On gauge transformation property of coordinate independent SO(9) vector states in SU(2) Matrix Theory

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Abstract

We investigate coordinate independent SO(9) vector states in SU(2) Matrix theory. There are 36 vector states, and we determine what representations of SU(2) they are decomposed into. Among them we find a unique set of states transforming in adjoint representation. We show that this set of states can appear as the linear term in the coordinate matrices in Taylor expansion of zero energy bound state wavefunction around the origin i.e. it satisfies the condition of full supersymmetry.

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1 Introduction

Construction of zero energy normalizable wavefunction is one of the long standing problems in Matrix theory, and attempts from various viewpoint have been made. One of such attempts is to investigate Taylor expansion in the coordinate matrices $X_i^a$ $(i = 1, \ldots, 9)$ around the origin. The first term of the expansion, which is independent of $X_i^a$, must be gauge and SO(9) invariant[1]. Construction of such states is not an easy mathematical problem, and in the case of SU(2) gauge group, it has been constructed in [2, 3], and shown that it is unique[3, 4].

To construct higher terms in the expansion we have to classify more general states independent of $X_i^a$. This is important not only for constructing zero energy bound states, but also for constructing general gauge invariant wavefunctions in Matrix theory. The number of states independent of $X_i^a$ is counted as follows: the fermionic matrix $\theta^a_{\alpha}$ has $16 \times (N^2 - 1)$ components for SU($N$) gauge group, and out of them we can construct $8(N^2 - 1)$ creation operators. Therefore we have $2^{8(N^2 - 1)}$ states. To deal with this large number of states, we need a systematic way of classification. This is also an interesting mathematical problem in its own right.

As a modest step toward this, in this paper we consider SO(9) vector states independent of $X_i^a$ in SU(2) Matrix theory. Such states appear in Taylor expansion of the zero energy wavefunction $|\psi\rangle$ as follows: As is always in supersymmetric systems, $|\psi\rangle$ is necessarily annihilated by supercharge $Q$, which is given by

$$Q = \text{tr} \left[ \Pi^i \gamma^i \theta + \frac{i}{2} [X^i, X^j] \gamma^{ij} \theta \right],$$

(1.1)

where $\Pi^i$ are the momenta conjugate to $X^i$. $|\psi\rangle$ must be gauge invariant, and moreover it has been shown in [1] that $|\psi\rangle$ is SO(9) invariant. Therefore its Taylor expansion in $X_i^a$ is given by

$$|\psi\rangle = |\phi\rangle + X_i^a |\phi_i^a\rangle + X_{i_1} X_{i_2} |\phi_{i_1 i_2}^{a_1 a_2}\rangle + \ldots$$

$$= \sum_{n=0}^{\infty} X_{i_1}^{a_1} \ldots X_{i_n}^{a_n} |\phi_{i_1 \ldots i_n}^{a_1 \ldots a_n}\rangle,$$

(1.2)

where $|\phi_{i_1 \ldots i_n}^{a_1 \ldots a_n}\rangle$ are states constructed by acting creation operators made of $\theta^a_{\alpha}$ on the vacuum for those operators. The leading term $|\phi\rangle$ has been constructed in [2, 3].

The equation $Q |\psi\rangle = 0$ is decomposed into three independent sequences $m = 0, 1, 2$ relating $|\phi_{i_1 \ldots i_{3n+m}}^{a_1 \ldots a_{3n+m}}\rangle$ and $|\phi_{i_1 \ldots i_{3(n+1)+m}}^{a_1 \ldots a_{3(n+1)+m}}\rangle$, and first two of those equations contain only one
\[ \phi_{a_1...a_n} \]:
\[ \gamma^i \theta^a \mid \phi_a \rangle = 0, \quad (1.3) \]
\[ \gamma^{i_1} \theta^{a_1} \mid \phi_{a_1 a_2} \rangle = 0, \quad (1.4) \]
\[ [X^i, X^{i_2}] \mid \phi_{a_1...a_{3n+m}} \gamma^{ij} \theta^a \rangle \mid \phi_{a_1...a_{3n+m}} \rangle = 0, \quad (1.5) \]

where \( n = 0, 1, 2, \ldots \). The first equation (1.3) restricts the first order term in \( X^a_i \). Since \( \mid \phi_a \rangle \) is an SO(9) vector, in this paper we investigate gauge transformation property of SO(9) vector states. In section 3 we enumerate SO(9) vector states and show that these states are classified into 15-, 11-, 7-, and 3-dimensional representation of SU(2). The 3-dimensional i.e. adjoint representation is a unique candidate of \( \mid \phi_a \rangle \). It is shown that it satisfies (1.3), and therefore \( \mid \phi_a \rangle \) can be proportional to \( \mid \phi_a \rangle \) with a nonzero coefficient. Section 2 is for preparing notation and method for investigating representations of SU(2) through reviewing analysis of SO(9) singlet case in [[4]]. Calculations are made with the help of symbolic manipulation program Mathematica and the package for \( \gamma \)-matrix algebra GAMMA[[5]].

\section{SO(9) singlets}

Before discussing SO(9) vector states, we quickly review SO(9) singlet states and show how we can extract representations of the gauge group from them.

States are constructed by acting creation operators made of \( \theta^a_\alpha \) on the vacuum. When we fix the adjoint index \( a \) of the gauge group, we obtain states in the following representations of SO(9):

- symmetric traceless repr. \( \mid ij \rangle \),
- antisymmetric repr. \( \mid ijk \rangle \),
- vector-spinor repr. \( \mid \alpha i \rangle \).

These satisfy \( \mid ii \rangle = 0 \) and \( (\gamma^i)_{\alpha \beta} \mid \beta i \rangle = 0 \) (Rarita-Schwinger condition).

Action of \( \theta_\alpha \) on these states (intertwining relations[[3, 4]]) in our notation \* is
\[ \theta_\alpha \mid ij \rangle = -\frac{1}{3} \left[ (\gamma^i)_{\alpha \beta} \mid \beta j \rangle + (\gamma^j)_{\alpha \beta} \mid \beta i \rangle \right] \], (2.1)

\*Our normalization is related to that in [[3, 4]] by \( \mid ij \rangle_{\text{there}} = -\frac{3}{2} \mid ij \rangle_{\text{here}}, \mid ijk \rangle_{\text{there}} = 3 \sqrt{\frac{2}{3}} \mid ijk \rangle_{\text{here}}, \) and \( \mid \alpha i \rangle_{\text{there}} = \mid \alpha i \rangle_{\text{here}} \).
\[ \theta_a |ijk\rangle = \frac{1}{\sqrt{3}} (\gamma^{[ij]}_{\alpha\beta} |\beta k\rangle), \quad (2.2) \]

\[ \theta_a |\beta i\rangle = -\frac{3}{4} (\gamma^i_{\alpha\beta} |ij\rangle - \frac{\sqrt{3}}{24} (\gamma^{ijkl} \gamma^i - 9\delta^{ij}\gamma^{kl})_{\alpha\beta} |jkl\rangle), \quad (2.3) \]

where \((\gamma^i)_{\alpha\beta}\) are 16 \times 16 real and symmetric SO(9) gamma matrices, and \([ijk]\) denotes antisymmetrization with the factor 1/3!.

Nonzero inner products of these states are

\[ \langle ij|kl \rangle = \frac{1}{2} (\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) - \frac{1}{9} \delta_{ij}\delta_{kl}, \quad (2.4) \]

\[ \langle ijk|lmn \rangle = \delta^i_l \delta^j_m \delta^k_n, \quad (2.5) \]

\[ \langle \alpha i|\beta j \rangle = \delta_{ij} \delta_{\alpha\beta} - \frac{1}{8} (\gamma^{ij})_{\alpha\beta}. \quad (2.6) \]

We have to take the adjoint indices into account and we obtain the above states for each index. For index \(a\) those states are denoted by \(|*\rangle_a\). Note that \(|*\rangle_a\) does not necessarily transform as adjoint representation of the gauge group.

