BILINEAR EQUATIONS IN HILBERT SPACE DRIVEN BY PATHS OF LOW REGULARITY

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Abstract. In the article, some bilinear evolution equations in Hilbert space driven by paths of low regularity are considered and solved explicitly. The driving paths are scalar-valued and continuous, and they are assumed to have a finite \( p \)-th variation along a sequence of partitions in the sense given by Cont and Perkowski \[Trans. Amer. Math. Soc. Ser. B, 6 \ (2019) \ 161–186\] (\( p \) being an even positive integer). Typical functions that satisfy this condition are trajectories of the fractional Brownian motion with Hurst parameter \( H = 1/p \).

A strong solution to the bilinear problem is shown to exist if there is a solution to a certain non–autonomous initial value problem. Subsequently, sufficient conditions for the existence of the solution to this initial value problem are given. The abstract results are applied to several stochastic partial differential equations with multiplicative fractional noise, both of the parabolic and hyperbolic type, that are solved explicitly in a pathwise sense.

1. Introduction. In the article, the evolution equation

\[
\begin{align*}
\mathrm{d}X_t &= AX_t \, \mathrm{d}t + BX_t \, \mathrm{d}\omega_t, \quad 0 \leq s < t \leq T, \\
X_s &= x_0,
\end{align*}
\]

in a Hilbert space \( V \) is studied. Here, \( A : V \supseteq \text{Dom } A \to V \) and \( B : V \supseteq \text{Dom } B \to V \) are (possibly unbounded) linear operators and \( \omega : [0,T] \to \mathbb{R} \) is a continuous function that has finite \( p \)-th variation along a sequence of partitions \( \{\pi_n\}_{n \in \mathbb{N}} \) of the interval \([0,T]\) in the sense of \([12]\) for some positive even integer \( p \). This means that the sequence of measures \( \{\mu_n\}_{n \in \mathbb{N}} \) on the measurable space \(([0,T],\mathcal{B}([0,T]))\) defined by

\[
\mu_n := \sum_{[t_j,t_{j+1}] \in \pi_n} \delta_{\omega_{t_j}} \left( |\omega_{t_{j+1}} - \omega_{t_j}|^p \right)
\]

converges weakly to a non-atomic measure \( \mu \) (\( \delta_u \) denotes the Dirac measure at the point \( u \in \mathbb{R} \)). The \( p \)-th variation is then defined as the function \( |\omega|^p_{[0,t]} : = \mu([0,t]) \).

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The notion of $p$-th variation along a sequence of partitions first appeared in Föllmer’s seminal paper [25] where the case $p = 2$ is considered. The concept is generalized to $p > 2$ by Cont and Perkowski in the paper [12].

Functions that have a finite $p$-th variation along a sequence of partitions very often arise as trajectories of stochastic processes. For example, it is well-known that almost every path of the Wiener process $(W_t)_{t \in [0,1]}$ has finite quadratic variation $[W]^2(t) = t$ along every non-random sequence of partitions $\pi = \{\pi_n\}_{n \in \mathbb{N}}$ of the interval $[0,1]$ whose mesh tends to zero; see, e.g., the remark following Theorem 7.4 in the recent paper [16]. However, it is also possible to consider processes of lower regularity. For example, almost every path of the fractional Brownian motion with the Hurst parameter $H \in (0,1)$ has finite $2$-th variation along a sequence of partitions whose mesh tends to zero; see, e.g., [24, 37, 56, 58, 65], but the non-integer case is not considered in this paper.

The term $BX_t \, d^\omega$ in equation (1) formally corresponds to the linear multiplicative noise term $BX_t \, \dot{\omega}$. It is natural to interpret equation (1) as the integral equation

$$X_t = x_0 + \int_s^t AX_r \, dr + \int_s^t BX_r \, d^\omega, \quad 0 \leq s < t \leq T; \quad (2)$$

however, the path $\omega$ is not of bounded variation (as $p$ is assumed to be larger than one) and therefore, the second integral above has to be given a rigorous meaning. Since the equation is bilinear, it is expected that the solution $X$ is such that $BX_t = (\partial_2 f)(t, \omega_t)$ for a suitable function $f : [0, T] \times \mathbb{R} \to V$ (throughout the article, the symbol $\partial_2 f$ is used to denote the $k$-th partial derivative of $f$ in the $i$-th variable). Hence, we only look for a solution $X$ for which $BX$ is a function of $t$ and $\omega_t$ and therefore the integral is defined only for such integrands. A key observation of Föllmer in [25] is that when the integrand is a sufficiently smooth functional of a path $\omega$, that is assumed to have a finite quadratic variation along a sequence of partitions $\pi$, then the integral with respect to $\omega$ can be defined as a pointwise limit of Riemann sums. As shown by Cont and Perkowski in [12], this remains true even when $\omega$ is a path that has a finite $p$-th variation along $\pi$ with $p$ being a positive even integer which is the case considered here.

In particular, following [12, 25], a function $[t \mapsto \tilde{f}(t, \omega_t)]$ is said to be integrable with respect to the path $\omega$ on the interval $[s, T]$ along the sequence of partitions $\pi$ if $\tilde{f} : [s, T] \times \mathbb{R} \to V$ has continuous derivatives in the second variable up to order $p - 2$ and if the Föllmer-type integral that is defined by the pointwise limit of the compensated Riemann sums

$$\int_s^t \tilde{f}(u, \omega_u) \, d^\pi \omega_u := \lim_{n \to \infty} \sum_{|t_j|, t_{j+1}| \in \pi_n} \sum_{k=1}^{p-1} (\partial_2^{k-1} \tilde{f})(t_j, \omega_{t_j}) \frac{1}{|t_j - t_{j+1}|} (\omega_{t_j+1} - \omega_{t_j})^k, \quad (3)$$

$s \leq t \leq T$, exists. As shown in Lemma 2.7 below, if $f \in \mathcal{C}^{1,p}([0, T] \times \mathbb{R}; V)$, then $[t \mapsto (\partial_2 f)(t, \omega_t)]$ is integrable with respect to $\omega$ in the above sense, and, moreover, its Föllmer-type integral can be explicitly computed via a certain change-of-variable formula. In fact, Lemma 2.7, which is a modification of [12, Theorem 1.5 and Theorem 1.10], is the key tool used in the paper.
Having given a rigorous meaning to equation (1), a main result of the present paper is that if \( v_{s,x_0} \) is a solution to the non-autonomous initial value problem

\[
\begin{cases}
\dot{v}(t) = \tilde{C}_s(t)v(t), & 0 \leq s < t \leq T, \\
v(s) = x_0,
\end{cases}
\tag{4}
\]

for a sufficiently regular initial state \( x_0 \in V \), where \( \tilde{C}_s \) is the family of linear operators defined by

\[
\tilde{C}_s(t) := GB(\omega_s - \omega_t) \left( A - \frac{1}{p!} [\omega]_p^\pi(t)B^p \right) GB(\omega_t - \omega_s), \quad 0 \leq s < t \leq T,
\]

with \( GB \) being the strongly continuous group of linear operators generated by \( B \) and \( [\omega]_p^\pi \) being the derivative of the \( p \)-th variation of \( \omega \) along \( \pi \) (that is assumed to exist), then the path \( X^\omega_{s,x_0} \) defined by

\[
X^\omega_{s,x_0}(t) := GB(\omega_t - \omega_s)v_{s,x_0}(t), \quad 0 \leq s < t \leq T,
\]

is a solution to the bilinear problem (1) (see Proposition 3.9 for the precise statement). It therefore follows that in order to find a solution to (1), it suffices to find a solution to problem (4). This is done in Proposition 3.12 and Proposition 3.16 where two sets of sufficient conditions for the family \( \tilde{C}_s \) to generate a strongly continuous evolution system of operators are given (see, e.g., [53] or [63]). These two sets of conditions are usually referred to as the parabolic and the hyperbolic case and, roughly speaking, they correspond to analyticity of the semigroup generated by \( \tilde{C}_s(t) \) for a fixed \( t \). It should be noted however that even though the solution to the initial value problem (4) found in Proposition 3.12 and Proposition 3.16 is unique, uniqueness of the solution to the bilinear equation (1) is still an open problem.

**Literature and related approaches.** Stochastic evolution equations with multiplicative noise have been investigated by many authors in various settings. A main source of inspiration for the work in the present paper is the article [13] where a correspondence between a linear stochastic evolution equation with linear multiplicative scalar white noise and a random evolution equation was established. In fact, it is a main purpose of the present paper to show that the techniques from [13] can be employed even if the stochastic bilinear equation is reinterpreted in the setting of the Itô-Föllmer calculus. Moreover, it turns out that by considering the \( p \)-th variation instead of the quadratic variation, a much larger variety of driving processes can be considered.

Explicit solutions to stochastic bilinear evolution equations driven by scalar processes that are different from the Wiener process have already been given in the literature. For example, in the articles [22] and [50], the driving process is a fractional Brownian motion with the Hurst parameter \( H > 1/2 \) and the stochastic integral is interpreted in the Skorokhod sense, i.e. as the Skorokhod integral composed with a suitable fractional transfer operator; see, e.g., [3]. Moreover, in the case of \( H > 1/2 \), the stochastic integral can also be defined pathwise due to high regularity of the driving path as a generalized Young integral that is given in terms of fractional derivatives; see, e.g., [67], and stochastic bilinear problems with this pathwise interpretation are studied in [31]. On the other hand, the literature on stochastic bilinear problems driven by fractional Brownian motions with the Hurst parameter \( H < 1/2 \) is much more scarce. In this case, explicit solutions to the bilinear problem
interpreted in the Skorokhod sense are analyzed in [61] but the pathwise approach of [31] cannot be used.

The problem of existence and uniqueness of pathwise solutions to more general (semi-)linear stochastic evolution equations with general multiplicative noise of the type \( G(X_t) \, dZ_t \) has also been treated in several papers. For example, in the article [49], the driving noise \( dZ \) is fractional with \( H > \frac{1}{2} \) in time and correlated in space and the authors prove existence and uniqueness of a pathwise mild solution by employing the generalized Young integration theory of [67]. Some improvements of these results can be also found in [35]. The case \( H < \frac{1}{2} \) has been also treated. Specifically, in [42], the authors considered the notion of compensated fractional derivatives and investigated the existence of solutions to stochastic differential equations driven by a scalar \( \beta \)-Hölder continuous path \( Z \) with \( \beta \in (\frac{1}{3}, \frac{1}{2}) \) by transforming the original equation into a system of two equations, one for the path component and the other for the area component, which are connected through the algebraic Chen relation. The results in [42] were extended to evolution equations in [29] and [30] which can be applied to the situation where \( Z \) is a Hilbert-space-valued fractional Brownian motion with the Hurst index \( H \in (\frac{1}{3}, \frac{1}{2}) \). See also the paper [39] that combines the rough evolution equation approach of [20, 35, 36] with the techniques of [30].

Generally speaking, rough path theory, introduced in [48] and further developed by many authors, has been successfully applied to ordinary differential equations (ODEs) driven by paths of low regularity; see, e.g., the monographs [27] and [28] and the references therein; and, in recent years, there has been enormous research activity in the field of partial differential equations (PDEs) driven by rough paths. Apart from the already mentioned works [20, 36], where a semigroup approach to rough PDEs is considered, we also refer to the works [7, 8], where flow transformations are used to treat fully non-linear rough PDEs; [34], where the theory of paracontrolled distributions is developed in order to solve some singular stochastic PDEs; and [18, 19, 41], where a variational approach to rough PDEs that is based on certain a-priori estimates and a rough Gronwall lemma is given.

While in some cases it is possible to extend the results to driving paths of arbitrarily low regularity (see, e.g., [33], where existence and uniqueness of rough ODEs is proved via a Picard iteration scheme for driving paths of any Hölder continuity) and the possibility of such extension is expected in some other cases (e.g. the articles [19, 36] mentioned above), the methods of rough path theory are generally very demanding from a technical point of view and only the cases \( H \in (\frac{1}{4}, \frac{1}{2}) \) (e.g. [7, 8, 20]) or \( H \in (\frac{1}{3}, \frac{1}{2}) \) (e.g. [18, 19, 34, 41]) are usually treated.

In contrast to this, the method used in the present article is simple and already provides explicit pathwise solutions to linear equations with linear multiplicative fractional noise of regularity \( H = \frac{1}{2k} \) for \( k \in \mathbb{N} \) which goes well beyond the case \( H \in (\frac{1}{4}, \frac{1}{2}) \). More specifically, instead of relying on the abstract technical results of the rough path theory, Föllmer’s pathwise stochastic calculus (or more precisely its generalization due to [12]) is used. It is however known that these two approaches are closely connected. In particular, it follows by [12, Lemma 4.7], that the \((p + 1)\)-tuple \( \mathbb{X} = (\mathbb{X}^0, \mathbb{X}^1, \ldots, \mathbb{X}^p) \) that is given by

\[
\begin{align*}
\mathbb{X}_{s,t}^0 & := 1, \\
\mathbb{X}_{s,t}^k & := \frac{1}{k!} (\omega_t - \omega_s)^k, \quad k = 1, 2, \ldots, p - 1,
\end{align*}
\]
\[
\mathcal{X}_{s,t}^p := \frac{1}{p!} (\omega_t - \omega_s)^p - \frac{1}{p!} \left( [\omega]^p_p(t) - [\omega]^p_p(s) \right),
\]
is a reduced rough path of finite \( p \)-variation (as defined in [12, Definition 4.6]) if \( \omega \) has finite \( p \)-th variation along the sequence of the dyadic Lebesgue partitions generated by \( \omega \). If it is assumed that \( f \) is a real-valued function of only one variable, i.e. \( f(t,x) \equiv f(x) \), then it follows by [12, Corollary 4.11] that the integral defined by formula (3) coincides with the rough path integral with respect to \( X \) (whose definition can be found in [12, Proposition 4.10]) where the derivatives \( f^{(k)}(\omega) \) play the role of higher-order Gubinelli’s derivatives. The interested reader can find a very thorough discussion of this connection in [27, section 5.3] for \( p = 2 \) and in [12, section 4.2] for \( p > 2 \).

Föllmer’s pathwise calculus has been developed by many authors; see, e.g., [5, 10, 11, 16, 25, 45, 54, 66]. Moreover, this calculus has already been used to solve some differential equations that are usually considered in the framework of Itô’s integration theory without relying on any probabilistic structure, see [40], and its importance has been recognized in mathematical finance; see, e.g., [15, 26, 47, 54] and the references therein. In all these reference, the case \( p = 2 \) is considered. For \( p > 2 \), see the very recent papers [12, 43, 60].

**Organization of the paper.** In Section 2, the notion of the \( p \)-th variation along a sequence of partitions is recalled in Definition 2.1 and two examples are given. Moreover, the integral with respect to paths with finite \( p \)-th variation along a sequence of partitions is defined and a change-of-variable formula is given in Lemma 2.7.

The bilinear equation (1) is treated in Section 3. Initially, the notion of a solution is defined and then, the reasoning is split into two parts - the first part in Subsection 3.1 contains the important case when \( A \) and \( G_B \) commute on a suitable domain; the second part in Subsection 3.2 contains the general non-commutative case.

The main result of Subsection 3.1 is Proposition 3.5. This is followed by three examples of stochastic (partial) differential equations with a multiplicative singular fractional noise. In Example 3.6, the equation is studied in dimension one. In particular, it shows that the solution to the bilinear problem (1) can be viewed as a generalized geometric fractional Brownian motion. The example is then extended in Example 3.7 to infinite dimensions by assuming that the operator \( A \) generates an analytic semigroup and \( B \) is essentially the identity operator and the particular case of a heat equation is given. Finally, the subsection is concluded by Example 3.8 where both \( A \) and \( B \) are unbounded operators.

The main result of Subsection 3.2 is Proposition 3.9 which links problem (1) to problem (4) without assuming that the operator \( A \) commutes with the group \( G_B \). The result is followed by Example 3.11 where it is applied to an equation for a stochastic harmonic oscillator with noisy damping. Then, the two cases when the family of operators \( \{ C_s(t), t \in [0,T] \} \) is parabolic and hyperbolic are treated separately in Proposition 3.12 and Proposition 3.16 where two sets of sufficient conditions for the existence of a solution to (1) are given. Each of these two results are followed by an example. In particular, Example 3.15 features a heat-type equation while in Example 3.18, a Schrödinger-type equation is given.

