Algebraic Framework for Quantization of Nonultralocal Models

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Abstract

Extension of the braid relations to the multiple braided tensor product of algebras that can be used for quantization of nonultralocal models is presented. The Yang–Baxter–type consistency conditions as well as conditions for the existence of the multiple coproduct (monodromy matrix), which can be used for construction of the commuting subalgebra, are given. Homogeneous and local algebras are introduced, and simplification of the Yang–Baxter–type conditions for them is shown. The Yang–Baxter–type equations and multiple coproduct conditions for homogeneous and local of order 2 algebras are solved.

1 Introduction

There is a well known procedure for solving a class of nonlinear differential equations – inverse scattering method [1]. For quantization of integrable field models like sine–Gordon (SG), nonlinear Schroedinger (NLS), and others, its quantum counterpart was developed [2]–[6]. The fundamental object of the quantum inverse scattering method is the quantum transition or monodromy matrix – the generating function of conserved quantities.

The starting point for derivation of the commutation relations of the elements of the monodromy matrix usually is the fact that the Poisson brackets for the Lax matrix $L$ of classical models quantized by the quantum inverse scattering method can be written by virtue of the classical $r$–matrix as

$$\{L(x,\lambda) \otimes L(y,\mu)\} = [r(\lambda,\mu), L(x,\lambda) \otimes \mathbf{1} + \mathbf{1} \otimes L(y,\mu)]\delta(x-y)$$

(1)
where \(\{,\} \) means the Poisson brackets of elements of \(M \times M\) matrices, \(r\) is a matrix \(M^2 \times M^2\) and \([,]\) means commutator of matrices. If one denotes the \(M^2 \times M^2\) matrices \(L \otimes 1\) and \(1 \otimes L\) as \(L_1\) and \(L_2\) then an alternative notation (and more appropriate in the following) for the set of the Poisson brackets (1) is

\[
\{L_1(x, \lambda), L_2(y, \mu)\} = [r_{12}(\lambda, \mu), L_1(x, \lambda) + L_2(y, \mu)]\delta(x - y)
\] (2)

The classical transition matrix \(\tau(x, y; \lambda)\) is defined by

\[
\partial_x \tau(x, y; \lambda) = L(x, \lambda)\tau(x, y; \lambda), \quad \tau(y, y; \lambda) = 1
\] (3)

and its Poisson brackets read

\[
\{\tau(x, y; \lambda) \otimes \tau(x, y; \mu)\} = [r_{12}(\lambda, \mu), \tau_1(x, y; \lambda) \tau_2(x, y; \mu)]
\] (4)

or

\[
\{\tau_1(x, y; \lambda), \tau_2(x, y; \mu)\} = [r_{12}(\lambda, \mu), \tau_1(x, y; \lambda)\tau_2(x, y; \mu)]
\] (5)

Quantization of relations (2) is given by the famous relations

\[
R_{12}(\lambda, \mu)L^n_1(\lambda)L^n_2(\mu) = L^n_2(\mu)L^n_1(\lambda)R_{12}(\lambda, \mu)
\] (6)

where the matrix \(R\) satisfies Yang–Baxter equation and \(L^n\) are quantum analogs of lattice regularized Lax operators. The \textit{quantum monodromy matrix} is defined in the quantum inverse scattering method as

\[
T^N(\lambda) := L^N(\lambda)L^{N-1}(\lambda)\ldots L^1(\lambda).
\] (7)

and one finds that the commutation relations of its elements are the same as those for the elements of \(L^n\).

\[
R_{12}(\lambda, \mu)T^N_1(\lambda)T^N_2(\mu) = T^N_2(\mu)T^N_1(\lambda)R_{12}(\lambda, \mu).
\] (8)

In derivation of relations for both classical and quantum monodromy matrices one assumes that the models are \textit{ultralocal}. For classical theories it means that the Poisson brackets of the Lax operators \(L(x, \lambda)\) are proportional to the \(\delta\)-function but not to its derivatives. In the quantum theories it means that the quantized operators \(L^n\) sitting in different sites \(n, m\) of the lattice commute

\[
L^n_1 L^m_2 = L^m_2 L^n_1 \text{ for } m \neq n.
\] (9)

Many integrable models like e.g. NLS and SG in the space–time coordinates are ultralocal but there are models like KdV, non–linear sigma models or SG in the light–cone coordinates where the Poisson brackets of the \(L\)-operators are proportional to the derivatives of the \(\delta\)-function. Thorough investigation of the classical nonultralocal models can be found in [8].

The first attempt to deal with quantization of nonultralocal models in the framework of the quantum inverse scattering method was done in the paper of Tsyplyaev [9] where a regularization to lattice ultralocal models was chosen and they were then quantized.
Quantization of relations

\[ \{T_1(\lambda), T_2(\mu)\} = [r_{12}(\lambda, \mu), T_1(\lambda)T_2(\mu)] + T_1(\lambda)s_{12}(\lambda, \mu)T_2(\mu) - T_2(\mu)s_{12}(\lambda, \mu)T_1(\lambda) \]

(10)

for classical monodromy matrix of periodic nonultralocal models that are the consequence of the nonultralocal relations

\[ \{L(x, \lambda) \otimes L(y, \mu)\} = [r(\lambda, \mu), L(x, \lambda, \mu) \otimes 1 + 1 \otimes L(y, \mu)]\delta(x - y) + \]

\[ [s(\lambda, \mu), L(x, \lambda, \mu) \otimes 1 - 1 \otimes L(y, \mu)]\delta(x - y) - 2s(\lambda, \mu)\delta'(x - y). \]

was suggested in [11]. The quantum version of the relations

\[ \{T_1(\lambda) \otimes T_2(\mu)\} = a_{12}T_1(\lambda)T_2(\mu) + T_1(\lambda)b_{12}T_2(\mu) - T_2(\mu)c_{12}T_1(\lambda) - T_1(\lambda)T_2(\mu)\delta_{12} \]

(11)

that include relations (10) is

\[ A_{12}(\lambda, \mu)T_1(\lambda)B_{12}(\lambda, \mu)T_2(\mu) = T_2(\mu)C_{12}(\lambda, \mu)T_1(\lambda)D_{12}(\lambda, \mu) \]

(12)

Another step was made in [11] where a classical nonultralocal model was discretized in such way that the nonultralocality was preserved and then it was quantized. The quantum relations for the discretized Lax operator remain nonultralocal in the sense that \( L^n \) sitting in the neighbouring sites do not commute.

\[ L_1^n L_2^{n+1} = L_2^{n+1} A_{12} L_1^n \]

