Berry-Esseen bounds for weighted averages of Poisson avoidance functionals

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Abstract

We consider functionals which are weighted averages of the avoidance function of a Poisson process. Using the approach to Stein’s method based on Malliavin calculus for Poisson functionals we provide explicit bounds for the Wasserstein distance between these standardized functionals and the standard normal distribution. Our approach relies on closed-form expressions for the action of some Malliavin type operators on avoidance functionals of Poisson processes. As a result we provide Berry-Esseen bounds in the CLT for the volume of the union of balls of a fixed radius around random Poisson centers or for the quantization error around points of a Poisson process. We also give Berry-Esseen bounds for avoidance functionals of empirical measures.

1 Introduction

Stein’s method of normal approximation has become one of the main tools for proving central limit theorems for general functions of independent random variables. In the last years the power of the method has been greatly expanded in a number of directions. In a series of papers ([10], [11], [12]) Nourdin, Peccati and co-authors showed how the ideas of Stein’s method could be combined with Malliavin calculus to produce explicit bounds for the normal approximation of smooth Gaussian functionals. More recently, in [13], the method was extended to cover normal approximation in Wasserstein distance of functionals of Poisson random measures. This approach has been successfully used for proving CLT’s for sequences of multiple Wiener-Itô integrals with respect to a Poisson measure (see [14], [17] and [8]) and in several instances in stochastic geometry, including CLT’s for the volume of the Poisson-Voronoi approximation of a compact convex set in Euclidean space (see [18]), for the length of a random geometric graph (see [17]) or for geometric functionals of intersection processes of a Poisson process of $k$-dimensional flats in $\mathbb{R}^d$ (see [8]).

The key to the Malliavin calculus approach to Stein’s method for Poisson functionals is that the Wasserstein distance between a standardized Poisson functional and the standard normal distribution is bounded by moments of some Malliavin operators acting on the functional. We refer to [7] and [14] for a complete account of the theory and to Sections 2 and 3 below for a succinct, self-contained description of the main facts about it. Now, if $\eta_\lambda$ is a homogeneous Poisson process with intensity $\lambda > 0$, $F_\lambda = F_\lambda(\eta_\lambda)$ satisfies $0 < \text{Var}(F_\lambda) < \infty$ and $X$ is a standard normal random variable then

$$
\text{d}_W \left( \frac{F_\lambda - \text{E}(F_\lambda)}{\sqrt{\text{Var}(F_\lambda)}}, X \right) \leq \frac{\text{Var}((DF_\lambda, -DL^{-1}(F_\lambda - \text{E}(F_\lambda))))^{1/2} + (\int \text{E}(D_z F_\lambda)^4 d\mu(z))^{1/2}}{\text{Var}(F_\lambda)},
$$

where $D$ denotes the difference operator and $L$ the Orstein-Uhlenbeck operator. More recently, in [2], a similar, more involved bound was proved for the case of $d_K$, the Kolmogorov metric. Both $DF_\lambda$

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and $-DL^{-1}(F_\lambda - E(F_\lambda))$ have a chaos decomposition that can be simply expressed in terms of the chaos decomposition of $F_\lambda$, namely, in the expansion

$$F_\lambda(\eta_\lambda) = E(F_\lambda(\eta_\lambda)) + \sum_{n=1}^{\infty} I_n(f_{n,\lambda}),$$

(1.2)

where $I_n(f_{n,\lambda})$ denotes the multiple Wiener-Itô integral of $f_{n,\lambda}$ with respect to $\eta_\lambda$ (we refer again to [17, 14] and Section 2 for details). This enables one to control the upper bound in (1.1) through the control of moments of products of multiple Wiener integrals of different orders. When the chaos expansion (1.2) consists of a finite number of terms this can be done through the use of diagram formulae, as discussed in [14, 17] or [8]. However, the kind of bound that one obtains from these diagram formulae is not tight enough for functionals with infinite terms in their chaos expansion. Still, the approach can be adapted through truncation arguments, as in [18] or [8] to prove CLT’s, but, to our best knowledge, no Berry-Esseen bounds can be derived from this type of approach in the case of infinite chaos expansions.

A more direct approach to obtain upper bounds for the right-hand side in (1.1) would be to exploit pathwise representations of the operators $D$ and $DL^{-1}$, not relying on $L_2$ expansions. There is indeed a simple pathwise representation for $D$ (see (2.3) below). A major breakthrough in the theory is given by [6], which provides a simple representation of the action of $L^{-1}$ through a so-called Mehler’s formula (see Theorem 3.2 in [6]). This representation greatly increases the usability of upper bounds like (1.1). In the cited reference, the representation is combined with a Poincaré inequality (see Proposition 2.5 in [6]) to produce simple bounds for the distance $(d_W$ or $d_K$) to normality of Poisson functionals (Theorems 1.1 and 1.2 in the cited reference).

In this paper we consider a special class of functionals for which the target functionals admit a particularly simple pathwise representation. We will assume throughout the paper that $\eta_\lambda$ is a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity $\lambda > 0$, defined on a probability space $(\Omega, \mathcal{F}, P)$, $A$ a bounded open set on $\mathbb{R}^d$ and $(\mathcal{B}, \mathcal{G}, \nu)$ a measure space. We further assume that, for each $s \in \mathcal{B}$, $Q(s)$ is a Borel set on $\mathbb{R}^d$ chosen in such a way that the map $(\omega, x, s) \mapsto 1(\eta_\lambda(x + \lambda^{-1/d}Q(s)) = 0)$ is $\mathcal{F} \otimes \beta^d \otimes \mathcal{G}$ measurable, where $\beta^d$ denotes the Borel $\sigma$-field on $\mathbb{R}^d$ and 1 stands for the indicator of a set. Under these assumptions, we will focus on the functional

$$F_\lambda(\eta_\lambda) := \int_{A \times B} 1(\eta_\lambda(x + \lambda^{-1/d}Q(s)) = 0) d(\ell \otimes \nu)(x, s).$$

(1.3)

This functional is a weighted average of values of the avoidance functional $A \mapsto 1(\eta_\lambda(A) = 0)$. The avoidance functional essentially contains all the information about $\eta_\lambda$ and, in fact, from Renyi’s Theorem and its generalizations (see, e.g., Theorem 9.2.XII in [H]) it is known that, with great generality, the distribution of a simple point process is determined by the values of the mean avoidance function $A \mapsto P(\eta(A) = 0)$ (we note that the term avoidance function is usually applied to this last functional; here we find it more convenient to make this change). We think of $x + \lambda^{-1/d}Q(s)$ as a suitably scaled neighborhood of $x$ of a particular shape. We show in this paper that the weighted averages of avoidance functionals like the one in (1.3) satisfy (see Theorems 3.4 and 3.6 below)

$$d_W \left( \frac{F_\lambda(\eta_\lambda) - E(F_\lambda(\eta_\lambda))}{\sqrt{\text{Var}(F_\lambda(\eta_\lambda))}} \right), X \right) \leq \frac{1}{\sqrt{X}} C_1 \frac{C_2(\lambda)}{\lambda}, \quad \lambda > 0,$$

(1.4)

where $X$ is a standard normal random variable, $C_1 = C_1(\{Q(s)\})$ is a constant that measures the size of certain four-wise intersections of translates of the sets $B(s)$ and $C_2(\lambda) = C_2(\lambda; \{Q(s)\})$ grows with $\lambda$ to a positive constant $C_2(\{Q(s)\})$ which measures measures the size of pairwise intersections of translates of the sets $B(s)$, with a similar result for $d_K$. Thus, CLT’s with rates for weighted averages of avoidance functionals can be proved from simple geometrical considerations about the family $\{Q(s)\}$. While our approach lacks the generality of the results in [B], the particular form of Mehler’s formula for the class of functionals considered here allows for a more direct treatment of the different terms that have to be controlled. Specifically, we do not use a general Poincaré inequality
but rely, instead, on an elementary bound for covariances of certain functionals or Poisson random measures (see Lemma 3.2 below). As a result we obtain bounds that are not covered by [6] and offer a transparent geometric interpretation, see Remark 3.5 below for further details.

We consider also the case in which we replace $\eta_i$ by an empirical measure $\nu_n = \sum_{i=1}^n \delta_{X_i}$, where $\{X_i\}$ are i.i.d. $\mathbb{R}^d$-valued r.v.’s. We show that, under suitable assumptions, for each $\delta \in (0, \frac{1}{2})$ there exists a positive constant, $C(\delta)$, such that

$$d_W \left( \frac{F_n(\nu_n) - E(F_n(\nu_n))}{\sqrt{\text{Var}(F_n(\nu_n))}}, X \right) \leq \frac{C(\delta)}{n^{1/4-\delta}}, \quad n \geq 1. \quad (1.5)$$

As an illustration of the power of (1.4) and (1.5) we show its application in two classical problems in stochastic geometry: germ-grain models and quantization. Germ-grain models are a common model in stochastic geometry (see, e.g., [19]). They deal with random sets which arise as the union of sets centered around random centers. A functional of interest in this setup is the volume of a transparent geometric interpretation, see Remark 3.5 below for further details.

Our second application deals with the problem of quantization, that is, of approximation of a random set. There are different approaches to this problem, depending on the way in which the locations of the finitely supported probability are chosen. Here we focus on quantization around random locations. More precisely, we consider the action of the key operators on avoidance functionals. Section 3 contains the main results of the paper, including the Berry-Esseen bounds for avoidance functionals of Poisson process. The focus is put on the case of homogeneous Poisson processes, for which the presentation is simpler, but we discuss the extension to general Poisson process and also to empirical measures. Section 4 presents the application to the germ-grain model and to random quantization errors. Finally, an Appendix provides some technical results used in some proofs in Section 3.
2 Malliavin calculus for avoidance functionals

As noted in the Introduction, the main result in this paper relies on the Malliavin calculus approach to Stein’s method for Poisson functionals introduced in [14], which provides an upper bound for the Wasserstein distance between the distribution of a Poisson functional and the standard normal distribution. The upper bound is expressed in terms of certain Malliavin type operators, which, in turn, are defined in terms of a Poisson chaos decomposition. For the sake of readability we summarize in this section the main facts about this chaos decomposition and the Malliavin type operators and refer to [7] and [13] for further details. Later in the section we particularize to the weighted averages of avoidance functionals, computing their chaos expansion and the action of the relevant Malliavin operators on it.

We proceed then assuming that \( \eta \) is a Poisson random measure on \( \mathbb{R}^d \) with intensity measure \( \mu \) (the theory is indeed much more general but this is enough for our purposes). A fundamental result is that any square integrable functional \( F(\eta) \) admits a chaos decomposition, namely, it can be expressed as

\[
F(\eta) = E(F(\eta)) + \sum_{n=1}^{\infty} I_n(f_n)
\]

in \( L^2 \) sense, where \( f_n(x_1, \ldots, x_n) \) is a real valued, square integrable function with respect to \( \mu^n \), symmetric in its \( n \) arguments and \( I_n \) denotes the multiple Wiener-Itô integral with respect to the compensated Poisson process, see, e.g. [14] or [7] for details. For symmetric \( f \in L_2((\mathbb{R}^d)^n, \mu^n) \cap L_1((\mathbb{R}^d)^n, \mu^n) \) we have

\[
I_n(f) = \sum_{I \subset [n]} (-1)^{|I|} \int_{(\mathbb{R}^d)^n} f(x_1, \ldots, x_n) d(\eta^I \otimes \mu^{[n]-I})(x_1, \ldots, x_n)
\]

where \([n]\) is short for \(\{1, \ldots, n\}\) and integration with respect to \( \eta^I \otimes \mu^{[n]-I} \) means summing \( f(x_1, \ldots, x_n) \) with the \( x_i \)'s in the \( I \) positions being all different and ranging in the support of \( \eta \) and integrating with respect to \( \mu \) all the other variables. We refer again to [7] for details.

