Dephasing of a non-relativistic quantum particle due to a conformally fluctuating spacetime

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Abstract
We investigate the dephasing suffered by a non-relativistic quantum particle within a conformally fluctuating spacetime geometry. Starting from a minimally coupled massive Klein–Gordon field, we derive an effective Schrödinger equation in the non-relativistic limit. The wavefunction couples to gravity through an effective nonlinear potential induced by the conformal fluctuations. The quantum evolution is studied through a Dyson expansion scheme up to second order. We show that only the nonlinear part of the potential can induce dephasing. This happens through an exponential decay of the off-diagonal terms of the particle density matrix. The bath of conformal radiation is modeled in three dimensions and its statistical properties are described in terms of a general power spectral density. Vacuum fluctuations at a low energy domain are investigated by introducing an appropriate power spectral density and a general formula describing the loss of coherence is derived. This depends quadratically on the particle mass and on the inverse cube of a particle-dependent typical cutoff scale. Finally, the possibilities for experimental verification are discussed. It is shown that current interferometry experiments cannot detect such an effect. However this conclusion may improve by using high mass entangled quantum states.

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1. Introduction

It is generally agreed that the underlying quantum nature of gravity implies that the
spacetime structure close to the Planck scale departs from that predicted by general relativity.
Unfortunately the quantum gravity domain is still beyond modern particle accelerators such as
the LHC. Nonetheless, finding experimental ways to test the quantum structure of spacetime
would be highly beneficial to the theoretical developments of our fundamental theories of
nature. In this respect, it has been suggested that quantum gravity could induce decoherence
on a quantum particle through its underlying Planck scale spacetime fluctuations [1–6].

As the sensitivity and performance of matter wave interferometers is increasing
[7–11], it is important to assess the theoretical possibility of a future experimental detection
of intrinsic, spacetime induced decoherence. The closely related dephasing effect due to a
random bath of classical GWs (e.g. of astrophysical origin) has been extensively studied e.g. in
[12]. The problem of the decoherence induced by spacetime fluctuations is difficult to study:
notwithstanding promising progress, mainly in loop quantum gravity and superstring theory
[13], a coherent and established QG theoretical framework is still missing. Thus theoretical
attempts for a prediction of the decoherence induced by spacetime fluctuations usually exploit
some kind of semiclassical framework. Such approaches typically represent the spacetime
metric close to the Planck scale by means of fluctuating functions. These are usually supposed
to mimic the vacuum quantum property of spacetime down to some cutoff scale \( \ell = \lambda L_P \),
where \( L_P \) is the Planck scale. The adimensional parameter \( \lambda \) marks the benchmark between
the fully quantum regime and the scale where the classical properties of spacetime start to
emerge [1]. The fact that classical fluctuating fields can be used to reproduce various genuine
quantum effects is well known, e.g. from the work of Boyer [14, 15] in the case of the EM field.
Frederick also applied the same technique to model spacetime fluctuations [16]. A classical
but fluctuating metric is often exploited in the literature in relation to problems related to the
microscopic behavior of the spacetime. This is often exploited in the literature in relation
to problems involving the microscopic behavior of the spacetime metric; e.g. a stochastic
metric was employed in [17, 18] to study the problem of gravitational collapse and big bang
singularities, while in [19] spacetime metric fluctuations were introduced and their ability to
induce a WEP violation studied.

A pioneering analysis of the problem of spacetime induced decoherence has been proposed
by Power & Percival (PP in the following) [1] in the case of a conformally modulated
Minkowski spacetime with conformal fluctuations traveling along one space dimension. This
was improved by Wang et al [3], who extended upon PP attempting to include the effect of
GWs. Conformal fluctuations are interesting as they are relatively easy to treat and offer a
convenient way to build ‘toy’ models to assess some of the problem’s features. They have an
important role in theoretical physics [20] and are sometimes invoked in the literature also in
relation to universal scalar fields [18, 21] that can arise naturally in some modified theories of
gravity such as scalar–tensor [22, 23].

Within a semiclassical approach that ‘replaces’ the true quantum environment by classical
fluctuating fields we should properly speak of dephasing of the quantum particle rather than
decoherence. In a remarkable paper [24] about quantum interference in the presence of
an environment, Stern and co-authors showed that a fully quantum approach that studies
decoherence by ‘tracing away’ the environment degrees of freedom in the quantum system
made up by particle + environment, and that in which the dephasing of the quantum particle
is due to a stochastic background field give equivalent results.

In this paper, we consider a conformally modulated four-dimensional spacetime metric
of the form \( g_{\alpha \beta} = (1 + A)^2 \eta_{\alpha \beta} \). Such a metric has been considered by PP [1], where the
dephasing problem was studied in the simple idealized case of a particle propagating in one dimension. By imposing Einstein’s equation on the metric $g_{ab}$, PP deduced a wave equation for $A$. Their procedure to derive an effective Newtonian potential interacting with the quantum probe started from the geodesic equation of a test particle. Even though this did not take properly into account the nonlinearity in the conformal factor $(1 + A)^2$, they found correctly that the change in the density matrix is given by $\delta \rho \propto M^2 T A_0^4 \tau^*$, where $M$ is the probing particle mass, $T$ is the flight time, $A_0$ is the amplitude of the conformal fluctuations and $\tau^*$ is their correlation time. This formula was used to set limits upon $\lambda$. However in doing this they did not treat the statistical properties of the fluctuations properly and this resulted in the wrong estimate $\lambda \propto (M^2 T / \delta \rho)^{1/7}$, as already noted by Wang et al in [3].

In their work, Wang and co-authors considered a metric of the kind $g_{ab} = (1 + A)^2 \gamma_{ab}$, where the conformal metric $\gamma_{ab}$ was supposed to encode GWs. This was done by exploiting the results in [25, 26] where a canonical geometrodynamics approach employing a conformal spacial 3-metric was studied. By exploiting an energy density balancing mechanism between the conformal and GWs parts of the total gravitational Hamiltonian the statistical properties of the conformal fluctuations were fixed. This corresponded to assume that each ‘quantum’ of the conformal field possessed a zero-point energy $-\hbar \omega$. Though an improvement over PP work, this approach is still one dimensional and too crude to make predictions. Moreover, as it shall be discussed extensively in a future report [27], the issue of energy balance between conformal fluctuations and GWs is a delicate one, and likely not to occur within the standard GR framework.

In the present work, we provide a coherent three-dimensional treatment of the problem of a slow massive test particle coupled to a conformally fluctuating spacetime. The conformal field $A$ is assumed to satisfy a simple wave equation. This will allow a direct comparison with PP result. We also note that such a framework is expected to arise naturally within a scalar–tensor theory of gravity. This issue will be discussed in a future report [27].

The work is organized as follows: in section 2 the correct non-relativistic limit of a minimally coupled Klein–Gordon (KG) field is deduced and an effective Newtonian potential depending nonlinearly on $A$ is identified in the resulting effective Schrödinger equation. In section 3, we set the general formalism to study the average quantum evolution through a Dyson expansion scheme for the particle density matrix $\rho$. In section 4, general results derived in appendix A are used to model the statistical and correlation properties of the fluctuations through a general, unspecified, power spectral density. In sections 5 and 6, we compute the average quantum evolution and derive a general expression for the evolved density matrix. We show in general that only a nonlinear potential can induce dephasing. The resulting dephasing formula implies an exponential decay of the density matrix off-diagonal elements and is shown to hold in general and independently of the specific spectral properties of the fluctuations. All we assume is that these obey a simple wave equation and that they are a zero mean random process. The overall dephasing predicted within the present three-dimensional model—equation (28)—is seen to be about two orders of magnitudes larger than in the one-dimensional case as derived by PP. Next we consider in section 7 the problem of vacuum fluctuations. To this end a power spectrum $S(\omega) \sim 1/\omega$ is introduced and we derive an explicit formula for the rate of change of the density matrix. This result improves over both Percival’s and Wang’s work in that its key ingredients are general enough to be potentially suited for a variety of physical situations. Finally the discussion in section 8 addresses the question of whether the dephasing due to conformal vacuum spacetime fluctuations could be detected. In other words, whether the proposed theory can be falsified or not. A possibility would be through matter wave interferometry employing large molecules. We consider this issue in the final part of this paper by estimating the probing particle resolution scale, setting
its ability to be affected by the fluctuations. The resulting formula for the dephasing indicates that the level of the effect is still likely to be beyond experimental capability, even for large molecules (e.g. fullerenes) [10]. A measurable effect could possibly result for larger masses, e.g. if entangles quantum states were employed [28].

2. Low velocity limit and effective Schrödinger equation

The problem we wish to solve is clearly defined: we consider a scalar field $A$ inducing conformal fluctuations on an otherwise flat spacetime geometry according to

$$g_{ab} = (1 + A)^2 \eta_{ab},$$

(1)

where $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski tensor. We will refer to $A$ as to the conformal field and this will be assumed to satisfy the wave equation $\partial^c \partial_c A = 0$. Solving this equation with random boundary conditions results in a randomly fluctuating field propagating in three-dimensional space. We assume this to be a small first-order quantity, i.e. $|A| = O(\varepsilon \ll 1)$. Equation (1) expresses the spacetime metric in the laboratory frame. We also suppose that the typical wavelengths of $A$ are effectively cut off at a scale set by $\ell = \lambda L_P$, where $L_P = (\hbar G/c^3)^{1/2} \approx 10^{-35}$ m is the Planck length. The dimensionless parameter $\lambda$ represents a structural property of spacetime: below $\ell$ a full quantum treatment of gravity would be needed so that, by definition, $\ell$ represents the scale at which a semiclassical approach that treats quantum effects by means of classical randomly fluctuating fields is supposed to be a valid approximation. The value for $\lambda$ is model dependent but it is generally agreed that $\lambda \gtrsim 10^2$ [3], so that $\ell$ is expected to be extremely small from a macroscopic point of view. This motivates the assumption that classical macroscopic bodies, including the objects making up the laboratory frame and also the observers, are unaffected by the fluctuations in $A$. This corresponds to the idea that a physical object is characterized by some typical resolution scale $L_R$ that sets its ability to ’feel’ the fluctuations: if $L_R \gg \ell$ these average out and do not affect the body, which simply follows the geodesic of the flat background metric. On the other hand, a microscopic particle can represent a successful probe of the conformal fluctuations if its resolution scale is small enough.

We are interested in the change of the phase induced on the wavefunction of a quantum particle by the fluctuating gravitational field. Various approaches to the problem of how spacetime curvature affects the propagation of a quantum wave exist in the literature; e.g. for a stationary, weak field and a non relativistic particle a Schrödinger-like equation can be recovered [29]. The more interesting case of time varying gravitational fields can be treated e.g. by eikonal methods that are usually restricted to weak fields with $g_{ab} = \eta_{ab} + h_{ab}$ and $|h_{ab}| \ll 1$ [30, 31]. Other approaches, e.g. in [19, 6], are based on the scheme developed by Kiefer [32] for the non-relativistic reduction of a KG field which is minimally coupled to a linearly perturbed metric.

The approach of PP in [1] and of Wang et al in [3] was to derive the geodesic equation in the weak field limit. Their treatments were however employing, incorrectly, the usual Newtonian limit scheme which is valid only for weak, linear and static perturbations [33]. This cannot be done in the present case as $\Omega^2 = (1 + A)^2$ induces a fast varying, nonlinear perturbation. In alternative to the Newtonian limit approach one could compute the geodesics of a conformally modulated Minkowski metric exactly and without making assumptions on the conformal factor. However this is ideally suitable for a zero size test particle or, more precisely, for a particle whose typical size is much less than the typical scale over which the spacetime geometry varies. This is not suitable for the situation we wish to study, where the spacetime fluctuations are assumed to vary on the very short scale $\ell = \lambda L_P$. 

