SOME TYPES OF COMPOSITION OPERATORS ON SOME SPACES OF FUNCTIONS

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ABSTRACT. We investigate some types of composition operators, linear and not, and conditions for some spaces to be mapped into themselves and for the operators to satisfy some good properties.

1. Introduction

The following non-linear operators, $C_f, x \mapsto f \circ x$ and $S_h, x(\cdot) \mapsto h(\cdot, x(\cdot))$ called, respectively, the (autonomous) composition operator and the (non-autonomous) superposition operator, have been widely studied. They appear especially in the process of solving certain non-linear integral equations. For instance, in [4] and [5], the authors show that existence and uniqueness results for solutions of non-linear integral equations of Hammerstein-Volterra type

$$x(t) = g(t) + \lambda \int_0^t k(t, s)f(x(s))ds, \quad (t \geq 0)$$

$$x(t) = g(t) + \lambda \int_0^t k(t, s)h(s, x(s))ds, \quad (t \geq 0)$$

and of Abel-Volterra type

$$x(t) = g(t) + \int_0^t \frac{k(t, s)f(x(s))}{|t-s|^\nu}ds, \quad (0 \leq t \leq 1)$$

are closely related to existence and uniqueness results for solutions of operator equations involving $S_h$ and $C_f$. Also, for example, for the integral equation of Volterra type in the Henstock setting it is proved, in [39], that the existence of a continuous solution depends, among other conditions, on the property of the involved non-autonomous superposition operator of mapping continuous functions into Henstock-integrable functions; in [25], the authors provide, in the Henstock-Kurzweil-Pettis setting, existence and closure results for integral problems driven by regulated functions, both in single- and set-valued cases [24]. Hence, in many fields of non-linear analysis and its applications (in particular to integral equations), the following problem becomes of interest:

Given a class $X$ of functions, find conditions on (eventually characterise) the functions under which the generated operators map the space $X$ into itself.

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The case of the operator $S_h$ is called in the literature Superposition Operator Problem ([10], [12], [13]) or, sometimes, Composition Operator Problem ([4], [7]) since it is also considered for the autonomous case $C_f$ and, also in this simpler form, it is sometimes unexpectedly difficult. In addition to the action spaces of the non-linear operators $C_f$ and $S_h$, also boundedness and continuity are properties which have been object of several studies: many results analysing such properties for composition operators on function spaces, among which $\text{Lip}$, $\text{Lip}_\gamma$, $\text{BV}$, $\text{BV}_p$, $\text{AC}$ and $W^{1,p}$, appeared in the last decades ([4], [5], [6], [7], [8], [9], [10], [12], [13], [15], [16], [17], [18], [19], [20], [21], [22], [23]).

This note is intended to serve as a survey on the state of art of some aspects and to describe some further properties of the non-linear operators $S_h$ and $C_f$ (left composition operator), and of the linear operator $C^R_f : x \mapsto x \circ f$ (right composition operator), discussing them and illustrating examples. Clearly, the theory is wide and far from being complete.

This note is so organised. In Section 2, we briefly introduce the investigated function spaces, and we recall some main properties. In Section 3, we analyse the non-linear operators $C_f$ and $S_h$, first on Lipschitz spaces and on some spaces of functions of bounded variations, providing the main results in literature, and with examples. Then, we focus on spaces of Baire functions. In particular, we show that when the operator $C_f$ maps the space of Baire functions into itself, then it is automatically continuous and bounded. We, also, characterise the non-linear operators $S_h$ which transform Baire one functions into maps of the same type, and we show how to easily construct a function $h$ which is not even Baire one but such that the associated operator $S_h$ maps the space of Baire functions into itself.

Section 4 is devoted to the linear composition operator $C^R_f$. We start by investigating Lipschitz spaces and some spaces of functions of bounded variations. In particular, our study shows that, unlike the case of left compositors, not all the investigated spaces have the same type of right compositors. Then, we study the linear operator $C^R_f$ on the space of Baire one functions and we develop some parallel results on the space of Baire two functions. Also, here, unlike the case of left Baire compositors which are the same for Baire classes of any order, and in particular for Baire one functions and Baire two functions, we show that it is not the case. Namely, we show that right Baire one compositors do not coincide with right Baire two compositors.

2. Preliminary definitions

In this section we collect some basic definitions and results, which will be needed in the sequel.

2.1. Lipschitz and $\gamma$-Lipschitz functions.

**Definition 2.1.1.** A function $f : [a, b] \to \mathbb{R}$ is said to be Lipschitz (or Lipschitz continuous) if there exists a constant $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y| \quad (a \leq x, y \leq b).$$

More generally, $f$ is said to be $\gamma$-Lipschitz (or $\gamma$-Lipschitz continuous or Hölder continuous), for $0 < \gamma \leq 1$, if there exists a constant $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|^{\gamma} \quad (a \leq x, y \leq b).$$
By $Lip([a, b])$, we denote the space of all Lipschitz functions on $[a, b]$, and, by
$Lip_\gamma([a, b])$, we denote the space of all $\gamma$-Lipschitz functions on $[a, b]$.
For $f \in Lip([a, b])$, let
$$Lip(f) = Lip(f; [a, b]) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$  
For $f \in Lip_\gamma([a, b])$, let
$$Lip_\gamma(f) = Lip_\gamma(f; [a, b]) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$  
The spaces $Lip([a, b])$ and $Lip_\gamma([a, b])$, endowed with the norms
$$\|f\|_{Lip} = f(a) + Lip(f)$$
and
$$\|f\|_{Lip_\gamma} = f(a) + Lip_\gamma(f),$$
are, respectively, Banach spaces. The following strict inclusion holds:
$$Lip_\alpha([a, b]) \subset Lip_\beta([a, b]) \quad (0 < \beta < \alpha \leq 1).$$

2.2. $p$-variation, Jordan variation, Riesz variation.

**Definition 2.2.1.** Let $f$ be a real valued function defined on $H \subseteq \mathbb{R}$. For $p > 0$ we
denote by $V_p(f, H)$ the $p$-variation of $f$ on $H$, that is the least upper bound of the sums
$$\sum_{i=1}^{n} |f(b_i) - f(a_i)|^p,$$
where $\{[a_i, b_i]\}_{i=1,...,n}$ is an arbitrary finite system of non-overlapping intervals with
$a_i, b_i \in H$, $i = 1, ..., n$.

If $H$ has a minimal as well as a maximal element, then $V_p(f, H)$ is the supremum of the sums
$$\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p,$$
where $\min(H) = t_0 < t_1 < \cdots < t_n = \max(H)$ and $t_i \in H$, $i = 0, ..., n$.

In the sequel of this paragraph, we consider $f$ as a function defined on a closed interval of the real line, that is $f : [a, b] \to \mathbb{R}$.

**Definition 2.2.2.** We define $BV_p([a, b]) = \{f : [a, b] \to \mathbb{R} : V_p(f, [a, b]) < +\infty\}$, i.e. $BV_p([a, b])$ is the space of functions of $p$-bounded variation on $[a, b]$.

**Definition 2.2.3.** When $p = 1$ the variation $V_1(f, [a, b])$ is called the Jordan variation of $f$ on $[a, b]$ and it is simply denoted by $V(f, [a, b])$. In particular, the space $BV_1([a, b]) = \{f : [a, b] \to \mathbb{R} : V(f, [a, b]) < +\infty\}$ is the space of functions of bounded Jordan variation on $[a, b]$ and it is simply denoted by $BV([a, b])$.