Full states are constructed by taking products of states from each index, and in the case of SU(2) gauge group, SO(9) singlet states are classified into the following 14 states \(|I\rangle (I = 1, 2, \ldots, 14)\) [4]:

\[ |1\rangle = |ij\rangle_1 |jk\rangle_2 |ki\rangle_3, \quad (2.7) \]

\[ |2\rangle = \epsilon^{ijklmnqr} |ijkl\rangle_1 |lmn\rangle_2 |pqr\rangle_3, \quad (2.8) \]

\[ |3\rangle = |ijkl\rangle_1 |jkl\rangle_2 |ij\rangle_3, \quad |4\rangle = |jkl\rangle_1 |ij\rangle_2 |ikl\rangle_3, \quad |5\rangle = |ij\rangle_1 |ikl\rangle_2 |jkl\rangle_3, \quad (2.9) \]

\[ |6\rangle = |\alpha i\rangle_1 |\alpha j\rangle_2 |ij\rangle_3, \quad |7\rangle = -|\alpha j\rangle_1 |ij\rangle_2 |\alpha i\rangle_3, \quad |8\rangle = |ij\rangle_1 |\alpha i\rangle_2 |\alpha j\rangle_3, \quad (2.10) \]

\[ |9\rangle = (\gamma^k)_{\alpha\beta} |\alpha i\rangle_1 |\beta j\rangle_2 |ijk\rangle_3, \quad |10\rangle = (\gamma^k)_{\alpha\beta} |\alpha i\rangle_1 |ijk\rangle_2 |\beta j\rangle_3, \quad |11\rangle = (\gamma^k)_{\alpha\beta} |ijk\rangle_1 |\alpha i\rangle_2 |\beta j\rangle_3, \quad (2.11) \]

\[ |12\rangle = (\gamma^{jkl})_{\alpha\beta} |\alpha i\rangle_1 |\beta i\rangle_2 |jkl\rangle_3, \quad |13\rangle = (\gamma^{jkl})_{\alpha\beta} |\alpha i\rangle_1 |jkl\rangle_2 |\beta i\rangle_3, \quad |14\rangle = (\gamma^{jkl})_{\alpha\beta} |jkl\rangle_1 |\alpha i\rangle_2 |\beta i\rangle_3. \quad (2.12) \]

At first glance it is not clear how these states transform under gauge transformation. It is read off by acting generators of the gauge group \(G^a = \frac{i}{2} f_{abc} \theta^b_{\alpha} \theta^c_{\alpha}\). Since \(G^a\) is always proportional

\footnote{Note that for later convenience we put an additional sign factor on \(S_7\) in [4]: |7\rangle = -S_7.}
to the structure constant $f_{123}$ in the SU(2) case, we use $g^a = G^a / i f_{123}$ instead of $G^a$ in the following. Representation matrix $M_{IJ}$ of $g^1$ on these states:

$$g^1 |I\rangle = M_{IJ} |J\rangle,$$  \hfill (2.13)

can be computed using the intertwining relations (2.1)-(2.3), and its components are real in our notation. Explicit expression of $M_{IJ}$ may be read off from the result given in \[4\]. In our notation $M$ is given by \[4\]

$$M = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{8} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{21} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-9 & 0 & 0 & 0 & -\frac{7}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{6} & -6\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{21} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{4} & -9\sqrt{3} & 9\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \hfill (2.14)

and representation matrices $N_{IJ}$ and $L_{IJ}$ corresponding to $g^2$ and $g^3$ respectively are given by the following interchange of the indices:

$$N_{IJ} = M_{IJ},$$  \hfill (2.15)

$$L_{IJ} = M_{IJ},$$  \hfill (2.16)

For example, $N_{3,11} = M_{4,9}, L_{1,7} = M_{1,6}$.

For an eigenvector $v_I$ and corresponding eigenvalue $\lambda$ of $M^T$:

$$v_I M_{IJ} = \lambda v_J,$$  \hfill (2.17)

$v_I |I\rangle$ gives an eigenvector of $g^1$ corresponding to the eigenvalue $\lambda$. As is well-known, for any unitary representation of SU(2), spectrum of eigenvalues of $G^1 / f_{123} = i g^1$ are given as a disjoint union of sets in the form of $(-j, -j + 1, \ldots, j - 1, j)$, where $j$ is nonnegative half integer. Since the spectrum of eigenvalues of $i M_{IJ}$ is given by

$$0, 0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, 6, -6,$$  \hfill (2.18)

there are one singlet and one ”spin 6” representation.

\[4\] There are some discrepancies between our result (2.14) and that read off from \[4\]. We think our result is correct because it gives an appropriate eigenvalue spectrum (2.18) and passes the test of hermiticity. However both results give the same eigenvectors of eigenvalue 0 and the conclusion about SO(9)×SU(2) singlet is common. We thank M. Hynek and M. Trzetrzelewski for correspondence about this point.
and we define indices $\pm$ of integer spins. Such representations can be constructed as symmetric tensor products of $\text{SO}(9)$ singlet states $^3$ and $^2$, and only the combination $v_1$ gives a state of eigenvalue 0 for $g^2$ and $g^3$. In other words, the following state $|S\rangle$: 

$$|S\rangle = \frac{6}{13} |ij\rangle_1 |jk\rangle_2 |ki\rangle_3 + |ikl\rangle_1 |jkl\rangle_2 |ij\rangle_3 + |ikl\rangle_1 |ij\rangle_2 |jkl\rangle_3 + |ij\rangle_1 |ikl\rangle_2 |jkl\rangle_3$$

(2.22)

is the unique SU(2)×SO(9) singlet state $^3$ and $^2$, and $|\phi\rangle$ must be proportional to $|S\rangle$.

To construct the spin 6 representation, and for later use, we consider general representations of integer spins. Such representations can be constructed as symmetric tensor products of adjoint (spin 1) representations with the tracelessness condition. i.e. a $(2n + 1)$-dimensional (spin $n$) representation is given by states $|a_1a_2\ldots a_n\rangle$ which satisfy

$$|a_1a_2\ldots a_{m-1}a_m a_{m+1}\ldots a_n\rangle = |a_m a_2 \ldots a_{m-1} a_1 a_{m+1} \ldots a_n\rangle, \quad |aaa\ldots a_n\rangle = 0,$$

(2.23)

and we define indices $\pm$ as follows:

$$|\pm a_2 \ldots a_n\rangle = \frac{1}{\sqrt{2}} (|2a_2 \ldots a_n\rangle \pm i |3a_2 \ldots a_n\rangle).$$

(2.24)

Then, from the tracelessness condition, $|+ - a_3 \ldots a_n\rangle = -\frac{1}{2} |11a_3 \ldots a_n\rangle$. This representation has the unique eigenvectors of $ig^1$ corresponding to the eigenvalue 0, $m$, and $-m$:

$$ig^1 |1\ldots 1\rangle = 0,$$

$$ig^1 |1\ldots 1\ldots + \rangle_m = -m |1\ldots 1\ldots + \rangle_m,$$

(2.25)

(2.26)
\[ ig^1 |1\ldots 1\ldots\ldots\rangle = m |1\ldots 1\ldots\ldots\rangle. \]  

(2.27)

Operators \( ig^\pm \equiv \frac{i}{\sqrt{2}} (g^2 \pm ig^3) \) increase and decrease numbers of + and − in these states:

\[ ig^+ |1\ldots 1\ldots\ldots\rangle = m |1\ldots 1\ldots\ldots\rangle, \]  

