Relative Gorenstein flat modules and Foxby classes and their model structures

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Abstract

A model structure on a category is a formal way of introducing a homotopy theory on that category, and if the model structure is abelian and hereditary, its homotopy category is known to be triangulated. So, a good way to both build and model a triangulated category is to build a hereditary abelian model structure.

Given a ring $R$ and a (non-necessarily semidualizing) left $R$-module $C$, we introduce and study new concepts of relative Gorenstein cotorsion and cotorsion modules: $G_C$-cotorsion and (strongly) $C_C$-cotorsion. As an application, we prove that there is a unique hereditary abelian model structure on the category of left $R$-modules, in which the cofibrations are the monomorphisms with $G_C$-flat cokernel and the fibrations are the epimorphisms with $C_C$-cotorsion kernel belonging to the Bass class $B_C(R)$.

In the second part, when $C$ is a semidualizing $(R,S)$-bimodule, we investigate the existence of abelian model structures on the category of left (resp., right) $R$-modules where the cofibrations are the epimorphisms (resp., monomorphisms) with kernel (resp., cokernel) belonging to the Bass (resp., Auslander) class $B_C(R)$ (resp., $A_C(R)$).

We also study the class of $G_C$-flat modules and the Bass class from the Auslander-Buchweitz approximation theory point of view. We show that they are part of weak AB-contexts. As the concept of weak AB-context can be dualized, we also give dual results that involve the class of $G_C$-cotorsion modules and the Auslander class.

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1 Introduction

Model categories were first introduced by Quillen in [39]. Roughly speaking, a model structure on a bi-complete category $\mathcal{A}$ is given by three classes of morphisms of $\mathcal{A}$, called cofibrations, fibrations and weak equivalences, from which it is possible to introduce a homotopy theory in $\mathcal{A}$. Model categories are interesting because they establish the theoretical framework for formally inverting the weak equivalences. We refer the reader to [32] for the definition and basic results.

Later, Hovey ([33]), and also Beligiannis and Reiten ([5, Chapter VIII]), proposed the notion of an abelian model structure on an abelian category $\mathcal{A}$, as a model structure that is compatible with the abelian context of the category. Hovey proved ([33, Theorem 2.2]) that there is a one-to-one correspondence between the class of abelian model structures and the class of Hovey triples (i.e., three classes of objects satisfying some properties). This establishes a relation between the theory of model categories and the representation theory via cotorsion pairs. If the abelian model structure is hereditary (i.e., the two complete cotorsion pairs induced by the Hovey triple are hereditary), then its homotopy category, $\text{Ho}(\mathcal{A})$, is the stable category of a Frobenius category, and so it is triangulated. This is an important situation in which one obtains a triangulated category from the point of view of model category theory. We refer the reader to [23] for a recent and interesting survey on hereditary abelian model structures.

The first purpose of this paper is to construct new hereditary abelian model structures on the category of left $R$-modules that involve the class of $G_C$-flat modules and the well-known Foxby classes, and use the homotopy categories of these model structures to further study these classes. A second purpose of this paper is to continue the study initiated in [6, 8, 9, 10] of relative Gorenstein homological algebra with respect to a non-necessary semidualizing module $C$. In particular, we will prove results that are new even in the case where $C$ is a semidualizing module.

The study of the Gorenstein homological dimension with respect to a semidualizing module $C$ goes back to Golod [27] when he introduced the finitely generated modules of finite $G_C$-dimension over a commutative noetherian ring $R$, as a relative theory to that of Auslander and Bridger [2]. Holm and Jørgensen in [30] extended Golod’s study of $G_C$-dimension of finitely generated modules to the case of arbitrary modules over commutative noetherian rings. They considered the $C$-Gorenstein projective and $C$-Gorenstein flat $R$-modules. These modules are the analogues of modules of $G_C$-dimension 0. Recently, these approaches have been unified in [10, 9] with the use of $G_C$-projective $R$-modules and $G_C$-flat $R$-modules.

We will also be interested in the case where the class of $G_C$-flat modules and the Bass class are part of weak Auslander-Buchweitz contexts in Hashimoto’s terminology [28]. This allows us, among other things, to deduce the existence of certain Auslander-Buchweitz approximations for $R$-modules of finite $G_C$-flat and $B_C$-injective dimensions, respectively.

This article is organized as follows.

In Section 2 we give the key definitions and preliminary results necessary for the rest of the paper.
In Section 3 we introduce and study two new concepts related to relative cotorsion modules: strongly $C_C$-cotorsion and $n$-$C_C$-cotorsion modules for a given integer $n \geq 1$. We are mainly interested in their links with other known classes of modules such as cotorsion modules (Proposition 3.5 and Corollary 3.10), as well as in their homological properties. It is investigated when these new classes are the right side class of a (perfect, complete, hereditary) cotorsion pair (Theorem 3.7).

In Section 4 we introduce and investigate a new concept of Gorenstein cotorsion modules: $G_C$-cotorsion. We characterize when the pair $(G_C\text{-flat}, G_C\text{-cotorsion})$ is a hereditary and perfect cotorsion pair (Theorem 4.1). Then, two different descriptions of the core of this cotorsion pair is given (Proposition 4.10 and Proposition 4.13). As applications, we show that $G_C\text{F}(R)$, the class of $G_C\text{-flat} R$-modules, is part of a left weak AB-context and that $G_C C(R)$, the class of $G_C\text{-cotorsion} R$-modules, is part of a right weak AB-context (Theorem 4.11). As a second application, we construct a new hereditary abelian model structure on the category of left $R$-modules, where the cofibrations are precisely the monomorphisms with $G_C\text{-flat}$ core and the fibrations are the epimorphisms with $C_C$-cotorsion kernel belonging to $B_C(R)$, the Bass class (Theorem 4.15).

In Section 5 we investigate when $B_C(R)$ is part of a right weak AB-context. Next, we find necessary and sufficient conditions to guarantee the existence of a hereditary abelian model structure whose fibrations are epimorphisms with kernel lying in $B_C(R)$ and whose cofibrations are monomorphisms with cokernel lying in $\text{Add}_R(C)$ (Theorem 5.8). At the end of this section we relate these model structures to the Frobenius category relative to $C$, the so called $C$-Frobenius category, recently introduced in [7]. It is shown that when the category of left $R$-modules is $C$-Frobenius, the homotopy category is an extension of the stable model category of a Frobenius ring. Of course, dual results concerning the Auslander class are investigated.

## 2 Preliminaries

Throughout this paper, $R$ and $S$ will be associative (non-necessarily commutative) rings with identity, and all modules will be, unless otherwise specified, unital left $R$-modules or left $S$-modules. When right $R$-modules need to be used, they will be denoted as $MR$, while in these cases left $R$-modules will be denoted by $rM$. We use $\mathcal{I}(R)$, $\mathcal{P}(R)$, $\mathcal{F}(R)$, and $\mathcal{C}(R)$ to denote the classes of injective, projective, flat, and cotorsion $R$-modules, respectively.

From now on $C$ will stand for an $R$-module, $S$ for its endomorphism ring, $S = \text{End}_R(C)$, and $C^+$ for the character module of $C$, $C^+ = \text{Hom}_R(C, \mathbb{Q}/\mathbb{Z})$. Notice that $C$ is an $(R, S)$-bimodule and $C^+$ an $(S, R)$-bimodule. We use $\text{Add}_R(C)$ (resp., $\text{Prod}_R(C)$) to denote the class of all $R$-modules which are isomorphic to direct summands of direct sums (resp., direct products) of copies of $C$, and we write $\text{add}_R(C)$ when such a direct sum is finite.

Given a class $\mathcal{X}$ of $R$-modules and a class $\mathcal{Y}$ of right $R$-modules, an $\mathcal{X}$-resolution of an $R$-module $M$ is an exact complex

$$\cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

where $X_i \in \mathcal{X}$. An $\mathcal{X}$-coresolution of $M$ is defined dually. A complex $X$ in $R$-Mod is called $(\mathcal{Y} \otimes_R -)$-exact (resp., $\text{Hom}_R(\mathcal{X}, -)$-exact, $\text{Hom}_R(-, \mathcal{X})$-exact) if $Y \otimes_R X$ (resp., $\text{Hom}_R(X, X)$, $\text{Hom}_R(X, X)$) is an exact complex for every $Y \in \mathcal{Y}$ (resp., $X \in \mathcal{X}$).
A class $\mathcal{X}$ of $R$-modules is projectively resolving if the class $\mathcal{X}$ is closed under extensions, kernels of epimorphisms, and $\mathcal{P}(R) \subseteq \mathcal{X}$. The class $\mathcal{X}$ is left thick if the class $\mathcal{X}$ is closed under extensions, kernels of epimorphisms, and direct summands. Injectively coresolving and right thick classes are defined dually. $\mathcal{X}$ is thick if it is left and right thick.

**Cotorsion pairs.** Given a class of $R$-modules $\mathcal{X}$ and an integer $n \geq 1$, we use the following standard notations:

$$\mathcal{X}^\perp_n = \{ M \in R\text{-Mod} | \text{Ext}^n_R(X, M) = 0, \forall X \in \mathcal{X}, \forall i = 1, \ldots, n \}.$$ 

$$\perp \mathcal{X}^n = \{ M \in R\text{-Mod} | \text{Ext}^n_R(M, X) = 0, \forall X \in \mathcal{X}, \forall i = 1, \ldots, n \}.$$ 

In particular, we set $\mathcal{X}^\perp = \mathcal{X}^{\perp 1}$, $\perp \mathcal{X} = \mathcal{X}^{\perp 1}$, $\mathcal{X}^{\perp \infty} = \cap_{n \geq 1} \mathcal{X}^n$ and $\perp \mathcal{X}^{\infty} = \cap_{n \geq 1} \perp \mathcal{X}^n$.

Given a class of $R$-modules $\mathcal{X}$, an $\mathcal{X}$-precover of a module $M$ is a morphism $f : X \to M$ with $X \in \mathcal{X}$, in such a way that $f_* : \text{Hom}_R(X', X) \to \text{Hom}_R(X', M)$ is surjective for every $X' \in \mathcal{X}$. The $\mathcal{X}$-precover $f$ is said to be special if it is an epimorphism and $\text{Ker}(f) \subseteq \mathcal{X}^\perp$, and it is said to be an $\mathcal{X}$-cover provided that every endomorphism $g : X \to X$ such that $fg = f$ is an automorphism of $X$. If every module has a $\mathcal{X}$-precover, a special $\mathcal{X}$-precover or a $\mathcal{X}$-cover, then the class $\mathcal{X}$ is said to be precovering, special precovering or covering, respectively. Special $\mathcal{X}$-preenvelopes, preenvelopes and envelopes can be defined dually.

A pair $(A, B)$ of classes of modules is called a cotorsion pair if $A^\perp = B$ and $A = \perp B$. If $(A, B)$ is a cotorsion pair, then the class $A \cap B$ is called the core of the cotorsion pair. A cotorsion pair $(A, B)$ is said to be hereditary if $A$ is projectively resolving, or equivalently, if $B$ is injectively coresolving. A cotorsion pair $(A, B)$ is said to be complete if any $R$-module has a special $B$-preenvelope, or equivalently, any $R$-module has a special $A$-precover. A cotorsion pair $(A, B)$ is said to be perfect if every module has an $A$-cover and a $B$-envelope. It is clear by its definition that a perfect cotorsion pair $(A, B)$ is complete. The converse holds when $A$ is closed under direct limits [26 Corollary 2.3.7]. A cotorsion pair $(A, B)$ is said to be cogenerated by a set if there is a set $\mathcal{X}$ such that $B = \mathcal{X}^\perp$. It is a well known fact that any cotorsion pair cogenerated by a set is complete (see [13] Theorem 7.4.1 for example).

The following two lemmas will be used in several places in the paper. The second lemma is known as Wakamatsu’s Lemma (see [13] Lemmas 2.1.1 and 2.1.2]).

**Lemma 2.1** Let $\mathcal{X}$ be a set of $R$-modules.

1. $\mathcal{X}^{\perp \infty} = M^\perp$ for some $R$-module $M$.

2. $\perp \mathcal{X}^{\infty} = \perp M$ for some $R$-module $M$.

**Proof.** The proof of (2) is dual to that of (1), so we only prove (1).

Let $X$ be the direct sum of all the modules in $\mathcal{X}$, consider any projective resolution of $X$,

$$\cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \to 0,$$

and let $K_{i+1} = \text{Ker}(f_i)$. Clearly, for any $i \geq 1$ and any $R$-module $A$, we have $\text{Ext}^i_R(K_i, A) \cong \text{Ext}^{i+1}_R(X, A)$. If we let $M = X \oplus (\oplus_{i \geq 1} K_i)$, then $\text{Ext}^i_R(M, A) \cong \text{Ext}^i_R(X, A) \oplus (\prod_{i \geq 1} \text{Ext}^i_R(K_i, A)) \cong \prod_{i \geq 1} \text{Ext}^i_R(X, A)$.

**Lemma 2.2** ([13], Lemmas 2.1.1 and 2.1.2]) Assume that $\mathcal{X} \subseteq R\text{-Mod}$ is closed under extensions.
1. If \( \varphi : X \to M \) is an epic \( X \)-cover of \( M \), then \( \varphi \) is a special \( X \)-precover.

2. If \( \varphi : M \to X \) is a monic \( X \)-envelope of \( M \), then \( \varphi \) is a special \( X \)-preenvelope.

Consequently, every covering class \( X \subseteq R\text{-Mod} \) closed under extensions and such that \( \mathcal{P}(R) \subseteq X \) is special precovering. Dually, every enveloping class \( X \subseteq R\text{-Mod} \) closed under extensions and such that \( \mathcal{I}(R) \subseteq X \) is special preenveloping.

