Topological Born-Infeld charged black holes in Einsteinian cubic gravity

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In this paper, we study four-dimensional topological black hole solutions of Einsteinian cubic gravity in the presence of nonlinear Born-Infeld electrodynamics and cosmological constant. First, we obtain the field equations which govern our solutions. Employing Abbott-Deser and Gauss formulas, we present the expressions of conserved quantities total mass and total charge of our topological black holes. Next, we put the expansion of metric function near horizon in field equation and solve the latter order by order to obtain the temperature as well as total mass. Moreover, we discuss the effect of different model parameters on the behavior of temperature. We find out that, for black solutions with planar and hyperbolic topologies on horizon, the temperature may be positive just for negative cosmological constant. However, spherical solutions may have positive temperature for positive, negative and zero values of cosmological constant. We reveal that the temperature decreases as charge of black holes and nonlinear parameter $b$ increase (large $b$ reproduces linear electrodynamics) for all topology types. Whereas the temperature of planar solutions is independent of cubic coupling $\lambda$, temperature of spherical (hyperbolic) solutions decreases (increases) as $\lambda$ grows. Also, it is shown that spherical solutions have a maximum value for temperature unlike the solutions with two other topologies. Next, we turn to calculate entropy and electric potential. We show that the first law of thermodynamics is satisfied for our solutions. Finally, we explore the thermal stability of our solutions. We reveal the influence of model parameters on stability of topological Born-Infeld charged black solutions of Einsteinian cubic gravity.

I. INTRODUCTION

General relativity has passed all tests successfully. The latest one was the detection of gravitational wave [1], almost one hundred years after Einstein predicted it. Despite these achievements, at situations where the spacetime is extremely curved, it is unavoidable to modify general relativity. The most natural modification is taking into account the higher-order curvature terms. The well-known higher-order Lovelock terms provide this kind of modification while they respect the constraints of early version of general relativity [2, 3]. However, the latter has no contribution in four dimensions. Recently, a cubic order curvature model with contribution in four dimensions called Einsteinian cubic gravity (ECG) has been proposed [4]. This model could attract many researcher’s attention very soon [5–19]. ECG respects the constraints that Lovelock theories do i.e. on a maximally symmetric background, it just propagates a transverse and massless graviton and in all dimensions it has the same relative coefficients of the different curvature invariants involved. Possessing the contribution in four dimensions along with other features have made this model intriguing and important. The latter property makes us able to see the effects of higher-order curvature modifications on $(2 + 1)$-dimensional holographic duals of gravity theory solutions.

The solutions in the context of ECG have been explored from different points of view. In [5], by constructing perturbative five-dimensional black hole solutions of ECG, the holographic entanglement Renyi entropy has been computed in the dual field theory. The first examples of black hole solutions in ECG have been obtained in [6], and thermal behaviors of them have been explored. In [7], the static and spherically symmetric generalizations of four-dimensional linearly charged and uncharged black hole solutions in ECG have been constructed and thermodynamics of them has been studied. The most general theory of gravity to cubic order in curvature called Generalized Quasi-Topological Gravity (GQTG) whose static spherically symmetric vacuum solutions are fully described by a single field equation has been constructed in [8]. In latter, the ECG as well as Lovelock and quasi-topological gravities have been recovered in four dimensions as special cases. General results corresponding to static and spherically symmetric black hole solutions of general higher-derivative gravities including GQTG have also been established [9]. It has been proved as well that the four-dimensional black hole solutions corresponding to an infinite family of ghost-free higher-order theories are universally stable below a certain mass [10]. In [11], by employing the continued fraction approximation, some interesting properties of ECG black hole solutions such as the innermost stable circular orbit of massive test bodies near a black hole and the shadow of an black hole have been computed. Some properties of a nonsupersymmetric conformal field theory in three dimensions which could be a holographic dual to four-dimensional ECG model has been explored in [12]. Euclidean AdS-Taub-NUT and bolt solutions with various base spaces in four- and six-dimensions constructed respectively in the context of ECG and GQTG have been studied as well and thermodynamics features of them have been explored [13].

In this paper, we study the four-dimensional topolog-
ical black hole solutions of ECG in the presence of nonlinear Born-Infeld (BI) electrodynamics and cosmological constant. To our knowledge, this is the first consideration of nonlinearly charged topological solutions in ECG. The importance of considering nonlinear BI electrodynamics is at least two fold. On the one hand, it resolves the singularity problem of Maxwell electrodynamics at the place of point charge [20]. On the other hand, it comes from the low energy limit of open superstring theory [21–23]. In addition, photon-photon interaction experiments have suggested that there is a nonlinear theory of electrodynamics in vacuum [24]. Black holes with different horizon’s topologies show drastically different thermodynamical properties as well. For instance, whereas Schwarzschild black holes with spherical horizon are not thermally stable, it has been argued that Schwarzschild-AdS black holes with planar or hyperbolic topologies at horizon are thermally stable and do not underlie Hawking-Page phase transition [25]. From holographic point of view, topological solutions are described as duals to thermal states of conformal field theories as well [26].

The map of this paper is as following: In next section, we will introduce the action of theory and find field equations as well as conserved quantities. In section III, we will calculate thermodynamical quantities and check the satisfaction of thermodynamics first law. We will explore thermal stability of our solutions in section IV. Last section is devoted to summary and concluding remarks.