**Remark 2.2.4.** It is well-known ([4], [5], [9]) that the space $BV_p([a, b])$, $p \geq 1$, ended with the norm $\|f\|_{BV_p} = |f(a)| + V_p(f, [a, b])^{\frac{1}{p}}$ is a Banach space. In particular, the space $BV([a, b])$ ended with the norm $\|f\|_{BV} = |f(a)| + V(f, [a, b])$ is a Banach space. Moreover, for $1 < p < q < \infty$, the following (strict) inclusions hold
$$BV([a, b]) \subset BV_p([a, b]) \subset BV_q([a, b]) \subset B([a, b])$$
where \( B([a, b]) = \{ f : [a, b] \to \mathbb{R} : f \text{ is bounded} \} \).

As it is well-known, the space \( BV([a, b]) \) is not closed under composition. For example, take \([a, b] = [0, 1]\) and \( f = g \circ h \) where \( g(x) = \sqrt{x} \) and \( h(x) \) be so defined

\[
h(x) = \begin{cases} 
0 & \text{if } x = 0 \\
x^2 \sin^2 \left( \frac{1}{x} \right) & \text{otherwise}.
\end{cases}
\]

In the case of continuous functions, we have the following definition.

**Definition 2.2.5.** Let \( f \) be continuous on \([a, b]\). Let \( G \) be the union of all open subintervals of \((a, b)\) on which \( f \) is either strictly monotonic or constant. The set of points of varying monotonicity of \( f \) is defined as

\[
K_f = [a, b] \setminus G.
\]

**Theorem 2.2.6.** ([33]: Theorem 2.3) For every \( f \in C([a, b]) \) and \( p \geq 1 \) we have

\[
V_p(f, K_f) = V_p(f, [a, b]).
\]

Let \( CBV_p([a, b]) = \{ f \in C([a, b]) : V_p(f, K_f) < \infty \} \).

**Corollary 2.2.7.** ([33]: Corollary 2.4) If \( p \geq 1 \) then \( CBV_p([a, b]) \) is the family of those \( f \in C([a, b]) \) for which \( V_p(f, [a, b]) < +\infty \).

As remarked in [33], no analogous statement as Theorem 2.2.6 holds if \( 0 < p < 1 \) since the only continuous functions \( f \) with \( V_p(f, [a, b]) < +\infty \) are constants. On the other hand, \( CBV_p([a, b]) \) contains, for example, the continuous, strictly monotone functions on \([a, b]\).

Now, we introduce another type of variation: the Riesz variation.

**Definition 2.2.8.** Let \( \mathcal{P} \) be the family of all partitions of the interval \([a, b]\). Given a real number \( p \geq 1 \), a partition \( P = \{ t_0, ..., t_m \} \) of \([a, b]\), and a function \( f : [a, b] \to \mathbb{R} \), the nonnegative real number

\[
RV_P(f, P) = RV_P(f, P, [a, b]) = \sum_{j=1}^{m} \frac{|f(t_j) - f(t_{j-1})|^p}{(t_j - t_{j-1})^{p-1}}
\]

is called the Riesz variation of \( f \) on \([a, b]\) with respect to \( P \). The (possibly infinite) number

\[
RV_p(f) = RV_p(f, [a, b]) = \sup \{ RV_P(f, P, [a, b]) : P \in \mathcal{P} \},
\]

where the supremum is taken over all the partitions of \([a, b]\), is called the total Riesz variation of \( f \) on \([a, b]\). In case \( RV_p(f) < \infty \) we say that \( f \) has bounded Riesz variation (or bounded \( p \)-variation in Riesz’s sense) on \([a, b]\), and we write \( f \in RBV_p([a, b]) \).

**Remark 2.2.9.** It is well-known ([4], [5]) that the space \( RBV_p([a, b]) \) equipped with the norm \( \| f \|_{RBV_p} = |f(a)| + RV_p(f) \) is a Banach space.

Functions of bounded Riesz variation are particularly interesting since they are related to Sobolev spaces, as we will see in the next paragraph (see Theorem 2.3.2).
2.3. Absolutely continuous functions.

**Definition 2.3.1.** A function \( f : [a, b] \to \mathbb{R} \) is said to be absolutely continuous if it satisfies the following property: for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( \{[a_k, b_k]\}_{k=1}^n \) is any finite collection of pairwise non-overlapping, closed subintervals of \([a, b]\) with \( \sum_{k=1}^n (b_k - a_k) < \delta \), then

\[
\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon.
\]

By \( AC([a, b]) \), we denote the space of all absolutely continuous functions on \([a, b]\). The following strict inclusion holds

\[
Lip([a, b]) \subset AC([a, b]).
\]

On the other hand, none of the \( Lip_\gamma([a, b]) \), \( 0 < \gamma < 1 \), is contained in \( AC([a, b]) \) (see, for example, [4]: Example 1.23). Moreover, \( AC([a, b]) \) is closed in \( BV([a, b]) \), and therefore it is a Banach space with respect to the \( BV \)-norm, which is equivalent to the \( W^{1,1} \)-norm

\[
\|f\|_{W^{1,1}} = \|f\|_{L^1} + \|f'\|_{L^1}.
\]

In fact, it coincides with \( W^{1,1}([a, b]) \) (see, for instance, [4]: Proposition 3.24; [5]: page 10, 1.1.16).

Recall the following useful relations.

(1) For \( 1 < q < p < +\infty \),

\[
Lip([a, b]) \subset W^{1,q}([a, b]) \subset W^{1,p}([a, b]) \subset AC([a, b]) \subset C([a, b]) \cap BV([a, b]) \subset B([a, b]).
\]

Alle the above inclusions are strict.

(2) Let \( 0 < \gamma < 1 \). Then

\[
Lip_\gamma([a, b]) \not\subseteq BV([a, b]).
\]

Moreover, there exists \( f \in \cap_{0 < \gamma < 1} Lip_\gamma([a, b]) \setminus BV([a, b]) \) ([4]: Example 1.23 and Example 1.24).

As already recalled above, the space \( RBV_p([a, b]) \) is basically the same of the Sobolev space \( W^{1,p}([a, b]) \), by the following well-known theorem.

**Theorem 2.3.2** (Riesz Theorem). ([5]: Theorem 1.3.5) Let \( 1 < p < \infty \). A function \( f : [a, b] \to \mathbb{R} \) belongs to \( RBV_p([a, b]) \) if and only if \( f \in AC([a, b]) \) and \( f' \in L^p([a, b]) \). Moreover, in this case the equality

\[
RV_p(f) = \|f'\|_{L^p([a,b])}^p = \int_a^b |f'(t)|^p \, dt
\]

holds, where \( RV_p(f) \) is the \( p \)-variation of \( f \) in the Riesz’s sense.

For \( p = 1 \), \( RBV_1([a, b]) = BV([a, b]) \). Hence, the previous theorem does not hold for \( p = 1 \) as a function in \( BV([a, b]) \) usually does not need to be continuous and therefore nor absolutely continuous.
2.4. Baire functions.

Recall that a topological space $X$ is called Polish space if it is separable and completely metrizable.
Let $X$ be a Polish space. Let $\mathcal{G}$ be the collection of all open subsets of $X$ and let $\mathcal{F}$ be the collection of all closed subsets of $X$. Recall that an $F_\sigma$ set is a countable union of closed sets, a $G_\delta$ set is a countable intersection of open sets, and a $G_{\delta\sigma}$ set is a countable union of $G_\delta$ sets [14]. In every metrizable spaces, every open set is an $F_\sigma$ set ([3]).
A real valued function $g : X \to \mathbb{R}$ is said to be Baire one if there exists a sequence \( \{g_k\}_{k \in \mathbb{N}} \) of continuous functions $g_k : X \to \mathbb{R}$ such that $\lim_{k \to +\infty} g_k(x) = g(x)$, for every $x \in X$. These functions are so called since they were first defined and studied by Baire ([11]). Clearly, each continuous function is of Baire class one.

