Frozen Waves: Stationary optical wavefields with arbitrary longitudinal shape, by superposing equal frequency Bessel beams: Frozen Waves$^{(†)}$

Michel Zamboni-Rached,

Department of Microwaves and Optics, Faculty of Electrical Engineering,
State University of Campinas, Campinas, SP, Brazil.

Abstract – In this paper it is shown how one can use Bessel beams to obtain a stationary localized wavefield with high transverse localization, and whose longitudinal intensity pattern can assume any desired shape within a chosen interval $0 \leq z \leq L$ of the propagation axis. This intensity envelope remains static, i.e., with velocity $v = 0$; and because of this we call “Frozen Waves” such new solutions to the wave equations (and, in particular, to the Maxwell equations). These solutions can be used in many different and interesting applications, as optical tweezers, atom guides, optical or acoustic bistouries, various important medical purposes, etc.

Keywords: Stationary wave fields; Localized solutions to the wave equations; Localized solutions to the Maxwell equations; X-shaped waves; Bessel beams; Slow light; Subluminal waves; Subsonic waves; Limited-diffraction beams; Finite-energy waves; Electromagnetic wavelets; Acoustic wavelets; Electromagnetism; Optics; Acoustics.

1. – Introduction

Since many years a theory of localized waves (LW), or nondiffraacting waves, has been developed, generalized, and experimentally verified in many fields as optics, microwaves and acoustics[1]. These waves have the surprising characteristics of resisting the diffraction effects for long distances, i.e., of possessing a large depth of field.

These waves can be divided into two classes, the localized beams, and the localized pulses. With regard to the beams, the most popular is the Bessel beam[1].

$^{(†)}$ Work supported by FAPESP (Brazil), ; previously available as e-print *****. E-mail address for contacts: mzamboni@dmo.fee.unicamp.br
Much work have been made about the properties and applications of single Bessel beams. By contrast, only a few papers have been addressed to the properties and applications of superpositions of Bessel beams with the same frequency, but with different longitudinal wave numbers. The few works on this subject have shown some surprising possibilities related with this type of superpositions, mainly the possibility of controlling the transverse shape of the resulting beam[2,3]. The other important point, i.e., that of controlling the longitudinal shape, has been very rarely analyzed, and the relevant papers have been confined to numerical optimization processes[4,5] to find out one appropriate computer-generated hologram.

In this work we develop a very simple method** that makes possible the control of the beam intensity longitudinal shape within a chosen interval $0 \leq z \leq L$, where $z$ is the propagation axis and $L$ can be much greater than the wavelength $\lambda$ of the monochromatic light which is being used. Inside such a space interval, we can construct a stationary envelope with many different shapes, including one or more high-intensity peaks (with distances between them much larger than $\lambda$). This intensity envelope remains static, i.e., with velocity $V = 0$; and because of this we call “Frozen Waves” such new solutions to the wave equations (and, in particular, to the Maxwell equations).

We also suggest a simple apparatus capable of generating these stationary fields.

Static wave solutions like these can have many different and interesting applications, as optical tweezers, atom guides, optical or acoustic bisturies, electromagnetic or ultrasound high-intensity fields for various important medical purposes, etc..**

2. – The mathematical methodology**

We start with the well known axis-symmetric Bessel beam

$$
\psi(\rho, z, t) = J_0(k_\rho \rho) e^{i\beta z} e^{-i\omega t}
$$

with

$$
k_\rho^2 = \frac{\omega^2}{c^2} - \beta^2
$$

** Patent pending.
where $\omega$, $k_\rho$, and $\beta$ are the angular frequency, the transverse and the longitudinal wave numbers, respectively. We also impose the conditions

$$\frac{\omega}{\beta} \geq 0 \quad \text{and} \quad k_\rho^2 \geq 0$$

(3)

to ensure forward propagation only, as well as a physical behavior of the Bessel function.

