JOINT DIAMONDS AND LAVER DIAMONDS

MIHA E. HABIČ

Abstract. We study the concept of jointness for guessing principles, such as ♦κ and various Laver diamonds. A family of guessing sequences is joint if the elements of any given sequence of targets may be simultaneously guessed by the members of the family. We show that, while equivalent in the case of ♦κ, joint Laver diamonds are nontrivial new objects. We give equiconsistency results for most of the large cardinals under consideration and prove sharp separations between joint Laver diamonds of different lengths in the case of θ-supercompact cardinals.

1. Introduction

The concept of a Laver function, introduced for supercompact cardinals in [10], is a powerful strengthening of the usual ♦-principle to the large cardinal setting. It is based on the observation that a large variety of large cardinal properties give rise to different notions of “large” set, intermediate between stationary and club, and these are then used to provide different guessing principles, where we require that the sequence guesses correctly on ever larger sets. This is usually recast in terms of elementary embeddings or extensions (if the large cardinal in question admits such a characterization), using various ultrapower constructions. For example, in the case of a supercompact cardinal κ, the usual definition states that a Laver function for κ is a function ℓ: κ → Vκ such that for any θ and any a ∈ Hθ+ there is a θ-supercompactness embedding j: V → M with critical point κ such that j(ℓ)(κ) = a (this ostensibly second order definition can be rendered in first order language by replacing the quantification over arbitrary embeddings with quantification over ultrapowers by measures on Pκ(θ), as in Laver’s original account).

Laver functions for other large cardinals were later defined by Gitik and Shelah (see [7]), Corazza (see [5]), Hamkins (see [8]) and others. The term Laver diamonds has been suggested to more strongly underline the connection between the large and small cardinal versions.

In this chapter we examine the notion of jointness for both ordinary and Laver diamonds. We shall give a simple example in section 2 for now let us just say that a family of Laver diamonds is joint if they can guess their targets simultaneously and independently of one another. Section 2 also introduces some terminology that will ease later discussion. Sections 3 and 4 deal with the outright existence or at least the consistency of joint Laver sequences for supercompact and strong cardinals, respectively. Our results will show that in almost all cases the existence of

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A joint Laver sequence of maximal possible length is simply equiconsistent with the particular large cardinal. The exception are the \( \theta \)-strong cardinals where \( \theta \) is a limit of small cofinality, for which we prove that additional strength is required for even the shortest joint sequences to exist. We also show that there are no nontrivial implications between the existence of joint Laver sequences of different lengths. Section 5 considers joint \( \Diamond_\kappa \)-sequences and their relation to other known principles. Our main result there shows that \( \Diamond_\kappa \) is simply equivalent to the existence of a joint \( \Diamond_\kappa \)-sequence of any possible length.

2. Jointness: A Motivating Example

All of the large cardinals we will be dealing with in this paper are characterized by the existence of elementary embeddings of the universe into certain inner models which have that cardinal as their critical point. We can thus speak of embeddings associated to a measurable, a \( \theta \)-strong, or a 17-huge cardinal \( \kappa \) etc. Since the definitions of (joint) Laver diamonds for these various large cardinals are quite similar, we give the following general definition as a framework to aid future exposition.

Definition 1. Let \( j \) be an elementary embedding of the universe witnessing the largeness of its critical point \( \kappa \) (e.g. a measurable, or a hugeness embedding) and let \( \ell \) be a function defined on \( \kappa \). We say that a set \( a \), the target, is guessed by \( \ell \) via \( j \) if \( j(\ell)(\kappa) = a \).

If \( A \) is a set or a definable class, say that \( \ell \) is an \( A \)-guessing Laver function (or Laver diamond) if for any \( a \in A \) there is an embedding \( j \), witnessing the largeness of \( \kappa \), such that \( \ell \) guesses \( a \) via \( j \). If there is an \( A \)-guessing Laver function for \( \kappa \), we shall say that \( \Delta_\kappa(A) \) holds.

To simplify the terminology even more, we shall associate to each type of large cardinal considered, a default set of targets \( A \) (for example, when talking about a measurable cardinal \( \kappa \), we will be predominantly interested in targets from \( H_{\kappa^+} \)). In view of this, whenever we neglect the mention of a particular class of targets, these default targets will be intended.

We will often specify the type of large cardinal embeddings we have in mind explicitly, by writing \( \Delta_\kappa^{\text{meas}} \), or \( \Delta_\kappa^{\theta \text{-sc}} \) etc. This is to avoid ambiguity; for example, we could conceivably start with a supercompact cardinal \( \kappa \) but only be interested in its measurable Laver functions. Even so, to keep the notation as unburdened as possible, we may sometimes omit the specific large cardinal property under consideration when it is clear from context.

As a further complication, the stated definition of an \( A \)-guessing Laver function is second-order, since we are quantifying over all possible embeddings \( j \). This is unavoidable for arbitrary \( A \). However, the default sets of targets we shall be working with are chosen in such a way that standard factoring arguments allow us to restrict our attention to ultrapower or extender embeddings. The most relevant definitions of Laver functions can therefore be recast in first-order language in the usual way.

Given the concept of a Laver diamond for a large cardinal \( \kappa \), we might ask when are two Laver functions different and how many distinct ones can the large cardinal carry. It is clear that the guessing behaviour of these functions is determined by their restrictions to large (in the sense of an appropriate large-cardinal measure)

\footnote{Different notation has been used by different authors to denote the existence of a Laver function. We chose here to follow Hamkins.}
sets; in other words, \(j(\ell)(\kappa)\) and \(j(\ell')(\kappa)\) equal one another if \(\ell\) and \(\ell'\) only differ on a small (nonstationary, say) subset of their domain. We definitely do not want to count these functions as distinct: they cannot even guess distinct targets! Instead, what we want are Laver functions whose targets, under a single embedding \(j\), can be chosen completely independently. Let us illustrate this situation with a simple example.

Suppose \(\ell: \kappa \to V_{\kappa}\) is a supercompactness Laver function. We can then define two functions \(\ell_0, \ell_1\) by letting \(\ell_0(\xi)\) and \(\ell_1(\xi)\) be the first and second components, respectively, of \(\ell(\xi)\), if this happens to be an ordered pair. These two are then easily seen to be Laver functions themselves, but have the additional property that, given any pair of targets \(a_0, a_1\), there is a single supercompactness embedding \(j\) such that \(j(\ell_0)(\kappa) = a_0\) and \(j(\ell_1)(\kappa) = a_1\). This additional trait, where two Laver functions are, in a sense, enmeshed, we call jointness.

**Definition 2.** Let \(A\) be a set or a definable class and let \(\kappa\) be a cardinal with a notion of \(A\)-guessing Laver function. A sequence \(\vec{\ell} = \langle \ell_\alpha; \alpha < \lambda \rangle\) of \(A\)-guessing Laver functions is an \(A\)-guessing joint Laver sequence if for any sequence \(\vec{a} = \langle a_\alpha; \alpha < \lambda \rangle\) of targets from \(A\) there is a single embedding \(j\), witnessing the largeness of \(\kappa\), such that each \(\ell_\alpha\) guesses \(a_\alpha\) via \(j\). If there is an \(A\)-guessing joint Laver sequence of length \(\lambda\) for \(\kappa\), we shall say that \(\mathbf{b}_{\kappa,\lambda}(A)\) holds.

In other words, a sequence of Laver diamonds is joint if, given any sequence of targets, these targets can be guessed simultaneously by their respective Laver diamonds.

We must be careful to distinguish between and entire sequence being jointly Laver or its members being pairwise jointly Laver. It is not difficult to find examples of three (or four or even infinitely many) Laver functions that are pairwise joint but not fully so. For example, given two joint Laver functions \(\ell_0\) and \(\ell_1\) we might define \(\ell_2(\xi)\) to be the symmetric difference of \(\ell_0\) and \(\ell_1\). It is easy to check that any two of these three functions can have their targets freely chosen, but the third one is uniquely determined by the other two.

Jointness also makes sense for ordinary diamond sequences, but needs to be formulated differently, since elementary embeddings do not (obviously) appear in that setting. Rather, we distil jointness for Laver diamonds into a property of certain ultrafilters and then apply this to more general filters and diamond sequences. We explore this further in section 5.

3. Joint Laver diamonds for supercompact cardinals

**Definition 3.** A function \(\ell: \kappa \to V_{\kappa}\) is a \(\theta\)-supercompactness Laver function for \(\kappa\) if it guesses elements of \(H_{\theta^+}\) via \(\theta\)-supercompactness embeddings with critical point \(\kappa\). This also includes the case of \(\kappa\) being measurable (as this is equivalent to it being \(\kappa\)-supercompact).

If \(\kappa\) is fully supercompact then a function \(\ell: \kappa \to V_{\kappa}\) is a Laver function for \(\kappa\) if it is a \(\theta\)-supercompactness Laver function for \(\kappa\) for all \(\theta\).

Observe that there are at most \(2^\kappa\) many \(\theta\)-supercompactness Laver functions for \(\kappa\), since there are only \(2^\kappa\) many functions \(\kappa \to V_{\kappa}\). Since a joint Laver sequence cannot have the same function appear on two different coordinates (as they could never guess two different targets), this implies that \(\lambda = 2^\kappa\) is the largest cardinal for which there could possibly be a joint Laver sequence of length \(\lambda\). Bounding
from the other side, a single \( \theta \)-supercompactness Laver function already yields a joint Laver sequence of length \( \theta \).

**Proposition 4.** If \( \mathfrak{S}^{\theta \text{-sc}}_\kappa \) holds then there is a \( \theta \)-supercompactness joint Laver sequence for \( \kappa \) of length \( \lambda = \min\{\theta, 2^\kappa\} \).

**Proof.** Let \( \ell \) be a Laver function for \( \kappa \). Fix a subset \( I \) of \( \mathcal{P}(\kappa) \) of size \( \lambda \) and a bijection \( f: \lambda \to I \). For \( \alpha < \lambda \) define \( \ell_\alpha: \kappa \to \mathcal{V}_\kappa \) by \( \ell_\alpha(\xi) = \ell(\xi)(f(\alpha) \cap \xi) \) if this makes sense and \( \ell_\alpha(\xi) = \emptyset \) otherwise. We claim that \( \langle \ell_\alpha; \alpha < \lambda \rangle \) is a joint Laver sequence for \( \kappa \).

To verify this let \( \vec{a} = \langle a_\alpha; \alpha < \lambda \rangle \) be a sequence of elements of \( H_{\theta^+} \). Then \( \vec{a} \circ f^{-1} \in H_{\theta^+} \), so by assumption there is a \( \theta \)-supercompactness embedding \( j: V \to M \) such that \( j(\ell)(\kappa) = \vec{a} \circ f^{-1} \). But now observe that, by elementarity, for any \( \alpha < \lambda \)

\[ j(\ell_\alpha)(\kappa) = j(\ell)(\kappa)(j(f(\alpha)) \cap \kappa) = j(\ell)(\kappa)(f(\alpha)) = a_\alpha \]  

Of course, if a given Laver function works for many degrees of supercompactness then the joint Laver sequence derived above will work for those same degrees. In particular, if \( \kappa \) is fully supercompact then this observation, combined with Laver’s original construction, gives us a supercompactness joint Laver sequence of length \( 2^\kappa \).

**Corollary 5.** If \( \kappa \) is supercompact then \( \mathfrak{S}^{\text{sc}}_{\kappa, 2^\kappa} \) holds.

Proposition 4 essentially shows that joint Laver sequences of maximal length exist automatically for cardinals with a high degree of supercompactness. Since we will be interested in comparing the strengths of the principles \( \mathfrak{S}_{\kappa, \lambda} \) for various \( \lambda \), we will in the remainder of this section be mostly concerned with cardinals \( \kappa \) which are not \( 2^\kappa \)-supercompact (but are at least measurable), so as to avoid situations where a single Laver function gives rise to the longest possible joint Laver sequence.

### 3.1. Creating long joint Laver diamonds

We now show that the existence of \( \theta \)-supercompactness joint Laver sequences of maximal length does not require strength beyond \( \theta \)-supercompactness itself. The following lemma is well-known.

**Lemma 6.** If \( \kappa \) is \( \theta \)-supercompact then \( \kappa \) has a Menas function, i.e. a function \( f: \kappa \to \kappa \) and a \( \theta \)-supercompactness embedding \( j: V \to M \) with \( \operatorname{cp}(j) = \kappa \) such that \( j(f)(\kappa) > \theta \).

**Theorem 7.** If \( \kappa \) is \( \theta \)-supercompact then there is a forcing extension in which \( \mathfrak{S}^{\theta \text{-sc}}_{\kappa, 2^\kappa} \) holds.

**Proof.** Since \( \theta \)-supercompactness of \( \kappa \) implies its \( \theta^{<\kappa} \)-supercompactness, we may assume that \( \theta^{<\kappa} = \theta \). Furthermore, we assume that \( 2^{\theta} = \theta^+ \), since this may be forced without adding subsets to \( \mathcal{P}_\kappa \theta \). Fix a Menas function \( f \) for \( \kappa \) as in lemma 6. Let \( \mathbb{P}_\kappa \) be the length \( \kappa \) Easton support iteration which forces with \( \mathbb{Q}_\gamma = \text{Add}(\gamma, 2^{\gamma}) \) at inaccessible closure points of \( f \), i.e. those inaccessible \( \gamma \) for which \( f[\gamma] \subseteq \gamma \). Finally, let \( \mathbb{P} = \mathbb{P}_\kappa * \mathbb{Q}_\kappa \). It is useful to note that forcing with \( \mathbb{P} \) does not change the value of \( 2^\kappa \). Let \( G * g \subseteq \mathbb{P} \) be generic; we will extract a joint Laver sequence from \( g \).

If \( g(\alpha) \) is the \( \alpha \)-th subset added by \( g \), we view it as a sequence of bits. Given any \( \xi < \kappa \) we may then view the segment of \( g(\alpha) \) between the \( \xi \)th bit and the next marker as the Mostowski code of an element of \( V_\kappa \) (employing bit doubling
or some other coding scheme which admits end-of-code markers). We then define \( \ell_\alpha: \kappa \rightarrow V_\kappa \) as follows: given an inaccessible \( \xi \), let \( \ell_\alpha(\xi) \) be the set coded by \( g(\alpha) \) at \( \xi \); otherwise let \( \ell_\alpha(\xi) = \emptyset \). We claim that \( \langle \ell_\alpha: \alpha < 2^\kappa \rangle \) is a joint Laver sequence.

Let \( \tilde{\alpha} = \langle a_\alpha: \alpha < 2^\kappa \rangle \) be a sequence of targets in \( H^{V[G][\theta]}_\theta \). Let \( j: V \rightarrow M \) be the ultrapower embedding by a normal measure on \( P_\kappa \theta \) which corresponds to \( f \), i.e. such that \( j(f)(\kappa) > \theta \). We will lift this embedding through the forcing \( P \) in \( V[G][\theta] \).

