Abstract—It has been shown that graph-cover pseudocodewords can be used to characterize the behavior of sum-product algorithm (SPA) decoding of classical codes. In this paper, we leverage and adapt these results to analyze SPA decoding of quantum stabilizer codes. We use the obtained insights to formulate modifications to the SPA that overcome some of its weaknesses.

I. INTRODUCTION

Graph covers have been shown to be a useful tool for analyzing sum-product algorithm (SPA) decoding of classical codes [1]. The task of analyzing the behavior of SPA decoding for quantum stabilizer codes is more challenging, especially because the degeneracy of quantum stabilizer codes needs to be taken into account. Despite these challenges, being able to understand and improve the behavior of the SPA is highly desirable, since it has been observed that the performance of the SPA is far from satisfactory when decoding quantum stabilizer codes, e.g., the toric cycle codes [5], [9].

In this paper, in a first step, we use graph-cover pseudocodewords to analyze the behavior of SPA. In particular, we can show that the decoding ability of the SPA is limited by the minimum distance of the normalizer label code, which is a serious problem for quantum LDPC codes, e.g., the toric codes [5] and MacKay’s bicycle codes [6], where the minimum distance of the normalizer label code is no larger than the row distance of the normalizer label code.

In a second step, we use the obtained insights to formulate modifications to the SPA that overcome some of its weaknesses. Taking advantage of the degeneracy of the quantum stabilizer code, the performance of the decoder is then limited by the minimum distance of the quantum stabilizer code instead of the minimum distance of the normalizer label code.

This paper is organized as follows. In Section II we review some basic notations including the stabilizer formalism and the standard SPA for quantum stabilizer codes. In Section III we analyze the performance of SPA for quantum stabilizer codes and give some other theoretical results about degenerate decoders of quantum stabilizer codes. In Section IV-A, we propose, first, some methods to improve the performance of the SPA for general quantum stabilizer codes and, second, a pseudocodeword-based decoder for quantum cycle codes. Finally, we show some simulation results in Section V.

II. BASICS

A. Quantum Stabilizer Formalism

We refer the readers to [7], [8] for a detailed introduction to quantum stabilizer codes, some recent developments of quantum error-correction codes, and more details of the notations. Moreover, see [3] for the use of pseudocodewords in the context of quantum stabilizer codes. Due to the page limitations, we only introduce the essential notations which are used throughout the paper.

Consider an $[n, k, d]$ quantum stabilizer code $C$ of length $n$, dimension $k$, and minimum distance $d$. The quantum stabilizer code $C$ may be characterized using the equivalent binary representation of its stabilizer, namely its binary stabilizer label code $B$, which is self-orthogonal under the symplectic inner product to guarantee the commutativity of generators of the stabilizer. The binary representation of a Pauli operator on $n$ qubits is a length-$2n$ binary vector $v = [v_1, ..., v_n] \in (\mathbb{F}_2^n)$, where each $v_i$ is obtained by mapping $I$, $X$, $Y$, and $Z$ onto $\mathbb{F}_2$ as follows

$I \mapsto [0,0], \ X \mapsto [1,0], \ Y \mapsto [1,1], \text{ and } Z \mapsto [0,1],$

and the weights of them are defined to be, respectively,

$\text{wt}([0,0]) \triangleq 0 \text{ and } \text{wt}([1,0]) = \text{wt}([1,1]) = \text{wt}([0,1]) \triangleq 1. $

In this paper, we make the following assumptions:

- the normalizer label code $N$ is the dual code of $B$ under the symplectic inner product (note that the self-orthogonality of $B$ implies that $B \subseteq N$);
- both $B$ and $N$ are binary linear codes of length $2n$ and of dimension $n-k$ and $n+k$, respectively;
- the weight of $v$ is $\text{wt}(v) \triangleq \sum_i \text{wt}(v_i)$;
- $d \triangleq \min_{v \in N} \text{wt}(v)$ and $t \triangleq \lfloor \frac{d+1}{2} \rfloor$;
- $d_N \triangleq \min_{v \in N} \text{wt}(v)$ and $t_N \triangleq \lfloor \frac{d_N}{2} \rfloor$.

A quantum stabilizer code $C$ is called a quantum cycle code if its normalizer label code $N$ is a cycle code, which means that the number of 1’s per column of the parity-check matrix $H$ describing $N$ is two. For example, the toric codes are quantum cycle codes [5], [9].

