Research Article

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On a comparison theorem for parabolic equations with nonlinear boundary conditions

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Abstract: In this article, a new type of comparison theorem for some second-order nonlinear parabolic systems with nonlinear boundary conditions is given, which can cover classical linear boundary conditions, such as the homogeneous Dirichlet or Neumann boundary condition. The advantage of our comparison theorem over the classical ones lies in the fact that it enables us to compare two solutions satisfying different types of boundary conditions. As an application of our comparison theorem, we can give some new results on the existence of blow-up solutions of some parabolic equations and systems with nonlinear boundary conditions.

Keywords: comparison theorem, nonlinear boundary conditions, blow up

MSC 2020: 35B51, 35K51, 35K57

1 Introduction

Mathematical models for various types of phenomena arising from physics, chemistry, biology and so on are often described as reaction diffusion equations which give typical examples of second-order nonlinear parabolic equations. It is widely known that comparison theorems yield very powerful tools for analyzing the second-order parabolic equations, e.g., for constructing super-solutions or sub-solutions, and for examining the asymptotic behavior of solutions. On the other hand, when one fixes right boundary conditions for the heat equations, it should be noted that if no artificial control of flux is given on the boundary, it is natural to consider the nonlinear boundary conditions from a physical point of view (cf. the Stefan-Boltzmann law). However, most of the existing results on comparison theorems for nonlinear diffusion equations deal with the standard linear boundary conditions such as Dirichlet or Neumann boundary conditions (see [1]).

Furthermore, all of them are concerned with the case in which the two solutions satisfying the same type of boundary conditions are compared. (In Bénilan and Díaz [2], a comparison theorem covering nonlinear boundary conditions is announced, but there it is assumed that compared two solutions satisfy the same nonlinear boundary condition.) Our comparison theorem, as described below, provides a new device which enables us to compare two solutions satisfying different types of (nonlinear) boundary conditions, e.g., we can compare two solutions satisfying the same equations but different types of

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boundary conditions, for instance, the homogeneous Dirichlet and Neumann boundary conditions (see Proposition 3.5).

The main objective of this article is to give a comparison theorem for a rather wide class of nonlinear systems of reaction diffusion equations with different types of nonlinear boundary conditions, i.e., the following system of equations for $U = (u^1, u^2, \ldots, u^m)$ given by

$$
\begin{aligned}
(P) \\
\frac{\partial u^k}{\partial t} - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} \left( a^k_{ij}(t, x) \frac{\partial u^k}{\partial x_i} \right) + \beta^k(t, x, u^t) - F^k(t, x, U) \equiv 0, \quad (t, x) \in Q_T, \\
- \sum_{i,j=1}^{N} a^k_{ij}(t, x) \frac{\partial u^k}{\partial x_i} \in y^k(t, x, u^t), \\
u^k(0, x) = a^k(x),
\end{aligned}
$$

where $\Omega$ is a general domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$, $Q_T = (0, T) \times \Omega$, $\Gamma_t = (0, T) \times \partial \Omega$, $v = v(x) = (v_1, \ldots, v_N)$ is the unit outward vector at $x \in \partial \Omega$, $u^k : Q_T \rightarrow \mathbb{R}$ $(k = 1, 2, \ldots, m)$ are the unknown functions.

As for the coefficients $a^k_{ij}$ $(k = 1, 2, \ldots, m)$, we assume

$$
\exists \lambda^k \geq 0 \quad \text{such that} \quad \lambda^k |\xi|^2 \leq \sum_{i,j=1}^{N} a^k_{ij}(t, x) \xi_i \xi_j \quad \forall \xi \in \mathbb{R}^N, \quad \text{a.e.} \quad (t, x) \in Q_T, \quad (1.1)
$$

We also assume that $F^k : Q_T \times \mathbb{R}^m \rightarrow \mathbb{R}^1$ $(k = 1, 2, \ldots, m)$ are (possibly multi-valued) nonlinear mappings; $\beta^k(t, x, \cdot)$ and $y^k(t, x, \cdot)$ $(k = 1, 2, \ldots, m)$ are maximal monotone graphs on $\mathbb{R}^1 \times \mathbb{R}^i$ for a.e. $(t, x)$. More precisely, there exist lower semi-continuous convex functions $j^k(t, x, r) : \mathbb{R} \times \mathbb{R} \rightarrow (-\infty, +\infty]$ and $\eta^k(t, x, r) : \mathbb{R} \times \mathbb{R} \rightarrow (-\infty, +\infty]$ such that $y^k = \partial j^k$ and $\beta^k = \partial \eta^k$, respectively. Here $\partial j^k$ and $\partial \eta^k$ denote subdifferentials of $j^k$ and $\eta^k$ with respect to $r \in \mathbb{R}$, respectively. We here present the notion of maximal monotone operators. Let $H$ be a real Hilbert space with inner product $(\cdot, \cdot)_H$ and let $A : H \rightarrow 2^H$ be a multi-valued mapping with domain $D(A) = \{u \in H; Au \neq \emptyset\}$. Then $A$ is said to be monotone if its graph $G(A) = \{[x, y]; x \in D(A), y \in Ax\}$ satisfies the following:

$$(u - v, \xi - \zeta)_H \geq 0, \quad \forall [u, \xi], \quad \forall [v, \zeta] \in G(A).$$

Moreover, a monotone operator $A$ is said to be maximal monotone, if $G(A)$ is not strictly contained in the graph of any other monotone operator. We also touch on the notion of subdifferential operators. Let $\Phi(H)$ be the set of all proper lower semicontinuous convex functionals $\phi : H \rightarrow (-\infty, +\infty]$, where proper means $D(\phi) = \{u \in H; \phi(u) < +\infty\} \neq \emptyset$, namely, $\phi \not\equiv +\infty$. For $\phi \in \Phi(H)$ and for each $u \in D(\phi)$, the subdifferential $\partial \phi(u)$ of $\phi$ at $u$ is defined by

$$
\partial \phi(u) = \{f \in H; \phi(v) - \phi(u) \geq (f, v - u)_H, \quad \forall v \in D(\phi)\}.
$$

Then $\partial \psi : H \rightarrow 2^H$ gives a possibly multivalued operator with domain $D(\partial \psi) = \{u \in D(\phi); \partial \psi(u) \neq \emptyset\}$, which is called by subdifferential operator. It is well known that the subdifferential operator becomes a maximal monotone operator on $H$ (see [3–5]).

The problem with this type of boundary conditions appears in models describing diffusion phenomena taking into consideration some nonlinear radiation law on the boundary (see Brézis [5] and Barbu [3]) and the solvability for (P) is examined in detail under various settings (see [3–6]). Here, we emphasize that the boundary condition in our setting can cover standard linear boundary conditions such as the homogeneous Dirichlet boundary condition and the homogeneous Neumann boundary condition. In fact for the simple case $-\frac{\partial u}{\partial v} = -\nabla u \cdot v = y(u)$, the homogeneous Neumann boundary condition can be described by taking $y(u) = 0$, the homogeneous Dirichlet boundary condition by taking $y(u) = 0$ with $y(u) = \mathbb{R}^1$ (multivalued) for $u = 0$ and $y(u) = \emptyset$ for $u \neq 0$ (see (3.18) and [3–5]). In this way, the multivaluedness of $y(\cdot)$
arises naturally in our setting, by which we are motivated to deal with multivalued mappings in the boundary conditions of \((P)\). In this article, we work with solutions of \((P)\) in the following sense.

**Definition 1.1.** A function \(U = (u^1, u^2, \ldots, u^m) : [0, T] \rightarrow \mathbb{R}^m\) is called a super-solution (resp. sub-solution) of \((P)\) on \([0, T]\) if and only if for all \(k \in \{1, 2, \ldots, m\},\)

\[
u^k \in C([0, T]; L^2(\Omega)) \cap L^\infty([0, T]; L^\infty(\Omega)) \cap W_{loc}^{1,2}(0, T; L^2(\Omega)) \cap L^2_{loc}(0, T; H^2(\Omega)),
\]

and there exist sections \(f^k, b^k, g^k \in L^2_{loc}((0, T]; L^2(\Omega))\) of \(F^k(t, x, U(t, x)), \beta^k(t, x, u^k(t, x))\) and \(g^k \in L^2_{loc}((0, T]; L^2(\partial\Omega))\) of \(y^k(t, x, u^k(t, x))\) satisfying \((P)\), i.e.,

\[
\begin{aligned}
\frac{\partial u^k}{\partial t} - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left( a_{ij}^k(t, x) \frac{\partial u^k}{\partial x_i} \right) + b^k(t, x) - f^k(t, x) \geq 0 \quad \text{(resp.} \leq 0) ,
\end{aligned}
\]

\[
\begin{aligned}
f^k(t, x, U) \in F^k(t, x, U(t, x)), \quad b^k(t, x) \in \beta^k(t, x, u^k(t, x)), \quad \text{a.e.} \quad (t, x) \in Q_T ,
\end{aligned}
\]

\[
\begin{aligned}
- \sum_{i,j=1}^{N} a_{ij}^k(t, x) \eta_{ij} \frac{\partial u^k}{\partial x_i} \leq g^k(t, x) \quad \text{(resp.} \geq),
\end{aligned}
\]

\[
\begin{aligned}
g^k(t, x) \in y^k(t, x, u^k(t, x)) \quad \text{a.e.} \quad (t, x) \in Q_T ,
\end{aligned}
\]

If \(U\) is a super- and sub-solution of \((P)\) on \([0, T]\) with the same sections \(f^k, b^k, g^k\), then \(U\) is called a solution of \((P)\) on \([0, T]\).