(13)

The spectrally dependent version of these relations appeared in the papers [12, 13] on integrable mappings for lattice KdV and related models. Relations for the quantized Lax operators read

\[ R_{12}^+(\lambda, \mu)L_1^n(\lambda)L_2^\pm(\mu) = L_2^\pm(\mu)R_1^n(\lambda)R_{12}^-(\lambda, \mu) \]

(14)

\[ L_1^n(\lambda)L_2^{n+1}(\mu) = L_2^{n+1}(\mu)S_{12}(\lambda, \mu)L_1^n(\lambda) \]

(15)

The matrices \( R^{\pm}, S \) satisfy conditions

\[ R_{12}^+(\lambda, \mu)R_{13}^+(\lambda, \nu)R_{23}^+(\mu, \nu) = R_{23}^+(\mu, \nu)R_{13}^+(\lambda, \nu)R_{12}^+(\lambda, \mu) \]

(16)

\[ R_{12}^+(\lambda, \mu)S_{13}(\lambda, \nu)S_{23}(\mu, \nu) = S_{23}(\mu, \nu)S_{13}(\lambda, \nu)R_{12}^+(\lambda, \mu) \]

(17)

and

\[ R_{12}^+(\lambda, \mu)S_{12}(\lambda, \mu) = S_{21}(\mu, \lambda)R_{12}^+(\lambda, \mu). \]

(18)

They guarantee that compact relations for the quantum monodromy matrix (8) can be obtained.

The reason why in the ultralocal case the quantum monodromy matrix (8) satisfies the same relations as its components \( L^n \) is that the algebra defined by the relations (8) for fixed \( n \) admits the matrix coproduct or, in other words, it can be extended into the bialgebra. The quantum monodromy matrix then actually is a representation of the multiple matrix coproduct of \( L \). Observation made in [14] is that similar situation occurs in the examples of quantized nonultralocal models but instead of normal matrix...
bialgebras (related to quantum groups) we must use their braided analogs (related to quantized braided groups [15]) because the commutation relations for neighbouring Lax operators $L^n$ can be interpreted as braiding relations between copies of an algebra that together with the matrix coproduct form the braided bialgebra. The quantum monodromy matrix can again be defined as a representation of the multiple matrix coproduct of generators. For this reason we believe that the multiple braided products of algebras investigated in this paper can be used for quantization of nonultralocal models.

2 Braided matrix bialgebras

We shall start with the simple braided product of algebras. Let $A$ is the complex associative algebra generated by unit element $1_A$ and $L^k_j$, $j, k \in \{1, 2, \ldots, M\}$ satisfying quadratic relations

$$A_{12} L_1 B_{12} L_2 = L_2 C_{12} L_1 D_{12}$$

where $L = \{L^k_j\}_{j,k=1}^M$ and $A, B, C, D$ are numerical matrices $M^2 \times M^2$ (i.e. $A_{12} = \{(A)_{i_1,j_2,j_1,j_2} \in \mathbb{C}\}_{i_1, j_2, j_1, j_2 = 1}^M$ and similarly for $B, C, D$). These algebras were introduced in [10] and include (algebras of functions on) quantum groups, quantum supergroups, braided groups, quantized braided groups, reflection algebras and others.

The consistency conditions for the algebra (19) are [10]

$$[A, A, A] = 0, \quad [D, D, D] = 0,$$

$$[A, C, C] = 0, \quad [D, B, B] = 0,$$

$$[A, B^+, B^+] = 0, \quad [D, C^+, C^+] = 0,$$

$$[A, C, B^+] = 0, \quad [D, B, C^+] = 0,$$

where we have introduced (constant) Yang–Baxter commutator

$$[R, S, T] := R_{12} S_{13} T_{23} - T_{23} S_{13} R_{12}$$

and the superscript $X^+$ means $PXP$ where $P$ is the permutation matrix $P_{ij} = \delta_i^j \delta_j^k$.

The first problem we shall solve is the existence of the coproduct, i.e. a coassociative map $\Delta : A \to A \otimes A$ that is homomorphism of the algebras. The algebras for which such a map can be found are called bialgebras.

There are two well known classes of algebras of the type (19) where matrix coproduct

$$\Delta(L^k_i) = L^k_i \otimes L^k_i$$

exists. They are the quantum groups where $A = D = R$, $B = C = 1$ and braided matrix groups [11] where $A = C = R^+$, $B = D = R$. In both cases the consistency conditions (20) reduce to the constant Yang–Baxter quation

$$[R, R, R] = 0$$

for the matrix R. The important difference between these two classes of bialgebras is in the definition of the product in $A \otimes A$. 
In the quantum groups, the multiplication in $A \otimes A$ is defined by the usual way namely

$$(a \otimes b) \cdot (c \otimes d) = ac \otimes bd$$

i.e.

$$m_{A \otimes A} = (m_A \otimes m_A) \circ (id \otimes \tau \otimes id),$$

where $\tau$ is the flip operator

$$\tau(a \otimes b) = b \otimes a,$$

and $m_A$ is the multiplication map in $A$.

In the braided groups the multiplication is defined in a more complicated way, namely

$$m_{\psi A \otimes A} = (m_A \otimes m_A) \circ (id \otimes \psi \otimes id) \circ (id \otimes \tau \otimes id),$$

(23)

where $\psi$ is the so called braiding i.e. isomorphism $\psi : A \otimes A \rightarrow A \otimes A$ satisfying several properties [17]. Clearly $m_{A \otimes A} = m_{\tau A \otimes A}$.

The algebras for which homomorphism $\Delta$ into the algebras with the product (23) can be introduced are called braided bialgebras. The algebras where it is of the form (22), we shall call braided matrix bialgebras.

