Dynamical resonance locking in tidally interacting binary systems

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ABSTRACT

We examine the dynamics of resonance locking in detached, tidally interacting binary systems. In a resonance lock, a given stellar or planetary mode is trapped in a highly resonant state for an extended period of time, during which the spin and orbital frequencies vary in concert to maintain the resonance. This phenomenon is qualitatively similar to resonance capture in planetary dynamics. We show that resonance locks can accelerate the course of tidal evolution in eccentric systems and also efficiently couple spin and orbital evolution in circular binaries. Previous analyses of resonance locking have not treated the mode amplitude as a fully dynamical variable, but rather assumed the adiabatic (i.e. Lorentzian) approximation valid only in the limit of relatively strong mode damping. We relax this approximation, analytically derive conditions under which the fixed point associated with resonance locking is stable, and further check these analytic results using numerical integrations of the coupled mode, spin, and orbital evolution equations. These show that resonance locking can sometimes take the form of complex limit cycles or even chaotic trajectories. We provide simple analytic formulae that define the binary and mode parameter regimes in which resonance locks of some kind occur (stable, limit cycle, or chaotic). We briefly discuss the astrophysical implications of our results for white dwarf and neutron star binaries as well as eccentric stellar binaries.

Key words: binaries, tides, dynamics

1 INTRODUCTION

The most frequent treatment of tidal effects in detached binaries relies on the weak-friction theory (Murray & Dermott 1999), which considers only the large-scale “equilibrium tide” (i.e., filling of the Roche potential) with dissipation parameterized by the tidal quality factor \( Q \) (Goldreich & Soter 1966). Such a treatment fails to account for the resonant excitation of internal stellar waves with intrinsic frequencies comparable to the tidal forcing—the so-called “dynamical tide” (Zahn 1975).

Many studies have considered the dynamical tide in different astrophysical contexts. There are two possible regimes: when the character of the excited modes is that of radially traveling waves, or when they represent standing waves. A traveling wave occurs when reflection is prohibited by a strong dissipative process at some radial location in the star or planet in question—e.g., rapid linear dissipation near the surface or nonlinear wave breaking.

The standing wave regime is the subject of this paper. This is applicable when a wave’s amplitude can be built up by many reflections. Existing calculations in this regime have used at least one of the following two approximations: 1) tides do not backreact on the spin of the star or planet in question (Lai 1994) Rathore et al. (2005) Fuller & Lai (2011), or 2) the mode amplitude is not treated as a dynamical variable, and instead has its amplitude set by the adiabatic approximation discussed in §6.1.1 (Witte & Savonije 1999) Fuller & Lai (2012b) Burkart et al. (2013).

In this paper we are interested in understanding the phenomenon of resonance locking, in which the orbital and spin frequencies vary in concert so as to hold the Doppler-shifted tidal forcing frequency \( k\Omega_{orb} - m\Omega_{spin} \) constant (Witte & Savonije 1999). Resonance locking is analogous to the phenomenon of capture into resonance in planetary dynamics (Goldreich 1965 Goldreich & Peale 1968); we provide a comparison in §10. Resonance locks can accelerate the course of tidal evolution, as we will show in §8. Moreover, recent studies (Fuller & Lai (2012) Burkart et al. (2012)) have proposed that resonance locks may have been observed in the Kepler system KOI-54 (Welsh et al. (2011), although this has been contested by O’Leary & Burkart (2013).

Since resonance locking involves a changing spin frequency, clearly it cannot occur under approximation (1) noted above. The domain of validity of approximation (2) is given in §6.1.1 (see also Burkart et al. (2013)). In this paper, we drop both of the above assumptions and examine resonance locks accounting for a dynamically evolving mode amplitude coupled to both the orbital and spin evolution. Our aim is to investigate the general dynamical properties of resonance locking, rather than to focus on a specific astrophysical application. Our key questions concern determining when...
resonance locks can occur and under what conditions they are dynamically stable.

This paper is structured as follows. We first describe the essential idea behind resonance locking in §2 and enumerate the approximations we make in order to limit the complexity of our analysis in §3. We then develop evolution equations for a single stellar or planetary eigenmode in §4.1 and determine the implied backreaction upon the binary orbit and stellar or planetary spin in §4.2. We establish the existence and assess the stability of fixed points in the evolution equations associated with resonance locking in §5. We describe two analytic approximations that a mode's amplitude follows in certain limits in §6.1 and then present example numerical integrations of resonance locks in §6.2. In §6.3 we discuss the possibility of chaos during resonance locking. We numerically and analytically determine the parameter regimes that lead to resonance locking in §7 and show that resonance locks can accelerate tidal evolution in §8. We apply our results to two example astrophysical systems—inspiraling compact object binaries and eccentric stellar binaries—in §9. We then conclude in §10.

2 BASIC IDEA

We first explain the essential mechanism behind resonance locking by considering the example situation of a circular white dwarf binary inspiraling due to the emission of gravitational waves (Burkart et al. 2013). Focusing on a particular white dwarf, and shifting to a frame of reference corotating with the white dwarf's spin, the tidal forcing frequency is \( \sigma = m(\Omega_{\text{orb}} - \Omega_{\text{spin}}) \), where \( m \) is the azimuthal spherical harmonic index and we temporarily assume \( \Omega_{\text{orb}} \gg \Omega_{\text{spin}} \). Due to the influence of gravitational waves, \( \Omega_{\text{orb}} \) gradually increases, and thus so does \( \sigma \). As such, \( \sigma \) sweeps towards resonance with the nearest normal mode, and this mode gains energy as it becomes increasingly resonant (Rathore et al. 2005).

Along with energy, however, comes angular momentum (for \( m \neq 0 \) modes). As the mode then damps, this angular momentum is transferred to the background rotation, increasing \( \Omega_{\text{spin}} \) and consequently decreasing \( \sigma \). Thus if the mode is capable of achieving a sufficient amplitude, fixed points can exist where \( \sigma \) (but not \( \Omega_{\text{orb}} \) or \( \Omega_{\text{spin}} \) individually) is held constant by tidal synchronization balancing orbital decay by gravitational waves, as illustrated in Figure 1.

This is the idea behind a tidal resonance lock.

The properties of such fixed points corresponding to resonance locking clearly depend on the rate of externally driven orbital evolution and the strength of tidal coupling to the mode in question. Furthermore, the mode’s damping rate influences both its maximum achievable amplitude as well as the rate at which it dissipates angular momentum into the background rotation profile. Lastly, since resonance locking involves a balance between orbital spin evolution, the ratio of the associated moments of inertia—roughly \( MR^2/\mu a^2 \) where \( \mu \) is the reduced mass and \( a \) is the semi-major axis—also plays a key role. We will see in subsequent sections that dimensionless parameters corresponding to these four quantities will entirely determine the dynamics of resonance locking.

The ideas we have presented here in the context of inspiraling white dwarf binaries carry over to the more general scenario where some generic physical process causing the forcing frequency \( \sigma \) or the eigenmode frequency \( \omega \) to evolve in one direction—e.g., gravitational waves, magnetic braking, the equilibrium tide, stellar evolution, etc.—is balanced by the resonant tidal interaction with a planetary or stellar normal mode causing \( \sigma \) to evolve in the opposite direction.

3 ESSENTIAL ASSUMPTIONS

Throughout this work, we invoke the following principal assumptions.

(i) As discussed in §1, we assume that the dynamical tide is composed of standing waves, and thus that linear dissipation (provided e.g. by radiative diffusion) is not strong enough to prohibit wave reflection.

(ii) As a mode’s amplitude grows, nonlinear processes can become important. For example, gravity waves break and catastrophically dissipate when their vertical displacement becomes comparable to their vertical wavelength (Lindzen 1966). Moreover, global parametric resonances can occur at smaller amplitudes, which transfer energy to a pair of daughter modes (e.g. Weinberg et al. 2012).

We completely neglect nonlinear hydrodynamical phenomena in this work. The applicability of this approximation depends on the particular star or planet’s structure and the binary’s parameters. For example, although Witte & Savonije (2002) considered resonance locking in solar-type binaries, Goodman & Dickson (1998) showed that the lack of a convective core in such stars allows geometric focusing to progressively amplify the local amplitude of an inward-propagating gravity wave to the point where wave breaking occurs over a wide range of binary parameters (see also Barker & Ogilvie 2010). This precludes the establishment of the standing waves required for a resonance lock.

(iii) We assume that the star or planet in question rotates as a rigid body, even though tidal torques are typically applied to the background rotation profile in a thin spherical shell where mode damping is strongest (Goldreich & Nicholson 1989a). We thus rely upon the action of an efficient angular momentum transport process. Whether such a process is available strongly depends on the application in question; see e.g. Burkart et al. (2013) and accompanying references. Without solid-body rotation, critical layers can develop where angular momentum is deposited, which would provide rapid, local dissipation and hence violate assumption (i).
(iv) We take the overall strength of a mode’s tidal forcing to be constant, and account for orbital evolution only as far as it affects the forcing frequency. We thus ignore, for example, that the magnitude of the tidal force depends on the binary’s semi-major axis, which necessarily must evolve if $\Omega_{\text{orb}}$ is to change. We determine when this assumption is valid in § 5.1; it would never be appropriate for long-timescale simulations that follow the evolution of many resonance locks (which are not performed in this paper).

(v) Lastly, to simplify our analysis, we only consider binary systems where the spin and orbital angular momenta are aligned. Generalization to spin-orbit-misaligned systems is straightforward (Lai 2012), and ultimately only introduces a Wigner $D$-matrix element into our definition of the tidal coupling coefficient $U$ in § 4.1.

