Flatness of Noetherian Hopf algebras over coideal subalgebras

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Abstract
It is proved in the paper that a Noetherian residually finite-dimensional Hopf algebra $H$ is a flat module over any right Noetherian right coideal subalgebra $A$. In the case when $A$ is a Hopf subalgebra we get faithful flatness. These results are obtained by verifying the existence of classical quotient rings of $A$ and $H$. It is also proved that the antipode of either right or left Noetherian residually finite-dimensional Hopf algebra is bijective. As a consequence, such a Hopf algebra is right and left Noetherian simultaneously.

Keywords Hopf algebras · Coideal subalgebras · Flatness

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1 Introduction
All algebras will be considered over a fixed field $k$. The structure of Hopf algebras as modules over Hopf subalgebras and, more generally, over coideal subalgebras is of fundamental importance. Freeness results on the module structure exist for pointed Hopf algebras [8], [16] and for finite-dimensional ones [9], [14], [21]. But the property of being a free module turns out to be too strong for other classes of Hopf algebras.

Commutative Hopf algebras are projective generators as modules over Hopf subalgebras (Takeuchi [27]) and are flat over right coideal subalgebras (Masuoka and Wigner [10]). Building upon the ideas of these papers Schneider [19] proved that any left or right Noetherian Hopf algebra is a faithfully flat module over central Hopf subalgebras. Some conditions on the algebras in conclusions of this kind are inevitable. Schauenburg [17] gave examples of Hopf algebras which are not faithfully flat over some Hopf subalgebras.

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An algebra is said to be residually finite dimensional if its ideals of finite codimension have zero intersection [12]. Many classes of Hopf algebras satisfy this condition. Any residually finite-dimensional Hopf algebra is flat over central right coideal subalgebras, and there are considerably better results in the case of Hopf subalgebras (see [22]). This shows once again that dealing with coideal subalgebras incurs extra complications.

The restriction to central subalgebras is clearly a serious limitation when it comes to noncommutative Hopf algebras. Unfortunately, the technique of central localizations used in [22] is not applicable in other situations. The main result of the present paper is

**Theorem 5.5** Let $A$ be a right Noetherian right coideal subalgebra of a residually finite-dimensional Noetherian Hopf algebra $H$. Then $A$ has a right Artinian classical right quotient ring, and $H$ is left $A$-flat. Moreover, if $A$ is a Hopf subalgebra, then $H$ is left and right faithfully $A$-flat.

The relevance of classical quotient rings (the Ore rings of fractions) to the question of flatness has been made clear in another article [23]. We will recall that result in Theorem 5.4 of the present paper, providing a more direct proof of the desired conclusion. Roughly speaking, it turns out that it suffices to verify exactness of the functor $\otimes_A H$ on the category of equivariant $A$-modules equipped with a compatible action of the dual Hopf algebra $H^\circ$, and information about equivariant modules is preserved under passage to quotient rings. Classical quotient rings are an important tool in the ring theory (see [5], [11]), and we find that this tool is really useful in questions concerning Hopf algebras.

Theorem 5.4 shows immediately that Theorem 5.5 holds when $A$ and $H$ are additionally assumed to be semiprime since then the classical quotient rings of $A$ and $H$ exist by the Goldie Theorem. Without this additional condition it is not easy to establish the existence of classical quotient rings, and here lies the main problem since the usual methods do not work.

The right coideal subalgebras of a Hopf algebra $H$ are module algebras for the dual Hopf algebra $H^\circ$. This suggests a reformulation of the problem in terms of module algebras. Inasmuch as the quotient rings are concerned, switching to the module structures is essential since those extend to quotient rings, while the comodule structures generally do not.

Let now $A$ be a right Noetherian $H$-semiprime $H$-module algebra. The first attempt to deal with its quotient ring was not fully successful. In [25] it was shown that $A$ has a semiprimary generalized quotient ring $Q$ constructed with respect to a certain filter of right ideals. The property of being semiprimary is close to being Artinian, but still it does not seem to allow one to deduce that $Q$ is a classical quotient ring. The latter conclusion was obtained in [25] only for some classes of Hopf algebras. In the present paper we will prove it assuming that the action of $H$ on $A$ is locally finite, i.e., each element of $A$ is contained in a finite-dimensional $H$-submodule:

**Theorem 5.2** Let $A$ be a right Noetherian $H$-semiprime $H$-module algebra such that the action of $H$ on $A$ is locally finite. Then $A$ has a right Artinian classical right quotient ring.

This result is sufficient to derive Theorem 5.5 since the action of $H^\circ$ on $H$, and therefore on all right coideal subalgebras of $H$, is locally finite.

It should be stressed that there are no restrictions on the Hopf algebra $H$ in Theorem 5.2. To achieve this generality we have to revise the approach of [25] where the antipode of $H$ was assumed to be bijective. Using a slightly modified filter $\mathcal{E}_H$ of right ideals, we are still
able to prove that the corresponding quotient ring $Q$ is semiprimary and $H$-semiprime. This is done in the first two sections of the present paper.

However, we do not need other parts of [25] since we provide completely different arguments to analyze the structure of $Q$ in Section 4 of the paper. In particular, we needn’t bother with the selfinjectivity of $Q$. Local finiteness of the action leads very quickly to decomposition of $Q$ as a direct product of $H$-simple algebras. Then we show that each $Q$-module has no nonzero $H'$-torsion elements, which is a crucial property in the verification that $Q$ is indeed a classical quotient ring.

The final results are presented in Section 5 of the paper. Most of them have been discussed already in this introduction. Combining our approach here with an already known result on the antipode proved in [20] we also obtain

**Theorem 5.3** Let $H$ be either right or left Noetherian residually finite-dimensional Hopf algebra. Then its antipode $S : H \to H$ is bijective. Hence $H$ is right and left Noetherian simultaneously.

By Theorem 5.5 applied to $A = H$ each residually finite-dimensional Noetherian Hopf algebra $H$ has an Artinian classical quotient ring. In the case when $H$ is a Noetherian affine PI Hopf algebra such a conclusion was deduced earlier by Wu and Zhang [28] as a consequence of Gorensteinness of $H$. As a matter of fact, the assumption of Theorem 5.5 is satisfied in this case, and so we obtain an alternative proof. Indeed, it was proved by Anan’ in [1] that each right Noetherian finitely generated PI algebra is residually finite dimensional.

One may wonder whether every Noetherian Hopf algebra is necessarily residually finite dimensional. There is ample evidence supporting this. Since the intersection $I$ of all ideals of finite codimension in a Hopf algebra $H$ is a Hopf ideal, to assert that $I = 0$ it suffices to check that the finite-dimensional representations of $H$ separate the elements in the first term of the coradical filtration of $H$. For example, for any finite-dimensional Lie algebra $\mathfrak{g}$ residuum finiteness of its universal enveloping algebra $U(\mathfrak{g})$ follows at once from Ado’s Theorem when $\text{char } k = 0$. The case of positive characteristic is easy since then $U(\mathfrak{g})$ is a finite module over its center. The group algebra $kG$ is known to be Noetherian when $G$ is a polycyclic-by-finite group. The fact that such a group $G$ is residually finite [15, Ch. 10, Lemma 2.11] implies that $kG$ is residually finite dimensional.

The quantized enveloping algebra $U_q(\mathfrak{g})$ of a complex semisimple Lie algebra $\mathfrak{g}$ is also residually finite dimensional. When $q$ is not a root of 1, this fact is proved in [6, Prop. 5.11].

In conclusion I would like to thank the referee for careful reading and several suggestions which improved the text.

**Terminology and Notation** All work will be done over a base field $k$. For a subset $X$ of a ring $R$ we denote by $\text{lann}_R X$ and $\text{rann}_R X$, respectively, the left and right annihilators of $X$ in $R$. An element $s \in R$ is called right regular if $\text{rann}_R s = 0$. Left regular elements are defined by the condition $\text{lann}_R s = 0$, and $s$ is called regular if it is both right and left regular.

A ring $Q$ containing $R$ as a subring is said to be a classical right quotient ring of $R$ if all regular elements of $R$ are invertible in $Q$ and each element of $Q$ can be written as $as^{-1}$ for some $a, s \in R$ with $s$ being regular. See [5] or [11] for information on related topics.
A ring is called \textit{semiprimary} if its Jacobson radical is nilpotent and the factor ring by the Jacobson radical is semisimple Artinian.

For general facts and definitions concerning Hopf algebras we refer to [12]. Let $H$ be a Hopf algebra over $k$. We denote by $\Delta$, $\varepsilon$, $S$ its comultiplication, counit and antipode, and we write $\Delta(h) = \sum h(1) \otimes h(2) \in H \otimes H$ for $h \in H$.

A \textit{right coideal} of $H$ is any subspace $U$ such that $\Delta(U) \subset U \otimes H$. A subalgebra of $H$ satisfying this condition is called a \textit{right coideal subalgebra}.

All algebras are assumed to be associative and unital. An $H$-module algebra $A$ is equipped with a left $H$-module structure such that

$$h(ab) = \sum h(1) a (h(2) b) \quad \text{for all } h \in H, a, b \in A.$$ 

We will use freely the following two identities which hold in such an algebra:

$$(ha)b = \sum (h(1) a) (h(2) b), \quad a \varepsilon(h)b = \sum S(h(1)) \left( (h(2) a) b \right).$$

It follows that $\text{ann}_A V$ is an $H$-submodule of $A$ for each $S(H)$-submodule $V$. If $V$ is an $H$-submodule, then $\text{rann}_A V$ is an $S(H)$-submodule, but we cannot be sure that $\text{rann}_A V$ is an $H$-submodule unless $S(H) = H$ (cf. [2, Cor. 2]). Similarly, the left annihilators of $S^2(H)$-submodules of $A$ are $S(H)$-submodules.

An $H$-module algebra $A$ is called \textit{H-simple} if $A \neq 0$ and $A$ has no $H$-stable ideals except the zero ideal and the whole $A$. An $H$-module algebra $A$ is \textit{H-prime} if $A \neq 0$ and $IJ \neq 0$ for all nonzero $H$-stable ideals $I$ and $J$ of $A$. And $A$ is \textit{H-semiprime} if $A$ contains no nonzero nilpotent $H$-stable ideals. By an ideal we mean a two-sided ideal. The action of $H$ on $A$ is said to be \textit{locally finite} if $\dim H \alpha < \infty$ for all $\alpha \in A$.

\section{The filter of right ideals}

In this section we introduce the filter $\mathcal{E}_H'$ and describe its properties. We largely follow [25, Section 5] with necessary modifications.

Recall that a (right) Gabriel topology on a ring $R$ is any set $\mathcal{G}$ of right ideals of $R$ satisfying the four conditions listed below where $I, J$ are assumed to be right ideals of $R$ and we use the notation $(I : a) = \{x \in R \mid ax \in I\}$:

$$(T1) \quad \text{If } J \in \mathcal{G} \text{ and } J \subset I \text{ then } I \in \mathcal{G};$$
$$(T2) \quad \text{If } I, J \in \mathcal{G} \text{ then } I \cap J \in \mathcal{G};$$
$$(T3) \quad \text{If } I \in \mathcal{G} \text{ then } (I : a) \in \mathcal{G} \text{ for each } a \in R;$$
$$(T4) \quad \text{If } J \in \mathcal{G} \text{ and } (I : a) \in \mathcal{G} \text{ for all } a \in J \text{ then } I \in \mathcal{G}.$$

With a Gabriel topology $\mathcal{G}$ one associates a hereditary torsion theory (see [26, Ch. VI, Th. 5.1]). A right $R$-module is said to be $\mathcal{G}$-torsion if each of its elements is annihilated by a right ideal in $\mathcal{G}$. The class of $\mathcal{G}$-torsion modules is closed under submodules, factor modules, coproducts, and extensions. An arbitrary right $R$-module $V$ has a largest $\mathcal{G}$-torsion submodule. This submodule consists of all elements of $V$ whose annihilators in $R$ belong to $\mathcal{G}$. A right $R$-module is called $\mathcal{G}$-torsionfree if it contains no nonzero $\mathcal{G}$-torsion submodules.

