t|^{2H} \right), \quad s, t \in [0, 1].
$$

Fractional Brownian motion is a popular model for long-range dependence in financial mathematics, economics and natural sciences. It is well known that $B^H$ has a continuous modification, and from now on we will assume that this

✩Yuliya Mishura thanks Aalto University School of Science for partial support of this research. Georgiy Shevchenko and Esko Valkeila were partially supported by the Academy of Finland, grant no. 21245.

∗Corresponding author

Email addresses: myus@univ.kiev.ua (Yuliya Mishura), zhora@univ.kiev.ua (Georgiy Shevchenko)
modification is chosen. For more information on fractional Brownian motion, see [8].

Dudley [4] showed that any functional $\xi$ of a standard Wiener process $W = \{W_t, t \in [0, 1]\}$ can be represented as an Itô stochastic integral $\int_0^1 \psi_t dW_t$, where $\psi$ is adapted to the natural filtration of $W$ and $\int_0^1 \psi_t^2 dt < \infty$ a.s. On the other hand, under an additional assumption $\int_0^1 E[\psi_t^2] dt < \infty$, only centered random variables with finite variances can be represented in this form and moreover $\psi$ is unique in this representation.

In this paper we study such questions for fractional Brownian motion. The integral we use is a (generalized) Lebesgue–Stieltjes integral, which is defined in a pathwise sense. Although the definition of the integral differs from the one for Wiener process, the results are similar in spirit to those of [4]. Precisely, our findings are as follows. We prove first that for any given distribution $F$ there exists an adapted process $\zeta$ such that $\int_0^1 \zeta_t dB^H_t$ has distribution $F$, i.e. $\xi$ can be represented as the integral $\int_0^1 g_t dB^H_t$, understood in an improper sense. We moreover show that $\xi = \int_0^1 g_t dB^H_t$ in a proper sense under additional assumption that $\xi$ is the final value of a Hölder continuous $\mathcal{F}$-adapted process. In addition, if $g$ is continuous, then this condition is not only sufficient, but also necessary. As a financial implication of these results, we get that in the fractional Black–Scholes model there exists a strong arbitrage and any contingent claim may be weakly hedged with an arbitrary hedging cost.

The paper has the following structure. In Section 2 we give some preliminaries on pathwise integration with respect to fBm. In Section 3 the main results are presented. In Section 4 we discuss applications of the results to mathematical finance and to zero integral problem, from which this research originates.

2. Preliminaries

We understand the integral with respect to fractional Brownian motion in a pathwise sense and define it as the generalized fractional Lebesgue–Stieltjes integral (see [11, 12]).

Consider two continuous functions $f$ and $g$, defined on some interval $[a, b] \subset [0, 1]$. For $\alpha \in (0, 1)$ define fractional derivatives

$$
(D_{a+}^\alpha f)(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(x-a)\alpha} + \alpha \int_a^x \frac{f(x) - f(u)}{(x-u)^{1+\alpha}} du \right) 1_{(a,b)}(x),
$$

$$
(D_{b-}^{1-\alpha} g)(x) = \frac{e^{i\pi\alpha}}{\Gamma(\alpha)} \left( \frac{g(x)}{(b-x)^{1-\alpha}} + (1-\alpha) \int_x^b \frac{g(x) - g(u)}{(x-u)^{2-\alpha}} du \right) 1_{(a,b)}(x).
$$

Assume that $D_{a+}^\alpha f \in L_p[a,b]$, $D_{b-}^{1-\alpha} g_{b-} \in L_q[a,b]$ for some $p \in (1, 1/\alpha)$, $q = p/(p-1)$, where $g_{b-}(x) = g(x) - g(b)$. 


Under these assumptions, the generalized Lebesgue–Stieltjes integral is defined as
\[ \int_a^b f(x)dg(x) = e^{-i\pi\alpha} \int_a^b (D_{a+}^{\alpha} f)(x)(D_{b-}^{1-\alpha} g)(x)dx. \]

It was shown in [11] that for any \( \alpha \in (1-H, 1) \) there exists the fractional derivative \( D_{a+}^{\alpha} f \in L_\infty[a,b] \). Hence, for \( f \) with \( D_{a+}^{\alpha} f \in L_1[a,b] \) we can define the integral w.r.t. fBm according to this formula:
\[ \int_a^b f_s dB^H_s = e^{-i\pi\alpha} \int_a^b (D_{a+}^{\alpha} f)(x)(D_{b-}^{1-\alpha} B^H)(x)dx. \]

In view of this, we will consider the following norm for \( \alpha \in (1-H, 1/2) \):
\[ \|f\|_{1,\alpha,[a,b]} = \int_a^b \left( \frac{|f(s)|}{(s-a)^{\alpha}} + \int_a^s \frac{|f(s) - f(z)|}{(s-z)^{1+\alpha}}dz \right) ds. \]

For simplicity we will abbreviate \( \|\cdot\|_{t,\alpha} = \|\cdot\|_{1,\alpha,[0,t]} \). The following estimate for \( t \leq 1 \) is clear:
\[ \left| \int_0^t f(s)dB^H_s \right| \leq K_\alpha(\omega)\|f\|_{t,\alpha}, \]
where \( K_\alpha(\omega) = \sup_{0 \leq u < s \leq 1} |D_{u-}^{1-\alpha} B^H_s| < \infty \) a.s.

We will need the following version of Itô formula for fractional Brownian motion. It was proved in [1] for convex functions \( F \), but a careful analysis of the proof shows that the following result is true.

**Theorem 2.1.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a function of locally bounded variation, \( F(x) = \int_0^x f(y)dy \). Then for any \( \alpha \in (1-H, 1/2) \) \( \|f(B^H_t)\|_{\alpha,1} < \infty \) a.s. and
\[ F(B^H_t) = \int_0^t f(B^H_s)dB^H_s. \]

Throughout the paper all unimportant constants will be denoted by \( C \), and their value may change from line to line. Random constants will be denoted by \( C(\omega) \).

3. Main results

3.1. Auxiliary construction

In this section we construct the essential ingredient of some results: an adapted process such that w.r.t. fractional Brownian motion it integrates (in improper sense) to infinity.

The key fact is the following well-known small ball estimate for fractional Brownian motion (see e.g. [2]): there is a constant \( c > 0 \), independent of \( \epsilon \) and \( T \), such that
\[ P \left( \sup_{t \in [0,T]} |B^H_t| < \epsilon \right) \leq e^{-cT\epsilon^{-1/H}} \text{ for } \epsilon \leq T^H. \]
Lemma 3.1. There exists an $\mathbb{F}$-adapted process $\varphi = \{\varphi_t, t \in [0,1]\}$ such that

- For any $t < 1$ and $\alpha \in (1 - H, 1/2)$ $\|\varphi\|_{\alpha,t} < \infty$ a.s., so integral $v_t = \int_0^t \varphi_s dB^H_s$ exists as a generalized Lebesgue–Stieltjes integral.

- $\lim_{t \to 1^-} v_t = \infty$ a.s.

Proof. Fix arbitrary $\gamma \in (1, 1/H)$ and $\beta \in (0, \frac{1}{\gamma H} - 1)$. Denote $\Delta_n = n^{-\gamma}/\zeta(\gamma)$, $\zeta(\gamma) = \sum_{n \geq 1} n^{-\gamma}$, and define $t_0 = 0$, $t_n = \sum_{k=1}^{n} \Delta_k$, $n \geq 1$, so that $t_n \to 1-$, $n \to \infty$. Denote also $f_\beta(x) = (1 + \beta) x^\beta \text{sign} x$, so that $\int_0^t f_\beta(z) dz = |x|^{1+\beta}$, $x \in \mathbb{R}$.

