On the construction of Riemannian three-spaces with smooth generalized inverse mean curvature flows *

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December 11, 2019

Abstract

Choose a smooth three-dimensional manifold \( \Sigma \) that is smoothly foliated by topological two-spheres, and also a smooth flow on \( \Sigma \) such that the integral curves of it intersect the leaves of the foliation precisely once. Choose also a smooth Riemannian three-metric \( h_{ij} \) on \( \Sigma \) such that the area of the foliating level sets is strictly increasing. Then, by altering suitably the lapse and shift of the flow but keeping the two-metrics induced on the leaves of the foliation fixed a large variety of Riemannian three-geometries is constructed on \( \Sigma \) such that the foliation, we started with, gets to be a smooth generalized inverse mean curvature foliation, the prescribed flow turns out to be a generalized inverse mean curvature flow. All this is done such that the scalar curvature of the constructed three-geometries is not required to be non-negative. Furthermore, each of the yielded Riemannian three-spaces is such that the Geroch energy is non-decreasing, and a quasi-local comparison of the surface area and the Geroch energy is also derived, which, whenever a minimal surface exists on \( \Sigma \) reduces to a quasi-local form of the Penrose inequality. If the metric \( h_{ij} \) we started with is asymptotically flat the constructed three-geometries will be so, and besides the positive energy theorem, if, in addition, a minimal surface exists on \( \Sigma \) the conventional global form of the Penrose inequality also holds.

*This is a written up version of a lecture given on 5th December 2019 at Institut Mittag-Leffler, Stockholm as part of the ongoing scientific program “General Relativity, Geometry and Analysis: beyond the first 100 years after Einstein”.

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1 Introduction

As general relativity is a metric theory of gravity it is highly non-trivial to assign, in a sensible way, mass, energy, linear and angular momenta to bounded spatial regions. Yet, since the early seventies, it is also part of the common suppositions that proper analytic characterization of highly energetic processes will be intractable unless suitable quasi-local, and possibly quasi-conserved, quantities can be found [9, 20, 5]. The first fundamental step towards the realization of the underlying objectives was made by Geroch and the ideas he proposed in [9] greatly inspired most of the later investigations. Since then considerable progress had been made in clearing up conditions guaranteeing global existence of inverse mean curvature foliations [11, 12, 1, 3]. This is essential in getting proofs of a quasi-local version of the positive energy theorem or that of, its close relative, the Penrose inequality [9, 13, 14, 19, 15, 16, 17, 8, 18, 29]. One of the most important results is the proof of the Riemannian Penrose inequality by Huisken and Ilmanen [11, 12] (see also [1, 3]) which, under suitable conditions, also yields a quasi-local proof of the positive energy theorem. Note that the later was proven originally by applying completely different techniques by Schoen, Yau and Witten [25, 26, 27, 31, 28]. In the aforementioned proof by Huisken and Ilmanen [11, 12], in order to prove the existence (in the weak sense) of inverse mean curvature flows and foliations, they required the involved timeslices to be maximal. In trying to get generalizations of this result by Huisken and Ilmanen one has to be able to show the existence of inverse mean curvature flows and foliations for timeslices such that the scalar curvature of the Riemannian three-metric on them is not necessarily non-negative.

The main purpose of the present paper is to introduce a new method that enables us to construct high variety of Riemannian three-spaces such that each admits a smooth generalized inverse mean curvature foliation and flow. As the main motivation for the use of the inverse mean curvature flow was to prove monotonicity of the Geroch energy, in our construction, using as a guiding principle, first conditions that can guarantee monotonous behavior of the Geroch energy are identified. The notion of inverse mean curvature flows is relaxed then slightly by allowing suitable rescaling of the flow vectors, and also by introducing, some geometrical adequate, non-trivial shift parts to the flows. As this slight generalization does not affect the location of the leaves of inverse mean curvature foliations it does not provide simplifications in the standard construction of inverse mean curvature foliations. Yet, by turning the coin to the other side, it turns out that the aforementioned generalization gets to be useful in constructing the desired type of Riemannian three-spaces. In doing so, we start with a smooth three-dimensional manifold that can be smoothly foliated by topological two-spheres. Choose then a smooth flow and a smooth Riemannian three-metric $h_{ij}$ such that the area of the foliating level sets is strictly increasing. Then,
by replacing the lapse and the shift, defined originally with respect to $h_{ij}$ and the
flow, by suitably chosen new lapse and shift—while keeping the two-metric, induced
on the leaves of the foliation, fixed—a large variety of smooth Riemannian metrics
are constructed on $\Sigma$ such that the original topological two-spheres get to provide a
smooth generalized inverse mean curvature foliation and the original flow gets to be
a generalized inverse mean curvature flow. Remarkably, the auxiliary flow—chosen
at the beginning of the construction to be merely smooth and regular but otherwise
arbitrary—gets by the end of the above outlined procedure to be a generalized inverse
mean curvature flow. It is also important to be emphasized here that our construction
does not require the scalar curvature of the three-metric to be non-negative. Thereby
the constructed Riemannian three-geometries will automatically accommodate non-
maximal timeslices which case could not be covered by the argument of Huisken and
Ilmanen [11, 12].

The most attractive property of the proposal made in this paper is that it does
not assume the global existence of or does not try to construct smooth inverse mean
curvature foliations which are definitely the most expensive ingredient in all the con-
ventional arguments [11, 12, 1, 3]. As opposed to this here we start with an arbitrarily
chosen, thereby, globally regular and smooth foliation and flow. The key observation
is that the construction of Riemannian three-spaces with all the desired properties
gets to be possible once the lapse and the shift of the metric $h_{ij}$ are adjusted appro-
riately. First the new shift is determined by solving an elliptic equation for a scalar
potential on the succeeding level surfaces individually, whereas the lapse is modified
by solving a hyperbolic equation on $\Sigma$. Note also that each of the yielded Rieman-
nian three-geometries is such that the Geroch energy—defined with respect to the
new metrics and the foliation we started with—is guaranteed to be non-decreasing.
Notably, a comparison of the surface area and the Geroch energy was also derived,
which, whenever a minimal surface exists on $\Sigma$, reduces to a quasi-local form of the
Penrose inequality. Whenever the original Riemannian three-metric $h_{ij}$ is guaranteed
to be asymptotically flat then the constructed three-spaces will also be so, and, be-
sides the positive energy theorem if, in addition, a minimal surface exists on $\Sigma$ the
conventional global form of the Penrose inequality also holds.

This paper is structured as follows: It starts, in section 2, by recalling the basic
notions and techniques we shall use. In section 3, a careful inspection of generic
variations of the Geroch energy is carried out. Subsection 3.1 is to determine those
conditions which guarantee monotonous behavior of the Geroch energy, whereas sub-
section 3.2 is to identify the alternative ways of getting control on monotonicity. In
section 4, first the conventional inverse mean curvature flow is put into a somewhat
new context. Our new proposal allowing us to construct the desired type of Riemann-
ian three-spaces is introduced in section 5. It starts by dynamical determination of
the shift, in subsection 5.1, and then by that of the lapse, in subsection 5.2. The quasi-
local comparison of the surface area and the Geroch energy, as well as, the quasi-local form of the Penrose inequality, along with some results relevant for asymptotic flat configurations are presented in subsection 5.3. The paper is closed by a summary in section 6.

2 Preliminaries

Start off by considering a smooth three-dimensional manifold $\Sigma$ endowed with a Riemannian metric $h_{ij}$. The topology of $\Sigma$ will be restricted by assuming that it can be foliated by a one-parameter family of topological two-spheres (which may degenerate at regular origins). These topological two-spheres will be denoted by $S^\rho$. They may be regarded as the $\rho = \text{const}$ level surfaces of a smooth real function $\rho: \Sigma \to \mathbb{R}$ with non-vanishing gradient $\partial_i \rho \neq 0$. Unless stated otherwise we shall assume that such a smooth real function has been chosen.

Given the transverse one-form field $\partial_i \rho$, with the help Riemannian metric $h_{ij}$ on $\Sigma$, a unit form field and a unit vector field can be defined as $\hat{n}_i = \partial_i \rho / (h^{kl} \partial_k \rho \partial_l \rho)^{1/2}$ and $\hat{n}^i = h^{ij} \hat{n}_j$, respectively. Both of these fields are normal to the $\rho = \text{const}$ level surfaces. Using them the operator $\hat{\gamma}^{ij} = \delta^{ij} - \hat{n}_i \hat{n}_j$, projecting fields to the tangent space of the level surfaces, gets also to be determined.

