Scalar and Tensor Parameters for Importing the Notation in Differential Geometry into Programming

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Abstract
This paper proposes a method for importing tensor index notation, including Einstein summation notation, into programming. This method involves introducing two types of parameters, i.e., scalar and tensor parameters. As an ordinary function, when a tensor parameter obtains a tensor as an argument, the function treats the tensor argument as a whole. In contrast, when a scalar parameter obtains a tensor as an argument, the function is applied to each component of the tensor. This paper shows that introducing these two types of parameters enables us to apply arbitrary functions to tensor arguments using index notation without requiring an additional description to enable each function to handle tensors. Furthermore, we show this method can be easily extended to define concisely the operators for differential forms such as the wedge product, exterior derivative, and Hodge star operator. It is achieved by providing users the method for controlling the completion of omitted indices.

Introduction
Tensor index notation invented by Ricci and Levi-Civita \cite{Ricci and Levi-Civita(1900)} has been playing a crucial role to develop differential geometry and the wide range of theoretical physics including the general theory of relativity. This is because this notation makes the description of tensor calculus intuitive.

Intuitive representation is important in programming languages as well as in mathematical notations. In computer science, a variety of programming languages have been evolving aiming to be more intuitive. For example, lexical scoping, high-order functions, and pattern-matching are features invented for that purpose.

Importing successful mathematical notations is the easiest method to improve programming languages. For example, decimal number system, function modularization, and infix notation for basic arithmetic operators have been implemented in the most of programming languages. However, the importation of
Mathematical notations into programming is not always easy. This is because the semantics of some mathematical notations are vague and complex to implement them as a part of programming languages. Tensor index notation is one of such notations.

Tensor calculus appears also in the various fields of computer science. Tensor calculus plays a fundamental role especially in computational physics and computer visions [Hartley and Zisserman(2003)]. Tensor analysis appears also in machine learning to handle multidimensional data. The importance of tensor calculus in computer science is increasing day by day.

There are several existing work for importing tensor index notation into programming. These existing work can be divided into two methods: a method that introduces special syntax and a method that introduces special operators.

In the first method, we describe the tensor calculation using syntax such as the Table expression of the Wolfram language. For example, $X_{ij} + Y_{ij}$ is represented as follows in Wolfram language. Here, it is assumed that all dimensions corresponding to each index of the tensor are a constant "M".

```
Table[X[[i,j]] + Y[[i,j]],{i,M},{j,M}]
```

We use the Sum expression inside the Table expression to describe the contraction of a tensor. For example, $X_{ij}Y_{jk}$ is represented as follows.

```
Table[Sum[X[[i,j]] * Y[[j,k]], {j,M}], {i,M},{k,M}]
```

This method has the advantage that we can use an arbitrary function defined for scalar values for tensor operations. For example, the following Wolfram program represents $\frac{\partial X_{ij}}{\partial x_k}$. "D" is the differential function in Wolfram language.

```
Table[D[X[[i,j]],x[[k]]],{i,M},{j,M},{k,M}]
```

Since this method has such advantages, a program using this method has been introduced by mathematicians in actual research. [Maeda et al.(2016)] [Maeda et al.(2010)]. However, this method has the disadvantage that we cannot directly apply functions to tensor arguments using index notation. As the result, the description of a program becomes more complicated than the description of the mathematical expressions. For example, a double loop consisting of the Table and Sum expressions appears in the program to express the contraction of a tensor in the Wolfram language, whereas the corresponding mathematical expression is flat. For existing work using this method, there are NumPy’s einsum operator [num(2018)] and Diderot’s EIN operator [Kindlmann et al.(2016)]. These operators have similar syntax with the Table expression.

In the second method, we describe the tensor calculation using operators specially prepared for tensors. For existing work using this method, there are
\[
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}
+ 
\begin{pmatrix}
10 \\
20 \\
30
\end{pmatrix}
= 
\begin{pmatrix}
11 & 21 & 31 \\
12 & 22 & 32 \\
13 & 23 & 33
\end{pmatrix}
= 
\begin{pmatrix}
11 \\
22 \\
33
\end{pmatrix}
\]

