\textbf{$n$-point functions at finite temperature}

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Abstract

We study $n$-point functions at finite temperature in the closed time path formalism. With the help of two basic column vectors and their dual partners we derive a compact decomposition of the time-ordered $n$-point functions with $2^n$ components in terms of $2^{n-1} - 1$ independent retarded/advanced $n$-point functions. This representation greatly simplifies calculations in the real-time formalism.

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I. INTRODUCTION

Finite temperature field theory has been extensively studied for a large number of problems in particle physics, solid state physics and the physics of the early universe. There exist real-time (RTF) and imaginary-time (ITF) formulations of finite temperature field theory [1–3]. The real-time formalism again comes in several variations [5]. To study the relations between those different formulations of the theory has been an interesting subject over the years [6–11]. In Ref. [8] Evans gave relations between the RTF 3-point Green functions and their analytically continued ITF counterparts. In Ref. [9] he noted that the situation was more complicated for 4-point and higher order $n$-point functions. Taylor studied the corresponding relations for 4-point functions in [12]. In [13,14] a column vector calculus for thermo-field dynamics was presented which simplifies the calculation of Feynman diagrams for matrix valued propagators. More recently in Ref. [15] the relation between the time-ordered and retarded/advanced [11] 3-point functions was reexamined in the closed time-path (CTP) formalism [3,14], and a simple decomposition of the 8-component real-time vertex tensor in terms of retarded/advanced 3-point functions was derived using outer products of the 2-component column vectors introduced in [13,14]. It was shown in [15,16] that, due to orthogonality relations between the column vectors, this representation greatly simplifies calculations in the real-time formalism.

The purpose of this short report is to establish relations between the time-ordered and retarded/advanced $n$-point functions by using the column vector technique. We will generalize the work in Ref. [15] on the 3-point functions to $n$-point functions. In doing so we shall show that all $2^n$ time-ordered functions can be expressed through linear combinations of $2^n - 1$ independent retarded/advanced functions.

In Sec. II we will review the column vector representation of the two-point function and study how it is decomposed into retarded/advanced propagators. In Sec. III we will study the analogous decomposition for the 3-point functions. General $n$-point functions will be studied in Sec. IV. Some conclusions are presented in Sec. V.
II. 2-POINT FUNCTIONS

To establish our notation and for later use we first consider the single-particle propagator for a bosonic field theory. In real time, the propagator has $2^2 = 4$ components since each of the two fields can take values on either branch of the real-time contour [2,3]. We generally follow the notation in Ref. [14].

The 4 components of the propagator can be combined into a $2 \times 2$ matrix

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix},$$

(1)

which can be rewritten [13,14] in terms of an outer product of 2-component column vectors:

$$D(p) = D_R(p) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 + n(p_0) \\ n(p_0) \end{pmatrix} - D_A(p) \begin{pmatrix} n(p_0) \\ 1 + n(p_0) \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$  

(2)

Here

$$n(p_0) = \frac{1}{e^\beta p_0 - 1}, \quad n(-p_0) = -(1 + n(p_0))$$

(3)

is the thermal Bose-Einstein distribution, and $D_{R,A}$ are the retarded and advanced propagators

$$D_R = D_{11} - D_{12} \quad D_A = D_{11} - D_{21}.$$  

(4)

Their spectral representations are

$$D_R(p) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\rho_-(\omega, p_0)}{p_0 - \omega + i\epsilon}, \quad D_A(p) = D_R^*(p).$$

(5)

$\rho_-(p)$ is the (real) spectral density in terms of which all propagator components can be expressed via spectral integrals.

Defining the following two column vectors

$$e_R(p) = \begin{pmatrix} 1 + n(p_0) \\ n(p_0) \end{pmatrix}, \quad e_A(p) = -\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

(6)

Eq. (2) can be written as
\[ D(p) = -\left( D_R(p) \, e_A(-p) \otimes e_R(p) + D_A(p) \, e_R(-p) \otimes e_A(p) \right). \tag{7} \]

We also define the “dual” vectors
\[
\tilde{e}_R(p) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \tilde{e}_A(p) = \begin{pmatrix} n(p_0) \\ -1 - n(p_0) \end{pmatrix}, \tag{8}\]

which satisfy \((\tau_3 \text{ is the third Pauli matrix})\)
\[
\tilde{e}_{R,A}(p) = -\tau_3 \, e_{A,R}(-p). \tag{9}\]