The intrinsic and extrinsic geometry of the $S^\rho$ level surfaces can then be represented by the induced Riemannian two-metric $\hat{\gamma}^{ij}$ and the extrinsic curvature $\hat{K}^{ij}$, defined via the relations,

$$\hat{\gamma}^{ij} = \hat{\gamma}^k_i \hat{\gamma}^l_j h_{kl} \quad \text{and} \quad \hat{K}^{ij} = \hat{\gamma}^l_i D_l \hat{n}_j - \frac{1}{2} \mathcal{L}_{\hat{n}} \hat{\gamma}^{ij}, \tag{2.1}$$

respectively. Here $D_i$ is the covariant derivative operator associated with $h_{ij}$ and $\mathcal{L}_{\hat{n}}$ denotes the Lie derivative with respect to the unit norm vector field $\hat{n}_i$. A $\rho = \text{const}$ level surface will be referred to be mean-convex if its mean curvature, $\hat{K}^{ij} \hat{n}_j = D_i \hat{n}^i$, is positive on $\mathcal{S}_\rho$.

Given a foliation $\mathcal{S}_\rho$, a smooth vector field $\rho^i$ on $\Sigma$ is called to be a flow, with respect to $\mathcal{S}_\rho$, if the integral curves of $\rho^i$ intersect each leaves precisely once, and also if $\rho^i$ is scaled such that $\rho^i \partial_i \rho = 1$ holds throughout $\Sigma$. Note that such a flow can always be chosen without referring to some background metric structure, and it is not unique as if a flow exists then infinitely many others do also exist. Nevertheless, whenever a Riemannian metric $h_{ij}$ is given, any smooth flow can always be characterized by its lapse, $\hat{N} = (\hat{n}^i \partial_i \rho)^{-1}$—which measures the normal separation of the surfaces $\mathcal{S}_\rho$—, and by its shift, $\hat{N}^i = \hat{\gamma}^i_j \rho^j$, via the relations

$$\rho^i = \hat{N} \hat{n}^i + \hat{N}^i. \tag{2.2}$$
Notably, if the area of the foliating level sets is strictly increasing a well-defined quasi-local orientation of the leaves \( \mathcal{S}_\rho \) emerges simply by regarding a flow \( \rho^i \) outward pointing if the area of the leaves is increasing with respect to it \[23, 24\]. This happens, for instance, if the integral of the product \( \hat{N}(\hat{K}_l^l) \) is greater than zero. To see this makes sense recall that the variation of the area \( A_\rho \) of the \( \rho = \text{const} \) level surfaces, with respect to the flow \( \rho^i \), reads as

\[
\mathcal{L}_\rho A_\rho = \int_{\mathcal{S}_\rho} L_\rho \hat{\epsilon} = \int_{\mathcal{S}_\rho} \left\{ \hat{N}(\hat{K}_l^l) + (\hat{D}_i\hat{N}^i) \right\} \hat{\epsilon} = \int_{\mathcal{S}_\rho} \hat{N}(\hat{K}_l^l) \hat{\epsilon},
\]

where, besides (2.2), the relations \( \mathcal{L}_n \hat{\epsilon} = (\hat{K}_l^l) \hat{\epsilon} \) and \( \mathcal{L}_N \hat{\epsilon} = \frac{1}{2} \hat{\gamma}^{ij} \mathcal{L}_\hat{N} \hat{\gamma}_{ij} \hat{\epsilon} = (\hat{D}_i\hat{N}^i) \hat{\epsilon} \), along with the vanishing of the integral of the total divergence \( \hat{D}_i\hat{N}^i \), were applied.

Note that \( \hat{N} \) does not vanish on \( \Sigma \) unless the Riemannian metric

\[
h^{ij} = \hat{\gamma}^{ij} + \hat{N}^{-2}(\rho^i - \hat{N}^i)(\rho^j - \hat{N}^j)
\]

gets to be singular. Notably as \( \hat{n}^i \) is a flow itself with \( \hat{N} \equiv 1 \) and it is natural to require that the (quasi-local) orientations by \( \hat{n}^i \) and \( \rho^i \) coincide, hereafter, we shall assume that \( \hat{N} \) is positive throughout \( \Sigma \). Combining the foregoing we get that if the integral of \( \hat{N}\hat{K}_l^l \) is greater than zero, the area is, indeed, increasing with respect to \( \rho^i \) and, in turn, that the flow \( \rho^i \) is outward pointing. The above argument can also be used to verify that for mean-convex surfaces the area is “piece-wise strictly increasing” as not only the total area but the area of any local surface element is increasing with respect to outward pointing flows.

3 Variation of the Geroch energy

In proceeding the variation of the (quasi-local) Geroch energy

\[
E_G = \frac{A_\rho^{1/2}}{64\pi^{3/2}} \int_{\mathcal{S}_\rho} \left[ 2 \hat{R} - (\hat{K}_l^l)^2 \right] \hat{\epsilon},
\]

\[1\] In most of the cases (3.5) is referred to as the (Riemannian) Hawking energy in spite of the fact that the Geroch and Hawking quasi-local energies have significantly different characters. For instance, the Hawking energy is known to depend, beside on the geometry of two-surface \( \mathcal{S}_\rho \) within \( \Sigma \), also on the way \( \mathcal{S}_\rho \) is embedded into an ambient space. As opposed to this the Geroch energy depends only on the geometry of \( \mathcal{S}_\rho \) within \( \Sigma \). In particular the Geroch energy is always smaller than or equal to the Hawking energy, and they are known to be equal only if for the extrinsic curvature \( K_{ij} \) of \( \Sigma \), defined w.r.t. an ambient space, the contraction \( \hat{\gamma}^{ij} K_{ij} \) vanishes on \( \mathcal{S}_\rho \) \[29\]. Based on these observations it is preferable to distinguish these two concepts and hereafter we shall refer to (3.5) as the Geroch energy. Accordingly, all the results covered by our paper will automatically apply to any three-dimensional manifold independent of the extrinsic geometry of \( \Sigma \).
will play central role. In this section we shall assume that the area \( \mathcal{A}_\rho \) is strictly increasing. Thereby, in identifying those conditions which guarantee that the Geroch energy is non-decreasing it suffices to show that so does the integral

\[
W(\rho) = \int_{\mathcal{J}_\rho} \left[ 2 \tilde{R} - (\tilde{K}_l^l)^2 \right] \tilde{\epsilon}.
\]

(3.6)

Note that if both \( \mathcal{A}_\rho \) and \( W(\rho) \) were non-decreasing, and for some \( \rho = \rho^* \), the integral \( W(\rho^*) \) was zero or positive then \( E_G \geq 0 \) would automatically hold to the exterior of \( \mathcal{J}_{\rho^*} \) in \( \Sigma \). As an important special case, it is worth to mention that \( E_G(\rho^*) = 0 \) if a regular origin occurs at \( \rho^* \). An origin is a point \( p \in \Sigma \) where the \( \mathcal{J}_\rho \) foliation degenerates, i.e. \( p = \lim_{\rho \rightarrow p^*} \mathcal{J}_\rho \) for some value \( \rho^* \) of \( \rho \). The origin at \( p \) can be regarded as regular if its tangent space \( T_p \Sigma \) is regular, i.e. \( T_p \Sigma \) is isomorphic to \( \mathbb{R}^3 \). It is straightforward to show then that the image, by the exponential map, of the canonical foliation of \( T_p \Sigma \approx \mathbb{R}^3 \) by concentric metric spheres centered at the origin in \( \mathbb{R}^3 \) yields a foliation \( \mathcal{J}_\rho \) (at least) in a sufficiently small neighborhood \( \mathcal{O}_p \) of a regular origin \( p \), such that \( W(\rho^*) = 0 \), and neither \( \mathcal{A}_\rho \) nor \( W(\rho) \) is decreasing in \( \mathcal{O}_p \).