The argument splits into two cases, depending on the size of \( \theta \). We deal first with the easier case when \( \theta \geq 2^\kappa \). In this case the poset \( j(P_\kappa) \) factors as \( j(P_\kappa) = P_\kappa \ast Q_\kappa \ast P_{\text{tail}} \). Since \( j(f)(\kappa) > \theta \), the next stage of forcing in \( j(P_\kappa) \) above \( \kappa \) occurs after \( \theta \), so \( P_{\text{tail}} \leq \theta \)-closed in \( M[G][\theta] \) and has size \( j(\kappa) \) there. Since \( M \) was an ultrapower, \( M[G][\theta] \) has at most \( |2^{j(\kappa)}| \leq (2^\kappa)\theta = \theta^+ \) many subsets of \( P_{\text{tail}} \), and so we can diagonalize against all of them to produce in \( V[G][\theta] \) an \( M[G][\theta] \)-generic \( G_{\text{tail}} \subseteq P_{\text{tail}} \) and lift \( j \) to \( j: V[G] \rightarrow M[j(G)] \), where \( j(G) = G \ast g \ast G_{\text{tail}} \).

Since \( M[j(G)] \) is still an ultrapower and thus closed under \( \theta \)-sequences in \( V[G][\theta] \), we get \( j[G] \in M[j(G)] \). Since \( j(Q_\kappa) \) is \( \leq \theta \)-directed closed in \( M[j(G)] \) it has \( q = \bigcup j[G] \) as a condition. Since \( M[j(G)] \) has both the sequence of targets \( \tilde{\alpha} \) and \( j(2^\kappa) \), we can further extend \( q \) to \( q^* \) by coding \( a_\alpha \) at \( \kappa \) in \( q(j(\alpha)) \) for each \( \alpha < 2^\kappa \). We again diagonalize against the \( \langle 2^{j(\kappa)} \rangle \leq \theta^+ \) many dense subsets of \( j(Q_\kappa) \) in \( M[j(G)] \) below the master condition \( q^* \) to get a \( M[j(G)] \)-generic \( g^* \subseteq j(Q_\kappa) \) and lift \( j \) to \( j: V[G][\theta] \rightarrow M[j(G)][g^*] \). Finally, observe that we have arranged the construction of \( g^* \) in such a way that \( g^*(j(\alpha)) \) codes \( a_\alpha \) at \( \kappa \) for all \( \alpha < 2^\kappa \) and, by definition, this implies that \( j(\ell_\alpha)(\kappa) = a_\alpha \) for all \( \alpha < 2^\kappa \). Thus we indeed have a joint Laver sequence for \( \kappa \) of length \( 2^\kappa \) in \( V[G][\theta] \).

It remains for us to consider the second case, when \( \kappa \leq \theta < 2^\kappa \). In this situation our assumptions on \( \theta \) imply that \( 2^\kappa = \theta^+ \). The poset \( j(P_\kappa) \) factors as \( j(P_\kappa) = P_\kappa \ast Q_\kappa \ast P_{\text{tail}} \), where \( Q_\kappa = \text{Add}(\kappa, (2^\kappa)^{M[G]}) \) is isomorphic but not necessarily equal to \( Q_\kappa \). Nevertheless, the same argument as before allows us to lift \( j \) to \( j: V[G] \rightarrow M[j(G)] \) where \( j(G) = G \ast g \ast G_{\text{tail}} \) and \( g \) is the isomorphic image of \( g \).

We seem to hit a snag with the final lift through the forcing \( Q_\kappa \), which has size \( 2^\kappa \) and thus resists the usual approach of lifting via a master condition, since this condition would simply be too big for the amount of closure we have. We salvage the argument by using a technique, originally due to Magidor [12], sometimes known as the “master filter argument”.

The forcing \( j(Q_\kappa) = \text{Add}(j(\kappa), 2^{j(\kappa)}) \) has size \( 2^{j(\kappa)} \) and is \( \leq \theta \)-directed closed and \( j(\kappa)^{+}\text{cc} \) in \( M[j(G)] \). Since \( M[j(G)] \) is still an ultrapower, \( |2^{j(\kappa)}| \leq \theta^+ = 2^\kappa \) and so \( M[j(G)] \) has at most \( 2^\kappa \) many maximal antichains of \( j(Q_\kappa) \). Let these be given in the sequence \( \langle Z_\alpha: \alpha < 2^\kappa \rangle \). Since each \( Z_\alpha \) has size at most \( j(\kappa) \), it is in fact contained in some bounded part of the poset \( j(Q_\kappa) \). Furthermore (and crucially), since \( j \) is an ultrapower by a measure on \( P_\kappa \theta \), it is continuous at \( 2^\kappa = \theta^+ \) and so there is for each \( \alpha \) a \( \beta_\alpha < 2^\kappa \) such that \( Z_\alpha \subseteq \text{Add}(j(\kappa), j(\beta_\alpha)) \).

In particular, each \( Z_\alpha \) is a maximal antichain in \( \text{Add}(j(\kappa), j(\beta_\alpha)) \). We will now construct in \( V[G][\theta] \) a descending sequence of conditions, deciding more and more of the antichains \( Z_\alpha \), which will generate a filter, the “master filter”, that will allow us to lift \( j \) to \( V[G][\theta] \) and also (lest we forget) witness the joint guessing property. We begin by defining the first condition \( q_0 \). Consider the generic \( g \) up to \( \beta_0 \). This piece has size \( \theta \) and so \( \bigcup j[g \upharpoonright \beta_0] \) is a condition in \( j(Q_\kappa) \). Let \( q_0 \) be the extension of \( \bigcup j[g \upharpoonright \beta_0] \) which codes the target \( a_\alpha \) at \( \kappa \) in \( q_0(j(\alpha)) \) for
each $\alpha < \beta_0$. This is still a condition in $j(Q_\kappa) \upharpoonright j(\beta_0)$ and we can finally let $q_0$ be any extension of $q'_0$ in this poset which decides the maximal antichain $Z_0$. Note that $q_0$ is compatible with every condition in $j[\mathcal{g}]$, since we extended the partial master condition $\bigcup j[\mathcal{g} \upharpoonright \beta_0]$ and made no commitments outside $j(Q_\kappa) \upharpoonright j(\beta_0)$. We continue in this way recursively, constructing a descending sequence of conditions $q_\alpha$ for $\alpha < \theta^+$, using the closure of $j(Q_\kappa)$ and $M[j(G)]$ to pass through limit stages. Now consider the filter $g^*$ generated by the conditions $q_\alpha$. It is $M[j(G)]$-generic by construction and also extends (or can easily be made to extend) $j[\mathcal{g}]$. We can thus lift $j$ to $j : V[G][\mathcal{g}] \rightarrow M[j(G)][g^*]$ and, since $P$ is $\theta^+$-cc and both $G_{\text{tail}}$ and $g^*$ were constructed in $V[G][\mathcal{g}]$, the model $M[j(G)][g^*]$ is closed under $\theta$-sequences which shows that $\kappa$ remains $\theta$-supercompact in $V[G][\mathcal{g}]$. Finally, as in the previous case, $g^*$ was constructed in such a way that $j(\ell_\alpha)(\kappa) = a_\alpha$ for all $\alpha < 2^\kappa$, verifying that these functions really do form a joint Laver sequence for $\kappa$. \hfill \Box

As a special case of theorem 7 we can deduce the corresponding result for measurable cardinals.

**Corollary 8.** If $\kappa$ is measurable then there is a forcing extension in which there is a joint Laver sequence for $\kappa$ of length $2^\kappa$.

It follows from the results of Hamkins [9] that the forcing $P$ from theorem 7 does not create any measurable or (partially) supercompact cardinals below $\kappa$, since it admits a very low gap. We could therefore have started with the least large cardinal $\kappa$ of interest and preserved its minimality through the construction.

**Corollary 9.** If $\kappa$ is the least $\theta$-supercompact cardinal then there is a forcing extension where $\kappa$ remains the least $\theta$-supercompact cardinal and $M^{\theta_{\text{sc}}}$ holds.

It is perhaps interesting to observe the peculiar arrangement of cardinal arithmetic in the model produced in the above proof. We have $2^\theta = \theta^+$ and, if $\theta \leq 2^\kappa$, also $2^\kappa = \theta^+$. In particular, we never produced a $\theta$-supercompactness joint Laver sequence of length greater than $\theta^+$ (assuming here, of course, that $\theta = \theta^{<\kappa}$ is the optimal degree of supercompactness). One has to wonder whether this is significant. Certainly the existence of long joint Laver sequences does not imply much about cardinal arithmetic, since, for example, if $\kappa$ is indestructibly supercompact, we can manipulate the value of $2^\kappa$ freely, while maintaining the existence of a supercompactness joint Laver sequence of length $2^\kappa$. On the other hand, even in the case of measurable $\kappa$, the consistency strength of $2^\kappa > \kappa^+$ is known to exceed that of $\kappa$ being measurable. The following question is therefore natural:

**Question 10.** If $\kappa$ is $\theta$-supercompact and $2^\kappa > \theta^+$, is there a forcing extension preserving these facts in which there is a joint Laver sequence for $\kappa$ of length $2^\kappa$?

We next show that the existence of joint Laver sequences is preserved under mild forcing. This will be useful later when we separate the existence of these sequences based on their lengths.

**Lemma 11.** Let $\kappa$ be $\theta$-supercompact, $\lambda$ a cardinal, and let $P$ be a poset such that either

1. $\lambda > \theta^{<\kappa}$ and $P$ is $\leq \lambda$-distributive, or
2. $|P| \leq \kappa$ and many $\theta$-supercompactness embeddings with critical point $\kappa$ lift through $P$. 


Then forcing with \( P \) preserves \( \mathbb{B}_{\kappa, \lambda}^{\theta, \text{-sc}} \).

We were intentionally vague in item (2) of the lemma. The hypotheses will definitely be satisfied if every \( \theta \)-supercompactness embedding should lift through \( P \), as is very often the case with forcing iterations up to the large cardinal \( \kappa \). The proof, however, will show that it is in fact sufficient that at least one ground model embedding associated to each target of the existing joint Laver sequence lift through \( P \). Furthermore, while the restriction \( |P| \leq \kappa \) in (2) is necessary for full generality, it can in fact be relaxed to \( |P| \leq \theta \) for a large class of forcings.

**Proof.** Under the hypotheses of (1) every ultrapower embedding by a measure on \( P_\kappa \theta \) lifts to the extension by \( P \) and no elements of \( H_\theta^+ \) or \( \lambda \)-sequences of such are added, so any ground model joint Laver sequence of length \( \lambda \) retains this property in the extension.

Now suppose that the hypotheses of (2) hold, let \( \langle \ell_\alpha; \alpha < \lambda \rangle \) be a joint Laver sequence for \( \kappa \) and let \( G \subseteq P \) be generic. We may also assume that the universe of \( P \) is a subset of \( \kappa \). Define functions \( \ell'_\alpha: \kappa \to V[G] \) by \( \ell'_\alpha(\xi) = \ell_\alpha(\xi)^{G, \xi} \) if this makes sense and \( \ell'_\alpha(\xi) = \emptyset \) otherwise. We claim that \( \langle \ell'_\alpha; \alpha < \lambda \rangle \) is a joint Laver sequence in \( V[G] \).

Let \( \vec{a} \) be a \( \lambda \)-sequence of targets in \( H_\theta^+ \). Since \( P \) is small we can find names \( \dot{a}_\alpha \) for these targets in \( H_\theta^+ \) and the sequence of these names is in \( V \). Let \( j: V \to M \) be a \( \theta \)-supercompactness embedding with critical point \( \kappa \) which lifts through \( P \) and which satisfies \( j(\ell_\alpha)(\kappa) = \dot{a}_\alpha \) for each \( \alpha \). It then follows that \( j(\ell'_\alpha)(\kappa) = a_\alpha \) in \( V[G] \), verifying the joint Laver diamond property there. \( \square \)

### 3.2. Separating joint Laver diamonds by length.

We next aim to show that it is consistent that there is a measurable Laver function for \( \kappa \) but no joint Laver sequences of length \( \kappa^+ \). The following proposition expresses the key observation for our solution, connecting the question to the number of normal measures problem.

**Proposition 12.** If there is a \( \theta \)-supercompactness joint Laver sequence for \( \kappa \) of length \( \lambda \) then there are at least \( 2^{\theta, \lambda} \) many normal measures on \( P_\kappa \theta \).

**Proof.** The point is that any normal measure on \( P_\kappa \theta \) realizes a single \( \lambda \)-sequence of elements of \( H_\theta^+ \) via the joint Laver sequence and there are \( 2^{\theta, \lambda} \) many such sequences of targets. \( \square \)

**Theorem 13.** If \( \kappa \) is measurable then there is a forcing extension in which there is a Laver function for \( \kappa \) but no joint Laver sequence of length \( \kappa^+ \).

**Proof.** After forcing as in the proof of theorem 7 if necessary, we may assume that \( \kappa \) has a Laver function. A result of Apter–Cummings–Hamkins \( \square \) then shows that \( \kappa \) still carries a Laver function in the extension by \( P = \text{Add}(\omega, 1) \ast \text{Coll}(\kappa^+, 2^{2^\kappa}) \) but only carries \( \kappa^+ \) many normal measures there. Proposition 12 now implies that there cannot be a joint Laver sequence of length \( \kappa^+ \) in the extension. \( \square \)

We can push this result a bit further to get a separation between any two desired lengths of joint Laver sequences. To state the sharpest result we need to introduce a new notion.

**Definition 14.** Let \( \kappa \) be a large cardinal supporting a notion of Laver diamond and \( \lambda \) a cardinal. We say that a sequence \( \vec{\ell} = \langle \ell_\alpha; \alpha < \lambda \rangle \) is an almost-joint Laver
sequence for $\kappa$ if $\ell \upharpoonright \gamma$ is a joint Laver sequence for $\kappa$ for any $\gamma < \lambda$. We say that $\mathbf{L}_{\kappa,<\lambda}$ holds if there is an almost-joint Laver sequence of length $\lambda$.

**Theorem 15.** Suppose $\kappa$ is measurable and let $\lambda$ be a regular cardinal satisfying $\kappa < \lambda \leq 2^\kappa$. If $\mathbf{L}_{\kappa,<\lambda}^{\text{meas}}$ holds then there is a forcing extension preserving this in which $\mathbf{L}_{\kappa,\lambda}^{\text{meas}}$ fails.

