An Efficient Method to Verify the Inclusion of Ellipsoids

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Abstract: We present a novel method for deciding whether a given $n$-dimensional ellipsoid contains another one (possibly with a different center). This method consists in constructing a particular concave function and deciding whether it has any value greater than $-1$ in a compact interval that is a subset of $[0,1]$. This can be done efficiently by a bisection algorithm method that is guaranteed to stop in a finite number of iterations. The initialization of the method requires $O(n^3)$ floating-point operations and evaluating this function and its derivatives requires $O(n)$. This can be also generalized to compute the smallest level set of a convex quadratic function containing a finite number of $n$-ellipsoids. In our benchmark with randomly generated ellipsoids, when compared with a classic method based on semidefinite programming, our algorithm performs 27 times faster for ellipsoids of dimension $n = 3$ and 2294 times faster for dimension $n = 100$. We illustrate the usefulness of this method with a problem in the control theory field.

Keywords: Ellipsoidal Inclusion, Ellipsoidal Calculus, Lagrangian Duality, S-Lemma

1. INTRODUCTION

For many problems in control and estimation theory, ellipsoidal sets represent a sensible compromise between expressiveness power and numerical tractability. Their simple characterization by a convex quadratic function allows the expression of involved control objectives and constraints as optimization problems that can be handled relatively efficiently by convex optimization. Also, they are suitable for characterizing uncertainties and disturbances, particularly when Gaussian noise is assumed. For a non-exhaustive list of classic applications on control, we refer to (Boyd et al., 1994; Kurzhanski and Vályi, 1997) and, for estimation, a few instances are (Schweppe, 1968; Bertsekas and Rhodes, 1971; Zolghadri, 1996).

More recently, with the development of modern control techniques, such as abstraction-based control design, neural-network-based control, and data-driven control, among others, the representation of mathematical concepts through ellipsoidal sets has been shown to be also useful in these contexts, e.g., developing barrier functions and local controllers (He et al., 2020; Egidio et al., 2022), assessing the safety of neural-networks (Fazlyab et al., 2019), and capturing data uncertainties in data-driven control methods (Bisoffi et al., 2022).

In view of all these applications and the growing necessity for efficient methods to perform numerical operations with ellipsoids, we present a novel approach to verify whether

one $n$-ellipsoid $E \subset \mathbb{R}^n$ is a subset of another ellipsoid $E_0 \subset \mathbb{R}^n$. Our contributions in this paper are listed as follows:

- we write the problem of inclusion of two ellipsoids as a concave minimization problem for which strong duality holds. Then, one can decide the inclusion by computing the maximum of the dual function, which is a scalar concave function that can be evaluated in $O(n)$ floating-point operations (FLOPs), as well as its derivative.
- we prove that the dual search domain for the maximum of the dual function can be restricted to a compact set contained in the interval $[0,1]$, which makes it suitable to be handled by bisection algorithms. Additionally, an early stop criterion is presented, which is triggered within a finite number of iterations of the bisection algorithm when strict inclusion holds.
- we generalize our algorithm to compute the smallest level set of a positive definite quadratic function containing a finite number of $n$-ellipsoids. This is a problem with applications in control theory, and we present an example, namely, calculating control forward invariant sets.
- we show that, in a benchmark consisting of randomly generated ellipsoids, when compared to the classic semidefinite programming-based method, our approach performs, on average, about 27 times faster for ellipsoids of dimension $n = 3$ and 2294 times faster in dimension $n = 100$.

Literature Review: The most simple case of testing the ellipsoid inclusion happens when they share the same center. As it will be discussed, this can be solved by comparing the eigenvalues of the Hessian matrices of the quadratic functions defining each of them. In the general case, when the ellipsoids do not share the same center, the

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inclusion problem can be reformulated (see (Boyd et al., 1994, p. 43)) as a linear matrix inequality (LMI) problem using the S-lemma (Pölk and Terlaky, 2007, Thm. 2.2).

This LMI problem yields a semidefinite program (SDP) and, thus, can be solved by one of many readily available SDP solvers. Nonetheless, the performance and accuracy of these solvers are often not ideal as they do not leverage the specific structure of the problem being solved. Therefore, in this work, we design a method that, by exploiting the structure of the ellipsoidal inclusion problem, outperforms general-purpose SDP solvers such as SDPA (Yamashita et al., 2010) and Mosek (ApS, 2019).

In a similar fashion, the authors of (Gilitschenski and Hanebeck, 2012, Prop. 2) reformulate the ellipsoid intersection problem as the minimization of a convex scalar function in a bounded interval. Their method requires $O(n^4)$ FLOPs and is based on algebraic geometry. For this same problem of intersection of n-ellipsoids, in (Ros et al., 2002), the authors present an algorithm to compute, among all ellipsoids that are convex combinations of two given ones, $E$ and $E_0$, the one with minimal volume and that contains $E \cap E_0$. Their method relies on the computation of the root of a polynomial of degree $2n - 1$ defined in a bounded interval, see (Ros et al., 2002, Thm. 3).

For the problem of inclusion of ellipsoids, however, no tailored procedure is available in the literature to the best of our knowledge.

**Notations:** We denote a matrix $A \in \mathbb{R}^{n \times m}$ by a capital letter and in bold, and the element of the $i$-th row and $j$-th column of $A$ (with $1 \leq i \leq n$ and $1 \leq j \leq m$) by $A_{ij}$. The Moore–Penrose inverse of a matrix $A$ is denoted as $A^\dagger$. Given a vector $v \in \mathbb{R}^n$, we define the support of vector $v$ as $S(v) = \{ i \in \{1, \ldots, n\} : v_i \neq 0 \}$. We denote by $\lambda_{\min}(A)$ the smallest eigenvalue in absolute value of the matrix $A \in \mathbb{R}^{n \times n}$. We denote by $\mathbb{S}_n^+$ the set of positive definite matrices of dimension $n$. Also, $A \succeq 0$ represents that $A \in \mathbb{S}_n^+$ and $A \succeq 0$, that $A \in \mathbb{S}_n^+$, i.e., $A$ is positive semidefinite. The convex hull of a set of vectors $v_1, \ldots, v_m \in \mathbb{R}^n$ is denoted as $\text{co}\{v_1, \ldots, v_m\}$. Given a set $S \subset \mathbb{R}^n$, we denote by $\text{int}(S)$ and $\partial S$, the interior and the boundary of $S$, respectively. Expressions containing the symbol “$\pm$” should be read twice replacing it by “$+$” and “$-$”.

