Superintegrability on N-dimensional spaces of constant curvature
from so$(N + 1)$ and its contractions

Francisco J. Herranz$^1,\dagger$ and Ángel Ballesteros$^2,\ddagger$

$^1$Departamento de Física, Escuela Politécnica Superior,
Universidad de Burgos, E-09006, Burgos, Spain.

$^2$Departamento de Física, Facultad de Ciencias,
Universidad de Burgos, E-09006, Burgos, Spain.

The Lie–Poisson algebra $so(N + 1)$ and some of its contractions are used to construct a family of superintegrable Hamiltonians on the $N$D spherical, Euclidean, hyperbolic, Minkowskian and (anti-)de Sitter spaces. We firstly present a Hamiltonian which is a superposition of an arbitrary central potential with $N$ arbitrary centrifugal terms. Such a system is quasi-maximally superintegrable since this is endowed with $2N - 3$ functionally independent constants of the motion (plus the Hamiltonian). Secondly, we identify two maximally superintegrable Hamiltonians by choosing a specific central potential and finding at the same time the remaining integral. The former is the generalization of the Smorodinsky–Winternitz system to the above six spaces, while the latter is a generalization of the Kepler–Coulomb potential, for which the Laplace–Runge–Lenz $N$-vector is also given. All the systems and constants of the motion are explicitly expressed in a unified form in terms of ambient and polar coordinates as they are parametrized by two contraction parameters (curvature and signature of the metric).

PACS numbers: 02.30.Ik; 02.20.Sv; 02.40.Ky

---

* Based on the contribution presented at the “XII International Conference on Symmetry Methods in Physics”, Yerevan (Armenia), July 2006.
To appear in Physics of Atomic Nuclei.

$\dagger$Electronic address: fjherranz@ubu.es

$\ddagger$Electronic address: angelb@ubu.es
I. INTRODUCTION

Let us consider the following potential on the \( N \)-dimensional (ND) Euclidean space [1]:

\[
U = \mathcal{F}(r) + \sum_{i=1}^{N} \frac{\beta_i}{x_i^2},
\]

where \( \beta_i \) are arbitrary real constants, \( x_i \) are Cartesian coordinates and \( \mathcal{F}(r) \) is an arbitrary smooth function depending on the Euclidean distance \( r = \left( \sum_{i=1}^{N} x_i^2 \right)^{1/2} \). This potential is known to be superintegrable and can be interpreted as the superposition of a central term with \( N \) centrifugal barriers associated to the \( \beta_i \)'s. Furthermore, two particular choices of \( \mathcal{F}(r) \) provide well known maximally superintegrable (MS) Euclidean Hamiltonians:

- If \( \mathcal{F}(r) = \omega^2 r^2 \) we obtain an isotropic harmonic oscillator with angular frequency \( \omega \). The \( N \) arbitrary centrifugal terms can be added to the oscillator potential keeping maximal superintegrability; the resulting potential is the Smorodinsky–Winternitz (SW) system [2–5].
- If \( \mathcal{F}(r) = -k/r \), where \( k \) is a real constant, we find the Kepler–Coulomb (KC) potential. In this case, only a maximum number of \((N - 1)\) centrifugal terms can be considered in order to preserve maximal superintegrability [1, 6, 7].

Our aim in this paper is to present a unified generalization of the superintegrable potential (1) and its two particular MS cases to the ND spherical, Euclidean, hyperbolic, Minkowskian and both de Sitter spaces. The approach we shall make use is based on the Lie–Poisson algebras associated to the Lie groups and subgroups involved in the construction of the above spaces as symmetrical homogeneous ones. Thus in the next section we introduce the basics on the Lie groups of isometries on these six ND spaces together with the two coordinate systems we shall deal with: \((N + 1)\) ambient coordinates in an auxiliary linear space and \( N \) intrinsic geodesic polar (spherical) coordinates. The kinetic energy which gives rise to the geodesic motion is studied in section III by starting from the metric. The generalization of the potential (1) is addressed in section IV in such a manner that general and global expressions for the Hamiltonian and its \( 2N - 3 \) functionally independent integrals of motion are explicitly given. Finally, last section is devoted to the study of the two MS Hamiltonians arising in the above family by choosing in an adequate way the radial function and by finding at the same time the remaining constant of the motion. Consequently, we obtain the generalization of the SW and generalized KC potentials for any value of the curvature and signature of the metric. We remark that these results generalized to arbitrary dimension \( N \) the 3D case recently studied in [8].
II. RIEMANNIAN SPACES AND RELATIVISTIC SPACETIMES

Let us consider a set of real Lie algebras $so_{\kappa_1,\kappa_2}(N+1)$ which come from $\mathbb{Z}_2^N$ graded contractions of $so(N+1)$, where $\kappa_1$ and $\kappa_2$ are two real contraction parameters. The non-vanishing Lie brackets of $so_{\kappa_1,\kappa_2}(N+1)$ in the basis spanned by $\{J_{\mu\nu}\} (\mu, \nu = 0, 1, \ldots, N; \mu < \nu)$ read [9]:

$$
\begin{align*}
[J_{ij}, J_{ik}] &= J_{jk}, & [J_{ij}, J_{jk}] &= -J_{ik}, & [J_{ik}, J_{jk}] &= J_{ij}, \\
[J_{1j}, J_{1k}] &= \kappa_2 J_{jk}, & [J_{1j}, J_{jk}] &= -J_{1k}, & [J_{1k}, J_{jk}] &= J_{1j}, \\
[J_{01}, J_{0k}] &= \kappa_1 J_{1k}, & [J_{01}, J_{1k}] &= -J_{0k}, & [J_{0k}, J_{1k}] &= \kappa_2 J_{01}, \\
[J_{0j}, J_{0k}] &= \kappa_1 \kappa_2 J_{jk}, & [J_{0j}, J_{jk}] &= -J_{0k}, & [J_{0k}, J_{jk}] &= J_{0j},
\end{align*}
$$

(2)

where $i, j, k = 2, \ldots, N$ and $i < j < k$. Both contraction parameters $\kappa_l$ ($l = 1, 2$) can take any real value. By scaling the Lie generators, each $\kappa_l$ can be reduced to either +1, 0 or −1; the limit $\kappa_l \to 0$ is equivalent to apply an Inönü–Wigner contraction.

