Quantum Walk and Quantum Billiards. Towards a better understanding of Quantum Chaos

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Quantum billiards have been simulated so far in many ways, but in this work a new approximation is considered. This study is based on the quantum billiard already obtained by others authors via a tensor product of two 1-D quantum walks. Chaotic and non chaotic billiards are tested

I. INTRODUCTION

The skeleton of this paper is the quantum walk [1] (hereafter QW) a well known mechanism of quantum spread based on tossing a coin and moving the particle one position to the left or to the right depending on whether we got a face or a cross in the previous coin toss.

The initial state represents a particle in the coordinate origin with two possible spin states. Its state is set to $|\Psi(t)\rangle = U(t)|x,u\rangle + D(t)|x,d\rangle$ and the evolution operator is determined by $\hat{U} = \hat{W} \otimes \hat{C}$ where $\hat{C}$ is a SU(2) matrix that mixes the probability amplitudes and $\hat{W} = |+><+| \otimes |x+1><x| + |><-| \otimes |x-1><x|$ is the displacement operator. Hereafter we are always going to consider that the matrix $\hat{C}$ has the following shape:

$$
\begin{pmatrix}
\cos \Theta & \sin \Theta \\
-\sin \Theta & \cos \Theta
\end{pmatrix}
$$

If we apply this operator over and over again we witness the propagation of the particle towards the ends of the line more quickly than in the classical random walk. Indeed this is an important characteristic that the QW takes advantage of. [1]

But we are not interested merely in the propagation, but in the bounce of the particle when it reaches both extremes as well. This is achieved with a new set of billiard operators that will be discussed later. A billiard is a closed area where the particles moves in, and the particle rebounds every time it reaches a wall. The particle moves inside the billiard according to the quantum rules of the quantum walk. And we are going to simulate such billiards with the aid of the QW that is perfectly explained at [1]

The goal of this paper is showing the difference between chaotic and non chaotic (so called integrable or regular) billiards. The corresponding spectra will be presented in the following sections. And we are going to reproduce somehow similar results that the ones that were obtained by Martin Sieber in Bristol [2] or the obtained by Arnd Bäcker in Regensburg [3] or the ones which are shown in [4]. By telling this I mean we are going to show that the distribution of distances between energy levels for a given $K$ is different in a chaotic billiard (Wigner distribution) from the one obtained in an integrable billiard (Lorentz distribution).

II. FIRST KIND OF BILLIARDS

The QW operator can be modified to include a bounce that happens every time the particle which propagates by the line reaches the borders. This is the simplest billiard you can make and its evolution operator (it is known that $\hat{U} = \hat{C} \hat{W}$) is described by:

$$
\hat{W} = |+><+| \sum_{n_I}^{n_D-1} |n+1><n| + |><-| \sum_{n_I+1}^{n_D} |n-1><n|
$$

Where $n_D$ stands for the right most point, and $n_I$ for the left most point. The particle moves on a line from the position $n$ to the position $n \pm 1$ depending on what was obtained in the coin toss. The other main idea involved here is that a particle with spin $\frac{1}{2}$ flips the sign of its spin every time it bounces. The first line of the equation right above almost describes a standard QW. It would describe it if it would be refered to every point in the grid, but that is not the case because it relates to all the points in the line except those from the borders.

![FIG. 1: 70 steps of this evolution](image)

We have to turn to the evolution equation if we want to get the spectrum. That equation tells us $|\psi(t+1)\rangle =$
\[ \hat{U} |\psi(t) > \] And because of the form of the billiard equation this problem cannot be solved analytically with Mathematica so we will solve it in a numeric manner. For instance for a grid that goes from -2 to 2 in steps of one by one we get the spectrum shown in Table 1.

In the upcoming lines we are going to obtain several spectra from different billiards. Several dispersion relations so to speak. But we are going to focus our attention on the eigenvalues obtained because we are actually interested in the tensor products of these spectra. It would be helpful to keep in mind that we are going to build the billiards via tensor products of the QW on a line or the QW in different paths with some other QW, like is already done in [5].

### III. SECOND KIND OF BILLIARDS

Here is explained a new kind of billiard in a line. Its main difference relies on the fact that the particle doesn’t flip its spin when it reaches the border but the particle indeed propagates by a different grid.