2. Preliminaries. In the first section, we recall the notion of the \( p \)-th variation of a continuous function along a sequence of partitions and some of its properties.
A change-of-variable formula that is the central tool used in the following sections is also given.

**Notation.** Let \(-\infty < a < b < \infty\). The symbol \(\mathcal{P}[a,b]\) denotes the set of finite partitions \(P = \{t_0, t_1, \ldots, t_N(P)\}, N(P) \in \mathbb{N}\), of the interval \([a,b]\) such that \(a = t_0 < t_1 < \ldots < t_{N(P)-1} < t_{N(P)} = b\). The oscillation of a function \(f \in \mathcal{C}([a,b])\) along the partition \(P \in \mathcal{P}[a,b]\) is defined by

\[
\text{osc} (f, P) := \max_{[t_j, t_{j+1}] \in P} \max_{x, y \in [t_j, t_{j+1}]} |f(x) - f(y)|
\]

where \([t_j, t_{j+1}] \in P\) means that both \(t_j\) and \(t_{j+1}\) belong to \(P\) and they are immediate successors.

Recall the definition of \(p\)-th variation along a sequence of partitions that is given in [12, Definition 1.1].

**Definition 2.1.** Let \(p > 0\) and \(0 \leq a < b < \infty\) and let \(\pi = \{\pi_n\}_{n \in \mathbb{N}} \subset \mathcal{P}[a,b]\) be a sequence of partitions of the interval \([a,b]\). A function \(S \in \mathcal{C}([a,b])\) is said to have a finite \(p\)-th variation along the sequence \(\pi\) if

\[
\lim_{n \to \infty} \text{osc}(S, \pi_n) = 0,
\]

and if the sequence of measures \(\{\mu_n\}_{n \in \mathbb{N}}\) on the measurable space \(([a,b], \mathcal{B}([a,b]))\) that is given by

\[
\mu_n := \sum_{[t_j, t_{j+1}] \in \pi_n} \delta_{t_j} |S(t_{j+1}) - S(t_j)|^p
\]

converges weakly to a finite non-atomic measure \(\mu\). Here, \(\delta_u\) denotes the Dirac measure at the point \(u \in \mathbb{R}\). If \(S\) has a finite \(p\)-th variation along the sequence \(\pi\), the notation \(S \in V_p(\pi)\) is used and the function \([S]_p^\pi : [a,b] \to [0,\infty)\) defined by

\[
[S]_p^\pi(t) := \mu([a,t])
\]

is called the \(p\)-th variation of \(S\) along \(\pi\).

**Remark 2.2.** The concept of \(p\)-th variation defined above is in general dependent on the sequence of partitions. In the case \(p = 2\), that is usually called the *quadratic variation*, this phenomenon is thoroughly investigated in [16, section 7] where the following result is given in Theorem 7.4. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. It is understood that all the following random processes and variables are defined on this space. Let \((W_t, t \in [0,1])\) be a Wiener process and let \((\mathcal{F}_t)_{t \in [0,1]}\) be the filtration generated by \(W\). Then for every continuous non-negative increasing stochastic process \((A_t, t \in [0,1])\) that starts at zero, there exists a refining sequence \(\pi = \{\pi_n\}_n\) of random partitions of the interval \([0,1]\) such that \(\{k2^{-n} | k = 0, 1, \ldots, 2^n\} \subseteq \pi_n\) holds for every \(n \in \mathbb{N}\) \(\mathbb{P}\)-almost surely and for which \(W \in V_2(\pi)\) \(\mathbb{P}\)-almost surely with \([W]_2^\pi(t) = A_t, t \in [0,1]\). In other words, any continuous non-negative increasing stochastic process can be viewed as the quadratic variation of the Wiener process along a suitable sequence of random partitions. On the other hand, as long as we restrict ourselves to sequences of partitions that consist of stopping times, the quadratic variation along such sequences is independent of the chosen partition. More precisely, it follows by [16, Proposition 2.3] that if \(\tau = \{\tau_n\}_n\) is a sequence of random partitions that consist of \((\mathcal{F}_t^W)\)-stopping times and that satisfies

\[
\lim_{n \to \infty} \text{osc}(W, \tau_n) = 0 \quad \mathbb{P}\text{-almost surely},
\]

then there exists a subsequence \(\tilde{\tau} \subseteq \tau\) such that \(W \in V_2(\tilde{\tau})\) with \([W]_2^\tilde{\tau}(t) = t, t \in [0,1], \mathbb{P}\text{-almost surely.}

**Remark 2.3.** Let us also stress that the concept of \(p\)-th variation along a sequence is very different from the usual notion of \(p\)-variation. In particular, it is well-known
deterministic functions with this property. Define the Faber-Schauder paths of random processes have finite variation along \( \tilde{\tau} \) (see, e.g., [57, Remark on p. 28]) that can be found in [51], shows that not only sample paths of random processes have finite \( p \)-th variation, but there are also some purely deterministic functions with this property. Define the Faber-Schauder functions by

\[
e_{0,0}(t) := \max\{0, \min\{t, 1-t\}\},
\]

\[
e_{n,k}(t) := 2^{-\frac{n}{2}} e_{0,0}(2^n t - k), \quad n \in \mathbb{N}, \ k \in \mathbb{Z},
\]

for \( t \in \mathbb{R} \) and consider the Takagi-Landsberg function \( \tau_H \) with \( H \in (0, 1) \) defined by

\[
\tau_H(t) := \sum_{n=0}^{\infty} 2^n (\frac{n}{2} - H) \sum_{k=0}^{2^n-1} e_{n,k}(t), \quad 0 \leq t \leq 1.
\]

Then by [51, Theorem 2.1], the function \( \tau_H \) has a finite \( 1/H \)-th variation along the sequence of dyadic partitions of the interval \([0, 1]\), i.e. along the sequence of partitions \( D = \{D_n\}_{n \in \mathbb{N}_0} \) where

\[
D_n := \{ k2^{-n} \mid k = 0, 1, \ldots, 2^n \}
\]

with \( [\tau_H]_H^D(t) = t E |Z_H|_{\hat{\pi}} \) for \( 0 \leq t \leq 1 \), where the random variable \( Z_H \) is defined by

\[
Z_H := \sum_{n=0}^{\infty} 2^{-n(1-H)} Y_n
\]

for a sequence \( \{Y_n\}_{n \in \mathbb{N}_0} \) that consists of independent, identically distributed random variables with the discrete uniform distribution on the set \( \{-1, 1\} \) defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

**Example 2.4.** This example, that can be found in [51], shows that not only sample paths of random processes have finite \( p \)-th variation, but there are also some purely deterministic functions with this property. Define the Faber-Schauder functions by

\[
e_{0,0}(t) := \max\{0, \min\{t, 1-t\}\},
\]

\[
e_{n,k}(t) := 2^{-\frac{n}{2}} e_{0,0}(2^n t - k), \quad n \in \mathbb{N}, \ k \in \mathbb{Z},
\]

for \( t \in \mathbb{R} \) and consider the Takagi-Landsberg function \( \tau_H \) with \( H \in (0, 1) \) defined by

\[
\tau_H(t) := \sum_{n=0}^{\infty} 2^n (\frac{n}{2} - H) \sum_{k=0}^{2^n-1} e_{n,k}(t), \quad 0 \leq t \leq 1.
\]

Then by [51, Theorem 2.1], the function \( \tau_H \) has a finite \( 1/H \)-th variation along the sequence of dyadic partitions of the interval \([0, 1]\), i.e. along the sequence of partitions \( D = \{D_n\}_{n \in \mathbb{N}_0} \) where

\[
D_n := \{ k2^{-n} \mid k = 0, 1, \ldots, 2^n \}
\]

with \( [\tau_H]_H^D(t) = t E |Z_H|_{\hat{\pi}} \) for \( 0 \leq t \leq 1 \), where the random variable \( Z_H \) is defined by

\[
Z_H := \sum_{n=0}^{\infty} 2^{-n(1-H)} Y_n
\]

for a sequence \( \{Y_n\}_{n \in \mathbb{N}_0} \) that consists of independent, identically distributed random variables with the discrete uniform distribution on the set \( \{-1, 1\} \) defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

**Example 2.5.** Consider the fractional Brownian motion \( W_H \) on the interval \([0, 1]\) defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Assume, that the Hurst parameter \( H \) belongs to the interval \([0, 1/2] \) and that the \( \sigma \)-algebra \( \mathcal{F} \) is generated by the process \( W_H \). Let \( \mathcal{H} \) be the reproducing kernel Hilbert space for the process \( W_H \); that is, \( \mathcal{H} \) is the completion of the space of step functions on the interval \([0, 1]\) with respect to the inner product

\[
\langle 1_{[0,s]}, 1_{[0,t]} \rangle_{\mathcal{H}} := R_H(s,t), \quad s, t \in [0, 1],
\]

where \( R_H \) is the covariance function of the fractional Brownian motion \( W_H \) given by

\[
R_H(s,t) = \frac{1}{2} \left( s^{2H} + t^{2H} - |s-t|^{2H} \right), \quad s, t \in [0, 1].
\]

It can be shown that the space \( \mathcal{H} \) consists of real-valued functions (since it is assumed that \( H \leq 1/2 \)) and there exists its characterization as an image of the space \( L^2(0,1) \) under a certain fractional integral, see [17, Theorem 3.3] and [55, section 4] for the details. On the space \( \mathcal{H} \), an isonormal Gaussian process, denoted again by \( W_H \), is given by the Wiener integral with respect to the process \( W_H \) that is defined as an extension of the map \( 1_{[0,t]} \mapsto W_H^t \). This means that a notion of a
stochastic integral as well as that of a stochastic derivative for $W^H$ can be defined using the Malliavin calculus, see [52]. In particular, the stochastic derivative $D^H$ (with respect to $W^H$) is defined by

$$D^H F = \sum_{i=1}^n (\partial_i f)(W^H(h_1), W^H(h_2), \ldots, W^H(h_n)) h_i$$

for a random variable $F$ that is given by $F = f(W^H(h_1), W^H(h_2), \ldots, W^H(h_n))$ for some $n \in \mathbb{N}$, $\{h_i\}_{i=1}^n \subset \mathcal{H}$, and $f : \mathbb{R}^n \to \mathbb{R}$ that is infinitely differentiable and such that $f$ and all its partial derivatives are bounded (the space of such $f$ is denoted by $\mathcal{S}_b$ in the sequel). It turns out that the operator $D^H : \mathcal{S}_b \to L^2(\Omega; \mathcal{H})$ is closable and its domain is the Sobolev-Watanabe space (with respect to $W^H$) $\mathcal{D}^{1,2}_{\mathcal{H}}$ that is defined as the closure of $\mathcal{S}_b$ with respect to the norm

$$\|F\|_{\mathcal{D}^{1,2}_{\mathcal{H}}} := (\mathbb{E}|F|^2 + \mathbb{E}\|D^H F\|_{\mathcal{H}}^2)^{\frac{1}{2}}, \quad F \in \mathcal{S}_b.$$ 

Similarly as in the above case of real-valued random variables $F$, this space as well as the stochastic derivative can be defined for $\mathcal{H}$-valued random variables (or even general Hilbert-space-valued random variables) in which case, the space is denoted by $\mathcal{D}^{1,2}_{\mathcal{H}}(\mathcal{H})$. The stochastic integral is defined as the $L^2$-adjoint of the stochastic derivative. More precisely, its domain $\text{Dom} \, \delta^H$ is the set of all $u \in L^2(\Omega; \mathcal{H})$ for which there exists a constant $c_u > 0$ such that the inequality

$$\left| \langle D^H F, u \rangle_{L^2(\Omega; \mathcal{H})} \right| \leq c_u \|F\|_{L^2(\Omega)}$$

is satisfied for every $F \in \mathcal{S}_b$. For $u \in \text{Dom} \, \delta^H$, the symbol $\delta^H(u)$ denotes the unique element of the space $L^2(\Omega)$ whose existence is ensured by the Riesz representation theorem, for which the equality

$$\langle F, \delta^H(u) \rangle_{L^2(\Omega; \mathcal{H})} = \langle u, D^H F \rangle_{L^2(\Omega; \mathcal{H})}$$

is satisfied for every $F \in \mathcal{S}_b$. The operator $\delta^H : \text{Dom} \, \delta^H \to L^2(\Omega)$ is then called the stochastic integral (with respect to $W^H$).

Assume now that $u \in \mathcal{D}^{1,2}(\mathcal{H})$ is such that there exists $q > \frac{1}{H}$ and constants $L_u, L_{Du} > 0$, and $\gamma > 1/2 - H$ such that the inequalities

$$\|u_t - u_s\|_{L^q(\Omega)} \leq L_u |t - s|^\gamma$$  \hspace{1cm} (5)

and

$$\|D^H u_t - D^H u_s\|_{L^q(\Omega; \mathcal{H})} \leq L_{Du} |t - s|^\gamma$$  \hspace{1cm} (6)

are satisfied for every $s, t \in [0, 1]$. Assume, moreover, that there exist constants $0 \leq \alpha < 2H$ and $L > 0$ such that the inequality

$$\sup_{s \in [0,1]} \|D^H u_t\|_{L^1(t; \Omega)} \leq Lt^{-\alpha}$$ \hspace{1cm} (7)

is satisfied for every $t \in (0, 1]$ and denote

$$i_t(u) := \int_0^t u_s \delta W^H_s := \delta(1_{[0,t]}u), \quad t \in [0,1].$$

Then it follows by slightly modifying the proof of [24, Theorem 4.1] that for every $t \in [0, 1]$, the convergence

$$\sum_{i=0}^{n} \left( \int_0^t u_s \delta W^H_s - \int_0^\frac{t}{n+1} u_s \delta W^H_s \right) \frac{1}{n+1} \longrightarrow_{n \to \infty} c_{\frac{1}{n}} \int_0^t |u_s|^\frac{1}{\gamma} \, ds,$$  \hspace{1cm} (8)
Lemma 3.1] and [12, Lemma 1.3] that there exists a subsequence \( \tilde{\pi} \) where \( c \to V \) and where \( \omega \to \) in a similar manner as [12, Theorem 1.5]. Let us first fix the following notation.

Let \( 0 < a < b < \infty \) and let \( p \in \mathbb{N} \) be even and \((V, \langle \cdot, \cdot \rangle_V, \| \cdot \|_V)\) be a Hilbert space. Let \( \omega \) be a path such that \( c \in V_\pi(\omega) \) with \( \|\omega\|_\pi \neq 0 \) for a sequence of partitions \( \pi = \{\pi_n\}_{n \in \mathbb{N}} \subset \mathcal{P}[a,b] \). Assume that \( \tilde{f} : [a, b] \times \mathbb{R} \to V \) is a continuous function whose partial derivatives \( \partial_k^{\tilde{f}} \), \( k = 0, 1, \ldots, p - 2 \), exist on \([a, b] \times \mathbb{R}\) and are continuous. We say that the function \( [t \mapsto \tilde{f}(t, \omega_t)] \) is integrable with respect to \( \omega \) on the interval \([a, b]\) along the sequence of partitions \( \pi \) if the pointwise limit of compensated Riemann sums

\[
\lim_{n \to \infty} \sum_{[t_j, t_{j+1}] \in \pi_n} \sum_{k=1}^{p-1} \frac{1}{k!} (\partial_k^{\tilde{f}})(t_j, \omega_{t_j})(\omega_{t_{j+1} \wedge t} - \omega_{t_j \wedge t})^k
\]

exists and is finite for every \( t \in [a, b] \). In this case, the limit will be denoted by \( \int_a^b \tilde{f}(u, \omega_u) \, d^p \omega_u \) and called the \((\text{Föllmer-type})\) integral of \([t \mapsto \tilde{f}(t, \omega_t)]\) with respect to \( \omega \) on \([a, b]\) along \( \pi \).

The following change-of-variable formula will be needed in the sequel. It is proved in a similar manner as [12, Theorem 1.5]. Let us first fix the following notation.