### 3.1 The variation of \( W(\rho) \)

We proceed by evaluating variations of \( W(\rho) \) with respect to generic flows. In doing so the key equation we shall apply relates the scalar curvatures of \( h_{ij} \) and \( \hat{\gamma}_{ij} \) as

\[
^{(3)}R = \tilde{R} - \left\{ 2 \mathcal{L}_\mathcal{K}(\tilde{K}_l^l) + (\tilde{K}_l^l)^2 + \tilde{K}_{kl}\tilde{K}^{kl} + 2 \tilde{N}^{-1} \hat{D}^l\hat{D}_l\tilde{N} \right\}.
\]

(3.7)

Note that this equation can be deduced from the Gauss-Codazzi relations (see, e.g. (A.1) in [21]), thereby, it holds on \( \Sigma \) without referring to any sort of field equations.

By varying \( W(\rho) \) with respect to an arbitrary flow we get

\[
\mathcal{L}_\rho W = -\int_{\mathcal{J}_\rho} \mathcal{L}_\rho \left[ (\tilde{K}_l^l)^2 \tilde{\epsilon} \right] = -\int_{\mathcal{J}_\rho} \left\{ \tilde{N} \mathcal{L}_\mathcal{K} \left[ (\tilde{K}_l^l)^2 \tilde{\epsilon} \right] + \mathcal{L}_\tilde{N} \left[ (\tilde{K}_l^l)^2 \tilde{\epsilon} \right] \right\}
\]

\[
= -\int_{\mathcal{J}_\rho} (\tilde{N}\tilde{K}_l^l) \left[ 2 \mathcal{L}_\mathcal{K} (\tilde{K}_l^l) + (\tilde{K}_l^l)^2 \right] \tilde{\epsilon} - \int_{\mathcal{J}_\rho} \hat{D}_l \left[ (\tilde{K}_l^l)^2 \tilde{N}^i \right] \tilde{\epsilon}
\]

\[
= -\int_{\mathcal{J}_\rho} (\tilde{N}\tilde{K}_l^l) \left[ (\tilde{R} - ^{(3)}R) - \tilde{K}_{kl}\tilde{K}^{kl} - 2 \tilde{N}^{-1} \hat{D}^l\hat{D}_l\tilde{N} \right] \tilde{\epsilon}, \quad (3.8)
\]

where on the first line \([22]\) and the Gauss-Bonnet theorem, on the second line again the relations \( \mathcal{L}_\mathcal{K} \tilde{\epsilon} = (\tilde{K}_l^l) \tilde{\epsilon} \) and \( \mathcal{L}_\tilde{N} \tilde{\epsilon} = (\hat{D}_l\tilde{N}^i) \tilde{\epsilon} \), whereas on the third line \((3.7)\) and the vanishing of the integral of \( \hat{D}_l \left[ (\tilde{K}_l^l)^2 \tilde{N}^i \right] \) were used. Applying then the Leibniz rule we get that

\[
\tilde{N}^{-1} \hat{D}^l\hat{D}_l\tilde{N} = \hat{D}^l (\tilde{N}^{-1} \hat{D}_l\tilde{N}) + \hat{N}^{-2} \tilde{\gamma}^{kl} (\hat{D}_k\tilde{N})(\hat{D}_l\tilde{N}), \quad (3.9)
\]
and—by introducing the trace-free part of \( \hat{K}_{ij} \) as \( \hat{K}_{ij} = \hat{K}_{ij} - \frac{\gamma_{ij}}{2} \hat{K}_{ll} \)—we also get that
\[
\hat{K}_{kl} \hat{K}^{kl} = \hat{K}_{kl} \hat{K}^{kl} + \frac{1}{2} \hat{K}_{ll}^2.
\] (3.10)

Combining these simple observations we get from (3.8)
\[
\mathcal{L}_\rho W = -\frac{1}{2} \int_{\mathscr{J}_\rho} (\hat{N} \hat{K}^l_l) \left[ 2 \hat{R} - (\hat{K}^l_l)^2 \right] \hat{\epsilon}
+ \int_{\mathscr{J}_\rho} (\hat{N} \hat{K}^l_l) \left[ (\alpha) R + \hat{K}_{kl} \hat{K}^{kl} + 2 \hat{N}^{-2} \hat{\gamma}^{kl} (\hat{D}_k \hat{N}) (\hat{D}_l \hat{N}) \right] \hat{\epsilon},
\] (3.11)
where the vanishing of the integral of the total divergence \( \hat{D}_l (\hat{N}^{-1} \hat{D}_l \hat{N}) \) was also used.

Notably, if the second term on the r.h.s. of (3.11) is non-negative the inequality
\[
\mathcal{L}_\rho W \geq -\frac{1}{2} \int_{\mathscr{J}_\rho} (\hat{N} \hat{K}^l_l) \left[ 2 \hat{R} - (\hat{K}^l_l)^2 \right] \hat{\epsilon}
\] (3.12)
holds. Note also that the r.h.s. of (3.12) is almost of the form \( W(\rho) \) so we are really close to the desired control on the variation of \( W(\rho) \). In proceeding recall that our primary aim is to single out those conditions which guarantee that the variation of the Geroch energy to be non-negative. It is straightforward to see that if in (3.12) the factor \( \hat{N} \hat{K}^l_l \) could be replaced by its average.

\[
\bar{\hat{N} \hat{K}^l_l} = \frac{\int_{\mathscr{J}_\rho} \hat{N} \hat{K}^l_l \hat{\epsilon}}{\int_{\mathscr{J}_\rho} \hat{\epsilon}}
\] (3.13)

things would simplify considerably as then the second term on the r.h.s. of (3.11) was non-negative provided that for the integral of the term in square brackets was so. Note also that the integrals in (3.13) had already been applied in (2.3), and it is immediate to see that
\[
\bar{\hat{N} \hat{K}^l_l} = \mathcal{L}_\rho \log[\mathscr{A}_\rho].
\] (3.14)

Accordingly, if the product \( \hat{N} \hat{K}^l_l \) could be replaced by its average—i.e. if the constancy of \( \hat{N} \hat{K}^l_l \) could be guaranteed on the individual \( \rho = \text{const} \) leaves—then (3.11) and (3.12), along with the non-negativity of integral of the second term on the r.h.s. of (3.11), would allow us to conclude that
\[
\left[ (64 \pi^{3/2}) / (\mathbf{\mathscr{A}})^{1/2} \right] \cdot \mathcal{L}_\rho E_G = \mathcal{L}_\rho W + \frac{1}{2} (\mathcal{L}_\rho \log[\mathscr{A}_\rho]) W \geq 0
\] (3.15)
with respect to the considered generic flow.

What has been established in the foregoing can also be summarized as follows.
Proposition 3.1. Consider a Riemannian three-space \((\Sigma, h_{ij})\) and assume that \(\Sigma\) is foliated by a one-parameter family of topological two-spheres \(\mathcal{F}_\rho\), the leaves of which are determined by a smooth function \(\rho : \Sigma \to \mathbb{R}\) the gradient \(\partial_i\rho\) of which may vanish only at isolated locations of regular origins. Assume also that a flow \(\rho^i = \hat{N} \hat{n}^i + \hat{N}^i\) has been chosen and the area of the foliating level sets is strictly increasing with respect to \(\rho^i\). Then, \(\mathcal{L}_\rho E_G \geq 0\) — i.e. the Geroch energy is non-decreasing — provided that the following two relations hold

\[
\hat{N} \hat{K}_l^l = \mathcal{L}_\rho \log[\mathcal{A}_\rho], \tag{3.16}
\]

\[
\int_{\mathcal{F}_\rho} \left[ R^{(3)} + \hat{K}_{kl} \hat{K}^{kl} + 2 \hat{N}^{-2} \hat{\gamma}^{kl} (\hat{D}_k\hat{N})(\hat{D}_l\hat{N}) \right] \hat{\epsilon} \geq 0. \tag{3.17}
\]

Note that (3.17) would allow \(R^{(3)}\) to be slightly negative, nevertheless, as in most of the inverse mean curvature based constructions \(\hat{K}_{kl}\) and \(\hat{N}\) get to be known only at the very end thereby the assumption \(R^{(3)} \geq 0\) is used in the pertinent considerations. As we would like to use the weakest possible restrictions we should keep in mind that it suffices to require (3.17) to be satisfied.

