MONOTONICITY RESULTS FOR THE FIRST STEKLOV EIGENVALUE ON COMPACT SURFACES

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Abstract. We show several results comparing sharp eigenvalue bounds for the first Steklov eigenvalue on surfaces under change of the topology. Among others, we obtain strict monotonicity in the genus. Combined with results of the second named author [Pe15] this implies the existence of free boundary minimal immersions from higher genus surfaces into Euclidean balls. Moreover, we can also give a new proof of a result by Fraser and Schoen that shows monotonicity in the number of boundary components.

1. Introduction

If $\Sigma$ is a compact surface with non-empty, compact boundary $\partial \Sigma$, the Dirichlet-to-Neumann operator $T$ is given by

$$T(u) = \partial_\nu \hat{u},$$

where $\hat{u}$ is the harmonic extension of $u \in C^\infty(\partial \Sigma)$ to $\Sigma$ and $\partial_\nu$ denotes the derivative along $\partial \Sigma$ in the direction of the outward pointing unit normal. This operator has purely discrete spectrum, which we will denote by

$$0 = \sigma_0(\Sigma) < \sigma_1(\Sigma) \leq \sigma_2(\Sigma) \leq \ldots.$$

Here, each eigenvalue is repeated as often as its multiplicity requires. For more on the Steklov spectrum, we refer to the nice survey [GP17].

Recently, there has been a lot of interest in maximization problems related to these eigenvalues. One motivation to study such problems is their connection to free boundary minimal surfaces in Euclidean balls, see [FS11]. In particular, spectacular results have been obtained for the first non-zero eigenvalue $\sigma_1$. Before we can discuss this in more detail, we need to introduce some notation.

For a compact, orientable surface of genus $\gamma$ and with $k$ boundary components, we write

$$\sigma^*(\gamma, k) = \sup_g \sigma_1(\Sigma, g) \text{ length}(\partial \Sigma, g)$$

where the sup runs through all smooth metrics $g$ on $\Sigma$.

Similarly, for non-orientable surfaces of non-orientable genus $\gamma$ and $k$ boundary components, we write

$$\sigma^*_K(\gamma, k) = \sup_g \sigma_1(\Sigma, g) \text{ length}(\partial \Sigma, g)$$

with the sup again taken over all smooth metrics $g$ on $\Sigma$.

Our main result relates these numbers under some change of the topology.
Theorem 1.1. (i) Assume that there is a smooth metric achieving $\sigma^*(\gamma, k)$. Then we have the strict inequalities

\begin{align*}
\sigma^*(\gamma, k + 1) &> \sigma^*(\gamma, k), \\
\sigma^*_K(2\gamma + 1, k) &> \sigma^*(\gamma, k).
\end{align*}

If we additionally assume that $k \geq 2$, we also have

\begin{align*}
\sigma^*(\gamma + 1, k - 1) &> \sigma^*(\gamma, k), \\
\sigma^*_K(2\gamma + 2, k - 1) &> \sigma^*(\gamma, k).
\end{align*}

(ii) Assume that there is a smooth metric achieving $\sigma^*_K(\delta, k)$, then

\begin{align*}
\sigma^*_K(\gamma, k + 1) &> \sigma^K(\gamma, k), \\
\sigma^*_K(\gamma + 1, k) &> \sigma^K(\gamma, k).
\end{align*}

The first of these inequalities (1.2) was shown by Fraser and Schoen using a different approach [FS16]. They also obtained that for each $k$ there are smooth metrics achieving $\sigma^*(0, k)$ for any $k \geq 1$. Very recently, Karpukhin extended this to any genus, i.e. he showed that there are smooth metrics achieving $\sigma^*(\gamma, k)$ for any $\gamma \geq 0$ and $k \geq 1$, [Ka18]. The key step in his proof is to show the non-strict version of (1.4) and to combine this with (1.2) to obtain

\begin{equation}
\sigma^*(\gamma + 1, k) > \sigma^*(\gamma, k).
\end{equation}

The existence of maximizers then follows from the main result of [Pe15].

All inequalities from Theorem 1.1 are obtained using the same gluing technique in the spirit of related results for the first eigenvalue of the Laplace operator on closed surfaces from [MS17]. Our technique gives a unified approach for all these problems.

As indicated above, it follows from Theorem 1.1 and [Pe15] that it is possible for any $\gamma \geq 0$ and $k \geq 1$ to find smooth metrics achieving $\sigma^*(\gamma, k)$. Since such maximizing metrics always arise as immersed free boundary minimal surfaces in Euclidean balls, we also get the existence of such immersions for any possible topological type of orientable surfaces.

Theorem 1.9. For any $\gamma \geq 0$ and $k \geq 1$, there is a smooth metric achieving $\sigma^*(\gamma, k)$. In particular, for any such $\gamma, k$, there is $N = N(\gamma, k)$ and a free boundary minimal immersion $\Sigma_{\gamma,k} \to \mathbb{B}^N$, where $\mathbb{B}^N \subset \mathbb{R}^n$ is the unit ball.

The same result was very recently independently obtained by Karpukhin [Ka18]. For $\gamma = 0$, this was obtained earlier by Fraser and Schoen [FS16]. We expect, that Theorem 1.1 can be used to obtain an analogous existence result also for non-orientable surfaces and plan to address this in a forthcoming paper.

The question for which topological types of surfaces one can find free boundary minimal surfaces in Euclidean balls, in particular the three ball, has recently also been addressed by very different techniques. Using an equivariant version of min-max theory Ketover obtained the existence of free boundary minimal surfaces in $\mathbb{B}^3$ of unbounded
genus and three boundary components \cite{Ke17, Ke17a}. Examples of the same topological type using desingularization techniques were found by Kapouleas and Li \cite{KL17}. Using related perturbation techniques, Folha, Pacard, and Zolotareva obtained the existence of examples in $\mathbb{B}^3$ with genus 1 and $k$ boundary components for $k$ large. Examples with high genus and connected boundary were constructed by Kapouleas and Wiygul \cite{KW17}.

Finally, we want to emphasize that we have been working on these results independently of Karpukhin. Although, there is a quite some overlap in the results, our conclusion are stronger and the techniques we use a very different from \cite{Ka18}.

**Notation.** If $I \subset \partial \Sigma$ is non-empty, we denote for $j \geq 0$ by $\sigma^D_j(\Sigma, I)$ the $j$-th Steklov eigenvalue with respect to Dirichlet conditions along $I$. Similarly, we denote by $\sigma^N_j(\Sigma, I)$ the $j$-th Steklov eigenvalue with respect to Neumann conditions along $I$. In any case, we always start enumerating eigenvalues at 0. In particular, for the Dirichlet problem we have that $\sigma^D_0(\Sigma, I) > 0$, whereas $\sigma^N_0(\Sigma, N) = 0$ for the Neumann problem. By abuse of notation, if $u \in H^{1/2}(\partial \Sigma)$ is Steklov eigenfunction, we also write $u$ for its harmonic extension to $\Sigma$. If $u \neq 0$ and $Tu = \sigma u$, that is, if $u$ is a non-trivial eigenfunction, we write $\sigma(u) = \sigma$.

## 2. Convergence of the spectrum

Let

$$R_{\varepsilon,h} = \left[-\varepsilon^2/2, \varepsilon^2/2\right] \times \left[-\varepsilon/2h, \varepsilon/2h\right] \subset \mathbb{R}^2$$

be a flat rectangle with side lengths $\varepsilon^2$ and $\varepsilon, h$ for $\varepsilon, h > 0$. Moreover, we write

$$I_\varepsilon = \left[-\varepsilon^2/2, \varepsilon^2/2\right] \times \{-\varepsilon/2h\} \cup \left[-\varepsilon^2/2, \varepsilon^2/2\right] \times \{\varepsilon/2h\} \subset \partial R_{\varepsilon,h}.$$ 

Given a compact surface $\Sigma$ with smooth, non-empty boundary, we can construct a new surface

$$\Sigma_{\varepsilon,h} = \Sigma \cup I_\varepsilon \ R_{\varepsilon,h}$$

by attaching $R_{\varepsilon,h}$ to $\Sigma$ along $I_\varepsilon$.

More precisely, we pick two distinct points $x_1, x_2 \in \partial \Sigma$. We also fix two curves $c_i \subset \partial \Sigma$ parametrised by arclength such that $c_i(0) = x_i$. Denote by $\Sigma_{\varepsilon,h}$ the surface obtained from $\Sigma$ by attaching $R_{\varepsilon,h}$ along the two components of $I_\varepsilon$ to $c_i([-\varepsilon^2/2, \varepsilon^2/2])$. For simplicity, we denote by $I_\varepsilon$ also the image of $I_\varepsilon$ in $\Sigma$ under the attaching map. We can do this with any orientation of the curves $c_{1,2}$ and the relevant boundary components $R_{\varepsilon,h}$. This will not affect any of our arguments below.

