Adversarial Link Prediction in Social Networks

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Abstract

Link prediction is one of the fundamental tools in social network analysis, used to identify relationships that are not otherwise observed. Commonly, link prediction is performed by means of a similarity metric, with the idea that a pair of similar nodes are likely to be connected. However, traditional link prediction based on similarity metrics assumes that available network data is accurate. We study the problem of adversarial link prediction, where an adversary aims to hide a target link by removing a limited subset of edges from the observed subgraph. We show that optimal attacks on local similarity metrics—that is, metrics which use only the information about the node pair and their network neighbors—can be found in linear time. In contrast, attacking Katz and ACT metrics which use global information about network topology is NP-Hard. We present an approximation algorithm for optimal attacks on Katz similarity, and a principled heuristic for ACT attacks. Extensive experiments demonstrate the efficacy of our methods.

Introduction

Link prediction is one of the most fundamental problems in social network analysis. The crux of link prediction as regards to a particular target link \((u,v)\) is to use an observed (sub)network to infer the likelihood of the existence of this link using a measure of similarity, or closeness, of \(u\) and \(v\) (Liben-Nowell and Kleinberg 2007; Wang et al. 2015; Al Hasan et al. 2006; Zhang and Chen 2018). For example, if \(u\) and \(v\) are individuals who have many friends, it may be natural to assume that they are themselves friends. Representational power of social networks implies very broad application of link prediction techniques, ranging from friend recommendations to inference of criminal and terrorist ties.

A crucial assumption in conventional similarity-based link prediction approaches is that the observed (sub)network is measured correctly. However, insofar as link prediction may reveal relationships which associated parties prefer to keep hidden—either for the sake of privacy, or to avoid being apprehended by law enforcement—it introduces incentives to manipulate network measurements in order to reduce perceived similarity scores for target links.

In order to systematically study the ability of an “adversary” to manipulate link prediction, we formulate attacks on link prediction as an optimization problem in which the adversary aims to minimize the score of a target link by removing a limited subset of edges from the observed subnetwork. Our first results show that optimal manipulation of local similarity metrics, that is, those which only depend on the endpoints and their neighbors, can be found in linear time, although specifics differ depending on the structure of the similarity metric. Next, we study two common similarity metrics that use global information about network topology, Katz and ACT. We show that optimally manipulating either is NP-Hard. However, we prove that the problem is monotone and submodular in the case of the Katz similarity metric, which means that we can achieve a \((1-1/e)\) approximation using a simple iterative greedy algorithm. Finally, we devise an algorithm for approximating the optimal solution in the context of the ACT similarity metric by first representing it in terms of effective resistance, and subsequently taking advantage of an analytic approximation of this quantity devised by Von Luxburg, Radl, and Hein (2014). We demonstrate the effectiveness of the approaches we develop through an extensive experimental evaluation.

Related Work Link prediction has been studied extensively, with the focus being largely on the design of node pair similarity metrics (Liben-Nowell and Kleinberg 2007; Wang et al. 2015). Such similarity metrics are commonly classified as neighbor-based (Leicht, Holme, and Newman 2006; Zhou, Liu, and Zhang 2009), path-based (Katz 1953; Liu, Jin, and Zhou 2009), and random-walk-based (Fouss et al. 2007), depending on the information used. The former corresponds to what we call local metrics, and the latter two are global metrics.

Recently there have been several efforts to study robustness of social network analysis tools. Zhang et al. (2016) consider robustness of link prediction to random noise. Waniek et al. (2018) study the vulnerability of centrality measures to adversarial manipulation. Waniek, Michalak, and Rahwan (2017)—the closest prior work to ours—also study the issue of hiding links, but consider both adding and removing edges, and use AUC and average precision as metrics. As a consequence of the complex problem setup, Waniek, Michalak, and Rahwan show that their problem is NP-Hard for essentially all similarity metrics, including local metrics. In contrast, we have a simpler model and exhibit a host of positive results.
Problem Statement

Similarity Metrics

A critical step in link prediction for a pair of nodes \( u \) and \( v \) is to calculate the similarity \( \text{Sim}(u, v) \) between these nodes. In the literature, a number of such similarity metrics have been proposed. For our purposes, we classify such metrics into two broad classes based on the information they use. The first class is local metrics that use only local information of the targeted link, such as the number of common neighbors and the degrees of nodes. The second class of global metrics uses global network information in computing similarity between a pair of nodes. We use the Katz similarity and the Average Commute Time (ACT) as two representatives of the global metrics.

Attack Model

At the high level, we consider attacks which aim to camouflage a target link (edge) by removing a small subset of edges from an observed part of the underlying graph, \( G = (V, E) \).

