Remote Tomography Via von Neumann-Arthurs-Kelly Interaction

S. M. Roy,1 Abhinav Deshpande,2 and Nitica Sakharwade2

1 Homi Bhabha Centre for Science Education, Tata Institute of Fundamental Research, Mumbai
2 Department of Physics, IIT Kanpur

(Dated: May 11, 2014)

Teletation usually involves entangled particles 1,2 shared by Alice and Bob, Bell-state measurement on particle 1 and system particle by Alice, classical communication to Bob, and unitary transformation by Bob on particle 2. We propose a novel method: interaction-based remote tomography. Alice arranges an entanglement generating von Neumann-Arthurs-Kelly interaction between the system and two apparatus particles, and then teleports the latter to Bob. Bob reconstructs the unknown initial state of the system not received by him by quadrature measurements on the apparatus particles.

PACS numbers: 03.65.Ta, 03.67.Lx, 06.20.Dk, 42.50.St

Keywords: quantum tracking, quantum tomography, teleportation, joint measurements, conjugate variables

Introduction. The idea of ‘quantum tracking’ of a single system observable by an apparatus observable first occurred in the measurement theory of Von Neumann, and generalized to two canonically conjugate observables by Arthurs and Kelly Jr. Suppose the initial state of the system-apparatus combine is factorized. If after interaction, the apparatus observable X has the same expectation value in the final state as the system observable A in the initial state, for arbitrary initial state of the system, then X is said to track A. This nomenclature was probably used first by Arthurs and Goodman who, as well as, Gudder, Hagler, and Stulpe proved the joint measurement uncertainty relation. The Arthurs-Kelly interaction can also enable exact measurements of some quantum correlations between position and momentum.

We shall be concerned here not with joint measurements but with the completely different idea of ‘remote quantum tomography’ which is akin to ‘quantum teleportation’ or the replication of an unknown quantum state of a particle at a distant location without physically transporting that particle. Teleportation, as first proposed by Bennett, Brassard, Crépeau, Jozsa, Peres and Wootters and generalized to infinite dimensional Hilbert spaces by Vaidman, usually involves four different technologies. (i) An EPR-pair is shared by observers A (Alice) and B (Bob) at distant locations. (ii) The system particle with unknown state is received by A who makes a Bell-state measurement on the joint state of that particle and the first particle of the EPR-pair and (iii) communicates the result via a classical channel to B, (iv) B then makes a unitary transformation depending on the classical information on the second particle of the EPR-pair to replicate the unknown system state. Teleportation has been experimentally realized, e.g. by Bouwmeester et al., and the methods and uses extensively reviewed, e.g. by Braunstein et al. The density matrix of the system particle can be constructed by quadrature measurements on the second particle of the EPR pair completing remote tomography.

Interaction-based Remote Tomography. We report here a completely new method for remote quantum tomography which replaces the above four technologies by the single step of an interaction between the system particle (say photon) and two apparatus photons. At location A, a system photon with unknown state interacts via a quantum optically generated Arthurs-Kelly interaction (see e.g. Stenholm) with two apparatus particles (say photons) in a known state. The apparatus photons are then sent to a distant observer B. B makes quantum tomographic quadrature measurements on the apparatus photons and reconstructs the exact initial density matrix of the system photon without ever having received that particle. (See Fig. 1). Practical implementation will require a quantum channel to send the two apparatus photons from location A to the distant location of B and a generalization of single photon Optical Homodyne Tomography (see e.g. and ) to two photons, both of which seem feasible and worthwhile. Instead of the usual method of preparing the apparatus photons in an initial entangled state and sharing them between A and B, this method of remote quantum tomography exploits the entanglement between the system photon and the apparatus photons generated by the three-particle Arthurs-Kelly interaction. Multiparticle interactions to generate entanglement have previously been exploited for quantum enhanced metrology. We proceed now to put the new method on a rigorous footing.