**Abelian model structures.** Given a bicomplete category \( A \), a model structure on \( A \) is given by three classes of morphisms of \( A \) called cofibrations, fibrations and weak equivalences, satisfying a set of axioms (see [32, Definition 1.1.3.]). If \( M \) is a model structure on \( A \), then it provides a general framework to study homotopy theory. By this we mean that we can associate to \( M \) the homotopy category, \( \text{Ho} A(M) \), which is defined by formally inverting the weak equivalences of \( M \). More precisely, \( \text{Ho} A(M) \) is obtained after localizing \( A \) at the class of weak equivalences (see [32, Sec. 1.2] for more details).

Hovey restricted the study of model structures to abelian categories. He defined in [33] the notion of an abelian model structure and showed in [33, Theorem 2.2] that there is a close link between some model structures on an abelian category and the cotorsion pairs in this category. Namely, an abelian model structure on any abelian category \( A \), is equivalent to a triple \( M = (Q, W, R) \) of classes of objects in \( C \) such that \( W \) is thick and \( (Q, W \cap R) \) and \( (Q \cap W, R) \) are complete cotorsion pairs. In this case, \( Q \) is precisely the class of cofibrant objects, \( R \) is precisely the class of fibrant objects, and \( W \) is the class of trivial objects of the model structure. Here, an abelian model structure in the sense of [32, Definition 1.1.3] is one in which: (a) a morphism is a (trivial) cofibration if and only if it is a monomorphism with (trivially) cofibrant cokernel; and (b) a morphism is a (trivial) fibration if and only if it is an epimorphism with (trivially) fibrant kernel.

Hovey’s correspondence shows that we can shift all our attention from morphisms (cofibrations, weak equivalences, and fibrations) to objects (cofibrant, trivial, and fibrant). In this paper, we identify any abelian model structure with such a triple and we often call it a Hovey triple.

A Hovey triple is hereditary if the two associated cotorsion pairs are hereditary. In general, it is difficult to prove that a particular category has a model structure. However, Gillespie [22] gave a new and useful method for constructing hereditary abelian model structures as follows.

**Theorem 2.3 ([22], Theorem 1.1)** Given two complete and hereditary cotorsion pairs \( (\tilde{Q}, \tilde{R}) \) and \( (Q, R) \) in an abelian category \( A \) such that \( \tilde{R} \subseteq R \) (or equivalently, \( \tilde{Q} \subseteq Q \)) and \( \tilde{Q} \cap \tilde{R} = Q \cap R \), there is a unique thick class \( W \) such that \( (Q, W, R) \) is a Hovey triple. Moreover, the class \( W \) is characterized by:

\[
W = \{ X \in A | \exists \text{ an exact sequence } 0 \to X \to R' \to Q' \to 0 \text{ with } R' \in \tilde{R}, Q' \in \tilde{Q} \}
\]

\[
= \{ X \in A | \exists \text{ an exact sequence } 0 \to R' \to Q' \to X \to 0 \text{ with } R' \in \tilde{R}, Q' \in \tilde{Q} \}.
\]

**Weak AB-contexts.** Given a full subcategory \( X \) of an abelian category \( A \), we denote by \( \text{res}(X) \) the full subcategory of objects in \( A \) having a finite \( X \)-resolution. Dually, we denote by \( \text{cores}(X) \) the full subcategory of objects in \( A \) having a finite \( X \)-coresolution.

Following [3], a triple \( (X, Y, \omega) \) of full subcategories of \( A \) is said to be a left weak Auslander-Buchweitz context (or a left weak AB-context for short) if the following three conditions are satisfied:
(AB1) $\mathcal{X}$ is left thick.
(AB2) $\mathcal{Y}$ is right thick and $\mathcal{Y} \subseteq \text{res}(\widehat{\mathcal{X}})$.
(AB3) $\omega = \mathcal{X} \cap \mathcal{Y}$ and $\omega$ is an injective cogenerator for $\mathcal{X}$, that is, $\omega \subseteq \mathcal{X} \cap \mathcal{X}^\perp_{\infty}$ and for any $X \in \mathcal{X}$ there exists an exact sequence $0 \to X \to W \to X' \to 0$ such that $X' \in \mathcal{X}$ and $W \in \omega$.

Right weak AB-contexts can be defined dually, and these were considered in [19] under the name weak co-AB-contexts.

Becerril, Mendoza, Pérez and Santiago introduced in [4] the notion of left Frobenius pairs and showed that there is a one-to-one correspondence between the class of left weak AB-contexts and that of Frobenius pairs. Recall that a pair $(\mathcal{X}, w)$ of full subcategories of $\mathcal{A}$ is a left Frobenius pair if $\mathcal{X}$ is left thick, and $w$ is closed under direct summands and an injective cogenerator for $\mathcal{X}$. Right Frobenius pairs $(\omega, \mathcal{Y})$ can be defined dually.

So, looking for left weak AB-contexts is the same as looking for left Frobenius pairs. The following two lemmas are the key to find our left and right weak AB-contexts in this paper.

**Lemma 2.4** The following assertions hold:

1. ([4, Theorem 5.4(1)]) If $(\mathcal{X}, \omega)$ is a left Frobenius pair in $\mathcal{A}$, then $(\mathcal{X}, \text{res}(\widehat{\omega}), \omega)$ is a left weak AB-context. Conversely, if $(\mathcal{X}, \mathcal{Y}, \omega)$ is a left weak AB-context in $\mathcal{A}$, then $(\mathcal{X}, \omega)$ is a left Frobenius pair.

2. ([37, Proposition 2.5]) If $(\mathcal{X}, \mathcal{Y})$ is a complete hereditary cotorsion pair in $\mathcal{A}$, then $(\mathcal{X}, \mathcal{X} \cap \mathcal{Y})$ is a left Frobenius pair.

In this paper we will also be interested in right weak AB-contexts. A dual result of Lemma 2.4 is therefore needed. For the reader’s convenience we state it here. It is not mentioned in either [4] or [37]. However, its proof is simply a dual to that of [4, Theorem 5.4(1)] and [37, Proposition 2.5].

**Lemma 2.5** The following statements hold:

1. If $(\omega, \mathcal{Y})$ is a right Frobenius pair in $\mathcal{A}$, then $(\omega, \text{cores}(\widehat{\omega}), \mathcal{Y})$ is a right weak AB-context. Conversely, if $(\omega, \mathcal{X}, \mathcal{Y})$ is a right weak AB-context in $\mathcal{A}$, then $(\omega, \mathcal{Y})$ is a right Frobenius pair.

2. If $(\mathcal{X}, \mathcal{Y})$ is a complete hereditary cotorsion pair in $\mathcal{A}$, then $(\mathcal{X} \cap \mathcal{Y}, \mathcal{Y})$ is a right Frobenius pair.

**Relative Gorenstein flat modules.** We recall the concept of $G_{\mathcal{C}}$-flat $R$-modules and all related notions.

**Definition 2.6** ([9]) Let $C$ be an $R$-module. An $R$-module $M$ is said to be $F_{\mathcal{C}}$-flat (resp., $I_{\mathcal{C}}$-injective) if $M^+ \in \text{Prod}_R(C^+)$ (resp., $M \in \text{Prod}_R(C)$).

We denote the class of all $F_{\mathcal{C}}$-flat modules as $F_{\mathcal{C}}(R)$ and that of all $I_{\mathcal{C}}$-injective modules as $I_{\mathcal{C}}(R)$.

**Remark 2.7** Note that an $R$-module $M$ is $F_{\mathcal{C}}$-flat if and only if $M^+$ is an $I_{\mathcal{C}^+}$-injective right $R$-module, and that the class of $F_{\mathcal{C}}$-flat modules is clearly closed under direct sums and direct summands.

**Examples 2.8** Here are some examples of $F_{\mathcal{C}}$-flat and $I_{\mathcal{C}^+}$-injective modules:
1. If $C$ is a flat generator of $R\text{-Mod}$, then
\[ F_C(R) = F(R) \text{ and } \text{Prod}_R(C^+) = I(R). \]

2. ([9 Proposition 3.3]) If $RC$ is finitely presented then
\[ F_C(R) = C \otimes_S F(S) \text{ and } \text{Prod}_R(C^+) = \text{Hom}_S(C, I(S)). \]

**Definition 2.9** ([31], Definition 2.1) An $(R, S)$-bimodule $C$ is semidualizing if the following conditions hold:
1. $RC$ and $CS$ both admit a degree-wise finite projective resolution.
2. $\text{Ext}^{\geq 1}_R(C, C) = \text{Ext}^{\geq 1}_S(C, C) = 0$.
3. The natural homothety maps $R \to \text{Hom}_S(C, C)$ and $S \to \text{Hom}_R(C, C)$ both are ring isomorphisms.

**Remark 2.10** Without any assumption on $C$ we always have that every $C$-flat module (resp., $C$-injective right $R$-module), in the sense of [31, Definition 5.1], is $F_C$-flat (resp., $I_C$-injective). However, when $C$ is semidualizing the classes of $F_C$-flat and $I_C$-injective modules coincide with those of $C$-flat and $C$-injective modules, respectively.

**Definition 2.11** ([10]) An $R$-module $C$ is said to be $w$-tilting if it satisfies the following two properties:
1. $C$ is $\Sigma$-self-orthogonal, that is, $\text{Ext}^{\geq 1}_R(C, C(I)) = 0$ for every set $I$.
2. There exists a $\text{Hom}_R(-, \text{Add}_R(C))$-exact $\text{Add}_R(C)$-coresolution $X : 0 \to R \to C_{-1} \to C_{-2} \to \cdots$

Dually, we define $\prod$-self-orthogonal and $w$-cotilting modules.

The notion of a $w^+$-tilting module was recently introduced in [9], where the authors studied Gorenstein flat modules with respect to a non-necessary semidualizing bimodule. It properly generalizes the concepts of semidualizing, Wakamatsu tilting and $w$-tilting modules (see [9 Proposition 4.3] and [10 Example 2.2]).

**Definition 2.12** ([9], Definition 4.1) An $R$-module $C$ is said to be $w^+$-tilting if it satisfies the following two properties:
1. $C$ is $\prod$-Tor-orthogonal, that is, $\text{Tor}^{\geq 1}_R((C^+)^I, C) = 0$ for every set $I$.
2. There exists a $(\text{Prod}_R(C^+) \otimes -)$-exact $F_C(R)$-coresolution $X : 0 \to R \to C_{-1} \to C_{-2} \to \cdots$

**Definition 2.13** ([9], Definition 4.4) Let $C$ be an $R$-module. An $R$-module $M$ is said to be $G_C$-flat if there exists an exact and $(\text{Prod}_R(C^+) \otimes_R -)$-exact sequence $X : \cdots \to F_1 \to F_0 \to C_{-1} \to C_{-2} \to \cdots$

with each $C_i \in F_C(R)$ and $F_j \in F(R)$, such that $M \cong \text{Im}(F_0 \to C_{-1})$. The $(\text{Prod}_R(C^+) \otimes_R -)$-exact exact sequence $X$ is called a complete $F_C$-flat resolution of $M$.

The class of all $G_C$-flat $R$-modules is denoted by $G_C F(R)$. 

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Recall [10] that an $R$-module $M$ is said to be $G_C$-projective if there exists an $\text{Hom}_R(\cdot, \text{Add}_R(C))$-exact exact sequence $\cdots \to P_i \to P_0 \to C_{-1} \to C_{-2} \to \cdots$, where $P_i \in \mathcal{P}(R)$, $C_j \in \text{Add}_R(C)$, and such that $M \cong \text{Im}(P_0 \to C_{-1})$.

It is still an open question whether or not (or at least when) every $G_C$-projective module is $G_C$-flat (even when $C = R$). The following result will be used later. It gives a partial answer to this question.

**Proposition 2.14** Assume that $RC$ has a degreewise finite projective resolution. The following statements hold:

1. $RC$ is $\prod$-$\text{Tor}$-orthogonal if and only if it is $\Sigma$-$\text{self-orthogonal}$.

2. Assume that $RC$ is $w$-$\text{tilting}$. Then, every $G_C$-projective $R$-module which has a degreewise finite projective resolution is $G_C$-flat. In particular, $RC$ is $w^+$-$\text{tilting}$.

**Proof.**

1. Since $RC$ has a degreewise finite projective resolution, this assertion follows from the isomorphism $\text{Tor}^R((C^+)^I, C) \cong \text{Ext}^R_C(C, C(I))^+$ for every set $I$ and $i \geq 1$, by [15] Theorem 3.2.13 and Remark 3.2.27.

2. Let $M$ be a $G_C$-projective $R$-module. If an exact sequence $\prod X : \cdots \to P_1 \to P_0 \to C_{-1} \to C_{-2} \to \cdots$, where the $P_i$s are all finitely generated and projective, each $C_j \in \text{add}_R(C)$, and $M \cong \text{Im}(P_0 \to C_{-1})$.

Now using [15] Theorems 3.2.11 and 3.2.22, we get isomorphisms of complexes

$$(C^+)^I \otimes_R X \cong (C^+ \otimes_R X)^I \cong (\text{Hom}_R(X, C^+))^I$$

for every set $I$. But $\text{Hom}_R(X, C)$ is exact so is $X \otimes_R X$ is exact for every $X \in \text{Prod}_R(C^+)$. Hence, $M$ is $G_C$-flat.

Finally, to see that $C$ is $w^+$-$\text{tilting}$ note that $C$ being $w$-$\text{tilting}$ means that $C$ and $R$ are $G_C$-projective. But since both $R$ and $C$ have a degreewise finite projective resolution, we get that they are also $G_C$-flat $R$-modules. Hence, $C$ is $w^+$-$\text{tilting}$.