Figure 1: Interpretation of “+” as an operator for tensor product

\[
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}^i
\cdot
\begin{pmatrix}
10 \\
20 \\
30
\end{pmatrix}^i
= 10 + 40 + 90 = 140
\]

\[
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}^i
\cdot
\begin{pmatrix}
10 \\
20 \\
30
\end{pmatrix}^j

\begin{pmatrix}
10 & 20 & 30 \\
20 & 40 & 60 \\
30 & 60 & 90
\end{pmatrix}^{ij}
\]

Figure 2: Interpretation of “+” as an operator for tensor product

Maxima [max(2016), Toth(2005), Solomonik and Hoefler(2015)], a computer algebra system that introduces index notation through the extension library tensor, and Ahalander’s work [Ahlander(2002)], which implements index notation on C++. These studies introduce index notation by implementing two special functions “+” and “·” that support tensor index notation. “+” is a function that sums the components of two tensors given as arguments. “·” is a function for tensor multiplication. It takes the tensor product of the two tensors given as arguments and takes the sum of the trace if there are pairs of the superscript and subscript with the same index variable. Using this method enables index notation to be represented directly in a program as the mathematical expressions. However, this method has the disadvantage that index notation can be used only by functions that are specially prepared to use it. One of the reason of this disadvantage is that index rules differ for each operator. For example, “+” and “·” work in the different way for the same arguments such as “\(A_i + B_i\)” and “\(A_i \cdot B_i\)”. “\(A_i + B_i\)” returns a vector, but “\(A_i \cdot B_i\)” returns a scalar.

This paper proposes a new method that takes the advantages but overcome the disadvantage of the above two methods. Our proposal enables us to apply the functions defined just for scalar values directly to tensor arguments using index notation without requiring any additional description. It also simplifies the definition of tensor specific operators such as “·”.

**Result**

This paper proposes a method that enables us to apply arbitrary user-defined functions to tensor arguments using index notation without requiring an addi-
tional description to enable each function to handle tensors. It is achieved by introducing two types of parameters, scalar parameters and tensor parameters. First, we introduce them. Second, we discuss the index reduction rules that are compatible with this idea. We simplify tensor index rules by removing a rule to handle the expressions such as “\(c = A_i B_i\)” that are valid only for the Cartesian coordinate system, in which the index can move up and down freely. It enables us to use the unified index rules for all user-defined functions. Third, we discuss the index completion rules. By designing the index completion rules for omitted indices properly, we become able to concisely define the operators even for the differential forms [Schutz(1980)] such as the wedge product, exterior derivative, and Hodge star operator.

The method proposed in this paper has already been implemented in a Scheme-like functional language. In reading this paper, one can think of that language as just an extended Scheme to support symbolic computation.

Scalar and Tensor Parameters

The basic contribution of this paper is that it introduces two types of parameters, scalar and tensor parameters. Scalar and tensor parameters are used to define two types of functions, scalar functions and tensor functions, respectively. Scalar functions are functions that are defined for scalar arguments. For example, “\(+\)”, “\(-\)”, “\(*\)”, and “\(/\)” are scalar functions. When a scalar function obtains a tensor as an argument, the function is applied to each component of the tensor. Tensor functions are functions that are defined for tensor arguments. For example, the functions that calculate tensor multiplication and matrix determinant are tensor functions. When a tensor function obtains a tensor as an argument, the function treats the tensor argument as a whole.

Figure 3 shows the definition of the \texttt{min} function as an example of a scalar function. The \texttt{min} function takes two numbers as arguments and returns the smaller one. “\$” is prepended to the beginning of the parameters of the \texttt{min} function. It means the parameters of the \texttt{min} function are scalar parameters. When a scalar parameter obtains a tensor as an argument, the function is applied to each component of the tensor as Figures 4 and 5. As Figure 4, if the indices of the tensors of the arguments are different, it returns the tensor product using the scalar function as the operator. As Figure 5, if the indices of the tensors given as arguments are identical, the scalar function is applied to each corresponding component. Thus the \texttt{min} function can handle tensors even though it is defined without considering tensors. The function name “\$\texttt{min}” is also prefixed by “\$”, but just as a convention of our system. Thus it can be ignored.

