Complete positivity of quantum dynamics is often viewed as a litmus test for physicality, yet it is well known that correlated initial states need not give rise to completely positive evolutions. This observation spurred numerous investigations over the past two decades attempting to identify necessary and sufficient conditions for complete positivity. Here we describe a complete and consistent mathematical framework for the discussion and analysis of complete positivity for correlated initial states of open quantum systems. Our key observation is that initial system-bath states with the same reduced state on the system must evolve under all admissible unitary operators to system-bath states with the same reduced state on the system, in order to ensure that the induced dynamical maps on the system are well-defined. Once this consistency condition is imposed, related concepts like the assignment map and the dynamical maps are uniquely defined. In general, the dynamical maps may not be applied to arbitrary system states, but only to those in an appropriately defined physical domain. We show that the constrained nature of the problem gives rise to not one but three inequivalent types of complete positivity. Using this framework we elucidate the limitations of recent attempts to provide conditions for complete positivity using quantum discord and the quantum data-processing inequality. The problem remains open, and may require fresh perspectives and new mathematical tools. The formalism presented herein may be one step in that direction.

I. INTRODUCTION

Completely positive (CP) maps have played an important role in the long and extensive history of the problem of the formulation and characterization of the dynamics of open quantum systems [1, 2]. They have become a widespread tool, e.g., in quantum information science [3]. Thus it is of interest to establish under which conditions CP maps can arise from a complete description of an open system that includes both the system and its environment or bath. To arrive at a map one first identifies an admissible set of initial system-bath states which is one-to-one with the corresponding set of system states via the partial trace. Then one jointly and unitarily evolves the system and bath, and then traces out the bath. It is well known that if the set of admissible initial system-bath states is completely uncorrelated (product states) with a fixed bath state then this “standard procedure” gives rise to a CP map description of the evolution of the system. The situation is far more complicated when the set of initial system-bath states contains correlated states. The basic reason for this is that when correlations are present in the initial system-bath state, a clean separation between system and bath is no longer possible, and the “standard procedure” alone no longer uniquely defines a map between the set of all initial and final system states.

Subsystem dynamical maps are uniquely defined by the joint unitary evolution of system and bath and the choice of the set of initial states and several studies have pointed out that entangled initial system-bath states can lead to non-CP maps [4-7]. Rodriguez-Rosario et al. highlighted the role of quantum discord [8] and showed that a CP map arises if the set of joint initial system-bath states is purely classically correlated, i.e., has vanishing quantum discord [9]. Subsequently Shabani & Lidar showed that, under certain additional constraints, the vanishing quantum discord condition is not just sufficient but also necessary for complete positivity [10,11]. More recently, Brodutch et al. [12] and Buscemi [13] demonstrated that this connection between complete positivity and discord does not generalize to all cases. Brodutch et al. did so by offering a counterexample in the form of a set of initial system-bath states, almost all of which are discordant, which nevertheless exhibit completely positive subdynamics. Buscemi followed this up by describing a general method for constructing examples yielding completely positive subdynamics, even though they may feature highly entangled states. It remains an open problem, therefore, to fully elucidate the relationship between structural features of the set of initial system-bath states and the behavior of the resulting dynamics, including whether or not the dynamics are completely positive.

We do not offer a complete solution to the problem in these pages. Rather, it is our purpose here to establish a complete and consistent mathematical framework for the discussion and analysis of linear subsystem dynamics, including the question of complete positivity for correlated initial states. Additionally, we develop the idea that, in the case of correlated initial states, there are several inequivalent definitions of complete positivity. Physical, mathematical, and operational concerns may recommend one flavor of complete positivity over the others, as we discuss. Our analysis builds on the notion of what we call “G-consistent operator spaces”, that represent the set of admissible initial system-bath states. G-consistency is necessary to ensure that initial system-bath states with the same reduced state on the system evolve under all admissible unitary operators to system-bath states with the same reduced
state on the system, ensuring that the dynamical maps on the system are well-defined.

The paper is organized as follows. In Section II, we examine the minimal conditions necessary for the existence of well-defined subsystem dynamical maps, and formally define $G$-consistent subsets. The more specific case of linear dynamical maps is considered in Section III, where we formally define $G$-consistent subspaces, consider properties of the uniquely defined “assignment maps” associated with these spaces, describe the subsystem dynamical maps related to a $G$-consistent subspace, and operator sum representations of these maps. Section IV develops some definitions of complete positivity, discusses the physical motivation of these ideas, and extends these definitions to apply to “assignment maps”. We present in Section V a number of examples culled from the existing literature on completely positive dynamics and show how they may be expressed in the present framework. Finally, a summary and discussion of open questions is offered in Section VI.

II. SUBSYSTEM DYNAMICAL MAPS

The time-evolution of a quantum subsystem is often described in terms of dynamical maps that transform the reduced state of the system at time $t_0$ to the reduced state at time $t_1$. However, the future state of a quantum system in contact with its environment depends upon not just the current state of the system, but the joint state of the system and environment, as well as the joint evolution. For this reason, such dynamical maps are not generally well-defined: each possible reduced state of the system may arise from infinitely many distinct system-bath states, which after joint evolution, can yield infinitely many distinct possible reduced system states at a future time $t_1$. We wish to describe a mathematical framework which is broad enough to contain the various dynamical map constructions of [9–13] and others. It is closely related to the assignment map approach first described by Pechukas [4], but is more general and places greater emphasis on the space of admissible initial system-bath states. In order to build this framework involving dynamical maps for subsystem evolution, it is necessary to remove the indeterminacy of the final system state from the problem, requiring that we begin by making an assumption about the set of admissible initial system-bath states, which we describe presently.

To set the stage, let us briefly review the setting of open quantum systems. For any Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$ and let $U(\mathcal{H})$ denote the unitary group on $\mathcal{H}$. For any $U \in U(\mathcal{H})$, the adjoint map $A \mapsto UAU^\dagger$ is the conjugation superoperator $A \mapsto UAU^\dagger$. Consider a system $S$ with a $d_S$-dimensional system Hilbert-space $\mathcal{H}_S$, a bath $B$ with a $d_B$-dimensional Hilbert space $\mathcal{H}_B$, and the joint system-bath Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_B$. Let $\mathcal{D}_S \subseteq \mathcal{B}(\mathcal{H}_S)$, $\mathcal{D}_B \subseteq \mathcal{B}(\mathcal{H}_B)$, and $\mathcal{D}_{SB} \subseteq \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B)$ denote the convex sets of density matrices on $\mathcal{H}_S$, $\mathcal{H}_B$, and $\mathcal{H}_S \otimes \mathcal{H}_B$, respectively. The joint system-bath state $\rho_{SB}(t) \in \mathcal{D}_{SB}$ evolves under a joint time-dependent unitary propagator $U(t)$ as $\text{Ad}_U(\rho_{SB}(0)) = \rho_{SB}(t)$. The reduced system and bath states are given via the partial trace operation by $\rho_S(t) = \text{Tr}_B[\rho_{SB}(t)] \in \mathcal{D}_S$ and $\rho_B(t) = \text{Tr}_S[\rho_{SB}(t)] \in \mathcal{D}_B$, respectively. The standard prescription for the system sub-dynamics is thus given by the following quantum dynamical process (QDP), acting on the initial system-bath state $\rho_{SB}(0)$:

$$\rho_S(t) = \text{Tr}_B[U(t)\rho_{SB}(0)U^\dagger(t)].$$

It is important to note that, in contrast to some authors (e.g., [13]), the map we seek is one that describes the time-evolution of the post-preparation state of the system. Evolution from an idealized state through experimental realization (i.e., state preparation) ending with unitary evolution of the system-bath state should be modeled with an additional “preparation” map precomposed with the evolution map described herein.

Definition 1 ($S$-dynamical map). Given $U \in U(\mathcal{H}_S \otimes \mathcal{H}_B)$, a subsystem $S$-dynamical map $\tau^S_B : \mathcal{D}_S \to \mathcal{D}_S$ is any map that satisfies $\tau^S_B(\text{Tr}_B \rho) = \text{Tr}_B(U(\rho)U^\dagger)$ for all $\rho \in \mathcal{S}$, i.e., that makes the diagram in Figure 1 commute.

Such a map can only be well-defined if, whenever $\rho_1, \rho_2 \in \mathcal{S}$ and $\text{Tr}_B \rho_1 = \text{Tr}_B \rho_2$, it follows that $\text{Tr}_B(U(\rho_1)U^\dagger) = \text{Tr}_B(U(\rho_2)U^\dagger)$. This defines a necessary property of $\mathcal{S}$ that we call $U$-consistency.

We typically want to define not just a single dynamical map, but a family of them based on a subsemigroup of unitary system-bath evolution operators (“semi” since we will set $t \geq 0$). This gives rise to the notion of $G$-consistency.

Definition 2 ($G$-Consistent Subset). Let $\mathcal{G} \subset U(\mathcal{H}_S \otimes \mathcal{H}_B)$ be a subsemigroup of the unitary group acting on the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_B$. A subset of system-bath states $\mathcal{S} \subset \mathcal{D}_{SB}$ will be called $G$-consistent if it is $U$-consistent for all $U \in \mathcal{G}$, i.e., if, whenever $\rho_1, \rho_2 \in \mathcal{S}$ are such that $\text{Tr}_B \rho_1 = \text{Tr}_B \rho_2$, $\text{Tr}_B(U(\rho_1)U^\dagger) = \text{Tr}_B(U(\rho_2)U^\dagger)$ for all $U \in \mathcal{G}$.

$\mathcal{G}$ represents the semigroup of “allowed” unitary evolutions of system and bath, in other words, the set of unitary evolutions, closed under compositions, for which the subsystem dynamical maps are of interest. For example, if the system and bath will only evolve according to a single fixed time-independent Hamiltonian $H$, then $\mathcal{G}$ is typically the one-parameter subsemigroup $\mathcal{G} = \{e^{-itH} : t \geq 0\}$. If the system and/or

FIG. 1. The set of admissible initial system-bath states $\mathcal{S} \subset \mathcal{D}_{SB}$ must be chosen such that a well-defined subsystem dynamical map $\tau^S_B : \mathcal{D}_S \to \mathcal{D}_S$ exists which makes the diagram commute, i.e., which satisfies $\tau^S_B(\text{Tr}_B \rho) = \text{Tr}_B(U(\rho)U^\dagger)$ for all $\rho \in \mathcal{S}$. This requirement defines the concept of a $U$-consistent subset [13].
bath are subject to control, then \( \mathcal{G} \) may be a larger subsemigroup representing all unitary operators that may be generated within the control scheme. Generally, the larger the semigroup \( \mathcal{G} \), the more restrictions \( \mathcal{G} \)-consistency places on the subset of admissible initial states \( \mathcal{S} \) and the smaller such a subset must be. Indeed, when \( \mathcal{G} = U(\mathcal{H}_s \otimes \mathcal{H}_B) \), the condition becomes that \( \Tr_B \) must describe a one-to-one correspondence between \( \mathcal{S} \) and \( \Tr_B \mathcal{S} \), i.e., for each system state \( \rho_s \in \mathcal{D}_s \), there exists at most\(^\text{*} \) one state \( \rho_B \in \mathcal{S} \) such that \( \Tr_B \rho_B = \rho_s \). 

Example 1. Štěmáčová and Buček [17] offered an example which demonstrates that not all subsets \( \mathcal{S} \subset \mathcal{D}_B \) are \( U(\mathcal{H}_s \otimes \mathcal{H}_B) \)-consistent, i.e., that there exist initial states \( \rho_1, \rho_2 \in \mathcal{D}_B \) and \( U \in \mathcal{U}(\mathcal{H}_s \otimes \mathcal{H}_B) \) such that \( \Tr_B \rho_1 = \Tr_B \rho_2 \), but \( \Tr_B(U \rho_1 U^\dagger) \neq \Tr_B(U \rho_2 U^\dagger) \). Specifically, they considered a system and bath comprising one qubit each, such that

\[
\begin{align*}
\rho_1 &= |\alpha|^2|00\rangle\langle 00| + |\beta|^2|11\rangle\langle 11| \tag{2a} \\
\rho_2 &= (\alpha|00\rangle + \beta|11\rangle)(\alpha^*|00\rangle + \beta^*|11\rangle) \tag{2b} \\
U &= -i(\sigma_x|00\rangle\langle 00| + |00\rangle\langle 00|) \simeq \text{CNOT}, \tag{2c}
\end{align*}
\]

which satisfy

\[
\begin{align*}
\Tr_B \rho_1 &= |\alpha|^2|00\rangle\langle 00| + |\beta|^2|11\rangle\langle 11| = \Tr_B \rho_2 \tag{3a} \\
\Tr_B(U \rho_1 U^\dagger) &= |\alpha|^2|00\rangle\langle 00| + |\beta|^2|11\rangle\langle 11| \tag{3b} \\
\Tr_B(U \rho_2 U^\dagger) &= (\alpha|0\rangle + \beta|1\rangle)(\alpha^*|0\rangle + \beta^*|1\rangle), \tag{3c}
\end{align*}
\]

so that \( \Tr_B(U \rho_1 U^\dagger) \neq \Tr_B(U \rho_2 U^\dagger) \). It follows that if \( \text{CNOT} \in \mathcal{G} \), then no \( \mathcal{G} \)-consistent subset \( \mathcal{S} \subset \mathcal{D}_B \) may contain both \( \rho_1 \) and \( \rho_2 \).

