On the factorisation formula for fundamental solutions in the inverse spectral transform

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Abstract
A factorization formula for wave functions, which is basic in the inverse spectral transform approach to initial-boundary value problems, is proved in greater generality than before. Applications follow. Related compatibility questions for the GBDT version of Bäcklund-Darboux transformation are treated too.

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1 Introduction

Zero curvature representation of the integrable nonlinear equations is a well known approach (see [11][14][31][50] and references in [14]), which was developed soon after the seminal Lax pairs appeared in [26]. Namely, many integrable nonlinear equations admit representation (zero curvature representation)

\[ G_t(x,t,z) - F_x(x,t,z) + [G(x,t,z), F(x,t,z)] = 0, \]  

\[ G_t := \frac{\partial}{\partial t} G, \quad [G,F] := GF - FG, \]
which is the compatibility condition of the auxiliary linear systems
\[
\frac{\partial}{\partial x} w(x,t,z) = G(x,t,z)w(x,t,z), \quad \frac{\partial}{\partial t} w(x,t,z) = F(x,t,z)w(x,t,z).
\] (1.2)

Here \( G \) and \( F \) are \( m \times m \) matrix functions, and \( z \) is the spectral parameter, which will be omitted sometimes in our notations.

Solution of integrable nonlinear equations is closely related to Lax pairs and zero curvature representations, which have been mentioned above, and has been a great breakthrough in the second half of the 20th century. An active study of the cases, which are close to integrable in a certain sense, followed (see, for instance, some references in [3, 6, 23]). Initial-boundary value problems for integrable nonlinear equations can be considered as an important example, where integrability is ”spoiled” by the boundary conditions. These problems are of great current interest, and inverse spectral transform (ISpT) method [15,21,32,33,37,38,41,44] is one of the fruitful approaches in this domain. Further we assume that \( x, t \) belong to a semi-strip
\[
\mathcal{D} = \{(x,t) : 0 \leq x < \infty, \ 0 \leq t < a\}. 
\] (1.3)

Normalize fundamental solutions of the auxiliary systems by the initial conditions
\[
\frac{d}{dx} W(x,t,z) = G(x,t,z)W(x,t,z), \quad W(0,t,z) = I_m; \quad (1.4)
\]
\[
\frac{d}{dt} R(x,t,z) = F(x,t,z)R(x,t,z), \quad R(x,0,z) = I_m, \quad (1.5)
\]
where \( I_m \) is the identity matrix of order \( m \). If condition (1.1) holds, the fundamental solution of (1.4) admits factorization
\[
W(x,t,z) = R(x,t,z)W(x,0,z)R(t,z)^{-1}, \quad R(t,z) := R(0,t,z). \quad (1.6)
\]

Formula (1.6) is one of the basic and actively used formulas in the inverse spectral transform method (see [32,33,37,38,41,44] and references therein). It was derived in [41,42] under some smoothness conditions (continuous differentiability of \( G \) and \( F \), in particular): see formulas (1.6) in [41], p.22 and in [32], p. 39.
Here we prove \((1.6)\) under weaker conditions and in much greater detail, which is important for applications. Namely, we prove the following theorem.

**Theorem 1.1** Let \(m \times m\) matrix functions \(G\) and \(F\) and their derivatives \(G_t\) and \(F_x\) exist on the semi-strip \(D\), let \(G, G_t,\) and \(F\) be continuous with respect to \(x\) and \(t\) on \(D\), and let \((1.1)\) hold. Then the equality

\[
W(x, t, z)R(t, z) = R(x, t, z)W(x, 0, z), \quad R(t, z) := R(0, t, z), \quad (1.7)
\]

is true.

Note that constructions similar to \((1.6)\) appear also in the theory of Knizhnik-Zamolodchikov equation (see Theorem 3.1 in \([46]\) and see also \([45]\)).

Theorem \([1.1]\) is proved in Section 2. Section 3 is dedicated to applications to initial-boundary value problems, and Theorem 3.2 on the evolution of the Weyl function for the ”focusing” modified Korteweg-de Vries (mKdV) equation is proved there as an example.

Related questions of the equality of mixed derivatives and application of this equality to the GBDT version (see \([17,18,20,30,33-36,39]\) and references therein) of the Bäcklund-Darboux transformation are treated in Section 4.

As usual, by \(\mathbb{N}\) we denote the set of positive integers, by \(\mathbb{C}\) we denote the complex plane, and by \(\mathbb{C}^m\) is denoted the \(m\)-dimensional coordinate space over \(\mathbb{C}\). By \(\Im z\) is denoted the imaginary part of \(z \in \mathbb{C}\), and \(\arg z\) is the argument of \(z\). By \(C^k(D)\) we denote functions and matrix functions, which are \(k\) times continuously differentiable on \(D\).

### 2 Proof of Theorem \([1.1]\)

The spectral parameter \(z\) is non-essential for the formulation of Theorem \([1.1]\) and for its proof and we shall omit it in this section. We shall need the proposition below.

**Proposition 2.1** Let the \(m \times m\) matrix function \(W\) be given on the semi-strip \(D\) by equation \((1.4)\), where \(G(x, t)\) and \(G_t(x, t)\) are continuous matrix functions in \(x\) and \(t\).
(i) Then the derivative \( W_t \) exists and matrix functions \( W \) and \( W_t \) are continuous with respect to \( x \) and \( t \) on the semi-strip \( \mathcal{D} \).

(ii) Moreover, the mixed derivative \( W_{tx} \) exists and the equality \( W_{tx} = W_{xt} \) holds on \( \mathcal{D} \).