A Symmetry Property. We shall use the Arthurs-Kelly system-apparatus interaction Hamiltonian, which is invariant under a class of simultaneous transformations on the system and apparatus specified below,

\[ H = K(\hat{q}\hat{p}_1 + \hat{p}\hat{p}_2) = K(\hat{q}_0\hat{p}_1,0 + \hat{p}_0\hat{p}_2,0) \]

where \( K \) is a coupling constant, \( \hat{q}, \hat{p} \) are position and momentum operators of the system, \( \hat{x}_1, \hat{x}_2 \) are two commuting position operators of the apparatus (e.g. two photons), with conjugate momenta \( \hat{p}_1, \hat{p}_2 \) which are coupled to \( \hat{q} \) and \( \hat{p} \) respectively. The rotated quadrature operators...
with subscript $\theta$ are defined using the rotation matrix $R$, 
\[
\begin{pmatrix}
\hat{q}_\theta \\
\hat{p}_\theta
\end{pmatrix}
= R
\begin{pmatrix}
\hat{q} \\
\hat{p}
\end{pmatrix}, \quad
\begin{pmatrix}
\hat{p}_{1,\theta} \\
\hat{p}_{2,\theta}
\end{pmatrix}
= R
\begin{pmatrix}
\hat{p}_1 \\
\hat{p}_2
\end{pmatrix}, \quad
R = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}.
\]
(2)

The operators $\hat{p}_{j,\theta}$ are seen to be just the commuting momentum operators of the apparatus particles corresponding to rotated co-ordinates $x_{j,\theta}$, for $j = 1, 2$,
\[
x_{1,\theta} + i x_{2,\theta} = \exp(-i\theta)(x_1 + i x_2), \quad \hat{p}_{j,\theta} = -i\partial/\partial x_{j,\theta}.
\]
(3)

We also define,
\[
\dot{x}_{j,\theta} + i \dot{x}_{2,\theta} = \exp(-i\theta)(\dot{x}_1 + i \dot{x}_2).
\]
(4)

Then, in the case of the apparatus being two photons with annihilation operators $a_i, i = 1, 2$,
\[
\dot{x}_{i,\theta} = a_i \exp(-i\theta)/\sqrt{2} + \mathrm{h.c.}, \quad \hat{p}_{i,\theta} = \dot{x}_{i,\theta} + \pi/2.
\]
(5)

**Exact Solution of the Schrödinger equation with generalized initial conditions.** We assume the constant $K$ to be so large that the free Hamiltonians of the system and the apparatus are negligible compared to $H$ during interaction time $T$. We start from an initial factorized state,
\[
\langle q | x_1, x_2 | \psi(t = 0) \rangle = \langle q | \phi \rangle \chi(x_1, x_2),
\]
(6)

where $\langle q | \phi \rangle$ is the unknown system wave function, and the apparatus wave function is chosen to be a product of two Gaussians, $\chi(x_1, x_2) = \chi_1(x_1)\chi_2(x_2)$,
\[
\chi_1(x_1) = \pi^{-1/4} b_1^{-1/2} \exp[-x_1^2/(2b_1^2)],
\chi_2(x_2) = \pi^{-1/4} (2b_2)\sqrt{2} \exp[-2b_2^2 x_2^2].
\]
(7)

Arthurs and Kelly chose $b_2 = b_1 = b$. We solve the Schrödinger equation with arbitrary $b_1, b_2$; we need $b_1 \neq b_2$ to utilise the symmetry of the Hamiltonian.

