The universe could be dark, ma non troppo

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Abstract.
We will present new insights into the dark phenomena, i.e., the recent acceleration of the universe linked to a sort of dark energy, and to the unknown dark matter. The study is based only on Einstein’s equations without cosmological constant, and on ordinary matter described as point masses. We will be limited, thereby, to the post-recombination epoch. We shall revise the outspread statement that a universe made of collision-less particles is well represented by dust, i.e; by an Einstein-de Sitter universe. Using well known results on the N-body problem expressed as a infinite series, and starting at zero order with the empty Milne universe, we shall get the aforementioned EdS universe at the first order, but at the next one we shall obtain a cosmological model whose energy density could explain the dark phenomena. No exotic dark components are necessary in principle, but we need to know the redshift of formation of the dominant particles in the present epoch, that we identify with the galaxies. Thus, assuming that redshift to be of the order of eleven, we shall get that the time evolution of the acceleration, and the supernovae luminous distance-redshift relation, are indistinguishable from the ones predicted by the $\Lambda$CDM model. However, if there was no realistic evolution model that could justify such an early galaxy formation epoch, then some quantity of dark energy would be necessary, ma non troppo.

1. Introduction
The necessity of a mysterious energy accounting for the recent acceleration of the universe comes from the outspread statement that a universe dominated by collision-less gravitating particles, should be described by a pressure-less perfect fluid, known as Einstein de Sitter universe (EdS); because, without a certain quantity of negative pressure, the universe will be by contrary decelerated. On the other hand, exotic matter is necessary for understanding the discrepancy between the content of mass traced by light and the traced by its gravitational effects. In this paper we accomplish a revision of the mentioned statement in two steps. We describe first properly the gravitational interaction of massive particles, arranged in a way so as satisfy the requirement of local isotropy everywhere, and then, we study the appropriate averaging in order to get a macroscopic description of the system of particles.

Our view is based on the fact that the energy and the metric tensors are untangled in the Einstein equations, and, their non linearity makes the averaging of the microscopic equations much more involved than in the linear case, as for example in the pure electromagnetic case. The average of both the energy and the metric tensors should be compatible with Einstein’s equations, if we want they represent the system of auto gravitating particles. In section 3, we use a well known iterative method to obtain the metric and the energy tensor for a system of
gravitating particles. In this way the average of the metric at each iteration is compatible with average of the energy tensor. We obtain the Einstein de Sitter universe in the first iteration, but the energy tensor corresponding to the second iteration is able to explain completely the observed dark energy phenomena (Subs. 5.1); and also, in a model dependent way, we can explain the dark matter content at cosmological scales (Subs. 5.2); and even, accepting an heuristic view, we can reproduce some aspects of the dark matter at galactic scales (Subs. 5.4).

This paper is on the wave of what generally is referred as backreaction. In the measure that we take care of the fact that, at each iteration, the averaged metric and the averaged energy tensor must satisfy the non linear Einstein equations, we are obtaining the backreaction of the metric on the macroscopic energy tensor. But, in the backreaction literature [2], the universe is considered as the real inhomogeneous universe, the averaging process is generally an spatial average, in order to reach a homogeneous universe. One study the backreaction of the small scale structure onto the behavior of the universe at great scale. The difference with our view is that we consider an unreal homogeneous and isotropic universe, made of particles, and study the pass to the continuous description by means of an statistical average. In our case it is the backreaction of the metric on the macroscopic energy tensor what is relevant.

2. Gravitating point particles in a cosmological setting

Let us start recalling the main features of the Milne’s cosmological model. It is a portion of a Minkowskian space, formed by a future light cone and a set of world lines inside it, whose intersection points with a hyperboloid of constant Minkowski proper time is a realization of an homogeneous Poisson process. Now we imagine that at some initial proper time $\tau_j$ the world lines represent the center of mass of galaxies, that will be considered as point particles. Havas-Goldberg [1] developed a method of solving the Einstein’s equations for a finite number of point particles, taking as energy tensor a distribution with support over the parametrized world lines of the particles. They introduced the Minkowski metric as an auxiliary one and used as parameter the Minkowskian proper time. The energy tensor takes the form

$$\mathbf{T}^{\mu\nu} = \sum_{j=1}^{N} M_j(\tau_j) v^\mu \cdot v^\nu$$

with $v^\mu = \frac{dx^\mu}{d\tau_j}$. An iterative method allows determining the terms of the series for the metric

$$g^{\mu\nu}(1) + g^{\mu\nu}(2) + \cdots$$

and the mass function:

$$M_j(\tau_j) = M_j^{(0)}(\tau_j) + M_j^{(1)}(\tau_j) + M_j^{(2)}(\tau_j) + \cdots$$

Using de Donder coordinate conditions one put Einstein equations in the form

$$\sum_{m=1}^{n} (\Box + 2\Lambda^{(m)}_{\mu\nu}) - 16\pi G \sum_{m=0}^{n-1} T^{(m)}_{\mu\nu}, \quad \Lambda^{(1)}_{\mu\nu} = 0$$

where the box represents the Dalambertian operator related to the auxiliary Minkowski metric, and appear two basic tensors:

$$T^{(m)}_{\mu\nu} := (\sqrt{-g}T^{\alpha}_{\mu})^{(m)}_{\alpha\beta} \eta_{\alpha\nu}$$

$$\gamma^{(m)}_{\mu\nu} := g^{(m)}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \eta^{\alpha\beta} g^{(m)}_{\alpha\beta}$$