The question now is, when the algebra (19) is a braided matrix bialgebra? We shall restrict ourselves to investigating the braidings of the form [16]

$$\psi(L^m_i \otimes L^n_k) := \psi_{ik}^{mn} j^l_r (L^r_j \otimes L^s_l)$$

(24)

where

$$\psi_{ik}^{mn} j^l_r := W_{id}^{aj} Y_{as}^{tb} Z_{sb}^{cn} X_{ck}^{md}.$$  

(25)

i.e. to the braidings that can be expressed also as the quadratic relation in $A \otimes A$.

$$L^{(1)}_1 X_{12} L^{(2)}_2 = W_{12} L^{(2)}_2 Y_{12} L^{(1)}_1 Z_{12},$$

(26)

where the elements of $L^{(1)}, L^{(2)} \in A \otimes A$ are defined as $L^{(1)}_i j = 1_A \otimes L^i_j$, $L^{(2)}_l j = L^l_j \otimes 1_A$, and

$$\tilde{X} := ((X^{t^2})^{-1})^{t_2} \iff X^{kn}_d \tilde{X}^{ml}_{kj} = \tilde{X}^{kn}_d X^{ml}_{kj} = \delta^m_n \delta^l_j$$

For this type of braiding we get:

**Lemma 1** The algebra $A$ given by (19) can be extended to the braided matrix bialgebra with the braiding (24) if

$$C = B^+, X = W = Z = B, Y^+ A = DY.$$  

(27)

Proof: The definition of matrix coproduct (22) can be rewritten as $\Delta(L) = L^{(2)} L^{(1)}$ and using (13, 26, 27), one can show that

$$A_{12} \Delta(L_1) B_{12} \Delta(L_2) = \Delta(L_2) C_{12} \Delta(L_1) D_{12}$$

(28)
Q.E.D.

The braided bialgebras given by the relations of the form

\[ A_{12}L_1B_{12}L_2 = L_2B_{21}L_1Y_{12}A_{12}Y_{12}^{-1} \] (29)

\[ L_1^{(1)}B_{12}L_2^{(2)} = B_{12}L_2^{(2)}Y_{12}L_1^{(1)}B_{12}, \] (30)

that follow from (27) are a slight generalization of QBG investigated in [15]. In that case \( Y = B^{-1} \).

Extension of this lemma to the multiple braided product will appear in the next Section.

3 Multiple braided product of algebras

As mentioned in the Introduction, the crucial object for construction of ultralocal models is the monodromy matrix. It actually is a representation of the multiple matrix coproduct \( \Delta^N(L) \) of generators of the quadratic algebra (6) with trivial braiding (9).

The examples in the Introduction suggest that commutation relations for quantized Lax operators of nonultralocal models yield algebras with nontrivial relations between operators localized in different lattice points i.e. nontrivial braiding, nevertheless, the monodromy matrix is again given by product of Lax operators i.e a representation of the multiple coproduct of an algebra. Therefore, it is worth to investigate the algebras and braidings that enable construction of multiple coproducts. In the following we shall present a generalization of the braided matrix bialgebras from the previous section.

The algebra we are going to investigate is generated by \( 1_B \) and \( N \times M^2 \) generators

\[(L^I)^k, \quad I \in \{1,2,\ldots,N\}, j,k \in \{1,2,\ldots,M\}\] (31)

satisfying quadratic relations

\[ L_1^JX_{12}^{JK}L_2^K = W_{12}^{JK}L_2^KY_{12}^{JK}L_1^JZ_{12}^{JK}, \quad J,K \in \{1,2,\ldots,N\} \] (32)

where \( X^{JK} \), \( W^{JK} \), \( Y^{JK} \), \( Z^{JK} \) for fixed \( J,K \in \{1,2,\ldots,N\} \) are numerical invertible matrices \( M^2 \times M^2 \) i.e. \( X^{JK} = X_{12}^{JK} = \{((X^{JK})_{i_1i_2}^{j_1j_2} \in C)^{M}_{i_1,j_1,i_2,j_2=1} \) and similarly for \( W \), \( Y \), \( Z \). Throughout this paper no summation over indices \( I,J,K,\ldots \) is assumed.

The partition of the set of the generators into \( N \) families of \( M \times M \) generators indicates that this algebra can be interpreted as the multiple braided tensor product of \( N \) algebras \( A_1 \otimes A_2 \otimes \ldots \otimes A_N \), where the relations (32) for \( J = K \) define the algebras \( A_J \) and for \( J \neq K \) they define the braid maps \( \psi_{JK} : A_J \otimes A_K \rightarrow A_K \otimes A_J \). If we want to interpret the algebra (32) as the multiple braided product of the same algebra \( A \) given by (19) then

\[ X^{JJ} = B, \quad W^{JJ} = A^{-1}, \quad Y^{JJ} = C, \quad Z^{JJ} = D. \] (33)

must hold for all \( J \).

The physical importance of the algebra (31,32) consists in possibility to desribe commutation relations of Lax operators both on finite and periodical infinite lattices.
It may happen that the algebra $\mathcal{B}$ can be overdetermined by the relations (32) because for fixed $J, K$, $J \neq K$ they actually define two braiding relations, namely (32) and

$$L_1^J Y_{21}^{KJ} L_2^K = (W_{21}^{KJ})^{-1} L_2^K \times_{21}^J L_1^J (Z_{21}^{KJ})^{-1} \quad (34)$$

To guarantee that these two formulae give the same braidings, we restrict the structure matrices $X$, $W$, $Y$, $Z$ by the conditions

$$X_{JK} = Y_{KJ} + 1, \quad (W_{JK})^{-1} = (W_{JK})^{-1} = (Z_{JK})^{-1} = (Z_{JK})^{-1}, \quad \text{for } J \neq K. \quad (35)$$

The quadratic algebras defined by relations of the form (19) or (32) must satisfy consistency conditions of the Yang–Baxter–type that follow from the requirement that no supplementary higher degree relations are necessary for unique transpositions of three and more elements [18]. For relations of the form (32) the conditions read (cf. [10])

$$\{[Z, Z, Z]\} = 0, \quad \{[W, W, W]\} = 0 \quad (36)$$

$$\{[Z, X, X]\} = 0, \quad \{[X, X, W]\} = 0 \quad (37)$$

$$\{[Z, Y^\dagger, Y^\dagger]\} = 0, \quad \{[Y^\dagger, Y^\dagger, W]\} = 0 \quad (38)$$

$$\{[Z, X, Y^\dagger]\} = 0, \quad \{[Y^\dagger, X, W]\} = 0 \quad (39)$$

where by $\{[R, S, T]\} = 0$ we mean that

$$[R, S, T]_{J_1 J_2 J_3} := [R_{J_1 J_2} S_{J_1 J_3} T_{J_2 J_3}] = R_{12}^{J_1 J_2} S_{13}^{J_1 J_3} T_{23}^{J_2 J_3} - T_{23}^{J_2 J_3} S_{13}^{J_1 J_3} R_{12}^{J_1 J_2} = 0 \quad (40)$$

for $J_i \in \{1, \ldots, N\}$, and

$$(Y^\dagger)^{JK} := (Y^{KJ})^+ = P Y^{KJ} P.$$  

The equations (36) – (39) can be derived by transposing elements of $L_{J_1}, L_{J_2}, L_{J_3}$ in

$$(L_{1}^{J_1} X_{12}^{J_2 J_2} L_{2}^{J_2}) X_{13}^{J_1 J_3} X_{23}^{J_2 J_3} L_{3}^{J_3} = L_{1}^{J_1} X_{12}^{J_1 J_2} X_{23}^{J_2 J_3} (L_{2}^{J_2} X_{23}^{J_2 J_3} L_{3}^{J_3}). \quad (41)$$

Comparing expressions obtained by different order of transpositions and using the properties of the Yang-Baxter commutator

$$[R, S, T] = 0 \iff [R^{-1}, S^{-1}, T^{-1}] = 0 \iff [T^\dagger, R^\dagger, S^\dagger] = 0 \quad (42)$$

we get (36) - (39).