Let $\tau_n = \min \left\{ t \geq t_{n-1} : \|B^H_t - B^H_{t_{n-1}}\|^{1/(1+\beta)} \geq n^{-1/(1+\beta)} \right\}$ and define

$$\varphi_t = \sum_{n=1}^{\infty} f_\beta(B^H_t - B^H_{t_n}) 1_{[\tau_n, \tau_{n+1})}(t).$$

First we establish estimate $\|\varphi\|_{\alpha,t} < \infty$ a.s. To that end, note that fractional Brownian motion $B^H$ is almost surely bounded on $[0,1]$ and write $\|\varphi\|_{\alpha,t_n} = I_1 + I_2$, where

$$I_1 = \int_0^{t_{n-1}} |\varphi_s| s^{-\alpha} ds \leq C(\omega),$$

$$I_2 = \int_0^t \int_0^s |\varphi_s - \varphi_u| (s-u)^{-1-\alpha} du ds$$

$$= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left( \int_0^{t_{k-1}} + \int_{t_{k-1}}^s \right) |\varphi_s - \varphi_u| (s-u)^{-1-\alpha} du ds.$$

Now estimate

$$\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \int_0^{t_{k-1}} |\varphi_s - \varphi_u| (s-u)^{-1-\alpha} du ds$$

$$\leq C(\omega) \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (s-u)^{-1-\alpha} du ds$$

$$\leq C(\omega) \sum_{k=1}^{n} (t - t_{k-1})^{-\alpha} ds \leq C(\omega) \sum_{k=1}^{n} \Delta_k^{-\alpha} < \infty,$$

Finally,

$$\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \int_0^{t_{k-1}} |\varphi_s - \varphi_u| (s-u)^{-1-\alpha} du ds$$

$$= \int_{\tau_n}^{t_n} \int_0^{\tau_k} |f_\beta(B^H_u - B^H_{t_k})| (s-u)^{-1-\alpha} du ds + I_k$$

$$\leq C(\omega) \int_{\tau_{k-1}}^{t_k} (s - \tau_k)^{-\alpha} ds + I_k \leq C(\omega) + I_k,$$
where

\[ I_k = \int_{t_{k-1}}^{T_k} \int_{t_{k-1}}^{t_k} \left| f_\beta(B^H_s - B^H_t) - f_\beta(B^H_u - B^H_{t_k}) \right| (s-u)^{-\alpha} ds \]

is finite almost surely by Theorem 2.1.

Now by the Itô formula, for \( t \in [t_{n-1}, t_n) \)

\[ v_t = \int_0^t \varphi_s dB^H_s = \sum_{k=1}^{n-1} \left| \Delta B^H_k \right|^{1+\beta} + \left| B^H_{t \wedge \tau_n} - B^H_{t_{n-1}} \right|^{1+\beta}, \]

where \( \Delta B^H_k = B^H_{t_k} - B^H_{t_{k-1}}, k \geq 1 \). It is easy to see that \( v_t \geq v_{t_n} \) for \( t \geq t_n \), so in order to prove that \( v_t \to \infty \), \( t \to 1 \), it is enough to show that \( v_{t_n} \to \infty \), \( n \to \infty \), which in turn is equivalent to \( \sum_{n=1}^{\infty} \left| \Delta B^H_n \right|^{1+\beta} = \infty \).

Observe that \( \left| \Delta B^H_n \right|^{1+\beta} \geq 1/n \) provided that \( \tau_n < t_n \). Therefore, defining

\[ A_n = \left\{ \sup_{t \in [t_{n-1}, t_n]} \left| B^H_t - B^H_{t_{n-1}} \right| < n^{-1/(1+\beta)} \right\}, n \geq 1, \]

it is enough to show that almost surely only finite number of the events \( A_n \) happens. Using the small ball estimate (3.1) and stationarity of increments of \( B^H \), we obtain

\[ P(A_n) = \mathbb{P} \left( \sup_{t \in [0, \Delta_n]} \left| B^H_t \right| < n^{-1/(1+\beta)} \right) \leq \exp \left\{ -c(\gamma)^{-1} n^{-\gamma + \frac{1}{2(1+\beta)}} \right\}, \]

so \( \sum_{n \geq 1} P(A_n) < \infty \) since \( \frac{1}{2(1+\beta)} > \gamma \). Thus, we get the desired statement from the Borel-Cantelli lemma.

**Remark 3.1.** It is easy to see that \( \| \varphi \|_{\alpha, t} < \infty \) even for a random \( t = t(\omega) < 1 \).

### 3.2. Stochastic integral with respect to fBm can have any distribution

The following result is about representation, not of a random variable, but rather of a distribution. From the financial point of view, it means that an investor can get any desired risk profile, using a self-financing portfolio (see Theorem 4.2). The key for its proof is Lemma 3.1, the rest of the proof goes exactly as in 4.

**Theorem 3.1.** For any distribution function \( F \) there exists an adapted process \( \zeta \) such that \( \| \zeta \|_{\alpha, 1} < \infty \) and the distribution function of \( \int_0^1 \zeta_s dB^H_s \) is \( F \).

**Proof.** It is clear that there exists a non-decreasing function \( g : \mathbb{R} \to \mathbb{R} \) such that \( g(B^H_{1/2}) \) has distribution \( F \). So it is enough to construct an adapted process \( \zeta \) such that \( \int_0^1 \zeta_s dB^H_s = g(B^H_{1/2}) \). Let \( \varphi \) be the process constructed in Lemma 3.1.

\[ v_t = \int_{1/2}^t \varphi_s dB^H_s. \]

Define \( \tau = \min \left\{ t \geq 1/2 : v_t = \left| g\left(B^H_{1/2}\right) \right| \right\} \). Since \( v_t \to \infty \) as \( t \to 1 \) a.s., we have \( \tau < 1 \) a.s. Now put

\[ \zeta_t = \varphi_t \text{sign} g(B^H_{1/2}) 1_{[1/2, \tau]}(t). \]
We have
\[ \|\xi\|_{\alpha,1} = \|\varphi\|_{\alpha,\tau} + \int_0^1 \int_0^{\tau} \frac{|\zeta_s|}{(t-s)^{1+\alpha}} ds \, dt \leq C(\omega) + C(\omega) \int_0^1 (t-\tau)^{-\alpha} \, dt < \infty. \]

Obviously,
\[ \int_0^1 \zeta_s dB^H_s = g(B^H_{1/2}), \]
whence the statement follows.

**Remark 3.2.** Observe that the process \( \zeta \) is adapted not only to the filtration \( F \), but also to the natural filtration of \( B^H \); moreover, this process is piecewise Hölder continuous, which implies that the integral is not only well-defined in the generalized Lebesgue–Stieltjes sense, but it is also a limit of integral sums.

**3.3. Any random variable is a stochastic integral in improper sense**

**Theorem 3.2.** For any \( \mathcal{F}_t \)-measurable variable \( \xi \) there exists an \( \mathcal{F} \)-adapted process \( \psi \) such that

- For any \( t < 1 \) and \( \alpha \in (1 - H, 1/2) \) \( \|\psi\|_{\alpha,t} < \infty \) a.s.
- \( \lim_{t \to 1} \int_0^t \psi_s dB^H_s = \xi \) a.s.

