TOPOLOGICAL ENTROPY OF MINIMAL GEOIDESICS
AND VOLUME GROWTH ON SURFACES

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ABSTRACT. Let $(M, g)$ be a compact Riemannian manifold of hyperbolic type, i.e. $M$ is a manifold admitting another metric of strictly negative curvature. In this paper we study the geodesic flow restricted to the set of geodesics which are minimal on the universal covering. In particular for surfaces we show that the topological entropy of the minimal geodesics coincides with the volume entropy of $(M, g)$ generalizing work of Freire and Mane.

1. INTRODUCTION AND MAIN RESULTS

Let $(M, g)$ be a compact Riemannian manifold (connected and $\partial M = \emptyset$) and $p: \tilde{M} \to M$ its universal Riemannian covering, saving $\pi: TM \to M$ for the canonical projection. In [10], A. Manning introduced the volume entropy (also called volume growth) $h(g)$ of $(M, g)$ defined by

$$h(g) := \lim_{r \to \infty} \frac{1}{r} \log \text{vol}B(x, r),$$

where $x \in \tilde{M}$ and $B(x, r)$ denotes the open ball with center $x$ and radius $r$. He proved that this limit exists and is independent of $x$. Let $h_{\text{top}}(\phi^t) = h_{\text{top}}(\phi_{SM}^t)$ denote the topological entropy of the geodesic flow $\phi^t$ of $g$ in the unit tangent bundle $SM = \{ g(v, v) = 1 \} \subset TM$. Manning proved the estimate

$$h_{\text{top}}(\phi_{SM}^t) \geq h(g).$$

In the case of nonpositive curvature he showed that equality holds. Subsequently this was generalized by A. Freire and R. Mane [4] to Riemannian metrics without conjugate points.

Let $\mathcal{M}$ be the closed and $\phi^t$-invariant subset of $SM$ consisting of all $v \in SM$ such that the geodesic $c_v$ with $\dot{c}_v(0) = v$ is globally length-minimizing, or simply a minimal geodesic. We denote by $\mathcal{M} = Dp(\tilde{\mathcal{M}})$ the projection of $\tilde{\mathcal{M}}$ to $SM$ and by $\phi_{\mathcal{M}}^t, \phi_{\tilde{\mathcal{M}}}^t$ the geodesic flow restricted to $\mathcal{M}, \tilde{\mathcal{M}}$, respectively. The following theorem of A. Katok and B. Hasselblatt ([7], Theorem 9.6.7) shows that it is enough to consider minimal geodesics to generate exponential complexity (provided $h(g) > 0$).

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Theorem 1.1. Let \((M, g)\) be a compact Riemannian manifold and \(\phi^t_{\mathcal{M}}\) be the geodesic flow \(\phi^t\) restricted to the minimal geodesics \(\mathcal{M} \subset SM\). Then
\[
h_{\text{top}}(\phi^t_{\mathcal{M}}) \geq h(g).
\]

Following W. Klingenberg [8] we say that a compact manifold \(M\) is of hyperbolic type, if there exists a metric of strictly negative curvature \(g_0\) on \(M\). We hope to prove in this case an inequality of the kind \(h_{\text{top}}(\phi^t_{\mathcal{M}}) \leq h(g)\), i.e., that equality holds in the above theorem. A first result in this direction is the following. We will introduce the notation \(h_{\text{top}}(\phi^t, F, \beta)\) and the notion of entropy expansiveness in Section 2.1.

Theorem 1.2. Let \((M, g)\) be a compact Riemannian manifold of hyperbolic type. Then there is a constant \(\beta\) depending only on \((M, g)\), such that for each compact set \(K \subset M\) we have
\[
h_{\text{top}}(\phi^t, \pi^{-1}(K) \cap \tilde{\mathcal{M}}, \beta) \leq h(g).
\]

Using a result of R. Bowen [1], which we shall prove below in the noncompact setting, we obtain the following.

Corollary 1.3. Let \((M, g)\) be a compact Riemannian manifold of hyperbolic type. If \(\phi^t_{\mathcal{M}}\) is \(\beta\)-entropy-expansive with \(\beta\) being the constant from Theorem 1.2, we have
\[
h_{\text{top}}(\phi^t_{\mathcal{M}}) = h(g).
\]

Presently we do not know if \(\phi^t_{\mathcal{M}}\) for Riemannian manifolds \((M, g)\) of hyperbolic type of arbitrary dimension is \(\beta\)-entropy-expansive. We shall prove, however, that in the two-dimensional case, \(\beta\)-entropy-expansiveness holds in the nonwandering set of \(\mathcal{M}\). This gives the following result.

Theorem 1.4. Let \((M, g)\) be a closed Riemannian surface. Then
\[
h_{\text{top}}(\phi^t_{\mathcal{M}}) = h(g).
\]

There are direct generalizations to Finsler metrics, cf. also Remark 4.6.

Note that for the 2-sphere \(S^2\), Theorem 1.4 is trivial (\(S^2\) has finite volume and no minimal geodesics). For the case of the 2-torus \(T^2\), the above theorem is due to E. Glasmachers [5]; here, however, the structure of the minimal geodesics is well-known due to G. A. Hedlund and V. Bangert. The key task in the proof of Theorem 1.4 for higher genus surfaces is to analyze the structure of \(\mathcal{M}\) in a sufficiently large subset to calculate the topological entropy.

The paper is organized as follows. In Section 2 we study topological entropy and local topological entropy for homeomorphisms of metric spaces and following the ideas of Bowen [1] we provide an estimate for the topological entropy. In Section 3, we give a detailed proof using the ideas provided by Katok and Hasselblatt that the topological entropy of the minimal geodesics is bounded below by the volume growth (Theorem 1.1). Moreover, we study the topological entropy of minimal geodesics on manifolds of hyperbolic type and give the proof of Theorem 1.2. Finally, in Section 4 we show that for surfaces the topological entropy of \(\phi^t_{\mathcal{M}}\) equals the volume growth of \(g\) (Theorem 1.4).
2. Topological entropy for homeomorphisms of metric spaces

In this section we study discrete dynamical systems. In order to apply our results to geodesic flows \( \phi^t, t \in \mathbb{R} \), observe that the topological entropy of \( \phi^t \) defined in the continuous setting coincides with that of the discrete system \( \phi^n, n \in \mathbb{Z} \), cf. Proposition 3.1.8 in [7].

2.1. **Bowen’s definition.** Here we recall Bowen’s definition of topological entropy. Let \( f : V \rightarrow V \) be a homeomorphism of a metric space \((V, d)\), not necessarily compact. For each \( n \in \mathbb{N} \), a metric on \( V \) is defined by

\[
d_n(x, y) := \max_{0 \leq j < n} d(f^j(x), f^j(y)).
\]

Let \( F \) be a subset of \( V \). We say that a set \( Y \subset V \) is \((n, \varepsilon)\)-spanning for \( F \) if the closed balls \( B_n(y, \varepsilon) = \{ y \in V : d_n(x, y) \leq \varepsilon \} \) with \( y \in Y \) cover \( F \). If \( Y \subset F \) and \( B_n(y, \varepsilon) \cap Y = \{ y \} \) for all \( y \in Y \), we say that \( Y \) is an \((n, \varepsilon)\)-separated subset of \( F \). Let \( r_n(F, \varepsilon) \) denote the minimal cardinality of \((n, \varepsilon)\)-spanning sets for \( F \) and let \( s_n(F, \varepsilon) \) denote the maximal cardinality of \((n, \varepsilon)\)-separated subsets of \( F \). It is easy to see that for any \( \varepsilon > 0 \) we have

\[
r_n(F, \varepsilon) \leq s_n(F, \varepsilon) \leq r_n(F, \varepsilon/2).
\]

Note that \( r_n(F, \varepsilon) < \infty \), if \( F \) is compact.