Let $A$ be a left $H$-module algebra. Denote by $\mathcal{E}$ the set of all essential right ideals of $A$. Recall that a right ideal is said to be \textit{essential} if it has nonzero intersection with each nonzero right ideal. It is well known that $\mathcal{E}$ satisfies (T1)–(T3).
In [25] we worked with the set \( \mathcal{E}_H \) of all right ideals \( I \) of \( A \) such that for each \( h \in H \) one has \( hJ \subseteq I \) for some \( J \in \mathcal{E} \). However, in the case when \( S(H) \neq H \) we do not get the necessary properties of this filter. For this reason we will use a slightly different filter of right ideals. Note that \( S(H) \) is a Hopf subalgebra of \( H \) since the antipode \( S : H \to H \) is a Hopf algebra antiendomorphism.

Denote by \( \mathcal{E}_H' \) the set of right ideals \( I \) of \( A \) having the property that for each \( h \in S(H) \) one has \( hJ \subseteq I \) for some right ideal \( J \in \mathcal{E} \) depending on \( I \) and \( h \). We will write \( \mathcal{E}_H'(A) \) instead of \( \mathcal{E}_H' \) when we need to indicate the algebra \( A \).

Since \( 1 \in S(H) \), each right ideal \( I \in \mathcal{E}_H' \) contains an essential right ideal, and therefore is itself essential. So \( \mathcal{E}_H \subseteq \mathcal{E}_H' \subseteq \mathcal{E} \). Clearly \( \mathcal{E}_H' = \mathcal{E}_H \) when \( S \) is surjective.

For a coalgebra \( C \) denote by \( \lbrack C, A \rbrack \) the vector space \( \text{Hom}_k(C, A) \) equipped with the convolution multiplication. If \( \dim C < \infty \), then \( \lbrack C, A \rbrack \cong A \otimes C^* \) as algebras, and if \( C \subseteq H \), there is an algebra homomorphism \( \tau : A \to \lbrack C, A \rbrack \) defined by the rule \( \tau(a)(c) = ca \). One can check that \( \lbrack C, A \rbrack = \tau(A)C^* \), and so \( C, A \) is finitely generated as a left \( \tau(A) \)-module.

When \( S \) is not bijective, we cannot derive the right hand version of this conclusion. In order to use the finiteness property in one of later arguments, we have to modify the previous construction.

Denote by \( \mathcal{F} \) the set of all finite-dimensional subcoalgebras of \( H \). Let \( C \in \mathcal{F} \), and let \( C^{\text{cop}} \) be \( C \) with the opposite comultiplication. The algebra \( \lbrack C^{\text{cop}}, A \rbrack \) is defined on the vector space \( \text{Hom}_k(C, A) \) by means of the multiplication

\[
(\xi \times \eta)(c) = \sum \xi(c_2)\eta(c_1), \quad \xi, \eta \in \text{Hom}_k(C, A), \quad c \in C.
\]

Clearly \( \lbrack C^{\text{cop}}, A \rbrack \cong A \otimes (C^*)^{\text{cop}} \). Define a map \( \rho : A \to \lbrack C^{\text{cop}}, A \rbrack \), \( a \mapsto \rho_a \), setting

\[
\rho_a(c) = S(c)a, \quad a \in A, \quad c \in C.
\]

This map is an algebra homomorphism since

\[
\rho_{ab}(c) = S(c)(ab) = \sum (S(c_2)a)(S(c_1)b) = (\rho_a \times \rho_b)(c)
\]

for all \( a, b \in A \) and \( c \in C \).

**Lemma 2.1** For \( C \in \mathcal{F} \) the algebra \( \lbrack C^{\text{cop}}, A \rbrack \) is a free \( A \)-module of finite rank with respect to the right action of \( A \) obtained via \( \rho : \)

\[
\xi \cdot \rho a = \xi \times \rho_a \quad \text{where} \quad \xi \in \text{Hom}_k(C, A), \quad a \in A.
\]

In particular, \( \rho \) is injective, and so the subalgebra \( \rho(A) \subseteq \lbrack C^{\text{cop}}, A \rbrack \) is isomorphic to \( A \), whenever \( C \neq 0 \).

**Proof** Clearly \( \text{Hom}_k(C, A) \cong C^* \otimes A \) is a free \( A \)-module of finite rank with respect to another right action of \( A \) such that

\[
(\xi a)(c) = \xi(c)a \quad \text{for} \quad \xi \in \text{Hom}_k(C, A), \quad a \in A, \quad c \in C.
\]

So it suffices to check that \( \cdot \rho \) is an isomorphic \( A \)-module structure. Define a linear transformation \( \Phi \) of \( \text{Hom}_k(C, A) \) setting

\[
(\Phi \xi)(c) = \sum S(c_1)\xi(c_2).
\]
Since
\[ (\Phi(\xi a))(c) = \sum S(c(1)) (\xi(c(2))a) = \sum (S(c(2))\xi(c(3))) (S(c(1))a) = \sum (\Phi\xi)(c(2)) \rho_a(c(1)) = (\Phi\xi \times \rho_a)(c) \]
for all \( c \in C \), we get \( \Phi(\xi a) = \Phi(\xi) \cdot \rho_a a \) for all \( \xi \in \text{Hom}_k(C, A) \) and \( a \in A \). The inverse transformation \( \Phi^{-1} \) is defined by the rule
\[ (\Phi^{-1}\xi)(c) = \sum c(1)\xi(c(2)). \]
Thus \( \Phi \) is bijective, and so \( \Phi \) is indeed an isomorphism between the two \( A \)-module structures on \( \text{Hom}_k(C, A) \).

For any right ideal \( I \) of \( A \) and a subcoalgebra \( C \subset H \) put
\[ IC = \tau^{-1}(\text{Hom}_k(C, I)) = \{ x \in A \mid Cx \subset I \}. \]
Since \( \tau : A \rightarrow [C, A] \) is an algebra homomorphism and \( \text{Hom}_k(C, I) \) is a right ideal of \([C, A]\), it is clear that \( IC \) is a right ideal of \( A \). Note that
\[ IS(C) = \rho^{-1}(\text{Hom}_k(C, I)) = \{ x \in A \mid S(C)x \subset I \}. \]

**Lemma 2.2** A right ideal \( I \) of \( A \) is in \( \mathcal{E}'_H \) if and only if \( IS(C) \in \mathcal{E} \) for each \( C \in \mathcal{F} \). Moreover, \( IS(C) \in \mathcal{E}'_H \) whenever \( I \in \mathcal{E}'_H \).

**Proof** Suppose that \( I \in \mathcal{E}'_H \). Given \( C \in \mathcal{F} \) and \( h \in S(H) \), let \( X \) be any basis of the finite-dimensional subspace \( S(C)h \subset S(H) \). Since \( X \) is finite and \( \mathcal{E} \) is closed under finite intersections of right ideals, there exists \( J \in \mathcal{E} \) such that \( gJ \subset I \) for all \( g \in X \). Then \( S(C)hJ \subset I \), that is, \( hJ \subset IS(C) \). This establishes the inclusion \( IS(C) \in \mathcal{E}'_H \subset \mathcal{E} \).

Conversely, since every element of \( H \) is contained in a finite-dimensional subcoalgebra, \( S(H) \) is the union of the subcoalgebras \( S(C) \) with \( C \in \mathcal{F} \), and obviously \( hIS(C) \subset I \) for all \( h \in S(C) \). This shows that \( I \in \mathcal{E}'_H \) whenever \( IS(C) \in \mathcal{E} \) for all \( C \in \mathcal{F} \).

**Lemma 2.3** The set \( \mathcal{E}'_H \) satisfies (T1)–(T3).

**Proof** Properties (T1), (T2) for \( \mathcal{E}'_H \) follow easily from the respective properties of \( \mathcal{E} \) since \( (I \cap J)_C = I_C \cap J_C \) and, in particular, \( I_C \subset J_C \) whenever \( I \subset J \).

Let us check (T3). Let \( I \in \mathcal{E}'_H \) and \( a \in A \). For each \( C \in \mathcal{F} \) the right ideal \( IS(C) \) is essential by Lemma 2.2, and we have to show that so is
\[ (I : a)_{S(C)} = \{ x \in A \mid a (S(C)x) \subset I \}. \]
Put \( K = \{ x \in A \mid (Ca)x \subset IS(C) \} \). Then \( K \in \mathcal{E} \) since \( Ca \) is a finite-dimensional subspace of \( A \), and \( \mathcal{E} \) satisfies (T2), (T3). To complete the proof it remains to show that \( K \subset (I : a)_{S(C)} \). This containment does hold because
\[ a (S(c)x) = \sum S(c(1)) ((c(2)a)x) \in S(C) IS(C) \subset I \]
for all \( x \in K \) and \( c \in C \).
Proposition 2.4 Suppose that $A$ is $H$-semiprime and satisfies ACC on right annihilators. Then $\mathcal{E}'_H$ is a Gabriel topology and $A$ is $\mathcal{E}'_H$-torsionfree as a module over itself with respect to right multiplications.

Proof Since $\mathcal{E}'_H$ satisfies (T1)–(T3), the set

$$N = \{a \in A \mid \text{rann}_A a \in \mathcal{E}'_H\}$$

is a right ideal of $A$. Obviously, $N$ is stable also under left multiplications in $A$. Suppose $a \in N$. Then $aI = 0$ for some $I \in \mathcal{E}'_H$. If $C \in \mathcal{F}$, then $I_{S(C)} \in \mathcal{E}'_H$ by Lemma 2.2. Now

$$(ca)x = \sum c_{(1)} (a S(c_{(2)})x) = 0$$

for all $c \in C$ and $x \in I_{S(C)}$ since $S(C)x \subset I$. Thus $bI_{S(C)} = 0$ for each $b \in Ca$. This shows that $Ca \subset N$. Since each element of $H$ is contained in a finite-dimensional subcoalgebra, we conclude that $N$ is an $H$-stable two-sided ideal of $A$.

Recall that the right singular ideal $\text{Sing}_A$ of $A$ is a two-sided ideal consisting of all elements of $A$ whose right annihilators are essential right ideals. Since $\mathcal{E}'_H \subset \mathcal{E}$, we have $N \subset \text{Sing}_A$. According to [11, Lemma 2.3.4] the ascending chain condition on right annihilators implies that $\text{Sing}_A$ is nilpotent. Hence so too is $N$, and the $H$-semiprimeness of $A$ yields $N = 0$.

Vanishing of $N$ means that lann$_A I = 0$ for each $I \in \mathcal{E}'_H$. In other words, $A$ is $\mathcal{E}'_H$-torsionfree. It remains to verify that $\mathcal{E}'_H$ satisfies (T4).

Let $I$ and $J$ be two right ideals of $A$ such that $J \in \mathcal{E}'_H$ and $(I : a) \in \mathcal{E}'_H$ for all $a \in J$. We have to show that $I \in \mathcal{E}'_H$. Let $C \in \mathcal{F}$. Then $J_{S(C)} \in \mathcal{E}'_H$, and we will check that $I_{S(C)} \in \mathcal{E}'_H$ too. For this we have to show that $I_{S(C)} \cap R \neq 0$ for each nonzero right ideal $R$ of $A$. But $J_{S(C)} \cap R \neq 0$, so that it suffices to consider the right ideals of the form $R = bA$ where $0 \neq b \in J_{S(C)}$. Fix such an element $b$ and put

$$K = \{x \in A \mid (S(C)b) x \subset I\}.$$

Here $S(C)b$ is a finite-dimensional subspace of $J$. Taking its basis, say $b_1, \ldots, b_n$, we get $K = \bigcap_{i=1}^n (I : b_i) \in \mathcal{E}'_H$ since $b_1, \ldots, b_n \in J$. If $y \in K_{S(C)}$, then $S(C)y \subset K$, and therefore

$$S(C)(by) = \sum (S(c_{(2)})b) (S(c_{(1)})y) \subset (S(C)b) K \subset I$$

for all $c \in C$, i.e. $by \in I_{S(C)}$. We see that

$$b K_{S(C)} \subset I_{S(C)} \cap bA.$$

But $K_{S(C)} \in \mathcal{E}'_H$ by Lemma 2.2. As we have proved already, all right ideals in $\mathcal{E}'_H$ have zero left annihilators. Hence $bK_{S(C)} \neq 0$, and therefore $I_{S(C)} \cap bA \neq 0$. \hfill $\Box$

Later we will have to work with $H$-module algebras which are not right Noetherian, but only right Goldie. The ACC on right annihilators is one of Goldie conditions. The second one is the ACC on direct sums of right ideals, which can be interpreted as the finiteness of the right uniform dimension. Our next aim is to show that in the presence of the Goldie conditions the filter $\mathcal{E}'_H$ is sufficiently large (see Lemma 2.6).