Certain properties of $G_C$-flat modules depend on the fact that the class $G_C\text{F}(R)$ is closed under extensions. But, unlike the class of $G_C$-projective and $G_C$-injective modules, it is unknown for what modules $C$ the class of $G_C$-flat modules is closed under extensions (see [9] Question 4). If we assume that $RC$ is $\prod$-$\text{Tor}$-orthogonal, then there are some situations under which this closure property holds:

(a) When $C$ is a flat generator of $R\text{-Mod}$. This situation follows from the equalities $\mathcal{F}_C(R) = \mathcal{F}(R)$ and $G_C\text{F}(R) = G\text{F}(R)$ by [9] Proposition 4.23 and the work of Šaroch and Štovíček [41] Theorem 4.11.

(b) When every $R$-module has finite $\mathcal{F}_C$-flat dimension (see [9] Corollary 7.5).

(c) When $\mathcal{F}_C(R)$ is closed under direct products ([9] Corollary 4.13]). In particular, when $S$ is right coherent and both $RC$ and $C_S$ are finitely presented (see Lemma 4.7).
This discussion naturally leads to the following question:

**Question:** Is the class of $G_C$-flat $R$-modules closed under extensions for any $\prod$-Tor-orthogonal $R$-module $C$?

Throughout, the following definition is needed.

**Definition 2.15 ([9], Definition 4.14)** Let $C$ be an $R$-module. A ring $R$ is said to be $G_C$-closed provided that the class of $G_C$-flat $R$-modules is closed under extensions.

**Foxby classes.** Associated to the bimodule $RC_S$ we have the Auslander and Bass classes, $A_C(S)$ and $B_C(R)$, respectively, defined as follows:

- $A_C(S)$ is the class of all $S$-modules $M$ satisfying:
  \[ \operatorname{Tor}_{\geq 1}^S(C, M) = 0 \]
  and the canonical map
  \[ \mu_M : M \to \operatorname{Hom}_R(C, C \otimes SM) \]
is an isomorphism of left $S$-modules.

- $B_C(R)$ consists of all $R$-modules $N$ satisfying:
  \[ \operatorname{Ext}_{\geq 1}^R(C, N) = \operatorname{Tor}_{\geq 1}^S(C, \operatorname{Hom}_R(C, N)) = 0 \]
  and the canonical map
  \[ \nu_N : C \otimes S \operatorname{Hom}_R(C, N) \to N \]
is an isomorphism of $R$-modules.

On the other hand, one can define the classes $A_C(R)$ and $B_C(S)$ of right $R$-modules and right $S$-modules, respectively.

We will refer to the modules in $A_C$ and $B_C$ as $A_C$-Auslander and $B_C$-Bass modules, respectively.

It is an important property of Auslander and Bass classes that they are equivalent under the following pair of functors ([20, Proposition 2.1]):

\[
\begin{align*}
\xymatrix{ B_C(R) & A_C(S) & A_C(R) & B_C(S) \\
\circ S & \operatorname{Hom}_R(C, -) & \operatorname{Hom}_{RS}(C, -) & \circ R C \ar[u] \end{align*}
\]

Consequently, Bass classes can be defined via Auslander classes and vice-versa:

\[ B_C(R) = C \otimes S A_C(S) \text{ and } A_C(R) = \operatorname{Hom}_{S}(C, B_C(S)). \]

Recall that an $R$-module $M$ is called self-small if the canonical morphism

\[ \operatorname{Hom}_R(M, M^{(I)}) \to \operatorname{Hom}_R(M, M)^{(I)} \]
is an isomorphism, for every set $I$. Examples of self-small modules are finitely generated modules. Note that the module $RC$ is self-small if and only if, for every set $I$, the canonical map $\mu_{S(I)} : S(I) \to \operatorname{Hom}_R(C, C \otimes S S^{(I)})$ is an isomorphism.

Foxby classes are expected to satisfy certain properties under dual assumptions. Inspired by this duality, we propose the dual notion to that of self-small.

**Definition 2.16** An $R$-module $RM$ is said to be self-co-small, if the canonical morphisms

\[ (M^+)^I \otimes R M \to (M^+ \otimes_R M)^I \text{ and } M^+ \otimes_R M \to \operatorname{Hom}_R(M, M)^+ \]

are isomorphisms for every set $I$. 


2 PRELIMINARIES

Remark 2.17

1. The module \( R \mathcal{C} \) is self-co-small if and only if the canonical morphism

\[
\nu_{(S^+)^I} : \text{Hom}_S(C, (S^+)^I) \otimes_R C \to (S^+)^I
\]

is an isomorphism for every set \( I \).

2. Any finitely presented \( R \)-module is self-co-small by [13, Theorems 3.2.11 and 3.2.22].

With respect to the terminology used in this paper, modules in the class \( \text{Add}_R(C) \) will be called \( \mathcal{P}_C \)-projective. By [6, Proposition 3.1], when \( C \) is self-small, the class of \( \mathcal{P}_C \)-projective modules coincides with that of \( C \)-projective modules, that is, modules in the class \( C \otimes_S \mathcal{P}(S) \) (see [31]).

It is straightforward to prove the following.

Lemma 2.18 When \( R \mathcal{C} \) is self-small the pair of functors

\[
\begin{align*}
\text{Add}_R(C) & \rightarrow \mathcal{P}(S) \\
\text{Hom}_R(C, -) & \rightarrow \mathcal{P}(S)
\end{align*}
\]

provides an equivalence of categories, and if \( R \mathcal{C} \) is self-co-small then

\[
\begin{align*}
\text{Prod}_R(C^+) & \rightarrow \mathcal{I}(S) \\
\text{Hom}_S(C, -) & \rightarrow \mathcal{I}(S)
\end{align*}
\]

is an equivalence of categories.

Corollary 2.19 The following assertions hold:

1. ([6 Proposition 3.1]) If \( C \) is self-small, then \( \text{Add}_R(C) = C \otimes_S \mathcal{P}(S) \).

2. If \( R \mathcal{C} \) is self-co-small, then \( \text{Prod}_R(C^+) = \text{Hom}_S(C, \mathcal{I}(S)) \).

Lemma 2.20 Let \( C \) be an \( R \)-module. The following assertions hold:

1. \( \mathcal{P}(S) \subseteq \mathcal{A}_C(S) \) if and only if \( R \mathcal{C} \) is \( \Sigma \)-self-orthogonal and self-small. In this case \( \text{Add}_R(C) \subseteq \mathcal{B}_C(R) \).

2. \( \mathcal{I}(S) \subseteq \mathcal{B}_C(S) \) if and only if \( R \mathcal{C} \) is \( \prod \)-Tor-orthogonal and self-cosmall. In this case \( \text{Prod}_R(C^+) \subseteq \mathcal{A}_C(R) \).

3. If \( R \mathcal{C} \) has a degreewise finite projective resolution, then \( \mathcal{F}(S) \subseteq \mathcal{A}_C(S) \) if and only if \( R \mathcal{C} \) is \( \prod \)-Tor-orthogonal. In this case \( \mathcal{F}_C(R) \subseteq \mathcal{B}_C(R) \).

Proof. 1. Follows by [10 Proposition 5.4(1)] and the fact \( \text{Add}_R(C) = C \otimes_S \mathcal{P}(S) \).

2. By the dual argument to that of [10 Proposition 5.4(1)] and the item 1.

3. Using Proposition 2.14, the "if" part follows by [6, Proposition 5.2] and the "only if" part follows by item 1 since \( \mathcal{P}(S) \subseteq \mathcal{F}(S) \).
3 RELATIVE COTORSION MODULES

Remark 2.21 1. By Lemma 2.20, the adjoint pair \((C \otimes - , \text{Hom}_R(-))\) is left semiorthogonal (in the sense of [20, Definition 2.1]) if and only if \(\mu C\) is \(\Sigma\)-self-small.

2. When \(C\) is considered as a right \(S\)-module, there is a version for each definition and result presented in this paper. For example, if \(R = \text{End}_S(C)\), then we have the following equalities \(\text{Add}_S(C) = \bigoplus R \otimes_R \mu C\) and \(\text{Prod}_S(C^+) = \text{Hom}_R(C, \mathcal{I}(R))\) when \(C_S\) is self-small and self-co-small, respectively.

Lemma 2.22 Let \(\mu C\) be self-co-small and \(\prod\)-Tor-orthogonal.

1. Any right \(R\)-module in \(A_{\mu C}(R)\) has a monic \(I_{\mu C}\)-injective preenvelope with cokernel in \(A_{\mu C}(R)\).

2. Assume that \(R\) has a degreewise finite projective resolution. Any \(R\)-module in \(B_{\mu C}(R)\) has an epic \(F_{\mu C}\)-flat cover with kernel in \(B_{\mu C}(R)\).

Proof. 1. By Lemma 2.20(2) we have \(\text{Prod}_R(C^+) \subseteq A_{\mu C}(R)\), so dual arguments to those of [6, Proposition 3.8], following similar steps to those of [6, Proposition 3.9], proves the statement.

2. Let \(M \in B_{\mu C}(R)\). Then, there exists by [9, Proposition 3.7(1)] an \(F_{\mu C}\)-flat cover \(\gamma : L \to M\) which is epic by [6, Proposition 3.8]. It remains to show that \(\text{Ker}(\gamma)\) is in \(B_{\mu C}(R)\). But this can be shown exactly as in [6, Proposition 3.8], using the fact that \(F_{\mu C}(R) \subseteq B_{\mu C}(R)\) by Lemma 2.20(3).

3 Relative cotorsion modules

In this section we introduce some classes of relative cotorsion modules. Besides their links with other known classes of modules (cotorsion, flat, \(F_{\mu C}\)-flat, etc.), we are interested in discovering the main homological properties of these new classes.

Recall that a module \(M\) is cotorsion if \(\text{Ext}_1^R(F, M) = 0\) for every flat module \(F\), equivalently, if \(\text{Ext}_i^R(F, M) = 0\) for every flat module \(F\) and every \(i \geq 1\) ([15, Definition 5.3.22]). In the following definition we extend the concept of cotorsion modules to our relative setting.

Definition 3.1 Given an \(R\)-module \(C\) and an integer \(n \geq 1\), an \(R\)-module \(M\) is said to be:

- \(n\)-\(C\)-cotorsion if \(\text{Ext}_i^R(N, M) = 0\) for all \(F_{\mu C}\)-flat modules \(N\) and all integer numbers \(i\) such that \(1 \leq i \leq n\).
- \(C\)-cotorsion if it is \(1\)-\(C\)-cotorsion.
- Strongly \(C\)-cotorsion if it is \(n\)-\(C\)-cotorsion for every \(n \geq 1\).

The classes of all \(C\)-cotorsion modules, of all \(n\)-\(C\)-cotorsion modules and of all strongly \(C\)-cotorsion modules will be denoted as \(C_{\mu C}(R)\), \(C_{\mu C}^n(R)\) and \(SC_{\mu C}(R)\), respectively.

Remarks 3.2 1. When \(R\) is a commutative noetherian ring and \(\mu C\) is a semidualizing \(R\)-module, strongly \(C\)-cotorsion modules coincide with the \(C\)-cotorsion modules defined in [12] and the strongly \(C\)-cotorsion modules defined in [13].
2. Given an integer \( n \geq 1 \), every \((n+1)\)-\(C\)-cotorsion \( R\)-module is \(n\)-\(C\)-cotorsion. Then, we have the following ascending sequence:

\[
\mathcal{S}C(R) \subseteq \cdots \subseteq C^{n+1}_C(R) \subseteq C^n_C(R) \subseteq \cdots \subseteq C_C(R),
\]

where \( \mathcal{S}C(R) \) can be written as \( \mathcal{S}C(R) = \cap_{n \geq 1} C^n_C(R) \).

Examples 3.3
1. Denote by \( C(R) \) the class of all cotorsion \( R\)-modules. If \( R \) is a flat generator, then \( C(R) = C_C(R) = \mathcal{S}C(R) \).

2. Every injective module is (strongly) \( C\)-cotorsion.

3. Assume that \( R \) is \( \prod\)-Tor-orthogonal. Given any \( \mathcal{F}_C\)-cover \( \varphi : F \rightarrow M \) (which exists by [9, Proposition 3.7]), \( \ker(\varphi) \) is \( C\)-cotorsion ([43, Proposition 2.2.7]).

4. Recall that a module \( M \) is called Gorenstein cotorsion if \( \text{Ext}^{1}_{\mathcal{R}}(G, M) = 0 \) for every Gorenstein flat module \( G \). If \( R \) is flat, then \( \mathcal{F}_C(R) \subseteq \mathcal{F}(R) \subseteq \mathcal{G}_F(R) \).

Hence, both cotorsion and Gorenstein cotorsion modules are \( C\)-cotorsion.

5. Assume that every \( \mathcal{F}_C\)-flat \( R\)-module has finite injective dimension. Then, \( \mathcal{I}(R)^{1 \infty} \subseteq \mathcal{S}C(R) \). In particular, every Gorenstein injective \( R\)-module is (strongly) \( C\)-cotorsion.

It is unknown whether or not the class of \( C\)-cotorsion \( R\)-modules is closed under cokernels of monomorphisms. However, when this happens, all the classes of relative cotorsion \( R\)-modules introduced above coincide.

Proposition 3.4 The following statements hold for any \( R\)-module \( C \).

1. The class \( \mathcal{S}C(R) \) is closed under cokernels of monomorphisms.

2. Let \( n \geq 1 \) be any integer number. The following assertions are equivalent:
   (a) The class \( C^n_C(R) \) is closed under cokernels of monomorphisms.
   (b) For every integer number \( k \geq n \), every \( k\)-\(C\)-cotorsion module is \((k+1)\)-\(C\)-cotorsion.
   (c) For every integer number \( k \geq n \), the class \( C^k_C(R) \) is closed under cokernels of monomorphisms.
   (d) Every \( n\)-\(C\)-cotorsion module is strongly \( C\)-cotorsion.

