Chaotic Mixing in Galactic Dynamics

Henry E. Kandrup

Department of Astronomy, Department of Physics, and Institute for Fundamental Theory, University of Florida, Gainesville, FL 32611

Abstract. This talk summarises what is currently understood about the phenomenon that has come to be known as chaotic mixing. The first part presents a concise statement as to what chaotic mixing actually is, and then explains why it should be important in galactic dynamics. The second discusses in detail what is currently known, describing the manifestations of chaotic mixing both for flows in time-independent Hamiltonian systems and for systems subjected to comparatively weak time-dependent perturbations. The third part discusses what one might expect if one were to allow for a strongly time-dependent potential, including possible implications for violent relaxation.

1. What Is Chaotic Mixing and Why Does it Matter?

Initially localised ensembles of initial conditions corresponding to chaotic orbits will, when integrated into the future, disperse exponentially at a rate set by the largest (short time) Lyapunov exponents $\chi$. By contrast, ensembles of initial conditions corresponding to regular orbits will only disperse as a power law in time. This much is obvious: After all, the defining characteristic of chaotic orbits is that they exhibit exponentially sensitive dependence on initial conditions. Not so obvious, perhaps, but also true is the following: At least in time-independent potentials, initially localised ensembles of chaotic initial conditions tend to evolve exponentially in time towards an invariant or near-invariant distribution, i.e., an equilibrium or near-equilibrium (cf. Kandrup & Mahon 1994, Mahon et al 1995, Kandrup 1998). In particular, as probed by lower order moments or by coarse-grained distribution functions, an initially localised ensemble of chaotic orbits typically evolves exponentially in time towards a near-uniform population of those phase space regions that are easily accessible, i.e., not impeded by cantori or an Arnold web.

This means that phase mixing can proceed much more efficiently for chaotic flows than for regular flows, where any approach towards a (near-) equilibrium typically proceeds as a power law in time. Chaotic flows should relax much more efficiently than do regular flows. It would thus seem that the phase mixing of chaotic flows, a phenomenon which Merritt & Valluri (1996) have termed chaotic mixing, could serve to provide an explanation of why various systems in
nature seem to approach an equilibrium or near-equilibrium as fast they do. In particular, chaotic mixing could help explain the remarkable efficacy of violent relaxation: Why do galaxies look ‘so relaxed’ when the nominal relaxation time \( t_R \) is typically much longer than \( t_H \), the age of the Universe? And, in particular, why can galaxies recover from a collision or close encounter as quickly and as efficiently as they apparently do?

2. Manifestations of Chaotic Mixing

2.1. Time-independent Hamiltonian systems

A localised ensemble of initial conditions corresponding to chaotic orbits will, when evolved into the future, begin by diverging in such a fashion that quantities like the dispersions in position or velocity diverge exponentially at a rate set by a characteristic short time Lyapunov exponent for the orbits, so that, e.g., \( \sigma_x(t) \propto \exp(+\chi t) \). Eventually, however, this divergence saturates, so that such quantities asymptote towards a near-constant value, which would suggest that the ensemble is approaching some equilibrium, or near-equilibrium, state. That this is actually so can be demonstrated by tracking the evolution of coarse-grained distribution functions. Thus, e.g., orbital data at fixed instants of time can be binned onto a \( k \times k \) grid so as to construct coarse-grained distribution functions like \( f(x,y,t) \) or \( f(x,v_x,t) \), and one can then ask whether it be true that such \( f(t) \)'s converge towards nearly time-independent distributions \( f_{niv} \). To address this question, one requires a definition of ‘distance’ between two distributions \( f_1 \) and \( f_2 \). The obvious choice entails a ‘pixel by pixel’ comparison in terms of a discrete \( L^p \) norm, so that, e.g., the distance between \( f_{niv}(x,y) \) and some \( f(x,y,t) \) satisfies

\[
Df(x,y,t) = \left[ \frac{\sum_a \sum_b |f(x,y,t) - f_{niv}(x,y)|^p}{\sum_a \sum_b |f_{niv}(x,y)|^p} \right]^{1/p}
\]

with \( p = 1 \) or 2.

