THE SUPERSYMMETRY METHOD OF RANDOM MATRIX THEORY

by

Martin R. Zirnbauer

1. Introduction

A prominent theme of modern condensed matter physics is electronic transport – in particular the electrical conductivity – of disordered metallic systems at very low temperatures. From the Landau theory of weakly interacting Fermi liquids one expects the essential aspects of the situation to be captured by the single-electron approximation. Mathematical models that have been proposed and studied in this context include random Schrödinger operators and band random matrices.

If the physical system has infinite size, two distinct possibilities exist: the quantum single-electron motion may either be bounded or unbounded. In the former case the disordered electron system is an insulator, in the latter case a metal with finite conductivity (if the electron motion is not critical but diffusive). Metallic behavior is expected for weakly disordered systems in three dimensions; insulating behavior sets in when the disorder is increased or the space dimension reduced.

The main theoretical tool used in the physics literature on the subject is the supersymmetry method pioneered by Wegner and Efetov (1979-1983). Over the past twenty years, physicists have applied the method in many instances, and a rather complete picture of weakly disordered metals has emerged. Several excellent reviews of these developments are available in print.

From the perspective of mathematics, however, the method has not always been described correctly, and what is sorely lacking at present is an exposition of how to implement the method rigorously. [Unfortunately, the correct exposition by Schäfer and Wegner (1980) was largely ignored or forgotten by later authors.] In this encyclopedia article an attempt will be made to help remedy the situation, by giving a careful review of the Wegner-Efetov supersymmetry method for the case of Hermitian band random matrices.

2. Gaussian Ensembles

Let \( V \) be a unitary vector space of finite dimension. A Hermitian random matrix model on \( V \) is defined by some probability distribution on \( \text{Herm}(V) \), the Hermitian linear operators on \( V \). You may fix some orthonormal basis of \( V \) and represent the elements \( H \) of \( \text{Herm}(V) \) by Hermitian square matrices.

Quite generally, probability distributions are characterized by their Fourier transform or characteristic function. In the present case this is

\[
\Omega(K) = \langle e^{i \text{Tr} HK} \rangle ,
\]

where the Fourier variable \( K \) is some other linear operator on \( V \), and \( \langle \ldots \rangle \) denotes the expectation value w.r.t. the probability distribution for \( H \). Later it will be important that, if \( \Omega(K) \) is an analytic function of \( K \), the matrix entries of \( K \) need not be from \( \mathbb{R} \) or \( \mathbb{C} \) but can be taken from the even part of some exterior algebra.

The probability distributions to be considered in this article are Gaussian with zero mean, \( \langle H \rangle = 0 \). Their Fourier transform is also Gaussian:

\[
\Omega(K) = e^{-\frac{1}{2} J(K,K)} ,
\]

where \( J \) is some quadratic form. We now describe \( J \) for a large family of hierarchical models that includes the case of band random matrices.

Let \( V \) be given a decomposition by orthogonal vector spaces:

\[
V = V_1 \oplus V_2 \oplus \ldots \oplus V_{|\Lambda|} .
\]

You should imagine that every vector space \( V_i \) corresponds to one site \( i \) of some lattice \( \Lambda \), and the total number of sites is \( |\Lambda| \). For simplicity, we take all dimensions to be equal: \( \text{dim} V_1 = \ldots = \text{dim} V_{|\Lambda|} = N \). Thus the dimension of \( V \) is \( N |\Lambda| \). The integer \( N \) is called the number of orbitals per site.

If \( \Pi_i \) is the orthogonal projector on the linear subspace \( V_i \subset V \), we take the bilinear form \( J \) to be

\[
J(K,K') = \sum_{i,j=1}^{|\Lambda|} J_{ij} \text{Tr}(\Pi_i K \Pi_j K') ,
\]

where the coefficients \( J_{ij} \) are real, symmetric, and positive. This choice of \( J \) implies invariance under the group \( \mathcal{U} \) of unitary transformations in each subspace:

\[
\mathcal{U} = \text{U}(V_1) \times \text{U}(V_2) \times \cdots \times \text{U}(V_{|\Lambda|}) .
\]

Clearly, \( \Omega(K) = \Omega(UKU^{-1}) \) or, equivalently, the probability distribution for \( H \) is invariant under conjugation \( H \mapsto UKU^{-1} \), for \( U \in \mathcal{U} \).

If \( \{ e^a_i \}_{a=1,...,N} \) is an orthonormal basis of \( V_i \), we define linear operators \( E_{ij}^{ab} : V_j \to V_i \) by \( E_{ij}^{ab} e_j^b = e_i^a \). By evaluating \( J(E_{ij}^{ab}, E_{j'j''}^{b'd'}) = J_{ij} \delta_{ij} \delta_{j'j''} \delta^{aa'} \delta^{bb'} \) one sees that the matrix entries of \( H \) are all statistically independent.
By varying the lattice \( \Lambda \), the number of orbitals \( N \), and the variances \( J_{ij} \), one obtains a large class of Hermitian random matrix models, two prominent subclasses of which are the following:

1. For \(|\Lambda| = 1\), one gets the Gaussian Unitary Ensemble (GUE). Its symmetry group is \( \mathcal{U} = U(N) \), the largest one possible in dimension \( N = \dim V \).

2. If \(|i - j|\) denotes a distance function for \( \Lambda \), and \( f \) a rapidly decreasing positive function on \( \mathbb{R}_+ \) of width \( W \), the choice \( J_{ij} = f(|i - j|) \) with \( N = 1 \) gives an ensemble of band random matrices with band width \( W \) and symmetry group \( \mathcal{U} = U(1)^{|\Lambda|} \).

Beyond being real, symmetric and positive, the variances \( J_{ij} \) are required to have two extra properties in order for all of the following treatment to go through:

- They must be positive as a quadratic form. This is to guarantee the existence of an inverse, which we denote by \( w_{ij} = (J^{-1})_{ij} \).
- The off-diagonal matrix entries of the inverse must be non-positive: \( w_{ij} \leq 0 \) for \( i \neq j \).