We define the following notions of topological entropy.

\[
\begin{align*}
  h_{\text{top}}(f, F, \varepsilon) &:= \limsup_{n \rightarrow \infty} \frac{1}{n} \log r_n(F, \varepsilon), \\
  h_{\text{top}}(f, F) &:= \lim_{\varepsilon \rightarrow 0} h_{\text{top}}(f, F, \varepsilon), \\
  h_{\text{top}}(f) &:= \sup_{F \subset V \text{ compact}} h_{\text{top}}(f, F).
\end{align*}
\]

Note that for any \( \varepsilon > 0 \) we have \( h_{\text{top}}(f, F, \varepsilon) \leq h_{\text{top}}(f, F) \) and if \( V \) is itself compact, we have \( h_{\text{top}}(f) = h_{\text{top}}(f, V) \). If we use \( s_n(F, \varepsilon) \) instead of \( r_n(F, \varepsilon) \), we obtain the same value for \( h_{\text{top}}(f, F) \). For details on topological entropy we refer to [12].

We need the following lesser known concept of local entropy introduced by Bowen [1]. For \( x \in V \) and \( \beta > 0 \) set

\[
Z_\beta(x) := \{ y \in V : d(f^n(x), f^n(y)) \leq \beta \ \forall \ n \in \mathbb{Z} \}.
\]

Then we call

\[
 h_{\text{top,loc}}(f, \beta) := \sup_{x \in V} h_{\text{top}}(f, Z_\beta(x))
\]

the \( \beta \)-local entropy of \( f \). We say that \( f \) is \( \beta \)-entropy-expansive for \( \beta > 0 \) if

\[
 h_{\text{top,loc}}(f, \beta) = 0.
\]
2.2. **An upper bound for the topological entropy of homeomorphisms.** In order to make use of the local entropy it will be important to compute entropy on coverings. We consider the following setting. Let \((\tilde{V}, \tilde{d})\) be a metric space and \(\Gamma\) a subgroup of isometries of \(\tilde{V}\) acting on \(\tilde{V}\). Assume that the quotient \(V := \tilde{V} / \Gamma\) is compact and equipped with a metric \(d\) such that the projection \(p: \tilde{V} \to V\) is a local isometry. Let \(\tilde{f}: \tilde{V} \to \tilde{V}\) be a homeomorphism which commutes with the group \(\Gamma\) and let \(f: V \to V\) be the projection defined by \(f(x) = p\tilde{f}p^{-1}(x)\) (this is well-defined since \(\tilde{f}, \Gamma\) commute). Note that \(f\) is a homeomorphism as well. Recall the following result.

**Proposition 2.1** (Theorem 8.12 in [12]). For each compact set \(K \subset \tilde{V}\) we have
\[
\htop(\tilde{f}, K) = \htop(f, p(K)).
\]
In particular, if \(p(K) = V\), then
\[
\htop(\tilde{f}, K) = \htop(f).
\]

We shall prove the following theorem which is a slight extension of a result of Bowen (see [1]). It allows us to estimate the topological entropy using coverings and will be crucial for our applications.

**Theorem 2.2.** Let \(K \subset \tilde{V}\) be a compact set such that \(p(K) = V\). Then for any \(\beta > 0\) we have
\[
\htop(f) \leq \htop(\tilde{f}, K, \beta) + \htop_{\text{loc}}(\tilde{f}, \beta).
\]

The proof of Theorem 2.2 rests on the following estimate.

**Lemma 2.3.** For any \(\varepsilon > 0, \delta > 0, \beta > 0\) there exists a constant \(c > 0\), such that
\[
r_n\left(\bar{B}_n(x, \beta, \delta)\right) \leq c e^{(a+\varepsilon)n} \quad \forall \ x \in K, \ n \in \mathbb{N},
\]
where \(a := \htop_{\text{loc}}(\tilde{f}, \beta)\).

We need the following elementary lemma (see [1]).

**Lemma 2.4.** Let \(F \subset \tilde{V}\) and consider integers \(0 = t_0 < t_1 < \ldots < t_r = n\). For \(\alpha > 0\) and \(0 \leq i < r\) let \(E_i\) be a \((t_{i+1} - t_i, \alpha)\)-spanning set for \(\tilde{f}^{t_i}(F)\). Then
\[
r_n(F; 2\alpha) \leq \prod_{i=0}^{r-1} \text{card}E_i.
\]

**Proof of Lemma 2.4.** For \((x_0, \ldots, x_{r-1}) \in E_0 \times \cdots \times E_{r-1}\) set
\[
B(x_0, \ldots, x_{r-1}) := \{x \in F \mid d(\tilde{f}^{t_i}(x), \tilde{f}^{t_i}(x_i)) \leq \alpha \ \forall \ 0 \leq i < r, t \in [0, t_{i+1} - t_i] \cap \mathbb{Z}\}.
\]
By assumption the \(B(x_0, \ldots, x_{r-1})\) cover \(F\) and using the triangle inequality we have \(d_n(x, y) \leq 2\alpha\) for all \(x, y \in B(x_0, \ldots, x_{r-1})\). Choosing from each nonempty set \(B(x_0, \ldots, x_{k-1})\) one element we obtain a \((n, 2\alpha)\)-spanning set. This yields the estimate. \(\square\)

**Proof of Lemma 2.3.** In the following fix positive numbers \(\varepsilon, \delta, \beta > 0\), a point \(x \in K\), an integer \(n \in \mathbb{N}\) and set \(F := \bar{B}_n(x, \beta)\). We describe the orbit \(\{x, \tilde{f}x, \ldots, \tilde{f}^{n-1}x\}\) by a finite collection of \(y\)'s in \(K\) and their sets \(Z_\beta(y)\).
Step 1. (Choice of \(y_1, \ldots, y_s \in K\) and appropriate neighborhoods \(V(y_i)\)) By the definition of \(a\) we find for all \(y \in K\) some integer \(m(y) \in \mathbb{N}\) and a \((m(y), \delta/2)\)-spanning set \(E(y)\) for \(Z_\beta(y)\) with

\[
\frac{1}{m(y)} \log \text{card} E(y) \leq a + \varepsilon.
\]

Define the open neighborhoods

\[
U(y) := \bigcup_{z \in E(y)} B_{m(y)}(z, \delta/2) \supset Z_\beta(y), \quad y \in K.
\]

For \(N \to \infty, R \setminus \beta\) the compact sets

\[
W_N(y, R) := \bigcap_{|j| \leq N} \tilde{f}^{-j} \tilde{B}(\tilde{f}^j y, R)
\]

decrease to the compact set \(Z_\beta(y)\). Thus we obtain \(N(y) \in \mathbb{N}, R(y) > \beta\), such that \(W_{N(y)}(y, R(y))\) is contained in the neighborhood \(U(y)\) of \(Z_\beta(y)\). Define

\[
V(y) := \text{Int} W_{N(y)}(y, R(y) - \beta), \quad y \in K.
\]

The triangle inequality implies that

\[
(\ast) \quad \forall z \in V(y) : \quad W_{N(y)}(z, \beta) \subset W_{N(y)}(y, R(y)) \subset U(y).
\]