Numerical experiments (Kandrup & Mahon 1994, Mahon et al 1995, Merritt & Valluri 1996, Kandrup 1998) indicate that, with respect to this measure of distance, \( f(t) \) does indeed converge towards an invariant, or near-invariant, \( f_{niv} \); and that, at least early on, this convergence is well fit by an exponential, i.e.,

\[
Df(t) \propto \exp(-\Lambda t).
\]

This is illustrated in Figure 1, which exhibits \( \sigma_x(t) \) and \( Df(x,y,t) \) for two different ensembles evolved in the three-dimensional dihedral potential (cf. Kandrup 1998). The ‘saturation’ observed in the plots of \( Df(t) \) is (at least primarily) a finite \( N \) effect: even if the coarse-grained \( f(t) \) and \( f_{niv} \) involved two different \( N \)-orbit samplings of the same smooth distribution, they would necessarily differ because of finite number statistics. Because the rate at which moments like \( \sigma_x \) grow is set by the Lyapunov exponents, one might speculate that \( \Lambda \) should also be related to \( \chi \). However, there does not always seem to be a clean one-to-one connection. One does find numerically that larger \( \chi \) correlates with larger \( \Lambda \), and that \( \chi \) is always larger than \( \Lambda \), i.e., the rate at which orbits diverge is larger
Figure 1. $\sigma_x(t)$ and $Df(x,y,t)$ for two different ensembles of chaotic orbits, with $E = 1$ and $E = 4$, evolved in the three-dimensional dihedral potential with $a = b = 1$.

than the rate at which they approach an equilibrium. However, there does not appear to exist a universal connection between $\chi$ and $\Lambda$ that holds for generic potentials admitting a coexistence of both regular and chaotic orbits. This is unlike the case of idealised systems like $K$-flows where (cf. Anosov 1967) $\Lambda$ and $\chi$ are directly related. Why this is the case is not obvious. However, it seems reasonable to conjecture that it is the nontrivial character of the phase space, i.e., the coexistence of both regular and chaotic orbits and the existence of topological obstructions such as the Arnold web, that renders phase mixing somewhat less efficient and substantially more complex than is the case for $K$-flows.

One thing that is evident is that the value of $\Lambda$ can depend on the details. For example, $\Lambda$ can depend on the choice of phase space coordinates that are being probed. Orbits can disperse faster in some directions than others, and this is reflected in the fact that convergence towards $f_{\text{niu}}$ can proceed at different rates for difference phase space surfaces. Thus, e.g., $f(x,y,t)$ and $f(x,z,t)$ can approach a (near-)equilibrium at very different rates. Similarly, the value of $\Lambda$ can depend on the level of coarse-graining. Depending on the form of the potential and the choice of initial conditions, finer coarse-grainings, i.e., a larger value of $k$ for the $k \times k$ grid, can yield either larger or smaller values of $\Lambda$.

It is important to emphasise that the distribution $f_{\text{niu}}$ towards which the orbit ensemble evolves initially does not in general correspond to a true microcanonical distribution, i.e., a uniform population of the entire phase space region that is in principle accessible to the orbits; and, as such it does not correspond to a strictly time-independent state. Indeed, if the ensemble be evolved for much longer times one discovers oftentimes that it will exhibit a slow secular evolution whereby it spreads further to probe phase space regions which were avoided completely at earlier times. The original approach towards $f_{\text{niu}}$ typically proceeds on a time scale shorter than $10t_D$, with $t_D$ a characteristic dynamical time. These much slower variations can proceed on a time scale as long as
$\sim 1000 t_D$ or even longer. What they reflect is a slow diffusion of chaotic orbits through cantori in $M$-dimensional potentials with $M - 1$ isolating integrals or along the Arnold web in potentials with no more than $M - 2$ isolating integrals. These topological obstructions do not serve as absolute impediments so as to prevent phase space transport. However, they can serve as partial barriers which significantly suppress the overall degree of phase space diffusion.

2.2. Systems with a weak time-dependence

Now consider how the idealised problem of chaotic mixing in a fixed time-independent potential can be impacted by weak time-dependent irregularities, perturbations so weak that, over time scales of interest, the values of the energy (and any other isolating integral associated with the unperturbed orbits) are very nearly conserved. The types of perturbations which one might expect to be operative in real galaxies include:

- **periodic driving**, which can mimic the effects of companion objects and/or systematic internal pulsations;
- **friction and white noise**, which can mimic discreteness effects, including gravitational Rutherford scattering between individual stars; and
- **coloured noise**, which can mimic the nearly random effects of a high density cluster environment or incoherent internal pulsations.