**Outline:** This paper is structured as follows. Section 2 includes the mathematical background needed throughout this paper. Section 3 is devoted to our main result: the optimization formulation of the inclusion test. In Section 4 we provide the details of the practical implementation, benchmarking with off-the-shelf solvers, and an example of application in control theory.

## 2. PRELIMINARY RESULTS

### 2.1 The Inclusion of Ellipsoids

Before presenting our main results, we first state some definitions and present existing results regarding the problem of verifying the inclusion of ellipsoids. An $n$-ellipsoid with center $c \in \mathbb{R}^n$ and shape defined by $P \succ 0$ is denoted as

\[ E(c, P) := \{ x \in \mathbb{R}^n : (x - c)^\top P (x - c) \leq 1 \}. \]

Naturally, the $n$-dimensional Euclidean ball of radius $r > 0$ and centered at $c$ is denoted as $B(c, r) := E(c, r^{-1}I_n)$.

For two ellipsoids $E = E(c, P)$ and $E_0 = E(c_0, P_0)$, the inclusion, the strict inclusion and the non-inclusion are denoted by the standard mathematical symbols $\subseteq$, $\subset$, and $\not\subseteq$. Besides these, an additional relation $\subseteq_0$ is considered in this paper and defined as follows

\[ E \subseteq_0 E_0 \iff E \subseteq E_0 \text{ and } \partial E \cap \partial E_0 \neq \emptyset, \]

which means that $E$ is included in $E_0$ and both ellipsoids have common points in their boundaries that we will call contact points. For studying the inclusion in our context, this situation denotes an extreme case and yields a particular interpretation of our algorithm to be presented.

Verifying whether one ellipsoid is included in another can be equivalently rewritten as verifying if a surrogate ellipsoid is included in a unit Euclidean ball centered at the origin. The next lemma formalizes this equivalence.

**Lemma 1.** Let matrices $P, P_0 \in \mathbb{S}_n^+$ and vectors $c, c_0 \in \mathbb{R}^n$ be given. The following equivalences hold

\[ E(c, P) \subseteq E(c_0, P_0) \iff E(\tilde{c}, \tilde{P}) \subseteq B(0, 1); \]

with

\[ \tilde{c} = L_0^\dagger (c - c_0), \quad \tilde{P} = L_0^{-1} P L_0^{-\top} \]

and $L_0$ defines the Cholesky factorization of $P_0 = L_0 L_0^\top$.

**Proof.** As $P_0 \succ 0$, we have that $L_0$ is regular. By applying the change of variables $\tilde{x} = L_0^\dagger (x - c_0)$, we have that $E(c, P)$ and $E(c_0, P_0)$ becomes respectively $E(\tilde{c}, \tilde{P})$ and $B(0, 1)$ in the space of $\tilde{x}$.

As a consequence, in this paper, we will equivalently study the problem of verifying if an ellipsoid is included in a Euclidean $n$-ball of radius 1 given that an appropriate change of variables transforming the original problem into this one always exists. Notice that this can be done under $O(n^3)$ arithmetic operations because of the required Cholesky factorization (Higham, 2009).

For the sake of completeness, before presenting the necessary and sufficient condition for inclusion on which our method is based, we will discuss other simpler criteria that allow us to sufficiently determine whether one ellipsoid is included in the other or not. These can be used as preliminary tests before running our algorithm to further speed up the execution time of an inclusion verification routine.

**Proposition 1.** Let matrices $P, P_0 \in \mathbb{S}_n^+$ and vectors $c, c_0 \in \mathbb{R}^n$ be given. The following are necessary conditions for $E(c, P) \subseteq E(c_0, P_0)$:

1. $c \in E(c_0, P_0)$,
2. $P \succeq P_0$.

Moreover, if $c = c_0$ then condition (2) is also sufficient.

**Proof.** Let us demonstrate each of these two statements.

1. (1) This trivially holds from the fact that $c \in E(c, P)$.
2. (2) To show a contradiction, assume that $E(c, P) \not\subseteq E(c_0, P_0)$ but also that there exists $v \in \mathbb{R}^n$ such that $v^\top P_0 v > v^\top P v = 1$, without loss of generality. Therefore, $x_+, x_- \in E(c, P)$, with $x_+ = c + v, x_- = c - v$. On the other hand, $x_+$ and/or $x_-$ are not
in $E(c_0, P_0)$, which can be verified by developing the left-hand side expression in the definition (1) as

$$
(x_c - c_0)^T P_0 (x_c - c_0) = (c - c_0 + v)^T P_0 (c - c_0 + v)
$$

$$
> (c - c_0)^T P_0 (c - c_0)
$$

$$
+ 2v^T P_0 (c - c_0) + 1
$$

This shows that at least $x_c \notin E(c_0, P_0)$ or $x_c \notin E(c_0, P_0)$, which is a contradiction.

To show the sufficiency of (2) for $c = c_0$, note that $(x - c)^T P (x - c) \geq (x - c)^T P_0 (x - c)$ for all $x \in \mathbb{R}^n$. Therefore, for all $x \in E(c, P)$ we have $1 \geq (x - c)^T P (x - c) \geq (x - c)^T P_0 (x - c)$, which implies that $x \in E(c_0, P_0)$.

Although simple, these tests allow us to efficiently decide about the inclusion of ellipsoids for some cases. Additionally, for the case $c = c_0$, one can show in a similar fashion that $P > P_0$ implies strict inclusion.

