NONDEGENERACY OF POSITIVE SOLUTIONS TO A KIRCHHOFF PROBLEM WITH CRITICAL SOBOLEV GROWTH

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Abstract. In this paper, we prove uniqueness and nondegeneracy of positive solutions to the following Kirchhoff equations with critical growth

$$- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u = u^5, \quad u > 0 \quad \text{in} \ \mathbb{R}^3,$$

where $a, b > 0$ are positive constants. This result has potential applications in singular perturbation problems concerning Kirchhoff equations.

Keywords: Kirchhoff equations; Positive solutions; Uniqueness; Nondegeneracy

1. INTRODUCTION AND MAIN RESULT

In this paper, we are concerned about the nonlocal Kirchhoff type problem

$$- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u = u^5, \quad u > 0 \quad \text{in} \ \mathbb{R}^3, \quad (1.1)$$

where $a, b > 0$ are constants, $\Delta = \sum_{i=1}^{3} \partial_{x_i}^2$ is the usual Laplacian operator in $\mathbb{R}^3$.

Denote by $D = D^{1,2}(\mathbb{R}^3)$ the completion of $C_0^\infty(\mathbb{R}^3)$ under the seminorm

$$\|\varphi\|_D^2 = \int_{\mathbb{R}^3} |\nabla \varphi|^2.$$

A (weak) solution to Eq. (1.1) is a function $u \in D$ satisfying

$$\left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi = \int_{\mathbb{R}^3} u^5 \varphi$$

for all $\varphi \in D$. By the Sobolev embedding $D \subset L^6(\mathbb{R}^3)$, all the integrals in the above equation are well defined.

Problem (1.1) and its variants have been studied extensively in the literature. Physician Kirchhoff [18] proposed the following time dependent wave equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \right) \frac{\partial^2 u}{\partial x^2} = 0$$

for the first time, in order to extend the classical D’Alembert’s wave equations for free vibration of elastic strings. [5] and Pohozaev [25] contributed some early research on the study of Kirchhoff
equations. Much attention was received until the work [23] of J.L. Lions. For more interesting results in this respect, we refer to e.g. [4, 7] and the references therein. From a mathematical point of view, the interest of studying Kirchhoff equations comes from the nonlocality of Kirchhoff type equations. For instance, the consideration of the stationary analogue of Kirchhoff’s wave equation leads to problem of the type

\[-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u = f(x, u) \quad \text{in } \Omega \quad (1.2)\]

where \(\Omega \subset \mathbb{R}^3\) is a smooth domain. Note that the term \(\int |\nabla u|^2 dx\) \(\Delta u\) depends not only on the pointwise value of \(\Delta u\), but also on the integral of \(|\nabla u|^2\) over the whole domain. In this sense, Eqs. (1.1) and (1.2) are no longer the usual pointwise equalities. This new feature brings new mathematical difficulties that make the study of Kirchhoff type equations particularly interesting.

In this paper, we study the uniqueness and nondegeneracy of positive solutions to problem (1.1). These quantitative properties play a fundamental role in singular perturbation problems. Let us briefly recall some results in this respect. Kwong [19] established uniqueness and nondegeneracy of positive solutions to the Schrödinger equations

\[-\Delta w + w = w^q, \quad w > 0 \quad \text{in } \mathbb{R}^N,\]

see also Chang et al. [6]; For quasilinear Schrödinger equations such as

\[-\Delta u - u\Delta |u|^2 + \omega u - |u|^{q-1} u = 0 \quad \text{in } \mathbb{R}^N,\]

where \(\omega > 0\) is a constant, \(q\) is an index denoting subcritical growth of the nonlinearity and \(N \geq 1\), see e.g. Selvitella [26], Xiang [28] and Adachi et al. [1]; For fractional Schrödinger equations such as

\[-(\Delta)^s w + w = w^q, \quad w > 0 \quad \text{in } \mathbb{R}^N,\]

