Chaotic ray dynamics in an optical cavity with a beam splitter

G. Puentes, A. Aiello, J. P. Woerdman
Huygens Laboratory, Leiden University, P.O. Box 9504, Leiden, The Netherlands
January 25, 2022

Abstract

We investigate the ray dynamics in an optical cavity when a ray splitting mechanism is present. The cavity is a conventional two-mirror stable resonator and the ray splitting is achieved by inserting an optical beam splitter perpendicular to the cavity axis. Using Hamiltonian optics, we show that such a simple device presents a surprisingly rich chaotic ray dynamics.

PACS numbers: 42.60.Da, 42.65.Sf, 42.15.-i

In this Letter we present a very simple optical cavity whose ray dynamics is nevertheless fully chaotic. Our starting point is the fact that a two-mirror optical cavity can be stable or unstable depending on its geometrical configuration. If a light ray is injected inside the cavity it will remain confined indefinitely when the configuration is stable but it will escape after a finite number of bounces when the cavity is unstable. Our interest is in a cavity which has both aspects of stability and instability (Fig 1). The cavity is modelled as a strip resonator made of two identical concave mirrors of radius of curvature $R$ separated by a distance $L$, where $L < 2R$ so that the cavity is globally stable. We then introduce a beam splitter (BS) inside the cavity, oriented perpendicular to the optical axis. In this way the BS defines two planar-concave subcavities: one on the left and one on the right with respect to the BS, with length $L_1$ and $L_2$, respectively. The main idea is that depending on the position of the BS the
left (right) subcavity becomes unstable for the reflected rays when $L_1$ ($L_2$) is bigger than $R$, while the cavity as a whole remains always stable ($L_1 + L_2 < 2R$).

Consideration of this system raises the nontrivial question whether there will be an "equilibrium" between the number of trapped rays and escaping rays. The trapped rays are those which bounce for infinitely long times due to the global stability of the cavity and the escaping ones are those which stay only for a finite time. If such equilibrium exists it could eventually lead to transient chaos since it is known in literature that instability (positive Lyapunov exponents) and mixing (confinement inside the system) form the skeleton of chaotic dynamics. In this Letter we show that under certain conditions such equilibrium can be achieved in our cavity and that chaotic ray dynamics is displayed.

In our system the BS plays a crucial role. It is modelled as a stochastic ray splitting element by assuming the reflection and transmission coefficients as random variables. Within the context of wave optics this model corresponds to the neglect of all interference phenomena inside the cavity, as required by the ray (zero-wavelength) limit. The stochasticity is implemented by using a Monte Carlo method to determine whether the ray is transmitted or reflected. When a ray is incident on the ray splitting surface of the BS, it is either transmitted through it, with probability $p$, or reflected with probability $1 - p$, where we assume $p = 1/2$ for a 50/50 beam splitter as shown in Fig. We then dynamically evolve a ray and at each reflection we use a random number generator with a uniform distribution to randomly decide whether to reflect or transmit the incident ray.

In the context of Hamiltonian optics, to characterize the trajectory of a ray we first choose a reference plane perpendicular to the optical axis $\hat{Z}$, coinciding with the surface of the BS. The intersection of a ray with this plane is specified by two parameters: the height $y$ above the optical axis and the angle $\theta$ between the trajectory and the same axis. We consider the
rays as point particles, as in standard billiard theory where the propagation of rays correspond to the trajectories of unit mass point particles moving freely inside the billiard and reflecting elastically at the boundary. In particular, we study the evolution of the transversal component of the momentum of the ray, i.e. \( v_y = |\vec{v}| \sin(\theta) \) so that we associate a ray of light with the two-dimensional vector \( \vec{r} = (y, v_y) \). It is important to stress that we use exact 2D-Hamiltonian optics, i.e. we do not use the paraxial approximation.

The evolution of a set of rays injected in the cavity with different initial conditions \((y_0, v_{y_0})\), is obtained by using a ray tracing algorithm. For each initial condition, the actual ray trajectory is determined by a random sequence \{...rrttttttttt..\} which specifies if the ray is reflected (r) or transmitted (t) by the BS. When one evolves the whole set of initial conditions, one can choose between two possibilities, either use the same random sequence for all rays in the set of initial conditions or use a different random sequence for each ray. In this Letter we use the same random sequence for all injected rays in order to uncover the dynamical randomness of the cavity.