### 3. Basic tools

#### 3.1. Green’s functions.—

A major goal of random-matrix theory is to understand the statistical behavior of the spectrum and the eigenstates of a random Hamiltonian \( H \). Spectral and eigenstate information can be extracted from the Green’s function, i.e. from matrix elements of the operator \( (z - H)^{-1} \) with complex parameter \( z \in \mathbb{C} \setminus \mathbb{R} \). For the models at hand, the good objects to consider are averages of \( \mathcal{U} \)-invariant observables such as

\[
\begin{align*}
(1) & \quad G^{(1)}_{ij}(z) = \left\langle \text{Tr}\Pi_{i}(z - H)^{-1} \right\rangle, \\
(2) & \quad G^{(2)}_{ij}(z_1, z_2) = \left\langle \text{Tr}\Pi_{i}(z_1 - H)^{-1}\Pi_{j}(z_2 - H)^{-1} \right\rangle.
\end{align*}
\]

The discontinuity of \( G^{(1)}_{ij}(z) \) across the real \( z \)-axis yields the local density of states. In the limit of infinite volume (\(|\Lambda| \to \infty\)), the function \( G^{(2)}_{ij}(z_1, z_2) \) for \( z_1 = E + i\varepsilon, z_2 = E - i\varepsilon \), real energy \( E \), and \( \varepsilon > 0 \) going zero, gives information on transport, e.g. the electrical conductivity by the Kubo-Greenwood formula.

Mathematically speaking, if \( G^{(2)}_{ij}(E + i\varepsilon, E - i\varepsilon) \) is bounded (for infinite volume) in \( \varepsilon \) and decays algebraically with distance \(|i - j|\) at \( \varepsilon = 0^+ \), the spectrum is absolutely continuous and the eigenstates are extended at energy \( E \). On the other hand, a pure point spectrum and localized eigenstates are signalled by the behavior \( G^{(2)}_{ij} \sim e^{-\lambda|i-j|} \) with positive Lyapunov exponent \( \lambda \).

#### 3.2. Green’s functions from determinants.—

For any pair of linear operators \( A, B \) on a finite-dimensional vector space \( V \), the following formula from basic linear algebra holds if \( A \) has an inverse:

\[
\frac{d}{dt}\left. \det(A + tB) \right|_{t=0} = \det(A) \text{Tr}(A^{-1}B) .
\]

Using it with \( A = z - H \) and \( z \in \mathbb{C} \setminus \mathbb{R} \), all Green’s functions can be expressed in terms of determinants; for example,

\[
G^{(2)}_{ij}(w, z) = \sum_{a,b=1}^N \frac{\partial^2}{\partial s \partial t} \left( \frac{\det((w-H)\det(z-H + tE_{ab}^{ij})}{\det(w-H - sE_{ab}^{ji})\det(z-H)} \right) \bigg|_{s=t=0}.
\]

It is clear that, given a formula of this kind, what one wants is a method to handle ensemble averages of ratios of determinants. This is what’s reviewed in the sequel.

#### 3.3. Determinants as Gaussian integrals.—

Let the Hermitian scalar product of the unitary vector space \( V \) be written as \( \phi_1, \phi_2 \mapsto (\phi_1, \phi_2) \), and denote the adjoint or Hermitian conjugate of a linear operator \( A \) on \( V \) by \( A^* \). If \( \text{Re} A := \frac{1}{2}(A + A^*) > 0 \), the standard Lebesgue integral of the Gaussian function \( \phi \mapsto e^{-(\phi, A\phi)} \) makes sense and gives

\[
\int e^{-(\phi, A\phi)} = \text{Det}A^{-1} ,
\]

where it is understood that we are integrating with the Lebesgue measure on the (normed vector space) \( V \) normalized by \( \int e^{-(\phi, \phi)} = 1 \). The same integral with \( \text{anti-commuting} \ \psi \text{ instead of the (commuting) } \phi \in V \) gives

\[
\int e^{-(\psi, A\psi)} = \text{Det}A .
\]

This basic formula from the field theory of fermionic particles is a consequence of the integration over anti-commuting variables actually being differentiation:

\[
\int d\bar{\psi}_1 d\psi_1 f(\bar{\psi}_1, \psi_1, \ldots) := \frac{\partial^2}{\partial \bar{\psi}_1 \partial \psi_1} f(\bar{\psi}_1, \psi_1, \ldots) .
\]

### 4. Fermionic variant

The supersymmetry method of random-matrix theory is a theme with many variations. The first variation to be described is the “fermionic” one. To optimize the notation, we now write \( d\mu_{N,f}(H) \) for the density of the Gaussian probability distribution of \( H \):

\[
\langle F(H) \rangle = \int F(H) d\mu_{N,f}(H) .
\]

All determinants and traces appearing below will be taken over vector spaces that are clear from the context.
Let $z_1, \ldots, z_n$ be any set of $n$ complex numbers, put $z := \text{diag}(z_1, \ldots, z_n)$ for later purposes, and consider

\begin{equation}
\Omega_{n,N}^{\text{ferm}}(z,J) = \int \prod_{\alpha=1}^{n} \text{Det}(z_{\alpha} - H) d\mu_{n,J}(H).
\end{equation}

The supersymmetry method expresses this average of a product of determinants in an alternative way, by integrating over a “dual” measure as follows.

