A note on the phase transition for independent alignment percolation

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We study the independent alignment percolation model on $\mathbb{Z}^d$ introduced by Beaton, Grimmett and Holmes. It is a model for random intersecting line segments defined as follows. First the sites of $\mathbb{Z}^d$ are independently declared occupied with probability $p$ and vacant otherwise. Conditional on the configuration of occupied vertices, consider the set of all line segments that are parallel to the coordinate axis, whose extremes are occupied vertices and that do not traverse any other occupied vertex. Declare independently the segments on this set open with probability $\lambda$ and closed otherwise. All the edges that lie on open segments are also declared open giving rise to a bond percolation model in $\mathbb{Z}^d$. We show that for any $d \geq 2$ and $p \in (0, 1]$ the critical value for $\lambda$ satisfies $\lambda_c(p) < 1$ completing the proof that the phase transition is non-trivial over the whole interval $(0, 1]$. We also show that the critical curve $p \mapsto \lambda_c(p)$ is continuous at $p = 1$.

Keywords: Percolation; renormalization; phase transition

1. Definition of the model

Alignment percolation has been recently introduced by Beaton, Grimmett and Holmes [2] as a model of random line segments in the hypercubic lattice $\mathbb{L}^d = (\mathbb{Z}^d, E^d)$ with $d \geq 2$. There, the authors define two versions of the model, referred to as the ‘one-choice model’ and the ‘independent model’ as we revisit below.

Fix $d \geq 2$ and let $\Omega := \{0, 1\}^{\mathbb{Z}^d}$. Given a configuration $\omega = \{\omega(v); v \in \mathbb{Z}^d\} \in \Omega$ a site $v \in \mathbb{Z}^d$ is said occupied when $\omega(v) = 1$ and vacant when $\omega(v) = 0$. For any parameter $p \in (0, 1]$ the independent Bernoulli site percolation model is the measure $\mathbb{P}^p$ on $\Omega$ under which $\{\omega(v); v \in \mathbb{Z}^d\}$ is a family of independent Bernoulli random variables with mean $p$. In other words, any site $v \in \mathbb{Z}^d$ is independently declared occupied or vacant with probability $p$ and $1 - p$, respectively.

Denote $\eta(\omega) = \{v \in \mathbb{Z}^d; \omega(v) = 1\}$. For a fixed configuration $\omega$, we say that a pair of sites $v_1, v_2 \in \mathbb{Z}^d$ is feasible when $v_1$ and $v_2$ differ only at a single coordinate and both $v_1$ and $v_2$ belong to $\eta$ but no other site in the line segment $v_1v_2$ that connects $v_1$ and $v_2$ belongs to $\eta$. We denote the set of feasible pairs by $F(\eta)$ and define a random graph $G = G(\omega)$ whose vertex set is $\eta(\omega)$ and whose edges are the feasible pairs; that is, $G(\omega) = (\eta, F(\eta))$.

In [2] two bond percolation models on $G$ were defined by specifying probability measures on $\{0, 1\}^{F(\eta)}$.

One-choice model. For every site $v \in \eta$ choose a feasible pair $f_v = vu \in F(\eta)$ uniformly among all $2d$ available possibilities. Declare $e = uv \in F(\eta)$ open if either $e = f_v$ or $e = f_v$. Otherwise, declare it closed.

Independent model. Fix $\lambda \in [0, 1]$ and declare each edge $e \in F(\eta)$ to be open independently with probability $\lambda$. 
Both versions described above can be regarded as dependent bond percolation models on \((\mathbb{Z}^d, \mathbb{E}^d)\) by declaring an edge \(e \in \mathbb{E}^d\) open if and only if the unique edge of \(F(\eta)\) that contains \(e\) is open. This gives rise to a random element \(\sigma\) in \(\Sigma := \{0, 1\}^{\mathbb{E}^d}\). In this paper, we are mainly interested in the connectivity properties of the random subgraph of \(\mathbb{Z}^d\) induced by the open edges in \(\sigma\).

Here we focus on the independent version of the model. In this context, let \(P^\omega_\lambda\) denote the measure on \(\Sigma\) induced by independent Bernoulli percolation on \(\{0, 1\}^{F(\eta)}\) with parameter \(\lambda\). The distribution of the pair \(\xi := (\omega, \sigma)\) is the probability measure \(P_{p, \lambda, d}\) on \(\Xi := \Omega \times \Sigma\) satisfying

\[
P_{p, \lambda, d}(A \times B) := \int_A P^{\omega}_\lambda(B) \, dP^\omega_p(\omega) \quad A \subset \Omega, \ B \subset \Sigma \text{ measurable},
\]

where \(\Omega\) and \(\Sigma\) are endowed with their respective cylinder \(\sigma\)-fields. We will often drop the dependency on \(d\) from the notation and write simply \(P_{p, \lambda}\). We will also abuse notation and write simply \(\xi(v)\) and \(\xi(e)\) in order to refer to the state of a site \(v\) and an edge \(e\), that is, \(\omega(v)\) and \(\sigma(e)\), respectively.

The set of open edges of \(\mathbb{L}^d\) is denoted by \(O = \{e \in \mathbb{E}^d; \xi(e) = 1\}\) and the set of closed edges by \(C = \mathbb{E}^d \setminus O\). A site \(v \in \mathbb{Z}^d\) percolates if it belongs to an infinite connected component of \(O\). We say that \(O\) percolates if there is a site \(v\) that percolates.

A standard coupling shows that \(\xi\) is monotone in \(\lambda\) in the stochastic sense, see Section 2. Thus, it is reasonable to expect that the model exhibits a non-trivial phase transition in the parameter \(\lambda\). More precisely, defining

\[
\theta(p, \lambda, d) := P_{p, \lambda, d}(o \text{ percolates}) \quad \text{and} \quad \lambda_c(p, d) := \sup\{\lambda \geq 0; \theta(p, \lambda, d) = 0\},
\]

where \(o\) denotes the origin in \(\mathbb{Z}^d\), it is expected that \(0 < \lambda_c(p, d) < 1\). This result has been partially established in [2, Theorem 2.4] where it is proved that \(\lambda_c(p, d) \geq \frac{p^*}{2d^2 - 1} > 0\) and also that there is a universal \(p^* > 0\) such that

\[
\lambda_c(p, d) < 1 \text{ if } p < p^* \text{ or } p > p_c^{\text{site}}(d). \tag{1}
\]

(Here, and in what follows, \(p_c^{\text{site}}(d)\) and \(p_c^{\text{bond}}(d)\) stand for the critical thresholds for independent site and bond percolation on \(\mathbb{Z}^d\).) Since \(p_c^{\text{site}}(d) \sim \frac{1}{2d^2}\) we know that there is some \(d_0(p^*)\) such that for \(d \geq d_0(p^*)\) we have \(\lambda_c(p, d) \in (0, 1)\) for every \(p \in (0, 1]\). However, since we do not have a good control on \(p^*\) it may be the case that \(p^* < p_c^{\text{site}}(d)\) meaning that (1) is not enough to not rule out that \(\lambda_c(p, d) = 1\) on \([p^*, p_c^{\text{site}}(d)]\), as illustrated in Figure 1.

