We study the discrete state structure of $\hat{c} = 1$ superconformal matter coupled to 2-D supergravity. Factorization properties of scattering amplitudes are used to identify these states and to construct the corresponding vertex operators. For both Neveu-Schwarz and Ramond sectors these states are shown to be organized in SU(2) multiplets.

The algebra generated by the discrete states is computed in the limit of null cosmological constant.
I. INTRODUCTION

Matrix models and its continuous counterpart, the Liouville model, have been subject of intense investigations during the last years as representations of 2-D gravity (see [1,2,3] for general references). The growing interest in 2-D gravity may be in part explained by the need of finding a non-perturbative scenario for solving different puzzles posed by perturbative string theory [4]. Gravity in two dimensions is also a tractable toy model for gravitation and it provides a possible description of random surfaces [5,2].

The coupling of conformal matter to 2-D gravity displays a very rich and non trivial structure. [6,7]. For bosonic matter with central charge \( c = 1 \), the theory can be thought as a string theory in two dimensions. Though a “first sight” counting might suggest that the tachyonic vacuum is the only degree of freedom, since there are no transverse modes, a more careful analysis indicates the existence of a full tower of discrete states. These are remnants of the massive string states in higher dimensions. Significant progress has been achieved in the understanding of this spectrum in the matrix formulation as well as in the Liouville approach [8,9].

In the supersymmetric case the situation is less clear. In particular there is no satisfactory supermatrix formulation (see however Refs. [10,11]). On the other hand, recent works attempt to understand the structure of the scattering amplitudes and the spectrum [12,13,14,15] in the super-Liouville model [16].

In this article we concentrate in the study of discrete states for \( \hat{c} = 1 \) (c=3/2) superconformal matter coupled to two dimensional gravity. By generalizing the analysis performed in [17] to the supersymmetric case, we first look at the pole structure of tachyonic scattering amplitudes. From the residues at the poles when two particles collide to the same point, we read the vertex operators corresponding to Neveu-Schwarz discrete states. A more complete analysis is then performed by studying the different constraints on polarization tensors imposed by the OPE with the superstress energy tensor. The analysis is further generalized to consider the Ramond sector as well. This is done in sections 2 and 3.

In section 4 we show, by choosing a “material” gauge (i.e. Liouville sector remains in the ground state), that the discrete states can be organized into multiplets of a global SU(2) algebra, as in the bosonic case [4,18]. The generators of the algebra are obtained by supercontour integrals of appropriate supercurrents.

In the last section we study the algebra generated by Neveu-Schwarz and Ramond discrete states and verify that, after proper redefinitions of the fields, it reduces to the algebra of the area preserving diffeomorphisms in two dimensions [4,19]. Two appendices are included. In appendix A we show a computation where three tachyons collide to the same point. In appendix B we define cocycle factors needed for a correct definition of OPE’s.
II. SCATTERING AMPLITUDES AND FACTORIZATION

Let us start with a brief review of the computations of Neveu-Schwarz amplitudes for two dimensional supergravity coupled to N=1 supermatter ($\hat{c} = 1$). In the conformal gauge, matter and super Liouville theories can be realized by two free superfields $X_0$ and $X_1$, the Liouville and the matter superfield:

$$X_0 = \phi + \theta \psi_0, \quad X_1 = x + \theta \psi_1.$$  \hspace{1cm} (2.1)

The total action can be expressed in terms of the two component superfield $X_\mu = (X_0, X_1)$ as follows

$$S = \frac{1}{4\pi} \int d^2 z d^2 \theta (D_\theta X \cdot D_\bar{\theta} X - i R Q \cdot X + 2 \mu e^{i \alpha \cdot X}),$$ \hspace{1cm} (2.2)

where $D = \partial_\theta + \theta \partial_z$ is the superderivative. The values of $Q_\mu = (-2i, 0)$ and $\alpha_\mu = (i, 0)$ are respectively fixed by requiring the vanishing of the total central charge and that the exponential term in the action has the correct conformal weight.

The two point function of the superfield $X_\mu$ is given by

$$G(1, 2) = \langle X_\mu(z_1, \theta_1) X_\nu(z_2, \theta_2) \rangle = -\eta_{\mu\nu} \log |z_{12}|^2$$ \hspace{1cm} (2.3)

where $z_{12} = z_1 - z_2 - \theta_1 \theta_2$.

The super energy-momentum tensor is

$$T(z, \theta) = -\frac{1}{2} D^2 X_\mu D X^\mu - \frac{i}{2} Q_\mu D^3 X^\mu \equiv \frac{1}{2} T_F + \theta T_B.$$ \hspace{1cm} (2.4)

The energy momentum tensor $T_B$ and the super current $T_F$, expressed in terms of the component fields $X_\mu = (\phi, x)$ and $\psi_\mu = (\psi_0, \psi_1)$, read

$$T_B = -\frac{1}{2} \partial X_\mu \partial X^\mu + \frac{1}{2} \psi_\mu \partial \psi^\mu - \frac{i}{2} Q_\mu \partial^2 X^\mu,$$ \hspace{1cm} (2.5)

$$T_F = -\psi_\mu \partial X^\mu - i Q_\mu \partial \psi^\mu$$ \hspace{1cm} (2.6)

and satisfy the N=1 superconformal operator product with $\hat{c} = 1$.

The gravitationally dressed tachyon vertex operator $\Psi(p)_{NS}$ of momentum $p$ is

$$\Psi(p)_{NS} = \int d^2 z d^2 \theta e^{i P \cdot X}$$ \hspace{1cm} (2.7)

where $P$ is the two component momentum $P_\mu = (-i \beta(p), p)$. By requiring that the conformal dimension of $\Psi_{NS}$ is $(\frac{1}{2}, \frac{1}{2})$, the “energy” $\beta(p)$ is determined to be

$$\beta^\pm = -1 \pm p.$$ \hspace{1cm} (2.8)
It corresponds to tachyons with chirality ±1. The calculation of the N-point function of these tachyons on a sphere is given by the path integral

\[ A_N = < \prod_{i=1}^{N} \Psi_{NS}(p_i) > = \int \frac{DX_0 DX_1}{V} e^{-S} \prod_{i=1}^{N} \left( d^2 z_i d^2 \theta_i e^{iP \cdot X(z_i, \theta_i)} \right) \]  \hspace{1cm} (2.9)

where \( V \) is the volume of the gauge symmetry group. By performing the standard integration over the zero mode of \( \phi \) and \( x \) this amplitude becomes