### 2.2 An optimization approach

Let us introduce the following optimization problem

$$
p^* = \min_{x \in \mathbb{R}^n} -x^T x
$$

s.t. \( (x - c)^T P (x - c) \leq 1 \),

which finds the point $x$ of maximum Euclidean norm within $E(c, P)$. Therefore, the optimal value $p^*$ is related to the problem of verifying the inclusion of an ellipsoid inside the Euclidean unit ball as follows:

$$
E(c, P) \subseteq B(0, 1) \iff p^* \geq -1.
$$

It is also straightforward to show that $p^* = -1$ if and only if $E(c, P) \subseteq B(0, 1)$. Although (5) is a non-convex optimization problem, it has some noteworthy properties. First, there always exists a (strictly) feasible point, given that the ellipsoid $E(c, P)$ contains its center in its interior. Also, it is always bounded, given that it is a minimization of a concave function within a convex set (Rockafellar, 1970, p. 342). Finally, the Lagrangian of this optimization problem is a quadratic function on $x$, given as

$$
\mathcal{L}(x, \beta) = -x^T x + \beta ((x - c)^T P (x - c) - 1)
$$

where $\beta$ is the Lagrange multiplier associated with the unique constraint of (5). Naturally, the Lagrange dual function $g : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$
g(\beta) = \min_{x \in \mathbb{R}^n} \mathcal{L}(x, \beta)
$$

and the dual optimization problem by

$$
d^* = \max_{\beta \in \mathbb{R}} g(\beta),
$$

where the dual domain is defined as

$$
D_g = \{ \beta \geq 0 : g(\beta) > -\infty \}.
$$

Notice that, because of its quadratic nature, the lower-boundedness of the Lagrangian function is closely related to the sign of its Hessian

$$
\nabla_x^2 \mathcal{L}(x, \beta) = \beta P - \mathbf{I}.
$$

As discussed in (Boyd et al., 2004, p. 458), for a given $\beta > 0$, this function is bounded from below if $\nabla_x^2 \mathcal{L}(x, \beta) \succ 0$ or if $\nabla_x^2 \mathcal{L}(x, \beta) \succeq 0$ and there exists $x \in \mathbb{R}^n$ such that $\nabla_x \mathcal{L}(x, \beta) = 0$. This implies that the domain of the dual function is either $D_g = [1/\lambda_{\min}(P), \infty)$ or $D_g = (1/\lambda_{\min}(P), \infty)$, depending on the matrix $P$ and the vector $c$ defining the Lagrangian (7).

The next section clarifies how the dual function (8) can be used to construct an efficient algorithm for verifying the inclusion of ellipsoids.

### 3. MAIN RESULTS

#### 3.1 An Algorithm to Test the Inclusion of Ellipsoids

Corresponding to the problem of verifying whether the inclusion $E(c, P) \subseteq B(0, 1)$ holds, we define the infinitely differentiable function $\ell_{c,P} : \mathbb{R} \rightarrow \mathbb{R}$ as

$$
\ell_{c,P}(\beta) := -\beta - \sum_{i \in S(\bar{c})} \beta_i \lambda_i \frac{\lambda_i}{\lambda_i - 1} - 1
$$

and its domain $I_{\bar{c}} := \left(0, \min\{1, \lambda_{\min}(P)\} \right)$.

Let $P \in S^n_+$ and $c \in \mathbb{R}^n$ be given. The function $\ell_{c,P}(\beta)$, given in (12), has the following properties:

1. $\ell_{c,P}(\beta)$ is concave in its domain $I_{\bar{c}}$;
2. $\forall \beta \in I_{\bar{c}}, \ell_{c,P}(\beta) < 0$;

Proof. For $\beta \in I_{\bar{c}}$, we decompose $\ell_{c,P}$ as

$$
\ell_{c,P}(\beta) = h_0(\beta) + \sum_{i \in S(\bar{c})} \beta_i \lambda_i h_i(\beta)
$$

with $h_0(\beta) = -\beta$ and $h_i(\beta) = -\beta/(\lambda_i - 1)$, $i \in S(\bar{c})$. Note that, $\beta_i \lambda_i \geq 0$ for all $i$, given that $P \succ 0$. Below we prove each property in the statement.

1. Evaluating the first and second derivatives of $\ell_{c,P}(\beta)$ for $\beta \in D_g$, we have

$$
\ell'_{c,P}(\beta) = -1 - \sum_{i \in S(\bar{c})} \beta_i \lambda_i (\lambda_i - 1)^2,
$$

$$
\ell''_{c,P}(\beta) = -2 \sum_{i \in S(\bar{c})} \beta_i \frac{\lambda_i^2}{(\lambda_i - 1)^3}.
$$

Since $P \succ 0$, we have $\lambda_i > 0$, which implies that the function $\ell_{c,P}(\beta)$ is strictly negative for all $\beta > \lambda_{\min}(P)^{-1}$. Thus, $\ell_{c,P}(\beta)$ is concave in $I_{\bar{c}}$.

2. The function $h_0(\beta) < 0$ in $I_{\bar{c}}$ and, for all $i \in S(\bar{c})$, the function $h_i(\beta) < 0$ in $(\lambda_i^{-1}, \infty) \supseteq I_{\bar{c}}$. We conclude that $\ell_{c,P}(\beta) < 0$ in $I_{\bar{c}}$.

The proof is concluded.

As demonstrated in the previous lemma, inside its domain $I_{\bar{c}}$ this function is concave. Hence, we can obtain

$$
\ell^*_{c,P} := \sup_{\beta \in I_{\bar{c}}} \ell_{c,P}(\beta)
$$

as the smallest $\beta$ that satisfies $\ell_{c,P}(\beta) \leq 0$ in $I_{\bar{c}}$. This value is the smallest $\beta$ for which $E(c, P) \subseteq B(0, 1)$, as verified by the algorithm.
Proof. First, let us recall that the optimal solution $\ell^*_{c,P}$ (Nocedal and Wright, 1999, Thm. 3.5), or by a bisection algorithm, which avoids the computation of the second derivative. The following theorem connects the function (12) with the ellipsoidal inclusion problem.