In Theorem 1 we bridge this gap proving that \(\lambda_c(p, d) < 1\) for every \(p \in (0, 1]\). For \(d \geq 3\) it is not hard to find an upper bound which holds uniformly over all the interval \([0, 1]\) by a simple comparison to Bernoulli percolation on the hexagonal lattice. For the case \(d = 2\), that seems to be more delicate, we employ renormalization methods in order to obtain the result.

**Theorem 1.** Let \(p_c^O = 1 - 2\sin(\pi/18)\) be the critical point for independent bond percolation on the hexagonal lattice\(^1\). For any \(p \in (0, 1]\) and \(d \geq 3\) we have that

\[
\lambda_c(p, d) \leq p_c^O. \tag{2}
\]

Moreover, there is \(\lambda_0 \in (0, 1)\) such that \(\lambda_c(p, 2) \leq \lambda_0\) for every \(p \in (0, 1]\).

\(^1\)For the hexagonal lattice the critical point is known exactly, see [5, Chapter 3]
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In [2] the authors conjecture that the function $p \mapsto \lambda_c(p, d)$ is non-increasing and continuous, as suggested by numerical simulation. As already mentioned there, one difficulty in obtaining results in this direction stems from the fact that it seems hard to build a coupling of the model with different values of $p$ that provides good control on how the connectivity changes, see Section 2. Currently, there is not much progress in proving monotonicity and continuity of $\lambda_c(\cdot, d)$, however, our renormalization arguments can be used to prove continuity as $p$ approaches 1. Notice that the estimates from [2] show that $\lim_{p \to 1} \lambda_c(p, d) \leq p_c^{\text{bond}}(d)$, since they provide a continuous curve that bounds $\lambda_c(p, d)$ from above, see Figure 1. We prove:

**Theorem 2.** Fix $d \geq 2$. For every $\lambda < p_c^{\text{bond}}(d)$ there exists $p_0(\lambda, d) \in (0, 1)$ such that

$$P_{p, \lambda, d}(\mathcal{O} \text{ percolates}) = 0 \quad \text{for every } p > p_0.$$  

(3)

Consequently, the function $p \mapsto \lambda_c(p, d)$ is continuous at $p = 1$.

Theorem 2 answers affirmatively a question posed in [2] (see the paragraph preceding Remark 2.6 therein).

Let us say a word about the proofs of Theorems 1 and 2. As mentioned above, the proof of Theorem 1 for $d \geq 3$ follows from embedding the hexagonal lattice in $\mathbb{Z}^d$ in an appropriate way and comparing alignment percolation on this lattice to Bernoulli percolation. This comparison does not work for $d = 2$ and we develop a multiscale renormalization to prove that $\lambda_c(\cdot, 2) < 1$.

Renormalization is a classical technique that has been successfully applied to a vast range of areas in mathematics and physics. Instead of providing a complete account on the applicability of the method, we would just like to mention that it has been successfully used to study models in which correlations decay suitably with distance. To mention a few examples, it has been employed to show the existence of a phase transition in dependent percolation e.g. random interlacements [9] and the Poisson cylinder

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**Figure 1.** The blue curves represent the lower and upper bounds for $\lambda_c(p, d)$ obtained in [2] (illustrated in Figure 2.1 therein). For $d \geq 3$, Theorem 1 complements the picture with an explicit upper bound; the critical curve is inside the region highlighted in gray. For $d = 2$ we do not have an explicit upper bound, but we ensure the critical curve is non-trivial in the interval $[p^*, p_c^{\text{site}}(2)]$. 

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model [10], to study the long-range behavior of random walks on random environments [3] and on detection problems [1].

The structure of the argument, as we use here, can be summarized in the following steps.

S1. (Recurrence inequality) Define a sequence of lengths $L_k$, called scales, increasing sufficiently fast. Then build a family of events $\{A_k(x): k \in \mathbb{N}, x \in \mathbb{Z}^d\}$ where each $A_k(x)$ is measurable with respect to the restriction of the process to a finite region having diameter of order $L_k$ around $x$. This family is required to be cascading, meaning that the occurrence of any of the events $A_{k+1}(x)$ in the scale $k+1$ implies the occurrence of two events $A_k(x_1)$ and $A_k(x_2)$ in scale $k$ for points $x_1$ and $x_2$. The events $A_k$ should be regarded as bad events and the goal is to show that their probabilities decay fast as we increase the scale $k$. Defining $q_{L_k}(p, \lambda, d) := \mathbb{P}_{p,\lambda,d}(A_k(x))$ which does not depend on $x$ (due to translation invariance) the cascading property together with the union bound can be used to deduce that

$$ q_{L_k+1}(p, \lambda, d) \leq g\left(\frac{L_{k+1}}{L_k}\right) \cdot [q_{L_k}(p, \lambda, d)^2 + \text{error}_k(p, \lambda, d)], \quad (4) $$

where $g$ is some suitable positive real function and the error accounts for the correlation between events of type $A_k(x_i)$. Moreover, if $x_1$ and $x_2$ are well-separated at scale $k$ but close enough at scale $k+1$ we may have a small error term and a function $g$ that does not increase too fast. In summary, inequality (4) relates $q_{L_k}$ at two successive scales by using the structure of cascading events and decay of correlations.

