Existence of symmetric central configurations

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Abstract

Central configurations have been of great interest over many years, with the earliest examples due to Euler and Lagrange. There are numerous results in the literature demonstrating the existence of central configurations with specific symmetry properties, using slightly different techniques in each. The aim here is to describe a uniform approach by adapting to the symmetric case the well-known variational argument showing the existence of central configurations. The principal conclusion is that there is a central configuration for every possible symmetry type. Finally the same argument is applied to the class of balanced configurations introduced by Albouy and Chenciner.

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Introduction

In the $n$-body problem, central configurations allow particularly simple motions. If the particles are released from a central configuration with zero initial velocity, the configuration will collapse to the centre of mass while maintaining the same shape up to rescaling. If they are given other particular initial velocities, each particle will follow an elliptical Kepler orbit, and the shape formed by the configuration will remain constant up to rescaling and rotation. They also occur as limiting configurations of parabolic motions [16] and (partial) collisions [9]. R. Moeckel has written a recent survey on the subject [17].

We consider the set of central configurations in $\mathbb{R}^d$. Of course, the most interesting cases are $d = 2$ and $d = 3$, but nothing is lost by considering general dimensions. Over the past few decades, many papers have been written demonstrating the existence of central configurations with various different symmetries, for example [3, 5, 6, 13, 14, 22, 23] (and references therein). The aim of this paper is to describe a uniform proof of all these existence results, using well-known arguments for the existence of symmetric solutions to variational problems. The main result is the following.

Theorem

Given any symmetric configuration of $n$ bodies in $\mathbb{R}^d$ and a corresponding symmetric distribution of masses, there is at least one central configuration of that symmetry type and with the given masses.

We state a more precise result as Theorem 1 below, after defining what is meant by symmetry type (or Burnside type), and a refinement using connected components. An example of a symmetric configuration with triangular ($D_3$) symmetry is illustrated in Figure 1: the relative sizes of the three orbits will depend on the relative masses of each. The proof of the theorem uses the well-known variational approach to existence of central configurations, adapted to the symmetric setting, and details are given in Section 2 below.
SYMMETRIC CENTRAL CONFIGURATIONS

A configuration of \( n \) points in \( \mathbb{R}^d \) is simply a set of \( n \) distinct points in \( \mathbb{R}^d \). A configuration with mass is a configuration where to each of the points there is ascribed a mass; we can represent such an object as a set of pairs \((x, m)\) with \( x \in \mathbb{R}^d \) and \( m > 0 \) although usually we suppress the mass in the notation. We denote the set of configurations of \( n \) points in \( \mathbb{R}^d \) by \( C(\mathbb{R}^d, n) \), or simply by \( C \).

The group \( O(d) \) of orthogonal transformations consists of rotations and rotation-reflections in \( \mathbb{R}^d \), the former having determinant equal to 1, the latter to \(-1\) (note that an orthogonal transformation of determinant \(-1\) is the product of a reflection and a rotation). It acts on the space of configurations in the obvious way: if \( C = \{x_1, \ldots, x_n\} \) and \( g \in O(d) \) then

\[
g \cdot C = \{gx_1, \ldots, gx_n\}.
\]

The action on configurations with mass is similar,

\[
g \cdot \{(x_1, m_1), \ldots, (x_n, m_n)\} = \{(gx_1, m_1), \ldots, (gx_n, m_n)\}.
\]

Now consider a finite subgroup \( G \) of the orthogonal group \( O(d) \). A configuration \( C \) (with mass) is a symmetric configuration (with mass) if the group \( G \) leaves the set invariant: \( g \cdot C = C \) (it will usually permute the points within the set). In particular, for the configurations with mass, this requires that the points \( x \) and \( gx \) in the configuration have the same mass; we call this an invariant mass distribution. Since all our arguments and results are independent of the mass provided it is an invariant mass distribution, we do not need to complicate the notation by incorporating the

Figure 1: A configuration with triangular \((D_3)\) symmetry, consisting of 12 points forming 3 orbits: two equilateral triangles and one semiregular hexagon. The theorem guarantees the existence of a configuration of this form, where the relative sizes will depend on the relative masses of each of the 3 orbits.

In Section 3 we give a few examples in 2 and 3 dimensions. We show for example the existence of nested and staggered (or dual) platonic solids, as well as (nested) cubeoctahedron and icosidodecahedron configurations, and discuss (in Example 7) why other Archimedean configurations are not likely to be central. In Section 4 we briefly describe the topological aspect of this problem. The final short section illustrates how the same techniques can be applied to balanced configurations, an extension of the idea of central configuration due to Albouy and Chenciner [1].