II. ACTION, FIELD EQUATIONS AND CONSERVED QUANTITIES

The Einsteinian cubic gravity (ECG) which is the most general dimension-independent gravity theory that consists of metric and Riemann tensor contractions for which linearized spectrum coincides with Einstein gravity one, in the presence of nonlinear electrodynamics could be written as [4]

$$S = -\frac{1}{16\pi} \int_M d^4x\sqrt{-g} \left( \sum_{i=1}^{3} \alpha_i L_i - 2\Lambda - \lambda\mathcal{P} + \mathcal{L}(F) \right),$$

up to cubic order in curvature. We write (1) in Planck units where we set $G = c = 1$. In above action, $\Lambda$ is cosmological constant which may be positive, zero or negative. Negative cosmological constant is usually set to $-3/l^2$ in four-dimensions where $l$ is the AdS radius. In action (1), $\alpha_i$’s and $\lambda$ are also some dimensionless constants. We will assume cubic coupling $\lambda \geq 0$ and fix $l$ to unity throughout this paper. Also, $L_i$’s stand for r-th order Lovelock terms where $L_1$ is Ricci scalar $R$ and $\alpha_1$ could be set to unity [2, 3]. The additional cubic contribution $\mathcal{P}$ is defined as [4]

$$\mathcal{P} = 12 R^e_{\ ab} R^d_{\ ef} f^e_{\ ad} f^b_{\ c} + R^e_{\ ab} R^d_{\ cf} f^e_{\ ad} f^b_{\ cf} - 12 R^a_{\ cde} R^d_{\ bef} R^e_{\ ab} + 8 R^b_{\ c} R^d_{\ c} R^e_{\ ab} f^e_{\ ad} f^b_{\ c}.$$

In this paper, we intend to consider Born-Infeld nonlinear electrodynamics for which

$$\mathcal{L}(F) = b^2 \left( 1 - \sqrt{1 + \frac{F}{b^2}} \right),$$

where $b$ is nonlinear parameter and $F = F_{ab} F^{ab}$ in which $F_{ab} = 2 \partial_a A_b$ and $A_a$ is the electromagnetic potential. As $b$ tends to infinity, $\mathcal{L}$ reduces to the linear Maxwell case i.e. $-F/4$. Whereas the second- and third-order Lovelock terms are respectively topological and trivial in four-dimensions, the new cubic term $\mathcal{P}$ has contribution in field equations [7].

We make the following ansatz for four-dimensional nonlinearly charged black solutions

$$ds^2 = -N^2(r) f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_k^2,$$

$$A = h(r) dt,$$

where

$$d\Omega_k^2 = \begin{cases} d\theta^2 + \sin^2(\theta) d\phi^2 & k = 1 \\ d\theta^2 + d\phi^2 & k = 0 \\ d\theta^2 + \sin^2(\theta) d\phi^2 & k = -1 \end{cases},$$

represents a 2-dimensional hypersurface with constant curvature $2k$ and area $A_k$. $k = 1, 0$ and $-1$ represent two-sphere $S^2$, plane $\mathbb{R}^2$ and hyperbolic $H^2$ topologies for the event horizon, respectively. Varying the action (1) with respect to $N(r)$, $f(r)$ and $h(r)$, we are left with three field equations. One of these field equations which arises from variation with respect to $f(r)$, is satisfied by $N(r) = const$. Then, we have two field equations

$$0 = -4\mathcal{H} - 2r^2 (k - \lambda r^2 - rf' - f) - \frac{12\lambda}{r} (r^3 f f'' + 2kr^2 f f''' - 4kr f f'') + r^3 f f' f'' + 4kr f f' + 4rf f'' - kr f'^2 - 4f^2 f' - 2rf^2 (rf'' - 2f''),$$

where $\mathcal{H} = b^2 - b^2 (1 - h^2/b^2)^{-1/2}$ which is reduced to $-h^2/2$ for Maxwell case. Note that prime denotes the derivative with respect to $r$. One could immediately solve the electrodynamic field equation (8) as

$$h(r) = -\frac{q}{r} \mathbf{F} \left( \frac{1}{2}, \frac{1}{4}, \frac{5}{4}, -\frac{q^2}{b^2r^4} \right),$$

where $\mathbf{F}(x, y, z, w)$ is the hypergeometric function and $q$ is an integration constant related to total electric charge of black hole. Expanding $h(r)$ about infinity $b$, the potential of Maxwell electrodynamics can be reproduced as $h(r) = -q/r + O(b^{-1})$. Substituting $h(r)$ from Eq. (9) to Einstein field equation (7) and then dividing expressions
by $2r^2$, one can easily integrate Einstein field equation once. After some manipulation, the result could be simplified as follows

$$
k r - m - \frac{r^3}{3} - rf + \frac{1}{6} b^2 r^3
\times \left[ 1 - F \left( -\frac{3}{4} - \frac{1}{2} - \frac{2}{4} - \frac{q^2}{b^2 r^2} \right) \right]
+ \frac{\lambda}{r^2} \left[ 6rf f'' (2k + r f' - 2f) - 2rf f'^2 (3k + rf') - 12f'f(k - f) \right] = 0, \tag{10}
$$

where $m$ is an integration constant related to the total mass of black hole.

