Conformal Riemannian $P$-Manifolds with Connections whose Curvature Tensors are Riemannian $P$-Tensors

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Abstract. The largest class of Riemannian almost product manifolds, which is closed with respect to the group of the conformal transformations of the Riemannian metric, is the class of the conformal Riemannian $P$-manifolds. This class is an analogue of the class of the conformal Kähler manifolds in almost Hermitian geometry. The main aim of this work is to obtain properties of manifolds of this class with connections, whose curvature tensors have similar properties as the Kähler tensors in Hermitian geometry.

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1. Introduction

A Riemannian almost product manifold $(M, P, g)$ is a differentiable manifold $M$ for which almost product structure $P$ is compatible with the Riemannian metric $g$ such that an isometry is induced in any tangent space of $M$. The systematic development of the theory of Riemannian almost product manifolds was started by K. Yano in [13], where basic facts of the differential geometry of these manifolds are given. In [9] A. M. Naveira gave a classification of Riemannian almost product manifolds with respect to the covariant differentiation $\nabla P$, where $\nabla$ is the Levi-Civita connection of $g$. This classification is very similar to the Gray-Hervella classification in [1] of almost Hermitian manifolds. Having in mind the results in [9], M. Staikova and K. Gribachev gave in [11] a classification of the Riemannian almost product manifolds with $\text{tr} P = 0$. In this case the manifold $M$ is even-dimensional.

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The geometry of a Riemannian almost product manifold \((M, P, g)\) is a geometry of both structures \(g\) and \(P\). There are important in this geometry the linear connections in respect of which the parallel transport determine an isomorphism of the tangent spaces with the structures \(g\) and \(P\). This is valid if and only if these structures are parallel with respect to such a connection. In the general case on a Riemannian almost product manifold there are countless number linear connections regarding which \(g\) and \(P\) are parallel. Such connections are called natural in [8].

In the present work we consider the class \(\mathcal{W}_1\) of conformal Riemannian \(P\)-manifolds (shortly \(\mathcal{W}_1\)-manifolds) from the Staikova-Gribachev classification. It is valid \(\mathcal{W}_1 = \mathcal{W}_3 \oplus \mathcal{W}_6\), were \(\mathcal{W}_3\) and \(\mathcal{W}_6\) are basic classes from the Naveira classification. The class \(\mathcal{W}_1\) is the largest class of Riemannian almost product manifolds, which is closed with respect to the group of the conformal transformations of the Riemannian metric. This class is an analogue of the class of conformal Kähler manifolds in almost Hermitian geometry.

The main aim of the present study is to obtain properties of a \(\mathcal{W}_1\)-manifold admisible a natural connection whose curvature tensor is a Riemannian \(P\)-tensor. The notion a Riemannian \(P\)-tensor, introduced in [7] on a Riemannian almost product manifold is an analogue of the notion of a Kähler tensor in Hermitian geometry. Natural connections whose curvature tensor is of Kähler type on almost contact B-metric manifolds are studied in [6].

The paper is organized as follows. In Sec. 2 we recall necessary facts about the Riemannian almost product manifolds, the class \(\mathcal{W}_1\), Riemannian \(P\)-tensors, natural connections. In Sec. 3 we obtain some properties of Riemannian \(P\)-tensors on a Riemannian almost product manifold. We obtain a presentation of such tensor on a 4-dimensional manifold by its scalar curvatures and we establish that any 4-dimensional Riemannian almost product manifolds is an almost Einstein manifold with respect to a Riemannian \(P\)-tensor. In Sec. 4 we consider \(\mathcal{W}_1\)-manifolds admisible a natural connection \(\nabla'\) whose curvature tensor \(R'\) is a Riemannian \(P\)-tensor. The main result here is Theorem 4.1 where the associated 1-forms of manifold are expressed by the scalar curvatures of \(R'\). We obtain properties of the manifold for the different cases from the classification of such connections given in [4]. We also discuss the cases when the manifold with such connection belongs to the Naveira classes \(\mathcal{W}_3\) and \(\mathcal{W}_6\). In Sec. 5 we consider two cases of a 4-dimensional manifold with connections whose curvature tensors are Riemannian \(P\)-tensors and we obtain explicit expressions of the curvature tensor of the Levi-Civita connection in these cases.

2. Preliminaries

Let \((M, P, g)\) be a Riemannian almost product manifold, i.e. a differentiable manifold \(M\) with a tensor field \(P\) of type \((1, 1)\) and a Riemannian metric \(g\) such that \(P^2 x = x, g(Px, Py) = g(x, y)\) for any \(x, y\) of the algebra \(\mathfrak{X}(M)\) of
the smooth vector fields on $M$. Further $x, y, z, u, w$ will stand for arbitrary elements of $\mathfrak{X}(M)$ or vectors in the tangent space $T_c M$ at $c \in M$.

In this work we consider manifolds $(M, P, g)$ with $\text{tr} P = 0$. In this case $M$ is an even-dimensional manifold. We assume that $\dim M = 2n$.