We also define the maximal existence time \(T_m(T)\) of a solution \(U\) by

\[
T_m(U) := \sup\{T > 0; \quad U \quad \text{is extended to} \quad [0, T] \quad \text{as a solution of} \quad (P) \quad \text{in the sense above}\}. 
\]

**Remark 1.2.** When the existence of solution is concerned, the assumption \(D(\beta^k) \cap D(y^k) \neq \emptyset\) is usually required for each \(k\) [3,5]. However, we do not apparently need this assumption to derive our comparison theorem, since the existence of solutions satisfying (1.3) is always assumed in our setting.

## 2 Main theorem and its proof

In this section, we state our comparison theorem for \((P)\) and give a proof of it. The idea of proof is standard and elementary, however, this type of comparison theorem can cover various types of nonlinear parabolic equations including those with classical linear boundary conditions. The applicability of this comparison theorem will be exemplified in the next section.

Consider the following two systems of equations:

\[
(P_1) \begin{cases}
\frac{\partial u^k}{\partial t} - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left( a_{ij}^k(t, x) \frac{\partial u^k}{\partial x_i} \right) + \beta_1^k(t, x, u^k) - F_1^k(t, x, U) \geq 0, \quad t > 0, \quad x \in \Omega, \\
- \sum_{i,j=1}^{N} a_{ij}^k(t, x) \eta_{ij} \frac{\partial u^k}{\partial x_i} \in y_1^k(t, x, u^k), \quad t > 0, \quad x \in \partial\Omega,
\end{cases}
\]

\[
\begin{aligned}
u^k(0, x) = a_1^k(x), \quad x \in \Omega,
\end{aligned}
\]

and

\[
(P_2) \begin{cases}
\frac{\partial u^k}{\partial t} - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left( a_{ij}^k(t, x) \frac{\partial u^k}{\partial x_i} \right) + \beta_2^k(t, x, u^k) - F_2^k(t, x, U) \geq 0, \quad t > 0, \quad x \in \Omega, \\
- \sum_{i,j=1}^{N} a_{ij}^k(x) \eta_{ij} \frac{\partial u^k}{\partial x_i} \in y_2^k(t, x, u^k), \quad t > 0, \quad x \in \partial\Omega,
\end{cases}
\]

\[
\begin{aligned}
u^k(0, x) = a_2^k(x), \quad x \in \Omega,
\end{aligned}
\]
Theorem 2.1. Let $U_1 = (u_1^1, u_1^2, \ldots, u_1^n)$ be a sub-solution of $(P)_1$ on $[0, T]$ and $U_2 = (u_2^1, u_2^2, \ldots, u_2^n)$ be a super-solution of $(P)_2$ on $[0, T]$, and let the following assumptions (A1)–(A4) be satisfied.

(A1) \( a_k^1(x) \leq a_k^2(x) \) a.e. \( x \in \Omega \) for all \( k \in \{1, 2, \ldots, m\} \).

(A2) For each \( k \in \{1, 2, \ldots, m\} \), one of the following (i)–(ii) holds true.

(i) \( \beta_k^1(t, x, \cdot) = \beta_k^2(t, x, \cdot) \) a.e. \( (t, x) \in Q_T \).

(ii) \( \sup\{b_1^2; b_2^2 \} \leq \inf\{b_1^1; b_2^1 \} \leq \sup\{b_1^1; b_2^1 \} \) a.e. \((t, x, r) \in Q_T \); \( \forall r_1 \in D(\beta_k^1(t, x, \cdot)), \forall r_2 \in D(\beta_k^2(t, x, \cdot)) \) with \( r_1 > r_2 \) a.e. \((t, x) \in Q_T \).

(A3) For each \( k \in \{1, 2, \ldots, m\} \), one of the following (i)–(iii) holds true.

(i) \( \gamma_k^1(t, x, \cdot) = \gamma_k^2(t, x, \cdot) \) a.e. \( (t, x) \in \Gamma_T \).

(ii) \( \sup\{g_1^2; g_2^2 \} \leq \inf\{g_1^1; g_2^1 \} \leq \sup\{g_1^1; g_2^1 \} \) a.e. \((t, x, r) \in \Gamma_T \); \( \forall r_1 \in D(\gamma_k^1(t, x, \cdot)), \forall r_2 \in D(\gamma_k^2(t, x, \cdot)) \) with \( r_1 > r_2 \) a.e. \((t, x) \in \Gamma_T \).

(iii) \( r_1^2 \leq r_2^2 \) \( \forall r_1 \in D(\gamma_k^1(t, x, \cdot)), r_2 \in D(\gamma_k^2(t, x, \cdot)) \) a.e. \((t, x) \in \Gamma_T \).

(A4) For each \( k \in \{1, 2, \ldots, m\} \), the following (i) and (ii) hold true.

(i) \( -\infty < \sup\{z; z \in F_k^1(t, x, U) \} \leq \inf\{z; z \in F_k^2(t, x, U) \} < +\infty \) a.e. \((t, x, U) \in Q_T \times \mathbb{R}^m \).

(ii) \( F_k^1(t, x, \cdot) \) or \( F_k^2(t, x, \cdot) \) is single-valued and satisfies the following structure condition $(SC)$ with \( F^k \) replaced by \( F_k^1 \) or \( F_k^2 \).

(SC) \( F_k(t, x, U) \) is differentiable for almost all \( U \in \mathbb{R}^m \) and satisfies

\[
\frac{\partial}{\partial u_j} F_k(t, x, U) \geq 0 \quad \text{for all } j \neq k \quad \text{for a.e. } (t, x, U) \in Q_T \times \mathbb{R}^m
\]  

(2.1)

and for any \( M > 0 \) there exists \( L_M > 0 \) such that

\[
\sup \left\{ \frac{\partial}{\partial u_j} F_k(t, x, U) \right\} \leq L_M.
\]  

(2.2)

Then, we have

\[
u_k^1(t, x) \leq u_k^1(t, x) \quad \forall k \in \{1, 2, \ldots, m\}, \quad \forall t \in [0, T], \quad \text{a.e. } x \in \Omega.
\]  

(2.3)

Proof. Let \( f_k^1, b_k^1, g_k^1 \) be the sections of \( F_k^1(U_1), \beta_k^1(u_1^1, \cdot), \gamma_k^1(u_1^1, \cdot) \) appearing in $(P)_1$, then \( w^k = u_k^1 - u_k^2 \) satisfies

\[
\begin{cases}
\partial_t w^k - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left( a_k^i(t, x) \frac{\partial w^k}{\partial x_i} \right) + b^k_1 - b^k_2 \leq f^1_k(U_1) - f^2_k(U_2), & (t, x) \in Q_T, \\
- \sum_{i,j=1}^N a_k^j(t, x) \frac{\partial w^k}{\partial x_j} \geq g^k_1 - g^k_2, & (t, x) \in Q_T, \\
w^k(0, x) = a_k^1(x) - a_k^2(x), & x \in \Omega.
\end{cases}
\]  

(2.4)

Multiplying (2.4) by \( (w^k)^+ = \max(w^k, 0) \), we have

\[
\int_{\Omega} \partial_t w^k (w^k)^+ dx - \int_{\Omega} \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left( a_k^i(t, x) \frac{\partial w^k}{\partial x_i} \right) (w^k)^+ dx + \int_{\Omega} (b^k_1 - b^k_2)(w^k)^+ dx \leq \int_{\Omega} (f^1_k(U_1) - f^2_k(U_2))(w^k)^+ dx.
\]

Here, we obtain

\[
\int_{\Omega} \partial_t w^k (w^k)^+ dx = \int_{\Omega} \partial_t w^k w^k dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} |w^k|^2 dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} (w^k)^2 dx,
\]
and by (1.1)

\[- \int_{\Omega} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}(a_{ij}^k(t, x) \frac{\partial w^k}{\partial x_i})(w^k) dx = \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}^k(t, x) \frac{\partial w^k}{\partial x_i} \frac{\partial(w^k)^+}{\partial x_j} dx - \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}^k(t, x) v_j \frac{\partial w^k}{\partial x_i} (w^k)^+ d\sigma \]

\[\geq \int_{|w^k(t, x)| \geq \epsilon} \sum_{i,j=1}^{N} a_{ij}^k(t, x) \frac{\partial w^k}{\partial x_i} \frac{\partial w^k}{\partial x_j} dx + \int_{\Omega} (g_1^k - g_2^k)(w^k)^+ d\sigma \]

\[= \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}^k(t, x) \frac{\partial(w^k)^+}{\partial x_i} \frac{\partial(w^k)^+}{\partial x_j} dx + \int_{\Omega} (g_1^k - g_2^k)(w^k)^+ d\sigma \]

\[\geq \lambda^k \int_{\Omega} \sum_{i,j=1}^{N} \frac{\partial(w^k)^+}{\partial x_i} \frac{\partial(w^k)^+}{\partial x_j} dx + \int_{\Omega} (g_1^k - g_2^k)(w^k)^+ d\sigma. \]