The three quantities that we have calculated to demonstrate the chaotic ray dynamics inside the cavity are the Poincaré Surface of Section (SOS), the exit basin diagrams and the escape time function. In all calculations we have assumed \( L_1 + L_2 = 0.16 \text{m} \) and the radius of curvature of the mirrors \( R = 0.15 \text{m} \); the diameter \( d \) of the two mirrors was \( d = 0.05 \text{m} \). In addition, the displacement \( \Delta \) of the BS with respect to the center of the cavity was chosen as 0.02m (unless specified otherwise), and the time was measured in number of bounces \( n \).

In Fig. 2, the successive intersections of a ray with initial transverse coordinates \( y_0 = 1 \times 10^{-5} \text{m}, \ v_{y_0} = 0 \) are represented by the black points in the SOSs. For \( \Delta = 0 \) the cavity configuration is symmetric and the dynamics is completely regular (Fig.2(a)); the on-axis trajectory represents an elliptic fixed point and nearby stable trajectories lie on continuous tori in phase space. In Fig.2(b), the BS is slightly displaced from the center (\( \Delta = 0.02 \text{m} \)), the same initial trajectory becomes unstable and spreads over a finite region of the phase space before escaping after a
large number of bounces \( (n = 75328) \). In view of the ring structure of Fig.2(b) we may qualify the motion as azimuthally ergodic. The fact that the ray-splitting mechanism introduced by the BS produces ergodicity is a well known result for a closed billiard. We find here an analogue phenomenon, with the difference that in our case the trajectory does not explore uniformly but only azimuthally the available phase space, as an apparent consequence of the openness of the system.

It is well known that chaotic hamiltonian systems with more than one exit channel exhibit irregular escape dynamics which can be displayed, e.g., by plotting the exit basin diagram. In our system, this diagram was constructed by defining a fine grid \((2200 \times 2200)\) of initial conditions \((y_0, v_{y_0})\). Each ray is followed until it escapes from the cavity. When it escapes from above \((v_y > 0)\) we plot a black dot in the corresponding initial condition, whereas when it escapes from below \((v_y < 0)\) we plot a white dot. This is shown in Fig.3 the uniform regions in the exit basin diagram correspond to rays which display a regular dynamics, whereas the dusty regions correspond to portions of phase space where there is sensitivity to initial conditions, since two initially nearby points can escape from opposite exits. Moreover, in Fig.3 one can see how the boundary between black and white regions becomes less and less smooth as one approaches the center of these diagrams. It is known that this boundary is actually a fractal set whose convoluted appearance is a typical feature of chaotic scattering systems.

Besides sensitivity to initial conditions, another fundamental ingredient of chaotic dynamics is the presence of infinitely long living orbits which are responsible for the mixing properties of the system. This set of orbits is usually called repeller, and is fundamental to generate a truly chaotic scattering system. To verify the existence of this set we have calculated the escape time or time delay function for a one-dimensional set of initial conditions specified by the initial position \(y_0\) (impact parameter) taken on the mirror \(M_1\) and the initial velocity \(v_{y_0} = 0\). The
escape time was calculated in the standard way, as the time (in number of bounces $n$) it takes a ray to escape from the cavity.

Fig. 4(a) shows the escape time function. The singularities of this function are a clear signature of the existence of long living orbits and the presence of peaks followed by flat regions are a signature of the exponential sensitivity to initial conditions. In order to verify the presence of an infinite set of long living orbits, we have zoomed in on the set of impact parameters $y_0$ in three different intervals (Fig. 4(b), (c) and (d)). Each zoom reveals the existence of new infinitely long living orbits. Infinite delay times correspond to orbits that are asymptotically close to an unstable periodic orbit. If we would continue to increase the resolution we would find more and more infinitely trapped orbits. The repeated existence of singular points is a signature of the mixing mechanism of the system due to the global stability of the cavity.

In conclusion, we have demonstrated that our simple optical system displays chaotic ray dynamics. It is important to stress that a key component for the development of chaos is the inclusion of non-paraxial rays which add the mixing properties to the system. In fact, it has been previously shown that paraxial ray dynamics can be unstable but not chaotic, in systems with stochastic perturbations. In our case, it is the stochastic ray splitting mechanism induced by the BS that destroys the regular motion of rays in the globally stable (but non-paraxial) cavity, as shown by the SOSs. Moreover, by calculating the exit basin diagrams we have found that they show fractal boundaries, which is a typical feature of chaotic ray dynamics. Finally, through the singularities in the escape time function, we have verified the presence of infinitely long living orbits, which in turns reveal the mixing mechanism of our optical cavity. An experimental confirmation of the fractal properties of the exit basin can be performed, e.g., in the way suggested in, by injecting a narrow laser beam into the cavity either in a regular or in a dusty region of phase space. In the former case one expects the beam to leave the cavity either from above or below, while in the latter case both exits should appear illuminated. This
proposed experiment is fully within the context of geometrical optics (interference plays no role) so that our stochastic model of the BS is adequate.