In this case, \( \mathcal{S}C(R) = C^k(R) \) for every \( k \geq n \).

Proof. (1) Straightforward.
(2) \( (a) \Rightarrow (b) \) Let \( X \) be \( n\)-\(C\)-cotorsion and consider a short exact sequence of \( R\)-modules

\[
0 \rightarrow X \rightarrow I \rightarrow L \rightarrow 0
\]

with \( I \) injective. By hypothesis, \( L \) is \( n\)-\(C\)-cotorsion, so the exact sequence

\[
0 = \text{Ext}^n_R(F, L) \rightarrow \text{Ext}^{n+1}_R(F, X) \rightarrow \text{Ext}^{n+1}_R(F, I) = 0
\]

shows that \( \text{Ext}^{n+1}_R(F, X) = 0 \) for every \( F \in \mathcal{F}_C(R) \), that is \( X \) is \((n+1)\)-\(C\)-cotorsion.

But then \( L \) is also \((n+1)\)-\(C\)-cotorsion, and repeating the argument we see that \( X \) is indeed \( k\)-\(C\)-cotorsion for every \( k \geq n \).
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(b) ⇒ (c) Let 0 → K → M → N → 0 be any exact sequence with K, M ∈ \( C^e_c(R) \), choose any \( F \in \mathcal{F}_C(R) \) and consider the induced exact sequence
\[
\text{Ext}^k(F, M) \to \text{Ext}^k(F, N) \to \text{Ext}^{k+1}(F, K).
\]
If \( k < n \) then \( \text{Ext}^k(F, M) = \text{Ext}^{k+1}(F, K) = 0 \) and then \( \text{Ext}^{k+1}(F, N) = 0 \).
If \( k \geq n \) then by repeatedly applying (2) we get that \( M \in \mathcal{C}^e_c(R) \) and \( K \in \mathcal{C}^{e+1}_c(R) \), so again \( \text{Ext}^k(F, M) = \text{Ext}^{k+1}(F, K) = 0 \) and then \( \text{Ext}^{k+1}(F, N) = 0 \).
Thus, in any case \( N \in \mathcal{C}^{e+1}_c(R) \).

Now, (c) ⇒ (a) and (b) ⇒ (d) are clear.

In light of Examples 2.6, it is natural to wonder whether there is a relation between \( C_c \)-cotorsion \( R \)-modules and cotorsion \( S \)-modules. The following result gives a useful relation.

**Proposition 3.5** Assume that \( C \) is a finitely presented \( R \)-module and let \( n \geq 1 \), be an integer. An \( R \)-module \( M \) is \( n \)-\( C_c \)-cotorsion if and only if \( M \in \mathcal{C}^{1+n}_c \) and \( \text{Hom}_R(C, M) \) is a cotorsion \( S \)-module. Consequently, \( \mathcal{B}_C(R) \cap \mathcal{C}^n_c(R) = C \otimes_S (\mathcal{A}_C(S) \cap \mathcal{C}(S)) \).

**Proof.** (⇒) Assume that \( M \) is \( n \)-\( C_c \)-cotorsion. Clearly \( M \in \mathcal{C}^{1+n} \), since \( R \) is \( \mathcal{F}_C \)-flat.

We prove now that \( \text{Hom}_R(C, M) \) is a cotorsion \( S \)-module.

Let \( F \) be any flat \( S \)-module and consider an exact sequence of \( S \)-modules
\[
0 \to K \to P \to F \to 0
\]
with \( P \) projective. Since the last sequence is pure we have that \( K \) is flat and that the sequence
\[
0 \to C \otimes_S K \to C \otimes_S P \to C \otimes_S F \to 0
\]
is exact. Then, (Example 2.6) \( C \otimes_S F \) is \( \mathcal{F}_C \)-flat, so \( \text{Ext}^1(C \otimes_S F, M) = 0 \).

Thus, we have the following commutative diagram with exact rows:
\[
\begin{array}{ccc}
\text{Hom}_R(C \otimes_S P, M) & \to & \text{Hom}_R(C \otimes_S K, M) \\
\downarrow{\cong} & & \downarrow{\cong} \\
\text{Hom}_S(P, \text{Hom}_R(C, M)) & \to & \text{Hom}_S(K, \text{Hom}_R(C, M)) & \to & \text{Ext}^1_S(F, \text{Hom}_R(C, M)) & \to & 0
\end{array}
\]
Therefore, \( \text{Ext}^1_S(F, \text{Hom}_R(C, M)) = 0 \) and hence \( \text{Hom}_R(C, M) \) is cotorsion.

(⇐) Consider an exact sequence of \( R \)-modules
\[
0 \to M \to I \to L \to 0,
\]
where \( I \) is injective. Since \( \text{Ext}^1_R(C, M) = 0 \), the induced sequence
\[
0 \to \text{Hom}_R(C, M) \to \text{Hom}_R(C, I) \to \text{Hom}_R(C, L) \to 0
\]
is exact.

Let \( C \otimes_S F \) be any \( \mathcal{F}_C \)-flat \( R \)-module and let’s proceed by induction on \( n \).

For \( n = 1 \): By the the implication (⇒), \( \text{Hom}_R(C, I) \) is cotorsion since \( I \) is \( C_c \)-cotorsion. Then, \( \text{Ext}^1_R(F, \text{Hom}_R(C, I)) = 0 \). Also, we have by hypothesis that \( \text{Ext}^1_R(F, \text{Hom}_R(C, M)) = 0 \), so the commutative diagram
\[
\begin{array}{ccc}
\text{Hom}_R(C \otimes_S F, I) & \to & \text{Hom}_R(C \otimes_S F, L) \\
\downarrow{\cong} & & \downarrow{\cong} \\
\text{Hom}_S(F, \text{Hom}_R(C, I)) & \to & \text{Hom}_S(F, \text{Hom}_R(C, L)) & \to & 0
\end{array}
\]
has exact rows and then $\text{Ext}^1_R(V, M) = \text{Ext}^1_R(C \otimes_S F, M) = 0$, that is, $M$ is $C_C$-cotorsion.

Assume now that $n > 1$ and that $M \in C_{n+1}$. By induction we know that $M$ is $n$-$C_C$-cotorsion, so we only need to prove that $\text{Ext}^{n+1}_R(C \otimes_S F, M) = 0$.

Since $R$ is injective (and then $(n+1)$-$C_C$-cotorsion), $\text{Hom}_R(C, I)$ is a cotorsion $S$-module and then the exact sequences

$$0 = \text{Ext}^k_R(C, I) \to \text{Ext}^k_R(C, L) \to \text{Ext}^{k+1}_R(C, M) = 0$$

and

$$0 = \text{Ext}^n_R(F, \text{Hom}_R(C, I)) \to \text{Ext}^n_R(F, \text{Hom}_R(C, L)) \to \text{Ext}^{n+1}_R(F, \text{Hom}_R(C, M)) = 0$$

show that $\text{Ext}^n_R(F, \text{Hom}_R(C, L)) = 0 = \text{Ext}^k_R(C, L)$ for every $k = 1, \ldots, n$. Using induction again, we get that $L$ is $n$-$C_C$-cotorsion, so the exact sequence

$$0 = \text{Ext}^n_R(C \otimes_S F, L) \to \text{Ext}^{n+1}_R(C \otimes_S F, M) \to \text{Ext}^{n+1}_R(C \otimes_S F, I) = 0,$$

gives that $\text{Ext}^{n+1}_R(C \otimes_S F, M) = 0$ as desired.

We now prove the equality $B_C(R) \cap C_C^0(R) = C \otimes_S (A_C(S) \cap C(S))$.

Let $M$ be an $R$-module. If $M \in B_C(R) \cap C_C^0(R)$, then $M = C \otimes_S F$ with $S F \in A_C(S)$. Moreover, since $M \in C_C^0(R)$, $F \cong \text{Hom}_S(C, C \otimes_S F) = \text{Hom}_S(C, M)$ is a cotorsion $S$-module by the above equivalence. Hence, $M = C \otimes_S F \in C \otimes_S (A_C(S) \cap C(S))$.

For the other inclusion, assume that $M = C \otimes_S F$ with $S F \in A_C(S) \cap C(S)$. Clearly, $M$ is in $B_C(R)$. On the other hand, we have that $\text{Hom}_R(C, M) \cong F$ is a cotorsion $S$-module and $\text{Ext}^i(C, M) = 0$ for every $i \geq 1$, since $M \in B_C(R)$. Using the equivalence proved above, we get that $M \in C_C^0(R)$.

\textbf{Corollary 3.6} Assume that $R C$ is finitely presented. An $R$-module $M$ is strongly $C_C$-cotorsion if and only if $M \in C_{\infty}$ and $\text{Hom}_R(C, M)$ is a cotorsion $S$-module. Consequently, $B_C(R) \cap SC_C(R) = C \otimes_S (A_C(S) \cap C(S)) = B_C(R) \cap C_S(R)$.

Given a regular cardinal number $\kappa$, following [16], Definition 2.1 and [36], Definition 3.6], a class $A$ of $R$-modules or complexes of $R$-modules is a $\kappa$-Kaplansky class if for every object $M \in A$ and for each $x \in M$, there exists a subobject $N$ of $M$ that contains $x$ with the property that $|N| \leq \kappa$ and both $N$ and $M/N$ are in $A$. We say that $A$ is a Kaplansky class if it is a $\kappa$-Kaplansky class for some regular cardinal number $\kappa$.

Two proofs were given by Bican, El Bashir and Enochs in [11], showing that the class of flat modules forms the left side of a perfect cotorsion pair $(F(R), C(R))$. In that paper, the authors solved what was known as the flat cover conjecture. In the following result, whose proof is inspired by that given by Enochs, we prove a relative version of that conjecture. This is the first ingredient to get our first Hovey triple in this paper.

\textbf{Theorem 3.7} Let $C$ be $\prod$-Tor-orthogonal $R$-module. The following statements hold.

1. $(C_C(R), C_C(R))$ is a complete cotorsion pair cogenerated by a set. Moreover, every $R$-module has a $C_C$-cotorsion envelope.

2. $(SC_C(R), SC_C(R))$ is a complete hereditary cotorsion pair cogenerated by a set.
3. The following assertions are equivalent:
   (a) \( (\mathcal{F}_C(R), \mathcal{F}_C(R)^\perp) \) is a perfect cotorsion pair cogenerated by a set.
   (b) Every \( R \)-module has a special \( \mathcal{F}_C \)-flat precover.
   (c) Every flat \( R \)-module is \( \mathcal{F}_C \)-flat.
   (d) \( R \) is an \( \mathcal{F}_C \)-flat \( R \)-module.

**Proof.** 1. Let \( F \in \mathcal{F}_C(R) \). If \( \kappa \geq |R| \) then for each \( x \in F \) (take \( N = Rx \) and \( f : N \hookrightarrow M \) in [13 Lemma 5.3.12]) there is a pure submodule \( F_0 \subseteq F \) with \( x \in F_0 \) such that \( |F_0| \leq \kappa \). Since the class \( \mathcal{F}_C(R) \) is closed under pure submodules and pure quotients by [9 Proposition 3.7(2)], \( F_0, F/F_0 \in \mathcal{F}_C(R) \).

The latter means that \( \mathcal{F}_C(R) \) is a \( \kappa \)-Kaplansky class. So, using transfinite induction, we can write \( F \) as the direct union of a continuous chain \( \left( F_\alpha \right)_{\alpha \leq \lambda} \) of pure submodules of \( F \) such that \( F_0, F_\alpha/F_\alpha \in \mathcal{F}_C(R) \) and \( |F_\alpha| \leq \kappa \) whenever \( \alpha + 1 < \lambda \).

Let \( X \) be the direct sum of all representatives of \( \mathcal{F}_C(R) \) with cardinality at most \( \kappa \). Clearly, \( \mathcal{C}_C(R) \subseteq X^\perp \), and [15 Theorem 7.3.4] gives that \( X^\perp \subseteq \mathcal{F}_C(R)^\perp = \mathcal{C}_C(R) \).

Thus, \( \mathcal{C}_C(R) = X^\perp \) and hence \( (\mathcal{C}_C(R), \mathcal{C}_C(R)) \) is a complete cotorsion pair by [15 Theorem 7.4.1].

The last statement follows from [16 Theorem 2.8].

2. If we prove that \( \mathcal{SC}_C(R) = M^\perp \) for some \( R \)-module \( M \), this assertion will follow from Proposition 5.3 and [15 Theorem 7.4.1].

By the proof of item 1 we have \( \mathcal{C}_C(R) = X^\perp \). We claim that \( \mathcal{SC}_C(R) = X^\perp \). Clearly, \( \mathcal{SC}_C(R) \subseteq X^\infty \). Conversely, take \( N \in X^\infty \) and let \( 0 \to N \to I \to L \to 0 \) be a short exact sequence of \( R \)-modules where \( I \) is injective. Note that \( L \in X^\infty \subseteq X^\perp = \mathcal{C}_C(R) \), so by the long exact sequence we get that

\[
0 = \text{Ext}^2_R(F, L) \to \text{Ext}^2_R(F, N) \to \text{Ext}^2_R(F, I) = 0
\]

for every \( F \in \mathcal{F}_C(R) \). Hence, \( \text{Ext}^2_R(F, N) = 0 \). Repeating this process, we get that \( \text{Ext}^2_R(F, N) = 0 \) for every \( i \geq 1 \). Therefore, \( \mathcal{SC}_C(R) = X^\perp = M^\perp \) for some \( R \)-module \( M \).

3. The implications \((a) \Rightarrow (b) \) and \((c) \Rightarrow (d) \) are obvious.

\((b) \Rightarrow (c) \) Let \( F \) be a flat \( R \)-module and consider a special \( \mathcal{F}_C \)-flat precover of \( F \):

\[
0 \to K \to X \to F \to 0.
\]

Since \( F \) is flat, this sequence is pure. But \( \mathcal{F}_C \) is closed under pure quotients so \( F \) is \( \mathcal{F}_C \)-flat.