**Notation.** By the symbol \( \mathcal{C}^{1,p}([a, b] \times \mathbb{R}; V) \) where \(-\infty < a < b < \infty\) and \( p \in \mathbb{N} \), and where \( V \) is a Hilbert space, we mean the set of functions \( f : [a, b] \times \mathbb{R} \to V \) such that the partial derivative \( \partial_1 f \) exists on \([a, b] \times \mathbb{R}\) and has a continuous extension to \([a, b] \times \mathbb{R}\); and, moreover, the partial derivatives \( \partial_2^k f \), \( k = 1, 2, \ldots, p \), exist on \([a, b] \times \mathbb{R}\) and are continuous.
Lemma 2.7. Let $0 \leq a < b < \infty$ and let $p \in \mathbb{N}$ be even and $(V, \langle \cdot, \cdot \rangle_V, \| \cdot \|_V)$ be a Hilbert space. Let $\omega$ be a path such that $\omega \in V_p(\pi)$ with $[\omega]_p^\pi \not= 0$ for a sequence of partitions $\pi = \{\pi_n\}_{n \in \mathbb{N}} \subset \mathcal{P}[a, b]$ whose mesh size

$$\|\pi_n\| := \max_{[t_j, t_{j+1}] \in \pi_n} |t_{j+1} - t_j|$$

tends to zero as $n \to \infty$. Assume further that $f$ is a function that belongs to the space $\mathcal{C}^{1,p}([a, b] \times \mathbb{R}; V)$. Then the function $[t \mapsto (\partial_2 f)(t, \omega_t)]$ is integrable with respect to $\omega$ on $[a, b]$ along $\pi$ and its (Föllmer-type) integral defined by

$$\int_a^t (\partial_2 f)(u, \omega_u) \, d\pi u = \lim_{n \to \infty} \sum_{[t_j, t_{j+1}] \in \pi_n} \sum_{k=1}^{p-1} \frac{1}{k!} (\partial_2^k f)(t_j, \omega_{t_j})(\omega_{t_{j+1} \land t} - \omega_{t_j \land t})^k$$

satisfies the formula

$$f(t, \omega_t) - f(a, \omega_a) = \int_a^t (\partial_1 f)(u, \omega_u) \, du + \frac{1}{p!} \int_a^t (\partial_2^p f)(u, \omega_u) \, d[\omega]_p^\pi(u)$$

$$+ \int_a^t (\partial_2 f)(u, \omega_u) \, d\pi u$$

(10)

for every $t \in (a, b)$ where the second integral is the Riemann-Stieltjes integral with respect to $[\omega]_p^\pi$.

Proof. Write

$$f(t, \omega_t) - f(a, \omega_a) = \sum_{[t_j, t_{j+1}] \in \pi_n} \left[ f(t_{j+1} \land t, \omega_{t_{j+1} \land t}) - f(t_j \land t, \omega_{t_j \land t}) \right]$$

$$= \sum_{[t_j, t_{j+1}] \in \pi_n} \left[ f(t_{j+1} \land t, \omega_{t_{j+1} \land t}) - f(t_j \land t, \omega_{t_j \land t}) \right]$$

$$+ \sum_{[t_j, t_{j+1}] \in \pi_n} \left[ f(t_j \land t, \omega_{t_j \land t}) - f(t_j \land t, \omega_{t_j \land t}) \right].$$

(11)

In order to treat the first sum notice that there is the estimate

$$\sum_{[t_j, t_{j+1}] \in \pi_n} \| f(t_{j+1} \land t, \omega_{t_{j+1} \land t}) - f(t_j \land t, \omega_{t_j \land t}) \|_V$$

$$- (\partial_1 f)(t_j \land t, \omega_{t_j \land t})(t_{j+1} \land t - t_j \land t) \|_V$$

$$\leq \sum_{[t_j, t_{j+1}] \in \pi_n} \| t_{j+1} \land t - t_j \land t \|_V$$

$$\times \sup_{r \in [t_j, t_{j+1}]} \| (\partial_1 f)(r, \omega_{t_j \land t}) - (\partial_1 f)(t_j \land t, \omega_{t_j \land t}) \|_V$$

by [21, (8.6.2) on p. 162]. The right-hand side of the above expression tends to zero as $n \to \infty$ since $\|\pi_n\| \to 0$ as $n \to \infty$ and the convergence of the first sum on the right-hand side of equation (11) to the first integral on the right-hand side of equation (10) follows. The second sum in equation (11) is treated in the same manner as in [12, Theorem 1.5].

3. Bilinear evolution equations. Throughout this section, $(V, \langle \cdot, \cdot \rangle_V, \| \cdot \|_V)$ is a Hilbert space, and $A : \text{Dom } A \subseteq V \to V$ and $B : \text{Dom } B \subseteq V \to V$ are two (not necessarily bounded) linear operators. Moreover, $\omega$ is a function such that $\omega \in V_p(\pi)$
for an even positive integer \( p \) and a sequence of partitions \( \pi = \{ \pi_n \}_{n \in \mathbb{N}} \subset \mathcal{P}[0,T], T > 0 \), whose mesh size tends to zero. The bilinear problem
\[
\begin{cases}
   \mathrm{d}X_t = AX_t \, \mathrm{d}t + BX_t \, \mathrm{d}\pi_t, & 0 \leq s < t \leq T, \\
   X_s = x_0
\end{cases}
\]
(BLP)
for \( x_0 \in V \) is considered in this section.

**Definition 3.1.** A function \( X : [s, T] \to V \) is said to be a *strong solution* to problem (BLP) if \( X \) takes values in the set \( \text{Dom} \: A \cap \text{Dom} \: B \), the function \( r \mapsto AX_r \) belongs to the space \( \mathcal{C}([s, T]; V) \), the function \( r \mapsto BX_r \) is integrable with respect to \( \omega \) on the interval \([s, T]\) along \( \pi \) in the sense of Definition 2.6, and if the equation
\[
   X_t = x_0 + \int_s^t AX_r \, \mathrm{d}r + \int_s^t BX_r \, \mathrm{d}\pi_r
\]
is satisfied for every \( t \in (s, T) \). The first integral in the above equation is the Bochner integral and the second integral is the Föllmer-type integral.

**Definition 3.2.** For every \( t \in [0, T] \), let \( K(t) : \text{Dom} \: K(t) \subset V \to V \) be a linear operator and consider the homogeneous initial value problem
\[
\begin{cases}
   \dot{v}(t) = K(t)v(t), & 0 \leq r < t \leq T, \\
   v(r) = v_0
\end{cases}
\]
(12)
for \( v_0 \in V \). Let \( \hat{V} \) be a Hilbert space continuously and densely embedded in \( V \). A function \( v \) is called a *\( \hat{V} \)-valued solution* to problem (12) if \( \hat{V} \subseteq \text{Dom} \: K(t) \) for every \( t \in [r, T]; \: v \in \mathcal{C}([r, T]; V) \cap \mathcal{C}([r, T]; \hat{V}) \); and if \( v \) satisfies both equations in (12) in the space \( V \).

Consider the following assumption.

**Assumption 3.3.**
- The \( p \)-th variation \( [\omega]_p^\pi \) of the path \( \omega \) satisfies \( [\omega]_p^\pi \neq 0 \) and it is continuously differentiable on the interval \([0, T] \). We denote its derivative by \( [\dot{\omega}]_p^\pi \).
- The operator \( B \) is the infinitesimal generator of a strongly continuous group of bounded linear operators acting on the space \( V \) that is denoted by \( G_B \).
- The family of the linear operators \( (C(t), t \in [0, T]) \) given by
\[
   C(t) := A - \frac{1}{p!} [\omega]_p^\pi(t) B^p
\]
(13)
satisfies \( \text{Dom} \: C(t) = \text{Dom} \: C \) for every \( t \in [0, T] \) where
\[
   \text{Dom} \: C := \text{Dom} \: A \cap \text{Dom} \: B^p.
\]
(14)

**Remark 3.4.** Note that the assumption of continuous differentiability of the function \( [\omega]_p^\pi \) is not too restrictive. Example 2.5 already provides many examples that satisfy this condition. In particular, let \( p \) be an even positive integer and let \( u : [0, 1] \to \mathbb{R} \) be a deterministic function that belongs to the Hölder space \( \mathcal{C}^\beta([0, 1]) \) with some \( \beta > 1/2 - 1/p \). Then \( u \) clearly satisfies conditions (5) - (7), so that we have, for every \( t \in [0, 1] \) \( P \)-almost surely,
\[
\int_0^t u_s \delta W_s^{1/p} \, \mathrm{d}s = c_p \int_0^t |u_s|^p \, \mathrm{d}s
\]
which is continuously differentiable in \( t \) since \( u \) is continuous.
3.1. The commutative case. In this section, a strong solution to (BLP) is found under the assumption that the operator $A$ commutes with the group $G_B$ (on an appropriate domain). Consider the homogeneous initial value problem

$$\begin{cases}
\dot{v}(t) = C(t)v(t), & 0 \leq s < t \leq T, \\
v(s) = x_0.
\end{cases} \quad (CP)$$

The next proposition connects the solution to problem (CP) to the solution to problem (BLP).

**Proposition 3.5.** Let Assumption 3.3 be verified. Let $\tilde{V}$ be a Hilbert space that is dense in $V$; continuously embedded in $\text{Dom } C$; closed under the action of the group $G_B$; and such that the following condition is satisfied:

1. The operator $A$ commutes with the group $G_B$ on $\tilde{V}$, i.e. the equation

$$\text{AG}_B(t)y = G_B(t)Ay$$

is satisfied for every $y \in \tilde{V}$ and $t \in \mathbb{R}$.

2. If there is a $\tilde{V}$-valued solution $v_{s,x_0}$ to problem (CP), the function $X_{s,x_0}^\omega : [s,T] \to V$ defined by

$$X_{s,x_0}^\omega(t) := G_B(\omega_t - \omega_s)v_{s,x_0}(t)$$

is a strong solution to the problem (BLP).

**Proof.** Define $f : [s,T] \times \mathbb{R} \to V$ by

$$f(t,x) := G_B(x - \omega_s)v_{s,x_0}(t).$$

For every $t \in [s,T]$, it holds that $v_{s,x_0}(t) \in \tilde{V} \subseteq \text{Dom } C \subseteq \text{Dom } B^p$ and therefore

$$(\partial_t^k f)(t,x) = B^k G_B(x - \omega_s)v_{s,x_0}(t)$$

for $(t,x) \in [s,T] \times \mathbb{R}, \ k = 1, 2, \ldots, p$ by [53, Theorem 1.2.4 c)]. Since $v_{s,x_0}$ belongs to $\mathcal{C}([s,T] ; \tilde{V})$ with $\tilde{V}$ being continuously embedded in $\text{Dom } C$, it follows that $\partial_t^k f \in \mathcal{C}([s,T] \times \mathbb{R}; V)$ for every $k = 1, 2, \ldots, p$. Moreover, since $v_{s,x_0} \in \mathcal{C}^1([s,T]; \tilde{V})$, it also follows that

$$(\partial_t f)(t,x) = G_B(x - \omega_s)\dot{v}_{s,x_0}(t)$$

for $(t,x) \in [s,T] \times \mathbb{R}$ and $\partial_t f \in \mathcal{C}([s,T] \times \mathbb{R}; V)$. Therefore, by Lemma 2.7 we have that

$$G_B(\omega_t - \omega_s)v_{s,x_0}(t) = x_0 + \int_s^t B^k G_B(\omega_r - \omega_s)v_{s,x_0}(r) \, d^k \omega_r$$

$$+ \frac{1}{p!} \int_s^t B^p G_B(\omega_r - \omega_s)v_{s,x_0}(r) \, d[\omega]_p^p(r)$$

$$+ \int_s^t G_B(\omega_r - \omega_s)\dot{v}_{s,x_0}(r) \, dr$$

holds. By using the fact that $v_{s,x_0}$ satisfies equation (CP), we obtain

$$\int_s^t G_B(\omega_r - \omega_s)v_{s,x_0}(r) \, dr = \int_s^t G_B(\omega_r - \omega_s)C(r)v_{s,x_0}(r) \, dr.$$

---

1. That is, $\tilde{V} \subseteq \text{Dom } C$ and there exists a finite positive constant $c$ such that for every $v \in \tilde{V}$ it holds that $\|v\| + |Av| + \|B^p v\| \leq c\|v\|$.

2. That is, for every $x \in \mathbb{R}$ it holds that $G_B(x)\tilde{V} \subseteq \tilde{V}$. 
Since \(v_{s,x_0}\) belongs to \(\mathcal{C}([s,T];\tilde{V})\) with \(\tilde{V}\) being continuously embedded into \(\text{Dom} \ C\), the function \(r \mapsto G_B(\omega_r - \omega_s)Av_{s,x_0}(r)\) belongs to \(\mathcal{C}([s,T];V)\) by continuity of \(\omega\). Consequently,

\[
\int_s^t G_B(\omega_r - \omega_s)C(r)v_{s,x_0}(r) \, dr = \int_s^t AG_B(\omega_r - \omega_s)v_{s,x_0}(r) \, dr - \frac{1}{p!} \int_s^t B^pG_B(\omega_r - \omega_s)v_{s,x_0}(r) \, d[\omega]_p^p(r)
\]

where the assumption (AB) and commutativity of \(B^p\) and \(G_B\) on \(\text{Dom} \ B^p\), see [53, Theorem 1.2.4 c]), were used. The claim follows. \(\square\)

**Example 3.6.** Let \(W^H = (W_i^H, t \in [0,1])\) be the fractional Brownian motion with Hurst parameter \(H \in (0,1)\) defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\). By Example 2.5 we know that \(\mathbb{P}\)-almost every sample path of \(W^H\) has a finite \(1/H\)-th variation along the sequence \(\tilde{D}\) (recall that \(\tilde{D}\) is a subsequence of the sequence \(D\) of dyadic partitions of the interval \([0,1]\)) that is given by

\[
[W^H]_{\tilde{D}}^\frac{1}{H}(t) = c_{\tilde{D}}t, \quad t \in [0,1],
\]

where \(c_{\tilde{D}} = \mathbb{E}|W_i^H|_{\tilde{D}}\). Consider the scalar problem

\[
\begin{cases}
    dX_t = aX_t + bX_t \, dB_t^\frac{1}{H}, & 0 < t \leq 1, \\
    X_0 = 1,
\end{cases}
\]

where \(a \in \mathbb{R}, b \in \mathbb{R} \setminus \{0\}\), and \(k \in \mathbb{N}\). The \(2k\)-th variation of \(W^H\) given by (15) is non-zero and continuously differentiable. Therefore, by appealing to Proposition 3.5 with \(\omega := W^\frac{1}{H}, V = \tilde{V} := \mathbb{R}, A := a,\) and \(B := b\) (so that \(G_B(x) = \exp\{bx\}, x \in \mathbb{R}\); \(C(t) = C\) for \(t \in [0,1]\) with \(C := a - \frac{c_{\tilde{D}}}{(2k)!} b^{2k}\); and \(v_{0,1}(t) = \exp\{Ct\}, t \in [0,1]\), we obtain that the process \((Y_{k,1}(t), t \in [0,1])\) defined by the formula

\[
Y_{k,1}(t) := \exp \left\{ bW_t^\frac{1}{H} + \left( a - \frac{c_{\tilde{D}}}{(2k)!} b^{2k} \right) t \right\}
\]

satisfies

\[
Y_{k,1}(t) = 1 + \int_0^t aY_{k,1}(r) \, dr + \int_0^t bY_{k,1}(r) \, dB_t^\frac{1}{H}
\]

for every \(t \in [0,1]\) \(\mathbb{P}\)-almost surely (the second index is used to distinguish between the solutions that are obtained in this and in the following two examples). The formula for the process \(Y_{1,1}\) is well-known from classical (Itô’s) stochastic calculus. In fact, if \(k = 1\), we have that \(W^\frac{1}{H}\) is the Wiener process (that we denote by \(W\) everywhere in the paper) and the process satisfies the equation

\[
Y_{1,1}(t) = 1 + \int_0^t aY_{1,1}(r) \, dr + \int_0^t bY_{1,1}(r) \, dW_t
\]

for every \(t \in [0,1]\) \(\mathbb{P}\)-almost surely where the integral \(\int_0^t \ldots \, dW_t\) is the usual Itô integral. On the other hand, already in the case \(k = 2\), the form of the solution differs from similar results that can be found in the literature. In particular, by [61, Theorem 3.4] (see also [62, Example 3.1]) it follows that the process \((\tilde{Y}_{2,1}(t), t \in [0,1])\) defined by

\[
\tilde{Y}_{2,1}(t) := \exp \left\{ bW_t^\frac{1}{H} + at - \frac{1}{2} b^2 \right\}
\]
satisfies the equation

\[ \mathcal{Y}_{2,1}(t) = 1 + \int_0^t a\mathcal{Y}_{2,1}(r) \, dr + \int_0^t b\mathcal{Y}_{2,1}(r) \, d\mathcal{W}_t^\frac{1}{2} \]

for every \( t \in [0, 1] \) \( \mathbb{P} \)-almost surely where the integral \( \int_0^t (...) \, d\mathcal{W}_r^\frac{1}{2} \) is the extension of the Skorokhod integral with respect to the fractional Brownian motion from Example 2.5 that is introduced in [9].