Proof. Consider system
\[
\frac{d}{dx} y = \hat{G}(x, y)y, \quad \hat{G}(x, y) = \hat{G}(x, y_{m+1}) := \begin{bmatrix} G(x, y_{m+1}) & 0 \\ 0 & 0 \end{bmatrix},
\]

where \( \hat{G} \) is an \( m+1 \times m+1 \) matrix function and \( y_{m+1} \) is the last entry of the column vector \( y \in \mathbb{C}^{m+1} \). Denote by \( W_j \) and \( e_j \) the \( j \)-th columns of \( W \) and \( I_m \), respectively (\( 1 \leq j \leq m \)). It easily follows from (1.4) that the solution of (2.1) with the initial condition
\[
y(0) = g = \begin{bmatrix} e_j \\ t \end{bmatrix}
\]
has the form
\[
y(x, g) = \begin{bmatrix} W_j(x, t) \\ t \end{bmatrix}.
\]

Putting \( G(x, t) = G(0, t) \) for \( -\varepsilon \leq x \leq 0 \) whereas \( t \geq 0 \), and putting \( G(x, t) = G(x, 0) + tG_t(x, 0) \) for \( -\varepsilon \leq t \leq 0 \) (\( \varepsilon > 0 \)) we extend \( G \) so that \( G \) and \( G_t \) remain continuous on the rectangles
\[
\mathcal{D}(a_1, a_2) = \{(x, t) : -\varepsilon \leq x \leq a_1, -\varepsilon \leq t \leq a_2 < a\}, \quad a_1, a_2 \in \mathbb{R}_+.
\]

Hence, it follows from the definition of \( \hat{G} \) in (2.1) that \( \hat{G}(x, y) \) and, as a consequence, the vector function \( \hat{G}(x, y)y \) are continuous on \( \mathcal{D}(a_1, a_2) \) together with their derivatives with respect to the entries of \( y \). Thus, according to the classical theory of ordinary differential equations (see, for instance, theorem on pp. 305-306 in [48]) the partial first derivatives of \( y(x, g) \) with respect to the entries of \( g \) exist in the interior \( \mathcal{D}_i(a_1, a_2) \) of \( \mathcal{D}(a_1, a_2) \). Moreover, \( y \) and its partial derivatives with respect to the entries of \( g \) are continuous. In particular, since by (2.2) we have \( g_{m+1} = t \), the functions \( y \) and \( y_t \) are continuous in all rectangles \( \mathcal{D}_i(a_1, a_2) \). Taking into account (2.3), we see that
$W$ and $W_t$ are continuous in the rectangles $D_i(a_1, a_2)$, and the statement (i) is true.

In view of (1.4) and considerations above the derivatives $W_x$, $W_{xt}$, and $W_t$ exist and are continuous in the rectangles $D_i(a_1, a_2)$. Hence, by a stronger formulation (see, for instance, [2, 47] or p. 201 in [28]) of the well-known theorem on mixed derivatives, $W_{tx}$ exists in $D_i(a_1, a_2)$ and $W_{tx} = W_{xt}$. Thus, the statement (ii) follows. □

Now, we can follow the scheme from Chapter 3 in [12] (see also Chapter 12 in [14]).

Proof of Theorem 1.1. According to statement (i) in Proposition 2.1 the matrix function $W_t$ exists and is continuous. Introduce $U(x, t)$ by the equality

$$U := W_t - FW.$$  \hfill (2.5)

By (1.4), (2.5), and statement (ii) in Proposition 2.1 we have

$$U_x = W_{tx} - F_x W - FW_x = W_{xt} - F_x W - FGW.$$  \hfill (2.6)

It is immediate also from (1.4) that

$$W_{xt} = (GW)_t = G_t W + GW_t.$$  \hfill (2.7)

Formulas (2.6) and (2.7) imply

$$U_x = G_t W + GW_t - F_x W - FGW = (G_t - F_x + GF - FG)W + GW_t - GFW.$$  \hfill (2.8)

It follows from (1.1), (2.8), and definition (2.5) that $U_x = GU$, that is, $U$ and $W$ satisfy the same equation. Taking into account $W(0, t) = I_2$, we derive $W_t(0, t) = 0$, and so by (2.5) we have $U(0, t) = -F(0, t)$. Finally, as

$$U_x = GU, \quad W_x = GW, \quad U(0, t) = -F(0, t), \quad W(0, t) = I_2,$$

we have $U(x, t) = -W(x, t)F(0, t)$ or, equivalently,

$$W_t(x, t) - F(x, t)W(x, t) = -W(x, t)F(0, t).$$  \hfill (2.9)

Put

$$Y(x, t) = W(x, t)R(t), \quad Z(x, t) = R(x, t)W(x, 0).$$  \hfill (2.10)
Recall that $R(t) = R(0, t)$. Therefore (1.5), (2.9), and (2.10) imply that

$$Y_t(x, t) = (F(x, t)W(x, t) - W(x, t)F(0, t))R(t) + W(x, t)F(0, t)R(t)$$

$$= F(x, t)Y(x, t), \quad Y(x, 0) = W(x, 0). \tag{2.11}$$

Formulas (1.5) and (2.10) imply that

$$Z_t(x, t) = F(x, t)Z(x, t), \quad Z(x, 0) = W(x, 0). \tag{2.12}$$

By (2.11) and (2.12) $Y = Z$, that is, (1.7) holds.

Remark 2.2 Though the case of continuous $F$ is more convenient for applications, it is immediate from the proof that the statement of Theorem 1.1 is true, when $F$ is differentiable with respect to $x$, and measurable and summable with respect to $t$ on all finite intervals from $\mathbb{R}_+$.

According to the proof of Theorem 1.1 the following remark is also true.

Remark 2.3 Theorem 1.1 holds on the domains more general than $\mathcal{D}$. In particular, it holds if we consider $(x, t) \in \mathcal{I}_1 \times \mathcal{I}_2$, where $\mathcal{I}_k$ ($k = 1, 2$) is the interval $[0, b_k)$ ($0 < b_k \leq \infty$).

Another interesting case of matrix factorizations related to boundary value problems is treated in [7,22].

3 Some applications

The matrix "focusing" $mKdV$ equation has the form

$$4v_t = v_{xxx} + 3(v_xv^*v + vv^*v_x), \tag{3.1}$$

where $v(x, t)$ is a $p \times p$ matrix function. Equation (3.1) is equivalent (see [8,14,49] and references therein) to zero curvature equation (1.1), where the $m \times m$ ($m = 2p$) matrix functions $G(x, t, z)$ and $F(x, t, z)$ are given by the formulas

$$G = izj + V, \quad j = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}, \quad V = \begin{bmatrix} 0 & v \\ -v^* & 0 \end{bmatrix}, \tag{3.2}$$

$$F = -iz^3j - z^2V - \frac{iz}{2}(V^2 + V_xj) + \frac{1}{4}(V_{xx} - 2V^3 - V_xV + VV_x). \tag{3.3}$$
At first we omit the variable $t$ in $V$ and $v$. The Weyl theory of the skew-self-adjoint Dirac system (also called Zakharov-Shabat or AKNS system)

$$
\frac{d}{dx}w(x, z) = (izj + V(x))w(x, z), \quad x \geq 0
$$

was treated in [11, 15, 32, 33] (see also preliminaries in [40]).

