Thirring sine-Gordon relationship by canonical methods

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Abstract

Using the canonical method developed for anomalous theories, we present the independent rederivation of the quantum relationship between the massive Thirring and the sine-Gordon models. The same method offers the possibility to obtain the Mandelstam soliton operators as a solution of Poisson brackets "equation" for the fermionic fields. We checked the anticommutation and basic Poisson brackets relations for these composite operators. The transition from the Hamiltonian to the corresponding Lagrangian variables produces the known Mandelstam’s result.

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1 Introduction

The connection between massive Thirring model of interacting fermions and sine-Gordon model with nonlinear scalar field is well known [1, 2, 3]. This Bose-Fermi equivalence has been obtained in [1] by performing the computations of the Green’s functions for both theories. After identification of some parameters, the Green’s functions became equal perturbation series, so that under these conditions this two theories are identical. An important step towards the obtaining of this result has been achieved in a pioneering paper [4]. In Ref. [2] Mandelstam has constructed the fermi fields as nonlocal functions of the sine-Gordon scalars. He showed that the corresponding operators create and annihilate the bare sine-Gordon solitons. These operators satisfy the proper commutation relations as well as the Thirring model field equations, which confirms Coleman’s result. This equivalence has been established on the quantum level and the relation between the Fermi and Bose fields is non-local. Beside the approaches mentioned above, Fermi-Bose equivalence was obtained in Refs. [5] and [6] using quantum mechanical interaction picture and Krein realization of the massless scalar field. The same problem has been considered in the papers [7].
In this paper we are going to derive above connection between massive Thirring and sine-Gordon models using canonical method [8, 9, 10]. Starting with the fermionic Thirring model we are going to construct the equivalent bosonic theory, which appears to be the sine-Gordon one. Our approach is different from the previously mentioned ones and naturally works in the Hamiltonian formalism. It gives a simpler proof of the same result.

We consider the formulation of the Thirring model with auxiliary vector fields, which on the equations of motion give the standard form of the Thirring model action. It is more convenient, regarding the fact that the method, which we use, is based on canonical formalism. The form of the action with auxiliary fields becomes linear in the fermionic current $j_\mu$. In Sec. 2.1 we are going to canonically quantize the fermionic field. So, it is useful to keep all parts which contain this field and to omit the bilinear part in auxiliary field. Such Lagrangian is invariant under local abelian gauge transformations. Consequently, the first class constraints (FCC) $j^\pm$ are present in the theory and satisfy abelian algebra as a Poisson bracket (PB) algebra. The quantum theory is anomalous, so that the central term appears in the commutator algebra of the operators $\hat{j}^\pm$, and the constraints become second class (SCC).

We define the effective bosonized theory, as a classical theory whose PB algebra of the constraints $J^\pm$ is isomorphic to the commutator algebra of the operators $\hat{j}^\pm$, in the quantized fermionic theory. Also, the bosonic Hamiltonian depends on $J^\pm$ in the same way as the fermionic Hamiltonian depends on $j^\pm$. The bosonized theory incorporates anomalies of the quantum fermionic theory at the classical level.

In Sec. 2.3 we find the effective Lagrangian for given algebra as its PB current algebra and given Hamiltonian in terms of the currents. The similar problems have been solved before in the literature [10] using canonical methods. We introduce the phase space coordinates $\varphi, \pi$ and parameterize the constraints $J^\pm$ by them. Then we find the expressions for the constraints $J^\pm$ in terms of phase space coordinates, satisfying given PB algebra, as well as for the Hamiltonian density $H_c$. We then use the general canonical method [8, 9, 10] for constructing the effective Lagrangian with the known representation of the constraints. Eliminating the momentum variable on its equation of motion we obtain thebose theory which is equivalent to the quantum fermi theory. Finally, returning omitted term, bilinear in $A_\mu$ and eliminating auxiliary vector field on its equation of motion we obtain the sine-Gordon model.

By the way, we obtain Hamiltonian bosonization formulae for the currents which depend on the momenta, while those for scalar densities depend only on the coordinates. Known Lagrangian bosonization rules can be obtained from the Hamiltonian ones, after eliminating the momenta.

The massless Thirring model is considered separately in Sec. 2.4. It is shown that in its quantum action exists one parameter which does not appear in the classical one.
Therefore, the quantum massless Thirring model is non-uniquely defined in agreement with Ref. [12].