4 FORMALISM

4.1 Mode amplitude evolution

We work in a frame of reference corotating with the spin of the body in question, where the rotation axis lies along the $\hat{z}$ direction. Following Schenk et al. (2002), we invoke a phase space eigenmode decomposition of the tidal response (Dyson & Schutz 1979). The standard stellar mode inner product for arbitrary vector fields $\eta$ and $\eta'$ is

$$\langle \eta, \eta' \rangle = \int \eta^* \cdot \eta' \rho \, dV,$$

(1)

and the anti-Hermitian Coriolis force operator is defined by $B \eta = 2\Omega_{\text{spin}} \hat{z} \times \eta$.

We consider the resonant interaction between a time-varying tidal potential evaluated on a Keplerian orbit and a single linear mode of our expansion. The mode has spherical harmonic indices $l$ and $m$, and has complex amplitude $q$. The differential equation describing the mode’s linear evolution is (Schenk et al. 2002)

$$\dot{q} + (i\omega + \gamma) q = \frac{i\omega}{\varepsilon} \langle \xi, a_{\text{node}} \rangle,$$

(2)

where $\omega$ is the mode’s eigenfrequency, $\gamma$ is its damping rate, $\xi$ is its Lagrangian displacement vector,

$$\varepsilon = 2\omega^2 \langle \xi, \xi \rangle + \omega \langle \xi, iB\xi \rangle$$

(3)

is a normalization factor (equal to the mode energy at unit amplitude), and $a_{\text{node}}$ is the time-dependent tidal acceleration vector. We assume $\gamma \ll \omega$ and, without loss of generality, we take $\omega > 0$.

Allowing for an arbitrary eccentricity, the projection of the tidal acceleration onto the mode $(\xi, a_{\text{node}})$ is proportional to

$$\left(\frac{a}{D}\right)^{l+1} e^{-i m f \psi_{\text{spin}}},$$

where $D$ is the binary separation, $a$ is the semi-major axis, $f$ is the true anomaly, and

$$(\psi_{\text{orb}}, \psi_{\text{spin}}) = \int \left(\Omega_{\text{orb}} - \Omega_{\text{spin}}\right) dt.$$

(4)

Assuming that changes in the orbital frequency and eccentricity occur on timescales much longer than an orbital period, we can expand the dependence on $D$ and $f$ in a Fourier series:

$$\left(\frac{a}{D}\right)^{l+1} e^{-i m f \psi_{\text{spin}}} \approx \sum_k X_k^m e^{-i k \psi_{\text{orb}} + i m \psi_{\text{spin}}},$$

(5)

where $X_k^m(\varepsilon)$ is a Hansen coefficient (Appendix B), with $X_k^m(\varepsilon = 0) = \delta_k^m$ for circular orbits. Since we are concerned with resonant mode-tide interaction, we henceforth consider only a single harmonic component of this expansion. The phase associated with this component is $\psi = k \psi_{\text{orb}} - m \psi_{\text{spin}}$.

Our mode amplitude equation from (2) then becomes

$$\dot{q} + (i\omega + \gamma) q = i\omega U e^{i\psi}.$$

(6)

The dimensionless mode-tide coupling strength associated with our harmonic component is

$$U = \left(\frac{M}{M'}\right)^{l+1} \left(\frac{R}{a}\right) \int_0^1 \frac{E_r}{\varepsilon} J_{lm} X_{lm} W_{lm},$$

(7)

where $M'$ is the companion mass,

$$J_{lm} = \frac{1}{M R^2} \left(\xi, \nabla \left(r^l Y_{lm}\right)\right)$$

(8)

is the mode’s linear tidal overlap integral (Press & Teukolsky 1977).

$$W_{lm} = \frac{4\pi}{2l + 1} Y_{lm}^* \left(\frac{\pi}{2}, 0\right)$$

(9)

is an order-unity angular coupling coefficient, and $E_r = GM^2/R$ is the gravitational energy scale. We also define the tidal forcing frequency to be $\sigma = \psi = k \Omega_{\text{orb}} - m \Omega_{\text{spin}}$, and take $\sigma > 0$ without loss of generality. We are free to choose the sign of $\sigma$ since we are considering a complex conjugate mode pair—one member of the pair is resonant with $\sigma$ while the other is resonant with $-\sigma$. Because $m$ can possess either sign, both prograde and retrograde modes are allowed for $\sigma > 0$. Since we are considering resonant interaction, our assumption is $\sigma \approx \omega$.

4.2 Forcing frequency evolution

We will first obtain the time derivative of the stellar or planetary spin frequency implied by equation (6). The canonical angular momentum associated with the mode together with its complex conjugate is given by (Appendix C)

$$J_{\text{mode}} = \frac{m \varepsilon}{\omega} |q|^2.$$

(10)

Differentiating with respect to time and substituting equation (6), we find

$$J_{\text{mode}} = -2\gamma J_{\text{mode}} + 2m \varepsilon U \text{Im} \left[q e^{i\psi}\right].$$

(11)

The first term in equation (11) results from damping, and is imparted to the stellar or planetary spin (Goldreich & Nicholson 1989b).\(^1\) We thus set

$$J_{\text{spin}} = 2\gamma J_{\text{mode}},$$

(12)

meaning that

$$\Omega_{\text{spin}} = \frac{2\gamma J_{\text{mode}}}{I_s + \alpha_{\text{spin}}},$$

(13)

where $I_s$ is the planet or star’s moment of inertia, and we have incorporated an additional term $\alpha_{\text{spin}}$ to account for processes that can change $\Omega_{\text{spin}}$ other than interaction with the mode in question—e.g., the equilibrium tide, magnetic braking, etc. We take $\alpha_{\text{spin}}$ to be constant.

\(^1\) It can be shown that the second term in equation (11) is exactly the angular momentum transfer rate from the orbit; see e.g., Weinberg et al. (2012). This justifies attributing the first term to changes in the spin.
The rate at which the orbital energy changes is given by (Weinberg et al. 2012)
\[ \dot{E}_{\text{orb}} = -2 \Omega_{\text{orb}} e \epsilon \text{Im} \left[ q e^{i\omega} \right]. \] (14)

We can convert equation (14) into an expression for \( \dot{\Omega}_{\text{orb}} \) using
\[ \frac{\dot{\Omega}_{\text{orb}}}{\Omega_{\text{orb}}} = -3 \frac{\dot{E}_{\text{orb}}}{\mu a^2 \Omega_{\text{orb}}}, \] (15)
which yields
\[ \dot{\Omega}_{\text{orb}} = \left( \frac{3k}{\mu a^2} \right) 2 \epsilon \mu a^2 \text{Im} \left[ q e^{i\omega} \right] + \alpha_{\text{orb}}, \] (16)
where we have again included an extra term \( \alpha_{\text{orb}} \) to account for e.g. orbital decay by gravitational waves.

It is useful to combine equations (13) & (16) to determine the time derivative of \( \delta \omega = \omega - \sigma \), which can be expressed as
\[ \frac{\delta \omega}{\omega} = -\Gamma_{\text{dr}} + \Gamma_{\text{tide}} \left( \frac{\gamma |q|^2}{\omega U^2} - r \text{Im} \left[ q e^{i\omega} \right] \right). \] (17)
Here we have combined both \( \alpha \) parameters from earlier into the “drift” rate
\[ \Gamma_{\text{dr}} = \frac{k \alpha_{\text{orb}} - m(1 - C) \alpha_{\text{spin}}}{\omega} = \frac{\partial \ln \omega}{\partial t}, \] (18)
where
\[ C = -\frac{1}{m} \frac{\partial \omega}{\partial \ln r}, \] (19)
allows for a rotationally dependent corotating-frame eigenmode frequency and \( \partial \omega / \partial t \) accounts for changes in the eigenmode frequency due to progressive modifications of the background hydrostatic profile from e.g. stellar evolution. The tidal backreaction rate is
\[ \Gamma_{\text{tide}} = \frac{2m^2 U^2 (1 - C) \epsilon}{I_\omega}, \] (20)
and parameterizes the strength of tidal coupling to the mode in question; it is related to the rate at which the mode can synchronize the star or planet at nonresonant amplitudes (\( |q| \sim |U| \)). Lastly, the moment of inertia ratio is
\[ r = \frac{k^2}{m^2} \frac{3I_\omega}{(1 - C) \mu a^2}. \] (21)
We assume that both \( \Gamma_{\text{tide}} \) and \( r \) are positive throughout this work, but allow \( \Gamma_{\text{dr}} \) to possess either sign.

5 RESONANCE LOCK FIXED POINTS

5.1 Existence of fixed points

We first remove the overall oscillatory time dependence of \( q \) by changing variables to \( Q = q e^{i\omega} / U \), so that equation (9) becomes
\[ \dot{Q} + \left( i \delta \omega + \gamma + \frac{U}{U} \right) Q = i \omega, \] (22)
where again \( \delta \omega = \omega - \sigma \). For simplicity, we now assume that \( \gamma \) is much larger than terms contributing to \( \dot{U}/U \), such as \( \dot{a} / a \) and \( \dot{e} / e \), so that equation (22) becomes
\[ \dot{Q} + (i \delta \omega + \gamma) Q = i \omega; \] (23)
\footnote{Tidal heating could contribute to the \( |q|^2 \) term in equation (17), due to heat deposited by the mode in question affecting the background star or planet; we neglect this for simplicity.}
as we will see in § 8, this essentially amounts to assuming \( \gamma \gg |\Gamma_{\text{dr}}| \). Equation (17) becomes
\[ \frac{\delta \omega}{\omega} = -\Gamma_{\text{dr}} + \Gamma_{\text{tide}} \left( \frac{\gamma |q|^2}{\omega U^2} - r \text{Im} \left[ Q e^{i\omega} \right] \right). \] (24)

Having thus eliminated direct dependence on the phase \( \psi \), our two dynamical variables are now \( Q \) and \( \delta \omega \). Since \( Q \) is complex, we have a third-order differential system.