**Proof.** We imitate the proof of theorem 13 but force instead with $\mathbb{P} = \text{Add}(\omega,1) \ast \text{Coll}(\lambda,2^\omega)$. The analysis based on [12] now shows that the final extension has at most $\lambda$ many normal measures on $\kappa$ and thus there can be no joint Laver sequences of length $\lambda$ there by proposition 12. That $\mathbf{L}_{\kappa,<\lambda}^{\text{meas}}$ still holds follows from (the proof of) lemma 11: part (2) implies that, by guessing names, the $\mathbf{L}_{\kappa,\lambda}^{\text{meas}}$-sequence from the ground model gives rise to one in the intermediate Cohen extension. Part (1) then shows that each of the initial segments of this sequence remains a joint Laver sequence in the final extension. □

We can also extend these results to $\theta$-supercompact cardinals without too much effort.

**Theorem 16.** If $\kappa$ is $\theta$-supercompact, $\theta$ is regular, and $\theta^{<\kappa} = \theta$ then there is a forcing extension in which $\mathbf{L}_{\kappa}^{\theta,\text{sc}}$ holds but $\mathbf{L}_{\kappa,\theta}^{\theta,\text{sc}}$ fails.

Of course, the theorem is only interesting when $\kappa \leq \theta < 2^\kappa$, in which case the given separation is best possible in view of proposition 4.

**Proof.** We may assume by prior forcing, as in theorem 7, that we have a Laver function for $\kappa$. We now force with $\mathbb{P} = \text{Add}(\omega,1) \ast \text{Coll}(\theta^+,2^\omega)$ to get an extension $V[g][G]$. By the results of [1] the extension $V[g][G]$ has at most $\theta^+$ many normal measures on $\mathcal{P}_\kappa \theta$ and therefore there are no joint Laver sequences of $\kappa$ of length $\theta^+$ there by proposition 12. It remains to see that there is a Laver function in $V[g][G]$. Let $\ell$ be a Laver function in $V$ and define $\ell' \in V[g][G]$ by $\ell'(\xi) = \ell(\xi)^g$ if $\ell(\xi)$ is an Add($\omega,1$)-name and $\ell'(\xi) = \emptyset$ otherwise. For a given $a \in H_{\theta^+}[V[g][G]] = H_{\theta^+}[V[g]$ we can select an Add($\omega,1$)-name $\dot{a} \in H_{\theta^+}$ and find a $\theta$-supercompactness embedding $j: V \rightarrow M$ such that $j(\ell)(\kappa) = \dot{a}$. The embedding $j$ lifts to $j: V[g][G] \rightarrow M[g][j(G)]$ since the Cohen forcing was small and the collapse forcing was $\leq \theta$-closed. But then clearly $j(\ell')(\kappa) = \dot{a}^g = a$, so $\ell'$ is a Laver function. □

**Theorem 17.** Suppose $\kappa$ is $\theta$-supercompact and let $\lambda$ be a regular cardinal satisfying $\theta^{<\kappa} < \lambda \leq 2^\kappa$. If $\mathbf{L}_{\kappa,<\lambda}^{\theta,\text{sc}}$ holds then there is a forcing extension preserving this in which $\mathbf{L}_{\kappa,\lambda}^{\theta,\text{sc}}$ fails.

**Proof.** The relevant forcing is $\text{Add}(\omega,1) \ast \text{Coll}(\lambda,2^{2^{\omega^{<\kappa}}})$. Essentially the argument from theorem 15 then finishes the proof. □

A question remains about the principles $\mathbf{L}_{\kappa,<\lambda}$, whether they are genuinely new or whether they reduce to other principles.

**Question 18.** Let $\kappa$ be $\theta$-supercompact and $\lambda < 2^\kappa$. Is $\mathbf{L}_{\kappa,<\lambda}^{\theta,\text{sc}}$ equivalent to $\mathbf{L}_{\kappa,\gamma}^{\theta,\text{sc}}$ holding for all $\gamma < \lambda$?

An almost-joint Laver sequence definitely gives instances of joint Laver diamonds at each particular $\gamma$. The reverse implication is particularly interesting in the case
when \( \lambda = \mu^+ \) is a successor cardinal. This is because simply rearranging the functions in a joint Laver sequence of length \( \mu \) gives joint Laver sequences of any length shorter than \( \mu^+ \). The question is thus asking whether \( \text{\textsc{d}}_{\kappa, \mu} \) suffices for \( \text{\textsc{d}}_{\kappa, \mu^+} \). The restriction to \( \lambda \leq 2^\kappa \) is necessary to avoid the following triviality.

**Proposition 19.** The principle \( \text{\textsc{d}}_{\kappa, \mu^+}^{\theta} \) fails for every cardinal \( \kappa \).

**Proof.** Any potential \( \text{\textsc{d}}_{\kappa, \mu^+}^{\theta} \)-sequence must necessarily have the same function appear on at least two coordinates. But then any initial segment of this sequence containing both of those coordinates cannot be joint, since it cannot guess distinct targets on those coordinates. \( \square \)

An annoying feature of the models produced in the preceding theorems is that in all of them the least \( \lambda \) for which there is no joint Laver sequence of length \( \lambda \) is \( \lambda = 2^\kappa \). One has to wonder whether this is significant.

**Question 20.** Is it relatively consistent that there is a \( \theta \)-supercompact cardinal \( \kappa \), for some \( \theta \), such that \( \text{\textsc{d}}_{\kappa}^{\theta} \) holds and \( \text{\textsc{d}}_{\kappa, \lambda}^{\theta} \) fails for some \( \lambda \) satisfying \( \lambda < 2^\kappa \)?

To satisfy the listed conditions, GCH must fail at \( \kappa \) (since we must have \( \kappa < \lambda < 2^\kappa \) by proposition 4). We can therefore expect that achieving the situation described in the question will require some additional consistency strength.

In the case of a measurable \( \kappa \) the answer to this question is positive: we will show in theorem 22 that, starting from sufficient large cardinal hypotheses, we can produce a model where \( \kappa \) is measurable and has a measurable Laver function but no joint Laver sequences of length \( \kappa^+ < 2^\kappa \). The proof relies on an argument due to Friedman and Magidor [6] which facilitates the simultaneous control of the number of measures at \( \kappa \) and the value of the continuum function at \( \kappa \) and \( \kappa^+ \).

Let us briefly give a general setup for the argument of [6] that will allow us to carry out our intended modifications without repeating too much of the work done there.

Fix a cardinal \( \kappa \) and suppose GCH holds up to and including \( \kappa \). Furthermore suppose that \( \kappa \) is the critical point of an elementary embedding \( j: V \rightarrow M \) satisfying the following properties:

- \( j \) is an extender embedding, i.e. every element of \( M \) has the form \( j(f)(\alpha) \) for some function \( f \) defined on \( \kappa \) and some \( \alpha < j(\kappa) \);
- \( (\kappa^{++})^M = \kappa^{++} \);
- there is a function \( f: \kappa \rightarrow V \), such that \( j(f)(\kappa) \) is, in \( V \) (and therefore also in \( M \)), a sequence of \( \kappa^{++} \) many disjoint stationary subsets of \( \kappa^{++} \cap \text{Cof} \kappa^+ \).

Given this arrangement, Friedman and Magidor define a forcing iteration \( \mathbb{P} \) of length \( \kappa + 1 \) (with nonstationary support) which forces at each inaccessible stage \( \gamma \leq \kappa \) with \( \text{Sacks}^*(\gamma, \gamma^{++}) \ast \text{Sacks}_{\text{id}+}^*(\gamma) \ast \text{Code}(\gamma) \). Here the conditions in \( \text{Sacks}_{\text{id}+}^* \) are perfect trees of height \( \gamma \), splitting on a club, and in which every splitting node of height \( \delta \) has \( \delta^{++} \) many successors; furthermore \( \text{Sacks}^*(\gamma, \gamma^{++}) \) is a large product of versions of \( \text{Sacks}(\gamma) \) where splitting is restricted to a club of singular cardinals and \( \text{Code}(\gamma) \) is a certain \( \leq \gamma \)-distributive notion of forcing coding information about the stage \( \gamma \) generics into the stationary sets given by \( j(\gamma) \).

Let \( G \subseteq \mathbb{P} \) be generic. In the interest of avoiding repeating the analysis of the forcing notion given in [6], we list some of the properties of the extension \( V[G] \) that we will use (but see [6] for proofs):
(1) \( P \) preserves cardinals and cofinalities, and increases the values of the continuum function by at most two cardinal steps. In particular, any inaccessible cardinals of \( V \) remain such in \( V[G] \);
(2) We have \( 2^\kappa = \kappa^{++} \) in \( V[G] \);
(3) \( P \) has the \( \kappa \)-Sacks property: for any function \( f : \kappa \to \text{Ord} \) in \( V[G] \) there is a function \( h \in V \) such that \( f(\alpha) \in h(\alpha) \) for all \( \alpha \) and \( |h(\alpha)| \leq \alpha^{++} \);
(4) the generic \( G \) is self-encoding in a strong way: in \( V[G] \) there is a unique \( M \)-generic for \( j(P)_{<j(\kappa)} \) extending \( j[G_{<\kappa}] \);
(5) If \( S_\kappa \) is the generic added by \( \text{Sacks}_{\kappa^{++}}(\kappa) \) within \( P \), then \( \bigcap j[S_\kappa] \) is a tuning fork: the union of \( \kappa^{++} \) many branches, all of which split off exactly at level \( \kappa \) and all of which are generic over \( j(G_\kappa) \);
(6) In \( V[G] \) there are exactly \( \kappa^{++} \) many \( M \)-generics for \( j(P) \) extending \( j[G] \), corresponding to the \( \kappa^{++} \) many branches in \( \bigcap j[S_\kappa] \). In particular, there are exactly \( \kappa^{++} \) many lifts \( j_\alpha \) of \( j \) in \( V[G] \), distinguished by \( j_\alpha(S_\kappa)(\kappa) = \alpha \) for \( \alpha < \kappa^{++} \).

**Proposition 21.** In the above setup, the iteration \( P \) adds a measurable Laver function for \( \kappa \).

**Proof.** Let \( G \subseteq P \) be generic. As we stated in item (1), for any \( \alpha < \kappa^{++} \) there is a lift \( j_\alpha : V[G] \to M[j_\alpha(G)] \) of \( j \) such that \( j_\alpha(S_\kappa)(\kappa) = \alpha \), where \( S_\kappa \) is the Sacks subset of \( \kappa \) added by the \( \kappa \)-th stage of \( G \). This shows that \( \hat{\ell}(\gamma) = S_\kappa(\gamma) \) is a \( \kappa^{++} \)-guessing measurable Laver function for \( \kappa \).

Note that all of the subsets of \( \kappa \) in \( M[j_\alpha(G)] \) (and \( V[G] \)) appear already in \( M[G] \).

Let \( \bar{e} = \langle e_\alpha ; \alpha < \kappa^{++} \rangle \) be an enumeration of \( H_{\kappa^{++}}^M[G] \) in \( M[G] \) and let \( \bar{e} \in M \) be a name for \( \bar{e} \). We can write \( \bar{e} = j(F)(\kappa) \) for some function \( F \), defined on \( \kappa \). Now define a function \( \ell : \kappa \to V_\kappa \) in \( V[G] \) by \( \ell(\gamma) = (F(\gamma)^G)(\hat{\ell}(\gamma)) \). This is, in fact, our desired Laver function; given an arbitrary element of \( H_{\kappa^{++}}^V[G] \), we can find it in the enumeration \( \bar{e} \). If \( \alpha \) is its index, then

\[
j_\alpha(\ell)(\kappa) = (j_\alpha(F)(\kappa)^{j_\alpha(G)})(j_\alpha(\hat{\ell})(\kappa)) = (j(F)(\kappa)^{j_\alpha(G)})(\alpha) = \bar{e}(\alpha) = e_\alpha \quad \Box
\]

**Theorem 22.** Suppose \( \kappa \) is \( (\kappa + 2) \)-strong and assume that \( V = L[\bar{E}] \) is the minimal extender model witnessing this. Then there is a forcing extension in which \( 2^\kappa = \kappa^{++} \), the cardinal \( \kappa \) remains measurable, \( \kappa \) carries a measurable Laver function but there are no measurable joint Laver sequences for \( \kappa \) of length \( \kappa^{++} \).

This finally answers question 20 in the positive.

**Proof.** Let \( j : V \to M \) be the ultrapower embedding by the top extender of \( \bar{E} \), the unique extender witnessing the \( (\kappa + 2) \)-strongness of \( \kappa \). In particular, every element of \( M \) has the form \( j(f)(\alpha) \) for some \( \alpha < j(\kappa) \), and \( M \) computes \( \kappa^{++} \) correctly. Furthermore, \( V \) has a canonical \( \check{\kappa}^{++}(\text{Cof}_{\kappa^{++}}) \)-sequence, which is definable over \( H_{\kappa^{++}} \). Since \( H_{\kappa^{++}} \in M \), this same sequence is also in \( M \) and, by definability, is of the form \( j(f)(\kappa) \) for some function \( f \). By having this diamond sequence guess the singletons \( \{ \xi \} \) for \( \xi < \kappa^{++} \), we obtain a sequence of \( \kappa^{++} \) many disjoint stationary subsets of \( \kappa^{++} \cap \text{Cof} \kappa^{+} \), and this sequence itself has the form \( j(f)(\kappa) \) for some function \( f \). We are therefore in a situation where the definition of the Friedman–Magidor iteration we described above makes sense. But first, we shall carry out some preliminary forcing.
Let $g \subseteq \text{Add}(\kappa^+, \kappa^{+3})$ be generic. Since this Cohen poset is $\leq \kappa$-distributive, the embedding $j$ lifts (uniquely) to an embedding $j^* : V[g] \to M[j(g)]$.\footnote{The lifted embedding will not be a $(\kappa + 2)$-strongness embedding and, in fact, $\kappa$ is no longer $(\kappa + 2)$-strong in $V[g]$. Nevertheless, the residue of strongness will suffice for our argument.} Let us examine the lifted embedding $j$. It is still an extender embedding. Additionally, since GCH holds in $V$, the forcing $\text{Add}(\kappa^+, \kappa^{+3})$ preserves cardinals, cofinalities, and stationary subsets of $\kappa^+$. Together this means that $M[j(g)]$ computes $\kappa^+$ correctly and the stationary sets given by the sequence $j(f)(\kappa)$ above remain stationary. Therefore we may still define the Friedman–Magidor iteration $\mathbb P$ over $V[g]$.

Let $G \subseteq \mathbb P$ be generic over $V[g]$. We claim that $V[g][G]$ is the model we want. We have $2^\kappa = \kappa^{++}$ in the extension, by item (2) of our list, and proposition 24 implies that $\kappa$ is measurable in $V[g][G]$ and $\mathcal{N}^{\text{meas}}_\kappa$ holds there. So it remains for us to see that $\mathcal{N}^{\text{meas}}_\kappa$ fails. By proposition 12 it suffices to show that $\kappa$ does not carry $2^\kappa = \kappa^{+3}$ many normal measures in $V[g][G]$.

