Ramsey Numbers for Non-trivial Berge Cycles

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Abstract

In this paper, we consider an extension of cycle-complete graph Ramsey numbers to Berge cycles in hypergraphs: for \( k \geq 2 \), a non-trivial Berge \( k \)-cycle is a family of sets \( e_1, e_2, \ldots, e_k \) such that \( e_1 \cap e_2, e_2 \cap e_3, \ldots, e_k \cap e_1 \) has a system of distinct representatives and \( e_1 \cap e_2 \cap \cdots \cap e_k = \emptyset \). In the case that all the sets \( e_i \) have size three, let \( B_k \) denotes the family of all non-trivial Berge \( k \)-cycles. The Ramsey numbers \( R(t, B_k) \) denote the minimum \( n \) such that every \( n \)-vertex 3-uniform hypergraph contains either a non-trivial Berge \( k \)-cycle or an independent set of size \( t \). We prove

\[
R(t, B_{2k}) \leq t^{1 + \frac{1}{2k-1} + \frac{4}{\log t}}
\]

and moreover, we show that if a conjecture of Erdős and Simonovits [12] on girth in graphs is true, then this is tight up to a factor \( t^{o(1)} \) as \( t \to \infty \).

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1 Introduction

Let $\mathcal{F}$ be a family of $r$-graphs and $t \geq 1$. The Ramsey numbers $R(t, \mathcal{F})$ denote the minimum $n$ such that every $n$-vertex $r$-graph contains either a hypergraph in $\mathcal{F}$ or an independent set of size $t$. For $k \geq 2$, a Berge $k$-cycle is a family of sets $e_1, e_2, \ldots, e_k$ such that $e_1 \cap e_2, e_2 \cap e_3, \ldots, e_k \cap e_1$ has a system of distinct representatives, and a Berge cycle is non-trivial if $e_1 \cap e_2 \cap \cdots \cap e_k = \emptyset$. Let $\mathcal{B}^r_k$ denote the family of non-trivial Berge $k$-cycles all of whose sets have size $r$. When $r = 2$, $\mathcal{B}^2_k = \{C_k\}$, where $C_k$ denotes the graph cycle of length $k$. In this paper, we let $\mathcal{B}^r_k = \mathcal{B}^3_k$.

It is a notoriously difficult problem to determine even the order of magnitude of $R(t, C_k)$ – the cycle-complete graph Ramsey numbers. Kim [18] proved $R(t, C_3) = \Omega(t^2 / \log t)$, which gives the order of magnitude of $R(t, C_3)$ when combined with the results of Ajtai, Komlós and Szemerédi [2] and Shearer [30]. The current state-of-the-art results on $R(t, C_3)$ are due to Fiz Pontiveros, Griffiths and Morris [13] and Bohman and Keevash [6], using the random triangle-free process, which determines $R(t, C_3)$ up to a small constant factor.

$$\left(\frac{1}{4} - o(1)\right) \frac{t^2}{\log t} \leq R(t, C_3) \leq (1 + o(1)) \frac{t^2}{\log t}. $$

The case $R(t, C_4)$ is the subject of a notorious conjecture of Erdős [7], where he conjectured that $R(t, C_4) = o(t^{2-\epsilon})$ for some $\epsilon > 0$. The current best upper bounds on $R(t, C_{2k})$ is

$$O\left(\left(\frac{t}{\log t}\right)^{k/(k-1)}\right),$$

which come from the work of Caro, Li, Rousseau and Zhang [9]. For $R(t, C_{2k+1})$, the best upper bound is

$$O\left(\frac{t^{(k+1)/k}}{\log^{1/k} t}\right)$$

due to Sudakov [31]. Recent results using pseudorandom graphs by Mubayi and the second author [26] give the best lower bounds on cycle-complete graph Ramsey numbers:

$$R(C_k, n) = \Omega\left(\frac{t^{(k-1)/(k-2)}}{\log^{2/(k-2)} t}\right).$$

In particular, via random block constructions, they show that

$$R(C_5, t) \geq (1 + o(1))t^{11/8}, \quad R(C_7, t) \geq (1 + o(1))t^{11/9}.$$
For $k \geq 3$, a loose $k$-cycle is a non-trivial Berge $k$-cycle, denoted $C_k^r$, with sets $e_1, e_2, \ldots, e_k$ of size $r$ such that $|e_1 \cap e_2| = 1$, $|e_2 \cap e_3| = 1$, $|e_k \cap e_1| = 1$, and for any other pairs of edges $e_i, e_j$, $e_i \cap e_j = \emptyset$. Ramsey type problems for loose cycles in $r$-graphs have been studied extensively [4, 10, 11, 14, 16–20, 24, 26]. For $r$-uniform hypergraphs with $r \geq 3$, Kostochka, Mubayi and the second author [19] proved for all $r \geq 3$, there exist constants $a, b > 0$ such that

$$\frac{at^2}{(\log t)^2} \leq R(t, C_k^r) \leq bt^2,$$

(1)

The following conjecture was proposed in [19]:

**Conjecture I.** For $r, k \geq 3$,

$$R(t, C_k^r) = t^{k-1} + o(1).$$

(2)