In [9] A.M. Naveira gives a classification of Riemannian almost product manifolds with respect to the tensor $F$ of type (0,3), defined by $F(x, y, z) = g((\nabla_x P)y, z)$, where $\nabla$ is the Levi-Civita connection of $g$. The tensor $F$ has the properties:

$$F(x, y, z) = F(x, z, y) = -F(x, Py, Pz), \quad F(x, y, Pz) = -F(x, Py, z).$$

Using the Naveira classification, in [11] M. Staikova and K. Gribachev give a classification of Riemannian almost product manifolds $(M, P, g)$ with $\text{tr} P = 0$. The basic classes of this classification are $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$. Their intersection is the class $\mathcal{W}_0$ of the Riemannian $P$-manifolds ([10]), determined by the condition $F = 0$. This class is an analogue of the class of Kähler manifolds in the geometry of almost Hermitian manifolds.

The class $\mathcal{W}_1$ from the Staikova-Gribachev classification contains the manifolds which are locally conformal equivalent to Riemannian $P$-manifolds. This class plays a similar role of the role of the class of the conformal Kähler manifolds in almost Hermitian geometry. We will say that a manifold from the class $\mathcal{W}_1$ is a $\mathcal{W}_1$-manifold.

The characteristic condition for the class $\mathcal{W}_1$ is the following:

$$\mathcal{W}_1 : F(x, y, z) = \frac{1}{2n} \left\{ g(x, y)\theta(z) - g(x, Py)\theta(Pz) + g(x, z)\theta(y) - g(x, Pz)\theta(Py) \right\},$$

where the associated 1-form $\theta$ is determined by $\theta(x) = g^{ij} F(e_i, e_j, x)$. Here $g^{ij}$ will stand for the components of the inverse matrix of $g$ with respect to a basis $\{e_i\}$ of $T_c M$ at $c \in M$. The 1-form $\theta$ is closed, i.e. $d\theta = 0$, if and only if $(\nabla_x \theta)y = (\nabla_y \theta)x$. Moreover, $\theta \circ P$ is a closed 1-form if and only if $(\nabla_x \theta)Py = (\nabla_y \theta)Px$.

In [11] it is proved that $\mathcal{W}_1 = \overline{\mathcal{W}}_3 \oplus \overline{\mathcal{W}}_6$, where $\overline{\mathcal{W}}_3$ and $\overline{\mathcal{W}}_6$ are the classes from the Naveira classification determined by the following conditions:

$$\overline{\mathcal{W}}_3 : \quad F(A, B, \xi) = \frac{1}{n} g(A, B)\theta^\nu(\xi), \quad F(\xi, \eta, A) = 0,$$

$$\overline{\mathcal{W}}_6 : \quad F(\xi, \eta, A) = \frac{1}{n} g(\xi, \eta)\theta^h(A), \quad F(A, B, \xi) = 0,$$

where $A, B, \xi, \eta \in \mathfrak{X}(M)$, $PA = A$, $PB = B$, $P\xi = -\xi$, $P\eta = -\eta$, $\theta^\nu(x) = \frac{1}{2} (\theta(x) - \theta(Px))$, $\theta^h(x) = \frac{1}{2} (\theta(x) + \theta(Px))$. In the case when $\text{tr} P = 0$, the above conditions for $\overline{\mathcal{W}}_3$ and $\overline{\mathcal{W}}_6$ can be written for any $x, y, z$ in the following form:

$$\overline{\mathcal{W}}_3 : \quad F(x, y, z) = \frac{1}{2n} \left\{ [g(x, y) + g(x, Py)] \theta(z) + [g(x, z) + g(x, Pz)] \theta(y) \right\}, \quad \theta(Px) = -\theta(x),$$

$$\overline{\mathcal{W}}_6 : \quad F(x, y, z) = \frac{1}{2n} \left\{ [g(x, y) - g(x, Py)] \theta(z) + [g(x, z) - g(x, Pz)] \theta(y) \right\}, \quad \theta(Px) = \theta(x).$$
In ([11], a tensor $L$ of type (0,4) with properties
\[ L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z), \tag{2.1} \]
\[ L(x, y, z, w) + L(y, z, x, w) + L(z, x, y, w) = 0 \tag{2.2} \]
is called a curvature-like tensor. Such a tensor on a Riemannian almost product manifold $(M, P, g)$ with the property
\[ L(x, y, Pz, Pw) = L(x, y, z, w) \tag{2.3} \]
is called a Riemannian $P$-tensor in ([7]. This notion is an analogue of the notion of a Kähler tensor in Hermitian geometry.

Let $S$ be a (0,2)-tensor on a Riemannian almost product manifold. In ([11] it is proved that
\[ \psi_1(S)(x, y, z, w) = g(y, z)S(x, w) - g(x, z)S(y, w) + S(y, z)g(x, w) - S(x, z)g(y, w) \]
is a curvature-like tensor if and only if $S(x, y) = S(y, x)$, and the tensor
\[ \psi_2(S)(x, y, z, w) = \psi_1(S)(x, y, Pz, Pw) \]
is curvature-like if and only if $S(x, Py) = S(y, Px)$. Obviously
\[ \psi_2(S)(x, y, Pz, Pw) = \psi_1(S)(x, y, z, w). \]
The tensors
\[ \pi_1 = \frac{1}{2} \psi_1(g), \quad \pi_2 = \frac{1}{2} \psi_2(g), \quad \pi_3 = \psi_1(\tilde{g}) = \psi_2(\tilde{g}) \]
are curvature-like, and the tensors $\pi_1 + \pi_2$, $\pi_3$ are Riemannian $P$-tensors.