S2. (Inductive step) The recursive inequality (4) suggests that if $q_{L_k}$ is small, if we have a good control of the error terms and if $g$ does not increase too fast, then $q_{L_{k+1}}$ should also be small. This is to say that a good upper bound for $q_{L_k}$ might be transferred to a good upper bound for $q_{L_{k+1}}$. In fact, assuming that $q_{L_{k_0}}(p, \lambda, d) \leq f(L_{k_0})$ for some choice of $k_0(p, \lambda, d)$ and some well-chosen positive real function $f(x)$ tending to zero sufficiently fast as $x \to \infty$, we show inductively that

$$ q_{L_k}(p, \lambda, d) \leq f(L_k) \quad \text{for every } k \geq k_0. \quad (5) $$

S3. (Trigger) Assume that the previous two steps have been carried on successfully and fix the index $k_0$ as given above. The goal of the triggering step is to find a choice of $p, \lambda$ so that

$$ q_{L_{k_0}}(p, \lambda, d) \leq f(L_{k_0}). \quad (6) $$

The fact that we have fixed $k_0$ leaves us with a finite size criterion, that is, we only need to control probabilities at the fixed scale $k_0$ which should behave nicely as functions of the parameters. Once we have found the good parameters so that (6) holds, we can trigger the induction in (5) and conclude that indeed (5) holds for every $k \geq k_0$ or, in other words, that $q_{L_k}(p, \lambda, d)$ indeed decays at least as fast as $f(L_k)$ as we increase $k$.

The proofs of Theorem 1 for $d = 2$ and of Theorem 2 follow essentially the structure above. Roughly speaking, $A_k(x)$ will be defined as being the event that there exists an open path (or dual path) crossing an annulus of size $L_k$ around $x$ (see Equations (18) and (21) for precise definitions and Figure 4 for an illustration). The trigger step S3 is performed differently in the proofs of Theorem 1 and Theorem 2. For the former, the triggering is performed fixing $p \in (0, 1)$ and taking $\lambda$ sufficiently close to 1, whereas for the latter we fix $\lambda < p_{c}^{\text{braid}}(d)$ and take $p$ sufficiently close to 1. Our application is reasonably simple and showcases the robustness of renormalization. In fact, the same type of arguments can be applied to other dependent percolation processes as long as they present a good decay of correlations which entails a small error term in (4), as will become clear in the proof of Lemma 1 below.
Throughout the text, we will use $c$ or $C$ to indicate positive constants whose values may change from line to line. Numbered constants like $c_0, c_1, \ldots$ or $k_0, k_1, \ldots$ will have their values fixed at the first time they appear and will remain fixed. We may indicate the dependence of constants on other parameters, for instance, $c_i(d)$ is a constant whose value depends on $d$ but not on any other parameter of the model.

Let us conclude this section by summarizing the structure of the paper. In Section 2 we discuss some properties of the model: monotonicity, uniqueness of the infinite cluster, the lattice condition, and decay of correlations. In Section 3.1 we start by presenting the proof of Theorem 1 for $d \geq 3$ and then move to the multiscale framework that will be used for proving Theorems 1 and 2. We complete their proofs on Sections 3.2 and 3.3, respectively.

## 2. Properties of the independent model

In this section we collect some useful properties fulfilled by independent alignment percolation model. Some of them were already mentioned in [2].

Let us first describe some monotonicity properties. Since we are concerned with percolation of the subgraph induced by the set of open edges, we will compare two configurations $\xi, \xi' \in \Xi$ using the natural partial order in $\Sigma$, that is, $\xi \preceq \xi'$ if and only if $\xi(e) \leq \xi'(e), \forall e \in E^d$.

Recall that $P_{p,\lambda}$ is the underlying probability distribution of the model and denote by $\xi_{p,\lambda}$ a random element distributed as $P_{p,\lambda}$. We may possibly omit one of the parameters in $\xi_{p,\lambda}$ when there is no risk of confusion.

For a fixed $p \in (0, 1]$, the standard monotone coupling may be used in order to construct the $\xi_{p,\lambda}$ in a monotone way in $\lambda$ (with respect to this partial order). For a fixed $\lambda \in [0, 1]$ the picture is more complex. This was already mentioned on [2] but we elaborate on this discussion here.

**Proposition 1.** Fix $\lambda \in (0, 1]$. For any $0 < p_1 < p_2 \leq 1$ there exists no coupling $(\xi_1, \xi_2)$ of random elements with marginal distributions $P_{p_i,\lambda}, i = 1, 2$ such that $\xi_{p_1} \preceq \xi_{p_2}$ almost surely or $\xi_{p_2} \preceq \xi_{p_1}$ almost surely.

**Proof.** Recall that $P_{p,\lambda}$ is the underlying probability distribution of the model and denote by $\xi_{p,\lambda}$ a random element distributed as $P_{p,\lambda}$. We may possibly omit one of the parameters in $\xi_{p,\lambda}$ when there is no risk of confusion.

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Conjecture 2.5 of [2] states that the function $\lambda_c(\cdot, d)$ should be continuous and strictly increasing. Proposition 1 does not invalidate this conjecture but shows that to compare the model for different values of $p$ may not be a simple task. A small observation is that we can identify a class of increasing events whose probability is decreasing in $p$. 
Proposition 2. Let \( E \subset \mathbb{Z}^d \) be finite. Denote by \( \xi(E) \) the restriction of \( \xi \) to the edges of \( E \) and write \( \xi(E) \equiv 1 \) if \( \xi(e) = 1 \) for every edge in \( E \). Then, for \( p_1 \leq p_2 \) it holds
\[
P_{p_1, \Lambda}(\xi(E) \equiv 1) \geq P_{p_2, \Lambda}(\xi(E) \equiv 1).
\]

Proof. We use a standard coupling for the underlying site percolation configuration. Consider a family of i.i.d. random variables \( \{U_v; v \in \mathbb{Z}^d\} \) with uniform distribution on \([0, 1]\) and for \( p \in (0, 1) \) define \( \eta_p \in \Omega \) by \( \eta_p(v) := \mathbb{1}\{U_v \leq p\} \) for every \( v \in \mathbb{Z}^d \). On \( \Omega \) we have a partial ordering defined by \( \eta \prec \eta' \) if and only if \( \eta(v) \leq \eta'(v) \) for every \( v \in \mathbb{Z}^d \). Notice that this coupling yields \( \eta_{p_1} \preceq \eta_{p_2} \). Denote by \( t(\eta_p, E) \) the number of feasible pairs of \( \eta_p \) that contain at least one edge that belongs to \( E \). We have
\[
P_{p, \Lambda}(\xi(E) \equiv 1) = E\left[P_{p, \Lambda}(\xi(E) \equiv 1 \mid \eta_p)\right] = E\left[\lambda(t(\eta_p, E))\right].
\]

Adding new sites to a configuration can only increase \( t(\cdot, E) \). Thus, we have
\[
P_{p_1, \Lambda}(\xi(E) \equiv 1) = E\left[\lambda(t(\eta_{p_1}, E))\right] \geq E\left[\lambda(t(\eta_{p_2}, E))\right] = P_{p_2, \Lambda}(\xi(E) \equiv 1). \quad \square
\]

Another known property of the model is that whenever \( \mathcal{O} \) percolates, the infinite cluster is unique \( P_{p, \Lambda, d}\text{-a.s.} \). The proof of this result follows from the classical argument of Burton and Keane [4], and is part of [2, Theorem 2.4]. This property may be useful, for instance to prove continuity of the critical curve but we will not use it in this paper.