To formalize, we consider two parties, the analyst who uses a similarity metric in order to infer existence of unobserved links, and the attacker, who aims to hide a target link \((u, v)\). We assume that the underlying graph \( G \) is not directly observable by the analyst (otherwise, link prediction is trivial), who must instead obtain partial information about it through a set of edge queries (e.g., measurements) \( Q = \{ (v_i, v_j) \} \) for a set of pairs of nodes, with each \( e \in Q \) labelled as an edge or a non-edge. Let \( E_Q = \{ (u_i, u_j) \in Q \} \) be the set of edges in \( G \), and \( E_Q^i = \{ (u_i, u_j) \in Q \} \) the set of non-edges in \( Q \). The attacker observes \( Q \) and knows the underlying graph \( G \), and chooses a subset of at most \( k \) edges \( E_a \subseteq E_Q \) to remove, thereby causing the associated pairs of nodes to appear as non-edges to the analyst (for example, the attacker makes these edges difficult to measure). After the attack, the analyst observes a modified subgraph \( G'_Q = (V, E_Q \setminus E_a) \), and uses this subgraph to make a prediction about the existence of unknown links between pairs of nodes \((u_i, u_j)\) based on their associated similarity score \( \text{Sim}(u_i, u_j) \). The attacker’s goal is to minimize the similarity score \( \text{Sim}(u, v) \) for the link \((u, v)\) they wish to camouflage. Thus, the attacker solves the following optimization problem:

\[
\min_{E_a \subseteq E_Q} \text{Sim}(u, v), \quad \text{s.t.} \ |E_a| \leq k. \tag{1}
\]

Before proceeding, we introduce some notation. For any two nodes \( u_i, u_j \in \bar{V} \), let \( d(u_i) \) and \( d(u_j) \) be their degrees and \( N(u_i, u_j) \) be the set of their common neighbors. Given a target pair of nodes \((u, v)\), the edges in \( E_Q \) can be divided into five mutually disjoint subsets, i.e., \( E_Q = \bigcup_{i=1}^{4} E_i \).

Specifically, \( E_1 \) and \( E_3 \) are sets of edges that are incident to \( v \); additionally, each edge in \( E_1 \) also connects to a node in \( N(u, v) \) (a common neighbor). \( E_2 \) and \( E_4 \) are edges that are incident to \( v \) and, in addition, an edge in \( E_2 \) also connects to a common neighbor. Finally, edges in \( E_5 \) are neither incident to \( u \) nor \( v \). We further define \( m = |E_Q| \) and \( m_i = |E_i| \). Note that since \( E_1 \) and \( E_2 \) are edges that connect to the common neighbors, \( m_1 = m_2 = |N(u, v)| \).

Attacking Local Similarity Metrics

In this section, we analyze attacks on link prediction with respect to the local metrics and show that these metrics can be attacked in linear time.

We begin by introducing some notation. Consider a tuple \((u, w, v)\) where \( u, v \) are the target nodes and \( w \) is a common neighbor of \( u \) and \( v \). As the attacker deletes links from \( E_Q \), there are four possible states of the tuples between \( u \) and \( v \). In state 1, both \((u, w)\) and \((w, v)\) are deleted. In state 2, \((u, w)\) is deleted while \((w, v)\) is not. In state 3, \((w, v)\) is deleted while \((u, w)\) is not. In state 4, neither \((u, w)\) nor \((w, v)\) is deleted. We use integer variables \( y_1, y_2, y_3, y_5 \) to denote the number of tuples in state 1, 2, 3, respectively. Furthermore, let \( y_4 \) and \( y_5 \) be the number of deleted edges from \( E_3 \) and \( E_4 \), respectively. In this way, the vector \((y_1, y_2, y_3, y_4, y_5)\) fully captures a strategy of the attacks on local metrics.

We divide the local metrics into two sub-classes: Common Degree Metric (CDM) and Target Degree Metric (TDM), as these will result in different optimal attack strategies. We now formally define these.

Definition 1. A local metric \( \text{Sim}(u, v) \) is a Common Degree Metric (CDM) if \( \text{Sim}(u, v) = \sum_{i=1}^{m_1} g(d(w_i)) \) such that \( g(d) \) is decreasing in \( d \), where \( w_i \in N(u, v) \).

An example of the CDM metric is the Adamic-Adar similarity, defined as \( \frac{|N(u, v)|}{\log d(u) + \log d(v)} \).

Definition 2. A local metric \( \text{Sim}(u, v) \) is a Target Degree Metric (TDM) if \( \text{Sim}(u, v) = f(y_1, y_2, y_3, y_4, y_5) \) such that \( f \) is decreasing in \( y_2 \) and \( y_3 \) and \( f \) is increasing in \( y_4 \) and \( y_5 \).

An important representative of the TDM class is Jaccard similarity, defined as \( \frac{|N(u, v)|}{d(u) + d(v) - |N(u, v)|} \). Next, we analyze attacks on these two subclasses.

Attacking CDM

By its definition, to minimize a CDM, we need to remove edges incident to common neighbors \( w \) of \( u \) and \( v \) in increasing order of degree \( d(w) \); we call this the RankCNDegree algorithm.

Next, we show that this algorithm is in fact optimal.

Theorem 1. The RankCNDegree algorithm finds an optimal attack on CDM.