(2.28)

\[ ig^+ |1\ldots 1\ldots\ldots\rangle = (-1)^m m |1\ldots 1\ldots\ldots\rangle, \]  

(2.29)

\[ ig^- |1\ldots 1\ldots\ldots\rangle = (-1)^m m |1\ldots 1\ldots\ldots\rangle, \]  

(2.30)

\[ ig^- |1\ldots 1\ldots\ldots\rangle = m |1\ldots 1\ldots\ldots\rangle. \]  

(2.31)

Now the construction of the spin 6 representation is straightforward. The eigenvector of \( iM^T \) corresponding to the eigenvalue 6 is

\[ (0, -3i, -108\sqrt{3}i, 108\sqrt{3}i, 0, 0, 0, 0, 0, -96, 0, 0, 8), \]  

(2.32)

which corresponds to \(|-\ldots -\ldots\rangle\). The rest of states are constructed by acting \( g^+ \) successively:

\[ |-\ldots -\ldots\rangle = -3i |2\rangle - 108\sqrt{3}i |3\rangle + 108\sqrt{3}i |4\rangle - 96 |11\rangle + 8 |14\rangle, \]

\[ |-\ldots -1\rangle = \frac{1}{6} ig^+ |-\ldots -\ldots\rangle, \]

\[ |-\ldots -11\rangle = -\frac{1}{5} ig^+ |-\ldots -1\rangle, \]

\[ |-\ldots 11\rangle = \frac{1}{4} ig^+ |-\ldots -11\rangle, \]

\[ \vdots \]  

(2.33)

Since it is not illuminating, we do not write down explicit expressions of all of these states. We only give \(|111111\rangle\), which is in the kernel of \( ig^1 \):

\[ |111111\rangle = \frac{1}{6!} (ig^+)^6 |-\ldots -\ldots\rangle \]

\[ = 324\sqrt{3}i |1\rangle + 432\sqrt{3}i |3\rangle + 432\sqrt{3}i |4\rangle - 702\sqrt{3}i |5\rangle \]

\[ = 432\sqrt{3}i \left( -\frac{13}{8} \nu^{(1)}_I |I\rangle + \nu^{(2)}_I |I\rangle \right), \]  

(2.34)

and is orthogonal to the state \((2.22)\). These states may appear in Taylor expansion of zero energy wavefunction in the form of \((X_{a_1}^{a_1}X_{a_2}^{a_2})(X_{a_3}^{a_3}X_{a_4}^{a_4})(X_{a_5}^{a_5}X_{a_6}^{a_6})|a_1a_2a_3a_4a_5a_6\rangle\).
3 SO(9) vectors

In this section we investigate SO(9) vector states, and as in the previous section we analyze their transformation property under gauge transformation from the viewpoint of the eigenvalue spectrum of the representation matrix.

There are 36 sets of states in SO(9) vector representation, and they are denoted by $|I, a, i\rangle$, where $I = 1, 2, \ldots, 12$. When $|I, 1, i\rangle$ takes the form of $|\ast_1\rangle_1 |\ast_2\rangle_2 |\ast_3\rangle_3$, $|I, 2, i\rangle$ and $|I, 3, i\rangle$ are given by $|\ast_1\rangle_2 |\ast_2\rangle_3 |\ast_3\rangle_1$ and $|\ast_1\rangle_3 |\ast_2\rangle_1 |\ast_3\rangle_2$ respectively. Note that $|I, a, i\rangle$ does not necessarily transform as SU(2) adjoint representation.

Explicitly, $|I, a, i\rangle$ are defined as follows:

$$|1, 1, i\rangle = |ijk\rangle_1 |jl\rangle_2 |kl\rangle_3,$$
$$|1, 2, i\rangle = |k\rangle_1 |ijk\rangle_2 |jl\rangle_3,$$
$$|1, 3, i\rangle = |jl\rangle_1 |kl\rangle_2 |ijk\rangle_3,$$ (3.1)

$$|2, 1, i\rangle = |ijk\rangle_1 |jlm\rangle_2 |klm\rangle_3,$$
$$|2, 2, i\rangle = |klm\rangle_1 |ijk\rangle_2 |jlm\rangle_3,$$
$$|2, 3, i\rangle = |jlm\rangle_1 |klm\rangle_2 |ijk\rangle_3,$$ (3.2)

$$|3, 1, i\rangle = |ij\rangle_1 |\alpha k\rangle_2 |\beta j\rangle_3 \langle \gamma^j \alpha \beta ,$$
$$|3, 2, i\rangle = -|\beta k\rangle_1 |ij\rangle_2 |\alpha k\rangle_3 \langle \gamma^j \alpha \beta ,$$
$$|3, 3, i\rangle = |\alpha k\rangle_1 |\beta k\rangle_2 |ij\rangle_3 \langle \gamma^j \alpha \beta ,$$ (3.3)

$$|4, 1, i\rangle = |jk\rangle_1 |\alpha i\rangle_2 |\beta j\rangle_3 \langle \gamma^k \alpha \beta ,$$
$$|4, 2, i\rangle = -|\beta j\rangle_1 |jk\rangle_2 |\alpha i\rangle_3 \langle \gamma^k \alpha \beta ,$$
$$|4, 3, i\rangle = |\alpha i\rangle_1 |\beta j\rangle_2 |jk\rangle_3 \langle \gamma^k \alpha \beta ,$$ (3.4)

$$|5, 1, i\rangle = |jk\rangle_1 |\alpha j\rangle_2 |\beta i\rangle_3 \langle \gamma^k \alpha \beta ,$$
$$|5, 2, i\rangle = -|\beta i\rangle_1 |jk\rangle_2 |\alpha j\rangle_3 \langle \gamma^k \alpha \beta ,$$
$$|5, 3, i\rangle = |\alpha j\rangle_1 |\beta i\rangle_2 |jk\rangle_3 \langle \gamma^k \alpha \beta ,$$ (3.5)

$$|6, 1, i\rangle = |jk\rangle_1 |\alpha j\rangle_2 |\beta k\rangle_3 \langle \gamma^i \alpha \beta ,$$
$$|6, 2, i\rangle = -|\beta k\rangle_1 |jk\rangle_2 |\alpha j\rangle_3 \langle \gamma^i \alpha \beta ,$$
$$|6, 3, i\rangle = |\alpha j\rangle_1 |\beta k\rangle_2 |jk\rangle_3 \langle \gamma^i \alpha \beta ,$$ (3.6)

$$|7, 1, i\rangle = |ijk\rangle_1 |\alpha j\rangle_2 |\alpha k\rangle_3 ,$$
$$|7, 2, i\rangle = -|\alpha k\rangle_1 |ijk\rangle_2 |\alpha j\rangle_3 ,$$
$$|7, 3, i\rangle = |\alpha j\rangle_1 |\alpha k\rangle_2 |ijk\rangle_3 ,$$ (3.7)

$$|8, 1, i\rangle = |ij\rangle_1 |\alpha l\rangle_2 |\beta j\rangle_3 \langle \gamma^k \alpha \beta ,$$
$$|8, 2, i\rangle = -|\beta l\rangle_1 |ijk\rangle_2 |\alpha l\rangle_3 \langle \gamma^k \alpha \beta ,$$
\[ |8, 3, i⟩ = |\alpha l⟩_1 |\beta l⟩_2 |ijk⟩_3 (\gamma^{jk})_{\alpha \beta}, \quad (3.8) \]

\[ |9, 1, i⟩ = |jkl⟩_1 |\alpha i⟩_2 |\beta l⟩_3 (\gamma^{jk})_{\alpha \beta}, \quad |9, 2, i⟩ = −|\beta l⟩_1 |jkl⟩_2 |\alpha i⟩_3 (\gamma^{jk})_{\alpha \beta}, \quad |9, 3, i⟩ = |\alpha i⟩_1 |\beta l⟩_2 |jkl⟩_3 (\gamma^{jk})_{\alpha \beta}, \quad (3.9) \]

