Solitons Reduced From Heterotic Fivebranes

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In view of the expectation that the solitonic sector of the lower dimensional world may be originated from the solitonic sector of string theory, various solitonic solutions are reduced from the heterotic fivebrane solutions in the ten-dimensional heterotic string theory. These solitons in principle can appear after proper compactifications, e.g. toroidal compactifications.

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1. Introduction

The study of solitons has long been pursued in various aspects by physicists as well as by mathematicians. In general, it involves the investigation of nonlinear evolution equations. In one spatial dimensional case, one can mainly deal with ordinary differential equations, which makes the situation relatively easier. But in higher dimensional cases, since one has to solve partial differential equations, the whole set of solutions are almost impossible to find. Nevertheless many interesting solutions are known and now string theory is not an exception anymore.

Lately the structures of the classical solitonic solutions of string theory have been actively investigated\[1\]. In particular, the heterotic fivebrane solution conjectured by Duff\[2\] and constructed by Strominger\[3\] is exceptionally interesting in the sense that it is dual to the fundamental string in a generalized sense of the electric-magnetic duality\[4\]. However, most of the solutions known so far are rather ten-dimensional solutions so that their fate in 4-d space-time after compactification is still elusive.

Thus it is important to address a question that what would be the implications of the physics of the fivebrane in ten-dimension on the physics in four-dimensional space-time after some proper compactification\[5\]. Some speculations were given by Strominger, too\[3\].

There may be some physical consequences due to the above duality. For example, the origin of the electric-magnetic duality in four-dimension might be such a string-fivebrane duality in ten-dimension. In other words, the monopole solution in four-dimension might be related to the fivebrane solution in ten-dimension. This aspect was already advocated by Harvey and Liu\[6\]. The dynamical similarities between these two systems are investigated classically in ref.[6]. Furthermore, we could conjecture that the solitonic sector in four-dimension is originated from the solitonic sector of ten-dimension.

In this paper, as a first step toward such structures in the solitonic sector of string theory, we attempt to investigate how these solutions appear in the (1 + 3)-dimensional subspace of the (1 + 9)-dimensional space-time. Perhaps this could provide some clues to consistent compactifications of the fivebrane system. By proper coordinate redefinitions

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1 This duality which interchanges Noether charge (e.g. electric charge) and topological charge (e.g. monopole charge) is in principle the foundation for the Montonen-Olive conjecture\[4\], which is yet to be confirmed rigorously.

2 Also with broken space-time supersymmetry ultimately, but such solutions are not known yet.
and the dimensional reduction imposing the Killing symmetries, we make the fields in this
subspace independent from the rest of the space so that they are more or less dimensionally
reduced solutions. In particular most of new solutions we present here are not based on
the instanton background in the transverse space[3].

Later, some remarks on the compactified solutions will also be given, whose detail
will be presented elsewhere. Also we can attempt to analyze the motion of strings in
the fivebrane geometry inside (1 + 3)-dimensional space-time, using the metric suggested
here[8].

This paper is organized as follows: In sect. 2, we review the derivations of the basic
fivebrane solutions. Then in sect.3, the hyperbolic monopoles and the neutral fivebrane
solutions are derived by imposing rotational symmetries. In sect.4 the (Euclidean) sine-
Gordon solitons are used as sources of new fivebrane solutions and in sect.5 the Euclidean
analogues of the \( \Phi^4 \) kinks are used to derive new fivebranes. In sect.6 using the conformal
mapping we attempted to analyze the instanton case and finally in sect.7 some speculation
on compactified fivebrane solutions is presented.

2. Heterotic Fivebranes

A fivebrane is a five dimensional extended object and the existence of such a higher
dimensional object is in some sense surprising. Nevertheless, such a solution exists in string
theory. First of all, let us review the derivations given in refs.[3][9].