The specification of a set \( \mathcal{S} \subset \mathcal{D}_B \) of admissible initial states may be thought of as a “promise” that the initial system-bath state will always lie in \( \mathcal{S} \). The constraint that \( \mathcal{S} \) must be \( \mathcal{G} \)-consistent can impose heavy restrictions on this set, often forcing \( \mathcal{S} \) to be very small relative to \( \mathcal{D}_B \). It will be instructive to consider the consequences of constraining to a \( \mathcal{G} \)-consistent subset \( \mathcal{S} \) in order to understand the properties of the resulting dynamical map and to place earlier studies in their proper context.

III. LINEAR DYNAMICAL MAPS

A. \( \mathcal{G} \)-consistent linear subspaces

Properties of the set of admissible initial states \( \mathcal{S} \) are closely related to properties of the resulting dynamical maps \( \tau_U^S \), and it is reasonable to wonder if there are either necessary or quite common properties of subsystem dynamics that should be incorporated into the framework we are developing. One such property, nearly ubiquitous in the open quantum system literature (see [14] for a notable exception we discuss in Section III.C), is that of convex-linearity of \( \tau_U^S \), i.e., the property that for any \( \rho_s, \sigma_s \in \mathcal{S} \), and any \( \alpha \in [0,1] \),

\[
\tau_U^S(\alpha \rho_s + (1-\alpha)\sigma_s) = \alpha \tau_U^S(\rho_s) + (1-\alpha)\tau_U^S(\sigma_s). \tag{4}
\]

Clearly, this statement about \( \tau_U^S \) requires that \( \Tr_B \mathcal{S} \) be a convex set. We will go a bit further and assume that \( \mathcal{S} \) itself is a convex subset of \( B(\mathcal{H}_s \otimes \mathcal{H}_B) \). Any subset \( \mathcal{S} \subset \mathcal{D}_B \) may be uniquely extended to the \( \mathbb{C} \)-linear subspace

\[
\mathcal{V} = \text{Span}_{\mathbb{C}} \mathcal{S} \subset B(\mathcal{H}_s \otimes \mathcal{H}_B) \tag{5}
\]

and the maps \( \Tr_B \big|_{\mathcal{S}} \) and \( \mathcal{A}_U \big|_{\mathcal{S}} \) may be likewise uniquely extended by linearity to \( \Tr_B \big|_{\mathcal{V}} \) and \( \mathcal{A}_U \big|_{\mathcal{V}} \). By assuming that \( \mathcal{S} \) is convex and \( \mathcal{G} \)-consistent, we may similarly extend the maps \( \tau_U^S : \Tr_B \mathcal{S} \to \mathcal{D}_s \) to \( \mathbb{C} \)-linear maps \( \Psi_U^S : \mathcal{V} \to B(\mathcal{H}_s) \) for all \( U \in \mathcal{G} \). It should be emphasized that, while \( \tau_U^S \) is defined only on states, the map \( \Psi_U^S \) is defined on the linear operator space \( \mathcal{V} \subset B(\mathcal{H}_s) \) which contains both states, and non-state operators.

It is readily verified that the subspace \( \mathcal{V} \) constructed as the convex span of a convex, \( \mathcal{G} \)-consistent \( \mathcal{S} \) exhibits the following properties:

1. \( \mathcal{V} \subset B(\mathcal{H}_s \otimes \mathcal{H}_B) \) is a \( \mathbb{C} \)-linear subspace.
2. \( \mathcal{V} \) is spanned by states, i.e., \( \text{Span}_{\mathbb{C}} (\mathcal{D}_B \cap \mathcal{V}) = \mathcal{V} \).
3. \( \mathcal{V} \) is \( \mathcal{G} \)-consistent, i.e., if \( X, Y \in \mathcal{V} \) are such that \( \Tr_B X = \Tr_B Y \) and if \( U \in \mathcal{G} \), then \( \Tr_B(U X U^\dagger) = \Tr_B(U Y U^\dagger) \).

Any \( \mathbb{C} \)-linear subspace \( \mathcal{V} \subset B(\mathcal{H}_s \otimes \mathcal{H}_B) \) will be called a \( \mathcal{G} \)-consistent subspace if it satisfies these three properties. Figure 2 illustrates how a \( \mathcal{G} \)-consistent subspace relates to \( \mathcal{D}_B \).

Note that it is implied by \( \mathbb{C} \)-linearity of \( \mathcal{V} \) and property 2 that \( \mathcal{V} \) is self-adjoint, i.e., \( X \in \mathcal{V} \) implies \( X^\dagger \in \mathcal{V} \). Note further that any \( \mathbb{C} \)-linear subspace \( \mathcal{V} \subset B(\mathcal{H}_s \otimes \mathcal{H}_B) \) which is self-adjoint and spanned by states is always \( \mathcal{G} \)-consistent when \( \mathcal{G} \) is the trivial group \( \mathcal{G} = \{ \mathbb{1} \} \).

Within any \( \mathbb{C} \)-linear subspace \( \mathcal{V} \subset B(\mathcal{H}_s \otimes \mathcal{H}_B) \), we may identify a further subspace \( \mathcal{V}_0 \subset \mathcal{V} \) by

\[
\mathcal{V}_0 := \ker( \Tr_B \big|_{\mathcal{V}} ) = \{ X \in \mathcal{V} : \Tr_B(X) = 0 \}. \tag{6}
\]
Then property [3] $G$-consistency, is equivalent to the property that $G \cdot Y_0 \subset \ker \Tr_B$, i.e., for any $X \in Y_0$ and $U \in G$, $\Tr_B(U X U^\dagger) = 0$. In other words, differences in states with the same reduced state cannot play any role in the final reduced state after evolution: $\Tr_B(\rho - \sigma) = 0$ for $\rho, \sigma \in V \cap D_{an}$ must imply $\Tr_B(U(\rho - \sigma) U^\dagger) = 0$ for all $U \in G$. The kernel of this idea goes back at least to [11] (section H.C), and possibly earlier.

B. Is dynamical map composition meaningful?

It should be noted that, for a given $G$-consistent subspace $V$, the dynamical maps $\{\Psi_U^V : U \in G\}$ do not comprise a semigroup in general. The reason is that $\Ad_U V$ need not be $V$ for all $U \in G$, so that the image of $\Psi_U^V$ need not lie in the domain of $\Psi_{U'}^V$ for all $U, U' \in G$. As a result, the composition $\Psi_{U'}^V \circ \Psi_U^V$ may be meaningless. If we consider the set of all dynamical maps for all $G$-consistent subspaces $V$, this problem of matching images and domains becomes even more of a challenge. Instead, we may contemplate a subclass of $G$-consistent subspaces which demonstrate the “internal consistency” [4] necessary to compose (or almost compose) the maps $\{\Psi_U^V : U \in G\}$. First, observe that a strict semigroup property for the subsystem dynamics defined by $V$ requires that the set of system-bath states after evolution by $U^\dagger$ be the same as the set of system-bath states before evolution by $U^\dagger$, for any $U, U' \in G$. Thus, it must hold that $\Ad_U V = V$ for all $U \in G$, i.e., that $V$ is an invariant subspace for the semigroup $G$. However, we may also consider a type of “weak invariance” as follows. Suppose that $G$ admits a subsemigroup $G'$ such that any open neighborhood of $G'$ in $G$ generates $G$. Suppose further that the $G'$-consistent subspace $V$ is $G'$-invariant. Then we may say that $V$ is weakly $G'$-invariant. For example, a Kraus subspace $V = B(H_S) \otimes \rho_0$ is $U(\mathbb{H}_S \otimes \mathbb{H}_B)$-consistent and $U(\mathbb{H}_S) \otimes \mathbb{1}$-invariant, and so may be considered weakly $U(\mathbb{H}_S \otimes \mathbb{H}_B)$-invariant. The point is that a $U \in G$ sufficiently close to $G'$ will leave $V$ almost invariant. A complete return to $V$ after application of $\Ad_U V$ can be achieved by a relaxation (i.e. a projection $P_Y$) back to $V$. Assuming a sufficiently strong, ongoing relaxation process back to $V$, the system-bath dynamics are no longer strictly unitary, but may be approximated by products $\prod_{i=1}^n P_Y U_i$, where $n$ is large and the $\{U_i\} \subset G$ are sufficiently close to $G'$. In this way, some of the construction of dynamical semigroups in the weak-coupling limit may be reproduced (at least abstractly) for more general weak links.

C. Assignment maps

Given a $G$-consistent subspace $V$, consider the quotient space $V/Y_0$ in which each element corresponds to an affine subspace of the form $X + Y_0 \subset V$ for some $X \in V$. Because of linearity, property [3] above is equivalent to the statement that if $Y, Z \in V$ are such that $\Tr_B Y = \Tr_B Z$, then $\Tr_B(U Y U^\dagger) = \Tr_B(U Z U^\dagger)$ for all $U \in G$. It follows that the linear maps $\Tr_B : V \rightarrow V / Y_0$ and $\Ad_U : V \rightarrow B(\mathbb{H}_S \otimes \mathbb{H}_B)$ uniquely define linear maps $\Tr_B : V / Y_0 \rightarrow V / Y_0$ and $\Ad_U : V / Y_0 \rightarrow B(\mathbb{H}_S \otimes \mathbb{H}_B) / \ker \Tr_B$. Moreover, the map $\Tr_B : V / Y_0 \rightarrow V / Y_0$ is a one-to-one correspondence, and so may be inverted, yielding the “assignment map” $\mathcal{A}_V$.

Definition 3 (Assignment Map). The assignment map $\mathcal{A}_V : \Tr_B V \rightarrow V / Y_0$ is defined by

$$\mathcal{A}_V(X) = \left( \Tr_B |_V \right)^{-1}(X) = \{ Y \in V : \Tr_B Y = X \}. \tag{7}$$

for any $X \in \Tr_B V$. For each $X \in \Tr_B V$, the output $\mathcal{A}_V(X)$ may be thought of either as an affine subspace of $V$, or as a point in the vector space $V/Y_0$.

This definition of the assignment map is a generalization of the original definition introduced by Pechukas [4]. A Pechukas-like assignment map (which assigns to each operator in $\Tr_B V$ a unique operator in $V$, rather than an affine subspace) is recovered if and only if the $G$-consistent subspace $V$ is $U(\mathbb{H}_S \otimes \mathbb{H}_B)$-consistent, as demonstrated in the following lemma.

Lemma 1. Let $V \subset B(\mathbb{H}_S \otimes \mathbb{H}_B)$ be a $C$-linear, self-adjoint subspace which is spanned by states, i.e., a $\{I\}$-consistent subspace. $V$ is $U(\mathbb{H}_S \otimes \mathbb{H}_B)$-consistent if and only if $\Tr_B |_V$ is injective, i.e., $\nu_0 := \ker \left( \Tr_B |_V \right) = \{0\}$. In this case, $V/Y_0 = V$, so that the assignment map carries each operator in $\Tr_B V$ to a unique operator in $V$, as in Pechukas’ original definition of the assignment map [4].

Proof. First, observe that if $\nu_0 = \{0\}$, then $G \cdot \nu_0 = \nu_0 \subset \ker \Tr_B$ for any subsemigroup $G \subset U(\mathbb{H}_S \otimes \mathbb{H}_B)$. Therefore $V$ is $G$-consistent for any $G \subset U(\mathbb{H}_S \otimes \mathbb{H}_B)$, and, in particular, is $U(\mathbb{H}_S \otimes \mathbb{H}_B)$-consistent. On the other hand, if $V = U(\mathbb{H}_S \otimes \mathbb{H}_B)$-consistent, then $\Span_C \left( \left\{U(\mathbb{H}_S \otimes \mathbb{H}_B) \cdot \nu_0 \right\} \subset \ker \Tr_B$ is an invariant subspace of the adjoint representation of $U(\mathbb{H}_S \otimes \mathbb{H}_B)$ on $\Sl(\mathbb{H}_S \otimes \mathbb{H}_B) = \ker \Tr$ [the zero trace operators in $B(\mathbb{H}_S \otimes \mathbb{H}_B)$]. Since ker $\Tr_B$ is a proper subspace of ker $\Tr$ (assuming $\dim \mathbb{H}_S > 1$), this invariant subspace generated by $\nu_0$ cannot be all of $\Sl(\mathbb{H}_S \otimes \mathbb{H}_B)$. The “only if” part of the lemma then follows from the irreducibility of the adjoint representation of $U(\mathbb{H}_S \otimes \mathbb{H}_B)$ on $\Sl(\mathbb{H}_S \otimes \mathbb{H}_B)$ [13] [2.4.4], which implies that the invariant subspace must be $\{0\}$, and therefore $\nu_0 = \{0\}$.