The quantum channel that we use in this paper is the quantum depolarizing channel (QDCh). Similar to the binary symmetric channel (BSC), the action of a QDCh with depolarizing probability $p$ is such that it acts independently on each

Supported in part by RGC GRF grant 2150965.
qubit: a qubit is either unchanged with probability $1 - p$, or affected by a unitary operator $X$, $Y$, or $Z$, each with probability $p/3$. Since we are decoding with respect to binary normalizer label codes, decoding is based on approximating the QDCh by two independent BSCs with crossover probability $2p/3$, i.e., the probability for having a bit-flip and a phase-flip is $2p/3$ independently for each qubit.

**Definition 1.** Given a syndrome $s \in \mathbb{F}_2^{n-k}$, let $s \mapsto t(s)$ be the mapping giving a coset representative of the coset of $N$ corresponding to the syndrome $s$. Note that if $e$ is the binary representation of the actual error, then $He^T = s^T$ and $e \in t(s) + N$.

A non-degenerate decoder $D_{ND}$ outputs a vector based on the syndrome $s$; a vector $v$ is considered a decoding error of $D_{ND}$ if $v \neq D_{ND}(vH^T)$. A degenerate decoder $D_D$ outputs a coset of $B$ based on the syndrome $s$; a vector $v$ is considered a decoding error of $D_D$ if $v \notin D_D(vH^T)$. The blockwise ML non-degenerate decoder and degenerate decoders $D_{ND}^{ML}$, $D_D^{ML}$, and $D_D^{ML^*}$ are defined to be, respectively,

$$D_{ND}^{ML}(s) \triangleq \arg \min_{v \in t(s)+N} \text{wt}(v), \quad D_D^{ML}(s) \triangleq D_{ND}^{ML}(s) + B,$$

$$D_D^{ML^*}(s) \triangleq \arg \max_{\ell \in B: \ell + B \in t(s) + N} p(\ell + B|s),$$

where $p(\ell + B|s)$ is the probability of the coset $\ell + B$ based on the syndrome $s$.

For the simulations in these papers, there is a decoding error if the output vector is not in the same coset of $B$ as the actual error or the output coset is not the same coset of $B$ as the actual error.

**B. SPA decoding, graph covers, and pseudocodewords**

SPA decoding of a quantum stabilizer code $C$ consists of the following steps: 1) running the SPA on a factor graph representing the coset of the normalizer label code $N$, where the coset is defined by the syndrome $s$ that is obtained from suitable quantum measurements; 2) outputting a vector $v$, 3) finding the coset of $B$ containing $v$. (For further details, see, e.g., Section IV.) In this paper, the factor graphs are normal factor graphs, where variables are associated with edges.

It was shown in [10] that fixed points of the SPA correspond to stationary points of the Bethe free energy function. As discussed in [1], for LDPC codes this means that the beliefs obtained at a fixed point of the SPA induce a pseudocodeword. The paper [1] also introduced the symbolwise graph-cover decoder, a decoder that finds the pseudocodeword with minimal Bethe free energy, or, equivalently, the pseudocodeword with the most pre-images in all $M$-covers of the base normal factor graph (after properly discounting for a channel-output-dependent term), when $M$ goes to infinity. For general codes, symbolwise graph-cover decoding is an approximation of the true behavior of SPA decoding. However, for cycle codes it was shown in [11] that SPA decoding is equivalent to symbolwise graph-cover decoding. Note that, although symbolwise graph-cover decoding is based on $M$-covers where $M$ goes to infinity, in many instances the study pseudocodewords induced by codewords in $M$-covers for small $M$ gives already many insights into the suboptimality of SPA decoding (see, e.g., the upcoming Fig. 2 that shows an $M$-cover for $M = 2$).

### III. Theoretical Analysis

In this section we characterize the performance of the non-degenerate and degenerate decoders defined in Definition 1. In particular, in Theorems 2 and 3 we prove that the minimum weight of decoding errors for the non-degenerate and degenerate decoders is $t_N + 1$ and $t + 1$, respectively. Moreover, in Theorems 4 and 5 we show two types of decoding errors limiting the performance of SPA decoding of quantum cycle codes.