Now, let us make a superposition of $2N + 1$ Bessel beams with the same frequency $\omega_0$, but with different (and still unknown) longitudinal wave numbers $\beta_n$:

$$\Psi(\rho, z, t) = e^{-i\omega_0 t} \sum_{n=-N}^{N} A_n J_0(k_\rho n \rho) e^{i \beta_n z},$$

(4)

where $A_n$ are constant coefficients. For each $n$, the parameters $\omega_0$, $k_\rho n$ and $\beta_n$ must satisfy Eq. (2), and, because of conditions (3), when considering $\omega_0 > 0$, we must have

$$0 \leq \beta_n \leq \frac{\omega_0}{c}$$

(5)

Now our goal is to find out the values of the longitudinal wave numbers $\beta_n$ and of the coefficients $A_n$ in order to reproduce approximately, inside the interval $0 \leq z \leq L$ (on the axis $\rho = 0$), a chosen longitudinal intensity pattern that we call $|F(z)|^2$. In other words, we want to have

$$\sum_{n=-N}^{N} A_n e^{i \beta_n z} \approx F(z) \quad \text{with} \quad 0 \leq z \leq L$$

(6)

Following Eq.(6), one might be tempted to take $\beta_n = 2\pi n/L$, thus obtaining a truncated Fourier series, expected to represent the desired pattern $F(z)$. Superpositions of Bessel beams with $\beta_n = 2\pi n/L$ has been actually used in some works to obtain a large set of transverse amplitude profiles[2]. However, for our purposes, this choice is not appropriate due two principal reasons: 1) It yields negative values for $\beta_n$ (when $n < 0$), which implies backwards propagating components (since $\omega_0 > 0$); 2) In the cases when $L >> \lambda_0$, which are of our interest here, the main terms of the series would correspond to very small values of $\beta_n$, which results in a very short field depth of the corresponding Bessel beams (when generated by finite apertures), impeding the creation of the desired envelopes far form the source.
Therefore, we need to make a better choice for the values of $\beta_n$, which allows forward propagation components only, and a good depth of field. This problem can be solved by putting

$$\beta_n = Q + \frac{2\pi}{L} n ,$$

(7)

where $Q > 0$ is a value to be chosen (as we shall see) according to the given experimental situation, and the desired degree of transverse field localization. Due to Eq. (5), we get

$$0 \leq Q \pm \frac{2\pi}{L} N \leq \frac{\omega_0}{c}$$

(8)

Inequation (8) determines the maximum value of $n$, that we call $N$, once $Q$, $L$ and $\omega_0$ have been chosen.

As a consequence, for getting a longitudinal intensity pattern approximately equal to the desired one, $F(z)$, in the interval $0 \leq z \leq L$, Eq. (4) should be rewritten as:

$$\Psi(\rho = 0, z, t) = e^{-i \omega_0 t} e^{i Q z} \sum_{n=-N}^{N} A_n e^{i \frac{2\pi}{L} n z},$$

(9)

with

$$A_n = \frac{1}{L} \int_0^L F(z) e^{-i \frac{2\pi}{L} n z} d z$$

(10)

Obviously, one obtains only an approximation to the desired longitudinal pattern, because the trigonometric series is necessarily truncated. Its total number of terms, let us repeat, will be fixed once the values of $Q$, $L$ and $\omega_0$ are chosen.

When $\rho \neq 0$, the wave field $\Psi(\rho, z, t)$ becomes

$$\Psi(\rho, z, t) = e^{-i \omega_0 t} e^{i Q z} \sum_{n=-N}^{N} A_n J_0(k_{\rho n} \rho) e^{i \frac{2\pi}{L} n z},$$

(11)

with

$$k_{\rho n}^2 = \omega_0^2 - \left( Q + \frac{2\pi n}{L} \right)^2$$

(12)

The coefficients $A_n$ will yield the amplitudes and the relative phases of each Bessel beam in the superposition.
Because we are adding together zero order Bessel functions, we can expect a high field concentration around $\rho = 0$.

3. – Some examples

In this section we shall present two examples of our methodology.