In Sec. 3. the same method will be applied for the construction of the fermionic Mandelstam’s operators. The algebra of the currents is the basic PB algebra. Commutation relations between the currents and fermionic fields completely define the fermionic fields. So, we first find the PB between $j_\pm$ and $\psi_\pm$. The corresponding commutation relations of the operators $\hat{j}_\pm$ and $\hat{\psi}_\pm$ are not anomalous. In order to obtain the bosonized expression for the fermions we ”solved” PB equation, which is isomorphic to the previous operator relation. We find the representation for the unknown fermion field using the known representation for the currents $J_\pm$. The solution depends on the phase space coordinates $\varphi$ and $\pi$ and represents Hamiltonian form of the Mandelstam’s creation and annihilation operators.

Sec. 4. is devoted to concluding remarks. The derivation of the central term, using normal ordering prescription, is presented in the Appendix A, and the field product regularization in the Appendix B.

2 Thirring Model

In this section the canonical method of bosonization will be applied to the Thirring model.

2.1 Canonical analysis of the theory

Thirring model [11] is a theory of massive Dirac field in two–dimensional space–time defined by the following Lagrangian

$$\mathcal{L}_{Th} = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi - \frac{g}{2} j_\mu j^\mu, \quad (2.1)$$

where $g$ is coupling constant, and $j^\mu \equiv \bar{\psi} \gamma^\mu \psi$ is the fermionic current. In two–dimensional space–time $\gamma$ – matrices are defined in terms of Pauli matrices $\sigma_1$, $\sigma_2$ and $\sigma_3$ as $\gamma^0 = \sigma_1$, $\gamma^1 = -i \sigma_2$, $\gamma^5 = -i \sigma_1 \sigma_2 = \sigma_3$ and obey standard relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \eta^{\mu\nu}, \quad \gamma^\mu \gamma_5 + \gamma_5 \gamma^\mu = 0.$$  

Metric tensor $\eta^{\mu\nu}$ is defined by $\eta^{00} = -\eta^{11} = 1$; $\eta^{01} = \eta^{10} = 0$. Axial–vector product $\gamma^\mu \gamma_5$ can be expressed in terms of $\gamma^\nu$ in a following way

$$\gamma^\mu \gamma_5 = -\epsilon^{\mu\nu} \gamma_\nu,$$

where $\epsilon^{\mu\nu}$ is totally antisymmetric tensor $\epsilon^{01} = -\epsilon^{10} = 1$. Weyl or chiral spinors are defined using $\gamma_5$ matrix

$$\gamma_5 \psi_\pm = \mp \psi_\pm,$$
which can be expressed with the help of chiral projectors $P_{\pm} \equiv \frac{1 \mp \gamma_5}{2}$ as

$$P_{\pm} \psi_{\pm} = \pm \psi_{\pm}.$$  

The definition of the projectors $P_{\pm}$ implies that Dirac spinor $\psi$ expressed in terms of Weyl spinors $\psi_{\pm}$ has a form

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}.$$  

Lagrangian given by Eq. (2.1) is on–shell equivalent to the following one

$$\mathcal{L} = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi + \frac{1}{2} j^\mu A_\mu + \frac{1}{8g} A^\alpha A_\alpha.$$  

(2.2)

Namely, equations of motion for the auxiliary field $A_\mu$, which are obtained from Lagrangian (2.2), have the form

$$\frac{1}{2} j^\mu + \frac{1}{4g} A^\mu = 0,$$

which, after substitution in (2.2), gives Lagrangian (2.1).

We are going to quantize the fermionic field, so we will consider the Lagrangian

$$\mathcal{L}_0 = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi + \frac{1}{2} j^\mu A_\mu,$$

(2.3)

keeping the terms with fermionic fields. The canonical method of bosonization will be applied to this Lagrangian. In terms of Weyl spinors $\psi_{\pm}$ and light–cone components of auxiliary field $A_\mu$, it reads

$$\mathcal{L}_0 = i\psi_-^* \dot{\psi}_- + i\psi_+^* \dot{\psi}_+ + i\psi_-^* \psi'_+ - i\psi_+^* \psi'_- - m(\psi_-^* \psi_+ + \psi_+^* \psi_-)$$

$$+ \frac{1}{2} (j_+ A_- + j_- A_+),$$  

(2.4)