Hence, we have

\[\frac{1}{2} \frac{d}{dt} \| (w^k)^+ (t) \|_{L^2}^2 + \int_{\Omega} (g_1^k - g_2^k)(w^k)^+ d\sigma + \int_{\Omega} (b_1^k - b_2^k)(w^k)^+ dx \leq \int_{\Omega} (f_1^k(U_1) - f_2^k(U_2))(w^k)^+ dx. \]  

(2.5)

Here we are going to show that

\[L_{\Omega} = \int_{\Omega} (g_1^k - g_2^k)(w^k)^+ d\sigma = \int_{[u_1^k = u_2^k]} (g_1^k - g_2^k)(u_1^k - u_2^k) d\sigma \geq 0. \] 

(2.6)

In fact, if (i) of (A3) is satisfied, then (2.6) is derived from the monotonicity of \( y^k \), and (iii) of (A3) implies \( (w^k)^+ |_{\Omega} = 0 \), which leads to \( L_{\Omega} = 0 \). As for the case where (ii) of (A3) is satisfied, \( u_1^k > u_2^k \) and \( g_1^k \in y^k(u_1^k), g_2^k \in y^k(u_2^k) \) imply that

\[(g_1^k - g_2^k)(u_1^k - u_2^k) \geq 0,\]

whence follows \( L_{\Omega} \geq 0 \).

In the same way as above, from (A3) we derive

\[\int_{\Omega} (b_1^k - b_2^k)(w^k)^+ dx \geq 0. \] 

(2.7)

Here we consider the case where \( F_1^k \) is singleton and satisfies (SC) with \( F^k \) replaced by \( F_1^k \).

Then by (i) of (A4) we obtain

\[\int_{\Omega} (f_1^k(U_1) - f_2^k(U_2))(w^k)^+ dx = \int_{\Omega} (F_1^k(U_1) - F_2^k(U_2))(w^k)^+ dx \]

\[= \int_{\Omega} (F_1^k(U_1) - F_2^k(U_2))(w^k)^+ dx + \int_{\Omega} (F_1^k(U_2) - F_2^k(U_2))(w^k)^+ dx \]

\[\leq \int_{\Omega} (F_1^k(U_1) - F_2^k(U_2))(w^k)^+ dx. \] 

(2.8)

Furthermore by virtue of (SC), there exists some \( \theta \in (0, 1) \) such that

\[I_1^k := \int_{\Omega} (F_1^k(U_1) - F_1^k(U_2))(w^k)^+ dx \]

\[= \int_{\Omega} \sum_{j=1}^{m} \frac{\partial}{\partial u_j} F_1^k(U_1 + \theta(U_1 - U_2))w^k dx \]

\[= \int_{\Omega} \sum_{j=1}^{m} \frac{\partial}{\partial u_j} F_1^k(U_2 + \theta(U_1 - U_2))w^k dx \]

\[= \int_{\Omega} m \frac{\partial}{\partial u_j} F_1^k(U_2 + \theta(U_1 - U_2))w^k dx. \]
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \sum_{k=1}^{m} \| (w)^{(k)}(t) \|_{L^2}^2 & \leq L_M \left( \sum_{k=1}^{m} \| (w)^{(k)}(t) \|_{L^2}^2 \right)^2 \\
& \leq 2 L_M \sum_{k=1}^{m} \| (w)^{(k)}(t) \|_{L^2}^2 \quad \forall t \in (0, T).
\end{align*}
\]

Thus, in view of (2.5), (2.6), (2.7), and (2.9), we finally obtain

\[
\sum_{k=1}^{m} \| (w)^{(k)}(t) \|_{L^2}^2 \leq \sum_{k=1}^{m} \| (w)^{(k)}(s) \|_{L^2}^2 e^{2L_M (t-s)} \quad 0 < s < t < T.
\]

Since \( w^k \in C([0, T]; L^2(\Omega)) \), letting \( s \to 0 \), we obtain by (A1)

\[
\sum_{k=1}^{m} \| (w)^{(k)}(t) \|_{L^2}^2 \leq \sum_{k=1}^{m} \| (a^{k}_1 - a^{k}_2)^{(k)} \|_{L^2}^2 e^{2L_M T} = 0 \quad \forall t \in [0, T],
\]

whence follows (2.3).

As for the case where \( F^k_1 \) is singleton and satisfies (SC) with \( F^k \) replaced by \( F^k_1 \), instead of (2.8) we can obtain

\[
\int_{\Omega} (f^k_1(U_1) - f^k_2(U_2))(w)^{(k)} dx \leq \int_{\Omega} (F^k_1(U_1) - F^k_2(U_2))(w)(w) dx.
\]

Then we can repeat the same argument as above with \( F^k_1 \) replaced by \( F^k_1 \).

\[
\square
\]

**Remark 2.2.**

1. If \( f^k_1(U_1) \leq f^k_2(U_2) \) is known \textit{a priori}, we need not assume (A4) for \( F^k_1 \) and \( F^k_2 \) in Theorem 2.1.
2. If \( g^k_1(U_1) \leq g^k_2(U_2) \) is known \textit{a priori}, we need not assume (A3) for \( y^k_1 \) and \( y^k_2 \) in Theorem 2.1.
3. If \( m = 1 \) in Theorem 2.1, then Assumption (2.1) is not needed.
4. When we discuss the existence of solutions for \( (P_i) \) (\( i = 1, 2 \)), we need to assume that \( \beta_i^k \) and \( \gamma_i^k \) are maximal monotone graphs. In Theorem 2.1, however, we need only the monotonicity of \( \beta_i^k \) and \( \gamma_i^k \), since the existence of solutions is always assumed in our setting.
5. The following condition gives a sufficient condition for (ii) of (A3).

\[
(iii) \begin{cases}
D(\gamma^k_1(t, x, \cdot)) \subset D(\gamma^k_2(t, x, \cdot)) & \text{a.e.} \ (t, x) \in \Gamma, \\
\inf_{g^k_1} g^k_1 \in \gamma^k_1(t, x, r) \geq \sup_{g^k_2} g^k_2 \in \gamma^k_2(t, x, r) & \forall r \in D(\gamma^k_1(t, x, \cdot)),
\end{cases}
\]

and the same assertion for (ii) of (A2) as above holds true.
3 Applications

In this section, we give a couple of examples of the application of our comparison theorem to some nonlinear problems. In particular, in Section 3.1, we give a simple proof of the existence of blowing-up solutions for nonlinear diffusion equations with nonlinear boundary conditions.

We also discuss in Section 3.2 the finite time blow up of solutions for a reaction diffusion system arising from a nuclear model with nonlinear boundary conditions, which consists of two equations possessing a nonlinear coupling term between two real-valued unknown functions.

3.1 Nonlinear heat equations with nonlinear boundary conditions

Consider the following nonlinear heat equations with nonlinear boundary conditions:

$$
\begin{align*}
\begin{cases}
\partial_t u_\gamma - \Delta u - F(u) & \geq 0, \quad t > 0, \quad x \in \Omega, \\
-\partial_t \gamma u & \geq 0, \quad t > 0, \quad x \in \partial \Omega,
\end{cases}
\end{align*}
$$

Here $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$ and $\partial_v$ denotes the outward normal derivative, i.e., $\partial_v u = \nabla \cdot u$. We further impose the following assumptions on $F$ and $\gamma$.

(a) $F : \mathbb{R}^1 \to 2^{\mathbb{R}^1}$ is a (possibly multi-valued) operator satisfying the following (i) and (ii).

(i) $0 \in F(0)$, \quad $\inf \{ z \in F(u) \} \geq |u|^{p-2}u$ \quad $\forall u \in \mathbb{R}^1$ \quad with $p > 2$, \quad (3.2)

(ii) $F(u) = F_0(u) + F^*_m(u) - F^*_m(u)$ \quad $\forall u \in \mathbb{R}^1$ \quad and

$F_0(\cdot)$ is singleton and locally Lipschitz continuous on $\mathbb{R}^1$, \quad (3.3)

(b) $\gamma : \mathbb{R}^1 \to 2^{\mathbb{R}^1}$ is a (possibly multi-valued) maximal monotone operator satisfying $0 \in \gamma(0)$.

In view of assumptions $0 \in F(0)$ and $0 \in \gamma(0)$, we immediately see that (3.1) possesses the trivial solution $v \equiv 0$ with sections $0 = f(v) \in F(v)$, $0 = g(v) \in \gamma(v)$. Let $u$ be any solution of (3.1) with $u_0(x) \geq 0$ with sections $f(u) \in F(u)$, $g(u) \in \gamma(u)$ satisfying the regularity required in Definition 1.1, whose existence is assured in Proposition 3.1, then applying Theorem 2.1 with $m = 1$, $F_1 = F_2 = F$, $\beta_1 = \beta_2 = 0$, $\gamma_1 = \gamma_2 = \gamma$; $a_1 = 0$, $a_2 = u_0$; $u_1 = v = 0$, $u_2 = u$, we conclude that $u \geq 0$ as far as $u$ exists. Here we use the fact that $0 = f(u) \leq \min \{ z \in F(u) \} \leq f(u_2)$ is assured a priori by (3.2) (see Remark 2.2).