The commutator of the two terms in $H$ in fact commutes with each of the terms. Hence,
\[
\exp(-iHt) = \exp(-iKt\hat{q}\hat{p}_1)\exp(-iKt\hat{q}\hat{p}_2)\exp(iK^2t^2\hat{p}_1\hat{p}_2/2).
\]
(8)

If we work in the $q, x_1, p_2$ representation, the three exponentials on the right-hand side successively translate $x_1, q, x_1$ acting on the initial wavefunction. Hence the exact solution of the Schrödinger equation is,
\[
\langle q, x_1, p_2 | t \rangle = \chi_1(x_1 - qKt + (1/2)p_2K^2t^2)\chi_2(p_2)\phi(q - p_2Kt),
\]
(9)

where $\chi_2$ denotes a Fourier transform of $\chi_2$. The coordinate space wave function is given by a Fourier transform. Choosing $KT = 1$ we obtain,
\[
\psi(q, x_1, x_2) = \int \psi(q, x_1, x_2, \xi) d\xi,
\]
(10)

where,
\[
\psi(q, x_1, x_2, \xi) = \phi(\xi) \exp(i(q - \xi)x_2)/(2\pi\sqrt{b_1b_2})
\exp(-2(q + \xi)(2b_1^2 - (q - \xi)^2)/(8b_1^2 b_2^2)).
\]
(11)

Tracing the system-apparatus density matrix over the system co-ordinate we obtain the apparatus density matrix at time $T$,
\[
\langle x_1, x_2 | \rho_{APP}(T) | x'_1, x'_2 \rangle = \int \psi(q, x_1, x_2, \xi)\psi^*(q, x'_1, x'_2, \xi) d\xi d\xi'.
\]
(12)

The probability densities $P_t(x_1)$ and $P_t(x_2)$ for $x_1$ and $x_2$ are obtained by integrating the diagonal elements of this density operator over $x_2$ and $x_1$ respectively. In fact $P_t(x_1)$ and $P_t(x_2)$ can be obtained from the Arthurs-Kelly expressions by $b^2 \to (b_1^2 + b_2^2)/2$ and $b^{-2} \to (b_1^{-2} + b_2^{-2})/2$ respectively. The resulting expectation values of $x_1, x_2$ equal those of $q, p$ respectively, but the dispersions are higher, $(\Delta x_1)^2 = (\Delta q)^2 + (b_1^2 + b_2^2)/2$, $(\Delta x_2)^2 = (\Delta p)^2 + (b_1^2 + b_2^2)/(8b_1^2 b_2^2)$.

Our key new results require $b_1 \neq b_2$. First, integrating the off-diagonal elements of the apparatus density matrix over $x_2, x_2'$,
\[
\int \langle x_1, x_2 | \rho_{APP}(T) | x'_1, x'_2 \rangle dx_2 dx_2' = \frac{1}{2\pi} \int \delta(q)^2 \exp(-x_1^2/(2b_1^2) + (x_1' - x_1)^2/(2b_2^2)) dq.
\]
(13)

This shows that we can extract the exact initial system position probability density from the final apparatus density matrix as the expectation value of an apparatus observable.
\[
|\langle q = x_1 | \phi \rangle|^2 = \lim_{b_2 \to 0} b_2 \sqrt{\pi} \int dx_2 dx_2'
\langle x_1, x_2 | \rho_{APP}(T) | x'_1, x'_2 \rangle
= \lim_{b_2 \to 0} Tr \rho_{APP}(T) Y(x_1),
\]
(14)

where $Y(x_1)$ is the apparatus observable,
\[
Y(x_1) = \frac{b_2}{\sqrt{\pi}} |x_1\rangle |x_1\rangle \int |x'_2\rangle \langle x'_2| dx'_2 dx'_2'
= 2b_2 \sqrt{\pi} \langle x_1\rangle (\langle \hat{p}_2 = 0 | \hat{p}_2 = 0 \rangle).
\]
(15)
Similarly, the exact initial system momentum probability density is an expectation value of an apparatus observable in the final apparatus density matrix,

\[ \langle p = x_2 | \phi \rangle^2 = \lim_{b_2 \to \infty} \frac{1}{2b_1 \sqrt{\pi}} \int dx_1 dx_1' \langle x_1, x_2 | \rho_{APP}(T) | x_1', x_2 \rangle = \lim_{b_2 \to \infty} Tr \rho_{APP}(T) Z(x_2), \quad (16) \]

where \( Z(x_2) \) is the apparatus observable,

\[ Z(x_2) = \frac{\sqrt{\pi}}{b_1} |(x_2)(x_2)|(|\hat{p}_1 = 0)\rangle \langle \hat{p}_1 = 0|). \quad (17) \]

