HEISENBERG ALGEBRA AND HILBERT SCHEMES OF POINTS  
ON PROJECTIVE SURFACES

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1. Introduction

The purpose of this paper is to throw a bridge between two seemingly unrelated  
subjects. One is the Hilbert scheme of points on projective surfaces, which has been  
intensively studied by various people (see e.g., [I, ES, Gö1, Gö2]). The other is the  
infinite dimensional Heisenberg algebra which is closely related to affine Lie algebras  
(see e.g., [K]).

We shall construct a representation of the Heisenberg algebra on the homology  
group of the Hilbert scheme. In other words, the homology group will become a  
Fock space. The basic idea is to introduce certain “correspondences” in the product  
of the Hilbert scheme. Then they define operators on the homology group by a  
well-known procedure. They give generators of the Heisenberg algebra, and the only  
thing we must check is that they satisfy the defining relation. Here we remark that  
the components of the Hilbert scheme are parameterized by numbers of points and  
our representation will be constructed on the direct sum of homology groups of all  
components. Our correspondences live in the product of the different components.  
Thus it is quite essential to study all components together.

Our construction has the same spirit with author’s construction [Na1, Na4] of repres-  
sentations of affine Lie algebras on homology groups of moduli spaces of “instantons” on  
ALE spaces which are minimal resolution of simple singularities. Certain corre-  
spondences, called Hecke correspondences, were used to define operators. These twist  
vector bundles along curves (irreducible components of the exceptional set), while  
ours twist around points. In fact, the Hilbert scheme of points can be considered  
as the moduli space of rank 1 vector bundles, or more precisely torsion free sheaves.  
Our construction should be considered as a first step to extend [Na1, Na4] to more  
general 4-manifolds. The same program was also proposed by Ginzburg, Kapranov  
and Vasserot [GKV].

Another motivation of our study is the conjecture about the generating function of  
the Euler number of the moduli spaces of instantons, which was recently proposed by  

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1The reason why we put the quotation mark will be explained in Remark A.6.
Vafa and Witten [VW]. They conjectured that it is a modular form for 4-manifolds under certain conditions. This conjecture was checked for various 4-manifolds using various mathematicians’ results. Among them, the most relevant to us is the case of K3 surfaces. Götsche and Huybrechts [GH] proved that the Betti numbers of moduli spaces of stable rank two sheaves are the same as those for Hilbert schemes. Götsche [Gö1] computed the Betti numbers of Hilbert schemes for general projective surfaces $X$. (The Hilbert schemes for $\mathbb{C}P^2$ were studied earlier by Ellingsrud and Strømme’s [ES].) If $X^{[n]}$ is the Hilbert scheme parameterizing $n$-points in $X$, the generating function of the Poincaré polynomials is given by

$$(1.1) \quad \sum_{n=0}^{\infty} q^n P_t(X^{[n]}) = \prod_{m=1}^{\infty} \frac{(1 + t^{2m-1}q^m)b_1(X)(1 + t^{2m+1}q^m)b_3(X)}{(1 - t^{2m-2}q^m)b_0(X)(1 - t^{2m}q^m)b_2(X)(1 - t^{2m+2}q^m)b_4(X)},$$

where $b_i(X)$ is the Betti number of $X$. Letting $t = -1$, we find the generating function of the Euler numbers is essentially the Dedekind eta function. In fact, the relation with the above formula and the Fock space was already pointed out in [VW]. Our result should be considered as a geometric realization of their indication.

The paper is organized as follows. In §2 we give preliminaries. We recall the definition of the convolution product in §2(i) with some modifications and describe some properties of the Hilbert scheme $X^{[n]}$ and the infinite Heisenberg algebra and its representations §§2(ii),2(iii). The definition of correspondences and the statement of the main result are given in §3. The proof will be given in §4. In the appendix, we study the particular case $X = \mathbb{C}^2$ in more detail. We give a description of $(\mathbb{C}^2)^{[n]}$ as a hyper-Kähler quotient of finite dimensional vector space by a unitary group action. It is very similar to the definition of quiver varieties [Na1]. The only difference is that we have an edge joining a vertex with itself. Using this description as a hyper-Kähler quotient, we compute the homology group of $(\mathbb{C}^2)^{[n]}$. We recover the formula (1.1) for $X = \mathbb{C}^2$. The difference between our approach and Ellingsrud-Strømme’s [ES] is only the description. Both use the torus action and study the fixed point set. But our presentation has a similarity in [Na2]. The appendix is independent of the other parts of this paper, but those similarities with author’s previous works explains motivation of this paper in part.

While the author was preparing this manuscript, he learned that the similar result was announced by Grojnowski [Gr]. He introduced exactly the same correspondence as ours.

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2. Preliminaries

2(i). Convolution Algebras. We need a slight modification of the definition of the convolution product in the homology groups given by Ginzburg [Gi] (see also [CG, Na1]).