Using (9), it is easy to show that $F_{rt} = h'(r)$ is

$$
F_{rt} = \frac{q}{r^2 \sqrt{1 + q^2/b^2 r^4}}. \tag{11}
$$

Then, one can calculate the total charge of solutions by using the Gauss law

$$
Q = \frac{1}{4\pi} \int r^2 L_{F} F_{\mu \nu} n^\mu u^\nu d\Omega_k, \tag{12}
$$

where $L_F = \partial L / \partial F$ and $n^\mu$ and $u^\nu$ are the unit spacelike and timelike normals to the hypersurface of radius $r$ given as $n^\mu = (\sqrt{-g_{tt}})^{-1} \, dt = (\sqrt{f(r)})^{-1} \, dt$ and $u^\nu = (\sqrt{g_{rr}})^{-1} \, dr = \sqrt{f(r)} \, dr$. Using Eqs. (11) and (12), we obtain the total charge of black hole as

$$
Q = \frac{q A_k}{16 \pi}. \tag{13}
$$

Moreover, according to Abbott and Deser definition [27], the total mass of black hole reads [28]

$$
M = \frac{m A_k}{8 \pi}. \tag{14}
$$

In next section, we turn to calculate thermodynamical quantities in order to check the thermodynamics first law.

III. THERMODYNAMICAL QUANTITIES AND THERMODYNAMICS FIRST LAW

In this section, we intend to calculate thermodynamics quantities and check the first law. For these purpose, we first have to compute field equations near the black hole horizon. Taylor expansion of metric function $f(r)$ near the black hole horizon $r_h$ is

$$
f(r) = \sum_{n=0}^{\infty} a_n (r - r_h)^n, \tag{15}
$$

in which $a_n = f^{(n)}(r_h)/n!$. Note that $a_0 = f(r_h) = 0$ and $a_1 = f'(r_h) = 2\kappa_g$, where $\kappa_g$ is surface gravity on the horizon and $f'(r_h) \geq 0$. Plugging above expansion into field equation (10), one receives following equation up to quadratic order of $r - r_h$:

$$
k r_h - m - 8\lambda \kappa_g^2 (2\kappa_g + \frac{3k}{r_h}) - \frac{\Lambda}{3} r_h^3
+ \frac{1}{6} r_h^3 b^2 \left[ 1 - F \left( -\frac{3}{4} - \frac{1}{2} - \frac{2}{4} - \frac{q^2}{b^2 r_h^2} \right) \right] + k - \Lambda r_h^2
- 2\kappa_g r_h - 24\lambda \kappa_g^2 + \frac{1}{2} r_h^2 b^2 \left( 1 - \sqrt{1 + \frac{q^2}{b^2 r_h^2}} \right) (r - r_h)
+ \left[ \frac{72 a_3 \lambda \kappa_g (k_0 + \frac{k}{r_h}) + 24 a_2 \lambda \kappa_g - a_2 r_h - 2\kappa_g - \Lambda r_h}{r_h} \right]
- \frac{24 a_2 \lambda \kappa_g}{r_h} (4 \kappa_g + \frac{3k}{r_h}) + \frac{1}{2} r_h b^2 \left[ 1 - \left( 1 + \frac{q^2}{b^2 r_h^2} \right)^{-\frac{1}{2}} \right]
+ \frac{72 \lambda \kappa_g^2}{r_h} k + \frac{96 \lambda \kappa_g^3}{r_h^2} (r - r_h)^2 + O((r - r_h)^3) = 0. \tag{16}
$$

Solving above equation order by order in terms of $r - r_h$ powers, up to second term, we get:

$$
k r_h - m - 8\lambda \kappa_g^2 \left( 2\kappa_g + \frac{3k}{r_h} \right) - \frac{\Lambda}{3} r_h^3
+ \frac{1}{6} r_h^3 b^2 \left[ 1 - F \left( -\frac{3}{4} - \frac{1}{2} - \frac{2}{4} - \frac{q^2}{b^2 r_h^2} \right) \right] = 0, \tag{17}
$$

$$
k + \frac{1}{2} r_h b^2 \left[ 1 - \sqrt{1 + \frac{q^2}{b^2 r_h^2}} \right] - \Lambda r_h^2
- 2\kappa_g r_h - 24\lambda \kappa_g^2 - \frac{k}{r_h} = 0. \tag{18}
$$

As expected, these relations reproduce the following RN-(A)dS results if $b \to \infty$ [7]

$$
k r_h - m - 8\lambda \kappa_g^2 \left( 2\kappa_g + \frac{3k}{r_h} \right) - \frac{\Lambda}{3} r_h^3 + \frac{q^2}{4r_h^2} = 0, \tag{19}
$$

$$
k - \frac{q^2}{4r_h^2} - \Lambda r_h^2 - 2\kappa_g r_h - 24\lambda \kappa_g^2 k = 0. \tag{20}
$$

It is notable to mention that since the third term in (16) is linear in terms of $a_3$, we can compute the latter in terms of $a_2$. Also, other higher order terms are linear with respect to other coefficients and therefore, all of them could finally be determined in terms of $a_2$. Consequently, the family of solutions have only one free parameter, $a_2$. This fact shows that the present model allows black solutions which have regular horizons where the regularity condition reduces the number of solutions from a two-parameter family to a one-parameter one.