**Theorem 2.** The minimum weight of decoding errors of $D_{ND}^{ML}$ is $t_N + 1$.

*Proof.* See Appendix A.

**Theorem 3.** The minimum weights of decoding errors of $D_D^{ML}$ and $D_D^{ML^*}$ both are $t + 1$.

*Proof.* See Appendix B.

**Theorem 4.** The minimum weight of errors that the SPA fails to decode for a toric code is 2. For a $[2L^2, 2, L]$ toric code with $L \geq 5$, the number of such weight-2 decoding errors of is $12L^2$.

*Proof.* For a $[2L^2, 2, L]$ toric code $C$ with $L < 5$, it cannot correct some weight-2 errors because of minimum distance. For a $[2L^2, 2, L]$ toric code $C$ for $L \geq 5$, there are two types of weight-2 errors that cannot be corrected using SPA decoding shown in Fig. 1. (Here and for other similar figures we use the drawing conventions listed in Table II, moreover, edges with pseudowights close to 0 are not drawn). After rescaling the pseudocodewords obtained from SPA decoding, we have the same pseudocodewords $\omega = [1, 1, 1, 1]$ for both cases, where we start from the left upper corner and go in clockwise direction. The SPA decoder either gets confused when the unscaled component is 0.5 at each edge or chooses the zero vector which has a wrong syndrome when the maximum unscaled component is less than 0.5. Since weight-1 errors can be decoded by SPA decoders, the dominating decoding errors of SPA decoding for toric codes are of weight 2. If we count the number of such weight-2 errors, there are 6 in each length-4 cycles and $12L^2$ in total.

For a quantum cycle code with even $d_N$, the minimum weight of decoding errors of SPA decoding is no larger than $d_N/2$ because of similar problems as in Fig. 1.

**Theorem 5.** The minimum weight of errors that SPA fails to decode for a toric code is no larger than $d_N$.

*Proof.* We want to show that there exist errors of weight $d_N$ that SPA fails to decode. Since the minimum weight of vectors in the normalizer label code $N$ is $d_N$, there exists a cycle
TABLE I: Drawing conventions for figures.

|        |        |
|--------|--------|
| empty  | $s_1 = 0$ for syndrome bit associated with $i$-th parity check |
| filled | $s_1 = 1$ for syndrome bit associated with $i$-th parity check |
| black  | channel introduced no error at that location |
| red    | channel introduced an error at that location |

of length $d_N$ in $\mathcal{N}$ and we assume the error is a path of length $d_N$ starting from any check involved in that cycle. Fig. 2 is a 2-cover of the relevant part of a toric code. We claim that the SPA cannot decode the above mentioned error. The reason is as follows. Besides the valid configuration (in red) in the base graph, there is another type of valid configuration (in blue) in some of the 2-covers. (The components of the pseudocodewords resulting from these valid configurations are shown next to the corresponding edges in Fig. 2.) The SPA converges to some pseudocodeword, which is approximately the weighted sum of all codewords in the base graph and graph covers, e.g., a rescaled pseudocodeword in Fig. 3. The SPA cannot output a vector with valid syndrome for all the possible such pseudocodewords, since the codeword (in red) in the base graph and the codeword (in blue) from some 2-cover have the same weight and similar contribution to the pseudocodeword.

More generally, for a quantum cycle code, the minimum weight of decoding errors of the SPA is no larger than $d_N$ for similar reasons.

IV. PSEUDOCODEWORD-BASED DECODING

If we want to improve the performance of SPA of quantum stabilizer cycle codes, or, more generally, quantum stabilizer codes, the first task is to address the problem mentioned in the proof of Theorem 4 by breaking the symmetry of the SPA to avoid ending up with pseudocodewords like the ones in Fig. 1. A. Reweighted SPA Decoding

Our first approach is to use the reweighted SPA decoding proposed in [12], which reweights message calculations. However, instead of uniformly reweighing the messages, we randomly select weights from a certain interval. We call the resulting algorithm randomly reweighted SPA (RR-SPA). Empirically, this method can improve the performance of SPA decoding of the toric codes, but there is not much improvement for MacKay’s bicycle codes.

