GROTHENDIECK'S INEQUALITIES FOR JB*-TRIPLES: PROOF OF THE BARTON–FRIEDMAN CONJECTURE

JAN HAMHALTER, ONDREJ F.K. KALENDA, ANTONIO M. PERALTA, AND HERMANN PFITZNER

Abstract. We prove that, given a constant \( K > 2 \) and a bounded linear operator \( T \) from a JB*-triple \( E \) into a complex Hilbert space \( H \), there exists a norm-one functional \( \psi \in E^* \) satisfying
\[
\|T(x)\| \leq K \|T\| \|x\|\psi,
\]
for all \( x \in E \). Applying this result we show that, given \( G > 8(1 + 2\sqrt{3}) \) and a bounded bilinear form \( V \) on the Cartesian product of two JB*-triples \( E \) and \( B \), there exist norm-one functionals \( \varphi \in E^* \) and \( \psi \in B^* \) satisfying
\[
|V(x, y)| \leq G \|V\| \|x\|\varphi \|y\|\psi,
\]
for all \( (x, y) \in E \times B \). These results prove a conjecture pursued during almost twenty years.

1. Introduction

In order to review the historical emplacement of a conjecture open for almost twenty years, we should turn back to the fifties, to a major contribution in functional analysis. Grothendieck’s inequalities and Grothendieck’s constants were named after A. Grothendieck, who established the first result in this direction in his celebrated “Résumé de la théorie métrique des produits tensoriels topologiques” (see [13]). Grothendieck’s original result proves the existence of a universal constant \( G > 0 \) (called Grothendieck’s constant), satisfying that for every couple \((\Omega_1, \Omega_2)\) of compact Hausdorff spaces and every bilinear form \( V \) on \( C(\Omega_1) \times C(\Omega_2) \) there exist two probability measures \( \mu_1 \) and \( \mu_2 \) on \( \Omega_1 \) and \( \Omega_2 \), respectively, such that
\[
|V(f, g)| \leq G\|V\| \left( \int_{\Omega_1} |f(t)|^2 d\mu_1(t) \right)^{\frac{1}{2}} \left( \int_{\Omega_2} |g(s)|^2 d\mu_2(s) \right)^{\frac{1}{2}}
\]
for all \( f \in C(\Omega_1) \) and \( g \in C(\Omega_2) \). In 1956, Grothendieck predicted the validity of the previous result when the space \( C(\Omega) \), of all complex valued continuous functions on a compact Hausdorff space \( \Omega \), is replaced with a general C*-algebra (cf. [13 §6, Question 4]). Grothendieck’s forethought was confirmed several years later. In subsequent remarkable contributions, G. Pisier [27] and U. Haagerup [14] established the so-called non-commutative Grothendieck inequality, which assures that

2010 Mathematics Subject Classification. 46L70, 17C65.
Key words and phrases. Grothendieck’s inequality, little Grothendieck inequality, JB*-triple, JBW*-triple.

The first two authors were in part supported by the Research Grant GACR 17-00941S. The first author was partly supported further by the project OP VV V Center for Advanced Applied Science CZ.02.1.01/0.0/0.0/16_019/000077. The third author was partially supported by Junta de Andalucía grant FQM375.
for every bounded bilinear form \( V \) on the cartesian product of two C\(^*\)-algebras \( A \) and \( B \), there exist two states \( \phi \in A^* \) and \( \psi \in B^* \) satisfying

\[
|V(x, y)| \leq 4 \| V \| \phi \left( \frac{xx^* + x^* x}{2} \right)^\frac{1}{2} \psi \left( \frac{yy^* + y^* y}{2} \right)^\frac{1}{2},
\]

for all \((x, y) \in A \times B\). Briefly speaking, at the cost of replacing probability measures with states and moduli of continuous functions with absolute values of the form \(|x|^2 = \frac{xx^* + x^* x}{2} \quad (x \in A)\), the Grothendieck’s inequality works for bounded bilinear forms on the Cartesian product of two C\(^*\)-algebras. That is, in the non-commutative setting, the pre-Hilbertian semi-norms of the form \( \|x\|_\phi^2 := \phi \left( \frac{xx^* + x^* x}{2} \right) \), where \( \phi \) runs through the set of all states on a C\(^*\)-algebra \( A \), are valid to factor all bounded bilinear forms.

There exists a class of complex Banach spaces, called JB\(^*\)-triples, which are determined by the holomorphic properties of their open unit balls (see Subsection 1.3 below for details). The class of JB\(^*\)-triples includes (among others) all C\(^*\)-algebras, and all complex Hilbert spaces. We therefore have a strictly wider class of complex Banach spaces than that determined by all C\(^*\)-algebras. The setting of JB\(^*\)-triples seemed an appropriate candidate to extend the Grothendieck’s inequality when in 1987 J.T. Barton and Y. Friedman explored this problem.

Although JB\(^*\)-triples lack an order structure like the one appearing in the setting of C\(^*\)-algebras, every JB\(^*\)-triple \( E \) admits a large collection of pre-Hilbertian semi-norms which arise naturally from the geometric structure and play a similar role to those determined by the states on a C\(^*\)-algebra. Barton and Friedman showed in [2] that for each norm-one functional \( \varphi \) in the dual, \( E^* \), of \( E \), and each norm-one element \( z \) in \( E^{**} \) with \( \varphi(z) = 1 \), the mapping \( x \mapsto \|x\|_\varphi = \varphi\{x, x, z\} \) defines a pre-Hilbert semi-norm on \( E \) which does not depend on the choice of the element \( z \). Let us observe that if \( \phi \) is a state on a C\(^*\)-algebra \( A \) and \( 1 \) denotes the unit element in \( A^{**} \), then \( \phi(1) = 1 \) and \( \|x\|_\phi^2 = \phi\{x, x, 1\} = \phi \left( \frac{xx^* + x^* x}{2} \right) \) for all \( x \in A \). Theorem 1.4 in [2] asserts the existence of a universal constant \( K \in [2, 3 + 2\sqrt{2}] \) satisfying the following property: for every bounded bilinear form \( V \) on the cartesian product of two JB\(^*\)-triples \( E \) and \( F \) there exist norm-one functionals \( \varphi \in E^* \) and \( \psi \in F^* \) satisfying

\[
(1) \quad |V(x, y)| \leq K \| V \| \|x\|_\varphi \|y\|_\psi,
\]

for all \((x, y) \in E \times F\). Building upon the results published in [2], Ch.-H. Chu, B. Iochum and G. Loupias gave an alternative proof of this result in [9, Theorem 6].

Grothendieck’s inequalities were revisited in the setting of real JB\(^*\)-triples at the beginning of this century, and it was pointed out in [23, 24] that the proof of [2, Theorem 1.3] contains a gap affecting also the arguments and conclusions in [9]. As a consequence of these difficulties, the original statement of Grothendieck’s inequality for JB\(^*\)-triples in [11] can not be considered as proved, and it is nowadays known as the Barton–Friedman conjecture.

The main results in [23, 25, 26] show that, at the cost of replacing semi-norms of the form \( \|\cdot\|_\varphi \) and \( \|\cdot\|_\psi \) with semi-norms of the form \( \|\cdot\|_{\varphi_1, \varphi_2} \), \( \|\cdot\|_{\psi_1, \psi_2} \) for convenient norm-one functionals \( \varphi_1, \varphi_2 \in E^* \) and \( \psi_1, \psi_2 \in F^* \), the conclusion in [11] is true for \( K > 4(1 + 2\sqrt{3}) \) (cf. [25, Theorem 6]). Let us remark that for \( \varphi_1, \varphi_2 \in E^* \) we set \( \|x\|_{\varphi_1, \varphi_2} := \|x\|_{\varphi_1}^2 + \|x\|_{\varphi_2}^2 \) \((x \in E)\). This result was applied to dissipate the
concerns affecting subsequent results in JB*-triple theory (for example, properties of the strong*-topology, characterization of weakly compact operators from a JB*-triple into a Banach space, etc.) whose proofs depended on the original form of Grothendieck’s inequality by Barton and Friedman. Despite these advances, the Barton–Friedman conjecture (i.e. the statement in (1)) was neither proven nor discarded.

In [24] the Barton-Friedmann conjecture was proved in some special cases – for Cartan factors and atomic JBW*-triples (i.e. \(\ell_\infty\)-sums of Cartan factors).

In 2012, G. Pisier wrote “The problem of extending the non-commutative Grothendieck theorem from C*-algebras to JB*-triples was considered notably by Barton and Friedman around 1987, but seems to be still incomplete” (cf. [28, Remark 8.3]). The recent monograph [7] deals with the Barton–Friedman conjecture under an equivalent reformulation in terms of the little Grothendieck inequality (see [7, Problem 5.10.13]). We refer to section 2 for more details on the little Grothendieck inequality. It is very well illustrated in [7, pages 337-346] how a proof to the Barton-Friedman conjecture, or equivalently, to the little Grothendieck inequality, might have important consequences and “restore the validity” of all subsequent works relying on the original Grothendieck inequality in (1).

In this paper we fill the gap by proving the Barton-Friedman conjecture. The main result reads as follows

**Theorem 1.1.** Suppose \( G > 8(1 + 2\sqrt{3}) \). Let \( E \) and \( B \) be JB*-triples. Then for every bounded bilinear form \( V : E \times B \to \mathbb{C} \) there exist norm-one functionals \( \varphi \in E^* \) and \( \psi \in B^* \) satisfying

\[ |V(x, y)| \leq G \|V\| \|x\|_\varphi \|y\|_\psi \]

for all \((x, y) \in E \times B\). \( \square \)

This theorem will be proved in Theorem 6.4 below.

The paper is organized as follows. In Subsection 1.1 we provide some background on JB*-triples. Subsection 1.2 deals with a representation of JBW*-triples in the form of a suitable direct sum (see Proposition 1.3).

Section 2 is devoted to the so-called little Grothendieck inequality. We recall where the gap was and indicate the strategy of our proof.

In the three following sections we prove the little Grothendieck inequality for individual summands from Proposition 1.3 and in the last section we glue the results together and provide proofs of the main results.

Along the paper, all Banach spaces will be over the field of complex numbers, the symbols \( S_X \) and \( B_X \) will stand for the unit sphere and the closed unit ball of a Banach space \( X \), respectively.