**Proof.** Process \( z_t = \tan E[\arctan \xi | \mathcal{F}_t] \) is \( \mathcal{F} \)-adapted and \( z_t \to \xi, t \to 1^- \), a.s. by the left continuity of \( \mathcal{F} \) and the martingale convergence.

Let \( \{t_n, n \geq 1\} \) be arbitrary increasing sequence of points from \([0, 1]\) converging to 1.

By Lemma 3.1 there exists an \( \mathcal{F} \)-adapted process \( \varphi^n \) on \([t_n, t_{n+1}]\) such that
\[ v^n_t = \int_{t_n}^t \varphi^n_s dB^H_s \to +\infty, t \to t_{n+1}^- . \]

Now denote \( \delta_n = z_{t_n} \) and \( \delta_n = \xi_n - \xi_{n-1}, n \geq 2, \delta_1 = \xi_1 \). Take \( \tau_n = \min \{t \geq t_n : v^n_t = |\delta_n|\} \) and define
\[ \psi_t = \sum_{n \geq 1} \varphi^n_{\tau_n} 1_{[t_n, \tau_n]}(t) \sign \delta_n, \quad x_t = \int_0^t \psi_s dB^H_s. \]

The finiteness of norm \( \|\psi\|_{\alpha,t} \) for \( t < 1 \) is proved as in Lemma 3.1 and Theorem 3.3. It is clear that \( x_{t_{n+1}} = \sum_{n=1}^{k-1} \delta_n = \xi_n \), so \( x_{t_{n+1}} \to \xi, n \to \infty \). Moreover, from the construction of process \( \psi \) it follows that for \( t \in [t_n, t_{n+1}] \) the value \( x_t \) is between \( \xi_{n-1} \) and \( \xi_n \), whence \( x_t \to \xi, t \to 1^- \).

**3.4. Which variables can be represented as stochastic integrals**

For some random variables \( \xi \) we can claim even more: the existence of an \( \mathcal{F} \)-adapted \( g_t \) such that \( \int_0^1 g_t dB^H_t \) is well-defined and is equal to \( \xi \). To establish the main result here, we need an auxiliary lemma.

**Lemma 3.2.** For \( 0 < s \leq t \leq 1 \)
\[ P(B^H_s B^H_t \leq 0) \leq C(t-s)^H t^{-H}. \]
Proof. As the distribution of $B^H$ is symmetric and continuous, it is enough to estimate $P(B^H_0 < 0 < B^H_t)$. By the self-similarity of $B^H$, 

$$P(B^H_0 < 0 < B^H_t) = P\left(\frac{s}{t} < 0 < \frac{s}{t} B^H_t\right).$$

If $s/t$ is small (less than $1/2$ say), then $|t-s|^H t^{-H} = |1-s/t|^H > 2^{-H}$, so (3.2) holds with $C = 2^H$. Thus, we only have to consider $s/t$ close to 1. Denote $u = s/t, \rho(u) = u^{-H} E \left[ B^H_u B^H_1 \right] = \left( u^{-H} + u^H - u^{-H} (1-u)^2 H \right)/2$. Write

$$1 - \rho(u) = u^{-H} \left( (1-u)^2 H - (1-u^H)^2 \right) \leq C(1-u)^2 H,$$

so we can estimate

$$P(B^H_0 < 0 < B^H_1) = \frac{1}{2\pi \left(1 - \rho(u)^2\right)^{1/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left\{ -\frac{x^2 + y^2 - 2\rho(u)xy}{2(1 - \rho(u)^2)} \right\} \, dx \, dy$$

$$= \frac{(1 - \rho(u)^2)^{1/2}}{2\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} \exp \left\{ -(x^2 + y^2 - 2\rho(u)xy)/2 \right\} \, dx \, dy$$

$$\leq C(1 - \rho(u))^{1/2} \int_{0}^{\infty} \int_{-\infty}^{\infty} \exp \left\{ -(x-y)^2/2 \right\} \, dx \, dy \leq C(1-u)^H.$$

A simple observation that $(1-u)^H = t^{-H} (t-s)^H$ concludes the proof. \qed

Theorem 3.3. Let for a random variable $\xi$ there exist a number $\alpha > 0$ and an $\mathbb{F}$-adapted almost surely $\alpha$-Hölder continuous process $\{z_t, t \in [0,1]\}$ such that $z_1 = \xi$. Then for any $\alpha \in (1 - H, (1 - H + a) \wedge 1/2)$ there exists an $\mathbb{F}$-adapted process $\psi$ such that $||\psi||_{\alpha,1} < \infty$ and $\int_{0}^{1} \psi_t dB^H_t = \xi$.

Proof. We can assume without loss of generality that $a < H$.

Step 1. Construction. Take some $\gamma > (1 - \alpha - H + a)^{-1} > 1$ and put $\Delta_n = n^{-\gamma}/\gamma(\gamma), t_0 = 0, t_n = \sum_{k=1}^{n} \Delta_k, n \geq 1$. For brevity, denote $\xi_n = z_{t_n}, \delta_n = |\xi_n - \xi_{n-1}|$.

We construct process $\psi$ recursively on intervals $(t_n, t_{n+1}]$. First, set $\psi_t = 0$ for $t \in [t_0, t_1)$ and fix some $\kappa \in \gamma(H-a), \gamma(1-\alpha)-1$. This is possible because $\gamma(H-a) < \gamma(1-\alpha)-1$ thanks to the choice of $\gamma$. Observe also that $a > H - \kappa/\gamma$ with such choice of $\kappa$.

Denote $\psi_t = \int_{0}^{t} \psi_t dB^H_t$. If $\psi$ is constructed on $[t_0, t_{n-1}]$ for some $n \geq 2$, we will show how to construct it on $(t_{n-1}, t_n)$. To this end, consider two cases.

Case A: $\psi_{t_{n-1}} = \xi_{n-2}$. Define

$$\tau_n = \min \left\{ t \geq t_{n-1} : n^\kappa \left| B^H_t - B^H_{t_{n-1}} \right| = \delta_{n-1} \right\} \wedge t_n$$

and set

$$\psi_t = n^\kappa \operatorname{sign}(B^H_t - B^H_{t_{n-1}}) \operatorname{sign}(\xi_{n-1} - \xi_{n-2}) 1_{t \leq \tau_n}$$

for $t \in [t_{n-1}, t_n)$. By the Itô formula,

$$v_{t_n} = v_{t_{n-1}} + n^\kappa \left| B^H_{t_n} - B^H_{t_{n-1}} \right| \operatorname{sign}(\xi_{n-1} - v_{t_{n-1}}).$$
so we have $v_{t_n} = \xi_{n-1}$ provided $\tau_n < t_n$.

Case B $v_{t_n-1} \neq \xi_{n-2}$. In this case we use a construction similar to that of Theorem 3.2. Namely, let $\varphi^n_t$ be an adapted process on $[t_{n-1}, t_n]$ such that $v^n_t := \int_{t_{n-1}}^t \varphi^n_s dB^H_s \to \infty$, $t \to t_n-$, define $\tau_n = \min \{ t \geq t_{n-1} : v^n_t = |\xi_{n-1} - v_{t_n-1}| \}$ and set $\psi_t = \varphi^n_t \operatorname{sign}(\xi_{n-1} - v_{t_n-1}) 1_{t \leq \tau_n}$ for $t \in [t_{n-1}, t_n)$. Then $v_{t_n} = \xi_{n-1}$.