By the compactness of \(K\) and \(p(K) = V\) we find \(y_1, \ldots, y_s \in K\) with

\[
\tilde{V} = \bigcup_{y \in \Gamma} \bigcup_{i=1}^s y V(y_i).
\]

Set

\[
n_0 := \max_{1 \leq i \leq s} \max \{N(y_i), m(y_i)\} \in \mathbb{N}.
\]

Step 2. (Description of \(F\) by the \(y_i\)’s) We claim the following:

\[
(\ast\ast) \quad \forall t \in [n_0, n - n_0] \cap \mathbb{Z} \exists i \in \{1, \ldots, s\}, \gamma \in \Gamma : \quad \tilde{f}^t(F) \subset \gamma U(y_i).
\]

**Proof of the claim.** We find \(\gamma, i\) with \(\tilde{f}^t x \in \gamma V(y_i)\), and hence

\[
\tilde{f}^t(F) = \bigcap_{j=0}^{n-1} \tilde{f}^{-j} \tilde{B}(\tilde{f}^j x, \beta) = \bigcap_{j=-t}^{n-t-1} \tilde{f}^{-j} \tilde{B}(\tilde{f}^j \tilde{f}^t x, \beta) \subset \bigcap_{j=-n_0}^{n_0} \tilde{f}^{-j} \tilde{B}(\tilde{f}^j \tilde{f}^t x, \beta)
\]

\[
= W_{n_0}(\tilde{f}^t x, \beta) = \gamma W_{n_0}(\gamma^{-1} \tilde{f}^t x, \beta) \subset \gamma U(y_i),
\]

where in the second line we used \(\Gamma \subset \text{Iso}(\tilde{V}, d)\) and that \(\tilde{f}\) commutes with \(\Gamma\), as well as \((\ast)\) and \(n_0 \geq N(y_i)\). \(\square\)
We obtain using the definition of $m$ and define $\varepsilon$. Letting $n$ using only $\varepsilon c$ and by Lemma 2.3 each of the sets in the above union can be $(n, \delta)$-spanning. Observe that $(n, \delta)$ is also $(t, \delta)$-spanning for $\tilde{f}^{i}(F)$.

Eventually we are in case (a) and the process stops. Moreover we have $t_{r-2} < n - n_{0} \leq t_{r-1} < n = t_{r}$ by $m(\gamma_{i_{r-2}}) \leq n_{0}$.

Note that $t_{k+1} = t_{k} \leq n_{0}$ for $k = 0, r - 1$ and by $(\ast \ast)$ the set $\gamma_{k}E(y_{i_{k}})$ is $(t_{k+1} - t_{k}, \delta/2)$-spanning for $\tilde{f}^{i_{k}}(F)$ for $k = 1, \ldots, r - 2$. Choose $E_{0}$, $E_{r-1}$ to be $(n_{0}, \delta/2)$-spanning for $\tilde{B}(x, \beta), \tilde{B}(\tilde{f}^{t_{r-1}}x, \beta)$, respectively of minimal cardinality, so $E_{0}$ is also $(t_{1} - t_{0}, \delta/2)$-spanning for $F$ and $E_{r-1}$ is also $(t_{r} - t_{r-1}, \delta/2)$-spanning for $\tilde{f}^{t_{r-1}}(F)$. Apply Lemma 2.4 to

$$E_{0}, E_{1} := \gamma_{1}E(y_{i_{1}}), \ldots, E_{r-2} := \gamma_{r-2}E(y_{i_{r-2}}), E_{r-1}$$

and define

$$\sqrt{c} := \sup_{y \in \mathcal{K}} r_{n_{0}}(\tilde{B}(y, \beta), \delta/2) < \infty.$$ 

We obtain using the definition of $m(\gamma_{i})$ and $\sum_{k=1}^{r-2} m(\gamma_{i_{k}}) \leq n - n_{0} \leq n$ that

$$r_{n}(F, \delta) \leq \text{card}E_{0} \cdot \left( \prod_{k=1}^{r-2} \text{card}E_{k} \right)^{\text{card}E_{r-1}} \leq c \cdot \sum_{k=1}^{r-2} \text{card}E(\gamma_{i_{k}}) \leq c \cdot \prod_{k=1}^{r-2} e^{(a+\varepsilon)m(\gamma_{i_{k}})} \leq c \cdot e^{(a+\varepsilon)n}.$$ 

Observe that $c$ depends only on $\delta, n_{0}, \beta$ and $n_{0}$ in turn is independent of $x, n$. \hfill $\square$

Now we are able to prove the theorem.

**Proof of Theorem 2.2.** Let $E_{n}$ be a minimal $(n, \beta)$-spanning set for $K$ and let $\varepsilon, \delta > 0$. Then

$$K \subset \bigcup_{x \in E_{n}} \tilde{B}_{n}(x, \beta),$$

and by Lemma 2.3 each of the sets in the above union can be $(n, \delta)$-spanned by using only $c e^{(a+\varepsilon)n}$ elements where $a = h_{\text{top}, \text{loc}}(\tilde{f}, \beta)$. Hence

$$r_{n}(K, \delta) \leq \text{card}E_{n} \cdot c e^{(a+\varepsilon)n} \leq r_{n}(K, \beta) \cdot c e^{(a+\varepsilon)n}$$

and

$$h_{\text{top}}(\tilde{f}, K, \delta) \leq h_{\text{top}}(\tilde{f}, K, \beta) + a + \varepsilon.$$ 

Letting $\varepsilon, \delta \to 0$, the claim follows using Proposition 2.1. \hfill $\square$
3. Bounds for topological entropy

3.1. Lower bound. We need the following theorem from the book [7] of Katok and Hasselblatt on the topological entropy of minimal geodesics on Riemannian manifolds. For the convenience of the reader we will provide here a complete proof of the result, which differs from the one in [7] in some details. Recall the notation $p: M \to \tilde{M}$ for the universal cover of $M$ and

\[ \mathcal{M} = \{ v \in \tilde{M} : \text{c}_v \text{ is a minimal geodesic } \} \subset \tilde{M}, \]
\[ \mathcal{M} = Dp(\mathcal{M}) \subset SM. \]

As usual, geodesics are parametrized by arc-length, unless otherwise specified.

**Theorem 3.1.** Let $(M, g)$ be a compact Riemannian manifold and $\phi_{\mathcal{M}}^t$ be the geodesic flow $\phi^t$ restricted to $\mathcal{M} \subset SM$. Then

\[ h_{\text{top}}(\phi_{\mathcal{M}}^t) \geq h(g). \]

For the proof of Theorem 3.1 we need a lemma similar to Lemma 2.4. Recall that $s_T(A, \delta)$ denotes the maximal cardinality of a $(T, \delta)$-separated subset of $A$.