The main dilemma we have to face now is rooted in the rigidity of the setup we started with. Namely, if both the Riemannian metric \(h_{ij}\) and the foliation are fixed then so are the mean curvature \(\hat{K}_{kl}\) and \(\hat{N}\) get to be known only at the very end thereby the assumption \(R^{(3)} \geq 0\) is used in the pertinent considerations. As we would like to use the weakest possible restrictions we should keep in mind that it suffices to require (3.17) to be satisfied.

3.2 The alternative ways of getting control on monotonicity

In advance of resolving the aforementioned problem it is rewarding to have a glance again of the mathematical structures we have by hand.

We have started with a Riemannian metric \(h_{ij}\) defined on a three-surface \(\Sigma\) foliated by topological two-spheres, which can always be fixed by choosing a smooth real function \(\rho : \Sigma \to \mathbb{R}\) with an almost nowhere vanishing gradient \(\partial_i\rho\). Indeed, the only “freedom” that remained is nothing but a simple relabeling \(\rho = \rho(\rho)\) of the leaves of the foliation which cannot yield more than the trivial replacement \(\hat{N} \to \hat{N}(d\rho/d\rho)\) in the lapse. Accordingly, the factor \(\hat{N} \hat{K}_l^l\) in (3.12) is not constant on the leave of the foliations but, at best, it is merely a smooth function there such that its integral is strictly positive.
\( h_{ij} \) reads as

\[
\begin{align*}
    ds^2 &= \hat{N}^2 d\rho^2 + \hat{\gamma}_{AB} \left( dx^A + \hat{N}^A d\rho \right) \left( dx^B + \hat{N}^B d\rho \right).
\end{align*}
\] (3.18)

In summing up, we can say that those Riemannian three-spaces \((\Sigma, h_{ij})\), where \( \Sigma \) can smoothly be foliated by topological two-spheres and a smooth flow had been chosen on \( \Sigma \), can be represented by either of the sets \( \{ h_{ij} ; \rho : \Sigma \to \mathbb{R}, \rho^i \} \) or \( \{ \hat{N}, \hat{N}^A, \hat{\gamma}_{AB} ; \rho : \Sigma \to \mathbb{R}, \rho^i = (\partial \rho)^i \} \). Therefore, in virtue of Proposition 3.1, in order to get the desired control on the monotonicity of the Geroch energy sensible choices for certain maximal subsets have to be made, whereas the missing ingredients have to be constructed by some process, in order to guarantee (3.16) and (3.17) to hold. For instance, if the Riemannian metric \( h_{ij} \) on \( \Sigma \) is preferred to be fixed then the flow and foliation have to be constructed. This is indeed the path laid down by Geroch in [9] by proposing the use of inverse mean curvature flow and foliation. Alternatively, we may start with a globally well-defined foliation \( \rho : \Sigma \to \mathbb{R} \) with a suitably chosen fixed evolutionary vector field \( \rho^i = (\partial \rho)^i \) and keeping only the smooth distribution of two-metrics, \( \hat{\gamma}_{AB} \), on the \( \rho = \text{const} \) leaves in \( \Sigma \). Our new proposal is to replace the lapse and the shift by new ones in order to guarantee conditions (3.16) and (3.17) to hold.

4 Inverse mean curvature flows and foliations

As indicated above, once the three-metric is fixed, in order to get the desired control on the variation of the Geroch energy the foliation has to be constructed dynamically. In virtue of Proposition 3.1 it is not incidental that each of the known attempts aiming to get foliations with non-decreasing Geroch energy [9, 13, 14, 19, 15, 8, 29] essentially starts by specifying a two-sphere and construct the other members of the desired foliation by flowing the initial two-sphere in \( \Sigma \) using an inverse mean curvature flow (IMCF) \( \rho^i \) proposed originally by Geroch in [9]. The simplest possible form of such a flow is

\[
\rho^i_{(\text{IMCF})} = (\hat{K}^i_l)^{-1} \hat{n}^i.
\] (4.19)

Whenever the global existence of this flow can be shown—as for the corresponding foliation \( \hat{N} \hat{K}^i_l \equiv 1 \) holds automatically—the Geroch energy is non-decreasing with respect to it provided that (3.17) is also guaranteed to hold. As in this process the

\footnote{Note that there is a much higher variate of dynamically determined foliations and flows to be applied. The so-called \( \beta \)-foliations proposed by Jacek Jezierski in [16, 17, 18]—generalizing the “conformal harmonic gauge fixing” (corresponding to the case \( \beta = 1 \)) introduced originally by Jerzy Kijowski [19] and studied in some details by Piotr Chruściel in [6]—are excellent examples of these types. Note, however, that, likewise in case of the IMCF, proving the global existence of \( \beta \)-foliations is also a notoriously difficult problem, and, as far as we know, this has not been done yet apart from simple spherically symmetric configurations.}
foliation \( \rho : \Sigma \to \mathbb{R} \) and, in turn, the lapse \( \hat{N} = (\hat{n}^i \partial_i \rho)^{-1} \) and \( \hat{K}_{kl} \), get to be known only at the very end of the construction the inequality \( (\rho^o R) \geq 0 \) is imposed (see, e.g. [11, 12]) to get (3.17) to hold in an obvious way.

In virtue of the observations made in subsection 3.1 this construction can also be carried out in a bit more relaxed way. First, by applying an area variation based on relabeling \( \rho = \rho(\rho) \) such that \( \hat{N} \hat{K}_{ll} \equiv 1 \) is replaced by the relation \( \hat{N} \hat{K}_{ll} = \hat{N} \hat{K}_{ll} = L \rho \log[A \rho] \), a rescaled IMCF can be introduced. In addition—however counter intuitive it looks like, especially in virtue of the insensitivity of the variation of the area and the Geroch energy to the shift part—we can also add a “shift part” to the rescaled IMCF to get what, hereafter, will be referred to as a generalized inverse mean curvature flow (GIMCF)

\[
\rho^i_{\text{(GIMCF)}} = \mathcal{L}_\rho \log[\omega_\rho](\hat{K}_{ll})^{-1} \hat{n}^i + \hat{N}^i.
\]

It is important to keep in mind that, by virtue of (2.1) and (2.2), \( \hat{N} \hat{K}_{ll} \) and \( \hat{N}^i \) are related by

\[
\hat{N} \hat{K}_{ll} = \frac{1}{2} \gamma^{ij} \mathcal{L}_\rho \hat{N}_{ij} - \hat{D}_i \hat{N}^i
\]

or—whenever local coordinates adopted to the foliation and the flow are applied—by

\[
\hat{D}_A \hat{N}^A = \mathcal{L}_\rho \log[\sqrt{\det(\hat{\gamma}_{AB})}] - \mathcal{L}_\rho \log[\omega_\rho].
\]

We shall return, in subsection 5.1, to the solubility of this equation for \( \hat{N}^A \). Note that if \( \hat{\gamma}_{AB} \) was not fixed \( \hat{N}^A \) could also be chosen to be arbitrary, nevertheless, in such a case, (4.22) would immediately impose a non-trivial restriction on the first term on the r.h.s. of (4.22), i.e. on \( \hat{\gamma}_{AB} \).

The generalized inverse mean curvature flow could also be used, in practice, as follows. Start by choosing a mean-convex topological two-sphere \( \mathcal{S} \) in \( \Sigma \), and an arbitrary but small positive real number \( A > 0 \) and set the initial value, \( (0) \hat{N} \), to be the positive function \( (0) \hat{N} = A \cdot (\hat{K}_{ll})^{-1} \) on \( \mathcal{S} \). Construct now an infinitesimally close two-surface \( \mathcal{S}' \) simply by Lie dragging the points of \( \mathcal{S} \) along the auxiliary flow \( \rho^i = \hat{N} \hat{n}^i \) in \( \Sigma \). By comparing the metric induced on \( \mathcal{S} \) and \( \mathcal{S}' \), respectively, both terms on

3Note that according to the terms used in [2] a flow yielded by the type of rescalings of \( \rho^i_{\text{(IMCF)}} \), as applied here, was referred to be a “generalized” inverse mean curvature flow. We shall prefer to reserve this term for slightly more general flows which, according to the discussions below, do also have non-trivial shift parts.