The goal of this section is to describe how the Steklov eigenvalues of $\Sigma_{\varepsilon,h}$ behave for $\varepsilon \to 0$.

Before we do this, we briefly want to discuss the topological type of $\Sigma_{\varepsilon,h}$ depending on how we attach the rectangle $R_{\varepsilon,h}$. As in the introduction, $\gamma$ denotes the genus if $\Sigma$ is orientable, and the non-orientable genus if $\Sigma$ is non-orientable. We write $k$ for the number of boundary components of $\Sigma$.

How the topological type changes by attaching $R_{\varepsilon,h}$ depends on the choices of the points $x_1, x_2$ and the orientations of the curves $c_i$.

We first discuss the case that $\Sigma$ is orientable with genus $\gamma$ and $k$ boundary components. If we take $x_1$ and $x_2$ to lie in the same component of the boundary we can
obtain two types of surfaces by attaching $R_{\varepsilon,h}$. We can attach $R_{\varepsilon,h}$ so that the resulting surface is orientable. Then $\Sigma_{\varepsilon,h}$ is orientable has genus $\gamma$ and $k + 1$ boundary components. If we reverse the orientation of one of the curves $c_i$, the resulting surface will be non-orientable, have $k$ boundary components and non-orientable genus $2\gamma + 1$.

If we assume that $k \leq 2$ there is also the option of taking $x_1$ and $x_2$ to lie in different components of $\partial \Sigma$. In this case, if we attach $R_{\varepsilon,h}$ such that we obtain an orientable surface, we find that $\Sigma_{\varepsilon,h}$ has genus $\gamma + 1$ and $k - 1$ boundary components. If we reverse the orientation of one of the curves $c_i$ the new surface will have non-orientable genus $2\gamma + 2$ and $k - 1$ boundary components.

If we start with a non-orientable surface $\Sigma$ of non-orientable genus $\gamma$ and $k$ boundary components, the topological type depends only on the location of the points $x_1$ and $x_2$. It will either have non-orientable genus $\gamma + 1$ and $k + 1$ boundary components, or non-orientable genus $\gamma + 1$ and $k$ boundary components.

We want to show that for $\varepsilon$ very small, there is good choice of $h$ bounded away from 0, such that the first Steklov eigenvalue of $\Sigma_{\varepsilon,h}$ is bounded from below by $\sigma_1(\Sigma)$ up to an error in $o(\varepsilon)$ as $\varepsilon \to 0$. This will then easily imply Theorem 1.1 since we also have good control on $\text{length}(\partial \Sigma_{\varepsilon,h})$. The very first step to perform this is to understand what happens to the spectrum of $\Sigma_{\varepsilon,h}$ as $\varepsilon \to 0$ without specifying the rate of convergence.

2.1. The Steklov eigenvalues of $R_{\varepsilon,h}$. In this section, we discuss the Steklov eigenvalues of $R_{\varepsilon,h}$ if we impose Dirichlet or Neumann conditions along $I_\varepsilon$. The coordinates we get from (2.2) will be natural later, but for the computation of the spectrum it is more convenient to use the coordinated we get by considering

$$R_{\varepsilon,h} = [-\varepsilon^2/2, \varepsilon^2/2] \times [0, \varepsilon h] \subset \mathbb{R}^2$$

We start with the observation, that the functions

$$f_\mu = \sin(\mu y) \cosh(\mu x)$$

are harmonic on $R_{\varepsilon,h}$. If we take $\mu = \mu_j = j\pi/(\varepsilon h)$ for $j \geq 1$, then $f_\mu$ vanishes along $I_\varepsilon$ and moreover, $\partial_y f_\mu = \mu \tanh(\varepsilon^2 \mu/2) f_\mu$ along $\partial R_{\varepsilon,h} \setminus I_\varepsilon$.

Similarly, one checks that

$$g_\mu = \sin(\mu y) \sinh(\mu x)$$

are Steklov eigenfunctions which vanish along $I_\varepsilon$. The corresponding eigenvalues in this case are given by $\mu_j \tanh^{-1}(\varepsilon^2 \mu_j/2)$.

We claim that $\{f_{\mu_j}, g_{\mu_j}\}_{j \in \mathbb{N}} \setminus \{0\}$ is a complete family of eigenfunctions for the Steklov problem on $R_{\varepsilon,h}$ with Dirichlet condition on $I_\varepsilon$. This follows easily from the symmetries of the hyperbolic functions. Write $\partial R_{\varepsilon,h}^- = \{x < 0\} \cap (\partial R_{\varepsilon,h} \setminus I_\varepsilon)$ for the left component of $\partial R_{\varepsilon,h} \setminus I_\varepsilon$, and $\partial R_{\varepsilon,h}^+ = \partial R_{\varepsilon,h} \setminus (I_\varepsilon \cup \partial R_{\varepsilon,h}^-)$ for the right component. After scaling, we can assume that we have $f_\mu = g_\mu$ on $\partial R_{\varepsilon,h}^-$. Using that $\cosh(x) = \cosh(-x)$ and $\sinh(x) = \sinh(-x)$, this implies that $f_\mu + g_\mu = 0$ on $\partial R_{\varepsilon,h}^-$. This implies that the span of $\{f_{\mu_j}, g_{\mu_j}\}_{j \in \mathbb{N}}$ contains all functions that vanish on $\partial R_{\varepsilon,h}^-$ and are equal to $\cos(\mu y)$ on $\partial R_{\varepsilon,h}^+$. The same argument gives the analogous statement with the roles of $\partial R_{\varepsilon,h}^+$ and $\partial R_{\varepsilon,h}^-$ interchanged. In particular, we find that $\{f_{\mu_j}, g_{\mu_j}\}_{j \in \mathbb{N}}$ in an $L^2$-basis of $L^2(R_{\varepsilon,h} \setminus I_\varepsilon)$. In conclusion, there are two different types of eigenvalues. Those corresponding to
$g_{\mu_j}$ become unbounded as $\varepsilon \to 0$, whereas those corresponding to $f_{\mu_j}$ converge to $\frac{j^2 \pi^2}{2 h^2}$ as $\varepsilon \to 0$.

The Steklov problem with Neumann conditions along $I_\varepsilon$ can be treated in the exact same way. This time, we use the functions

\begin{align*}
(2.5)\quad f_\mu &= \cos(\mu y) \cosh(\mu x) \\
(2.6)\quad g_\mu &= \cos(\mu y) \sinh(\mu x),
\end{align*}

with $\mu = \mu_j = j\pi/(\varepsilon h)$ for $j \geq 0$.

2.2. Pointwise estimates on eigenfunctions. In this section, we prove a pointwise bound on eigenfunctions on $\Sigma_{\varepsilon,h}$ in the region, where we attach the rectangle $R_{\varepsilon,h}$.

**Lemma 2.7.** Let $u_{\varepsilon,h}$ be a $\sigma_{\varepsilon,h}$ eigenfunction on $\Sigma_{\varepsilon,h}$ with $\|u_{\varepsilon,h}\|_{L^2(\partial\Sigma_{\varepsilon,h})} = 1$. Assume that $\sigma_{\varepsilon,h}$ is uniformly bounded, then we have the following two estimates

\begin{align*}
(2.8)\quad \sup_{\partial\Sigma \setminus I_r} |u_{\varepsilon,h}| &\leq C \log \left(\frac{1}{r}\right) \\
(2.9)\quad \sup_{I_\varepsilon} |u_{\varepsilon,h}| &\leq C \log \left(\frac{1}{\varepsilon}\right),
\end{align*}

where $C$ is a constant that is independent of $\varepsilon$ and $h$.

**Proof.** Without loss of generality, we focus on the component of $I_\varepsilon$ that contains $x_1$. We take a conformal chart at the neighborhood of $x_1$ so that the metric $g$ is isometric to $e^{2\omega}(dx^2 + dy^2)$ and $x_1$ is sent to 0. Then $u_{\varepsilon,h}$ satisfies the equation

\begin{align*}
\begin{cases}
\Delta u_{\varepsilon,h} = 0 & \text{in } \mathbb{D}_2^+ \\
-\partial_y u_{\varepsilon,h} = \sigma_{\varepsilon} e^{\omega} u_{\varepsilon,h} & \text{on } [-2,2] \times \{0\} \setminus [-\varepsilon^2/2,\varepsilon^2/2] \times \{0\}
\end{cases}
\end{align*}

at the neighbourhood of 0. By standard elliptic estimates [Ta11, Chapter 5.7], we have

\begin{align*}
(2.10)\quad \|u_{\varepsilon,h}\|_{C^1(\overline{\mathbb{D}_2^+ \setminus \mathbb{D}_4^+})} &\leq C \left( \|u_{\varepsilon,h}\|_{L^2([-2,2] \setminus [-\varepsilon^2/4,\varepsilon^2/4] \times \{0\})} + \|\nabla u_{\varepsilon,h}\|_{L^2(\overline{\mathbb{D}_2^+ \setminus \mathbb{D}_4^+})} \right).
\end{align*}

We set

$v(r, \theta) = u_{\varepsilon,h}(re^{i\theta})$ and $f(r) = \int_0^\pi v(r, \theta) d\theta$.

