May 11, 2010

HAMILTON-JACOBI THEORY IN $k$-SYMPLECTIC FIELD THEORIES

M. DE LEÓN, D. MARTÍN DE DIEGO, J.C. MARRERO, M. SALGADO, AND S. VILARIÑO

Abstract. In this paper we extend the geometric formalism of Hamilton-Jacobi theory for Mechanics to the case of classical field theories in the $k$-symplectic framework.

Contents

1. Introduction 1
2. Geometric preliminaires 2
3. $k$-symplectic Hamiltonian field theory and the Hamilton-Jacobi equation 3
   3.1. $k$-symplectic Hamiltonian field theory 3
   3.2. The Hamilton-Jacobi equation 4
4. The Hamilton-Jacobi problem and the $k$-symplectic Lagrangian field theory 8
   4.1. $k$-symplectic Lagrangian field theory 8
   4.2. Hamilton-Jacobi problem on $T^1_k Q$ 9
5. Example 12
Acknowledgments 13
References 13

1. INTRODUCTION

The usefulness of Hamilton-Jacobi theory in Classical Mechanics is well-known, giving an alternative procedure to study and, in some cases, to solve the evolution equations [1]. The use of symplectic geometry in the study of Classical Mechanics has permitted to connect the Hamilton-Jacobi theory with the theory of lagrangian submanifolds and generating functions.

At the beginning of the 1900s an analog of Hamilton-Jacobi equation for field theory has been developed [24], but it has not been proved to be as powerful as the theory which is available for mechanics [4, 5, 21, 22, 23, 25].

Our goal in this paper is to describe this equation in a geometrical setting.

Let us recall that there are two different ways to describe a field theory, say multisymplectic and $k$-symplectic geometry. A multisymplectic structure abstracts the canonical geometry of bundles of exterior forms, in the same way that symplectic geometry captures the essential facts on cotangent bundles [6]. On the contrary, a $k$-symplectic structure is locally equivalent to the Whitney sum $(T^1_k)^* Q = T^* Q \oplus \cdots \oplus T^* Q \oplus k$.
\( \oplus T^*Q \) of \( k \)-copies of the cotangent bundle \( T^*Q \). In any case, given a Hamiltonian function, both geometric structures produce the field equations.

The aim of this paper is to extend the Hamilton-Jacobi theory to field theories just in the context of \( k \)-symplectic manifolds (we remit to [11] for a description in the multisymplectic setting). The dynamics for a given hamiltonian function \( H \) is interpreted as a family of vector fields (a \( k \)-vector field) on the phase space \( (T_k^1)^*Q \). The Hamilton-Jacobi equation is of the form

\[
d(H \circ \gamma) = 0,
\]

where \( \gamma = (\gamma_1, \ldots, \gamma_k) \) is a family of closed 1-forms on \( Q \). Therefore, we recover the classical form

\[
H(q^1, \frac{\partial W^1}{\partial q^1}, \ldots, \frac{\partial W^k}{\partial q^1}) = \text{constant}.
\]

where \( \gamma_i = dW_i \). It should be noticed that our method is inspired in a recent result by Cariñena et al [7] (this method has also used to develop a Hamilton-Jacobi theory for nonholonomic mechanical systems [10]; see also [12] [8]).

The paper is structured as follows. In Section 2, we recall the notion of \( k \)-vector field and their integral sections. In Section 3 we discuss \( k \)-symplectic Hamiltonian field theory and the Hamilton-Jacobi equation in that context. The corresponding result in the lagrangian description of the field theory is obtained in Section 4. Finally, an example is discussed in Section 5, with the aim to show how the method works.

## 2. Geometric preliminaires

In this section we briefly recall some well-known facts about tangent bundles of \( k \)-velocities (we refer the reader to [13] [14] [18] [19] [20] for more details).

Let \( \tau_M : TM \rightarrow M \) be the tangent bundle of \( M \). Let us denote by \( T_k^1M \) the Whitney sum \( TM \oplus \kern-.1667em \ldots \oplus TM \) of \( k \) copies of \( TM \), with projection \( \tau : T_k^1M \rightarrow M \), \( \tau(v_1, \ldots, v_k) = x \), where \( v_A \in T_xM, 1 \leq A \leq k \). \( T_k^1M \) can be identified with the manifold \( J_0^1(R^k, M) \) of the \( k \)-velocities of \( M \), that is, 1-jets of maps \( \eta : R^k \rightarrow M \) with source at \( 0 \in R^k \), say

\[
\begin{align*}
J_0^1(R^k, M) & \equiv TM \oplus \kern-.1667em \ldots \oplus TM \\
J_0^k \eta & \equiv (v_1, \ldots, v_k)
\end{align*}
\]

where \( x = \eta(0) \), and \( v_A = T\eta(0)(\frac{\partial}{\partial v_A}) \). Here \((t^1, \ldots, t^k)\) denote the standard coordinates on \( R^k \). \( T_k^1M \) is called the tangent bundle of \( k \)-velocities of \( M \) or simply \( k \)-tangent bundle for short, see [19].

Denote by \((x^i, v^i)\) the fibred coordinates in \( TM \) from local coordinates \((x^i)\) on \( M \). Then we have fibred coordinates \((x^i, v_A), 1 \leq i \leq m, 1 \leq A \leq k \), on \( T_k^1M \), where \( m = \text{dim} \ M \).

**Definition 2.1.** A section \( X : M \rightarrow T_k^1M \) of the projection \( \tau \) will be called a \( k \)-vector field on \( M \).

Since \( T_k^1M \) is the Whitney sum \( TM \oplus \ldots \oplus TM \) of \( k \) copies of \( TM \), we deduce that to give a \( k \)-vector field \( X \) is equivalent to give a family of \( k \)-vector fields \( X_1, \ldots, X_k \) on \( M \) by projecting \( X \) onto each factor. For this reason we will denote a \( k \)-vector field by \((X_1, \ldots, X_k)\).