The conjecture is true for $k = 3$ due to [11]. It is shown in [28] that $R(t, C_3^3) \leq t^{4+o(1)}$. Mérour [24] showed $R(t, C_k^3) = O(t^{1+1/((k-1)/2)})$ for $k \geq 3$ and $R(t, C_k^r) = O(t^{1+1/\lfloor k/2 \rfloor})$ for $r \geq 4$ and every odd integers $k \geq 5$, improving earlier results of Collier-Cartaino, Graber and Jiang [10]. Conjecture [1] motivates our current study of non-trivial Berge $k$-cycles. In support of the above conjecture, we prove the following result for non-trivial Berge cycles of even length:

**Theorem 1.** For $k \geq 3$, and $t$ large enough,

$$R(t, B_{2k}) \leq t^{2k-1 + \frac{4}{\sqrt{\log t}}}. $$

Erdős and Simonovits [12] conjectured that there exists an $n$-vertex graph of girth more than $2k$ with $\Theta(n^{1+1/k})$ edges. This notoriously difficult conjecture remains open, except when $k \in \{2, 3, 5\}$, largely due to the existence of generalized polygons [3, 32, 33]. Towards this conjecture, Lazebnik, Ustimenko and Woldar [22] gave the densest known construction, which has $\Omega(n^{1+2/(3k-2)})$ edges. We prove the following theorem relating this conjecture to lower bounds on Ramsey numbers for non-trivial Berge cycles:

**Theorem 2.** Let $k \geq 2$, $r \geq 3$. Suppose there exists an $n$-vertex graph of girth more than $2k$ with $cn^{1+1/k}$ edges for any integer $n$ large enough and some positive constant $c$. Then for $t$ large enough and some positive constant $c_{k,r}$ dependent on $k$ and $r$,

$$R(t, B_k^r) \geq c_{k,r} \left( \frac{t}{\log t} \right)^{\frac{1}{r-1}}.$$

(3)

This shows that if the Erdős-Simonovits Conjecture is true, then Theorem [1] is tight up to a
$t^{o(1)}$ factor. Indeed, following the proof of Theorem 2, the known construction of Lazebnik, Ustimenko and Woldar [22] would give a weaker lower bound of $\Omega((t/\log t)^{(3k-2)/(3k-4)})$.

Let $B_k$ be the family of 3-uniform Berge $k$-cycles without non-triviality. Random graphs together with the Lovász local lemma give $R(t, B_k) \geq t^{(2k^2/(2k^2-k-2)-o(1))}$. We prove the following theorem, which gives a substantially better lower bound for $B_4$ if the Erdős-Simonovits Conjecture is true.

**Theorem 3.** Suppose there exists an $n$-vertex graph of girth more than $8$ with $c_1n^{5/4}$ edges for any integer $n$ large enough and some positive constant $c_1$. Then for $t$ large enough and some positive constant $c_2$,

$$R(t, B_4) \geq \left(\frac{c_2 t}{\sqrt{\log t}}\right)^{16/13}.$$

In fact, this is also a lower bound for $R(t, \{B_2, B_3, B_4\})$. A natural 3-uniform analog of the Erdős-Simovits conjecture is that there exist $n$-vertex $\{B_2, B_3, \ldots, B_k\}$-free 3-graphs with $n^{1+1/[k/2]-o(1)}$ edges. This is true for $k = 3$ due to Ruzsa and Szemerédi [29]. The proof of Theorem 3 makes use of the fact that there exist $n$-vertex $\{B_2, B_3, B_4\}$-free 3-graphs with $\Omega(n^{3/2})$ edges, that is, the conjecture is true for $k = 4$, which is due to Lazebnik and the second author [23]. More generally, following the proof of Theorem 3 if the 3-uniform analog of the Erdős-Simonovits Conjecture is true, then we have $R(t, \{B_2, B_3, \ldots, B_{2k}\}) \geq t^{2k^2/(2k^2-k-2)-o(1)}$ and $R(t, \{B_2, B_3, \ldots, B_{2k+1}\}) \geq t^{2k(k-1)/(2k^2-3k-1)-o(1)}$, which are substantially better than the lower bounds obtained by random graphs.

We prove Theorem 1 in Section 5, Theorem 2 in Section 2 and Theorem 3 in Section 3. Theorem 2 is valid for all values of $k \geq 2$ and $r \geq 3$, while Theorem 1 only works for even values of $k$ and $r = 3$. We believe that Theorem 1 should extend to odd values of $k$ and all $r \geq 3$.

**Conjecture II.** For all $r, k \geq 3$,

$$R(t, B_k^r) \leq t^{\frac{k}{r}+o(1)}.$$  

(4)

**Notation and terminology.** For a hypergraph $H$, let $V(H)$ denote the vertex set of $H$, $v(H) = |V(H)|$ and let $|H|$ be the number of edges in $H$. If all edges of $H$ have size $r$, we say $H$ is an $r$-uniform hypergraph, or an $r$-graph for short. For $v \in V(H)$, let $d_H(v) = |\{e \in H : v \in e\}|$ be the degree of $v$ in $H$. We denote the average degree of $H$ by $\overline{d}(H)$, denote the minimum degree of $H$ by $\delta(H)$, and the maximum degree of $H$ by $\Delta(H)$. For $u, v \in V(H)$, let $d_H(u, v) = |\{w : uvw \in H\}|$ denote the codegree of the pair $\{u, v\}$. An
independent set in a hypergraph is a set of vertices containing no edge of the hypergraph. Let \( \alpha(H) \) denote the largest size of an independent set in a hypergraph \( H \).

