Non-symplectic symmetries and bi-Hamiltonian structures of the rational Harmonic Oscillator

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Abstract. The existence of bi-Hamiltonian structures for the rational Harmonic Oscillator (non-central harmonic oscillator with rational ratio of frequencies) is analyzed by making use of the geometric theory of symmetries. We prove that these additional structures are a consequence of the existence of dynamical symmetries of non-symplectic (non-canonical) type. The associated recursion operators are also obtained.

Keywords: Dynamical symmetries, Superintegrability, bi-Hamiltonian structures, Recursion operators.

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1 Non-symplectic symmetries

It is well known that there is a close relation [1] between integrability and the existence of alternatives structures (see e.g. [2] for a recent paper) and also that integrable systems are systems endowed with a great number of symmetries. The purpose of this letter is to analyze, in the particular case of the $n = 2$ harmonic oscillator, how these additional structures arise from the existence of dynamical symmetries of non-symplectic (non-canonical) type.

Let $(M, \omega_0, H)$ be a Hamiltonian system and $\Gamma_H$ the associated Hamiltonian vector field, defined by $i(\Gamma_H) \omega_0 = dH$. A (infinitesimal) dynamical symmetry of this system is a vector field $Y \in \mathfrak{X}(M)$ such that $[Y, \Gamma_H] = 0$. When $Y$ is a dynamical but non-symplectic symmetry of the system, then we have that (i) the dynamical vector field $\Gamma_H$ is bi-Hamiltonian, and (ii) the function $Y(H)$ is the new Hamiltonian, and therefore it is a constant of motion.

A sketch of the proof [3]-[6] of this statement is as follows: The vector field $Y$ does not preserve $\omega_0$ and, as it is a non-canonical transformation, it determines a new 2-form $\omega_Y = \mathcal{L}_Y \omega_0$ ($\mathcal{L}_Y$ denotes de Lie derivative with respect to $Y$). As $Y$ is a symmetry, $[Y, \Gamma_H] = 0$, then $\mathcal{L}_Y \circ i_{\Gamma_H} = i_{\Gamma_H} \circ \mathcal{L}_Y$, and, consequently,

$$i_{\Gamma_H} \omega_Y = i_{\Gamma_H} \mathcal{L}_Y \omega_0 = \mathcal{L}_Y i_{\Gamma_H} \omega_0 = \mathcal{L}_Y (dH) = d(Y(H)).$$

Therefore, the 2-form $\omega_Y$ is admissible for the dynamical vector field $\Gamma_H$, i.e. $\mathcal{L}_{\Gamma_H} \omega_Y = 0$, which is weakly bi-Hamiltonian with respect to the original symplectic 2-form $\omega_0$ and the new structure $\omega_Y$. Of course the particular form of $\omega_Y$ depends on $Y$ and, in some cases, it can be just a constant multiple of $\omega_0$ (trivial bi-Hamiltonian system). In some other cases $\omega_Y$ may be a degenerate 2-form with a nontrivial kernel. In any case, the vector field $\Gamma_H$ is a dynamical system solution of the following two equations

$$i(\Gamma_H) \omega_0 = dH, \quad \text{and} \quad i(\Gamma_H) \omega_Y = d[Y(H)].$$

Therefore the function $H_Y = Y(H)$, that must be a constant of motion, can be considered as a new Hamiltonian for $\Gamma_H$.

2 Bi-Hamiltonian structures of the rational Harmonic Oscillator

The two-dimensional harmonic oscillator

$$H = \frac{1}{2} (p_x^2 + p_y^2) + \frac{1}{2} (\lambda_1^2 x^2 + \lambda_2^2 y^2)$$

