The ‘most classical’ states of Euclidean invariant elementary quantum mechanical systems

László B. Szabados
Wigner Research Centre for Physics,
H-1525 Budapest 114, P. O. Box 49, EU
e-mail: lbszab@rmki.kfki.hu

March 9, 2023

Abstract

Complex techniques of general relativity are used to determine all the states in two and three dimensional momentum spaces in which the equality holds in uncertainty relations for non-commuting basic observables of Euclidean invariant elementary quantum mechanical systems, even with non-zero intrinsic spin. It is shown that while there is a 1-parameter family of such states for any two components of the angular momentum vector operator with any angle between them, such states exist for a component of the linear and angular momenta only if these components are orthogonal to each other and hence the problem is reduced to the two-dimensional Euclidean invariant case. We also show that the analogous states exist for a component of the linear momentum and of the centre-of-mass vector only if the angle between them is zero or an acute angle. No such state (represented by a square integrable and differentiable wave function) can exist for any pair of components of the centre-of-mass vector operator. Therefore, the existence of such states depends not only on the Lie algebra, but on the choice of its generators as well.

Keywords: smallest uncertainty states, Euclidean invariant quantum mechanical systems, edth operators

1 Introduction

The so-called canonical coherent states of the quantum mechanical system with the Heisenberg algebra as its algebra of basic observables are usually interpreted as the ‘most classical’ states of the system. These are precisely the states that yield equality in the uncertainty relation for the canonically conjugate observables. (For a review, see e.g. [1, 2]). The states analogous to these, called the coherent states, have already been introduced for systems with more complicated algebra of basic observables, e.g. with the Lie algebras $su(2)$, $su(1, 1)$ (see e.g. [3]-[8]), with the Lie algebra $e(n)$ of the Euclidean group $E(n)$ in $n \geq 2$ dimensions (see e.g. [9]-[18]), or even with more the general ones (see [19]).

In particular, in the case of the Euclidean groups, these states were constructed on the circle $S^1$ and on the 2-sphere $S^2$ [11]-[12], [13]-[16], [18], and even on general $n$-spheres $S^n$ [14], by constructing $n$ operators that are analogous to the annihilation operators of the Heisenberg
system. In these investigations, the basic observables were the position variable and the orbital angular momentum, but the spin part of the total angular momentum was a priori assumed to be vanishing.

In our previous paper [20] we determined all the states of SU(2)-invariant quantum mechanical systems, in which the uncertainties of two components of angular momentum operators in two arbitrarily chosen directions, \( J(\alpha) \) and \( J(\beta) \), yield equality in the uncertainty relation. It turned out that allowing the angle between the two components of the angular momentum vector to be arbitrary a new quantum mechanical phenomenon emerges: the expectation values of the two components of the angular momentum behave in a symmetric way only if the classical parameter space of the solutions is extended to be a larger space that is homeomorphic to the non-trivial Riemann surface known in connection with the complex function \( \sqrt{z} \).

In the present paper, we extend these investigations to \( E(3) \)-invariant (and, as an illustration of the general strategy, the much simpler \( E(2) \)-invariant) elementary quantum mechanical systems. (Following the work of Newton and Wigner [21], we call the system elementary if its states belong to the carrier space of some unitary, irreducible representation of its symmetry group. Here by \( E(3) \) we mean the quantum mechanical Euclidean group, i.e. the semidirect product of SU(2) and the translation group \( \mathbb{R}^3 \), rather than the isometry group of the Euclidean 3-space [22].) No time evolution equation for the states is used; it is only based on the general kinematical structure of the theory. Since the components of the linear momentum, as the generators of space translations, are already among the elements of Lie algebras \( e(2) \) and \( e(3) \), moreover we think that the momentum space is a more natural arena for the formulation of quantum theory than the configuration space, we search for these states in the position representation. (Since in these special cases both the momentum and configuration spaces are modeled by some Euclidean space, the momentum and position representations are usually considered to be physically equivalent. However, physically, and hence also mathematically, these two spaces are different: The former has an extra structure as it is a vector space with the physically distinguished point \( p^i = 0 \), the latter is only an affine space, in which the origin is not distinguished and any two of its points are equivalent.) No a priori restriction is imposed on the unitary, irreducible representations, thus, in particular, the intrinsic spin is not assumed to be zero.

If \( \mathbf{A} \) and \( \mathbf{B} \) are any two not commuting observables and \( \phi \) is a normalized state of the system, then, as is well known, the necessary and sufficient condition of the equality in the uncertainty relation, \( \Delta_\phi \mathbf{A} \Delta_\phi \mathbf{B} \geq |\langle [\mathbf{A}, \mathbf{B}] \rangle_\phi|/2 \), is that \( \phi \) is the solution of the eigenvalue equation

\[
(\mathbf{A} - i\lambda \mathbf{B})\phi = (\langle \mathbf{A} \rangle_\phi - i\lambda \langle \mathbf{B} \rangle_\phi)\phi \tag{1.1}
\]

for some non-zero real number \( \lambda \). Here e.g. \( \langle \mathbf{A} \rangle_\phi \) is the expectation value of \( \mathbf{A} \), and the standard deviation in the state \( \phi \) is given by \( \Delta_\phi \mathbf{A} = \sqrt{\langle \mathbf{A}^2 \rangle_\phi - (\langle \mathbf{A} \rangle_\phi)^2} \). It follows from (1.1) that \( |\lambda| = \Delta_\phi \mathbf{A}/\Delta_\phi \mathbf{B} \), and, in what follows, we choose \( \lambda \) to be positive. We will call these states the ‘most classical’ states (with respect to the observables \( \mathbf{A} \) and \( \mathbf{B} \)). (1.1) shows why it is almost impossible to find a non-trivial state in which the equality in the uncertainty relations would hold for all the pairs \( (\mathbf{A}, \mathbf{B}) \), \( (\mathbf{B}, \mathbf{C}) \) and \( (\mathbf{C}, \mathbf{A}) \) from the three non-commuting observables \( \mathbf{A} \), \( \mathbf{B} \) and \( \mathbf{C} \): the state \( \phi \) would have to be the eigenvector of the three non-self-adjoint operators \( \mathbf{A} - i\lambda_1 \mathbf{B}, \mathbf{B} - i\lambda_2 \mathbf{C} \) and \( \mathbf{C} - i\lambda_3 \mathbf{A} \) at the same time for some non-zero reals \( \lambda_1, \lambda_2 \) and \( \lambda_3 \). It is equation (1.1) that we will solve in the carrier space of unitary, irreducible representations of \( E(2) \) and \( E(3) \) when the basic observables belong to the Lie algebra \( e(2) \) and \( e(3) \), respectively.

We find that the existence and the properties of the most classical states of elementary systems depend not only on the algebra of observables, but also on the actual choice of the pair of observables. We show that, in contrast to the \( E(2) \)-invariant systems, in \( E(3) \)-invariant systems the existence of the most classical states is not guaranteed for any two non-commuting observables; and even if these exist, then the expectation values of the observables could be considerably restricted relative to their classical values.
In particular, (1) for the component $p(\alpha)$ and $J(\beta)$ of the linear and total angular momenta in the direction $\alpha^i$ and $\beta^j$, respectively, the most classical states exist precisely when the two directions are orthogonal to each other, and the expectation value of the former is zero and that of the latter is an integer/half-odd-integer times of $\hbar$ for systems with integer/half-odd-integer spin; (2) for the components $J(\alpha)$ and $J(\beta)$ of the angular momentum the most classical states always exist (see the third paragraph above, and [20] for the details); (3) for the component $p(\alpha)$ and $C(\beta)$ of the linear momentum and centre-of-mass, respectively, these states exist precisely when the angle between the directions $\alpha^i$ and $\beta^j$ is zero or an acute angle, and the range of the expectation value of $p(\alpha)$ is restricted by this angle; (4) for the components $C(\alpha)$ and $C(\beta)$ of the centre-of-mass no such state exists at all which could be represented by any differentiable, square integrable wave function. In the cases when these states exist, they form infinite parameter families. These are explicitly given in closed form, and they are considerably more general than the previously known ones.

In section 2 we determine the most classical states of the $E(2)$-invariant systems and discuss their properties. Then, section 3 is devoted to the $E(3)$-invariant systems: first we summarize the unitary, irreducible representations of $e(3)$, and then we determine the most classical states in the cases mentioned above. The results are summarized and discussed in section 4. In deriving the above results, complex techniques of general relativity are used. To make the paper (mostly) self-contained, these ideas are summarized in the Appendix. For a more detailed discussion of these ideas and techniques, see e.g. [23] [24] [25] [26] [27] [28].

Our conventions are mostly those of [23] (and identical with those of [20]). In particular, round brackets around indices denote symmetrization. No abstract indices are used, every index is a concrete (name, or component) index.

2 $E(2)$-invariant elementary systems

The (abstract) Lie algebra $e(2)$ is spanned by the elements $p^1$, $p^2$ and $J$ satisfying the commutation relations $[p^1, p^2] = 0$, $[p^1, J] = i\hbar p^2$ and $[p^2, J] = -i\hbar p^1$. Thus $P^2 := (p^1)^2 + (p^2)^2$ is a Casimir operator in its enveloping algebra; and hence an irreducible representation of $e(2)$ is labeled by the value $P \geq 0$ of $P$. The representation space is $\mathcal{H} := L_2(S, P d\phi)$, where $S := \{(p^1, p^2) \in \mathbb{R}^2 | (p^1)^2 + (p^2)^2 = P^2\}$, the circle of radius $P$ in the classical momentum space $\mathbb{R}^2$, is analogous to the ‘mass shell’ (or rather the ‘kinetic energy shell’), endowed with the Euclidean scalar product $\delta_{ij}, i, j = 1, 2$. We lower and raise the small Latin indices freely by the Kronecker delta $\delta_{ij}$ and its inverse, respectively. Here, $(p^1, p^2)$ are Cartesian coordinates in the momentum space $\mathbb{R}^2$. In the measure $P d\varphi$ the angle $\varphi \in [0, 2\pi)$ is defined by $(p^1, p^2) = (P \cos \varphi, P \sin \varphi)$.

An element of the group $E(2)$ is represented by a translation $\xi^i \in \mathbb{R}^2$ and the angle $\chi \in [0, 2\pi)$ of a rotation; and the action of such an element $(\xi^i, \chi)$ of $E(2)$ on the function $\phi \in \mathcal{H}$ is $(U(\xi^i, \chi)\phi)(\varphi) := \exp(i\hbar \xi^i / \hbar) \phi(\varphi - \chi)$. Clearly, this $U$ provides a unitary, irreducible representation of $E(2)$, and yields the representation of the abstract Lie algebra $e(2)$ as $p_i \phi = p_i \phi$ and $J_\phi = i\hbar (d\phi/d\varphi)$ as well. $p_i$ are bounded Hermitian operators, and $J$ is a formally self-adjoint (unbounded) operator on the dense subspace of the smooth functions in $\mathcal{H}$.

Applying (1.1) to the present operators, for the (normalized) state $\phi$ we obtain $(p_1 - i\lambda_1 J_\phi)\phi = ((p_1)\phi - i\lambda_1 (J)\phi)\phi$ and $(p_2 - i\lambda_2 J)\phi = ((p_2)\phi - i\lambda_2 (J)\phi)\phi$. However, in the above representation of the operators, these equations yield $P(\cos \varphi / \lambda_1 - \sin \varphi / \lambda_2)\phi = ((p_1)\phi / \lambda_1 - (p_2)\phi / \lambda_2)\phi$, which could hold true for every $\varphi$ only if $\phi = 0$. Thus, there is no state in which the equality would hold in both inequalities $\Delta_\phi p_1 \Delta_\phi J \geq \hbar |(p_2)\phi| / 2$ and $\Delta_\phi p_2 \Delta_\phi J \geq \hbar |(p_1)\phi| / 2$. Only one such equation can be required to hold (as we already noted following equation (1.1)).

Thus, let $\alpha_i \in \mathbb{R}^2$ and form the operator $p(\alpha) := \alpha^i p_i$. Although the parameters $\alpha_i$ could be arbitrary, it is more natural to consider only those of the form $(\alpha_1, \alpha_2) = (\cos \alpha, \sin \alpha)$, where $\alpha \in [0, 2\pi)$, because in this case the operator $p(\alpha)$ is e.g. $p_1$ in the appropriately rotated basis. Then
\[ [p(\alpha), J] = -i\hbar p^+(\alpha), \] where \( p^+(\alpha) := \alpha_2 p_1 - \alpha_1 p_2 \). Hence \( \Delta_\phi p(\alpha) \Delta_\phi J \geq \hbar |\langle p^+(\alpha) \rangle_\phi| / 2 \), in which, according to \([11]\), the equality holds precisely when \( \langle p(\alpha) - i\lambda J \rangle_\phi = \langle \langle p(\alpha) \rangle_\phi - i\lambda \langle J \rangle_\phi \rangle_\phi \).

In the above explicit momentum representation, this condition takes the form of the differential equation

\[
\lambda \hbar \frac{d\phi}{d\varphi} = \left( \langle p(\alpha) \rangle_\phi - P \cos(\varphi - \alpha) - i\lambda \langle J \rangle_\phi \right) \phi. \tag{2.1}
\]

Its solution is

\[
\phi = A \exp \left( -\frac{P}{\lambda \hbar} \sin(\varphi - \alpha) + \frac{1}{\lambda \hbar} \left( \langle p(\alpha) \rangle_\phi - i\lambda \langle J \rangle_\phi \right) \varphi \right),
\]

where \( A \) is a constant. However, this function is periodic in \( \varphi \) with period \( 2\pi \) precisely when \( \langle p(\alpha) \rangle_\phi = 0 \) and \( \langle J \rangle_\phi = \ell \hbar, \ell \in \mathbb{Z} \), and hence

\[
\phi = A \exp \left( -\frac{P}{\lambda \hbar} \sin(\varphi - \alpha) - i\ell \varphi \right). \tag{2.2}
\]

The constant \( A \) is fixed by the normalization condition

\[
1 = \int_0^{2\pi} |\phi|^2 P \, d\varphi = 2\pi P |A|^2 I_0 \left( -\frac{2P}{\lambda \hbar} \right), \tag{2.3}
\]

where \( I_k \) is the \( k \)th modified Bessel function of the first kind, \( k = 0, 1, 2, \ldots \); which can be given by

\[
I_k(x) = \frac{1}{2\pi} \int_0^{2\pi} \exp(x \cos(\omega)) \cos(k\omega) d\omega = \sum_{l=0}^{\infty} \frac{1}{l!(l+k)!} \left( \frac{x}{2} \right)^{2l+k}. \tag{2.4}
\]

In particular, \( I_k(x) \) is an even function if \( k \) is even, and it is odd if \( k \) is odd. They satisfy the recurrence relations

\[
x(I_k(x) - I_{k+2}(x)) = 2(k+1)I_{k+1}(x).
\]

The asymptotic expansion of \( I_k(x) \) for large \( x \) is

\[
I_k(x) \sim (2\pi x)^{-1/2} \exp(x)(1 - 4k^2 - 1)/(8x) + \ldots \] (see e.g. \([29]\), pp 375-377).

Note also that \( I_k(T/dx)I_0 \), where \( T_k \) is the Chebyshev polynomial of the first kind \([30]\).

In particular, \( I_1 = (dI_0/dx) \) and \( I_2 = 2(d^2I_0/dx^2) - I_0 \) hold. We will use these formulae below, and in the next section, too.

Then it is straightforward to calculate the expectation values and uncertainties: as we should, we do, in fact, recover \( \langle p(\alpha) \rangle_\phi = 0 \) and \( \langle J \rangle_\phi = \ell \hbar \) from \([2.2]\), too. In addition, by integration by parts, we obtain that

\[
\langle p(\alpha) \rangle_\phi = \lambda \frac{\hbar}{2} \langle p^+(\alpha) \rangle_\phi, \quad \langle J \rangle_\phi = \langle J \rangle_\phi + \frac{1}{\lambda^2} \langle (p(\alpha))^2 \rangle_\phi;
\]

and hence

\[
\Delta_\phi p(\alpha) \Delta_\phi J = \lambda \frac{\hbar}{2} \langle p^+(\alpha) \rangle_\phi, \quad \Delta_\phi J = \frac{\hbar}{\lambda} \langle p^+(\alpha) \rangle_\phi. \tag{2.5}
\]

Thus, \( \Delta_\phi p(\alpha) \) and \( \Delta_\phi J \) do, indeed, saturate the inequality \( \Delta_\phi p(\alpha) \Delta_\phi J \geq \hbar |\langle p^+(\alpha) \rangle_\phi| / 2 \), and we recover \( \lambda = \Delta_\phi p(\alpha) / \Delta_\phi J \), too. Finally, to determine \( \langle p(\alpha) \rangle_\phi \), we need to calculate \( \langle p_1 \rangle_\phi \) and \( \langle p_2 \rangle_\phi \) explicitly. Using

\[
\int_0^{2\pi} \exp(x \cos(\omega)) \sin(k\omega) d\omega = 0, \quad k \in \mathbb{Z}, \tag{2.6}
\]

by integration by parts we find that

\[
\langle p_1 \rangle_\phi = -\sin \alpha 2\pi |A|^2 P^2 I_1 \left( -\frac{2P}{\lambda \hbar} \right) = \sin \alpha 2\pi |A|^2 P^2 I_1 \left( \frac{2P}{\lambda \hbar} \right). \tag{2.7}
\]

By \( 0 = \langle p(\alpha) \rangle_\phi = \cos \alpha \langle p_1 \rangle_\phi + \sin \alpha \langle p_2 \rangle_\phi \), \( \langle p_2 \rangle_\phi \) can be determined, too. (Similar calculations yield that \( \langle (p_1)^2 \rangle_\phi = \langle (p_2)^2 \rangle_\phi = P^2 / 2 \), and hence that \( \langle \delta^j p_i p_j \rangle_\phi = P^2 \), as we expected.) Hence,

\[
\langle \Delta_\phi p(\alpha) \rangle_\phi = \pi \hbar \lambda P^2 |A|^2 I_1 \left( \frac{2P}{\lambda \hbar} \right).
\]
Thus, using the normalization condition (2.3), finally we obtain that

$$\Delta_{\phi} \mathbf{p}(\alpha) \Delta_{\phi} \mathbf{J} = \frac{1}{2} \mathcal{P} \frac{I_1(2\frac{P}{\lambda})}{I_0(\frac{2P}{\lambda})}.$$ (2.8)

Since $I_0(0) = 1$ and $I_1(0) = 0$, in the $\lambda \to \infty$ limit the product uncertainty $\Delta_{\phi} \mathbf{p}(\alpha) \Delta_{\phi} \mathbf{J}$ tends to zero. On the other hand, since $\lim_{x \to \infty}(I_1(x)/I_0(x)) = 1$, in the $\lambda \to 0$ limit it tends to a finite value: $\Delta_{\phi} \mathbf{p}(\alpha) \Delta_{\phi} \mathbf{J} \to \mathcal{P} \hbar/2$.