As a rule of thumb the vectors \(e_{R,A}\) are associated with legs which carry outflowing momenta while the \(\tilde{e}_{R,A}\) are associated with inflowing momenta. If the propagator \(D(p) = G^{(2)}(p, -p)\) is represented by a Feynman diagram in which the momentum \(p\) flows from left to right, the left leg carries the inflowing momentum \(p\) (i.e. the outflowing momentum \(-p\)) while the right leg carries the inflowing momentum \(-p\) (i.e. the outflowing momentum \(p\)). This structure is reflected by the signs of the momentum arguments of the column vectors \(e_{R,A}\) in Eq. (7).

With the column vector contraction rule \[13,14\]
\[
\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2 \tag{10}\]
the basis vectors (3) and their dual partners (8) satisfy the orthogonality relations
\[
e_{\alpha}(p) \cdot \tilde{e}_{\beta}(p) = \delta_{\alpha \beta} \quad (\alpha, \beta = R, A). \tag{11}\]

Throughout this paper we will use latin letters \(a, b, c, \ldots = 1, 2\) for the usual thermal indices and greek letters \(\alpha, \beta, \gamma, \ldots = R, A\) for retarded/advanced indices.

The self-energy \(\Pi\), defined through the Schwinger-Dyson equation \(D^{-1} = D_0^{-1} - \Pi\), is given by \[15\]
\[
\Pi(p) = \Pi_R(p) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 + n(p_0) \\ -n(p_0) \end{pmatrix} - \Pi_A(p) \begin{pmatrix} n(p_0) \\ -1 - n(p_0) \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{12}\]
which can be rewritten as
\( \Pi(p, -p) = -\left( \Pi_R(p) \tilde{e}_R(p) \otimes \tilde{e}_A(-p) + \Pi_A(p) \tilde{e}_A(p) \otimes \tilde{e}_R(-p) \right). \) \hspace{1cm} (13)

The two momentum arguments denote the two inflowing momenta in the legs connected to the self-energy. Let us define its \( R/A \) components via

\[ \Pi_{\alpha\beta}(p, -p) \equiv \left( e_\alpha(p) \otimes e_\beta(-p) \right) \bullet \Pi(p, -p) \quad (\alpha, \beta = R, A). \] \hspace{1cm} (14)

The \( \bullet \) denotes the contraction between the 2 vectors \( e \) in the outer product shown in Eq. (14) with the corresponding 2 dual vectors \( \tilde{e} \) carrying the same momentum argument in Eq. (13). Using (11) one finds

\[ \Pi_{RR} = \Pi_{AA} = 0, \quad \Pi_{RA} = \Pi_{AR}^* = -\Pi_R = -\Pi_A^*. \] \hspace{1cm} (15)

Note that this differs from the notation used in Ref. [11] who give \( \Pi_{RA} = \Pi_{AR} = 0 \) and \( \Pi_{RR} = \Pi_R = \Pi_{AA}^* \). The reason for this discrepancy is that we use the convention that both momenta in \( \Pi(p, -p) \) are inflowing, and our \( R \) and \( A \) indices indicate retarded and advanced boundary conditions for these momenta (see below). Aurenche and Becherrawy, on the other hand, use for the two-point function (contrary to their convention for the 3-point function which agrees with ours) one in- and one outgoing momentum. Following (9) this implies that on the leg with the outgoing momentum the \( R \) and \( A \) indices should be reversed relative to our notation, in agreement with the above findings.

**III. 3-POINT VERTEX FUNCTIONS**

The RTF \( n \)-point Green functions have \( 2^n \) components \( G_{a_1a_2...a_n}(p_1, p_2, \ldots, p_n) \) where \( a_i = 1 \) or 2. We denote the truncated \( n \)-point function, from which the \( n \) external propagators have been removed, by \( \Gamma_{a_1a_2...a_n}(p_1, p_2, \ldots, p_n) \) and call it the \( n \)-point vertex. We adopt the convention that all momenta flow into the \( n \)-point vertex such that \( p_1 + p_2 + \ldots + p_n = 0 \) due to energy-momentum conservation.