We now discuss other properties that were not presented in [2]. Denote by \( \mu \) the marginal of \( P_{p, \Lambda} \) on \( \Sigma \) and for \( \Lambda \subset \mathbb{Z}^d \) finite let \( \mu_{\Lambda} \) be defined as the restriction of \( \mu \) to cylinder events on \( \Lambda \). Moreover, for \( x, y \in \{0, 1\}^\Lambda \) define \( x \lor y, x \land y \in \{0, 1\}^\Lambda \) as
\[
(x \lor y)(e) := \max\{x(e), y(e)\} \quad \text{and} \quad (x \land y)(e) := \min\{x(e), y(e)\} \quad \text{for every } e \in \Lambda.
\]

It is worth noticing that \( \mu_{\Lambda} \) fails to satisfy the so-called lattice condition
\[
\mu_{\Lambda}(x \lor y) \mu_{\Lambda}(x \land y) \geq \mu_{\Lambda}(x) \mu_{\Lambda}(y) \quad \text{for every } x, y \in \Sigma.
\] (8)

Let us abuse notation and write \( e_i \) in order to refer to the edge \( (o, e_i) \). Consider \( \Lambda = \{\pm e_1, \pm e_2\} \subset \mathbb{Z}^d \).

Let \( x, y \) be configurations on \( \{0, 1\}^\Lambda \) given by
\[
x(e) = \mathbb{1}\{\pm e_1\}(e) \quad \text{and} \quad y(e) = \mathbb{1}\{e_2\}(e),
\]

It is straightforward to check that
\[
\mu_{\Lambda}(x \lor y) \mu_{\Lambda}(x \land y) - \mu_{\Lambda}(x) \mu_{\Lambda}(y)
= [p\lambda^3(1 - \lambda)] \cdot [p\lambda^2(1 - \lambda)^3] - [p\lambda^2(1 - \lambda)^2 + (1 - p)\lambda(1 - \lambda)] \cdot [p\lambda^2(1 - \lambda)^2] < 0.
\]

Proving (8) for every finite \( \Lambda \) is a common strategy to show that the measure \( \mu \) satisfies the FKG inequality, also known as positive association. However, the FKG inequality is not equivalent to the lattice condition; for background on positive association and its relation to the lattice condition we refer to [6, Chapter 2]. This raises the question whether the model satisfies the FKG inequality.

Although the independent alignment percolation model does not exhibit great monotonicity properties, it does present fast correlation decay. This fact will allow us to implement the multiscale renormalization approach outlined in Section 1.
For $x \in \mathbb{Z}^d$ and $L > 0$ let us define the $l_{\infty}$ ball and sphere of radius $L$ centered in $x$ as being

$$B(x, L) := \{z \in \mathbb{Z}^d; \|z - x\|_{\infty} \leq L\} \quad \text{and} \quad \partial B(x, L) := \{z \in \mathbb{Z}^d; \|z - x\|_{\infty} = |L|\}.$$  

Notice that balls are just lattice cubes. For $B \subset \mathbb{Z}^d$, we say that an event $A$ is supported on edges of $B$ if $A$ belongs to the $\sigma$-algebra generated by $\{\xi(e); e = \{u, v\} \in \mathbb{E}^d, u, v \in B\}$.

For two measurable functions $f_1$ and $f_2$ on $\Xi$ we denote $\text{Cov}_{p,\lambda}(f_1, f_2)$ their covariance under $\mathbb{P}_{p,\lambda}$. For events $A_1$ and $A_2$ we write

$$\text{Cov}_{p,\lambda}(A_1, A_2) = \text{Cov}_{p,\lambda}(\mathbf{1}_{A_1}, \mathbf{1}_{A_2}).$$

**Lemma 1** (Decay of correlations). Let $L > 0$ and $p \in (0,1)$. For sites $x_1, x_2 \in \mathbb{Z}^d$ satisfying $D = \|x_1 - x_2\|_{\infty} > 2L$, let $A_i$, $i = 1, 2$, be events supported on edges of $B_i := B(x_i, L)$, respectively. Then,

$$|\text{Cov}_{p,\lambda}(A_1, A_2)| \leq 4 \cdot (2L + 1)^{d-1} \cdot e^{-\alpha(p) \cdot (D-2L)} \quad (9)$$

with $\alpha(p) := -\log(1-p) > 0$.

**Proof.** The assumption on $D$ implies $\text{dist}(B_1, B_2) = D - 2|L| \geq 1$ and, in particular, $B_1 \cap B_2$ is empty. Write $x_{1,t}$ for the $t$-th coordinate of the vector $x_1$. Choose the smallest $1 \leq t \leq d$ such that $D = \|x_{1,t} - x_{2,t}\|$ and notice that we can separate $B_1$ from $B_2$ by a hyperplane orthogonal to $e_t$ the $t$-th vector of the canonical basis of $\mathbb{R}^d$. Denote by $\pi_t(B)$ the projection of a set $B \subset \mathbb{R}^d$ into the subspace $e_t^\perp$ (perpendicular to $e_t$) and define

$$\Pi_t = \pi_t(B_1) \cap \pi_t(B_2).$$

If we have $\Pi_t = \emptyset$ then $A_1$ and $A_2$ are independent and their covariance is zero. Otherwise, for every $z \in \Pi_t$ define $I_z = [a_z, b_z] \cap \mathbb{Z}^d$ as the unique line segment supported on the line $\pi_t^{-1}(z)$ with $a_z \in B_1, b_z \in B_2$ and such that $(I_z \setminus \{a_z, b_z\}) \cap (B_1 \cup B_2) = \emptyset$. Define the event

$$C = \{ \exists z \in \Pi_t \text{ such that } \omega(v) = 0, \forall v \in I_z \} \quad (10)$$

and notice that on $C^c$ the state of edges in $B_1$ and $B_2$ are independent, see Figure 2. Indeed, for $\omega \in C^c$ notice that if $F_t(\omega)$ denotes the feasible edges intersecting $B_t$ and $f_t = \{u_i, v_i\} \in F_t(\omega)$ then $|f_1 \cap f_2| \leq 1$. Thus, omitting the dependency on $p$ and $\lambda$ we have that

$$\mathbb{P}(A_1 \cap A_2 \cap C^c) = \mathbb{P}(A_1 \cap A_2) \mathbb{P}(C^c) - \mathbb{P}(A_1) \mathbb{P}(A_2 | C^c) \mathbb{P}(C^c) - \mathbb{P}(A_2) \mathbb{P}(A_1 | C^c) \mathbb{P}(C^c).$$