(1)

has the two one-degree of freedom energies, $I_1 = E_x$ and $I_2 = E_y$, as fundamental constants of motion. The superintegrability of the rational case, $\lambda_1 = m \lambda_0$, $\lambda_2 = n \lambda_0$, with $m, n \in \mathbb{N}$,
can be proved by making use of a complex formalism [7, 8]. Let $K_x, K_y$, be the following two functions:

$$K_x = p_x + i m \lambda_0 x$$
$$K_y = p_y + i n \lambda_0 y$$

then the Hamiltonian $H$ and the canonical symplectic form $\omega_0$ become

$$H = \frac{1}{2} \left( K_x K_x^* + K_y K_y^* \right) ,$$

and

$$\omega_0 = \frac{i}{2 m \lambda_0} dK_x \wedge dK_x^* + \frac{i}{2 n \lambda_0} dK_y \wedge dK_y^* .$$

We have

$$\{K_x, K_x^*\} = 2 i m \lambda_0 , \quad \{K_y, K_y^*\} = 2 i n \lambda_0 ,$$

and therefore, the evolution equations are

$$\frac{d}{dt} K_x = i m \lambda_0 K_x , \quad \frac{d}{dt} K_y^* = -i n \lambda_0 K_y^* .$$

Hence, the complex function $J$ defined as

$$J = K_x^n (K_y^*)^m$$

is a constant of motion that determines two different real first integrals, $I_3 = \text{Im}(J)$ and $I_4 = \text{Re}(J)$, which are polynomials in the momenta of degree $m + n - 1$ and $m + n$, respectively. As an example, for the isotropic case, $\lambda_1 = \lambda_2 = \lambda_0$, we obtain

$$\text{Re}(J) = p_x p_y + \lambda_0^2 x y ,$$
$$\text{Im}(J) = \lambda_0 (x p_y - y p_x) .$$

$\text{Im}(J)$ is just the angular momentum, and $\text{Re}(J)$ is the non-diagonal component of the Fradkin tensor [9]. For the first non-isotropic case, $\lambda_1 = \lambda_0$, $\lambda_2 = 2 \lambda_0$, we arrive to

$$\text{Re}(J) = p_x^2 p_y + \lambda_0^2 (4 y p_x - x p_y) x ,$$
$$\text{Im}(J) = (x p_y - y p_x) p_x + \lambda_0^2 x^2 y .$$

This complex procedure provides not just the fundamental constant $I_3$, but the pair $(I_3, I_4)$; although the ‘partner’ function $I_4$ is not independent (is a function of $I_1$, $I_2$, $I_3$), we will see that it plays an important rôle, since it is closely concerned with the bi-Hamiltonian formalism. In fact, we will take the complex function $J$ as our starting point for the search of symmetries, but $J$ means not only one but two functions, $I_3$ and $I_4$.

The Noether theorem in the Hamiltonian formalism states that all constants of motion arise from canonical symmetries of the Hamiltonian function. Moreover, in differential geometric terms, the infinitesimal symmetries are simply those corresponding to the Hamiltonian vector fields, with respect to the canonical structure $\omega_0$, defined by the constants of motion. In this particular case, the above complex function $J$ given by (2) arises from a symmetry of (1) represented by the complex vector field $X_J$ defined by

$$i(X_J) \omega_0 = dJ , \quad X_J(H) = 0 . \quad (3)$$

In the following, and for easy of notation, we will suppose $\lambda_0 = 1$. 

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**Proposition 1** The complex vector field $X_J$, defined by (3) as the canonical infinitesimal symmetry associated to $J$, can be written as a linear combination of two dynamical but non-symplectic symmetries of $\Gamma_H$.

**Proof:** Let us denote by $Y_{xm}$ and $Y_{yn}$ the Hamiltonian vector fields of $K_x$ and $K_y$

$$i(Y_{xm})\omega_0 = dK_x, \quad i(Y_{yn})\omega_0 = dK_y,$$

with coordinate expressions

$$Y_{xm} = \frac{\partial}{\partial x} - i m \frac{\partial}{\partial p_x}, \quad Y_{yn} = \frac{\partial}{\partial y} - i n \frac{\partial}{\partial p_y}.$$