The first order equations are

$$\Box \gamma^{(1)}_{\mu\nu} = -16\pi G T^{(0)}_{\mu\nu}$$

$$\eta^{\rho\lambda} \partial_{\lambda} \gamma^{(1)}_{\mu\nu} = 0$$
The field equation to first order gives the metric
\[ g^{(1)}_{\mu\nu}(x) = -4Gm \sum_{j=1}^{N} \frac{(\eta_{\mu\alpha}v_j^\alpha v_j^\beta - \frac{1}{2}\eta_{\mu\nu})r}{\eta(x - z_j, v_j)_{r}} \]  
(7)

and the coordinate condition produces the motion and the mass to order zero:
\[ \frac{dv_j^\alpha}{d\tau_j} = 0, \quad \dot{M}_j^{(0)} = 0 \quad \Rightarrow \quad M_j^{(0)} = m_j \]  
(8)

The second order equations are
\[ \square(\gamma^{(1)}_{\mu\nu} + \gamma^{(2)}_{\mu\nu}) + 2\Lambda^{(2)}_{\mu\nu} = -16\pi G(T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu}) \]
\[ \eta^{\mu\lambda}\partial_\lambda(\gamma^{(1)}_{\mu\nu} + \gamma^{(2)}_{\mu\nu}) = 0 \]  
(9)
(10)

the coordinate condition produces the motion and the mass to order one:
\[ \frac{d}{d\tau_j}(mv_j^\alpha(\eta_{\mu\alpha} + g^{(1)}_{\mu\alpha}) + M_j^{(1)}v_j^\alpha\eta_{\mu\alpha}) - \frac{1}{2}mv_j^\alpha v_j^\beta\partial_\lambda g^{(1)}_{\rho\lambda} = 0 \]  
(11)
\[ \dot{M}_j^{(1)} = -\frac{1}{2} \frac{d}{d\tau_j}(mg^{(1)}_{\alpha\beta}v_j^\alpha v_j^\beta) \]  
(12)

We shall assume the same mass for all the particles, hence from the last equation we get the mass function in the form
\[ M^{(1)} = -\frac{1}{2}mg^{(1)}_{\alpha\beta}v^\alpha v^\beta + C^{(1)}, \quad C^{(1)} = ma \]  
(13)

where we have written the constant of integration in the form $ma$. This is the basic result for solving the dark problems, and we proceed to obtain the sum appearing in the expression of the metric. A regularization is necessary because the metric is infinite on the world lines of the particles. This problem was solved with different methods and give for the metric on a particle $k$ the expression
\[ g^{(1)}_{\mu\nu}(z_k(\tau)) = -4Gm \sum_{j\neq k}^{n} \frac{\eta_{\mu\alpha}v_j^\alpha v_j^\beta - \frac{1}{2}\eta_{\mu\nu})r}{\eta(x - z_j, v_j)_{r}} + 4m(v_k^\alpha v_k^\beta + \bar{v}_k^\alpha \bar{v}_k^\beta)\eta_{\mu\alpha}\eta_{\nu\beta}. \]  
(14)

A major simplification comes from the fact that the world lines of the particles are straight lines, for $(z^0 = \tau, z^i = 0)$ is solution of the equations of motion (8,11), therefore only the sum in the right hand of (14) should be considered. Moreover, as all the world lines are equivalent due to the Lorentz invariance of the evolution equations, we only need to calculate the physical metric over a world line of reference $L$, and, as we want to obtain the mass function given in (13) we only need the component $g^{(1)}_{00}$ of the metric tensor, because using coordinates in which the particle of reference is at rest $v^a = 0$ for $a = 1, 2, 3$. Then the metric at any point $p_o$ over the line of reference $L$ after the birth of the galaxies will be the result of the sum
\[ g^{(1)}_{00}(p_o) = -Gm \sum_{j} \frac{\gamma^2_j - 1/2}{r_j\gamma_j(1 + \bar{v}_j)} \]  
(15)

At any point $p_o$ over the reference line, the sum in equation (15), is always over a finite number of particles, even if the universe is made of infinite particles, for only intervene the world lines
intersecting the past light cone of \( p_0 \) limited by the initial hyperboloid. We assume that the intersection of the world lines of the particles with the hyperboloid \( \tau = \tau_i \) define a uniform Poisson random process, the sum becomes an integral over the hyperboloid. This integral is in fact the statistical average. A change of variables (see appendix) allow us to integrate over the hyperplane \( t = \text{const.} \).

\[
\langle g_{00}^{(1)}(p_0) \rangle = -4GNm \int_0^{r_{\text{max}}} \frac{t}{(t^2 - r^2)^2} r_j \gamma_j (1 + v_j) \frac{4\pi r^2 dr}{4} 
\]

The statistical average of the \( g_{00} \) metric component can be found in the Appendix of [2]

\[
\langle g_{00}^{(1)} \rangle = -\frac{\Omega_{\text{bai}}}{4} H_0^2 t^2 g(t, t_i) \quad \langle g_{00}^{(1)} \rangle = 0
\]

\[
g(t, t_i) = \left( 1 + \frac{t_i}{t} \right)^3 - 4\frac{t_i^2}{t^2} \left( \frac{3}{2} (1 + \frac{t_i}{t}) - t_i \right)
\]

\[
\frac{dt}{ds} = 1 - \frac{1}{2} \langle g_{00}^{(1)} \rangle = 1 + \frac{\Omega_{\text{bai}}}{8} H_0^2 t^2 g(t, t_i)
\]

The function \( g(t, t_i) \) and its first derivative vanishes at the initial time \( t_i \), and tends rapidly to unity when \( t \) increases. This is relevant to explain the acceleration, without use of a cosmological constant, as we will show below. Substituting into (13) we get the statistical average of the mass, up to the first order:

\[
M(t) = m \frac{dt}{ds} + C^{(1)} \quad C^{(1)} \equiv m\alpha
\]

\[
M(t) = m \left( 1 + \alpha + \frac{\Omega_{\text{bai}}}{8} H_0^2 t^2 g(t, t_i) \right)
\]

This is all we need from the N-body gravitational problem. Now we pass to averaging the field equations in order to obtain a cosmological model.