We will show here that the \( n \)-point vertex functions can be conveniently expressed as sums of outer products of \( n \) column vectors in a very similar way as Eq. (13). For the
are straightforward analytical continuations such a representation was given in Eq. (35) of Ref. [[15]]. The analogous representation of the truncated 3-point function such a representation was given in Eq. (35) of Ref. [[15]]. The indices of the 3-point functions on the r.h.s. of (17) indicate whether a retarded or advanced propagator (\(D_R\) or \(D_A\), respectively) is attached to the corresponding leg.

As shown in Ref. [[11]], the retarded/advanced 3-point functions on the r.h.s. of (17) are straightforward analytical continuations \(i\omega_j \to p_j^0 + i\epsilon_j \ (j = 1, 2, 3)\) of the ITF 3-point function such a representation was given in Eq. (35) of Ref. [[15]]. The analogous representation of the truncated 3-point function such a representation was given in Eq. (35) of Ref. [[15]]. The indices of the 3-point functions on the r.h.s. of (17) indicate whether a retarded or advanced propagator (\(D_R\) or \(D_A\), respectively) is attached to the corresponding leg.

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function, with \( \epsilon_j > 0 \) for each momentum corresponding to a leg labelled by \( R \) and \( \epsilon_j < 0 \) for each momentum corresponding to a leg labelled by \( A \). Momentum conservation requires \( \epsilon_1 + \epsilon_2 + \epsilon_3 = 0 \) which implies \( \Gamma_{AAA} = \Gamma_{RRR} = 0 \). This explains why these two \( R/A \) functions do not appear on the r.h.s. of (17). While \( \Gamma_{AA...A} = 0 \) follows directly from the general identity [3]

\[
\sum_{a_1,a_2,...,a_n=1}^{2} \Gamma_{a_1a_2...a_n} = 0, \tag{19}
\]

the vanishing of the \( n \)-point vertex with only retarded legs, \( \Gamma_{RR...R} = 0 \), is a result [18] of the KMS condition and is therefore only true in global thermal equilibrium.

Comparison of (17) with (16) yields the identifications

\[
\Gamma_{Ri} = \Gamma_{RAA}, \quad \Gamma_{R} = \Gamma_{ARA}, \quad \Gamma_{Ro} = \Gamma_{AAR} \tag{20}
\]

as well as the identities

\[
\begin{align*}
\Gamma_{ARR}(p_1,p_2,p_3) &= -\frac{(1 + n_2)(1 + n_3) - n_2n_3}{(1 + n_1) - n_1} \Gamma_{RAA}(p_1,p_2,p_3), \tag{21a} \\
\Gamma_{RAR}(p_1,p_2,p_3) &= -\frac{(1 + n_1)(1 + n_3) - n_1n_3}{(1 + n_2) - n_2} \Gamma_{ARA}(p_1,p_2,p_3), \tag{21b} \\
\Gamma_{RRA}(p_1,p_2,p_3) &= -\frac{(1 + n_1)(1 + n_2) - n_1n_2}{(1 + n_3) - n_3} \Gamma_{AAR}(p_1,p_2,p_3). \tag{21c}
\end{align*}
\]

The latter are manifestations of the general relation [18]

\[
\Gamma_{\alpha_1...\alpha_n}(p_1,\ldots,p_n) = (-1)^{n_1 + 1} \frac{\prod_{\{i|\alpha_i=R\}} (1 + n_i)}{\prod_{\{i|\alpha_i=A\}} n_i} \Gamma^*_{\bar{\alpha}_1...\bar{\alpha}_n}(p_1,\ldots,p_n) \tag{22}
\]

which results from the KMS condition in thermal equilibrium. Here \( \bar{\alpha} = A, R \) if \( \alpha = R, A \), respectively. The second equality in (22) is easily proven using the identity (A3).
IV. $n$-POINT VERTEX FUNCTIONS FOR $n \geq 4$

Running through the manipulations of the previous Section in reverse order it is now easy to generalize Eq. (16) to higher order $n$-point functions. We illustrate the procedure for $n=4$; its generalization to arbitrary $n$ is then straightforward.