The mechanism involved is the same. You toss a coin and depending on the result obtained you move to the right or to the left, but instead of moving one unit you move two units by the grid A and you continue the walk until you reach the borders. There the particle bounces and it moves only one unit in this rebound, so the particle is in a different grid now. If the particle was before propagating through the grid A, the grid of the even numbers so to speak, now it begins to travel by the grid B, the grid of the odd numbers. The idea is maybe better understood with this graph.

![Image of a grid with arrows indicating the movement of a particle](image)

It should not be forgotten that the real walk goes by a line, one and only one, although the particle takes different positions on the grid depending on the number of bounces it has undergone.

The spectrum that is got now involves a matrix too big to be shown here, even for small numbers. So we adjoint the unitary operator such that \[ |\psi(t + 1) >= \hat{U}|\psi(t) > \] with \( \hat{U} = \hat{W}\hat{C} \) and we trust that it is not a hard task to obtain the spectrum with the plane wave technique relating to put the probability amplitudes as plane waves and replacing them in the evolution equation right above.

\[
\hat{W} = \sum_{P_D} |p + 2, u > < p, u| + |I_D, u > < P_D, u| \quad (2)
\]

\[
\sum_{P_L} |p - 2, d > < p, d| + |I_L, d > < P_l, d|\]
\[
\sum_{I_L} |i - 2, u > < i, u| + |P_L, u > < I_l, u|
\]
\[
\sum_{I_D} |i + 2, d > < i, d| + |P_D, d > < I_D, d|
\]

The demonstration that this operator is unitary may not be straightforward so we attach it here.

If you did: \( U^*U \) you would get \( \sum_{P_l} |p + 2, u > < p + 2, u| + |P_l, u > < P_l, u| + \sum_{I_D} |i - 2, u > < i - 2, u| + |I_D, u > < I_D, u| + \sum_{P_l} |p - 2, d > < p - 2, d| + |P_D, d > < P_D, d| + \sum_{I_l} |p - 2, d > < p - 2, d| + |I_l, d > < I_l, d|.

You would be led to \( \hat{I}_D \otimes (|u > < u| + |d > < d|) \) It finally results in the identity matrix (as was expected). And it should be convenient to remember that the \( p \) and \( i \) goes in steps of two units, in such a manner that the posterior element to \( p \) is \( p + 2 \).

![Image of a grid with arrows indicating the movement of a particle](image)

**FIG. 2:** 70 steps of this evolution.

### IV. THIRD KIND OF BILLIARDS

This is the last type of billiard that is going to be introduced here. But previously is appropriate to remind the definition of quantum chaos as the quantum counterpart of a classic system with chaotic behavior. And in a
classical billiard the chaotic or non chaotic behaviour depends on whether or not two very close initial trajectories finish in the neighbourhood of the same point.

It seems clear that the presence of curves in one side of the billiard is linked to a chaotic performance and this is why a series of curved line billiard are exposed. All of them are based on the two kind of billiards presented so far. But the curves are achieved with different functions like cos, cosh, tanh, ...

For example an operator which describes the evolution in a sinusoidal path is the next:

\[
\tilde{W} = |+><+| \otimes (3)
\]

\[
\sum_{\alpha_i+\text{step}} |\sin(\alpha + \text{step}), \alpha + \text{step}><\sin(\alpha), \alpha| + |+-><-| \otimes \sum_{\alpha_i} |\sin(\alpha - \text{step}), \alpha - \text{step}><\sin(\alpha), \alpha|
\]

The step represents the leap between one point in the grid and the next one. And the part of the operator written in red stands for the sin evolution. Because such a sinusoidal path can not be achieved in only one dimension, so we need that tensor product. If we did not put the part in red we would get a QW in a line that goes from -1 to 1 repeating positions as the time goes up.