The heterotic fivebrane is a solution to the equations of the supersymmetric vacuum
for the heterotic string

\[
\delta \psi_M = \left( \partial_M + \frac{i}{4} \Omega_{MAB} \Gamma^{AB} \right) \epsilon = 0,
\]

\[
\delta \lambda = \left( \Gamma^A \partial_A \phi + \frac{i}{8} H_{AMC} \Gamma^{ABC} \right) \epsilon = 0,
\]

\[
\delta \chi = F_{AB} \Gamma^{AB} \epsilon = 0,
\]

where \( \psi_M \), \( \lambda \) and \( \chi \) are the gravitino, dilatino and gaugino, and the generalized connection
is given by

\[
\Omega_{AB}^M \equiv \omega_{AB}^M - H_{AB}^M,
\]

where \( \omega \) is the usual spin connection. Now \( H \) satisfies the following anomaly equation:

\[
dH = \alpha' \left( \text{tr} R \wedge R - \frac{1}{30} \text{Tr} F \wedge F \right) + O(\alpha'^2).
\]
In the above we have properly rescaled all the field variables and that the string coupling 
\( g_s = e^{-\phi} \) and \( \alpha' \) are only independent couplings. In the heterotic string theory \( \alpha' \) is 
proportional to \( \kappa^2/g_{\text{YM}} \), where \( \kappa \) is the gravitational coupling constant.

The corresponding low-energy effective action for the heterotic string is

\[
S = \frac{1}{\kappa^2} \int d^{10}x \sqrt{-g} e^{2\phi} \left( R + 4 \partial_\mu \phi \partial^\mu \phi - \frac{1}{3} H^2 - \frac{1}{30} \alpha' \text{Tr} F^2 + \cdots \right),
\]

(2.6)

where the dots include the fermionic part of the action that are not relevant for our purpose now.

In \((1+9)\)-dimension we have Majorana-Weyl fermions, which decompose down to chiral spinors according to \( \text{SO}(1,9) \supset \text{SO}(1,5) \otimes \text{SO}(4) \) for \( M^{1,9} \to M^{1,5} \times M^4 \) decomposition. For such spinors the dilatino equation eq.(2.2) is satisfied by

\[
H_{\mu\nu\lambda} = \pm \epsilon_{\mu\nu\lambda\sigma} \partial^\sigma \phi,
\]

(2.7)

where \( \mu, \nu, \ldots \) are indices for the transverse space \( M^4 \) and \( \phi = \phi(x^\mu) \), while we shall use indices \( a, b, \ldots \) for \( M^{1,5} \). The dilaton itself is determined by solving the anomaly equation.

Then other equations are solved by constant chiral spinors \( \epsilon_\pm \) and the metric

\[
g_{ab} = \eta_{ab}, \quad g_{\mu\nu} = e^{-2\phi} \delta_{\mu\nu}
\]

(2.8)

such that

\[
\delta \psi_\mu = (\nabla_\mu + \frac{1}{2} \Gamma_{\mu\nu} \partial^\nu \phi) \epsilon_\pm = \partial_\mu \epsilon_\pm = 0,
\]

\[
\delta \psi_a = \nabla_a \epsilon_\pm = \partial_a \epsilon_\pm = 0,
\]

(2.9)

and

\[
\delta \chi = F^\pm_{\mu\nu} \Gamma^{\mu\nu} \epsilon_\pm = -F^\pm_{\mu\nu} \Gamma^{\mu\nu} \epsilon_\pm = 0,
\]

(2.10)

where eq.(2.10) is achieved using the instanton configuration for the (anti)self-dual YM equation in the flat Euclidean space \( \mathbb{R}^4 \)

\[
F^\pm_{\mu\nu} = \pm \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^\pm_{\rho\sigma}
\]

(2.11)

for an SU(2) subgroup of \( E_8 \times E_8 \) or SO(32).

Solutions of eq.(2.11) are basic ingredients to build fivebrane solutions. For example, the instanton solutions lead to the Strominger’s fivebrane solutions. There are two relevant fivebrane solutions. One is the “gauge” solution and the other is the “symmetric” solution. We shall first derive the former, then the latter. In this case \( \phi = \phi(r^2) \) now, i.e. no angular
dependence, where \( r^2 = \sum (x^i)^2 \). With a finite instanton scale size \( \lambda \), from eqs.\((2.3),(2.7)\) we obtain

\[
e^{-2\phi} = e^{-2\phi_0} + 8\alpha' \frac{(r^2 + 2\lambda^2)}{(r^2 + \lambda^2)^2},
\]

where \( \phi_0 \) is the value of the dilaton at spatial infinity. Thus we have a fivebrane living in \( M^{1,5} \) which is a point-like object in \( M^4 \). This is the gauge solution.