**Theorem 1.** Let an ellipsoid $E(c, P)$ be given. Define the function $\ell_{c,P}(\beta)$ as in (12) and consider its supremum $\ell^*_{c,P}$ over the domain $\mathcal{I}_P$. The following equivalences always hold:

$$
E(c, P) \subset \text{int}(B(0,1)) \iff \ell^*_{c,P} > -1;
$$

$$
E(c, P) \subseteq B(0,1) \iff \ell^*_{c,P} = -1.
$$

**Proof.**

We can rewrite

$$
g(\beta) = -\beta - \sum_{i \in S(\ell)} c_i^T \frac{\lambda_i \beta}{\lambda_i \beta - 1}.
$$

Therefore the dual function $g(\beta) = \ell_{c,P}(\beta)$ for all $\beta \in D_\beta = \mathcal{I}_P$, and, hence, $d^* = \ell^*_{c,P}$, concluding the proof.

**Remark 1.** The inclusion $E(c, P) \subseteq B(0,1)$ is equivalent to

$$
\min_{\beta \geq 0} \max_{x \in \mathbb{R}^n} \mathcal{L}(x, \beta) \geq -1
$$

which, in turn, is equivalent to

$$
\exists \beta \geq 0, \forall x \in \mathbb{R}^n : \mathcal{L}(x, \beta) \geq -1.
$$

We can rewrite

$$
\mathcal{L}(x, \beta) + 1 = [x^T 1 F(\beta)[x^T 1]^T
$$

with

$$
F(\beta) = (\beta P - 1 - \beta P c c^T P (\beta P - I)^+ P c).
$$

Therefore, the first case (left) the inclusion of $E(c, P)$ is contained in $\mathcal{E}_0$ (orange) if and only if the maximum of $\ell_{c,P}(\beta)$ is greater than $-1$. Due to the fact that all matrices in this last expression are diagonal, one can rewrite

$$
g(\beta) = -\beta - \sum_{i \in S(\ell)} c_i^T \frac{\lambda_i \beta}{\lambda_i \beta - 1}.
$$

This shows the equivalence to

$$
\exists \beta \geq 0 : F(\beta) \geq 0.
$$

Note that this last inequality is the LMI condition proposed by (Boyd et al., 1994, Sec. 3.7.1) characterizing the inclusion $E(c, P) \subseteq B(0,1)$. This shows the equivalence and the connection between this approach and ours. A numerical comparison, in terms of computational time and memory required to verify inclusions by both methods, is presented in Section 4.

Figure 1 illustrates the results of Theorem 1. The ellipsoid $E$ (blue) is contained in $\mathcal{E}_0$ (orange) if and only if the maximum of $\ell_{c,P}(\beta)$ is greater than $-1$. Due to the fact that all matrices in this last expression are diagonal, one can rewrite

$$
g(\beta) = -\beta - \sum_{i \in S(\ell)} c_i^T \frac{\lambda_i \beta}{\lambda_i \beta - 1}.
$$

This shows the equivalence to

$$
\exists \beta \geq 0 : F(\beta) \geq 0.
$$

Theorem 1 is the foundation of our algorithm to verify the inclusion of ellipsoids. We recall that, although one ellipsoid is considered to be a Euclidean $n$-ball $B(0,1)$, Lemma 1 provides a change of variables that always allows us to transform the general problem into the one tackled in Theorem 1.

In Figure 1, an illustration of the results of Theorem 1 is provided. There, three cases of ellipsoids $E = E(c, P)$ (blue) and $E_0 = E(c_0, P_0)$ (orange) are depicted, along with the corresponding functions $\ell_{c,P}$ in the interval $[1/\lambda_{\text{min}}(P), 1]$, where $c$ and $P$ are given in (3). Notice that, for the first case (left) the inclusion of $E$ (blue ellipsoid) within $\mathcal{E}_0$ (orange) does not hold and the corresponding function $\ell_{c,P}$ is always strictly below $-1$. For the second case (middle), the inclusion holds, and therefore, the maximum of $\ell_{c,P}$ is greater than $-1$. Finally, a third

by classic optimization algorithms such as Newton’s method, which guarantees a quadratic convergence rate to $\ell^*_{c,P}$ (Nocedal and Wright, 1999, Thm. 3.5), or by a bisection algorithm, which avoids the computation of the second derivative. The following theorem connects the function (12) with the ellipsoidal inclusion problem.

**Theorem 1.** Let an ellipsoid $E(c, P)$ be given. Define the function $\ell_{c,P}(\beta)$ as in (12) and consider its supremum $\ell^*_{c,P}$ over the domain $\mathcal{I}_P$. The following equivalences always hold:

$$
E(c, P) \subset \text{int}(B(0,1)) \iff \ell^*_{c,P} > -1;
$$

$$
E(c, P) \subseteq B(0,1) \iff \ell^*_{c,P} = -1.
$$

**Proof.** First, let us recall that the optimal solution $p^*$ of the primal optimization problem (5) allows us to decide whether $E(c, P)$ is included or not inside $B(0,1)$, i.e., the inclusion $E(c, P) \subset \text{int}(B(0,1))$ (resp. $E(c, P) \subseteq B(0,1)$) holds if and only if $p^* > -1$ (resp. $p^* = -1$). Let us show that this problem has no duality gap, that is, $p^* = d^*$ where $d^*$ is the optimal solution of the dual problem (9).

Notice that Slater’s constraint qualification (Boyd et al., 2004, p. 226) holds given that $x = c$ is a strictly feasible point. Although the primal problem is not convex due to the concave objective function, Slater’s condition implies strong duality in this case, given that this problem is the minimization of a quadratic function subject to a single quadratic inequality, see (Boyd et al., 2004, Sec. B4) for a detailed proof.