We shall see that for certain type of matrices $X, Y, Z, W$ the large systems of equations (36) – (33) reduces to much smaller. Note e.g. that if $Y^{JK} = (X^{KJ})^+$ then $Y^\dagger = X$ and the equations (38), (39) are equivalent to (37). Another reductions will appear in the following.

In the algebra (32) the following theorem, that will be useful for the construction of the multiple matrix coproduct, holds.
Theorem 1 Let the matrices of generators $L^J$ satisfy (32), where

\[
X^{1,K-1} = Z^{1K}, \quad W^{JN} = X^{J+1,N},
\]

(43)

\[
Z^{J+1,1}Y^{J1} = 1, \quad Y^{NK}W^{N,K-1} = 1,
\]

(44)

\[
Z^{J+1,K}Y^{JK}W^{J,K-1} = X^{J+1,K-1},
\]

(45)

for $J = 1, 2, \ldots, N-1$, $K = 2, 3, \ldots, N$. Then the elements of the matrix

\[
T^N := L^NL^{N-1} \ldots L^1
\]

(46)

satisfy

\[
T^N X^{1N} T^N_2 = W^{NN}L^N_2 Y^{NN}T^N_1(2)W^{1,N-1}L^{N-1}_2 T^N_2(2)Y^{11}L^1_1 Z^{11}
\]

(47)

Proof: It consists in essentially direct, even though sometimes rather involved calculation. We shall present the main steps of the calculation.

First we shall show that (the subscript (12) is omitted in the following formulas)

\[
T^N X^{1N} T^N_2 = W^{NN}L^N_2 Y^{NN}T^N_1(2)W^{1,N-1}L^{N-1}_2 T^N_2(2)Y^{11}L^1_1 Z^{11}
\]

(48)

where we introduce

\[
T^N_1(K) := \prod_{J=N}^K L^J I Z^{J,N-K+2}Y^{J-1,N-K+2},
\]

(49)

\[
T^N_2(K) := \prod_{J=N-K}^1 Y^{K-1,J+1}W^{K-1,J}L^J_2,
\]

(50)

and

\[
\prod_{J=K}^M x^J := x^K \ldots x^M
\]

(51)

Indeed, the exchange relations for $L^J$ (32) imply that if $W^{JN} = X^{J+1,N}$ for $J = 1, 2, \ldots, N-1$ then

\[
T^N_1X^{1N}L^N_2 = W^{NN}L^N_2 Y^{NN}T^N_1(2)L^1_1 Z^{1N}
\]

(52)

and if $X^{1,K-1} = Z^{1K}$ for $K = 2, 3, \ldots, N$ then

\[
L^1_1 Z^{1N} T^N_2 = W^{1,N-1}L^{N-1}_2 T^N_2(2)Y^{11}L^1_1 Z^{11}
\]

(53)

The equation (48) then follows from (52,53).

Due to (32) and (43)–(45), the exchange relations between $T^N(K)$ and $L$ read

\[
T^N_1(K)W^{K-1,N-K+1}L^{N-K+1} =
\]

\[
W^{N,N-K+1}L^{N-K+1}Y^{N,N-K+1}T^N_1(K+1)L^K_1 Z^{K,N-K+1}
\]

(54)

\[
L^K_1 Z^{K,N-K+1} T^N_2(K) = W^{K,N-K}L^{N-K}T^N_2(K+1)Y^{K1}L^K_1 Z^{K1}
\]

(55)
and thus
\[ T_1^N(K)W^{K-1,N-K+1}L_2^{N-K+1}T_2^N(K) = W^{N,N-K+1}L_2^{N-K+1}Y^{N,N-K+1}T_1^N(K+1)W^{K,N-K}L_2^{N-K}T_2^N(K+1)Y^1L_1^1Z^{1,1}. \quad (56) \]

The crucial point of the proof is that left-hand side of (56) is equal to the middle part of the right-hand side up to \( K \to K + 1 \). As
\[ T_1^N(N) = L_1^1Z^{N,N-1,2}, \quad T_2^N(N) = 1, \]
we get from (48) after \( N - 1 \) steps
\[ T_1^N X^{1N} T_2^N = \left( \prod_{K=1}^{N-1} W^{N,N-K+1}L_2^{N-K+1}Y^{N,N-K+1}W^{N,N-K} \right) \times \]
\[ L_1^1Z^{N,N-1,2}W^{N-1,21}L_2^1 \left( \prod_{K=N-1}^{1} Y^{K,1}L_1^KZ^{1,1} \right) = \]
\[ W^{NN} \left( \prod_{K=1}^{N-1} L_2^{N-K+1}Y^{N,N-K+1}W^{N,N-K} \right) L_2^{1}Y^{N1}L_1^N \left( \prod_{K=N-1}^{1} Z^{K+1,1}Y^{K,1}L_1^K \right) Z_1^1 \]

Using (44) we get (47). Q.E.D.

When the relation (53) holds then the algebra \( \mathcal{B} \) can be interpreted as the multiple braided tensor product of \( N \) algebras \( \mathcal{A} \) and the conds (43)–(45) together with \( X^{1N} = B \), \( Y^{N1} = C \) can be interpreted as conds for the existence of the multiple coproduct of \( L \). This, on the other hand, can be used for construction of the monodromy matrix for nonultralocal models.

A simple way to satisfy the conditions (36)–(39), (43)–(45), and (35) for \( W, X, Y, Z \) is to choose
\[ X^{JK} = R(J, K + 1), \quad Y^{JK} = R(K, J + 1)^+, \]
\[ W^{JK} = (R(K + 1, J + 1)^+)^{-1}, \quad Z^{JK} = R(J, K), \]
where \( R \) is a "unitary" solution of the Yang–Baxter equation
\[ \{[R, R, R]\} = 0, \quad R(K, J)^{-1} = R(J, K)^+ \quad \text{for} \ J \neq K. \]

The algebraic relations for \( \mathcal{B} \) in this case are a generalization of the braid group relations
\[ R_{21}(K + 1, J + 1)L_1^1R_{12}(J, K + 1)L_2^K = L_2^K R_{21}(K, J + 1)L_1^1R_{12}(J, K). \]

Another solution of the conditions (36)–(39), (43)–(45), and (35) is given in the next Chapter.