**Example 3.7.** More generally, consider the problem

\[
\begin{aligned}
&\frac{dX_t}{dt} = AX_t \, dt + bX_t \, d\mathcal{W}_t, & 0 < t \leq 1, \\
&X_0 = x_0,
\end{aligned}
\]

(16)

where \( A : \text{Dom } A \subseteq V \to V \) is the infinitesimal generator of a strongly continuous semigroup \( S_A \) of bounded linear operators acting on a Hilbert space \( V \) and \( b \in \mathbb{R} \setminus \{0\} \).

As in the above Example 3.6, it holds that \( [W^{1/2k}]_{2k}(t) = c_{2k} t \) which is a non-zero and continuously differentiable function. The operator \( B := b \text{Id}_V \) generates a strongly continuous group \( G_{b\text{Id}_V} \) given by \( G_{b\text{Id}_V}(t) = e^{bt} \text{Id}_V \) for \( t \in \mathbb{R} \). The system of operators \( C(t) \), that is defined by (13), is independent of \( t \) and it is given by \( C(t) = C \) where

\[ C := A - \frac{c_{2k}}{(2k)!} b^{2k} \text{Id}_V \]

on \( \text{Dom } A \) that equals \( \text{Dom } C \) defined by (14). Thus, Assumption 3.3 is satisfied. We will show that the remaining assumptions of Proposition 3.5 are satisfied with \( \tilde{V} = \text{Dom } A \) equipped with the graph norm of the operator \( A \). Since the operator \( A \) generates a strongly continuous semigroup, the space \( \text{Dom } A \) is dense in \( V \) by the Hille-Yosida theorem; see, e.g., [53, Theorem 1.5.3]. Moreover, \( \text{Dom } A \) is closed under the action of the group \( G_{b\text{Id}_V} \) since the action of this group is a multiplication by \( e^{bt} \). By the same argument, it follows that condition (AB) is satisfied as well. Finally, the operator \( C \) generates a strongly continuous semigroup \( S_C \) by [53, Theorem 3.1.1] that is given by

\[ S_C(t) = \exp \left\{ - \frac{c_{2k}}{(2k)!} b^{2k} t \right\} S_A(t) \]

for \( t \geq 0 \) by [53, formula (1.2) on p. 77]. Therefore, by [63, Theorem 3.2.2], for every \( x_0 \in \text{Dom } A \), there is a \( \text{Dom } A \)-valued solution to the problem

\[
\begin{aligned}
&\dot{v}(t) = C v(t), & 0 < t \leq 1, \\
v(0) = x_0,
\end{aligned}
\]

that is given by \( v_{0,x_0}(t) = S_C(t)x_0 \). Consequently, for every \( x_0 \in \text{Dom } A \), it follows by Proposition 3.5 that the process \( (Y_{k,2}(t), t \in [0, 1]) \) defined by

\[ Y_{k,2}(t) := \exp \left\{ bW_t^\frac{1}{2} - \frac{c_{2k}}{(2k)!} b^{2k} t \right\} S_A(t)x_0 \]

satisfies the equation

\[ Y_{k,2}(t) = x_0 + \int_0^t AY_{k,2}(r) \, dr + \int_0^t bY_{k,2}(r) \, dW_r^\frac{1}{2} \]

for every \( t \in [0, 1] \) \( \mathbb{P} \)-almost surely.
An example to which this result can be applied is the initial-boundary value problem for the heat equation that is formally described by

$$(\partial_t u)(t, x) = (\Delta_x u)(t, x) + b(t, x)\dot{W}^{1/2}_t$$

for $(t, x) \in [0, 1] \times \mathcal{O}$ where $\mathcal{O} \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is a bounded domain with smooth boundary $\partial \mathcal{O}$ and $\Delta_x$ is the Laplace operator. The equation is subject to the initial condition $u(0, x) = u_0(x)$, $x \in \mathcal{O}$, and to the Dirichlet boundary condition

$$u(t, x) = 0$$

or the Neumann boundary condition

$$\frac{\partial u}{\partial \nu}(t, x) = 0$$

for $(t, x) \in [0, 1] \times \partial \mathcal{O}$. In the Neumann problem, the symbol $\frac{\partial}{\partial \nu}$ denotes the conormal derivative. The above problem is rigorously interpreted as equation (16) by setting $V := L^2(\mathcal{O})$, $x_0 := u_0$, and $A := \Delta_x$ on Dom $A := W^{2,2}(\mathcal{O}) \cap W_0^{1,2}(\mathcal{O})$ for the Dirichlet problem or on Dom $A := \{ f \in W^{2,2}(\mathcal{O}) \mid \frac{\partial f}{\partial \nu} = 0 \text{ on } \partial \mathcal{O} \}$ for the Neumann problem and solved by the above method if $u_0 \in \text{Dom } A$.

**Example 3.8.** Consider the formal equation

$$(\partial_t u)(t, x) = a(\partial_t^2 u)(t, x) + b(\partial_x u)(t, x)\dot{W}^{1/2k}_t$$

for $(t, x) \in [0, 1] \times \mathbb{R}$ with the initial condition $u(0, x) = u_0(x)$ for $x \in \mathbb{R}$ where $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$. Here again, the process $W^{1/2k}$ is the formal time derivative of the fractional Brownian motion with the Hurst parameter $H = 1/2k$ for some $k \in \mathbb{N}$ that is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Rigorously, the above problem can be written in our framework by setting $\omega := W^{1/2k}$, $V := L^2(\mathbb{R})$, $A := a\partial^2$ on Dom $A := W^{2,2}(\mathbb{R})$ (where $[(a\partial^2)f](x) := af''(x)$ for $f \in W^{2,2}(\mathbb{R})$), $B := b\partial$ on Dom $B := W^{1,2}(\mathbb{R})$ (where $[(b\partial)f](x) := bf'(x)$ for $f \in W^{1,2}(\mathbb{R})$). Below, it is shown that this equation can be solved by our method if $k = 1$ and $a > b^2/2$ or if $k$ is an even positive integer, i.e. if the driving signal is either the Wiener process or a fractional Brownian motion with Hurst parameter $H = 1/4m$ for $m \in \mathbb{N}$.

As in Example 2.5, we have that $[W^{1/2k}]_{2k}^2(t) = c_{2k}t$ which is a non-zero and continuously differentiable function. The operator $b\partial$ generates a strongly continuous group $G_{b\partial}$ on the space $L^2(\mathbb{R})$ that is given by

$$[G_{b\partial}(t)f](x) = f(x + bt)$$

for $t, x \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$; see, e.g., [23, Proposition 1 on p. 66]. The operators $C(t)$ defined by (13) are independent of $t$ and they are given by $C(t) = C_k$ where

$$C_k = a\partial^2 - \frac{c_{2k}}{(2k)!}b^{2k}\partial^{2k}$$

(17)

on the space Dom $C = W^{2k,2}(\mathbb{R})$. Hence, Assumption 3.3 is satisfied. The space $\tilde{V} := W^{2k,2}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. Since $G_{b\partial}(t)$ is simply a shift operator for every $t \in \mathbb{R}$, it maps $W^{2k,2}(\mathbb{R})$ back into itself. Moreover, there is the equality

$$[G_{b\partial}(t)(a\partial^2)f](x) = af''(x + bt) = [(a\partial^2)G_{b\partial}(t)f](x)$$

for every $f \in W^{2k,2}(\mathbb{R})$ and every $t, x \in \mathbb{R}$ so that condition (AB) is satisfied as well. It remains to show that the corresponding homogeneous initial value problem
(CP) has a $W^{2k,2}(\mathbb{R})$-valued solution. Below, we split the reasoning into two cases that are treated separately.

The Wiener case ($k = 1$). If $k = 1$, the operator $C_1$ is given by $C_1 = c\partial^2$ with $c := a - \frac{k^2}{2}$ and it is defined on Dom $C_1 = W^{2,2}(\mathbb{R})$. If $c > 0$, then the operator $-c\partial^2$ is strongly elliptic, see [53, Definition 7.2.1], and hence, by [53, Theorem 7.2.7 and Remark 7.2.9] it follows that the operator $c\partial^2$ generates an analytic semigroup $S_{c\partial^2}$ of bounded linear operators on the space $L^2(\mathbb{R})$. This is of course the scaled heat semigroup given by

$$[S_{c\partial^2}(t)f](x) = \frac{1}{\sqrt{4\pi ct}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4ct}} f(y) \, dy$$

for $t > 0$, $x \in \mathbb{R}$, and $f \in L^2(\mathbb{R})$. Hence, it follows by [63, Theorem 3.2.2] that for every $f \in W^{2,2}(\mathbb{R})$ there is a $W^{2,2}(\mathbb{R})$-valued solution to the problem

$$\begin{cases}
\dot{v}(t) = C_1 v(t), & 0 < t \leq 1, \\
v(0) = f,
\end{cases}$$

that is given by $v_{0,f}(t) = S_{c\partial^2}(t)f$. Consequently, by Proposition 3.5, we obtain that the process $(Y_{1,3}(t), t \in [0,1])$ defined by

$$Y_{1,3}(t) := G_{b\partial}(W_t) S_{c\partial^2}(t)f$$

satisfies the equation

$$Y_{1,3}(t) = f + \int_0^t a\partial^2 Y_{1,3}(r) \, dr + \int_0^t b\partial Y_{1,3}(r) \, d\mathcal{B}W_r$$

for every $t \in [0,1] \ \mathbb{P}$-almost surely. Moreover, in this case, there is the explicit formula

$$Y_{1,3}(t,x) = \frac{1}{\sqrt{4\pi ct}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4ct}} f(y + bw_t) \, dy, \quad t \in [0,1], x \in \mathbb{R}.$$ 

The singular fractional Brownian motion case ($k > 1$). In this case, the operator $-C_k$ that is given by (17) is strongly elliptic if $k$ is an even integer. For $k = 2m$, $m \in \mathbb{N}$, it follows by [53, Theorem 7.2.7 and Remark 7.2.9] that the operator $C_{2m}$ generates an analytic semigroup $S_{C_{2m}}$ of bounded linear operators acting on the space $L^2(\mathbb{R})$. Therefore, as in the Wiener case above, we have by Proposition 3.5 that for every $f \in W^{4m,2}(\mathbb{R})$, the process $(Y_{2m,3}(t), t \in [0,1])$ defined by

$$Y_{2m,3}(t) := G_{b\partial}(W_t^{1/m}) S_{C_k}(t)f$$

satisfies the equation

$$Y_{2m,3}(t) = f + \int_0^t a\partial^2 Y_{2m,3}(r) \, dr + \int_0^t b\partial Y_{2m,3}(r) \, d\mathcal{B}W_r^{1/m}$$

for every $t \in [0,1] \ \mathbb{P}$-almost surely.

3.2. The non-commutative case. In what follows, we show that the commutativity assumption (AB) in Proposition 3.5 can be weakened in the spirit of [13] (see also [14, Section 6.5]). Set $z_{s,\omega}(t) := G_B(\omega_t - \omega_s)$ for $t \in [s,T]$ and note that for every $t \in [s,T]$, the operator $z_{s,\omega}(t)$ is invertible with $z_{s,\omega}^{-1}(t) = G_B(\omega_s - \omega_t)$. Consider the homogeneous initial value problem

$$\begin{cases}
\dot{v}(t) = z_{s,\omega}^{-1}(t) C(t) z_{s,\omega}(t) v(t), & 0 \leq s < t \leq T, \\
v(s) = x_0.
\end{cases} \quad \text{(NCP)}$$
Similarly as in Proposition 3.5 it is now shown that if problem (NCP) has a solution, then the bilinear problem (BLP) also has a solution; however, without assuming that $A$ and $GB$ commute. More precisely, there is the following result:

**Proposition 3.9.** Let Assumption 3.3 be verified. Let $\tilde{V}$ be a Hilbert space that is dense in $V$; continuously embedded in Dom $C$; closed under the action of the group $G_B$; and such that the restriction of the group $G_B$ to $\tilde{V}$ forms a strongly continuous group on $\tilde{V}$. If there exists a $\tilde{V}$-valued solution $v_{s,x_0}$ to problem (NCP), then the function $X_{s,x_0}^\omega : [s,T] \to V$ defined by

$$X_{s,x_0}^\omega(t) := z_{s,\omega}(t)v_{s,x_0}(t)$$

is a strong solution to the problem (BLP).

**Remark 3.10.** Note that if condition (AB) is satisfied, then problem (NCP) becomes problem (CP).

**Proof of Proposition 3.12.** Define $f : [s,T] \times \mathbb{R} \to V$ by

$$f(t,x) := G_B(x - \omega_s)v_{s,x_0}(t).$$

As in the proof of Proposition 3.5, we apply Lemma 2.7 to obtain

$$z_{s,\omega}(t)v_{s,x_0}(t) = x_0 + \int_s^t Bz_{s,\omega}(r)v_{s,x_0}(r) \, d\pi \omega_r + \frac{1}{p!} \int_s^t B^p z_{s,\omega}(r)v_{s,x_0}(r) \, d[\omega]_p^r(r) + \int_s^t z_{s,\omega}(r)\dot{v}_{s,x_0}(r) \, dr.$$

By using the fact that $v_{s,x_0}$ satisfies equation (NCP), we obtain

$$\int_s^t z_{s,\omega}(r)\dot{v}_{s,x_0}(r) \, dr = \int_s^t C(r)z_{s,\omega}(r)v_{s,x_0}(r) \, dr.$$

Since $v_{s,x_0}$ belongs to $\mathcal{C}([s,T]; \tilde{V})$ with $\tilde{V}$ being continuously embedded into Dom $C$ and since the restriction of the group $G_B$ to $\tilde{V}$ forms a strongly continuous group there, the function $[r \mapsto Az_{s,\omega}(r)v_{s,x_0}(r)]$ belongs to $\mathcal{C}([s,T]; \tilde{V})$ by continuity of $\omega$. Consequently,

$$\int_s^t C(r)z_{s,\omega}(r) \, dr = \int_s^t Az_{s,\omega}(r)v_{s,x_0}(r) \, dr - \frac{1}{p!} \int_s^t B^p z_{s,\omega}(r)v_{s,x_0}(r) \, d[\omega]_p^r(r)$$

which proves the claim.

We give the following example to which Proposition 3.9 can be directly applied.

**Example 3.11.** Consider the formal equation

$$\ddot{x}(t) + 2\gamma(1 + \tilde{\omega}_t)\dot{x}(t) + k^2 x(t) = 0,$$

for $t \in [0,1]$ subject to the initial condition $x(0) = x_0$ and $\dot{x}(0) = v_0$ with $x_0, v_0 \in \mathbb{R}$. This equation describes the dynamics of the force-free harmonic oscillator of unit mass, intrinsic frequency $k > 0$, and a damping rate $2\gamma > 0$ that is subject to random perturbations $\omega$ (for concrete physical applications, see, e.g., [32] and the references therein). It is assumed that $\omega$ is a continuous function that has finite $p$-th variation along $\pi$ for some positive even integer $p$ and a sequence of partitions $\pi \subset P[0,1]$ whose mesh size tends to zero. It is also assumed that $[\omega]_p^\pi \in \mathcal{C}([0,1])$ and $\omega_0 = 0$. 