Therefore, we must now show that $\ell^*_{c,P} = d^*$. Notice that, the minimization problem within the definition of the dual function (8) is convex for $\beta \in D_\beta$ and, therefore, its global minimizers $x^*$ for a given $\beta \in D_\beta$ are all points satisfying the first-order optimality condition

$$
\nabla_x \mathcal{L}(x^*, \beta) = 2(\beta P - I)x^* + 2\beta Pc = 0.
$$

If $\beta > 1/\lambda_{\text{min}}(P)$, then the matrix $(\beta P - I)$ can be inverted and there is a unique minimizer $x^*(\beta) = -(\beta P - I)^{-1}\beta Pc$. However, when $\beta = 1/\lambda_{\text{min}}(P)$, two cases may occur: 1) there is no $x^*$ such that (17) holds and therefore, $1/\lambda_{\text{min}}(P)$ does not belong to the domain $D_\beta$ of the dual function $g(\beta)$; 2) there exist infinitely many $x^*$ satisfying (17), which are given by

$$
x^*(\beta) = -(\beta P - I)^{-1} \beta Pc + \kappa v
$$

where $N \in \mathbb{R}^{n \times m}$ has columns spanning the $m$-dimensional nullspace of $(P - \lambda_{\text{min}}(P)I)$ and $v \in \mathbb{R}^m$. Each of these minimizers is a global minimizer, without loss of generality, let us choose $x^*(\beta) = -(\beta P - I)^{-1}\beta Pc$ (i.e., $v = 0$). Naturally, this minimizer must satisfy (17) by assumption. Substituting $x^*(\beta)$ into (17) and performing a few algebraic manipulations yields

$$
-\beta(\beta P - I)^+ P c = \beta P c - \beta^2 P(\beta P - I)^+ P c.
$$

Hence, the dual function (8) can be rewritten as

$$
g(\beta) = \mathcal{L}(x^*(\beta), \beta)
$$

$$
= \beta(c^T P c - 1) - \beta^2 c^T P(\beta P - I)^+ P c
$$

$$
= -\beta - \beta c^T (\beta P - I)^+ P c
$$

$$
= -\beta - \beta E^T (\beta D - I)^+ D E,
$$

where (19) was used to obtain the last equation and the last equation uses the spectral decomposition $P = V D V^T$ and the transformation $E = V^T c$. Moreover,
case (right) illustrates the case when $1/\lambda_{\min}(P) \in D_p$. This happens because, after transforming $\mathcal{E}_0$ into the unit Euclidean ball centered at the origin, the center of the transformed $\mathcal{E}$ is perpendicular to its greatest semi-axis (or semi-axes), which is the eigenvector associated to $\lambda_{\min}(P)$.

Before introducing our general algorithm, let us present an additional property of the scalar function $\ell_{c,P}(\beta)$, defined in (12), which will be important for implementation purposes.

**Proposition 2.** Let $P \in S^n_+$ and $c \in \mathbb{R}^n$ be given. The function $\ell_{c,P}(\beta)$, defined in (12) satisfies

$$
\ell_{c,P}(\beta) < -1, \quad \forall \beta > \max \{\lambda_{\min}(P)^{-1}, 1 - c^Tc\}.
$$

**Proof.** By contradiction, assume that there exists $\beta_0 > 1 - c^Tc$ such that $\beta_0 > \lambda_{\min}(P)^{-1}$ and $\ell_{c,P}(\beta_0) \geq -1$. From the proof of Theorem 1, we have that $\ell_{c,P}(\beta_0) = g(\beta_0)$, which implies

$$
\ell_{c,P}(\beta_0) = \min_{x \in \mathbb{R}^n} \mathcal{L}(x, \beta_0)
\leq \mathcal{L}(c, \beta_0) = -c^Tc - \beta_0 < -1
$$

which is a contradiction.

Besides providing a useful upper bound on the interval on which $\ell_{c,P}(\beta) \geq -1$, reducing the search space for $\beta$ that maximizes $\ell_{c,P}(\beta)$, Proposition 2 also provides a sufficient condition for $\mathcal{E}(c, P) \not\subseteq B(0,1)$. Indeed, if $1 - c^Tc < 1/\lambda_{\min}(P)$, then $\ell_{c,P}(\beta) < -1$ for all $\beta > 1/\lambda_{\min}(P)$ and, therefore, the inclusion does not hold.

Finally, these results can be joined together into Algorithm 1. It is important to highlight that this algorithm starts at lines 1-2 by verifying the tests of Propositions 1 and 2 through the function `isPreTestConclusive()`. Whenever these tests are not conclusive, the algorithm proceeds to evaluate the necessary and sufficient condition from Theorem 1. To do so, notice that the lines 3-5 constitute the algorithm of the initialization and can be computed within $O(n^3)$ FLOPs, due to the Cholesky Factorization (Higham, 2009) and the spectral decomposition (Banks et al., 2022). Line 6 consists of the maximization of a concave scalar function $\ell_{c,P}(\beta)$, defined in (12), on the interval

$$
\mathcal{T}_{c,P} := [\lambda_{\min}(P)^{-1}, 1 - c^Tc].
$$

**Algorithm 1:** Test the inclusion of an ellipsoid $\mathcal{E} = E(c, P)$ in an another ellipsoid $\mathcal{E}_0 = E(c_0, P_0)$.

1. If `isPreTestConclusive()` then
2. \hspace{1cm} return `PreTestConclusive()`;
3. \hspace{1cm} end if
4. \hspace{1cm} $\mathcal{L}_0$ ← Cholesky factorization of $P_0 = \mathcal{L}_0\mathcal{L}_0^T$;
5. \hspace{1cm} $(\bar{c}, \bar{P})$ ← $(\mathcal{L}_0^Tc - \mathcal{L}_0^Tc_0), \mathcal{L}_0^{-1}P_0\mathcal{L}_0^T$;
6. \hspace{1cm} $V, D$ ← Spectral decomposition of $\bar{P} = VDV^T$;
7. \hspace{1cm} $\ell_{c,P}^* \leftarrow \max_{\beta \in \mathcal{T}_{c,P}} \ell_{c,P}(\beta)$;
8. \hspace{1cm} if $\ell_{c,P}^* > -1$ then
9. \hspace{1cm} \hspace{1cm} return $\mathcal{E} \subseteq \mathcal{E}_0$;
10. \hspace{1cm} \hspace{1cm} \hspace{1cm} else if $\ell_{c,P}^* = -1$ then
11. \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} return $\mathcal{E} \subseteq \mathcal{E}_0$;
12. \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} else
13. \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} return $\mathcal{E} \not\subseteq \mathcal{E}_0$;

The upper bound of the interval $\mathcal{T}_{c,P}$ is determined from Proposition 2. This maximization problem can be solved without difficulty by a bisection algorithm or, more efficiently, by Newton’s method. Notice that, for the former, one needs to compute the first and second derivatives of $\ell_{c,P}(\beta)$, defined respectively in (14) and (15), which can be done in $O(n)$ FLOPs.