These effects can be addressed computationally by generating numerical solutions to a *Langevin equation* of the form (cf. van Kampen 1981)

$$\frac{d^2 r}{dt^2} = -\nabla V_0 (r) - \nabla V_1 (r, \omega t) - \eta (r, v) v + F(r, v, t). \quad (3)$$

Here the first term on the right hand side represents the unperturbed potential, whereas the second allows for time-dependent perturbations with period $\tau = 2\pi / \omega$. The third term is a dynamical friction and the fourth corresponds to random kicks, which are described probabilistically. Assuming in the usual fashion that the kicks correspond to Gaussian noise, everything about $F$ is characterised by the first two moments, which are assumed to satisfy

$$\langle F_a (t) \rangle = 0 \quad \text{and} \quad \langle F_a (t_1) F_b (t_2) \rangle = K_{ab} (r, v, t_1 - t_2) \quad (a, b = x, y, z), \quad (4)$$

with $K_{ab}$ the autocorrelation function. Picking $K_{ab}$ proportional to a Dirac delta yield white noise, corresponding to instantaneous kicks with a vanishing autocorrelation time. As a typical example of coloured noise, $K$ can be taken to sample the Ornstein-Uhlenbeck process, for which

$$K_{ab} \propto \exp (-|t_1 - t_2| / t_c). \quad (5)$$

It turns out that the relevant parameters for characterising the effects of noisy perturbations are the autocorrelation time $t_c$ and diffusion constant $D$, defined respectively by the relations

$$t_c \equiv \frac{\int dt K(t)}{\int dt K(t)} \quad \text{and} \quad D \equiv \int dt K(t) dt. \quad (6)$$

It is clear dimensionally that $D \sim F^2 t_c$, where $F$ represents the characteristic amplitude of the random forces modeled by the coloured noise.
Time-dependent perturbations of the form incorporated in this Langevin equation can impact chaotic mixing (Pogorelov & Kandrup 1999, Kandrup, Pogorelov, & Sideris 2000, Siopis & Kandrup 2000) both (1) by accelerating the approach towards a near-equilibrium in a single phase space region and (2) by facilitating diffusion through cantori or along an Arnold web to achieve a true equilibrium. The perturbations act via a resonant coupling between the natural frequencies of the orbits and the natural frequencies of the perturbation. The largest effects arise when the perturbing frequencies are comparable to $t_D^{-1}$, although there can also be couplings via harmonics and subharmonics.

That periodic driving acts via a resonant coupling is hardly surprising. That noise should also act via a resonant coupling is perhaps less obvious but also true. The important point here is that the autocorrelation function $K(t)$ determines the frequencies for which the perturbation has power, since the spectral density is related simply to the Fourier transform of $K$. White noise has a flat spectral density, with power at all frequencies, and can thus couple to all frequencies. Coloured noise with a finite autocorrelation time $t_c$ cuts off at high frequencies so that, for $t_c \gg t_D$, the noise has a comparatively minimal effect.

In the absence of time-dependent irregularities, phase mixing within a single nearly disjoint phase space region is limited by Liouville’s Theorem, which constrains significantly the degree to which a swarm of orbits can ‘fuzz out’ on small scales, e.g. by guaranteeing that phase space trajectories do not cross. Even very weak perturbations can wiggle the orbits enough to allow such ‘fuzzing’ to occur.

Diffusion between nearly disjoint phase space regions is also facilitated by the fact that perturbations can wiggle the original orbits, thus helping them find appropriate avenues of escape through cantori or along the Arnold web from one chaotic phase space region to another. Overall, the diffusion of point masses through such phase space holes is well modeled as a Poisson process, whereby orbits escape the original region at a near-constant rate. In this sense, the physics is fundamentally similar to the problem of effusion of gas through a tiny hole in elementary statistical physics. The introduction of noise or periodic driving accelerates this process by continually wiggling the orbits.

