Emergent of Majorana Fermion mode and Dirac Equation in Cavity Quantum Electrodynamics

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Abstract

We present the results of low lying excitation of coupled optical cavity arrays. We derive the Dirac equation for this system and explain the existence of Majorana fermion mode in the system. We present quite a few analytical relations between the Rabi frequency oscillation and the atom-photon coupling strength to achieve the different physical situation of our study and also the condition for massless excitation in the system. We present several analytical relations between the Dirac spinor field, order and disorder operators for our systems. We also show that the Luttinger liquid physics is one of the intrinsic concept in our system.

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**Introduction:** The recent experimental success in engineering strong interaction between the photons and atoms in high quality micro-cavities opens up the possibility to use light matter system as quantum simulators for many body physics [1-21]. Many interesting results are coming out to understand the complicated quantum many body system. The Bose-Hubbard model, quantum spin model and the other exotic quantum phases of the quantum many body system have already been studied [3-7]. The further application of the basic principle of cavity-QED system is the circuit QED [22-26]. When a qubit (qubits) coupled to the high quality LC circuits, it presents the same physical picture with few extra achievement over the conventional cavity-QED system. A focus on the coupled cavities is one of the most potential candidate for an efficient quantum simulator due to the control of the microcavities parameters and success of fabrication of large scale cavity arrays [25-26]. This is the very brief discussion of the presence status of the cavity QED system.

In the present study one of our goal is to predict the presence of Majorana fermions in our model system. Before we proceed further, we would like to describe very briefly about the apparence of Majorana fermions in quantum condensed matter system. Majorana had introduced a special kind of fermions which are their own antiparticle, i.e., the neutral particle [25, 26]. He had introduced this particle to describe neutrons. In recent years, there are several candidates of Majorana fermions in quantum condensed matter system like quantum Hall system with filling fraction $\frac{5}{2}$ [27, 28]. Kitaev at first found the existence of Majorana fermion mode in one dimensional model [29]. Many research group have already been proposed the physically existence of MFs at the edge state of 1D system like electrostatic defects lines in superconductor, quasi-one dimensional superconductor and cold atom trapped in one dimension [30, 31]. Majorana fermions are obey non-Abelian statics both in 2D and 1D, allowing of certain gate operation required in quantum computation [32]. Due to the non-local character, the qubit built out of Majorana fermions are insensitive to local parity conserving perturbation [32-36]. The search for experimentally accessible systems that are described by a Dirac equation has received much attention in recent years [32-36]. In this research paper, we present an extensive derivation of Dirac equation and also the existence of Majorana fermions mode in an optical cavity array. We also present the analytical relation between the Rabi frequency oscillation and the atom-photon coupling strengths to mimic the transverse Ising model, Dirac equation, magnetic ordered state, quantum paramagnetic state and massless excitation. Quantum state engineering of the optical cavity array system
is in the state of art due to the rapid technical development of this field [1] therefore one can achieve these quantum phases in the laboratory.

**The Model Hamiltonian and Majorana Fermion Modes:**

The Hamiltonian of our present study consists of three parts:

\[ H = H_A + H_C + H_{AC} \]  \hspace{1cm} (1)

The Hamiltonians are the following

\[ H_A = \sum_{j=1}^{N} \omega_e |e_j><e_j| + \omega_{ab} |b_j><b_j| \]  \hspace{1cm} (2)

where \( j \) is the cavity index. \( \omega_{ab} \) and \( \omega_e \) are the energies of the state \( |b> \) and the excited state respectively. The energy level of state \( |a> \) is set as zero. \( |a> \) and \( |b> \) are the two stable state of a atom in the cavity and \( |e> \) is the excited state of that atom in the same cavity. The following Hamiltonian describes the photons in the cavity,

\[ H_C = \omega_C \sum_{j=1}^{N} \sigma_j^{z} + J_C \sum_{j=1}^{N} (\sigma_j^{+}\sigma_{j+1}^{-} + h.c), \]  \hspace{1cm} (3)

where \( \sigma_j^{z} \) is the photon creation (annihilation) operator for the photon field in the \( j \)'th cavity, \( \omega_C \) is the energy of photons and \( J_C \) is the tunneling rate of photons between neighboring cavities. The interaction between the atoms and photons and also by the driving lasers are described by

\[ H_{AC} = \sum_{j=1}^{N} [(\frac{\Omega_a}{2} e^{-i\omega_a t} + g_a a_j)|e_j><a_j| + h.c] + [a \leftrightarrow b]. \]  \hspace{1cm} (4)