Let $U^* \in V[g][G]$ be a normal measure on $\kappa$ and let $j^* : V[g][G] \to N[j^*(g)][j^*(G)]$ be its associated ultrapower embedding. This embedding restricts to $j^* : V \to N$. Since $V$ is the core model from the point of view of $V[g][G]$, the embedding $j^*$ arises as the ultrapower map associated to a normal iteration of extenders on the sequence $\bar{E}$.

We first claim that the first extender applied in this iteration is the top extender of $\bar{E}$. Let us write $j^* = j_1 \circ j_0$, where $j_0 : V \to N_0$ results from the first applied extender. Clearly $j_0$ has critical point $\kappa$. Now suppose first that $j_0(\kappa) < \kappa^{++}$. Of course, $j_0(\kappa)$ is inaccessible in $N_0$ and, since $N$ is an inner model of $N_0$, also in $N$. But $j_0(\kappa)$ is not inaccessible in $N[j^*(g)][j^*(G)]$, since $2^\kappa = \kappa^{++}$ there. This is a contradiction, since passing from $N$ to $N[j^*(g)][j^*(G)]$ preserves inaccessibility, by item (1) of our list.

It follows that we must have $j_0(\kappa) \geq \kappa^{++}$. We will argue that the extender $E$ applied to get $j_0$ witnesses the $(\kappa + 2)$-strongness of $\kappa$, so it must be the top extender of $\bar{E}$ and $j_0 = j_1$. Using a suitable indexing of $\bar{E}$, the extender $E$ has index $(j_0(\kappa))^{N_0} > \kappa^{++}$, and the coherence of the extender sequence implies that the sequences in $V$ and in $N_0$ agree up to $\kappa^{++}$. By the acceptability of these extender models it now follows that $H^V_{\kappa^{++}} = H^{N_0}_{\kappa^{++}}$, or, equivalently, $V_{\kappa^{++} + 2} \subseteq N_0$.

Finally, we claim that the iteration giving rise to $j^*$ ends after one step, meaning that $j^* = j_1$. Suppose to the contrary that $j_1$ is nontrivial. By the normality of the iteration, the critical point of $j_1$ must be some $\lambda > \kappa$. We can find a function $h \in V[g][G]$, defined on $\kappa$, such that $j^*(h)(\kappa) = \lambda$, since $j^*$ is given by a measure ultrapower of $V[g][G]$. By the $\kappa$-Sacks property of $\mathbb P$ (see item (3)) we can cover the function $h$ by a function $\bar{h} \in V[g]$; in fact, since the forcing to add $g$ was $\leq \kappa$-closed, we have $\bar{h} \in V$. Now

$$\lambda = j^*(h)(\kappa) \in j^*(\bar{h})(\kappa) = j_1(j(\bar{h}))(j_1(\kappa)) = j_1(j(\bar{h})(\kappa))$$

and $A = j(\bar{h})(\kappa)$ has cardinality at most $\kappa^{++}$ in $M$. In particular, since $\kappa^{++} < \lambda$, we have $\lambda \in j_1(A) = j_1[A]$, which is a contradiction, since $\lambda$ was the critical point of $j_1$.

We can conclude that any embedding $j^*$ arising from a normal measure on $\kappa$ in $V[g][G]$ is a lift of the ground model $(\kappa + 2)$-strongness embedding $j$. But there are exactly $\kappa^{++}$ many such lifts: the lift to $V[g]$ is unique, and there are $\kappa^{++}$ many
possibilities for the final lift to $V[g][G]$, according to item (6). Therefore there are only $\kappa^{++}$ many normal measures on $\kappa$ in $V[g][G]$. □

Ben Neria and Gitik have recently announced that the consistency strength required to achieve the failure of GCH at a measurable cardinal carrying a unique normal measure is exactly that of a measurable cardinal $\kappa$ with $o(\kappa) = \kappa^{++}$ (see [3]). Their method is flexible enough to allow us to incorporate it into our proof of theorem 22, reducing the consistency strength hypothesis required there from a $(\kappa + 2)$-strong cardinal $\kappa$ to just $o(\kappa) = \kappa^{++}$. We have chosen to present the proof based on the original Friedman–Magidor argument since it avoids some complications arising from using the optimal hypotheses.

3.3. (Joint) Laver diamonds and the number of normal measures. The only method of controlling the existence of (joint) Laver diamonds we have seen is by controlling the number of large cardinal measures, relying on the rough bound given by proposition 12. One has to wonder whether merely the existence of sufficiently many measures guarantees the existence of (joint) Laver diamonds. We focus on the simplest form of the question, concerning measurable cardinals.

**Question 23.** Suppose $\kappa$ is measurable and there are at least $2^\lambda$ many normal measures on $\kappa$ for some $\lambda \geq \kappa$. Does there exist a measurable joint Laver sequence for $\kappa$ of length $\lambda$?

As the special case when $\lambda = \kappa$, the question includes the possibility that having $2^\kappa$ many normal measures, the minimum required, suffices to give the existence of a measurable Laver function for $\kappa$. Even in this very special case it seems implausible that simply having enough measures would automatically yield a Laver function. Nevertheless, in all of the examples of models obtained by forcing and in which we have control over the number of measures that we have seen, Laver functions have existed. On the other hand, Laver functions and joint Laver sequences also exist in canonical inner models that have sufficiently many measures. These models carry long Mitchell-increasing sequences of normal measures that we can use to obtain ordinal-guessing Laver functions. We can then turn these into actual Laver functions by exploiting the coherence and absoluteness of these models.

**Definition 24.** Let $A$ be a set (or class) of ordinals and let $\vec{\ell}$ be an $A$-guessing Laver function for some large cardinal $\kappa$. Let $\triangleleft$ be some well-order (one arising from an $L$-like inner model, for example). We say that $\triangleleft$ is suitable for $\vec{\ell}$ if, for any $\alpha \in A$, there is an elementary embedding $j$, witnessing the largeness of $\kappa$, such that $\vec{\ell}$ guesses $\alpha$ via $j$ and $j(\triangleleft) \upharpoonright (\alpha + 1) = \triangleleft \upharpoonright (\alpha + 1)$; that is, the well-orders $j(\triangleleft)$ and $\triangleleft$ agree on their first $\alpha + 1$ many elements.

If $\mathcal{J}$ is a class of elementary embeddings witnessing the largeness of $\kappa$, we say that $\triangleleft$ is supersuitable for $\mathcal{J}$ if $j(\triangleleft) \upharpoonright j(\kappa) = \triangleleft \upharpoonright j(\kappa)$ for any $j \in \mathcal{J}$.

We could, for example, take the class $\mathcal{J}$ to consist of all ultrapower embeddings by normal measures on $\kappa$ or, more to the point, all ultrapower embeddings arising from a fixed family of extenders. We should also note that, for the notion to make sense, the order type of $\triangleleft$ must be quite high: at least $\sup A$ in the case of well-orders suitable for an $A$-guessing Laver function and at least $\sup_{j \in \mathcal{J}} j(\kappa)$ for supersuitable well-orders (the latter would also make sense if the order type of $\triangleleft$ were smaller than $\kappa$, but that case is not of much interest).
Clearly any supersuitable well-order is suitable for any ordinal-guessing Laver function \( \ell \), provided that the class \( J \) includes the embeddings via which \( \ell \) guesses its targets. The following obvious lemma describes the way in which suitable well-orders will be used to turn ordinal-guessing Laver functions into set-guessing ones.

**Lemma 25.** Let \( A \) be a set (or class) of ordinals and let \( \ell \) be an \( A \)-guessing Laver function for some large cardinal \( \kappa \). Let \( \prec \) be a well-order such that \( \text{otp}(\prec) \subseteq A \). If \( \prec \) is suitable for \( \ell \), then there is a \( B \)-guessing Laver function for \( \kappa \), where \( B \) is the field of \( \prec \).

**Proof.** We can define a \( B \)-guessing Laver function by simply letting \( \ell(\xi) \) be the \( \ell(\xi) \)th element of \( \prec \). Then, given a target \( b \in B \), we can find its index \( \alpha \) in the well-order \( \prec \) and an embedding \( j \) such that \( j(\ell)(\kappa) = \alpha \). Since \( \prec \) is suitable for \( \ell \), the orders \( \prec \) and \( j(\prec) \) agree on their \( \alpha \)th element and so \( \ell \) guesses \( b \) via \( j \). \( \square \)

It follows from the above lemma that in any model with a sufficiently supersuitable well-order, being able to guess ordinals suffices to be able to guess arbitrary sets.

**Lemma 26.** Let \( X \) be a set (or class) of ordinals and let \( J \) be a class of elementary embeddings of \( L[X] \) with critical point \( \kappa \) such that \( j(X) \cap j(\kappa) = X \cap j(\kappa) \) for any \( j \in J \). Then \( \leq_X \), the canonical order of \( L[X] \), is supersuitable for \( J \).

**Proof.** This is obvious; the order \( \leq_X \) \( \{j(\kappa)\} \) is definable in \( L_{j(\kappa)}[X] \), but by our coherence hypothesis this structure is just the same as \( L_{j(\kappa)}[j(\kappa)] \). \( \square \)

We are mostly interested in this lemma in the case when \( X = E \) is an extender sequence and \( L[E] \) is an extender model in the sense of [14]. In particular, we want \( E \) to be acceptable (a technical condition which implies enough condensation properties in \( L[E] \) to conclude \( H^L[E] = L_\lambda[E] \)), coherent (meaning that if \( j : L[E] \rightarrow L[F] \) is an ultrapower by the oth extender of \( E \) then \( F \upharpoonright (\alpha + 1) = E \upharpoonright \alpha \), and to use Jensen indexing (meaning that the index of an extender \( E \) on \( E \) with critical point \( \kappa \) is \( j_E(\kappa)^+ \), as computed in the ultrapower).

**Corollary 27.** Let \( V = L[E] \) be an extender model. Then the canonical well-order is supersuitable for the class of ultrapower embeddings by the extenders on the sequence \( E \).

**Proof.** This is immediate from the preceding lemma and the fact that our extender sequences are coherent and use Jensen indexing. \( \square \)

**Theorem 28.** Let \( V = L[E] \) be an extender model. Let \( \kappa \) be a cardinal such that every normal measure on \( \kappa \) appears on the sequence \( E \). If \( o(\kappa) \geq \kappa^+ \) then \( \Sigma^\text{meas}_\kappa \) holds. Moreover, if \( o(\kappa) = \kappa^{++} \) then \( \Sigma^\text{meas}_{\kappa,\kappa^+} \) and even \( \Sigma^\text{meas}_\kappa (H_{\kappa^{++}}) \), holds.

In particular, the above theorem implies \( \Sigma^\text{meas}_\kappa \) holds in the least inner model with the required number of measures and the same holds for \( \Sigma^\text{meas}_{\kappa,\kappa^+} \). This provides further evidence that the answer to question [22] which remains open, might turn out to be positive.

**Proof.** We can argue for the two cases more or less uniformly: let \( \lambda \in \{\kappa^+,\kappa^{++}\} \) such that \( \lambda \leq o(\kappa) \). The function \( \ell(\xi) = o(\xi) \) is a \( \lambda \)-guessing measurable Laver function for \( \kappa \). By the acceptability of \( E \) we have that \( H_\lambda = L_\lambda[E] \). The canonical
well-order $\leq E \cap L(\vec{E})$ has order type $\lambda$ and, by corollary 27, is supersuitable for the class of ultrapower embeddings by normal measures on $\kappa$. It follows that $\leq E$ is suitable for $\vec{E}$, so, by lemma 25, there is an $H_\lambda$-guessing measurable Laver function for $\kappa$.

To finish the proof we still need to produce a joint measurable Laver sequence for $\kappa$, in the case that $o(\kappa) = \kappa^{++}$. This is done in exactly the same way as in proposition 3 one simply uses the $H_{\kappa^{++}}$-guessing Laver function to guess the whole sequence of targets for a joint Laver sequence. □

We should also mention that, if we restrict to a smaller set of targets, having enough normal measures does give us Laver functions.

**Lemma 29.** Let $\kappa$ be a regular cardinal and $\gamma \leq \kappa$ and suppose that $\langle \mu_\alpha; \alpha < \gamma \rangle$ is a sequence of distinct normal measures on $\kappa$. Then there is a sequence $\langle X_\alpha; \alpha < \gamma \rangle$ of pairwise disjoint subsets of $\kappa$ such that $X_\alpha \in \mu_\beta$ if and only if $\alpha = \beta$.

**Proof.** We prove the lemma by induction on $\gamma$. In the base step, $\gamma = 1$, we simply observe that, since $\mu_0 \neq \mu_1$, we must have a set $X_0 \subseteq \kappa$ such that $X_0 \in \mu_0$ and $\kappa \setminus X_0 \in \mu_1$.

The successor step proceeds similarly. Suppose that the lemma holds for sequences of length $\gamma$ and fix a sequence of measures $\langle \mu_\alpha; \alpha < \gamma + 1 \rangle$. By the induction hypothesis we can find pairwise disjoint sets $\langle Y_\alpha; \alpha < \gamma \rangle$ such that each $Y_\alpha$ picks out a unique measure among those with indices below $\gamma$. Again, since $\mu_\gamma$ is distinct from all of the other measures, we can find sets $Z_\alpha \in \mu_\gamma \setminus \mu_\alpha$ for each $\alpha < \gamma$. Then the sets $X_\alpha = Y_\alpha \setminus Z_\alpha$ for $\alpha < \gamma$ and $X_\gamma = \bigcap_\alpha Z_\alpha$ are as required.

In the limit step suppose that the lemma holds for all $\delta < \gamma$. We can then fix sequences $\langle X_\alpha^\delta; \alpha < \delta \rangle$ for each $\delta < \gamma$ as above. The argument proceeds slightly differently depending on whether $\gamma = \kappa$ or not. If $\gamma < \kappa$ we can simply let $X_\alpha = \bigcap_{\alpha < \delta < \gamma} X_\alpha^\delta \in \mu_\alpha$. If, on the other hand, we have $\gamma = \kappa$ then first let $Y_\alpha = \Delta_{\alpha < \delta < \kappa} X_\alpha^\delta \in \mu_\alpha$. Observe that the $Y_\alpha$ are almost disjoint: $Y_\alpha \cap Y_\beta$ is bounded in $\kappa$ for any $\alpha, \beta < \kappa$. Now consider

$$X_\alpha = Y_\alpha \setminus \bigcup_{\beta < \alpha} (Y_\alpha \cap Y_\beta)$$

for $\alpha < \kappa$. Since $Y_\alpha \cap Y_\beta$ is bounded for all $\beta < \alpha$, we still have $X_\alpha \in \mu_\alpha$. Furthermore, we obviously have $X_\alpha \cap X_\beta = \emptyset$ for $\beta < \alpha$ and this implies that the $X_\alpha$ are pairwise disjoint. □

**Theorem 30.** Let $\kappa$ be a measurable cardinal and $\gamma < \kappa^+$ an ordinal. There is a $\gamma$-guessing measurable Laver function for $\kappa$ if and only if there are at least $|\gamma|$ many normal measures on $\kappa$.