Proof. To see why the RankCNDegree algorithm is optimal, we first note that deleting edges from \( E_3 \) and \( E_4 \) would have no effect on \( \text{Sim}(u, v) \). Thus, the attacker will only delete edges from \( E_1 \) and \( E_2 \). Second, since \( g(d(w_i)) > 0 \), \( \forall w_i \in N(u, v) \), the attacker will always delete exactly \( k \) edges. Third, since \( g(d(w_i)) \) is decreasing in \( d(w_i) \) for CDM, it is optimal to delete edges incident to common neighbors in order of increasing degree.

Combining this result with the fact that one can select \( k \) smallest items in linear time yields the following.

Corollary 1. An optimal attack on CDM can be found in time \( O(m_1) \).
Attacking TDM

Next, we show that TDM has enough structure to enable optimal attack in $O(k)$ time.

Lemma 1. An optimal attack on TDM satisfies $y^*_1 = y^*_4 = y^*_5 = 0$ and $y^*_2 + y^*_3 = k$.

Proof. For TDM, as $f(y_1, y_2, y_3, y_4, y_5)$ is increasing in $y_4$ and $y_5$, a rational attacker will always set $y^*_4 = 0$ and $y^*_5 = 0$ in order to minimize $f$.

Next, we prove $y^*_1 = 0$. Suppose the attacker chooses to delete $(u, w_i)$ for some $w_i \in N(u, v)$, the edge $(w_i, v)$ becomes an edge in $E_4$. Further deleting $(w_i, v)$ will increase $y_5$, thus increasing $f$, since $f$ is increasing in $y_5$. Similarly, if an attacker deletes $(w_i, v)$, he will never delete $(u, w_i)$. Thus, a rational attacker will never delete $(u, w_i)$ and $(w_i, v)$ simultaneously. That is, $y^*_1 = 0$.

As $f$ is decreasing in $y_2$ and $y_3$, the attacker will maximize $y_2$ and $y_3$ in order to maximize $f$. Note that the budget constraint is $2y_1 + y_2 + y_3 + y_4 + y_5 \leq k$. Thus, an optimal attack strategy satisfies $y^*_2 + y^*_3 = k$.

Lemma 1 states that in an optimal attack on TDM, the attacker will always choose $k$ edges from $E_1 \cup E_2$ to delete. The following theorem specifies how the attacker will choose the $k$ edges in an optimal attack.

Theorem 2. The optimal attack on TDM selects arbitrary $y^*_2$ links from $E_1$ and $(k - y^*_2)$ links from $E_2$ to delete with the constraint that for any selected link $(u, w_i) \in E_1$ and $(v, w_2) \in E_2$, $w_1 \neq w_2$. The value of $y^*_2$ is the solution of a single-variable integer optimization problem.

Proof. From Lemma 1, we know that an optimal attack strategy satisfies $y^*_1 = y^*_4 = y^*_5 = 0$. Thus, problem (1) is reduced to two-variable integer optimization problem:

$$\min_{y_2, y_3} f(y_2, y_3), \text{ s.t. } y_2 + y_3 \leq k, y_2, y_3 \in [0, k].$$

Since $y^*_2 + y^*_3 = k$, we can substitute $y_3$ in problem (2) by $k - y_2$, yielding

$$\min_{y_2} f(y_2, k - y_2), \text{ s.t. } y_2 \in [0, k],$$

which is a single-variable integer optimization problem. Solving problem (3) yields the optimal attack strategy $y^*_2$, and $y^*_3 = k - y^*_2$.

Corollary 2. An optimal attack on TDM can be found in time $O(k)$.

Problem Formulation for Katz Similarity

The Katz similarity [Katz 1953] between a pair of nodes $(u, v)$ is defined as

$$Katz(u, v) = \sum_{l=1}^{\infty} \beta^l |path_{uv}^l| = (\beta A + \beta^2 A^2 + \beta^3 A^3 + \cdots)_{uv},$$

where $|path_{uv}^l|$ denotes the number of walks of length $l$ between $u$ and $v$, $\beta$ is a parameter and $(\cdot)_{uv}$ denotes the entry in the $u$th row and $v$th column of a matrix.

Now, we investigate the effect of deleting one edge on Katz$(u, v)$. Assume the $i$th edge connects nodes $p$ and $q$. Let $A_{pq} = A_{qp} = x_i$. Then, the adjacency matrix $A$ and the degree matrix $D$ are fully captured by the vector $x$. Thus, Katz$(u, v)$ is a function of $x$, written as Katz$(x)$.

Theorem 3. Katz$(x)$ is an increasing function of $x$.

Proof. Let $A$ and $A'$ be the corresponding adjacency matrices of $x$ and $x'$. If $x \preceq x'$, we have $A \preceq A'$. Now, consider the $j$th term of the Katz similarity matrix $K$, which is $\beta^j A^j$. As every entry in $A$ is non-negative and $\beta > 0$, we have $\beta^j A^j \preceq \beta^j A'^j$, for every $j$. Thus, Katz$(x)$ $\preceq$ Katz$(x')$.