## 2 Proof of Theorem 2

We will use the following lemma to get a large bipartite subgraph with large minimum degree and small maximum degree:

**Lemma 4.** Let \( k \geq 3, c > 0 \), and let \( G \) be an \( n \)-vertex graph of girth more than \( 2k \) with more than \( 2cn^{1+1/k} \) edges. Then there exists a bipartite subgraph \( G' \) of \( G \) such that \( \delta(G') \geq cn^{1/k} \), \( \Delta(G') \leq n^{1/k}/ck^{-1} \), and \( v(G') \geq ckn \).

**Proof.** A maximum cut of \( G \) gives a bipartite subgraph with at least \( cn^{1+1/k} \) edges. A subgraph \( G' \) of this bipartite subgraph of minimum degree at least \( cn^{1/k} + 1 \) may be obtained by repeatedly removing vertices of degree at most \( cn^{1/k} \). Let \( \Delta := \Delta(G') \) be the maximum degree of \( G' \), and let \( v \) be a vertex of maximum degree, then the number of vertices at distance \( k \) from \( v \) is at least \( \Delta c^{k-1} n^{(k-1)/k} \), since \( G \) has girth larger than \( 2k \). In particular, \( \Delta c^{k-1} n^{(k-1)/k} \leq n \) and so \( \Delta \leq n^{1/k}/ck^{-1} \). The number of vertices in \( G' \) is at least \( ckn \), since \( G' \) has minimum degree at least \( cn^{1/k} + 1 \) and girth larger than \( 2k \). \( \square \)

Let \( r \geq 2 \), a *star* with vertex set \( V \) is an \( r \)-graph on \( V \) consisting of all edges containing a fixed vertex of \( V \), i.e., the edge set of a star is \( \{e \subset V : |e| = r, v \in e\} \) for some vertex \( v \in V \). Let integers \( d \geq m \) and let \( S_{d,m} \) be a \( d \)-vertex \( r \)-graph consisting of \( m \) vertex-disjoint stars of size \( \lfloor d/m \rfloor \) or \( \lceil d/m \rceil \).

**Lemma 5.** Let integer \( r \geq 2 \), and let integers \( d \geq m \). The probability that a uniformly chosen set of \( s \) vertices of \( S_{d,m} \) is independent is at most

\[
\exp \left( -\frac{m(s - rm)}{2d} \right).
\]

**Proof.** Let the vertex sets of these stars be \( V_1, V_2, \ldots, V_m \). The probability that a uniformly chosen set of \( s_i \) vertices in \( V_i \) is independent in \( S_{d,m} \) is at most \( 1 - \frac{s_i}{[d/m]} \leq 1 - ms_i/2d \) if \( s_i \geq r \), and is 1 if \( s_i < r \). Hence, this probability is at most \( 1 - m(s_i - r)/2d \) for \( 0 \leq s_i \leq d \). Therefore a uniformly chosen set \( I \subset S_{d,m} \) of \( s \) vertices with \( |I \cap V_i| = s_i \) is independent with probability at most

\[
\prod_{i=1}^{m} \left( 1 - \frac{m(s_i - r)}{2d} \right) \leq \exp \left( -\sum_{i=1}^{m} \frac{m(s_i - r)}{2d} \right) = \exp \left( -\frac{m(s - rm)}{2d} \right).
\]
Now we are ready to prove Theorem 2.

**Proof of Theorem 2.** It suffices to show that for \( n \) large enough, there exists an \( n \)-vertex \( B^r_k \)-free \( r \)-graph with independence number \( O(n^{1 - \frac{1}{k}} \log n) \). Let \( G \) be an \( n \)-vertex graph of girth more than \( 2k \) with \( 2cn^{1+1/k} \) edges for some positive constant \( c \). By Lemma 4, there exists a bipartite subgraph \( G' \) of \( G \) with at least \( N = e^k n \) vertices, minimum degree at least \( cn^{1/k} \) and maximum degree at most \( n^{1/k} / c^{k-1} \). Let \( X, Y \) be the parts of this bipartite graph where \( |Y| \geq |X| \). Let \( m = 8 \log n / e^k \). We form an \( r \)-graph \( H \) with vertex set \( Y \) by placing a random copy of \( S_{d(x),m} \) on the vertex set \( N_G'(x) \), the neighborhood of \( x \) in \( G' \), independently for each \( x \in X \). Since \( G' \) has girth more than \( 2k \), it is straightforward to check that \( H \) does not contain any non-trivial Berge \( k \)-cycle. We now compute the expected number of independent sets of size \( t = rmn^{1-1/k}/e^{k+1} \) in \( H \). Clearly, \( \log t \geq (1 - 1/k) \log n \). If \( H \) has no independent set of size \( t \) with positive probability, then since \( v(H) \geq N/2 \), we find that

\[
R(t, B^r_k) \geq N/2 \geq \frac{c^k}{2} \left( \frac{c^{2k+1}t}{8r \log n} \right)^{\frac{k}{k-1}} \geq c_{k,r} \left( \frac{t}{\log t} \right)^{\frac{k}{k-1}},
\]

for some positive constant \( c_{k,r} \). This is enough to prove Theorem 2.

