MAGNUS-TYPE INTEGRATOR FOR NON-AUTONOMOUS SPDEs DRIVEN BY MULTIPLICATIVE NOISE

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Abstract. This paper aims to investigate numerical approximation of a general second order non-autonomous semilinear parabolic stochastic partial differential equation (SPDE) driven by multiplicative noise. Numerical approximations of autonomous SPDEs are thoroughly investigated in the literature, while the non-autonomous case is not yet understood. We discretize the non-autonomous SPDE by the finite element method in space and the Magnus-type integrator in time. We provide a strong convergence proof of the fully discrete scheme toward the mild solution in the root-mean-square $L^2$ norm. The result reveals how the convergence orders in both space and time depend on the regularity of the noise and the initial data. In particular, for multiplicative trace class noise we achieve convergence order $O\left(h^2 \left(1 + \max(0, \ln \left(t_m/h^2\right)) + \Delta t^{\frac{1}{2}}\right)\right)$. Numerical simulations to illustrate our theoretical finding are provided.

1. Introduction. We consider the numerical approximations of the following semilinear parabolic non-autonomous SPDE driven by multiplicative noise

\[
\begin{aligned}
&\frac{dX}{dt} = [A(t)X + F(t,X)]dt + B(t,X)dW(t), & &\text{in } \Lambda \times (0,T], \\
&X(0) = X_0, & &\text{in } \Lambda,
\end{aligned}
\]

in the Hilbert space $L^2(\Lambda)$, where $\Lambda$ is a bounded domain of $\mathbb{R}^d$, $d = 1, 2, 3$ and $T \in (0, \infty)$. The family of unbounded linear operators $A(t)$ are not necessarily self-adjoint. Each $A(t)$ is assumed to generate an analytic semigroup $S_t(s) := e^{A(t)s}$. The nonlinear functions $F$ and $B$ are respectively the drift and the diffusion parts. Precise assumptions on $A(t)$, $F$ and $B$ to ensure the existence of the unique mild solution of (1) are given in the next section. The random initial data is denoted

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by $X_0$. We denote by $(\Omega, \mathcal{F}, P)$ a probability space with a filtration $(\mathcal{F}_t)_{t \in [0, T]} \subset \mathcal{F}$ that fulfills the usual conditions (see [23, Definition 2.1.11]). The noise term $W(t)$ is assumed to be a $Q$-Wiener process defined in the filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \in [0, T]})$, where the covariance operator $Q : H \rightarrow H$ is assumed to be linear, self-adjoint and positive definite. It is well known [23] that the noise can be represented as

$$W(t, x) = \sum_{i=0}^{\infty} \sqrt{q_i} e_i(x) \beta_i(t),$$  

where $(q_i, e_i)_{i \in \mathbb{N}}$ are the eigenvalues and eigenfunctions of the covariance operator $Q$, and $(\beta_i)_{i \in \mathbb{N}}$ are independent and identically distributed standard Brownian motions. The deterministic counterpart of (1) finds applications in many fields such as quantum fields theory, electromagnetism, nuclear physics (see e.g. [3] and references therein). It is worth to mention that models based on SPDEs can offer a more realistic representation of the system than models based only on PDEs, due to uncertainty in the input data. In many situations it is very hard to exhibit explicit solutions of SPDEs. Numerical algorithms are therefore excellent tools to provide good approximations. Numerical approximations of (1) based on implicit, explicit Euler methods and exponential integrators with $A(t) = A$, where $A$ is self-adjoint are thoroughly investigated in the literature, see e.g. [11, 14, 13, 28, 29, 17, 27] and the references therein. If we turn our attention to the case of time independent operator $A(t) = A$, with $A$ not necessary self-adjoint, the list of references become remarkably short, see e.g., [16, 20]. To the best of our knowledge numerical approximations of (1) with time dependent linear operator $A(t)$ are not yet investigated in the scientific literature, due to the complexity of the associated evolution operators $U(t, s), 0 \leq s \leq t \leq T$. Our aim in this paper is to fill that gap and propose an explicit numerical scheme to approximate (1). We use the finite element method for spatial discretization and Magnus-type integrator for temporal discretization. Magnus-type integrator is based on a truncation of Magnus expansion, which was first proposed in [19] to represent the solution of non-autonomous homogeneous differential equation in the exponential form. Magnus expansion was further studied in [1, 2, 3]. The first numerical method based on Magnus expansion was proposed in [10] for deterministic time-dependent homogeneous Schrödinger equation. The study in [10] was extended in [5] for partial differential equation of the following form

$$u'(t) = A(t)u(t) + b(t), \quad 0 < t \leq T, \quad u(0) = u_0.$$  

To build our novel scheme, we follow [5] and apply the Magnus-type integrator method to the semi-discrete problem (40) and obtain the fully discrete scheme (52), called stochastic Magnus-type integrators (SMTI). We investigate the strong convergence of the new fully discrete scheme toward the mild solution. Due to the complexity of the evolution operators $U(t, s)$ and their corresponding semi discrete version $U_h(t, s)$, novel technical estimates are provided to achieve convergence orders comparable of that of autonomous SPDEs [16, 14, 20]. The result indicates how the convergence orders in both space and time depend on the regularity of the initial data and the noise. In particular for multiplicative trace class noise, we achieve optimal convergence orders of $O\big(h\beta + \Delta t^{\min(\beta, 1)/2}\big)$, where $\beta$ is the regularity parameter, defined in Assumption 2.1.
The rest of this paper is organised as follows. Section 2 provides the general setting, the fully discrete scheme and the main result. In Section 3, we provide some preparatory results and we present the proof of the main result. Section 4 provides some numerical experiments to sustain our theoretical result.

2. Mathematical setting, numerical scheme and main result.

2.1. Notations and main assumptions. Let \((H, \langle \cdot, \cdot \rangle_H, \|\cdot\|)\) and \((U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)\) be two separable Hilbert spaces. We denote by \(L^2(\Omega, U)\) the Banach space of all equivalence classes of square-integrable \(U\)-valued random variables. Let \(L(U, H)\) be the space of bounded linear mappings from \(U\) to \(H\) endowed with the usual operator norm \(\|\cdot\|_{L(U, H)}\). By \(L_2(U, H) := HS(U, H)\), we denote the space of Hilbert-Schmidt operators from \(U\) to \(H\) equipped with the norm

\[ \|l\|_{L_2(U, H)}^2 := \sum_{i=1}^{\infty} \|l\psi_i\|^2, \quad l \in L_2(U, H), \] (4)

where \((\psi_i)_{i=1}^{\infty}\) is an orthonormal basis of \(U\). Note that this definition is independent of the orthonormal basis of \(U\). For simplicity, we use the notations \(L(U) := L(U)\) and \(L_2(U) := L_2(U)\). For all \(l \in L(U, H)\) and \(l_1 \in L_2(U, H)\) we have \(l l_1 \in L_2(U, H)\) and

\[ \|l_1\|_{L_2(U, H)} \leq \|l\|_{L(U, H)} \|l_1\|_{L_2(U)}. \] (5)

The space of Hilbert-Schmidt operators from \(Q^{1/2}(H)\) to \(H\) is denoted by \(L_0^1 := L_2(Q^{1/2}(H), H) = HS(Q^{1/2}(H), H)\). As usual, \(L_0^0\) is equipped with the norm

\[ \|l\|_{L_0^0} := \|lQ^{1/2}\|_H = \left( \sum_{i=1}^{\infty} \|lQ^{1/2}e_i\|^2 \right)^{1/2}, \quad l \in L_0^0, \] (6)

where \((e_i)_{i=1}^{\infty}\) is an orthonormal basis of \(H\). This definition is independent of the orthonormal basis of \(H\). For an \(L_0^0\)-predictable stochastic process \(\phi : [0, T] \times \Lambda \to L_0^0\) such that

\[ \int_0^t \mathbb{E}\|\phi Q^{1/2}\|_{HS}^2 ds < \infty, \quad t \in [0, T], \] (7)

the following relation called Itô isometry holds

\[ \mathbb{E} \left\| \int_0^t \phi dW(s) \right\|^2 = \int_0^t \mathbb{E} \|\phi\|_{L_2}^2 ds = \int_0^t \mathbb{E} \|\phi Q^{1/2}\|_{HS}^2 ds, \quad t \in [0, T], \] (8)

see e.g. [22, Step 2 in Section 2.3.2] or [23, Proposition 2.3.5]. In the sequel of this paper, we consider \(H = L^2(\Lambda, \mathbb{R})\). To guarantee the existence of a unique mild solution of (1) and for the purpose of the convergence analysis, we make the following assumptions.

Assumption 2.1. The initial data \(X_0 : \Omega \to H\) is assumed to be measurable and satisfies \(X_0 \in L^2\left(\Omega, \mathcal{D}\left((-A(0))^\frac{\beta}{2}\right)\right)\), \(0 \leq \beta \leq 2\).

Assumption 2.2. (i) As in [5, 6, 9], we assume that \(\mathcal{D}(A(t)) = D, 0 \leq t \leq T\) and the family of linear operators \(A(t) : D \subset H \to H\) to be uniformly sectorial on \(0 \leq t \leq T\), i.e. there exist two constants \(c > 0\) and \(\theta \in \left(\frac{1}{2}, \pi\right)\) such that

\[ \left\|\left(\lambda I - A(t)\right)^{-1}\right\|_{L(L^2(\Lambda))} \leq \frac{c}{|\lambda|}, \quad \lambda \in S_\theta, \] (9)
where \( S_0 := \{ \lambda \in \mathbb{C} : \lambda = \rho e^{i \theta}, \rho > 0, 0 \leq |\phi| \leq \theta \} \). As in [9], by a standard scaling argument, we assume \(-A(t)\) to be invertible with bounded inverse.