**Lemma 3.2.** Let $(V, d)$ be a metric space, $\phi^t: V \to V$ a continuous flow and $A \subset V$. For times $0 = t_0 < t_1 < \cdots < t_m = T$ and $\delta > 0$ we have

\[ \prod_{i=1}^{m} s_{t_i - t_{i-1}}(\phi^{t_{i-1}}A, \delta) \geq s_T(A, 2\delta). \]

**Proof of Lemma 3.2.** Let $L$ be a maximal $(T, 2\delta)$-separated subset of $A$ and let $L_i$ be maximal $(t_i - t_{i-1}, \delta)$-separated subsets of $\phi^{t_{i-1}}(A)$ for $i = 1, \ldots, m$. For $(x_1, \ldots, x_m) \in L_1 \times \cdots \times L_m$ set

\[ B(x_1, \ldots, x_m) := \{ z \in L \mid d(\phi^{t_1 + s_{t_i - t_{i-1}}}z, \phi^{t_i}x_i) \leq \delta \forall 1 \leq i \leq m, t \in [0, t_i - t_{i-1}) \}. \]

Since $L$ is $(T, 2\delta)$-separated, the triangle inequality implies $\text{card}B(x_1, \ldots, x_m) \leq 1$. Therefore, since the cardinalities of the $L_i$ are maximal implying that they are also $(t_i - t_{i-1}, \delta)$-spanning,

\[ \text{card}L = \text{card}\left( \bigcup_{(x_1, \ldots, x_m)} B(x_1, \ldots, x_m) \right) \leq \prod_{i=1}^{m} \text{card}L_i. \]

**Proof of Theorem 3.1.** Fix $x \in \tilde{M}$, $\epsilon > 0$ and write

\[ \delta := \text{inj}(M) > 0, \quad \epsilon := h(g), \quad a := \sup_{y \in \tilde{M}} \text{vol}B(y, 2\delta), \quad b := h_{\text{top}}(\phi_{SM}^t) \]

We have the following: there exists a sequence $T_k \to \infty$ such that

\[ \text{vol}B(x, T_k + \delta/2) - \text{vol}B(x, T_k) \geq e^{h(1-\epsilon)T_k}, \]

for otherwise adding up the volume of the annuli $B(x, T_k + \delta/2) \sim B(x, T_k)$ with $T_{k+1} = T_k + \delta/2$ starting at $T_0$ sufficiently large would yield that the exponential growth rate is less than $h \cdot (1 - \epsilon)$.
Let $N_k$ be a maximal $2\delta$-separated set in the annulus $B(x, T_k + \delta/2) \sim B(x, T_k)$, then we have for all $k \in \mathbb{N}$
\[ a \cdot \text{card} N_k \geq \text{vol} \left( \bigcup_{y \in N_k} B(y, 2\delta) \right) \geq \text{vol} B(x, T_k + \delta/2) - \text{vol} B(x, T_k) \geq e^{h(1-\epsilon)T_k}. \]
For $y \in N_k$ let $c_y : [0, d(x, y)] \to \tilde{M}$ be a minimal geodesic segment with $c(0) = x$ and $c(d(x, y)) = y$. Now, if $y_1, y_2 \in N_k$ with $y_1 \neq y_2$ we have
\[ d(c_{y_1}(T_k), c_{y_2}(T_k)) \geq d(y_1, y_2) - d(y_1, c_{y_1}(T_k)) - d(y_2, c_{y_2}(T_k)) > \delta, \]
so the sets
\[ \mathcal{S}_k := \{ c_y(0) : y \in N_k \} \]
are $(T_k, \delta)$-separated with respect to the metric $d_1$ on $SM$, defined as
\[ d_1(v, w) = \max_{t \in [0, 1]} d(c_v(t), c_w(t)). \]
In $SM$ the sets $S_k := Dp(\mathcal{S}_k)$ are $(T_k, \delta/2)$-separated. Define the decreasing sequence of compact sets
\[ \mathcal{M}_k := Dp \left\{ v \in SM : c_v : [-\sqrt{T_k}, \sqrt{T_k}] \to \tilde{M} \text{ is minimal} \right\}, \quad \bigcap_{k \in \mathbb{N}} \mathcal{M}_k = \mathcal{M}. \]
In order to find large separated sets in $\mathcal{M}$ we shall find them in the sets $\mathcal{M}_k$, observing that for $t \in [\sqrt{T_k}, T_k - \sqrt{T_k}]$ we have
\[ \phi^t S_k \subset \mathcal{M}_k. \]
Assume $k$ is large enough, such that
\[ s_{\sqrt{T_k}}(S_k, \delta/4) \leq e^{2b\sqrt{T_k}}, \quad \sqrt{T_k} \geq \frac{2b}{\epsilon h}. \]
We apply Lemma 3.2 and obtain
\[ s_{T_k - \sqrt{T_k}}(\phi^{\sqrt{T_k}T}S_k, \delta/4) \cdot s_{\sqrt{T_k}}(S_k, \delta/4) \geq s_{T_k}(S_k, \delta/2) \geq \text{card} N_k \geq \frac{1}{a} e^{h(1-\epsilon)T_k}, \]
showing that
\[ s_{T_k - \sqrt{T_k}}(\phi^{\sqrt{T_k}T}S_k, \delta/4) \geq \frac{1}{a} e^{h(1-\epsilon)T_k - 2b\sqrt{T_k}} \geq \frac{1}{a} e^{h(1-2\epsilon)T_k}. \]
Let now
\[ T \in (0, T_k - \sqrt{T_k}), \quad m_k = \left\lfloor \frac{T_k - \sqrt{T_k}}{T} \right\rfloor \in \mathbb{N}. \]
Applying Lemma 3.2 again gives
\[ \left( \prod_{i=0}^{m_k-1} s_T(\phi^{iT+\sqrt{T_k}T}S_k, \delta/8) \right) \cdot s_{T_k - \sqrt{T_k} - m_k T}(\phi^{m_k T + \sqrt{T_k}T}S_k, \delta/8) \geq s_{T_k - \sqrt{T_k}}(\phi^{\sqrt{T_k}T}S_k, \delta/4) \geq \frac{1}{a} e^{h(1-2\epsilon)T_k}, \]
and hence
\[
\prod_{i=0}^{m_k-1} s_T(\phi_i T + \sqrt{\tau_k} S_k, \delta / 8) \geq \left( \frac{1}{a} e^{h(1-2e)T_k} \right)^{s_{T_k-m_k} T(\phi^{m_k}_{T + \sqrt{\tau_k} S_k}, \delta / 8)} \geq \frac{1}{a} e^{h(1-2e)T} \geq \frac{1}{a} e^{h(1-2e)T_k - 2bT_k},
\]
where in the last step we assumed that \( T \) is large, so that \( s_T(SM, \delta / 8) \leq e^{2bT} \).

Hence one of the factors in the last product has to be “large”, i.e., for some \( i \in \{0, \ldots, m_k - 1\} \) we have

\[
s_T(\phi_i T + \sqrt{\tau_k} S_k, \delta / 8) \geq \frac{1}{a} e^{h(1-2\varepsilon)T} \geq \frac{1}{a} e^{h(1-2\varepsilon)T} e^{\frac{2bT}{m_k}}.
\]

Note also that \( \phi_i T + \sqrt{\tau_k} S_k \subset \mathcal{M}_k \), so when letting \( k \to \infty \) while fixing \( T \) and using \( m_k \to \infty \), we find a \((T, \delta / 8)\)-separated set in \( \mathcal{M} = \bigcap_k \mathcal{M}_k \) of cardinality at least

\[
\frac{1}{a} e^{h(1-2\varepsilon)T} \lim_{k \to \infty} e^{\frac{2bT}{m_k}} = \frac{1}{a} e^{h(1-2\varepsilon)T}.
\]

This proves the theorem:

\[
h_{\text{top}}(\phi^f_{\mathcal{M}_k}) \geq h_{\text{top}}(\phi^f_{\mathcal{M}_k}, \delta / 8) \geq h - 2\varepsilon.
\]