Notice that we dropped the dependence in $\varepsilon$ and $h$ for ease of notation. Then,

\begin{align*}
\frac{1}{r} \partial_r (rf'(r)) &= \frac{1}{r} \int_0^\pi \partial_r (r\partial_r v(r, \theta))d\theta = -\frac{1}{r^2} \int_0^\pi \partial_{\theta\theta} v(r, \theta) d\theta \\
&= \frac{1}{r^2} (\partial_{\theta} v(r, 0) - \partial_{\theta} v(r, \pi)) \\
&= \frac{1}{r} \sigma_{\varepsilon} e^{\omega(r,0)} (u_{\varepsilon,h}(r, 0) + u_{\varepsilon,h}(r, \pi)).
\end{align*}
We integrate twice so that for any \( r \in (\varepsilon^2/2, 1] \),
\[
(2.11) \quad |f(r)| \leq |f(1)| + \ln \left( \frac{1}{r} \right) \left( \sigma_\varepsilon \int_{[-1,1][[-r,r]} e^{\omega(t,0)} |u_{\varepsilon,h}(t,0)| dt + |f'(1)| \right).
\]
By (2.10) and since \((u_{\varepsilon,h})\) is uniformly bounded in \(H^1(\Sigma_{\varepsilon,h})\), we get a constant \(C\) independent of \(\varepsilon\) and \(h\) such that
\[
(2.12) \quad f(r) \leq C \left( 1 + \ln \left( \frac{1}{r} \right) \right).
\]
In particular for \(r = \varepsilon^2\),
\[
(2.13) \quad f(\varepsilon^2) \leq C \left( 1 + 2 \ln \left( \frac{1}{\varepsilon} \right) \right).
\]
In order to deduce the estimate (2.8) for \(r \geq \varepsilon^2\) from the first of these two bounds, we consider the functions \(u_r(x) = u_{\varepsilon,h}(rx)\), \(\omega_r(x) = \omega(rx)\). Then, \(u_r\) solves the equation
\[
\begin{cases}
\Delta u_r = 0 & \text{in } \mathbb{D}_3^+ \setminus \mathbb{D}_1^{3/2} \\
-\partial_y u_r = r\sigma_{\varepsilon,h}e^{\omega_r}u_r & \text{on } [-3,3] \times \{0\} \setminus [-1/2,1/2] \times \{0\}
\end{cases}
\]
By standard elliptic boundary estimates we have
\[
\|u_r - f(r)\|_{L^\infty(\mathbb{D}_3^+ \setminus \mathbb{D}_1^{3/2})} \leq C \left( \|ru_r\|_{L^2([-3,3] \times \{0\} \setminus [-1/2,1/2] \times \{0\})} + \|\nabla u_r\|_{L^2(\mathbb{D}_3^+ \setminus \mathbb{D}_1^{3/2})} \right).
\]
The right-hand term is uniformly bounded in \(\varepsilon\) and \(h\) since \((u_{\varepsilon,h})\) is uniformly bounded in \(H^1(\Sigma_{\varepsilon,h})\) and we have
\[
\|ru_{\varepsilon,h}\|_{L^2([-3,3] \times \{0\} \setminus [-1/2,1/2] \times \{0\})} \leq r^{1/2}\|u_{\varepsilon,h}\|_{L^2(\partial\Sigma_{\varepsilon,h})},
\]
and
\[
\|\nabla u_r\|_{L^2(\mathbb{D}_3^+ \setminus \mathbb{D}_1^{3/2})} \leq \|\nabla u_{\varepsilon,h}\|_{L^2(\Sigma)}.
\]
Combining this with (2.12), we obtain (2.8).

We are left with showing (2.9). This time we set \(\tilde{u}_{\varepsilon,h}(x) = u_{\varepsilon,h}(2\varepsilon^2 x)\), \(\tilde{\omega}(x) = \omega(2\varepsilon^2 x)\), \(\Omega = \mathbb{D}_3^+ \cup ([-1,1] \times [-1,0])\) and \(\Omega' = \mathbb{D}_2^+ \cup ([-1,1] \times [-2,0])\). Then \(\tilde{u}_{\varepsilon,h}\) satisfies the equation
\[
\begin{cases}
\Delta \tilde{u}_{\varepsilon,h} = 0 & \text{in } \Omega \\
-\partial_y \tilde{u}_{\varepsilon,h} = 2\varepsilon^2 \sigma_{\varepsilon,h} e^{\tilde{\omega}} \tilde{u}_{\varepsilon,h} & \text{on } [-2,2] \times \{0\} \setminus [-1,1] \times \{0\} \\
-\partial_x \tilde{u}_{\varepsilon,h} = 2\varepsilon^2 \sigma_{\varepsilon,h} \tilde{u}_{\varepsilon,h} & \text{on } \{1\} \times [-2,2] \\
-\partial_{\varepsilon,h} \tilde{u}_{\varepsilon,h} = 2\varepsilon^2 \sigma_{\varepsilon,h} \tilde{u}_{\varepsilon,h} & \text{on } \{1\} \times [-2,2]
\end{cases}
\]
By elliptic estimates [Ro11, Theorem 3.14 (ii)], we have
\[
\|\tilde{u}_{\varepsilon,h} - f(\varepsilon^2)\|_{L^\infty(\Omega')} \leq C \left( \varepsilon^2 \|	ilde{u}_{\varepsilon,h}\|_{L^2(J)} + \|
abla \tilde{u}_{\varepsilon,h}\|_{L^2(\Omega')} \right)
\]
where we set \(J = ([[-2,2] \setminus [-1,1]) \times \{0\}) \cup ([1,1] \times [-2,2])\). The right-hand term is uniformly bounded in \(\varepsilon\) and \(h\) since \((u_{\varepsilon,h})\) is uniformly bounded in \(H^1(\Sigma_{\varepsilon,h})\) and we have
\[
\|\varepsilon^2 \tilde{u}_{\varepsilon,h}\|_{L^2(J)} \leq \varepsilon \|u_{\varepsilon,h}\|_{L^2(\partial\Sigma_{\varepsilon,h})} \quad \text{and} \quad \|
abla \tilde{u}_{\varepsilon,h}\|_{L^2(\Omega')} \leq \|\nabla u_{\varepsilon,h}\|_{L^2(\Sigma)}.
\]
Thanks to (2.13), we get (2.9). \(\Box\)
2.3. A mixed boundary value problem. For the proof of Theorem 2.15 we need estimates for solutions of the following mixed boundary value problem.

Let $u$ be a Steklov eigenfunction on $\Sigma_{\epsilon,h}$. We consider the unique (weak) solution $v_{\epsilon,h}$ to the following mixed boundary value problem.

\begin{equation}
\begin{cases}
\Delta v_{\epsilon,h} = 0 & \text{in } R_{\epsilon,h} \\
v_{\epsilon,h} = \hat{u}_{\epsilon,h} & \text{on } I_{\epsilon} \\
\partial_{\nu} v_{\epsilon,h} = 0 & \text{on } \partial R_{\epsilon,h} \setminus I_{\epsilon}.
\end{cases}
\end{equation}

Existence of solutions follows from standard variational techniques. Since $u$ is smooth up to the boundary, except maybe at the corners of $R_{\epsilon,h}$, standard regularity results ensure that $v_{\epsilon,h}$ is continuous up to the boundary and smooth up to the boundary except maybe in the corners. Since $v_{\epsilon,h}$ is harmonic, it attains its maximum on the boundary, see \cite[Theorem 2.2]{GT83} At each point in the interior of $\partial R_{\epsilon,h} \setminus I_{\epsilon}$ the assumptions of the Hopf boundary lemma \cite[Lemma 3.4]{GT83} are satisfied. Since $\partial_{\nu} v_{\epsilon,h} = 0$ at these points, the maximum has to be located on $I_{\epsilon}$. In particular, this implies uniqueness of solutions.