From (17) and (18), we could determine $m$ and surface
gravity $\kappa_g$ as functions of model parameters:

$$\kappa_g = \frac{r_h^3}{24\lambda k} \left[ 1 + \frac{12k\lambda b^2}{r_h^4} \left( 1 - \sqrt{1 + \frac{q^2}{b^2r_h^2}} \right) \right]$$

$$- \frac{24k\lambda}{r_h^2} + \frac{24k^2\lambda^2}{r_h^4} - \frac{r_h^3}{24\lambda k}.$$ (21)

$$\frac{m}{r_h} = k - \frac{24\lambda \kappa_g^2}{r_h^2} - \frac{\Lambda r_h^2}{3} - \frac{16\lambda \kappa_g^3}{r_h^4} + \frac{b^2r_h^2}{6} \left[ 1 - F \left( -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, -\frac{q^2}{b^2r_h^2} \right) \right].$$ (22)

It is notable to point out that $m$ is related to total mass $M$ via Eq. (14). If $\lambda$ and $b$ tends to zero and infinity, respectively, the above equations reduce to RN-(A)ds black hole ones:

$$\kappa_g = \frac{k}{2r_h} - \frac{q^2}{8r_h^3} - \frac{\Lambda r_h}{2},$$

$$\frac{m}{r_h} = k - \frac{\Lambda r_h^2}{3} + \frac{q^2}{4r_h^2}.$$ (22)

The Hawking temperature of our solution can be written in terms of surface gravity [29]

$$T = \frac{\kappa_g}{2\pi},$$ (23)

where $\kappa_g$ has been introduced in Eq. (21). Using (21)
and (23), one could find the temperature as

\[
T = \frac{r_h^3}{48\pi\lambda k} \left[ 1 + \frac{12k\lambda b^2}{r_h^3} \left( 1 - \sqrt{1 + \frac{q^2}{b^2r_h^4}} \right) \right] - \frac{24k\lambda}{r_h^3} + \frac{24k^2\lambda}{r_h^3} \frac{1}{2} - \frac{r_h^3}{48\pi\lambda k}. \quad (24)
\]

Since the temperature \( T \) needs to be positive or zero to be physically reliable, we have to impose some constraints on the physical quantities and parameters. Eq. (24) shows that \( T \) is not equal or greater than zero for all values of \( k \) and \( \Lambda \). We will seek for conditions for which \( T \) may be positive or zero. For \( k = 0 \), the temperature reads as

\[
T (k = 0) = -\frac{r_h}{\pi l} \left[ \frac{b^2}{8} \left( \sqrt{1 + \frac{q^2}{b^2r_h^4}} - 1 \right) + \frac{\Lambda}{4} \right]. \quad (25)
\]

It is obvious that planar Born-Infeld charged black solutions \((k = 0)\) have \( T \geq 0 \) just for \( \Lambda < 0 \) i.e. AdS case. Also, Eq. (25) shows that temperature does not depend on \( \lambda \) for this kind of black holes. For \( k = 1 \), the temperature (24) may be non-negative, for positive, zero and negative cosmological constant. In the case of \( k = -1 \), for positive values of \( \lambda \) (as we have assumed before), (24) shows that the temperature may be positive just for AdS case i.e. \( \Lambda < 0 \).

Now, we are going to disclose the effects of each model parameter \( \lambda, b \) and \( q \) on temperature. From Eq. (24), one can see that for \( k = 1 \) \((k = -1)\), the first term is positive (negative) while the second term is negative\(^2\) (positive). Increasing parameters \( b \) and \( q \) enhances the magnitude of following term in Eq. (24).

\[
12k\lambda b^2 \left( 1 - \sqrt{1 + \frac{q^2}{b^2r_h^4}} \right).
\]

The latter term is negative (positive) for \( k = 1 \) \((k = -1)\) and appears in first term which is positive (negative). So, as \( b \) or \( q \) enhances, \( T \) lowers for both \( k = 1 \) and \(-1\) cases. As well, the same behavior occurs for \( k = 0 \) (Eq. (25)). Note that in latter case \( \Lambda \) is negative. In order to exhibit these effects, we have plotted \( T \) versus \( r_h \) for different values of \( b \) and \( q \) in Fig. 1 for the case \( k = 1 \) and \( \Lambda = 0 \), as an example. For \( k = 0 \), the temperature is independent of \( \lambda \) as one can see in Eq. (25). For \( k = 1 \) and \(-1\), the influence of \( \lambda \) on temperature has been represented in Fig. 2. Fig. 2(a) (2(b)) shows that growing \( \lambda \) makes the temperature value lower (higher) for \( k = 1 \) \((k = -1)\). It means that as influences of cubic terms grow, the temperature of spherical (hyperbolic) black holes decreases (increases). For large size black solutions, the temperature reads as

\[
T (r_h \to \infty) = -\frac{\Lambda r_h}{4\pi} + O(r_h^{-1}),
\]