Therefore, to summarize, solution (2.2) is parameterized by the integer $\ell \in \mathbb{Z}$ and the continuous parameter $\lambda \in (0, \infty)$, the latter being the ratio of the two uncertainties. This ratio is unrestricted, although the expectation values themselves are discrete: $\langle \mathbf{p}(\alpha) \rangle_\phi = 0$ and $\langle \mathbf{J} \rangle_\phi = \ell \hbar$. Although there is no state in the family of states (2.2) in which the product uncertainty would actually take its minimum, by (2.8) in the limit $\lambda \to \infty$ we can approximate such an ideal state as much as we wish. (Another candidate to be a measure of the ‘overall uncertainty’ could be the sum of the square of the uncertainties, $\hbar^2(\Delta_{\phi} \mathbf{p}(\alpha))^2 + \mathcal{P}^2(\Delta_{\phi} \mathbf{J})^2$, which gives $(\mathcal{P}P/2)(\hbar \lambda^2 + \mathcal{P}^2/\lambda)(I_1/I_0)$. This would yield a different notion of the ‘most classical’ states.) The role of the parameter $\alpha \in [0, 2\pi)$ is only to specify which component of the linear momentum is considered, and this can be chosen freely. The product uncertainty does not depend on $\alpha$, as it should not.

### 3 $E(3)$-invariant elementary systems

#### 3.1 The unitary, irreducible representations of $e(3)$

The Lie algebra $e(3)$ is generated by $\mathbf{p}^i$ and $\mathbf{J}_i$, $i = 1, 2, 3$, satisfying $[\mathbf{p}^i, \mathbf{p}^j] = 0$, $[\mathbf{p}_i, \mathbf{J}_j] = i\hbar \varepsilon_{ijk} \mathbf{P}^k$ and $[\mathbf{J}_i, \mathbf{J}_j] = i\hbar \varepsilon_{ijk} \mathbf{J}^k$, where the lowering and raising of the small Latin indices are defined by $\delta_{ij}$ and its inverse, and $\varepsilon_{ijk}$ is the alternating Levi-Civita symbol. The two Casimir operators are $\mathbf{P}^2 := \delta_{ij} \mathbf{p}^i \mathbf{p}^j$ and $\mathbf{W} := \mathbf{J}_i \mathbf{p}^i = \mathbf{p}^i \mathbf{J}_i$. Thus, in an irreducible representation of $e(3)$, the Casimir operators are of the form $\mathbf{P}^2 \mathbf{I}$ and $\mathbf{w} \mathbf{I}$, respectively, with some non-negative $\mathbf{P}^2$ and real $\mathbf{w}$, where $\mathbf{I}$ is the identity operator.

The part of $\mathbf{J}_i$ that $\mathbf{W}$ does not fix is $\mathbf{C}_i := -\varepsilon_{ikl} \mathbf{J}_k \mathbf{p}^l + i\hbar \mathbf{p}_i$. (Since in itself $-\varepsilon_{ikl} \mathbf{J}_k \mathbf{p}^l$ is not self-adjoint even if $\mathbf{p}^i$ and $\mathbf{J}_i$ are, the term $i\hbar \mathbf{p}_i$ is needed to ensure the self-adjointness of $\mathbf{C}_i$.) Then

$$[\mathbf{p}_i, \mathbf{C}_k] = -i\hbar (\delta_{ik} \mathbf{P}^2 - \mathbf{p}_i \mathbf{p}_k), \quad [\mathbf{C}_i, \mathbf{C}_k] = -i\hbar \left( \varepsilon_{ikl} \mathbf{p}^l \mathbf{W} + \mathbf{C}_i \mathbf{p}_k - \mathbf{C}_k \mathbf{p}_i \right).$$ (3.1)

In an irreducible representation, the part $\mathbf{C}_i$ of $\mathbf{J}_i$ can be interpreted as $\mathbf{P}^2$-times the centre-of-mass vector operator. $\mathbf{W}$ represents the intrinsic spin, or rather the helicity, with respect to the 3-momentum, while $\mathbf{L}^i := \varepsilon_{ijk} C_j \mathbf{p}_k / \mathbf{P}^2 = \mathbf{J}^i - \mathbf{w} \mathbf{p}^i / \mathbf{P}^2$ is the orbital angular momentum. $\mathbf{J}_i \mathbf{J}^i$ is a Casimir operator only of the $\mathfrak{su}(2)$ subalgebra, and $\mathcal{P}^2 \mathbf{J}_i \mathbf{J}^i = (\mathbf{w}^2 - \hbar^2 \mathbf{P}^2) \mathbf{I} + \mathbf{C}_i \mathbf{C}^i$ holds.

The unitary, irreducible representations of $E(3)$ and $e(3)$ in their traditional, purely algebraic bra-ket formalism are known and are summarized e.g. in [31] [12] [17]. Nevertheless, they can also be derived and presented in a geometric form using Weyl spinor fields on 2-spheres of the momentum space, analogously to those on the mass-shell for the Poincaré group (see [22]). However, a special feature of the spinor fields in the latter formalism, viz. their N-type character (see Appendix A.2 or for a more detailed discussion, [23] [24]), is not manifest. Moreover, and more importantly, the naive, apparently ‘obvious’ decomposition of the total angular momentum operator into its spin and orbital parts is not well defined. This drawback is cured by the use of the space of the square integrable cross section of the complex line bundles $\mathcal{O}(-2\mathbf{s})$ as the representation space. These two forms of the representations are motivated by the spinorial ideas and techniques of general relativity [23] [24], and, in the case of $SU(2)$-invariant systems
considered in [20], their use has already made considerably easier to solve the resulting partial differential equations in closed form. In Appendix A.1 we summarize the complex and spinorial tools that we use, and Appendix A.2 is a summary of the key results on the unitary, irreducible representations of $E(3)$ in these two forms. These appendices make the present paper essentially self-contained. For a more detailed discussion of these ideas, see [25, 23, 24], or, for a concise summary of the necessary background, see Appendix A.1 of [20].

Thus, the concrete realization of the unitary, irreducible representations of $e(3)$ that we use here is based on the complex line bundle $\mathcal{O}(−2s)$ over the 2-sphere $S$ of radius $P$, where $2s \in \mathbb{Z}$ is fixed. This 2-sphere, $S := \{ p^i \in \mathbb{R}^3 \, | \, p^i p^i \delta_{ij} = P^2 \}$, is thought of as the ‘kinetic energy shell’ in the classical momentum 3-space $(\mathbb{R}^3, \delta_{ij})$, in which $p^i$ are Cartesian coordinates. The carrier space of the representation is the Hilbert space $\mathcal{H}_s = L_2(S, dS)$ of the square-integrable cross sections of the line bundle, where the measure $dS$ is the natural metric area element on $S$. The integral of the first Chern class of $\mathcal{O}(−2s)$ is 2s, which is a topological invariant, and hence $s$ characterizes the global non-triviality (or ‘twist’) of the bundle (see e.g. [28]).

In this representation, the $e(3)$ Casimir operators have the form $P^2 \phi = P^2 \phi$ and $W \phi = \hbar P s \phi$ (see Appendix A.2), and the basic observables and the centre-of-mass operator act on $\phi$ as

$$p^i \phi = p^i \phi, \quad (3.2)$$

$$J^i \phi = s \hbar \frac{p^i}{P} \phi + L^i \phi = s \hbar \frac{p^i}{P} \phi + P \hbar (m^i \partial' \phi - \tilde{m}^i \delta \phi), \quad (3.3)$$

$$C_i \phi = i \hbar (P^2 m_i \partial' \phi + P^2 \tilde{m}_i \delta \phi - p_i \phi). \quad (3.4)$$

In particular, the spin of the system is encoded into the twist $s$ of the line bundle $\mathcal{O}(−2s)$. For the definition of the complex null tangents $m^i$ and $\tilde{m}^i$ of $S$, the operators $\partial$ and $\partial'$ and the related concepts, and the derivation of (3.2)-(3.4), see the appendices A.1, A.2. In this representation, the $su(2)$ Casimir operator is $J_i J^i$, where $J^i = s^2 \hbar^2 \partial + P^2 \hbar^2 (\partial' + \delta' \partial) \phi$. The square of the centre-of-mass vector operator deviates from this only in a term proportional with the identity operator: $C_i C^i \phi = P^2 J_i J^i \phi + P^2 \hbar^2 (1 - s^2) \delta I$.

### 3.2 The most classical states for the $(p^i, J_i)$ system

If $p^i$ and $J_i$ are considered to be the basic observables in $e(3)$, then there are two kinds of non-commuting observables: one linear and one angular momentum component, and two different angular momentum components. We discuss these cases separately.

#### 3.2.1 The most classical states for the observables $(p(\alpha), J(\beta))$

For any $\alpha_i, \beta_i \in \mathbb{R}^3$ satisfying $\alpha_i \alpha_j \delta_{ij} = \beta_i \beta_j \delta_{ij} = 1$, we form the operators $p(\alpha) := \alpha_i p^i$ and $J(\beta) := \beta_i J_i$, i.e. the components of the basic observables determined by the directions $\alpha_i$ and $\beta_i$. However, without loss of generality, we may assume that e.g. $\beta_1 = \beta_2 = 0$ and $\beta_3 = 1$, because by an appropriate rotation of the Cartesian coordinate system this can always be achieved. Using the commutators of the basic observables, in any normalized state $\phi$ we obtain $[p(\alpha), J(\beta)] = i \hbar \alpha_i \beta_j \varepsilon_{ijk} p^k$. Hence, we assume that $\alpha_3 \neq \pm 1$, because otherwise $\alpha_i = \pm \beta_i$ would be allowed, and for these $p(\alpha)$ and $J(\beta)$ would commute. Then

$$\Delta_\phi p(\alpha) \Delta_\phi J(\beta) \geq \frac{1}{2} \| \langle [p(\alpha), J(\beta)] \rangle \| \phi = \frac{1}{2} \| \alpha_i \beta_j \varepsilon_{ijk} \langle p^k \rangle \| \phi, \quad (3.5)$$

in which, by (1.1), the condition of the equality is

$$\langle \phi | p(\alpha) - i \lambda J(\beta) \rangle \phi = \left( \langle \phi | p(\alpha) \rangle - i \lambda \langle J(\beta) \rangle \phi \right) \phi \quad (3.6)$$

6
for some positive $\lambda$. By (3.2)-(3.3), in the unitary, irreducible representation labeled by $P$ and $s$, this condition takes the form

$$-i\lambda P\hbar \beta^i \left( m_i \delta^i \phi - \bar{m}_i \delta \phi \right) + (\alpha_i \beta^i - i\hbar \beta^i \rho^i \frac{\partial}{\partial P}) \phi = \left( \langle p(\alpha) \rangle_\phi - i\lambda (J(\beta))_\phi \right) \phi. \quad (3.7)$$

Using $\beta^i = (0, 0, 1)$ and the explicit form of the operators $\delta$ and $\delta'$ and of the Cartesian components of the vectors $m_i$ and $\bar{m}_i$ expressed by the polar coordinates $(\theta, \varphi)$ (see Appendix A.1), we obtain the differential equation

$$\frac{\partial \ln \phi}{\partial \varphi} = \frac{P}{\lambda h} \sqrt{1 - \alpha_3^2 \sin \theta \cos(\varphi - \alpha)} + \left( \frac{P}{\lambda h} \alpha_3 \cos \theta - \frac{1}{\lambda h} \langle p(\alpha) \rangle_\phi \right) + i \left( \frac{1}{h} (J_3)_\phi + s \right), \quad (3.8)$$

where we parameterized $(\alpha_1, \alpha_2)$ as $\sqrt{1 - \alpha_3^2}(\cos \alpha, \sin \alpha)$. Its solution is

$$\phi = A \exp \left( \frac{P}{\lambda h} \sqrt{1 - \alpha_3^2 \sin \theta \sin(\varphi - \alpha)} + \left( \frac{P}{\lambda h} \alpha_3 \cos \theta - \frac{1}{\lambda h} \langle p(\alpha) \rangle_\phi \right) \varphi + i \left( \frac{1}{h} (J_3)_\phi + s \right) \varphi \right),$$

where $A = A(\theta)$ is an arbitrary complex function of $\theta$. This $\phi$ is periodic in $\varphi$ with $2\pi$ period precisely when the second term is vanishing for any value of $\theta$, and, in the third term, the coefficient of $\varphi$ between the round brackets is an integer, say $\ell$. This holds precisely when

$$\alpha_3 = 0, \quad \langle p(\alpha) \rangle_\phi = 0, \quad \langle J_3 \rangle_\phi = (\ell - s) h, \quad \ell \in \mathbb{Z}. \quad (3.9)$$

Then the solution of (3.8) takes the form

$$\phi = A \exp \left( \frac{P \sin \theta}{\lambda h} \sin(\varphi - \alpha) + i \ell \varphi \right). \quad (3.10)$$

Although its structure for any given $\theta$ is just that of solution (2.2) for the $E(2)$-invariant system, it depends on one arbitrary complex function, $A = A(\theta)$, restricted only by the normalization condition

$$1 = \int_S |\phi|^2 dS = 2\pi P^2 \int_0^\pi |A|^2 I_0 \sin \theta d\theta. \quad (3.11)$$

Here $I_0$ is the $0\text{th}$ modified Bessel function of the first kind, whose argument now is $2P \sin \theta / (\lambda h)$. Using (2.3), (2.6), (3.10) and this normalization condition, it is straightforward to calculate $\langle p_1 \rangle_\phi$ and $\langle J_3 \rangle_\phi$, and we do, in fact, recover $\langle p(\alpha) \rangle_\phi = 0$ and $\langle J_3 \rangle_\phi = (\ell - s) h$ (and, as we expect, $\langle \delta^j p_1 p_2 \rangle_\phi = P^2$; too); and, in addition, we find that

$$\sin \alpha \langle p_1 \rangle_\phi - \cos \alpha \langle p_2 \rangle_\phi = -2\pi P^3 \int_0^\pi |A|^2 I_1 \sin^2 \theta d\theta.$$

Using $\alpha_3 = 0$, $\langle J_3 \rangle^2_\phi = \langle J_3 \phi, J_3 \phi \rangle$ and the recurrence relation $x(I_0(x) - I_2(x)) = 2I_1(x)$, similar calculations yield

$$(\Delta_\phi p(\alpha))^2 = \lambda^2 (\Delta_\phi J_3)^2 = \pi \lambda h P^3 \int_0^\pi |A|^2 I_1 \sin^2 \theta d\theta. \quad (3.12)$$

Therefore, the product uncertainty is $\Delta_\phi p(\alpha) \Delta_\phi J_3 = (\Delta_\phi p(\alpha))^2 / \lambda$; and the equality does, in fact, hold in (3.5).

Next we clarify the $\lambda \to \infty$ and $\lambda \to 0$ limits of uncertainties. Using the recurrence relation and re-expressing $I_1$ in terms of $I_0$ and $I_2$, and recalling that the Bessel functions $I_k(x)$ are non-negative for $x \geq 0$, by the normalization condition (3.11) equation (3.12) gives that

$$(\Delta_\phi p(\alpha))^2 = \pi P^4 \int_0^\pi |A|^2 (I_0 - I_2) \sin^3 \theta d\theta \leq \pi P^4 \int_0^\pi |A|^2 I_0 \sin \theta d\theta = \frac{1}{2} P^2;$$

7
Taking into account the asymptotic form of $I_1$ that, by (3.11), the asymptotic behaviour of $|A|^2$ is determined by that of $I_0$: with the notation $x_0 := 2P/(\lambda h)$, for large $x_0$ it is

$$|A|^2 \sim \frac{1}{2\pi^2 P^2} \exp\left(-x_0 \sin \theta \right) \left(\frac{8x_0 \sqrt{2\pi x_0 \sin \theta}}{1 + 8x_0 \sin \theta} + \cdots \right).$$

Taking into account the asymptotic form of $I_1$ for large $x$, we find that $(\Delta_\phi p(\alpha))^2 = \hbar \lambda/\pi + O(\lambda^2)$, and hence $(\Delta_\phi J(\beta))^2$ diverges, while the product uncertainty tends to the finite value $hP/\pi$ in the $\lambda \to 0$ limit.

Therefore, the most classical states for the pair $(p(\alpha), J(\beta))$ of observables can exist only if the two directions are orthogonal to each other, $\alpha_i \beta_i = \alpha_3 = 0$. However, with $\alpha_3 = 0$ the operators $p(\alpha)$ and $J(\beta)$ generate an $e(2)$ sub-Lie algebra in $e(3)$: with the definition $p^{\pm}(\alpha) := \alpha_i \beta_j \varepsilon_{ijk} p^k$, one has, in fact, that $[p(\alpha), J(\beta)] = i\hbar p^{\pm}(\alpha)$, $[p^\pm(\alpha), J(\beta)] = -i\hbar p(\alpha)$ and $[p(\alpha), p^\pm(\alpha)] = 0$. Thus, it is not a surprise that the states (3.10) are similar to the ones obtained for the $E(2)$-invariant systems and depend on an integer $\ell$ and the parameter $\lambda$ (although they depend on one almost free function of $\theta$, too, which is restricted only by the normalization condition). The range of $\lambda$ is unrestricted, which can be any positive number though the expectation values $\langle p(\alpha) \rangle_\phi$ and $\langle J(\beta) \rangle_\phi$ could take only discrete values. $\langle J(\beta) \rangle_\phi/\hbar$ is half-odd-integer iff $s$ is. The product uncertainty $\Delta_\phi p(\alpha) \Delta_\phi J(\beta)$ can be made as small as we wish by choosing $\lambda$ large enough, i.e. when we approach an eigenstate of $J(\beta)$; but, in the $\lambda \to 0$ limit (i.e. when we approach an eigenstate of $p(\alpha))$ the product uncertainty cannot be made smaller than the finite value $hP/\pi$.