3.2. **Upper bound for manifolds of hyperbolic type.** Recall that a compact Riemannian manifold \((M, g)\) is of hyperbolic type, if there exists a metric of strictly negative curvature \( g_0 \) on \( M \); from now on we assume the existence of such a fixed \( g_0 \) on the compact manifold \( M \). When we lift objects such as \( g, g_0 \) from \( M \) to the universal cover \( \tilde{M} \) we will frequently denote them by the same letters. In the following we write \( d \) for the metric on \( \tilde{M} \) induced by \( g \) and \( d_{g_0} \) for the one induced by the background metric \( g_0 \). Due to the compactness of \( M \) the two metrics on \( \tilde{M} \) are equivalent, i.e., there exists a constant \( C > 0 \) such that

\[
\frac{1}{C} d(p, q) \leq d_{g_0}(p, q) \leq Cd(p, q) \quad \forall p, q \in \tilde{M}.
\]

We write \( d_1 \) for the metric on \( S\tilde{M} \) defined by

\[
d_1(v, w) := \max_{t \in [0,1]} d(c_v(t), c_w(t))
\]

and \( d_H(A, B) \) for the Hausdorff metric on sets \( A, B \subset \tilde{M} \) with respect to \( d \).

The following theorem is fundamental for the study of \( \mathcal{M} \) in manifolds of hyperbolic type. It was proved by Morse in dimension 2 and by Klingenberg in arbitrary dimensions.

**Theorem 3.3** (Morse Lemma, cf. [8] or [9]). Let \((M, g)\) be a manifold of hyperbolic type. Then there is a constant \( r_0 = r_0(g, g_0) > 0 \) with the following properties.

(i) If \( c: [a, b] \to \tilde{M} \) and \( \alpha: [a_0, b_0] \to \tilde{M} \) are minimizing geodesic segments with respect to \( g, g_0 \), respectively, joining \( c(a) = \alpha(a_0) \) to \( c(b) = \alpha(b_0) \), then

\[
d_H(c[a, b], \alpha[a_0, b_0]) \leq r_0.
\]
(ii) For any minimizing $g$-geodesic $c : \mathbb{R} \to \tilde{M}$ there is a $g_0$-geodesic $\alpha : \mathbb{R} \to M$ and conversely for any $g_0$-geodesic $\alpha : \mathbb{R} \to \tilde{M}$ a minimizing $g$-geodesic $c : \mathbb{R} \to \tilde{M}$ with

$$d_H(\alpha(\mathbb{R}), c(\mathbb{R})) \leq r_0.$$ 

In this subsection we prove the following theorem stated in the introduction as Theorem 1.2. As a consequence we immediately obtain Corollary 1.3 in the introduction using the results in Subsection 2.2.

**Theorem 3.4.** Let $(M, g)$ be a compact Riemannian manifold of hyperbolic type and $K \subset \tilde{M}$ a compact set in the universal cover $\tilde{M}$. Let

$$\mathcal{F} = SK \cap \hat{\mathcal{A}},$$

where $SK = \pi^{-1}(K) \cap S\tilde{M}$. Then there is some constant $\beta = \beta(g, g_0)$ such that

$$h_{top}(\phi^t, \mathcal{F}, \beta) \leq h(g).$$

In order to prove the theorem, we construct spanning sets for $\mathcal{F}$. Let $K \subset \tilde{M}$ be a compact set with diam $K = a$. For $r > a$ consider

$$K_r := \{z \in \tilde{M} : r - a \leq d(z, K) \leq r\}.$$

Let $K^e_r, K^c_r$ be minimal $\varepsilon$-spanning sets for $K, K_r$, respectively. For $y \in K^e_r, z \in K^c_r$, let $\alpha_{yz} : \mathbb{R} \to \tilde{M}$ be the $g_0$-geodesic connecting $y$ and $z$ such that $\alpha_{yz}(0) = y$ and $\alpha_{yz}(d_{g_0}(y, z)) = z$. By the Morse Lemma, there exists a (unit speed) minimizing $g$-geodesic $c_{yz} : \mathbb{R} \to \tilde{M}$ $r_0$-close to $\alpha_{yz}(\mathbb{R})$. Set

$$P_r := \{c_{yz}(0) : y \in K^e_r, z \in K^c_r\} \subset \hat{\mathcal{A}}.$$

**Lemma 3.5.** $P_r$ is a $(r - 1, \beta)$-spanning set for $\mathcal{F}$ with respect to the metric $d_1$ where $\beta$ is given by $\beta := 5r_0 + (2C^2 + 1)\varepsilon$.

**Proof of Lemma 3.5.** Let $c : \mathbb{R} \to \tilde{M}$ be a minimizing $g$-geodesic with $c(0) \in K$. Then $c(r) \in K_r$ and we can choose $y \in K^e_r, z \in K^c_r$ with

$$d(y, c(0)) \leq \varepsilon, \quad d(z, c(r)) \leq \varepsilon.$$

Let $a$ be the $g_0$-geodesic connecting $c(0)$ and $c(r)$ parametrized such that $a(0) = c(0)$ and $a(d_{g_0}(y, z)) = c(r)$. By convexity of the function $t \mapsto d_{g_0}(a(t), a_{yz}(t))$ due to nonpositive curvature we find

$$d_{g_0}(a(t), a_{yz}(t)) \leq \max\{d_{g_0}(c(0), y), d_{g_0}(c(r), z)\} \leq C\varepsilon \quad \forall t \in [0, d_{g_0}(y, z)].$$

Let $A = c[0, r]$ and $B = c_{yz}[0, r']$ be the subsegment of $c_{yz}$ lying $r_0$-close to $\alpha_{yz}[0, d_{g_0}(y, z)]$ with respect to the $g$-Hausdorff metric $d_H$. ($c_{yz}$ is a minimal geodesic in the $r_0$-tube around $\alpha_{yz}$, showing that $B$ exists). Using the Morse Lemma we find (omitting for the moment the intervals $[0, d_{g_0}(y, z)]$ for $a, a_{yz}$)

$$d_H(A, B) \leq d_H(A, a) + d_H(a, a_{yz}) + d_H(a_{yz}, B) \leq 2r_0 + C^2\varepsilon.$$

By definition of the Hausdorff distance, for $t \in [0, r]$ there is some $t' \in \mathbb{R}$ with $d(c(t), c_{yz}(t')) \leq 2r_0 + C^2\varepsilon$. By minimality of $c, c_{yz}$ we find with $d(c(0), c_{yz}(0)) \leq r_0 + \varepsilon$ that $|t - t'| \leq 3r_0 + (C^2 + 1)\varepsilon$ and hence

$$d(c(t), c_{yz}(t)) \leq d(c(t), c_{yz}(t')) + d(c_{yz}(t'), c_{yz}(t)) \leq 2r_0 + C^2\varepsilon + 3r_0 + (C^2 + 1)\varepsilon.$$
Therefore, taking $\beta := 5r_0 + (2C^2 + 1)\varepsilon$ we obtain

\[ d_1(\dot{c}(t), \dot{c}_{yz}(t)) = \max_{s \in [0,1]} d(c(t + s), c_{yz}(t + s)) \leq \beta \quad \forall t \in [0, r - 1]. \]

We can now prove the theorem.