Resonance locking corresponds to a fixed point in the evolution equations. We thus set time derivatives to zero in equation (23) to derive
\[ Q_t = \frac{\omega}{\delta \omega_t - i\gamma}, \] (25)
and hence
\[ \frac{\text{Re}(Q)}{\text{Im}(Q)} = \frac{\omega}{\delta \omega_t^2 + \gamma^2} \times \left[ \delta \omega_t \right]. \] (26)
Similarly, setting \( \delta \omega = 0 \) in equation (24) and using the fact that \( (\gamma/\omega) |Q|^2 = \text{Im}(Q) \), we have
\[ \Gamma_{\text{dr}} = (1 - r) \Gamma_{\text{tide}} \text{Im}(Q). \] (27)
We can use equations (26) & (27) to derive
\[ \delta \omega_t^2 = \frac{\omega^2 \Gamma_{\text{tide}} (1 - r)}{\Gamma_{\text{dr}}} - \gamma^2. \] (28)

So far we have not determined the sign of \( \delta \omega_t \), and indeed there is one fixed point for each sign; however, we will show in § 5.2.2 that one is always unstable.

These fixed points exist if equations (26) & (27) can be solved for \( Q_t \) and \( \sigma_t \), which is possible if (assuming \( \Gamma_{\text{tide}} > 0 \))
\[ \frac{\gamma}{\omega} \Gamma_{\text{tide}} < \frac{1 - r}{\Gamma_{\text{dr}}}; \] (29)
this in particular requires
\[ (1 - r) \Gamma_{\text{dr}} > 0. \] (30)
If \( \Gamma_{\text{dr}} > 0 \), then equation (20) reduces to
\[ |k| < |m| \left( \frac{(1 - C) \mu a^2}{3I_\omega} \right)^{1/2}. \] (31)
As a result, Fuller & Lai (2012) refered to the quantity on the right-hand side of equation (31) as the “critical” harmonic.

The existence of the fixed points also relies on the “weak-tide” limit, which we define to be \( |\delta \omega_t| < \Delta \omega / 2 \), where \( \Delta \omega \) is the eigenmode frequency spacing near the mode in question. This means that the fixed point must lie within the eigenmode’s domain of influence, so that the contribution of other eigenmode resonances can be legitimately neglected. If this is not the case, then resonance locking is not possible.

For example, if \( \Gamma_{\text{dr}} \) is very small, equation (27) requires a commensurately small value of \( Q_t \). However, this could be impossible to achieve in practice, since it would require a very large value of the detuning \( \delta \omega \), allowing the possibility for a neighboring mode to come into resonance. The actual outcome in such a situation is that the tidal interaction would dominate the dynamics, and the drift processes contributing to \( \Gamma_{\text{dr}} \) would be irrelevant—i.e., the “strong-tide” limit.

A necessary condition for the weak-tide limit, and thus for a resonance lock to be able to occur, is \( |\delta \omega_t| \ll \omega \), which evaluates to
\[ |\Gamma_{\text{dr}}| \gg \Gamma_{\text{tide}} (1 - r) / \omega. \] (32)
We will use this as a convenient, although very liberal, proxy for the real requirement of \(|\delta \omega_\ell| < \Delta \omega/2\), so as to avoid handling the additional parameter \(\Delta \omega\). Calculations of dynamical tidal evolution that use the adiabatic approximation (§ 6.1.1) with many modes—e.g. Wittenberg & Savonije (1999), Fuller & Lin (2012)—already naturally account for the true requirement.

5.2 Fixed point stability

5.2.1 Linearization

We will now perform a linear stability analysis about each fixed point. First, note that the presence of the nonanalytic functions \(\cdot\) and \(\Im\) in equation (23) necessitates treating the real and imaginary parts of \(\dot{Q}\) separately. Thus let

\[
\dot{\zeta} = [\Re(Q) \quad \Im(Q) \quad (\sigma - \omega)/\omega]^T,
\]

and

\[
\frac{d\zeta}{dt} = f(\zeta),
\]

where \(f\) represents equations (23) & (24) and satisfies \(\nabla \cdot f = -2\gamma\). We can then derive a time-evolution equation for \(\delta \omega = \zeta - \zeta_0\) by invoking results from § 5.1.

Proceeding, to linear order we have

\[
\frac{d}{dt} \delta \zeta = \Delta \delta \zeta + \alpha(\delta \zeta),
\]

and \(\beta = \Gamma_d / (1 - r)\).

5.2.2 Eigenvalues

We assume in this section that equation (29) is satisfied, so that the resonance locking fixed point formally exists, and that equation (32) is also satisfied, so that we are in the weak-tide limit and the fixed point physically exists. The characteristic polynomial \(P\) for the eigenvalues \(\lambda\) of \(A\) is

\[
P(\lambda) = \lambda^3 + 2\gamma \lambda^2 + P_1 \lambda + P_0,
\]

where

\[
P_1 = \frac{\omega_\gamma \Gamma_{\text{tide}}}{\beta} - \frac{\omega \beta \delta \omega_\ell}{\gamma}, \quad P_0 = 2\omega \Gamma_{\text{tide}} \delta \omega_\ell.
\]

A fixed point is asymptotically stable if all eigenvalues satisfy \(\Re(\lambda) < 0\). A standard theorem then states that, for this to occur, it is necessary (but not sufficient) that all of the coefficients of \(\lambda^i\) \((i \geq 0)\) in equation (37) possess the same sign. We thus must have that \(P_{1,0} > 0\).

First, we see that

\[
+ 1 = \text{sign}(P_0) = \text{sign}(\Gamma_d) \text{sign}(\delta \omega_\ell).
\]

Since equations (26) & (27) admit two solutions, corresponding to \(\delta \omega_\ell = \pm |\delta \omega_\ell|\), the criterion that \(P_0 > 0\) simply allows us to select the solution that could potentially be stable. Henceforth we will focus on the solution that satisfies equation (39), which we refer to as the “lagging” fixed point; similarly, the “leading” fixed point is the one that fails to satisfy equation (39).

Equation (39) permits the following intuitive interpretation: the forcing frequency \(\sigma\) is “pushed” towards an eigenfrequency in the direction specified by the sign of \(\Gamma_d\), but the approaching eigenmode “pushes” \(\sigma\) in the opposite direction, and resonance locking occurs when these “forces” cancel out (§ 2). In particular, if \(\Gamma_d > 0\), we find that \(\sigma_i\) should be smaller than \(\omega\), meaning \(\delta \omega_\ell = \omega - \sigma_i > 0\), consistent with equation (39).

If it remains to analyze \(P_1\) and to ascertain when it is also positive. Moreover, since positivity of the characteristic polynomial’s coefficients is only a necessary condition for stability, we must then further examine the Hurwitz matrix associated with \(P\) to establish when its leading principal minors are also positive (e.g. Gradshiteyn & Ryzhik 2007). We perform this analysis in Appendix A in the limit of \(\gamma \ll |\delta \omega_\ell|\); the result is that the lagging fixed point is stable if and only if

\[
\Gamma_{\text{d}} < 0 \quad \text{or} \quad 0 < \Gamma_{d} < (1 - r) \left(\frac{\Gamma_{\text{tide}}}{\omega}\right)^{1/3}
\]

(subject to the assumptions we have made thus far). In particular, the lagging fixed point is thus always stable for \(r > 1\) per equation (39).

Figure 2 shows the stability region of the lagging fixed point as a function of the damping rate \(\gamma\) and the frequency drift rate \(\Gamma_d\) for two example values of the backreaction rate \(\Gamma_{\text{d}}\) and the moment of inertia ratio \(r\). Stability was determined by numerically solving for the eigenvalues \(\lambda\) using equation (37), and is indicated by green shading, while instability is white; regions where the fixed point does not exist are shaded dark gray. Equations (29), (32), & (40), which are displayed as blue dashed, black dotted, and magenta dotted lines, closely correspond to the green region’s boundaries. The values used in the top panel were chosen based on our white dwarf binary application in § 5.1. In the bottom panel we have \(r > 1\), which by equation (59) implies \(\Gamma_d < 0\).

The instability boundary defined by equation (40) is in fact a supercritical Hopf bifurcation (Wiggins 2003), corresponding to the loss of stability of a complex conjugate eigenvalue pair. Past the bifurcation, inside the unstable region of parameter space, this unstable complex conjugate pair splits into two unstable real eigenvalues, as shown in Figure 3. In other words, the fixed point switches from being an unstable spiral to an unstable node. Determining the parameter space manifold on which this splitting occurs will be useful in § 7.2; we can accomplish this by setting the discriminant of equation (37) to zero (thus requiring a repeated root), yielding

\[
0 = 36\gamma P_0 - 32\gamma^3 P_0 + 4\gamma^7 P_1^2 - 4P_1^2 - 27P_0^2.
\]

Although this equation cannot be solved analytically, we can nonetheless determine the asymptotic dependence of \(\Gamma_d\) on \(\gamma\) in the limit \(\gamma \to 0\). Newton’s polygon for equation (41) shows that this asymptotic dependence is linear: \(\Gamma_d = A\gamma\). Substituting this into equation (41), setting \(\gamma = 0\), and solving for \(A\), we have that the lagging fixed point is a spiral if

\[
\Gamma_{d} < \gamma \left(\frac{1 - r}{P_0^{2/3}}\right) \left(\frac{\Gamma_{\text{tide}}}{\omega}\right)^{1/3}.
\]
6 Trajectories

Here we will show several examples of trajectories that can be produced by our dynamical equations from § 5.1. First, however, in § 6.1 we will discuss two different analytic approximations that the trajectories follow in certain limits.