4Notably, the \( \beta \)-foliations introduced by Jezierski [16, 17, 18], via setting up “gauge conditions” in a completely coordinate dependent way (see, e.g. eqs. (3) and (6) in [16]), when \( \beta = 0 \) and when the area radius, corresponding to the relation in the present case \( \mathcal{L}_\rho(\log[\omega_\rho]) = 2/\rho \), is used—(this later condition has never been spelled out properly in [16, 17, 18])—can be seen to be equivalent to a restricted class of the above introduced generalized inverse mean curvature flows.
the r.h.s. of (4.22) can be evaluated on $\mathcal{S}'$. In performing the succeeding steps we have to update both the lapse and the shift such that the relation $\hat{N}\hat{K}^i_l = \overline{\hat{N}}\overline{\hat{K}}^i_l$ gets to be maintained in each of these steps. In doing so update first the lapse on $\mathcal{S}'$ by setting

$$\hat{N} = L_\rho \log [A_\rho] \cdot (\hat{K}^i_l)^{-1},$$

where $L_\rho \log [A_\rho]$ is the positive real number determined via the previous infinitesimal step. The key point here is that one can also update the shift on $\mathcal{S}'$—such that $\hat{N}\hat{K}^i_l = \overline{\hat{N}}\overline{\hat{K}}^i_l$ holds there—simply by solving (4.22) for $\hat{N}^A$ as shown in subsection 5.1.

The succeeding infinitesimal step can then be done by Lie dragging the points of $\mathcal{S}'$ along the flow $\rho^i = \hat{N} \hat{n}^i + \hat{N}^i$ with lapse and shift determined on $\mathcal{S}'$ as indicated above. This way we get the next (infinitesimally close) two-surface $\mathcal{S}''$, and by performing analogous infinitesimal steps ultimately we get a one-parameter family of two-surfaces $\mathcal{S}_\rho$ foliating a one-sided neighborhood of $\mathcal{S}$ in $\Sigma$ such that the product $\hat{N}\hat{K}^i_l$ is guaranteed to be a positive constant on each of the individual leaves.

It is important to emphasize that the vanishing of $L_\rho \log [A_\rho]$, which in the conventional setup corresponds to the vanishing of $\hat{K}^i_l$, could get on the way of applicability of the use of these generalized inverse mean curvature flows. In particular, as $L_\rho \log [A_\rho]$ vanishes at minimal and maximal surfaces they do represent naturally limits to the domains in $\Sigma$, where a GIMCF can be applied. Note, however, that the occurrence of minimal and maximal surfaces depends on the choice we make for a time slice in the ambient space. For instance, while the bifurcation surface of the Schwarzschild spacetime is a minimal surface on the standard Schwarzschild $t_{Schw} = const$ time-slices, the Kerr-Schild $t_{KS} = const$ time-slices of the same spacetime can be foliated by metric spheres with area radius ranging from zero to infinity such that neither of the $r = const$ leaves is extremal. In this respect it is also important to keep in mind that the use of generalized inverse mean curvature foliations and flows does not require $\Sigma$ to be a ‘time symmetric’ or maximal, or in other words, the three-scalar curvature, $^{(3)}R$, is not required to be non-negative.

Note also that the most serious issue, namely, the global existence and regularity of foliations yielded by inverse mean curvature flows\textsuperscript{5} does not get to be relaxed as the leaves of foliations constructed by (4.19) or by (4.20) do coincide. Accordingly, the introduction of generalized inverse mean curvature flows may appear to be completely superfluous. The rest of this paper is to convince the readers that things are in order. More precisely, it is shown that a large variety of Riemannian three-spaces can be constructed such that each will be endowed with a smooth inverse mean curvature foliation and flow.

\textsuperscript{5}The level of the involved technicalities gets to be transparent in the proof of the Riemannian Penrose inequality by Huisken and Ilmanen [11, 12], or in that of the corresponding higher dimensional generalization by Bray [11], Bray and Lee [3].
5 The new construction

Our aim is now to construct Riemannian three-metrics on $\Sigma$ such that both of the conditions (3.16) and (3.17) in Proposition3.1 are guaranteed to hold. The most preferable property of the proposed new approach is that it starts by choosing a globally well-defined smooth foliation and flow on $\Sigma$. As indicated above, these later are the most expensive ingredients in the conventional construction of inverse mean curvature foliations. Obviously, we have to pay some price for these conveniences. Indeed, both the lapse and the shift have to be altered properly in order to get a unit norm vector field $\hat{n}^i = \hat{N}^{-1}(\rho^i - \hat{N}^i)$, and, in turn, a suitable Riemannian metric $h_{ij} = \hat{\gamma}^{ij} + \hat{n}_i \hat{n}_j$ on $\Sigma$ such that both of the conditions (3.16) and (3.17) get also to be guaranteed to hold on $\Sigma$.

5.1 Altering the shift

As already indicated, given the insensitivity of the variation of the area and that of the Geroch energy to the specific form of the shift the use of generalized inverse mean curvature flows may appear to be entirely counter-intuitive to use GIMCFs. Nevertheless, our construction starts by the determination of a suitable shift vector field on the individual leaves of the prescribed foliation. Indeed, as demonstrated below, while treating the foliation $\rho : \Sigma \to \mathbb{R}$, the flow field $\rho^i = (\partial_\rho)^i$ and the metric $\hat{\gamma}_{AB}$ induced on the $\rho = \text{const}$ leaves as prescribed fields (4.22) can always be used to determine the shift such that condition (3.16) holds throughout $\Sigma$.

Before applying (4.22) it is important to have a self-consistency check. Clearly, the integral of both sides of (4.22), when evaluated on any of the $\mathcal{S}_\rho$ leaves, must vanish. The integral of the total divergence on the l.h.s. is obviously zero, whereas the integral of the r.h.s. can also be seen to vanish by virtue of the relations

$$\int_{\mathcal{S}_\rho} \mathcal{L}_\rho \log \left[ \sqrt{\det(\hat{\gamma}_{AB})} \right] \tilde{\varepsilon} = \int_{\mathcal{S}_\rho} \mathcal{L}_\rho \left[ \sqrt{\det(\hat{\gamma}_{AB})} \right] \varepsilon = \mathcal{L}_\rho \left[ \int_{\mathcal{S}_\rho} \tilde{\varepsilon} \right] = \mathcal{L}_\rho [\mathcal{A}_\rho],$$

where $\varepsilon = \tilde{\varepsilon}/\sqrt{\det(\hat{\gamma}_{AB})}$ and its $\rho$-invariance were applied.

Notably, solving (4.22) is easier than it may appear for the first glance. To see this recall that as the first Betti number of topological two-spheres is zero they admit only the trivial harmonic form. This, along with the Hodge decomposition, allows us—for any fixed $\rho = \rho_0$ value—the shift vector to be represented on $\mathcal{S}_{\rho_0}$ by a pair of smooth functions $\chi$ and $\eta$ via the relation

$$\hat{N}^A = \hat{D}^A \chi + \hat{e}^{AB} \hat{D}_B \eta.$$  \hspace{1cm} (5.24)

The first and the second terms on the r.h.s. of (5.24) are the longitudinal and transversal parts of $\hat{N}^A$, with potentials $\chi$ and $\eta$, respectively. Using this representation, (4.22)
can be seen to take the form of the elliptic equation

\[ \hat{D}^A \hat{D}_A \chi = \mathcal{L}_\rho \log \left[ \sqrt{\det(\hat{\gamma}_{AB})} \right] - \mathcal{L}_\rho \log[\mathcal{A}_\rho] \]  

(5.25)

for \( \chi \) on \( \mathcal{J}_\rho \). Solutions to (5.25)—they are unique up to the “monopole” part—can, in principle, be constructed out of the coefficients of the expansion of the r.h.s. of (5.25) in terms of the eigenfunction of the Laplacian \( \hat{D}^A \hat{D}_A \) [4]. As all the coefficients and source terms in this elliptic equation smoothly depend on the smooth distribution of the metric \( \hat{\gamma}_{AB} \) solutions to (5.25) do exist and they do also smoothly depend on the \( \rho \)-coordinate. Accordingly, we have the following.