**Definition 2.2.** An integral section of the \( k \)-vector field \( X = (X_1, \ldots, X_k) \), passing through a point \( x \in M \), is a map \( \psi : U_0 \subset R^k \rightarrow M \), defined on some neighborhood...
$U_0$ of $0 \in \mathbb{R}^k$, such that

$$
\psi(0) = x, \ T \psi \left( \left. \frac{\partial}{\partial t} \right|_t \right) = X_A(\psi(t)) \quad \text{for every } t \in U_0, \ 1 \leq A \leq k
$$
or, what is equivalent, $\psi$ satisfies that $X \circ \psi = \psi^{(1)}$, being $\psi^{(1)}$ is the first prolongation of $\psi$ to $T^k_kM$ defined by

$$
\psi^{(1)} : \quad U_0 \subset \mathbb{R}^k \longrightarrow T^k_kM
\quad t \longrightarrow \psi^{(1)}(t) = j_0^1 \psi_t,
$$

where $\psi_t(s) = \psi(t + s)$.

A $k$-vector field $X = (X_1, \ldots, X_k)$ on $M$ is said to be integrable if there is an integral section passing through every point of $M$.

In local coordinates, we have

$$
\psi^{(1)}(t^1, \ldots, t^k) = \left( \psi^{i}(t^1, \ldots, t^k), \left. \frac{\partial \psi^{i}}{\partial t^A} \right|^{(1)}(t^1, \ldots, t^k) \right), \ 1 \leq A \leq k, \ 1 \leq i \leq m, \quad (2.1)
$$

and then $\psi$ is an integral section of $(X_1, \ldots, X_k)$ if and only if the following equations holds:

$$
\left. \frac{\partial \psi^{i}}{\partial t^A} \right|^{(1)} = X_A^i \circ \psi \quad 1 \leq A \leq k, \ 1 \leq i \leq m,
$$

being $X_A = X_A^i \frac{\partial}{\partial q^i}$.

Notice that, in case $k = 1$, Definition 2.2 coincides with the definition of integral curve of a vector field.

3. $k$-symplectic Hamiltonian field theory and the Hamilton-Jacobi equation

In this section, we shall recall the $k$-symplectic Hamiltonian formulation for classical field theories, (see [9, 20] for more details). Later, we shall describe the Hamilton-Jacobi problem in this setting.

3.1. $k$-symplectic Hamiltonian field theory. Let $Q$ be a configuration manifold with local coordinates $(q^i)$, $1 \leq i \leq n$ and $T^*Q$ its cotangent bundle with fibered coordinates $(q^i, p_i)$. Denote by $\pi_Q : T^*Q \to Q$ the canonical projection. Define the Liouville 1-form or canonical 1-form $\theta_Q$ by

$$
(\theta_Q)_\alpha(Y) = \alpha(T\pi_Q(Y)), \quad \text{where } Y \in T_\alpha(T^*Q)
$$
The canonical 2-form $\omega_Q$ on $T^*Q$ is the symplectic form $\omega_Q = -d\theta_Q$. Therefore, we have

$$
\theta_Q = p_idq^i, \quad \omega_Q = dq^i \wedge dp_i. \quad (3.1)
$$

Denote

$$
(T^k_k)^*Q = T^*Q \oplus \cdots \oplus T^*Q
$$
the Whitney sum of $T^*Q$ with itself $k$ times. We introduce coordinates $(q^i, p_i^1, \ldots, p_i^k)$ and the canonical projections

$$
\Pi_Q : (T^k_k)^*Q \longrightarrow Q, \quad \Pi_A : (T^k_k)^*Q \longrightarrow T^*Q
$$
where $A$ indicates the summand $A$-th in the Whitney sum.

We can endow $(T^k_k)^*Q$ with a $k$-symplectic structure given by the family of $k$ canonical presymplectic forms $(\omega^1, \ldots, \omega^k)$, where

$$
\omega^A = \Pi_A^*(\omega_Q).$$
Therefore, we have
\[ \omega^A = dq^i \wedge dp^A_i. \]

Denote also by \( \theta^A = \Pi^*_A(\theta_Q) \).

**Remark 3.1.** Let us recall that a \( k \)-symplectic structures is given by a family of \( k \) two-forms satisfying some compatibility conditions (see [2, 3, 15, 16, 17, 18]).

Consider a Hamiltonian \( H : (T^*_k)^*Q \rightarrow \mathbb{R} \). The field equations are then obtained as follows.

Consider the mapping
\[ \flat : T^1_k((T^*_k)^*Q) \rightarrow T^*(T^*_k)^*Q \]
\[ z = (z_1, \ldots, z_k) \mapsto \flat(z) = \text{trace}(i_{z_A} \omega^B) = \sum_{A=1}^k i_{z_A} \omega^A \]

Then, we look for the solutions of the equation
\[ \flat(Z) = dH. \]  
\[ (3.2) \]

Notice that \( Z = (Z_1, \ldots, Z_k) \) is a \( k \)-vector field on \((T^*_k)^*Q\), that is, each \( Z_A \) is a vector field on \((T^*_k)^*Q\).

Using a local coordinates system \((q^i, p^A_i)\) on \((T^*_k)^*Q\), each \( Z_A \) is locally given by
\[ Z_A = Z^i_A \partial / \partial q^i + (Z^j_A) B_i \partial / \partial p^B_i. \]

Therefore, we obtain that the equation \((3.2)\) is locally expressed as follows:
\[ Z^i_A = \frac{\partial H}{\partial p^A_i}, \quad \sum_{A=1}^k (Z^j_A)_i = -\frac{\partial H}{\partial q^i}. \]  
\[ (3.3) \]

Now, if \( Z \) is integrable, an integral section of \( Z \)
\[ \sigma(t^1, \ldots, t^k) = (\sigma^i(t^1, \ldots, t^k), \sigma^A_i(t^1, \ldots, t^k)) \]
satisfies the Hamilton equations
\[ \frac{\partial \sigma^i}{\partial t^A} = \frac{\partial H}{\partial p^A_i} \circ \sigma, \quad \sum_{A=1}^k \frac{\partial \sigma^A_i}{\partial t^A} = -\frac{\partial H}{\partial q^i} \circ \sigma. \]  
\[ (3.4) \]

Let us observe that if \( Z = (Z_1, \ldots, Z_k) \in \ker \flat \) then
\[ Z^i_B = 0, \quad \sum_{A=1}^k (Z^j_A)_i = 0. \]  
\[ (3.5) \]