3. The macroscopic energy tensor

In this section we will make the transition from a discrete system to a continuum and that implies to define a kind of averaging of the "microscopic" equations, in our case, the field equations (1). We have already given in Eq. (17) the statistical average of the first order metric. The average of non linear quantities as \( A_{\mu\nu}^{(2)} \) reduces to the product of averages of first order quantities, for our statistical process is a uniform Poisson random process. It remains to define the average of distributional tensor densities of the kind \( \sqrt{-g} P^{\mu\nu} = \sum \int \tilde{p}^{\mu\nu}(\lambda) \delta(4)(x^\rho - z^\rho(\lambda)) d\lambda \) with support on world lines of particles, where the parameter \( \lambda \) stand for the Minkowskian time \( \tau \) or for the proper time \( s \). Two comments will help to introduce a convenient procedure:

- The world lines of the particles are straight lines at each iteration.
- The surface \( \tau = \text{const.} \) coincides with a surface \( s = \text{const.} \); but with \( \tau > s \), as can be derived from the expression of \( dt/\text{ds} \) given above in (19).

Let \( A_x \) be a convenient neighborhood of the point \( x \), defined by two neighbor hypersurfaces: \( \tau = t + \Delta t/2 \) , \( \tau = t - \Delta t/2 \) (that are also hypersurfaces of constant proper time: \( s = s(t) + \Delta s/2, s = s(t) - \Delta s/2 \) and a thin time like cone; and let \( S \) be the intersection of the hypersurface \( \tau = t \) with \( A_x \) as shown in Figure 1.

Let \( \varphi_x \) be the characteristic function of the set \( A_x \): \( \varphi_x(u) = 1 \) if \( u \in A_x \), \( \varphi_x(u) = 0 \) otherwise. The average of a distributional tensor density \( \sqrt{-g} P^{\mu\nu} \) is defined as follows
Figure 1: Element of volume \( A_x \) centered at a point over the reference line \( L \), limited by two surfaces \( \tau = \text{const.} \). We show the 3-dimensional surface \( S \) necessary to get the density.

\[
\langle \sqrt{-g} \, P^{\mu \nu} \rangle (x) = \lim_{\Delta \lambda \to 0} \frac{\langle \sqrt{-g} \, P^{\mu \nu}, \varphi \rangle}{\text{Vol}^{(3)}(S) \Delta \lambda},
\]

where the numerator is the action of the distribution on the characteristic function, and \( \lambda \) denotes \( \tau \) or \( s \). This average coincides with the statistical average of the quantities \( P^{\mu \nu} \) in the case of a uniform random Poisson process over the surface \( \lambda = \text{const.} \).

Let us consider the distribution \( \sqrt{-g} T^{\rho \sigma} = \sum_{j=1}^{N} m_j u^\rho_j u^\sigma_j \delta^{(4)}(x^\rho - z^\rho_j(s_j)) ds_j \), where now the support are curves parametrized with the proper time. The value that this distribution takes on a test function is \( \langle \sqrt{-g} T^{\rho \sigma} \rangle (x) = \lim_{s \to 0} \frac{\langle \sqrt{-g} T^{\rho \sigma}, \varphi \rangle}{\text{Vol}^{(3)}(S) \Delta s} = nm u^\rho \). The energy density for a universe made of collision-less particles is then:

\[
\rho_{\text{micro}} = T_{00} = \rho \sqrt{-g} T^{00},
\]

so, \( \rho_{\text{micro}} = \sqrt{-g} T^{00} \). Averaging this quantity we get the macroscopic energy density \( \rho = \langle \rho_{\text{micro}} \rangle \)

\[
\rho = \lim_{\Delta t \to 0} \frac{\langle \sqrt{-g} T^{00}, \varphi \rangle}{\text{Vol}^{(3)}(S) \Delta t} = n(t) M(t)
\]

with the mass given in Eq. (21).
4. Construction of a sequence of FLRW cosmological models

Let us first recall some basic facts. Any space-time which admits a motion pattern, i.e., a set of world lines defining an observer, for which there exist local isotropy at each point, is a FLRW cosmological model [3]; and any FLRW model is completely characterized by a choice for the index of curvature \( k \), and giving the energy density as a function of the expansion factor \( \rho_r(a) \), that we shall write in the form

\[
\rho_r(a) = \frac{3H_0^2}{8\pi G} \left( \frac{\Omega_m}{a^3} + f(a) \right),
\]

The Friedmann equation

\[
\frac{da}{ds} = H_0 \sqrt{\frac{\Omega_m}{a} + \Omega_k + a^2 f(a)}, \quad \Omega_k = -k/H_0^2 R_0^2
\]

determines the expansion factor as function of the cosmological time \( s \). Henceforth we shall take \( k = 0 \). Standard calculations produce the pressure and the acceleration of the model

\[
p(a) = \frac{3H_0^2}{8\pi G} \left( -f(a) + \frac{1}{3} a \frac{df}{da} \right)
\]
\[
\frac{\ddot{a}}{a} = -\frac{\dot{H}_0^2}{H_0^2} \left( \frac{\Omega_m}{2a^3} - f(a) + \frac{1}{2} a \frac{df}{da} \right)
\]