(ii) Similarly to [6, 9, 5, 22], we require the following Lipschitz conditions: there exists a positive constant \( K_1 \) such that
\[
\| (A(t) - A(s)) (A(0))^{-1} \|_{L(H)} \leq K_1 |t - s|, \quad s, t \in [0, T], \tag{10}
\]
\[
\| (-A(0))^{-1} (A(t) - A(s)) \|_{L(D, H)} \leq K_1 |t - s|, \quad s, t \in [0, T]. \tag{11}
\]

(iii) Since we are dealing with non smooth initial value, we follow [21, Theorem 6.1, Chapter 5, Proposition 1.]

\[
\text{Remark 1. } \text{As a consequence of Assumption 2.2 (i) and (iii), for all } \alpha \geq 0 \text{ and } \delta \in [0, 1], \text{ there exists a constant } C_1 > 0 \text{ such that the following estimates hold uniformly for all } t \in [0, T]
\]
\[
\left\| (-A(t))^{\alpha} e^{sA(t)} \right\|_{L(H)} \leq C_1 s^{-\alpha}, \quad s > 0, \tag{14}
\]
\[
\left\| (-A(t)^{-\delta} (I - e^{sA(t)}) \right\|_{L(H)} \leq C_1 s^\delta, \quad s \geq 0, \tag{15}
\]

\text{see e.g. [9, (2.1)].}

**Proposition 1.** [21, Theorem 6.1, Chapter 5] Let \( \Delta(T) := \{(t, s) : 0 \leq s \leq t \leq T\} \).

Under Assumption 2.2 there exists a unique evolution system [21, Definition 5.3, Chapter 5] \( U : \Delta(T) \rightarrow L(H) \) such that

(i) There exists a positive constant \( K_0 \) such that
\[
\| U(t, s) \|_{L(H)} \leq K_0, \quad 0 \leq s \leq t \leq T. \tag{16}
\]

(ii) \( U(t, s) \in C^1([s, T]; L(H)), 0 \leq s \leq T, \)
\[
\frac{\partial U}{\partial t}(t, s) = A(t)U(t, s), \quad 0 \leq s < t \leq T, \tag{17}
\]
\[
\| A(t)U(t, s) \|_{L(H)} \leq \frac{K_0}{t - s}, \quad 0 \leq s < t \leq T. \tag{18}
\]

(iii) \( U(t, \cdot) \in C^1([0, t]; H), 0 < t \leq T, v \in D(A(0)) \) and
\[
\frac{\partial U}{\partial s}(t, s)v = -U(t, s)A(s)v, \quad 0 \leq s \leq t \leq T, \tag{19}
\]
\[
\| A(t)U(t, s)A(s)^{-1} \|_{L(H)} \leq K_0, \quad 0 \leq s \leq t \leq T. \tag{20}
\]

Note that \( D \left((-A(t))^{\frac{\alpha}{2}}\right) = R \left((-A(t))^{\frac{-\alpha}{2}}\right) = (-A(t))^{\frac{-\alpha}{2}} H \subset H, \alpha \geq 0, \) where for an operator \( G, \ R(G) \) stands for its range, see e.g., [24, Section 12.4.2]. We equip \( \mathcal{H}_0^\alpha := D \left((-A(t))^{\frac{\alpha}{2}}\right), \alpha \in \mathbb{R} \) with the norm \( \|u\|_{\alpha, t} := \|(-A(t))^{\frac{\alpha}{2}} u\|. \) Due to (12)-(13) and for ease of notation, we simply write \( \mathcal{H}^\alpha \) and \( \| . \|_\alpha. \) Note that \( \mathcal{H}^\alpha \) equipped with the inner product
\[
\langle u, v \rangle_\alpha := \langle (-A(0))^{\frac{\alpha}{2}} u, (-A(0))^{\frac{\alpha}{2}} v \rangle_H, \quad u, v \in \mathcal{H}_0^\alpha \tag{21}
\]
is a separable $\mathbb{R}$-Hilbert space, since $H = L^2(\Lambda, \mathbb{R})$ is a $\mathbb{R}$-separable Hilbert space. See also [12, Section 2] or [7, Section 2].

We follow [25] and assume that the nonlinear operators $F$ and $B$ satisfy the following Lipschitz condition.

**Assumption 2.3.** The nonlinear operator $F : [0, T] \times H \rightarrow H$ is assumed to be $\beta/2$-Hölder continuous with respect to the first variable and Lipschitz continuous with respect to the second variable, i.e. there exists a positive constant $K_3$ such that

$$
\|F(s, 0)\| \leq K_3, \quad \|F(t, u) - F(s, v)\| \leq K_3 \left( |t - s|^\frac{\beta}{2} + \|u - v\| \right),
$$

for all $s, t \in [0, T]$ and $u, v \in H$.

**Assumption 2.4.** We assume the diffusion function $B : [0, T] \times H \rightarrow L_2^\beta$ to be $\beta/2$-Hölder continuous with respect to the first variable and Lipschitz continuous with respect to the second variable, i.e. there exists a positive constant $K_4$ such that

$$
\|B(s, 0)\|_{L_2^\beta} \leq K_4, \quad \|B(t, u) - B(s, v)\|_{L_2^\beta} \leq K_4 \left( |t - s|^\frac{\beta}{2} + \|u - v\| \right),
$$

for all $s, t \in [0, T]$ and $u, v \in H$.

The following theorem ensures the existence of a unique mild solution of (1).

**Theorem 2.5.** [25, Theorem 1.3] Let Assumptions 2.1, 2.2 (i)–(ii), 2.3 and 2.4 be fulfilled. Then there exists a unique predictable stochastic process $^1X : [0, T] \times \Omega \rightarrow \mathcal{H}^\gamma$ (with $\gamma \in \min\{0, \min(1, \beta)\}$), called mild solution of (1) and satisfying

$$
X(t) = U(t, 0)X_0 + \int_0^t U(t, s)F(s, X(s))ds + \int_0^t U(t, s)B(s, X(s))dW(s),
$$

$\mathbb{P}$-a.s. for all $t \in [0, T]$, where $U(t, s)$ is the evolution system of Proposition 1. Moreover, for all $t \in [0, T]$, $X(t) \in L^2(\Omega, \mathcal{H}^\gamma)$ and there exists a positive constant $K_5$ such that

$$
\sup_{0 \leq t \leq T} \|X(t)\|_{L^2(\Omega, \mathcal{H}^\gamma)} \leq K_5 \left( 1 + \|X_0\|_{L^2(\Omega, \mathcal{H}^\gamma)} \right).
$$

Additionally, the solution process $X(t)$, $t \in [0, T]$ is continuous with respect to $(\mathbb{E}[\|\cdot\|_{L_2^\gamma}])^{\frac{1}{\beta}}$.

To achieve optimal convergence order in space for multiplicative noise when $\beta \in [1, 2]$, we require the following further assumption, also used in [7, 12, 27, 16, 20, 14].

**Assumption 2.6.** We assume that there exists a positive constant $c_1 > 0$, such that $B(s, \mathcal{H}^{\beta-1}) \subset HS \left( Q^{\frac{1}{2}}(H), \mathcal{H}^{\beta-1} \right)$ and

$$
\left\| \left( -A(0) \right)^{\frac{\beta-1}{2}} B(s, v) \right\|_{L_2^\beta} \leq c_1 \left( 1 + \|v\|_{\mathcal{H}^{\beta-1}} \right), \quad v \in \mathcal{H}^{\beta-1}, \quad s \in [0, T],
$$

where $\beta$ comes from Assumption 2.1.

\[1^1\text{up to modifications}\]
2.2. Fully discrete scheme and main result. In the rest of this work, for the sake of simplicity, we assume the family of linear operators $A(t)$ in (1) to be of second order. To have a more precise description of $A(t)$, let us introduce the following second order differential operator

$$A(t) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left( q_{i,j}(t, x) \frac{\partial}{\partial x_j} \right) - \sum_{j=1}^{d} q_j(t, x) \frac{\partial}{\partial x_j} + q_0(t, x) I,$$

(27)

where $q_{i,j}, q_j$ and $q_0$ are $C^1$ functions in $[0, T] \times \overline{X}$ and $q_{i,j}$ satisfies the following ellipticity condition

$$\sum_{i,j=1}^{d} q_{i,j}(t, x) \xi_i \xi_j \geq c_1 |\xi|^2, \quad (t, x) \in [0, T] \times \overline{X}, \quad \xi \in \mathbb{R}^d,$$

(28)

where $c_1 > 0$ is a uniform constant. Furthermore, we assume that there exist $c_2 \geq 0$ and $0 < \gamma \leq 1$ such that the following Hölder continuity holds

$$|q_{i,j}(t, x) - q_{i,j}(s, x)| \leq c_2 |t - s|^{\gamma}, \quad 0 \leq t, s \leq T, \quad x \in \Lambda.$$ 

As in [4, Chapter III], we introduce two spaces $\mathbb{H}$ and $V$, such that $\mathbb{H} \subset V$, depending on the boundary conditions of $-A(t)$. For Dirichlet boundary conditions, we take

$$V = \mathbb{H} = H_0^1(\Lambda) = \overline{C_c^\infty(\Lambda)}^{H^1(\Lambda)}.$$ 

(29)

For Robin boundary condition, we take $V = H^1(\Lambda)$ and

$$\mathbb{H} = \{ v \in H^2(\Lambda) : \partial v/\partial v_A + \alpha_0 v = 0, \text{ on } \partial \Lambda \}, \quad \alpha_0 \in \mathbb{R},$$

(30)

where $\partial v/\partial v_A$ stands for the differentiation along the outer conormal vector $v_A$, pointing at $n = (n_i)_{i=1}^d$, given by

$$\partial v/\partial v_A = \sum_{i,j=1}^{d} n_i(x) q_{i,j}(x) \frac{\partial v}{\partial x_j}, \quad x \in \partial \Lambda.$$ 

(31)

One can easily check that [4, Chapter III, (11.14')] the bilinear operator $a(t)$, associated to $-A(t)$ defined by $a(t)(u, v) = \langle -A(t)u, v \rangle_H$, $u \in D(A(t))$, $v \in V$ satisfies

$$a(t)(v, v) \geq \lambda_0 \| v \|_V^2, \quad v \in V, \quad t \in [0, T],$$

(32)

where $\lambda_0$ is a positive constant, independent of $t$. Note that $a(t)(\cdot, \cdot)$ is bounded in $V \times V$ ([4, Chapter III, (11.13)]), so the following operator $A(t) : V \to V^*$ defined through

$$a(t)(u, v) = \langle -A(t)u, v \rangle \quad u, v \in V, \quad t \in [0, T],$$

is well defined, where $V^*$ is the dual space of $V$ and $\langle \cdot, \cdot \rangle$ the duality pairing between $V^*$ and $V$. Identifying $H$ to its adjoint space $H^*$ by the Riesz representation theorem, we get the following continuous and dense inclusions

$$V \subset H \subset V^*, \quad \text{and therefore} \quad \langle u, v \rangle_H = \langle u, v \rangle, \quad u \in H, \quad v \in V.$$ 

So if we want to replace $\langle \cdot, \cdot \rangle_H$ by the scalar product of $\langle \cdot, \cdot \rangle$ in $H$, we therefore need to have $A(t)u \in H$, for $u \in V$. So the domain of $-A(t)$ is defined as

$$D := D(-A(t)) = D(A(t)) = \{ u \in V, \ A(t)u \in H \}.$$ 

It is well known that [4, Chapter III, (11.11) & (11.11')] in the case of Dirichlet boundary conditions $D = H_0^1(\Lambda) \cap H^2(\Lambda)$ and in the case of Robin boundary conditions $D = \mathbb{H}$ given by (30). We write the restriction of $A(t) : V \to V^*$ to $D(A(t))$
We consider the projection $P$ by splitting the domain $\Lambda$ in finite triangles. Let $A_h > 0$ for all maximal length realization of $A$ defined by $(\cdot)^\alpha$. In the abstract form (1), the nonlinear functions $F : H \rightarrow H$ and $B : H \rightarrow HS(Q \hat{\varphi}(H), H)$ are defined by
\begin{equation}
(F(t,v))(x) = f(x, t, v(x)), \quad (B(t,v)u(x) = b(x, t, v(x))u(x), \quad (34)
\end{equation}
for all $x \in \Lambda$, $v \in H$ and $u \in Q \hat{\varphi}(H)$, where $f : \Lambda \times \mathbb{R} \rightarrow \mathbb{R}$ and $b : \Lambda \times \mathbb{R} \rightarrow \mathbb{R}$ are continuously differentiable functions with globally bounded derivatives.