The curvature tensor $R$ of $\nabla$ is determined by $R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z$ and the corresponding tensor of type (0,4) is defined as follows $R(x, y, z, w) = g(R(x, y)z, w)$. We denote the Ricci tensor and the scalar curvature curvature of $R$ by $\rho$ and $\tau$, respectively, i.e. $\rho(y, z) = g^{ij} R(e_i, y, z, e_j)$ and $\tau = g^{ij} \rho(e_i, e_j)$. The associated Ricci tensor $\rho^*$ and the associated scalar curvature $\tau^*$ of $R$ are determined by $\rho^*(y, z) = g^{ij} R(e_i, y, z, Pe_j)$ and $\tau^* = g^{ij} \rho^*(e_i, e_j)$. In a similar way there are determined the Ricci tensor $\rho(L)$ and the scalar curvature $\tau(L)$ for any curvature-like tensor $L$ as well as the associated quantities $\rho^*(L)$ and $\tau^*(L)$.

In ([8], a linear connection $\nabla'$ on a Riemannian almost product manifold $(M, P, g)$ is called a natural connection if $\nabla' P = \nabla' g = 0$.

In ([2] it is established that the natural connections $\nabla'$ on a $\mathcal{W}_1$-manifold $(M, P, g)$ form a 2-parametric family, where the torsion $T$ of $\nabla'$ is determined by
\[ T(x, y, z) = \frac{1}{2n} \{ g(y, z)\theta(Px) - g(x, z)\theta(Py) \}
+ \lambda \{ g(y, z)\theta(x) - g(x, z)\theta(y) + g(y, Pz)\theta(Px) - g(x, Pz)\theta(Py) \} \tag{2.4}
+ \mu \{ g(y, Pz)\theta(x) - g(x, Pz)\theta(y) + g(y, z)\theta(Px) - g(x, z)\theta(Py) \}, \]
where $\lambda, \mu \in \mathbb{R}$. 
Let us recall the following statements.

**Theorem 2.1 ([4]).** Let $R'$ be the curvature tensor of a natural connection $\nabla'$ on a $W_1$-manifold $(M, P, g)$. Then the following relation is valid:

$$R = R' - g(p, p)\pi_1 - g(q, q)\pi_2 - g(p, q)\pi_3 - \psi_1(S') - \psi_2(S''),$$

where

$$p = \lambda\Omega + (\mu + \frac{1}{2n}) P\Omega, \quad q = \lambda P\Omega + \mu\Omega, \quad g(\Omega, x) = \theta(x), \quad (2.5)$$

$$S'(y, z) = \lambda (\nabla'\theta) z + (\mu + \frac{1}{2n}) (\nabla'\theta) Pz - \frac{1}{2n} \{\lambda\theta(y)\theta(Pz) + \mu\theta(y)\theta(z)\}, \quad (2.6)$$

$$S''(y, z) = \lambda (\nabla'\theta) z + \mu (\nabla'\theta) Pz + \frac{1}{2n} \{\lambda\theta(Py)\theta(z) + \mu\theta(Py)\theta(Pz)\}.$$

**Theorem 2.2 ([4]).** Let $R'$ be the curvature tensor of a natural connection $\nabla'$ on a $W_1$-manifold $(M, P, g) \notin W_3 \cup W_6$. Then the all possible cases are as follows:

i) If $\nabla'$ is the connection $D$ determined by $\lambda = \mu = 0$, then $R'$ is a Riemannian $P$-tensor if and only if the 1-form $\theta$ is not closed and the 1-form $\theta \circ P$ is closed;

ii) If $\nabla'$ is the connection $\tilde{D}$ determined by $\lambda = 0, \mu = -\frac{1}{2n}$, then $R'$ is a Riemannian $P$-tensor if and only if the 1-form $\theta$ is closed and the 1-form $\theta \circ P$ is not closed;

iii) If $\nabla'$ is a connection for which $\lambda^2 - \mu^2 - \frac{\mu}{2n} \neq 0$, then $R'$ is a Riemannian $P$-tensor if and only if the 1-forms $\theta$ and $\theta \circ P$ are closed;

iv) If $\nabla'$ is a connection for which $\lambda \neq 0, \lambda^2 - \mu^2 - \frac{\mu}{2n} = 0$ and $R'$ is a Riemannian $P$-tensor then the 1-forms $\theta$ and $\theta \circ P$ are not closed.

Let us remark that the connection $D$ determined by $\lambda = \mu = 0$ is investigated in [3].