The authors of Ref. [3–5] have derived an effective spin model by considering the following physical processes: A virtual process regarding emission and absorption of photons between the two stable states of neighboring cavity yields the resulting effective Hamiltonian as

\[ H_{xy} = \sum_{j=1}^{N} B \sigma_j^{z} + \sum_{j=1}^{N} (\frac{J_1}{2} \sigma_j^{+}\sigma_{j+1}^{-} + \frac{J_2}{2} \sigma_j^{-}\sigma_{j+1}^{+} + h.c) \]  \hspace{1cm} (5)

When \( J_2 \) is real then this Hamiltonian reduces to the XY model. Where \( \sigma_j^{z} = |b_j><b_j| - |a_j><a_j| \), \( \sigma_j^{+} = |b_j><a_j| \), \( \sigma_j^{-} = |a_j><b_j| \).

\[ H_{xy} = \sum_{i=1}^{N} (B \sigma_i^{z} + J_1(\sigma_i^{z}\sigma_{i+1}^{-} + \sigma_i^{-}\sigma_{i+1}^{+})) \]
With $J_x = (J_1 + J_2)$ and $J_y = (J_1 - J_2)$.

We follow the references [3, 37], to present the analytical expression for the different physical parameters of the system.

$$ B = \frac{\delta_1}{2} - \beta $$

$$ \beta = \frac{1}{2} \left[ \frac{\Omega_b^2}{4\Delta_b} (\Delta_b - \frac{\Omega_b^2}{4\Delta_b} - \frac{\Omega_b^2}{4(\Delta_a - \Delta_b)}) - \gamma_b g_b^2 - \gamma_t g_a^2 \right] + \gamma_1 g_a^2 \left( a \leftrightarrow b \right) $$

$$ J_1 = \frac{\gamma_2}{4} \left( \frac{\Omega_a^2 g_b^2}{\Delta_a^2} + \frac{\Omega_b^2 g_a^2}{\Delta_b^2} \right), \quad J_2 = \frac{\gamma_2}{2} \left( \frac{\Omega_a \Omega_b g_a g_b}{\Delta_a \Delta_b} \right). $$

Where $\gamma_{a,b} = \frac{1}{N} \sum_k \frac{1}{\omega_{a,b} - \omega_k}$, $\gamma_1 = \frac{1}{N} \sum_k \frac{1}{(\omega_a + \omega_b)/2 - \omega_k}$ and $\gamma_2 = \frac{1}{N} \sum_k \frac{e^{ik}}{(\omega_a + \omega_b)/2 - \omega_k}$. $\delta_1 = \omega_{ab} - (\omega_a - \omega_b)/2$, $\Delta_a = \omega_e - \omega_a$, $\Delta_b = \omega_e - \omega_a - (\omega_{ab} - \delta_1)$. $\delta_a^k = \omega_e - \omega_{k} + \omega_e - \omega_k - (\omega_{ab} - \delta_1)$, $\omega_k = \omega_e + J_c \sum_k \cos k$. $g_a$ and $g_b$ are the couplings of respective transition to the cavity mode, $\Omega_a$ and $\Omega_b$ are the Rabi frequency of laser with frequency $\omega_a$ and $\omega_b$.

The system reduces to Ising model with transverse field at $J_1 = J_2$, i.e., $J_x$ become $J_1 + J_2$ and $J_y = 0$. The effective Hamiltonian become the transverse Ising model which studied in the previous literature [38–40]. Here our main motivation is to use some of important results of this model Hamiltonian to discuss the relevant physics of array of cavity QED system.