1 SYMMETRIC CONFIGURATIONS

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mass and just write configurations as in (1) rather than (2). If we wish to refer to the mass of the particle at the point \( x \) we will write it as \( m(x) \).

The symmetric configurations therefore form the subset \( C^G := \text{Fix}(G, C) \) of \( C \). Let \( C \) be a symmetric configuration. If \( x \in C \) then so is \( gx \), and therefore so is the orbit of \( x \), which is the set of images of \( x \) under the elements of \( G \):

\[
G \cdot x = \left\{ gx \in \mathbb{R}^d \mid g \in G \right\}.
\]

If \( G \) acts on a finite set, then the set can be partitioned into a disjoint union of orbits. However, different orbits may have different ‘geometry’, and this is made precise by the orbit type of an orbit defined as follows. The isotropy subgroup \( G_x \) of \( x \) is the subgroup of \( G \) consisting of those transformations fixing \( x \):

\[
G_x = \left\{ g \in G \mid gx = x \right\}.
\]

In particular, if \( x = 0 \) then \( G_x = G \). It is a simple exercise to show that if \( y = gx \) then \( G_y = gG_xg^{-1} \); that is the isotropy subgroups of two points in the same orbit are conjugate. Thus to each orbit is associated a conjugacy class of subgroups of \( G \), called the orbit type of the orbit. For a subgroup \( H \) of \( G \), one denotes the conjugacy class containing \( H \) by \( (H) \), and for \( x \in \mathbb{R}^d \), the orbit type of \( x \) is therefore \( (G_x) \). The number of points in an orbit of type \( (H) \) is then \( |G|/|H| \) (where \( |H| \) is the order of a group \( H \)).

Notice in particular that if an orbit has type \( (H) \) say, then at least one of the points \( x \) of the orbit has isotropy subgroup \( G_x = H \) and thus lies in the fixed point subspace

\[
\text{Fix}(H, \mathbb{R}^d) = \left\{ x \in \mathbb{R}^d \mid hx = x, \ \forall h \in H \right\},
\]

which is a linear subspace of \( \mathbb{R}^d \).

As a simple example consider the dihedral subgroup \( D_3 \) of \( O(2) \); this is the symmetry group of the equilateral triangle in the plane. See Figures 1 and 2(a). There are, at this point in the discussion (ignoring the refinement below), three types of orbit: the origin with orbit type \( (D_3) \), an orbit of type \( (Z_3) \) consisting of 3 points forming an equilateral triangle (each vertex of the triangle is fixed by a reflection) and finally a ‘generic’ orbit of type \( (1) \) consisting of 6 points forming a semiregular hexagon all with trivial isotropy. In this way we can write a general \( D_3 \)-symmetric configuration as an integer combination of orbit types (so many orbits of each orbit type):

\[
C = \epsilon(D_3) + a(Z_3) + b(1),
\]

where \( \epsilon \in \{0, 1\} \) (since the only point with isotropy \( D_3 \) is the origin, so there can be at most one such orbit), while \( a, b \in \mathbb{N} = \{0, 1, 2, \ldots\} \). A similar discussion applies to \( D_4 \), see Figure 2(b), but note that there are two non-conjugate reflections in \( D_4 \), here denoted \( \kappa \) and \( \kappa' \).

Extending this example to \( \mathbb{R}^3 \), we let \( D_3 \) act as before on the \((x, y)\)-coordinates, and \( Z_2 \) act by reflection in the \((x, y)\)-plane, so \( \tau(x, y, z) = (x, y, -z) \) (the Schoenflies notation for this subgroup of \( O(3) \) is \( D_{3h} \)). There are now a total of 6 orbit types: the three considered above in the plane \( z = 0 \), but now with isotropy type enhanced by \( Z_2^\tau \), so for example the orbit type of the 3-point orbit is \( (Z_2^\tau \times Z_2^\kappa) \), where \( \kappa \) is a reflection in \( D_3 \), and three new ones which are, firstly a pair of opposite points on the \( z\)-axis at \((0, 0, \pm z)\) for some \( z \neq 0 \), with isotropy type \( (D_3) \), secondly orbits of 6 points forming a triangular prism, which has isotropy type \( (Z_3^\kappa) \), and finally the ‘generic’ orbit consisting of 12 points arranged at the vertices of a semiregular hexagonal prism, with isotropy type \( (1) \). Thus a general symmetric configuration with symmetry \( D_{3h} \) is of the form

\[
C = \epsilon(D_3 \times Z_2^\tau) + a(Z_2^\tau \times Z_2^\kappa) + b(Z_2^\tau) + c(D_3) + d(Z_3^\kappa) + e(1),
\]

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with again $\epsilon \in \{0,1\}$ and $a, b, c, d, e \in \mathbb{N}$.

