Prime and Primitive Ideals of Ultragraph Leavitt Path Algebras

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Abstract. Let $G$ be an ultragraph and let $K$ be a field. We describe prime and primitive ideals in the ultragraph Leavitt path algebra $L_K(G)$. We identify the graded prime ideals in terms of downward directed sets and then we characterize the non-graded prime ideals. We show that the non-graded prime ideals of $L_K(G)$ are always primitive.

1. Introduction

Let $E$ be a (directed) graph. The Leavitt path algebra $L_K(E)$, which is a purely algebraic analogue of graph Cuntz-Krieger $C^*$-algebra $C^*(E)$ [17, 10], was introduced in [1, 2]. The algebras $L_K(E)$ are generalizations of the Leavitt algebras $L(n, 1)$ [21]. The study of the structure of prime and primitive Leavitt path algebras have been the subject of a series of papers in recent years (see [5, 6, 3]). For a unital commutative ring $R$, the graded prime (primitive) ideals of $L_R(E)$ are characterized in [19] via special subsets of the vertex (called maximal tails). Furthermore, the structure of non-graded prime and primitive ideals of $L_K(E)$ have been identified in [16]. It was shown in [16] that there is a one-to-one correspondence between non-graded prime (primitive) ideals of $L_K(E)$ and maximal tails containing a loop without exits and the prime spectrum of $K[x, x^{-1}]$.

Ultragraph Leavitt path algebras have been widely studied, see [11, 8, 7, 12, 13, 22, 14]. Ultragraph Leavitt path algebra $L_R(G)$ was introduced in [15] as the algebraic version of ultragraph $C^*$-algebra $C^*(G)$ [24]. The algebras $L_R(G)$ are generalizations of the Leavitt path algebras $L_R(E)$. The structure of ultragraph Leavitt path algebras are more complicated, because in ultragraphs the range of each edge is allowed to be a nonempty set of vertices rather than a single vertex. The class of ultragraph Leavitt path algebras is strictly larger than the class of Leavitt path algebras of directed graphs. Also, every Leavitt path algebra of a directed graph can be embedded as a subalgebra in a unital ultragraph Leavitt path algebra.

The aim of this paper is to give a complete description of the prime ideals as well as the primitive ideals of $L_K(G)$. We start in Section 2 by recalling the definition of the quotient ultragraph $G/(H, S)$ and its Leavitt path algebra $L_K(G/(H, S))$. In Section 3, we characterize the graded prime ideals...
ideals in terms of the downward directed sets. To describe the structure of non-graded prime ideals, we investigate the structure of the closed ideals of $L_K(\mathcal{G}/(H,S))$ which contain no nonzero set idempotents. In Section 4, we give a complete description of primitive ideals. We show that a graded prime ideal $I_{(H,B_H)}$ is primitive if and only if the quotient ultragraph $\mathcal{G}/(H,B_H)$ satisfies Condition (L). Finally, we prove that every non-graded prime ideal in $L_K(\mathcal{G})$ is primitive.

2. Preliminaries

In this section, we briefly review the basic definitions and properties of ultragraphs [24], quotient ultragraphs [20] and ultragraph Leavitt path algebras [15].

An ultragraph $\mathcal{G} = (G^0, G^1, r_\mathcal{G}, s_\mathcal{G})$ consists of a countable set of vertices $G^0$, a countable set of edges $G^1$, the source map $s_{\mathcal{G}} : G^1 \to G^0$ and the range map $r_{\mathcal{G}} : G^1 \to \mathcal{P}(G^0) \setminus \{\emptyset\}$, where $\mathcal{P}(G^0)$ denotes the collection of all subsets of $G^0$. By an algebra in $\mathcal{P}(X)$, we mean a collection of subsets of $X$ which is closed under the set operations $\cup$, $\cap$ and $\setminus$. We write $G^0$ for the smallest algebra in $\mathcal{P}(G^0)$ containing $\{v\}, r_{\mathcal{G}}(e) : v \in G^0$ and $e \in G^1$.

Definition 2.1. A subcollection $H \subseteq G^0$ is hereditary if

1. $\{s_{\mathcal{G}}(e)\} \in H$ implies $r_{\mathcal{G}}(e) \in H$ for all $e \in G^1$.
2. $A \cup B \in H$ for all $A, B \in H$.
3. $A \in H, B \in G^0$ and $B \subseteq A$, imply $B \in H$.

The subcollection $H \subseteq G^0$ is saturated if, whenever $0 < |s_{\mathcal{G}}^{-1}(v)| < \infty$ satisfies $r_{\mathcal{G}}(e) \in H$ for all $e \in s_{\mathcal{G}}^{-1}(v)$, we have $\{v\} \in H$. Let $H \subseteq G^0$ be a saturated hereditary subcollection. We define the breaking vertices of $H$ to be the set

$$B_H := \left\{ v \in G^0 : |s_{\mathcal{G}}^{-1}(v)| = \infty \text{ and } 0 < |s_{\mathcal{G}}^{-1}(v) \cap \{e : r_{\mathcal{G}}(e) \notin H\}| < \infty \right\}.$$ 

If $H \subseteq G^0$ is a saturated hereditary subcollection and $S \subseteq B_H$, then $(H,S)$ is called an admissible pair in $\mathcal{G}$.

2.1. Quotient ultragraph. In order to define the quotient of ultragraphs we need to recall and introduce some notations. Let $(H,S)$ be an admissible pair in ultragraph $\mathcal{G} = (G^0, G^1, r_\mathcal{G}, s_\mathcal{G})$. Given $A \in \mathcal{P}(G^0)$, set $\overline{A} := A \cup \{w' : w \in A \cap (B_H \setminus S)\}$. The smallest algebra in $\mathcal{P}(\overline{G^0})$ containing $\{\{v\}, \{w'\}, r_{\mathcal{G}}(e) : v \in G^0, w \in B_H \setminus S \text{ and } e \in G^1\}$ is denoted by $\overline{G^0}$. Let $\sim$ be a relation on $\overline{G^0}$ defined by $A \sim B$ if and only if there exists $V \in H$ such that $A \cup V = B \cup V$. Then, by [20, Lemma 3.5], $\sim$ is an equivalent relation on $\overline{G^0}$ and the operations

$$[A] \cup [B] := [A \cup B], \quad [A] \cap [B] := [A \cap B] \text{ and } [A] \setminus [B] := [A \setminus B]$$

are well-defined on the equivalent classes $\{[A] : A \in \overline{G^0}\}$. It can be shown that $[A] = [B]$ if and only if both $A \setminus B$ and $B \setminus A$ belong to $H$. 