For the case of measurable matrix function $v$ such that

$$
\sup_{0 < x < \infty} \|v(x)\| \leq M, \quad (3.5)
$$

the Weyl matrix function $\varphi$ of system (3.4) is uniquely defined in the semi-plane $\Im z < -M$ by the inequality

$$
\int_0^\infty \begin{bmatrix} \varphi(z)^* & I_p \end{bmatrix} W(x, z)^* W(x, z) \begin{bmatrix} \varphi(z) \\ I_p \end{bmatrix} dx < \infty, \quad \Im z < -M < 0,
$$

where $W$ is the normalized by $W(0, z) = I_m$ fundamental solution of (3.4). Weyl functions are constructed using pairs of meromorphic $p \times p$ matrix functions $P_1(z)$, $P_2(z)$, which are nonsingular and have property-$j$, that is,

$$
P(z)^* P(z) > 0, \quad P(z)^* j P(z) \leq 0, \quad P : \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}.
$$

**Theorem 3.1** [32] There is a unique Weyl function of such a system (3.4) that (3.5) holds. This Weyl function is holomorphic in the semi-plane $\Im z < -M$. It is given by the equality

$$
\varphi(z) = \lim_{r \to \infty} \left( A_{11}(r, z)P_1(r, z) + A_{12}(r, z)P_2(r, z) \right) \\
\quad \times \left( A_{21}(r, z)P_1(r, z) + A_{22}(r, z)P_2(r, z) \right)^{-1} \quad (\Im z < -M),
$$

$$
A(r, z) = \left\{ A_{kp}(r, z) \right\}_{k,p=1}^2 := W(r, \bar{z})^*, \quad (3.8)
$$

where the pairs $\{P_1, P_2\}$ are arbitrary pairs satisfying (3.7).

Our next theorem on the evolution of the Weyl function in the case of the focusing mKdV follows from Theorems 1.1 and 3.1. The case of the defocusing mKdV was earlier treated in [11, 42, 44].
Theorem 3.2 Let a $p \times p$ matrix function $v \in C^1(D)$ have a continuous partial second derivative $v_{xx}$, and let $v_{xxx}$ exist. Assume that $v$ satisfies mKdV (3.1) and that the inequalities
\[
\sup_{(x,t) \in D} \|v(x,t)\| \leq M, \quad \sup_{(x,t) \in D} (\|v_x(x,t)\| + \|v_{xx}(x,t)\|) < \infty \quad (3.10)
\]
hold.

Then the evolution $\varphi(t,z)$ of the Weyl function of the skew-self-adjoint Dirac system (1.4), where $G$ has the form (3.2), is given by the equality
\[
\varphi(t,z) = \left( R_{11}(t,z) \varphi(0,z) + R_{12}(t,z) \right) \left( R_{21}(t,z) \varphi(0,z) + R_{22}(t,z) \right)^{-1} \quad (3.11)
\]
in the semi-plane $\Im z < -M < 0$. Here the block matrix function
\[
R(t,z) = \{ R_{kn}(t,z) \}_{k,n=1}^2 = R(0,t,z) \quad (3.12)
\]
is defined by the boundary values $v(0,t)$, $v_x(0,t)$, and $v_{xx}(0,t)$ via formulas (1.5) and (3.3).

Proof. As $V^* = -V$ and $(V_x j)^* = V_x j$, it is immediate from (3.3) that $F(x,t,z)^* + F(x,t,z) = 0$. Hence, it follows from (1.5) that
\[
\partial_t (R(x,t,z)^* R(x,t,z)) = 0.
\]
Therefore, using equalities $R(x,0,z) = I_m$ and (3.12), we get
\[
R(x,t,z)^* R(x,t,z) = I_m, \quad R(t,z)^* R(t,z) = I_m,
\]
or, equivalently,
\[
R(x,t,z)^* = R(x,t,z)^{-1}, \quad R(t,z)^* = R(t,z)^{-1}. \quad (3.13)
\]
In view of (3.13) rewrite (1.7) in the form
\[
A(x,t,z) R(x,t,z) = R(t,z) A(x,0,z), \quad (3.14)
\]
where $A(x,t,z) := W(x,t,z)^*$ (compare with (3.9)). Let $P(x,z)$ satisfy (3.7) and put
\[
\tilde{P}(x,t,z) = \begin{bmatrix} \tilde{P}_1(x,t,z) \\ \tilde{P}_2(x,t,z) \end{bmatrix} = R(x,t,z) P(x,z). \quad (3.15)
\]
By (3.14) and (3.15) we have

$$\mathcal{A}(x, t, z)\tilde{P}(x, t, z) = R(t, z)\mathcal{A}(x, 0, z)P(x, z). \quad (3.16)$$

Now, taking into account that \(P(x, z)\) is a nonsingular pair with property-\(j\), we show that \(\tilde{P}(x, t, z)\) is a nonsingular pair with property-\(j\) too. According to (1.5), (3.3), and (3.10) we get

$$\frac{\partial}{\partial t} (R(x, t, z)^* jR(x, t, z)) = R(x, t, z)^* \left(i(\bar{z}^3 - z^3)I_m + O(z^2)\right) R(x, t, z) \quad (3.17)$$

for \(z \to \infty\). Formula (3.17) implies that for some

$$M_1 > M > 0 \quad (M \geq \sup_{(x,t) \in D} \|v(x, t)\|), \quad (3.18)$$

and for all \(z\) from the domain

$$D_1 = \{z : z \in \mathbb{C}, \quad \Im z < -M_1, \quad 0 > \arg z > -\pi/4\} \quad (3.19)$$

we have

$$\frac{\partial}{\partial t} (R(x, t, z)^* jR(x, t, z)) \leq 0,$$

and so

$$R(x, t, z)^* jR(x, t, z) \leq j. \quad (3.20)$$

Relations (3.7), (3.15), and (3.20) imply that

$$\tilde{P}(x, t, z)^* \tilde{P}(x, t, z) > 0, \quad \tilde{P}(x, t, z)^* j\tilde{P}(x, t, z) \leq 0 \quad (z \in D_1). \quad (3.21)$$

Clearly, it suffices to prove (3.11) for values of \(z\) from \(D_1\). (According to (3.18) and (3.19) the domain \(D_1\) belongs to the semi-plane \(\Im z < -M\).)