Let us now turn our attention to the space discretization of the problem (1). We start by splitting the domain $\Lambda$ in finite triangles. Let $T_h$ be the triangulation with maximal length $h$ satisfying the usual regularity assumptions, and $V_h \subset V$ be the space of continuous functions that are piecewise linear over the triangulation $T_h$.

We consider the projection $P_h$ from $H = L^2(\Lambda)$ to $V_h$ defined for every $u \in H$ by
\begin{equation}
\langle P_h u, \chi \rangle_H = \langle u, \chi \rangle_H, \quad u \in H, \chi \in V_h. \quad (35)
\end{equation}
For all $t \in [0, T]$, the discrete operator $A_h(t) : V_h \rightarrow V_h$ is defined by
\begin{equation}
\langle -A_h(t)\phi, \chi \rangle_H = \langle ((-A(t))^{1/2}\phi, ((-A(t))^{1/2}\chi \rangle_H = a(t)(\phi, \chi), \quad \phi, \chi \in V_h. \quad (36)
\end{equation}
Note that $(-A(t))^{1/2}$ is the adjoint of $((-A)(t))^{1/2}$. As in [15, (2.9)] or [4, 8], it follows that there exist constants $C_2 > 0$ and $\delta \in (\frac{1}{4}\pi, \pi)$ such that
\begin{equation}
\| (\lambda I - A_h(t))^{-1} \|_{L(H)} \leq \frac{C_2}{|\lambda|}, \quad \lambda \in S_{\theta} \quad (37)
\end{equation}
holds uniformly for $h > 0$ and $t \in [0, T]$. It also holds that for any $t \in [0, T]$, $A_h(t)$ generates an analytic semigroup $S^t_h(s) = e^{sA_h(t)}$, $s \in [0, T]$ and the smooth properties (14) and (15) hold for $A_h$ uniformly for $h > 0$ and $t \in [0, T]$, i.e., for all $\alpha \geq 0$ and $\delta \in [0, 1]$, there exists a positive constant $C_3$ such that the following estimates hold uniformly for $h > 0$ and $t \in [0, T]$, see e.g., [4, 8]
\begin{equation}
\| (-A_h(t))^{\alpha}e^{sA_h(t)} \|_{L(H)} \leq C_3 s^{-\alpha}, \quad s > 0, \quad (38)
\end{equation}
\begin{equation}
\| (-A_h(t))^{-\delta}(I - e^{sA_h(t)}) \|_{L(H)} \leq C_3 s^\delta, \quad s \geq 0. \quad (39)
\end{equation}
The semi-discrete version of (1) consists of finding $X^h(t) \in V_h$, $t \in [0, T]$ such that $X^h(0) := P_hX_0$ and
\begin{equation}
dX^h(t) = [A_h(t)X^h(t) + P_hF(t, X^h(t))]dt + P_hB(t, X^h(t))dW(t), \quad t \in (0, T). \quad (40)
\end{equation}
Let us consider the following linear non-autonomous differential equations
\begin{equation}
y'(t) = A(t)y(t), \quad t \geq t_0, \text{ with } y(t_0) \text{ given.} \quad (41)
\end{equation}
It is well known that if $A(t)$ is a one dimensional matrix, the solution of (41) is given by
\[ y(t) = \exp \left( \int_{t_0}^{t} A(s) ds \right) y(t_0), \quad t \geq t_0. \]  
\hfill (42)

If $A(t)$ is an $n \times n$ matrix with $n \in \mathbb{N}$ and commute, i.e. $A(t_1)A(t_2) = A(t_2)A(t_1)$, $t_1, t_2 \geq t_0$, then the solution of (41) is again given by (42).

It was shown by Magnus [19, Theorem III] that in the general case where the matrices $A(t)$, $t \in [0,T]$ are time dependents and does not commute, the solution of (41) can be given in the following exponential form
\[ y(t) = e^{\Theta(t-t_0)}y(t_0), \quad t \geq t_0, \]  
\hfill (43)

where $\Theta(t)$ called Magnus expansion is given by the following series [19, (3.28)]
\[ \Theta(t) = \int_{t_0}^{t} A(\tau) d\tau + \frac{1}{2} \int_{t_0}^{t} \left[ A(\tau), \int_{t_0}^{\tau} A(\sigma) d\sigma \right] d\tau \\
+ \frac{1}{4} \int_{t_0}^{t} \left( \int_{t_0}^{\tau} \left[ \int_{t_0}^{\sigma} A(\mu) d\mu, A(\sigma) \right] d\sigma, A(\tau) \right) d\tau \\
+ \frac{1}{12} \int_{t_0}^{t} \left( \int_{t_0}^{\tau} A(\sigma) d\sigma, \int_{t_0}^{\tau} \left[ \int_{t_0}^{\sigma} A(\mu) d\mu, A(\tau) \right] d\sigma \right) d\tau + \cdots. \]  
\hfill (44)

Here the Lie-product $[u,v]$ of $u$ and $v$ is given by $[u,v] = uv - vu$.

For deterministic problems, numerical methods based on this expansion received some attentions since one decade, see e.g. [3, 5, 10, 18]. For the time-dependent Schrödinger equation [5], the Magnus expansion (44) was truncated after the first term and the integral was approximated by the mid-point rule. This mid-point rule approximation of $\Theta(t)$ was also used in [10] to obtain a second-order Magnus type integrator for non-autonomous deterministic parabolic partial differential equation (PDE). Note that the convergence analysis in [5, 10] was only done in time.

Throughout this paper, we take $t_m = m \Delta t \in [0,T]$, where $T = M \Delta t$ for $m, M \in \mathbb{N}$, $m \leq M$. To build our numerical method, we follow the case of linear problem and write the solution of our semi-discrete problem (40) at time $t_{m+1}$ as follows
\[ X^h(t_{m+1}) = e^{\Theta_h(t_{m+1} - t_m)} X^h(t_m) + \int_{t_m}^{t_{m+1}} e^{\Theta_h(s-t_m)} P_h F(s, X^h(s)) ds \]
\hfill (45)

where $\Theta_h$ is given by (44) with $A(t)$ replaced by $A_h(t)$. Let $X^h_m$ be the numerical approximation of $X^h(t_m)$. To approximate the first term of (45), we truncate (44) after the first term and use the rectangle rule with left point rule to approximate the remaining integral. This yields
\[ e^{\Theta_h(t_{m+1} - t_m)} X^h(t_m) \approx e^{ \int_{t_m}^{t_{m+1}} A_h(s) ds } X^h_m \approx e^{\Delta t A_h(t_m)} X^h_m. \]  
\hfill (46)

The second integral in (45) is approximated as follows
\[ \int_{t_m}^{t_{m+1}} e^{\Theta_h(s-t_m)} P_h F(s, X^h(s)) ds \approx \int_{t_m}^{t_{m+1}} e^{\Theta_h(s-t_m)} P_h F(t_m, X^h_m) ds. \]  
\hfill (47)

To approximate $\Theta_h(t_{m+1} - s)$, we truncate (44) after the first term and this yields
Substituting (48) in (47) yields the following approximation
\[
\int_{t_m}^{t_{m+1}} e^{\Theta_h(t_{m+1} - s)} P_h F(s, X^h(s)) ds \approx \int_{t_m}^{t_{m+1}} e^{A_h(t_m)(t_{m+1} - s)} P_h F(t_m, X^h_m) ds. \tag{49}
\]
The last term in (45) is approximated as follows
\[
\int_{t_m}^{t_{m+1}} e^{\Theta_h(t_{m+1} - s)} P_h B(s, X^h(s)) dW(s) \\
\approx \int_{t_m}^{t_{m+1}} e^{\Theta_h(t_{m+1} - s)} P_h B(t_m, X^h_m) dW(s) \\
\approx \int_{t_m}^{t_{m+1}} e^{A_h(t_m)\Delta t} P_h B(t_m, X^h_m) dW(s). \tag{50}
\]
Substituting (50), (49) and (46) in (45) yields the following fully discrete scheme for (1), called stochastic Magnus-type integrators (SMTI)
\[
X^h_{m+1} = e^{\Delta t A_h}(t_m) X^h_m + \int_{t_m}^{t_{m+1}} e^{(t_{m+1} - s) A_h(t_m)} P_h F(t_m, X^h_m) ds \\
+ \int_{t_m}^{t_{m+1}} e^{\Delta t A_h(t_m)} P_h B(t_m, X^h_m) dW(s). \tag{51}
\]
Note that the numerical scheme (51) can be written as follows
\[
X^h_{m+1} = e^{\Delta t A_h,m} X^h_m + \Delta t \varphi_1(\Delta t A_h,m) P_h F(t_m, X^h_m) \\
+ e^{\Delta t A_h,m} P_h B(t_m, X^h_m) \Delta W_m, \quad m = 0, \ldots, M, \tag{52}
\]
\[
X^h_0 = P_h X_0, \quad A_h,m := A_h(t_m), \quad \text{where the linear operator } \varphi_1(\Delta t A_h,m) \text{ is given by}
\]
\[
\varphi_1(\Delta t A_h,m) := \frac{1}{\Delta t} \int_0^{\Delta t} e^{(t-s) A_h,m} ds, \tag{53}
\]
and for any \( M \in \mathbb{N}, \Delta t = T/M, t_m = m\Delta t, m = 0, 1, \ldots, M \) and
\[
\Delta W_m := W((m + 1)\Delta t) - W(m\Delta t). \tag{54}
\]
We also note that an equivalent formulation of the numerical scheme (52), easy for simulation is given by
\[
X^h_{m+1} = X^h_m + P_h B(t_m, X^h_m) \Delta W_m \\
+ \Delta t \varphi_1(\Delta t A_h,m) \left[ A_h,m \left\{ X^h_m + P_h B(t_m, X^h_m) \Delta W_m \right\} + P_h F(t_m, X^h_m) \right]. \tag{55}
\]
With the numerical method in hand, we can now state its strong convergence result toward the mild solution, which is in fact our main result. In the rest of this paper \( C \) is a generic constant independent of \( h, m, M \) and \( \Delta t \) that may change from one place to another.

**Theorem 2.7.** [Main result] Let Assumptions 2.1, 2.2, 2.3 and 2.4 be fulfilled.

(i) If \( 0 \leq \beta < 1 \), then the following error estimate holds
\[
(\mathbb{E} \Vert X(t_m) - X^h_m \Vert^2)^{1/2} \leq C \left( h^\beta + \Delta t^{2} \right). \tag{56}
\]
Lemma 3.1. \[26, 4\] proof.

Preparatory results.

