An extrapolation method for projection data filtration in pulsed X-ray tomography

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Abstract. This paper deals with an inverse problem that consists of an attenuation coefficient identification for the non-stationary radiation transfer equation. To solve the problem, we propose a method that uses several pulses of radiation to extrapolate ideal projection data corresponding to a non-scattering medium. Numerical experiments on the Shepp-Logan phantom show that the method proposed improves the reconstruction quality.

1. Introduction

The problem of filtering the scattered radiation in X-ray tomography is still relevant despite the long-standing and successful research improving the quality of images [1, 2, 3]. The X-ray radiation scattered by the patient’s body becomes one of the main factors that degrade the diagnostic quality of tomographic images. Therefore, the development of the new methods for suppressing scattered radiation is urgent.

Until recently, the mathematical formulation of the X-ray tomography problems mainly used stationary models of radiation transfer processes, which is due to the complexity of designing short-pulsed X-ray sources and high-time resolution detectors. To date, successful promising studies are being carried out on the development of devices generating ultra-short X-ray pulses [6]. This, along with the invention of high-time resolution detectors, gives prospects to construct the new schemes of tomographic scanning.

This paper considers a method to filter the scattering radiation in projection data. We use the serial irradiation of a medium with short pulses of different duration to construct an extrapolation approximation of “ideal” projection profiles. The tomography problem was formalized as an inverse problem for the non-stationary radiation transfer equation and consisted in finding an attenuation coefficient with the known solution on the domain boundary.

Within the framework of the single scattering approximation that is often used in the medical tomography, we have proved vanishing a scattered component of the radiation flux density with a probe pulse duration tending to zero. The results obtained uses the majorizing function that allows us to extrapolate the solution of the transport equation without scattering and to construct a method for solving a tomography problem.

The method proposed was tested on numerical experiments to correct the projection data using the Shepp-Logan phantom filled with a proportionally scattering medium [7]. We have investigated the influence of the proportionality coefficient value of a scattering medium on the quality of the projection data correction.
2. Statement of the inverse problem

Let us consider the following integro-differential radiation transfer equation [8]

\[
\left( \frac{1}{c} \frac{\partial}{\partial t} + \omega \cdot \nabla_{r} + \mu(r) \right) I(r, \omega, t) = \sigma(r) \int_{\Omega} p(r, \omega' \cdot \omega') I(r, \omega', t) d\omega',
\]

where \( I(r, \omega, t) \) is the radiation flux density at the point \( r \in G \subset \mathbb{R}^3 \), on the direction \( \omega \in \Omega = \{ \omega \in \mathbb{R}^3 : |\omega| = 1 \} \) at the time instant \( t \in [0, T] \), \( \mu \) is the attenuation factor, \( \sigma \) is the scattering coefficient, \( c \) is the velocity of photons, and \( p \) is the scattering phase function.

Let an irradiated object be entirely contained within a cylinder of diameter \( d \). Let \( \Pi_\omega \) be a plain tangent to the boundary of the domain \( G \) and perpendicular to the direction \( \omega \), \( \Pi_\omega = \{ r \in \mathbb{R}^3 : r \cdot \omega = d/2 \} \). We assume that the medium is irradiated with a series of pulses depending on the direction \( \omega^* \in \Omega^* = \{ \omega = (\omega_1, \omega_2, 0) \in \Omega : \omega_3 = 0 \} \). The object scanning is carried out by synchronous rotation of planes with the radiation sources \( \Pi_{-\omega^*} \) and the detectors \( \Pi_{+\omega^*} \). We will use the following notation:

\[
X = G \times \Omega \times [0, T], \quad X_0 = G \times \Omega \times \{ t = 0 \},
\]

\[
\Omega_{-\omega^*} = \{ \omega \in \Omega : -\omega^* \cdot \omega > 0 \}, \quad Y^- = \Pi_{-\omega^*} \times \Omega_{-\omega^*} \times [0, T], \quad X^- = Y^- \cup X_0.
\]

Let us introduce the function

\[
h(z, \omega, t) = \begin{cases} 
0, & \text{for } (z, \omega, t) \in X_0, \\
h_{\text{ext}}(z, \omega, t), & \text{for } (z, \omega, t) \in Y^-,
\end{cases}
\]

and supplement equation (1) with the unified initial-boundary condition:

\[
I|_{X^-} = h(r, \omega, t). \tag{2}
\]

For brevity we will omit the parametric dependence of a direct problem solution on the \( \omega^* \) direction.