A real valued function $g : X \to \mathbb{R}$ is said to be Baire two if there exists a sequence \( \{g_k\}_{k \in \mathbb{N}} \) of Baire one functions $g_k : X \to \mathbb{R}$ such that $\lim_{k \to +\infty} g_k(x) = g(x)$, for every $x \in X$. And, in general, a real valued function $g : X \to \mathbb{R}$ is said to be Baire class $n$ if there exists a sequence $\{g_k\}_{k \in \mathbb{N}}$ of functions of Baire class $n - 1$, $g_n : X \to \mathbb{R}$, such that $\lim_{n \to +\infty} g_n(x) = g(x)$, for every $x \in X$.

Denote by $\mathcal{B}_0(X)$ the collection of real valued continuous function on $X$, that is $\mathcal{B}_0(X) = C(X)$, by $\mathcal{B}_1(X)$ the collection of real valued Baire one functions of $X$, and, in general, by $\mathcal{B}_n(X)$, the collection of real valued Baire $n$ functions on $X$. Then, the following inclusions (all strict) hold:

$$C(X) = \mathcal{B}_0(X) \subset \mathcal{B}_1(X) \subset \cdots \subset \mathcal{B}_n(X) \subset \mathcal{B}_{n+1}(X) \subset \cdots.$$ 

Several equivalent definitions of Baire class one functions have been obtained so far. It is known that $g$ is Baire one if and only if for every open set $A$, $g^{-1}(A)$ is an $F_\sigma$ set, and, moreover, $g$ is Baire two if and only if for every open set $A$, $g^{-1}(A)$ is a $G_{\delta\sigma}$ set (see for instance [31], [40]).

In particular, given $g : X \to \mathbb{R}$, the following are equivalent:

1. $g$ is Baire one
2. for every open subset $A$ of $\mathbb{R}$, $g^{-1}(A)$ is an $F_\sigma$ set
3. for every closed set $C$ in $X$, the restriction $g|_C$ has a point of continuity in $C$.

Clearly, if a function $g : X \to \mathbb{R}$ has countably many discontinuity points then it is Baire one. In particular, if $g : [a, b] \to \mathbb{R}$ is monotone, or of bounded variation, then $g$ is Baire one. In general, the functions of Baire class one play an important role in applications. For example, the semi-continuous functions, the derived functions, all belong to this class ([14], [32]).

If $g$ is Baire one, then the set of points of continuity of $g$ is a residual subset of $X$. This last property is not a characterisation as the following example shows ([38]: page 148, Example IV). Interesting very recent results concerning fixed points of Baire functions and the so called equi-Baire property are in [1] and [2].

**Example 2.4.1.** Let $X = [0, 1]$. Let $C$ be the Cantor ternary set. The set $C$ has Lebesgue measure zero and is of first category since it is nowhere dense. Let $\mathcal{C}_0$ be the collection of the points of $P$ which are not endpoints of the complementary intervals. Let $f = \chi_C$ and $g = \chi_{\mathcal{C}_0}$. Then $f$ and $g$ are continuous at points of $[0, 1] \setminus C$ and discontinuous at points of $C$. But $f$ is Baire one as it is the
characteristic function of a closed set but $g$ is not Baire one as $g_C$ is discontinuous at every point.

Another well-known example of non Baire one function is the following: the Dirichlet function.

**Example 2.4.2.** Let $X = [0, 1]$. The Dirichlet function is the map $g(\cdot) = \chi_{\mathbb{Q} \cap [0,1]}(\cdot)$. List all the rationals in $[0,1]$ as $r_1, r_2, \ldots, r_k, \ldots$. Define, for each $n \in \mathbb{N}$,

$$g_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, \ldots, r_n\} \\ 0 & \text{otherwise} \end{cases}$$

As $g_n$ has finitely many discontinuity points, it is of Baire class one. The Dirichlet map is the pointwise limit of the sequence $\{g_n\}_{n \in \mathbb{N}}$. So it is Baire two but not Baire one.

The following is a beautiful, natural characterisation of a Baire one function.

**Theorem 2.4.3.** ([35]: Theorem 1) Suppose $f : X \to Y$ is a mapping between complete separable metric spaces $(X, d_X)$ and $(Y, d_Y)$. Then the following statements are equivalent.

1. For any $\epsilon > 0$, there exists a positive function $\delta$ on $X$ such that $d_Y(f(x), f(y)) < \epsilon$ whenever $d_X(x, y) < \min\{\delta(x), \delta(y)\}$.
2. The function $f$ is of Baire class one.

The function $\delta$ of the Theorem 2.4.3 can be chosen to be Baire one as the following theorem shows.

**Theorem 2.4.4.** ([34]: Corollary 33) Suppose $f : X \to Y$ is a mapping between complete separable metric spaces $(X, d_X)$ and $(Y, d_Y)$. Then the following statements are equivalent.

1. For any $\epsilon > 0$, there exists a positive Baire class one function $\delta$ on $X$ such that $d_Y(f(x), f(y)) < \epsilon$ whenever $d_X(x, y) < \min\{\delta(x), \delta(y)\}$.
2. The function $f$ is of Baire class one.

3. **Two types of non-linear operators: (left) composition operators and superposition operators**

Recall that an operator between two normed spaces is said to be bounded if it maps bounded sets into bounded sets. Unlike the case of a linear operator, in the non-linear case, the two properties of being bounded and of being continuous are not equivalent. They are not even, in general, related: a non-linear operator may be continuous without being bounded, or bounded without being continuous.

**Definition 3.0.1.** Let $J$ be an arbitrary interval. Let $f : \mathbb{R} \to \mathbb{R}$. The operator $g \mapsto f \circ g$ where $g : J \to \mathbb{R}$ is an arbitrary function, is called the (autonomous) composition operator generated by the function $f$. It is usually denoted by $C_f$. Hence, for each $g : J \to \mathbb{R}$, $C_f(g) : J \to \mathbb{R}$ is defined as $C_f(g)(\cdot) = f(g(\cdot))$.

**Definition 3.0.2.** Let $J$ be an arbitrary interval. Let $h : J \times \mathbb{R} \to \mathbb{R}$. The operator $g(\cdot) \mapsto h(\cdot, g(\cdot))$, 
where $g : J \to \mathbb{R}$ is an arbitrary function, is called the (non-autonomous) superposition operator the generated by the function $h$. It is usually denoted by $S_h$. Hence, for each $g : J \to \mathbb{R}$, $S_h(g) : J \to \mathbb{R}$, is defined as $S_h(g)(\cdot) = h(\cdot, g(\cdot))$.

### 3.1. Composition Operators.

Composition operators on Lipschitz functions and on some spaces of functions of bounded variations

**Theorem 3.1.1.** ([4]: Theorem 5.9) The operator $C_f$ maps the space $BV([a, b])$ into itself if and only if the corresponding function $f$ is locally Lipschitz on $\mathbb{R}$, i.e. for each $r > 0$, there exists $k(r) > 0$ such that

$$
|f(u) - f(v)| \leq k(r)|u - v|, \quad (u, v \in \mathbb{R}, |u|, |v| \leq r).
$$

**Theorem 3.1.2.** ([5]: Theorem 3.4.1; [4]: Theorem 5.24) Let $1 < p < \infty$, $0 < \gamma \leq 1$. The following conditions are equivalent.

(a) The function $f : \mathbb{R} \to \mathbb{R}$ satisfies the local Lipschitz condition $(\ast)$.