(i) If \(1 \leq \beta < 2\) and moreover if Assumption 2.6 is satisfied, then the following error estimate holds
\[
\left( \mathbb{E}\|X(t_m) - X^h(t_m)\|^2 \right)^{\frac{1}{2}} \leq C \left( h^{\beta} + \Delta t^{\frac{1}{2}} \right),
\] (57)

(ii) If \(\beta = 2\) and if Assumption 2.6 is fulfilled, then the following error estimate holds
\[
\left( \mathbb{E}\|X(t_m) - X^h(t_m)\|^2 \right)^{\frac{1}{2}} \leq C \left[ h^2 \left( 1 + \max(0, \ln(t_m/h^2)) \right) + \Delta t^{\frac{1}{2}} \right].
\] (58)

3. Proof of the main result. The proof of the main result needs some preparatory results.

3.1. Preparatory results. The following lemmas will be useful in our convergence proof.

Lemma 3.1. \[26, 4\] Let Assumption 2.2 be fulfilled. Then for any \(\gamma \in [0, 1]\), the following estimates hold uniformly in \(h > 0\) and \(t \in [0, T]\)
\[
K^{-1}\|(-A_h(0))^{-\gamma} v\| \leq \|((-A_h(t))^{-\gamma} v\| \leq K\|((-A_h(0))^{-\gamma} v\|, \quad v \in V_h,
\] (59)
\[
K^{-1}\|(-A_h(0))^{\gamma} v\| \leq \|((-A_h(t))^{\gamma} v\| \leq K\|((-A_h(0))^{\gamma} v\|, \quad v \in V_h \cap D,
\] (60)
where \(K\) is a positive constant independent of \(t\) and \(h\).

Remark 2. From \[4, \text{Chapter III}\] it is well known that there exists a unique evolution system \(U_h : \Delta(T) \rightarrow L(H)\), satisfying \[21, (6.3), \text{Page 149}\]
\[
U_h(t, s) = S^h_s(t - s) + \int_s^t S^h_s(t - \tau) R^h(\tau, s) d\tau,
\] (61)
where \(S^h_s(t) := e^{A_h(s)t}\), \(R^h(t, s) := \sum_{m=1}^{\infty} R^h_m(t, s)\), with \(R^h_m(t, s)\) satisfying the following recurrence relation \[21, (6.22), \text{Page 153}\]
\[
R^h_{m+1} = \int_s^t R^h_s(t, \tau) R^h_m(\tau, s) d\tau, \quad m \geq 1
\] (62)
and \(R^h_1(t, s) := (A_h(s) - A_h(t)) S^h_s(t - s)\). Note also that from \[21, (6.6), \text{Chapter 5, Page 150}\], the following identity holds
\[
R^h(t, s) = R^h_1(t, s) + \int_s^t R^h_s(t, \tau) R^h(\tau, s) d\tau.
\] (63)
The mild solution of (40) is therefore given by
\[
X^h(t) = U_h(t, 0) P_h X_0 + \int_0^t U_h(t, s) P_h F(s, X^h(s)) ds + \int_0^t U_h(t, s) P_h B(s, X^h(s)) dW(s).
\] (64)

Lemma 3.2. Under Assumption 2.2, the evolution system \(U_h : \Delta(T) \rightarrow H\) satisfies the following

(i) \(U_h(., s) \in C^1([s, T]; L(H))\), \(0 \leq s \leq T\) and
\[
\frac{\partial U_h}{\partial t}(t, s) = A_h(t) U_h(t, s), \quad 0 \leq s \leq t \leq T;
\] (65)
\[
\|A_h(t) U_h(t, s)\|_{L(H)} \leq \frac{C}{t-s}, \quad 0 \leq s < t \leq T.
\] (66)
Lemma 3.3. [26] or [4, Chapter III]. Let Assumption 2.2 be fulfilled.

(i) The following estimates hold
\[ \|R^h(t,s)\|_{L(H)} \leq C, \quad \|R^h_m(t,s)\|_{L(H)} \leq \frac{C}{m!} (t-s)^{m-1}, \quad m \geq 1, \]
\[ \|R^h(t,s)\|_{L(H)} \leq C, \quad \|U_h(t,s)\|_{L(H)} \leq C, \quad 0 \leq s \leq t \leq T. \]

(ii) For any \(0 \leq \alpha \leq 1, 0 \leq \beta \leq 1\) and \(0 \leq s \leq t \leq T\), the following estimates hold
\[ \|(-A_h(r))^\alpha U_h(t,s)\|_{L(H)} \leq C(t-s)^{-\alpha}, \quad r \in [0,T], \]
\[ \|U_h(t,s)(-A_h(r))\|_{L(H)} \leq C(t-s)^{-\alpha}, \quad r \in [0,T], \]
\[ \|(-A_h(r))\gamma U_h(t,s)(-A_h(s))\|_{L(H)} \leq C(t-s)^{-\alpha}, \quad r \in [0,T]. \]

(iii) For any \(0 \leq s \leq t \leq T\) the following useful estimates hold
\[ \| (U_h(t,s) - I) (-A_h(s))^{-\gamma} \|_{L(H)} \leq C(t-s)^{-\gamma}, \quad 0 \leq \gamma \leq 1, \]
\[ \| (-A_h(r))^{-\gamma} (U_h(t,s) - I) \|_{L(H)} \leq C(t-s)^{-\gamma}, \quad 0 \leq \gamma \leq 1. \]

The following space and time regularity of the semi-discrete problem (40) will be useful in our convergence analysis.

Lemma 3.4. Let Assumptions 2.1, 2.2 (i)-(ii), 2.3 and 2.4 be fulfilled with the corresponding \(0 \leq \beta < 1\). Then for all \(\gamma \in [0, \beta]\) the following estimates hold
\[ \|(-A_h(r))^\frac{1}{2} X^h(t)\|_{L^2(\Omega, H)} \leq C, \quad 0 \leq r, t \leq T, \]
\[ \|X^h(t_2) - X^h(t_1)\|_{L^2(\Omega, H)} \leq C(t_2 - t_1)^{\frac{1}{2}}, \quad 0 \leq t_1 \leq t_2 \leq T. \]

Moreover if Assumption 2.6 is fulfilled, then (76) and (77) hold for \(\beta = 1\).

Proof. We first show that \(\sup_{t \in [0,T]} \|X^h(t)\|_{L^2(\Omega, H)} \leq C\). Taking the norm in both sides of (64) and using the inequality \((a + b + c)^2 \leq 3a^2 + 3b^2 + 3c^2, a, b, c \in \mathbb{R}_+\) yields
\[ \|X^h(t)\|_{L^2(\Omega, H)}^2 \leq 3 \|U_h(t,0)P_hX_0\|_{L^2(\Omega, H)}^2 + 3 \left\| \int_0^t U_h(t,s)P_h F(s, X^h(s)) \, ds \right\|_{L^2(\Omega, H)}^2 ds + 3 \left\| \int_0^t U_h(t,s)P_h B(s, X^h(s)) \, dW(s) \right\|_{L^2(\Omega, H)}^2 \]
\[ := I_0 + I_1 + I_2. \]

Using Lemma 3.3 (i) and the uniform boundedness of \(P_h\), it holds that
\[ I_0 \leq 3\|X_0\|_{L^2(\Omega, H)}^2 \leq C. \]
Using again Lemma 3.3 (i), Assumption 2.3 and the uniform boundedness of \( P_h \), it holds that
\[
I_1 \leq 3 \left( \int_0^t \| U_h(t, s) P_h F(s, X^h(s)) \|_{L^2(\Omega, H)} \right)^2 \leq C \left( \int_0^t \left( C + \| X^h(s) \|_{L^2(\Omega, H)} \right) ds \right)^2.
\]
Using Hölder inequality yields
\[
I_1 \leq C + C \int_0^t \| X^h(s) \|_{L^2(\Omega, H)}^2 ds.
\] (80)
Applying the itô-isometry, using Lemma 3.3 (ii) and Assumption 2.4, it holds that
\[
I_2 = 3 \int_0^t E \| U_h(t, s) P_h B(s, X^h(s)) \|_{L^2(\Omega, H)}^2 ds \leq C + C \int_0^t \| X^h(t) \|_{L^2(\Omega, H)}^2 ds.
\] (81)
Substituting (81), (80) and (79) in (78) yields
\[
\| X^h(t) \|_{L^2(\Omega, H)}^2 \leq C + C \int_0^t \| X^h(s) \|_{L^2(\Omega, H)}^2 ds.
\] (82)
Applying the continuous Gronwall lemma to (82) yields
\[
\| X^h(t) \|_{L^2(\Omega, H)}^2 \leq C, \quad t \in [0, T].
\] (83)
Let us now prove (76). Pre-multiplying (64) by \((-A_h(r))^{-\frac{1}{2}}\), taking the norm in both sides and using triangle inequality yields
\[
\left\| \left(-A_h(r)\right)^{-\frac{1}{2}} X^h(t) \right\|_{L^2(\Omega, H)} \leq \left\| \left(-A_h(r)\right)^{-\frac{1}{2}} U_h(t, 0) \right\|_{L^2(\Omega, H)} \leq \int_0^t \left\| \left(-A_h(r)\right)^{-\frac{1}{2}} U_h(t, s) P_h X^h(s) \right\|_{L^2(\Omega, H)} ds + \int_0^t \left\| \left(-A_h(r)\right)^{-\frac{1}{2}} U_h(t, s) P_h B(s, X^h(s)) \right\|_{L^2(\Omega, H)} \right. \leq I_0 + I_1 + I_2.
\] (84)
Inserting \((-A_h(0))^{-\frac{1}{2}}(-A_h(0))^{-\frac{1}{2}}\), using Lemmas 3.3 (ii) 3.1, it holds that
\[
I_0 \leq \|(-A_h(r))^{-\frac{1}{2}} U_h(t, 0)(-A_h(0))^{-\frac{1}{2}} \|_{L(H)} \|(-A_h(0))^{-\frac{1}{2}} X_0\| \leq C.
\] (85)
Using Lemmas 3.1, 3.3 (ii), Assumption 2.3 and (83) yields
\[
I_1 \leq C \int_0^t \left\| \left(-A_h(s)\right)^{-\frac{1}{2}} U_h(t, s) \right\|_{L(H)} \sup_{t \in [0, T]} \left\| F(s, X^h(s)) \right\|_{L^2(\Omega, H)} ds \leq C \sup_{s \in [0, T]} (1 + \| X^h(s) \|_{L^2(\Omega, H)}) \int_0^t (t - s)^{-\frac{1}{2}} ds \leq C.
\] (86)
Applying the Itô-isometry, using Lemmas 3.1, 3.3 (ii), Assumption 2.4 and (83) yields
\[
II^2_2 = \int_0^t \mathbb{E} \left\| (-A_h(0))^{\frac{3}{2}} U_h(t, s) P_h B(s, X^h(s)) \right\|_{L^2}^2 \, ds
\leq \ C \sup_{s \in [0,T]} \left( 1 + \|X^h(s)\|_{L^2(\Omega, H)}^2 \right) \int_0^t (t-s)^{-\gamma} \, ds \leq C. \quad (87)
\]
Substituting (87), (86) and (85) in (84) completes the proof of (76). The proof of (77) follows from (64). In fact from (64), we have
\[
\|X^h(t_2) - X^h(t_1)\|_{L^2(\Omega, H)} \leq \|(U_h(t_2, 0) - U_h(t_1, 0)) P_h X_0\|_{L^2(\Omega, H)}
+ \int_0^{t_1} \|(U_h(t_2, s) - U_h(t_1, s)) P_h F(s, X^h(s))\|_{L^2(\Omega, H)} \, ds
+ \int_{t_1}^{t_2} \|(U_h(t_2, s) P_h F(s, X^h(s))\|_{L^2(\Omega, H)} \, ds
+ \int_0^{t_1} (U_h(t_2, s) - U_h(t_1, s)) P_h B(s, X^h(s)) \, dW(s) \|_{L^2(\Omega, H)}
+ \int_{t_1}^{t_2} (U_h(t_2, s) P_h B(s, X^h(s)) \, dW(s) \|_{L^2(\Omega, H)}
:= III_0 + III_1 + III_2 + III_3 + III_4. \quad (88)
\]
Inserting an appropriate power of \(-A_h(t_1)\), using Lemmas 3.3 (ii)-(iii) and [20, Lemma 1] yields
\[
III_0 = \|(U_h(t_2, t_1) - I) U_h(t_1, 0) P_h X_0\|_{L^2(\Omega, H)}
\leq \| (U_h(t_2, t_1) - I)(-A_h(t_1))^{-\frac{3}{2}} \|_{L(H)}
\times \| (-A_h(t_1))^{\frac{3}{2}} U_h(t_1, 0)(-A_h(t_1))^{-\frac{3}{2}} \|_{L(H)}\| (-A_h(t_1))^{\frac{3}{2}} P_h X_0\|_{L^2(\Omega, H)}
\leq C(t_2 - t_1)^{\frac{\alpha}{2}}. \quad (89)
\]
Using Assumption 2.4, (76), Lemma 3.3 (ii) and (iii) yields
\[
III_1 \leq \int_0^{t_1} \|(U_h(t_2, t_1) - I) U_h(t_1, s)\|_{L(H)} \| P_h F(s, X^h(s))\|_{L^2(\Omega, H)} \, ds
\leq C \int_0^{t_1} \|(U_h(t_2, t_1) - I)(-A_h(t_1))^{-\frac{3}{2}} \|_{L(H)}\| (-A_h(t_1))^{\frac{3}{2}} U_h(t_1, s)\|_{L(H)} \, ds
\leq C \int_0^{t_1} (t_2 - t_1)^{\frac{3}{2}} (t_1 - s)^{-\frac{\alpha}{2}} \, ds
\leq C(t_2 - t_1)^{\frac{\alpha}{2}}. \quad (90)
\]
Using Lemma 3.3 (i) and Assumption 2.3, it holds that
\[
III_2 \leq C \int_{t_1}^{t_2} \sup_{s \in [0,T]} \| F(s, X^h(s))\|_{L^2(\Omega, H)} \, ds \leq C(t_2 - t_1). \quad (91)
\]
Using the Itô-isometry, Assumption 2.6, (76), Lemma 3.3 (ii)-(iii) and following the same lines as the estimate of $III_1$ yields