**Proof.** First suppose that $\Delta^\text{meas}_\kappa(\gamma)$ holds. Then, just as in proposition 122 each target $\alpha < \gamma$ requires its own embedding $j$ via which it is guessed and this gives us $|\gamma|$ many distinct normal measures.

Conversely, suppose that we have at least $|\gamma|$ many normal measures on $\kappa$. We can apply lemma 29 to find a sequence of pairwise disjoint subsets distinguishing these measures. By reorganizing the measures and the distinguishing sets we may assume that they are given in sequences of length $\gamma$. We now have normal measures $\langle \mu_\alpha; \alpha < \gamma \rangle$ and sets $\langle X_\alpha; \alpha < \gamma \rangle$ such that $\mu_\alpha$ is the unique measure concentrating on $X_\alpha$; we may even assume that the $X_\alpha$ partition $\kappa$. Let $f_\alpha$ for $\alpha < \gamma$ be the
representing functions for \( \alpha \), that is, \( j(f_\alpha)(\kappa) = \alpha \) for any ultrapower embedding \( j \) by a normal measure on \( \kappa \). We can now define a \( \gamma \)-guessing Laver function \( \ell \) by letting \( \ell(\xi) = f_\alpha(\xi) \) where \( \alpha \) is the unique index such that \( \xi \in X_\alpha \). This function indeed guesses any target \( \alpha < \gamma \): simply let \( j: V \to M \) be the ultrapower by \( \mu_\alpha \). Since \( \mu_\alpha \) concentrates on \( X_\alpha \) we have \( j(\ell)(\kappa) = j(f_\alpha)(\kappa) = \alpha \).

**Corollary 31.** Let \( \kappa \) be a measurable cardinal and fix a subset \( A \subseteq H_{\kappa^+} \) of size at most \( \kappa \). Then there is an \( A \)-guessing measurable Laver function for \( \kappa \) if and only if there are at least \( |A| \) many normal measures on \( \kappa \).

**Proof.** The forward direction follows just as before: each target in \( A \) gives its own normal measure on \( \kappa \). Conversely, if there are at least \( |A| \) many normal measures on \( \kappa \) then, by theorem [30], there is an \( |A| \)-guessing measurable Laver function \( \bar{\ell} \). Fix a bijection \( f: |A| \to A \). We may assume, moreover, that \( A \subseteq \mathcal{P}(\kappa) \). Then we can define an \( A \)-guessing Laver function \( \ell \) by letting \( \ell(\xi) = f(\bar{\ell}(\xi)) \cap \xi \). This definition works: to guess \( f(\alpha) \) we let \( \bar{\ell} \) guess \( \alpha \) via some \( j \). Then \( j(\ell)(\kappa) = j(f(\alpha)) \cap \kappa = f(\alpha) \).

Lemma [29] can be recast in somewhat different language, giving it, and the subsequent results, a more topological flavour.

Given a cardinal \( \kappa \) let \( \mathcal{M}(\kappa) \) be the set of normal measures on \( \kappa \). We can topologize \( \mathcal{M}(\kappa) \) by having, for each \( X \subseteq \kappa \), a basic neighbourhood \( [X] = \{ \mu \in \mathcal{M}(\kappa) : X \in \mu \} \) (this is just the topology induced on \( \mathcal{M}(\kappa) \) by the Stone topology on the space of ultrafilters on \( \kappa \)). Lemma [29] can now be restated to say that any subspace of \( \mathcal{M}(\kappa) \) of size at most \( \kappa \) is discrete and, moreover, the discretizing family of subsets of \( \kappa \) witnessing this can be taken to be pairwise disjoint. One might thus hope to show the existence of Laver functions by exhibiting even larger discrete subspaces of \( \mathcal{M}(\kappa) \). In pursuit of that goal we obtain the following generalization of corollary [31].

**Theorem 32.** Let \( \kappa \) be a measurable cardinal and \( A \subseteq \mathcal{P}(\kappa) \). Then \( \mathcal{S}_\kappa^{\mathit{meas}}(A) \) holds if and only if there are for each \( a \in A \) a set \( S_a \subseteq \kappa \) and a normal measure \( \mu_a \) on \( \kappa \) such that \( \{ \mu_a : a \in A \} \) is discrete in \( \mathcal{M}(\kappa) \), as witnessed by \( \{ S_a : a \in A \} \), and \( S_a \cap S_b \subseteq \{ \xi : a \cap \xi = b \cap \xi \} \).

Note that we could have relaxed our hypothesis to \( A \subseteq H_{\kappa^+} \) by working with Mostowski codes.

**Proof.** Assume first that \( \ell \) is a measurable \( A \)-guessing Laver function for \( \kappa \). Then we can let \( S_a = \{ \xi : \ell(\xi) = a \cap \xi \} \). Obviously we have \( j(\ell)(\kappa) = a \) if and only if the measure derived from \( j \) concentrates on \( S_a \). It follows that the measures \( \mu_a \) derived this way form a discrete subspace of \( \mathcal{M}(\kappa) \) and we obviously have \( S_a \cap S_b \subseteq \{ \xi : a \cap \xi = b \cap \xi \} \).

Conversely, assume we have such a discrete family of measures \( \mu_a \) and a discretizing family of sets \( S_a \). We can define an \( A \)-guessing measurable Laver function \( \ell \) by letting \( \ell(\xi) = a \cap \xi \) where \( a \) is such that \( \xi \in S_a \). This is well defined by the coherence condition imposed upon the \( S_a \) and it is easy to see that \( \ell \) satisfies the guessing property.

This topological viewpoint presents a number of questions which might suggest an approach to question [28]. For example, it is unclear whether, given a discrete family of normal measures one can find an almost disjoint discretizing family as in
the above theorem. Even more pressingly, we do not know whether it is possible for \( \mathcal{M}(\kappa) \) to have no discrete subspaces of size \( \kappa^+ \) (while itself having size at least \( \kappa^+ \)).

3.4. Laver trees. Thus far we have thought of joint Laver diamonds as simply matrices or sequences of Laver diamonds. To better facilitate the reflection properties required for the usual forcing iterations using prediction, we would now like a different representation. A reasonable attempt seems to be trying to align the joint Laver sequence with the full binary tree of height \( \kappa \).

**Definition 33.** Let \( \kappa \) be a large cardinal supporting a notion of Laver diamond. A Laver tree for \( \kappa \) is a labelling of the binary tree such that the labels along the branches form a joint Laver sequence. More precisely, a Laver tree is a function \( D: \langle \kappa \rangle \rightarrow V \) such that for any sequence of targets \( \langle a_s; s \in \kappa \rangle \) there is an elementary embedding \( j\colon \langle |t| \rangle(t) \) if this makes sense and \( D(t) = \emptyset \) otherwise. We claim this defines a Laver tree for \( \kappa \). Indeed, let \( \bar{a} = \langle a_s; s \in \kappa \rangle \) be a sequence of targets. Since \( \theta \geq 2^\kappa \) we get \( \bar{a} \in H_{\theta^+} \), so there is a \( \theta \)-supercompactness embedding \( j \) such that \( j(D)(s) = a_s \) for all \( s \in \kappa \). Therefore, given any \( s \in \kappa \), we have \( j(D)(s) = j(t)(\kappa)(s) = a_s \).

Proposition 34. Suppose \( \kappa \) is \( \theta \)-supercompact and \( \theta \geq 2^\kappa \). Then a \( \theta \)-supercompactness Laver tree for \( \kappa \) exists if and only if a \( \theta \)-supercompactness Laver function for \( \kappa \) does (if and only if \( \mathbb{S}_{\kappa, 2^\kappa}^{\theta, \text{sc}} \) holds).

Proof. The forward implication is trivial, so we focus on the reverse implication. Let \( \ell \) be a Laver function. For any \( t \in \langle \kappa \rangle \) define \( D(t) = \ell(|t|)(t) \) if this makes sense and \( D(t) = \emptyset \) otherwise. We claim this defines a Laver tree for \( \kappa \). Indeed, let \( \bar{a} = \langle a_s; s \in \kappa \rangle \) be a sequence of targets. Since \( \theta \geq 2^\kappa \) we get \( \bar{a} \in H_{\theta^+} \), so there is a \( \theta \)-supercompactness embedding \( j \) such that \( j(D)(s) = a_s \). Therefore, given any \( s \in \kappa \), we have \( j(D)(s) = j(t)(\kappa)(s) = a_s \).

In other situations, however, the existence of a Laver tree can have strictly higher consistency strength than merely a \( \theta \)-supercompact cardinal.

**Definition 35.** Let \( X \) be a set and \( \theta \) a cardinal. A cardinal \( \kappa \) is \( \theta \)-strong with closure \( \theta \) if there is an elementary embedding \( j: V \rightarrow M \) with critical point \( \kappa \) such that \( \mathbf{^\theta M} \subseteq M \) and \( X \in M \).

**Proposition 36.** Suppose \( \kappa \) is \( \theta \)-supercompact and there is a \( \theta \)-supercompactness Laver tree for \( \kappa \). Then \( \kappa \) is \( \theta \)-strong with closure \( \theta \) for any \( X \subseteq H_{\theta^+} \) of size at most \( 2^\kappa \).

Proof. Suppose \( D: \langle \kappa \rangle \rightarrow V \) is a Laver tree and fix an \( X \subseteq H_{\theta^+} \) of size at most \( 2^\kappa \). Let \( f: \kappa \rightarrow X \) enumerate \( X \). We can then find a \( \theta \)-supercompactness embedding \( j: V \rightarrow M \) with critical point \( \kappa \) such that \( j(D)(s) = f(s) \) for all \( s \in \kappa \). In particular, \( X = j(D)[\kappa] \) is an element of \( M \), as required.

If \( 2^\kappa \leq \theta \) then \( \theta \)-strongness with closure \( \theta \) for all \( X \subseteq H_{\theta^+} \) of size \( 2^\kappa \) amounts to just \( \theta \)-supercompactness and proposition 36 gives the full equivalence of Laver functions and Laver trees. But if \( \theta < 2^\kappa \) then \( \theta \)-strongness with closure \( \theta \) can have additional consistency strength. For example, we might choose \( X \) to be a

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3Not to be confused with particularly bushy trees used as conditions in Laver forcing.
normal measure on $\kappa$ to see that $\kappa$ must have nontrivial Mitchell rank (by iterating this idea we can even deduce that $o(\kappa) = (2^\kappa)^+$). In the typical scenario where $2^\kappa = 2^\theta = \theta^+$, we can also reach higher and choose $X$ to be a normal measure on $\mathcal{P}_\kappa \theta$ and see that $\kappa$ must also have nontrivial $\theta$-supercompactness Mitchell rank. We can use this observation to show that there might not be any Laver trees, even in the presence of very long joint Laver sequences.

**Theorem 37.** Suppose GCH holds and let $\kappa$ be $\theta$-supercompact where either $\theta = \kappa$ or $\text{cf}(\theta) > \kappa$. Then there is a cardinal-preserving forcing extension in which $\kappa$ remains $\theta$-supercompact, has a $\theta$-supercompactness joint Laver sequence of length $2^\kappa$, but is also the least measurable cardinal. In particular, $\theta < 2^\kappa$ and $\kappa$ has no $\theta$-supercompactness Laver trees in the extension.

**Proof.** We may assume by prior forcing as in the proof of theorem 7, that $\kappa$ has a Laver function. Additionally, by performing either Magidor’s iteration of Prikry forcing (see [11]) or applying an argument due to Apter and Shelah (see [2]), depending on whether $\theta = \kappa$ or not, we may assume that, in addition to being $\theta$-supercompact, $\kappa$ is also the least measurable cardinal.

We now apply corollary 9 and arrive at a model where $\kappa$ carries a $\theta$-supercompactness joint Laver sequence of length $2^\kappa$, but is also the least measurable. It follows that there can be no ($\theta$-supercompactness or even measurable) Laver trees for $\kappa$, since, by the discussion above, their existence would imply that $\kappa$ has nontrivial Mitchell rank, implying that there are many measurables below $\kappa$. □

**Proposition 36** can be improved slightly to give a jump in consistency strength even for $I$-Laver trees where $I$ is not the whole set of branches. A simple modification of the proof given there yields the following result, together with the corresponding version of theorem 37.

**Theorem 38.** Suppose $\kappa$ is $\theta$-supercompact and there is a $\theta$-supercompactness $I$-Laver tree for $\kappa$ for some $I \subseteq ^\kappa 2$ of size $2^\kappa$. If $I$ is definable (with parameters) over $H_{\kappa^+}$ then $\kappa$ is $X$-strong with closure $\theta$ for any $X \subseteq H_{\theta^+}$ of size at most $2^\kappa$.

The above theorem notwithstanding, we shall give a construction which shows that the existence of an $I$-Laver tree does not yield additional consistency strength, provided that we allow $I$ to be sufficiently foreign to $H_{\kappa^+}$. The argument will rely on being able to surgically alter a Cohen subset of $\kappa^+$ in a variety of ways. To this end we fix some notation beforehand.

**Definition 39.** Let $f$ and $g$ be functions. The graft of $f$ onto $g$ is the function $g \sideset{\wr}{\circ} f$, defined on $\text{dom}(g)$ by
\[(g \sideset{\wr}{\circ} f)(x) = \begin{cases} f(x): & x \in \text{dom}(g) \cap \text{dom}(f) \\ g(x): & x \in \text{dom}(g) \setminus \text{dom}(f) \end{cases} \]
Essentially, the graft replaces the values of $g$ with those of $f$ on their common domain.

**Theorem 40.** Let $\lambda$ be a regular cardinal and assume $\diamondsuit_\lambda$ holds. Suppose $M$ is a transitive model of ZFC (either set- or class-sized) such that $\lambda \in M$ and $M^{< \lambda} \subseteq M$ and $|\mathcal{P}(\lambda)^M| = \lambda$. Then there are an unbounded set $I \subseteq \lambda$ and a function $g: \lambda \rightarrow$

\footnote{Of course, if $\theta > \kappa$, this arrangement requires a strong failure of GCH at $\kappa$. In fact, $2^\kappa = \theta^+$ in the Apter–Shelah model.}
H_\lambda such that, given any f : I \to H_\lambda, the graft g \upharpoonright f is generic for Add(\lambda, 1) over M \[3\]

The hypothesis of \Diamond_\lambda is often automatically satisfied. Specifically, our assumptions about M imply that 2^{<\gamma} = \lambda. If \lambda = \kappa^+ is a successor, this gives 2^\kappa = \kappa^+ which already implies \Diamond_\lambda by a result of Shelah [13].