The quadratic Casimir for $so_{\kappa_1,\kappa_2}(N+1)$, associated to the Killing–Cartan form, is given by

$$
\mathcal{C} = \kappa_2 J_{01}^2 + \sum_{j=2}^{N} J_{0j}^2 + \kappa_1 \sum_{j=2}^{N} J_{1j}^2 + \kappa_1 \kappa_2 \sum_{i,j=2}^{N} J_{ij}^2.
$$

Next from the Lie group $SO_{\kappa_1,\kappa_2}(N+1)$ with Lie algebra (2) we construct the following ND symmetrical homogeneous space:

$$
S^N_{[\kappa_1,\kappa_2]} = SO_{\kappa_1,\kappa_2}(N+1)/SO_{\kappa_2}(N), \quad SO_{\kappa_2}(N) = \langle J_{ij}; i, j = 1, \ldots, N \rangle.
$$

(4)

The parameter $\kappa_1$ turns out to be the constant sectional curvature of the space, while $\kappa_2$ determines the signature of the metric as $\text{diag}(+1, \kappa_2, \ldots, \kappa_2)$. In this way, we find that $S^N_{[\kappa_1,\kappa_2]}$ comprises well known spaces of constant curvature:

- When $\kappa_2$ is positive, say $\kappa_2 = +1$, we recover the three classical Riemannian spaces. These are the spherical $\kappa_1 > 0$, Euclidean $\kappa_1 = 0$, and hyperbolic spaces $\kappa_1 < 0$: $S^N_{[+]} = SO(N+1)/SO(N)$, $S^N_{[0]} = ISO(N)/SO(N)$ and $S^N_{[-]} = SO(N,1)/SO(N)$. The first two rows of (2) span the rotation Lie subalgebra $so(N)$, while the $N$ generators $J_{0i}$ ($i = 1, \ldots, N$) appearing in the last two rows play the role of translations. The curvature can be written as $\kappa_1 = \pm 1/R^2$ where $R$ is the radius of the space ($R \to \infty$ for the Euclidean case).

- When $\kappa_2$ is negative we find a Lorentzian metric corresponding to relativistic spacetimes; namely, the anti-de Sitter $\kappa_1 > 0$, Minkowskian $\kappa_1 = 0$, and de Sitter spaces $\kappa_1 < 0$: $S^N_{[+]} = SO(N-1,2)/SO(N-1,1)$, $S^N_{[0]} = ISO(N-1,1)/SO(N-1,1)$ and $S^N_{[-]} = SO(N,1)/SO(N-1,1)$. The generators $J_{01}$, $J_{0j}$, $J_{1j}$ and $J_{ij}$ ($i, j = 2, \ldots, N$)
are identified with time translation, space translations, boosts and spatial rotations, respectively. The first row of (2) is a rotation subalgebra $so(N-1)$ and the two first rows span the Lorentz subalgebra $so(N-1,1)$. The two contraction parameters can be expressed as $\kappa_1 = \pm 1/\tau^2$, where $\tau$ is the (time) universe radius, and $\kappa_2 = -1/c^2$, where $c$ is the speed of light.

- Finally, the contraction $\kappa_2 = 0$ gives rise to Newtonian (non-relativistic) spacetimes with a degenerate metric. Since we shall construct superintegrable systems on $\mathbb{S}^N_{[\kappa_1,\kappa_2]}$, for which the kinetic energy is provided by the metric, hereafter we assume $\kappa_2 \neq 0$.

In what follows we introduce an explicit model of the space $\mathbb{S}^N_{[\kappa_1,\kappa_2]}$ in terms of $(N+1)$ ambient coordinates and also of $N$ intrinsic geodesic quantities.

The vector representation of $so_{\kappa_1,\kappa_2}(N+1)$ is given by $(N+1) \times (N+1)$ real matrices [9]:

$$
\begin{align*}
J_{01} &= -\kappa_1 e_{01} + e_{10}, & J_{0j} &= -\kappa_1 \kappa_2 e_{0j} + e_{j0}, \\
J_{1j} &= -\kappa_2 e_{1j} + e_{j1}, & J_{jk} &= -e_{jk} + e_{kj}, \quad j, k = 2, \ldots, N,
\end{align*}
$$

where $e_{ij}$ is the matrix with entries $(e_{ij})_{lm} = \delta^1_l \delta^m_j$. Any generator $X \in so_{\kappa_1,\kappa_2}(N+1)$ fulfills

$$
X^T \mathbb{I}_\kappa + \mathbb{I}_\kappa X = 0, \quad \mathbb{I}_\kappa = \text{diag}(+1, \kappa_1, \kappa_1 \kappa_2, \ldots, \kappa_1 \kappa_2),
$$

so that any element $G \in SO_{\kappa_1,\kappa_2}(N+1)$ verifies $G^T \mathbb{I}_\kappa G = \mathbb{I}_\kappa$. Then $SO_{\kappa_1,\kappa_2}(N+1)$ is a group of isometries of $\mathbb{I}_\kappa$ acting on a linear ambient space $\mathbb{R}^{N+1} = (x_0, x_1, \ldots, x_N)$ through matrix multiplication. The origin $O$ in $\mathbb{S}^N_{[\kappa_1,\kappa_2]}$ has ambient coordinates $O = (1, 0, \ldots, 0)$ which is invariant under the (Lorentz) rotation subgroup $SO_{\kappa_2}(N)$ (4). The orbit of $O$ corresponds to the homogeneous space $\mathbb{S}^N_{[\kappa_1,\kappa_2]}$ which is contained in the “sphere” provided by $\mathbb{I}_\kappa$:

$$
\Sigma \equiv x_0^2 + \kappa_1 x_1^2 + \kappa_1 \kappa_2 \sum_{j=2}^{N} x_j^2 = 1.
$$

The $(N+1)$ ambient coordinates $x = (x_0, x_1, \ldots, x_N)$, subjected to (7), are also called Weierstrass coordinates. The metric on $\mathbb{S}^N_{[\kappa_1,\kappa_2]}$ follows from the flat ambient metric in $\mathbb{R}^{N+1}$ in the form:

$$
\text{ds}^2 = \frac{1}{\kappa_1} \left( \text{d}x_0^2 + \kappa_1 \text{d}x_1^2 + \kappa_1 \kappa_2 \sum_{j=2}^{N} \text{d}x_j^2 \right)_{\Sigma}.
$$

A differential realization of $so_{\kappa_1,\kappa_2}(N+1)$ (2), coming directly from the vector representation (5), is given by

$$
\begin{align*}
J_{01} &= \kappa_1 x_1 \partial_0 - x_0 \partial_1, & J_{0j} &= \kappa_1 \kappa_2 x_j \partial_0 - x_0 \partial_j, \\
J_{1j} &= \kappa_2 x_j \partial_1 - x_1 \partial_j, & J_{jk} &= x_k \partial_j - x_j \partial_k,
\end{align*}
$$

where $j, k = 2, \ldots, N$ and $\partial_\mu = \partial/\partial x_\mu$. 
Next we parametrize the \((N + 1)\) ambient coordinates \(x\) of a generic point \(P\) in terms of \(N\) intrinsic quantities \((r, \theta, \phi_3, \ldots, \phi_N)\) on the space \(S^N_{[\kappa_1][\kappa_2]}\) through the following action of \(N\) one-parametric subgroups of \(SO_s(N + 1)\) on the origin \(O\):

\[
x = \exp(\phi_{N-1} J_{N-1}) \exp(\phi_{N-2} J_{N-2}) \ldots \exp(\phi_3 J_{23}) \exp(\theta J_{12}) \exp(r J_{01}) O. \tag{10}
\]

