CURVES OF GENUS 2, CONTINUED FRACTIONS, 
AND SOMOS SEQUENCES

ALFRED J. VAN DER POORTEN

Abstract. We detail the continued fraction expansion of the square root of monic sextic polynomials. We note in passing that each line of the expansion corresponds to addition of the divisor at infinity, and interpret the data yielded by the general expansion. In particular we obtain an associated Somos sequence defined by a three-term recurrence relation of gap 6.

In the present note I study the continued fraction expansion of the square root of a sextic polynomial, inter alia obtaining integer sequences generated by recursions

\[ A_{h-3}A_{h+3} = aA_{h-2}A_{h+2} + bA_h^2. \]

Specifically, see \[ for the case \( (T_h) = (\ldots, 2, 1, 1, 1, 1, 1, 1, 2, 3, \ldots) \), where I illustrate how the continued fraction expansion data readily allows one to recover the genus 2 curve \( C : Y^2 = (X^3 - 4X + 1)^2 + 4(X - 2) \) giving rise to the sequence.

1. SOME BRIEF REMINDERS

A reminder exposition on continued fractions in quadratic function fields appears as §4 of \[. However, the naive reader needs little more than that a continued fraction expansion of a quadratic irrational integer \( \omega \) is a two-sided sequence of lines, \( h \) in \( \mathbb{Z} \),

\[ \frac{\omega + P_h}{Q_h} = a_h - \frac{\omega + P_{h+1}}{Q_{h+1}} \; ; \quad \text{in brief} \; \omega_h = a_h - \rho_h, \]

with \( (\omega + P_{h+1})(\omega + P_h) = -Q_hQ_{h+1} \) defining the integer sequences \( (P_h) \) and \( (Q_h) \). Necessarily, one must have, say, \( Q_0 \) divides \( (\omega + P_0)(\omega + P_0) \) in which case the integrality of the sequence \( (a_h) \) of partial quotients guarantees that always \( Q_h \) divides \( (\omega + P_h)(\omega + P_h) \). If the partial quotient \( a_h \) is always chosen as the integer part of \( \omega_h \) then \( \omega_0 \) reduced implies that all the \( \omega_h \) and \( \rho_h \) are reduced; and always \( a_h \) also is the integer part of \( \rho_h \). Then conjugation retrieves the left hand half of the expansion of \( \omega_0 \) from that of \( \rho_0 \). In the function field case, one reads ‘polynomial’ for ‘integer’.

2. CONTINUED FRACTION EXPANSION OF THE SQUARE ROOT OF A SEXTIC

We suppose the base field \( \mathbb{F} \) is not of characteristic 2 or 3 because those cases requires changes throughout the exposition; indeed, nontrivial changes in the case

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characteristic 2. We study the continued fraction expansion of the square root of a sextic polynomial $D \in \mathbb{F}[X]$. Set
\begin{equation}
C : Y^2 = D(X) := (X^3 + fX + g)^2 + 4u(X^2 - vX + w),
\end{equation}
and for brevity write $A = X^3 + fX + g$ and $R = u(X^2 - vX + w)$. Set $Z = \frac{1}{2}(Y + A)$ and notice that $Z^2 = AZ - R = 0$. The other root of this equation is $\overline{Z}$.

Suppose that $(X^2 - v_0X + w_0)$ divides the norm
\begin{equation}
Z_0\overline{Z}_0 = -R + d_0(X + e_0)(A + +d_0(X + e_0)),
\end{equation}
and that $Z_0$ has been so chosen that all its partial quotients are of degree 1. Such a choice is 'generic' if the base field is infinite.