From Theorem 3 we know that deleting a link will decrease Katz$(x)$. As the attacker’s goal is to minimize Katz$(x)$, he would always delete up to $k$ edges from $E_Q$. Thus, poisoning attack with Katz similarity can be modelled as the following optimization problem, which we term Prob-Katz:

$$\min \text{Katz}(x), \text{ s.t. } \sum_{i=1}^{m} x_i = m - k, x \in \{0, 1\}^m.$$

Problem Formulation for ACT

The ACT between two nodes, ACT$(u, v)$, is intuitively the expected time for a simple random walker to travel from a node $u$ to node $v$ on a graph and return to $u$. Since ACT$(u, v)$ is a distance metric, the associated similarity metric would be its inverse, and the attacker’s aim is to maximize ACT$(u, v)$. ACT$(u, v)$ can be defined as

$$ACT(u, v) = V_G(L_{uu}^1 + L_{uv}^1 - 2L_{uv}^1),$$

where $V_G$ is the volume of the graph which is twice of the number of edges (Fouss et al. 2007).

Directly optimizing ACT$(u, v)$ is hard. In fact, deleting an edge may either increase or decrease ACT$(u, v)$. Fortunately, Ghosh, Boyd, and Saberi [2008] show that when edges are unweighted (as in our setting), ACT$(u, v)$ can be defined in terms of Effective Resistance (ER): ACT$(u, v) = V_G\text{ER}(u, v)$. It is also not difficult to see that both the volume $V_G$ and ER can be represented in terms of $x$, a vector of indicators of which edges remain after a subset are deleted.

We begin by investigating the effect of deleting an edge on ER$(x)$. We use a well-known result by Doyle and Snell [2000] to this end.

Lemma 2 ([Doyle and Snell 2000]). The effective resistance between two nodes will strictly increase when an edge is deleted.
The following theorem is then an immediate corollary.

**Theorem 4.** ER($x$) is a decreasing function of $x$. That is, for any two binary vectors $x \leq x'$, ER($x$) $\geq$ ER($x'$).

Theorem 4 maximizing ER($x$) would always entail deleting all allowed edges. Let $t$ be the maximum number of edges that can be deleted. Then, maximizing ER($x$) can be formulated as the following optimization problem, which we term **Prob-ER**:

$$\max_x \text{ER}(x), \quad \text{s.t.} \quad \sum_{i=1}^{m} x_i = m - t, x \in \{0,1\}^m.$$  

However, while ER($x$) increases as we delete edges, volume $V_G = 2 \sum_{i=1}^{m} x_i$ decreases. Fortunately, since volume is linear in the number of deleted edges, the problem of optimizing ACT to that of solving **Prob-ER** by solving the latter for $t = \{0, \ldots, k\}$, and choosing the best of these in terms of ACT. Similarly, hardness of **Prob-ER** implies hardness of optimizing ACT. Consequently, the rest of this section focuses on solving **Prob-ER**.

**Hardness Results**

We present the hardness results for both **Prob-Katz** and **Prob-ER**. We first show that **Prob-Katz** and **Prob-ER** can be unified as the same optimization problem, named **Prob-Unified**, with the difference that weights in **Prob-Katz** are all non-negative while weights in **Prob-ER** can be arbitrary. Consequently, hardness of **Prob-Katz** implies hardness of **Prob-ER**.

We define the optimization problem **Prob-Unified** as

$$\min_x C_w(x), \quad \text{s.t.} \quad \sum_{i=1}^{m} x_i = m - k, x \in \{0,1\}^m,$$

where the objective function $C_w(x)$ is constructed as follows. Let $x \in \{0,1\}^m$ be independent binary variables. Let $c(t, x)$ denote a term formed by choosing $t$ variables from $x$ and taking the product. Clearly, there are $\binom{m}{t}$ different terms in the form of $c(t, x)$. Denote the $i$th term by $c_i(t, x)$, where $i = 1, 2, \ldots, \binom{m}{t}$. Assign a weight $w_i$ to the term $c_i(t, x)$. Let a vector $w$ denote all the weights. Then the function $C_w(x)$ parametrized by $w$ is defined as:

$$C_w(x) = w_0 + \sum_{i=1}^{\binom{m}{t}} w_i \cdot c_i(t, x).$$

**Theorem 5.** **Prob-Katz** are instances of **Prob-Unified** where all weights are non-negative.

**Proof.** We need to show that Katz($x$) is in the form of $C_w(x)$ and all the weights $w$ are non-negative.

Consider an arbitrary walk $a$ of length $l$ between node $u$ and $v$. Use a binary variable $y_i$ to represent an edge $e_i$ that appears in $a$, denoted as $e_i \in a$. Then the contribution made by the walk $a$ to the Katz similarity can be represented as $\beta^l \prod_{i \in a} y_i$. If $e_i \in E_Q$, let $y_i = x_i$; otherwise, let $y_i = 1$. Further note that an edge $e_i$ may appear multiple times in $a$. However, $x_i^s = x_i$ for any non-negative integer $s$ since $x_i \in \{0,1\}$. As a result, the contribution of the walk $a$ can be represented as $\beta^l \prod_{i \in E_Q} x_i$, which is in the form of $\beta^l c_i(t, x)$ for some $t$ and $i$. From the definition of Katz similarity, we have

$$\text{Katz}(x) = \sum_{a} (\beta^l \prod_{i \in a} y_i) = \sum_{a} (\beta^l \prod_{i \in E_Q} x_i),$$

which is in the form of $C_w(x)$. Especially, the weight of each term is $\beta^l$ for some length $l$. Since $\beta^l > 0$, we have all the weights $w$ are non-negative. Thus, **Prob-Katz** are instances of **Prob-Unified** with the specification that all weights are non-negative.