\((d) \Rightarrow (a) \) By (1), [13 Theorem 5.2.3], and [9 Proposition 3.7], we only need to show that \( \mathcal{F}_C(R) = \mathcal{C}_C(R) \).

Clearly, \( \mathcal{F}_C(R) \subseteq \mathcal{C}_C(R) \). Conversely, take \( X \in \mathcal{C}_C(R) \) and consider an \( \mathcal{F}_C \)-flat cover \( f : F \to X \), which exists by [9 Proposition 3.7(1)]. Since the class \( \mathcal{F}_C(R) \) is closed under direct sums and summands and \( R \) is \( \mathcal{F}_C \)-flat, we get that \( \mathcal{P}(R) \subseteq \mathcal{F}_C(R) \).

Hence, the morphism \( f \) is surjective and \( K = \text{Ker}(f) \) is \( \mathcal{C}_C \)-cotorsion by Lemma 2.2 (recall that \( \mathcal{F}_C \) is closed under extensions by [9 Proposition 3.7]). But since \( X \in \mathcal{C}_C(R) \), \( X \) is a direct summand of \( F \in \mathcal{F}_C(R) \). Hence, \( X \in \mathcal{F}_C(R) \) and thus \( \mathcal{F}_C(R) = \mathcal{C}_C(R) \).

4 Relative Gorenstein flat model structures

In this section we prove that the class of \( G_C \)-flat modules is the left hand class of a perfect hereditary cotorsion pair. Consequently, every module has a \( G_C \)-F-cover.
Using Hovey’s one-to-one correspondence between abelian model structures and Hovey triples, we obtain a model structure on $R$-$\text{Mod}$ whose cofibrant objects are precisely the $G_C$-flat modules. But first we introduce and investigate some properties of Gorenstein cotorsion modules with respect to $C$.

**Definition 4.1** Let $C$ be an $R$-module. An $R$-module $M$ is said to be $G_C$-cotorsion if $\text{Ext}_R^1(N, M) = 0$ for all $G_C$-flat modules $N$.

We use $G_C C(R)$ to denote the class of all $G_C$-cotorsion $R$-modules.

**Remarks 4.2** When $C = R$, $G_C C(R)$ coincides with the class of all Gorenstein cotorsion modules $GC(R) = G\mathcal{F}(R)^\perp$ ($G\mathcal{F}(R)$ is the class of all Gorenstein flat modules).

$G_C$-cotorsion modules are both cotorsion and (strongly) $C_C$-cotorsion, as the following result shows.

**Proposition 4.3** Let $C$ be a $\prod$-Tor-orthogonal $R$-module. Then, every $G_C$-cotorsion module is $C_C$-cotorsion. Moreover, if $R$ is $G_C F$-closed and $C$ is $w^+$-tilting, then

$$G_C C(M) = C(R) \cap SC_C(R) \cap H_C(R),$$

where $H_C(R)$ is the class of $R$-modules $M$ such that the complex $\text{Hom}_R(X, M)$ is exact for all complete $F_C$-flat sequences $X$.

**Proof.** The first statement follows from the inclusion $F_C(R) \subseteq G_C F(R)$ by [9] Corollary 4.9. Now we prove the equality.

($\subseteq$) We know that $F(R) \subseteq G_C F(R)$ by [9] Proposition 4.17, so we have

$$G_C C(R) = G_C F(R)^\perp \subseteq F(R)^\perp = C(R).$$

On the other hand, we also know that $F_C(R) \subseteq G_C F(R)$, so

$$G_C F(R)^\perp \subseteq F_C(R)^\perp = SC_C(R).$$

Thus, if $M$ is any $G_C$-cotorsion module, to prove that $M$ is strongly $C_C$-cotorsion we only need to prove that $\text{Ext}_R^2(G, M) = 0$ for every $G \in G_C F(R)$. Choose then any such $G$ and any projective resolution

$$\cdots \to P_1 \to P_0 \to G \to 0$$

of $G$. If we call $K_i = \text{Ker}(P_{i-1} \to P_{i-2}) \forall i \geq 1$, we see by [9] Theorem 4.21] that $K_i \in G_C F(R) \forall i \geq 0$ and all the $P_i$’s are $G_C$-flat. Then, we have, for every $i \geq 2$,

$$\text{Ext}_R^1(G, M) \cong \text{Ext}_R^1(K_{i-1}, M) = 0.$$

Finally, choose any complete $F_C$-flat resolution

$$X : \cdots \to F_1 \to F_0 \to C_{-1} \to C_{-2} \to \cdots$$

and let us see that $\text{Hom}_R(X, M)$ is exact for any $G_C$-cotorsion module $M$.

By [9] Corollaries 4.9 and 4.19] every image $I_i = \text{Im}(F_{i+1} \to F_i)$ and every kernel $K_i = \text{Ker}(C_i \to C_{i-1})$ are $G_C$-flat, so $\text{Ext}_R^1(I_i, M) = 0 = \text{Ext}_R^1(K_i, M)$ for all $i \geq 0$ and all $j \leq -1$, which implies that $\text{Hom}_R(X, M)$ is exact.

($\supseteq$) Let $M \in C(R) \cap SC_C(R) \cap H_C(R)$ and $N$ be $G_C$-flat. Then, there exists a complete $F_C$-flat resolution $X$ of $N$ as above. Consider the short exact sequence $0 \to I_0 \to F_0 \to N \to 0$. Since this sequence is $\text{Hom}_R(-, M)$-exact by the hypotheses and $M$ is cotorsion, the exactness of the sequence

$$0 \to \text{Hom}_R(N, M) \to \text{Hom}_R(F_0, M) \to \text{Hom}_R(I_0, M) \to \text{Ext}_R^1(N, M) \to 0$$

shows that $\text{Ext}_R^1(N, M) = 0$. Thus, $M$ is $G_C$-cotorsion. $\blacksquare$
Now we are in position to investigate when the classes $G_C^F(R)$ and $G_C(C)$ are covering and enveloping, respectively. In fact, we will prove more than that: we will characterize exactly when $(G_C^F(R), G_C(C))$ is a perfect and hereditary cotorsion pair. We will use the following lemma.

**Lemma 4.4** Assume that $R$ is $G_C^F$-closed and $C$ is $w^+$-tilting. Then, the class of $G_C$-flat $R$-modules is a Kaplansky class.

**Proof.** Let $\mathcal{A}$ be the class of all $(\prod_R(C^+) \otimes_R -)$-exact exact complexes of $R$-modules with components in $\mathcal{A} := \mathcal{F}(R) \cup \mathcal{F}_C(R)$. Since $\mathcal{F}_C(R)$ and $\mathcal{F}(R)$ are closed under direct summands, pure submodules and pure quotients (see [9, Proposition 3.7 (2)]), so is the class $\mathcal{A}$. On the other hand, as in the proof of [17, Theorem 3.7], we get that the class $\mathcal{A}$ is closed under pure subcomplexes and pure quotients (in the sense of [17, Definition 3.1]). Using now [17, Proposition 3.4], we get that the class $\mathcal{A}$ is a Kaplansky class. But since $G_C^F(R)$ is the class of 0-syzygies of exact sequences in $\mathcal{A}$ by [9, Corollary 5.2], we get that it is also a Kaplansky class, as desired. ■

**Corollary 4.5** Assume that $R$ is $G_C^F$-closed and $C$ is $w^+$-tilting. Then, $G_C^F(R)$ is preenveloping if and only if it is closed under direct products.

**Proof.** Follows from [16, Theorem 2.5] since the class $G_C^F$ is closed under direct limits by [9, Proposition 4.15]. ■

**Theorem 4.6** Let $C$ be a $\prod$-Tor-orthogonal $R$-module. The following assertions are equivalent:

1. $(G_C^F(R), G_C(C))$ is a perfect hereditary cotorsion pair cogenerated by a set.
2. $R$ is $G_C^F$-closed and $C$ is $w^+$-tilting.

In this case, $G_C^F(R)$ is covering and $G_C(C)$ is enveloping.

**Proof.** 2. $\Rightarrow$ 1. Using Lemma 4.4 $G_C^F(R)$ is a Kaplansky class. By [9, Propositions 4.15 and 4.17], it is closed under direct limits and contains all projective $R$-modules. Therefore, since $G_C^F(R)$ is closed under extensions, we get that $(G_C^F(R), G_C(C))$ is a perfect cotorsion pair by [16, Theorem 2.9]. Moreover, since the class $G_C^F(R)$ is projectively resolving by [9, Theorem 4.21], our pair is hereditary. Finally, as in the proof of Theorem 3.7(1), our pair is cogenerated by a set.

1. $\Rightarrow$ 2. By hypothesis, $\mathcal{C}G_C(C) = G_C^F(R)$, so the class $G_C^F(R)$ is closed under extensions and since $R \in G_C^F(R)$, we get that $C$ is $w^+$-tilting by [9, Proposition 4.17]. ■

**Lemma 4.7** Assume that $\mu C$ is finitely presented. Then, $\mathcal{F}_C(R)$ is closed under direct products if and only if $S$ is right coherent and $C_S$ is finitely presented.

In this case, if $C$ is $\prod$-Tor-orthogonal, then $R$ is $G_C^F$-closed.

**Proof.** $(\Rightarrow)$ Since $\mu C$ is finitely presented, Example 2.8 says that $\mathcal{F}_C(R) = C \otimes_S \mathcal{F}(S)$. Then, for any $\mathcal{F}_C$-flat $R$-module $C \otimes_S F$ we have that the canonical homomorphism $\tau_{C_S F} : C \otimes_S \text{Hom}_R(C, C \otimes_S F) \to C \otimes_S F$ is an isomorphism by [15, Theorem 3.2.14]. In particular, since $C^I \in \mathcal{F}_C(R)$ for any set $I$ by the hypotheses, $\tau_{C^I}$ is always an isomorphism. But $\tau_{C^I}$ is nothing but the composition of the canonical morphisms

$$C \otimes_S \text{Hom}_R(C, C^I) \xrightarrow{\beta} C \otimes_S \text{Hom}_R(C, C)^I = C \otimes_S S^I \xrightarrow{\cdot 0} (C \otimes_S S)^I = C^I$$
and $\beta$ is an isomorphism. Thus, $\alpha$ is also an isomorphism for any set $I$ and then $C_S$ is finitely presented.

Now, since $C^I \in \mathcal{F}_C(R)$, there must exist some $F \in \mathcal{F}(S)$ such that $C^I = C \otimes_S F$. Then, we can use [13] Theorem 3.2.14 again to get the following chain of isomorphisms:

$$sF \cong S \otimes_S F = \text{Hom}_R(C, C) \otimes_S F \cong \text{Hom}_R(C, C \otimes_S F) = \text{Hom}_R(C, C^I) \cong \text{Hom}_R(C, C^I) = S^I.$$ 

Therefore, $sS^I$ is flat for any set $I$ and then $S$ is right coherent.

($\Leftarrow$) By Example 2.2 we know that $\mathcal{F}_C(R) = C \otimes_S \mathcal{F}(S)$. Thus, choose any family $\{F_i; \ i \in I\}$ of flat $S$-modules and let us prove that $\prod_{i \in I} C \otimes_S F_i \in \mathcal{F}_C(R)$. But $C_S$ is finitely presented so $\prod_{i \in I} C \otimes_S F_i \cong C \otimes_S (\prod_i F_i)$, and $\prod_i F_i$ is flat because $S$ is right coherent.

Finally, the fact that $R$ is $G_C$-F-closed follows from [9] Corollary 4.13. 

**Corollary 4.8** Let $C$ be a semidualizing $(R, S)$-bimodule such that $S$ is right coherent. Then, the following assertions hold:

1. $(G_C F(R), G_C C(R))$ is a perfect and hereditary cotorsion pair.
2. $G_C F(R)$ is special precovering.
3. $G_C C(R)$ is special preenveloping.

**Proof.** Follows from Theorem 4.6, Lemma 2.2 and Lemma 4.7.

**Corollary 4.9** Assume that $R$ is $G_C$-F-closed and that $C$ is a $w^+$-tilting $R$-module. Then, the following assertions are equivalent:

1. Every $R$-module is $G_C$-cotorsion.
2. Every $G_C$-flat $R$-module is $G_C$-cotorsion.
3. Every $G_C$-flat $R$-module is projective.
4. $R$ is left perfect and every $G_C$-flat $R$-module is flat.
5. $R$ is left perfect and every cotorsion $R$-module is $G_C$-cotorsion.

**Proof.** 1. $\Rightarrow$ 2. Clear.
2. $\Rightarrow$ 3. Let $M$ be a $G_C$-flat $R$-module and consider an exact sequence $0 \to K \to P \to M \to 0$ where $P$ is projective. Since $P$ is $G_C$-flat by [9] Proposition 4.17 and the class $G_C F(R)$ is closed under kernels of epimorphisms by [9] Proposition 4.15, $K$ is $G_C$-flat and then $G_C$-cotorsion by hypothesis. Therefore, the above sequence is split and $M$ is projective.
3. $\Rightarrow$ 4. Every flat module is $G_C$-flat by [9] Proposition 4.17(4)], so it is projective by hypothesis. Then, $R$ is left perfect and then every $G_C$-flat module is flat.
4. $\Rightarrow$ 5. Since both $(\mathcal{F}_C(R), C(R))$ and $G_C F(R), G_C C(R))$ are cotorsion pairs, we have $\mathcal{C} = \mathcal{F}(R)^\perp \subseteq G_C F(R)^\perp = G_C C(R)$, where the middle inclusion holds because $G_C F(R) \subseteq \mathcal{F}(R)$.
5. $\Rightarrow$ 1. Since $R$ is perfect, every $R$-module is cotorsion by [13] Proposition 3.3.1]. Hence, every $R$-module is $G_C$-cotorsion.

