A Note on 3-quasi-Sasakian Geometry

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Abstract. 3-quasi-Sasakian manifolds were recently studied by the authors as a suitable setting
unifying 3-Sasakian and 3-cosymplectic geometries. In this paper some geometric properties of this
class of almost 3-contact metric manifolds are briefly reviewed, with an emphasis on those more
related to physical applications.

Keywords: Almost contact metric 3-structures, 3-Sasakian manifolds, 3-cosymplectic manifolds.
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1. INTRODUCTION

The class of 3-quasi-Sasakian manifolds is the analogue in the setting of 3-structures of
the class of quasi-Sasakian manifolds, introduced by Blair [3] and later studied among
others by Tanno [15], Kanemaki [13], Olszak [14]. More recent are the examples of ap-
plications of quasi-Sasakian manifolds to string theory found by Friedrich and his col-
laborators [2, 10]. Just like quasi-Sasakian manifolds include Sasakian and cosymplectic
manifolds, so 3-quasi-Sasakian manifolds unify 3-Sasakian and 3-cosymplectic geometry.
A 3-quasi-Sasakian manifold can arise, for example, as the product of a 3-Sasakian
manifold and a hyper-Kähler manifold (see Sect. 3 or [8]). The setting of 3-structures
has been recently the object of a wider interest from both mathematicians and physi-
cists due to the important role acquired by the 3-Sasakian and the related quaternionic
structures in supergravity and superstring theory, where they appear in the so called hy-
permultiplet solutions (see e. g. [1, 2, 6, 17]). This note contains a concise review of the
main properties of 3-quasi-Sasakian manifolds, recently studied by the authors in [8],
together with some relevant properties of the two important subclasses of 3-Sasakian
and 3-cosymplectic manifolds which were compared in [9].

2. 3-QUASI-SASAKIAN GEOMETRY

An almost contact metric manifold is a $(2n + 1)$-dimensional manifold $M$ endowed
with a field $\phi$ of endomorphisms of the tangent spaces, a vector field $\xi$, called Reeb
vector field, a 1-form $\eta$ satisfying $\phi^2 = -I + \eta \otimes \xi$, $\eta(\xi) = 1$ (where $I: TM \to TM$
is the identity mapping) and a compatible Riemannian metric $g$ such that $g(\phi X, \phi Y) =
g(X, Y) - \eta(X)\eta(Y)$ for all $X, Y \in \Gamma(TM)$. The manifold is said to be normal if
the tensor field $N^{(1)} = [\phi, \phi] + 2d\eta \otimes \xi$ vanishes identically. The 2-form $\Phi$ on $M$
defined by $\Phi(X, Y) = g(\phi X, \phi Y)$ is called the fundamental 2-form of the almost contact
metric manifold \((M, \phi, \xi, \eta, g)\). Normal almost contact metric manifolds such that both \(\eta\) and \(\Phi\) are closed are called \textit{cosymplectic manifolds} and those such that \(d\eta = \Phi\) are called \textit{Sasakian manifolds}. The notion of quasi-Sasakian structure unifies those of Sasakian and cosymplectic structures. A \textit{quasi-Sasakian manifold} is defined as a normal almost contact metric manifold whose fundamental 2-form is closed. A quasi-Sasakian manifold \(M\) is said to be of rank \(2p\) (for some \(p \leq n\)) if \((d\eta)^p \neq 0\) and \(\eta \wedge (d\eta)^p = 0\) on \(M\), and to be of rank \(2p + 1\) if \(\eta \wedge (d\eta)^p \neq 0\) and \((d\eta)^{p+1} = 0\) on \(M\) (cf. [3, 15]). Blair proved that there are no quasi-Sasakian manifolds of even rank. Just like Blair and Tanno did, we will only consider quasi-Sasakian manifolds of constant (odd) rank.