\[ A_N = \left( \frac{\mu}{2\pi} \right)^s \alpha^{-1} \Gamma(-s) \int \frac{D\tilde{X}_0 D\tilde{X}_1}{\tilde{V}} e^{-\tilde{S}} \left( \int d\theta e^{i\alpha \tilde{X}_0} \right)^s \int \prod_{i=1}^{N} \left( d^2 z_i d^2 \theta_i e^{i\beta \tilde{X}_0} e^{ip_i \tilde{X}_1} \right) \]  \hspace{1cm} (2.10)

where \( \tilde{S} \) is the free action (i.e. \( \mu = 0 \)) and

\[ \sum_{i=1}^{N} (P_i + s\alpha + Q)_\mu = 0, \]  \hspace{1cm} (2.11)

giving the energy momentum conservation. Notice that, in order to have a non zero result, \( N + s \) must be even since the tachyon vertex is a fermion operator on the world-sheet. This amplitude has been computed in Ref. [12,14] where the two dimensional integrals have been explicitly evaluated. Like in the bosonic case, the amplitude factorizes in N-external leg factors and exhibits leg poles when the value of the external tachyon momenta is an integer: \( p_i = r + 1, \ r = 0,1,... \) Here we study the factorization of the N-tachyon amplitude and we identify these legs poles as corresponding to higher level states which are present in the spectrum for discrete value of the momenta.

We consider therefore the following correlation function

\[ A_{N,s} = \left< \prod_{i=2}^{N+s} \left( d^2 z_i d^2 \theta_i e^{iP_i \cdot X(z_i, \theta_i)} \right) \right>_0 \]

\[ = \prod_{i=1}^{N+s} e^{-\sum_{i\neq j} P_i \cdot P_j G(i,j)} \]  \hspace{1cm} (2.12)

where \( < \cdots >_0 \) means in the free theory, \( P_{N+j} = (\alpha, 0), \ j = 1,...s \) and \( G(i,j) \) is given in (2.3). This amplitude presents singularities when two of the tachyons collide to a same point (we will comment on the pinching of more tachyons at the end).

Let us consider the kinematical configuration where all tachyons except one, have the same positive chirality. By considering the contribution to the amplitude from the integration region where one tachyon with momentum \( p_1 \) and negative chirality collides with an other tachyon of momentum \( p_2 \) (and positive chirality), we find

\[ < ... >_0 = \int d\theta d\theta_1 \sum_{m=0}^{\infty} \frac{1}{2(P_1 \cdot P_2 + m + 1)} \left( \frac{1}{m!} \right)^2 F_m(2, j) \]  \hspace{1cm} (2.13)
where

\[ F_m(2, j) = < : e^{iP_2 \cdot X} | \partial_{\bar{D}}^m D_1 |^2 e^{iP_1 \cdot X} : \prod_{j=3}^N e^{iP_j \cdot X} >_1 | = 2 \] (2.14)

(double dots indicate normal ordering and \( \partial \bar{\partial} X \equiv 0 \)). Therefore the poles in the amplitude are found for

\[ P_1 \cdot P_2 + (m + 1) = 0 \] (2.15)

where \( n = 2m + 1 \ (n = 1, 3, ... \) ) is the number of superderivatives in the corresponding vertex operator. The intermediate state momentum is \( P = P_1 + P_2 \) and satisfies the level \( n \) mass shell condition

\[ \frac{1}{2} P (P + Q) + \frac{n-1}{2} = 0. \] (2.16)

As expected we find only odd level intermediate states (i.e. bosonic) in this factorization, due to the fermionic nature of the tachyon vertex. Since the kinematical constraint (2.11) fixes the value of \( P_1 \)

\[ p_1 = -\frac{N + s - 2}{2} = -\frac{t + 1}{2}, \quad \beta_1 = \frac{N + s - 4}{2} = \frac{t - 1}{2} \] (2.17)

where \( t = N + s - 3 \) is an odd number \( \geq 1 \), the condition (2.13) fixes the value of the momentum of the other tachyon and we find a pole in the amplitude at level \( n \) when \( p_2 \) is given by

\[ p_2 = \frac{t + n}{2t}, \quad \beta_2 = \frac{n - t}{2t}. \] (2.18)

Therefore the momentum of the intermediate state is

\[ p = \frac{n - t^2}{2t}, \quad \beta(p) = \frac{t^2 - 2t + n}{2t} = -1 + \sqrt{\left(\frac{n - t^2}{2t}\right)^2 + n}. \] (2.19)

This is precisely the momentum of a state of level \( n \) with positive chirality.

Notice that from the explicit result of the N-tachyon amplitude, we know that the amplitude presents leg poles when the momenta of the positive chirality tachyons are \( p_i = r + 1 \), where \( r \) is a non-negative integer. Then from all possible intermediate states, only those corresponding to \( p_2 \) integer survive, the others being null states. From (2.18), the number of tachyons \( N \) is then related to the level \( n \) by \( n = (2r + 1)t \) and the intermediate state momentum (2.19) is an integer.

Let us first consider levels \( n = 1 \) in the N=4 amplitude for \( s = 0 \) (i.e. \( t = 1 \)). The residue of the pole \( n = 1 \) (i.e. \( p_2 = 1 \)) is given by \( F_0(2, j) \) and it is reproduced by the vertex operator (we omit barred derivatives in the following)
where $\epsilon_\mu = i P_\mu = (0, -i)$ and $P = (0, 0)$. In terms of the component fields this vertex reads $V_1^+ = -i \partial x$. Notice that $\epsilon_\mu$ satisfies the polarization condition $\epsilon_\mu (P + Q)_\mu = 0$.

For $t > 1$ (i.e. $N > 4$ or $s \neq 0$) the value of $p^2$ at the pole, $(2.18)$, is not an integer and therefore we expect to find only null level 1 intermediate states. In fact for general $t$ the residue is reproduced by a vertex $(2.20)$ with polarization and momentum given by

$$
\epsilon_\mu = (\frac{t - 1}{2}, -i \frac{t + 1}{2}), \quad P = (-i \frac{(t - 1)^2}{2t}, 1 - \frac{t^2}{2t}).
$$

For $t \neq 1$, this vertex is a total derivative

$$
V_1(t) = \frac{t}{t - 1} \theta (\epsilon \frac{(t - 1)^2}{2t} - i \frac{t^2}{2t} - x).
$$

The residue of the pole $n = 3$ (i.e. $F^1(2, j)$ in eq. $(2.14)$) is reproduced by the vertex operator

$$
V_3 = \int d\theta (\epsilon_\mu DDX^\mu + \epsilon_\mu \nu DDX^\mu DXX^\nu) e^{i P X}
$$

where $\epsilon_\mu = i P_\mu$, $\epsilon_\mu \nu = -P_\mu P_\nu$ and $P = (P_1 + P_2)$ which are given by $(2.17)$ and $(2.19)$ with $n = 3$. Non null intermediate states are found for $t = 1$ and $t = 3$ corresponding to poles in the amplitude at $p^2 = 2$ and $p^2 = 1$ respectively. In the first case the momentum and polarizations are

$$
P = (-i, 1), \quad \epsilon_\mu = (0, -i), \quad \epsilon_{00} = 0, \quad \epsilon_{11} = -1, \quad \epsilon_{01} = \epsilon_{10} = 0.
$$

