ESTIMATES FOR THE VOLUME OF A LORENTZIAN MANIFOLD

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Abstract. We prove new estimates for the volume of a Lorentzian manifold and show especially that cosmological spacetimes with crushing singularities have finite volume.

0. Introduction

Let \( N \) be a \((n + 1)\)-dimensional Lorentzian manifold and suppose that \( N \) can be decomposed in the form

\[
N = N_0 \cup N_- \cup N_+,
\]

where \( N_0 \) has finite volume and \( N_- \) resp. \( N_+ \) represent the critical past resp. future Cauchy developments with not necessarily a priori bounded volume. We assume that \( N_+ \) is the future Cauchy development of a Cauchy hypersurface \( M_1 \), and \( N_- \) the past Cauchy development of a hypersurface \( M_2 \), or, more precisely, we assume the existence of a time function \( x^0 \), such that

\[
N_+ = x^0 = t_1, \quad M_1 = \{x^0 = t_1\},
\]

\[
N_- = x^0 = t_2, \quad M_2 = \{x^0 = t_2\},
\]

and that the Lorentz metric can be expressed as

\[
ds^2 = \frac{e^{2\psi}}{\nu^0} \left\{-dx^0{}^2 + \sigma_{ij}(x)dx^i dx^j \right\},
\]

where \( x = (x^i) \) are local coordinates for the space-like hypersurface \( M_1 \) if \( N_+ \) is considered resp. \( M_2 \) in case of \( N_- \).

The coordinate system \((x^\alpha)_{0 \leq \alpha \leq n}\) is supposed to be future directed, i.e. the past directed unit normal \((\nu^\alpha)\) of the level sets

\[
M(t) = \{x^0 = t\}
\]

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is of the form

\[(\nu^\alpha) = -e^{-\psi}(1, 0, \ldots, 0).\]

If we assume the mean curvature of the slices \(M(t)\) with respect to the past directed normal—cf. Section 2 for a more detailed explanation of our conventions—is strictly bounded away from zero, then, the following volume estimates can be proved

**Theorem 0.1.** Suppose there exists a positive constant \(\epsilon_0\) such that

\[(0.6) \quad H(t) \geq \epsilon_0 \quad \forall t_1 < t < T_+,
and
(0.7) \quad H(t) \leq -\epsilon_0 \quad \forall T_- < t < t_2,
then

\[(0.8) \quad |N_+| \leq \frac{1}{\epsilon_0} |M(t_1)|,
and
(0.9) \quad |N_-| \leq \frac{1}{\epsilon_0} |M(t_2)|.

These estimates also hold locally, i.e. if \(E_i \subset M(t_i), i = 1, 2,\) are measurable subsets and \(E_1^+, E_2^-\) the corresponding future resp. past directed cylinders, then,

\[(0.10) \quad |E_1^+| \leq \frac{1}{\epsilon_0} |E_1|,
and
(0.11) \quad |E_2^-| \leq \frac{1}{\epsilon_0} |E_2|.

1. **Proof of Theorem 0.1**

In the following we shall only prove the estimate for \(N_+\), since the other case \(N_-\) can easily be considered as a future development by reversing the time direction.

Let \(x = x(\xi)\) be an embedding of a space-like hypersurface and \((\nu^\alpha)\) be the past directed normal. Then, we have the Gauß formula

\[(1.1) \quad x^\alpha_{ij} = h_{ij}\nu^\alpha.\]
where \((h_{ij})\) is the second fundamental form, and the Weingarten equation

\[
\nu^i_\alpha = h^k_i x^\alpha_k.
\]

We emphasize that covariant derivatives, indicated simply by indices, are always full tensors.

The slices \(M(t)\) can be viewed as special embeddings of the form

\[
x(t) = (t, x^i),
\]

where \((x^i)\) are coordinates of the initial slice \(M(t_1)\). Hence, the slices \(M(t)\) can be considered as the solution of the evolution problem

\[
\dot{x} = -e^\psi \nu, \quad t_1 \leq t < T_+,
\]

with initial hypersurface \(M(t_1)\), in view of \([1.3]\).

From the equation \([1.4]\) we can immediately derive evolution equations for the geometric quantities \(g_{ij}, h_{ij}, \nu, \) and \(H = g_{ij} h_{ij}\) of \(M(t)\), cf. e.g. \([3]\), Section 4], where the corresponding evolution equations are derived in Riemannian space.

For our purpose, we are only interested in the evolution equation for the metric, and we deduce

\[
\dot{g}_{ij} = \langle \dot{x}_i, x_j \rangle + \langle x_i, \dot{x}_j \rangle = -2e^\psi h_{ij},
\]

in view of the Weingarten equation.