where the chiral currents $j_\pm$ are defined by $j_\pm \equiv \sqrt{2} \psi_\pm^* \psi_\pm$, and fields $A_\pm \equiv (1/\sqrt{2})(A_0 \pm A_1)$. Time and space coordinate, are respectively $\tau \equiv x^0$ and $\sigma \equiv x^1$, and corresponding derivatives are $\dot{\psi} \equiv \partial \psi / \partial \tau$ and $\dot{\psi}' \equiv \partial \psi / \partial \sigma$. Now we will investigate Hamiltonian structure of the theory defined by Lagrangian (2.4). This Lagrangian is already in Hamiltonian form. It is linear in time derivatives of basic Lagrangian variables $\psi_+$ and $\psi_-$, whose conjugate momenta are $\pi_\pm = i\psi_\pm^*$. Variables without time derivatives, $A_+$ and $A_-$, are Lagrange multipliers and the primary constraints corresponding to them are the FCC

$$j_\pm \equiv \sqrt{2} \psi_\pm^* \psi_\pm = -i\sqrt{2} \pi_\pm \psi_\pm.$$  

(2.5)

From Eq. (2.4) we can conclude that canonical Hamiltonian density of the Thirring model takes a form

$$\mathcal{H}_c = -i(\psi_-^* \psi'_- - \psi_+^* \psi'_+) + m(\psi_-^* \psi_+ + \psi_+^* \psi_-) = t_+ - t_- + m(\rho_+ + \rho_-),$$  

(2.6)

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where we introduced the energy–momentum tensor $t_{\pm}$ and chiral densities $\rho_{\pm}$ by the relations

$$t_{\pm} \equiv i \psi_{\pm}^{*} \psi_{\pm}' = \pi_{\pm} \psi_{\pm}' , \quad \rho_{\pm} \equiv \psi_{\pm}^{*} \psi_{\pm} = - i \pi_{\pm} \psi_{\pm}.$$  

Total Hamiltonian is defined as

$$H_T = \int d\sigma H_T ,$$  \hspace{1cm} (2.7)

where total Hamiltonian density, $H_T$, is

$$H_T = t_{\pm} - t_{\mp} + m(\rho_{\pm} + \rho_{\mp}) - \frac{1}{2}(j_{\mp} A_{\pm} + j_{\pm} A_{\mp}).$$  \hspace{1cm} (2.8)

Starting with basic PB

$$\{\psi_{\pm}(\sigma), \pi_{\pm}(\bar{\sigma})\} = \delta(\sigma - \bar{\sigma}) ,$$  \hspace{1cm} (2.9)

it is easy to show that currents $j_{\pm}$ satisfy two independent abelian PB algebras

$$\{j_{\pm}(\sigma), j_{\pm}(\bar{\sigma})\} = 0 , \quad \{j_{\mp}(\sigma), j_{\pm}(\bar{\sigma})\} = 0 .$$  \hspace{1cm} (2.10)

Using Eq. (2.9), we can find PB of the currents $j_{\pm}$ with the quantities $t_{\pm}$ and $\rho_{\pm}$

$$\{j_{\pm}(\sigma), t_{\pm}(\bar{\sigma})\} = - j_{\pm}(\sigma) \delta'(\sigma - \bar{\sigma}) , \quad \{j_{\pm}(\sigma), t_{\mp}(\bar{\sigma})\} = 0 ,$$  \hspace{1cm} (2.11)

$$\{j_{\pm}(\sigma), \rho_{\pm}(\bar{\sigma})\} = - i \sqrt{2} \rho_{\pm} \delta(\sigma - \bar{\sigma}) , \quad \{j_{\pm}(\sigma), \rho_{\mp}(\bar{\sigma})\} = i \sqrt{2} \rho_{\pm} \delta(\sigma - \bar{\sigma}).$$  \hspace{1cm} (2.12)

The last relations imply

$$\{H(c)(\sigma), j_{\pm}(\bar{\sigma})\} = \pm j_{\pm}(\sigma) \delta'(\sigma - \bar{\sigma}) \mp i \sqrt{2}(\rho_{+} - \rho_{-}),$$  \hspace{1cm} (2.13)

which help us to obtain

$$\dot{j}_{+} = \{j_{+}, H_T\} = j'_{+} + im\sqrt{2}(\rho_{+} - \rho_{-}),$$  \hspace{1cm} (2.14)

$$\dot{j}_{-} = \{j_{-}, H_T\} = -j'_{-} - im\sqrt{2}(\rho_{+} - \rho_{-}).$$  \hspace{1cm} (2.15)

Taking the sum of the equations (2.14) and (2.15), we get

$$\partial_{-} j_{+} + \partial_{+} j_{-} = \partial_{\mu} j^{\mu} = 0 ,$$  \hspace{1cm} (2.16)

which implies that the current $j^{\mu}$ is conserved.