Notice that, as $H = I_1 + I_2$ with $|K_x|^2 = 2I_1$ and $|K_y|^2 = 2I_2$, we have

$$\Gamma_H = \text{Re}(K_x^* Y_{xm} + K_y Y_{yn}^*).$$

Then, the complex vector field $X_J$, canonical infinitesimal symmetry of the Harmonic oscillator, can be written as the following linear combination

$$X_J = n Y + m Y',$$

where the $Y, Y'$, are given by

$$Y = (K_x^{(n-1)} K_y^{*m}) Y_{xm}, \quad Y' = (K_x^n K_y^{*(m-1)}) Y_{yn}^*.$$

The important point is that these two vector fields, $Y$ and $Y'$, are neither locally-Hamiltonian with respect to $\omega_0$

$$\mathcal{L}_Y\omega_0 \neq 0, \quad \mathcal{L}_{Y'}\omega_0 \neq 0,$$

nor infinitesimal symmetries of the Hamiltonian

$$\mathcal{L}_Y H \neq 0, \quad \mathcal{L}_{Y'} H \neq 0.$$

Concerning the Lie bracket of $Y$ with the dynamical vector field $\Gamma_H$, it is given by

$$[Y, \Gamma_H] = (K_x^{n-1} K_y^{*m}) [Y_{xm}, \Gamma_H] - \Gamma_H (K_x^{n-1} K_y^{*m}) Y_{xm}$$

but as

$$[Y_{xm}, \Gamma_H] = -i Y_{xm}, \quad [Y_{yn}^*, \Gamma_H] = i Y_{yn}^*,$$

and

$$\Gamma_H (K_x^{n-1} K_y^{*m}) = (n - 1)(i m)(K_x^{n-1} K_y^{*m}) + m(-i n)(K_x^{n-1} K_y^{*m}) = -i m (K_x^{n-1} K_y^{*m}),$$

we arrive to

$$[Y, \Gamma_H] = 0.$$
Thus \( Y \) is a dynamical but non-symplectic (non-canonical) symmetry of \( \Gamma_H \). It can be proved, in a similar way, that this property is also true for \( Y' \). Notice that, in the language of 1-forms, this property arises from the fact that \( dJ \) splits as a sum of two non-closed 1-forms that, nevertheless, remain invariant under \( \Gamma_H \), that is, \( dJ = n \phi_1 + m \phi_2, d\phi_r \neq 0, \mathcal{L}_{\Gamma_H}(\phi_r) = 0, r = 1, 2 \).

Two new structures can be obtained from \( \omega_0 \) by Lie derivation with respect to \( Y \) and \( Y' \). If we denote by \( \omega_Y \) and \( \omega'_Y \) these two new 2-forms, then we obtain

\[
\omega_Y = -m (K_x^{(n-1)}K_y^{*(m-1)}) dK_x \wedge dK_y^*, \quad \omega'_Y = n (K_x^{(n-1)}K_y^{*(m-1)}) dK_x \wedge dK_y^*.
\]

In the following we will denote by \( \Omega \) the complex 2-form defined as

\[
\Omega = dK_x \wedge dK_y^* = \Omega_1 + i \Omega_2
\]

where the two real 2-forms, \( \Omega_1 = \text{Re}(\Omega) \) and \( \Omega_2 = \text{Im}(\Omega) \), take the form

\[
\Omega_1 = m n dx \wedge dy + dp_x \wedge dp_y, \quad \Omega_2 = m dx \wedge dp_y + n dy \wedge dp_x.
\]

Notice that \( \omega_Y \) and \( \omega'_Y \) satisfy the relation \( n \omega_Y + m \omega'_Y = 0 \). Actually, this can be considered as a consequence of the fact that \( X_J \) is locally Hamiltonian with respect to the canonical form \( \omega_0 \).

**Proposition 2** The dynamical vector field \( \Gamma_H \) of the rational Harmonic Oscillator is a bi-Hamiltonian system with respect to \((\omega_0, \omega_Y)\).