For the other one we find

$$
P = (-i, -1), \quad \epsilon_\mu = (1, -2i) \quad \epsilon_{00} = 1, \quad \epsilon_{11} = -4, \quad \epsilon_{01} = \epsilon_{10} = -2i.
$$

The operators for higher level intermediate states may be constructed following the same steps. In order to obtain even level (i.e. fermionic) intermediate states a more general factorization of the amplitude must be taken into account. In fact, fermionic states may appear only when an odd number of tachyons collides to a same point on the world-sheet. In appendix A we will compute the contribution to the amplitude from the integration region where a negative chirality tachyon collides with two positive chirality tachyons. We will find that the intermediate states have always negative chirality and therefore the residue vanishes (the left side blob of the amplitude has two states with negative chirality). This is a general feature of a factorization which involves more than two tachyons and therefore the fermionic intermediate states decouple in this amplitude.
The scattering of Ramond fields may also be considered. In this case it is convenient to bosonize the world-sheet fermion in order to construct the Ramond vertex \[20\]

\[\begin{align*}
e^{i\sigma} &= \frac{1}{\sqrt{2}}(\psi_0 + i\psi_1), \\
e^{-i\sigma} &= \frac{1}{\sqrt{2}}(\psi_0 - i\psi_1).
\end{align*}\]

We define the propagator \( \mathcal{G}[\sigma(z)\sigma(z')] = -\log(-(z - z')) \). The Ramond vertex operator in the \(-\frac{1}{2}\) picture is

\[V_{-\frac{1}{2}}^\alpha = e^{-\frac{i}{2} \rho} e^{\frac{i}{2} i\sigma} e^{iP \cdot X}\]

where \( \alpha = \pm 1 \), \( \rho \) is the bosonized ghost current and \( P = (-i\beta, p) \). The energy \( \beta \) is determined by imposing the mass shell condition \( \frac{1}{2}P(P + Q) + \frac{1}{2} = 0 \) and that this state is annihilated by \( G_0 \) (this is equivalent to enforce the Dirac equation). The two requirements are satisfied if

\[\beta = -1 + \alpha p.\]

Recall that in the computation of the amplitude vertices in the "\( \frac{1}{2} \)"-picture should be considered since the total ghost charge must add to \(-2\). The amplitude for \( N - 2M \) NS-states (tachyons) and \( 2M \) R-states (eq. (2.26)) has been computed in Ref. [12]. Like in the previous case, the amplitude is nonvanishing only if all the states, except one, have positive chirality. When the negative chirality state is a NS-tachyon (R), \( N + s \) must be even (odd) in order to conserve the total fermionic charge. Besides the leg poles for integer value of the tachyon momentum, the amplitude has leg poles when the value of the R state momentum is \( p_i = l + \frac{1}{2}, \quad l = 0, 1, ... \). These poles may be identified as corresponding to higher level discrete R states which can be seen in factorization of the amplitude when one NS state and one R state of opposite chirality collide to the same point. Since the derivation is completely similar to the NS case we give only the result. Intermediate states of level \( n \) are found when \( n \) is an even integer (the mass shell condition and momentum are given by (2.16) and (2.19) respectively). When a negative chirality NS tachyon collides with the R state, the level \( n \) intermediate state corresponds to the R-leg pole \( p_2 = l + \frac{1}{2} \) for \( n = 2lt \) and has momentum \( p = l - \frac{t}{2} \) (notice that \( t = N + s - 3 \) is odd in this case). When a negative chirality R state collides with a NS tachyon, we find an intermediate state for \( n = (2r + 1)t \) with a momentum \( p = \frac{2r + 1 - t}{2} \), corresponding to a NS leg pole at \( p_1 = r + 1 \) (in this case \( t \) is even). Therefore the momentum of the intermediate state is always half-integer.

\[1\] The minus sign inside the logarithm is due to the \( \psi \)-propagator we are working with

\[<\psi_\mu(z)\psi_\nu(z')>=\frac{\delta_{\mu\nu}}{z-z'}.\]
For example, the \( n = 0 \) level state is found only in the first case and corresponds to the R-leg pole at \( p_2 = \frac{1}{2} \). Its vertex operator is given by (2.26) with \( \alpha = -1 \) and \( P = (i\frac{1}{2}, -\frac{1}{2}) \). At level \( n = 2 \) there are two states of momentum \( P = (-\frac{i}{2}, \pm \frac{1}{2}) \) corresponding to the Ramond leg pole \( p_2 = \frac{3}{2} \) and to the tachyon leg pole \( p_1 = 1 \) respectively. The corresponding vertex operators are

\[
V_2^+(p = \pm \frac{1}{2}) = \{e^{\pm i\frac{3}{2}g} \mp i(\partial \sigma + \partial x)e^{\pm i\frac{1}{2}g}\}e^{\pm i\frac{3}{4}+\frac{1}{2}}
\]  

(2.28)

where \( \pm \) is the chirality of the Ramond state (2.26).

III. HIGHER LEVEL OPERATORS

In order to gain a better comprehension of these intermediate states it is useful to look at the OPE of the superstress energy tensor with vertices of general polarizations. The requirement of conformal invariance imposes certain conditions on these polarizations. In two dimensions the situation is rather peculiar since the number of constraints plus possible gauge symmetries equals the number of components of the polarization tensors. Thus, in principle, no degrees of freedom are left. This is true except for some particular values of momentum for which either the constraints are relaxed or the gauge transformations become linearly dependent, therefore leaving space for new states which are responsible for the singularities of the N-tachyon amplitude.