In the two previous sections we have considered an algorithm \( \eta \rightarrow T^{(0)} \rightarrow g^{(1)} \rightarrow T^{(1)} \rightarrow g^{(2)} \rightarrow \cdots \) which allows to obtain a sequence of approximate space-times:

\[
\left( M^+, \eta + \langle g^{(1)} \rangle, \langle T^{(0)} \rangle \right), \left( M^+, \eta + \sum_{n=1}^{\infty} \langle g^{(n)} \rangle, \sum_{k=0}^{\infty} \langle T^{(k)} \rangle \right) \left( M^+, \eta + \sum_{n=1}^{\infty} \langle g^{(n)} \rangle, \sum_{k=0}^{\infty} \langle T^{(k)} \rangle \right)
\]

In fact, we have obtained the first, and partially the second one, because we know the macroscopic energy tensor to the first order but not the metric to second order \( g^{(2)} \). If the series \( \sum_{k=1}^{\infty} g^{(k)} \) converges to a metric \( g^{\text{lim}} \) it will be diffeomorphic to a FLRW metric \( g_F \) in the region \( s \geq s_i \), because by construction it will be locally isotropic everywhere. We will not need to determine the whole change of coordinates \( t = t(s, \bar{x}^m) \), \( x^i = x^i(s, \bar{x}^m) \) that transforms de Donder coordinates \( (t,x^m) \) into the standard cosmological coordinates \( (s,\bar{x}^m) \); the problem is quite simple if one takes into account that:

- We know that necessarily \( n(t(a)) = n_o/a^3 \).
- We only need the function \( t = t(a) \) which relates the time like de Donder coordinate to the expansion factor of the universe; for, on the reference world line \( L \) we know \( t = t(s) \), given by Eq. (19), and \( s = s(a) \) given by Eq. (27).

We can get the function \( t = t(a) \) as a series \( \sum_{k=0}^{\infty} t^{(k)}(a) \), though as we will show here we shall only need the term \( t^{(0)} \). Let us write (19) in the form

\[
\frac{dt}{ds} = \sum_{k=0}^{\infty} F^{(k)}(t), \quad F^{(0)}(t) = 1, \quad F^{(1)}(t) = -\frac{1}{2} \frac{\dot{g}^{(1)}}{g_0}
\]

from Eq. (27), with \( k = 0 \), we get \( \frac{ds}{da} = \frac{1}{H_0 a} \left( \frac{\Omega_m}{a^3} + f(a) \right)^{-1/2} \), and from the relation \( \frac{dt}{da} = \frac{dt}{ds} \frac{ds}{da} \) we get the equation that determines \( t(a) \):

\[
\frac{dt}{da} = \frac{1}{H_0 a} \left( \frac{\Omega_m}{a^3} + \frac{\Omega_o}{a^3} \sum_{k=1}^{\infty} F^{(k)}(t(a)) \right)^{-1/2}
\]
To order zero we obtain

$$H_0t^{(0)} = \frac{2a^{3/2}}{3\Omega_m^{1/2}}$$

(33)

But, each element of the sequence is also locally isotropic everywhere, hence we can associate to each term of the sequence \(M^+, \eta + \sum_{n=1}^n (g^{(n)}), \sum_{k=0}^{n-1} (l^{(k)})\), the FLRW model that corresponds to the energy density \(\sum_{k=0}^{n-1} \rho^{(k)}(t(a))\). We shall denote these space-times as:

\[(M, \rho^{(0)}_F), (M, \rho^{(0)}_F + g^{(1)}_F), \cdots\]

(34)

Henceforth we shall refer all densities to the critical density. So, we shall write: \(\rho_m = \frac{3H_0^2}{8\pi G} \Omega_{ba}, \quad n_o m = \frac{3H_0^2}{8\pi G} \Omega_{ba}, \quad n_o (m + \alpha) = \frac{3H_0^2}{8\pi G} \Omega_m, \quad \Omega_m = \Omega_{ba}(1 + \alpha)\). We have obtained the two first terms of the sequence (34):

(i) \(\rho^{(0)}_F(a) = \frac{3H_0^2}{8\pi G} \Omega_{ba}(1 + \alpha)\), hence the first one is an Einstein de Sitter universe.

(ii) \(\rho^{(0)}_F(a) + \rho^{(1)}_F(a)\) is obtained by replacing \(n(t) \rightarrow n_o/a^3\), and substituting \(t = t^{(0)}(a)\), into equation (25). Using henceforth the notation \(\rho_p(a) = \rho^{(0)}_F(a) + \rho^{(1)}_F(a)\), we shall write the energy density (25) in the standard form of Eq. (26):

$$\rho_p(a) = \frac{3H_0^2}{8\pi G} \left(\frac{\Omega_m}{a^3} + f(a)\right), \quad f(a) = \frac{g(a, a_i)}{18a_i^2(1 + \alpha)}$$

(35)

$$g(a, a_i) = \left(1 + \frac{a_1^3}{a^3}\right)^3 - 4\frac{a_1^3}{a^3}\left(\frac{3}{2} \left(1 + \frac{a_1^3}{a^3}\right) - \frac{a_1^{3/2}}{a^{3/2}}\right)$$

(36)

where the function \(g(a, a_i)\) vanishes at \(a = a_i\) and tends rapidly to a constant for increasing \(a\). We assume that the series \(\sum_{k=0}^{\infty} \rho^{(k)}_F(a)\) converge to the true metric \(\rho^{lim}_F(a)\), though this has never been proved. But, the goal of this work is to show how well the two first terms of the series agree with the observations.