This gauge solution is valid only for \( \lambda \gg \sqrt{\alpha'} \). Nevertheless, there is another fivebrane solution with \( \lambda = 0 \), which is called the elementary fivebrane or the “neutral” solution \([10],[9]\). For this neutral solution the YM fields vanishes and as a result, the dilaton is given by the same form as following “symmetric” solution.

The symmetric solution can be derived by setting the RHS of eq.\((2.5)\) zero. Then compared to the gauge solution we derived before, the differences are

\[
e^{-2\phi} = e^{-2\phi_0} + \frac{Q}{r^2}, \quad Q = n\alpha',
\]

and

\[
F_{\mu\nu} = R_{\mu\nu}(\Omega),
\]

where \( F \) and \( R \) are both self-dual. This symmetric solution is known to be exact \([11]\).

Note that this solutions satisfy the scale symmetry

\[
\phi \rightarrow \phi + \ln \sigma, \\
r \rightarrow \sigma^{-1}r, \\
\lambda \rightarrow \sigma^{-1}\lambda,
\]

where \( \sigma \) is a constant. The \( \lambda = 0 \) case is not related to the \( \lambda \neq 0 \) case in terms of this scale symmetry, but it retains a similar scale symmetry without the last property.

Now we would like to call the reader’s attention to the fact that any solution of eq.\((2.11)\) in principle leads to a fivebrane solution, as long as the anomaly equation eq.\((2.5)\) provides a nontrivial solution for the dilaton. In particular many lower dimensional solutions to the self-dual YM equation are known \([12]\) so that in principle we can relate all these solitonic solutions to the heterotic fivebranes.
3. Rotationally Symmetric Cases

The Bogomol’nyi equation can be reduced from the SDYM equation by requiring one Killing symmetry along one of the cartesian coordinates. Then the BPS monopole solution is the SO(3) rotational symmetric solution, which is used in ref. [5]. If we claim other rotational symmetries in some subspaces of the transverse space, we can obtain other type of fivebrane solutions closely related to the solutions derived in the previous section.

First, there is a $S^1$-invariant instanton solution. The basic observation is that for $(x^\mu) = (x^1, x^2, u, v)$ we can introduce a set of cylindrical coordinates

$$u = \rho \cos \theta, \quad v = \rho \sin \theta,$$

then the metric for $M^4$ can be rewritten as

$$ds^2_4 = e^{-2\phi} \left( (dx^1)^2 + (dx^2)^2 + (d\rho)^2 + \rho^2 (d\theta)^2 \right),$$

Since $r^2 = (x^1)^2 + (x^2)^2 + \rho^2$, now the dilaton $\phi$ will be defined in terms of the coordinates $x^1, x^2, \rho$.

In this coordinate system the YM vector fields can be identified as

$$A_\mu dx^\mu = A_1 dx^1 + A_2 dx^2 + A_\rho d\rho + \rho A_\theta d\theta,$$

Now we can adopt the Killing reduction of the self-dual YM equation to the lower dimensional integrable systems [12]. If we require a Killing symmetry such that the YM vector fields do not depend on the $\theta$-coordinate, i.e $\partial_\theta A_\mu = 0$, and define a scalar field as $\Phi \equiv A_\theta$, then the self-dual YM equation becomes

$$F_{12} = D_\rho \Phi, \quad F_{\rho 1} = D_2 \Phi, \quad F_{2\rho} = D_3 \Phi.$$ (3.4)

This is a set of Bogomol’nyi equations except the condition $\rho \geq 0$. Note that the above procedure is in fact equivalent to getting the hyperbolic monopoles from the self-dual YM equation using the conformal equivalence of $R^4 - R^2 \sim S^1 \times H^3$ [13], where $H^3$ is the hyperbolic space of an upper half plane $(x^1, x^2, \rho)$. Thus we basically recover a fivebrane with an extra $S^1$-rotational invariance which behaves like a monopole in the subspace.