Definition 2.2. Let \((H, S)\) be an admissible pair in \(\mathcal{G}\). The \textit{quotient ultragraph of} \(\mathcal{G}\) by \((H, S)\) is the quadruple \(\mathcal{G}/(H, S) := (\Phi(G^0), \Phi(G^1), r, s)\), where
\[
\Phi(G^0) := \{\{v\}, \{w'\} : v \in G^0, \{v\} \not\in H \text{ and } w \in B_H \setminus S\},
\]
\[
\Phi(G^1) := \{e \in G^1 : r_G(e) \not\in H\},
\]
and \(s : \Phi(G^1) \to \Phi(G^0)\) and \(r : \Phi(G^1) \to \{[A] : A \in \overline{G^0}\}\) are the maps defined by \(s(e) := \{s_G(e)\}\) and \(r(e) := [r_G(e)]\) for every \(e \in \Phi(G^1)\), respectively.

We denote by \(\Phi(G^0)\) the smallest algebra in \(\{[A] : A \in \overline{G^0}\}\) containing \(\Phi(G^0) \cup \{r(e) : e \in \Phi(G^1)\}\).

One can see that \(\Phi(G^0) = \{[A] : A \in \overline{G^0}\}\). If \(A, B \in \overline{G^0}\) and \([A] \cap [B] = [A]\), then we write \([A] \subseteq [B]\). Also, we write \([v]\) instead of \(\{v\}\) for every vertex \(v \in G^0 \setminus H\).

A \textit{path} in \(\mathcal{G}/(H, S)\) is a finite sequence \(\alpha = e_1 e_2 \cdots e_n\) of edges with \(s(e_{i+1}) \subseteq r(e_i)\) for \(1 \leq i \leq n - 1\). We consider the elements of \(\Phi(G^0)\) as the paths of length zero. We let \(\text{Path}(\mathcal{G}/(H, S))\) denotes the set of all paths in \(\mathcal{G}/(H, S)\). We define \([A]^* := [A]\) and \(\alpha^* := e_n^* e_{n-1}^* \cdots e_1^*\), for every \([A] \in \Phi(G^0)\) and \(\alpha = e_1 e_2 \cdots e_n \in \text{Path}(\mathcal{G}/(H, S))\). The maps \(r, s\) extend to \(\text{Path}(\mathcal{G}/(H, S))\) in an obvious way.

2.2. \textbf{Leavitt path algebra.} A vertex \([v] \in \Phi(G^0)\) is called an \textit{infinite emitter} if \(|s^{-1}([v])| = \infty\) and a \textit{sink} if \(|s^{-1}([v])| = 0\). A \textit{singular vertex} is a vertex that is either a sink or an infinite emitter and we denote the set of singular vertices by \(\Phi_{sg}(G^0)\).

Definition 2.3. Let \(\mathcal{G}/(H, S)\) be a quotient ultragraph and let \(K\) be a field. A \textit{Leavitt} \(\mathcal{G}/(H, S)\)-\textit{family} in a \(K\)-algebra \(X\) is a set \(\{q_A, t_e, t_e^* : [A] \in \Phi(G^0)\text{ and } e \in \Phi(G^1)\}\) of elements in \(X\) such that
\begin{enumerate}
  \item \(q_{[\emptyset]} = 0\), \(q_A q_B = q_{A \cap B}\) and \(q_A \cup [B] = q_{[A] + q[B] - q[A \cap B]}\);
  \item \(q(e)t_e = t_e q(e) = t_e^* q(s(e)) = t_e^*\) and \(q(e) t_e^* = t_e^* q(s(e)) = t_e^*\);
  \item \(t_e^* t_f = \delta_{e,f} q(r(e))\);
  \item \(q_{[v]} = \sum_{s(e) = [v]} t_e t_e^*\) whenever \([v] \in \Phi(G^0) \setminus \Phi_{sg}(G^0)\).
\end{enumerate}

The \textit{Leavitt path algebra of} \(\mathcal{G}/(H, S)\), denoted by \(L_K(\mathcal{G}/(H, S))\), is defined to be the \(K\)-algebra generated by a universal Leavitt \(\mathcal{G}/(H, S)\)-family.

Let \(\mathcal{G}\) be an ultragraph. If we consider the quotient ultragraph \(\mathcal{G}/(\emptyset, \emptyset)\), then \([A] = \{A\}\) For every \(A \in G^0\). Thus we can consider the ultragraph \(\mathcal{G}\) as the quotient ultragraph \(\mathcal{G}/(\emptyset, \emptyset)\). So it makes sense to talk about the \textit{ultragraph Leavitt path algebra} \(L_K(\mathcal{G})\) and define it as \(L_K(\mathcal{G}/(\emptyset, \emptyset))\). In fact, the definition of ultragraph Leavitt path algebras ([15, Definition 2.1]) is an special case of the Definition 2.3.

By [15, Theorem 2.15], \(L_K(\mathcal{G}/(H, S))\) is of the form
\[
\text{span}_K \{t_\alpha q_A t_\beta^* : \alpha, \beta \in \text{Path}(\mathcal{G}/(H, S))\text{ and } r(\alpha) \cap [A] \cap r(\beta) \neq \emptyset\},
\]
where \( t_\alpha := t_{e_1}t_{e_2} \cdots t_{e_n} \) if \( \alpha = e_1e_2 \cdots e_n \) and \( t_\alpha := q[A] \) if \( \alpha = [A] \). Also, \( L_K(\mathcal{G}/(H,S)) \) is a \( \mathbb{Z} \)-graded ring by the grading

\[ L_K(\mathcal{G}/(H,S))_n = \text{span}_K \{ t_\alpha q[A]t_\beta^* : |\alpha| - |\beta| = n \} \quad (n \in \mathbb{Z}). \]

Throughout the article we denote the universal Leavitt \( \mathcal{G} \)-family and \( \mathcal{G}/(H,S) \)-family by \( \{s,p\} \) and \( \{t,q\} \), respectively. Also, we suppose that \( L_K(\mathcal{G}) = L_K(s,p) \) and \( L_K(\mathcal{G}/(H,S)) = L_K(t,q) \).

3. Prime ideals

In this section, we give a complete description of the prime ideals of \( L_K(\mathcal{G}) \). We first characterize the primeness of a graded ideal in terms of the downward directed sets and then we characterize the non-graded prime ideals of \( L_K(\mathcal{G}) \).

3.1. Graded prime ideals. We recall the definition of downward directed sets from [20, Definition 5.3]. Let \( \mathcal{G} \) be an ultragraph and \( A,B \in \mathcal{G}^0 \). We write \( A \geq B \) if either \( B \subseteq A \) or there is a path \( \alpha \) of positive length such that \( s_\mathcal{G}(\alpha) \in A \) and \( B \subseteq r_\mathcal{G}(\alpha) \). For the sake of simplicity, we will write \( \{v\} \geq \{w\} \) instead of \( \{v\} \supseteq \{w\} \). A nonempty subset \( M \) of \( \mathcal{G}^0 \) is said to be downward directed if for every \( A,B \in M \) there exists \( \emptyset \neq C \in M \) such that \( A,B \geq C \).

**Lemma 3.1.** Let \( \mathcal{G}/(H,S) \) be a quotient ultragraph. Then every nonzero graded ideal of \( L_K(\mathcal{G}/(H,S)) \) contains idempotent \( q[A] \) for some \( [0] \neq [A] \in \Phi(\mathcal{G}^0) \).