### 1.1. Basic notions and nomenclature

The aim of extending the celebrated Riemann mapping theorem to complex Banach spaces of arbitrary dimension led W. Kaup to classify bounded symmetric domains in arbitrary complex Banach spaces (see [21]). It was proved by L. Harris that the open unit ball of a C*-algebra is a bounded symmetric domain (cf. [17]). It should be recalled that a domain \( D \) in a complex Banach space is symmetric if for each \( a \in D \) there is a biholomorphic map \( S_a \) of \( D \) onto itself with \( S_a = S_a^{-1} \), such that \( a \) is an isolated fixed point of \( S_a \). However, the open unit balls of all C*-algebras do not exhaust all examples, namely, infinite dimensional complex Hilbert spaces enjoy the same property, but
they are never C*-algebras. The celebrated contribution due to W. Kaup shows that the biholomorphic images of the open unit balls of JB*-triples cover all possible examples of bounded symmetric domains (cf. [21] or [8] Theorem 2.5.27).

A JB*-triple is a complex Banach space E equipped with a (continuous) triple product \( \langle \cdot, \cdot, \cdot \rangle : E^3 \to E \), which is symmetric and bilinear in the outer variables and conjugate-linear in the middle one, and satisfies the following algebraic–analytic axioms (where given \( a, b, c \in E \), \( L(a, b) \) stands for the (linear) operator on \( E \) given by \( L(a, b)(x) = \langle a, b, x \rangle \), for all \( x \in E \)):

\( JB^*-1 \) \( L(x, y)L(a, b) = L(L(x, y)(a), b) - L(a, L(y, x)(b)) + L(a, b)L(x, y) \), for all \( a, b, x, y \in E \);

\( JB^*-2 \) The operator \( L(a, a) \) is a hermitian operator with nonnegative spectrum for each \( a \in E \);

\( JB^*-3 \) \( \|a, a, a\| = \|a\|^3 \) for all \( a \in E \).

The space \( B(H, K) \) of all bounded linear operators between complex Hilbert spaces \( H \) and \( K \), which is rarely a C*-algebra, is always a JB*-triple when equipped with the triple product defined by \( \langle x, y, z \rangle = \frac{1}{2}(xy^*z + zy^*x) \). The same triple product provides a structure of JB*-triple for every C*-algebra. Moreover, if \( H \) is a complex Hilbert space, it can be canonically identified with \( B(\mathbb{C}, H) \), so the above triple product produce induces a structure of JB*-triple on \( H \).

Moreover, every JB*-algebra \( B \) (see, e.g. [30] or [16, Section 3.8]) is a JB*-triple under the triple product defined by \( \langle x, y, z \rangle = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^* \) \((x, y, z \in B) \) (see [8] Lemma 3.1.6) or [3] Theorem 4.1.45). We recall that a JB*-algebra is a complex Jordan Banach algebra \( A \) equipped with an algebra involution \( * \) satisfying the following three conditions

\[ \|a \circ b\| \leq \|a\| \|b\|, \quad \|a^*\| = \|a\|, \quad \text{and} \quad \|\{a, a^*, a\}\| = \|a\|^3, \]

for all \( a, b \in A \), where that \( \{a, a^*, a\} = 2(a \circ a^*) \circ a - a^* \circ a \).

A formidable result due to Kaup asserts that a linear bijection between JB*-triples is a triple isomorphism if and only if it is an isometry (cf. [21] Proposition 5.5)).

Given \( a, b \in E \) the symbol \( Q(a, b) \) will stand for the conjugate-linear operator given by \( Q(a, b)(x) = \langle a, x, b \rangle \). We shall write \( Q(a) \) for \( Q(a, a) \).

An element \( e \in A \) is said to be a tripotent if \( e = \{e, e, e\} \). Every projection in a C*-algebra \( A \) is a tripotent when the latter is regarded as a JB*-triple. Actually, tripotents in \( A \) are precisely partial isometries.

For each tripotent \( e \in E \), the eigenvalues of the mapping \( L(e, e) \) are contained in the set \( \{0, \frac{1}{2}, 1\} \). Given \( i \in \{0, 1, 2\} \), the linear operator \( P_i(e) : E \to E \) is defined by

\[ P_0(e) = L(e, e)(2L(e, e) - id_E) = Q(e)^2, \]

\[ P_1(e) = 4L(e, e)(id_E - L(e, e)) = 2(L(e, e) - Q(e)^2), \]

and \( P_2(e) = (id_E - L(e, e))(id_E - 2L(e, e)) \).

It is known that \( P_0(e), P_1(e) \) and \( P_2(e) \) are contractive linear projections (see [12] Corollary 1.2), which are called the Peirce projections associated with \( e \). Furthermore, the range of \( P_1(e) \) is the eigenspace, \( E_1(e) \), of \( L(e, e) \) corresponding to the eigenvalue \( \frac{1}{2} \), and

\[ E = E_2(e) \oplus E_1(e) \oplus E_0(e) \]
is known as the Peirce decomposition of $E$ relative to $e$ (see [12, §8, Definition 1.2.37] or [6, §4.2.2] and [7, §5.7] for more details). If $E$ is a unital $C^*$-algebra and $e \in E$ a tripotent, then $e$ is a partial isometry with initial projection $p_i$ and final projection $p_f$. The Peirce projections are given by the following identities

$$P_2(e)(x) = p_f x p_i, \quad P_1(e)(x) = p_f x (1-p_i) + (1-p_f) x p_i, \quad P_0(e)(u) = (1-p_f)x(1-p_i),$$

where $x$ runs through $E$.

A tripotent $e$ is called complete if $E_0(e) = \{0\}$. If $E = E_2(e)$, or equivalently, if $\{e, e, x\} = x$ for all $x \in E$, we say that $e$ is unitary.

For each tripotent $e$ in a JB$^*$-triple, $E$, the Peirce-2 subspace $E_2(e)$ is a unital $B^*$-algebra with unit $e$, product $a \circ_e b := \{a, e, b\}$ and involution $a^{**} := \{e, a, e\}$ (cf. [8, §1.2 and Remark 3.2.2]). As we noticed above, every JB$^*$-algebra is a JB$^*$-triple with respect to the product

$$\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*.$$

By Kaup’s theorem (see [21, Proposition 5.5]) the triple product on $E_2(e)$ is uniquely determined by the expression

$$\{a, b, c\} = (a \circ_{e} b^{**}) \circ_{e} c + (c \circ_{e} b^{**}) \circ_{e} a - (a \circ_{e} c) \circ b^{**},$$

for every $a, b, c \in E_2(e)$. Therefore, unital JB$^*$-algebras are in one-to-one correspondence with JB$^*$-triples admitting a unitary element.

We shall make use of the following natural partial order defined on the set of tripotents in a JB$^*$-triple $E$. Two tripotents $e, v$ in $E$ are called orthogonal (denoted by $e \perp v$) if $\{e, e, v\} = 0 \Leftrightarrow \{v, v, e\} = 0 \Leftrightarrow e \in E_0(v) \Leftrightarrow v \in E_0(e)$. Suppose $e, u$ are tripotents in $E$, we say that $e \leq u$ if $u - e$ is a tripotent which is orthogonal to $e$. By [12, Corollary 1.7] $e \leq u$ if and only if any of the equivalent conditions holds:

(a) $P_2(e)(u) = e$;
(b) $\{e, u, e\} = e$;
(c) $e$ is a projection (i.e. a self-adjoint idempotent) in the JB$^*$-algebra $E_2(u)$.

A JBW$^*$-triple is a JB$^*$-triple which is also a dual Banach space. In the triple setting, JBW$^*$-triples play the role of von Neumann algebras in the class of $C^*$-algebras. A fundamental result in the theory of JB$^*$-triples proves that every JBW$^*$-triple admits a unique (isometric) predual and its product is separately weak$^*$-continuous (see [3]). JBW$^*$-algebras, von Neumann algebras, and complex Hilbert spaces are examples of JBW$^*$-triples for the triple products presented above (cf. [8, Example 2.5.33 and Lemma 3.1.6]).

The complete tripotents of a JB$^*$-triple $E$ are precisely the extreme points of its closed unit ball (cf. [4, Lemma 4.1] and [22, Proposition 3.5] or [8, Theorem 3.2.3]). Therefore every JBW$^*$-triple contains a huge set of complete tripotents.

The theory of JBW$^*$-triples is deeply indebted with the study on the predual of JBW$^*$-triples developed by F. Friedman and B. Russo in [12]. Among the many influencing results established in this reference, it is shown that for each non-zero functional $\varphi$ in the predual, $M_\varphi$, of a JBW$^*$-triple $M$, there is a unique tripotent $s(\varphi) \in M$, called the support tripotent of $\varphi$, such that $\varphi = \varphi \circ P_2(s(\varphi))$, and $\varphi |M_2(s(\varphi))$ is a faithful positive functional on the JBW$^*$-algebra $M_2(s(\varphi))$ (cf. [12, Proposition 2], or [7, Proposition 5.10.57]). We recall that a functional $\varphi$ in the
dual space of a JB$^*$-algebra $B$ is called faithful if $\varphi(a) = 0$ for $a \geq 0$ implies $a = 0$. We know from [12] part (b) in the proof of Proposition 2 that
\begin{equation}
\text{if } u \text{ is a tripotent in } M \text{ with } \|\varphi\| = \varphi(u), \text{ then } u \geq e(\varphi).
\end{equation}

It is now time to recall the definition of the pre-Hilbert semi-norm appearing in Grothendieck’s inequalities, which were introduced by J.T. Barton and Y. Friedman in [2]. Suppose $\varphi$ is a functional in the predual of JBW$^*$-triple $M$. By [2, Proposition 1.2] the mapping $M \times M \to \mathbb{C}$, $(x,y) \mapsto \varphi\{x,y,s(\varphi)\}$ is a positive semi-definite sesquilinear form on $M$. In particular, the Cauchy-Schwarz inequality holds. The associated pre-Hilbert semi-norm is denoted by $\|x\|_\varphi := \sqrt{\varphi\{x,x,s(\varphi)\}}$ for every tripotent $x \in M$.

It is further known that $\|x\|_\varphi^2 = \varphi\{x,x,s(\varphi)\} = \varphi\{x,z\}$, whenever $z$ is an element in $M$ satisfying $\varphi(z) = \|z\| = 1$. In particular, $\|x\|_\varphi^2 = \varphi\{x,x,u\}$ for every tripotent $u \in M$ with $u \geq s(\varphi)$. Moreover, as a consequence of the fact that $\|\{x,y,z\}\| \leq \|x\|\|y\|\|z\|$ for all $x,y,z$ in a JB$^*$-triple, we get
\begin{equation}
\|x\|_\varphi \leq \sqrt{\|\varphi\|\|x\|}.
\end{equation}

1.2. A representation of JBW$^*$-triples. Key tools we use to prove our main results include structure results of JBW$^*$-triples obtained by G. Horn and E. Neher in [18, (1.7)], [19, (1.20)], and recently revisited in [15] to decompose every JBW$^*$-triple $M$ in a suitable way. Before formulating the variant we need give the following easy lemma on decomposing special JBW$^*$-triples.