$$III_1^2 \leq C(t_2 - t_1)\beta. \quad (92)$$

Using the Itô-isometry and following the same lines as that of $III_2$ yields

$$III_2^2 \leq C(t_2 - t_1). \quad (93)$$

Substituting (93), (92), (91), (90) and (89) in (88) completes the proof of (77). \(\square\)

Let us consider the following deterministic problem: find $u \in V$ such that

$$u' = A(t)u, \quad u(\tau) = v, \quad \tau \geq 0, \quad t \in (\tau, T].$$

The corresponding semi-discrete problem in space is: find $u_h \in V_h$ such that

$$u_h' (t) = A_h(t)u_h, \quad u_h(\tau) = P_hv, \quad \tau \geq 0, \quad t \in (\tau, T].$$

Let us define the operator

$$T_h(t, \tau) := U(t, \tau) - U_h(t, \tau)P_h,$$

so that $u(t) - u_h(t) = T_h(t, \tau)v$. The following lemma will be useful in our convergence analysis.

Lemma 3.5. [26] Let $r \in [0, 2]$ and $0 \leq \gamma \leq r$. Let Assumption 2.2 be fulfilled. Then the following error estimate holds for the semi-discrete approximation (95)

$$\|u(t) - u_h(t)\| = \|T_h(t, \tau)v\| \leq Ch^r (t - \tau)^{-\frac{r-\gamma}{2}} \|v\|, \quad v \in \mathcal{H}^r. \quad (97)$$

Proposition 2. [Space error] Let Assumptions 2.1, 2.2, 2.3 and 2.4 be fulfilled. Let $X(t)$ and $X^h(t)$ be the mild solution of (1) and (40) respectively.

(i) If $0 \leq \beta < 1$, then the following error estimate holds

$$\|X(t) - X^h(t)\|_{L^2(\Omega, H)} \leq Ch^\beta, \quad 0 \leq t \leq T. \quad (98)$$

(ii) If $1 \leq \beta < 2$ and furthermore if Assumption 2.6 is fulfilled, then the following error estimate holds

$$\|X(t) - X^h(t)\|_{L^2(\Omega, H)} \leq Ch^\beta, \quad 0 \leq t \leq T, \quad (99)$$

(iii) If $\beta = 2$ and furthermore if Assumption 2.6 is fulfilled, then the following error estimate holds

$$\|X(t) - X^h(t)\|_{L^2(\Omega, H)} \leq Ch^2 \left(1 + \max \left(0, \ln(t/h^2)\right)\right), \quad 0 < t \leq T. \quad (100)$$

Proof. Subtracting (64) from (24), taking the $L^2$ norm and using triangle inequality yields

$$\|X(t) - X^h(t)\|_{L^2(\Omega, H)} \leq \|U(t, 0)X_0 - U_h(t, 0)P_hX_0\|_{L^2(\Omega, H)}$$

$$+ \left\|\int_0^t \left[U(t, s)F(s, X(s)) - U_h(t, s)P_hF(s, X^h(s))\right] \, ds\right\|_{L^2(\Omega, H)}$$

$$+ \left\|\int_0^t \left[U(t, s)B(s, X(s)) - U_h(t, s)P_hB(s, X^h(s))\right] \, dW(s)\right\|_{L^2(\Omega, H)}$$

$$=: IV_0 + IV_1 + IV_2. \quad (101)$$

Using Lemma 3.5 with $r = \gamma = \beta$ yields

$$IV_0 \leq Ch^\beta \|X_0\|_{L^2(\Omega, H^\beta)} \leq Ch^\beta. \quad (102)$$
Using Lemma 3.5 with \( r = \beta, \gamma = 0 \), Assumption 2.3, Lemmas 3.4 and 3.3 yields

\[
IV_1 \leq \int_0^t \|U(t,s)F(s,X(s)) - U(t,s)F(s,X^h(s))\|_{L^2(\Omega,H)} \, ds
+ \int_0^t \|U(t,s)F(s,X(s)) - U_h(t,s)P_hF(s,X^h(s))\|_{L^2(\Omega,H)} \, ds
\leq C \int_0^t \|X(s) - X^h(s)\|_{L^2(\Omega,H)} \, ds + Ch^\beta \int_0^t (t-s)^{-\frac{\beta}{2}} ds
\leq Ch^\beta + C \int_0^t \|X(s) - X^h(s)\|_{L^2(\Omega,H)} \, ds.
\]

Using the Itô-isometry property, Lemma 3.4, Lemma 3.5 with \( r = \beta \) and \( \gamma = \frac{2-\beta}{2} \), Assumption 2.6 yields

\[
IV_2^2 = \int_0^t \|U(t,s)B(s,X(s)) - U_h(t,s)P_hB(s,X^h(s))\|_{L^2(\Omega,H)}^2 \, ds
\leq \int_0^t \|U(t,s)B(s,X(s)) - U(t,s)B(s,X^h(s))\|_{L^2(\Omega,H)}^2 \, ds
+ \int_0^t \|U(t,s)B(s,X^h(s)) - U_h(t,s)P_hB(s,X^h(s))\|_{L^2(\Omega,H)}^2 \, ds
\leq C \int_0^t \|X(s) - X^h(s)\|_{L^2(\Omega,H)}^2 \, ds + Ch^{2\beta} \int_0^t (t-s)^{-1+\beta} ds
\leq Ch^{2\beta} + C \int_0^t \|X(s) - X^h(s)\|_{L^2(\Omega,H)}^2 \, ds.
\]

Substituting (104), (103) and (102) in (101) yields

\[
\|X(t) - X^h(t)\|_{L^2(\Omega,H)} \leq Ch^{2\beta} + C \int_0^t \|X(s) - X^h(s)\|_{L^2(\Omega,H)}^2 \, ds.
\]

Applying the continuous Gronwall lemma to (105) yields

\[
\|X(t) - X^h(t)\|_{L^2(\Omega,H)} \leq Ch^\beta.
\]

For non commutative operators \( H_j \) on a Banach space, we introduce the following notation for the composition of operators

\[
\prod_{j=t}^k H_j = \begin{cases} H_k H_{k-1} \cdots H_t & \text{if } k \geq t, \\ I & \text{if } k < t. \end{cases}
\]

The following lemma will be useful in our convergence proof.

**Lemma 3.6.** [26] Let Assumption 2.2 be fulfilled. Then the following estimate holds

\[
\left\| \left( \prod_{j=t}^m e^{tA_{h,j}} \right) (A_{h,t})^\gamma \right\|_{L(H)} \leq Ct_{m-l}^{-\gamma}, \quad \gamma = 1
\]

\[
\left\| \left( -A_{h,k} \right)^{\gamma_1} \left( \prod_{j=t}^m e^{tA_{h,j}} \right) (A_{h,t})^{-\gamma_2} \right\|_{L(H)} \leq Ct_{m-l}^{-\gamma_1}, \quad \gamma_1, \gamma_2 > 0.
\]
Lemma 3.7. (i) For all $\alpha \geq 0$, the following estimate holds
\[ \| R^h(t, s)(-A_h(s))^{\alpha} \|_{L(H)} \leq C(t - s)^{-\alpha}, \quad t, s \in [0, T]. \] \hspace{1cm} (110)

(ii) For all $\alpha \in [0, 1]$, the following estimate holds
\[ \| (U_h(t_j, t_{j-1}) - e^{\Delta t A_h, j^{-1}})(-A_h, j^{-1})^{-\alpha} \|_{L(H)} \leq C\Delta t^{1+\alpha}. \] \hspace{1cm} (111)

(iii) For all $\alpha \in [0, 1)$, the following estimate holds
\[ \| (U_h(t_j, t_{j-1}) - e^{\Delta t A_h, j^{-1}})(-A_h, j^{-1})^{\alpha} \|_{L(H)} \leq C\Delta t^{1-\alpha}. \] \hspace{1cm} (112)

(iv) For all $\alpha \in [0, 1)$, the following estimate holds
\[ \| (-A_h, j^{-1})^{-\alpha}(U_h(t_j, t_{j-1}) - e^{\Delta t A_h, j^{-1}}) \|_{L(H)} \leq C\Delta t^{1+\alpha}. \] \hspace{1cm} (113)