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The following result is inspired by an argument due to Estrada, Iacob and Pérez in [18 Proposition 4.1].

**Proposition 4.10** Assume that $R$ is $G_{C}F$-closed and that $C$ is a $w^{+}$-tilting $R$-module. Then,
\[ G_{C}F(R) \cap G_{C}C(R) = \mathcal{F}_{C}(R) \cap C_{C}(R). \]

**Proof.** $(\subseteq)$ Assume that $M \in G_{C}F(R) \cap G_{C}C(R)$. Then, $M \in C_{C}(R)$ by Proposition 4.3. Moreover, since $M$ is $G_{C}$-flat, there exists by [9 Proposition 4.10] an exact sequence of $R$-modules $0 \to M \to V \to G \to 0$ where $V$ is $\mathcal{F}_{C}$-flat and $G$ is $G_{C}$-flat. This short exact sequence splits since $G$ is $G_{C}$-flat and $M$ is $G_{C}$-cotorsion. Hence, $M \in \mathcal{F}_{C}(R)$.

$(\supseteq)$ Assume now that $M \in \mathcal{F}_{C}(R) \cap C_{C}(R)$. Clearly $M \in G_{C}F(R)$ (recall that $\mathcal{F}_{C}(R) \subseteq G_{C}F(R)$ by [9 Corollary 4.9]). Let us prove that $M \in G_{C}C(R)$.

By Theorem 4.6, $M$ has a special $G_{C}C(R)$-preenvelope
\[ 0 \to M \to X \to G \to 0. \]

Since $M$ and $G$ are $G_{C}$-flat modules and $R$ is $G_{C}F$-closed, $X$ is $G_{C}$-flat as well and then $X \in G_{C}F(R) \cap G_{C}C(R) \subseteq \mathcal{F}_{C}(R) \cap C_{C}(R)$ by the first inclusion. Now, since $M$ and $X$ are $\mathcal{F}_{C}$-flat and $G$ is $G_{C}$-flat, $G \in \mathcal{F}_{C}(R)$ by [9 Theorem 7.12]. Then, the short exact sequence splits. Hence, $M$ is $G_{C}$-cotorsion. \[ \square \]

We are now ready to show that $G_{C}F(R)$ (resp., $G_{C}C(R)$) is part of a left (resp., right) weak AB-context. Note that $\mathcal{F}_{C}(R) \cap C_{C}(R) = \mathcal{F}_{C}(R) \cap SC_{C}(R)$. The claim concerning the class $G_{C}F(R)$ in the following result was proved by Sather-Wagstaff, Sharif and White in [42], when the ring $R$ is commutative and noetherian and $C$ is a semidualizing $R$-module. Here we use a different and recent approach and also we show this result with significantly lower restrictions on $R$ and $C$.

**Theorem 4.11** If $R$ is $G_{C}F$-closed and $C$ is a $w^{+}$-tilting $R$-module, then
\[ \left( G_{C}F(R), res(\mathcal{F}_{C}(R) \cap C_{C}(R)), \mathcal{F}_{C}(R) \cap C_{C}(R) \right) \]

is a left weak AB-context, and
\[ \left( \mathcal{F}_{C}(R) \cap C_{C}(R), cores(\mathcal{F}_{C}(R) \cap C_{C}(R)), G_{C}C(R) \right) \]

is a right weak AB-context.

**Proof.** By Lemma 2.3(1), it suffices to show that $(G_{C}F(R), \mathcal{F}_{C}(R) \cap C_{C}(R))$ is a left Frobenius pair. But this follows by Theorem 4.6, Proposition 4.11 and Lemma 2.3(2). The claim concerning the class $G_{C}C(R)$ follows similarly using Lemma 2.5. \[ \square \]

An immediate consequence of Theorem 4.11 is that the modules in $res(G_{C}F(R))$ (resp., $cores(G_{C}C(R))$) have some special approximations in the sense of Auslander and Buchweitz [9]. The following result follows from [9 Theorem 1.1] and its dual. For more interesting consequences of Theorem 4.11 we refer to [28 Theorem 1.12.10].

**Corollary 4.12** Assume that $R$ is $G_{C}F$-closed and $R_{C}$ is $w^{+}$-tilting.
1. For every $M \in \text{res}(\widehat{\text{GC}}F(R))$ there exist exact sequences
\[ 0 \to Y_M \to X_M \to M \to 0 \text{ and } 0 \to M \to Y^M \to X^M \to 0 \]
with $X_M, X^M \in \text{GC}F(R)$ and $Y_M, Y^M \in \text{res}(\mathcal{F}_C(R) \cap \mathcal{C}_C(R))$.

2. For every very $N \in \text{cores}(\widehat{\text{GC}}C(R))$ there exist exact sequences
\[ 0 \to Y^N \to X^N \to N \to 0 \text{ and } 0 \to N \to Y_N \to X_N \to 0 \]
with $Y_N, Y^N \in \text{GC}C(R)$ and $X_N, X^N \in \text{cores}(\mathcal{F}_C(R) \cap \mathcal{C}_C(R))$.

**Proposition 4.13** Assume that $C$ is $w$-tilting $R$-module with a degreewise finite projective resolution. Then, $(\{ \cdot \in \mathcal{C}_C(R) \mid \cdot \in \mathcal{B}_C(R) \})$, $(\mathcal{B}_C(R) \cap \mathcal{C}_C(R))$ is a hereditary complete cotorsion pair cogenerated by a set.

**Proof.** First of all, note that $C$ is $[\cdot, \cdot]$-orthogonal by Proposition 4.11 and $\mathcal{B}_C(R) \cap \mathcal{C}_C(R) = \mathcal{B}_C(R) \cap \mathcal{S}_C(R)$ by Corollary 3.6. Now using [5, Theorem 3.10], we see that $\mathcal{B}_C(R) = X^1_1 \infty$ for some set $X_1$. Similarly, $\mathcal{S}_C(R) = X^2_\infty$ for some set $X_2$ by Theorem 5.4(2). Then, $\mathcal{B}(R) \cap \mathcal{C}(R) = X^2_1 \infty \cap X^2_\infty = (X_1 \cup X_2)^1 \infty$. Thus, $\mathcal{B}(R) \cap \mathcal{C}(R) = M^1$ for some $R$-module $M$ by Lemma 2.1. Hence, our pair is hereditary and cogenerated by a set and then complete by [15, Theorem 7.4.1].

Under strong conditions we get a different description of the core of the cotorsion pair $(\text{GC}F(R), \text{GC}C(R))$ which is the last ingredient to get our first model structure.

**Proposition 4.14** If $R$ is $\text{GC}F$-closed and $C$ is a $w^+$-tilting $R$-module admitting a degreewise finite projective resolution, then
\[ \text{GC}F(R) \cap \text{GC}C(R) = \hat{\{ \cdot \in \mathcal{B}_C(R) \cap \mathcal{C}_C(R) \}} \cap (\mathcal{B}_C(R) \cap \mathcal{C}_C(R)) \]

**Proof.** $(\subseteq)$ By Proposition 4.10, $\text{GC}F(R) \cap \text{GC}C(R) \subseteq \text{GC}C(R) \subseteq \mathcal{F}_C(R) \cap \mathcal{C}_C(R)$. So, $\mathcal{F}_C(R) \subseteq \mathcal{B}_C(R)$ by Lemma 2.20(3). Then, $M \in \mathcal{F}_C(R) \subseteq \mathcal{B}_C(R)$ and hence $\text{GC}F(R) \cap \text{GC}C(R) \subseteq \mathcal{B}(R) \cap \mathcal{C}(R)$.

On the other hand, we have $\mathcal{B}(R) \cap \mathcal{C}(R) \subseteq \mathcal{C}(R)$, which implies that $\hat{\mathcal{B}_C(R)} \subseteq \hat{\mathcal{B}_C(R)} \cap \mathcal{C}_C(R)$. But $\mathcal{F}_C(R) \subseteq \hat{\mathcal{B}_C(R)}$. Hence, $\text{GC}F(R) \cap \text{GC}C(R) \subseteq \hat{\mathcal{B}(R) \cap \mathcal{C}(R)}$ by Proposition 4.10.

$(\supseteq)$ Conversely, let $M \in \hat{\mathcal{B}(R) \cap \mathcal{C}(R)} \cap (\mathcal{B}(R) \cap \mathcal{C}(R))$. By Proposition 4.10, we only need to show that $M \in \mathcal{F}_C(R) \cap \mathcal{C}_C(R)$.

Clearly, $M \in \mathcal{C}_C(R)$ and since $M \in \mathcal{B}_C(R)$, Lemma 2.22(2) says that there exists an epic $\mathcal{F}_C$-flat cover $\gamma : \mathcal{F}_C \to M$ with $K := \text{Ker}(\gamma) \in \mathcal{B}_C(R)$. Wakamatsu Lemma says that $K \in \mathcal{C}_C(R)$, that is $K \in \mathcal{B}(R) \cap \mathcal{C}(R)$. Since $M \in \hat{\mathcal{B}(R) \cap \mathcal{C}(R)}$, the short exact sequence $0 \to K \to F \to M \to 0$ splits. Thus, $M \in \mathcal{F}_C(R)$, as desired.

Modules in $\mathcal{H}_C := \mathcal{B}_C(R) \cap \mathcal{C}_C(R)$ will be called $\mathcal{H}_C$-cotorsion modules. They will play the role of fibrant objects in the following model structure.

**Theorem 4.15** Let $R$ be $\text{GC}F$-closed and $C$ be a $w$-tilting $R$-module admitting a degreewise finite projective resolution. Then, there exists a unique hereditary abelian model structure on $R$-$\text{Mod}$, called the $\text{GC}$-$\text{flat}$ model structure, as follows:
- The cofibrant objects coincide with the $\text{GC}$-$\text{flat}$ $R$-modules.
- The fibrant objects coincide with the $\mathcal{H}_C$-cotorsion $R$-modules.
- The class of trivially cofibrant objects coincide with $\hat{\mathcal{B}_C(R) \cap \mathcal{C}_C(R)}$.
- The trivially fibrant objects coincide with the $\mathcal{G}_C$-cotorsion $R$-modules.
Proof. It follows from Theorem 3.7 and Theorem 4.6 that the pairs
\[(\text{GcF}(R), \text{GcC}(R)) \text{ and } \left(\text{Add}_R(R) \cap \text{C}(R), \text{B}_C(R) \cap \text{C}(R)\right)\]
are complete and hereditary cotorsion pairs. By Proposition 4.13, these cotorsion pairs have the same core. Let us now show that \(\text{GcC}(R) \subseteq \text{B}_C(R) \cap \text{C}(R)\).

The inclusion \(\text{GcC}(R) \subseteq \text{C}(R)\) holds by Proposition 4.13. On the other hand, as we did in the proof Proposition 2.14(2), we can get an addition\(R\)-coresolution of \(R\):
\[0 \to R \xrightarrow{t} C_0 \xrightarrow{i_1} C_1 \xrightarrow{i_2} \cdots\]
which is \(\text{Hom}_R(-, C)\)-exact (or equivalent, \(\text{Hom}_R(-, \text{Add}_R(C))\)-exact since each \(C_i\) is finitely generated). Then, \(\text{B}_C(R) = (C \oplus (\oplus_{i \geq 0} \text{Coker} t_i))^\perp\) by [6] Theorem 3.10.

Now by [10] Corollary 2.13, each \(\text{Coker}(t_i)\) is \(\text{GcC}\)-projective. But since each \(\text{Coker}(t_i)\) has a degreewise finite projective resolution, it is also a \(\text{GcF}\)-flat module by Proposition 2.14. It follows that \(C \oplus (\oplus_{i \geq 0} \text{Coker} t_i) \in \text{GcF}(R)\) which implies that \(\text{GcC}(R) \subseteq (C \oplus (\oplus_{i \geq 0} \text{Coker} t_i))^\perp = \text{B}_C(R)\). Thus, Theorem 2.3 gives the desired Hovey triple.

Remark 4.16 1. Under Hovey’s one-to-one correspondence between abelian model structures and Hovey triples [6], the \(\text{GcF}\)-flat model structure is described as follows:

- A morphism \(f\) is a cofibration (trivial cofibration) if and only if it is a monomorphism with \(\text{GcC}\)-flat cokernel \((\text{Coker}(f) \in \text{Add}_R(R) \cap \text{C}(R))\).
- A morphism \(g\) is a fibration (trivial fibration) if and only if it is an epimorphism with \(\text{H}_C\)-cotorsion kernel \((\text{Ker}(g) \in \text{GcC}(R))\).

Moreover, this model structure is cofibrantly generated in the sense of [32] Section 2.1. This can be seen by [32] Lemma 6.7 and Corollary 6.8 since the pairs \((\text{Add}_R(R) \cap \text{C}(R), \text{B}_C(R) \cap \text{C}(R))\) and \((\text{GcF}(R), \text{GcC}(R))\) are cogenerated by a set.

2. When \(C\) is \(\Sigma\)-self-orthogonal admitting a degreewise finite projective resolution, \(C\) is \(w\)-tilting and \(R\) is \(\text{GcF}\)-closed whenever the \(\text{GcF}\)-flat model structure exists. This can be easily seen by Theorem 4.6 and Proposition 2.14 since the \(\text{GcF}\)-flat model structure implies that \((\text{GcF}(R), \text{GcC}(R))\) is a complete and hereditary cotorsion pair.