**Lemma 1.2.** Let $V$ be a von Neumann algebra, $p \in V$ a projection and $(z_j)_{j \in J}$ an orthogonal family of projections in the center of $pVp$ with sum equal to $p$. Then
\[ pV = \bigoplus_{j \in J} z_j V. \]

More precisely, the mapping
\[ L : x \mapsto (z_jx)_{j \in J} \]
is an onto isometry witnessing the above equality.

**Proof.** The mapping $L$ is clearly a one-to-one linear mapping with $\|L\| \leq 1$. Moreover, for any $a,b,c \in pV$ and $j \in J$ we have
\[ \{z_ja, z_jb, z_jc\} = \frac{1}{2} (z_jab^*z_jc + z_jcb^*z_ja) = \frac{1}{2} (z_jab^*c + z_jcb^*a) = z_j \{a,b,c\}, \]
where in the second equality we used the fact that the elements $ab^*$ and $cb^*$ belong to $pVp$ and hence they commute with $z_j$.

It follows that $L$ is a triple homomorphism. Since $L$ is injective, it is an isometry by [8, Theorem 3.4.1].

Finally, it is clear that the range contains all elements with only finitely many nonzero coordinates. Since $L$ is weak$^*$-to-weak$^*$ continuous, it follows that $L$ is onto. □

The promised representation result follows. For definitions and basic results on types of projections in von Neumann algebras we refer to [29, Chapter V].
Proposition 1.3. Let $M$ be any JBW$^*$-triple. Then $M$ is (isometrically) JB$^*$ triple isomorphic to a JBW$^*$-triple of the form

$$
\left( \bigoplus_{k \in \Lambda} L^\infty(\mu_k, C_k) \right) \oplus L^\infty N \oplus L^\infty p_1 V \oplus L^\infty p_2 V \oplus L^\infty p_3 V,
$$

where

- $(\mu_k)_{k \in \Lambda}$ is a (possibly empty) family of probability measures;
- Each $C_k$ is a finite dimensional JB$^*$-triple (actually a finite dimensional Cartan factor) for any $k \in \Lambda$;
- $N$ is a JBW$^*$-algebra;
- $V$ is a von Neumann algebra, $p_1, p_2, p_3 \in V$ are projections such that $p_1$ is properly infinite, $p_2 V p_2$ is a von Neumann algebra of type $II_1$ and $p_3 V p_3$ is a finite von Neumann algebra of type $I$.

Proof. By \cite[Proposition 9.1]{13} $M$ is (isometrically) JB$^*$ triple isomorphic to a JBW$^*$-triple of the form

$$
\left( \bigoplus_{k \in \Lambda} L^\infty(\mu_k, C_k) \right) \oplus L^\infty N \oplus L^\infty pV,
$$

where $(\mu_k)_{k \in \Lambda}$, $(C_k)_{k \in \Lambda}$ and $N$ have the properties given in the statement and, moreover, $V$ is a von Neumann algebra and $p \in V$ is a projection.

It remains to refine this decomposition a bit. The summand $pV$ can be decomposed as a direct sum of two summands of the form $p_1 V$ and $p_2 V$, where $p_1$ is a properly infinite projection and $p_2$ is a finite projection (cf. \cite[Proposition 6.3.7]{20} or \cite[Theorem 10.1]{15}).

Further, by \cite[Theorem V.1.19]{29} there are orthogonal central projections $z_1, z_2$ in $p_2 V p_2$ with $z_1 + z_2 = p$ such that $z_1 p_2 V p_2$ is of type $I$ and $z_2 p_2 V p_2$ of type $II_1$.

To complete the proof set $p_2 = z_2 p_2', p_3 = z_1 p_2'$ and use Lemma 1.2. \qed

2. Little Grothendieck inequality

The difficulties around Barton-Friedman conjecture are essentially due to a gap in the proof of the so-called little Grothendieck inequality stated in \cite[Theorem 1.3]{2}. As pointed out in \cite{25} only the following statement was actually proved.

Lemma 2.1. (\cite[Lemma 3]{25}, \cite[Theorem 1.3]{2}) Let $M$ be a complex JBW$^*$-triple, $H$ a complex Hilbert space, and let $T : M \to H$ be a norm-attaining weak$^*$-continuous linear operator. Then there exists a norm-one normal functional $\varphi \in M_*$ satisfying

$$
\|T(x)\| \leq \sqrt{2} \|T\| \|x\|_\varphi,
$$

for all $x \in M$. \qed

In \cite{25} it was observed that the assumption of norm-attaining, tacitly used in \cite{2}, need not to be satisfied. Via approximating operators by norm-attaining ones the following perturbed version of \cite[Theorem 1.3]{2} was proved.

Theorem 2.2. \cite[Theorem 3]{25} Let $K > \sqrt{2}$ and $\varepsilon > 0$. Then, for every JBW$^*$-triple $M$, every complex Hilbert space $H$, and every weak$^*$-continuous linear operator $T : M \to H$, there exist norm-one functionals $\varphi_1, \varphi_2 \in M_*$ such that
inequality
\[ \|T(x)\| \leq K \|T\| \sqrt{\|x\|_{\varphi_1}^2 + \varepsilon \|x\|_{\varphi_2}^2} \]
holds for all \( x \in M \). \qed

This version is enough for many structure results on JBW*-triples, but the question whether the perturbation is necessary, remained to be challenging. We can get rid of the perturbation if we assume that the JBW*-triple \( M \) contains a unitary element, or equivalently, when \( M \) is a (unital) JBW*-algebra, as witnessed by the following theorem.

**Theorem 2.3.** \([25, \text{Theorem } 4]\) Let \( K > 2 \) and let \( M \) be a JBW*-triple admitting a unitary element \( u \). Then for every complex Hilbert space and every weak*-continuous linear operator \( T : M \to H \) there exists a norm-one functional \( \varphi \in M_* \) such that \( s(\varphi) \leq u \) and
\[ \|T(x)\| \leq K \|T\| \|x\|_{\varphi}, \]
for all \( x \in M \). \qed

We are going to extend this theorem to general JBW*-triples by analyzing behaviour of the seminorms \( \|\cdot\|_{\varphi_1,\varphi_2} \) for a pair of normal functionals which do not have necessarily norm one. More specifically, we are going to prove the following theorem.

**Theorem 2.4.** Let \( M \) be a JBW*-triple. Then given any two functionals \( \varphi_1, \varphi_2 \) in \( M_* \), there exists a norm-one functional \( \psi \in M_* \) such that
\[ \|x\|_{\varphi_1,\varphi_2} \leq \sqrt{2} \cdot \sqrt{\|\varphi_1\| + \|\varphi_2\| \cdot \|x\|_{\psi}}, \]
for all \( x \in M \). Furthermore, given \( K > 2 \), for every complex Hilbert space \( H \), and every weak*-to-weak continuous linear operator \( T : M \to H \), there exists a norm-one functional \( \psi \in M_* \) satisfying
\[ \|T(x)\| \leq K \|T\| \|x\|_{\psi} \]
for all \( x \in M \).

This theorem will be proved in Theorem 6.1 below.

Observe that, once we establish the first estimate in this theorem, the second part follows easily from Theorem 2.2 (note that \( \sqrt{\|\varphi_1\| + \varepsilon \|x\|_{\varphi_2}^2} = \|x\|_{\varphi_1,\varphi_2} \)).

The first estimate will be proved using the representation from Proposition 1.3. We will prove it for individual summands and then we will glue the results together using the following proposition which is a finer version of [24, Theorem 2.12].

**Proposition 2.5.** Let \( \{M_\alpha\}_{\alpha \in \Lambda} \) be a family of JBW*-triples for which there exists a positive constant \( G \) satisfying that for every \( \alpha \in \Lambda \), and every couple of normal functionals \( \varphi_{1,\alpha}, \varphi_{2,\alpha} \in (M_\alpha)_* \), there exists a norm-one functional \( \varphi_\alpha \in (M_\alpha)_* \) satisfying
\[ \|x\|_{\varphi_{1,\alpha},\varphi_{2,\alpha}} \leq G \sqrt{\|\varphi_{1,\alpha}\| + \|\varphi_{2,\alpha}\| \cdot \|x\|_{\varphi_\alpha}}, \]
for all \( x \in M_\alpha \). Let \( M = \bigoplus_{\alpha \in \Lambda} M_\alpha \). Then for every couple of normal functionals \( \varphi_1, \varphi_2 \in M_* \) there exists a norm-one functional \( \varphi \in M_* \) satisfying
\[ \|x\|_{\varphi_1,\varphi_2} \leq G \sqrt{\|\varphi_1\| + \|\varphi_2\| \cdot \|x\|_{\varphi}}, \]
for all \( x \in M \).
Lemma 2.6. Let \( \varphi_1, \varphi_2 \in M_* \) be given. For \( \alpha \in \Lambda \) and \( j = 1, 2 \) denote by \( \varphi_{j, \alpha} \) the restriction of \( \varphi_j \) to \( M_\alpha \) (or, more precisely, the composition of \( \varphi_j \) with the canonical embedding of \( M_\alpha \) into \( M \)). By the assumption there is a norm-one functional \( \varphi_\alpha \in (M_\alpha)_* \) with

\[
\|x\|_{\varphi_1, \varphi_2} \leq G\sqrt{\|\varphi_1\| + \|\varphi_2\|} \|x\|_{\varphi_\alpha}, \text{ for } x \in M_\alpha.
\]

Further, set

\[
c_\alpha = \frac{\|\varphi_1\| + \|\varphi_2\|}{\|\varphi_1\| + \|\varphi_2\|}, \quad \alpha \in \Lambda,
\]

and observe that \( \sum_{\alpha \in \Lambda} c_\alpha = 1 \). Thus the functional \( \varphi \in M_* \) defined by

\[
\varphi((x_\alpha)_{\alpha \in \Lambda}) = \sum_{\alpha \in \Lambda} c_\alpha \varphi_\alpha(x_\alpha) \text{ for } x = (x_\alpha)_{\alpha \in \Lambda} \in M,
\]

has norm one. Moreover, for each \( x \in M \) we have

\[
\|x\|^2_{\varphi_1, \varphi_2} = \sum_{\alpha \in \Lambda} \|x_\alpha\|^2_{\varphi_1, \varphi_2} \leq \sum_{\alpha \in \Lambda} G^2(\|\varphi_1\| + \|\varphi_2\|) \|x_\alpha\|^2_{\varphi_\alpha} = G^2(\|\varphi_1\| + \|\varphi_2\|) \|x\|^2_{\varphi},
\]

The individual summands will be addressed in the three following sections, in the last section we glue the results together and show that a solution to the Barton–Friedman conjecture follows.