Proof. Let \langle f_\alpha : \alpha < \gamma \rangle, with \alpha \to H_{(\alpha)}, be a \Diamond_\lambda-sequence and fix an enumeration \langle D_\alpha : \alpha < \lambda \rangle of the open dense subsets of Add(\lambda, 1) in M. We shall construct by recursion a descending sequence of conditions p_\alpha \in Add(\lambda, 1) and an increasing sequence of sets I_\alpha as approximations to g and I. Specifically, we shall use \Diamond_\lambda to guess pieces of any potential function f and ensure along the way that the modified conditions p_\alpha f meet all of the listed dense sets.

Suppose we have built the sequences \langle p_\alpha : \alpha < \gamma \rangle and \langle I_\alpha : \alpha < \gamma \rangle for some \gamma < \lambda. Let I_\gamma^* = \bigcup_{\alpha < \gamma} I_\alpha. Let p_\gamma^* \in M be an extension of \bigcup_{\alpha < \gamma} p_\alpha such that I_\gamma^* \subseteq \text{dom}(p_\gamma^*) \in \lambda; such an extension exists in M by our assumption on the closure of M.

Let us briefly summarize the construction. We shall surgically modify the condition p_\gamma by grafting the function given by \Diamond_\lambda onto it. We shall then extend this modified condition to meet one of our dense sets, after which we will undo the surgery. We will be left with a condition p_\gamma which is one step closer to ensuring that the result of one particular grafting g \upharpoonright f is generic. At the same time we also extend I_\gamma^* by adding a point beyond the domains of all the conditions constructed so far.

More precisely, let \tilde{p}_\gamma^* = p_\gamma^* \upharpoonright (f_\gamma \upharpoonright I_\gamma^*). This is still a condition in M. Let \tilde{p}_\gamma be any extension of this condition inside D_\eta_\gamma, where \eta_\gamma is the least such that \tilde{p}_\gamma^* \notin D_\eta_\gamma, and satisfying \text{dom}(\tilde{p}_\gamma) \in \lambda. Finally, we undo the initial graft and set p_\gamma = \tilde{p}_\gamma \upharpoonright (p_\gamma^* \upharpoonright I_\gamma^*). Note that we have p_\gamma \leq p_\gamma^*. We also extend our approximation to I with the first available point, letting I_\gamma = I_\gamma^* \cup \{\min(\lambda \setminus \text{dom}(p_\gamma))\}.

Once we have completed this recursive construction we can set I = \bigcup_{\gamma < \lambda} I_\gamma and g = \bigcup_{\gamma < \lambda} p_\gamma. Let us check that these do in fact have the desired properties.

Let f : I \to H_\lambda be a function. We need to show that g \upharpoonright f is generic over M. Using \Diamond_\lambda, we find that there are stationarily many \gamma such that f_\gamma = f \upharpoonright \gamma. Note also that there are club many \gamma such that I_\gamma^* \subseteq \gamma is unbounded, and together this means that S = \{\gamma : f_\gamma \upharpoonright I_\gamma^* = f \upharpoonright (I \cap \gamma)\} is stationary. The conditions \tilde{p}_\gamma for \gamma \in S extend each other and we have \bigcup_{\gamma \in S} \tilde{p}_\gamma = g \upharpoonright f. Furthermore, since the sets D_\alpha are open, the construction of \tilde{p}_\gamma ensures that eventually these conditions will meet every such dense set, showing that g \upharpoonright f really is generic.

The construction in the above proof is quite flexible and can be modified to make the set I generic in various ways as well (for example, we can arrange for I to be Cohen or dominating over M etc.).

Theorem 41. If \kappa is \theta-supercompact then there is a forcing extension in which there is a \theta-supercompactness I-Laver tree for \kappa for some I \subseteq \kappa^2 of size 2^{\kappa}.

Proof. If \theta \geq 2^\kappa then even a single Laver function for \kappa gives rise to a full Laver tree, by proposition [24] and we can force the existence of a Laver function by theorem [7]. We thus focus on the remaining case when \kappa \leq \theta < 2^\kappa.

\footnote{Here we take the forcing-equivalent version of Add(\lambda, 1) which adds a function g : \lambda \to H_\lambda by initial segments.}
We make similar simplifying assumptions as in theorem 7. Just as there we assume that $\theta = \theta^{<\kappa}$. Furthermore, we may assume that $2^\theta = \theta^+$, since this can be forced without adding subsets to $\mathcal{P}_\theta$ and affecting the $\theta$-supercompactness of $\kappa$. Note that these cardinal arithmetic hypotheses imply that $2^\gamma = \theta^+$. Let $\mathbb{P}$ be the length $\kappa$ Easton support iteration which adds, in a recursive fashion, a labelling of the tree $^{<\kappa}2$ of the extension. Specifically, let $\mathbb{P}$ force with $\mathbb{Q}_\gamma = \text{Add}(\gamma, 1)$ at each inaccessible $\gamma < \kappa$ stage $\gamma$. Let $G \subseteq \mathbb{P}$ be generic and let $G_\gamma$ be the piece added at stage $\gamma$. Using suitable coding, we can see each $G_\gamma$, in $V[G]$, as a function $G_\gamma: \gamma \rightarrow H_{\gamma^+}$; in particular, we should note that the iteration $\mathbb{P}$ does not add any nodes to the tree $(\leq \gamma)2|V|G$ at stage $\gamma$ or later, so that $G_\gamma$ really does label the whole level $\gamma 2$. Thus $G$ induces a map $D: <^{<\kappa}2 \rightarrow V_{\kappa}[G]$, by extending the $G_\gamma$ in any way we like to the entire tree. We shall show that $D$ is an $I$-Laver tree for some specifically chosen $I$.

Fix a $\theta$-supercompactness embedding $j: V \rightarrow M$ in $V$. Note that $M[G]^0 \subseteq M[G]$ in $V[G]$ as well, since the forcing $\mathbb{P}$ is $\theta^+$-cc. Furthermore, in $V[G]$ we still have $2^\theta = \theta^+$, which implies $\check{\theta}^+$ by a result of Shelah [13]. Now apply theorem [10] to $M[G]$ and $\lambda = \theta^+$ to obtain an $I \subseteq <^{<\kappa}2$ of size $\theta^+$ and a function $g: <^{<\kappa}2 \rightarrow H_{\theta^+}$ such that for any $f: I \rightarrow H_{\theta^+}$ in $V[G]$, the graft $g \upharpoonright f$ is generic over $M[G]$. We claim that $D$ is an $I$-Laver tree.

To check the guessing property, fix a sequence of targets $\vec{a} = \langle a_s; s \in I \rangle$ in $V[G]$. We shall lift the embedding $j$ to $V[G]$. Let us write $j(\mathbb{P}) = \mathbb{P} \ast \mathbb{Q}_{\kappa} \ast \mathbb{P}_{\text{tail}}$. We know that $g \upharpoonright \vec{a}$ is $M[G]$-generic for $\mathbb{Q}_{\kappa}$, so we only need to find the further generic for $\mathbb{P}_{\text{tail}}$. We easily see that $M[G][g \upharpoonright \vec{a}]^\theta \subseteq M[G][g \upharpoonright \vec{a}]$ in $V[G]$, that $\mathbb{P}_{\text{tail}}$ is $\leq \theta$-closed in that model, and that $M[G][g \upharpoonright \vec{a}]$ only has $\theta^+$-many subsets of $\mathbb{P}_{\text{tail}}$. We can thus diagonalize against these dense sets in $\theta^+$-many steps and produce a generic $G_{\text{tail}}$ for $\mathbb{P}_{\text{tail}}$. Putting all of this together, we can lift $j$ to $j: V[G] \rightarrow M[j(G)]$ in $V[G]$, where $j(G) = G \ast (g \upharpoonright \vec{a}) \ast G_{\text{tail}}$. Now consider $j(D)$. This is exactly the labelling of the tree $<^{<\kappa^+}2$ in $M[j(G)]$ given by $j(G)$. Furthermore, for any $s \in I$, we have $j(D)(s) = (g \upharpoonright \vec{a})(s) = a_s$, verifying the guessing property.

Given a Laver tree for $\kappa$, it is easy to produce a joint Laver sequence of length $2^\kappa$ from it by just reading the labels along each branch of the Laver tree. The resulting sequence then exhibits a large degree of coherence. We might wonder about the possibility of reversing this process, starting with a joint Laver sequence and attempting to fit it into a tree. A sequence for which this can be done might be called treeable. But, taken literally, this notion is not very robust. For example, all functions in a treeable joint Laver sequence must have the same value at 0. This means that we could take a treeable sequence and modify it in an inessential way to destroy its treeability. To avoid such trivialities, we relax the definition to only ask for coherence modulo bounded perturbations.

**Definition 42.** Let $\kappa$ be a regular cardinal and $\vec{f} = \langle f_\alpha; \alpha < \lambda \rangle$ a sequence of functions defined on $\kappa$. The sequence $\vec{f}$ is treeable if there are a bijection $e: \lambda \rightarrow 2^\kappa$ and a tree labelling $D: <^\kappa 2 \rightarrow V$ such that, for all $\alpha < \lambda$, we have $D(e(\alpha) \upharpoonright \xi) = f_\alpha(\xi)$ for all but boundedly many $\xi < \kappa$.

Given an $I \subseteq 2^\kappa$, the sequence $\vec{f}$ is $I$-treeable if the above holds for a bijection $e: \lambda \rightarrow I$.

**Lemma 43.** Let $\kappa$ be a regular cardinal and assume that $(2^{<\kappa})^+ \leq 2^\kappa$. Let $G \subseteq \text{Add}(\kappa, 2^\kappa)$ be generic. Then $G$ is not treeable.
Proof. Let us write $G = \langle g_\alpha; \alpha < 2^\kappa \rangle$ as a sequence of its slices. Now suppose that this sequence were treeable and let $\dot{c}$ and $\dot{D}$ be names for the indexing function and the labelling of $\kappa^{<\kappa}2$, respectively. Our cardinal arithmetic assumption implies that the name $\dot{D}$ only involves conditions from a bounded part of the poset $\text{Add}(\kappa, 2^\kappa)$, so we may assume that the labelling $D$ exists already in the ground model. Let $p$ be an arbitrary condition and $\alpha < \kappa$. Since we assumed that $G$ was forced to be treeable, there is a name $\dot{\gamma}$ for an ordinal, beyond which $G_\alpha$ agrees with $D \upharpoonright f(\alpha)$. By strengthening $p$ if necessary, we may assume that the value of $\dot{\gamma}$ has been decided.

We now inductively construct a countable descending sequence of conditions below $p$, deciding longer and longer initial segments of $\dot{\gamma}(\alpha)$, in such a way that, for some $\delta > \gamma$, their union $p^* \leq p$ decides $\dot{\gamma}(\alpha) \upharpoonright \delta$ but does not decide $G_\alpha(\delta)$. Then $p^*$ can be further extended to a condition forcing $G_\alpha(\delta) \neq D(\dot{\gamma}(\alpha) \upharpoonright \delta)$, which contradicts the fact that $p$ forces that $G$ is treeable. \hfill $\Box$

Corollary 44. If $\kappa$ is $\theta$-supercompact then there is a forcing extension in which there is a nontreeable $\theta$-supercompactness joint Laver sequence for $\kappa$ of length $2^\kappa$.

Proof. The joint Laver sequence constructed in theorem 47 was added by forcing with $\text{Add}(\kappa, 2^\kappa)$, so it is not treeable by lemma 43. \hfill $\Box$

4. Joint Laver diamonds for strong cardinals

Definition 45. A function $\ell: \kappa \to V_\kappa$ is a $\theta$-strongness Laver function if it guesses elements of $V_\theta$ via $\theta$-strongness embeddings with critical point $\kappa$.

If $\kappa$ is fully strong then a function $\ell: \kappa \to V_\kappa$ is a Laver function for $\kappa$ if it is a $\theta$-strongness Laver function for $\kappa$ for all $\theta$.

As in the supercompact case, $2^\kappa$ is the largest possible cardinal length of a $\theta$-strongness joint Laver sequence for $\kappa$, just because there are only $2^\kappa$ many functions $\ell: \kappa \to V_\kappa$.

The set of targets $V_\theta$ is a bit unwieldy and lacks some basic closure properties, particularly in the case when $\theta$ is a successor ordinal. The following lemma shows that, modulo some coding, we can recover a good deal of closure under sequences.

Lemma 46. Let $\theta$ be an infinite ordinal and let $I \in V_\theta$ be a set. If $\theta$ is successor ordinal or $\text{cf}(\theta) > |I|$ then $V_\theta$ is closed under a coding scheme for sequences indexed by $I$. Moreover, this coding is $\Delta_0$-definable.

Proof. If $\theta = \omega$ then the $I$ under consideration are finite. Since $V_\omega$ is already closed under finite sequences we need only deal with $\theta > \omega$.

Fix in advance a simply definable flat pairing function $\langle \cdot, \cdot \rangle$ (flat in the sense that any infinite $V_\theta$ is closed under it; the Quine–Rosser pairing function will do).

Let $\vec{a} = \langle a_i; i \in I \rangle$ be a sequence of elements of $V_\theta$. For each $i \in I$ we can find an (infinite) ordinal $\theta_i < \theta$ such that $a_i \cup \{i\} \subseteq V_{\theta_i}$. Now let $\vec{a}_i = \{[i, b]; b \in a_i\} \subseteq V_{\theta_i}$, and finally define $\vec{a} = \bigcup_{i \in I} \vec{a}_i$. We see that $\vec{a} \subseteq V_{\sup, \theta}$ and, under our hypotheses, $\sup, \theta_i < \theta$. It follows that $\vec{a} \subseteq V_{\sup, \theta_i+1} \subseteq V_\theta$ as required. \hfill $\Box$

Proposition 47. Let $\kappa$ be $\theta$-strong with $\kappa + 2 \leq \theta$ and let $\lambda \leq 2^\kappa$ be a cardinal. If there is a $\theta$-strongness Laver function for $\kappa$ and $\theta$ is either a successor ordinal or $\lambda < \text{cf}(\theta)$ then there is a $\theta$-strongness joint Laver sequence of length $\lambda$ for $\kappa$.

In particular, if $\theta$ is a successor then a single $\theta$-strongness Laver function already yields a joint Laver sequence of length $2^\kappa$, the maximal possible.
Proof. We aim to imitate the proof of proposition 4. To that end, fix an $I \subseteq P(\kappa)$ of size $\lambda$ and a bijection $f : \lambda \to I$. If $\ell$ is a Laver function for $\kappa$, we define a joint Laver sequence by letting $\ell_\alpha(\xi)$, for each $\alpha < \lambda$, be the element of $\ell(\xi)$ with index $f(\alpha) \cap \xi$ in the coding scheme described in lemma 46.