Let \(g = \det(g_{ij})\), then,

\[
\dot{g} = gg^{ij} \dot{g}_{ij} = -2e^\psi H g,
\]

and thus, the volume of \(M(t), |M(t)|\), evolves according to

\[
\frac{d}{dt}|M(t)| = \int_{M(t_1)} \frac{d}{dt} \sqrt{g} = -\int_{M(t)} e^\psi H,
\]

where we shall assume without loss of generality that \(|M(t_1)|\) is finite, otherwise, we replace \(M(t_1)\) by an arbitrary measurable subset of \(M(t_1)\) with finite volume.

Now, let \(T \in [t_1, T_+]\) be arbitrary and denote by \(Q(t_1, T)\) the cylinder

\[
Q(t_1, T) = \{(x^0, x): t_1 \leq x^0 \leq T\},
\]

then,
(1.9) \[ |Q(t_1, T)| = \int_{t_1}^{T} \int_{M} e^{\psi}, \]

where we omit the volume elements, and where, \( M = M(x^0) \).

By assumption, the mean curvature \( H \) of the slices is bounded from below by \( \epsilon_0 \), and we conclude further, with the help of (1.7),

(1.10) \[
|Q(t_1, T)| \leq \frac{1}{\epsilon_0} \int_{t_1}^{T} \int_{M} e^{\psi} H
= \frac{1}{\epsilon_0} \{ |M(t_1)| - |M(T)| \}
\leq \frac{1}{\epsilon_0} |M(t_1)|.
\]

Letting \( T \) tend to \( T_+ \) gives the estimate for \( |N_+| \).

To prove the estimate (1.10), we simply replace \( M(t_1) \) by \( E_1 \).

If we relax the conditions (0.6) and (0.7) to include the case \( \epsilon_0 = 0 \), a volume estimate is still possible.

**Theorem 1.1.** If the assumptions of Theorem 0.1 are valid with \( \epsilon_0 = 0 \), and if in addition the length of any future directed curve starting from \( M(t_1) \) is bounded by a constant \( \gamma_1 \) and the length of past any directed curve starting from \( M(t_2) \) is bounded by a constant \( \gamma_2 \), then,

(1.11) \[ |N_+| \leq \gamma_1 |M(t_1)| \]

and

(1.12) \[ |N_-| \leq \gamma_2 |M(t_2)|. \]

**Proof.** As before, we only consider the estimate for \( N_+ \).

From (1.6) we infer that the volume element of the slices \( M(t) \) is decreasing in \( t \), and hence,

(1.13) \[ \sqrt{g(t)} \leq \sqrt{g(t_1)} \quad \forall t_1 \leq t. \]

Furthermore, for fixed \( x \in M(t_1) \) and \( t > t_1 \)

(1.14) \[ \int_{t_1}^{t} e^{\psi} \leq \gamma_1 \]

because the left-hand side is the length of the future directed curve 

(1.15) \[ \gamma(\tau) = (\tau, x) \quad t_1 \leq \tau \leq t. \]
Let us now look at the cylinder \( Q(t_1, T) \) as in \((1.8)\) and \((1.9)\). We have

\[
|Q(t_1, T)| = \int_{t_1}^{T} \int_{M(t_1)} e^{ψ} \sqrt{g(t, x)} \leq \int_{t_1}^{T} \int_{M(t_1)} e^{ψ} \sqrt{g(t, x)}
\]

\[
\leq γ_1 \int_{M(t_1)} \sqrt{g(t_1, x)} = γ_1 |M(t_1)|
\]

by applying Fubini’s theorem and the estimates \((1.13)\) and \((1.14)\). □

2. Cosmological spacetimes

A cosmological spacetime is a globally hyperbolic Lorentzian manifold \( N \) with compact Cauchy hypersurface \( S_0 \), that satisfies the timelike convergence condition, i.e.

\[
\bar{R}_{αβνρ} v^α v^β \geq 0 \quad ∀ \langle ν, ν \rangle = -1.
\]

If there exist crushing singularities, see \([1]\) or \([2]\) for a definition, then, we proved in \([2]\) that \( N \) can be foliated by spacelike hypersurfaces \( M(τ) \) of constant mean curvature \( τ, -∞ < τ < ∞ \),

\[
N = \bigcup_{0 ≠ τ ∈ \mathbb{R}} M(τ) \cup \mathcal{C}_0,
\]

where \( \mathcal{C}_0 \) consists either of a single maximal slice or of a whole continuum of maximal slices in which case the metric is stationary in \( \mathcal{C}_0 \). But in any case \( \mathcal{C}_0 \) is a compact subset of \( N \).

In the complement of \( \mathcal{C}_0 \) the mean curvature function \( τ \) is a regular function with non-vanishing gradient that can be used as a new time function, cf. \([3]\) for a simple proof.