This project is part of the program of FOM and is also supported by the EU under the IST-ATESIT contract. We thank S. Oemrawsingh for useful contributions to software development.

References

[1] A. E. Siegman, *Lasers* (University Science Books, Mill Valley, CA, 1996).

[2] P. Cvitanović *et al.*, *Classical and Quantum Chaos* (www.nbi.dk/ChaosBook/, 2002).

[3] L. Couchman, E. Ott, and T. M. Antonsen, Jr., Phys. Rev. A 46, 6193 (1992).

[4] E. Ott. *Chaos in Dynamical Systems* (Cambridge University Press, 2002), 2nd ed.

[5] S. Bleher, C. Grebogi, E. Ott, and R. Brown, Phys. Rev. A 38, 930 (1988).

[6] S. Ree and L. E. Reichl, Phys. Rev. E 65, 055205(R) (2002).

[7] D. Sweet, E. Ott and J. A. Yorke, Nature 399, 315 (1999).

[8] P. Gaspard, *Chaos, Scattering and Statistical Mechanics* (Cambridge University Press, 1998), 1st ed.

[9] S. Bleher and C. Grebogi and E. Ott, Physica D 46, 87-121 (1990).

[10] A. Aiello, M.P. van Exter, and J. P. Woerdman, Phys. Rev. E 68, 046208 (2003).

[11] S. Longhi, Phys. Rev. E 65, 027601 (2002).

[12] G. Puentes, A. Aiello, and J. P. Woerdman, submitted to Phys. Rev. E (2003).
List of Figure Captions

Fig. 1. Schematic diagram of the cavity model; R indicates the radius of curvature of the mirrors. Two subcavities of length $L_1$ and $L_2$ are coupled by a BS. The total cavity is globally stable for $L = L_1 + L_2 < 2R$. $\Delta = L_1 - L/2$ represents the displacement of the BS with respect to the center of the cavity. When a ray hits the surface of the BS, which we choose to coincide with the reference plane, it can be either reflected or transmitted with equal probability; for a 50/50 beam splitter $p = 1/2$.

Fig. 2. SOS for (a) $\Delta = 0$: the ray dynamics is stable and thus confined on a torus in phase space. (b) $\Delta = 0.002m$, the dynamics becomes unstable and the ray escapes after $n = 75328$ bounces. Note the ring structure in this plot.

Fig. 3. Exit basin for $\Delta = 0.02m$. The fractal boundaries are a typical feature of chaotic scattering systems.

Fig. 4. (a) Escape time as a function of the initial condition $y_0$. (b) Blow up of a small interval along the horizontal axis in (a). (c) and (d) Blow ups of consecutive intervals along the set of impact parameters $y_0$ shown in (b).
Figure 1: Schematic diagram of the cavity model; R indicates the radius of curvature of the mirrors. Two subcavities of length $L_1$ and $L_2$ are coupled by a BS. The total cavity is globally stable for $L = L_1 + L_2 < 2R$. $\Delta = L_1 - L/2$ represents the displacement of the BS with respect to the center of the cavity. When a ray hits the surface of the BS, which we choose to coincide with the reference plane, it can be either reflected or transmitted with equal probability; for a 50/50 beam splitter $p = 1/2$.

Figure 2: SOS for (a) $\Delta = 0$: the ray dynamics is stable and thus confined on a torus in phase space. (b) $\Delta = 0.002m$, the dynamics becomes unstable and the ray escapes after $n = 75328$ bounces. Note the ring structure in this plot.
Figure 3: Exit basin for $\Delta = 0.02m$. The fractal boundaries are a typical feature of chaotic scattering systems.
Figure 4: (a) Escape time as a function of the initial condition $y_0$. (b) Blow up of a small interval along the horizontal axis in (a). (c) and (d) Blow ups of consecutive intervals along the set of impact parameters $y_0$ shown in (b).