It is now easy to verify that the functions $\ell_\alpha$ form a joint Laver sequence: given a sequence of targets $\vec{a}$, we can replace it, by using $f$ and lemma 46, with a coded version $\tilde{a} \in V_\theta$. We can then use $\ell$ to guess $\tilde{a}$ and the $\theta$-strongness embedding obtained this way will witness the joint guessing property of the $\ell_\alpha$. □

Again, as in the supercompact case, if the Laver diamond we started with works for several different $\theta$ then the joint Laver sequence derived above will also work for those same $\theta$. In particular, if $\kappa$ is strong then combining the argument from proposition 47 with the Gitik–Shelah construction of a strongness Laver function in [7] gives an analogue of corollary 5.

Corollary 48. If $\kappa$ is strong then there is a strongness joint Laver sequence for $\kappa$ of length $2^\kappa$.

Proposition 47 implies that in most cases (that is, for most $\theta$) we do not need to do any additional work beyond ensuring that there is a $\theta$-strongness Laver function for $\kappa$ to automatically also find the longest possible joint Laver sequence. For example, if $\theta$ is a successor or if $\text{cf}(\theta) > 2^\kappa$ then a single $\theta$-strongness Laver function yields a joint Laver sequence of length $2^\kappa$. To gauge the consistency strength of the existence of $\theta$-strongness joint Laver sequences for $\kappa$ we should therefore only focus on the consistency strength required for a single Laver diamond, and, separately, on $\theta$ of low cofinality.

Forcing constructions giving a single $\theta$-strongness Laver diamond for a cardinal $\kappa$ are quite well known (one can simply add, after a suitable preparation, a Cohen subset to $\kappa$; a simple master condition argument shows that this adds a Laver function). We therefore get an immediate corollary to proposition 47.

Corollary 49. Let $\kappa$ be $\theta$-strong with $\kappa + 2 \leq \theta$. If $\theta$ is either a successor ordinal or $\text{cf}(\theta) \geq \kappa^+$ then there is a forcing extension in which $\Delta^\theta_{\kappa, 2^\kappa}$ holds.

Moving on to the case of $\theta$ of low cofinality, the question of the necessity of the hypotheses of proposition 47 remains very attractive.

Question 50. Suppose there is a $\theta$-strongness Laver function for $\kappa$ (with $\theta$ possibly being a limit of low cofinality). Is there a $\theta$-strongness joint Laver sequence of length $\kappa$? Or even of length $\omega$?

We give a partial answer to this question. In contrast to the supercompact case, some restrictions are in fact necessary to allow for the existence of joint Laver sequences for $\theta$-strong cardinals. The existence of even the shortest of such sequences can surpass the existence of a $\theta$-strong cardinal in consistency strength.

To give a better lower bound on the consistency strength required, we introduce a notion of Mitchell rank for $\theta$-strong cardinals, inspired by Carmody [4].

Definition 51. Let $\kappa$ be a cardinal and $\theta$ an ordinal. The $\theta$-strongness Mitchell order $\triangleleft$ for $\kappa$ is defined on the set of $(\kappa, V_\theta)$-extenders, by letting $E \triangleleft F$ if $E$ is an element of the (transitive collapse of the) ultrapower of $V$ by $F$.

Unsurprisingly, this Mitchell order has properties analogous to those of the usual Mitchell order on normal measures on $\kappa$ or the $\theta$-supercompactness Mitchell order.
on normal fine measures on \( P_\theta \), as studied by Carmody. In particular, the \( \theta \)-strongness Mitchell order is well-founded and gives rise to a notion of a \( \theta \)-strongness Mitchell rank. Having \( \theta \)-strongness Mitchell rank at least 2 implies that many cardinals below \( \kappa \) have reflected versions of \( \theta \)-strongness; for example, if \( \kappa \) has \((\kappa + \omega)\)-strongness Mitchell rank at least 2, then there are stationarily many cardinals \( \lambda < \kappa \) which are \((\lambda + \omega)\)-strong (and much more is true).

We should mention a bound on the \( \theta \)-strongness Mitchell rank of a cardinal \( \kappa \).

If \( j : V \rightarrow M \) is the ultrapower by a \((\kappa, V_\theta)\)-extender then any \((\kappa, V_\theta)\)-extenders in \( M \) appear in \( V_\kappa^{M_\kappa} \). It follows that these extenders are represented by a function \( f : V_\kappa \rightarrow V_\kappa \) and a seed \( a \in V_\theta \). In particular, there are at most \( \beth_\theta \) many such extenders, counted in \( V \). Any given extender therefore has at most \( \beth_\theta \) many predecessors in the Mitchell order, so the highest possible \( \theta \)-strongness Mitchell rank of \( \kappa \) is \( \beth_\theta \).

**Theorem 52.** Let \( \kappa \) be a \( \theta \)-strong cardinal, where \( \theta \) is a limit ordinal, and \( \text{cf}(\theta) \leq \kappa < \theta \). If there is a \( \theta \)-strongness joint Laver sequence for \( \kappa \) of length \( \text{cf}(\theta) \) then \( \kappa \) has maximal \( \theta \)-strongness Mitchell rank.

**Proof.** We first show that the existence of a short \( \theta \)-strongness joint Laver sequence implies a certain degree of hypermeasurability for \( \kappa \). Let \( \ell = (\ell_\alpha : \alpha < \text{cf}(\theta)) \) be the joint Laver sequence. If \( \tilde{a} = (a_\alpha : \alpha < \text{cf}(\theta)) \) is any sequence of targets in \( V_\theta \), there is, by definition, a \( \theta \)-strongness embedding \( j : V \rightarrow M \) with critical point \( \kappa \) such that \( j(\ell_\alpha)(\kappa) = a_\alpha \). But we can recover \( \ell \) from \( j(\tilde{\ell}) \) as an initial segment, since \( \tilde{\ell} \) is so short. Therefore we actually get the whole sequence \( \tilde{a} \in M \), just by evaluating that initial segment at \( \kappa \). Now consider any \( a \subseteq V_\theta \). We can resolve \( a \) into a \( \text{cf}(\theta) \)-sequence of elements \( a_\alpha \) of \( V_\theta \) such that \( a = \bigcup_\alpha a_\alpha \). Our argument then implies that \( a \) is an element of \( M \).

Now let \( E \) be an arbitrary \((\kappa, V_\theta)\)-extender. Since \( E \) can be represented as a family of measures on \( \kappa \) indexed by \( V_\theta \), it is coded by a subset of \( V_\theta \) (using the coding scheme from lemma [40] for example). Applying the argument above, there is a \( \theta \)-strongness embedding \( j : V \rightarrow M \) with critical point \( \kappa \) such that \( E \in M \). It follows that \( \kappa \) is \( \theta \)-strong in \( M \), giving \( \kappa \) nontrivial \( \theta \)-strongness Mitchell rank in \( V \).

The argument in fact yields more: given any collection of at most \( \beth_\theta \) many \((\kappa, V_\theta)\)-extenders, we can code the whole family by a subset of \( V_\theta \) and, again, obtain an extender whose ultrapower contains the entire collection we started with. It follows that, given any family of at most \( \beth_\theta \) many extenders, we can find an extender which is above all of them in the \( \theta \)-strongness Mitchell order. Applying this fact inductively now shows that \( \kappa \) must have maximal \( \theta \)-strongness Mitchell rank.

Just as in the case of \( \theta \)-supercompactness we can also consider \( \theta \)-strongness Laver trees. In view of propositions [47] and [47] it is not surprising that again, for most \( \theta \), a single \( \theta \)-strongness Laver diamond yields a full Laver tree.

**Proposition 53.** Suppose \( \kappa \) is \( \theta \)-strong where \( \kappa + 2 \leq \theta \) and \( \theta \) is either a successor ordinal or \( \text{cf}(\theta) > 2^\kappa \). Then a \( \theta \)-strongness Laver tree for \( \kappa \) exists if and only if a \( \theta \)-strongness Laver function for \( \kappa \) does (if and only if \( \Delta_{\kappa, 2^\kappa}^{\text{sc}} \) holds).

**Proof.** We follow the proof of proposition [47]. Note that, since \( \theta \geq \kappa + 2 \), we have \( \kappa^2 \in V_\theta \), so \( V_\theta \) is closed under sequences indexed by \( \kappa^2 \) via the coding scheme given
by lemma [H6] If $\ell$ is a $\theta$-strongness Laver function for $\kappa$ we define a Laver tree by letting $D(t)$ be the element with index $t$ in the sequence coded by $\ell(|t|)$, if this makes sense. It is now easy to check that this truly is a Laver tree: given any sequence of targets $\vec{a}$ we simply use the Laver function $\ell$ to guess it (or, rather, its code), and the embedding $j$ obtained this way will witness the guessing property for $D$.

5. Joint diamonds

Motivated by the joint Laver sequences of the previous sections we now apply the jointness concept to smaller cardinals. Of course, since we do not have any elementary embeddings of the universe with critical point $\omega_1$, say, we need a reformulation that will make sense in this setting as well.

As motivation, consider a measurable Laver function $\ell$ and let $a \subseteq \kappa$. By definition there is an elementary embedding $j : V \to M$ such that $j(\ell)(\kappa) = a$. Let $U$ be the normal measure on $\kappa$ derived from this embedding. Observe that $a$ is represented in the ultrapower by the function $f_a(\xi) = a \cap \xi$ and thus, by Łoś’s theorem, we conclude that $\ell(\xi) = a \cap \xi$ for $U$-almost all $\xi$. Therefore $\ell$ is (essentially) a $\diamondsuit_\kappa$-sequence. Similarly, if we are dealing with a joint Laver sequence $\langle \ell_\alpha; \alpha < \lambda \rangle$ there is for every sequence $\langle a_\alpha; \alpha < \lambda \rangle$ of subsets of $\kappa$ a normal measure on $\kappa$ with respect to which each of the sets $\{ \xi < \kappa; \ell_\alpha(\xi) = a_\alpha \cap \xi \}$ has measure one.

This understanding of jointness seems amenable to transfer to smaller cardinals. There are still no normal measures on $\omega_1$, but perhaps we can weaken that requirement slightly.

Recall that a filter on a regular cardinal $\kappa$ is normal if it is closed under diagonal intersections, and uniform if it extends the cobounded filter. It is a standard result that the club filter on $\kappa$ is the least normal uniform filter on $\kappa$ (in fact, a normal filter is uniform if it extends the club filter). It follows that any subset of $\kappa$ contained in a (proper) normal uniform filter is stationary. Conversely, if $S \subseteq \kappa$ is stationary, then it is easy to check that $S$, together with the club filter, generates a normal uniform filter on $\kappa$. Altogether, we see that a set is stationary iff it is contained in a normal uniform filter. This observation suggest an analogy between Laver functions and $\diamondsuit_\kappa$-sequences: in the same way that Laver functions guess their targets on positive sets with respect to some large cardinal measure, $\diamondsuit_\kappa$-sequences guess their targets on positive sets with respect to some normal uniform filter. Extending the analogy, in the same way that a joint Laver sequence is a collection of Laver functions that guess sequences of targets on positive sets with respect to a common large cardinal measure (corresponding to the single embedding $j$), a collection of $\diamondsuit_\kappa$-sequences will be joint if they guess sequences of targets on positive sets with respect to a common normal uniform filter.

We will adopt the following terminology: if $\kappa$ is a cardinal then a $\kappa$-list is a function $d : \kappa \to \mathcal{P}(\kappa)$ with $d(\alpha) \subseteq \alpha$.

**Definition 54.** Let $\kappa$ be an uncountable regular cardinal. A $\diamondsuit_{\kappa, \lambda}$-sequence is a sequence $\vec{d} = \langle d_\alpha; \alpha < \lambda \rangle$ of $\kappa$-lists such that for every sequence $\langle a_\alpha; \alpha < \lambda \rangle$ of subsets of $\kappa$ there is a (proper) normal uniform filter $F$ on $\kappa$ such that for every $\alpha$ the guessing set $S_\alpha = S(d_\alpha, a_\alpha) = \{ \xi < \kappa; d_\alpha(\xi) = a_\alpha \cap \xi \}$ is in $F$.

An alternative, apparently simpler attempt at defining jointness would be to require that all the $\kappa$-lists in the sequence guess their respective targets on the
same stationary set. However, a straightforward diagonalization argument shows that, with this definition, there are no \( \diamondsuit_{\kappa, \kappa} \)-sequences at all. Upon reflection, the definition we gave above corresponds more closely to the one in the case of Laver diamonds and, hopefully, is not vacuous.

We will not use the following proposition going forward, but it serves to give another parallel between \( \diamondsuit \)-sequences and Laver diamonds. It turns out that \( \diamondsuit \)-sequences can be seen as Laver functions, except that they work with generic elementary embeddings.

**Proposition 55.** Let \( \kappa \) be an uncountable regular cardinal and \( d \) a \( \kappa \)-list. Then \( d \) is a \( \diamondsuit_{\kappa} \)-sequence iff there is, for any \( a \subset \kappa \), a generic elementary embedding \( j: V \to M \) with critical point \( \kappa \) and \( M \) wellfounded up to \( \kappa^+ \) such that \( j(d)(\kappa) = a \).

**Proof.** Suppose \( d \) is a \( \diamondsuit_{\kappa} \)-sequence and fix a target \( a \subset \kappa \). Let \( S(d, a) = \{ \xi < \kappa ; d(\xi) = a \cap \xi \} \) be the guessing set. By our discussion above there is a normal uniform filter \( \mathcal{F} \) on \( \kappa \) with \( S \in \mathcal{F} \). Let \( G \) be a generic ultrafilter extending \( \mathcal{F} \) and \( j: V \to M \) the generic ultrapower by \( G \). Then \( M \) is wellfounded up to \( \kappa^+ \) and \( \kappa = \text{id}_{\mathcal{G}} \). Since \( S \in G \), Loś's theorem now implies that \( j(d)(\kappa) = a \).

Conversely, fix a target \( a \subset \kappa \) and suppose that there is a generic embedding \( j \) with the above properties. We can replace \( j \) with the induced normal ultrapower embedding and let \( U \) be the derived ultrafilter in the extension. Since \( j(d)(\kappa) = a \) it follows that \( S(d, a) \in U \). But since \( U \) extends the club filter, \( S(d, a) \) must be stationary. \( \square \)

Similarly to the above proposition, a sequence of \( \kappa \)-lists is joint if they can guess any sequence of targets via a single generic elementary embedding.

We would now like to find a “bottom up” criterion deciding when a collection of subsets of \( \kappa \) (namely, some guessing sets) is contained in a normal uniform filter. The following key lemma gives such a criterion, which is completely analogous to the finite intersection property characterizing containment in a filter.

**Definition 56.** Let \( \kappa \) be an uncountable regular cardinal. A family \( \mathcal{A} \subset \mathcal{P}(\kappa) \) has the diagonal intersection property if for any function \( f: \kappa \to \mathcal{A} \) the diagonal intersection \( \bigtriangleup_{\alpha < \kappa} f(\alpha) \) is stationary.