Recall that the uniform dimension $\text{udim} M$ of a module $M$ is the largest number of nonzero submodules forming a direct sum, and $\text{udim} M < \infty$ if no infinite direct sum of nonzero submodules exist. If $R$ is a subring of a ring $T$, let $T_R$ be $T$ regarded as a right $R$-module with respect to the action of $R$ on $T$ by right multiplications. Our argument is based on a ring-theoretic observation contained in [25, Lemma 5.4] which we repeat here with a condensed proof:
Lemma 2.5 Suppose that $R$ is a subring of a ring $T$ such that $\text{udim } T_R < \infty$. Then

$$xT + \text{rann}_T x$$

is an essential submodule of $T_R$ for any element $x \in T$ with $\text{rann}_T x = \text{rann}_T x^2$.

Proof Suppose that $V \subset T_R$ is a submodule such that $V \cap (xT + \text{rann}_T x) = 0$. Then $x^i V \cong V$ for each $i \geq 0$, and the sum $\sum_{i=0}^{\infty} x^i V \subset T_R$ is direct. The finiteness of the uniform dimension entails $V = 0$.

Lemma 2.6 Suppose that $\text{udim } A_A < \infty$. Then

$$uA + \text{rann}_A u \in \mathcal{E}_H'$$

for any element $u \in A$ with $\text{rann}_A u = \text{rann}_A u^2$. In particular, $uA \in \mathcal{E}_H'$ whenever $u$ is right regular in $A$.

Proof Put $I = uA + \text{rann}_A u$. We want to apply Lemma 2.5 with $T = [C^{\text{cop}}, A]$ and $R = \rho(A)$ where $C \in \mathcal{F}$. By Lemma 2.1 $R \cong A$ and $T$ is a free right $R$-module of finite rank. Hence $\text{udim } T_R < \infty$.

Making use of the identification $T \cong A \otimes (C^*)^{\text{op}}$ let $x = u \otimes 1 \in T$. We have $xT \cong uA \otimes (C^*)^{\text{op}}$ and $\text{rann}_T x \cong (\text{rann}_A u) \otimes (C^*)^{\text{op}}$. Hence

$$xT + \text{rann}_T x = \text{Hom}_k(C, I) \cong I \otimes (C^*)^{\text{op}}.$$ 

Since $x^2 = u^2 \otimes 1$, we deduce that $\text{rann}_T x^2 = (\text{rann}_A u^2) \otimes (C^*)^{\text{op}} = \text{rann}_T x$. All assumptions of Lemma 2.5 thus hold, and $\text{Hom}_k(C, I)$ is then an essential right $R$-submodule of $T$.

Recall that

$$I_{S(C)} = \rho^{-1}(\text{Hom}_k(C, I)).$$

Let $0 \neq b \in A$. If $\rho_b = 0$ in $T$ then $b \in I_{S(C)}$ since $\rho_b \in \text{Hom}_k(C, I)$. If $\rho_b \neq 0$ then

$$\rho_b R \cap \text{Hom}_k(C, I) \neq 0;$$

since $\rho_b R = \rho(bA)$, there exists $a \in bA$ such that $0 \neq \rho a \in \text{Hom}_k(C, I)$. In the latter case $0 \neq a \in I_{S(C)}$. Thus $I_{S(C)} \cap bA \neq 0$ in any case, and so $I_{S(C)} \in \mathcal{E}$. Lemma 2.2 completes the proof.

As we have seen, under the hypothesis of Proposition 2.4 all right ideals in $\mathcal{E}'_H$ have zero left annihilators. It will be important later that the right annihilators are zero as well, but we can prove this only under stronger restrictions:

Lemma 2.7 Suppose that $A$ is $S^2(H)$-semiprime and satisfies ACC on right annihilators. Then $\text{rann}_A I = 0$ for each $I \in \mathcal{E}'_H$.

Proof For each right ideal of $A$ its right annihilator in $A$ is a two-sided ideal. By the ACC the set

$$\{\text{rann}_A I \mid I \in \mathcal{E}'_H\}$$

has a maximal element, say $K$. But this set is directed by inclusion since the set $\mathcal{E}'_H$ is directed by inverse inclusion according to property (T2) and since the correspondence $I \mapsto$ Springer
rann_A I reverses inclusions. Therefore K is the largest among all right annihilators of right ideals in \( E_H \). We have to show that \( K = 0 \).

Now pick \( I \in \mathcal{E}'_H \) such that \( K = \text{rann}_A I \). If \( a \in K, C \in \mathcal{F}, c \in C, x \in IS(C) \), then
\[
x \left(S^2(c)a\right) = \sum S^2(c(2)) ((S(c(1)))x) a = 0
\]
since \( S(C)x \subseteq I \) and \( Ia = 0 \). This shows that \( S^2(c)a \in \text{rann}_A IS(C) \). Lemma 2.2 tells us that \( IS(C) \subseteq \mathcal{E}'_H \). Hence \( \text{rann}_A IS(C) \subseteq K \) by the choice of \( K \), and it follows that \( S^2(C)K \subseteq K \). Since \( H \) is the union of subcoalgebras \( C \in \mathcal{F} \), we conclude that \( K \) is stable under the action of \( S^2(H) \).

The left annihilator \( L = \text{lann}_A K \) is a two-sided ideal as well. It is stable under the action of \( S(H) \) since \( K \) is an \( S^2(H) \)-submodule of \( A \). Hence \( K \cap L \) is an \( S^2(H) \)-stable ideal. Since \( (K \cap L)^2 \subset KL = 0 \), we deduce that \( K \cap L = 0 \).

Since \( KL \subset K \cap L \), it follows that \( KL = 0 \) too. But \( L \in \mathcal{E}'_H \) since \( L \) contains any right ideal \( I \in \mathcal{E}'_H \) such that \( K = \text{rann}_A I \). By Proposition 2.4 \( \text{lann}_A L = 0 \), which entails \( K = 0 \), as required.

**Lemma 2.8** Let \( I \) be an ideal of finite codimension in \( H \). Then \( S^n(H) + I = H \) for all \( n > 0 \).

**Proof** Recall that the finite dual \( H^* \) of \( H \) is a Hopf algebra consisting of all linear functions \( H \to k \) vanishing on an ideal of finite codimension in \( H \). The antipode \( S^o \) of \( H^* \) is defined by the rule \( S^o(f) = f \circ S \) for \( f \in H^* \).

By [20, Th. A] \( S^o \) is injective since \( H^* \) is always residually finite dimensional and, as a consequence, weakly finite. If \( f \in H^* \) is any linear function vanishing on \( S^n(H) + I \), then \( f \in H^o \) and \( S^o(f) = 0 \), whence \( f = 0 \) by injectivity of \( S^o \). This yields the desired conclusion.

**Corollary 2.9** Let \( V \) be a locally finite-dimensional \( H \)-module, that is, each element of \( V \) is contained in a finite-dimensional submodule. Then each \( S^n(H) \)-submodule of \( V \) is an \( H \)-submodule.

**Proof** Let \( I \) be the annihilator of the \( H \)-submodule \( Hv \) generated by some element \( v \in V \). Since \( \dim Hv < \infty \), this ideal of \( H \) has finite codimension. By Lemma 2.8 \( S^n(H) + I = H \), and we get \( Hv = S^n(H)v \) since \( Iv = 0 \). Hence \( Hv \) is contained in each \( S^n(H) \)-submodule of \( V \) containing \( v \).

**Corollary 2.10** Suppose that \( A \) is \( H \)-semiprime and satisfies ACC on right annihilators. If the action of \( H \) on \( A \) is locally finite, then \( \text{rann}_A I = 0 \) for each \( I \in \mathcal{E}'_H \).

**Proof** By Corollary 2.9 each \( S^2(H) \)-stable ideal of \( A \) is stable under the action of the whole \( H \). Therefore there is no difference between the \( H \)-semiprimeness and the \( S^2(H) \)-semiprimeness of \( A \), and so Lemma 2.7 applies.

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**3 The quotient ring**

Let \( R \) be a ring and \( \mathcal{G} \) a filter of right ideals satisfying the axioms (T1)–(T4) of a Gabriel topology. The abelian groups \( \text{Hom}_R(I, R) \) with \( I \in \mathcal{G} \) form an inductive system, and in the
case when $R$ is $\mathcal{G}$-torsionfree as a right $R$-module the localization of $R$ with respect to $\mathcal{G}$ is defined as the limit

$$R_\mathcal{G} = \lim_{I \in \mathcal{G}} \text{Hom}_R(I, R).$$

If $\alpha : I \to R$ and $\beta : J \to R$ are two right $R$-linear maps, where $I, J \in \mathcal{G}$, then $\alpha \circ \beta$ is defined on $\beta^{-1}(I)$. The $R$-module $J/\beta^{-1}(I)$ is $\mathcal{G}$-torsion since it embeds in the $\mathcal{G}$-torsion module $R/I$. Since $R/J$ is $\mathcal{G}$-torsion, so is $R/\beta^{-1}(I)$ too, which means that $\beta^{-1}(I) \in \mathcal{G}$. Thus $\alpha \circ \beta$ represents an element of $R_\mathcal{G}$, and this one is taken to be the product of the two elements represented by $\alpha$ and $\beta$ respectively. In this way $R_\mathcal{G}$ acquires a ring structure. We call $R_\mathcal{G}$ with this structure the quotient ring of $R$ with respect to $\mathcal{G}$.

The ring $R$ is identified with the subring of $R_\mathcal{G}$ consisting of all elements represented by left multiplications in $R$. If $q \in R_\mathcal{G}$ is an arbitrary element represented by $\alpha : I \to R$, where $I \in \mathcal{G}$, then $q\alpha(x) = \alpha(x)$ for all $x \in I$; hence $qI \subset R$, and $qI \neq 0$ unless $q = 0$. In particular, each nonzero right $R$-submodule of $R_\mathcal{G}$ has a nonzero intersection with $R$, so that $R_\mathcal{G}$, regarded as a right $R$-module, is an essential extension of $R$. It follows that $R_\mathcal{G}$, along with $R$, is $\mathcal{G}$-torsionfree.

Suppose now that $\mathcal{G}$ is a Gabriel topology on a left $H$-module algebra $A$ such that $A$ is $\mathcal{G}$-torsionfree. The right ideals in $\mathcal{G}$ form a neighbourhood base of $0$ for a topology making $A$ into a topological algebra. If all elements of $H$ operate on $A$ as continuous transformations, then the action of $H$ on $A$ is said to be $\mathcal{G}$-continuous, and it is known in this case that the action extends to the quotient ring $A_\mathcal{G}$ [13, Th. 3.13]. It will be important for us that the conclusion of that theorem remains valid under a slightly weaker assumption when continuity of the action is required only for elements of $S(H)$:

**Lemma 3.1** Suppose that all elements of $S(H)$ operate on $A$ as $\mathcal{G}$-continuous transformations. Then $A_\mathcal{G}$ is a left $H$-module algebra with respect to an action of $H$ extending the given action on $A$.