### 3.2. The Hamilton-Jacobi equation

The standard formulation of the Hamilton-Jacobi problem for Hamiltonian Mechanics consist of finding a function \( S(t, q^i) \) (called the principal function) such that
\[ \frac{\partial S}{\partial t} + H(q^i, \frac{\partial S}{\partial q^i}) = 0. \]  
\[ (3.6) \]

If we put \( S(t, q^i) = W(q^i) - t \)-constant, then \( W : Q \rightarrow \mathbb{R} \) (called the characteristic function) satisfies
\[ H(q^i, \frac{\partial W}{\partial q^i}) = \text{constant}. \]  
\[ (3.7) \]

Equations \((3.6)\) and \((3.7)\) are indistinctly referred as the Hamilton-Jacobi equation in Hamiltonian Mechanics.
In the framework of the $k$-symplectic formalism, a Hamiltonian is a function $H \in \mathcal{C}^\infty((T^1_k)\ast Q)$. In this context, the Hamilton-Jacobi problem consists of finding $k$ functions $W^1, \ldots, W^k : Q \to \mathbb{R}$ such that

$$H(q^i, \frac{\partial W^1}{\partial q^i}, \ldots, \frac{\partial W^k}{\partial q^i}) = \text{constant}. \quad (3.8)$$

In this section we give a geometric version of the Hamilton-Jacobi equation (3.8).

Let $\gamma : Q \rightarrow (T^1_k)\ast Q$ be a closed section of $\Pi_Q : (T^1_k)\ast Q \rightarrow Q$. Therefore, $\gamma = (\gamma^1, \ldots, \gamma^k)$ where each $\gamma^A$ is an ordinary closed 1-form on $Q$. Thus we have that every point has an open neighborhood $U \subset Q$ where there exists $k$ functions $W^A \in \mathcal{C}^\infty(U)$ such that $\gamma^A = dW^A$.

Now, let $Z$ be a $k$-vector field on $(T^1_k)\ast Q$. Using $\gamma$ we can construct a $k$-vector field $Z_{\gamma}$ on $Q$ such that the following diagram is commutative

$$\begin{array}{c}
(T^1_k)\ast Q \\
\downarrow \gamma \downarrow \\
Q\\
\end{array} \quad \begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\end{array} \quad \begin{array}{c}
(T^1_k)\ast Q \ast Q \\
\downarrow T_{\gamma}^1 \Pi_Q \\
T_{\gamma}^1 Q \\
\end{array}$$

that is,

$$Z_{\gamma} := T_{\gamma}^1 \Pi_Q \circ Z \circ \gamma.$$

Let us remember that for an arbitrary differentiable map $f : N \rightarrow M$, the induced map $T^1_k f : T^1_k N \rightarrow T^1_k M$ is defined by $T^1_k f(v_{1x}, \ldots, v_{kx}) = (T_x f(v_{1x}), \ldots, T_x f(v_{kx}))$.

Notice that the $k$-vector field $Z$ defines $k$ vector fields on $(T^1_k)\ast Q$, say $Z = (Z_1, \ldots, Z_k)$. In the same manner, the $k$-vector field $Z_{\gamma}$ determines $k$ vector fields on $Q$, say $Z_{\gamma} = (Z_{\gamma}^1, \ldots, Z_{\gamma}^k)$.

In local coordinates, if each $Z_A$ is locally given by

$$Z_A = Z_A^i \frac{\partial}{\partial q^i} + (Z_A^p) \frac{\partial}{\partial p^i},$$

then $Z_{\gamma}^A$ has the following local expression:

$$Z_{\gamma}^A = (Z_A^\gamma \circ \gamma) \frac{\partial}{\partial q^i}. \quad (3.9)$$

**Theorem 3.2.** *(Hamilton-Jacobi Theorem)* Let $Z$ be a solution of the Hamilton equations (3.2) and $\gamma : Q \rightarrow (T^1_k)\ast Q$ be a closed section of $\Pi_Q : (T^1_k)\ast Q \rightarrow Q$, that is, $\gamma = (\gamma^1, \ldots, \gamma^k)$ where each $\gamma^A$ is an ordinary closed 1-form on $Q$. If $Z$ is integrable then the following statements are equivalent:

(i) If $\sigma : U \subset \mathbb{R}^k \rightarrow Q$ is an integral section of $Z_{\gamma}$ then $\gamma \circ \sigma$ is a solution of the Hamilton equations;

(ii) $d(H \circ \gamma) = 0$.

**Proof.** The closeness of the 1-forms $\gamma^A = \gamma^A_i dq^i$ states that

$$\frac{\partial \gamma^A_i}{\partial q^i} = \frac{\partial \gamma^B_j}{\partial q^i}. \quad (3.10)$$

$(i) \Rightarrow (ii)$
Let us suppose that \( \gamma \circ \sigma(t) = (\sigma^i(t), \gamma^A_i(\sigma(t))) \) is a solution of the Hamilton equations for \( H \), then

\[
\frac{\partial \sigma^i}{\partial t} \bigg|_t = \frac{\partial H}{\partial p^i_j} \bigg|_{\gamma(\sigma(t))} \quad \text{and} \quad \sum_{A=1}^{k} \frac{\partial (\gamma^A_i \circ \sigma)}{\partial t^A} \bigg|_t = - \frac{\partial H}{\partial q^i_j} \bigg|_{\gamma(\sigma(t))} .
\]

(3.11)

Now, we will compute the differential of the function \( H \circ \gamma : Q \to \mathbb{R} \):

\[
d(H \circ \gamma) = \left( \frac{\partial H}{\partial q^i_j} \circ \gamma + \left( \frac{\partial H}{\partial p^i_j} \circ \gamma \right) \frac{\partial \gamma^A_i}{\partial q^i_j} \right) dq^i .
\]

(3.12)