The proof for the summands \( N \) and \( p_1 V \) is given in Corollary 3.4 and it is done by a refinement of the proof of Theorem 2.3 using some ideas from [15]. The proof for the remaining cases is done by showing that in these cases any seminorm of the form \( \|\cdot\|_{\varphi_1, \varphi_2} \) attains its maximum on \( B_M \) and then applying Lemma 2.1. The last step of this approach is explained in the following lemma.

Lemma 2.6. Let \( \varphi_1, \varphi_2 \in M_* \) be two normal functionals such that the seminorm \( \|\cdot\|_{\varphi_1, \varphi_2} \) attains its maximum on \( B_M \). Then there is a norm-one functional \( \psi \in M_* \) such that

\[
\|x\|_{\varphi_1, \varphi_2} \leq \sqrt{2} \sqrt{\|\varphi_1\| + \|\varphi_2\|} \|x\|_{\varphi}
\]

for all \( x \in M \).

Proof. Set

\[
N_{\varphi_1, \varphi_2} = \{x \in M : \|x\|_{\varphi_1, \varphi_2} = 0\}.
\]

On the quotient space \( M/N_{\varphi_1, \varphi_2} \), the semi-norm \( \|\cdot\|_{\varphi_1, \varphi_2} \) becomes a pre-Hilbert norm. Let \( H_{\varphi_1, \varphi_2} \) be the completion of the so-defined pre-Hilbert space and let \( \pi_{\varphi_1, \varphi_2} \) be the natural quotient map viewed as a map from \( M \) into \( H_{\varphi_1, \varphi_2} \). The separate weak*-continuity of the triple product and [4] ensure that \( \pi_{\varphi_1, \varphi_2} \) is a weak*-continuous linear operator with norm at most \( \sqrt{\|\varphi_1\| + \|\varphi_2\|} \). Finally, we may apply Lemma 2.1 to the operator \( T = \pi_{\varphi_1, \varphi_2} \). \( \square \)
3. JBW*-triples in which Peirce-2 subspaces of tripotents are upward directed

In this section we particularize our study to JBW*-triples satisfying that Peirce-2 subspaces of tripotents are upward directed by inclusion. The idea stems from [15] where such JBW*-triples were considered in order to have a mild substitute for the lack of an order, see e.g. [15] Proposition 6.5. Let us begin with a series of technical lemmata.

**Lemma 3.1.** Let \( \varphi_1, \varphi_2 \) be two functionals in the predual of a JBW*-triple \( M \). Suppose there exists a tripotent \( p \in M \) such that \( s(\varphi_1) \leq p \) and \( s(\varphi_2) \leq p \). Then the functional \( \psi = \frac{\varphi_1 + \varphi_2}{\| \varphi_1 \| + \| \varphi_2 \|} \) satisfies \( \| \psi \| = 1 \), \( s(\psi) \leq p \), and

\[
\| x \|_{\varphi_1, \varphi_2} = \sqrt{\| \varphi_1 \| + \| \varphi_2 \| \cdot \| x \|_{\psi}}, \quad x \in M.
\]

**Proof.** Set \( e = s(\varphi_2) \) and \( u = s(\varphi_1) \). By the assumption we have \( u \leq p \) and \( e \leq p \). Further, \( \varphi_2(p) = \varphi_2(e) = \| \varphi_2 \| \) and \( \varphi_1(p) = \varphi_1(u) = \| \varphi_1 \| \), so \( \psi(p) = 1 \). Since clearly \( \| \psi \| \leq 1 \), we deduce that \( \| \psi \| = \psi(p) = 1 \) and hence \( s(\psi) \leq p \) (cf. (3)).

Finally, for \( x \in M \) we have

\[
\| x \|_{\varphi_1, \varphi_2}^2 = \psi(\{ x, x, s(\psi) \}) = \psi(\{ x, x, p \}) = \frac{\varphi_1(\{ x, x, p \}) + \varphi_2(\{ x, x, p \})}{\| \varphi_1 \| + \| \varphi_2 \|}.\]

In our next proposition we show that the semi-norm given by a normal functional whose support tripotent is contained in the Peirce-2 subspace of another tripotent \( p \) in a JBW*-triple \( M \) can be bounded by the semi-norm given by a positive functional in the predual of the JBW*-algebra \( M_2(p) \).

**Proposition 3.2.** Let \( M \) be a JBW*-triple and let \( \varphi \in M_* \). Assume that \( p \in M \) is a tripotent such that \( s(\varphi) \in M_2(p) \). Then there exists a functional \( \hat{\varphi} \in M_* \) such that \( \| \hat{\varphi} \| = \| \varphi \| \), \( s(\hat{\varphi}) \leq p \) and \( \| x \|_{\hat{\varphi}} \leq \sqrt{2} \| x \|_{\varphi} \) for all \( x \in M \).

**Proof.** We mimic the approach in the proof of [15] Lemma 7.7. By the arguments in the first paragraph in the proof of [5] Proposition 2.4 (see also [10] Lemma 3.9) we can find a unital JB*-algebra \( B \) and an isometric triple embedding \( \pi : M \to B \) such that \( \pi(p) \) is a projection in \( B \). We can therefore assume that \( M \) is a JB*-subtriple of \( B \) and \( p \) is a projection in \( B \). The triple product in \( B \) (and in \( M \)) is uniquely determined by the expressions \( \{ a, b, c \} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^* \) \((a, b, c \in B)\).

Set \( u = s(\varphi) \). Define \( G : B \to B \) by \( G(x) = P_2(u)(x \circ u) \). Having in mind that \( 1 - p \perp u \), and hence for each \( x \in M \), we have \( P_2(u)(x \circ u) = P_2(u)(x, 1, u) = P_2(u)(x, p, u) \in M \), we deduce that \( G \) maps \( M \) into \( M_2(u) \) and its restriction to \( M \) is weak*-to-weak* continuous.

Set \( \hat{\varphi} = \varphi \circ G|_M \). Then \( \hat{\varphi} \in M_* \) and \( \| \hat{\varphi} \| \leq \| \varphi \| \) (as Peirce projections are contractive and hence clearly \( \| G \| \leq 1 \)).

Moreover,

\[
\hat{\varphi}(p) = \varphi(P_2(u)(p \circ u)) = \varphi P_2(u)(p, p, u) = \varphi(u) = \| \varphi \|,\]

[10] J. Hamhalter, O.F.K. Kalenda, A.M. Peralta, and H. Pfitzner
Lemma 4.1. will be obtained after a series of lemmata.

Proof. will show that this property can be carried over to the space $L^B$. Henceforth, let $C$ be such a JB$^*$-triple. Since $C$ is finite dimensional, every bounded linear operator from $C$ into a Hilbert space attains its norm. In particular, any seminorm $\| \cdot \|_{\varphi_1,\varphi_2}$ attains its maximum on the unit ball $B_C$. We will show that this property can be carried over to the space $L^\infty(\mu, C)$. This goal will be obtained after a series of lemmata.

Lemma 4.1. The mapping $C \times C^* \to [0, \infty)$, $(x, \varphi) \mapsto \|x\|\varphi$ is continuous.
Proof. The set

$$A := \{(\varphi, e) \in C^* \times S_C : \varphi(e) = \|\varphi\|, \{e, e, e\} = e\}$$

is clearly closed. Moreover, the mapping $\Phi : C \times A \to [0, \infty)$ given by

$$\Phi(x, \varphi, e) = \varphi \{x, x, e\}$$

is continuous and $\Phi(x, \varphi, e) = \|x\|_\varphi^2$ for $x \in C$ and $(\varphi, e) \in A$.

Assume now that $(x_n, \varphi_n)$ is a sequence in $C \times C^*$ converging to an element $(x, \varphi)$. We will show that $\|x_n\|_{\varphi_n} \to \|x\|_\varphi$. Otherwise, up to passing to a subsequence, we may assume that $\|x_n\|_{\varphi_n} \to e \neq \|x\|_\varphi$ (note that the sequence $(\|x_n\|_{\varphi_n})_n$ is bounded). Let $e_n = s(\varphi_n)$ for $n \in \mathbb{N}$. We may assume, without loss of generality, that the sequence $(e_n)$ converges to some $e \in C$. Since $(\varphi_n, e_n) \in A$ for each $n$, necessarily $(\varphi, e) \in A$ as well. Thus

$$\|x_n\|_{\varphi_n}^2 = \Phi(x_n, \varphi_n, e_n) \to \Phi(x, \varphi, e) = \|x\|_\varphi^2,$$

a contradiction which completes the proof. \[\square\]

Lemma 4.2. The set valued mapping $\Psi : \mathbb{C}^2 \to 2^{BC}$ defined by

$$\Psi(\varphi_1, \varphi_2) = \{x \in BC : \|x\|_{\varphi_1, \varphi_2} = \max_{y \in BC} \|y\|_{\varphi_1, \varphi_2}\}$$

is upper semi-continuous and compact-valued. Consequently, there is a Borel-measurable selection $H$ from $\Psi$.

Proof. Taking into account that $S_C$ is compact, by \cite{11} Lemma 3.1.1 it is enough to show that the set

$$\{(\varphi_1, \varphi_2, x) \in (\mathbb{C}^*)^2 \times BC : \|x\|_{\varphi_1, \varphi_2} = \max_{y \in BC} \|y\|_{\varphi_1, \varphi_2}\}$$

is closed. But this easily follows from Lemma 4.1 as this set equals

$$\bigcap_{y \in BC} \{(\varphi_1, \varphi_2, x) \in (\mathbb{C}^*)^2 \times BC : \|x\|_{\varphi_1, \varphi_2} \geq \|y\|_{\varphi_1, \varphi_2}\}.$$

Since $\Psi$ has clearly nonempty values, the final statement follows, for example, from the Kuratowski-Ryll-Nardzewski theorem (see \cite{11} Theorem 18.13). \[\square\]

Let $(\Omega, \Sigma, \mu)$ be a probability space, and let $M = L^\infty(\mu, C)$. Then $M$ is a JBW*-triple (with the triple product defined pointwise) and $M_* = L^1(\mu, C^*)$.