For a locally compact topological space $X$, let $H^\lf_{\ast}(X)$ denote the homology group of possibly infinite singular chains with locally finite support (the Borel-Moore homology) with rational coefficients. The usual homology group of finite singular chains will be denoted by $H^\ast(X)$. If $X = X \cup \{\infty\}$ is the one point compactification of $X$, we have $H^\lf_{\ast}(X)$ is isomorphic to the relative homology group $H_{\ast}(X, \{\infty\})$. If $X$ is an $n$-dimensional oriented manifold, we have the Poincaré duality isomorphism

$$H^n_{\ast}(X) \cong H^{n-\ast}(X), \quad H_{\ast}(X) \cong H^{n-\ast}(X),$$

where $H^\ast$ and $H^c_\ast$ denote the ordinary cohomology group and the cohomology group with compact support respectively.

Let $M_1, M_2, M_3$ be oriented manifolds of dimensions $d_1, d_2, d_3$ respectively, and $p_{ij}: M_1 \times M_2 \times M_3 \to M_i \times M_j$ be the natural projection. We define a convolution product

$$*: \left(H^{1f}_{i_1}(M^1) \otimes H_{i_2}(M^2)\right) \otimes \left(H^{1f}_{d_2-i_2}(M^2) \otimes H_{i_3}(M^3)\right) \to H^{1f}_{i_1}(M^1) \otimes H_{i_3}(M^3)$$

by

$$(c_1 \otimes c_2) * (c'_2 \otimes c_3) \overset{\text{def.}}{=} c_2 \cap c'_2 \ c_1 \otimes c_3,$$

where $c_2 \cap c'_2 \in Z$ is the natural pairing between $H_{i_2}(M^2)$ and $H^{1f}_{d_2-i_2}(M^2) \cong H^{i_2}(M^2)$.

Suppose $Z$ is a submanifold in $M_1 \times M_2$ such that

$$\text{the projection } Z \to M^1 \text{ is proper.}$$

Then the fundamental class $[Z]$ defines an element in

$$[Z] \in H^\lf_{\dim Z}(\overline{M}_1 \times M_2, \{\infty\} \times M_2) = \bigoplus_{i+j=\dim Z} H^{1f}_{i_1}(M^1) \otimes H_{j}(M^2),$$

where $\overline{M}_1 = M_1 \cup \{\infty\}$ is the one point compactification of $M_1$ and we have used the Künneth formula. More generally, if $[Z]$ is a cycle whose support $Z$ satisfies (2.2), the same construction works. Using (2.1), we get an operator, which is denoted also by $[Z]$,

$$[Z]: H^{1f}_{j}(M^2) \to H^{1f}_{j+\dim Z-d_2}(M^1).$$
2(ii). Hilbert Schemes of Points on Surfaces. Let $X$ be a nonsingular quasi-projective surface defined over the complex number $\mathbb{C}$. Let $X^{[n]}$ be the component of the Hilbert scheme of $X$ parameterizing the ideals of $\mathcal{O}_X$ of colength $n$. It is smooth and irreducible [Fo]. Let $S^n X$ denotes the $n$-th symmetric product of $X$. It parameterizes formal linear combinations $\sum n_i [x_i]$ of points $x_i$ in $X$ with coefficients $n_i \in \mathbb{Z}_{>0}$ with $\sum n_i = n$. There is a canonical morphism

$$\pi: X^{[n]} \to S^n X; \quad \pi(J) \overset{\text{def}}{=} \sum_{x \in X} \text{length}(\mathcal{O}_X / J)_x [x].$$

It is known that $\pi$ is a resolution of singularities.

The symmetric power $S^n X$ has a natural stratification into locally closed subvarieties as follows. Let $\nu$ be a partition of $n$, i.e., a sequence $n_1, n_2, \ldots, n_r$ such that $n_1 \geq n_2 \geq \cdots \geq n_r$, $\sum n_i = n$.

Then $S^n X$ is defined by

$$S^n X \overset{\text{def}}{=} \{ \sum n_i [x_i] \in S^n X \mid x_i \neq x_j \text{ for } i \neq j \}.$$

It is known that $\pi$ is semi-small with respect to the stratification $S^n X = \bigcup S^n X [I]$, that is

1. for each $\nu$, the restriction $\pi: \pi^{-1}(S^n X) \to S^n X$ is a locally trivial fibration,
2. $\text{codim} S^n X = 2 \dim \pi^{-1}(x)$ for $x \in S^n X$.

Moreover, it is also known that $\pi^{-1}(x)$ is irreducible.

2(iii). The Infinite Dimensional Heisenberg Algebra. We briefly recall the definition of the infinite dimensional Heisenberg algebra and its representations. See [K, §9.13] for detail.

The infinite dimensional Heisenberg algebra $\mathfrak{s}$ is generated by $p_i, q_i \ (i = 1, 2, \ldots)$ and $c$ with the following relations:

\begin{align}
[&p_i, p_j] = 0, \quad [q_i, q_j] = 0 \\
[&p_i, q_j] = \delta_{ij} c.
\end{align}

For every $a \in \mathbb{C}^*$, the Lie algebra $\mathfrak{s}$ has an irreducible representation on the space $R = \mathbb{C}[x_1, x_2, \ldots]$ of polynomials in infinitely many indeterminates $x_i$ defined by

$$p_i \mapsto a \frac{\partial}{\partial x_i}, \quad q_i \mapsto x_i, \quad c \mapsto a \text{Id}.$$

This representation has a highest weight vector $1$, and $R$ is spanned by elements

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} = q_1^{j_1} q_2^{j_2} \cdots q_n^{j_n} 1.$$
We extend $s$ by a derivation $d_0$ defined by

$$[d_0, q_j] = jq_j, \quad [d_0, p_j] = -jp_j.$$ 

The above representation $R$ extends by $d_0 \mapsto \sum_j jx_j \frac{\partial}{\partial x_j}$.