In the limit, \( b_1 \to 0, b_2 \to \infty \), we have faithful tracking of both system position and system momentum, since \( Y(x_1) \) tracks the position projectors \( |\hat{q} = x_1 > < \hat{q} = x_1| \) for all \( x_1 \) and \( Z(x_2) \) tracks the system momentum projectors \( |\hat{p} = x_2 > < \hat{p} = x_2| \) for all \( x_2 \).

Further, the Wigner function of the initial system state can be calculated exactly from the final apparatus density matrix,

\[ W(x_1, x_2) = \lim_{b_1 \to 0, b_2 \to \infty} \frac{b_2}{2\pi b_1} \int dx_1' dx_2' \langle x_1, x_2 | \rho_{APP}(T) | x_1', x_2 \rangle. \quad (18) \]

We now show that we can indeed measure a continuous infinity of apparatus observables on the final state to obtain the initial Wigner function of the system particle.

**Rotated quadratures and Quantum Tomography.** In order to harness the symmetry property mentioned above, we need a corresponding symmetry property of the initial apparatus state, \( \chi(x_1, x_2) = \chi(x_1, x_2, \theta) \). Therefore we are forced to use initial apparatus states very different from Arthurs and Kelly. We need,

\[ 2b_1 b_2 = 1; \quad \chi(x_1, x_2) = \chi(x_1, x_2, \theta), \]

\[ = \pi^{-1/2} b_1^{-1} \exp \left[-(x_1^2 + x_2^2)/(2b_1^2)\right]. \quad (19) \]

For this choice, the system-apparatus initial state can be rewritten for arbitrary \( \theta \) as,

\[ \langle \hat{q}_\theta = q_\theta | \hat{x}_{1, \theta} = x_{1, \theta}, \hat{x}_{2, \theta} = x_{2, \theta} | \psi(t = 0) \rangle = \langle \hat{x}_{1, \theta} = x_{1, \theta}, \hat{x}_{2, \theta} = x_{2, \theta} | \psi(t = 0) \rangle = \chi(x_{1, \theta}, x_{2, \theta}), \quad (20) \]

with the obvious notation \( (\hat{q}_\theta - q_\theta | \hat{q}_\theta = q_\theta) = 0 \). Since the Hamiltonian \( H_0 \) and the initial apparatus states have the exact same form in terms of the rotated variables as in terms of the original variables, we can repeat the previous calculations with \( \hat{q}_\theta, \hat{p}_\theta, q_\theta, p_\theta, x_{1, \theta}, x_{2, \theta} \) replacing \( \hat{q}, \hat{p}, q, p, x_1, x_2 \) respectively. Hence the matrix elements of \( \rho_{APP} \) are obtained by replacing in the previously obtained expressions:

\[ q, p, x_1, x_2, x_1', x_2' \to \hat{q}_\theta, \hat{p}_\theta, q_\theta, x_1, x_2, x_1', x_2'. \]

Thus, we obtain for arbitrary \( \theta \),

\[ \left\langle \hat{q}_\theta = q_\theta | \phi \right\rangle^2 = \lim_{b_1 \to 0} Tr \rho_{APP}(T) Y_\theta(u), \quad (21) \]

\[ Y_\theta(u) = \sqrt{\frac{\pi}{b_1}} |\hat{x}_{1, \theta} = u, \hat{x}_{2, \theta} = u \rangle \langle \hat{x}_{1, \theta} = u, \hat{x}_{2, \theta} = u| \quad (22) \]

Since, \( \hat{q}_\theta = \hat{q} + \pi/2 \) the initial system probability densities for it are obtained from above just by replacing \( \theta \to \theta + \pi/2 \).