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A wave approach that starts from a relativistic KG field does not suffer from the limitations of geodesic approach: this could be suitable for localized particles, while the KG approach does not assume any wave profile. For these reasons we expect the two methods to be inequivalent. Moreover, the wave approach is conceptually clearer also because the coupling between gravity and a scalar field is well understood and in the appropriate non-relativistic weak field limit an effective Schrödinger equation emerges naturally. This will be our approach below.

We describe the quantum particle of mass $M$ by means of a minimally coupled KG field $\phi$:

$$g^{ab} \nabla_a \nabla_b \phi = \frac{M^2 c^2}{\hbar^2} \phi,$$

where $\nabla_a$ is the covariant derivative of the physical metric $g_{ab}$. Using $g_{ab} = \Omega^2 \eta_{ab}$ this equation can easily be made explicit [33] and reads

$$\left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \right) \phi = \frac{\Omega^2 M^2 c^2}{\hbar^2} \phi - 2 \partial_a (\ln \Omega) \partial^a \phi,$$

(2)

i.e. the wave equation for a massive scalar field plus a perturbation due to $A$ describing the coupling to the conformally fluctuating spacetime. We remark that, had we considered the alternative meaningful scenario of a conformally coupled scalar field, then the equation

$$g^{ab} \nabla_a \nabla_b \phi - R\phi/6 - M^2 c^2 \phi/\hbar^2 = 0$$

would read explicitly

$$\left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \right) \phi = \frac{\Omega^2 M^2 c^2}{\hbar^2} \phi - 2 \partial_a (\ln \Omega) \partial^a \phi - \phi \Omega^{-1} \partial^c \partial_c \Omega.$$

Since it is $\Omega^{-1} \partial^c \partial_c \Omega = (1 - A) \partial^c \partial_c A + O(\epsilon^3)$ we see that if $A$ is assumed to satisfy the wave equation then the curvature term has no effect: in this case the minimally and conformally coupled KG equations are equivalent up to second order in $A$. We also note that, by introducing the auxiliary field $\Phi := \Omega \phi$, equation (2) turns out to be equivalent to

$$\left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \right) \Phi = \frac{\Omega^2 M^2 c^2}{\hbar^2} \Phi.$$

In principle, if a solution for $\Phi$ were known, then the physical scalar field representing the particle would follow formally as $\phi = \Omega^{-1} \Phi = (1 - A + A^2) \Phi$, up to second order in $A$. However in studying the dephasing problem we will only find an averaged solution for the average density matrix representing the quantum particle. Therefore, even if a solution in this sense is known in relation to $\Phi$, it would not be obvious how to obtain the corresponding averaged density matrix related to $\phi$, which is what we are interested in.

In view of the above considerations we work directly with equation (2) and now proceed in deriving its suitable non-relativistic limit. We will make two assumptions:

(i) the particle is slow i.e., if $\tilde{p} = M \tilde{v}$ is its momentum in the laboratory frame, we have

$$\frac{\tilde{v}}{c} \ll 1;$$

(ii) the effect of the conformal fluctuations is small, i.e. the induced change in momentum $\delta p = M \delta v$ is small compared to $M \tilde{v}$:

$$\frac{\delta v}{\tilde{v}} \ll 1.$$
In view of these assumptions we can write
\[
\phi = \psi \exp(-iMc^2t/\hbar),
\]
where the field \( \psi \) is close to be a plane wave of momentum \( \tilde{p} \). As a consequence we have
\[
\frac{\partial^2 \psi}{\partial t^2} \approx -\frac{1}{\hbar^2} \left( \frac{\tilde{p}^2}{2M} \right)^2 \psi.
\]
Using this and multiplying by \( \hbar^2/2M \), equation (2) yields
\[
\left[ i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2M} \nabla^2 - \frac{Mc^2}{8} \left( \frac{\tilde{v}}{c} \right)^4 \right] \psi
= \left( A + \frac{A^2}{2} \right) Mc^2 \psi - \frac{\hbar^2}{M} \partial^a \phi \partial_a \ln(1 + A) \times \exp(iMc^2t/\hbar). \tag{3}
\]
Leaving the term \( T_4 \) aside for the moment, the orders of the three underlined terms must be carefully assessed. We have
\[
T_1 \sim M\tilde{v}^2, \quad T_2 \sim Mc^2 \left( \frac{\tilde{v}}{c} \right)^4, \quad \langle T_3 \rangle \sim \epsilon^2 Mc^2,
\]
where an average has been inserted since \( T_3 \) is fluctuating. It follows that
\[
\frac{T_2}{T_1} \sim \left( \frac{\tilde{v}}{c} \right)^2 \ll 1.
\]
Thus, in the non-relativistic limit, \( T_2 \) is negligible in comparison to \( T_1 \). This is the case in a typical interferometry experiment where it can be \( \tilde{v} \approx 10^2 \text{ m s}^{-1} \) \cite{11}, so that \( (\tilde{v}/c)^2 \approx 10^{-12} \). Next we have
\[
\frac{\langle T_3 \rangle}{T_1} \sim \left( \frac{\epsilon c}{\tilde{v}} \right)^2.
\]
The request that the conformal fluctuations have a small effect thus gives the condition
\[
\frac{\langle T_3 \rangle}{T_1} \ll 1 \iff \epsilon^2 \sim \langle A^2 \rangle \ll \left( \frac{\tilde{v}}{c} \right)^2. \tag{4}
\]
That this condition is effectively satisfied can be checked \textit{a posteriori} after the model is complete. It depends on the statistical properties of the conformal field and the particle ability to probe them. This will be related to a particle resolution scale. At the end of the discussion in section 8.3 we will show that (4) is satisfied if, e.g., the particle resolution scale is given by its Compton length.

Under these conditions the non-relativistic limit of equation (3) yields
\[
-\frac{\hbar^2}{2M} \nabla^2 \psi + \left( A + \frac{A^2}{2} \right) Mc^2 \psi + T_4 = i\hbar \frac{\partial \psi}{\partial t}, \tag{5}
\]
where
\[
T_4 := -\frac{\hbar^2}{M} \partial^a \phi \partial_a \ln(1 + A) \times \exp(iMc^2t/\hbar).
\]
In order to assess the correction due to this term we split it into two contributions by writing separately the time and space derivatives. Using the fact that \[
\frac{\partial \phi}{\partial t} \approx -\frac{i}{\hbar} \frac{M c^2}{\bar{\hbar}} \psi \exp(-i \frac{M c^2 t}{\bar{\hbar}})
\]
it is easy to see that
\[
T_4 = -i\hbar (\dot{\mathcal{A}} - \mathcal{A} \dot{\mathcal{A}}) \psi - i\hbar \tilde{v}(\mathcal{A} \dot{x} - \mathcal{A} \ddot{x}) \psi,
\]
(6)

where \(\dot{\mathcal{A}} := \partial \mathcal{A}/\partial t\), \(\mathcal{A} \dot{x} := \partial \mathcal{A}/\partial x\) and where we assumed the particle velocity in the laboratory to be along the \(x\)-axis. In appendix B we show that if \(\mathcal{A}\) is (i) a stochastic isotropic perturbation and (ii) effectively fast varying over a typical length \(\lambda_A = \kappa \bar{\hbar}/(M c)\) related to the particle resolution scale, then \(T_4\) reduces to
\[
T_4 = (\mathcal{A} - \mathcal{A}^2) \frac{M c^2}{\kappa} \psi.
\]
(7)

Here \(\kappa \sim 1\) is dimensionless and its precise value is unimportant. The important point is that \(T_4\) yields a positive extra nonlinear term in \(\mathcal{A}\) that adds up to what we already have in (5).

Finally we get the effective Schrödinger equation
\[
-\frac{\hbar^2}{2M} \nabla^2 \psi + V \psi = i\hbar \frac{\partial \psi}{\partial t},
\]
where the nonlinear fluctuating potential \(V\) is defined by
\[
V := (C_1 \mathcal{A} + C_2 \mathcal{A}^2) M c^2.
\]
(8)

The values of the constants \(C_1\) and \(C_2\) depend on \(\kappa\). For \(\kappa = 1\) it would be \(C_1 = 0\) and \(C_2 = 3/2\). For generality we will leave them unspecified in the following treatment and consider \(\kappa\) as a constant of order one.

### 3. Average quantum evolution

#### 3.1. Dyson expansion for short evolution time

We now have a rather well-defined problem: that of the dynamics of a non-relativistic quantum particle under the influence of the nonlinear stochastic potential (8). The Schrödinger equation for a free particle is suitable to describe the interference patterns that could result e.g. in an interferometry experiment employing cold molecular beams. When the particle in the beam propagates through an environment, we are dealing with an open quantum system. This in general suffers decoherence, resulting in a loss of visibility in the fringes pattern [9, 11]. This is a well-defined macroscopic quantity. In the present semiclassical treatment the environment due to spacetime fluctuations is represented, down to the semiclassical scale \(\ell\), by a sea of random radiation encoded in \(\mathcal{A}\) and resulting in the fluctuating potential \(V\). An estimate of the overall dephasing can be obtained by considering the statistical averaged dynamics of a single quantum particle interacting with \(V\). In practice we will need (i) to solve for the dynamics of a single particle of mass \(M\) and (ii) calculate the averaged wavefunction by averaging over the fluctuations. The outcome of (i) would be some sort of ‘fluctuating’ wavefunction carrying, beyond the information related to the innate quantum behavior of the system that related to the fluctuations in the potential. The outcome of (ii) is to yield a general statistical result describing what would be obtained in an experiment where many identical particles propagate through the same fluctuating potential.
We thus consider the Hamiltonian operator $\hat{H}(t) = \hat{H}^0 + \hat{H}'(t)$, where $\hat{H}^0$ is the kinetic part while

$$\hat{H}'(t) = \int d^3x \, V(x, t) |x\rangle \langle x|$$

is the perturbation due to the fluctuating potential energy. Here $|x\rangle \langle x|$ is the projection operator on the space spanned by the position operator eigenstate $|x\rangle$. Indicating the state vector at time $t$ with $\psi_t$, the related Schrödinger equation reads

$$\hat{H}(t) \psi_t = i\hbar \frac{\partial \psi_t}{\partial t}.$$

Using the density matrix formalism, the general solution can be expressed through a Dyson series as $[34]$

$$\rho_T = \rho_0 + \hat{U}_1(T) \rho_0 + \rho_0 \hat{U}_1^\dagger(T) + \hat{U}_2(T) \rho_0 + \hat{U}_1(T) \rho_0 \hat{U}_1^\dagger(T) + \rho_0 \hat{U}_2^\dagger(T) + \cdots,$$

where $\rho_0$ is the initial density matrix and the propagators $\hat{U}_1(T)$ and $\hat{U}_2(T)$ are given by

$$\hat{U}_1(T) := -\frac{i}{\hbar} \int_0^T \hat{H}(t') dt',$$

$$\hat{U}_2(T) := -\frac{i}{\hbar^2} \int_0^T dt' \int_0^{t'} dt'' \hat{H}(t') \hat{H}(t'').$$

In truncating the series to second order we assume that the system evolves for a time $T$ such that $T \ll T^*$, where $T^*$ is defined as the typical timescale required to have a significant change in the density matrix $\rho$.