The terms in decomposition (2.1) are uncorrelated. In fact, for symmetric \( f \in L_2((\mathbb{R}^d)^n, \mu^n) \), \( g \in L_2((\mathbb{R}^d)^m, \mu^m) \) we have

\[
E(I_m(f)I_{m}(g)) = I(m = n)n!(f, g)_n
\]

where \((f, g)_n\) denotes the usual inner product in \( L_2((\mathbb{R}^d)^m, \mu^m) \). In particular, we see from (2.1) that

\[
\text{Var}(F(\eta)) = \sum_{n=1}^{\infty} n!\|f_n\|_2^2
\]

and also that the functions \( f_n \) in the chaos expansion (2.1) are unique. In fact, these functions can be expressed in a remarkably simple way in terms of the difference operator, \( D_z \), defined by the equation

\[
D_z F(\eta) = F(\eta + \delta_z) - F(\eta),
\]

with \( \delta_z \) meaning Dirac’s measure on \( z \). Thus, \( D_z F \) is the difference between the functional evaluated on the random set given by the support of \( \eta \) supplemented by \( z \) and its evaluation on the support of \( \eta \). In terms of this operator, the functions \( f_n \) in (2.1) are (up to a scaling constant) the expected values of the iterated difference operator acting on \( F(\eta) \). More precisely,

\[
f_n(x_1, \ldots, x_n) = \frac{1}{n!} E(D_{x_1, \ldots, x_n} F(\eta)).
\]

Here \( D_{x_1, \ldots, x_n} F(\eta) = D_{x_1} D_{x_2}, \ldots, x_n F(\eta) \). A useful fact about the operator \( D_{x_1, \ldots, x_n} \) is that

\[
D_{x_1, \ldots, x_n} F(\eta) = \sum_{I \subset \{1, \ldots, n\}} (-1)^{|I|} D_I F(\eta) + \sum_{i \in I} \delta_{x_i},
\]
see, e.g., [18].

Alternatively, the difference operator can be expressed in terms of an orthogonal expansion. In fact, for $F$ as in (2.1)

$$D_z F(\eta) = \sum_{n=1}^{\infty} nI_{n-1}(f_n(z, \cdot)).$$

(2.7)

Other important Malliavin operators are expressed through similar orthogonal expansions. These include the Ornstein-Uhlenbeck operator, $L$, given by

$$LF(\eta) = -\sum_{n=1}^{\infty} nI_n(f_n),$$

and the inverse Ornstein-Uhlenbeck operator, $L^{-1}$, for which

$$L^{-1}F(\eta) = -\sum_{n=1}^{\infty} \frac{1}{n}I_n(f_n).$$

The domain of $L$ consists of the $L_2$ functionals such that $\text{Var}(F(\eta)) = \sum_{n=1}^{\infty} n!n!\|f_n\|^2 < \infty$ while that of $L^{-1}$ is the set of centered $L_2$ functionals.

The operator $D_zL^{-1}$ will be of special interest for us. For $F$ in the domain of $L^{-1}$ we have

$$-D_zL^{-1}F = \sum_{n=1}^{\infty} I_{n-1}(f_n(z, \cdot)).$$

(2.8)

Our next results provide a simple closed form for $D_zL^{-1}F(\eta)$ in the case of avoidance functionals. Part d) in the next Lemma can be obtained from Mehler’s formula (Theorem 3.2 in [6]). Here we provide a self-contained elementary proof.

**Lemma 2.1** If $\eta$ is a Poisson point process on $\mathbb{R}^d$ with intensity measure $\mu$, $A \subset \mathbb{R}^d$ a Borel set such that $\mu(A) < \infty$ and $F(\eta) = 1(\eta(A) = 0) - e^{-\mu(A)}$ then,

(a) the $f_n$ functions in the chaos expansion (2.1) are given by

$$f_n(x_1, \ldots, x_n) = \frac{(-1)^n}{n!}e^{-\mu(A)} \prod_{i=1}^{n} 1_A(x_i), \quad n \geq 1.$$  

(2.9)

(b)

$$I_n(f_n) = \frac{e^{-\mu(A)}}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{\eta(A)}{k} k!\mu(A)^{n-k},$$

(c)

$$D_zF(\eta) = -1_A(z)F(\eta) \quad \text{and}$$

(d)

$$D_zL^{-1}F(\eta) = 1_A(z) \int_0^1 \psi(t) e^{-\mu(A)t} dt.$$  

**Proof.** We compute first the $f_n$ kernels in the chaos expansion (2.1). For convenience, given a point measure $\rho$ we rewrite the set $(\rho(A) = 0)$ as $(A \cap \rho = \emptyset)$. From (2.6) we see that

$$f_n(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{I \subset \{1, \ldots, n\}} (-1)^{|I|+|I|} P(A \cap (\eta \cup (\cup_{i \in I}\{x_i\})) = \emptyset).$$
Now, if \( x_i \in A \) for all \( i \in \{1, \ldots, n\} \) then all the terms in the expansion of \( f_n(x_1, \ldots, x_n) \) vanish except for the case \( I = \emptyset \) and, consequently, \( f_n(x_1, \ldots, x_n) = \frac{(-1)^n}{n!} P(A \cap \eta = \emptyset) \). Assume, on the contrary, that there is some index \( i \in \{1, \ldots, n\} \) such that \( x_i \notin A \). Then, for every \( J \subset \{1, \ldots, n\} \setminus \{i\} \) we have \( A \cap (\eta \cup \cup_{j \in J} \{x_j\}) = A \cap (\eta \cup \cup_{j \in J \cup \{i\}} \{x_j\}) \). Hence,

\[
 f_n(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{J \subset \{1, \ldots, n\} \setminus \{i\}} (-1)^{n+|J|} [P(A \cap (\eta \cup \cup_{j \in J} \{x_j\}) = \emptyset) - P(A \cap (\eta \cup \cup_{j \in J \cup \{i\}} \{x_j\}) = \emptyset)] = 0.
\]

Combining these facts we obtain (2.9).

We turn now to (b). Fix \( I \subset [n] \) with \( |I| = k \), say. Then

\[
 \int_{(\mathbb{R}^d)^n} f(x_1, \ldots, x_n) d(\eta^{(I)} \otimes \mu^{(n-I)})(x_1, \ldots, x_n) = \frac{(-1)^n}{n!} \int_{(\mathbb{R}^d)^n} e^{-\mu(A)} \mu(A)^{n-k} \binom{n}{k} \binom{\eta(A)}{k} d\eta(x_1, \ldots, x_n)
\]

and (2.2) becomes

\[
 I_n(z) = e^{-\mu(A)} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \mu(A)^{n-k} \binom{n}{k} \binom{\eta(A)}{k},
\]

proving (b). Part (c) is obvious. Finally, to prove (d) using (2.8) and linearity of the stochastic integral we see that

\[
 D_k L^{-1} F(\eta) = 1_A(z) e^{-\mu(A)} \int_{0}^{1} \frac{e^{-(1-x)\eta(A)\mu(A)}}{(1-x)^{n+1}} dx.
\]

Here we have used that \( \frac{1}{n+1} \) completes the proof.

From this Lemma 2.1 we obtain a simple description of the action of the operators \( D \) and \( DL^{-1} \) over the avoidance functionals defined in (2.3). The proof is a simple consequence of Lemma 2.1 and linearity of the operators \( D \) and \( DL^{-1} \). We omit details.

**Corollary 2.2** Assume \( \eta \) is a Poisson point process on \( \mathbb{R}^d \) with intensity measure \( \mu \) defined on the probability space \((\Omega, \mathcal{F}, P), A \) a bounded open set on \( \mathbb{R}^d \) and \((\mathcal{B}, \mathcal{G}, \nu)\) a measure space. Assume further that, for each \( s \in \mathcal{B}, Q(s) \) is a Borel set on \( \mathbb{R}^d \) chosen in such a way that the map \((\omega, x, s) \mapsto 1(\eta(x + Q(s)) = 0) \) is \( \mathcal{F} \otimes \mathcal{B} \otimes \mathcal{G} \) measurable. Then, if

\[
 F(\eta) = \int_{A \times \mathcal{B}} 1(\eta(x + Q(s)) = 0) d(\ell \otimes \nu)(x, s).
\]
and
\[ \int_{(A \times B)^2} e^{-\mu(Q(z,s) \cup Q(y,t))} dx d\nu(s) dy d\nu(t) < \infty, \]
we have
\[ D_z F(\eta) = -\int_{A \times B} 1(\eta(x + Q(s)) = 0) 1(z \in x + Q(s)) d(\ell \otimes \nu)(x, s), \]
and
\[ D_z L^{-1}(F(\eta) - E(F(\eta))) = \int_{A \times B} \left( \int_0^1 t^{\eta(x+Q(s))} e^{-\mu(x+Q(s)t)} dt \right) 1(z \in x + Q(s)) d(\ell \otimes \nu)(x, s). \]

3 Normal approximation for avoidance functionals

In this section we provide an upper bound for the Wasserstein and Kolmogorov distance between the distribution of the standardized avoidance functional and the standard normal distribution. Our approach is based on the following result, which is a simplified version of Theorem 3.1 in [13] and Theorem 3.1 in [2].

**Theorem 3.1** If \( \eta \) is a Poisson point process with nonatomic, \( a \)-finite intensity measure \( \mu \), \( F = F(\eta) \) satisfies \( 0 < \text{Var}(F) < \infty \) and \( X \sim \mathcal{N}(0, 1) \) then
\[
d_W \left( \frac{F - E(F)}{\sqrt{\text{Var}(F)}}, X \right) \leq \frac{\text{Var}(\int A(x) B(x) d\mu(x))^{1/2}}{\text{Var}(F)} + \frac{E(\int A(x)^2 | B(x) | d\mu(x))}{\text{Var}(F)^{3/2}}
\]
\[
d_K \left( \frac{F - E(F)}{\sqrt{\text{Var}(F)}}, X \right) \leq \frac{\text{Var}(\int A(x) B(x) d\mu(x))^{1/2}}{\text{Var}(F)} + \frac{4\sqrt{2\pi} E(\int A(x)^2 | B(x) | d\mu(x))}{\text{Var}(F)^{3/2}}
\]
\[
+ \frac{1}{2} \frac{\text{Var}(\int A(x)^2 | B(x) | d\mu(x))^{1/2}}{\text{Var}(F)^{3/2}}
\]
\[
+ \frac{E(\int A(x)^2 B(x)^2 d\mu(x)) + \int |D_\eta C(x) D_x C(y)| d\mu(x) d\mu(y)^{1/2}}{\text{Var}(F)}
\]
where \( A(x) = D_x F \), \( B(x) = -D_x L^{-1}(F - E(F)) \) and \( C(x) = A(x) | B(x) | \).

**Proof.** The first inequality is simply Theorem 3.1 in [13] applied to \( \tilde{F} = \frac{F - E(F)}{\sqrt{\text{Var}(F)}} \), after noting that \( E(\int A(x) B(x) d\mu(x)) = \text{Var}(F) \). For the second we use Theorem 3.1 in [2] to get
\[
d_K \left( \frac{F - E(F)}{\sqrt{\text{Var}(F)}}, X \right) \leq \frac{\text{Var}(\int A(x) B(x) d\mu(x))^{1/2}}{\text{Var}(F)} + \frac{\sqrt{\pi} E(\int A(x)^2 | B(x) | d\mu(x))}{\text{Var}(F)^{3/2}}
\]
\[
+ \frac{1}{2} \frac{\text{Var}(\int A(x)^2 | B(x) | d\mu(x))^{1/2}}{\text{Var}(F)^{3/2}}
\]
\[
+ \frac{\sup_{t \in \mathbb{R}} E(\int A(x) B(x) | D_x 1(F > t) d\mu(x))}{\text{Var}(F)}
\]
For the third term in the last upper bound we use Schwarz’s inequality and the fact that \( \sqrt{a} + \sqrt{b} \leq \sqrt{a + b} \) for \( a, b \geq 0 \), to obtain
\[
E(\int F - E(F) | A(x)^2 | B(x) | d\mu(x)) \leq \left( \frac{E(\int A(x)^2 | B(x) | d\mu(x))^{1/2}}{\text{Var}(F)^{3/2}} \right)^{1/2}
\]
\[
= \left( \frac{\text{Var}(\int A(x)^2 | B(x) | d\mu(x)) + E(\int A(x)^2 | B(x) | d\mu(x))^{2}}{\text{Var}(F)^{3/2}} \right)^{1/2}
\]
\[
\leq \left( \frac{\text{Var}(\int A(x)^2 | B(x) | d\mu(x))}{\text{Var}(F)^{3/2}} + E(\int A(x)^2 | B(x) | d\mu(x)) \right)^{1/2}
\]
To deal with the fourth term we assume that the corresponding term in the statement is finite (there is nothing to prove otherwise). Then we can use Lemma 2.2 and Proposition 2.3 in [6] and Schwarz’s inequality to see that $C$ is in dom($\delta$), with $\delta$ the Skorohod integral operator, and

$$E(\int A(x)B(x)|D_x\mathbf{1}(F > t)d\mu(x)) = E(\mathbf{1}(F > t)\delta(C)) \leq (E(\mathbf{1}(F > t)\delta(C)^2))^{1/2} \leq E(\delta(C)^2)^{1/2} \leq (E(\int A(x)^2B(x)^2d\mu(x) + \int|D_yC(x)D_xC(y)|d\mu(x)d\mu(y))^{1/2}. \quad (3.2)$$

This completes the proof. \hfill $\Box$

We plan to apply now Theorem 3.1 to avoidance functionals. The next result will be essential for this task.