### 3.2.2 A summary of the most classical states for the observables $(J(\alpha), J(\beta))$

The case of two angular momentum components, $J(\alpha) := \alpha^i J_i$ and $J(\beta) := \beta^i J_i$ for any two unit vectors $\alpha^i$ and $\beta^i$ for which $\alpha_i \beta_i \neq \pm 1$, has already been clarified in [20]; both the expectation values and the standard deviations and the corresponding wave functions have been determined. Thus, for the sake of completeness, we only summarize the key results and present those details that we need in subsection 3.3.2.

Condition (1.11) of the equality in the uncertainty relation for $J(\alpha)$ and $J(\beta)$ is

$$\langle J(\alpha) - i\lambda J(\beta) \rangle_\phi = \left(\langle J(\alpha) \rangle_\phi - i\lambda \langle J(\beta) \rangle_\phi \right) \phi =: \hbar C \phi$$

for some $\lambda > 0$. By (3.3) in the irreducible representation labeled by $P$ and $s$, this takes the explicit form

$$P(\alpha^i - i\lambda \beta^i) \left( m_j \delta^i_j \phi - m_j \delta^i \phi + s P_i \phi \right) = C \phi.$$  

(3.14)

Using spinorial techniques of general relativity, and in particular the principal spinors of the spinor form of the complex spatial vector $\alpha_i - i\lambda \beta_i$, we have already determined the complex eigenvalue $C$. It is given by

$$C = m \sqrt{1 - \lambda^2 - 2i\lambda \alpha_3},$$

(3.15)

where $m = -j, -j+1, \ldots, j$, and $j$ is such that $j = |s|, |s|+1, \ldots$. Hence, by (3.14), the expectation values are

$$\langle J(\alpha) \rangle_\phi = m \frac{\hbar}{\sqrt{2}} \sqrt{1 - \lambda^2 + \sqrt{(1 - \lambda^2)^2 + 4\lambda^2 \alpha_3^2}},$$

(3.16)

$$\lambda \langle J(\beta) \rangle_\phi = \text{sign}(\alpha_3) m \frac{\hbar}{\sqrt{2}} \sqrt{\lambda^2 - 1 + \sqrt{(1 - \lambda^2)^2 + 4\lambda^2 \alpha_3^2}}.$$  

(3.17)

if $\alpha_3 \neq 0$; these are $m\hbar \sqrt{1 - \lambda^2}$ and 0, respectively, if $\alpha_3 = 0$ and $\lambda < 1$; while these are 0 and $m\hbar \sqrt{\lambda^2 - 1}$, respectively, if $\alpha_3 = 0$ and $\lambda > 1$. Thus, these expectation values depend on the
discrete ‘quantum number’ \( m \), and the two continuous parameters \( \alpha_3 \) and \( \lambda \). The expectation values are zero if \( \alpha_3 = 0 \) and \( \lambda = 1 \).

However, for given non-zero \( m \), \( \langle J(\alpha) \rangle_{\phi} \) is a continuous function on the parameter space \( P := \{ (\alpha_3, \lambda) \mid \alpha_3 \in (-1, 1), \lambda \in (0, \infty) \} \), but \( \langle J(\beta) \rangle_{\phi} \) is not continuous at \( \alpha_3 = 0 \) for \( \lambda > 1 \).

Since physically neither angular momentum component is distinguished over the other, in \cite{20} we concluded that the two expectation values should behave in a symmetric way. In fact, the standard deviations showed this symmetry: they turned out to be continuous on \( \lambda \to 0 \) and \( \lambda \to \infty \) limits. Hence, the proper parameter space \( \mathcal{R} \) on which the expectation values (and the states also) should depend is homeomorphic to the Riemann surface known in connection with the function \( \sqrt{z} \) in complex analysis. This \( \mathcal{R} \) is obtained from \( P \) by cutting it along the \( \lambda \geq 1 \) segment of the \( \alpha_3 = 0 \) axis, and identifying the resulting edges with the corresponding opposite edges in a second copy of \( P \) that has been cut in the same way. The branch point of \( \mathcal{R} \) corresponds to the point \( \alpha_3 = 0 \) and \( \lambda = 1 \) of \( P \), and we refer to this as the exceptional case, and to any other as the generic case.

In \cite{20}, we determined the eigenfunctions of the eigenvalue equation (3.14) in closed form, too. Since physically neither angular momentum component is distinguished over the other, in subsection 3.3.2 we summarize its key points. Parameterizing the components of \( \alpha \) by \( \alpha_3 \) via \( (\alpha_1, \alpha_2) = \sqrt{1 - \alpha_3^2}(\cos \alpha_1, \sin \alpha_1) \), where \( \alpha \in [0, 2\pi) \), and writing (3.14) in the complex stereographic coordinates \( (\zeta, \bar{\zeta}) \), we find that it is more convenient to use the coordinates \( (\xi, \bar{\xi}) \) defined by \( \xi := \exp[-i\alpha] \zeta \). In these coordinates, (3.14) takes the form

\[
X(\ln \phi) = C - \frac{1}{2} s\left( \sqrt{1 - \alpha_3^2} \xi - \alpha_3 + i\lambda \right) - \frac{1}{2} s\left( \sqrt{1 - \alpha_3^2} \bar{\xi} - \alpha_3 + i\lambda \right),
\]

where the complex vector field \( X \) on the left is defined by

\[
X := \left( \frac{1}{2} \sqrt{1 - \alpha_3^2}(1 - \xi^2) + (\alpha_3 - i\lambda)\xi \right) \frac{\partial}{\partial \xi} - \left( \frac{1}{2} \sqrt{1 - \alpha_3^2}(1 - \bar{\xi}^2) + (\alpha_3 + i\lambda)\bar{\xi} \right) \frac{\partial}{\partial \bar{\xi}}.
\]

Introducing the notation

\[
\xi_{\pm} := \frac{\alpha_3 - i\lambda \pm \sqrt{1 - \lambda^2 - 2i\lambda \alpha_3}}{\sqrt{1 - \alpha_3^2}},
\]

in the generic case the general local solution of (3.18) is

\[
\phi = \phi_0 \left( \frac{(\xi - \xi_-)(\xi - \xi_+)}{(\xi - \xi_+)(\xi - \xi_-)} \right)^{m/2} \left( \frac{(\xi - \xi_-)(\xi - \xi_-)}{(\xi - \xi_+)(\xi - \xi_-)} \right)^{s/2},
\]

where \( \phi_0 \) is an arbitrary smooth complex function of

\[
w := \frac{(\xi - \xi_+)(\xi - \xi_+)}{(\xi - \xi_-)(\xi - \xi_-)}.
\]

In particular, \( \phi_0 \) could be

\[
\frac{(\xi - \xi_+)^a(\xi - \xi_+)^a(\xi - \xi_-)^b(\xi - \xi_-)^b}{(1 + \xi\xi)^{a+b}} = w^a \left( \frac{\xi_+ - \xi_-}{\xi_+ - \xi_-} \right)^{a+b}
\]

with arbitrary real \( a \) and \( b \). However, with such a general \( \phi_0 \) (3.21) is only a local solution of (3.14); we still have to ensure that \( \phi \) be well defined even on small circles surrounding the poles, and be square integrable, too. These requirements restrict the structure of \( \phi_0 \), and, in particular, \( a \) and \( b \) are restricted to be only non-negative integer or half-odd-integer for which \( a + b = j \).
If $\lambda = 1$ and $\alpha_3 = 0$ (exceptional case), then $\xi_\pm = -i$ and the general \textit{local} solution of equation (3.14) is
\begin{equation}
\phi = \phi_0 \left( \frac{i + \xi}{i + \xi} \right)^s, \tag{3.23}
\end{equation}
where we have used that in the exceptional case the eigenvalue $C$ is zero, and $\phi_0$ is an arbitrary smooth complex function of
\begin{equation}
v := \frac{1}{i + \xi} + \frac{1}{i + \xi}. \tag{3.24}
\end{equation}
In particular, $\phi_0$ could be
\begin{equation}
\left( \frac{(\xi + i)(\xi + i)}{(1 + \xi)} \right)^a = \frac{1}{(1 - iv)^a}
\end{equation}
with \textit{arbitrary} real $a$. However, by the requirement that the corresponding $\phi$ be well defined and square integrable, this $a$ is restricted to be a non-negative integer or half-odd-integer $j$ for which $j = |s| + n$ holds for some $n = 0, 1, 2, \ldots$.

To summarize, for \textit{any} pair of angular momentum components in any irreducible representation of the $su(2)$ subalgebra labeled by $s$ and $j$, there is a 1-parameter family of most classical states which, in addition, depend on the discrete ‘quantum number’ $m = -j, -j + 1, \ldots, j$, too. There is \textit{no} restriction on $\alpha_3$, i.e. on the angle between these two components.

### 3.3 The most classical states for the $(p^i, C_i)$ system

Since the total angular momentum $J_i$ can equivalently be given by the Casimir operator $W$ and the centre-of-mass vector operator $C_i$, it is natural to ask for the most classical states with respect to the pairs of observables $(p(\alpha), C(\beta))$ and $(C(\alpha), C(\beta))$. We consider these cases separately.

#### 3.3.1 The most classical states for the observables $(p(\alpha), C(\beta))$

For any $\alpha^i, \beta^i \in \mathbb{R}^3$ satisfying $\alpha^i \alpha^j \delta_{ij} = \beta^i \beta^j \delta_{ij} = 1$, we form $p(\alpha) := \alpha^i p_i$ and $C(\beta) := \beta^i C_i$. For these
\begin{equation}
\Delta_p p(\alpha) \Delta_p C(\beta) \geq \frac{1}{2} |[p(\alpha), C(\beta)]|_{\phi} = \frac{\hbar}{2} P^2 \alpha^i \beta^i - \langle p(\alpha) p(\beta) \rangle_{\phi}, \tag{3.25}
\end{equation}
in which the equality holds precisely when, for some \textit{positive} $\lambda$,
\begin{equation}
(p(\alpha) - i\lambda C(\beta))\phi = \left( \langle p(\alpha) \rangle_{\phi} - i\lambda \langle C(\beta) \rangle_{\phi} \right) \phi
\end{equation}
holds. Using (3.2) and (3.4), this condition is the differential equation
\begin{equation}
P^2 \beta^i (m_i \delta^i \phi + \bar{m}_i \phi) + \left( \frac{\alpha_i}{\lambda \hbar} - \beta_i \right) P^i \phi = \frac{1}{\lambda \hbar} \left( \langle p(\alpha) \rangle_{\phi} - i\lambda \langle C(\beta) \rangle_{\phi} \right) \phi. \tag{3.26}
\end{equation}
As earlier, without loss of generality, we assume that $\beta^i = (0, 0, 1)$ and we use the parameterization $(\alpha_1, \alpha_2) = \sqrt{1 - \alpha_3^2} (\cos \alpha, \sin \alpha)$. Then, in the polar coordinates $(\theta, \varphi)$, this equation takes the form
\begin{equation}
\frac{\partial \ln \phi}{\partial \theta} = -\frac{1}{\sin \theta} \frac{\langle p(\alpha) \rangle_{\phi} - i\lambda \langle C(\beta) \rangle_{\phi}}{P \lambda \hbar} + \frac{\alpha_3 - \lambda \hbar}{\lambda \hbar} \cot \theta + \frac{1}{\lambda \hbar} \sqrt{1 - \alpha_3^2} \cos(\varphi - \alpha). \tag{3.27}
\end{equation}
Its solution is
\begin{equation}
\phi = A \exp \left( \frac{\theta}{\lambda \hbar} \sqrt{1 - \alpha_3^2} \cos(\varphi - \alpha) - i\frac{\langle C_3 \rangle_{\phi}}{P \hbar} \ln \cot \frac{\theta}{2} \right) \sin^a \theta \tan^b \frac{\theta}{2}, \tag{3.28}
\end{equation}
where $A = A(\varphi)$ is an arbitrary periodic function of $\varphi$, and now the powers $a$ and $b$ are $a := \alpha_3/\lambda \hbar - 1$ and $b := -\langle p(\alpha) \rangle_{\phi}/P \lambda \hbar$. 

10
Since the area element on $S$ in the polar coordinates is $dS = P^2 \sin \theta \, d\theta \wedge d\varphi$ and the modulus of the first two factors in (3.28) is bounded on $S$, the condition of the square-integrability of $\phi$ is equivalent to the existence of the integral $\int_0^\pi \sin^{2a+1} \theta \tan^{2b}(\theta/2) \, d\theta$. This condition is equivalent to $a > -1$ and $a + 1 > b > -a - 1$; i.e. to

$$
\alpha_3 > 0, \quad -P\alpha_3 < \langle \mathbf{p}(\alpha) \rangle_\phi < P\alpha_3. \quad (3.29)
$$

Hence, by the requirement of the square-integrability of the wave function and $\alpha_3 = \alpha_i\beta^i$, the angle between the directions $\alpha^i$ and $\beta^i$ must be zero or an acute angle; and the range of the expectation value $\langle \mathbf{p}(\alpha) \rangle_\phi$ is restricted by $\alpha_3$: $|\langle \mathbf{p}(\alpha) \rangle_\phi| < \alpha_3 P \leq P$. However, in this interval $\langle \mathbf{p}(\alpha) \rangle_\phi$ is freely specifiable.

Since $A = A(\varphi)$ is periodic, we may write it as $\sum_{m \in \mathbb{Z}} A_m \exp(im(\varphi - \alpha))$ with some complex constants $A_m$. Then, using (2.6), the normalization condition for $\phi$ is

$$
1 = \int_S |\phi|^2 dS = 2\pi P^2 \sum_{m \in \mathbb{Z}} \sum_{k=0}^{\infty} A_{k+m} A_m \int_0^\pi I_k \sin^{2a+1} \theta \tan^{2b} \theta \frac{d\theta}{2}. \quad (3.30)
$$

Here $I_k(x)$ is the $k$th modified Bessel function of the first kind, and its argument now is $2\theta\sqrt{1 - a^2}/(\lambda h)$. In particular, if in the expansion of $A(\varphi)$ there were only one non-zero coefficient, say $A_m$, then only $I_0$ would appear in (3.30).

Then we can calculate the expectation values and uncertainties of the basic observables. However, for states with general $A(\varphi)$, these calculations are rather lengthy and technically involved, without yielding much more insight into the nature of the problem than in the special case when $A(\varphi)$ has the form $A_m \exp(im(\varphi - \alpha))$ for a single $m$. Thus, for the sake of simplicity, we assume that $A(\varphi)$ has this special form.

A direct consequence of (3.28) is that $\Delta_\varphi C(\beta) = \Delta_\varphi \mathbf{p}(\alpha)/\lambda$. Thus, we need to calculate only $\Delta_\varphi \mathbf{p}(\alpha)$. However, the calculation of $\langle (\mathbf{p}(\alpha))^2 \rangle_\phi$ is a bit more complicated: using

$$
(\alpha_i \mathbf{p}^i)^2 = P^2 \left(1 - \alpha_3^2 \sin \theta \cos(\varphi - \alpha) + \alpha_3 \cos \theta \right)^2
$$

$$
= P^2 \left(1 - \frac{\alpha_3^2}{2} \sin^2 \theta \left(1 + \cos(2(\varphi - \alpha))\right) + 2\alpha_3 \sqrt{1 - \alpha_3^2} \sin \theta \cos(\varphi - \alpha) + \alpha_3^2 \cos^2 \theta \right)
$$

and equation (2.3) for the modified Bessel functions, we find that

$$
\langle (\mathbf{p}(\alpha))^2 \rangle_\phi = 2\pi P^4 |A_m|^2 \int_0^\pi \frac{1}{2} \left(1 - \alpha_3^2 \sin^2 \theta + \alpha_3^2 \cos^2 \theta \right) I_0 + 2\alpha_3 \sqrt{1 - \alpha_3^2} \sin \theta \cos(\varphi - \alpha) I_1 + \frac{1}{2} \left(1 - \alpha_3^2 \sin^2 \theta I_2 \right) \sin^{2a+1} \theta \tan^{2b} \frac{d\theta}{2}.
$$

Then, expressing $I_1$ and $I_2$ by $I_0$ and its derivatives, by integration by parts, and using elementary trigonometric identities, we obtain

$$
\langle (\Delta_\varphi \mathbf{p}(\alpha))^2 \rangle_\phi = \frac{1}{2} \hbar \frac{\lambda_i P^2}{2} \sqrt{1 - \alpha_3^2} \sin \theta \cos(\varphi - \alpha) \int_0^\pi \frac{1}{2} \hbar \sqrt{1 - \alpha_3^2} \sin \theta \cos \varphi \, d\varphi \, I_0 \sin^{2a+1} \theta \tan^{2b} \frac{d\theta}{2}. \quad (3.31)
$$

$\langle \mathbf{p}(\alpha) \mathbf{p}(\beta) \rangle_\phi = \langle \mathbf{p}(\alpha) \mathbf{p}(\beta) \rangle_\phi$ can be calculated in a similar way. It is given by

$$
\langle \mathbf{p}(\alpha) \mathbf{p}(\beta) \rangle_\phi = 2\pi P^4 |A_m|^2 \int_0^\pi \frac{1}{2} \hbar \sin^2 \theta + \frac{\langle \mathbf{p}(\alpha) \rangle_\phi}{P} \cos \theta \, I_0 \sin^{2a+1} \theta \tan^{2b} \frac{d\theta}{2}, \quad (3.32)
$$

which is just the second term between the big round brackets in (3.31). Thus, by $\Delta_\varphi C(\beta) = \Delta_\varphi \mathbf{p}(\alpha)/\lambda$ and equations (3.31)-(3.32), we have the explicit expression for the product uncertainty, too:

$$
\Delta_\varphi \mathbf{p}(\alpha) \Delta_\varphi C(\beta) = \frac{1}{2} \hbar \left(\alpha_i \beta^i P^2 - \langle \mathbf{p}(\alpha) \mathbf{p}(\beta) \rangle_\phi \right). \quad (3.33)
$$
This depends on \( \lambda \) only through the state \( \phi \).