\[ |10, 1, i⟩ = |jkl⟩_1 |\alpha i⟩_2 |\beta i⟩_3 (\gamma^{jk})_{\alpha \beta}, \quad |10, 2, i⟩ = −|\beta i⟩_1 |jkl⟩_2 |\alpha i⟩_3 (\gamma^{jk})_{\alpha \beta}, \quad |10, 3, i⟩ = |\alpha i⟩_1 |\beta i⟩_2 |jkl⟩_3 (\gamma^{jk})_{\alpha \beta}, \quad (3.10) \]

\[ |11, 1, i⟩ = |jkl⟩_1 |\alpha j⟩_2 |\beta k⟩_3 (\gamma^{il})_{\alpha \beta}, \quad |11, 2, i⟩ = −|\beta k⟩_1 |jkl⟩_2 |\alpha j⟩_3 (\gamma^{il})_{\alpha \beta}, \quad |11, 3, i⟩ = |\alpha j⟩_1 |\beta k⟩_2 |jkl⟩_3 (\gamma^{il})_{\alpha \beta}, \quad (3.11) \]

\[ |12, 1, i⟩ = |jkl⟩_1 |\alpha m⟩_2 |\beta m⟩_3 (\gamma^{ijkl})_{\alpha \beta}, \quad |12, 2, i⟩ = −|\beta m⟩_1 |jkl⟩_2 |\alpha m⟩_3 (\gamma^{ijkl})_{\alpha \beta}, \quad |12, 3, i⟩ = |\alpha m⟩_1 |\beta m⟩_2 |jkl⟩_3 (\gamma^{ijkl})_{\alpha \beta}. \quad (3.12) \]

In the following, components of vectors corresponding to the above states are placed in the order in (3.11)-(3.12). For example, \((v_{(1,1)}, v_{(1,2)}, v_{(1,3)}, v_{(2,1)}, v_{(2,2)}, v_{(2,3)}, \ldots)\) for \(v_{(Ia)}\). Under this rule, representation matrix of \(g^1\):

\[ g^1 |I, a, i⟩ = M_{(Ia),(Jb)} |J, b, i⟩ \],

which is calculated by using the intertwining relations, is given as follows:

\[
\begin{align*}
M_{(1,1),(Jb)} & = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\
& = (13/9, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),
\end{align*}
\]

\[
\begin{align*}
M_{(1,2),(Jb)} & = (0, 0, 0, 0, 0, 1/(9\sqrt{3}), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\
& = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),
\end{align*}
\]

\[
\begin{align*}
M_{(1,3),(Jb)} & = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\
& = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),
\end{align*}
\]

\[
\begin{align*}
M_{(2,1),(Jb)} & = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\
& = (22/27, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),
\end{align*}
\]

\[
\begin{align*}
M_{(2,2),(Jb)} & = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\
& = (8/27, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),
\end{align*}
\]

\[
\begin{align*}
M_{(2,3),(Jb)} & = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),
\end{align*}
\]
\[ M_{(3,1),(J_b)} = \begin{pmatrix} 8/27, 0, 0, 2/27, 0, 0, 1/27, 0, 0, 5/27, 0, 0, -4/27, 0, 0, -1/27, 0, 0 \end{pmatrix}, \]

\[ M_{(3,2),(J_b)} = \begin{pmatrix} 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \end{pmatrix}, \]

\[ M_{(3,3),(J_b)} = \begin{pmatrix} 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \end{pmatrix}, \]

\[ M_{(4,1),(J_b)} = \begin{pmatrix} 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \end{pmatrix}, \]

\[ M_{(4,2),(J_b)} = \begin{pmatrix} 0, 0, 2\sqrt{3}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \end{pmatrix}, \]

\[ M_{(4,3),(J_b)} = \begin{pmatrix} 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \end{pmatrix}, \]

\[ M_{(5,1),(J_b)} = \begin{pmatrix} 0, 0, -6\sqrt{3}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \end{pmatrix}, \]

\[ M_{(5,2),(J_b)} = \begin{pmatrix} 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \end{pmatrix}, \]

\[ M_{(5,3),(J_b)} = \begin{pmatrix} 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \end{pmatrix}, \]

\[ M_{(6,1),(J_b)} = \begin{pmatrix} 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \end{pmatrix}, \]

\[ M_{(6,2),(J_b)} = \begin{pmatrix} 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \end{pmatrix}, \]

\[ M_{(6,3),(J_b)} = \begin{pmatrix} 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \end{pmatrix}, \]

\[ M_{(7,1),(J_b)} = \begin{pmatrix} 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \end{pmatrix}, \]

\[ M_{(7,2),(J_b)} = \begin{pmatrix} 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \end{pmatrix}, \]
\[ M_{(7,3),(J_b)} = (0, 0, 0, 0, 0, 0, 1/(4\sqrt{3}), 0, 0, -1/(4\sqrt{3}), 0, 0, -1/(4\sqrt{3}), 0, 0, 1/(2\sqrt{3}), 0, 0, -1/6, 0, 0, 1/3, 0, 0, -17/24, 0, 0, 1/24, 0, 0, -1/6, 0, 0, 1/8, 0), \]
\[ M_{(8,1),(J_b)} = (-18, 0, 0, -51, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \]
\[ M_{(8,2),(J_b)} = (0, 0, 0, 0, 0, 0, 0, 0, 3\sqrt{3}/2, 0, 0, 2\sqrt{3}, 0, 0, -14/3, 0, 0, 1/3, 0, 0, -1/3, 0, 0, -1/12, 0, 0, 8/3, 0, 0, 0), \]
\[ M_{(8,3),(J_b)} = (0, 0, 0, 0, 0, 0, 0, 0, 3\sqrt{3}/2, 0, 0, 0, 0, 0, 2\sqrt{3}, 0, 0, 14/3, 0, 0, -1/3, 0, 0, 1/12, 0, 0, 1/3, 0, 0, 8/3, 0, 0, 0), \]
\[ M_{(9,1),(J_b)} = (0, 0, 0, -18, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \]
\[ M_{(9,2),(J_b)} = (0, 0, 0, 0, 0, 0, 0, 0, 3\sqrt{3}/2, 0, 0, 0, 0, 0, 0), \]
\[ M_{(9,3),(J_b)} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 7\sqrt{3}/2, 0, 0, 0, 0, 0, 0, 0, 0, 0), \]
\[ M_{(10,1),(J_b)} = (0, 0, 0, 0, -18, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \]
\[ M_{(10,2),(J_b)} = (0, 0, 0, 0, 0, 0, 0, 0, 7\sqrt{3}/2, 0, 0, 0, 0, 0, 0), \]
\[ M_{(10,3),(J_b)} = (0, 0, 0, 0, 0, 0, 0, 0, 3\sqrt{3}/2, 0, 0, 0, 0, 0, 0, 0, 0), \]
\[ M_{(11,1),(J_b)} = (0, 0, 0, 0, -9, 9, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \]
\[ M_{(11,2),(J_b)} = (0, 0, 0, 0, 0, 0, 0, 1/(4\sqrt{3}), 0, 0, 11/(4\sqrt{3}), 0, 0, -1/(4\sqrt{3}), 0, 0, 0, 10/(4\sqrt{3}), 0, 0, -11/6, 0, 0, -5/6, 0, -29/24, 0, 0, 17/24, 0, 0, -13/6, 0, 0, -1/24), \]
\[ M_{(11,3),(J_b)} = (0, 0, 0, 0, 0, 0, -1/(4\sqrt{3}), 0, 0, 1/(4\sqrt{3}), 0, 0, -11/(4\sqrt{3}), 0, 0, 10/(4\sqrt{3}), 0, 0, 11/6, 0, 0, -5/6, 0, 0, 17/24, 0, 0, -29/24, 0, 0, 13/6, 0, 0, 1/24, 0), \]
\[ M_{(12,1),(J_b)} = (0, 0, 0, 0, 0, 0, 0, 144, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \]
\[ M_{(12,2),(J_b)} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 9\sqrt{3}/2, 0, 0, -15\sqrt{3}), \]
This matrix also gives actions of $g^2$ and $g^3$ on $|I, a, i\rangle$:

$$g^2 |I, a, i\rangle = M_{(I,a-1),(J,b-1)} |J, b, i\rangle, \quad g^3 |I, a, i\rangle = M_{(I,a+1),(J,b+1)} |J, b, i\rangle,$$  \hfill (3.15)

where indices $a, b, \ldots$ are defined modulo 3.