5. What is dark matter and what is dark energy

It is convenient to express Eqs. (35-36) in terms of the redshift. Substituting \(a = \frac{1}{1+z}\) one gets:

$$\rho_p(z, z_i) = \frac{3H_0^2}{8\pi G} \left(\Omega_{ba}(1 + \alpha)(1 + z)^3 + f_z(z, z_i)\right), \quad f_z(z, z_i) = \frac{(1 + z_i)^3}{18(1 + \alpha)} g_z(z, z_i)$$

(37)

The component \(f_z(z, z_i)\), is zero at the formation redshift \(z_i\) and tends to a constant when \(z\) diminishes, hence, it is likely that it could give predictions similar to the derived from the cosmological constant. But as it is not a constant, this component could be an explanation of what is been denoted by a dark energy. On the other hand, it is evident that the parameter \(\alpha\) is related with the dark matter at cosmological scales: in terms of the critical density, the dark matter density is \(\Omega_{dm} = \Omega_{ba}\alpha\). Let us explain separately the meaning of these components.

5.1. Dark energy

The energy density \(\rho_p(z, z_i)\) depends on three parameters, namely, \(\Omega_{ba}, \alpha, z_i\). We have obtained the moduli distance-redshift relation corresponding to this cosmological model. One can fit this expression to the supernovae data to get the parameter’s values, to assess the goodness of the fitting, and compare with the \(ΛCDM\) model. But, here we shall proceed in a less systematic way to obtain a set of parameters that agree quite well with the supernova data. We shall take
(Ω_{ba}, Ω_{dm}) = (0.049, 0.268) from the ΛCDM concordance model, and derive from them our parameter α = Ω_{dm}/Ω_{ba}. Our third parameter z_i can be derived using the fact that our energy density component f_z(z, z_i) tends rapidly to a constant for small redshifts, and, identifying the value of this function at zero redshift, with the energy density Ω_A = 0.683 (related to the cosmological constant) obtained in the concordance model. The equation f_z(0, z_i) = Ω_A is a relation of the form F_i(Ω_{ba}, Ω_{dm}, Ω_A, z_i) = 0 that determines the redshift of galaxy formation z_i = 10.76. With these values our component f_z(z, z_i) is of the form shown in Fig. (2). Our cosmological model for the galactic phase is determined by the energy density given in Eq. (37), whose main consequences are the following:

(i) The moduli distance-redshift relation up to redshift 2, that we show in in Fig. (4). The agreement with the concordance model is quite impressive.

(ii) The deceleration factor as function of redshift, taking into account Eq. (29), that we show in Fig (3), where we see again the good agreement with the prediction of the concordance model. We have now a reason why the acceleration of the universe is a recent phenomena: It is due to the gravitational interaction between the galaxies, hence it must occur at redshift lesser than the galaxy formation redshift.

We still have a matter pending, that of justifying a non zero constant of integration α in our equations for the mass (13) and (21), that now will be convenient to express as function of the redshift. Using Eq. (33) and a = 1/(1 + z) we get:

\[ M(z, z_i) = m + M^{(1)}(z, z_i) \, , \, \, M^{(1)}(z_i, z_i) = mα \]  

\[ M^{(1)}(z, z_i) = m \left( α + \frac{(1 + z_i)^3}{18(1 + α)(1 + z)^3} g_z(z, z_i) \right) \]

The first summand is the baryonic mass, and, we shall call gravitational mass to the second one.

5.2. Dark matter at large scales

The only way to justify α ≠ 0 is to assume that before galaxy formation there was a phase dominated also by particles, "stars", formed at redshift z_i', that collapsed at redshift z_i to form the galactic phase. We shall consider this phase described by a FLRW model too. This is certainly a rough model of evolution, used only to convey the main idea of this work, though qualitatively produces interesting results. In this phase we can write for the total mass of a
Figure 4: Our prediction for the moduli distance redshift relation is indistinguishable from the $\Lambda$ CDM model.

star expressions like Eqs. (38, 39), but with magnitudes refered now to the star phase, that we denote by quotations marks:

$$M'(z, z'_i) = m' + M'^{(1)}(z, z'_i), \quad m' \alpha' = M'^{(1)}(z'_i, z'_i)$$

$$M'^{(1)}(z, z'_i) = m' \left( \alpha' + \frac{(1 + z'_i)^3}{18(1 + \alpha')(1 + z)^3} g_z(z, z'_i) \right)$$

We shall require at galaxy formation redshift $z_i$:

- Continuity baryonic mass: $m = N'm'$, where $N'$ number of stars in a galaxy. Multiplying by the number density of galaxies at the formation redshift $n(z_i)$ we have $n(z_i)m = n(z_i)N'm'$, from which we get the number densities of stars $n'(z_i) = N'n(z_i)$.