Then from (3.10), (3.11) and (3.12) we obtain

\[
d(H \circ \gamma)(\sigma(t)) = \left( \frac{\partial H}{\partial q^i_j} \bigg|_{\gamma(\sigma(t))} + \frac{\partial H}{\partial p^i_j} \bigg|_{\gamma(\sigma(t))} \frac{\partial \gamma^A_i}{\partial q^i_j} \bigg|_{\gamma(\sigma(t))} \right) dq^i(\sigma(t))
\]

\[
= - \sum_{A=1}^{k} \frac{\partial (\gamma^A_i \circ \sigma)}{\partial t^A} \bigg|_t + \frac{\partial \sigma^j}{\partial t^A} \bigg|_t \left( \frac{\partial \gamma^A_i}{\partial q^i_j} \bigg|_{\gamma(\sigma(t))} \right) dq^i(\sigma(t))
\]

\[
= - \sum_{A=1}^{k} \frac{\partial (\gamma^A_i \circ \sigma)}{\partial t^A} \bigg|_t + \frac{\partial \sigma^j}{\partial t^A} \bigg|_t \frac{\partial \gamma^A_i}{\partial q^i_j} \bigg|_{\gamma(\sigma(t))} dq^i(\sigma(t)) = 0 .
\]

the last term being zero by the chain rule. Since \( Z \) is integrable, the \( k \)-vector field \( Z^\gamma \) is integrable, then for each point \( q \in Q \) we have an integral section \( \sigma : U_0 \subset \mathbb{R}^k \to Q \) of \( Z^\gamma \) passing through this point, then

\[
d(H \circ \gamma) = 0 .
\]

(ii) \(\Rightarrow\) (i)

Let us suppose that \( d(H \circ \gamma) = 0 \) and \( \sigma \) is an integral section of \( Z^\gamma \). Now we will prove that \( \gamma \circ \sigma \) is a solution to the Hamilton field equations, that is (3.11) is satisfied.

Since \( d(H \circ \gamma) = 0 \), from (3.12) we obtain

\[
0 = \frac{\partial H}{\partial q^i_j} \circ \gamma + \left( \frac{\partial H}{\partial p^i_j} \circ \gamma \right) \frac{\partial \gamma^A_i}{\partial q^i_j} .
\]

(3.13)

From (3.13) and (3.9) we know that

\[
Z^\gamma_A = \left( \frac{\partial H}{\partial p^i_j} \circ \gamma \right) \frac{\partial }{\partial q^i}
\]

and then since \( \sigma \) is an integral section of \( Z^\gamma \) we obtain

\[
\frac{\partial \sigma^j}{\partial t^A} = \frac{\partial H}{\partial p^i_j} \circ \gamma \circ \sigma .
\]

(3.14)

On the other hand, from (3.10), (3.13) and (3.14) we obtain

\[
\sum_{A=1}^{k} \frac{\partial (\gamma^A_i \circ \sigma)}{\partial t^A} = \sum_{A=1}^{k} \frac{\partial \gamma^A_i}{\partial q^i_j} \circ \sigma \frac{\partial }{\partial q^i_j} = \sum_{A=1}^{k} \frac{\partial \gamma^A_i}{\partial q^i_j} \circ \sigma \left( \frac{\partial H}{\partial p^i_j} \circ \gamma \circ \sigma \right)
\]

\[
= \sum_{A=1}^{k} \frac{\partial \gamma^A_i}{\partial q^i_j} \circ \sigma \left( - \frac{\partial H}{\partial q^i_j} \circ \gamma \circ \sigma \right) = - \frac{\partial H}{\partial q^i_j} \circ \gamma \circ \sigma .
\]

and thus we have proved that \( \gamma \circ \sigma \) is a solution to the Hamilton field equations.

\( \square \)
Remark 3.3. In the particular case $k = 1$ the above theorem can be found in [11].

Theorem 3.4. Let $Z$ be a solution of the Hamilton equations (3.2), and $\gamma : Q \rightarrow (T^*_k Q)^\ast$ be a closed section of $\Pi_Q : (T^*_k)^\ast Q \rightarrow Q$, that is, $\gamma = (\gamma^1, \ldots, \gamma^k)$ where each $\gamma^A$ is an ordinary closed 1-form on $Q$. Then, the following statements are equivalent:

(i) $Z \circ \gamma - T^1_k \gamma(Z^\gamma) \in \ker b$
(ii) $d(H \circ \gamma) = 0$.

Proof. We know that if $Z_A$ and $\gamma^A$ are locally given by

$$Z_A = Z^i_A \frac{\partial}{\partial q^i} + (Z^B_A, \frac{\partial}{\partial p^B}) , \quad \gamma^A = \gamma^i_A dq^i.$$

then $Z^\gamma_A = (Z^A \circ \gamma) \frac{\partial}{\partial q^i}$. Thus a direct computation shows that $Z \circ \gamma - T^1_k \gamma(Z^\gamma) \in \ker b$ is locally written as

$$\left((Z^i_A \circ \gamma - (Z^j_A \circ \gamma) \frac{\partial \gamma^B}{\partial q^j} \right) \frac{\partial}{\partial p^B} \circ \gamma = (Y^i_A \circ \gamma \left( \frac{\partial}{\partial p^B} \circ \gamma \right). \quad (3.15)$$

where $(Y^i_A)^4 = 0$.

Now, we are ready to prove the result.

Assume that (i) holds, then from (3.3), (3.5) and (3.15) we obtain that

$$0 = \sum_{A=1}^k \left((Z^i_A \circ \gamma - (Z^j_A \circ \gamma) \frac{\partial \gamma^A}{\partial q^i} \right)$$

$$= - \left( \frac{\partial H}{\partial q^i} \circ \gamma \left( \frac{\partial \gamma^A}{\partial q^i} \right) + \left( \frac{\partial H}{\partial p^B_j} \circ \gamma \right) \frac{\partial \gamma^A}{\partial q^i} \right)$$

$$= - \left( \frac{\partial H}{\partial q^i} \circ \gamma \left( \frac{\partial \gamma^A}{\partial q^i} \right) \right)$$

where in the last identity we are using the closeness of $\gamma$ (see (3.10)). Therefore, $d(H \circ \gamma) = 0$ (see (3.12)).

The converse is proved in a similar way by reversing the arguments.

Remark 3.5. In the particular case $k = 1$ the above theorem can be found in [11].

Remark 3.6. It should be noticed that if $Z$ and $Z^\gamma$ are $\gamma$-related, that is, $Z_A = T^1_k \gamma(Z^\gamma)_A$, then $d(H \circ \gamma) = 0$, but the converse does not hold.

