Ground state and nodal solutions for fractional Orlicz problems with lack of regularity and without the Ambrosetti-Rabinowitz condition

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Abstract

We consider a non-local Shr¨ odinger problem driven by the fractional Orlicz g-Laplace operator as follows

\[ (-\Delta g)^\alpha u + g(u) = K(x)f(x,u), \quad \text{in } \mathbb{R}^d, \]  

(P)

where \( d \geq 3 \), \((-\Delta g)^\alpha\) is the fractional Orlicz g-Laplace operator, \( f : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R} \) is a measurable function and \( K \) is a positive continuous function. Employing the Nehari manifold method and without assuming the well-known Ambrosetti-Rabinowitz and differentiability conditions on the non-linear term \( f \), we prove that the problem (P) has a ground state of fixed sign and a nodal (or sign-changing) solutions.

Keywords: Ground state, Nodal solutions, Fractional Orlicz-Sobolev spaces, Nehari method, Generalized subdifferential.

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1 Introduction

Recently, much attention has been focused on the study of non-linear problems involving non-local operators. These types of operators arise in several areas such as the description of many physical phenomena (see [29, Part II - Chapters 12 and 13]).

In this paper we consider the following non-local Shr¨ odinger equation

\[ (-\Delta g)^\alpha u + g(u) = K(x)f(x,u), \quad \text{in } \mathbb{R}^d, \]  

(P)

where \( d \geq 3 \), \( f : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R} \) is a measurable function, \( K \) is a positive continuous function, and \((-\Delta g)^\alpha\) is the fractional Orlicz g-Laplace operator introduced in [13] and defined as

\[ (-\Delta g)^\alpha u(x) = \text{p.v.} \int_{\mathbb{R}^d} g \left( \frac{|u(x) - u(y)|}{|x - y|^\alpha} \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} \frac{dy}{|x - y|^{d+\alpha}}, \]  

(1.1)

where p.v. being a commonly used abbreviation for "in the principle value sense", \( \alpha \in (0, 1) \) and \( G \) is an N-function such that \( g = G' \). We recall the definition of N-function and its properties later in Section 2. The variational setting for the fractional g-Laplace operator is the fractional Orlicz-Sobolev space \( W^{\alpha,G}(\mathbb{R}^d) \), which is introduced in [13]. For more details on the fractional Orlicz-Sobolev spaces, we refer the reader to [3, 4, 5, 14, 15, 21] and the references therein.

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Observe that when \( \alpha = 1 \) and \( g(t) = |t|^{p-2}t, \ p > 1, \) the problem (P) turns into the classical p-Laplacian problem
\[
-\text{div}(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = K(x)f(x, u).
\]
When \( \alpha = 1 \) and \( g(t) = a(|t|)t, \) the problem (P) transmute into the Orlicz g-Laplacian problem
\[
-\text{div}(a(|\nabla u|)|\nabla u|^{p-2}\nabla u) + a(|u|)u = K(x)f(x, u), \] where \( \text{div}(a(|\nabla u|)|\nabla u|) \) is the Orlicz g-Laplace operator.

When \( \alpha \in (0, 1) \) and \( g(t) = |t|^{p-2}t, \) the problem (P) transformed into the fractional p-Laplacian problem
\[
-(\Delta_{p})^{\alpha}u + |u|^{p-2}u = K(x)f(x, u), \] where \( (\Delta_{p})^{\alpha}u \) is the fractional p-Laplace operator.

In the last decades, the existence of ground state and nodal solutions for the above problems (classical p-Laplacian, Orlicz g-Laplacian and fractional p-Laplacian problems) have been studied extensively. We do not intend to review the huge bibliography, we just emphasize that the Nehari method is a very effective tool for proving the existence of solutions of such problems, see [1, 2, 7, 8, 9, 11, 12, 16, 17, 21, 22, 23, 27, 30, 34] and the references therein.

In [8], S. Barile and G. M. Figueiredo have studied the following equation
\[
-\text{div}(a(|\nabla u|)|\nabla u|^{p-2}\nabla u) + V(x)b(|u|)|u|^{p-2} = K(x)f(u), \] in \( \mathbb{R}^d, \) (P1)
where \( d \geq 3, \ 2 \leq p < d, \ a, \ b, \) are \( C^1 \) real functions and \( V, K \) are continuous positives functions. By assuming the well-Known Ambrosetti-Rabinowitz (AR for short), differentiability \( (f \in C^1) \) conditions on the non-linear term \( f \) and by using a minimization argument coupled with a quantitative deformation lemma, they proved the existence of a least energy sign-changing solution for equation (P1) with two nodal domains.

In [21], G. M. Figueiredo considered the following equation
\[
-M \left( \int_{\Omega} g(|\nabla u|)dx \right) \Delta_g u = f(u), \] in \( \Omega, \) (P2)
where \( \Omega \) is a bounded domain in \( \mathbb{R}^d, \ M \) is \( C^1 \) function and \( \Delta_g u := \text{div}(a(|\nabla u|)|\nabla u|) \) is the Orlicz g-Laplace operator \( (g(t) = a(|t|)t). \) By considering the (AR) and differentiability conditions on the non-linear term \( f \) and following the approaches employed in [8], the author proved the existence of a least energy nodal solution for equation (P2).

In [2], V. Ambrosio and T. Isernia have studied the following fractional equation
\[
-\alpha \Delta^{\alpha}u + V(x)u = K(x)f(u), \] in \( \mathbb{R}^d, \) (P3)
where \( \alpha \in (0, 1) \) and \( d > 2\alpha, \ (\Delta)^{\alpha}u \) is the fractional Laplace operator and \( V, K \) are continuous functions. Under the (AR) and differentiability conditions on the non-linear term \( f \) and by applying a minimization method combined with a quantitative deformation lemma, they proved the existence of a least energy nodal solution. In [7], the authors extended the result obtained in [2] to a non-local generalized fractional Orlicz equations of type (P3) (under the assumptions considered in [2, 8, 21] on the non-linear term).

In [22], without the (AR) and differentiability conditions on the non-linear term \( f, \ G. \ M. \ Figueiredo \) and J. R. Santos Júnior established the existence of least energy sign-changing solutions for problem (P3). Indicate that the authors developed a new approach based on the topological degree. In [1], by following closely the approach developed in [22] and under the same assumptions on the non-linear term \( f, \ V. \ Ambrosio \) et al. proved the existence of a least energy nodal solution for equation (P3).

The goal of this paper is to prove the existence of a ground state and a least energy nodal weak solution for the generalized fractional Orlicz problem (P) without the (AR) and differentiability conditions on the non-linear term \( f. \) We mean by weak solution of problem (P) a critical point \( u \in W^{\alpha,G}(\mathbb{R}^d) \) of the energy functional \( J \) associated to problem (P):
\[
J'(u)v = 0, \] for all \( v \in W^{\alpha,G}(\mathbb{R}^d) \) (the functional \( J \) will be defined later in Section 4). Our approaches are based on the Nehari method and the generalized subdifferential. The main features and difficulties involved in the study of the problem (P) are listed as follows:
The non-local character of the fractional Orlicz g-Laplacian. More precisely, in contrast with the classic case \[8, 9, 11, 12, 17\], in the fractional Orlicz framework we do not have the following decompositions

\[ J(u) = J(u^+) + J(u^-), \text{ and } J'(u^-)u^- = J'(u^+)u^+ = 0, \]

where \( u = u^+ + u^- \)

for \( u \) belonging to a subset \( \mathcal{M} \) of Nehari manifold \( \mathcal{N} \) of \( J \) (the definitions of \( u^+, u^- \), and the sets \( \mathcal{N}, \mathcal{M} \) will be specified later in Section 4). Such facts make the use of the minimization method more difficult. To overcome this difficulty, we use a new estimate which is inspired by the work \[32\].

Unlike with the hypotheses on the non-linear term \( f \) stated in \[2, 7, 31, 32\], in our present work, we do not require the differentiability of \( f \) (see hypotheses \((Hf)\) Section 2). So, we do not hope the existence of a differentiable structure in the sets \( \mathcal{N} \) and \( \mathcal{M} \). For more details about this subject, we refer the reader to \[10\]. To surmount the lack of differentiability, we use the idea employed in \[25\], which is based on the non-smooth multiplier rule of Clarke \[18, \text{Theorem 10.47, p. 221}\].

Since we do not assume the (AR) condition on \( f \), it is not trivial that minimizing sequences are bounded. We overcome this difficulty, with adaptation of arguments employed in \[27\]. Moreover, we invoke Miranda’s theorem \[28\] to show that \( \mathcal{M} \neq \emptyset \).

To the best of our knowledge, there is only two papers in the literature, see \[7, 31\], devoted for the existence and multiplicity of sign-changing solutions for fractional Orlicz problems under the (AR) condition. There are no results treating the existence and multiplicity of sign-changing solutions for fractional Orlicz problems without the (AR) condition.

This paper is organized as follows. In Section 2, firstly, we set the variational framework related to the problem \(\mathbf{P}\) and we establish some properties about N-functions. Secondly, we present the definition of the generalized subdifferential. In Section 3, we set the hypotheses on the weight function \( K \) and the non-linear term \( f \), and we state our main result (Theorem 3.2). In Section 4, we present the energy functional associated to problem \(\mathbf{P}\) and we prove some technical lemmas. In Section 5, we show the existence of a ground state solution for problem \(\mathbf{P}\). Finally, in Section 6 we prove the existence of a least energy nodal weak solution.

2 Mathematical preliminaries

2.1 Framework setting: Fractional Orlicz-Sobolev Spaces

In this subsection, we recall some necessary properties about N-functions and the fractional Orlicz-Sobolev spaces. For more details we refer the reader to \[5, 6, 13, 33\].

In order to construct a fractional Orlicz-Sobolev space setting for problem \(\mathbf{P}\), we consider the following assumptions on \( G \) and \( g \):

\((H_G)\) \quad \( g : \mathbb{R} \to \mathbb{R} \) is an odd, continuous and non-decreasing function and \( G : \mathbb{R} \to \mathbb{R}^+ \) defined by

\[
G(t) = \int_0^t g(s) \, ds,
\]

such that \( G \) and \( g \) satisfy the following assumptions

\((g_1)\) \quad \( g(t) > 0 \), for all \( t > 0 \), \( g(0) = 0 \) and \( \lim_{t \to +\infty} g(t) = +\infty \).

\((g_2)\) \quad There exist \( g^-, g^+ \in (1, d) \) such that

\[
g^- \leq \frac{g(t)t}{G(t)} \leq g^+, \quad \text{for all } t > 0.
\]

\((g_3)\) \quad \( g \in C^1(\mathbb{R}^+_*) \) and \( g^- - 1 \leq \frac{g'(t)t}{g(t)} \leq g^+ - 1 \), for all \( t > 0 \).
(g_4) \int_0^1 \frac{G^{-1}(t)}{t^{1/\alpha}} dt < \infty \text{ and } \int_1^{+\infty} \frac{G^{-1}(t)}{t^{1/\alpha}} dt = \infty.

G \text{ is an N-function: } G \text{ is even, positive, continuous and convex function, Moreover } \frac{G(t)}{t} \to 0 \text{ as } t \to 0 \text{ and } \frac{G(t)}{t} \to +\infty \text{ as } t \to +\infty \text{ (see [33]).}

The conjugate N-function of G denoted \( \tilde{G} \) is defined by

\[ \tilde{G}(t) = \int_0^t \tilde{g}(s) \, ds, \]

where \( \tilde{g} : \mathbb{R} \to \mathbb{R} \) is given by \( \tilde{g}(t) = \sup\{s : g(s) \leq t\} \). Since g is a continuous function, it comes that \( \tilde{g}(\cdot) = g^{-1}(\cdot) \).

Involving the functions G and \( \tilde{G} \), we have the Young’s inequality

\[ st \leq G(s) + \tilde{G}(t). \quad (2.2) \]

We say that an N-function G satisfies the \( \triangle_2 \)-condition, if there exists C > 0 such that

\[ G(2t) \leq CG(t), \text{ for all } t > 0. \quad (2.3) \]

The assumption (g_2) implies that G and \( \tilde{G} \) satisfy the \( \triangle_2 \)-condition (see [33]).