The eigenvalue spectrum of $i M^T$ is

$$0, 0, 0, 0, 1, 1, 1, −1, −1, −1, 2, 2, 2, −2, −2, −2,$$

$$3, 3, 3, −3, −3, −3, 4, 4, −4, −4, 5, 5, −5, −5, 6, −6, 7, −7,$$  \hfill (3.16)

and we see there is one spin 7, one spin 5, one spin 3 and one spin 1 representation.

As a check of the correctness of the above $M$, we have confirmed that $i f_{123} M_{(J,b),(K_c)} P_{(Ia),(K_c)} \delta_{ij} = \langle I, a, i|G^4|J, b, j\rangle$ is hermitian, where nonzero components of $P_{(Ia),(Jb)} \delta_{ij} \equiv \langle I, a, i|J, b, j\rangle = P_{(Jb),(Ia)} \delta_{ji}$ are

$$P_{(1,1),(1,1)} = P_{(1,2),(1,2)} = P_{(1,3),(1,3)} = 77/3,$$

$$P_{(2,1),(2,1)} = P_{(2,2),(2,2)} = P_{(2,3),(2,3)} = 196/9,$$

$$P_{(2,1),(2,2)} = P_{(2,1),(2,3)} = P_{(2,2),(2,3)} = 56/9,$$

$$P_{(3,1),(3,1)} = P_{(3,2),(3,2)} = P_{(3,3),(3,3)} = 6776/9,$$

$$P_{(3,1),(4,1)} = P_{(3,2),(4,2)} = P_{(3,3),(4,3)} = 77,$$

$$P_{(3,1),(5,1)} = P_{(3,2),(5,2)} = P_{(3,3),(5,3)} = 77,$$

$$P_{(3,1),(6,1)} = P_{(3,2),(6,2)} = P_{(3,3),(6,3)} = −154/9,$$

$$P_{(4,1),(4,1)} = P_{(4,2),(4,2)} = P_{(4,3),(4,3)} = 792,$$

$$P_{(4,1),(5,1)} = P_{(4,2),(5,2)} = P_{(4,3),(5,3)} = 99,$$

$$P_{(4,1),(6,1)} = P_{(4,2),(6,2)} = P_{(4,3),(6,3)} = 176,$$

$$P_{(5,1),(5,1)} = P_{(5,2),(5,2)} = P_{(5,3),(5,3)} = 792,$$

$$P_{(5,1),(6,1)} = P_{(5,2),(6,2)} = P_{(5,3),(6,3)} = 176,$$

$$P_{(6,1),(6,1)} = P_{(6,2),(6,2)} = P_{(6,3),(6,3)} = 6281/9,$$

$$P_{(7,1),(7,1)} = P_{(7,2),(7,2)} = P_{(7,3),(7,3)} = 455/3,$$
\[ P_{(7,1),(8,1)} = P_{(7,2),(8,2)} = P_{(7,3),(8,3)} = 322/3, \]
\[ P_{(7,1),(9,1)} = P_{(7,2),(9,2)} = P_{(7,3),(9,3)} = -28, \]
\[ P_{(7,1),(10,1)} = P_{(7,2),(10,2)} = P_{(7,3),(10,3)} = -28, \]
\[ P_{(8,1),(8,1)} = P_{(8,2),(8,2)} = P_{(8,3),(8,3)} = 8288/3, \]
\[ P_{(8,1),(9,1)} = P_{(8,2),(9,2)} = P_{(8,3),(9,3)} = 364, \]
\[ P_{(8,1),(10,1)} = P_{(8,2),(10,2)} = P_{(8,3),(10,3)} = 364, \]
\[ P_{(9,1),(9,1)} = P_{(9,2),(9,2)} = P_{(9,3),(9,3)} = 2016, \]
\[ P_{(9,1),(10,1)} = P_{(9,2),(10,2)} = P_{(9,3),(10,3)} = 84, \]
\[ P_{(9,1),(11,1)} = P_{(9,2),(11,2)} = P_{(9,3),(11,3)} = -420, \]
\[ P_{(9,1),(12,1)} = P_{(9,2),(12,2)} = P_{(9,3),(12,3)} = 1596, \]
\[ P_{(10,1),(10,1)} = P_{(10,2),(10,2)} = P_{(10,3),(10,3)} = 2016, \]
\[ P_{(10,1),(11,1)} = P_{(10,2),(11,2)} = P_{(10,3),(11,3)} = 420, \]
\[ P_{(10,1),(12,1)} = P_{(10,2),(12,2)} = P_{(10,3),(12,3)} = -1596, \]
\[ P_{(11,1),(11,1)} = P_{(11,2),(11,2)} = P_{(11,3),(11,3)} = 966, \]
\[ P_{(11,1),(12,1)} = P_{(11,2),(12,2)} = P_{(11,3),(12,3)} = 1596, \]
\[ P_{(12,1),(12,1)} = P_{(12,2),(12,2)} = P_{(12,3),(12,3)} = 47712. \] (3.17)

Let us construct representations of the gauge group from higher ones. First, the eigenvector of \( iM^T \) corresponding to the eigenvalue 7:

\[
u^{(7,7)} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \sqrt{3}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \] (3.18)

gives the state \([- - - - - - , i] \) in the spin 7 representation:

\[ | - - - - - - , i \rangle = u^{(7,7)}_{(I_a)} | I, a , i \rangle. \] (3.19)

Here states in this representation are denoted by \(| a_1 \ldots a_7 , i \rangle \). States in the lower representations are also denoted analogously. Other states in this representation are obtained by successive action of \( ig^+ \) on \([- - - - - - , i] \). We write down only the following 3 states in eigenspaces of \( iM^T \) corresponding to the eigenvalues 5, 3, and 1 respectively:

\[ | - - - - - - 11 , i \rangle = -\frac{1}{42} (ig^+)^2 | - - - - - - , i \rangle = u^{(7,5)}_{(I_a)} | I, a , i \rangle, \] (3.20)
\[ | - - - 1111 , i \rangle = \frac{1}{840} (ig^+)^4 | - - - - - - , i \rangle = u^{(7,3)}_{(I_a)} | I, a , i \rangle, \] (3.21)
Next let us construct the spin 5 representation. The eigenspace of \( iM^T \) corresponding to the eigenvalue 5 is 2 dimensional, and in this space the following \( u^{(5,5)} \) is orthogonal to \( u^{(7,5)} \) i.e. 
\[ \bar{u}^{(7,5)}_{(Ia)} P_{(Ia),(Jb)} u^{(5,5)}_{(Jb)} = 0; \]
\[ u^{(5,5)} = (0, 0, 0, 0, 0, 0, 88, -88i, 0, -88, -2i, 0, 2, 88i, 0, 2, -2i, \]
\[ 0, -118\sqrt{3}, -118i\sqrt{3}, 0, 29\sqrt{3}, 29i\sqrt{3}, 0, -44\sqrt{3}, 11i\sqrt{3}, \]
\[ 0, 11\sqrt{3}, -44i\sqrt{3}, 0, -58\sqrt{3}, 58i\sqrt{3}, 0, 5\sqrt{3}, -5i\sqrt{3}). \]