Figure 6 shows the definition of the “\*” function as an example of a tensor function. “\*” is a function for multiplying tensors. “\%” is prepended to the beginning of the parameters of the “\*” function. It means the parameters of the “\*” function are tensor parameters. As with ordinary functions, when a tensor is provided to a tensor parameter, the function treats the tensor argument as

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[Schutz(1980)]

https://www.egison.org
(define $\text{min}$ (lambda ([x y]) (if (less-than? x y) x y)))

Figure 3: Definition of the \text{min} function

\[
\begin{bmatrix}
1 & 10 \\
2 & 20 \\
3 & 30
\end{bmatrix}_i \rightarrow \begin{bmatrix}
\text{min}(1, 10) & \text{min}(1, 20) & \text{min}(1, 30) \\
\text{min}(2, 10) & \text{min}(2, 20) & \text{min}(2, 30) \\
\text{min}(3, 10) & \text{min}(3, 20) & \text{min}(3, 30)
\end{bmatrix}_{ij} = \begin{bmatrix}
1 & 1 & 1 \\
2 & 2 & 2 \\
3 & 3 & 3
\end{bmatrix}
\]

Figure 4: Application of the \text{min} function to the vectors with different indices

\[
\begin{bmatrix}
1 & 10 \\
2 & 20 \\
3 & 30
\end{bmatrix}_i \rightarrow \begin{bmatrix}
\text{min}(1, 10) & \text{min}(1, 20) & \text{min}(1, 30) \\
\text{min}(2, 10) & \text{min}(2, 20) & \text{min}(2, 30) \\
\text{min}(3, 10) & \text{min}(3, 20) & \text{min}(3, 30)
\end{bmatrix}_{ii} = \begin{bmatrix}
1 & 10 \\
2 & 20 \\
3 & 30
\end{bmatrix}
\]

Figure 5: Application of the \text{min} function to the vectors with identical indices

(\text{define $\cdot$} (lambda ([t1 t2]) (contract + (* t1 t2))))

Figure 6: Definition of the \\text{"\cdot\"} function

\[
\begin{bmatrix}
1 & 10 \\
2 & 20 \\
3 & 30
\end{bmatrix}_i \cdot \begin{bmatrix}
10 & 20 & 30 \\
20 & 40 & 60 \\
30 & 60 & 90
\end{bmatrix}_i = \begin{bmatrix}
10 & 20 & 30 \\
20 & 40 & 60 \\
30 & 60 & 90
\end{bmatrix}_ij = 10 + 40 + 90 = 140
\]

Here we introduce a more mathematical example. When the mathematical expression in Figure 8 is expressed in a general way in the Wolfram language, it becomes a program such as the one shown in Figure 9. The same expression can be expressed in our system as shown in Figure 10. Our system supports two types of indices, both superscripts and subscripts. A subscript is represented

\[
\begin{bmatrix}
1 & 10 \\
2 & 20 \\
3 & 30
\end{bmatrix}_i \cdot \begin{bmatrix}
10 & 20 & 30 \\
20 & 40 & 60 \\
30 & 60 & 90
\end{bmatrix}_i = \begin{bmatrix}
10 & 20 & 30 \\
20 & 40 & 60 \\
30 & 60 & 90
\end{bmatrix}_ij
\]

a whole. When a tensor with indices is provided, it is passed to the tensor function maintaining its indices.

In Figure 6 "+" and \\text{"\cdot\"} are scalar functions for addition and multiplication, respectively. \text{contract} is a primitive function to contract a tensor that has pairs of a superscript and subscript with identical symbols. Figure 7 shows the example for calculating the inner product of two vectors using the \\text{"\cdot\"} function. We can use the \\text{"\cdot\"} function for any kind of tensor multiplication such as tensor product and matrix multiplication as well as inner product.
\[
R_{jkl}^i = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^m_{jl} \Gamma^i_{mk} - \Gamma^m_{jk} \Gamma^i_{ml}
\]

Figure 8: Formula of Riemann curvature tensor

\[
R=\text{Table}[D[\Gamma[[i,j,l]],x[[k]]] - D[\Gamma[[i,j,k]],x[[l]]] + \sum_{m} \Gamma[[m,j,l]][\Gamma[[i,m,k]]] - \Gamma[[m,j,k]][\Gamma[[i,m,l]]],
\{i,M\},
\{j,M\},
\{k,M\},
\{l,M\}]
\]

Figure 9: Wolfram program that represents the formula in Figure 8

\[
(\text{define } \_\_1_{i,j,k,l}_2)
(\text{with-symbols } \_\_1)
(+ (- (\partial/\partial \Gamma~i_j_l x~k) (\partial/\partial \Gamma~i_j_k x~l))
  (- (. \Gamma~m_j_l \Gamma~i_m_k) (. \Gamma~m_j_k \Gamma~i_m_l))))
\]