Recall that an exact category (in the sense of Quillen) is an additive category with an exact structure, that is, a distinguished class of ker-coker sequences which are called confullations, subject to certain axioms ([34] Appendix A). For example, a full additive subcategory of an abelian category that is closed under extensions is an exact category such that confullations are short exact sequences with terms in the subcategory. All the full additive subcategories in this paper will be considered exact with respect to this canonical exact structure.

Recall that a Frobenius category is an exact category with enough injectives and projectives and such that the projective objects coincide with the injective objects. Given a Frobenius category \(\mathcal{A}\), we can form the stable category \(\mathcal{A}^\perp := \mathcal{A}/\sim\), which has the same objects as \(\mathcal{A}\) and \(\text{Hom}(X, Y) = \text{Hom}(X, Y)/\sim\), where \(f \sim g\) if and only if \(f-g\) factors through a projective-injective object.
The main fact about a Frobenius category $\mathcal{A}$ is that the stable category is canonically triangulated and it encodes the corresponding relative homological algebra on $\mathcal{A}$.

The first part of the following result was recently proved by Hu, Geng, Wu and Li in [29] Theorem 4.3 when $R$ is a commutative noetherian ring and $C$ is a semidualizing $R$-module. We obtain it here with a different approach and fewer assumptions.

**Corollary 4.17** Assume that $R$ is $G_C$-closed and that $C$ is $w$-tilting admitting a degreewise finite projective resolution. Then, the category $\mathcal{A}_{c,f} := G_C F(R) \cap B_C(R) \cap C_C(R)$ is a Frobenius category. The projective-injective objects are exactly the objects in $W_{c,f} = \mathcal{F}_C(R) \cap C_C(R)$. Moreover, the homotopy category of the $G_C$-flat model structure is triangle equivalent to the stable category $G_C F(R) \cap B_C(R) \cap C_C(R)$.

**Proof.** Use Theorem [4.15] and Proposition [4.10] together with [21] Proposition 5.2(4) and Lemma 4.7. The last assertion follows from [21] Corollary 5.4. See also [23] Proposition 4.2 and Theorem 4.3.

The Gorenstein flat model structure goes back to Gillespie and Hovey [25, Theorem 3.12] when the ring is Iwanaga-Gorenstein. Recently, Šťovíček proved in [11] the existence of this model structure over any arbitrary ring.

**Corollary 4.18** (The Gorenstein flat model structure) For any finitely generated projective generator $C$ of $R$-$\text{Mod}$, there exists a unique hereditary abelian model structure on $R$-$\text{Mod}$ where $G_C F(R) = \mathcal{G}_F(R)$ is the class of cofibrant objects and $B_C(R) \cap C_C(R) = \mathcal{C}(R)$ is the class of fibrant objects.

In this case, the category $G_C F(R) \cap C(R)$ is a Frobenius category where the projective-injective objects are exactly the flat-cotorsion $R$-modules. Moreover, the homotopy category of the Gorenstein flat model structure is triangle equivalent to the stable category $G_C F(R) \cap C(R)$.

**Proof.** Clearly $\mathcal{F}_C(R) = \mathcal{F}(R)$ since $\mu C$ is a projective generator. Hence, $\mathcal{C}(R) = \mathcal{C}(R) \cap G_C F(R) = G_C F(R) = \mathcal{G}_C(R)$ by [9] Proposition 4.23 and [21] Corollary 4.12. This implies that $R$ is $G_C$-closed. So, by Theorem 4.15 we have the desired model structure.

By the above corollary, a projective generator $\mu C$ is an example of a case where the $G_C$-flat model structure exists. At the end of this section we will give another example of $C$ being neither projective nor a generator.

Recall from [10] that the FP-injective dimension of an $R$-module $M$, denoted by $\text{FP-id}_R(M)$, is defined to be the smallest non-negative integer $n$ such that, for every finitely presented $R$-module $F$, the equality $\text{Ext}^{n+1}_R(F, M) = 0$ holds. In particular, $M$ is called FP-injective if $\text{FP-id}_R(M) = 0$. We write $\mathcal{FI}(R)$ to denote the class of all FP-injective $R$-modules. Note that $\mathcal{FI}(R) = \mathcal{I}(R)$ when $R$ is a (left) noetherian ring.

**Lemma 4.19** Let $C$ be a semidualizing $(R, R)$-bimodule. Then, $\mathcal{F}_C(R) = \mathcal{FI}(R)$ if and only if $R$ is left and right coherent, and both $\mu C$ and $C_R$ are FP-injective.

**Proof.** ($\Rightarrow$) Clearly $\mu C$ is FP-injective. Moreover, $R$ is left coherent by [10] Theorem 3.2 since $\mathcal{F}_C(R)$ is closed under direct limits, and $R$ is right coherent by Lemma 4.7 since $\mathcal{FI}(R)$ is closed under direct products.
4 RELATIVE GORENSTEIN FLAT MODEL STRUCTURES

Now we prove that $C_R$ is FP-injective. By [13 Theorem 1], it suffices to show that $C^+$ is a flat $R$-module. We know that $R^+$ is an injective $R$-module, so $R^+ \in \mathcal{FI}(R) = \mathcal{F}_C(R)$ and hence $R^+ = C \otimes_R F$ for some flat module $F \in \mathcal{F}(R)$. Thus, $C^+ \cong (R \otimes_R C)^+ \cong \text{Hom}_R(RC, R^+) \cong \text{Hom}_R(RC, C \otimes_R F) \cong F \in \mathcal{F}(R)$.

$(\Leftarrow)$ Let $X$ be a left $R$-module. If $X \in \mathcal{F}_C(R)$, then $X^+ \oplus Y = (C^+)^J$ for some module $Y$ and some set $I$. But since $\mathcal{R}C$ is FP-injective and $R$ is left coherent, $(C^+)^J$ is flat by [13 Theorem 1] and [15 Theorem 3.2.24]. Hence, $X^+$ is flat and then $X \in \mathcal{FI}(R)$.

Conversely, assume that $X \in \mathcal{FI}(R)$. Then, $X^{++} \in \mathcal{I}(R)$ by [13 Theorem 1] which implies that $X^{++}$ is a direct summand of $(R^+)^J$ for some set $J$.

Note also that $(C^+)^J$ is a flat $R$-module since $C_R$ is FP-injective and $R$ is right coherent. Hence $X^{++} \cong C \otimes_R \text{Hom}_R(C, X^{++})$ is a direct summand of $C \otimes_R \text{Hom}_R(RC, (R^+)^J) \cong C \otimes_R (C^+)^J \in C \otimes_R \mathcal{F}(R) = \mathcal{F}_C(R)$.

Hence $X^{++} \in \mathcal{F}_C(R)$. But $X$ is a pure submodule of $X^{++}$ and $\mathcal{F}_C(R)$ is closed under pure submodules by [9 Theorem 3.8]. Therefore, $X \in \mathcal{F}_C(R)$ and thus $\mathcal{F}_C(R) = \mathcal{FI}(R)$.

Recall ([38 Definition 2.1]) that an $R$-module $M$ is Gorenstein FP-injective if there exists a $\text{Hom}_R(\mathcal{FI}(R), -)$-exact exact sequence of injective $R$-modules $\cdots \to E_1 \to E_0 \to E_{-1} \to \cdots$, with $M = \text{Im}(E_0 \to E_{-1})$.

**Corollary 4.20** Let $R$ be left and right coherent with $FP - \text{id}(R) < \infty$. If $C$ is a semidualizing $(R, R)$-bimodule such that $\mathcal{R}C$ and $C_R$ are FP-injective, then there exists a hereditary abelian model structure on $R$-$\text{Mod}$ such that each $R$-module is cofibrant and the fibrant objects are the Gorenstein FP-injective $R$-modules in $B_C(R)$.

**Proof.** Clearly $C$ is $w$-tilting and using Lemma 4.17 we see that $R$ is $G_C F$-closed. By Lemma 4.19 $\mathcal{F}_C(R) = \mathcal{FI}(R)$ and then $S\mathcal{C}_C(R) = \mathcal{FI}^{=\infty}$ is the class of Gorenstein FP-injective modules ([38 Theorem 2.4]). Consequently, using Corollary 4.9 we get that $B_C(R) \cap S\mathcal{C}_C(R) = B_C(R) \cap S\mathcal{C}_C(R)$ is the class of Gorenstein FP-injective that are in $B_C(R)$. Finally, since $\mathcal{I}(R) \subseteq \mathcal{F}_C(R)$ and $\text{Proj}_R(C^+) \subseteq \mathcal{F}(R_R)$, every $R$-module is $G_{C}$-flat. Therefore, Theorem 4.15 gives the desired model structure. ■

**Example 4.21** ([7], Example 3.5) Take the quiver $Q : \bullet \to \bullet \to \cdots \to \bullet$ with $n \geq 1$ vertices and $R = kQ$ the path algebra over an algebraic field $k$. By [23 example 3.5] there are two semidualizing $(R, R)$-bimodules

$$C_1 = R \text{ and } C_2 = R^* = \text{Hom}_R(R, k).$$

Note that $C_2$ is neither a flat nor a generator module. There are two hereditary abelian model structures on $R$-$\text{Mod}$ which are as follows:

$$\mathcal{M}_1 = (G_{C_1}F(R), \mathcal{W}_1, \mathcal{B}_{C_1}(R) \cap \mathcal{C}_{C_1}(R)) = (\mathcal{GP}(R), \mathcal{GP}(R)^+, R$-Mod),$$

$$\mathcal{M}_2 = (G_{C_2}F(R), \mathcal{W}_2, \mathcal{B}_{C_2}(R) \cap \mathcal{C}_{C_1}(R)) = (R$-Mod, $^\bot \mathcal{GI}(R), \mathcal{GI}(R))$$

where $\mathcal{GP}(R)$ and $\mathcal{GI}(R)$ denote the class of Gorenstein projective and injective $R$-modules, respectively.
5 Auslander and Bass model structures

In this section, we study the existence of an abelian model structure on \( R\text{-Mod} \) (resp., \( \text{Mod-}R \)) such that \( \mathcal{B}_C(R) \) (resp., \( \mathcal{A}_C(R) \)) is the class of fibrant (resp., cofibrant) objects.

In order to apply Hovey’s one-to-one correspondence between abelian model structures and Hovey triples, we first need to know when the Bass (resp., Auslander) class \( \mathcal{B}_C(R) \) (resp., \( \mathcal{A}_C(R) \)) forms the right (resp., left) hand of a complete and hereditary cotorsion pair. This has been recently proved in [6, Corollary 3.12 and Theorem 3.16(ii)].

Proposition 5.1 Let \( C \) be an \( R \)-module. The following assertions hold:

1. If \( R \) is self-small and \( \Sigma \)-self-orthogonal, then \( \mathcal{B}_C(R) \) is a complete and hereditary cotorsion pair.

Proof. 1. See [6, Corollary 3.12].

2. The proof is almost dual to that of [6, Corollary 3.12], we only mention a few specific points concerning this dualization. We split the proof into three parts:

- Any \( M \in \mathcal{A}_C(R) \) has a monic \( \mathcal{I}_C^+ \)-injective preenvelope with cokernel in \( \mathcal{A}_C(R) \) by Lemma 2.22.

- \( (\mathcal{A}_C(R), \mathcal{A}_C(R)^\perp) \) is a complete cotorsion pair: given any \( (\text{Prod}_R(C^+) \otimes_R \text{−}) \)-exact exact sequence \( 0 \to R \to C_0 \to C_1 \to \cdots \) with each \( C_i \) \( \mathcal{F}_C \)-flat and applying the functor \( \text{Hom}(\text{−}, Q/\mathbb{Z}) \), we get a \( \text{Hom}(\text{Prod}_R(C^+), \text{−}) \)-exact exact sequence \( \cdots \to C_{i+1}^+ \to C_i^+ \to R^+ \to 0 \) where each \( C_i^+ \) is \( \mathcal{I}_C^+ \)-injective. If we let \( K_i = \text{Coker}(t_i) \) with \( i \geq 0 \) and \( K = C \oplus (\oplus_{i \geq 0} K_i) \), then \( K^+ \cong C^+ \oplus (\prod_{i \geq 0} \text{Ker}(t_i)^+) \). But \( C^+ \) is a \( \Sigma \)-cotilting right \( R \)-module so by dualizing the proof of [6, Theorem 3.10] we get \( \mathcal{A}_C(R) = \mathcal{I}_C^+(K^+) \). Thus, this part follows by [15, Theorem 7.4.1].

- The hereditary property follows by [10, Proposition 5.4.(4)].

Proposition 5.2 Let \( C \) be an \( R \)-module. The following assertions hold.

1. If \( R \) is self-small and \( \Sigma \)-self-orthogonal, then \( \mathcal{B}_C(R) \) is a complete and hereditary cotorsion pair.

Proof. 1. Follows by [6, Proposition 4.9(1)] and Lemma 2.19(2).

2. Note that \( C^+ \) is \( \prod \)-self-orthogonal ([6, Proposition 3.5]). So, this equality follows by Lemma 2.20(2) and the dual argument to that of [6, Proposition 4.9(1)].

The following is the first main result of this section. It shows that \( \mathcal{B}_C(R) \) (resp., \( \mathcal{A}_C(R) \)) is part of a right (left) AB-context. Its proof is similar to that of Theorem 4.11 and Corollary 4.12.

Theorem 5.3 Let \( C \) be an \( R \)-module. The following assertions hold.
1. If $R\Sigma$ is self-small and $\Sigma$-self-orthogonal, then the triple
\[
\left(\text{Add}_R(C), \text{cores}(\overline{\text{Add}_R(C)}), \text{B}_C(R)\right)
\]
is a right weak $AB$-context in $R\text{-Mod}$. In this case, for every $N \in \text{cores}(\overline{\text{B}_C(R)})$ there exist exact sequences
\[
0 \to Y^N \to X^N \to N \to 0 \text{ and } 0 \to N \to Y_N \to X_N \to 0
\]
with $Y_N, Y^N \in B_C(R)$ and $X_N, X^N \in \text{cores}(\text{Add}_R(C))$.