**Proof of Theorem 3.4.** We have for any $x \in K$

\[ \text{card} K^\varepsilon_r \leq C_\varepsilon \cdot \text{vol} B(x, r + a + \varepsilon/2), \]

where $C_\varepsilon := \left( \inf_{y \in M} \text{vol} B(y, \varepsilon/2) \right)^{-1}$, which implies that

\[ \text{card} P_r \leq \text{card} K^\varepsilon_r \cdot \text{card} K^\varepsilon_r \leq C_\varepsilon \cdot \text{vol} B(x, r + a + \varepsilon/2). \]

Hence

\[
\begin{align*}
    h_{\text{top}}(\phi^t, \mathcal{F}, \beta) &\leq \lim_{r \to \infty} \frac{1}{r - 1} \log \text{card} P_r \\
&\leq \lim_{r \to \infty} \frac{1}{r - 1} \log \text{vol} B(x, r + a + \varepsilon/2) \\
&= \lim_{r \to \infty} \frac{1}{r - 1} \frac{r + a + \varepsilon/2}{r + a + \varepsilon/2} \log \text{vol} B(x, r + a + \varepsilon/2) \\
&= h(g).
\end{align*}
\]

4. THE TWO-DIMENSIONAL CASE

In this section we prove $h_{\text{top}}(\phi^t, \mathcal{F}, \beta) \leq h(g)$ for closed, orientable Riemannian surfaces $(M, g)$. Let us first discuss Theorem 3.4 in the case of genus 1, i.e., $M$ is the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Here the background metric is euclidean. Observe that in the proof of Lemma 3.5, we only used the convexity of the background distance function, which also holds in the flat torus (flat $\mathbb{R}^2$, respectively), and the Morse Lemma. The analog of the Morse Lemma holds on $T^2$ as well, due to Hedlund (cf. [6], Lemma 7.1). Since on the 2-sphere, $\mathcal{M} = \emptyset$, and surfaces of genus $\geq 2$ are of hyperbolic type, Theorem 3.4 follows for all orientable closed surfaces.

The next step will be to apply Theorem 2.2, i.e., we have to show that the local entropy $h_{\text{top}, \text{loc}}(\phi^t, \beta)$ vanishes for all $\beta > 0$, which requires knowledge of the structure in strips of width $\beta$ in $\mathcal{M}$. This is the agenda of the next subsection, where the assumption $\dim M = 2$ becomes essential.

4.1. **Structure of the minimals.** Let $M$ be a closed orientable surface of genus $\geq 1$ with universal cover $\tilde{M}$ diffeomorphic to $\mathbb{R}^2$. As a Riemannian background metric on $M$ we chose some metric of constant curvature $-1$ (genus $\geq 2$) or 0 (genus = 1), respectively.

By the Morse Lemma (which also holds for $M = T^2$ as noted above), we can assign to each minimal geodesic $c: \mathbb{R} \to \tilde{M}$ a geodesic $\alpha$ (unparametrized, but oriented) with respect to the background metric, such that $c(\mathbb{R})$ lies in bounded distance from $\alpha$. Hence, each minimal geodesic $c_v(\mathbb{R}) \subset \tilde{M}$ with $v \in \mathcal{M}$ divides $\tilde{M}$ into two connected components, that we denote by $\tilde{M}^+(v), \tilde{M}^-(v)$,
with $\tilde{M}^+(v)$ lying left of $c_v$ with respect to the orientation of $c_v$ given by its parametrization.

Moreover, we let $\mathcal{G}$ be the set of unparametrized, but oriented geodesics with respect to the background metric and for $\alpha \in \mathcal{G}$ write $\tilde{\mathcal{U}}_\alpha \subset \tilde{\mathcal{M}} \subset S\tilde{M}$ for the set of $v \in \tilde{\mathcal{M}}$, such that $c_v$ lies in finite distance from $\alpha$, moving to $\infty$ in the same direction as $\alpha$.

**Definition 4.1.** For $\alpha \in \mathcal{G}$ set

$\tilde{\mathcal{U}}_\alpha^+ := \{ v \in \tilde{\mathcal{U}}_\alpha : \forall \omega \in \tilde{\mathcal{U}}_\alpha : \pi \omega \in c_v(\mathbb{R}) \Rightarrow c_v[0, \infty) \subset \tilde{M}^+(v) \}$,

$\tilde{\mathcal{U}}_\alpha^- := \{ v \in \tilde{\mathcal{U}}_\alpha : \forall \omega \in \tilde{\mathcal{U}}_\alpha : \pi \omega \in c_v(\mathbb{R}) \Rightarrow c_v[0, \infty) \subset \tilde{M}^-(v) \}$,

$\tilde{\mathcal{U}}_\alpha := \tilde{\mathcal{U}}_\alpha^+ \cup \tilde{\mathcal{U}}_\alpha^-$,  $\mathcal{M} := \bigcup_{\alpha \in \mathcal{G}} \tilde{\mathcal{U}}_\alpha^0 \subset \tilde{\mathcal{M}}$,  $\mathcal{M}^0 := \partial \mathcal{M}$.  

**Remark 4.2.**

(i) It follows from the definition that no two geodesics from $\tilde{\mathcal{U}}_\alpha^+$ (resp. $\tilde{\mathcal{U}}_\alpha^-$) intersect transversely in $\tilde{M}$.

(ii) One can show (but we will not use) that $\mathcal{M}^0$ is closed and $\phi^t$-invariant.

(iii) If $M$ has genus $\geq 2$, then it is known that $\tilde{\mathcal{U}}_\alpha^\pm$ are never empty. In fact, the intersection $\tilde{\mathcal{U}}_\alpha^- \cap \tilde{\mathcal{U}}_\alpha^+$ contains the velocity vectors of the bounding geodesics of $\tilde{\mathcal{U}}_\alpha$ (cf. Theorem 8 in [11]). For $M = T^2$, this follows from Proposition 4.3.

By Remark 4.2 (i), the sets $\tilde{\mathcal{U}}_\alpha^0$ have a simple structure in $\tilde{M}$, so when calculating $h_{top}(\phi^t)$ we would like to stick to $\mathcal{M}^0$. For this it is important that $\mathcal{M}^0$ is “sufficiently large”.

**Proposition 4.3.** If $M$ is the 2-torus $T^2$, then $\mathcal{M}^0 = \mathcal{M}$.

**Proof.** If $\alpha \in \mathcal{G}$ is periodic, then $\tilde{\mathcal{U}}_\alpha = \tilde{\mathcal{U}}_\alpha^+ \cup \tilde{\mathcal{U}}_\alpha^-$, which follows from the results of Morse and Hedlund, cf. Theorems XI, XIII, XV and XVII in [6].

If $\alpha \in \mathcal{G}$ is not periodic, then $\tilde{\mathcal{U}}_\alpha = \tilde{\mathcal{U}}_\alpha^- \cap \tilde{\mathcal{U}}_\alpha^+$, which follows from the results of Bangert, cf. Theorem (6.9) in [2].

For higher-genus surfaces, the statement of Proposition 4.3 is not at all clear to us. However, we have to following sufficient statement.

Let $\Omega \subset SM$ denote the nonwandering set of $\phi^t$ restricted to $\tilde{\mathcal{M}}$.

**Theorem 4.4.** Let $M$ have genus $\geq 2$. Then $\mathcal{M}^0 \subset SM$ contains the nonwandering set $\Omega$ of $\phi^t$.