2.4. The limit spectrum. Denote by

$$0 = \sigma_0(h) < \sigma_1(h) \leq \sigma_2(h) \leq \ldots$$

the reordered union of the Steklov spectrum of $\Sigma$ and the sequence $\left(\frac{j^2 \pi^2}{2h^2}\right)_{j \in \mathbb{N}\setminus\{0\}}$. We can now state the result describing the Steklov spectrum of $\Sigma_{\epsilon,h}$ for $\epsilon \to 0$.

**Theorem 2.15.** The Steklov spectrum of $\Sigma_{\epsilon,h}$ converges locally uniformly in $h$ to $(\sigma_j(h))_{j \in \mathbb{N}}$, i.e. for any $a,b$ with $0 < a < b$, any $\delta > 0$ and $j \in \mathbb{N}$ there is $\epsilon_0 > 0$ such that for any $h \in [a,b]$ and any $\epsilon \leq \epsilon_0$, we have

\begin{equation}
|\sigma_j(\Sigma_{\epsilon,h}) - \sigma_j(h)| \leq \delta.
\end{equation}

Moreover, let $\epsilon_l \to 0$ and $h_l \to h > 0$ as $l \to +\infty$, we let $u^l_j$ be a sequence of eigenfunctions associated to $\sigma_j(\Sigma_{\epsilon_l,h_l})$ such that $\|u^l_j\|_{L^2(\partial \Sigma_{\epsilon_l,h_l})} = 1$. Then, up to a subsequence, we have

1. $u^l_j \to u^l$ in $L^2(\partial \Sigma)$ as $l \to \infty$, where $u^l$ is a null function or an eigenfunction associated to $\sigma_j(h)$ on $\Sigma$, and

2. $\int_{\partial R_{\epsilon_l,h_l}} |u_l - e_l|^2 \to 0$, where $e_l$ is a null function or a Dirichlet eigenfunction on $R_{\epsilon_l,h_l}$, associated to $\sigma^l_j = \frac{j^2 \pi^2}{h_l^2} \tanh\left(\frac{j^2 \pi ^2}{2h_l^2}\right)$ for some fixed $j \geq 1$, such that $\sigma^l_j \to \sigma_j(h)$ as $l \to +\infty$.

Moreover, we have $\lim_{l \to \infty} \|e_l\|_{L^2(\partial R_{\epsilon_l,h_l})} + \|u^l\|_{L^2(\partial \Sigma)} = 1$.

The main tool for the proof is the following general result for the convergence of spectra of a family of operators in the presence of a so-called coupling map This is essentially contained in \cite{Po03}. Below, we state a slightly more general version, taken from \cite{MS17}. The main difference is that this version takes care of the dependence of the height parameter $h$.

Suppose we are given separable Hilbert spaces $\mathcal{H}_{\epsilon,h}$ and $\mathcal{H}'_{\epsilon,h}$, equipped with quadratic forms $q_{\epsilon,h}$ and $q'_{\epsilon,h}$, respectively. We assume that these quadratic forms are non-negative.
and closed. Then there is a unique self-adjoint operator associated to \( q_{\varepsilon,h} \) which will henceforth be referred to as \( Q_{\varepsilon,h} \), similarly we have \( Q'_{\varepsilon,h} \) associated to \( q'_{\varepsilon,h} \). Note, that the spectrum of \( Q_{\varepsilon,h} \) and \( Q_{\varepsilon,h}' \) is purely discrete.

The \( k \)-th eigenvalues of \( q_{\varepsilon,h} \) and \( q'_{\varepsilon,h} \) are henceforth denoted by \( \sigma_k(\varepsilon,h) \) and \( \sigma_k(\varepsilon,h)' \), respectively. Let \( L_k(\varepsilon,h) \) denote the direct sum of the eigenspaces of \( Q_{\varepsilon,h} \) corresponding to the first \((k+1)\)-eigenvalues. Finally, we denote by \( \text{dom}(q_{\varepsilon,h}) \) the domain of \( q_{\varepsilon,h} \).

**Lemma 2.17** ([MS17 Lemma A.5]). For each \( \varepsilon, h > 0 \) let \( \Phi_{\varepsilon,h} : \text{dom}(q_{\varepsilon,h}) \to \text{dom}(q'_{\varepsilon,h}) \)
be a linear map such that all \( u_{\varepsilon} \in L_k(\varepsilon,h) \) with \( \sup_{\varepsilon}(\|u_{\varepsilon}\|_{H_{\varepsilon,h}} + q_{\varepsilon,h}(u_{\varepsilon})) < \infty \) satisfy the following two conditions.

1. \( \lim_{\varepsilon \to 0}(\|\Phi_{\varepsilon,h}u_{\varepsilon}\|_{H_{\varepsilon,h}'}) - \|u_{\varepsilon}\|_{H_{\varepsilon,h}}) = 0 \), locally uniformly in \( h \),
2. \( q'_{\varepsilon,h}(\Phi_{\varepsilon,h}u_{\varepsilon}) \leq q_{\varepsilon,h}(u_{\varepsilon}) \).

Moreover, assume that \( \sigma_k(\varepsilon,h) \leq C \) for any \( \varepsilon > 0 \), fixed \( k \), and \( h \in [h_0,h_1] \subset (0,\infty) \). Then we have

\[
\sigma_k(\varepsilon) \leq \sigma_k(\varepsilon,h) + o(1),
\]
where the \( o(1) \) term is locally uniform in \( k \) and \( h \in (0,\infty) \).

**Proof of Theorem 2.15.** We proceed in several steps, we first show the (easy) asymptotic upper bound for the Steklov eigenvalues of \( \Sigma_{\varepsilon,h} \) in terms of \( (\sigma_j(h))_{j \in \mathbb{N}} \). Afterwards we verify the assumptions of Lemma 2.17 which will imply the asymptotic lower bound.

**Step 1: Upper bound on eigenvalues**

Let \( w_{\varepsilon,h} \) be a Steklov eigenfunction with eigenvalue \( \sigma(w_{\varepsilon,h}) \) on \( R_{\varepsilon,h} \), which has Dirichlet boundary conditions along \( I_{\varepsilon} \). Clearly, we can extend \( w_{\varepsilon,h} \) by zero to all of \( \Sigma_{\varepsilon,h} \) and obtain a test function on \( \Sigma_{\varepsilon,h} \), that we still call \( w_{\varepsilon,h} \) and satisfies

\[
\int_{\Sigma_{\varepsilon,h}} |\nabla w_{\varepsilon,h}|^2 = \int_{\partial R_{\varepsilon,h}\setminus I_{\varepsilon}} |\nabla w_{\varepsilon,h}|^2 = \sigma(w_{\varepsilon,h}) \int_{\partial R_{\varepsilon,h}\setminus I_{\varepsilon}} |w_{\varepsilon,h}|^2 = \sigma(w_{\varepsilon,h}) \int_{\partial \Sigma_{\varepsilon,h}} |w_{\varepsilon,h}|^2.
\]

Given \( x \in \partial \Sigma \), let \( \eta \) be a cut-off function that is 1 near \( x \) and has

\[
\int_{\Sigma} |\nabla \eta|^2 \leq \delta
\]
for some small \( \delta > 0 \). Such a choice is possible for \( \delta \) arbitrarily small, since the capacity of \( x \) vanishes.

If we take \( u \) to be a Steklov eigenfunction of \( \Sigma \) with eigenvalue \( \sigma \) we can construct a new test function \( \eta u \) that extends by zero to all of \( \Sigma_{\varepsilon,h} \) and has

\[
\int_{\Sigma_{\varepsilon,h}} |\nabla (\eta u)|^2 \leq \int_{\Sigma} |\nabla u|^2 + C \max\{1, |\sup u|^2\} \left( \int_{\Sigma} |\nabla \eta|^2 \right)^{1/2} \leq \sigma \int_{\partial \Sigma_{\varepsilon,h}} |u|^2 + C \max\{1, |\sup u|^2\} \delta^{1/2}.
\]
Since the functions \( \nu \) and \( w_{\varepsilon,h} \) have disjoint support, it easily follows from the variational characterisation of the eigenvalues, that we have

\[
\sigma_j(\Sigma_{\varepsilon,h}) \leq \sigma_j(h) + o(1)
\]

for \( \varepsilon \to 0 \) and the \( o(1) \) term locally uniformly in \( j \) and \( h \).

**Step 2: Lower bound on eigenvalues**

We want to apply Lemma 2.17 to the Hilbert spaces \( H_{\varepsilon,h} = L^2(\partial\Sigma_{\varepsilon,h}) \) and \( H'_{\varepsilon,h} = L^2(\partial\Sigma \setminus I_\varepsilon) \oplus L^2(\partial R_{\varepsilon,h} \setminus I_\varepsilon) \). The corresponding quadratic forms are defined as follows.

