A Reduced-Speed-of-Light Formulation of the Magnetohydrodynamic-Particle-in-Cell Method

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ABSTRACT

A reduced-speed-of-light (RSOL) approximation is a useful technique for magnetohydrodynamic (MHD)-particle-in-cell (PIC) simulations. With a RSOL, some “in-code” speed-of-light $\tilde{c}$ is set to much lower values than the true $c$, allowing simulations to take larger timesteps (which are restricted by the Courant condition given the large CR speeds). However, due to the absence of a well-formulated RSOL implementation from the literature, naive substitution of the true $c$ with a RSOL, the CR properties in MHD-PIC simulations (e.g. CR energy or momentum density, gyro radius) vary artificially with respect to each other and with respect to the converged ($\tilde{c} \rightarrow c$) solutions with different choices of a RSOL. Here, we derive a new formulation of the MHD-PIC equations with a RSOL, and show that (1) it guarantees all steady-state properties of the CR distribution function and background plasma/MHD quantities are independent of the RSOL $\tilde{c}$ even for $\tilde{c} \ll c$, (2) ensures that the simulation can simultaneously represent the real physical values of CR number, mass, momentum, and energy density, (3) retains the correct physical meaning of various terms like the electric field, and (4) ensures the numerical timestep for CRs can always be safely increased by a factor $\sim c/\tilde{c}$. This new RSOL formulation should enable greater self-consistency and reduced CPU cost in simulations of CR-MHD interactions.

Key words: cosmic rays — plasmas — methods: numerical — MHD — galaxies: evolution — ISM: structure

1 INTRODUCTION

Cosmic rays (CRs) are relativistic charged particles which can exchange energy and momentum with the surrounding medium, and thus can potentially play a significant role in the kinematic and thermal evolution of astrophysical fluids, as well as being interesting in their own right as probes of high-energy astro-particle physics. CRs are expected to be in equipartition with the turbulent and magnetic energy density in the ISM (Boulares & Cox 1990), and have been shown to be influential at galactic and extragalactic scales in a number of recent numerical studies, with implications for galaxy formation (Chan et al. 2019; Su et al. 2019; Hopkins et al. 2020a; Buck et al. 2020), galactic outflows/winds (Ruszkowski et al. 2017; Farber et al. 2018; Hopkins et al. 2020b), cosmic inflows and virial shocks (Ji et al. 2021) and the phase structure of the circumgalactic medium (Sale et al. 2016; Butsky et al. 2020; Ji et al. 2020; Wang et al. 2020). In simulations of CRs on ISM or star formation or galaxy scales, as well as classical models of Galactic CR transport which are compared to Solar system CR experiments (Strong & Moskalenko 2001; Jónhannesson et al. 2016; Evoli et al. 2017), a fluid or Fokker-Planck type approximation for CRs is necessarily adopted, usually with some simplified assumptions for the effective “diffusion coefficient” or “streaming speed” (or some combination thereof). The studies above, and others comparing different assumptions for these CR “transport coefficients” (Butsky & Quinn 2018; Hopkins et al. 2021) have shown that the effects of CRs and their interactions with the background plasma, let alone Solar system observables, are highly dependent on these coefficients. Therefore, it is critical to obtain a better understanding of CR transport physics and CR “feedback” effects (coupling to the gas via magnetic fields) at a microscopic level.