**Theorem 5.1.** Consider a three-dimensional smooth manifold \( \Sigma \). Assume that \( \Sigma \) is foliated by a one-parameter family of topological two-spheres \( \mathcal{J}_\rho \), determined by a smooth function \( \rho : \Sigma \to \mathbb{R} \) the gradient \( \partial_\rho \rho \) of which may vanish only at an isolated regular origin. Assume that a smooth flow \( \rho^i \) and a Riemannian three-metric, \( h_{ij} \), have also been chosen such that the integral curves of \( \rho^i \) intersect each of the \( \rho = \text{const} \) level surfaces precisely once, and such that a smooth distribution two-metrics \( \hat{\gamma}_{AB} \) is induced on the \( \mathcal{J}_\rho \) leaves of the foliation by \( h_{ij} \). Then, there exists (up to its monopole part) a unique smooth solution to (5.25) on \( \Sigma \) such that for the corresponding smooth vector field \( \hat{N}^A \) (4.22) holds regardless of the choice made for the potential \( \eta \).

Note that the above result does not require the increase of the area of the foliating two-surfaces. Note also that the most important implication of Theorem 5.1 is that whenever \( \hat{N}^A \) is constructed out of \( \hat{\gamma}_{AB} \) as described above then irrespective of the choice we can make for \( \hat{N} > 0 \) the corresponding metric, determined by (3.18), will satisfy condition (3.16) automatically throughout \( \Sigma \). Note also that as for \( \hat{N}^A \) (4.22) holds then regardless of the choice of \( \hat{N} > 0 \) the three-metric made out of the datum \( \hat{N}, \hat{N}^A, \hat{\gamma}_{AB} \) will be such that the (so far generic) smooth flow we started with (by the end of the above outlined process) gets to be a generalized inverse mean curvature flow and the foliation \( \mathcal{J}_\rho \) will get to be a generalized inverse mean curvature foliation.

### 5.2 Choosing the lapse

It remained to show that a function \( \hat{N} \) can also be chosen such that (3.17) holds for the constructed new three-metric on \( \Sigma \).

As shown below (3.17) imposes only a very mild integral condition on \( \hat{N} \). To see this notice first that, in virtue of (3.7), (3.9) and (3.10),

\[ ^{(3)}R = \hat{R} - \left\{ 2 \mathcal{L}_\rho(\hat{R}^l_l) + \frac{3}{2} (\hat{R}^l_l)^2 + \hat{K}^k_l K_k^l + 2 \left[ \hat{D}^l (\hat{N}^{-1} \hat{D}_l \hat{N}) + \hat{N}^{-2} \hat{\gamma}^{kl} (\hat{D}_k \hat{N})(\hat{D}_l \hat{N}) \right] \right\} \]  

(5.26)
holds, which implies that (3.17) is satisfied if and only if
\[ \int \mathcal{L}_p \left[ \hat{R} - \left\{ 2 \mathcal{L}_n(\hat{K}^l_l) + \frac{3}{2} (\hat{K}^l_l)^2 \right\} \right] \right| \geq 0. \] (5.27)

In demonstrating that (5.27) can indeed be rephrased as an integral equation for \( \hat{N} \) we shall use
\[ \hat{K}_{kl} = \frac{1}{2} \mathcal{L}_n \hat{\gamma}_{kl} = \hat{N}^{-1} \left[ \frac{1}{2} \mathcal{L}_p \hat{\gamma}_{kl} - \hat{D}_{(k} \hat{N}_{l)} \right] = \hat{N}^{-1} \hat{K}_{kl}, \] (5.28)
along with its contraction
\[ \hat{K}^l_l = \hat{\gamma}^{kl} \hat{K}_{kl} = \hat{N}^{-1} \left[ \frac{1}{2} \hat{\gamma}^{kl} \mathcal{L}_p \hat{\gamma}_{kl} - \hat{D}_l \hat{N}^l \right] = \hat{N}^{-1} \hat{K}, \] (5.29)
where
\[ \hat{K} = \hat{N} \hat{K}^l_l = \frac{1}{2} \hat{\gamma}^{kl} \mathcal{L}_p \hat{\gamma}_{kl} - \hat{D}_l \hat{N}^l = \mathcal{L}_p \log[\omega^\mu]. \] (5.30)
Using the relation \( \hat{n}^i = \hat{N}^{-1} [\rho^i - \hat{N}^i] \) we also get
\[ \mathcal{L}_n (\hat{K}^l_l) = \hat{N}^{-1} \left[ \mathcal{L}_p (\hat{K}^l_l) - \mathcal{L}_n (\hat{K}^l_l) \right] = \hat{K}^{-1} (\hat{K} \hat{N}^{-1}) \left[ \mathcal{L}_p (\hat{K}^l_l) - \mathcal{L}_n (\hat{K}^l_l) \right] \]
\[ = \hat{K}^{-1} (\hat{K}^l_l) \left[ \mathcal{L}_p (\hat{K}^l_l) - \mathcal{L}_n (\hat{K}^l_l) \right] = \frac{1}{2} \hat{K}^{-1} \left[ \mathcal{L}_p (\hat{K}^l_l)^2 - \mathcal{L}_n (\hat{K}^l_l)^2 \right], \] (5.31)
where \( \mathcal{L}_X (\hat{K}^l_l)^2 = 2 (\hat{K}^l_l) \mathcal{L}_X (\hat{K}^l_l) \), that holds for any smooth vector field \( X^a \) on \( \Sigma \), was also used.

By substituting (5.31) into the second term in the integrand of (5.27) and using the replacement \( (\hat{K}^l_l)^2 = \hat{K}^2 \hat{N}^{-2} \) we get
\[ 2 \mathcal{L}_n (\hat{K}^l_l) + \frac{3}{2} (\hat{K}^l_l)^2 = \hat{K}^{-1} \left\{ \mathcal{L}_p [\hat{K}^2 \hat{N}^{-2}] - \hat{N}^A \hat{D}_A [\hat{K}^2 \hat{N}^{-2}] \right\} + \frac{3}{2} [\hat{K}^2 \hat{N}^{-2}]. \] (5.32)

Using then
\[ \mathcal{L}_p [\hat{K}^2 \hat{N}^{-2}] = 2 (\hat{K} \mathcal{L}_p \hat{K} \hat{N}^{-2}) + \hat{K}^2 (\mathcal{L}_p \hat{N}^{-2}), \] (5.33)
along with (4.22) and \( \hat{K} = \mathcal{L}_p \log[\omega^\mu], \) and also by applying some simple algebra and dropping the total divergence \( \hat{D}_A (\hat{N}^A [\hat{K}^2 \hat{N}^{-2}]) \), we get
\[ 2 \mathcal{L}_n (\hat{K}^l_l) + \frac{3}{2} (\hat{K}^l_l)^2 = \hat{K}^{-1} \left\{ \mathcal{L}_p [\hat{K}^2 \hat{N}^{-2}] + (\hat{D}_A \hat{N}^A) [\hat{K}^2 \hat{N}^{-2}] \right\} + \frac{3}{2} [\hat{K}^2 \hat{N}^{-2}] \]
\[ = 2 (\mathcal{L}_p \hat{K} \hat{N}^{-2}) + \hat{K} (\mathcal{L}_p \hat{N}^{-2}) + \mathcal{L}_p \left( \log \left[ \sqrt{\det(\hat{\gamma}_{AB})} \right] \right) [\hat{K} \hat{N}^{-2}] + \frac{1}{2} [\hat{K}^2 \hat{N}^{-2}]. \] (5.34)
By assuming from now on that $\rho$ is the area “radial coordinate”, i.e. $A_\rho = 4\pi \rho^2$, we immediately get

$$\dot{K} = \mathcal{L}_\rho \log[A_\rho] = \frac{2}{\rho} \quad \text{and} \quad \mathcal{L}_\rho \dot{K} = -\frac{2}{\rho^2}.$$  \hspace{1cm} (5.35)