Then it is easy to see

$$\text{tr}_R q^{d_0} = \prod_{j=1}^{\infty} \frac{1}{1 - q^j}. \tag{2.5}$$

The representation $R$ carries a unique bilinear form $B$ such that $B(1, 1) = 1$ and $p_i$ is the adjoint of $q_i$, provided $a \in \mathbb{R}$. In fact, distinct monomials are orthogonal and we have

$$B(x_1^{i_1} \ldots x_n^{i_n}, x_1^{j_1} \ldots x_n^{j_n}) = a \sum j_k \prod j_k! \tag{2.6}$$

We also need the infinite dimensional Clifford algebra $\text{Cl}$ (see e.g., [Fr]). It is generated by $\psi_i, \psi^*_i (i = 1, 2, \ldots)$ and $c$ with the relations

$$\psi_i \psi_j + \psi_j \psi_i = 0, \quad \psi^*_i \psi^*_j + \psi^*_j \psi^*_i = 0 \tag{2.7}$$

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{ij} c.$$

This algebra has a representation on the exterior algebra $F = \wedge^* V$ of an infinite dimensional vector space $V = \mathbb{C} dx^1 \oplus \mathbb{C} dx^2 \oplus \cdots$ defined by

$$\psi_i \mapsto d x^i \wedge, \quad \psi_i^* \mapsto \frac{\partial}{\partial x_i}, \quad c \mapsto \text{Id},$$

where $\lrcorner$ denotes the interior product. This has the highest weight vector $1$ and spanned by

$$dx_{i_1} \wedge \cdots \wedge dx_{i_n} = \psi_{i_1} \cdots \psi_{i_n} 1, \quad (i_1 > i_2 > \cdots > i_n).$$

We extend $\text{Cl}$ by $d$ defined by

$$[d, \psi_i] = i\psi_i, \quad [d, \psi^*_i] = -i\psi^*_i.$$ 

It acts on $F$ by

$$d(dx_{i_1} \wedge \cdots \wedge dx_{i_n}) = (\sum i_k) dx_{i_1} \wedge \cdots \wedge dx_{i_n}.$$ 

The character is given by

$$\text{tr}_F q^d = \prod_{j=1}^{\infty} (1 + q^j).$$
3. Main Construction

3(i). Definitions of Generators. Let $X$ as in §2(ii). Take a basis of $H^{1f}_*(X)$ and assume that each element is represented by a (real) closed submanifold $C^a$. (a runs over $1, 2, \ldots, \dim H^{1f}_*(X).$) Take a dual basis for $H^*_a(X) \cong H^{4-n}_c(X)$, and assume that each element is represented by a submanifold $D^a$ which is compact. (Those assumptions are only for the brevity. The modification to the case of cycles is clear.)

For each $a = 1, 2, \ldots, \dim H^{1f}_*(X)$, $n = 1, 2, \ldots$ and $i = 1, 2, \ldots$, we introduce cycles of products of the Hilbert schemes by

$$E^a_i(n) \stackrel{\text{def}}{=} \{ (\mathcal{J}_1, \mathcal{J}_2) \in X^{[n-i]} \times X^{[n]} \mid \mathcal{J}_1 \supset \mathcal{J}_2 \text{ and } \text{Supp}(\mathcal{J}_1/\mathcal{J}_2) = \{ p \} \text{ for some } p \in D^a \},$$

$$F^a_i(n) \stackrel{\text{def}}{=} \{ (\mathcal{J}_1, \mathcal{J}_2) \in X^{[n+i]} \times X^{[n]} \mid \mathcal{J}_1 \subset \mathcal{J}_2 \text{ and } \text{Supp}(\mathcal{J}_2/\mathcal{J}_1) = \{ p \} \text{ for some } p \in C^a \}.$$  

The dimensions are given by

$$\dim \mathbb{R} E^a_i(n) = 4(n-i) + 2(i-1) + \dim \mathbb{R} D^a,$$

$$\dim \mathbb{R} F^a_i(n) = 4n + 2(i-1) + \dim \mathbb{R} C^a.$$  

This follows from the fact $X^{[n]} \to S^n X$ is semi-small (see §2(ii)). Since the projections $E^a_i(n) \to X^{[n-i]}$ and $F^a_i(n) \to X^{[n+i]}$ are proper, we have classes

$$[E^a_i(n)] \in \bigoplus_{k,l} H^{1f}_k(X^{[n-i]}) \otimes H_l(X^{[n]}),$$

$$[F^a_i(n)] \in \bigoplus_{k,l} H^{1f}_k(X^{[n+i]}) \otimes H_l(X^{[n]}).$$  

Our main result is the following:

**Theorem 3.1.** The following relations hold in $\bigoplus_{k,l,m,n} H^{1f}_k(X^{[m]}) \otimes H_l(X^{[n]})$.

(3.2)  

$$[E^a_i(n-j)] \ast [E^b_j(n)] = (-1)^{\dim D^a \dim D^b} [E^b_j(n-i)] \ast [E^a_i(n)]$$