This gives \((i = 2, \ldots, N - 1)\):

\[
x_0 = C_{\kappa_1}(r), \\
x_1 = S_{\kappa_1}(r) C_{\kappa_2}(\theta), \\
x_i = S_{\kappa_1}(r) S_{\kappa_2}(\theta) \prod_{s=3}^{i} \sin \phi_s \cos \phi_{i+1}, \\
x_N = S_{\kappa_1}(r) S_{\kappa_2}(\theta) \prod_{s=3}^{N} \sin \phi_s,
\]

where hereafter a product \(\prod_{i}\) such that \(s > i\) is assumed to be equal to 1. The \(\kappa\)-dependent trigonometric functions \(C_{\kappa}(x)\) and \(S_{\kappa}(x)\) are defined by [10, 11]:

\[
C_{\kappa}(x) = \begin{cases} 
\cos \sqrt{\kappa} x, & \kappa > 0, \\
1, & \kappa = 0, \\
\cosh \sqrt{-\kappa} x, & \kappa < 0.
\end{cases}
S_{\kappa}(x) = \begin{cases} 
\frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} x, & \kappa > 0, \\
x, & \kappa = 0, \\
\frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa} x, & \kappa < 0.
\end{cases}
\tag{12}
\]

Notice that here \(\kappa \in \{\kappa_1, \kappa_2\}\). The \(\kappa\)-tangent is defined by \(T_{\kappa}(x) = S_{\kappa}(x)/C_{\kappa}(x)\) and its contraction \(\kappa = 0\) gives \(T_0(x) = x\).

The canonical parameters \((r, \theta, \phi_3, \ldots, \phi_N)\) dual to \((J_{01}, J_{12}, J_{23}, \ldots, J_{N-1}N)\) are called geodesic polar coordinates. In order to explain their (physical) geometrical role, let us consider a (time-like) geodesic \(l_1\) and other \((N - 1)\) (space-like) geodesics \(l_j (j = 2, \ldots, N)\) in \(S^N_{[\kappa_1][\kappa_2]}\) which are orthogonal at the origin \(O\) (each translation \(J_{0i}\) moves \(O\) along \(l_i\)). Then [8, 12]:

- The radial coordinate \(r\) is the distance between the point \(P\) and the origin \(O\) measured along the geodesic \(l\) that joins both points. In the Riemannian spaces with \(\kappa_1 = \pm 1/R^2\), \(r\) has dimensions of length, \([r] = [R]\); notice that the dimensionless coordinate \(r/R \equiv \phi_1\) is usually taken instead of \(r\), and so considered as an ordinary angle [13]. In the relativistic spacetimes with \(\kappa_1 = \pm 1/c^2\), \(r\) has dimensions of a time-like length: \([r] = [\tau]\).
- The coordinate \(\theta\) is an ordinary angle in the three Riemannian spaces \((\kappa_2 = +1)\), say \(\theta \equiv \phi_2\), while it corresponds to a rapidity in the relativistic spacetimes \((\kappa_2 = -1/c^2)\) with dimensions \([\theta] = [c]\). For the six spaces, \(\theta\) parametrizes the orientation of \(l\) with respect to the basic (time-like) geodesic \(l_1\).
- The remaining \((N - 2)\) coordinates \(\phi_3, \phi_4, \ldots, \phi_N\) are ordinary angles for the six spaces and correspond to the polar angles of \(l\) relative to the reference flag at the origin \(O\) spanned by \(\{l_1, l_2\}, \{l_1, l_2, l_3\}, \ldots, \{l_1, \ldots, l_{N-1}\}\).
In the Riemannian cases \((r, \theta, \phi_3, \ldots, \phi_N)\) parametrize the complete space, while in the relativistic spacetimes these only cover the time-like region limited by the light-cone on which \(\theta \to \infty\). The flat contraction \(\kappa_1 = 0\) gives rise to the usual spherical coordinates in the Euclidean space \((\kappa_2 = 1)\).

By introducing (11) in (8), we obtain the metric in \(S^N_{[\kappa_1\kappa_2]}\) expressed in geodesic polar coordinates:

\[
ds^2 = dr^2 + \kappa_2 S^2_{\kappa_1}(r) \left\{ d\theta^2 + S^2_{\kappa_2}(\theta) \sum_{i=3}^{N} \left( \prod_{s=3}^{i-1} \sin^2 \phi_s \right) d\phi_i^2 \right\}. \tag{13}
\]

### III. FREE MOTION

The metric (13) gives rise to the kinetic energy \(T\) of a particle written in terms of the velocities \(\dot{r}, \dot{\theta}, \dot{\phi}_3, \ldots, \dot{\phi}_N\) which corresponds to the free Lagrangian of the geodesic motion on the space \(S^N_{[\kappa_1\kappa_2]}\); namely

\[
T = \frac{1}{2} \left( \dot{r}^2 + \kappa_2 S^2_{\kappa_1}(r) \left( \dot{\theta}^2 + S^2_{\kappa_2}(\theta) \sum_{i=3}^{N} \left( \prod_{s=3}^{i-1} \sin^2 \phi_s \right) \dot{\phi}_i^2 \right) \right). \tag{14}
\]

Then the canonical momenta are obtained through \(p = \partial T / \partial \dot{q}\):

\[
\begin{align*}
p_r &= \dot{r}, \\
p_\theta &= \kappa_2 S^2_{\kappa_1}(r) \dot{\theta}, \\
p_{\phi_i} &= \kappa_2 S^2_{\kappa_1}(r) S^2_{\kappa_2}(\theta) \left( \prod_{s=3}^{i-1} \sin^2 \phi_s \right) \dot{\phi}_i,
\end{align*}
\]

where \(i = 3, \ldots, N\). Hence the free Hamiltonian in the geodesic polar phase space \((q; p) = (r, \theta, \phi_3, \ldots, \phi_N; p_r, p_\theta, p_{\phi_3}, \ldots, p_{\phi_N})\) with respect to the canonical Lie–Poisson bracket,

\[
\{f, g\} = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right), \tag{16}
\]

is given by

\[
T = \frac{1}{2} \left( p_r^2 + \frac{1}{\kappa_2 S^2_{\kappa_1}(r)} \left( p_\theta^2 + \frac{1}{S^2_{\kappa_2}(\theta)} \sum_{i=3}^{N} \frac{p_{\phi_i}^2}{\prod_{s=3}^{i-1} \sin^2 \phi_s} \right) \right). \tag{17}
\]