For an independent \( t \)-set \( I \) in \( H \), \( I \cap N_G'(x) \) is an independent set in \( S_{d(x),m} \) for all \( x \in X \). Since these events are independent, setting \( s(x) = |I \cap N_G'(x)| \), and applying Lemma 5 gives:

\[
\mathbb{P}(I \text{ independent in } H) \leq \prod_{x \in X} \exp \left( -\frac{m(s(x) - rm)}{2d(x)} \right) = \exp \left( -\sum_{x \in X} \frac{ms(x)}{2d(x)} + \sum_{x \in X} \frac{rm^2}{2d(x)} \right).
\]

For every \( x \in X \), \( cn^{1/k} \leq d(x) \leq n^{1/k} / e^{k-1} \) and therefore

\[
\mathbb{P}(I \text{ independent in } H) \leq \exp \left( -\frac{c^{k-1}m \sum_{x \in X} s(x)}{2n^{1/k}} + \frac{|X|rm^2}{2cn^{1/k}} \right).
\]

Now \( \sum_{x \in X} s(x) \) is precisely the number of edges of \( G' \) between \( X \) and \( I \). Since every vertex in \( I \) has degree at least \( cn^{1/k} \), this number of edges is at least \( cn^{1/k}t = rmn / e^k \). Consequently, using \( |X| < n/2 \),

\[
\mathbb{P}(I \text{ independent in } H) \leq \exp \left( -\frac{c^kmt}{2} + \frac{c^kmt}{4} \right) = \exp \left( -\frac{c^kmt}{4} \right).
\]
The expected number of independent sets of size \( t \) is at most
\[
\binom{n}{t} \exp \left( -\frac{c^kmt}{4} \right) < \exp \left( t \log n - \frac{c^kmt}{4} \right) = \exp (-t \log n).
\]
This is vanishing as \( n \to \infty \), and the proof of Theorem 2 is complete.

\section{Proof of Theorem 3}

Lazebnik and the second author \cite{23} showed that there exist \( n \)-vertex \( B_4 \)-free 3-graphs with \((1/6 + o(1))n^{3/2}\) triples. More specifically, for \( n \) large enough, there exists a linear \( n \)-vertex \( B_4 \)-free 3-graphs \( J_n \) with \( n^{3/2}/10 \) triples and maximum degree at most \( n^{1/2} \). We want to find an upper bound for the probability that a random \( s \)-set is independent in \( J_n \). We make use of the following lemma, where we make no effort to optimize the constants.

\textbf{Lemma 6.} Let \( n, s \) be integers such that \( s < \sqrt{n}/2 \). For \( n \) large enough, the probability that a uniformly chosen set of \( s \) vertices of \( J_n \) is independent is at most
\[
\exp \left( -\frac{s^3 - 216}{80n^{3/2}} \right).
\]
When \( s \geq \sqrt{n}/2 \), the probability is at most \( 639/640 \).

\textit{Proof.} This is trivial when \( s < 6 \). When \( 6 < s < \sqrt{n}/2 \), let \( X \) be the uniformly chosen \( s \)-set. For any edge \( e \in E(J_n) \), let \( A_e \) be the event that \( e \in X \). Then by inclusion-exclusion principle, for \( n \) large enough, the probability that \( X \) is not independent is at least
\[
\sum_{e \in E(J_n)} \mathbb{P}(A_e) - \sum_{\{e, f\} \in E(J_n)} \mathbb{P}(A_e \wedge A_f) \geq \frac{1}{s} \binom{n^{3/2}}{s} \left( \frac{n - 3}{10} \right) - n \binom{(n^{1/2})}{s} \binom{n - 5}{2} - \binom{n^{3/2}/10}{s} \binom{n - 6}{2} \binom{n - 6}{s} \geq s^3 \frac{1}{40n^{3/2}} \left( 1 - \frac{4s^3}{n^{3/2}} \right) \geq \frac{s^3}{80n^{3/2}}.
\]
Therefore, for \( s > 6 \) and \( n \) large enough, the probability that \( X \) is independent is at most
\[
1 - \frac{s^3}{80n^{3/2}} \leq \exp \left( -\frac{s^3}{80n^{3/2}} \right) < \exp \left( -\frac{s^3 - 216}{80n^{3/2}} \right),
\]
When $s \geq \sqrt{n}/2$, the probability is at most

$$1 - \left(\frac{\sqrt{n}/2}{80n^{3/2}}\right)^3 = \frac{639640}{640^3}.$$ 

Now we are ready to prove Theorem 3.