**Proof.** Let \( I \) be a nonzero graded ideal of \( L_K(\mathcal{G}/(H,S)) \). Then the quotient map \( \pi : L_K(\mathcal{G}/(H,S)) \to L_K(\mathcal{G}/(H,S))/I \) is a graded homomorphism. Suppose that \( q[A] \notin I \) for every \( [0] \neq [A] \in \Phi(\mathcal{G}^0) \). Then, by [15, Theorem 3.2], \( \phi \) is injective, which is impossible. \( \Box \)

**Lemma 3.2.** Let \( I \) be an ideal of \( L_K(\mathcal{G}) \). Consider \( H_1 := \{A \in \mathcal{G}^0 : p_A \in I\} \). If \( I \) is prime, then \( \mathcal{G}^0 \setminus H_1 \) is downward directed.

**Proof.** Let \( X := L_K(\mathcal{G})/I \) and denote by \( \bar{x} \) the image of \( x \in C^*(\mathcal{G}) \) in \( X \). For every \( A,B \in \mathcal{G}^0 \setminus H_1 \), both \( \bar{x} \bar{p}_A X \) and \( \bar{x} \bar{p}_B X \) are nonzero ideals of \( X \). Suppose that \( I \) is a prime ideal of \( L_K(\mathcal{G}) \). It follows that \( X \) is a prime ring. Thus \( \bar{x} \bar{p}_A X \bar{p}_B X \) is a nonzero ideal of \( X \) and consequently \( \bar{p}_A \bar{p}_B \neq 0 \). Since \( X \) is of the form

\[ \text{span}_K \{ \tilde{s}_\alpha \tilde{p}_C \tilde{s}_\beta^* : C \in \mathcal{G}^0, \alpha, \beta \in \text{Path}(\mathcal{G}) \text{ and } r_\mathcal{G}(\alpha) \cap C \cap r_\mathcal{G}(\beta) \neq \emptyset \}, \]

there exist \( \alpha, \beta \in \text{Path}(\mathcal{G}) \) and \( C \in \mathcal{G}^0 \) such that \( p_A(s_\alpha p_C s_\beta^*)p_B \neq 0 \), which would mean that \( s_\mathcal{G}(\alpha) \in A \) and \( s_\mathcal{G}(\beta) \in B \). Thus if we set \( D := r_\mathcal{G}(\alpha) \cap C \cap r_\mathcal{G}(\beta) \), then we can deduce that \( A,B \geq D \). Therefore \( \mathcal{G}^0 \setminus H_1 \) is downward directed. \( \Box \)
Let $(H, S)$ be an admissible pair in $\mathcal{G}$. For any $w \in B_H$, set

$$p^H_w := p_w - \sum_{s_{\varphi}(e) = w, \; r_{\varphi}(e) \notin H} s_e s_e^*,$$

and we define $I_{(H, S)}$ as the (two-sided) ideal of $L_K(\mathcal{G})$ generated by the idempotents $\{p_A : A \in H\} \cup \{p^H_w : w \in S\}$. By [15, Theorem 4.4], $L_K(\mathcal{G}/(H, S)) \cong L_K(\mathcal{G})/I_{(H, S)}$ and the correspondence $(H, S) \mapsto I_{(H, S)}$ is a bijection from the set of all admissible pairs of $\mathcal{G}$ to the set of all graded ideals of $L_K(\mathcal{G})$.

**Proposition 3.3.** If $I_{(H, S)}$ is a prime ideal of $L_K(\mathcal{G})$, then $|B_H \setminus S| \leq 1$

**Proof.** Assume to the contrary that $w, z \in B_H \setminus S$. Let $I$ and $J$ be two ideals of $L_K(\mathcal{G}/(H, S))$ generated by $q_{[w]}$ and $q_{[z]}$, respectively. Since $[w], [z]$ are two sinks in $\mathcal{G}/(H, S)$ and $q_{[w]}q_{[z]} = 0$, we have that $q_{[w]}L_K(\mathcal{G}/(H, S))q_{[z]} = 0$. Thus $IJ = 0$, contradicting the primeness of $L_K(\mathcal{G}/(H, S))$. □

**Theorem 3.4.** Let $\mathcal{G}$ be an ultragraph. Set

$$X_1 = \{I_{(H, B_H)} : G^0 \setminus H \text{ is downward directed}\}$$

and

$$X_2 = \{I_{(H, B_H \setminus \{w\})} : w \in B_H \text{ and } A \supseteq w \text{ for all } A \in G^0 \setminus H\}.$$ Then $X_1 \cup X_2$ is the set of all graded prime ideals of $L_K(\mathcal{G})$.

**Proof.** Let $X$ be the set of all graded prime ideals of $L_K(\mathcal{G})$. We prove that $X = X_1 \cup X_2$.

Assume $I_{(H, B_H)} \in X_1$. We show that the zero ideal of $L_K(\mathcal{G}/(H, B_H))$ is prime. Since $\{0\}$ is a graded ideal, it suffices to prove that for every nonzero graded ideals $I, J$ of $L_K(\mathcal{G}/(H, B_H))$, $IJ \neq \{0\}$ (see [23, Proposition II.1.4]). If $I$ and $J$ are such ideals, then by Lemma 3.1, there exist nonzero idempotents $q_{[A]} \in I$ and $q_{[B]} \in J$. Since $A, B \in G^0 \setminus H$ and $G^0 \setminus H$ is downward directed, there exists $C \in G^0 \setminus H$ such that $A, B \supseteq C$. Thus $q_{[C]} \in I \cap J$ and therefore $\{0\}$ is a prime ideal. Since $L_K(\mathcal{G}/(H, B_H)) \cong L_K(\mathcal{G})/I_{(H, B_H)}$, we deduce that $I_{(H, B_H)}$ is prime. Hence $X_1 \subseteq X$.

Let $I_{(H, B_H \setminus \{w\})} \in X_2$. Then $G^0 \setminus H$ is downward directed, because $A \supseteq w$ for all $A \in G^0 \setminus H$. Now, in a similar way as before we obtain that $I_{(H, B_H \setminus \{w\})}$ is prime. Thus $X_2 \subseteq X$.

To establish the reverse inclusion, let $I_{(H, S)} \in X$. From Proposition 3.3 we have that either $S = B_H$ or $S = B_H \setminus \{w\}$ for some $w \in B_H$. If $S = B_H$, then, by Lemma 3.2, $G^0 \setminus H$ is downward directed. Hence $I_{(H, S)} \in X_1$.