To clarify the dependence of \( \Delta_\phi \langle p(\alpha)\rangle \Delta_\phi C(\beta) \) on \( \lambda \) with given \( \langle p(\alpha)\rangle \), let us recall that, by the normalization condition, \( |A_m|^2 \) depends on \( \lambda \), too. Hence, by (3.30) and (3.32)

\[
\langle p(\alpha)p(\beta)\rangle = P^2 \int_0^\pi I_0 \left( \frac{\langle p(\alpha)\rangle}{p} \cos \theta + \frac{1}{2} \lambda h \sin^2 \theta \right) \sin^{2a+1} \theta \tan^{2b} \frac{\theta}{2} d\theta
\]

Since \( I_0 \sin^{2a+1} \theta \tan^{2b}(\theta/2) \) is non-negative on the interval \([0, \pi]\) and \( \sin^2 \theta, \cos \theta \leq 1 \), the integrand in the numerator of (3.33) is not greater than \( \langle p(\alpha)\rangle |P + \lambda h/2 I_0 \sin^{2a+1} \theta \tan^{2b}(\theta/2) \); and it is not less than \( -\langle p(\alpha)\rangle |P I_0 \sin^{2a+1} \theta \tan^{2b}(\theta/2) \). Hence, \( -P |\langle p(\alpha)\rangle| \leq \langle p(\alpha)p(\beta)\rangle \leq P |\langle p(\alpha)\rangle| + \lambda h P^2/2 \), yielding

\[
\frac{1}{2} h P \left( \alpha_3 P - |\langle p(\alpha)\rangle| - \frac{1}{2} \lambda h P \right) \leq \Delta_\phi \langle p(\alpha)\rangle \Delta_\phi C(\beta) \leq \frac{1}{2} h P \left( \alpha_3 P + |\langle p(\alpha)\rangle| \right).
\]

Since by (3.29) \( \alpha_3 P - |\langle p(\alpha)\rangle| \) is strictly positive, by \( \Delta_\phi \langle p(\alpha)\rangle = \lambda \Delta_\phi C(\beta) \) this implies that, in the \( \lambda \to 0 \) limit, \( \Delta_\phi C(\beta) \) diverges and \( \Delta_\phi \langle p(\alpha)\rangle \) tends to zero; and \( \lim_{\lambda \to 0} \Delta_\phi \langle p(\alpha)\rangle = \Delta_\phi C(\beta) \) is not less than \( \frac{1}{2} h P (\alpha_3 P - |\langle p(\alpha)\rangle|) \) but less than \( h P^2 \alpha_3 \). Hence this limit is finite and positive.

The calculation of the \( \lambda \to \infty \) limit is a bit longer. By (2.41)

\[
I_0 \sin^{2(a+1)} \theta \tan^{2b} \frac{\theta}{2} = \left( 1 + \left( \frac{\theta/\lambda - \alpha_3^2}{\lambda h} \right)^2 + O(\lambda^{-4}) \right) 2^{2(a+1)} (\sin^2 \frac{\theta}{2} a + b + 1) (\cos^2 \frac{\theta}{2} a - b + 1),
\]

where the powers are

\[
a + b + 1 = \frac{1}{\lambda h} (\alpha_3 P - |\langle p(\alpha)\rangle|) =: A > 0, \quad a - b + 1 = \frac{1}{\lambda h} (\alpha_3 P + |\langle p(\alpha)\rangle|) =: B > 0.
\]

Hence, for large \( \lambda \), the denominator of (3.33) is

\[
\int_0^\pi I_0 \sin^{2a+1} \theta \tan^{2b} \frac{\theta}{2} d\theta = \int_0^\pi \frac{1}{\sin \theta} \sin^{2(a+1)} \theta \tan^{2b} \frac{\theta}{2} d\theta + O(\lambda^{-2})
\]

\[
= \frac{1}{2} 2^{2(a+1)} \int_0^{\pi/2} (\sin x)^{2a+2b+1} (\cos x)^{2a-2b+1} dx + O(\lambda^{-2})
\]

\[
= \frac{1}{2} 2^{2(a+1)} \beta \left( \frac{A}{\lambda}, \frac{B}{\lambda} \right) + O(\lambda^{-2}) = \frac{1}{2} 2^{2(a+1)} \frac{\Gamma \left( \frac{A}{\lambda} \right) \Gamma \left( \frac{B}{\lambda} \right)}{\Gamma \left( \frac{A+B}{\lambda} \right)} + O(\lambda^{-2}),
\]

where \( \beta(x, y) \) is Euler’s beta function, which can also be re-expressed by the \( \Gamma \) function as above (see [29], p. 258). (Here, to avoid confusion, we denoted the beta function by \( \beta(x, y) \) instead of the standard \( B(x, y) \).) Using also the property \( \Gamma(x+1) = x \Gamma(x) \) of the \( \Gamma \) function, in a similar way we obtain that the integrals in the numerator of (3.33) are

\[
\int_0^\pi I_0 \sin \theta \theta \sin^{2(a+1)} \theta \tan^{2b} \frac{\theta}{2} d\theta = \frac{AB}{(A+B)(A+B+\lambda)} 2^{2(a+1)} \frac{\Gamma \left( \frac{A}{\lambda} \right) \Gamma \left( \frac{B}{\lambda} \right)}{\Gamma \left( \frac{A+B}{\lambda} \right)} + O(\lambda^{-2}),
\]

\[
\int_0^\pi \frac{\cos \theta}{\sin \theta} \sin^{2(a+1)} \theta \tan^{2b} \frac{\theta}{2} d\theta = \frac{1}{2} \frac{B-A}{B+A} 2^{2(a+1)} \frac{\Gamma \left( \frac{A}{\lambda} \right) \Gamma \left( \frac{B}{\lambda} \right)}{\Gamma \left( \frac{A+B}{\lambda} \right)} + O(\lambda^{-2}).
\]

Substituting these into (3.33) we find that

\[
\langle p(\alpha)p(\beta)\rangle = \alpha_3 P^2 - \frac{2}{h} \left( \alpha_3^2 P^2 - |\langle p(\alpha)\rangle|^2 \right) \frac{1}{\lambda} + O(\lambda^{-2}).
\]

Hence, in the \( \lambda \to \infty \) limit, \( \Delta_\phi \langle p(\alpha)\rangle^2 \) tends to \( \alpha_3^2 P^2 - |\langle p(\alpha)\rangle|^2 \), while both \( \Delta_\phi C(\beta) \) and \( \Delta_\phi \langle p(\alpha)\rangle \Delta_\phi C(\beta) \) tend to zero as \( 1/\lambda \).
Therefore, to summarize, by \((3.29)\) and \(\alpha_3 = \alpha_i \beta^i\), the most classical states exist for \(p(\alpha)\) and \(C(\beta)\) only if the angle between the directions \(\alpha^i\) and \(\beta^i\) is zero or an acute angle. If this is the right or a blunt angle, then no such state exists. There is no restriction on the spin \(s\). The solutions depend on one free function, and the solution even with \(A = A_m \exp(i m(\varphi - \alpha))\) forms a three-parameter family, parameterized by the expectation values, \(\langle p(\alpha) \rangle_\phi\) and \(\langle C(\beta) \rangle_\phi\), and by \(\lambda\). However, while the range of the expectation value \(\langle C(\beta) \rangle_\phi\) and the positive parameter \(\lambda\) is unrestricted, \(\langle p(\alpha) \rangle_\phi\) is restricted by \((3.29)\), but no more. The closer the \(\alpha^i\) to be orthogonal to \(\beta^i\), the shorter the interval around zero from which \(\langle p(\alpha) \rangle_\phi\) can take values. In the limit \(\alpha_3 \to 0\), i.e. when \(\alpha^i\) is getting to be orthogonal to \(\beta^i\), the second condition in \((3.29)\) reduces to \(\langle p(\alpha) \rangle_\phi \to 0\) (compare with the conditions \((3.9)\) in the \((p^i, J^i)\) system). As functions of \(\lambda\), the asymptotic behaviour of the standard deviations and the product uncertainty is similar to that of the standard deviations and the product uncertainty in the \((p^i, J^i)\) case.

### 3.3.2 The non-existence of the most classical states for the pairs of observables \((C(\alpha), C(\beta))\)

The philosophy of the calculation is similar to the previous ones: we form \(C(\alpha) := \alpha^i C_i\) and \(C(\beta) := \beta^i C_i\), in which \(\beta^i = (0, 0, 1)\) and the unit vector \(\alpha_i\) is not parallel with \(\beta_i\), i.e. \(\alpha_3 \neq \pm 1\). For these observables
\[
\Delta_\phi C(\alpha) \Delta_\phi C(\beta) \geq \frac{\hbar}{2} |h Ps \alpha^i \beta^j \varepsilon_{ijk} \langle p^k \rangle_\phi - \alpha^i \beta^j \langle C_i p_j - C_j p_i \rangle_\phi| \tag{3.34}
\]
follows. Here, the equality holds precisely when
\[
\left( C(\alpha) - i \lambda C(\beta) \right)_\phi = \left( \langle C(\alpha) \rangle_\phi - i \lambda \langle C(\beta) \rangle_\phi \right)_\phi \tag{3.35}
\]
for some \(\lambda > 0\). This, in the unitary, irreducible representation labeled by \(P\) and \(s\), is equivalent to
\[
P(\alpha^i - i \lambda \beta^i) \left( m_i \delta^i + m_i \delta^j - \frac{p_i}{\ell} \phi \right) = -\frac{i}{\ell} \left( \langle C(\alpha) \rangle_\phi - i \lambda \langle C(\beta) \rangle_\phi \right)_\phi = : -i C \phi. \tag{3.36}
\]
Since \(C(\alpha) - i \lambda C(\beta)\) does not commute with the Casimir operator \(J_i J^i\) of the subalgebra \(su(2) \subset \mathfrak{e}(3)\), the spinorial method of \([20]\) cannot be used directly to solve the eigenvalue problem \((3.35)\). Thus, our strategy might be to find first the general local solution \(\phi\) of \((3.36)\), and then to determine the eigenvalue \(C\) from the requirement that \(\phi\) be globally well defined and square integrable on \(S\). In fact, this strategy could have been followed successfully in the determination of the most classical states for the observables \((J(\alpha), J(\beta))\) in subsection 3.2.2 (or in \([20]\)). Nevertheless, following this strategy, we show that there is no such states represented by any square integrable wave function that would also be differentiable on \(S\) except only at isolated points.

- **The general local solutions**

As in subsection 3.2.2, we write equation \((3.36)\) in the rotated complex stereographic coordinates \((\xi, \bar{\xi})\). It is
\[
Y(\ln \phi) = -i C - \frac{1}{2} s \sqrt{1 - \alpha_3^2} (\xi - \bar{\xi}) + \frac{1}{1 + \xi \bar{\xi}} \left( \sqrt{1 - \alpha_3^2} (\xi + \bar{\xi}) + (\alpha_3 - i \lambda)(\xi \bar{\xi} - 1) \right), \tag{3.37}
\]
where the complex vector field \(Y\) on its left hand side is defined by
\[
Y := \left( \frac{1}{2} \sqrt{1 - \alpha_3^2} (1 - \xi^2) + (\alpha_3 - i \lambda) \xi \right) \frac{\partial}{\partial \xi} + \left( \frac{1}{2} \sqrt{1 - \alpha_3^2} (1 - \bar{\xi}^2) + (\alpha_3 - i \lambda) \bar{\xi} \right) \frac{\partial}{\partial \bar{\xi}}. \tag{3.38}
\]
It might be worth noting that the vector field \(Y\) is orthogonal to \(X\), introduced in subsection 3.2.2 with respect to the metric \((A, J)\) on \(S\). Also, it has vanishing Lie bracket with \(X\), and
where we used the notation introduced in (3.20). Also, a simple calculation shows that

\[ \phi = \ln \left( \frac{\xi - \xi_+}{\xi - \xi_-} \right) \]

hence, in particular, the functions \( v \) and \( u \) defined below and satisfying \( X(u) = 1, X(v) = 0, Y(u) = 0 \) and \( Y(v) = 1 \) form a local complex coordinate system adapted to the vector fields \( Y \) and \( X \) on the complexified \( S \).

We solve (3.37) by rewriting the terms on the right in the form of derivatives in the direction \( Y \). Since

\[-\sqrt{1 - \alpha_3^2 \xi - \alpha_3 + i\lambda} = Y \left( \ln \left( \frac{1}{2} \sqrt{1 - \alpha_3^2 (1 - \xi^2) + (\alpha_3 - i\lambda)\xi} \right) \right),\]

the second term on the right of (3.37) can be written in the form

\[ Y \left( \ln \left( \frac{\xi - \xi_+}{\xi - \xi_-} \right)^{s/2} \right),\]

where we used the notation introduced in (3.20). Also, a simple calculation shows that

\[ Y \left( \ln(1 + i\xi \bar{\xi}) \right) = \sqrt{1 - \alpha_3^2 \xi + \xi^2} + (\alpha_3 - i\lambda)\frac{\xi \xi - 1}{1 + \xi^2} + \frac{1}{2} Y \left( \ln((\xi - \xi_+)(\xi - \xi_-)(\bar{\xi} - \xi_+)(\bar{\xi} - \xi_-)) \right),\]

and hence the last term on the right of (3.37) is

\[ Y \left( \ln \left( \frac{1 + \xi \bar{\xi}}{\sqrt{(\xi - \xi_+)(\xi - \xi_-)(\bar{\xi} - \xi_+)(\bar{\xi} - \xi_-)}} \right) \right).\]

Thus, if we knew the solution \( v \) of \( Y(v) = 1 \), then the first term on the right of (3.37) would have the form \( Y(-iCv) \) and hence we already would have a particular solution of (3.37). To have its general solution, we need the general solution \( \phi_0 \) of the homogeneous equation \( Y(\phi_0) = 0 \), too.

In the generic case (i.e. when \( (1 - \lambda)^2 + \alpha_3^2 > 0 \)), see subsection 3.2.2, the solution of \( Y(v) = 1 \) is precisely the function \( v := u_1 + u_2 \) (up to the addition of the solution of the homogeneous equation \( Y(\phi_0) = 0 \)), where \( u_1 \) and \( u_2 \) are given by

\[ u_1 = -\frac{1}{2\sqrt{1 - \lambda^2 - 2i\alpha_3}} \ln \frac{\xi - \xi_+}{\xi - \xi_-}, \quad u_2 = -\frac{1}{2\sqrt{1 - \lambda^2 - 2i\alpha_3}} \ln \frac{\bar{\xi} - \xi_+}{\bar{\xi} - \xi_-}, \]

(3.39)

The general solution \( \phi_0 \) of \( Y(\phi_0) = 0 \) is an arbitrary smooth complex function of \( u := u_1 - u_2 \), i.e. \( \phi_0 \) is an arbitrary smooth complex function of

\[ z := \frac{(\xi - \xi_-)(\bar{\xi} - \xi_+)}{(\xi - \xi_-)(\bar{\xi} - \xi_+)}. \]

(3.40)

Thus, \( \phi_0 \) depends on \( \xi \) and \( \bar{\xi} \) only through \( z \): \( \phi_0 = \phi_0(z(\xi, \bar{\xi})) \). Therefore, in the generic case, the general local solution of equation (3.37) is

\[ \phi = \phi_0 \left( \frac{(\xi - \xi_+)(\bar{\xi} - \xi_+)}{(\xi - \xi_-)(\bar{\xi} - \xi_-)} \right)^{M/2} \left( \frac{(\xi - \xi_+)(\xi - \xi_-)}{(\xi - \xi_-)(\xi - \xi_-)} \right)^{s/2} \frac{1 + \xi \bar{\xi}}{\sqrt{(\xi - \xi_+)(\xi - \xi_-)(\bar{\xi} - \xi_+)(\bar{\xi} - \xi_-)}} \]

\[ = \phi_0 \left( 1 + \xi \bar{\xi} \right) \left( \xi - \xi_+ \right)^{\frac{1}{2}(M+s-1)} \left( \xi - \xi_- \right)^{\frac{1}{2}(M-s-1)} \left( \bar{\xi} - \xi_+ \right)^{\frac{1}{2}(-M+s-1)} \left( \bar{\xi} - \xi_- \right)^{\frac{1}{2}(-M-s-1)}, \]

(3.41)

where the power \( M \) is defined by

\[ M := \frac{iC}{\sqrt{1 - \lambda^2 - 2i\alpha_3}}. \]

(3.42)

Note that, in general, this might be complex: \( M = M_1 + iM_2 \), where \( M_1, M_2 \in \mathbb{R} \).
In the exceptional case (i.e. when \( \alpha_3 = 0 \) and \( \lambda = 1 \)), the solution of \( Y(v) = 1 \) is \( v = u_1 + u_2 \), while that of \( Y(\phi_0) = 0 \) is an arbitrary smooth complex function of \( u = u_1 - u_2 \), where now

\[
u_1 := \frac{1}{i + \xi}, \quad u_2 := \frac{1}{i + \xi},
\]

(3.43)

Hence, the local solution of (3.37) is

\[
\phi = \phi_0 \exp \left(-\text{i}C \left( \frac{1}{1 + \xi} + \frac{1}{i + \xi} \right) \right) \left( \frac{1 + \xi}{1 + \xi} \right) s \frac{1 + \xi \xi}{(1 + \xi)(i + \xi)},
\]

(3.44)

where \( \phi_0 \) is still an arbitrary smooth complex function of \( u \), and hence \( \phi_0 \) depends on \( \xi \) and \( \bar{\xi} \) only through \( u \).