Step 2. We argue that almost surely there is $N(\omega)$ such that $v_{t_n} = \xi_{n-1}$ for $n \geq N(\omega)$. As in the proof of Lemma 3.1, we will use Borel–Cantelli lemma and the small ball estimate (3.1). For brevity, we will omit the phrase “almost surely” in the rest of the proof.

Define events

$$C_n = \left\{ \sup_{t \in [t_{n-1}, t_n]} n^\kappa |B^H_t - B^H_{t_{n-1}}| \leq \delta_{n-1}, n \geq 2. \right\}$$

We are going to show that only finite number of $C_n$ happens. Take some $b \in (H - \kappa/\gamma, a)$. By our assumption,

$$\delta_{n-1} = |z_{n-1} - z_{n-2}| \leq C(\omega) \Delta_{n-1}^a \leq C(\omega) \Delta_n^a.$$ 

There exists $N_1(\omega)$ such that $C(\omega) \Delta_n^a \leq \Delta_b^a$ for $n \geq N_1(\omega)$, therefore

$$\delta_{n-1} \leq \Delta_n^b \quad \text{for } n \geq N_1(\omega). \quad (3.3)$$

So it is enough to prove that only finite number of events

$$D_n = \left\{ \sup_{t \in [t_{n-1}, t_n]} n^\kappa |B^H_t - B^H_{t_{n-1}}| \leq \Delta_n^b \right\}.$$

happens. The increments of fBm $B^H$ are stationary, hence by the small ball estimate for $n$ sufficiently large

$$P(D_n) = P \left( \sup_{t \in [0, \Delta_n]} n^\kappa |B^H_t| < \Delta_n^b \right)$$

$$= P \left( \sup_{t \in [0, \Delta_n]} |B^H_t| < \zeta(\gamma)^{-b} n^{-b \gamma - \kappa} \right) \leq \exp \left\{ -c \zeta(\gamma)^{b/H-1} n^{-\gamma+(b \gamma + \kappa)/H} \right\}.$$

Since $b > H - \kappa/\gamma$, equivalently, $-\gamma+(b \gamma + \kappa)/H > 0$, we have that $\sum_{n=1}^{\infty} P(D_n) < \infty$. So by the Borel–Cantelli lemma, only finite number of events $D_n$ happens.

As we have already noted, this implies that the same is true for $C_n$. Thus, for some $N(\omega)$ we have $\sup_{t \in [t_{n-1}, t_n]} n^\kappa |B^H_t - B^H_{t_{n-1}}| > \delta_{n-1}$ for all $n \geq N(\omega)$. This implies that $v_{t_M} = \xi_{M-1}$ no matter whether we have Case A or B on $(t_{M-1}, t_M)$, moreover, we have Case A on $(t_{n-1}, t_n]$ and $v_{t_n} = \xi_{n-1}$ for all $n \geq N(\Omega) + 1$.

Step 3. Now we prove that $\|\psi\|_{\alpha,1} < \infty$ a.s.

Let for $n \geq 2$

$$A_n = \{ \text{We have Case A on the interval } [t_{n-1}, t_n] \} , \quad B_n = A^c_n.$$
Write $\psi_t = \psi_t^A + \psi_t^B$, where
\[
\psi_t^A = \psi_t \sum_{n=2}^{\infty} 1_{(t_{n-1}, t_n]}(t) 1_{A_n}
\]
and $\psi_t^B$ is defined similarly. By Step 2, only finite numbers of the events $B_n$ take place, hence the finiteness of $\|\psi_t^B\|_{\alpha,1}$ can be proved as in Lemma 3.1 and Theorem 3.1.

It remains to prove that $\|\psi_t^A\|_{\alpha,1} < \infty$. Clearly, it is enough to show that $E \left[ \|\psi_t^A\|_{\alpha,1} \right] < \infty$. Write
\[
E \left[ \|\psi_t^A\|_{\alpha,1} \right] = I_1 + I_2,
\]
where
\[
I_1 = \int_0^1 E \left[ |\psi_t| t^{-\alpha} \right] dt \leq C \sum_{n=2}^{\infty} t_{n-1} \int_{t_{n-1}}^{t_n} n^\kappa dt
\]
and $I_2$ is defined similarly. By Step 2, only finite numbers of the events $B_n$ take place, hence the finiteness of $\|\psi_t^B\|_{\alpha,1}$ can be proved as in Lemma 3.1 and Theorem 3.1.

Now estimate the terms individually, denoting $\sigma_n(t) = \text{sign}(B_t^H - B_{t_{n-1}}^H)$:
\[
\sum_{n=2}^{\infty} t_{n-1} \int_{t_{n-1}}^{t_n} E \left[ |\psi_t - \psi_s| (t-s)^{-1-\alpha} \right] ds dt
\]
\[
\leq C \sum_{n=2}^{\infty} n^\kappa \int_{t_{n-1}}^{t_n} (t-s)^{-1-\alpha} ds dt
\]
\[
\leq C \sum_{n=2}^{\infty} n^\kappa \int_{t_{n-1}}^{t_n} (t-s)^{-1-\alpha} ds dt
\]
\[
I_2' := \sum_{n=2}^{\infty} t_{n-1} \int_{t_{n-1}}^{t_n} E \left[ |\psi_t - \psi_s| (t-s)^{-1-\alpha} \right] ds dt
\]
\[
\leq \sum_{n=2}^{\infty} n^\kappa \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t} E \left[ |\sigma_n(t) 1_{t \leq \tau_n} - \sigma_n(s) 1_{s \leq \tau_n} | (t-s)^{-1-\alpha} \right] dt ds
\]
\[
\leq \sum_{n=2}^{\infty} n^\kappa \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t} \left[ P((B_s^H - B_{t_{n-1}}^H)(B_t^H - B_{t_{n-1}}^H) \leq 0) + E[1_{s \leq \tau_n \leq t}] \right] (t-s)^{1+\alpha} ds dt.
\]
By the stationarity of fBm increments and Lemma 3.2
\[ P\left( (B_s^H - B_t^H)(B_t^H - B_{t_n}^H) \leq 0 \right) \leq C(t - t_{n-1})^{-H} (t - s)^H. \]

Also observe that
\[
\int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t} 1_{s \leq \tau_n \leq t} (t - s)^{-1-\alpha} ds \, dt \leq C \int_{\tau_n}^{t_n} (t - \tau_n)^{-\alpha} \leq C(t_n - \tau_n)^{1-\alpha} \leq C\Delta_n^{1-\alpha}.
\]

Then we can continue estimation:
\[
R_2 \leq C \sum_{n=2}^{\infty} n^\kappa \left( \int_{t_{n-1}}^{t_n} (t - t_{n-1})^{-H} \int_{t_{n-1}}^{t} (t - s)^{H-1-\alpha} ds \, dt + \Delta_n^{1-\alpha} \right)
\]
\[
\leq C \sum_{n=2}^{\infty} n^\kappa \left( \int_{t_{n-1}}^{t_n} (t - t_{n-1})^{-\alpha} ds + n^{-\gamma(1-\alpha)} \right) \leq C \sum_{n=2}^{\infty} n^{\kappa - \gamma(1-\alpha)} < \infty.
\]

Concluding, \( \|\psi\|_{n,1} < \infty \), as required.

\[\square\]

Remark 3.3. It is easy to see that the assumption of Theorem 3.3 is equivalent to the following one: there exists a number \( a > 0 \), an increasing sequence \( \{t_n, n \geq 1\} \) of points converging to 1 and a sequence of random variables \( \{\xi_n, n \geq 1\} \) such that \( \xi_n \) is \( \mathcal{F}_{t_n} \)-measurable for any \( n \geq 1 \) and
\[
|\xi_n - \xi| = O(|t_n - 1|^a)
\]
a.s. as \( n \to \infty \). (Clearly, this condition is implied by the assumption of Theorem 3.3; vice versa it can be proved by a clever linear interpolation.)