where \(0 < s < 1 \leq N\) and \(q\) is an index denoting subcritical growth of the nonlinearity, see e.g. Frank and Lenzmann [12] and Frank, Lenzmann and Silvestre [13], Full and Valdinoci [10]. For elliptic equations with critical growth

\[-(\Delta)^s w = w^{\frac{N+2s}{N-2}}, \quad w > 0 \quad \text{in } \mathbb{R}^N,\quad (1.3)\]

see Ambrosetti et al. [3] and Dávila et al. [8] for \(s = 1\) and \(0 < s < 1\), respectively. In particular, Ambrosetti and Malchiodi [2] provides a systematical research on nondegeneracy of ground states to various types of elliptic problems together with applications in singular perturbation problems.

It is also known that the uniqueness and nondegeneracy of ground states are of fundamental importance when one deals with orbital stability or instability of ground states. It mainly removes the possibility that directions of instability come from the kernel of the related linear operator. The uniqueness and nondegeneracy of ground states also play an important role in blow-up analysis for the corresponding standing wave solutions in the corresponding time-dependent equations, see e.g. [12, 13] and the references therein.

For Kirchhoff problems, not much is known in this respect. Recently, Li et al. [21] established uniqueness and nondegeneracy for positive solutions to Kirchhoff equations with subcritical growth. More precisely, they proved that the following Kirchhoff equation

\[-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + u = u^p, \quad u > 0 \quad \text{in } \mathbb{R}^3,\]

where \(1 < p < 5\), has a unique nondegenerate positive radial solution. As a counterpart to this result, we have the following theorem for Kirchhoff equations with critical Sobolev growth.
Theorem 1.1. For any positive constants \(a, b > 0\), there exists a unique positive solution \(u \in D\) to equation (1.1) up to scalings and translations. Moreover, \(u\) is nondegenerate in the sense that

\[
\text{Ker} \mathcal{L}_+ = \text{span} \{ u_{x_1}, u_{x_2}, u_{x_3}, u/2 + x \cdot \nabla u \},
\]

where \(\mathcal{L}_+: D \to D\) is defined as

\[
\mathcal{L}_+ \varphi = -\left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta \varphi - 2b \left( \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi \right) \Delta u - 5u^4 \varphi
\]

for all \(\varphi \in D\).

Remark that our result can also be viewed as an extension of the nondegeneracy result of Ambrosetti [3] for equations (1.3) with \(s = 1\), since in the case \(b = 0\), problem (1.1) is reduced to (1.3) with \(s = 1\).

Our notations are standard. For simplicity, we write by \(\int u\) the integral \(\int_{\mathbb{R}^3} u \, dx\) unless otherwise stated. We also write \(u(x) = u(|x|)\) whenever \(u\) is a radially symmetric function.

2. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Let

\[
Q(x) = \frac{3^{1/4}}{(1 + |x|^2)^{1/2}}
\]

be the unique positive radial solution with \(Q(0) = 3^{1/4}\) that satisfies

\[-\Delta Q = Q^5 \quad \text{in} \quad \mathbb{R}^3,
\]

see e.g. Ambrosetti et al. [3]. For any \(\lambda > 0\) and \(x_0 \in \mathbb{R}^3\), it is direct to verify that the functions \(x \mapsto \lambda^{-1/2}Q((x - x_0)/\lambda)\) are also solutions to the above equation.