Using twice that for any event $E$ one has $|\mathbb{P}(E) - \mathbb{P}(E \cap C^c)| \leq \mathbb{P}(C)$, we can bound

$$|\mathbb{P}(A_1)\mathbb{P}(A_2) - \mathbb{P}(A_1 \cap C^c)\mathbb{P}(A_2 | C^c)| = |\mathbb{P}(A_1)\mathbb{P}(A_2) - \mathbb{P}(A_2 \cap C^c)\mathbb{P}(A_1 | C^c) + \mathbb{P}(A_2 \cap C^c)| \mathbb{P}(A_1) - \mathbb{P}(A_1 \cap C^c)| \leq (1 + \mathbb{P}(C^c)) \cdot \mathbb{P}(C),$$

which leads to the estimate

$$|\text{Cov}(A_1, A_2)| = |\mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1)\mathbb{P}(A_2)| \leq \mathbb{P}(C) + \mathbb{P}(C^c) + \mathbb{P}(A_1 \cap A_2 \cap C^c) - \mathbb{P}(A_1 \cap C^c)\mathbb{P}(A_2 \cap C^c)$$
The projections of boxes $B_i$ into the hyperplane $e_i^\perp$ form the gray region $\Pi_t = \pi_t(B_1) \cap \pi_t(B_2)$. On the event that for each $z \in \Pi_t$ the line segment $I_z = [a_z, b_z] \cap \mathbb{Z}^d$ contains an occupied site (red cross) events $A_1$ and $A_2$ are independent.

\[
= (2 + \mathbb{P}(C^c))\mathbb{P}(C) + |\mathbb{P}(A_1 \mid C^c)\mathbb{P}(A_2 \mid C^c) \cdot [\mathbb{P}(C^c) - \mathbb{P}(C^c)^2]| \\
\leq 2\mathbb{P}(C) + \mathbb{P}(C^c)\mathbb{P}(C) + \mathbb{P}(C^c) - \mathbb{P}(C^c)^2 \\
= 4\mathbb{P}(C) - 2\mathbb{P}(C)^2.
\]

Finally, we use the union bound and the fact that $|\Pi_t| \leq (2L + 1)^{d-1}$ to estimate

\[
\mathbb{P}(C) \leq (2L + 1)^{d-1} \cdot (1 - p)^{D - 2L}.
\]

Remark 1. Lemma 1 is stated for $\mathbb{P}_{p, \lambda}$. However, it can be adapted for the ‘one-choice model’ and more generally for any measure $\mathbb{P}_\omega$ on the feasible pairs $F(\eta)$ satisfying that random variables \(\{\xi(f_i); i \in I\}\) are mutually independent for any endpoint-disjoint family \(\{f_i; i \in I\} \subset F(\eta)\). To see why this is true, just consider instead of the event $C$ appearing in (10) the event $\tilde{C} = \{\exists f_1 \in F_1(\omega), f_2 \in F_2(\omega) \text{ such that } |f_1 \cap f_2| \geq 1\}$.

To prevent $\tilde{C}$ from occurring we just have to ensure each line $I_z$ with $z \in \pi_t(B_1) \cup \pi_t(B_2) =: \tilde{\Pi}_t$ contains 2 sites in $\eta(\omega)$ lying in the region between the two boxes. The decay of correlations will follow once one notes that $|\tilde{\Pi}_t| \leq 2(2L + 1)^{d-1}$ and changing $\alpha(p)$ accordingly. Moreover, we notice that the bound on (9) is independent of $\lambda$.

3. Phase transition for alignment percolation

In this section we study the features of the phase transition for independent alignment percolation. We first consider the case $d \geq 3$, in which no renormalization is needed in order to prove Theorem 1. Indeed, a construction that appeared in [7] fits very nicely to this case.

Proof of Theorem 1, case $d \geq 3$. We show that it is possible embed the hexagonal lattice in $\mathbb{Z}^d$ in such a way that the state of their edges are independent. Notice that it suffices to prove the claim on the case $d = 3$, since for larger $d$ one can run the same argument on the subset $\{x \in \mathbb{Z}^d; x_i = 0, \forall 4 \leq i \leq d\}$. 

Figure 2. The projections of boxes $B_i$ into the hyperplane $e_i^\perp$ form the gray region $\Pi_t = \pi_t(B_1) \cap \pi_t(B_2)$. On the event that for each $z \in \Pi_t$ the line segment $I_z = [a_z, b_z] \cap \mathbb{Z}^d$ contains an occupied site (red cross) events $A_1$ and $A_2$ are independent.
Thus, let us consider \( d = 3 \) and for a fixed \( k \in \mathbb{Z} \) we write
\[
V_1 := \{(x, y, z) \in \mathbb{Z}^3; x + y + z = k\} \quad \text{and} \quad V_2 := \{(x, y, z) \in \mathbb{Z}^3; x + y + z = k + 1\}.
\]

The desired subgraph of \( \mathbb{L}^3 \) is obtained by considering the vertex set \( V_1 \cup V_2 \) and the edges \( uv \) with \( u \in V_1, v \in V_2 \) and \( \|u - v\|_\infty = 1 \). As shown in Figure 3, this defines a subgraph of \( \mathbb{Z}^3 \) that is isomorphic to the hexagonal lattice. Under the measure \( P_\lambda^\omega \), the state of each edge is independent since a line parallel to one of the canonical directions intersects the planes \( V_1 \) and \( V_2 \) in precisely one point each. Therefore, the alignment percolation restricted to such a graph is just independent Bernoulli percolation whose critical point equals \( p_\infty = 1 - 2 \sin(\pi/8) \).

The rest of this section will be dedicated to the development of a multiscale scheme that will help us to establish the proof of Theorem 1 in the case \( d = 2 \) and of the continuity of the critical curve at the right edge of the interval \([0, 1]\).

### 3.1. Multiscale renormalization

In this section we build a multiscale renormalization scheme that allows us to prove Theorem 1 (in the case \( d = 2 \)) and Theorem 2. A key step will be the use of the correlation decay in Lemma 1.