This gives \(- - - - - , i\) in the spin 5 representation:
\[ | - - - - - , i\rangle = u^{(5,5)}_{(Ia)} | I, a, i\rangle. \] (3.27)

Other states in this representation are obtained by successive action of \( ig^+ \) on \(- - - - - , i\). We just show the following 2 states in eigenspaces of \( iM^T \) corresponding to the eigenvalues 3 and 1 respectively:
\[ | - - - 11, i\rangle = -\frac{1}{20} (ig^+)^2 | - - - - - , i\rangle = u^{(5,3)}_{(Ia)} | I, a, i\rangle, \] (3.28)
\[ | -1111, i\rangle = \frac{1}{120} (ig^+)^4 | - - - - - , i\rangle = u^{(5,1)}_{(Ia)} | I, a, i\rangle, \] (3.29)

where
\[ u^{(5,3)} = \frac{1}{20} \cdot (0, 0, 0, 0, 0, 0, 0, 320, 320i, 0, 400, 130i, 0, 130, 400i, 0, -950, -950i, \]
This gives $|--,-,i\rangle$ in the spin 3 representation:

$$|--,-,i\rangle = u_{(1a)}^{(3,3)} |I, a, i\rangle.$$  (3.33)

Other states in this representation are obtained by successive action of $ig^+$ on $|--,-,i\rangle$. We show the following state in the eigenspace of $iM^T$ corresponding to the eigenvalue 1:

$$|11, i\rangle = -\frac{1}{6} (ig^+)^2 |--,-,i\rangle = u_{(1a)}^{(3,1)} |I, a, i\rangle,$$  (3.34)

where

$$u_{(3,1)}^{(3,1)} = \frac{1}{2} \cdot (0, 0, 0, 0, 0, 0, 98, -98i, 0, -62, -118i, 0, 118, 62i, 0, 28, -28i, 0, 100\sqrt{3}, 100i\sqrt{3}, 0, -20\sqrt{3}, -20i\sqrt{3}, 0, 3\sqrt{3}, -33i\sqrt{3}, 0, -33\sqrt{3}, 3i\sqrt{3}, 0, 12\sqrt{3}, -12i\sqrt{3}, 0, -3\sqrt{3}, 3i\sqrt{3}).$$  (3.35)

Finally, let us construct the spin 1 representation. The eigenspace of $iM^T$ corresponding to the eigenvalue 1 is 4 dimensional, and in this space the following $u^{(1,1)}$ is orthogonal to $u^{(7,1)}$, $u^{(5,1)}$ and $u^{(3,1)}$:

$$u^{(1,1)} = \frac{1}{\sqrt{2}} \cdot (0, 0, 0, 0, 0, 0, 148, -148i, 0, -112, 112i, 0, -112, 112i, 0, -112, 112i, 0, 0, 0, 0, 0, 0, 13\sqrt{3}, -13i\sqrt{3}, 0, -13\sqrt{3}, 13i\sqrt{3}, 0, 182\sqrt{3}, -182i\sqrt{3}, 0, -13\sqrt{3}, 13i\sqrt{3}).$$  (3.36)
This gives \(|-i, i\rangle\) in the spin 1 representation:

\[
|-i, i\rangle = u^{(1,1)}_{(ia)} |I, a, i\rangle. \tag{3.37}
\]

From \(|-i, i\rangle\) we can construct \(|1, i\rangle\) and \(|+, i\rangle\). Then we see that those three states are written in the following compact form \((a = 1, 2, 3)\):

\[
|a, i\rangle = 148 |3, a, i\rangle - 112 (|4, a, i\rangle + |5, a, i\rangle + |6, a, i\rangle) + 13\sqrt{3} (|9, a, i\rangle - |10, a, i\rangle + 14 |11, a, i\rangle - |12, a, i\rangle). \tag{3.38}
\]

Now it is easy to confirm the full transformation property of these states:

\[
G^a |b, i\rangle = -i f_{abc} |c, i\rangle. \tag{3.39}
\]

Actually we can easily construct a set of states in \((\text{adjoint}) \times \text{vector}\) representation of \(\text{SU}(2) \times \text{SO}(9)\) from the singlet state \(|S\rangle\):

\[
f_{abc} \theta^b_{\alpha} (\gamma^i)_{\alpha \beta} \theta^c_{\beta} |S\rangle, \tag{3.40}
\]

and this should be proportional to \(|a, i\rangle\). Indeed, we have confirmed that

\[
|a, i\rangle = -\frac{351}{4 f_{123}} f_{abc} \theta^b_{\alpha} (\gamma^i)_{\alpha \beta} \theta^c_{\beta} |S\rangle. \tag{3.41}
\]

Since only \(|a, i\rangle\) transforms as adjoint representation of the gauge group, \(|\phi^a_i\rangle\) must be proportional to \(|a, i\rangle\), and now the question is if it satisfies \((1.3)\) or not. To calculate \(\gamma^i \theta^a |a, i\rangle\), we first note that \(\gamma^i \theta^2 |I, 2, i\rangle\) and \(\gamma^i \theta^3 |I, 3, i\rangle\) are given from \(\gamma^i \theta^1 |I, 1, i\rangle\) by the following replacement:

\[
(\gamma^i \theta^2)_{a} |I, 2, i\rangle = (\gamma^i \theta^1)_{a} |I, 1, i\rangle |s_1| s_2 |s_3 \rangle \rightarrow |s_2| s_3 |s_1 \rangle,
\]

\[
(\gamma^i \theta^3)_{a} |I, 3, i\rangle = (\gamma^i \theta^1)_{a} |I, 1, i\rangle |s_1| s_2 |s_3 \rangle \rightarrow |s_1| s_2 |s_3 \rangle,
\]

and we need the following Fierz transformations to rewrite \(\gamma^i \theta^2 |I, 2, i\rangle\) and \(\gamma^i \theta^3 |I, 3, i\rangle\) in the same form as \(\gamma^i \theta^1 |I, 1, i\rangle\):

\[
(\Gamma_1)_{a\beta} |\delta k\rangle_1 |\beta i\rangle_2 |\gamma j\rangle_3 (\Gamma_2)_{\gamma\delta} = \frac{1}{16} \sum_{n=0}^{4} \frac{1}{n!} \times (\gamma^{i_1 \ldots i_n})_{a\beta} |\beta k\rangle_1 |\gamma i\rangle_2 |\delta j\rangle_3 (\Gamma_1^T \gamma^{i_1 \ldots i_n} \Gamma_2^T)_{\gamma\delta}, \tag{3.44}
\]
\[(\Gamma_1)_{\alpha\beta} |\gamma\rangle_1 |\delta\rangle_2 |\beta\rangle_3 (\Gamma_2)_{\gamma\delta} = \frac{1}{16} \sum_{n=0}^{4} \frac{1}{n!} (-1)^{\frac{1}{2}n(n-1)} \times (\gamma^{i_{1}\ldots i_{n}})_{\alpha\beta} |\beta\rangle_1 |\gamma\rangle_2 |\delta\rangle_3 (\Gamma_2^T)_{\gamma^{i_{1}\ldots i_{n}}\Gamma_1} \gamma_{\delta}. \quad (3.45)\]