We need a more concrete description of the elements in $M_*$. Assume $g \in M_* = L^1(\mu, C^*)$. Let $u = s(g)$. Then $u$ is a tripotent in $M$, hence $u(\omega)$ is a tripotent in $C$ for almost all $\omega \in \Omega$. Under these circumstances we have

$$\|g\| = g(u) = \text{Re} \int \langle g(\omega), u(\omega) \rangle \, d\mu(\omega) = \int \text{Re} \langle g(\omega), u(\omega) \rangle \, d\mu(\omega) \leq \int \|g(\omega)\| \cdot \|u(\omega)\| \, d\mu(\omega) \leq \int \|g(\omega)\| \, d\mu(\omega) = \|g\|$$

So, we have everywhere equalities, hence $\langle g(\omega), u(\omega) \rangle = \|g(\omega)\| \text{ almost everywhere}$, and thus $u(\omega) \geq s(g(\omega))$ almost everywhere (cf. \cite{13}).

It follows that for almost all $\omega$ we have

$$\|x\|_{\|g(\omega)\|}^2 = \langle g(\omega), \{x, x, u(\omega)\} \rangle, \text{ for all } x \in C.$$
GROTHENDIECK’S INEQUALITIES

Therefore, given \( f \in M \) we have

\[
\|f\|^2_g = \langle g, \{f, f, u\} \rangle = \int \langle g(\omega), \{f(\omega), f(\omega), u(\omega)\} \rangle \, d\mu(\omega) = \int \|f(\omega)\|^2_{g(\omega)} \, d\mu(\omega).
\]

Let \( g_1, g_2 \in L^1(\mu, C^\ast) \). Let \( H \) be the Borel-measurable selection from \( \Psi \) given by (4.2). We set \( f(\omega) = H(g_1(\omega), g_2(\omega)) \). Then \( \|f\|_\infty \leq 1 \). Let \( h \in L^\infty(\mu, C) \) be an element of the unit ball. Then

\[
\|h\|^2_{g_1, g_2} = \int \|h(\omega)\|^2_{g_1(\omega), g_2(\omega)} \, d\mu(\omega) \leq \int \|f(\omega)\|^2_{g_1(\omega), g_2(\omega)} \, d\mu(\omega) = \|f\|^2_{g_1, g_2}.
\]

Therefore the pre-Hilbert semi-norm \( \|\cdot\|_{g_1, g_2} \) attains its maximum on the closed unit ball of \( L^\infty(\mu, C) \) (at \( f \)).

The previous arguments combined with Lemma 2.6 provide the following solution to the little Grothendieck problem for JBW\(^\ast\)-triples of the form \( L^\infty(\mu, C) \).

**Proposition 4.3.** Let \( (\Omega, \Sigma, \mu) \) be a probability space, and let \( M = L^\infty(\mu, C) \), where \( C \) is a finite dimensional JB\(^\ast\)-triple. Then for every couple of normal functionals \( g_1, g_2 \in M \), the pre-Hilbert semi-norm \( \|\cdot\|_{g_1, g_2} \) attains its maximum on the closed unit ball of \( L^\infty(\mu, C) \), and thus there exists a norm-one functional \( h \in M \), satisfying

\[
\|f\|_{g_1, g_2} \leq \sqrt{2} \sqrt{\|g_1\| + \|g_2\|} \|f\|_h,
\]

for all \( f \in M \).

5. **Right ideals associated with finite projections in a von Neumann algebra**

The aim of this section is to solve the little Grothendieck problem for the summands \( p_2V \) and \( p_3V \) from Proposition 1.3. They require different methods, but some tools are common for both cases. The first lemma shows how to express the hilbertian semi-norms using polar decomposition of the functional.

**Lemma 5.1.** Let \( V \) be a von Neumann algebra, \( p \in V \) a finite projection and \( \varphi \in (pV)^\ast \). Then there is a positive functional \( \psi \) on \( pVp \) and a unitary element \( u \in V \) such that \( \|\psi\| = \|\varphi\| \), \( \varphi(x) = \psi(xu^pu^xu^pu) \) for \( x \in pV \), \( s(\psi)u^* = s(\varphi) \), and

\[
\|x\|_\varphi^2 = \frac{1}{2} (\psi(x^u^pu^xu^pu^*xu^pu^*xu^pu^*)) \quad \text{for all } x \in pV.
\]

**Proof.** Let \( v = s(\varphi) \). Then \( v \), being a tripotent in \( pV \), is a partial isometry in \( V \) with final projection \( q \leq p \). Denote by \( r \) the initial projection. Further, since \( p \) is finite, \( q \) is finite as well, hence \( v \) can be extended to a unitary operator \( \tilde{v} \in V \) (cf. [29, Proposition V.1.38]).

Set \( \psi(x) := \varphi(x\tilde{v}) \) for \( x \in pV \). Since \( x \mapsto x\tilde{v} \) is an isometry of \( pV \) onto \( pV \), we deduce that \( \|\psi\| = \|\varphi\| \). Further, since

\[
\psi(\tilde{v}) = \varphi(\tilde{v}) = \varphi(v) = \|\varphi\| = \|\psi\|,
\]

we deduce that \( s(\psi) \leq q \) (cf. [3]), hence \( \psi|_{pVp} \) is a positive functional on \( pVp \). It remains to observe that one can take \( u = \tilde{v}^* \). Indeed, for any \( x \in pV \) we have

\[
\psi(xup) = \psi(x\tilde{v}^*p) = \psi(qx\tilde{v}^*p) = \psi(qx\tilde{v}^*q) = \psi(qxr\tilde{v}^*) = \varphi(qxr) = \varphi(x).
\]

In particular,

\[
\|\varphi\| = \|\psi\| = \psi(s(\psi)) = \psi(s(\psi)p) = \psi(s(\psi)u^*up) = \varphi(s(\psi)u^*),
\]
which shows that \( s(\psi)u^* \geq s(\varphi) \) (cf. \([3]\)). But \( s(\psi) \leq q \) implies that \( s(\psi)u^* = s(\varphi) \).

Finally, for any \( x \in pV \) we have

\[
\|x\|_\varphi^2 = \varphi(\{x, x, v\}) = \frac{1}{2}\varphi(xx^*v + vx^*x) = \frac{1}{2}\psi(xx^*vup + vx^*xup)
\]

\[
= \frac{1}{2}(\psi(xx^*) + \psi(pu^*x^*xup)),
\]

where in the last equality we used that \( vup = vv^*p = qp = q \) and \( s(\psi) \leq q \) to obtain the first term and

\[
\psi(xx^*xup) = \psi(qu^*x^*xup) = \psi(qpu^*x^*xup) = \psi(pu^*x^*xup),
\]

to obtain the second term. \(\square\)

The key result for algebras of type \( II_1 \) is established in the next lemma.

**Lemma 5.2.** Let \( V \) be a von Neumann algebra of type \( II_1 \) and let \( p \in V \) be a projection. Then for each couple of functionals \( \varphi_1, \varphi_2 \in (pV)_* \), the pre-Hilbert semi-norm \( \|\|_{\varphi_1, \varphi_2} \) attains its maximum on the closed unit ball of \( pV \).

**Proof.** For \( j = 1, 2 \) let \( \psi_j \) be a positive functional in \((pV)_* \) and \( u_j \in V \) a unitary element provided by Lemma \([5, 3]\) for \( \varphi_j \). Then

\[
\|x\|_{\varphi_1, \varphi_2}^2 = \frac{1}{2}(\psi_1(xx^*) + \psi_1(pu_1^*x^*xu_1p) + \psi_2(xx^*) + \psi_2(pu_2^*x^*xu_2p))
\]

for any \( x \in pV \).

By the Krein-Milman theorem and the weak*-compactness of the the closed unit ball of \( pV \), the supremum of this semi-norm on the closed unit ball of \( pV \) is attained if and only if it is attained at an extreme point of this closed unit ball. Note that a tripotent in \( pV \) is a partial isometry in \( V \) with final projection below \( p \), the tripotent is complete (i.e. it is an extreme point of the closed unit ball) if and only if its final projection equals \( p \). Therefore the supremum of the semi-norm over the unit ball equals \( \sqrt{C} \), where

\[
C = \sup \left\{ \frac{1}{2}(\psi_1(xx^*) + \psi_1(pu_1^*x^*xu_1p) + \psi_2(xx^*) + \psi_2(pu_2^*x^*xu_2p)); xx^* = p \right\}
\]

\[
= \sup \left\{ \frac{1}{2}(\psi_1(p) + \psi_2(p)) + \frac{1}{2}(\psi_1(pu_1^*x^*xu_1p) + \psi_2(pu_2^*x^*xu_2p)); xx^* = p \right\}.
\]

Let \( T \) be the center-valued trace on \( V \) (cf. \([29\) Theorem V.2.6]). If \( x \in V \) is such that \( xx^* = p \), then \( 0 \leq x^*x \leq 1 \) and \( T(x^*x) = T(p) \). Hence

\[
C \leq \sup \left\{ \frac{1}{2}(\psi_1(p) + \psi_2(p)) + \frac{1}{2}(\psi_1(pu_1^*yu_1p) + \psi_2(pu_2^*yu_2p)); 0 \leq y \leq 1, T(y) = T(p) \right\}.
\]

The supremum on the right-hand side is attained, as it is a supremum of an affine weak*-continuous functional over the convex weak*-compact set

\[
K = \{ y \in V; 0 \leq y \leq 1, T(y) = T(p) \}.
\]

So, the supremum is attained at an extreme point of \( K \). Now, we claim that every extreme point of \( K \) is a projection. Indeed, assume that, say, \( y \in K \) is not a projection. Since \( 0 \leq y \leq 1 \), we may consider the spectral measure \( E \) of \( y \). Since \( y \) is not a projection, there is some \( \delta \in (0, \frac{1}{2}) \) such that \( q = E([\delta, 1 - \delta]) \neq 0 \). Since \( V \) is of type \( II_1 \), there is a projection \( r \leq q \) with \( r \sim q - r \). Set

\[
v = y + \delta(2r - q), \quad w = y - \delta(2r - q).
\]
Then \( y = \frac{1}{2}(v + w) \), \( T(v) = T(w) = T(y) = T(p) \) (as \( T(r) = \frac{1}{2}T(q) \) by [24 Corollary V.2.8]). Moreover
\[
v \geq y - \delta q \geq 0, \quad \text{and} \quad v \leq y + \delta q \leq 1,
\]
and similarly for \( w \). It follows that \( v, w \in K \), so \( y \) is not an extreme point of \( K \). This finishes the proof of the claim.