**Lemma 57.** Let \( \kappa \) be uncountable and regular and let \( \mathcal{A} \subset \mathcal{P}(\kappa) \). The family \( \mathcal{A} \) is contained in a normal uniform filter on \( \kappa \) iff \( \mathcal{A} \) satisfies the diagonal intersection property.

**Proof.** The forward direction is clear, so let us focus on the converse. Consider the family of sets

\[
E = \left\{ C \cap \bigtriangleup_{\alpha < \kappa} f(\alpha) ; f \in {}^\kappa \mathcal{A}, C \subset \kappa \text{ club} \right\}
\]

We claim that \( E \) is directed under diagonal intersections: any diagonal intersection of \( \kappa \) many elements of \( E \) includes another element of \( E \). To see this take \( C_\alpha \cap \bigtriangleup_{\beta < \kappa} f_\alpha(\beta) \in E \) for \( \alpha < \kappa \). Let \( \langle \cdot, \cdot \rangle \) be a pairing function and define \( F: \kappa \to \lambda \) by \( F(\langle \alpha, \beta \rangle) = f_\alpha(\beta) \). A calculation then shows that

\[
\bigtriangleup_{\alpha < \kappa} \big( C_\alpha \cap \bigtriangleup_{\beta < \kappa} f_\alpha(\beta) \big) = \bigtriangleup_{\alpha < \kappa} \big( C_\alpha \cap \bigtriangleup_{\beta < \kappa} f_\alpha(\beta) \big) \supseteq \bigtriangleup_{\alpha < \kappa} D \cap \bigtriangleup_{\alpha < \kappa} F(\alpha)
\]

where \( D \) is the club of closure points of the pairing function.

It follows that closing \( E \) under supersets yields a normal uniform filter on \( \kappa \). By considering constant functions \( f \) we also see that every \( a \in \mathcal{A} \) is in this filter. \( \square \)
Lemma 57 will be the crucial tool for verifying $\diamondsuit_{\kappa,\lambda}$. More specifically, we shall often apply the following corollary.

**Corollary 58.** A sequence $\vec{d} = \langle d_\alpha; \alpha < \lambda \rangle$ is a $\diamondsuit_{\kappa,\lambda}$-sequence iff every subsequence of length $\kappa$ is a $\diamondsuit_{\kappa,\kappa}$-sequence.

**Proof.** The forward implication is obvious; let us check the converse. Let $\vec{a} = \langle a_\alpha; \alpha < \lambda \rangle$ be a sequence of targets and let $S_\alpha$ be the corresponding guessing sets. By lemma 57 we need to check that the family $S = \{ S_\alpha; \alpha < \lambda \}$ satisfies the diagonal intersection property. So fix a function $f: \kappa \to S$ and let $r = \{ \alpha; S_\alpha \in f[\kappa] \}$. By our assumption $\vec{d}|r$ is a $\diamondsuit_{\kappa,|r|}$-sequence, so $f[\kappa]$ is contained in a normal uniform filter and, in particular, $\Delta_\alpha f(\alpha)$ is stationary. \qed

This characterization leads to fundamental differences between joint diamonds and joint Laver diamonds. While the definition of joint diamonds was inspired by large cardinal phenomena, the absence of a suitable analogue of the diagonal intersection property in the large cardinal setting provides for some very surprising results.

**Definition 59.** Let $\kappa$ be an uncountable regular cardinal. A $\diamondsuit_{\kappa}$-tree is a function $D: <\kappa \to P(\kappa)$ such that for any sequence $\langle a_s; s \in \kappa \rangle$ of subsets of $\kappa$ there is a (proper) normal uniform filter on $\kappa$ containing all the guessing sets $S_s = S(D, a_s) = \{ \xi < \kappa; D(s|\xi) = a_s \cap \xi \}$.

This definition clearly imitates the definition of Laver trees. We also have a correspondence in the style of proposition 55: a $\diamondsuit_{\kappa}$-tree acts like a Laver tree for $\kappa$ using generic elementary embeddings.

The following theorem, the main result of this section, shows that, in complete contrast to our experience with joint Laver diamonds, $\diamondsuit_{\kappa}$ already implies all of its stronger, joint versions.

**Theorem 60.** Let $\kappa$ be an uncountable regular cardinal. The following are equivalent:

1. $\diamondsuit_{\kappa}$
2. $\diamondsuit_{\kappa,\kappa}$
3. $\diamondsuit_{\kappa,2^\kappa}$
4. there exists a $\diamondsuit_{\kappa}$-tree.

**Proof.** For the implication (1) $\implies$ (2), let $d: \kappa \to P(\kappa)$ be a $\diamondsuit_{\kappa}$-sequence and fix a bijection $f: \kappa \to \kappa \times \kappa$. Define $d_\alpha(\xi) = (f[d(\xi)])_\alpha \cap \alpha = \{ \eta < \alpha; (\alpha, \eta) \in f[d(\xi)] \}$

We claim that $\langle d_\alpha; \alpha < \kappa \rangle$ is a $\diamondsuit_{\kappa,\kappa}$-sequence.

To see this take a sequence of targets $\langle a_\alpha; \alpha < \lambda \rangle$ and let $S_\alpha = \{ \xi < \kappa; d_\alpha(\xi) = a_\alpha \cap \xi \}$ be the guessing sets. The set $T = \{ \xi < \kappa; f^{-1}\left(\bigcup_{\alpha<\kappa} \{\alpha\} \times a_\alpha\right) \cap \xi = d(\xi) \}$ is stationary in $\kappa$. Let $\mathcal{F}$ be the filter generated by the club filter on $\kappa$ together with $T$. This is clearly a proper filter and, since a set $Y$ is $\mathcal{F}$-positive iff $Y \cap T$ is stationary, Fodor’s lemma immediately implies that it is also normal. We claim
that we have $S_\alpha \in \mathcal{F}$ for all $\alpha < \kappa$, so that $\mathcal{F}$ witnesses the defining property of a $\diamondsuit_{\kappa, \kappa}$-sequence. Since $f[\xi] = \xi \times \xi$ for club many $\xi < \kappa$, the set

$$T' = \left\{ \xi < \kappa; d(\xi) = f^{-1}\left[ \bigcup_{\alpha < \kappa} \{\alpha\} \times a_\alpha \right] \cap f^{-1}[\xi \times \xi] \right\}$$

is just the intersection of $T$ with some club and is therefore in $\mathcal{F}$. But now observe that

$$T' = \left\{ \xi < \kappa; d(\xi) = f^{-1}\left[ \bigcup_{\alpha < \xi} \{\alpha\} \times (a_\alpha \cap \xi) \right] \right\} = \left\{ \xi < \kappa; \forall \alpha < \xi \colon a_\alpha \cap \xi = (f[d(\xi)])_\alpha = d_\alpha(\xi) \right\} = \bigtriangleup S_\alpha$$

We see that $T' \in \mathcal{F}$ is, modulo a bounded set, contained in each $S_\alpha$ and can thus conclude, since $\mathcal{F}$ is uniform, that $S_\alpha \in \mathcal{F}$ for all $\alpha < \kappa$.

Instead of proving (2) $\implies$ (3) it will be easier to show (2) $\implies$ (4) directly. Since the implications (4) $\implies$ (3) $\implies$ (1) are obvious, this will finish the proof.

Fix a $\diamondsuit_{\kappa, \kappa}$-sequence $\bar{d}$. We proceed to construct the $\diamondsuit_{\kappa}$-tree $D$ level by level; in fact the only meaningful work will take place at limit levels. At a limit stage $\gamma < \kappa$ we shall let the first $\gamma$ many $\diamondsuit_{\kappa}$-sequences anticipate the labels and their positions. Concretely, consider the sets $d_\alpha(\gamma)$ for $\alpha < \gamma$. For each $\alpha$ we interpret $d_{2\alpha}(\gamma)$ as a node on the $\gamma$-th level of $\subseteq \kappa 2$ and let $D(d_{2\alpha}(\gamma)) = d_{2\alpha+1}(\gamma)$, provided that there is no interference between the different $\diamondsuit_{\kappa}$-sequences. If it should happen that for some $\alpha \neq \beta$ we get $d_{2\alpha}(\gamma) = d_{2\beta}(\gamma)$ but $d_{2\alpha+1}(\gamma) \neq d_{2\beta+1}(\gamma)$ we scrap the whole level and move on with the construction higher in the tree. At the end we extend $D$ to be defined on the nodes of $\subseteq \kappa 2$ that were skipped along the way in any way we like.

We claim that the function $D$ thus constructed is a $\diamondsuit_{\kappa}$-tree. To check this let us fix a sequence of targets $\bar{a} = \langle a_s; s \in \kappa 2 \rangle$ and let $S_\alpha$ be the guessing sets. By lemma 47 it now suffices to check that $\bigtriangleup_{\alpha < \kappa} S_{s_\alpha}$ is stationary for any sequence of branches $\langle s_\alpha; \alpha < \kappa \rangle$.

For $\alpha < \kappa$ let $T_{2\alpha} = \{ \xi; s_{2\alpha}^{-1}[[1]] \cap \xi = d_{2\alpha}(\xi) \}$ and $T_{2\alpha+1} = \{ \xi; A^s_{\alpha} \cap \xi = d_{2\alpha+1}(\xi) \}$. Since our construction was guided by a $\diamondsuit_{\kappa, \kappa}$-sequence, there is a normal uniform filter on $\kappa$ which contains every $T_\alpha$. In particular, $T = \bigtriangleup_{\alpha < \kappa} T_\alpha$ is stationary. By a simple bootstrapping argument there is a club $C$ of limit ordinals $\gamma$ such that all $s_\alpha \upharpoonright \gamma$ for $\alpha < \gamma$ are distinct. Let $\gamma \in C \cap T$. We now have $s_{2\alpha}^{-1}[[1]] \cap \gamma = d_{2\alpha}(\gamma)$ and $s_{2\alpha} \cap \gamma = d_{2\alpha+1}(\gamma)$ for all $\alpha < \gamma$. But this means precisely that the construction of $D$ goes through at level $\gamma$ and that $\gamma \in \bigcap_{\alpha < \gamma} S_{s_\alpha}$ and it follows that $\bigtriangleup_{\alpha < \kappa} S_{s_\alpha}$ is stationary.

We can again consider the treeability of joint diamond sequences, as we did in definition 42. We get the following analogue of corollary 43.

**Theorem 61.** If $\kappa$ is an uncountable regular cardinal and GCH holds then after forcing with $\text{Add}(\kappa, 2^\kappa)$ there is a nontreeable $\diamondsuit_{\kappa, 2^\kappa}$-sequence.

**Proof.** Let $\mathbb{P} = \text{Add}(\kappa, 2^\kappa)$ and $G \subseteq \mathbb{P}$ generic; we refer to the $\alpha$-th subset added by $G$ as $G_\alpha$. We will show that the generic $G$, seen as a sequence of $2^\kappa$ many $\diamondsuit_{\kappa}$-sequences in the usual way, is a nontreeable $\diamondsuit_{\kappa, 2^\kappa}$-sequence.

Showing that $G$ is a $\diamondsuit_{\kappa, 2^\kappa}$-sequence requires only minor modifications to the usual proof that a Cohen subset of $\kappa$ codes a $\diamondsuit_{\kappa}$-sequence. Thus, we view each
$G_\alpha$ as a sequence, defined on $\kappa$, with $G_\alpha(\xi) \subseteq \xi$. Fix a sequence $\langle \dot{a}_\alpha; \alpha < 2^\kappa \rangle$ of names for subsets of $\kappa$, a name $\dot{f}$ for a function from $\kappa$ to $2^\kappa$ and a name $\dot{C}$ for a club in $\kappa$ as well as a condition $p \in P$. We will find a condition $q \leq p$ forcing that $\dot{C} \cap \Delta_{\alpha < \kappa} S_{f(\alpha)}$ is nonempty, where $S_{f(\alpha)}$ names the set $\{ \xi < \kappa; \dot{a}_{f(\alpha)} \cap \xi = G_\alpha(\xi) \}$; this will show that $G$ codes a $\diamondsuit_{\kappa, 2^\kappa}$-sequence by Lemma 57.

We build the condition $q$ in $\omega$ many steps. To start with, let $p_0 = p$ and let $\gamma_0$ be an ordinal such that $\text{dom}(p_0) \subseteq 2^\kappa \times \gamma_0$. We now inductively find ordinals $\gamma_n$, sets $B_n \subseteq \gamma_n$, functions $f_n$ and a descending sequence of conditions $p_n$ satisfying $\text{dom}(p_n) \subseteq 2^\kappa \times \gamma_n$ and $p_{n+1} \Vdash \gamma_n \in \dot{C}$ as well as $p_{n+1} \Vdash \dot{f} \in \gamma_n = f_n$ and $p_{n+1} \Vdash \dot{a}_{f_n(\alpha)} = B_n^\alpha$ for $\alpha < \gamma_n$. Let $\gamma = \sup_n \gamma_n$ and $p_\omega = \bigcup_n p_n$ and $f_\omega = \bigcup_n f_n$ and $B_\omega = \bigcup_n B_n^\alpha$. The construction of these ensures that $\text{dom}(p_\omega) \subseteq 2^\kappa \times \gamma$ and $p_\omega$ forces that $\dot{f} \in \gamma = f_\omega$ and $\dot{a}_{f(\alpha)} \cap \gamma = B_\omega^\alpha$ for $\alpha < \gamma$ as well as $\gamma \in \dot{C}$. To obtain the final condition $q$ we now simply extend $p_\omega$ by placing the code of $B_\omega^\alpha$ on top of the $f(\alpha)$-th column for all $\alpha < \gamma$. It now follows immediately that $q \Vdash \gamma \in \dot{C} \cap \Delta_{\alpha < \kappa} S_{f(\alpha)}$.

It remains to show that the generic $\diamondsuit_{\kappa, 2^\kappa}$-sequence is not treeable. This follows from Lemma 53. \hfill $\square$

In the case of Laver diamonds we were able to produce models with quite long joint Laver sequences but no Laver trees simply on consistency strength grounds (see Theorem 67). In other words, we have models where there are long joint Laver sequences, but none of them are treeable. The situation seems different for ordinary diamonds, as Theorem 60 tells us that treeable joint diamond sequences exist as soon as a single diamond sequence exists. While Theorem 61 shows that it is at least consistent that there are nontreeable such sequences, we should ask whether this is simply always the case.

**Question 62.** Is it consistent for a fixed $\kappa$ that every $\diamondsuit_{\kappa, 2^\kappa}$-sequence is treeable? Is it consistent that all $\diamondsuit_{\kappa, 2^\kappa}$-sequences are treeable for all $\kappa$?

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