**Lemma 3.2** If $\eta$ is a Poisson random measure with intensity measure $\mu$ and $A, B, C$ and $D$ are measurable sets of finite $\mu$-measure then

$$E\left(\mathbf{1}(\eta(A) = 0)\int_0^1 u_1^{\eta(B)}e^{-\mu(B)u}du, \mathbf{1}(\eta(C) = 0)\int_0^1 v_1^{\eta(D)}e^{-\mu(D)v}dv\right) = e^{-\mu(A \cup B)} \frac{1 - e^{-\mu(A \cap B)}}{\mu(A \cap B)},$$

(= $e^{-\mu(A \cup B)}$ if $\mu(A \cap B) = 0$) and

$$\text{Cov}\left(\mathbf{1}(\eta(A) = 0)\int_0^1 u_1^{\eta(B)}e^{-\mu(B)u}du, \mathbf{1}(\eta(C) = 0)\int_0^1 v_1^{\eta(D)}e^{-\mu(D)v}dv\right) \leq e^{-\mu(A \cup B \cap C \cup D)}1((A \cup B) \cap (C \cup D) \neq \emptyset).$$

**Proof.** From the fact $1(\eta(A) = 0)u_1^{\eta(B)} = 1(\eta(A) = 0)u_1^{\eta(B \cap A^C)}$ and independence of $\eta(A)$ and $\eta(B \cup A^C)$ we see that

$$E\left(\mathbf{1}(\eta(A) = 0)\int_0^1 u_1^{\eta(B)}e^{-\mu(B)u}du\right) = e^{-\mu(A)}\int_0^1 E(u_1^{\eta(B \cap A^C)})e^{-\mu(B)u}du$$

$$= e^{-\mu(A)}\int_0^1 e^{-\mu(B \cap A^C)(1-u)}e^{-\mu(B)u}du$$

$$= e^{-\mu(A \cup B)}\int_0^1 e^{-\mu(A \cap B)u}du$$

and prove the first part of the Lemma. For the upper bound for the covariance we note that we have independence (hence null covariance) if $(A \cup B) \cap (C \cup D) = \emptyset$. On the other hand, arguing as above we see that

$$E\left(\mathbf{1}(\eta(A) = 0)\int_0^1 u_1^{\eta(B)}e^{-\mu(B)u}du\mathbf{1}(\eta(C) = 0)\int_0^1 v_1^{\eta(D)}e^{-\mu(D)v}dv\right)$$

$$= E\left(\mathbf{1}(\eta(A \cup C) = 0)\int_{(0,1)^2} u_1^{\eta(A \cap B \cap C \cap D \cap C)}v_1^{\eta(A \cap B \cap C \cap D \cap D)}(uv)e^{-\mu(B)(1-u)}e^{-\mu(D)(1-v)}du dv\right)$$

$$= e^{-\mu(A \cup C)}\int_{(0,1)^2} e^{-\mu(A \cap B \cap C \cap D \cap C)(1-u)}e^{-\mu(A \cap B \cap C \cap D \cap D)(1-v)}e^{-\mu(A \cap B \cap C \cap D)(1-uv)}$$

$$\times e^{-\mu(B)u}e^{-\mu(D)v}du dv$$

$$= e^{-\mu(A \cup B \cup C \cup D)}\int_{(0,1)^2} e^{-\mu(B \cap (A \cup C))(1-u)}e^{-\mu(D \cap (A \cup B))(1-v)}e^{-\mu(A \cap B)(1-uv)}du dv$$

$$\leq e^{-\mu(A \cup B \cup C \cup D)},$$
where the last upper bound follows from the fact that the exponent in the last integral is nonpositive for every $u, v \in (0, 1)$. \qed

Now we focus on the following setup. $\eta_\lambda$ will denote a homogenous Poisson process on $\mathbb{R}^d$ with intensity $\lambda$ defined on $(\Omega, \mathcal{F}, P)$, $\mathcal{A}$ nonempty, bounded open set on $\mathbb{R}^d$ and $(B, \mathcal{G}, \nu)$ a measure space and will assume that, for every $s \in B$, $Q(s)$ is a Borel set on $\mathbb{R}^d$ chosen in such a way that the map $(\omega, x, s) \mapsto 1(\eta_\lambda(x + \lambda^{-1/d}Q(s)) = 0)$ is $\mathcal{F} \otimes \beta^d \otimes \mathcal{G}$ measurable and such that

$$
\int_B e^{-\ell(Q(s))}d\nu(s) < \infty. \tag{3.3}
$$

We will write $Q_\lambda(x, s) = x + \lambda^{-1/d}Q(s)$ and $Q(x, s) = Q_1(x, s)$. Now, we consider the avoidance functional

$$
F_\lambda(\eta_\lambda) = \int_{A \times B} 1(\eta_\lambda(Q_\lambda(x, s)) = 0)d(\ell \otimes \nu)(x, s). \tag{3.4}
$$

Our next results provides simple expressions for the mean and variance of $F_\lambda(\eta_\lambda)$. For a simpler statement we introduce the functions

$$
V(z) := \int_{B \times B} e^{-\ell(Q(0,s)) + \ell(Q(z,t))}\{e^{\ell(Q(0,s) \cap (Q(z,t))} - 1\}d\nu(s)d\nu(t), \quad z \in \mathbb{R}^d,
$$

and

$$
C_1(\lambda) := \int_{U_\lambda} V(z)dxdz, \quad \lambda > 0, \tag{3.5}
$$

where $U_\lambda = \{(x, z) : x \in A, z \in \lambda^{1/d}(A - x)\}$. We observe that $C_1(\lambda)$ is a nondecreasing function that satisfies

$$
\lim_{\lambda \to \infty} C_1(\lambda) = C_1 := \ell(A) \int_{\mathbb{R}^d} V(z)dz. \tag{3.6}
$$

**Lemma 3.3** Under $(3.3)$, if $F_\lambda(\eta_\lambda)$ is the avoidance functional defined in $(3.4)$ and the constant $C_1$ in $(3.6)$ is finite then $F_\lambda(\eta_\lambda)$ has finite second moment and

$$
E(F_\lambda(\eta_\lambda)) = \ell(A) \int_B e^{-\ell(Q(s))}d\nu(s)
$$

and

$$
\lambda \text{Var}(F_\lambda(\eta_\lambda)) = C_1(\lambda),
$$

with $C_1(\lambda)$ defined by $(3.5)$.

**Proof.** For computing the mean we use Fubini’s theorem and the fact that $\lambda\ell(Q_\lambda(x, s)) = \ell(Q(s))$ to get

$$
E(F(\eta_\lambda)) = \int_{A \times B} P(\eta_\lambda(Q_\lambda(x, s)) = 0)dxd\nu(s) = \int_{A \times B} e^{-\lambda\ell(Q_\lambda(x, s))}dxd\nu(s) = \ell(A) \int_B e^{-\ell(Q(s))}d\nu(s).
$$

Turning now to the variance we use again Fubini’s theorem to obtain

$$
E(F(\eta_\lambda)^2) = \int_{A \times A} \int_{B \times B} P(\eta_\lambda(Q_\lambda(x, s) \cup Q_\lambda(y, t)) = 0)d\nu(s)d\nu(t) dxdy
$$

$$
= \int_{A \times A} \int_{B \times B} e^{-\lambda\ell(Q_\lambda(x, s) \cup Q_\lambda(y, t))}d\nu(s)d\nu(t) dxdy
$$

$$
= \int_{A \times A} \int_{B \times B} e^{-\lambda\ell(Q_\lambda(y-x, t))}d\nu(s)d\nu(t) dxdy, \tag{3.7}
$$

9
the last equality coming from translation invariance of Lebesgue measure. The change of variable \( y = x + \lambda^{-1/2}z \) and the fact that \( \lambda\ell(Q_\lambda(0, s) \cup Q_\lambda(\lambda^{-1/2}z, t)) = \ell(Q(0, s) \cup Q(z, t)) \) yield

\[
E(F(\eta_\lambda)^2) = \lambda^{-1} \int_{U_\lambda} \left[ \int_{B \times B} e^{-\ell(Q(0, s) \cup Q(z, t))} d\nu(u) d\nu(v) \right] dxdz.
\]

A similar computation shows that

\[
(E(F(\eta_\lambda)))^2 = \lambda^{-1} \int_{U_\lambda} \left[ \int_{B \times B} e^{-\ell(Q(0, s)) + \ell(Q(z, t))} d\nu(u) d\nu(v) \right] dxdz.
\]

and, combining the last two equations, we conclude that

\[
\text{Var}(F(\eta_\lambda)) = \lambda^{-1} \int_{U_\lambda} V(z) dxdz.
\]

We are ready now for the main results of this section, namely, explicit upper bounds for the Wasserstein or Kolmogorov distance between the law of standardized Poisson avoidance functionals and the standard normal distribution. We consider first the Wasserstein case and we introduce the constants

\[
C_{2,a} = \ell(A) \int_{(\mathbb{R}^d)^2} W_a(z_1, z_2) dz_1 dz_2,
\]

\[
C_{2,b} = \ell(A) \int_{(\mathbb{R}^d)^3} W_b(z_1, z_2, z_3) dz_1 dz_2 dz_3
\]

(3.8)

with

\[
W_a(z_1, z_2) = \int_{B^3} e^{-\ell(Q_2 \cap 0) - \ell(Q_3 \cap 0)} \lambda^{-1/2} \prod_{i=0}^2 \nu(ds_i),
\]

\[
W_b(z_1, z_2, z_3) = \int_{B^4} e^{-\ell(Q_2 \cap 0)} \lambda^{-1/2} \prod_{i=0}^3 \nu(ds_i),
\]

where, in the last two integrals \( Q_i \) is short notation for \( Q(z_i, s_i) \) and \( z_0 = 0 \). Note that \( C_{2,a} \) and \( C_{2,b} \) weighted averages of the size of different three- or four-wise intersections of translates of the sets \( Q(s) \). As we show next, control of these averages is all that is needed to give CLT’s with rates in Wasserstein distance for avoidance functionals.

**Theorem 3.4** Assume \( \eta_\lambda \) is a homogenous Poisson process on \( \mathbb{R}^d \) with intensity \( \lambda \) defined on \((\Omega, \mathcal{F}, P)\). \( \mathcal{A} \) a nonempty, bounded open set on \( \mathbb{R}^d \) and \((\mathcal{B}, \mathcal{G}, \nu)\) a measure space. Assume further that, for each \( s \in \mathcal{B} \), \( Q(s) \) is a Borel set on \( \mathbb{R}^d \) chosen in such a way that the map \((\omega, x, s) \mapsto 1(\eta_\lambda(x + \lambda^{-1/2}Q(s)) = 0)\) is \( \mathcal{F} \otimes \beta^d \otimes \mathcal{G} \) measurable. If

\[
F_\lambda(\eta) = \int_{\mathcal{A} \times \mathcal{B}} 1(\eta(x + \lambda^{-1/2}Q(s)) = 0) d(\ell \otimes \nu)(x, s),
\]