Figure 2. Resonance locking fixed point stability analysis. Regions where the fixed point is asymptotically stable are shaded green, while unstable regions are white, and regions where the fixed point does not exist are dark gray. We determined the stability regions by numerically evaluating the eigenvalues $\lambda_i$ of the matrix $A$ determined by linearizing the equations of motion about the fixed point (equation (33) and enforcing $\text{Re}(\lambda_i) < 0$ together with equation (32). The analytic results in equations (29), (32), & (40) are displayed as blue dashed, black dotted, and magenta dot-dashed lines. The lower-right gray triangle in both panels corresponds to where the weak-tide limit is certainly violated, and thus the fixed point does not physically exist; see § 5.1. In the bottom panel we have $r > 1$, which by equation (30) implies $\Gamma_{ad} < 0$.

6.1 Analytic approximations

6.1.1 Adiabatic approximation

In this paper, we define the adiabatic approximation to be the situation where the mode amplitude can instantaneously adjust to a changing forcing frequency, and thus we can set the $\dot{Q}$ term in equation (23) to zero and assume $\sigma$ is constant. Equation (23) can then be solved exactly:

$$ Q_{ad} = \frac{\omega}{\delta \omega^2 - \Gamma^2}. $$

This approximation, also referred to as the Lorentzian approximation due to the form of equation (43), is frequently employed in the literature.

The domain of validity of the adiabatic approximation can be determined by comparing the maximum possible mode growth rate to the growth rate implied by equation (43); when the latter exceeds the former, the adiabatic approximation is no longer valid. We can determine the maximum rate at which a mode amplitude $Q$ can grow by providing a perfect resonance to equation (23), i.e., by setting $\sigma = \omega$. Dropping the damping term and setting $\sigma = 0$, the particular solution is

$$ Q(t) = i \omega t, $$

which implies that $|\dot{Q}_{\text{max}}| = \omega$. Next, in order to estimate the time derivative of $Q_{ad}$, we take $\sigma \approx \Gamma_{ad} \omega$, which gives us

$$ |\dot{Q}_{ad}| \approx |\Gamma_{ad}| \left( \frac{\omega^2}{\delta \omega^2 + \gamma^2} \right). $$

We then equate $|\dot{Q}_{ad}|$ and $|\dot{Q}_{\text{max}}|$ and solve for $\delta \omega$, finding

$$ \delta \omega_{\text{ad}}^2 \approx |\Gamma_{ad}| \omega - \gamma^2; $$

if $|\Gamma_{ad}| \omega < \gamma^2$, then the adiabatic approximation is always valid.

6.1.2 No-backreaction approximation

Although directly solving equations (23) & (24) outside the adiabatic limit requires numerical integration, we can nonetheless produce an approximate analytic expression for $Q(t)$ in the limit that backreaction of the mode on the tidal forcing frequency $\sigma$ is unimportant (Reisenegger & Goldreich 1994; Rathore et al. 2005). This
approximation subsumes the adiabatic approximation, but is also more complicated.

Since we are already assuming that the mode damping rate is weak enough for the resonance locking fixed point to exist (equation 39), we can simply take $\gamma \to 0$. Subject to this simplification, we can solve equation 36, yielding
\[ q(t) \approx i\omega U e^{-\int_0^t e^{i\omega t - Q(t)} dt}, \] (47)
with $t_0 \ll (\omega \Gamma_\delta)^{-1/2}$. If we then approximate $\psi$ as
\[ \psi \approx \psi_0 + \omega t + \omega \Gamma_\delta t^2 / 2, \]
where we have set resonance to occur (i.e. $\psi = \omega$) at $t = 0$, then equation (47) becomes a closed-form solution to the equations of motion.

Since the integral in equation (47) approaches a constant for $t \gg (\omega \Gamma_\delta)^{-1/2}$, and since we wish to estimate the maximum value of $|Q|$, we can simply extend the domain of integration to $(-\infty, +\infty)$, yielding (using e.g. the method of stationary phase)
\[ q(t) \approx -(1-i)U \sqrt{\frac{\pi \omega}{\Gamma_\delta}} e^{-\omega t + \psi_0} \] (48)
for $t \gg (\omega \Gamma_\delta)^{-1/2}$. We thus find that
\[ |Q|_{\text{max}} \approx \sqrt{\frac{2\pi \omega}{\Gamma_\delta}}, \] (49)
We plot this maximal value of $|Q|$ in the middle panels of Figures 4–7 as a dash-dotted black line.

6.2 Numerical results

Figures 4–7 show full numerical solutions to equations (23) & (24) for several different choices of our four parameters $\gamma$, $\Gamma_\delta$, $\Gamma_{\text{tide}}$, and $r$. In each case, the mode amplitude is initialized to the adiabatic approximation in the regime where it should be valid (§6.1.1).

First, Figure 4 gives a simple example of resonance locking into a stable fixed point. Next, we hold $\Gamma_{\text{tide}}$, $\gamma$, and $r$ constant, but increase $\Gamma_\delta$ by a factor of $\sim 10$, thus making $\sigma$ sweep towards resonance more quickly. In this case the fixed point is no longer stable, since equation (40) is no longer satisfied, and Figure 5 shows the limit cycle resonance lock that then occurs. The system passes through resonance in between the two unstable fixed points, and then oscillates back and forth through resonance. This oscillation pumps the mode amplitude up as the system is repelled by the fixed points. Eventually, the oscillation ceases and the mode’s angular momentum discharges into rotation causing the system to travel back away from resonance and decay back onto the adiabatic approximation. The cycle then begins again. This limit cycle is in fact precisely the stable periodic orbit generated by the supercritical Hopf bifurcation (§5.2.2).

Figure 6 shows the resulting evolution again holding all parameters constant except for $\Gamma_\delta$, which we increase by another factor of $\sim 10$. The resonance locking fixed point is now sufficiently unstable that it suppresses the mode amplitude’s growth and prevents resonance locking from occurring. Near resonance the mode amplitude still grows appreciably, but after the lock fails to hold, damping causes the mode amplitude to decay exponentially. We have found the linear character of the lagging fixed point to be the principal distinguishing factor between a limit cycle occurring and a failed resonance capture due to fixed point suppression. Specifically, we find that if the lagging fixed point is an unstable node, meaning all its eigenvalues are real (Figure 5), then it acts to suppress the mode amplitude and leads to a failed capture. If instead the lagging fixed point is an unstable spiral, meaning its eigenvalues contain an unstable complex conjugate pair (§5.2.2), then it is able to “pump” a trajectory to high amplitude and allow the formation of a limit cycle. This distinction, which is valid within a factor of $\sim 3$ in $\Gamma_\delta$, will be critical in §7.2.

Figure 4. Numerical integration of mode and orbital evolution equations for the case of resonance locking into a stable fixed point. Parameters used were $\Gamma_\delta/\omega = 10^{-9}$, $\gamma/\omega = 10^{-4.5}$, $\Gamma_{\text{tide}}/\omega = 10^{-10}$, and $r = 0.5$. The adiabatic approximation is shown in the top panel as a dashed black line (§6.1.1), while the actual system trajectory is purple. The red circle shows the lagging fixed point (§5). Individual timeseries for the mode amplitude $Q$ and the forcing frequency $\sigma$ are shown in the bottom two panels. The dash-dotted black line in the mode amplitude panel shows equation (49), which gives the maximum amplitude attainable under the no-backreaction approximation (§6.1.2). The lagging fixed point’s eigenvalues are $\lambda/\omega \in (-0.031 \pm 1.2i) \times 10^{-5}, -1.6 \times 10^{-6}$.

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Figure 5. Resonance lock limit cycle for a case in which the fixed points exist (red and black circles) but are linearly unstable. Parameters used were $\Gamma_d/\omega = 10^{-7.4}$, $\gamma/\omega = 10^{-4.5}$, $\Gamma_{ide}/\omega = 10^{-10}$, and $r = 0.5$. Conventions used are the same as in Figure 4. Color shows time, ranging from purple at $t = 0$ to light blue. The red & black circles show the lagging and leading fixed points, respectively (§ 5). System sweeps through resonance without initially being captured. However, the system then oscillates through resonance several times, pumping up the mode’s amplitude. Eventually, the oscillation ceases and the mode’s angular momentum discharges into rotation causing the system to travel back away from resonance and start over. The lagging fixed point’s eigenvalues are $\lambda/\omega \in \{ -3.9 \times 10^{-4}, (1.6 \pm 0.49) \times 10^{-4} \}$.

6.3 Chaos

Lastly, Figure 6 shows a chaotic trajectory. This situation is very similar to the limit cycle evolution shown in Figure 5 in that the resonance locking fixed point is unstable but $\Gamma_d$ is not so large that resonance locking doesn’t occur altogether; the essential difference is that the mode amplitude profile resulting from the adiabatic approximation (dashed line) is capped by damping not far above the fixed points, unlike in Figure 5 where it ascends much higher. This corresponds to the fact that the choice of parameters for Figure 7 lies close (logarithmically speaking) to the bifurcation manifold in parameter space where equation 29 ceases to be satisfied and the fixed points no longer exist. This appears to be a key ingredient for chaos, as we will explain below, which is why we have changed $\gamma$ to a larger value than that used in Figures 4–6 (so that equation 29 is closer to not being satisfied).