Let $S = B_H \setminus \{w\}$ and $A \in G^0 \setminus H$. We show that $A \supseteq w$. Clearly the result holds for $w \in A$, so let $w \notin A$. By the primeness of $L_K(\mathcal{G}/(H, S))$, we must have $IJ \neq \{0\}$ for every nonzero ideals $I$ and $J$ of $L_K(\mathcal{G}/(H, S))$. Thus

$$L_K(\mathcal{G}/(H, S))q_{[A]}L_K(\mathcal{G}/(H, S))q_{[w]}L_K(\mathcal{G}/(H, S)) \neq \{0\},$$
and hence \(q[A]L_K(\mathcal{G}/(H,S))q[w] \neq \{0\}\). Since \([w'] = 0\), there exists a path \(\alpha\) of positive length such that \(q[A]t_\alpha q[w] \neq 0\). Therefore \(s(\alpha) \subseteq [A]\) and \([w'] \subseteq r(\alpha)\), which implies that \(s_G(\alpha) \in A\) and \(w \in r_G(\alpha)\). Thus \(A \supseteq w\) and consequently \(I_{(H,S)} \in X_2\). This proves that \(X \subseteq X_1 \cup X_2\).

From [15, Theorem 5.4], we know that the ultragraph \(\mathcal{G}\) satisfies Condition (K) if and only if every ideal of \(L_K(\mathcal{G})\) is graded. The following corollary now follows from [15, Theorem 5.4] and Theorem 3.4.

**Corollary 3.5.** If the ultragraph \(\mathcal{G}\) satisfies Condition (K), then Theorem 3.4 gives a complete description of the prime ideals of \(L_K(\mathcal{G})\).

### 3.2. Non-graded prime ideals

Let \(\mathcal{G}/(H,S)\) be a quotient ultragraph and let \(\gamma = e_1e_2 \cdots e_n\) be a path of positive length. If \(s(\gamma) \subseteq r(\gamma)\), then \(\gamma\) is called a loop. Denote \(\gamma^1 := \{e_1, e_2, \ldots, e_n\}\). We say that \(\gamma\) has an exit if either \(r(e_i) \neq s(e_{i+1})\) for some \(1 \leq i \leq n\) or there exist an edge \(f \in \Phi(\mathcal{G}^1)\) and an index \(i\) such that \(s(f) \subseteq r(e_i)\) but \(f \neq e_{i+1}\). The quotient ultragraph \(\mathcal{G}/(H,S)\) satisfies Condition (L) if every loop in \(\mathcal{G}/(H,S)\) has an exit.

**Lemma 3.6.** Let \(\mathcal{G}/(H,S)\) be a quotient ultragraph. If \(\gamma = e_1e_2 \cdots e_n\) is a loop in \(\mathcal{G}/(H,S)\) without exits, then \(I_{\gamma,0}\) and \(K[x, x^{-1}]\) are Morita equivalent as rings, where \(I_{\gamma,0}\) is an ideal of \(L_K(\mathcal{G}/(H,S))\) generated by \(\{q_{s(e_i)} : 1 \leq i \leq n\}\).

**Proof.** Let \(\gamma = e_1e_2 \cdots e_n\) be a loop with no exits in \(\mathcal{G}/(H,S)\) and \([v] = s(\gamma)\). Consider

\[
(I_{\gamma,0}, q[v]I_{\gamma,0}q[v], I_{\gamma,0}q[v], q[v]I_{\gamma,0}, \psi, \phi),
\]

where \(\psi(m \otimes n) = mn\) and \(\phi(n \otimes m) = nm\). It can be shown that this is a (surjective) Morita context. Thus \(I_{\gamma,0}\) and \(q[v]I_{\gamma,0}q[v]\) are Morita equivalent.

Now we show that \(q[v]I_{\gamma,0}q[v] \cong K[x, x^{-1}]\). Since \(\gamma\) is a loop without exits, we deduce that

\[
I_{\gamma,0} = \text{span}_K \{t_{e_i}q_{s(e_i)}t_{e_i^*} : s(e_i) \subseteq r(\mu) \cap r(\nu) \text{ and } 1 \leq i \leq n\},
\]

and \(q_{s(e_i)} = (t_{e_i} \cdots t_{e_n})q[v](t_{e_n^*} \cdots t_{e_i^*})\) for \(1 \leq i \leq n\). This implies that

\[
q[v]I_{\gamma,0}q[v] = \text{span}_K \{(t_\gamma)^m(t_{\gamma^*})^n : m, n \geq 1\}.
\]

Let \(E\) be the graph with one vertex \(w\) and one loop \(f\). Then \(\{p_w := q[w], s_f := t_\gamma, s_f^* := t_{\gamma^*}\}\) is a Leavitt \(E\)-family in \(q[v]I_{\gamma,0}q[v]\). It follows from the universality of \(L_K(E)\) and the graded uniqueness theorem that \(L_K(E) \cong q[v]I_{\gamma,0}q[v]\). Since \(L_K(E) \cong K[x, x^{-1}]\), we conclude that \(I_{\gamma,0}\) and \(K[x, x^{-1}]\) are Morita equivalent.

To prove the next lemma, we use the fact that \(L_K(\mathcal{G}/(H,S))\) can be estimated by the Leavitt path algebras of finite graphs. So, we recall this approximation from [15, Section 3] and then we prove Lemma 3.8.

Let \(\mathcal{G}/(H,S)\) be a quotient ultragraph and \(F\) be a finite subset of \(\Phi(\mathcal{G}^0)\cup \Phi(\mathcal{G}^1)\). We construct a finite graph \(G_F\) as follows. Let \(F^0 := F \cap \Phi(\mathcal{G}^0)\) and
Let \( I \) such that \( G \) is a loop in \( I \) if and only if \( \{ \cdot \} \) of \( \Phi \) \( \Phi(R_F)^{n} \) and \( 0 \neq s^{-1}(\{v_i\}) \subseteq F^1 \) for \( 1 \leq i \leq m \), and

\[
\Gamma_{0} := \{ \omega \in \{0, 1\}^{n} \setminus \{0^n\} : \) where vertices \( \{v_1, \ldots, [v_m] \} \) exist such that \( R(\omega) = \bigcup_{i=1}^{m} [v_i] \) and \( \emptyset \neq s^{-1}(\{v_i\}) \subseteq F^1 \) for \( 1 \leq i \leq m \), \]

and

\[
\Gamma_{F} := \{ \omega \in \{0, 1\}^{n} \setminus \{0^n\} : R(\omega) \neq [\emptyset] \) and \( \omega \notin \Gamma_{0} \}. \]