A natural question is what random variables satisfy the assumption of Theorem 3.3. Below we give some examples of such random variables.

Example 3.1. \( \xi = F(B_{s_1}^H, \ldots, B_{s_k}^H) \), where \( F: \mathbb{R}^n \to \mathbb{R} \) is locally Hölder continuous with respect to each variable. In this case we can set \( z_t = F(B_{s_{1/\Delta t}}^H, \ldots, B_{s_{1/\Delta t}}^H) \), which is clearly Hölder continuous.

Example 3.2. \( \xi = G(\{B_s^H, s \in [0,1]\}) \), where \( G: C[0,1] \to \mathbb{R} \) is locally Hölder continuous with respect to the supremum norm on \( C[0,1] \). In the case one can set \( z_t = G(\{B_{s_{1/\Delta t}}^H, s \in [0,1]\}) \).

Example 3.3. \( \xi = 1_A, A \in \mathcal{F} \). Indeed, for some increasing sequence \( \{t_n, n \geq 1\} \) of points converging to 1, in view of the right continuity of \( F \), the set \( A \) can be approximated by some \( \mathcal{F}_{t_n} \)-measurable sets \( A_n \) in probability. Hence, some subsequence of the indicator functions \( 1_{A_n} \) (without loss of generality, the sequence itself) converges almost surely. But then \( |1_{A_n} - 1_A| = 0 \) a.s. for all \( n \) large enough, so \( 3.3 \) is obvious.

Consequently, any simple \( \mathcal{F} \)-measurable function also satisfies the assumption of Theorem 3.3.
In view of financial applications, the three examples given above and their transformations are enough, because they cover virtually all possible derivative securities: European options, Asian options, barrier options, lookback options, digital options etc.

Further we will show that the assumption of Theorem 3.3 is not only natural, but also is close to be a criterion: it is a necessary condition under additional assumption that \( \psi \) is continuous.

**Theorem 3.4.** Let \( \xi \) be an \( \mathcal{F}_t \)-measurable random variable and let there exist an \( \mathcal{F}_t \)-adapted continuous process \( \psi \) such that for some \( \alpha > 1 - H \) \( \| \psi \|_{\alpha, 1} < \infty \) a.s. and \( \int_0^t \psi_s dB^H_s = \xi \). Then the assumption of Theorem 3.3 is satisfied.

**Proof.** Thanks to the Garsia–Rodemich–Rumsey inequality [6], it follows from continuity of \( \psi \) and estimate \( \| \psi \|_{\alpha, 1} < \infty \) that \( \psi \) is almost surely Hölder continuous of any order \( a < \alpha \). We also know that \( B^H \) is almost surely Hölder continuous of any order \( b < H \). Then by the well known property of the Lebesgue–Stieltjes integral (which is Young integral in this situation), \( z_t = \int_0^t \psi_s dB^H_s \) is almost surely Hölder continuous of any order \( c < \alpha \). \( \mathcal{F}_t \)-adaptedness of \( z \) is obvious.

For completeness, we give the following example showing that there exist random variables which do not satisfy the assumption of Theorem 3.3 even in the case where the filtration \( \mathcal{F} \) is generated by \( B^H \).

**Example 3.4.** Assume that \( \mathcal{F} = \{ \mathcal{F}_t = \sigma(B^H_s, s \in [0, t]), t \in [0, 1] \} \). It is well known (see [4]) that there exists a Wiener process \( W \) such that its natural filtration coincides with \( \mathcal{F} \). Define \( \xi = \int_1^1 g(t) dW_t \), where \( g(t) = (1 - t)^{-1/2} |\log(1 - t)|^{-1} \). We will show that \( \xi \) does not satisfy the assumption of Theorem 3.3. Roughly, the idea is that the best, at least in the mean-square sense, \( \mathcal{F}_t \)-adapted approximation of \( \xi \) is \( z_t = \int_0^t g(t) dW_t \), but \( z \) is not Hölder continuous at 1.

Without loss of generality we assume that \( W \) is defined on the classical Wiener space, i.e. \( \Omega = C[0, 1] = \{ \omega(t), t \in [0, 1] \} \), \( W_t(\omega) = \omega(t) \), \( P \) is the Wiener measure, \( \mathcal{F}_t \) is the \( P \)-completion of the \( \sigma \)-algebra generated by events of the form \( \{ \omega(u) \in B \}, u \leq t, B \in \mathcal{B}(\mathbb{R}) \). Arguing by contradiction, put \( t_n = 1 - 1/(n + 1) \), \( n \geq 1 \) and let \( \{ \zeta_n, n \geq 1 \} \) be a sequence of random variables such that \( \zeta_n \) is \( \mathcal{F}_{t_n} \)-measurable for each \( n \geq 1 \), and for some \( a > 0 \)

\[
\sup_{n \geq 1} n^a |\xi(\omega) - \zeta_n(\omega)| < \infty \quad \text{a.s.}
\]

Decompose \( \xi = \zeta_n + \eta_n \), where \( \xi_n = \int_0^{t_n} f(t) dW_t \) is \( \mathcal{F}_{t_n} \)-measurable, \( \eta_n = \int_{t_n}^1 f(t) dW_t \) is independent of \( \mathcal{F}_{t_n} \). Then we have

\[
\sup_{n \geq 1} n^a |\eta_n(\omega) + \alpha_n(\omega)| < \infty \quad \text{a.s.},
\]

(3.5)
where $\alpha_n(\omega) = \xi_n - \zeta_n$ is $\mathcal{F}_{t_n}$-measurable. Define the following bijective transformation on $\Omega$:

$$
\psi_n(\omega)(t) = \begin{cases} 
\omega(t), & t \in [0, t_n], \\
2\omega(t_n) - \omega(t), & t \in (t_n, 1].
\end{cases}
$$

(we reflect the path after the point $t_n$). It is clear that $\psi_n$ is measurable and preserves the measure $P$. In particular, $\sup_{n \geq 1} n^a |\eta_n(\psi_n(\omega)) + \alpha_n(\psi_n(\omega))| < \infty$ a.s. It easy to check that $\eta_n(\psi_n(\omega)) = -\eta_n(\omega)$ a.s., and $\alpha_n(\psi_n(\omega)) = \alpha_n(\omega)$ a.s. due to $\mathcal{F}_{t_n}$-measurability. Therefore, $\sup_{n \geq 1} n^a |\eta_n(\omega) - \alpha_n(\omega)| < \infty$ a.s. Combining this relation with (3.5) and using the triangle inequality, we get

$$
M(\omega) := \sup_{n \geq 1} n^a |\eta_n(\omega)| + |\eta_n(\omega) - \alpha_n(\omega)| < \infty
$$
a.s. And since the family $\{\eta_n, n \geq 1\}$ is Gaussian, $M(\omega)$ has Gaussian tails thanks to Fernique’s theorem \[5\]. In particular, $E[M(\omega)^2] < \infty$. It follows that

$$
\sup_{n \geq 1} n^{2a} E[\eta_n^2] = \sup_{n \geq 1} \frac{n^{2a}}{\log(n + 1)} \leq E[M(\omega)^2] < \infty,
$$

which is absurd. Consequently, $\xi$ does not satisfy the assumption of Theorem 3.3.