We now come to the main topic of our paper, i.e. 3-quasi-Sasakian geometry, which is framed into the more general setting of almost 3-contact geometry. An \textit{almost 3-contact metric manifold} is a \((4n + 3)\)-dimensional smooth manifold \(M\) endowed with three almost contact structures \((\phi_1, \xi_1, \eta_1), (\phi_2, \xi_2, \eta_2), (\phi_3, \xi_3, \eta_3)\) satisfying the following relations, for any even permutation \((\alpha, \beta, \gamma)\) of \(\{1, 2, 3\}\),

\[
\phi_\gamma = \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta \phi_\alpha + \eta_\alpha \otimes \xi_\beta, \\
\xi_\gamma = \phi_\alpha \xi_\beta = -\phi_\beta \xi_\alpha, \quad \eta_\gamma = \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha,
\]

and a Riemannian metric \(g\) compatible with each of them. It is well known that in any almost 3-contact metric manifold the Reeb vector fields \(\xi_1, \xi_2, \xi_3\) are orthonormal with respect to the compatible metric \(g\) and that the structural group of the tangent bundle is reducible to \(Sp(n) \times I_3\). Due to a general result (cf. [12, Prop. 3.6.2]), it follows that any 3-quasi-Sasakian manifold is a \textit{spin manifold}, i.e. there exists a double cover of the orthonormal frame bundle \(SO(TM)\), which is non-trivial on the fibers of the latter, by a principal bundle with structure group a spin group. Then, to each representation of the spin group corresponds an associated vector bundle whose sections are called \textit{spinor fields} (see [7] for details). The existence of spinor fields is required in quantum theories which encompass fermions.

Moreover, by putting \(\mathcal{H} = \bigcap_{\alpha=1}^3 \ker(\eta_\alpha)\) one obtains a \(4n\)-dimensional \textit{horizontal} distribution on \(M\) and the tangent bundle splits as the orthogonal sum \(TM = \mathcal{H} \oplus \mathcal{V}\), where \(\mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle\) is the \textit{vertical} distribution.

**Definition 2.1** A 3-quasi-Sasakian manifold is an almost 3-contact metric manifold \((M, \phi, \xi, \eta, g)\) such that each almost contact structure is quasi-Sasakian.

The class of 3-quasi-Sasakian manifolds includes as special cases the well-known 3-Sasakian and 3-cosymplectic manifolds.

The following theorem combines the results obtained in Theorems 3.4 and 4.2 of [8].

**Theorem 2.2** Let \((M, \phi, \xi, \eta, g)\) be a 3-quasi-Sasakian manifold. Then the 3-dimensional distribution \(\mathcal{V}\) generated by \(\xi_1, \xi_2, \xi_3\) is integrable. Moreover, \(\mathcal{V}\) defines a totally geodesic and Riemannian foliation of \(M\) and for any even permutation \((\alpha, \beta, \gamma)\)
of \( \{1,2,3\} \) and for some \( c \in \mathbb{R} \)
\[ [\xi_\alpha, \xi_\beta] = c \xi_\gamma. \]

Using Theorem 2.2 we may divide 3-quasi-Sasakian manifolds in two classes according to the behaviour of the leaves of the foliation \( \mathcal{Y} \): those 3-quasi-Sasakian manifolds for which each leaf of \( \mathcal{Y} \) is locally \( SO(3) \) (or \( SU(2) \)) (which corresponds to take in Theorem 2.2 the constant \( c \neq 0 \)), and those for which each leaf of \( \mathcal{Y} \) is locally an abelian group (this corresponds to the case \( c = 0 \)).

The preceding theorem also allows to define a canonical metric connection on any 3-quasi-Sasakian manifold. Indeed, let \( \nabla^B_\mathcal{Y} \) be the Bott connection associated to \( \mathcal{Y} \), that is the partial connection on the normal bundle \( TM/\mathcal{Y} \cong \mathcal{H} \) of \( \mathcal{Y} \) defined by \( \nabla^B_\mathcal{Y} Z := [V, Z]_\mathcal{H} \) for all \( V \in \Gamma(\mathcal{Y}) \) and \( Z \in \Gamma(\mathcal{H}) \). Following [16] we may construct an adapted connection on \( \mathcal{H} \) putting
\[ \tilde{\nabla}_{XY} := \begin{cases} \nabla^B_{XY}, & \text{if } X \in \Gamma(\mathcal{Y}); \\ (\nabla_{XY})_\mathcal{H}, & \text{if } X \in \Gamma(\mathcal{H}). \end{cases} \]