Let A and B are two N-functions, we say that A is essentially stronger than B (B \( \ll \) A in symbols), if for every positive constant k, we have

\[ \lim_{t \to +\infty} \frac{B(kt)}{A(t)} = 0. \]

Another important function related to the N-function G, is the Sobolev conjugate N-function denoted \( G_* \) and defined by

\[ G_*^{-1}(t) = \int_0^t \frac{G^{-1}(s)}{s^{1/\alpha}} \, ds, \quad t > 0. \]

\[ G_*(t) = \int_0^t g_*(s) \, ds \]

and according to assumption (g_2),

\[ g_* \leq g_*(t) \leq g_*^+, \text{ for all } t > 0, \text{ where } g_*^+ = \frac{dg^+}{d-g^+} \text{ and } g_*^- = \frac{dg^-}{d-g^-}. \quad (2.4) \]

Since G satisfies the \( \triangle_2 \)-condition, the Orlicz space \( L^G(\mathbb{R}^d) \) is the vectorial space of all measurable function \( u : \mathbb{R}^d \to \mathbb{R} \) satisfies

\[ \tilde{\rho}(u) := \int_{\mathbb{R}^d} G(u) \, dx < \infty \text{ (see [33]).} \]

\( L^G(\mathbb{R}^d) \) is a Banach space under the Luxemburg norm

\[ \|u\|_{(G)} = \inf \left\{ \lambda > 0 : \tilde{\rho}(\frac{u}{\lambda}) \leq 1 \right\}. \]

Next, we introduce the fractional Orlicz-Sobolev space. We denote by \( W^{\alpha,G}(\mathbb{R}^d) \) the fractional Orlicz-Sobolev space defined by

\[ W^{\alpha,G}(\mathbb{R}^d) = \left\{ u \in L^G(\mathbb{R}^d) : \tilde{p}(\alpha; u) < \infty \right\}, \quad (2.5) \]

where \( \tilde{p}(\alpha; u) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G\left( \frac{u(x) - u(y)}{|x-y|^\alpha} \right) \frac{dx dy}{|x-y|^{d'}}. \)

The space \( W^{\alpha,G}(\mathbb{R}^d) \) is equipped with the norm,

\[ \|u\|_{\alpha,G} = \|u\|_{(G)} + [u]_{(\alpha,G)}, \quad (2.6) \]
In the sequel we will use $\| \cdot \|_{C}$ and reflexive Banach space. Moreover, $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{\alpha,G}(\mathbb{R}^d)$ (see [13, Proposition 2.10]).

Let
\[
\rho(\alpha; u) := \tilde{\rho}(u) + \overline{\rho}(\alpha; u) \quad \text{and} \quad \|u\| = \inf \left\{ \lambda > 0 : \frac{u}{\lambda} \leq 1 \right\}. \tag{2.8}
\]

Evidently, $\|u\|$ is an equivalent norm to $\|u\|_{\alpha,G}$ with the relation
\[
\frac{1}{2} \|u\|_{\alpha,G} \leq \|u\| \leq 2 \|u\|_{\alpha,G}, \quad \text{for all } u \in W^{\alpha,G}(\mathbb{R}^d).
\]

In the sequel we will use $\| \cdot \|$ as a norm for the space $W^{\alpha,G}(\mathbb{R}^d)$.

As we mentioned in Section 1, the fractional $g$-Laplace operator is defined by
\[
(-\triangle_g)^{\alpha} u(x) = \text{p.v.} \int_{\mathbb{R}^d} g\left(\frac{|u(x) - u(y)|}{|x - y|^{\alpha}}\right) \frac{u(x) - u(y)}{|x - y|^{d+\alpha}} \, dy
= \text{p.v.} \int_{\mathbb{R}^d} g\left(\frac{|u(x) - u(y)|}{|x - y|^{\alpha}}\right) \, dy \quad \text{(since } g \text{ is odd).} \tag{2.9}
\]

$(-\triangle_g)^{\alpha}$ is well defined between $W^{\alpha,G}(\mathbb{R}^d)$ and its topological dual space $(W^{\alpha,G}(\mathbb{R}^d))^* = W^{-\alpha,\tilde{G}}(\mathbb{R}^d)$. In fact, in [13, Theorem 6.12], the following representation formula is provided
\[
\langle (-\triangle_g)^{\alpha} u, v \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g\left(\frac{|u(x) - u(y)|}{|x - y|^{\alpha}}\right) \frac{v(x) - v(y)}{|x - y|^{d+\alpha}} \, dx \, dy \tag{2.10}
\]

for all $u, v \in W^{\alpha,G}(\mathbb{R}^d)$. Where $\langle \cdot, \cdot \rangle$ is the duality brackets for the pair $\left(W^{\alpha,G}(\mathbb{R}^d), W^{-\alpha,\tilde{G}}(\mathbb{R}^d)\right)$.

In what follows, we give some theorems and lemmas related to N-functions and the fractional Orlicz-Sobolev space.

**Theorem 2.1.** (see [2])
Under the assumption $(H_G)$, the continuous embedding $W^{\alpha,G}(\mathbb{R}^d) \hookrightarrow L^B(\mathbb{R}^d)$ holds, for all $B$ N-function satisfies the $\Delta_2$-condition and $B \ll \tilde{G}^*$.

Under the assumptions $(g_1) - (g_3)$, some elementary inequalities and properties listed in the following lemmas are valid. See [5, 6, 21, 24].

**Lemma 2.2.** If $G$ is an N-function, then
\[
G(a + b) \geq G(a) + G(b), \quad \text{for all } a, b \geq 0.
\]

**Lemma 2.3.** Under the assumptions $(g_1) - (g_3)$, the functions $G$ and $\tilde{G}$ satisfy the following inequalities
\[
\tilde{G}(G(t)) \leq G(2t) \quad \text{and} \quad \tilde{G}\left(\frac{G(t)}{t}\right) \leq G(t) \quad \forall \, t \geq 0. \tag{2.11}
\]

**Lemma 2.4.** Assume that the assumptions $(g_1) - (g_2)$ hold, then
\begin{enumerate}
  \item \(\min\{a^g, a^{g^+}\} G(t) \leq G(at) \leq \max\{a^g, a^{g^+}\} G(t), \quad \text{for all } a, t \geq 0.
  \item \(\min\{a^{g-1}, a^{g^+ - 1}\} g(t) \leq g(at) \leq \max\{a^{g-1}, a^{g^+ - 1}\} g(t), \quad \text{for all } a, t \geq 0.
  \item \(\min\{a^{g^-}, a^{g^+_s}\} G_s(t) \leq G_s(at) \leq \max\{a^{g^-}, a^{g^+_s}\} G_s(t), \quad \text{for all } a, t \geq 0.
  \item \(\min\{a^{g^- - s}, a^{g^+_s - s}\} \hat{G}(t) \leq \hat{G}(at) \leq \max\{a^{g^--s}, a^{g^+_s - s}\} \hat{G}(t), \quad \text{for all } a, t \geq 0.
\end{enumerate}
Lemma 2.5. Assume that the assumptions \((g_1)-(g_3)\) hold, then

1. \(\min\{\|u\|_{L^G}, \|u\|_{L^G}^+\} \leq \tilde{\rho}(u) \leq \max\{\|u\|_{L^G}, \|u\|_{L^G}^+\} \), for all \(u \in L^G(\mathbb{R}^d)\).
2. \(\min\{\|u\|_{L^G}, \|u\|_{L^G}^+\} \leq \tilde{\rho}(u) \leq \max\{\|u\|_{L^G}, \|u\|_{L^G}^+\} \), for all \(u \in L^G(\mathbb{R}^d)\).
3. \(\min\{\|u\|_{L^G}, |u|^+\} \leq \tilde{\rho}(u) \leq \max\{\|u\|_{L^G}, |u|^+\} \), for all \(u \in L^G(\mathbb{R}^d)\).
4. \(\min\{\|u\|_{L^G}, |u|^+\} \leq \rho(\alpha; u) \leq \max\{\|u\|_{L^G}, |u|^+\} \), for all \(u \in L^G(\mathbb{R}^d)\).

2.2 The ”generalized subdifferential”

A main tool used in the present paper is the subdifferential theory of Clark [11, 19] for locally Lipschitz functionals. Let \(X\) be a Banach space, \(X^*\) its topological dual, and let \(\langle \cdot, \cdot \rangle_X\) denote the duality brackets for the pair \((X, X^*)\).

Definition 2.6. Let the functional \(\phi : X \to \mathbb{R}\). We say that \(\phi\) is locally Lipschitz if, for every \(x \in X\), there exists an open neighborhood \(U(x)\) of \(x\) and \(k_x > 0\) such that

\[
|\phi(u) - \phi(v)| \leq k_x \|u - v\|_X \quad \text{for all } u, v \in U(x).
\]

Definition 2.7. Given a locally Lipschitz function \(\phi : X \to \mathbb{R}\), the ”generalized directional derivative” of \(\phi\) at \(u \in X\) in the direction \(v \in X\), denoted by \(\hat{\partial}\phi(u; v)\), is defined by

\[
\hat{\partial}\phi(u; v) = \lim_{t \to 0} \frac{\phi(x + tv) - \phi(x)}{t},
\]

Definition 2.8. The ”generalized subdifferential” of \(\phi\) at \(u \in X\) is the set \(\partial\phi(u) \subseteq X^*\) given by

\[
\partial\phi(u) = \{\phi^* \in X^* : \langle \phi^*, v \rangle_X \leq \hat{\partial}\phi(u; v) \quad \text{for all } v \in X\}.
\]

The Hahn-Banach theorem implies that \(\partial\phi(u) \neq \emptyset\) for all \(u \in X\), it is convex and \(w^*\)-compact (in weak topology sense). If \(\phi\) is also convex, then it coincides with the subdifferential in the sense of convex functionals (see [29]). If \(\phi \in C^1(X, \mathbb{R})\), then \(\partial\phi(u) = \{\phi'(u)\}\). Note that the generalized subdifferential has a remarkable calculus, similar to that in the classical derivative (see [11, 19, 20]).

In the following section, we state our hypotheses (on \(f\) and \(K\)) and main result.

3 Hypotheses on \(f\) and \(K\), and main result

Next, we set the hypotheses on the weight function \(K(\cdot)\) and the reaction function \(f(\cdot, \cdot)\).

\((H_K)\) \(K : \mathbb{R}^d \to \mathbb{R}\) is a continuous function and satisfies

1. \((K_1)\) \(K(x) > 0\), for all \(x \in \mathbb{R}^d\) and \(K \in L^\infty(\mathbb{R}^d)\).
2. \((K_2)\) If \(\{A_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^d\) is a sequence of Borel sets such that the Lebesgue measure \(mes(A_n) \leq R\), for all \(n \in \mathbb{N}\) and some \(R > 0\), then

\[
\lim_{r \to +\infty} \int_{A_n \cap B_r(0)} K(x) \, dx = 0, \quad \text{uniformly in } n \in \mathbb{N}.
\]

\((H_f)\) \(f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}\) is a measurable function such that for a.a. \(x \in \mathbb{R}^d\), \(f(x, 0) = 0\), \(f(x, \cdot)\) is locally Lipschitz and

\[
(f_1) \lim_{|s| \to 0^+} \frac{f(x, s)}{g(s)} = 0 \quad \text{uniformly in } x \in \mathbb{R}^d.
\]
Therefore,

\[ \lim_{|s| \to +\infty} \frac{f(x, s)}{g_*(s)} = 0 \text{ uniformly in } x \in \mathbb{R}^d. \]

\[ \lim_{s \to +\infty} \frac{F(x, s)}{|s|^{\theta}} = +\infty \text{ uniformly in } x \in \mathbb{R}^d, \text{ where } F(x, s) = \int_0^{s} f(x, t) dt. \]

\[ 0 < (g^+ - 1)f(x, s)s < f^+(x, s)s^2, \text{ for a.a. } x \in \mathbb{R}^d, \text{ all } f^+(x, s) \in \partial_s f(x, s), \text{ and all } |s| > 0. \]

**Remark 3.1.**  • Under hypothesis (f) and by the generalized subdifferential calculus of Clarke [19, p. 48], for a.a. \( x \in \mathbb{R}^d \), we have

\[ s \mapsto \frac{f(x, s)}{|s|^{\theta - 1}} \text{ is increasing on } (0, +\infty) \text{ and on } (-\infty, 0) \]

and

\[ s \mapsto f(x, s)s - g^+ F(x, s) \text{ is increasing on } [0, +\infty) \text{ and decreasing on } (-\infty, 0]. \]

• The assumption \((H_f)\) is weaker than the \((AR)\) condition. Indeed, the function \( f(s) = |s|^{\theta - 2} \text{ln}(1 + |s|) \) (for the sake of simplicity, we drop the \( x \)-dependence) satisfies hypotheses \((H_f)\) but not the \((AR)\) condition.

• By hypothesis \((f)\) and the fact that \( f(x, 0) = 0 \), for a.a. \( x \in \mathbb{R}^d \), we have

\[ f(x, s) \geq 0 \leq 0, \text{ for a.a. } x \in \mathbb{R}^d \text{ and all } s \geq 0 \leq 0. \]

Therefore,

\[ F(x, t) = \int_0^t f(x, s) ds \geq 0, \text{ for a.a. } x \in \mathbb{R}^d \text{ and all } t \geq 0. \]

**Theorem 3.2.** Suppose that the hypotheses \((H_f)\), \((H_G)\) and \((H_K)\) hold, then problem \([P]\) has a ground state solution \( \bar{u} \in W^{\alpha, G}(\mathbb{R}^d) \) of fixed sign and a nodal weak solution \( \tilde{u} \in W^{\alpha, G}(\mathbb{R}^d) \).

In the following section, we give the energy functional associated to problem \([P]\) and some technical lemmas.

4 Energy functional and technical lemmas

We start by given the energy functional associated to problem \([P]\).