Since we are here concerned with only non-negative solutions, the typical model of $F$ and $\gamma$ is given by $F(u) = |u|^{p-2}u$ and $\gamma(u) = |u|^{q-2}u$. For this special case, when $q < p$, i.e., the nonlinearity inside the region is stronger than that at the boundary, it might be straightforward to prove that there exist solutions of (3.1) which blow up in finite time by applying the same strategy as that in [7]. Although, it is difficult to apply such a method to (3.1) for the case where $q \geq p$, and to derive the existence of blow-up solutions for this case by using the variational structure, one would need some complicated classifications on parameters $(p, q)$ with heavy calculations (cf. [8]). We emphasize that our method for showing the existence of blow-up solutions relying on Theorem 2.1 provides us a much simpler device with wider applicability.

First we state the local existence result for (3.1).

**Proposition 3.1.** Let $u_0 \in L^\infty(\Omega)$, then there exists $T_0 = T_0(\|u_0\|_{L^\infty}) > 0$ such that (3.1) possesses a solution $u$ satisfying the following regularity:

$$
|u| \in C([0, T_0]; L^2(\Omega)) \cap L^\infty(0, T_0; L^\infty(\Omega)), \quad |\sqrt{t} \partial_t u|, \sqrt{t} \Delta u \in L^2(0, T_0; L^2(\Omega)). \quad (3.4)
$$
Moreover, let $T_m = T_m(u)$ be the maximal existence time of $u$, then the following alternative holds:

- $T_m = +\infty$ or
- $T_m < +\infty$, $\lim_{t \to T_m} \|u(t)\|_{L^\infty} = +\infty$.

**Proof.** Since $\gamma$ is assumed to be maximal monotone, there exists a lower semi-continuous convex function $j : \mathbb{R} \to (-\infty, +\infty)$ such that $j(r) \geq 0$, and $\partial j(u) = \gamma(u)$ (see [4]).

Define the functional $\varphi$ on $L^2(\Omega)$ by

$$\varphi(u) = \left\{ \begin{array}{ll}
\frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\Omega} |u|^2 \, dx + \int_{\partial \Omega} j(u) \, d\sigma & \text{if } u \in D(\varphi) = \{ u \in H^1(\Omega); j(u) \in L^1(\partial \Omega) \}, \\
\infty & \text{if } u \in L^2(\Omega) \setminus D(\varphi).
\end{array} \right.$$ 

Then we can see that $\varphi$ is a lower semi-continuous convex function on $L^2(\Omega)$ and the subdifferential operator $\partial \varphi$ associated with $\varphi$ is given as follows (see [3–5]):

$$\partial \varphi(u) = -\Delta u + u, \quad \partial \varphi(u) = \{ u \in H^2(\Omega); -\partial_j u(x) \in \gamma(u(x)) \text{ a.e. on } \partial \Omega \}.$$ 

Furthermore, the following elliptic estimate for $\partial \varphi$ holds, i.e., there exist some constants $c_1, c_2 > 0$ such that

$$|u|_{H^2} \leq c_1 + \|u\|_{H^1}^2 + c_2 \quad \forall u \in D(\partial \varphi). \quad (3.5)$$

Then by putting $B(u) = -u - F(u)$, (3.1) can be reduced to the following abstract evolution equation in $H = L^2(\Omega)$:

$$\begin{cases}
\frac{d}{dt} u(t) + \partial \varphi(u(t)) + B(u(t)) \ni 0, & t > 0, \\
u(0) = u_0.
\end{cases} \quad (CP)$$

In order to show the existence of time local solutions of $(P)^r$ belonging to $L^\infty(\Omega)$, we rely on “$L^\infty$-Energy Method” developed in [6]. To this end, we introduce another maximal monotone graph $\beta_M(\cdot) = \partial \eta_M(\cdot)$ on $\mathbb{R}^1 \times \mathbb{R}^1$ by

$$\beta_M(r) = \begin{cases}
\emptyset & |r| > M, \\
(-\infty, 0] & r = -M, \\
0 & |r| < M, \\
[0, +\infty) & r = M,
\end{cases} \quad \eta_M(r) = \begin{cases}
0 & |r| \leq M, \\
+\infty & |r| > M.
\end{cases}$$

The realizations of $\beta_M$ and $\eta_M$ in $H = L^2(\Omega)$ are given by

$$\beta_M(u) = \partial I_{K_M}(u) = \begin{cases}
\emptyset & |u(x)| > M, \\
(-\infty, 0] & u(x) = -M, \\
0 & |u(x)| < M, \\
[0, +\infty) & u(x) = M,
\end{cases} \quad I_{K_M}(u) = \begin{cases}
0 & u \in K_M = \{ u \in L^2(\Omega); |u(x)| \leq M \text{ a.e. } x \in \Omega \}, \\
+\infty & u \in L^2(\Omega) \setminus K_M.
\end{cases}$$

Here we put

$$\varphi_M(u) = \varphi(u) + I_{K_M}(u).$$

Then we can obtain

$$\partial \varphi_M(u) = \partial \varphi(u) + \beta_M(u) \quad \forall u \in D(\partial \varphi_M) = D(\partial \varphi) \cap K_M. \quad (3.6)$$
In fact, since the Yosida approximation \( (\beta_M)\alpha(\cdot) \) of \( \beta_M(\cdot) \) is given by
\[
(\beta_M)\alpha(u) = \begin{cases} 
\frac{u(x) + M}{\lambda} & u(x) \leq -M, \\
0 & |u(x)| < M, \\
\frac{u(x) - M}{\lambda} & u(x) \geq M,
\end{cases}
\]
we easily see
\[
(\partial \varphi(u), (\beta_M)\alpha(u))_L^2 = \int_\Omega \left( -\Delta u + u(\beta_M)\alpha(u) \right) dx \geq \int_\Omega (\beta_M)\alpha(u) |\nabla u(x)|^2 dx + \int_{\partial \Omega} -\partial_u u(x)(\beta_M)\alpha(u(x)) d\sigma \geq 0. \tag{3.7}
\]
Here we used the fact that \( u(\beta_M)\alpha(u) \geq 0 \), \( (\beta_M)\alpha(u) \geq 0 \), \( -\partial_u u(x) \in \gamma(u(x)) \) and \( 0 \in \gamma(0) \) implies that \( \gamma(u) \subset (-\infty, 0] \) if \( u \leq 0 \) and \( \gamma(u) \subset [0, +\infty) \) if \( u \geq 0 \).

Consequently (3.7) together with Theorem 4.4 and Proposition 2.17 in [4] assures that \( \partial \varphi + \partial I_M \) becomes maximal monotone. Hence, since \( \varphi(u) + \partial I_M(u) \in \partial \varphi_M(u) \) is obvious, we can conclude that (3.6) holds true.

Now consider the following auxiliary equation:
\[
(CP)_M \left\{ \begin{array}{l}
\frac{d}{dt} u(t) + \partial \varphi_M(u(t)) + B(u(t)) \ni 0, \\
u(0) = u_0,
\end{array} \right.
\tag{3.8}
\]
where we choose \( M > 0 \) such that
\[
M := \|u_0\|_L^\infty + 2.
\tag{3.9}
\]
Then we easily see that \( u_0 \in D(\partial \varphi_M)^{L^2} = K_M \).

Define a monotone increasing function \( \ell(\cdot) : [0, \infty) \to [0, \infty) \) by
\[
\ell(r) = r + \sup \{ |z| ; z \in F(r), \ |r| \leq r \}.
\tag{3.10}
\]
Here we note that \( \ell(\cdot) \) takes a finite value for any finite \( r \), which is assured by assumption \( D(F) = D(F_M) = D(F_m) = R^1 \) and then we obtain
\[
\sup \{ |z| ; z \in B(u(x)) \} \leq \ell(|u(x)|).
\tag{3.11}
\]
Hence, we obtain
\[
||B(u)||_L^2 = \sup \{ ||z||_L^2 ; z \in B(u) \} \leq \ell(\|u\|_\infty^2)\Omega^1 \leq \ell(M)\Omega^1 \quad \forall u \in D(\partial \varphi_M),
\tag{3.12}
\]
since \( u \in D(\partial \varphi_M) \) implies \( ||u||_L^\infty \leq M \). Now we are going to check some assumptions required in [9]. It is easy to see that (3.11) assures assumption (A5) of Theorem III and (A6) of Theorem IV in [9] by taking \( H = L^2(\Omega) \). Furthermore, the compactness assumption (A1), the set \( \{ u ; \varphi_M(u) \leq L \} \) is compact in \( H = L^2(\Omega) \), is obviously satisfied, since \( \Omega \) is bounded; and the demiclosedness assumption (A2) is also assured, since the maximal monotone parts \( F_M^\alpha \) are always demiclosed in \( L^2(\Omega) \). Thus, we can apply Theorem III and Corollary IV of [9] to conclude that (3.1) admits a solution \( u \) on \([0, T]\) for any \( T > 0 \) satisfying (3.4) with \( T_0 \) replaced by \( T \).

