Abstract

We show the light-front representation of the field theoretical Bethe-Salpeter equation (BSE) in the ladder approximation using the quasi potential reduction. We discuss the equivalence of the covariant ladder Bethe-Salpeter equation with an infinite set of coupled equations for the Green’s functions of the different light-front Fock-states.

1 Introduction

Recently, we have proposed a quasi-potential reduction of the field-theoretical Bethe-Salpeter equation to study two-particle bound states on the light-front for composite systems of bosons and fermions [1]. The reduction scheme eliminates the relative light-front time between the particles, and the global propagation of the intermediate system appears leading to Dirac’s [4] concept of representing the dynamics of the quantum system for light-front times $x^+ = t + z$. We have derived two-body equations for which the effective interaction is irreducible with respect to the light-front two-body propagation. The global light-front time projection makes explicit the intermediate propagation within higher Fock-space sectors, and the mixing with the valence component of the wave-function. The effective two-body equation for the valence component of the light-front wave-function from the quasi-potential reduction has an effective interaction which contains the coupling with higher-Fock state components. In lowest order, the two-body bound-state equation is the one first derived by Weinberg [2].

In view of the possible application of the proposed scheme to study the structure of the bound-state, for example, with electromagnetic probes [1 3], it is timely to reveal the
physical aspects of the dynamics generated by the quasi-potential reduction with respect to the mixing of the lowest Fock component with higher Fock components of the light-front wave function. We exemplify our procedure using a model for interacting bosons \( \phi_1 \), \( \phi_2 \) and \( \sigma \) for which the interaction Lagrangian is given by,

\[
\mathcal{L}_I = g_S \phi_1^\dagger \phi_1 \sigma + g_S \phi_2^\dagger \phi_2 \sigma .
\]  

(1)

2 Projection on the Light-Front Hyperplane

The two-boson transition matrix \( T(K) \) in relativistic field theory, for a given total four-momentum of the system is the solution of

\[
T(K) = V(K) + V(K)G_0(K)T(K),
\]

(2)

where \( V(K) \) is two-body irreducible. Neglecting self-energy parts the disconnected Green’s function, written in light-front coordinates, e.g., \( k_i = (k_i^- := k_i^0 - k_i^3 \ , \ k_i^+ := k_i^0 + k_i^3 \ , \ \vec{k}_{i\perp}) \), is given by:

\[
\langle k^- \mid G_0(K) \mid k^- \rangle = - \frac{1}{2\pi} \frac{\delta (k^-_1 - k^-_1)}{k^+_1 (K^- \! - \! \vec{k}_{1\perp}^\dagger)} \frac{\delta (k^-_1 - k^-_1)}{k^+_1 (K^- \! - \! \vec{k}_{1\perp}^\dagger)} \left( k^-_1 - \frac{\pi^2 + m^2 - io}{K^- \! - \! k^+_1} \right),
\]

(3)

where the “hat” indicates operator form in the “kinematic” momentum \((k^+_1, \vec{k}_{1\perp})\), and \( \vec{k}_{1\perp}^\dagger = \frac{\pi^2 + m^2 - io}{K^- \! - \! k^+_1} \). The basis states for functions of the kinematical light-front momentum are defined by \( \langle x^-_1 \! - \! \vec{x}_{1\perp} \mid k^+_1 \! - \! \vec{k}_{1\perp}^\dagger \rangle = e^{-i(\vec{k}_{1\perp} \cdot \vec{x}_{1\perp})} \), which are eigenfunctions of the momentum operators \((\vec{k}^+_1, \vec{k}_{1\perp})\) and the free energy operator \( \vec{k}^-_{1\perp} \). The states \( \mid k^+_1 \! - \! \vec{k}_{1\perp} \rangle \) form a complete basis in the space of functions of the kinematical variables, e.g., \( \int \frac{dk^+_1 \!/ (2\pi)^3 \mid k^+_1 \! - \! \vec{k}_{1\perp} \rangle \langle k^+_1 \! - \! \vec{k}_{1\perp}}{c = 1} \) with normalization \( \langle k^+_1 \! - \! \vec{k}_{1\perp} \! \mid k^+_1 \! - \! \vec{k}_{1\perp} \rangle = 2(2\pi)^3 \delta (k^+_1 \! - \! k^+_1) \delta (\vec{k}_{1\perp} \! - \! \vec{k}_{1\perp}) \).