The following lemma, due to [Miller1981], will prove useful in showing the connection between **Prob-ER** and **Prob-Unified**.

**Lemma 3** ([Miller1981]). Let $A$ and $A+B$ be non-singular matrices and let $B$ have rank $r > 0$. Let $B = B_1 + \cdots + B_r$, where each $B_i$ has rank 1 and each $C_{i+1} = A + B_1 + \cdots + B_i$ is non-singular; for $t = 1, \ldots, r$. If $C_1 = A$, then

$$C_{t+1}^{-1} = C_t^{-1} - g_t C_t^{-1} B_t C_t^{-1}, t = 1, \ldots, r,$$

where $g_t = \frac{1}{1 + \text{Tr}(C_t^{-1} B_t)}$ and $\text{Tr}(\cdot)$ denotes the trace of a matrix.

**Theorem 6.** **Prob-ER** are instances of **Prob-Unified** with unrestricted weights.

**Proof.** Let $x_i \in \{0,1\}$ be associated with an edge $e_i \in E_Q$ that connects two nodes $p$ and $q$ in the graph $G_Q$. Let $B_i$ be a corresponding symmetric matrix, where $(B_i)_{pq} = (B_i)_{qp} = -1$, $(B_i)_{pp} = (B_i)_{qq} = 1$, and the rest of entries of $B_i$ are zero. Then, if the edge $e_i$ is deleted, the Laplacian matrix $L$ of the graph $G_Q$ becomes $L + B_i$. When all the edges from $E_Q$ are deleted, denote the resulting Laplacian matrix as $L_{EQ}$.

Let $x_i$ denote a state of the edge $e_i$ such that $x_i = 0$ means the edge is deleted. Then the Laplacian matrix of the graph can be written as $L = (L_{EQ} + \sum_{i=1}^{m} x_i B_i)$. Its pseudo-inverse is

$$L^{-1} = (L_{EQ} - E + \sum_{i=1}^{m} x_i B_i)^{-1} + E,$$

where $E$ is an $n \times n$ matrix with each entry being $\frac{1}{2}$.

We now use the result in Lemma 3 to derive the expression of $(L_{uuv}^l + L_{vvu}^l - 2L_{uv}^l)$. Let $C_1 = (L_{E_0} - E)$ and $B = x_1 B_1 + x_2 B_2 + \cdots + x_m B_m$. Let $C_t = (L_{E_0} - E) + x_1 B_1 + x_2 B_2 + \cdots + x_t B_t$. Then $L^{-1} = C_m + E$. The rank of each matrix $x_i B_i$ has rank 1 when $x_i = 1$. The rank of matrix $B$ is an integer since each $B_i$ is similar to that of solving **Prob-ER** by solving the latter for $t = 1, \ldots, r$. If $C_1 = A$, then

$$C^{-1} = C^{-1} - g_t C^{-1} B_t C^{-1}, t = 1, \ldots, r,$$

where $g_t = \frac{1}{1 + \text{Tr}(C^{-1} B_t)}$ and $\text{Tr}(\cdot)$ denotes the trace of a matrix.

Using the fact that $x_i^s = x_i$ for any non-negative integer $s$, we can derive the form of $L^{-1}$, which is in a similar form of $C^{-1}(x)$, except that the weights are matrices. As a result, $L_{uu}^l + L_{vv}^l - 2L_{uv}^l$ is in the form of $C^{-1}(x)$ where the weights can be zero, positive, and negative. Now, $\text{ER}(x) = L_{uu}^l + L_{vv}^l - 2L_{uv}^l$. Thus, $-\text{ER}(x)$ is also in the form of $C^{-1}(x)$.
Since $\text{Prob-ER}$ is equivalent to
\[
\min_{\mathbf{x}} \ -E_R(\mathbf{x}) \quad \text{s.t.} \quad \sum_{i=1}^{m} x_i = m - k, \quad \mathbf{x} \in \{0, 1\}^{m},
\]
it follows that $\text{Prob-ER}$ are instances of $\text{Prob-Unified}$. \hfill \Box

The unified problem formulation facilitates the following result.

**Theorem 7.** $\text{Prob-Katz}$ and $\text{Prob-ER}$ are both NP-hard.

**Proof.** Since $\text{Prob-Katz}$ involves a stronger restriction on the weights in $\text{Prob-Unified}$, it will suffice to prove hardness of $\text{Prob-Katz}$.

We introduce the weighted maximum coverage problem, denoted as $\text{MaxCover}$. Let $U = \{a_1, a_2, \ldots, a_n\}$ be the universe of elements and $S = \{S_1, S_2, \ldots, S_m\}$ be a family of sets, where each $S_i$ is a set of the elements from $U$. Additionally, each $a_i \in U$ is associated with a non-negative weight $w(a_i)$. The problem MaxCover is to select a subset $S_Q \subseteq S$ of size $k$ such that the sum of weights $\text{Sum}(S_Q) = \sum_{i=1}^{m} w(a_i) : a_i \in \cup_{S_i \in S_Q} S_i$ is maximized.