The above formal equation is interpreted in our framework as the bilinear equation
\[
\begin{aligned}
\frac{dX_t}{dt} &= AX_t + BX_t d\pi \omega_t, \\ X_0 &= (x_0, v_0)^	op,
\end{aligned}
\tag{18}
\]
with
\[
A := \begin{pmatrix} 0 & 1 \\ -k^2 & -2\gamma \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 0 & 0 \\ 0 & -2\gamma \end{pmatrix}
\]
considered as bounded linear operators on the space \( V = \mathbb{R}^2 \). The matrix \( B \) generates a strongly continuous group \((G_B(x), x \in \mathbb{R})\) that is given by the matrix exponential
\[
G_B(x) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\gamma x} \end{pmatrix}
\]
and \((C(t), t \in [0, 1])\) is defined as
\[
C(t) = \begin{pmatrix} 0 & 1 \\ -k^2 & -2\gamma - (2\gamma)^p |\dot{\omega}|_p(t) \end{pmatrix}.
\]
Consequently, the system of operators \((\tilde{C}_0(t), t \in [0, 1])\) is given by
\[
\tilde{C}_0(t) = \begin{pmatrix} 0 & e^{-2\gamma \omega t} \\ -k^2 e^{2\gamma \omega t} & -2\gamma - (2\gamma)^p |\dot{\omega}|_p(t) \end{pmatrix}
\]
and since this matrix has continuous entries, it follows that the \( \mathbb{R}^2 \)-valued solution to problem (NCP) exists. Thus assumptions of Proposition 3.9 are verified with \( \tilde{V} = \mathbb{R}^2 \) and it follows that the problem (18) admits a strong solution.

In the following two subsections, we give sufficient conditions for the existence of a strong solution to problem (BLP) by solving equation (NCP) in two cases: in the parabolic case and in the hyperbolic case.

3.2.1. The parabolic non-commutative case. In what follows we give some sufficient conditions for the solvability of problem (NCP) in what is usually called the parabolic case. For a linear operator \( K : \text{Dom} \, K \subseteq V \to V \), the symbol \( \rho(K) \) denotes the resolvent set, i.e. the set of all \( \lambda \in \mathbb{C} \) such that the operator \((\lambda I - K)\) has bounded inverse. Here, \( I \) denotes the identity operator. We say that a family of operators \((K(t), t \in [0, T])\) is parabolic\(^3\) if the following three conditions are satisfied:

\((P1)\) The family \((K(t), t \in [0, T])\) consists of closed linear operators on \( V \) that are defined on a common domain \( D \) which is independent of \( t \) and dense in \( V \).

\((P2)\) There exists \( \lambda_K \in \mathbb{R} \) such that, for every \( t \in [0, T] \), the resolvent set \( \rho(K(t)) \) contains the half-plane
\[
\mathbb{C}^{\lambda_K^+} := \{ \lambda \in \mathbb{C} \mid \Re[\lambda] \geq \lambda_K \}
\]
and there exists a constant \( M_K > 0 \) such that the inequality
\[
\| \lambda I - K(t) \|_{L(V)} \leq \frac{M_K}{1 + |\lambda|}
\]
is satisfied for every \( \lambda \in \mathbb{C}^{\lambda_K^+} \) and \( t \in [0, T] \).

\(^3\)The term parabolic is used since assumptions (P1) and (P2) imply that, for fixed \( t \in [0, T] \), the operator \( K(t) \) generates an analytic semigroup.
(P3) There exist constants $L_K > 0$ and $0 < \alpha \leq 1$ such that inequality
\[
\left\| (\lambda_0 I - C(t))^{-1} \right\|_{\mathcal{L}(V)} \leq L_K |t-s|^\alpha
\]
is satisfied for every $s, t \in [0, T]$.

**Proposition 3.12.** Assume that $\omega$ satisfies the following two regularity conditions:

- (p1) The path $\omega$ belongs to $\mathcal{C}^{\gamma_1}([0, T])$ for some $0 < \gamma_1 \leq 1$.
- (p2) The $p$-th variation $[\omega]_p^\pi$ is non-zero and belongs to the space $\mathcal{C}^{1,\gamma_2}([0, T])$ for some $0 < \gamma_2 \leq 1$.

Assume that the operator $B$ satisfies the following condition:

- (p3) The operator $B$ is the infinitesimal generator of a strongly continuous group of bounded linear operators $G_B$ acting on the space $V$.

Assume that the family $(C(t), t \in [0, T])$ defined by formula (13) and the space $\text{Dom } C$ defined by formula (14) satisfy the following three conditions:

- (p4) $\text{Dom } C$ is dense in $V$.
- (p5) For every $t \in [0, T]$, the linear operator $C(t)$ is closed and it holds that $\text{Dom } C(t) = \text{Dom } C$.
- (p6) There exists $\lambda_0 \in \mathbb{R}$ such that for every $t \in [0, T]$, the resolvent set $\rho(C(t))$ contains the half-plane $C^+_{\lambda_0}$; and there exists a finite positive constant $k_1$ such that the inequality
\[
\left\| (\lambda_0 I - C(t))^{-1} \right\|_{\mathcal{L}(V)} \leq \frac{k_1}{1 + |\lambda|}
\]
holds for every $\lambda \in C^+_{\lambda_0}$ and $t \in [0, T]$.

Assume, moreover, the following:

- (p7) There exists a finite positive constant $k_2$ such that the inequality
\[
\|v\|_V + \|Av\|_V + \|B^p v\|_V \leq k_2 \left( \|v\|_V + \left\| (\lambda_0 I - C(t))v \right\|_V \right)
\]
holds for every $v \in \text{Dom } C$ and $t \in [0, T]$.

- (p8) The restriction of the group $G_B$ to $\text{Dom } C$ is a strongly continuous group of bounded linear operators acting on $\text{Dom } C$ that is endowed with the graph inner product of the operator $(\lambda_0 I - C(s))$.

Finally, assume that $B$ and the family $(C(t), t \in [0, T])$ are connected in the following manner:

- (p9) There exist $K \subseteq \text{Dom } B$ and a function $L : [0, T] \to \mathcal{L}(V)$ such that $K$ is the core of $B$, $L$ is bounded, and the equality
\[
(\lambda_0 I - C(t))B(\lambda_0 I - C(t))^{-1}z = (B + L(t))z
\]
is satisfied for every $z \in K$ and $t \in [0, T]$.

For $\alpha \geq 0$ and $t \in [0, T]$, denote by $D_{\alpha}(t)$ the domain of the fractional power\(^4\)

\[(\lambda_0 I - C(t))^\alpha\]

endowed by its graph inner product. If there exists $\varepsilon > 0$ such that $x_0 \in D_{1+\varepsilon}(s)$, then there is a strong solution to the bilinear problem (BLP).

**Remark 3.13.** Proposition 3.12 is reminiscent of [13, Proposition 1] and the proof technique is taken from there. There are, however, several reasons why it seems necessary to include the full proof here.

\(^4\)The fractional powers can be defined since assumptions (p4), (p5), and (p6) imply that $C(r)$ generates an analytic semigroup of bounded linear operators on $V$ and that the operator $\lambda_0 I - C(r)$ is positive; see, e.g., [53, Section 2.6].
The first difference is that Proposition 3.12 allows for a large number of driving paths \( \omega \) while \([13, \text{Proposition 1}]\) only deals with the case when \( \omega \) is a path of the Wiener process.

The second difference is that in \([13, \text{Proposition 1}]\), the operator \( C(t) \) does not depend on \( t \) since the usual quadratic variation of the Wiener process has constant derivative. As it is seen from the following proof, the dependence of \( C(t) \) on \( t \) causes some technical difficulties.

Finally, in \([13, \text{Proposition 1}]\), it could be assumed that \( \text{Dom} \ A \subseteq \text{Dom} \ B^2 \) since typically, the operator \( A \) exhibits “worse” behaviour than \( B^2 \); however, in our case, the situation is in many cases reversed. This is already seen in Example 3.8 where \( B \) is a first-order differential operator so that \( B^p \) is \( p \)-th order differential operator (recall that \( p \) is a positive even integer so that \( p \geq 2 \)) and \( A \) is a second-order differential operator so that \( \text{Dom} \ B^p \subseteq \text{Dom} \ A \). Because of this, the roles of \( \text{Dom} \ A \) and \( \text{Dom} \ B^p \) are symmetric in Proposition 3.12.

**Proof of Proposition 3.12.** Without loss of generality, it is assumed that \( \lambda_0 = 0 \). We divide the proof into several steps.

**Step 1:** Let \( s \in [0, T] \) be fixed for the rest of the proof. First it is shown that the space \( \text{Dom} \ C \) is closed under the action of the group \( G_B \) so that the operator

\[
\tilde{C}_s(t) := z_{s, \omega}(t)C(t)z_{s, \omega}(t) = G_B(\omega_t - \omega_s)^{-1}C(t)G_B(\omega_t - \omega_s)
\]

on \( \text{Dom} \ \tilde{C}_s(t) = \text{Dom} \ C \) is well-defined for every \( t \in [0, T] \). Indeed, note that the equality

\[
C(t)(\lambda I - B)^{-1}C(t)^{-1} = (\lambda I - (B + L(t)))^{-1}
\]

(19)

holds for every \( t \in [0, T] \) and \( \lambda \in \rho(B) \cap \rho(B + L(t)) \). This is proved in the same manner as \([13, \text{formula (5)}]\) by using assumption (p9). By \([53, \text{Theorem 3.1.1}]\), we have that for every \( t \in [0, T] \), the operator \( B + L(t) \) is the infinitesimal generator of a strongly continuous group of bounded linear operators acting on the space \( V \) that we denote by \( G_{B+L(t)} \) and that satisfies

\[
\|G_{B+L(t)}(x)\|_{X(V)} \leq M \exp \left\{ (m + M\|L(t)\|_{X(V)})\|x\| \right\}
\]

for every \( x \in \mathbb{R} \) where \( M \geq 1 \) and \( m \geq 0 \) are constants such that the inequality

\[
\|G_B(x)\|_{X(V)} \leq M \exp\{m|x|\}
\]

holds for every \( x \in \mathbb{R} \). This follows by the fact that the operator \( B \) generates a strongly continuous group \( G_B \) by assumption (p3). It follows from (19) that the space \( \text{Dom} \ C \) is closed under the action of the group \( G_B \) and that for every \( x \in \mathbb{R} \) and \( t \in [0, T] \), the equality

\[
C(t)G_B(x)C(t)^{-1} = G_{B+L(t)}(x)
\]

(20)

is satisfied.

Before we continue, let us fix the following notation that will simplify the exposition. Set

\[
\Omega_{0,T} := \{ x \in \mathbb{R} \mid x = |\omega_u - \omega_v|, \ u, v \in [0, T] \}
\]

and note that this set is compact since the path \( \omega \) is continuous. In particular, the number \( \sup \Omega_{0,T} \) is finite. Moreover, for fixed \( u, v \in [0, T] \), we have

\[
\|G_B(\omega_u - \omega_v)\|_{X(V)} \leq M \exp \{m|\omega_u - \omega_v|\} \leq M \exp\{m \sup \Omega_{0,T} =: k_\omega\}
\]
Step 2: In the second step, it is shown that the family \( (\tilde{C}_s(t), t \in [0, T]) \) is parabolic. Indeed, it is clear by assumptions (p5) and (p6) that the family \( (\tilde{C}_s(t), t \in [0, T]) \) satisfies conditions (P1) and (P2). We will show that the condition (P3) is satisfied as well. To this end, let \( u, v \in [0, T] \) be arbitrary. We have that the inequality
\[
\| \tilde{C}_s(u) \tilde{C}_s(v)^{-1} - I \|_{\mathcal{L}(V)} \leq k_s^2 \| C(u) G_B(\omega_u - \omega_v) C(v)^{-1} - G_B(\omega_u - \omega_v) \|_{\mathcal{L}(V)}
\]
is satisfied and adding and subtracting the term \( C(v) G_B(\omega_u - \omega_v) C(v)^{-1} \) inside the norm yields the inequality
\[
\| \tilde{C}_s(u) \tilde{C}_s(v)^{-1} - I \|_{\mathcal{L}(V)} \leq k_s^2 \| (I) + (II) \|
\]
where the expressions (I) and (II) are defined by
\[
(I) := \| [C(u) - C(v)] G_B(\omega_u - \omega_v) C(v)^{-1} \|_{\mathcal{L}(V)},
\]
\[
(II) := \| C(v) G_B(\omega_u - \omega_v) C(v)^{-1} - G_B(\omega_u - \omega_v) \|_{\mathcal{L}(V)}.
\]
For the term (I) it follows that
\[
(21) \quad (I) = \frac{1}{p!} \left| \tilde{\omega} \right|^p_p(u) - \left| \tilde{\omega} \right|^p_p(v) \| B^p G_B(\omega_u - \omega_v) C(v)^{-1} \|_{\mathcal{L}(V)}
\]
\[
\leq k_\omega \frac{p!}{p!} (k_1 + 1) k_2 \left| \tilde{\omega} \right|^p_p \| \omega \|_{\mathcal{L}^{\gamma_2}([0, T])} |u - v|^\gamma_2
\]
where we successively used Hölder continuity of the derivative \( \tilde{\omega}_p \) from assumption (p2), the fact that \( B^p \) and \( G_B \) commute on \( \text{Dom } B^p \) by [53, Theorem 1.2.4 c)], equivalence of graph norms of \( C(r) \) from assumption (p7), and the uniform estimate on the resolvent from assumption (p6). In order to treat the term (II), note that by [53, equality (1.2) on p. 77] there is the equality
\[
\| [G_{B+L(t)}(x) - G_B(x)] z = \int_0^x G_B(x - \xi) L(t) G_{B+L(t)}(\xi) z d\xi
\]
for every \( t \in [0, T], x > 0, \) and \( z \in \text{Dom } B, \) and, consequently, the estimate
\[
(22) \quad \| G_{B+L(t)}(x) - G_B(x) \|_{\mathcal{L}(V)} \leq L(t) \| \mathcal{L}(V) \| \sup_{|\xi| \leq |x|} \| G_B(\xi) \|_{\mathcal{L}(V)} \| G_{B+L(t)}(\xi) \|_{\mathcal{L}(V)} \| x \|
\]
for every \( t \in [0, T] \) and \( x \in \mathbb{R} \) is obtained. Denote \( k_L := \sup_{r \in [0,T]} \| L(r) \|_{\mathcal{L}(V)} \) where the supremum is finite by assumption (p9). By using equality (20), estimate (22), and Hölder continuity of the path \( \omega \) from assumption (p1) successively, we obtain the following:
\[
(II) \leq k_\omega k_L M \exp \left\{ (m + Mk_L) \sup \{ \Omega_{0,T} \} \left| \omega \right|_{\mathcal{L}^{\gamma_1}([0, T])} |u - v|^\gamma_1 \right\}.
\]
Altogether, we obtain from inequalities (21) and (23) that there are constants \( \kappa_1, \kappa_2 > 0 \) such that the inequality
\[
(I) + (II) \leq \kappa_1 |u - v|^\gamma_1 + \kappa_2 |u - v|^\gamma_2
\]
holds. This proves that the system \( (\tilde{C}_s(t), t \in [0, T]) \) satisfies (P3). Note also that if \( \gamma_0 := \min \{ \gamma_1, \gamma_2 \}, \) by [63, formula (5.5) on p. 118], there exists a constant \( \kappa_0 > 0 \) such that
\[
\| (\tilde{C}_s(u) - \tilde{C}_s(v)) \tilde{C}_s(r)^{-1} \|_{\mathcal{L}(V)} \leq \kappa_0 |u - v|^\gamma_0
\]
holds for every \( u, v, r \in [0, T]. \)
Before continuing with the next step note that \( \tilde{C}_s(s) = C(s) \) so that this operator is the infinitesimal generator of an analytic semigroup \( S_{\tilde{C}_s(s)} = S_{C(s)} \) of bounded linear operators acting on the space \( V \). Consequently, for \( \alpha \geq 0 \), it holds that \( (-\tilde{C}_s(s))^\alpha = (-C(s))^\alpha \) with the domain \( \text{Dom}(-\tilde{C}_s(s))^\alpha = \text{Dom}(-C(s))^\alpha = D_\alpha(s) \). This space becomes a Hilbert space when endowed with the (graph) inner product of \( (-\tilde{C}_s(s))^\alpha = (-C(s))^\alpha \) given by

\[
\langle z_1, z_2 \rangle_{D_\alpha(s)} := \langle z_1, z_2 \rangle_V + \langle (\tilde{C}_s(s))^\alpha z_1, (\tilde{C}_s(s))^\alpha z_2 \rangle_V, \quad z_1, z_2 \in D_\alpha(s).
\]

We denote the induced norm by \( \| \cdot \|_{D_\alpha(s)} \).