The general form of the vertex operator for a state of the level \( n \) is

\[
\int d^2\theta V_R V_L e^{iPX}
\]

(3.1)

where \( V_L \) is given by a sum of all possible terms of the form

\[
\epsilon_{\mu_1...\mu_s} \prod_{j=1}^{s} D^{n_j} X^{\mu_j}
\]

such that \( \sum_{j=1}^{s} n_j = n \), \( n = 1, 2, 3, ... \)  

(3.2)

(in the following we shall consider only the left part of the vertex). The vertex has conformal dimension \((\frac{1}{2}, \frac{1}{2})\) if (2.16) is satisfied. The two solutions

\[
\beta = -1 \pm \sqrt{p^2 + n}
\]

(3.3)

correspond to \((+)\) and \((-)\) states. Moreover, the polarization tensors \( \epsilon_{\mu_1...\mu_s} \) must satisfy constraints coming from the requirement that the vertex operator is a primary superfield. In the following we will find these relations and the explicit solutions for the \( n = 1, n = 2 \) and \( n = 3 \) case.
At level \( n = 1 \) the general form of the vertex operator is
\[
V_1 = \int d\theta \epsilon_\mu D X^\mu e^{iP X}
\] (3.4)
and, in this case, eq. (2.16) reads \( P(P + Q) = 0 \). The polarization tensor \( \epsilon_\mu \) must satisfy
\[
\epsilon_\mu (P + Q)_\mu = 0.
\] (3.5)
This condition is solved by \( \epsilon_\mu = a P_\mu \). In the operator language this solution corresponds to the state
\[
W_1 = G_{-1/2} |p >
\] (3.6)
which is null and therefore decouples (as usual we call \( G_r \) and \( L_m \) the Fourier modes of the supercurrent \( T_F \) and \( T_B \) respectively). Indeed this state corresponds to the gauge symmetry \( \epsilon_\mu \rightarrow \epsilon_\mu + P_\mu \). Then for a generic value of the momenta there is no physical degree of freedom at this level. There are two cases where this reasoning is not valid. The gauge symmetry does not exist when \( P = 0 \) or the constraint relaxes when \( P = -Q \). In correspondence of these two values of the momentum we have two physical states
\[
V_1^+ = \lim_{p \rightarrow 0} \frac{V_1}{p} = \int d\theta D X_1 = \partial x
\] (3.7)
and
\[
V_1^- = \int d\theta D X_1 e^{-2X_0} = (\partial x + 2\psi_0 \psi_1)e^{-2\phi}.
\] (3.8)
The first one is exactly the vertex (2.20). The second one cannot be obtained in the factorization of the N-tachyon amplitude considered previously (i.e. with one negative chirality tachyon and the remaining ones of positive chirality) since they decouple in this amplitude.

At level \( n = 2 \) the vertex operator is given by
\[
V_2 = (\epsilon_\mu D D X^\mu + \epsilon_\mu \nu D X^\mu D X^{\nu}) e^{iP X}
\] (3.9)
where \( P(P + Q) + 1 = 0 \), \( \epsilon_{\mu \nu} = -\epsilon_{\nu \mu} \) and
\[
\epsilon_\nu - 2i(P + Q)_\mu \epsilon_{\mu \nu} = 0. \] (3.10)
The only solution is
\[
\epsilon_\mu = (-4 + P Q) P_\mu + Q_\mu
\]
\[
\epsilon_{\mu \nu} = \frac{i}{2} (P_\mu \epsilon_\nu - P_\nu \epsilon_\mu).
\] (3.11)
These polarizations correspond in the operator language to the state

\[ W_2 = G_{-1/2}(\epsilon \cdot \psi e^{iP \cdot X}) \]  

which is null. It is easy to check that neither the gauge symmetry degenerates nor the constraints relax for physical states. This result is consistent with the Kac-determinant argument [15].

At level \( n = 3 \) we have

\[ V_3 = (\epsilon_\mu D D D X^\mu + \epsilon_{\mu\nu} D D X^\mu D X^\nu) e^{iP \cdot X} \]  

where \( P(P + Q) + 2 = 0 \). In general it is possible to include a term of the form \( \epsilon_{\mu\nu\rho} D X^\mu D X^\nu D X^\rho \). However, due to the fact that \( \epsilon_{\mu\nu\rho} \) must be completely antisymmetric this term is not present in two dimensions. The following constraints must be satisfied

\[ \epsilon_{\mu\mu} + i \epsilon_{\mu}(P + 2Q)^\mu = 0, \]  

\[ \epsilon_{\mu} - i \epsilon_{\mu\nu}(P + Q)^\nu = 0, \]  

\[ \epsilon_{\nu\mu} = \epsilon_{\mu\nu}. \]

The number of degrees of freedom is therefore reduced to two by these equations. This is equal to the number of gauge symmetries generated by the two independent null states

\[ W_3^{(a)} = (G_{-3/2} + 2G_{-1/2}L_{-1})|p > \]  

\[ W_3^{(b)} = G_{-1/2}(\epsilon \cdot \partial X e^{iP \cdot X}) + L_{-1}(\epsilon \cdot \psi e^{iP \cdot X}) \]

where

\[ \epsilon_\mu(P + Q)_\mu = 0. \]

The vertex operators \( V_3^{(a)} \) and \( V_3^{(b)} \) corresponding to these states are given by (3.13) with the following polarizations

\[ \epsilon_\mu^{(a)} = i\left(\frac{3}{2}P_\mu - \frac{1}{2}Q_\mu\right), \quad \epsilon_{\mu\nu}^{(a)} = -\frac{1}{2}\eta_{\mu\nu} - P_\mu P_\nu \]

\[ \epsilon_\mu^{(b)} = i\left((-4 + P \cdot Q)P_\mu + 2Q_\mu\right), \quad \epsilon_{\mu\nu}^{(b)} = \frac{i}{2}(P_\mu \epsilon_\nu^{(b)} + P_\nu \epsilon_\mu^{(b)}). \]

It is easy to check, by using (2.16), that the polarizations (3.19a) and (3.19b) satisfy (3.14-3.16).
Therefore the most general vertex operator for a state of the level \( n = 3 \) is null and it is given by a linear combination of \( V_3^{(a)} \) and \( V_3^{(b)} \). However for the particular value of momentum \( P = (-i, \pm 1) \), the two operators \((3.17a)\) and \((3.17b)\) become linearly dependent and it is possible to find a combination of them with zero polarization tensors, therefore trivially satisfying eqs. \((3.14)-(3.16)\). As in the \( n = 1 \) level, a non null vertex operator can be found by taking a particular limit to this value of the momentum

\[
V_3^+(p = \pm 1) = \lim_{p \to \pm 1} \frac{V_3^{(a)} + \frac{1}{4} V_3^{(b)}}{p \mp 1}.
\]  

(3.20)

By using eqs. \((3.19a)\) and \((3.19b)\) the polarization tensors are

\[
\epsilon_\mu = \left( \mp \frac{1}{4}, -\frac{i}{4} \right), \quad \epsilon_{00} = \mp \frac{1}{4}, \quad \epsilon_{11} = \pm \frac{5}{4}, \quad \epsilon_{10} = \epsilon_{01} = -\frac{i}{2}.
\]  

(3.21)

They satisfy the polarization equations \((3.14)-(3.16)\). The two vertex operators \((3.20)\) coincide with those found in the factorization of the 4-tachyon amplitude up to null operators. In fact, by using \((3.19a)-(3.19b)\) and \((2.24a)-(2.24b)\) we find

\[
\frac{1}{2} V_3^+ = V_3^{(a)} - \frac{3}{2} V_3^{(b)}.
\]  

Notice that by performing the gauge transformation which corresponds to the addition of the null state

\[
\mp \frac{1}{8} V_3^{(b)}
\]

to \( V_3^+ (p = \pm 1) \), it is possible to change to a state possessing only matter excitations (the so called material gauge). This state has \( \epsilon_1 = i, \epsilon_{11} = \pm 1 \) and all the other components equal to zero.