We start by defining the sequence of scales \( L_0, L_1, \ldots \) along which we analyze the model. Let
\[
L_0 := 100 \quad \text{and} \quad L_{k+1} = 100L_k,
\]
that is, \( L_k = 100^{k+1} \) is a sequence growing exponentially fast. This same sequence is used in the proofs of both Theorem 1 in the case \( d = 2 \) and Theorem 2. The family of cascading events
\{A_k(x)\}_{x \in \mathbb{Z}^d, k \in \mathbb{Z}^+} \text{ will be defined properly for each proof, but in both cases } A_k(x) \text{ will be the event in which there is ‘some connection’ between the inner and outer boundaries of annular region } B(x, 10L_k) \setminus B(x, L_k), \text{ as illustrated in Figure 4. Once these events are defined, Steps S1 and S2 in Section I can be carried on exactly the same way in both proofs.}

For establishing the cascading property we will need a certain control for the relative position of boxes from two consecutive scales. For each } k \text{ it is possible to choose two collections } \mathcal{L}_k^1 \subset \partial B(o, L_{k+1}) \text{ and } \mathcal{L}_k^2 \subset \partial B(o, 5L_{k+1}) \text{ of points such that
}
\begin{align}
\partial B(o, L_{k+1}) &\subset \bigcup_{x \in \mathcal{L}_k^1} B(x, L_k) \quad \text{and} \quad \partial B(o, 5L_{k+1}) \subset \bigcup_{x \in \mathcal{L}_k^2} B(x, L_k) \tag{12}
\end{align}
and satisfying
\begin{align}
c \leq |\mathcal{L}_k^i| \leq C \quad \text{for } i = 1, 2, \tag{13}
\end{align}
for positive constants } c = c(d) \text{ and } C = C(d). \text{ This fact is straightforward and we omit its proof; intuitively, we are just stating that in order to cover the boundary of a } d\text{-dimensional ball of radius } R \text{ with balls of radius } r \text{ we need approximately } (R/r)^{d-1} \text{ balls. The bounds in (13) show that in our renormalization argument we can take the function } g \text{ appearing in (4) as a constant depending only on } d. \text{ Roughly speaking } \{B(x, L_k) : x \in \mathcal{L}_k\} \text{ will play the role of the regions where we will witness the occurrence of } A_k(x_i) \text{ at scale } k \text{ which are implied by the occurrence } A_{k+1}(o) \text{ as explained in Step S1, see Figure 4.}

We now state two properties that once fulfilled by the events } A_k(x) \text{ will allow us to implement Steps S1 and S2.

P1. } P_{p,\lambda,d}(A_k(x)) \text{ does not dependent upon the choice of } x \text{ which allows us to define}
\begin{align}
q_k(p, \lambda, d) := P_{p,\lambda,d}(A_k(o)).
\end{align}

P2. The occurrence of event } A_{k+1}(o) \text{ implies that there are } x_i \in \mathcal{L}_k^i \text{ such that } A_k(x_i) \text{ also occur, for } i = 1, 2.

In words, P1 is a translation invariance property and P2 guarantees that the events are cascading. Recall that the events are said to be cascading if the occurrence of such an event at scale } k + 1 \text{ implies the occurrence of similar events at the previous scale } k. \text{ An instance of event satisfying these properties is illustrated in Figure 4.

The events } A_k(x) \text{ to be defined below will satisfy Properties P1 and P2. The key point is that these properties imply the validity of Steps S1 and S2 as we show next:

Lemma 2 (Step S1 – Recurrence Inequality). Let } \{A_k(x) : x \in \mathbb{Z}^d, k \in \mathbb{N}\} \text{ be a collection of events on } \Xi \text{ satisfying Properties P1 and P2. Then,
\begin{align}
q_{k+1}(p, \lambda, d) \leq c_0(d)q_k(p, \lambda, d)^2 + c_1(d) \cdot l_k^{d-1} e^{-\alpha(p)\cdot L_k}, \tag{14}
\end{align}
where } c_0 \text{ and } c_1 \text{ are positive constants depending only on } d.

Proof. The boxes } B(x_i, 10L_k) \text{ given by property P2 are well-separated since
\begin{align}
\|x_1 - x_2\|_\infty \geq 5L_{k+1} - L_{k+1} = 4L_{k+1} \quad \text{for every } x_i \in \mathcal{L}_k^i.
\end{align}
Phase transition for independent alignment percolation

Figure 4. Illustration of the event $A_{k+1}(o)$. It entails the existence of a path (dual for Theorem 1 and primal for Theorem 2) connecting the inner and outer boundaries of the annulus $B(o, 10L_{k+1}) \setminus B(o, L_{k+1})$. We emphasize the cascading property: occurrence of $A_{k+1}(o)$ implies the occurrence of $A_k(x_1)$ and $A_k(x_2)$ for some $x_i \in L^i_k$ with $i = 1, 2$. Events $A_k(x_i)$ are supported on the annuli $B(x_i, 10L_k) \setminus B(x_i, L_k)$.

Using the fact that $A_k(x)$ is supported on the set of edges inside $B(x, 10L_k)$, we have by Lemma 1 that there exists a constant $c_2 = c_2(d) > 0$ such that

$$|\text{Cov}_{p,\lambda}(A_k(x_1), A_k(x_2))| \leq c_2 \cdot L_k^{d-1} \cdot e^{-\alpha(p)(4L_{k+1}-20L_k)} \leq c_2 \cdot L_k^{d-1} \cdot e^{-\alpha(p)L_k}$$

for every $k$. By (13) we know that $|L^i_k| \leq c_3$ for some constant $c_3(d) > 0$. Thus, we can write

$$q_{k+1}(p, \lambda, d) \leq \sum_{x_1 \in L^1_k, x_2 \in L^2_k} \mathbb{P}_{p,\lambda,d}(A_k(x_1) \cap A_k(x_2))$$

$$\leq |L^1_k| \cdot |L^2_k| \cdot [q_k(p, \lambda, d)^2 + c_2(d) \cdot L_k^{d-1} e^{-\alpha(p)L_k}]$$

$$\leq c_3(d)^2 q_k(p, \lambda, d)^2 + c_3(d)^2 \cdot c_2(d) \cdot L_k^{d-1} e^{-\alpha(p)L_k}.$$ (15)

The proof is finished by defining $c_0 = c_3^2$ and $c_1 = c_3^2 \cdot c_2$.