MaxCover is a well-studied NP-hard problem [Hochbaum and Pathria 1998]. Now, our task is to prove that MaxCover $\leq_p$ $\text{Prob-Katz}$.

Given an instance of MaxCover, let an element $a_i \in U$ be a walk of length $l$. Assign the weight of $a_i$ as $w(a_i) = \beta^l$. Associate each set $S_j \in S$ with a binary variable $y_j$, where $y_j = 1$ means that the set $S_j$ is selected. Let $I_{a_i}$ be the index set such that $a_i \in S_j$ if and only if $j \in I_{a_i}$. Then, the weight $w(a_i)$ in the sum of weights $\text{Sum}(S_Q)$ can be represented as $\sum_{j \in I_{a_i}} (1 - y_j) \cdot w(a_i)$. Thus, $\text{Sum}(S_Q) = \sum_{i=1}^{m} (\sum_{j \in I_{a_i}} y_j) \cdot w(a_i)$.

Since $y_j \in \{0, 1\}$ are binary variables, $\sum_{j \in I_{a_i}} y_j = 1 - \prod_{j \in I_{a_i}} (1 - y_j)$. Then $x_j = 1 - y_j$, for $j = 1, \ldots, m$. Then, $\text{Sum}(S_Q) = \sum_{i=1}^{m} (1 - \prod_{j \in I_{a_i}} (1 - y_j)) \cdot w(a_i) = 1 - \sum_{i=1}^{m} w(a_i) + \sum_{i=1}^{m} \left( \sum_{j \in I_{a_i}} x_j \right) \cdot w(a_i)$. Moreover, the constraint $\sum_{j=1}^{m} y_j = k$ is equivalent to $\sum_{j=1}^{m} x_j = m - k$. Thus, the MaxCover problem
\[
\max_{\mathbf{y}} \text{Sum}(S_Q) \quad \text{s.t.} \quad \sum_{i=1}^{m} x_i = m - k
\]
is equivalent to
\[
\min_{\mathbf{x}} \sum_{i=1}^{n} (w(a_i) \cdot \prod_{j \in I_{a_i}} x_j) \quad \text{s.t.} \quad \sum_{j=1}^{m} x_j = m - k,
\]
which is an instance of $\text{Prob-Katz}$ since every $w(a_i)$ is non-negative. As a result, MaxCover $\leq_p$ $\text{Prob-Katz}$. Thus, we can conclude both $\text{Prob-Katz}$ and $\text{Prob-ER}$ are NP-hard. \hfill \Box

**Practical Attack Strategies**

While computing an optimal attack on Katz and ACT is NP-Hard, we now devise approximate approaches which are highly effective in practice.

**Attacking Katz Similarity** To attack Katz similarity, we transform the attacker’s optimization problem into that of maximizing a monotone increasing submodular function. We define a set function $f(S_p)$ as follows. Let $S_p \subseteq E_Q$ be a set of edges that an attacker chooses to delete. Let $A_p$ be the adjacency matrix of the graph $G_Q$ after all the edges in $S_p$ are deleted. Define
\[
f(S_p) = \beta A_p + \beta^2 A_p^2 + \beta^3 A_p^3 + \cdots
\]
Since there is a one-to-one mapping between the set $S_p$ and the matrix $A_p$, the function $f(S_p)$ is well-defined. We note that $f(S_p)$ gives the Katz similarity matrix of the graph $G$ after all the edges in $S_p$ are deleted. We further define a set function
\[
g_{uw}(S_p) = (K - f(S_p))_{uw},
\]
where $K = f(\emptyset)$ (the Katz similarity matrix when no edges are deleted) and $(\cdot)_{uv}$ denotes the $u$th row and $v$th column of a matrix. Clearly, when $S_p = \emptyset$, $g_{uw}(S_p) = 0$.

Then, $\text{Prob-Katz}$ is equivalent to
\[
\max_{S_p \subseteq E} g_{uw}(S_p), \quad \text{s.t.} \quad |S_p| = k
\]

**Theorem 8.** The set function $g_{uw}(S_p)$ is monotone increasing and submodular.

**Proof.** To prove that $g_{uw}$ is monotone increasing, we need to show that $\forall S_p \subseteq S_Q \subseteq Q$, $g_{uw}(S_p) \leq g_{uw}(S_Q)$. It is equivalent to show $(f(S_p))_{uv} \geq (f(S_Q))_{uv}$. We note that $(f(S_p))_{uv}$ and $(f(S_Q))_{uv}$ are the Katz similarity between $u$ and $v$ after the edges in $S_p$ and $S_Q$ are deleted, respectively. Theorem 7 states that the Katz similarity will decrease as more edges are deleted. Since $S_p \subseteq S_Q$, we have $f(S_p) \geq f(S_Q)$. Thus, $g_{uw}(S_p) \leq g_{uw}(S_Q)$.