Before we proceed further, we would like to discuss in detail the analytical relation between the different coupling constants of cavity QED system to achieve this Hamiltonian. In the microcavity array, the condition for $J_1 = J_2$ achieve when

$$ \Omega_a^2 g_b^2 \Delta_b^2 + \Omega_b^2 g_a^2 \Delta_a^2 = 2\Omega_a \Omega_b g_a g_b \Delta_a \Delta_b. $$

The above condition implies that $\Omega_a = \Omega_b \frac{g_a}{g_b} \frac{\Delta_a}{\Delta_b}$. The only constraint is that $\Delta_a \neq \Delta_b$, the magnetic field diverge when $\Delta_a = \Delta_b$. At the same time, $\Omega_a = \Omega_b$ and $g_a = g_b$ are also not possible because this limit also leads to the condition $\Delta_a = \Delta_b$. Suppose we consider, $\Omega_a = \alpha_1 \Omega_b$, $g_a = \alpha_2 g_b$ and $\Delta_a = \alpha_3 \Delta_b$. These relations implies that $\alpha_1^2 + \alpha_2^2 \alpha_3^2 = 2\alpha_1 \alpha_2 \alpha_3$. $\alpha_1 = \alpha_2 \alpha_3$, $\alpha_1, \alpha_2$ and $\alpha_3$ are the numbers. These analytical relations help to implement the
transverse Ising model Hamiltonian but $\alpha_1$, $\alpha_2$ and $\alpha_3$ should not be equal to 1.

The quantum state engineering of cavity QED is in the state of art due to the rapid progress of technological development of this field [1]. Therefore one can achieve this limit to get the desire quantum state.

$$H_T = B \sum_{j=1}^{N}(\sigma_z(j) + \lambda \sigma_x(j)\sigma_x(j+1)),$$

(11)

where $\lambda = \frac{J_1 + J_2}{B}$. The transverse Ising model was studied widely in the literature and also exhibit a quantum phase transition between the magnetically ordered state to the quantum paramagnetic phase for $\lambda > 1$ and $\lambda < 1$ respectively [38–40].

Now we express the condition for the magnetic order phase and quantum paramagnetic phase in terms of the physical parameters of the optical cavity QED system which gives us the relevant physics of the system.

The condition for the magnetic ordered system can be expressed as

$$\frac{\gamma_2}{4} \left( \frac{\Omega_a g_b^2}{\Delta_a^2} + \frac{\Omega_b g_a^2}{\Delta_b^2} \right) + \frac{\gamma_2}{2} \left( \frac{\Omega_a \Omega_b g_a g_b}{\Delta_a \Delta_b} \right) > \omega_{ab} - \frac{\omega_a - \omega_b}{2} - 2\beta$$

(12)

The condition for the quantum paramagnetic phase is

$$\frac{\gamma_2}{4} \left( \frac{\Omega_a g_b^2}{\Delta_a^2} + \frac{\Omega_b g_a^2}{\Delta_b^2} \right) + \frac{\gamma_2}{2} \left( \frac{\Omega_a \Omega_b g_a g_b}{\Delta_a \Delta_b} \right) < \omega_{ab} - \frac{\omega_a - \omega_b}{2} - 2\beta$$

(13)

When the applied magnetic field is absent, the effective Ising model has two degenerate ground states. The ground states are $|A> = \Pi_j |\rightarrow j>$, $|B> = \Pi_j |\leftarrow j>$. For a finite magnetic field but less than $J_1 + J_2$, the system has a tendency to flip the pseudo spin. At that phase one can write down the true eigen state, $|\psi_A> = \frac{1}{\sqrt{2}}(|A> + |B>), |\psi_B> = \frac{1}{\sqrt{2}}(|A> - |B>)$. Now our main intention is to recast this spin model in spinless fermion model through Jordan-Wigner transformation which relate the spin operators to the spinless fermion operators. We use the following relation:

$$\sigma_z = 2c^\dagger(j)c(j) - 1, \sigma_x(j)\sigma_x(j+1) = (c^\dagger(n) - c(n))(c^\dagger(n + 1) - c(n + 1)).$$

One can write the Hamiltonian after the Jordan-Wigner transformation as

$$H = 2 \sum_{j=1}^{N} c^\dagger(j)c(j) + \lambda(c^\dagger(j) - c(j))(c^\dagger(j + 1) - c(j + 1))$$

(14)

We solve this Hamiltonian, to get the energy spectrum by taking the Fourier transform.

$$c(j) = \frac{1}{\sqrt{N}} \sum_k e^{-ika}, c^\dagger(j) = \frac{1}{\sqrt{N}} \sum_k e^{ika}.$$ Where $c_k$ and $c_k^\dagger$ are the fermionic annihilation
and creation operator in momentum space.