(3.9)

holds and the constant \( C_1 \) in (3.9) is finite, then \( F_\lambda(\eta_\lambda) \) has finite second moment and

\[
d_W\left(\frac{F_\lambda(\eta_\lambda) - E(F_\lambda(\eta_\lambda))}{\sqrt{\text{Var}(F_\lambda(\eta_\lambda))}}, X\right) \leq \frac{1}{\sqrt{\lambda}} \left( \frac{C_{2,a}^2}{C_1(\lambda)} + \frac{C_{2,b}^{1/2}}{C_1(\lambda)} \right), \quad \lambda > 0,
\]

where \( X \) denotes a standard normal random variable and \( C_{2,a} \) and \( C_{2,b} \) are given by (3.8).
Proof. From Corollary 2.2 and Fubini’s theorem we see that
\[
\int_{\mathbb{R}^d} (D_z F_\lambda(\eta)) \| D_z L^{-1}(F - E(F)) \| dz \\
= \int_{(A \times B)^3} 1(\eta_\lambda(\cup_{i=0}^1 Q_\lambda(x_i, s_i)) = 0) \left( \int_0^1 \eta_\lambda(Q_\lambda(x_2, s_2)) e^{-u \lambda t} \lambda t Q_\lambda(x_2, s_2) du \right) \\
\times \ell(\cap_{i=0}^2 Q_\lambda(x_i, s_i)) \prod_{i=0}^2 (\ell \times \nu)(x_i, s_i)
\]
and, therefore, using Lemma 3.2 (and the fact that \(1 - e^{-x} \leq x, x \geq 0\)),
\[
E(\int_{\mathbb{R}^d} (D_z F_\lambda(\eta))^2 | D_z L^{-1}(F - E(F)) | dz) \\
\leq \int_{(A \times B)^3} e^{-\lambda t} \lambda t \prod_{i=0}^2 (\ell \times \nu)(x_i, s_i)
\]
\[
= \int_{A^3} \left[ \int_{B^3} e^{-\lambda t} \lambda t \prod_{i=0}^2 (\ell \times \nu)(x_i, s_i) \right] \prod_{i=0}^2 dx_i.
\]
Hence, if we change variables, \(x_i = x_0 + \lambda^{-1/d} z_i, i = 1, 2\), denote \(V_\lambda = \{(x_0, z_1, z_2) : x_0 \in A, z_i \in \lambda^{1/d}(A - x_0), i = 1, 2\}\), and observe that \(\lambda t \cap_{i=0}^2 Q_\lambda(x_i, s_i) = \ell \cap_{i=0}^2 Q_\lambda(z_i, s_i)\) (with \(z_0 = 0\)) and similarly for \(\ell \cap_{i=0}^2 Q_\lambda(x_i, s_i)\), we see that
\[
E(\int_{\mathbb{R}^d} (D_z F_\lambda(\eta))^2 | D_z L^{-1}(F - E(F)) | dz) = \frac{1}{\lambda^2} \int_{V_\lambda} W_a(z_1, z_2) dx_0 dz_1 dz_2 \leq \frac{C_{2,a}}{\lambda^2}.
\] (3.9)
In a similar fashion, we obtain from Corollary 2.2 that \(\text{Var}(\langle D F_\lambda(\eta), D L^{-1}(F_\lambda(\eta) - E F_\lambda(\eta)) \rangle)\) equals the variance of
\[
\int_{A^3 \times B^2} 1(\eta_\lambda(Q_\lambda(x_0, s_0)) = 0) \left( \int_0^1 t \eta_\lambda(Q_\lambda(x_1, s_1)) e^{-\lambda t} \lambda t Q_\lambda(x_0, s_0) dt \right) \lambda t \prod_{i=0}^1 dx_i d\nu(s_i),
\]
which, using the covariance inequality of Lemma 3.2, is upper bounded by
\[
\int_{A^3 \times B^3} e^{-\lambda t} \lambda t (\cap_{i=0}^2 Q_t) \lambda t (\cap_{i=0}^3 Q_t) 1((\cup_{i=0}^1 Q_t) \cap (\cup_{i=0}^2 Q_t) \neq \emptyset) \prod_{i=0}^3 dx_id\nu(s_i),
\]
with \(Q_i = Q_\lambda(x_i, s_i), i = 0, \ldots, 3\) in this last integral. A change of variable as above yields now
\[
\text{Var}(\langle D F_\lambda(\eta), D L^{-1}(F_\lambda(\eta) - E F_\lambda(\eta)) \rangle) \leq \frac{C_{2,b}}{\lambda^3}.
\] (3.10)
Now, the conclusion follows from Theorem 8.1, Lemma 3.2 and (3.9). □

Remark 3.5 Theorem 3.4 should be compared to Proposition 1.3 in [6]. We observe that we do not need, for instance, to assume uniform upper bounds for (moments of) \(D_z F_\lambda(\eta)\) in order to have a CLT with rate \(\lambda^{-1/2}\) in Wasserstein distance (of course, the gain comes from the fact that we have restricted ourselves to a particular class of functionals and rely on more specific covariance bounds).

□

A slightly cleaner version of the upper bound in Theorem 3.4 (at the cost of slightly worse constants) can be obtained as follows. From Schwarz inequality we see that
\[
\int E(\|D_z F\|^2 | D_z L^{-1}(F - E(F))) d\mu(z) \leq \left( \int E(D_z F)^4 d\mu(z) \right)^{1/2} \\
\times \left( \int E |D_z L^{-1}(F - E(F))|^2 d\mu(z) \right)^{1/2},
\]
while, on the other hand, \( \int E[D_zL^{-1}(F - E(F))]^2d\mu(z) = \sum_{n=1}^{\infty}(n - 1)!\|f_n\|_n^2 \leq \sum_{n=1}^{\infty}n!\|f_n\|_n^2 = \text{Var}(F) \). From this, arguing as in the last proof, we see that under the assumptions of Theorem 3.4, for \( K(\lambda) \) 

\[
d_W \left( \frac{F_\lambda(y_\lambda) - E(F_\lambda(y_\lambda))}{\sqrt{\text{Var}(F_\lambda(y_\lambda))}}, X \right) \leq \frac{1}{\sqrt{C_1(\lambda)}} \frac{C_2}{2}, \quad \lambda > 0, \tag{3.11}
\]

with \( C_2 = C_{2,c}^{1/2} + C_{2,b}^{1/2} \),

\[
C_{2,c} = \ell(A) \int_{(\mathbb{R}^d)^3} W_c(z_1, z_2, z_3)dz_1dz_2dz_3, \tag{3.12}
\]

and

\[
W_c(z_1, z_2, z_3) = \int_{B^4} e^{-\ell(i_i^2, Q)}\ell(\cap_{i=0}^5 3Q(z_i, s_i)) \prod_{i=0}^5 \nu(ds_i),
\]

where, as before, in the last integral, \( z_0 = 0 \).

We consider next the case of the Kolmogorov distance. Apart from the above defined \( C_{2,a}, C_{2,b} \) and \( C_{2,c} \), the relevant constants in this case are

\[
C_{2,d} = \ell(A) \int_{(\mathbb{R}^d)^5} W_d(z_1, \ldots, z_5)dz_1 \cdots dz_5, \tag{3.13}
\]

with

\[
W_d(z_1, \ldots, z_5) = \int_{B^5} e^{-\ell(i_i^2, Q(z_i, s_i))\ell(\cap_{i=0}^5 3Q(z_i, s_i))} \times 1((\cup_{i=0}^2 Q(z_i, s_i) \cap (\cup_{i=0}^5 3Q(z_i, s_i)) \neq \emptyset) \prod_{i=0}^5 \nu(ds_i),
\]

and

\[
C_{2,e} = \ell(A) \int_{(\mathbb{R}^d)^4} W_c(z_1, z_2, z_3)dz_1dz_2dz_3, \tag{3.14}
\]

with

\[
W_c(z_1, z_2, z_3) = \int_{B^4} e^{-\ell(i_i^2, Q(z_i, s_i))\ell(\cap_{i=0}^5 3Q(z_i, s_i))} \prod_{i=0}^5 \nu(ds_i).
\]

Again \( z_0 = 0 \) in the above integrals. With this notation we have the following result.

**Theorem 3.6** Under the assumptions of Theorem 3.4 for \( \lambda > 0 \),

\[
d_R \left( \frac{F_\lambda(y_\lambda) - E(F_\lambda(y_\lambda))}{\sqrt{\text{Var}(F_\lambda(y_\lambda))}}, X \right) \leq \frac{1}{C_1(\lambda)} \frac{1}{\sqrt{A}} \left( \frac{4 + \sqrt{2\pi}}{8} C_{2,a} + C_{2,b}^{1/2} + \frac{1}{2} \frac{C_{2,c}^{1/2}}{\lambda^{1/2} C_1(\lambda)^{1/2}} + (C_{2,c} + 9C_{2,e})^{1/2} \right),
\]

with \( C_1(\lambda) \) as in (3.3) and constants \( C_{2,a} \) to \( C_{2,e} \) defined as in (3.8), (3.12), (3.13) and (3.14).

**Proof.** We use the second inequality in Theorem 3.1. The first two terms can be handled as in Theorem 3.3. For the third term we use Fubini’s Theorem and Lemma 3.2 to get

\[
\text{Var} \left( \int_{(\mathbb{R}^d)^2}(D_zF_\lambda(y_\lambda))^2D_zL^{-1}(F_\lambda(y_\lambda) - EF_\lambda(y_\lambda))|\lambda d\mu(z) \right)
\]

\[
\leq \int_{(A \times B)^6} \lambda(\cap_{i=0}^2 Q(x_i, s_i))\lambda(\cap_{i=0}^5 3Q(x_i, s_i))e^{-\lambda(i_i^2, Q(x_i, s_i))} \times 1((\cup_{i=0}^2 Q(x_i, s_i) \cap (\cup_{i=0}^5 3Q(x_i, s_i)) \neq \emptyset) \prod_{i=0}^5 (dx_i d\nu(s_i)).
\]
Again, the change of variables, \( x_i = x_0 + \lambda^{-1/d} z_i, \ i = 1, \ldots, 5 \), yields

\[
\text{Var}\left( \int_{R^d} (D_x F_\lambda(x)) \big| D_x L^{-1}(F_\lambda(x)) - EF_\lambda(x) \right| d\lambda z) \leq \frac{C_2 d}{\lambda^3}. \tag{3.15}
\]

Finally, we turn to the fourth term. We note that the proof of Lemma 3.2 shows that

\[
E \left( 1(\eta(A) = 0) \int_0^1 u^\eta(B) e^{-\mu(B)u} du 1(\eta(C) = 0) \int_0^1 v^\eta(D) e^{-\mu(D)v} dv \right) \leq e^{-\mu(A \cup B \cup C \cup D)}.
\]

Using this fact, Fubini’s Theorem and arguing as above we see that

\[
E \left( \int_{R^d} (D_x F_\lambda(x))^2 (D_x L^{-1}(F_\lambda(x)) - EF_\lambda(x))^2 d\lambda z \right) \leq \frac{C_2 e}{\lambda^3}. \tag{3.16}
\]

For the double integral we use that \( D_x(UV) = (D_x U)(D_x V) + (D_x U)V + U(D_x V) \) to get that \( D_y(C(x)) = D^2_{x,y} F D^2_{x,y} (F - EF) + D^2_{x,y} F D_{y} L^{-1}(F - EF) + D_x F D_{x} L^{-1}(F - EF) \) and similarly for \( D_y(C(x)) \). The expected value of the double integrals of the nine terms in the resulting cross-product can be handled similarly. For instance, using the pathwise expressions for \( D_x F_\lambda(x) \) and \( D_x L^{-1}(F_\lambda(x)) - EF_\lambda(x) \) we obtain similar pathwise expressions for \( D^2_{x,y} F_\lambda(x) \) and \( D^2_{x,y} L^{-1}(F_\lambda(x)) - EF_\lambda(x) \), namely,

\[
D^2_{x,y} F_\lambda(x) = \int_{A \times B} 1(\eta(L(x_1, s_1) = 0) 1(x \in Q_\lambda(x_1, s_1)) 1(y \in Q_\lambda(x_1, s_1)) dx_1 d\nu(s_1),
\]

\[
D^2_{x,y} L^{-1}(F_\lambda(x)) - EF_\lambda(x) = \int_{A \times B} 1(\eta(L(x_2, s_2) = 0) 1(x \in Q_\lambda(x_2, s_2)) 1(y \in Q_\lambda(x_2, s_2)) dx_2 d\nu(s_2).
\]