B. Initial-message-reweighted SPA of Quantum Stabilizer Codes

In order to introduce our second approach, we recall that the SPA is based on the log-likelihood ratios (LLRs) $\gamma_i \triangleq \log \left( \frac{p(e_i = 1)}{p(e_i = 0)} \right)$ and the syndrome $s$. Our second approach is Algorithm 1 called initial-message-reweighted SPA (IMR-SPA), that also runs the SPA, however, now the LLRs are reweighted, i.e., $\gamma_i$ is replaced by $w_i \gamma_i$, where $w_i$ is a weighting factor randomly generated from some interval. Empirically, it is observed that the RR-SPA and the IMR-SPA have similar performance for the toric codes. From an analysis point of view, the IMR-SPA is preferable compared to the RR-SPA, because after suitable adaptations, we can apply the Bethe free energy framework [1], [10], [11] to analyze the IMR-SPA.

We briefly explain why the IMR-SPA helps to improve the performance of SPA decoding of quantum stabilizer codes. Namely, assume that we know the actual error vector $\tilde{e}$ for analysis. For SPA decoding, using the LLR vector $\gamma$ with the syndrome $s$ is equivalent to using the LLR vector $\tilde{\gamma}$, where
\begin{algorithm}
\caption{Initial-message-reweighted SPA (IMR-SPA)}
\begin{algorithmic}[1]
\Require the syndrome \( s \), the max. number of iterations for SPA, and the reweighting range \([a, b]\).
\Ensure \( v + B \).
\Inputs\end{algorithmic}
\begin{algorithmic}[1]
\State Use SPA to obtain an output vector \( v \).
\If{\( Hv^T = s^T \) (equivalently \( v \in t(s) + N \))}
\State Output \( v + B \).
\Else
\While{\( Hv^T \neq s^T \)}
\State For the \( i \)-th variable, randomly generate a weighting factor \( w_i \in [a, b] \) and reweight the LLR to the SPA from \( \gamma_i \) to be \( w_i \gamma_i \).
\State Use SPA to obtain an output vector \( v \).
\{Set the max. number of trial times if necessary.\}
\EndWhile
\EndIf
\end{algorithmic}
\end{algorithm}

\[ \tilde{z}_i \triangleq (-1)^{\tilde{e}_i} \gamma_i, \] with the syndrome \( 0 \). SPA decoding succeeds when it converges to the all-zero vector based on the LLR vector \( \tilde{\gamma} \) and the syndrome \( 0 \). The IMR-SPA changes the LLR vector for the standard SPA from \( \tilde{z}_i \) to be \( w_i \tilde{z}_i \) and hence may move some \( \tilde{\gamma} \) from the “bad” region to the “good” region in which the SPA converges to the all-zero vector.

\section{Pseudocodeword-based Decoder of Quantum Cycle Codes}

For quantum cycle codes, the IMR-SPA decpoding can improve the minimum weight of decoding errors beyond \( d_N/2 \), but it is still limited by the problems mentioned in Theorem 5. Therefore, we propose a pseudocodeword-based decoder described in Algorithm 2 abbreviated as SPA+PCWD, to further improve the performance of SPA decoding for quantum cycle codes. When SPA decoding fails to output a vector with valid syndrome, we hope to make use of the SPA pseudocodeword to obtain one with valid syndrome. There are two difficulties in this task: 1) parts of components of the pseudocodeword contributed by valid configurations of graph covers are misleading for the decoder and should be removed; 2) the valid configurations of the base graph are mixed together in the pseudocodeword and we need to find a way to pick a valid configuration out of them. The main idea of the decoder is to start from any check with \( s_i = 1 \) and always follow the edge with largest possible component for the next step until reaching another check \( s_j = 1 \). When the error paths are not too close to each other, the weight of each edge is contributed by all the valid paths where shorter paths have larger weight and this algorithm is likely to find a codeword with largest probability by a greedy search at each step. We use a simple example to explain the procedure of Algorithm 2

\textbf{Example 6.} Consider a \([2L^2, 2, L]\) toric code of \( L = 9 \). With \( p = 4/(4L^2) \), we obtain a rescaled SPA pseudocodeword \( \tilde{\omega} \triangleq \omega/0.0836 \) after 100 iterations in Fig. 4 after rescaling, where pseudowights less than \( 0.001 \cdot (\max_i \tilde{\omega}_i) \) are ignored. First, we can remove the parts of the components contributed by codewords from graph covers and obtain the pseudocodeword in Fig. 4. Then, we start from the left upper check \( s_1 = 1 \) and follow the edges with largest possible component until reaching \( s_2 = 1 \) to obtain one of the paths in Fig. 5. Set all the weights in that path to be 1, output the vector; and set \( s_1 = s_2 = 0 \). If the syndrome \( s \) has not become the all-zero vector, repeat the above steps until it becomes the all-zero vector.