The effect of the environment upon a large collection of identically prepared systems is found by taking the average over the fluctuating potential as explained above. Formally and up to second order we have

$$\langle \rho_T \rangle = \langle \rho_0 + \hat{U}_1(T) \rho_0 + \rho_0 \hat{U}_1^\dagger(T) + \hat{U}_2(T) \rho_0 + \hat{U}_1(T) \rho_0 \hat{U}_1^\dagger(T) + \rho_0 \hat{U}_2^\dagger(T) \rangle.$$

The average density matrix $\langle \rho_T \rangle$ will describe the average evolution of the system including the effect of dephasing.

It is straightforward to show that, up to second order in the Dyson’s expansion, the kinetic and potential parts of the Hamiltonian give independent, additive contributions to the average evolution of the density matrix, i.e. $\langle \rho_T \rangle = [\langle \rho_T \rangle_0 + [\langle \rho_T \rangle_1]$, where

$$\langle \rho_T \rangle_0 := \rho_0 + [\hat{U}_1(T)]_0 \rho_0 + \rho_0 [\hat{U}_1^\dagger(T)]_0$$

$$+ [\hat{U}_2(T)]_0 \rho_0 + [\hat{U}_1(T)]_0 \rho_0 \hat{U}_1^\dagger(T)_0 + \rho_0 \hat{U}_2^\dagger(T)_0,$$

$$\langle \rho_T \rangle_1 := \langle \rho_0 + [\hat{U}_1(T)]_1 \rho_0 + \rho_0 [\hat{U}_1^\dagger(T)]_1$$

$$+ [\hat{U}_2(T)]_1 \rho_0 + [\hat{U}_1(T)]_1 \rho_0 \hat{U}_1^\dagger(T)_1 + \rho_0 \hat{U}_2^\dagger(T)_1 \rangle.$$

Here the kinetic propagators $[\hat{U}_1(T)]_0$ and $[\hat{U}_2(T)]_0$ depend solely on $\hat{H}^0$, while the potential propagators $[\hat{U}_1(T)]_1$ and $[\hat{U}_2(T)]_1$ depend only on $\hat{H}'(t)$. In the following section we estimate the dephasing by calculating the term $[\langle \rho_T \rangle_1$ alone.

4. The conformal field and its correlation properties

We now set the statistical properties of the conformal field $A$. This is assumed to represent a real, stochastic process having a zero mean. We further assume it to be isotropic. In appendix A we review a series of important results concerning stochastic processes, in particular in relation
to real stochastic signals satisfying the wave equation. The main quantity characterizing the process is the power spectral density $S(\omega)$. In the case of an isotropic bath of random radiation, field averages such as $\langle A^2 \rangle$, $\langle |\nabla A|^2 \rangle$ and $\langle (\partial_t A)^2 \rangle$ can be found in terms of $S(\omega)$, e.g.

$$\langle A^2 \rangle = \frac{1}{(2\pi)^3} \int d^3k S(k),$$

where $kc = \omega$. In appendix A, we show how the conformal field can be resolved into components traveling along all possible space directions according to

$$A(\mathbf{x}, t) = \int d\mathbf{k} A_\mathbf{k}(\mathbf{k} \cdot \mathbf{x}/c - t),$$

where $d\mathbf{k}$ indicates the elementary solid angle. The capacity of the fluctuations to maintain correlation is encoded in the autocorrelation function $C(\tau)$. In the same appendix we prove a generalization of the usual Wiener–Khintchine (WK) theorem, valid for the case of a spacetime-dependent process satisfying the wave equation, and linking the autocorrelation function to the power spectral density according to

$$C(\tau) = \frac{1}{(2\pi c)^3} \int d\omega \omega^2 S(\omega) \cos(\omega \tau).$$

(10)

This allows us to prove that wave components traveling along independent space directions are uncorrelated, i.e.

$$\langle A_\mathbf{k}(t) A_{\mathbf{k}'}(t + \tau) \rangle = \delta(\mathbf{k}, \mathbf{k}') C(\tau).$$

(11)

4.1. Summary of the correlation properties of the conformal fluctuations

The main statistical properties of the directional stochastic waves $f_\mathbf{k}$ are summarized by

$$\langle f_\mathbf{k}(t) \rangle = 0,$$

$$\langle f_\mathbf{k}(t) f_\mathbf{k}(t') \rangle = \delta(\mathbf{k}, \mathbf{k}') R(t - t'),$$

(13)

(14)

i.e. each component has zero mean and fluctuations traveling along different space directions are perfectly uncorrelated. These two properties imply that odd products of directional components have also a zero mean, i.e.

$$\langle f_\mathbf{k_1}(t_1) f_\mathbf{k_2}(t_2) f_\mathbf{k_3}(t_3) \rangle = 0.$$

(15)
In the following dephasing calculation we will need to evaluate means involving products of four directional components. To this purpose we need to introduce the second-order correlation function \( R''(t - t') \) according to
\[
\langle [\hat{f}_k(t)]^2 [\hat{f}_{k'}(t')]^2 \rangle = 1 + \delta(\hat{k}, \hat{k}') \left[ R''(t - t') - 1 \right].
\] (16)

This definition is compatible with the fact that the mean is one when components traveling in different directions are involved, i.e. \( \langle [\hat{f}_k(t)]^2 [\hat{f}_{k'}(t')]^2 \rangle = 1 \) if \( \hat{k} \neq \hat{k}' \).

5. Dephasing calculation outline

To calculate the dephasing suffered by the probing particle we must evaluate the average of all the individual terms in equation (9). The relevant propagators are
\[
[U_1(T)]_1 := -\frac{i}{\hbar} \int_0^T dt' \hat{H}^1(t'),
\]
\[
[U_2(T)]_1 := -\frac{1}{\hbar^2} \int_0^T dt \int_0^{t'} dt' \hat{H}^1(t) \hat{H}^1(t').
\]

The interaction Hamiltonian is given by
\[
\hat{H}^1(t) = \int d^3x V(x, t) |x\rangle \langle x|,
\]
where the potential energy is
\[
V(x, t) = C_1 M c^2 A_0 \int d\hat{k} f_\hat{k}(t - x \cdot \hat{k}/c) + C_2 M c^2 A_0^2 \left[ \int d\hat{k} f_\hat{k}(t - x \cdot \hat{k}/c) \right]^2.
\]

5.1. First-order terms of the Dyson expansion

We evaluate the two first-order terms in the Dyson expansion. For a more compact notation, we do not show the argument of the directional components \( \hat{f}_k \). The contribution of the linear part of the potential \( C_1 M c^2 A \) vanishes trivially since \( \langle A \rangle = 0 \). The quadratic part gives
\[
\langle \hat{U}_1(T) \rho_0 \rangle = -\frac{i C_2 M c^2 A_0^2}{\hbar} \int_0^T dt \int d^3x |x\rangle \langle x| \left[ \int d\hat{k} f_\hat{k}(t - x \cdot \hat{k}/c) \right]^2 \rho_0.
\]

Using \( A_\hat{k}(t) = \sqrt{C_0} f_\hat{k}(t) \) and \( \langle A^2 \rangle = 4\pi C_0 \equiv 4\pi A_0^2 \) it is seen that the average yields \( 4\pi \). Since \( \int d^3x |x\rangle \langle x| = \hat{I} \) and integrating over \( T \) we find
\[
\langle \hat{U}_1(T) \rho_0 \rangle = -\frac{4\pi C_2 M c^2 A_0^2 T}{\hbar} \rho_0.
\] (17)

The calculation of the other first-order term proceeds in the same way. Since \( \hat{U}_1^\dagger(T) = -\hat{U}_1(T) \), it yields the same result as in (17) but with the opposite sign (more in general, all the odd terms in the Dyson expansion have an \( i \) factor and also yield a vanishing contribution). We thus see that at first order in the Dyson expansion there is no net dephasing and \( \langle \hat{U}_1(T) \rho_0 + \rho_0 \hat{U}_1^\dagger(T) \rangle = 0 \).

5.2. Second-order terms of the Dyson expansion

The second-order calculation is more complicated. A fundamental point is that the linear part of the potential does again give a vanishing contribution. Dephasing will be shown to come as a purely nonlinear effect due to the nonlinear potential term \( \sim A^2 \).
5.2.1. (Non-)contribution of the linear part of the potential. To have an idea of how things work we consider e.g. the average of the term $\hat{U}_2 \rho_0$. This has the following structure:

$$\langle \hat{U}_2 \rho_0 \rangle \sim \int \! d\tau \int \! d\tau' \int \! d^3 y \langle y | \hat{V} \int \! d^3 y' \langle y' | \langle V(y, \tau) V(y', \tau') \rangle \rho_0. $$

The interesting part is the average $\langle V(y, \tau) V(y', \tau') \rangle$. This is given by $\langle A(y, \tau) A(y', \tau') \rangle + \langle A(y, \tau) A^2(y', \tau') \rangle + \langle A^2(y, \tau) A^2(y', \tau') \rangle$.

The second term vanishes in virtue of property (15). This is seen using the directional decomposition (12) and writing

$$\langle A(y, \tau) A^2(y', \tau') \rangle = A_0^2 \int \! d\mathbf{k}_1 \int \! d\mathbf{k}_2 \int \! d\mathbf{k}_3 \langle f_{\mathbf{k}_1} f_{\mathbf{k}_2} f_{\mathbf{k}_3} \rangle \rho_0 = 0.$$ 

The first term derives from the linear part of the potential. It results in the contribution:

$$\langle A(y, \tau) A(y', \tau') \rangle \Rightarrow \int_0^T \! d\tau \int_0^T \! d\tau' \int \! d^3 y \langle y | \hat{V} \int \! d^3 y' \langle y' | \langle \delta(\mathbf{k}, \mathbf{k}') R(t - t' + y' \cdot \mathbf{k} - y \cdot \mathbf{k}) \rangle \rho_0.$$ 

For convenience of notation we set $c = 1$ in the arguments of the directional functions $f_{\mathbf{k}}$. Using equation (14) the average yields the two-point correlation function according to $\delta(\mathbf{k}, \mathbf{k}') R(t - t' + y' \cdot \mathbf{k} - y \cdot \mathbf{k})$. Integrating with respect to $\mathbf{k}$ yields

$$\langle A(y, \tau) A(y', \tau') \rangle \Rightarrow \int_0^T \! d\tau \int_0^T \! d\tau' \int \! d^3 y \langle y | \hat{V} \int \! d^3 y' \langle y' | \int \! d\mathbf{k} R(t - t' + \mathbf{k} \cdot (y' - y)) \rho_0.$$ 

The corresponding matrix element is found by inserting $|\mathbf{x}\rangle$ and $|\mathbf{x}'\rangle$ respectively on the left and on the right. Using $\langle \mathbf{x} | \hat{V} | \mathbf{x}' \rangle = \delta(\mathbf{x} - \mathbf{y})$ and exploiting the properties of the delta function we find

$$\langle A(y, \tau) A(y', \tau') \rangle \Rightarrow K \times \int_0^T \! d\tau \int_0^T \! d\tau' \int \! d\mathbf{k} R(t - t').$$

where $K$ is a constant given by

$$K = -\frac{C_i^2 A_0^2 M^2 c^3 \rho_{\text{av}}(0)}{\hbar^2},$$

and where $\rho_{\text{av}}(0) := \langle \mathbf{x} | \rho_0 | \mathbf{x}' \rangle$. The similar terms coming from $\langle \rho_0 \hat{U} \rangle$ will contribute in the same way as in (18), thus yielding an extra factor 2. Finally, through a similar calculation it is found that the terms $\sim \langle A(y, \tau) A(y', \tau') \rangle$ coming from $\langle \hat{U}_1 \rho_0 \hat{U}_1 \rangle$ contribute according to

$$\langle A(y, \tau) A(y', \tau') \rangle \Rightarrow -K \times \int_0^T \! d\tau \int_0^T \! d\tau' \int \! d\mathbf{k} R[t - t' + \mathbf{k} \cdot (\mathbf{x}' - \mathbf{x})].$$

Bringing all together, the overall contribution deriving from the linear part $C_i M c^3 A$ of the effective potential is found to be proportional to the expression:

$$I := \int_0^T \! d\tau \left[ 2 \int_0^T \! d\tau' R(t - t') - \int_0^T \! d\tau' R(t - t' + \mathbf{k} \cdot \Delta \mathbf{x}) \right],$$

where $\Delta \mathbf{x} := \mathbf{x}' - \mathbf{x}$. In appendix C we prove that this vanishes provided $R(t)$ is an even function and the drift time $T$ is much larger than the time needed by the fluctuations to propagate through the distance $|\Delta \mathbf{x}|$, i.e. if $T \gg \mathbf{k} \cdot \Delta \mathbf{x}$, where $c = 1$. This condition is certainly satisfied in a typical interferometry experiment where the drift time $T$ can be of the
order of $\sim 1$ ms and $cT$ is indeed much larger than the typical space separations $|\Delta x|$ relevant to quantify the loss of contrast in the measured interference pattern.