This allows to use similar arguments as above to conclude that

\[
E \left( \int_{R^d} (D^2_{x,y} F_\lambda(x))^2 (D^2_{x,y} L^{-1}(F_\lambda(x)) - EF_\lambda(x))^2 d\lambda x dy \right) \leq \frac{C_2 e}{\lambda^3} \tag{3.17}
\]

\[
\int_{A \times B} e^{-\mu_\lambda(Q_\lambda(x,s))} d\nu(s) < \infty, \tag{3.20}
\]

while if further

\[
\tilde{\tilde{C}}_1(\lambda) := \int_{U_\lambda} V_\lambda(x,z) dz < \infty, \tag{3.21}
\]

where

\[
V_\lambda(x,z) = \int_{B \times B} e^{-(\mu_\lambda(Q_\lambda(x,s)) + \mu_\lambda(Q_\lambda(x + \lambda^{-1/d} z, t))) \left[ e^{\mu_\lambda(Q_\lambda(x,s)) \cap Q_\lambda(x + \lambda^{-1/d} z, t))} - 1 \right] d\nu(s) d\nu(t),
\]

\[
\int_{A \times B} e^{-\mu_\lambda(Q_\lambda(x,s))} d\nu(s) < \infty, \tag{3.20}
\]

while if further

\[
\tilde{\tilde{C}}_1(\lambda) := \int_{U_\lambda} V_\lambda(x,z) dz < \infty, \tag{3.21}
\]

where

\[
V_\lambda(x,z) = \int_{B \times B} e^{-(\mu_\lambda(Q_\lambda(x,s)) + \mu_\lambda(Q_\lambda(x + \lambda^{-1/d} z, t))) \left[ e^{\mu_\lambda(Q_\lambda(x,s)) \cap Q_\lambda(x + \lambda^{-1/d} z, t))} - 1 \right] d\nu(s) d\nu(t),
\]
then $F_\lambda(\eta_\lambda)$ has finite second moment and an application of Theorem 3.1 and the argument leading to (3.11) yields

$$
\lim_{\lambda \to \infty} \frac{\tilde{C}_1(\lambda)}{\tilde{C}_1(\lambda)} = \int_{\mathbb{B}^4} e^{-f(x)(Q_0(x,s)) + \ell(Q(z,t))} \left[ e^{f(x)(Q_0(x,s) \cap (Q_0(x,s)) \cap (Q_0(x,s)) - 1)\right] d\nu(s) d\nu(t).
$$

Similarly, if we can exchange limits and integration for $\tilde{C}_{2,a}(\lambda)$ and $\tilde{C}_{2,b}(\lambda)$ we will get

$$
\lim_{\lambda \to \infty} \tilde{C}_{2,a}(\lambda) = \int_{\mathbb{A} \times \mathbb{R}^d} W_a(x,z_1,z_2,z_3) dxdz_1 dz_2 dz_3 := \tilde{C}_{2,a}
$$

and

$$
\lim_{\lambda \to \infty} \tilde{C}_{2,b}(\lambda) = \int_{\mathbb{A} \times \mathbb{R}^d} W_b(x,z_1,z_2,z_3) dxdz_1 dz_2 dz_3 := \tilde{C}_{2,b}
$$

with

$$
\tilde{W}_a(z_0,z_1,z_2,z_3) = \int_{\mathbb{B}^4} f(z_0) e^{-f(z_0)(Q_0(x,s))} \ell(Q_0(x,s)) \nu(ds_1),
$$

$$
\tilde{W}_b(z_0,z_1,z_2,z_3) = \int_{\mathbb{B}^4} f(z_0) e^{-f(z_0)(Q_0(x,s))} \ell(Q_0(x,s)) \nu(ds_1),
$$

where, as before, $Q_i$ is short notation for $Q(z_i,s_i)$ and $z_0 = 0$. Hence, provided that (3.23), (3.24) and (3.25) hold with $\tilde{C}_1 \in (0, \infty)$, $\tilde{C}_{2,a} < \infty$ and $\tilde{C}_{2,b} < \infty$, we have that for each $\varepsilon > 0$ there exists $\lambda_0 > 0$ such that

$$
d_W \left( F_\lambda(\eta_\lambda) - E(F_\lambda(\eta_\lambda)) \right) \leq \frac{(1 + \varepsilon) C_{2,a} + C_{2,b}}{C_1} \lambda \geq \lambda_0,
$$

and we get a proper Berry-Esseen bound in this nonhomogeneous setup. Similar considerations apply to the case of Kolmogorov distance. □
We conclude this section with a discussion about the extension of Theorem 3.3 to the case of empirical measures. Again, for a cleaner presentation, we restrict ourselves to the case of uniform empirical measures. We are interested in Berry-Esseen bounds for \( F_n(\nu_n) \), where \( F_n \) is the avoidance functional of Theorem 3.2. We will show that, under suitable assumptions, Berry-Esseen bounds for \( F_n(\nu_n) \) can be obtained from Berry-Esseen bounds for \( F_n(\eta_n) \) where \( \eta_n \) is a Poisson process with intensity \( n \). This processes fit into the setup of Remark 3.7 having intensity measure \( d\mu_n(x) = \frac{1}{|A|} 1_A(x) dx \). Hence, if, for each \( s \), \( Q(s) \) is a bounded open set that contains 0 then for \( x \in A \) and large enough \( n \) we have \( \mu_n(Q_n(x,s)) = n\ell(Q_n(x,s)) = \ell(Q(s)) \), \( \mu_n(Q_n(x,s) \cup Q_n(x + n^{-1/d}z, t)) = \ell(Q(0,s) \cup Q(z,t)) \) and similarly for the related quantities in the integrals in \( \tilde{C}_{2,a}(n) \) and \( \tilde{C}_{2,b}(n) \). To keep this discussion simpler we make the assumption that \( \nu \) is a finite measure; for some \( K > 0 \) and all \( s \in B \), \( Q(s) \subset B(0,K) \).

Then, by dominated convergence we get
\[
\lim_{n \to \infty} \tilde{C}_1(n) = C_1, \quad \lim_{n \to \infty} \tilde{C}_{2,a}(n) = C_{2,a}, \quad \text{and} \quad \lim_{n \to \infty} \tilde{C}_{2,b}(n) = C_{2,b}
\]
with \( C_1, C_{2,a} \) and \( C_{2,b} \) as in (3.10) and (3.11) and, therefore, since this constants are finite (by 3.27) we have that for some positive constant
\[
d_W \left( \frac{F_n(\eta_n) - E(F_n(\eta_n))}{\sqrt{\Var(F_n(\eta_n))}} \right), X \leq \frac{C}{\sqrt{n}} \quad n \geq 1.
\]

To derive a Berry-Esseen bound for \( F_n(\nu_n) \) from (3.28) we use the well known fact that \( \eta_n \overset{d}{=} \sum_{i=1}^{N_n} \delta_{X_i} \), if \( N_n \) is a Poisson random variable with mean \( n \) independent of the \( X_i \)’s (with \( \eta_n \) the null measure if \( N_n = 0 \)). We introduce the constant
\[
\alpha_n := \int_{A \times B} \left( 1 - \frac{\ell(Q_n(x,s) \cap A)}{\ell(A)} \right)^n \frac{\ell(Q_n(x,s) \cap A)}{\ell(A)} dx dv(s)
\]
and note that \( \alpha_n \leq \tilde{K} \nu(B) < \infty \), with \( \tilde{K} = \ell(B(0,K)) \), since we are assuming (3.27). From Lemmas 3.10 and 3.11 below we see that under these assumptions there exist constants \( n_0, D \) (depending only on \( K \) such that for \( n \geq n_0 \) and \( |k - n| < n \) we have
\[
\left| E(F_n(\nu_n) - F_n(\nu_k) - \frac{\alpha_n}{n}(k - n)) \right| \leq D \left( \frac{k-n}{n} \right)^2
\]
and
\[
\Var(F_n(\nu_n) - F_n(\nu_k)) \leq D \left( \frac{k-n}{n} \right)^2 + \frac{1}{n^{1/2+\nu}}
\]
If we take now \( \nu \in (0,\frac{1}{6}) \) and set \( L_n = n^{1/2+\nu} \) we see that for \( n \geq n_0 \)
\[
E \left( \sqrt{n}(F_n(\nu_n) - F_n(\eta_n)) - \alpha_n \frac{N_n-n}{\sqrt{n}} \right)^2 = \sum_{|k-n| \leq L_n} P(N_n = k) n E \left( (F_n(\nu_n) - F_n(\nu_k)) - \frac{\alpha_n}{n}(k - n) \right)^2
\]
\[
= \sum_{|k-n| \leq L_n} P(N_n = k) n E \left( (F_n(\nu_n) - F_n(\nu_k)) - \frac{\alpha_n}{n}(k - n) \right)^2
\]
\[
= \sum_{|k-n| \leq L_n} P(N_n = k) n E \left( (F_n(\nu_n) - F_n(\nu_k)) - \frac{\alpha_n}{n}(k - n) \right)^2
\]
\[
\leq D \left( \frac{1}{n^{1/2+\nu}} + D \left( \frac{1}{n^{1/2+\nu}} + C n \ell(B(0,K)) \right)^{1/2}
\]
\[
\leq C \left( \frac{1}{n^{1/2+\nu}} + n \ell(B(0,K)) \right)^{1/2},
\]
for some constant \( C \), where the last bound comes from Hölder’s inequality and the fact that, since \( F_n(\nu_n) \) and \( F_n(\eta_n) \) are bounded above by \( \ell(A) \nu(B) \) and \( E((N_n-n)/\sqrt{n})^4 = 3 + \frac{1}{n} \), \( E((F_n(\nu_n) - F_n(\eta_n))^4) \).
\( F_n(\eta_n) - \frac{1}{\sqrt{n}}(N_n - n) \) is a bounded sequence. On the other hand, from Chernoff’s inequality we know that
\[
P(|N_n - n| > L_n) \leq e^{-nh(L_n/n)} + e^{-nh(-L_n/n)}
\]
with \( h(u) = (1 + u) \log(1 + u) - u, \ u \geq -1 \). A Taylor expansion shows that \( h(u) \sim \frac{u^2}{2} \) as \( u \to 0 \) (the ratio tends to one) which means that for some positive constant \( c \) and large enough \( n \) we have \( h(L_n/n) \geq cL_n^2/n^2, \ h(-L_n/n) \geq cL_n^2/n^2 \) and, consequently,
\[
P(|N_n - n| > L_n) \leq 2e^{-cn^\nu}.
\]
Hence, \( n(P(|N_n - n| > L_n))^{1/2} \) vanishes at a faster rate than \( n^{-(1/2 - \nu)} \) and we see that for some constants (that we, again, call \( n_0 \) and \( C \)) we have
\[
E\left( \frac{\sqrt{n} \cdot (F_n(\nu_n) - F_n(\eta_n)) - \alpha_n \frac{N_n - n}{\sqrt{n}}}{\sqrt{\alpha_n \frac{N_n - n}{\sqrt{n}}} \cdot \sqrt{n}} \right)^2 \leq \frac{C}{n^{1/2 - \nu}}, \ n \geq n_0.
\]
This last bound shows that, on one hand,
\[
\lim_{n \to \infty} n \cdot \text{Var}(F_n(\nu_n)) = C_1 - \left( \int_{B} e^{-\ell(Q(s))/\ell(A)} \ell(Q(s)) d\nu(s) \right)^2 =: D_1,
\]
with \( C_1 \) as in \( \text{(3.6)} \), and on the other hand
\[
d_W\left( \sqrt{n} \cdot (F_n(\nu_n) - EF_n(\nu_n)), \sqrt{n} \cdot (F_n(\eta_n) - E\nu_n)) \right) \leq \frac{C''}{n^{1/2}}, \ n \geq n_1.
\]
A further use of Theorem 3.8 (a straightforward modification of the argument in the proof of Theorem 3.8) yields that for some constants \( C'', n_1 \)
\[
d_W\left( \sqrt{n} \cdot (F_n(\eta_n) - EF_n(\eta_n)) + \alpha_n \frac{N_n - n}{\sqrt{n}}, X' = \frac{X'}{\alpha_n} \right) \leq \frac{C''}{n^{1/2}}, \ n \geq n_1,
\]
where \( X' \) is a centered normal r.v. with variance \( D_1 \). Finally, combining \( \text{(3.31)} \) and \( \text{(3.32)} \) we conclude that for every \( \delta \in (0, 1) \) there exists a constant \( C(\delta) \) such that
\[
d_W\left( \sqrt{n} \cdot (F_n(\nu_n) - EF_n(\nu_n)), X' = \frac{X'}{\alpha_n} \right) \leq \frac{C(\delta)}{n^{1/4 - \delta}}, \ n \geq 1.
\]
With a simple rescaling we obtain the next result that summarizes this discussion.

**Theorem 3.8** Assume that \( A \subset \mathbb{R}^d \) is a bounded open set, \( \{X_n\}_{n \geq 1} \) are i.i.d. uniform r.v.’s on \( A \), defined on \( (\Omega, \mathcal{F}, P) \), \( \nu_n = \sum_{i=1}^{n} \delta_{X_i}, (B, \mathcal{G}, \nu) \) a measure space and suppose further that, for each \( s \in B, Q(s) \) is a Borel set on \( \mathbb{R}^d \) chosen in such a way that the map \( (\omega, x, s) \to 1(\nu_n(x + n^{-1/d}Q(s)) = 0) \) is \( \mathcal{F} \otimes \beta^d \otimes \mathcal{G} \) measurable. If
\[
F_n(\eta) = \int_{A \times B} 1(\eta(x + n^{-1/d}Q(s)) = 0)d(\ell \otimes \nu)(x, s),
\]
and \( \text{(3.27)} \) holds we have \( \text{(3.30)} \) and for each \( \nu \in (0, \frac{1}{4}) \) there exists a constant, \( C(\delta) \) such that
\[
d_W\left( \frac{F_n(\nu_n) - E(F_n(\nu_n))}{\sqrt{\text{Var}(F_n(\nu_n))}}, X' = \frac{X'}{\alpha_n} \right) \leq \frac{C(\delta)}{n^{1/4 - \delta}}, \ n \geq 1,
\]
where \( X \) denotes a standard normal random variable.