3. Some properties of the Riemannian $P$-tensors on Riemannian almost product manifolds

**Lemma 3.1.** Let $L$ be a Riemannian $P$-tensor on a Riemannian almost product manifold $(M, P, g)$. Then the following properties are valid:

$$L(x, Py, Pz, w) = L(Px, Py, z, w) = L(x, y, z, w), \quad (3.1)$$

$$L(Px, y, z, w) = L(x, Py, z, w) = L(x, y, Pz, w) = L(x, y, z, Pw). \quad (3.2)$$

**Proof.** Equalities (2.1), (2.2) and (2.3) imply

$$L(Px, Py, z, w) = L(x, y, z, w), \quad (3.3)$$

and the following equality follows from (2.2) and (3.3):

$$L(x, Py, Pz, w) = -L(y, z, x, w) - L(z, Px, Py, w).$$
In the latter equality, we substitute $Px$ and $Py$ for $x$ and $y$, respectively. Then, according to (3.3), we obtain

$$L(x, Py, Pz, w) = -L(Py, z, Px, w) - L(z, x, y, w).$$

The summation of the latter two equalities, bearing in mind (2.2) and (2.3), implies $L(x, Py, Pz, w) = L(x, y, z, w)$. Equalities (3.1) follow from the last equality and (3.3) as well as (3.2) — from (3.1) and (2.3). □

**Theorem 3.2.** A curvature-like tensor $L$ on 4-dimensional Riemannian almost product manifold is a Riemannian $P$-tensor if and only if $L$ has the following form:

$$L = \frac{1}{8} \{\tau(L)(\pi_1 + \pi_2) + \tau^*(L)\pi_3\}.$$

**Proof.** Let $L$ be a Riemannian $P$-tensor on a 4-dimensional Riemannian almost product manifold $(M, P, g)$ and $\{E_1, E_2, PE_1, PE_2\}$ be an orthonormal adapted basis (12) of $T_c M$, $c \in M$. Taking into account (2.3) and Lemma 3.1 we obtain that the non-zero components of $L$ are expressed by the sectional curvature $\nu = L(E_1, E_2, E_1, E_2)$ of the 2-plane $\{E_1, E_2\}$ and its associated sectional curvature $\nu^* = L(E_1, E_2, E_1, PE_2)$.

Let the arbitrary vectors $x, y, z, w$ have the form

$$x = x^1 E_1 + x^2 E_2 + x^3 PE_1 + x^4 PE_2,$$

$$y = y^1 E_1 + y^2 E_2 + y^3 PE_1 + y^4 PE_2,$$

$$z = z^1 E_1 + z^2 E_2 + z^3 PE_1 + z^4 PE_2,$$

$$w = w^1 E_1 + w^2 E_2 + w^3 PE_1 + w^4 PE_2.$$

Taking into account the linearity of $L$, equalities (2.1), (2.3) and Lemma 3.1 we get

$$L(x, y, z, w) = \nu \{a(x, y)a(z, w) + b(x, y)b(z, w)\} + \nu^* \{a(x, y)b(z, w) + b(x, y)a(z, w)\},$$

where

$$a(x, y) = x^1 y^2 + x^3 y^4 - x^2 y^1 - x^4 y^3,$$

$$b(x, y) = x^1 y^4 + x^3 y^2 - x^2 y^3 - x^4 y^1.$$

Since the basis $\{E_1, E_2, PE_1, PE_2\}$ is orthonormal, the following equalities are valid

$$(\pi_1 + \pi_2)(x, y, z, w) = -a(x, y)a(z, w) - b(x, y)b(z, w),$$

$$(\pi_3)(x, y, z, w) = -a(x, y)b(z, w) - b(x, y)a(z, w)$$

and then (3.5) takes the form

$$L(x, y, z, w) = -\nu(\pi_1 + \pi_2)(x, y, z, w) - \nu^*\pi_3(x, y, z, w).$$

Equality (3.6) implies the following form of the Ricci tensor of $L$:

$$\rho(L)(y, z) = -2\nu g(y, z) - 2\nu^* g(y, Pz).$$

Then we obtain the following formulae for the scalar curvatures of $L$:

$$\tau(L) = -8\nu, \quad \tau^*(L) = -8\nu^*.$$
Thus, (3.6) implies (3.4).

Vice versa, let \( L \) be a curvature-like tensor of the form (3.4). Then, according to \( \pi_1 + \pi_2 \) and \( \pi_3 \) are Riemannian \( P \)-tensors, the tensor \( L \) is also a Riemannian \( P \)-tensor. \( \Box \)

Bearing in mind [10], a 2-plane \( \alpha \in T_c M \) of a Riemannian almost product manifold \((M, P, g)\) is called totally real (respectively, invariant) if \( \alpha \) and \( P\alpha \) are orthogonal (respectively, \( \alpha \) and \( P\alpha \) coincide). By Lemma 3.1 we establish that all totally real basic 2-planes in the orthonormal adapted basis \( \{E_1, E_2, PE_1, PE_2\} \) have a sectional curvature \( \nu = L(E_1, E_2, E_1, E_2) \) with respect to the Riemannian \( P \)-tensor \( L \) and the invariant 2-planes have zero sectional curvatures with respect to \( L \).

Since equality (3.7) for a Riemannian \( P \)-tensor \( L \) implies
\[
\rho(L)(y,z) = \frac{1}{4} \left\{ \tau(L)g(y,z) + \tau^*(L)g(y,Pz) \right\},
\]
the manifold \((M, P, g)\) is almost Einstein with respect to \( L \).

So, the following statement is valid.

**Proposition 3.3.** Each 4-dimensional Riemannian almost product manifold is almost Einstein, which is of point-wise constant totally real sectional curvatures and zero invariant sectional curvatures with respect to arbitrary Riemannian \( P \)-tensor.