This idea of writing a $G$-invariant set as an integer combination of orbit types goes back to Burnside [2] in the early days of group theory, so we call this the Burnside type of a symmetric configuration. Many details about properties of Burnside types set can be found in [11].

In order to treat equilateral triangles and their ‘duals’ as distinct types, we need to refine the Burnside type to what we call the topological Burnside type. This is most easily illustrated with the simple $D_3$ example shown in Figure 2(a). While the two triangles (red and blue) have the same orbit type, they cannot be continuously deformed one into the other whilst maintaining that orbit type, and so belong to different connected components of the set of orbits with orbit type $(\mathbb{Z}_2)$. We denote these as $(\mathbb{Z}_2)$ and $(\mathbb{Z}_2)′$. A similar phenomenon occurs in $\mathbb{R}^3$ with tetrahedra and their duals. In contrast, Figure 2(b) shows that the square and its dual have distinct isotropy types, here denoted $(\mathbb{Z}_2^x)$ and $(\mathbb{Z}_2^y)$, where $\kappa$ is the reflection in the $x$-axis and $\kappa'$ the reflection in the diagonal $y = x$, and $\mathbb{Z}_2^x$ denotes the group of order 2 generated by $\kappa$; for $D_4$ the topology does not refine the Burnside type.

We therefore define the topological Burnside type by distinguishing connected components of the set of orbits of type $(H)$ into connected components, writing them as $(H), (H)′$ etc, or more generally $(H)^\alpha$ for $\alpha$ in some index set.

We are now in a position to state a more precise version of the theorem above.

**Theorem 1**  Given any finite subgroup $G$ of $O(d)$ and any topological Burnside type $\Gamma$ for $G$, there is at least one central configuration in each connected component of the set $C(\Gamma)$.

The set $C(\Gamma)$ fails to be connected only if one of the fixed point spaces is 1-dimensional and the number of orbits of the corresponding type is greater than 1, for then reordering those points corresponds to different connected components. See Section 4 for more details.

The precise central configuration whose existence is given by the theorem will depend on the values of the masses of the particles (recall that we assume the mass distribution to be invariant: that is, points in the same orbit have equal mass). From these existence theorems, under non-degeneracy conditions which are probably generic, one can apply the implicit function theorem to obtain central configurations with non-symmetric mass distributions, at least for nearby values of the masses, though the configurations will no longer be symmetric in general. Moreover when the central configurations are degenerate, then one expects to see symmetry-breaking bifurcations, as for example in [14].
2 Proof of the theorem

This type of theorem is usually presented by representing the configurations as ordered \(n\)-tuples and then using the permutation group acting by permuting the points as in [15, 18]; this approach is needed particularly if collisions are involved, such as in [21]. However, this is not necessary for our problem and here we proceed directly on the configurations as sets, as described above, which removes the need for introducing permutations.

Without loss of generality, we restrict attention to configurations whose centre of mass is at the origin: \(\sum_{x \in C} m(x) x = 0\). Central configurations are determined by two functions defined for any configuration (with mass). First the potential,

\[
U(C) = \sum_{\{x, y\} \subset C} \frac{m(x) m(y)}{\|x - y\|},
\]

where \(m(x)\) is the mass of the particle at the point \(x\) and the sum is over all unordered pairs of distinct points in the configuration \(C\). The other function is the total inertia about the origin,

\[
I(C) = \sum_{x \in C} m(x) \|x\|^2.
\]

We can take as our definition the following, a configuration \(C\) is a central configuration if \(C\) is a critical point of \(U\) when restricted to a level set of \(I\). Since both functions are homogeneous (of degrees -1 and 2 respectively) it follows that if a configuration \(C\) is central then so is any homothetic configuration \(\lambda C = \{\lambda x \mid x \in C\}\) for any \(\lambda \neq 0\), and consequently we can restrict attention to the level set \(I = 1\) for convenience. See for example [17] for details. Let \(C_1 = C \cap \{I = 1\}\).

Since both \(U\) and \(I\) depend only on the distances between the particles (and their masses), they are both invariant under the orthogonal group \(O(d)\). To prove the theorem, we use the principle of symmetric criticality, first established (in greater generality) by Palais [20]:

**Principle of symmetric criticality** Suppose a finite group \(G\) acts smoothly on a manifold \(M\) and suppose \(f : M \rightarrow \mathbb{R}\) is a smooth invariant function. Let \(x \in M^G := \text{Fix}(G, M)\). Then \(x\) is a critical point of \(f\) if and only if it is a critical point of the restriction \(f|_{M^G}\).