2. If $R\Sigma$ is self-co-small and $\prod$-Tor-orthogonal, then the triple
\[
\left(\text{A}_C(R), \text{res}(\text{Prod}_R(C^+)), \text{Prod}_R(C^+)\right)
\]
is a left weak $AB$-context in $\text{Mod}$-$R$. In this case, for every $M \in \text{res}(\overline{\text{A}_C(R)})$, there exist exact sequences of right $R$-modules
\[
0 \to Y_M \to X_M \to M \to 0 \text{ and } 0 \to M \to Y^M \to X^M \to 0
\]
with $X_M, X^M \in \text{A}_C(R)$ and $Y_M, Y^M \in \text{res}(\overline{\text{Prod}_R(C^+)})$.

Remark 5.4 Using all left and right $AB$-contexts constructed in this paper, one would use Lemmas 2.4 and 2.5, together with [73, Theorems B and C], to describe more left and right $AB$ contexts in the category of complexes of $R$-modules, and in the category of representations of a left rooted quiver $Q$ with values in the category of $R$-modules.

Now we investigate when the class $\text{Add}_R(C)$ (resp., $\text{Prod}_R(C^+)$) is the right (resp., left) hand class of a complete cotorsion pair.

Proposition 5.5 Let $\Sigma R\Sigma$ be self-small, $\Sigma$-self-orthogonal and such that the canonical map $R \to \text{Hom}_S(C, C)$ is an isomorphism. Then, $\left(\Sigma \Sigma \text{Add}_R(C), \Sigma \Sigma \text{Add}_R(C)\right)$ is a complete cotorsion pair if and only if $S$ is left perfect and right coherent, and $C_S$ is finitely presented and FP-injective.

Proof. First of all, notice that following the argument of [35, Proposition 3.9] (keeping in mind Lemma 2.13)), one easily sees that the condition of $R\Sigma$ being finitely generated can be substituted by that of $\Sigma R\Sigma$ being self-small.

1. $\Rightarrow$ 2. By hypothesis, the class $\Sigma \Sigma \text{Add}_R(C) = \Sigma \Sigma \text{Add}_R(C)\Sigma$ is closed under direct products and then $S$ is left perfect and right coherent, and $C_S$ is finitely presented by [35, Theorem 3.1 and Proposition 3.9]. On the other hand, we have $R^+ \in (\Sigma \Sigma \text{Add}_R(C))^\perp = \text{Add}_R(C)$. Then, $R^+ = C \otimes_S P$ for some projective $S$-module $P$. Hence, $C^+ \cong (R \otimes_R C)^+ \cong \text{Hom}_R(C, R^+) \cong \text{Hom}_R(C, C \otimes_S P) \cong P$. Thus, $C_S$ is FP-injective by [13, Theorem 1].

2. $\Leftarrow$ 1. First we prove that every $R$-module $M$ has a special $P_C$-projective preenvelope.

By [35, Theorem 3.1, and Proposition 3.9], $\text{Add}_R(C)$ is enveloping and by Lemma 2.2 it suffices to show that each $P_C$-envelope $f : M \to X$ is a monomorphism. If we prove that every injective module is $P_C$-projective, then clearly the map $f$ will be a monomorphism. But since the class $\text{Add}_R(C)$ is closed under direct summands, and using [35, Theorem 3.1 and Proposition 3.9] again we get that it is also closed under...
direct products, we only need to check that the injective cogenerator \( R \)-module \( R^+ \) is \( \mathcal{P}_C \)-projective.

By [13] Theorem 1, \( C^+ \) is a flat \( S \)-module and then projective since \( S \) is left perfect. So, by [15] Theorem 3.2.11, \( R^+ \cong \mathrm{Hom}_S(C, C)^+ \cong C \otimes_S C^+ \) is \( \mathcal{P}_C \)-projective. Now, it only remains to show that \( (\mathcal{I} \operatorname{Add}_R(C))^+ = \mathcal{I} \operatorname{Add}_R(C) \).

The inclusion \( \mathcal{I} \operatorname{Add}_R(C) \subseteq (\mathcal{I} \operatorname{Add}_R(C))^+ \) is clear. For the converse, let \( X \in (\mathcal{I} \operatorname{Add}_R(C))^+ \) and consider a special \( \mathcal{P}_C \)-projective preenvelope of \( X \): \( 0 \to X \to F \to L \to 0 \). Hence, \( \mathrm{Ext}_R^1(L, X) = 0 \) and then the short exact sequence splits. Thus, \( X \) is a direct summand of \( F \) and so \( X \in \mathcal{I} \operatorname{Add}_R(C) \).

To prove the dual of Proposition 5.5, we need the following lemma.

**Lemma 5.6** Assume that \( R \) is self-co-small and \( C \) is \( \mathcal{C}_S \)-finitely presented. Then, the following assertions are equivalent:

1. Every right \( R \)-module \( M \) has an \( \mathcal{I}_{C^+} \)-injective (pre)cover.
2. Every direct sum (limit) of \( \mathcal{I}_{C^+} \)-injective right \( R \)-modules is \( \mathcal{I}_{C^+} \)-injective.
3. \( S \) is a right noetherian ring.

**Proof.** 1. \( \Rightarrow \) 2. Consider a family \( (X_i)_{i \in I} \) of \( \mathcal{I}_{C^+} \)-injective right \( R \)-modules and let \( M = \bigoplus_{i \in I} X_i \). By hypothesis, there exists an \( \mathcal{I}_{C^+} \)-injective precover \( f : X \to M \). For each \( i \in I \), there exists a morphism \( g_i : X_i \to X \) such that \( fg_i = \lambda_i : X_i \to M \) is the canonical injection. Now take the morphism \( g : M \to X \) such that \( g \lambda_i = g_i \) for each \( i \in I \). Hence, \( fg \lambda_i = fg_i = \lambda_i \) for each \( i \in I \). Thus, \( fg = 1_M \) and then \( f : X \to M \) is a split epimorphism. Hence, \( M \) is a direct summand of \( X \) which is \( \mathcal{I}_{C^+} \)-injective. Hence, \( M = \bigoplus_{i \in I} X_i \) is \( \mathcal{I}_{C^+} \)-injective as well.

2. \( \Rightarrow \) 1. For every set \( I \), the right \( R \)-module \( (C^+)^I \cong (C^{(I)})^+ \) is pure-injective, so is every \( \mathcal{I}_{C^+} \)-injective. Then, \( C^+ \) is \( \Sigma \)-pure-injective by the hypotheses, that is, \( (C^+)^{(K)} \) is pure-injective for every set \( K \). Hence, \( \operatorname{Prod}_R(C^+) \) is precovering by [1] Proposition 6.10. Moreover, by the hypotheses and [15] Corollary 5.2.7 the class \( \operatorname{Prod}_R(C^+) \) is covering.

2. \( \Rightarrow \) 3. By [15] Theorem 3.1.17 we need to prove that every direct sum of injective right \( S \)-modules is injective. But since every injective right \( S \)-module is a direct summand of some \( (S^+)^I \), it suffices to show that any direct sum of copies of \( (S^+)^I \) is an injective right \( S \)-module, for any set \( I \).

Let \( K \) be a set. Since \( R \) is self-co-small, \( ((S^+)^I)^{(K)} = ((\operatorname{Hom}_R(C, C)^+)^I)^{(K)} \cong (C^+)^I \otimes_R C^{(K)} \cong ((C^+)^I)^{((K)} \otimes_R C \in \mathcal{I}(S) \) by Lemma 2.18(2) since \( (C^+)^I \) is \( \mathcal{I}_{C^+} \)-injective by hypothesis.

**Proposition 5.7** Assume that \( R \) is self-co-small and \( \prod \)-Tor-orthogonal, that \( C \) is \( \mathcal{C}_S \)-finitely presented and that the canonical map \( R \to \operatorname{Hom}_S(C, C) \) is an isomorphism. Then, \( (\operatorname{Prod}_R(C^+), \operatorname{Prod}_R(C^+)^\perp) \) is a complete cotorsion pair if and only if \( S \) is right noetherian and \( C \) is injective.

**Proof.** Follow a dual argument to that of Proposition 5.5 using Lemma 5.6.
Theorem 5.8 (The $\mathcal{B}_C$-Bass model structure) Let $C$ be a semidualizing $(R, S)$-bimodule. The following assertions are equivalent:

1. There exists a unique hereditary abelian model structure on $R\text{-Mod}$ such that:
   - The fibrant objects coincide with the $\mathcal{B}_C$-Bass $R$-modules.
   - The class of cofibrant objects coincides with $\perp \text{Add}_R(C)$.
   - The trivially fibrant objects coincide with the $\mathcal{P}_C$-projective $R$-modules.
   - The class of trivially cofibrant objects coincides with $\perp \mathcal{B}_C(R)$.

2. $S$ is left perfect and right coherent, $C_S$ is FP-injective and $\text{Add}_R(C)$ is closed under cokernels of monomorphisms.

In this case, the category $\mathcal{A}_{c,f} = \perp \text{Add}_R(C) \cap \mathcal{B}_C(R)$ is a Frobenius category. The projective-injective objects exactly the $\mathcal{I}_C^+$-injective right $R$-modules. Moreover, the homotopy category of the $\mathcal{B}_C$-Bass model structure is triangle equivalent to the stable category $\perp \text{Add}_R(C) \cap \mathcal{B}_C(R)$.

Proof. 1. $\Rightarrow$ 2. It follows from the hypothesis that $(\mathcal{Q}, \mathcal{R}) = (\perp \text{Add}_R(C), \text{Add}_R(C))$ is a complete hereditary cotorsion pair. So, the assertion follows by Corollary 5.5.

The rest of this result can be proven as in Theorem 4.15 and Corollary 4.17.

Dually, we have:

Theorem 5.9 (The $\mathcal{A}_C$-Auslander model structure) Let $C$ be a semidualizing $(R, S)$-bimodule. The following assertions are equivalent:

1. There exists a unique hereditary abelian model structure on $\text{Mod}\text{-}R$ such that:
   - The cofibrant objects coincide with the $\mathcal{A}_C$-Auslander right $R$-modules.
   - The class of fibrant objects coincide with $\text{Prod}_R(C^+)\perp$.
   - The trivially cofibrant objects coincide with the $\mathcal{I}_C^+$-injective right $R$-modules.
   - The trivially fibrant objects coincide with the $R$-modules in $\mathcal{A}_C(R)\perp$.

2. $S$ is right noetherian, $C_S$ is injective and $\text{Prod}_R(C^+)$ is closed under kernels of epimorphisms.

In this case, the category $\mathcal{A}_{c,f} = \mathcal{A}_C(R) \cap \text{Prod}_R(C^+)$ is a Frobenius category. The projective-injective objects exactly the $\mathcal{I}_C^+$-injective right $R$-modules. Moreover, the homotopy category of the $\mathcal{A}_C$-Auslander model structure is triangle equivalent to the stable category $\mathcal{A}_C(R) \cap \text{Prod}_R(C^+)$.

Let $C$ be a semidualizing $(R, R)$-bimodule. Recall [7, Definition 5.1] that an extension closed full subcategory $\mathcal{F}$ of $R\text{-Mod}$ is said to be a $C$-Frobenius category provided that it has enough projective and enough injective objects and $\mathcal{I}(\mathcal{F}) = C \otimes_R \mathcal{P}(\mathcal{F})$ where $\mathcal{I}(\mathcal{F})$ and $\mathcal{P}(\mathcal{F})$ denote the full subcategories of the projective and injective objects of $\mathcal{F}$, respectively. In particular, $R\text{-Mod}$ is $C$-Frobenius if and only if the injective $R$-modules coincide with the $\mathcal{P}_C$-projective modules. Note that $R$ is quasi-Frobenius if and only if $R\text{-Mod}$ is $R$-$R$-Frobenius.

Lemma 5.10 Assume that $C$ is a semidualizing $(R, R)$-bimodule. Then, $R\text{-Mod}$ is $C$-Frobenius if and only if the projective $R$-modules coincide with the $\mathcal{I}_C^+$-injectives.

Proof. Straightforward.
Corollary 5.11 Assume that $C$ is a semidualizing $(R, R)$-bimodule. The following assertions are equivalent:

1. There exists a unique hereditary Hovey triple $\mathcal{M}_Z = (R-\text{Mod}, \mathcal{B}_C(R), \mathcal{B}_C(R))$.
2. $R$-Mod is $C$-Frobenius.
3. There is a unique hereditary Hovey triple $\mathcal{M}_P = (\mathcal{A}_C(R), \mathcal{A}_C(R)^\perp, R-\text{Mod})$.

In this case, the categories $\mathcal{B}_C(R)$ and $\mathcal{A}_C(R)$ are Frobenius categories whose projective-injective objects are exactly the $\mathcal{P}_C$-projective and $\mathcal{I}_C^+$-injective $R$-modules respectively. Moreover, we have the triangle equivalences

$$\text{Ho}(\mathcal{M}_Z) \cong \mathcal{B}_C(R) \text{ and } \text{Ho}(\mathcal{M}_P) \cong \mathcal{A}_C(R).$$

When $C = R$, the above two model structures coincide. Their homotopy category is known as the stable module category. The stable module category is the main object of study in modular representation theory and is discussed in detail by Hovey in [32, Section 2.2].

Corollary 5.12 ([32, Section 2.2]) The following assertions are equivalent:

1. There exists a unique hereditary Hovey triple $(R-\text{Mod}, \mathcal{P}(R), R-\text{Mod})$.
2. $R$ is quasi-Frobenius.
3. There exists a unique hereditary Hovey triple $(R-\text{Mod}, \mathcal{I}(R), R-\text{Mod})$.

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