Thus, the Lorentz metric can be expressed in Gaussian coordinates \( (x^α) \) with \( x^0 = τ \) as in \((1.3)\). We choose arbitrary \( τ_2 < 0 < τ_1 \) and define

\[
N_0 = \{ (τ, x): τ_2 \leq τ \leq τ_1 \},
\]

\[
N_- = \{ (τ, x): -∞ < τ \leq τ_2 \},
\]

\[
N_+ = \{ (τ, x): τ_1 \leq τ < ∞ \}.
\]

Then, \( N_0 \) is compact, and the volumes of \( N_-, N_+ \) can be estimated by

\[
|N_+| \leq \frac{1}{τ_1} |M(τ_1)|,
\]
and
\begin{equation}
|N_-| \leq \frac{1}{|\tau_2|} |M(\tau_2)|.
\end{equation}

Hence, we have proved

**Theorem 2.1.** A cosmological spacetime $N$ with crushing singularities has finite volume.

**Remark 2.2.** Let $N$ be a spacetime with compact Cauchy hypersurface and suppose that a subset $N_- \subset N$ is foliated by constant mean curvature slices $M(\tau)$ such that
\begin{equation}
N_- = \bigcup_{0 < \tau \leq \tau_2} M(\tau)
\end{equation}
and suppose furthermore, that $x^0 = \tau$ is a time function—which will be the case if the timelike convergence condition is satisfied—so that the metric can be represented in Gaussian coordinates $(x^\alpha)$ with $x^0 = \tau$.

Consider the cylinder $Q(\tau, \tau_2) = \{ \tau \leq x^0 \leq \tau_2 \}$ for some fixed $\tau$. Then,
\begin{equation}
|Q(\tau, \tau_2)| = \int_{\tau}^{\tau_2} \int_M e^\psi = \int_{\tau}^{\tau_2} H^{-1} \int_M He^\psi,
\end{equation}
and we obtain in view of \((1.7)\)
\begin{equation}
\tau_2^{-1} \{|M(\tau)| - |M(\tau_2)|\} \leq |Q(\tau, \tau_2)|,
\end{equation}
and conclude further
\begin{equation}
\lim_{\tau \to 0} |M(\tau)| \leq \tau_2 |N_-| + |M(\tau_2)|,
\end{equation}
i.e.
\begin{equation}
\lim_{\tau \to 0} |M(\tau)| = \infty \implies |N_-| = \infty.
\end{equation}
3. The Riemannian case

Suppose that $N$ is a Riemannian manifold that is decomposed as in [13] with metric

\begin{equation}
\tilde{ds}^2 = e^{2\psi} \{dx^0 + \sigma_{ij}(x^0, x)dx^i dx^j\}.
\end{equation}

The Gauß formula and the Weingarten equation for a hypersurface now have the form

\begin{equation}
x^\alpha_{ij} = -h_{ij}\nu^\alpha,
\end{equation}

and

\begin{equation}
\nu^\alpha_i = h_i^k x^\alpha_k.
\end{equation}

As default normal vector—if such a choice is possible—we choose the outward normal, which, in case of the coordinate slices $M(t) = \{x^0 = t\}$ is given by

\begin{equation}
(\nu^\alpha) = e^{-\psi}(1, 0, \ldots, 0).
\end{equation}

Thus, the coordinate slices are solutions of the evolution problem

\begin{equation}
\dot{x} = e^{\psi} \nu,
\end{equation}

and, therefore,

\begin{equation}
\dot{g}_{ij} = 2e^{\psi} h_{ij},
\end{equation}

i.e. we have the opposite sign compared to the Lorentzian case leading to

\begin{equation}
\frac{d}{dt}|M(t)| = \int_M e^{\psi} H.
\end{equation}

The arguments in Section 1 now yield

**Theorem 3.1.** (i) Suppose there exists a positive constant $\epsilon_0$ such that the mean curvature $H(t)$ of the slices $M(t)$ is estimated by

\begin{equation}
H(t) \geq \epsilon_0 \quad \forall t_1 \leq t \leq T_+,
\end{equation}

and

\begin{equation}
H(t) \leq -\epsilon_0 \quad \forall T_- < t \leq t_2,
\end{equation}

then
\[(3.10)\quad |N_+| \leq \frac{1}{\epsilon_0} \lim_{t \to T_+} |M(t)|,\]

and

\[(3.11)\quad |N_-| \leq \frac{1}{\epsilon_0} \lim_{t \to T_-} |M(t)|.\]

(ii) On the other hand, if the mean curvature \(H\) is negative in \(N_+\) and positive in \(N_-\), then, we obtain the same estimates as Theorem 0.1, namely,

\[(3.12)\quad |N_+| \leq \frac{1}{\epsilon_0} |M(t_1)|,\]

and

\[(3.13)\quad |N_-| \leq \frac{1}{\epsilon_0} |M(t_2)|.\]

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