4. Discussion and applications

4.1. Application to finance

On the time interval $[0, 1]$ consider a fractional Black–Scholes, or simply $(B, S)$-, market with a risky asset (stock) $S$ and a non-risky asset (bond) $B$, which solve the following equations:

$$
dB_t = r_t B_t dt,
$$

$$
BS_t = \mu S_t dt + \sigma S_t dB_t^H,
$$

equivalently, assuming $B_0 = 1$,

$$
B_t = \exp \left\{ \int_0^t r_s ds \right\},
$$

$$
S_t = S_0 \exp \left\{ \mu t + \sigma B_t^H \right\}. \tag{4.1}
$$

The interest rate $r$ can be random. For the technical simplicity we assume that it is absolutely bounded by a non-random constant.

Let $\mathbb{F}$ be the filtration generated by $B$ and $S$: $\mathcal{F}_t = \sigma \{ B_u, S_u, u \leq t \} = \sigma \{ B_u, B_u^H, u \leq t \}$. Due to continuity of $B$ and $B^H$, $\mathbb{F}$ is left-continuous.

We remind standard notions from financial mathematics.

**Definition 4.1.** A portfolio, or trading strategy, is an $\mathbb{F}$-predictable process $\Pi = (\Pi_t)_{t \in [0, 1]} = (\pi_t^0, \pi_t^1)_{t \in [0, 1]}$, where $\pi_t^0$ denotes the number of bonds, and $\pi_t^1$
denotes the number of shares owned by an investor at time $t$. The value of the portfolio $\Pi$ at time $t$ is
\[
V^\Pi_t = \pi^0_t B_t + \pi^1_t S_t.
\]

The portfolio is called self-financing if
\[
dV^\Pi_t = \pi^0_t dB_t + \pi^1_t dS_t,
\]
that is, changes in the portfolio value are only due to changes in asset prices, so there is no external capital inflows and outflows.

**Remark 4.1.** Condition (4.2) is understood in the sense that
\[
V^\Pi_t = V^\Pi_0 + \int_0^t \left( \pi^0_u dB_u + \pi^1_u dS_u \right) = V^\Pi_0 + \int_0^t \left( (\pi^0_u \mu_u B_u + \pi^1_u \mu_u S_u) du + \pi^1_u S_u dB^H_u \right),
\]
i.e. for a self-financing strategy, we assume that the latter integrals are well-defined as Lebesgue and generalized Lebesgue–Stieltjes integrals correspondingly.

**Remark 4.2.** Thanks to the left continuity of $F$, the property of $F$-predictability of process $\pi$ is equivalent to $F$-adaptedness.

Further, for any $F$-adapted process $\pi^1$ such that $\int_0^1 \pi^1_u S_u dB_u^H$ is well defined and for any initial capital $V_0$ it is possible to construct a self-financing portfolio $\Pi$ such that its risky part is $\pi^1$ and $V^\Pi_0 = V_0$.

Define the discounted value of a portfolio
\[
C^\Pi_t = V^\Pi_t B_t^{-1}.
\]
It is easy to check that
\[
dC^\Pi_t = \pi^1_t dX_t,
\]
where $X_t = S_t B_t^{-1}$ is the discounted risky asset price process. We stress once more that the integrability with respect to $X$ is understood in the following sense: we say that integral
\[
\int_0^t a_s dX_s = \int_0^t \left( a_s (\mu - r_s) X_s ds + \sigma a_s X_s dB^H_s \right)
\]
exists if $a_s (\mu - r_s) X_s$ is Lebesgue integrable on $[0, t]$ and $\| a X \|_{\alpha, t} < \infty$ for some $\alpha \in (1 - H, 1/2)$.

**Definition 4.2.** A self-financing portfolio $\Pi$ is *arbitrage* if $V^\Pi_0 = 0$, $V^\Pi_1 \geq 0$ a.s., and $P \left( V^\Pi_1 > 0 \right) > 0$. It is called *strong arbitrage* if additionally there exists a constant $c > 0$ such that $V^\Pi_1 \geq c$ a.s.
Definition 4.3. A contingent claim on fractional \((B, S)\)-market is a non-negative \(\mathcal{F}_1\)-measurable random variable \(\xi\).

Contingent claim \(\xi\) is called attainable, or hedgeable, if there exists a self-financing portfolio \(\Pi\), which is called a hedge, or replicating portfolio, for \(\xi\), such that \(V_{\Pi}^1 = \xi\) a.s.

We will call \(\xi\) weakly hedgeable if there exists a portfolio \(\Pi\) (a weak hedge), self-financing on each interval \([0, t]\), \(t < 1\), such that \(\lim_{t \to 1^-} V_{\Pi}^t = \xi\) a.s.

The initial portfolio value \(V_{\Pi}^0\) is called a hedging cost or a weak hedging cost correspondingly.

Rogers [10] showed that fractional \((B, S)\)-market model admits arbitrage in an unconstrained case, like ours. For more information on the arbitrage possibilities in these models, see [2] and references therein.

We have the following result on strong arbitrage, which we do not prove this immediately, as it will follow from the stronger result further (Theorem 4.2).

Theorem 4.1. The fractional \((B, S)\)-market model admits strong arbitrage.

Remark 4.3. The Ito formula from Theorem 2.1 is not enough to provide a strong arbitrage. Indeed, it gives
\[
\int_0^1 f(B_H^s) = F(B_H^1).
\]

But \(B_H^1\) is Gaussian, so it can be arbitrarily close to 0 with a positive probability, and since \(F\) is continuous, \(F(B_H^1)\) is also arbitrarily close to 0 with a positive probability, so strong arbitrage is impossible in this case.

Now we establish an auxiliary result, similar to Lemma 3.1.

Lemma 4.1. There exists an \(\mathbb{F}\)-adapted process \(\phi = \{\phi_t, t \in [0, 1]\}\) such that for any \(t < 1\) integral \(v_t = \int_0^t \phi_s dX_s\) is well defined in the above sense and \(\lim_{t \to 1^-} v_t = \infty\) a.s.

Proof. Put \(\phi_s = X_s^{-1}\varphi_s\), where \(\varphi\) is defined in Lemma 3.1. Since \(\phi_s(\mu - r_s)X_s = \varphi_s(\mu - r_s)\) is almost surely bounded in \(s\), and \(\phi_s X_s = \varphi_s\), we have that integral \(\int_0^t \phi_s dX_s\) is well defined. Moreover, integral \(\int_0^1 \varphi_s(\mu - r_s)ds\) is finite, while \(\int_0^1 \varphi_s dB_H^s \to \infty\), \(t \to 1^-\), so by (4.3) we have \(v_t \to \infty\), \(t \to 1^-\).