This connection can be also extended to a connection on all \( TM \) by requiring that \( \tilde{\nabla} \xi_\alpha = 0 \) for each \( \alpha \in \{1,2,3\} \). Some properties of this global connection have been considered in [9] for any almost 3-contact metric manifold. Now combining Theorem 2.2 with [9, Theorem 3.6] we have:

**Theorem 2.3** Let \((M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)\) be a 3-quasi-Sasakian manifold. Then there exists a unique metric connection \( \tilde{\nabla} \) on \( M \) satisfying the following properties:

(i) \( \tilde{\nabla} \eta_\alpha = 0, \tilde{\nabla} \xi_\alpha = 0 \) for each \( \alpha \in \{1,2,3\} \),
(ii) \( \tilde{T}(X,Y) = 2 \Sigma_{\alpha=1}^3 d\eta_\alpha(X,Y) \xi_\alpha \) for all \( X,Y \in \Gamma(TM) \).

3. **THE RANK OF A 3-QUASI-SASAKIAN MANIFOLD**

For a 3-quasi-Sasakian manifold one can consider the ranks of the three structures \((\phi_\alpha, \xi_\alpha, \eta_\alpha, g)\). The following theorem assures that these three ranks coincide.

**Theorem 3.1** ([8]) Let \((M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)\) be a 3-quasi-Sasakian manifold of dimension \( 4n + 3 \). Then the 1-forms \( \eta_1, \eta_2 \) and \( \eta_3 \) have the same rank \( 4l + 3 \) or \( 4l + 1 \), for some \( l \leq n \), according to \( [\xi_\alpha, \xi_\beta] = c \xi_\gamma \) with \( c \neq 0 \), or \( [\xi_\alpha, \xi_\beta] = 0 \), respectively.

According to Theorem 3.1, we say that a 3-quasi-Sasakian manifold \((M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)\) has rank \( 4l + 3 \) or \( 4l + 1 \) if any quasi-Sasakian structure has such rank. We may thus classify 3-quasi-Sasakian manifolds of dimension \( 4n + 3 \), according to their rank. For any \( l \in \{0, \ldots, n\} \) we have one class of manifolds such that \( [\xi_\alpha, \xi_\beta] = c \xi_\gamma \) with \( c \neq 0 \), and one class of manifolds with \( [\xi_\alpha, \xi_\beta] = 0 \). The total number of classes amounts then to \( 2n + 2 \). In the following we will use the notation \( \mathcal{E}^{4m} := \{ X \in \Gamma(\mathcal{H}) \mid iXd\eta_\alpha = 0 \} \), while \( \mathcal{E}^{4l} \) will be the orthogonal complement of \( \mathcal{E}^{4m} \) in \( \Gamma(\mathcal{H}) \), \( \mathcal{E}^{4l+3} := \mathcal{E}^{4l} \oplus \Gamma(\mathcal{Y}) \), and \( \mathcal{E}^{4m+3} := \mathcal{E}^{4m} \oplus \Gamma(\mathcal{Y}) \).

We now consider the class of 3-quasi-Sasakian manifolds such that \( [\xi_\alpha, \xi_\beta] = c \xi_\gamma \) with \( c \neq 0 \) and let \( 4l + 3 \) be the rank. In this case, according to [3], we define for each
structure \((\phi_{\alpha}, \xi_{\alpha}, \eta_{\alpha}, g)\) two \((1, 1)\)-tensor fields \(\psi_{\alpha}\) and \(\theta_{\alpha}\) by putting

\[
\psi_{\alpha}X = \begin{cases} 
\phi_{\alpha}X, & \text{if } X \in \mathcal{E}^{4l+3}; \\
0, & \text{if } X \in \mathcal{E}^{4m}; 
\end{cases} 
\quad \theta_{\alpha}X = \begin{cases} 
0, & \text{if } X \in \mathcal{E}^{4l+3}; \\
\phi_{\alpha}X, & \text{if } X \in \mathcal{E}^{4m}.
\end{cases}
\]