Thus we have here the important result that the linear part of the potential does not induce in general any dephasing up to second order in Dyson expansion. In fact we show in the following section that dephasing results purely as an effect of the nonlinear potential term $C_2 M c^2 A^2$.

5.2.2. Contribution of the nonlinear part of the potential. This calculation requires estimating averages of the kind $\langle A^2(y, t) A^2(y', t') \rangle$, which will bring in the second-order correlation function $R''$ defined in (16). This is straightforward but algebraically lengthy. Proceeding in a similar way as done above, exploiting the statistical properties (13)–(16) and the already mentioned result $I = 0$ in relation to (19), then the general result for the density matrix evolution and valid up to second order in the Dyson expansion follows as

$$
\rho_{xx}(T) = \rho_{xx}(0) - \frac{32 C_2^2 \pi^2 M^2 c^4 A_0^2 \rho_{xx}(0)}{\hbar^2} \times \left[ \int_0^T dt \int_0^T dt' R^2(t - t') - \frac{1}{16 \pi^2} \int dk \int d\mathbf{k} \int_0^T dt \int_0^T dt' \right.
$$

$$
\times \left. R(t - t' - \mathbf{k} \cdot \Delta x/c) \times R(t - t' - \mathbf{k} \cdot \Delta x/c) \right].
$$

(20)

Remarkably, the second-order correlation function does not play any role: the first-order correlation function $R(\tau)$ and thus the power spectral density $S(\omega)$ completely determine the system evolution up to second order. Equation (20) implies that the diagonal elements of the density matrix are left unchanged by time evolution. This is seen by setting $\Delta x = 0$ which yields immediately $\rho_{xx}(T) = \rho_{xx}(0)$ for every $T$.

6. General density matrix evolution for large drift times

To verify that we have dephasing with an exponential decay of the off-diagonal elements we need further simplify the result (20) by analyzing its behavior for appropriately large evolution times. To this end we start from the following identity:

$$
\int_0^T dt \int_0^T dt' g(t - t') = \frac{1}{2\pi} \int_0^T dt \int_0^T dt' \int_{-\infty}^\infty d\omega \bar{g}^{(2)}(\omega) e^{i\omega(t - t')} = \frac{1}{2\pi} \int_{-\infty}^\infty d\omega \bar{g}^{(2)}(\omega) \left[ \frac{\sin(\omega T/2)}{\omega/2} \right]^2.
$$

where $\bar{g}(\omega)$ denotes the Fourier transform of the function $g(t)$. Denoting $[0, \Delta \omega]$ as a frequency interval where $\bar{g}(\omega)$ is slow varying, it is straightforward to show that the above identity reduces to

$$
\int_0^T dt \int_0^T dt' g(t - t') \approx \bar{g}(0) T,
$$

(21)

for $T \gtrsim (\Delta \omega)^{-1}$. Note that for this to happen $g(t)$ does not even need being an even function. This condition translates what we mean by appropriately long evolution time. In section 8.2 we will show that it is equivalent to $T \gtrsim \tau_\star$, where $\tau_\star$ is the fluctuations correlation time. This is defined below.

Equation (21) can now be used to evaluate the time integrals appearing in (20). This is done by identifying in one case $g(t) := R^2(t)$ and in the other $g_{t\tau}(t) := R(t + \tau) R(t + \tau')$,
where \( \tau \) and \( \tau' \) stand respectively for \(-\mathbf{k} \cdot \Delta \mathbf{x}/c\) and \(-\mathbf{\hat{k}} \cdot \Delta \mathbf{x}/c\), and where the normalized correlation function can be expressed, using the generalized WK theorem (10), as

\[
R(\tau) \equiv \frac{C(\tau)}{C_0} = \frac{1}{C_0(2\pi c)^2} \int_0^{\infty} d\omega \omega^2 S(\omega) \cos(\omega \tau).
\]

Note that the integration frequency has a cutoff at \( \omega_c = \omega_P / \lambda \), where the Planck frequency is \( \omega_P := 2\pi / T_0 = 1.166 \times 10^{42} \text{ s}^{-1} \). This is consistent with the fact that below the scale \( \ell = \lambda L_P \) the approximation of randomly fluctuating fields breaks down. In alternative this may simply correspond to the fact that the probing particle is insensitive to the short wavelengths as a result of its own finite resolution scale \( L_R \). Application of (21) to \( g(t) := R^2(t) \) yields the result

\[
\int_0^T dt \int_0^T dt' R^2(t - t') = \tau_c T,
\]

where the correlation time is defined as

\[
\tau_c := \frac{\hat{\mathcal{F}}[R^2(t)](0)}{\int_0^{\infty} d\omega \omega^4 S^2(\omega)}.
\]

\( \hat{\mathcal{F}} \) denoting Fourier transform. On the other hand, application of (21) to \( g_{\tau\tau}(t) := R(t + \tau)R(t + \tau') \) gives the result

\[
\int_0^T dt \int_0^T dt' R(t - t' + \tau)R(t - t' + \tau') = \tau_c \Gamma[\omega_c(\tau - \tau')] T,
\]

where the characteristic function \( \Gamma \) is defined as

\[
\Gamma(\omega_c t) := \frac{\int_0^{\infty} d\omega \omega^4 S^2(\omega) \cos(\omega t)}{\int_0^{\infty} d\omega \omega^4 S^2(\omega)}.
\]

This is dimensionless and satisfies the following general properties:

- \( \Gamma(\omega_c t) = \Gamma(-\omega_c t) \),
- \( \Gamma(0) = 1 \),
- \( \Gamma(\omega_c t) < 1 \), for \( t \neq 0 \),
- \( \Gamma(\omega_c t) \to 0 \), for \( t \to \infty \).

Note that both the correlation time \( \tau_c \) and the characteristic function \( \Gamma \) depend on the fluctuations power spectral density only.

The results (22) and (24) can now be used in equation (20) to yield the next result

\[
\rho_{xx}(T) = \rho_{xx}(0) \left[ 1 - \frac{32C_2^2\pi^2 M^2 c^4 A_0^4 \tau_c T}{\hbar^2} \times F(\Delta \mathbf{x}) \right],
\]

where

\[
F(\Delta \mathbf{x}) := 1 - \frac{1}{16\pi^2} \int d\mathbf{k} \int d\mathbf{\hat{k}} \Gamma[\omega_c(\mathbf{\hat{k}} - \mathbf{k}) \cdot \Delta \mathbf{x}/c].
\]

This equation is important and represents one of the main results of this paper. It implies that dephasing due to conformal fluctuations does indeed occur in general and independently of the precise power spectrum characterizing the fluctuations. Without the need to evaluate the angular integrals, this follows from the properties of the characteristic function \( \Gamma \). The fact that \( \Gamma(\omega_c t) < 1 \) implies \( 0 < F(\Delta \mathbf{x}) \leq 1 \) with (i) \( F(\Delta \mathbf{x} = 0) = 0 \) and (ii) \( F(\Delta \mathbf{x} \to \infty) = 1 \) as special limiting cases. As a consequence the diagonal elements are unaffected while the off-diagonal elements decay exponentially according to

\[
\rho_{xx} := \frac{\rho_{xx}(T) - \rho_{xx}(0)}{T} = -\left[ \frac{32C_2^2\pi^2 M^2 c^4 A_0^4 \tau_c}{\hbar^2} \times F(\Delta \mathbf{x}) \right] \rho_{xx}(0).
\]
providing of course that $T$ is small enough so that the change in the density matrix is small. Finally, if $\delta\rho := \rho_{xx}(T) - \rho_{xx}(0)$, we can define the *dephasing* as $|\delta\rho/\rho_0|$. Thanks to the property $F(\Delta x \to \infty) = 1$, this converges for large spacial separations to the constant maximum value

$$\left|\frac{\delta\rho}{\rho_0}\right| = \frac{32C^2\pi^2 M^2 c^4 A_0^4 \tau_s T}{\hbar^2}. \quad (28)$$

This result based on the present three-dimensional analysis of the conformal fluctuations can be compared to the analogue one-dimensional result that PP found in [1]. Using a Gaussian correlation function from the outset they found

$$\left|\frac{\delta\rho}{\rho_0}\right|_{1D} = \frac{\sqrt{\pi} M^2 c^4 A_0^4 \tau_g T}{\sqrt{2}\hbar^2},$$

where $\tau_g$ stands for some characteristic correlation time of the fluctuations. Identifying approximately $\tau_g \approx \tau_s$, we have $(32C^2\pi^2)/(\sqrt{\pi/2}) \approx 250$, assuming $C^2 \sim 1$. Thus the present three-dimensional analysis is seen to predict a dephasing 2 orders of magnitude larger than in the idealized one-dimensional case.

### 6.1. A remark on the validity of the Dyson expansion

We have found that the change in the density matrix is given by

$$\left|\frac{\delta\rho}{\rho_0}\right| \sim \left(\frac{Mc^2}{\hbar}\right)^2 \tau_s T \times A_0^4.$$ 

In order for the expansion scheme to be effective, the propagation time $T$ must be short enough to guarantee that $|\delta\rho/\rho_0|$ is small. How short depends of course on the statistical properties of the fluctuations, encoded in $\tau_s$, and on the probing particle mass $M$. A fullerene C$_{70}$ molecule with $(M_{C_{70}} \approx 10^{-24} \text{ kg})$ gives $M c^2/\hbar \approx 10^{27} \text{ s}^{-1}$. Therefore the approach is consistent only if correlation time $\tau_s$, the flight time $T$ and field squared amplitude $A_0^2$ are appropriately small. We will come back on this issue in section 8.2, where it is shown that, in the case of vacuum fluctuations (introduced in the next session), it is $\tau_s \approx \lambda T_P$ and $A_0^2 \sim 1/\lambda^2$. For a flight time $T \approx 1 \text{ ms}$, typical of interferometry experiments, this results in $|\delta\rho/\rho_0| \approx 10^6/\lambda^3$. For any reasonable value of $\lambda \gtrsim 10^3$ the density matrix change is indeed small and the Dyson expansion scheme well-posed up to second order. In appendix D we estimate the fourth-order term in the expansion, which will also yield a term proportional to $A_0^4$. It will be shown that its contribution in fact vanishes under quite general circumstances. This puts the result (26) on an even stronger basis.