The global light-front time propagator of the free two-boson system between the hyperplanes \( x^+_1 = x^+_2 = x^+ \) and \( x^+_1 = x^+_2 = x^+ \), is the Fourier transform of

\[
g_0(K) = \mid G_0(K) \mid = \int dk^+_1 \! dk^-_1 \langle k^-_1 \! - \! \vec{k}_{1\perp} \mid G_0(K) \mid k^-_1 \! - \! \vec{k}_{1\perp} \rangle = \frac{i g_0^{(2)}(K)}{k^+_1 (K^+ \! - \! k^+_1^\dagger)} ,
\]

(4)

where \( K^+ > 0 \) is used without any loss of generality. In Eq. (4) the vertical bar \( \mid \) indicates that the dependence on \( k^-_1 \) is integrated out. The bar on the left of the Green’s function represents integration on \( k^-_1 \) in the bra-state, the bar on the right in the ket-state.
The free two-body light-front Green’s function, $g^{(2)}_0(K)$ in Eq. (4), is a particular case of the light-front Green’s function for $N$ particles:

$$g^{(N)}_0(K) = \left[ \prod_{j=1}^{N} \theta(\hat{k}^+_j) \theta(K^+ - \hat{k}^+_j) \right] \left( K^- - \tilde{K}_0^{(N)}^- + i\sigma \right)^{-1},$$  \hspace{1cm} (5)

where $\tilde{K}_0^{(N)}^- = \sum_{j=1}^{N} \hat{k}_j^-$ is the free light-front Hamiltonian. The difference between the free two-body light-front Green’s function and $g_0(K)$, Eq. (4), is the phase-space factor for particles 1 and 2.

The covariant two-body propagator is the solution of

$$G(K) = G_0(K) + G_0(K)V(K)G(K) = G_0(K) + G_0(K)T(K)G_0(K).$$  \hspace{1cm} (6)

It gives the individual time propagation of the two-body system. The interacting propagator between hyperplanes of $x^+ = \text{const.}$, is given by $g(K) \equiv |G(K)|$.

Our goal is the decomposition of the kernel of the integral equation of the global two-body propagator, $g(K)$, with an effective interaction irreducible with respect to the light-front propagation of the intermediate two-body system. For that purpose, we use the quasi-potential reduction of Ref. [5], where the transition matrix $T(K)$ and the Bethe-Salpeter amplitude $|\Psi_B\rangle$ of the covariant BSE can be obtained with the help of an auxiliary Green’s function $\tilde{G}_0(K)$, which we choose as

$$\tilde{G}_0(K) := G_0(K)|g_0^{-1}(K)|G_0(K),$$  \hspace{1cm} (7)

because it allows to make explicit the two-boson propagation on the global light-front time in the intermediate states. The transition matrix satisfies

$$T(K) = W(K) + W(K)\tilde{G}_0(K)T(K),$$  \hspace{1cm} (8)
$$W(K) = V(K) + V(K)|G_0(K) - \tilde{G}_0(K)|W(K).$$  \hspace{1cm} (9)

We introduce the auxiliary three-dimensional transition matrix

$$t(K) = g_0(K)^{-1}|G_0(K)T(K)G_0(K)|g_0(K)^{-1}.$$  \hspace{1cm} (10)

The interaction in the integral equation for $t(K)$, has the desired property to be irreducible in respect light-front two-body propagation. The auxiliary transition matrix is the solution of a three-dimensional integral equation, derived from Eq. (8):

$$t(K) = w(K) + w(K)g_0(K)t(K),$$  \hspace{1cm} (11)
$$w(K) = g_0(K)^{-1}|G_0(K)W(K)G_0(K)|g_0(K)^{-1}.$$  \hspace{1cm} (12)
From Eqs. (3) and (11), one has:

\[ g(K) = g_0(K) + g_0(K)w(K)g(K) \quad \text{(13)} \]

The dynamics of the interacting two-particle system can be fully described by its propagation between hyperplanes \( x^+ = x^0 + x^3 = \text{const.} \) in light-front dynamics [4], and covariant the transition matrix can be expressed in terms of \( t(K) \) [1].