Let \( J : W^{\alpha, G}(\mathbb{R}^d) \to \mathbb{R} \) be the energy functional associated to problem \([P]\) defined by

\[ J(u) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G \left( \frac{u(x) - u(y)}{|x - y|^\alpha} \right) \frac{dx dy}{|x - y|^\alpha} + \int_{\mathbb{R}^d} G(u) \, dx - \int_{\mathbb{R}^d} K(x)F(x, u) \, dx, \text{ for all } u \in W^{\alpha, G}(\mathbb{R}^d). \]

In view of hypotheses \((H_G), (H_f)\) and \((H_K), J \in C^1(W^{\alpha, G}(\mathbb{R}^d)) \) and

\[ \langle J'(u), v \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{u(x) - u(y)}{|x - y|^\alpha} \right) \frac{v(x) - v(y)}{|x - y|^\alpha} \, dx dy + \int_{\mathbb{R}^d} g(u) v \, dx - \int_{\mathbb{R}^d} K(x) f(x, u) v \, dx, \]

for all \( u, v \in W^{\alpha, G}(\mathbb{R}^d) \).
Definition 4.1. We say that \( u \in W^{\alpha,G}(\Omega) \) is a weak solution of problem (P) if
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{w(x) - w(y)}{|x-y|^{\alpha}} \right) v(x) - v(y) \, dx \, dy + \int_{\mathbb{R}^d} g(u) v(\partial^+u) \, dx = \int_{\mathbb{R}^d} K(x) f(x, u) v(\partial^+u) \, dx, \quad \text{for all} \ W^{\alpha,G}(\mathbb{R}^d).
\]

Let \( \mathcal{N} \) be the Nehari manifold for \( J \), defined by
\[
\mathcal{N} = \left\{ u \in W^{\alpha,G}(\mathbb{R}^d) : u \neq 0, \langle J'(u), u \rangle = 0 \right\}
\]
where \( \langle \cdot, \cdot \rangle \) is the duality brackets for the pair \( (W^{\alpha,G}(\mathbb{R}^d)^*, W^{\alpha,G}(\mathbb{R}^d)) \). Since we seek nodal solutions, we consider the following subset of \( \mathcal{N} \)
\[
\mathcal{M} = \left\{ w \in W^{\alpha,G}(\mathbb{R}^d) : w^+ \neq 0, w^- \neq 0, \langle J'(w), w^+ \rangle = \langle J'(w), w^- \rangle = 0 \right\}.
\]
Recall that \( w^+ = \max \{w, 0\}, w^- = \min \{w, 0\} \) for \( w \in W^{\alpha,G}(\mathbb{R}^d) \). Evidently, \( w^\pm \in W^{\alpha,G}(\mathbb{R}^d) \), \( w = w^+ + w^- \) and \( |w| = |w^+| + |w^-| \).

In what follows, we give some technical lemmas and results which are crucial for the proof of Theorem 4.2.

Lemma 4.2. Assume that hypothesis \((g_3)\) holds. Then, the function defined by
\[
s \mapsto G(s) - \frac{1}{g^+(s)} g(s) s
\]
is non-decreasing on \((0, +\infty)\) and non-increasing on \((-\infty, 0)\).

Proof. From \((g_3)\), we have,
\[
g(s) - \frac{1}{g^+} g'(s) s - \frac{1}{g^+} g(s) \geq \left( \frac{g^+ - 1}{g^+} - \frac{g^- - 1}{g^+} \right) g(s) \geq 0, \quad \text{for all} \ s \in (0, +\infty).
\]
Which gives that,
\[
s \mapsto G(s) - \frac{1}{g^+} g(s) s \text{ is non-decreasing on } (0, +\infty).
\]
Exploiting the fact that \( G(s) - \frac{1}{g^+} g(s) s \) is even on \( \mathbb{R} \), we deduce that
\[
s \mapsto G(s) - \frac{1}{g^+} g(s) s \text{ is non-increasing on } (-\infty, 0).
\]
This ends the proof.

Lemma 4.3. [Lemma 3.2]
Assume that the hypotheses \((f_1) - (f_2)\) and \((H_K)\) hold. Let \( \{u_n\}_{n \in \mathbb{N}} \) be a sequence such that \( u_n \rightharpoonup u \) in \( W^{\alpha,G}(\mathbb{R}^d) \), then
\[
\begin{align*}
(1) \quad & \lim_{n \to +\infty} \int_{\mathbb{R}^d} K(x) F(x, u_n) \, dx = \int_{\mathbb{R}^d} K(x) F(x, u) \, dx. \\
(2) \quad & \lim_{n \to +\infty} \int_{\mathbb{R}^d} K(x) f(x, u_n) u_n \, dx = \int_{\mathbb{R}^d} K(x) f(x, u) u \, dx. \\
(3) \quad & \lim_{n \to +\infty} \int_{\mathbb{R}^d} K(x) f(x, u_n^+) u_n^+ \, dx = \int_{\mathbb{R}^d} K(x) f(x, u^+) u^+ \, dx.
\end{align*}
\]

Lemma 4.4. [Lemma 4.3]
Assume that the hypotheses \((H_f)\), \((H_G)\) and \((H_K)\) hold. Let \( w \in \mathcal{M} \), then
\[
\langle J'(w^\pm), w^\pm \rangle \leq \langle J'(w), w^\pm \rangle.
\]
Lemma 4.5. Assume that the hypotheses \((H_f), (H_G)\) and \((H_K)\) hold. Then, for all \(w \in W^{\alpha,G}(\mathbb{R}^d)\) such that \(w^\pm \neq 0\), we have

\[
J(w^+) + J(w^-) < J(w).
\]

Proof. Let \(w \in W^{\alpha,G}(\mathbb{R}^d)\) such that \(w^\pm \neq 0\). We argue by contradiction, suppose that

\[
J(w) \leq J(w^+) + J(w^-).
\]

Firstly, we observe that

\[
J(w) = J(w^+) + J(w^-) + \int_{\text{supp}(w^-)} \int_{\text{supp}(w^+)} G \left( \frac{w^+(x) - w^-(y)}{|x - y|^{\alpha}} \right) \frac{dxdy}{|x - y|^d} + \int_{\text{supp}(w^+)} \int_{\text{supp}(w^-)} G \left( \frac{w^-(x) - w^+(y)}{|x - y|^{\alpha}} \right) \frac{dxdy}{|x - y|^d}.
\]

By (4.1) and the fact that \(G(\cdot)\) is an even positive function, then

\[
\begin{cases}
\int_{\text{supp}(w^-)} \int_{\text{supp}(w^+)} G \left( \frac{w^+(x) - w^-(y)}{|x - y|^{\alpha}} \right) \frac{dxdy}{|x - y|^d} = 0,
\int_{\text{supp}(w^+)} \int_{\text{supp}(w^-)} G \left( \frac{w^-(x) - w^+(y)}{|x - y|^{\alpha}} \right) \frac{dxdy}{|x - y|^d} = 0.
\end{cases}
\]

Thus, one has

\[
w^+(x) = w^-(y) \text{ and } w^-(x) = w^+(y), \text{ for a.a. } x, y \in \mathbb{R}^d.
\]

Therefore, \(w^\pm = 0\) in \(\mathbb{R}^d\) (since \(\text{supp}(w^+) \cap \text{supp}(w^-) = \emptyset\)). Which is a contradiction. This ends the proof. \(\square\)

5 Ground state solutions

In this section, we prove the existence of a ground state solution for problem \((P)\).

Proposition 5.1. Assume that the hypotheses \((H_f), (H_G)\) and \((H_K)\) hold, then for all \(u \in W^{\alpha,G}(\mathbb{R}^d)\setminus\{0\}\), there exists a unique \(t_u > 0\) such that \(t_u u \in \mathcal{N}\).

Proof. Let \(u \in W^{\alpha,G}(\mathbb{R}^d)\setminus\{0\}\), consider the following fibering map \(h_u : (0, +\infty) \to \mathbb{R}\) defined by

\[
h_u(t) = \langle J(tu), tu \rangle \text{ for all } t > 0.
\]

From hypotheses \((f_1)\) and \((f_2)\), for all \(\varepsilon > 0\) there exists \(C_\varepsilon > 0\) such that

\[
f(x, s) \leq \varepsilon g(s) + C_\varepsilon g_*(s), \text{ for a.a. } x \in \mathbb{R}^d \text{ and all } s \in \mathbb{R}.
\]

Using (5.1), Lemma 2.2 and assumptions \((g_2)\) and \((K_1)\), we get

\[
h_u(t) \geq g^{-t^g_\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G \left( \frac{u(x) - u(y)}{|x - y|^{\alpha}} \right) \frac{dxdy}{|x - y|^d} + \int_{\mathbb{R}^d} g(tu) tudx
\]

\[
- \varepsilon \|K\|_\infty \int_{\mathbb{R}^d} g(tu) tudx - C_\varepsilon K\|_\infty \int_{\mathbb{R}^d} g_*(tu) t^\varepsilon t^g dx
\]

\[
\geq g^{-t^g_\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G \left( \frac{u(x) - u(y)}{|x - y|^{\alpha}} \right) \frac{dxdy}{|x - y|^d} + (1 - \varepsilon \|K\|_\infty) g^{-t^g_\varepsilon} \int_{\mathbb{R}^d} G(u) dx
\]

\[
- g^+_* t^g_\varepsilon \int_{\mathbb{R}^d} G_*(u) dx
\]

\[
= g^{-t^g_\varepsilon} \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G \left( \frac{u(x) - u(y)}{|x - y|^{\alpha}} \right) \frac{dxdy}{|x - y|^d} + (1 - \varepsilon \|K\|_\infty) \int_{\mathbb{R}^d} G(u) dx \right]
\]

\[
- g^+_* t^g_\varepsilon \int_{\mathbb{R}^d} G_*(u) dx.
\]
Choosing $\varepsilon \in (0, \frac{1}{\|K\|_\infty})$ in (5.2) and using the fact that $g^+ < g^-$, we find $t_0 > 0$ small enough such that

$$h_u(t) > 0, \forall t \in (0, t_0).$$

Let $A \subset \text{supp}(u)$ such that the Lebesgue measure of $A$ is positive, $\text{mes}(A) > 0$. Using (5.2), (5.3), Lemmas 2.4, 2.5, assumptions $(g_2)$, and $(K_1)$, for $t$ large, we find

$$\frac{h_u(t)}{t^{g^+}} \leq g^+ \max \left\{ \|u\|^{g^+}, \|u\|^{g^+} \right\} - \int_{\mathbb{R}^d} K(x) \frac{f(x, tu)}{t^{g^+}} dx \leq g^+ \max \left\{ \|u\|^{g^+}, \|u\|^{g^+} \right\} - \int_{\mathbb{R}^d} K(x) \frac{F(x, tu)}{t^{g^+}} dx \leq g^+ \max \left\{ \|u\|^{g^+}, \|u\|^{g^+} \right\} - \int_{\mathbb{R}^d} K(x) \frac{F(x, tu)}{|tu|^{g^+}} |u|^{g^+} dx$$

(5.3)

In light of hypothesis $(f_3)$, we see that

$$\lim_{t \to +\infty} \frac{F(x, tu)}{|tu|^{g^+}} |u|^{g^+} = +\infty,$$ uniformly for all $x \in A$. (5.4)

By assumption $(K_1)$ and Fatou’s lemma, it follows that

$$\int_A K(x) \frac{F(x, tu)}{|tu|^{g^+}} |u|^{g^+} dx \to +\infty$$ as $t \to +\infty$. (5.5)

From (5.3) and (5.5), we get

$$\limsup_{t \to +\infty} \frac{h_u(t)}{t^{g^+}} \leq -\infty.$$

Therefore, there is $t_1 > 0$ large enough such that

$$h_u(t) < 0, \forall t \in (t_1, +\infty).$$

According to Bolzano’s theorem, there exists $t_u > 0$ such that $h_u(t_u) = 0$. Hence, $(J'(t_u), t_u) = 0$, that is, $t_u u \in \mathcal{N}$.

In what follows, we prove the uniqueness of $t_u > 0$. Let $t_1, t_2$ be the two different positive number such that $t_i u \in \mathcal{N}$, $i = 1, 2$.

Firstly, we consider the case $u \in \mathcal{N}$. Without loss of generality, we may take $t_1 = 1$ and $t_1 \neq t_2$. Thus,

$$\langle J'(u), u \rangle = 0$$

(5.6)

and

$$\langle J'(t_2 u), u \rangle = 0.$$ (5.7)

From (5.6), it yields that

$$\int_{\mathbb{R}^d} K(x) \frac{f(x, u)}{|u|^{g^+}} |u|^{g^+} - 1 u dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{u(x) - u(y)}{|x - y|^\alpha} \right) \frac{u(x) - u(y)}{|x - y|^{\alpha + d}} dxdy + \int_{\mathbb{R}^d} g(u) u dx.$$ (5.8)

If $t_2 < t_1 = 1$, then, by (5.7) and Lemma 2.3, we obtain

$$\int_{\mathbb{R}^d} K(x) \frac{f(x, t_2 u)}{|t_2 u|^{g^+}} |u|^{g^+} - 1 u dx \geq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{u(x) - u(y)}{|x - y|^\alpha} \right) \frac{u(x) - u(y)}{|x - y|^{\alpha + d}} dxdy + \int_{\mathbb{R}^d} g(u) u dx.$$ (5.9)
Subtracting (5.9) from (5.8), using (3.1) and hypothesis (K1), we infer that
\[
0 < \int_{\mathbb{R}^d} K(x)|u|^\gamma - 1 u \left( \frac{f(x, u)}{|u|^\gamma - 1} - \frac{f(x, t_2 u)}{|t_2 u|^\gamma - 1} \right) \, dx \leq 0. \tag{5.10}
\]
Which is a contradiction.