- Continuity of gravitational mass:

$$m\alpha = M^{(1)}(z_i, z_i) = N'M'^{(1)}(z_i, z'_i)$$

This assumption, supports taken $\alpha \neq 0$, considering $m\alpha$ as the gravitational mass inherited from the previous phase: The stars that form a galaxy have being increasing its mass due to the gravitational interaction with all the stars crossing its past lift cone; in the same way that the mass of a galaxy augments in the galactic phase. As a consequence, the total mass is conserved $M(z_i, z_i) = N'M'(z_i, z'_i)$, and multiplying by $n(z_i)$ we obtain that the energy density is conserved: $n(z_i)M(z_i, z_i) = n(z_i)N'M'(z_i, z'_i) = n'(z_i)M'(z_i, z'_i)$. Taking into account Eqs. (38, 39), and (40, 41) we can write:

$$\alpha = \alpha' + \frac{(1 + z'_i)^3}{18(1 + \alpha')(1 + z)^3} g_z(z_i, z'_i)$$
But Eq. (41) has introduced another constant of integration $\alpha'$ for the star phase. We can sustain again a non zero value for it by assuming that, before the star formation, there was a phase dominated also by free particles, that will be referred as "free atoms",
5.4. Heuristic approach to dark matter at galactic scales

In order to explain dark matter at galactic scales it is necessary to consider inhomogeneous perturbations of our homogeneous model; but, we can get some insights from an "heuristic viewpoint", suggested by the vacuole solutions (Einstein-Strauss, Kottler). These are exact solutions in which, inside a vacuole, an inhomogeneity is isolated from the cosmological expansion.

So, we assume a gradual transition from a total screening from the cosmological effects at the center of the inhomogeneity, to a total cosmological influence. Let us make the conjecture that the energy density at distance \( r \) from the center is well described by the expression:

\[
\rho(r, z) = \mu_{ba}(r) + \chi(r)\mu_{ba}(r)\left(\alpha + \frac{(1 + z_i)^3}{18(1 + \alpha)(1 + z)^3}g_z(z, z_i)\right),
\]

where \( \chi(r) = 1 - \frac{\Phi_{ba}(r)}{\Phi_{ba}(0)} \), \( \mu_{ba} \) is the baryonic mass density, and \( \Phi_{ba}(r) \) the galactic newtonian gravitational potential. With this we can calculate the total mass, the gravitational mass and the fraction of gravitational mass contained inside a sphere of radius \( r \); and, the rotation velocity at radius \( r \):

\[
M(r, z) = \int_0^r \rho(r, z)(u)4\pi u^2du, \quad M_{GR}(r, z) = \int_0^r (\rho(r, z) - \mu_{ba}(r))4\pi u^2du
\]

\[
f_{GR}(r, z) = \frac{M_{GR}(r, z)}{M_{ba}(r) + M_{GR}(r, z)}
\]

\[
v^2(r) = \frac{GM(r, z)}{r}
\]

Baryons and dark matter has been recently disentangled [6] in the spiral lens B1933+503, observed at redshift \( z = 0.755 \). We shall use a Plummer’s model, with baryonic density \( \mu_{ba} = \frac{3M}{4\pi R^2}(1 + \frac{r^2}{R^2})^{-5/2} \) and newtonian potential \( \Phi_{ba} = -\frac{GM}{R}(1 + \frac{r^2}{R^2})^{-1/2} \). We have taken \( M = 2.6 \times 10^{11}M_\odot \) and scale factor \( R = 12Kpc \) in order to get the intersection of dark matter and baryonic mass profiles just at \( r = 10Kpc \), as it is manifest in Fig. (12) of [6]. We have represented in Fig. (5) the baryonic and the gravitational mass inside a radius \( r \) as calculated with Eqs. (50, 51). They agree qualitatively with the observations of baryonic and dark matter reported in [6].

The rotation curve calculated using Eq. (52) is shown in Fig. (6), and contrasted with some observations taken from [6]. We see that the baryonic component fails to explain the observed velocities, but adding the gravitational mass we get a good description.

The dependence on redshift of the gravitational mass fraction profile, given by Eq. (51), is illustrated in Fig. (7) for the redshifts \( z = 0.755 \) and \( z = 0.138 \). The fraction \( f_{GR}(r, z) \) is greater for the older galaxies (lower \( z \)), for the content of gravitational mass increases with the age of the observed galaxy, as consequence of Eqs. (38 , 39). In Fig. (8) we give the graph of \( f_{GR}(8.2, z) \), calculated with Eq. (51) to compare with the observed dark matter fraction in galaxies B 1933+503 and and SDSSJ2141-000, recently reported in [6] and [7] respectively. In both cases \( 8.2Kpc \approx 2.2R_d \) with \( R_d \) equal to the disk scale length. We conclude that our conjecture gives qualitative good results and could be a guide to start the study of the inhomogeneous case.

6. Conclusions

6.1. The more solid result

The equation (24) for the energy density, \( \rho = n(t)M(t) \), comes from the second iteration of the Einstein equations (first order in the energy tensor, second order in the metric). It is
the basic result of this work, because it means that the mass of a particle, formed at instant $t_i$, is the monotonic increasing function $M(t)$ given in Eq. (21), or, expressed in terms of redshift, in Eqs. (38, 39). These equations say that the mass of a particle depends on the interaction with all the particles over its past light cone; and, as show Eqs. (19-20) it is composed of the baryonic mass plus the contribution of the metric. We have denoted this component as gravitational mass. This is clearly on the line of the Mach principle on the inertial mass, but here appears not as a principle, but as consequence of the Einstein equations. It has been an agreeable surprise to find a similar sentence to the outlined above in the Einstein’s address to the Prussian Academy in 1921 [8]: The general theory of relativity teaches that the inertia of a given body is greater if there are more ponderable masses in proximity to it; thus it seems very natural to reduce the total effect of inertia of a body to the action and reaction between it and the other bodies in the universe. Dark matter and dark energy can be explained with this ignored gravitational mass. Recalling

Figure 5: Gravitational and baryonic mass inside a sphere of radius $r$ predicted by our model for the galaxy B 1933+503. It is similar to the DM observations showed in Fig 12 of [6].

