Abstract

Although we know that black holes are characterized by a temperature and an entropy, we do not yet have a satisfactory microscopic “statistical mechanical” explanation for black hole thermodynamics. I describe a new approach that attributes the thermodynamic properties to “would-be gauge” degrees of freedom that become dynamical on the horizon. For the (2+1)-dimensional black hole, this approach gives the correct entropy.

I. Introduction

It has now been more than twenty years since Bekenstein [1] and Hawking [2] demonstrated that black holes are thermodynamic objects, with characteristic temperatures and entropies. The evidence for black hole thermodynamics is convincing: the same temperatures and entropies can be obtained from a wide variety of approaches, ranging from the study of quantum field theory in black hole backgrounds [3, 4] to semiclassical path integration [5, 6] to appeals to the consistency of standard thermodynamics in the presence of black holes [7, 8]. But despite considerable effort, we do not yet have a satisfactory “statistical mechanics” of black hole thermodynamics; we cannot explain the temperature and entropy in terms of microscopic degrees of freedom. The entropy of the Universe is, apparently,

$$S = \frac{1}{4} \sum \text{(areas of black hole horizons)} + \sum \text{(statistical mechanical entropies of everything else)}. \quad (1)$$
This lack of symmetry is disturbing.

Let us suppose that black hole thermodynamics has an undiscovered statistical mechanical origin. It is reasonable to expect that the missing microscopic degrees of freedom should come from quantum gravity—after all, the relevant thermodynamic quantities can be obtained from a path integral for gravity with no additional couplings. But (2+1)-dimensional gravity then presents a paradox. Bañados, Teitelboim, and Zanelli showed in 1992 that general relativity in three spacetime dimensions admits black hole solutions \[7\], and these black holes have thermodynamic properties not unlike those in 3+1 dimensions \[8\]. But the physical degrees of freedom of (2+1)-dimensional gravity are fairly well understood (see, for example, \[9\]), and it is easy to see that there are simply not enough degrees of freedom in the conventional formulation to account for the predicted entropy. Something is missing.

The goal of this article is to describe a new approach, discovered independently by Balachandran, Chandar, and Momen \[10, 11\] and by me \[12, 13\], that may provide the missing degrees of freedom. The basic argument is that in the presence of a black hole horizon, certain field excitations that are normally discarded as “pure gauge” become physical. In 2+1 dimensions, these degrees of freedom may be counted, and given some reasonable assumptions about quantization, they yield the correct entropy. A (3+1)-dimensional version of this counting argument does not yet exist, but work is in progress.

II. Gauge-Fixing, Boundaries, and Physical Degrees of Freedom

General relativity has a large gauge group, the group of spacetime diffeomorphisms, and the number of physical degrees of freedom is correspondingly smaller than it first appears. Let us briefly review two methods for identifying the physical degrees of freedom:

1. The York splitting \[14\]: In an \((n+1)\)-dimensional spacetime, let \(g_{ij}\) denote the induced metric on an \(n\)-dimensional spacelike hypersurface \(\Sigma\). Any small fluctuation \(\delta g_{ij}\) of the metric can be decomposed as

\[
\delta g_{ij} = h_{ij}^{TT} + \delta \phi g_{ij} + (L \xi)_{ij},
\]

where

\[
(L \xi)_{ij} = \nabla_i \xi_j + \nabla_j \xi_i - \frac{1}{n} g_{ij} \nabla_k \xi^k,
\]

and the transverse traceless deformation \(h_{ij}^{TT}\) satisfies

\[
g^{ij} h_{ij}^{TT} = 0, \quad (L^i h^{TT})_i = -2 \nabla^j h_{ij}^{TT} = 0.
\]
is essentially the deformation $L_{\xi} g_{ij}$ of $g_{ij}$ induced by the infinitesimal diffeomorphism generated by the vector field $\xi^i$. The true dynamical degrees of freedom are therefore, at least infinitesimally, the transverse traceless variations $h^{ij}_{TT}$. In particular, when $n = 2$, the kernel of $L^\dagger$ is finite dimensional—it is the space of quadratic differentials $[15]$—and the physical configuration space has only finitely many degrees of freedom.