**The non-existence of differentiable wave functions: The generic case**

The coefficient of \( \phi_0 \) in (3.41), i.e.

\[
F := (1 + \xi \xi)(\xi - \xi_+)^{1/2} \left( \xi + \xi_- \right)^{1/2} \left( M - s - 1 \right) \left( M - s - 1 \right) \left( \xi - \xi_- \right)^{1/2} \left( M - s - 1 \right) \left( \xi - \xi_- \right)^{1/2} \left( M - s - 1 \right),
\]

(3.45)

is bounded in the \( \xi \rightarrow \infty \) limit, but, depending on the value of \( s \) and \( M \), it is either singular or finite (or zero) at the points \( \xi_+, \xi_-, \xi_- \) and \( \xi_- \). In addition, \( F \) may have discontinuities along lines in the \((\xi, \bar{\xi})\)-plane. Then the role of the still free function \( \phi_0 \) of \( z \) in (3.41) would be to ensure the square integrability of \( \phi \) in such a way that the zeros of \( \phi_0 \) would compensate the singularities of \( F \) to obtain square integrable \( \phi \). (\( \phi_0 \) may have singularities at the zeros of \( F \), and may have jumps to compensate the potential discontinuities of \( F \).) We show that this strategy yields that there is no \( \phi_0 \) that could yield a square integrable \( \phi \) which would, in addition, be differentiable everywhere on \( S \) except possibly at the points \( \xi_\pm, \xi_\pm \).

First we show that \( (\xi, \bar{\xi}) \mapsto z \), given by (3.40), is a \( \mathbb{C} \rightarrow \mathbb{C} \) surjective map, and hence the domain of \( \phi_0 \) should be the entire complex \( z \)-plane except perhaps curves or finitely many isolated points. By (3.40) the value \( z = 1 \) is the image of the whole real axis \( \xi = \bar{\xi} \). Hence we should show only that any \( z \neq 1 \) is the image of some \( \xi \). Using \( \xi_+ \xi_- = -1 \) and the notation \( \xi := r \exp(\text{i}\chi) \), (3.40) gives

\[
\begin{align*}
r^2(z - 1) - r \exp(\text{i}\chi)(z \xi_- - \xi_+) - r \exp(-\text{i}\chi)(z \xi_+ - \xi_-) - (z - 1) &= 0, \\
r^2(\bar{z} - 1) - r \exp(-\text{i}\chi)(\bar{z} \xi_- - \bar{\xi}_+) - r \exp(\text{i}\chi)(\bar{z} \xi_+ - \bar{\xi}_-) - (\bar{z} - 1) &= 0.
\end{align*}
\]

Using \( z \neq 1 \), these yield that

\[
\exp(2\text{i}\chi) = \frac{(z - 1)(\bar{z} \xi_- - \bar{\xi}_+) - (\bar{z} - 1)(z \xi_+ - \xi_-)}{(z - 1)(\bar{z} \xi_- - \xi_+) - (z - 1)(\bar{z} \xi_+ - \xi_-)}.
\]

(3.46)

The first is an explicit expression of the phase of \( \xi \) in terms of \( z \) and the constants \( \xi_\pm \) without any further restriction on them; and the second has a unique (positive) solution since its discriminant is positive. Hence, any \( z \in \mathbb{C} \) has some pre-image under the map (3.40). Therefore, \( \phi_0 \) should be defined on the whole complex plane (except possibly along lines and at isolated points determined by \( F \)) and depends only on \( z \), but not on \( \bar{z} \). Note also that \( z \) maps the points \( \xi_-, \xi_+ \) of the complex \( \xi \)-plane into the zero of the complex \( z \)-plane, it maps the whole real axis \( \xi = 1 \) into 1, and the points \( \xi_+, \xi_- \) to infinity. Also, the point \( z = 1 \) corresponds to the infinity \( \xi = \infty \) of the \( \xi \)-plane. (N.B.: The map \( (\xi, \bar{\xi}) \mapsto z \) is the product of one fraction linear transformation and the complex conjugate of another one.)

By (3.20) it is easy to see that, in the generic case, \( \xi_+, \bar{\xi}_+, \xi_- \) and \( \bar{\xi}_- \) are different points of the complex \( \xi \)-plane and none of them is zero. Hence, if \( \xi_0 \) denotes any of these points,
then there is a positive number, \( R > 0 \), such that \( U_R(\xi_0) := \{ \xi \in \mathbb{C} \mid |\xi - \xi_0| < R \} \) is an open neighbourhood of \( \xi_0 \), which does not contain any of the others. Since \( S \) is compact, the square integrability of \( \phi \) is equivalent to its local square integrability. Since \( F \) is finite on the real axis where \( z = 1 \), moreover \( z \to 1 \) if \( \xi \to \infty \), in the \( \xi \to \infty \) limit \( \phi_0(z(\xi, \bar{\xi})) \to \phi_0(1) \). Thus, \( \phi_0 \) remains bounded in the \( \xi \to \infty \) limit, too. Therefore, by (3.46) the square integrability of \( \phi \) is equivalent to the square integrability of \( \phi_0(\xi - \xi_\pm)^p \) and of \( \phi_0(\xi - \xi_\pm)^p \) on \( U_R(\xi_\pm) \), where \( p \) is the corresponding (in general complex) power in \( F \). Nevertheless, although \( (\xi, \bar{\xi}) \to \phi_0(z(\xi, \bar{\xi})) \) is smooth even on the real axis of the complex \( \xi \)-plane, \( z \to \phi_0(z) \) is not necessarily differentiable at \( z = 1 \), because the map \( (\xi, \bar{\xi}) \to z(\xi, \bar{\xi}) \) shrinks the whole real axis and the point at infinity of the complex \( \xi \)-plane to the single point 1 of the \( z \)-plane.

Since \( \xi_+, \xi_+^-, \xi_- \), and \( \xi_- \) are all different points and none of them is zero, by (3.40) \( z \) depends in the limit \( \xi \to \xi_\pm \) essentially only on \( \xi - \xi_\pm \), and hence in this limit \( \phi_0 = \phi_0(z(\xi, \bar{\xi})) \) depends essentially only on \( \xi - \xi_\pm \). In a similar way, in the limit \( \bar{\xi} \to \xi_\pm \), the function \( \phi_0(z(\xi, \bar{\xi})) \) depends essentially only on \( \xi - \bar{\xi}_\pm \). Thus first let us suppose that \( \phi_0 \simeq (\xi - \xi_\pm)^q \) in the limit \( \xi \to \xi_\pm \) for some \( q, \alpha \in \mathbb{C} \). If we write \( \xi - \xi_\pm = r \exp(i\chi) \) and \( p =: p_1 + ip_2 \) and \( q =: q_1 + iq_2 \) with \( p_1, p_2, q_1, q_2 \in \mathbb{R} \), then

\[
\phi_0(\xi - \xi_\pm)^q = \alpha(\xi - \xi_\pm)^{p+q} = \alpha r^{p_1+q_1} \exp(-p_2 + q_2) \exp\left(i\left((p_1 + q_1) \chi + (p_2 + q_2) \ln r\right)\right). \tag{3.46}
\]

Hence, on the neighbourhood \( U_R(\xi_\pm) \), one has \(|\phi_0(\xi - \xi_\pm)^2| \simeq |\alpha|^2 r^{2(p_1+q_1)} \exp(-2(p_2 + q_2)\chi)\). Then the \( \epsilon \to 0 \) limit of

\[
\int_\epsilon^R \int_0^{2\pi} |\phi_0(\xi - \xi_\pm)|^2 r d\chi dr = \frac{1 - \exp(-4\pi(p_2 + q_2))}{2(p_2 + q_2)} \left|\frac{\alpha^2}{2(p_1 + q_1 + 1)}\right|^R \tag{3.40}
\]

is finite precisely when \( q_1 + p_1 + 1 > 0 \). Since for sufficiently small \( R \) the deviation of the metric area element \( \mathcal{A} \) of \( S \) from the ‘coordinate area element’ \( r d\chi dr \) on \( U_R(\xi_\pm) \) can be neglected, we obtain that \( \phi_0(\xi - \xi_\pm)^p \) is square integrable on \( U_R(\xi_\pm) \) precisely when \( q_1 > -p_1 - 1 \). In a similar way, in a neighbourhood of \( \xi_\pm \), i.e. in the \( \xi \to \xi_\pm \) limit, if \( \phi_0 \simeq \alpha(\xi - \xi_\pm)^q \), then for the square integrability of \( \phi_0(\xi - \xi_\pm)^p \) on \( U_R(\xi_\pm) \) we obtain the same condition: \( q_1 > -p_1 - 1 \).

In particular, we have the following

A. in the \( \xi \to \xi_+ \) limit if \( \phi_0 \simeq \alpha(\xi - \xi_+)^a \) with \( a =: a_1 + ia_2, a_1, a_2 \in \mathbb{R} \), then the condition \( q_1 > -p_1 - 1 = -(M_1 + s + 1)/2 \) above shows that there are two possibilities: i. if \( M_1 + s \leq -1 \), then, in the \( \xi \to \xi_+ \) limit, \( \phi_0 \) must tend to zero as \( \alpha(\xi - \xi_+)^{a_1} \), where \( a_1 > -(M_1 + s + 1)/2 \geq 0 \); or ii. if \( M_1 + s > -1 \), then in this limit \( \phi_0 \) might in principle diverge, but there is an upper bound on the rate of its divergence: \( \phi_0 \simeq \alpha(\xi - \xi_+)^{a_1} \), where now \(-a_1 < (M_1 + s + 1)/2 \).

B. in the \( \xi \to \bar{\xi}_+ \) limit if \( \phi_0 \simeq \alpha(\xi - \bar{\xi}_+)^b \) with \( b =: b_1 + ib_2, b_1, b_2 \in \mathbb{R} \), then either i. \( M_1 - s \leq -1 \), and then in this limit \( \phi_0 \) must tend to zero as \( \alpha(\xi - \bar{\xi}_+)^{b_1} \), where \( b_1 > -(M_1 - s + 1)/2 \geq 0 \); or ii. \( M_1 - s > -1 \), when \( \phi_0 \) might in principle diverge as \( \alpha(\xi - \bar{\xi}_+)^{b_1} \), but there is an upper bound on the rate of its divergence, which is \(-b_1 < (M_1 - s + 1)/2 \).

C. in the \( \xi \to \xi_- \) limit if \( \phi_0 \simeq \alpha(\xi - \xi_-)^c \) with \( c =: c_1 + ic_2, c_1, c_2 \in \mathbb{R} \), then either i. \( M_1 - s \geq 1 \), in which case \( \phi_0 \) must tend to zero as \( \alpha(\xi - \xi_-)^{c_1} \), where \( c_1 > (M_1 - s - 1)/2 \geq 0 \); or ii. \( M_1 - s < 1 \), in which case \( \phi_0 \) might diverge, but the upper bound on the rate of its divergence is \(-c_1 < (M_1 - s + 1)/2 \).

D. in the \( \xi \to \bar{\xi}_- \) limit if \( \phi_0 \simeq \alpha(\xi - \bar{\xi}_-)^d \) with \( d =: d_1 + id_2, d_1, d_2 \in \mathbb{R} \), then either i. \( M_1 + s \geq 1 \), in which case \( \phi_0 \) must tend to zero as \( \alpha(\xi - \bar{\xi}_-)^{d_1} \) with \( d_1 > (M_1 + s - 1)/2 \geq 0 \); or ii. \( M_1 + s < 1 \), when \( \phi_0 \) might diverge as \( \alpha(\xi - \bar{\xi}_-)^{d_1} \), but there is an upper bound on the rate of its divergence: \(-d_1 < (M_1 - s - 1)/2 \).
However, since \( \phi_0 \) depends on \( \xi \) and \( \xi \) only through \( z \), and moreover by (3.40) \( z \to 0 \) if \( \xi \to \xi_+ \) or \( \xi \to \xi_- \) and \( z \to \infty \) if \( \xi \to \xi_+ \) or \( \xi \to \xi_- \), the asymptotic behaviour of \( \phi_0 \) in the limits \( \xi \to \xi_+, \xi_- \) is the same, and in the limits \( \xi \to \xi_+, \xi_- \) also. Hence \( a = d \) and \( b = c \) must hold, and, in particular,

\[
\lim_{\xi \to \xi_+} \phi_0(z(\xi, \bar{\xi})) = \lim_{\xi \to \xi_-} \phi_0(z(\xi, \bar{\xi})), \quad \lim_{\xi \to \xi_+} \phi_0(z(\xi, \bar{\xi})) = \lim_{\xi \to \xi_-} \phi_0(z(\xi, \bar{\xi})).
\] (3.47)

These constraints must be taken into account when we determine the asymptotic form of \( \phi_0 \) from the above results in cases A. – D.

Next, let us evaluate the consequences of the requirement that the wave function \( \phi \) be well defined on the whole \( S \) except only at the points \( \xi_\pm, \xi_\pm \). This means that \( \phi \) must be periodic on small closed complex paths even if they surround (but do not cross) a singularity. On the path \( \xi = \xi_\pm + r \exp(i\chi), \chi \in [0, 2\pi) \), for sufficiently small \( r \) the wave function \( \phi \) is periodic in \( \chi \) with period \( 2\pi \) precisely when \( \phi_0(\xi - \xi_\pm)^p \) is periodic. By (3.40) this is equivalent to \( q_2 = -p_2 \) and \( q_1 + p_1 = n \in \mathbb{Z} \). However, by the condition \( q_1 > -p_1 - 1 \) of the local square integrability, the second condition is, in fact, \( p_1 + q_1 = n = 0, 1, 2, \ldots \). The analogous argumentation in the case of paths surrounding the singularities \( \xi_\pm \) gives the same result. Therefore, in particular cases A. – D, we have that

\[
\begin{align*}
a_1 &= -\frac{1}{2}(M_1 + s + 1) + n_1, & a_2 &= -\frac{1}{2}M_2, \\
b_1 &= -\frac{1}{2}(M_1 - s + 1) + n_2, & b_2 &= \frac{1}{2}M_2, \\
c_1 &= \frac{1}{2}(M_1 - s + 1) + n_3, & c_2 &= \frac{1}{2}M_2, \\
d_1 &= \frac{1}{2}(M_1 + s + 1) + n_4, & d_2 &= \frac{1}{2}M_2,
\end{align*}
\]

where \( n_1, n_2, n_3, n_4 = 0, 1, 2, \ldots \). But by \( a = d \) and \( b = c \) these immediately imply that both \( M \) and the powers \( a, b, c, d \) must be real; that

\[
a = \frac{1}{2}(1 + n_1 + n_4) \geq \frac{1}{2}, \quad b = \frac{1}{2}(1 + n_2 + n_3) \geq \frac{1}{2};
\] (3.48)

and that \( 2M = n_1 + n_2 - n_3 - n_4 \) and \( 2s = n_1 - n_2 + n_3 - n_4 \) hold. Thus, both \( M \pm s \) are integer. Therefore, \( \phi_0(z) \) must tend to zero as \( 1/z^a \) if \( z \to \infty \), and also to zero as \( z^b \) if \( z \to 0 \). Hence, in particular, \( \phi_0 \) must be bounded on the complex \( z \)-plane.

Now, let us suppose that both \( M \pm s \) are odd: \( M + s = 2k_1 + 1 \) and \( M - s = 2k_2 + 1 \) for some \( k_1, k_2 \in \mathbb{Z} \). Then, apart from the isolated points \( \xi_\pm, \xi_\pm \), the function \( F \) given by (3.45) is smooth on \( S \). Hence, to ensure the smoothness of \( \phi \) on \( S - \{\xi_\pm, \xi_\pm\} \), too, the function \( \phi_0 \) must also be well defined and differentiable on the whole complex \( z \)-plane except possibly at \( z = 0, 1 \). But at the same time \( \phi_0(z) \) must tend to zero both when \( z \to 0 \) and when \( z \to \infty \), i.e. it is bounded. But then the next Lemma forces \( \phi_0 \) to be identically zero:

**Lemma.** Let \( \phi_0 : \mathbb{C} - \{0\} \to \mathbb{C} \) be continuous and differentiable on \( \mathbb{C} - \{0, 1\} \). If \( \phi_0(z) \) tends to zero as \( 1/z^a \) in the \( z \to \infty \) limit for some \( a > 0 \), and if it tends to zero as \( z^b \) in the \( z \to 0 \) limit for some \( b > 0 \), then \( \phi_0 \) is identically zero.

**Proof.** Let \( n \in \mathbb{N} \) such that \( 1/n < a \). Then \( \sqrt{z - 1}\phi_0 \) tends to zero both in the \( z \to 0 \) and in the \( z \to \infty \) limits, and it is zero at \( z = 1 \). Let \( R(n) \) be the Riemann surface associated naturally with the \( \sqrt{z - 1} \) function: it is the union of \( n \) copy of the complex \( z \)-plane which has been cut along the \( z \geq \pm 1 \) portion of the real axis, and in which the \( \text{Im}(z) > 0 \) (upper) side edge of the cut in the \( k \)th copy has been identified with the \( \text{Im}(z) < 0 \) (lower) side edge of the cut in the \( (k + 1) \)th copy for any \( k \in \mathbb{N} \) modulo \( n \). Then \( \sqrt{z - 1}\phi_0 \) extends to \( R(n) - \{0, 1\} \)
to be a differentiable function, which tends to zero in the $z \to 0$, $z \to 1$ and $z \to \infty$ limits. Thus, for any $\epsilon > 0$, there exists a positive number, $r > 0$, such that on the neighbourhood $U_{\epsilon}(0) := \{ \xi \in \mathbb{C} | |\xi| < \epsilon \}$ and $U_{\epsilon}(1) := \{ \xi \in \mathbb{C} | |\xi - 1| < \epsilon \}$ of 0 and 1, respectively, the modulus $|\sqrt{z - 1} \phi_0|$ is smaller than $\epsilon$. Also, there is a positive number, $R > 0$, such that this modulus is smaller than $\epsilon$ on $U_R(\infty) := \{ \xi \in \mathbb{C} | |\xi| > R \}$. These neighbourhoods determine open neighbourhoods of 0, 1 and $\infty$, respectively, in $\mathcal{R}(n)$, too, such that $|\sqrt{z - 1} \phi_0| < \epsilon$ on these neighbourhoods. Thus, $\max \{ |\sqrt{z - 1} \phi_0(z)| \mid z \in \partial U_{\epsilon}(0) \cup \partial U_{\epsilon}(1) \cup \partial U_R(\infty) \} \leq \epsilon$. But $K := \mathcal{R}(n) - U_{\epsilon}(0) - U_{\epsilon}(1) - U_R(\infty)$ is compact and $\partial U_{\epsilon}(0) \cup \partial U_{\epsilon}(1) \cup \partial U_R(\infty) = \partial K$, and hence, by the maximum modulus principle (see e.g. [32]), $|\sqrt{z - 1} \phi_0(z)| \leq \epsilon$ for any $z \in K$, too. Therefore, $|\sqrt{z - 1} \phi_0|$ must be smaller than an arbitrarily chosen small $\epsilon > 0$ on the whole $\mathcal{R}(n)$, implying that $\phi_0$ is identically vanishing.