Introducing an auxiliary unitary vector space $\mathbb{C}^n$, one associates with every site $i$ of the lattice $\Lambda$ an object $Q_i \in \text{Herm}(\mathbb{C}^n)$, the space of Hermitian $n \times n$ matrices. If $dQ_i$ for $i = 1, \ldots, |\Lambda|$ are Lebesgue measures on $\text{Herm}(\mathbb{C}^n)$, one puts $DQ = \text{const} \times \prod_i dQ_i$ and

\begin{equation}
d\nu_{n,J}(Q) := e^{-\frac{i}{2} \sum_{j} (J^{-1})_{jj} \text{Tr} Q_{j}} DQ.
\end{equation}

The multiplicative constant in $DQ$ is fixed by requiring the density to be normalized: $\int d\nu_{n,J}(Q) = 1$. By completing the square, this Gaussian probability measure has the characteristic function

\[ e^{\sum_{j} \text{Tr} Q_{j} K_j} \int d\nu_{n,J}(Q) = e^{-\frac{1}{2} \sum_{j} J_{jj} \text{Tr} K_{j}}, \]

where the Fourier variables $K_1, \ldots, K_{|\Lambda|}$ are $n \times n$ matrices with matrix entries taken from $\mathbb{C}$ or another commutative algebra.

The key relation of the fermionic variant of the supersymmetry method is that the expectation of the product of determinants \[5\] has another expression as

\begin{equation}
\Omega_{n,N}^{\text{ferm}}(z,J) = \int \prod_{j=1}^{|\Lambda|} \text{Det}^N(z - iQ_j) d\nu_{n,J}(Q)
\end{equation}

(i = $\sqrt{-1}$). The strategy of the proof is quite simple: one writes the determinants in both expressions for $\Omega_{n,N}^{\text{ferm}}$ as Gaussian integrals over $\prod_{j=1}^{|\Lambda|} \text{Det}^N(z - iQ_j)$ and $\nu_{n,J}(Q)$ which are

\begin{equation}
\nu_{n,J}(Q) = e^{-\frac{i}{2} \sum_{j} (J^{-1})_{jj} \text{Tr} Q_{j}} DQ.
\end{equation}

for both expressions of $\Omega_{n,N}^{\text{ferm}}$. In other words, although the probability distributions $d\mu_{n,J}(H)$ and $d\nu_{n,J}(Q)$ are distinct (they are defined on different spaces), their characteristic functions coincide when evaluated on the Fourier variables $K = \sum_{\alpha} \psi_{\alpha}(\psi_{\alpha}, \bullet)$ for $H$ and $(K_{i})_{\alpha\beta} = (\psi_{\alpha}, \Pi_{i} \psi_{\beta})$ for $Q_i$. This establishes the claimed equality of the expressions \[5\] and \[7\] for $\Omega_{n,N}^{\text{ferm}}(z,J)$.

What’s the advantage of passing to the alternative expression by $d\nu_{n,J}(Q)$? The answer is that, while $H$ is made up of independent random variables, the new variables $Q_i$, called the Hubbard-Stratonovich field, are correlated: they interact through the “exchange” constants $w_{ij} = (J^{-1})_{ij}$. If that interaction creates enough collectivity, a kind of mean-field behavior results.

For the simple case of GUE ($|\Lambda| = 1$, $w_{11} = N/\lambda^2$) with $z_1 = \ldots = z_n = E$, one gets the relation

\[ \langle \text{Det}^N(E - H) \rangle = \int \text{Det}^N(E - iQ) e^{-\frac{N}{2\lambda^2} \text{Tr}^2 Q} dQ, \]

the right-hand side of which is easily analyzed by the steepest descent method in the limit of large $N$.

For band random matrices in the so-called ergodic regime the physical behavior turns out to be governed by the constant mode $Q_1 = \ldots = Q_{|\Lambda|}$ - a fact that can be used to establish GUE universality in that regime.

### 5. Bosonic variant

The bosonic variant of the present method, due to Wegner, computes averages of products of determinants placed in the denominator:

\begin{equation}
\Omega_{n,N}^{\text{bos}}(z,J) = \int \prod_{\alpha=1}^{n} \text{Det}^{-1}(z_{\alpha} - H) d\mu_{n,J}(H),
\end{equation}

where we now require $\Im z_{\alpha} \neq 0$ for all $\alpha = 1, \ldots, n$. Complications relative to the fermionic case arise from the fact that the integrand in \[8\] has poles. If one replaces the anti-commuting vectors $\psi_{\alpha}$ by commuting ones $\phi_{\alpha}$, and then simply repeats the previous calculation in a naive manner, one arrives at

\begin{equation}
\Omega_{n,N}^{\text{bos}}(z,J) = \int \prod_{j=1}^{|\Lambda|} \text{Det}^{-N}(z - Q_{j}) d\nu_{n,J}(Q),
\end{equation}

where the integral is still over $Q_j \in \text{Herm}(\mathbb{C}^n)$. The calculation is correct, and relation \[5\] therefore holds true, provided that the parameters $z_1, \ldots, z_n$ all lie in the same half (upper or lower) of the complex plane. To obtain information on transport properties, however, one needs parameters in both the upper and lower halves; see the paragraph following \[2\]. The general case to be addressed below is $\Im z_{\alpha} > 0$ for $\alpha = 1, \ldots, p$, and $\Im z_{\alpha} < 0$ for $\alpha = p + 1, \ldots, n$. Careful inspection of the steps leading to equation \[9\] reveals a convergence problem for $0 < p < n$. In fact, \[9\] with $Q_j \in \text{Herm}(\mathbb{C}^n)$ turns out to be false in that range. Learning how to resolve this problem is the main step toward mathematical mastery of the method. Let us therefore give the details.

If $s_{\alpha} := \text{sgn} \Im z_{\alpha}$, the good (meaning convergent) Gaussian integral to consider is

\[ \int e^{i \sum_{\alpha} s_{\alpha} \langle \psi_{\alpha}(\psi_{\alpha} - H) \rangle} = \prod_{\alpha=1}^{n} \text{Det}^{-1}( - is_{\alpha}(z_{\alpha} - H)) \]

To avoid carrying around trivial constants, we now assume $i^{(n-2p)|\Lambda|} = 1$. Use of the characteristic function
of the distribution for \( H \) then gives

\[
\Omega_{\alpha N}^{{\text{bos}}}(z,J) = \int e^{i \sum_{ij} s_{ij} \phi_{ij}(\phi,\phi_T)} \\
\times e^{-\frac{k}{2} \sum_{ij} \alpha_{ij} s_{ij} (\phi_{ij},\alpha_{ij})} \mathcal{D}M = e^{\frac{1}{2} \sum_{ij} w_{ij} Tr(sM_i sM_j)}
\]

The difficulty of analyzing this expression stems from the \( \text{"hyperbolic"} \) nature (due to the indefiniteness of the signs \( s_{ij} = \pm 1 \)) of the term quartic in the \( \phi_{ij}, \phi_{ij} \).