Assume that there exists \( \varepsilon > 0 \) is such that \( x_0 \in D_{1+\varepsilon}(s) \). Define the function \( f_{s,x_0} : [s,T] \to V \) by

\[
f_{s,x_0}(t) := [\tilde{C}_s(t) - \tilde{C}_s(s)]S_{\tilde{C}_s(s)}(t - s)x_0.
\]

**Step 3:** In this step, it is shown that \( f_{s,x_0} \) belongs to the space \( \mathscr{C}^\gamma([s,T]; V) \) where \( \gamma =: \min\{\varepsilon, \gamma_1, \gamma_2\} \). To this end, let \( u, v \in [s,T] \) be such that \( u > v \). There is the inequality

\[
\|f_{s,x_0}(u) - f_{s,x_0}(v)\|_V \leq (\text{III}) + (\text{IV})
\]

where

\[
(\text{III}) := \|\tilde{C}_s(u) - \tilde{C}_s(v)\|S_{\tilde{C}_s(s)}(u - s)x_0\|_V,
\]

\[
(\text{IV}) := \|\tilde{C}_s(v) - \tilde{C}_s(s)\|S_{\tilde{C}_s(s)}(u - s) - S_{\tilde{C}_s(s)}(v - s)]x_0\|_V.
\]

For (III), we have by commutativity of the semigroup \( S_{\tilde{C}_s(s)} \) on \( D_{1+\varepsilon}(s) \) ensured by [53, Theorem 2.6.13 b)] that

\[
(\text{III}) \leq \|\tilde{C}_s(u) - \tilde{C}_s(v)\|\tilde{C}_s(s)^{-1}\|S_{\tilde{C}_s(s)}(u - s)x_0\|_V
\]

\[
\leq \|\tilde{C}_s(u) - \tilde{C}_s(v)\|\tilde{C}_s(s)^{-1}\|S_{\tilde{C}_s(s)}(\gamma)\|\tilde{C}_s(s)x_0\|_V
\]

from which it follows by (24) that there exists a finite positive constant \( c_1 \) such that

\[
(\text{III}) \leq c_1(u - v)^{\gamma_0}. \tag{25}
\]

Similarly, for (IV), there is the estimate

\[
(\text{IV}) \leq \|\tilde{C}_s(v) - \tilde{C}_s(s)\|\tilde{C}_s(s)^{-1}\|S_{\tilde{C}_s(s)}(u - s) - S_{\tilde{C}_s(s)}(v - s)]x_0\|_V
\]

where the first term is bounded by \( \kappa_0 T^{\gamma_0} \) by (24) and for the second term, we obtain

\[
\|\tilde{C}_s(s)[S_{\tilde{C}_s(s)}(u - s) - S_{\tilde{C}_s(s)}(v - s)]x_0\|_V
\]

\[
\leq c_\varepsilon(u - v)^\varepsilon \sup_{\tau \in [0,T]} \|S_{\tilde{C}_s(s)}(\tau)\|_V \|(-\tilde{C}_s(s))^{1+\varepsilon}x_0\|_V
\]

with some finite positive constant \( c_\varepsilon \) by using [53, Theorem 2.6.13 d)] and commutativity of the semigroup \( S_{\tilde{C}_s(s)} \) with its generator \( \tilde{C}_s(s) \) on \( D_{1+\varepsilon}(s) \). Thus it is seen that there exists a finite positive constant \( c_2 \) such that

\[
(\text{IV}) \leq c_2(u - v)^\varepsilon. \tag{26}
\]

From inequalities (25) and (26) it follows that

\[
\|f_{s,x_0}(u) - f_{s,x_0}(v)\|_V \leq c_1(u - v)^{\gamma_0} + c_2(u - v)^\varepsilon
\]

which shows that indeed \( f_{s,x_0} \in \mathscr{C}^\gamma([s,T]; V) \).

**Step 4:** We have shown that the family \( (\tilde{C}_s(t), t \in [0,T]) \) satisfies conditions (P1)-(P3) and that the function \( f_{s,x_0} \) belongs to the space \( \mathscr{C}^\gamma([s,T]; V) \). Moreover,
since \( f_{s,x_0}(s) = 0 \), it follows by [53, Theorem 5.7.1 and Remark 5.7.2] that there is a (unique) function \( u_{s,x_0} \in \mathcal{C}^{1,\gamma'}([s,T];V) \) with \( \gamma' < \gamma \); \( u_{s,x_0}(t) \in \text{Dom } C \) for every \( t \in [s,T] \) (we can include the point \( s \) in the interval since \( 0 \in \text{Dom } C \)); and such that it satisfies the equation

\[
\begin{aligned}
\dot{u}(t) &= \tilde{C}_s(t)u(t) + f_{s,x_0}(t), \quad 0 \leq s < t \leq T, \\
u(s) &= 0.
\end{aligned}
\]  

(27)

If we now define a function \( v_{s,x_0} : [s,T] \to V \) by

\[
v_{s,x_0}(t) := u_{s,x_0}(t) + S_{\tilde{C}_s(s)}(t-s)x_0,
\]

(28)

it follows that \( v_{s,x_0} \in \mathcal{C}^1([s,T];V) \), \( v_{s,x_0}(t) \in \text{Dom } C \) for every \( t \in [s,T] \), and it is easily verified that \( v_{s,x_0} \) satisfies the equation

\[
\begin{aligned}
\dot{v}(t) &= \tilde{C}_s(t)v(t), \quad 0 \leq s < t \leq T, \\
v(s) &= x_0.
\end{aligned}
\]

(29)

**Step 5:** Now it is shown that \( v_{s,x_0} \in \mathcal{C}([s,T];D_1(s)) \). Note first that all the graph norms of \( \tilde{C}_s(t) \), \( t \in [0,T] \), are equivalent. Indeed, for \( u,v \in [0,T] \) and \( z \in \text{Dom } C \), there is the following estimate:

\[
\begin{align*}
\|z\|_V + \|\tilde{C}_s(z)z\|_V &= \|z\|_V + \|\tilde{C}_s(u) - \tilde{C}_s(v)\|_V + \|\tilde{C}_s(v)z\|_V \\
&\leq \|z\|_V + \|	ilde{C}_s(u) - \tilde{C}_s(v)\|_V + \|\tilde{C}_s(v)^{-1}\|_{\mathcal{L}(V)}\|\tilde{C}_s(v)z\|_V \\
&\leq (1 + \kappa_0 T^\gamma\|z\|_V + \|\tilde{C}_s(v)z\|_V)
\end{align*}
\]

by using inequality (24). We now consider the norm \( \| \cdot \|_{D_1(s)} \) defined by

\[
\|z\|_{D_1(s)} := \|z\|_V + \|C(s)z\|_V, \quad z \in D_1(s),
\]

which is equivalent to the norm \( \| \cdot \|_{D_1(s)} \). For \( u,v \in [s,T] \), we have that

\[
\|v_{s,x_0}(u) - v_{s,x_0}(v)\|_{D_1(s)} = \|v_{s,x_0}(u) - u_{s,x_0}(v)\|_V + \|\tilde{C}_s(u)v_{s,x_0}(u) - v_{s,x_0}(v)\|_V \\
&\leq (1 + \kappa_0 T^\gamma)(\|V\| + \|VI\|)
\]

where

\[
(V) := \|v_{s,x_0}(u) - v_{s,x_0}(v)\|_V,
\]

\[
(VI) := \|\tilde{C}_s(u)v_{s,x_0}(u) - v_{s,x_0}(v)\|_V.
\]

The term \( (V) \) tends to zero as \( u \to v \) since \( v_{s,x_0} \in \mathcal{C}([s,T];V) \). For \( (VI) \), we obtain by adding and subtracting the term \( \tilde{C}_s(v)v_{s,x_0}(v) \) inside the norm that

\[
(VI) \leq \|\tilde{C}_s(u)v_{s,x_0}(u) - \tilde{C}_s(v)v_{s,x_0}(v)\|_V + \|\tilde{C}_s(v) - \tilde{C}_s(u)\|_V
\]

is satisfied. Moreover, for the second term above, we have by (24) that

\[
\|\tilde{C}_s(v) - \tilde{C}_s(u)\|_V \leq \|\tilde{C}_s(v) - \tilde{C}_s(u)\|_V + \|\tilde{C}_s(v) - \tilde{C}_s(u)\|_V
\]

\[
\leq \kappa_0 \|u - v\|_V \sup_{\tau \in [s,T]} \|\tilde{C}_s(\tau)v_{s,x_0}(\tau)\|_V
\]

so that the estimate

\[
(VI) \leq \|\tilde{C}_s(u)v_{s,x_0}(u) - \tilde{C}_s(v)v_{s,x_0}(v)\|_V + \kappa_0 \|u - v\|_V \sup_{\tau \in [s,T]} \|\tilde{C}_s(\tau)v_{s,x_0}(\tau)\|_V
\]

is obtained. Now, since \( v_{s,x_0} \in \mathcal{C}^1([s,T];V) \) and \( \dot{v}_{s,x_0}(t) = \tilde{C}_s(t)v_{s,x_0}(t) \) holds for every \( t \in (s,T) \), we have that the map \( t \mapsto \tilde{C}_s(t)v_{s,x_0}(t) \) belongs to \( \mathcal{C}([s,T];V) \).
Therefore, the supremum in the estimate above is finite and (VI) tends to zero as \( u \to v \). Consequently, we have that \( v_{s,x_0} \in \mathcal{C}((s,T]; D_1(s)) \).

**Step 6:** Finally, all the previous steps are put together. Clearly, Assumption 3.3 is satisfied by assumptions (p2), (p3), and (p5). Set \( \tilde{V} := D_1(s) \). Since \( D_1(s) = \text{Dom } C \), it follows that \( D_1(s) \) is dense in \( V \) by assumption (p4), continuously embedded in \( \text{Dom } C \) (and thus in \( V \)) by assumption (p7), closed under the action of the group \( G_B \) by Step 1, and such that the restriction of \( G_B \) to \( D_1(s) \) is a strongly continuous group there by assumption (p8). Moreover, the function \( v_{s,x_0} \) satisfies equation (NCP) by Step 4 and belongs to \( \mathcal{C}^1([s,T]; V) \cap \mathcal{C}([s,T]; D_1(s)) \) by Step 4 and Step 5. Therefore, \( v_{s,x_0} \) is a \( D_1(s) \)-valued solution to problem (NCP) and by Proposition 3.9 it follows that the function \( X^\omega_{s,x_0} : [s,T] \to V \) defined by \( X^\omega_{s,x_0} (t) := G_B(\omega_t - \omega_s) v_{s,x_0} (t), \ t \in [s,T], \) is a strong solution to problem (BLP).

**Remark 3.14.** Let us comment on the structure of the strong solution to (BLP) in Proposition 3.12. Set \( \Delta (T) := \{ (t, r) \in [0,T]^2 | t \geq r \} \). The family of operators \( \{ \tilde{C}_s(r), r \in [0,T] \} \) satisfies conditions (P1)-(P3), there exists a unique strongly continuous evolution system \( \tilde{U}_s \) corresponding to this family by [53, Theorem 5.6.1]. That is, there exists a function \( U_s : \Delta (T) \to \mathcal{L}(V) \) such that it satisfies the evolution property

\[
U_s(r,r) = I, \quad 0 \leq r \leq T,
\]

\[
U_s(t,r) = U_s(t,x)U_s(x,r), \quad 0 \leq r \leq x \leq t \leq T;
\]

is strongly continuous, i.e. for every \( v \in V \) the function \( U_s(\cdot, \cdot) v : \Delta (T) \to V \) is continuous; and such that it corresponds to the family \( \{ \tilde{C}_s(t), t \in [0,T] \} \), i.e. the following holds:

(i) There exists a constant \( c_s > 0 \) such that \( \| U_s(t,r) \| \mathcal{L}(V) \leq c_s \) is satisfied for every \( (t, r) \in \Delta (T) \).

(ii) For every \( 0 < r < t \leq T \), the range of \( U_s(t,r) \) is the space \( \text{Dom } C \).

(iii) For every \( 0 \leq r < t \leq T \), the derivative \( \partial_t U_s(t,r) \) is a bounded linear operator on the space \( V \), it is strongly continuous on \( 0 \leq r < t \leq T \) and for every \( v \in V \) satisfies

\[
[\partial_t U_s(\cdot, r)v](t) = \tilde{C}_s(t)U_s(t,r)v, \quad 0 \leq r < t \leq T.
\]

(iv) For every \( v \in \text{Dom } C \) and \( t \in (0,T] \), the function \( U_s(t,\cdot) v : [0,t] \to V \) is differentiable and satisfies

\[
[\partial_2 U_s(t, \cdot) v](r) = -U_s(t,r)\tilde{C}_s(r)v, \quad 0 \leq r \leq t.
\]

It is immediate (or by [53, Theorem 5.6.8]) that for every \( x_0 \in V \), the function \( \tilde{v}_{s,x_0} \) defined by \( \tilde{v}_{s,x_0}(t) := U_s(t,s)x_0 \) is the unique function that belongs to the space \( \mathcal{C}^1([s,T]; V) \cap \mathcal{C}([s,T]; V) \) and satisfies equation (NCP). This function also satisfies \( \tilde{v}_{s,x_0}(t) \in \text{Dom } C \) for every \( t \in (s,T] \) and if we assume that \( x_0 \in \text{Dom } C \) then we also have that \( v_{s,x_0}(s) \in \text{Dom } C \) and the argument in Step 4 of the proof of Proposition 3.12 was needed in order to find a function \( v_{s,x_0} \) that satisfies equation (NCP) and that belongs to the space \( \mathcal{C}^1([s,T]; V) \). Such a function exists under the stronger assumption that \( x_0 \in D_1+\varepsilon(s) \) for some \( \varepsilon > 0 \) and it is given by formula (28). However, by uniqueness of the solution to problem (NCP), we have that \( \tilde{v}_{s,x_0} \equiv v_{s,x_0} \). This means that the strong solution \( X^\omega_{s,x_0} \) to problem (BLP) is in fact given by

\[
X^\omega_{s,x_0} (t) = G_B(\omega_t - \omega_s)U_s(t,s)x_0, \quad t \in [s,T].
\]
Example 3.15. Consider the formal equation

\[(\partial_t u)(t, x) = (\partial^2_x u)(t, x) + g(x)(\partial_x u)(t, x)\omega_t\]

for \((t, x) \in [0, 1] \times \mathbb{R}\) with the initial condition \(u(0, x) = u_0(x)\) for \(x \in \mathbb{R}\) where \(g\) is a given function. Assume that \(g\) and its derivatives up to order three are all bounded and uniformly continuous real functions on \(\mathbb{R}\) and that \(0 < c_1 \leq g(x) \leq c_2 < \infty\) holds for every \(x \in \mathbb{R}\) and some constants \(c_1, c_2\). It is moreover assumed that \(\omega\) is a function that belongs to the space \(C^{\gamma_1}(\mathbb{R})\) for some \(\gamma_1 \in (0, 1)\) and that has finite 4-th variation along some sequence of partitions \(\pi \in \mathcal{P}[0, 1]\) whose mesh size tends to zero. It is also assumed that \(|\omega|^4\) is non-zero, belongs to \(C^{1, \gamma_2}(\mathbb{R})\) for some \(\gamma_2 \in (0, 1)\) and its first derivative is strictly positive.