Fix \( y \in \text{ext } K \) where the supremum is attained. Then \( y \) is a projection satisfying \( T(y) = T(p) \), so \( y \sim p \) by [25 Corollary V.2.8]. Therefore there is \( x \in V \) with \( xx^* = p \) and \( x^*x = y \). We finally observe that the supremum \( C \) is attained at this element \( x \).

The following technical lemma enables us, roughly speaking, to reduce the case \( pV \) for a finite projection \( p \) to the case \( pV \) where the whole \( V \) is finite.

**Lemma 5.3.** Let \( V \) be a von Neumann algebra and \( p \leq t \) two projections in \( V \) such that \( p \) is finite. Consider the JBW*-triple \( M = pV \) and its subtriple \( N = pVt \). Let \( \varphi_1, \varphi_2 \in M_* \) be two functionals such that \( s(\varphi_j) \in N \) for \( j = 1, 2 \). Then
\[
\sup \{ ||x||_{\varphi_1, \varphi_2} ; x \in B_M \} = \sup \{ ||x||_{\varphi_1, \varphi_2} ; x \in B_N \}.
\]

**Proof.** We use some ideas from the proof of [24 Proposition 2.8]. Let \( W = tVt \). Then \( W \) is a von Neumann algebra, a \( C^* \)-subalgebra of \( V \) and \( t \) is its unit. Set \( u_j' = s(\varphi_j) \) for \( j = 1, 2 \).

Both these tripotents are partial isometries in \( W \) with final projection below \( p \). Since \( p \) is finite, by [29 Proposition V.1.38] these partial isometries can be extended to unitary elements \( u_1'', u_2'' \in W \). Set
\[
u_j = pu_j'' \quad \text{for} \quad j = 1, 2.
\]

Then \( u_1, u_2 \) are partial isometries in \( W \) with final projection equal to \( p \). In particular, they are complete tripotents in \( N \) and also in \( M \).

Moreover,
\[
u_j' \leq u_j \quad \text{for} \quad j = 1, 2,
\]
where we use the standard order on tripotents. Indeed, it is enough to observe that
\[
\{u_j', u_j, u_j''\} = u_j'u_j'u_j'' = u_j'(u_j'')^*pu_j' = u_j'(u_j'')^*p_j(u_j')u_j' = u_j'(u_j')^*u_j' = u_j'.
\]

Further, define functionals \( \zeta_j \in W_* \) by \( \zeta_j(x) = \varphi_j(u_jx) \) for \( x \in W \). Clearly \( ||\zeta_j|| \leq ||\varphi_j|| \) and, moreover,
\[
\zeta_j(t) = \varphi_j(u_jt) = \varphi_j(u_j) = ||\varphi_j||,
\]
hence \( \zeta_j \) is positive (and \( s(\zeta_j) \leq t \)).

Given \( x \in M \), set \( x_1 = xt \) and \( x_2 = x(1 - t) \). Note that
\[
\{x, x, u_j\} = \frac{1}{2}(xx^*u_j + u_jx^*x) = \frac{1}{2}(x_1x_1^*u_j + x_2x_2^*u_j + u_jx_1^*x_1 + u_jx_1^*x_2)
\]
where we used that \( x_1x_2 = x_2x_1 = 0 \) and \( u_jx_2 = 0 \) (the initial and the final projections of \( u_j \) both are below \( t \)). Since \( \frac{1}{t}u_jx_1^*, x_2 \in pV(1 - t) \subset M_1(u_j) \), we deduce
\[
P_2(u_j) \{x, x, u_j\} = \frac{1}{2}P_2(u_j)(x_1x_1^*u_j + x_2x_2^*u_j + u_jx_1^*x_1)
\]
Using the fact that \( s(\varphi_j) = u'_j \leq u_j \) we infer that

\[
\|x\|^2_{\varphi_1, \varphi_2} = \frac{1}{2} \varphi_1(x_1', x_1 u_1 + x_2 x_2' u_1 + u_1' x_1) + \frac{1}{2} \varphi_2(x_1 x_1' u_2 + x_2 x_2' u_2 + u_2' x_1) \\
= \frac{1}{2} (\varphi_1(x_1 x_1' u_1 + x_2 x_2' u_1) + \varphi_2(x_1 x_1' u_2 + x_2 x_2' u_2) + \varphi_2(x_1 x_1)).
\]

By the Krein-Milman theorem and the weak*-compactness of \( B_M \) (and \( B_N \)), the supremum of this semi-norm over any of these balls equals the supremum over its extreme points, i.e., over completes tripotents. Further note that a complete tripotent in \( M \) (in \( N \)) is a partial isometry in \( V \) (in \( W \)) with final projection equal to \( p \), i.e., an element \( x \in M \) (\( x \in N \)) satisfying \( xx^* = p \). Since for \( x \in M \) we have \( xx^* = x_1 x_1' + x_2 x_2' \), we have

\[
\sup\{\|x\|^2_{\varphi_1, \varphi_2} : x \in B_N\} \leq \sup\{\|x\|^2_{\varphi_1, \varphi_2} : x \in B_M\} \\
= \sup\{\|x\|^2_{\varphi_1, \varphi_2} : x \in M, xx^* = p\} \\
= \frac{1}{2} \sup \{\varphi_1(pu_1) + \varphi_2(pu_2) + \varphi_1(x_1' x_1) + \varphi_2(x_2' x_1) : x \in M, x_1 x_1' + x_2 x_2' = p\} \\
= \frac{1}{2} \sup \{\varphi_1(pu_1) + \varphi_2(pu_2) + \varphi_1(y' y) + \varphi_2(y' y) : y \in N, yy^* \leq p\} \\
= \frac{1}{2} \sup \{\varphi_1(pu_1) + \varphi_2(pu_2) + \varphi_1(y' y) + \varphi_2(y' y) : y \in B_N\} \\
= \frac{1}{2} \sup \{\varphi_1(pu_1) + \varphi_2(pu_2) + \varphi_1(y' y) + \varphi_2(y' y) : y \in N, yy^* = p\} \\
\leq \sup\{\|x\|^2_{\varphi_1, \varphi_2} : x \in B_N\},
\]

where we used that \( y \mapsto (\varphi_1(y' y) + \varphi_2(y' y))^{1/2} \) is a weak*-continuous pre-hilbertian semi-norm, hence the supremum can be computed over extreme points. \( \square \)

We are now in a position to present a solution to the little Grothendieck problem for the summand \( p_2 V \) from Proposition 1.3.

**Proposition 5.4.** Let \( V \) be a von Neumann algebra and \( p \in V \) a projection such that \( pVp \) is of type \( II_1 \). Then for any \( \varphi_1, \varphi_2 \in (pV)' \), the semi-norm \( \|\cdot\|_{\varphi_1, \varphi_2} \) attains its maximum on the unit ball of \( pV \) and therefore there exists a norm-one functional \( \psi \in (pV)' \), satisfying

\[
\|x\|_{\varphi_1, \varphi_2} \leq \sqrt{2} \cdot \sqrt{\|\varphi_1\| + \|\varphi_2\| \cdot \|x\|_{\psi}}, \text{ for all } x \in pV.
\]

**Proof.** For \( j = 1, 2 \) let \( \psi_j \) be a positive functional on \( pVp \) and \( u_j \in V \) a unitary element provided by Lemma 5.3 for \( \varphi_j \). Set

\[
t = p \vee u_1 pu_1^* \vee u_2 pu_2^*
\]

and \( W = tvt \). Then, being the supremum of three projections equivalent to \( p \), is a finite projection (cf. [29, Theorem V.1.37]). Moreover, the central carrier (also called the central support) of \( p \) in \( W \) equals \( t = 1_W \) (just observe that if \( z \) is a central projection in \( W \) with \( zp = 0 \), then \( zu_j pu_j^* = zu_j pu_j^* z = 0 \) for all \( j = 1, 2 \), and hence \( z = 0 \)).

We claim that \( W \) is of type \( II_1 \). Indeed, assume that \( r \in W \) is a nonzero abelian projection. Since the central carrier of \( p \) equals \( 1_W \) [29, Lemma V.1.25] yields a nonzero projections \( r_1 \leq r \) such that \( r_1 \sim p \). Since \( r_1 \) is abelian, \( p \) is abelian, too, which contradicts the assumption that \( pVp \) is of type \( II_1 \).
Moreover, for \( j = 1, 2 \) we have 
\[
 s(\varphi_j) = s(\psi_j)u_j^*,
\]
so the initial projection is 
\[
u_j s(\psi_j)u_j^* \leq u_j p u_j^* \leq t,
\]
hence \( s(\varphi_j) \in p V t = p W \). By Lemma 5.2 the pre-Hilbert semi-norm \( \| \cdot \|_{\varphi_1, \varphi_2} \) attains its maximum on the closed unit ball of \( p V t \). We deduce from Lemma 5.3 that \( \| \cdot \|_{\varphi_1, \varphi_2} \) actually attains its maximum on the closed unit ball of \( p V \). Thus, by Lemma 2.6 there is a norm-one functional \( \psi \in (p V)^* \), such that 
\[
\| x \|_{\varphi_1, \varphi_2} \leq \sqrt{2} \cdot \sqrt{\| \varphi_1 \| + \| \varphi_2 \|} \cdot \| x \|_{\psi}, \quad x \in p V.
\]
\[\square\]

So, we have solved the case of the summand \( p_2 V \) from Proposition 2.3 and we turn our attention to the remaining summand \( p_3 V \).

Henceforth, for each natural \( n \), the symbol \( M_n \) stand for the C*-algebra of all \( n \times n \)-matrices with complex entries. Given \( 1 \leq k \leq n \), we shall denote by \( U(M_n) \) the set of all unitary matrices in \( M_n \), and by \( P_k(M_n) \) the set of all projections of rank \( k \).

**Lemma 5.5.** The following assertions hold:

(a) Any two projections \( q_1, q_2 \in P_k(M_n) \) are unitarily equivalent;
(b) \( P_k(M_n) \) is a compact set;
(c) given \( r \in P_k(M_n) \) there is a Borel measurable function \( v : P_k(M_n) \to U(M_n) \) such that
\[
r = v(q)^* q v(q) \quad \text{for all } q \in P_k(M_n).
\]

**Proof.** (a) This is well known and easy to see.

(b) It is clear that \( U(M_n) \) is a compact set and that the mapping 
\[
u \mapsto uru^*, \quad u \in U(M_n),
\]
where \( r \in P_k(M_n) \) is fixed, is a continuous map of \( U(M_n) \) onto \( P_k(M_n) \). Thus, \( P_k(M_n) \) is compact.