$^3$ Note that the difference between this monopole and the monopole solution given in ref. [5], where $\partial_4 A_\mu = 0$ is required, is that for the latter $r^2$ is not the same as that of the fivebrane, but for the former it is so.
In the hyperbolic monopole case we required the rotational symmetry in the $uv$-plane. We can also solve the SDYM equation with $S^2$ rotational symmetry. Let
\[ x^2 = \rho \sin\theta \cos\varphi, \quad x^3 = \rho \sin\theta \sin\varphi, \quad x^4 = \rho \cos\theta. \]  
(3.5)

In this coordinate system now the YM vector fields are
\[ A_\mu dx^\mu = A_1 dx^1 + A_\rho d\rho + \rho A_\theta d\theta + \rho \sin\theta A_\varphi d\varphi, \]
where we can introduce a scalar field $\Phi \equiv \rho A_\theta$. Then the SDYM equations, requiring that the fields do not depend on the angular variables, reduce to
\[ F_{x\rho} = 0, \quad D_x \Phi = 0 = D_\rho \Phi, \quad A_\varphi = 0. \]  
(3.7)

This is rather a trivial system, but it still provides a source for a fivebrane. The solutions of these equations are pure gauge solutions, which can be gauged away so that we can set all the gauge fields zero. Then the anomaly equation eq.(2.5) simply becomes $dH = 0$. Thus we recover the elementary fivebrane solution\[10\][9].

4. (Euclidean) Sine-Gordon Case

The (anti)self-dual YM equations have an interesting reduction to the two-dimensional solitonic system, namely the sine-Gordon equation. Here we shall attempt a new reduction of the (A)SDYM equation to the Euclidean sine-Gordon equation for the gauge group $SU(2)$ and the Euclidean signature, then to solve the anomaly equation eq.(2.5) for this solution. The usual sine-Gordon system can be recovered by further reducing this system, incorporating the time dimension in $M^{1,5}$.

For the Euclidean signature we can introduce two sets of complex coordinates for convenience, although one can use the real coordinates, as
\[ z = x + iy, \quad \overline{z} = x - iy, \quad w = u + iv, \quad \overline{w} = u - iv, \]
(4.1)

where $(x, y, u, v)$ are the cartesian coordinates. In this coordinate system the SDYM equations will be written as
\[ F_{z\overline{z}} - F_{w\overline{w}} = 0, \quad F_{z\overline{w}} = 0, \quad F_{\overline{z}w} = 0, \]
(4.2)

Note that the identification of the scalar fields are different in the $S^1$- and $S^2$- symmetric cases.
while the ASDYM equations are

\begin{equation}
F_{zz} + F_{ww} = 0, \quad F_zw = 0, \quad F_{zw} = 0. \tag{4.3}
\end{equation}

For the gauge group SU(2) with the generators \( J_\pm = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2), \quad J_3, \) which are in the adjoint representation such that \((J_a)_{bc} = -i\epsilon_{abc},\) we can introduce an ansatz for the gauge fields as

\begin{equation}
A_z = f_1 J_3, \quad A_\overline{z} = f_2 J_3, \quad A_w = g_1 J_+ + g_3 J_-, \quad A_{\overline{w}} = g_2 J_- + g_4 J_+. \tag{4.4}
\end{equation}

With such identifications the SDYM equations reduce to

\begin{align}
f_1 &= \partial_z \ln g_2 = -\partial_\overline{z} \ln g_4, \\
f_2 &= -\partial_\overline{z} \ln g_1 = \partial_z \ln g_3, \\
0 &= \partial_z f_2 - \partial_\overline{z} f_1 - g_1 g_2 + g_3 g_4, \tag{4.5}
\end{align}

and the conditions that \( \partial_\overline{z} f_1 = \partial_w f_2 = 0, \partial_w g_2 = \partial_\overline{z} g_3, \partial_w g_4 = \partial_\overline{z} g_1.\) The last conditions can be simply satisfied by requiring two Killing symmetries along \((u,v)\) directions such that none of the fields depend on the \((u,v)\)-coordinates. For the ASDYM equation we obtain more or less the same set of equations.