4. \( \mathcal{W}_1 \)-manifolds with a natural connection whose curvature tensor is a Riemannian \( P \)-tensor

It is known ([11]), that the curvature tensor of a linear connection \( \nabla' \) with torsion \( T \) satisfies the second Bianchi identity
\[
\mathcal{S}_{x,y,z} \{((\nabla'_xR')(y,z,u) + R'(T(x,y),z),u) = 0, \tag{4.1}
\]
where \( \mathcal{S}_{x,y,z} \) is stand for the cyclic sum by \( x, y, z \).

Now, let \( \nabla' \) be a natural connection on \( \mathcal{W}_1 \)-manifold \((M, P, g)\). Because \( \nabla'g = 0 \), equality (4.1) implies
\[
\mathcal{S}_{x,y,z} \{((\nabla'_xR')(y,z,u,w) + R'(T(x,y),z,u,w) \} = 0. \tag{4.2}
\]

Using (2.4) for the torsion tensor \( T \) of type (1,2) of \( \nabla' \), we have
\[
T(x,y) = \left\{ \lambda \theta(x) + \left( \mu + \frac{1}{2n} \right) \theta(Px) \right\} y + [\lambda \theta(Px) + \mu \theta(x)] Py
- \left\{ \lambda \theta(y) + \left( \mu + \frac{1}{2n} \right) \theta(Py) \right\} x - [\lambda \theta(Py) + \mu \theta(y)] Px,
\]
which implies immediately
\[
T(x,y) - PT(Px,y) = \frac{1}{2n} \{\theta(Px)y - \theta(x)Py\}. \tag{4.3}
\]
Let the curvature tensor $R'$ of $\nabla'$ be a Riemannian $P$-tensor. 
In (4.2), we substitute $Px$ and $Pw$ for $x$ and $w$, respectively. Then, we subtract the result equality from (4.2). Using Lemma 3.1 for $R'$, we get

\[
(\nabla_x R') (y, z, u, w) + R' (T(x, y) - PT(Px, y), z, u, w)
\]

\[- (\nabla_{P_x} R') (y, z, u, Pw) - R' (T(x, z) - PT(Px, z), y, u, w) = 0.
\]

Hence, applying Lemma 3.1, the property (4.3) and the linearity of $R'$, we obtain

\[
(\nabla_x R') (y, z, u, w) - (\nabla_{P_x} R') (y, z, u, Pw)
\]

\[+ \frac{1}{n} \{ \theta(Px) R' (y, z, u, w) - \theta(x) R' (y, Pz, u, w) \} = 0.
\]

(4.4)

Let $R'$ have a Ricci tensor $\rho'$ and scalar curvatures $\tau'$ and $\tau''$. Applying a contraction by $g^{ij}$ to $y = e_i$ and $w = e_j$, from (4.4) we get

\[
(\nabla_x \rho') (z, u) - (\nabla_{P_x} \rho') (z, Pu) + \frac{1}{n} \{ \theta(Px) \rho' (z, u) - \theta(x) \rho' (Pz, u) \} = 0.
\]

After that, applying a contraction by $g^{ks}$ to $z = e_k$ and $u = e_s$, we have

\[
d\tau' (x) - d\tau'' (Px) + \frac{1}{n} \{ \theta(Px) \tau' - \theta(x) \tau'' \} = 0,
\]

which implies

\[
d\tau' (Px) - d\tau'' (x) + \frac{1}{n} \{ \theta(x) \tau' - \theta(Px) \tau'' \} = 0.
\]

(4.5)

(4.6)

Equalities (4.5) and (4.6) determine a linear system for the 1-forms $\theta(x)$ and $\theta(Px)$ with a determinant $\Delta = (\tau'')^2 - (\tau')^2$.

4.1. Case $(M, P, g) \notin \overline{W}_3 \cup \overline{W}_6$

In this case, for the considered $W_1$-manifold is valid $\theta(x) \neq \pm \theta(Px)$. Let $\Delta \neq 0$, i.e. $|\tau''| \neq |\tau'|$. Then the linear system has the following solution:

\[
\theta(x) = \frac{n}{2} \left\{ d \ln \frac{|\tau''| + \tau'}{\tau'' - \tau'} (x) - d \ln \left( \frac{\tau''}{\tau''} - \frac{\tau'}{\tau'} \right) (Px) \right\},
\]

\[
\theta(Px) = \frac{n}{2} \left\{ d \ln \frac{|\tau''| + \tau'}{\tau'' - \tau'} (Px) - d \ln \left( \frac{\tau''}{\tau''} - \frac{\tau'}{\tau'} \right) (x) \right\}.
\]

(4.7)

Let $\Delta = 0$, i.e. $|\tau''| = |\tau'|$. Then, from (4.5) we have

\[
\tau' \{ \theta(Px) - \varepsilon \theta(x) \} = n \{ \varepsilon d\tau' (x) - d\theta' (Px) \}, \quad \varepsilon = \pm 1.
\]