If the assumptions of the Theorem 4 are strengthened to
\[ W^{JK} = X^{J+1,K} = Z^{J+1,K-1}, \quad Y^{JK} = (X^{J+1,K-1})^{-1} \]
then we can derive the exchange relations also for transition matrices
\[ T^{L,M} := L^1L^2 \cdots L^M, \ L > M. \quad (59) \]

They read
\[ T_1^{L_1M_1} X_1^{M_1} L_2 T_2^{L_2M_2} X_2^{L_2+1,M_2} \]
4 Homogeneous and local algebras

The structure coefficients in the commutation relations for $L^n, L^m$ in the examples of nonultralocal models usually do not depend explicitly both on $n$ and $m$ but only on their difference (the equally distant neighbours commute in the same way irrespectively of the site). That is why in the following we shall consider a special case of the algebras from the previous section, namely such that their structure matrices $X, W, Y, Z$ do not depend on both $J$ and $K$ but on their difference only

$$X^{JK} = X^{J−K}, Y^{JK} = Y^{J−K}, W^{JK} = W^{J−K}, Z^{JK} = Z^{J−K}$$

(61)

for $J,K = 1, \ldots, N$. We shall call these algebras homogeneous.

In the homogeneous algebras the multiple coproduct conditions (43)–(45) are equivalent to

$$X^I = Z^{I−1} = W^{I−1} \text{ for } I = −N + 2, \ldots, N − 1, I \neq 1$$

(62)

$$Y^I = (Z^{I+1})^{−1} = (W^{I+1})^{−1} \text{ for } I = −N + 1, \ldots, N − 2, I \neq −1$$

(63)

$$X^1 = Z^0Y^{−1}W^0$$

(64)

and the conds (33) now read

$$Y^I = (X^{−I})^+, \quad Z^{−I} = (Z^I)^+ \iff (Z^I)^{−1} = (Z^{−I})^+ \text{ for } I \neq 0.$$  

(65)

Moreover we get $Y^0 = X^0$ from (62, 63, 65).

The reduction of the set of the Yang–Baxter–type equations for homogeneous algebras describes:

**Theorem 2** If the structure matrices $X, Y, W, Z$ in (32) satisfy (61, 62, 63, 65) and $P(X^{1−N})^{−1}P =: Z^N$ then the set of the Yang–Baxter–type equations (36–39) is equivalent to

$$\begin{align*}
[Z^K, Z^{I+K}, Z^I] &= 0, \\
[Z^I, Z^I, Z^I] &= 0, \quad [W^0, Z^K, Z^K] = 0, \\
[W^0, W^0, W^0] &= 0, \quad [X^I, X^I, W^0] = 0, \quad [Z^0, X^1, X^1] = 0, \\
[Z^I, Z^I, X^I] &= 0, \quad [X^I, Z^I, Z^I] = 0
\end{align*}$$

(66–69)

where $0 \leq K \leq N$, $0 \leq I \leq N − 1$, $0 \leq i + K \leq N$.

For the proof of the theorem we shall use two following lemmas

**Lemma 2** If we denote $(R^\#)^{JK} := (R^+)^{KJ} = P(R^KJ)^{−1}P$ and analogously $S^\#, T^\#$, then (60) that

$$[R, S, T]^{JK} = 0 \iff [R^\#, T, S]^{JK} = 0 \iff [S, R, T^\#]^{JK} = 0$$

(70)
Proof: Follows directly from the definition of Yang–Baxter commutator (40).

Corollary: \[
[R, S, T]^{IJK} = 0 \iff [S^#, T^#, R]^{KJL} = 0 \iff [T^#, R^#, S^#]^{JKL} = 0 \iff [T^#, S^#, R^#]^{KJL} = 0,
\]

Lemma 3 If \((Q^#)^{JK} = Q^{JK}\) then \([Q, Q, Q]_{IJK} = 0\) is equivalent to \([Q, Q, Q]_{IJK} = 0\), for \(I \geq J \geq K\).

Proof: For any \(I', J', K'\) there is a permutation \(\pi\) such that \(\pi(I') \geq \pi(J') \geq \pi(K')\), and from Lemma 2 we get \([Q, Q, Q]_{I', J', K'} = 0\), \(\iff [Q, Q, Q]_{\pi(I'), \pi(J'), \pi(K')} = 0\).

Proof of the Theorem 2: First, from (62, 63, 65) we get
\[
Y^{JK} = Y^{J-K} = (X^{K-J})^+ = (X^{KJ})^+
\]
so that \(X = Y^+\) and both (38) and (39) are equivalent to (37).

Second, if we introduce \(Z^{-N} = W^{-N} := X^{1-N} = P(Z^N)^{-1}P\) then due to (62)
\[
X^{JK} = Z^{J-K-1} = W^{J-K-1}\text{ for } J - K \neq 1
\]
and the set of equations (37) can be rewritten as
\[
[Z, X, X]^{I, J, K} = [Z^{I-J}, Z^{I-K-1}, Z^{J-K-1}] = [Z, Z, Z]^{I,J,K+1} = 0 \quad (70)
\]
for \(I, J \neq K + 1\),
\[
[X, X, W]^{I, J, K} = [W^{I-J-1}, W^{I-K-1}, W^{J-K}] = [W, W, W]^{I-1,J,K} = 0 \quad (71)
\]
for \(J, K \neq I - 1\), and
\[
[Z, X, X]^{K+1, K+1, K} = [Z^0, X^1, X^1] = 0, \quad (72)
\]
\[
[X, X, W]^{I, I-1, I-1} = [X^1, X^1, W^0] = 0, \quad (73)
\]
\[
[Z, X, X]^{I, J, I-1} = [Z^{I-J}, X^1, Z^{J-I}] = 0, \quad (74)
\]
\[
[Z, X, X]^{I, I, I-1} = [Z^{I-J}, Z^{I-J}, X^1] = 0,
\]
\[
[X, X, W]^{K+1, J, K} = [Z^{K-J}, X^1, Z^{J-K}] = 0,
\]
\[
[X, X, W]^{K+1, K+, J} = [X^1, Z^{J-K}, Z^{J-K}] = 0,
\]
for \(I \neq 1, K \neq N, I \neq J, K \neq J\). The last four equations are equivalent to
\[
[Z^L, Z^L, X^1] = 0, \quad [X^1, Z^L, Z^L] = 0, \quad L = 1, 2, \ldots, N - 1 \quad (74)
\]
due to Lemma 2 and \((Z^\#)^{JK} = Z^{JK}\) for \(J \neq K\).