(b) The operator $C_f$ maps the space $BV_p([a, b])$ into itself.

(c) The operator $C_f$ maps the space $BV([a, b])$ into itself.

(d) The operator $C_f$ maps the space $AC([a, b])$ into itself.

(e) The operator $C_f$ maps the space $RBV_p([a, b])$ into itself.

(f) The operator $C_f$ maps the space $Lip,([a, b])$ into itself.

Moreover, in this case the operator $C_f$ is automatically bounded.

**Theorem 3.1.3.** ([5]: Theorem 3.1.7) Under the hypothesis $(\ast)$, the operator $C_f$ is automatically continuous in $BV([a, b])$.

**Theorem 3.1.4.** ([5]: Theorem 3.4.2) Under the hypothesis $(\ast)$, the operator $C_f$ is automatically continuous in $RBV_p([a, b])$, $1 < p < \infty$.

Note that the equivalence between condition (a) and (d) of Theorem 3.1.2 is a particular case of the following more general result involving Sobolev spaces, which follows from Theorem 1 in [37]:

**Theorem 3.1.5.** Let $1 \leq q \leq p < \infty$, and let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function. Then the composition operator $C_f$ maps the space $W^{1,p}([a, b])$ into $W^{1,q}([a, b])$ if and only if $f$ satisfies the local Lipschitz condition $(\ast)$. Moreover, the operator $C_f$ is bounded and the following inequality holds:

$$
\|C_f(g)\|_{W^{1,q}} \leq b(M)(1 + \|g\|_{W^{1,p}}),
$$

where $\|g\|_{W^{1,p}} \leq M$ and $b(M)$ is a constant depending on $M$.

Some spaces behave well with respect to the composition operator, as the following results show:

**Theorem 3.1.6.** ([4]: Theorem 5.20) The operator $C_f$ maps $C([a, b])$ into itself if and only if $f$ is continuous on $\mathbb{R}$. In this case, the operator $C_f$ is automatically bounded and continuous in the norm $\| \cdot \|_C$.

**Remark 3.1.7.** Let $0 < \gamma < 1$. As example 5.25 in [4] shows, there exists a composition operator $C_f$ that maps $Lip,([0, 1])$ into itself but is not continuous. In order to have the continuity of $C_f$, extra properties have to be satisfied by the
generating function $f$. In [28] the authors prove that $C_f$ is continuous on $\text{Lip}_\gamma([a,b])$ if and only if $f \in C^1(\mathbb{R})$. In Theorem 5.26 of [4], the authors prove that the continuity of $C_f$, defined from $\text{Lip}_\gamma([a,b])$ into itself, is equivalent to its uniform continuity on bounded subsets.

Coherently with the terminology used in [40] by Zhao in the case of Baire functions, given a space of real functions $X$ defined on a real interval $J$, we say that a function $f : \mathbb{R} \to \mathbb{R}$ is a left $X$ compositor if $f \circ g \in X$ for every $g \in X$. Hence we can re-write Theorem 3.1.2 as a characterisation of left compositors for some spaces.

**Theorem 3.1.8.** Let $1 < p < \infty$, $0 < \gamma \leq 1$. Let $f : \mathbb{R} \to \mathbb{R}$. The following statements are equivalent.

1. The function $f$ satisfies the local Lipschitz condition $(\ast)$.
2. The function $f$ is a left $BV_p([a,b])$ compositor.
3. The function $f$ is a left $BV([a,b])$ compositor.
4. The function $f$ is a left $AC([a,b])$ compositor.
5. The function $f$ is a left $RBV_p([a,b])$ compositor.
6. The function $f$ is a left $Lip_\gamma([a,b])$ compositor.

**Remark 3.1.9.** Hence, the collections of left $AC([a,b])$, $BV_p([a,b])$ ($1 \leq p < \infty$), $RBV_p([a,b])$ ($1 < p < \infty$), $\text{Lip}_\gamma([a,b])$ ($0 < \gamma \leq 1$), respectively, compositors, are all the same, namely they all coincide with the collection of all maps satisfying $(\ast)$.

From Theorem 3.1.8 and the fact that the composition, the sum and the product of two functions satisfying the local Lipschitz condition $(\ast)$ still satisfies the local Lipschitz condition $(\ast)$, we have the following proposition.

**Proposition 3.1.10.** Let $1 < p < \infty$, $0 < \gamma \leq 1$. Let $X = BV_p([a,b])$, $BV([a,b])$, $AC([a,b])$, $RBV_p([a,b])$, $\text{Lip}_\gamma([a,b])$. Then, the following hold.

1. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are left $X$ compositors then so is the sum $f + g$.
2. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are left $X$ compositors then so is the product $fg$.
3. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are left $X$ compositors then so is the composition $f \circ g$.

In particular, for these spaces, the collection of left compositors is a vector space and an algebra.

**Composition operators on spaces of Baire functions**

Let $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. If $g$ is a Baire one function and $f$ is continuous, then the composition function $f \circ g$ is Baire one but, as it is well-known, the composition of two Baire one functions is not necessarily Baire one. Here is a well-known example.

**Example 3.1.11.** ([40]: Example 1) Let $f : \mathbb{R} \to \mathbb{R}$ be so defined

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n}, n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$
and \( g : \mathbb{R} \to \mathbb{R} \) be the Riemann function so defined

\[
g(x) = \begin{cases} 
\frac{1}{q} & \text{if } x = \frac{p}{q}, \text{ } p \text{ and } q \text{ are co-prime integers and } 0 < q \\
1 & \text{if } x = 0 \\
0 & \text{otherwise.}
\end{cases}
\]

Then \( f \circ g \) is the Dirichlet function, that is not Baire one.

Thus, as done above for other spaces, it is natural to ask which functions \( f \) have the property that their composition with any Baire one function still is of Baire class one, that is \( C_f \) maps the space of Baire one functions in itself.

**Remark 3.1.12.** Zhao, in [40], calls a function \( f : \mathbb{R} \to \mathbb{R} \) for which \( C_f(\mathcal{B}_1) \subseteq \mathcal{B}_1 \) a left Baire one compositor, that is, \( f \) is a left Baire one compositor if and only if \( f \circ g \) is Baire one whenever \( g \) is a Baire one function.

The following result follows from Theorem 3 of [27], in the case of Baire one functions and, more generally, from Theorem D of [30], for Baire functions of class \( n \).

**Theorem 3.1.13.** Let \( n \in \mathbb{N} \). Let \( f : \mathbb{R} \to \mathbb{R} \). The following are equivalent:

(a) The function \( f \) is continuous.
(b) The operator \( C_f \) maps the space \( \mathcal{B}_n \) into itself.

A subset \( A \) of \( \mathcal{B}_n \) is bounded if each element \( h : [a, b] \to \mathbb{R} \) in \( A \) is bounded, that is \( \|h\|_{\infty} = \sup_{x \in [a, b]} |h(x)| < \infty \), and there exists \( M > 0 \) with \( \|h\|_{\infty} \leq M, \forall h \in A \).

We have that, in the case of Baire functions, the composition operator behaves well. Namely, the following holds.

**Theorem 3.1.14.** Let \( n \in \mathbb{N} \). Let \( f : \mathbb{R} \to \mathbb{R} \). If \( C_f \) maps the space \( \mathcal{B}_n \) into itself then it is automatically continuous with respect to pointwise convergence, and it is bounded.