D. How constrained are the domains of dynamical maps?

Some authors have expressed a desire to keep $\Tr_B V = B(\mathbb{H}_S)$, so that the domain of the dynamical maps $\Psi_U^V$ is unconstrained. Lemma [1] shows that, when $G$ is all of $U(\mathbb{H}_S \otimes \mathbb{H}_B)$, any $G$-consistent subspace must be severely constrained to have dimension no higher than $B(\mathbb{H}_S)$. In Lemma [2] we demonstrate that, if $G$ contains non-local (entangling) operators, then any $G$-consistent subspace $V$ must necessarily be a proper subspace of $B(\mathbb{H}_S \otimes \mathbb{H}_B)$, and therefore is constrained in some way. Specifically, Lemma [2] says that if $G$ contains non-local unitaries, and $V = B(\mathbb{H}_S \otimes \mathbb{H}_B)$, then $\nu_0 = ker \Tr_B$ is such that $G \cdot \nu_0 \not\subset ker \Tr_B$, so that $V$ is not $G$-consistent.
Thus, the only $G$-consistent subspaces must be proper subspaces of $B(H_S \otimes H_B)$.

**Lemma 2.** The full kernel

$$\ker \text{Tr}_B := \{ X \in B(H_S \otimes H_B) : \text{Tr}_B X = 0 \}$$

(8)
is $G$-invariant if and only if $G$ is a semigroup of the group $U(H_S) \otimes U(H_B)$ of local unitary operators.

(essentially this same question, posed and answered in different language, was the subject of [19].)

**Proof.** First, observe that “if” is trivial, i.e., if $U \in U(H_S) \otimes U(H_B)$, then $A \otimes 1 \subset \ker \text{Tr}_B$ since, if $U = U_s \otimes U_b$, then $\text{Tr}_B(U X U^\dagger) = U_s[ \text{Tr}_B X ] U_b^\dagger = 0$ for any $X \in \ker \text{Tr}_B$.

To prove “only if”, i.e., if $A \otimes 1 \subset \ker \text{Tr}_B$, then $U \in U(H_S) \otimes U(H_B)$, note first that $B(H_S) \otimes 1$ is the orthogonal complement to $\ker \text{Tr}_B$ in the Hilbert-Schmidt geometry. This may be seen from the fact that $B(H_S) \otimes 1 = B(H_S) \otimes 1 \oplus \ker \text{Tr}_B$ since any $X \in B(H_S) \otimes 1$ may be uniquely decomposed as $X = X_{ker} \otimes 1 + X_{ker}$ where $X_{ker} = X - \text{Tr}_B(X) \otimes 1 / d_B \in \ker \text{Tr}_B$, and $B(H_S) \otimes 1$ and $\ker \text{Tr}_B$ are orthogonal subspaces [if $A \in B(H_S)$ and $X \in \ker \text{Tr}_B$, then $\langle A \otimes 1, X \rangle_{HS} = \langle A, \text{Tr}_B X \rangle_{HS} = 0$].

Now, $U \in U(H_S) \otimes U(H_B)$ satisfies $A \otimes 1 \subset \ker \text{Tr}_B$ if and only if $0 = (A \otimes 1) \text{HS} = \langle X, A \otimes 1 \rangle_{HS}$ for all $X \in \ker \text{Tr}_B$ and $A \in B(H_S)$, i.e., if and only if $A \otimes 1 \subset B(H_S) \otimes 1 \subset B(H_S) \otimes \ker \text{Tr}_B$. In other words, the condition is that, for any $A \in B(H_S)$, there exists $B \in B(H_B)$ such that $U^\dagger (A \otimes 1) U = B \otimes 1$. This condition implies, in particular, that $U$ belongs to the normalizer of $U(H_S) \otimes 1$, which is $U(H_S) \otimes U(H_B)$ [20].

It follows from Lemma 2 that any $C$-linear subspace $\mathcal{V} \subset B(H_S \otimes H_B)$ which is spanned by states (i.e., a $1$-consistency subspace) is $U(H_S) \otimes U(H_B)$-consistent. This is because, for any such $\mathcal{V}$, $V_0 = \ker (\text{Tr}_B |_{\mathcal{V}}) \subset \ker \text{Tr}_B$ is mapped by $U(H_S) \otimes U(H_B)$ into $\ker \text{Tr}_B$ by (Lemma 2), which is the condition for $U(H_S) \otimes U(H_B)$-consistency. In particular, for any $G \subset U(H_S) \otimes U(H_B)$, $\mathcal{V} = B(H_S \otimes H_B)$ is a valid $G$-consistent subspace and always gives rise to completely positive dynamics [21], since for any $U = U_s \otimes U_b \in G$, $\Psi^H_{\mathcal{V}}(X) = U_s X U_b^\dagger$ for any $X \in \mathcal{V}$, which is trivially completely positive.

With $G$-consistency imposing such strong constraints on the space of admissible system-bath operators, the desire to keep the domain of the dynamical maps unconstrained seems misplaced. If it is reasonable to promise that the initial system-bath state will never lie outside a low-dimensional subspace $\mathcal{V}$, it should be reasonable to also promise that the initial system state will never lie outside a proper subspace $\text{Tr}_B \mathcal{V} \subset B(H_S)$. Building on this observation we now proceed to identify a unique linear dynamical map for the subsystem.

**E. Linear Dynamical Maps from $G$-Consistent Subspaces**

Note that the quotient space $\mathcal{V}/\mathcal{V}_0$ admits some additional structure that will be useful in characterizing the assignment map. First, observe that $\mathcal{V}_0$ is a self-adjoint $C$-linear subspace. An affine subspace $X + \mathcal{V}_0 \subset \mathcal{V}/\mathcal{V}_0$ contains a Hermitian operator if $X + \mathcal{V}_0 = Y + \mathcal{V}_0$ with $Y = Y^\dagger$, which holds if and only if the affine subspace is self-adjoint, i.e., $(X + \mathcal{V}_0)^\dagger = X + \mathcal{V}_0$. Such an affine subspace will then be spanned by Hermitian operators, i.e., if $(X + \mathcal{V}_0)^\dagger = X + \mathcal{V}_0$ and $X \subset H_X$ is the set of Hermitian elements in $X + \mathcal{V}_0$, then $X + \mathcal{V}_0$ is the $C$-affine hull of $H_X$. We may also identify a closed convex cone $(\mathcal{V}/\mathcal{V}_0)^+$ of “positive” elements in $\mathcal{V}/\mathcal{V}_0$ as the collection of affine subspaces in $\mathcal{V}/\mathcal{V}_0$ that each contain at least one positive operator. Thus $\mathcal{V}/\mathcal{V}_0$ is an ordered vector space with a conjugate-linear involution $\dagger$. The partial trace $\text{Tr}_B$ maps each affine subspace in $(\mathcal{V}/\mathcal{V}_0)^+$ to a positive operator in $\text{Tr}_B \mathcal{V}$. Likewise, we call the assignment map $A_{\mathcal{V}}: \text{Tr}_B \mathcal{V} \rightarrow \mathcal{V}/\mathcal{V}_0$ a “positive” map if $A_{\mathcal{V}}$ maps every positive operator in $\text{Tr}_B \mathcal{V}$ to an element of $(\mathcal{V}/\mathcal{V}_0)^+$.

**Lemma 3.** The assignment map $A_{\mathcal{V}} : \text{Tr}_B \mathcal{V} \rightarrow \mathcal{V}/\mathcal{V}_0$ defined in Eq. (7) is $C$-linear, $\dagger$-linear, i.e., $A_{\mathcal{V}}(X^\dagger) = A_{\mathcal{V}}(X)^\dagger$, and trace-preserving.

**Proof.** $C$-linearity of $A_{\mathcal{V}}$ follows from the linearity of $\text{Tr}_B |_{\mathcal{V}}$. Similarly, for any $X \in B(H_S \otimes H_B)$, $\text{Tr}_B(X + \mathcal{V}_0) = \text{Tr}_B(X)$ and $(X + \mathcal{V}_0)^\dagger = X^\dagger + \mathcal{V}_0$, so that $\text{Tr}_B((X + \mathcal{V}_0)^\dagger) = \text{Tr}_B(X^\dagger) = \text{Tr}_B(X)^\dagger$. If $\text{Tr}_B(X + \mathcal{V}_0) = \text{Tr}_B(Y + \mathcal{V}_0)$, then $X - Y \in \mathcal{V}_0$, so $X + \mathcal{V}_0 = Y + \mathcal{V}_0$. Therefore $A_{\mathcal{V}}(\text{Tr}_B(X^\dagger))$ can only be $X^\dagger + \mathcal{V}_0 = A_{\mathcal{V}}(\text{Tr}_B(X))^\dagger$, so that $A_{\mathcal{V}}$ is $\dagger$-linear. Finally, $\text{Tr}(A_{\mathcal{V}}(X)) = \text{Tr}(\text{Tr}_B[A_{\mathcal{V}}(X)]) = \text{Tr}(X)$ for any $X \in \text{Tr}_B \mathcal{V}$, so that $A_{\mathcal{V}}$ is trace-preserving.

We are now able to define a unique $C$-linear dynamical map for the subsystem, presented in the following lemma.

**Lemma 4.** Let $\mathcal{V} \subset B(H_S \otimes H_B)$ be a $G$-consistent subspace. For any $U \in G$ there is a unique map $\Psi_{\mathcal{V}}^U : \text{Tr}_B \mathcal{V} \rightarrow B(H_S)$ such that the diagram in Fig. 3 commutes, i.e., such that, for any operator $X \in \mathcal{V}$, $\Psi_{\mathcal{V}}^U(\text{Tr}_B X) = \text{Tr}_B(U X U^\dagger)$. That map $\Psi_{\mathcal{V}}^U : \text{Tr}_B \mathcal{V} \rightarrow B(H_S)$, given by $\Psi_{\mathcal{V}}^U = \text{Tr}_B \circ A_{\mathcal{V}}$, is $C$-linear, $\dagger$-linear, and trace-preserving over the domain $\text{Tr}_B \mathcal{V}$, and acts as the dynamical map for system states in $\text{Tr}_B(\mathcal{D}_{5b} \cap \mathcal{V}) \subset \mathcal{D}_5$. 

FIG. 3. For any $G$-consistent subspace $\mathcal{V}$ and unitary evolution operator $U \in G \subset U(H_S \otimes H_B)$, this commutative diagram uniquely defines the $C$-linear, $\dagger$-linear, trace-preserving map $\Psi_{\mathcal{V}}^U = \text{Tr}_B \circ A_{\mathcal{V}}$, which acts as the time evolution operator for system states in $\text{Tr}_B(\mathcal{D}_{5b} \cap \mathcal{V}) \subset \mathcal{D}_5$. $G$-consistency is the condition that, for every $U \in G$, a map $A_{\mathcal{V}} : \mathcal{V} \rightarrow B(H_S \otimes H_B)/\ker \text{Tr}_B$ exists that makes the top rectangle commute.
The uniqueness of $\Psi_U^V$ is due to the uniqueness of the assignment map $A_V$ and the maps $\Ad_U^0 : V / V_0 \to \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B)$ and $\Tr^0 : \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B) / \ker \Tr_B \to \mathcal{B}(\mathcal{H}_S)$. Since these maps are all $\mathbb{C}$-linear and $\dagger$-linear, $\Psi_U^V$ is as well. The $\dagger$-linearity of $\Psi_U^V$ is equivalently expressed as $\Psi_U^V$ being Hermiticity-preserving, which is sometimes shortened to just “Hermitian”. For any state $\rho_S \in \Tr_B(\mathcal{D}_S \cap V)$, the affine subspace $A_V(\rho_S)$ must contain a state in $\mathcal{D}_S$, so the transformation $\rho_S \mapsto \Psi_U^V(\rho_S)$ reflects the unitary evolution of a valid system-bath state in $V$, and therefore $\Psi_U^V$ is the dynamical map for such a state.

**Definition 4 (Physical Domain).** We call the convex set $\Tr_B(\mathcal{D}_S \cap V) \subset \mathcal{D}_S$ the physical domain because, as described in Lemma 4, these are the system states for which the maps $\Psi_U^V$ act as the physical dynamical maps. This is called the “compatibility domain” in [6]. The “promise” that initial system-bath states will lie in $V \cap \mathcal{D}_S$ implies a “promise” that initial system-states will lie in the physical domain.

We stress a few key points concerning this construction:

1. The linearity of the maps $A_V$ and $\Psi_U^V$ is due to our choice to assume that the set of admissible initial system-bath states belong to a linear subspace. Depending on the physical processes involved, other choices are possible, leading to non-linear assignment maps and non-linear evolution operators $\Psi_U^V$ [2, 17, 22].

2. The map $\Psi_U^V$ does not generally act as the evolution operator for all system states, or even for all system states in $\mathcal{D}_S \cap \Tr_B V$. Any state $\rho \in \mathcal{D}_S \cap \Tr_B V$ which does not lie in the convex “physical domain” $\Tr_B(\mathcal{D}_S \cap V)$ is mapped by the assignment map to an affine subspace of $\mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B)$ which contains no valid system-bath states in $\mathcal{D}_S$. Since the transformation by $\Psi_U^V$ of such a system state is not tied to the unitary evolution of a valid system-bath state, it is empty of any physical meaning. Such a map $\Psi_U^V$ should never be described without clearly indicating the physical domain of system states on which it can meaningfully be applied. To apply the map outside this domain is to overinterpret the mathematics.

3. For each $U \in \mathcal{G}$, the definition of $\Psi_U^V$ depends entirely upon the $\mathcal{G}$-consistent subspace $V$. Different consistent subspaces, even if they include some shared fiducial state $\rho_{SB}$, will yield different maps $\Psi_U^V$.

We illustrate these concepts in Fig. 4.