In recent years, hybrid magnetohydrodynamic (MHD)-particle-in-cell (PIC) simulations have been developed to investigate the interactions between CRs and background plasma, where CRs are evolved kinetically with the PIC treatment while the ionized gas is treated a MHD fluid (Bai et al. 2015; Mignone et al. 2018). This is possible because, for many astrophysical applications including all those above, the background plasma (1) is non-relativistic (bulk speeds $u \ll c$), (2) is sufficiently well-ionized that ideal MHD is a good approximation, and (3) has gyro radii which are vastly smaller the CRs (e.g. for typical ISM protons and electrons, the gyro radii are factors $\sim 10^2 - 10^3$ times smaller than the $\sim$ GeV CRs carrying most of the CR energy density). Compared with previous pure PIC simulations where both CRs and ions (and sometimes electrons as well) are simulated as particles (e.g., Spitkovsky 2005; Gargaté & Spitkovsky 2011; Kunz et al. 2014), the hybrid MHD-PIC method can simulate CR kinetic effects and feedbacks to gas by resolving the CR gyro-radius ($\sim$ AU for GeV CRs), without needing to explicitly resolve the extremely tiny background plasma ion inertial length of $\sim 10^{-6}$ AU or even smaller background electron skin depth of $\sim 10^{-8}$ AU. This method has been proven capable of capturing the CR streaming instability (Kulsrud & Pearce 1969; Skilling 1971; Bai et al. 2019) and accounting for the ion-neutral damping effects (Kulsrud & Cesarsky 1971; Zweibel & Shull 1982; Plotnikov et al. 2021; Bambic et al. 2021). Therefore, the MHD-PIC simulations have a unique advantage in following the evolution of CRs and their feedback/coupling to gas in many astrophysical contexts on much greater scales and over much longer durations than pure PIC simulations.

 Even with this approximation, however, one difficulty in simulating CRs is that CRs are relativistic with a velocity close to the speed of light $c$, far greater than other characteristic MHD velocities (e.g., the sound speed $s$, the Alfven speed $v_A$), which severely limits the simulation timestep (for any standard Courant-type condition) and thus makes simulations computationally prohibitive. For instance, the speed of light $c$ is involved in computing the Lorentz force acting on CRs as follows:

$$F_L \equiv q [E + (v/c) \times B],$$

where $q$ is the CR charge, $v$ the CR velocity, $B$ the magnetic field ($E$ electric field), and $c$ the true speed of light possibly with $c \gg c_s, v_A$, etc. For convenience, an artificial-speed-of-light (ASOL) $\tilde{c}$ is usually substituted to simplify the true speed of light $c$ in Eq. (1), and the Lorentz factor is accordingly rewritten as $\gamma = c/\sqrt{\tilde{c}^2 - v^2}$ (Bai et al. 2015; Mignone et al. 2018; van Marle et al. 2018; Bai...
momentum density and mass/number density contain factors of \( \sim \) properties. For instance, since the expressions of the CR energy density, hereinafter refereed to as a reduced-speed-of-light (RSOL), might not be corrected by "post-simulation" rescaling once the system of CRs data is time-evolved for long periods. The "back-reaction" is defined in the lab (simulation) frame. We expect Lorentz forces to affect the fluid (\( \nabla \times \left( \vec{E} + \vec{B} \right) \)) in the laboratory frame, however, when dealing with multi-scale problems where physical processes have different characteristic lengths / times / energies (\( \sim \)).

These challenges motivate us to develop an accurate formulation of the MHD-PIC equations with a RSOL, which resolves all of the issues above. The paper is organized as follows. §2 DERIVATION

Begin by considering the general Vlasov equation for a population of CRs:

\[
\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_x f + F \cdot \nabla_p f = \frac{\partial f}{\partial t}_{\text{coll}} \tag{2}
\]

where \( \nabla_{x,p} \) represent the gradients in position and momentum space, respectively, \( F \) is the force, \( f \) is the distribution function, the term on the right-hand-side represents collisional processes (which we will neglect for now, and return to later), \( t \) is time, and this is defined in the lab (simulation) frame. We expect Lorentz forces (Eq. (1)) to dominate in the regimes of interest, but note that our derivation is robust to the form of \( F \).