Using $\Gamma_{AAAA}(p_1, p_2, p_3, p_4) = 0 = \Gamma_{RRRR}(p_1, p_2, p_3, p_4)$, the analogue of (17) reads

$$
\Gamma(p_1, p_2, p_3, p_4) = \Gamma_{RAAA}(p_1, p_2, p_3, p_4) \bar{e}_R(p_1) \otimes \bar{e}_A(p_2) \otimes \bar{e}_A(p_3) \otimes \bar{e}_A(p_4)
+ \Gamma_{ARRR}(p_1, p_2, p_3, p_4) \bar{e}_A(p_1) \otimes \bar{e}_R(p_2) \otimes \bar{e}_R(p_3) \otimes \bar{e}_R(p_4)
+ \Gamma_{ARAA}(p_1, p_2, p_3, p_4) \bar{e}_A(p_1) \otimes \bar{e}_R(p_2) \otimes \bar{e}_A(p_3) \otimes \bar{e}_A(p_4)
+ \Gamma_{RARR}(p_1, p_2, p_3, p_4) \bar{e}_R(p_1) \otimes \bar{e}_A(p_2) \otimes \bar{e}_R(p_3) \otimes \bar{e}_R(p_4)
+ \Gamma_{AARA}(p_1, p_2, p_3, p_4) \bar{e}_A(p_1) \otimes \bar{e}_A(p_2) \otimes \bar{e}_R(p_3) \otimes \bar{e}_A(p_4)
+ \Gamma_{RARR}(p_1, p_2, p_3, p_4) \bar{e}_R(p_1) \otimes \bar{e}_R(p_2) \otimes \bar{e}_A(p_3) \otimes \bar{e}_A(p_4)
+ \Gamma_{ARAR}(p_1, p_2, p_3, p_4) \bar{e}_A(p_1) \otimes \bar{e}_A(p_2) \otimes \bar{e}_A(p_3) \otimes \bar{e}_R(p_4)
+ \Gamma_{ARRA}(p_1, p_2, p_3, p_4) \bar{e}_A(p_1) \otimes \bar{e}_R(p_2) \otimes \bar{e}_A(p_3) \otimes \bar{e}_R(p_4)
+ \Gamma_{RARA}(p_1, p_2, p_3, p_4) \bar{e}_R(p_1) \otimes \bar{e}_R(p_2) \otimes \bar{e}_A(p_3) \otimes \bar{e}_R(p_4)
+ \Gamma_{ARRA}(p_1, p_2, p_3, p_4) \bar{e}_A(p_1) \otimes \bar{e}_R(p_2) \otimes \bar{e}_A(p_3) \otimes \bar{e}_R(p_4).
$$

(23)

The $R/A$ vertex functions satisfy the general definition

$$
\Gamma_{\alpha_1 \alpha_2 \ldots \alpha_n}(p_1, p_2, \ldots, p_n) = \left( e_{\alpha_1}(p_1) \otimes e_{\alpha_2}(p_2) \otimes \ldots \otimes e_{\alpha_n}(p_n) \right) \bullet \Gamma(p_1, p_2, \ldots, p_n)
$$

(24)

(with $\alpha_i = R, A$) in terms of contractions with the basis vectors \(\mathbf{e}\). Eq. (26) gives the relations
7. Combining (23) with (25) we get

\[
\Gamma_{ARRR}(p_1, p_2, p_3, p_4) = \frac{(1 + n_2)(1 + n_3)(1 + n_4) - n_2n_3n_4 \Gamma^*_{RAAA}(p_1, p_2, p_3, p_4)}{(1 + n_1) - n_1}, \tag{25a}
\]

\[
\Gamma_{RARR}(p_1, p_2, p_3, p_4) = \frac{(1 + n_1)(1 + n_3)(1 + n_4) - n_1n_3n_4 \Gamma^*_{RAAA}(p_1, p_2, p_3, p_4)}{(1 + n_2) - n_2}, \tag{25b}
\]

\[
\Gamma_{RRAR}(p_1, p_2, p_3, p_4) = \frac{(1 + n_1)(1 + n_2)(1 + n_4) - n_1n_2n_4 \Gamma^*_{RAAR}(p_1, p_2, p_3, p_4)}{(1 + n_3) - n_3}, \tag{25c}
\]