Proof. From the integral equation (63), we have
\[ R^h(t, s)(-A_h(s))^{\alpha} = e^{A_h(s)(t-s)}(-A_h(s))^{\alpha} \]
\[ + \int_s^t R^h_1(t, \tau)R^h(\tau, s)(-A_h(s))^{\alpha} d\tau. \] \hspace{1cm} (114)

Taking the norm in both sides of (114), using (39) and Lemma 3.3 yields
\[ \| R^h(t, s)(-A_h(s))^{\alpha} \|_{L(H)} \leq \| e^{A_h(s)(t-s)}(-A_h(s))^{\alpha} \|_{L(H)} \]
\[ + \int_s^t \| R^h_1(\tau, s) \|_{L(H)} \| R^h(\tau, s)(-A_h(s))^{\alpha} \|_{L(H)} d\tau \]
\[ \leq C(t - s)^{-\alpha} + C \int_s^t \| R^h(\tau, s)(-A_h(s))^{\alpha} \|_{L(H)} d\tau. \] \hspace{1cm} (115)

Applying the continuous Gronwall lemma to (115) yields
\[ \| R^h(t, s)(-A_h(s))^{\alpha} \|_{L(H)} \leq C(t - s)^{-\alpha}. \] \hspace{1cm} (116)

This completes the proof of (i). From (61) and (63), we have
\[ U_h(t_j, t_{j-1}) - e^{\Delta t A_h, j^{-1}} \]
\[ = \int_{t_{j-1}}^{t_j} e^{(t_j - \tau)A_h(\tau)} R_h(\tau, t_{j-1}) d\tau \]
\[ = \int_{t_{j-1}}^{t_j} e^{(t_j - \tau)A_h(\tau)} R_h^1(\tau, t_{j-1}) d\tau \]
\[ + \int_{t_{j-1}}^{t_j} e^{(t_j - \tau)A_h(\tau)} \left[ \int_{t_{j-1}}^{\tau} R_h^1(\tau, s)R^h(s, t_{j-1}) ds \right] d\tau \]
\[ = \int_{t_{j-1}}^{t_j} e^{(t_j - \tau)A_h(\tau)} (A_h(\tau) - A_h(t_{j-1})) e^{A_h, j^{-1}(-t_j - \tau)} d\tau \]
\[ + \int_{t_{j-1}}^{t_j} e^{(t_j - \tau)A_h(\tau)} \left[ \int_{t_{j-1}}^{\tau} R_h^1(\tau, s)R^h(s, t_{j-1}) ds \right] d\tau. \] \hspace{1cm} (117)
Therefore, from (117), for all \( \alpha \in [0, 1] \), using (39) and Lemma 3.3, it holds that
\[
\left\| (U_h(t_j, t_{j-1}) - e^{\Delta t A_h (t_j - t_{j-1})}) (-A_{h,j-1})^{-\alpha} \right\|_{L(H)} 
\leq \int_{t_{j-1}}^{t_j} \left\| e^{(t_j - \tau)A_h(\tau)} (A_h(\tau) - A_h(t_{j-1})) (-A_{h,j-1})^{-1} \right\|_{L(H)} d\tau 
+ \int_{t_{j-1}}^{t_j} \left\| e^{(t_j - \tau)A_h(\tau)} \right\|_{L(H)} \left\| (A_h(\tau) - A_h(t_{j-1})) (-A_{h,j-1})^{-1} \right\|_{L(H)} d\tau 
\times \left\| e^{A_{h,j-1}(\tau - t_{j-1})} (-A_{h,j-1})^{-1-\alpha} \right\|_{L(H)} d\tau + C \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{\tau} d\tau ds 
\leq C \int_{t_{j-1}}^{t_j} (\tau - t_{j-1})^\alpha d\tau + C \Delta t^2 \leq C \Delta t^{1+\alpha}.
\] (118)

This completes the proof of (ii). The proof of (iii) and (iv) are similar to that of (ii) using (i).

\[\square\]

The following lemma can be found in [15]

**Lemma 3.8.** For all \( \alpha_1, \alpha_2 > 0 \) and \( \alpha \in [0, 1] \), there exist two positive constants \( C_{\alpha_1, \alpha_2} \) and \( C_{\alpha, \alpha_2} \) such that
\[
\Delta t \sum_{j=1}^{m} t_{m-j+1}^{1+\alpha_2} t_j^{1+\alpha_2} \leq C_{\alpha_1, \alpha_2} t_m^{1+\alpha_1+\alpha_2},
\] (119)
\[
\Delta t \sum_{j=1}^{m} t_{m-j+1}^{-\alpha} t_j^{1+\alpha_2} \leq C_{\alpha, \alpha_2} t_m^{-\alpha+\alpha_2}.
\] (120)

**Proof.** The proof of (119) follows from the comparison with the integral
\[
\int_0^t (t - s)^{-1+\alpha_1} s^{-1+\alpha_2} ds.
\] (121)
The proof of (120) is a consequence of (119).

\[\square\]

The following lemma is fundamental in our convergence analysis.

**Lemma 3.9.** Let Assumption 2.2 be fulfilled. Then for all \( 1 \leq i \leq m \leq M \).

(i) The following estimate holds
\[
\left\| \left( \prod_{j=i}^{m} U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=i-1}^{m-1} e^{\Delta t A_{h,j}} \right) \right\|_{L(H)} \leq C \Delta t^{1-\epsilon},
\] (122)
where \( \epsilon > 0 \) is a positive number small enough.

(ii) The following estimate also holds
\[
\left\| \left( \prod_{j=i}^{m} U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=i-1}^{m-1} e^{\Delta t A_{h,j}} \right) (-A_{h,i-1})^{-\epsilon} \right\|_{L(H)} \leq C \Delta t.
\] (123)
Proof. First of all note that
\[
\left( \prod_{j=i}^{m} U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=i}^{m-1} e^{\Delta t A_{h,j}} \right) = \left( \prod_{j=i}^{m} U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=i}^{m} e^{\Delta t A_{h,j-1}} \right). \tag{124}
\]

Using the telescopic sum, (124) can be written as follows
\[
\left( \prod_{j=i}^{m} U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=i}^{m} e^{\Delta t A_{h,j-1}} \right) = \sum_{k=1}^{m-i+1} \left( \prod_{j=i+k}^{m} U_h(t_j, t_{j-1}) \right) \left( U_h(t_{i+k-1}, t_{i+k-2}) - e^{\Delta t A_{h,i+k-2}} \right) \cdot \left( \prod_{j=i}^{i+k-2} e^{\Delta t A_{h,j-1}} \right). \tag{125}
\]

Writing down explicitly the first term of (125) gives
\[
\left( \prod_{j=i}^{m} U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=i}^{m} e^{\Delta t A_{h,j-1}} \right) = \left( \prod_{j=i+1}^{m} U_h(t_j, t_{j-1}) \right) \left( U_h(t_i, t_{i-1}) - e^{\Delta t A_{h,i-1}} \right) + \sum_{k=2}^{m-i+1} \left( \prod_{j=i+k}^{m} U_h(t_j, t_{j-1}) \right) \left( U_h(t_{i+k-1}, t_{i+k-2}) - e^{\Delta t A_{h,i+k-2}} \right) \cdot \left( \prod_{j=i}^{i+k-2} e^{\Delta t A_{h,j-1}} \right). \tag{126}
\]

Taking the norm in both sides of (126), using Lemma 3.3, Lemma 3.7 (ii) and Lemma 3.6 yields
\[
\left\| \left( \prod_{j=i}^{m} U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=i}^{m} e^{\Delta t A_{h,j-1}} \right) \right\|_{L(H)} \leq \left\| U_h(t_{m-i+1}, t_i) \right\|_{L(H)} \left\| U_h(t_i, t_{i-1}) - e^{\Delta t A_{h,i-1}} \right\|_{L(H)} + \sum_{k=2}^{m-i+1} \left\| U_h(t_m, t_{i+k-1}) \right\|_{L(H)} \\
\times \left\| \left( U_h(t_{i+k-1}, t_{i+k-2}) - e^{\Delta t A_{h,i+k-2}} \right) (-A_{h,i+k-2})^{-1+\epsilon} \right\|_{L(H)} \times \left\| (-A_{h,i+k-2})^{1-\epsilon} \left( \prod_{j=i}^{i+k-2} e^{\Delta t A_{h,j-1}} \right) \right\|_{L(H)}.
\]
Iterating the numerical scheme (51) by substituting $X^e$ in two parts as follows. 

Theorem 2.7. Proof of Theorem 2.7. Iterating the mild solution (129) yields

$$\|X(t) - X^h_m\|_{L^2(\Omega,H)} \leq \|X(t) - X^h(t_m)\|_{L^2(\Omega,H)} + \|X^h(t_m) - X^h_m\|_{L^2(\Omega,H)} =: V + VI.$$  

This completes the proof of (i). The proof of (ii) is similar to that of (i) using (109) and Lemma 3.8.

With the above preparatory results in hand, we can now prove our main result.

3.2. Proof of Theorem 2.7. Using triangle inequality, we split the fully discrete error in two parts as follows.

$$\|\Delta X(t)\| L^2(\Omega,H) \leq \|\Delta X(t) - \Delta X^h(t_m)\| L^2(\Omega,H) + \|\Delta X^h(t_m) - \Delta X^h_m\| L^2(\Omega,H) =: V + VI.$$  

The space error $V$ is estimated in Lemma 3.5. It remains to estimate the time error $VI$. Note that the mild solution of (40) can be written as follows.

$$X^h(t_m) = U_h(t_m, t_{m-1})X^h(t_{m-1}) + \int_{t_{m-1}}^{t_m} U_h(t_m, s)P_hF(s, X^h(s)) \, ds$$

Iterating the mild solution (129) yields

$$X^h(t_m) = \left( \prod_{j=1}^{m} U_h(t_j, t_{j-1}) \right) P_hX_0 + \int_{t_{m-1}}^{t_m} U_h(t_m, s)P_hF(s, X^h(s)) \, ds$$

Iterating the numerical scheme (51) by substituting $X^h_j$, $j = m - 1, \cdots, 1$ only in the first term of (51) by their expressions yields

$$X^h_m = \left( \prod_{j=0}^{m-1} e^{tA_{h,j}} \right) X_0 + \int_{t_{m-1}}^{t_m} e^{(t_m-s)A_{h,m-1}}P_hF(t_{m-1}, X^h_m) \, ds$$

+ \int_{t_{m-1}}^{t_m} e^{tA_{h,m-1}}P_hB(t_{m-1}, X^h_m) \, dW(s)$$

+ \sum_{k=1}^{m-1} \int_{t_{m-k}}^{t_{m-k-1}} \left( \prod_{j=m-k+1}^{m} e^{tA_{h,j}} \right) e^{(t_{m-k}-s)A_{h,m-k-1}}P_hF(t_{m-k-1}, X^h_{m-k-1}) \, ds
\[ + \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h,j}} \right) e^{\Delta t A_{h,m-k-1}} P_h B \left( t_{m-k-1}, X_{m-k-1}^h \right) dW(s). \]