The serial irradiation is proposed to be dependent on direction \( \omega^* \) and described by square impulses with the pulse duration \( \epsilon \).

\[
h(\xi, \omega, t) = \begin{cases} 
1/\epsilon, & (\xi, \omega, t) \in \Pi_{-\omega^*} \times \Omega_{-\omega^*} \times (0, \epsilon), \\
0, & (\xi, \omega, t) \not\in \Pi_{-\omega^*} \times \Omega_{-\omega^*} \times (0, \epsilon).
\end{cases}
\]

From the physical point of view, such a definition of radiation source restricts only the pulse duration without assuming collimation or spatial localization of radiation sources, which are often used in tomography to suppress the scattering effects in a medium.

We investigate the following inverse problem. The problem is to find the function \( \mu \) from (1), (2) with the additional condition

\[
\int_{\Pi_{\omega^*} \times \Omega^*} \frac{d/c + \epsilon}{d/c} I(\eta, \omega^*, t) dt = H(\eta, \omega^*), \quad (\eta, \omega^*) \in \Pi_{\omega^*} \times \Omega^*, \tag{3}
\]

where \( c, d, \epsilon, h, H \) are given.

Thus, to find the attenuation coefficient, only the averaged values of the flux density are needed over the interval equal to the pulse width shifted by the travel time of the ballistic leaving the probed signal from the source to the receiver, which somewhat reduces the requirements for the temporal resolution of the detectors.
3. Estimation of the scattered radiation contribution. The profiles extrapolation procedure

To construct a method for solving the problem, we estimate the contribution of the scattered radiation to the total radiation flux depending on a probe pulse width. In the paper we restrict only a single scattering approximation. In this framework the direct problem solution can be represented as a sum of ballistic and scattered components

\[ I(\eta, \omega^*, t) = I_0(\eta, \omega^*, t) + I_1(\eta, \omega^*, t), \quad \eta \in \Pi_{\omega^*}. \]  

Here \( I_0 \) means the ballistic term, and \( I_1 \) denotes the single scattered one.

The following estimations are valid

\[ \frac{d}{c + \varepsilon} \int_{d/c}^{d/c+\varepsilon} I_0(\eta, \omega^*, t) dt = \exp \left( - \int_0^d \mu(\eta - \tau \omega) d\tau \right), \]  

\[ \frac{d}{c + \varepsilon} \int_{d/c}^{d/c+\varepsilon} I_1(\eta, \omega^*, t) dt \leq \text{Const} \left( 1 + \varepsilon \ln \left| 1 + \frac{1}{\varepsilon} \right| - \frac{1}{\varepsilon} \ln \left| 1 + \varepsilon \right| \right), \]  

where \( \varepsilon = c \varepsilon / d \).

The latter inequality shows that the integral of the scattered term tends to zero with a lower pulse duration. In this case the integral of the ballistic component gives an exponential law of attenuation like the case of a medium without scattering.

Such a behavior is clearly visible in figure 1. The graph shows the profiles of the outgoing radiation depending on the probe pulse duration. It can be seen that decreasing a pulse width leads to resulting projections tending to an "ideal" projection without scattering. These projections almost coincide at a pulse width of 3 picoseconds. In practice, using such sources with ultrashort pulses is difficult because they are fairly expensive. Our idea is to measure the radiation fluxes created by two sources with a larger pulse duration and then to use extrapolation to obtain profiles corresponding to the "zero" pulse width.

Let a medium to be irradiated with two sources of different durations \( \varepsilon_1 \) and \( \varepsilon_2 \) and \( H(\varepsilon_1), H(\varepsilon_2) \) are corresponding outgoing radiation fluxes. With a single scattering approximation we can write down

\[ H(\varepsilon) = \int_{d/c}^{d(1+\varepsilon)/c} I(\eta, \omega^*, t, \varepsilon) dt = \exp \left( - \int_0^d \mu(\eta - \tau \omega^*) d\tau \right) + C \Phi(\varepsilon), \]

where

\[ \Phi(\varepsilon) = \left( 1 + \varepsilon \ln \left| 1 + \frac{1}{\varepsilon} \right| - \frac{1}{\varepsilon} \ln \left| 1 + \varepsilon \right| \right). \]  