In a way similar to the proofs of (3.13) and (3.20) we derive

$$\mathcal{A}(x, t, z) = W(x, t, z)^{-1}, \quad W(x, t, z)^* jW(x, t, z) \geq j \quad (\Im z < -M). \quad (3.22)$$

It is immediate from (3.22) that

$$\mathcal{A}(x, t, z)^* j\mathcal{A}(x, t, z) \leq j \quad (\Im z < -M). \quad (3.23)$$
Hence, inequalities (3.7) and (3.21) imply
\[
\det \left( A_{21}(x, 0, z)P_1(x, z) + A_{22}(x, 0, z)P_2(x, z) \right) \neq 0 \quad (\Im z < -M), \quad (3.24)
\]
\[
\det \left( A_{21}(x, t, z)\widetilde{P}_1(x, t, z) + A_{22}(x, t, z)\widetilde{P}_2(x, t, z) \right) \neq 0 \quad (z \in D_1). \quad (3.25)
\]
In view of (3.24) rewrite (3.16) as
\[
A(x, t, z)\widetilde{P}(x, t, z) = R(t, z) \begin{bmatrix} \phi(x, 0, z) \\ I_p \end{bmatrix} \left( A_{21}(x, 0, z)P_1(x, z) + A_{22}(x, 0, z)P_2(x, z) \right), \quad (3.26)
\]
\[
\phi(x, 0, z) := (A_{11}(x, 0, z)P_1(x, z) + A_{12}(x, 0, z)P_2(x, z)) 
\times \left( A_{21}(x, 0, z)P_1(x, z) + A_{22}(x, 0, z)P_2(x, z) \right)^{-1}. \quad (3.27)
\]
According to (3.24)-(3.26) we get
\[
(\phi(x, 0, z) := \begin{pmatrix} A_{11}(x, 0, z)P_1(x, z) + A_{12}(x, 0, z)P_2(x, z) \\ A_{21}(x, 0, z)P_1(x, z) + A_{22}(x, 0, z)P_2(x, z) \end{pmatrix})^{-1}
\times \left( A_{21}(x, 0, z)P_1(x, z) + A_{22}(x, 0, z)P_2(x, z) \right)^{-1} \quad (z \in D_1).
\]
As \(\widetilde{P}(x, t, z)\) satisfies (3.21) for \(z \in D_1\), using (3.8) we derive
\[
\varphi(t, z) = \lim_{x \to \infty} \left( A_{11}(x, t, z)\widetilde{P}_1(x, t, z) + A_{12}(x, t, z)\widetilde{P}_2(x, t, z) \right)^{-1} \quad (z \in D_1).
\]
In a similar way we derive from (3.8) and (3.27) that
\[
\varphi(0, z) = \lim_{x \to \infty} \phi(x, 0, z) \quad (\Im z < -M). \quad (3.30)
\]
Let us show that
\[
\det \left( R_{21}(t, z)\varphi(0, z) + R_{22}(t, z) \right) \neq 0 \quad (z \in D_1). \quad (3.31)
\]
Indeed, it follows from (3.7), (3.28), and (3.27) that
\[
[\phi(x, 0, z)^* \quad I_p] \begin{bmatrix} \phi(x, 0, z) \\ I_p \end{bmatrix} \leq 0. \quad (3.32)
\]
By (3.30) and (3.32) the inequality
\[
\begin{bmatrix}
\varphi(0, z)^* & I_p
\end{bmatrix} j
\begin{bmatrix}
\varphi(0, z) & I_p
\end{bmatrix} \leq 0.
\] (3.33)
is true. Finally, inequalities (3.20) and (3.33) imply
\[
\begin{bmatrix}
\varphi(0, z)^* & I_p
\end{bmatrix} R(t, z) j R(t, z) \begin{bmatrix}
\varphi(0, z) & I_p
\end{bmatrix} \leq 0 \quad (z \in D_1).
\] (3.34)

It is immediate from (3.34) that (3.31) holds. Relations (3.28)-(3.31) imply (3.11) in the domain \(D_1\). Hence, by analyticity equality (3.11) holds in the semi-plane \(\Im z < -M\). ■

In a way similar to [24] and to more general constructions for self-adjoint systems in [12,44] (see also some references therein), one can use structured operators to solve inverse problem for system (3.4) too. Namely, to recover \(v\), which satisfies condition (3.5), from the Weyl function \(\varphi\) we use operators \(S_l\) (acting in \(L^2_{p}(0, l), 0 < l < \infty\)) of the form
\[
S_l f = f(x) + \frac{1}{2} \int_0^l \int_{[x-r]}^{x+r} s'(\frac{\lambda + x - r}{2}) s'\left(\frac{\lambda + r - x}{2}\right)^* dz f(r) dr.
\] (3.35)

Here \(s'(x) := \frac{d}{dx} s(x)\). Below we give the procedure from [32] modified in accordance with [15,33].

First, we recover a \(p \times p\) matrix function \(s(x)\) with the entries from \(L^2(0, l)\) (i.e., \(s(x) \in L^2_{p \times p}(0, l)\)) via the Fourier transform. That is, we put
\[
s(x) = \frac{i}{2\pi} e^{-\eta x} \lim_{a \to \infty} \int_{-a}^a e^{i\xi x} z^{-1} \varphi(z/2) d\xi \quad (z = \xi + i\eta, \quad \eta < -2M),
\] (3.36)
the limit l.i.m. being the limit in \(L^2_{p \times p}(0, l)\). Formula (3.36) has sense for any \(l < \infty\), and so the matrix function \(s(x)\) is defined on the non-negative real semi-axis \(x \geq 0\). Moreover, \(s\) is absolutely continuous, it does not depend on the choice of \(\eta < -2M\), \(s'\) is bounded on any finite interval, and \(s(0) = 0\). To define the operator \(S_l\) we substitute \(s'(x)\) into (3.35).