Proof of Theorem 3 Let $G$ be an $n$-vertex graph of girth more than 8 with $2c_1n^{5/4}$ edges for some positive constant $c_1$. By Lemma 4, there exists a bipartite subgraph $G'$ of $G$ with at least $N = c_1n^{1/4}$ vertices, minimum degree at least $c_1n^{1/4}$ and maximum degree at most $n^{1/4}/c_1^3$. Let $X$, $Y$ be the parts of this bipartite graph where $|Y| \geq |X|$. We form a 3-graph $H$ with vertex set $Y$ by placing a random copy of $J_{d(x)}$ on the vertex set $N_{G'}(x)$, the neighborhood of $x$ in $G$, independently for each $x \in X$. Since $G$ has girth more than $2k$, it is straightforward to check that $H$ does not contain any Berge 4-cycle. Let $m = 8c_1^{1/4}\sqrt{\log n}$, and let $t = mn^{13/16}$. Clearly, $\log t > 13\log n/16$. If $H$ has no independent sets of size $t$ with positive probability, then since $v(H) \geq N/2$, we conclude that

$$R(t, B_4) \geq N/2 \geq \frac{c_1^4}{2} \left(\frac{t}{8c_1^{1/4}\sqrt{\log n}}\right)^{16/13} \geq c_2 \left(\frac{t}{\sqrt{\log t}}\right)^{16/13},$$

for some positive constant $c_2$. This is enough to prove Theorem 3.

Let $A$ be a $t$-set in $Y$, and let $X_A = \{x \in X||N_{G'}(x) \cap A| \geq \sqrt{t}/2\}$, $\overline{X}_A = X \setminus A$. We now evaluate the probability that $A$ is independent in $H$ in two cases.

Case 1: When $|X_A| < n^{5/6}$. Since the induced bipartite subgraph of $G'$ on $X_A \cup A$ has girth 8, the number of edges of $G'$ between $X_A$ and $A$ is less than $(n^{5/6})^{5/4} = n^{25/24}$. If $A$ is independent in $H$, then $N_{G'}(x) \cap A$ is also independent in $J_{d(x)}$ for all $x \in X$. Since these events are independent, setting $s(x) = |N_{G'}(x) \cap A|$, and applying Lemma 6 gives

$$\mathbb{P}(A \text{ independent in } H) \leq \prod_{x \in X_A} \exp\left(-\frac{s(x)^3 - 216}{80d(x)^{3/2}}\right)$$

$$= \exp\left(-\sum_{x \in X_A} \frac{s(x)^3}{80d(x)^{3/2}} + \sum_{x \in \overline{X}_A} \frac{27}{10d(x)^{3/2}}\right).$$

For every $x \in X$, $c_1n^{1/4} \leq d(x) \leq n^{1/4}/c_1^3$ and hence together with Jenson’s inequality we
have
\[ P(A \text{ independent in } H) \leq \exp \left( -\frac{c_1^{9/2} \sum_{x \in X_A} s(x)}{80n^{3/8}} + \frac{27|X_A|}{10c_1^{3/2}n^{3/8}} \right) \]
\[ \leq \exp \left( -\frac{c_1^{9/2} \left( \sum_{x \in X_A} s(x) \right)^3}{80n^{3/8}|X_A|^2} + \frac{27|X_A|}{10c_1^{3/2}n^{3/8}} \right). \]

Note that \( \sum_{x \in X_A} s(x) \) is exactly the number of edges of \( G' \) between \( X_A \) and \( A \), which is at least \( tc_1n^{1/4} - n^{25/24} = (1 - o(1))c_1mn^{17/16}. \) Also note that \( |X_A| < N/2 = c_1^4n/2. \) Consequently,
\[ P(A \text{ independent in } H) \leq \exp \left( -\frac{(1 - o(1))m^{3} n^{13/16}}{20c_1^{1/2}} + \frac{27c_1^{5/2}n^{5/8}}{20} \right) \]
\[ < \exp \left( -\frac{m^{3} n^{13/16}}{32c_1^{1/2}} \right). \]

**Case 2:** When \( |X_A| \geq n^{5/6} \). Applying Lemma 6 gives
\[ P(A \text{ independent in } H) \leq \frac{(639/640)|X_A|}{\exp(-n^{5/6}/640)} < \exp \left( -\frac{m^{3} n^{13/16}}{32c_1^{1/2}} \right). \]

In both cases we have \( P(A \text{ independent in } H) < \exp \left( -\frac{m^{3} n^{13/16}}{32c_1^{1/2}} \right). \) Therefore the expected number of independent sets of size \( t \) in \( H \) is at most
\[ \binom{n}{t} \exp \left( -\frac{m^{3} n^{13/16}}{32c_1^{1/2}} \right). \]
\[ < \exp \left( mn^{13/16} \log n - \frac{m^{3} n^{13/16}}{32c_1^{1/2}} \right) = \exp \left( -mn^{13/16} \log n \right). \]
This is vanishing as \( n \to \infty \), which completes the proof of Theorem 3.

\[ \square \]

## 4 Degrees, codegrees and independent sets

We make use of the following elementary lemma, whose proof is a standard probabilistic argument, included for completeness:

**Lemma 7.** Let \( d \geq 1 \), and let \( H \) be a 3-graph of average degree at most \( d \). Then
\[ \alpha(H) \geq \frac{2\nu(H)}{3d^2}. \]
Proof. Let $X$ be a subset of $V(H)$ whose elements are chosen independently with probability $p = d^{-1}/2$. We can get an independent set by deleting a vertex for each edge of $H$ contained in $X$. Then the expected size of such independent set is at least

$$pv(H) - p^3|H| = pv(H) - \frac{p^3dv(H)}{3} = \frac{2v(H)}{3d^{1/2}}.$$ 

Hence, there must exist an independent set of size at least the desired lower bound, which completes the proof.