### 7. Explicit dephasing in the case of vacuum fluctuations

The result (26) is quite general. The only ingredients entering the analysis so far have been (i) a spacetime metric $g_{ab} = (1 + A)^2 \eta_{ab}$ with $|A| = O(\epsilon \ll 1)$, (ii) a randomly fluctuating conformal field $A$ satisfying the wave equation $\partial^2 A = 0$ and (iii) isotropic fluctuations characterized by an arbitrary power spectral density $S(\omega)$. The dephasing then occurs as a result of the nonlinearity in the effective potential $V = Mc^2[C_1 A + C_2 A^2]$.

A particularly interesting case, potentially related to the possibility of detecting experimental signs of quantum gravity, is that in which the fluctuations in $A$ are the manifestation, at the appropriate semiclassical scale $\ell = \lambda L_P$, of underlying vacuum quantum fluctuations close to the Planck scale. Strictly speaking the presence of the probing particle perturbs the genuine quantum vacuum state. For this reason it would be appropriate to
talk of effective vacuum, i.e. up to the presence of the test particle. By its nature, the present semiclassical analysis cannot take in account the backreaction of the system on the environment. Therefore we simply assume that the modifications on the vacuum state can be neglected as long as the probing particle mass is not too large and the evolution time short. We thus model the effective vacuum properties of the conformal field \( A \) at the semiclassical scale on the basis of the properties that real vacuum is expected to possess at the same scale. It is a fact that vacuum looks the same to all inertial observers far from gravitational fields. In particular, its energy density content should be Lorentz invariant. This can obtained through an appropriate choice of the power spectrum \( S(\omega) \).

### 7.1. Isotropic power spectrum for ‘vacuum’ conformal fluctuations

According to the above discussion we expect the average properties of \( A \) above the scale \( \ell \) to be Lorentz invariant. In particular, the interesting quantities derived in appendix A.5

\[
\langle A^2 \rangle = \frac{1}{(2\pi)^3} \int d^3k S(k),
\]

\[
\langle |\nabla A|^2 \rangle = \frac{1}{(2\pi)^3} \int d^3k k^2 S(k),
\]

\[
\langle (\partial_t A)^2 \rangle = \frac{1}{(2\pi)^3} \int d^3k \omega^2(k) S(k),
\]

should be invariant. As discussed in appendix A, for a stationary, isotropic signal, the averages \( \langle \cdot \rangle \) can in fact be carried out through suitable spacetime integrations over an appropriate averaging scale \( L \gg \ell \). In alternative they can be expressed as in the above integrals depending on the power spectrum and adopting a high-energy cutoff set by \( k_\lambda := 2\pi/(\lambda L_P) \).

The problem of the Lorentz invariance of the above quantities has been discussed in details by Boyer [14] within his random electrodynamics framework. He showed that the choice \( S \sim 1/\omega \) is unique in guaranteeing an energy spectrum \( \varrho(\omega) \propto \omega^3 \), also shown to be the only possible choice for a Lorentz invariant energy spectrum of a massless field. In the present case we take

\[
S(k) := \frac{\hbar G}{2c^2 \omega(k)}.
\]

The combination of the constants \( \hbar, G \) and \( c \) gives the correct dimensions for a power spectrum (i.e. \( L^3 \)), while the factor 1/2 guarantees that the resulting zero-point energy density is equivalent to that resulting from the superposition of zero-point contributions \( \hbar \omega/2 \). In particular this element makes the connection between the present stochastic approach and quantum theory. With this choice for the power spectrum the integrals (30) and (31) are indeed Lorentz invariant as they are related to the energy density of \( A \). The same holds for the integral in (29) as \( d^3k/\omega(k) \) is a Lorentz invariant measure [35]. A final important point, which should not be overlooked, is that Lorentz invariance is preserved provided the cutoff \( k_\lambda \) is given by the same number for all inertial observers, as also discussed in details by Boyer. In other words, this means that the critical length that sets the border line between the random field approach and the full quantum gravity regime is supposed to be the same for any inertial observer. It represents some kind of structural property of spacetime and not an observer-dependent property. Accordingly it must not be transformed under Lorentz transformations. It is important to note that this requirement will naturally be satisfied later when we employ an effective cutoff set by the particle Compton length.
Using (32) the normalized correlation function can be found explicitly from the generalized Wiener–Khintchine theorem to be

\[ R(\tau) := \frac{C(\tau)}{C_0} = 2 \left[ \frac{\sin \omega_c \tau}{\omega_c \tau} + \frac{\cos \omega_c \tau - 1}{\omega^2_c \tau^2} \right]. \tag{33} \]

The peak of the autocorrelation function is linked to the fluctuations squared amplitude and gives explicitly

\[ C_0 \equiv A_0^2 = \frac{1}{8\pi \lambda^2}, \tag{34} \]

implying \( \langle A^2 \rangle = 1/(2\lambda)^2 \). The correlation time and characteristic function follow from equations (23) and (25) as

\[ \tau_* = \frac{2}{3} \lambda T_p \tag{35} \]

and

\[ \Gamma(\sigma) = \frac{3}{\sigma^3} \sin(\sigma) + \frac{6\cos(\sigma)}{\sigma^2} - \frac{6\sin(\sigma)}{\sigma^3}, \tag{36} \]

where \( \sigma = \omega_c \tau \) is a dimensionless variable. The plot of the squared normalized correlation function \( R^2(t - t') \) is shown in figure 1: \( t - t' = \tau_* \) corresponds to the first secondary peak in the curve, where the correlation in the fluctuations is reduced to \( \sim 70\% \). This fully motivates the choice of \( \tau_* \) to represent the correlation time.

The explicit form of the characteristic function can be used in (27) to evaluate the remaining angular integrals and find the detailed expression for the density matrix evolution valid for all \( (x, x') \). Isotropy implies that the result must depend on \( |x - x'| \) only. The integration is straightforward and yields the result

\[ F(\sigma) := 1 - \frac{3}{2\sigma^2} \left( 1 - \frac{\sin \sigma \cos \sigma}{\sigma} \right). \tag{37} \]

Substituting the results (34)–(37) into (26) yields the explicit result for the dephasing, valid for vacuum fluctuations described by \( S \sim 1/\omega \).
Figure 2. Plot of the function $F(\sigma)$ in the range $\sigma = 0..10$, where $\sigma = 2\pi |x - x'|/(\lambda L_P)$. The curve tends very rapidly to the limiting value of 1 and for spatial separations $|x - x'|$ which are slightly larger than $\ell = \lambda L_P$ the dephasing converges rapidly to its maximum value.

\[
\left| \frac{\delta \rho_{xx}}{\rho_0} \right| = \frac{1}{3\lambda^3} \left( \frac{M}{M_P} \right)^2 \left( \frac{T}{T_P} \right) \times F \left( \frac{2\pi |x - x'|}{\ell} \right). \tag{38}
\]

where we considered $C_2 \sim 1$ and where

\[ M_P := \frac{\hbar}{c^2 T_P} = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8} \text{ kg} = 1.310 \times 10^{19} \text{ amu} \]

is the Planck mass. The function $F$ is plotted in figure 2. It enjoys the properties $F(0) = 0$ and $F(\sigma) \to 1$ for $\sigma \gg 1$, so that for $|x - x'| \gtrsim 10\ell$ the decoherence rate converges rapidly to its maximum value.

8. Discussion

8.1. Probing particle resolution scale and effective dephasing

Equation (38) is an important result. It gives the dephasing in the density matrix of a quantum particle propagating in space under the only action of a randomly fluctuating potential due to spacetime vacuum conformal fluctuations. The fact that it predicts an exponential decay of the off-diagonal elements of the system density matrix (which is the distinctive feature of quantum decoherence) is interesting as a further confirmation that certain effects involving quantum fluctuations can be mimicked by means of a semi-classical treatment in the spirit of Boyer [14, 15].

A significant feature of our dephasing formula is the quadratic dependence on the probing particle mass $M$, which comes as a consequence of the underlying nonlinearity. The coefficient $1/(3\lambda^3)$ sets the overall strength of the effect. It is proportional to $A_0^4$ and to the fluctuations correlation time $\tau_c$: the more intense the fluctuations, the larger the dephasing and the longer the various directional components stay correlated, the higher their ability to induce dephasing. We have found $\tau_c \approx \lambda T_P$, in such a way that the correlation time directly depends on the spacetime intrinsic cutoff parameter $\lambda$. According to this picture all the wavelength down to the cutoff $\ell = \lambda L_P$ should be able to affect the probing particle. However an atom or molecule
Figure 3. Fourier transform of $R^2(\sigma)$ as a function of the frequency $\omega$ in units of the cutoff frequency $\omega_c$.

is likely to possess its own resolution scale $L_R$. Thus, whenever $\tau_\sigma < L_R$, the ability of the fluctuations to affect the particle would be reduced, as they would effectively average out. To characterize this feature of the problem we write, in analogy to $\ell = \lambda L_P$,

$$L_R = \lambda_R L_P,$$

and use $\lambda_R$ as a new, particle-dependent, cutoff parameter. In general it is $\lambda_R > \lambda$. The new effective correlation time is now given by $\tau_\sigma \approx \lambda_R T_P$. The distance traveled by the fluctuations during a correlation time is $L_\sigma = c\tau_\sigma \approx 2L_R/3$. Thus the effective correlation distance $L_\sigma$ basically corresponds to the particle resolution scale: short wavelengths that do not keep their correlation up to the scale $L_R$ average out and cannot affect the probing particle. The new, effective dephasing results by substituting $\lambda$ with $\lambda_R$ into (38):

$$\frac{\delta \rho_{xx}}{\rho_0} = \frac{1}{3} \left( \frac{L_P}{L_R} \right)^3 \left( \frac{M}{M_P} \right)^2 \left( \frac{T}{T_P} \right) \times F \left( \frac{2\pi |x-x'|}{L_R} \right).$$

8.2. Validity of the long drift time regime

We recall that this result holds for ‘long’ drift times $T$, i.e. when $T \gtrsim \Delta \omega^{-1}$, where $\Delta \omega$ is an appropriate frequency range over which the Fourier transforms of $R^2(t)$ and $R(t + \tau)R(t + \tau')$ vary little. We are now in the position to make this precise and define clearly the limits of applicability of the theory. To this end we consider the Fourier transform of $R^2(\omega, \tau)$.
$$\delta[R^2(\omega, \tau)](\omega) \approx \frac{1}{\omega_0} \delta[R^2(\sigma)](\omega/\omega_0),$$

with $R(\sigma)$ given in (33). Its plot is displayed in figure 3. The spectrum falls to 0 for $\omega \gg 2\omega_c$. The value of the peak at $\omega = 0$ is precisely $4\pi/3$, verifying that $\tau_* \approx \delta[R^2(\omega_0, \tau)](0) = 4\pi/(3\omega_0)$. The smaller box shows a zoom in the region $\sigma \in [0, 1/100]$: the curve is slow varying in this range since $\delta[R^2(\omega_0, \tau)](0) = 4\pi/3 \approx 4.19$ and $\delta[R^2(\omega_0, \tau)](1/100) \approx 4.13$. Similarly it is possible to check that the Fourier transform of $R(\sigma + \eta)R(\sigma + \eta')$, where the adimensional parameters $\eta$ and $\eta'$ depend on space direction and locations as $\eta := -\omega_0 \hat{k} \cdot \Delta x/s$ and $\eta' := -\omega_0 \mathbf{K} \cdot \Delta x/c$, enjoys a similar property: for every choice of $\eta$ and $\eta'$, the resulting Fourier transform is slow varying in the range $\sigma \in [0, 1/100]$. Following this discussion we choose $\Delta \omega \approx [0, \omega_0/100]$. We can now quantify the concept of ‘long drift time’ by $T \gtrsim 100/\omega_0$. From $\omega_0 = 2\pi/(\lambda_R T_p) = 4\pi/(3\tau_*)$ this yields the condition $T \gtrsim 25 \tau_*$. 