A sinusoidal trajectory could be also obtained with the well known evolution operator \( U = \hat{W} \hat{C} \) with any SU(2) matrix as the coin operator and where the \( W \) has been defined as follows:

\[
\tilde{W} = |u><u| \otimes (4)
\]

\[
\left( \sum_{P_I} |\sin(p + 2\text{step}), p + 2\text{step}><\sin p, p| + |\sin I_D, I_D><\sin P_D, P_D| + \right)
\]

\[
+ |d><d| \otimes \left( \sum_{P_I} |\sin(p - 2\text{step}), p - 2\text{step}><\sin p, p| + |\sin I_I, I_I><\sin P_I, P_I| + \right) + |u><u| \otimes \left( \sum_{I_I} |\sin(i - 2\text{step}), i - 2\text{step}><\sin i, i| + |\sin P_I, P_I><\sin I_I, I_I| + \right)
\]

\[
+ |d><d| \otimes \left( \sum_{I_I} |\sin(i + 2\text{step}), i + 2\text{step}><\sin i, i| + |\sin P_D, P_D><\sin I_D, I_D| + \right)
\]

I would like to highlight that this operator does not resemble a 2-D QW, but a QW through the trajectory of a sin function instead. In a 2-D QW you would need two coins in order to move independently through the two axis of the plane and here we only use one unique

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**TABLE I: Kind 1 Billiard Spectrum**

| \( \sin \Theta \) | \(- \cos \Theta\) | \(0\) | \(e^{ik} \sin \Theta \) | \(-e^{ik} \cos \Theta\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) |
|---|---|---|---|---|---|---|---|---|---|---|
| \(e^{-ik} \cos \Theta\) | \(e^{-ik} \sin \Theta\) | \(0\) | \(0\) | \(e^{ik} \sin \Theta\) | \(-e^{ik} \cos \Theta\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) |
| \(0\) | \(0\) | \(0\) | \(e^{-ik} \sin \Theta\) | \(e^{-ik} \cos \Theta\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) |
| \(0\) | \(0\) | \(0\) | \(0\) | \(e^{ik} \sin \Theta\) | \(-e^{ik} \cos \Theta\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) |
| \(0\) | \(0\) | \(0\) | \(0\) | \(e^{-ik} \cos \Theta\) | \(e^{-ik} \sin \Theta\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) |
| \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(e^{ik} \sin \Theta\) | \(-e^{ik} \cos \Theta\) | \(0\) | \(0\) | \(0\) | \(0\) |
| \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) |

**FIG. 3: Kinds of billiards**
coin in order to keep the particle in the desired sin trajectory. When $\alpha$ increases to $\alpha + \text{step}$, $\sin(\alpha)$ goes up to $\sin(\alpha + \text{step})$ and, likewise, when $\alpha$ decreases to $\alpha - \text{step}$, $\sin(\alpha)$ goes to $\sin(\alpha - \text{step})$. Therefore you replicate the sin shape.

These two operators, that of course are unitary matrices, can be replicated with $\cosh$, $\tanh$ and $\cos$ or any curved $f(x)$ actually instead of the $\sin$ function in order to get a curved line. The foretold functions are the three examples that are going to be analysed in here.

Regarding to the spectrum we can still use the one plotted in the Table 1 and the spectrum obtained in the billiard kind 2 that was not written in this work. But replacing the $e^{ik}$ by $e^{ik} \sin(\alpha) e^{i \sin(\alpha + \text{step})} / e^{i \sin(\alpha)} e^{ik}$, and $e^{-ik}$ by $e^{ik} \sin(\alpha) e^{i \sin(\alpha - \text{step})} / e^{i \sin(\alpha)} e^{-ik}$. And, of course, if you use cos, cosh or tanh instead of sin, the spectra you get depend on $K_{\cos(\alpha)}$, $K_{\cosh(\alpha)}$, $K_{\tanh(\alpha)}$ in addition to $K_{\alpha}$ and they look like this:

The spectra that are obtained with this technique are only slices of the real spectrum. Because the real spectrum depends also on $\alpha$, but on $K_{\alpha}$ and it does depend on $\cos(\alpha)$ and on $K_{\cos(\alpha)}$. It is a cube of numbers so to speak, and you find different slices in it depending on whether or not you are in an odd or even position and on whether or not you are close to $\alpha_I$ or $\alpha_D$. These slices, each one of them, does not depend on $\alpha$ so they can be regarded as independent matrices since they depend only on $\cos(\alpha)$

What is shown here represents the spectrum for a fixed value of $\alpha$ of the billiard that relies on two different $K$ variables, $K_{\cos(\alpha)}$ and $K_{\alpha}$. There are four different kind of slices depending on the value of $\alpha$. Depending on if $\alpha$ is equal to $\alpha_D$ or $\alpha_I$ or if $\alpha$ is an even step or an odd one. But once you have chosen your slice that slice does not depend explicity on $\alpha$

All these spectra were obtained over a grid of 5 points for billiards of kind 1 and a grid of 6 points for billiards of kind 2. And all of them are worked for an even $\alpha$ (Remember that there are different spectra depending on the value of $\alpha$ you choose)

V. THE GOAL. THE 2-D BILLIARDS

These billiards are obtained by a tensor product of the billiards we have displayed so far. They are arranged in such a way that some of them are chaotic systems in the sense that are the counterpart of classical chaotic ones. And the others should be non chaotic (or regular) according to the previous definition.