**Proof.** As \( C_f \) maps the space \( \mathcal{B}_n \) into itself, by Theorem 3.1.13, \( f \) is continuous. Assume that \( \{g_k\}_{k \in \mathbb{N}} \) is a sequence in \( \mathcal{B}_n \) converging pointwise to a function \( g \) of Baire class \( n \). Then \( C_f(g_k) = f \circ g_k \) is a sequence of Baire functions of class \( n \) pointwise converging to the Baire class \( n \) function \( C_f(g) = f \circ g \).

Let \( A \) be a bounded subset of \( \mathcal{B}_n \) and let \( M > 0 \) be such that \( \|h\|_{\infty} \leq M, \forall h \in A \).

As \( f \) is continuous on \( \mathbb{R} \), the restriction of \( f \) to the compact \( [-M, M] \) admits a maximum. Call this maximum \( L \). Then, for each \( h \in A \), it is \( \|C_f(h)\|_{\infty} = \|f \circ h\|_{\infty} \leq L \). Hence, \( C_f(A) \) is bounded. As \( A \) was an arbitrary bounded subset of \( \mathcal{B}_n \), the thesis follows.

\[ \square \]

A natural question arises: what about right compositors in all the previous spaces? Clearly, right composition \( g \mapsto g \circ f \), defined with a suitable \( f \) and on suitable spaces, is linear.

This question is investigated in Section 4.
3.2. Superposition operators.

Superposition operators on Lipschitz functions and some spaces of functions of bounded variations

As for the case of composition operators, we are interested in investigating which spaces are mapped by $S_h$ into themselves, and, in general, in the properties of $S_h$. Natural sufficient conditions are the following:

**Proposition 3.2.1.** ([10]: Proposition 3.1) Let $h : [a, b] \times \mathbb{R} \to \mathbb{R}$, $0 < \gamma < 1$, and $1 < p < +\infty$.

1. If $h(\cdot, y) \in \text{Lip}$ uniformly w.r.t. $y$ then $S_h$ maps $\text{Lip}$ into itself.
2. If $h(\cdot, y) \in \text{BV}$ uniformly w.r.t. $y$ and $h(x, \cdot) \in \text{Lip}$ uniformly w.r.t. $x$ then $S_h$ maps $\text{BV}$ into itself.
3. If $h(\cdot, y) \in \text{AC}$ uniformly w.r.t. $y$ and $h(x, \cdot) \in \text{Lip}$ uniformly w.r.t. $x$ then $S_h$ maps $\text{AC}$ into itself.
4. If $h(\cdot, y) \in \text{Lip}_\gamma$ uniformly w.r.t. $y$ and $h(x, \cdot) \in \text{Lip}$ uniformly w.r.t. $x$ then $S_h$ maps $\text{Lip}_\gamma$ into itself.
5. If $h(\cdot, y) \in \text{BV}_p$ uniformly w.r.t. $y$ and $h(x, \cdot) \in \text{Lip}$ uniformly w.r.t. $x$ then $S_h$ maps $\text{BV}_p$ into itself.

Other known results are the following.

**Theorem 3.2.2.** ([4]: Theorem 6.1) Let $h : [a, b] \times \mathbb{R} \to \mathbb{R}$. The operator $S_h$ maps $C([a, b])$ into itself if and only if $h$ is continuous on $[a, b] \times \mathbb{R}$. In this case, the operator $S_h$ is automatically bounded and continuous in the norm $\| \cdot \|_\infty$.

**Theorem 3.2.3.** ([4]: Theorem 6.4) Let $0 < \gamma \leq 1$. Let $h : [a, b] \times \mathbb{R} \to \mathbb{R}$. The operator $S_h$ maps $\text{Lip}_\gamma([a, b])$ into itself and is bounded w.r.t. the norm $\| \cdot \|_{\text{Lip}_\gamma}$, if and only if $h$ satisfies the mixed local Hölder-Lipschitz condition

$$|h(s, u) - h(t, v)| \leq k(r)|s - t|^{\gamma} + |u - v| \quad (a \leq s, t \leq b, |u|, |v| \leq r).$$

In particular, the function $h$ is then necessarily continuous on $[a, b] \times \mathbb{R}$.

**Theorem 3.2.4.** ([17]: Theorem 3.8) Suppose that $h : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a given function. The following conditions are equivalent:

(i) the non-autonomous superposition operator $S_h$ maps the space $\text{BV}([a, b])$ into itself and is locally bounded;

(ii) for every $r > 0$ there exists a constant $M_r > 0$ such that for every $k \in \mathbb{N}$, every finite partition $a = t_0 < \cdots < t_k = b$ of the interval $[a, b]$ and every finite sequence $u_0, u_1, \ldots, u_k \in [-r, r]$ with $\sum_{i=1}^{k} |u_i - u_{i-1}| \leq r$, the following inequalities hold

$$\sum_{i=1}^{k} |h(t_i, u_i) - h(t_{i-1}, u_i)| \leq M_r \quad \text{and} \quad \sum_{i=1}^{k} |h(t_{i-1}, u_i) - h(t_{i-1}, u_{i-1})| \leq M_r.$$

In [36], the author presents necessary and sufficient conditions for the continuity of a non-autonomous superposition operator in the $\text{BV}([a, b])$ case:

**Theorem 3.2.5.** ([36]: Theorem 10) Suppose that $h : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a function such that the superposition operator $S_h$ maps the space $\text{BV}([a, b])$ into itself. Let $x \in \text{BV}([a, b])$ be fixed. The following conditions are equivalent:
(i) the superposition operator $S_h$ is continuous at $x$;
(ii) for each $t \in [a, b]$, the function $u \in \mathbb{R} \mapsto h(t, u) - h(t, x(t))$ is continuous at $u = x(t)$ and for every $\epsilon > 0$ there exists $\delta > 0$ such that, for every $k \in \mathbb{N}$, every partition $a = t_0 < \cdots < t_k = b$ of the interval $[a, b]$, and every finite sequence $u_0, u_1, \ldots, u_k \in [-\delta, \delta]$ with $\sum_{i=1}^{k} |u_i - u_{i-1}| \leq \delta$, we have
\[
\sum_{i=1}^{k} |[h(t_i, u_i + x_i) - h(t_{i-1}, u_i + x_{i-1})] - [h(t_i, x_i) - h(t_{i-1}, x_{i-1})]| \leq \epsilon,
\]
and
\[
\sum_{i=1}^{k} |h(t_{i-1}, u_i + x_{i-1}) - h(t_{i-1}, u_{i-1} + x_{i-1})| \leq \epsilon,
\]
where $x_i = x(t_i), i \in \{0, \ldots, k\}$.

The following result is a special case of Theorem 3.8 in [17] (when $X = BV_\varphi$ with the Young function $\varphi(t) = t^p$).

**Theorem 3.2.6.** Let $p \geq 1$. Suppose that $h : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a given function. Consider the following conditions:

(a) for every $r > 0$ there exists a constant $M_r > 0$ such that for every $k \in \mathbb{N}$, every finite partition $a = t_0 < \cdots < t_k = b$ of the interval $[a, b]$ and every finite sequence $u_0, u_1, \ldots, u_k \in [-r, r]$ with $\sum_{i=1}^{k} |u_i - u_{i-1}|^p \leq r$ the following inequalities hold
\[
\sum_{i=1}^{k} |h(t_i, u_i) - h(t_{i-1}, u_i)|^p \leq M_r \text{ and } \sum_{i=1}^{k} |h(t_{i-1}, u_i) - h(t_{i-1}, u_{i-1})|^p \leq M_r.
\]

(b) the non-autonomous superposition operator $S_h$ maps the space $BV_p([a, b])$ into itself and is locally bounded.

Then, (a) implies (b). Moreover, $S_h$ is locally bounded.