Lemma 8. Let $H$ be a 3-graph on $n$ vertices, and $0 < \epsilon < 1/2$. Then there exists an induced subgraph $G$ of $H$ satisfying the following properties:

1. $v(G) \geq n \frac{1}{\log_2(\frac{1}{\epsilon})}$,
2. $\Delta(G) \leq \frac{d(G)}{\epsilon}$.

Proof. Let $H = G^{(0)}$. We do the following for $i \geq 0$. If $\Delta(G^{(i)}) \leq d(G^{(i)})/\epsilon$, we let $G = G^{(i)}$. Otherwise, iteratively delete vertices of $G^{(i)}$ with degree at least $d(G^{(i)})$. Each deleted vertex will result in the loss of at least $d(G^{(i)})$ edges. So we can delete at most

$$\frac{|G^{(i)}|}{d(G^{(i)})} = \frac{v(G^{(i)}) \cdot d(G^{(i)})}{3 \cdot d(G^{(i)})} = \frac{v(G^{(i)})}{3} < \frac{v(G^{(i)})}{2}$$

vertices in this step. Let $G^{(i+1)}$ be the subgraph induced by the remaining vertices. Then we have $v(G^{(i+1)}) > v(G^{(i)})/2$. If $\Delta(G^{(i+1)}) \leq d(G^{(i+1)})/\epsilon$, then we let $G = G^{(i+1)}$. Otherwise, we have

$$d(G^{(i+1)}) \leq \epsilon \Delta(G^{(i+1)}) < \epsilon d(G^{(i)}).$$

Let $K = 2 \log_{1/\epsilon} n$. We must obtain an induced subgraph $G$ with $\Delta(G) \leq d(G)/\epsilon$ after at most $K$ repetitions. Otherwise, after $K$ repetitions, since the average degree decreases by at least a factor of $\epsilon$ after each repetition, the remaining graph $G^{(K)}$ will have no edge, which satisfies the condition $\Delta(G^{(K)}) \leq d(G^{(K)})/\epsilon$. Suppose after $m \leq K$ repetitions we have the desired induced subgraph $G$ with $\Delta(G) < d(G)/\epsilon$. Since the number of vertices decreases by at least a factor of 2, we also have

$$v(G) > \frac{n}{2^m} \geq n \frac{1}{\log_2(\frac{1}{\epsilon})}.$$ 

This completes the proof.
We use the following slightly weaker version of a lemma due to Méroueh [24]; the lemma is in fact valid for 3-graphs $H$ with no loose $k$-cycles:

**Lemma 9.** Let $H$ be a $\mathcal{B}_k$-free 3-graph. Then there exists a subgraph $H^*$ of $H$ such that $|H^*| > |H|/(3k^2)$ and each edge of $H^*$ contains a pair of codegree 1.

**Proof.** Given a 3-graph $G$ and a pair of vertices $x, y$, we say that $\{x, y\}$ is $G$-light if $d_G(x, y) < k$. Let $G_1 = H$, and let $H_1$ consist of all edges of $G_1$ containing a $G_1$-light pair, and let $G_2 = G_1 \setminus H_1$. For $i \geq 2$, let $H_i$ consist of all edges of $G_i$ containing a $G_i$-light pair, and let $G_{i+1} = G_i \setminus H_i$. Suppose for contradiction that $G_k$ is not empty. Let $e_1 = \{v_1, v_2, v_3\}$ be an edge in $G_k$, then by definition, $\{v_2, v_3\}$ is not a $G_{k-1}$-light pair, and hence, there exists an edge $e_2 = \{v_2, v_3, v_4\}$ such that $v_4 \neq v_1$. For $2 \leq i \leq k-1$, let $e_i = \{v_i, v_{i+1}, v_{i+2}\}$ be an edge in $G_{k-1}$. By definition, $\{v_{i+1}, v_{i+2}\}$ is not a $G_{k-1}$-light pair, and hence, there exists an edge $e_{i+1} = \{v_{i+1}, v_{i+2}, v_{i+3}\}$ in $G_k$ such that $v_{i+3}$ is distinct from all $v_j$, $1 \leq j \leq i$. Therefore, we have a tight path of length $k$ in $G_1 = H$, that is, a hypergraph consisting of $k+2$ distinct vertices $v_i$, $1 \leq i \leq k+2$, and $k$ edges $e_i = \{v_i, v_{i+1}, v_{i+2}\}$, $1 \leq i \leq k$. This is also a non-trivial Berge $k$-cycle. Indeed, when $k$ is even, $\{v_2, v_4, \ldots, v_k, v_{k+1}, v_1, \ldots, v_3\}$ forms a system of distinct representatives of $\{e_1 \cap e_2, e_2 \cap e_4, e_4 \cap e_6, \ldots, e_{k-2} \cap e_k, e_k \cap e_{k-1}, e_{k-1} \cap e_{k-3}, \ldots, e_3 \cap e_1\}$, and when $k$ is odd, $\{v_2, v_4, \ldots, v_{k+1}, v_k, v_{k-2}, \ldots, v_3\}$ forms a system of distinct representatives of $\{e_1 \cap e_2, e_2 \cap e_4, e_4 \cap e_6, \ldots, e_{k-3} \cap e_{k-1}, e_{k-1} \cap e_k, e_k \cap e_{k-2}, \ldots, e_3 \cap e_1\}$. This results in a contradiction, since $H$ is $\mathcal{B}_k$-free. Therefore, $G_k$ must be empty, and hence $H$ can be partitioned into $k-1$ subgraphs $H_i$, $1 \leq i \leq k-1$, such that each $H_i$ consists of edges containing a $G_i$-light pair, which is also $H_i$-light. Let $H'$ be a subgraph $H_i$ with the most edges, then by the pigeonhole principle,