Figure 6: Predicted rotation curve considering baryonic plus gravitational mass (upper curve), and only baryonic mass (lower curve) in galaxy B 1933+503. Observed velocities taken from [6]. It is similar to Fig 8 in [6].

Figure 7: Gravitational mass fraction inside a sphere of radius $r$ in a galaxy similar to B 1933+503 for two different redshifts.

Figure 8: Gravitational mass fraction inside a sphere of 8.2 Kpc as function of redshift, and observed values in galaxies B 1933+503 ($z=0.755$), and SDSSJ2141-0001 ($z=0.138$).
the Eqs. (38, 37)

\[ M(z, z_i) = m + M^{(1)}(z, z_i), \quad M^{(1)}(z, z_i) = m \alpha, \quad z < z_i \]

\[ \rho(z, z_i) = n(z) M(z, z_i) = \frac{3H_0^2}{8\pi G} \left( \Omega_{ba}(1 + \alpha)(1 + z)^3 + \frac{(1 + z_i)^3}{18(1 + \alpha)(1 + z)^3} g(z, z_i) \right) \]

will facilitate the following summary. The last addend in the right hand of Eq. (54), represented in Fig. (2), is responsible of the acceleration of the universe, due to its rapid tendency to a constant value. This fact justifies the cosmological constant, only as the limit value of the gravitational energy density. There is no need of a mysterious dark energy, is just the metric, i.e., it is just gravitation. Figs. 3, 4 show great agreement of our predictions with the obtained with the \( \Lambda CDM \) model, and therefore, the agreement with observations. From the supernova experiment we can infer the first three parameters of our model: \((\Omega_{ba}, \Omega_{ba\alpha}, z_i) = (0.049, 0.268, 10.76)\). So we predict galaxy formation at redshift \( \approx 11 \) when the universe was 400 million years old. These results are based only on the existence of a phase of the universe, locally isotropic everywhere, dominated by auto-gravitating particles (galaxies).

6.2. Model dependent results

In order to explain the dark matter component at large scales we have needed two more phases dominated by particles. In subsection 5.2 the constant \( m \alpha \) in Eq. (53) has been explained as the gravitational mass acquired by the particles forming the previous phases, namely, stars and atoms. Multiplying this constant by the number density of galaxies one explains the constant \( \Omega_{ba\alpha} \) appearing in Eq. (54). This can explain the dark matter component at cosmological scales, without necessity of exotic particles. Our model depends on six parameters: the baryonic mass density \( \Omega_{ba} \), the gravitational mass densities in the galactic and the star phases \( \Omega_{ba\alpha} \) and \( \Omega_{ba\alpha'} \), the redshifts of formation of galaxies and stars: \( z_i < z_i' \), and the redshift of the effective decoupling of atoms and radiation \( z_i'' \). Requiring continuity of total energy density through the surfaces of formation of the stars and galaxies, and assuming \( z_i'' = 100 \) as it is indicated in [4], we get the two remaining parameters \((\Omega_{ba\alpha'}, z_i') = (0.267, 20.7)\).

Therefore, the major part of the gravitational mass was produced in the atomic phase \((\Omega_{ba\alpha'} = 0.267 < \Omega_{ba\alpha} = 0.268)\), and the redshift of formation of the star phase was \( z_i' \approx 21 \) when the universe was a hundred millions years old.

6.3. Results dependent on a conjecture

We have inferred funny results, making a conjecture, about the explanation of dark matter phenomena at galactic scales. Figures (5 - 8) show predictions for rotation curves, gravitational mass fraction profile, and its redshift dependence that seems to be in agreement with the observations.

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Appendix

Let us show how we have got the statistical average of the \( g^{(1)}_{00} \) metric component, given in equations (17, 18). We must calculate the sum appearing in (9). We recall that the worldlines are parametrized with the Minkowskian time \( \tau \). Let \( p_o \) be a point on a world line of reference \( L \), and choose coordinates \( \{x^\alpha\} \) such that the tangent vector at \( p_o \) to this line be \( v = \partial_t \), and the coordinates of \( p_o = (t, 0, 0, 0) \). Let \( \Sigma_t = \{x \in M/\{x^0 = t = \text{const.}\} \).
\[ \Sigma_r = \{ x \in M / t^2 - r^2 = \tau = \text{const.} \}, \]
and denote by \( C_{p_o}^{\text{past}} \) the past light cone at \( p_o \), limited by
the hiperboloid \( \Sigma_r \), which represents the birth of the galaxies at time \( \tau_i \). The summands in (9)
are referred to the retarded points \( p_j \) where a particle world line \( L_j \) intersects the cone \( C_{p_o}^{\text{past}} \).