It appears that the details of the perturbation are comparatively unimportant, so that, e.g., details that would be difficult (if not impossible) to extract from observations are largely irrelevant. For example, different forms of noise, both additive and multiplicative, characterised by comparable amplitudes and autocorrelation times tend to have virtually identical effects; and the presence or absence of dynamical friction, which is a slowly varying systematic perturbation, tends to be nearly irrelevant. All that seems to matter are the amplitude of the perturbation, i.e., how hard the orbits are kicked, and the autocorrelation time $t_c$, which sets the natural time scale for the perturbation.

Quite generally, the overall efficacy of the perturbation as probed by (1) the rate at which an ensemble approaches a near-equilibrium or (2) the rate at which orbits diffuse through topological obstructions appears to scale logarithmically with the diffusion constant $D$. For very large and very small autocorrelation times, the precise value of $t_c$ is nearly irrelevant: for $t_c \to 0$ one recovers the results appropriate for white noise; for $t_c \to \infty$ the noise becomes almost
Figure 2. (a) The convergence rate $\Lambda$ for an ensemble of chaotic orbits in the lowest energy shell for the triaxial Dehnen potential with $\gamma = 1$, $c/a = 1/2$ and $(a^2 - b^2)/(a^2 - c^2) = 1/2$, allowing for coloured noise with diffusion constant $D = 2.5 \times 10^{-4}$ and variable autocorrelation time $t_c$. (b) $\Lambda$ for a different ensemble evolved in the same potential with the same energy. (c) $\tau_{0.05}$, the time required for $L^2$ convergence towards $f_{niv}$ at the 5% level on a $20 \times 20$ grid for the ensemble in (a). (d) The same for the ensemble in (b). In each panel, the dashed line corresponds to the ensemble evolved without any perturbations.

completely irrelevant. For intermediate values of the autocorrelation time, the efficacy of the perturbation also scales logarithmically in $t_c$.

It should be stressed that the perturbations do not act simply by making the orbits more chaotic. Unless the perturbations are of comparatively large amplitude, so large than energy is no longer approximately conserved, the size of a typical Lyapunov exponent remains nearly unchanged.

But how large must these perturbations be to have a significant effect? The answer here is that even very small perturbations can have a surprisingly large effect. For example, white noise corresponding to a diffusion constant $D \sim 10^{-6}$ in natural units, and hence a relaxation time $t_R \sim 10^6 t_D$, or coloured noise corresponding to kicks with $D \sim 10^{-3}$ and time scale $t_c \sim 10 t_D$ can increase the rate at which $f(t)$ approaches a near-invariant $f_{niv}$ by a factor of three and the rate of diffusion through an Arnold web by an order of magnitude or even more.

This behaviour is illustrated in Figures 2 and 3 which, respectively, exhibit the effects of varying the diffusion constant $D$ for fixed $t_c$ and the effects of varying $t_c$ for fixed $D$. The right and left panels exhibit results for two different orbit ensembles, in each case corresponding to a collection of orbits evolving in the lowest energy shell of the $\gamma = 1$ triaxial analogues of the Dehnen potentials considered by Merritt & Fridman (1996). In each figure, the top panels exhibit the initial convergence rate $\Lambda$ expressed in units of $t_D^{-1}$. The lower panels exhibit $\tau_{0.05}$, the time required for $L^2$ convergence towards $f_{niv}$ at the 5% level. The ensemble in the left hand panels corresponds seemingly to a ‘typical’ ensemble of chaotic orbits in this potential. The ensemble in the right hand panel is a somewhat less typical ensemble for which the later time approach towards
Figure 3. (a) The convergence rate $\Lambda$ for an ensemble of chaotic orbits in the lowest energy shell for the triaxial Dehnen potential with $\gamma = 1$, $c/a = 1/2$ and $(a^2 - b^2)/(a^2 - c^2) = 1/2$, allowing for coloured noise with autocorrelation time $t_c = t_D$ and variable diffusion constant $D$. (b) $\Lambda$ for a different ensemble evolved in the same potential with the same energy. (c) $\tau_{0.5}$, the time required for $L^2$ convergence towards $f_{niv}$ at the 5% level on a $20 \times 20$ grid for the ensemble in (a). (d) The same for the ensemble in (b). In each panel, the dashed line corresponds to the ensemble evolved without any perturbations.