$$|H'| > \frac{|H|}{k}.$$  

Now consider a graph $J$ whose vertex set is the set of 3-edges of $H'$, and two 3-edges of $H'$ form an edge of $J$ if they share an $H'$-light pair. It is easy to see that $J$ has maximum degree at most $3k - 6$. Then we can greedily take an independent set of $J$ of size at least $v(J)/(3k - 5)$, and this independent set correspond to a subgraph $H^*$ of $H'$ such that

$$|H^*| > \frac{|H'|}{3k - 5} > \frac{|H|}{3k^2},$$

and each edge of $H^*$ contains a pair of codegree 1. $\square$
5 Proof of Theorem [1]

A key ingredient of the proof of Theorem [1] is a supersaturation theorem for cycles in graphs: we make use of the following result proved by Simonovits [8] (see Morris and Saxton [25] for stronger supersaturation):

**Lemma 10.** For every \( n, k \geq 2 \), there exist constants \( \gamma, b_0 > 0 \) such that for every \( b \geq b_0 \), any \( n \)-vertex graph \( G \) with at least \( bn^{1+1/k} \) edges contains at least \( \gamma b^{2k} n^2 \) copies of \( C_{2k} \).

We next give a simple lemma which says that if a graph has many cycles of length \( 2k \) containing a fixed edge, then it has many edges.

**Lemma 11.** Let \( G \) be a graph containing \( m \) cycles of length \( 2k \), each containing an edge \( e \in G \). Then \( |G| \geq m^{1/(k-1)}/2 \).

**Proof.** For each cycle \( C \) of length \( 2k \) containing \( e \), let \( M(C) \) be the perfect matching of \( C \) containing \( e \). Fixing a matching \( M \subset G \) of size \( k \) containing \( e \), at most \( (k-1)!2^{k-1} \) cycles \( C \) have \( M(C) = M \). It follows that the number of distinct matchings \( M \subset G \) of size \( k \) containing \( e \) is at least \( m/(k-1)!2^{k-1} \), and therefore

\[
\frac{|G| - 1}{k - 1} \geq \frac{m}{(k-1)!2^{k-1}}.
\]

We conclude \( |G|^{k-1} \geq m/2^{k-1} \) and therefore \( |G| \geq m^{1/(k-1)}/2 \).

Now we are ready to prove Theorem [1]

**Proof of Theorem [1].** It suffices to show that for every large enough integer \( n \), an \( n \)-vertex \( B_{2k} \)-free 3-graph \( H \) contains an independent set of size at least \( n^{(2k-1)/(2k)-5/(2\sqrt{\log n})} \). By Lemma [8] with \( \epsilon = \exp (-\sqrt{\log_2 n}) \), we find an induced subgraph \( H_0 \) of \( H \) with \( n_0 \) vertices, average degree \( d_0 \) and maximum degree \( D_0 \) such that \( n_0 \geq n^{1-2/\sqrt{\log n}} \) and \( D_0 < d_0/\epsilon \). By Lemma [9] there is a subgraph \( H_1 \) of \( H_0 \) with at least \( |H_0|/(4k^2) \) edges such that each edge of \( H_1 \) contains a pair of codegree 1 in \( H_1 \). Let \( \chi : V(H_1) \to \{1, 2, 3\} \) be a random 3-coloring and let \( H_2 \) consist of all triples in \( H_1 \) such that the pair of vertices of colors 1 and 2 has codegree 1 in \( H_1 \) and the last vertex in the triple has color 3. The probability that an edge in \( H_1 \) is also an edge in \( H_2 \) is at least \( 1/27 \), and therefore the expected number of edges in \( H_2 \) is at least \( |H_1|/27 \geq |H_0|/(108k^2) \). Fix a coloring so that \( |H_2| \geq |H_0|/(108k^2) \). Consider the bipartite graph \( G \) comprising all pairs of vertices of colors 1 and 2 contained in an edge of \( H_2 \). Thus, \( |G| = |H_2| \) and \( G \) has average degree \( d_G \geq d_0/(108k^2) \). For convenience, let
\[ b > 0 \text{ be defined by } d_G = 2bn_0^{1/k} \text{ so } |G| = bn_0^{1+1/k}. \] By Lemma 10 there exist constants \( \gamma, b_0 > 0 \) such that if \( b > b_0 \), then \( G \) must contain at least \( \gamma b^{2k}n_0^2 \) copies of \( C_{2k} \). Notice that we must have \( 1/\epsilon > b_0 \) when \( n \) is large enough. The proof is split into two cases.