2.1. Proof of uniqueness.

Proof of uniqueness. Let \(u \in D\) be an arbitrary positive solution to Eq. (1.1) and let

\[
c = a + b \int |\nabla u|^2.
\]

Then the function \(\bar{u}(x) = u(\sqrt{c}x)\) solves

\[-\Delta \bar{u} = \bar{u}^5 \quad \text{in} \quad \mathbb{R}^3.
\]

Hence, the uniqueness of \(Q\) implies that \(\bar{u}(x) = \lambda^{-1/2}Q((x - x_0)/\lambda)\) for some \(x_0 \in \mathbb{R}^3\) and some \(\lambda > 0\). That is,

\[
u(x) = \lambda^{-1/2}Q \left( \frac{x}{\sqrt{c}} - x_0 \right) / \lambda.
\]

This gives \(\int |\nabla u|^2 = \sqrt{c} \int |\nabla Q|^2\). Hence

\[
c = a + b\sqrt{c} \int |\nabla Q|^2,
\]

which yields

\[
\sqrt{c} = \frac{1}{2} \left( b \| \nabla Q \|^2_2 + b^2 \| \nabla Q \|^2_4 + 4a \right).
\]

Hence,

\[
u(x) = \lambda^{-1/2}Q \left( \frac{x}{b \| \nabla Q \|^2_2 + b^2 \| \nabla Q \|^2_4 + 4a} - x_0 \right) / \lambda
\]

for some \(x_0 \in \mathbb{R}^3\) and some \(\lambda > 0\). This proves that (1.1) has a unique positive energy solution up to translations and scalings.
We point out that \( c \) depends only on \( a, b \) since \( Q \) is explicitly given. In other words, \( c \) is independent of the choice of the positive solution \( u \). As a consequence, we conclude that all the positive energy solutions to (1.1) is given by

\[
S = \left\{ \lambda^{-1/2} Q \left( \frac{x}{b \|Q\|^2 + \sqrt{b^2 \|Q\|^2 + 4a}} - y \right) \right\} : \lambda > 0, y \in \mathbb{R}^3 \}.
\]

2.2. Proof of nondegeneracy. With no loss of generality, we assume that \( u(x) = u(|x|) \) is the unique positive radial energy solution to Eq. (1.1) with \( \lambda = 1 \) in (2.1). Still write \( c = a + b \int |\nabla u|^2 \).

Keep in mind that \( c \) depends only on \( a, b \). Define \( A : D \to D \) by

\[
A \varphi = -c \Delta \varphi - 5u^4 \varphi.
\]

It is straightforward to derive from [2, Chapter 5] that

\[
\text{Ker} A = \text{span} \{ u/2 + x \cdot \nabla u, u_{x_1}, u_{x_2}, u_{x_3} \}.
\]

(2.2)

Note that \( u/2 + x \cdot \nabla u = u/2 + ru'(r) \) with \( r = |x| \) is a radial function in \( D \). For simplicity, denote \( D_{\text{rad}} = \{ v \in D : v(x) = v(|x|) \} \).

To prove the nondegeneracy of \( \mathcal{L}_+ \), first we prove

**Proposition 2.1.** Let \( \mathcal{L}_+ \varphi = 0 \) and \( \varphi \in D_{\text{rad}} \). Then \( \varphi = \lambda (u/2 + x \cdot \nabla u) \) for some \( \lambda \in \mathbb{R} \).

**Proof.** Direct computation shows that \( u/2 + x \cdot \nabla u = u/2 + ru'(r) \) is indeed a radial solution to equation \( \mathcal{L}_+ \varphi = 0 \). We have to prove that \( u/2 + x \cdot \nabla u \) is the unique radial solution to equation \( \mathcal{L}_+ \varphi = 0 \) in \( D_{\text{rad}} \) up to a constant.

Let \( \varphi \in D_{\text{rad}} \) satisfy \( \mathcal{L}_+ \varphi = 0 \). It is equivalent to

\[
A \varphi = 2b \left( \int \nabla u \cdot \nabla \varphi \right) \Delta u.
\]