Now defining

\begin{equation}
g_1 = -g_2 = e^{-\frac{i}{2}\psi}, \quad g_3 = -g_4 = e^{\frac{i}{2}\psi}, \tag{4.6}
\end{equation}

we obtain the Euclidean version of the sine-Gordon equation,

\begin{equation}
\partial_z \partial_\overline{z} \psi - 2\sin \psi = \frac{1}{4}(\partial_x^2 + \partial_y^2)\psi - 2\sin \psi = 0. \tag{4.7}
\end{equation}

The above is related, redefining \(y = it,\) to the \((m^2 = 8)\) sine-Gordon equation

\begin{equation}
(\partial_t^2 - \partial_x^2)\varphi + \frac{m^2}{\lambda} \sin \lambda \varphi = 0, \tag{4.8}
\end{equation}

where the coupling constant \(\lambda\) can be rescaled away since we are not interested in quantizing this system here.

In this background the anomaly equation eq.(2.5) becomes up to the first order of \(\alpha'\)

\begin{equation}
(\partial_x^2 + \partial_y^2)e^{-2\phi} = 4\alpha' \left[ \sin \psi \left( \partial_x^2 + \partial_y^2 \right) \psi + \cos \psi \left( (\partial_x \psi)^2 + (\partial_y \psi)^2 \right) \right]. \tag{4.9}
\end{equation}
Using the above sine-Gordon equation we can easily solve this equation to obtain a solution

\[ e^{-2\phi} = e^{-2\phi_0} + 4\alpha'(1 - \cos\psi), \quad (4.10) \]

where \( \psi \) satisfies the sine-Gordon equation and \( \phi_0 \) is the value of the dilaton \( \phi \) at \( x, y = \pm\infty \).

Due to the Derrick’s theorem\[14\] applied to the Euclidean sine-Gordon theory, there is no finite-action static solution for \( \psi \). Nevertheless, we can have infinite-action static solutions, which do not generate any tunnelling effect. In fact we can easily find the following solution:

\[ \psi = 4Q \tan^{-1} \left[ \gamma e^{\alpha x + \beta y} \right], \quad (4.11) \]

where \( \gamma \) is an arbitrary irrelevant constant so that we can set \( \gamma = 1 \) without loss of generality, and \( \alpha^2 + \beta^2 = 8 \). \( Q = \pm1 \) is the soliton charge. This solution is related to the soliton solutions of the sine-Gordon equation eq.(4.8),

\[ \varphi = 4Q \tan^{-1} \left[ \exp \frac{m}{\sqrt{1 - c^2}} \left( -\frac{x - ct}{\sqrt{1 - c^2}} \right) \right], \quad (4.12) \]

identifying

\[ y = it, \quad c = i\tilde{c}, \quad \alpha = \frac{m}{\sqrt{1 + \tilde{c}^2}}, \quad \beta = \frac{m\tilde{c}}{\sqrt{1 + \tilde{c}^2}}, \quad m = 2\sqrt{2}. \quad (4.13) \]

It is straightforward to show that the corresponding action of the Euclidean sine-Gordon theory is indeed infinite for these solutions. However, this cannot be a reason to abandon these solutions for our purpose because this action is not an essential ingredient for fivebrane solutions. Note that the SDYM equation is not an equation of motion so that the action for any reduced system from the SDYM equation is not relevant to us. Due to the self-dual YM structure, the corresponding fivebrane solutions can still saturate the necessary Bogomol’nyi bound for the energy density. Strictly speaking, the fivebrane is not an instanton related to the tunnelling effect because we work on the \( (1 + 9) \) dimensional spacetime. From this point of view, whether the action of the heterotic string is finite or not is not really a relevant issue to us. We are just interested in looking for some solitonic solutions.

Using eq.\( (4.11) \), now the dilaton eq.\( (4.10) \) becomes

\[ e^{-2\phi} = e^{-2\phi_0} + 16\alpha' \frac{e^{2(\alpha x + \beta y)}}{(e^{2(\alpha x + \beta y) + 1})^2}. \quad (4.14) \]
Note that this solution does not have any singularity and depends on the $x, y$-coordinates explicitly, not just on $x^2 + y^2$. This dilaton solution does not care about the sign of the soliton charge $Q = \pm 1$, while the YM fields depend on the charge $Q = \pm 1$. We can also express the YM fields eq.(4.14) in terms of eq.(4.11) as follows:

\begin{align*}
A_z &= -Q(\beta + i\alpha) \frac{e^{(\alpha x + \beta y)}}{e^{2(\alpha x + \beta y)} + 1} J_3, \\
A_{\overline{z}} &= Q(\beta - i\alpha) \frac{e^{(\alpha x + \beta y)}}{e^{2(\alpha x + \beta y)} + 1} J_3, \\
A_w &= \frac{1 - e^{2(\alpha x + \beta y)} - i2Qe^{(\alpha x + \beta y)}}{e^{2(\alpha x + \beta y)} + 1} J_+ + \frac{1 - e^{2(\alpha x + \beta y)} + i2Qe^{(\alpha x + \beta y)}}{e^{2(\alpha x + \beta y)} + 1} J_-, \\
A_{\overline{w}} &= \frac{1 - e^{2(\alpha x + \beta y)} - i2Qe^{(\alpha x + \beta y)}}{e^{2(\alpha x + \beta y)} + 1} J_- + \frac{1 - e^{2(\alpha x + \beta y)} + i2Qe^{(\alpha x + \beta y)}}{e^{2(\alpha x + \beta y)} + 1} J_+.
\end{align*}

The fact that there are all the four dimensional YM fields indicates that the solutions we have here are still fivebrane solutions.

Now let us count the zero modes. In the two-dimension parametrized by $(x, y)$ coordinates the soliton solutions eq.(4.11) generate four zero modes, which are two for the two translational symmetries of the $x, y$-directions, one for the $(\alpha^2 + \beta^2 = 8)$ “scaling” symmetry and one for the O(2) rotational symmetry of $(\alpha x + \beta y)$. This last O(2) symmetry is due to the fact that the O(2) rotation of $(x, y)$ can be compensated by O(2) rotation of $(\alpha, \beta)$. Since the two Killing symmetries, $(\partial_u, \partial_v)$, generate four extra zero modes for the fivebrane, the fivebrane solution still has 120 bosonic zero modes, including 112 zero modes due to $E_8 \rightarrow SU(2) \times E_7$, like in the “gauge” solution case. We expect that the fermionic zero modes counting is also similar to the “gauge” solution case.

Note that the time-independent part of the sine-Gordon system can be easily obtained by further imposing one more Killing symmetry, incorporating the time-dimension from $M^{1,5}$. The corresponding fivebrane solutions can be easily reduced from the Euclidean case.

5. $\Phi^4$ case

After we obtained the sine-Gordon system, we can easily reduce the previous system to the $\Phi^4$ system, which has different type of kink solutions in (1+1)-dimensional case. In the Euclidean case we can again obtain analogues of these solutions, though they are again infinite-action solutions.
For this purpose we use

\[ g_1 = g_2 = e^{-i \sqrt{\lambda} \Phi}, \quad g_3 = g_4 = e^{i \sqrt{\lambda} \Phi}, \]  

(5.1)

for the same ansatz eq.(4.4) and by truncating at the leading order of \( \lambda \), we obtain the field equation for the \( \Phi^4 \) scalar field theory as

\[ (\partial_x^2 + \partial_y^2) \Phi + 8 \Phi - \frac{4}{3} \lambda \Phi^3 = 0. \]  

(5.2)

Again solving now the familiar anomaly equation, we obtain the dilaton

\[ e^{-2\phi} = e^{-2\phi_0} + 2\alpha'^2 \left( 1 - \frac{\lambda}{12} \Phi^2 \right), \]  

(5.3)

and the YM fields

\[ A_x = -\frac{\sqrt{\lambda}}{2} \partial_y \Phi J_3, \quad A_x = \frac{\sqrt{\lambda}}{2} \partial_x \Phi J_3, \]  

\[ A_u = \sqrt{2\lambda} \left( \Phi - \frac{\lambda}{24} \Phi^3 \right) J_2, \quad A_v = i2\sqrt{2} \left( 1 - \frac{\lambda}{8} \Phi^2 \right) J_1, \]  

(5.4)

where

\[ \Phi = Q \sqrt{\frac{6}{\lambda}} \tanh(\alpha x + \beta y), \quad Q = \pm 1, \]  

(5.5)

and \( \alpha^2 + \beta^2 = 4. \)

Note that although eq.(5.3) does not depend on \( \lambda \) explicitly for solutions eq.(5.5), the results we have make sense only for small \( \lambda \).