Other non null states may be found when the constraints eqs.\((3.14)-(3.16)\) relax. This happens when \( P = (3i, \pm 1) \). For each value of \( P \) there is a \((-)\) state with a polarization given by

\[
\epsilon_\mu = (0, 0), \quad \epsilon_{00} = -\epsilon_{11} = \pm i \epsilon_{01} = \pm i \epsilon_{10} = 1.
\]  

(3.22)

Also in this case it is possible to make a gauge transformation to the material gauge by adding the null state

\[
\frac{1}{8} (V_3^{(a)} - \frac{11}{4} V_3^{(b)}).
\]

The above analysis can be repeated for the Ramond sector of the theory. Again physical states are allowed only for some particular value of the momenta. In the following we explicitly work out the polarization conditions and their solutions for the level \( n = 2 \) and compare the result with the states found previously in the factorization (eq.\((2.28)\)). The general form of the vertex operator is

\[
V_2 = e^{-\frac{i}{2} \left( \epsilon^\alpha \partial X^- + \epsilon_{-\alpha} \partial X^+ + \omega^\alpha \partial \sigma e^{i\frac{3}{2} \sigma} + \nu^\alpha e^{i\frac{5}{2} \sigma} \right)} e^{iP \cdot X}
\]  

(3.23)

where \( \alpha = \pm 1 \) (we have introduced the complex notation \( X^\pm = \frac{1}{\sqrt{2}} (\phi \pm ix) \)). The mass shell condition for this state is given by \((2.10)\) with \( n = 2 \). The polarization \( \epsilon^\alpha_{-\alpha}, \omega^\alpha \) and \( \nu^\alpha \) must satisfy certain conditions coming from the requirement of superconformal invariance \((G_0 \text{ and } G_1 \text{ must annihilate the state})\). By using \((2.7)\) we find
\[ v^{+} = -i\epsilon^{+}(P_{+} + \frac{1}{2}Q_{+}), \quad \omega^{+} = \epsilon^{+}(P_{+} + \frac{1}{2}Q_{+}), \]
\[ \epsilon^{+} = (P_{+} + \frac{3}{2}Q_{+})\omega^{+} + i(P_{+} + \frac{3}{2}Q_{+})v^{+} \] \hspace{1cm} (3.24a)

and

\[ v^{-} = -i\epsilon^{-}(P_{+} + \frac{1}{2}Q_{+}), \quad \omega^{-} = -\epsilon^{-}(P_{+} + \frac{1}{2}Q_{+}), \]
\[ \epsilon^{-} = i\omega^{-} - (P_{+} + \frac{3}{2}Q_{+})\omega^{-} \] \hspace{1cm} (3.24b)

where \( P_{\pm} = \frac{1}{\sqrt{2}}(P_{0} \pm iP_{1}) = \sqrt{2}(\beta \mp p) \). The solutions are

\[ \epsilon^{+} = 2iP_{+}, \quad \epsilon^{-} = i(P_{-} - \frac{1}{2}Q_{-}), \]
\[ v^{+} = (P_{+} + \frac{1}{2}Q_{+})(P_{-} - \frac{1}{2}Q_{-}), \quad \omega^{-} = -i(1 + Q_{+}P_{-}) \] \hspace{1cm} (3.25a)

and

\[ \epsilon^{+} = 2iP_{-}, \quad \epsilon^{-} = i(P_{+} - \frac{1}{2}Q_{+}), \]
\[ v^{-} = (P_{+} + \frac{1}{2}Q_{+})(P_{-} - \frac{1}{2}Q_{-}), \quad \omega^{+} = i(1 + Q_{+}P_{+}) \] \hspace{1cm} (3.25b)

They correspond to the null states \( G_{0}G_{-1}|p, - > \) and \( G_{0}G_{-1}|p, + > \) respectively, where \( |p, \alpha > \) is the state (2.20) with \( P(P + Q) = -1 \). However there is a particular value of the momenta for which the gauge transformation degenerates. In the first case (eq.(3.25a)) this happens for \( P_{+} = 0 \) and \( P_{-} = \frac{1}{2}Q_{-} \), i.e. \( P = (-i, 1) \); in the second (eq.(3.25b)) for \( P_{-} = 0 \) and \( P_{+} = \frac{1}{2}Q_{+} \), i.e. \( P = (-i, -1) \), leading to (+) states (2.28).

Other non null states are found when the constraints (3.24a) and (3.24b) relax. This happens when \( P_{+}Q_{+} = 3 \) and \( P_{-}Q_{-} = 3 \) (i.e \( P = (i\frac{3}{2}, \mp \frac{1}{2}) \)), respectively. For these values of the momentum there are two physical states. Up to a gauge transformation their vertex is

\[ V^{-}_{2}(p = \mp \frac{1}{2}) = e^{-\frac{i}{2}}\{i(\partial x - \partial \sigma)e^{\mp \frac{i}{2}\sigma} \pm e^{\mp i\frac{3}{2}\sigma}e^{\mp \frac{i}{2}x - \frac{5}{2}\phi} \}. \] \hspace{1cm} (3.26)