**Proof.** Let $x \in BV_p([a, b])$ be such that $\|x\|_{BV_p} \leq r$. Let $M_r$ be a constant for which (a) is satisfied and let $a = t_0 < \cdots < t_k = b$ be a finite partition on $[a, b]$. Then,
\[
\sum_{i=1}^{k} |h(t_i, x(t_i)) - h(t_{i-1}, x(t_{i-1}))|^p
\]
\[
= \sum_{i=1}^{k} |[h(t_i, x(t_i)) - h(t_{i-1}, x(t_i))] - [-h(t_{i-1}, x(t_i)) + h(t_{i-1}, x(t_{i-1}))]|^p
\]
\[
\leq 2^{p-1} \sum_{i=1}^{k} |[h(t_i, x(t_i)) - h(t_{i-1}, x(t_i))] + [-h(t_{i-1}, x(t_i)) + h(t_{i-1}, x(t_{i-1}))]|^p
\]
\[
= 2^{p-1} \sum_{i=1}^{k} |h(t_i, x(t_i)) - h(t_{i-1}, x(t_i))|^p + \sum_{i=1}^{k} |h(t_{i-1}, x(t_i)) + h(t_{i-1}, x(t_{i-1}))|^p
\]
\[
\leq 2^{p-1} 2M_r
\]
\[
= 2^p M_r.
\]
Hence, $S_h(x) \in BV_p([a,b])$. Moreover, note that from (a) it follows that $f$ is locally bounded, that is, for each $r > 0$, there exists $r' > 0$ such that, for each $(t,u) \in [a,b] \times [-r,r]$, it is $|h(t,u)| \leq r'$. Let $A$ be a bounded subset of $BV_p([a,b])$ and let $r > 0$ be such that $\|x\|_{BV_p} \leq r$, $\forall x \in A$. Then, for all $x \in A$,

$$\|S_h(x)\|_{BV_p} = |S_h(x)(a)| + V_p(S_h(x),[a,b])^{\frac{1}{p}}$$

$$= |h(a,x(a))| + V_p(S_h(x),[a,b])^{\frac{1}{p}}$$

$$\leq |h(a,x(a))| + 2M_p^{\frac{1}{p}}$$

$$\leq r' + 2M_p^{\frac{1}{p}} \quad \text{(since } |x(a)| < r\text{)},$$

i.e. $S_h$ is locally bounded with $R := r' + 2M_p^{\frac{1}{p}}$. □

As far as we know, up to now, no characterisation is known of the functions $h$ for the associated operator $S_h$ to map $BV_p([a,b])$ into itself, with $p > 1$.

Other interesting results concerning the operator $S_h$ in the spaces mentioned above and in other spaces, like, for instance, Sobolev spaces and Besov spaces, also in higher dimension, can be found, for example, in [10], [12], [13].

**Superposition operators on some spaces of Baire functions**

**Theorem 3.2.7.** Let $h : [0,1] \times \mathbb{R} \to \mathbb{R}$. Then, the following statements are equivalent.

1. For every positive function $g \in B_1([0,1])$, for every positive $\epsilon$, there exists a positive function $\lambda$ on $[0,1]$, of Baire class one, such that $|t - s| < \min\{\lambda(t), \lambda(s)\}$ implies $|h(t,g(t)) - h(s,g(s))| < \epsilon$.

2. For every $g \in B_1([0,1])$, the function $h(\cdot,g(\cdot)) \in B_1([0,1])$.

**Proof.** The proof follows from Theorem 2.4.3. □

**Proposition 3.2.8.** If $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous then, for every $g \in B_1([0,1])$, the function $h(\cdot,g(\cdot)) \in B_1([0,1])$. Hence, the non-autonomous superposition operator $S_h$ maps the space $B_1([0,1])$ into itself.

**Proof.** This follows from the fact that the composition of a continuous function with a Baire one function is a Baire one function. □

**Remark 3.2.9.** Proposition 3.2.8 cannot be reverted. A function $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ can have the property that $h(\cdot,g(\cdot))$ is Baire one for each Baire one function $g$ without even being itself of Baire class one. Here is an example. Let $h(x,y) : [0,1] \times \mathbb{R} \to \mathbb{R}$ be so defined:

$$h(x,y) = \begin{cases} 1 & \text{if } (x,y) = (0,y) \text{ with } y \text{ rational} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the function $h_0(y) = h(0,y)$, that is the restriction of $f$ to the $y$ axis, is the characteristic function of the set $\{(0) \times \mathbb{R}\} \cap \mathbb{Q}$, the Dirichlet function, that, of course, is not of Baire class one.
Let \( g \in \mathcal{B}_1([0,1]) \). We distinguish two cases: \( g(0) \) is rational, \( g(0) \) is irrational.

If \( g(0) \) is rational then
\[
h(t, g(t)) = \begin{cases} 
1 & \text{if } t = 0 \\
0 & \text{otherwise.}
\end{cases}
\]

Hence, \( h(\cdot, g(\cdot)) \) has only a discontinuity at \( t = 0 \) and therefore it is of Baire class one.

If \( g(0) \) is irrational then \( h(t, g(t)) = 0 \) for all \( t \in [0,1] \). Hence, \( h(\cdot, g(\cdot)) \) is constant, and therefore continuous and, hence, of Baire class one.

### 4. A Type of Linear Operators: Right Composition Operators

**Definition 4.0.1.** Let \( J \) be an arbitrary interval. Let \( f : J \to \mathbb{R} \). The operator
\[
g \mapsto g \circ f
\]
where \( g : \mathbb{R} \to \mathbb{R} \) is an arbitrary function, is the right composition operator generated by \( f \). We, hereby, denote it by \( C^R_J f \). Hence, for each \( g : \mathbb{R} \to \mathbb{R} \), \( C^R_J f(g) : J \to \mathbb{R} \), is defined as
\[
C^R_J f(g)(\cdot) = g(f(\cdot)).
\]

As in the case of non-linear operators, we are interested in finding conditions on \( f \) in order for \( C^R_J f \) to map a space of functions \( X \) into itself.

**Right composition operators on Lipschitz functions and on some spaces of functions of bounded variations**

In [29] right \( BV \) compositors are completely characterised.

**Definition 4.0.2.** ([29]) Without loss of generality, take \([a,b] = [0,1]\). For a positive integer \( N \), let
\[
J_N = \{ X \subseteq [0,1] : X \text{ can be expressed as a union of } N \text{ intervals} \}
\]
(where the intervals may be open or closed at either end and singletons are allowed as degenerate closed intervals). Since any interval is a union of two subintervals, \( J_N \subseteq J_{N+1} \). A function \( f : [0,1] \to \mathbb{R} \) is said to be of \( N \)-bounded variation if \( f^{-1}([c,d]) \in J_N \) for all \([c,d] \subset \mathbb{R}\). These functions are also called pseudo-monotone functions (see [5]). Clearly, every monotone function is pseudo-monotone, indeed it belongs to \( BV([0,1]) \). Let \( BV(N) \) be the set of all functions \( f : [0,1] \to [0,1] \) of \( N \)-bounded variation, and \( BV'(N) \) the set of all bounded functions \( f : [0,1] \to \mathbb{R} \) of \( N \)-bounded variation.

**Remark 4.0.3.** Clearly, for every \( N \in \mathbb{N} \), the following inclusion holds:
\[
BV(N) \subseteq BV([0,1]).
\]
The inclusion is strict as Example 4.0.5 shows.

**Lemma 4.0.4.** ([29]: Lemma 1) Every function in \( BV'(N) \) is of bounded variation.

The converse of Lemma 4.0.4 does not hold as the following example shows.