\[
\Gamma_{RRRA}(p_1, p_2, p_3, p_4) = \frac{(1 + n_1)(1 + n_2)(1 + n_3) - n_1n_2n_3 \Gamma^*_{AAAR}(p_1, p_2, p_3, p_4)}{(1 + n_1) - n_1}, \tag{25d}
\]

\[
\Gamma_{AARR}(p_1, p_2, p_3, p_4) = \frac{(1 + n_3)(1 + n_4) - n_3n_4 \Gamma^*_{RRAA}(p_1, p_2, p_3, p_4)}{(1 + n_1)(1 + n_2) - n_1n_2}, \tag{25e}
\]

\[
\Gamma_{ARRA}(p_1, p_2, p_3, p_4) = \frac{(1 + n_2)(1 + n_3) - n_2n_3 \Gamma^*_{RAAR}(p_1, p_2, p_3, p_4)}{(1 + n_1)(1 + n_4) - n_1n_4}, \tag{25f}
\]

\[
\Gamma_{ARAR}(p_1, p_2, p_3, p_4) = \frac{(1 + n_2)(1 + n_4) - n_2n_4 \Gamma^*_{RARA}(p_1, p_2, p_3, p_4)}{(1 + n_1)(1 + n_3) - n_1n_3}, \tag{25g}
\]

which reduce the number of independent retarded/advanced 4-point functions to $2^{4-1} - 1 = 7$. Combining (23) with (25) we get

\[
\Gamma(p_1, p_2, p_3, p_4) = \Gamma_{RAAA} \epsilon_R(p_1) \otimes \epsilon_A(p_2) \otimes \epsilon_A(p_3) \otimes \epsilon_A(p_4)
\]

\[+ \frac{(1 + n_2)(1 + n_3)(1 + n_4) - n_2n_3n_4 \Gamma^*_{RAAA} \epsilon_R(p_1) \otimes \epsilon_R(p_2) \otimes \epsilon_R(p_3) \otimes \epsilon_R(p_4)}{(1 + n_1) - n_1} \]

\[+ \Gamma_{ARRA} \epsilon_A(p_1) \otimes \epsilon_R(p_2) \otimes \epsilon_A(p_3) \otimes \epsilon_A(p_4)
\]

\[+ \frac{(1 + n_1)(1 + n_3)(1 + n_4) - n_1n_3n_4 \Gamma^*_{ARRA} \epsilon_R(p_1) \otimes \epsilon_R(p_2) \otimes \epsilon_R(p_3) \otimes \epsilon_R(p_4)}{(1 + n_2) - n_2} \]

\[+ \Gamma_{ARAR} \epsilon_A(p_1) \otimes \epsilon_A(p_2) \otimes \epsilon_R(p_3) \otimes \epsilon_R(p_4)
\]

\[+ \frac{(1 + n_1)(1 + n_2)(1 + n_4) - n_1n_2n_4 \Gamma^*_{ARAR} \epsilon_R(p_1) \otimes \epsilon_R(p_2) \otimes \epsilon_R(p_3) \otimes \epsilon_R(p_4)}{(1 + n_3) - n_3} \]

\[+ \Gamma_{RRAR} \epsilon_R(p_1) \otimes \epsilon_R(p_2) \otimes \epsilon_A(p_3) \otimes \epsilon_A(p_4)
\]

\[+ \frac{(1 + n_3)(1 + n_4) - n_3n_4 \Gamma^*_{RRAR} \epsilon_A(p_1) \otimes \epsilon_A(p_2) \otimes \epsilon_R(p_3) \otimes \epsilon_R(p_4)}{(1 + n_1)(1 + n_2) - n_1n_2} \]

\[+ \Gamma_{RARR} \epsilon_R(p_1) \otimes \epsilon_A(p_2) \otimes \epsilon_A(p_3) \otimes \epsilon_R(p_4)
\]

\[+ \frac{(1 + n_4) - n_4 \Gamma^*_{RARR} \epsilon_A(p_1) \otimes \epsilon_A(p_2) \otimes \epsilon_R(p_3) \otimes \epsilon_R(p_4)}{(1 + n_1)(1 + n_3) - n_1n_3} \]

\[+ \Gamma_{RRRA} \epsilon_R(p_1) \otimes \epsilon_R(p_2) \otimes \epsilon_R(p_3) \otimes \epsilon_A(p_4)
\]
\[
+ \frac{(1 + n_2)(1 + n_4) - n_2 n_4}{(1 + n_1)(1 + n_3) - n_1 n_3} \Gamma_{RARA} \tilde{e}_A(p_1) \otimes \tilde{e}_R(p_2) \otimes \tilde{e}_A(p_3) \otimes \tilde{e}_R(p_4).
\]  