5.1. Fyodorov’s method. — The integrand for \( \Omega^{{\text{bos}}} \) is naturally expressed in terms of \( n \times n \) matrices \( M_i \) with matrix elements \( (M_i)_{\alpha\beta} = (\phi_{\alpha\beta},\Pi_i \phi_{\beta}) \). These matrices lie in \( \text{Herm}^+(\mathbb{C}^n) \), i.e. they are non-negative as well as Hermitian. Fyodorov’s idea was to introduce them as the new variables of integration. To do that step recall the basic fact that, given two differentiable spaces \( X \) and \( Y \) and a smooth map \( \psi : X \rightarrow Y \), a distribution \( \mu \) on \( X \) is pushed forward to a distribution \( \psi_\#(\mu) \) on \( Y \) by \( \psi_\#(\mu)(f) := \mu(f \circ \psi) \), where \( f \) is any test function on \( Y \).

We apply this universal principle to the case at hand by identifying \( X \) with \( V^n \), and \( Y \) with \( \text{Herm}^+(\mathbb{C}^n)^{[A]} \), and \( \psi \) with the mapping that sends

\[
(\phi_1, \ldots, \phi_n) \in X \quad \text{to} \quad (M_1, \ldots, M_{[A]}) \in Y
\]

by \( (M_i)_{\alpha\beta} = (\phi_{\alpha\beta},\Pi_i \phi_{\beta}) \). On \( X = V^n \) we are integrating with the product Lebesgue measure normalized by \( \int e^{-\sum_{ij} \phi_{ij} \phi_{ij}} = 1 \). We now want the push forward of this flat measure (or distribution) by the mapping \( \psi \). In general, the push forward of a measure is not guaranteed to have a density but may be singular (like a Dirac \( \delta \)-distribution). This is in fact what happens if \( N < n \).

The matrices \( M_i \) then have less than the maximal rank, so they fail to be positive but possess zero eigenvalues, which implies that the flat measure on \( X \) is pushed forward by \( \psi \) into the boundary of \( Y \). For \( N \geq n \), on the other hand, the push forward measure does have a density on \( Y \); and that density is \( \prod_{i=1}^{[A]} (\text{Det} M_i)^{-N-n} dM_i \), as is seen by transforming to the eigenvalue representation and comparing Jacobians. The \( dM_i \) are Lebesgue measures on \( \text{Herm}^+(\mathbb{C}^n) \), normalized by the condition

\[
\int_{M_i > 0} e^{-\text{Tr} M_i (\text{Det} M_i)^{N-n} dM_i} = \int e^{-\sum_{ij} \phi_{ij} \phi_{ij}} = 1
\]

Assembling the sign information for \( \text{Im} z_\alpha \) in a diagonal matrix \( s := \text{diag}(s_1, \ldots, s_n) \), and pushing the integral over \( X \) forward to an integral over \( Y \) with measure \( DM := \prod_i dM_i \), we obtain Fyodorov’s formula:

\[
\Omega^{{\text{bos}}}_{\alpha N}(z,J) = \int e^{i \sum_{ij} s_{ij} \phi_{ij}(\phi,\phi_T)} \\
\times e^{-\frac{k}{2} \sum_{ij} \alpha_{ij} s_{ij} (\phi_{ij},\alpha_{ij})} \mathcal{D}M = e^{\frac{1}{2} \sum_{ij} w_{ij} Tr(sM_i sM_j)}
\]

This formula has a number of attractive features. One is ease of derivation, another is ready generalizability to the case of non-Gaussian distributions. The main disadvantage of the formula is that it does not apply to the case of band random matrices (because of the restriction \( N \geq n \)); nor does it combine nicely with the fermionic formula \( (12) \) to give a supersymmetric formalism, as one formula is built on \( J \alpha \) and the other on \( w_{ij} \).

Note that \( (11) \) clearly displays the dependence on the signature of \( \text{Im} z_\alpha \): you cannot remove the \( s_1, \ldots, s_n \) from the integrand without changing the domain of integration \( Y = \text{Herm}^-(\mathbb{C}^n)^{[A]} \). This important feature is missing from the naive formula \( (7) \).

Setting \( q = n - p \), let \( U(p,q) \) be the pseudo-unitary group of complex \( n \times n \) matrices \( T \) with inverse \( T^{-1} = sT^*s \). Since \( |\text{Det} T| = 1 \) for \( T \in U(p,q) \), the integration domain \( Y \) and density \( DM = \prod_i dM_i \) of Fyodorov’s formula are invariant under \( U(p,q) \) transformations \( M_i \rightarrow TM_i T^* \), and so is actually the integrand in the limit where all parameters \( z_1, \ldots, z_n \) become equal. Thus the elements of \( U(p,q) \) are global symmetries in that limit. This observation holds the key to another method of transforming the expression \( (11) \).

5.2. The method of Schäfer and Wegner. — To rescue the naive formula \( (7) \), what needs to be abandoned is the integration domain \( \text{Herm}(\mathbb{C}^n) \) for the matrices \( Q_i \). The good domain to use was constructed by Schäfer and Wegner, but was largely forgotten in later physics work.