(c) Fix \( r \in P_k(M_n) \) and consider the continuous mapping used in (b). The inverse of this mapping admits a Borel measurable selection by the Kuratowski-Ryll-Nardzewski theorem (cf. [1 Theorem 18.13]). Denote the selection by \( v \). Then
\[
v(q) r v(q)^* = q \quad \text{for all } q \in P_k(M_n),
\]
hence the assertion follows. \(\square\)

**Lemma 5.6.** Let \( W = L^\infty(\mu, M_n) \) for a probability measure \( \mu \) and \( n \in \mathbb{N} \).

(a) An element \( f \in W \) is a projection if and only if \( f(\omega) \) is a projection in \( M_n \) for \( \mu \)-almost all \( \omega \);
(b) Any projection \( f \in W \) is unitarily equivalent to a projection \( g \in W \) such that \( g(\omega) \in \{0, r_1, \ldots, r_{n-1}, I\} \) for \( \mu \)-almost all \( \omega \), where \( r_j \in M_n \) is a fixed projection of rank \( j \) for \( 1 \leq j < n \).

**Proof.** (a) This assertion follows immediately from definitions.

(b) Let \( f \in W \) be a projection. For \( k \in \{0, \ldots, n\} \) let 
\[
A_k = \{ \omega; \dim \operatorname{ran} f(\omega) = k \}.
\]
By Lemma 5.5(b) each $A_k$ is $\mu$-measurable, being a preimage of a compact set. Further, for each $k \in \{1, \ldots, n\}$ let $v_k : P_k(M_n) \to U(M_n)$ be the mapping provided by Lemma 5.5(c) for the projection $r_k$. Set

$$u(\omega) = \begin{cases} I & \omega \in A_0 \cup A_n, \\ v_k(\omega) & \omega \in A_k, 0 < k < n. \end{cases}$$

Then $u$ is a unitary element of $W$ and $g = u^*fu$ is a projection satisfying the required properties. □

**Lemma 5.7.** Let $W = L^\infty(\mu, M_n)$ for a probability measure $\mu$ and $n \in \mathbb{N}$. Let $p \in W$ be a projection. Then the JB$^*$-triple $pW$ is JB$^*$-triple isomorphic to

$$\bigoplus_{1 \leq k \leq n} L^\infty(\mu_k, r_k M_n),$$

where $\mu_k$ is a finite non-negative measure and $r_k \in M_n$ is a projection of rank $k$ for each $k \in \{1, \ldots, n\}$.

**Proof.** For each $k \in \{0, \ldots, n\}$ let $r_k \in M_n$ be a projection of rank $k$ (note that $r_0 = 0$ and $r_n = I$). By Lemma 5.6 $p$ is unitarily equivalent to a projection $g$ such that $g(\omega) \in \{r_0, \ldots, r_n\}$ $\mu$-almost everywhere. Then $pW$ is triple-isomorphic to $gW$. Further, for $k = 0, \ldots, n$ set

$$A_k = \{\omega; g(\omega) = r_k\}.$$

Then

$$gW = \bigoplus_{1 \leq k \leq n} L^\infty(\mu|_{A_k}, r_k M_n),$$

which completes the proof. □

**Lemma 5.8.** Let $V$ be a finite von Neumann algebra of type I and let $p \in V$ be a projection. Then the JB$^*$-triple $pV$ is JB$^*$-triple isomorphic to

$$\bigoplus_{j \in J} \ell_\infty L^\infty(\mu_j, p_j M_{n_j}),$$

where $\mu_j$ is a probability measure, $n_j \in \mathbb{N}$ and $p_j \in M_{n_j}$ is a projection for $j \in J$.

**Proof.** By combining [29] Theorem V.1.27 and [29] Corollary V.2.9 we get an orthogonal family $(z_\alpha)_{\alpha \in \Lambda}$ of central projections in $V$ with sum equal to 1 such that $z_\alpha V$ is isomorphic to $A_\alpha \otimes M_{n_\alpha}$, where $A_\alpha$ is a $\sigma$-finite abelian von Neumann algebra and $n_\alpha \in \mathbb{N}$ for $\alpha \in \Lambda$. Each $A_\alpha$, being $\sigma$-finite, is isomorphic to $L^\infty(\mu_\alpha)$ for some probability measure $\mu_\alpha$. Thus $pV = \bigoplus_{\alpha \in \Lambda} pz_\alpha V$ is isomorphic to

$$\bigoplus_{\alpha \in \Lambda} \ell_\infty z_\alpha pL^\infty(\mu_\alpha, M_{n_\alpha}).$$

We conclude by applying Lemma 5.7 to each summand. □

The following proposition solves the case of the summand $p_3V$ from Proposition 1.3.
6. Proof of Grothendieck's inequalities for JB*-triples

Now we are ready to prove the Barton-Friedmann conjecture. We start by re-stating and proving the little Grothendieck inequality given in Theorem 2.4

Theorem 6.1. Let $M$ be a JBW*-triple. Then given any two functionals $\varphi_1, \varphi_2$ in $M_*$, there exists a norm-one functional $\psi \in M_*$ such that

$$\|x\|_{\varphi_1, \varphi_2} \leq \sqrt{2} \cdot \sqrt{\|\varphi_1\| + \|\varphi_2\|} \cdot \|x\|_{\psi}$$

for all $x \in M$. Furthermore, given $K > 2$, for every complex Hilbert space $H$, and every weak*-to-weak continuous linear operator $T : M \to H$, there exists a norm-one functional $\psi \in M_*$ satisfying

$$\|T(x)\| \leq K \|T\| \|x\|_{\psi}$$

for all $x \in M$. 

Proof. For $j = 1, 2$ let $\psi_j$ be a positive functional on $pVp$ and $u_j \in V$ a unitary element provided by Lemma 5.1 for $\varphi_j$. Set

$$t = p \vee u_1pu_1^* \vee u_2pu_2^*$$

and $W = tvt$. Then $t$, being the supremum of three projections equivalent to $p$, is a finite projection. Moreover, the central carrier of $p$ in $W$ equals $t = 1_W$.

We claim that $W$ is of type $I$. Indeed, assume that $r \in W$ is a nonzero projection. Since the central carrier of $p$ equals $1_W$, [29] Lemma V.1.7] yields that there are two nonzero projections $r_1 \leq r$ and $r_1 \leq p$ such that $r_1 \sim p_1$. Since $pVp$ is of type $I$, there is a nonzero abelian projection $p_2 \leq p_1$. Then there is a projection $r_2 \leq r_1$ equivalent to $p_2$. Therefore $r_2$ is abelian and $r_2 \leq r_1 \leq r$, which completes the proof of the claim.

Moreover, for $j = 1, 2$ we have $s(\varphi_j) = s(\psi_j)u_j^*$, so the initial projection is $u_j^*s(\psi_j)u_j^* \leq u_jpu_j^* \leq t$, hence $s(\varphi_j) \in pVp = pW$. By Lemma 5.8 $pW = pWt$ is JB*-triple isomorphic to $\bigoplus_{j \in J} L^\infty(\mu_j, p_jM_{n_j})$, where $\mu_j$ is a probability measure, $n_j \in \mathbb{N}$ and $p_j \in M_{n_j}$ is a projection for $j \in J$. For each $j \in J$, let $\varphi_{1,j} = \varphi_1|_{L^\infty(\mu_j, p_jM_{n_j})}$ and $\varphi_{2,j} = \varphi_2|_{L^\infty(\mu_j, p_jM_{n_j})}$. Proposition 4.3 assures that the pre-Hilbert semi-norm $\|\varphi_1, \varphi_2|_{L^\infty(\mu_j, p_jM_{n_j})} = \|\varphi_1, \varphi_2|_{L^\infty(\mu_j, p_jM_{n_j})}$ attains its maximum on the closed unit ball of $L^\infty(\mu_j, p_jM_{n_j})$ at some point $x_j$. It follows that the semi-norm $\|\varphi_1, \varphi_2|_{L^\infty(\mu_j, p_jM_{n_j})}$ attains its maximum on the closed unit ball of $pW = pWt$ at the point $(x_j)_{j \in J}$. By Lemma 5.8, we therefore can apply Lemma 5.8 to deduce that $\|\varphi_1, \varphi_2|_{L^\infty(\mu_j, p_jM_{n_j})}$ attains its maximum on the closed unit ball of $pW$. Finally, Lemma 2.6 yields a norm-one functional $\psi \in (pV)_*$ such that

$$\|x\|_{\varphi_1, \varphi_2} \leq \sqrt{2} \sqrt{\|\varphi_1\| + \|\varphi_2\|} \|x\|_{\psi}, \text{ for all } x \in pV.$$
Proof. The first statement follows from the results of the previous section. Indeed, consider the decomposition of $M$ from Proposition 1.3. The statement for individual summands follows from Proposition 4.3, Corollary 3.4, Proposition 5.4, and Proposition 5.9, respectively. Finally, Proposition 2.5 completes the argument.

Let us prove the second statement. Fix $K > 2$. Let $\varepsilon > 0$ be such that $K > 2(1 + \varepsilon)$. By Theorem 2.2 there are norm-one functionals $\varphi_1, \varphi_2 \in M_*$ such that for any $x \in M$ we have
\[
\|T(x)\| \leq \sqrt{K} \|T\| \sqrt{\|x_1\|_{\varphi_1}^2 + \varepsilon \|x_2\|_{\varphi_2}^2} = \sqrt{K} \|T\| \|x\|_{\varphi_1, c_2}.
\]
By the first part of the theorem we get a norm-one functional $\psi \in M_*$ such that for $x \in M$ we have
\[
\|x\|_{\varphi_1, c_2} \leq \sqrt{2 \|\varphi_1\| + \varepsilon \|\varphi_2\|} \|x\|_{\psi} = \sqrt{2(1 + \varepsilon)} \|x\|_{\psi}.
\]
By combining the two inequalities we get
\[
\|T(x)\| \leq \sqrt{2(1 + \varepsilon)K} \|x\|_{\psi} \leq K \|x\|_{\psi}
\]
for $x \in M$. This completes the proof. \hfill \Box

Given a bounded linear operator $T$ from a JB$^*$-triple $E$ into a complex Hilbert space $H$ we can always consider its bitranspose $T^{**} : E^{**} \rightarrow H$, which is a weak$^*$-continuous linear operator from a JBW$^*$-triple into a complex Hilbert space. We therefore arrive, via Theorem 6.1, to a proof of the little Grothendieck inequality with one control functional.

