The hyperbolic Brownian plane

Thomas Budzinski

ENS Paris

July 7th, 2016
Planar maps

Definitions

- A planar map is a locally finite, connected graph embedded in the plane in such a way that:
  - no two edges cross, except at a common endpoint,
  - every compact subset of the plane intersects finitely many vertices and edges,
- considered up to orientation-preserving homeomorphism.

- The faces of the map are the connected components of its complementary. The degree of a face is the number of half-edges adjacent to this face.
Planar maps

Definitions

- A planar map is a locally finite, connected graph embedded in the plane in such a way that:
  - no two edges cross, except at a common endpoint,
  - every compact subset of the plane intersects finitely many vertices and edges,
- The faces of the map are the connected components of its complementary. The degree of a face is the number of half-edges adjacent to this face.

\[ \begin{align*}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{triangle.png} \\
= \\
\includegraphics[width=0.3\textwidth]{loop.png} \\
\neq \\
\includegraphics[width=0.3\textwidth]{non-planar.png}
\end{array}
\end{align*} \]
Definition

- A **triangulation of the plane** is an infinite planar map in which all the faces have degree 3. It may contain loops and multiple edges.
- A **triangulation with a hole of perimeter** $p$ is a finite map in which all the faces have degree 3 except the external face, which has degree $p$.
- A **rooted** triangulation is a triangulation with a distinguished oriented edge. From now on, all the triangulations will be rooted.

Examples: a rooted triangulation with a hole of perimeter 6.
Definition

If $t$ is a triangulation of a $p$-gon and $T$ a triangulation of the plane, we write $t \subset T$ if $T$ may be obtained by "filling" the hole of $t$ with an infinite triangulation.
Theorem (≈ Angel-Schramm, 2003)

There is a random triangulation of the plane $\mathbb{T}$, called the $\text{UIPT}$ (Uniform Infinite Planar Triangulation), such that for any triangulation $t$ with a hole of perimeter $p$, we have

$$\mathbb{P}(t \subset \mathbb{T}) = C_p \lambda_c^{|t|},$$

where $|t|$ is the number of vertices of $t$ and we have $\lambda_c = \frac{1}{12\sqrt{3}}$ and $C_p = 2\sqrt{3} \frac{p(2p)!}{p!^2} 3^p$. 
Picture by N. Curien.
Condition on $t \subset \mathbb{T}$, and let $e$ be an edge of $\partial t$:

$P(\text{Case I}) = P(t+f \subset \mathbb{T})$

$P(t \subset \mathbb{T}) = C_p + \frac{1}{\lambda} |t| + C_p \lambda |t| c$.

$P(\text{Case II})$ and $P(\text{Case III})$ are also explicitly known, and depend only on $p$. 

Thomas Budzinski

The hyperbolic Brownian plane
Condition on $t \subset T$, and let $e$ be an edge of $\partial t$:

\[ \text{Case I} \]

Then
\[
\mathbb{P}(\text{Case I}) = \frac{\mathbb{P}(t+f \subset T)}{\mathbb{P}(t \subset T)} = \frac{C_{p+1} \lambda_c^{\vert t \vert + 1}}{C_p \lambda_c^{\vert t \vert}} = \frac{C_{p+1}}{C_p} \lambda_c.
\]
Condition on $t \subset \mathbb{T}$, and let $e$ be an edge of $\partial t$:

- **Case I**

  \[
  \mathbb{P}(\text{Case I}) = \frac{\mathbb{P}(t+f \subset \mathbb{T})}{\mathbb{P}(t \subset \mathbb{T})} = \frac{C_{p+1} \lambda_c^{|t+1|}}{C_p \lambda_c^{|t|}} = \frac{C_{p+1}}{C_p} \lambda_c.
  \]

- **Case II;** (here $i = 2$)

  \[
  \mathbb{P}(\text{Case II;}) \quad \text{and} \quad \mathbb{P}(\text{Case III;}) \quad \text{are also explicitly known, and depend only on $p$.}
  \]
Peeling process and consequences