We take

\[
q_{\varepsilon,h}(u) = \int_{\Sigma_{\varepsilon,h}} |\nabla \hat{u}_{\varepsilon,h}|^2,
\]

for \( \hat{u}_{\varepsilon,h} \) the harmonic extension of \( u \in H^{1/2}(\partial\Sigma_{\varepsilon,h}) \). The domain of \( q'_{\varepsilon,h} \) is

\[
H^{1/2}(\partial\Sigma \setminus I_\varepsilon) \oplus H^{1/2}(\partial R_{\varepsilon,h} \setminus I_\varepsilon).
\]

Given an element \( u = (u_1, u_2) \in \text{dom}(q'_{\varepsilon,h}) \), we define \( q'_{\varepsilon,h}(u) = \int_{\Sigma} |\nabla \tilde{u}_{\varepsilon,h}|^2 + \int_{R_{\varepsilon,h}} |\nabla \tilde{u}_{\varepsilon,h}| \). Here, \( \tilde{u}_{\varepsilon,h} \) is the unique harmonic function on \( R_{\varepsilon,h} \) such that \( \tilde{u}_{\varepsilon,h} = u_2 \) on \( \partial R_{\varepsilon,h} \setminus I_\varepsilon \) and \( u = 0 \) on \( I_\varepsilon \).

The coupling map is given by

\[
\Phi_{\varepsilon,h}: u \mapsto (\hat{u}_{\varepsilon,h}|_{\partial\Sigma \setminus I_\varepsilon}, u|_{\partial R_{\varepsilon,h} \setminus I_\varepsilon} - v_{\varepsilon,h}),
\]

where we denote by \( \hat{u}_{\varepsilon,h} \) the harmonic extension of \( u \) to \( \Sigma_{\varepsilon,h} \) and by \( v_{\varepsilon,h} \) the solution to the mixed boundary value problem (2.14) with boundary values \( u \).

The first assumption of Lemma 2.17 is a consequence of Lemma 2.7. More precisely, it follows from (2.9) that

\[
|v_{\varepsilon,h}| \leq C|\log(\varepsilon)|
\]

on \( I_\varepsilon \). As explained in Section 2.3, the maximum principle implies that \( |v_{\varepsilon,h}| \) attains its maximum on \( I_\varepsilon \). In particular, we have \( |v_{\varepsilon,h}| \leq C|\log(\varepsilon)| \) on all of \( R_{\varepsilon,h} \). Therefore,

\[
\int_{\partial R_{\varepsilon,h} \setminus I_\varepsilon} |v_{\varepsilon,h}| \leq C\varepsilon h|\log(\varepsilon)|,
\]

which gives precisely the first assumption from Lemma 2.17. The second assumption follows from the following claim.

**Claim 2.22.** Let \( u \in H^1(R_{\varepsilon,h}) \) and \( v_{\varepsilon,h} \) be a solution to (2.14) with boundary values \( u = u \), then we have

\[
\int_{R_{\varepsilon,h}} |\nabla(u - v_{\varepsilon,h})|^2 \leq \int_{R_{\varepsilon,h}} |\nabla u|^2.
\]

**Proof.** Since \( u - v_{\varepsilon,h} = 0 \) on \( I_\varepsilon \) and \( \partial_\nu v_{\varepsilon,h} = 0 \) on \( \partial R_{\varepsilon,h} \setminus I_\varepsilon \), it follows from integration by parts that

\[
\int_{R_{\varepsilon,h}} \nabla(\hat{u}_{\varepsilon,h} - v_{\varepsilon,h}) \cdot \nabla v_{\varepsilon,h} = \int_{\partial R_{\varepsilon,h}} (\hat{u}_{\varepsilon,h} - v_{\varepsilon,h}) \partial_\nu v_{\varepsilon,h} = 0.
\]

Therefore,
\[
\int_{R_{\epsilon,h}} |\nabla (u - v_{\epsilon,h})|^2 = \int_{R_{\epsilon,h}} \nabla (u - v_{\epsilon,h}) \cdot \nabla u - \int_{R_{\epsilon,h}} \nabla (u - v_{\epsilon,h}) \cdot \nabla v_{\epsilon,h}
\]
\[
= \int_{R_{\epsilon,h}} \nabla (u - v_{\epsilon,h}) \cdot \nabla u
\]
\[
= \int_{R_{\epsilon,h}} |\nabla u|^2 - \int_{R_{\epsilon,h}} \nabla v_{\epsilon,h} \cdot \nabla u
\]
\[
= \int_{R_{\epsilon,h}} |\nabla u|^2 - \int_{R_{\epsilon,h}} \nabla v_{\epsilon,h} \cdot \nabla u + \int_{R_{\epsilon,h}} \nabla v_{\epsilon,h} \cdot \nabla (u - v_{\epsilon,h})
\]
\[
= \int_{R_{\epsilon,h}} |\nabla u|^2 - \int_{R_{\epsilon,h}} |\nabla v_{\epsilon,h}|^2
\]
\[
\leq \int_{R_{\epsilon,h}} |\nabla u|^2,
\]
which is exactly the assertion. \(\square\)

Thus, we conclude that
\[
\sigma_j(\Sigma_{\epsilon,h}) \geq \sigma_j(h) - o(1),
\]
where the \(o(1)\) term is locally uniform in \(j\) and \(h\).

**Step 3: Behaviour of eigenfunctions**

Let \(\epsilon_i \to 0\) and \(h_i \to h > 0\). For simplicity, we write \(\Sigma_i = \Sigma_{\epsilon_i,h_i}\), and \(R_i = R_{\epsilon_i,h_i}\), and \(I_i = I_{\epsilon_i}\). Moreover, let \(u^j_i\) be normalized \(\sigma_j(\Sigma_i)\)-eigenfunctions. It follows from the first step, that \(\sigma_j(\Sigma_i)\) is uniformly bounded. Therefore, \(u^j_i\) has
\[
\int_{\Sigma} |\nabla u^j_i|^2 \leq C.
\]

By standard elliptic boundary estimates this implies that \(u^j_i\) is uniformly bounded in \(H^1(\Sigma)\). Since the embedding \(H^1(\Sigma) \to L^2(\Sigma)\) is compact, we can extract a non-relabeled subsequence, such that \(u^j_i \to w^j \in L^2(\Sigma)\). Moreover, it also follows from standard elliptic estimates, that the convergence is smooth in every compact subset of \(\Sigma \setminus \{x_1, x_2\}\). In particular, we see that \(\partial_{\nu} w^j = \sigma_j(h) w^j\) along \(\partial \Sigma \setminus \{x_1, x_2\}\). By a capacity argument this easily implies that \(w^j\) is a \(\sigma_j(h)\) eigenfunction on \(\Sigma\).

In order to prove the second assertion, we take \(\{\phi^i_1\}\) a complete orthonormal system of Dirichlet eigenfunctions on \(R_i\) with corresponding eigenvalues \(\sigma^D_i(R_i, I_i)\). For \(i \in \mathbb{N}\) we can test the equation for \(u_i - v_i\) with \(\phi^i_1\) and find
\[
\sigma(u_i) \int_{\partial R_i} u_i \phi^i_1 = \int_{\partial R_i} \partial_{\nu}(u_i - v_i) \phi^i_1
\]
\[
= \int_{R_i} \nabla (u_i - v_i) \cdot \nabla \phi^i_1
\]
\[
= \sigma^D_i(R_i, I_i) \int_{\partial R_i} (u_i - v_i) \phi^i_1.
\]
By Lemma 2.7 and the maximum principle, we have
\[ \left| \int_{\partial R_i \setminus I_i} v_i \phi_i^j \right| \leq \left( \int_{\partial R_i \setminus I_i} |v_i|^2 \right)^{1/2} \leq C(h \varepsilon_l)^{1/2} |\log(\varepsilon_l)| = o(1). \]