Writing \( (M_k)_{\alpha\beta} = (\phi_{\alpha\beta},\Pi_k \phi_{\beta}) \) as before, consider the function

\[
F_M(Q) = e^{\frac{1}{2} \sum w_{ij} Tr(sQ_i + i\zeta)(sQ_j + i\zeta)} - \sum_k Tr M_k Q_k
\]

viewed as a holomorphic function of

\[
Q = (Q_1, \ldots, Q_{[A]}) \in \text{End}(\mathbb{C}^n)^{[A]}.
\]

If the Gaussian integral \( \int F_M(Q) dQ \) with holomorphic density \( DQ = \prod_i dQ_i \) is formally carried out by completing the square, one gets the integrand of \( (10) \). This is just what we want, as it would allow us to pass to a \( Q \)-matrix formulation akin to the one of Section 4. But how can that formal step be made rigorous? To that end, one needs to (i) construct a domain on which \( |F_M(Q)| \) decays rapidly so that the integral exists, and (ii) justify completion of the square and shifting of variables.

To begin, take the absolute value of \( F_M(Q) \). Putting \( \frac{1}{2}(Q_j + Q_j^*) =: \Re Q_j \) and \( \frac{i}{2}(Q_j - Q_j^*) =: \Im Q_j \), you have

\[
|F_M| = e^{-\frac{1}{2}(f_1 + f_2 + f_3)}
\]

with

\[
f_1(Q) = \sum_{ij} w_{ij} Tr(s\Im Q_i + z)(s\Im Q_j + z) + \text{c.c.},
\]

\[
f_2(Q) = -2 \sum_{ij} w_{ij} Tr(s\Re Q_i)(s\Re Q_j),
\]

\[
f_3(Q) = 4 \sum_i Tr(M_i + s\Im z \sum_j w_{ij}) \Re Q_i.
\]

The factor \( e^{-f_1/2} \) is now obviously bounded by some constant not depending on \( z \) or \( Q \), hence \( F_M(Q) \) is holomorphic on the whole region \( \Re Q_j > 0 \). The factor \( e^{-f_2/2} \) is bounded by the Gaussian integral \( \int e^{-\frac{1}{2} Tr Q Q^*} dQ \), which is bounded by 1. The factor \( e^{-f_3/2} \) is bounded above by some constant not depending on \( z \) or \( Q \), hence \( F_M(Q) \) is holomorphic on the whole region \( \Re Q_j > 0 \). Therefore, the Gaussian integral \( \int F_M(Q) dQ \) is well defined.
These expressions suggest making the following choice of integration domain for \( Q_i \) (\( i = 1, \ldots, |A| \)). Pick some real constant \( \lambda > 0 \) and put
\[
\Re Q_i = \lambda T_i T_i^*, \quad \Im Q_i = P_i := \begin{pmatrix} P_i^+ & 0 \\ 0 & P_i^- \end{pmatrix},
\]
with \( T_i \in U(p,q), \) \( P_i^+ \in \text{Herm}(\mathbb{C}^p), \) \( P_i^- \in \text{Herm}(\mathbb{C}^q). \) The set of matrices \( Q_i \) so defined is referred to as the Schäfer-Wegner domain \( X_{\lambda}^{p,q}. \) The range of the field \( Q = (Q_1, \ldots, Q_{|A|}) \) is the direct product \( X_{\lambda}^{p,q} \times \cdots \times X_{\lambda}^{p,q}. \) The required shift, which is bounded from below by the constant \(-2\lambda^2 n \sum \Im w_{ij},\) is positive, as \( \Im T_i T_i^* > 0 \) and the trace of a product of two positive Hermitian matrices is always positive. Third,
\[
f_3(Q) = 4 \lambda \sum i j \text{Tr} (M_i + s \Im m \sigma \sum j w_{ij}) T_i T_i^*
\]
is positive, as \( \ldots \) is positive Hermitian. As long as \( s \Im m \sigma > 0 \) the function \( f_3 \) goes to infinity for all possible directions of taking the \( T_i \) to infinity on \( U(p,q). \)

Thus when the matrices \( Q_i \) are taken to vary on the Schäfer-Wegner domain \( X_{\lambda}^{p,q}, \) the absolute value \( |F_{\lambda}| = e^{-\frac{1}{4}(f_1+f_2+f_3)} \) decays rapidly at infinity. This establishes the convergence of \( \int_{X} F_M(Q) DQ. \)

Next, let us count dimensions. The mapping \( T \mapsto TT^* \) for \( T \in U(p,q) =: G \) is invariant under right multiplication of \( T \) by elements of the unitary subgroup \( H := U(p) \times U(q) =: \text{Cartan embedding} \) of \( G/H \) into \( G. \) The real manifold \( G/H \) has dimension \( 2pq \) and so does its image under the Cartan embedding. Augmenting this by the dimension of \( \text{Herm}(\mathbb{C}^p) \) and \( \text{Herm}(\mathbb{C}^q) \) (from \( P_i \)), one gets \( \dim X_{\lambda}^{p,q} = 2pq + p^2 + q^2 = (p+q)^2 = n^2, \) which is as it should be.

Finally, why can one shift variables and do the Gaussian integral over \( Q \) (with translation-invariant \( DQ \)) by completing the square? This question is legitimate as the Schäfer-Wegner domain \( X_{\lambda}^{p,q} \) lacks invariance under the required shift, which is \( Q_i \mapsto Q_i - sz + \sum j J_{ij} s M_{js}. \)

To complete the square in (12), introduce a parameter \( t \in [0,1] \) and consider the family of shifts
\[
Q_i \mapsto Q_i + t(-isz + \sum j J_{ij} s M_{js}).
\]
For fixed \( t, \) this shift takes \( X_{\lambda}^{p,q} \) into another domain, \( X(T) \). Inspection shows that the function \( f_{\lambda} \) still decays rapidly (uniformly in the \( M_j \)) on \( X(T) \), as long as \( t < 1 \). Without changing the integral one can add pieces to \( X(T) \) (for \( t < 1 \)) at infinity to arrange for the chain \( X \rightarrow X(T) \) to be a cycle. Because \( X(T) \) is homeomorphic to \( X(0) = X \), this cycle is a boundary: there exists a manifold \( Y(t) \) of dimension \( \dim X + 1 \) such that \( \partial Y(t) = X \rightarrow X(T) \). Viewed as a holomorphic differential form of degree \( (n^2-1)|A| \) in the complex space \( \text{End}(\mathbb{C}^n)^{|A|}, \) the integrand \( \omega := F_M(Q)DQ \) is closed (i.e. \( d\omega = 0 \)). Therefore, by Stokes’ theorem,
\[
\int_{X} \omega - \int_{X(T)} \omega = \int_{\partial Y(t)} \omega = \int_{Y(t)} d\omega = 0,
\]
which proves \( \int_{X(T)} F_M(Q)DQ = \int_{X} F_M(Q)DQ, \) independent of \( t. \) (This argument does not go through for the non-rigorous choice \( sQ_i := T_i P_i T_i^{-1} \) usually made!)