\(^3\) Note that we have assumed \( \lambda \) to be positive

| \( \Lambda \) | \( \lambda \) | \( b \) | \( q \) | \( T_{\text{max}} \) |
|---|---|---|---|---|
| 0 | 0.01 | 0.2 | 0.5 | 0.10 |
| 0.05 | 0.06 |
| +3 | 0.01 | 0.2 | 0.5 | 0.05 |
| 0.05 | 0.03 |
| 0 | 0.01 | 0.50 | 1 | 0.08 |
| 0.95 | 0.07 |
| +3 | 0.01 | 0.50 | 1 | 0.05 |
| 0.95 | 0.04 |
| 0 | 0.01 | 0.6 | 1.2 | 0.07 |
| 1.6 | 0.06 |
| +3 | 0.01 | 0.6 | 1.2 | 0.046 |
| 1.6 | 0.038 |

| Table I: The behavior of maximum temperature \( T_{\text{max}} \) for various values of model parameters. Note that \( T_{\text{max}} \) exists just for \( k = 1 \) with positive and zero \( \Lambda \). |
has no effect in our calculations so far. However, it has contribution in entropy. Using the metric (4) with \( N = 1 \), one determines
\[
S = \frac{A_k}{4} \left( 1 - 24\frac{\kappa_2^2}{r_h^2} \left( \frac{2k}{\kappa_3 r_h} + 1 \right) + \frac{4k\alpha_2}{r_h^2} \right), \tag{28}
\]
in which \( \kappa_3 \) could be replaced by (21). It is remarkable to mention that, above relation reduces to Bekenstein–Hawking area law if higher-order terms disappear (\( \lambda = \alpha_2 = 0 \)).

Now, we turn to calculate the electric potential. The electric potential \( U \), measured at infinity with respect to horizon is defined by
\[
U = A_\mu \chi^\mu \big|_{r \rightarrow \infty} - A_\mu \chi^\mu \big|_{r = r_h}, \tag{29}
\]
where \( \chi = \partial_t \) is the null generator of the horizon. Using (9) and (29), one finds
\[
U = \frac{q}{r_h} F \left( \frac{1}{2} - \frac{1}{4} - \frac{5}{4} - \frac{q^2}{b^2 r_h^2} \right). \tag{30}
\]

In order to check the first law of thermodynamics, we first write a Smarr-type formula. Using (13), (14) and (22), the Smarr-type formula \( M(r_h, Q) \) can be written as
\[
M(r_h, Q) = \frac{A_k}{8\pi} \left[ \frac{4k\alpha_2}{r_h} (6k + 4\kappa_3 r_h) \right. \\
+ \frac{1}{6} b^2 r_h^3 \left( 1 - \left( -\frac{3}{4} - \frac{1}{2} - \frac{1}{4} - \frac{(16\pi Q)^2}{b^2 A_k^2 r_h^4} \right) \right) \right]. \tag{31}
\]

According to (28), \( r_h = r_h(S, Q) \) and in general \( M = M(S, Q) \). We can then consider \( S \) and \( Q \) as a complete set of extensive quantities for mass. Therefore, temperature \( T \) and electric potential \( U \) are defined as conjugate intensive quantities for \( S \) and \( Q \), respectively. So, the first law of thermodynamics could be written as:
\[
dM = TdS + UdQ. \tag{32}
\]

In order to check the satisfaction of thermodynamics first law, we use
\[
dM = \left( \frac{\partial M}{\partial r_h} \right)_Q dr_h + \left( \frac{\partial M}{\partial Q} \right)_{r_h} dQ, \\
dS = \left( \frac{\partial S}{\partial r_h} \right)_Q dr_h + \left( \frac{\partial S}{\partial Q} \right)_{r_h} dQ. \tag{33}
\]

Using Eqs. (13), (21), (23), (28), (30), (31) and (33), our calculations show that (32) is satisfied\(^2\). Therefore, the first law of thermodynamics is satisfied for our topological black solutions.

In next section, we will study the thermal stability of our solutions in canonical ensemble.

IV. THERMAL STABILITY OF SOLUTIONS

In this section, we intend to study thermal stability of the four-dimensional topological ECBI black solutions in canonical ensemble. In this ensemble where the charge is a fixed parameter, the positivity of heat capacity \( C = T/(\partial^2 M/\partial S^2)_Q \) guarantees the local stability [31]. Therefore, it is sufficient to check that \( (\partial^2 M/\partial S^2)_Q \) is positive in order to explore the stability of solutions in the ranges where temperature is positive as well. Since in (31), we calculate \( M(r_h, Q) \), in order to obtain \( (\partial^2 M/\partial S^2)_Q \), we have to use the chain rule as below
\[
\left( \frac{\partial^2 M}{\partial S^2} \right)_Q = \frac{\partial M}{\partial r_h} \frac{\partial^2 r_h}{\partial S^2} + \left( \frac{\partial S}{\partial r_h} \right)^{-2} \frac{\partial^2 M}{\partial r_h^2}. \tag{34}
\]

where the first term in the right hand side of above equation may be computed using the following equation
\[
\frac{\partial^2 r_h}{\partial S^2} = \left( \frac{\partial r_h}{\partial S} \right) \frac{\partial^2 r_h}{\partial S^2} \frac{\partial (\partial S)}{\partial r_h} = \left( \frac{\partial S}{\partial r_h} \right)^{-1} \frac{\partial (\partial S)}{\partial r_h} \frac{1}{\partial r_h}. \tag{35}
\]