**Proof.** Let $\tilde{\Omega} = D\phi^{-1}(\Omega) \subset \tilde{\mathcal{M}}$ be the lifted nonwandering set of $\phi^t$ and $\Gamma$ be the group of deck transformations of the covering $p: \tilde{M} \rightarrow M \equiv \tilde{M}/\Gamma$. Choose any $v \in \tilde{\Omega} \cap \tilde{\mathcal{U}}_\alpha$ (such $\alpha \in \mathcal{G}$ exist for any $v \in \tilde{\mathcal{M}}$ by the Morse Lemma) and $U_n = B(v, 1/n) \cap \tilde{\mathcal{U}} \subset S\tilde{M}$ for $n \in \mathbb{N}$. By definition of $\Omega$ there exists $\gamma_n \in \Gamma - \{ id \}$ and $t_n > 0$ such that $D\gamma_n \phi^{t_n} U_n \cap U_n \neq \emptyset$. In particular there is some $v_n \in U_n$ such that $w_n := D\gamma_n \phi^{t_n} v_n \in U_n$. Assume $v \notin \tilde{\mathcal{U}}_\alpha^0$, so there are two minimals $c^\pm : \mathbb{R} \rightarrow \tilde{M}$ in $\tilde{\mathcal{U}}_\alpha$ with $c^\pm(0) = c_v(t^\pm)$ and $c^\pm(0, \infty) \subset \tilde{M}^\pm(v)$. 


We assume for intuition the setting of $\tilde{M} = \{z \in \mathbb{C} : |z| < 1\}$ (the Poincaré disk model), where $\tilde{M}(\infty) \cong S^1$ is the boundary at infinity of $\tilde{M}$. Every $\gamma \in \Gamma$ extends naturally to $\tilde{M}(\infty)$. Moreover, every $\alpha \in \mathcal{G}$ and with $\alpha$ all geodesics $c_\nu, \nu \in \tilde{\mathcal{M}}_\alpha$ have two limit points $\alpha(\pm \infty) = c_\nu(\pm \infty) \in \tilde{M}(\infty)$.

Suppose first that $c_{\nu_0}(\infty) = c_{\nu_0}(\infty) = \alpha(\infty) \in \tilde{M}(\infty)$ for some $n$. Then $\alpha(\infty) = c_{\nu_0}(\infty) = \gamma_n c_{\nu_0}(\infty) = \gamma_n \alpha(\infty)$, so $\alpha(\infty)$ is the point at $+\infty$ for some periodic minimal axis of $\gamma_n$. If $c_\nu$ is itself periodic, Theorems 10 and 13 in [11] show that in fact there are no minimal geodesics in $\tilde{\mathcal{M}}_\alpha$ intersecting $c_\nu$ transversely, i.e., $\nu \in \tilde{\mathcal{M}}^0_\alpha$. If $c_\nu$ is not periodic, it is asymptotic to some periodic minimal $c_0$ in $+\infty$, approaching its limit from “the right side” (i.e., from $\tilde{M}^+(c_0)$), say, by Theorem 10 in [11]. Now $c^+$ is also asymptotic in $+\infty$ to that same minimal $c_0$ (Theorem 13 in [11]). But two asymptotic minimal in $\tilde{M}$ cannot intersect transversely (Theorem 6 in [11]), so we obtain a contradiction.

Assume now that $c_{\nu_n}(\infty) \neq c_{\nu_n}(\infty)$ for all $n \in \mathbb{N}$. Interchanging $\nu_n, \nu_n$ and maybe taking a subsequence, we may assume that $\nu_n \rightarrow \nu$ and $c_{\nu_n}(\infty) \neq \alpha(\infty)$ for all $n$. Moreover we can assume that the $c_{\nu_n}(\infty)$ lie in one connected component of $\tilde{M}(\infty) - \{\alpha(-\infty), \alpha(\infty)\}$, say $c_{\nu_n}(\infty) \in \tilde{M}(\infty) \cap \tilde{M}^+(\nu)$. Now, by $c_{\nu_n}(t^+) \rightarrow c_\nu(t^+)$ and the assumptions on the points at infinity of $c_{\nu_n}, c^+$, there have to be two intersections of $c_{\nu_n}, c^+$ for large $n$, contradicting the minimality of both geodesics.

\[ 4.2. \textbf{Entropy in strips of finite width.} \text{ In this section we will show that the local entropy of the geodesic flow in the nonwandering set } \Omega \subset \mathcal{M}^0 \text{ of } \phi^t_{\tilde{\mathcal{M}}} \text{ is vanishing. We work in the universal cover and write } \tilde{\Omega} := \text{D}^0 \tilde{\Omega} \subset S\tilde{M} \text{ for the lifted nonwandering set of } \phi^t_{\tilde{\mathcal{M}}}. \text{ Recall} \]

\[ d_1(v, w) = \max_{t \in [0,1]} d(c_\nu(t), c_\nu(t)) \quad v, w \in \tilde{S}\tilde{M}, \]

\[ Z_\beta(v) = \{w \in \tilde{\Omega} : d(c_\nu(t), c_\nu(t)) \leq \beta \forall t \in \mathbb{R} \subset \tilde{S}\tilde{M}, \quad v \in \tilde{\Omega}. \]

\textbf{Lemma 4.5.} \text{Let } M \text{ be a closed orientable surface of genus } \geq 1, \text{ and } \tilde{\Omega} \subset \tilde{\mathcal{M}} \text{ the lifted nonwandering set of } \phi^t_{\tilde{\mathcal{M}}}. \text{ Then for any } v_0 \in \tilde{\Omega} \text{ and any } \beta > 0 \text{ we have} \]

\[ h_{\text{top}}(\phi^t_{\tilde{\Omega}}, Z_\beta(v_0)) = 0. \]

\textit{Hence the geodesic flow restricted to } $\tilde{\Omega}$ \textit{is } $\beta$-entropy-expansive for any $\beta > 0$.

\textbf{Proof.} \text{Fix } v_0 \in \tilde{\Omega}, \beta > 0 \text{ and some small } \delta > 0. \text{ By Proposition 4.3 and Theorem 4.4 we find } \alpha \in \mathcal{G} \text{ with } v_0 \in \tilde{\mathcal{M}}^0_\alpha \text{ and hence } Z_\beta(v_0) \subset \tilde{\mathcal{M}}^0_\alpha. \text{ We shall prove that } (T - 1, 2\delta)-\text{spanning sets } E \text{ of minimal cardinality for } Z_\beta(v_0) \cap \tilde{\mathcal{M}}^+ \text{ have cardinality growing at most linearly in } T. \text{ The same arguments work for } Z_\beta(v) \cap \tilde{\mathcal{M}}^- \text{ and hence give the proposition.}
Write
\[ A := Z_0(v_0) \cap \hat{M}_a^+, \]
\[ K_T := \{ x \in \hat{M} : d(x, c_{v_0}[0, T]) \leq \beta \}, \]
\[ \mathcal{F}(c_v, \delta) := \{ x \in \hat{M} : d(x, c_v(\mathbb{R})) < \delta \}, \quad v \in A. \]
The sets \( A, K_T \) are compact.