Note that, for each \(\alpha \in \{1, 2, 3\}\) we have \(\phi_{\alpha} = \psi_{\alpha} + \theta_{\alpha}\). Next, we define a new (pseudo-Riemannian, in general) metric \(\bar{g}\) on \(M\) setting

\[
\bar{g}(X, Y) = \begin{cases} 
-d\eta_{\alpha}(X, \phi_{\alpha}Y), & \text{for } X, Y \in \mathcal{E}^{4l}; \\
g(X, Y), & \text{elsewhere}.
\end{cases}
\]

This definition is well posed by virtue of normality and of \([8, \text{Lemma 5.3}]\). \((M, \phi_{\alpha}, \xi_{\alpha}, \eta_{\alpha}, \bar{g})\) is in fact a hyper-normal almost 3-contact metric manifold, in general non-3-quasi-Sasakian. We are now able to formulate the following decomposition theorem, proven in \([8]\).

**Theorem 3.2** Let \((M^{4n+3}, \phi_{\alpha}, \xi_{\alpha}, \eta_{\alpha}, g)\) be a 3-quasi-Sasakian manifold of rank \(4l + 3\) with \([\xi_{\alpha}, \xi_{\beta}] = 2\bar{\xi}_{\gamma}\). Assume \([\theta_{\alpha}, \theta_{\beta}] = 0\) for some \(\alpha \in \{1, 2, 3\}\) and \(\bar{g}\) positive definite on \(\mathcal{E}^{4l}\). Then \(M^{4n+3}\) is locally the product of a \(3\)-Sasakian manifold \(M^{4l+3}\) and a hyper-Kählerian manifold \(M^{4m}\) with \(m = n - 1\).

We now consider the class of 3-quasi-Sasakian manifolds such that \([\xi_{\alpha}, \xi_{\beta}] = 0\) and let \(4l + 1\) be the rank. In this case we define for each structure \((\phi_{\alpha}, \xi_{\alpha}, \eta_{\alpha}, g)\) two \((1, 1)\)-tensor fields \(\psi_{\alpha}\) and \(\theta_{\alpha}\) by putting

\[
\psi_{\alpha}X = \begin{cases} 
\phi_{\alpha}X, & \text{if } X \in \mathcal{E}^{4l}; \\
0, & \text{if } X \in \mathcal{E}^{4m+3}; 
\end{cases} 
\quad \theta_{\alpha}X = \begin{cases} 
0, & \text{if } X \in \mathcal{E}^{4l}; \\
\phi_{\alpha}X, & \text{if } X \in \mathcal{E}^{4m+3}.
\end{cases}
\]

Note that for each \(\alpha\) the maps \(-\psi_{\alpha}^2\) and \(-\theta_{\alpha}^2 + \eta_{\alpha} \otimes \xi_{\alpha}\) define an almost product structure which is integrable if and only if \([-\psi_{\alpha}^2, -\psi_{\alpha}^2] = 0\) or, equivalently, \([\psi_{\alpha}, \psi_{\alpha}] = 0\). Under this assumption the structure turns out to be 3-cosymplectic:

**Theorem 3.3** ([8]) Let \((M, \phi_{\alpha}, \xi_{\alpha}, \eta_{\alpha}, g)\) be a 3-quasi-Sasakian manifold of rank \(4l + 1\) such that \([\xi_{\alpha}, \xi_{\beta}] = 0\) for any \(\alpha, \beta \in \{1, 2, 3\}\) and \([\psi_{\alpha}, \psi_{\beta}] = 0\) for some \(\alpha \in \{1, 2, 3\}\). Then \(M\) is a 3-cosymplectic manifold.