Next, denote the \(p \times 2p\) block rows of \(W\) by \(\omega_1\) and \(\omega_2\):
\[
\omega_1(x) = [I_p \quad 0] W(x, 0), \quad \omega_2(x) = [0 \quad I_p] W(x, 0).
\] (3.37)
It follows from (3.2) and (3.4) that \( W(x,0)^*W(x,0) = I_m \). Hence, by (3.2), (3.4), and (3.37) we have

\[ v(x) = \omega_1'(x) \omega_2(x)^*, \tag{3.38} \]

and \( \omega_1, \omega_2 \) satisfy the equalities

\[ \omega_1(0) = [I_p \ 0], \quad \omega_1 \omega_1^* \equiv I_p, \quad \omega_1' \omega_1^* \equiv 0, \quad \omega_1 \omega_2^* \equiv 0. \tag{3.39} \]

It is immediate that \( \omega_1 \) is uniquely recovered from \( \omega_2 \) using (3.39).

Finally, we obtain \( \omega_2 \) via the formula

\[ \omega_2(l) = [0 \ I_p] - \int_0^l \left( S_l^{-1} s'(x) \right)^* [I_p \ s(x)] dx \quad (0 < l < \infty), \tag{3.40} \]

where \( S_l^{-1} \) is applied to \( s' \) columnwise.

**Theorem 3.3** Assume that \( \varphi \) is the Weyl function of system (3.4), where \( j \) and \( V \) have the form (3.2) and \( v \) satisfies (3.5). Then \( v \) is recovered from \( \varphi \) via formulas (3.38)–(3.40), where \( s \) and \( S_l \) are given by equalities (3.35) and (3.36). All the mentioned above relations are well-defined and the inequalities \( S_l \geq I \) hold.

Another inverse problem, where condition (3.5) on \( v \) is substituted by a condition on \( \varphi \), is also solved in [15, 33, 40] using the same procedure.

**Remark 3.4** One can apply Theorems 3.2 and 3.3 to recover solutions of mKdV. Theorems on the evolution of the Weyl functions constitute also the first step in proofs of uniqueness and existence of the solutions of nonlinear equations via ISpT method (see, for instance, [40]).

4 Factorization of the fundamental solution via Darboux matrix

Various versions of Bäcklund-Darboux transformation and commutation methods are widely used in spectral theory, differential equations and nonlinear integrable equations (see, for instance, [9, 10, 12, 13, 16, 19, 25, 27, 29, 51] and [40]).
numerous references therein). In this section we consider a so called GBDT version of the Bäcklund-Darboux transformation (see references in Introduction and some basic notations and results in Appendix). The statement of Theorem 4.2 is formulated and proved here in greater generality than before.

One can apply Theorem A.1 on GBDT to construct solutions and wave functions of nonlinear integrable equations. For this purpose we use auxiliary linear systems for integrable nonlinear equation, namely, linear systems:

\[
\begin{align*}
  w_x &= Gw, \quad w_t = Fw; \\
  G(x, t, z) &= -\sum_{k=0}^{r} z^k q_k(x, t) - \sum_{s=1}^{l} \sum_{k=1}^{r_s} (z - c_s)^{-k} q_{sk}(x, t), \\
  F(x, t, z) &= -\sum_{k=0}^{R} z^k Q_k(x, t) - \sum_{s=1}^{L} R_{s} \sum_{k=1}^{R_s} (z - C_s)^{-k} Q_{sk}(x, t),
\end{align*}
\]

and zero curvature (compatibility condition) representation (1.1) of the integrable nonlinear equation itself. We consider nonlinear equations on the domain \((x, t) \in I_1 \times I_2\), where \(I_k (k = 1, 2)\) is the interval \([0, b_k), (0 < b_k \leq \infty)\).

By Theorem 1.1 and Remark 2.3 the following corollary is true.

**Corollary 4.1** Let coefficients \(\{q_k(x, t)\}\) and \(\{q_{sk}(x, t)\}\) be differentiable with respect to \(t\) and let coefficients \(\{Q_k(x, t)\}\) and \(\{Q_{sk}(x, t)\}\) be differentiable with respect to \(x\) on the domain \(I_1 \times I_2\). Assume also that matrix functions \(\{q_k(x, t), q_{sk}(x, t)\}, \{Q_k(x, t), Q_{sk}(x, t)\}, \{\frac{\partial}{\partial t} q_k(x, t)\},\) and \(\{\frac{\partial}{\partial t} q_{sk}(x, t)\}\) are continuous with respect to \(x\) and \(t\), and that zero curvature equation (1.1), where \(G\) and \(F\) are given by (4.2) and (4.3), holds. Then there is the fundamental solution of (4.1) normalized by the condition

\[
w(0, 0, z) = I_m. \quad (4.4)
\]

**Proof.** Put

\[
w(x, t, z) = W(x, t, z)R(t, z). \quad (4.5)
\]

By (1.4), (1.5), and (1.7) we see that \(w\) given by (4.5) satisfies (4.1). According to the second relations in (1.4) and (1.5) equality (4.4) holds too. \(\blacksquare\)
Further assume that \( G, F, \) and coefficients in (4.2) and (4.3) satisfy conditions of Corollary 4.1, and that \( w \) is given by (4.5).

When we deal with two auxiliary linear systems, we fix \( n \in \mathbb{N} \), three \( n \times n \) parameter matrices, namely \( A_1, A_2, \) and \( S(0, 0) \), and two \( n \times m \) parameter matrices, namely, \( \Pi_1(0, 0) \) and \( \Pi_2(0, 0) \). These matrices are chosen so that they satisfy the matrix identity

\[
A_1 S(0, 0) - S(0, 0) A_2 = \Pi_1(0, 0) \Pi_2(0, 0)^\ast. \tag{4.6}
\]