**Proof:** Notice that

\[
i(\Gamma_H)\omega_Y = -m (K_x^{(n-1)}K_y^{*(m-1)}) i(\Gamma_H)\Omega,
\]

and as

\[
i(\Gamma_H)\Omega = \Gamma_H(K_x) dK_y^* - \Gamma_H(K_y^*) dK_x = i m K_x dK_y^* + i n K_y^* dK_x,
\]

we obtain that

\[
i(\Gamma_H)\omega_Y = -i m d(K_x^n K_y^m).
\]

Thus, \( \Gamma_H \) is Hamiltonian vector field with respect to \( \omega_Y \) with \( K_x^n K_y^m \) as Hamiltonian function. Moreover, we can also compute the action of \( Y \) on \( H \); a direct calculation gives

\[
H_Y \equiv Y(H) = -i m (K_x^n K_y^m).
\]

To conclude, we have found that the integral of motion \( J \) determines the following bi-Hamiltonian system

\[
i(\Gamma_H)\omega_0 = dH, \quad i(\Gamma_H)\omega_Y = dH_Y.
\]

Remark first that \( \Gamma_H \) is bi-Hamiltonian with respect to two different structures: the canonical symplectic form \( \omega_0 \) and another one, \( \omega_Y \), which is complex. If we write \( \omega_Y =
ω₄ + i ω₃, then Γₜ can be considered as a bi-Hamiltonian system with respect to the following three real forms (ω₀, ω₃, ω₄) (i.e. it is a three-Hamiltonian system). The ω₀-Hamilton equation determined by J,

\[ i(X_J) \omega_0 = dJ , \]
is also complex; thus it determines two real Hamiltonian equations

\[ i(X_4) \omega_0 = dI_4 , \quad i(X_3) \omega_0 = dI_3 , \]

with \( X_4, X_3 \), given by \( X_J = X_4 + i X_3 \).

As a second remark, the complex 2-form \( \Omega = dK_x \land dK_y^* \) is well defined but it is not symplectic. In fact, it can be proved that \( \Omega_1 = \text{Re}(\Omega) \) and \( \Omega_2 = \text{Im}(\Omega) \) satisfy

\[ \Omega_1 \land \Omega_1 = \Omega_2 \land \Omega_2 = mn (dx \land dy \land dp_x \land dp_y) , \quad \text{and} \quad \Omega_1 \land \Omega_2 = 0 , \]

so we obtain

\[ \Omega \land \Omega = (\Omega_1 \land \Omega_1 - \Omega_2 \land \Omega_2) + 2 i \Omega_1 \land \Omega_2 = 0 . \]

Thus, the degenerate character of \( \Omega \) is directly related with its complex nature. Moreover, the kernel of \( \Omega \) is the distribution generated by \( Y_{xm} \) and \( Y_{yn}^* \),

\[ \text{Ker } \Omega = \{ f Y_{xm} + g Y_{yn}^* \mid f, g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C} \} , \]

therefore it satisfies

\[ [\text{Ker } \Omega, \Gamma_t] \subset \text{Ker } \Omega . \]

Finally, the 2-form \( \omega_Y \) is also degenerate. We obtain, \( \text{Ker } \omega_Y = \text{Ker } \Omega \), because of the relation between \( \Omega \) and \( \omega_Y \). However \( \omega_3 \) and \( \omega_4 \), defined as \( \omega_Y = \omega_4 + i \omega_3 \), are symplectic real forms. Moreover, the form \( \omega_0 + \Omega \) is symplectic because of \( \{ K_x, K_y^* \} = 0 \).

### 3 Recursion operators

The bi-Hamiltonian structure \( (\omega_0, \omega_Y) \) defines a complex recursion operator \( R_Y \) by

\[ \omega_Y(X,Y) = \omega_0(R_Y X, Y) , \quad \forall X, Y \in \mathcal{X}(M) , \]
or, equivalently, \( R_Y = \bar{\omega}_0^{-1} \circ \omega_Y \). Since it is complex, it can be written as \( R_Y = R_4 + i R_3 \), so that \( R_4 \) and \( R_3 \) satisfy the relations

\[ \omega_3(X,Y) = \omega_0(R_3 X, Y) , \quad \text{and} \quad \omega_4(X,Y) = \omega_0(R_4 X, Y) . \]

Thus, we have that \( R_3 \) and \( R_4 \) are given by \( R_3 = \bar{\omega}_0^{-1} \circ \bar{\omega}_3 \) and \( R_4 = \bar{\omega}_0^{-1} \circ \bar{\omega}_4 \).