(3.3)  

$$[F^a_i(n+j)] \ast [F^b_j(n)] = (-1)^{\dim C^a \dim C^b} [F^b_j(n-i)] \ast [F^a_i(n)]$$

(3.4)  

$$[E^a_i(n+j)] \ast [F^b_j(n)] = (-1)^{\dim D^a \dim C^b} [F^b_j(n-i)] \ast [E^a_i(n)] + \delta_{ab} \delta_{ij} c_i [\Delta(n)],$$

where $\Delta(n)$ is the diagonal of $X^{[n]}$, and $c_i$ is a nonzero integer depending only on $i$ (independent of $X$).
In particular, for each fixed $a$, the map
\[ p_i \mapsto \sum_n [E^a_i(n)], \quad q_i \mapsto \sum_n [F^a_i(n)] \quad \text{when } \dim C^a \text{ is even} \]
\[ \psi_i^* \mapsto \sum_n [E^a_i(n)], \quad \psi_i \mapsto \sum_n [F^a_i(n)] \quad \text{when } \dim C^a \text{ is odd} \]
defines a homomorphism from the Heisenberg algebra and the Clifford algebra respectively.

Considering $[E^a_i(n)], [F^a_i(n)]$ as operators on $\bigoplus_{k,n} H_{lf}^k(X[n])$, we have a representation of the product of Heisenberg algebras and Clifford algebras. Comparing Göttsche's Betti number formula and the character formula, we get the following:

**Theorem 3.5.** The direct sum $\bigoplus_{k,n} H^I_k(X^{[n]})$ of homology groups of $X^{[n]}$ is the highest weight module where the highest weight vector $v_0$ is the generator of $H^I_0(X^{[0]}) \cong \mathbb{Q}$.

**Remark 3.6.** The author does not know the precise values of $c_i$'s. It is easy to get $c_1 = 1$, $c_2 = -2$, but general $c_i$ become difficult to calculate.

### 4. Proof of Theorem 3.1

**4(i). Proof of Relations (I).** Consider the product $X^{[n-i-j]} \times X^{[n-j]} \times X^{[n]}$ and let $p_{12},$ etc. be as in §2(i). The intersection $p^{-1}_1 (E^a_i(n-j)) \cap p^{-1}_3 (E^b_j(n))$ consists of triples $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$ such that

\[ \mathcal{J}_1 \supset \mathcal{J}_2 \supset \mathcal{J}_3 \quad (4.1) \]

\[ \text{Supp}(\mathcal{J}_1/\mathcal{J}_2) = \{p\}, \quad \text{Supp}(\mathcal{J}_2/\mathcal{J}_3) = \{q\} \text{ for some } p \in D^a, q \in D^b. \quad (4.2) \]

Replacing $D^b$ by $\tilde{D}^b$ in the same homology class, we may assume $\dim D^a \cap \tilde{D}^b = \dim D^a + \dim \tilde{D}^b - 4$. (If the right hand side is negative, the set is empty.) Let $U$ be the open set in the intersection consisting points with $p \neq q$ in (4.2). Outside the singular points of $p^{-1}_1 (E^a_i(n-j)), p^{-1}_3 (E^b_j(n))$, the intersection is transverse along $U$. The complement $p^{-1}_1 (E^a_i(n-j)) \cap p^{-1}_3 (E^b_j(n)) \setminus U$ consists of $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$ with (4.1) and

\[ \text{Supp}(\mathcal{J}_1/\mathcal{J}_2) = \text{Supp}(\mathcal{J}_2/\mathcal{J}_3) = \{p\} \text{ for some } p \in D^a \cap \tilde{D}^b. \]

Its dimension is at most

\[ 4n - 2i - 2j - 4 + \dim D^a + \dim D^b - 4, \]

which is strictly smaller than the dimension of the intersection

\[ 4n - 2i - 2j - 4 + \dim D^a + \dim D^b. \]
Now consider the product $X^{[n-i-j]} \times X^{[n-i]} \times X^{[n]}$. The intersection $p_{12}^{-1}(E_j(n-i)) \cap p_{23}^{-1}(E_i(n))$ consists of triples $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$ such that

\begin{equation}
\mathcal{J}_1 \supset \mathcal{J}_2 \supset \mathcal{J}_3
\end{equation}

\begin{equation}
\text{Supp}(\mathcal{J}_1/\mathcal{J}_2) = \{q\} \quad \text{Supp}(\mathcal{J}_3/\mathcal{J}_2) = \{p\} \quad \text{for some } q \in D^b, \ p \in D^a.
\end{equation}

Let $U'$ be the open set in the intersection consisting points with $p \neq q$ in (4.4). The intersection is again transverse along $U$ outside singular sets. The complement $p_{12}^{-1}(E_j(n-i)) \cap p_{23}^{-1}(E_i(n)) \setminus U'$ has dimension is at most

\[4n - 2i - 2j - 4 + \dim D^a + \dim D^b - 4,\]

which is also strictly smaller than the dimension of the intersection.