Algorithm 1 can be early-stopped whenever a $\beta \in \mathcal{T}_{c,P}$ is found such that $\ell_{c,P}(\beta) > -1$. However, as discussed in the next subsection, computing this maximum is an efficient way to obtain a distance between the boundaries of the two ellipsoids (when the inclusion holds) or by how much $\mathcal{E}_0$ must be inflated so it contains $\mathcal{E}$.

### 3.2 Consequences of Theorem 1

The first consequence of Theorem 1 that we discuss in this paper is the fact that whenever $E(c, P) \subseteq B(0,1)$, knowing $\beta > 0$ that maximizes $\ell_{c,P}(\beta)$ allows us to fully characterize contact points between $E(c, P)$ and $\partial B(0,1)$.

**Corollary 1.** If $E(c, P) \subseteq B(0,1)$, then, all contact points

$$
\bar{x} \in \{x : (x-c)^T P (x-c) = x^T x = 1\} \quad (26)
$$

satisfy

$$
\bar{x} = -(\beta^*P - I)^+ \beta^*Pc + Nu \quad (27)
$$

for some $v \in \mathbb{R}^m$ where $N \in \mathbb{R}^{n \times m}$ has columns spanning the $m$-dimensional nullspace of $(P - \beta^*I)$ and

$$
\beta^* = \arg \max_{\beta \in \mathcal{T}_{c,P}} \ell_{c,P}(\beta)
$$

**Proof.** By the fact that strong duality holds, the contact points $\bar{x}$ are optimal solutions of the primal problem (5) and, thus, satisfy the equation (17) for the optimal $\beta^*$ of the dual problem and, thus, satisfy (27).

**Corollary 1** provides a complete characterization of the contact points between the boundary of the two ellipsoids. Notice that, there exist a unique contact point whenever $\beta^* > 1/\lambda_{\min}(P)$, which ensures that the matrix $(P - \beta^*I)$ has full rank. However, if $\beta^* = 1/\lambda_{\min}(P)$, the uniqueness is no longer guaranteed. In this case, a contact point can be found by selecting $v = \alpha n_0$ with $n_0 \in \mathbb{R}^m$ and finding the scalar $\alpha$ that solves the quadratic equation generated by evaluating $\bar{x}^T \bar{x} = 1$.

As shown in Theorem 1, the maximal value $\ell_{c,P}^*$ being greater or lesser than $-1$ provides some geometrical insights regarding the inclusion. Besides that, its magnitude yields additional information, as the next corollary shows.

**Corollary 2.** For any $c, c_0 \in \mathbb{R}^n$ and $P, P_0 \in S^n_+$, let $\bar{c}$ and $\bar{P}$ be defined as in (3) and $\ell_{\bar{c},\bar{P}}^*$, defined in (16), the following inclusions hold

$$
E(c, P) \subseteq E(c_0, -\ell_{c,P}^*P) \quad (28)
$$

$$
E(d, -\ell_{\bar{c},\bar{P}}^*P) \subseteq E(c_0, P_0) \quad (29)
$$

with $d = (-\ell_{\bar{c},\bar{P}}^*)^{-1/2}(c - c_0) + c_0$.

**Proof.** By Lemma 1, we have that $E(c, P) \subseteq E(c_0, \gamma^{-1}P_0)$ if and only if $E(\gamma^{-1/2}\bar{c}, \gamma \bar{P}) \subseteq B(0,1)$ with $\bar{c}$, $\bar{P}$ defined in (3). By its definition in (12) we have

$$
\ell_{\bar{c},\bar{P}}^* = -\beta^* - \sum_{i \in S(\bar{c})} c_i^2 \frac{\lambda_i \beta^*}{\lambda_i \beta^* - 1} \quad (30)
$$
Let $\beta^*_\gamma = \gamma^{-1} \beta^*$ and $\gamma = -\ell^*_{c,p}$. By the property (2) in Lemma 2, we have $\beta^*_\gamma > 0$. Tediuous but simple algebraic manipulations where (30) is used, yield $\ell^*_{-1/2,\varepsilon^0}(\beta^*_\gamma) = -1$ and $\ell^*_{-1/2,\varepsilon^0}(\beta^*_\gamma) = 0$ (recall (14)), which shows that $\beta^*_\gamma$ is the maximizer of $\ell^*_{-1/2,\varepsilon^0}(\beta)$. Therefore, Theorem 1 ensures that $E(c, P) \subseteq \hat{E}(c, \gamma^{-1} P_0)$, which is the inclusion (28) in the statement. As a consequence, (29) also holds as the ellipsoids therein are the same as those in (28) after a translation of $-c_0$, a uniform scaling of $(\ell^*_{c,p})^{-1/2}$, and a translation of $c_0$.

Notice that, different interpretations can be given to the inclusion (28) in Corollary 2:

- $\ell^*_{c,p} > -1$: The ellipsoid $E(c_0, P_0)$ can be compressed by, at most, a factor of $(\ell^*_{c,p})^{-1/2} < 1$ and will still contain $E(c, P)$;
- $\ell^*_{c,p} = -1$: Corollary 2 becomes trivial, as we have $\gamma = 1$ and there is already a contact point between the boundaries of $E(c_0, P_0)$ and $E(c, P)$;
- $\ell^*_{c,p} < -1$: The ellipsoid $E(c_0, P_0)$ must be inflated by, at least, a factor of $(\ell^*_{c,p})^{-1/2} > 1$ to contain $E(c, P)$.

The same applies to (29) but considering inflation and compression of $E(c, P)$ with respect to the $c_0$.

Intuitively, Corollary 2 provides a simple method for determining what is the ellipsoid of minimum volume centered in $c_0$ and with a shape defined by the spectrum of $P_0$ that contains $E(c, P)$. Alternatively, we can also compute the least sub-level set of the quadratic function $x \mapsto (x - c_0)^T P_0 (x - c_0)$ containing the $n$-ellipsoid $E(c, P)$. This interpretation will be explored in the following sections, with applications in control design problems.