For $h = 0, 1, 2, \ldots$ we denote the complete quotients of $Z_0$ by
\begin{equation}
Z_h = (Z + d_h(X + e_h))/u_h(X^2 - v_hX + w_h),
\end{equation}
noting that the $Z_h$ all are reduced, namely deg $Z_h > 0$ but deg $\overline{Z}_h < 0$. The upshot is that the $h$-th line of the continued fraction expansion of $Z_0$ is
\begin{equation}
\frac{Z + d_h(X + e_h)}{u_h(X^2 - v_hX + w_h)} = \frac{X + v_h}{u_h} - \frac{Z + d_{h+1}(X + e_{h+1})}{u_h(X^2 - v_hX + w_h)}.
\end{equation}
Then evident recursion formulas yield
\begin{equation}
f + d_h + d_{h+1} = -v_h^2 + w_h
\end{equation}
\begin{equation}
g + d_he_h + d_{h+1}e_{h+1} = v_hw_h
\end{equation}
and
\begin{equation}
-u_hu_{h+1}(X^2 - v_hX + w_h)(X^2 - v_{h+1}X + w_{h+1})
\end{equation}
\begin{equation}
= (Z + d_{h+1}(X + e_{h+1}))(\overline{Z} + d_{h+1}(X + e_{h+1})).
\end{equation}
Hence, noting that $Z\overline{Z} = -u(X^2 - vX + w)$ and $Z + \overline{Z} = A = X^3 + fX + g$, we may equate coefficients in (6) to see that
\begin{equation}
\text{[4]: } X^4)
\end{equation}
\begin{equation}
d_{h+1} = -u_hw_{h+1}.
\end{equation}
Given that, we obtain, after in each case dividing by $-u_hw_{h+1}$,
\begin{equation}
\text{[5]: } X^3)
\end{equation}
e_{h+1} = -v_h - v_{h+1};
\begin{equation}
\text{[6]: } X^2)
\end{equation}
\begin{equation}
f + d_{h+1} = v_hw_{h+1} + (w_h + w_{h+1}) + u/d_{h+1};
\end{equation}
\begin{equation}
\text{[7]: } X^1)
\end{equation}
\begin{equation}
(f + d_{h+1})e_{h+1} + (g + d_{h+1}e_{h+1}) = -v_hw_{h+1} - v_{h+1}w_h - uv/d_{h+1};
\end{equation}
\begin{equation}
\text{[8]: } X^0)
\end{equation}
\begin{equation}
(g + d_{h+1}e_{h+1})e_{h+1} = w_hw_{h+1} + uw/d_{h+1}.
\end{equation}
The $X^2$ equation readily becomes
\begin{equation}
-d_h = f - w_h + v_h^2 + d_{h+1} = v_h(v_h + v_{h+1}) + w_{h+1} + u/d_{h+1},
\end{equation}
\*At any rate, sufficiently many partial quotients so as to make the following discussion useful.
so \( d_{h+1}(v_h e_{h+1} - w_{h+1}) = d_h d_{h+1} + u \). With similar manipulation of the next two equations we felicitously obtain

\[
\begin{align*}
(7a) & \quad d_{h+1}(v_h e_{h+1} - w_{h+1}) = d_h d_{h+1} + u; \\
(7b) & \quad -v_h d_{h+1}(v_h e_{h+1} - w_{h+1}) = d_h d_{h+1}(e_h + e_{h+1}) - u v; \\
(7c) & \quad w_h d_{h+1}(v_h e_{h+1} - w_{h+1}) = d_h d_{h+1} e_e_{h+1} + u w.
\end{align*}
\]

That immediately yields

\[
\begin{align*}
(8a) & \quad d_h d_{h+1}(e_h + e_{h+1} + v_h) = u(v - v_h); \\
(8b) & \quad d_h d_{h+1}(e_e e_{h+1} - w_h) = -u(w - w_h).
\end{align*}
\]

Incidentally, by

\[-d_{h+1} = f - w_h + v_h^2 + d_h = v_h(v_{h-1} + v_h) + w_{h-1} + u/d_h,
\]

we also discover that, mildly surprisingly,

\[
(9) \quad d_h d_{h+1} + u = d_{h+1}(v_h e_{h+1} - w_{h+1}) = d_h(v_h e_h - w_{h-1}).
\]

3. A RIDICULOUS COMPUTATION

It is straightforward to notice that the three final equations \(7\) yield

\[
e_h^2(v_{h-1}v_h + w_{h-1} + w_h) + e_h(v_{h-1}w_h + v_h w_{h-1}) + w_{h-1}w_h = -u(e_h^2 + v e_h + w)/d_h.
\]

Remarkably, by \(8\)

\[
(d_{h-1} d_h + u)(d_h d_{h+1} + u) = d_h^2(v_{h-1}e_h - w_h)(v_h e_h - w_{h-1})
\]

\[
= e_h^2 v_{h-1} v_h - e_h (v_{h-1}w_{h-1} + v_h w_h) + w_{h-1} w_h
\]

and so, because

\[-(v_{h-1} w_{h-1} + v_h w_h) = v_{h-1} w_h + v_h w_{h-1} - (w_{h-1} + w_h)(v_{h-1} + v_h) = v_{h-1} w_h + v_h w_{h-1} + e_h(w_{h-1} + w_h),
\]

we obtain the surely useful identity

\[
(10) \quad (d_{h-1} d_h + u)(d_h d_{h+1} + u) = -u d_h(e_h^2 + v e_h + w).
\]

This just one of the nine such identities provided by the equations \(7\), and \(8\).