It follows that \(C\) is a central configuration with symmetry \(G\) if and only if it is a critical point of the restriction of \(U\) to the submanifold \(C_1^G\) of \(C_1\).

The manifold structure of \(C\) is defined as follows. Let \(C \in \mathcal{C}\) be a configuration with \(n\) points say, and let \(\varepsilon > 0\) be the closest Euclidean distance between any pair of distinct points in \(C\). To form a local chart in a neighbourhood of \(C\) choose an ordering of the points in \(C\), say \((x_1, \ldots, x_n)\) and consider the \(\varepsilon\)-ball \(B\) in \((\mathbb{R}^d)^n\); the local chart is then the map \(B \rightarrow C\) taking an \(n\)-tuple to the underlying set. Compatibility between different choices of ordering is easy to check. This is equivalent to expressing \(C\) as the orbit space of the free action of the permutation group \(S_n\) on \((\mathbb{R}^d)^n \setminus \Delta\) where \(\Delta\) is the subset where two or more points coincide.

Now \(I : C \rightarrow \mathbb{R}\) is a smooth non-singular invariant function so that \(C_1 = I^{-1}(1)\) is a smooth \(G\)-invariant submanifold. It is standard fact that if a finite (or compact) group acts on a manifold, then the fixed point spaces are closed submanifolds. Thus we have a smooth function \(U : C_1^G \rightarrow \mathbb{R}\), and we want to show it must have a critical point, for then the result follows by the principle of symmetric criticality.

Because of the form of \(I\) (positive definite quadratic form), if \(C_j\) is a sequence of configurations in \(C_1\) or in \(C_1^G\) that doesn’t contain a limit point, then the minimal distance between pairs of points must tend to zero. Consequently, \(U \rightarrow \infty\) on such a sequence. It follows that \(U\) must attain a minimum somewhere on \(C_1\), or \(C_1^G\) respectively, and this minimum is the desired critical point. \(\square\)
**Remark 2** As is well-known, if $C$ is a central configuration in $\mathbb{R}^d$ and $e > d$ then $C$ is also a central configuration in $\mathbb{R}^e$, when embedded in $\mathbb{R}^d \times \{0\} \subset \mathbb{R}^e$. To see this write $\mathbb{R}^e = \mathbb{R}^d \times \mathbb{R}^{e-d}$, and consider the 2-element subgroup of $O(e)$ generated by $g = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. Then $\text{Fix}(g, \mathbb{R}^e) = \mathbb{R}^d$, and the result follows from the principle of symmetric criticality.

**Remark 3** We have been considering an ambient space of arbitrary dimension $d$. If $d > 3$ the relevance of the inverse square law is debatable, and for physical reasons should probably be replaced by an inverse power $(d-1)$ law. The potential would then be of the form

$$U(C) = \sum_{\{x,y\} \subset C} \frac{m(x)m(y)}{\|x-y\|^{d-2}}.$$ 

However, Theorem 1 only relies on the symmetry of the function $U$ and the fact that as the configuration approaches a collision, so $U \to \infty$. It follows that the approach would also apply with this gravitational law, and indeed with any other potential depending only on the shape of the configuration and the masses, in a symmetric fashion.

### 3 Examples

#### 3.1 Dimension 2

Here it is straightforward to list the different types of symmetric configuration. The only finite subgroups of $O(2)$ are the cyclic groups $C_k$ (of order $k$) and the dihedral groups $D_k$ (of order $2k$).

If $G = C_k$ is cyclic, then there are two Burnside types (and no topological refinement), namely $(C_k)$ and $(\mathbb{1})$. The corresponding fixed point subspaces are the origin and $\mathbb{R}^2$. A symmetric configuration is then a set of $n = kr$ points, forming $r$ regular $k$-gons centred on the origin, together with possibly a point at the origin. The Burnside type is

$$\Gamma = \varepsilon (C_k) + r (\mathbb{1})$$

(3)

Now suppose $G = D_k$. In this case there are four topological Burnside types, as described above for $k = 3$ or 4.

$$\begin{cases} (D_k), (\mathbb{Z}_2^k), (\mathbb{Z}_2^k)' & \text{for } k \text{ odd} \\ (D_k), (\mathbb{Z}_2^k), (\mathbb{Z}_2^k)' & \text{for } k \text{ even} \end{cases}$$

(4)

To treat the two cases together, denote these topological orbits as $(D_k), (A), (B)$ and $(\mathbb{1})$. Then a general symmetric configuration would have topological Burnside type

$$\Gamma = \varepsilon (D_k) + a(A) + b(B) + c(\mathbb{1}).$$

Geometrically, this would consist of $\varepsilon$ points at the origin, $a$ regular nested $k$-gons, $b$ staggered (or twisted by $\pi/k$) regular $k$-gons and $c$ semiregular $2k$-gons, all centred at the origin. A semiregular $2k$-gon is the orbit of a point in the complement of the axes of reflection.