Write \( e_0 = u/2 + x \cdot \nabla u \) for simplicity. Since \( D_{\text{rad}} \) is a Hilbert space, denote by \( D_0 \) the orthogonal complement of \( \mathbb{R} e_0 \) in \( D_{\text{rad}} \). Then \( \varphi = \lambda e_0 + \tilde{\varphi} \) for some \( \lambda \in \mathbb{R} \) and \( \tilde{\varphi} \in D_0 \). By a direct computation, we find that \( \int \nabla u \cdot \nabla e_0 = 0 \). This implies \( u \in D_0 \). Moreover, note that (2.2) implies that \( A \) is invertible on \( D_0 \). It follows from \( A e_0 = 0 \) and \( \int \nabla u \cdot \nabla e_0 = 0 \) that \( \tilde{\varphi} \) satisfies

\[
A \tilde{\varphi} = 2b \left( \int \nabla u \cdot \nabla \tilde{\varphi} \right) \Delta u = -2b\frac{c}{c} \left( \int \nabla u \cdot \nabla \tilde{\varphi} \right) u^5.
\]

Claim that \( \int \nabla u \cdot \nabla \tilde{\varphi} = 0 \). To this end, first note that \( A u = -4u^5 \). Thus, the above identity implies

\[
A \left( \tilde{\varphi} - \frac{b}{2c} \left( \int \nabla u \cdot \nabla \tilde{\varphi} \right) u \right) = 0.
\]

We know \( u \in D_0 \). Hence, (2.2) yields

\[
\tilde{\varphi} - \frac{b}{2c} \left( \int \nabla u \cdot \nabla \tilde{\varphi} \right) u = 0.
\]

This yields

\[
\nabla \tilde{\varphi} = \frac{b}{2c} \left( \int \nabla u \cdot \nabla \tilde{\varphi} \right) \nabla u.
\]

It follows

\[
\int \nabla u \cdot \nabla \tilde{\varphi} = \frac{b}{2c} \left( \int \nabla u \cdot \nabla \tilde{\varphi} \right) \int |\nabla u|^2.
\]

Since \( c > b \int |\nabla u|^2 \), we have \( b \int |\nabla u|^2/(2c) < 1/2 \). We deduce from the above equation that \( \int \nabla u \cdot \nabla \tilde{\varphi} = 0 \). This proves the claim.
Therefore, \( \mathcal{A} \tilde{\varphi} = 0 \). Recall \( \tilde{\varphi} \in D_0 \). Applying (2.2) gives \( \tilde{\varphi} = 0 \). Thus, we obtain \( \varphi = \lambda e_0 \).

The proof is complete. \( \square \)

Now we can prove the nondegeneracy part of Theorem 1.1.

**Proof of nondegeneracy.** Since we have proved Proposition 2.1, the rest proof for the nondegeneracy of \( \mathcal{L}_+ \) is standard. We refer the readers to [2, Chapter 5] for details. \( \square \)

**References**

[1] S. Adachi, M. Shibata and T. Watanabe, A note on the uniqueness and the non-degeneracy of positive radial solutions for semilinear elliptic problems and its application. Preprint at arXiv: 1602.07086 [math.AP].

[2] A. Ambrosetti and A. Malchiodi, Perturbation Methods and Semilinear Elliptic Problems on \( \mathbb{R}^N \). Birkhäuser Verlag, 2006.

[3] A. Ambrosetti, J. Garcia Azorero and I. Peral, Perturbation of \( \Delta u + u^{(N+2)/(N-2)} = 0 \), the scalar curvature problem in \( \mathbb{R}^N \), and related topics. J. Funct. Anal. 165(1999), 117-149.

[4] A. Arosio and S. Panizzi, On the well-posedness of the Kirchhoff string. Trans. Amer. Math. Soc. 348(1996), 305-330.

[5] S. Bernstein, Sur une classe d’équations fonctionelles aux dérivées partielles. Bull. Acad. Sci. URSS. Sér. 4(1940), 17-26.

[6] S.-M. Chang, S. Gustafson, K. Nakanishi and T.-P. Tsai, Spectra of linearized operators for NLS solitary waves. SIAM J. Math. Anal. 39 (2007/08), no. 4, 1070-1111.

[7] P. D’Ancona and S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data. Invent. Math. 108(1992), 247-262.