- Allows to discover $\mathbb{T}$, almost "face by face", in a Markovian way.
- Very flexible: the choice of $e$ may be adapted to the information we are looking for:
  - growth in $r^4$ [Angel],
  - critical probabilities for percolation [Angel, Angel-Curien, Richier],
  - subdiffusivity of the random walk [Benjamini-Curien]
\( \lambda \)-Markovian triangulations

**Definition**

A random triangulation of the plane \( T \) is \( \lambda \)-Markovian if there are constants \( (C_p)_{p \geq 1} \) such that for any triangulation \( t \) with a hole of perimeter \( p \) we have

\[
\mathbb{P}(t \subset T) = C_p(\lambda) \lambda^{|t|}.
\]
λ-Markovian triangulations

Definition

A random triangulation of the plane $T$ is $\lambda$-Markovian if there are constants $(C_p)_{p \geq 1}$ such that for any triangulation $t$ with a hole of perimeter $p$ we have

$$\mathbb{P}(t \subset T) = C_p(\lambda)\lambda^{|t|}.$$ 

Proposition (Curien 2014, B. 2016)

If $\lambda > \lambda_c$ then there is no $\lambda$-Markovian triangulation. If $0 < \lambda \leq \lambda_c$ then there is a unique one (in distribution), that we write $T_{\lambda}$. Besides we have

$$C_p(\lambda) = \frac{1}{\lambda} \left( 8 + \frac{1}{h} \right)^{p-1} \sum_{q=0}^{p-1} \binom{2q}{q} h^q,$$

where $h \in (0, \frac{1}{4}]$ is such that $\lambda = \frac{h}{(1+8h)^{3/2}}$. 

Thomas Budzinski

The hyperbolic Brownian plane
Hyperbolic behaviour

- Exponential volume growth [Curien]
- Anchored expansion: if $A$ is a finite, connected set of vertices containing the root, then $|\partial A| \geq c|A|$ [Curien].
- The simple random walk has positive speed [Curien, Angel-Nachmias-Ray].
A planar map can be seen as a (discrete) metric space, equipped with its graph distance and the counting measure on its vertices.

The set of all (classes of) locally compact measured metric spaces can be equipped with the local Gromov-Hausdorff-Prokhorov distance.

Theorem (Curien-Le Gall 14, B. 16)

Let $\mu_T$ be the counting measure on the set of vertices of $T$. We have the following convergence in distribution for the local Gromov-Hausdorff-Prokhorov distance:

$$\left( \frac{1}{n} T, \frac{1}{n} 4 \mu_T \right) \rightarrow_{d} \rightarrow_{a} +\infty P$$

where $P$ is a random (pointed) measured metric space homeomorphic to the plane called the Brownian plane.
A planar map can be seen as a (discrete) metric space, equipped with its graph distance and the counting measure on its vertices.

The set of all (classes of) locally compact measured metric spaces can be equipped with the local Gromov-Hausdorff-Prokhorov distance.

Theorem (Curien-Le Gall 14, B. 16)

Let \( \mu_T \) be the counting measure on the set of vertices of \( T \). We have the following convergence in distribution for the local Gromov-Hausdorff-Prokhorov distance:

\[
\left( \frac{1}{n} T, \frac{1}{n^4} \mu_T \right) \xrightarrow{(d)} P \quad \text{as } n \to +\infty
\]

where \( P \) is a random (pointed) measured metric space homeomorphic to the plane called the Brownian plane.
For $\lambda < \lambda_c$ fixed $\frac{1}{n} T_\lambda$ cannot converge because $T_\lambda$ "grows too quickly".
For $\lambda < \lambda_c$ fixed $\frac{1}{n} \mathbb{T}_\lambda$ cannot converge because $\mathbb{T}_\lambda$ "grows too quickly".

We look for $(\lambda_n) \to \lambda_c$ such that $\frac{1}{n} \mathbb{T}_{\lambda_n}$ converges.
Scaling limit of $T_\lambda$?

- For $\lambda < \lambda_c$ fixed $\frac{1}{n}T_\lambda$ cannot converge because $T_\lambda$ "grows too quickly".
- We look for $(\lambda_n) \to \lambda_c$ such that $\frac{1}{n}T_{\lambda_n}$ converges.