Figure 10: A program by the proposed method that represents the formula in Figure 8

by "\_\_1\_2". A superscript is represented by "~\_1\_2". Note that a double loop consisting of the Table and Sum expressions appears in the program in the Wolfram language, whereas the program in our system is flat, similarly to the mathematical expression. This is achieved by using tensor index notation in the program. In particular, the reason that the loop structure by the Sum expression in the Wolfram language does not appear in our expression to express \( \Gamma^m_{jl} \Gamma^i_{mk} - \Gamma^m_{jk} \Gamma^i_{ml} \) is that the "\_\_1\_2" function can handle Einstein summation notation.

The part that we would like the reader to pay particular attention to in this example is the program "\( (\partial/\partial \Gamma~i_j_l x~1) \)" expressing \( \Gamma_{jkl} \) in the first term on the right-hand side. In the Wolfram language, the differential function "\( \partial/\partial \)" is applied to each tensor component, but the differential function "\( \partial/\partial \)" is applied directly to the tensors in our proposed method.

The differential function "\( \partial/\partial \)" is defined in a program as a scalar function. When a tensor is provided as an argument to a scalar function, the function is applied automatically to each component of the tensor. Therefore, when defining a scalar function, it is sufficient to consider only a scalar as its argument. That is, in the definition of the "\( \partial/\partial \)" function, the programmer need only write the program for the case in which the argument is a scalar value. Despite that, the program "\( (\partial/\partial \Gamma~i_j_l x~1) \)" returns a fourth-order tensor.

Thus, we can naturally import tensor index notation including Einstein notation into programming if we clearly distinguish between tensor functions such as "\_\_1\_2" and scalar functions such as "\_\_1\_2" and "\( \partial/\partial \)".
Reduction Rules for Tensors with Indices

In this section, we discuss the index reduction rules that are compatible with the idea of scalar and tensor parameters. Tensors are combined by the unified way by scalar and tensor functions as we explained in the previous section. Therefore, we consider the reduction rules just for a single tensor.

First, to access the components of a tensor, we add integer indices to the tensor. An arbitrary number of indices can be added to one tensor, though adding a number of indices larger than the rank of the target tensor results in an error. In our system, a tensor is expressed by enclosing its components with "[" and "]". We express a higher-order tensor by nesting this description, as we do for an $n$-dimensional array.

\[
\begin{bmatrix}
\begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 
\end{bmatrix}
\end{bmatrix}_{2}\begin{bmatrix} 21 \\ 22 \\ 23 
\end{bmatrix}
\]

We can use symbols as well as integers as indices. In the proposed system, unbound variables are treated as symbols. We declare the indices of a tensor by using the symbols for the indices. If multiple indices of the same symbol appear, our system converts it to the tensor composed of diagonal components for these indices. After this conversion, the leftmost index symbol remains. For example, the indices "$i\,j\,i$" convert to "$i\,j$".

\[
\begin{bmatrix}
\begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 
\end{bmatrix}
\end{bmatrix}_{i\,j}\begin{bmatrix} 21 \\ 22 \\ 23 
\end{bmatrix}
\]

When three or more subscripts of the same symbol appear, our system converts it to the tensor composed of diagonal components for all these indices.

\[
\begin{bmatrix}
\begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 
\end{bmatrix}
\end{bmatrix}_{i\,i\,i}
\]

Superscripts and subscripts behave symmetrically. When only superscripts are used, they behave in exactly the same manner as when only subscripts are used.

\[
\begin{bmatrix}
\begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 
\end{bmatrix}
\end{bmatrix}^{1\,1}\begin{bmatrix} 11 \\ 12 \\ 13 
\end{bmatrix}
\]

\[
\begin{bmatrix}
\begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 
\end{bmatrix}
\end{bmatrix}^{i\,j\,i}\begin{bmatrix} 12 \\ 22 \\ 32 
\end{bmatrix}
\]

\[
\begin{bmatrix}
\begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 
\end{bmatrix}
\end{bmatrix}^{i\,j\,i}\begin{bmatrix} 12 \\ 22 \\ 32 
\end{bmatrix}
\]

\[
\begin{bmatrix}
\begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 
\end{bmatrix}
\end{bmatrix}^{i\,j}\begin{bmatrix} 12 \\ 22 \\ 32 
\end{bmatrix}
\]
\[ E({A, xs}) = \]
if \( e(xs) = [] \) then
\( \{A, xs\} \)
elsif \( e(xs) = \{\{k,j\}, \ldots\} \) \& \( p(k, xs) = p(j, xs) \) then
\( E(\{\text{diag}(k, j, A), \text{remove}(j, xs)\}) \)
elsif \( e(xs) = \{\{k,j\}, \ldots\} \) \& \( p(k, xs) \neq p(j, xs) \) then
\( E(\{\text{diag}(k, j, A), \text{update}(k, 0, \text{remove}(j, xs))\}) \)