Since the fixed points are so close to the peak of the adiabatic profile, the pumping process that occurs due to repulsion from the fixed points cannot allow the mode to acquire a very large amplitude. As a result, when the mode’s angular momentum eventually...
drains into the background spin, the system does not retreat back from resonance very far, and has little time to decay back onto the adiabatic approximation before resonance is reached again. The initial condition upon entering resonance is consequently somewhat different each cycle, leading to the potential for chaos.

We show a three-dimensional projection of the orbit from Figure 7 in Figure 8. The lagging fixed point is shown by a small red sphere, while its unstable plane corresponding to eigenvalues \( \lambda/\omega = (0.98 \pm 2.4i) \times 10^{-4} \) is also displayed.

We now present numerical evidence that the path depicted in Figure 8 follows a strange attractor. We emphasize that our evidence is not rigorous. A strange attractor of a dynamical system, also known as an attracting chaotic invariant set, is a set that (Wiggins 2003):

(i) is compact,
(ii) is invariant under the dynamical equations,
(iii) is attracting,
(iv) has sensitive dependence on initial conditions, and
(v) is topologically transitive.

Figure 8 appears to begin to trace out a bounded, attracting, invariant set, which we denote \( \Lambda \); this addresses conditions (i) – (iii). In Figure 9, we estimate the largest Lyapunov exponent of the trajectory in Figure 8 to be \( 6 \times 10^{-3} \). Since this is positive, trajectories that begin together deviate exponentially as time passes. This then points toward condition (iv) being satisfied. Lastly, since color indicates time in Figure 8, the fact that dark purple (early times) is tightly and randomly interwound with light blue (late times) leads us to believe that condition (v) is likely also satisfied. Again, we have presented only suggestive evidence; further study is required to fully address the presence of a strange attractor.

In addition, we note that our preliminary investigations show that the chaos present results from the Hopf orbit undergoing a period-doubling cascade, and is similar in several ways to the Rössler attractor (Rössler 1976). This requires further study.

7 ACHIEVING RESONANCE LOCKS

7.1 Numerical results

In order to ascertain the conditions that lead to resonance locks, we performed sample integrations of equations (23) & (24) numerically. We initialized each integration with

\[
\delta \omega_0 = 10 \times \max(\delta \omega_1, \delta \omega_{ad}, \gamma)
\]
Numerous other potential resonance lock criteria exist; however, we have found this to be the most reliable.

Figure 9. Estimation of largest Lyapunov exponent for the chaotic trajectory in Figures 7 & 8. We initialized two numerical integrations of our dynamical equations on the adiabatic solution (§ 6.1.1) with slightly perturbed initial detuning frequencies: δω0 = 0.95δω and 0.950001δω. The blue curve shows the norm of the difference between the resulting values of the reduced mode amplitude Q as a function of time. The trajectories initially deviate exponentially, demonstrating chaos, with a rough functional form of $e^{\gamma t}$ for $\tau \approx 2 \times 10^4 / \omega$ (dashed magenta line). We thus estimate the largest Lyapunov exponent (Wiggins 2003) for the trajectories to be $\approx 1 / \tau \approx 6 \times 10^{-5} \omega > 0$, which is close to the damping rate of $\gamma = 10^{-4} \omega$.

and $Q_0$ set by the adiabatic approximation from equation (43). We then performed each integration until $t_1 = 2\delta\omega_0 / \Gamma_\text{dr}$. We determined that a resonance lock occurred if, assuming $r < 1$,

$$\min \frac{\delta\omega}{\delta\omega_0} > -0.5,$$

where $\min \delta\omega$ is the minimum value of $\delta\omega$ attained over the final 10% of integration. A similar formula was used for $r > 1$, but accounting for the fact that resonance is then approached from the left (in terms of $\delta\omega$; see § 5.1). Note that these conditions account for stable, limit cycle, and chaotic forms of resonance locking (§ 6.2).

Figure 10 shows our results. Light blue regions indicate that a resonance lock did occur, while white indicates the reverse, and dark gray indicates that the fixed point does not exist (§ 5.1). The green lines are the analytic formula from equation (53) below for the boundary between successful and failed resonance locking. The thin purple lines are the equivalent condition for resonance locks to occur when $r > 1$, from equation (53). Both analytic approximations, which we will develop in the next section, show good agreement with numerical results.

### 7.2 Analytic approximations

We now seek to obtain an analytical understanding of our numerical results in Figure 10. We will thus attempt to assemble a set of analytic approximations to determine what values of our four parameters $\gamma$, $\Gamma_\text{dr}$, $\Gamma_\text{ide}$, and $r$ (§ 3) lead to resonance locks, and what values do not. We define resonance locking in this context to be any behavior such that $\sigma$ does not increase without bound as $t \to \infty$; this definition comprises locking into a stable fixed point (Figure 3), limit cycles about an unstable fixed point (Figure 5), and chaotic behavior like in Figure 7 but does not include the behavior seen e.g. in Figure 6.

We see by inspecting equations (25) & (43) that the adiabatic solution exactly passes through the resonance locking fixed point. As a result, a sufficient condition for a resonance lock to be achieved is $|\delta\omega| > |\delta\omega_0|$, which evaluates to

$$\Gamma_\text{dr}^2 \lesssim (1 - r) |\Gamma_\text{ide}| \gamma,$$

(50)

together with fixed point stability (§ 5.2 equation 40). However, this is a very conservative estimate of the resonance locking regime. To develop a set of necessary and sufficient criteria for resonance locks, recall that the time derivative of $\delta\omega$ is given by (equation 24)

$$\delta\omega / \omega = -\Gamma_\text{dr} + \Gamma_\text{ide}(g_1 - rg_2),$$

(51)

where $g_1$ and $g_2$ are

$$g_1 = \frac{\gamma}{\omega} |Q|^2, \quad g_2 = \text{Im}[Q].$$

(52)

First, assume $r \gg 1$. By equation (40) we see that $\Gamma_\text{dr} < 0$ and thus that the system approaches resonance from the left (in terms of $\delta\omega$; note that in Figures 4-7 the abscissa is $t = -\delta\omega = -\omega_0$). Since the lagging fixed point is always stable in this situation (§ 5.2.2), we simply need the resonance passage to provide sufficient amplitude to reach the fixed point in order for a stable resonance lock to take hold. Invoking the no-backreaction approximation results from § 6.1.2 to substitute a maximum value of $\text{Im}[Q]$ into equation (51), dropping $g_1$, and setting $\delta\omega \sim 0$ leads to the following condition for resonance locking:

$$\frac{\Gamma_\text{dr}}{\Gamma_\text{ide}} > 3r \sqrt{-\frac{\pi \omega}{\Gamma_\text{dr}}}.$$

(53)

Although our analysis is strictly valid only for $r$ much larger than unity, we find it to work well even for $r \gtrsim 1$, as can be seen in the bottom row of Figure 10 (where $r = 1.5$). We have inserted an additional factor of 3 on the right-hand side of equation (53) to match our numerical results.

Next, assume $r \ll 1$. By equation (40) this means $\Gamma_\text{dr} > 0$, so the system approaches the resonance locking fixed point from the right (in terms of $\delta\omega$). Here, however, the fixed point is not always stable, as we found in § 5.2.3. In § 6.2 we argued that resonance locks in the form of limit cycles or chaotic trajectories could occur when the fixed point was an unstable spiral (with a complex conjugate pair of eigenvalues), but that resonance capture failed when the fixed point was an unstable node (with all real eigenvalues); this was true within a factor of $\sim 3$ in terms of the value of $\Gamma_\text{dr}$. Thus a necessary condition for resonance locks (to within a factor of $\sim 3$) is equation (42), which specifies where the fixed point is a spiral. Next, we drop $g_1$ and again substitute the no-backreaction approximation results from § 6.1.2 to find that resonance passage can deliver a system to the resonance locking fixed point if

$$\frac{\gamma}{\Gamma_\text{dr}} > \frac{\Gamma_\text{dr}}{2 \pi \Gamma_\text{ide}}.$$

Augmented with equation (42), this approximately becomes the following condition for resonance locking:

$$\frac{\gamma}{\Gamma_\text{dr}} > \frac{\Gamma_\text{dr}}{6 \pi \Gamma_\text{ide}} + \frac{1}{3} \left( \frac{\omega}{\Gamma_\text{ide}} \right)^{1/3}.$$  \quad (r \ll 1)

(54)

Similar to our formula for $r > 1$, our analysis is valid only for $r$ very close to zero; however, we again find it to work well even for $r \lesssim 1$, as can be seen in the right panels of the top two rows of Figure 10 (where $r = 0.5$). We have inserted an additional factor of 1/3 on the right-hand side of equation (54) to match our numerical results.
8 TIDAL EVOLUTION DURING RESONANCE LOCKS

8.1 Accelerating tidal evolution
Here we generalize the energetic arguments made in Burkart et al. (2013) to estimate the orbital and spin evolution during a resonance lock. During a lock, the reduced mode amplitude $Q$ is roughly given by its value at the fixed point, i.e., equation (27); note, however, that this approximation is very crude for limit cycles and chaotic orbits (e.g. Figures 5 & 7) since in those cases the real and imaginary parts of $Q$ have a more complicated dependence on time. Using equations (16) & (27) together with our definitions of $\Gamma_\alpha$, $\Gamma_\text{tide}$, and $r$ from equations (18) – (21), we can derive

$$\dot{\Omega}_\text{orb} = \left(\frac{1}{1-r}\right) \left[ \alpha_{\text{orb}} - \frac{r}{k} \left( m\omega_\alpha + \frac{\partial \omega}{\partial t} \right) \right], \quad \text{(55)}$$