Third, it follows from Lemma 3 and \((Z^\#)^{JK} = Z^{JK}, (W^\#)^{JK} = W^{JK}\) for \(J \neq K\) that the set \((36)\) is equivalent to

\[
[W^{I-J}, W^{I-K}, W^{J-K}] = 0, \quad [Z^{I-J}, Z^{I-K}, Z^{J-K}] = 0 \quad I \geq J \geq K
\]

and this is equivalent to the system

\[
[W^0, W^0, W^0] = 0, \quad [W^0, Z^{I-K}, Z^{I-K}] = 0, \quad [Z^{I-J}, Z^{I-J}, W^0] = 0,
\]

\[
[Z^{I-J}, Z^{I-K}, Z^{J-K}] = 0
\]

for \(I \geq J \geq K\), because \(W^J = Z^J\) for \(J \neq 0\).

Comparing the sets of equations \((70)–(73), (74), (76,77)\) with \((66)–(69)\) completes the proof.

In summary, we have found that the homogeneous algebras admitting the multiple coproduct are determined by the matrices \(W^0, X^1, Z^0, Z^1, \ldots, Z^N\) that satisfy

\[
X^1 = Z^0(X^1)^+W^0
\]

and the consistency conditions \((66)–(69)\).

The relations for the homogeneous algebra then read

\[
L^J_1(Z^1_{21})^{-1}L^J_2 = W^0_1L^J_2(Z^0_{12})^{-1}L^J_1, \quad 1 \leq J \leq N
\]

\[
L^{J+1}_1X^1_{12}L^J_2 = Z^1_{12}L^J_2(Z^0_{12})^{-1}L^{J+1}_1Z^1_{12}, \quad 1 \leq J \leq N - 1
\]

\[
L^{I+K}_1Z^{K-1}_2L^J_2 = Z^K_{12}L^J_2(Z^0_{12})^{-1}L^{I+K}_1Z^K_{12}, \quad 1 \leq J \leq N - K, \quad 2 \leq K \leq N - 1,
\]

For models with periodic boundary conditions the definition of homogeneous algebras must be modified in such way that the relations \((79)–(81)\) hold for all \(J,K = 1,\ldots,N\) and the summation \(J + 1, J + K, \ldots\) is done \(\text{mod } N\). Besides that, the condition of periodicity in the infinite lattices

\[
X^{JK} = X^{J,K+N} = X^{J+N,K},
\]

and analogously for \(Y,W,Z\), yield in the homogeneous algebras

\[
X^I = X^{I+N}, \quad Y^I = Y^{I+N}, \quad Z^I = Z^{I+N}, \quad W^I = W^{I+N}.
\]

From \((85)\) and the definition of \(Z^N\) we then get

\[
Z^{N-I} = P(Z^I)^{-1}P, \quad I = 1,\ldots,N - 1
\]

\[
Z^N = P(X^1)^{-1}P.
\]

Next we want to formalize the fact that the models we consider are local in the sense that they have nontrivial commutation relations only for several neighbouring operators. We shall call algebras \((32)\) local of order \(Q\) if

\[
L^J_1L^K_2 = L^K_2L^J_1 \quad \text{for } |J - K| > Q.
\]
The local algebras of order 0 can be called ultralocal and they are the ordinary (i.e. unbraided) multiple tensor products of algebras $\mathcal{A}$. Let us stress here that we speak about locality of commutation relations and not about the locality of hamiltonians.

One can see from (79)–(81) the homogeneous algebra are local of order $Q > 0$ iff

$$Z^I = 1 \text{ for } I \geq Q$$

and ultralocal iff

$$Z^I = 1 \text{ for } I \geq 1, \ X^1 = 1.$$  

For models with the periodic boundary conditions the definition of local algebras is modified to the form

$$L_1^J L_2^K = L_2^K L_1^J \text{ for } |(J - K) \mod N| > Q.$$ \hspace{1cm} (86)

### 5 Spectrally dependent algebras

In order that we may apply the above introduced algebras for quantization of nonultralocal models we must consider spectral dependent generators and structure matrices i.e. algebras given by

$$L_1^I (\lambda_1) X_{12}^{JK} (\lambda_1, \lambda_2) L_2^K (\lambda_2) = W_{12}^{JK} (\lambda_1, \lambda_2) L_2^K (\lambda_2) Y_{12}^{IK} (\lambda_1, \lambda_2) L_1^I (\lambda_1) Z_{12}^{JK} (\lambda_1, \lambda_2)$$ \hspace{1cm} (87)

The importance of these algebra consists in the fact that we can find a commuting subalgebra that can be used for construction of quantum hamiltonian of a model together with conserved quantities.

It is easy to check that all formulas in Chapters (2,3) remain valid when we replace generators

$$L \to L(\lambda), \ L_1 \to L_1(\lambda_1), \ L_2 \to L_2(\lambda_2),$$

matrices

$$M_{12}^{JK} \to M_{12}^{JK}(\lambda_1, \lambda_2)$$

for $M = X, Y, W, Z$ and the set of Yang–Baxter commutators (40) by

$$\{ [R, S, T] \} : = \{ [R, S, T]^{J_1 J_2 J_3} (\lambda_1, \lambda_2, \lambda_3) :=$$

$$R_{12}^{J_1 J_2} (\lambda_1, \lambda_2) S_{13}^{J_1 J_3} (\lambda_1, \lambda_3) T_{23}^{J_2 J_3} (\lambda_2, \lambda_3) - T_{23}^{J_2 J_3} (\lambda_2, \lambda_3) S_{13}^{J_1 J_3} (\lambda_1, \lambda_3) R_{12}^{J_1 J_2} (\lambda_1, \lambda_2) \}$$ \hspace{1cm} (88)

where $J_i \in \{1, \ldots, N\}, \ \lambda_i \in \mathbb{C}.$

Particularly it means that under the ”spectrally modified” conditions of the Theorem 1 the multiple coproduct of spectrally dependent generators

$$T^N(\lambda) : = L^N(\lambda)L^{N-1}(\lambda) \ldots L^1(\lambda)$$ \hspace{1cm} (89)