Now we are going to show that there exists \( T_0 > 0 \) such that
\[
||u(t)||_L^\infty \leq M + 1 \quad \forall t \in [0, T_0],
\tag{3.13}
\]
whence follows \( \beta_M(u(t)) = 0 \) for all \( t \in [0, T_0] \), which implies that \( u \) turns out to be the desired solution of the original equation (3.1) on \([0, T_0]\).

To see this, multiplying \((CP)_M\) by \( |u|^{r-2}u \), we obtain by (3.10)
\[
\frac{1}{r} \frac{d}{dt} ||u(t)||_{L^r} + (r - 1) \int_\Omega |u|^{r-2} |\nabla u(t)|^2 dx + \int_{\partial \Omega} g(t, x)|u|^{r-2} u(t) d\sigma \leq \ell(\|u(t)\|_{L^\infty}) ||u(t)||_{L^r} |||u||_{L^1} |\Omega|^{1/r},
\]
where we choose
\[
M := \|u_0\|_L^\infty + 2.
\tag{3.11}
\]
Then we easily see that \( u_0 \in D(\partial \varphi_M)^{L^2} = K_M \).

Define a monotone increasing function \( \ell(\cdot) : [0, \infty) \to [0, \infty) \) by
\[
\ell(r) = r + \sup \{ |z| ; z \in F(r), \ |r| \leq r \}.
\tag{3.9}
\]
Here we note that \( \ell(\cdot) \) takes a finite value for any finite \( r \), which is assured by assumption \( D(F) = D(F_M) = D(F_m) = R^1 \) and then we obtain
\[
\sup \{ |z| ; z \in B(u(x)) \} \leq \ell(|u(x)|).
\tag{3.10}
\]
Hence, we obtain
\[
||B(u)||_L^2 = \sup \{ ||z||_L^2 ; z \in B(u) \} \leq \ell(\|u\|_\infty^2)\Omega^1 \leq \ell(M)\Omega^1 \quad \forall u \in D(\partial \varphi_M),
\tag{3.12}
\]
since \( u \in D(\partial \varphi_M) \) implies \( ||u||_L^\infty \leq M \). Now we are going to check some assumptions required in [9]. It is easy to see that (3.11) assures assumption (A5) of Theorem III and (A6) of Theorem IV in [9] by taking \( H = L^2(\Omega) \). Furthermore, the compactness assumption (A1), the set \( \{ u ; \varphi_M(u) \leq L \} \) is compact in \( H = L^2(\Omega) \), is obviously satisfied, since \( \Omega \) is bounded; and the demiclosedness assumption (A2) is also assured, since the maximal monotone parts \( F_M^\alpha \) are always demiclosed in \( L^2(\Omega) \). Thus, we can apply Theorem III and Corollary IV of [9] to conclude that (3.1) admits a solution \( u \) on \([0, T]\) for any \( T > 0 \) satisfying (3.4) with \( T_0 \) replaced by \( T \).

Now we are going to show that there exists \( T_0 > 0 \) such that
\[
||u(t)||_L^\infty \leq M + 1 \quad \forall t \in [0, T_0],
\tag{3.13}
\]
whence follows \( \beta_M(u(t)) = 0 \) for all \( t \in [0, T_0] \), which implies that \( u \) turns out to be the desired solution of the original equation (3.1) on \([0, T_0]\).

To see this, multiplying \((CP)_M\) by \( |u|^{r-2}u \), we obtain by (3.10)
where \( g(t, x) \in \gamma(u(t, x)) \) and so \( g(t, x)|u|^{r-2}u(t, x) \geq 0 \). Hence,
\[
\frac{d}{dt}\|u(t)\|_{L^r} \leq \ell(\|u(t)\|_{L^\infty})|\Omega|^{1/r}.
\]

Letting \( r \to \infty \), we obtain (see [6])
\[
\|u(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty} + \int_0^t \ell(\|u(s)\|_{L^\infty})ds.
\]
(3.13)

Then Lemma 2.2 of [6] assures that if we set
\[
T_0 = \frac{1}{2\ell(\|u_0\|_{L^\infty} + 1)},
\]
(3.14)
then (3.12) holds true.

In order to prove the alternative part, assume that \( T_m < \infty \) and \( \liminf_{t \to T_m} \|u(t)\|_{L^\infty} = M_0 < \infty \). Then there exists a sequence \( \{t_n\}_{n \in \mathbb{N}} \) such that
\[
t_n \to T_m \quad \text{as} \quad n \to \infty \quad \text{and} \quad \|u(t_n)\|_{L^\infty} \leq M_0 + 1 \quad \forall n \in \mathbb{N}.
\]
(3.15)
Hence, in view of (3.14), the definition of \( T_0 \), regarding \( u(t_n) \) as initial data, we find that \( u \) can be continued up to \( t_n + \frac{1}{2(2(M_n + 2))} \), which becomes strictly larger than \( T_m \) for sufficiently large \( n \) such that \( T_m - t_n < \frac{1}{4d(M_0 + 2)} \). This leads to a contradiction. Thus, the alternative assertion is verified.

Remark 3.2.

(1) One can prove that under the same assumptions in Proposition 3.1, problem \((P)_F^\gamma\) with the boundary condition replaced by the homogeneous Dirichlet (resp. Neumann) boundary condition, denoted by \((P)_D^\gamma\) (resp. \((P)_N^\gamma\)), admits a time local solution \( u \) satisfying (3.4), which is denoted by \( u_D^\gamma \) (resp. \( u_N^\gamma \)). To do this, it suffices to repeat the same arguments as those in the proof of Proposition 3.1 with obvious modifications such as \( \int_\Omega \varphi(x)dx = 1 \).

(2) If assumption (F) is satisfied with \( F_m = 0 \), then the solution of \((P)_F^\gamma\) (or \((P)_D^\gamma, (P)_N^\gamma\)) given in Proposition 3.1 is unique.

Our result on the existence of solutions of (3.1) which blow up in finite time can be formulated in terms of the following eigenvalue problem:
\[
\begin{cases}
-\Delta \phi = \lambda \phi, & x \in \Omega, \\
\phi = 0, & x \in \partial\Omega.
\end{cases}
\]
(3.16)

Let \( \lambda_1 > 0 \) be the first eigenvalue of (3.16) and \( \phi_1 \) be the associated positive eigenfunction normalized by \( \int_\Omega \phi_1(x)dx = 1 \).

We here consider the following fully studied problem:
\[
\begin{aligned}
(P)_D^p & \\
\begin{cases}
\partial_t u - \Delta u = |u|^{p-2}u, & t > 0, \quad x \in \Omega, \\
u = 0, & t > 0, \quad x \in \partial\Omega, \\
u(0, x) = u_0(x) \geq 0, & x \in \Omega.
\end{cases}
\end{aligned}
\]

It is well known that \((P)_D^p\) admits the unique time local solution \( u_D^p \) for any \( u_0 \in L^\infty(\Omega) \) and \( T_m(u_D^p) < \infty \) if \( p > 2 \) and \( u_0 \) satisfies
\[
u_0 \in L^\infty(\Omega), \quad 0 \leq u_0(x) \quad \text{a.e.} \quad x \in \Omega, \quad \text{and} \quad \int_\Omega u_0(x)\phi_1(x)dx > \lambda_1^{1/p-1},
\]
(3.17)
which is proved by the so-called Kaplan’s method.

By comparing the solution \( u \) of (3.1) with \( u_D^p \), we obtain the following result.
Proposition 3.3. Assume that \( u_0 \) satisfies (3.17) and let \( u_p^P \) be any solution of (3.1), then \( T_m(u_p^P) \leq T_m(u_p^D) < \infty \), i.e., \( u_p^P \) blows up in finite time.

Proof. We apply Theorem 2.1 with \( m = 1 \), \( a_{i,j} = \delta_{i,j} \) and \( \beta_i = \beta_j = 0 \), \( a_1 = a_2 = u_0 \). Then (A1) and (A2) are automatically satisfied. As for (A4), we take \( F(x,t,u) = |u|^{p-2}u \) and \( F_\delta(t,x,u) = F(u) \), then (3.2) assures (i) of (A4), and it is clear that \( F_1 \) satisfies (SC), since \( F_1 \) is of \( C^1 \)-class with respect to \( u \). As for the boundary conditions, we set

\[
y_i(r) = y_0^P(r) = \begin{cases} \mathbb{R}^1 & \text{for } r = 0, \\ \emptyset & \text{for } r \neq 0, \end{cases}
\]

(3.18)

\[
y_i(r) = y_0^P(r) = \begin{cases} y(r) & \text{for } r > 0, \\ \infty & \text{for } r = 0, \\ \emptyset & \text{for } r < 0. \end{cases}
\]

(3.19)

Then we can easily see that \( y_i \) is monotone, i.e., \( (z_1 - z_2)(r_1 - r_2) \geq 0 \) for all \( |r_1, z_1|, |r_2, z_2| \in y_i \). In fact, this is obvious when \( r > 0 \) or \( r = 0 \) (\( i = 1, 2 \)). Let \( r_1 > 0 \) and \( r_2 = 0 \), then \( z_2 \in y(0) \) or \( z_2 \in (-\infty, 0) \). If \( z_2 \in y(0) \), the monotonicity of \( y \) assures the assertion; and if \( z_2 \in (-\infty, 0) \), since \( 0 \in y(0) \) implies \( z_1 \geq 0 \), we obtain \( z_1 - z_2)(r_1 - r_2) \geq z_1 r_1 \geq 0 \).