Since the constraints $j_{\pm}$ are the first class ones, classical theory, defined by Lagrangian (2.3), has local abelian symmetry, whose generators $j_{\pm}$ satisfy abelian PB algebra, given by Eq. (2.10).
2.2 Quantization of the fermionic theory

Passing from the classical to the quantum theory can be obtained by introducing the operators $\hat{\psi}$ and $\hat{\pi}$, instead of corresponding classical fields. Poisson brackets are replaced by corresponding commutators, and operator product is defined using normal ordering prescription, whose details are explained in Appendix A,

$$\hat{j}_\pm \equiv -i\sqrt{2}: \hat{\pi}_\pm \hat{\psi}_\mp : , \quad \hat{t}_\pm \equiv : \hat{\pi}_\pm \hat{\psi}_\mp', \quad \hat{\rho} \equiv -i: \hat{\pi}_\pm \hat{\psi}_\mp : .$$

Algebra of the operators $\hat{j}_\pm$, $\hat{t}_\pm$ and $\hat{\rho}_\pm$ takes a form (Appendix A)

\begin{align}
\{ \hat{j}_\pm(\sigma), \hat{j}_\mp(\bar{\sigma}) \} &= \pm 2i\hbar \kappa \delta'(\sigma - \bar{\sigma}), \quad \{ \hat{j}_+(\sigma), \hat{j}_-(\bar{\sigma}) \} = 0, \\
\{ \hat{j}_\pm(\sigma), \hat{t}_\mp(\bar{\sigma}) \} &= -i\hbar \hat{j}_\mp(\bar{\sigma}) \delta'(\sigma - \bar{\sigma}), \quad \{ \hat{j}_+(\sigma), \hat{t}_-(\bar{\sigma}) \} = 0, \\
\{ \hat{j}_\pm(\sigma), \hat{\rho}_\mp(\bar{\sigma}) \} &= \hbar \sqrt{2}\hat{\rho}_\mp \delta(\sigma - \bar{\sigma}), \quad \{ j_\pm(\sigma), \rho_\mp(\bar{\sigma}) \} = -\hbar \sqrt{2}\rho_\mp \delta(\sigma - \bar{\sigma}),
\end{align}

where $\kappa \equiv \frac{\hbar}{2\pi}$.

The current operators with different chirality commute, as well as the corresponding variables in the classical theory. The difference between classical and quantum algebra is appearance of the central term in the commutator current algebra (2.17). As its consequence, the operators $\hat{j}_\pm$ are the second class constraints operators. This leads to the existence of the anomaly. Namely, symmetry of the classical theory, whose generators are the first class constraints $j_\pm$, is no longer symmetry at the quantum level.

2.3 Effective bosonic theory

Now we will introduce new variables $J_\pm, \Theta_\pm, R_\pm$ and postulate their PB algebra to be isomorphic to commutator algebra in the quantum fermionic theory, given by Eqs. (2.17), (2.18), and (2.19)

\begin{align}
\{ J_+(\sigma), J_-(\bar{\sigma}) \} &= \pm 2\kappa \delta'(\sigma - \bar{\sigma}), \quad \{ J_+(\sigma), J_-(\bar{\sigma}) \} = 0, \\
\{ \Theta_\pm(\sigma), J_\mp(\bar{\sigma}) \} &= J_\pm(\sigma) \delta'(\sigma - \bar{\sigma}), \quad \{ \Theta_\pm(\sigma), J_\mp(\bar{\sigma}) \} = 0, \\
\{ J_\pm(\sigma), R_\mp(\bar{\sigma}) \} &= -i\sqrt{2}R_\mp(\sigma) \delta(\sigma - \bar{\sigma}), \quad \{ J_\pm(\sigma), R_\mp(\bar{\sigma}) \} = i\sqrt{2}R_\mp(\sigma) \delta(\sigma - \bar{\sigma}).
\end{align}