Since $C_n < D_n$ it follows that dihedral configurations also have cyclic symmetry. Conversely, it is unknown whether there exists a central configurations with cyclic symmetry that does not in fact have dihedral symmetry.
3.2 Dimension 3

This case is more complex, resulting in many more types of central configuration. A description of all the possible symmetry types is given in [15], although some adaptation is needed as in that reference the action is restricted to the sphere: in particular the origin did not appear and nested polyhedra are not possible.

**Example 4** Consider \( G = \mathbb{Z}_2 \times \mathbb{Z}_2 \) with one generator \( \tau \) acting by reflection in the \((x,y)\)-plane, and the other \( \rho \) by rotation by \( \pi \) about the \( z \)-axis. The Schoenflies notation is \( C_{2h} \). There are 4 Burnside types: \((C_{2h}), (\mathbb{Z}_2^0), (\mathbb{Z}_2^1)\) and \((\emptyset)\). The Burnside type \( \Gamma = \varepsilon(C_{2h}) + a(\mathbb{Z}_2^0) + b(\mathbb{Z}_2^1) + c(\emptyset) \) consists of \( \varepsilon \) points at the origin, \( 2a \) points symmetrically placed along the \( z \)-axis, \( 2b \) points in the \((x,y)\)-plane placed symmetrically with respect to the origin and \( 4c \) points in space, placed in \( G \)-orbits (these each forming the vertices of a rectangle). For any symmetric distribution of masses among these points, the theorem tells us that there is at least one central configuration with the points in such a configuration.

**Example 5** Consider the subgroup \( D_{nh} < O(3) \). As a group this is isomorphic to \( \mathbb{Z}_2 \times D_k \), and is generated by the reflection \( \tau \) in the \((x,y)\)-plane (giving the \( \mathbb{Z}_2 \) factor) and the usual dihedral group acting on the \((x,y)\)-plane and leaving the 'vertical' \( z \)-axis fixed. Among the orbit types are the origin with orbit type \((D_{nh})\), the horizontal lines of reflection forming two components (as in 2 dimensions) with orbit type \((\mathbb{Z}_2^0 \times \mathbb{Z}_2^0)\) and \((\mathbb{Z}_2^1 \times \mathbb{Z}_2^0)\) or \((\mathbb{Z}_2^0 \times \mathbb{Z}_2^1)\) (accordingly as \( k \) is even or odd) giving orbits of regular \( k \)-gons and their duals, and the prisms with \( k \)-fold symmetry with the \( 2k \) vertices lying in the vertical planes of reflection and with \( z \neq 0 \) and orbit type \((\mathbb{Z}_2^0)\) and \((\mathbb{Z}_2^1)\) or \((\mathbb{Z}_2^k)\) as above. Consider in particular the configurations with \( 3k \) points with Burnside type

\[
C = \begin{cases} 
1(\mathbb{Z}_2^0) + 1(\mathbb{Z}_2^1 \times \mathbb{Z}_2^0) & \text{if } n \text{ is even} \\
1(\mathbb{Z}_2^0) + 1(\mathbb{Z}_2^1 \times \mathbb{Z}_2^1) & \text{if } n \text{ is odd}
\end{cases}
\]

This is chosen so that the \( k \)-gon in the plane \( z = 0 \) is staggered (dual) relative to the polygons in the other horizontal planes. The theorem then implies there must be a central configuration of this symmetry type, so proving a conjecture of Corbera and Llibre [6] on 'double antiprisms'. A similar result is available if the three \( k \)-gons are aligned rather than staggered.