Figure 10. Resonance lock regimes based on numerical solution of mode, spin, and orbit evolution equations from § 5.1. Light blue shading indicates that resonance locking occurs (including stable fixed point locks, limit cycles, and chaotic locks, as in Figures 4, 5 & 7), while white indicates the reverse (as in Figure 6). Above each dashed blue line, the resonance locking fixed point does not exist (§ 5.1; equation 29). The green and purple lines correspond to our analytic formulae for the resonance locking regime (§ 7.2; equations 54 & 53 respectively). The dash-dotted magenta line indicates the upper boundary of the domain of stability of the resonance lock fixed point (§ 5.2; equation 40), while the dotted black line shows where the weak-tide limit is certainly violated (§ 5.1; equation 35). Limit cycles and chaotic orbits (as in Figures 5 & 7) preferentially occur in the regions between the green and magenta lines (where the fixed point is unstable but locks still occur).
where again \( \alpha_{\text{orb}} \) and \( \alpha_{\text{spin}} \) represent the contributions to \( \Omega_{\text{orb}} \) and \( \Omega_{\text{spin}} \) from slowly varying processes other than resonant interaction with the normal mode in question (see equations [13] & [15]). This can be converted into an energy transfer rate by equation [13]. Performing a similar derivation for the spin frequency, we have

\[
\Omega_{\text{spin}} = \left( \frac{r}{1-r} \right) \left[ -\alpha_{\text{spin}} + \frac{1}{\ell (k\alpha_{\text{orb}} - \partial \omega / \partial \ell \tau) \ell} \right].
\] (56)

Examining equations [55] & [56], we see that a resonance lock acts to accelerate the orbital and spin evolution given by the non-resonant processes contributing to \( \alpha_{\text{orb}} \) and \( \alpha_{\text{spin}} \), which are due e.g. to gravitational wave orbital decay, the equilibrium tide, etc. Moreover, since the time derivative of the eccentricity is simply a linear combination of \( \dot{\Omega}_{\text{orb}} \) and \( \dot{\Omega}_{\text{spin}} \) [Witte & Savonije 1999], resonance locking also accelerates circularization. The degree of acceleration depends on how close the moment of inertia ratio \( r \) is to unity; we estimate under what conditions \( r \sim 1 \) in § 8.2. This acceleration of tidal evolution is what led [Witte & Savonije 2002] to conclude that resonance locks solve the solar-binary problem [Melbon et al. 2006], although they neglected essential nonlinear effects that obviate their results (see § 3).

The presence of \( \alpha_{\text{spin}} \) in the evolution equation for \( \dot{\Omega}_{\text{orb}} \) (and \( \alpha_{\text{orb}} \) in the equation for \( \dot{\Omega}_{\text{spin}} \)) implies that a resonance lock efficiently couples orbital and spin evolution together, as well as to stellar evolution through the rate of change of the eccentricity \( \partial \omega / \partial \ell \). For example, if gravitational waves in an inspiraling white dwarf binary cause orbital decay, a resonance lock will cause tidal synchronization to occur on a gravitational wave timescale (§ 9.2). Similarly, resonance locking can cause stellar spin down by magnetic braking or eigenfrequency evolution due to tidal heating to backreact on the binary orbit.

### 8.2 Conditions for rapid tidal evolution

Here we estimate the “typical” value of \( r \) (defined in equation [21]) expected to occur in a given binary, so as to assess whether resonance locks significantly accelerate tidal evolution (§ 8.1). First, consider a binary in a circular, spin-aligned orbit. For the lowest-order \( l = 2 \) spherical harmonic of the tidal potential, only one temporal Fourier component of the tidal forcing exists: \( k = m = 2 \) (§ 4.1). With only a single forcing component, it is thus generally unlikely to find \( r \sim 1 \). The exception is when considering a star whose companion’s mass is \( \ll M \), in which case \( r \) may be \( \sim 1 \) even for a circular orbit.

For an eccentric orbit, however, there is significant power at many harmonics \( k \) of the orbital frequency. In this case, a resonance lock persists until it is disrupted when another mode, driven by a different Fourier component, also comes into resonance and upsets the balance of spin and orbital evolution thereby enforcing \( \delta \omega = 0 \). Whether the existing resonance lock can withstand a second passage depends on how “robust” it is; we can estimate the degree of “robustness” using the maximum amplitude achieved under the no-backreaction approximation from § 6.1.2⁴

\[
|q|_{\text{max}} \approx |U| \left( \frac{2\pi \omega \Omega_{\text{orb}}}{|a|} \right)^{1/2}.
\] (57)

If we equate \( \omega \approx \kappa \Omega_{\text{orb}} \), having taken \( m\Omega_{\text{orb}} \ll |\kappa \Omega_{\text{orb}}| \) for simplicity, then \( |q|_{\text{max}} \) depends on the harmonic index \( k \) as

\[
|q|_{\text{max}} \propto \kappa^k X^k_{\text{orb}}.
\] (58)

where \( X \) is a Hansen coefficient and \( b > 0 \) is a constant that accounts for power law dependences on the tidal overlap integral and other mode-dependent quantities entering into equation [57] (§ 4.1). Using our scaling derived in Appendix B, equation [58] becomes

\[
|q|_{\text{max}} \propto \kappa^{b-1/2} \exp[-kg(e)],
\] (59)

where the full form of \( g(e) \) is given in equation [12].

The longest-lived resonance locks in an eccentric binary will be those with the largest values of \( |q|_{\text{max}} \). Thus we can estimate the “typical” value of \( b \) for finding the value of \( k \) that maximizes \( |q|_{\text{max}} \).

\[
\arg \max_k |q|_{\text{max}} \approx \frac{b-1/2}{g(e)}.
\] (60)

In order to produce a simple order-of-magnitude estimate, we take \( b = 1/2 \) to find that the most robust resonance locks will have \( r \sim 1 \), and thus significantly increase the rate of tidal evolution, \( \Gamma_{\text{gw}} \)

\[
1 - e^2 \sim \left( \frac{10^5 \Gamma_{\text{gw}} \kappa}{\mu a^2} \right)^{1/3}.
\] (61)

When this condition is not satisfied, then either the longest-lived resonance locks will not provide much acceleration of tidal evolution, or there will be no long-lived locks at all.

### 9 ASTROPHYSICAL APPLICATIONS

In §§ 9.1 & 9.2 we determine where tidal resonance locks may be able to occur by applying the criteria we have developed—equations [29] & [55]—to inspiraling compact object binaries and eccentric stellar binaries.

#### 9.1 Inspiring compact object binaries

We consider the case of a circular, spin-orbit-aligned binary consisting of either two white dwarfs or two neutron stars. The equivalent of the drift rate \( \Gamma_{\text{dr}} \) in this case is

\[
\Gamma_{\text{dr}} = \frac{m^2 \Omega_{\text{orb}}}{\omega T_{\text{gw}}},
\] (62)

where \( T_{\text{gw}} \) is the gravitational wave orbital decay time given by [Peters 1964]

\[
t_{\text{gw}} = \omega_\star^{-1} \frac{5}{96} \frac{(1+M'/M)^{1/3}}{M'/M} \beta_\star^2 \left( \frac{\omega_\star}{\Omega_{\text{orb}}} \right)^{8/3}.
\] (63)

Here \( \omega_\star^2 = GM/R^3 \) is the dynamical frequency and \( \beta_\star^2 = GM/Rc^2 \) is the ratio of the escape velocity to the speed of light. We consider only the \( l = m = 2 \) component of the tidal response.

For a fiducial double-white dwarf binary, we use the 0.6\( \bigcirc \), \( T_{\text{gw}} = 5,500 \) K carbon/oxygen white dwarf model described in [Burkart et al. 2013], with a radius of \( R = 0.013 \bigcirc \), and a moment of inertia of \( I_e = 0.18MR^2 \). For the damping rate \( \gamma \) and tidal overlap

---

⁴ Our results are insensitive to the expression by which a resonance lock’s “robustness” is quantified, so long as it is proportional to \( U \).

⁵ Note that whether \( r > 1 \) or \( r < 1 \) is determined by the sign of \( \Gamma_{\text{dr}} \), which is independent of our estimates here; see § 5.1.
Dynamical resonance locking

Figure 11. Resonance locking regimes for a fiducial white dwarf binary. Yellow shading indicates that the resonance locking fixed points exists (equation 29), but that resonance locks are nonetheless not possible. Orange shading indicates the complete resonance locking regime, derived from our numerical results in § 7. The dashed black line on the top panel is a simple, schematic system trajectory for a double-white dwarf binary. The system begins in the upper right with a long orbital period and a small rotation rate. Once orbital decay causes the system to reach the orange region, a resonance lock occurs and the forcing frequency \( \sigma \) is held approximately constant. This was already demonstrated in Burkart et al. (2013), using the adiabatic approximation (§ 6.1.1), which is valid for determining when the resonance locking fixed points exist.

Eventually, however, the system exits the orange region. At this point resonance locking is no longer possible because of the short gravitational wave inspiral time. This novel prediction comes from our analysis in this work.\(^6\) Note that this prediction neglects nonlinear hydrodynamical phenomena (§ 3): in reality the wave amplitude eventually becomes large enough to cause wave breaking, as shown in Burkart et al. (2013); Fuller & Lai (2012a). This may be a more stringent constraint on the existence of resonance locks in many close white dwarf binaries.