Next, we prove $g_{uw}$ is submodular. Let $e \in E_i \setminus S_q$ be an edge between node $i$ and node $j$ in the graph. Let $G$ be an $n \times n$ matrix where $G_{ij} = G_{ji} = 1$ and the rest of the entries are 0. Then we have the set $S_p \cup e$ is associated with $A_p - G$ and $S_Q \cup e$ is associated with $A_Q - G$. For a set $S$, let $\Delta(e|S) = f(S \cup e) - f(S)$. Then we need to show
\[
\Delta(e|S_p) \leq \Delta(e|S_Q).
\]
Denote the $t$th item of $\Delta(e|S)$ as $\Delta^{(t)}(e|S)$. In the following, we will first prove $\Delta^{(t)}(e|S_p) \leq \Delta^{(t)}(e|S_Q)$ by induction. Assume that the inequality holds for $t = s$ (it’s straightforward to verify the case for $t = 1$ and $t = 2$). That is
\[
\beta^s [(A_p - G)^s \cdot (A_p - G)^s - (A_Q - G)^s \cdot (A_Q - G)^s] \leq 0.
\]
When $t = s + 1$, we have
\[
\Delta^{(s+1)}(e|S_p) - \Delta^{(s+1)}(e|S_Q) / \beta^{s+1}
\]
\[
= (A_p - G)^{s+1} - (A_p)^{s+1} - (A_Q - G)^{s+1} + (A_Q)^{s+1}
\]
\[
= (A_p - G)^s A_p - (A_p)^{s+1} - (A_Q - G)^s A_Q + (A_Q)^{s+1}
\]
\[
- [(A_p - G)^{s+1} - (A_Q - G)^{s+1}] G
\]
\[
\leq (A_p - G)^s A_p - (A_p)^{s+1} - (A_Q - G)^s A_Q + (A_Q)^{s+1}
\]
The inequality comes from the fact that \((A_p - G) \geq (A_q - G)\) when \(G \geq 0\). Furthermore, since \(S_p \subset S_q\), we have \(A_p = A_q + F\) for some \(F \geq 0\). Thus,

\[
(\Delta(s+1)(e|S_p) - \Delta(s+1)(e|S_q)) / \beta^{s+1} \\
\leq (A_p - G)^s(A_q + F) - (A_q)^s(A_q + F) \\
= [(A_p - G)^s - (A_q)^s]A_q \\
\leq 0
\]

By induction, we have \(\Delta(t)(e|S_p) \leq \Delta(t)(e|S_q)\) for \(t = 1, 2, 3, \cdots\). Note that when \(\beta\) is chosen to be less than the reciprocal of the maximum of the eigenvalues of \(A_q - G\), the sum will converge. Thus, \(\Delta(e|S_p) \leq \Delta(e|S_q)\).

Since Prob-Katz is equivalent to maximizing a monotone increasing submodular function under a cardinality constraint, we can achieve a \((1 - 1/e)\) approximation by applying a simple iterative greedy algorithm in which we delete one edge at a time that maximizes the marginal impact on the objective. We call this resulting algorithm Greedy-Katz.

**Attacking ACT** From the analysis of minimizing Katz similarity, it is natural to investigate submodularity of the effective resistance or ACT as a function of the set of edges. Unfortunately, counter examples show that the effective resistance is neither submodular nor supermodular. Consequently, we need to leverage a different kind of structure for ER.

Our first step is to approximate the objective function \(ER(u, v)\) based on the results by Von Luxburg, Radl, and Hein (2014), who show that \(ER(u, v)\) can be approximated by \(1 / (d(u) + d(v))\) for large geometric graphs as well as random graphs with given expected degrees. Consequently, we use the approximation \(ER(u, v) \approx ER_{ap}(u, v) = 1 / (d(u) + d(v))\).

Our goal is to solve the following optimization problem to approximate optimal ER:

\[
\max_{E_0 \subset E_0} ER_{ap}(u, v), \quad \text{s.t.} \ |E_0| = k. \tag{6}
\]

In contrast with Prob-ER, this is easy to optimize, as shown in the following theorem.

**Theorem 9.** The optimal solution to maximizing \(ER_{ap}(u, v)\) is to delete \(k\) edges that connect to node \(u\) if \(d(u) < d(v)\); otherwise, delete \(k\) edges that connect to node \(v\).

**Proof.** Let \(E(u)\) and \(E(v)\) be the set of edges that connect to \(u\) and \(v\), respectively. If \(e \notin E(u) \cup E(v)\), deleting edge \(e\) will have no effect on \(ER_{ap}(u, v)\). Since the attacker’s goal is to maximize \(ER_{ap}(u, v)\), a rational attacker will only choose edges from \(E(u) \cup E(v)\) to delete.