2. A constraint analysis $[16]$: The momentum constraint of canonical general relativity takes the form

$$H^i = -2\nabla_j \pi^{ij} = 0,$$

where $\pi^{ij}$ is the momentum conjugate to $g_{ij}$. The canonical generator of spatial diffeomorphisms, on the other hand, is

$$G[\xi] = \int_{\Sigma} d^n x (\nabla_i \xi_j + \nabla_j \xi_i) \pi^{ij} = \int_{\Sigma} d^n x \xi_i H^i.$$

That is, the standard Poisson brackets of the canonical variables imply that

$$\{G[\xi], g_{ij}\} = \nabla_i \xi_j + \nabla_j \xi_i = L_{\xi} g_{ij}$$

along with the corresponding expression for the momentum. Invariance under spatial diffeomorphisms thus follows from the vanishing of the constraints, and variations of the metric of the form (7) are thus nonphysical. A similar argument relates diffeomorphisms generated by timelike vector fields to the Hamiltonian constraint $H$, at least on shell.

Note, however, that both of these arguments implicitly assumed that $\Sigma$ had no boundary. In the presence of a boundary, the splitting (2) is well-defined and unique only if one chooses boundary conditions that make the operator $L^\dagger L$ self-adjoint. The simplest such choice is

$$\xi^i |_{\partial \Sigma} = 0,$$

which restricts the “pure gauge” degrees of freedom to those generated by vector fields that vanish on the boundary. Similarly, equation (9) required an integration by parts; if $\Sigma$ has a boundary, we find instead

$$G[\xi] = \int_{\Sigma} d^n x \xi_i H^i + 2 \int_{\partial \Sigma} d^{n-1} x \xi_i \pi^{i\perp}.$$  

The last term in (9) vanishes only for vector fields satisfying the boundary conditions (8). For vector fields that do not vanish on $\partial \Sigma$, the constraints no longer imply invariance, but merely relate diffeomorphisms to a new set of variables

$$O[\xi] = \int_{\partial \Sigma} d^{n-1} x \xi_i \pi^{i\perp}$$

at the boundary. Balachandran et al. have shown that these are physical variables, that is, that they commute with the constraints, thus providing a new set
of boundary observables [10, 11]. In the corresponding quantum theory, we thus expect a new set of operators associated with the boundary, and a new set of physical degrees of freedom, the “would-be pure gauge” degrees of freedom associated with vector fields that do not vanish at the boundary.

Additional evidence for new degrees of freedom associated with boundaries comes from a variety of sources. For example, Esposito et al. have shown that in the one-loop computation of the partition function for Euclidean gravity on a four-ball, the gauge and ghost contributions do not cancel, as they do for a closed manifold, but instead give a necessary contribution to the path integral [17]. Baez et al. [18] and Smolin [19] have investigated the loop variable approach to quantum gravity in the presence of boundaries, and have also found evidence for boundary degrees of freedom. Finally, in the nongravitational context it is well known that Chern-Simons theory on a manifold with boundary induces a dynamical Wess-Zumino-Witten (WZW) theory on the boundary [20, 21], whose degrees of freedom can be understood as “would-be gauge” degrees of freedom of precisely the kind described above [22, 23, 24].

III. The Horizon as a Boundary

Such “extra” boundary degrees of freedom are natural candidates for the microscopic degrees of freedom responsible for black hole thermodynamics. In particular, if we treat the horizon of a black hole as a boundary, we will obtain new states whose number might be related to the black hole’s entropy. There is an obvious objection to this viewpoint, however: the event horizon of a black hole is not, in fact, a boundary.

But while the horizon is not a true boundary of spacetime, it is a surface upon which we impose boundary conditions. Any quantum mechanical statement about black holes is necessarily a statement about conditional probabilities: for instance, “If spacetime contains an event horizon of a certain size, then we should see Hawking radiation with a certain spectrum.” To impose the condition (“spacetime contains an event horizon of a certain size”), we must fix “boundary” data, requiring the existence of a hypersurface with appropriate geometric properties—vanishing expansion of outgoing null geodesics, for example.

In a path integral approach to quantization, there is an obvious way to do this: we can split spacetime $M$ into two pieces, say $M_1$ and $M_2$, along a hypersurface $\Sigma$, and perform separate path integrals in $M_1$ and $M_2$ with suitable boundary conditions at $\Sigma$. Such a procedure has been studied extensively, particularly in the context of two-dimensional conformal field theory [13, 25, 26], as has the converse problem of “sewing,” that is, integrating over boundary data on $\Sigma$ to recover the path integral on $M$. In particular, it is known that in order to “sew” properly, the actions on $M_1$ and $M_2$ must sometimes be supplemented by boundary terms, and these boundary terms may break gauge invariance and give dynamics to “pure
gauge” degrees of freedom at the boundary.