In the neutron star case, the gravitational wave time is much shorter than for white dwarf binaries at comparable values of \( R/a \), since neutron stars are much more relativistic objects. Resonance passage thus happens very quickly, which prevents modes from reaching amplitudes large enough to allow locking. As a result, resonance locks are never possible, even though the resonance locking fixed point exists for \( P_{\text{orb}} \lesssim 50 \text{ ms} \).

To get a sense of the degree to which equation (54) fails to be satisfied for neutron star binaries, we first compute the following quantities with \( \Omega_{\text{spin}} = 0 \):

\[
\frac{\gamma}{\omega} = 10^{-7} T_8^{-2} \left( \frac{P_{\text{orb}}}{50 \text{ ms}} \right)^3 \frac{\Gamma_{\text{dr}}}{\omega} = 6 \times 10^{-5} \left( \frac{P_{\text{orb}}}{50 \text{ ms}} \right)^{-5/3} \frac{\Gamma_{\text{side}}}{\omega} = 10^{-11} \left( \frac{P_{\text{orb}}}{50 \text{ ms}} \right)^{-6} \quad r = 0.002 \left( \frac{P_{\text{orb}}}{50 \text{ ms}} \right)^{-4/3}.
\]

We thus set \( r \approx 0 \). Substituting the remaining values into equation (54) and simplifying, we have

\[
10T_8^2 \lesssim 10^{-7} \left( \frac{P_{\text{orb}}}{50 \text{ ms}} \right)^{1/3}.
\]

This shows that resonance locking fails to occur by \( \approx 8 \) orders of magnitude at a wide range of orbital periods. The conclusion that resonance locks cannot occur in neutron star binaries is thus very robust.

9.2 Eccentric binaries

In this section we estimate whether resonance locking can occur in eccentric stellar binaries (Witte & Savonije 1999). For a fiducial system, we take parameters consistent with KOI-54 (Welsh et al. 2011), where it has been recently suggested that one or more of the observed tidally excited pulsations may be the signatures of resonance locks (Fuller & Lai 2012b; Burkart et al. 2012) see however O’Leary & Burkart (2013). This system consists of two similar A stars with \( M \approx 2.5M_\odot, R \approx 2.2R_\odot \), and \( T_{\text{eff}} \approx 8,500 \text{ K} \). The binary’s orbital parameters are \( e = 0.83 \) and \( P_{\text{orb}} = 43 \text{ days} \). Burkart et al. (2012) estimated damping rates for such stars to be

\[
\gamma \sim 0.1 \left( \frac{\omega}{\omega_{\text{pc}}} \right)^{-4} \text{ Myr}^{-1}.
\]

\(^6\) Burkart et al. (2013) did assess where the adiabatic approximation (which was inaccurately referred to as the “secular approximation”) became invalid; however, this is an incomplete consideration that fails to account for fixed point instability, limit cycles, etc.
For overlap integrals $I$, we take the following scaling derived for A stars from Burkart et al. (2013):

$$I \sim 10^{-5} \left( \frac{\omega}{\omega_a} \right)^{11/6}. \quad (66)$$

The drift rate $\Gamma_{dr}$ in this case comes from the equilibrium tide’s influence on each star’s spin and on the overall orbital frequency. We account only for the equilibrium tide’s effect on the orbital frequency for simplicity. Parameterizing the equilibrium tide’s energy transfer rate by its quality factor $Q_{eq}$, so that (Goldreich & Soter 1966)

$$|E_{eq}| \sim \frac{E_{tide} \Omega_{orb}}{Q_{eq}}, \quad (67)$$

we have the approximate formula

$$\Gamma_{dr} \sim \frac{E_{tide} k}{Q_{eq} \sigma^2 \omega}. \quad (68)$$

where the energy contained in the equilibrium tide is roughly $E_{tide} \sim \lambda (M/\bar{M})^2 (R/\bar{R})^{0.5} E_a$, and the constant $\lambda$ (related to the apsidal motion constant) is $\sim 3 \times 10^{-3}$ for an A star.

Witte & Savonije (1999) invoked the adiabatic approximation (§ 6.1.1) to show that resonance locking could occur in various fiducial eccentric binary systems; Burkart et al. (2012) performed a similar analysis for KOI-54. However, as we have established in this work, this only means that the resonance locking fixed point exists, and not necessarily that resonance locking actually occurs. For the latter, we need to apply our criterion from equation (54).

As in § 9.1 we proceed to compute the values of our four parameters that affect the possibility of resonance locks in the current situation. We fix $r$, but assume $0 \ll r < 1$. We also assume that the star is nonrotating. We then find

$$\frac{\gamma}{\omega} = 10^{-11} \left( \frac{P_{orb} \text{ 40 day}}{1} \right)^{5/3},$$

$$\frac{\Gamma_{dr}}{\omega} = 3 \times 10^{-21} \left( \frac{P_{orb} \text{ 40 day}}{1} \right)^{-1} \left( \frac{Q_{eq} \text{ 100^3}}{1} \right)^{-1},$$

$$\frac{\Gamma_{tide}}{\omega} = 6 \times 10^{-18} \left( \frac{P_{orb} \text{ 40 day}}{1} \right)^{-1} \left( \frac{Q_{eq} \text{ 100^3}}{1} \right)^{-1}. \quad (69)$$

Substituting into equation (54), noting that the first term on the right-hand side of equation (54) is much smaller than the second in this case (unlike in § 9.1), we find that locks are present if

$$1 > 10^{-4} \left( \frac{1}{1} \right)^{-1} \left( \frac{P_{orb} \text{ 40 day}}{1} \right)^{4.1} \left( \frac{Q_{eq} \text{ 100^3}}{1} \right)^{-1}. \quad (69)$$

It thus appears that resonance locks are indeed possible in eccentric binaries, subject to the validity of the assumptions enumerated in § 3 (e.g., solid-body rotation).

10 CONCLUSION

We have studied tidally induced resonance locking in close (but detached) binary systems. In a resonance lock, the detuning frequency $\delta \omega = \omega - \sigma$ between a stellar or planetary eigenmode frequency $\omega$ and a particular Fourier harmonic of the tidal driving frequency $\sigma = k \Omega_{orb} - m \Omega_{spin}$ is held constant (§ 2 Witte & Savonije 1999). This happens when a slowly varying physical process causing $\delta \omega$ to evolve in one direction is balanced by resonant interaction with the eigenmode in question causing $\delta \omega$ to evolve in the reverse direction. The slowly varying process could be, e.g., orbital decay due to gravitational waves causing $\Omega_{orb}$ to increase, magnetic braking causing $\Omega_{spin}$ to decrease, stellar evolution altering $\omega$, or nonresonant components of the tidal response (the “equilibrium tide”) affecting both $\Omega_{orb}$ and $\Omega_{spin}$ simultaneously.

Our primary goal has been to understand the dynamical properties and stability of resonance locks without relying on simplifying approximations for the mode amplitude evolution used in previous calculations. We defer detailed implications of these results to future papers. We have derived a novel set of equations allowing for a dynamically evolving mode amplitude coupled to the evolution of both $\Omega_{spin}$ and $\Omega_{orb}$ (§ 4). In particular, we do not assume that the mode amplitude is given by a Lorentzian profile resulting from the adiabatic approximation (§ 6.1.1) used in previous work, but instead solve the fully time-dependent mode amplitude equation.

In § 5 we analyzed the stability of the dynamical fixed points associated with resonance locks. We analytically derived when such fixed points exist (equation 29), there are either two fixed points or none for a given eigenmode. Although one of these eigenmodes is always unstable, the other can be stable when certain restrictions on binary and mode parameters are met (equation 40 in § 5.2). One of the important conclusions of this analysis is that resonance locks can exist and be stable even when the adiabatic approximation for the mode amplitude evolution is invalid (which happens, e.g., in the limit of moderately weak damping).

In § 6.2 we analyzed the properties of resonance locks using direct numerical integration of our dynamical equations. In the simplest case in which a resonance lock fixed point exists and is stable, two possibilities arise: either a resonance passage is able to pump the mode’s amplitude up sufficiently high to reach the fixed point and be captured into it, creating the resonance lock (Figure 2), or the system instead sweeps through resonance without locking.

The more interesting situation is when both fixed points are unstable. In this case, we showed that resonance locking can nonetheless occur in some cases in a time-averaged sense. In these situations the mode amplitude and detuning frequency $\delta \omega$ execute limit cycles or even chaotic trajectories around the fixed points (see Figures 5 & 7). We presented evidence in § 6.2 suggesting that resonance locking may in fact correspond to a strange attractor for certain parameter values; see Figure 8.

In order to determine when resonance locking of some kind occurs (either stable, limit cycle, or chaotic), we performed numerical integrations over wide ranges of parameter values in § 7.1. Using analytic approximations from § 6.1 we then developed approximate analytic formulae that explain our numerical results and define the binary and mode parameter regimes in which resonance locks of some kind occur. The key results are equations (53) & (54) and Figure 10. Future studies of tidal evolution that do not include our full set of coupled mode-spin-orbit evolution equations can nonetheless utilize our results to assess whether resonance locks can occur.

One of the interesting consequences of resonance locks highlighted by Witte & Savonije (1999) and Witte & Savonije (2002) using numerical simulations in the adiabatic approximation is that locks can produce a significant speed up of orbital and spin evolution. We have explained this analytically in § 8.1. In particular, we have demonstrated that resonance locks generically act to produce orbital and spin evolution on a timescale that is somewhat shorter.
than the slowly varying physical process whose influence drives
the system into a lock. The magnitude of this acceleration depends
on the effective moment of inertia ratio \( r \) defined in equation (21)
and is large for \( r \sim 1 \). The latter condition can only be satisfied in
eccentric binaries or binaries with high mass ratios. For the case
of an eccentric orbit, we derived a rough condition for significant
acceleration of the rate of orbital and spin evolution in § 4.