If \( t_2 > t_1 = 1 \), then, by (5.7) and Lemma 2.4, we get
\[
\int_{\mathbb{R}^d} K(x)f(x, t_2 u) |u|^\gamma - 1 u \, dx \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{u(x) - u(y)}{|x - y|^\alpha} \right) u(x) - u(y) \, dx \, dy + \int_{\mathbb{R}^d} g(u) \, u \, dx. \tag{5.11}
\]
Subtracting (5.11) from (5.9), using (3.1) and hypothesis (K1), we deduce that
\[
0 \leq \int_{\mathbb{R}^d} K(x)|u|^\gamma - 1 u \left( \frac{f(x, u)}{|u|^\gamma - 1} - \frac{f(x, t_2 u)}{|t_2 u|^\gamma - 1} \right) \, dx < 0. \tag{5.12}
\]
Which is a contradiction too. Therefore \( t_1 = t_2 = 1 \).

Secondly, for the case \( u \notin \mathcal{N} \). Let \( v = t_1 u \in \mathcal{N}, t_1 \neq 1 \) and \( t_2 u = \frac{t_2}{t_1} t_1 u = \frac{t_2}{t_1} v \in \mathcal{N} \). Applying the same arguments as above, we prove that \( \frac{t_2}{t_1} = 1 \).

This ends the proof. \( \square \)

**Proposition 5.2.** Assume that the hypotheses \((H_f), (H_G)\) and \((H_K)\) hold. Then for all \( u \in \mathcal{N} \),
\[
J(tu) \leq J(u), \text{ for all } t > 0.
\]

**Proof.** Let \( u \in \mathcal{N} \) and consider the fibering map \( k_u : (0, +\infty) \rightarrow \mathbb{R} \) defined by
\[
k_u(t) = J(tu) \text{ for all } t > 0.
\]
By the hypotheses \((f_1)\) and \((f_2)\), for all \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that
\[
F(x, s) \leq \varepsilon G(s) + C_\varepsilon G^*(s) \text{, for a.a. } x \in \mathbb{R}^d \text{ and all } |s| > 0. \tag{5.13}
\]
Using (5.13), Theorem 2.1, Lemmas 2.2-2.3 and hypothesis (K1), then
\[
k_u(t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G \left( \frac{tu(x) - tu(y)}{|x - y|^\alpha} \right) \, dx \, dy + \int_{\mathbb{R}^d} G(tu) \, dx - \int_{\mathbb{R}^d} K(x) \, F(x, tu) \, dx
\geq t^{\gamma^+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G \left( \frac{u(x) - u(y)}{|x - y|^\alpha} \right) \, dx \, dy + (1 - \varepsilon \|K\|_\infty) \int_{\mathbb{R}^d} G(tu) \, dx
- C_\varepsilon \|K\|_\infty \int_{\mathbb{R}^d} G^*(tu) \, dx
\geq t^{\gamma^+} (1 - \varepsilon \|K\|_\infty) \rho(\alpha; u) - t^{\gamma^-} C_\varepsilon \|K\|_\infty \int_{\mathbb{R}^d} G^*(u) \, dx.
\]
Taking \( \varepsilon \in (0, \frac{1}{\|K\|_\infty}) \), having in mind that \( g^+ < g^- \), there is \( t_0 > 0 \) sufficiently small such that
\[
0 < k_u(t), \text{ for all } t \in (0, t_0). \tag{5.14}
\]
Let \( A \subset \text{supp}(u) \) such that the Lebesgue measure of \( A \) is positive, \( mes(A) > 0 \).
It’s clear that, for \( t \) large enough,
\[
\frac{k_u(t)}{t^{\gamma^+}} \leq g^+ \max \left\{ \|u\|^{\gamma^+}, \|u\|^{\gamma^-} \right\} - \int_{\mathbb{R}^d} K(x) \frac{F(x, tu)}{|tu|^{\gamma^+}} \, d|u|^{\gamma^+} \, dx. \tag{5.15}
\]
Under (5.5),
\[
\limsup_{t \to +\infty} k_u(t) \leq -\infty.
\]
Therefore, the map \(k_u(.)\) has a global maximum \(t_u > 0\). So, \(t_u\) is a critical point for \(k_u(.)\):
\[
\langle J'(t_uu), u \rangle = 0, \quad t_uu \in \mathcal{N}.
\]
By Proposition 5.1 and the fact that \(u \in \mathcal{N}\), we deduce that \(t_u = 1\).
Hence,
\[
J(tu) \leq J(u), \quad \text{for all } t > 0.
\]
This completes the proof.

In order to prove the existence of a ground state solution for problem (15), we consider the following minimization problem
\[
m_0 := \inf_{\mathcal{N}} J.
\]

**Proposition 5.3.** Suppose that the hypotheses \((H_f), (H_G)\) and \((H_K)\) are satisfied, then \(0 < m_0\).

**Proof.** Let \(u \in W^{\alpha,G}(\mathbb{R}^d) \setminus \{0\}\) such that \(|u| \leq 1\). Using assumption \((K_1)\), (5.13), Lemma 2.4 and Theorem 2.1, we get
\[
J(u) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{G(u(x) - u(y))}{|x - y|^{\alpha}} \, dx \, dy + \int_{\mathbb{R}^d} G(u) \, dx - \int_{\mathbb{R}^d} K(x)F(x, u) \, dx
\]
\[
\geq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{G(u(x) - u(y))}{|x - y|^{\alpha}} \, dx \, dy + (1 - \varepsilon\|K\|_{\infty}) \int_{\mathbb{R}^d} G(u) \, dx
\]
\[
- C_{\varepsilon}\|K\|_{\infty} \int_{\mathbb{R}^d} G_*(u) \, dx
\]
\[
\geq (1 - \varepsilon\|K\|_{\infty}) \rho(\alpha; u) - C_{\varepsilon}\|K\|_{\infty} \int_{\mathbb{R}^d} G_*(u) \, dx
\]
\[
\geq (1 - \varepsilon\|K\|_{\infty}) \|u\|_{G_*} - C_{\varepsilon}\|K\|_{\infty} \max \left\{ \|u\|_{G_*}, \|u\|_{G_*} \right\}
\]
\[
\geq (1 - \varepsilon\|K\|_{\infty}) \|u\|_{G_*} - C_{\varepsilon}\|K\|_{\infty} C_1 \|u\|_{G_*}, \quad \text{for all } \varepsilon > 0.
\]

Where \(C_{\varepsilon}\) is the constant given by (5.13). Taking \(\varepsilon \in (0, \frac{1}{\|K\|_{\infty}})\) and using the fact that \(g^+ < g_*^\varepsilon\), we find \(\varrho \in (0, 1)\) small enough and \(\eta > 0\) such that
\[
J(u) \geq \eta, \quad \text{for all } \|u\| = \varrho.
\]
Let \(u \in \mathcal{N}\), choosing \(t_u > 0\) such that \(|t_uu| = \varrho\). By Proposition 5.2, \(J(u) \geq J(t_uu) \geq \eta > 0\). Therefore, \(m_0 > 0\). Thus the proof.

The infimum of \(J\) is attained on \(\mathcal{N}^\prime:\)

**Proposition 5.4.** Under hypotheses \((H_f), (H_G)\) and \((H_K)\), there exists \(\tilde{u} \in \mathcal{N}^\prime\) such that \(J(\tilde{u}) = m_0\).

**Proof.** Let \(\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{N}\) such that
\[
J(u_n) \to m_0, \quad \text{as } n \to +\infty.
\]
(5.16)
Firstly, we prove that \(\{u_n\}_{n \in \mathbb{N}}\) is bounded in \(W^{\alpha,G}(\mathbb{R}^N)\). We argue by contradiction, assume that there exists a subsequence, denoted again by \(\{u_n\}_{n \in \mathbb{N}}\) such that
\[
\|u_n\| \to +\infty \quad \text{as } n \to +\infty,
\]
let
\[
v_n := \frac{u_n}{\|u_n\|} \quad \text{for all } n \in \mathbb{N}.
\]
(5.17)
Since \( \{v_n\}_{n \in \mathbb{N}} \) is bounded in \( W^{\alpha,G}(\mathbb{R}^d) \), which is a reflexive space, there exists \( v \in W^{\alpha,G}(\mathbb{R}^d) \) such that

\[
v_n \rightharpoonup v \quad \text{in} \quad W^{\alpha,G}(\mathbb{R}^d),
\]

and

\[
v_n(x) \to v(x) \quad \text{as} \quad n \to +\infty, \quad \text{for a.a. in} \quad \mathbb{R}^d.
\]

(5.18)

Let prove that \( v \neq 0 \). Since \( \{u_n\}_{n \in \mathbb{N}} \in \mathcal{N} \), according to Proposition 5.2 and Lemma 2.5, for all \( t \geq 0 \), we have

\[
J(u_n) = J(\|u_n\|v_n) \geq J(tv_n)
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G\left(\frac{tv_n(x) - tv_n(y)}{|x - y|^\alpha}\right) \frac{dxdy}{|x - y|^d} + \int_{\mathbb{R}^d} G(tv_n)dx - \int_{\mathbb{R}^d} K(x)F(x, tv_n)dx
\]

\[
\geq \min \left\{ \|tv_n\|g^-, \|tv_n\|g^+ \right\} - \int_{\mathbb{R}^d} K(x)F(x, tv_n)dx
\]

\[
= \min \left\{ t^{g^-}, t^{g^+} \right\} - \int_{\mathbb{R}^d} K(x)F(x, tv_n)dx.
\]

(5.19)

Assume that \( v_n \rightharpoonup v = 0 \), by Lemma 4.3 we obtain

\[
\int_{\mathbb{R}^d} K(x)F(x, tv_n)dx \to 0, \quad \text{for all} \quad t > 0.
\]

Passing to the limit in (5.19), we get

\[
+\infty > m_0 \geq \min \left\{ t^{g^-}, t^{g^+} \right\}, \quad \text{for all} \quad t > 0.
\]

Thus the contradiction, therefore, \( v \neq 0 \).

Using (5.17) and Lemma 2.5, we see that

\[
J(u_n) = J(\|u_n\|v_n)
\]

\[
\leq g^+ \max \left\{ \|u_n\|g^-, \|v_n\|g^-, \|u_n\|g^+, \|v_n\|g^+ \right\} - \int_{\mathbb{R}^d} K(x)F(x, \|u_n\|v_n)dx
\]

\[
\leq g^+ \max \left\{ \|u_n\|g^-, \|v_n\|g^+ \right\} - \int_{\mathbb{R}^d} K(x)F(x, \|u_n\|v_n)dx,
\]

which is equivalent to

\[
\frac{J(u_n)}{\max \{\|u_n\|g^-, \|u_n\|g^+ \}} \leq g^+ - \int_{\mathbb{R}^d} \frac{K(x)F(x, \|u_n\|v_n)}{\max \{\|u_n\|g^-, \|u_n\|g^+ \}}dx.
\]

(5.20)

Exploiting hypotheses (f3), (K1), Fatou’s lemma and the fact that \( v \neq 0 \), we obtain

\[
\liminf_{n \to +\infty} \int_{\mathbb{R}^d} K(x) \frac{F(x, \|u_n\|v_n)}{\max \{\|u_n\|g^-, \|u_n\|g^+ \}}dx = \liminf_{n \to +\infty} \int_{\mathbb{R}^d} K(x) \frac{F(x, \|u_n\|v_n)}{\|u_n\|g^+}dx
\]

\[
= \liminf_{n \to +\infty} \int_{\mathbb{R}^d} K(x) \frac{F(x, \|u_n\|v_n)}{(\|u_n\|v_n)^{g^+}}|v_n|^{g^+}dx
\]

\[
= +\infty.
\]

(5.21)

By (5.20), it yields that

\[
\frac{J(u_n)}{\max \{\|u_n\|g^-, \|u_n\|g^+ \}} \to -\infty \quad \text{as} \quad n \to +\infty,
\]

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which leads to a contradiction with (5.10). Therefore, \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( W^{\alpha,G}(\mathbb{R}^d) \). Up to a subsequence, 
\[
    u_n \rightharpoonup \hat{u} \text{ in } W^{\alpha,G}(\mathbb{R}^d),
\]
and 
\[
    u_n(x) \to \hat{u}(x) \text{ as } n \to +\infty, \text{ for a.a. } x \in \mathbb{R}^d.
\]
(5.22)

Suppose that \( \hat{u} = 0 \), then
\[
    0 < m_0 \leq J(u_n) \xrightarrow{n \to +\infty} J(0) = 0,
\]
which is a contradiction. Thus, \( \hat{u} \neq 0 \). According to Proposition 5.1, there is a unique \( t_{\hat{u}} > 0 \) such that 
\[
    t_{\hat{u}} \hat{u} \in \mathbb{N}.
\]
(5.23)

By (5.22), Proposition 5.2, Lemma 4.3 and Fatou’s lemma, it follows that
\[
    m_0 = \lim_{n \to +\infty} J(u_n) \geq \liminf_{n \to +\infty} J(t_{\hat{u}}u_n) \geq J(t_{\hat{u}}\hat{u}) \geq m_0.
\]
(5.24)

Therefore, \( m_0 = J(t_{\hat{u}}\hat{u}) = \inf J \).