We have proved that in the limit,

\[ b_1 \to 0, b_2 = 1/(2b_1) \to \infty, \quad (23) \]

we can recover exactly the initial system probability densities of arbitrary Hermitian linear combinations \( \hat{q}_\theta \).

\[ \langle \hat{q}_\theta = u | \rho_S | \hat{q}_\theta = u \rangle = \left| \langle \hat{q}_\theta = u | \phi \rangle \right|^2, \quad (24) \]

and hence the initial Wigner function, by measuring expectation values of Hermitian operators in the same final state of the apparatus after interaction.

**Reconstruction of the initial Density Matrix of the System from the final Apparatus Density Matrix.** Quantum tomography is completed by calculating the Wigner function \( W(q, p) \) as an inverse Radon transform,

\[ W(q, p) = (2\pi)^{-2} \int_0^\infty \eta d\eta \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} du \exp(i\eta(u - (q \cos \theta + p \sin \theta))) |\hat{q}_\theta = u, \rho_S | \hat{q}_\theta = u \rangle, \quad (25) \]

and from that the density operator,

\[ \langle q | \rho_S | q' \rangle = (2\pi)^{-1} \int_0^{\pi} |q - q'| d\theta (\sin \theta)^{-2} \exp((-i(q^2 - q'^2) \cot \theta)/2) \int_{-\infty}^{\infty} du \exp(iu(q - q')/\sin \theta) |\hat{q}_\theta = u | \rho_S | \hat{q}_\theta = u \rangle. \quad (26) \]

**Accounting for time evolution of the apparatus photons during transit time \( \tau \) to distant location \( B \).** Note that

\[ Tr \rho_{APP}(T) Y_\theta(u) = Tr \rho_{APP}(T + \tau) \times \exp(-iH_\tau \tau) Y_\theta(u) \exp(iH_\tau \tau), \quad (27) \]

where the Hamiltonian \( H_\tau = \omega(a_1 \dagger a_1 + a_2 \dagger a_2 + 1) \), if the photons have the same frequency \( \omega \). Hence the \( \langle \hat{q}_\theta = u | \rho_S | \hat{q}_\theta = u \rangle \) are equivalently given by replacing \( \rho_{APP}(T), \hat{x}_{1, \theta}, \hat{x}_{2, \theta} \to \rho_{APP}(T + \tau), \cos(\omega \tau) \hat{x}_{1, \theta} - \sin(\omega \tau) \hat{p}_{1, \theta}, \cos(\omega \tau) \hat{p}_{2, \theta} + \sin(\omega \tau) \hat{x}_{2, \theta} \) respectively. We just have to measure different quadratures for the apparatus photons depending on the transit time \( \tau \).
Quantitative comparisons for the third excited state of the oscillator.

Our exact theorems are for the limit $b_1 \to 0$. The purpose here is to estimate how small this parameter has to be for reasonably accurate reconstruction of the initial state which, in this example, is chosen to be the highly non-classical third excited of the oscillator. The wave function in the position basis is

$$\phi(q) = (2q^3 - 3q) \exp\left(-\frac{q^2}{2}\right)/(\sqrt{3}\pi^{1/4}).$$ \hspace{1cm} (28)

The Wigner function is a function of $q^2 + p^2 \equiv d$

$$W(d) = \exp(-d)[4d^3 - 18d^2 + 18d - 3]/(3\pi).$$ \hspace{1cm} (29)

In the figure we make quantitative comparisons between the Wigner function, our reconstructed Wigner function with $2b_1b_2 = 1$ (for $b_1 = \{0.1, 0.3\}$) and the Arthurs-Kelly Probability distribution. It is worth noting that for $b_1 = \frac{1}{\sqrt{2}}$, the reconstructed Wigner function is equal to the Arthurs-Kelly distribution which differs greatly from the true Wigner function. Towards practical utility, note that for $b_1 = .1$ the reconstructed Wigner function and the position probability derived from it are already very close to the actual, though the theorem of exact equality is only in the limit $b_1 \to 0$.