8.3. Some numerical estimates and outlook

In summary, we have studied the dephasing of a non-relativistic quantum particle induced by a conformally modulated spacetime $g_{ab} = (1 + A)^2 \eta_{ab}$, where $A$ is a random scalar field satisfying the wave equation. The important case of vacuum fluctuations can be characterized by a suitable power spectrum $S \sim 1/\omega$. If $L_R = \lambda_R L_p$ is the probing particle resolution scale, the dephasing for $|\mathbf{x} - \mathbf{x}'| \gg L_R$ converges rapidly to

$$\frac{\Delta \rho}{\rho_0} \approx \frac{1}{3} \left( \frac{L_p}{L_R} \right)^2 \left( \frac{M}{M_p} \right)^2 \left( \frac{T_p}{T} \right). \quad (39)$$

The effective correlation time of the fluctuations is given by $\tau_* \approx \lambda_R T_p$. The above result holds for ‘long’ drift times satisfying $T \gtrsim (10 \sim 10^2) \lambda_R T_p$.

To conclude we want to give some numerical estimates of the dephasing that could be expected in a typical matter wave interferometry experiment, e.g. like those described in [9], where fullerene molecules have been employed with drift times of the order of $T =: T_{ex} \approx 10^{-3}$ s. Consider e.g. a C$_{70}$ molecule with $M = M_{C_{70}} \approx 1.24 \times 10^{-24}$ kg. In comparison to the Planck units we have

$$T_{ex} \approx 10^{10} T_p, \quad M_{C_{70}} \approx 10^{-17} M_p.$$ 

Thus, it is clear that the most critical factor controlling the strength of the effect is set by the probing particle mass, together with the effective resolution cutoff scale. Using these data in (39) we can estimate

$$\frac{\Delta \rho}{\rho_0} \approx \frac{10^6}{\lambda_R} \Rightarrow \lambda_R \approx \left( \frac{10^6}{|\Delta \rho/\rho_0|} \right)^{1/3}.$$ 

This could be used to estimate $\lambda_R$ if we were able to identify within an experiment a residual amount of dephasing that cannot be explained by other standard mechanisms (e.g. environmental decoherence, internal degrees of freedom). Figure 4 plots $\lambda_R$ against $|\Delta \rho/\rho_0|$: a dephasing due to conformal fluctuations in the range 1–0.1% would imply a resolution parameter in the range $\lambda_R \approx 10^3$ to $10^4$. This would represent a lower bound on $\lambda_R$, as interferometry experiments will get more and more precise in measuring and modeling environmental decoherence. Estimating the present typical uncertainty of typical interferometry experiments as $|\Delta \rho/\rho_0| \approx 0.01\%$ we get

$$\lambda_R \gtrsim 10^3 \quad \text{for C}_{70}. \quad \text{P M Bonifacio et al}$$
We remark that such order of magnitude estimates are consistent with a small change in the density matrix and the second-order Dyson expansion approach.

A value for $\lambda_R$ as small as $10^3$ would probably approach the intrinsic spacetime structural limit set by $\lambda$, i.e. $\ell = \lambda L_P$. It is interesting to ask what amount of dephasing our model predicts, independently of experimental data. To this end we need to prescribe theoretically the particle resolution scale $L_R$. Though no obvious choice exists, an interesting possibility would be to set it equal to the particle Compton length, i.e.

$$L_R = \frac{h}{Mc}.$$  

This choice is obviously Lorentz invariant and also motivated by the fact that the Compton length represents a fundamental uncertainty in the position of a non-relativistic quantum particle. Indeed, by Heisenberg uncertainty principle, $\Delta x \approx h/Mc$ would imply $\Delta p \gtrsim Mc$, implying an uncertainty in the energy of the same order of the rest mass $Mc^2$. In such a situation QFT would become relevant. Alternatively it can also be argued that wavelengths shorter than $h/Mc$ would have enough energy to create a particle of mass $M$ from the vacuum.

With this choice, equation (39) becomes

$$\left| \frac{\delta \rho}{\rho_0} \right| = \frac{1}{24\pi^3} \left( \frac{M}{M_P} \right)^5 \left( \frac{T}{T_P} \right).$$  

(40)

This can be used to estimate the amount of dephasing induced by vacuum conformal fluctuations.

In the case of $C_{70}$ the Compton wavelength is $\approx 10^{-18}$ m $\approx 10^{18}L_P$, corresponding to $\lambda_R \approx 10^{18}$. For a propagation time of $\approx 1$ ms this gives a dephasing

$$\left| \frac{\delta \rho}{\rho_0} \right| (M_{C_{70}}, 1 \text{ ms}) \approx 10^{-44},$$

which would be negligible and far beyond the possibility of experimental detection. Thus, in order to achieve an amount of dephasing within the current experimental accuracy, much heavier quantum particles are needed. In atomic mass units $C_{70}$ has a mass $M_{C_{70}} \approx 10^3$ amu.
Equation (40) applied to a particle with mass $M \approx 10^{11}$ amu and with a drift time $T \approx 100$ ms gives the estimate

$$\left| \frac{\delta p}{\rho_0} \right| (10^{11} \text{ amu}, 100 \text{ ms}) \approx 10^{-2}.$$ 

A drift time of $\sim 100$ ms could possibly be achieved in a space based experiment. On the other hand, the need of a quantum particle as heavy as $10^{11}$ amu poses an extraordinary challenge. A possibility would be to employ quantum entangled states. This is already being considered in the literature, e.g. in [28], where entangled atomic states are studied and suggested as a possible improved probe for future detection of spacetime induced dephasing.

In relation to the issue of a possible future experimental detection of dephasing due to vacuum effects, it is important to note that even the vacuum fluctuations of the EM field can influence the fringe visibility of a neutral particle if this has a permanent electric or magnetic dipole moment [38]. The amount of decoherence due to this effect, e.g. in a typical interference experiment, is expected to depend quadratically on the dipole moment of the particle. Fullerene molecules such as C$_{70}$ (or even the spherically shaped C$_{60}$) have symmetric charge distributions and possess no permanent dipoles [39]. In this case no extra effect would be expected. Drugged versions of fullerene where, e.g., Na or Li atoms are combined with C$_{60}$ to form heavier molecules such as Na$_{10}$C$_{60}$ or Li$_{10}$C$_{60}$ can possess a permanent electric dipole moment between $\sim 10$ and $\sim 20$ D, depending on the number of drugging atoms [40]. While in this case some decoherence due to EM vacuum effects would be theoretically expected, the mass of such molecules is still far too low for any gravitational effect due to conformal fluctuations to be detectable. Only in the case of more complex and heavier quantum systems, e.g. gold clusters or entangled states, with a permanent dipole the two effects would theoretically both contribute to the overall dephasing. However, the dephasing due to a permanent dipole moment is in general predicted to be typically of the order of the square of the typical dipole length in units of the total length of the trajectory [38] In this sense, as the mass of the quantum probe is increased, we expect the gravitational induced dephasing to represent the dominant effect.

The last important point that needs verification is that the condition (4) given earlier at the beginning of this paper is indeed verified: that was required in order for the change in momentum due to the fluctuations to be smaller than the laboratory particle momentum $\vec{p} = M\vec{v}$. It reads $\varepsilon^2 \sim \langle A^2 \rangle \ll (\vec{v}/c)^2$. The field effective mean quadratic amplitude interacting with the particle is given by $\langle A^2 \rangle \sim \lambda_R^{-2}$. Thus we have the condition

$$\frac{1}{\lambda_R} \equiv \frac{L_P}{L_R} \ll \frac{\vec{v}}{c}.$$ 

By using the expression for the Planck length and with $L_R$ given by the particle Compton length, this yields a condition on the particle mass $M$:

$$\frac{M}{M_P} \ll \frac{2\pi \vec{v}}{c}.$$ 

For typical laboratory velocities $\vec{v}/c \approx 10^{-6}$ and, since $M_P \approx 10^{19}$ amu, this condition is met for particle masses up to $M \approx 10^{13}$ amu, including the case of C$_{70}$ molecules or the heavier entangled quantum states discussed above. This limit would be reduced for slower particles.

We conclude by remarking that the theory described in this paper is quite general, in the sense that as a starting input it only needs a conformally modulated metric and a scalar field satisfying the wave equation. Of course, it is important to identify in concrete which theories of gravity can actually yield such a scenario. This important problem will be the object of a future report [27], where a general framework for the study of fluctuating fields close to...
the Planck scale is introduced and applied to standard GR as well scalar–tensor theories of gravity.

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Appendix A. Stochastic scalar waves and generalized Wiener–Khintchine (WK) theorem

A.1. General solution to the wave equation

In this appendix we work out some general results holding for scalar stochastic waves. Working in units with $c = 1$, the solution to the wave equation $\left(-\partial^2_t + \nabla^2\right) f(x, t) = 0$ can be written as

$$f(x, t) = \frac{1}{(2\pi)^n} \int d^n k \hat{f}(k, t) e^{ik \cdot x},$$

where

$$\hat{f}(k, t) = \int d^n x f(x, t) e^{-ik \cdot x}$$

and $f \in L^2(\mathbb{R}^3)$. The Fourier coefficients take the general form

$$\hat{f}(k, t) = a(k) e^{-ik \cdot t} + b(k) e^{ik \cdot t}$$

for some complex functions $a(k)$ and $b(k)$ and where $k := |k|$.

The wave can be written as

$$f(x, t) = \frac{1}{(2\pi)^n} \int d\hat{k} \int_0^{\infty} dk k^{n-1} [a(k) e^{i\hat{k} \cdot x / c - t} + b(-k) e^{-i\hat{k} \cdot x / c - t}],$$

where $d\hat{k}$ represent the elementary solid angle in momentum space. Using spherical coordinates in momentum space with $k(\theta, \varphi, k)$ and employing the notation $a_k(k) := a(\theta, \varphi; k)$ and $b_{-k}(k) := b(\pi - \theta, \varphi + \pi; k)$, we define the directional wave along $\hat{k}(\theta, \varphi)$ as

$$f_k(\hat{k} \cdot x / c - t) := \frac{1}{(2\pi)^n} \int_0^{\infty} dk k^{n-1} [a_k(k) e^{i\hat{k} \cdot x / c - t} + b_{-k}(k) e^{-i\hat{k} \cdot x / c - t}],$$

where for clarity we have restored the speed of light. The general solution can thus be written according to the directional decomposition

$$f(x, t) = \int d\hat{k} f_k(\hat{k} \cdot x / c - t).$$

(A.1)

A.2. Stochastic waves

In this section we review some elementary properties of stochastic signals of one time variable $t$.