The important point is that the complex 2-form \( \Omega = dK_x \land dK_y^* \) can be decomposed as \( \Omega = \Omega_1 + i \Omega_2 \), where both 2-forms, \( \Omega_1 \) and \( \Omega_2 \), are symplectic. Hence, we have, in addition to \( R_3 \) and \( R_4 \), two other recursion operators \( R_1 \) and \( R_2 \) associated with the bi-Hamiltonian structures provided by \( \Omega_1 \) and \( \Omega_2 \), respectively.
Proposition 3 The tensor fields $R_1$ and $R_2$ are invertible operators which anticommute and satisfy $R_2^2 = R_1^2$.

Proof: As $\Omega_1$ and $\Omega_2$ are symplectic forms, the operators $R_1$ and $R_2$ are invertible. Their coordinate expressions are given by

$$R_1 = \frac{\partial}{\partial y} \otimes dp_x - \frac{\partial}{\partial x} \otimes dp_y + mn \left( \frac{\partial}{\partial p_x} \otimes dy - \frac{\partial}{\partial p_y} \otimes dx \right), \quad (4)$$

$$R_2 = m \left( \frac{\partial}{\partial y} \otimes dx + \frac{\partial}{\partial p_x} \otimes dp_y \right) + n \left( \frac{\partial}{\partial x} \otimes dy + \frac{\partial}{\partial p_y} \otimes dp_x \right). \quad (5)$$

Therefore,

$$R_2 R_1 = mn \text{Id}, \quad R_1 R_2 = mn \text{Id}.$$ 

Moreover we have

$$R_2 R_1 = n \left( \frac{\partial}{\partial x} \otimes dp_x - m^2 \frac{\partial}{\partial p_x} \otimes dx \right) - m \left( \frac{\partial}{\partial y} \otimes dp_y - n^2 \frac{\partial}{\partial p_y} \otimes dy \right),$$

and $R_1 R_2 = -R_2 R_1$; therefore $R_2 R_1 + R_1 R_2 = 0$.

We recall that the relation between $\omega_Y$ and $\Omega$ is $\omega_Y = \omega_4 + i \omega_3 = -m K \Omega$, with the complex function $K$ given by $K = K^{(n-1)}_x K^{(m-1)}_y = K_r + i K_i$. Thus, the above two tensor fields, $R_3$ and $R_4$, are given by

$$R_4 = -m (K_r R_1 - K_i R_2), \quad (6)$$

$$R_3 = -m (K_r R_2 + K_i R_1), \quad (7)$$

and, making use of the preceding proposition, we arrive to

$$R_4^2 = R_3^2 = r \text{Id}, \quad \text{with} \quad r = m^2 (mn)^2 |K|^2,$$

and

$$R_4 R_3 = m^2 |K|^2 R_1 R_2, \quad R_3 R_4 = m^2 |K|^2 R_2 R_1,$$

where the modulus of $K$ is function of the two first integrals, $I_1$ and $I_2$,

$$|K|^2 = K_r^2 + K_i^2 = (2I_1)^{(n-1)} (2I_2)^{(m-1)}.$$ 

Thus, the two tensor fields, $R_3$ and $R_4$, anticommute as well.

Proposition 4 The complex operator $R_Y = R_4 + i R_3$ is such that $\text{Image}(R_Y) = \text{Ker} R_Y$.

Proof: Let $X$ be a generic vector field on the phase space

$$X = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial p_x} + d \frac{\partial}{\partial p_y},$$
with \( a, b, c, d \), arbitrary functions. Then, since \( R_Y = R_4 + iR_3 \) is given by \( R_Y = -mK(R_1 + iR_2) \), the subspace \( \text{Image}(R_Y) \) is given by

\[
\text{Image}(R_Y) = -\left\{ Z \in \mathfrak{X}(\mathbb{R}^2 \times \mathbb{R}^2) \mid Z = -mK(R_1(X) + iR_2(X)) \right\}.
\]

We obtain

\( R_1(X) + iR_2(X) = -(d - i n b)Y_{xm} + (c + i m a)Y_{yn}^* \).