Corollary 3.7. Let $Z$ be a solution of (3.2), and $\gamma$ a closed section of $\Pi_Q : (T^*_k)^\ast Q \rightarrow Q$, as in the above theorem. If $Z$ is integrable then the following statements are equivalent:

(i) $Z \circ \gamma - T^1_k \gamma(Z^\gamma) \in \ker b$;
(ii) $d(H \circ \gamma) = 0$;
(iii) If $\sigma : U \subset \mathbb{R}^k \rightarrow Q$ is an integral section of $Z^\gamma$ then $\gamma \circ \sigma$ is a solution of the Hamilton equations.

The equation

$$d(H \circ \gamma) = 0 \quad (3.16)$$
can be considered as the geometric version of the Hamilton-Jacobi equation for \(k\)-symplectic field theories. Notice that in local coordinates, equation (3.16) reads us
\[
H(q^i, \gamma_A(q)) = \text{constant}.
\]
which when \(\gamma_A = dW_A\), where \(W_A: Q \to \mathbb{R}\) is a function, takes the more familiar form
\[
H(q^i, \frac{\partial W_A}{\partial q^i}) = \text{constant}.
\]

4. The Hamilton-Jacobi problem and the \(k\)-symplectic Lagrangian field theory

4.1. \(k\)-symplectic Lagrangian field theory. Consider now the Lagrangian formalism. Let \(L \in \mathcal{C}^\infty(T^1_kQ)\) be a regular Lagrangian function, that is, the Hessian matrix \(\left(\frac{\partial^2 L}{\partial v^i_A \partial v^j_B}\right)\) has maximal rank. We can endow \(T^1_kQ\) with a \(k\)-symplectic structure given by the family of \(k\) 2-forms \(\omega^A_L\), where
\[
\omega^A_L = F^L_*(\omega^A).
\]
and \(FL: T^1_kQ \to (T^1_k)^*Q\) is the Legendre transformation introduced by Günther [9]. In local coordinates \(FL(q^i, v^i_A) = (q^i, \frac{\partial L}{\partial v^i_A})\). Thus we have
\[
\omega^A_L = dq^i \wedge d\left(\frac{\partial L}{\partial v^i_A}\right).
\]
Denote also by \(\theta^A_L = F^L_*(\theta^A)\).

We define the Lagrangian energy function as \(E_L = \Delta(L) - L\) where \(\Delta \in \mathfrak{X}(T^1_kQ)\) is the Liouville vector field, that is, the infinitesimal generator of the flow
\[
\psi: \mathbb{R} \times T^1_kQ \to T^1_kQ \quad ; \quad \psi(s, v^i_1, \ldots, v^i_k) = (e^s v^i_1, \ldots, e^s v^i_k).
\]
As in the Hamiltonian formalism, we consider the mapping
\[
\flat_L: T^1_k(T^1_kQ) \to T^*(T^1_kQ)
\]
\[
z = (z_1, \ldots, z_k) \mapsto \flat_L(z) = \text{trace} (i_{z_A} \omega^B_L) = \sum_{A=1}^{k} i_{z_A} \omega^B_L.
\]
First we study the kernel of \(\flat_L\).

Let \(Z = (Z_1, \ldots, Z_k)\) be a \(k\)-vector field on \(T^1_kQ\), that is, each \(Z_A\) is a vector field on \(T^1_kQ\) locally given
\[
Z_A = Z_A^i \frac{\partial}{\partial q^i} + (Z_A)^B_i \frac{\partial}{\partial v^i_B}
\]
then
\[
\flat_L(Z_1, \ldots, Z_k) = \left[ (\frac{\partial^2 L}{\partial q^i \partial v^A} - \frac{\partial^2 L}{\partial q^j \partial v^B} Z^j_A - \frac{\partial^2 L}{\partial v^B \partial v^A} (Z_A)^i_B) dq^i \right. + \left. \frac{\partial^2 L}{\partial v^i_B \partial v^A} Z^i_A dv^j_B \right].
\]
(4.1)

Therefore, since \(L\) is regular, \((Z_1, \ldots, Z_k) \in \ker \flat_L\) if and only if
\[
Z_A^i = 0, \quad \frac{\partial^2 L}{\partial v^i_B \partial v^A} (Z_A)^i_B = 0.
\]
(4.2)
Now, we look for the solutions of the equation
\[ \gamma_L(Z) = dE_L \] (4.3)
which is locally expressed as follows:
\[ Z^i_A = v^i_A, \quad \frac{\partial^2 L}{\partial q^i \partial v^A_j} v^j_A + \frac{\partial^2 L}{\partial v^A_i \partial v^A_j} (Z_A)^j_B = \frac{\partial L}{\partial q^i}. \] (4.4)

Then, if \( Z \) is integrable, an integral section
\[ \sigma(t^1, \ldots, t^k) = (q^i(t^1, \ldots, t^k), v^i_A(t^1, \ldots, t^k)) \]
satisfies the Euler-Lagrange equations
\[ v^i_A = \frac{\partial q^i}{\partial t^k}, \quad \sum_{k=1}^k \frac{d}{dt} \left( \frac{\partial L}{\partial v^A_i} \right) = \frac{\partial L}{\partial q^i}. \] (4.5)

4.2. Hamilton-Jacobi problem on \( T^1_kQ \).

In this section we formulate the Hamilton-Jacobi problem on the tangent bundle of \( k \)-velocities.

In the section 3.2 we comment that the Hamilton-Jacobi problem in the \( k \)-symplectic framework consists in finding \( k \) functions \( W^1, \ldots, W^k : Q \to \mathbb{R} \) such that
\[ H(q^i, \frac{\partial W^1}{\partial q^i}, \ldots, \frac{\partial W^k}{\partial q^i}) = \text{constant} \] (4.6)

In a geometric terms, equation (4.6) can be written as \( H \circ (dW^1, \ldots, dW^k) = \text{constant} \), where \( (dW^1, \ldots, dW^k) \) is a section of the tangent bundle of \( k \)-covelocities, \( (T^1_k)^*Q \). As we have seen in the section 5.2 we look for a closed section \( \gamma = (\gamma^1, \ldots, \gamma^k) \) of \( \Pi_Q \) such that \( H \circ \gamma = \gamma^*H = \text{constant} \). Let us observe that the section \( \gamma \) is closed, and hence locally exact, \( \gamma^A = dW^A \). The condition \( d\gamma = 0 \) can be alternatively be expressed in terms of the canonical forms \( (\omega^1, \ldots, \omega^k) \) in the form \( \gamma^*\omega^A = 0 \), \( A = 1, \ldots, k \), so that one can reformulate the Hamilton-Jacobi geometric problem in the form: find a section \( \gamma = (\gamma^1, \ldots, \gamma^k) : Q \to (T^1_k)^*Q \) of \( \Pi_Q \) such that
\[ \gamma^*H = \text{constant}, \quad \gamma^*\omega^A = 0, \quad A = 1, \ldots, k. \] (4.7)