satisfies

$$T_1^N (\lambda_1) X_{12}^{1N} (\lambda_1, \lambda_2) T_2^N (\lambda_2) = W_{12}^{NN} (\lambda_1, \lambda_2) T_2^N (\lambda) Y_{12}^{N1} (\lambda_1, \lambda_2) T_1^N (\lambda_1) Z_{12}^{11} (\lambda_1, \lambda_2).$$ \hspace{1cm} (90)
Beside that it was shown in [10, 12, 13] that if the numerical matrix function $K(\lambda)$ satisfies
\[(W_{12}^{N_2})^{t_1t_2}(\lambda_1, \lambda_2)K_{11}^{t_1}(\lambda_1)\tilde{X}_{12}(\lambda_1, \lambda_2)K_{22}^{t_2}(\lambda_2) = K_{22}^{t_2}(\lambda)\tilde{Y}_{12}(\lambda_1, \lambda_2)K_{11}^{t_1}(\lambda_1)((Z_{12}^{11}(\lambda_1, \lambda_2))^{t_1t_2})^{-1}, \quad (91)\]
where $t_1$, $t_2$ mean the transposition in the first respectively second pair of indices and
\[\tilde{X}_{12}(\lambda_1, \lambda_2) = \left(((X_{12}^{N_1}(\lambda_1, \lambda_2))^t_1\right)^{-1}, \quad (92)\]
\[\tilde{Y}_{12}(\lambda_1, \lambda_2) = \left(((Y_{12}^{N_1}(\lambda_1, \lambda_2))^t_2\right)^{-1}, \quad (93)\]
then the algebra elements $t(\lambda) = K^\mu_2(\lambda)T^\mu_j(\lambda) = tr[K(\lambda)T(\lambda)]$ commute
\[[t(\lambda_1), t(\lambda_2)] = 0.\]

Relations of the spectral–dependent ultralocal algebras
\[R_{12}(\lambda, \mu)L^J_1(\lambda)L^J_2(\mu) = L^J_2(\mu)L^J_1(\lambda)R_{12}(\lambda, \mu)\]
\[L^J_1(\lambda)L^K_2(\mu) = L^K_2(\mu)L^J_1(\lambda), \quad J \neq K.\]
are special case of (87) where
\[X^{JK} = Y^{JK} = 1, \quad W^{IJ} = Z^{IJ} = 1 \quad \text{for} \ I \neq J, \ W^{JJ} = R^{-1}, \ Z^{JJ} = R.\]
The quantum nonultralocal model [12] mentioned in the Introduction is an example of the spectrally dependent algebra that is homogeneous and local of order 1 because the relations (14, 15) are special case of (79)–(81) where
\[Z^0 = R^{-}(\lambda_1, \lambda_2),\]
\[W^0 = R^+(\lambda_1, \lambda_2)^{-1},\]
\[X^1 = S(\lambda_1, \lambda_2),\]
\[Z^I = 1 \quad \text{for} \ 1 \leq I \leq N - 1.\]

If we want to find a spectrally dependent, homogeneous algebra that is local of order $Q > 1$ we must find matrices $W^0, X^1, Z^0, Z^1, \ldots, Z^{Q-1}$ satisfying the consistency conditions (86–89) where $Z^I = 1$ for $Q \leq J \leq N$ in the non–periodic, and $Q \leq J \leq N – Q$ in the periodic case.

We shall present a homogeneous algebra that is local of order 2. For that purpose we need four spectral dependent matrices $W^0(\lambda, \mu), Z^0(\lambda, \mu), X^1(\lambda, \mu), Z^1(\lambda, \mu)$ that satisfy
\[W^0(\lambda, \mu) = PX^1(\mu, \lambda)^{-1}PZ^0(\lambda, \mu)^{-1}X^1(\lambda, \mu), \quad (94)\]
and
\[[Z^0, Z^0, Z^0] = 0, \quad (95)\]
\[[Z^1, 1, Z^1] = 0, \quad (96)\]
\[[Z^0, Z^1, Z^1] = 0, \ [Z^1, Z^1, Z^0] = 0, \quad (97)\]
\[
\begin{align*}
[Z^0, X^1, X^1] &= 0, \quad [X^1, Z^1, Z^1] = 0, \quad [Z^1, Z^1, X^1] = 0, \quad (98) \\
[W^0, Z^1, Z^1] &= 0, \quad [Z^1, Z^1, W^0] = 0, \quad [X^1, X^1, W^0] = 0, \quad (99) \\
[W^0, W^0, W^0] &= 0, \quad (100)
\end{align*}
\]

where \([ , , ]\) now means the spectral dependent Yang–Baxter commutator

\[
[R, S, T] := R_{12}(\lambda_1, \lambda_2)S_{13}(\lambda_1, \lambda_3)T_{23}(\lambda_2, \lambda_3) - T_{23}(\lambda_2, \lambda_3)S_{13}(\lambda_1, \lambda_3)R_{12}(\lambda_1, \lambda_2).
\]

The nontrivial commutation relations of the spectrally dependent, local of the order 2, and homogeneous algebra then read

\[
L_1^J(\lambda)Z_{21}^J(\mu, \lambda)^{-1}L_2^J(\mu) = W_{12}^0(\lambda, \mu)L_2^J(\mu)Z_{12}^J(\lambda, \mu)^{-1}L_1^J(\lambda)Z_{12}^J(\lambda, \mu), \quad (101)
\]

\[
L_1^{J+1}(\lambda)X_{12}^J(\mu, \lambda)L_1^J(\mu) = Z_{12}^J(\lambda, \mu)L_2^J(\mu)L_1^{J+1}(\lambda)Z_{12}^J(\lambda, \mu), \quad (102)
\]

\[
L_1^{J+2}(\lambda)Z_{12}^J(\lambda, \mu)L_1^J(\mu) = L_2^J(\mu)L_1^{J+2}(\lambda), \quad (103)
\]

To solve the Yang–Baxter–type system of equations \((93)–(100)\), we have started with the six–vertex rational solution of the Yang–Baxter equation

\[
Z^0(\lambda, \mu) = \begin{pmatrix}
\lambda - \mu + \eta & 0 & 0 & 0 \\
0 & \lambda - \mu & \eta & 0 \\
0 & \eta & \lambda - \mu & 0 \\
0 & 0 & 0 & \lambda - \mu + \eta
\end{pmatrix}, \quad (104)
\]

and diagonal solution of the equation \((16)\)

\[
Z^1(\lambda, \mu) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & b(\lambda) & 0 & 0 \\
0 & 0 & c(\mu) & 0 \\
0 & 0 & 0 & \kappa b(\lambda)c(\mu)
\end{pmatrix}, \quad (105)
\]

where \(b, c\) are arbitrary functions and \(\kappa, \eta\) are constants. They satisfy the equations \((55)–(57)\)