Next, we shall prove that \( t_{\hat{u}} = 1 \). Since \( \{u_n\}_{n \in \mathbb{N}} \in \mathcal{N} \), then
\[
    \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{u_n(x) - u_n(y)}{|x-y|^\alpha} \right) \frac{u_n(x) - u_n(y)}{|x-y|^{\alpha+d}} dxdy + \int_{\mathbb{R}^d} g(u_n)u_n dx = \int_{\mathbb{R}^d} K(x)f(x,u_n)u_n dx, \text{ for all } n \in \mathbb{N}.
\]
By (5.22), Lemma 1.4 and Fatou’s lemma, we deduce that
\[
    \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{\hat{u}(x) - \hat{u}(y)}{|x-y|^\alpha} \right) \hat{u}(x) - \hat{u}(y) \frac{\hat{u}(x) - \hat{u}(y)}{|x-y|^{\alpha+d}} dxdy + \int_{\mathbb{R}^d} g(\hat{u})\hat{u} dx \leq \int_{\mathbb{R}^d} K(x)f(x,\hat{u})\hat{u} dx,
\]
(5.25)

Suppose that \( t_{\hat{u}} > 1 \). From (5.23) and Lemma 2.4 one has
\[
    \int_{\mathbb{R}^d} K(x)f(x,t_{\hat{u}}\hat{u})|\hat{u}|^{q^*-1}\hat{u} \leq \int_{\mathbb{R}^d} g \left( \frac{\hat{u}(x) - \hat{u}(y)}{|x-y|^\alpha} \right) \hat{u}(x) - \hat{u}(y) \frac{\hat{u}(x) - \hat{u}(y)}{|x-y|^{\alpha+d}} dxdy + \int_{\mathbb{R}^d} g(\hat{u})\hat{u} dx.
\]
(5.26)

Putting together (5.25) and (5.26), using (5.1) and hypothesis \((K_1)\), we find that
\[
    0 \leq \int_{\mathbb{R}^d} K(x)|u|^{q^*-1}u \left[ \frac{f(x,u)}{|u|^{q^*-1}u} - \frac{f(x,t_{u}u)}{|t_{u}u|^{q^*-1}} \right] dx < 0.
\]
(5.27)

Thus the contradiction. Therefore, \( 0 < t_u \leq 1 \).
Suppose that $t \neq 1$. Using Proposition 5.4, Lemmas 4.2 and 4.3, Fatou’s lemma and hypothesis $(K_1)$, we see that

$$m_0 = J(t \hat{u}) = J(t \hat{u}) - \frac{1}{g^+} (J'(t \hat{u}), t \hat{u})$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G \left( \frac{t^2 \hat{u}(x) - t \hat{u}(y)}{|x-y|^{\alpha}} \right) - \frac{1}{g^+} \left( \frac{t^2 \hat{u}(x) - t \hat{u}(y)}{|x-y|^{\alpha}} \right) t \hat{u}(x) - t \hat{u}(y) \right) \frac{dx dy}{|x-y|^d}$$

$$+ \int_{\mathbb{R}^d} G(t \hat{u}) - \frac{1}{g^+} g(t \hat{u}) t \hat{u} dx + \int_{\mathbb{R}^d} K(x) \left[ \frac{1}{g^+} f(x, t \hat{u}) t \hat{u} - F(x, t \hat{u}) \right] dx$$

$$\leq \lim_{n \to +\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G \left( \frac{u_n(x) - u_n(y)}{|x-y|^{\alpha}} \right) - \frac{1}{g^+} \left( \frac{u_n(x) - u_n(y)}{|x-y|^{\alpha}} \right) u_n(x) - u_n(y) \right) \frac{dx dy}{|x-y|^d}$$

$$+ \int_{\mathbb{R}^d} G(u_n) - \frac{1}{g^+} g(u_n) u_n dx + \int_{\mathbb{R}^d} K(x) \left[ \frac{1}{g^+} f(x, u_n) u_n - F(x, u_n) \right] dx$$

$$= \lim_{n \to +\infty} \inf J(u_n) = J(\hat{u})$$

Which is a contradiction. Thus, $t = 1$. Hence,

$$m_0 = J(\hat{u}) = \inf_{\mathcal{N}} J.$$

This completes the proof. 

In the following we prove that $\hat{u}$ is a critical point of the functional $J$.

**Proposition 5.5.** Assume that the hypotheses $(H_f)$, $(H_G)$ and $(H_K)$ hold. Then, $\hat{u}$ is a critical point of $J$. Hence, $\hat{u}$ is a least energy weak solution of problem (1).

**Proof.** Let consider the functional $\varphi : W^{\alpha,G}(\mathbb{R}^d) \to \mathbb{R}$ defined by

$$\varphi(u) = (J'(u), u) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{u(x) - u(y)}{|x-y|^{\alpha}} \right) \frac{u(x) - u(y)}{|x-y|^{\alpha}} \frac{dx dy}{|x-y|^d}$$

$$+ \int_{\mathbb{R}^d} g(u) dx - \int_{\mathbb{R}^d} K(x) f(x, u) dx.$$

By hypotheses $(H_f)$, $\varphi$ is locally Lipschitz (see [19], Theorem 2.7.2, p. 221]).

Let $u \in W^{\alpha,G}(\mathbb{R}^d)$, for all $\varphi^*_u \in \partial \varphi(u)$, there is $f^*(x, u) \in \partial_u f(x, u)$ such that

$$\langle \varphi^*_u, v \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{u(x) - u(y)}{|x-y|^{\alpha}} \right) \frac{v(x) - v(y)}{|x-y|^{\alpha}} \frac{dx dy}{|x-y|^d}$$

$$+ \int_{\mathbb{R}^d} g(u) dx - \int_{\mathbb{R}^d} K(x) f(x, u) dx$$

$$+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{u(x) - u(y)}{|x-y|^{\alpha}} \right) \frac{|v(x) - v(y)|^2}{|x-y|^{2\alpha}} \frac{dx dy}{|x-y|^d}$$

$$+ \int_{\mathbb{R}^d} g(u) v^2 dx - \int_{\mathbb{R}^d} K(x) f^*(x, u) v^2 dx, \text{ for all } v \in W^{\alpha,G}(\mathbb{R}^d). \quad (5.28)$$

From Proposition 5.4 we have

$$J(\hat{u}) = m_0 = \inf \left\{ J(u) : \varphi(u) = 0, u \in W^{\alpha,G}(\mathbb{R}^d) \setminus \{0\} \right\}.$$
According to the non-smooth multiplier rule of Clarke [18, Theorem 10.47, p. 221], there exists $\lambda_0 \geq 0$ such that

$$0 \in \partial (J + \lambda_0 \varphi)(\hat{u}).$$

By the subdifferential calculus of Clarke [19, p. 48], it follows that

$$0 \in \partial J(\hat{u}) + \lambda_0 \partial \varphi(\hat{u}).$$

Thus,

$$0 = J'(\hat{u}) + \lambda_0 \varphi^*_\hat{u} \text{ in } (W^{\alpha,G}(\mathbb{R}^d))^*, \quad \text{for all } \varphi^*_\hat{u} \in \partial \varphi(\hat{u}). \quad (5.29)$$

Since $\hat{u} \in \mathcal{N}$, we have

$$0 = \langle J'(\hat{u}), \hat{u} \rangle + \lambda_0 (\varphi^*_\hat{u}, \hat{u}) = \lambda_0 (\varphi^*_\hat{u}, \hat{u}), \quad \text{for all } \varphi^*_\hat{u} \in \partial \varphi(\hat{u}) \quad (5.30)$$

Using (5.28), hypotheses (f4), (g3) and the fact that $\hat{u} \in \mathcal{N}$, we get

$$\langle \varphi^*_\hat{u}, \hat{u} \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{g}'(\hat{u}) (\hat{u}(x) - \hat{u}(y)) \frac{[\hat{u}(x) - \hat{u}(y)]^2}{|x - y|^{2\alpha}} \frac{dxdy}{x - y|d}$$

$$+ \int_{\mathbb{R}^d} \hat{g}'(\hat{u}) \hat{u}^2 dx - \int_{\mathbb{R}^d} K(x) f^*(x, \hat{u}) \hat{u}^2 dx$$

$$\leq |g^+ - 1| \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{g} \frac{(\hat{u}(x) - \hat{u}(y))}{|x - y|^\alpha} \frac{\hat{u}(x) - \hat{u}(y)}{|x - y|^\alpha} dxdy \right.$$ 

$$+ \int_{\mathbb{R}^d} \hat{g}(\hat{u}) \hat{u} dx - \int_{\mathbb{R}^d} K(x) f^*(x, \hat{u}) \hat{u}^2 dx$$

$$\leq \int_{\mathbb{R}^d} K(x) [(g^+ - 1) f(x, \hat{u}) \hat{u} - f^*(x, \hat{u}) \hat{u}^2] dx$$

$$< 0 \quad (5.31)$$

According to (5.28), $\lambda_0 = 0$. Therefore, from (5.30), we deduce that

$$J'(\hat{u}) = 0 \text{ in } (W^{\alpha,G}(\mathbb{R}^d))^*.$$

Hence, $\hat{u}$ is a critical point of $J$, so, it is a weak solution of problem [P].

Thus the proof. \(\square\)

6 Least energy nodal solution

In this section, we establish the existence of least energy nodal solution for problem [P] and we show that the ground state solution $\hat{u}$, obtained in Proposition 5.5, is of fixed sign. Finally, we give the proof of Theorem 3.2.

In order to find a least energy nodal solution for problem [P], we look for a minimizer of the energy functional $J$ on the constraint $\mathcal{M}$. Let consider the following minimization problem

$$m_1 = \inf_{\mathcal{M}} J. \quad (M-1)$$

**Proposition 6.1.** Assume that the hypotheses $(H_f)$, $(H_G)$ and $(H_K)$ hold. Let $w \in W^{\alpha,G}(\mathbb{R}^d)$ such that $w^\pm \neq 0$, then there exists a unique pair $t_{w^+}, s_{w^-} > 0$ such that

$$t_{w^+} w^+ + s_{w^-} w^- \in \mathcal{M}.$$ 

**Proof.** Let $\xi : (0, +\infty) \times (0, +\infty) \rightarrow \mathbb{R}^2$ be a continuous vector field given by

$$\xi(t, s) = (\xi_1(t, s), \xi_2(t, s)), \quad \text{for all } t, s \in (0, +\infty) \times (0, +\infty)$$

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where

\[ \xi_1(t, s) = \langle J'(tw^+ + sw^-), tw^+ \rangle \]  and \[ \xi_2(t, s) = \langle J'(tw^+ + sw^-), sw^- \rangle. \]

Arguing as in the proof of Proposition 5.1, there exist \( r_1 > 0 \) small enough and \( R_1 > 0 \) large enough such that

\[ \xi_1(t, t) > 0, \quad \xi_2(t, t) > 0, \quad \text{for all} \quad t \in (0, r_1), \]
\[ \xi_1(t, t) < 0, \quad \xi_2(t, t) < 0, \quad \text{for all} \quad t \in (R_1, +\infty). \quad (6.1) \]

Note that \( \xi_1(t, s) \) is non-decreasing in \( s \) on \((0, +\infty)\) for fixed \( t > 0 \) and \( \xi_2(t, s) \) is non-decreasing in \( t \) on \((0, +\infty)\) for fixed \( s > 0 \) (see [7, Proof of Lemma 4.7]). Then, there are \( r > 0 \), \( R > 0 \) with \( r < R \) such that

\[ \xi_1(r, s) > 0, \quad \xi_1(R, s) < 0, \quad \text{for all} \quad s \in (r, R], \]
\[ \xi_2(t, r) > 0, \quad \xi_2(t, R) < 0, \quad \text{for all} \quad t \in (r, R]. \]

Applying the Miranda’s theorem [22] on \( \xi \), there exist some \( t_{w^+}, s_{w^-} \in (r, R] \) such that \( \xi_1(t_{w^+}, s_{w^-}) = \xi_2(t_{w^+}, s_{w^-}) = 0 \). Which means that \( t_{w^+} w^+ + s_{w^-} w^- \in \mathcal{M} \).