Let \( r_j \) be its radial coordinate and \( r \) the radial coordinate of the intersection point \( L_j \cap \Sigma_i \).
Thus, the tangent vector to the world line \( L_j \) at \( p_j \) will be \( v_j = \gamma_j (\partial_t + \vec{v_j} \partial_r) \), with \( \vec{v_j} = \vec{t_j} \),
and \( \gamma_j = (t^2 - r^2)^{-1/2} \). This is a unit vector with respect the Minkowski metric. With these
notations we can write \( g_{00}^{(1)} (p_o) \) as follows:

\[
g_{00}^{(1)} (p_o) = -Gm \sum_j \frac{\gamma_j^2 - 1/2}{r_j \gamma_j (1 + \vec{v_j})} \quad \text{(A.1)}
\]

We have seen in section (3.1.1) that a Milne Universe has a uniform number density over
hipersurfaces \( \Sigma_r \) of constant Minkowski proper time, and in consequence a non uniform density
\( n^{(0)} (r, t) \), on the hiperboloid \( \Sigma_r \). We shall make a statistical average by
assuming a uniform random process over \( \Sigma_r \). This is equivalent to consider a point process on
\( \Sigma_t \) with the density \( n^{(0)} (r, t) \) given above. The statistical average is then given by the integral

\[
\langle g_{00}^{(1)} (p_o) \rangle = -4GNm \int_0^{r_{mx}} \frac{t \gamma_j^2 - 1/2}{(t^2 - r^2)^2} \frac{1}{r_j \gamma_j (1 + \vec{v_j})} 4\pi r^2 dr \quad \text{(A.2)}
\]

where \( r_{mx} \) is the radial coordinate of the point intersection of \( \Sigma_t \) with the world line \( L_{lim} \)
as shown in Figure A1. The set of worldliness like \( L_{lim} \) defines the boundary of the particles that
have interacted with the reference particle \( L \) at time \( t \).

To perform the integral we need the relation between the radial coordinates \( r_j \) and \( r \). It is
straightforward to derive it from the Figure A1: \( r_j = \frac{r^*}{r + t_i} \). The result of the integration is

\[
\langle g_{00}^{(1)} (p_o) \rangle = -8\pi GNm \left( \frac{2}{3} t^2 (t^2 - r_{mx}^2)^{-3/2} - (t^2 - r_{mx}^2)^{-1/2} + \frac{1}{3t} \right) \quad \text{(A.3)}
\]

It remains to get the value of \( r_{mx} \). We will shall show that it depends on both the present time
\( t \) and the initial time \( \tau_i \). Let \( r^* \) be the radial coordinate of the point \( p_{lim} = L_{lim} \cap \Sigma_i \). From
the figure it is easy to get

\[
r_{mx} = t + \frac{r^*}{t - r^*} \quad \text{(A.4)}
\]

Next we solve the system \( \vec{t} + \vec{r} = \vec{t} , \vec{t}^2 - \vec{r}^2 = \vec{t}^2_i \) to get the radial coordinate \( r^* = \frac{t^2 - \tau_i^2}{t^2} \), and
substituting into the previous equation we get

\[
r_{max} = \frac{t^2 - \tau_i^2}{t^2 + \tau_i^2} \quad \text{(A.5)}
\]

But, the intersection of the reference world line \( L \) with the initial hiperboloid \( \Sigma_{\tau_i} \) is a point
with coordinate time \( t_i = \tau_i \), thus we can write finally \( r_{max} = \frac{t^2 - t_i^2}{t^2 + t_i^2} \). It is straightforward to
get the relations:

\[
t^2 - r_{mx}^2 = \frac{4t^4 t^2}{(t^2 + t_i^2)^2} , \ (t^2 - r_{mx}^2)^{3/2} = \frac{8t^6 t^3}{(t^2 + t_i^2)^3} , \ (t^2 - r_{mx}^2)^{1/2} = \frac{2t^2 t_i}{t^2 + t_i^2} \quad \text{(A.6)}
\]

Substituting into the expression (A.3), and writing \( \mu_i = Nm \) for the initial mass density we get

\[
\langle g_{00}^{(1)} (p_o) \rangle = -8\pi G \mu_i \left( \frac{1}{12} t_i^2 \left( 1 + \frac{t_i^2}{t^2} \right)^3 - \frac{t_i^2}{2} \left( \frac{1}{2} \left( 1 + \frac{t_i^2}{t^2} \right) - \frac{t_i}{3} \right) \right) \quad \text{(A.7)}
\]
Figure A1: This figure shows the meaning of the radial coordinates used in the appendix:

(i) \( r_{mx} \) is the radial coordinate of the point \( L_{lim} \cap \Sigma_t \)
(ii) \( r^* \) is radial coordinate of the point \( L_{lim} \cap \Sigma_{r_i} \)
(iii) \( r_j \) is the radial coordinate of the point \( L_j \cap C_{\nu}^{past} \): the retarded position of particle \( j \).
(iv) \( r \) is the radial coordinate of the point \( L_j \cap \Sigma_t \)

References
[1] Havas P and Goldberg J 1962 Phys. Rev. 128 398.
[2] Clarkson Ch, Ellis G, Larena J, Umeh O 2011: Does the growth of structure affect our dynamical models of the universe?. arXiv:1109.2314v1[astro-ph.CO].
[3] Portilla M 2014 Universe made of baryonic gravitating particles behaves as a \( \Lambda \)CDM Universe arXiv:1405.0399 [gr-qc].
[4] Rindler W 2006 Relativity: Special, General and Cosmological (Oxford)chapter 6 pp 170-180.
[5] Scott D and Moss A 2009 Mon. Not. R. Astron. Soc. 397 445.
[6] Jeon et al. 2012 Ap.J. 754 34.
[7] Suyu S 2012 Ap.J. 750 10.
[8] Dutton S 2011 Mon. Not. R. Astron. Soc. 1 23.
[9] Einstein A Sidelights on Relativity: Geometry and experience. (Dover) pp 42.