Thus, in this case, there are no most classical states.

Next suppose that both $M + s = 2k_1$ and $M - s = 2k_2$. Then by (3.45)

\[
F = \frac{1}{\sqrt{(\xi - \xi_+)(\xi - \xi_-)(\xi - \xi_-)(\xi - \xi_-)}} (\xi - \xi_+)^{s_1} (\xi - \xi_-)^{s_2} (\xi - \xi_-)^{-k_2} (\xi - \xi_-)^{-k_1} (1 + \xi \xi_0). 
\]

The first factor has jumps on straight lines parallel with the positive real axis that start, respectively, at $\xi_0, \xi_\pm$. Hence, to be able to obtain a wave function $\phi$ that is smooth on $S - \{\xi_\pm, \xi_\pm\}$, there should exist a smooth function $\Phi$ on $S - \{\xi_\pm, \xi_\pm\}$ satisfying

\[
\Phi \sqrt{(\xi - \xi_+)(\xi - \xi_-)(\xi - \xi_-)(\xi - \xi_-)} = \phi_0. \quad (3.49)
\]

By the definition (3.40) of $z$, this condition can be rewritten in the form

\[
\Phi(\xi - \xi_+)(\xi - \xi_-) = \sqrt{z} \phi_0 =: \phi_1.
\]

Since the expression on the left is smooth on $S - \{\xi_\pm, \xi_\pm\}$ and the map $(\xi, \xi_0) \mapsto z$ is surjective and smooth on the $\xi$-plane except at $\xi_-$ and $\xi_+$, $\phi_1$ is a well defined differentiable function on the whole complex $z$-plane except possibly at $z = 0, 1$. By (3.48) $\phi_1$ is still bounded: in the $z \to 0$ limit it tends to zero as $z^b$ with $b \geq 3/2$, and although in the $z \to \infty$ limit it does not necessarily tend to zero, but by (3.48) it is still bounded.

However, (3.49) can be rewritten in the other equivalent form

\[
\Phi(\xi - \xi_+)(\xi - \xi_-) = \frac{\phi_0}{\sqrt{z}} =: \phi_2,
\]

and the previous analysis can be repeated. We obtain that $\phi_2(z)$ tends to zero in the $z \to \infty$ limit as $1/z^a$ with $a \geq 3/2$, while it remains bounded in the $z \to 0$ limit. Then, however, $\phi_1 \phi_2 = (\phi_0)^2$ satisfies the conditions of the Lemma above, and hence $\phi_0$ must be identically zero, which is a contradiction.

Next let us suppose that $M + s = 2k_1$ and $M - s = 2k_2 + 1$. Then by (3.45)

\[
F = \frac{1}{\sqrt{(\xi - \xi_+)(\xi - \xi_-)}} (\xi - \xi_+)^{s_1} (\xi - \xi_-)^{s_2} (\xi - \xi_-)^{-k_2} (\xi - \xi_-)^{-k_1} (1 + \xi \xi_0).
\]

Apart from the first factor, this is smooth and non-zero on $S - \{\xi_\pm, \xi_\pm\}$, but the first factor is not only singular at isolated points, but also discontinuous on certain lines in the $\xi$-plane. Hence, to obtain a wave function $\phi$ that is smooth on $S - \{\xi_\pm, \xi_\pm\}$, there should exist a function $\Phi$ which is smooth there and satisfies

\[
\Phi \sqrt{(\xi - \xi_+)(\xi - \xi_-)} = \phi_0. \quad (3.50)
\]
We show that if (3.50) were satisfied, then there would exist a Riemann surface \( \tilde{\mathcal{R}} \) and a complex analytic extension \( \phi_0 \) of \( \phi \) from the complex \( z \)-plane to \( \tilde{\mathcal{R}} \) which would take its maximum at some point of \( \tilde{\mathcal{R}} \), in contrast to the statement of the maximum modulus principle of complex analysis (see e.g. [22]).

The factor \( \sqrt{\xi - \xi_+} \) on the left of (3.50) has a jump along the straight line \( \gamma_1(x) := x + iy_+, x \geq x_+ \), where \( x_+ + iy_+ := \xi_+ \) defines the real and imaginary parts of \( \xi_+ \). In fact, if \( \xi_0 = x_0 + iy_+ \), \( x_0 > x_+ \), is any point of \( \gamma_1 \) and \( \xi \to \xi_+ \) denotes the limit at \( \xi_0 \) along any complex path through \( \xi_0 \) in the \( y > y_+/y < y_+ \) side of the line \( \gamma_1 \), then \( \lim_{\xi \to \xi_+} \sqrt{\xi - \xi_+} = (x_0 - x_+) \) and \( \lim_{\xi \to \xi_0} \sqrt{\xi - \xi_+} = -(x_0 - x_+) \). Since \( \Phi \) is continuous and \( \sqrt{\xi - \xi_-} \) is continuous in a neighbourhood of \( \gamma_1 \), by (3.50) the function \( \phi_0 \) must be discontinuous along \( \gamma_1 \):

\[
\lim_{\xi \to \xi_+} \phi_0(z(\xi, \bar{\xi})) = - \lim_{\xi \to \xi_0} \phi_0(z(\xi, \bar{\xi})).
\]

Analogously, the factor \( \sqrt{\xi - \xi_-} \) in (3.50) has a similar discontinuity at any point \( \xi_0 = x_0 - iy_- \), \( x_0 > x_- \), of the straight line \( \gamma_2(x) := x - iy_- \), \( x \geq x_- \). Thus \( \phi_0 \) has an analogous jump along the straight line \( \gamma_2 \), too.

Taking the \( \xi \) and \( \bar{\xi} \) derivative of (3.50) we find, respectively, that

\[
\sqrt{\xi - \xi_+} \frac{\partial \Phi}{\partial \xi} \bigg|_{\xi = \xi_+} + \frac{1}{2} \frac{\Phi}{(\xi - \xi_+)(\xi - \xi_-)} = \phi_0' \bigg( \frac{\xi_+ - \xi_0}{(\xi - \xi_+)(\xi - \xi_-)} \bigg),
\]

\[
\sqrt{\xi - \xi_-} \frac{\partial \Phi}{\partial \bar{\xi}} \bigg|_{\xi = \xi_-} + \frac{1}{2} \frac{\Phi}{(\xi - \xi_+)(\xi - \xi_-)} = \phi_0' \bigg( \frac{\xi_0 - \xi_-}{(\xi - \xi_+)(\xi - \xi_-)} \bigg),
\]

where \( \phi_0' \) denotes the derivative of \( \phi_0 \) with respect to \( z \). The coefficients of \( \sqrt{\xi - \xi_\pm} \) on the left of these equations are continuous along the straight line \( \gamma_1 \) (except at \( \xi_\pm \)), and these left hand sides have well defined limits from both the \( y > y_+ \) and the \( y < y_+ \) sides of \( \gamma_1 \). Since, on the right of these equations, the coefficients of \( \phi_0' \) are continuous (except at \( \xi_\pm \)), this implies, first, that \( \phi_0' \) has a well defined limit at the image under \( z \) of any point of \( \gamma_1 \) from both sides of \( \gamma_1 \), and, second, that these limits are \(-1\)-times of each other. Taking higher derivatives of (3.50), an analogous argumentation gives that any \( n \)th order derivative \( \phi_0^{(n)} \) of \( \phi_0 \) behaves in the same way:

\[
\lim_{\xi \to \xi_+} \phi_0^{(n)}(z(\xi, \bar{\xi})) = - \lim_{\xi \to \xi_0} \phi_0^{(n)}(z(\xi, \bar{\xi})).
\]

\( \phi_0^{(n)} \) has an analogous jump along the straight line \( \gamma_2 \), too.

Let \( \mathcal{R}_1 \) denote the complex \( \xi \)-plane which is cut along the straight lines \( \gamma_1 \) and \( \gamma_2 \), and let \( \mathcal{R}_2 \) denote another copy of \( \mathcal{R}_1 \). Then we form the Riemann surface \( \tilde{\mathcal{R}} \) by identifying the \( y > y_+ \) side edge along \( \gamma_1 \) in \( \mathcal{R}_1 \) with the \( y < y_+ \) side edge along \( \gamma_1 \) in \( \mathcal{R}_2 \); and the \( y < y_+ \) side edge along \( \gamma_1 \) in \( \mathcal{R}_1 \) with the \( y > y_+ \) side edge along \( \gamma_1 \) in \( \mathcal{R}_2 \). We make the analogous identification of the edges of the other straight line \( \gamma_2 \), too. \( \xi_+ \) and \( \xi_- \) will be the two branch points in \( \mathcal{R} \). Then we define \( \Phi \) on the second copy \( \mathcal{R}_2 \) to be equal to \( \Phi \) on \( \mathcal{R}_1 \), and \( \phi_0 \) (or, more precisely, \( \phi_0 \circ z \)) on \( \mathcal{R}_2 \) to be \(-1\) times of \( \phi_0 \) on \( \mathcal{R}_1 \). With these definitions we have extended \( \Phi \) and \( \phi_0 \) from \( \mathcal{R}_1 \) to \( \tilde{\mathcal{R}} \), and by (3.51) these are smooth and (3.50) still holds on \( \tilde{\mathcal{R}} \).

Finally, let \( \tilde{\mathcal{R}}_1 \) and \( \tilde{\mathcal{R}}_2 \) be two copies of the complex \( z \)-plane which have been cut along the image of the straight lines \( \gamma_1 \) and \( \gamma_2 \), and then construct the Riemann surface \( \tilde{\mathcal{R}} \) from these two in the way analogous to how \( \tilde{\mathcal{R}} \) was constructed. This \( \tilde{\mathcal{R}} \) can be considered as the image of \( \mathcal{R} \) under the map \( z \) in a wider sense. In fact, under the action of the map \( z \), the branch points \( \xi_+ \) and \( \xi_- \) of \( \mathcal{R} \) are pushed out to infinity of \( \tilde{\mathcal{R}} \), and the branch point 1 of \( \mathcal{R} \) corresponds to the whole real axis and the infinity of \( \tilde{\mathcal{R}} \). Thus, \( \mathcal{R} \) and \( \tilde{\mathcal{R}} \) are not homeomorphic to each other. In particular, while \( \mathcal{R} \) is connected, \( \tilde{\mathcal{R}} \) consists of two disconnected components \( \tilde{\mathcal{R}}' \) and \( \tilde{\mathcal{R}}'' \), each of them being homeomorphic to the cylinder \( S^1 \times \mathbb{R} \), and which are ‘connected’ to each other only at infinity.
The function \( \phi_0 \) on \( \mathcal{R} \) defines a continuous function \( \tilde{\phi}_0 \) both on \( \tilde{\mathcal{R}}' - \{0\} \) and \( \tilde{\mathcal{R}}'' - \{0\} \), and this \( \tilde{\phi}_0 \) is differentiable on \( \tilde{\mathcal{R}}' - \{0,1\} \) and \( \tilde{\mathcal{R}}'' - \{0,1\} \). Moreover, in the \( z \to 0 \) limit \( \tilde{\phi}_0(z) \) tends to zero as \( z^b \) with \( b \geq 1/2 \), and in the \( z \to \infty \) limit it tends to zero as \( 1/z^a \) with \( a \geq 1/2 \). Then, repeating the argumentation of the proof of the Lemma above, we conclude that \( \tilde{\phi}_0 \), and hence \( \phi_0 \) itself, must be identically vanishing.

The previous analysis can be repeated if \( M + s \) is odd and \( M - s \) is even with the same conclusion, and hence there are no most classical states in this case either.

**The non-existence of differentiable wave functions: The exceptional case**

The logic of the proof of the non-existence of differentiable wave functions in the exceptional case is similar to that in the generic case. Thus we only sketch the key steps.

First we show that \( (\xi, \bar{\xi}) \mapsto u \) is a surjective \( \mathcal{C} \to \mathcal{C} \) map, and hence the domain of \( \phi_0 \) should be the entire complex \( u \)-plane except possibly curves or finitely many isolated points. By (3.43) \( u := u_1 - u_2 = 0 \) on the real axis \( \xi = \bar{\xi} \) of the complex \( \xi \)-plane, and hence we should only show the existence of a pre-image of any \( u \neq 0 \). However, with the notation \( \xi := r \exp(i\chi) \) the definition of \( u \) gives

\[
\begin{align*}
    r^2 + r \left( \exp(i\chi) \left( i + \frac{1}{u} \right) + \exp(-i\chi) \left( i - \frac{1}{u} \right) \right) - 1 &= 0, \\
    r^2 + r \left( \exp(-i\chi) \left( -i + \frac{1}{\bar{u}} \right) + \exp(i\chi) \left( -i - \frac{1}{\bar{u}} \right) \right) - 1 &= 0;
\end{align*}
\]

implying that

\[
\begin{align*}
    \exp(2i\chi) &= \frac{u + \bar{u} - 2iu\bar{u}}{u + \bar{u} + 2iu\bar{u}}, \\
    r^2 + \frac{1}{2} r \left( \exp(i\chi) \left( \frac{1}{u} - \frac{1}{\bar{u}} \right) + \exp(-i\chi) \left( \frac{1}{\bar{u}} - \frac{1}{u} \right) \right) - 1 &= 0.
\end{align*}
\]

These show that \( (\xi, \bar{\xi}) \mapsto u \) is, in fact, a surjective \( \mathcal{C} \to \mathcal{C} \) map. This maps the points \( \xi = \pm i \) of the complex \( \xi \)-plane into infinity of the complex \( u \)-plane.

If the (complex) eigenvalue \( C \) in (3.44) were non-zero, then solution (3.44) would have an essential singularity at both \( \xi = i \) and \( -i \), and with these singularities \( \phi \) could not be locally square integrable. Therefore, \( C \) must be vanishing and the local solution is

\[
\phi = \phi_0 (1 + \xi \bar{\xi}) (\xi + i)^{-s-1} (\bar{\xi} + i)^{-s-1}. \tag{3.52}
\]

The coefficient of \( \phi_0 \) in (3.52),

\[
F := (1 + \xi \bar{\xi}) (\xi + i)^{-s-1} (\bar{\xi} + i)^{-s-1},
\]

is bounded in the limit \( \xi \to \infty \), and it is non-singular and non-zero everywhere except at the two points \( \xi = \pm i \). In the limit \( \xi \to -i \) the function \( \phi_0 \) depends essentially only on \( \xi + i \), and hence, in this limit, we may write \( \phi_0 \simeq \alpha (\xi + i)^a \) for some \( a = a_1 + ia_2 \in \mathcal{C} \), \( a_1, a_2 \in \mathcal{R} \), and \( \alpha \in \mathcal{C} \). Similarly, in the limit \( \xi \to i \), we have that \( \phi_0 \simeq \alpha (\xi + i)^b \), \( b = b_1 + ib_2 \) for some \( b_1, b_2 \in \mathcal{R} \).

On a small enough open neighbourhood \( U_\mathcal{R}(-i) \) of \( -i \) we write \( \xi = -i + r \exp(i\chi) \), and, on this neighbourhood,

\[
\phi(\xi + i)^{-s-1} \simeq \alpha (\xi + i)^a \exp(-a_2 \chi) \exp\left(i((a_1 + s - 1)\chi + a_2 \ln r)\right). \tag{3.53}
\]

Hence, repeating the argumentation that we had in the generic case, we find that the condition of the square integrability of \( \phi \) yields that \( a_1 + s > 0 \). Also, the square integrability of \( \phi \) on the neighbourhood \( U_\mathcal{R}(i) \) of \( i \) yields \( b_1 - s > 0 \).

Since \( u \to \infty \) in both limits \( \xi \to \pm i \), the asymptotic form of \( \phi_0 \) in both these limits are the same. Hence, \( a = b \), yielding that \( a_1 = b_1 > 0 \), i.e. \( \phi_0 \) must tend to zero if \( \xi \to \pm i \), i.e. if \( u \to \infty \).
If, in addition, the wave function \( \phi \) is required to be well defined on the whole \( S \) except only at \( \xi = \pm i \), then we find that \( a \) and \( b \) must be real and \( a = -s + 1 + n_1, b = s + 1 + n_2 \), where \( n_1, n_2 = 0, 1, 2, \ldots \). Then by the equality \( a = b \) these yield that the two integers are not independent: \( 2s = n_1 - n_2 \). Therefore, to summarize, \( \phi_0(u) \) must tend to zero in the \( u \to \infty \) limit as \( 1/a^2 \), where \( a = 1 + \frac{1}{2}(n_1 + n_2) \) and \( n_1, n_2 = 0, 1, 2, \ldots \) such that \( 2s = n_1 - n_2 \). In particular, it must be bounded on the complex \( u \)-plane.