Compare (4.6) with a similar matrix identity (A.2) for parameter matrices \( A_k, \Pi_k(0), \) and \( S(0) \) in Appendix. Matrix functions \( \Pi_1(x, t), \Pi_2(x, t), \) and \( S(x, t) \) are determined by the initial values \( \Pi_1(0, 0), \Pi_2(0, 0), \) and \( S(0, 0) \), respectively, differential equations (A.3)–(A.5) with respect to derivatives in \( x \) and similar equations with respect to derivatives in \( t \). That is, \( \Pi_1, \Pi_2, \) and \( S \) satisfy equations:

\[
(\Pi_1)_x = \sum_{k=0}^{r} A_1^k \Pi_1 Q_k + \sum_{s=1}^{r_s} \sum_{k=1}^{r_k} (A_1 - c_s I_n)^{-k} \Pi_1 Q_k, \tag{4.7}
\]

\[
(\Pi_2^\ast)_x = -\left( \sum_{k=0}^{r} q_k \Pi_1 A_2^k + \sum_{s=1}^{r_s} \sum_{k=1}^{r_k} q_k \Pi_2^\ast Q_k (A_2 - c_s I_n)^{-k} \right), \tag{4.8}
\]

\[
S_x = \sum_{k=1}^{r} \sum_{j=1}^{k} A_1^{k-j} \Pi_1 q_k \Pi_2^\ast A_2^{-1} - \sum_{s=1}^{r_s} \sum_{k=1}^{r_k} \sum_{j=1}^{r_j} (A_1 - c_s I_n)^{j-k-1} \times \Pi_1 q_k \Pi_2^\ast (A_2 - c_s I_n)^{-j}, \tag{4.9}
\]

which coincide with (A.3)–(A.5), and additional equations with respect to \( t \):

\[
(\Pi_1)_t = \sum_{k=0}^{R} A_1^k \Pi_1 Q_k + \sum_{s=1}^{R_s} (A_1 - C_s I_n)^{-k} \Pi_1 Q_k, \tag{4.10}
\]

\[
(\Pi_2^\ast)_t = -\left( \sum_{k=0}^{R} Q_k \Pi_2^\ast A_2^k + \sum_{s=1}^{R_s} \sum_{k=1}^{r_k} Q_k \Pi_2^\ast (A_2 - C_s I_n)^{-k} \right), \tag{4.11}
\]

\[
S_t = \sum_{k=1}^{R} \sum_{j=1}^{k} A_1^{k-j} \Pi_1 q_k \Pi_2^\ast A_2^{-1} - \sum_{s=1}^{R_s} \sum_{k=1}^{r_k} \sum_{j=1}^{r_j} (A_1 - C_s I_n)^{j-k-1} \times \Pi_1 q_k \Pi_2^\ast (A_2 - C_s I_n)^{-j}. \tag{4.12}
\]
We require
\[
\{c_k\} \cap \sigma(A_k) = \emptyset, \quad \{C_s\} \cap \sigma(A_k) = \emptyset \quad (k = 1, 2),
\]  
(4.13)
where \(\sigma(A)\) is the spectrum of \(A\). Then, Theorem 4.1 provides expressions for derivatives \((w_A(x, t, z))_x\) and \((w_A(x, t, z))_t\):
\[
(w_A)_x = \tilde{G}w_A - w_AG, \quad (w_A)_t = \tilde{F}w_A - w_AF,
\]  
(4.14)
where \(\tilde{G}\) has the same structure as \(G\) and is given by formulas (A.9)-(A.13). Similarly \(\tilde{F}\) has the same structure as \(F\), namely,
\[
\tilde{F}(x, t, z) = -\sum_{k=0}^{R} z^k \tilde{Q}_k(x, t) - \sum_{s=1}^{L} \sum_{k=1}^{R_s} (z - C_s)^{-k} \tilde{Q}_{sk}(x, t),
\]  
(4.15)
where coefficients are given by (A.10)-(A.13) after substitution \(Q_k, Q_{sk}, \tilde{Q}_k, \tilde{Q}_{sk}, R, R_s, L, \) and \(L_s\) instead of \(q_k, q_{sk}, \tilde{q}_k, \tilde{q}_{sk}, r, r_s, l, \) and \(l_s, \) respectively, in those formulas. The matrix function \(w_A\) in (4.14) has the form
\[
w_A(x, t, z) = I_m - \Pi_2(x, t)^*S(x, t)^{-1}(A_1 - zI_n)^{-1}\Pi_1(x, t)
\]  
(4.16)
(compare with (A.7)). By (4.1) and (4.14) we have
\[
\tilde{w}_x = \tilde{G}\tilde{w}, \quad \tilde{w}_t = \tilde{F}\tilde{w}, \quad \tilde{w}(x, t, z) := w_A(x, t, z)w(x, t, z).
\]  
(4.17)

The following theorem shows that \(\tilde{G}\) and \(\tilde{F}\) satisfy zero curvature equation
\[
\tilde{G}_t - \tilde{F}_x + [\tilde{G}, \tilde{F}] = 0
\]  
(4.18)
on the domain \(\mathcal{D}_S\) of the points of invertibility of \(S\):
\[
\mathcal{D}_S = \mathcal{I}_1 \times \mathcal{I}_2 \setminus \mathcal{O}_S, \quad \mathcal{O}_S = \{(x, t) : \det S(x, t) = 0\}.
\]  
(4.19)

**Theorem 4.2** Let \(G\), \(F\), and coefficients in (4.2) and (4.3) satisfy conditions of Corollary 4.1. Assume that \(\Pi_1, \Pi_2, \) and \(S\) satisfy (4.6)-(4.12). Then, the coefficients \(\{\tilde{q}_k\}, \{\tilde{q}_{sk}\}\) and \(\{\tilde{Q}_k\}, \{\tilde{Q}_{sk}\}\) of \(\tilde{G}\) and \(\tilde{F}\), respectively, are continuous on \(\mathcal{D}_S\) together with derivatives \(\frac{\partial}{\partial x}\tilde{q}_k\) and \(\frac{\partial}{\partial x}\tilde{q}_{sk}\). Moreover, zero curvature equation (4.18) holds on \(\mathcal{D}_S\).
Proof. By (A.10)-(A.13) and (4.7)-(4.12) the differentiability and continuity statements of our theorem for the coefficients of $\tilde{G}$ and $\tilde{F}$ are true. Moreover, it follows from (1.1) that coefficients $\{\frac{\partial}{\partial x} Q_k\}$ and $\{\frac{\partial}{\partial x} Q_{sk}\}$, and hence also coefficients $\{\frac{\partial}{\partial x} Q_k\}$ and $\{\frac{\partial}{\partial x} Q_{sk}\}$, are continuous on $D_S$. Therefore, the matrix functions $\tilde{G}$, $\tilde{G}_t$, $\tilde{F}$, and $\tilde{F}_x$ are continuous on the domain $D_S$.