3.1. The special case \( u = 0 \). Consider now the case in which \( R \), the remainder term \( u(X^2 - v X + w) \), is replaced by \(-v(X - w)\). In effect \( u \leftarrow 0 \) except that \( uv \leftarrow v \), \( uw \leftarrow vw \). For instance, \(10\) becomes

\[
(10) \quad d_{h-1} d_h d_{h+1} = -v(e_h + w),
\]

and, we’ll need this, we now have

\[
\begin{align*}
(8a) & \quad e_h + e_{h+1} + v_h = v/d_h d_{h+1}; \\
(8b) & \quad e_h e_{h+1} - w_h = -v u/d_h d_{h+1}.
\end{align*}
\]

Indeed, we find that

\[
(11) \quad d_{h-1} d_h^2 d_{h+1} + d_{h+2} = v^2(e_h e_{h+1} + w(e_h + e_{h+1}) + w^2) = v^2(w_h - w v_h + w^2)
\]

and therefore that

\[
(12) \quad d_{h-2} d_h^3 d_{h+1} d_{h+2} =
\]

\[
v^4(w_{h-1} w_h + w^2(v_{h-1} v_h + (w_{h-1} + w_h)) - w(v_{h-1} w_h + w_{h-1} v_h) - w^3(v_{h-1} + v_h) + w^4),
\]

\]
This last expression is transformed by the equations (6) to become
\begin{equation}
(13) \quad v^4 \left( (g + d_h e_h) e_h - vw/d_h + w^2(f + d_h) + \\
+ w \left(f + d_h e_h + (g + d_h e_h) + v/d_h \right) + w^3 e_h + w^4 \right) = v^4 (e_h + w) \left( (g + d_h e_h) + w(f + d_h) + w^3 \right).
\end{equation}
Thus
\begin{equation}
(14) \quad d_{h-2} d_{h-1} d_h^3 d_{h+1} = -v^3 \left( (g + d_h e_h) + w(f + d_h) + w^3 \right).
\end{equation}
But wait, there's more! By (10) we know that 
\begin{equation}
(15) \quad d_{h-2} d_{h-1} d_h^3 d_{h+1} = -v^3 d_{h-1} d_{h+2} - v^3 (g + w f + w^3).
\end{equation}

**Theorem 1.** Set \( D(X) = A^2 + 4R := (X^3 + f X + g)^2 - 4v(X - w) \) and let \( Z = \frac{1}{2}(Y + A) \), so \( Z^2 - AZ - R = 0 \). Denote by
\begin{equation}
Z_h = \frac{Z + d_h (X + e_h)}{u_h (X^2 - v_h X + w_h)},
\end{equation}
h \( \in \mathbb{Z} \), the complete quotients of the continued fraction expansion of \( Z_0 \); here \( Q_0(X) = u_0(X^2 - v_0 X + w_0) \) must divide the norm \( d_0^2 (X + e_0)^2 + d_0 (X + e_0) A - R \).

Denote by \( (T_h) \) a sequence defined by appropriate initial values and the recursive relation
\begin{equation}
(16) \quad T_{h-1} T_{h+1} = d_h T_h^2.
\end{equation}
Then
\begin{equation}
(17) \quad T_{h-3} T_{h+3} = v^2 T_{h-2} T_{h+2} - v^3 (g + w f + w^3) T_h^2.
\end{equation}

**Proof.** It suffices to check that, given (10), we need only multiply (15) by \( T_h^2 \).

**Remark.** The reader should note the evident tight analogy with the corresponding result for quartic polynomials detailed in [6]. On the other hand, the results of [6] continue to make sense even in *singular* cases, when there are partial quotients of degree greater than one. That’s not quite so here: surely, \( T_{k-3} = T_{k-2} = 0 \) is usually not compatible with (17), suggesting that then our division by \( e_h + w \) in the course of our ‘ridiculous argument’ may be an improper division by zero.