After straightforward calculation using these we obtain
\[
(\gamma^{i\alpha})_{\alpha} |3, a, i\rangle = -\frac{11}{3} |1, \alpha\rangle - \frac{11}{24} |2, \alpha\rangle - \frac{11}{24} |3, \alpha\rangle + \frac{11}{24} |5, \alpha\rangle + \frac{11}{24} |6, \alpha\rangle - \frac{11}{48} |8, \alpha\rangle + \frac{11}{48} |9, \alpha\rangle + \frac{11}{144} |12, \alpha\rangle - \frac{11}{144} |13, \alpha\rangle, \quad (3.46)
\]
\[
(\gamma^{i\alpha})_{\alpha} |4, a, i\rangle = -\frac{1}{24} |1, \alpha\rangle + \frac{1}{4} |2, \alpha\rangle - \frac{5}{8} |3, \alpha\rangle - \frac{3}{8} |4, \alpha\rangle - \frac{7}{12} |5, \alpha\rangle + \frac{13}{24} |6, \alpha\rangle - \frac{7}{48} |7, \alpha\rangle - \frac{1}{24} |8, \alpha\rangle - \frac{1}{16} |9, \alpha\rangle + \frac{1}{288} |11, \alpha\rangle + \frac{1}{72} |12, \alpha\rangle + \frac{1}{144} |13, \alpha\rangle - \frac{1}{1152} |15, \alpha\rangle, \quad (3.47)
\]
\[
(\gamma^{i\alpha})_{\alpha} |5, a, i\rangle = -\frac{1}{24} |1, \alpha\rangle - \frac{5}{8} |2, \alpha\rangle + \frac{1}{4} |3, \alpha\rangle + \frac{3}{8} |4, \alpha\rangle + \frac{13}{24} |5, \alpha\rangle - \frac{7}{12} |6, \alpha\rangle + \frac{7}{48} |7, \alpha\rangle + \frac{1}{16} |8, \alpha\rangle + \frac{1}{24} |9, \alpha\rangle + \frac{1}{288} |11, \alpha\rangle - \frac{1}{144} |12, \alpha\rangle - \frac{1}{72} |13, \alpha\rangle - \frac{1}{1152} |15, \alpha\rangle, \quad (3.48)
\]
\[
(\gamma^{i\alpha})_{\alpha} |6, a, i\rangle = -\frac{7}{12} |1, \alpha\rangle + \frac{7}{24} |2, \alpha\rangle + \frac{7}{24} |3, \alpha\rangle + \frac{1}{8} |5, \alpha\rangle + \frac{1}{8} |6, \alpha\rangle - \frac{1}{16} |8, \alpha\rangle + \frac{1}{16} |9, \alpha\rangle - \frac{1}{48} |11, \alpha\rangle + \frac{1}{144} |12, \alpha\rangle - \frac{1}{576} |15, \alpha\rangle, \quad (3.49)
\]
\[
(\gamma^{i\alpha})_{\alpha} |9, a, i\rangle = \frac{1}{\sqrt{3}} \left[ -\frac{1}{12} |1, \alpha\rangle + \frac{1}{2} |2, \alpha\rangle - \frac{5}{4} |3, \alpha\rangle + \frac{5}{4} |4, \alpha\rangle + \frac{5}{6} |5, \alpha\rangle - \frac{5}{12} |6, \alpha\rangle + \frac{1}{24} |7, \alpha\rangle + \frac{1}{3} |8, \alpha\rangle + \frac{1}{6} |10, \alpha\rangle + \frac{1}{8} |9, \alpha\rangle + \frac{1}{48} |11, \alpha\rangle + \frac{1}{18} |12, \alpha\rangle - \frac{1}{72} |13, \alpha\rangle - \frac{1}{12} |14, \alpha\rangle + \frac{5}{576} |15, \alpha\rangle \right], \quad (3.50)
\]
\[
(\gamma^{i\alpha})_{\alpha} |10, a, i\rangle = \frac{1}{\sqrt{3}} \left[ -\frac{1}{12} |1, \alpha\rangle - \frac{1}{2} |3, \alpha\rangle + \frac{5}{4} |2, \alpha\rangle + \frac{5}{4} |4, \alpha\rangle - \frac{5}{6} |6, \alpha\rangle + \frac{11}{12} |5, \alpha\rangle + \frac{1}{24} |7, \alpha\rangle + \frac{1}{3} |9, \alpha\rangle \right] \]
\[
(\gamma^i\theta^a)_{\alpha} |11, a, i\rangle = -\frac{1}{6} |10, \alpha\rangle + \frac{1}{8} |8, \alpha\rangle - \frac{1}{48} |11, \alpha\rangle - \frac{1}{72} |12, \alpha\rangle \\
+ \frac{1}{18} |13, \alpha\rangle + \frac{1}{12} |14, \alpha\rangle - \frac{5}{576} |15, \alpha\rangle, \\
\]
(3.51)

\[
(\gamma^i\theta^a)_{\alpha} |12, a, i\rangle = \frac{1}{\sqrt{3}} \left[ \frac{31}{12} |1, \alpha\rangle - \frac{3}{2} |2, \alpha\rangle - \frac{3}{2} |3, \alpha\rangle - \frac{7}{12} |5, \alpha\rangle \\
- \frac{7}{12} |6, \alpha\rangle + \frac{5}{24} |8, \alpha\rangle - \frac{1}{6} |10, \alpha\rangle - \frac{5}{24} |9, \alpha\rangle \\
+ \frac{1}{24} |11, \alpha\rangle - \frac{1}{18} |12, \alpha\rangle + \frac{1}{18} |13, \alpha\rangle + \frac{1}{12} |14, \alpha\rangle \\
- \frac{1}{72} |15, \alpha\rangle \right], \quad (3.52)
\]

\[
|1, \alpha\rangle = |\alpha i_1|\gamma j_2|\delta j_3|\gamma^i_{\gamma\delta}, \\
|2, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta j_1|\gamma i_2|\gamma j_3, \\
|3, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta j_1|\gamma i_2|\gamma^i_{\gamma\delta}, \\
|4, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta k_1|\gamma k_2|\delta k_3|\gamma^i_{\gamma\delta}, \\
|5, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta k_1|\gamma i_2|\delta k_3|\gamma^i_{\gamma\delta}, \\
|6, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta k_1|\gamma k_2|\delta i_3|\gamma^i_{\gamma\delta}, \\
|7, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta k_1|\gamma l_2|\delta l_3|\gamma^i_{\gamma\delta}, \\
|8, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta l_1|\gamma i_2|\delta l_3|\gamma^i_{\gamma\delta}, \\
|9, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta l_1|\gamma l_2|\delta i_3|\gamma^i_{\gamma\delta}, \\
|10, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta l_1|\gamma i_2|\delta j_3|\gamma^i_{\gamma\delta}, \\
|11, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta m_1|\gamma m_2|\delta m_3|\gamma^i_{\gamma\delta}, \\
|12, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta m_1|\gamma i_2|\delta m_3|\gamma^i_{\gamma\delta}, \\
|13, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta m_1|\gamma m_2|\delta i_3|\gamma^i_{\gamma\delta}, \\
|14, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta m_1|\gamma i_2|\delta j_3|\gamma^i_{\gamma\delta}, \\
|15, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta m_1|\gamma n_2|\delta n_3|\gamma^i_{\gamma\delta}. \\
\]
(3.54)