Let us suppose that we want an optical wave field with $\lambda_0 = 0.632 \mu m$, that is, with $\omega_0 = 2.98 \times 10^{15} \text{Hz}$, whose longitudinal pattern (along its $z$-axis) in the range $0 \leq z \leq L$ is given by the function

$$F(z) = \begin{cases} 
-4 \frac{(z - l_1)(z - l_2)}{(l_2 - l_1)^2} & \text{for } l_1 \leq z \leq l_2 \\
1 & \text{in } l_3 \leq z \leq l_4 \\
-4 \frac{(z - l_5)(z - l_6)}{(l_6 - l_5)^2} & \text{for } l_5 \leq z \leq l_6 \\
0 & \text{elsewhere} 
\end{cases} \quad (13)$$

where $l_1 = L/10$, $l_2 = 3L/10$, $l_3 = 4L/10$, $l_4 = 6L/10$, $l_5 = 7L/10$ and $l_6 = 9L/10$. In other words, the desired longitudinal shape, in the range $0 \leq z \leq L$, is a parabolic function for $l_1 \leq z \leq l_2$, a unitary step function for $l_3 \leq z \leq l_4$, and again a parabola in the interval $l_5 \leq z \leq l_6$, it being zero elsewhere (in the interval $0 \leq z \leq L$). In this example, let us put $L = 0.5 \text{m}$.

We can then calculate the coefficients $A_n$, which appear in the superposition (11), by inserting Eq.(13) into Eq.(10). Let us choose, for instance, $Q = 0.9998 \omega_0/c$: This choice allows the maximum value $N = 158$ of $n$, as one can infer from Eq.(8). Let us specify that, in such a case, one is not obliged to use just $N = 158$, but one can adopt for $N$ any values smaller than it; more in general, any value smaller than that calculated via Eq.(8). Of course, on using the maximum value allowed for $N$, one will get a better result.

In the present case, let us adopt the value $N = 20$. In Fig.1(a) we compare the intensity of the desired longitudinal function $F(z)$ with that of the Frozen Wave (FW), $\Psi(\rho = 0, z, t)$, obtained from Eq.(9) by using the mentioned value $N = 20$.

One can verify that a good agreement between the desired longitudinal behaviour and our approximate Frozen Wave is already obtained with $N = 20$. Obviously, the use of
Figure 1: (a) Comparison between the intensity of the desired longitudinal function \( F(z) \) and that of our Frozen Wave (FW), \( \Psi(\rho = 0, z, t) \), obtained from Eq. (9). The solid line represents the function \( F(z) \), and the dotted one our FW. (b) 3D-plot of the field intensity of the FW chosen in this case by us.

higher values for \( N \) will improve the approximation.

Fig.1(b) shows the 3D-intensity of our FW, given by Eq. (11). One can observe that this field possesses the desired longitudinal pattern, while being endowed with a good transverse localization.

We can expect that, for a desired longitudinal pattern of the field intensity, on choosing smaller values of the parameter \( Q \) one will get FWs with higher transverse width (for the same number of terms in the series (11)), because of the fact that the Bessel beams in (11) will possess a larger transverse wave number, and consequently higher transverse concentrations. We can verify this expectation on considering, for instance, a desired longitudinal pattern, in the range \( 0 \leq z \leq L \), given by the function

\[
F(z) = \begin{cases} 
-4 \frac{(z - l_1)(z - l_2)}{(l_2 - l_1)^2} & \text{in } l_1 \leq z \leq l_2 \\
0 & \text{in the otherwise}
\end{cases},
\]

with \( l_1 = L/2 - \Delta L \) and \( l_2 = L/2 + \Delta L \). Such a function has a parabolic shape, with
the peak centered at $L/2$ and a width of $2\Delta L$. By adopting $\lambda_0 = 0.632 \mu m$ (that is, $\omega_0 = 2.98 \times 10^{15} \text{Hz}$), let us use the superposition (11) with two different values of $Q$: we shall obtain two different FWs that, in spite of having the same longitudinal intensity pattern, will have different transverse localizations. Namely, let us consider $L = 0.5 \text{m}$ and $\Delta L = L/50$, and the two values $Q = 0.99996 \omega_0/c$ and $Q = 0.99980 \omega_0/c$. In both cases the coefficients $A_n$ will be the same, calculated from Eq. (10), on using this time the value $N = 30$ in the superposition (11). The results are shown in Figures (2a) and (2b).