To obtain (34), one could also use Eqs. (21), (28) and (31). Since the explicit form of \( (\partial^2 M/\partial S^2)_Q \) is complicated, we avoid writing it here. However, the main results are depicted in Figs. 3 and 4 for \( k = 1 \), Fig. 5 for \( k = 0 \) and Fig. 6 for \( k = -1 \). Note that in latter figures, we plot \( C = A_k (\partial^2 M/\partial S^2)_Q \). The black solutions are thermally stable provided \( C > 0 \) in the regions where \( T > 0 \) as well. Note that temperature is positive for all values of parameters for which Figs. 3-6 are plotted. As we discussed before, temperature may be positive for \( i \) \( k = +1 : \Lambda = +, 0, - \), \( ii \) \( k = 0 : \Lambda < 0 \) and \( iii \) \( k = -1 : \Lambda < 0 \). For other values of \( k \) and \( \Lambda \), temperature is always negative. As an example, the behavior of temperature \( T \) for a specific set of parameters related to Fig. 3(a) is shown in Fig. 7. One could see that the behavior of temperature in terms of \( \lambda \) and \( b \) coincides with what discussed before i.e. temperature decreases as latter parameters increase. For the sake of brevity, we avoid exhibiting temperature figures for all sets of parameters corresponding to Figs. 3-6. In the following, we will study the stability for \( k = +1, 0 \) and \(-1 \) cases separately:

- \( k = +1 \): For this case, temperature may be positive for positive, zero and negative values of \( \Lambda \). For \( \Lambda = 0 \) and \(-3 \), we exhibit the behavior of \( C \) with respect to \( \lambda \) and \( b \) in Fig. 3. For \( \Lambda > 0 \), the behavior of \( C \) is shown in Fig. 4. Fig. 3(a) shows that, for

\(^2\) Note that \( A_k \propto r_h^2 \) and therefore the last term in entropy (28) would be a constant proportional to \( \alpha_2 \). Hence, this term is dropped when one calculates the derivatives of entropy.
Λ = 0, increasing λ would support the thermal stability of the solution for some fixed values of b. On the other hand, by increasing b for some ranges of λ, the system becomes unstable and as we enhance b further, it reaches the stability again. For Λ = −3 (Fig. 3(b)), it could be seen that increasing both parameters λ and b leads to stability of the solution. To study the stability for Λ = +3 case, one could see Fig. 4. For clearness, we plot positive and negative C cases separately. Figs. 4(a) and 4(b) show that increasing λ could leads the system to unstable phase for some ranges of b (4(a)) whereas for some other ranges, this increment may weaken the unstability. However, for some fixed values of λ, the system goes from stable phase (Fig. 4(a)) to unstable one (Fig. 4(b)) as b increases.

• k = 0: In this case, we may have positive temperature for Λ = −3. The behavior of C is plotted in Fig. 5 for this case. The behavior is splitted to positive and negative C cases to be more clear. Here, increasing λ supports the stability and weaken the unstability. As positive C panel shows, increase of λ makes C greater. Furthermore, for negative C values, the unstability is weakened as λ enhances. The effect of b is different. Whereas, for positive C values, the enhancement of b supports the stability, C becomes more negative as b increases for negative C values.

• k = −1: Here, we may choose Λ = −3 in order to have physically reliable temperature (T > 0). We could study the stability from Fig. 6 in this case. The latter figure shows that, λ may have different effects on stability. For some ranges of b where it is smaller (the system is more nonlinear), growing λ may decrease positive C. However, for some other ranges where b is greater, the enhancement
V. SUMMARY AND CONCLUDING REMARKS

In present paper, we studied four-dimensional topological Born-Infeld (BI) charged black hole solutions in the context of Einsteinian cubic gravity (ECG) in the presence of cosmological constant $\Lambda$. ECG is the most general gravity up to cubic order in curvature which is independent of dimension and its linearized spectrum coincides with general relativity one. Also, both theoretical (open superstring theory) and experimental (photon-photon interaction experiments) evidences suggest nonlinear theories of electrodynamics such as BI model. Topological solutions are known as duals to conformal field theories thermal states as well and could show extremely different thermodynamical features. To our knowledge, this is the first study of topological solutions in Einsteinian cubic gravity with nonlinear electrodynamics.