Combining the previous two estimates implies that
\[ (\sigma(u_l) - \sigma^D_l(R_l, I_l)) \int_{\partial R_l} (u_l - v_l) \phi_i^j \leq o(1) \]
uniformly in \( i \) as \( l \to \infty \), since we assume \( \sigma(u_l) \) to be uniformly bounded. Therefore, for any \( i \) such that \( \lim_{l \to \infty} (\sigma(u_l) - \sigma^D_l(R_l, I_l)) \neq 0 \), we need to have
\[ \int_{\partial R_l} (u_l - v_l) \phi_i^j \to 0 \]
uniformly in \( i \). Since \( \sigma^D_l(R_l, h_l) \to \frac{(j+1)^2\pi^2}{2h^2} \) as \( l \to \infty \), there can be at most one \( i_0 \) such that \( \lim_{l \to \infty} (\sigma(u_l) - \sigma^D_l(R_l, I_l)) \neq 0 \). If there is such an \( i_0 \), we have
\[ \int_{\partial R_l} (u_l - v_l - \alpha_l \phi_{i_0}^j) \to 0 \]
form some real numbers \( \alpha_l \). Using again Lemma 2.7 the maximum principle, and the equation of \( v_l \), we find
\[ \int_{\partial R_l} (u_l - v_l - \alpha_l \phi_{i_0}^j)^2 = \int_{\partial R_l \setminus I_l} (u_l - \alpha_l \phi_{i_0}^j)^2 + o(1) \]
as \( l \to \infty \). If we choose \( e_l = \alpha_l \phi_{i_0}^j \), we get the second assertion. The remaining claim follows from
\begin{align*}
|\|e_l\|_{L^2(\partial R_l)} + \|u_l\|_{L^2(\partial \Sigma)} - \|u_l\|_{L^2(\partial \Sigma)}| &= |\|e_l\|_{L^2(\partial R_l \setminus I_l)} + \|u_l\|_{L^2(\partial \Sigma \setminus I_l)} - \|u_l\|_{L^2(\partial \Sigma \setminus I_l)} + o(1)| \\
&\leq |\|u_l - u\|_{L^2(\partial \Sigma \setminus I_l)} + \|e_l - u_l\|_{L^2(\partial R_l \setminus I_l)} - \|u_l\|_{L^2(\partial \Sigma \setminus I_l)}| + o(1) \\
&= o(1)
\end{align*}
as \( l \to \infty \).

3. Concentration of eigenfunctions

In this section we give a more precise description of how eigenfunctions on \( \Sigma_{\varepsilon, h} \) behave for small \( \varepsilon \). We then apply this to show that it is possible to find for \( \varepsilon \) sufficiently small a parameter \( h_\varepsilon \), such that the first Steklov eigenvalue of \( \Sigma_{\varepsilon, h_\varepsilon} \) has multiplicity at least two.

Recall that we write \( \mu_j(h) = \frac{(j+1)^2\pi^2}{2h^2} \) for the limits of the Dirichlet eigenvalues of the Steklov operator on \( R_{\varepsilon, h} \). Moreover, let \( h_\varepsilon \) be the unique value, for which \( \lambda_0(h_\varepsilon) = \sigma_1(\Sigma) \).

In order to formulate our next results, we need to introduce some more notation. We choose \( 0 < h_0 < h_1 \), such that we have
\[ 0 < \mu_0(h_1) < \sigma_1(\Sigma) < \mu_0(h_0) < \mu_1(h_1) < \sigma_2(\Sigma). \]
It then follows from Theorem 2.15 that for \( \varepsilon \) small enough and \( h \in [h_0, h_1] \) there are exactly \( \text{mult}(\sigma_1(\Sigma)) + 1 \) eigenvalues of \( \Sigma_{\varepsilon,h} \) contained in the interval \((0, \mu_1(h_1) - \delta_0)\), for some small \( \delta_0 > 0 \), which we fix now once and for all. Denote the direct sum of the eigenspaces, associated to these eigenvalues, by \( E_{\varepsilon,h} \).

**Proposition 3.1.** There is \( K \geq 1 \), such that for any \( \delta_1, \delta_2 > 0 \), there is \( \varepsilon_0 > 0 \) (that also depends on \( \delta_0 \) from above) such that for \( \varepsilon \leq \varepsilon_0 \) and \( h \in [h_0, h_1] \setminus (h_* - \delta_1, h_* + \delta_1) \), there is a normalized eigenfunction \( w_{\varepsilon,h} \in E_{\varepsilon,h} \) on \( \Sigma_{\varepsilon,h} \), such that

\[
\int_{\partial \Sigma} |w_{\varepsilon,h}|^2 \leq \delta_2.
\]

and such that for any normalized \( u_{\varepsilon,h} \in E_{\varepsilon,h} \cap \langle w_{\varepsilon,h} \rangle^\perp \), we have

\[
\int_{\partial \Sigma} |u_{\varepsilon,h}|^2 \geq 1 - \frac{\delta_2^{1/2}}{K}.
\]

Note that the second assertion in particular implies that \( w_{\varepsilon,h} \) is unique up to sign if we choose \( \delta_2 < 1/2 \).

**Proof.** To see that there is at least one such function, let \( \delta_2 < 1 \) assume that we have a normalized eigenfunction \( w_{\varepsilon,h} \in E_{\varepsilon,h} \) with eigenvalue \( \sigma(w_{\varepsilon,h}) \) and

\[
\delta_2 \leq \int_{\partial \Sigma} |w_{\varepsilon,h}|^2.
\]

Thanks to the first bound \((2.8)\) from Lemma 2.7 we can choose \( r \) very small, depending on \( \delta_2 \) such that

\[
\int_{I_r} |w_{\varepsilon,h}|^2 \leq \delta_2/2,
\]

for all \( \varepsilon \) such that \( 2\varepsilon^2 \leq r \). Therefore, we get that

\[
\int_{\partial \Sigma \setminus I_r} |w_{\varepsilon,h}|^2 = \int_{\partial \Sigma} |w_{\varepsilon,h}|^2 - \int_{I_r} |w_{\varepsilon,h}|^2 \geq \delta_2/2.
\]

By elliptic boundary estimates, \( w_{\varepsilon,h} \) converges smoothly to a Steklov eigenfunction \( u \) of \( \Sigma \) in compact subsets of \( \Sigma \setminus I_r/2 \). It follows from \((3.3)\), that \( u \neq 0 \). In particular, the eigenvalue needs to satisfy

\[
\lim_{\varepsilon \to 0} \sigma(w_{\varepsilon,h}) = \sigma_1(\Sigma).
\]

Since \( \delta_1 > 0 \) it follows that we can choose \( \varepsilon_0 \) small enough, such that

\[
|\sigma(w_{\varepsilon,h}) - \sigma_1(\Sigma)| \leq \frac{1}{4} |\sigma_1(\Sigma) - \mu_0(h)|
\]

for all \( h \in [h_0, h_1] \setminus (h_* - \delta_1, h_* + \delta_1) \). At the same time, it follows from Theorem 2.15 that \( \Sigma_{\varepsilon,h} \) has an eigenvalue \( \sigma_{\varepsilon} \) with

\[
|\sigma_{\varepsilon} - \mu_0(h)| \leq \frac{1}{4} |\sigma_1(\Sigma) - \mu_0(h)|
\]

if we choose \( \varepsilon_0 \) sufficiently small. It follows from \((3.6)\) and \((3.7)\), that \( \sigma(w_{\varepsilon,h}) \neq \sigma_{\varepsilon} \). In particular, there is an eigenfunction, for which \((3.4)\) can not hold.
We still have to show, that (up to decreasing \( \varepsilon_0 \)) we also have (3.1). Thanks to (2.9) from Lemma 2.7 it suffices to show that

(3.9) \[
\int_{\partial R_{\varepsilon,h} \setminus I_{\varepsilon}} |u_{\varepsilon,h}|^2 \leq \delta_2/2.
\]

for any normalized \( u_{\varepsilon,h} \in E_{\varepsilon,h} \cap \langle w_{\varepsilon,h} \rangle^\perp \) and \( \varepsilon \leq \varepsilon_0 \).

above is new, below is old

Denote by \( e_{\varepsilon,h} \) a normalized \( \sigma_D^0(R_{\varepsilon,h}) \)-eigenfunction. It follows from Theorem 2.15 and the second bound (2.9) in Lemma 2.7 that for any \( \delta_3 > 0 \) we can choose \( \varepsilon_0 \) small enough, such that

(3.10) \[
\int_{\partial R_{\varepsilon,h} \setminus I_{\varepsilon}} |w_{\varepsilon,h} - e_{\varepsilon,h}|^2 \leq \delta_3.
\]

Let \( v_{\varepsilon,h} \) be the solution to (2.14) with boundary values given by \( u_{\varepsilon,h} \). We write \( f_{\varepsilon,h} = u_{\varepsilon,h} - v_{\varepsilon,h} \).