**Case 1.** \( b \geq 1/\epsilon \). By the pigeonhole principle, there exists an edge \( e \) such that the number of \( C_{2k} \) containing \( e \) in \( G \) is at least
\[
\frac{2k\gamma b^{2k}n_0^2}{|G|} = 2k\gamma b^{2k-1}n_0^{1-\frac{1}{k}}.
\]
Let \( G' \) be the union of all \( 2k \)-cycles in \( G \) containing \( e \). Then by Lemma 11 for some constant \( c \),
\[
|G'| \geq cb^{2+\frac{1}{k}}n_0^{\frac{1}{k}} = \frac{1}{2}cb^{1+\frac{1}{k}}d_G \geq \frac{1}{216k^2}c\epsilon^{-1-\frac{1}{k}}d_0 > D_0
\]
provided \( n \) is large enough. Let \( C \) be a \( 2k \)-cycle in \( G \) containing \( e \). Then there exist edges \( e_1 \cup \{v_1\}, e_2 \cup \{v_2\}, \ldots, e_{2k} \cup \{v_{2k}\} \) in \( H_2 \) where \( e_1, e_2, \ldots, e_{2k} \in C \) and \( v_1, v_2, \ldots, v_{2k} \) have color 3. Since \( H_2 \) is \( \mathcal{B}_{2k} \)-free, for some vertex \( z \) we have \( v_1 = v_2 = \cdots = v_{2k} = z \). Since each cycle \( C \) in \( G' \) contain \( e \), they must have the same \( z \). Now the degree of \( z \) in \( H_2 \) is at least \( |G'| > D_0 \), which contradicts the fact that \( H_0 \) has maximum degree at most \( D_0 \).

**Case 2.** \( b < 1/\epsilon \). In this case, \( d_G < 2n_0^{1/k}/\epsilon \) and so \( d_0 < (216k^2/\epsilon)n_0^{1/k} \). By Lemma 7 on \( H_0 \),
\[
\alpha(H) \geq \alpha(H_0) \geq \frac{2n_0}{3d_0^2} \geq \frac{2}{3} \left( \frac{216k^2}{\epsilon} \right)^{-\frac{1}{k}} n_0^2 \geq \frac{1}{9}\sqrt[2k]{n}^{\frac{2k-1}{2k}} - \frac{5k-2}{2k}\sqrt[2k]{n} \geq n^{\frac{2k-1}{2k}} - \frac{5}{2\sqrt[2k]{n}}.
\]
Now let \( n = t^{\frac{2k-1}{2k}} + \frac{5}{2\sqrt[2k]{n}} \). Clearly, \( \log n > \frac{2k}{2k-1} \log t \). Hence, an \( n \)-vertex \( \mathcal{B}_{2k} \)-free 3-graph \( H \) contains an independent set of size
\[
n^{2k-1 + \frac{5}{2\sqrt[2k]{n}}} = t^{\left( \frac{2k}{2k-1} + \frac{5}{2\sqrt[2k]{n}} \right)} > t
\]
provided \( n \) is large enough. Therefore, we have \( R(t, \mathcal{B}_{2k}) < t^{\frac{2k}{2k-1} + \frac{5}{2\sqrt[2k]{n}}} \).

In fact, by more careful computation, we can obtain a slightly better upper bound \( R(t, \mathcal{B}_{2k}) < t^{\frac{2k}{2k-1} + \frac{c}{\sqrt[2k]{n}}} \), where \( c > \frac{5k-2}{2k-1} \cdot \sqrt[2k]{n} \).

### 6 Concluding remarks

- Notice that Theorem 2 is valid for odd values of \( k \), we believe that Theorem 1 should extend to odd values of \( k \). An obstacle to applying the same idea as in the proof for
even values of $k$ is that we don’t have “good” supersaturation for odd cycles. New ideas may be required to complete the proof for odd values.

- It seems likely that Theorem 1 can be extended to $r$-uniform hypergraphs with $r \geq 4$, however when following the proof of Theorem 1, two obstacles arise. The first is that one requires supersaturation for Berge cycles in $r$-uniform hypergraphs for $r \geq 3$ (in other words, an $r$-uniform version of Lemma 8). A second obstacle is that an $r$-uniform analog of Lemma 9 is not straightforward: for instance if an edge $e$ in an $r$-graph is contained in $m$ Berge cycles of length $2k$, then the number of edges may be as low as $m^{1/(2k-1)}$: take a graph $2k$-cycle, and replace one edge with the hyperedge $e$, and each other edge with $m^{1/(2k-1)}$ hyperedges. We believe these technical obstacles may be overcome (some of the ideas in the recent paper of Mubayi and Yepremyan [27] may apply).

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