Now we proceed to deduce a symplectic realization of the Lie generators of \(so_{\kappa_1, \kappa_2}(N+1)\). In ambient coordinates \(x_\mu\) and momenta \(p_\mu\) this comes from the vector fields (9) through the replacement \(\partial_\mu \to -p_\mu\):

\[
\begin{align*}
J_{01} &= x_0 p_1 - \kappa_1 x_1 p_0, \quad J_{0j} = x_0 p_j - \kappa_1 \kappa_2 x_j p_0, \\
J_{ij} &= x_i p_j - \kappa_2 x_j p_i, \quad J_{jk} = x_j p_k - x_k p_j,
\end{align*} \tag{18}
\]
where \( j, k = 2, \ldots, N \). The metric (8) provides the kinetic energy in the ambient velocities \( \dot{x}_\mu \) so that the momenta \( p_\mu \) read
\[
p_0 = \dot{x}_0 / \kappa_1, \quad p_1 = \dot{x}_1, \quad p_j = \kappa_2 \dot{x}_j, \quad j = 2, \ldots, N.
\]
(19)

By computing the velocities \( \dot{x}_\mu \) in the parametrization (11), and introducing the momenta (15) and (19), we can find the relationship between the ambient momenta and the geodesic polar ones (see [8] for \( N = 3 \)), which in turn allows us to obtain the generators (18) written in geodesic polar coordinates and conjugated momenta; these are:

- Translation generators \( (i = 2, \ldots, N - 1) \):

\[
J_{01} = C_{\kappa_2}(\theta)p_r - \frac{S_{\kappa_2}(\theta)}{T_{\kappa_1}(r)} p_\theta,
\]
\[
J_{0i} = \kappa_2 \frac{S_{\kappa_2}(\theta) \prod_{m=3}^{i+1} \sin \phi_m}{\tan \phi_{i+1}} p_r + \frac{C_{\kappa_2}(\theta) \prod_{m=3}^{i+1} \sin \phi_m}{T_{\kappa_1}(r) \tan \phi_{i+1}} p_\theta - \frac{p_{\phi_{i+1}}}{T_{\kappa_1}(r) S_{\kappa_2}(\theta) \prod_{l=3}^{i+1} \sin \phi_l},
\]
\[
J_{0N} = \kappa_2 S_{\kappa_2}(\theta) \prod_{m=3}^{N} \sin \phi_m p_r + \frac{C_{\kappa_2}(\theta) \prod_{m=3}^{N} \sin \phi_m}{T_{\kappa_1}(r)} p_\theta + \sum_{s=3}^{N} \cos \phi_s \prod_{l=3}^{N} \sin \phi_l p_{\phi_s}.
\]

- Rotation generators \( (i = 2, \ldots, N - 1 \text{ and } i < j = 3, \ldots, N - 1) \):

\[
J_{1i} = \cos \phi_{i+1} \prod_{m=3}^{i} \sin \phi_m p_\theta + \cos \phi_{i+1} \prod_{m=3}^{i} \sin \phi_m \frac{\cos \phi_s \prod_{l=3}^{s} \sin \phi_l p_{\phi_s}}{T_{\kappa_2}(\theta) \prod_{l=3}^{s} \sin \phi_l} - \sin \phi_{i+1} \frac{p_{\phi_{i+1}}}{T_{\kappa_2}(\theta) \prod_{l=3}^{i+1} \sin \phi_l},
\]
\[
J_{1N} = \prod_{m=3}^{N} \sin \phi_m p_\theta + \sum_{s=3}^{N} \cos \phi_s \prod_{m=3}^{N} \sin \phi_m \frac{p_{\phi_s}}{T_{\kappa_2}(\theta) \prod_{l=3}^{s} \sin \phi_l},
\]
\[
J_{ij} = \sin \phi_{i+1} \cos \phi_{j+1} \prod_{m=i+1}^{j} \sin \phi_m p_{\phi_{i+1}} - \frac{\cos \phi_{i+1} \sin \phi_{j+1} \prod_{l=i+1}^{j} \sin \phi_l p_{\phi_{j+1}}}{\prod_{l=i+1}^{j} \sin \phi_l},
\]
\[
+ \cos \phi_{i+1} \cos \phi_{j+1} \prod_{s=3}^{i+1} \sin \phi_s \frac{p_{\phi_s}}{T_{\kappa_2}(\theta) \prod_{l=3}^{s} \sin \phi_l},
\]
\[
J_{iN} = \sin \phi_{i+1} \prod_{m=i+1}^{N} \sin \phi_m p_{\phi_{i+1}} + \cos \phi_{i+1} \prod_{s=i+1}^{N} \sin \phi_s \frac{p_{\phi_s}}{T_{\kappa_2}(\theta) \prod_{l=i+1}^{s} \sin \phi_l},
\]
(21)

where from now on a sum \( \sum_{s=a}^{b} \) such that \( a > b \) is assumed to be equal to 0.

Then the following statement holds.

**Proposition 1.** (i) The generators (20) and (21) fulfill the commutation relations (2) with respect to the Lie–Poisson bracket (16).

(ii) All of them Poisson commute with \( T \) (17).
The first point can be proven by direct computations, while the second one comes from the fact that the kinetic energy can be obtained from the Casimir $C$ (3) as $2\kappa_2 T = C$ by introducing the above realization of the generators.

Therefore all the $N(N+1)/2$ generators of $so_{\kappa_1,\kappa_2}(N+1)$ give rise to integrals of motion for $T$. In order to characterize the maximal superintegrability of the geodesic motion on the space $S_{[\kappa_1][\kappa_2]}^N$ let us define two sets of $(N-1)$ functions, coming from the rotation generators, which are quadratic in the momenta:

$$\mathcal{J}^{(l)} = \sum_{j=2}^{l} J_{ijj}^2 + \kappa_2 \sum_{i,j=2}^{l} J_{ij}^2, \quad l = 2, \ldots, N, \quad \mathcal{J}_N^{(N)} = \mathcal{J}_{N}^{(N)} = \sum_{i,j=N-k+1}^{N} J_{ij}^2, \quad k = 2, \ldots, N-1.$$  \hspace{1cm} (22)

And it can be shown:

**Proposition 2.** (i) The $N$ functions $\{\mathcal{J}^{(2)}, \mathcal{J}^{(3)}, \ldots, \mathcal{J}^{(N)}, T\}$ are mutually in involution. The same property holds for the second set $\{\mathcal{J}_N^{(N)}, \mathcal{J}_{N-1}^{(N-1)}, \ldots, \mathcal{J}_2^{(2)}, T\}$.