Consider now the Lagrangian \( k \)-symplectic formalism. Let \( L \in \mathcal{C}^\infty(T^1_kQ) \) be a regular Lagrangian function and \( \theta^A_L, \omega^A_L, A = 1, \ldots, k \) the associated Lagrangian forms. A literal translation of the above formulation of the Hamilton-Jacobi problem for the tangent bundle of \( k \)-covelocities to the tangent bundle of \( k \)-velocities would be: to find a section \( X = (X_1, \ldots, X_k) : Q \to T^1_k \) of \( \tau \) such that
\[ X^*E_L = \text{constant}, \quad X^*\omega^A_L = 0, \quad A = 1, \ldots, k, \] (4.8)
where \( E_L \) denotes the energy function associated to \( L \). The last family of conditions \( X^*\omega^A_L = 0, \quad A = 1, \ldots, k \) implies that the section \( X \) is associated (at least locally) with a mapping \( W = (W^1, \ldots, W^k) : Q \to \mathbb{R}^k \) by means of the relation \( X^*\theta^A_L = dW^A \). In fact,
\[ 0 = X^*\omega^A_L = -d(X^*\theta^A_L) \]
then, the 1-form \( X^*\theta^A_L \) is closed and therefore locally exact, thus there is a function \( W^A : Q \to \mathbb{R} \) defined on a neighborhood of each point of \( Q \), such that, \( X^*\theta^A_L = dW^A \). Locally, this means
\[ \frac{\partial L}{\partial v^A_i} \circ X = \frac{\partial W^A}{\partial q^i}. \]
Theorem 4.1. Let $X = (X_1, \ldots, X_k)$ be an integrable k-vector field on $Q$ such that $X^*\omega^A_L = 0$. Then, the following statements are equivalent:

(i) If $\sigma: U \subset \mathbb{R}^k \to Q$ is an integral section of $X$ then $\sigma^{(1)}$ is a solution of the Euler-Lagrange equations;

(ii) $d(E_L \circ X) = 0$.

Proof. Since $X^*\omega^A_L = 0$ we have that the 1-forms $X^*\theta^A_L$ are closed. The closeness of this 1-forms states that

$$\frac{\partial^2 L}{\partial q^i \partial v^j_A} \circ X + \left( \frac{\partial^2 L}{\partial v^j_B \partial v^i_A} \circ X \right) \frac{\partial X^B_j}{\partial q^i} = 0$$

(4.9)

where $X(q) = (q, X^B_j(q))$.

In first place, let us suppose that $\sigma(t) = (\sigma^i(t))$ is an integral section of $X$

$$X^A_1 \circ \sigma = \frac{\partial \sigma^i}{\partial t}$$

(4.10)

such that $\sigma^{(1)}$ is a solution of the Euler-Lagrange equations.

We will prove that $d(E_L \circ X) = 0$ along $\sigma$. Now, we compute the differential of the function $E_L \circ X: Q \to \mathbb{R}$. In local coordinates we obtain that

$$E_L \circ X = X^A_1 \left( \frac{\partial L}{\partial v^i_A} \circ X \right) - L \circ X$$

then

$$d(E_L \circ X) = \left( X^A_1 \left( \frac{\partial^2 L}{\partial q^i \partial v^j_A} \circ X \right) + \left( \frac{\partial^2 L}{\partial v^j_B \partial v^i_A} \circ X \right) \frac{\partial X^B_j}{\partial q^i} \right) dt$$

(4.11)

Therefore, from (4.5), (4.9), (4.10) and (4.11) we obtain

$$d(E_L \circ X)(\sigma(t)) = \left( X^A_1(\sigma(t)) \left( \frac{\partial^2 L}{\partial q^i \partial v^j_A} \big|_{X(\sigma(t))} \right) + \left( \frac{\partial^2 L}{\partial v^j_B \partial v^i_A} \big|_{X(\sigma(t))} \right) \frac{\partial X^B_j}{\partial q^i} \big|_{X(\sigma(t))} \right) dt$$

$$= \left( X^A_1(\sigma(t)) \left( \frac{\partial^2 L}{\partial q^i \partial v^j_A} \big|_{X(\sigma(t))} \right) + \left( \frac{\partial^2 L}{\partial v^j_B \partial v^i_A} \big|_{X(\sigma(t))} \right) \frac{\partial X^B_j}{\partial q^i} \big|_{X(\sigma(t))} \right) dt$$

$$= \left( \frac{\partial \sigma^i}{\partial t} \big|_{q} \frac{\partial^2 L}{\partial q^i \partial v^j_A} \big|_{\sigma(t)} + \frac{\partial^2 L}{\partial v^j_B \partial v^i_A} \big|_{\sigma(t)} \right) dt$$

(4.12)

Thus $d(E_L \circ X) = 0$ along $\sigma$ and since the k-vector field $X$ is integrable, for each point $q \in Q$ we have an integral section $\sigma$ of $X$ passing through this point, then

$$d(E_L \circ X) = 0.$$

The converse is proved in a similar way by reversing the arguments. 

Theorem 4.2. Let $Z$ be a solution of the Euler-Lagrange equations (4.7) and $X: Q \to T_k^1Q$ be a section of $\tau: T_k^1Q \to Q$, such that $X^*\omega^A_L = 0$. Then, the following statements are equivalent:

(i) $Z \circ X - T_k^1X(X) \in \ker \gamma_L$

(ii) $d(E_L \circ X) = 0$. 

Proof. A direct computation shows that if \( Z_A \) and \( X_A \) are locally given by
\[
Z_A = v^i_A \frac{\partial}{\partial q^i} + (Z_A)^i_B \frac{\partial}{\partial v^i_B}, \quad X_A = X^i_A \frac{\partial}{\partial q^i}
\]
then
\[
Z_A \circ X - TX(X_A) = \left( (Z_A)^j_B \circ X - X^j_A \frac{\partial X^i_B}{\partial q^i} \right) \left( \frac{\partial}{\partial v^j_B} \circ X \right) (4.12)
\]
Now, we are prepared to prove the result.