**Example 4.0.5.** ([5]: Example 2.1.2) Let
\[
f(x) = \begin{cases} 
x^2 \sin^2 \left( \frac{1}{x} \right) & \text{if } 0 < x \leq 1 \\
0 & \text{if } x = 0.
\end{cases}
\]
Then, $f$ is in $BV([0,1])$ since $f'$ exists and is bounded. But $f$ is not pseudo-monotone as $f^{-1}([0]) = \{0\} \cup \{\frac{1}{n\pi} : n \in \mathbb{N}\}$.

**Theorem 4.0.7.** ([29]: Theorem 3; [5]: Theorem 2.1.4) For $f : [0,1] \to [0,1]$, the composition $g \circ f$ belongs to $BV((0,1))$ for all $g \in BV'([0,1])$ if and only if $f \in BV'(N)$ for some $N$. Moreover, if $f \in BV'(N)$, then $C^R_f$ is bounded.

All previous results are proved, in [5] and [29], on the unit interval $[0,1]$, but, of course, it is the same thing if we work on any interval $[a,b]$. In this general setting, we also prove the following results:

**Theorem 4.0.7.** The following conditions are equivalent:

(a) The function $f : [a,b] \to [a,b]$ satisfies a Lipschitz condition on $[a,b]$.

(b) The operator $C^R_f$ maps the space $\text{Lip}([a,b])$ into itself.

Moreover, the operator $C^R_f$ is automatically bounded.

**Proof.** (b) $\Rightarrow$ (a) is straightforward: take $g$ in $\text{Lip}([a,b])$ to be the identity. Then, by hypothesis, $C^R_f(g) \in \text{Lip}([a,b])$. As $g \circ f = f$, we are done.

(a) $\Rightarrow$ (b) Let $g \in \text{Lip}([a,b])$. Then, for every $x, y$ in $[a,b]$, $$(g \circ f)(x) - (g \circ f)(y) \leq \text{Lip}(g) \text{Lip}(f)|x - y|.$$ Hence $C^R_f(g) \in \text{Lip}([a,b])$.

Next, we prove the continuity (boundedness). Let $\{g_n\}_{n \in \mathbb{N}}$ and $g$ be functions in $\text{Lip}([a,b])$ with $\lim_{n \to +\infty} \|g_n - g\|_{\text{Lip}} = 0$.

Recall that $$\|g_n - g\|_{\text{Lip}} = |(g_n - g)(a)| + \text{Lip}(g_n - g).$$

Hence, $$\lim_{n \to +\infty} \|C^R_f(g_n) - C^R_f(g)\|_{\text{Lip}} = 0$$

as, in particular, $$\lim_{n \to +\infty} \|g_n - g\|_{\infty} = 0$$

and, therefore, $$\lim_{n \to +\infty} (g_n \circ f)(a) = (g \circ f)(a),$$

and, moreover, $$0 \leq \text{Lip}((g_n - g) \circ f) \leq \text{Lip}(g_n - g) \text{Lip}(f) \to 0$$ as $n \to +\infty$. \hfill \Box

**Proposition 4.0.8.** Let $0 < \gamma < 1$. If the function $f : [a,b] \to [a,b]$ satisfies a Lipschitz condition on $[a,b]$ then the operator $C^R_f$ maps the space $\text{Lip}_\gamma([a,b])$ into itself. Moreover, in this case the operator $C^R_f$ is automatically bounded.

**Proof.** Let $g \in \text{Lip}_\gamma([a,b])$. Then, for every $x, y$ in $[a,b]$, $$(g \circ f)(x) - (g \circ f)(y) = |g(f(x)) - g(f(y))| \leq \text{Lip}_\gamma(g) \text{Lip}(f)\gamma|x - y|.$$ Hence $C^R_f(g) \in \text{Lip}_\gamma([a,b])$.

Next, we prove the continuity (boundedness). Let $\{g_n\}_{n \in \mathbb{N}}$ and $g$ be functions in $\text{Lip}_\gamma([a,b])$ with $\lim_{n \to +\infty} \|g_n - g\|_{\text{Lip}_\gamma} = 0$. 


Recall that
\[ \|g_n - g\|_{\text{Lip}_R} = \| (g_n - g)(a) \| + \text{Lip}_R(g_n - g). \]
Hence,
\[ \lim_{n \to +\infty} \|C_R^f(g_n) - C_R^f(g)\|_{\text{Lip}_R} = 0 \]
as, in particular,
\[ \lim_{n \to +\infty} \|g_n - g\|_{\infty} = 0 \]
and, therefore,
\[ \lim_{n \to +\infty} (g_n \circ f)(a) = (g \circ f)(a), \]
and, moreover,
\[ 0 \leq \text{Lip}_\gamma((g_n - g) \circ f) \leq \text{Lip}_\gamma(g_n - g)\text{Lip}(f) \gamma \to 0 \text{ as } n \to +\infty. \]

**Proposition 4.0.9.** If the function \( f : [a, b] \to [a, b] \) is absolutely continuous and non-decreasing, then the operator \( C_R^f \) maps the space \( AC([a, b]) \) into itself. Moreover, in this case the operator \( C_R^f \) is automatically bounded.

**Proof.** Let \( g \in AC([a, b]) \) and let \( \epsilon > 0 \). Then, there exists \( \delta > 0 \) such that for all collections \( \{[a_1, b_1], \ldots, [a_n, b_n]\} \) of pairwise non-overlapping subintervals of \([a, b]\), condition
\[ \sum_{k=1}^n (b_k - a_k) < \delta \]
implies that
\[ \sum_{k=1}^n |g(b_k) - g(a_k)| < \epsilon. \]
As \( f \in AC([a, b]) \) there exists \( \nu > 0 \) such that, for all collections \( \{[a_1', b_1'], \ldots, [a_n', b_n']\} \) of pairwise non-overlapping subintervals of \([a, b]\), the condition
\[ \sum_{k=1}^n (b_k' - a_k') < \nu \]
implies that
\[ \sum_{k=1}^n |f(b_k') - f(a_k')| < \delta \quad (\bullet). \]
We may assume that the intervals \([a_k, b_k]\) are all non-degenerated and that \( a \leq a_1' < b_1' < \cdots < a_n' < b_n' \leq b \) and, as \( f \) is non-decreasing, it is \( a \leq f(a_1') \leq f(b_1') \leq \cdots \leq f(a_n') \leq f(b_n') \leq b \). Hence \( \{f(a_1'), f(b_1'), \ldots, f(a_n'), f(b_n')\} \) is a collection of pairwise non-overlapping subintervals of \([a, b]\) satisfying condition (\(\bullet\)), and then it follows that
\[ \sum_{k=1}^n |g(f(b_k')) - g(f(a_k'))| < \epsilon. \]
As \( \epsilon \) is arbitrary \( g \circ f \) is absolutely continuous, that is \( g \circ f \in AC([a, b]) \).
Next, we prove the continuity (boundedness). Let \( \{g_n\}_{n \in \mathbb{N}} \) and \( g \) be functions in \( AC([a, b]) \) with
\[ \lim_{n \to +\infty} \|g_n - g\|_{AC} = 0. \]
As 
\[ \|C^R_f(g_n) - C^R_f(g)\|_{AC} = \|g_n \circ f - g \circ f\|_{BV} \]
\[ = \|(g_n \circ f)(a) - (g \circ f)(a)\| + V((g_n - g) \circ f, [a, b]) \]
\[ \leq \|(g_n \circ f)(a) - (g \circ f)(a)\| + V(g_n - g, [a, b]) \]
\[ \leq \|(g_n \circ f)(a) - (g \circ f)(a)\| + \|g_n - g\|_{AC} \]
and
\[ \lim_{n \to +\infty} \|(g_n \circ f)(a) - (g \circ f)(a)\| = 0, \]
then
\[ \lim_{n \to +\infty} \|C^R_f(g_n) - C^R_f(g)\|_{AC} = 0. \]
Hence, the thesis. □

Hence, we have a condition, both necessary and sufficient, for the operator \(C^R_f\) to map \(\text{Lip}([a, b])\) into itself, and we showed sufficient conditions for the operator \(C^R_f\) to map \(\text{Lip}_n([a, b])\) and \(AC([a, b])\) into themselves, respectively.