First, we introduced the action of theory and obtained field equations. Integrating field equations, we received the electromagnetic potential as well as a second order differential equation for metric function. Then, we presented the total mass and total charge expressions by using Abbott-Deser and Gauss formulas, respectively. Putting the expansion of metric function near horizon in second order field equation, we received the temperature ($T$) and total mass ($M$) equations by solving the latter order by order. We showed that the temperature is physically reliable ($T \geq 0$) just for some special cases. While it may be positive for negative, zero and positive values of cosmological constant for spherical solutions ($k = 1$), for

in $\lambda$ may make the system stable. Furthermore, increasing $b$ weakens thermal stability.
solutions with hyperbolic or planar topologies on horizon, the temperature may be positive just for negative cosmological constant. Furthermore, we revealed the influence of cubic coupling \( \lambda \), Born-Infeld parameter \( b \) and black hole charge on the behavior of temperature. We found out that the temperature decreases as both nonlinear parameter \( b \) and \( q \) (related to total charge) enhance. This occurs for all types of topologies. It is remarkable to mention that, linear Maxwell electrodynamics is reproduced as \( b \) goes to infinity. Also, increasing \( \lambda \) makes the temperature of spherical (hyperbolic) solutions lower (higher) while temperature of planar solutions is independent of cubic coupling \( \lambda \). Furthermore, for all topologies, the temperature of black solutions tends to \(-\Delta r_h/4\pi\) for large values of horizon radius \( r_h \), i.e., large size black holes. In addition, spherical solutions have a maximum value for temperature whereas hyperbolic and planar solutions do not have this upper bound. This maximum temperature lowers for greater values of \( \lambda \), \( b \) and \( q \). In order to check the satisfaction of thermodynamics first law, we turned to compute entropy and electric potential. We used the Wald formula in order to obtain the entropy of our topologically nonlinearly charged solutions. We showed that the thermodynamics first law is satisfied for our black solutions. Finally, we investigated the thermal stability of our solutions. In order to reveal the effects of model parameters on local thermal stability of solutions, we studied the behavior of \((\partial^2 M/\partial S^2)_Q\) in the ranges for which temperature is positive. For planar solutions \((k = 0)\), when the temperature is positive just in the presence of negative cosmological constant, the enhancement of cubic coupling \( \lambda \) supports stability. This kind of behavior is also seen for spherical solutions \((k = 1)\) with zero cosmological constant. Moreover, increase of nonlinear parameter \( b \) weakens the stability for hyperbolic solutions \((k = -1)\) for which \( T \geq 0 \) provided \( \Lambda < 0 \). This also occurs for spherical solutions with \( \Lambda > 0 \). Spherical solutions with \( \Lambda < 0 \) may lead to stability by increasing both cubic coupling \( \lambda \) and nonlinear parameter \( b \).

In this work, we considered Born-Infeld model as nonlinear electrodynamics. It is interesting to explore the effects of other known nonlinear models on these kinds of solutions. In addition, it is worthwhile to explore the critical behavior of nonlinearly charged solutions of Einsteinian cubic gravity in both extended [32] and non-extended [33] thermodynamics phase spaces.