Define the finite graph \( G_{F} = (G_{F}^{0}, G_{F}^{1}, r_{F}, s_{F}) \), where

\[
G_{F}^{0} := F^0 \cup F^1 \cup \Gamma_{F}, \]

\[
G_{F}^{1} := \{ (e, f) \in F^1 \times F^1 : s(f) \subseteq r(e) \}
\cup \{ (e, [v]) \in F^1 \times F^0 : [v] \subseteq r(e) \}
\cup \{ (e, \omega) \in F^1 \times \Gamma_{F} : \omega_i = 1 \) whenever \( e = e_i \}, \]

with \( s_{F}(e, f) = s_{F}(e, [v]) = s_{F}(e, \omega) = e \) and \( r_{F}(e, f) = f, r_{F}(e, [v]) = [v], r_{F}(e, \omega) = \omega \). By [15, Lemma 3.1], the elements

\[
P_{e} := t_{e}e^{*}, \quad P_{[v]} := q_{[v]}(1 - \sum_{e \in F^1} t_{e}e^{*}), \quad P_{\omega} := q_{R(\omega)}(1 - \sum_{e \in F^1} t_{e}e^{*}), \]

\[
S_{(e, f)} := t_{e}P_{f}, \quad S_{(e, [v])} := t_{e}P_{[v]}, \quad S_{(e, \omega)} := t_{e}P_{\omega}, \]

\[
S_{(e, f)^*} := P_{f}t_{e}^{*}, \quad S_{(e, [v])^{*}} := P_{[v]}t_{e}^{*}, \quad S_{(e, \omega)^{*}} := P_{\omega}t_{e}^{*}, \]

form a Leavitt \( G_{F} \)-family in \( L_{K}(G/(H, S)) \) such that

\[
L_{K}(G_{F}) \cong L_{K}(S, P) = \langle q_{[v]}, t_{e}, t_{e}^{*} : [v] \in F^0, e \in F^1 \rangle. \]

Remark 3.7. Let \( \gamma := e_{1}e_{2} \cdots e_{n} \) be a loop in \( G/(H, S) \) and \( F \) be a finite subset of \( \Phi_{sg}(G_{F}^{0}) \cup \Phi(G_{F}^{1}) \) containing \( \{e_1, e_2, \ldots, e_n\} \). Then \( \tilde{\gamma} := (e_1, e_2) \cdots (e_n, e_1) \) is a loop in \( G_{F} \). Since the elements of \( F^0 \cup \Gamma_{F} \) are sinks in \( G_{F} \), every loop in \( G_{F} \) is of the form \( \beta \) where \( \beta \) is a loop in \( G/(H, S) \). By using the argument of [20, Lemma 4.3], we can show that \( \gamma \) is a loop without exits in \( G/(H, S) \) if and only if \( \tilde{\gamma} \) is a loop without exits in \( G_{F} \).

The set of vertices in the loops without exits of \( G/(H, S) \) is denoted by \( P_{c}(G/(H, S)) \). Also, we denote by \( I_{P_{c}(G/(H, S))} \) the ideal of \( L_{K}(G/(H, S)) \) generated by the idempotents associated to the vertices in \( P_{c}(G/(H, S)) \).

Lemma 3.8. Let \( G/(H, S) \) be a quotient ultragraph. If \( z \in L_{K}(G/(H, S)) \setminus I_{P_{c}(G/(H, S))} \), then there exist \( x, y \in L_{K}(G/(H, S)) \) and \( [\emptyset] \neq [A] \in \Phi(G_{F}^{0}) \) such that \( \text{xyz} = q_{[A]} \).

Proof. Let \( \{F_{n}\} \) be an increasing sequence of finite subsets of \( (G_{F}^{H})_{sg} \cup \Phi(G_{F}^{1}) \) such that \( \bigcup_{n=1}^{\infty} F_{n} = \Phi_{sg}(G_{F}^{0}) \cup \Phi(G_{F}^{1}) \). Then

\[
\bigcup_{n} L_{K}(G_{F_{n}}) = \langle q_{[v]}, t_{e}, t_{e}^{*} : [v] \in \Phi_{sg}(G_{F}^{0}), e \in \Phi(G_{F}^{1}) \rangle = L_{K}(G/(H, S)). \]
Now, let \( z \in L_K(G/(H,S)) \setminus I_{P_e(G/(H,S))} \). There exists \( F_n \) such that \( z \in L_K(G_{F_n}) \). We show that \( z \notin I_{P_e(G_{F_n})} \). By Remark 3.7, we have

\[
P_e(G_{F_n}) = \{ \tilde{\alpha} : \alpha \in P_e(G/(H,S)) \text{ and } \gamma_0 \subseteq G^1_{F_n} \}.
\]

Suppose that \( \gamma := e_1 e_2 \cdots e_n \in P_e(G/(H,S)) \). For every \( 1 \leq i \leq n \) we have \( P_{e_i} = t_{e_i} t_{e_i}^* \in I_{P_e(G/(H,S))} \). Thus \( I_{P_e(G/(H,S))} \) contains generators of \( I_{P_e(G_{F_n})} \). This implies that \( I_{P_e(G_{F_n})} \subseteq L_K(G_{F_n}) \cap I_{P_e(G/(H,S))} \) and consequently \( z \in L_K(G_{F_n}) \setminus I_{P_e(G_{F_n})} \).

By [4, Proposition 5.2], there exist \( x', y' \in L_K(G_{F_n}) \) and \( w \in G^0_{F_n} \) such that \( x' z y' = P_w \). We distinguish three cases.

1. Let \( w = [v] \in F^0_n \). Since \( [v] \in \Phi_{sg}(G^0) \), there exists \( f \in \Phi(G^1) \setminus F^1_n \) such that \( [v] = s(f) \). Set \( x = t_f x' \) and \( y = y' t_f \). Then \( xyz = q_r(f) \).
2. If \( w = e \in F^1_n \), then \( t_e x' z y' t_e = q_r(e) \).
3. Let \( w = \omega \in \Gamma_{F_n} \). Thus there exists a vertex \( [v] \subseteq R(\omega) \) such that either \([v] \) is a sink or there is an edge \( f \in \Phi(G^1) \setminus F^1_n \) with \( s(f) = [v] \).

In the former case, we deduce that \( q[v] x' z y' = q[v] Q \omega = q[v] \) and in the later case \( t_f x' z y' t_f = t_f Q \omega t_f = q_r(f) \).

Proposition 3.9 is an immediate consequence of Lemma 3.8.

**Proposition 3.9.** Let \( G/(H,S) \) be a quotient ultragraph. If \( I \) is an ideal of \( L_K(G/(H,S)) \) with \( \{ [A] \neq [0] : q_{[A]} \in I \} = \emptyset \), then \( I \subseteq I_{P_e(G/(H,S))} \).

**Remark 3.10.** Let \( G/(H,S) \) be a quotient ultragraph, \( x \in L_K(G/(H,S)) \) and let the ideal of \( L_K(G/(H,S)) \) generated by \( I_{H,S,x} \cup \{ x \} \) is denoted by \( I_{H,S,x} \). Suppose that \( \gamma \) is a loop in \( G/(H,S) \) without exits and \( f(x) \) is a polynomial in \( K[x, x^{-1}] \). It can be shown that \( I_{f(t_{x'})} = I_{f(t_{x'})} \), where \( x' \) is a permutation of \( \gamma \).