Thus, for $\tau' \neq 0$ the following equalities are valid:

\[
\theta(Px) - \theta(x) = n \{ d\ln \tau' (x) - d\ln \tau' (Px) \} \quad \text{for} \quad \tau' = \tau'' \neq 0,
\]

\[
\theta(Px) + \theta(x) = -n \{ d\ln \tau' (x) + d\ln \tau' (Px) \} \quad \text{for} \quad \tau' = -\tau'' \neq 0.
\]

(4.8)

Hence we establish the truthfulness of the following
Theorem 4.1. Let the curvature tensor $R'$ of a natural connection $\nabla'$ on a $\mathcal{W}_1$-manifold $(M, P, g) \notin \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ be a Riemannian $P$-tensor with scalar curvatures $\tau'$ and $\tau^{*'}$. Then for the 1-forms $\theta$ and $\theta \circ P$ are valid equalities (4.7) for $|\tau^{*'}| = |\tau'|$ and equalities (4.8) for $|\tau^{*'}| \neq |\tau'|$.

From Theorem 2.2 and Theorem 4.1 we obtain the following

Theorem 4.2. Let the curvature tensor $R'$ of a natural connection $\nabla'$ on a $\mathcal{W}_1$-manifold $(M, P, g) \notin \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ be a Riemannian $P$-tensor with scalar curvatures $\tau'$ and $\tau^{*'}$. Then

i) For the connection $\nabla' = D$ determined by $\lambda = \mu = 0$, are valid the properties:

a) If $|\tau^{*'}| \neq |\tau'|$, then $d \ln \left| \frac{\tau^{*'} + \tau'}{\tau^{*'} - \tau'} \right| \circ P$ is a closed 1-form;

b) If $|\tau^{*'}| = |\tau'| \neq 0$, i.e. $\tau^{*'} = \epsilon \tau'$, then $d \theta = \epsilon \text{nd}(d \ln \tau' \circ P)$.

ii) For the connection $\nabla' = \tilde{D}$ determined by $\lambda = 0$ and $\mu = -\frac{1}{2n}$, are valid the properties:

a) If $|\tau^{*'}| \neq |\tau'|$, then $d \ln \left| (\tau^{*'})^2 - (\tau')^2 \right| \circ P$ is a closed 1-form;

b) If $|\tau^{*'}| = |\tau'| \neq 0$, i.e. $\tau^{*'} = \epsilon \tau'$, then $d \theta \circ P = \epsilon \text{nd}(d \ln \tau' \circ P)$.

iii) For the connections $\nabla'$ determined by $\lambda^2 - \mu^2 - \frac{\mu}{2n} = 0$, are valid the properties:

a) If $|\tau^{*'}| \neq |\tau'|$, then $d \ln \left| \frac{\tau^{*'} + \tau'}{\tau^{*'} - \tau'} \right| \circ P$ and $d \ln \left| (\tau^{*'})^2 - (\tau')^2 \right| \circ P$ are closed 1-forms;

b) If $|\tau^{*'}| = |\tau'| \neq 0$, then $d \ln \tau' \circ P$ is a closed 1-form.

Proof. Let $\nabla'$ be the connection $D$ determined by $\lambda = \mu = 0$. Then, according to Theorem 2.2 the 1-form $\theta \circ P$ is closed. If $|\tau^{*'}| \neq |\tau'|$, then by virtue of Theorem 4.1 are valid equalities (4.7). We differentiate the second equality of (4.7) and because of $d \theta \circ P = 0$ and $d \circ d = 0$, we obtain $d \left( d \ln \left| \frac{\tau^{*'} + \tau'}{\tau^{*'} - \tau'} \right| \circ P \right) = 0$, i.e. the property a) from i) is valid. If $|\tau^{*'}| = |\tau'| \neq 0$, then by virtue of Theorem 4.1 are valid equalities (4.8). After that by a differentiation we have $d \theta = \epsilon \text{nd}(d \ln \tau' \circ P)$, i.e. the property b) from i) is valid.

In a similar way we establish the properties from ii) and iii).

4.2. Case $(M, P, g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$

In this case, for the considered $\mathcal{W}_1$-manifold is valid $\theta(x) = \pm \theta(Px)$.

Theorem 4.3. Let the curvature tensor $R'$ of a natural connection $\nabla'$ on a $\mathcal{W}_1$-manifold $(M, P, g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ be a Riemannian $P$-tensor with scalar curvatures $\tau'$ and $\tau^{*'}$. Then

i) For $(M, P, g) \in \overline{\mathcal{W}}_3$ the following properties are valid:

a) If $|\tau^{*'}| \neq |\tau'|$, then $d \left( \tau^{*'} - \tau' \right) \circ P$ is a closed 1-form and the 1-form $\theta(x)$ has the form

$$\theta(x) = \frac{n}{2} \left\{ d \ln \left| \tau^{*'} + \tau' \right| (x) - d \ln \left| \tau^{*'} + \tau' \right| (Px) \right\};$$
b) If \( \tau^* = \tau' \neq 0 \), then the 1-form \( \theta(x) \) has the form

\[
\theta(x) = \frac{n}{2} \left\{ d \ln |\tau^*| (x) - d \ln |\tau'| (Px) \right\},
\]

and if \( \tau^* = -\tau' \neq 0 \), then \( d\tau' \circ P \) is a closed 1-form.