### 3 Light-Front Two-Particle Green’s Function

The integral equation satisfied by the light-front Green’s function, derived from Eq. (13), is

\[ g^{(2)}(K) = g_0^{(2)}(K) + g_0^{(2)}(K)\nu(K)g^{(2)}(K) \quad \text{(14)} \]

where \( g^{(2)}(K) \equiv -i\hat{\omega}g(K)\hat{\Omega}, \nu(K) = i\hat{\omega}^{-1}w(K)\hat{\Omega}^{-1} \) and the phase space operator is \( \hat{\Omega} := \sqrt{k_1^+(K^+ - \bar{k}_1^+)} \),

The interaction \( w(K) \), Eq. (12), expanded according to Eq. (9) in first order of the driving term \( V(K) \), is given by

\[ w^{(2)}(K) = g_0(K)^{-1}|G_0(K)V(K)G_0(K)|g_0(K)^{-1} \quad \text{(15)} \]

The matrix element \( \langle k_1^+\bar{k}_1^+|w^{(2)}(K)|k_2^+\bar{k}_2^+ \rangle \) is obtained from Eq. (15):

\[ \langle k_1^+\bar{k}_1^+|w^{(2)}(K)|k_2^+\bar{k}_2^+ \rangle = (igs)^2 \frac{\theta(k_1^+ - k_2^+)}{(k_1^+)^2} \frac{i}{K^- - K_0^{(3)-}} + io + [k' \leftrightarrow k_1] \quad \text{(16)} \]

where \( K_0^{(3)-} = \frac{\bar{k}_1^2 + m^2}{k_1^+} - \frac{(\bar{k}_1^2 - \bar{k}_2^2 + m^2)}{K^- - k_2^+} - \frac{(\bar{k}_1^2 - \bar{k}_2^2 + m^2)}{k_1^+ - k_2^+} \).

The second order term in the expansion of \( w(K) \), is given by

\[ w^{(4)}(K) = g_0(K)^{-1}|G_0(K)V(K)G_0(K)V(K)G_0(K)|g_0(K)^{-1} - g_0(K)^{-1}|G_0(K)V(K)G_0(K)V(K)G_0(K)|g_0(K)^{-1} \quad \text{(17)} \]

The second term in the r.h.s of Eq. (17) comes from the iteration of \( w^{(2)}(K) \), which is \( w^{(2)}g_0(K)w^{(2)} \). The subtraction of the iterated term in Eq. (17) cancels the corresponding terms, such that the matrix element \( \langle k_1^+\bar{k}_1^+|w^{(4)}(K)|k_2^+\bar{k}_2^+ \rangle \) is two-body irreducible. The final form is:

\[ \langle k_1^+\bar{k}_1^+|w^{(4)}(K)|k_2^+\bar{k}_2^+ \rangle = \frac{(igs)^4}{2(2\pi)^3} \int dp_1^+ dp_1^+ \theta(p_1^+) \theta(K^+ - p_1^+) \frac{\theta(K^+ - p_1^+)}{p_1^+} \]
\[
\times \frac{\theta (k_1^+ - p_1^+)}{(k_1^+ - p_1^+)} K^- - \frac{p_{1\perp}^2 + m_1^2}{p_1^-} - \frac{i}{k_1^+ - k_1^+} \left[ (\vec{K}_{1\perp} - p_{1\perp})^2 + m_2^2 + \mu^2 \right] + \mu_1 + \mu_2 \]

\[
\times \frac{\theta (p_1^+ - k_1^+)}{(p_1^+ - k_1^+)} K^- - \frac{k_{1\perp}^2 + m_1^2}{k_1^+ - k_1^+} - \frac{i}{k_1^+ - k_1^+} \left[ (\vec{K}_{1\perp} - p_{1\perp})^2 + m_2^2 + \mu^2 \right] + \mu_1 + \mu_2 \]

\[
+ \ [k_1'^+ \leftrightarrow k_1] . \tag{18}
\]

Introducing the interaction between light-front states which creates or destroys a quantum of the intermediate boson, \( \sigma \) defined by the matrix elements

\[
\langle q|k_\sigma|v\rangle = -2(2\pi)^3 \delta (q + k_\sigma - k) \frac{g_s}{\sqrt{q^+ k_{\sigma^+}^+}} \theta (k_\sigma^+) \]

\[
\langle q|v|k_\sigma k \rangle = -2(2\pi)^3 \delta (k + k_\sigma - q) \frac{g_s}{\sqrt{q^+ k_{\sigma^+}^+}} \theta (k_\sigma^+) , \tag{19}
\]

for the model defined by the Lagrangian of Eq. \( \text{(1)} \), we can rewrite the first and second order terms \( w^{(2)} \) and \( w^{(4)} \), in the form of the effective interaction terms \( \nu^{(2)} (K) \) and \( \nu^{(4)} (K) \), respectively, as:

\[
\langle k_1'^+ \vec{k}_{1\perp} | \nu^{(2)} (K) | k_1^+ \vec{k}_{1\perp} \rangle = \langle k_1'^+ \vec{k}_{1\perp} | v g_0^{(3)} (K) v | k_1^+ \vec{k}_{1\perp} \rangle . \tag{20}
\]

\[
\langle k_1'^+ \vec{k}_{1\perp} | \nu^{(4)} (K) | k_1^+ \vec{k}_{1\perp} \rangle = \langle k_1'^+ \vec{k}_{1\perp} | v g_0^{(3)} (K) v g_0^{(4)} (K) v g_0^{(3)} (K) v | k_1^+ \vec{k}_{1\perp} \rangle . \tag{21}
\]