**F. Operator Sum Representations**

The map $\Psi_U^V$ may be extended to a $\mathbb{C}$-linear, $\dagger$-linear, trace-preserving map $\Phi_U^V$ on $\mathcal{B}(\mathcal{H}_S)$, for example by defining $\Phi_U^V$ to be zero on $\mathcal{B}(\mathcal{H}_S) / \Tr_B V$, the orthogonal complement of $\Tr_B V$. Such an extension, though arbitrary, may be desirable because the resulting $\Phi_U^V$ admits an operator sum representation (OSR) with real-valued coefficients

$$\Phi_U^V(X) = \sum_k a_k E_k X E_k^\dagger;$$

for $a_k \in \mathbb{R}$ [23, 24].

In the case of an OSR such that $a_k = 1$ for all $k$, we call such a representation a Kraus OSR.

**Lemma 5.** The assignment map $A_V : \Tr_B V \to V / V_0$ associated to any $\mathcal{G}$-consistent subspace $V$ admits an OSR with real-valued coefficients of the form $A_V(X) = \sum_k a_k Q_k X Q_k^\dagger + V_0$.

**Proof.** This may be shown, for example, by extending $A_V$ to a $\mathbb{C}$-linear, $\dagger$-linear map $\tilde{A}_V : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B) / V_0$ and applying Choi’s method to $\tilde{A}_V$ as follows. First, choose orthonormal bases $\{|i\}^{d_S}_{i=1}$ and $\{|\alpha\}^{d_B}_{\alpha=1}$ for $\mathcal{H}_S$ and $\mathcal{H}_B$, observe that $\tilde{A}_V \otimes \id_{\mathcal{B}(\mathcal{H}_B)}(|\xi\rangle\langle\xi|)$ is spanned by Hermitian operators, where $|\xi\rangle = \sum_i |i\rangle \otimes |\alpha\rangle \in \mathcal{H}_S \otimes \mathcal{H}_B$. Choose any Hermitian operator $A$ in this space, choosing $A \geq 0$ if possible, and eigendecompose

$$\tilde{A}_V \otimes \id_{\mathcal{B}(\mathcal{H}_B)}(|\xi\rangle\langle\xi|) \ni A = \sum_k a_k |q_k\rangle\langle q_k|.$$  

The operators $Q_k \in \mathcal{B}(\mathcal{H}_S; \mathcal{H}_S \otimes \mathcal{H}_B) \simeq \mathcal{H}_S \otimes \mathcal{H}_B \otimes \mathcal{H}_S^*$
defined by
\[ Q_k = \sum_{j,\alpha, i} (j, \alpha, i | q_k) | j, \alpha \rangle \langle i | \]  

yield an OSR for the extended assignment map: \( \hat{A}_V(X) = \sum_k a_k Q_k X E_k^\dagger \). These \( Q_k \) are essentially partial transposes of the \( | q_k \rangle \in H_\mathcal{S} \otimes H_\mathcal{B} \otimes H_\mathcal{S} \), transforming the “column vectors” in the second \( H_\mathcal{S} \) to “row vectors” in \( H_\mathcal{S} \).

It follows that \( \Phi^\dagger_V(X) = \sum_{k,\alpha} a_k E_{k\alpha} X E_{k\alpha}^\dagger \), where \( E_{k\alpha} = (\alpha | U Q_k \in B(H) \). In special cases, this general recipe may not be the most efficient for obtaining an OSR. For example, in the case of a Kraus map where \( A_V(X) = X \otimes \rho_\mathcal{B} \) and \( \rho_\mathcal{B} = \sum_\alpha \rho_\alpha | \alpha \rangle \langle \alpha | \), an OSR for \( A_V \) is given by the operators \( Q_\alpha = \sqrt{\rho_\alpha} \otimes | \alpha \rangle \), so that \( \Phi^\dagger_V(X) = \sum_{\alpha,\beta} E_{\alpha,\beta} X E_{\alpha,\beta}^\dagger \) where \( E_{\alpha,\beta} = | \beta \rangle U Q_\alpha \). It bears repeating that, regardless of these choices for constructing the OSR, the resulting map \( \Phi^\dagger_V \) only acts as the subsystem dynamical map due to \( U \) on system states in the “physical domain” \( Tr_\mathcal{B}(D_{3\mathcal{B}} \cap \mathcal{V}) \subset D_\mathcal{S} \).

IV. COMPLETE POSITIVITY

Having described, for each allowed joint system-bath unitary evolution \( U \in \mathcal{G} \), the unique dynamical map \( \Psi_U : Tr_\mathcal{B} \mathcal{V} \rightarrow B(H_\mathcal{S}) \) which is \( \mathbb{C} \)-linear, \( \dagger \)-linear, and trace-preserving, we turn to the question of complete positivity.

A. Notions of Complete Positivity

Because \( A_V \) and \( \Psi_U \) are generally defined on a subspace of \( B(H_\mathcal{S}) \), rather than the full algebra, we may consider three generally nonequivalent definitions of complete positivity of the subsystem dynamics; one which is essentially the original definition of Stinespring [25], and two more that we introduce.

Definition 5 (Complete Positivity). Let \( \mathcal{K} \) and \( \mathcal{H} \) be Hilbert spaces and let \( \mathcal{R} \subset B(\mathcal{K}) \) be a self-adjoint \( \mathbb{C} \)-linear subspace spanned by positive operators. A \( \mathbb{C} \)-linear, \( \dagger \)-linear map \( F : \mathcal{R} \rightarrow B(\mathcal{H}) \) is

1. Completely Positive (CP) [25] if \( F \otimes \text{id} : \mathcal{R} \otimes B(H_\mathcal{W}) \rightarrow B(\mathcal{H} \otimes H_\mathcal{W}) \) is a positive map for all finite dimensional Hilbert spaces \( H_\mathcal{W} \), i.e., every positive operator in

\[ \mathcal{R} \otimes B(H_\mathcal{W}) = \text{Span}_\mathbb{C}\{A \otimes B : A \in \mathcal{R}, B \in B(H_\mathcal{W})\} \]  

is mapped to a positive operator in \( B(\mathcal{H} \otimes H_\mathcal{W}) \). \( H_\mathcal{W} \) may be thought of as the state space of a non-interacting, non-evolving “witness” system;

2. Completely Positively Extensible (CPE) if \( F \) admits a completely positive extension \( \hat{F} : B(\mathcal{K}) \rightarrow B(\mathcal{H}) \), i.e., if there exists a CP map \( \hat{F} : B(\mathcal{K}) \rightarrow B(\mathcal{H}) \) such that \( \hat{F}|_\mathcal{R} = F \);  

3. Completely Positively Zero Extensible (CPZE) if \( F \circ P_\mathcal{R} : B(\mathcal{K}) \rightarrow B(\mathcal{H}) \) is completely positive, where \( P_\mathcal{R} : B(\mathcal{K}) \rightarrow \mathcal{R} \) is the orthogonal projection onto \( \mathcal{R} \) with respect to the Hilbert-Schmidt inner product.

It should be noted that a CPE map exhibits the property that the CP extension, being a CP map defined on the entire algebra \( B(H_\mathcal{S}) \), admits a Kraus OSR [24, 26]. As such, the existence of a Kraus OSR for the map \( F : \mathcal{R} \rightarrow B(\mathcal{H}) \) is an alternate characterization of CPE-ness.

It is straightforward to see that if a map \( F : \mathcal{R} \rightarrow B(\mathcal{H}) \) is CPZE, then it is CPE, since the zero extension is just one possible extension of \( F \). Likewise if a map \( F : \mathcal{R} \rightarrow B(\mathcal{H}) \) is CPE, it is also CP, since restricting the CP extension \( \hat{F} : B(\mathcal{K}) \rightarrow B(\mathcal{H}) \) to the subspace \( \mathcal{R} \subset B(\mathcal{H}) \) does not break complete positivity. The following two theorems, translated into this terminology of consistent subspaces and CP/CPE/CPZE maps, may be considered partial converses of these statements.

Theorem 1 (Arveson [27]). If \( \mathcal{V} \in Tr_\mathcal{V} \), then every CP map with domain \( Tr_\mathcal{V} \) can be extended to a CP map on \( B(H_\mathcal{S}) \), i.e., every CP map on \( Tr_\mathcal{V} \) is CPE.

Theorem 2 (Choi & Effros [28]). If \( Tr_\mathcal{V} \) is a unital \( C^* \)-subalgebra of \( B(H_\mathcal{S}) \), then the orthogonal projection \( P_{Tr_\mathcal{V}} \) is completely positive, and therefore every CP map with domain \( Tr_\mathcal{V} \) is CPZE.

That the property of being CPE does not necessarily imply CPZE-ness is demonstrated by the following counterexample.

Example 2. Suppose \( H_\mathcal{S} \) is 3-dimensional with orthonormal basis \( \{ | i \rangle \}_{i=0}^7 \). Fix some \( \rho \in D_\mathcal{B} \) and let

\[ \mathcal{V} = \text{Span}_\mathbb{C}\{|i\rangle \otimes \rho : (0,1) \neq (i,j) \neq (1,0)\}. \]  

The assignment map \( A_V \) associated with this \( \mathcal{V} \) is CPE but not CPZE, demonstrating that the hypothesis of Theorem 2 that \( Tr_\mathcal{V} \) be a subalgebra of \( B(H_\mathcal{S}) \) is necessary.

Proof. \( Tr_\mathcal{V} \) is given by

\[ Tr_\mathcal{V} \mathcal{V} = \text{Span}_\mathbb{C}\{|i\rangle \otimes \rho : (0,1) \neq (i,j) \neq (1,0)\}. \]  

The orthogonal projection onto \( Tr_\mathcal{V} \mathcal{V} \) has Choi matrix

\[ \text{Choi}(P_{Tr_\mathcal{V}} \mathcal{V}) = \sum_{0 \leq i, j < 2} |i\rangle \langle i| \otimes |j\rangle \langle j|. \]  

This matrix has eigenvalue \( -|00\rangle - |11\rangle + \sqrt{2} |22\rangle \) with eigenvalue \( 1 - \sqrt{2} < 0 \), so it is not positive, and therefore \( P_{Tr_\mathcal{V}} \mathcal{V} \) is not CP. This demonstrates that there exist self-adjoint subspaces \( Tr_\mathcal{V} \mathcal{V} \subset B(H_\mathcal{S}) \) containing the identity and which are not \( C^* \) subalgebras for which the orthogonal projection \( P_{Tr_\mathcal{V}} \mathcal{V} \) is not CP. The zero-extended assignment map \( \hat{A} = A_V \circ P_{Tr_\mathcal{V}} \mathcal{V} \) is simply \( \hat{A}(\sigma) = (P_{Tr_\mathcal{V}} \mathcal{V}(\sigma)) \otimes \rho \). Therefore, for any \( X \in B(H_\mathcal{S} \otimes H_\mathcal{W}) \),

\[ \hat{A} \otimes \text{id}_{B(H_\mathcal{W})}(X) = (P_{Tr_\mathcal{V}} \mathcal{V} \otimes \text{id}(X)) \otimes \rho, \]
so that $\tilde{A}$ is not CP because $\mathcal{P}_{\text{Tr}_B V}$ is not CP, and therefore $\mathcal{A}_V$ is not CPZE. However, for any $X \in \text{Tr}_B V \otimes \mathcal{B}(\mathcal{H}_W)$,

$$\mathcal{A}_V \otimes \text{id}(X) = X \otimes \rho,$$

which is a positive map for all finite-dimensional witnesses, so that $\mathcal{A}_V$ is CP. Moreover, since $\mathbb{1} \in \text{Tr}_B V$, Theorem 1 implies that $\mathcal{A}_V$ is CPE.

One observation to make about $\mathcal{G}$-consistency, that will be useful in applying these notions of complete positivity, is that, when a $\mathcal{G}$-consistent subspace $V$ is tensored with the operator algebra of a “witness system” as in the definition of complete positivity above, the resulting operator space is still appropriately consistent. In other words, $\mathcal{G}$-consistency is stable in the following sense:

**Lemma 6.** If $\mathcal{G} \subset U(\mathcal{H}_S \otimes \mathcal{H}_A)$ and $V \subset B(\mathcal{H}_S \otimes \mathcal{H}_A)$ is a $\mathcal{G}$-consistent subspace, then for any finite-dimensional Hilbert space $\mathcal{H}_W$, $V \otimes \mathcal{B}(\mathcal{H}_W) \subset B(\mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_W)$ is a $\mathcal{G} \otimes U(\mathcal{H}_W)$-consistent subspace. In particular, it is a $\mathcal{G} \otimes \mathbb{1}_{\mathcal{H}_W}$-consistent subspace.

**Proof.** First, observe that, if $V \otimes \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_A)$ is a $\mathcal{G}$-consistent subspace, then it is $\mathcal{C}$-linear and spanned by states. It follows that $V \otimes \mathcal{B}(\mathcal{H}_W)$ is $\mathcal{C}$-linear. Furthermore, since $V$ and $\mathcal{B}(\mathcal{H}_W)$ are each spanned by states, $V \otimes \mathcal{B}(\mathcal{H}_W)$ is spanned by states of the form $\rho \otimes \sigma$ with $\rho \in V$ and $\sigma \in \mathcal{D}_W$.