As we have remarked before, 3-Sasakian and 3-cosymplectic manifolds belong to the class of 3-quasi-Sasakian manifolds, having respectively rank \(4n + 3 = \dim(M)\) and rank 1. We now briefly collect some additional properties of these two important subclasses. We have seen that the vertical distribution \(\mathcal{V}\) is integrable already in any 3-quasi-Sasakian manifold. Ishihara ([11]) has shown that if the foliation defined by \(\mathcal{V}\) is regular then the space of leaves is a quaternionic-Kählerian manifold. Boyer, Galicki and Mann have proved the following more general result.

**Theorem 3.4** ([5]) Let \((M^{4n+3}, \phi_{\alpha}, \xi_{\alpha}, \eta_{\alpha}, g)\) be a 3-Sasakian manifold such that the Killing vector fields \(\xi_1, \xi_2, \xi_3\) are complete. Then

(i) \(M^{4n+3}\) is an Einstein manifold of positive scalar curvature equal to \(2(2n + 1)(4n + 3)\).

(ii) Each leaf of the foliation \(\mathcal{V}\) is a 3-dimensional homogeneous spherical space form.
The space of leaves $M^{4n+3}/\mathcal{V}$ is a quaternionic-Kählerian orbifold of dimension $4n$ with positive scalar curvature equal to $16n(n+2)$.

We consider now the horizontal distribution: on the one hand, in the 3-Sasakian subclass $\mathcal{H}$ is never integrable. On the other hand, in any 3-cosymplectic manifold $\mathcal{H}$ is integrable since each $\eta_\alpha$ is closed. Furthermore, the projectability with respect to $\mathcal{V}$ is always granted, as the following theorem shows.

**Theorem 3.5 ([9])** Every regular 3-cosymplectic manifold projects onto a hyper-Kählerian manifold.

As a corollary, it follows that every 3-cosymplectic manifold is Ricci-flat.

In [9] the horizontal flatness of such structures has been studied. In particular it has been proven to be equivalent to the existence of Darboux-like coordinates, that is local coordinates $\{x_1, \ldots, x_{4n}, z_1, z_2, z_3\}$ with respect to which, for each $\alpha \in \{1, 2, 3\}$, the fundamental 2-forms $\Phi_\alpha = d\eta_\alpha$ have constant components and $\xi_\alpha = a^{\alpha}_{1} \frac{\partial}{\partial z_1} + a^{\alpha}_{2} \frac{\partial}{\partial z_2} + a^{\alpha}_{3} \frac{\partial}{\partial z_3}$, $a^{\alpha}_{\beta}$ being functions depending only on the coordinates $z_1, z_2, z_3$. Consequently, in view of Theorem 3.4 and Theorem 3.5 we have the following result.

**Theorem 3.6 ([9])** A 3-Sasakian manifold does not admit any Darboux-like coordinate system. On the other hand, a 3-cosymplectic manifold admits a Darboux-like coordinate system around each of its points if and only if it is flat.

### 4. FINAL REMARKS

A number of natural questions arose during the development of our work on 3-quasi-Sasakian manifolds. We have seen that 3-Sasakian manifolds do not admit any Darboux coordinate system, while on 3-cosymplectic manifolds such coordinate exist if and only if the manifold is flat, so it is natural to wonder whether these coordinates do not exist on any 3-quasi-Sasakian manifold of rank greater than one. Another important topic would be to study the projectability of 3-quasi-Sasakian manifolds for understanding the general relation between this class and the quaternionic structures, since the 3-Sasakian manifolds project over quaternionic-Kähler structures while the structure of the leaf space turns out to be globally hyper-Kählerian in the 3-cosymplectic case. Finally, as both 3-Sasakian and 3-cosymplectic manifolds are Einstein manifolds a natural question would be to ask whether all 3-quasi-Sasakian manifolds are Einstein. However, since we have already found an example of an $\eta$-Einstein, non-Einstein 3-quasi-Sasakian manifold in [8], the natural problem now becomes to establish if there is any 3-quasi-Sasakian manifolds which is not $\eta$-Einstein. We will try to address some of these questions in the next future.

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