To give a rough sense of the possible application of our results,
we applied them to three sample astrophysical systems in § 4:
inspiring white dwarf and neutron star binaries in § 9.1 and
eccentric binaries with early-type stars in § 9.2. As has been argued
previously using the adiabatic approximation for mode amplitudes,
resonance locks are likely very common in white dwarf binaries
and eccentric stellar binaries. They cannot, however, occur in neu-
tron star binaries since orbital decay by gravitational wave emission
is too rapid. A future application that may be of considerable inter-
est is tidal circularization during high-eccentricity migration of hot
Jupiters.

The theory of resonance locking that we have developed bears
some similarity to resonance capture in planetary dynamics [Murray
& Dermott 1999]. In the case of both mean-motion resonances
(Goldreich 1965) and spin-orbit resonances (Goldreich & Peale
1968), the generic equation governing the evolution of the relevant
angle \( \Psi \) towards resonance is

\[
\dot{\Psi} = -F \sin \Psi + G. \tag{70}
\]

For mean-motion resonances, \( \Psi \) defines the angle between the
mean anomalies of two orbiting bodies, while for spin-orbit res-
onances, \( \Psi \) is related to the difference between a body’s mean
and true anomalies (relevant only for eccentric orbits). In both cases,
\( G \) provides a frequency drift term, resulting from orbital decay for
mean-motion resonances and from a net tidal torque for spin-orbit
resonances.

Equation (70) can be compared to our equation describing the
evolution of the detuning frequency \( \delta \omega = \omega - \sigma \) in equation (24).
Both specify the second time derivative of a resonance angle, and
both contain a frequency drift term describing how resonance is ap-
proached. The essential difference is that in place of the pendulum
restoring force present in equation (70), tidal resonance locking in-
stead contains two terms providing the complicated interaction with
a stellar or planetary normal mode. Thus, although there are qual-
titative similarities between resonance capture in planetary dynamics
and resonance locking, no formal mathematical analogy exists.

It is important to reiterate that resonance locks can only oc-
cur under a specific set of conditions (§ 4). They are not relevant
to all close binaries. In particular, the dynamical tide must be com-
posed of global radial standing waves, with damping times much
longer than radial wave travel times. For this reason, an efficient
angular momentum transport process must maintain approximate
solid body rotation; if not, critical layers may develop where mode
angular momentum is deposited into the background rotation pro-
file, which would lead to efficient local wave damping. In addition,
we have restricted our analysis to linear perturbation theory. In
practice, this represents a restriction on the maximum mode am-
plitudes that are allowed, since nonlinear instabilities can act on
large-amplitude waves. For example, in the case of binaries con-
taining solar-type stars with radiative cores, wave breaking in the
core likely prohibits the establishment of global standing waves
(Goodman & Dickson 1998), thus also precluding resonance locks
from developing.

\section*{ACKNOWLEDGEMENTS}

The authors are pleased to thank Edgar Knobloch, Keaton Burns,
Eugene Chiang, and Scott Tremaine for useful discussions and
guidance. J.B. is an NSF Graduate Research Fellow. E.Q. was sup-
ported by a Simons Investigator award from the Simons Foundation,
the David and Lucile Packard Foundation, and the Thomas
Alison Schneider Chair in Physics at UC Berkeley. P.A. is an Alfred
P. Sloan Fellow, and received support from the Fund for Excellence
in Science and Technology from the University of Virginia.

\section*{APPENDIX A: DERIVING FIXED POINT STABILITY
CONDITIONS}

Here we will determine the stability region for the resonance lock-
ing fixed point described in § 5. The characteristic polynomial \( P \) in
question is given in equation (37).

First, \( P_t > 0 \) reduces to

\[
\delta \omega_t < \frac{(1-r)^2 \gamma \Gamma_{\text{tide}}}{r \Gamma_{\text{c}}} \tag{A1}
\]

We immediately see that this is satisfied if \( \Gamma_{\text{c}} < 0 \) (assuming \( \Gamma_{\text{tide}} > 0 \)), since by equation (39) we then have \( \delta \omega_t < 0 \). If instead \( \delta \omega_t > 0 \),
then we can take \(|\delta \omega_t| \gg \gamma \) to derive

\[
\Gamma_{\text{c}} < \gamma \left( \frac{1-r}{P_t} \right)^{1/3} \tag{A2}
\]

Next, the Hurwitz matrix for \( P \) is

\[
H = \begin{pmatrix} 2\gamma & P_t \\ 1 & P_t \end{pmatrix}. \tag{A3}
\]

The remaining condition for stability of the fixed point is that the
three leading principal minors of \( H \) must be positive; this formally
yields three additional inequalities. However, one is \( \gamma > 0 \) which
is always satisfied, and the other two are actually identical:

\[
2\gamma P_t > P_0, \tag{A4}
\]

which expands to

\[
\gamma \Gamma_{\text{tide}} (1-r)^2 > \Gamma_{\text{c}}^2 \delta \omega_t. \tag{A5}
\]

We now see that \( \Gamma_{\text{c}} < 0 \) implies asymptotic stability. If \( \Gamma_{\text{c}} > 0 \),
then the inequality reduces to (again assuming \(|\delta \omega_t| \gg \gamma \))

\[
\Gamma_{\text{c}} < \gamma (1-r) \left( \frac{\Gamma_{\text{tide}}}{\omega} \right)^{1/3}. \tag{A6}
\]

Since equation (A6) is more restrictive than equation (A2) (since \( r < 1 \) for \( \Gamma_{\text{c}} > 0 \); see equation (30)), equation (A6) is the condition
for asymptotic stability when \( \Gamma_{\text{c}} > 0 \).

\section*{APPENDIX B: HANSEN COEFFICIENT SCALING}

Here we will determine how the Hansen coefficients \( X_{lm}^k \) scale with
harmonic index \( k \). These coefficients are defined to satisfy

\[
\left( \frac{a}{D(t)} \right)^{1+1} e^{-i m f(t)} = \sum_{k=-\infty}^{\infty} X_{lm}^k e^{-i k \Omega_{\text{orb}} t}, \tag{B1}
\]

where \( D \) is the binary separation and \( f \) is the true anomaly. For
\(|k| \gg |m| \Omega_{\text{peri}}/\Omega_{\text{orb}} \), where \( \Omega_{\text{peri}} \) is the effective orbital frequency
at periapse, we have that \( X_{lm}^k \ll X_{00}^k \). We can express \( X_{lm}^k \) in general
as an integral over the eccentric anomaly $E$ (Burkart et al. 2012):

$$X_k = \frac{1}{\pi} \int_0^{\pi} \cos(k(E-e\sin E))\,dE$$

$$= J_k(ek)$$

where $J$ is a Bessel function. We can expand $J_k(ek)$ as (Abramowitz & Stegun 1972)

$$J_k(ek) \approx \frac{1}{\sqrt{k}} \exp[-k\gamma(e)],$$

where

$$g(e) = \frac{1}{2} \ln \left( \frac{1+\gamma}{1-\gamma} \right) - \gamma$$

$$= \frac{\gamma^3}{3} + \frac{\gamma^5}{5} + \frac{\gamma^7}{7} + \ldots$$

and $\gamma = \sqrt{1-e^2}$.

**APPENDIX C: CANONICAL ANGULAR MOMENTUM**

Here we will derive the canonical angular momentum associated with a stellar perturbation. The Lagrangian density for a stellar perturbation is (Friedman & Schutz 1978a)

$$\mathcal{L} = \frac{1}{2} \rho \left( |\xi|^2 + \xi \cdot B \xi - \xi \cdot C \xi \right),$$

where $\xi$ is the Lagrangian displacement vector, the Coriolis force operator $B$ was defined in § 4.1, and the Hermitian operator $C$ is defined in e.g. Schenk et al. (2002). The $z$ component of the canonical angular momentum is then

$$J = -\left< \partial_\phi \xi, \frac{\partial \mathcal{L}}{\partial \xi} \right>$$

$$= -\left< \partial_\phi \xi, \dot{\xi} + B\xi/2 \right>. $$

As in § 4.1, we perform a phase space expansion of the Lagrangian displacement vector and its time derivative (Schenk et al. 2002), so that

$$\left< \xi, \dot{\xi} \right> = \sum_A q_A \left< \xi_A, \dot{\xi}_A \right>,$$

where $A$ runs over all rotating-frame stellar eigenmodes and both frequency signs. The canonical angular momentum then becomes

$$J = \frac{1}{2} \sum_{AB} \sum_A q_A^* q_B m_A \left[ 2\omega_B \left< \xi_A, \xi_B \right> + \left< \xi_A, iBE_B \right> \right].$$

Since $\left< \partial_\phi \xi, \dot{\xi} \right>$ is real valued, we can re-express equation (C4) as

$$J = \frac{1}{2} \sum_{AB} q_A^* q_B \left[ m_A \omega_B \left< \xi_A, \xi_B \right> + m_B \left< \xi_A, iBE_B \right> \right].$$

We have that $\left< \xi_A, \xi_B \right> \propto \delta_{m_A, m_B}$, so $J$ reduces to

$$J = \frac{1}{2} \sum_A q_A^* q_A \omega_A^2,$$

where we have used the orthogonality relation (Friedman & Schutz 1978b)

$$\omega_A^2 \left< \xi_A, \xi_B \right> + \left< \xi_A, iBE_B \right> = \delta_{AB} \frac{\omega_A}{\omega_A} \left< \xi_A, \xi_A \right> \delta_{m_A, m_B}. $$

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