Following the standard practice for radiation-hydrodynamics and newly-developed methods which evolve the Fokker-Planck equation for a population of CRs (Skinner & Ostriker 2013; Hopkins et al. 2022), we multiply this equation by \( 1/c \) but then ad-hoc replace the value \( 1/c \) only in front of the time derivative \( \partial_t f \) with a different value \( \hat{c} \), to introduce the RSOL.

\[
\frac{1}{c} \frac{\partial f}{\partial t} + \frac{v}{c} \cdot \nabla_x f + F = \frac{1}{c} \frac{\partial f}{\partial t}_{\text{coll}} \tag{3}
\]

This is equivalent to taking \( \hat{\eta}_f \bar{f}_{\text{sol}} = (\hat{c}/c) \hat{\eta}_f \bar{f}_{\text{true}} \) i.e., the time variation of \( f \) is systematically slowed by a factor of \( c/\hat{c} \), equivalent to a rescaling of time "as seen by" the CRs. This is what allows us, fundamentally, to take larger timesteps by a factor \( c/\hat{c} \), as the time variation is slower. But this also ensures that one still recovers exactly the correct steady-state solutions (\( \partial_t f \rightarrow 0 \)) for the distribution function \( f \), independent of the choice of \( \hat{c} \).

Now we can turn this into the equation for a “single” CR group. More accurately for a MHD-PIC method which Monte-Carlo samples the dynamics of a CR population, we assume that \( f \) is a sum of \( \delta \)-functions each representing a group or “packet” of CRs (which are represented in-code by some “super-particles” or other tracer field):

\[
f(x, p, s, ... \bigm| \hat{N}_j) = \sum_j N_j W(x - \langle x \rangle_j) \delta(p - \langle p \rangle_j) f_s(x_j), \ldots \tag{4}
\]

where \( s \) denotes the species (with individual CR mass \( m_s \), etc., \( j \) and \( N_j \) is the label and total number of individual CR particles of each species \( s_j \), \( W \), \( \delta \) and \( f_s \) are the spatial kernel weighting functions, \( \delta \)-function and species function respectively, and \( \langle \rangle \) represents an average of individual CR particles over a whole CR “packet”. Inserting this into Eq. (3) and integrating over \( d^3 x \) \( d^3 p \) \( ds \ldots \) times \( x, \ p, \ s, \ldots \), we obtain the evolution equation for each “packet.”

Ignoring collisions, we have \( d_t N_j = 0, d_t s_j = 0, d_t \langle m_s \rangle_j = 0 \), where \( d_t \delta f_j = d\delta f_j/dt = \dot{\rho}_j \) represents the Lagrangian time derivative with each packet of some quantity \( \dot{\rho} \), i.e. CR number, and individual charge, species, mass are conserved along the trajectories of individual CRs (since we have neglected spallation and other catastrophic processes).

For \( x \), we obtain: \( c^{-1} \dot{x}_j(x) = c^{-1} \langle v \rangle_j \), or \( \langle v \rangle_j \) is the effective CR advection speed of CR packets across the grid (which we can define as \( \vec{v}_{\text{eff}} \)), which is reduced by our introduction of \( \hat{c} < c \). If \( |\langle v \rangle_j| \approx c \), then trivially \( |\dot{x}_j(x)\rangle = \hat{c} \), so the momentum equation is a reduced speed of light (RSOL) as desired.

The momentum equation (using various properties of \( \delta \)-functions and cross-products to simplify the algebra):

\[
\frac{1}{c} \hat{d}_t \langle p \rangle_j \bigm| \langle F \rangle_j = \frac{1}{c} \frac{\langle v \rangle_j}{c} \left[ E + \langle v \rangle_j \times B \right] \tag{5}
\]

where \( E = -(u/c) \times B \) to leading order here (with \( u \) the fluid velocity, and \( p \) the fluid density), and in the last equality we have inserted the assumption that \( F \) comes primarily from Lorentz forces, but our expression \( \hat{d}_t \langle p \rangle_j = (\hat{c}/c) \langle F \rangle_j \) is general and allows for the introduction of any other forces (e.g. gravity).