An important example of this phenomenon is Chern-Simons theory. Consider for simplicity an abelian U(1) Chern-Simons theory, described by the action

$$I_M[A] = \frac{k}{2\pi} \int_M d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho,$$

(11)

where $M$ is a closed three-manifold. This action is invariant under gauge transformations

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda,$$

(12)

and leads to Euler-Lagrange equations

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0.$$

(13)

The space of classical solutions is thus the space of flat connections modulo gauge transformations. The corresponding quantum theory is fairly simple, and it may be shown to have a finite-dimensional Hilbert space (see, for example, [27]).

Let us now split $M$ into two pieces along a surface $\Sigma$, and consider the action (11) restricted to, say, $M_1$. On a manifold with boundary, the variation of the action gives

$$\delta I_M[A] = (\text{Euler-Lagrange equations}) - \frac{k}{2\pi} \int_\Sigma d^2x n_\rho \epsilon^{\rho\mu\nu} A_\mu \delta A_\nu,$$

(14)

and the boundary term in (14) means that there are typically no classical extrema. It is known, at least in many examples, that in order to ensure proper “sewing” of transition amplitudes, we must add boundary terms to the action in a way that guarantees that extrema exist for a sufficiently large set of boundary data. In Chern-Simons theory, the standard approach is to choose a complex structure on $\Sigma$ and to fix the boundary value of the component $A_z$, which is canonically conjugate to $A_{\bar{z}}$. The boundary term in (14) can then be cancelled by a boundary action

$$I_\Sigma[A] = \frac{k}{2\pi} \int_\Sigma d^2x A_z A_{\bar{z}}.$$

(15)

Observe now that the action

$$I_{M_1}[A] = I_{M_1}[A] + I_\Sigma[A]$$

(16)

is no longer invariant under the gauge transformations (12) unless $\Lambda$ vanishes at the boundary. This feature should look familiar: it is a gauge theoretical analog of the gravitational phenomenon we saw in the preceding section. We can make this noninvariance explicit by decomposing $A_\mu$ as

$$A_\mu = \bar{A}_\mu + \partial_\mu \Lambda,$$

(17)
where $\bar{A}_\mu$ is a gauge-fixed potential; then

$$I'_{M_1}[\bar{A}] = I'_{M_1}[\bar{A}] + \frac{k}{2\pi} \int_{\Sigma} d^2x \left( \partial_z \Lambda \partial_{\bar{z}} \Lambda + 2 \bar{A}_z \partial_{\bar{z}} \Lambda \right).$$  \hspace{1cm} (18)

The would-be gauge transformation $\Lambda$ has thus become a dynamical field on $\Sigma$, with an action that can be recognized as a chiral Wess-Zumino-Witten action. This is a dramatic result: we have gone from a Chern-Simons theory with a finite-dimensional Hilbert space to a theory that includes an infinite-dimensional Hilbert space describing boundary degrees of freedom.

An analogous process occurs in the nonabelian case. Let $A = A_\mu^a T^a dx^\mu$ denote a connection one-form for a nonabelian gauge group $G$ with generators $T^a$. Then the Chern-Simons action

$$I'_{M_1}[A] = \frac{k}{4\pi} \int_{M_1} Tr \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) + \frac{k}{4\pi} \int_{\Sigma} Tr A_z A_{\bar{z}}$$  \hspace{1cm} (19)

appropriate for fixing $A_z$ at $\Sigma$ again splits into two pieces; under the decomposition

$$A = g^{-1} dg + g^{-1} \tilde{A} g,$$  \hspace{1cm} (20)

the action becomes \[22, 23\]

$$I'_{M_1}[^\text{\bar{A}}, g] = I'_{M_1}[\bar{A}] + k I_{\text{WZW}}[^\text{\bar{A}}, A_z],$$  \hspace{1cm} (21)

where $I_{\text{WZW}}[^g, \bar{A}_z]$ is now the action of a nonabelian chiral WZW model on the boundary $\Sigma$.

$$I_{\text{WZW}}[^g, \bar{A}_z] = \frac{1}{4\pi} \int_{\Sigma} Tr \left( g^{-1} \partial_z g g^{-1} \partial_{\bar{z}} g - 2 g^{-1} \partial_z g g_{\bar{z}} \right) + \frac{1}{12\pi} \int_{M} Tr \left( g^{-1} dg \right)^3.$$  \hspace{1cm} (22)

Witten has shown that when this WZW action is included, Chern-Simons theory “sews” properly at the boundary $\Sigma$ \[23\].