**Theorem (B. 16)**

Let $(\lambda_n)_{n \geq 0}$ be a sequence of numbers in $(0, \lambda_c]$ such that

$$\lambda_n = \lambda_c \left(1 - \frac{2}{3n^4}\right) + o\left(\frac{1}{n^4}\right).$$

Then

$$\left(\frac{1}{n}T_{\lambda_n}, \frac{1}{n^4}\mu_{T_{\lambda_n}}\right) \xrightarrow{(d)} P^h$$

where $P^h$ is a random (pointed) measured metric space homeomorphic to the plane that we call the **hyperbolic Brownian plane**.
For $r \geq 0$ we write $\overline{B}_r(\mathcal{P})$ for the *hull* of radius $r$ of $\mathcal{P}$, that is, the reunion of its ball of radius $r$ and all the bounded connected components of its complementary.
For $r \geq 0$ we write $\overline{B_r(P)}$ for the hull of radius $r$ of $P$, that is, the reunion of its ball of radius $r$ and all the bounded connected components of its complementary.

**Theorem (Curien-Le Gall 14)**

- There is a natural notion of "perimeter" of $\overline{B_r(P)}$, that we write $P_r(P)$, and $(P_r(P))_{r \geq 0}$ is a time-reversed stable branching process (in particular it is càdlàg with only negative jumps).
For $r \geq 0$ we write $\overline{B}_r(\mathcal{P})$ for the *hull* of radius $r$ of $\mathcal{P}$, that is, the reunion of its ball of radius $r$ and all the bounded connected components of its complementary.

**Theorem (Curien-Le Gall 14)**

- There is a natural notion of "perimeter" of $\overline{B}_r(\mathcal{P})$, that we write $P_r(\mathcal{P})$, and $(P_r(\mathcal{P}))_{r \geq 0}$ is a time-reversed stable branching process (in particular it is càdlàg with only negative jumps).
- If $V_r(\mathcal{P})$ is the volume of $\overline{B}_r(\mathcal{P})$, then

\[
(V_r(\mathcal{P}))_{r \geq 0} = \left( \sum_{t_i \leq r} \xi_i |\Delta P_r(\mathcal{P})|^2 \right)_{r \geq 0},
\]

where $(t_i)$ is a measurable enumeration of the jumps of $(P_r(\mathcal{P}))_{r \geq 0}$, and the $\xi_i$ are i.i.d. with density $\frac{e^{-1/2x}}{\sqrt{2\pi x^5}} \mathbb{1}_{x > 0}$. 

Thomas Budzinski  
The hyperbolic Brownian plane
For all $r \geq 0$, the random variable $\overline{B}_{r}(\mathcal{P}^{h})$ has density

$$e^{-2V_{2r}(\mathcal{P})} e^{P_{2r}(\mathcal{P})} \int_{0}^{1} e^{-3P_{2r}(\mathcal{P})x^2} \, dx$$

with respect to $\overline{B}_{r}(\mathcal{P})$. 
We use the convergence of \( T \) to \( P \) and the absolute continuity relations between \( T \) and \( T_\lambda \):

\[
\frac{\mathbb{P}(B_r(T_\lambda) = t)}{\mathbb{P}(B_r(T) = t)} = \frac{C_p(\lambda)}{C_p(\lambda_c)} \left( \frac{\lambda}{\lambda_c} \right)^{|t|}
\]
Sketch of proof

- We use the convergence of $T$ to $P$ and the absolute continuity relations between $T$ and $T_\lambda$:

\[
\frac{\mathbb{P}(B_r(T_\lambda) = t)}{\mathbb{P}(B_r(T) = t)} = \frac{C_p(\lambda)}{C_p(\lambda_c)} \left( \frac{\lambda}{\lambda_c} \right)^{|t|}
\]

- Two main tools:
  - precise asymptotics for the $C_p(\lambda)$,
  - a reinforcement of the convergence of $T$ to $P$. 