Thereby (5.34) may also be written as

$$2 \mathcal{L}_\rho (\dot{K}_l^l) + \frac{3}{2} (\dot{K}_l^l)^2 = -\frac{2}{\rho} \dot{N}^{-2} + \frac{3}{2} (\mathcal{L}_\rho \dot{N}^{-2}) + \mathcal{L}_\rho \left( \log \left[ \sqrt{\det(\tilde{\gamma}_{AB})} \right] \right) \left[ \frac{2}{\rho} \dot{N}^{-2} \right]$$

$$= \left\{ \mathcal{L}_\rho \left( \log \left[ \frac{2}{\rho} \dot{N}^{-2} \right] \right) + \mathcal{L}_\rho \left( \log \left[ \sqrt{\det(\tilde{\gamma}_{AB})} \right] \right) \right\} \left[ \frac{2}{\rho} \dot{N}^{-2} \right].$$  \hspace{1cm} (5.36)

Returning to the integral inequality (5.27), by referring to the Gauss-Bonnet Theorem, we get

$$8\pi \geq \int_{\mathcal{S}_\rho} \left[ 2 \mathcal{L}_\rho (\dot{K}_l^l) + \frac{3}{2} (\dot{K}_l^l)^2 \right] \tilde{\epsilon}$$

$$= \int_{\mathcal{S}_\rho} \mathcal{L}_\rho \left[ \log \left\{ \frac{2}{\rho} \dot{N}^{-2} \sqrt{\det(\tilde{\gamma}_{AB})} \right\} \right] \left( \frac{2}{\rho} \dot{N}^{-2} \right) \tilde{\epsilon}$$

$$= \int_{\mathcal{S}_\rho} \mathcal{L}_\rho \left[ \frac{2}{\rho} \dot{N}^{-2} \sqrt{\det(\tilde{\gamma}_{AB})} \right] \tilde{\epsilon} = \mathcal{L}_\rho \left[ \int_{\mathcal{S}_{\rho_0}} \frac{2}{\rho} \dot{N}^{-2} \tilde{\epsilon} \right],$$  \hspace{1cm} (5.37)

where in the last but one step the $\rho$-invariant two-form $\tilde{\epsilon} = \tilde{\epsilon}/\sqrt{\det(\tilde{\gamma}_{AB})}$ was applied.

Appealing then to the fact that both $\rho$ and $\dot{N}$ are positive, and integrating (5.37) we get

$$\int_{\mathcal{S}_\rho} \frac{2}{\rho} \dot{N}^{-2} \tilde{\epsilon} - \int_{\mathcal{S}_{\rho_0}} \frac{2}{\rho_0} \dot{N}^{-2} \tilde{\epsilon} \leq \int_{\rho_0}^{\rho} 8\pi \, d\rho = 8\pi [\rho - \rho_0]$$  \hspace{1cm} (5.38)

which implies that (3.17) is satisfied if and only if the integral inequality

$$\int_{\mathcal{S}_\rho} \dot{N}^{-2} \tilde{\epsilon} \leq 4\pi \rho [\rho - \rho_0] + \rho \rho_0 \int_{\mathcal{S}_{\rho_0}} \dot{N}^{-2} \tilde{\epsilon} = A_\rho - \sqrt{A_{\rho_0} \cdot A_\rho} \left[ 1 - \frac{1}{4\pi} \int_{\mathcal{S}_{\rho_0}} \dot{N}^{-2} \tilde{\epsilon} \right]$$  \hspace{1cm} (5.39)

holds for $\dot{N}$, where $\tilde{\epsilon}$ is the volume element on the $\mathcal{S}_\rho$ leaves such that $\int_{\mathcal{S}_\rho} \tilde{\epsilon} = 4\pi$.

What have been done in the current and in the previous subsection may also be summarized as follows. Some tiny alterations of the three-metric $h_{ij}$, we started with, were performed such that an essentially arbitrarily chosen smooth foliation $\rho : \Sigma \to \mathbb{R}$ and flow $\rho^i = (\partial_\rho)^i$, along with the smooth two-metric $\tilde{\gamma}_{AB}$, induced on the $\rho = \text{const}$ level surfaces, are kept to be fixed but $\dot{N}$ and $\dot{N}^A$ were deformed. All these were done such that for the yielded three-metric, made out of the new datum $\dot{N}$, $\dot{N}^A$, $\tilde{\gamma}_{AB}$, both conditions (3.16) and (3.17) of Proposition 3.1 hold. By summarizing what has been shown above we have the following.
Theorem 5.2. Assume that the conditions of Theorem 5.1 hold, and that the area of the foliating level sets is strictly increasing. Assume also that the constructed $\hat{N}^A$ is such that $\hat{K} = \hat{N}K^I_i = \mathcal{L}_p \log[\mathcal{A}_p]$ throughout $\Sigma$. Then, for any smooth positive function $\hat{N} : \Sigma \to \mathbb{R}$ such that the integral inequality (5.39) holds, the three-metric made out of the datum $\hat{N}, \hat{N}^A, \hat{\gamma}_{AB}$ will be such that the prescribed smooth foliation and flow we started with, by the end of the above outlined process, get to be generalized inverse mean curvature foliation and flow, and, in turn, for each of the yielded Riemannian three-spaces the Geroch energy is non-decreasing.

5.3 Some quasi-local and global results

It is important to know if for the constructed three-geometries, besides the monotonic behavior of the Geroch energy, some other favorable properties also hold. In this section a quasi-local form of the Penrose inequality, along with the positive energy theorem, will be shown to hold.

5.3.1 “Quasi-local” Penrose inequality

Start by noticing that, by virtue of $\mathcal{A}_\rho = \int_{\mathcal{S}_\rho} \hat{\epsilon}$, (5.39) may also be rephrased as

$$\left(\mathcal{A}_{\rho_0} \cdot \mathcal{A}_\rho\right)^{1/2} \left[1 - \frac{1}{4\pi} \int_{\mathcal{S}_{\rho_0}} \hat{N}^{-2} \hat{\epsilon}\right] \leq \int_{\mathcal{S}_\rho} \left[1 - \hat{N}^{-2}\right] \hat{\epsilon}, \quad (5.40)$$

which immediately suggests that there must be an intimate relation between (5.40) and the Penrose inequality. To see that this is indeed the case it is rewarding to rewrite (3.5)—by applying the area “radial coordinate” $\rho$—as

$$E_G = \frac{\mathcal{A}_\rho^{1/2}}{64\pi^{3/2}} \int_{\mathcal{S}_\rho} \left[2 \hat{R} - (\hat{K}_i^I)^2 \hat{\epsilon}\right] = \frac{\mathcal{A}_{\rho_0}^{1/2}}{64\pi^{3/2}} \left[16\pi - \int_{\mathcal{S}_\rho} \hat{K}^2 \hat{N}^{-2} \hat{\epsilon}\right]$$

$$= \frac{\mathcal{A}_{\rho_0}^{1/2}}{64\pi^{3/2}} \left[\left(\frac{2}{\rho}\right)^2 (4\pi \rho^2) - \int_{\mathcal{S}_\rho} \left(\frac{2}{\rho}\right)^2 \hat{N}^{-2} \hat{\epsilon}\right] = \frac{1}{4\pi^{1/2} \mathcal{A}_{\rho_0}^{1/2}} \int_{\mathcal{S}_\rho} \left[1 - \hat{N}^{-2}\right] \hat{\epsilon}, \quad (5.41)$$

where, in the last two steps, the relation $\mathcal{A}_\rho = 4\pi \rho^2 = \int_{\mathcal{S}_\rho} \hat{\epsilon}$ was used repeatedly. Combining then (5.40) and (5.41), and assuming that (in the non-flat case) $\hat{N} > 1$, apart from regular origin(s), we immediately get that

$$E_G = \frac{1}{4\pi^{1/2} \mathcal{A}_{\rho_0}^{1/2}} \int_{\mathcal{S}_{\rho_0}} \left[1 - \hat{N}^{-2}\right] \hat{\epsilon} \geq \frac{1}{4\pi^{1/2} \mathcal{A}_{\rho_0}^{1/2}} \left[1 - \frac{1}{4\pi} \int_{\mathcal{S}_{\rho_0}} \hat{N}^{-2} \hat{\epsilon}\right], \quad (5.42)$$

or, in a more familiar form, that $\mathcal{A}_{\rho_0} \left[1 - \frac{1}{4\pi} \int_{\mathcal{S}_{\rho_0}} \hat{N}^{-2} \hat{\epsilon}\right] \leq 16\pi E_G^2$. As this later inequality holds for any $\rho > \rho_0$ value it should hold in the limiting $\rho \to \rho_0^+$ case, as
Notably, it allows one to compare the surface area and the quasi-local Geroch energy for each of the individual leaves of the generalized inverse mean curvature foliation in either of the constructed Riemannian three-spaces. Remarkably, if $\mathcal{S}_{\rho_0}$ is a minimal surface with $\hat{N} \to 0$ while $\rho \to \rho^\pm_0 (>0)$ (5.43) reduces to the quasi-local form of the Penrose inequality

$$\mathcal{A}_{\rho_0} \leq 16\pi E_G^2|_{\rho_0}. \quad (5.44)$$

Notice that this form of the Penrose inequality is much stronger than the conventional one used in the literature as it relates the area of a minimal surface with the quasi-local Geroch energy evaluated there.