In the limit \( t \rightarrow 1 \), one encounters the expression
\[
\int_{X(1)} F_M(Q)DQ = \int_{X} dV_{n,J} (iQ) \times e^{-\frac{1}{4}\sum j J_{ij} \text{Tr} (s M_{js} M_{js}) + i\sum k \text{Tr} (sz M_{ks})}
\]
with \( dV_{n,J} \) as in (6). The normalization integral over \( X \) is defined by taking the Hermitian matrices \( P_i \) to be the inner variables of integration. The outer integrals over the \( T_i \) then demonstrably exist, and one can fix the (otherwise arbitrary) normalization of \( DQ \) by setting \( \int_{X} dV_{n,J} (iQ) = 1. \) Making that choice, and comparing with (10), one has proved
\[
\Omega_{n,N}^{\text{bos}} = \int_{\phi,\bar{\phi}} \left( \int_{X} F_M(Q)_{\mu\nu} \right)(Q) DQ .
\]

The final step is to change the order of integration over the \( Q \)- and \( \phi \)-variables, which is permitted since the \( Q \)-integral converges uniformly in \( \phi. \) Doing the Gaussian \( \phi \)-integral and shifting \( Q_k \rightarrow Q_k - iz \), one arrives at the Schäfer-Wegner formula for \( \Omega_{n,N}^{\text{bos}}: \)
\[
\Omega_{n,N}^{\text{bos}}(z,w^{-1}) = \int_{X} e^{\sum j w_{ij} \text{Tr} (s Q_i) + i\sum k \text{Tr} (s z M_{ks})} \times e^{-N\sum k \text{Tr} (s Q_k - iz) DQ},
\]
which is a rigorous version of the naive formula (2). Compared to Fyodorov’s formula, it has the disadvantage of not being manifestly invariant under global hyperbolic transformations \( Q_i \mapsto T Q_i T^* \) (the integration domain \( X \) isn’t invariant). Its best feature is that it does apply to the case of band random matrices with one orbital per site (\( N = 1 \)).
6. Supersymmetric variant

We are now in a position to tackle the problem of averaging ratios of determinants. For concreteness, we shall discuss the case where the number of determinants is two for both the numerator and the denominator, which is what is needed for the calculation of the function \( G^{(2)}_{ij}(z_1, z_2) \) defined in equation (4). We will consider the case of relevance for the electrical conductivity: \( z_1 = E + i\varepsilon, z_2 = E - i\varepsilon \), with \( E \in \mathbb{R} \) and \( \varepsilon > 0 \).

A \( Q \)-integral formula for \( G^{(2)}_{ij}(z_1, z_2) \) can be derived by combining the fermionic method for

\[
\left\langle \text{Det}(z_1 - H) \text{Det}(z_2 - H + t_2 E_{ij}^{ab}) \right\rangle
\]

with the Schäfer-Wegner bosonic formalism for

\[
\left\langle \text{Det}^{-1}(z_1 - H - t_1 E_{ij}^{ba}) \text{Det}^{-1}(z_2 - H) \right\rangle,
\]

and eventually differentiating with respect to \( t_1, t_2 = 0 \) and summing over \( a, b \); see Section 5.2. All steps are formally the same as before, but with traces and determinants replaced by their supersymmetric analogs. Having given a great many technical details in Sections 4 and 5, we now just present the final formula for the generating function of

\[
\left\langle \text{Det}(z_1 - H) \text{Det}(z_2 - H + t_2 E_{ij}^{ab}) \right\rangle
\]


\[
\text{Det}(z_1 - H - t_1 E_{ij}^{ba}) \text{Det}(z_2 - H) \right\rangle
\]

by combining the Schäfer-Wegner bosonic method with the fermionic variant — is written as

\[
\left( \text{Det}(z_1 - H) \text{Det}(z_2 - H + t_2 E_{ij}^{ab}) \right) \\
\text{Det}(z_1 - H - t_1 E_{ij}^{ba}) \text{Det}(z_2 - H) \right\rangle
\]

\[
= \int DQ e^{\frac{1}{2} \sum_{i \in \Lambda} \text{STr}(sQ_i Q_i^\dagger)}
\times e^{-\text{STr} \ln \left( \sum_{r} \langle \tilde{Q}_r - i\varepsilon z \rangle \otimes E_{ij}^{ab} + \text{in} E_{ij}^{a} \otimes E_j^{b} - t_2 E_{ij}^{ab} \otimes E_{ij}^{ab} \right)},
\]

where the second supertrace includes a sum over sites and orbitals, and on setting \( t_1 = t_2 = 0 \) becomes

\[
e^{-N \sum \text{STr} \ln \langle \tilde{Q}_r - i\varepsilon z \rangle} = \prod_r \text{SDet}^{-N}(Q_r - i\varepsilon z).
\]

The superintegral ‘measure’ \( DQ = \prod_r DQ_r \) is the flat Berezin form, i.e. the product of differentials for all the commuting matrix entries in \( \langle Q \rangle_{BB} \) and \( \langle Q \rangle_{FF} \), times the product of derivatives for all the anti-commuting matrix entries in \( \langle Q \rangle_{BF} \) and \( \langle Q \rangle_{FB} \).

To prove the formula (14), two new tools are needed, a brief account of which is as follows.