This equation allows us to express the Radon ray transform of the function \( \mu \) based on the corresponding output signals.

\[ \exp \left( - \int_0^d \mu(\eta - \tau \omega^*) d\tau \right) = H(\varepsilon_1) - \frac{H(\varepsilon_1) - H(\varepsilon_2)}{\Phi(\varepsilon_1) - \Phi(\varepsilon_2)} H(\varepsilon_1). \]  

The inverse problem solution reduces to the inversion of the Radon transform

\[ \int_0^d \mu(\eta - \tau \omega^*) d\tau = - \ln \left| H(\varepsilon_1) - \frac{H(\varepsilon_1) - H(\varepsilon_2)}{\Phi(\varepsilon_1) - \Phi(\varepsilon_2)} \Phi(\varepsilon_1) \right|. \]  

To find function \( \mu \), the wide-known convolution and back projection algorithm, can be used.
4. Numerical experiments and discussion

In this section, we present the results of numerical experiments using the Shepp-Logan medical phantom filled with proportional scattering media for changing the proportionality coefficient. To describe the serial irradiation of the medium, we used a pulsed source depending on the direction $\omega^\ast$ with pulse durations of 100 and 300 picoseconds ($\epsilon_1 = 300, \epsilon_2 = 100$).

The algorithm proposed was tested in two steps. At the first one, we calculated output radiation profiles for given parameters of the medium with different probe pulses using Monte Carlo method. At the second step, we corrected the projection profiles based on formula (8) and then compared the results obtained with an ideal profile corresponding to a medium without scattering.

The experiment parameters were chosen as close as possible to the parameters of real tomographs. The simulated tomograph consists of 101 detectors with 200 angular directions. For each detector, we simulate a $10^5$ photon trajectories and tracked up to 10 acts of photon-matter interactions.

Some results of the numerical experiments are shown in figure 2 and 3. Figure 2 corresponds to a medium with 50% scattering level, and 90% in figure 3. The plots show raw data corresponding to the profiles obtained by irradiating the medium with 100 ps. and 300 ps. width pulses (the dash-dotted lines). The bold line is the profile of ideal projection and the broken one is the profile obtained with extrapolation according to formula (8).

The graph shows that an increase in the duration of the probe pulse leads to a shift of the corresponding projections upward along the ordinate axis. At the same time, the qualitative behavior is preserved. As a result, the reconstructed values of the attenuation coefficient are expected to be greatly underestimated. In turn, the use of extrapolation gives a projection that
Figure 2. Profiles of the outgoing radiation for the Shepp-Logan phantom filled with 50% of a scattered medium. Raw data corresponding to the profiles obtained by irradiating the medium with 100 ps. and 300 ps. width pulses marked by broken lines. The bold line is the profile of ideal projection and the dash-dotted one is the profile obtained with extrapolation according to formula (8).

is much closer to the "ideal" one.

Note that the extrapolated projection is below the ideal one, so the reconstruction of the attenuation coefficient from such projections appears to be slightly overestimated values. An increase in the value of the scattering proportionality coefficient leads to degradation of the extrapolation quality. As an example, a graph with 90% scattering level shows that applying extrapolation gives no significant improvement in comparison with the use of raw data corresponding to 100 ps. pulse width.

5. Conclusion
In this paper, when constructing an extrapolation method to correct the projection data, we use only two sets of measurements with a different pulse duration. Good results are obtained for low scattering media that are typical of the medical tomography. Increase in the number of measurements with different duration pulses is clear to improve the quality and accuracy of extrapolation. However, additional measurements increase the radiation dose for a patient. Nevertheless, theoretical studies in this area are promising, and we associate our future investigations with this field. First, we will focus on the choice of an optimal multiparameter expansion of the function $H$ in the extrapolation formula from the point of view of approximation accuracy.
Figure 3. Profiles of the outgoing radiation for the Shepp-Logan phantom filled with 90% scattered medium. Raw data corresponding to the profiles obtained by irradiating the medium with 100 ps. and 300 ps. width pulses are marked by broken lines. The bold line is the profile of ideal projection and the dash-dotted one is the profile obtained with extrapolation according to formula (8).

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