**Example 6** Consider the symmetry group \( T_d \) of the regular tetrahedron, which has order 24. There are 5 orbit types: \((T_d)\) (the origin), \((S_3)\) (radial lines through the vertices of the tetrahedron or its dual), \((\mathbb{Z}_2 \times \mathbb{Z}_2)\) (mid-points of the 6 edges, forming an octahedron), \((\mathbb{Z}_2)\) (other points on the edges, forming an orbit of 12 points) and \((\emptyset)\) (generic points, orbits of 24 points). The theorem tells us that for any non-negative integers \( \varepsilon, a, b, c, d, e \), there is a symmetric configuration of Burnside type

\[
C = \varepsilon(T_d) + a(S_3) + b(S_3)' + c(\mathbb{Z}_2 \times \mathbb{Z}_2) + d(\mathbb{Z}_2) + e(\emptyset).
\]

Here as usual \( \varepsilon \in \{0,1\} \) determines whether or not there is a point at the origin, \( a \) is the number of nested tetrahedra and \( b \) the number of nested dual tetrahedra, etc.

Similar results apply to the other groups \( O_h \) and \( I_h \), from which we deduce the existence theorem of [5] on nested Platonic solids. One can also deduce the existence of two types of Archimedean solid: the cubeoctahedron and the icosidodecahedron.

The vertices of the cubeoctahedron lie at the mid-points of the cube (or of the octahedron) and has octahedral symmetry \( O_h \); it is uniquely determined by the orbit type \((\mathbb{Z}_2 \times \mathbb{Z}_2)\), a subgroup
generated by reflections in two orthogonal planes, and this shows that this is also a central configuration. The icosidodecahedron consists of 30 vertices placed at the mid-points of the edges of the dodecahedron (or of the icosahedron) and is similarly determined uniquely by the analogous orbit type \((\mathbb{Z}_2 \times \mathbb{Z}_2)\), but now as a subgroup of the icosahedral group \(I_h\), and this shows it too is a central configuration. Similarly, nested cubeoctahedra and nested icosidodecahedra also form central configurations.

**Example 7** On the other hand, other Archimedean solids do not (in all likelihood) form central configurations. This is because their symmetry group does not determine their shape. For example, consider the family of truncated tetrahedra. These are obtained by shaving off the 4 vertices of a regular tetrahedron, replacing them with 3 vertices each and 4 new equilateral triangles as faces. The original faces of the tetrahedron then become semiregular hexagons. As more is shaved off, the ratio between the lengths of the semiregular hexagons varies (increases say), and when the two lengths are equal, the hexagon is regular, and this truncated tetrahedron is an Archimedean solid. Let \(\rho > 0\) denote the ratio of the sides of the semiregular hexagon. As \(\rho \to 0\) so the orbit tends to a tetrahedron, and as \(\rho \to \infty\) the 12 vertices merge in pairs to form an octahedron. The Archimedean truncated tetrahedron of course corresponds to \(\rho = 1\). The theorem implies that there is at least one value of \(\rho > 0\) which forms a central configuration, and numerical calculations (using Maple) show this is unique and has the value \(\rho = 0.855\) which does not correspond to the Archimedean shape (the edge between two semiregular hexagons being shorter than the edges of the equilateral triangles).

It is to be expected that a similar phenomenon happens for the other Archimedean shapes: namely that they fail to be central configurations (except of course the cubeoctahedron and icosidodecahedron discussed above). The distinguishing feature of these two particular shapes among the Archimedean ones is that they are *edge regular*, which means that all the edges are equivalent under the symmetry group, while in the others there are two distinct types of edge, and the symmetry alone does not force them to be of equal length.

**Example 8** Consider the subgroup of \(SO(3)\) with 4 elements consisting of the identity, and the rotations by \(\pi\) about each of the \(x\), \(y\) and \(z\)-axes. The Schoenflies notation is \(D_2\). There are 5 types of orbit for this group. Firstly the point at the origin which is fixed by the whole group. Secondly, a pair of opposite points on the \(z\)-axis, and these have isotropy equal to the subgroup of order 2 generated by the corresponding rotation \(R_z\), third and fourth are the corresponding pairs of points on the \(x\)- and \(y\)-axes, and finally a generic orbit consisting of the 4 points \((x, y, z), (x, -y, -z), (-x, y, -z), (-x, -y, z)\) which are distinct provided at most one of the coordinates is 0, and which has trivial isotropy. Any symmetric configuration has Burnside type

\[
\Gamma = \varepsilon(G) + a(R_z) + b(R_y) + c(R_x) + d(\mathbb{I})
\]

In particular, with \(\varepsilon = 0\), \(a = k\), \(b = p\), \(c = \ell\), \(d = 0\) we reclaim the result of Jiang and Zhao \[12\] using their notation. And of course we can let \(\varepsilon = 1\) or \(d > 0\) to obtain a more general result.