\section{V. SIMULATION RESULTS}

Fig. 6 shows some simulation results of SPA+PCWD decoding of toric codes described in Algorithm 2 where we use at most 100 iterations of SPA to obtain SPA pseudocodewords. According to the simulation results in [4], the performance of the original SPA gets worse as the code block length of toric codes increases. As shown in Fig. 6 the performance of the SPA+PCWD improves as the code block length of toric codes increases and the SPA+PCWD has similar performance as the neural belief-propagation decoder [4]. Fig. 7 shows the weight distribution of the decoding errors of SPA+PCWD decoding of toric codes, where the minimum weight of errors increases as the block length increases. We also observed that the IMR-SPA and the RR-SPA have similar performance as the SPA+PCWD.
The maximum number of IMR trials is 10. The IMR-SP A has better performance than the neural belief-propagation decoder for a MacKay’s bicycle code with the total row weight 11 and the depolarizing probability less than $10^{-6}$. As shown in Fig. 6, the average word error rate (WER) of IMR-SP A is lower than that of SPA, except for toric codes with $L < 9$. However, there are at least 20 errors for points with probability less than $10^{-6}$ and at least 100 errors for other points are collected.

Fig. 7: Weight distribution of decoding errors of SPA+PCWD for toric codes (solid: average, dashed: minimum).

for toric codes with $L < 9$, but unfortunately they are limited by some weight-4 errors for $L \geq 9$.

Fig. 8 shows some simulation results of the IMR-SP A of a $[256, 32]$ MacKay’s bicycle code with the total row weight 11, 9, 7, 5, and 3 for toric codes (solid: average, dashed: minimum). As shown in Fig. 8, the average word error rate (WER) of IMR-SP A is lower than that of SPA, except for toric codes with $L < 9$. However, there are at least 20 errors for points with probability less than $10^{-6}$ and at least 100 errors for other points are collected. The IMR-SP A has better performance than the neural belief-propagation decoder for a MacKay’s bicycle code with the total row weight 11 and the depolarizing probability less than $10^{-6}$. As shown in Fig. 6, the average word error rate (WER) of IMR-SP A is lower than that of SPA, except for toric codes with $L < 9$. However, there are at least 20 errors for points with probability less than $10^{-6}$ and at least 100 errors for other points are collected.

APPENDIX A

PROOF OF THEOREM 2

Omitted. The idea is to show each coset of $\mathcal{N}$ contains at most one vector of weight less than or equal to $t_N$ and there exists some coset of $\mathcal{N}$ containing two vectors of weights less than or equal to $t_N + 1$, one of which is a decoding error.

APPENDIX B

PROOF OF THEOREM 3

Let the set of decoding errors and the minimum weight of decoding errors for $D_A^B$ be $E_A^B \triangleq \mathbb{F}_2^{2n} \setminus (\cup_{s=1}^{D_A^B} \{s\})$ and $d_A^B \triangleq \min_{v \in E_A^B} \text{wt}(v)$. We first show that $d_D^{ML^*} \leq t + 1$ and then $d_D^{ML^*} \geq t + 1$.

- ($d_D^{ML^*}, d_D^{ML} \leq t + 1$) Since $d \triangleq \min_{v \in E_A^B} \text{wt}(v)$, there exists $v \in \mathcal{N} \setminus B$ such that $\text{wt}(v) = d$. There exist $v_1, v_2 \in \mathbb{F}_2^{2n}$ such that $v_1 + v_2 = v$, $\text{wt}(v_2) = t + 1$, and $\text{wt}(v_1) = d - \text{wt}(v_2) \leq d_N - (t + 1) \leq t + 1$. Then we have $v_1 \in t(s) + \ell_1 + B$ and $v_2 \in t(s) + \ell_2 + B$, for some $s$ and $\ell_1 \neq \ell_2 \in \mathcal{N}$. Then at most one of them can be in $D_D^{ML^*}(s)$ or $D_D^{ML}(s)$. Hence $v_{i_1} \in E_{D_D}^{ML^*}$ and $v_{i_2} \in E_{D_D}^{ML}$ for some $i_1, i_2$ and $d_{D_D}^{ML^*} \leq \text{wt}(v_{i_1}) \leq t_N + 1$ and $d_{D_D}^{ML} \leq \text{wt}(v_{i_2}) \leq t_N + 1$.