6.1. Gaussian superintegrals. — There exists a supersymmetric generalization of the Gaussian integral formulas given in Section 3.2 if \( A, D (B, C) \) are linear operators or matrices with commuting (resp. anti-commuting) entries, and \( \Re \lambda > 0 \), one has

\[
\text{SDet}^{-1}(A B C D) = \int e^{-\langle \hat{\Phi}, A \hat{\Phi} \rangle - \langle \hat{\Phi}, B \hat{\Psi} \rangle - \langle \hat{\Psi}, C \hat{\Psi} \rangle - \langle \hat{\Psi}, D \hat{\Psi} \rangle}.
\]

Verification of this formula is straightforward. Using it, one writes the last factor in (14) as a Gaussian superintegral over four vectors: \( \phi_1, \phi_2, \psi_1, \) and \( \psi_2 \). The integrand then becomes Gaussian in the matrices \( Q_r \).

6.2. Shifting variables. — The next step in the proof is to do the ‘Gaussian’ integral over the supermatrices \( Q_r \). By definition, in a superintegral one first carries out the Fermi integral, and afterwards the ordinary integrations. The Gaussian integral over the anti-commuting parts \( \langle Q \rangle_{BF} \) and \( \langle Q \rangle_{FB} \) is readily done by completing the square and shifting variables using the fact that fermionic integration is differentiation:

\[
\int d\xi f(\xi - \xi') = \frac{\partial}{\partial \xi} f(\xi - \xi') = \int d\xi f(\xi).
\]

Similarly, the Gaussian integral over the Hermitian matrices \( \langle Q \rangle_{FF} \) is done by completing the square and shifting. The integral over \( \langle Q \rangle_{BB} \), however, is not Gaussian, as the domain is not \( \mathbb{R}^{6n} \) but the Schäfer-Wegner domain. Here, more advanced calculus is required: these integrations are done by using a supersymmetric change-of-variables theorem due to Berezin to make the necessary shifts by nilpotents. (There is not enough space to describe this here, so please consult Berezin’s book.)
Without difficulty one finds the result to agree with the left-hand side of Eq. (14), thereby establishing that formula.

7. Approximations

All manipulations so far have been exact and, in fact, rigorous (or can be made so with little extra effort). Now we turn to a sequence of approximations that have been used by physicists to develop a quantitative understanding of weakly disordered quantum dots, wires, films etc. While physically satisfactory, not all of these approximations are under full mathematical control. We will briefly comment on their validity as we go along.

7.1. Saddle-point manifold. — We continue to consider \( G_{ij}^{(2)}(E+i\epsilon, E-i\epsilon) \) and focus on \( E=0 \) (the center of the energy band) for simplicity. By varying the exponent on the right-hand side of (14) at \( t_1 = t_2 = 0 \), one gets the following equation:

\[
\sum_j w_{ij} s Q_j s - N Q_j^{-1} = 0 ,
\]

which is called the saddle-point equation.

Let us now assume translational invariance, \( w_{ij} = f(|i-j|) \). Then, if \( \lambda = \sqrt{N/\sum_j w_{ij}} \), the saddle-point equation has \( i \)-independent solutions of the form

\[
Q_i = \lambda \begin{pmatrix} q_{BB} & 0 \\ 0 & q_{FF} \end{pmatrix} ,
\]

where for \( q_{FF} \) there are three possibilities: two isolated points \( q_{FF} = \pm 1 \) (unit matrix) coexist with a manifold

\[
q_{FF} = \begin{pmatrix} \cos \theta_1 & \sin \theta_1 e^{i\phi} \\ \sin \theta_1 e^{-i\phi} & -\cos \theta_1 \end{pmatrix},
\]

which is 2-dimensional, whereas the solution space for \( q_{BB} \) consists of a single connected 2-manifold:

\[
q_{BB} = \begin{pmatrix} \cosh \theta_0 & \sinh \theta_0 e^{i\phi} \\ \sinh \theta_0 e^{-i\phi} & \cosh \theta_0 \end{pmatrix} .
\]

The solutions \( q_{FF} = \pm 1 \) are usually discarded in the physics literature. (The argument is that these break supersymmetry and therefore get suppressed by fermionic zero modes. For the simpler case of the one-point function \( \Pi \) and in three space dimensions, such suppression has recently been proved by Disertori, Pinson and Spencer.) Other solutions for \( q_{BB} \) are ruled out by the requirement \( \Re Q_j > 0 \) for the Schäfer-Wegner domain.

The set of matrices (16) and (15) – the saddle-point manifold – is diffeomorphic to the product of a two-hyperboloid \( H^2 \) with a two-sphere \( S^2 \). Moving along that manifold \( M := H^2 \times S^2 \) leaves the \( \phi \)-field integrand (14) unchanged (for \( \varepsilon_1 = \varepsilon_2 = t_1 = t_2 = 0 \)).

One can actually anticipate the existence of such a manifold from the symmetries at hand. These are most transparent in the starting point of the formalism as given by the characteristic function \( \langle e^{-i\Tr HK} \rangle \) with

\[
K = \hat{\phi}_1 \otimes \phi_1 - \hat{\phi}_2 \otimes \phi_2 + \psi_1 \otimes \psi_1 + \bar{\psi}_2 \otimes \psi_2 .
\]

The signs of this quadratic expression are what is encoded in the signature matrix \( s = \text{diag}(1,-1,1,1) \) (recall that the first two entries are forced by \( \Im \varepsilon_1 > 0 \) and \( \Im \varepsilon_2 < 0 \)). The quadratic form \( K \) is invariant under the product of two Lie groups: \( U(1,1) \) acting on the \( \phi \)-s, and \( U(2) \) acting on the \( \psi \)-s. This invariance gets transferred by the formalism to the \( Q \)-side; the saddle-point manifold \( M \) is in fact an orbit of the group action of \( G := U(1,1) \times U(2) \) on the \( Q \)-field. In the language of physics, the degrees of freedom of \( M \) correspond to the Goldstone bosons of a broken symmetry.