4. NUMERICAL EXPERIMENTS

4.1 Implementation details

To solve the optimization problem in Line 6 of Algorithm 1, we propose a bisection algorithm (Algorithm 2) with stopping criteria guaranteeing inclusion without contact point ($E \subseteq \text{int}(E_0)$) and non-inclusion ($E \not\subseteq E_0$). Note that the inclusion with contact points ($E \subseteq E_0$) cannot be decided numerically, motivating the definition of $E \subseteq \text{int}(E_0)$, which means that, given a machine precision $\epsilon$ of the computer, the algorithm cannot determine if $E \subset \text{int}(E_0)$ or $E \not\subseteq E_0$.

Proposition 3. Algorithm 2 is correct and terminates in a finite number of steps.

Proof. Let $\beta^* = \arg\max_{\beta \in \mathcal{T}_P} \ell_{c,p}(\beta)$. Whenever $\beta^* \notin \mathcal{T}_P$ we have $\ell_{c,p}(1 - c_1^T c) > 0$ and $\ell_{c,p}(1 - c^T c) < -1$ implies, by Proposition 2, that $E \not\subseteq E_0$, which the algorithm deals with in line 5. Let us consider the case $\beta^* \in \mathcal{T}_P$. In an arbitrary interval $[l, u] \subseteq \mathcal{T}_P$ containing $\beta^*$ for which $l > 1/\lambda_{\min}(P)$, the function $\ell_{c,p}$ is locally $L$-smooth with $L = \max_{\beta \in [l, u]} |\ell_{c,p}'(\beta)| = \ell_{c,p}(\beta)$ (since $\ell_{c,p}'$ is positive and increasing in $[l, u]$). Therefore, due to (Nesterov et al., 2018, Lemma 1.2.3) we have

$$\ell_{c,p}(\beta^*) - \frac{L}{2} (\beta - \beta^*)^2 \leq \ell_{c,p}(\beta) \leq \ell_{c,p}(\beta^*) \quad \forall \beta \in [l, u]$$

and since $\beta^* \in [l, u]$, we obtain the following lower and upper bounds on $\ell_{c,p}$ for all $\beta \in [l, u]$:

$$\ell_{c,p}(\beta) \leq \ell_{c,p}(\beta^*) \leq \ell_{c,p}(\beta) - \frac{\ell_{c,p}''(0)}{2} (u - l)^2. \quad (31)$$

At each iteration $k$, lines 14 or 16 ensure that $\beta^* \in [l_k, u_k]$. If at a given iteration $k$, either $\ell_{c,p}(\beta_k) \geq -1$, or $\ell_{c,p}(\beta_k) - \ell_{c,p}''(l_k)(u_k - l_k)^2/2 < -1$, one can respectively conclude by (31) that $\ell_{c,p} < -1$ or $\ell_{c,p} > -1$.

Whenever $\ell_{c,p} > -1$, the continuity of $\ell_{c,p}(\beta)$ ensures that the condition in line 9 will be satisfied at a given iteration, given that the interval $[l_k, u_k]$ converges to $\beta^*$ as $k \to \infty$. In the opposite case, when $\ell_{c,p} < -1$, Algorithm 2 also stops by satisfying the condition in line 11. This holds by the same argument that $[l_k, u_k]$ converges to $\beta^*$, which also implies that $(u_k - l_k)^2 \to 0$ and $\ell_{c,p}'(l_k) \to \ell_{c,p}'(\beta^*) < 0$ as $k \to \infty$, concluding the proof. Finally, if $\ell_{c,p} = 1$, the stop criterion of the main loop (i.e., $u_k - l_k \leq \epsilon$, for a given precision $\epsilon > 0$) will be satisfied and the algorithm returns $E \not\subseteq E_0$, implying that for this precision $\epsilon$ the inclusion with contact points may hold.

Finally, note that evaluating $\ell_{c,p}(\beta)$ and its derivatives are computationally inexpensive operations. Once $c^T i \in \mathcal{S}(\varepsilon)$, and the eigenvalues of $P$ have been computed, the exact number of FLOPs to evaluate the function $\ell_{c,p}$ for a given $\beta$ is $5|\mathcal{S}(\varepsilon)|$, which, in the worst case, represents $5n$ FLOPs. Similarly, by performing some preliminary computations, the evaluation of $\ell_{c,p}'$ and $\ell_{c,p}''$ at a given $\beta$ requires respectively $5|\mathcal{S}(\varepsilon)|$ and $6|\mathcal{S}(\varepsilon)|$ FLOPs. Note that the evaluation of $\ell_{c,p}$ and its derivative share several algebraic operations, which can be used to reduce the total number of computations.

Algorithm 2: Test the inclusion of an ellipsoid $E = E(c, P)$ in the ball $E_0 = B(0, 1)$. Returns either $E \subset \text{int}(E_0)$, $E \not\subseteq E_0$, or $E \not\subseteq E_0$.

1. $l_0, u_0 \leftarrow \lambda_{\min}(P)^{-1}, 1 - c_1^T c$;
2. if $\ell_{c,p}(l_0) > -1$ or $\ell_{c,p}(u_0) > -1$ then
   3. return $E \subset \text{int}(E_0)$;
4. else if $\ell_{c,p}(\beta_k) < -1 + \ell_{c,p}'(l_k)(u_k - l_k)^2/2$ then
   5. return $E \not\subseteq E_0$;
11. else if $\ell_{c,p}(\beta_k) < -1$ then
   12. return $E \not\subseteq E_0$;
13. else if $\ell_{c,p}(\beta_k) > 0$ then
   14. $l_{k+1}, u_{k+1} \leftarrow \beta_k, u_k$;
15. if $\ell_{c,p}(\beta_k) > 0$ then
   16. $l_{k+1}, u_{k+1} \leftarrow l_k, \beta_k$;
17. $k \leftarrow k + 1$;
18. return $E \subseteq E_0$;
The underlying SDP problem tries to find SDP A (Yamashita et al., 2010) and Mosek (ApS, 2019).