For the uniqueness of the pairs \((t_{w^+}, s_{w^-})\), we argue by contradiction. Suppose that there exist two different pairs \((t_1, s_1)\) and \((t_2, s_2)\) such that

\[ t_1 w^+ + s_1 w^- \in \mathcal{M} \] and \[ t_2 w^+ + s_2 w^- \in \mathcal{M}. \]

We distinguish two cases:

(A): If \( w \in \mathcal{M} \). Without loss of generality, we may take \((t_1, s_1) = (1, 1)\) and assume that \( t_2 \leq s_2 \), we have

\[ \int_{\mathbb{R}^d} K(x) f(x, w^+) w^+ dx = A^+(w) \quad (6.2) \]

and

\[ \int_{\mathbb{R}^d} K(x) f(x, w^-) w^- dx = A^-(w). \quad (6.3) \]

Where

\[
A^+(w) = \int_{\text{supp}(w^+)} \int_{\text{supp}(w^+)} g \left( \frac{w^+(x) - w^+(y)}{|x-y|^\alpha} \right) \frac{w^+(x) - w^+(y)}{|x-y|^\alpha + d} dx dy \\
+ \int_{\text{supp}(w^-)} \int_{\text{supp}(w^+)} g \left( \frac{w^+(x) - w^-(y)}{|x-y|^\alpha} \right) \frac{w^+(x)}{|x-y|^\alpha + d} dx dy \\
+ \int_{\text{supp}(w^+)} \int_{\text{supp}(w^-)} g \left( \frac{w^-(x) - w^+(y)}{|x-y|^\alpha} \right) \frac{-w^+(y)}{|x-y|^\alpha + d} dx dy \\
+ \int_{\mathbb{R}^d} g(w^+) w^+ dx \quad (6.4)
\]

and

\[
A^-(w) = \int_{\text{supp}(w^-)} \int_{\text{supp}(w^-)} g \left( \frac{w^-(x) - w^-(y)}{|x-y|^\alpha} \right) \frac{w^-(x) - w^-(y)}{|x-y|^\alpha + d} dx dy \\
+ \int_{\text{supp}(w^-)} \int_{\text{supp}(w^-)} g \left( \frac{w^+(x) - w^-(y)}{|x-y|^\alpha} \right) \frac{-w^-(y)}{|x-y|^\alpha + d} dx dy \\
+ \int_{\text{supp}(w^+)} \int_{\text{supp}(w^-)} g \left( \frac{w^-(x) - w^+(y)}{|x-y|^\alpha} \right) \frac{-w^+(x)}{|x-y|^\alpha + d} dx dy \\
+ \int_{\mathbb{R}^d} g(w^-) w^- dx. \quad (6.5)
\]
Since the map $s \mapsto g(s)$ is non-decreasing on $(0, +\infty)$ and on $(-\infty, 0)$ and $t_2 \leq s_2$, we infer that

$$
\begin{aligned}
g \left( \frac{t_2w^+(x) - s_2w^-(y)}{|x-y|^\alpha} \right) t_2w^+(x) &\geq g \left( \frac{t_2w^+(x) - t_2w^-(y)}{|x-y|^\alpha} \right) t_2w^+(x), \\
g \left( \frac{s_2w^-(x) - t_2w^+(y)}{|x-y|^\alpha} \right) (-t_2w^+(y)) &\geq g \left( \frac{t_2w^-(x) - t_2w^+(y)}{|x-y|^\alpha} \right) (-t_2w^+(y)), \\
g \left( \frac{t_2w^+(x) - s_2w^-}{|x-y|^\alpha} \right) (-s_2w^-) &\leq g \left( \frac{s_2w^+(x) - s_2w^-}{|x-y|^\alpha} \right) (-s_2w^-), \\
g \left( \frac{s_2w^-(x) - t_2w^+(y)}{|x-y|^\alpha} \right) s_2w^- (x) &\leq g \left( \frac{s_2w^-(x) - s_2w^+}{|x-y|^\alpha} \right) s_2w^- (x),
\end{aligned}
$$

for a.a. $x, y \in \mathbb{R}^d$.

By Lemma 2.4 and since $t_1w^+ + s_1w^- \in \mathcal{M}$, and $t_2w^+ + s_2w^- \in \mathcal{M}$, it yields that

$$
\int_{\mathbb{R}^d} K(x) \frac{f(x, t_2w^+)}{\min\{t_2^-, t_2^+\}} dx \geq A^+(w) \quad (6.7)
$$

and

$$
\int_{\mathbb{R}^d} K(x) \frac{f(x, s_2w^-)}{\max\{s_2^-, s_2^+\}} dx \leq A^-(w). \quad (6.8)
$$

In what follows, we will show that the following five cases cannot happen:

1. $t_2 < s_2 = 1$.
2. $s_2 > t_2 = 1$.
3. $0 < t_2 \leq s_2 < 1$.
4. $1 < t_2 \leq s_2$.
5. $0 < t_2 < 1 < s_2$.

Suppose that one of the cases (1), (3) or (5), holds. According to (6.1), (6.2), (6.7) and hypothesis $(K_1)$, we get

$$
0 \leq \int_{\mathbb{R}^d} K(x)|w^+|^{g^+-1}w^+ \left[ \frac{f(x, t_2w^+)}{|t_2w^+|^{g^+-1}} - \frac{f(x, w^+)}{|w^+|^{g^+-1}} \right] dx < 0.
$$

Thus the contradiction. Then, the cases (1), (3) and (5) cannot be realized.

Suppose that case (2) or (4) holds. According to (6.1), (6.3), (6.8) and hypothesis $(K_1)$, one has

$$
0 \leq \int_{\mathbb{R}^d} K(x)|w^-|^{g^+-1}w^- \left[ \frac{f(x, w^-)}{|w^-|^{g^+-1}} - \frac{f(x, s_2w^-)}{|s_2w^-|^{g^+-1}} \right] dx < 0.
$$

Which is a contradiction too. Then, the cases (2) and (4) cannot be realized. We deduce that $(t_1, s_1) = (1, 1) = (t_2, s_2)$.

(B): If $w \not\in \mathcal{M}$. Let $v = t_1w^+ + s_1w^- \in \mathcal{M}$, $v^+ = t_1w^+$ and $w^- = s_1w^-$, so $(t_1, s_1) \neq (1, 1)$. It is clear that

$$
t_2w^+ + s_2w^- = \frac{t_2}{t_1}t_1w^+ + \frac{s_2}{s_1}s_1w^- = \frac{t_2}{t_1}v^+ + \frac{s_2}{s_1}v^- \in \mathcal{M}.
$$

Arguing as in the case (A), we conclude that

$$
\frac{t_2}{t_1} = \frac{s_2}{s_1} = 1.
$$

This completes the proof. □
Remark 6.2. Under the hypotheses \((H_f), (H_G)\) and \((H_K)\),

$$0 < m_0 = \inf_{\mathcal{N}} J = \inf_{\mathcal{M}} J = m_1.$$ 

Proposition 6.3. Assume that the hypotheses \((H_f), (H_G)\) and \((H_K)\) hold. Then, for all \(w \in \mathcal{M}\),

$$J(tw^+ + sw^-) \leq J(w), \text{ for all } t, s > 0.$$ 

Proof. Let \(w \in \mathcal{M}\) and consider the fibering map \(\mu_w : (0, +\infty) \times (0, +\infty) \to \mathbb{R}\) defined by

$$\mu_w(t, s) = J(tw^+ + sw^-) \text{ for all } t, s > 0.$$ 

In light of Proposition \(6.1\)

$$\mu_w(0, 0) = J(0) = 0 < m_1 \leq \mu_w(1, 1) = J(w). \quad (6.9)$$ 

Let \(t, s > 0\) large enough, using Lemmas 2.4 and 2.5 we obtain

$$\mu_w(t, s) \leq \max \left\{ \|tw^+ + sw^-\|g^-, \|tw^+ + sw^-\|g^+ \right\} - \int_{\mathbb{R}^d} K(x)F(x, tw^+ + sw^-)dx \leq 2^{g^-1} \max \left\{ \|t\|g^- \|w^+\|g^- + |s|g^\|w^-\|g^- + |t|g^\|w^+\|g^- + |s|g^\|w^-\|g^- \right\} \leq 2^{g^-1} \max \left\{ \max \{|t|g^- \}, |s|g^- \right\} \left\{ \|w^+\|g^- + \|w^-\|g^- \right\} \max \left\{ |t|g^\|w^+\|g^- + |s|g^\|w^-\|g^- \right\} - \int_{\mathbb{R}^d} K(x)F(x, tw^+ + sw^-)dx \leq 2^{g^-1} \max \left\{ |t|g^\|w^+\|g^- + |s|g^\|w^-\|g^- \right\} - \int_{\mathbb{R}^d} K(x)F(x, tw^+ + sw^-)dx \leq 2^{g^-1} \max \left\{ |t|g^\|w^+\|g^- + |s|g^\|w^-\|g^- \right\} - \int_{\mathbb{R}^d} K(x)F(x, tw^+ + sw^-)dx.$$ 

It follows that

$$\frac{\mu_w(t, s)}{\max \{|t|g^\|w^+\|g^- + |s|g^\|w^-\|g^- \}} \leq 2^{g^-1} \max \left\{ \|w^+\|g^- + \|w^-\|g^- , \|w^+\|g^+ + \|w^-\|g^+ \right\} - \int_{\mathbb{R}^d} K(x)F(x, tw^+ + sw^-) \max \{|t|g^\|w^+\|g^- + |s|g^\|w^-\|g^- \} dx. \quad (6.11)$$ 

By assumption \((f_3)\) and the fact that \(\text{supp}(w^+) \cap \text{supp}(w^-) = \emptyset\), we infer that

$$\lim_{|(t, s)| \to +\infty} \frac{F(x, tw^+ + sw^-)}{\max \{|t|g^\|w^+\|g^- + |s|g^\|w^-\|g^- \}} = +\infty, \text{ for a.a. } x \in \mathbb{R}^d.$$ 

(6.12)

Applying \((6.11)\) and \((6.12)\), we deduce that

$$\limsup_{|(t, s)| \to +\infty} \mu_w(t, s) \leq -\infty.$$ 

According to \((6.3)\), the map \(\mu_w(\cdot, \cdot)\) has a global maximum \((t_{w^+}, s_{w^-}) \in (0, +\infty) \times (0, +\infty)\). \((t_{w^+}, s_{w^-})\) is a critical point for \(\mu_w(\cdot, \cdot)\), that is,

$$\langle J'(t_{w^+}w^+ + s_{w^-}w^-), w^+ \rangle = 0$$

and

$$\langle J'(t_{w^+}w^+ + s_{w^-}w^-), w^- \rangle = 0.$$
By Proposition 6.3 and the fact that \( w \in \mathcal{M} \),
\[
(t_{w^+}, s_{w^-}) = (1, 1).
\]

Hence,
\[
J(tw^+ + sw^-) \leq J(t_{w^+}w^+ + s_{w^-}w^-) = J(w), \text{ for all } t, s > 0.
\]

This ends the proof.

**Proposition 6.4.** Assume that hypotheses \((H_f), (H_G)\) and \((H_K)\) hold. Let \( \{w_n\}_n \subset \mathcal{M} \) such that \( w_n \rightharpoonup w \) in \( W^{\alpha,G}(\mathbb{R}^d) \), then \( w^\pm \neq 0 \).

**Proof.** We claim that there is \( \varrho > 0 \) such that
\[
\varrho \leq \|v^\pm\|, \quad \text{for all } v \in \mathcal{M}.
\]

Indeed, by using Lemma 4.4,
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g\left(\frac{v^\pm(x) - v^\pm(y)}{|x - y|^\alpha}\right) \frac{v^\pm(x) - v^\pm(y)}{|x - y|^\alpha} dx dy + \int_{\mathbb{R}^d} g(v^\pm)v^\pm dx \leq \int_{\mathbb{R}^d} K(x)f(x, v^\pm)v^\pm dx.
\]

Exploiting (2.4), (5.1), hypotheses \((g_2)\) and \((K_1)\), for all \( \varepsilon > 0 \) we get
\[
[g^- - g^+\varepsilon\|K\|_\infty] \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G\left(\frac{v^\pm(x) - v^\pm(y)}{|x - y|^\alpha}\right) \frac{dx dy}{|x - y|^\alpha} dx dy + \int_{\mathbb{R}^d} G(v^\pm) dx
\]
\[
\leq g^+_* C_\varepsilon\|K\|_\infty \int_{\mathbb{R}^d} G_*(v^\pm) dx.
\]

Without lose of generality, we may assume that \( 0 \neq \|v\| < 1 \). By Lemma 2.5 and Theorem 2.1, we deduce that
\[
[g^- - g^+\varepsilon\|K\|_\infty]\|v^\pm\|g^+ \leq g^+_* \tilde{C}C_\varepsilon\|K\|_\infty\|v^\pm\|g^+.
\]

Choosing \( \varepsilon \) small enough, we conclude that
\[
\left(\frac{C_1}{C_2}\right)^{\frac{1}{g^+ - g^-}} \leq \|v^\pm\|
\]

where \( C_1 = g^- - g^+\varepsilon\|K\|_\infty > 0 \) and \( C_2 = g^+_* \tilde{C}C_\varepsilon\|K\|_\infty > 0 \). Consequently, there exists a positive radius \( \varrho > 0 \) such that \( \|v^\pm\| \geq \varrho \), with \( \varrho = \left(\frac{C_1}{C_2}\right)^{\frac{1}{g^+ - g^-}} \). Thus, the claim.