(ii) The $2N-1$ functions $\{\mathcal{J}^{(2)}, \mathcal{J}^{(3)}, \ldots, \mathcal{J}^{(N)} \equiv \mathcal{J}_N^{(N)}, \mathcal{J}_{N-1}^{(N-1)}, \ldots, \mathcal{J}_3^{(3)}, \mathcal{J}_2^{(2)}, J_{0j}, T\}$, where $j$ is fixed ($j = 1, \ldots, N$), are functionally independent.

Consequently, $T$ is MS and its independent integrals of motion come from the (Lorentz) rotation subgroup plus one from the translation generators.

IV. QUASI-MAXIMALLY SUPERINTEGRABLE POTENTIALS

Let us consider the following potential $U(q) = U(r, \theta, \phi_3, \ldots, \phi_N)$ defined on $S_{[\kappa_1][\kappa_2]}^N$:

$$U = F'(x_0) + \sum_{s=1}^{N} \beta_i \frac{x_{s}^2}{r^2} \quad \text{(23)}$$

$$= F(r) + \frac{1}{S_{\kappa_1}^2(r)} \left( \frac{\beta_1 C_{\kappa_2}(\theta)}{S_{\kappa_2}^2(\theta)} \prod_{s=3}^{N} \sin^2 \phi_s \cos^2 \phi_{s+1} + \sum_{i=2}^{N-1} \frac{\beta_i}{S_{\kappa_2}^2(\theta) \prod_{s=3}^{N} \sin^2 \phi_s} \right),$$

where $F(C_{\kappa_1}(r)) \equiv F(r)$ is an arbitrary smooth function and $\beta_i$ are arbitrary real constants. This corresponds to the superposition of a (curved) central potential $F(r)$, only depending on the geodesic distance $r$, with $N$ centrifugal barriers associated with the $\beta_i$-terms. Therefore this is the generalization of the Euclidean potential (1) to $S_{[\kappa_1][\kappa_2]}^N$.

The Hamiltonian $H = T + U$, with kinetic energy (17) and potential (23), has $N(N-1)/2$ integrals of the motion quadratic in the momenta which come from the rotation generators (21) $(i, j = 2, \ldots, N)$:

$$I_{1i} = J_{1i}^2 + 2\beta_1 \kappa_2 \frac{x_i^2}{x_1^2} + 2\beta_2 \kappa_2 \frac{x_i^2}{x_2^2}, \quad I_{ij} = J_{ij}^2 + 2\beta_1 \kappa_2 \frac{x_i^2}{x_j^2} + 2\beta_2 \kappa_2 \frac{x_i^2}{x_j^2}. \quad \text{(24)}$$
In the geodesic polar phase space these are given by \((i, j = 2, \ldots, N - 1)\):

\[
I_{1i} = J_{1i}^2 + 2\beta_1\kappa_2^2 T^2_{\kappa_2}(\theta) \prod_{s=3}^{N} \sin^2 \phi_s \cos^2 \phi_{i+1} + \frac{2\beta_1\kappa_2}{T^2_{\kappa_2}(\theta) \prod_{s=3}^{N} \sin^2 \phi_s \cos^2 \phi_{i+1}},
\]

\[
I_{1N} = J_{1N}^2 + 2\beta_1\kappa_2^2 T^2_{\kappa_2}(\theta) \prod_{s=3}^{N} \sin^2 \phi_s + \frac{2\beta_N\kappa_2}{T^2_{\kappa_2}(\theta) \prod_{s=3}^{N} \sin^2 \phi_s},
\]

\[
I_{ij} = J_{ij}^2 + 2\beta_1\kappa_2 \prod_{s=i+1}^{N} \sin^2 \phi_s \cos^2 \phi_{j+1} + \frac{2\beta_j\kappa_2 \cos^2 \phi_{i+1}}{\prod_{s=i+1}^{N} \sin^2 \phi_s \cos^2 \phi_{j+1}},
\]

\[
I_{iN} = J_{iN}^2 + 2\beta_1\kappa_2 \prod_{s=i+1}^{N} \sin^2 \phi_s \cos^2 \phi_{i+1} + \frac{2\beta_N\kappa_2 \cos^2 \phi_{i+1}}{\prod_{s=i+1}^{N} \sin^2 \phi_s}.
\]

(25)

Obviously, neither all these constants are in involution, nor they are functionally independent. Similarly to (22), we define two sets of \((N - 1)\) functions:

\[
Q^{(l)} = \sum_{j=2}^{l} I_{1j} + \kappa_2 \sum_{i,j=2}^{l} I_{ij}, \quad l = 2, \ldots, N,
\]

\[
Q_{(N)} = Q^{(N)}, \quad Q_{(k)} = \sum_{i,j=N-k+1}^{N} I_{ij}, \quad k = 2, \ldots, N - 1.
\]

(26)

And superintegrability properties of \(H\) are determined by:

**Proposition 3.** (i) The \(N\) functions \(\{Q^{(2)}, Q^{(3)}, \ldots, Q^{(N)}, H\}\) are mutually in involution. The same holds for the set \(\{Q_{(N)}, Q_{(N-1)}, \ldots, Q_{(2)}, H\}\).

(ii) The \(2N - 2\) functions \(\{Q^{(2)}, Q^{(3)}, \ldots, Q^{(N)} \equiv Q_{(N)}, \ldots, Q_{(3)}, Q_{(2)}, H\}\) are functionally independent.

Notice that the difference with respect to the free motion described in proposition 2 is that now one constant of the motion is left to ensure maximal superintegrability (for \(T\) this role was played by one of the translations generators). In this sense we shall say that \(H\) is quasi-maximally superintegrable. Nevertheless, some specific choices for the arbitrary radial function \(F(r)\) lead to an additional integral thus providing MS potentials. Next we present the two relevant cases which correspond to the SW and the generalized KC systems on the space \(S^N_{[\kappa_1, \kappa_2]}\).