- ($d_D^{ML^*} \geq t + 1$) For any syndrome $s \in \mathbb{F}_2^{2n-k}$, there is at most one coset of $B$ in $t(s) + \mathcal{N}$ containing vectors of weights smaller or equal to $t$, otherwise suppose there are $v_1, v_2 \in t(s) + \mathcal{N}$ with $\text{wt}(v_1), \text{wt}(v_2) \leq t$ such that $v_1 \in t(s) + \ell_1 + B$ and $v_2 \in t(s) + \ell_2 + B$ for $\ell_1 \neq \ell_2 \in \mathcal{N}$, and then $v_1 + v_2 \in \mathcal{N} \setminus B$ with $\text{wt}(v_1 + v_2) \leq 2t < d$. A contradiction arises. Hence all vectors of weight no larger than $t$ are in $\cup_{s=1}^{D_D^{ML^*}} \{s\}$ and not in $E^{ML}$, which implies $d_{D_D}^{ML^*} \geq t + 1$.

In general, $D_D^{ML^*}$ is more optimal compared with $D_D^{ML^*}$ and hence $d_D^{ML} \geq d_D^{ML^*} \geq t + 1$ which implies $d_D^{ML} = t + 1$. 
REFERENCES

[1] P. O. Vontobel, “Counting in graph covers: A combinatorial characteriza-
tion of the bethe entropy function,” IEEE Trans. Inf. Theory, vol. 59, no. 9, pp. 6018–6048, 2013.

[2] Z. Babar, P. Botsinis, D. Alanis, S. X. Ng, and L. Hanzo, “Fifteen years of quantum LDPC coding and improved decoding strategies,” IEEE Access, vol. 3, pp. 2492–2519, 2015.

[3] J. X. Li and P. O. Vontobel, “LP decoding of quantum stabilizer codes,” in Proc. IEEE 2018 Int. Symp. Inf. Theory, Vail, Colorado, USA, June 17 – 22 2018, pp. 1306–1310.

[4] Y.-H. Liu and D. Poulin, “Neural belief-propagation decoders for quantum error-correcting codes,” arXiv preprint arXiv:1811.07835, 2018.

[5] J.-P. Tillich and G. Zémor, “Quantum LDPC codes with positive rate and minimum distance proportional to the square root of the blocklength,” IEEE Trans. Inf. Theory, vol. 60, no. 2, pp. 1193–1202, 2014.

[6] D. J. MacKay, G. Mitchison, and P. L. McFadden, “Sparse-graph codes for quantum error correction,” IEEE Trans. Inf. Theory, vol. 50, no. 10, pp. 2315–2330, 2004.

[7] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information. UK: Cambridge University Press, 2010.

[8] D. A. Lidar and T. A. Brun, Quantum Error Correction. Cambridge University Press, 2013.

[9] J. X. Li and P. O. Vontobel, “Factor-graph representations of stabilizer quantum codes,” in Proc. 54th Allerton Conf. on Communication, Control, and Computing, Allerton House, Monticello, IL, USA, Sep. 28–30 2016, pp. 1046–1053.

[10] J. S. Yedidia, W. T. Freeman, and Y. Weiss, “Constructing free-energy approximations and generalized belief propagation algorithms,” IEEE Trans. Inf. Theory, vol. 51, no. 7, pp. 2282–2312, Jul. 2005.

[11] H. D. Pfister and P. O. Vontobel, “On the relevance of graph covers and zeta functions for the analysis of SPA decoding of cycle codes,” in Proc. IEEE 2013 Int. Symp. Inf. Theory, Istanbul, Turkey, July 7 – 12 2013, pp. 3000–3004.

[12] H. Wymeersch, F. Penna, and V. Savić, “Uniformly reweighted belief propagation: A factor graph approach,” in Proc. IEEE Int. Symp. Inf. Theory, Saint-Petersburg, Russia, July, 31 – August, 5 2011, pp. 2000–2004.