There exists a homeomorphism between $U$ and $U'$ given by

\[U \ni (\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3) \mapsto (\mathcal{J}_1, \mathcal{J}_2', \mathcal{J}_3) \in U',\]

where $\mathcal{J}_2'$ is a sheaf such that

\[\mathcal{J}_1/\mathcal{J}_2' = \mathcal{J}_2/\mathcal{J}_3, \quad \mathcal{J}_2'/\mathcal{J}_3 = \mathcal{J}_1/\mathcal{J}_2.\]

Such $\mathcal{J}_2'$ exists since supports of $\mathcal{J}_1/\mathcal{J}_2$ and $\mathcal{J}_2/\mathcal{J}_3$ are different points $p$ and $q$.

Taking account of orientations and the estimate of the dimension of the complements, we get the relation (3.2). The proof of (3.3) is exactly the same.

**4(ii). Proof of Relations (II).** The proof of (3.4) is almost similar to the above. Consider the product $X^{[n-i+j]} \times X^{[n+j]} \times X^{[n]}$. The intersection $p_{12}^{-1}(E_i(n+j)) \cap p_{23}^{-1}(F_j(n))$ consists of triples $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$ such that

\begin{equation}
\mathcal{J}_1 \supset \mathcal{J}_2 \subset \mathcal{J}_3
\end{equation}

\begin{equation}
\text{Supp}(\mathcal{J}_1/\mathcal{J}_2) = \{p\}, \quad \text{Supp}(\mathcal{J}_3/\mathcal{J}_2) = \{q\} \quad \text{for some } p \in D^a, \ q \in C^b.
\end{equation}

Replacing $C^b$ by $\tilde{C}^b$ in the same homology class, we may assume $\dim D^a \cap \tilde{C}^b = \dim D^a + \dim \tilde{C}^b - 4$. (If the right hand side is negative, the set is empty.) Since $\{D^a\}$ and $\{C^b\}$ are dual bases each other, the equality holds if and only if $a = b$. Let $U$ be the open set in the intersection consisting points with $p \neq q$ in (4.6).
Next consider the product \( X^{[n-i+j]} \times X^{[n-i]} \times X^{[n]} \). The intersection \( p_{12}^{-1}(F^b_i(n - i)) \cap p_{23}^{-1}(E^a_i(n)) \) consists of triples \((J_1, J_2, J_3)\) such that

\[
J_1 \subset J_2 \supset J_3
\]  

(4.7)

\[
\text{Supp}(J_2' / J_1) = \{q\}, \quad \text{Supp}(J_3' / J_2) = \{p\} \text{ for some } q \in C^b, \ p \in D^a.
\]  

(4.8)

Let \( U' \) be the open set in the intersection consisting points with \( p \neq q \) in (4.8).

There exists a homeomorphism between \( U \) and \( U' \) given by

\[
U \ni (J_1, J_2, J_3) \mapsto (J_1, J_2', J_3) \in U',
\]

where \( J_2' \) is

\[
(J_1 \oplus J_3) / \{(f, f) \mid f \in J_2\}.
\]

The inverse map is given by \( J_2 = J_1 \cap J_3 \).

Let \( U_c, U_{cc} \) be the complement of \( U \) and \( U' \) respectively. If \((J_1, J_3)\) is in the image \( p_{13}(U_c) \) or \( p_{13}(U_{cc}) \), then \( J_1 \) and \( J_3 \) are isomorphic outside a point \( p \in D^a \cap C^b \). In particular, it is easy to check

\[
\dim p_{13}(U_c), \ \dim p_{13}(U_{cc}) \leq 4n - 2i + 2j - 4 + \dim D^a + \dim C^b,
\]

where the right hand side is the expected dimension of the intersection. The equality holds only if \( i = j \) and \( \dim D^a + \dim C^b = 4 \). Moreover, \( \{D^a\} \) and \( \{C^b\} \) are dual bases, when \( \dim D^a + \dim C^b = 4 \), the intersection \( D^a \cap C^b \) is empty unless \( a = b \). Thus we have checked (3.4) when \( i \neq j \) or \( a \neq b \).

Now assume \( i = j \) and \( a = b \). Then \( p_{13}(U_{cc}) \) has smaller dimension and \( p_{13}(U_c) \) is union of the diagonal \( \Delta(n) \) and smaller dimensional sets. Hence the left hand side of (3.4) is a multiple of \( [\Delta(n)] \). In order to calculate the multiple, we may restrict the intersection on the open set where

1. \( J_1 \) and \( J_3 \) are contained in the open stratum \( \pi^{-1}(S_{0,1,\ldots,1}) \),
2. \( \text{Supp} \mathcal{O} / J_1, \ \text{Supp} \mathcal{O} / J_3 \) do not intersect with \( D^a \cap C^a \).