**IV. SU(2) CURRENT ALGEBRA AND DISCRETE STATES**

In the bosonic \( c = 1 \) theory, further insight on the structure of the discrete states can be gained by realizing that they, together with some tachyon states, fit into multiplets of an SU(2) algebra. The highest weight vectors of the SU(2) multiplets are tachyon states with momentum \( p = \sqrt{2}j; \ j = 1/2, \ 1, \ 3/2, \cdots \) and discrete states are obtained applying
the lowering operator of the SU(2) algebra. All primary states of the $c = 1$ theory at the
SU(2) compactification radius can be obtained with this procedure. We will now show how
to make a similar analysis for the $N=1$ case. In the bosonic theory the generators have the
key property that they do not change the conformal properties of the state on which they
act. This is ensured because they are contour integrals of conformal weight 1 currents. It
is easy to show that the supercontour integral of a superconformal superfield $j_{1/2}(z, \theta)$ of
weight $h = 1/2$ commutes with the super Virasoro algebra. In fact
\[
\left[ \oint \frac{dz'}{2\pi i} d\theta'(z', \theta') T_{z\theta}(z', \theta'), \oint \frac{dz}{2\pi i} d\theta j_h(z, \theta) \right] = \oint \frac{dz}{2\pi i} d\theta [\epsilon \partial + \frac{1}{2}(D\epsilon)D + h(\partial \epsilon)]j_h(z, \theta)
\]
and the r.h.s is the contour integral of a total derivative if $h = 1/2$. The SU(2) generators
are given by the following contour integrals
\[
H_\pm = \pm \sqrt{2} \oint \frac{dz}{2\pi i} d\theta e^{\pm iX_1(z, \theta)} \quad H_0 = \oint \frac{dz}{2\pi i} d\theta iD X_1(z, \theta).
\]
In the Neveu-Schwarz sector the primary fields for the discrete states can be constructed
using this algebra as follows. The highest weight vector is the matter part of a tachyon
\[
\phi_{jj}(z, \theta) = e^{ijX_1(z, \theta)}
\]
where the momentum must be integer so that the action of $H_\pm$ is well defined. The states
are obtained as
\[
\phi_{jm}(z, \theta) = k(j, m)(H_-)^{j-m}\phi_{jj} \quad k(j, m) = \sqrt{\frac{(j + m)!}{(j - m)!}(2j)!}
\]
where $j = 0, 1...$ and $m = -j, -j + 1, ..j$. They have conformal weight $j^2/2$ and momentum
$p = m$. These are precisely the weights and momenta for the states in the NS sector
prescribed by the Kac formula argument of ref. [13]. By adding the Liouville field part with
the two possible dressings we obtain
\[
V_{jm}^\pm = \int d\theta \Phi_{jm}^\pm
\]
where
\[
\Phi_{jm}^\pm = \phi_{jm} e^{(-1\pm j)X_0}.
\]
According to our previous convention, the level of these states is $n = j^2 - m^2$.
In the Ramond case it is convenient to include the Liouville field from the beginning since the highest weight vectors of the SU(2) algebra are the Ramond ground states with half-integer momentum. 

\[ R_{jj}^\pm = e^{-\frac{1}{2}\rho(z)}e^{\pm \frac{1}{2}i\sigma(z)}e^{ijx(z)}e^{(-1\pm j)\phi}. \]  

(4.7)

Notice that if \( j \) is half integer the action of \( H_{\pm} \) is well defined as can be seen by using the bosonized form of these operators

\[ H_{\pm} = \oint \frac{dz}{2\pi i} \left( e^{i\sigma(z)} - e^{-i\sigma(z)} \right) e^{\pm ix(z)}. \]  

(4.8)

The rest of the multiplet is obtained acting with \( H_- \)

\[ R_{jm}^\pm = k(j, m)(H_-)^{j-m}R_{jj}^\pm. \]  

(4.9)

It is easy to see, analysing the weight and momentum of the matter part, that all the states predicted by the Kac formula [15] for the Ramond case are obtained in this way.

### V. ALGEBRA OF DISCRETE STATES

The description of discrete states as multiplets of SU(2) is very useful since the study of the interaction between these modes is greatly simplified demanding SU(2) covariance [9,7]. Their interaction is given by the quadratic piece in the \( \beta \)-function which in turn depends on the coefficient of term \( 1/z \) of their OPE

\[ O_{j_1 m_1}(z)O_{j_2 m_2}(z') \sim \frac{1}{z-z'}F_{j_1 m_1, j_2 m_2}^{j_3 m_3}O_{j_3 m_3}(z') \]  

(5.1)

where \( O \) denotes a NS or R state [1,3,13]. We first take the plus sign on both fields on the l.h.s., other cases will be analysed later on. Conservation of matter momentum implies that \( m_3 = m_1 + m_2 \) and conservation of Liouville momentum (valid on the bulk) says that we have also the plus sign on the r.h.s. of (5.1) and that \( j_3 = j_1 + j_2 - 1 \). As the left hand side of (5.1) transforms as a product of two SU(2) representations we have

\[ F_{j_1 m_1, j_2 m_2}^{j_3 m_3} = C_{j_3, j_1, j_2, m_1, m_2}^{j_1, j_2, -1, m_3} g(j_1, j_2) \]  

(5.2)

where \( C_{j_1, m_1, j_2, m_2}^{j_3, j_1, j_2, m_3} \) is the Clebsh-Gordan coefficient for the product of two SU(2) representations, which for the above values of \( j_3 \) and \( m_3 \) reads

\footnote{We use the vertex in the \(-\frac{1}{2}\)-picture. As noted in [20] there are infinitely many operators with different ghost number for each physical state.}
\[ C_{j_1, m_1, j_2, m_2}^{j_3, m_3} = \frac{N(j_3, m_3) j_2 m_1 - j_1 m_2}{N(j_1, m_1) N(j_2, m_2) \sqrt{j_3(j_3 + 1)}} \]  \tag{5.3}

where

\[ N(j, m) = \left[ \frac{(j + m)! (j - m)!}{(2j - 1)!} \right]^{1/2}. \]  \tag{5.4}

The value of \( g(j_1, j_2) \) can be found performing explicitly the OPE for specific values of \( m_1 \) and \( m_2 \). We choose \( m_1 = j_1 - 1 \), \( m_2 = j_2 \). We consider separately the three possibilities, i.e. two NS states, one NS and one R and finally two R states in (5.1), since the corresponding functions \( g(j_1, j_2) \) are different. Moreover it is necessary to include cocycle factors to ensure proper commutation relations. Our cocycles are constructed in appendix B. Let us start with two NS states

\[ V_{j_1, j_1-1}(z)V_{j_2, j_2}(z') = \frac{1}{\sqrt{2j_1}} \int d\theta \Phi^+_{j_1, j_1} \int d\theta' \Phi^+_{j_2, j_2} \]

\[ = -\frac{1}{\sqrt{j_1}} \int d\theta d\theta' \int \frac{du}{2\pi i} e^{-i\theta(\theta + \theta')} \int e^{i j_1 x(z, \theta) + (-j_1 + j_1) \phi(z, \theta)} \int e^{i j_2 x(z', \theta') + (-j_2 + j_2) \phi(z', \theta')} \int e^{i j_3 x + (-j_3) \phi}. \]

No derivatives of the exponential appear because we are interested in the most divergent term in the OPE. Performing the \( \theta \)-integrations and the rescaling \( u = (z - z')y \) we obtain

\[ \frac{1}{z - z'} \frac{1}{\sqrt{j_1}} \int \frac{dy}{2\pi i} e^{i j_1 x(1 + y)^{-j_2} \int d\theta' \Phi^+_{j_3, j_3}(z', \theta)} \int d\theta' \Phi^+_{j_3, j_3}(z', \theta) \]  \tag{5.5}