**Remark 3.9** Theorem 3.8 provides non-trivial Berry-Esseen bounds for avoidance functionals of empirical measures. The assumptions are stronger than those for the case of Poisson functionals.
These assumptions can possibly be relaxed. On the other hand, we believe that the rate in Theorem 3.8 is not optimal and we conjecture that, under suitable assumptions,

$$d_W \left( \frac{F_n(\nu_n) - E(F_n(\nu_n))}{\sqrt{\text{Var}(F_n(\nu_n))}} , X \right) \leq \frac{C}{n^{1/2}}, \quad n \geq 1,$$  \quad (3.33)

for some constant $C$. The present approach does not yield (3.33) and we think that it would be of interest to devote future research to design an approach that gives this conjectured rate for empirical functionals.

We end the section with two technical lemmas used in the proof of Theorem 3.8.

**Lemma 3.10** Under the assumptions of Theorem 3.8 there exists a constant, $D > 0$ such that for $n \geq 1$

$$\left| E(F_n(\nu_n) - F_n(\nu_k) - \frac{\nu_n}{n}(k-n)) \right| \leq D\left(\frac{k-n}{n}\right)^2.$$  \quad (3.34)

**Proof.** Set $b_n(x,s) = n\ell(Q_n(x,s) \cap A)/\ell(A)$. We note first that $b_n(x,s) \leq \ell(Q(s))/\ell(A) \leq K/\ell(A)$ ($K = \ell(B(0,K))$) for each $x \in A$ and also that

$$E(F_n(\nu_n) - F_n(\nu_k)) = \int_{A \times B} \left((1 - \frac{b_n(x,s)}{n})^n - (1 - \frac{b_n(x,s)}{n})^k\right) dx d\nu(s).$$  \quad (3.35)

Next, for $0 \leq x \leq 1$ and $l > 0$ we have

$$|(1-x)^l - (1-lx)| \leq \frac{l^2}{2} x^2.$$  \quad (3.36)

Hence, for $k > n$ (3.35) yields $|1 - (1 - b_n(x,s)/n)^{k-n} - (k-n)b_n(x,s)/n| \leq \frac{1}{2} \left(\frac{k-n}{n}\right)^2 K^2/\ell(A)^2$ for all $x$ and $s$ and, since $0 \leq 1 - b_n(x,s)/n \leq 1$, we can use (3.34) to obtain

$$\left| E(F_n(\nu_n) - F_n(\nu_k) - \frac{\nu_n}{n}(k-n)) \right| \leq \frac{1}{2} \frac{K^2\nu(B)}{\ell(A)^2} \left(\frac{k-n}{n}\right)^2.$$  \quad (3.37)

If $0 \leq k \leq n$ (3.35) implies

$$|(1 - b_n(x,s)/n)^{n-k} - 1 - (k-n)b_n(x,s)/n| \leq \frac{1}{2} \frac{K^2\nu(A)}{\ell(A)^2} \left(\frac{k-n}{n}\right)^2,$$

which, in turn, implies

$$|(1 - b_n(x,s)/n)^n - (1 - b_n(x,s)/n)^{k}| \leq \frac{K}{\ell(A)} \left\lvert \frac{k-n}{n} \right\rvert + \frac{K^2\nu(A)}{\ell(A)^2} \left(\frac{k-n}{n}\right)^2.$$  \quad (3.38)

From these last two estimates we obtain the conclusion. \quad \Box

**Lemma 3.11** Under the assumptions of Theorem 3.8 there exist constants, $D > 0$, $n_0$ such that for $n \geq n_0$ and $|k-n| \leq n$ we have

$$\text{Var}(F_n(\nu_n) - F_n(\nu_k)) \leq D\left(\frac{k-n}{n}\right)^2.$$  \quad (3.39)

**Proof.** We assume for simplicity $\ell(A) = 1$, $\nu(B) = 1$, the general case following with straightforward changes. If we set $a_n(x,y,t) = n\ell((Q_n(x,s) \cup Q_n(y,t)) \cap A)$ and $b_n(x,s) = n\ell(Q_n(x,s) \cap A)$ then $a_n(x,y) \leq 2K$, $b_n(x) \leq K$ with $K = \ell(B(0,K))$ and

$$\text{Var}(F_n(\nu_n) - F_n(\nu_k)) = \int_{A \times B} \left((1 - \frac{a_n(x,y,t)}{n})^n + (1 - \frac{a_n(x,s,y,t)}{n})^{k-n} - 2(1 - \frac{b_n(x,s)}{n})^{k-n} \right. \\
- (1 - \frac{b_n(x,s)}{n})^n (1 - \frac{b_n(y,t)}{n})^n (1 - \frac{b_n(x,s)}{n})^{k-n} (1 - \frac{b_n(y,t)}{n})^{k-n} \\
\left. - 2(1 - \frac{b_n(x,s)}{n})^{k-n}\right) dx d\nu(s) dy d\nu(t).$$
Let us assume that \( k > n \) and set
\[
 V_{n,k} = \int_{(A \times B)^2} \left[ \left( 1 - \frac{a_n(x,s,y,t)}{n} \right)^n \left( 1 + e^{-\frac{k-n}{n}a_n(x,s,y,t)} \left( 1 - \frac{(k-n)a_n^2(x,s,y,t)}{2n^2} \right) \right) 
 - 2e^{-\frac{k-n}{n}b_n(x,s)} \left( 1 - \frac{(k-n)b_n^2(x,s)}{2n^2} \right) \right] \, dx \, dy \, d\nu(s) \, d\nu(t).
\]

Then, from Lemma 4.3, we see that \( |\text{Var}(F_n(\nu_n) - F_n(\nu_k)) - V_{n,k}| \leq A(2\tilde{K})(7 + A(2\tilde{K}))\frac{k-n}{n}. \)
Similarly, if we set
\[
 \tilde{V}_{n,k} = \int_{(A \times B)^2} \left[ \left( 1 - \frac{a_n(x,s,y,t)}{n} \right)^n \left( 1 + e^{-\frac{k-n}{n}a_n(x,s,y,t)} - 2e^{-\frac{k-n}{n}b_n(x,s)} \right) \right] \, dx \, dy \, d\nu(s) \, d\nu(t),
\]
then \( |V_{n,k} - \tilde{V}_{n,k}| \leq 5\tilde{K}^2\frac{k-n}{n}. \) Next, we note that \( \tilde{V}_{n,k} = V_{n,k,1} + V_{n,k,2} \) with
\[
 V_{n,k,1} = \int_{(A \times B)^2} \left[ \left( 1 - \frac{b_n(x,s,y,t)}{n} \right)^n \left( 1 - \frac{b_n(y,s,t)}{n} \right)^n \left( e^{-\frac{k-n}{n}a_n(x,s,y,t)} - e^{-\frac{k-n}{n}(b_n(x,s) + b_n(y,t))} \right) \right] \, dx \, dy \, d\nu(s) \, d\nu(t)
\]
and
\[
 V_{n,k,2} = \int_{(A \times B)^2} \left[ \left( 1 - \frac{a_n(x,s,y,t)}{n} \right)^n \left( 1 - \frac{b_n(x,s,y,t)}{n} \right)^n \times \left( 1 + e^{-\frac{k-n}{n}a_n(x,s,y,t)} - 2e^{-\frac{k-n}{n}b_n(x,s)} \right) \right] \, dx \, dy \, d\nu(s) \, d\nu(t).
\]

Since
\[
 0 \leq e^{-\frac{k-n}{n}a_n(x,s,y,t)} - e^{-\frac{k-n}{n}(b_n(x,s) + b_n(y,t))} \leq \frac{k-n}{n}(b_n(x,s) + b_n(y,t) - a_n(x,s,y,t))
\]
we see that for each \( x, s, t \) the integrand in \( V_{n,k,1} \) (as a function of \( y \)) vanishes outside \( B(x, 2\tilde{K}/n^{1/d}) \), a set with volume \( 2^d\tilde{K}/n \), and is bounded by \( \frac{k-n}{n}\tilde{K} \). Hence, \( V_{n,k,1} \leq 2^d\tilde{K}^2\frac{k-n}{n} \). Finally, to deal with \( V_{n,k,2} \) we can use Lemma 4.4 to see that, provided \( n > 4\tilde{K} \),
\[
 \left| \left( 1 - \frac{b_n(x,s,y,t)}{n} \right)^n - e^{-a_n(x,s,y,t)} \right| \leq (A(2\tilde{K}) + 2\tilde{K}^2)\frac{1}{n},
\]
and
\[
 \left| \left( 1 - \frac{b_n(x,s,y,t)}{n} \right)^n - e^{-b_n(x,s) + b_n(y,t)} \right| \leq (2A(\tilde{K}) + \tilde{K}^2 + (A(\tilde{K}) + \frac{1}{2}\tilde{K}^2)^2)\frac{1}{n}.
\]
Now, the fact that
\[
 |1 + e^{-\frac{k-n}{n}a_n(x,s,y,t)} - 2e^{-\frac{k-n}{n}b_n(x,s)}| \leq 2\tilde{K}\frac{k-n}{n},
\]
and the argument used in the bound for \( V_{n,k,1} \) allow us to conclude that \( V_{n,k,2} \leq C\frac{k-n}{n} \) for some constant \( C \). The case \( k < n \) follows as in the proof of Lemma 3.10. \( \square \)

4 Applications.

In this section we show the power of Theorem 3.4 through its application to two classical models in stochastic geometry: germ-grain models and quantization. We deal first with a particular type of germ grain model given by the union of balls of a fixed radius around Poisson centers, which we truncate to a bounded box to keep the volume of that union finite. More precisely, we consider the functional
\[
 G_\lambda(\eta) = \ell \left( \bigcup_{z \in \eta} B(z,t) \right) \cap \left[ -\frac{\lambda^{1/d}}{2}, \frac{\lambda^{1/d}}{2} \right]^d,
\]
where \( \eta \) is homogeneous Poisson process of unit intensity on \( \mathbb{R}^d \) and \( B(z,t) \) is the ball of radius \( t > 0 \) around \( z \). As before, we write \( X \) for a standard normal random variable. With this notation we have the following.
Theorem 4.1 With the above notation, we have \( E(G_\lambda(\eta)) = \lambda (1 - e^{-\omega_d t^d}) \), where \( \omega_d \) denotes the volume of the \( d \)-dimensional unit ball and

\[
\lim_{\lambda \to \infty} \frac{1}{\lambda} \text{Var}(G_\lambda(\eta)) = e^{-2\omega_d t^d} \int_{B(0,2t)} (e^{\ell(B(0,t) \cap B(z,t))} - 1)dz.
\]

Furthermore, for each \( \lambda_0 > 0 \) there exists a finite constant, \( C(d,t,\lambda_0) \), depending only on \( d, t \) and \( \lambda_0 \) such that

\[
\max \left( d_W \left( \frac{G_\lambda(\eta) - E(G_\lambda(\eta))}{\sqrt{\text{Var}(G_\lambda(\eta))}} , X \right), d_K \left( \frac{G_\lambda(\eta) - E(G_\lambda(\eta))}{\sqrt{\text{Var}(G_\lambda(\eta))}} , X \right) \right) \leq \frac{C(d,t,\lambda_0)}{\sqrt{\lambda}} , \quad \lambda \geq \lambda_0.
\]