So, by (6.13),
\[
\|w^\pm_n\| \geq \varrho, \quad \text{for all } n \in \mathbb{N}.
\]

According to Lemma 4.3,
\[
\langle J'(w^\pm_n), w^\pm_n \rangle \leq \langle J'(w_n), w^\pm_n \rangle = 0.
\]

By \((g_2)\) and Lemma 2.6, we get
\[
g^- \min\{\|w^\pm_n\|g^-, \|w^\pm_n\|g^+ \} \leq \int_{\mathbb{R}^d} K(x)f(x, w^\pm_n) w^\pm_n dx.
\]

Putting together (6.15) and (6.16), we find
\[
g^- \min\{g^-, g^+\} \leq g^- \min\{\|w^\pm_n\|g^-, \|w^\pm_n\|g^+ \} \leq \int_{\mathbb{R}^d} K(x)f(x, w^\pm_n) w^\pm_n dx.
\]

On the other hand, in light of Lemma 4.3 one has
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^d} K(x)f(x, w^\pm_n) w^\pm_n dx = \int_{\mathbb{R}^d} K(x)f(x, w^\pm) w^\pm dx.
\]

Combining (6.17) with (6.18), we get
\[
0 < g^- \min\{g^-, g^+\} \leq \int_{\mathbb{R}^d} K(x)f(x, w^\pm) w^\pm dx,
\]

thus, \( w^\pm \neq 0 \). This ends the proof.
In the following proposition, we prove that the infimum of $J$ is attained on $\mathcal{M}$.

**Proposition 6.5.** Assume that the hypotheses $(H_f)$, $(H_G)$ and $(H_K)$ hold. Then, there exists $\bar{w} \in \mathcal{M}$ such that $J(\bar{w}) = m_1$.

**Proof.** Let $\{w_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$ such that

$$J(w_n) \to_{n \to +\infty} m_1.$$  

Arguing as in the proof of Proposition 5.3, we deduce that $\{w_n\}_{n \in \mathbb{N}}$ is bounded in $W^{\alpha,G}(\mathbb{R}^d)$. By passing to a subsequence if necessary, $w_n \to \hat{w}$ in $W^{\alpha,G}(\mathbb{R}^d)$, $w_n(x) \to \hat{w}(x)$ as $n \to +\infty$, for a.a. $x \in \mathbb{R}^d$ (6.19)

and

$$w_n^\pm(x) \to \hat{w}_n^\pm(x) \text{ as } n \to +\infty, \text{ for a.a. } x \in \mathbb{R}^d. \quad (6.20)$$

Applying Proposition 6.4, we see that $\hat{w}_n^\pm \neq 0$. So, according to Proposition 6.1, there is a unique pair $t_{\hat{w}_n^+}, s_{\hat{w}_n^-} > 0$ such that $t_{\hat{w}_n^+} + s_{\hat{w}_n^-} \hat{w}_n \in \mathcal{M}$, that is,

$$\langle J'(t_{\hat{w}_n^+} + s_{\hat{w}_n^-} \hat{w}_n), \hat{w}_n^+ \rangle = 0 \text{ and } \langle J'(t_{\hat{w}_n^+} + s_{\hat{w}_n^-} \hat{w}_n), \hat{w}_n^- \rangle = 0. \quad (6.21)$$

Since $\{w_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$, by (6.20), Proposition 6.3, Lemma 4.3 and Fatou’s lemma, we obtain

$$m_1 = \lim_{n \to +\infty} J(w_n) \geq \liminf_{n \to +\infty} J(t_{\hat{w}_n^+} + s_{\hat{w}_n^-} \hat{w}_n^+)$$

$$\geq J(t_{\hat{w}_n^+} + s_{\hat{w}_n^-} \hat{w}_n^-)$$

$$\geq m_1. \quad (6.22)$$

Therefore,

$$m_1 = \inf_{\mathcal{M}} J = J(t_{\hat{w}_n^+} + s_{\hat{w}_n^-} \hat{w}_n^-). \quad (6.23)$$

We show that $t_{\hat{w}_n^+} = s_{\hat{w}_n^-} = 1$ and we produced it in two steps.

**Step 1:** $0 < t_{\hat{w}_n^+}, s_{\hat{w}_n^-} \leq 1$. Indeed, using (6.20), Lemma 4.3 and Fatou’s lemma, we find that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{\hat{w}(x) - \hat{w}(y)}{|x - y|^\alpha} \right) \frac{\hat{w}_n^+(x) - \hat{w}_n^+(y)}{|x - y|^\alpha + d} \, dx \, dy + \int_{\mathbb{R}^d} g(\hat{w}_n^+) \hat{w}_n^+ \, dx \leq \int_{\mathbb{R}^d} K(x) f(x, \hat{w}_n^+) \hat{w}_n^+ \, dx. \quad (6.24)$$

From (6.21), we have

$$\int_{\mathbb{R}^d} K(x) f(x, t_{\hat{w}_n^+} + \hat{w}_n^+) \hat{w}_n^+ \, dx$$

$$= \int_{\text{supp}(\hat{w}_n^+)} \int_{\text{supp}(\hat{w}_n^+)} g \left( \frac{t_{\hat{w}_n^+} \hat{w}_n^+(x) - t_{\hat{w}_n^+} \hat{w}_n^+(y)}{|x - y|^\alpha} \right) \frac{t_{\hat{w}_n^+} \hat{w}_n^+(x) - t_{\hat{w}_n^+} \hat{w}_n^+(y)}{|x - y|^\alpha + d} \, dx \, dy$$

$$+ \int_{\text{supp}(\hat{w}_n^-)} \int_{\text{supp}(\hat{w}_n^+)} g \left( \frac{t_{\hat{w}_n^+} \hat{w}_n^+(x) - s_{\hat{w}_n^-} \hat{w}_n^-(y)}{|x - y|^\alpha} \right) \frac{t_{\hat{w}_n^+} \hat{w}_n^+(x)}{|x - y|^\alpha + d} \, dx \, dy$$

$$+ \int_{\text{supp}(\hat{w}_n^+)} \int_{\text{supp}(\hat{w}_n^-)} g \left( \frac{s_{\hat{w}_n^-} \hat{w}_n^-(x) - t_{\hat{w}_n^+} \hat{w}_n^+(y)}{|x - y|^\alpha} \right) \frac{s_{\hat{w}_n^-} \hat{w}_n^-(x) - t_{\hat{w}_n^+} \hat{w}_n^+(y)}{|x - y|^\alpha + d} \, dx \, dy$$

$$+ \int_{\mathbb{R}^d} g(t_{\hat{w}_n^+} + \hat{w}_n^+) t_{\hat{w}_n^+} \, dx. \quad (6.25)$$

Without loss of generality, we suppose that $t_{\hat{w}_n^+} \geq s_{\hat{w}_n^-}$. By (6.25), Lemma 2.4 and the fact that the map $s \mapsto g(s)$ is non-decreasing function on $\mathbb{R}$, it yields that

$$\int_{\mathbb{R}^d} K(x) f(x, t_{\hat{w}_n^+} + \hat{w}_n^+) t_{\hat{w}_n^+} \, dx \leq \max\{t_{\hat{w}_n^+}, t_{\hat{w}_n^+}^+\} \sigma,$$
where

\[
0 \leq \sigma = \int_{\text{supp}(\hat{w}^+)} \int_{\text{supp}(\hat{w}^-)} g \left( \frac{\hat{w}^+(x) - \hat{w}^-(y)}{|x-y|^\alpha} \right) \frac{\hat{w}^+(x) - \hat{w}^-(y)}{|x-y|^\alpha + d} \, dx \, dy \\
+ \int_{\text{supp}(\hat{w}^-)} \int_{\text{supp}(\hat{w}^+)} g \left( \frac{\hat{w}^-(x) - \hat{w}^+(y)}{|x-y|^\alpha} \right) \frac{\hat{w}^+(x) - \hat{w}^-(y)}{|x-y|^\alpha + d} \, dx \, dy \\
+ \int_{\text{supp}(\hat{w}^+)} \int_{\text{supp}(\hat{w}^-)} g \left( \frac{\hat{w}^-(x) - \hat{w}^+(y)}{|x-y|^\alpha} \right) \frac{-\hat{w}^-(y)}{|x-y|^\alpha + d} \, dx \, dy \\
+ \int_{\mathbb{R}^d} g(\hat{w}^+) \hat{w}^+ \, dx \\
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{\hat{w}(x) - \hat{w}(y)}{|x-y|^\alpha} \right) \frac{\hat{w}^+(x) - \hat{w}^-(y)}{|x-y|^\alpha + d} \, dx \, dy \\
+ \int_{\mathbb{R}^d} g(\hat{w}^+) \hat{w}^+ \, dx.
\]

Arguing by contradiction, and suppose that \( t_{\hat{w}^+} > 1 \), then

\[
\int_{\mathbb{R}^d} K(x) \frac{f(x, t_{\hat{w}^+} \hat{w}^+) f_{\hat{w}^-}}{t_{\hat{w}^+}} \leq \sigma. \tag{6.26}
\]

Putting together (6.24) and (6.26), using (K1) and (3.1), we obtain

\[
0 \leq \int_{\mathbb{R}^d} K(x) (\hat{w}^+)^2 \left[ \frac{f(x, \hat{w}^+)}{(\hat{w}^+)^2 - 1} - \frac{f(x, t_{\hat{w}^+} \hat{w}^+) (t_{\hat{w}^+} / \hat{w}^+)^2 - 1}{(t_{\hat{w}^+} / \hat{w}^+)^2 - 1} \right] \, dx < 0
\]

which is a contradiction. Therefore, \( 0 < t_{\hat{w}^+}, s_{\hat{w}^-} \leq 1 \).

Step 2: \( t_{\hat{w}^+} = s_{\hat{w}^-} = 1 \). Indeed, we argue by contradiction and suppose that \( (t_{\hat{w}^+}, s_{\hat{w}^-}) \neq (1, 1) \). By (3.2), (6.19), (6.20), Lemma 4.2 and Fatou’s lemma, it follows that

\[
m_1 \leq J(t_{\hat{w}^+} \hat{w}^+ + s_{\hat{w}^-} \hat{w}^-) \quad \text{(since } t_{\hat{w}^+} \hat{w}^+ + s_{\hat{w}^-} \hat{w}^- \in M) \\
= J(t_{\hat{w}^+} \hat{w}^+ + s_{\hat{w}^-} \hat{w}^-) - \frac{1}{g^+} (J(t_{\hat{w}^+} \hat{w}^+ + s_{\hat{w}^-} \hat{w}^-), t_{\hat{w}^+} \hat{w}^+ + s_{\hat{w}^-} \hat{w}^-) \\
< J(\hat{w}) - \frac{1}{g^+} \langle J'(\hat{w}), \hat{w} \rangle \\
\leq \liminf_{n \to +\infty} \left[ J(w_n) - \frac{1}{g^+} \langle J'(w_n), w_n \rangle \right] \\
= \liminf_{n \to +\infty} J(w_n) \quad \text{(since } w_n \in M) \\
= m_1.
\]

Which is a contradiction, thus, \( t_{\hat{w}^+} = s_{\hat{w}^-} = 1 \). Hence, according to (6.24), it comes that

\[
m_1 = \inf_{M} J = J(t_{\hat{w}^+} \hat{w}^+ + s_{\hat{w}^-} \hat{w}^-) = J(\hat{w}).
\]

This ends the proof.

\[\square\]

**Proposition 6.6.** Under the hypotheses \((H_f), (H_G) and (H_K)\), \( \hat{w} \) is a critical point for \( J \), that is, \( \hat{w} \) is a least energy weak nodal solution of problem \( (1) \).

**Proof.** We consider the functionals \( \varphi_{\pm} : W^{1,G}(\mathbb{R}^d) \to \mathbb{R} \) defined by

\[
\varphi_{\pm}(w) = \langle J'(w), w^\pm \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{w(x) - w(y)}{|x-y|^\alpha} \right) \frac{w^\pm(x) - w^\pm(y)}{|x-y|^\alpha + d} \, dx \, dy \\
+ \int_{\mathbb{R}^d} g(w) w^\pm \, dx - \int_{\mathbb{R}^d} K(x) f(x, w^\pm) w^\pm \, dx.
\]

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Thus, according to the non-smooth multiplier rule of Clarke [18, Theorem 10.47, p. 221], there exist \( f^*(x, w^\pm) \in \partial_{w^\pm} f(x, w^\pm) \) such that

\[
\langle \varphi_{w^\pm}^*, v \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{w(x) - w(y)}{|x-y|^\alpha} \right) \frac{v^\pm(x) - v^\pm(y)}{|x-y|^{2\alpha+d}} \, dx \, dy
+ \int_{\mathbb{R}^d} g(w^\pm)v^\pm \, dx - \int_{\mathbb{R}^d} K(x)f(x, w^\pm) \, v^\pm \, dx
+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{w(x) - w(y)}{|x-y|^\alpha} \right) [v(x) - v(y)] [w^\pm(x) - w^\pm(y)] \, dx \, dy
+ \int_{\mathbb{R}^d} g'(w^\pm)(v^\pm)^2 \, dx - \int_{\mathbb{R}^d} K(x)f^*(x, w^\pm)(v^\pm)^2 \, dx,
\]  

(6.27)

for all \( v \in W^{\alpha, G}(\mathbb{R}^d) \).