By Young’s inequality, for any \( \delta_4 > 0 \)

(3.11) \[
\int_{\partial R_{\varepsilon,h} \setminus I_{\varepsilon}} |f_{\varepsilon,h}|^2 \leq (1 + \delta_4) \int_{\partial R_{\varepsilon,h} \setminus I_{\varepsilon}} \left( f_{\varepsilon,h} - \left( \int_{\partial R_{\varepsilon,h}} u_{\varepsilon,h} e_{\varepsilon,h} \right) e_{\varepsilon,h} \right)^2 + \left( 1 + \frac{1}{\delta_4} \right) \int_{\partial R_{\varepsilon,h} \setminus I_{\varepsilon}} |e_{\varepsilon,h}|^2 \left( \int_{\partial R_{\varepsilon,h} \setminus I_{\varepsilon}} f_{\varepsilon,h} e_{\varepsilon,h} \right)^2.
\]

In order to estimate the second summand, we note that if we choose \( \varepsilon \) sufficiently small, then

\[
\left| \int_{\partial R_{\varepsilon,h} \setminus I_{\varepsilon}} f_{\varepsilon,h} e_{\varepsilon,h} \right| \leq \int_{\partial R_{\varepsilon,h} \setminus I_{\varepsilon}} |(f_{\varepsilon,h} - u_{\varepsilon,h})(e_{\varepsilon,h} - w_{\varepsilon,h})| + \int_{\partial R_{\varepsilon,h} \setminus I_{\varepsilon}} |(f_{\varepsilon,h} - u_{\varepsilon,h})w_{\varepsilon,h}| + \int_{\partial R_{\varepsilon,h} \setminus I_{\varepsilon}} |u_{\varepsilon,h}(e_{\varepsilon,h} - w_{\varepsilon,h})| + \int_{\partial R_{\varepsilon,h} \setminus I_{\varepsilon}} u_{\varepsilon,h} w_{\varepsilon,h} \leq 2\delta_3^{1/2} + C \varepsilon \log \left( \frac{1}{\varepsilon} \right) + \delta_2^{1/2} \leq 4\delta_3^{1/2} + \delta_2^{1/2},
\]

where we have used that \( u_{\varepsilon,h} \) and \( w_{\varepsilon,h} \) are orthogonal in \( L^2(\partial \Sigma_{\varepsilon,h}) \) and Lemma 2.7.

For the first summand we use the Poincaré inequality. Given \( \delta_5 > 0 \), we can choose \( \varepsilon \) small enough, such that \( \sigma_D^0(R_{\varepsilon,h}, I_{\varepsilon}) \geq (1 - \delta_5)\mu_1(h) \) and \( \sigma_D^0(R_{\varepsilon,h}, I_{\varepsilon}) \leq (1 + \delta_5)\mu_0(h) \).
We then conclude that
\[
\int_{\partial R_{\varepsilon,h} \setminus I_\varepsilon} \left( f_{\varepsilon,h} - \left( \int_{\partial R_{\varepsilon,h}} f_{\varepsilon,h} e_{\varepsilon,h} \right) e_{\varepsilon,h} \right)^2 \leq \frac{1}{(1 - \delta_5)\mu_1(h)} \int_{\partial R_{\varepsilon,h}} \left| \nabla \left( f_{\varepsilon,h} - \left( \int_{\partial R_{\varepsilon,h}} f_{\varepsilon,h} e_{\varepsilon,h} \right) e_{\varepsilon,h} \right) \right|^2 \leq \frac{1 + \delta_4}{(1 - \delta_5)\mu_1(h)} \int_{R_{\varepsilon,h}} |\nabla f_{\varepsilon,h}|^2 + \frac{(1 + \delta_4)(1 + \delta_5)\mu_0(h)}{\delta_4(1 - \delta_5)\mu_1(h)} \left( \int_{\partial R_{\varepsilon,h}} f_{\varepsilon,h} e_{\varepsilon,h} \right)^2.
\]

(3.12)

Since \( f_{\varepsilon,h} = 0 \) on \( I_\varepsilon \) and \( \partial_\nu f_{\varepsilon,h} = \sigma(u_{\varepsilon,h}) \) along \( \partial R_{\varepsilon,h} \setminus I_\varepsilon \) it follows from integration by parts and Lemma 2.7 that
\[
\int_{R_{\varepsilon,h}} |\nabla f_{\varepsilon,h}|^2 = \sigma(u_{\varepsilon,h}) \int_{\partial R_{\varepsilon,h} \setminus I_\varepsilon} u_{\varepsilon,h} f_{\varepsilon,h}
\]
\[
\leq \sigma(u_{\varepsilon,h}) \int_{\partial R_{\varepsilon,h} \setminus I_\varepsilon} |f_{\varepsilon,h}|^2 + C \log \left( \frac{1}{\varepsilon} \right)
\]
\[
\leq \sigma(u_{\varepsilon,h}) \int_{\partial R_{\varepsilon,h} \setminus I_\varepsilon} |f_{\varepsilon,h}|^2 + \delta_3^{1/2}.
\]

(3.13)

We now first choose \( \delta_4 \) and \( \delta_5 \) small enough, such that for \( \varepsilon \leq \varepsilon_0 \)
\[
\frac{(1 + \delta_4)\sigma(u_{\varepsilon,h})}{(1 - \delta_5)\mu_1(h)} \leq \theta < 1,
\]

(3.14)

where \( \theta \) depends only on \( \delta_0 \).

Combining this with all estimates from above, we find
\[
(1 - \theta) \int_{\partial R_{\varepsilon,h} \setminus I_\varepsilon} |f_{\varepsilon,h}|^2 \leq \left( 1 + \frac{1}{\delta_4} \right) \left( \delta_2^2 + 3\delta_3^2 \right) + \frac{(1 + \delta_4)\delta_3^{1/2}}{\delta_4(1 - \delta_5)\mu_1(h)} \frac{4(1 + \delta_4)(1 + \delta_5)\mu_0(h)(\delta_3^{1/2} + \delta_2^{1/2})}{\delta_4(1 - \delta_5)\mu_1(h)}.
\]

If we take \( \delta_3 \) sufficiently small and use that \( \|f_{\varepsilon,h} - u_{\varepsilon,h}\|_{L^2(\partial \Sigma_{\varepsilon,h} \setminus I_\varepsilon)} \leq \delta_3^{1/2} \), this implies the claim. \( \square \)

We now apply the above proposition to exhibit a particularly useful choice of the parameter \( h \) for \( \varepsilon \) small.

**Proposition 3.15.** For sufficiently small but fixed \( \varepsilon \) there is \( h = h_\varepsilon \in (h_0, h_1) \), such that the multiplicity of \( \sigma_1(\Sigma_{\varepsilon,h}) \) is at least two.

**Proof.** Let \( \delta_1 > 0 \) such that \( [h_0, h_1] \setminus (h_* - \delta_1, h_* + \delta_1) \neq \emptyset \). We apply Proposition 3.1 with \( \delta_2 = 1/k \) and get some \( \varepsilon_k \) such that for any \( \varepsilon \leq \varepsilon_k \) and \( h \in [h_0, h_1] \setminus (h_* - \delta_1, h_* + \delta_1) \)
there is a normalized eigenfunction $w_{\varepsilon,h} \in E_{\varepsilon,h}$ which is unique up to sign and has

$$\int_{\partial \Sigma} |w_{\varepsilon,h}|^2 \leq 1/k. \tag{3.16}$$

For $\varepsilon$ fixed, the family of metrics on $\Sigma_{\varepsilon,h}$ depend analytically on $h$. Therefore, analytic perturbation theory, see [Kat95] Chapter 7, implies that for $\varepsilon$ fixed, there is family of normalized eigenfunctions $v_{\varepsilon}(h)$ that depends analytically on $h$ and satisfies $v_{\varepsilon}(h_0) = w_{\varepsilon,h_0}$. Since any normalized eigenfunction $u_{\varepsilon,h}$ orthogonal to $w_{\varepsilon,h}$ needs to have

$$\int_{\partial \Sigma} |w_{\varepsilon,h}|^2 \geq 1 - 1/k \tag{3.17}$$

by Proposition 5.3 it follows that $v_{\varepsilon}(h) = w_{\varepsilon,h}$ for $h \in [h_0, h_* - \delta_1]$. Moreover, since $v_{\varepsilon}(h) = w_{\varepsilon,h}$ for $h \in [h_0, h_* - \delta_1]$ it follows that

$$m_{\varepsilon}(h) = \int_{\partial \Sigma} |v_{\varepsilon}(h)|^2$$

is analytic. Moreover, since $v_{\varepsilon}(h) = w_{\varepsilon,h}$ in $[h_0, h_* - \delta_1]$ it follows that

$$m_{\varepsilon}(h) \to 0 \text{ in } [h_0, h_* - \delta_1]. \tag{3.18}$$

Moreover, it follows from Lemma 2.7 that $m_{\varepsilon}(h) \leq 2$ for any $h \in [h_0, h_1]$. Thus, up to taking a subsequence, $m_{\varepsilon}$ converges locally uniformly in $(h_0, h_1)$ to an analytic function $m_*$. Since $m_*$ is analytic, it follows from (3.18) that $m$ vanishes identically. In particular, we need to have $m_{\varepsilon}(h_1) \leq 1/4$ for $\varepsilon$ sufficiently small. This implies that we need to have $v_{\varepsilon}(h_1) = w_{\varepsilon,h_1}$ up to changing the sign.