The Hamiltonian reduce to

\[
H = 2 \sum_{k>0} (1 + \lambda \cos k)(c_k^\dagger c_k + c_{-k}^\dagger c_{-k}) + 2i\lambda \sum_{k>0} \sin k (c_k^\dagger c_{-k}^\dagger + c_k c_{-k})
\]  

(15)

Now our main task is to express the Hamiltonian in the diagonalized form. We follow the Bogoliubov transformation.

\[
\eta_k = \alpha_k c_k + i\beta_k c_{-k}^\dagger \quad \text{and} \quad \eta_{-k} = \alpha_k c_{-k} - i\beta_k c_k^\dagger, \quad k > 0.
\]

The operator \(\eta_k\) and \(\eta_k^\dagger\) are the fermionic operators. We use the following relations,

\[
\{\eta_k, \eta_p^\dagger\} = \delta_{k,p}, \quad \{\eta_k, \eta_p\} = 0, \quad \{\eta_k^\dagger, \eta_p\} = 0.
\]

This relation implies, \(\alpha_k^2 + \beta_k^2 = 1\). One can also revert the relation between \(c_k\) and \(\eta_k\). We also parameterize \(\alpha_k = \cos \theta_k\) and \(\beta_k = \sin \theta_k\).

One can express the transformed Hamiltonian in two parts

\[
H = H_A + H_B
\]  

(16)

\[
H_A = \sum_{k>0} [-2(1 + \lambda \cos k)(\alpha_k^2 - \beta_k^2) + 4\lambda \sin k \alpha_k \beta_k] \\
\quad (\eta_k^\dagger \eta_k \eta_{-k}^\dagger \eta_{-k})
\]

(17)

\[
H_B = \sum_{k>0} [4i(1 + \lambda \cos k)\alpha_k \beta_k + 2i\lambda \sin k(\alpha_k^2 - \beta_k^2)] \\\n\quad (\eta_k^\dagger \eta_{-k}^\dagger \eta_k \eta_{-k})
\]

(18)

To express this Hamiltonian in the diagonal form, we find the following relation

\[
4(B + \frac{2\Omega_b^2 g_a^2}{\Delta b^2} \cos k)\alpha_k \beta_k + 2\frac{2\Omega_b^2 g_a^2}{\Delta b^2} \sin k(\alpha_k^2 - \beta_k^2) = 0.
\]

Finally this gives the condition,

\[
tan 2\theta_k = \frac{2\gamma_2 \Omega_b^2 g_a^2}{2\gamma_2 \Omega_b^2 g_a^2 \cos k - (\delta_1 - 2\beta)\Delta b^2}
\]  

(19)

\[2\alpha_k \beta_k = \sin 2\theta_k, \quad \alpha_k^2 - \beta_k^2 = \cos 2\theta_k.
\]

Now we analysis the spectrum:

\[
H = 2 \sum_k \Omega_K \eta_k^\dagger \eta_k
\]  

(20)

We can express this energy spectrum in terms of Rabi frequency oscillation, atom-photon coupling strength,

\[
\Omega_k = \frac{2}{\delta_1/2 - \beta} \sqrt{\left(\frac{\delta_1/2 - \beta}{2}\right)^2 + \gamma_2^2 \frac{\Omega_b^4 g_a^4}{\Delta a^4} + 2(\delta_1/2 - \beta)\gamma_2 \frac{\Omega_b^2 g_a^2}{\Delta b^2}}
\]  

(21)
The minimum occurs at \( k = \pm \pi \), \( \Omega_{k=\pm\pi} = 2|1 - \lambda| = 2|1 - \frac{2\gamma_2}{\delta_1 - 2\beta} \Omega_b^2 g_a^2 | \). We are interested in the continuum limit and also restore the lattice spacing \( \alpha \) and measure the momentum w.r.t the minimum value, \( k = \pi + k' \alpha \). The energy expression which contains the physical dimension of energy is \( E(k') = \frac{\Omega_b}{2\alpha} \). In this limit, \( E(k') = \sqrt{\left( \frac{1 - \lambda}{\alpha} \right)^2 + \lambda k'^2} \). If \( \lambda \) is close to a critical value \( \lambda \sim 1 \), we then have the dispersion of a particle with mass \( m = \frac{\lambda}{\alpha} \).