Finally computing the contour integral and considering the appropriate cocycle factor \( (-1)^{j_1+1} \) we have

\[ F_{j_1, j_1-1, j_2, j_2}^{j_1+1, j_2-1, j_1+1, j_2-1} = -\frac{(j_1 + j_2 - 1)!}{\sqrt{j_1(j_1 + 1)!}(j_2 - 1)!}. \]  \tag{5.6}

Upon comparing this with (5.2) we find \( g(j_1, j_2) \) and replacing this value back in (5.2) we find

\[ F_{j_1, m_1, j_2, m_2}^{j_1, j_2-1, m_1 + m_2} = \frac{N(j_3, m_3)(j_3 - 1)! \sqrt{j_3}}{N(j_1, m_1)(j_1 - 1)! \sqrt{j_1} N(j_2, m_2)(j_2 - 1)! \sqrt{j_2}} (j_2 m_1 - j_1 m_2). \]  \tag{5.7}

In the OPE of one NS state with one R state it is convenient to use the bosonized form of the NS vertex (4.3)

\[ V_{jj}^+ = \frac{1}{\sqrt{2}} \left[ (-1 + 2j)e^{i\sigma} - e^{-i\sigma} \right] e^{ijx} e^{(-1+j)\phi}. \]  \tag{5.8}
As in the previous case, we calculate $g(j_1, j_2)$ from the operator product

$$V_{j_1 j_1 -1}^+ (z) R_{j_2 j_2}^+ (0) = \frac{1}{\sqrt{2j_1}} H_- (V_{j_1 j_1}^+) (z) R_{j_2 j_2}^+ (0).$$

(5.9)

After a calculation similar to the one sketched above, we obtain

$$F_{j_1, m_1, j_2, m_2}^{j_1 + j_2 - 1, m_1 + m_2} = \frac{N(j_3, m_3)(j_3 - 1)!\sqrt{3}}{N(j_1, m_1)(j_1 - 1)!\sqrt{j_1} N(j_2, m_2)(j_2 - 1)!\sqrt{j_2}} (j_2 m_1 - j_1 m_2).$$

(5.10)

When the operator product of two R states (4.9) is considered, a NS state with superghost number $-1$ is found. This operator is related through picture changing\[1] to the conventional NS operator of superghost number zero (4.14). Eq. (5.1) for this case reads

$$R_{j_1 m_1}^+ (z) R_{j_2 m_2}^+ (z') \sim \frac{1}{z - z'} F_{j_1, m_1, j_2, m_2}^{j_1 m_1, j_2 m_2} V_{j_1 m_1}^{-}.$$ 

(5.11)

Performing explicitly the operator product (5.11) for the case $m_1 = j_1 - 1$, $m_2 = j_2$ (in order to extract $g(j_1, j_2)$) and considering the cocycle factor $(-1)^{j_1 + j_2}$, we finally obtain

$$F_{j_1, m_1, j_2, m_2}^{j_1 + j_2 - 1, m_1 + m_2} = \frac{N(j_3, m_3)(j_3 - 1)!\sqrt{3}}{N(j_1, m_1)(j_1 - 1/2)!\sqrt{j_1} N(j_2, m_2)(j_2 - 1/2)!\sqrt{j_2}} \frac{1}{2^2} (j_2 m_1 - j_1 m_2).$$

(5.12)

After the following redefinitions

$$O_{j_m}^+ \rightarrow \tilde{N}(j, m) O_{j, n}^+ \quad \text{with} \quad \tilde{N}(j, m) = \begin{cases} N(j, m)(j - 1)!\sqrt{j} & j \in \mathbb{Z} \\ 2^{1/4} N(j, m)(j - 1/2)!\sqrt{j} & j \in \mathbb{Z} + 1/2 \end{cases}$$

(5.13)

the structure constants are

$$F_{j_1, m_1, j_2, m_2}^{j_1 + j_2 - 1, m_1 + m_2} = (j_2 m_1 - j_1 m_2)$$

(5.14)

for all cases.

A similar analysis can be performed for the operator product coefficients of the fields $O^-$. When we have two minus signs in (5.14), conservation of Liouville momentum implies that $j_3 = j_1 + j_2 + 1$. As this representation does not appear in the product $j_1 \otimes j_2$ we conclude that

$$O^-(z) O^- (z') \sim 0 \times \frac{1}{z - z'}.$$ 

(5.15)

1For example the tachyon vertex in the $-1$ picture reads $V_{j_j}^{j_j} (-1) = e^{ijx} e^{(-1+j)d}$ (as explained in [20]).
Using similar arguments for the case of $O^+O^-$ we see that
\[
O^+_{1m_1}(z)O^-_{2m_2}(z') \sim \begin{cases} 
0 \times \frac{1}{z-z'} & \text{for } j_2 \leq j_1 - 1 \\
\frac{1}{z-z'} F^+_{j_1,m_1,j_2,m_2} O^-_{j_3m_3} & \text{for } j_2 > j_1 - 1 
\end{cases}
\] (5.16)
where $m_3 = m_1 + m_2$, $j_3 = j_2 - j_1 + 1$. In principle we could find these OPE coefficients using SU(2) covariance and performing explicitly some operator products. We note, however, that they can be obtained by using associativity of the OPE in the three point function
\[
\langle (O^+_{r_1m_1}O^+_{r_2m_2})O^-_{r_3m_3} \rangle = \langle O^+_{r_1m_1}(O^+_{r_2m_2}O^-_{r_3m_3}) \rangle 
\] (5.17)
then
\[
F^+_{j_1m_1,j_2m_2} \langle O^+_{j_3m_3}O^+_{j_1m_1}O^-_{j_3m_3} \rangle = F^+_{j_1m_1,j_2m_2} \langle O^+_{j_1m_1}O^+_{j_2m_2}O^-_{j_3m_3} \rangle 
\] (5.18)
where $r_i$ is the superghost number ($r_1 + r_2 + r_3 = -2$) and $j_3 = j_1 + j_2 - 1$, $m_3 = m_1 + m_2$. By deforming contours we find that
\[
\langle O^+_{rj}O^-_{2-r} \rangle = (-1)^{j-m} \langle O^+_{rj}O^-_{2-r} \rangle = (-1)^{j-m} s(j) \frac{1}{(z-z')^2}. 
\] (5.19)
Where $s(j)$ is a factor whose explicit expression is unnecessary for our purposes. Thus after renormalising the operators
\[
O^-_{rj} \rightarrow \frac{1}{(-1)^{j-m}s(j)N(j,m)} O^-_{rj}. 
\] (5.20)
the structure constants are
\[
F^+_{j_1m_1,j_2m_2} = -(j_3m_1 + j_1m_3), 
\] (5.21)
We find that, after renormalizing the operators, the algebra is the same as in the bosonic case. The fact that we had three different cases, NS-NS, NS-R and R-R, and only two possible ways of redefining the operators is a check of our computations. Our work also provides an alternative, and more explicit, derivation of the result of ref. [19]. We have computed the algebra only for the left sector (or open string), for the closed string we should join the right one.

