The volume growth of complete gradient shrinking Ricci solitons

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Abstract

We prove that any gradient shrinking Ricci soliton has at most Euclidean volume growth. This improves a recent result of H.-D. Cao and D. Zhou by removing a condition on the growth of scalar curvature.

A complete Riemannian manifold $M^n$ of dimension $n$ is called gradient shrinking Ricci soliton if there exists $f \in C^\infty (M)$ and a constant $\rho > 0$ such that

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij},$$

where $R_{ij}$ denotes the Ricci curvature tensor and $\nabla_i \nabla_j f$ denotes the Hessian of $f$. We can scale the metric on $M$ such that $\rho = \frac{1}{2}$, which will always be assumed in this paper i.e.

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2} g_{ij}. \quad (1)$$

Gradient Ricci solitons have been intensively studied in the context of the Ricci flow ([H1], [H2]) and are natural generalizations of Einstein metrics. Since often the limit of dilations of singularities in the Ricci flow is a Ricci soliton, it is useful to have a good knowledge of their geometry.

In a recent paper, H.-D. Cao and D. Zhou have studied the volume growth rate of complete noncompact gradient shrinking solitons [C-Z]. They proved, assuming the normalization (1), that if a compete gradient shrinking Ricci soliton has scalar curvature bounded above by

$$R (x) \leq \alpha r^2 (x) + A (r (x) + 1),$$
for some $0 \leq \alpha < \frac{1}{4}$, then there exists $C > 0$ such that for $r > 0$ sufficiently large

$$Vol(B_p(r)) \leq Cr^n,$$

where $Vol(B_p(r))$ is the volume of the geodesic ball of radius $r$ at $p$.

A standard example of gradient shrinking soliton is the Gaussian soliton, which is the flat space $(\mathbb{R}^n, dx^2)$ with $R = 0$ and $f(x) = \frac{1}{4} |x|^2$. Having in mind the Gaussian soliton, we see that this volume growth is optimal. On the other hand, the above assumption on the growth of scalar curvature is mild, since by a result in [C-C-Z] any complete gradient shrinking soliton satisfies $R(x) \leq \frac{1}{4} (r(x) + c)^2$.

One question raised by Cao-Zhou is if this assumption can be dropped. In this short note we prove that indeed the same volume growth holds without any assumptions on the gradient shrinking soliton. We establish the following.

**Theorem 1** Let $M^n$ be a complete noncompact gradient shrinking Ricci soliton normalized as in (1). Then there exist constants $C > 0$ and $\delta > 0$ such that for any $r \geq \delta$

$$Vol(B_p(r)) \leq Cr^n.$$

**Proof.** We first recall the following standard properties of a shrinking Ricci soliton. Taking the trace in (1) it follows that

$$R + \Delta f = \frac{n}{2}.$$ 

Using the Bianchi identities, it can be proved that there exists a constant $C_0$ such that

$$R + |\nabla f|^2 - f = C_0.$$ 

Clearly, we can normalize $f$ such that $C_0 = 0$. This will always be assumed in this note, therefore

$$R + |\nabla f|^2 - f = 0.$$ 

We also recall a known result of B.L. Chen, which states that any complete shrinking Ricci soliton has nonnegative scalar curvature i.e. $R \geq 0$, see [C].

To prove the theorem, the following asymptotic estimate for the potential function will be instrumental, see [C-Z]:

$$\frac{1}{4} (r(x) - c)^2 \leq f(x) \leq \frac{1}{4} (r(x) + c)^2,$$ 

(2)
where \( r(x) = d(p, x) \) is the distance from a fixed point \( p \in M \) and \( c \) is a constant depending on \( n \) and the geometry of \( B_p(1) \).

Note that when the Ricci curvature of \( M \) is bounded this is a result of Perelman, [P]. When the Ricci curvature is non-negative, that \( f \) has at most quadratic growth was pointed out in Lemma 2.3 of [N-W], based on the results in [Ni]. The precise upper bound in (2) was observed in [C-C-Z]. The lower bound can be deduced from the argument in [F-M-Z], using \( R \geq 0 \).

Let
\[
\rho(x) = 2\sqrt{f(x)}.
\]
Then, by (2) we know that
\[
r(x) - c \leq \rho(x) \leq r(x) + c
\]
We denote
\[
D(r) = \{ x : \rho(x) < r \}
\]
\[
V(r) = \text{vol}(D(r)) = \int_{D(r)} dv
\]
\[
\chi(r) = \int_{D(r)} R dv
\]
Our goal is to prove that for any \( r \geq \delta \),
\[
V(r) \leq Cr^n,
\]
which clearly proves the theorem because \( \rho(x) \) and \( r(x) \) are equivalent.

We have the following inequality, established in [C-Z]:
\[
\frac{V(r)}{r^n} - \frac{V(r_0)}{r_0^n} \leq 4 \frac{\chi(r)}{r^{n+2}}.
\] (3)
This holds for any \( r > r_0 > \sqrt{2(n+2)} \). For completeness, we include its proof below.

Using \( R + \Delta f = \frac{n}{2} \), we get
\[
2 \int_{D(r)} \Delta f = nV(r) - 2\chi(r).
\] (4)
On the other hand, using that $R + |\nabla f|^2 - f = 0$ and the co-area formula it results that

$$2 \int_{D(r)} \Delta f = 2 \int_{\partial D(r)} \nabla f \cdot \frac{\nabla \rho}{|\nabla \rho|} = 4 \int_{\partial D(r)} \frac{|\nabla f|^2}{|\nabla \rho|} \quad (5)$$

Therefore we arrived at the following identity (Lemma 3.1 in [C-Z])

$$nV(r) - rV'(r) = 2\chi(r) - \frac{4}{r}\chi'(r).$$

Multiply this identity by $r^{-n-1}$ and integrate from $r_0$ to $r$ we get

$$\frac{V(r)}{r^n} - \frac{V(r_0)}{r_0^n} = 4 \int_{r_0}^r s^{-n-2}\chi'(s) \, ds - 2 \int_{r_0}^r (n-1)s^{-n-1}\chi(s) \, ds.$$

We integrate the right hand side by parts to obtain

$$\frac{V(r)}{r^n} - \frac{V(r_0)}{r_0^n} = 4 \left( \frac{\chi(r)}{r^{n+2}} - \frac{\chi(r_0)}{r_0^{n+2}} \right) + 2 \int_{r_0}^r s^{-n-3}\chi(s) \left( 2(n+2) - s^2 \right) \, ds.$$

Finally, (3) follows from the observation that $\chi(s)$ is non-negative, because $R \geq 0$.

We now finish the proof. Notice that (5) implies that

$$\int_{D(r)} \Delta f \geq 0,$$

hence, by (4) we get that

$$\chi(r) \leq \frac{n}{2} V(r).$$

Plugging this in (3) it follows that

$$V(r) \leq \left( \frac{V(r_0)}{r_0^n} \right) r^n + 4 \frac{\chi(r)}{r^2} \leq \left( \frac{V(r_0)}{r_0^n} \right) r^n + 2n \frac{V(r)}{r^2}.$$

Clearly, choosing $r > 2\sqrt{n}$ we obtain

$$V(r) \leq 2 \left( \frac{V(r_0)}{r_0^n} \right) r^n,$$

which proves the theorem.
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