Now let us suppose that \( s \) is an integer. Then, apart from the points \( \pm i \), the function \( F \) is smooth on \( S \). Hence, if we want the wave function \( \phi \) to be differentiable on \( S \) \( \setminus \{ \pm i \} \), then \( \phi_0 \) have to be well defined on the \( \xi \)-plane minus the points \( \pm i \). Therefore, \( \phi_0 \) must be a differentiable bounded function on the entire \( u \)-plane. Hence, by Liouville’s theorem, it must be constant. But since \( \phi_0 \) must be vanishing at \( u = \infty \), it would have to be identically zero. This is a contradiction.

If \( s \) is half odd integer, i.e. \( s = S + 1/2 \) for some \( S \in \mathbb{Z} \), then

\[
F = \frac{1}{\sqrt{(\xi + i)(\xi + i)}}(\xi + i)^S(\xi + i)^{-S-1}(1 + \xi \xi).
\]

Again, as in the generic case, the existence of a wave function \( \phi \) that is differentiable on \( S \) \( \setminus \{ \pm i \} \) is equivalent to the existence of a differentiable function \( \Phi \) on \( S \) \( \setminus \{ \pm i \} \) satisfying

\[
\Phi \sqrt{(\xi + i)(\xi + i)} = \phi_0.
\]

However, by the definition of \( u \) we can rewrite this condition in the form

\[
\Phi \sqrt{\xi - \xi} = \sqrt{u} \phi_0.
\]

On the upper half of the complex \( \xi \)-plane, where \( y := \text{Im}(\xi) > 0 \), we have that \( \sqrt{\xi - \xi} = -(1 + i)\sqrt{y} \), while on the lower half plane that \( \sqrt{\xi - \xi} = (1 + i)\sqrt{|y|} \). Hence, \( \Phi \sqrt{\xi - \xi} \) is continuous on the whole complex \( \xi \)-plane, and it is smooth everywhere outside the real axis \( \xi = \xi \). Since, however, the image of this real axis in the \( u \)-plane is 0, we conclude that \( \sqrt{u} \phi_0 \) is differentiable on the \( u \)-plane except its origin. Clearly, \( \sqrt{u} \phi_0 \) is vanishing at \( u = 0 \), and, in the \( u \to \infty \) limit, it falls off as \( (1/|u|)(1+n_1+n_2)/2 \). In particular, it is bounded on the whole \( u \)-plane. But then the Lemma above implies that \( \sqrt{u} \phi_0 \) is identically zero, yielding that \( \phi_0 \) itself is identically vanishing. This is a contradiction.

Therefore, there are no differentiable, normalizable most classical states neither in the generic nor in the exceptional case.

## 4 Summary and conclusions

It is well known (see e.g. [1, 2]) that in a Heisenberg system, generated by \( \{q, p, I\} \), (1) the most classical states always exist, and (2) they can be parameterized by the expectation values, \( \langle q \rangle_{\phi} \) and \( \langle p \rangle_{\phi} \), and by the quotient of the standard deviations, \( \lambda = \Delta\phi p/\Delta\phi q \). (3) There is no restriction on the expectation values, they can take any value; and hence (4) the set of the pairs \( (\langle q \rangle_{\phi}, \langle p \rangle_{\phi}) \) is in a one-to-one correspondence with the points of the phase space \( T^*\mathbb{R} \simeq \mathbb{R}^2 \) of the ‘corresponding’ classical system. (5) Although the quotient \( \Delta\phi p/\Delta\phi q \) can be any positive number, the product of the standard deviations is a universal constant: \( \Delta\phi p \Delta\phi q = \hbar/2 \). Thus, the most classical states are the minimal uncertainty states (i.e. in which the function \( F: \phi \mapsto \Delta\phi p\Delta\phi q \) on the unit sphere in the Hilbert space of the states takes its minimum), too.

As a result of the present investigations, we found that \( E(3) \) invariant elementary quantum mechanical systems provide pairs of observables such that, for the most classical states with respect to them, each of the above properties is failed to be satisfied in some of these examples.
In particular: (1) most classical states exist for the pair \((p(\alpha), J(\beta))\) only when the direction \(\alpha\) is orthogonal to \(\beta\), and for \((p(\alpha), C(\beta))\) only when the angle between these directions is zero or an acute angle, but no such state (which could be represented by a differentiable wave function) exists at all for \((C(\alpha), C(\beta))\). (2) The most classical states, when they exist, depend not only on the expectation values (or the parameters fixing the expectation values), but also on an (almost) completely free functions, \(A(\theta), \phi_0(w)\) and \(A(\varphi)\) in the \((p(\alpha), J(\beta))\), \((J(\alpha), J(\beta))\) and \((p(\alpha), C(\beta))\) cases, respectively, i.e. on infinitely many parameters, too. (3) The expectation values can be considerably restricted: in the \((p(\alpha), J(\beta))\) and \((p(\alpha), C(\beta))\) cases, respectively, the expectation value of the linear momentum must be zero, \(\langle p(\alpha) \rangle_\varphi = 0\), and restricted by \(|\langle p(\alpha) \rangle_\varphi| \leq P_{\alpha,\beta}\); hence (4) these values do not exhaust all their classically allowed values. (5) Although, as in the case of the Heisenberg system, the quotient of the standard deviations of the two observables (i.e. the range of \(\lambda\)) is not restricted at all, their product depends on \(\lambda\) without a minimum at any finite value of \(\lambda\). Thus, there are no minimal uncertainty states when the most classical states exist. (This last property has already been realized in \[33\] in the special case of the most classical states for the two angular momentum operators \((J_1, J_2)\) in the \(su(2)\) algebra.) However, when no most classical states exist, then, in principle, minimal uncertainty states might exist, but these could not saturate the uncertainty relation: in these the strict inequality would have to hold.

These results show explicitly that the most classical states are associated only with pairs of observables, rather than with the whole algebra of its (basic) observables: for a certain choice of a pair of observables from this algebra the most classical states exist, but for a different choice they do not at all.

It is known that, in simultaneous measurements of conjugate (non-commuting) observables of the Heisenberg system \[34\], the final states after the most accurate (or ideal) measurements are the most classical (pure), rather than more general mixed states \[35\]. Hence, assuming that this link between the ideal simultaneous measurements and the most classical states is characteristic not only to Heisenberg systems, the lack of the most classical states for a pair of observables indicates that the final state of the system after the simultaneous measurement of these observables should necessarily be mixed.

5 Acknowledgments

Thanks are due to the referee for his critical remarks and for several useful references.

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors. The author has no conflicts to disclose.

A Appendix

In Appendix A.1 mostly to fix the notations, we summarize the necessary differential geometric background that we need both in the main part of the paper and in Appendix A.2. Then, in Appendix A.2 we present the unitary, irreducible representations of the Euclidean group \(E(3)\) in the form that we use. As far as we can see, it is these representations that fit most naturally to the Euclidean group; and they are spanned precisely by the spin weighted spherical harmonics. In deriving these, we use the analogous representations of \(SU(2)\), given in Appendix A.1.4 of \[20\] (where a more detailed quantum mechanical interpretation of the geometric notions is also given). These representations are analogous in their spirit to those of the Poincaré group summarized in \[22\].
A.1 Complex coordinates and the edth operators on $S$

Let $p^i, i = 1, 2, 3$, denote Cartesian coordinates in $\mathbb{R}^3$ and $S := \{ p^i \in \mathbb{R}^3 | P^2 := \delta_{ij}p^ip^j = \text{const} \}$, the metric 2-sphere of radius $P$. The complex stereographic coordinates, projected from the north pole, are defined on $U_n := S - \{ (0,0,P) \}$, the sphere minus its north pole, by $\zeta := \exp(i\varphi) \cot(\theta/2)$, where $(\theta, \varphi)$ are the standard spherical polar coordinates. In terms of $(\zeta, \bar{\zeta})$, the Cartesian coordinates of the point $p^i \in U_n$ are

$$p^i = P\left( \frac{\zeta + \bar{\zeta}}{1 + \zeta \bar{\zeta}}, i\frac{\bar{\zeta} - \zeta}{1 + \zeta \bar{\zeta}}, \frac{\zeta \bar{\zeta} - 1}{1 + \zeta \bar{\zeta}} \right). \quad (A.1)$$

The outward pointing unit normal to $S$ at the point $p^i$ is $n^i := p^i/P$. This normal is completed to a basis by the complex vector field

$$m^i := \frac{1}{\sqrt{2}} \left( \frac{1 - \zeta^2}{1 + \zeta \bar{\zeta}}, \frac{1 + \zeta^2}{1 + \zeta \bar{\zeta}}, \frac{2\zeta}{1 + \zeta \bar{\zeta}} \right) \quad (A.2)$$

and its complex conjugate $\bar{m}^i$. These are orthogonal to $n^i$, null (i.e. $m^i \bar{m}_i = 0$), normalized with respect to each other (i.e. $m^i \bar{m}_i = 1$), and $p^i \bar{m}^j \varepsilon_{ijk} = iP$ holds. (Recall that, in the present paper, the metric on $S^3$ is chosen to be the positive definite $\delta_{ij}$, rather than the negative definite spatial part of the Minkowski metric $\eta_{ab} := \text{diag}(1,-1,-1,-1)$, where $a,b = 0, i$.) The vector field $m^i$, as a differential operator, is given by

$$m^i \left( \frac{\partial}{\partial p^j} \right) = \frac{1}{\sqrt{2}P}(1 + \zeta \bar{\zeta}) \left( \frac{\partial}{\partial \zeta} \right). \quad (A.3)$$

Hence, $\zeta$ is a local anti-holomorphic coordinate on $U_n$. Also in these coordinates, the line element of the metric and the corresponding area element on $S$ of radius $P$, respectively, are

$$dh^2 = \frac{4P^2}{(1 + \zeta \bar{\zeta})^2} d\zeta d\bar{\zeta} = P^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad dS = \frac{-2iP^2}{(1 + \zeta \bar{\zeta})^2} d\zeta \wedge d\bar{\zeta} = P^2 \sin \theta d\theta \wedge d\varphi. \quad (A.4)$$

There are analogous constructions on $U_s := S - \{ (0,0,-P) \}$, on the 2-sphere minus the south pole, too; and the structures defined on $U_n$ are related to those introduced on $U_s$ smoothly on the overlap $U_n \cap U_s$.

Considering $\mathbb{R}^3$ to be the $p^0 = P$ hyperplane of the Minkowski space $\mathbb{R}^{1,3}$ with the Cartesian coordinates $p^0 = (p^0, p^i)$ and the flat metric $\eta_{ab}$, the 2-sphere $S$ is just the intersection of the $p^0 = P$ hyperplane with the null cone of the origin $p^0 = 0$ in $\mathbb{R}^{1,3}$. Hence, for any $p^i \in S$, there is a spinor $\pi^A$, the `spinor constituent’ of $p^i$, such that $p^i = \sigma^i_{AA} \pi^A \pi^A$, and $\pi^A$ is unique only up to the phase ambiguity $\pi^A \mapsto \exp(i\gamma)\pi^A$, $\gamma \in [0, 2\pi)$. (Here $\sigma^a_{AA}$ are the standard $SL(2,C)$ Pauli matrices (including the factor $1/\sqrt{2}$), according to the conventions of [23]. Note, however, that we lower and raise the small Latin indices by $\delta_{ij}$ and its inverse. The capital Latin indices $A, B, \ldots = 0, 1$ are concrete spinor name indices, referring to some constant normalized spin frame in $\mathbb{R}^3$. The spinor name indices are raised and lowered by the standard symplectic metric $\varepsilon^{AB}$ and its inverse [23][24].) Since $P^2 = \delta_{ij}p^ip^j = -\varepsilon_{ij} \sigma^a_{AA} \sigma^b_{BB} \pi^A \pi^A \pi^B \pi^B = -(\eta_{ab}\sigma^a_{AA}\sigma^b_{BB} - \sigma^0_{AA}\sigma^0_{BB})\pi^A \pi^A \pi^B \pi^B = (\sigma^a_{AA} \pi^A)^2$, the norm of $\pi^A$ with respect to $\sqrt{2}\sigma^a_{AA}$ is $(\sqrt{2}P)^{1/2}$. However, $\pi^A$ as a spinor field is well defined only on $S$ minus one point (see e.g. [24][25]). Thus $\pi^A$ on $U_n$ and the analogous one on $U_s$ are only locally defined ‘spinorial coordinates’ on $S$.

The complex line bundles $O(-2s)$ over $S$ can be introduced using the totally symmetric N-type spinor fields of rank $2|s|$ on $S$: if $s = -|s| \leq 0$ then these spinor fields are unprimed and their principal spinor at the point $p^i = \sigma^i_{AA} \pi^A \pi^A$ is $\pi^A$, and if $s = |s| > 0$ then the spinor fields are primed and their principal spinor is $\bar{\pi}^A$. (Recall that e.g. $\lambda^A$ is called a $2|s|$-fold principal spinor of the totally symmetric spinor $\phi^{A_1 \ldots A_2 |s|}$ if $\phi^{A_1 \ldots A_2 |s|} \lambda^{A_1} = 0$ holds, in which case $\phi^{A_1 \ldots A_2 |s|}$ necessarily has the form $\phi \lambda^{A_1} \ldots \lambda^{A_2 |s|}$ for some $\phi$. These spinors are called null...
or of type N, see e.g. [23 24]. Hence, e.g. on the domain $U_n$, these spinor fields have the form
\[ \phi^{A_1 \ldots A_{2|s|}} = \phi^{A_1 \ldots A_{2|s|}} \] and
\[ \chi^{A'_1 \ldots A'_{2|s|}} = \chi^{A'_1 \ldots A'_{2|s|}} \], where $\phi$ and $\chi$ are complex functions on $U_n$. Thus, the fibers of these bundles are one complex dimensional, and the line bundle $\mathcal{O}(-2s)$ is just the abstract bundle of these fibers over $S$. $U_n$ and $U_s$ are local trivialization domains of $\mathcal{O}(-2s)$, and the functions $\phi$ for $s = -|s|$ (and $\chi$ for $s = |s|$) are local cross sections of $\mathcal{O}(-2s)$ on $U_n$, $\mathcal{O}(-2s)$ is globally trivializable precisely when $s = 0$.

The phase ambiguity $\pi^A \mapsto \exp(i\gamma)\pi^A$ in the principal spinor yields the ambiguity $\phi \mapsto \exp(-2i|s|\gamma)\phi$, where $\gamma$ is an arbitrary $[0, 2\pi]$-valued locally defined function. The analogous ambiguity in the function $\chi$ is $\chi \mapsto \exp(2is\gamma)\chi$. Therefore, despite this ambiguity, the Hermitian scalar product of any two cross sections, representing e.g. $\phi^{A_1 \ldots A_{2|s|}}$ and $\psi^{A_1 \ldots A_{2|s|}}$ and given by
\[ \langle \phi^{A_1 \ldots A_{2|s|}}, \psi^{A_1 \ldots A_{2|s|}} \rangle_s := \int_S \bar{\phi}\psi dS, \tag{A.5} \]
is well defined. The space of the square-integrable cross sections of $\mathcal{O}(-2s)$ is a Hilbert space, and is also denoted by $\mathcal{H}_s$. One can show that this scalar product is just $(\sqrt{2P})^{-2|s|}$ times the familiar, standard $L_2$-scalar product of the two spinor fields (see [22]). In quantum mechanics (see Appendix [2,2]), these square integrable totally symmetric spinor fields, or, equivalently, the corresponding cross sections of the line bundle $\mathcal{O}(-2s)$, play the role of the wave functions, while their domain, $\mathcal{S}$, is analogous to the mass-shell in the momentum space.

On $U_n$, we introduce the spinor field $o^A := (\sqrt{2P})^{-1/2}\pi^A$, which is completed by a spinor field $\iota^A$ on $U_n$ to be the Newman–Penrose spinor basis $\{o^A, \iota^A\}$ such that $o_A\iota^A = 1$, $m^A o_A = -o^A\iota^A$ and $m^A o_A = -\iota^A o^A$. Then it is easy to see that $p^A o_A = P(\pi_A o_A - o^A\iota^A)/\sqrt{2}$ also holds. Recalling that a scalar is said to have the spin weight $\frac{1}{2}(p-q)$ if under the rescaling $\{o^A, \iota^A\} \mapsto \{\lambda o^A, \lambda^{-1}\iota^A\}$, where $\lambda$ is any nowhere vanishing complex function on the domain of the spin frame, the scalar $\phi$ transforms as $\phi \mapsto \lambda^p \lambda^q \phi$ (see e.g. [23 24]). Thus, $\mathcal{O}(-2s)$ is just the bundle of spin weighted scalars of weight $s$ on $\mathcal{S}$. In particular, the components of the vectors $m^A$ and $\bar{m}^A$ are of type $(1, -1)$ and $(-1, 1)$, respectively, while those of $p^A$ are sums of a $(1,1)$ and a $(-1,-1)$ type scalar. Thus, the spin weight of $m^A$, $\bar{m}^A$ and $p^A$ is $1$, $-1$ and $0$, respectively.