Thus, \( \text{Image}(R_Y) \) is made up of linear combinations of \( Y_{xm} \) and \( Y_{yn}^* \) with arbitrary complex functions as coefficients. But, since \( \text{Ker}R_1 = \text{Ker}Y \) and \( \text{Ker}\omega_Y \) coincides with \( \text{Ker}Y \), which is also spanned by \( Y_{xm} \) and \( Y_{yn}^* \), we arrive to \( \text{Image}(R_Y) = \text{Ker}R_Y \). Consequently \( R_Y^2 = R_Y \circ R_Y = 0 \).

Given a bi-Hamiltonian system on a manifold \( M \), \( i(\Gamma)\omega_0 = dH_0 \) and \( i(\Gamma)\omega_1 = dH_1 \), the point is that the tensor field \( R \), that was just defined by the relation between \( \omega_1 \) and \( \omega_0 \), induces a sequence of structures. Starting with the basic Hamiltonian system \( (\omega_0, \Gamma_0 = \Gamma, dH_0) \) we can construct a sequence of 2-forms \( \omega_k \), of vector fields \( \Gamma_k \), and of 1-forms \( \alpha_k \), \( k = 1, 2, \ldots \), defined by \( \bar{\omega}_k = \bar{\omega}_0 \circ R^k \), \( \Gamma_k = R^k(\Gamma_0) \), and \( \alpha_k = R^k(dH_0) \). Then it follows that

\[
i(\Gamma_0)\omega_1 = i(\Gamma_1)\omega_0 = dH_1,
\]

\[
i(\Gamma_0)\omega_2 = i(\Gamma_1)\omega_1 = i(\Gamma_2)\omega_0 = \alpha_2,
\]

where

\[
\bar{\omega}_1 = \bar{\omega}_0 \circ R,
\]

\[
\Gamma_1 = R(\Gamma_0),
\]

\[
dH_1 = R^*(dH_0),
\]

\[
\bar{\omega}_2 = \bar{\omega}_1 \circ R,
\]

\[
\Gamma_2 = R(\Gamma_1),
\]

\[
\alpha_2 = R^*(dH_1),
\]

The 1-form \( \alpha_2 \) is not necessarily exact, but if there is \( H_2 \) such that \( \alpha_2 = dH_2 \), then the vector field \( \Gamma_1 \) is a bi-Hamiltonian system as well. An interesting case is when \( \alpha_2 \) is not exact but there exist a nonvanishing function \( F_2 \) and another function \( H_2 \) such that \( \alpha_2 = F_2 dH_2 \). Then \( F_2^{-1} \) is an integrating factor for \( \alpha_2 \), and the vector field \( \Gamma_1 \) is said to be quasi-bi-Hamiltonian [11],[12].

Coming back to the rational harmonic oscillator as a bi-Hamiltonian system, \( i(\Gamma_H)\omega_0 = dH_0 \) and \( i(\Gamma_H)\omega_Y = dH_Y \), the situation is as follows:

(i) The action of \( R_Y \) is such that \( \Gamma_H \equiv \Gamma_0 \) becomes \( \Gamma_1 = R_Y(\Gamma_H) = -i m X_j \),

(ii) \( dH_0 \) transforms into \( dH_Y = R_Y^*(dH_0) = -i dJ \), and

(iii) \( \omega_0 \) becomes \( \omega_Y \) such that \( \bar{\omega}_Y = \bar{\omega}_0 \circ R_Y \).

We have proved that \( R_Y^2 = 0 \) because of Proposition 4; therefore, \( \Gamma_1 \) transforms into the new field, \( \Gamma_2 = R_Y(\Gamma_1) = R_Y^2(\Gamma_H) = 0 \), while \( dH_Y \) transforms into \( \alpha_2 = R_Y^*(dH_Y) = R_Y^2(dH_0) = 0 \). Hence, it follows that the equation

\[
i(\Gamma_H)\omega_2 = i(\Gamma_1)\omega_1 = i(\Gamma_2)\omega_0 = \alpha_2
\]

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becomes
\[ i(\Gamma_1) \omega_1 = 0. \]

Notice that this last equation corresponds to the property \( i(X_J) \omega_Y = 0 \).