We have found several matrices \(X^1, W^0\) that solve \((93)–(100)\). The first class of solutions is given by \(X^1\) that is equal to \(Z^0\) modified by an arbitrary function \(\zeta\)

\[
X^1(\lambda, \mu) = \begin{pmatrix}
\lambda - \mu + \eta & 0 & 0 & 0 \\
0 & \lambda - \mu & \eta \zeta(\mu) & 0 \\
0 & \eta & (\lambda - \mu)\zeta(\mu) & 0 \\
0 & 0 & 0 & (\lambda - \mu + \eta)\zeta(\mu)
\end{pmatrix}, \quad (106)
\]

and from \((24)\) we get the twisted and gauge transformed rational solution of Yang–Baxter equation

\[
W^0(\lambda, \mu) = -\frac{1}{(\lambda - \mu)^2 - \eta^2} \begin{pmatrix}
\frac{\lambda - \mu + \eta}{\zeta(\lambda)} & 0 & 0 & 0 \\
0 & \frac{\lambda - \mu}{\zeta(\lambda)} & \frac{\eta \zeta(\mu)}{\zeta(\lambda)} & 0 \\
0 & \eta & (\lambda - \mu)\zeta(\mu) & 0 \\
0 & 0 & 0 & (\lambda - \mu + \eta)\frac{\zeta(\mu)}{\zeta(\lambda)}
\end{pmatrix}, \quad (107)
\]

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Note that for $\zeta = 1$ $W^0(\lambda, \mu) = P(Z^0(\mu, \lambda))^{-1} P$.

The second class of solutions is given by

$$X^1(\lambda, \mu) = \begin{pmatrix} \lambda + \mu + \rho + \eta & 0 & 0 & 0 \\ 0 & \lambda + \mu + \rho & \eta \zeta(\mu) & 0 \\ 0 & \eta \sigma & \zeta(\mu)(\lambda + \mu + \rho) & 0 \\ 0 & 0 & 0 & \zeta(\mu)(\lambda + \mu + \rho + \eta) \end{pmatrix}$$

(108)

where $\sigma^2 = 1$, $\zeta$ is an arbitrary function and $\rho$ is a constant and

$$W^0(\lambda, \mu) = \frac{1}{(\lambda - \mu)^2 - \eta^2} \begin{pmatrix} \frac{\lambda - \mu - \eta}{\zeta(\lambda)} & 0 & 0 & 0 \\ 0 & \frac{\lambda - \mu}{\zeta(\lambda)} & -\frac{\eta \zeta(\mu)}{\zeta(\lambda)} & 0 \\ 0 & -\eta & (\lambda - \mu) \zeta(\mu) & 0 \\ 0 & 0 & 0 & (\lambda - \mu - \eta) \frac{\zeta(\mu)}{\zeta(\lambda)} \end{pmatrix},$$

(109)

For $\zeta = 1$ $W^0(\lambda, \mu) = (Z^0(\lambda, \mu))^{-1}$.

The third class of solutions is

$$X^1(\lambda, \mu) = \begin{pmatrix} \rho - \mu & 0 & 0 & 0 \\ 0 & \rho - \mu & 0 & 0 \\ 0 & \eta & \zeta(\mu)(\rho - \mu) & 0 \\ 0 & 0 & 0 & \zeta(\mu)(\rho - \mu) \end{pmatrix}$$

(110)

$$W^0(\lambda, \mu) = \frac{\rho - \mu}{(\rho - \lambda)(\lambda - \mu)^2 - \eta^2} \begin{pmatrix} \lambda - \mu - \eta & 0 & 0 & 0 \\ 0 & \frac{(\lambda - \mu)}{\zeta(\lambda)} & -\frac{\eta \zeta(\mu)}{\zeta(\lambda)} & 0 \\ 0 & -\eta & (\lambda - \mu) \zeta(\mu) & 0 \\ 0 & 0 & 0 & \frac{(\lambda - \mu - \eta) \zeta(\mu)}{\zeta(\lambda)} \end{pmatrix},$$

(111)

where $\zeta$ is an arbitrary function and $\rho$ is a constant. Note that this class generalizes the solution given in [12].

For twisted and gauge transformed version of $Z^0$ one can obtain an analogical set of $X^1$ and $W^0$.

The commutation relations for monodromy matrix are

$$T_{12}^N(\lambda) X_{12}^1(\lambda, \mu) T_{21}^N(\mu) = W_{12}^0(\lambda, \mu) T_{21}^N(\mu) X_{21}^1(\lambda, \mu) T_{12}^N(\lambda) Z_{12}^0(\lambda, \mu).$$

(112)

for the periodic lattices, and

$$T_{12}^N(\lambda) T_{21}^N(\mu) = W_{12}^0(\lambda, \mu) T_{21}^N(\mu) T_{12}^N(\lambda) Z_{12}^0(\lambda, \mu).$$

(113)

for the finite lattices.

6 Conclusions

Motivated by examples of quantized nonultralocal models we have extended the concept of the braided bialgebra to the multiple braided tensor product of algebras and we have given arguments for its applications in the quantizing of nonultralocal models.
Conditions for the existence of the multiple braided coproduct in the algebra (19) are given in the Theorem 1. The multiple coproducts, when represented by operators, are quantum monodromy matrices that generate hamiltonian and conserved quantities of a quantum model.

Further we have introduced the homogeneous and local algebras because one can expect that the commutation relations of Lax operators for many quantized nonultralocal models on lattice depend only on their distance in the lattice. Rather complicated Yang–Baxter–type consistency conditions for structure matrices of the braided tensor product of algebras simplify essentially in this case – see Theorem 2. Nevertheless, even then solving the Yang–Baxter–type set of equations for the structure matrices of the algebras remain nontrivial task.

We have presented three classes of solutions of the Yang–Baxter–type equations that determine a homogeneous algebra that is local of order 2. They are of the six vertex form and depend rationally on the spectral parameters. The commutation relations for the monodromy matrix are given as well.

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Note: Shortly after this work was published in the electronic preprint version (hep-th/9412142), another paper on this topic, motivated by Chern–Simons theory, appeared [20]. The algebra investigated there is spectral dependent homogeneous algebra (79)–(81) where $Z^I$ are independent of $I$ for $I \geq 1$.

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