One can observe that both FWs have the (same) longitudinal intensity pattern, but the one with the smaller $Q$ is endowed with the higher transverse localization.

Figure 2: (a) The Frozen Wave with $Q = 0.99996 \omega_0/c$ and $N = 30$, approximately reproducing the chosen longitudinal pattern represented by Eq. (14). (b) A different Frozen wave, now with $Q = 0.99980 \omega_0/c$ (but still with $N = 30$) forwarding the same longitudinal pattern. We can observe that in this case (with a lower value for $Q$) a higher transverse localization is obtained.

4. – Generation of Frozen Waves

Concerning the generation of Frozen Waves, we have to recall that the superpositions (11), which define them, consists of sums of Bessel beams. Let us also recall that a Bessel beam, when generated by finite apertures (as it must be, in any real situations), maintains
its nondiffracting properties till a certain distance only (its field depth), given by

\[ Z = \frac{R}{\tan \theta} , \tag{15} \]

where \( R \) is the aperture radius and \( \theta \) is the so-called axicon angle, related with the longitudinal wave number by the known expression\[1\]

\[ \cos \theta = \frac{c \beta}{\omega}. \]

So, given an apparatus whatsoever capable of generating a single (truncated) Bessel beam, we can use an array of such apparatuses to generate a sum of them, with the appropriate longitudinal wave numbers and amplitudes/phases (as required by Eq.(11)), thus producing the desired FW. Here, it is worthwhile to notice that we shall be able to generate the desired FW in the the range \( 0 \leq z \leq L \) if all Bessel beams entering the superposition (11) are able to reach the distance \( L \) resisting the diffraction effects. We can guarantee this if \( L \leq Z_{\text{min}} \), where \( Z_{\text{min}} \) is the field depth of the Bessel beam with the smallest longitudinal wave number \( \beta_{n=-N} = Q - 2\pi N/L \), that is, with the shortest depth of field. In such a way, once we have the values of \( L, \omega_0, Q, N \), from Eq.(15) and the above considerations it results that the radius \( R \) of the finite aperture has to be

\[ R \geq L \sqrt{\frac{\omega_0^2}{c^2 \beta_{n=-N}^2}} - 1 \tag{16} \]

The simplest apparatus capable of generating a Bessel beam is that adopted by Durnin et al.[6], which consists in an annular slit located at the focus of a convergent lens and illuminated by a cw laser. Then, an array of such annular rings, with the appropriate radii and transfer functions able to yield both the correct longitudinal wave numbers* and the coefficients \( A_n \) of the fundamental superposition (11), can generate the desired FW. This questions will be analyze in more detail elsewhere.

Obviously, other powerful tools, like the computer generated holograms (ROACH’s approach, for instance), may be used to generated our FWs.

5. – Conclusions

In this work we have shown how Bessel beams can be used to obtain stationary localized wave fields, with high transverse localization, whose longitudinal intensity pattern

*Once a value for \( Q \) has been chosen.
can assume any desired shape within a chosen space interval \(0 \leq z \leq L\). The produced envelope remains static, i.e., with velocity \(V = 0\), and because of this we have called Frozen Waves such news solutions.

The present results can find applications in many fields:** For instance, in the optical tweezers modelling, since we can construct stationary optical fields with a great variety of shapes, capable, e.g., of trapping particles or tiny objects at different locations. This topic is being studied and will be reported elsewhere.

Acknowledgements

The author is very grateful to Erasmo Recami, Hugo E. H. Figueroa, Marco Mattiuzi, C. Dartora and V. Abate for continuous discussions and collaboration. This work was supported by FAPESP (Brazil).

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