Proof. We observe first that taking \( = (0,1)^d, B \) a set with a single element (which we denote 0), \( \mathcal{G} \) the (only) \( \sigma \)-field on \( B, \nu \) the probability measure concentrated on 0 and \( Q(0) \) the open ball centered at \( 0 \in \mathbb{R}^d \) with radius \( t \), we have \( G_\lambda(\eta) \overset{d}{=} \lambda (1 - F_\lambda(\eta \lambda)) \) with \( \eta \lambda \) a Poisson process on \( \mathbb{R}^d \) with constant intensity \( \lambda \) and \( F_\lambda \) as in Theorem 3.3. From Lemma 3.3 we obtain \( E(F_\lambda(\eta \lambda)) = e^{-\omega_d t^d} \) and

\[
\lambda \text{Var}(F_\lambda(\eta \lambda)) = e^{-2\omega_d t^d} \int_{U_\lambda} (e^{\ell(B(0,t) \cap B(z,t))} - 1)dxdz = C_1(\lambda),
\]

where \( U_\lambda = \{(x, z) : z \in (0,1)^d, z \in \lambda^{1/d}((0,1)^d - x)\} \). We note also that \( C_1(\lambda) \) grows to \( C_1 \) as \( \lambda \to \infty \) with

\[
C_1 = e^{-2\omega_d t^d} \int_{\mathbb{R}^d} (e^{\ell(B(0,t) \cap B(z,t))} - 1)dz = e^{-2\omega_d t^d} \int_{B(0,2t)} (e^{\ell(B(0,t) \cap B(z,t))} - 1)dz < \infty,
\]

since \( B(0, t) \cap B(z, t) = \emptyset \) (hence, the last integrand vanishes) if \( \|z\| > 2t \). Thus, we can apply (3.11) and it suffices to show finiteness of the constants \( C_{2,c} \) and \( C_{2,b} \). Now, we have

\[
C_{2,c} = \int_{(\mathbb{R}^d)^3} e^{-\ell(B(0,t) \cup (\cup_{i=1}^3 B(z_i,t)))} \ell(B(0,t) \cap B(z_1,t))dz_1dz_2dz_3 = \int_{B(0,2t)^3} e^{-\ell(B(0,t) \cup (\cup_{i=1}^3 B(z_i,t)))} \ell(B(0,t) \cap (\cup_{i=1}^3 B(z_i,t)))dz_1dz_2dz_3 < \infty,
\]

since, as before, \( B(0,t) \cap B(z_1,t) = \emptyset \) if \( \|z\| > 2t \). On the other hand, writing \( D = \{(z_1, z_2, z_3) : (B(0,t) \cup B(z_1,t)) \cap (B(0,t) \cup B(z_1,t)) \neq \emptyset\} \), we have

\[
C_{2,b} = \int_{D} e^{-\ell(B(0,t) \cup (\cup_{i=1}^3 B(z_i,t)))} \ell(B(0,t) \cap B(z_1,t))\ell(B(z_2,t) \cap B(z_3,t))dz_1dz_2dz_3 \\
\leq \int_{B(0,2t) \times B(0,6t)^2} e^{-\ell(B(0,t) \cup (\cup_{i=1}^3 B(z_i,t)))} \ell(B(0,t) \cap B(z_1,t))\ell(B(z_2,t) \cap B(z_3,t))dz_1dz_2dz_3 < \infty,
\]

where the last bound comes from the fact that if \( \ell(B(0,t) \cap B(z_1,t)) > 0 \) then \( \|z\| < 2t \) and, if this is the case and \( (z_1, z_2, z_3) \in D \), then \( z_2 \) or \( z_3 \) must have norm less than \( 4t \) and if, furthermore, \( \ell(B(z_2,t) \cap B(z_3,t)) > 0 \), then the other point must have norm less than \( 6t \). Finally, we take \( C(d,t,\lambda_0) = (C_{2,c}^{1/2} + C_{2,b}^{1/2})/C_1(\lambda_0) \) and the result follows for the case of the Wasserstein metric. For the result in Kolmogorov distance we note that the constant \( C_{2,d} \) is upper bounded by

\[
\int_{B(0,6t)^2} e^{-\ell(B(0,t))^2}dz_1 \cdots dz_5 < \infty,
\]

with a similar bound proving finiteness of \( C_{2,c} \). \( \square \)
Remark 4.2 From the computations in the proof of Theorem 4.1 we see that $C_{2b} \leq 8^d \omega_d^d \lambda^d e^{-\omega_d t}$. Similarly, $C_{2b} \leq 72^d \omega_d^d \lambda^d e^{-\omega_d t}$. Now, if we take $\lambda > (2t)^d$, then for each $z \in (t/\lambda^{1/d}, 1 - t/\lambda^{1/d})$ we have $\{x\} \times B(0, t) \subset U_z$ and, as a consequence,

$$C_1(\lambda) \geq e^{-2\omega_d t^d} (1 - \frac{2t}{\lambda^{1/d}}) \int_{B(0,t)} (e^{\ell(B(0,t) \cap B(z,t))} - 1)dz \geq e^{-2\omega_d t^d} (1 - \frac{2t}{\lambda^{1/d}})^d \omega_d t^d (e^{\omega_d \left(\frac{t}{\lambda^{1/d}}\right)^d} - 1),$$

since, for each $z \in B(0, t)$ \(B(z/2, t/2) \subset (B(0, t) \cap B(z, t))\). From these estimates we see that for $\lambda > (2t)^d$

$$d_W \left( \frac{G_{\lambda}(\eta) - E(G_{\lambda}(\eta))}{\text{Var}(G_{\lambda}(\eta))}, X \right) \leq \frac{C(d, t)}{\sqrt{\lambda}},$$

with

$$C(d, t) = \frac{8^d/2 e^{-2\omega_d t^d} (1 + 3\omega_d^{1/2} t^{d/2})}{e^{-2\omega_d t^d} (1 - \frac{2t}{\lambda^{1/d}})^d \omega_d t^d (e^{\omega_d \left(\frac{t}{\lambda^{1/d}}\right)^d} - 1)}.$$ Of course, the constant is not optimal, but we see how easily we can get a simple explicit upper bound in Theorem 4.3.

Next, we provide a Berry-Esseen bound for the volume of the union of balls centered around the points of a empirical measure. The proof is a simple application of Theorem 3.8.

Theorem 4.3 With the notation of Theorem 4.3 if $X_1, \ldots, X_n$ are i.i.d. uniform r.v.’s on $[-\frac{1}{2} \delta, \frac{1}{2} \delta]$ and $\nu = \sum_{i=1}^n \delta_{X_i}$, we have

$$\lim_{n \to \infty} E(G_n(\nu_n)) = (1 - e^{-\omega_d t^d})$$

and

$$\lim_{n \to \infty} \frac{1}{n} \text{Var}(G_n(\nu_n)) = e^{-2\omega_d t^d} \left[ \int_{B(0,2t)} (e^{\ell(B(0,t) \cap B(z,t))} - 1)dz - \omega_d t^{2d} \right].$$

Furthermore, for each $\delta \in (0, \frac{1}{4})$ there exists a finite constant, $C(d, t, \delta)$, depending only on $d$, $t$ and $\delta$ such that

$$d_W \left( \frac{G_n(\nu_n) - E(G_n(\nu_n))}{\text{Var}(G_n(\nu_n))}, X \right) \leq \frac{C(d, t, \delta)}{n^{1/4 - \delta}}, \quad n \geq 1.$$ 

Theorems 4.1 and 4.3 provide improvements over known results. [16] contains CLT’s that include both the functionals $G_{\lambda}(\eta)$ and $G_n(\nu_n)$ in this paper. Yet, to our knowledge, we are giving the first Berry-Esseen bounds for functionals of this type.

We turn now to the quantization error functional. As noted in the Introduction, quantization deals with the issue of approximation of a continuous measure by another measure concentrated on a finite set and this problem has applications in a variety of fields. We focus on the case of quantization around random Poisson centers. More precisely, we assume that $\eta_\lambda$ is a Poisson process on $\mathbb{R}^d$ with constant intensity $\lambda$ and consider

$$H(\eta_\lambda) = \int_{[0,1]^d} \min_{z \in \eta_\lambda} \|x - z\|^p dx = \sum_{z \in \eta_\lambda} \int_{C(z, \eta_\lambda)} \|x - z\|^p dx,$$

where $C(z, \eta_\lambda)$ is the Voronoi cell around $z$, that is, the set of points $x \in [0,1]^d$ which are closer to $z$ than to any other point in the support of $\eta_\lambda$.  

20
A key fact about the functional $H$ is that for any random point measure on $\mathbb{R}^d$, $\eta$,

$$H(\eta) = \int_{[0,1]^d} \min_{z \in \eta} \|x - z\|^p dx$$

$$= \int_{[0,1]^d} \left( p \int_0^\infty s^{p-1} 1(\min_{z \in \eta} \|x - z\| > s) ds \right) dx$$

$$= \int_{(0,1)^d \times (0,\infty)} 1(\eta(B(x,s)) = 0) dx d\nu(s),$$  \hspace{1cm} (4.2)

where $\nu$ is the Borel measure on $(0, \infty)$ given by $d\nu(s) = ps^{p-1} ds$. While the last expression in (4.1) can be used to compute moments of $F(\eta_s)$ via Campbell’s Theorem (see [20]) we see from (4.2) that the quantization error functional is, up to a scaling factor, an avoidance functional as those considered in Theorem 3.4. More precisely, we have that

$$H(\eta_s) = \lambda^{-p/d} F_\lambda(\eta_s)$$  \hspace{1cm} (4.3)

if $A = (0,1)^d$, $B = (0,\infty)$ equiped with the Borel $\sigma$-field, $d\nu(s) = ps^{p-1} ds$, $s \in (0,\infty)$ and $Q(s) = B(0,s)$. A simple consequence of this fact and Lemma 3.3 is that

$$\mathbb{E}(H(\eta_s)) = \lambda^{-\frac{p}{d}} \int_0^\infty ps^{p-1} e^{-\ell(B(0,s))} ds = \lambda^{-\frac{p}{d}} \omega_d \Gamma(1 + \frac{p}{d}),$$

where $\omega_d$ denotes the volume of the $d$-dimensional unit ball and

$$\text{Var}(F(\eta_s)) = \lambda^{-1 - \frac{2p}{d}} C_1(\lambda)$$

with $C_1(\lambda) = \int_{U_\lambda} V(z) dx dz$, $U_\lambda = \{(x, z) : x \in (0,1)^d, z \in \lambda^{1/d}((0,1)^d - x)\}$ and

$$V(z) = \int_{(0,\infty) \times (0,\infty)} p u^{p-1} v^{p-1} e^{-(\ell(B(0,u)) + \ell(B(z,v)))} (e^{\ell(B(0,u) \cap B(z,v))} - 1) du dv.$$

As in Lemma 3.3 we have that $C_1(\lambda)$ grows to $C_1 = \int_{\mathbb{R}^d} G(z) dz$. We prove next finiteness of $C_1$

Lemma 4.4 With the above notation $G(z)$ is integrable on $\mathbb{R}^d$ and

$$\lim_{\lambda \to \infty} \lambda^{1 + \frac{p}{d}} \text{Var}(H(\eta_s)) = \int_{\mathbb{R}^d} G(z) dz.$$

Proof. To show integrability of $G$ (the limit follows from monotone convergence), we observe that if $u + v < \|z\|$ then $B(0,u) \cap B(z,v) = \emptyset$ and $e^{\ell(B(0,u) \cap B(z,v))} = 1 = 0$. On the other hand we always have $e^{\ell(B(0,u) \cap B(z,v))} \leq e^{\min(\ell(B(0,u)),\ell(B(z,v)))}$. Therefore,

$$V(z) \leq \int_{(0,\infty) \times (0,\infty) \setminus (\frac{u}{2}, \frac{u}{2}) \times (\frac{u}{2}, \frac{u}{2})} p u^{p-1} v^{p-1} e^{-\max(\ell(B(0,u)),\ell(B(z,v)))} du dv$$

$$= 2 \int_{\frac{u}{2}}^\infty pu^{p-1} e^{-\omega_d u^d} \left[ \int_0^u p u^{p-1} du \right] du$$

$$= 2p \int_{\frac{u}{2}}^\infty pu^{p-1} e^{-\omega_d u^d} du.$$

From this bound and the coarea formula we conclude that

$$\int_{\mathbb{R}^d} V(z) dz \leq d\omega_d \int_0^\infty t^{d-1} \left[ \int_0^\infty u^{2p-1} e^{-\omega_d u^d} du \right] dt$$

$$= \omega_d t^{d+1} \int_0^\infty u^{2p+1} e^{-\omega_d u^d} du$$

$$= p^{d+1} \omega_d \frac{2p}{d} \int_0^\infty x^{2p+1} e^{-x^2} dx = \frac{2p}{d} 2^{d+1} \omega_d \Gamma(\frac{2p}{d} + 1) < \infty.$$
We are ready now for the main result about the Poisson quantization error. In this case, and for the sake of brevity, we restrict ourselves to the case of Wasserstein distance but a similar analysis would yield an equivalent result in terms of Kolmogorov’s distance.