It remains only to prove $\mathcal{G} \otimes U(\mathcal{H}(\mathcal{H}_W))$-consistency. To that end, let $X \in \ker \left( \text{Tr}_B \left|_{V \otimes \mathcal{B}(\mathcal{H}_W)} \right. \right) = V_0 \otimes \mathcal{B}(\mathcal{H}_W)$, $U \in \mathcal{G}$, and $V \in U(\mathcal{H}(\mathcal{H}_W))$. $X$ may be expanded as $X = \sum_i Y_i \otimes Z_i$ where $\{Y_i\} \subset V_0$ and $\{Z_i\} \subset \mathcal{B}(\mathcal{H}_W)$. Then

$$\text{Tr}_B \left[ (U \otimes V)X(U \otimes V)^\dagger \right] = \sum_i \text{Tr}_B (U_i Y_i U_i^\dagger) \otimes (V_i Z_i V_i^\dagger) = 0$$

(18)

since $Y_i \in V_0$ and $V$ is $\mathcal{G}$-consistent. Then $(U \otimes V)X(U \otimes V)^\dagger \in \ker \left( \text{Tr}_B \left|_{\mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_W)} \right. \right)$, so $V \otimes \mathcal{B}(\mathcal{H}_W)$ is $\mathcal{G} \otimes U(\mathcal{H}(\mathcal{H}_W))$-consistent. It follows trivially that $V \otimes \mathcal{B}(\mathcal{H}_W)$ is $\mathcal{G} \otimes \mathbb{1}_{\mathcal{H}_W}$-consistent.

**B. Defining Completely Positive Assignment Maps**

Let $V$ be a $\mathcal{G}$-consistent subspace, let $V_0 = \ker \left( \text{Tr}_B \left|_{V} \right. \right)$, and let $\mathcal{H}_W$ be a finite-dimensional Hilbert space. We would like to apply the above notions of complete positivity not only to the dynamical maps $\mathcal{W}_U$, but also to the assignment map $\mathcal{A}_V : \text{Tr}_B V \rightarrow V/V_0$. To do so, we will need to identify the positive elements within $V/V_0 \otimes \mathcal{B}(\mathcal{H}_W)$. To that end, we begin by establishing a natural isomorphism between $V/V_0 \otimes \mathcal{B}(\mathcal{H}_W)$ and a space in which the positive elements are readily identifiable.

**Lemma 7.** The space $V/V_0 \otimes \mathcal{B}(\mathcal{H}_W)$ is naturally isomorphic to $V \otimes \mathcal{B}(\mathcal{H}_W)/V_0 \otimes \mathcal{B}(\mathcal{H}_W)$.

\[ \ker \left( \text{Tr}_B \left|_{\mathcal{B}(\mathcal{H}_W)} \right. \right) = V_0 \otimes \mathcal{B}(\mathcal{H}_W). \]

By invoking Lemma 7 in the same way that we defined a positive cone $(V/V_0)^+$ comprising those affine subspaces in $V/V_0$ that each contain at least one positive operator, we are now able to describe a positive cone in $V/V_0 \otimes \mathcal{B}(\mathcal{H}_W)$, thereby defining a matrix ordering for $V/V_0$.

**Definition 6.** An element in $V/V_0 \otimes \mathcal{B}(\mathcal{H}_W)$ will be considered positive, and therefore belonging to $(V/V_0 \otimes \mathcal{B}(\mathcal{H}_W))^+$, if the corresponding affine subspace in $(V \otimes \mathcal{B}(\mathcal{H}_W))/V_0 \otimes \mathcal{B}(\mathcal{H}_W)$ contains at least one positive operator.

**Lemma 8.** This matrix ordering of $V/V_0$ (and the analogous matrix ordering for $B(\mathcal{H}_S \otimes \mathcal{H}_A)/\ker \text{Tr}_B$) is such that all
maps in Figure 3 are completely positive, with the exception of $\mathcal{A}_2$ and $\Psi^+_2$. Indeed, it is the minimal matrix ordering necessary for this (minimal in the sense that the fewest elements of $\mathcal{V}/\mathcal{V}_0 \otimes \mathcal{B}(\mathcal{H}_w)$ are considered “positive”).

Proof. Fix some finite-dimensional $\mathcal{H}_w$ and let $B \in (\mathcal{V}/\mathcal{V}_0 \otimes \mathcal{B}(\mathcal{H}_w))^+$. It follows that $h^{-1}(B)$ is positive in $\mathcal{V} \otimes \mathcal{B}(\mathcal{H}_w)/\mathcal{V}_0 \otimes \mathcal{B}(\mathcal{H}_w)$ and therefore that there exists a positive $A \in \mathcal{V} \otimes \mathcal{B}(\mathcal{H}_w)$ such that $p \otimes \text{id}(A) = B$, where $p : \mathcal{V} \to \mathcal{V}/\mathcal{V}_0$ is the natural projection map. So the positive cone $(\mathcal{V}/\mathcal{V}_0 \otimes \mathcal{B}(\mathcal{H}_w))^+$ is precisely the image through $p \otimes \text{id}$ of the positive cone $(\mathcal{V} \otimes \mathcal{B}(\mathcal{H}_w))^+$. Consequently, $(\mathcal{V}/\mathcal{V}_0 \otimes \mathcal{B}(\mathcal{H}_w))^+$ is the minimal positive cone (for all $\mathcal{H}_w$) to make $p : \mathcal{V} \to \mathcal{V}/\mathcal{V}_0$ a completely positive map. As we have given $\mathcal{B}(\mathcal{H}_w \otimes \mathcal{H}_w)/\ker \text{Tr}_B$ the analogous matrix ordering, the same is true of the natural projection $p' : \mathcal{B}(\mathcal{H}_w \otimes \mathcal{H}_w) \to \mathcal{B}(\mathcal{H}_w \otimes \mathcal{H}_w)/\ker \text{Tr}_B$. Thus, minimality is proved, and it only remains to examine complete positivity of the other maps.

Suppose $B \in (\mathcal{V}/\mathcal{V}_0 \otimes \mathcal{B}(\mathcal{H}_w))^+$ is positive. Then there exists $A \in \mathcal{V} \otimes \mathcal{B}(\mathcal{H}_w)$ which is positive and is such that $p \otimes \text{id}(A) = B$. Complete positivity of $\text{Ad}_U$ and $p'$ imply that $(p' \otimes \text{id}) \circ (\text{Ad}_U \otimes \text{id})(A) = (p' \otimes \text{Ad}_U)(p \otimes \text{id}(A)) = (p' \otimes \text{Ad}_U)(A)$ is positive. But $(p' \otimes \text{Ad}_U)(p \otimes \text{id}(A)) = (\text{Ad}_U^0 \circ p') \otimes \text{id}(A) = \text{Ad}_U^0 \otimes \text{id}(B)$, so that $\text{Ad}_U^0 \otimes \text{id}$ is a positive map for all $\mathcal{H}_w$, and therefore $\text{Ad}_U^0$ is completely positive. Likewise, $\text{Tr}_B \otimes \text{id}(A) = \text{Tr}_B \otimes \text{id}(B)$ is positive, so that $\text{Tr}_B$ is completely positive. And complete positivity of $\text{Tr}_B^0$ is proved analogously: for any positive $A \in \mathcal{B}(\mathcal{H}_w \otimes \mathcal{H}_w)/\ker \text{Tr}_B$ there exists positive $A \in (p' \otimes \text{id})^{-1}(B)$, and therefore $\text{Tr}_B \otimes \text{id}(B) = \text{Tr}_B \otimes \text{id}(A)$ is positive, so that $\text{Tr}_B^0$ is completely positive.

Then $A_V : \text{Tr}_B \mathcal{V} \to \mathcal{V}/\mathcal{V}_0$ is completely positive if and only if $A_V \otimes \text{id}_{\mathcal{B}(\mathcal{H}_w)} : (\text{Tr}_B \mathcal{V}) \otimes \mathcal{B}(\mathcal{H}_w) \to \mathcal{V}/\mathcal{V}_0 \otimes \mathcal{B}(\mathcal{H}_w)$ is positive for all finite-dimensional $\mathcal{H}_w$, i.e., if and only if $A_V \otimes \text{id}_{\mathcal{B}(\mathcal{H}_w)}$ maps positive operators in $\text{Tr}_B \mathcal{V} \otimes \mathcal{B}(\mathcal{H}_w)$ to elements in $(\mathcal{V}/\mathcal{V}_0 \otimes \mathcal{B}(\mathcal{H}_w))^+$. Equivalently, $A_V$ is completely positive if and only if $\text{Tr}_B \circ (H_{2\mathcal{V}} \cap \{V \otimes \mathcal{B}(\mathcal{H}_w)\}) = D_{\mathcal{V}} \cap \{\text{Tr}_B \mathcal{V} \otimes \mathcal{B}(\mathcal{H}_w)\}$ for all finite dimensional $\mathcal{H}_w$, i.e., if and only if every system-witness state in $\text{Tr}_B \mathcal{V} \otimes \mathcal{B}(\mathcal{H}_w)$ is “covered” by a system-bath-witness state in $\mathcal{V} \otimes \mathcal{B}(\mathcal{H}_w)$.

We summarize the different notions of complete positivity and their interrelations in Fig. 6.

Determining which $\mathcal{G}$-consistent subspaces give rise to completely positive (CP, CPE, or CPZE) assignment maps is a rich and complex problem. However, as we will discuss later, when $\text{Tr}_B \mathcal{V} = \mathcal{B}(\mathcal{H}_w)$, a theorem of Pechukas [4] generalized by Jordan, et al., [6], shows that the assignment map for a $\mathcal{U}(\mathcal{H}_w \otimes \mathcal{H}_w)$-consistent subspace $\mathcal{V}$ is positive if and only if it is CPZE and $\mathcal{V}$ is of the form $\mathcal{V} = \mathcal{B}(\mathcal{H}_w) \otimes \rho_0$ for some fixed $\rho_0$ in $D_{\mathcal{V}}$. We now show that in the simple case where $\text{Tr}_B \mathcal{V}$ is one-dimensional, the question of complete positivity of the assignment map may be answered comprehensively.

Lemma 9. If $\mathcal{V} \subset \mathcal{B}(\mathcal{H}_w \otimes \mathcal{H}_w)$ is a $\mathcal{G}$-consistent subspace such that $\text{Tr}_B \mathcal{V}$ is 1-dimensional, then the assignment map $A_V : \text{Tr}_B \mathcal{V} \to \mathcal{V}/\mathcal{V}_0$ is CPZE.

Proof. Let $\rho \in D_{\mathcal{V}} \cap \mathcal{V}$ and $\rho_s = \text{Tr}_B \rho = D_{\mathcal{V}} \cap \text{Tr}_B \mathcal{V}$, so that $\text{Tr}_B \mathcal{V} = \mathcal{C}_\rho_s$. The assignment map is then $A_V(z\rho_s) = z\rho + \mathcal{V}_0$ for any $z \in \mathbb{C}$. Let $A : \mathcal{B}(\mathcal{H}_w) \to \mathcal{B}(\mathcal{H}_w \otimes \mathcal{H}_w)/\mathcal{V}_0$ be the zero-extension of $A_V$, i.e., for any $X \in \mathcal{B}(\mathcal{H}_w)$

$$A(X) = \frac{\text{Tr}_B^2(X)}{\text{Tr}_B^2(\rho_s^2)} \rho_s + \mathcal{V}_0.$$  

Let $\mathcal{H}_w$ be a finite-dimensional witness Hilbert space. Then, for any $Y \in \mathcal{B}(\mathcal{H}_w \otimes \mathcal{H}_w)$.

$$h^{-1} \circ A \otimes \text{id}(Y) = \rho \otimes \frac{\text{Tr}_B^2([\sqrt{\rho_s} \otimes 1]Y(\sqrt{\rho_s} \otimes 1))}{\text{Tr}_B^2(\rho_s^2)} + \mathcal{V}_0 \otimes \mathcal{B}(\mathcal{H}_w),$$

where $h : \mathcal{B}(\mathcal{H}_w \otimes \mathcal{H}_w) \otimes \mathcal{B}(\mathcal{H}_w)/\mathcal{V}_0 \otimes \mathcal{B}(\mathcal{H}_w) \to \mathcal{B}(\mathcal{H}_w \otimes \mathcal{H}_w)/\mathcal{V}_0 \otimes \mathcal{B}(\mathcal{H}_w)$ is the isomorphism constructed analogously to that in Lemma 7. It follows that $A \otimes \text{id}_{\mathcal{B}(\mathcal{H}_w)}$ is a positive map for any finite-dimensional witness $\mathcal{H}_w$. Thus, $A_V$ is CPZE.

It is an open problem to find some generalizations of these results to arbitrary $G$-consistent subspaces. In particular, Example 2 shows that Lemma 8 does not extend to higher dimensional $\text{Tr}_B \mathcal{V}$ without modification.