**Proof** The continuity assumption means that for each $h \in S(H)$ and each $I \in \mathcal{G}$ there exists $I_h \in \mathcal{G}$ such that $hI_h \subset I$. If $C \in \mathcal{F}$, then $S(C)$ is a finite-dimensional subspace of $S(H)$; therefore by (T2) for each $I \in \mathcal{G}$ there exists $K \in \mathcal{F}$ such that $hK \subset I$ for all $h \in S(C)$, i.e. $K \subset I_{S(C)}$ in the notation of Section 2. Note that the latter inclusion and (T1) imply that $I_{S(C)} \in \mathcal{G}$.

Given any $h \in H$ and a right $A$-linear map $\alpha : I \to A$ where $I \in \mathcal{G}$, let $C_h \in \mathcal{F}$ be the smallest subcoalgebra containing $h$, and define $h\alpha : I_{S(C_h)} \to A$ by the rule

$$(h\alpha)(x) = \sum h_{(1)} \alpha(S(h_{(2)}x), \quad x \in I_{S(C_h)}).$$

As in [3, Th. 18], one checks that the map $h\alpha$ is $A$-linear. Since $I_{S(C_h)} \in \mathcal{G}$, this map represents an element of $A_\mathcal{G}$.

Thus for $h \in H$ and $q \in A_\mathcal{G}$ we can define $hq$ to be the element of $A_\mathcal{G}$ represented by $h\alpha$ where $\alpha$ is any representative of $q$. If $g$ is a second element of $H$, then $(gh)q = g(hq)$ since the two $A$-linear maps $(gh)\alpha$ and $g(h\alpha)$ agree on the right ideal

$$\{x \in A \mid S(C_h)S(C_g)x \subset I\} = (I_{S(C_h)})_{S(C_g)} \in \mathcal{G}.$$ 

Let $q, q' \in A_\mathcal{G}$ be represented by $A$-linear maps $\alpha : I \to A$ and $\beta : J \to A$ where $I, J \in \mathcal{G}$. For each $c \in C_h$ the map $c\alpha$ is defined on $I_{S(C_h)} \in \mathcal{G}$ and $c\beta$ is defined on $J_{S(C_h)} \in \mathcal{G}$. Since $\{c\beta \mid c \in C_h\}$ is a finite-dimensional subspace of $A$-linear maps.
$J_{S(C_h)} \rightarrow A$, there exists $K \in \mathcal{G}$ such that $K \subset J_{S(C_h)}$ and $(c\beta)(K) \subset I_{S(C_h)}$ for all $c \in C_h$. If $x \in K$ and $c \in C_h$, then

$$\beta(S(c)x) = \sum S(c(1))c(2)\beta(S(c(3))x) = \sum S(c(1))((c(2)\beta)(x)) \in S(C_h)I_{S(C_h)}$$

whence $\beta(S(c)x) \in I$, and so $r \circ \beta$ is defined at $S(c)x$. We get

$$(h(\alpha \circ \beta))(x) = \sum h(1)\alpha(\beta(S(h(2))x)) = \sum h(1)\alpha(S(h(2))h(3)\beta(S(h(4))x))$$

Thus $h(\alpha \circ \beta)$ agrees with $\sum (h(1)\alpha \circ (h(2)\beta)$ on $K$. This means that

$$h(qq') = \sum (h(1)q)(h(2)q'),$$

i.e. the $H$-module structure on $A_{\mathcal{G}}$ is compatible with the multiplication.

It was proved in Lemma 2.2 that $I_{S(C)} \in \mathcal{E}^\prime_H$ for each $I \in \mathcal{E}^\prime_H$ and each $C \in \mathcal{F}$. This fact shows that all elements of $S(H)$ operate on $A$ as $\mathcal{E}^\prime_H$-continuous transformations.

**Proposition 3.2** Suppose that $A$ is $H$-semiprime and right Noetherian. Then the quotient ring $Q$ of $A$ with respect to the filter of right ideals $\mathcal{E}^\prime_H$ is a semiprimary $H$-semiprime $H$-module algebra.

**Proof** Note that Proposition 2.4 applies to $A$ since $A$ satisfies ACC on arbitrary right ideals. Hence $\mathcal{E}^\prime_H$ is a Gabriel topology and $A$ is $\mathcal{E}^\prime_H$-torsionfree. By Lemma 3.1 $Q$ is a left $H$-module algebra with respect to an action of $H$ extending the given action on $A$.

By general properties of quotient rings $Q$ is an essential extension of $A$ in the category of right $A$-modules. In particular, each nonzero right ideal of $Q$ has nonzero intersection with $A$. If $I$ is a nilpotent $H$-stable ideal of $Q$, then $I \cap A$ is a nilpotent $H$-stable ideal of $A$. Then $I \cap A = 0$ by $H$-semiprime of $A$, and we must have $I = 0$. Therefore $Q$ is $H$-semiprime.

The less obvious part of the proof is to show that $Q$ is semiprimary. Here the arguments follow [25] with the filter $\mathcal{E}^\prime_H$ replaced by $\mathcal{E}^\prime_H$ everywhere. We indicate below the main steps referring to [25] for other details.

**Claim 1** $Q$ is right Goldie, i.e., $Q$ satisfies ACC on right annihilators and has finite right uniform dimension.

This is a general property of quotient rings of right Noetherian rings (see [25, Lemma 6.1]).

**Claim 2** For a right ideal $I$ of $Q$ one has $I \in \mathcal{E}^\prime_H(Q)$ if and only if $I \cap A \in \mathcal{E}^\prime_H(A)$.

This is an analog of [25, Lemma 6.2]. Setting $J = I \cap A$, we have

$$J_{S(C)} = I_{S(C)} \cap A \quad \text{for each } C \in \mathcal{F}.$$ 

Therefore $I_{S(C)}$ is an essential right ideal of $Q$ if and only if $J_{S(C)}$ is an essential right ideal of $A$ (see [4, Prop. 2.32(a)]). It remains to apply the characterization of the filters $\mathcal{E}^\prime_H(A)$, $\mathcal{E}^\prime_H(Q)$ given in Lemma 2.2.
Claim 3 If $I \in \mathcal{E}'_H(Q)$ then each right $Q$-linear map $I \to Q$ is induced by a left multiplication in $Q$.

It is well known that the quotient ring $Q$ coincides with its own localization with respect to the filter $\mathcal{G}$ of right ideals $I$ such that $I \cap A \in \mathcal{E}'_H(A)$ [26, Ch. X, §2]. By Claim 2 we have $\mathcal{G} = \mathcal{E}'_H(Q)$, and therefore bijectivity of the canonical map $Q \to Q\mathcal{G}$ amounts to Claim 3.

Claim 4 For any $u \in Q$ satisfying $\text{rann}_Q u = \text{rann}_Q u^2$ there exists an idempotent $e \in Q$ such that $u$ is an invertible element of the ring $eQe$ with unity $e$.

Put $Y = \text{rann}_Q u$. The right ideal $I = uQ + Y$ is in $\mathcal{E}'_H(Q)$ by Lemma 2.6, and the assumption about $u$ implies that the sum here is direct. The projection of $I$ onto $uQ$ is a right $Q$-linear map $I \to Q$. By Claim 3 it is the restriction to $I$ of the left multiplication by some element $e \in Q$. Then $eu = u$ and $eY = 0$. Since $(ue - u)I = 0$ and $(e^2 - e)I = 0$, it follows that $ue = u$ and $e^2 = e$ by torsionfreeness of $Q$.

There is also a right $Q$-linear map $I \to Q$ such that $uq \mapsto eq$ for all $q \in Q$ and $q \mapsto 0$ for all $q \in Y$. It is the restriction to $I$ of the left multiplication by another element $v \in Q$. Then $vu = e$ and $vY = 0$ by the choice of $v$. The equalities $uv = e$ and $ev = ve = v$ follow again from the fact that $I$ has zero left annihilator. Thus $u, v \in eQe$, and $v$ is the inverse of $u$ in the ring $eQe$, as asserted in Claim 4.

Now we are able to continue as in the final part of the proof of [25, Lemma 6.4]. As explained there in detail, Claims 1 and 4 imply that each right ideal of $Q$ has the form $aQ + K$ where $a \in Q$ is an idempotent and $K$ is a nil right ideal of $Q$. Since the ring $Q$ is right Goldie, it has a largest nilpotent ideal $N$, and each nil right ideal of $Q$ is contained in $N$. In particular, $K \subset N$. This shows that each right ideal of the factor ring $Q/N$ is generated by an idempotent. Hence $Q/N$ is semisimple Artinian, and $N$ is the Jacobson radical of $Q$.

4 Semiprimary module algebras

We wish to know that the ring $Q$ in Proposition 3.2 is actually a classical quotient ring of $A$. In spite of an effort made in [25] this conclusion in its full generality remains unproved. We will verify it under the additional assumption that the action of $H$ on $A$ is locally finite.

We will need only the properties of $Q$ established in Proposition 3.2, while the precise choice of the filter $\mathcal{E}'_H$ will not be significant any longer. Therefore our assumptions about $Q$ in this section are slightly more general. The final conclusion is presented in Proposition 4.9.

The first important step, where the local finiteness comes into play, consists in decomposing $Q$ as a direct product of $H$-simple algebras. This is done in Proposition 4.2, and here we have to apply one earlier result on freeness of equivariant modules.

Let $B$ be any $H$-module algebra. A right $B$-module $M$ is said to be $H$-equivariant if $M$ is equipped with a left $H$-module structure such that

$$h(va) = \sum (h(1)v)(h(2)a) \quad \text{for all } h \in H, v \in M, a \in B.$$  

Suppose that $B$ is semilocal, i.e., the factor ring of $B$ by the Jacobson radical is semisimple Artinian. Then the set $\text{Max} B$ of maximal ideals of $B$ is finite, and the ring $B/P$ is simple.
Artinian for each \( P \in \text{Max } B \). If \( M \) is a finitely generated right \( B \)-module, then \( M/MP \) is a right \( B/P \)-module of finite length, and we define the rank of \( M \) at \( P \) as
\[
 r_P(M) = \frac{\text{length } M/MP}{\text{length } B/P}.
\]
In conformance with the notation of Section 2 \( PH = \{ a \in B \mid Ha \subseteq P \} \) is the largest \( H \)-stable ideal of \( B \) contained in \( P \). Thus \( PH = 0 \) if and only if \( P \) contains no nonzero \( H \)-stable ideals of \( B \). Now let us recall [21, Lemma 7.5]:

**Lemma 4.1** Let \( B \) be a semilocal \( H \)-module algebra and \( M \) an \( H \)-equivariant finitely generated right \( B \)-module. Suppose that there exists \( P \in \text{Max } B \) such that \( PH = 0 \) and \( r_P(M) \geq r_{P'}(M) \) for all \( P' \in \text{Max } B \). Then \( M^n \) is a free \( B \)-module for some integer \( n > 0 \).

Lemma 4.1 can be applied, in particular, when \( B \) is \( H \)-simple, but actually in some cases it can be used to show that \( B \) is \( H \)-simple when this is not known beforehand. This is what we are going to do next.

**Proposition 4.2** Suppose that \( Q \) is a semiprimary \( H \)-semiprime \( H \)-module algebra containing an \( H \)-stable subalgebra \( A \) such that the action of \( H \) on \( A \) is locally finite and \( I \cap A \neq 0 \) for each nonzero right ideal \( I \) of \( Q \). Then there is an isomorphism of \( H \)-module algebras
\[
Q \cong Q_1 \times \ldots \times Q_n
\]
where \( Q_1, \ldots, Q_n \) are \( H \)-simple \( H \)-module algebras.

**Proof** Recall that every semiprimary ring satisfies DCC on finitely generated right ideals [26, Ch. VIII, Prop. 5.5]. In particular, \( Q \) has a minimal nonzero \( H \)-stable finitely generated right ideal, say \( M \). For each \( a \in A \) the \( H \)-stable right ideal \((Ha)Q \) is finitely generated by the local finiteness of the action on \( A \). Since each nonzero right ideal of \( Q \) contains nonzero elements of \( A \), each nonzero \( H \)-stable right ideal of \( Q \) contains therefore a nonzero \( H \)-stable finitely generated right ideal of \( Q \).