As a corollary, similarly to Theorem 3.1, we have the following result.

Theorem 4.2. For any distribution function \(F\) there is a self-financing portfolio \(\Pi\) with \(V_{\Pi}^0 = 0\) such that its discounted terminal capital \(C_{\Pi}^1\) has distribution \(F\).

Proof. As in the proof of Theorem 3.1 let \(g\) be such that \(g(B_H^{1/2})\) has distribution \(F\), \(\phi\) be as in Lemma 3.1 and \(v_t = \int_{1/2}^t \phi_s dX_s\). Set
\[
\tau = \min \left\{ t \geq 1/2 : v_t = g(B_H^{1/2}) \right\}, \quad \pi_t^1 = \phi_{[1/2, \tau]}(t)
\]
Then it is possible to construct a self-financing portfolio \( \Pi = (\pi^0, \pi^1) \) with \( V_0^\Pi = 0 \). Clearly, \( \pi^0_t = 0, t \leq 1/2 \), so \( C_{1/2}^\Pi = 0 \). Further,

\[
C_1^\Pi = C_{1/2}^\Pi + \int_{1/2}^1 \pi^1_s dX_s = \int_{1/2}^t \phi_s dX_s = g(B_{1/2}^H),
\]

as required.

If we let \( F \) to be the distribution function of some constant \( A > 0 \) and observe that \( B_1^{-1} \) is greater than a non-random positive constant due to our assumption on \( r_t \), then we derive the result about strong arbitrage.

Now we are ready to state main results of this section concerning hedging of contingent claims in the fractional \((B,S)\)-market.

**Theorem 4.3.** Any contingent claim \( \xi \) in the fractional \((B,S)\)-market is weakly hedgeable. Moreover, its weak hedging cost can be any real number.

**Proof.** As in Theorem 3.2, for any \( V_0 \in \mathbb{R} \) there is an \( \mathbb{F} \)-adapted process \( \pi^1 \) such that

\[
\int_0^t \pi^1_s dX_s \to \xi B_1^{-1} - V_0, \ t \to 1 -
\]

Then we can construct a self-financing portfolio \( \Pi = (\pi^0, \pi^1) \) such that \( V_0^\Pi = V_0 \). We have

\[
V_t^\Pi = B_t C_t^\Pi = B_t \left(V_0 + \int_0^t \pi^1_s dX_s\right) \to \xi, \ t \to 1 -
\]

as required.

**Theorem 4.4.** Assume that for a contingent claim \( \xi \) there exists an \( \mathbb{F} \)-adapted almost surely Hölder continuous process \( \{z_t, t \in [0,1]\} \) with \( z_1 = \xi \). Then \( \xi \) is hedgeable and its hedging cost can be any real number.

**Proof.** As in the previous theorem, it is for enough to construct an \( \mathbb{F} \)-adapted process \( \pi^1 \) such that \( \int_0^1 \pi^1_t dX_t \) is well defined and

\[
V_0 + \int_0^1 \pi^1_t dX_t = \xi B_1^{-1}.
\]

To that end we slightly modify the construction from Theorem 3.3. Namely, we first take some \( \gamma > (1 - \alpha - H + a)^{-1} > 1 \) and put \( \Delta_n = n^{-\gamma}/\zeta(\gamma) \), \( t_n = \sum_{k=1}^{n} \Delta_k \), \( \Delta B^H_{t_n} = B^H_{t_n} - B^H_{t_{n-1}} \), \( \xi_n = z_{t_n} B_{t_n}^{-1} \). We will also use the notation \( B^H_{x,y} = B^H_{x} - B^H_{y} \).

Then we construct process \( \pi^1 \) recursively on intervals \([t_n, t_{n+1})\), starting by setting \( \pi^1_0 = 0 \) on \([0,t_1)\). Then we take some \( \kappa \in (\gamma(H - a), \gamma(1 - \alpha) - 1) \).

Denote \( v_t = \int_0^t \pi^1_s dX_s \). If \( \psi \) is constructed up to \( t_n \), we define

\[
\tau_n = \min \left\{ t \geq t_n : n^\kappa \left| \int_{t_n}^t (\mu - r_s) \text{sign} B^H_{s,t_n} ds + \sigma B^H_{s,t_n} \right| = |v_{t_n} - \xi_n| \right\} \land t_{n+1}
\]

15
and set
\[ \pi_t^1 = \kappa \text{sign} B_{t,t_n}^H \text{sign}(\xi_n - v_{t_n})1_{t < \tau_n} X_t^{-1}. \]
for \( t \in [t_n, t_{n+1}) \).

The rest of proof is the same as in Theorem 3.3, so we do not repeat it fully, making only important remarks. The Step 2 of proof will be still true, since \( \int_{t_n}^t (\mu - r_s) \text{sign} B_{s,t_n}^H ds \) is of order \( (t - t_n) \) which is negligible compared to the quantities involved in this step. In the Step 3, we should not consider expectations immediately. Instead, we note that \( X^{-1} \) is almost surely Hölder continuous of any order less than \( H \) and estimate for \( t, s \in [0, \tau_n) \)
\[
|\psi_s - \psi_t| \leq C(\omega)n^\kappa \left( 1_{B_{t,t_n}^H B_{s,t_n}^H < 0} + |X_t^{-1} - X_s^{-1}| \right)
\leq C(\omega)n^\kappa \left( 1_{B_{t,t_n}^H B_{s,t_n}^H < 0} + (t - s)^d \right),
\]
where \( d \in (\alpha, H) \) is such that \( \kappa - \gamma(1 - d + H - \alpha) < -1 \). For other \( t, s \) we write simply
\[
|\psi_s - \psi_t| \leq C(\omega).
\]
Then we take expectation only of the term involving \( 1_{B_{t,t_n}^H B_{s,t_n}^H < 0} \) (without the random constant) and we know that it is finite. The rest of terms are easily checked to be finite exactly as in Theorem 3.3, and multiplication by a random constant cannot make things infinite.

4.2. Zero integral

Assume that process \( g \) is adapted to \( F \), and the integral \( \int_0^1 g_s dB_s^H \) is well defined. We are interested in the following question:

- If \( \int_0^1 g_s dB_s^H = 0 \) a.s., is it true that \( g = 0 \) almost everywhere with respect to \( P \otimes \lambda \)?

Recall the following fact for standard Brownian motion \( W \). Assume that \( \int_0^1 h_s^2 ds < \infty \). Then we have the following equivalence from the Itô-isometry
\[
\int_0^1 h_s dW_s = 0 \iff h = 0 \quad P \otimes \lambda \text{-a.e.}
\]
If the integrability assumption is replaced by \( \int_0^1 h_s^2 ds < \infty \), then the conclusion is false: it is proved in [3] that one can construct an adapted process \( h \) such that \( \int_0^{1/2} h_s dW_s = 1 \) and \( \int_{1/2}^1 h_s dW_s = -1 \), so \( h \) cannot equal zero identically.

Similarly, in the fractional Brownian framework, thanks to Theorem 3.3 we can construct \( g \) adapted to the filtration generated by the fractional Brownian motion \( B^H \) such that \( \int_0^{1/2} g_s dB_s^H = 1 \) and \( \int_{1/2}^1 g_s dB_s^H = -1 \). This gives a negative answer to the question we are interested in.