If we choose

$$h_{\varepsilon} = \inf\{h \mid v_{\varepsilon}(h) \text{ is a } \sigma_1(\Sigma_{\varepsilon,h})\text{-eigenfunction}\},$$

we need to have $\text{mult} \sigma_1(\Sigma_{\varepsilon,h_{\varepsilon}}) \geq 2$. Indeed, if $\text{mult} \sigma_1(\Sigma_{\varepsilon,h_{\varepsilon}}) = 1$ the first eigenspace is spanned by $v_{\varepsilon,h_{\varepsilon}}$. But this remains the case for $h$ slightly smaller than $h_{\varepsilon}$, which contradicts the definition of $h_{\varepsilon}$. \hfill \Box

4. Proof of Theorem 1.1

In this final section we give the proof of Theorem 1.1. More precisely, we show that one can find a good lower bound for $\sigma_1(\Sigma_{h_{\varepsilon}})$, where $h_{\varepsilon}$ come from Proposition 3.15.

4.1. Reduction to Neumann boundary conditions. For $\varepsilon > 0$ sufficiently small we apply Proposition 3.15 and choose $h = h_{\varepsilon}$ with $\text{mult}(\sigma_1(\Sigma_{\varepsilon,h_{\varepsilon}})) \geq 2$, and $h_{\varepsilon} \in [h_0, h_1]$. From now on, we simply write $R_{\varepsilon} := R_{\varepsilon,h_{\varepsilon}}$, $\Sigma_{\varepsilon} := \Sigma_{\varepsilon,h_{\varepsilon}}$, and $\sigma_{\varepsilon} := \sigma_1(\Sigma_{\varepsilon})$.

**Lemma 4.1.** For $\varepsilon$ sufficiently small, we have

$$\sigma_{\varepsilon} \leq \sigma^D_0(R_{\varepsilon}, I_{\varepsilon}) = \sigma^N_1(R_{\varepsilon}, I_{\varepsilon}).$$

**Proof.** Let $v_{\varepsilon}$ be a $\sigma^D_0(R_{\varepsilon}, I_{\varepsilon})$-eigenfunction, which we extend by 0 to all of $\Sigma_{\varepsilon}$. Let $B_{\varepsilon} \subset \Sigma$ be a neighbourhood of $I_{\varepsilon}$. Since the capacity of a point relative to any ball vanishes, we can find a cut-off function $\psi : \Sigma \setminus B_{\varepsilon} \to [0, 1]$, such that $\psi = 0$ near $\partial B_{\varepsilon}$ and

$$\int_{\Sigma \setminus B_{\varepsilon}} |\nabla \psi|^2 \leq \frac{\sigma_{\varepsilon}}{2} \int_{\partial \Sigma \setminus I_{\varepsilon}} |\psi|^2.$$
Since $\psi$ vanishes near $I_\varepsilon$, we can extend it by 0 to all of $\Sigma_\varepsilon$. Consider the two dimensional space spanned by $v_\varepsilon$ and $\psi$. Since these two functions have disjoint supports, one easily checks that
\[
\int_{\Sigma_\varepsilon} |\nabla \varphi|^2 \leq \max\{\sigma_\varepsilon/2, \sigma_0^D(R_\varepsilon, I_\varepsilon)\}\int_{\partial\Sigma_\varepsilon} |\varphi|^2
\]
for any function $\varphi$ in this space. It follows, that
\[
\sigma_\varepsilon \leq \sigma_0^D(R_\varepsilon, I_\varepsilon).
\]

Combining the previous lemma with mult($\sigma_1(\Sigma)$) $\geq 2$ we can find a useful lower bound for $\sigma_1(\Sigma_\varepsilon)$.

**Lemma 4.2.** For $\varepsilon$ sufficiently small, we have
\[
\sigma_1^N(\Sigma, I_\varepsilon) \leq \sigma_\varepsilon.
\]

**Proof.** Since mult($\lambda_1(\Sigma_\varepsilon)$) $\geq 2$, we can choose a $\sigma_\varepsilon$-eigenfunction $u_\varepsilon$ on $\Sigma_\varepsilon$ satisfying
\[
\int_{\partial R_\varepsilon \setminus I_\varepsilon} u_\varepsilon = 0.
\]
Since $u_\varepsilon$ is orthogonal to the constant functions in $L^2(\partial\Sigma_\varepsilon)$, this implies
\[
\int_{\partial\Sigma \setminus I_\varepsilon} u_\varepsilon = \int_{\partial\Sigma_\varepsilon} u_\varepsilon - \int_{\partial R_\varepsilon \setminus I_\varepsilon} u_\varepsilon = 0.
\]
In particular, $u_\varepsilon\big|_{\Sigma}$ is an admissible test function for $\sigma_1^N(\Sigma, I_\varepsilon)$. We have
\[
\int_{\Sigma} |\nabla u_\varepsilon|^2 = \sigma_\varepsilon \int_{\partial\Sigma \setminus I_\varepsilon} |u_\varepsilon|^2 + \sigma_\varepsilon \int_{I_\varepsilon} |u_\varepsilon|^2 - \int_{R_\varepsilon} |\nabla u_\varepsilon|^2.
\]
Since $\int_{\partial R_\varepsilon \setminus I_\varepsilon} u_\varepsilon = 0$, we have
\[
\sigma_\varepsilon \int_{\partial R_\varepsilon \setminus I_\varepsilon} |u_\varepsilon|^2 \leq \frac{\sigma_\varepsilon}{\sigma_1^N(R_\varepsilon, I_\varepsilon)} \int_{R_\varepsilon} |\nabla u_\varepsilon|^2 \leq \int_{R_\varepsilon} |\nabla u_\varepsilon|^2,
\]
where we use Lemma 4.2. Inserting (4.5) into (4.4) implies
\[
\int_{\Sigma} |\nabla u_\varepsilon|^2 \leq \sigma_\varepsilon \int_{\partial\Sigma \setminus I_\varepsilon} |u_\varepsilon|^2.
\]
To conclude the argument, we need to show that $u_\varepsilon$ does not vanish identically on $\partial\Sigma \setminus I_\varepsilon$. If this were the case, it follows from the last inequality, that $u_\varepsilon$ vanishes identically on $\Sigma$. In particular, this implies
\[
\int_{R_\varepsilon} |\nabla u_\varepsilon|^2 = \sigma_\varepsilon \int_{\partial R_\varepsilon} |u_\varepsilon|^2.
\]
Moreover, we also get that the trace $u_\varepsilon|_{I_\varepsilon}$ of $u_\varepsilon$ has to vanish identically. Combining these observations with Lemma 4.1 implies that $u_\varepsilon$ is a non-trivial $\sigma_0^D(R_\varepsilon, I_\varepsilon)$ eigenfunction. This contradicts (4.3). \□
4.2. Conclusion. We are now prepared to proof Theorem 1.1

Proof of Theorem 1.1 By scaling, we may assume that length(∂Σ) = 1. For ε sufficiently small, we take h_ε given by Proposition 3.15 and consider Σ_ε = Σ_ε,h_ε. It follows from Lemma 4.2 that we have

σ_1(Σ_ε) ≥ σ_1^N(Σ, I_ε).

Let u_ε be a normalized µ_1^N(Σ, I_ε)-eigenfunction. By elliptic estimates, e.g. [Ro11, Theorem 3.14 (ii)], we find that u_ε is uniformly bounded. In particular, we find that

\left| \int_{∂Σ} u_ε \right| \leq C_ε^2.

Since u_ε is normalized, this implies by the variational characterization of the first eigenvalue, that

σ_1(Σ) \leq \frac{\int_{Σ} |∇u_ε|^2}{\int_{∂Σ} |u_ε|^2 - (\int_{∂Σ} u_ε)^2} \leq \frac{σ_1^N(Σ, I_ε)}{1 - O(ε^4)} \leq \frac{σ_1 Σ_ε}{1 - O(ε^4)}.

If we combine this with (4.7) length(∂Σ_ε) ≥ 1 + 2h_0ε + O(ε^2),

we obtain that

σ_1(Σ_ε) length(∂Σ_ε) > σ_1(Σ) length(∂Σ).

for ε sufficiently small. In a final step, we can approximate the metric on Σ_ε,h be a smooth metric, so that we still have the strict inequality from above.

□

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