Let \( H \) be a saturated hereditary subset of \( G^0 \) and \( \gamma = e_1 e_2 \cdots e_n \) be a loop in \( G \). We say that \( \gamma = e_1 e_2 \cdots e_n \) is a loop in \( G^0 \setminus H \) if \( r_G(\gamma) \in G^0 \setminus H \). Also, \( \gamma \) has an exit in \( G^0 \setminus H \) if either \( r_G(e_i) \setminus s_G(e_{i+1}) \in G^0 \setminus H \) for some \( 1 \leq i \leq n \) or there exists an edge \( f \in G^1 \) and an index \( i \) such that \( r_G(f) \in G^0 \setminus H \) and \( s_G(f) \subseteq r_G(e_i) \) but \( f \neq e_{i+1} \). One can see that the quotient ultragraph \( G/(H,B_H) \) satisfies Condition (L) if and only if every loop in \( G^0 \setminus H \) has an exit in \( G^0 \setminus H \).

**Theorem 3.11.** Let \( G \) be an ultragraph and let \( I \) be an ideal of \( L_K(G) \). Denote \( H := H_I \). Then \( I \) is a non-graded prime ideal if and only if

1. \( G^0 \setminus H \) is downward directed,
2. \( G^0 \setminus H \) contains a loop \( \gamma \) without exits in \( G^0 \setminus H \) and
3. \( I = I_{(H,B_H,f(\gamma))} \), where \( f(x) \) is an irreducible polynomial in \( K[x, x^{-1}] \).

**Proof.** Let \( I \) be a non-graded prime ideal and \( S := \{ w \in B_H : p^H_w \in I \} \). It follows from Lemma 3.2 that \( G^0 \setminus H \) is downward directed. Consider the
that contradiction with the primeness of $I$.

In view of Lemma 3.1, this means that $\bar{I}$ is a non-graded ideal. Thus by the Cuntz-Krieger uniqueness theorem [15, Theorem 3.6], $\mathcal{G}/(H, S)$ contains a loop $\gamma$ without exits. Let $w \in B_H \setminus S$ and $X := \frac{L_K(\mathcal{G}/(H, S))}{\bar{I}}$. Since $\gamma$ has no exits and $[w']$ is a sink, we get that $(q_{[w]} + \bar{I})X(q_{\gamma} + \bar{I}) = \{0\}$, in contradiction with the primeness of $X$. Hence $S = B_H$.

Since $\mathcal{G}^0 \setminus H$ is downward directed, we get that $\gamma$ is unique (up to permutation). Thus, by Proposition 3.9, we have $I \subset I_{\gamma^0}$. From Lemma 3.6, we know that $L_{\gamma^0}$ is Morita equivalent to $K[x, x^{-1}]$ by the Morita correspondence $J \mapsto q_{\gamma}Jq_{\gamma}$. Since the primeness is preserved by the Morita correspondence, we deduce that $q_{\gamma}I_{q_{\gamma}}$ is a prime ideal of $K[x, x^{-1}]$. Thus there exists an irreducible polynomial $f(x)$ in $K[x, x^{-1}]$ such that $q_{\gamma}I_{q_{\gamma}}$ is generated by $f(x)$. Hence $I = I_{\langle f(t_{\gamma}) \rangle}$. Since $\bar{I} \subset L_K(\mathcal{G}/(H, B_H)) = \frac{L_K(\mathcal{G})}{I_{(H, B_H)}}$ and

$$q_A = p_A + I_{(H, B_H)} \quad \text{for } A \in \Phi(\mathcal{G}^0),$$

$$s_e = s_e + I_{(H, B_H)} \quad \text{for } e \in \Phi(G^1),$$

$$s_e^* = s_e^* + I_{(H, B_H)} \quad \text{for } e \in \Phi(G^1),$$

we deduce that $I = I_{\langle H, B_H, f(t_{\gamma}) \rangle}$.

Conversely, assume that the above three conditions hold. Denote by $\bar{I}$ the image of $I$ in the quotient $L_K(\mathcal{G})/I_{(H, B_H)}$. Hence $\bar{I} = I_{\langle f(t_{\gamma}) \rangle}$. Since $\mathcal{G}^0 \setminus H$ is downward directed, $\gamma$ is unique (up to permutation). It follows from Proposition 3.9 that $\bar{I} \subset I_{\gamma^0}$. Thus $q_{\gamma}I_{q_{\gamma}}$ is an ideal of $K[x, x^{-1}]$ generated by $f(x)$. As $f(x)$ is an irreducible polynomial in $K[x, x^{-1}]$, the ideal $q_{\gamma}I_{q_{\gamma}}$ is prime and therefore, by Lemma 3.6, $\bar{I}$ is a prime ideal of $L_K(\mathcal{G})/I_{(H, B_H)}$. Since $\{0\} \neq \bar{I} \subset \gamma^0$, we get that $\bar{I}$ is a non-graded ideal. Consequently, $I$ is a non-graded prime ideal of $L_K(\mathcal{G})$.

\section{4. Primitive ideals}

In this section, we characterize primitive ideals of $L_K(\mathcal{G})$. We determine primitive quotient ultragraph Leavitt path algebras and then we characterize graded primitive ideals. Finally, we see that the non-graded prime ideals of $L_K(\mathcal{G})$ are always primitive.

4.1. **Graded primitive ideals.** A ring $R$ is said to be left (right) primitive, if it has a faithful simple left (right) $R$–module. Since $L_K(\mathcal{G}/(H, S)) = L_K(\mathcal{G})/(H, S)^{op}$, we deduce that $L_K(\mathcal{G}/(H, S)$ is left primitive if and only if it is right primitive. So we simply say it is primitive.
Lemma 4.1. Let $G/(H, B_H)$ be a quotient ultragraph and 

\[ X := \{ [A] \in \Phi(G^0) \setminus \{ [\emptyset] \} : [A] \cap [v] = [\emptyset] \} \text{ for every vertex } [v] \in \Phi(G^0) \} . \]

If $G^0 \setminus H$ is downward directed, then $X$ is closed under finite intersections.

Proof. Suppose that $G^0 \setminus H$ is downward directed. We show that if $[A] \in X$ and $A \supseteq C$ for some $C \in G^0 \setminus H$, then $C \subseteq A$. If $C \not\subseteq A$, then there is a path $\alpha$ of positive length such that $s_G(\alpha) \in A$ and $C \subseteq t_G(\alpha)$. So $\{ s_G(\alpha) \} \in H$. Thus, by the hereditary property of $H$, $C \in H$, which is impossible.

Now, let $[A_1], \ldots, [A_n] \in X$ and $A := A_1 \cap \cdots \cap A_n$. Set $C_1 = A_1$. Since $G^0 \setminus H$ is downward directed, there exists $C_2 \in G^0 \setminus H$ such that $C_1, A_2 \supseteq C_2$. Therefore $C_2 \subseteq C_1 \cap A_2$. Similarly, there exists $C_3 \in G^0 \setminus H$ such that $C_3 \subseteq C_2 \cap A_2$. Repeating this process we find $C_n \subseteq G^0 \setminus H$ such that $C_n \subseteq A$. Hence $A \in G^0 \setminus H$ and thus $[A] \neq [\emptyset]$. Since $[A] \cap [v] = [\emptyset]$ for all $[v] \in \Phi(G^0)$, we conclude that $[A] \in X$. \(\square\)

Let $A$ be a unital ring. By [9, Theorem 1], $A$ is left primitive if and only if there is a left ideal $M \neq A$ of $A$ such that for every nonzero two-sided ideal $I$ of $A$, $M + I = A$. We use this to prove the following theorem.