Subtracting (131) from (130) yields

\[ X^h(t_m) - X^h_m = \left( \prod_{j=1}^{m} U_h(t_j, t_{j-1}) \right) P_h X_0 - \left( \prod_{j=0}^{m-1} e^{\Delta t A_{h,j}} \right) P_h X_0 \]

\[ + \int_{t_{m-1}}^{t_m} \left[ U_h(t_m, s) P_h F \left( s, X^h(s) \right) - e^{(t_m-s)A_{h,m-1}} P_h F \left( t_{m-1}, X_{m-1}^h \right) \right] ds \]

\[ + \int_{t_{m-1}}^{t_m} \left[ U_h(t_m, s) P_h B \left( s, X^h(s) \right) - e^{\Delta t A_{h,m-1}} P_h B \left( t_{m-1}, X_{m-1}^h \right) \right] dW(s) \]

\[ + \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k+1}^{m} U_h(t_j, t_{j-1}) \right) U_h(t_{m-k}, s) P_h F \left( s, X^h(s) \right) ds \]

\[ - \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h,j}} \right) e^{(t_{m-k}-s)A_{h,m-k-1}} P_h F \left( t_{m-k-1}, X_{m-k-1}^h \right) ds \]

\[ + \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k+1}^{m} U_h(t_j, t_{j-1}) \right) U_h(t_{m-k}, s) P_h B \left( s, X^h(s) \right) dW(s) \]

\[ - \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h,j}} \right) e^{\Delta t A_{h,m-k-1}} P_h B \left( t_{m-k-1}, X_{m-k-1}^h \right) dW(s) \]

\[ =: VI_1 + VI_2 + VI_3 + VI_4 + VI_5. \] (132)

Taking the norm in both sides of (132) yields

\[ \| X^h(t_m) - X^h_m \|_{L^2(\Omega, H)}^2 \leq 25 \sum_{i=1}^{5} \| VI_i \|_{L^2(\Omega, H)}. \] (133)

In what follows, we estimate separately \( \| VI_i \|_{L^2(\Omega, H)}, \ i = 1, \ldots, 5. \)

3.2.1. Estimate of \( VI_1, VI_2 \) and \( VI_3. \) Using Lemma 3.9, it holds that

\[ \| VI_1 \|_{L^2(\Omega, H)} \leq \left\| \left( \prod_{j=1}^{m} U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=0}^{m-1} e^{\Delta t A_{h,j}} \right) \right\|_{L(H)} \| X_0 \|_{L^2(\Omega, H)} \]

\[ \leq C \Delta t^{1-\epsilon}. \] (134)

Using triangle inequality yields

\[ \| VI_2 \|_{L^2(\Omega, H)} \]

\[ \leq \int_{t_{m-1}}^{t_m} \| U_h(t_m, s) P_h F \left( s, X^h(s) \right) \|_{L^2(\Omega, H)} ds \]
Applying the Itô-isometry, using Assumption 2.4, (38), Theorem 2.5 and Lemma 3.3, it holds that

\[ \Vert V I \Vert_{L^2(\Omega, H)} \leq C \int_{t_{m-1}}^{t_m} \Vert X^h(t_{m-1}) - X_{m-1}^h \Vert_{L^2(\Omega, H)} ds + C \int_{t_{m-1}}^{t_m} ds \]

Using (38), Lemma 3.3, Assumption 2.3 and Theorem 2.5, it holds that

\[ \Vert V I_2 \Vert_{L^2(\Omega, H)} \leq C \int_{t_{m-1}}^{t_m} K_2(t_{m-1}) ds + C \int_{t_{m-1}}^{t_m} \Vert X^h(t_{m-1}) - X_{m-1}^h \Vert_{L^2(\Omega, H)} ds + C \int_{t_{m-1}}^{t_m} ds \]

Applying the Itô-isometry, using Assumption 2.4, (38), Theorem 2.5 and Lemma 3.3 yields

\[ \Vert V I_3 \Vert_{L^2(\Omega, H)}^2 \]

\[ \leq 9 \int_{t_{m-1}}^{t_m} E \Vert U_h(t_{m-1}) P_h^2 B(s) \Vert_{L^2}^2 ds \]

\[ + 9 \int_{t_{m-1}}^{t_m} E \Vert e^{\Delta t A_{h,m-1}} [P_h B(t_{m-1}, X_{m-1}^h) - P_h B(t_{m-1}, X^h(t_{m-1}))] \Vert_{L^2}^2 ds \]

\[ + 9 \int_{t_{m-1}}^{t_m} E \Vert e^{\Delta t A_{h,m-1}} P_h F(t_{m-1}, X^h(t_{m-1})) \Vert_{L^2}^2 ds \]

\[ \leq C \int_{t_{m-1}}^{t_m} ds + C \int_{t_{m-1}}^{t_m} \Vert X^h(t_{m-1}) - X_{m-1}^h \Vert_{L^2(\Omega, H)}^2 ds + C \int_{t_{m-1}}^{t_m} ds \]

\[ \leq C \Delta t + C \Delta t \Vert X^h(t_{m-1}) - X_{m-1}^h \Vert_{L^2(\Omega, H)}^2. \tag{136} \]

3.2.2. Estimate of VI₄. To estimate VI₄, we split it in five terms as follows.

\[ VI_4 = \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k+1}^{m} U_h(t_j, t_{j-1}) \right) U_h(t_{m-k}, s) \]

\[ \left[ P_h F(s, X^h(s)) - P_h F(t_{m-k-1}, X^h(t_{m-k-1})) \right] ds \]

\[ + \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k+1}^{m} U_h(t_j, t_{j-1}) \right) \]

\[ \left[ U_h(t_{m-k}, s) - U_h(t_{m-k}, s_{m-k-1}) \right] P_h F(t_{m-k-1}, X^h(t_{m-k-1})) ds \]

\[ + \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k}^{m} U_h(t_{j}, t_{j-1}) \right) \left( \prod_{j=m-k}^{m} e^{\Delta t A_{h,j}} \right) \]

\[ P_h F(t_{m-k-1}, X^h(t_{m-k-1})) ds \]

\[ + \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h,j}} \right) \left( e^{\Delta t A_{h,m-k-1}} - e^{\Delta t A_{h,s}} \right) A_{h,m-k-1} \]

\[ P_h F(t_{m-k-1}, X^h(t_{m-k-1})) ds \]
Using triangle inequality and Assumption 2.3 yields

\[ \| V_{I41} \|_{L^2(\Omega, \mathcal{H})} \leq \int_{t_{m-k-1}}^{t_{m-k}} \left( \int_{t_{m-k-1}}^{t_{m-k}} (s - t_{m-k-1})^{\frac{\alpha}{2}} ds \right) ds \]

Using Lemma 3.3 yields

\[ \| V_{I41} \|_{L^2(\Omega, \mathcal{H})} \leq C \Delta t^{\frac{\alpha}{2}} + C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} (s - t_{m-k-1})^{\frac{\min(\beta, 1)}{2}} ds \leq C \Delta t^{\frac{\min(\beta, 1)}{2}}. \]

Using Triangle inequality and inserting an appropriate power of \( A_{h,m-1} \) yields

\[ \| V_{I42} \|_{L^2(\Omega, \mathcal{H})} \leq \int_{t_{m-k-1}}^{t_{m-k}} \left( \int_{t_{m-k-1}}^{t_{m-k}} (s - t_{m-k-1})^{\frac{\alpha}{2}} ds \right) ds \]

Using Lemma 3.3, Assumption 2.3 and Theorem 2.5 yields

\[ \| V_{I42} \|_{L^2(\Omega, \mathcal{H})} \leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( t_{m-k} - t_{m-k-1} \right)^{1-\epsilon} (s - t_{m-k-1})^{1-\epsilon} ds \]

\[ \leq C \Delta t^{1-\epsilon} \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} t_{k}^{1+\epsilon} ds \]

\[ \leq C \Delta t^{1-\epsilon} \sum_{k=1}^{m-1} \Delta t_k^{1+\epsilon} \leq C \Delta t^{1-\epsilon}. \]
Using triangle inequality and inserting appropriate power of $A_{n,m-1}$ yields

$$
\|VI_{43}\|_{L^2(\Omega,H)} \leq \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left\| \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h,j}} \right) \left( e^{(s-t_{m-k})A_{h,m-k-1}} - I \right) e^{(t_{m-k}-s)A_{h,m-k-1}} \right\|_{L(H)} \times \|P_h F \left( t_{m-k-1}, X^h(t_{m-k-1}) \right) \|_{L^2(\Omega,H)} ds
$$

$$
\leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left\| \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h,j}} \right) (-A_{h,m-k-1})^{1-\epsilon} \right\|_{L(H)} \times \left\|e^{(s-t_{m-k-1}A_{h,m-k-1})} \right\|_{L(H)} ds
$$

Using Lemma 3.6, Assumption 2.3, Theorem 2.5, (38) and (39) yields

$$
\|VI_{43}\|_{L^2(\Omega,H)} \leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} t_k^{-1+\epsilon} (s - t_{m-k-1})^{1-\epsilon} ds
$$

$$
\leq C \Delta t^{1-\epsilon} \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} t_k^{-1+\epsilon} \Delta t ds \leq C \Delta t^{1-\epsilon}. \quad (140)
$$

Using triangle inequality and inserting an appropriate power of $A_{h,m-1}$ yields

$$
\|VI_{44}\|_{L^2(\Omega,H)} \leq \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left\| \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h,j}} \right) \left( I - e^{(s-t_{m-k-1})A_{h,m-k-1}} \right)e^{(t_{m-k}-s)A_{h,m-k-1}} \right\|_{L(H)} \times \|P_h F \left( t_{m-k-1}, X^h(t_{m-k-1}) \right) \|_{L^2(\Omega,H)} ds
$$

$$
\leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left\| \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h,j}} \right) (-A_{h,m-k})^{1-\epsilon} \right\|_{L(H)} \times \left\|e^{(s-t_{m-k-1}A_{h,m-k-1})} \right\|_{L(H)} ds
$$

Using Lemma 3.6, (38), (39), Assumption 2.3 and Lemma 3.3 yields

$$
\|VI_{44}\|_{L^2(\Omega,H)} \leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} t_k^{-1+\epsilon} (s - t_{m-k-1})^{1-\epsilon} ds
$$

$$
\leq C \Delta t^{1-\epsilon} \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} t_k^{-1+\epsilon} ds \leq C \Delta t^{1-\epsilon}. \quad (141)
$$
Using Lemma 3.3 and Assumption 2.3 yields

$$
\|VI_{45}\|_{L^2(\Omega,H)} \leq C \sum_{k=1}^{m-1} \int_{t_{m-k}}^{t_{m-k-1}} \|X^h(t_{m-k-1}) - X^h_{m-k-1}\|_{L^2(\Omega,H)} ds
$$

$$
\leq C \Delta t \sum_{k=0}^{m-1} \|X^h(t_k) - X^h_k\|_{L^2(\Omega,H)}. \tag{142}
$$

Substituting (142), (141), (140), (139) and (138) in (137) yields

$$
\|VI_4\|_{L^2(\Omega,H)} \leq C \Delta t^{\min(\beta,1)/2} + C \Delta t \sum_{k=0}^{m-1} \|X^h(t_k) - X^h_k\|_{L^2(\Omega,H)}. \tag{143}
$$