The fluctuating conformal field at a given space location $x$ represents an example of a stochastic process $f(t)$. Its properties can be defined in a statistical sense. If the possible values of $f$ obey a probability density $p_t(f)$ the statistical average at time $t$ is defined as

$$\langle f(t) \rangle := \int p_t(f) f \, df.$$
The variance is
\[ \langle [f(t)]^2 \rangle := \int p_t(f) f^2 \, df. \]

Denoting the probability density of having the particular outcomes \( f(t) \) and \( f(t') \) by \( p_{t'}(f, f') \), then the autocorrelation or two-points correlation function of \( f(t) \) is defined as
\[ R(t - t') = \langle f(t) f(t') \rangle := \int p_{t'}(f, f') \, f f' \, df \, df'. \]

If the probability of having the value \( f(t') \) is completely independent from the previous outcome \( f(t) \) then \( p_{t't}(f, f') = p_t(f) p_t'(f') \) and the stochastic process described by \( f(t) \) is said to be perfectly uncorrelated. In this case \( \langle f(t) f(t') \rangle = \langle f(t) \rangle \langle f(t') \rangle \).

Higher order autocorrelation functions can also be defined. The second-order correlation function is given by
\[ R''(t - t') = \langle [f(t)]^2 [f(t')]^2 \rangle := \int p_{t't}(f, f') f^2 f'^2 \, df \, df'. \]

In the case of conformal fluctuations, we assume that the stochastic process is stationary, i.e. all average properties do not depend on time, and isotropic. This implies that different directional components have the same statistical properties. Finally, by assuming the stochastic process to be ergodic, and since it satisfies the wave equation, then statistical averages can be replaced by time or space averages taken on any given sample function representing the process.

A.3. Generalized Wiener–Khintchine theorem for a stochastic process satisfying the wave equation

We now derive a generalization of the Wiener–Khintchine theorem linking the power spectrum to the autocorrelation function. We assume that a particular sample function representing the stochastic process can be written as in (A.1), in such a way that the wave equation is satisfied. Some care must be taken in using the Fourier expansions relations of the previous section. Indeed, for a given \( t, f \) is not in general a square integrable function belonging to \( L^2(\mathbb{R}^3) \). To circumvent this problem, given a sample function of the process and for any given \( t \), we define
\[ \tilde{f}^L(k, t) := \int_{D_L} d^n x f(x, t) e^{-ik \cdot x}, \]

where \( D_L \) is a three-dimensional cubic domain of side \( L \). Then, the function
\[ f^L(x, t) := \frac{1}{(2\pi)^n} \int d^n k \, \tilde{f}^L(k, t) e^{ik \cdot x} \]  \hspace{1cm} (A.2)

satisfies the wave equation provided that
\[ \tilde{f}^L(k, t) = a^L(k) e^{-ik \cdot \tau} + b^L(k) e^{ik \cdot \tau}. \]  \hspace{1cm} (A.3)

These expressions will be used later while taking the limit with \( L \to \infty \).

Consider now a complex, ergodic, time-dependent stochastic function \( f(x, t) \) in an \( n \)-dimensional space with time \( t \), satisfying the wave equation \((\partial_{t}^2 - \nabla^2) f(x, t) = 0\). The autocorrelation function of \( f(x, t) \) for any two events \((x_1, t_1)\) and \((x_2, t_2) = (x_1 + \xi, t_1 + \tau)\) is a function of \( \xi \) and \( \tau \) given by
\[ C(\xi, \tau) = \langle f(x_1, t_1) f(x_2, t_2) \rangle. \]
and having the property \( C(-\xi, -\tau) = C(\xi, \tau)^* \). For any fixed choice of \( x_1 \) and \( t_1 \) it also satisfies the wave equation, i.e. \((\partial^2_t - \nabla^2_\xi) C(\xi, \tau) = 0\). Assuming that, for any \( \tau \), \( C(\xi, \tau) \) belongs to \( L^2(\mathbb{R}^3) \) we have

\[
C(\xi, \tau) = \frac{1}{(2\pi)^{3n}} \int d^3k [\alpha(k) e^{-i k \tau} + \beta(k) e^{i k \tau}] e^{i k \cdot \xi}. \tag{A.4}
\]

Note that \( \alpha(k)^* = \alpha(k) \), \( \beta(k)^* = \beta(k) \), i.e. both \( \alpha(k) \) and \( \beta(k) \) are real. For \( \tau = 0 \) (A.4) reduces to the case of a stochastic process of one \( n \)-dimensional space variable:

\[
C(\xi, 0) = \frac{1}{(2\pi)^3} \int d^3k [\alpha(k) + \beta(k)] e^{i k \cdot \xi}.
\]

The standard WK theorem then implies a power spectrum:

\[
S(k) = \alpha(k) + \beta(k). \tag{A.5}
\]

To determine \( \alpha(k) \) and \( \beta(k) \) we consider the stochastic process for some fixed time \( t_0 \). From equations (A.2) and (A.3) we have

\[
f^L(x, t_0) = \frac{1}{(2\pi)^n} \int d^3k f^L(k, t_0) e^{i k \cdot x}, \tag{A.6}
\]

with

\[
f^L(k, t_0) = a^L(k) e^{-i k t_0} + b^L(k) e^{i k t_0}. \tag{A.7}
\]

We are thus dealing with one space variable stochastic process for which the usual WK theorem holds. Exploiting the fact that the stochastic process is stationary we define mean power spectral density as

\[
S(k) := \lim_{L, T \to \infty} \frac{1}{T} \int_0^T dt_0 \frac{1}{L^n} \langle \tilde{f}^L(k, t_0)^* \tilde{f}^L(k, t_0) \rangle.
\]

Using equations (A.6) and (A.7) this reduces to

\[
S(k) = \lim_{L \to \infty} \frac{1}{L^n} (|a^L(k)|^2 + |b^L(k)|^2).
\]

Comparing with (A.5) we have

\[
\alpha(k) = \lim_{L \to \infty} \frac{1}{L^n} (|a^L(k)|^2)
\]

\[
\beta(k) = \lim_{L \to \infty} \frac{1}{L^n} (|b^L(k)|^2).
\]

**A.3.1. Real stochastic process.** If \( f(x, t) \) is real it is easily verified that \( S(k) \) is even, i.e. \( S(k) = S(-k) \) and that \( C(\xi, \tau) \) is real. Moreover the power spectrum is then given simply by

\[
S(k) \equiv \lim_{L \to \infty} \frac{2}{L^n} (|a^L(k)|^2).
\]

In this case we get the generalized Wiener–Khintchine theorem for a real, stationary stochastic scalar field in the form:

\[
C(\xi, \tau) = \frac{1}{(2\pi)^n} \int d^3k S(k) \cos(k \cdot \xi - k \tau), \tag{A.8}
\]

where we have restored the speed of light.
A.4. Correlation properties of wave components in different directions

The results that we have come to establish allow us to show that wave components traveling in different directions are uncorrelated. Evaluating \( \langle f^*(\mathbf{x}, t_1) f(\mathbf{x}, t_2) \rangle \) using equation (A.1) we have

\[
C(\xi, \tau) = \langle f^*(\mathbf{x}, t) f(\mathbf{x} + \xi, t + \tau) \rangle
= \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\mathbf{k}'}{(2\pi)^3} \langle f^*_k(\mathbf{k} \cdot \mathbf{x} - t) f_k(\mathbf{k}' \cdot \mathbf{x} - t + \mathbf{k}' \cdot \frac{\xi}{c} - \tau) \rangle.
\]

(A.9)

Using the WK theorem in its form as given by (A.4), restoring the speed of light, swapping \( \mathbf{k} \) to \(-\mathbf{k}\) in the second integral and splitting the \( d^3k \) integral in its angular and magnitude parts we can write as well

\[
C(\xi, \tau) = \int \frac{d\mathbf{k} \cdot \xi}{c - \tau) C(\mathbf{k} \cdot \xi/c - \tau),
\]

(A.10)

where we defined the correlation function in the direction \( \mathbf{k} \) as

\[
C_k(\mathbf{k} \cdot \xi/c - \tau) := \frac{1}{(2\pi)^3} \int_0^\infty dk k^{n-1} \left[ \alpha_k(k) e^{i\mathbf{k} \cdot \mathbf{x}/c - \tau} + \beta_{-\mathbf{k}}(k) e^{-i\mathbf{k} \cdot \mathbf{x}/c - \tau} \right],
\]

with \( \alpha_k(k) := \alpha(k) \) and \( \beta_{-\mathbf{k}}(k) := \beta(-\mathbf{k}) \). The two equations (A.9) and (A.10) must be equivalent. This implies at once the following equation:

\[
\langle f^*_k(\mathbf{k} \cdot \mathbf{x} - t) f_k(\mathbf{k}' \cdot \mathbf{x}/c - \tau) \rangle = \int \frac{d\mathbf{k} \cdot \xi}{c - \tau) C_k(\mathbf{k} \cdot \xi/c - \tau) or, equivalently, since \( \mathbf{k} \cdot \mathbf{x}/c \) has the dimensions of a time,
\]

(A.11)

We thus see that the fluctuating field can be resolved into components along different directions represented by completely uncorrelated functions of just one time variable.

A.5. Isotropic power spectrum and field averages

In the relevant case of a real and isotropic signal in three-dimensional space the spectral density must also be isotropic, i.e. \( S(\mathbf{k}) := S(k) \). We can then introduce an isotropic correlation function as

\[
C_k(\tau) \to C(\tau) := \frac{1}{(2\pi)^3} \int_0^\infty dk k^2 S(k) \cos(kc\tau),
\]

as deduced from (A.8) and (A.10), and (A.11) reads simply \( \langle f_k(t) f_k(t + \tau) \rangle = \delta(\mathbf{k}, \mathbf{k}') C(\tau) \). Using this with equation (A.11) we have

\[
\langle f^2 \rangle = \frac{1}{(2\pi)^3} \int d^3k S(k).
\]

It is also useful to have two expressions for the average of the squared time and space derivatives of the field, as these are directly related to the field energy density. Using equation (A.2) and the ergodic property it is straightforward to show that

\[
\langle (\partial_t f)^2 \rangle = \frac{1}{(2\pi)^3} \int d^3k k^2 S(k)
\]

and

\[
\langle (\nabla f)^2 \rangle = \frac{1}{(2\pi)^3} \int d^3k k^2 S(k).
\]
Appendix B. Treatment of the term $T_4$

We consider the correction term $T_4 = -i\hbar(\dot{A} - A\dot{A})\psi - i\hbar\tilde{v}(A_x - AA_x)\psi$, derived in section 2. We write $A$, in the isotropic and real case, as $\approx \int d\hat{k} \int_{2\pi/L_R}^{2\pi/L_R} d\hat{k} k^2 a(k) e^{i\hat{k} \cdot \mathbf{x}}$, (B.1)

where the upper cutoff is set by the particle resolution scale $L_R$. The power spectrum is basically proportional to the square of the Fourier component $a(k)$. For $S(k) \sim 1/k$ we have $a(k) \sim k^{-1/2}$, so that the effective coefficient appearing in the expansion is $k^2 a(k) \sim k^{3/2}$. Thus the short wavelengths close to the cutoff give the most important contribution. For this reason we can approximate the field as $A \approx \int d\hat{k} \int_{2\pi/L_R}^{2\pi/L_R} d\hat{k} k^{3/2} \Delta(k - k_A) e^{i\hat{k} \cdot \mathbf{x}}$, (B.2)

where the function $\Delta(k - k_A)$ is peaked around a typical wave number $k_A$. This can in principle be selected in such a way that the average properties of (B.1) are equivalent to those of (B.2). Effectively we get $A \approx \int d\hat{k} k_A^{3/2} e^{i\hat{k} \cdot \mathbf{x}}$, (B.3)

so that the conformal field is approximated as a fast varying and isotropic random signal characterized by a single typical wavelength $\lambda_A \equiv \frac{2\pi}{k_A}$. In relation to the fluctuations ability to affect the particle, this will be close to the particle resolution scale, i.e. we put $\lambda_A \equiv \kappa L_R$, with $\kappa \gtrsim 1$. From (B.3) we now have

$$\dot{A} \approx -ik_A c A = -\frac{2\pi i}{\kappa L_R} c A = -\frac{iM c^2}{\kappa \hbar} A,$$

where we used $L_R = \hbar/Mc$. The space derivatives yield

$$A_x \approx i k_A^{5/2} \int d\hat{k} \hat{k}_x e^{i\hat{k} \cdot \mathbf{x}} = 0.$$

Using these two relations in (6) yields the result (7).