It follows that \( M \) is minimal in the set of all nonzero \( H \)-stable right ideals of \( Q \). If \( I \) is any two-sided \( H \)-stable ideal of \( Q \), then either \( MI = M \) or \( MI = 0 \) since \( MI \) is an \( H \)-stable right ideal and \( M I \subset M \).

Pick any maximal ideal \( P \in \text{Max } Q \) for which \( r_P(M) \) attains the maximum value. Note that \( M \neq MP \) since \( M \neq 0 \). Then \( MP_H \neq M \) too, and therefore \( MP_H = 0 \). Thus \( M \) is an \( H \)-equivariant finitely generated right \( Q/PH \)-module and \( P/P_H \) is a maximal ideal of the factor algebra \( Q/P_H \) satisfying the hypotheses of Lemma 4.1. We conclude that a direct sum of several copies of \( M \) is a free \( Q/P_H \)-module.

But then the assignment \( I \mapsto MI \) gives an injection of the lattice of \( H \)-stable ideals of the \( H \)-module algebra \( Q/P_H \) into the lattice of \( H \)-stable right ideals of \( Q \) contained in \( M \).

Since \( 0 \) and \( M \) are the only two elements of the latter lattice, we deduce that the algebra \( Q/P_H \) is \( H \)-simple.

Now \( T = QM \) is a two-sided \( H \)-stable ideal of \( Q \) with the property that \( TP_H = 0 \). Since \( (T \cap P_H)^2 = 0 \), we must have \( T \cap P_H = 0 \) by the \( H \)-semiprimeness of \( Q \). On the other hand, \( T + P_H = Q \) since this sum is an \( H \)-stable ideal of \( Q \) properly containing \( P_H \). The Chinese remainder theorem yields \( Q \cong Q_1 \times Q' \) where \( Q_1 = Q/P_H \) and \( Q' = Q/T \).

Clearly \( Q' \) is a semiprimary \( H \)-module algebra, and the projection \( \pi \) of \( Q \) onto \( Q' \) is a homomorphism of \( H \)-module algebras. Hence \( \pi(A) \) is an \( H \)-stable subalgebra of \( Q' \) on
which the action of $H$ is locally finite. Each $H$-stable right ideal of $Q'$ can be written as $\pi(J)$ where $J$ is an $H$-stable right ideal of $Q$ lying in the kernel of the other projection $Q \rightarrow Q_1$. If $J \neq 0$, then $J \cap A \neq 0$ by the hypothesis, and it follows that $\pi(J) \cap \pi(A) \neq 0$ since the map $\pi|_J$ is injective. Also $Q'$ is $H$-semiprime since so is $Q$.

Thus $Q'$ satisfies the same assumptions as $Q$, but has fewer maximal ideals. We have seen that $Q_1$ is $H$-simple. Proceeding by induction on the cardinality of the set Max $Q$, we may assume that $Q'$ is a direct product of finitely many $H$-simple $H$-module algebras, and the proof is completed. 

\begin{lemma} \label{lem:4.3}
The $H$-module algebra $Q$ in Proposition 4.2 is in fact $S(H)$-semiprime and each direct factor $Q_i$ is $S(H)$-simple.
\end{lemma}

\begin{proof}
By an argument given in the proof of Proposition 4.2 any $H$-module algebra isomorphic to a direct factor of $Q$ satisfies the assumptions imposed on $Q$ in the statement of Proposition 4.2. Therefore it suffices to consider the case when $n = 1$ and $Q$ is $H$-simple.

For any $S(H)$-stable ideal $I$ of $Q$ its left annihilator is an $H$-stable ideal of $Q$. Since $Q$ is $H$-simple, we must have lann$_Q I = 0$ whenever $I \neq 0$. In particular, a nonzero $S(H)$-stable ideal cannot be nilpotent. So $Q$ is $S(H)$-semiprime.

Now we can apply Proposition 4.2, replacing $H$ with its Hopf subalgebra $S(H)$. It shows that $Q$ is isomorphic as an $S(H)$-module algebra to a direct product of several $S(H)$-simple $S(H)$-module algebras. The direct factors may be identified with minimal nonzero $S(H)$-stable ideals of $Q$. If $I_1, I_2$ are two different such ideals, then $I_1 I_2 \subset I_1 \cap I_2 = 0$, but this is impossible since lann$_Q I_2 = 0$, as we have seen already. Hence $Q$ is $S(H)$-simple.
\end{proof}

Let $B$ be an arbitrary $H$-module algebra. For a right $H$-comodule $U$ and a right $B$-module $V$ we will consider the vector space $U \otimes V$ as a right $B$-module with respect to the twisted action of $B$ defined by the rule

$$ (u \otimes v)a = \sum u(0) \otimes v \left(S(u(1))a\right), \quad u \in U, \ v \in V, \ a \in B, $$

where $\sum u(0) \otimes u(1) \in U \otimes H$ is the symbolic notation for the image of $u$ under the comodule structure map $U \rightarrow U \otimes H$.

We will also need similar tensoring operations on left modules. Given a left $B$-module $V$ and $U$ as above, there is a left $B$-module structure on the vector space $V \otimes U$ defined by the rule

$$ a(v \otimes u) = \sum \left(u(1)\right)a v \otimes u(0), \quad u \in U, \ v \in V, \ a \in B. $$

\begin{lemma} \label{lem:4.4}
Suppose that $\mathcal{G}$ is a right Gabriel topology on $A$ such that all elements of $S(H)$ operate on $A$ as $\mathcal{G}$-continuous transformations. If $V$ is a $\mathcal{G}$-torsion right $A$-module, then so is $U \otimes V$ for any right $H$-comodule $U$.
\end{lemma}

\begin{proof}
Let $u \in U$ and $v \in V$. We have $\nu I = 0$ for some $I \in \mathcal{G}$. Since $\sum u(0) \otimes u(1) \in U \otimes C$ for some $C \in \mathcal{G}$, it follows from the formula for the action of $A$ in $U \otimes V$ that $u \otimes v$ is annihilated by the right ideal $I_\mathcal{G}(C)$ of $A$. But $I_\mathcal{G}(C) \in \mathcal{G}$, and therefore $u \otimes v$ lies in the $\mathcal{G}$-torsion submodule of $U \otimes V$. Since such elements span the whole $U \otimes V$, the conclusion follows.
\end{proof}
Lemma 4.5 Suppose that $B$ is an $S(H)$-simple $H$-module algebra. If $K$ is a simple right ideal of $B$ and $V$ is any nonzero right $B$-module, then $K$ embeds in the right $B$-module $U \otimes V$ for some finite-dimensional right $H$-module $U$.

Proof We may regard $H$ as a right $H$-comodule with respect to the comultiplication in $H$. If $a \in B$ annihilates $H \otimes V$, then
\[ \sum h(1) \otimes v(S(h(2))a) = 0 \quad \text{for all } h \in H \text{ and } v \in V, \]
and applying the map $\varepsilon \otimes \text{id} : H \otimes V \to V$, we get $v(S(h)a) = 0$. Hence $S(H)a$ is contained in the annihilator $I$ of the $B$-module $V$, i.e., $a$ lies in the largest $S(H)$-stable ideal $IS(H)$ of $B$ contained in $I$. Since $V \neq 0$, we have $I \neq B$, but then $IS(H) = 0$ by the $S(H)$-simplicity of $B$.

This shows that $H \otimes V$ is a faithful $B$-module. Therefore there exists an element $t \in H \otimes V$ such that $tK \neq 0$. We have $tK \cong K$ since $K$ is a simple right $B$-module, and $t \in C \otimes V$ for some $C \in \mathcal{F}$ since $H$ is the union of finite-dimensional subcoalgebras. Then $tK \subseteq C \otimes V$, and we may take $U = C$, a subcomodule of $H$. \hfill \square

Lemma 4.6 Suppose that $Q$ is a semiprimary $H$-semiprime $H$-module algebra containing an $H$-stable subalgebra $A$ on which the action of $H$ is locally finite. Suppose also that $\mathcal{G}$ is a right Gabriel topology on $A$ such that all elements of $S(H)$ operate on $A$ as $\mathcal{G}$-continuous transformations and the following two properties hold:

(a) $\text{lann}_Q I = 0$ for each $I \in \mathcal{G}$,
(b) for each $q \in Q$ there exists $I \in \mathcal{G}$ such that $qI \subseteq A$.

Then all right $Q$-modules are $\mathcal{G}$-torsionfree as right $A$-modules, and therefore $IQ = Q$ for each $I \in \mathcal{G}$.

Proof If in (b) $q \neq 0$, then $qI \neq 0$ by (a). This shows that each nonzero right ideal of $Q$ has nonzero intersection with $A$. So the assumptions of Proposition 4.2 are satisfied, and we conclude that $Q \cong Q_1 \times \ldots \times Q_n$ where $Q_1, \ldots, Q_n$ are $S(H)$-simple $H$-module algebras by Lemma 4.3.

By (a) $Q$ is a $\mathcal{G}$-torsionfree right $A$-module. Hence so are all right ideals of $Q$. Let $M$ be any right $Q$-module. We have $M \cong M_1 \oplus \ldots \oplus M_n$ where $M_i$, for each $i$, is a $Q_i$-module on which $Q$ acts via the projection $Q \to Q_i$. To prove that $M$ is $\mathcal{G}$-torsionfree it suffices to consider the case when $M = M_i$ for some $i$.

Denote by $N$ the largest $\mathcal{G}$-torsion $A$-submodule of $M$. If $x \in N$ and $q \in Q$, then the coset $xq + N$ is annihilated by a right ideal in $\mathcal{G}$, according to (b). Since the $A$-module $M/N$ is $\mathcal{G}$-torsionfree, it follows that $Nq \subseteq N$ for each $q \in Q$. In other words, $N$ is a $Q$-submodule of $M$. Assuming $M$ to be a $Q_1$-module, we conclude that so is $N$.

Since $Q_i$ is semiprimary, it has a simple right ideal, say $K$. If $N \neq 0$, then, by Lemma 4.5, $K$ embeds in $U \otimes N$ for some right $H$-comodule $U$. In this case $K$ has to be $\mathcal{G}$-torsion by Lemma 4.4. But this is impossible since $K$ is isomorphic to a right ideal of $Q$, and therefore $K$ is $\mathcal{G}$-torsionfree, as we have observed already. Thus $N = 0$, and $M$ is indeed $\mathcal{G}$-torsionfree.

In particular, the right $Q$-module $Q/IQ$ is $\mathcal{G}$-torsionfree for any right ideal $I$ of $A$. On the other hand, $Q/IQ$ is $\mathcal{G}$-torsion whenever $I \in \mathcal{G}$ since the right $A$-modules $Q/A$ and $A/I$ are $\mathcal{G}$-torsion. In this case we must have $Q/IQ = 0$. \hfill \square
The conclusion of Lemma 4.6 implies that $Q$ is a perfect right localization of $A$ (see [26, Ch. XI, Th. 2.1]). However, the final goal has not been reached yet.

By Lemma 4.6 no nonzero element of a right $Q$-module is annihilated by a right ideal in $\mathcal{G}$. We will need a similar conclusion for left $Q$-modules. This will require more delicate arguments since $\mathcal{G}$ is not a left Gabriel topology. For a left $A$-module $M$ put

$$T_{\mathcal{G}}(M) = \{x \in M \mid x \text{ is annihilated by a right ideal in } \mathcal{G}\}.$$

For each right ideal $I \in \mathcal{G}$ the set $\text{Ann}_M I = \{x \in M \mid Ix = 0\}$ is a submodule of $M$. Since $\mathcal{G}$ is a filter, the set of all such submodules is directed by inclusion. Hence

$$T_{\mathcal{G}}(M) = \bigcup_{I \in \mathcal{G}} \text{Ann}_M I$$

is a submodule too. However, we cannot be sure that $T_{\mathcal{G}}(M)$ is a $Q$-submodule when $M$ is a left $Q$-module.