Then it is clear that the multiple is a constant independent of \( n \) and \( X \), which we denoted by \( c_i \). The only thing left is to show \( c_i \neq 0 \). We may assume \( X = \mathbb{C}^2 \) and \( n = i \). We consider the quotient of \( (\mathbb{C}^2)^{[i]} \) devided by the action of \( \mathbb{C}^2 \) which comes from the parallel translation. Thus \( c_i \) is equal to the self-intersection number of \( [\pi^{-1}(i[0])] \) in \( (\mathbb{C}^2)^{[i]} / \mathbb{C}^2 \). Since \( \pi : (\mathbb{C}^2)^{[i]} \to S^i \mathbb{C}^2 \) has irreducible fibers, \( [\pi^{-1}(i[0])] \) is the generator of \( H_{2i-2}((\mathbb{C}^2)^{[i]} / \mathbb{C}^2) \). Now our assertion follows from a general result which holds for any semi-small morphism [CG, 7.7.15]. That is the non-degeneracy of the intersection form on the top degree of the fiber.
Appendix A. Hilbert Schemes of Points on the Plane

In this appendix, we describe the Hilbert scheme \((\mathbb{C}^2)^{[n]}\), or more generally framed moduli spaces of torsion free sheaves on \(\mathbb{C}P^2\), as a hyper-Kähler quotient of a finite dimensional vector space with respect to a unitary group action, and then compute its homology.

A(i). The framed moduli space of torsion free sheaves on \(\mathbb{C}P^2\). Let \(V\) and \(W\) be vector spaces over the complex field whose dimensions are \(n\) and \(r\). Let

\[
M \overset{\text{def}}{=} \{(B_1, B_2, i, j) \mid B_1, B_2 \in \text{Hom}(V, V), i \in \text{Hom}(W, V), j \in \text{Hom}(W, V)\}.
\]

Consider the following complex ADHM equation:

\[
\mu_C(B_1, B_2, i, j) \overset{\text{def}}{=} [B_1, B_2] + ij = 0.
\]

Let \([z_0 : z_1 : z_2]\) be the homogeneous coordinates of \(\mathbb{C}P^2\). We consider the following homomorphisms of sheaves over \(\mathbb{C}P^2\)

\[
V \otimes \mathcal{O}(-1) \overset{\sigma}{\rightarrow} (V \oplus V \oplus W) \otimes \mathcal{O} \overset{\tau}{\rightarrow} V \otimes \mathcal{O}(1),
\]

where

\[
\sigma = \begin{pmatrix} B_1 \otimes z_0 - 1_V \otimes z_1 \\ B_2 \otimes z_0 - 1_V \otimes z_2 \\ j \otimes z_0 \end{pmatrix}, \quad \tau = \begin{pmatrix} - (B_2 \otimes z_0 - 1_V \otimes z_2) & B_1 \otimes z_0 - 1_V \otimes z_1 & i \otimes z_0 \end{pmatrix}.
\]

Here \(1_V\) denotes the identity map of \(V\). By the complex ADHM equation, this is a complex, that is \(\tau \sigma = 0\).

If \(\sigma\) is injective and \(\tau\) is surjective as sheaf homomorphisms, then \(E \overset{\text{def}}{=} \ker \tau / \text{im} \tau\) is a torsion free sheaf on \(\mathbb{C}P^2\) with

\[
\text{rank } E = r, \quad c_1(E) = 0, \quad c_2(E) = n.
\]

Remark that the injectivity of \(\sigma\) is weaker than the requirement that \(\sigma\) induces injective homomorphisms between stalks over any points. This stronger condition makes the resulting \(E\) locally free.

Over the line \(l = \{z_0 = 0\}\), the sheaf \(E\) is naturally identified with \(W \otimes \mathcal{O}_l\).

Conversely suppose we are given a torsion free sheaf \(E\) over \(\mathbb{C}P^2\) which has a trivialization \(E|_l \cong W \otimes \mathcal{O}_l\). We then define a vector space \(V\) by

\[
V \overset{\text{def}}{=} H^1(\mathbb{C}P^2, E(-2)).
\]
Using the vanishing of $H^0(l, E(-1))$ and $H^1(l, E(-1))$, we obtain the isomorphism $H^1(\mathbb{C}P^2, E(-2)) \cong H^1(\mathbb{C}P^2, E(-1))$. Thus the multiplications of $z_1$, $z_2$ define endomorphisms $B_1$, $B_2$ of $V$. We define $i$ and $j$ as natural homomorphisms

\[ i: W \cong H^0(l, E) \to V \cong H^1(\mathbb{C}P^2, E(-1)), \]

\[ j: V \cong H^1(\mathbb{C}P^2, E(-2)) \to W \cong H^1(l, E(-2)), \]

which are induced from the short exact sequences of sheaves

\[ 0 \to O(-1) \to O \to O_l \to 0, \]

\[ 0 \to O(-3) \to O(-2) \to O_l(-2) \to 0. \]

Then we have the following [OSS]:

**Theorem A.1.** Let $\mathcal{M}(r, 0, n)$ be the moduli space of torsion free sheaves $E$ over $\mathbb{C}P^2$ with a trivialization $E|_l \cong W \otimes O_l$ which has $c_1(E) = 0$, $c_2(E) = n$. The above correspondence defines a bijection between $\mathcal{M}(r, 0, n)$ and the quotient of the set of $(B_1, B_2, i, j) \in \mathcal{M}$ satisfying

1. (complex ADHM equation) $\mu_C(B_1, B_2, i, j) = [B_1, B_2] + ij = 0$
2. $\sigma$ is injective and $\tau$ is surjective,

by the natural action of $\text{GL}(V)$. Here the Chern classes are given by $c_1(E) = 0$, $c_2(E) = \dim V$.