Appendix C. An integral identity

In this appendix we prove that the result

$$I := \int_0^T dt \left[ 2 \int_0^t dt' f(t - t') - \int_0^T dt' f(t - t' - \hat{k} \cdot \Delta \mathbf{x}) \right] = 0,$$

used in section 5.2.1, holds for an arbitrary even function $f$ and in the limit $\hat{k} \cdot \Delta \mathbf{x}/T \to 0$.

Using $c = 1$, we let for simplicity $\Delta := \hat{k} \cdot \Delta \mathbf{x}$. Defining the variable $\tau := t - t'$ we have

$$\int_0^t dt' f(t - t') = \int_0^\tau d\tau f(\tau),$$

while, with $\tau := t - t' - \Delta$,

$$\int_0^T dt' f(t - t' - \Delta) = \int_{t - \Delta - T}^{t - \Delta} d\tau f(\tau) = \int_0^{t - \Delta} d\tau f(\tau) - \int_{t - \Delta}^{t - \Delta - T} d\tau f(\tau).$$
Introducing the primitive of \( f(t) \)

\[ F(t) := \int_0^t d\tau \ f(\tau) \]

we can rewrite \( I \) as

\[ I = 2 \int_0^T dt \ F(t) - \int_0^T dt \ F(t - \Delta) + \int_0^T dt \ F(t - \Delta - T). \]

Performing a further change of variable \( z := t - \Delta \) the second integral reads

\[ \int_0^T dt \ F(t - \Delta) = \int_{-\Delta}^{T-\Delta} dz \ F(z) := \int_{0}^{T-\Delta} dz \ F(z) - \int_{-\Delta}^{0} dz \ F(z). \]

Performing a similar operation on the third integral we obtain

\[ I = 2 \int_0^T dt \ F(t) - \int_{-\Delta}^{T-\Delta} dz \ F(z) + \int_{-\Delta}^{0} dz \ F(z) + \int_{0}^{T-\Delta} dz \ F(z) - \int_{-\Delta}^{0} dz \ F(z). \]

Now we use the information \( T \gg \Delta \) and \( f(t) = f(-t) \). As an elementary consequence we have \( F(t) = -F(-t) \), implying that

\[ \int_0^T dt \ F(t) = \int_0^{-T} dt \ F(t). \]

Approximating \( T \pm \Delta \approx T \) and swapping the sign of the upper integration bounds appropriately we have

\[ I = 2 \int_0^T dt \ F(t) - \left[ \int_0^T dz \ F(z) - \int_0^\Delta dz \ F(z) \right] - \left[ \int_0^T dz \ F(z) - \int_0^\Delta dz \ F(z) \right]. \]

The integrals from 0 to \( \Delta \) can all be neglected exploiting again the fact that \( T \gg \Delta \) and we obtain

\[ I \approx 2 \int_0^T dt \ F(t) - 2 \int_0^T dt \ F(t) = 0. \]

The result is exact in the limit \( \Delta/T \to 0 \).

**Appendix D. Fourth-order term in the Dyson expansion**

The fourth-order propagator in the Dyson expansion is

\[ \hat{U}_4(T) := \left( \frac{-i}{\hbar} \right)^4 \int_0^T dt^{(1)} \int_0^{t^{(1)}} dt^{(2)} \int_0^{t^{(2)}} dt^{(3)} \int_0^{t^{(3)}} dt^{(4)} \hat{H}(t^{(1)}) \hat{H}(t^{(2)}) \hat{H}(t^{(3)}) \hat{H}(t^{(4)}), \]

implying a fourth-order term in the expression for the density matrix evolution given by

\[ \langle \hat{U}_4 \rho_0 \rangle \sim \int_0^T dt^{(1)} \int_0^{t^{(1)}} dt^{(2)} \int_0^{t^{(2)}} dt^{(3)} \int_0^{t^{(3)}} dt^{(4)} \int d^3 \chi^{(1)} |y^{(1)}(\chi^{(1)})| d^3 \chi^{(2)} |y^{(2)}(\chi^{(2)})| \cdots \int d^3 \chi^{(4)} |y^{(4)}(\chi^{(4)})| \times \langle V(y^{(1)}, t^{(1)}) V(y^{(2)}, t^{(2)}) V(y^{(3)}, t^{(3)}) V(y^{(4)}, t^{(4)}) \rangle \rho_0. \]

Since the potential is \( V = M c^2(C_1 A + C_2 A^2) \) the average yields one term proportional to \( A_0^2 \):

\[ \langle V(y^{(1)}, t^{(1)}) V(y^{(2)}, t^{(2)}) V(y^{(3)}, t^{(3)}) V(y^{(4)}, t^{(4)}) \rangle \]

\[ \Rightarrow T_4[A_0^2] \sim \left( \frac{M c^2 A_0}{\hbar} \right)^4 \int d^4 k^{(1)} \cdots \int d^4 k^{(4)} |f_k(\tau^{(1)}) f_k(\tau^{(2)}) f_k(\tau^{(3)}) f_k(\tau^{(4)})|. \]
where \( \tau^{(i)} : = t^{(i)} - y^{(i)} \cdot \mathbf{k}^{(i)} \). This requires knowledge of the four-point correlation function, involving the average of the product of four directional components evaluated at different points. For a real random process having a zero mean and Gaussian distribution the four-point function reduces to [36, 37]

\[
\langle f_k(\tau^{(1)}) f_k(\tau^{(2)}) f_k(\tau^{(3)}) f_k(\tau^{(4)}) \rangle = \langle f_k(\tau^{(1)}) f_k(\tau^{(2)}) \rangle \langle f_k(\tau^{(3)}) f_k(\tau^{(4)}) \rangle \\
+ \langle f_k(\tau^{(1)}) f_k(\tau^{(3)}) \rangle \langle f_k(\tau^{(2)}) f_k(\tau^{(4)}) \rangle \\
+ \langle f_k(\tau^{(1)}) f_k(\tau^{(4)}) \rangle \langle f_k(\tau^{(2)}) f_k(\tau^{(3)}) \rangle.
\]

We can now use equation (14) to express the two-point correlations functions:

\[
\langle f_k(\tau^{(1)}) f_k(\tau^{(2)}) f_k(\tau^{(3)}) f_k(\tau^{(4)}) \rangle = \delta(\mathbf{k}^{(1)}, \mathbf{k}^{(2)}) R(\tau^{(1)} - \tau^{(2)}) \delta(\mathbf{k}^{(3)}, \mathbf{k}^{(4)}) R(\tau^{(3)} - \tau^{(4)}) \\
+ \delta(\mathbf{k}^{(1)}, \mathbf{k}^{(3)}) R(\tau^{(1)} - \tau^{(3)}) \delta(\mathbf{k}^{(2)}, \mathbf{k}^{(4)}) R(\tau^{(2)} - \tau^{(4)}) \\
+ \delta(\mathbf{k}^{(1)}, \mathbf{k}^{(4)}) R(\tau^{(1)} - \tau^{(4)}) \delta(\mathbf{k}^{(2)}, \mathbf{k}^{(3)}) R(\tau^{(2)} - \tau^{(3)}).
\]

This implies that the term \( T_4[A_{\mathbf{k}}] \) deriving from \( \langle \hat{U}_4 \mathcal{A}_0 \rangle \) has the structure:

\[
T_4[A_{\mathbf{k}}] \sim 3 \left( \int d\mathbf{k}^{(1)} \int d\mathbf{k}^{(2)} \int_0^T dt^{(1)} \int_0^T dt^{(2)} \delta(\mathbf{k}^{(1)}, \mathbf{k}^{(2)}) R(\tau^{(1)} - \tau^{(2)}) \right)^2,
\]

where all upper bounds in the time integration can be set equal to \( T \) by appropriate normal ordering [34]. By carrying out one of the two angular integrations we have

\[
T_4[A_{\mathbf{k}}] \sim 3 \left( \int d\mathbf{k}^{(1)} \int_0^T dt^{(1)} \int_0^T dt^{(2)} R(t^{(1)} - t^{(2)} - \mathbf{k}^{(1)} \cdot (y^{(1)} - y^{(2)})) \right)^2.
\]

The double time integral can be simplified using the general result (21) and we have

\[
T_4[A_{\mathbf{k}}] \sim 3 \left( \int d\mathbf{k}^{(1)} \tilde{\mathcal{G}}[R(t + \tau)](0) T \right)^2,
\]

where \( \tilde{\mathcal{G}} \) denotes Fourier transform and \( \tau := -\mathbf{k}^{(1)} \cdot (y^{(1)} - y^{(2)}) \). The Fourier transform can be evaluated using that \( R(t + \tau) = C(t + \tau)/C_0 \). Then

\[
\tilde{\mathcal{G}}[R(t + \tau)](\omega) = \frac{1}{C_0} \int_{-\infty}^{\infty} dt \mathcal{C}(t + \tau) e^{-i\omega t}
\]

\[
= \frac{1}{C_0(2\pi c)^3} \int_0^{\omega_c} d\omega \int_{-\infty}^{\infty} dt \omega^2 S(\omega') e^{-i\omega t} \cos \omega' (t + \tau)
\]

\[
= \frac{1}{2C_0(2\pi c)^3} \int_0^{\omega_c} d\omega \int_{-\infty}^{\infty} dt \omega^2 S(\omega') e^{-i\omega t} [e^{i\omega(t+\tau)} + e^{-i\omega(t+\tau)}]
\]

\[
= \frac{1}{2C_0(2\pi c)^3} \int_0^{\omega_c} d\omega \omega^2 S(\omega') \int_{-\infty}^{\infty} dt [e^{i\omega(t+\tau)} + e^{i\omega(t-\tau)}] e^{i\omega t} + e^{i\omega t} e^{-i\omega t}.
\]

Integrating with respect to \( t \) gives

\[
\tilde{\mathcal{G}}[R(t + \tau)](\omega) = \frac{1}{2C_0(2\pi c)^3} \int_0^{\omega_c} d\omega \omega^2 S(\omega') \delta(\omega' - \omega) e^{i\omega \tau} + \delta(-\omega' - \omega) e^{-i\omega \tau}.
\]

This vanishes for \( \omega > \omega_c \). For \( \omega = 0 \) we have

\[
\tilde{\mathcal{G}}[R(t + \tau)](0) = \frac{1}{2C_0(2\pi c)^3} \int_0^{\omega_c} d\omega \omega^2 S(\omega') [\delta(\omega') e^{i\omega \tau} + \delta(-\omega') e^{-i\omega \tau}].
\]

Carrying out the frequency integral and using the properties of the \( \delta \) function we have

\[
\tilde{\mathcal{G}}[R(t + \tau)](0) = \frac{1}{2C_0(2\pi c)^3} \lim_{\omega \to 0} \omega^2 S(\omega).
\]

In the interesting case \( S \sim 1/\omega \) this tends to 0, in such a way that \( T_4[A^{(2)}_0] \to 0 \) and the fourth-order Dyson expansion term does not give any extra contribution proportional to \( A^{(2)}_0 \).
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