Note that $T_{\mathcal{G}}(M)$ is stable under all endomorphisms of $M$. In particular, if $D$ is a skew field contained in the endomorphism ring $\text{End}_A M$, then $T_{\mathcal{G}}(M)$ is a vector space over $D$.

**Lemma 4.7** In addition to the hypothesis of Lemma 4.6 assume that $\text{rann}_A I = 0$ for each $I \in \mathcal{G}$. Then $T_{\mathcal{G}}(M) = 0$ for each left $Q$-module $M$.

**Proof** If $I \in \mathcal{G}$, then $A \cap \text{rann}_Q I = \text{rann}_A I = 0$. Since $\text{rann}_Q I$ is a right ideal of $Q$ having zero intersection with $A$, we must have $\text{rann}_Q I = 0$. This shows that $T_{\mathcal{G}}(Q) = 0$, and therefore $T_{\mathcal{G}}(L) = 0$ for each left ideal $L$ of $Q$.

It is clear that $T_{\mathcal{G}}$ is a left exact functor. Thus for each exact sequence of left $Q$-modules

$$0 \to M' \to M \to M'' \to 0$$

there is an exact sequence of left $A$-modules

$$0 \to T_{\mathcal{G}}(M') \to T_{\mathcal{G}}(M) \to T_{\mathcal{G}}(M'')$$

and it follows that $T_{\mathcal{G}}(M) = 0$ whenever $T_{\mathcal{G}}(M') = 0$ and $T_{\mathcal{G}}(M'') = 0$. Since the Jacobson radical $J$ of $Q$ is nilpotent, each left $Q$-module has a finite chain of submodules with factors annihilated by $J$. Hence it suffices to prove that $T_{\mathcal{G}}(M) = 0$ when $JM = 0$. Since the factor ring $Q/J$ is semisimple Artinian, any such a module $M$ is semisimple. Since $T_{\mathcal{G}}$ is an additive functor, the proof of the equality $T_{\mathcal{G}}(M) = 0$ reduces further to the case when $M$ is simple.

There are finitely many isomorphism classes of simple $Q$-modules. Let $V_1, \ldots, V_p$ be a full set of pairwise nonisomorphic simple left $Q$-modules. For each $i$ the endomorphism ring $E_i = \text{End}_Q V_i$ is a skew field and $V_i$ is a finite-dimensional vector space over $E_i$. Then $T_{\mathcal{G}}(V_i)$ is a vector subspace of $V_i$. Put

$$\mu = \max_{i=1,\ldots,p} \frac{\dim_{E_i} T_{\mathcal{G}}(V_i)}{\dim_{E_i} V_i}.$$

We will show that $\mu = 0$. This will yield $T_{\mathcal{G}}(V_i) = 0$ for all $i$, and the proof of the lemma will be completed.

**Claim 1** Suppose that $M$ is a left $Q$-module and $D$ is a skew field contained in the endomorphism ring $\text{End}_Q M$. If $\dim_D M < \infty$, then

$$\dim_D T_{\mathcal{G}}(M) \leq \mu \cdot \dim_D M.$$

Consider first the case when $M$ is an isotypic semisimple left $Q$-module. In other words, $M$ is a direct sum of a possibly infinite family of copies of some simple module $V_i$. With
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$S = \text{Hom}_Q(V_i, M)$ we have $M \cong S \otimes_{E_i} V_i$ as $Q$-modules and as $\text{End}_Q M$-modules. Here $Q$ acts on $V_i$, while $\text{End}_Q M$ acts on $S$. It follows that

$$T_{\mathcal{G}}(M) \cong S \otimes_{E_i} T_{\mathcal{G}}(V_i)$$

as $\text{End}_Q M$-modules. We get $\dim_D M = (\dim_D S)(\dim_{E_i} V_i)$ and

$$\dim_D T_{\mathcal{G}}(M) = (\dim_D S) \left(\dim_{E_i} T_{\mathcal{G}}(V_i)\right).$$

The assumption $\dim_D M < \infty$ implies that $\dim_D S < \infty$, and the claim follows from the inequality

$$\dim_{E_i} T_{\mathcal{G}}(V_i) \leq \mu \cdot \dim_{E_i} V_i.$$

In the general case we proceed as follows. If $M'$ is any submodule of $M$ stable under the action of $D$, then $D$ embeds in $\text{End}_Q M'$ and in $\text{End}_Q M''$ where we put $M'' = M/M'$. Then

$$\dim_D M = \dim_D M' + \dim_D M'', \hspace{1cm} (**)$$

where the last inequality follows from the exact sequence of $D$-vector spaces ($\ast$). If Claim 1 is true for the $Q$-modules $M'$ and $M''$, it is clear that Claim 1 is true for $M$ as well.

We can use this argument with $M' = JM$. Since $J$ is nilpotent, verification of Claim 1 is thus reduced to the case when $JM = 0$, and so $M$ is semisimple. But then the isotypic components of $M$ are stable under all endomorphisms, and the proof reduces in a similar way to the case considered at the beginning. Thus Claim 1 has been verified.

Claim 2 If in Claim 1 the equality $\dim_D T_{\mathcal{G}}(M) = \mu \cdot \dim_D M$ is attained, then

$$\dim_D T_{\mathcal{G}}(M') = \mu \cdot \dim_D M'$$

for any $Q$-submodule $M'$ of $M$ stable under the action of $D$.

Indeed, $\dim_D T_{\mathcal{G}}(M') \leq \mu \cdot \dim_D M'$ and $\dim_D T_{\mathcal{G}}(M'') \leq \mu \cdot \dim_D M''$ where $M'' = M/M'$. If one of these two inequalities were strict, then we would get

$$\dim_D T_{\mathcal{G}}(M) < \mu \cdot \dim_D M$$

from (**)\), a contradiction. Thus both inequalities are in fact equalities.

Claim 3 Suppose that $B$ is an $S(H)$-simple $H$-module algebra, $L$ is a simple left ideal of $B$, and $V$ is any nonzero left $B$-module. There exists a finite-dimensional right $S(H)$-comodule $U$ such that $L$ embeds as a $B$-submodule in $V \otimes U$ with the twisted left $B$-module structure.

This is an analog of Lemma 4.5 with a similar proof. Considering $S(H)$ as a right $H$-comodule with respect to the comultiplication in $H$, the twisted left $B$-module $V \otimes S(H)$ is faithful. Hence $L$ embeds in $V \otimes S(H)$, and therefore in $V \otimes U$ where $U = S(C)$ for some finite-dimensional subcoalgebra $C$ of $H$.

We are now in a position to complete the proof of Lemma 4.7. Among the simple left $Q$-modules $V_1, \ldots, V_p$ we pick $V_j$ such that

$$\dim_{E_j} T_{\mathcal{G}}(V_j) = \mu \cdot \dim_{E_j} V_j.$$

Let $Q_1, \ldots, Q_n$ be the $H$-simple direct factors of $Q$ given by Proposition 4.2. Then $V_j$ is a $Q_i$-module for some $i$. Since $Q_i$ is semiprimary, it has a simple left ideal, say $L$. By Lemma 4.3 $Q_i$ is $S(H)$-simple. Therefore Claim 3 shows that $L$ is isomorphic to a submodule of the twisted $Q_i$-module $M = V_j \otimes U$ for some finite-dimensional right $S(H)$-comodule $U$. 

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Now note that $T_{\mathcal{G}}(V_j) \otimes U \subset T_{\mathcal{G}}(M)$. Indeed, if $v \in T_{\mathcal{G}}(V_j)$ and $u \in U$, then $Iv = 0$ for some $I \in \mathcal{G}$ and $\sum u(0) \otimes u(1) \in U \otimes S(C)$ for some $C \in \mathcal{F}$. It follows then from the formula for the twisted action of $Q$ in $V_j \otimes U$ that $v \otimes u$ is annihilated by the right ideal $I_{S(C)}$ of $A$. Since $I_{S(C)} \in \mathcal{G}$, we get $v \otimes u \in T_{\mathcal{G}}(M)$.

The skew field $E_j = \text{End}_Q V_j$ embeds in $\text{End}_Q M$ in a natural way, and

$$\dim_{E_j} M = (\dim_{E_j} V_j) (\dim_k U),$$

$$\dim_{E_j} T_{\mathcal{G}}(M) \geq (\dim_{E_j} T_{\mathcal{G}}(V_j)) (\dim_k U).$$

Hence $\dim_{E_j} T_{\mathcal{G}}(M) \geq \mu \cdot \dim_{E_j} M$. By Claim 1 the opposite inequality is also true, and so we must have an equality here. But then Claim 2 shows that

$$\dim_{E_j} T_{\mathcal{G}}(M') = \mu \cdot \dim_{E_j} M'$$

for each $Q$-submodule $M'$ of $M$ stable under the action of $E_j$.

Take $M'$ to be the sum of all $Q$-submodules of $M$ isomorphic to $L$. Obviously, $M'$ is stable under all endomorphisms of $M$, and so the previous equality must hold. Note that $M' \neq 0$. On the other hand, $L$ is isomorphic to a right ideal of $Q$. As we have seen, this entails $T_{\mathcal{G}}(L) = 0$. Since $M'$ is an isotypic semisimple $Q$-module, we get $T_{\mathcal{G}}(M') = 0$. It follows that $\mu = 0$, and we are done.

Lemma 4.8 Let $R$ be a semiprime right Goldie subring of a semisimple Artinian ring $S$. Suppose that $\mathcal{G}$ is a set of right ideals of $R$ with the two properties:

(a) $\text{lann}_S I = 0$ for each $I \in \mathcal{G}$,
(b) for each $x \in S$ there exists $I \in \mathcal{G}$ such that $xI \subset R$.

Then $S$ is a classical right quotient ring of $R$, and each right ideal $I \in \mathcal{G}$ contains a regular element of $R$.

Proof By the Goldie theorem the ring $R$ has a semisimple Artinian classical right quotient ring $Q$. It is known from the proof of Goldie’s theorem that $Q$ is the localization of $R$ with respect to the filter $\mathcal{E}$ of all essential right ideals of $R$, and a right ideal of $R$ is essential if and only if it contains a regular element of $R$.

Considering $S$ as a right $R$-module, denote by $T$ the set of all elements $x \in S$ whose right annihilator in $R$ belongs to $\mathcal{E}$. Then $T$ is a right $R$-submodule of $S$, and $T \cap R = 0$ since no nonzero element of $R$ is annihilated by a regular element of $R$. On the other hand, (a) and (b) imply that each nonzero right $R$-submodule of $S$ has nonzero intersection with $R$. Hence $T = 0$.

Thus we have shown that $\text{lann}_S I = 0$ for each $I \in \mathcal{E}$. In other words, $\text{lann}_S u = 0$ for each regular element $u$ of $R$, so that all regular elements of $R$ are left regular in $S$. But left regular elements of a right or left Artinian ring are invertible. We conclude that regular elements of $R$ are invertible in $S$.

By the universality property of Ore localizations, the embedding $R \to S$ extends to a ring homomorphism $\varphi : Q \to S$, and $\varphi$ is injective since $\text{Ker} \varphi \cap R = 0$. So $Q$ is identified with a subring of $S$.

Let $I \in \mathcal{G}$. By (a) $I$ has zero left annihilator in $Q$. Since $Q$ is semisimple Artinian, its right ideal $IQ$ is generated by an idempotent, say $e$. Noting that $(1 - e)I = 0$, we deduce that $e = 1$. Thus $IQ = Q$, which means precisely that $I$ contains a regular element of $R$.

It follows now from (b) that for each $x \in S$ we have $xu \in R$ for some regular element $u$ of $R$, whence $x = (xu)u^{-1} \in Q$. We conclude that $S = Q$. 

\[ \square \]
Proposition 4.9 Suppose that $Q$ is a semiprimary $H$-semiprime $H$-module algebra containing a right Noetherian $H$-stable subalgebra $A$ on which the action of $H$ is locally finite. Suppose also that $\mathcal{G}$ is a right Gabriel topology on $A$ such that all elements of $S(H)$ operate on $A$ as $\mathcal{G}$-continuous transformations and the following two properties hold:

(a) $\text{lann}_Q I = 0$ and $\text{rann}_A I = 0$ for each $I \in \mathcal{G}$,
(b) for each $q \in Q$ there exists $I \in \mathcal{G}$ such that $qI \subset A$.