If $\delta_s$ denotes the (Cartesian components of the) covariant derivative operator of the induced Levi-Civita connection acting on the spinor fields on $\mathcal{S}$, then e.g. for $s = -|s|$ the $\iota^A$-spinor components of the covariant directional derivatives of the spinor fields, defined by $\delta_s o^A := m^A \delta_s ((\sqrt{2P})^{-|s|}\phi o_A \ldots o_{A_2})\iota^{A_1 \ldots A_{2|s|}}$ and $\delta_s' o^A := m^A \delta_s ((\sqrt{2P})^{-|s|}\phi o_A \ldots o_{A_2})\iota^{A_1 \ldots A_{2|s|}}$, give just the edth and edth-prime operators of Newman and Penrose [26] acting on the appropriate line bundles (see also [23 24 23 27]). $\delta_s$ and $\delta_s'$ acting on cross sections of $\mathcal{O}(-2s)$ for $s = |s|$ are defined analogously. $\delta_s$ increases, and $\delta_s'$ decreases the spin weight by one. In the complex stereographic coordinates on $U_n$, the explicit form of these operators, acting on a function $\phi$ of spin weight $s$, is
\[ \delta_s o^A = \frac{1}{\sqrt{2P}} \left( -\frac{1}{2} \iota^A (1 + \cot \phi) \frac{\partial \phi}{\partial \zeta} + s \cot \phi \right), \quad \delta_s' o^A = \frac{1}{\sqrt{2P}} \left( -\frac{1}{2} \iota^A (1 - \cot \phi) \frac{\partial \phi}{\partial \zeta} - s \cot \phi \right) \tag{A.6} \]
while, in the more familiar polar coordinates $(\theta, \varphi)$, these operators take the form
\[ \delta_s o^A = -\frac{1}{\sqrt{2P}} e^{i\varphi} \left( \frac{\partial \phi}{\partial \theta} - i \frac{1}{\sin \theta} \frac{\partial \phi}{\partial \varphi} - s \cot \frac{\theta}{2} \right), \quad \delta_s' o^A = -\frac{1}{\sqrt{2P}} e^{-i\varphi} \left( \frac{\partial \phi}{\partial \theta} + i \frac{1}{\sin \theta} \frac{\partial \phi}{\partial \varphi} + s \cot \frac{\theta}{2} \phi \right). \tag{A.7} \]
If no confusion arises, simply we write $\delta$ and $\delta'$ instead of $\delta_s$ and $\delta_s'$. These operators link the spinors $o^A$ and $\iota^A$: $\delta o^A = 0$, $\delta' o^A = \iota^A/\sqrt{2P}$, $\delta o^A = -\iota^A/\sqrt{2P}$ and $\delta' o^A = 0$; which imply $\delta p^A = m^A$, $\delta m^A = 0$ and $\delta m^A = -p^A/P^2$. 

24
A.2 The unitary, irreducible representations of $E(3)$

We *a priori* assume that the spectrum of the operator $p^i$ is $\mathbb{R}^3$, the classical momentum space, and we endow it with the 3-metric $\delta_{ij}$. Thus $(\mathbb{R}^3, \delta_{ij})$ is a flat Riemannian 3-manifold with the globally defined Cartesian coordinates $p^i$ and the physically distinguished origin $p^i = 0$. The action of $SU(2)$ is given by $p^i \mapsto A_{ij}^i(A)p^j$, where $A_{ij}^i(A) := -\sigma_{A}^{AB}A_{ij}^{A}B\sigma_{ij}^{AB}$, in which $A_{ij}^{A}B \in SU(2)$ and the over-bar denotes complex conjugation. (The $(-)$ sign in the expression of $A_{ij}^i(A)$ is due to our convention that, in the present paper, we lower and raise the small Latin indices by the positive definite $\delta_{ij}$ and its inverse.) The surfaces of transitivity of $SU(2)$ are $S := \{ p^i \in M | p^2 := \delta_{ij}p^ip^j = \text{const} \}$ (the ‘kinetic energy-shell’, being analogous to the mass-shell in the representation theory of the Poincaré group), which are 2-spheres for $P > 0$ and the single point $p^i = 0$ for $P = 0$.

The irreducible representations of $E(3)$ are labeled by the value of the two Casimir operators, $P$ and $w$, and hence there are two disjoint cases: when $P > 0$ and when $P = 0$. Thus, first let us suppose that $P > 0$, and fix a point $\vec{p}^{\prime} \in S$. According to the method of induced representations, first we should find the representations of the stabilizer subgroup for $\vec{p}^{\prime}$ in $E(3)$. This is $U(1) \subset SU(2)$, and, by Schur’s lemma, all of its irreducible representations are one-dimensional, and these are labeled by $s = 0, \pm\frac{1}{2}, \pm 1, \ldots$. If $\vec{p}$ is the spinor constituent of $\vec{p}^{\prime}$, then this one-dimensional representation space is chosen to be spanned by the spinor of the form $\vec{p}^{\prime A_1} \cdots \vec{p}^{\prime A_{2s}}$, if $s = -|s| \leq 0$, and $\vec{p}^{\prime A_1} \cdots \vec{p}^{\prime A_{2s}}$, if $s = |s| > 0$. Then the next step is the generation of the representation space for the whole group $E(3)$ from this one dimensional space by the elements of $SU(2)$ that do not leave $\vec{p}^{\prime}$ fixed, and by the translations in $E(3)$. Clearly, this is just the construction of the bundle of totally symmetric unprimed N-type spinors $\phi^{A_1 \ldots A_{2s}}$ on $S$ if $s = -|s|$, and of the totally symmetric primed N-type spinors $\chi^{A_1 \ldots A_{2s}}$ on $S$ if $s = |s|$. The resulting bundles are isomorphic to the line bundles $\mathcal{O}(-2s)$ with the corresponding $s$.

Next we determine the explicit form of the representation of $E(3)$ by operators $U(\xi^i, A_{AB})$ acting on the spinor fields. The action of $SU(2)$ e.g. on the spinor field $\phi^{A_1 \ldots A_{2s}}$ is defined by $(U(0, A_{AB})\phi)^{A_1 \ldots A_{2s}}(p^i) := A_{B}^{A_1} \cdots A_{B_{2s}}^{A_2} \phi^{B_{1} \ldots B_{2s}}(A(A^{-1})^i_j p^j)$; while the action of the translation with $\xi^i$ is $(U(\xi^i, \delta_{AB})\phi)^{A_1 \ldots A_{2s}}(p^i) := \exp(ip_\xi^i/\hbar)\phi^{A_1 \ldots A_{2s}}(p^i)$, the multiplications by the phase factor $\exp(ip_\xi^i/\hbar)$. These transformations provide a representation of $E(3)$, and these are unitary with respect to the scalar product (A.5). Moreover, since the functions $\exp(ip_\xi^i/\hbar)$ on $S$ span a dense subspace in the space of the square integrable functions on $S$, this representation is irreducible, too. The $L_2$ space of the spinor fields with given $s$, i.e. the carrier space of the unitary, irreducible representation of $E(3)$ with given $P$ and $w$, will be denoted by $H_s$. As we will see below, $w$ is linked to $s$, because we will find that $w = \hbar P$s.

In this representation, still for $s = -|s| \leq 0$, the operators $p^i$ and $J_{ij} := \varepsilon_{ijk}J^k$ are defined to be the densely defined self-adjoint generators of these transformations: If $\xi^i = uT^i$, then $p^i$ is defined by $(i/\hbar)T^ip^i\phi^{A_1 \ldots A_{2s}} := \frac{d}{du}((U(uT^i, A_{AB})\phi)^{A_1 \ldots A_{2s}})_{u=0}$. Here the limit in the definition of the derivative is meant in the strong topology of $H_s$. Evaluating this, we find that $p^i$ is the multiplication operator. We already determined even the specific form of the self-adjoint generators $J_{ij}$ of the action of $SU(2)$ in [20], and we do not repeat that derivation here. We obtain

$$p_i \phi^{A_1 \ldots A_{2s}} = p_i \phi^{A_1 \ldots A_{2s}}, \quad (A.8)$$

$$J_{ij} \phi^{A_1 \ldots A_{2s}} = \frac{i\hbar}{2}(p_j \frac{\partial}{\partial p^j} - p_i \frac{\partial}{\partial p^i}) \phi^{A_1 \ldots A_{2s}} + \sqrt{2} \hbar s \varepsilon_{ij} \delta_k^{(A_1}(B_1 \delta^{A_2 \ldots A_{2s}}_{B_2 \ldots B_{2s}})) \phi^{B_1 \ldots B_{2s}]}, \quad (A.9)$$

Here $\sigma_{A}^{AB}$ are the standard $SU(2)$ Pauli matrices (including the factor $1/\sqrt{2}$), which are just the unitary spinor form of the three non-trivial $SL(2, \mathbb{C})$ Pauli matrices, and, with the present sign convention, these are given by $\sigma_{A}^{AB} = \delta_{ij}e^{AC} \sigma_{ij}^{CB} \sqrt{2} \sigma_{0B}B$. Since $p_i$ is a multiplication operator and $S$ is compact, it is well defined and is bounded on the whole of $H_s$. On the other hand, $J_{ij}$ is well defined only on the *dense subspace* of the smooth spinor fields in $H_s$. Then
it is a straightforward calculation to check that these operators do, indeed, satisfy the defining
commutation relations of $e(3)$ on the appropriate dense subspaces; and hence that (A.8) and
(A.9) provide a unitary irreducible representation of the Lie algebra $e(3)$.

By (A.8) the Casimir operator $P^2$ on $H_s$ is simply the multiplication by $P^2$. In [20] we
already calculated the action of $p_i J_i$ on the spinor fields $\phi^{A_1 \ldots A_{|2|}}$. However, by (A.8) in the
irreducible representation labeled by $P$ this is just the Casimir operator $W$. Hence,

$$W \phi^{A_1 \ldots A_{|2|}} = p_i J_i \phi^{A_1 \ldots A_{|2|}} = \hbar P s \phi^{A_1 \ldots A_{|2|}}.$$  \hspace{1cm} (A.10)

Therefore, $W$ is, in fact, proportional to the identity operator, and the value of $W$ on $H_s$ is
$w = \hbar P s$. Thus the representation can be labeled equally well by $P$ and $w$ or by $P$ and $s$.
Repeating the previous analysis if $s = |s| > 0$, we find the same expression for both $J_{ij}$ and $W$.

Therefore, we obtained the unitary, irreducible representations of the Euclidean group $E(3)$ just
in the geometrical form that we have for the Poincaré group, and summarized in [22].

In [20], we also calculated the contractions $m^i J_i$ and $\bar{m}^i J_i$. Using these, equation (A.10) and
the fact that on $U_n$ the spinor field is $\phi^{A_1 \ldots A_{|2|}} = \phi^{\pi A_1 \ldots \pi A_{|2|}}$, we obtained

$$J_i \phi^{A_1 \ldots A_{|2|}} = \left( m_i \bar{m}^j + \bar{m}_i m^j + \frac{1}{P^2} p_i p^j \right) J_j \phi^{A_1 \ldots A_{|2|}}$$
$$= \hbar P m_i \delta^j \phi - \bar{m}_i \delta \phi + s \frac{m_i}{P^2} \phi \pi A_1 \ldots \pi A_{|2|}.$$  \hspace{1cm} (A.11)

Therefore, defining the action of $J_i$ on the spin weighted function $\phi$ with spin weight $s$ simply
by $J_i \phi := (\sqrt{2} P)^{-|s|} (J_i \phi^{A_1 \ldots A_{|2|}}) \pi A_1 \ldots \pi A_{|2|}$, we obtain

$$J_i \phi = \hbar P \left( m_i \delta^j \phi - \bar{m}_i \delta \phi + s \frac{m_i}{P} \phi \right).$$  \hspace{1cm} (A.12)

The first two terms together on the right, denoted by $L^j$, are the orbital, while the third is the
spin part of the angular momentum. The former can also be written as $P^2 \varepsilon_{ijk} L^k = C_i p_j - C_j p_i$, from which the expression

$$C_i \phi = i \hbar \left( P^2 m_i \delta^j \phi + P^2 \bar{m}_i \delta \phi - p_i \phi \right)$$  \hspace{1cm} (A.13)

for the centre-of-mass vector operator follows.

By (A.11) it is easy to compute the only Casimir operator $J_i J^i$ of the $su(2)$ subalgebra. It is
given by

$$J_i J^i \phi = \hbar^2 \left(- P^2 (\partial \delta^j + \delta \partial^j) \phi + s^2 \phi \right).$$  \hspace{1cm} (A.13)

where $\partial \delta^j + \delta \partial^j$ is just the Laplace operator on $\mathcal{S}$. Hence the spectrum of $J_i J^i$ is $j(j + 1) \hbar^2$, $j = |s|, |s| + 1, \ldots$, as we expected, and, for fixed $s$ and $j$, the $SU(2)$-irreducible subspaces in $H_s$
are spanned by the spin weighted spherical harmonics $Y_{j,m}$, $m = -j, -j + 1, \ldots, j$.

If $P = 0$, then $p^i$ is represented by the identity operator. In this representation, the value $w$ of $W$ is
necessarily zero, but this representation is not irreducible with respect to $SU(2)$. It is an infinite direct sum of unitary irreducible representations of $SU(2)$ labeled by the value $j$ of the Casimir operator of $su(2)$.

References

[1] J. R. Klauder, B.-S. Skagerstam, Coherent states, Applications in physics and mathematical physics, in Coherent States, World Scientific, Singapore 1985

[2] W.-M. Zhang, D. H. Feng, R. Gilmore, Coherent states: Theory and some applications, Rev. Mod. Phys. 62 867 (1990)
[3] J. M. Radcliffe, Some properties of coherent spin states, J. Phys. A: Gen. Phys. 4, 313 (1971)

[4] F. T. Arecchi, E. Courtens, R. Gilmore, H. Thomas, Atomic coherent states in quantum optics, Phys. Rev. A 6, 2211 (1972)

[5] E. H. Lieb, The classical limit of quantum spin systems, Commun. Math. Phys. 31, 327 (1973)

[6] C. Aragone, G. Guerri, S. Salamo, J. L. Tani, Intelligent spin states, J. Phys. A: Math., Nucl. Gen., 7, L149 (1974)

[7] A. O. Barut, L. Girardello, New ‘coherent’ states associated with non-compact groups, Commun. Math. Phys. 21, 41 (1971)

[8] X. Wang, B. C. Sanders, S.-h. Pan, Entangled coherent states for systems with $SU(2)$ and $SU(1,1)$ symmetries, J. Phys. A: Math. Gen. 33, 7451 (2000)

[9] S. de Bièvre, Coherent states over symplectic homogeneous spaces, J. Math. Phys. 30, 1401 (1989)

[10] C. J. Isham, J. R. Klauder, Coherent states for $n$-dimensional Euclidean groups $E(n)$ and their application, J. Math. Phys. 32, 607 (1991)

[11] K. Kowalski, J. Rembieliński, L. C. Papaloucas, Coherent states for a quantum particle on a circle, J. Phys. A: Math. Gen. 29, 4149 (1996), quant-ph/9801029

[12] K. Kowalski, J. Rembieliński, Quantum mechanics on a sphere and coherent states, J. Phys. A: Math. Gen. 33, 6035 (2000), quant-ph/9912094

[13] K. Kowalski, J. Rembieliński, On the uncertainty relations and squeezed states for the quantum mechanics on a circle, J. Phys. A: Math. Gen. 35, 1405 (2002)

[14] B. C. Hall, J. J. Mitchell, Coherent states on spheres, J. Math. Phys. 43, 1211 (2002), Erratum: 46, (2005) 059901, quant-ph/0109086

[15] P. L. García de León, J. P. Gazeau, Coherent state quantization and phase operator, Phys. Lett. A 361, 301 (2007)

[16] R. Fresneda, J. P. Gazeau, D. Noguera, Quantum localisation on the circle, J. Math. Phys. 59, 052105 (2018)

[17] R. A. Silva, T. Jacobson, Particle on the sphere: Group-theoretic quantization in the presence of a magnetic monopole, J. Phys. A: Math. Theor. 54, 235303 (2021), arXiv: 2011.04888 [quant-ph]

[18] J. Guerrero, H. M. Moya-Cessa, Coherent states for equally spaced, homogeneous waveguide arrays, arXiv:2112.01673 [quant-ph]

[19] A. M. Perelomov, Coherent states for arbitrary Lie group, Commun. Math. Phys. 26, 222 (1972)

[20] L. B. Szabados, An odd feature of the ‘most classical’ states of $SU(2)$ invariant quantum mechanical systems, J. Math. Phys. 64 (2023), doi: 10.1063/5.0109611, arXiv: 2106.08695

[21] T. D. Newton, E. P. Wigner, Localized states for elementary systems, Rev. Mod. Phys. 21, 400 (1949)
[22] R. F. Streater, A. S. Wightman, *PCT, Spin and Statistics, and All That*, W. A. Benjamin, INC, New York 1964

[23] R. Penrose, W. Rindler, *Spinors and Spacetime*, vol 1, Cambridge University Press, Cambridge 1984

[24] S. A. Hugget, K. P. Tod, *An Introduction to Twistor Theory*, London Mathematical Society Student Texts 4, 2nd edition, Cambridge University Press, Cambridge 1994

[25] M. Eastwood, P. Tod, Edth – a differential operator on the sphere, Math. Proc. Camb. Phil. Soc. 92, 317 (1982)

[26] E. T. Newman, R. Penrose, Note on the Bondi–Metzner–Sachs group, J. Math. Phys. 7, 863 (1966)

[27] J. N. Goldberg, A. J. Macfarlane, E. T. Newman, F. Rohrlich, E. C. G. Sudarshan, Spin-s spherical harmonics and δ, J. Math. Phys. 8, 2155 (1967)

[28] J. Frauendiener, L. B. Szabados, The kernel of the edth operators on higher-genus spacelike 2-surfaces, Class. Quantum Grav. 18, 1003 (2001), [gr-qc/0010089](https://arxiv.org/abs/gr-qc/0010089)

[29] *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Eds. M. Abramowitz and I. A. Stegun, US National Bureau of Standards, 1972

[30] E. W. Weisstein, *Modified Bessel Function of the First Kind*, From MathWorld–A Wolfram Web Resource. [https://mathworld.wolfram.com/ModifiedBesselFunctionoftheFirstKind.html](https://mathworld.wolfram.com/ModifiedBesselFunctionoftheFirstKind.html) (cited on 2020, September)

[31] W.-K. Tung, *Group Theory in Physics*, World Scientific, Singapore 1985

[32] W. Schlag, *A Concise Course in Complex Analysis and Riemann Surfaces*, URL: [https://gauss.math.yale.edu/~ws442/complex.pdf](https://gauss.math.yale.edu/~ws442/complex.pdf)

[33] C. Aragone, E. Chalbaud, S. Salamó, On intelligent spin states, J. Math. Phys. 17, 1963 (1976)

[34] E. Arthurs, J. L. Kelly, On the simultaneous measurements of a pair of conjugate observables, Bell Syst. Tech. J. 44, 725 (1965)

[35] C. Y. She, H. Heffner, Simultaneous measurement of noncommuting observables, Phys. Rev. 152, 1103 (1966)