Next, for any two edges \(e_1, e_2 \in E(u)\), deleting one of them will have the same effect on \(ER_{ap}(u, v)\). Similarly, deleting any edge in \(E(v)\) will also have the same effect on \(ER_{ap}(u, v)\). Based on this observation, we let \(z_1\) and \(z_2\) be the number of edges deleted from \(E(u)\) and \(E(v)\), respectively. Let \(d_u = |E(u)|\) and \(d_v = |E(v)|\). Then problem (6) is equivalent to

\[
\max_{z_1, z_2} \frac{1}{d_u - z_1} + \frac{1}{d_v - z_2}
\]

\[
\text{s.t. } z_1 + z_2 \leq k, \quad z_1, z_2 \in [0, k] \tag{7}
\]

The optimal solution to problem (7) is \((z_1, z_2) = (k, 0)\) if \(d_u < d_v\) and \((z_1, z_2) = (0, k)\) otherwise. \(\square\)

The full algorithm for attacking ACT, which we term *Local-ACT*, thus tries \(t\) between 0 and \(k\) (the upper bound on the number of edges that can be deleted), finds the optimal solution to problem (6) as prescribed by Theorem 9, and finally chooses the solution with the largest ACT value.

**Experiments**

In this section, we numerically evaluate the effect of poisoning attacks on link prediction by measuring the similarities among nodes when edges are deleted from the network. Our experiments use two classes of networks: 1) randomly generated scale-free networks and 2) a Facebook friendship network (Leskovec and Krevl 2014). In our scale-free networks, the degree distribution satisfies \(P(k) \propto k^{-\gamma}\), where \(\gamma\) is a parameter. We restrict attention to target pairs \(u\) and \(v\) with \(|N(u, v)| > k\). All similarity scores are scaled to 1.0 when no edges are deleted.

**Local Metrics**

For the ease of presentation, we term the optimal attack strategies on CDM and TDM as OPT-CDM and OPT-TDM, respectively. We compare the performances of these with a heuristic algorithm proposed recently by Waniek, Michalak, and Rahwan (2017), which is named Unbiased-Deletion as it randomly deletes \(k\) links from \(E_1 \cup E_2\) with the constraint that no two selected links connect to the same common neighbor. We note that Unbiased-Deletion is equivalent to OPT-TDM for certain TDM such as Jaccard.

The results for scale-free and Facebook networks are shown in Fig. [1 and 2] respectively. For scale-free networks, we choose two scales \((n = 1000, 2000)\) and two values of the parameter \(\gamma\) \((\gamma = 2.0, 2.5)\), which are typical for social networks (Leskovec and Krevl 2014). For Facebook networks, we choose four scales, where the number of nodes \((n)\) and edges \((m)\) are specified in Fig. [2].

We can observe from Fig. [1 and 2] that Unbiased-Deletion can indeed be quite suboptimal. Moreover, the adversary can have a substantial impact on perceived likelihood of links (i.e., similarity scores) by deleting fewer than 0.5% of edges.

**Katz Similarity**

Next, we evaluate the performance of Greedy-Katz and compare it with Unbiased-Deletion. The results in Fig. [3] show that the Greedy-Katz is very effective in decreasing Katz similarity for both types of networks. For example, deleting 4 edges greedily in a scale-free network \((n = 500, \gamma = 2.5)\) can cause the Katz similarity to decrease by 80%. We note that the gap between Greedy-Katz and Unbiased-Deletion becomes larger as the size of the network grows. This is because for larger networks, Unbiased-Deletion has a lower chance of deleting the best edges.
ACT Distance Metric

Finally, we compare the performance of Local-ACT to Unbiased-Deletion for scale-free and Facebook networks. The results are shown in Fig. 1. From the results, we can see that both Local-ACT and Unbiased-Deletion can effectively increase the ACT distance. Moreover, our Local-ACT algorithm significantly outperforms Unbiased-Deletion for scale-free and Facebook networks with various scales.

Conclusion

We investigate the problem of minimizing the similarity between a target pair of nodes, with the goal of hiding the corresponding link, by deleting a limited number of edges. We divide similarity metrics associated with potential links into two broad classes: local metrics and global metrics. For local metrics, we show that an optimal attack can be computed in linear time, although the attack algorithm can be different for different classes of local metrics. Moreover, our experiments demonstrate that the resulting optimal attacks are significantly stronger than a recently proposed heuristic for a similar problem.

For global metrics, we prove that both the problem of minimizing Katz and that of maximizing ACT, or, equivalently, minimizing its inverse (two common such metrics) are NP-Hard. We then propose an efficient greedy algorithm (Greedy-Katz) and a principled heuristic algorithm (Local-ACT) for the two problems, respectively. Our experiments show that our approximate algorithms for Katz and ACT are highly effective in practice and, in particular, significantly outperform a recently proposed heuristic.

Figure 1: Comparison between optimal algorithms and Unbiased-Deletion on scale-free networks.

Figure 2: Comparison between optimal algorithms and Unbiased-Deletion on Facebook networks.

Figure 3: Comparison between Greedy-Katz and Unbiased-Deletion. Solid line: Greedy-Katz. Dotted line: Unbiased-Deletion. Lines in the same color show results on the same network.

Figure 4: Comparison between Local-ACT and Unbiased-Deletion. Solid line: Local-ACT. Dotted line: Unbiased-Deletion. Lines in the same color show results on the same network.
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