Since \( y(r) \subset y_i(r) \) for all \( r \geq 0 \) and \( u_p^P(t,x) \geq 0 \) a.e. \( (t,x) \in T \), which is assured by \( u_p^P(t,x) \geq 0 \) a.e. \( (t,x) \in T \), \( u_p^P(t,x) \) satisfies \( -\partial_i u_p^P(t,x) \in y_i(u_p^P(t,x)) \) a.e. \( (t,x) \in T \).

On the other hand, \( -\partial_i u_p^D(t,x) \in y_i(u_p^D(t,x)) \) implies \( u_p^D(t,x) \in D(y_i) = \{0\} \) and \( -\partial_i u_p^D(t,x) \in \mathbb{R} \), i.e., \( u_p^D(t,x) \) obeys the homogeneous Dirichlet boundary condition (see [3–5]).

Thus, since \( D(y_i) = \{0\} \) and \( D(y_i) \subset [0, +\infty) \), (iii) of (A2) is satisfied. Consequently, applying Theorem 2.1, we find that

\[
0 \leq u_p^D(t,x) \leq u_p^P(t,x) \quad \forall t \in [0, T) \text{ a.e. } x \in \Omega,
\]

(3.20)

where \( T = \min(T_m(u_p^P), T_m(u_p^D)) \), whence follows

\[
|u_p^D(t)|_{L^\infty} \leq |u_p^P(t)|_{L^\infty} \quad \forall t \in [0, T).
\]

(3.21)

Here suppose that \( T_m(u_p^D) < T_m(u_p^P) \), then it follows from (3.21) that

\[
\lim_{t \to T_m(u_p^P)} |u_p^P(t)|_{L^\infty} = +\infty,
\]

which contradicts the definition of \( T_m(u_p^P) \). Hence, we conclude that \( T_m(u_p^P) \leq T_m(u_p^D) < +\infty \). \( \square \)

As the special case where \( F(u) = |u|^{p-2}u \), we obtain the following (see (2) of Remark 3.2).

Corollary 3.4. Assume that \( u_0 \) satisfies (3.17) and let \( u_p^P \) be the unique solution of (3.1) with \( F(u) = |u|^{p-2}u \), denoted by \( (P)_p^P \), then \( T_m(u_p^P) \leq T_m(u_p^D) < \infty \), i.e., \( u_p^P \) blows up in finite time.

We next consider another typical classical boundary condition, namely, the following problem with the homogeneous Neumann boundary condition:

\[
(P)_p^N \begin{cases} \partial_i u - \Delta u = |u|^{p-2}u, \quad t > 0, \quad x \in \Omega, \\
\partial_i u = 0, \quad t > 0, \quad x \in \partial \Omega, \\
u(0, x)u = u_0(x) \geq 0, \quad x \in \Omega. \end{cases}
\]

Then it is also well known that \( (P)_p^N \) admits the unique positive local solution \( u_p^N \) for any \( 0 \leq u_0 \in L^\infty(\Omega) \) and \( T_m(u_p^N) < \infty \) if \( u_0 \) is not identically zero in \( \Omega \).

Let \( u_p^N \) be any solution of \( (P)_p^N \) (see Remark 3.2), and we apply Theorem 2.1 with \( m = 1 \), \( a_{i,j} = \delta_{i,j} \) and \( \beta_i = \beta_j = 0 \), \( y_1 = y_2 = y^N \) : \( 0 \), \( a_1 = a_2 = u_0 \). Then (A1), (A2) and (A3) are automatically satisfied. As for (A4),
we take $F_i(t, x, u) = |u|^{p-2}u$ and $F_j(t, x, u) = F(u)$, then (3.2) assures (i) of (A4), and it is clear that $F_i$ satisfies (SC). Then we obtain
\[
\|u_p^m(t)\|_{L^\infty} \leq \|u_p^N(t)\|_{L^\infty} \quad \forall t \in [0, T) \quad \text{with} \quad T = \min(T_m(u_p^m), T_m(u_p^N)),
\]
whence follows
\[
T_m(u_p^N) \leq T_m(u_p^m). \tag{3.23}
\]
We now compare $(P)_p^N$ with $(P)_p^r$, i.e., $(P)_r^p$ with $F(u) = |u|^{p-2}u$. Let $u_p^r$ be the unique non-negative solution of $(P)_r^p$ (cf. (2) of Remark 3.2). We apply Theorem 2.1 with $m = 1$, $a_{i,j} = \delta_{i,j}$ and $\beta_1 = \beta_2 = 0$, $a_1 = a_2 = u_0$, $F_i(u) = F_j(u) = |u|^{p-2}u$. Then (A1), (A2) and (A4) are satisfied. As for (A3), define $\gamma(r)$ and $\gamma_e(r)$ by
\[
y_i(r) = \gamma_i(r) = \begin{cases} 
\gamma(r) & \text{for } r > 0, \\
(-\infty, 0] & \text{for } r = 0, \\
\emptyset & \text{for } r < 0,
\end{cases}
\]
\[
y_e(r) = \gamma_e(r) = \begin{cases} 
0 & \text{for } r > 0, \\
(-\infty, 0] & \text{for } r = 0, \\
\emptyset & \text{for } r < 0.
\end{cases}
\]
Then we can show that $\gamma_i, \gamma_e$ are monotone by the same reasoning as that for (3.19).

Moreover, since $\gamma(r) \subset \gamma_i(r)$ and $0 = \gamma_e(r) \subset \gamma_i(r)$ for $r \geq 0$, and $u_p^r(t, x), u_p^N(t, x) \geq 0 \ a.e. \ (t, x) \in \Gamma_T$ are assured by $u_p^r(t, x), u_p^N(t, x) \geq 0 \ a.e. \ (t, x) \in Q_T$, we obtain $-\partial_r u_p^r(t, x) \in \gamma_i(u_p^r(t, x))$ and $-\partial_r u_p^N(t, x) \in \gamma_e(u_p^N(t, x))$ for a.e. $(t, x) \in \Gamma_T$.

Furthermore, for any $r_1 \in D(\gamma_i), r_2 \in D(\gamma_e)$ with $r_2 < r_1$, since $D(\gamma_i) = [0, +\infty)$ and $r_2 < r_1$ implies $0 < r_1$ and $0 \in \gamma(0)$ is assumed, we have
\[
\sup\{g_2; g_2 \in \gamma_e(r_2)\} \leq 0 \leq \inf\{g_1; g_1 \in \gamma(r_1)\}.
\]
Hence, (ii) of (A3) is satisfied. Consequently, applying Theorem 2.1, we find that
\[
0 \leq u_p^r(t, x) \leq u_p^N(t, x) \quad \forall t \in [0, T) \quad \text{a.e.} \quad x \in \Omega,
\]
where $T = \min(T_m(u_p^r), T_m(u_p^N))$, whence follows
\[
T_m(u_p^N) \leq T_m(u_p^r) \quad \text{and} \quad \|u_p^r(t)\|_{L^\infty} \leq \|u_p^N(t)\|_{L^\infty} \quad \forall t \in [0, T_m(u_p^N)). \tag{3.24}
\]
Thus, putting arguments above all together, we obtain the following observations.

**Proposition 3.5.** Let $u_p^r$ be any solution of $(P)_r^p$ and let $u_p^N$ be the unique solution of $(P)_r^N$ ($*= D, y, N$). Then the following hold.

(i) $T_m(u_p^r) \leq T_m(u_p^0), T_m(u_p^r) \leq T_m(u_p^r), T_m(u_p^N) \leq T_m(u_p^N).

(ii) $T_m(u_p^N) \leq T_m(u_p^r) \leq T_m(u_p^r).

**Remark 3.6.** By virtue of (3.20), we can also derive some results on the strong maximum principle (see [1]) for $(P)_r^p$.

### 3.2 Reaction diffusion system arising from nuclear reactor

In this subsection, we exemplify the applicability of Theorem 2.1 for systems of parabolic equations. We consider the following reaction diffusion system, which consists of two equations possessing a nonlinear coupling term between two real-valued unknown functions (cf. [10–12]).
Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial \Omega$. Moreover, $u_1$, $u_2$ are real-valued unknown functions, $a$ and $b$ are given positive constants. As for the parameters appearing in the boundary condition, we assume $\alpha_i \in [0, \infty)$, $\gamma_i \in (1, \infty)$ ($i = 1, 2$). We note that the boundary condition for $u_i$ becomes the homogeneous Neumann boundary condition when $\alpha_i = 0$, and the Robin boundary condition when $\alpha_i > 0$ and $\gamma_i = 2$. We further assume that the given initial data $u_{i0}$, $u_{20}$ are nonnegative and belong to $L^\infty(\Omega)$.