**V. MAXIMALLY SUPERINTEGRABLE POTENTIALS**

**A. Smorodinsky–Winternitz potential**

The harmonic oscillator potential on \(S^N_{[\kappa_1, \kappa_2]}\) is obtained through the following choice for the function \(F\):

\[
F'(x_0) = \beta_0 \left( \frac{1 - x_0^2}{\kappa_1 x_0^2} \right) = \beta_0 \left( \frac{x_1^2 + \kappa_2 \sum_{i=2}^{N} x_i^2}{x_0^2} \right), \quad F(r) = \beta_0 T_{\kappa_1}(r),
\]

(27)
where $\beta_0$ is an arbitrary real parameter ($\beta_0 = \omega^2$). This is just the Higgs oscillator [14, 15] formerly obtained in the curved Riemannian spaces. We can add the $N$ arbitrary centrifugal terms (23) to (27) thus obtaining the generalization of the SW system, $H^{SW} = T + U^{SW}$, to the space $S^{N}_{[\kappa_1][\kappa_2]}$:

$$U^{SW} = \beta_0 T_{\kappa_1}(r) + \frac{1}{S_{\kappa_1}(r)} \left( \frac{\beta_1}{C_{\kappa_2}(\theta)} + \sum_{i=2}^{N-1} \frac{\beta_i}{S_{\kappa_2}(\theta) \prod_{s=3}^{i} \sin^2 \phi_s \cos^2 \phi_{i+1}} + \frac{\beta_N}{S_{\kappa_2}(\theta) \prod_{s=3}^{N} \sin^2 \phi_s} \right).$$ (28)

The contraction $\kappa_1 = 0$ (with $\kappa_2 = +1$) of (28) reproduces the flat SW potential given in the Introduction but here written in polar coordinates. Notice that, under this contraction, the $N$ ambient coordinates $x_i$ ($i = 1, \ldots, N$) coincide with the Cartesian ones, while $x_0 = 1$ (see (11)). The 2D and 3D SW systems on the spherical and hyperbolic spaces have been constructed by following different approaches [16–20], and for the three ND Riemannian spaces altogether these can be found in [12, 21, 22]. Less developed are the SW Hamiltonians on relativistic spacetimes since, to our knowledge, only very recent results cover the (1+1)D [23, 24] and (2+1)D cases [8]. Moreover, SW-type systems on certain 2D [23] and ND [25, 26] spaces of nonconstant curvature have been, again very recently, studied.

The SW Hamiltonian on $S^{N}_{[\kappa_1][\kappa_2]}$ has additional constants of motion to those given in proposition 3. Similarly to what happened with the geodesic motion, any of the translation generators (20) gives rise to an integral quadratic in the momenta ($i = 2, \ldots, N$):

$$I_{0i} = J_{0i}^2 + 2\beta_0 \frac{x_i^2}{x_0^2} + 2\beta_1 \frac{x_0^2}{x_i^2}, \quad I_{0i} = J_{0i}^2 + 2\beta_0 \kappa_2 \frac{x_i^2}{x_0^2} + 2\beta_1 \kappa_2 \frac{x_0^2}{x_i^2},$$ (29)

to be compared with (24). In polar coordinates, these are ($i = 2, \ldots, N - 1$):

$$I_{0i} = J_{0i}^2 + 2\beta_0 \frac{T_{\kappa_1}(r) C_{\kappa_2}(\theta)}{T_{\kappa_1}(r) C_{\kappa_2}(\theta)},$$

$$I_{0i} = J_{0i}^2 + 2\beta_0 \kappa_2 \frac{T_{\kappa_1}(r) S_{\kappa_2}(\theta) \prod_{s=3}^{i} \sin^2 \phi_s \cos^2 \phi_{i+1}}{T_{\kappa_1}(r) S_{\kappa_2}(\theta) \prod_{s=3}^{i} \sin^2 \phi_s \cos^2 \phi_{i+1}},$$

$$I_{0N} = J_{0N}^2 + 2\beta_0 \kappa_2 \frac{T_{\kappa_1}(r) S_{\kappa_2}(\theta) \prod_{s=3}^{N} \sin^2 \phi_s + 2\beta_1 \kappa_2}{T_{\kappa_1}(r) S_{\kappa_2}(\theta) \prod_{s=3}^{N} \sin^2 \phi_s}. \quad (30)$$

From this set of $N$ additional integrals, we establish the superintegrability of $H^{SW}$.

**Proposition 4.** (i) The $N$ functions (30) Poisson commute with $H^{SW}$.

(ii) The $2N - 1$ functions $\{Q^{(2)}, Q^{(3)}, \ldots, Q^{(N)} \equiv Q_{(N)}, \ldots, Q_{(3)}, Q_{(2)}, I_{0j}, H\}$, where $j$ is fixed ($j = 1, \ldots, N$), are functionally independent.

Therefore, the known result concerning maximal superintegrability of the SW system on the three ND Riemannian spaces of constant curvature also holds for the relativistic spacetimes covering in a unified way the complete family $S^{N}_{[\kappa_1][\kappa_2]}$. 


B. Generalized Kepler–Coulomb potential

The KC potential \([18–20, 24, 27–31]\) on the space \(\mathbb{S}_{[\kappa_1, \kappa_2]}^N\) is achieved by choosing

\[
\mathcal{F}'(x_0) = -k \frac{x_0}{\sqrt{(1 - x_0^2)/\kappa_1}} = -k \frac{x_0}{\sqrt{x_1^2 + \kappa_2 \sum_{j=2}^{N} x_j^2}}, \quad \mathcal{F}(r) = -\frac{k}{T_{\kappa_1}(r)}, \tag{31}
\]

where \(k\) is an arbitrary real parameter. Such a potential is known to be MS on the three ND Riemannian spaces. Nevertheless, in this case it is not possible to add \(N\) arbitrary centrifugal terms keeping this property as it does happen with the SW potential in such a manner that, at least, one of the \(\beta_i\)-terms must vanishes (see [8] for the 3D case). In this way, we find that, in principle, there are \(N\) possible generalized KC (GKC) potentials, which can be understood as the superposition of the proper KC potential (31) together with \((N - 1)\) centrifugal terms appearing within (23). Explicitly, these are \((j = 2, \ldots, N - 1)\):

\[
\mathcal{U}_i^{GKC} = -\frac{k}{T_{\kappa_1}(r)} + \frac{1}{S_{\kappa_1}^2(r) \beta_{i}^{2}} \left( \frac{\beta_{i}}{\prod_{s=3}^{N} \sin^2 \phi_s \cos^2 \phi_{l+1}} + \frac{\beta_{N}}{\prod_{s=3}^{N} \sin^2 \phi_s} \right),
\]

\[
\mathcal{U}_j^{GKC} = -\frac{k}{T_{\kappa_1}(r)} + \frac{1}{S_{\kappa_1}^2(r) \beta_{j}^{2}} \left( \frac{\beta_{j}}{\prod_{s=3}^{N} \sin^2 \phi_s \cos^2 \phi_{l+1}} + \frac{\beta_{N}}{\prod_{s=3}^{N} \sin^2 \phi_s} \right), \tag{32}
\]

\[
\mathcal{U}_N^{GKC} = -\frac{k}{T_{\kappa_1}(r)} + \frac{1}{S_{\kappa_1}^2(r) \beta_{N}^{2}} \left( \frac{\beta_{N}}{\prod_{s=3}^{N} \sin^2 \phi_s \cos^2 \phi_{l+1}} \right).
\]

The contraction of a given \(\mathcal{U}_i^{GKC}\) to the Euclidean case gives the known result \([1, 6, 7]\) \(\mathcal{U}_i^{GKC} = -k/r + \sum_{l=1,l \neq i}^{N} \beta_l/x_l^2\) as commented in the Introduction.