Then $Q$ is a classical right quotient ring of $A$.

Proof We proceed in several steps.

Claim 1 If $M$ is any maximal ideal of $Q$, then $M \cap A$ is a prime ideal of $A$.

Recall that for any nonzero right module over a right Noetherian ring the set of annihilators of nonzero submodules contains a prime ideal. For example, all maximal elements of this set are prime ideals [5, Prop. 3.12]. Let $P$ be any prime annihilator of a nonzero submodule of the right $A$-module $Q/M$. Thus $P$ is a prime ideal of $A$ such that $M \cap A \subset P$ and $yP \subset M$ for some $y \in Q$, $y \notin M$.

For each $I \in \mathcal{G}$ put

$L(I) = \{ x \in Q \mid xIP \subset M \}$.

It is clear that $L(I)$ is a left ideal of $Q$ and $M \subset L(I)$, so that $L(I)/M$ is a left ideal of the factor ring $Q/M$. Since the simple ring $Q/M$ is Artinian, the set of left ideals $\{ L(I) \mid I \in \mathcal{G} \}$ has a maximal element, say $L_0$. Moreover, since the correspondence $I \mapsto L(I)$ reverses inclusions, this set is directed by inclusion, whence $L_0$ is in fact its largest element. Thus $L(I) \subset L_0$ for each $I \in \mathcal{G}$.

Note that $L(A) = \{ x \in Q \mid xP \subset M \}$ is a right $A$-submodule of $Q$. Let $x \in L(A)$ and $q \in Q$. By (b) there exists $I \in \mathcal{G}$ such that $qI \subset A$. Then

$xqIP \subset xAP \subset xP \subset M$,

yielding $xq \in L(I) \subset L_0$. This shows that $L(A)Q \subset L_0$. Note that $L(A)Q$ is a two-sided ideal of $Q$ containing $M$. But $L(A) \not\subset M$ by the choice of $P$, whence $L(A)Q \not\subset M$. Since $Q/M$ is a simple ring, we must have $L(A)Q = Q$. It follows that $L_0 = Q$ too.

Now pick $I \in \mathcal{G}$ such that $L_0 = L(I)$. Since $1 \in L_0$, we get $IP \subset M$. This means that the image of $P$ in $Q/M$ is contained in the left $A$-submodule $T_{\mathcal{G}}(Q/M)$, in the notation of Lemma 4.7. By that lemma $T_{\mathcal{G}}(Q/M) = 0$, which entails $P \subset M$. Hence $P = M \cap A$, and Claim 1 is thus proved.

Claim 2 Denote by $N$ the prime radical of $A$ and by $J$ the Jacobson radical of $Q$. Then $N = J \cap A$.

Let $M_1, \ldots, M_k$ be all the maximal ideals of $Q$. Then $J = \bigcap M_i$. By Claim 1 $P_i = M_i \cap A$ is a prime ideal of $A$ for each $i$. We have $J \cap A = N'$ where $N' = \bigcap P_i$. Since $J$ is nilpotent, $N'$ is a nilpotent ideal of $A$. Hence $N'$ is contained in each prime ideal of $A$, and therefore $N' = N$.

Claim 3 The factor ring $Q/J$ is a classical right quotient ring of $A/N$.
The ring $S = Q/J$ is semisimple Artinian, while $R = A/N$ is semiprime right Noetherian. Let $\pi : Q \to S$ be the canonical homomorphism. By Claim 2 $R$ is identified with the subring $\pi(A)$ of $S$.

Consider the set $\{\pi(I) \mid I \in \mathcal{G}\}$ of right ideals of $R$. If $q \in Q$, then it follows from (b) that $\pi(q)\pi(I) \subseteq \pi(A)$ for some $I \in \mathcal{G}$. For each $I \in \mathcal{G}$ we have $IQ = Q$ by Lemma 4.6, whence $\pi(I)S = S$, and therefore $\pi(I)$ has zero left annihilator in $S$. Thus we meet the hypothesis of Lemma 4.8, and Claim 3 follows.

Denote by $C$ the set of all elements $u \in A$ which are regular modulo $N$, i.e., whose images $\pi(u)$ in the ring $\pi(A) \cong A/N$ are regular elements of that ring. If $u \in C$, then $\pi(u)$ is invertible in $Q/J$ by Claim 3; this implies that $u$ is invertible in $Q$ since $J$ is the Jacobson radical of $Q$. This shows that all elements of $C$ are regular in $A$. Conversely, each regular element of a right Noetherian ring is regular modulo the prime radical (see [5, Lemma 11.8] or [11, 4.1.3]). Thus $C$ is the set of all regular elements of $A$.

By Lemma 4.8 each right ideal in the set $\{\pi(I) \mid I \in \mathcal{G}\}$ contains a regular element of the ring $\pi(A)$. Therefore $I \cap C \neq \emptyset$ for all $I \in \mathcal{G}$. It follows now from condition (b) that for each $q \in Q$ there exists $u \in C$ such that $qu \in A$. We have seen already that all elements of $C$ are invertible in $Q$. These properties characterize $Q$ as a classical right quotient ring of $A$.

### 5 Final results

In the first result of this section we complete the work on Noetherian $H$-semiprime $H$-module algebras done in the preceding sections. This will then be used to derive results on bijectivity of the antipode and on flatness over coideal subalgebras.

**Theorem 5.1** Let $A$ be a right Noetherian $H$-semiprime $H$-module algebra such that the action of $H$ on $A$ is locally finite. Then $A$ has a right Artinian classical right quotient ring.

**Proof** Let $Q$ be the quotient ring of $A$ with respect to the filter of right ideals $\mathcal{G} = \mathcal{E}'_H$. By Proposition 2.4 $\mathcal{G}$ is a Gabriel topology and $A$ is $\mathcal{G}$-torsionfree. Since $Q$ is an essential extension of $A$ in the category of right $A$-modules, $Q$ is $\mathcal{G}$-torsionfree as well. Combined with Corollary 2.10 this amounts to condition (a) in the statement of Proposition 4.9. Condition (b) is satisfied by the construction of $Q$. By Proposition 3.2 $Q$ is semiprimary and $H$-semiprime. Thus the assumptions of Proposition 4.9 are fulfilled, and therefore $Q$ is a classical right quotient ring of $A$. Since $A$ is right Noetherian, so too is $Q$. Since $Q$ is also semiprimary, it has to be right Artinian (see [26, Ch. VIII, Prop. 1.12]).

From [24, Th. 1.1] we deduce that the quotient ring $Q$ in Theorem 5.1 is quasi-Frobenius, but this fact will not be needed.

The dual Hopf algebra $H^\circ$ consists of all linear functions $H \to k$ vanishing on an ideal of finite codimension in $H$. There is an action of $H^\circ$ on $H$ defined by the rule

$$f \mapsto h = \sum f(h_{(2)}) h_{(1)}, \quad f \in H^\circ, \; h \in H.$$  

It makes $H$ into a left $H^\circ$-module algebra. Right coideals of $H$ are stable under this action of $H^\circ$. Since each element of $H$ is contained in a finite-dimensional subcoalgebra, the action of $H^\circ$ on $H$ is locally finite.
Lemma 5.2 Suppose that $H$ is a residually finite-dimensional Hopf algebra. Then $H$ is an $H^\circ$-simple $H^\circ$-module algebra. Each right coideal subalgebra $A$ of $H$ is an $H^\circ$-prime $H^\circ$-module algebra.

Proof Since $H$ is residually finite dimensional, the $H^\circ$-submodules of $H$ are precisely the right coideals. Now, if $I$ is a right ideal of $H$ such that $\Delta(I) \subset I \otimes H$, then $I$ may be regarded as a Hopf module. By the structure of Hopf modules (see [12, 1.9.4])

$I = I^{\text{co}H}H$ where $I^{\text{co}H} = \{h \in I \mid \Delta(h) = h \otimes 1\}$.

Since $H^{\text{co}H} = k$, we deduce that $I^{\text{co}H}$ equals either 0 or $k$. Hence either $I = 0$ or $I = H$. In other words, 0 and $H$ are the only two $H^\circ$-stable right ideals of $H$. In particular, $H$ is $H^\circ$-simple.

Any right coideal subalgebra $A$ of $H$ is an $H^\circ$-stable subalgebra, and so is itself an $H^\circ$-module algebra. Suppose that $I$ is a nonzero $H^\circ$-stable ideal of $A$. Then $IH$ is an $H^\circ$-stable ideal of $H$, whence $IH = H$. If $J$ is another nonzero $H^\circ$-stable ideal of $A$, then $JH = H$ too. It follows that $IJH = H$, and therefore $IJ \neq 0$. Thus $A$ is $H^\circ$-prime.

Theorem 5.3 Let $H$ be either right or left Noetherian residually finite-dimensional Hopf algebra. Then its antipode $S : H \to H$ is bijective. Hence $H$ is right and left Noetherian simultaneously.

Proof According to [20, Th. A] $S$ is bijective whenever $H$ can be embedded into a left perfect ring $Q$ such that $Q$ is an essential extension of $H$ as a right $H$-module. By Lemma 5.2 $H$ is $H^\circ$-simple, and therefore $H^\circ$-semiprime, as an $H^\circ$-module algebra. If $H$ is right Noetherian, the required embedding is provided already by Proposition 3.2 (semiprimary rings are left perfect).

If $H$ is left Noetherian, we consider the Hopf algebra $H^{\text{op},\text{cop}}$ obtained from $H$ by taking the opposite multiplication and comultiplication. It has the same antipode $S$, but is right Noetherian. So we can refer to the case already treated.

Bijectivity of $S$ implies that $S$ is an antiautomorphism of $H$ as a Hopf algebra. In particular, $H \cong H^{\text{op}}$ as algebras. Therefore the right hand properties of $H$ are equivalent to the left hand ones.

In the next theorem we repeat results which can be found in [23, Th. 1.8, Cor. 1.9]. However, we present only the part concerned with flatness and provide a proof which bypasses the category equivalences considered in [23].

Theorem 5.4 Let $A$ be a right coideal subalgebra of a residually finite-dimensional Hopf algebra $H$. Suppose that $A$ and $H$ have right Artinian classical right quotient rings $Q(A)$ and $Q(H)$. Then $H$ is left $A$-flat. Moreover, if $A$ is a Hopf subalgebra, then $H$ is left and right faithfully $A$-flat.

Proof First we note that the embedding $H \hookrightarrow Q(H)$ enables us to apply [20, Th. A] and conclude that the antipode $S : H \to H$ is bijective. This fact will be used without further notice. Next, the action of $H^\circ$ on $A$ and $H$ extends to $Q(A)$ and $Q(H)$ by [25, Th. 2.2].
Claim 1 \( Q(A) \) is an \( H^\circ \)-simple \( H^\circ \)-module algebra.

If \( I \) and \( J \) are two nonzero \( H^\circ \)-stable ideals of \( Q(A) \), then \( I \cap A \) and \( J \cap A \) are nonzero \( H^\circ \)-stable ideals of \( A \). Since \( A \) is \( H^\circ \)-prime by Lemma 5.2, we deduce that \( IJ \neq 0 \). Thus \( Q(A) \) is \( H^\circ \)-prime.

Since the action of \( H^\circ \) on \( A \) is locally finite, we can now apply Proposition 4.2. It shows that \( Q(A) \) is a direct product of finitely many \( H^\circ \)-simple \( H^\circ \)-module algebras \( Q_1, \ldots, Q_n \). Since \( Q(A) \) is \( H^\circ \)-prime, we cannot have \( n > 1 \). Hence \( n = 1 \), and so \( Q(A) = Q_1 \) is indeed \( H^\circ \)-simple.

Claim 2 The inclusion \( A \hookrightarrow H \) extends to a ring homomorphism \( Q(A) \to Q(H) \).