The equations of this system with linear boundary conditions were proposed in [13] to describe the diffusion phenomenon of neutron and heat in nuclear reactors, where $u_1$ and $u_2$ represent the neutron density and the temperature, respectively. However, we here consider this system with nonlinear boundary conditions of power type as above, since from a physical point of view, it seems to be more natural to consider the nonlinear boundary condition rather than the linear ones. In fact, the linear boundary conditions such as Dirichlet or Neumann type can be realized only when some artificial controls of the flux are given on the boundary. For a large scale system such as nuclear reactors, however, it is extremely difficult to give such a control, so actually in reactors no control is given for the flux on the boundary.

When there is no artificial control of the flux on the boundary, there exists a well-known radiation model in physics, called the Stefan-Boltzmann law, which says that the total radiant heat power emitted from the boundary is proportional to the fourth power of the temperature when $N = 3$, which is far from linear.

The existence and uniqueness of non-negative local solutions of (NR) belonging to $L^\infty(\Omega)$ is shown in [12] for the case where $\gamma_i = 2$, where it is also proved that (NR) possesses a positive stationary solution $\bar{U} = (\bar{u}_1, \bar{u}_2)$, which works as the threshold to separate global existence and finite time blow up for the case where $\gamma_1 = \gamma_2 = 2$, i.e., roughly speaking, if the initial data stay below $\bar{U}$, then the corresponding solution exists globally, and if the initial data are larger than $\bar{U}$, then the corresponding solution blows up in finite time. As for the case where $\gamma_i \neq 2$, however, this method for showing the existence of blow-up solutions does not work well.

Nevertheless, it is possible to show that (NR) with $\gamma_i \neq 2$ admits blow-up solutions by applying the same strategy as that in the previous subsection. Along the same lines as before, we first consider the following Dirichlet problem for (NR).

\[
\begin{aligned}
\partial_t u_1 - \Delta u_1 &= u_1 u_2 - b u_1, & t > 0, & x \in \Omega, \\
\partial_t u_2 - \Delta u_2 &= a u_1, & t > 0, & x \in \Omega, \\
u_1 = u_2 = 0, & t > 0, & x \in \partial \Omega, \\
u_1(0, x) = u_{10}(x) \geq 0, & \nu_2(0, x) = u_{20}(x) \geq 0, & x \in \Omega.
\end{aligned}
\]

We first note that for every $U_0 = (u_{10}, u_{20}) \in L^\infty(\Omega) = \{u_1, u_2\}; u_i \geq 0, u_i \in L^\infty(\Omega)$ ($i = 1, 2$), (NR) or (NR)$^D$ possess a unique solution $U(t) = (u_1(t), u_2(t)) \in L^\infty(\Omega)$ satisfying the blow-up alternative with respect to $L^\infty$-norm such as in Proposition 3.1. We are going to show this result for a more general equation:

\[
\begin{aligned}
\partial_t u_1 - \Delta u_1 &= u_1 u_2 - b u_1, & t > 0, & x \in \Omega, \\
\partial_t u_2 - \Delta u_2 &= a u_1, & t > 0, & x \in \Omega, \\
\partial_t u_1 + \gamma_1 u_1 &\partial_t u_2 + \gamma_2 u_2 = 0, & t > 0, & x \in \partial \Omega, \\
u_1(0, x) = u_{10}(x) \geq 0, & \nu_2(0, x) = u_{20}(x) \geq 0, & x \in \Omega,
\end{aligned}
\]

where $\gamma_i : \mathbb{R}^1 \to \mathbb{R}^1$ are maximal monotone operators ($i = 1, 2$). To do this, we can repeat much the same arguments as those in the proof of Proposition 3.1.

Let $H = L^2(\Omega) \times L^2(\Omega)$ with inner product $(U, V)_H = (u_1, v_1)_{L^2} + (u_2, v_2)_{L^2}$ for $U = (u_1, u_2)$, $V = (v_1, v_2)$, and put $|V|^2 = |V_1|^2 + |V_2|^2$. Let $j_i : \mathbb{R}^1 \to (-\infty, +\infty]$ be lower semi-continuous convex functions such that $\partial j_i = \gamma_i$ ($i = 1, 2$). For the Dirichlet (resp. Neumann) boundary condition, we put $j_i(0) = 0$ and $j_i(r) = +\infty$ for $r \neq 0$ (resp. $j_i(r) = 0$, $\forall r \in \mathbb{R}^1$).
Then we define
\[
\varphi(U) = \frac{1}{2} \int_{\Omega} (|\nabla U(x)|^2 + |U(x)|^2) \, dx + \sum_{i=1}^{2} \int_{\Omega} j_i(u_i(x)) \, dx \quad U \in D(\varphi),
\]
and put
\[
\phi(U) = \frac{1}{2} \int_{\Omega} (|\nabla U(x)|^2 + |U(x)|^2) \, dx + \sum_{i=1}^{2} \int_{\Omega} j_i(u_i(x)) \, dx \quad U \in H(\varphi),
\]
where \( D(\varphi) = \{ U; \ u_i \in H^1(\Omega) \ \text{j}_i(u_i) \in L(\Omega) \ (i = 1, 2) \} \).

For the homogeneous Dirichlet (resp. Neumann) boundary condition case, we take
\[
\partial \Omega = \partial \Omega^0 \ (\partial \Omega^0 \ (\partial \Omega^0) (\partial \Omega^0) \partial \Omega^0),
\]
where \( \partial \Omega^0 \ (\partial \Omega^0) (\partial \Omega^0) \partial \Omega^0 \).

Furthermore, the elliptic estimate (3.5) holds true for all \( U \in D(\partial \varphi) \).

Then by putting
\[
\phi(U) = \frac{1}{2} \int_{\Omega} (|\nabla U(x)|^2 + |U(x)|^2) \, dx + \sum_{i=1}^{2} \int_{\Omega} j_i(u_i(x)) \, dx \quad U \in H(\varphi),
\]
we can easily show that \( (\text{NR})' \) can be reduced to the following abstract evolution equation in \( H \).

\[
(\text{CP})' \left\{ \begin{array}{ll}
\frac{d}{dt} U(t) + \partial \varphi(U(t)) + B(U(t)) = 0, & t > 0,
\end{array} \right.
\]
\[
U(0) = U_0 = (u_{10}, u_{20}).
\]

In order to apply "\( L^\infty \)-Energy Method," we again introduce the following cut-off functions \( I_{K_{1,M}}(\cdot) \ (i = 1, 2) \):
\[
I_{K_{1,M}}(U) = \left\{ \begin{array}{ll}
0 & U \in K_{1,M} = \{ U = (u_1, u_2) \in H; |u_i(x)| \leq M \quad \text{a.e.} \ x \in \Omega \},
\end{array} \right.
\]
and put
\[
\phi_{M}(U) = \varphi(U) + I_{K_{1,M}}(U) + I_{K_{2,M}}(U).
\]

Then we obtain
\[
\partial \varphi(U) = \partial \varphi(U) + \partial I_{K_{1,M}}(U) + \partial I_{K_{2,M}}(U) \quad \forall U \in D(\partial \varphi) \cap K_{1,M} \cap K_{2,M}.
\]

Consider the following auxiliary equation:
\[
(\text{CP})' \left\{ \begin{array}{ll}
\frac{d}{dt} U(t) + \partial \varphi_{M}(U(t)) + B(U(t)) = 0, & t > 0,
\end{array} \right.
\]
\[
U(0) = U_0
\]
where we choose \( M > 0 \) such that
\[
M = \| U_0 \|_{L^\infty} + 2 = \| u_{10} \|_{L^\infty} + \| u_{20} \|_{L^\infty} + 2.
\]

Then as in the proof of Proposition 3.1, we can easily show that \( (\text{CP})' \), which is equivalent to the following \( (\text{NR})' \), admits a unique global solution \( U(t) = (u_1(t), u_2(t)) \).

\[
(\text{NR})' \left\{ \begin{array}{ll}
\partial_t u_1 - \Delta u_1 + \beta_M (u_1) = u_1 u_2 - u_1, & t > 0, \ x \in \Omega,
\partial_t u_2 - \Delta u_2 + \beta_M (u_2) = a u_1, & t > 0, \ x \in \Omega,
\partial_t u_1 + \gamma_1 (u_1) = \partial_\nu u_1 + \gamma_2 (u_2) = 0, & t > 0, \ x \in \partial \Omega,
\end{array} \right.
\]
\[
u_1(0, x) = u_{10}(x) \geq 0, \quad u_2(0, x) = u_{20}(x) \geq 0, \quad x \in \Omega.
\]

Then in parallel with (3.13), multiplying the first and second equations of \( (\text{NR})' \) by \( |u_1|^{-2} u_1 \) and \( |u_2|^{-2} u_2 \), we can obtain
\[
\| U(t) \|_{L^\infty} \leq \| U_0 \|_{L^\infty} + \int_0^t \epsilon \left( \| U(s) \|_{L^\infty} \right) ds \quad \text{with} \ \epsilon(r) = ar + r^2,
\]
where \( \| U \|_{L^\infty} = \| (u_1, u_2) \|_{L^\infty} = \| u_1 \|_{L^\infty} + \| u_2 \|_{L^\infty}. \) Then we can repeat the same arguments as those in the proof of Proposition 3.1. Furthermore, multiplying the first and second equations of \( (\text{NR})' \) by \( u_i \) for \( u_i = \max(-u_i, 0) \) and \( u_i^2 = \max(-u_i, 0) \), we can easily deduce
Then by Gronwall’s inequality, we obtain \( u_i(t) = u_2(t) = 0 \) for all \( t \), i.e., \((u_1, u_2)\) is a non-negative solution (see [12]) (the non-negativity of solutions can also be derived from application of Theorem 2.1 for \((NR)^v\) with the coupling term \(u_1u_2\) replaced by \(u_1^2u_2\)).