Assume that (i) holds, then from (4.2), (4.4) and (4.12) we obtain that
\[
0 = \left( (Z_A)^j_B \circ X - X^j_A \frac{\partial X^i_B}{\partial q^i} \right) \left( \frac{\partial^2 L}{\partial v^j_B \partial v^i_A} \circ X \right)
\]
where in the last identity we are using the closeness of \( X^* \theta^A_L \) (see (4.9)). Therefore,
\[
d(E_L \circ X) = 0 \quad (4.11)
\]
The converse is proved in a similar way by reversing the arguments.

\[\square\]

Corollary 4.3. Let \( Z \) be a solution of (4.3), and \( X \) an integrable \( k \)-vector field on \( Q \) such that \( X^* \omega^A_L = 0 \). Then the following statements are equivalent:

(i) \( Z \circ X - T^k_1 X(X) \in \ker \flat_L \);
(ii) \( d(E_L \circ X) = 0 \);
(iii) If \( \sigma: U \subset \mathbb{R}^k \rightarrow Q \) is an integral section of \( X \) then \( \sigma^{(1)} \) is a solution of the Euler-Lagrange equations.

The equation
\[
d(E_L \circ X) = 0 \quad (4.13)
\]
can be considered as the geometric Lagrangian version of the Hamilton-Jacobi equation for \( k \)-symplectic field theories. Notice that in local coordinates, the equation (4.13) reads us
\[
E_L(q^i, X^A(q^i)) = \text{constant}. \quad (4.14)
\]
If \( X^* \omega^A_L = 0 \), then \( 0 = X^* \omega^A_L = -X^* d\theta^A_L = -d(X^* \theta^A_L) \), we have that every point has an open neighborhood \( U \subset Q \) where there exists \( k \) functions \( W^A \in \mathcal{C}^\infty(U) \) such that
\[
dW^A = X^* \theta^A_L = \left( \frac{\partial L}{\partial v^i_A} \circ X \right) dq^i
\]
and then in local coordinates this means
\[
\frac{\partial L}{\partial v^i_A} \circ X = \frac{\partial W^A}{\partial q^i}.
\]
If the Lagrangian \( L \) is regular, then the Legendre transformation \( FL \) is a local diffeomorphism, then in a neighborhood of each point of \( T^k_1 Q \) we have \( H = E_L \circ FL^{-1} \). Therefore, if we consider the section \( \gamma = (X^* \theta^1_L, \ldots, X^* \theta^k_L): Q \rightarrow (T^k_1)^* Q \), locally given by
\[
\gamma(q) = (q, \frac{\partial W^A}{\partial q^i} q^i)
\]
we have
\[ H(\gamma(q)) = H(q^i, \frac{\partial W^A}{\partial q^i}) = H(q^i, \left. \frac{\partial L}{\partial \dot{q}^i} \right|_{X(q)}) = H \circ FL(X(q)) \]
\[ = E_L(X(q)) = E_L(q^i, X^i_A(q^B)) = \text{constant} \]
and thus (4.14) takes the form
\[ H(q^i, \frac{\partial W^A}{\partial q^i}) = \text{constant}, \]
where \( H = E_L \circ FL^{-1}. \)

5. Example

Vibrating string. In this example we consider the theory of a vibrating string. Coordinates \((t^1, t^2)\) are interpreted as the time and the distance along the string, respectively.

Let us denote by \((q, p^1, p^2)\) the coordinates of \((T^1_2)^*R\) and let us consider the Hamiltonian
\[
H : \quad (T^1_2)^*R \to \mathbb{R} \\
(q, p^1, p^2) \mapsto \frac{1}{2} \left( \frac{(p^1)^2}{\sigma} - \frac{(p^2)^2}{\tau} \right)
\]
where \(\sigma\) and \(\tau\) are certain constants of the mechanical system. In a real string, these constants represent the linear mass density, that is, a measure of mass per unit of length and Young’s module of the system related to the tension of the string, respectively.

Let \(\gamma : R \to (T^1_2)^*R\) be the section of \(\pi R\) defined by \(\gamma(q) = (aq dq, bqdq)\) where \(a\) and \(b\) are two constants such that \(\tau a^2 = \sigma b^2\). This section \(\gamma\) satisfies the condition \(d(H \circ \gamma) = 0\), therefore, the condition (i) of the Theorem 3.2 holds.

The 2-vector field \(Z^\gamma = (Z^\gamma_1, Z^\gamma_2)\) is locally given by
\[
Z^\gamma_1 = \frac{a}{\sigma} \frac{\partial}{\partial q}, \quad Z^\gamma_2 = \frac{b}{\tau} \frac{\partial}{\partial q}
\]
If \(\psi : \mathbb{R}^2 \to \mathbb{R}\) is an integral section of \(Z^\gamma\), then
\[
\frac{\partial \psi}{\partial t^1} = \frac{a}{\sigma} \psi, \quad \frac{\partial \psi}{\partial t^2} = -\frac{b}{\tau} \psi,
\]
thus
\[
\psi(t^1, t^2) = C \exp \left( \frac{a}{\sigma} t^1 - \frac{b}{\tau} t^2 \right), \quad C \in \mathbb{R}
\]
By Theorem 3.2 one obtains that the map \(\phi = \gamma \circ \psi,\) locally given by
\[
(t^1, t^2) \mapsto (\psi(t^1, t^2), a\psi(t^1, t^2), b\psi(t^1, t^2)),
\]
is a solution of the Hamilton equations associated to \(H\), that is,
\[
0 = \frac{\partial \psi}{\partial t^1} + b \frac{\partial \psi}{\partial t^2}
\]
\[
\frac{a}{\sigma} \psi = \frac{\partial \psi}{\partial t^1}
\]
\[
-\frac{b}{\tau} \psi = \frac{\partial \psi}{\partial t^2}
\]
Let us observe that from this system one obtains that \( \psi \) is a solution of the motion equation of the vibrating string, that is,

\[
\sigma \partial_{11} \psi - \tau \partial_{22} \psi = 0,
\]

(5.1)

where \( \psi(t^1, t^2) \) denotes the displacement of each point of the string as function of the time \( t^1 \) and the position \( t^2 \).