This is a special case of [23, Lemma 1.7]. We do not offer any improvements in its proof.

Let further \( V \) be a right \( A \)-module, and put \( M = V \otimes H \). We will view \( M \) as an \( H^\circ \)-equivariant right \( A \)-module with the actions of \( A \) and \( H^\circ \) defined as follows:

\[
(v \otimes h) a = \sum v a(1) \otimes h a(2), \quad f \to (v \otimes h) = v \otimes (f \to h)
\]

for \( v \in V, h \in H, a \in A \) and \( f \in H^\circ \).

In this way \( \otimes H \) becomes a functor from the category of right \( A \)-modules to the category of \( H^\circ \)-equivariant right \( A \)-modules. Note that the action of \( H^\circ \) on \( M \) is locally finite since so is the action of \( H^\circ \) on \( H \).

Claim 3 The right \( Q(A) \)-module \( M \otimes_A Q(A) \) is projective. Moreover, \( M^n \otimes_A Q(A) \) is a free \( Q(A) \)-module for some integer \( n > 0 \).

Since \( Q(A) \) is an extension of \( A \) in the category of \( H^\circ \)-module algebras, there is a well-defined action of \( H^\circ \) on \( M \otimes_A Q(A) \) such that

\[
f \to (x \otimes q) = \sum (f(1) \to x) \otimes (f(2) \to q) \quad \text{for } f \in H^\circ, x \in M, q \in Q(A).
\]

It makes \( M \otimes_A Q(A) \) an \( H^\circ \)-equivariant right \( Q(A) \)-module. If \( U \) is any \( H^\circ \)-stable subspace of \( M \), then the \( A \)-submodule \( U A \) generated by \( U \) is \( H^\circ \)-stable too, and

\[
F_U = (U A) \otimes_A Q(A)
\]

is an \( H^\circ \)-equivariant right \( Q(A) \)-module which may be identified with a submodule of \( M \otimes_A Q(A) \) since \( Q(A) \) is left \( A \)-flat by standard properties of classical quotient rings. If \( \dim U < \infty \), then \( F_U \) is finitely generated. Each \( H^\circ \)-equivariant finitely generated right \( Q(A) \)-module is projective since \( Q(A) \) is \( H^\circ \)-simple, so that Lemma 4.1 can be applied with a suitable choice of maximal ideal \( P \). Thus

\[
\{ F_U \mid U \text{ a finite-dimensional } H^\circ \text{-submodule of } M \}
\]

is a directed set of projective submodules of the right \( Q(A) \)-module \( M \otimes_A Q(A) \). The union of this family of submodules gives the whole module since the action of \( H^\circ \) on \( M \) is locally finite. We arrive at the first conclusion of Claim 3, observing that inductive direct limits of flat modules are flat and that all flat right modules over a right Artinian ring are projective.

Take \( n \) to be the greatest common divisor of the lengths of simple factor rings of the right Artinian ring \( Q(A) \). It follows from Lemma 4.1 and the Krull-Schmidt Theorem that for each \( H^\circ \)-equivariant finitely generated right \( Q(A) \)-module \( K \) the \( Q(A) \)-module \( K^n \) is free for exactly this value of \( n \) which does not depend on \( K \). A basis of \( M^n \otimes_A Q(A) \) over \( Q(A) \) can then be obtained by a suitable application of Zorn’s Lemma (see [21, Th. 7.6]).
Flatness of Noetherian Hopf algebras over coideal subalgebras

Denote by \( \left( \sum va_{(1)} \otimes g a_{(2)} \otimes h - v \otimes g \otimes ah \mid v \in V, \ a \in A, \ g, h \in H \right) \).

Claim 4 There is an isomorphism of right \( H \)-modules \( M \otimes_A H \cong (V \otimes_A H) \otimes H \) where \( H \) is assumed to act by right multiplications on the last tensorands.

We have \( M \otimes_A H \cong (V \otimes H \otimes H) / R \) where \( R \) is the subspace of \( V \otimes H \otimes H \) spanned by
\[
\left\{ \sum va_{(1)} \otimes g a_{(2)} \otimes h - v \otimes g \otimes ah \mid v \in V, \ a \in A, \ g, h \in H \right\}.
\]

Denote by \( R' \) the subspace of \( V \otimes H \otimes H \) spanned by
\[
\left\{ va \otimes g \otimes h - \sum v \otimes g S^{-1}(a_{(2)}) \otimes a_{(1)}h \mid v \in V, \ a \in A, \ g, h \in H \right\}.
\]

In fact \( R' \subset R \) since
\[
va \otimes g \otimes h = \sum va_{(1)} \otimes g S^{-1}(a_{(3)})a_{(2)} \otimes h \equiv \sum v \otimes g S^{-1}(a_{(2)}) \otimes a_{(1)}h
\]
modulo \( R \), and \( R \subset R' \) since
\[
v \otimes g \otimes ah = \sum v \otimes ga_{(3)}S^{-1}(a_{(2)}) \otimes a_{(1)}h \equiv \sum va_{(1)} \otimes ga_{(2)} \otimes h
\]
modulo \( R' \). Hence \( R = R' \), and therefore
\[
M \otimes_A H \cong (V \otimes H \otimes H) / R' \cong V \otimes_A X
\]
where \( X = H \otimes H \) regarded as a left \( A \)-module with respect to the action of \( A \) defined by the rule
\[
a \mapsto (g \otimes h) = \sum g S^{-1}(a_{(2)}) \otimes a_{(1)}h, \quad a \in A, \ g, h \in H.
\]

The linear transformation \( \xi \) of \( H \otimes H \) defined by \( \xi(g \otimes h) = \sum S^{-1}(g_{(2)}) \otimes g_{(1)}h \) has the inverse transformation \( g \otimes h \mapsto \sum S(g_{(1)}) \otimes g_{(2)}h \). Since
\[
\xi(ag \otimes h) = \sum S^{-1}(g_{(2)})S^{-1}(a_{(2)}) \otimes a_{(1)}g_{(1)}h = a \mapsto (\xi(g \otimes h))
\]
\( \xi \) gives an isomorphism of \( A \)-modules \( Y \cong X \) where \( Y = H \otimes H \) with the action of \( A \) by left multiplications on the first tensorand. It follows that
\[
M \otimes_A H \cong V \otimes_A X \cong V \otimes_A Y \cong (V \otimes_A H) \otimes H.
\]

Since \( \xi \) is right \( H \)-linear with respect to the action of \( H \) by right multiplications on the second tensorand of \( H \otimes H \), we get an isomorphism of right \( H \)-modules, as stated in Claim 4.

We are ready now to verify flatness of \( H \) over \( A \). Let \( 0 \to V' \to V \to V'' \to 0 \) be an exact sequence of right \( A \)-modules. It gives rise to an exact sequence of right \( A \)-modules
\[
0 \to M' \to M \to M'' \to 0
\]
with the action of \( A \) as specified earlier in the case of \( M \). Since the right quotient ring \( Q(A) \) is left \( A \)-flat, the sequence
\[
0 \to M' \otimes_A Q(A) \to M \otimes_A Q(A) \to M'' \otimes_A Q(A) \to 0
\]
is exact as well. This sequence of right \( Q(A) \)-modules splits since all terms in it are projective by Claim 3. Applying the functor \( ? \otimes_{Q(A)} Q(H) \), we get an exact sequence
\[
0 \to M' \otimes_A Q(H) \to M \otimes_A Q(H) \to M'' \otimes_A Q(H) \to 0.
\]
Note that $M \otimes_A Q(H) \cong (M \otimes_A H) \otimes_H Q(H) \cong (V \otimes_A H) \otimes Q(H)$ in view of Claim 4, and this isomorphism is functorial in $V$. Therefore the previous exact sequence can be rewritten as

$$0 \to (V' \otimes_A H) \otimes Q(H) \to (V \otimes_A H) \otimes Q(H) \to (V'' \otimes_A H) \otimes Q(H) \to 0.$$ 

Since the final tensoring in all terms here is performed over the ground field, we deduce that the sequence $0 \to V' \otimes_A H \to V \otimes_A H \to V'' \otimes_A H \to 0$ is exact. Thus the functor $? \otimes_A H$ on the category of right $A$-modules is exact, which means that $H$ is indeed left $A$-flat.

There remains the question of faithful flatness. Dealing with it is based on the following observation:

**Claim 5** If $V \neq 0$, but $V \otimes_A H = 0$, then $A$ has an $H^\circ$-stable right ideal $I$ such that $I \neq 0$ and $I \neq A$.

Let $M$ be as defined earlier. Applying Claim 4, we obtain $M \otimes_A H = 0$. It follows that $M \otimes_A Q(H) = 0$, and $F \otimes_{Q(A)} Q(H) = 0$ where we put $F = M \otimes_A Q(A)$. This is only possible when $F = 0$ since $F^n$ is a free $Q(A)$-module for some $n > 0$, according to Claim 3. Thus $M \otimes_A Q(A) = 0$. This means that any finite subset of $M$ is annihilated by a regular element of $A$.

Now recall that $M$ is an $H^\circ$-equivariant right $A$-module and the action of $H^\circ$ on $M$ is locally finite. Since $M \neq 0$, there exists a finite-dimensional $H^\circ$-submodule $0 \neq U \subset M$. Put $I = \{a \in A \mid Ua = 0\}$. Then $I$ is an $H^\circ$-stable right ideal of $A$. It contains a regular element of $A$ since $I$ coincides with the annihilator of any finite basis of $U$. Hence $I \neq 0$.

On the other hand, $1 \notin I$ since $U \neq 0$. Thus Claim 5 has been verified.

Suppose now that $A$ is a Hopf subalgebra. Recall that the $H^\circ$-submodules of $H$ are precisely the right coideals of $H$. Since $\Delta(A) \subset A \otimes A$, the $H^\circ$-submodules of $A$ are precisely the right coideals of $A$. By the Hopf module argument recalled in the proof of Lemma 5.2 any Hopf algebra contains no nontrivial right ideals which are simultaneously right coideals. This means that $A$ has no $H^\circ$-stable right ideals other than 0 and the whole $A$. Then, by Claim 5, $V \otimes_A H \neq 0$ for each nonzero right $A$-module $V$. We have seen already that $H$ is left $A$-flat, and so we obtain faithful flatness on the left.

We have mentioned that $S : H \to H$ is bijective. Since $A$ satisfies the same assumptions as $H$, its antipode is also bijective. But the antipode of $A$ is the restriction of $S$ to $A$. Thus $S$ is an antiautomorphism of $H$ mapping $A$ onto itself. From this it is clear that $H$ is faithfully $A$-flat on both sides.

**Remark** By a result of Masuoka and Wigner [10, Th. 2.1] for any Hopf algebra $H$ with bijective antipode flatness of $H$ over its Hopf subalgebra $A$ implies faithful flatness. However, this implication was proved in [10] via a chain of several equivalent conditions. In the proof of Theorem 5.4 we have avoided that argument, using Claim 5 instead.

**Theorem 5.5** Let $A$ be a right Noetherian right coideal subalgebra of a residually finite-dimensional Noetherian Hopf algebra $H$. Then $A$ has a right Artinian classical right quotient ring, and $H$ is left $A$-flat. Moreover, if $A$ is a Hopf subalgebra, then $H$ is left and right faithfully $A$-flat.

**Proof** By Lemma 5.2 $A$ is an $H^\circ$-prime $H^\circ$-module algebra. Since the action of $H^\circ$ on $A$ is locally finite, we can apply Theorem 5.2 and conclude that $A$ has a right Artinian classical right quotient ring $Q(A)$. The same result can be applied to $H$ in place of $A$. 

\[\square\]
Thus we meet the hypothesis of Theorem 5.4, and everything follows.

**Corollary 5.6** Retain all assumptions of Theorem 5.5. If $A$ is a Hopf subalgebra, then $H$ is a projective generator in the categories of right and left $A$-modules.

**Proof** This follows from a result of Schneider [18, Cor. 1.8]. See also Masuoka and Wigner [10, Th. 2.1, Cor. 2.9].

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