As has been explained already before the spectra we get should have different sorts of spacing between levels for every value of $K$ you want to choose, depending on whether the billiard is chaotic or not. But we observe a very random set of distributions of such distances instead.

In previous work, in systems studied not only with Schrödinger equation but also with discrete techniques such as the Bose - Hubbard hamiltonian, that difference was clear. And these billiards should go from less chaotic to more chaotic according to this scheme:
But since in the ongoing study the behaviour of the billiard depends on several parameters, such as which parameters $K_{\cos(\alpha_1)}$, $K_{\cos(\alpha_2)}$, $K_{\alpha_1}$, $K_{\alpha_2}$ you select in the representation (in this study we are going to focus on $K_{\cos(\alpha)}$), the angle of the coins, the number of points in the grid, the shape itself of the billiard and the kind of billiard chosen to simulate the bounce (Remember we have defined two main kind of billiards). I have to conclude that the results I get are quite different than the ones present in the currently available bibliography. Even to the point that is impossible to assign a difference that help us to distinguish chaotic billiards from non chaotic billiards.

Now I attach some billiards and the corresponding spacing between the energy levels for some fixed $K_{\cos(\alpha)}$.

In the making of the distribution that is got from the spacing we have not taken into account the two gaps that appear in the spectrum. Such gap could also be considered and then ignored in the graph displayed, but it would provide a worse picture, a picture where this separation would be worse observed. In the captions is shown the geometry of the billiards involved. The numbers 1 or 2 stand for the kind of billiard that was considered.

And as seen in the piece of spectrum shown above, in this very billiard the differences in the spacing between consecutive levels of energy are equally spread (more or less) in such a way that it leads to the next distribution.

In the following figures we show the repulsion between adjacent energy levels in a regular billiard and the Poisson distribution that emerges of such set of distances. And, in the other hand, the periodic distances that are found between two consecutive levels in a chaotic billiard and the resulting Wigner distribution are shown as well. Although these differences are not well appreciated in this footage.

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And here, according to [2], we can better distinguish between the chaotic billiard and the regular one. The spacing in the levels of the chaotic billiard seems to be more equally spread than the spacing in the regular billiard. Here is far better appreciated than in the previous image.

But the separations between adjacent levels can lead us to a very different distribution. We get very weird results if we change the angle that defines the SU(2) matrix which is our coin. Here some examples are attached that are quiet unlike the “standard” ones.
VI. CONCLUSIONS

In the set of plots that is shown above several facts appear.

1. In this approximation, via quantum walk, there are not well set differences between the quantum counterparts of chaotic and non-chaotic classical billiards. These differences used to be established in the spacing between two consecutive energy levels. And, in general, we don’t observe that here. Although in some cases (depending mainly on the coin angle) our results match with the previous ones we can find in the bibliography.

2. The spacing between levels depends on several factors, namely: $\Theta$ the angle of the coin, the geometry involved, the kind of spectrum implemented (what we have called Kind 1 and Kind 2) and the value of $K$ chosen.

3. We conclude that in this scheme at least one arrow must be wrong.

4. The QW with E field applied was studied by Bru and companies in [6] and we observe here that if we work with $\hat{U} = \hat{E}\hat{W}\hat{C}$ we obtain always (actually not always. With the SU(2) matrix’s angles as $\pi$ or $\pi/2$ this effect does not occur) a distribution corresponding to a regular billiard $E = exp(i F_i x)$ where $x$ stands for the distance to the origin and $F_i$ can be any number. It resembles the electric potential.

5. As it was forementioned in [Conclusion 1] the results match, for a certain set of $\Theta_1$, $\Theta_2$ and other parameters, with those obtained with a well potential in the Schrödinger equation as is shown in [7]. We would like to emphasize that these results are somehow wider than the existing previous ones.

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