**Theorem 6.2.** Let $E$ be a JB$^*$-triple, $H$ a complex Hilbert space, and $K > 2$. Then for every bounded linear operator $T : E \rightarrow H$, there exists a norm-one functionals $\psi \in E^*$ satisfying
\[
\|T(x)\| \leq K \|T\| \|x\|_{\psi},
\]
for all $x \in E$. \hfill \Box

The previous Theorems 6.1 and 6.2 restore the equilibrium and the validity of original statements concerning the little Grothendieck inequality in the case of JB$^*$-triples in [2, 9]. It also provides a complete solution to [7, Problem 5.10.13], [25, Remark 3], and [28, Remark 8.3]. We shall next trace back the original sources to see how our results can be also employed to provide a complete proof to the Barton–Friedmann conjecture concerning Grothendieck’s inequality for bilinear forms on JB$^*$-triples.

**Theorem 6.3.** Suppose $G > 8(1 + 2\sqrt{3})$. Let $M$ and $N$ be JBW$^*$-triples. Then for every separately weak$^*$-continuous bilinear form $V : M \times N \rightarrow \mathbb{C}$ there exist norm-one functionals $\varphi \in M_*$ and $\psi \in N_*$ satisfying
\[
|V(x, y)| \leq G \|V\| \|x\|_{\varphi} \|y\|_{\psi}
\]
for all $(x, y) \in M \times N$.

**Proof.** Thanks to our previous Theorem 6.1 we can recover a trick from [9, Theorem 6] and [25, Remark 3]. A brief argument is included here for completeness reasons. Let us find a weak$^*$-to-weak continuous linear operator $R : M \rightarrow N_*$ defined by $V(a, b) = \langle R(a), b \rangle$ ($a, b \in M \times N$). Clearly $\|R\| \leq \|V\|$. By [9, Lemma 5] $R$ factors through a complex Hilbert space, more precisely, there exists a complex Hilbert space $H$ and bounded linear operators $T : M \rightarrow H$, $S : H \rightarrow N_*$ satisfying...
Let $\tilde{G} = \left( \frac{G}{2(1 + 2\sqrt{3})} \right)^{\frac{1}{2}} > 2$. By applying Theorem 6.1 to the weak*-continuous linear operators $T : M \to H$ and $S^* : N \to H$ we find two norm-one functionals $\varphi \in M_*$ and $\psi \in N_*$ satisfying

\[ \|T(x)\| \leq \tilde{G} \|T\| \|x\|_\varphi, \quad \text{and} \quad \|S^*(y)\| \leq \tilde{G} \|S^*\| \|y\|_\psi \]

for all $(x, y) \in M \times N$. We therefore have

\[ |V(x, y)| = |\langle R(x), y \rangle| = |\langle T(x), S^*(y) \rangle| \leq \tilde{G}^2 \|T\| \|S\| \|x\|_\varphi \|y\|_\psi \]

\[ \leq G \|V\| \|x\|_\varphi \|y\|_\psi \]

for all $(x, y) \in M \times N$. \[ \square \]

Since every bounded bilinear form on the cartesian product of two JB*-triples admits a norm-preserving separately weak*-continuous extension to the cartesian product of the corresponding bidual spaces (cf. [25, Lemma 1]), Theorem 6.3 implies the following statement (restating of Theorem 1.1 from Introduction).

**Theorem 6.4.** Suppose $G > 8(1 + 2\sqrt{3})$. Let $E$ and $B$ be JB*-triples. Then for every bounded bilinear form $V : E \times B \to \mathbb{C}$ there exist norm-one functionals $\varphi \in E^*$ and $\psi \in B^*$ satisfying

\[ |V(x, y)| \leq G \|V\| \|x\|_\varphi \|y\|_\psi \]

for all $(x, y) \in E \times B$. \[ \square \]

**Remark 6.5.** The optimal values of the constants in question remain to be unknown. However, it seems that our method cannot give a better constant in Theorem 6.1. One factor $\sqrt{2}$ appears due to the use of Lemma 2.1 and a second factor $\sqrt{2}$ appears due to estimates of semi-norms $\|\cdot\|_{\varphi_1, \varphi_2}$ by a semi-norm generated by one functional. Let us consider a JBW*-triple represented as in Proposition 1.3. The individual summands have different behaviour.

(i) The JBW*-algebra $N$ is covered by the already known Theorem 2.3.

(ii) The summand $p_1 V$ is covered by Corollary 3.4. This approach can be applied to $N$ as well (note that Corollary 3.4 can be viewed as a generalization of Theorem 2.3).

(iii) The remaining summand, i.e.,

\[ \left( \bigoplus_{k \in \Lambda} L^\infty(\mu_k, C_k) \right) \oplus^\ell \infty p_2 V \oplus^\ell \infty p_3 V, \]

has a special property. It follows from our arguments that in this case $\|\cdot\|_{\varphi_1, \varphi_2}$ attains its maximum on the unit ball for any two normal functionals $\varphi_1, \varphi_2$.

This analysis confirms that there are two basic tools – attaining the norm and some kind of order on tripotents.
References

[1] Aliprantis, C. D., and Border, K. C. Infinite dimensional analysis: A hitchhiker’s guide. 3rd ed., 2006.

[2] Barton, T., and Friedman, Y. Grothendieck’s inequality for JB*-triples and applications. *Journal of the London Mathematical Society* 2, 3 (1987), 513–523.

[3] Barton, T., and Timoney, R. M. Weak*-continuity of Jordan triple products and its applications. *Math. Scand.* 59, 2 (1986), 177–191.

[4] Braun, R., Kaup, W., and Upmeier, H. A holomorphic characterization of Jordan C*-algebras. *Mathematische Zeitschrift* 161, 3 (1978), 277–290.

[5] Bunce, L. J., Fernández-Polo, F. J., Martínez Moreno, J., and Peralta, A. M. A Saitô-Tomita-Lusin theorem for JB*-triples and applications. *Q. J. Math.* 57, 1 (2006), 37–48.

[6] Cabrera García, M., and Rodríguez Palacios, A. Non-associative normed algebras. Vol. 1, vol. 154 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2014. The Vidav-Palmer and Gelfand-Naimark theorems.

[7] Cabrera García, M., and Rodríguez Palacios, A. Non-associative normed algebras. Vol. 2, vol. 167 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2018. Representation theory and the Zel’manov approach.

[8] Chu, C.-H. Jordan structures in geometry and analysis, vol. 190 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2012.

[9] Fabian, M. J. Gâteaux differentiability of convex functions and topology. *Canadian Mathematical Society Series of Monographs and Advanced Texts*. John Wiley & Sons, Inc., New York, 1997. Weak Asplund spaces, A Wiley-Interscience Publication.

[10] Friedman, Y., and Russo, B. Structure of the predual of a JBW*-triple. *J. Reine Angew. Math.* 356 (1985), 67–89.

[11] Haagerup, U. The Grothendieck inequality for bilinear forms on C*-algebras. *Advances in Mathematics* 56, 2 (1985), 93–116.

[12] Kadison, R. V., and Ringrose, J. R. *Fundamentals of the theory of operator algebras*. Vol. II, vol. 16 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997. Advanced theory, Corrected reprint of the 1986 original.

[13] Hanche-Olsen, H., and Størmer, E. *Jordan operator algebras*, vol. 21. Pitman Advanced Publishing Program, 1984.

[14] Horn, G. Classification of JBW*-triples of type I. *Mathematische Zeitschrift* 196, 2 (1987), 271–291.

[15] Hamhalter, J., Kalenda, O. F. K., Peralta, A. M., and Pfitzner, H. Measures of weak non-compactness in preduals of von Neumann algebras and JBW*-triples. preprint 2019, arXiv:1901.08056v1.

[16] Horváth, J. *Measures of weak non-compactness in preduals of von Neumann algebras and JBW*-triples*. Preprint 2019, arXiv:1901.08056v1.

[17] Horn, G., and Neifer, E. Classification of continuous JBW*-triples. *Transactions of the American Mathematical Society* 306, 2 (1988), 553–578.

[18] Kadison, R. V., and Ringrose, J. R. *Fundamentals of the theory of operator algebras*. Vol. II, vol. 16 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997. Advanced theory, Corrected reprint of the 1986 original.

[19] Kaup, W. A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces. *Mathematische Zeitschrift* 183, 4 (1983), 503–529.

[20] Kaup, W., and Upmeier, H. Jordan algebras and symmetric Siegel domains in Banach spaces. *Mathematische Zeitschrift* 157, 2 (1977), 179–200.

[21] Peralta, A. M. Little Grothendieck’s theorem for real JB*-triples. *Mathematische Zeitschrift* 237, 3 (2001), 531–545.

[22] Peralta, A. M. New advances on the Grothendieck’s inequality problem for bilinear forms on JB*-triples. *Math. Inequal. Appl.* 8 (2005), 7–21.
[25] Peralta, A. M., and Rodríguez-Palacios, A. Grothendieck’s inequalities for real and complex JBW*-triples. *Proceedings of the London Mathematical Society* 83, 3 (2001), 605–625.

[26] Peralta, A. M., and Rodríguez-Palacios, A. Grothendieck’s inequalities revisited. In *North-Holland Mathematics Studies*, vol. 189. Elsevier, 2001, pp. 409–423.

[27] Pisier, G. Grothendieck’s theorem for noncommutative C*-algebras, with an appendix on Grothendieck’s constants. *Journal of Functional Analysis* 29, 3 (1978), 397–415.

[28] Pisier, G. Grothendieck’s theorem, past and present. *Bulletin of the American Mathematical Society* 49, 2 (2012), 237–323.

[29] Takesaki, M. *Theory of operator algebras. I*. Springer-Verlag, New York-Heidelberg, 1979.

[30] Wright, J. D. M. Jordan C*-algebras. *Michigan Math. J.* 24, 3 (1977), 291–302.

Czech Technical University in Prague, Faculty of Electrical Engineering, Department of Mathematics, Technická 2, 166 27, Prague 6, Czech Republic

*E-mail address*: hamhalte@math.feld.cvut.cz

Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis, Sokolovská 86, 186 75 Praha 8, Czech Republic

*E-mail address*: kalenda@karlin.mff.cuni.cz

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain.

*E-mail address*: aperalta@ugr.es

Université d’Orléans, BP 6759, F-45067 Orléans Cedex 2, France

*E-mail address*: pfitzner@univ-orleans.fr