**Theorem 4.5** If \( \eta_\lambda \) is a Poisson random measure on \( \mathbb{R}^d \) with constant intensity \( \lambda \) and

\[
H(\eta_\lambda) = \int_{[0,1]^d} \min_{z \in \eta_\lambda} \|x - z\|^p dx
\]

then there exist positive constants \( C, \lambda_0 \) such that

\[
d_W \left( \frac{H(\eta_\lambda) - E(H(\eta_\lambda))}{\sqrt{\text{Var}(H(\eta_\lambda))}} , X \right) \leq C \sqrt{\lambda}, \quad \lambda \geq \lambda_0
\]

where \( X \) denotes a standard normal random variable.

**Proof.** In view of Theorem 3.4, in its version (3.11) and Lemma 4.4 all we have to do is to prove finiteness of the constants \( C_2,c \) and \( C_2,b \) in (3.8). This follows from Lemmas 4.6 and 4.7 below.

As we mentioned in the Introduction a CLT for \( H(\eta_\lambda) \) and related functionals can be found in [20]. As for the case of germ-grain models, Theorem 4.5 is, to our knowledge, the first Berry-Esseen bound for this type of functional.

We conclude with the Lemmas used in the proof of Theorem 4.5.

**Lemma 4.6** If \( F_\lambda(\eta_\lambda) \) is defined as in (4.3) and \( C_2,c \) the related constant defined in (3.12) then \( C_2,c < \infty \).

**Proof.** We note that

\[
C_2,c = \int_{\mathbb{R}^d} \left( \prod_{i=1}^4 ps_{s_i}^{p-1} \right) e^{-\ell(\bigcup_{i=1}^4 B(0, s_i))} \ell (\bigcap_{i=1}^4 B(y_i, s_i)) \prod_{i=1}^4 dy_i \prod_{i=1}^4 ds_i, \tag{4.4}
\]

where we fix \( y_1 = 0 \). A simple computation gives that

\[
C_1 = \int_{\mathbb{R}^d} E \left[ A(z) B(z)^3 \right] dz
\]

with

\[
A(z) = \int_0^\infty ps^{p-1} 1(\eta_1(B(0, s)) = 0) 1(\|z\| \leq s) ds
\]

and

\[
B(z) = \int_{\mathbb{R}^d \times (0,\infty)} ps^{p-1} 1(\eta_1(B(y, s)) = 0) 1(\|y - z\| \leq s) dy ds.
\]

We observe now that \( B(z) \equiv B(0) \) for all \( z \in \mathbb{R}^d \) (change variable \( y - z = x \) and use the fact that a shift of \( \eta_1 \) is still a Poisson process on \( \mathbb{R}^d \) with constant unit intensity). Hence, by Schwarz inequality,

\[
C \leq \left( E(B(0)^6) \right)^{1/2} \int_{\mathbb{R}^d} \left( E[A(z)^2] \right)^{1/2} dz \tag{4.5}
\]

Next, we show that \( B(0) \) has finite moments of all orders. In fact, let us define

\[
R := \inf \{ s > 0 : \ell(y \in \mathbb{R}^d : \eta_1(B(y, s)) = 0, \|y\| \leq s) = 0 \}.
\]
R is the minimal radius s such that every point in the ball B(0, s) has at least a point of the Poisson process η1 within s distance. On the other hand
\[
B := B(0) = \int_0^\infty ps^{p-1} \ell(y \in \mathbb{R}^d : \eta_1(B(y, s)) = 0, \|y\| \leq s)ds
\]
and, therefore, it suffices to show that R has finite moments of all orders. We can prove this choosing a partition of the surface of the (d − 1)-dimensional unit sphere (the boundary of B(0,1)) into N = N(d) measurable regions, R1, . . . , RN, of equal area and diameter less than one (a proof that this can be done, together with estimates on the minimal N(d) number of regions needed in the partition, can be found, for instance, in Lemma 21 in [3]; see also [9]). We set then S_i = \cup_{t \in [0,1]} tR_i and T_i = \inf\{t > 0 : \eta_1(tS_i) > 0\}. Then S1, . . . , SN is a partition of B(0,1) into regions of equal volume and diameter less than one. On the other hand, T1, . . . , TN are i.i.d. random variables with P(T_i > t) = P(\eta_1(tS_i) = 0) = e^{-t(S_i)} = e^{-t^d/N}. Hence,
\[
P(\max_{1 \leq i \leq N} T_i > t) \leq Ne^{-t^d/N}
\]
(4.6) But now, if we take \( t \geq \max_{1 \leq i \leq N} T_i \), then for each i there is a point of \( \eta_1 \) in tS_i. The ball of radius t centered at that point covers tS_i (tS_i has diameter less than t). Thus, for every x in the ball B(0, t) there is a point of \( \eta_1 \) within t distance and, consequently, \( t \geq R \) and this shows that \( R \leq \max_{1 \leq i \leq N} T_i \) and, combined with (4.6), that
\[
E(R^d) \leq \int_0^\infty qt^{q-1}Ne^{-t^d/N}dt < \infty,
\]
(4.7) as claimed. Finally, to see that \( \int_{\mathbb{R}^d} (E[A(z)^2])^{1/2}dz < \infty \) and complete the proof in view of (4.5), we write \( A(z) = \int_{\{\|z\|<\infty\}} ps^{p-1}1(\eta(B(0, s)) = 0)ds \). Hence,
\[
A(z)^2 = \int_{\{\|z\|<\infty\}} p^2(st)^{p-1}1(\eta(B(0, s) \cup B(0, t)) = 0)dsdt
\]
and
\[
E(A(z)^2) = \int_{\{\|z\|<\infty\}} p^2(st)^{p-1}e^{-cd(\max(s,t))d}dsdt
\]
\[
\leq 2\int_{\|z\|<\infty} pt^{p-1}e^{-cdt^d}dt
\]
\[
\leq 2\int_{\|z\|<\infty} pt^{p-1}e^{-cdt^d}dt.
\]
Since \( \int_{\mathbb{R}} t^{p-1}e^{-cdt^d}dt \approx x^{2p-d}e^{-cx^d} \) as \( x \to \infty \) (in the sense that the ratio tends to a positive constant; this follows from l’Hôpital’s rule, for instance) we see that \( E(A(z)^2)^{1/2} \leq K\|z\|^{p-d/2}e^{-\frac{cd}{2}\|z\|^d} \) for some constant K and large enough \( \|z\| \). But this shows that \( \int_{\mathbb{R}^d} (E[A(z)^2])^{1/2}dz < \infty \) and completes the proof. □

Finally, we prove the last technical result of this section.

**Lemma 4.7** If \( F_\lambda(\eta_\lambda) \) is defined as in (4.3) and \( C_{2,b} \) the related constant defined in (3.8) then \( C_{2,b} < \infty \).

**Proof.** We note that now, fixing \( y_1 = 0 \), writing \( B_i = B(y_i, s_i), i = 1, \ldots, 4 \) and
\[
g(s_1, s_2, s_3, s_4) = \int_{\mathbb{R}^d} e^{-\ell(\cup_{i=1}^4 B_i)} \ell(B_1 \cap B_2) \ell(B_3 \cap B_4)1((B_1 \cup B_2) \cap (B_3 \cup B_4) \neq \emptyset) \prod_{i=2}^4 dy_i,
\]

23
we have
\[ C_{2,b} = \int_{(0,\infty)^4} g(s_1, s_2, s_3, s_4) \prod_{i=1}^{4} p_{s_i} \, ds_i. \]

Next, observe that \( \ell(B_1 \cap B_2) = 0 \) if \( \|y_2\| > s_1 + s_2 \), whereas if \( \|y_2\| \leq s_1 + s_2 \) then \( \|z\| \leq s_1 + 2s_2 \) for every \( z \in B_1 \cup B_2 \). Similarly, \( \ell(B_3 \cap B_4) > 0 \) implies that \( \|y_3 - y_4\| < s_3 + s_4 \). Hence, if \( \ell(B_1 \cap B_2) > 0 \), \( \ell(B_3 \cap B_4) > 0 \) and \( (B_1 \cup B_2) \cap (B_3 \cup B_4) \neq \emptyset \) then \( \|y_i\| \leq 2(s_1 + s_2 + s_3 + s_4), \ i = 2, 3, 4. \) This shows that
\[ g(s_1, s_2, s_3, s_4) \leq 2^4 \ell(B(0, \sum_{i=1}^{4} s_i)^3 \ell(B(0, s_1 \wedge s_2)) \ell(B(0, s_3 \wedge s_4)) e^{-\ell(B(0, s_i = 1))}. \]

Thus, it suffices to prove that for positive \( q_i, i = 1, \ldots, 4, \)
\[ \int_{\{0 < s_1 < s_2 < s_3 < s_4\}} e^{-c_d s_i} \prod_{i=1}^{4} s_i \, ds_i < \infty. \]

Indeed, from iterated integration we see that the last integral equals
\[ \int_{0}^{\infty} e^{-c_d s_i} \frac{s_i q_1 + q_2 + q_3 + q_4 + 3}{(q_1 + 1)(q_2 + 2)(q_1 + q_2 + q_3 + 3)} \, ds_i < \infty \]
and the result follows. \( \square \)

Appendix.

The following technical results have been used in the proof of auxiliary Lemmas needed for the proof of Theorem 3.8.

**Lemma 4.8** If \( 0 < 2x < n \) and \( 0 < 2y < n \) then
\[ \left| \left(1 - \frac{x}{n}\right)^n - e^{-x} \left(1 - \frac{x^2}{2n}\right) \right| \leq \frac{A(x)}{n^2}, \]
\[ \left(1 - \frac{x}{n}\right)^n - e^{-x} \left(1 - \frac{x^2}{2n}\right) \leq \frac{x^2 y^2}{4n^2} + \frac{A(x)}{n^2} + \frac{A(x) A(y)}{n^4}, \]
where \( A(x) = \frac{2}{3} x^3 + \frac{9}{8} x^4. \)

**Proof.** We observe first that for \( 0 < x < n \) we have
\[ 0 \leq e^{-x} - \left(1 - \frac{x}{n}\right)^n = \int_{x}^{n \log(1 - \frac{x}{n})} e^{-t} \, dt. \quad (4.8) \]

Now, from the series expansion \( -n \log(1 - \frac{x}{n}) = n \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{x}{n}\right)^k \), \( 0 < x < n \) we see that \( 0 \leq -n \log(1 - \frac{x}{n}) - (x + \frac{x^2}{2n}) \leq \frac{9}{8} \sum_{k=3}^{\infty} \left(\frac{x}{n}\right)^k = \frac{n}{3} \frac{x^3}{1-x/n}. \) Hence, if \( 0 < 2x < n \) we have
\[ x + \frac{x^2}{2n} \leq -n \log(1 - \frac{x}{n}) \leq x + \frac{x^2}{2n} + \frac{9}{8} \frac{x^3}{n}. \quad (4.9) \]

Noting that \( \int_{x}^{x + \frac{x^2}{2n}} e^{-t} \, dt = e^{-x} \left(1 - e^{-\frac{x^2}{2n}}\right) \) we obtain from (4.8) and (4.9) that
\[ \left| \left(1 - \frac{x}{n}\right)^n - e^{-x} e^{-\frac{x^2}{2n}} \right| \leq \frac{x^2 y^2}{3 n^2}, \quad \text{if} \ 0 < 2x < n. \quad (4.10) \]

We can easily check that \( |e^{-x} - (1 - x)| \leq \frac{x^2}{4} \) for \( x \geq 0 \) and this entails that \( |e^{-\frac{x^2}{2n}} - (1 - \frac{x^2}{n})| \leq \frac{x^4}{3 n^2}. \) Combining this with (4.10) we obtain the first inequality in the statement and from this, trivially, we get the second. \( \square. \)
Lemma 4.9 If $0 < 2x < n$ and $0 < k - n < n$ then

$$\left| (1 - \frac{x}{n})^{k-n} - e^{-\frac{k-n}{n}x} (1 - \frac{k-n}{2n}x) \right| \leq A(x) \frac{k-n}{n} \frac{1}{n^2},$$

where $A(x) = \frac{2}{3} x^3 + \frac{1}{8} x^4$.

**Proof.** Apply Lemma 4.8 to $x' = \frac{k-n}{n}x, \ n' = k - n$ and note that $A$ is increasing on $[0, \infty)$, hence, $A(x') \leq A(x)$. □

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