\( K \) also has a number of supersymmetries, mixing \( \phi \)-s with \( \psi \)-s. At the infinitesimal level, these combine with the generators of \( G \) to give a Lie superalgebra of symmetries \( \mathfrak{g} := \mathfrak{u}(1,1|2) \). One therefore expects some kind of saddle-point supermanifold, say \( M \), on the \( Q \)-side.

\( M \) can be constructed by extending the above solution \( q_0 := \text{diag}(q_{BB}, q_{FF}) \) of the dimensionless saddle-point equation \( sq = q^{-1} \) to the full \( 4 \times 4 \) supermatrix space. Putting \( q = q_0 + q_1 \) with \( q_1 = \begin{pmatrix} 0 & q_{BF} \\ q_{FB} & 0 \end{pmatrix} \), and linearizing in \( q_1 \), one gets

\[
s q_1 s = -q_0^{-1} q_1 q_0^{-1} .
\]

The solution space of this linear equation for \( q_1 \) has dimension four for all \( q_0 \in M \). Based on it, one expects four Goldstone fermions to emerge along with the four Goldstone bosons of \( M \).

For the simple case under consideration, one can introduce local coordinates and push the analysis to non-linear order, but things get quickly out of hand (when done in this way) for more challenging, higher-rank cases. Fortunately, there exists an alternative, coordinate-independent approach, as the mathematical object to be constructed is completely determined by symmetry!

7.2. Riemannian symmetric superspace. — The linear equation (17) associates with every point \( x \in M \) a four-dimensional vector space of solutions \( V_x \). As the point \( x \) moves on \( M \) the vector spaces \( V_x \) turn and twist; thus they form what is called a vector bundle \( V \) over \( M \). (The bundle at hand turns out to be non-trivial, i.e. there exists no global choice of coordinates for it.)

A section of \( V \) is a smooth mapping \( s : M \to V \) such that \( s(x) \in V_x \) for all \( x \in M \). The sections of \( V \) are to be
finite frequency

multiplied in the exterior sense, as they represent anti-commuting degrees of freedom; hence the proper object to consider is the exterior bundle, $\Lambda V$.

It is a beautiful fact that there exists a unique action of the Lie superalgebra $g$ on the sections of $\Lambda V$ by first-order differential operators, or derivations for short. (Be advised however that this canonical $g$-action is not well-known in physics or mathematics.)

The manifold $M$ is a symmetric space, i.e. a Riemannian manifold with $G$-invariant geometry. Its metric tensor, $g$, uniquely extends to a second-rank tensor field (still denoted by $g$) which maps pairs of derivations of $\Lambda V$ to sections of $\Lambda V$, and is invariant with respect to the $g$-action. This collection of objects — the symmetric space $M$, the exterior bundle $\Lambda V$ over it, the action of the Lie superalgebra $g$ on the sections of $\Lambda V$, and the $g$-invariant second-rank tensor $g$ — form what the author calls a Riemannian symmetric superspace, $\mathcal{M}$.

7.3. Non-linear sigma model. — According to the Landau-Ginzburg-Wilson paradigm of the theory of phase transitions, the large-scale physics of a statistical mechanical system near criticality is expected to be controlled by an effective field theory for the long-wave length excitation of the order parameter of the system.

Wegner is credited for the profound insight that the LGW paradigm applies to the random-matrix situation at hand, with the role of the order parameter being taken by the matrix $Q$. He argued that transport observables (such as the electrical conductivity) are governed by slow spatial variations of the $Q$-field inside the saddle-point manifold. Efetov skilfully implemented this insight in a supersymmetric variant of Wegner’s method.

While the direct construction of the effective continuum field theory by gradient expansion of $\mathcal{M}$ is not an entirely easy task, the outcome of the calculation is pre-determined by symmetry. On general grounds, the effective field theory has to be a non-linear sigma model for the Goldstone bosons and fermions of $\mathcal{M}$: if $\{\phi^A\}$ are local coordinates for the bundle $V$ with metric $g_{AB}(\phi)$, the action functional is

$$ S = \sigma \int d^d x \partial_\mu \phi^A g_{AB}(\phi) \partial_\mu \phi^B. $$

The coupling parameter $\sigma$ has the physical meaning of bare (i.e. unrenormalized) conductivity. In the present model $\sigma = NW^2 a^{2-d}$, where $W$ is essentially the width of the band random matrix in units of the lattice spacing $a$ (the short-distance cutoff of the continuum field theory). $S$ is the effective action in the limit $z_1 = z_2$. For a finite frequency $\omega = z_1 - z_2$, a symmetry-breaking field of the form $i \omega \int d^d x f(\phi)$, where $v = \lambda^{-1} a^{d-1}$ is the local density of states, has to be added to $S$.

By perturbative renormalization group analysis, i.e. by integrating out the rapid field fluctuations, one finds for $d = 2$ that $\sigma$ decreases on increasing the cutoff $a$. This property is referred to as asymptotic freedom in field theory. On its basis one expects exponentially decaying correlations, and hence localization of all states, in two dimensions. However, a mathematical proof of this conjecture is not currently available.

In three dimensions and for a sufficiently large bare conductivity, the renormalization flow goes toward the metallic fixed point ($\sigma \to \infty$), where $G$-symmetry is broken spontaneously. A rigorous proof of this important conjecture (existence of disordered metals in three space dimensions) is not available either.

7.4. Zero-mode approximation. — For a system in a box of linear size $L$, the cost of exciting fluctuations in the sigma model field is estimated as the Thouless energy $E_{\text{Th}} = \sigma \sqrt{L^2}$. In the limit of small frequency, $|\omega| \ll E_{\text{Th}}$, the physical behavior is dominated by the constant modes $\phi^A(x) = \phi^A$ (independent of $x$). By computing the integral over these modes, Efetov found the energy-level correlations in the small-frequency limit to be those of the Gaussian Unitary Ensemble.

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MARTIN R. ZIRNBAUER, Institut für Theoretische Physik, Universität zu Köln, Zülpicher Str. 77, 50937 Köln, Germany
E-mail: zirn@thp.uni-koeln.de