[8] J. Dávila, M. del Pino and Y. Sire, Nondegeneracy of the bubble in the critical case for nonlocal equations. Proc. Amer. Math. Soc. 141(2013), 3865-3870.

[9] Y. Deng, S. Peng and W. Shuai, Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in \( \mathbb{R}^3 \). J. Funct. Anal. 269(2015), 3500-3527.

[10] M. Fall and E. Valdinoci, Uniqueness and nondegeneracy of positive solutions of \((-\Delta)^s u + u = 0\) when \( s \) is close to 1. Comm. Math. Phys. 329(2014), 383-404.

[11] G.M. Figueiredo, N. Ikoma, N. and J. R. Santos Júnior, Existence and concentration result for the Kirchhoff type equations with general nonlinearity. Arch. Rational Mech. Anal. 213(2014), 931-979.

[12] R.L. Frank and E. Lenzmann, Uniqueness of non-linear ground states for fractional Laplacians in \( \mathbb{R}^3 \). J. Funct. Anal. 269(2015), 3500-3527.

[13] R.L. Frank, E. Lenzmann and L. Silvestre, Uniqueness of radial solutions for the fractional Laplacian. Commun. Pur. Appl. Math. 69(2016), 1671-1726.

[14] Z. Guo, Ground states for Kirchhoff equations without compact condition. J. Differential Equations 259(2015), 2884-2902.

[15] Y. He, Concentrating bounded states for a class of singularly perturbed Kirchhoff type equations with a general nonlinearity. J. Differential Equations 261(2016), 6178-6220.

[16] Y. He and G. Li, Standing waves for a class of Kirchhoff type problems in \( \mathbb{R}^3 \) involving critical Sobolev exponents. Calc. Var. Partial Differential Equations 54(2015), 3067-3106.

[17] X. He and W. Zou, Existence and concentration behavior of positive solutions for a Kirchhoff equation in \( \mathbb{R}^3 \). J. Differential Equations 252(2012), 1813-1834.

[18] G. Kirchhoff, Mechanik, Teubner, Leipzig, 1883.

[19] M. K. Kwong, Uniqueness of positive solutions of \( \Delta u - u + u^p = 0 \) in \( \mathbb{R}^n \). Arch. Rational Mech. Anal. 105(1989), no. 3, 243-266.

[20] Y. Li, F. Li and J. Shi, Existence of a positive solution to Kirchhoff type problems without compactness conditions. J. Differential Equations 253(2012), 2285-2294.
[21] G. Li, P. Luo, S. Peng, C. Wang and C.-L. Xiang, *Uniqueness and Nondegeneracy of positive solutions to Kirchhoff equations and its applications in singular perturbation problems*. Preprint at arXiv:1703.05459 [math.AP].

[22] G. Li and H. Ye, *Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$*. J. Differential Equations 257(2014), 566-600.

[23] J.L. Lions, *On some questions in boundary value problems of mathematical physics*. Contemporary Development in Continuum Mechanics and Partial Differential Equations, in: North-Holland Math. Stud., vol. 30, North-Holland, Amsterdam, New York, 1978, pp. 284-346.

[24] K. Perera and Z. Zhang, *Nontrivial solutions of Kirchhoff-type problems via the Yang index*. J. Differential Equations 221(2006), 246-255.

[25] S.I. Pohozaev, *A certain class of quasilinear hyperbolic equations*. Mat. Sb. (N.S.) 96(138) (1975), 152-166, 168 (in Russian).

[26] A. Selvitella, *Nondegeneracy of the ground state for quasilinear Schrödinger equations*. Calc. Var. Partial Differential Equations 53(2015), 349-364.

[27] J. Wang, L. Tian, J. Xu and F. Zhang, *Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth*. J. Differential Equations 253(2012), 2314-2351.

[28] C.-L. Xiang, *Remarks on Nondegeneracy of Ground States for Quasilinear Schrödinger Equations*. Discrete Contin. Dyn. Syst. 36(2016), 5789-5800.

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