**Right composition operators on some spaces of Baire functions**

**Definition 4.0.10.** ([40]) A function \(f : \mathbb{R} \to \mathbb{R}\) is called \(k\)-continuous if for every positive function \(\epsilon\) there is a positive function \(\delta\) such that for any \(x, y \in \mathbb{R}\), \(|x - y| < \min\{\delta(x), \delta(y)\}\) implies \(|f(x) - f(y)| < \min\{\epsilon(f(x)), \epsilon(f(y))\}\).

Note that every continuous function is \(k\)-continuous.

**Proposition 4.0.11.** ([26]: Lemma 3.3) The following properties hold:

1. If \(f\) and \(g\) are \(k\)-continuous functions, then so is the sum \(f + g\).
2. If \(f\) and \(g\) are \(k\)-continuous functions, then so is the product \(fg\).

In [40], right Baire one compositors have been completely characterised:

**Theorem 4.0.12.** ([40]: Theorem 1) Let \(f : \mathbb{R} \to \mathbb{R}\) be a function. Then the following statements are equivalent.

1. For any closed subset \(A\) of \(\mathbb{R}\), \(f^{-1}(A)\) is an \(F_\sigma\) set.
2. For any \(F_\sigma\) set \(A\), \(f^{-1}(A)\) is an \(F_\sigma\) set.
3. For every positive Baire class one function \(\epsilon(\cdot)\), there is a positive function \(\delta(\cdot)\) on \(\mathbb{R}\) such that \(|x - y| < \min\{\delta(x), \delta(y)\}\) implies \(|f(x) - f(y)| < \min\{\epsilon(f(x)), \epsilon(f(y))\}\).
4. \(f\) is a right Baire one compositor.

Moreover, another characterisation of right Baire one compositor, involving \(k\)-continuous functions, is given in [26]:

**Theorem 4.0.13.** ([26]: Theorem 2.6) A function \(f : \mathbb{R} \to \mathbb{R}\) is a right Baire one compositor if and only if \(f\) is \(k\)-continuous.

**Theorem 4.0.14.** The following two conditions are equivalent:

(a) The function \(f : \mathbb{R} \to \mathbb{R}\) is \(k\)-continuous.

(b) The operator \(C^R_f\) maps the space \(B_1\) into itself.

Proof. This follows from Theorem 4.0.12 and Theorem 4.0.13. □
Remark 4.0.15. Clearly, since each open set is an $F_\sigma$, from Theorem 4.0.12 it follows that every continuous function is a right Baire one compositor. The collection of Baire one compositors lies strictly between the collection of Baire one functions and the collection of continuous functions. For instance, the Reimann function, as Example 3.1.11 shows, is a Baire one function but it is not a right Baire one compositor.

Remark 4.0.16. The composition of two right Baire one compositors is a right Baire one compositor. Therefore, if $g : \mathbb{R} \to \mathbb{R}$ is a right Baire one compositor, then, for each $c \in \mathbb{R}$, $c + g$ and $cg$ are right Baire one compositors as well because they are the compositions of $g$ and the function $h(x) = c + x$ and $k(x) = cx$, respectively.

However, Zhao writes in [40] that it is still not clear whether the addition (product) of a continuous function and a right Baire one compositor is a right Baire one compositor. We can give a positive answer to the problem posed by Zhao, combining Proposition 4.0.11 with Theorem 4.0.13:

Theorem 4.0.17. Let $f$ be a continuous function and let $g$ be a right Baire one compositor. Then $f + g$ and $fg$ are right Baire one compositors.

Proof. Let $f$ be a continuous function. Then $f$ is also $k$–continuous. Let $g$ be a right Baire one compositor, then, by Theorem 4.0.13, $g$ is a $k$–continuous function. Hence, from the properties in Proposition 4.0.11 it follows that $f + g$ and $fg$ are $k$–continuous functions. By Theorem 4.0.13 this is equivalent to say that $f + g$ and $fg$ are right Baire one compositors.

Theorem 4.0.18. Let $f : \mathbb{R} \to \mathbb{R}$. The following conditions are equivalent:

1. For any $G_\delta$ set $C$, $f^{-1}(C)$ is a $G_{\delta \sigma}$.
2. For any $G_{\delta \sigma}$ set $C$, $f^{-1}(C)$ is a $G_{\delta \sigma}$.
3. The operator $C_f^B$ maps the space $B_2$ into itself (that is, the function $f : \mathbb{R} \to \mathbb{R}$ is a right Baire two compositor).

Proof. (1) and (2) are clearly equivalent as every $G_\delta$ set is a $G_{\delta \sigma}$ set.

(2) $\Rightarrow$ (3) Let $A$ be an open set and let $g \in B_2$. Then $g^{-1}(A)$ is a $G_{\delta \sigma}$. Hence, by the hypothesis it follows that $(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A))$ is a $G_{\delta \sigma}$. Hence, $g \circ f \in B_2$.

(3) $\Rightarrow$ (1) Suppose $f$ is a right Baire two compositor and $C$ a $G_\delta$ set. Let $g$ be the characteristic function of $C$. As $C$ is both $F_{\sigma \delta}$ and $G_{\delta \sigma}$, $g$ is Baire two. Hence $g \circ f$ is Baire two. Therefore $f^{-1}(C)$ is a $G_{\delta \sigma}$ set because $f^{-1}(C) = (g \circ f)^{-1}(0, \frac{1}{2})$. \hfill $\Box$

The following example shows that there exist functions which are right Baire two compositors but not right Baire one compositors.

Example 4.0.19. It is well-known that $Q$ is an $F_\sigma$ set, but not a $G_\delta$ set. As every $F_\sigma$ set is a $G_{\delta \sigma}$ set, then $Q$ is a $G_{\delta \sigma}$ set. Moreover, $\mathbb{R} \setminus Q = \mathbb{R} \setminus \bigcup_{q \in Q} \{q\} = \cap_{q \in Q}(\mathbb{R} \setminus \{q\})$ is a $G_\delta$ set but not an $F_\sigma$ set. As every $G_\delta$ set is a $G_{\delta \sigma}$ set, then $\mathbb{R} \setminus Q$ is a $G_{\delta \sigma}$ set. Consider the function $f = \chi_{\mathbb{R} \setminus Q}$. Then $f^{-1}(\{1\}) = \mathbb{R} \setminus Q$. As $\{1\}$ is closed, from condition (1) of Theorem 4.0.12, it follows that $f$ is not a right Baire one compositor. Now, let $C$ be a $G_\delta$ set. Then

$$f^{-1}(C) = \begin{cases} \mathbb{R} & \text{if } 0, 1 \in C \\ \mathbb{R} \setminus Q & \text{if } 1 \in C, 0 \notin C \\ Q & \text{if } 0 \in C, 1 \notin C \\ \emptyset & \text{if } 0, 1 \notin C \end{cases}$$
and hence $f^{-1}(C)$ is a $G_{δσ}$ set. From condition (1) of Theorem 4.0.18, it follows that $f$ is a right Baire two compositor.

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