C. Why Complete Positivity?

Definition 5 advocates different notions of complete positivity. In order to properly address the question of complete positivity of system dynamics, we must first ask ourselves: what aspects of complete positivity make it a physically interesting property? There may be several valid answers to this...
\[ \mathcal{V}/\mathcal{V}_0 \otimes \mathcal{B}(\mathcal{H}_w) \xrightarrow{A_{V} \otimes \text{id}} \mathcal{B}(\mathcal{H}_w \otimes \mathcal{H}_w) \]
\[ \mathcal{A}_{V} \otimes \text{id} \geq 0 \]
\[ \text{Tr}_w \mathcal{V} \otimes 1 \otimes \mathcal{B}(\mathcal{H}_w) \]
\[ \mathcal{V} \otimes (\mathcal{W} \otimes \mathcal{B}(\mathcal{H}_w)) \]
\[ \mathcal{A}_{V} \otimes \text{id} \geq 0 \]
\[ \text{Tr}_w \mathcal{V} \otimes 1 \otimes \mathcal{B}(\mathcal{H}_w) \]

Fig. 7. It is possible for non-positivity in the assignment map (possibly tensored with the identity on a witness matrix algebra) to be hidden by the positivity of \( A_{V} \) and \( \text{Tr}_w \mathcal{V} \) for all \( U \in \mathcal{G} \), resulting in dynamical maps which appear positive or completely positive according to the mathematical definitions. However, strictly speaking, these dynamical maps do not satisfy the first answer to the question “why complete positivity?” An example is considered in Example 3.

question and it is possible that there will be no general consensus as to which is most compelling. Here are two:

1. The existence of a non-interacting, non-evolving “witness” (for which the initial system-witness state can be entangled) should not be enough to break the positivity of the evolution. This is frequently claimed as a mandate for complete positivity.

2. As mentioned above, CP maps are sometimes representable by Kraus OSRs \( \Phi(\rho) = \sum E_i \rho E_i^\dagger \), and Kraus OSRs are ubiquitous in quantum information theory. For example, such an OSR can be thought of as a convex combination \( \Phi(\rho) = \sum p_i \rho_i \) of Kraus operators of the environment, yielding outcome state \( \rho_i \) with probability \( p_i \).

1. **Witnessed Positivity**

If the importance of complete positivity in quantum dynamics is to do with maintaining positivity in the presence of a witness, i.e., point 1 above, then the key question, as we now show, is whether the assignment map is CP. If we consider some dynamical map \( \Psi'_U : \mathcal{T}_{\mathcal{V}} \mathcal{V} \rightarrow \mathcal{B}(\mathcal{H}_w) \) for \( U \in \mathcal{G} \) and a finite-dimensional witness Hilbert space \( \mathcal{H}_w \), then

\[
\Psi'_U \otimes \text{id}_{\mathcal{B}(\mathcal{H}_w)} = \text{Tr}_w \circ \text{Ad}_U \circ \text{id}_{\mathcal{H}_w} \circ (A_{V} \otimes \text{id}_{\mathcal{B}(\mathcal{H}_w)})
\]

(23)

is the dynamical map \( \Psi'_U \otimes \text{id}_{\mathcal{B}(\mathcal{H}_w)} \) for the \( \mathcal{G} \otimes 1_{\mathcal{H}_w} \)-consistent subspace \( \mathcal{V} \otimes \mathcal{B}(\mathcal{H}_w) \) (see Lemma 6). If \( \Psi'_U \otimes \text{id}_{\mathcal{B}(\mathcal{H}_w)} \) is positive for all finite-dimensional \( \mathcal{H}_w \), then, mathematically (by definition), it is considered completely positive. However, it is possible that there exists a witness state space \( \mathcal{H}_w \) such that \( \Psi'_U \otimes \text{id}_{\mathcal{B}(\mathcal{H}_w)} \) is positive, while \( A_{V} \otimes \text{id}_{\mathcal{B}(\mathcal{H}_w)} \) is non-positive, as illustrated in Fig. 7.

**Observation 1**. In such a situation (where \( \Psi'_U \otimes \text{id}_{\mathcal{B}(\mathcal{H}_w)} \geq 0 \) but \( A_{V} \otimes \text{id}_{\mathcal{B}(\mathcal{H}_w)} \nless 0 \)), the positivity of \( \Psi'_U \otimes \text{id}_{\mathcal{B}(\mathcal{H}_w)} \) should be considered “non-physical” as it arises from assigning some states in \( \mathcal{T}_{\mathcal{V}} \mathcal{V} \otimes \mathcal{B}(\mathcal{H}_w) \) to non-positive operators in \( \mathcal{V} \otimes \mathcal{B}(\mathcal{H}_w) \) and “evolving” those non-positive operators.

If the desire was to show that all system-witness states in \( \mathcal{T}_{\mathcal{V}} \mathcal{V} \otimes \mathcal{B}(\mathcal{H}_w) \cap \mathcal{D}_{\mathcal{V}} \) evolve to states in \( \mathcal{D}_{\mathcal{V}} \), then we have not fulfilled this goal. It is only satisfied if the assignment map \( A_{V} \otimes \text{id}_{\mathcal{B}(\mathcal{H}_w)} \) is itself a positive map. This obviously holds for all finite dimensional witnesses \( \mathcal{H}_w \) if and only if \( A_{V} \) is completely positive (CP).

We now give an explicit example of such non-physical complete positivity. This is a special case of a class of examples considered in [29] to illustrate non-positive assignment maps.

**Example 3**. Let \( d_s = \dim \mathcal{H}_s, d_h = \dim \mathcal{H}_w, \rho = \frac{1}{d_s} \mathcal{I} \), and \( \sigma = |\psi\rangle \langle \psi| \otimes |\phi\rangle \langle \phi| \) for any states \( |\psi\rangle \in \mathcal{H}_s \) and \( |\phi\rangle \in \mathcal{H}_w \). Then \( \mathcal{V} = \text{Span}_{\mathcal{G}_w} \{ \rho, \sigma \} = \mathcal{U}(\mathcal{H}_s \otimes \mathcal{H}_w) \)-consistent subspace.

The corresponding assignment map \( A_{V} \) is not positive (see also Fig. 5), but \( \Psi_U = \mathcal{T}_{\mathcal{V}} \circ \text{Ad}_U \circ A_{V} \) is CPZE for all \( U \in \mathcal{U}(\mathcal{H}_s \otimes \mathcal{H}_w) \).

**Proof**. First, note that \( A_{V} \) is not positive, because for \( a \geq 0 \) and \( -\frac{a}{d_s^2} \leq b < -\frac{a}{d_s^4} \), \( a \rho + b \sigma \geq 0 \) but \( \mathcal{T}_{\mathcal{V}}(a \rho + b \sigma) = \frac{a}{d_s^2} \mathcal{I} + b |\psi\rangle \langle \psi| \geq 0 \).

On the other hand, let \( \mathcal{P} : \mathcal{B}(\mathcal{H}_s) \rightarrow \mathcal{T}_{\mathcal{V}} \mathcal{V} \) via the orthogonal projection onto \( \mathcal{T}_{\mathcal{V}} \mathcal{V} \) and let \( \{ |i\rangle \} \) be an orthonormal basis for \( \mathcal{H}_s \) with \( |1\rangle = |\psi\rangle \). Then \( \mathcal{P}(|i\rangle \langle j|) = 0 \) for \( i \neq j \), \( \mathcal{P}(|1\rangle \langle 1|) = |1\rangle \langle 1| \), and \( \mathcal{P}(|i\rangle \langle i|) = (1 - |1\rangle \langle 1|)/(d_s - 1) \) for \( i \neq 1 \). For any \( U \in \mathcal{U}(\mathcal{H}_s \otimes \mathcal{H}_w) \), the Choi matrix for the map \( \Psi'_U \circ \mathcal{P} \) is

\[
\mathcal{T}_{\mathcal{V}}(U |\sigma U^\dagger|) \otimes |1\rangle \langle 1| + \frac{1 - \mathcal{T}_{\mathcal{V}}(U |\sigma U^\dagger|)}{d_s - 1} \otimes (1 - |1\rangle \langle 1|) \]

(24)

because \( |1\rangle \langle 1| \) is mapped to \( \sigma \) via the assignment map and \( 1 - |1\rangle \langle 1| \) is mapped to \( d_s \rho - \sigma \). Since \( \mathcal{T}_{\mathcal{V}}(U |\sigma U^\dagger|) \in \mathcal{D}_{\mathcal{V}} \), \( 1 - \mathcal{T}_{\mathcal{V}}(U |\sigma U^\dagger|) \) is also positive, so this Choi matrix is obviously positive. Therefore \( \Psi'_U \circ \mathcal{P} \) is CP, and \( \Psi'_U \) is CPZE.

2. **Kraus OSR for Dynamical Maps**

On the other hand, if the existence of a Kraus OSR for the dynamical maps (point 2 above) is of primary importance, then the key question is whether \( \Psi'_U \) is CP, but rather whether \( \Psi'_U \) is completely positively extensible (CPE) for each \( U \in \mathcal{G} \). This is because

**Lemma 10**. For any \( \mathcal{G} \)-consistent subspace \( \mathcal{V} \) and any \( U \in \mathcal{G} \), the dynamical map \( \Psi'_U \) admits a Kraus OSR if and only if \( \Psi'_U \) is CPE.

**Proof**. If \( \Psi'_U \) is CPE, the CP extension \( \hat{\Psi}'_U \) is defined over the entire algebra \( \mathcal{B}(\mathcal{H}_w) \), so that Choi’s theorem [24] may be invoked to conclude that \( \Psi'_U \) admits a Kraus OSR. Likewise, if \( \Psi'_U \) may be written in terms of a Kraus OSR, then that OSR defines a CP extension \( \hat{\Psi}'_U : \mathcal{B}(\mathcal{H}_w) \rightarrow \mathcal{B}(\mathcal{H}_w) \) so that \( \Psi'_U \) is CPE.

It should be noted that, while the map \( \Psi'_U \) is uniquely defined by the choice of \( \mathcal{G} \)-consistent subspace \( \mathcal{V} \) and the choice of \( U \in \mathcal{G} \), neither its CP extension \( \hat{\Psi}'_U \) (if one exists) nor the Kraus OSR thereof is uniquely defined. It should also be noted...
that any Kraus OSR obtained in this way is still subject to the restrictions mentioned earlier: it cannot be meaningfully applied to any state outside of the physical domain $\text{Tr}_B(\mathcal{D}_{SB} \cap \mathcal{V})$.

\section*{V. EXAMPLES}

In this section we apply the general framework developed above to earlier work on the topic of complete positivity. In many cases we reinterpret this earlier work in light of the notion of $G$-consistency, CPE, and CPZE maps.

\subsection*{A. Kraus (1971)}

The standard Kraus map \[26\] is built from a $U(\mathcal{H}_S \otimes \mathcal{H}_B)$-consistent subspace of the form $\mathcal{V} = B(\mathcal{H}_S) \otimes \rho_B$ for some fixed bath state $\rho_B \in \mathcal{D}_B$, so that the admissible initial states $\rho_{SB} \in \{\rho_S \otimes \rho_B : \rho_S \in \mathcal{D}_S\} = \mathcal{D}_{SB} \cap \mathcal{V}$ are uncorrelated. As was mentioned at the end of Section \[11\] the assignment map $A : \text{Tr}_B \mathcal{V} \rightarrow \mathcal{V} / \mathcal{V}_0$ associated with this subspace $\mathcal{V}$ admits an OSR $A'(X) = \sum_i Q_i X Q_i^\dagger$ with operators $Q_i = \sqrt{A_i} \otimes |\nu\rangle$ where $A_{\nu} = \sum_\nu A_{\nu}\langle \nu | \nu \rangle$ is the eigendecomposition of $A_{\nu}$. This OSR is evidently completely positive. It follows that, for any joint unitary evolution $U \in U(\mathcal{H}_S \otimes \mathcal{H}_B)$, the subdynamical map $P_{Y\mu}^U$ is completely positive and admits the OSR

$$\rho_S(t) = P_{Y\mu}^U[\rho_S(0)] = \sum_{\mu,\nu} E_{\mu\nu} \rho_S(0) E_{\mu\nu}^\dagger$$

for $E_{\mu\nu} = \langle \mu | U Q_{\nu}^\dagger = \sqrt{A_{\nu}} \langle \mu | U | \nu \rangle$, (25b)

where the operation elements $E_{\mu\nu}$ are the Kraus operators. It is worth noting that, in this case, the “physical domain” includes all system states, i.e., $\text{Tr}_B(\mathcal{D}_{SB} \cap \mathcal{V}) = \mathcal{D}_S$. A theorem of Pechukas \[4\], the proof of which was supplied by Pechukas in the case of a 2-dimensional system, and by Jordan, Shaji, and Sudarshan \[3\] for general finite-dimensional systems, shows that the only $U(\mathcal{H}_S \otimes \mathcal{H}_B)$-consistent subspaces exhibiting this property (that $\text{Tr}_B(\mathcal{D}_{SB} \cap \mathcal{V}) = \mathcal{D}_S$) are those of the form $\mathcal{V} = B(\mathcal{H}_S) \otimes \rho_B$ for a fixed $\rho_B \in \mathcal{D}_B$.