Let us find the expressions for the currents $J_\pm$, energy–momentum tensor $\Theta_\pm$, and chiral densities $R_\pm$ in terms of scalar field $\varphi$ and its conjugate momenta $\pi$, with the PB

$$\{ \varphi(\sigma), \pi(\bar{\sigma}) \} = \delta(\sigma - \bar{\sigma}).$$

Assuming that the currents $J_\pm$ are linear in the momentum $\pi$, it is easy to show that the expression

$$J_\pm = \pm \pi + \kappa \varphi'$$

(2.23)
is a solution of \((\ref{eq:2.20})\). Supposing that the energy–momentum tensor \(\Theta_{\pm}\) is quadratic in the currents \(J_{\pm}\), we can immediately obtain its bosonic representation from algebra \((\ref{eq:2.21})\)

\[
\Theta_{\pm} = \pm \frac{1}{4\kappa} J_{\pm} J_{\pm}.
\]  \(\text{(2.24)}\)

Bosonic representation for scalar densities, \(R_{\pm}\), can be obtained from algebra \((\ref{eq:2.22})\). Assuming that scalar densities are momentum independent, we have

\[
R_{\pm} = M \exp (\pm i\sqrt{2}\varphi),
\]  \(\text{(2.25)}\)

where \(M\) is constant. The scalar densities as well as parameter \(M\) have dimension of mass.

Total Hamiltonian density of the effective bosonic theory is defined by the analogy with total Hamiltonian density of the fermionic theory, given by Eq. \((\ref{eq:2.8})\)

\[
\mathcal{H}_T = \Theta_{+} - \Theta_{-} + m(R_{+} + R_{-}) - \frac{1}{2} (J_{+} A_{-} + J_{-} A_{+}),
\]  \(\text{(2.26)}\)

and the Lagrangian of the effective bosonic theory has the form

\[
\mathcal{L}_{Th}^{T} = \pi \dot{\varphi} - \mathcal{H}_T.
\]  \(\text{(2.27)}\)

Substituting Eq. \((\ref{eq:2.26})\) in Eq. \((\ref{eq:2.27})\), and using Eqs. \((\ref{eq:2.23})\), \((\ref{eq:2.24})\) and \((\ref{eq:2.25})\), we obtain

\[
\mathcal{L}_{Th}^{T} = \pi \dot{\varphi} - \frac{1}{2\kappa} \pi^2 - \frac{1}{2} \kappa \varphi'^2 - 2mM \cos \sqrt{2}\varphi + \frac{1}{\sqrt{2}} (-\pi A_1 + \kappa \varphi' A_0) \).
\]  \(\text{(2.28)}\)

On the equation of motion for the momentum \(\pi\)

\[
\pi = \kappa (\dot{\varphi} - \frac{A_1}{\sqrt{2}}).
\]  \(\text{(2.29)}\)

this Lagrangian takes a form

\[
\mathcal{L}_{Th}^{T} = \frac{1}{2} \kappa \partial_\mu \varphi \partial^\mu \varphi + \frac{\kappa}{\sqrt{2}} e^{\mu\nu} A_\mu \partial_\nu \varphi - 2mM \cos (\sqrt{2} \varphi) + \frac{1}{4} \kappa A_1^2.
\]  \(\text{(2.30)}\)

It is possible to add to the effective Lagrangian some local functional, depending on the fields \(A_{\pm}\). In order to obtain Lorentz invariant action of the effective bosonic theory, we will choose additional term in the form

\[
\Delta \mathcal{L}_{Th}^{T} = -\frac{1}{4} \kappa A_1^2.
\]

Adding counterterm \(\Delta \mathcal{L}_{Th}^{T}\) to the Lagrangian \((\ref{eq:2.30})\), and returning the term bilinear in \(A_\mu\) we get

\[
\mathcal{L}_{eff}^{T} = \frac{1}{2} \kappa \partial_\mu \varphi \partial^\mu \varphi + \frac{\kappa}{\sqrt{2}} e^{\mu\nu} A_\mu \partial_\nu \varphi - 2mM \cos (\sqrt{2} \varphi) + \frac{1}{8g} A^\mu A_\mu.
\]  \(\text{(2.31)}\)
Now we can eliminate auxiliary field $A_\mu$ from Lagrangian (2.31), using its equation of motion

$$A^\mu = -\frac{4g\kappa}{\sqrt{2}} \epsilon^{\mu\nu} \partial_\nu \varphi.$$  \hspace{1cm} (2.32)

Substituting this equation back into Lagrangian (2.31), we get

$$L_{\text{Th eff}}^T = \frac{1}{2} \kappa (1 + 2g\kappa) \partial_\mu \varphi \partial^\mu \varphi - 2mM \cos (\sqrt{2} \varphi).$$  \hspace{1cm} (2.33)