If \( \lambda = 1 \), it becomes the massless particle \( E(k') \sim k' \).

In the cavity QED system, we can express the condition for massless excitation of the system as \( (\delta_1/2 - \beta)\Delta_b^2 = \gamma_2 \Omega_b^2 g_a^2 \). In terms of fields, \( \eta(a) = \frac{1}{\sqrt{N}} \sum_k e^{ika} \eta_k \), \( \eta(a)^\dagger = \frac{1}{\sqrt{N}} \sum_k e^{-ika} \eta_k^\dagger \).

\( \chi_1(a) = (1/2)(\eta(a) + \eta(a)^\dagger) \), \( \chi_2(a) = (1/2i)(\eta(a) - \eta(a)^\dagger) \)
\( \chi_1(a)^\dagger = \chi_1(a), \chi_2(a)^\dagger = \chi_2(a) \).

\( \{\chi_1(x_1), \chi_2(x_2)\} = \delta_{x_1,x_2}\delta_{1,2} \). Therefore, \( \chi_1 \) and \( \chi_2 \) are satisfying the all properties of neutral fermionic fields what Majorana proposed.

The authors of [15] have investigated the low lying excitation of one dimensional array of circuit QED (cktQED) with each cktQED being in the ultra strong coupling regime and they have found the Majorana bound state. But the starting Hamiltonian of our system is completely different.

Now we calculate the energy density of the system using Eq. 21. We would like to integrate the dispersion spectrum \( \Omega_k \) to get the energy density. The analytical expression for energy density is

\[
\epsilon_0 = \frac{2}{\pi} \left( 1 + \frac{2\gamma_2}{\delta_1 - 2\beta} \frac{\Omega_b^2 g_a^2}{\Delta_b^2} \right) \cdot \frac{E \left( \frac{\pi}{2}, \sqrt{1 - \gamma} \right)}{E \left( \frac{\pi}{2}, \sqrt{1 - \gamma^2} \right)}
\]

(22)

Where \( \gamma = \frac{(\delta_1 - 2\beta)\Delta_b^2 - 2\gamma_2 \Omega_b^2 g_a^2}{(\delta_1 - 2\beta)\Delta_b^2 + 2\gamma_2 \Omega_b^2 g_a^2} \), \( E \left( \frac{\pi}{2}, \sqrt{1 - \gamma^2} \right) \) is the complete Elliptic integral of 2nd kind. After a little bit of calculation, we obtain the analytical expression in the asymptototic limit for energy density.

\[
\epsilon_0 = 1 + 1/2 \left( \ln |\frac{(\delta_1 - 2\beta)\Delta_b^2 + 2\gamma_2 \Omega_b^2 g_a^2}{(\delta_1 - 2\beta)\Delta_b^2 - 2\gamma_2 \Omega_b^2 g_a^2}| - 1/2 \right) \gamma^2 \\
+ 3/16 \left( \ln |\frac{(\delta_1 - 2\beta)\Delta_b^2 + 2\gamma_2 \Omega_b^2 g_a^2}{(\delta_1 - 2\beta)\Delta_b^2 - 2\gamma_2 \Omega_b^2 g_a^2}| - 13/12 \right) \gamma^4
\]

(23)

**Derivation of Dirac Equation and Condition of Massless Excitations:**

During the derivation of Dirac equation we rotate the y-axis of the spin basis through \( \pi/2 \). Through this rotation \( z \)-axis become \( x \)-axis and \( x \)-axis become \( -z \)-axis. The physics of the system remain the same. The starting Hamiltonian of our system now becomes
\[ H = \sum_n [\lambda s_z(n)s_z(n+1) + s_x(n)]. \tag{24} \]