A space time interpretation of these discrete higher level states is needed both in the open an closed string case. In the latter, the GSO-projection must be included. Actually, in $d = 2$ it is not necessary to make this projection, since there is no true tachyon, but, once it is done, the theory should become topological, as it is conjectured in ref. [12]. It could be interesting to compute the effective action in this case.
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APPENDIX A

In this appendix we study the factorization process where one tachyon of momentum $p_1$ and positive chirality collides with two tachyons, one of momentum $p_2$ and positive chirality and an other of momentum $p_3$ and negative chirality, (i.e. the contribution from the region where $z_2 \rightarrow z_1$ and $z_3 \rightarrow z_1$ in (2.12)).

\[ <...> = \int \prod_{i=1}^{N} dz_i d\theta_i \sum_{n,k=0}^{n,k} \frac{1}{n!k!} z_{21}^{-\nu_{12} + n} z_{31}^{-\nu_{13} + k} z_{32}^{-\nu_{32}} F_{nk}(1,j) |2\rangle |1\rangle |3\rangle |1\rangle \]

where $\nu_{ij} = -P_i \cdot P_j$, $z_{12} = z_1 - z_2 - \theta_2 \theta_1$ and similarly for $z_{31}$, $z_{32}$. $F_{nk}(1,j)$ is (the left part of)

\[ F_{nk}(1,j) = \langle e^{iP_1 \cdot X} e^{iP_2 \cdot X} e^{iP_3 \cdot X} \prod_{j=4}^{N} e^{iP_j \cdot X} |0\rangle |2=1,3=1 \rangle \]

By performing the rescaling $z_3 - z_1 = uv$ where $u = z_2 - z_1$ and integrating over $u$, the poles are found for $\nu = l + 1$, where $l = 0, 1, \ldots$ and $\nu = \nu_{12} + \nu_{13} + \nu_{23}$. The momentum of the corresponding intermediate states is given by $P = (-i(p_1 + p_2 - 2 + \frac{t-1}{2}), p_1 + p_2 - \frac{t+1}{2})$. By using the pole condition

\[ \nu = 2 - t - (p_1 + p_2)(1 - t) = l + 1 \]

it is easy to show that this is a momentum of a level $n = 2l$ state with negative chirality. Therefore the residue at these poles should be zero. We have checked this for the intermediate tachyon ($l = 0$). In this case the residue is given by

\[ A_0 = \int |d_z d\theta_1 d\theta_2 |^{N} d_z d\theta_1 d\theta_2 |v - 1|^{-\nu_{31}} \{ -\frac{\nu_{23}}{v - 1} \frac{\nu_{31}}{v} \theta_1 D_2 + (\nu_{23} + \nu_{31} - 1) \theta_1 D_3 \}^2 \langle e^{iP_1 \cdot X} e^{iP_2 \cdot X} e^{iP_3 \cdot X} \prod_{j=4}^{N} e^{iP_j \cdot X} |0\rangle |2=1,3=1 \rangle \]

The result of the integration over $v$ is zero. For example the first term gives

\[ \nu_{23} \int d^2v |v|^{-2\nu_{31}} |v - 1|^{-2\nu_{23}} = \]

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\[ \nu_{23} \Gamma(-\nu_{31} + 1) \Gamma(-\nu_{23}) \Gamma(\nu_{31} + \nu_{23}) = 0 \] (A6)

since \( \nu_{23} + \nu_{31} = 1 \) due to energy momentum conservation and the pole condition (\( \nu = 1 \)). The other two terms are zero for the same reason.

**APPENDIX B: COCYCLE OPERATORS**

The operators (4.5, 4.9) do not commute inside radial ordered correlation functions. The commutation of the exponential part of them, \( e^{r \rho} e^{i \alpha \sigma} e^{i m x} e^{i \beta \phi} \), gives rise to a factor \((-1)^{x_1 \cdot x_2}\) where \( x = (r, \alpha, m, \beta) \) and the scalar product is defined with the metric \((-, +, +, -)\). Notice that \( x_1 \cdot x_2 \) is an integer since the operators are mutually local. In order to ensure the correct commutation relations cocycle operators are needed. \([21]\). It is convenient to express \( x \) as a linear combination with integer coefficient of some basis vectors

\[ x = n_i e_i = \frac{1}{2} (m + \beta + \alpha + r) e_1 + \frac{1}{2} (m + \beta - \alpha - r) e_2 - 2 \beta e_3 + (\beta + r) e_4 \] (B1)

where

\[ e_1 = (0, 1, 1, 0) \quad e_2 = (0, -1, 1, 0) \quad e_3 = (-\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}) \quad e_4 = (1, -1, 0, 0). \]

Then

\[ x_1 \cdot x_2 = n_i^1 G_{ij} n_j^2 \quad \text{with} \quad G_{ij} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \] (B2)

Let us define the cocycle function \( \epsilon(x, y) = \pm 1 \) with the following properties

\[ \epsilon(x, y) = (-1)^{x_1 \cdot y_1} \epsilon(y, x) \] (B3)

\[ \epsilon(x, y) \epsilon(x + y, z) = \epsilon(x, y + z) \epsilon(y, z) \] (B4)

Choosing \( \epsilon(x_1, x_2) = (-1)^{n_i M_{ij} n_j} \) these properties are ensured if \( M_{ij} - M_{ji} \equiv G_{ij} (mod \ 2) \). By taking

\[ M_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \] (B5)

we have

\[ \epsilon(x_1, x_2) = (-1)^{\beta_1 (\beta_2 - m_2) + r_1 (r_2 - \alpha_2)}. \] (B6)

The cocycle operator is \([21]\).
\[ c(x_1) = \sum_{x_2} \epsilon(x_1, x_2) |x_2\rangle \langle x_2| \] (B7)

where \(|x_2\rangle\) is a state with eigenvalues of ghost, fermion, matter and Liouville charges given by \(x_2\). The operators \(\mathcal{O}_x\) are redefined to \(\tilde{\mathcal{O}}_x = \mathcal{O}_x c(x)\) so that

\[ \tilde{\mathcal{O}}_{x_1} \tilde{\mathcal{O}}_{x_2} = \epsilon(x_1, x_2) \mathcal{O}_{x_1 + x_2} c(x_1 + x_2). \] (B8)

This redefinition is implicitly understood when we compute the algebra of discrete states.
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