Here we prepare the following lemma concerning the existence of blow-up solutions of \((NR)^D\).

**Proposition 3.7.** Assume that \((u_{i0}, u_{20})\) belongs to \(L^\infty(\Omega)\) and satisfies

\[
\int_\Omega (au(x) + bu(x) - \frac{1}{2}u^2(x))\phi_1(x)dx \geq 0, \quad \int_\Omega u_{20}(x)\phi_1(x)dx > 2(b + \lambda_1).
\]

Then the solution \((u(t), u_2(t))\) of \((NR)^D\) blows up in finite time. Here, \(\lambda_1\) and \(\phi_1\) are the first eigenvalue and its associate normalized positive eigenfunction of (3.16).

**Proof.** Suppose that \(U(t)\) is a global solution. Then multiplying the first and second equations of \((NR)^D\) by \(\phi_1\), we obtain

\[
\frac{d}{dt}\left( \int_\Omega u_i\phi_1 dx \right) + (b + \lambda_1)\int_\Omega u_i\phi_1 dx = \int_\Omega u_1u_2\phi_1 dx, \quad (3.26)
\]

\[
\frac{d}{dt}\left( \int_\Omega u_2\phi_1 dx \right) + \lambda_1\int_\Omega u_2\phi_1 dx = a\int_\Omega u_1\phi_1 dx. \quad (3.27)
\]

Following [14], we set

\[
y(t) = \int_\Omega u_2(t)\phi_1 dx, \quad z(t) = y'(t) + (b + \lambda_1)y(t) - \frac{1}{2}\int_\Omega u_2^2(t)\phi_1 dx.
\]

Then by (3.27) and (3.26), we obtain

\[
y''(t) = -\lambda_1y'(t) + a\int_\Omega u_1(t)\phi_1 dx = -\lambda_1y'(t) - (b + \lambda_1)\int_\Omega au_1\phi_1 dx + \int_\Omega au_2u_1\phi_1 dx. \quad (3.28)
\]

We substitute \(au_1 = \partial_1u_2 - \Delta u_2\) in (3.28), then by integration by parts we have

\[
y''(t) + (b + 2\lambda_1)y'(t) + \lambda_1(b + \lambda_1)y(t) = \frac{1}{2}\frac{d}{dt}\left( \int_\Omega u_2^2\phi_1 dx \right) + \int_\Omega |\nabla u_1|^2\phi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\phi_1 dx,
\]

whence follows

\[
z'(t) \geq -\lambda_1z(t).
\]

Therefore, we obtain \(z(t) \geq z(s)e^{-\lambda_1(t-s)}\) for \(0 < s < t\). Here (3.27) and (3.25) yield

\[
z(s) = y'(s) + (b + \lambda_1)y(s) - \frac{1}{2}\int_\Omega u_2^2(s)\phi_1 dx
\]

\[
= \int_\Omega \left( au_1(s) + bu_2(s) - \frac{1}{2}u_2^2(s) \right)\phi_1 dx
\]

\[
\rightarrow \int_\Omega \left( au_{i0} + bu_{20} - \frac{1}{2}u_{20}^2 \right)\phi_1 dx \geq 0 \quad \text{as } s \to 0,
\]
since \( u_i(t), u_2(t) \in C([0, 1]; L^2(\Omega)) \cap L^\infty(0, 1; L^\infty(\Omega)) \). Hence, we see that \( z(t) \geq 0 \) for all \( t > 0 \), i.e., we have
\[
y'(t) \geq -(b + \lambda_0)y(t) + \frac{1}{2} \int_\Omega u_2^2(t) \phi_1 \, dx \geq -(b + \lambda_0)y(t) + \frac{1}{2} y^2(t) \geq \frac{1}{2} y(t)(y(t) - \lambda b + \lambda_0)). \tag{3.29}
\]

Then (3.29) assures that \( y(t) \) blows up in finite time if \( y(0) > \lambda b + \lambda_0 \). \qed

In order to make it clear that solutions of parabolic systems differ according to their boundary conditions imposed, we here denote the unique solutions of (NR) and (NR)\(^D\) by \( U^f(t) = (u^f_i(t), u^f_2(t)) \) and \( U^O(t) = (u^O_i(t), u^O_2(t)) \) with the same initial data \( U_0 \in L^\infty(\Omega) \), respectively.

We are going to compare \( U^f(t) \) with \( U^O(t) \) by applying Theorem 2.1. for \( U_1 = U^O, U_2 = U^f \). Let
\[
m = 2; \quad a_1 = a_2 = 1; \quad a_i = a_i^2 = u_2 = u_0; \quad \beta_i = \beta_2 = \beta_2 = \beta_2 = 0; \quad F_1(U) = F^f_1(U) = F^O(U) = u_0 u_2 - b u_1; \quad F_2(U) = F^O_2(U) = F^f_2(U) = F^f_2(U) = a u_1;
\]
\[
y^p(r) = y^f_1(r) = y^O_2(r), \quad y^f_2(r) = \begin{cases} a |r| r^{-2} & \text{for } r > 0, \\ 0 & \text{for } r < 0, \end{cases}
\]
where \( y^O \) is the maximal monotone graph defined by (3.18). Then (A1), (A2) and (i) of (A4) are obviously satisfied. Moreover as in the proof of Proposition 3.3, we can see that \( u^O_i(t) \) and \( u^f_i(t) \) obey the homogeneous Dirichlet boundary condition, and that \( -\partial_t u^O_i \in \gamma^O_i(u^O_i) \) and \( -\partial_t u^f_i \in \gamma^f_i(u^f_i) \) hold, since \( u^O_i(t) \) and \( u^f_i(t) \) are non-negative solutions. Therefore, \( D(\beta^f_i) = D(\beta^O_i) = D(\beta_0) = \{0\} \) and \( D(y^f_2(t)) = D(y^O_2(t)) = \Omega_{\to} \) assure (iii) of (A3).

Hence, to apply Theorem 2.1, it suffices to check (ii) of (A4), i.e., \( F^f_1(U) = u_0 u_2 - b u_1, \quad F^O(U) = a u_1 \) satisfies (SC). Since \( F^f, F^O \in C^1(R^2), (3.3) \) is obvious. As for (3.2), we obtain
\[
\frac{\partial}{\partial u_i} F^f_2(U) = a > 0, \quad \frac{\partial}{\partial u_2} F^O(U) = u_1 \geq 0.
\]

Consequently, applying Theorem 2.1, we conclude
\[
T_m(U^f) \leq T_m(U^O) \quad \text{and} \quad 0 \leq u^O_i(t, x) \leq u^f_i(t, x), \quad 0 \leq u^f_i(t, x) \leq u^O_i(t, x) \quad \forall t \in [0, T_m(U^f)) \quad \text{a.e.} \quad x \in \Omega.
\]

Thus by virtue of Proposition 3.7, we have the following corollary.

**Corollary 3.8.** Assume that \( (u_{i0}, u_{o2}) \) belongs to \( L^\infty_{\text{loc}}(\Omega) \) and satisfies (3.25). Then the unique solution \( U(t) = (u_i(t), u_2(t)) \) of (NR) blows up in finite time.

**Remark 3.9.** The existence of \( (u_{i0}, u_{o2}) \) satisfying (3.25) is assured when \( a > 0 \). For instance, if \( u_{i0} \geq \frac{1}{20} u_{i0}^2 \) and \( u_{o2} \) is sufficiently large, then (3.25) is satisfied.

For the case where \( a = 0 \), however, there are no initial data \( (u_{i0}, u_{o2}) \) satisfying (3.25). In fact, \( a = 0 \) implies that \( \sup_{t, \Omega} |u(t)|_{L^\infty} \leq |u_{i0}|_{L^\infty} \), then \( u(t) \) satisfies \( \partial_t u_i - \Delta_i u_i \leq |u_{o2}|_{L^\infty} u_i(t) \), whence follows \( |u(t)|_{L^\infty} \leq |u_{o2}|_{L^\infty} e^{\int_{t_0}^t |u_{o2}_{r2}|_{L^\infty} \, dt'}. Consequently, every local solution can be continued globally.

**Remark 3.10.** The assertion of Corollary 3.8 holds true for more general equation (NR)\(^r\), provided that \( 0 \in \gamma_i(0) (i = 1, 2) \) is satisfied.

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