Thus, taking into account (4.17) we see that $\tilde{w}$, $\tilde{w}_x$, $\tilde{w}_t$, and $\tilde{w}_{xt}$ exist and are continuous on $D_S$. Hence, the conditions of the stronger formulation of the theorem on mixed derivative, which is already used in the proof of Theorem 1.1 are fulfilled and $\tilde{w}_{xt} = \tilde{w}_{tx}$ in the interior $(D_S)_i$ of $D_S$. Using (4.17) rewrite the equality $\tilde{w}_{xt} = \tilde{w}_{tx}$ in the form

$$(\tilde{G}_t + \tilde{G}\tilde{F})\tilde{w} = (\tilde{F}_x + \tilde{F}\tilde{G})\tilde{w}. \quad (4.20)$$

It follows from (4.1) and (4.4) that $w$ is invertible on $I_1 \times I_2$, and it follows from (A.14) that $w_A$ is invertible on $D_S$. Thus $\tilde{w} = w_A w$ is invertible on $D_S$. Now, it follows from (4.20) that (4.18) holds in $(D_S)_i$. By continuity, (4.18) holds on $D_S$. ■

Remark 4.3 Theorem 4.2 is basic to construct solutions and wave functions of integrable equations via GBDT. The corresponding normalized wave functions $\tilde{w}$ (see [17, 18, 33, 34, 36, 37, 39]) have the form:

$$\tilde{w}(x, t, z) = \tilde{w}(x, t, z)w_A(0, 0, z)^{-1} = w_A(x, t, z)w(x, t, z)w_A(0, 0, z)^{-1}. \quad (4.21)$$

Compatibility of the equations (4.7)-(4.12) is a separate question. In full generality it will be addressed elsewhere, and here we consider an important and characteristic example of the compatibility of equations (4.7) and (4.10), when $G$ and $F$ are polynomials.

Proposition 4.4 Let $G$ and $F$ be polynomials

$$G(x, t, z) = -\sum_{k=0}^{R} z^k q_k(x, t), \quad F(x, t, z) = -\sum_{s=0}^{R} z^s Q_s(x, t), \quad (4.22)$$

such that the conditions of Corollary 4.1 are satisfied. Then, the corresponding equations (4.7) and (4.10), which determine $\Pi_1$, are compatible.
Proof. When $G$ and $F$ are polynomials, equations (4.7) and (4.10) take the form

\begin{align}
(P_1)_x &= \sum_{k=0}^{r} A_k^1 \Pi_1 q_k, \\
(P_1)_t &= \sum_{s=0}^{R} A_s^1 \Pi_1 Q_s.
\end{align}

(4.23)

Denote the $p$-th column of $\Pi_1$ by $(\Pi_1)_p$ and introduce a block vector with $(\Pi_1)_p$ as its blocks:

\[ \vec{\Pi}_1 = \begin{bmatrix} (\Pi_1)_1 \\ \vdots \\ (\Pi_1)_m \end{bmatrix} \in \mathbb{C}^{mn}. \]

(4.24)

Equations (4.23) can be rewritten in an equivalent form in terms of $\vec{\Pi}_1$:

\begin{align}
(\vec{\Pi}_1)_x &= \gamma \vec{\Pi}_1, \\
(\vec{\Pi}_1)_t &= \Gamma \vec{\Pi}_1, \\
\gamma(x,t) &= \sum_{k=0}^{r} q_k(x,t)^T \otimes A_k^1, \\
\Gamma(x,t) &= \sum_{s=0}^{R} Q_s(x,t)^T \otimes A_s^1,
\end{align}

(4.25)

(4.26)

where $q^T$ denotes the transpose of a matrix $q$, and $q \otimes A$ is the Kronecker product of matrices $q$ and $A$. As $\{q_k\}$ and $\{Q_k\}$ satisfy conditions of Corollary 4.1, so $\gamma$ and $\Gamma$ satisfy the differentiability and continuity conditions of Theorem 1.1 and to prove the compatibility it remains only to show that zero curvature equation

\[ \gamma_t - \Gamma_x + [G, f] = 0 \]

holds. For that purpose consider the block in the $i$-th block row and in the $p$-th block column in (4.27). We get an equality

\[ \sum_{k=0}^{r} \left( \frac{\partial}{\partial t} q_k \right)_{pi} A_k^1 - \sum_{s=0}^{R} \left( \frac{\partial}{\partial x} Q_s \right)_{pi} A_s^1 + \left( \sum_{k=0}^{r} (q_k)^T_i \otimes A_k^1 \right) \left( \sum_{s=0}^{R} (Q_s^T)_p \otimes A_s^1 \right) \\
- \left( \sum_{s=0}^{R} (Q_s^T)_i \otimes A_s^1 \right) \left( \sum_{k=0}^{r} (q_k^T)_p \otimes A_k^1 \right) = 0, \]

(4.28)
where \((q_k)_i^T\) is the transpose of the \(i\)-th column of \(q_k\). Rewrite (4.28) as

\[
\sum_{k=0}^{r} \left( \frac{\partial}{\partial t} q_k \right)_{pi} A_1^k - \sum_{s=0}^{R} \left( \frac{\partial}{\partial x} Q_s \right)_{pi} A_1^s + \sum_{k=0}^{r} \sum_{s=0}^{R} \left( (Q_s q_k)_{pi} - (q_k Q_s)_{pi} \right) A_1^{k+s} = 0.
\]

It is immediate that independently from the choice of \(A_1\) the equality above follows from the equality

\[
\sum_{k=0}^{r} \left( \frac{\partial}{\partial t} q_k \right)_{pi} z^k - \sum_{s=0}^{R} \left( \frac{\partial}{\partial x} Q_s \right)_{pi} z^s + \sum_{k=0}^{r} \sum_{s=0}^{R} \left( (Q_s q_k)_{pi} - (q_k Q_s)_{pi} \right) z^{k+s} = 0.
\]

(4.29)

In other words equation (4.27) follows from

\[
\sum_{k=0}^{r} z^k \frac{\partial}{\partial t} q_k - \sum_{s=0}^{R} z^s \frac{\partial}{\partial x} Q_s + \sum_{k=0}^{r} \sum_{s=0}^{R} z^{k+s} \left( Q_s q_k - q_k Q_s \right) = 0.
\]

(4.30)

Notice that in view of (4.22) formula (4.30) is equivalent to (1.1). Thus, (4.30) holds, and so (4.27) holds too. ■

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A Appendix. GBDT for system depending rationally on spectral parameter

In this appendix we consider the GBDT version of the Bäcklund-Darboux transformation (BDT) for a general case of first order system depending rationally on the spectral parameter \(z\):

\[
w_x = Gw, \quad G(x, z) = -\left( \sum_{k=0}^{r} z^k q_k(x) + \sum_{s=1}^{l} \sum_{k=1}^{r_s} (z - c_s)^{-k} q_{sk}(x) \right), \quad (A.1)
\]

where \(x \in \mathcal{I}\), and the coefficients \(q_k(x)\) and \(q_{sk}(x)\) are \(m \times m\) locally integrable matrix functions. To simplify notations we assume that \(\mathcal{I}\) is either interval \([0, b]\) \((0 < b < \infty)\) or interval \([0, b]\) \((0 < b \leq \infty)\). In our presentation of
GBDT we follow Section 3 of the review [39]. Further references one can find in Introduction and [39].