\subsection*{B. Pechukas (1994)}

To illustrate the fact that subsystem dynamics need not be completely positive, Pechukas offered an example involving a two-state system and bath and a $U(\mathcal{H}_S \otimes \mathcal{H}_B)$-consistent subspace defined by a $\mathbb{C}$-linear, $\dagger$-linear assignment map of the form

$$\rho_S \mapsto \left[ \rho_S \left( \rho_{SB}^{eq} \right)^{-1} \rho_{SB}^{eq} + \rho_{SB}^{eq} \left( \rho_S^{eq} \right)^{-1} \rho_S \right] / 2$$

where $\rho_{SB}^{eq} = \text{Tr}_B(\rho_{SB}^{eq})$ are fixed “equilibrium states” and $\rho_S^{eq} > 0$ (not just $\geq 0$). If $\rho_{SB}^{eq}$ is not a tensor product state, and the domain of this map is taken to be all of $B(\mathcal{H}_S)$, then the image will be a well-defined $U(\mathcal{H}_S \otimes \mathcal{H}_B)$-consistent subspace of $B(\mathcal{H}_S \otimes \mathcal{H}_B)$. However, the assignment map cannot be positive under these assumptions, let alone completely positive, because pure states $\rho_S$ are not mapped to product states $\rho_S \otimes \rho_B$ as a positive assignment map must do (this last point may be seen as a consequence of the fact that a bipartite state $\rho_{SB}$ with a pure reduced state $\rho_S$ possesses zero mutual information, i.e., saturates subadditivity of the von Neumann entropy \[30\], and must therefore be a product state \[31\]).

\subsection*{C. Alicki (1995)}

Commenting on Pechukas’ theorem concerning complete positivity, Alicki \[14\] suggested abandoning either consistency (i.e. $\text{Tr}_B \circ A_{\nu} = \text{id}$) or positivity of the assignment map, rather than giving up complete positivity of the dynamical maps. To illustrate the possible loss of consistency, he describes a scheme for turning a CP map $T : B(\mathcal{H}_S) \rightarrow B(\mathcal{H}_S)$ into a CPE assignment map with domain given by the fixed points of $T$. Suppose $T$ is represented as $T(\rho_S) = \sum_n V_n \rho_S V_n^\dagger$ and fix some $\rho_{SB}^{eq} \in \mathcal{D}_{SB}$. Then one can postulate a $\mathbb{C}$-linear assignment map

$$A_{\nu}(\rho_S) = \sum_n V_n \rho_S V_n^\dagger \otimes \frac{\text{Tr}_B[(V_n^\dagger V_n \otimes \mathbb{1}) \rho_{SB}^{eq}]}{\text{Tr}[(V_n^\dagger V_n \otimes \mathbb{1}) \rho_{SB}^{eq}]}$$

$$= \sum_{ij\alpha} Q_{ij\alpha}^\dagger \rho_S Q_{ij\alpha}$$

where $Q_{ij\alpha} \in B(\mathcal{H}_S; \mathcal{H}_S \otimes \mathcal{H}_B)$ is the operator

$$Q_{ij\alpha} = V_n \otimes \langle i | (V_n^\dagger \otimes \mathbb{1}) \rho_{SB}^{eq} | \alpha \rangle / \sqrt{\text{Tr}[(V_n^\dagger V_n \otimes \mathbb{1}) \rho_{SB}^{eq}]}$$

\(11\)
so that $A_Y$ is CPE. Since $Tr_{\rho} A_Y = T$, the correct domain of definition of this assignment map is the $C$-linear space of fixed points of $T$ in $B(H_{\rho})$. Letting $V'$ be the image of this domain through $A_Y$, we may define a $U(H_{\rho} \otimes H_{\alpha})$-consistent subspace by $\mathcal{V} = \text{Span}_{C}(D_{\mathcal{S}} \cap V')$. Since the assignment map $A_Y$ is CPE, it is clear that the dynamical maps $\Psi_Y : Tr_{\rho} V \rightarrow B(H_{\rho})$ are also CPE for all $U \in U(H_{\rho} \otimes H_{\alpha}).$ It is also clear that, although this construction is described by Alicki as involving a loss of consistency, it is entirely valid within the framework of $G$-consistent subspaces. It should be noted, however, that this construction depends not only on the map $T$ and the “equilibrium state” $\rho_{SB}$, but also on the representation of $T$. In general, two equivalent OSRs $T = \sum R_{m} \cdot R_{m}^{*}$ will not yield the same consistent subspace $\mathcal{V}$, thus will not yield the same assignment map $A_{Y}$ or dynamical maps $\Psi_{Y}$.

Alicki goes on to consider a generally nonlinear assignment map

$$A(\rho_{S}) = \sum \lambda_{n} P_{n} \otimes \frac{Tr_{\rho_{SB}}(P_{n} \otimes I)}{Tr_{\rho_{SB}}(P_{n} \otimes I)}$$

(29)

where again $\rho_{SB} \in D_{SB}$ is a fixed “equilibrium state”, and $\rho_{S} = \sum \lambda_{n} P_{n} P_{n}$ is the spectral decomposition of $\rho_{S}$. Then $Tr_{\rho} A = id$ on all of $D_{S}$, but $A : D_{S} \rightarrow D_{SB}$ is convex-linear if and only if $\rho_{SB} \in S$ is a tensor product state. In either case, the image of this assignment map $S = A(D_{S}) \subset D_{SB}$ is a valid $U(H_{\rho} \otimes H_{\alpha})$-consistent subset as described in Section [I] contained within the subset of zero-discord states in $D_{SB}$ [8] since any state in the image of $A$ is of the form $\sum \lambda_{n} P_{n} \otimes \sigma_{n}$ for orthogonal projectors $\{P_{n}\}$ and states $\{\sigma_{n}\} \subset D_{SB}$. As such, $\rho_{SB} \in S$ only if it has this very special form. It is clear that the resulting dynamical maps $\tau_{S}$ will be positive, but they will also generally not be convex-linear. However, the physical meaning of $\rho_{SB} \in S$ is unclear when $\rho_{SB} \notin S$, and moreover $\tau_{S} \otimes id$ and $A \otimes id$ have no meaning when $\tau_{S}$ and $A$ are nonlinear, so that complete positivity has no meaning either.

D. Štefamachović and Bužek (2001)

In addition to Example [1] Štefamachović and Bužek [17] offer a second example which demonstrates that, if we fix any $\rho_{S}(0), \rho_{S}(T) \in D_{S}$, then it is possible to choose $H_{\alpha}, a U(H_{\rho} \otimes H_{\alpha})$-consistent subspace $\mathcal{V} \subset B(H_{\rho} \otimes H_{\alpha})$, and a unitary transformation $U \in U(H_{\rho} \otimes H_{\alpha})$ such that the corresponding dynamical map $\Psi_{Y}$ takes $\rho_{S}(0)$ to $\rho_{S}(T)$. Indeed, they show this can be done with a Kraus-like consistent subspace $\mathcal{V} = B(H_{\rho}) \otimes \rho_{S}$, where $H_{\rho} \simeq H_{\rho}$ and $\rho_{S} = \rho_{S}(T)$, and where the unitary transformation $U = \sum |i\rangle \langle j| \otimes |j\rangle \langle i|$ is the swap operator $U |\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$.

E. Jordan, Shaji, and Sudarshan (2004)

Jordan, Shaji, and Sudarshan [6] develop a two qubit example (one “system” qubit and one “bath” qubit) in detail in order to examine quantum dynamics in the presence of initial entanglement. Their example can be expressed in terms of a $G$-consistent subspace, where $G \subset U(H_{\rho} \otimes H_{\alpha})$ is the one-parameter subsemigroup generated by the Hamiltonian $H = (\omega/2)Z \otimes X$ where $Z$ and $X$ are Pauli operators. The $G$-consistent subspace $\mathcal{V} \subset B(H_{\rho} \otimes H_{\alpha})$ is then the 14-dimensional orthogonal complement (with respect to the Hilbert-Schmidt inner product) of $\text{Span}_{C}\{\alpha I + Y \otimes X, \beta I - X \otimes X\}$ for some fixed choice of $\alpha, \beta \in (-1, 1)$. The kernel $\mathcal{V}_{0} = \ker (Tr_{\rho} |\psi\rangle)$ is then the 10-dimensional subspace

$$\mathcal{V}_{0} = \text{Span}_{C}\{I \otimes X, I \otimes Y, I \otimes Z, X \otimes Y, X \otimes Z, Y \otimes Y, Y \otimes Z, Z \otimes X, Z \otimes Y, Z \otimes Z\}.$$ (30)

It is straightforward to check that $G \cdot \mathcal{V}_{0} \subset \ker Tr_{\rho}$ as required. The quotient space $\mathcal{V}/\mathcal{V}_{0}$ is then 4-dimensional, so that $Tr_{\rho} \mathcal{V} = B(H_{\rho})$. However, as pointed out in [6], unless $\alpha = \beta = 0$, the assignment map $A_{Y} : Tr_{\rho} \mathcal{V} \rightarrow \mathcal{V}/\mathcal{V}_{0}$ is not positive. If it were positive, then $A_{Y}([0]|0\rangle\langle 0|)$ would be an affine subspace of $B(H_{\rho} \otimes H_{\alpha})$ containing a state of the form $[0]|0\rangle\langle 0|$ for some $\rho_{S} \in D_{S}$. But unless $\alpha = \beta = 0$, no state of that form lies in $\mathcal{V}$ since such states are not orthogonal to $\alpha I + Y \otimes X$ and $\beta I - X \otimes X$. It follows that the affine subspace $A_{Y}([0]|0\rangle\langle 0|)$ contains only unit trace, non-positive operators, so that $A_{Y}$ is non-positive. Moreover it is shown that the $\Psi'_{Y}([0]|0\rangle\langle 0|)$ need not be positive for $U \in G$ and therefore $\Psi'_{Y}$ is not completely positive for all $U \in G$. On the other hand, if $\alpha = \beta = 0$, then $\mathcal{V}$ contains the $U(H_{\rho} \otimes H_{\alpha})$-consistent subspace $\mathcal{V} = B(H_{\rho}) \otimes I$ for which $\mathcal{V} = \mathcal{V} \oplus \mathcal{V}_{0}$. In that case, the assignment map is not only positive, but completely positive.

F. Carteret, Terno, and Życzkowski (2008)

Carteret et al. [7], consider an example with a one qubit system and bath and a $U(H_{\rho} \otimes H_{\alpha})$-consistent subspace of the form

$$\mathcal{V} = \text{Span}_{C}\left\{\sigma_{1} \otimes I, \sigma_{2} \otimes I, \sigma_{3} \otimes I, I + a \sum_{i=1}^{3} \sigma_{i} \otimes \sigma_{i}\right\},$$

(31)

where $\{\sigma_{i}\}$ are the Pauli operators and $-1 < a < 1/3$ is a fixed constant. For $a \neq 0$, the associated assignment map $A_{Y}$ is non-positive, admitting a physical domain $Tr_{\rho} (D_{\mathcal{S}} \cap \mathcal{V})$ which, within the Bloch sphere, is the ball centered at $1/2$ with radius $\sqrt{(1+a)(1-3a)}$ for $a \geq 0$ and with radius $1 + a$ for $a \leq 0$ (this latter point about $a < 0$ was missed in [7]). The authors note that the dynamical map $\Psi_{Y}$ is trivially completely positive for any $U \in U(H_{\rho} \otimes H_{\alpha})$ that commutes with $\sum \sigma_{i} \otimes \sigma_{i}$. They then consider unitary transformations of the form

$$U = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & \sin \theta & 0 \\
0 & -\sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},$$

(32)
pointing out that the $\theta = \pi/4$ case yields a CP dynamical map $\Psi^U$ and incorrectly claiming that $\theta = \pi$ yields a non-CP dynamical map (this case also yields a CP $\Psi^U$, because for $\theta = \pi$, $U = \sigma_x \otimes \sigma_x$ commutes with $\sum \sigma_i \otimes \sigma_i$). However, it may be shown that the Choi matrix for the dynamical map $\Psi^U\pi$ associated with such a $U$ is given by

$$\text{Choi}(\Psi^U\pi) = \frac{1}{2} [I + \cos^2(\theta)\sigma_z \otimes \sigma_z + a \sin(2\theta) I \otimes \sigma_z + \cos(\theta)(\sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y)]$$

(33)

so that other values for $\theta$ and $a$ can yield non-CP maps, for example the case $\theta = \pi/6$ and $a > 1/2\sqrt{3}$.

G. Rodríguez-Rosario, Modi, Kuah, Shaji, and Sudarshan (2008)

Rodríguez-Rosario et al. [9], considered $U(\mathcal{H}_S \otimes \mathcal{H}_B)$-consistent subspaces of the form

$$\mathcal{V} = \text{Span}_C\{|i\rangle\langle i| \otimes \sigma_i\},$$

(34)

where $\{|i\rangle\} \subset \mathcal{H}_S$ is an orthonormal system for the basis and $\{|\sigma_i\rangle\} \subset \mathcal{D}_\text{a}$ are bath states. Then $\mathcal{V} \cap \mathcal{D}_\text{a}$ comprises only zero-discord states, i.e., those that exhibit only classical correlations with respect to some measurement basis [8]. They showed that, for such a $\mathcal{V}$ and any $U \in U(\mathcal{H}_S \otimes \mathcal{H}_B)$, the corresponding dynamical map $\Psi^U\pi$ is always CPE. In fact, more is true: the assignment map $A_\mathcal{V} : \text{Tr}_\mathcal{V} \rightarrow \mathcal{V}$ is CPZE since $A_\mathcal{V} \circ \mathcal{P}_{\text{Tr}_\mathcal{V}}(X) = \sum_{i,\alpha} E_{i,\alpha} X E_{i,\alpha}^\dagger$ is CP, where $\mathcal{P}_{\text{Tr}_\mathcal{V}} : \mathcal{B}(\mathcal{H}_S) \rightarrow \text{Tr}_\mathcal{V}$ is the orthogonal projection onto $\text{Tr}_\mathcal{V}$ and $E_{i,\alpha} = |i\rangle\langle i| \otimes \sqrt{\sigma_i |\alpha\rangle\langle \alpha|}$ for some orthonormal basis $\{|\alpha\rangle\} \subset \mathcal{H}_B$.