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[1] B. P. Abbott, et al. (LIGO Scientific Collaboration, Virgo Collaboration), Observation of gravitational waves from a binary black hole merger, Phys. Rev. Lett. 116, 061102 (2016), [arXiv:1602.03837]
[2] D. Lovelock, Divergence-free tensorial concomitants, Aequationes mathematicae 4, 127 (1970).
[3] D. Lovelock, The Einstein Tensor and Its Generalizations, J. Math. Phys. (N.Y.) 12, 498 (1971).
[4] P. Bueno and P. A. Cano, Einsteinian cubic gravity, Phys. Rev. D 94, 104005 (2016), [arXiv:1607.06463]
[5] A. Dey, P. Roy, and T. Sarkar, On holographic Renyi entropy in some modified theories of gravity, J. High Energy Physics 04, 098 (2018), [arXiv:1609.02290]
[6] R. A. Hennigar and R. B. Mann, Black holes in Einsteinian cubic gravity, Phys. Rev. D 95, 064055 (2017), [arXiv:1610.06675]
[7] P. Bueno and P. A. Cano, Four-dimensional black holes in Einsteinian cubic gravity, Phys. Rev. D 94, 124051 (2016), [arXiv:1610.08019]
[8] R. A. Hennigar, D. Kubiznak, and R. B. Mann, Generalized quasi-topological gravity, Phys. Rev. D 95, 104042 (2017), [arXiv:1703.01631]
[9] P. Bueno and P. A. Cano, On black holes in higher-derivative gravities, Class. Quant. Grav. 34, 175008 (2017), [arXiv:1703.04625]
[10] P. Bueno and P. A. Cano, Universal black hole stability in four dimensions, Phys. Rev. D 96, 024034 (2017), [arXiv:1704.02967]
[11] R. A. Hennigar, M. B. J. Poshteh, and R. B. Mann, Shadows, Signals, and Stability in Einsteinian Cubic Gravity, Phys. Rev. D 97, 064041 (2018), [arXiv:1801.03223]
[12] P. Bueno, P. A. Cano, and A. Ruiperez, Holographic studies of Einsteinian cubic gravity, J. High Energy Physics 03, 150 (2018), [arXiv:1802.00018]
[13] P. Bueno, P. A. Cano, R. A. Hennigar, and R. B. Mann, NUTs and bolts beyond Lovelock, J. High Energy Physics 10, 095 (2018), [arXiv:1808.01671]
[14] Y.-Z. Li, Holographic Studies of The Generic Massless Cubic Gravities, Phys. Rev. D 99, 066014 (2019), [arXiv:1901.03349]
[15] M. R. Mehdizadeh and A. H. Ziaie, Traversable wormholes in Einsteinian cubic gravity, Modern Physics Letters A, 2050017 (2019), [arXiv:1903.10907]
[16] P. Bueno, P. A. Cano, and R. A. Hennigar, (Generalized) quasi-topological gravities at all orders, Class. Quant. Grav. 37, 015002 (2020), [arXiv:1909.07983]
[17] P. A. Cano and D. Pereñiguez, Extremal Rotating Black Holes in Einsteinian Cubic Gravity, Phys. Rev. D 101, 044016 (2020), [arXiv:1910.10721]
[18] D. J. Burger, W. T. Emond, and N. Moynihan, Rotating Black Holes in Cubic Gravity, [arXiv:1910.11618].
[19] A. M. Frassino and J. V. Rocha, Charged black holes in Einsteinian cubic gravity and non-uniqueness, [arXiv:2002.04071].
[20] M. Born and L. Infeld, Foundations of the new field theory, Proc. R. Soc. A 144, 425 (1934).
[21] E. S. Fradkin and A. Tseytlin, Effective field theory from quantized string, Phys. Lett. B 163, 123 (1985).
[22] R. R. Metsaev, M. A. Rakhmanov and A. A. Tseytlin, The Born-Infeld action as the effective action in the open superstring theory, Phys. Lett. B 193, 207 (1987).
[23] E. Bergshoeff, E. Sezgin, C. Pope and P. Townsend, The Born-Infeld action from conformal invariance of the open superstring, Phys. Lett. B 188, 70 (1987).
[24] D. L. Burke et al., Positron Production in Multiphoton Light-by-Light Scattering, Phys. Rev. Lett. 79, 1626 (1997);
C. Bamber et al., Studies of nonlinear QED in collisions of 46.6-GeV electrons with intense laser pulses, Phys. Rev. D 60, 092004 (1999);
D. Tommasini, A. Ferrando, H. Michinel and M. Seco, Detecting photon-photon scattering in vacuum at exawatt lasers, Phys. Rev. A 77, 042101 (2008), [arXiv:0802.0101];
D. Tommasini, A. Ferrando, H. Michinel and M. Seco, Precision tests of QED and non-standard models by searching photon-photon scattering in vacuum with high power lasers, J. High Energy Phys. 0911, 043 (2009), [arXiv:0909.4663];
O. J. Pike, F. Mackenroth, E. G. Hill and S. J. Rose, A photon–photon collider in a vacuum hohlraum, Nature Photonics 8, 434 (2014).
[25] D. Birmingham, Topological Black Holes in Anti-de Sitter Space, Class. Quant. Grav. 16, 1197 (1999), [hep-th/9808032].
[26] R. Emparan, AdS/CFT Duals of Topological Black Holes and the Entropy of Zero-Energy States, J. High Energy Physics 9906, 036 (1999), [hep-th/9906040].
[27] L. F. Abbott and S. Deser, Stability of gravity with a cosmological constant, Nucl. Phys. B 195, 76 (1982).
[28] T. K. Dey, Born-Infeld black holes in the presence of a cosmological constant, Phys. Lett. B 595, 484 (2004), [hep-th/0406169];
R. G. Cai, D. W. Pang and A. Wang, Born-Infeld Black Holes in (A)dS Spaces, Phys. Rev. D 70, 124034 (2004), [hep-th/0410158];
D. C. Zou, Z. Y. Yang, R. H. Yue and P. Li, Thermodynamics of Gauss-Bonnet-Born-Infeld black holes in AdS space, Mod. Phys. Lett. A 26, 515 (2011), [arXiv:1011.3184];
P. Li, R. H. Yue and D. C. Zou, Thermodynamics of Third Order Lovelock-Born-Infeld Black Holes, Commun. Theor. Phys. 56, 845 (2011), [arXiv:1110.0064].
[29] S. W. Hawking, Black hole explosions?, Nature 248, 30 (1974).
[30] R. M. Wald, Black Hole Entropy is Noether Charge, Phys. Rev. D 48, 3427 (1993), [gr-qc/9307038];
V. Iyer and R. M. Wald, Some Properties of Noether Charge and a Proposal for Dynamical Black Hole Entropy, Phys. Rev. D 50, 846 (1994), [gr-qc/9403028];
T. Jacobson, G. Kang and R. C. Myers, On Black Hole Entropy, Phys. Rev. D 49, 6587 (1994), [gr-qc/9312023].
[31] M. Cvetic and S. S. Gubser, Phases of R-charged Black Holes, Spinning Branes and Strongly Coupled Gauge Theories, J. High Energy Phys. 04, 024 (1999), [hep-th/9902195];
M. M. Caldarelli, G. Cognola and D. Klemm, Thermodynamics of Kerr-Newman-AdS Black Holes and Conformal Field Theories, Class. Quant. Gravit. 17, 399 (2000), [hep-th/9908022];
S. S. Gubser and I. Mitra, The evolution of unstable black holes in anti-de Sitter space, J. High Energy Phys., 08, 018 (2001).
[32] D. Kubiznak and R. B. Mann, P-V criticality of charged AdS black holes, J. High Energy Physics 1207, 033 (2012), [1205.0559].
[33] A. Dehyadegari, A. Sheykhi and A. Montakhab, Critical behaviour and microscopic structure of charged AdS black holes via an alternative phase space, Phys. Lett. B 768, 235 (2017), [1607.05333].