We can make this more clear by defining the following for notation purposes. Take quantities such as the Lorentz factors \( \beta \equiv |\vec{v}| \) and \( \gamma \), and likewise the energy \( E \) and momentum \( p \) of individual CRs to have their true values, so they are intrinsic properties of each
CR particle. Then we have:

$$\mathbf{v}^{\text{eff}}_j \equiv \frac{\partial_t (\mathbf{x})_j + \frac{q_j}{e} (\mathbf{v})_j}{\mathbf{v}^{\text{eff}}_j} \mathbf{v}^{\text{eff}}_j = \frac{\mathbf{e}}{c} (\mathbf{v})_j \tag{6}$$

$$\langle \mathbf{v}^{\text{eff}}_j \rangle \equiv \frac{\langle \mathbf{v}^{\text{eff}}_j \rangle}{e} = \frac{\mathbf{e}}{c} \tag{7}$$

$$\langle \mathbf{v}^{\text{eff}}_j \rangle_0 = \frac{1}{\sqrt{\langle \mathbf{v}^{\text{eff}}_j \rangle_0 \langle \mathbf{v}^{\text{eff}}_j \rangle_0}} \tag{8}$$

$$\langle \mathbf{E} \rangle_j \equiv \langle \mathbf{E} \rangle_j (m_{sj})^2 \tag{9}$$

$$\langle \mathbf{p} \rangle_j \equiv \langle \mathbf{p} \rangle_j (m_{sj}) c = \langle \mathbf{v} \rangle_j (m_{sj}) \mathbf{v}^{\text{eff}}_j = \frac{c}{e} \langle \mathbf{v} \rangle_j (m_{sj}) \mathbf{v}^{\text{eff}}_j \tag{10}$$

By definition, then the total CR number, mass, momentum, and energy represented by a given “packet” are: \(N_j, M_j \equiv N_j (m_{sj}), E_j \equiv N_j \langle \mathbf{E} \rangle_j, P_j \equiv N_j \langle \mathbf{p} \rangle_j\). We can then re-write the momentum equation for the “CR packet” as:

$$\frac{d}{dt} \langle \mathbf{v}^{\text{eff}}_j \rangle = \left( \frac{\mathbf{e}}{c} \right) \left( \frac{\langle q \rangle_j}{m_{sj} e} \right) \left( \mathbf{E} + \mathbf{v} \times \mathbf{B} \right) = \frac{1}{\mathbf{v}^{\text{eff}}_j} \frac{d}{dt} \langle \mathbf{v} \rangle_j \mathbf{v}^{\text{eff}}_j \tag{11}$$

So this is exactly the “true” acceleration, reduced by \(\mathbf{e}/c\). We now have a momentum equation which can evolve \(\mathbf{v}^{\text{eff}}_j\), the effective advection speed of a CR “packet” across the grid, entirely written in terms of background MHD quantities (which retain exactly their usual meaning) and well-defined properties of the “packet.”

We can derive the “back-reaction” force on the gas by going back to the original Vlasov equation, calculate the force or change in momentum for \(\mathbf{e}/c\), integrate over the distribution function, and apply the equal-and-opposite change to gas, as in Hopkins et al. (2022). Or, because we are assuming tight coupling between non-relativistic ions and magnetic fields, we could also calculate a current effect and use Ampere’s law, to get to the force on the gas, as in Zweibel (2017); Thomas & Pfrommer (2019), and obtain the identical answer. The important thing is that just like the radius case, the “normal” value of \(\mathbf{e}\) is what appears here, which ensures that the force on the gas/magnetic fields has its correct behavior (identical to \(\mathbf{e} = c\)) when the CR distribution function is in steady state. This gives:

$$\frac{d}{dt} (\rho \mathbf{u}) = - \int d^3 p f F = - \sum_j n_j F_j = - \sum_j n_j (q_j) \left( \mathbf{E} + \mathbf{v} \times \mathbf{B} \right)$$

$$= - \sum_j \frac{1}{2} \int d^3 \mathbf{v} d\mathbf{v} J_{\mathbf{v}} = - \sum_j \int d\mathbf{v} J_{\mathbf{v}}$$

$$= - \sum_j M_j \int \left( \frac{\mathbf{e}}{c} \right) \frac{1}{\mathbf{v}^{\text{eff}}_j} \frac{d}{dt} \langle \mathbf{v} \rangle_j \mathbf{v}^{\text{eff}}_j \tag{12}$$

where \(n_j \equiv N_j/V_j\) is the local CR number density with \(V_j\) some effective differential volume assigned to the “packet.”