**Figure 11:** Pseudo code of index reduction

The index reduction rules thus far are the same as those of the existing work \cite{Ahlander2002}.

Let us consider a case in which the same symbols are used for a superscript and a subscript. In this case, the tensor is automatically contracted using “+” in the existing research. In contrast, our system just converts it to the tensor composed of diagonal components, as in the above examples. However, in that case, the summarized indices become a *supersubscript*, which is represented by “\(_i\)”.

\[
\begin{bmatrix}
11 & 12 & 13 \\
21 & 22 & 23 \\
31 & 32 & 33
\end{bmatrix}^i_i
\]

Even when three or more indices of the same symbol appear that contain both supersubscripts and subscripts, our system converts it to the tensor composed of diagonal components for all these indices.

\[
\begin{bmatrix}
11 & 21 & 31 \\
13 & 41 & 1
\end{bmatrix}^i_i
\]

The reason not to contract it immediately is to parameterize an operator for contraction. The components of supersubscripts can be contracted by using the `contract` expression. The `contract` expression receives a function to be used for contraction as the first argument, and a target tensor as the second argument.

\[
(\text{contract} + \begin{bmatrix}
11 & 22 & 33 \end{bmatrix}^i_i)
\]

Figure 11 shows pseudo code of index reduction as explained in this section. \( E(A, xs) \) is a function for reducing a tensor with indices. \( A \) is an array that consists of tensor components. \( xs \) is a list of indices appended to \( A \). For example, \( E(A, xs) \) works as follows with the tensor whose indices are “\(_i\_j\_i\)”. We use “\(_1\)”, “\(_-1\)”, and “\(_0\)” to represent a superscript, subscript, and supersubscript, respectively.

\[
E(\begin{bmatrix}
11 & 21 \end{bmatrix} \begin{bmatrix}
13 & 41 \end{bmatrix} \begin{bmatrix}
15 & 61 \\
17 & 81 \end{bmatrix} \end{bmatrix}^i_i, \\
\begin{bmatrix}
1, 1, \{i,1\}, \{1,-1\}\end{bmatrix} = \\
\begin{bmatrix}
11 & 51 & 16 & 81 \end{bmatrix}, \begin{bmatrix}
i, 0, \{j,1\}\end{bmatrix}
\]

---

8
Next, we explain the helper functions used in Figure 11. \( \text{e}(\text{xs}) \) is a function for finding pairs of identical indices from \( \text{xs} \). \( \text{diag}(k, j, \mathbf{A}) \) is a function for creating the tensor that consists of diagonal components of \( \mathbf{A} \) for the \( k \)-th and \( j \)-th order. \( \text{remove}(k, \text{xs}) \) is a function for removing the \( k \)-th element from \( \text{xs} \). \( \text{p}(k, \text{xs}) \) is a function for obtaining the value of the \( k \)-th element of the assoc list \( \text{xs} \). \( \text{update}(k, p, \text{xs}) \) is a function for updating the value of the \( k \)-th element of the assoc list \( \text{xs} \) to \( p \). These functions work as follows.

\[
\begin{align*}
\text{e}([[i, 1], [j, -1], [i, 1]]) & = [[1, 3]] \\
\text{diag}(1, 2, [[[11 12]], [[21 22]]]) & = [[11 22]] \\
p(2, [[i, 1], [j, -1]]) & = -1 \\
\text{remove}(2, [[i, 1], [j, -1]]) & = [[i, 1]] \\
\text{update}(2, 0, [[i, 1], [j, -1]]) & = [[i, 1], [j, 0]]
\end{align*}
\]

**Implementation of Scalar and Tensor Parameters**

In this section, we explain how to implement scalar and tensor parameters.