For each of the \(N\) potentials \(\mathcal{U}_i^{GKC}\) there exists an additional constant of the motion given by \((i = 1, \ldots, N)\):

\[
L_i = \sum_{l=1,l \neq i}^{N} J_{0l} J_{li} + k \frac{\kappa_2 x_i}{\sqrt{x_1^2 + \kappa_2 \sum_{j=2}^{N} x_j^2}} = 2\kappa_2 \sum_{l=1,l \neq i}^{N} \beta_l \frac{x_0 x_i}{x_l^2}, \tag{33}
\]

where \(J_{li} = -J_{il}\) if \(i < l\). In the geodesic polar phase space these integrals turn out to be \((j = 2, \ldots, N - 1)\):

\[
L_1 = -\sum_{l=2}^{N} J_{0l} J_{1l} + k \kappa_2 \frac{C_{\kappa_2}(\theta)}{T_{\kappa_1}(r) S_{\kappa_2}^2(\theta)} \left( \frac{\beta_{i}}{\prod_{m=3}^{N} \sin^2 \phi_m \cos^2 \phi_{l+1}} + \frac{\beta_{N}}{\prod_{m=3}^{N} \sin^2 \phi_m} \right),
\]

\[
-2\kappa_2 \frac{C_{\kappa_2}(\theta)}{T_{\kappa_1}(r) S_{\kappa_2}^2(\theta)} \left( \sum_{l=2}^{N} \frac{\beta_{i}}{\prod_{m=3}^{N} \sin^2 \phi_m \cos^2 \phi_{l+1}} + \frac{\beta_{N}}{\prod_{m=3}^{N} \sin^2 \phi_m} \right),
\]

\[
\frac{\beta_{i}}{\prod_{m=3}^{N} \sin^2 \phi_m \cos^2 \phi_{l+1}} + \frac{\beta_{N}}{\prod_{m=3}^{N} \sin^2 \phi_m} \right),
\]

\[
\frac{\beta_{i}}{\prod_{m=3}^{N} \sin^2 \phi_m \cos^2 \phi_{l+1}} + \frac{\beta_{N}}{\prod_{m=3}^{N} \sin^2 \phi_m} \right),
\]
\[ L_j = \sum_{l=1; l\neq j}^{N} J_{ql} J_{lj} + k \kappa_2 S_{\kappa_2} (\theta) \prod_{s=3}^{j} \sin \phi_s \cos \phi_{j+1} - 2 \kappa_2 \prod_{s=3}^{j} \frac{\sin \phi_s \cos \phi_{j+1}}{T_{\kappa_1} (r) S_{\kappa_2} (\theta)} \left( \beta_1 T_{\kappa_2}^2 (\theta) \right) + \sum_{l=2; l\neq j}^{N-1} \frac{\beta_l}{\prod_{m=3}^{l} \sin^2 \phi_m \cos^2 \phi_{l+1}} + \frac{\beta_N}{\prod_{m=3}^{N} \sin^2 \phi_m} \right), \quad (34) \]

\[ L_N = \sum_{l=1}^{N} J_{ql} J_{lN} + k \kappa_2 S_{\kappa_2} (\theta) \prod_{s=3}^{N} \sin \phi_s \]

\[ -2 \kappa_2 \frac{\prod_{s=3}^{N} \sin \phi_s}{T_{\kappa_1} (r) S_{\kappa_2} (\theta)} \left( \beta_1 T_{\kappa_2}^2 (\theta) + \sum_{l=2}^{N-1} \frac{\beta_l}{\prod_{m=3}^{l} \sin^2 \phi_m \cos^2 \phi_{l+1}} \right). \]

The MS of each Hamiltonian \( \mathcal{H}_i^{\text{GKC}} = \mathcal{T} + \mathcal{U}_i^{\text{GKC}} \) (\( i \) fixed and \( i = 1, \ldots, N \)) is stated as:

**Proposition 5.** (i) The function \( L_i \) (34) Poisson commutes with \( \mathcal{H}_i^{\text{GKC}} \).

(ii) The \( 2N - 1 \) functions \( \{ Q^{(2)}, Q^{(3)}, \ldots, Q^{(N)} \} \equiv Q_{(N)}, \ldots, Q_{(3)}, Q_{(2)}, L_i, \mathcal{H}_i^{\text{GKC}} \) are functionally independent.

We remark that for the three Riemannian cases with \( \kappa_2 = +1 \), the \( N \) GKC Hamiltonians are all equivalent providing the superposition of the KC potential with \( (N - 1) \) centrifugal barriers. In contrast, for the three relativistic spacetimes with \( \kappa_2 < 0 \), \( \mathcal{U}_1^{\text{GKC}} \) is formed by a time-like KC potential with \( (N - 1) \) space-like centrifugal barriers, while the remaining \( (N - 1) \) potentials \( \mathcal{U}_j^{\text{GKC}} \) \( (j = 2, \ldots, N) \) are all equivalent and composed by the time-like KC potential, a time-like centrifugal barrier with parameter \( \beta_1 \), and other \( (N - 2) \) space-like ones. In any case, to consider initially \( N \) possible GKC Hamiltonians affords for a direct understanding of the appearance of the Laplace–Runge–Lenz vector on \( S^N_{[\kappa_1]\kappa_2} \) as the following statements show.

**Proposition 6.** Let us take the Hamiltonian \( \mathcal{H}_i^{\text{GKC}} = \mathcal{T} + \mathcal{U}_i^{\text{GKC}} \) (\( i \) fixed and \( i = 1, \ldots, N \)) with \( \beta_j = 0 \) \( (j \neq i) \). Then

(i) The two functions \( L_i, L_j \) Poisson commute with \( \mathcal{H}_i^{\text{GKC}} \).

(ii) The set \( \{ Q^{(2)}, Q^{(3)}, \ldots, Q^{(N)} \} \equiv Q_{(N)}, \ldots, Q_{(3)}, Q_{(2)}, \mathcal{H}_i^{\text{GKC}} \) together with either \( L_i \) or \( L_j \) are \( 2N - 1 \) functionally independent functions.

**Proposition 7.** Let \( \beta_i = 0 \ \forall i \), then:

(i) The \( N \) GKC potentials reduce to its common KC potential on \( S^N_{[\kappa_1]\kappa_2} \): \( \mathcal{U}_i^{\text{GKC}} \equiv \mathcal{U}^{\text{KC}} = -k/\ T_{\kappa_1} (r) \).