With these insights, we can also incorporate collisional effects. Returning to Eq. (3), ignoring the terms in \(F\), we simply have \(d\phi_j = \langle \mathbf{e}/c \rangle d\phi_j/\text{local}\) for some \(\phi\) and relevant collisional effect (e.g. catastrophic losses). So for example if catastrophic losses remove CRs at a “true” rate \(\gamma_\ell = -(\mathbf{v} \cdot \mathbf{e}) n_{\ell} / n_{\ell} (\text{where } n_{\ell}\text{ is the local nucleon number density}), the total number/mass/momentum/energy of a “packet” is reduced accordingly as \(d\phi_j \rightarrow -(\mathbf{v} \cdot \mathbf{e}) n_{\ell} / n_{\ell} \phi_j\) for \(\phi_j = \langle \mathbf{v} \rangle, M_j, E_j, P_j\).

Upon examination, a few points are clear. First, as intended, the manner in which \(\mathbf{e}\) enters is equivalent to a rescaling of time as seen by the CRs, so it guarantees that we can always increase the timestep by \(\mathbf{e}/c\). This means that the gyro-frequency is reduced in the lab frame (necessarily) by the same factor (as \(\omega_\ell^{\text{eff}} = \Omega_\ell (\mathbf{e}/c))\), so the actual effective time it requires to complete a gyro orbit is longer. However, this means that for a given true \(v_j\) or equivalently CR energy or \(\gamma_j\), the gyro radius on the grid is entirely independent of \(\mathbf{e}\). We can explicitly confirm either if we think about the RSOL as a rescaling in time, or simply examine the “effective” transport speeds across the grid, that the Courant criteria for CR “advection” (gyro orbits) become e.g. \(\Delta t < C/\omega_\ell^{\text{eff}} = (\mathbf{e}/c) \Delta t = c\); likewise for advection “through a cell” (which restricts both CR and MHD timesteps), \(\Delta t < C \Delta x/\mathbf{v}^{\text{eff}} = (\mathbf{e}/c) \Delta x = \mathbf{c}\). Just like with radiation-hydrodynamics, the “conserved momentum” here upon MHD-CR exchange is \(\rho \mathbf{u} + (\mathbf{e}/c) \mathbf{P}\) (where \(\mathbf{P} \equiv \sum_j P_j\)), not \(\rho \mathbf{u} + \mathbf{P}\) (which is only true if \(\mathbf{e} = c\)). From the form of the Vlasov equation and force on gas written in terms of the distribution function \(f\), it is also immediately obvious that so long as the distribution function is in steady-state, i.e. \(\partial_t f \rightarrow 0\), then we are guaranteed to recover the correct steady-state solutions for the distribution function, i.e. the correct steady state distribution of CR velocities, phase and pitch angles, momenta/Lorentz factors, etc. as well as the correct force on the gas. We are also guaranteed that processes such as scattering occur on the correct length scales: e.g. scattering rates (or e.g. collisional/catastrophic loss rates) will be slowed by a factor \(\mathbf{e}/c\), but so will transport speeds, so by the time a CR has traveled a given distance \(\ell\) down a magnetic field line, it will have an identical integrated scattering/catastrophic loss probability, independent of \(\mathbf{e}\).