ii) For \((M, P, g) \in \mathcal{W}_6\) the following properties are valid:

a) If \( |\tau^*| \neq |\tau'| \), then \( d(\tau^* + \tau') \circ P \) is a closed 1-form and the 1-form \( \theta(x) \) has the form

\[
\theta(x) = \frac{n}{2} \left\{ d \ln |\tau^*| - d \ln |\tau' - \tau'| (Px) \right\};
\]

b) If \( |\tau^*| = |\tau'| \neq 0 \), then \( d\tau' \circ P \) is a closed 1-form and if \( \tau^* = -\tau' \neq 0 \), then 1-form \( \theta(x) \) has the form

\[
\theta(x) = -\frac{n}{2} \left\{ d \ln |\tau'| (x) + d \ln |\tau'| (Px) \right\}.
\]

Proof. Let \((M, P, g) \in \mathcal{W}_3\), i.e. \( \theta(Px) = -\theta(x) \). Then, according to (4.6), it is follows

\[
\theta(x) (\tau^* + \tau') = n \left\{ d\tau^*(x) - d\tau'(Px) \right\}.
\]

Hence, for \( |\tau^*| \neq |\tau'| \) we obtain the equalities

\[
\theta(x) = \frac{n}{2} \frac{d\tau^*(x) - d\tau'(Px)}{\tau^* + \tau'}, \quad \theta(x) = -\frac{n}{2} \frac{d\tau^*(Px) - d\tau'(x)}{\tau^* + \tau'},
\]

which imply the property a) from i).

If \( |\tau^*| = |\tau'| \neq 0 \), then (4.9) implies the property b) from i).

In a similar way, if \((M, P, g) \in \mathcal{W}_6\), i.e. \( \theta(Px) = \theta(x) \), then (4.6) implies the property a) and b) from ii).

\[\square\]

5. Two cases of 4-dimensional \( \mathcal{W}_1 \)-manifolds

According to Theorem 2.1 and Theorem 3.2, we obtain the following

Theorem 5.1. Let the curvature tensor \( R' \) of a natural connection \( \nabla' \) on a 4-dimensional \( \mathcal{W}_1 \)-manifold \((M, P, g)\) is a Riemannian \( P \)-tensor with scalar curvatures \( \tau' \) and \( \tau^* \). Then the curvature tensor \( R \) of the Levi-Civita connection has the following form

\[
R = \frac{1}{8} \left\{ \tau'(\pi_1 + \pi_2) + \tau^* \pi_3 \right\} - \psi_1(S') - \psi_2(S'') - g(p, p)\pi_1 - g(q, q)\pi_2 - g(p, q)\pi_3.
\]

(5.1)

Further we consider the cases when \( \nabla' \) is the connection \( D \) or the connection \( \tilde{D} \).
5.1. Case  $\nabla' = D$

Let $\nabla'$ be the natural connection $D$ on a 4-dimensional $W_1$-manifold $(M, P, g)$ whose curvature tensor $R'$ is a Riemannian $P$-tensor. Taking into account that $\lambda = \mu = 0$ and $n = 2$, from (2.5), (2.6) and (2.7) we have

$$g(p, p) = \frac{\theta(\Omega)}{16}, \quad g(q, q) = g(p, q) = 0, \quad S'(y, z) = \frac{(D_y \theta) P z}{4}, \quad S''(y, z) = 0.$$  

Then from (5.1) we obtain the relation

$$R = \frac{1}{8} \{\tau' (\pi_1 + \pi_2) + \tau^* \pi_3\} - \frac{\theta(\Omega)}{16} \pi_1 - \psi_1(S'), \quad (5.2)$$  

which implies immediately

$$\rho = \rho' - \frac{3\theta(\Omega)}{16} g - \text{tr} S' g - 2S', \quad \tau = \tau' - \frac{\theta(\Omega)}{4} - 6 \text{tr} S', \quad \tau^* = \tau^{**} - 2 \text{tr} \tilde{S}' , \quad (5.3)$$  

where $\tilde{S}' (y, z) = S'(y, P z)$.

In [3], it is get the equality

$$D_y z = \nabla_y z + \frac{1}{2n} \{g(y, z) P \Omega - \theta(P z) y\}.$$  

From the latter equality, we obtain the following

$$(D_y \theta) z = (\nabla_y \theta) z + \frac{1}{4} \{\theta(y) \theta(P z) - g(y, z) \theta(P \Omega)\}.$$  

Then $S'$ is expressed as follows:

$$S'(y, z) = \frac{1}{4} (\nabla_y \theta) z + \frac{1}{16} \{\theta(y) \theta(P z) - g(y, z) \theta(P \Omega)\}.$$  

Hence, the following equalities follows immediately:

$$\text{tr} S' = \frac{\text{div} P \Omega}{4} + \frac{\theta(\Omega)}{16}, \quad \text{tr} \tilde{S}' = \frac{\text{div} \Omega}{4} - \frac{3\theta(P \Omega)}{16}, \quad (5.4)$$  

where $\text{div} \Omega$ is stand for the divergence of $\Omega$.