By Proposition 6.3

\[ J(\tilde{w}) = m_1 = \inf \{ J(w) : w \in W^{\alpha, G}(\mathbb{R}^d), w^\pm \neq 0, \varphi_+(w) = \varphi_-(w) = 0 \}. \]

Thus, according to the non-smooth multiplier rule of Clarke [18, Theorem 10.47, p. 221], there exist \( \lambda_+, \lambda_- \geq 0 \) such that

\[ 0 \in \partial(J + \lambda_+ \varphi_+ + \lambda_- \varphi_-)(\tilde{w}). \]

The subdifferential calculus of Clarke [19, p. 48], gives that

\[ 0 \in \partial J(\tilde{w}) + \lambda_+ \partial \varphi_+(\tilde{w}) + \lambda_- \partial \varphi_-(\tilde{w}). \]

Then,

\[ 0 = J'(\tilde{w}) + \lambda_+ \varphi_{w^+}^* + \lambda_- \varphi_{w^-}^* \quad \text{in} \quad (W^{\alpha, G}, (\mathbb{R}^d))^*, \quad \text{for all} \quad \varphi_{w^+}^* \in \partial \varphi_+(\tilde{w}) \quad \text{and} \quad \varphi_{w^-}^* \in \partial \varphi_-(\tilde{w}). \]

(6.28)

Since \( \tilde{w} \in \mathcal{M} \),

\[ 0 = \langle J'(\tilde{w}), \tilde{w} \rangle + \lambda_+ \langle \varphi_{w^+}^*, \tilde{w} \rangle + \lambda_- \langle \varphi_{w^-}^*, \tilde{w} \rangle = \lambda_+ \langle \varphi_{w^+}^*, \tilde{w} \rangle + \lambda_- \langle \varphi_{w^-}^*, \tilde{w} \rangle, \]

(6.29)

for all \( \varphi_{w^+}^* \in \partial \varphi_+(\tilde{w}) \) and all \( \varphi_{w^-}^* \in \partial \varphi_-(\tilde{w}) \).

Let observe that

\[ \text{sign} \left( \tilde{w}(x) - \tilde{w}(y) \right) = \text{sign} \left( \tilde{w}^\pm(x) - \tilde{w}^\pm(y) \right), \quad \text{for a.a.} \quad x, y \in \mathbb{R}^d. \]

(6.30)

Using (6.27), (6.30), assumption \((g_3)\) and the fact that \( \tilde{w} \in \mathcal{M} \), we obtain

\[
\langle \varphi_{w^\pm}^*, \tilde{w} \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g'(\tilde{w}(x) - \tilde{w}(y)) \, \frac{[\tilde{w}(x) - \tilde{w}(y)][\tilde{w}^\pm(x) - \tilde{w}^\pm(y)]}{|x-y|^{2\alpha+d}} \, dx \, dy
+ \int_{\mathbb{R}^d} g(\tilde{w}^\pm)^2 \, dx - \int_{\mathbb{R}^d} K(x)f'(x, \tilde{w}^\pm)(\tilde{w}^\pm)^2 \, dx
\leq \left[ g^+ - 1 \right] \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{\tilde{w}(x) - \tilde{w}(y)}{|x-y|^\alpha} \right) \frac{\tilde{w}^\pm(x) - \tilde{w}^\pm(y)}{|x-y|^{\alpha+d}} \, dx \, dy
+ \int_{\mathbb{R}^d} g(\tilde{w}^\pm) \, dx \right] - \int_{\mathbb{R}^d} K(x)f^*(x, \tilde{w}^\pm)(\tilde{w}^\pm)^2 \, dx
= \int_{\mathbb{R}^d} K(x) \left( [g^+ - 1] f(x, \tilde{w}^\pm) \tilde{w}^\pm - f^*(x, \tilde{w}^\pm)(\tilde{w}^\pm)^2 \right) \, dx.
\]

(6.31)

By (6.31), hypotheses \((f_4)\) and \((K_1)\), we infer that

\[ \langle \varphi_{w^+}^*, \tilde{w} \rangle < 0 \quad \text{and} \quad \langle \varphi_{w^-}^*, \tilde{w} \rangle < 0. \]

(6.32)

According to (6.29), it comes that \( \lambda_\pm = 0 \). Therefore, from (6.28), we conclude that \( \tilde{w} \) is a critical point of \( J \). Hence, \( \tilde{w} \) is a nodal weak solution for problem \((P)\).

This completes the proof. 

\[ \square \]
Proposition 6.7. Assume that the hypotheses \((H_f), (H_G)\) and \((H_K)\) hold. Then, the ground state solution \(\hat{u}\) of problem \((P)\) has a fixed sign. Moreover,

\[
m_0 = J(\hat{u}) = \inf_{\mathcal{N}} J \leq \inf_{\mathcal{M}} J = J(\hat{w}) = m_1.
\]

Proof. We argue by contradiction. Suppose that \(\hat{u}^\pm \neq 0\), then

\[
m_0 = \inf_{\mathcal{N}} J \geq \inf_{\mathcal{M}} J = m_1.
\]

(6.33)

Since \(\mathcal{M} \subset \mathcal{N}\),

\[
m_0 = \inf_{\mathcal{N}} J \leq \inf_{\mathcal{M}} J = m_1.
\]

(6.34)

Putting together (6.33) and (6.34), we get

\[
m_0 = J(\hat{u}) = \inf_{\mathcal{N}} J = \inf_{\mathcal{M}} J = J(\hat{w}) = m_1.
\]

(6.35)

On the other hand, since \(\hat{w} \in \mathcal{M}\), \(\hat{w}^\pm \neq 0\). Then, from Proposition 5.1 there is a unique pair \(t_{\hat{w}^+}, s_{\hat{w}^-} > 0\) such that \(t_{\hat{w}^+} \hat{w}^+ \in \mathcal{N}\) and \(s_{\hat{w}^-} \hat{w}^- \in \mathcal{N}\).

By Lemma 4.5 and Proposition 6.3, it follows that

\[
2m_0 \leq J(t_{\hat{w}^+} \hat{w}^+ + s_{\hat{w}^-} \hat{w}^-) \\
< J(t_{\hat{w}^+} \hat{w}^+ + s_{\hat{w}^-} \hat{w}^-) \\
\leq J(\hat{u}) \\
= \inf_{\mathcal{M}} J = m_1.
\]

(6.36)

Which is a contradiction with (6.35). Therefore, \(\hat{u}\) has a fixed sign, and

\[
m_0 = J(\hat{u}) = \inf_{\mathcal{N}} J < \inf_{\mathcal{M}} J = J(\hat{w}) = m_1.
\]

Thus the proof.

Proof of Theorem 3.2: Theorem 3.2 is a consequence of the Propositions 5.5, 6.6 and 6.7.

References

[1] V. Ambrosio, G. M. Figueredo, T. Isernia, G. M. Bisci, Sign-changing solutions for a class of zero mass nonlocal Schrödinger equations. Adv. Nonlinear Stud. 19 (2018), 113-132.

[2] V. Ambrosio, T. Isernia, Sign-changing solutions for a class of fractional Schrödinger equations with vanishing potentials. Lincei Mat. Appl. 29 (2018), 127-152.

[3] S. Bahrouni, A. Salort, Neumann and Robin type boundary conditions in Fractional Orlicz-Sobolev spaces. ESAIM Control Optim. Calc. Var. 27 (2021) S15.

[4] A. Bahrouni, S. Bahrouni, M. Xiang, On a class of nonvariational problems in fractional Orlicz-Sobolev spaces. Nonlinear Anal. 190 (2020), 111595.

[5] S. Bahrouni, H. Ounaies, Embedding theorems in the fractional Orlicz-Sobolev space and applications to non-local problems. Discrete Contin. Dyn. Syst. 40 (2020), 2917-2944.

[6] S. Bahrouni, H. Ounaies, L. S. Tavares, Basic results of fractional Orlicz-Sobolev space and applications to non-local problems. Topol. Methods Nonlinear Anal. 55 (2020), 681-695.

[7] A. Bahrouni, H. Missaoui, H. Ounaies, Least-energy nodal solutions of nonlinear equations with fractional Orlicz-Sobolev spaces. (2021). https://arxiv.org/abs/2105.03368.
[8] S. Barile, G. M. Figueredo, Existence of least energy positive negative and nodal solutions for a class of $p&Q$-problems with potentials vanishing at infinity. *J. Math. Anal. Appl.* **427** (2015), 1205-1233.

[9] T. Bartsch, Z. Liu, T. Weth, Sign Changing Solutions of Superlinear Schrödinger Equations. *Comm. Partial Differential Equations* **29** (2012), 25-42.

[10] T. Bartsch and T. Weth, A note on additional properties of sign changing solutions to superlinear elliptic equations. *Topol. Methods Nonlinear Anal.* **22** (2003), 1-14.

[11] A. M. Batista, M. F. Furtado, Positive and nodal solutions for a nonlinear Schrödinger-Poisson system with sign-changing potentials. *Nonlinear Anal.* **39** (2018), 142-156.

[12] H. Berestycki, P. L. Lions, Nonlinear scalar field equations, I Existence of a ground state. *Arch. Ration. Mech. Anal.* **82** (1983), 313–346.

[13] J. Fernández Bonder, A. M. Salort, Fractional order Orlicz-Sobolev space. *J. Funct. Anal.* **277** (2019), 333-367.

[14] J. Fernández Bonder , A. M. Salort, H. Vivas, Global Hölder regularity for eigenfunctions of the fractional $g$-Laplacian. arXiv:2112.00830.

[15] J. Fernández Bonder , A. M. Salort, H. Vivas, Interior and up to the boundary regularity for the fractional $g$-Laplacian: the convex case. arXiv preprint [arXiv:2008.05543], (2020).

[16] X. Chang, Z. Nie, Z. Q. Wang, Sign-Changing Solutions of Fractional $p$-Laplacian Problems. *Adv. Nonlinear Stud.* **19** (2019), 29-53.

[17] S. Chen, X. Tang, Ground state sign-changing solutions for elliptic equations with logarithmic nonlinearity. *Acta Math. Hungar.* **157** (2019), 27-38.

[18] F. Clarke, Functional Analysis, Calculus of Variation and Optimal control. *Grad. Texts in Math.* **264**, Springer, London, (2013).

[19] F. H. Clarke, Optimization and Nonsmooth Analysis. *Canad. Math. Soc. Se. Monog. Adv. Texts* John Wiley & Sons, New York, (1983).

[20] F. J. S. A. Corrêa, M. L. M. Carvalho, J. V. A. Goncalves, E. D. Silva, Sign changing solutions for quasilinear superlinear elliptic problems. *Quart. J. Math.* **68** (2017), 391-420.

[21] G. M. Figueredo, J. A. Santos, Existence of least energy solution with two nodal domains for a generalized Kirchoff problem in an Orlicz-Sobolev space. *Math. Nachr.* **290** (2017), 583-603.

[22] G. M. Figueiredo, J. R. Santos Júnior, Existence of a least energy nodal solution for a Schrödinger-Kirchhoff equation with potential vanishing at infinity. *J. Math. Phys.* **56** (2015), 051506.

[23] G. M. Figueiredo, R. G. Nascimento, Existence of a nodal solution with minimal energy for a Kirchhoff equation. *Math. Nachr.* **288** (2015), 48-60.

[24] N. Fukagai, M. Ito, K. Narukawa, Positive solutions of quasilinear elliptic equations with critical Orlicz-Sobolev nonlinearity on $\mathbb{R}^N$. *Funkcial. Ekvac.* **49** (2006), 235-267.

[25] L. Gasiński, N. S. Papageorgiou, Constant sign and nodal solutions for superlinear double phase problems. *Adv. Calc. Var.*, **14** (2019), 1-14.

[26] L. Gasiński, N. S. Papageorgiou, Nonlinear Analysis. *Ser. Math. Anal. Appl.* **9**, Chapman & Hall/CRC, Boca Raton, (2006).

[27] T. Isernia, Sign-changing solutions for a fractional Kirchhoff equation. *Nonlinear Anal.* **190** (2020), 111623.

[28] C. Miranda, Unósservazione sul teorema di Brouwer. *Boll. Unione Mat. Ital.* **3** (1940), 57.
[29] G. Molica Bisci, V. Rădulescu, R. Servadei, Variational Methods for Nonlocal Fractional Problems, with a Foreword by Jean Mawhin, Encyclopedia of Mathematics and its Applications. *Cambridge University Press* **162** Cambridge, (2016).

[30] G. Molica. Bisci, V. Rădulescu, Ground state solutions of scalar field fractional Schrödinger equations. *Calc. Var. Partial Differential Equations* **54** (2015), 2985-3008.

[31] P. Ochoa, A. Silva, M. J. S. Marziani, Existence and multiplicity of solutions for a Dirichlet problem in Fractional Orlicz-Sobolev spaces. (2021), https://arxiv.org/abs/2112.14520.

[32] N. S. Papageorgiou, V. D. Rădulescu, D. D. Repǒvs, Ground state and nodal solutions for a class of double phase problems. *Z. Angew. Math. Phys.* **71** (2020), 1-15.

[33] M. M. Rao, Z. D. Ren, Theory of Orlicz Spaces. Marcel Dekker, Inc, New York (1991).

[34] L. Xu, H. Chen, Positive, negative and least energy nodal solutions for Kirchhoff equations in $\mathbb{R}^N$. *Complex. Var. Elliptic. Equ.* **66** (2021), 1676-1698.