We can replace the second condition by what is similar to one introduced in [Na1, 3.5], [Na5, 3.8]. (In fact, the condition used there is the ‘transpose’ of the following.) The proof is easy and omitted.

**Lemma A.2.** The injectivity of $\sigma$ holds always and the surjectivity of $\tau$ is equivalent to saying that there exists no proper subspace $S$ of $V$ such that $B_k(S) \subset S$ ($k = 1, 2$) and $i(W) \subset S$.

Now we put hermitian metrics on $V$ and $W$. Then $\mathcal{M}$ is regarded as a Kähler manifold and we can consider the real moment map with respect to the natural action by $U(V)$:

\[ \mu_\mathbb{R}(B_1, B_2, i, j) \overset{\text{def}}{=} [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger - j^\dagger j, \]

where $(\cdot)^\dagger$ is the hermitian adjoint.

Thanks to the above lemma, the following can be proved exactly as in [Na1, 3.5]:

**Proposition A.3.** Fix a negative real number $\zeta_\mathbb{R}$. Then the condition (2) in Theorem A.1 holds if and only if there exists $g \in \text{GL}(V)$ such that

\[ \mu_\mathbb{R}(gB_1g^{-1}, gB_2g^{-1}, gi, jj^{-1}) = -\zeta_\mathbb{R}. \]

Moreover such a $g$ is unique up to $U(V)$. 
Hence the moduli space can be identified with
\[ \mu_C^{-1}(0) \cap \mu_R^{-1}(-\zeta_R) / U(V). \]
The map \( \mu = (\mu_R, \mu_C) \) is a hyper-Kähler moment map in the sense of [HKLR], where the vector space \( M \) has a structure of a \( \mathbb{H} \)-module by
\[ J(B_1, B_2, i, j) \overset{\text{def}}{=} (-B_1^\dagger, B_2^\dagger, -j^\dagger, i^\dagger). \]
Thus the moduli space \( \mathcal{M}(r, 0, n) \) is a hyper-Kähler quotient.

Since the action of \( U(V) \) on \( \mu_C^{-1}(0) \cap \mu_R^{-1}(-\zeta_R) \) is free, we have (cf. [Na3, 1.3])

**Theorem A.4.** The moduli space \( \mathcal{M}(r, 0, n) \) is a smooth hyper-Kähler manifold of real dimension \( 4nr \). Moreover the metric is complete.

We would like to remark the relation with the original ADHM description. This was already explained in [Na3] in more detail. So our description is sketchy. The framed moduli space of SU\((r)\)-instanton with \( c_2 = n \) is described as a hyper-Kähler quotient
\[ \{ (B_1, B_2, i, j) \in \mu_C^{-1}(0) \cap \mu_R^{-1}(0) \mid \text{stabilizer in } U(V) \text{ is trivial} \} / U(V). \]
This is also a smooth hyper-Kähler manifold, but the metric is not complete. The metric completion \( \mu_C^{-1}(0) \cap \mu_R^{-1}(0) / U(V) \) is isomorphic to the framed moduli space of ideal instantons, or Uhlenbeck’s (partial) compactification. This space has singularities, and there exists a natural morphism from our space \( \mathcal{M}(r, 0, n) \), which is a resolution. These two moduli spaces, of torsion free sheaves and of ideal instantons, naturally appear for more general projective surfaces. The construction of the morphism for general cases was done by J. Li [Li].

Suppose \( r = 1 \). The double dual \( E^\vee \vee \) of a rank 1 torsion-free sheaf \( E \) is locally free and has \( c_1(E^\vee \vee) = 0 \). Hence \( E^\vee \vee = \mathcal{O} \) and \( E \) is an ideal \( \mathcal{J} \) of colength finite. This shows that \( \mathcal{M}(1, 0, n) \) is isomorphic to the Hilbert scheme \( (\mathbb{C}^2)^n \).

**Remark A.6.** (1) Note that the \( K3^{[n]} \) was the first example of higher dimensional compact hyper-Kähler manifold given by Fujiki and Beauville [Be]. It seems natural to conjecture that \( X^{[n]} \) has a hyper-Kähler structure if \( X \) has a hyper-Kähler structure. For example, when \( X \) is an ALE space, this is true. When \( X^{[n]} \) is projective, the existence of a holomorphic symplectic structure is enough thanks to the solution of the Calabi conjecture by Yau [Ya]. Although there are some extensions of Calabi conjectures to noncompact manifolds (e.g., [BK, TY]), our case \( (\mathbb{C}^2)^n \) is uncovered probably. This is because our manifolds do not have quadratic curvature decay, and the cones of the ends are not smooth manifolds.

(2) In [KN], the ADHM description for instantons on ALE spaces was given. This is similar to (A.5), but the vector space \( V \) now becomes a representation of quivers of affine type. The modification “\( \mu_R^{-1}(0) \rightarrow \mu_R^{-1}(-\zeta_R) \)” used in this appendix can be
also applied to the case of ALE spaces. It gives the ADHM description of torsion free sheaves, instead of instantons. In fact, the corresponding moduli spaces were already studied in [Na1]. More precisely, the parameter $\zeta$ used in [KN] was tracefree, while one used in [Na1] was not. Thus the action of affine Lie algebra, constructed in [Na1], is not on the homology of moduli spaces of instantons, but of torsion free sheaves. In [Na4], we persisted in genuine vector bundles and we give only representations of finite dimension Lie algebras.