As with ordinary parameters, when a tensor parameter obtains a tensor as an argument, the function treats the tensor as it is. It means the implementation of tensor parameters is same with the ordinary parameters.

In contrast, when a scalar parameter obtains a tensor as an argument, the function is applied to each component of the tensor. A function with scalar parameters is converted to a function only with tensor parameters by using the tensor-map function as follows. In this way, we can implement scalar parameters.

\[
(\lambda [x y] \ldots) 
;=>(\lambda [x y] 
  (\text{tensor-map} \ (\lambda [x] 
    (\text{tensor-map} \ (\lambda [y] \ldots) 
      y) 
    x))
\]

As the name implies, the tensor-map function applies the function of the first argument to each component of the tensor provided as the second argument. If the result of applying the function of the first argument to each component of the tensor provided as the second argument is a tensor with indices, it moves those indices to the end of the tensor that is the result of evaluating the tensor-map function.

Let’s review the \texttt{min} function defined in Figure 3 as an example. This \texttt{min} function can handle tensors as arguments as follows.

\[
\begin{align*}
\text{min} \ [1 2 3]_{i} & \ [10 20 30]_{j} \\
;=[[11 111] [[22 222] [33 233]]]_{i,j}
\end{align*}
\]

\[
\begin{align*}
\text{min} \ [1 2 3]_{i} & \ [10 20 30]_{i} \\
;=[[1 1 1] [12 22] [33 33]]_{i}
\end{align*}
\]

Note that the tensor indices of the first evaluated result are “\( i \_ j \)”. If the tensor-map function simply applies the function to each component of the tensor, the result of this program will be similar to
“[[[1 1 1]]_j [[[2 2 2]]_j [[[3 3 3]]_j]]_i”]. However, as explained above if the results of applying the function to each component of the tensor are tensors with indices, it moves those indices to the end of the tensor that is the result of evaluating the tensor-map function. This is the reason that the indices of the evaluated result are “_i_j”. This mechanism enables us to directly apply scalar functions to tensor arguments using index notation as the above example.

Next, let’s review the “..” function defined in Figure 6 as an example of a tensor function. When a tensor with indices is given as an argument of a tensor function, it is passed to the tensor function maintaining its indices. Note that we can directly apply tensor functions to tensor arguments using index notation as in the following example.

( ( [1 2 3]_i [10 20 30]_i )
;140
( ( [1 2 3]_i [10 20 30]_j )
;1 ( [10 20 30] [20 40 60] [30 60 90] )_i_j
( ( [1 2 3]_i [10 20 30]_i )
;1 ( 10 40 90 )_i

Since a tensor parameter is used only when defining a function that contracts tensors, in most cases only scalar parameters are used.

Inverted Scalar Arguments

The “∂/∂” function appearing in Figure 10 is a scalar function. However, “∂/∂” is not a normal scalar function. “∂/∂” is a scalar function that inverts indices of the tensor given as its second argument. For example, the program “∂/∂ Π_i-j-k x_i” returns the fourth-order tensor with the indices “_1-j-k-l”.

To define scalar functions such as “∂/∂”, we use inverted scalar parameters. Inverted scalar parameters are represented by “.$$$. A program that uses inverted scalar parameters is transformed as follows. Here, the flip-indices function is a primitive function for inverting the indices of a tensor provided as an argument upside down. Supersubscripts remain as supersubscripts even if they are inverted.

(define $$∂/∂ (lambda [f $$x] ...))
=>(define $$∂/∂ (lambda [%f %x]
(tensor-map (lambda [%f] (tensor-map (lambda [%x] ...)
(flip-indices x)))

In the following example, we apply “∂/∂” to tensors.

(∂/∂ ( [(* r (sin θ)) (* r (cos θ))]_i [r θ]_j )
;11 [((sin θ) (* r (cos θ))] [((cos θ) (* -1 r (sin θ))]_i-j

https://github.com/egison/egison/blob/master/lib/math/analysis/derivative.egi
\[
\frac{\partial}{\partial \theta} \left[ (\sin \theta) (\cos \theta) \right] \_1 \left( r \theta \right) \_1 \\
;\left[ \left( \sin \theta \right) (\cos \theta) \right] ^\_i 
\]\n
The with-symbols Expression

The with-symbols expression is syntax for generating new local symbols, such as the Module expression in the Wolfram language. One-character symbols that are often used as indices of tensors such as “i”, “j”, and “k” are often used in another part of a program. Generating local symbols using with-symbols expressions enables us to avoid variable conflicts for such symbols.