Given a Burnside symmetry type, it is natural (and useful) to have a measure of the complexity of the corresponding set of configurations, using topological invariants. This information can be used to find a lower bound on the number of central configurations of the given type, using Morse theory if all critical points are non-degenerate or more generally using Lusternik-Schnirelman category. However, we do not address that question here.
4 Topology

Let \( G \) be a given finite subgroup of \( O(d) \) and let \( \Gamma \) be a Burnside type for \( G \). We are interested in the topology of \( C(\Gamma) \). In this short section we give some indications of what this topology is and how it can be used.

Consider the quotient space \( X = \mathbb{R}^d / G \), which is in general a singular space. The image of the points with orbit type \((H)\) in \( X \) forms a submanifold of \( X \), which we denote \( X_H \), and together the \( X_H \) form a stratification of \( X \). Note that if \( H \) and \( H' \) are conjugate then \( X_H = X_{H'} \). See for example [7] for details on group actions and stratifications of their orbit space.

Thus, if a particular symmetric configuration consists of \( a \) orbits of type \((H)\), then it is determined by \( a \) points in \( X_H \). Denote by \( \mathcal{I} = \mathcal{I}(G) \) the collection of all conjugacy classes of isotropy subgroups of \( G \), and by \( \mathcal{T} \) its refinement into topological classes \((\mathcal{H}^a)\). For a conjugacy class \((H) \in \mathcal{I} \) we work with a representative \( H \). Thus we write \( a_H \) rather than \( a((H)) \) for the number of orbits of type \((H)\).

The following decomposition of \( C(\Gamma) \) is immediate from the discussion above.

**Proposition 9** Let \( G < O(d) \) be a finite subgroup and \( \Gamma = \sum a_H(H) \) a given Burnside type, where the sum is over \((H) \in \mathcal{I} \), and \( a_H \in \mathbb{N} \). Then there is a diffeomorphism

\[
C(\Gamma) \cong \prod_{(H) \in \mathcal{I}} C(X_H, a_H),
\]

where \( \prod \) denotes the Cartesian product and \( C(X_H, a) \) is the configuration space of \( a \) points in \( X_H \).

Note that the connected components \( X_H^a \) of \( X_H \) correspond to the topological Burnside types with orbit type \((H)\). The expression above is readily refined to give

\[
C(\Gamma) = \prod_{(H)^a \in \mathcal{T}} C(X_H^a, a_H^a).
\]

One can identify each \( X_H \) with a quotient of a subspace of \( \mathbb{R}^d \) as follows. For each isotropy subgroup \( H \) let \( V = V(H) = \text{Fix}(H, \mathbb{R}^d) \) and let \( V^o = V^o(H) \) be the subset of points whose isotropy is precisely \( H \). It follows from the relation \( G_{g, x} = gG_xg^{-1} \), that for \( x \in V^o \) one has \( g x \) is also in \( V^o \) if and only if \( g \in N_G(H) \), the normaliser of \( H \) in \( G \). It follows from this that \( X_H = V^o(H) / N_G(H) \). This is illustrated in Figure 3 for the dihedral group \( D_n \) acting on \( \mathbb{R}^2 \).

We finish this section with some observations and an example.

- The trivial case where \( V(H) = \{0\} \) does not contribute to the topology of \( C(\Gamma) \).
• The simplest non-trivial case is when the fixed point space is 1-dimensional: \( \dim(V(H)) = 1 \). This has already been mentioned in Section 1, and is well-known. If \( a > 1 \) the space \( X_H(a) \) is a disjoint union of contractible connected components, corresponding to different orderings of the points.

• If a fixed point space has dimension 2, there are two different possibilities. Let \( H \) be the isotropy subgroup in question and \( V \) the fixed point space (of dimension 2). The first possibility is that \( V \setminus V^o \) is a (finite) union of 1-dimensional subspaces, and in this case each component of \( V^o \) is diffeomorphic to the plane. The contribution to \( \mathcal{C}(\Gamma) \), if \( \Gamma \) includes \( a(H)^a \) is then diffeomorphic to \( C(\mathbb{R}^2, a) \). The topology of this space is well-known: its fundamental group is the pure braid group on \( a \) strings, while all its higher homotopy groups vanish [8].

The second possibility is \( V^o \) is a punctured plane, and the contribution to \( \mathcal{C}(\Gamma) \) from \( a \) orbits of type \((H)\) is equivalent to \( C(\mathbb{R}^2, a + 1) \).