IV. A (2+1)-Dimensional Model

The discussion has so far been rather general. We have seen that there are, plausibly, new degrees of freedom associated with black hole horizons; and we have seen that for a particular theory that is not gravity, similar degrees of freedom lead to interesting dynamics. The next step should be to put these two ingredients together to find the dynamics of our gravitational boundary degrees of freedom.

Unfortunately, this is not easy. The basic difficulty can be seen by comparing equations (2) and (20). For the gauge theory, the splitting of $A$ into “physical” and “gauge” degrees of freedom is a local decomposition, valid for arbitrary finite gauge transformations. For gravity, on the other hand, the decomposition has only
been written infinitesimally; because diffeomorphisms move points, the analog of (20) for finite diffeomorphisms is highly nonlocal.

In 2+1 dimensions, this difficulty can be avoided. As first shown by Achúcarro and Townsend [29], general relativity in three spacetime dimensions can be rewritten as a Chern-Simons theory, with diffeomorphisms replaced by ordinary gauge transformations. In particular, if the cosmological constant is negative—as required for the existence of a black hole [7]—then general relativity is equivalent to an SO(2, 1)×SO(2, 1) Chern-Simons theory. As is the case for other Chern-Simons theories, (2+1)-dimensional gravity therefore induces a Wess-Zumino-Witten action on spatial boundaries, and we can study the dynamics of the “would-be pure gauge” degrees of freedom in some detail.

The results of such an analysis can be summarized as follows [12]:

1. The diffeomorphisms and local Lorentz transformations of general relativity are equivalent to SO(2, 1)×SO(2, 1) gauge transformations of the Chern-Simons theory [30].

2. On a manifold with boundary, (2+1)-dimensional gravity in its Chern-Simons form induces an SO(2, 1)×SO(2, 1) WZW action on the boundary. With some reasonable assumptions about quantization, the states of this boundary theory can be written down explicitly.

3. While almost all of the diffeomorphisms of the boundary become dynamical, the theory retains a remnant of diffeomorphism invariance: the diffeomorphisms generated by Killing vectors—which are in the kernel of \( L \) and thus missing from equation (2)—generate invariances that must be respected by the physical states.

4. When this physical state condition is imposed, the number of states is finite, and can be estimated by standard number theoretical arguments. Given a fixed horizon size, the resulting number of states is the exponential of the correct Bekenstein-Hawking entropy,

\[
n(r_+) = \exp \left\{ \frac{2\pi r_+}{4\hbar G} \right\}. \tag{23}\n\]

Let me now give a few more details. (For a full description, see [12].) If we write the cosmological constant as \( \Lambda = -1/\ell^2 \), we can define two SO(2, 1) gauge fields,

\[
A^a = \omega^a + \frac{1}{\ell}e^a, \quad \tilde{A}^a = \omega^a - \frac{1}{\ell}e^a. \tag{24}\n\]

Here \( e^a = e_\mu^a dx^\mu \) is a triad, \( \omega^a = \frac{1}{2}e^{abc} \omega_{\mu bc} dx^\mu \) is a spin connection, and

\[
k = \frac{\ell \sqrt{2}}{8G}. \tag{25}\n\]
in the normalizations of reference \[12\]. The Einstein-Hilbert action of general relativity is then

\[ I_{\text{grav}} = I_{\text{CS}}[A] - I_{\text{CS}}[\tilde{A}], \]

(26)

where \( I_{\text{CS}}[A] \) is the Chern-Simons action \[19\]. As described above, this action must be supplemented with appropriate boundary terms if the horizon is treated as a boundary. The nature of these terms depends on the choice of boundary conditions; for a black hole, we can demand that \( \partial M \) be a null surface and that the expansion \( \theta^+ \) of outgoing null geodesics vanish, so that \( \partial M \) is an apparent horizon. The resulting action induces an \( \text{SO}(2,1) \times \text{SO}(2,1) \) chiral WZW theory on \( \partial M \) in a manner exactly analogous to the appearance of the abelian WZW action in \[18\].