The hyperbolic Brownian plane
Asymptotics for the absolute continuity relations

**Proposition**

Fix \( r > 0 \). Let \((\lambda_n), (p_n), \) and \((v_n)\) be such that:

- \( \lambda_n = \lambda_c \left( 1 - \frac{2}{3n^4} \right) + o\left( \frac{1}{n^4} \right) \),
- \( \frac{v_n}{n^4} \to 3v \),
- \( \frac{p_n}{n^2} \to \frac{3}{2}p \).

Let \( t_n \) be a possible value of \( \overline{B_{rn}}(\mathbb{T}) \) such that \( t_n \) has \( v_n \) vertices and a hole of perimeter \( p_n \). Then

\[
\frac{\mathbb{P}(\overline{B_{rn}}(\mathbb{T}\lambda_n) = t_n)}{\mathbb{P}(\overline{B_{rn}}(\mathbb{T}) = t_n)} \to e^{-2v} e^{p} \int_{0}^{1} e^{-3px^2} dx.
\]
Theorem

The three following convergences hold jointly in distribution as $n \to +\infty$:

$$\begin{align*}
\frac{1}{n} T &\quad \to \quad \mathcal{P} \\
\left(\frac{1}{n^4} |\overline{B_{rn}(T)}| \right)_{r \geq 0} &\quad \to \quad (3V_r(\mathcal{P}))_{r \geq 0} \\
\left(\frac{1}{n^2} |\partial B_{rn}(T)| \right)_{r \geq 0} &\quad \to \quad \left(\frac{3}{2}Pr(\mathcal{P})\right)_{r \geq 0}.
\end{align*}$$
Reinforced convergence to the Brownian plane

Theorem

The three following convergences hold jointly in distribution as \( n \to +\infty \):

\[
\begin{align*}
\frac{1}{n} T & \rightarrow P \\
\left( \frac{1}{n^4} |B_{rn}(\mathbb{T})| \right)_{r \geq 0} & \rightarrow (3V_r(\mathcal{P}))_{r \geq 0} \\
\left( \frac{1}{n^2} |\partial B_{rn}(\mathbb{T})| \right)_{r \geq 0} & \rightarrow \left( \frac{3}{2} Pr(\mathcal{P}) \right)_{r \geq 0}.
\end{align*}
\]

- First two marginals: follows from Gromov-Hausdorff-Prokhorov convergence.

Thomas Budzinski

The hyperbolic Brownian plane
Reinforced convergence to the Brownian plane

Theorem

The three following convergences hold jointly in distribution as $n \to +\infty$:

\[
\begin{align*}
\frac{1}{n} T & \rightarrow \mathcal{P} \\
\left( \frac{1}{n^4} |Br_n(T)| \right)_{r \geq 0} & \rightarrow \left( 3 V_r(\mathcal{P}) \right)_{r \geq 0} \\
\left( \frac{1}{n^2} |\partial Br_n(T)| \right)_{r \geq 0} & \rightarrow \left( \frac{3}{2} Pr(\mathcal{P}) \right)_{r \geq 0}.
\end{align*}
\]

- First two marginals: follows from Gromov-Hausdorff-Prokhorov convergence.
- Joint convergence of the last two marginals. [Curien-Le Gall]
Theorem

The three following convergences hold \textbf{jointly} in distribution as 
\( n \to +\infty \) : 

\[
\left\{ \begin{array}{l}
\frac{1}{n} \mathbb{T} \rightarrow \mathcal{P} \\
\left( \frac{1}{n^4} |\overline{B_{rn}(\mathbb{T})}| \right)_{r \geq 0} \rightarrow (3 V_r(\mathcal{P}))_{r \geq 0} \\
\left( \frac{1}{n^2} |\partial \overline{B_{rn}(\mathbb{T})}| \right)_{r \geq 0} \rightarrow \left( \frac{3}{2} P_r(\mathcal{P}) \right)_{r \geq 0}.
\end{array} \right.
\]

- First two marginals: follows from Gromov-Hausdorff-Prokhorov convergence.
- Joint convergence of the last two marginals. [Curien-Le Gall]
- To conclude: show that \((P_r(\mathcal{P}))_{r \geq 0}\) is determined by 
  \((V_r(\mathcal{P}))_{r \geq 0}\).
THANK YOU!