3.2.3. Estimate of $VI_5$. To estimate $VI_5$, we split it in four terms as follows

$$
VI_5 = \sum_{k=1}^{m-1} \int_{t_{m-k}}^{t_{m-k-1}} \left( \prod_{j=m-k+1}^{m} U_h(t_j, t_{j-1}) \right) U_h(t_{m-k}, s) \left[ P_h B(s, X^h(s)) - P_h B(t_{m-k-1}, X^h(t_{m-k-1})) \right] dW(s)
$$

$$
+ \sum_{k=1}^{m-1} \int_{t_{m-k}}^{t_{m-k-1}} \left( \prod_{j=m-k+1}^{m} U_h(t_j, t_{j-1}) \right) \left[ U_h(t_{m-k}, s) - U_h(t_{m-k}, t_{m-k-1}) \right] P_h B(t_{m-k-1}, X^h(t_{m-k-1})) dW(s)
$$

$$
+ \sum_{k=1}^{m-1} \int_{t_{m-k}}^{t_{m-k-1}} \left( \prod_{j=m-k+1}^{m} U_h(t_j, t_{j-1}) \right) \left[ \frac{\partial}{\partial s} \left( \prod_{j=m-k+1}^{m} e^{\Delta t A_{h,j}} \right) \right] P_h B(t_{m-k-1}, X^h(t_{m-k-1})) dW(s)
$$

$$
+ \sum_{k=1}^{m-1} \int_{t_{m-k}}^{t_{m-k-1}} \left( \prod_{j=m-k+1}^{m} e^{\Delta t A_{h,j}} \right) \left[ P_h B(t_{m-k-1}, X^h(t_{m-k-1})) - P_h B(t_{m-k-1}, X^h_{m-k-1}) \right] dW(s)
$$

$$
=: VI_{51} + VI_{52} + VI_{53} + VI_{54}. \tag{144}
$$

Using the Itô-isometry property, Lemma 3.3, Assumption 2.4 and Lemma 3.4 yields

$$
\|VI_{51}\|_{L^2(\Omega,H)}^2 = \sum_{k=1}^{m-1} \int_{t_{m-k}}^{t_{m-k-1}} \mathbb{E} \left[ \left\| U_h(t_m, s) \left[ P_h B(s, X^h(s))

\right.

\left. - P_h B(t_{m-k-1}, X^h(t_{m-k-1})) \right] \right\|_{L^2(H)}^2 ds
$$

$$
\leq C \sum_{k=1}^{m-1} \int_{t_{m-k}}^{t_{m-k-1}} (s - t_{m-k-1})^\beta ds
$$

$$
+ C \sum_{k=1}^{m-1} \int_{t_{m-k}}^{t_{m-k-1}} \|X^h(s) - X^h(t_{m-k-1})\|_{L^2(\Omega,H)}^2 ds
$$

$$
\leq C \Delta t^\beta + C \sum_{k=1}^{m-1} \int_{t_{m-k}}^{t_{m-k-1}} (s - t_{m-k-1})^{\min(\beta,1)} ds
$$

$$
\leq C \Delta t^{\min(\beta,1)}. \tag{145}
$$
Applying the Itô-isometry, using Lemma 3.3, Assumption 2.4 and Lemma 3.4 yields

\[
\|V I_{52}\|_{L^2(\Omega, H)}^2 = \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \mathbb{E} \left\| U_h(t_m, t_{m-k}) U_h(t_{m-k}, s) \right\|_{L^2(H)}^2 ds \\
\leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left\| U_h(t_m, t_{m-k}) (-A_{h,m-k}) \frac{1}{s-t} \right\|_{L(H)}^2 ds \\
\times \left\| (-A_{h,m-k}) \frac{1}{s-t} U_h(t_{m-k}, s) (-A_{h,m-k}) \frac{1}{s-t} \right\|_{L(H)}^2 ds \\
\leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} t_k^{-1+\epsilon} (s-t_{m-k-1})^{1-\epsilon} ds \\
\leq C \Delta t^{1-\epsilon} \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} t_k^{-1+\epsilon} ds \leq C \Delta t^{1-\epsilon}. \tag{146}
\]

Applying the Itô-isometry, Lemma 3.9, Assumption 2.4 and Lemma 3.4 yields

\[
\|V I_{53}\|_{L^2(\Omega, H)}^2 = \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \mathbb{E} \left[ \left( \prod_{j=m-k}^{m} U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=m-k}^{m} e^{\Delta t A_{h,j}} \right) \right] ds \\
\leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \Delta t^{1-\epsilon} ds \leq C \Delta t^{1-\epsilon}. \tag{147}
\]

Applying the Itô-isometry, Lemma 3.6 and Assumption 2.4 yields

\[
\|V I_{54}\|_{L^2(\Omega, H)}^2 = \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \mathbb{E} \left( \prod_{j=m-k}^{m} e^{\Delta t A_{h,j}} \right) ds \\
\leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left\| P_h B \left( t_{m-k-1}, X^h(t_{m-k-1}) \right) - P_h B \left( t_{m-k-1}, X^h_{m-k-1} \right) \right\|_{L^2(\Omega, H)}^2 ds \\
\leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left\| X^h(t_{m-k-1}) - X^h_{m-k-1} \right\|_{L^2(\Omega, H)} ds \\
\leq C \Delta t \sum_{k=0}^{m-1} \left\| X^h(t_k) - X^h_{k} \right\|_{L^2(\Omega, H)}^2. \tag{148}
\]

Substituting (148), (147), (146) and (145) in (144) yields

\[
\|V I_5\|_{L^2(\Omega, H)}^2 \leq C \Delta t^{\min(\beta,1)} + C \Delta t \sum_{k=0}^{m-1} \left\| X^h(t_k) - X^h_{k} \right\|_{L^2(\Omega, H)}^2. \tag{149}
\]

Substituting (149), (143), (136), (135) and (134) in (132) yields

\[
\left\| X^h(t_m) - X^h_m \right\|_{L^2(\Omega, H)} \leq C \Delta t^{\min(\beta,1-\epsilon)} + C \Delta t \sum_{k=0}^{m-1} \left\| X^h(t_k) - X^h_{k} \right\|_{L^2(\Omega, H)}^2. \tag{150}
\]
Applying the discrete Gronwall lemma to (150) yields
\[ \|X^h(t_m) - X^h_m\|_{L^2(Ω, H)} \leq C\Delta t^{\min(\beta, 1-\epsilon)/2}. \] (151)

Note that to achieve optimal convergence 1/2 when \( \beta \geq 1 \), we only need to re-estimate \( \|V_{52}\|_{L^2(Ω, H)} \) and \( \|V_{53}\|_{L^2(Ω, H)} \) by using Assumption 2.6 and Lemma 3.9 (ii). This is straightforward. The proof of Theorem 2.7 is therefore completed.

4. Numerical experiments. We consider the following stochastic dominated advection-diffusion reaction equation with constant diagonal diffusion tensor
\[ dX = \left[ (1 + e^{-t}) (\Delta X - \nabla \cdot (qX)) - \frac{e^{-t}X}{|X| + 1} \right] dt + XdW, \quad X(0) = 0, \] (152)

with mixed Neumann-Dirichlet boundary conditions on \( Λ = [0, L_1] × [0, L_2] \). The Dirichlet boundary condition is \( X = 1 \) at \( Γ = \{(x, y) : x = 0\} \) and we use the homogeneous Neumann boundary conditions elsewhere. The eigenfunctions \( \{e_{i,j}\} = \{e_{i,j}^{(1)} \otimes e_{i,j}^{(2)}\}_{i,j ≥ 0} \) of the covariance operator \( Q \) are the same as for the Laplace operator \( -\Delta \) with homogeneous boundary condition, given by
\[ e_0^{(i)}(x) = \sqrt{\frac{T}{L_i}}, \quad e_i^{(i)}(x) = \sqrt{\frac{T}{L_i}} \cos \left( \frac{i\pi}{L_i} x \right), \quad i ∈ N, \]
where \( l ∈ \{1, 2\} \), \( x ∈ Λ \). We assume that the noise can be represented as
\[ W(x, t) = \sum_{(i,j) ∈ N^2} \sqrt{λ_{i,j}} e_{i,j}(x) β_{i,j}(t), \] (153)

where \( β_{i,j}(t) \) are independent and identically distributed standard Brownian motions, \( λ_{i,j} \), \( (i,j) ∈ N^2 \) are the eigenvalues of \( Q \), with
\[ λ_{i,j} = (i^2 + j^2)^{-(β+δ)}, \quad β > 0, \] (154)
in the representation (153) for some small \( δ > 0 \). To obtain trace class noise, it is enough to have \( β + δ > 1 \). In our simulations, we take \( β ∈ \{1.5, 2\} \) and \( δ = 0.001 \). In (34), we take \( b(x, u) = u, x ∈ Λ \) and \( u ∈ R \). Therefore, from [12, Section 4] it follows that the operators \( B \) defined by (34) fulfills Assumption 2.4 and Assumption 2.6.

The function \( F \) is given by \( F(t,v) = -\frac{e^{-t}}{1 + |v|}, \) \( t ∈ [0, T], v ∈ H \) and obviously satisfies Assumption 2.3. The linear differential operator \( A(t) \) is given by
\[ A(t) = (1 + e^{-t}) (\Delta(., -\nabla v(.))), \quad t ∈ [0, T], \] (155)
where \( v \) is the Darcy velocity. We obtain the Darcy velocity field \( v = (q_i) \) by solving the following system
\[ \nabla \cdot v = 0, \quad v = -k \nabla p, \] (156)
with Dirichlet boundary conditions on \( Γ_D = \{0, L_1\} × [0, L_2] \) and Neumann boundary conditions on \( Γ_N = (0, L_1) × \{0, L_2\} \) such that
\[ p = \begin{cases} 1 & \text{in } \{0\} × [0, L_2] \\ 0 & \text{in } \{L_1\} × [0, L_2] \end{cases} \]
and \( -k \nabla p(x, t) \cdot n = 0 \) in \( Γ_N \). Here, we use a constant permeabily tensor \( k \) and obtained almost a linear presure \( p \). Clearly \( D(A(t)) = D(A(0)), t ∈ [0, T] \) and \( D((-A(t))^α) = D((-A(0))^α), t ∈ [0, T], 0 ≤ α ≤ 1 \). The function \( q_{i,j}(t, x) \) defined in (27) is given by \( q_{i,i}(t, x) = 1 + e^{-t} \), and \( q_{i,j}(t, x) = 0, \) \( i ≠ j \). Since \( q_{i,i}(t, x) \) is bounded below by \( 1 + e^{-T} \), it follows that the ellipticity condition (28) holds and
therefore as a consequence of Section 2.2, it follows that $A(t)$ is sectorial. Obviously Assumption 2.2 is fulfilled.

In Figure 1, we can observe the convergence of the the stochastic Magnus scheme for two noise’s parameters. Indeed the order of convergence in time is $0.57$ for $\beta = 1$ and $0.54$ for $\beta = 2$. These orders are close to the theoretical orders $0.5$ obtained in Theorem 2.7 for $\beta = 1$ and $\beta = 2$.

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