The equalities (5.3) and (5.4) imply

$$\tau' = \tau + \frac{3 \text{div} P \Omega}{2} + \frac{9 \theta(\Omega)}{8}, \quad \tau^{**} = \tau^* + \frac{\text{div} \Omega}{2} - \frac{3 \theta(P \Omega)}{8}. \quad (5.5)$$  

By virtue of (5.2) and (5.5) we obtain the truthfulness of the following

**Theorem 5.2.** Let $(M, P, g)$ be a 4-dimensional $W_1$-manifold. If $(M, P, g)$ admit the natural connection $D$ determined by $\lambda = \mu = 0$ whose curvature tensor is a Riemannian $P$-tensor, then the curvature tensor $R$ of the Levi-Civita connection has the following form

$$R = \frac{1}{32} \{4\tau + 6 \text{div} P \Omega + 3 \theta(\Omega)\} (\pi_1 + \pi_2) \quad + \frac{1}{64} \{8\tau^* + 4 \text{div} \Omega - 3 \theta(P \Omega)\} \pi_3 - \frac{\theta(\Omega)}{16} \pi_1 - \psi_1(S').$$
5.2. Case $\nabla' = \tilde{D}$

Let $\nabla'$ be the natural connection $\tilde{D}$ on a 4-dimensional $\mathcal{W}_1$-manifold $(M, P, g)$ whose curvature tensor $R'$ is a Riemannian $P$-tensor. Taking into account that $\lambda = 0$, $\mu = -\frac{1}{4}$ and $n = 2$, from (2.5), (2.6) and (2.7) we have

\[
g(p, p) = g(p, q) = 0, \quad g(q, q) = \frac{1}{16} \theta(\Omega),
\]

\[
S'((y, z)) = \frac{1}{16} \theta(y) \theta(z), \quad S''((y, z)) = -\frac{1}{4} \left( \tilde{D}_y \theta \right) Pz - \frac{1}{16} \theta(Py) \theta(Pz).
\]

Then from (5.1) we obtain the relation

\[
R = \frac{1}{8} \left\{ \tau' \left( \pi_1 + \pi_2 \right) + \tau^* \pi_3 \right\} - \frac{\theta(\Omega)}{16} \pi_2 - \psi_1(S') + \psi_2(S''),
\]

which implies immediately

\[
\rho = \rho' + \frac{\theta(\Omega)}{16} g - \text{tr} S' g - 2S' - \text{tr} \tilde{S}' g + 2S'',
\]

\[
\tau = \tau' + \frac{\theta(\Omega)}{4} - 6\text{tr} S' + 2\text{tr} S'', \quad \tau^* = \tau^* - 2\text{tr} \tilde{S}' - 2\text{tr} \tilde{S}'',
\]

where $\tilde{g}(y, z) = g(y, Pz)$ and $\tilde{S}''(y, z) = S''(y, Pz)$.

According to [2], we have $D = \nabla + Q$, where $Q(x, y, z) = T(z, y, x)$. Then, using [2.4], $\lambda = 0$, $\mu = -\frac{1}{4}$ and $n = 2$, we get

\[
\tilde{D}_y \theta z = (\nabla_y \theta) z - \frac{1}{4} \left\{ \theta(z) Py - g(y, Pz) \theta(\Omega) \right\},
\]

which implies

\[
\left( \tilde{D}_y \theta \right) z = (\nabla_y \theta) z - \frac{1}{4} \left\{ \theta(z) Py - g(y, Pz) \theta(\Omega) \right\}.
\]

Then we obtain the following expression of $S''$:

\[
S''((y, z)) = -\frac{1}{4} (\nabla_y \theta) Pz - \frac{1}{16} g(y, z) \theta(\Omega).
\]

The latter equality and the first equality of (5.6) imply

\[
\begin{align*}
\text{tr} S' &= \frac{\theta(\Omega)}{16}, \quad \text{tr} \tilde{S}' = \frac{\theta(P\Omega)}{16}, \\
\text{tr} S'' &= -\frac{\text{div} P\Omega + \theta(\Omega)}{4}, \quad \text{tr} \tilde{S}'' = -\frac{\text{div} \Omega}{4}.
\end{align*}
\]

(5.9)

From equalities (5.8) and (5.9) we have

\[
\begin{align*}
\tau' &= \tau + \frac{\text{div} P\Omega}{2} + \frac{5\theta(\Omega)}{8}, \quad \tau^* = \frac{\text{div} \Omega}{2} + \frac{\theta(P\Omega)}{8}.
\end{align*}
\]

(5.10)

By virtue of (5.7) and (5.10) we obtain the truthfulness of the following
Theorem 5.3. Let $(M, P, g)$ be a 4-dimensional $W_1$-manifold. If $(M, P, g)$ admit the natural connection $\tilde{D}$ determined by $\lambda = 0$ and $\mu = -\frac{1}{4}$ whose curvature tensor is a Riemannian $P$-tensor, then the curvature tensor $R$ of the Levi-Civita connection has the following form

$$R = \frac{1}{64} \{8\tau + 4\text{div}P\Omega + 5\theta(\Omega)\} (\pi_1 + \pi_2)$$

$$+ \frac{1}{64} \{8\tau^* - 4\text{div}\Omega + \theta(P\Omega)\} \pi_3 - \frac{\theta(\Omega)}{16} \pi_2 - \psi_1(S') - \psi_2(S'').$$

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