For example, consider the symmetric 5-body configurations, with symmetry \( D_1 \approx \mathbb{Z}_2 \) acting by reflection in a line (see Figure 2(c,d) of [13]). The orbit types are \((D_1)\) and \((\mathbb{I})\), and for a total of 5 bodies there are 3 possibilities,

\[
1(D_1) + 2(\mathbb{I}), \quad 3(D_1) + 1(\mathbb{I}), \quad \text{and} \quad 5(D_1).
\]

The \( 5(D_1) \) are the collinear Moulton configurations of 5 bodies. For \( \Gamma = 1(D_1) + 2(\mathbb{I}) \), the resulting space \( \mathcal{C}(\Gamma) \) is homotopic to the circle, so \( U \) must have at least two critical points, as illustrated in [13].

• Higher dimensional fixed points spaces will contribute to higher homotopy groups and cohomology, but the correspondence is not so easily understood.

• Even though the set \((\mathbb{R}^d)^o\) of points in \( \mathbb{R}^d \) with trivial isotropy may not be connected, the quotient \((\mathbb{R}^d)^o/G\) is always connected. This is because the complement of \((\mathbb{R}^d)^o\) is a union of linear subspaces, and so the only way \((\mathbb{R}^d)^o\) is disconnected is through hyperplanes, and these only arise as fixed point sets for reflections (a matrix with \((d-1)\) eigenvalues equal to \(+1\) and one equal to \((-1)\)), and the reflection then identifies the two sides of the corresponding hyperplane. The following example shows that \((\mathbb{R}^d)^o\) may not be contractible.

**Example 10** Consider finally Example 8 above, and \( \mathcal{C}(\Gamma) \) for \( \Gamma = \varepsilon(G) + a(R_2) + b(R_y) + c(R_z) + d(\mathbb{I}) \). The orbit types \((G),(R_2),(R_y)\) and \((R_z)\) give spaces of dimension 1 or less, so correspond to connected components of \( \mathcal{C}(\Gamma) \). However the generic orbit, with orbit type \((\mathbb{I})\), consists of 4 points in the complement of the coordinate axes. Its contribution to \( \mathcal{C}(\Gamma) \) is \( \mathcal{C}(X_1, d) \), and one can show that the stratum \( X_1 = (\mathbb{R}^3)^o/G \) is the thrice punctured sphere. Thus the space \( \mathcal{C}(\Gamma) \) is homotopic to the space of \( d \) points in the thrice-punctured sphere. Using Morse theory one can show that for \( d = 1 \), and assuming critical points are non-degenerate, there must be at least three critical points: one minimum and 2 saddle points. (In fact if \( d = 1 \) and \( a = b = c = 0 \) then there are 5 critical points: two minima occurring at tetrahedral configurations and 3 saddles occurring at squares in the coordinate hyperplanes.)

### 5 Balanced Configurations

Balanced configurations were introduced by Albouy and Chenciner [1, 4] (see also [17]), and one definition is as follows. Consider an ordered configuration \((x_1, \ldots, x_n)\) in \((\mathbb{R}^d)^o\) with \( \sum m_i x_i = 0 \) and
let $X$ be the matrix whose columns are the position vectors of the points $x_1, \ldots, x_n$ (a $d \times n$ matrix), and let $\mu$ be the $n \times n$ diagonal matrix with $\mu_{jj} = m_j$. Then define the two matrices

$$B = MX X^T, \quad S = X^T MX.$$ 

These matrices have the same spectrum, with multiplicities, apart from the possible 0 eigenvalue, and this spectrum (including multiplicities) is called the \textit{inertia spectrum} of the configuration [4]. It is easy to see that the eigenvalues of the matrix are all non-negative, and that the inertia function $I$ is the trace of either matrix $\text{tr}(B) = \text{tr}(S)$. Let $C(\sigma)$ denote the space of all configurations with inertia spectrum $\sigma$. A configuration is \textit{balanced} if it is a critical point of the potential function $U$ restricted to $C(\sigma)$. Since $I$ is constant on $C(\sigma)$ it follows that any central configuration is also a balanced configuration. The variational argument used in this paper shows that there is always a balanced configuration in each non-empty $C(\sigma)$.

**Theorem 11** Given any finite subgroup $G$ of $O(d)$, let $\sigma$ be the inertia spectrum of some symmetric configuration. Then $C(\sigma)^G$ is a non-empty closed subset of $C(\sigma)$ and there is a symmetric balanced configuration in each component of $C(\sigma)^G$, and indeed on $C(\sigma)(\Gamma)$, for any topological Burnside type $\Gamma$ for which $C(\sigma)(\Gamma)$ is non-empty.

It is interesting to ask what the effect of symmetry is on the inertia spectrum of a symmetric configuration; that is, for a given (topological) Burnside type $\Gamma$, what is the possible spectrum of $S$?

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