Theorem 4.2. Let $G/(H, S)$ be a quotient ultragraph. Then $L_K(G/(H, S))$ is primitive if and only if one of the following holds:

(i) $S = B_H, G/(H, S)$ satisfies Condition (L) and $G^0 \setminus H$ is downward directed.

(ii) $S = B_H \setminus \{ w \}$ for some $w \in B_H$ and $A \supseteq w$ for all $A \in G^0 \setminus H$.

Proof. First, suppose that $L_K(G/(H, S))$ is primitive. Then $L_K(G/(H, S))$ is prime and thus $I_{(H, S)}$ is a prime ideal of $L_K(G)$. From Proposition 3.3 we deduce that $|B_H \setminus S| \leq 1$. So either $|B_H \setminus S| = 1$ or $S = B_H$. Thus (ii) follows by Theorem 3.4 or (i) holds by Theorem 3.4 and appalling Lemma 3.6, respectively.

Conversely, suppose that (i) holds. Applying Theorem 3.4, the graded ideal $I_{(H, S)}$ is prime which implies that $L_K(G/(H, S))$ is a prime ring. By [18, Lemmas 2.1 and 2.2], there exists a prime unital $K$-algebra $R$ such that $L_K(G/(H, S))$ embeds in $R$ as a two-sided ideal and primitivity of $R$ gives the primitivity of $L_K(G/(H, S))$ and vice versa. So it is enough to show that $R$ is primitive. Suppose that $X$ is the set defined in Lemma 4.1. We distinguish two cases depending on $X$.

Case 1. Let $X = \emptyset$ and let $[v]$ be a vertex in $G/(H, S)$. Define $H = \{ [u] \in \Phi(G^0) : v \geq u \}$. Since $\Phi(G^0)$ is countable we can write $H = \{ [v_1], [v_2], \ldots \}$. We claim that there exists a sequence $\{ \lambda_i \}_{i=1}^\infty \in \text{Path}(G/(H, S))$ such that for every $i \in \mathbb{N}$, $\lambda_{i+1} = \lambda_i \mu_i$ for some path $\mu_i \in \text{Path}(G/(H, S))$ and also $v_1 \geq s_G(\mu_i)$. Note that for every $[A] \in \Phi(G^0)$ we define $s_G([A]) = r_G([A]) = A$.

Set $\lambda_1 = [v_1]$, since $G/(H, S)$ is downward directed, there exists $A_2 \in G^0 \setminus H$ such that $v_2 \geq A_2$ and $r_G(\lambda_1) \geq A_2$. If $A_2 \subseteq r_G(\lambda_1)$, then $A_2 = \{ v_1 \}$.
and take \( \mu_1 = [A_2] \). If \( A_2 \not\subseteq r_G(\lambda_1) \), then there is a path \( \mu_1 \) of positive length such that \( s_G(\mu_1) \in r_G(\lambda_1) \) and \( A_2 \subseteq r_G(\mu_1) \). Now define \( \lambda_2 = \lambda_1 \mu_1 \). Also \( v_1 \geq s_G(\mu_1) \), \( v_2 \geq A_2 \) and \( A_2 \subseteq r_G(\lambda_2) \). Now by induction we assume that there exist \( \lambda_1, \ldots, \lambda_n \) with the previous properties. Corresponding there exist \( A_k \in G^0 \setminus H \) for \( k \in \{1, 2, \ldots, n\} \) such that \( v_k \geq A_k \) and \( A_k \subseteq r_G(\lambda_k) \).

Since \([A_n] \neq [0]\) and \( X = \emptyset \), there is a vertex \([u_n] \in \Phi(G^0)\) such that \( u_n \subseteq A_n \). Hence \( v_n \geq u_n \) and \([u_n] \subseteq r(\lambda_n)\), one can show that there is a path \( \mu_n \in \text{Path}(G) \) such that \( s_G(\mu_n) = u_n \). Now define \( \lambda_{n+1} = \lambda_n \mu_n \) and corresponding \( A_{n+1} \in G^0 \setminus H \) with the same property as before.

Now set

\[
M = \sum_{i=1}^{\infty} R(1 - t_{\lambda_i} t_{\lambda_i^*}),
\]

which is a left ideal of \( R \). If \( 1 \in M \), then \( 1 = \sum_{i=1}^{n} r_i (1 - t_{\lambda_i} t_{\lambda_i^*}) \) for some \( n \in \mathbb{N} \) and \( r_1, \ldots, r_n \in R \). Since \( t_{\lambda_j} t_{\lambda_k} t_{\lambda_k} = t_{\lambda_j} t_{\lambda_k^*} (k \geq j) \), we have

\[
t_{\lambda_n} t_{\lambda_n^*} = \sum_{i=1}^{n} r_i (1 - t_{\lambda_i} t_{\lambda_i^*}) t_{\lambda_n} t_{\lambda_n^*} = 0,
\]

which is a contradiction. Hence \( 1 \notin M \) and \( M \) is a proper left ideal of \( R \).

By [9, Theorem 1], to prove that \( R \) is primitive, it is enough to show that \( M + I = R \) for every nonzero two-sided ideal \( I \) of \( R \). Take \( I_1 = I \cap L_K(G/(H, S)) \). But \( R \) is prime, so \( I_1 \) is a nonzero two sided ideal of \( L_K(G/(H, S)) \). Since \( G/(H, S) \) satisfies Condition (L), by the Cuntz-Krieger uniqueness theorem [15, Theorem 3.6], there exists \([0] \neq [A] \in \Phi(G^0)\) such that \( q[A] \in I_1 \). By downward directedness, there exists \( C \in G^0 \setminus H \) for which \( v \geq C \) and \( A \geq C \). As \( X = \emptyset \), there is a vertex \([z] \in \Phi(G^0)\) such that \( z \in C \).

Hence \( v \geq z \) and thus \( z = v_n \) for some \( n \geq 1 \). We know that \( v_n \geq s_G(\mu_n) \), where \( \lambda_{n+1} = \lambda_n \mu_n \). Consequently, \( A \geq C \geq v_n \geq s_G(\mu_n) \geq r_G(\lambda_{n+1}) \).

Since \( q[A] \in I_1 \), we deduce that \( q_{r(\lambda_{n+1})} \in I_1 \) and \( t_{\lambda_{n+1}} t_{\lambda_{n+1}^*} \in I_1 \). Thus

\[
1 = (1 - t_{\lambda_{n+1}^*} t_{\lambda_{n+1}}) + t_{\lambda_{n+1}^*} t_{\lambda_{n+1}} \in M + I,
\]

which implies that \( M + I = R \).

Case 2. Let \( X \neq \emptyset \). Define \( M = \sum_{[A] \in X} R(1 - q[A]) \), which is a left ideal of \( R \).