### 3 COMPARISON TO NAIVE RSOL SUBSTITUTION

While “pure PIC” methods are quite mature, MHD-PIC methods for applications such as CR dynamics have only been widely-developed relatively recently. As previously discussed, the ASOL method can be interpreted as changing the unit system while retaining the same dimensionless numbers of a problem. While when dealing with multi-scale problems where prohibitively small timestep might become an issue, simply treating the RSOL \(\mathbf{e}\) as if the “true” \(\mathbf{e}\) were reduced for the CRs specifically, does enable much larger timesteps, but leads to several conceptual difficulties and inconsistencies in an effort to match the physical properties in real systems, including: (1) it is impossible to simultaneously reproduce the true CR energy and/or momentum and/or number densities, regardless of whatever limit the system is in, since for example in their formulation \(e_j \equiv dE_j/dx = n_j \gamma_j (m_{sj})^2 \mathbf{v}^{\text{eff}}_j\), so either \(n_j\) or \(e_j\) must be incorrect (and likewise for momentum density); (2) the transition between relativistic and ultra-relativistic behavior occurs at the incorrect CR energy/momentum; (3) this leads to a momentum equation (in our notation) \(d\mathbf{v}/dt \langle \mathbf{v} \rangle_j \mathbf{v}^{\text{eff}}_j \rightarrow \langle \mathbf{v} \rangle_j (m_{sj}) c (e \mathbf{v} + \mathbf{v}^{\text{eff}}_j \times \mathbf{B})\), which is different from Eq. (11) by \((\mathbf{e}/c)^2\) in the electric-field term and \((\mathbf{e}/c)^3\) in the magnetic-field term, breaking the physical correspondence between the two unless one redefines the electric field \(\mathbf{E}\) to be different from that assumed in the MHD derivation; (4) likewise if one rescales this momentum equation to keep \((q/m_{sj}) c\) fixed, then this leads not to a rescaling of \(\Omega\) with \(\mathbf{e}\), but gives the same \(\Omega\) (requiring the same timestep as simulations with \(\mathbf{e} = c\) for gyro orbits), instead with a smaller gyro radius \(\Omega\); (5) there is no guarantee that when the distribution function is in steady-state (even for the simplest homogeneous gas and CR configuration), the distribution function and/or the net back-reaction force on the gas will actually be correct, compared to their value with \(\mathbf{e} = c\). As a result, in these approaches, one obtains certain results that vary systematically with \(\mathbf{e}\) and must be extrapolated (e.g. by running a large suite of simulations with different \(\mathbf{e}\)) to estimate their correct \(\mathbf{e} \rightarrow c\) values. In contrast, our approach allows for formal convergence of steady-state properties with \(\mathbf{e} \ll c\).

We finally note that our proposed RSOL formulation actually derives the “correction terms” which are needed to add to the electric field, CR acceleration equation, and “back-reaction” force equation, and generically guarantees that the CR distribution function will
give the correct steady-state answer. However, simply replacing $c$ with RSOL $\tilde{c}$, by definition, will not solve the correct CR distribution function in steady state. The previous method is equivalent to a simultaneous rescaling of both the velocities and spatial units of the original problem, so by decreasing $c$, it ends up with solving a different problem from what is intended to solve originally, specifically a problem with a different range of spatial scales and a different ratio of $v_{\perp}\tilde{c}$, $c_{\perp}\tilde{c}$ and $u/c$. Whereas our proposed method, equivalent to a rescaling of time (“as seen by”) the CRs only, can obtain the right answer for the actually desired dimensionless numbers of the problem simultaneously (and all physical properties as well, e.g., CR mass, momentum and energy densities), so long as the CR distribution function is in steady-state. Therefore, whenever a RSOL is needed for enabling larger timesteps, our proposed RSOL method should always be implemented in order to achieve the best possible consistency within simulated systems.

4 NUMERICAL IMPLEMENTATION & TIMESTEP CONSTRAINTS

The form of the RSOL proposed here can be immediately and trivially implemented in any MHD-PIC code with minimal effort, as it amounts to a straightforward renormalization of the time-derivative terms which would be computed anyways. We have explored existing implementations in the code GIZMO in Ji et al. (2022), but note it immediately applies to any other MHD-PIC implementation, for example both the “full-” and “a-” methods in Bai et al. (2015, 2019). In these methods the MHD (non-relativistic ion/neutrall/electron) dynamics are solved in some fluid limit with whatever solver is most useful, additional physics (e.g. radiation transport or dust as in 7) can be added as well, and then the CRs are integrated on this background with a “super-particle” approach. In this, one simply identifies each super-particle with a “packet” above, and immediately obtain their evolution equations. The only difference between our approach and one with no RSOL is the insertion of appropriate factors of $\tilde{c}/c$ in the code.