(ii) The \( N \) functions \( (j = 2, \ldots, N - 1) \):

\[
L_1 = -\sum_{l=2}^{N} J_{ql} J_{1l} + k \kappa_2 C_{\kappa_2} (\theta),
\]

\[
L_j = \sum_{l=1; l\neq j}^{N} J_{ql} J_{lj} + k \kappa_2 S_{\kappa_2} (\theta) \prod_{s=3}^{j} \sin \phi_s \cos \phi_{j+1}, \quad (35)
\]
\[ L_N = \sum_{i=1}^{N-1} J_{0i} J_{iN} + k \kappa_2 S_{\kappa_2}(\theta) \prod_{s=3}^{N} \sin \phi_s, \]

Poisson commute with \( \mathcal{H}^{\text{KC}} = \mathcal{T} + \mathcal{U}^{\text{KC}}. \)

(iii) The set \( \{ Q^{(2)}, Q^{(3)}, \ldots, Q^{(N)} \} \equiv Q_{(N)}, \ldots, Q_{(3)}, Q_{(2)}, \mathcal{H}^{\text{KC}} \) together with any of the components \( L_i \) \((i = 1, \ldots, N)\) are \(2N - 1\) functionally independent functions.

We stress that (35) are the components of the Laplace–Runge–Lenz \(N\)-vector on \(S^N_{[\kappa_1]\kappa_2}\); these are transformed as a vector under the action of the generators of the subgroup \(SO_{\kappa_2}(N)\) (4) (either rotations for \(\kappa_2 > 0\) or Lorentz transformations for \(\kappa_2 < 0\)).

Proofs and details of all the results here presented will be given elsewhere, together with a physical/geometrical description of the MS SW and GKC Hamiltonians on each particular space \(S^N_{[\kappa_1]\kappa_2}\).

\section*{ACKNOWLEDGEMENTS}

This work was partially supported by the Ministerio de Educación y Ciencia (Spain, Project FIS2004-07913) and by the Junta de Castilla y León (Spain, Project VA013C05).

[1] N.W. Evans, \textit{Phys. Rev. A} \textbf{41}, 5666–5676 (1990).
[2] J. Fris, V. Mandrosov, Y.A. Smorodinsky, M. Uhlir, and P. Winternitz, \textit{Phys. Lett.} \textbf{16}, 354–356 (1965).
[3] N.W. Evans, \textit{Phys. Lett. A} \textbf{147}, 483–486 (1990).
[4] N.W. Evans, \textit{J. Math. Phys.} \textbf{32}, 3369–3375 (1991).
[5] C. Grosche, G.S. Pogosyan, and A.N. Sissakian, \textit{Fortschr. Phys.} \textbf{43}, 453–521 (1995).
[6] M.A. Rodríguez and P. Winternitz, \textit{J. Math. Phys.} \textbf{43}, 1309–1322 (2002).
[7] E.G. Kalnins, G.C. Williams, W. Miller, and G.S. Pogosyan, \textit{J. Phys. A: Math. Gen.} \textbf{35}, 4755–4773 (2002).
[8] F.J. Herranz and A. Ballesteros, \textit{SIGMA} \textbf{2}, 010(22) (2006).
[9] F.J. Herranz and M. Santander, \textit{J. Phys. A: Math. Gen.} \textbf{30}, 5411–5426 (1997).
[10] F.J. Herranz, R. Ortega and M. Santander, \textit{J. Phys. A: Math. Gen.} \textbf{33}, 4525–4551 (2000).
[11] F.J. Herranz and M. Santander, \textit{J. Phys. A: Math. Gen.} \textbf{35}, 6601–6618 (2002).
[12] A. Ballesteros, F.J. Herranz, M. Santander, and T. Sanz-Gil, \textit{J. Phys. A: Math. Gen.} \textbf{36}, L93–L99 (2003).
[13] A.A. Izmest‘ev, G.S. Pogosyan, and A.N. Sissakian, J. Math. Phys., 40, 1549–1573 (1999).
[14] P.W. Higgs, J. Phys. A: Math. Gen. 12, 309–323 (1979).
[15] H.I. Leemon, J. Phys. A: Math. Gen. 12, 489–501 (1979).
[16] C. Grosche, G.S. Pogosyan, and A.N. Sissakian, Fortschr. Phys. 43, 523–563 (1995).
[17] E.G. Kalnins, W. Miller, and G.S. Pogosyan, J. Math. Phys. 38, 5416–5433 (1997).
[18] E.G. Kalnins, W. Miller, and G.S. Pogosyan, J. Phys. A: Math. Gen. 33, 6791–6806 (2000).
[19] E.G. Kalnins, J.M. Kress, G.S. Pogosyan, and W. Miller, J. Phys. A: Math. Gen. 34, 4705–4720 (2001).
[20] M.F. Rañada and M. Santander, J. Math. Phys. 40, 5026–5057 (1999).
[21] F.J. Herranz, A. Ballesteros, M. Santander, and T. Sanz-Gil, in Superintegrability in Classical and Quantum Systems, Eds. P. Tempesta et al, CRM Proceedings and Lecture Notes 37, (AMS, Providence), 75–89 (2004); math-ph/0501035.
[22] A. Ballesteros and F.J. Herranz, J. Phys. A: Math. Theor. 40, F51–F59 (2007).
[23] A. Ballesteros, F.J. Herranz, and O. Ragnisco, J. Phys. A: Math. Gen. 38, 7129–7144 (2005).
[24] J.F. Cariñena, M.F. Rañada, M. Santander, and T. Sanz-Gil, J. Nonlinear Math. Phys. 12, 230–252 (2005).
[25] O. Ragnisco, A. Ballesteros, F.J. Herranz, and F. Musso, SIGMA 3, 026(20) (2007).
[26] A. Ballesteros, A. Enciso, F.J. Herranz, and O. Ragnisco, A maximally superintegrable system on an \( n \)-dimensional space of nonconstant curvature, arXiv:math-ph/0612080.
[27] E. Schrödinger, Proc. R. Ir. Acad. A 46, 9–16 (1940).
[28] E.G. Kalnins, W. Miller, and G.S. Pogosyan, J. Math. Phys. 41, 2629–2657 (2000).
[29] A. Nersessian and G. Pogosyan, Phys. Rev. A 63, 020103(4) (2001).
[30] E.G. Kalnins, W. Miller, and G.S. Pogosyan, Phys. Atomic Nuclei 65, 1086–1094 (2002).
[31] J.F. Cariñena, M.F. Rañada, and M. Santander, J. Math. Phys. 46, 052702(18) (2005).