The with-symbols expression takes a list of symbols as its first argument. These symbols are valid only in the expression given in the second argument of the with-symbols expression.

\[
\text{(with-symbols \{i\} (contract + (* [1 2 3] \_i [10 20 30] \_i)))} \\
;60
\]

It acts in a special way when the evaluation result of the body of the with-symbols expression contains the symbols generated by the with-symbols expressions. In that case, the result tensor is transposed to shift those symbols backward and remove them. In the following evaluation result, the matrix is transposed because “j” is shifted backward before it removed. This mechanism is useful to handle differential forms that will be discussed in the next section.

\[
\text{(with-symbols \{j\} \left[ \left[ \left[ 1 \ 2 \right] \left[ 3 \ 4 \right] \right] \_j \_i \right]\left[ \left[ 1 \ 3 \right] \left[ 2 \ 4 \right] \right] \_i \_i} 
\]

Index Completion Rules for Tensors with Omitted Indices

By designing the index completion rules for omitted indices properly, we can extend our method explained so far to express a calculation handling the differential forms [Schutz(1980)].

Differential Forms

In mathematics, we treat a n-th order tensor as a k-form when only \((n - k)\) indices are appended to it. For example, we treat a third order tensor “\(\omega \_i \_j\)” as a matrix-valued 1-form. We import the same convention into programming. Then, let’s consider the index completion rules that goes with this convention.

Let A and B be scalar-valued 2-forms.

In default, we complement the same indices to each argument of a scalar function as follows.

\[
(+ A B) \\
;=>(+ A_{t1 \_t2} B_{t1 \_t2})
\]

[http://reference.wolfram.com/language/ref/Module.html](http://reference.wolfram.com/language/ref/Module.html)
Most of cases, if we complement indices as above, we can represent the operation for differential forms. However, this completion is not suitable for functions specially defined for differential forms. For example, in the case of the wedge product, we would like to append the different indices to each argument as follows.

\[(\wedge A B) \Rightarrow (\wedge A_{t1\_t2} B_{t3\_t4})\]

We introduce the “!" operator for that purpose. If the “!" operator is prepended to the function application, the omitted indices are complemented by the latter method. The function for calculating the wedge product is defined as follows using the “!" operator.

\[(\text{define } \wedge (\lambda [X \ Y] !(. X Y)))\]

We can define also the exterior derivative as follows. The flip function is a function for swapping the arguments of a two-argument function. It is used to transpose the result.

\[(\text{define } d (\lambda [X] !((\text{flip } \partial / \partial) \text{ params } X)))\]

Hodge star operator is defined as follows.

\[(\text{define } \ast (\lambda [A] (\text{let } \{[k (\text{df-order } A)]\}
\text{(with-symbol } \{i\ j\}
\text{(*) (sqrt \{\text{abs (M-det g_#_#)}\)}
\{\varepsilon[i_1\ldots i_n] \cdot g_{j_1\ldots j_k}
\text{(map (lambda [x] g[i\_x\][j\_x]) (between 1 k)))))))))\]

It is a direct translation of the following mathematical formula for Hodge star operator.\(^6\)

\[\ast A = \sqrt{\text{det}|g|} \cdot \varepsilon_{i_1\ldots i_n} \cdot A_{j_1\ldots j_k} \cdot g^{i_1j_1} \cdots g^{i_kj_k} \cdot e^{i_{k+1} \wedge \ldots \wedge e^{i_n}}\]

In the above program, the df-order function is a function that returns \(n\) when it obtains an \(n\)-form. The M.det function is a function for calculating the determinant of the argument matrix. \(\varepsilon\) is the Levi-Civita symbols. "#" used in the indices as “g_#_#" represents a dummy symbol. All instances of “#” are treated as different symbols. In the writing of a program that deals with high-order tensors, the number of symbols used for indices increases. A dummy symbol is introduced to suppress that. Using this mechanism makes it easier to distinguish indices that are important in the program, thereby also improving the readability of the program.

Thus, we can define concisely the operators for differential forms by controlling the completion of omitted indices as the above method. We can see the sample programs that use the functions defined above in Egison Mathematics Notebook\(^7\).