6. Instanton Membrane

The solutions we have derived so far are new fivebrane solutions. On the contrary, the instanton solutions of the self-dual YM equation in general lead to \( \partial_\theta A_\mu \neq 0 \) for the coordinates given by eq.(3.1), even though the relevant ingredients to construct the fivebrane solution, namely, \( \phi, \ H_{\mu\nu\lambda}, \) are independent from \( \theta \). This raises a question speculated in ref.[3], that is, keeping the instanton structure in the transverse space, whether we can still reduce it to lower dimensional objects.

For the same three-dimensional subspace, now if we do not require \( \partial_\theta A_\mu = 0 \), we get the structure suggested by Strominger, that is, the instanton lies in one “internal” and three “external” dimensions. But this case does not quite define a compactified internal
space. Nevertheless, we can attempt to define reasonable subspaces, which can be more or less independent from the rest of the space. Among others\textsuperscript{5}, there is at least one interesting case. Note that $M^4$ is conformally equivalent to $M^1 \times S^3$. Using this, we can get a membrane out of a fivebrane in a $(1 + 3)$-dimensional subspace. In this case the instanton lies in three “internal” dimensions.

Let us introduce the four dimensional radial coordinates as

\begin{align}
  x^1 &= rsin\zeta sin\theta cos\varphi, \\
  x^2 &= rsin\zeta sin\theta sin\varphi, \\
  x^3 &= rsin\zeta cos\theta, \\
  x^4 &= rcos\zeta,
\end{align}

(6.1)

and

\[ r = ae^{\xi/a}, \]

then the metric for $M^1 \times S^3$ is given by

\[ ds_4^2 = e^{-2\phi}e^{2\xi/a} \left[ d\xi^2 + a^2 \left( d\zeta^2 + sin^2\zeta d\theta^2 + sin^2\zeta sin^2\theta d\varphi^2 \right) \right], \]

(6.2)

where $a$ is the radius of $S^3$ and the conformal factor depends only on $\xi$.

We can construct a metric for $(1+3)$-dimensional subspace as

\[ ds_{(1,3)}^2 = -dt^2 + (dx^5)^2 + (dx^6)^2 + d\eta^2, \]

(6.3)

where a new coordinate $\eta$ is introduced such as $d\eta^2 = e^{-2\phi}e^{2\xi/a}d\xi^2$, since this conformal factor is a function of $\xi$ only. In this case the “internal” space has a so-called “warp factor” in contrast to the cases of known examples, where the warp factor depending on the internal space appears in front of the space-time metric\textsuperscript{[15]}.

Though the vector potential $A_\mu$ depends on $\theta$-coordinate, it is not surprising because the instanton lies partly over $S^3$. For small $a$ we can in principle reduce the instanton solution in $M^4$ down to $M^1$ by doing a harmonic expansion over $S^3$. In general there are nonvanishing massive modes.

\textsuperscript{5} Most of them are rather physically awkward.
7. Discussion

One of the most important issues to make the fivebrane solution of heterotic string theory really relevant to the physical world is to investigate what to happen after the compactification from ten-dimension. Since the fivebrane is dimensionally too large to fit into the four-dimension naturally, we should expect that any compactified solution ever exists, it should show up as some lower dimensional objects which could fit into the four-dimension.

In general one could expect that the fivebranes could appear as particles, strings or membranes in the (1+3) dimensional spacetime. In this letter we have presented an explicit construction of solitonic solutions in the lower dimensional subspace of the transverse space $M^4$, reduced from the heterotic fivebrane solutions in ten-dimension. These solitons can survive after, for example, toroidal compactifications, because they do not depend on the other coordinates explicitly.

As given in sect.6, the example of putting instanton partially into the internal space suggests that other compactified solutions of fivebranes would have space-dependent warp factors in the metric of internal space. For example, one can in principle map $M^{1,9}$ conformally to $M^{1,3} \times S^2 \times S^4$ or $M^{1,3} \times S^3 \times S^3$, we get a string or a membrane respectively. Note that both internal spaces are Kähler and the sizes of the internal spaces are controlled by two radii now. Perhaps to confirm whether these really allow compactified fivebrane solutions may not be an easy exercise.

We also expect that the origin of the electric-magnetic duality in four-dimensional world is originated from the string-fivebrane duality in ten-dimension in such a way that the solitonic sector of the four-dimensional effective field theory might be coming from the fivebrane sector of the string theory.

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