The formal equation can be given rigorous meaning in our framework as follows. Set \(V := L^2(\mathbb{R})\), \(A := \partial^2\) on \(\text{Dom } A := W^{2,2}(\mathbb{R})\), \(B := g\partial\) on \(\text{Dom } B := W^{1,2}(\mathbb{R})\) (where \([(g\partial)f](x) = g(x)f'(x)\) for \(f \in W^{1,2}(\mathbb{R})\)). Clearly, the function \(\omega\) satisfies conditions (p1) and (p2). The operator \(g\partial\) generates a strongly continuous group \(G_{g\partial}\) of bounded linear operators on the space \(L^2(\mathbb{R})\) by Lemma A.1 and Lemma A.2 so that (p3) is satisfied. Moreover, we have that the operator \(B^4 = (g\partial)^4\) has the domain \(\text{Dom } B^4 = W^{4,2}(\mathbb{R})\) and is given by

\[(g\partial)^4f = q_1f' + q_2f''' + q_3f''' + q_4f^{(4)}\]

for \(f \in W^{4,2}(\mathbb{R})\) where

\[
q_1(x) := g(x)[g'(x)]^3 + [g(x)]^3g'''(x) + 4[g(x)]^2g'(x)g''(x),
q_2(x) := 7[g(x)]^3g'(x)^2 + 4[g(x)]^3g''(x),
q_3(x) := 6[g(x)]^3g'(x),
q_4(x) := [g(x)]^4,
\]

for \(x \in \mathbb{R}\). Consequently, \(\text{Dom } C\) defined by (14) is the space \(W^{4,2}(\mathbb{R})\). This space is dense in \(L^2(\mathbb{R})\) so that (p4) is satisfied. The operator \(C(t)\) defined by (13) is

\[C(t) = -\frac{1}{4t}[\omega^\pi_4(t)q_1\partial + \left(1 - \frac{1}{4t}[\omega^\pi_4(t)q_2\partial^2 - \frac{1}{4t}[\omega^\pi_4(t)q_3\partial^3 - \frac{1}{4t}[\omega^\pi_4(t)q_4\partial^4
\]

for \(t \in [0, 1]\), and the equality \(\text{Dom } C(t) = W^{4,2}(\mathbb{R}) = \text{Dom } C\) holds for every \(t \in [0, 1]\) since both \([\omega]^\pi_4\) and \(q_4\) are strictly positive. Clearly, the operator \(C(t)\) is also closed for every \(t \in [0, 1]\) and therefore, (p5) is satisfied. Now, because of boundedness and positivity of \([\omega]^\pi_4\) and \(q_4\), the operators \(-C(t), t \in [0, 1]\), are uniformly strongly elliptic in the sense of [4, p. 659], uniformly in \(t \in [0, 1]\). Therefore, by the fundamental a-priori estimate in [4, Theorem 12.2], assumption (p6) is satisfied with some \(\lambda_0 \in \mathbb{R}\). By the same theorem, it also follows that the graph norms of \((\lambda_0I - C(t)), t \in [0, 1]\), are all equivalent with the Sobolev norm \(\|\cdot\|_{W^{4,2}(\mathbb{R})}\) and that the equivalence constants can be chosen to be independent of \(t \in [0, 1]\). Since the same is true for \((g\partial)^4\) by the closed graph theorem (see, e.g., [6, Corollary 2.15]), assumption (p7) is also satisfied. Assumption (p8) is satisfied by the Sobolev tower property; see, e.g., [23, Proposition 2.5.2 (ii)]; and it remains to show that assumption (p9) is satisfied as well.

To this end, consider the operator \(T_g := g''\partial + 2g'\partial^2\) on \(\text{Dom } T_g := W^{2,2}(\mathbb{R})\) (i.e. \(T_gf)(x) = g''(x)f'(x) + 2g'(x)f''(x)\) for \(f \in W^{2,2}(\mathbb{R})\)). It is straightforward to check that for every \(t \in [0, 1]\), the equality

\[B(\lambda I - C(t))z = (\lambda I - C(t))Bz + T_gz\]
holds for every $\lambda \in \rho(C(t))$ and $z \in W^{5,2}(\mathbb{R})$. Consequently, for $t \in [0,1]$ and $\tilde{z} \in W^{1,2}(\mathbb{R})$, $(\lambda_0 I - C(t))^{-1} \tilde{z}$ belongs to the space $W^{5,2}(\mathbb{R})$ and the equality

$$B \tilde{z} = (\lambda_0 I - C(t))B(\lambda_0 I - C(t))^{-1} \tilde{z} + T_\varphi(\lambda_0 I - C(t))^{-1} \tilde{z}$$

holds. Consequently, assumption (p9) is satisfied with $Q$. Consequently, assumption (p9) is satisfied with $Q$.

(H2) There exists a family $(Q_t)$ of bounded linear operators on $\mathcal{L}(V)$ such that $Q_t \leq 1$ for every $t \in [0,1]$, which is bounded by [4, Theorem 12.2] again.

In what follows, we give some sufficient conditions for the solvability of problem (NCP) in what is sometimes called the hyperbolic case.

3.2.2. The hyperbolic case. In what follows, we give some sufficient conditions for the solvability of problem (NCP) in what is sometimes called the hyperbolic case. For the purposes of the following lines, assume that there exists a Hilbert space $Y$ that is continuously and densely embedded in the Hilbert space $V$. We say that a family $(K(t), t \in [0,T])$ of infinitesimal generators of strongly continuous semigroups is hyperbolic if the following three conditions are satisfied:

(H1) The family $(K(t), t \in [0,T])$ is $(M_K, \lambda_K)$-stable in $V$, i.e. there exist constants $M_K \geq 1$ and $\lambda_K \in \mathbb{R}$ such that $(\lambda_K, \infty) \subset \rho(K(t))$ holds for every $t \in [0,T]$ and the inequality

$$\left\| \prod_{i=1}^{k} (\lambda I - K(t_i))^{-1} \right\|_{\mathcal{L}(V)} \leq \frac{M_K}{(\lambda - \lambda_K)^k}$$

is satisfied for every $\lambda > \lambda_K$ and every finite sequence $0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq T$, $k \in \mathbb{N}$.

(H2) There exists a family $(Q(t), t \in [0,T])$ of isomorphisms of $Y$ onto $V$ such that $Q(t)$ is strongly continuously differentiable on $[0,T]$ to $\mathcal{L}(Y;V)$ and there exists a family $(k(t), t \in [0,T])$ of bounded linear operators such that $k(t)$ is strongly continuous on $[0,T]$ to $\mathcal{L}(V)$ and the equality

$$Q(t)K(t)Q(t)^{-1} = K(t) + k(t)$$

holds for every $t \in [0,T]$.

(H3) For every $t \in [0,T]$, it holds that $Y \subseteq Dom K(t)$. Moreover, $K(t)$ is strongly continuous on $[0,T]$ to $\mathcal{L}(Y;V)$.

Proposition 3.16. Assume that $\omega$ satisfies the condition

(h1) The $p$-th variation $[\omega]_p^2$ is non-zero and belongs to $C^1([0,T])$.

Assume that the operator $B$ satisfies the conditions

(h2) The operator $B$ is the infinitesimal generator of a strongly continuous group of bounded linear operators $G_B$ acting on the space $V$.

(h3) For every $x \in \mathbb{R}$, it holds that $\|G_B(x)\|_{\mathcal{L}(V)} \leq 1$.

Assume further that there exists a Hilbert space $\tilde{V}$ that is dense in $V$, continuously embedded in $Dom C$, closed under the action of the group $G_B$; and such that the following conditions are satisfied:

(h4) The restriction of the group $G_B$ to $\tilde{V}$ forms a strongly continuous group on $\tilde{V}$.

(h5) For every $x \in \mathbb{R}$, it holds that $\|G_B(x)\|_{\mathcal{L}(\tilde{V})} \leq 1$. 


Moreover, it is assumed that the family \((C(t), t \in [0, T])\) defined by (13) satisfies the following conditions:

\((h6)\) For every \(t \in [0, T]\), the linear operator \(C(t)\) is closed and it holds that \(\text{Dom } C(t) = \text{Dom } C\).

\((h7)\) There exists a constant \(\lambda_1 \in \mathbb{R}\) such that the resolvent set \(\rho(C(t))\) contains the ray \((\lambda_1, \infty)\) for every \(t \in [0, T]\) and such that the inequality

\[\|\lambda I - C(t)\|_{\mathcal{L}(V)} \leq \frac{1}{\lambda - \lambda_1}\]

is satisfied for every \(\lambda > \lambda_1\) and every \(t \in [0, T]\).

\((h8)\) There exists an isomorphism \(Q \in \mathcal{L}(\tilde{V}; V)\) such that

\((a)\) for every \(t \in [0, T]\), it holds that \(C(t)Q^{-1}(\text{Dom } C) \subseteq \tilde{V};\)

\((b)\) there exists \(\lambda > \lambda_1\) such that \(Q(\lambda - C(t))^{-1}(\tilde{V}) \subseteq \text{Dom } C\) for every \(t \in [0, T];\)

\((c)\) for every \(x \in \mathbb{R}\) and every \(y \in \tilde{V}\), it holds that

\[QG_B(x)y = G_B(x)Qy;\]

and there exists a strongly continuous function \(L : [0, T] \to \mathcal{L}(V)\) such that the equality

\[QC(t)Q^{-1}v = C(t)v + L(t)v\]

is satisfied for every \(t \in [0, T]\) and every \(v \in \text{Dom } C\).

If \(x_0 \in \tilde{V}\), then there exists a strong solution to the bilinear problem \((BLP)\).

**Proof.** Define the family of linear operators

\[\tilde{C}_s(t) := z_s^{-1}(t)C(t)z_s(t), \quad t \in [0, T],\]

on the domain \(\text{Dom } \tilde{C}_s(t) := z_s^{-1}(t)\text{Dom } C\).

**Step 1:** It is shown first that the family \((\tilde{C}_s(t), t \in [0, T])\) is stable in \(V\), i.e. that this family satisfies condition \((H1)\). Indeed, it follows that for every \(t \in [0, T]\), the space \(\text{Dom } \tilde{C}_s(t)\) is dense in \(V\) since \(\tilde{V}\) is dense in \(V\) and \(\tilde{V} \subseteq \text{Dom } \tilde{C}_s(t)\). Moreover, the operator \(\tilde{C}_s(t)\) is closed because \(C(t)\) is closed. Furthermore, as in the proof of Proposition 3.12, there is the inclusion \(\rho(C(t)) \subseteq \rho(\tilde{C}_s(t))\) and the equality

\[\begin{align*}
(\lambda I - \tilde{C}_s(t))^{-1} &= z_s^{-1}(t)(\lambda I - C(t))^{-1}z_s(t) \\
&= \frac{1}{\lambda - \lambda_1}
\end{align*}\]  

(30)

is satisfied for every \(\lambda \in \rho(C(t))\). Hence, it follows by assumptions \((h3)\) and \((h7)\) that \((\lambda_1, \infty) \subseteq \rho(\tilde{C}_s(t))\) and, moreover, there is the estimate

\[\|\lambda I - \tilde{C}_s(t)\|_{\mathcal{L}(V)} \leq \|z_s(t)\|_{\mathcal{L}(V)}\|\lambda I - C(t)\|_{\mathcal{L}(V)}^{-1}\|z_s(t)\|_{\mathcal{L}(V)} \leq \frac{1}{\lambda - \lambda_1}\]

for every \(\lambda > \lambda_1\). Therefore, by the Hille-Yosida theorem [53, Corollary 1.3.8], it follows that the operator \(\tilde{C}_s(t)\) generates a strongly continuous semigroup \(S_{\tilde{C}_s(t)}(u)\) of bounded linear operators acting on \(V\) such that \(\|S_{\tilde{C}_s(t)}(u)\|_{\mathcal{L}(V)} \leq e^{\lambda_1 u}\) holds for every \(u \geq 0\). Consequently, the family \((\tilde{C}_s(t), t \in [0, T])\) of generators is \((1, \lambda_1)\)-stable in \(V\) by the remark preceding [53, Theorem 5.2.2].

**Step 2:** By using assumption \((h8)\), it can be shown that \(\text{Dom } \tilde{C}_s(t) = \text{Dom } \tilde{C}_s^Q(t)\) where

\[
\text{Dom } \tilde{C}_s^Q(t) := \{v \in V \mid Q^{-1}u \in \text{Dom } \tilde{C}_s(t), \tilde{C}_s(t)Q^{-1}v \in \tilde{V}\}
\]
is the natural domain of the operator $Q\hat{C}_s(t)Q^{-1}$; see, e.g., the proof of [63, Proposition 4.2.2]. Now, we prove that there exists a strongly continuous function $\hat{L} : [0, T] \to \mathcal{L}(\tilde{V})$ such that

$$Q\hat{C}_s(t)Q^{-1}v = \hat{C}_s(t)v + \hat{L}(t)v$$

holds for every $t \in [0, T]$ and every $v \in \text{Dom } \hat{C}_s(t)$, i.e. that the family $(\hat{C}_s(t), t \in [0, T])$ satisfies condition (H2). To this end, let $t \in [0, T]$ and $v \in \text{Dom } \hat{C}_s(t)$ be arbitrary. Then, there exists a $\tilde{v} \in \text{Dom } C$ such that $v = G_B^{-1}(\omega_t - \omega_s)\tilde{v}$ and we obtain

$$Q\hat{C}_s(t)Q^{-1}v = Q\hat{C}_s(t)Q^{-1}G_B^{-1}(\omega_t - \omega_s)\tilde{v}$$

$$= Q\hat{C}_s(t)G_B^{-1}(\omega_t - \omega_s)Q^{-1}\tilde{v}$$

$$= QG_B^{-1}(\omega_t - \omega_s)C(t)Q^{-1}\tilde{v}$$

$$= G_B^{-1}(\omega_t - \omega_s)QC(t)Q^{-1}\tilde{v}$$

$$= G_B^{-1}(\omega_t - \omega_s)[C(t) + L(t)]\tilde{v}$$

$$= G_B^{-1}(\omega_t - \omega_s)C(t)G_B(\omega_t - \omega_s)v + G_B^{-1}(\omega_t - \omega_s)L(t)G_B(\omega_t - \omega_s)v$$

$$= \hat{C}_s(t)v + G_B^{-1}(\omega_t - \omega_s)L(t)G_B(\omega_t - \omega_s)v$$

by using assumption (h8).

**Step 3:** Finally, it is shown that $\hat{C}_s(\cdot)$ is strongly continuous on $[0, T]$ to $\mathcal{L}(\tilde{V}; V)$. To this end, let $z \in \tilde{V}$ and $u, v \in [0, T]$. By using commutativity of $G_B$ and $B^p$ on $\tilde{V} \subseteq \text{Dom } B^p$, the following inequality is obtained:

$$||(\hat{C}_s(u) - \hat{C}_s(v))z||_V \leq \frac{1}{p^\pi} ||[\omega]^\pi_p(u) - [\omega]^\pi_p(v)||_{BPz} ||V$$

$$+ ||G_B(\omega_u - \omega_s)^{-1}AG_B(\omega_v - \omega_s)z - G_B(\omega_v - \omega_s)^{-1}AG_B(\omega_v - \omega_s)z||_V.$$ 

The second term can be estimated as follows:

$$||G_B(\omega_u - \omega_s)^{-1}AG_B(\omega_v - \omega_s)z - G_B(\omega_v - \omega_s)^{-1}AG_B(\omega_v - \omega_s)z||_V$$

$$\leq ||G_B(\omega_u - \omega_v)||_{XP(V)} ||G_B(\omega_u - \omega_v)^{-1}AG_B(\omega_u - \omega_v)z - AG_B(\omega_v - \omega_s)z||_V$$

$$\leq ||G_B(\omega_u - \omega_v)^{-1}||_{XP(V)} ||AG_B(\omega_u - \omega_v)z - AG_B(\omega_v - \omega_s)z||_V$$

$$+ ||G_B(\omega_u - \omega_v)^{-1} - I||AG_B(\omega_v - \omega_s)z||_V$$

$$\leq c||G_B(\omega_u - \omega_v)z - G_B(\omega_v - \omega_s)z||_V$$

$$+ ||G_B(\omega_u - \omega_v)^{-1} - I||AG_B(\omega_v - \omega_s)z||_V$$

where the continuous embedding of $\tilde{V}$ into $\text{Dom } C$ is used in the last inequality. As $u \to v$, the last two terms converge to zero, the first term by strong continuity of the restriction of the group $G_B$ on $\tilde{V}$ (assumption (h4)) and the second term by strong continuity of the group $G_B$ on $V$ (assumption (h2)). Since by assumption (h1), the $p$-th variation $[\omega]^\pi_p$ has continuous derivative, it follows that $||(\hat{C}_s(u) - \hat{C}_s(v))z||_V$ converges to zero as $u \to v$ which proves the claim.