We claim that \( M \) is proper, otherwise let \( 1 = \sum_{i=1}^{n} r_i (1 - q[A_i]) \in M \), where \([A_1], \ldots, [A_n] \in X \) and \( r_1, \ldots, r_n \in R \). Then \( q[A] = \sum_{i=1}^{n} r_i (1 - q[A_i]) q[A] = 0 \), where \( A := A_1 \cap \cdots \cap A_n \). By Lemma 4.1 \([A] \neq [0]\), so \( q[A] \neq 0 \), a contradiction.

Consider \( I \) and \( I_1 \) as before. Choose \([A] \in X, C \in G^0 \setminus H \) and \([\emptyset] \neq [B] \in \Phi(G^0)\) such that \( q[B] \in I_1 \), \( B \geq C \) and \( A \geq C \). As we have seen in the proof of Lemma 4.1, \([C] \in X \). Since \( B \geq C \), we have \( q[C] \in I_1 \). Therefore

\[
1 = (1 - q[C]) + q[C] \in M + I.
\]

Hence \( M + I = R \) and consequently \( R \) is primitive.

Finally, suppose that (iii) holds. Then \( G^0 \setminus H \) is downward directed. By Theorem 3.4, \( L_K(G/(H, S)) \) is a prime ring. We claim that \( G/(H, S) \) satisfies Condition (L), now with a similar argument as in the previous part if we
take \( M = R(1 - q[w]) \), one can show that \( M + I = R \) for every nonzero two-sided ideal \( I \) of \( R \) and consequently \( R \) is primitive.

To prove the claim, let \( \gamma = e_1e_2 \cdots e_n \) be a loop in \( \mathcal{G}/(H,S) \). By (ii) we have \( s_\mathcal{G}(\gamma) \geq w \), so either \( w = s_\mathcal{G}(\gamma) \) or \( w \neq s_\mathcal{G}(\gamma) \). Thus \( r(e_n) \neq s(e_1) \) or there is a path \( \alpha \) such that \( s_\mathcal{G}(\alpha) = s_\mathcal{G}(\gamma) \) and \( \{w\} \subseteq r_\mathcal{G}(\alpha) \), respectively. Therefore \( \gamma \) has an exit in \( \mathcal{G}/(H,S) \).

A two-sided ideal \( I \) of a ring \( R \) is called a left (right) primitive ideal if \( R/I \) is a left (right) primitive ring. We note that a graded ideal \( I_{(H,S)} \) of \( L_K(\mathcal{G}) \) is left primitive if and only if it is right primitive.

Theorem 4.2 immediately yields the following.

**Corollary 4.3.** Let \( \mathcal{G} \) be an ultragraph. A graded ideal \( I_{(H,S)} \) of \( L_K(\mathcal{G}) \) is primitive if and only if one of the following holds:

(i) \( S = B_H \), \( \mathcal{G}^0 \setminus H \) is downward directed and every loop in \( \mathcal{G}^0 \setminus H \) has an exit in \( \mathcal{G}^0 \setminus H \).

(ii) \( S = B_H \setminus \{w\} \) for some \( w \in B_H \) and \( A \geq w \) for all \( A \in \mathcal{G}^0 \setminus H \).

**4.2. Non-graded primitive ideals.** It is well known that every primitive ideal of a ring is prime. The next theorem shows that every non-graded prime ideal of \( L_K(\mathcal{G}) \) is primitive.

**Theorem 4.4.** Let \( \mathcal{G} \) be an ultragraph. A non-graded prime ideal of \( L_K(\mathcal{G}) \) is prime if and only if it is primitive.

**Proof.** Let \( I \) be a non-graded prime ideal of \( L_K(\mathcal{G}) \). By Theorem 3.11, \( I = I_{(H,B_H,f(s_\gamma)))} \), where \( \mathcal{G}^0 \setminus H \) is a downward directed set containing a loop \( \gamma \) without exits in \( \mathcal{G}^0 \setminus H \) and \( f(x) \) is an irreducible polynomial in \( K[x,x^{-1}] \). Let \( \bar{I} = I/I_{(H,B_H)} \) and \( s_\mathcal{G}(\gamma) = v \). As in the proof of Theorem 3.11, the ideal \( q[v]\bar{I}q[v] \) of \( K[x,x^{-1}] \) generated by \( f(x) \) is a maximal ideal. Hence \( K[x,x^{-1}]/q[v]\bar{I}q[v] \cong K \). On the other hand, we have

\[
\mathcal{P}_v \left( \frac{L_K(\mathcal{G})}{I} \right) \mathcal{P}_v \cong \frac{q[v]L_K(\mathcal{G}/(H,B_H))q[v]}{q[v]Iq[v]} \cong \frac{K[x,x^{-1}]}{q[v]Iq[v]},
\]

where \( \mathcal{P}_v = p_v + I \) and \( q[v] = p_v + I_{(H,B_H)} \). Since every field is primitive, we see that \( L_K(\mathcal{G})/I \) has a primitive corner and so, \( I \) is a primitive ideal of \( L_K(\mathcal{G}) \).

**Example 4.5.** Consider the ultragraph \( \mathcal{G} \) given below.
Set $H_1 = \{v\}$ and $H_2 = \{rg(e) \setminus \{v, v_1\}\}$. We have that
$B_{H_1} = \{w\}$ and $B_{H_2} = \emptyset$. Let $H$ be a saturated hereditary subcollection of $\mathcal{G}$. It can be shown that $\mathcal{G}^0 \setminus H$ is downward directed if and only if $H = H_1$ or $H = H_2$. It follows from Theorem 3.4 that $I_{(H_1, \{w\})}$ and $I_{(H_2, \emptyset)}$ are (graded) prime ideals of $L_K(\mathcal{G})$. Since $A \geq w$ for every $A \in \mathcal{G}^0 \setminus H_1$, by Corollary 4.3 (ii), $I_{(H_1, \emptyset)}$ is a primitive ideal of $L_K(\mathcal{G})$. $I_{(H_1, \{w\})}$ is not primitive because $f$ is a loop without exits in $\mathcal{G}^0 \setminus H_1$. If $g(x)$ is an irreducible polynomial in $K[x, x^{-1}]$, then, by Theorem 3.11 and Theorem 4.4, $I_{(H_1, \{w\}, g(s_f))}$ is a non-graded prime (primitive) ideal in $L_K(\mathcal{G})$. Since every loop in $\mathcal{G}^0 \setminus H_2$ has an exit in $\mathcal{G}^0 \setminus H_2$, by Corollary 4.3 (i), $I_{(H_2, \emptyset)}$ is primitive. Finally, let $X$ be the set of prime ideals and $Y$ be the set of primitive ideals in $L_K(\mathcal{G})$. Then we have

$$X = \{I_{(H_1, \{w\})}, I_{(H_1, \emptyset)}, I_{(H_2, \emptyset)}, I_{(H_1, \{w\}, g(s_f))}\}$$

and $Y = X \setminus \{I_{(H_1, \{w\})}\}$, where $g(x)$ is an irreducible polynomial in $K[x, x^{-1}]$.

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