Conclusions and Outlook. (i) We have shown that the generation of entanglement by the Arthurs-Kelly Hamiltonian between an unknown state of a system photon and chosen initial state of two apparatus photons enables a one-step remote tomographic reconstruction of the unknown initial state of the system photon, instead of the usual four step process. This ‘interaction based remote tomography’ is practically feasible because the technology of generating this interaction quantum optically is well established.

(ii) Remote Tomography requires the measurement of the two photon observable $Y_{\theta}(u)$. Since this is a product of two commuting quadrature operators for the apparatus photons, each of the kind usually measured for a single photon, the measurement should be possible by generalizing optical homodyning to the two teleported photons. This generalization will by itself be a stimulating development.

(iii) The Arthurs-Goodman result on impossibility of simultaneous accurate tracking of position and momentum by commuting observables of the apparatus is not violated. The secret is that the apparatus observables tracking position and momentum do not commute,

$$[Y(x_1), Z(x_2)] \neq 0.$$

This is not a problem since we are only interested in faithful tomography of the initial system state, from repeated measurements on the teleported apparatus particles, and not in the simultaneous measurement of position and momentum.

(iv) The final density operator of the system can also be exactly calculated and it can be seen that $<q>_T =<q>_0$, $\Delta q_T^2 \equiv \Delta q_0^2 + 2b_2^2$, since the final system state is different from the initial state, and depends on the
initial states of both the system and the apparatus, the no-cloning\(^\text{13}\) and no-hiding theorems\(^\text{14}\) are respected.

(v) If the initial system \(S_1\) is entangled with another system \(S_2\), the apparatus photons after interaction with \(S_1\) become entangled with \(S_2\), leading to interaction-based teleportation of entanglement\(^\text{15}\).

**Acknowledgements.** SMR thanks Sam Braunstein for many helpful suggestions including the name ‘remote tomography’, and Arun Pati, Ujjwal Sen and Aditi Sen De for discussions. AD and NS thank the NIUS program of the Homi Bhabha Centre for Science Education; SMR thanks the Indian National Science Academy for the INSA Senior Scientist award.

---

1. J. Von Neumann, *Math. Foundations of Quantum Mechanics*, Princeton University Press (1955).
2. E. Arthurs and J. L. Kelly, Jr., *Bell System Tech. J.* 44, 725 (1965); K. Husimi, *Proc. Phys. Math. Soc. Japan*, 22, 264 (1940); S. L. Braunstein, C. M. Caves and G. J. Milburn, *Phys. Rev. A* 43, 1153 (1991); S. Stenholm, *Ann. Phys.* 218, 233 (1992); P. Busch, T. Heinonen and P. Lahti, *Phys. Reports* 452, 155 (2007).
3. E. Arthurs and M. S. Goodman, *Phys. Rev. Lett.* 60, 2447 (1988); S. Gudder, J. Hagler, and W. Stulpe, *Found. Phys. Lett.* 1, 287 (1988).
4. C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres and W. K. Wootters, *Phys. Rev. Lett.* 70, 1895 (1993).
5. L. Vaidman, *Phys. Rev. A* 49, 1473 (1994).
6. D Bouwmeester, J-W Pan, K Mattle, M Eibl, H Weinfurter and A Zeilinger, *Nature* 390, 575 (1997); A. Furusawa et al, *Science* 282, 706 (1998).
7. S. L. Braunstein and P. van Loock, *Rev. Mod. Phys.* 77, 513 (2005); S. Pirandola, S. Mancini, S. Lloyd, and S. L. Braunstein, *Nature Physics* 4, 726 (2008); G. Brassard, S. Braunstein, R. Cleve, *Physica D* 120, 43 (1998).
8. S. M. Roy, manuscript in preparation.