Therefore, the effective interaction \( \nu (K) \) up to second order in the expansion is the sum of Eqs. \( \text{(20)} \) and \( \text{(21)} \):

\[
\nu (K) \approx v \left( g_0^{(3)} (K) + g_0^{(3)} (K) v g_0^{(4)} (K) v g_0^{(3)} (K) \right) v . \tag{22}
\]

We identify in Eq. \( \text{(22)} \), the propagation of the intermediate three-particle system, with corrections up to second order in the coupling constant. It is easy to imagine that the interacting three-particle Green’s function should be obtained if the expansion of the auxiliary interaction \( w (K) \) is performed to all orders in the ladder using Eqs. \( \text{(1)} \) and \( \text{(12)} \). Thus, the effective interaction should be \( \nu (K) = v g^{(3)} (K) v \). From Eq. \( \text{(22)} \), one sees that \( g^{(3)} \) is as function of the four-particle Green’s function, which should be coupled to the five-body one, and so on. Therefore, one can construct a hierarchy of coupled equations for Green’s functions with any number of particles.

5
4 Hierarchy Equations and Summary

The light-front Green’s function of the two-body system expressing the covariant BS equation is written as:

\[
\begin{align*}
g^{(2)}(K) &= g^{(2)}_0(K) + g^{(2)}_0(K)vg^{(3)}(K)vg^{(2)}(K), \\
g^{(3)}(K) &= g^{(3)}_0(K) + g^{(3)}_0(K)vg^{(4)}(K)vg^{(3)}(K), \\
g^{(4)}(K) &= g^{(4)}_0(K) + g^{(4)}_0(K)vg^{(5)}(K)vg^{(4)}(K), \\
&\quad\ldots, \\
g^{(N)}(K) &= g^{(N)}_0(K) + g^{(N)}_0(K)vg^{(N+1)}(K)vg^{(N)}(K),
\end{align*}
\]

(23)

This set of equations contains all two-body irreducible diagrams with exception of those including closed loops of bosons \(\Phi_1\) and \(\Phi_2\) and part of the cross-ladder diagrams. The truncation of the light-front Fock space allows only two bosons states with two particles \(\Phi_1\) and \(\Phi_2\) in the intermediate states, without any restriction on the number of bosons \(\sigma\), which excludes the complete representation of the crossed ladder diagrams. It also resembles the iterated resolvent method of Ref.[6]. To get the two-body propagator for light-front times in the covariant ladder approximation, the kernel of the hierarchy equations should be restricted.

A systematic expansion can be obtained by truncating the light-front Fock space up to \(N\) particles in the intermediate states (boson 1, boson 2 and \(N - 2\) \(\sigma\)’s) in the set of Eqs.(23), which amounts to substituting \(g^{(N)}(K) \rightarrow g^{(N)}_0(K)\). By restricting to up to four-particles in the intermediate state propagation, we get the following nonperturbative equation for the Green’s function:

\[
\begin{align*}
g^{(2)}(K) &= g^{(2)}_0(K) + g^{(2)}_0(K)vg^{(3)}(K)vg^{(2)}(K), \\
g^{(3)}(K) &= g^{(3)}_0(K) + g^{(3)}_0(K)vg^{(4)}(K)vg^{(3)}(K).
\end{align*}
\]

(24)

(25)

The kernel of Eq.(24) still contains an infinite sum of light-front diagrams, that are obtained by solving Eq.(25). To get the ladder approximation up to order \(g_4^4\), or up to second order in the quasi-potential expansion, with the effective interaction of Eq.(23), only the free and first order terms are kept in Eq.(25), while restricting it only to the ladder.

In summary, we discussed the general framework for constructing the light-front two-body Green’s function with quasi-potential reduction of the Bethe-Salpeter equation in the light-front. We displayed a set of coupled hierarchy equations which gives the two-body propagator in several cases, including the ladder approximation. Finally we pointed out that the truncation in the hierarchy can be performed consistently by approximating the N-body Green’s function with the free operator, i.e., the light-front N-body intermediate state is accounted in lowest order in the two-body propagation.
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