Lemma 3 (Step S2 – Inductive step). Let $\{A_k(x); x \in \mathbb{Z}^d, k \in \mathbb{N}\}$ be a collection of events in $\Xi$ satisfying Properties P1 and P2. There is $\beta > 0$ and $k_0 = k_0(p, d)$ such that if for some $k_1 \geq k_0$ one has

$$q_{k_1}(p, \lambda, d) \leq e^{-L_{k_1}^\beta}$$ (16)

then

$$q_k(p, \lambda, d) \leq e^{-L_k^\beta}$$ holds for every $k \geq k_1$. (17)
**Proof.** Take $\beta = \frac{1}{2} \log_2 \frac{2}{\log 100}$, so that $100 \beta < 2$. Choose $k_0(d, p)$ as the smallest integer $k$ such that for $k \geq k_0$ we have

$$c_0(d) \cdot e^{-(2-100 \beta) k} \leq \frac{1}{2} \quad \text{and} \quad c_1(d) \cdot L_k^{d-1} e^{-\alpha(p) (L_k + L_{k+1})} \leq \frac{1}{2},$$

where $c_0(d), c_1(d)$ are given by Lemma 2. This is always possible since the left-hand sides of both inequalities tend to zero as $k \to \infty$. Suppose there is some $k(p, \lambda, d) \geq k_0$ satisfying the inequality in (17). Then, we can write

$$\frac{q_{k+1}(p, \lambda, d)}{e^{-L_{k+1}^d}} \leq c_0(d) \cdot q_k(p, \lambda, d)^2 e^{L_{k+1}^d} + c_1(d) \cdot L_k^{d-1} e^{-\alpha(p) L_k + L_{k+1}^d} \leq c_0(d) \cdot e^{-2L_{k+1}^d} + c_1(d) \cdot L_k^{d-1} e^{-\alpha(p) L_k + L_{k+1}^d} \leq c_0(d) \cdot e^{-(2-100 \beta) L_k^d} + c_1(d) \cdot L_k^{d-1} e^{-\alpha(p) L_k + L_{k+1}^d} \leq 1,$$

where we have used the definition of $k_0$ in the last inequality. This means that the inequality in (17) carries on to the next scale $k + 1$. The result follows by induction. $\square$

**Remark 2.** Notice that $k_0(p, d)$ is a decreasing function of $p$ since $\alpha(p)$ is increasing. Thus, for any fixed $\epsilon > 0$ we can replace $k_0(p, d)$ by $k_0(\epsilon, d)$ uniformly over $p \in [\epsilon, 1]$.

### 3.2. Phase transition for alignment percolation, case $d = 2$

We now focus on the case $d = 2$. Consider the dual graph $(\mathbb{L}^2)^*$ of $\mathbb{L}^2$, defined as the graph with vertex set $\mathbb{Z}^2 + (1/2, 1/2)$ and edge set $(\mathbb{E}^2)^*$ connecting sites at Euclidean distance 1. Therefore, $(\mathbb{L}^2)^*$ and $\mathbb{L}^2$ are isomorphic. Moreover, to each edge $e^*$ in $(\mathbb{E}^2)^*$ there exists a unique edge $e$ in $\mathbb{E}^2$, so that $e$ and $e^*$ intersect at right angle. Given any configuration $\xi \in \Xi$, we denote $\mathcal{O}^*$ (resp. $\mathcal{C}^*$) the set of dual edges whose corresponding primal edge is open (resp. closed). Notice however that, unlike when we consider independent bond percolation, for alignment percolation the distributions of $\mathcal{O}^*$ and $\mathcal{C}$ are not the same. Notice also that any event can be defined in terms of the statuses of either the primal or the dual edges. Define for $x \in \mathbb{Z}^2$ events

$$A_k(x) := \{ \text{there is an open circuit in } B(x, 10L_k) \setminus B(x, L_k) \text{ surrounding } B(x, L_k) \} \cap \mathcal{E},$$

which can be seen as the event on which there exists a dual path of $\mathcal{C}^*$ edges from the inside of $B(x, L_k)$ to the outside of $B(x, 10L_k)$. The key fact is that the events $A_k(x)$ satisfy Properties P1 and P2, allowing us to use Lemmas 2 and 3. The reader is invited to consult Figure 4 for an illustration of the event $A_{k+1}(o)$.

Recall that $p^*$ was defined in (1). In order to establish a meaningful bound on the probability of the events $A_k(x)$ we need to perform the triggering step. This is the content of the following:

**Lemma 4 (Step S3 – Trigger for Theorem 1).** For any $p \in [p^*/2, 1]$ there are $k_0 > 0$ and $\lambda_0 \in (0, 1)$ such that for every $\lambda \geq \lambda_0$ and $k \geq k_0$

$$q_k(p, \lambda) \leq e^{-L_k^d}.$$  (19)
Proof. Let $k_0(p)$ be given as in Lemma 3. By Remark 2 we can take $k_0$ uniformly for $p \in [p^*/2, 1]$. Now, let us check that we can take $\lambda$ sufficiently close to 1 in order to ensure that $q_{k_0}(p, \lambda) \leq \exp[-L_{k_0}^\beta]$. Let $N(k_0)$ be the total number of edges from $\mathbb{E}^d$ with some extremity in $B(o, 10L_{k_0})$ and define $\lambda_0 = (1 - \exp[-L_{k_0}^\beta])^{N(k_0)-1}$. Notice that we can write

$$q_{k_0}(p, \lambda) = \mathbb{P}_{p, \lambda}(A_{k_0}(\omega)) \leq 1 - \mathbb{E}_{p, \lambda} \left[ \mathbb{P}_{p, \lambda}(\text{all edges inside } B(o, 10L_{k_0}) \text{ are open } | \eta) \right].$$

Since $k_0$ is fixed, the number of edges of $F(\eta)$ that have some extremity inside $B(o, 10L_{k_0})$ is bounded from above by $N(k_0)$. Thus, we have for every $\lambda \geq \lambda_0$,

$$q_{k_0}(p, \lambda) \leq 1 - \lambda N(k_0) \leq 1 - \lambda_0 N(k_0) = \exp[-L_{k_0}^\beta].$$

The result follows from Lemma 3.

We are now ready to give the proof of Theorem 1 in the case $d = 2$. The idea is as follows: the bound (19) shows that, for large enough $k$, it is hard to have long dual paths of length $L_k$ as soon as $\lambda$ is taken large. We can build on this fact to show that dual circuits around the origin will also be unlikely. This ultimately shows that the origin belongs to an infinite connected component with positive probability.