\[
|1, \alpha\rangle = |\alpha i_1|\gamma j_2|\delta j_3|\gamma^i_{\gamma\delta}, \\
|2, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta j_1|\gamma i_2|\gamma j_3, \\
|3, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta j_1|\gamma i_2|\gamma^i_{\gamma\delta}, \\
|4, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta k_1|\gamma k_2|\delta k_3|\gamma^i_{\gamma\delta}, \\
|5, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta k_1|\gamma i_2|\delta k_3|\gamma^i_{\gamma\delta}, \\
|6, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta k_1|\gamma k_2|\delta i_3|\gamma^i_{\gamma\delta}, \\
|7, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta k_1|\gamma l_2|\delta l_3|\gamma^i_{\gamma\delta}, \\
|8, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta l_1|\gamma i_2|\delta l_3|\gamma^i_{\gamma\delta}, \\
|9, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta l_1|\gamma l_2|\delta i_3|\gamma^i_{\gamma\delta}, \\
|10, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta l_1|\gamma i_2|\delta j_3|\gamma^i_{\gamma\delta}, \\
|11, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta m_1|\gamma m_2|\delta m_3|\gamma^i_{\gamma\delta}, \\
|12, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta m_1|\gamma i_2|\delta m_3|\gamma^i_{\gamma\delta}, \\
|13, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta m_1|\gamma m_2|\delta i_3|\gamma^i_{\gamma\delta}, \\
|14, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta m_1|\gamma i_2|\delta j_3|\gamma^i_{\gamma\delta}, \\
|15, \alpha\rangle = (\gamma^i)_{\alpha\beta}|\beta m_1|\gamma n_2|\delta n_3|\gamma^i_{\gamma\delta}. \\
\]
(3.55)

\[
(\gamma^i\theta^a)_{\alpha} |a, i\rangle = -13 |8, \alpha\rangle + 13 |9, \alpha\rangle - 26 |10, \alpha\rangle - \frac{239}{72} |11, \alpha\rangle
\]
(3.53)
Using (3.72), (3.74) and (3.75), we can see that the right hand side of (3.69) is in fact zero:

\begin{equation}
-\frac{13}{3} |12, \alpha\rangle + \frac{13}{3} |13, \alpha\rangle + 13 |14, \alpha\rangle - \frac{551}{288} |15, \alpha\rangle.
\end{equation}

In this calculation we did not take \( \gamma^{12\ldots9} = 1 \) into account. If we use it we see that the above 15 states \( |I, \alpha\rangle \) are not independent. First note that the following hold from \( \gamma^{12\ldots9} = 1 \):

\begin{equation}
\gamma^{i_1\ldots i_n} = (-1)^{\frac{n(n-1)}{2}} (9 - n)! \epsilon^{i_1\ldots i_9} \gamma^{i_{n+1}\ldots i_9},
\end{equation}

and

\begin{equation}
(\gamma^{1i\ldots in})_{\alpha\beta}(\gamma^{1i\ldots inj_1\ldots j_m})_{\gamma\delta} = (-1)^{\frac{m(m+2n-1)}{2}} (9 - n - m)! (\gamma^{1j_1\ldots j_m k_3\ldots k_9 \ldots n-m})_{\alpha\beta} (\gamma^{k_1\ldots k_9 \ldots n-m})_{\gamma\delta}.
\end{equation}

With (3.71) and the Rarita-Schwinger condition, we can show the following:

\begin{align}
|15, \alpha\rangle &= (\gamma^{ijklm})_{\alpha\beta} |\beta m\rangle_1 |\gamma n\rangle_2 |\delta n\rangle_3 (\gamma^{ijkl})_{\gamma\delta} \\
&= -4(\gamma^{ijk})_{\alpha\beta} |\beta m\rangle_1 |\gamma n\rangle_2 |\delta n\rangle_3 (\gamma^{ijkm})_{\gamma\delta} \\
&= -4 |11, \alpha\rangle.
\end{align}

Furthermore, from the Rarita-Schwinger condition, \( 0 = (\gamma^{ijkl})_{\alpha\beta} |\beta m\rangle_1 |\gamma i\rangle_2 |\delta j\rangle_3 (\gamma^{kl})_{\gamma\delta} \), and by expanding the product of the gamma matrices,

\begin{align}
0 &= (\gamma^{ijklm})_{\alpha\beta} |\beta m\rangle_1 |\gamma i\rangle_2 |\delta j\rangle_3 (\gamma^{kl})_{\gamma\delta} + 2(\gamma^{ijk})_{\alpha\beta} |\beta m\rangle_1 |\gamma i\rangle_2 |\delta j\rangle_3 (\gamma^{km})_{\gamma\delta} \\
&\quad + (\gamma^{ijkl})_{\alpha\beta} |\beta j\rangle_1 |\gamma i\rangle_2 |\delta j\rangle_3 (\gamma^{kl})_{\gamma\delta} - (\gamma^{ijk})_{\alpha\beta} |\beta i\rangle_1 |\gamma i\rangle_2 |\delta j\rangle_3 (\gamma^{kl})_{\gamma\delta} \\
&= |8, \alpha\rangle - |9, \alpha\rangle + 2 |10, \alpha\rangle + (\gamma^{ijklm})_{\alpha\beta} |\beta m\rangle_1 |\gamma i\rangle_2 |\delta j\rangle_3 (\gamma^{kl})_{\gamma\delta}.
\end{align}

Applying (3.71) to the last term of the above and expanding the product of gamma matrices, we obtain

\begin{equation}
0 = |8, \alpha\rangle - |9, \alpha\rangle + 2 |10, \alpha\rangle + \frac{1}{3} |12, \alpha\rangle - \frac{1}{3} |13, \alpha\rangle - |14, \alpha\rangle + \frac{1}{12} |15, \alpha\rangle.
\end{equation}

Similarly, from \( 0 = (\gamma^{ijklm})_{\alpha\beta} |\beta n\rangle_1 |\gamma i\rangle_2 |\delta j\rangle_3 (\gamma^{klm})_{\gamma\delta} \), we obtain

\begin{equation}
0 = 3 |8, \alpha\rangle - 3 |9, \alpha\rangle + 6 |10, \alpha\rangle - |11, \alpha\rangle - |12, \alpha\rangle + |13, \alpha\rangle + 3 |14, \alpha\rangle.
\end{equation}

Using (3.72), (3.73) and (3.74), we can see that the right hand side of (3.69) is in fact zero:

\begin{equation}
\gamma^{i\theta^a} |a, i\rangle = 0.
\end{equation}

Therefore \( |\phi_1^a\rangle \) can be proportional to \( |a, i\rangle \) with a nonzero coefficient.
4 Discussion

We have found that 36 vector states are classified into spin 7, spin 5, spin 3 and spin 1 representation of SU(2). In addition, we have shown that the linear term in $X^i_a$ in Taylor expansion of the zero energy wavefunction around the origin is proportional to the spin 1 representation.

Clearly brute-force calculation as the one we have made in this paper is too inefficient to construct states in higher representations of SO(9), and we have to invent more useful method to analyze them. We have found that $|a,i\rangle$ is given by acting $\theta^a_\alpha$ on the singlet $|S\rangle$. Other states may be constructed similarly. For example, the traceless part of the following state:

$$
\sum_{(a_1a_2a_3)} [f_{a_1b_1c_1} \theta^1_{a_1} (\gamma^i)_{a_1\beta_1} \theta^1_{\beta_1}] [f_{a_2b_2c_2} \theta^2_{a_2} (\gamma^j)_{a_2\beta_2} \theta^2_{\beta_2}] [f_{a_3b_3c_3} \theta^3_{a_3} (\gamma^k)_{a_3\beta_3} \theta^3_{\beta_3}] |S\rangle, \tag{4.1}
$$

where $\sum_{(a_1a_2a_3)}$ is summation over permutations of $a_1a_2a_3$, must be proportional to $|a_1a_2a_3,i\rangle$. The question is if the proportionality constant is zero or not. We did not check it, and similar problems arise for other representations. In general it is an interesting problem if there are states which cannot be constructed from $|S\rangle$ or not.

Before constructing each states explicitly it is desirable to know multiplicities of representations in the $2^{N^2-1}$-dimensional vector space[6]. Related counting has been made in [7].

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