As we discussed in Section III D whenever the semigroup $\mathcal{G}$ contains nonlocal unitary operators, all $\mathcal{G}$-consistent subspaces must be constrained in some way. Moreover, if we accept that such constraints on the initial-system-bath states are inevitable, there seems to be little reason to disallow constraints on $\text{Tr}_\mathcal{V}$, which is the domain of the assignment map $A_\mathcal{V}$ and of the dynamical maps $\Psi^U\pi$ for all $U \in \mathcal{G}$. The authors of [9] seem to take another position, however, stating in the introduction of that paper that “the dynamical map is well-defined if it is positive on a large enough set of states such that it can be extended by linearity to all states of the system.” It may be observed, as the authors themselves do, that for the $\text{dim}(\mathcal{H}_S)$-dimensional consistent subspaces $\mathcal{V}$ in (34), the linear domain of definition, $\text{Tr}_\mathcal{V}$, of the dynamical maps consists only of operators which are diagonal in the fixed basis $\{|i\rangle\}$. Such dynamical maps are, in fact, not ill-defined at all, contrary to the position adopted in [9]. They are well-defined maps with well-defined domains which are low-dimensional subspaces of $\mathcal{B}(\mathcal{H}_S)$. In addition, they are associated with a well-defined, CPE assignment map $A_\mathcal{V} : \text{Tr}_\mathcal{V} \rightarrow \mathcal{V}$, so that the complete positivity of the dynamical maps has a good, physical basis related to the unitary evolution of system-bath-witness states for arbitrary finite-dimensional witnesses (see Section IV C).

H. Shabani and Lidar (2009)

Shabani and Lidar [10] expanded the class of consistent subspaces from the zero-discord subspaces considered in [9], to the class of all valid $U(\mathcal{H}_S \otimes \mathcal{H}_B)$-consistent subspaces of the form

$$\mathcal{V} = \text{Span}_C\{|i\rangle\langle j| \otimes \phi_{ij}\},$$

(35)

where $\{|i\rangle\}$ is an orthonormal basis for $\mathcal{H}_S$ and $\{|\phi_{ij}\rangle\} \subset \mathcal{B}(\mathcal{H}_B)$ are bath operators. Ref. [10] showed that, within this class of consistent subspaces, the dynamical maps $\Psi^U\pi$ are CPE for all $U \in U(\mathcal{H}_S \otimes \mathcal{H}_B)$ if and only if $\mathcal{V} \cap \mathcal{D}_\text{a}$ comprises only zero-discord states.

In light of the framework developed here, it is clear that the results of [10] do not amount in general to necessary and sufficient conditions for complete positivity, but are restricted to the class of consistent subspaces of the form (35). This lack of generality is a point that has been largely overlooked by the community, including by two of the present authors. Thus the present work amounts to a retraction and correction of some of the claims made in [10]. The issue is that, while the form (35) is general enough to describe any given state $\rho_{SB}$, it is not general enough to describe every $\mathcal{G}$-consistent subspace. As a simple example, the 2-dimensional $U(\mathcal{H}_S \otimes \mathcal{H}_B)$-consistent subspace described in Example 2 is not of the form (35), nor is it contained in any subspace of this form.

Let us also comment on [11], which does not address the question of whether the maps it constructs are CP or not, but states in its Theorem 3, that “the most general form of a quantum dynamical process irrespective of the initial system-bath state (in particular arbitrarily entangled initial states are possible) is always reducible to a Hermitian map from the initial system to the final system state.” We note that this only holds for $\mathcal{G}$-consistent subspaces; when the set of admissible initial states $\mathcal{S}$ is not $\mathcal{G}$-consistent, there is no subsystem $\mathcal{S}$-dynamical map (for some $U \in \mathcal{G}$).

I. Brodutch, Datta, Modi, Rivas, and Rodríguez-Rosario (2013)

To demonstrate that vanishing discord is not necessary for complete positivity, Brodutch et al. [12] showed that $U(\mathcal{H}_S \otimes \mathcal{H}_B)$-consistent subspaces $\mathcal{V}$ exist for which almost all states in $\mathcal{V} \cap \mathcal{D}_\text{a}$ are discordant, but the dynamical maps $\Psi^U\pi$ are nonetheless CPZE (and therefore CPE) for all $U \in U(\mathcal{H}_S \otimes \mathcal{H}_B)$. The counterexamples described in [12] are subspaces of the form

$$\mathcal{V} = \text{Span}_C(\rho_{01} \cup \{|i\rangle\langle i| \otimes \sigma_i\}_{i=2}^n)$$

(36)

where $\rho_{01} = |0\rangle\langle 0| \otimes \sigma_0 + |1\rangle\langle 1| \otimes \sigma_1 + |+\rangle\langle +| \otimes \sigma_+$. $\{|i\rangle\}$ is an orthonormal basis for $\mathcal{H}_S$, and $\{|\sigma_+, \sigma_0, \ldots, \sigma_n\rangle\} \subset \mathcal{D}_\text{a}$ are bath states.

It should be mentioned that, while Brodutch, et al. were correct to criticize the generality of the conclusion concerning the necessity of vanishing discord for complete positivity in
[10], their claim of nonlinearity appearing in [10] is not valid. By expressing the construction of [10] in the language of $G$-consistent subspaces as in [5], it should be clear that, while the initial system-bath states lie in a constrained subspace— as they always must for $G$-consistency, i.e., to obtain well-defined dynamical maps (see Lemma 2)—all spaces and maps in the Shabani-Lidar framework are $\mathbb{C}$-linear.

J. Buscemi (2013)

Buscemi [13] has devised a general method for constructing $U(H_S \otimes H_A)$-consistent subspaces $V$ which yield CPE dynamical maps $\Psi'_V$ for all $U \in U(H_S \otimes H_A)$. The technique involves first introducing a “reference” subsystem $R$ and choosing a fixed tripartite state $\rho_{RSB}$ that saturates the strong subadditivity inequality for von Neumann entropy; in other words the conditional mutual information, $I(R : B|S)_\rho$, is zero. A subspace $V' \subset B(H_S \otimes H_A)$ is then formed as

$$V' = \{ \text{Tr}_B([L \otimes 1_B] \rho_{RSB}) : L \in B(H_A) \}$$

(37)

and the corresponding $U(H_S \otimes H_A)$-consistent subspace is $V = \text{Span}_C(D_{SB} \cap V')$. Buscemi proves that such a construction always leads to CPE dynamical maps. While there is no claim that all $U(H_S \otimes H_A)$-consistent subspaces yielding CPE dynamical maps must be formed in this way, the technique does yield a rich set of examples, including the zero-discord subspaces of Rodríguez-Rosario et al. [9] and the examples of Brodutch et al. [12], as well as examples featuring highly entangled states. It should also be noted that, although Buscemi’s method predicts that the dynamical maps will be CPE in these examples, it does not appear to predict the fact that the assignment maps are CP (indeed, CPZE) in the examples of [9] and [12].

VI. SUMMARY & OPEN QUESTIONS

We have formulated a general framework for open quantum system dynamics in the presence of initial correlations between system and bath. It is based around the notion of $G$-consistent operator spaces $V \subset B(H_S \otimes H_A)$ representing the set of admissible initial system-bath states. $G$-consistency is necessary to ensure that initial system-bath states with the same reduced state on the system evolve under all admissible unitary operators $U \in G$ to system-bath states with the same reduced state on the system, ensuring the dynamical maps $\Psi'_V$ are well-defined. Once such a $G$-consistent subspace is chosen, related concepts like the assignment map and the dynamical maps are uniquely defined. In general, the dynamical maps may not be applied to arbitrary system states, but only to those in the physical domain $\text{Tr}_R(D_{SB} \cap V)$. Interestingly, non-positive assignment maps can give rise to completely positive dynamical maps, but this is not a situation that is physically acceptable since it involves the propagation of unphysical system-bath “states”.

These $G$-consistent subspaces may be highly constrained, with the result that complete positivity of the assignment and dynamical maps may be defined in several distinct ways, relating to different interpretations and desirable features of complete positivity for open system dynamics. Indeed, we have identified two new types of completely positive maps, which we have called CPE and CPZE. It is important to distinguish between these flavors of complete positivity and whether one is considering complete positivity of the assignment map or of the dynamical maps. We have shown that various special cases considered in earlier work can be unified within the framework of $G$-consistency, and that earlier discussions of complete positivity are better understood using CPE or CPZE maps.

Having laid out this framework of $G$-consistent subspaces, we have left many important questions open. The formalism is general enough to encompass other proposed frameworks, most notably the assignment map framework introduced by Pechukas [4], and various worked examples. These various results and special cases illustrate perhaps the biggest open question: what structural features characterize the subclass of $G$-consistent subspaces which give rise to completely positive dynamics (in any reasonable flavor)? Recent work [9, 10] suggested that the answer might lie with quantum discord. However, more recently it became apparent that the focus on quantum discord is too restrictive [12, 13], and here we have shown that earlier work has illuminated some small corners of the space of $G$-consistent subspaces, but a complete analysis remains to be seen.

ACKNOWLEDGMENTS

This research was supported by the ARO MURI grant W911NF-11-1-0268 and by NSF grant numbers PHY-969969 and PHY-803304.

[1] K. Kraus, States, Effects, and Operations (Springer, Berlin, 1983).
[2] H.-P. Breuer and F. Petruccione, The Theory of Open Quantum Systems (Oxford University Press, Oxford, 2002).
[3] M. Nielsen and I. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, England, 2000).
[4] F. Pechukas, Phys. Rev. Lett. 73, 1060 (1994).
[5] P. Pechukas, Phys. Rev. Lett. 75, 3021 (1995).
[6] T. F. Jordan, A. Shaji, and E. C. G. Sudarshan, Phys. Rev. A 70, 052110 (2004).
[7] H. A. Carteret, D. R. Terno, and K. Zyczkowski, Phys. Rev. A 77, 042113 (2008).
[8] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. 88, 017901 (2002).
[9] C. A. Rodríguez-Rosario, K. Modi, A. Kuah, A. Shaji, and E. Sudarshan, J. Phys. A 41, 205301 (2008).
[10] A. Shabani and D. A. Lidar, Phys. Rev. Lett. 102, 100402 (2009).
[11] A. Shabani and D. A. Lidar, Phys. Rev. A 80, 012309 (2009).
[12] A. Brodutch, A. Datta, K. Modi, A. Rivas, and C. A. Rodríguez-Rosario, Phys. Rev. A 87, 042301 (2013).
[13] F. Buscemi, Phys. Rev. Lett. 113, 140502 (2014).
[14] R. Alicki, Phys. Rev. Lett. 75, 3020 (1995).
[15] This diagram is similar to diagrams in [2, Fig. 3.2] and [12, 32, 33], but emphasizes the role of the set $S$ of admissible initial states.
[16] It should be noted that this notion of $G$-consistency is more general than the notion of assignment map consistency described by Alicki [14] and considered in several later works [12, 29, 33]. The typical interpretation of assignment map consistency is equivalent to $U(H_S \otimes H_B)$-consistency with the additional requirement that $\text{Tr}_B S = D_S$, i.e., every state of the subsystem must be covered by exactly one system-bath state in $S$.
[17] P. Štelmachovič and V. Bužek, Phys. Rev. A 64, 062106 (2001).
[18] R. Goodman and N. R. Wallach, Symmetry, Representations, and Invariants (Springer, New York, 2009).
[19] H. Hayashi, G. Kimura, and Y. Ota, Phys. Rev. A 67, 062109 (2003).
[20] M. D. Grace, J. Dominy, R. L. Kosut, C. Brif, and H. Rabitz, New J. Phys. 12, 015001 (2010), special Issue: Focus on Quantum Control.
[21] D. Salgado and J. L. Sanchez-Gomez, (2002), arXiv:quant-ph/0211164.
[22] K. M. F. Romero, P. Talkner, and P. Hänggi, Phys. Rev. A 69, 052109 (2004).
[23] J. de Pillis, Pacific J. Math. 23, 129 (1967).
[24] M.-D. Choi, Linear Algebra. Appl. 10, 285 (1975).
[25] W. F. Stinespring, Proc. Amer. Math. Soc. 6, 211 (1955).
[26] K. Kraus, Ann. Physics 64, 311 (1971).
[27] W. Arveson, Acta Math. 123, 141 (1969).
[28] M.-D. Choi and E. G. Effros, J. Funct. Anal. 24, 156 (1977).
[29] C. A. Rodríguez-Rosario, K. Modi, and A. Aspuru-Guzik, Phys. Rev. A 81, 012313 (2010).
[30] H. Araki and E. H. Lieb, Comm. Math. Phys. 18, 160 (1970).
[31] L. W. Bruch and H. Falk, Phys. Rev. A 2, 1598 (1970).
[32] C. A. Rodriguez, The Theory of Non-Markovian Open Quantum Systems, Ph.D. thesis, The University of Texas at Austin (2008).
[33] K. Modi, C. A. Rodríguez-Rosario, and A. Aspuru-Guzik, Phys. Rev. A 86, 064102 (2012).