**Proof of Theorem 1, case $d = 2$.** We now show that for any $p \in [p^*/2, 1]$ and $\lambda \geq \lambda_0$ given by Lemma 4 we have that $O$ percolates $\mathbb{P}_{p, \lambda}$-a.s. Since we are working on the plane, we have that if $O$ does not percolate there must be a sequence $\gamma_n$ of disjoint circuits in $(L^2)^*$ that surround the origin with $\gamma_n \subset C^*$ and

$$\text{dist} (o, \gamma_n \cap (\mathbb{R}^+ \times \{0\})) \rightarrow \infty. \quad (20)$$

The reader might find it useful to consult Figure 5 for an illustration of such a sequence of circuits. For each $k \geq 0$ consider the points $\{x_{k,i} \subset \mathbb{R}^+ \times \{0\}$ defined by

$$x_{k,i} := (10L_k + (i-1)2L_k, 0) \quad \text{for } 1 \leq i \leq 501.$$ 

This choice of points ensures that

- $B(x_{k,i}, 10L_k) \subset \mathbb{R}^+ \times \mathbb{R}$;
- $B(x_{k,i}, L_k)$ and $B(x_{k,i+1}, L_k)$ are adjacent;
- $x_{k+1,i}$ is to the left of $x_{k,501}$.

Then, the family of boxes $\{B(x_{k,i}, L_k); k \geq 0, 1 \leq i \leq 501\}$ covers the half-line $(10L_0, \infty) \times \{0\}$. Choose some ordering $\hat{A}_l$ for the collection of events $\{A_k(x_{k,i}); k \geq 0, 1 \leq i \leq 501\}$. The existence of the sequence of circuits $\gamma_n \subset C^*$ satisfying (20) implies

$$\mathbb{P}_{p, \lambda} (\{O \text{ percolates}\})^c \leq \mathbb{P}_{p, \lambda} (\hat{A}_l, \text{i.o.}).$$

However, we have by Lemma 4 that

$$\sum_{l \geq 1} \mathbb{P}_{p, \lambda} (\hat{A}_l) = \sum_{k \geq 0} \sum_{i=1}^{501} q_k(p, \lambda) \leq 501 \sum_{k \geq 0} \exp[-L_k^\beta] < \infty$$

and Borel-Cantelli lemma implies that $\mathbb{P}_{p, \lambda} (\hat{A}_l, \text{i.o.}) = 0$. We conclude that for $p \in [p^*/2, 1]$ and $\lambda \geq \lambda_0$ the set $O$ percolates $\mathbb{P}_{p, \lambda}$-a.s. By [2, Theorem 2.4], we can find $\lambda_0$ such that if $p \in (0, p^*/2)$
and \( \lambda \geq \tilde{\lambda}_0 \) then \( \mathbb{P}_{\lambda} \)-a.s. Increasing \( \lambda_0 \) if needed, we can assume \( \lambda_0 \geq \tilde{\lambda}_0 \) and then we have \( \lambda_c(p,2) \leq \lambda_0 \) uniformly on \( p \in (0,1] \).

3.3. Continuity at \( p = 1 \)

We now employ a similar renormalization argument to prove Theorem 2. The sequence of scales we choose is still (11), but now we consider events

\[
A_k(x) = \{ B(x,L_k) \text{ is connected to } \partial B(x,10L_k) \text{ by edges in } \mathcal{O} \}.
\]

Once again, it is straightforward to check that events \( A_k(x) \) satisfy properties \( \mathbf{P1} \) and \( \mathbf{P2} \).

**Proof of Theorem 2.** Fix \( \lambda < p_{\mathbb{C}}^{\text{bond}}(d) \). We first need to complete the triggering step for events \( A_k(x) \).

By Remark 2 we can take \( \tilde{k}_0 \) depending only on \( d \) for \( p \in [1/2,1] \). Denoting by \( V = V(d,k) \) the set of vertices of \( B(o,10L_k) \) and by \( \mathbb{P}_\lambda^b \) the independent bond percolation measure on \( \mathbb{L}^d \), we can write

\[
q_k(p,\lambda,d) \leq \mathbb{P}_{p,\lambda}(\xi(V) \neq 1) + \mathbb{P}_{p,\lambda}(\xi(V) = 1, A_k(o))
= (1 - p^{|V|}) + p^{|V|} \cdot \mathbb{P}_\lambda^b(A_k(o)).
\]

Since \( \lambda < p_{\mathbb{C}}^{\text{bond}}(d) \) we have exponential decay for the radius of an open cluster, cf. [5, Theorem 5.4]. Thus, there is a positive constant \( \psi(\lambda) \) such that

\[
\mathbb{P}_\lambda^b(A_k(o)) \leq c(d)L_{k}^{d-1} \cdot \exp[-\psi(\lambda)L_k].
\]

Let us now pick \( \tilde{k}_0(\lambda,d) > 0 \) such that

\[
c(d)L_{k}^{d-1} \cdot \exp[-\psi(\lambda)L_k] \leq \frac{1}{2} \exp[-L_{k}^\beta] \quad \text{for } k \geq \tilde{k}_0(\lambda,d).
\]

Taking \( k_1(\lambda,d) = \max\{\tilde{k}_0(\lambda,d),k_0(d)\} \), we have

\[
q_{k_1}(p,\lambda,d) \leq (1 - p^{|V(k_1)|}) + \frac{1}{2} \exp[-L_{k_1}^\beta].
\]

(22)
Since $\lambda$ and $d$ are fixed we can pick $p_0 = p_0(\lambda, d)$ such that $1 - p|V(k_1)| \leq \frac{1}{2} \exp[-L_{k_1}^\beta]$, for any $p \geq p_0(\lambda, d)$. Plugging into (22) we get

$$q_{k_1}(p, \lambda, d) \leq \exp[-L_{k_1}^\beta] \text{ for } p \geq p_0(\lambda, d),$$

concluding the trigger step. This implies that there is no percolation for these values of $p, \lambda$ and $d$. In fact, from (23) and Lemma 3 we get $q_k(p, \lambda, d) \leq \exp[-L_k^\beta]$ whenever $k \geq k_1$ and since

$$\theta(p, \lambda, d) \leq \lim_{k \to \infty} q_k(p, \lambda, d) = 0$$

we have

$$\theta(p, \lambda, d) = 0 \text{ for } p \geq p_0,$$

that implies (3). In other words, this means that for any $\lambda < \lambda_c^{\text{bond}}(d)$ the critical curve restricted to the interval $[p_0(\lambda, d), 1]$ must be above the horizontal segment of height $\lambda$. The continuity of $\lambda_c(\cdot, d)$ at $p = 1$ follows from the fact that $\lim_{p \to 1} \lambda_c(p, d) \leq \lambda_c^{\text{bond}}(d)$, as explained in the paragraph before the statement of Theorem 2.

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