### 5.3.2 Some of the global aspects

Restricting now attention to asymptotically flat configurations recall first that a three-metric $h_{ij}$ is asymptotically flat if in the asymptotic region it approaches the Euclidean metric not slower than $\rho^{-1}$. This condition, in virtue of the results in subsection 2.2.1 of [7], can also be rephrased in terms of the variables $\hat{N}, \hat{N}^A, \hat{\gamma}_{AB}$ by requiring them to fall off as

$$\hat{N} - 1 \sim O(\rho^{-1}), \quad \hat{N}^A \sim O(\rho^{-3}) \quad \text{and} \quad \hat{\gamma}_{AB} - \rho^2 \hat{\gamma}_{AB} \sim O(\rho^{-1}), \quad (5.45)$$

where $\hat{\gamma}_{AB}$ denotes the unit sphere metric. As the asymptotic form (5.25) reads as $\hat{\gamma}^{AB} \partial_A \partial_B \chi = 0$ its solution should be of order zero in $\rho$. This, in virtue of (5.24), implies that $\hat{D}^A \chi \sim \rho^{-2}, \hat{\gamma}^{AB} \partial_B \chi$ should fall off not slower than $\rho^{-3}$. Accordingly, if the other potential $\eta$ is chosen suitably the altered shift will fall off sufficiently fast to allow the constructed three-metric to be asymptotically flat. Clearly, the inequality (5.39) imposes only a very mild restriction on $\hat{N}$ thereby the first condition in (5.45) will also hold for $\hat{N}$. As the metric $\hat{\gamma}_{AB}$, induced on the leaves of the foliation $\mathcal{S}_\rho$, was kept to be fixed the third condition in (5.45) will automatically hold if the original three-metric is asymptotically flat. By combining all the above observations we get that the constructed three-spaces will be asymptotically flat if the original was so. This, along with the monotonous behavior of the Geroch energy, verifies then the last statement of the theorem below.

In deriving the “global” form of the Penrose inequality recall first that for an asymptotically flat configuration the Arnowitt-Deser-Misner (ADM) total energy $E_{ADM}$ can be obtained by taking the $\rho \to \infty$ limit of the Geroch energy (3.5) [13]. As Theorem 5.2 guarantees that the Geroch energy is non-decreasing, in virtue of (5.44) the
global form of the Penrose inequality,
\[ \mathcal{A}_{\rho_0} \leq 16\pi E_{\text{ADM}}^2, \]  
(5.46)
can also be seen to hold.

By combining the results of this section we get the following.

**Theorem 5.3.** Consider a Riemannian three-space \((\Sigma, h_{ij})\), and assume that the conditions of Theorems 5.1 and 5.2 hold. Then for any of the constructed Riemannian three-spaces

(i) on any of the individual leaves \(\mathcal{S}_\rho\) of the generalized inverse mean curvature foliation the surface area \(A_\rho\), multiplied by the factor \[1 - \frac{1}{4\pi} \int_{\mathcal{S}_\rho} \hat{N}^{-2} \hat{e},\] is always less than or equal to \(16\pi E_{\hat{G}}^2\),

(ii) if \(\mathcal{S}_{\rho_0}\), for some \(\rho_0 > 0\), is a minimal surface then the quasi-local form \(5.44\) of the Penrose inequality holds, and

(iii) if, in addition, the original Riemannian three-metric is asymptotically flat the positive energy theorem holds, and if, in addition, a minimal surface exists on \(\Sigma\), the global form of the Penrose inequality \(5.46\) also holds.

### 6 Final remarks

A new method was introduced that enables us to construct high variety of Riemannian three-spaces such that each admits a smooth generalized inverse mean curvature foliation and flow. In doing so we started with a smooth Riemannian three-metric \(h_{ij}\) on a smooth three-dimensional manifold \(\Sigma\). The latter was assumed to be smoothly foliated by topological two-spheres the area of which was also assumed to be strictly increasing. In addition, a smooth flow was also chosen. The desired type of Riemannian three-spaces were constructed by replacing the lapse and the shift by suitably chosen new lapse and shift such that both conditions (3.16) and (3.17) of Proposition 3.1 hold. Condition (3.16) guarantees that the prescribed foliating two-spheres get to form a generalized inverse mean curvature foliation and the original flow gets to be a generalized inverse mean curvature flow, whereas if, in addition, condition (3.17) holds, as well, then the Geroch energy is also guaranteed to be non-decreasing. Remarkably, a “normalized” area of the leaves was found to be bounded by the pertinent Geroch energy expression. This quasi-local comparison, whenever a minimal surface exists on \(\Sigma\), reduces to a quasi-local form of the Penrose inequality. If the metric \(h_{ij}\) we started with is asymptotically flat the constructed three-geometries will be so, and, in addition, to the positive energy theorem, the conventional global form of the
Penrose inequality was also shown to hold if a minimal surface is guaranteed to exist on $\Sigma$.

The main message conveyed by Theorems 5.1 and 5.2 is that the variety of Riemannian three-spaces that can be constructed by the proposed new method is really large. To see this it is rewarding to inspect the elementary steps made in producing the desired type of Riemannian three-spaces. In constructing the new shift (5.25), and in turn, (4.22) were solved. In doing so only one of the potentials in (5.24) gets to be determined whereas the other potential remains freely specifiable. The choice of the lapse is limited only by the integral inequality (5.39) which enlarges the variability of the constructed three-spaces considerably.

Concerning the scalar curvature $^{(3)}R$ of the constructed three-spaces no stronger assumption was made than the integral inequality (3.17). In particular, our construction does not require the scalar curvature of the three-metric to be non-negative, and, in turn, the constructed Riemannian three-geometries are ready to accommodate non-maximal “timeslices”. Therefore, it is of primary importance to know if in realistic situations slices with almost everywhere non-negative scalar curvature exists. To this end it is important that even in Minkowski spacetime there are smooth spherically symmetric spacelike hypersurfaces such that their scalar curvature, apart from the “symmetry center”, is everywhere negative and tends to zero in approaching spacelike infinity. This timeslice, along with a flood of slightly perturbed ones, clearly demonstrates that the new method proposed in the present paper does indeed possesses the desired capability to host three-spaces with arbitrary scalar curvature.

It is important to emphasize that as far as the construction of generalized inverse mean curvature foliations and flows is concerned the topology of the base manifold—apart from assuming that $\Sigma$ can smoothly be foliated by mean convex topological two-spheres which may degenerate at regular origin(s)—was not restricted. Accordingly, the first part of the proposed method—as it was described in details in subsection 5.1—does apply to three-manifolds possessing either of the topologies $\mathbb{R}^3$, $S^3$, $\mathbb{R} \times S^2$ or $S^1 \times S^2$ with one, two or no origin(s) at all.

Notably, no use of Einstein’s equations or any other field equations concerning the metric $h_{ij}$ or that of an ambient space had been applied anywhere in our construction. Note also that merely the Riemannian character of the three-metric $h_{ij}$ on $\Sigma$ was used. Accordingly, if $\Sigma$ happens to be a hypersurface in some ambient four-dimensional space the signature of the pertinent metric could be either Lorentzian or Euclidean. The most important implication of all these observations is that the proposed new construction does essentially apply to all the geometrized theories of gravity.
Acknowledgments

The author is grateful to Lars Anderson and Robert Wald for helpful correspondence. Special thanks are due to Bahar Kirik Rácz for careful reading and helpful corrections. This project was supported by the POLONEZ programme of the National Science Centre of Poland (under the project No. 2016/23/P/ST1/04195) which has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 665778.

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