The Harmonic Oscillator can be considered as a complex and weakly bi-Hamiltonian system, or alternatively, as endowed with two different real bi-Hamiltonian structures
\[ i(\Gamma_H) \omega_0 = dH_0, \quad i(\Gamma_H) \omega_4 = m dI_3, \quad i(\Gamma_H) \omega_3 = -m dI_4. \]

One real structure gives rise to \( \Gamma_{13} \) defined by \( \Gamma_{13} = R_3(\Gamma_H) \), and the other one to \( \Gamma_{14} = R_4(\Gamma_H) \). They are such that
\[ i(\Gamma_{14}) \omega_0 = i(\Gamma_H) \omega_4 = m dI_3, \quad i(\Gamma_{13}) \omega_0 = i(\Gamma_H) \omega_3 = -m dI_4. \]

Moreover taking into account that \( R_2^4 = r \text{Id} \), we arrive to
\[ \Gamma_{24} = R_4(\Gamma_{14}) = R_4^2(\Gamma_H) = r \Gamma_H, \]
\[ \omega_{24} = R_4^* (m dI_3) = R_4^2 (dH_0) = r dH_0, \]
\[ \hat{\omega}_{24} = \hat{\omega}_4 \circ R_4 = \hat{\omega}_0 \circ R_4^2 = r \hat{\omega}_0, \]

and similar results for \( R_3 \).

Now, making use of all these relations, we can prove the following final proposition concerning the properties of the vector fields \( X_3 \) and \( X_4 \).

**Proposition 5** Let \( X_3 \) and \( X_4 \) denote the two infinitesimal canonical symmetries generating the two constants of motion \( I_3 \) and \( I_4 \). Then \( X_3 \) and \( X_4 \) are quasi-bi-Hamiltonian systems. Moreover, \( \omega_4(X_3, \Gamma_H) = \omega_3(X_4, \Gamma_H) = 0 \).

**Proof:** The rational Harmonic Oscillator is endowed with the two constants of motion \( I_3 \) and \( I_4 \) which means, via the Hamiltonian Noether theorem, the existence of two symmetries. They are geometrically represented by two vector fields, \( X_3 \) and \( X_4 \), that can be uniquely determined as solutions of the following two equations
\[ i(X_3) \omega_0 = dI_3, \quad i(X_4) \omega_0 = dI_4. \]

Then we have
\[ \Gamma_{13} = R_3(\Gamma_H) = -m X_4, \quad \Gamma_{14} = R_4(\Gamma_H) = m X_3. \]

Hence, if we denote by \( f_{34} \) the function \( f_{34} = m(mn)^2 |K|^2 \), we arrive to
\[ i(X_3) \omega_0 = dI_3, \quad i(X_3) \omega_4 = f_{34} dH_0, \]

and
\[ i(X_4) \omega_0 = dI_4, \quad i(X_4) \omega_3 = -f_{34} dH_0. \]

So, both \( X_3 \) and \( X_4 \) are quasi-bi-Hamiltonian systems.
A direct consequence of this property is that the dynamical vector field $\Gamma_H$ is orthogonal to $X_3$ with respect to the symplectic structure $\omega_4$,
\[ i(X_3)i(\Gamma_H)\omega_0 = 0, \quad \text{and} \quad i(X_3)i(\Gamma_H)\omega_4 = 0. \]
Similarly, we obtain
\[ i(X_4)i(\Gamma_H)\omega_0 = 0, \quad \text{and} \quad i(X_4)i(\Gamma_H)\omega_3 = 0. \]
Finally, $X_3$ and $X_4$ are orthogonal vector fields with respect to both structures, $\omega_3$ and $\omega_4$:
\[ i(X_3)i(X_4)\omega_3 = 0, \quad \text{and} \quad i(X_3)i(X_4)\omega_4 = 0. \]

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