\(^6\)https://ncatlab.org/nlab/show/Hodge+star+operator
\(^7\)https://www.egison.org/math/
Demonstration

In this section, we present a program for calculating the Riemann curvature tensor \( \text{Fleisch}(2011), \text{Schutz}(1980), \text{Ollivier}(2011) \), the fourth-order tensor that expresses the curvature of a manifold, to demonstrate our proposal. Figure [12] is the program for calculating the Riemann curvature tensor of \( S^2 \) using the formula of curvature form.

In our system, when binding a tensor to a variable, we can specify the type of indices in the variable name. For example, we can bind different tensors to “\( g_{--} \)”, “\( g_{--} \)”, “\( \Gamma_{--} \)”, and “\( \Gamma_{--} \)”. This feature is also implemented in Maxima \([\text{max}(2016)]\). This feature simplifies variable names.

In Figure [12], some of the tensors are bound to a variable with symbolic indices such as “\( \Gamma_{i--j} \)”. It is automatically desugared as follows. This syntactic sugar renders a program closer to the mathematical expression. \text{transpose} is a function for transposing the tensor in the second argument as specified in the first argument.

\[
\text{transpose}(\Gamma_{i--j} \ldots)
\]

\[
\Rightarrow (\text{define } \Gamma_{i--j} \ldots)
\]

The \text{df-normalize} function is used in the definition of the curvature form “\( \Omega_{i--j} \)”. It is a function for normalizing a differential form to an anti-symmetric tensor. It is defined in our system as a tensor function.

Christoffel symbols of the first and second kind, and curvature form are defined in mathematics as follows. We can see our proposal succeeds to express these formulas directly in the program.

\[
\Gamma_{ijk} = \frac{1}{2} \left( \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{kj}}{\partial x^i} \right)
\]

\[
\Gamma^{i}_{jk} = g^{im} \Gamma_{mjk}
\]

\[
\Omega^{i}_{j} = d\omega^{i}_{j} + \omega^{i}_{k} \wedge \omega^{k}_{j}
\]

Discussion

We introduced several forms of syntax into a language to introduce the new concepts of scalar and tensor functions. However, we think it is possible to introduce the concepts of scalar and tensor functions even using a static type system or the overloading feature of object-oriented programming. For example, in a static type system, whether the parameter of a function is a scalar or tensor parameter can be specified when specifying the type of the function. Although we could not discuss this in this paper, it is an interesting research topic to think about how to incorporate the ideas proposed in this paper into existing programming languages naturally.

In particular, it is of substantial significance to incorporate this method into programming languages such as Formura \([\text{Muranushi et al.}(2016)]\) and Diderot \([\text{Kindlmann et al.}(2016)]\).
Parameters and metric tensor

```
(define $x \theta \varphi)
```

```
(define $g_{-} [1 0 r^{-2} 1])
(define $g^{-} [1 0 0 1])
```

Christoffel symbols

```
(define $\Gamma_{i \ j \ k} \left( \partial x_{j}/\partial x_{k} \right)
\left( \partial x_{i}/\partial x_{k} \right)
\left( \partial x_{i}/\partial x_{j} \right))
```

```
(define $\Gamma^{i \ j \ k} \left( \text{with-symbols \{m,n\}} \right)
\left( \Gamma_{m \ j \ k} \right))
```

Connection form

```
(define $\omega^{i \ j} \left( \text{with-symbols \{k\}} \right)
\left( \Gamma_{i \ j \ k} \right))
```

Curvature form

```
(define $\Omega^{i \ j} \left( \text{with-symbols \{k\}} \right)
\left( \text{df-normalize} \left( \omega^{i \ j} \right) \right)
```

Figure 12: A program for calculating Riemann curvature tensor of $S^2$ using the formula of curvature form

that have a compiler that generates code for executing tensor calculation in parallel. For example, incorporating this method into Formura would enable us to describe physical simulation using not only the Cartesian coordinate system but also more general coordinate systems such as the polar and spherical coordinate system in simple programs.

By the way, index notation as discussed in this paper is a notation invented over a century ago. Especially, it is well known that Einstein summation notation was invented by Einstein when he was working on general relativity theory. In addition to index notation, there might still be many notations in mathematics that are useful, but not yet introduced into programming. There might also be notations that describe the formulas of existing theories more concisely, but that mathematicians have not discovered yet.

We contend that it is very useful for those researching programming languages who are familiar with many programming paradigms and can flexibly create new programming languages to learn a wider range of mathematics for the future of programming languages.

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