We recast the Hamiltonian in the following form because it will help us to use the order and disorder operator directly to the derivation of the equation of motion and finally the Dirac equation. Here we present an extensive derivation of Dirac equation for our model system. Kramers-Wannier symmetry for two dimensional Ising model reveals for the case of one-dimensional quantum Ising chain through the dual lattice to the original one (site index \( n \)). Here we introduce the order and disorder operator following [39, 40]. These operators are defining the sites of the dual lattice, i.e., we define the operator between the nearest-neighbor site of the original lattice. The analytical relation between the Pauli operators and \( \mu \) operators are the following:

\[ \mu_z^2 = 1 = \mu_x^2, \tag{25} \]
\[ \mu_z(n-1/2)\mu_z(n+1/2) = \sigma_x(n). \tag{26} \]
\[ \mu_x(n+1/2) = \sigma_z(n)\sigma_z(n+1), \tag{27} \]
\[ \mu_z(n+1/2) = \Pi_{j=1}^{n}\sigma_z(j). \tag{28} \]
\[ \sigma_z(n) = \Pi_{j=0}^{n-1}\mu_x(j + 1/2), \tag{29} \]
\[ [\mu_x(n+1/2), \mu_z(n' + 1/2)] = 2\delta_{n,n'}. \tag{30} \]
\[ [\mu_z(n+1/2), \mu_z(n' + 1/2)] = 0, \tag{31} \]
\[ [\mu_z(n+1/2), \sigma_x(n')] = 0 \tag{32}. \]

The operator \( \mu_z(n+1/2) \) acting on the original spin of the lattice makes a spin flip of all those spin placed on the left hand side of spin at the site \( n \). Therefore \( \mu_z(n+1/2) \) is a kink operator, it introduce the disorder in the system. It is very clear from the above analytical relation of the operators that \( \mu_x \) is related with the allignment of the spin operator.

Here we define the Dirac spinor, \( \chi_1(n) = \sigma_z(n)\mu_z(n + 1/2) \) and \( \chi_2(n) = \sigma_z(n)\mu_z(n - 1/2) \).

Now our main task is to find the equation of motion for the operators, \( \sigma_3(n) \) and \( \mu_3(n) \) which help us to build the Dirac equation.

The equation of motion for the \( \sigma_z(n) \) is the following:

\[ \frac{\partial \sigma_z(n)}{\partial \tau} = [H, \sigma_z(n)] = \sigma_x(n)\sigma_z(n) \]
The equation of motion for $\mu_z(n+1/2)$ is the following:

$$\frac{\partial \mu_z(n+1/2)}{\partial \tau} = \lambda \mu_z(n+1/2) \mu_z(n+1/2)$$

$$= \lambda \sigma_z(n) \sigma_z(n+1/2) \mu_z(n+1/2)$$

(34)

(35)

Now we use the properties of the $\sigma$ and $\mu$ operators to derive the equation of motion for the Majorana fields $\chi_1(n)$ and $\chi_2(n)$.

$$\frac{\partial \chi_1(n)}{d\tau} = \frac{\partial \sigma_z(n)}{\partial \tau} \mu_z(n+1/2) + \sigma_z(n) \frac{\partial \mu_z(n)}{\partial \tau}.$$  

$$= \sigma_x(n) \sigma_z(n) \mu_z(n+1/2) + \lambda \sigma_z(n) \sigma_z(n+1) \mu_z(n+1/2).$$

(36)

(37)

$$\frac{\partial \chi_1(n)}{d\tau} = -\sigma_z(n) \mu_z(n-1/2) \mu_z(n+1/2)$$

$$+ \lambda \sigma_z(n) \sigma_z(n+1) \mu_z(n+1/2).$$

(38)

$$\frac{\partial \chi_1(n)}{d\tau} = -\chi_2(n) + \lambda \chi_2(n+1).$$

(39)