(3) There is yet another modification of (A.5). In stead of changing the value of the real moment map $\mu_R$, we can change the value of the complex moment map to get $\mu_R^{-1}(0) \cap \mu_C^{-1}(-\zeta_C) / U(V)$. It is a deformation of $\mu_R^{-1}(-\zeta_R) \cap \mu_C^{-1}(0) / U(V)$. Changing the identification $\mathbb{R}^4 \cong \mathbb{C}^2$, we can identify this space also with the moduli space of torsion free sheaves.

**A(ii). Topology of $(\mathbb{C}^2)^{[n]}$.** Our next task is to compute the homology group of the Hilbert scheme $(\mathbb{C}^2)^{[n]}$.

We consider $(\mathbb{C}^2)^{[n]}$ as a hyper-Kähler quotient as in the previous section. As in [ES, Na2], the idea is to use the torus action given by

$$(B_1, B_2, i, j) \mapsto (t_1 B_1, t_2 B_2, t_1 i, t_2 j) \quad (t_1, t_2) \in S^1 \times S^1.$$ 

This action commutes with the $U(V)$-action and induces an action on $M(1,0,n)$. If $U(V). (B_1, B_2, i, j) \in M(1,0,n)$ is a fixed point of the torus action, there exists a homomorphism $\lambda: \mathbb{C}^* \to GL(V)$ such that

$$(t_1 B_1, t_2 B_2, t_1 i, t_2 j) = (\lambda(t_1, t_2) B_1 \lambda(t_1, t_2)^{-1}, \lambda(t_1, t_2) B_2 \lambda(t_1, t_2)^{-1}, \lambda(t_1, t_2) i, j \lambda(t_1, t_2)^{-1}).$$

If $V = \bigoplus V(k,l)$ is the weight space decomposition, the above equation can be written as

$$B_1(V(k,l)) \subset V(k+1,l), \quad B_2(V(k,l)) \subset V(k,l+1),$$

$$i(W) \subset V(1,0), \quad j(V(k,l)) = 0 \text{ unless } (k,l) = (0,-1).$$

Since $\bigoplus_{k \geq 1, l \geq 0} V(k,l)$ contains $\text{Im } i$ and $B_k$-invariant, the condition in Lemma A.2 implies it is equal to $V$. In particular, $j = 0$. The situation is visualized as

$$\begin{array}{c}
\mathbb{C} = W \xrightarrow{i} V(1,0) \xrightarrow{B_1} V(2,0) \xrightarrow{B_1} V(3,0) \xrightarrow{B_1} \cdots \\
\downarrow \quad \downarrow \quad \downarrow \\
V(1,1) \xrightarrow{B_1} V(2,1) \xrightarrow{B_1} V(3,1) \xrightarrow{B_1} \cdots \\
\downarrow \quad \downarrow \quad \downarrow \\
\vdots \quad \vdots \quad \vdots
\end{array}$$
The easy argument using the induction on $k$ or $l$ shows

**Lemma A.7.**  
(1) $\dim V(k,l) = 1$ or $0$,  
(2) $\sum_l \dim V(k,l)$ is monotone non-increasing with respect to $k$.

The fixed point is uniquely determined by $\dim V(k,l)$, and hence by the partition  
\[(\sum_l \dim V(1,l), \sum_l \dim V(2,l), \ldots)\]
of $n$.

The map  
\[F(U(V)_*(B_1, B_2, i, j)) \overset{\text{def}}{=} \|B_1\|^2 + \varepsilon \|B_2\|^2 + \|i\|^2 + \varepsilon \|j\|^2\]
is the moment map for the torus action coupled with a certain element in the Lie algebra of the torus. If we take $\varepsilon > 0$ sufficiently small and generic, the critical point $p$ of $F$ is a fixed point. Then the tangent space $T_p((\mathbb{C}^2)^n)$ is a torus module and the Morse index of $F$ at $p$ is given by the sum of the dimension of weight spaces $T(k,l)$ such that (a) $k < 0$, or (b) $k = 0$ and $l < 0$. Calculating the dimension, we find  
\[
\text{Index of } F \text{ at } p = 2 \sum_{l \geq 1} \sum_k \dim V(k,l) = 2n - 2 \sum_k \dim V(k,0).
\]

If $p$ corresponds to a partition of $n$ as above, this index is equal to  
\[2n - 2(\text{number of parts}).\]

Thus we get

**Theorem A.8.** The homology group $H_*(((\mathbb{C}^2)^n])$ has no torsion and vanishes in odd degrees. The Betti number of $(\mathbb{C}^2)^n]$ is given by  
\[b_{2i}((\mathbb{C}^2)^n]) = \text{number of partitions of } n \text{ into } n - i \text{ parts.}\]

**Corollary A.9.** If $P_t((\mathbb{C}^2)^n]) = \sum t^i b_i((\mathbb{C}^2)^n])$ denotes the Poincaré polynomial, its generating function is given by  
\[
\sum_{n \geq 0} q^n P_t((\mathbb{C}^2)^n]) = \prod_{m=1}^\infty \frac{1}{1 - t^{2m-2} q^m}.
\]
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