$f_{niv}$ was especially slow: even though the initial rates $\Lambda$ for the right and left ensembles are comparable initially, the right hand ensemble reaches a near-invariant $f_{niv}$ much more slowly. The obvious point then is that even very weak noise, corresponding to very large $t_c$ ($100t_D$ or longer) and/or very small $D$ ($2.5 \times 10^{-4}$ or less) can dramatically accelerate the approach towards $f_{niv}$.

3. Implications for Violent Relaxation

Chaotic mixing has obvious implications for the rate at which various irregularities can disperse in a time-independent, or nearly time-independent, potential. More important, however, is the fact that it could serve as an important ingredient in a satisfactory theory of violent relaxation. As described in Lynden-Bell (1967), violent relaxation relies on phase mixing which, e.g., will cause an initially localised ensemble of points to disperse. The important point, then, is that numerical experiments involving regular motions, in the spirit of Lynden-Bell's balls rolling in a pig-trough, yield comparatively inefficient phase mixing, whereas chaotic motions in the same potential can yield a rapid, and effective, phase mixing (cf. Kandrup 1999).

It is, however, clear that realistic galactic potentials cannot be completely chaotic, and one might anticipate that, unless chaotic orbits largely dominate the flow, chaotic mixing need not prove all that important for violent relaxation. Nevertheless, there are several lines of reasoning that would suggest that chaotic mixing could prove of significant importance anyway: First of all, one knows that, generically, time-dependent potentials typically admit larger measures of chaotic orbits with sensitive dependence on initial conditions than do
time-independent potentials. This is hardly surprising given the observation that they typically have one fewer global isolating integral. Time-dependent potentials also allow for the possibility of transitions between regularity and chaos. In particular, an orbit which is regular most of the time can become chaotic for part of the time (cf. Kandrup & Drury 1998). As discussed above, it is clear that even very weak time-dependences can dramatically expedite phase mixing by facilitating both a ‘fuzzing’ of orbits in a given, nearly disjoint, phase space region and a diffusion between different phase space regions (Pogorelov & Kandrup 1999, Kandrup & Siopis 2000). For comparatively weak perturbations, one can think of the phase space as nearly unaltered and simply envision orbits being ‘assisted’ in their phase space diffusion. However, larger perturbations can also blur, move, and (in some cases) even remove topological obstructions like the Arnold web, again facilitating accelerated phase space transport.

One final caveat should, however, be stressed: Chaotic mixing is likely to be comparatively unimportant for structure formation in the early Universe, at least within the context of the standard Friedmann cosmologies. It would appear that the overall expansion of the Universe largely suppresses exponential chaos (Kandrup & Drury 1998), so that chaotic mixing should be largely inoperative until a primordial irregularity has ‘pinched off’ from the universal expansion.

Acknowledgments. I am pleased to acknowledge useful interactions with my collaborators, Ilya Pogorelov, Ioannis Sideris, and, especially, Christos Siopis and Elaine Mahon. This research was supported in part by NSF AST-0070809.

References

Anosov, D. V. 1967, Proc. Steklov Inst. Math., 90, 1
Kandrup, H. E. 1998, MNRAS, 301, 960
Kandrup, H. E. 1999, ASP Conference Series, 182, 197
Kandrup, H. E., Drury, J. 1998, Ann. N. Y. Acad. Sci. 867, 306
Kandrup, H. E., Mahon, M. E. 1994, Phys. Rev. E, 49, 3735
Kandrup, H. E., Pogorelov, I. V., Sideris, I. V. 2000, MNRAS, 311, 719
Kandrup, H. E., Siopis, C. 2000, preprint in preparation
Lynden-Bell, D 1967, MNRAS, 136, 101
Mahon, M. E., Abernathy, R. A., Bradley, B. O., Kandrup, H. E. 1995, MNRAS, 275, 443
Merritt, D., Fridman, T. 1996, ApJ, 460, 136
Merritt, D., Valluri, M. 1996, ApJ, 471, 82
Pogorelov, I. V., Kandrup, H. E., 1999, Phys. Rev. E, 60, 1567
Siopis, C., Kandrup, H. E. 2000, MNRAS, in press
van Kampen, N. G. 1981, Stochastic Processes in Physics and Chemistry. North Holland: Amsterdam