4.2 Performances

Fig. 2 shows the average execution times (in seconds) for testing ellipsoid inclusion \( E(c, P) \subseteq E(c_0, P_0) \) as a function of ellipsoid dimension \( n \).

| \( n \) | Algorithm 1 | LMIs+SDPA | LMIs+Mosek |
|-----|-------------|-----------|------------|
| 3   | 8.2        | 192.6     | 153.5      |
| 10  | 26.5       | 733.2     | 577.8      |
| 30  | 118.4      | 4886.6    | 3792.1     |
| 100 | 865.8      | 51822.6   | 39608.1    |

Let us now compare the performance of Algorithm 1 with those of solving the LMI condition (23) provided by Boyd et al. (1994) with two conventional SDP solvers, namely, SDPA (Yamashita et al., 2010) and Mosek (ApS, 2019). The underlying SDP problem tries to find \( \beta \geq 0 \) such that the matrix \( F(\beta) \succeq 0 \) and, although a single variable is being searched for, the problem deals with an SDP restriction of size \( n + 1 \).

Fig. 2 shows the average execution times (in seconds) for 200 randomly generated problems, each consisting of two ellipsoids \( E = E(c, P) \) and \( E_0 = E(c_0, P_0) \). These were generated in a way that, in 100 cases, we have \( E \subseteq \text{int}(E_0) \), and \( E \nsubseteq E_0 \) in the remaining ones. Also, we made sure that the conditions in Proposition 1 do not hold, so running the bisection Algorithm 2 was required each time. The comparison with the SDP solvers was repeated for \( n \)-ellipsoids with \( n \in \{3, 10, 30, 100\} \) to evaluate how these approaches scale up. These were performed in an Intel(R) Xeon(R) W-2295 CPU @ 3.00GHz with Julia 1.7.3 and using the optimization toolbox JuMP (Dunning et al., 2017). We computed that, in this benchmark, the execution times for Algorithm 1 were lesser than those for SDPA and Mosek in all cases. On average, our algorithm outperforms 27, 49, 162, and 2294 times faster than the SDP A and Mosek in all cases. On average, our algorithm

| \( n \) | \( \mathcal{E} \subseteq \text{int}\mathcal{E}_0 \) | \( \mathcal{E} \nsubseteq \mathcal{E}_0 \) |
|-----|----------------|----------------|
| 3   | Alg. 1 | SDPA | MOSEK | Alg. 1 | SDPA | MOSEK |
| 10  | 13.8 | 203.3  | 152.1  |
| 30  | 36.4 | 733.1  | 576.0  |
| 100 | 901.9 | 51822.6 | 39606.2 |

set for additive disturbances. This problem has been extensively studied in the literature (e.g., see (Blanchini, 1999, Section 4.1)) and, therefore, the goal of this section is not to provide a new method to tackle it but rather to demonstrate the results from Corollary 2.

For that, consider an LTI system

\[
\dot{x} = Ax + Bu + Hw
\]

where \( x(t) \in \mathbb{R}^2 \) is the state variable, and the signals \( u(t), w(t) \in \mathbb{R} \) are the control and the additive disturbance. This system is defined by the matrices

\[
A = \begin{pmatrix} 0 & 1 \\ 0.1 & 0.3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}, \quad H = \begin{pmatrix} -0.3 \\ 0.6 \end{pmatrix}
\]

and is controlled by a state-feedback LQR controller \( u(t) = Kx(t) \) synthesized for LQR parameters \( Q = I \) and \( R = 1 \), see (Boyd et al., 1994, p. 114) for details on the LQR problem. The corresponding solution of the Algebraic Riccati Equation and feedback gain are

\[
P = \begin{pmatrix} 36.10 & 42.36 \\ 42.36 & 72.98 \end{pmatrix}, \quad K = (4.24 730),
\]

which allows us to define the closed loop matrix \( A_c = A - BK \) and a Lyapunov function \( v(x) = x^TPx \).

Bounded additive disturbances: Due to the additive disturbance, the descent condition for this Lyapunov function does not hold everywhere. For an arbitrary \( w \in \mathbb{R} \), this can be verified as

\[
\dot{v}(x, w) := 2x^T(PA_c)x + 2x^TPHw \geq -(x - Gw)^T S(x - Gw) + r(w)
\]

and is controlled by a state-feedback LQR controller \( u(t) = Kx(t) \) synthesized for LQR parameters \( Q = I \) and \( R = 1 \), see (Boyd et al., 1994, p. 114) for details on the LQR problem. The corresponding solution of the Algebraic Riccati Equation and feedback gain are

\[
P = \begin{pmatrix} 36.10 & 42.36 \\ 42.36 & 72.98 \end{pmatrix}, \quad K = (4.24 730),
\]

which allows us to define the closed loop matrix \( A_c = A - BK \) and a Lyapunov function \( v(x) = x^T Px \).

4.3 Applications in Control Theory

This section illustrates one possible application in the control theory of linear time-invariant (LTI) systems, namely, the computation of a control forward-invariant
In summary, Corollary 2 allows the verification that the Lyapunov function \( v(x) = x^TPx \) strictly decreases in \( \mathbb{R}^2 \setminus \mathcal{V} \) despite the persistent disturbance. This verification is done efficiently by performing the following numerical operations: one Cholesky decomposition, one spectral decomposition, and one bisection algorithm for maximizing a concave scalar function in a compact interval.

5. CONCLUSION AND FUTURE WORK

We presented a new method to verify the inclusion of \( n \)-ellipsoids, which consists in the maximization of a scalar concave and smooth function (12). This function and its derivatives can be computed in \( \mathcal{O}(n) \) floating-point operations and the interval (25) where its maximum lies is is done efficiently by performing the following numerical operations: one Cholesky decomposition, one spectral decomposition, and one bisection algorithm for maximizing a concave scalar function in a compact interval.

The source codes for the numerical experiments carried out in this paper are available in the following repository:

https://github.com/egidio1n/EllipsoidInclusion.jl

For future work, we plan to generalize this approach to verify the emptiness of the intersection of ellipsoids and also other quadrics. We also seek to apply these results to model and data-driven abstraction-based control.

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