Acknowledgments

We acknowledge the partial financial support of Ministerio de Innovación y Ciencia, Project MTM2007-62478, MTM2008-00689, MTM2008-03606-E, MTM2009-13383 and project Ingenio Mathematica(i-MATH) No. CSD2006-00032 (Consolider-Ingenio2010).

References

[1] R.A. Abraham, J.E. Marsden: Foundations of Mechanics (Second Edition), Benjamin-Cummings Publishing Company, New York, 1978.

[2] A. Awane: \( k \)-symplectic structures. J. Math. Phys. 33 (1992), 4046-4052.

[3] A. Awane, M. Goze: Pfaffian systems, \( k \)-symplectic systems. Kluwer Academic Publishers, Dordrecht 2000.

[4] M.C. Bertí, B.M. Pimentel, P.J. Pompeia: Hamilton-Jacobi approach for first order actions and theories with higher order derivatives. Annals of Physics 323 (2008), 527–547.

[5] D. Bruno: Constructing a class of solutions for the Hamilton-Jacobi equations in field theory. J. Math. Phys. 48, 112902 (2007).

[6] F. Cantrijn, A. Ibort and M. de León: On the geometry of multisymplectic manifolds. J. Austral. Math. Soc. (Series A) 66 (1999), 303–330.

[7] J. F. Cariñena, X. Gracia, G. Marmo, E. Martínez, M.C. Muños-Lecanda, N. Román-Roy: Geometric Hamilton-Jacobi theory. Int. J. Geom. Methods Mod. Phys. 3 (2006), no. 7, 1417–1458.

[8] J. F. Cariñena, X. Gracia, G. Marmo, E. Martínez, M.C. Muños-Lecanda, N. Román-Roy: Geometric Hamilton-Jacobi Theory for Nonholonomic Dynamical Systems. arXiv:0908.2453.

[9] C. Günther: The polysymplectic Hamiltonian formalism in field theory and calculus of variations I: The local case. J. Differential Geom. 25 (1987), 23-53.

[10] M. de León, David Iglesias-Ponte, D. Martín de Diego: Towards a Hamilton-Jacobi theory for nonholonomic mechanical systems. J. Phys. A 41 (2008), no. 1, 015205, 14 pp

[11] M. de León, J.C. Marrero, D. Martín de Diego: A geometric Hamilton-Jacobi theory for classical field theories. In Variations, Geometry and Physics in honour of Demeter Krupka sixty-fifth birthday, O. Krupkova and D. J. Saunders (Editors), Nova Science Publishers Inc., New York 2009, pp. 129–140.

[12] M. de León, J.C. Marrero, D. Martín de Diego: Linear almost Poisson structures and Hamilton-Jacobi equation. Applications to nonholonomic Mechanics. arXiv:0801.4358.

[13] M. de León, I. Méndez, M. Salgado: \( p \)-almost tangent structures. Rend. Circ. Mat. Palermo Serie II XXXVII (1988), 282-294.

[14] M. de León, I. Méndez, M. Salgado: Integrable \( p \)-almost tangent structures and tangent bundles of \( p^t \)-velocities. Acta Math. Hungar. 58(1-2) (1991), 45-54.

[15] M. de León, E. Merino, M. Salgado: \( k \)-cosymplectic manifolds and Lagrangian field theories. J. Math. Phys. 42 (2001), no. 5, 2092–2104.

[16] M. de León, E. Merino, M. Salgado: Stable almost cotangent structures. Boll. Un. Mat. Ital. B (7) 11 (1997), no. 3, 509–529.

[17] M. de León, E. Merino, J.A. Oubina, P.R. Rodrigues, M. Salgado: Hamiltonian systems on \( k \)-cosymplectic manifolds. J. Math. Phys. 39 (1998), no. 2, 876–893.

[18] M. de León, E. Merino, J.A. Oubina, J.A. Oubina, P.R. Rodrigues, M. Salgado: Hamiltonian systems on \( k \)-cosymplectic manifolds. J. Math. Phys. 39 (1998), no. 2, 876–893.

[19] A. Morimoto: Liftings of some types of tensor fields and connections to tangent \( p^t \)-velocities. Nagoya Math. J. 40 (1970), 13-31.

[20] F. Munteanu, A. M. Rey, M. Salgado: The Günther’s formalism in classical field theory: momentum map and reduction. J. Math. Phys. 45(5) (2004) 1730–1751.

[21] C. Paufier, H. Romer: De Donder-Weyl equations and multisymplectic geometry. In: XXXIII Symposium on Mathematical Physics (Torvín, 2001). Rep. Math. Phys. 49 no. 2-3 (2002), 325–334.
[22] C. Paufler, H. Romer: Geometry of Hamiltonian n-vector fields in multisymplectic field theory. J. Geom. Phys. 44 no. 1 (2002), 52–69.
[23] G. Rosen: Hamilton-Jacobi functional theory for the integration of classical field equations. International Journal of Theoretical Physics 4, 4 (1971), 281–285.
[24] H. Rund: The Hamilton-Jacobi Theory in the Calculus of Variations. Hazell, Watson and Viney Ltd., Aylesbury, Buckinghamshire, U.K. 1966.
[25] L. Vitagliano: The Hamilton-Jacobi formalism for higher order field theories. http://arxiv.org/abs/1003.5236v1

M. de León: Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), Consejo Superior de Investigaciones Científicas, Serrano 123, 28006 Madrid, Spain
E-mail address: mdeleon@icmat.es

D. Martín de Diego: Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M), Consejo Superior de Investigaciones Científicas, C/ Serrano 123, 28006 Madrid, Spain
E-mail address: david.martin@icmat.es

J.C. Marrero: Departamento de Matemática Fundamental, Facultad de Matemáticas, Universidad de la Laguna, Spain
E-mail address: jcmarrer@ull.es

M. Salgado: Departamento de Xeometría e Topoloxía, Facultade de Matemáticas, Universidade de Santiago de Compostela, 15782-Santiago de Compostela, Spain
E-mail address: modesto.salgado@usc.es

S. Vilariño: Departamento de Matemáticas, Facultad de Ciencias, Universidad de A Coruña, Campus de A Zapateira, 15008-A Coruña, Spain
E-mail address: silvia.vilarino@udc.es