This boundary action is completely characterized by a current algebra \[31\]

\[ [J^a_m, J^b_n] = i f^{ab}_{
\cdot\cdot\cdot} c J^c_{m+n}, \quad [\tilde{J}^a_m, \tilde{J}^b_n] = i \tilde{f}^{ab}_{
\cdot\cdot\cdot} c \tilde{J}^c_{m+n}, \]

(27)

where \( \tilde{g} \) is the Cartan-Killing metric, and the zero-modes \( J^a_0 \) and \( \tilde{J}^a_0 \) of the currents are fixed by the boundary data on \( \partial M \). Because the group is noncompact, the quantization of the \( \text{SO}(2,1) \) WZW model is not completely understood, but in the large \( k \)—i.e., small \( \Lambda \)—limit, the model may be approximated by a theory of six independent bosonic string oscillators. Such a system has an infinite number of states, which can be generated from a vacuum \(|0\rangle\) that satisfies

\[ J^a_n |0\rangle = \tilde{J}^a_n |0\rangle = 0 \quad \text{for} \ n > 0 \]

(28)

by acting with \( J^a_{-n} \) and \( \tilde{J}^a_{-n} \).

We have not yet applied the physical state condition, however. Recall that the horizon fields can be interpreted as “would-be pure gauge” excitations that become dynamical at a boundary. It is apparent from equation \( (2) \) (or equations \( (6) \) and \( (10) \)), however, that the excitations corresponding to Killing vectors remain genuine gauge degrees of freedom. We must therefore impose a remaining Wheeler-DeWitt equation, requiring that physical states be invariant under those diffeomorphisms that are generated by Killing vectors. For the \((2+1)\)-dimensional black hole background, this requirement is that

\[ \hat{L}_0 |\text{phys}\rangle = 0, \]

(29)

where \( \hat{L}_0 \) is the zero-mode of the Virasoro generator associated with the affine algebra \[27\].

Now, \( \hat{L}_0 \) has a standard expression in terms of the currents \[27\]:

\[ \hat{L}_0 = \sum_{i=1}^{6} N_i + \text{(current zero-mode pieces)}, \]

(30)

where the \( N_i \) are number operators (six because there are three components of \( J^a \) and three components of \( \tilde{J}^a \)). Imposing \( (29) \) thus fixes this sum of number
operators in terms of zero-modes, which are in turn determined by boundary data at the black hole horizon. After a bit of manipulation, we find the condition

$$
\sum_{i=1}^{6} N_i = \left( \frac{r_+}{4G} \right)^2,
$$

(31)

where $r_+$ is the horizon radius.

For large values of $r_+$, the number of states satisfying this condition can be found by a number theoretical argument that dates back to Ramanujan and Hardy \[32\]. The result is equation (23). But the factor in the exponent in (23) is precisely the correct Bekenstein-Hawking entropy for a (2+1)-dimensional black hole \[8\], so our “would-be pure gauge” degrees of freedom do, in fact, explain the entropy.

While this argument is quite convincing, it is not entirely satisfactory, due to limitations in our current understanding of WZW models for noncompact groups. Witten has suggested a more rigorous approach to the quantization of Euclidean gravity in 2+1 dimensions, which leads naturally to a connection with SU(2) WZW models \[33\]. It should be possible to investigate the entropy of the (2+1)-dimensional black hole in this context; preliminary results indicate that the expression (23) for the number of states can be reproduced.

V. Conclusion

These results for the entropy of the (2+1)-dimensional black hole are exciting, but they are also frustrating. The methods described here do not generalize to 3+1 dimensions, and while we may argue that the horizon degrees of freedom still exist, it is not clear how to count them. A useful step in this direction would be to find the appropriate symplectic structure for the “would-be pure gauge” degrees of freedom on the horizon; Epp has recently made some progress in this direction \[34\]. We must also understand the appropriate boundary terms at the horizon in the conventional metric formulation of general relativity, a problem for which recent results of Teitelboim should be relevant \[35\]. It would also be interesting to look at the abstract quantization of the group \( \text{Diff} S^2 \) of diffeomorphisms of the two-sphere, which might relate to our problem in the same way coadjoint orbit quantization of \( \text{Diff} S^1 \) relates to WZW theory \[36\].

Finally, it is interesting to note that the techniques used to count horizon states in 2+1 dimensions are remarkably similar to the methods that string theorists have recently used to determine black hole entropy \[37\]. The physical starting points seem very different, but the appearance of such similar mathematics suggests that there may be hidden connections.
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