With this formulation, we see also that when initializing the system, one should simply initialize the correct $N_j$ such that e.g. $N = \sum N_j = M_{\text{CR total}}/(m_5)$, where $(m_5)$ is the true individual CR mass. Or equivalently $N = \langle E \rangle_j = E_{\text{CR total}}/(\langle \gamma \rangle_j (m_5) c^3)$. This ensures that the correct total CR energy and momentum densities $\epsilon$ and $P$ are also initialized – these are all automatically consistent with one another. Inconsistency only arises if we incorrectly identify $\langle \gamma \rangle_j (m_5) v_{\text{eff}}$ as the individual CR momentum, but this is not correct because the relation between $v_{\text{eff}}$ and the momentum as it appears in conservation and dynamics equations is changed by the introduction of the RSOL. As detailed above, quantities like $p$, $v$, $\gamma$, $\beta$, $E$, etc. all retain their usual intrinsic physical meaning (that they would have with $\tilde{c} = c$), but the actual advection velocity $\partial_t (x_j)$ of a “super-particle” is $v_{\text{eff}}$, and the CR momentum equation is modified with the appropriate $\tilde{c}/c$ factors.

The result of this is that all timestep constraints for the CRs also rescale by $\tilde{c}/c$, as all the time-derivative terms for CRs rescale with this. Obviously, one requires $\Delta t < C \Omega_{\text{f}}^{-1} = C (r_g/c)(c/\tilde{c}) C\langle (r_g)/(c) \rangle_j$, where $(r_g)$ is the gyro radius, and $C$ is a Courant factor, or more generally $\Delta t < C v_{\text{eff}}||d(\text{eff})||$, and if there are some catastrophic losses, similarly $\Delta t < C N_j/\langle N_j \rangle_j \sim (c/\tilde{c})/(\sigma v_j) n_0$. And for the Courant condition of CRs propagating through the gas with some cell size $\Delta x$, we have $\Delta t < C \Delta x/||d|| = C \Delta x/||v|| = (c/\tilde{c}) C \Delta x/||\text{v}||$. For the case with non-relativistic CRs, even though it does not gain significant benefit from the RSOL regarding computational time saving, this RSOL implementation is still recommended: it naturally returns to the ASOL formulation under such regimes, and can be safely and conveniently extrapolated to relativistic regimes where a RSOL might be needed.

Finally, just like any other RSOL implementation, convergence to correct solutions require choosing a sufficiently large value of $\tilde{c}$ compared to other (e.g. MHD) signal speeds in the problem. Although formally steady-state solutions are independent of $\tilde{c}$ for all $\tilde{c}$ with this method, if one chose $\tilde{c}$ too small compared to e.g. some rms turbulent velocity of the MHD fluid, $\tilde{c} \ll \langle |\delta u| \rangle$, then the plasma would artificially “outpace” the CRs and the system will not actually be able to converge to a steady-state distribution function $f$ in the first place. This is also consistent with the requirement of sufficient scale separation in order to ensure, e.g., a much smaller gyro-period than the eddy turnover / Alfvén crossing timescales.

5 CONCLUDING REMARKS

In this paper, we propose an accurate, well-defined formulation of a RSOL which can be applied to MHD-PIC simulations. The major advantages of the proposed formulation are that: (1) The timestep can be safely increased by $\tilde{c}/c$ as intended (regardless of if the timestep restriction from gyro orbits, advection over the grid, grid response to CRs, or collisional terms dominates). But meanwhile (2) the key properties of CRs in numerical simulations, such as energy density, momentum density, mass/number density and CR gyro-radius, are independent of the choices of $\tilde{c}$, and (3) they can simultaneously reproduce their true, physical values (those one would have with $\tilde{c} = c$). And perhaps most importantly (4) solutions for CR bulk properties, distribution function, and their interactions with/forces on the magnetic fields and background fluid are independent of $\tilde{c}$ as well (even if $\tilde{c} < c$) so long as the CR distribution function $f$ approaches a local steady-state. These advantages naturally recommend the formulation here as a standard and straightforward implementation for MHD-PIC simulations whenever a RSOL is required.

DATA AVAILABILITY STATEMENT

There are no new data associated with this article.

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