**Step 1.** For \( \delta > 0 \) and \( v, w \in A \) with \( c_v[0, T] \subset \mathcal{F}(c_v, \delta/3) \) there exists \( s_0 \in \mathbb{R} \) such that
\[ d(c_v(t + s_0), c_w(t)) \leq \delta \quad \forall t \in [0, T]. \]

**Proof.** By assumption for any \( t \in [0, T] \) there is some \( s(t) \in \mathbb{R} \) with
\[ d(c_v(s(t)), c_w(t)) \leq \delta_0 := \delta/3. \]
Using the minimality of \( c_v, c_w \) one finds
\[ s(t) - s(0) \leq 2\delta_0 + t, \quad t \leq 2\delta_0 + s(t) - s(0). \]
Hence with \( s_0 := s(0) \) we have
\[ d(c_v(t + s_0), c_w(t)) \leq d(c_v(t + s(0)), c_v(s(t))) + d(c_v(s(t)), c_w(t)) \leq |s(t) - s(0) - t| + \delta_0 \leq 3\delta_0 = \delta. \]

**Step 2.** Let \( F(v, \delta) := \{ \phi^{j\delta} v : j \in \mathbb{Z}, |j| \leq 2(1 + \beta/\delta) \} \) for \( v \in A \). Then for \( v, w \in A \) with \( c_v[0, T] \subset \mathcal{F}(c_v, \delta/3) \) there exists \( v_j = \phi^{j\delta} v \in F(v, \delta) \) such that
\[ d(c_{v_j}(t), c_w(t)) \leq 2\delta \quad \forall t \in [0, T]. \]

**Proof.** Let \( s_0 \) be as in step 1 and \( j, r \in [0, \delta) \) with \( s_0 = j\delta + r \). Then
\[ d(c_v(t + j\delta), c_w(t)) \leq d(c_v(t + j\delta), c_v(t + s_0)) + d(c_v(t + s_0), c_w(t)) \leq |t + j\delta - t - s_0| + \delta \leq 2\delta. \]
By definition of \( A \) we have \( d(\pi v, \pi w) \leq 2\beta \) and hence again by step 1
\[ |s_0| = d(c_v(s_0), \pi v) \leq d(c_v(s_0), \pi w) + d(\pi w, \pi v) \leq \delta + 2\beta, \]
showing
\[ |j| \leq \frac{|s_0| + |r|}{\delta} \leq \frac{\delta + 2\beta + \delta}{\delta} = 2(1 + \beta/\delta). \]

**Step 3.**
\[ h_{top}(\phi^t, A) = 0. \]

**Proof.** Consider the family of (oriented) unparametrized curves
\[ \mathcal{A} := \{ c_v(\mathbb{R}) \subset \hat{M} : v \in A \}. \]
\( \mathcal{A} \) is ordered by Remark 4.2 (i) \((c < c' \iff c' \subset \hat{M}^+(c)) \) and we construct a sequence of geodesics \( c_1 < ... < c_n < ... \in \mathcal{A} \). By closedness of \( A \) we find a \(<\)-smallest geodesic \( c_1 \) in \( \mathcal{A} \). If \( c_1, ..., c_n \) are already chosen, take \( c_{n+1} \in \mathcal{A} \) to be the \(<\)-smallest geodesic \( c_{n+1} > c_n \), such that the compact segment \( c_n \cap K_T \) is
we find (cf. Corollary 8.6.1 (ii) in [12]).

Using Theorem 2.2 and Lemma 4.5 we find

\[ D_n := \tilde{M}^+(c_n) \cap B(p_n, \delta/3) \]

lies in the open strip between \( c_n < c_{n+1} \) (for \( \delta \) small, such that \( c_n \) does not return to \( D_n \) by minimality). Moreover all half disks \( D_n \) are contained in a \( \delta/3 \)-neighborhood of \( K_T \) and disjoint, since the \( c_i \) are ordered. As the volume of \( D_n \) is bounded from below by some constant \( C(\delta) \) using standard comparison theorems and the compactness of \( M \), and the volume of the \( K_T \)-neighborhood is finite, growing linearly with \( T \), the above construction stops at some finite \( N(T) \), again \( N(T) \) growing at most linearly. On the other hand, by construction for any \( c \in \mathcal{A} \) we find some \( i \in \{1,\ldots,N(T)\} \) such that \( c \cap K_T \subset \mathcal{F}(c_i, \delta/3) \).

Choose the parameterization of the \( c_i \) such that \( v_i := \tilde{c}_i(0) \in A \). Now by step 2 the set

\[ E(T, \delta) := \bigcup_{i=1}^{N(T)} F(v_i, \delta) \]

is \((T-1,2\delta)\)-spanning for \( A \) with respect to \( d_1 \) with cardinality

\[ \text{card} E(T, \delta) = N(T) \cdot \text{card} F(v_i, \delta) = N(T) \cdot (4(1 + \beta \delta) + 1). \]

Hence for any \( \delta > 0 \) we have

\[ h_{\text{top}}(\phi^t, A, 2\delta) \leq \lim_{T \to \infty} \frac{\log \text{card} E(T, \delta)}{T-1} = \lim_{T \to \infty} \frac{\log N(T)}{T-1} = 0, \]

and by letting \( \delta \to 0 \), the claim follows.

This concludes the proof of Lemma 4.5.

We can now prove Theorem 1.4 from the introduction.

Proof of Theorem 1.4. First observe that it is enough to consider orientable surfaces of genus \( \geq 1 \). In the nonorientable case, we just move to the two-sheeted orientable cover. In the case of genus \( = 0 \) (i.e., the 2-sphere) we already noted that \( \mathcal{A} = \emptyset \) and \( h(g) = 0 \).

By Theorem 3.1 we have \( h_{\text{top}}(\phi_{\mathcal{A}}^t) \geq h(g) \). To show the reverse inequality take any compact set \( K \subset M \) with \( p(K) = M \). By Theorem 3.4 (which holds also for \( M = T^2 \), as noted at the beginning of Section 4) there is some \( \beta > 0 \) with

\[ h_{\text{top}}(\phi^t, \tilde{\Omega} \cap SK, \beta) \leq h_{\text{top}}(\phi^t, \tilde{\mathcal{A}} \cap SK, \beta) \leq h(g). \]

Using Theorem 2.2 and Lemma 4.5 we find

\[ h_{\text{top}}(\phi^t) \leq h_{\text{top}}(\phi^t, \tilde{\Omega} \cap SK, \beta) + h_{\text{top,loc}}(\phi^t, \tilde{\Omega} \cap SK, \beta) \]

\[ = h_{\text{top}}(\phi^t, \tilde{\Omega} \cap SK, \beta) \leq h(g). \]

But since \( \Omega \) is has full measure with respect to any invariant probability on \( \mathcal{A} \), we find (cf. Corollary 8.6.1 (ii) in [12])

\[ h_{\text{top}}(\phi_{\mathcal{A}}^t) = h_{\text{top}}(\phi_{\mathcal{A}}^t) \leq h(g). \]

\[ \square \]
Remark 4.6. Theorem 1.4 also holds if one replaces the Riemannian metric $g$ by a Finsler metric $F$ (not necessarily reversible). The Morse Lemma only requires that the two norms $F, F_0 = \sqrt{g_0}$ are equivalent. The volume entropy of $F$ can be defined by
\[
h(F) = \lim_{r \to \infty} \frac{\log \text{vol}_{g_0} B(x, r)}{r},
\]
where $\text{vol}_{g_0}$ is the volume measured in some Riemannian background metric $g_0$ and $B(x, r)$ is the ball defined by the end-points of Finsler geodesic rays of length $\leq r$ initiating form $x$. Theorems 3.1 and 3.4 continue to hold. Arguing along these lines yields Theorem 1.4 in the Finsler case. This also has implications for Tonelli Lagrangian systems, as on high enough energy levels the arising Euler–Lagrange flow is a reparametrization of a Finsler geodesic flow, cf. [3].

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