Nevertheless, in some special cases we can conclude that the integrand \( g \) is zero when the integral \( \int_0^1 g_s dB_s^H = 0 \). One can show this for integrands with finite fractional chaotic expansion and for simple predictable integrands.
First consider \( g \) with a finite fractional Wiener–Itô expansion. We give only brief explanation here, the details can be found e.g. in [3, Chapter 3]. Let \( \phi(t, s) = H(2H - 1) |t - s|^{2H - 2} \) and define a scalar product of functions 
\( f, g : [0, 1] \to \mathbb{R} \)
\[ \langle f, g \rangle_H = \int_{[0,1]^2} \phi(t, s)f(t)g(s)dt ds \]
and the corresponding norm \( \|f\|_H = \langle f, f \rangle_H^{1/2} \). The space \( L^2_H([0, 1]) \) is the space of functions \( f : [0, 1] \to \mathbb{R} \) such that \( \|f\|_H < \infty \). Consider the \( n \)th symmetric tensor power of \( L^2_H([0, 1]) \):
\[ \hat{L}^2_H([0, 1]^n) = L^2_H([0, 1]) \otimes \cdots \otimes L^2_H([0, 1]) \]
It inherits a Hilbertian structure from \( L^2_H([0, 1]) \): for \( f, g \in \hat{L}^2_H([0, 1]^n) \)
\[ \langle f, g \rangle_{\hat{L}^2_H([0, 1]^n)}^2 = \int_{[0,1]^{2n}} f(t_1, \ldots, t_n)g(s_1, \ldots, s_n)\phi(t_1, s_1) \cdots \phi(t_n, s_n)dt_1 \cdots dt_n ds_1 \cdots ds_n. \]
For a function \( f \in \hat{L}^2_H([0, 1]^n) \) it is possible to define the iterated stochastic integral
\[ I_n(f) = \int_{[0,1]^n} f(t_1, \ldots, t_n)dB_{t_1}^H \circ \cdots \circ dB_{t_n}^H \]
(we use the symbol \( \circ \) here to emphasize that this integral differs from the path-wise iterated integral.) Now let \( g = \{g_t, t \in [0, 1]\} \) have finite fractional chaos expansion of the form
\[ g_t = \sum_{k=0}^n I_k(f_k(\cdot, t)) \]
with \( f_k(\cdot, t) \in \hat{L}^2_H([0, 1]^k) \), \( k \leq n \). Thanks to finiteness of expansion, the process \( g_t \) has a stochastic derivative and
\[ D_s g_t = \sum_{k=1}^n kI_{k-1}(f_k(\cdot, s, t)). \]
Assume that \( f_k \in \hat{L}^2_H([0, 1]^{k+1}) \), \( k \leq n \). Then the process \( g \) belongs to the domain of the divergence integral \( \delta \) and
\[ \delta(g) = \sum_{k=1}^n I_{k+1}(\tilde{f}_k), \]
where \( \tilde{f}_k \) is the symmetrization of \( f \). Also assume that
\[ T(g) = \int_{[0,1]^2} |D_s g_t| \phi(t, s)ds dt < \infty \]
a.s. A checkable sufficient condition for this is
\[ \mathbb{E} \left[ T(g)^2 \right] = \sum_{k=1}^n k! \int_{[0,1]^4} \langle f_k(\cdot, s, t), f_k(\cdot, u, v) \rangle_H \phi(t, s)\phi(u, v)ds dt du dv < \infty. \]
Then it is known (see \[3\), Proposition 4.1]) that there is a relation between pathwise and divergence integrals:

$$\int_0^1 g_t dB_t^H = \delta(g) + \int_{[0,1]^2} D_s g_t \phi(s, t) ds dt.$$ 

If \(\int_0^1 g_t dB_t^H = 0\), then

$$\delta(u) = - \int_{[0,1]^2} D_s u_t \phi(s, t) ds dt. \quad (4.4)$$

The chaotic expansion of the right-hand side is

$$- \int_{[0,1]^2} D_s u_t \phi(s, t) ds dt = \sum_{k=0}^{n-1} (k+1) I_k(h_k),$$

where

$$h_k(t_1, \ldots, t_k) = - \int_{[0,1]^2} f_{k+1}(t_1, \ldots, t_k, s, t) \phi(t, s) ds dt.$$

Since the Wiener–Itô expansion is unique, the corresponding coefficients of the left-hand and the right-hand sides of equality (4.4) are the same. In particular, taking \((n + 1)\)th terms of expansion, we get \(I_{n+1}(\hat{f}_n) = 0 \) a.s., whence \(\hat{f}_n = 0 \) \(\lambda\)-a.e., consequently, \(f_n = 0 \) \(\lambda\)-a.e. Using a backward induction, we get that \(f_k = 0 \) \(\lambda\)-a.e. for all \(k \leq n\), concluding that \(g = 0 \) \(P \otimes \lambda\)-a.e.

Assume now that \(g\) is a simple predictable process of the form

$$g = \sum_{k=1}^{m} \alpha_k 1_{(t_{k-1}, t_k]},$$

where \(0 = t_0 < t_1 < \cdots < t_m = 1, \alpha_k \in \mathcal{F}_{t_{k-1}}\). If

$$\int_0^1 g_t dB_t^H = \sum_{k=1}^{m} \alpha_k \Delta B_{t_k}^H = 0,$$

then it was proved in \[8\), Theorem 2.5.1] that \(\alpha_k = 0 \) a.s. A proof uses the representation of fractional Brownian motion with respect to standard Brownian motion on the finite interval considered in \[9\].

References

[1] E. Azmoodeh, Y. Mishura, E. Valkeila, On hedging European options in geometric fractional Brownian motion market model, Stat. Decis. 27 (2) (2009) 129–143.

[2] C. Bender, T. Sottinen, E. Valkeila, Fractional processes as models in stochastic finance, in: G. Di Nunno, B. Øksendal (Eds.), Advanced mathematical methods for finance, Springer, Berlin, 2011, pp. 75–103.
[3] F. Biagini, Y. Hu, B. Øksendal, T. Zhang, Stochastic calculus for fractional Brownian motion and applications, Springer, London, 2008.

[4] R. Dudley, Wiener functionals as Ito integrals, Ann. Probab. 5 (1977) 140–141.

[5] X. Fernique, Intégrabilité des vecteurs gaussiens., C. R. Acad. Sci., Paris, Sér. A 270 (1970) 1698–1699.

[6] A. Garsia, E. Rodemich, Monotonicity of certain functionals under rearrangement, Ann. Inst. Fourier 24 (2) (1974) 67–116.

[7] W. Li, Q.-M. Shao, Gaussian processes: Inequalities, small ball probabilities and applications, North-Holland/ Elsevier, Amsterdam, 2001.

[8] Y. Mishura, Stochastic calculus for fractional Brownian motion and related processes, Springer, Berlin, 2008.

[9] I. Norros, E. Valkeila, J. Virtamo, An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions, Bernoulli 5 (4) (1999) 571–587.

[10] L. Rogers, Arbitrage with fractional Brownian motion, Math. Finance 7 (1) (1997) 95–105.

[11] S. Samko, A. Kilbas, O. Marichev, Fractional integrals and derivatives: theory and applications, New York, NY: Gordon and Breach, 1993.

[12] M. Zähle, Integration with respect to fractal functions and stochastic calculus. I, Probab. Theory Relat. Fields 111 (3) (1998) 333–374.