As GBDT is a so called iterated BDT we fix an integer $n > 0$. Next, we fix five matrices, namely, $n \times n$ matrices $A_k (k = 1, 2)$ and $S(0)$, and $n \times m$ matrices $\Pi_k(0) (k = 1, 2)$. It is required that these matrices form an $S$-node, that is, the identity

$$A_1 S(0) - S(0) A_2 = \Pi_1(0) \Pi_2(0)^*$$  \hspace{1cm} (A.2)

holds. Matrix functions $\Pi_k(x)$ are introduced via initial values $\Pi_k(0)$ and linear differential equations:

\begin{align*}
(\Pi_1)_x &= \sum_{k=0}^r A_k^1 \Pi_1 q_k + \sum_{s=1}^l \sum_{k=1}^{r_s} (A_1 - c_s I_n)^{-k} \Pi_1 q_{sk}, \hspace{1cm} (A.3) \\
(\Pi_2^*)_x &= -\left( \sum_{k=0}^r q_k \Pi_2^* A_k^2 + \sum_{s=1}^l \sum_{k=1}^{r_s} q_{sk} \Pi_2^* (A_2 - c_s I_n)^{-k} \right), \hspace{1cm} (A.4)
\end{align*}

where $\{q_k\}$ and $\{q_{sk}\}$ are coefficients from $G$. Compare (A.1) with (A.4) to see that $\Pi_2^*$ can be viewed as a generalized eigenfunction of the system $u_x = Gu$.

Matrix function $S(x)$ is introduced via $\frac{d}{dx}S$ by the equality

\begin{align*}
S_x &= \sum_{k=1}^r \sum_{j=1}^k A_1^{k-j} \Pi_1 q_k \Pi_2^* A_2^{j-1} - \sum_{s=1}^l \sum_{k=1}^{r_s} \sum_{j=1}^k (A_1 - c_s I_n)^{j-k-1} \times \Pi_1 q_{sk} \Pi_2^* (A_2 - c_s I_n)^{-j}. \hspace{1cm} (A.5)
\end{align*}

Equality (A.5) is chosen so that the identity $\left( A_1 S - SA_2 \right)_x = \left( \Pi_1 \Pi_2^* \right)_x$ holds. Hence, taking into account (A.2) we have

$$A_1 S(x) - S(x) A_2 = \Pi_1(x) \Pi_2(x)^*, \hspace{1cm} x \in \mathcal{I}. \hspace{1cm} (A.6)$$

By Theorem A.1 below, the Darboux matrix for system (A.1) has the form (1.2):

$$w_A(x, z) = I_m - \Pi_2(x)^* S(x)^{-1} (A_1 - z I_n)^{-1} \Pi_1(x). \hspace{1cm} (A.7)$$
In other words, \( w_A \) satisfies the equation
\[
\frac{d}{dx} w_A(x, z) = \tilde{G}(x, z) w_A(x, z) - w_A(x, z) G(x, z),
\] (A.8)
where \( \tilde{G} \) has the same structure as \( G \):
\[
\tilde{G}(x, z) = -\left( \sum_{k=0}^{r} z^k \tilde{q}_k(x) + \sum_{s=1}^{l} \sum_{k=1}^{r_s} (z - c_s)^{-k} \tilde{q}_{sk}(x) \right).
\] (A.9)

The transformed coefficients \( \tilde{q}_k \) and \( \tilde{q}_{sk} \) are given by the formulas
\[
\tilde{q}_k = q_k - \sum_{j=k+1}^{r} \left( q_j Y_{j-k-1} - X_{j-k-1} q_j + \sum_{i=k+2}^{j} X_{j-i} q_j Y_{i-k-2} \right),
\] (A.10)
\[
\tilde{q}_{sk} = q_{sk} + \sum_{j=k}^{r_s} \left( q_{sj} Y_{s,k-j-1} - X_{s,k-j-1} q_{sj} - \sum_{i=k}^{j} X_{s,i-j-1} q_{sj} Y_{s,k-i-1} \right),
\] (A.11)
where \( X_k(x), Y_k(x), X_{sk}(x), \) and \( Y_{sk}(x) \) are expressed in terms of the matrices \( A_k \) and matrix functions \( S(x) \) and \( \Pi_k(x) \):
\[
X_k = \Pi_2^* S^{-1} A_k^k \Pi_1, \quad X_{sk} = \Pi_2^* S^{-1} (A_1 - c_s I_n)^k \Pi_1, \quad (A.12)
\]
\[
Y_k = \Pi_2^* A_k^2 S^{-1} \Pi_1, \quad Y_{sk} = \Pi_2^* (A_2 - c_s I_n)^k S^{-1} \Pi_1. \quad (A.13)
\]

Denote the spectrum of matrix \( A \) by \( \sigma(A) \).

**Theorem A.1** Let first order system (A.1) and five matrices \( S(0), A_k, \) and \( \Pi_k \) \( (k = 1, 2) \) be given. Assume that the identity (A.2) holds and that \( \{c_s\} \cap \sigma(A_k) = \emptyset \) \( (k = 1, 2) \). Then, in the points of invertibility of \( S \), the transfer matrix function \( w_A \) given by (A.7), where \( S \) and \( \Pi_k \) are determined by (A.3)–(A.5), satisfies equation (A.8), where \( \tilde{G} \) is determined by the formulas (A.9)–(A.13).

**Remark A.2** The matrix function \( w_A \) is invertible, since it can be derived from (A.6) that
\[
w_A(x, z)^{-1} = I_m + \Pi_2(x)^* (A_2 - z I_n)^{-1} S(x)^{-1} \Pi_1(x).
\] (A.14)
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