4. **A Cute Example**

The example
\begin{equation}
(18) \quad T_{h-3} T_{h+3} = T_{h-2} T_{h+2} + T_h^2,
\end{equation}
with \( T_0 = T_1 = T_2 = T_3 = T_4 = T_5 = 1 \) is readily seen to derive from the genus 2 curve
\begin{equation}
(19) \quad C : Y^2 = (X^3 - 4X + 1)^2 + 4(X - 2).
\end{equation}
To indeed see this, we first note that of course we need \( d_1 = d_2 = d_3 = d_4 = 1 \) to produce the initial values from \( T_0 = T_1 = 1 \). Since, plainly, \( T_{-1} = T_6 = 2 \), clearly \( d_0 = 2 \). By the Theorem, we expect to require \( v^2 = 1 \) and \(-v^3 (g + w f + w^3) = 1 \). Without loss of generality, we may take \( v = -1 \). From (10) we then read off that
\( e_1 = 2 - w \) and \( e_2 = 1 - w \).
Thus, by (4) and (5) we have
\[ f + 2 = -v_1^2 + w_1 \quad \text{and} \quad g + 3 - 2w = v_1 w_1. \]
But from (8a) and (8b) we evaluate \( v_1 \) and \( w_1 \) in terms of \( w \) as
\[ 3 - 2w + v_1 = -1 \quad \text{and} \quad (2 - w)(1 - w) - w_1 = w. \]
Substituting appropriately we find that \( 1 = g + fw + w^3 = 6w - 11 \) so, as already announced, \( v = -1, w = 2, g = 1, \) and \( f = -4. \)

Furthermore, we have \( v_1 = 0 \) and \( v_0 + v_1 + e_1 = 0, \) so \( v_0 = 0; \) then \( f + 3 = -v_0^2 + w_0 \) yields \( w_0 = -1. \) Noting that \( g + 2e_0 + e_1 = 0, \) we find that \( e_0 = -1/2. \)
Thus the relevant continued fraction expansion commences
\[
Z_0 := \frac{Z + 2X - 1}{X^2 - 1} = X - \frac{Z + X}{X^2 - 1} \]
\[
= -X - \frac{Z + X - 1}{-(X^2 - 2)}
\]
\[
= X + 1 - \frac{Z + X - 1}{X^2 - X - 1}
\]
\[
= X - \frac{Z + X}{-(X^2 - 2)}
\]
\[
= X - \frac{Z + 2X - 1}{X^2 - 1}
\]
\[
\ldots
\]
providing a useful check on our allegations and displaying an expected symmetry.

Denote by \( M \) the divisor class defined by the pair of points \((\varphi, 0)\) and \((\varphi, 0)\) — here, \( \varphi \) is the golden ratio, a happenstance that will please adherents to the cult of Fibonacci — and by \( S \) the divisor class at infinity. Then the sequence \((T_h) = (\ldots, 2, 1, 1, 1, 1, 1, 1, 2, 3, 4, 8, 17, 50, \ldots)\) may be thought of as arising from the points \( M - S, M, M + S, M + 2S, \ldots \) on the Jacobian of the curve \( C \) displayed at (19). Evidently, \( M - S = -M \) so \( 2M = S \) on \( \text{Jac}(C) \).

Incidentally, this closing aside identifying the continued fraction expansion with stepwise addition on the Jacobian is gratuitous. However, concerned readers might contemplate the introduction to Cantor’s paper [4] and the instructive discussion by Kristin Lauter in [5]. A central theme of the paper [1] is a generalisation of the phenomenon to Padé approximation in arbitrary algebraic function fields.

5. Comments

I consider the argument given in (3) above to be quite absurd and am ashamed to have spent a great deal of time in extracting it. Such are the costs of truly low lowbrow arguments; see [2] for heights of ‘brow’. The only saving grace is my mildly ingenious use of symmetry in the argument’s later stages. I do not know whether there is an appealing result of the present genre if \( u \neq 0; \) but see my remarks below.

I should admit that I realised, but only after having successfully selected \( u = 0, \) that Noam Elkies had suggested to me at ANTS, Sydney 2002, that an identity of the genre (17) would exist, but had in fact specified just the special case \( \deg R = 1. \)

Mind you, with some uninteresting effort one can show (say by counting free parameters) that over an algebraic extension of the base field there is a birational
transformation which transforms the given curve to one where $\deg R = 1$. That does not truly better the present theorem.

On the other hand, a dozen years ago\(^1\), David Cantor\(^4\) mentions that his results lead readily to Somos sequences both in genus 1 and 2; the latter with gap 8 (see \(^4\) for the relevant notions). The latter consequence is not obvious; however, recently, Cantor has told me a rather ingenious idea which clearly yields the result for all hyperelliptic curves $Y^2 = E(X)$, $E$ a quintic, say with constant coefficient 1. In brief, Cantor’s result is more general than mine but does not deal with all cases I handle here; nor does it produce the expected recursion formulae of gap 6.

The most serious disappointment is that the best argument I can produce here is just a much more complex version of that of \(^6\) for genus 1. Seemingly, a different Ansatz, a new view on the issues, is needed if my methods are to yield results in higher genus.

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Centre for Number Theory Research, 1 Bimbil Place, Killara, Sydney, NSW 2071, Australia

E-mail address: alf@math.mq.edu.au (Alf van der Poorten AM)

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\(^1\)I have a revision of his manuscript dated November, 1992.