**Step 4:** Since the system $(\hat{C}_s(t), t \in [0, T])$ satisfies conditions (H1)-(H3), it follows by the theorem in [44] that for every $x_0 \in \tilde{V}$ there exists a unique function $v_{s,x_0}$ that belongs to the space $\mathcal{C}([s, T]; \tilde{V}) \cap \mathcal{C}^1([s, T]; V)$ and that satisfies equation (NCP). Since assumptions (h1), (h2), (h6), the assumptions made on $\tilde{V}$, and assumption (h7) allow to use Proposition 3.9, we conclude that the bilinear problem (BLP) admits a strong solution. □
Remark 3.17. Let us comment on the structure of the strong solution to (BLP) in Proposition 3.16. By the theorem in [44], there exists a unique strongly continuous evolution system \( U_s \) on \( \Delta(T) \) to \( \mathcal{L}(V) \) with the following properties:

(i) For every \((t, r) \in \Delta(T)\), it holds that \( \|U_s(t, r)\|_{\mathcal{L}(V)} \leq e^{\lambda_1(t-r)} \).

(ii) For every \((t, r) \in \Delta(T)\), \( U_s(t, r)\hat{V} \subseteq \hat{V} \) and \( U_s \) is strongly continuous on \( \Delta(T) \) to \( \mathcal{L}(\hat{V}) \).

(iii) For every \( v \in \hat{V} \) and \( r \in [0, T) \), the function \( U_s(\cdot, r)v : [r, T] \to V \) is continuously differentiable and it holds that

\[
[\partial_t U_s(\cdot, r)v](t) = \tilde{C}_s(t)U_s(t, r)v, \quad r \leq t \leq T.
\]

(iv) For every \( v \in \hat{V} \) and \( t \in [0, T) \), the function \( U_s(t, \cdot)v : [0, t] \to V \) is continuously differentiable and it holds that

\[
[\partial_x U_s(t, \cdot)v](r) = -U_s(t, r)\tilde{C}_s(r)v, \quad 0 \leq r \leq t.
\]

Consequently, for \( x_0 \in \hat{V} \), the unique solution to problem (NCP) is given by \( v_{s,x_0}(t) = U_s(t, s)x_0 \). Therefore, the strong solution \( X_{s,x_0}^\omega \) to the bilinear problem (BLP) constructed in Proposition 3.16 is in fact given by

\[
X_{s,x_0}^\omega(t) = G_B(\omega_t - \omega_s)U_s(t, s)x_0, \quad t \in [s, T].
\]

Example 3.18. Consider the formal equation

\[
(\partial_t u)(t, x) = \frac{1}{8}(\partial^4_x u)(t, x)+i\left[(\partial^2_x u)(t, x) + (\partial^2_x g)(t, x) - g(x)u(t, x)\right] + (\partial_x u)(t, x)\dot{W}_t^{1/4}
\]

for \((t, x) \in [0, 1] \times \mathbb{R}\) with the initial condition \( u(0, x) = u_0(x) \) for \( x \in \mathbb{R} \) where \( g \in \mathcal{C}_b^0(\mathbb{R}) \) is strictly positive and where \( i \) is the imaginary unit. This formal equation can be given rigorous meaning in our framework as follows. Set \( V := L^2(\mathbb{R}; \mathbb{C}), \)

\( A := \frac{1}{8}\partial^4 + i(\partial^4 + \partial^2 - gI) \) on \( \text{Dom} \ A = W^{4,2}(\mathbb{R}; \mathbb{C}), \)

\( B := \partial \) on \( \text{Dom} \ B = W^{1,2}(\mathbb{R}; \mathbb{C}), \)

\( \omega := W^{1/2} \) is the fractional Brownian motion with the Hurst parameter \( H = 1/4 \).

As before, if we take the sequence of partitions \( \pi \) to be either the sequence of dyadic partitions \( \tilde{D} \) or the sequence \( \tilde{E} \), we have by Example 3.4 that \( |W^{1/2}_\pi|^2(t) = 3t \) which is continuously differentiable so that assumption \((h1)\) is satisfied. Moreover, as in Example 3.8, we have that \( B = \partial \) generates a strongly continuous group \( G_\partial \) of left-shift operators on \( L^2(\mathbb{R}; \mathbb{C}) \) so that assumption \((h2)\) is also satisfied and assumption \((h3)\) is easily verified.

For the family of operators \((C(t), t \in [0, 1])\), we have for every \( t \in [0, 1] \) that \( C(t) = C \) where the operator \( C \) is defined on \( \text{Dom} \ C = W^{4,2}(\mathbb{R}; \mathbb{C}) \) by

\[
[Cf](x) := i\left[f^{(4)}(x) + f''(x) - g(x)f(x)\right], \quad f \in W^{4,2}(\mathbb{R}; \mathbb{C}).
\]

Set \( \hat{V} := W^{6,2}(\mathbb{R}; \mathbb{C}) \). Clearly, we have that this space is dense in \( L^2(\mathbb{R}; \mathbb{C}) \), continuously embedded in \( \text{Dom} \ C \), and closed under the action of the group \( G_\partial \).

Moreover, by the Sobolev tower property from [23, Proposition 5.2 (ii)], assumption \((h4)\) is satisfied and assumption \((h5)\) can also be easily verified. Assumption \((h6)\) is clearly satisfied as well.

Moreover, since the operator \( C_0 \) defined by \( C_0 := \partial^4 + \partial^2 - gI \) is self-adjoint on \( W^{4,2}(\mathbb{R}) \), it follows that \( C = iC_0 \) is skew-adjoint on \( W^{4,2}(\mathbb{R}; \mathbb{C}) \) and therefore, it generates a unitary, strongly continuous group of bounded linear operators on \( L^2(\mathbb{R}; \mathbb{C}) \) by Stone’s theorem; see, e.g., [53, Theorem 1.10.8]. Thus, assumption \((h7)\) is satisfied with some \( \lambda_1 \in \mathbb{R} \) by the Hille-Yosida theorem; see, e.g., [53, Corollary 1.3.8].
Finally, it remains to verify assumption (h8). To this end, set \( Q := (I - \partial^2)^3 \) on \( W^{6,2}(\mathbb{R}; \mathbb{C}) \). This operator provides an isomorphism between \( W^{6,2}(\mathbb{R}; \mathbb{C}) \) and \( L^2(\mathbb{R}; \mathbb{C}) \); see, e.g. [2, section 1.1.4] and [64, Theorem 2.5.6]. Furthermore, it holds that \( C(I - \partial^2)^{-3}f \in \tilde{V} \) for \( f \in \text{Dom } C; (I - \partial^2)^3(\lambda I - C)^{-1}f \in \text{Dom } C \) for \( f \in \tilde{V} \) and \( \lambda > \lambda_1 \); and, moreover,

\[
(I - \partial^2)^3G_\theta(x)f = \tilde{G}_\theta(x)(I - \partial^2)^3f
\]

for \( f \in W^{6,2}(\mathbb{R}; \mathbb{C}) \) and \( x \in \mathbb{R} \). Finally, consider the operator \( T_\theta \) defined by

\[
T_\theta f := i \left\{ \frac{1}{8} [\partial^2(gf) - g\partial^2 f] - 3 [\partial^4(gf) - g\partial^4 f] + \frac{1}{4} [\partial^6(gf) - g\partial^6 f] \right\}
\]

for \( f \in \text{Dom } T_\theta \) : \( W^{6,2}(\mathbb{R}; \mathbb{C}) \). It is straightforward to verify that the equality

\[
(I - \partial^2)^3Cf = C(I - \partial^2)^3f + T_\theta f
\]

holds for every \( f \in W^{10,2}(\mathbb{R}; \mathbb{C}) \). Consequently, we have for \( f \in W^{4,2}(\mathbb{R}; \mathbb{C}) \) that \((I - \partial^2)^{-3}f \) belongs to \( W^{10,2}(\mathbb{R}; \mathbb{C}) \) and the equality

\[
(I - \partial^2)^3C(I - \partial^2)^{-3}f = Cf + T_\theta (I - \partial^2)^{-3}f
\]

is satisfied. Thus, the conditions of Proposition 3.16 are verified and therefore, if \( u_0 \in W^{6,2}(\mathbb{R}; \mathbb{C}) \), then there is a strong solution to the problem

\[
\begin{align*}
\dot{X}_t &= \frac{1}{8} \partial^4 + i (\partial^4 + \partial^2 - gI) \dot{X}_t \, dt + \partial X_t \, dW^4_t, \quad 0 < t \leq 1, \\
X_0 &= u_0.
\end{align*}
\]

Appendix A. Strongly continuous group generated by \( g\partial \). Assume that \( g \in C(\mathbb{R}) \) is a function for which there exists two constants \( c_1, c_2 \) such that

\[
0 < c_1 \leq g(x) \leq c_2 < \infty
\]

holds for every \( x \in \mathbb{R} \). Here, it is shown that the operator \( g\partial : W^{1,2}(\mathbb{R}) \to L^2(\mathbb{R}) \) defined by \([g\partial f](x) := g(x)f^1(x)\) for \( x \in \mathbb{R} \) generates a strongly continuous group \((G_{g\partial}(t), t \in \mathbb{R})\) of bounded linear operators acting on the space \( L^2(\mathbb{R}) \) given by

\[
[G_{g\partial}f](x) = f \left( h^{-1}(h(x) + t) \right), \quad x \in \mathbb{R}, f \in L^2(\mathbb{R}),
\]

where \( h : \mathbb{R} \to \mathbb{R} \) is the function defined by

\[
h(x) := \int_c^x \frac{1}{g(r)} \, dr, \quad x \in \mathbb{R},
\]

for arbitrary fixed \( c \in \mathbb{R} \). Although this fact seems to be well-known, cf., e.g., [13, page 113], we were unable to find the proof in our (operator semigroup) setting so we include it here for the sake of completeness. It should be noted that this group of operators is closely connected to the translation equation (see, e.g., [1, Section 1, Chapter 6]), one-parameter Lie groups (see, e.g., [38, Section 6 of Chapter 8, p. 293 - 299]), and the theory of generalized shift operators (see, e.g., [46] and the many references therein). The proof is split into two lemmas.

Lemma A.1. The family of operators \((G_{g\partial}(t), t \in \mathbb{R})\) is a strongly continuous group of bounded linear operators acting on the space \( L^2(\mathbb{R}) \).

Proof. We begin with some observations. Fix \( c \in \mathbb{R} \). Since \( g \) is continuous and bounded from below by the positive constant \( c_1 \), the function \( h \) is well-defined and continuously differentiable with positive derivative, it is also increasing, and therefore injective. By the mean value theorem and the intermediate value theorem, \( h \) is also surjective and, consequently, invertible on the whole real line and we denote its inverse by \( h^{-1} \). Moreover, since \( h \) is strictly monotone, the inverse \( h^{-1} \)
is continuous. Now, for every $t \in \mathbb{R}$, the operator $G_{g\vartheta}(t)$ is linear and bounded on $L^2(\mathbb{R})$ by the following estimate:

$$\|G_{g\vartheta}(t)f\|_{L^2(\mathbb{R})} := \int_{\mathbb{R}} |f(h^{-1}(h(x) + t))|^2 \, dx$$
$$= \int_{\mathbb{R}} |f(x)|^2 \frac{g(h^{-1}(h(x) - t))}{g(x)} \, dx \leq \frac{c_2}{c_1} \|f\|_{L^2(\mathbb{R})},$$  \tag{31}

It is straightforward to verify that the system $(G_{g\vartheta}(t), t \in \mathbb{R})$ is a group. Moreover, by estimate (31), we have that the function $G_{g\vartheta} : \mathbb{R} \rightarrow \mathcal{L}(L^2(\mathbb{R}))$ is uniformly bounded and hence, in order to show its strong continuity, it is enough to show that $G_{g\vartheta}(t)f$ tends to $f$ as $t \downarrow 0$ in the topology of $L^2(\mathbb{R})$ for $f$ from a dense subset of $L^2(\mathbb{R})$, cf. [23, Exercise I.5.9 (5)]. Let $f \in \mathcal{C}_c(\mathbb{R})$. We have that

$$\lim_{t \downarrow 0} \|G_{g\vartheta}(t)f - f\|_{\infty} = \limsup_{t \downarrow 0} \sup_{x \in \mathbb{R}} |f(h^{-1}(h(x) + t)) - f(x)| = 0$$

by (uniform) continuity of $f$ and continuity of $h^{-1}$ and $h$ on $\mathbb{R}$ which implies that $G_{g\vartheta}(t)f$ tends to $f$ as $t \downarrow 0$ in the topology of $L^2(\mathbb{R})$.

Lemma A.2. The infinitesimal generator of the group $(G_{g\vartheta}(t), t \in \mathbb{R})$ is the operator $g\vartheta$.

Proof. Let $u \in L^2(\mathbb{R})$ and assume that there exists $z \in L^2(\mathbb{R})$ such that

$$\lim_{s \downarrow 0} \left\| G_{g\vartheta}(s)u - u \right\|_{L^2(\mathbb{R})} = 0.$$  

Let $a \in \mathbb{R}$ be arbitrary and define a function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(t) := \int_0^a [G_{g\vartheta}(r)u](t) \, dr.$$  

Assume, for simplicity, that $a > 0$. The case $a < 0$ can be done in a very similar manner. By using the group property of $G_{g\vartheta}$, we have for $0 < s < a$ that

$$\frac{1}{s} (G_{g\vartheta}(s)F) = \frac{1}{s} \int_0^a G_{g\vartheta}(s + r)u \, dr - \frac{1}{s} \int_0^a G_{g\vartheta}(r)u \, dr.$$  \tag{32}

Consequently, we obtain that

$$\frac{1}{s} (G_{g\vartheta}(s)F) = \int_0^a G_{g\vartheta}(r) \left[ \frac{G_{g\vartheta}(s)u - u}{s} \right] \, dr \quad \text{as} \quad \int_0^a G_{g\vartheta}(r)z \, dr$$

where the convergence follows by the dominated convergence theorem and estimate (31). For this limiting function, there is the expression

$$\int_0^a [G_{g\vartheta}(r)z](x) \, dr = \int_0^a z \left( h^{-1}(h(x) + r) \right) \, dr = \int_x^{h^{-1}(h(x) + a)} \frac{z(r)}{g(r)} \, dr, \quad x \in \mathbb{R}.$$  

On the other hand, we also have from equation (32) that

$$\frac{1}{s} (G_{g\vartheta}(s)F) = \frac{1}{s} \int_s^{s+a} G_{g\vartheta}(r)u \, dr - \frac{1}{s} \int_0^a G_{g\vartheta}(r)u \, dr$$
$$= \frac{1}{s} \int_0^{a+s} G_{g\vartheta}(r)u \, dr - \frac{1}{s} \int_0^a G_{g\vartheta}(r)u \, dr$$

By strong continuity of the group $G_{g\vartheta}$ and Lebesgue’s differentiation theorem, it follows that the right-hand side of the above equality tends to

$$u \left( h^{-1}(h(\cdot) + a) \right) - u(\cdot)$$
in $L^2(\mathbb{R})$ as $s \downarrow 0$. Consequently, it follows that the inequality
\begin{equation}
 u \left( h^{-1}(h(x) + a) \right) - u(x) = \int_x^{h^{-1}(h(x) + a)} \frac{z(r)}{g(r)} \, dr \tag{33}
\end{equation}
is satisfied with almost every $x \in \mathbb{R}$. Finally, we have that for every $x, y \in \mathbb{R}$ there exists $a \in \mathbb{R}$ such that $y = h^{-1}(h(x) + a)$ and therefore, possibly after modifying $u$ on a null set, we have that the equation
\begin{equation}
 u(y) - u(x) = \int_x^y \frac{z(r)}{g(r)} \, dr
\end{equation}
is satisfied for almost every $x, y \in \mathbb{R}$ by equality (33) which shows that $u \in W^{1,2}(\mathbb{R})$ and it holds that $z = gu' = (g\partial f)$.

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