Now the equations of motion for $\chi_2(n)$ are

$$\frac{\partial \chi_2(n)}{d\tau} = \frac{\partial \sigma_z(n)}{\partial \tau} \mu_z(n-1/2)$$

$$+ \sigma_z(n) \frac{\partial \mu_z(n-1/2)}{\partial \tau}.$$  

(40)

$$\frac{\partial \chi_2(n)}{d\tau} = \sigma_x(n) \sigma_z(n) \mu_z(n-1/2)$$

$$+ \lambda \sigma_z(n) \sigma_z(n-1) \sigma_z(n) \mu_z(n-1/2).$$

(41)

$$\frac{\partial \chi_2(n)}{d\tau} = \mu_z(n-1/2) \mu_z(n+1/2) \sigma_z(n) \mu_z(n-1/2)$$

$$+ \lambda \sigma_z(n-1) \mu_z(n-1/2).$$

(42)

After a little bit of calculations and using the relation between the disorder operators (Eq. 23-30), we finally arrive the equation of motion of $\chi_2(n)$ as,

$$\frac{\partial \chi_2(n)}{d\tau} = -\chi_1(n) + \lambda \chi_1(n-1).$$

(43)
These two fields, $\chi_1(n)$ and $\chi_2(n)$ satisfy the following relations, \(\{\chi_1(n_1), \chi_2(n_2)\} = 2\delta_{n_1, n_2}\). One can write down the above equation in the following compact form,

\[
(\gamma^0 \frac{\partial}{\partial t} + \gamma^3 \frac{\partial}{\partial r} + m)\chi = 0. \tag{44}
\]

where $\chi^\dagger = (\chi_1, \chi_2)$ and $m = \frac{1-\lambda}{\alpha}, \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Now we prove the presence of Luttinger liquid physics is intrinsic to the optical cavity array system by comparing the analytical relation of Majorana fermion operators with the order and disorder operator with the free Dirac field in Abelian bosonization theory. Here we express the analytical relation between the Majorana operators and the disorder and Pauli operators:

\[
\chi_1(n) = \sigma_z(n)\mu_z(n + 1/2) = -\mu_z(n + 1/2)\sigma_z(n) \tag{45}
\]

\[
\chi_2(n) = \sigma_z(n)\mu_z(n - 1/2) = \mu_z(n - 1/2)\sigma_z(n) \tag{46}
\]

\[
\sigma_z(n)\chi_2(n) = \mu_z(n - 1/2) = \chi_2(n)\sigma_z(n) \tag{47}
\]

\[
\sigma_z(n)\chi_1(n) = \mu_z(n + 1/2) = -\chi_1(n)\sigma_z(n) \tag{48}
\]

\[
\sigma_z(n) = \mu_z(n - 1/2)\chi_2(n) = \chi_2(n)\mu_z(n - 1/2) \tag{49}
\]

\[
\mu_z(n + 1/2)\chi_1(n) = -\chi_1(n)\mu_z(n + 1/2) = \sigma_z(n) \tag{50}
\]

The above relations can be extended to account for arbitrary space separation between different operators. Then one obtains the following sets of commutation relations.

\[
\sigma_z(x_1)\mu_z(x_2) = \mu_z(x_2)\sigma_z(x_1)\text{sign}(x_1 - x_2) \tag{51}
\]

\[
\sigma_z(x_1)\chi_2(x_2) = \chi_2(x_2)\sigma_z(x_1)\text{sign}(x_1 - x_2) \tag{52}
\]

\[
\mu_z(x_1)\chi_2(x_2) = -\chi_2(x_2)\sigma_z(x_1)\text{sign}(x_1 - x_2) \tag{53}
\]

It is very clear from the above analytical relations that $\chi_1^\dagger = \chi_1$ and $\chi_2^\dagger = \chi_2$. The above relation has similarity with the free Dirac field in Abelian bosonization theory, where Dirac field operator is a local product of two phase exponential depending on the scalar field and its dual $[41, 42]$, as one study the Luttinger liquid physics in Abelian bosonization theory. Therefore the Luttinger liquid physics is the intrinsic to the optical microcavity system.
Conclusions

We have presented an extensive derivation of Dirac equation and the existence of Majorana fermion modes for the optical cavity array with the relation between Rabi frequency oscillation and the atom photon coupling strength. We have presented the condition for massless excitation. We have also presented several analytical relations between the Majorana field, order and disorder operator.

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