(26)

The generalization is obvious. We have chosen the $R/A$ vertices with the smallest number of $R$ indices as independent $n$-point functions; among them the functions with only a single $R$ index are the fully retarded linear response functions \cite{18}. Eq. (22) permits, however, to select any other set of $2^{n-1} - 1$ independent $n$-point functions and to rewrite Eq. (26) and its generalizations accordingly.

V. CONCLUSIONS

We have given relations for arbitrary $n$ between the real-time thermal $n$-point functions in the single-time representation and the retarded/advanced $n$-point vertex functions introduced by Aurenche and Becherrawy \cite{11}. These relations were expressed in a very compact way in terms of an orthogonal basis of two 2-component column vectors and their dual partners. The advantage of the $R/A$ representation is that it diagonalizes the single particle propagator, shifting the thermal distribution functions from the propagators to the vertices \cite{11}. In our column vector representation the thermal distribution functions appear only inside the column vectors. Their orthogonality properties implement certain relations among the thermal distribution functions which lead to immediate cancellations between various different contributions \cite{13} to complex Feynman diagrams; in other approaches to real-time finite temperature field theory these calculations often occur at a much later stage of the calculation. Thus our representation leads to appreciable simplifications for the Feynman diagram calculus in the real-time formulation of thermal field theory.

Our column vector representation gives the thermal components of the time-ordered $n$-point functions as linear combinations of $2^{n-1} - 1$ independent retarded/advanced $n$-point functions and their complex conjugates. Although we have here considered only bosonic $n$-point functions, the generalization of our results to fermionic fields is straightforward and involves only the simple substitution $n \rightarrow -n_f$ in the column vectors. The useful relation between thermal distribution functions given in the Appendix has been presented for the
general case, too. Similar relations as those given here for the truncated \( n \)-point vertex functions hold for the connected \( n \)-point Green functions; the only change is that in that case the signs of all lower components in the column vectors \( \tilde{e} \) must be reversed.

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**APPENDIX A: A USEFUL RELATION**

Let us write the thermal equilibrium distributions as

\[
  n(x_i) = \frac{1}{e^{x_i} - \eta_i}, \quad n(-x_i) = -\eta_i - n(x_i). \tag{A1}
\]

where \( x_i = \beta p_0^i \), \( \eta_i = 1 \) for bosons, and \( \eta_i = -1 \) for fermions. Defining

\[
  \tilde{n} \left( \sum_{i=1}^{k} x_i \right) = \frac{1}{\exp \left( \sum_{i=1}^{k} x_i \right) - \prod_{i=1}^{k} \eta_i} \tag{A2}
\]

one can prove the identity

\[
  \prod_{i=1}^{k} (1 + \eta_i n(x_i)) - \prod_{i=1}^{k} \eta_i n(x_i) = \frac{\prod_{i=1}^{k} n(x_i)}{\tilde{n} \left( \sum_{i=1}^{k} x_i \right)}. \tag{A3}
\]

It is easy to check the identity for \( k = 2 \) and prove it for \( k > 2 \) by induction.

Due to fermion number conservation, an \( n \)-point function with \( n \) external legs has always an even number of fermionic legs. This implies

\[
  \prod_{i=1}^{n} \eta_i = 1. \tag{A4}
\]
Since momentum conservation also requires

\[ \sum_{i=1}^{n} x_i = 0, \]  

(A5)

Eq. (A3) yields the identity

\[ \prod_{i=1}^{n} (1 + \eta_i n(x_i)) - \prod_{i=1}^{n} n(x_i) = 0 \]  

(A6)

if the product goes over all legs of an \( n \)-point function. Previously used identities like

Eq. (34) in Ref. [15] are simple consequences of Eq. (A6). The physical interpretation of

Eq. (A6) is that the probability for creating and for destroying \( n \) particles in the heat bath are equal.
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This can be seen by comparing the column vector representations of the propagator (2) and self-energy (12) and generalizing from the 2-point to the $n$-point case.

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