This Lagrangian, after rescaling the scalar field $\varphi$, $\varphi \rightarrow \frac{\sqrt{2}}{\beta} \varphi$, takes a form

$$L_{\text{Th eff}}^T = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - 2mM \cos (\beta \varphi),$$  \hspace{1cm} (2.34)

where $\beta$ is defined by

$$\beta \equiv \left[ \frac{1}{2} \kappa (1 + 2g\kappa) \right]^{-1/2}.$$  \hspace{1cm} (2.35)

As it is usual, we will add constant term to the Lagrangian (2.34), in order to have vanishing energy for the vacuum configuration $\varphi = 0$ and obtain

$$L_{\text{Th eff}}^T = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - 2mM \left[ \cos (\beta \varphi) - 1 \right].$$  \hspace{1cm} (2.36)

This is the Lagrangian of the sine–Gordon theory. Therefore, Thirring model is equivalent to the sine–Gordon theory, if there exists following relation between parameters

$$\frac{4\pi}{\beta^2 \hbar} = 1 + \frac{g\hbar}{\pi} = 1 + 2g\kappa,$$  \hspace{1cm} (2.37)

which is consequence of (2.35). Result given by Eq. (2.37), obtained by canonical method, is in agreement with one in Ref. [1]. Coleman has obtained this result by direct computation of Green’s functions in both Thirring and sine–Gordon model using perturbative technique. The relation (2.37) is the main result in this chapter. From its form, we can easy conclude that free Thirring model ($g = 0$) is equivalent to sine–Gordon theory with

$$\beta^2 = \frac{4\pi}{\hbar}.$$  \hspace{1cm} (2.38)

It is worth to emphasize that relation (2.37) implies duality between Thirring and sine–Gordon model. Namely, from this relation it follows that the large values of Thirring coupling constant $g$ corresponds to the small value of sine–Gordon parameter $\beta$.

### 2.4 One parameter class solutions of the massless Thirring model

We shall consider separately the massless Thirring model. This case is specially interesting, because corresponding quantum theory is non-uniquely defined. Namely, in the quantum action of the theory exists one parameter, which does not appear in the classical one [12].
Since the Thirring sine-Gordon relationship was already established, we will show the existence of this parameter starting with the corresponding sine-Gordon model.

Firstly, we split vector field from Lagrangian (2.30) to the quantum and external part

\[ A_\mu = a_\mu + A^{ex}_\mu. \]

Here the field \( a_\mu \) plays the role of our auxiliary field and \( A^{ex}_\mu \) is the Hagen’s external source. Then, omitting the mass term and the local functional dependence on the vector fields, we obtain from (2.30)

\[
\mathcal{L}^{Th}(\varphi, a + A^{ex}) = \frac{1}{2} \kappa \partial_\mu \varphi \partial^\mu \varphi + \frac{\kappa}{\sqrt{2}} \epsilon^{\mu\nu}(a_\mu + A^{ex}_\mu) \partial_\nu \varphi.
\] (2.39)

The invariance of the Thirring model under replacement

\[
j^\mu \rightarrow j_5^\mu, \quad A^{ex}_\mu \rightarrow A^{ex}_{5\mu}, \quad g \rightarrow -g,
\] (2.40)

corresponds to the Lagrangian \( \mathcal{L}^{Th}(\varphi_5, a_5 + A^{ex}_{5\mu}) \). Note that we introduce new auxiliary fields \( \varphi_5 \) and \( a_5 \), while the external fields are related by the dual transformation \( A^{ex}_{5\mu} = \epsilon_{\mu\nu} A^{ex\nu} \).

The symmetry of the massless Thirring model given by (2.40) allows us to introduce one-parameter Lagrangian

\[
\mathcal{L}(\xi, \eta) = \xi \mathcal{L}^{Th}(\varphi, a + A^{ex}) + \eta \mathcal{L}^{Th}(\varphi_5, a_5 + A^{ex}_{5\mu}) + \frac{1}{8g}(a_\mu a^\mu - a_{5\mu} a^\mu_5),
\] (2.41)

with \( \xi + \eta = 1 \). The parameters \( \xi \) and \( \eta \) obey this constraint because the Lagrangian given by Eq. (2.41) has to correspond to the Lagrangian of the massless Thirring model. The terms quadratic in auxiliary fields, in fact replaced the last term of the eq. (2.31). The second part was adding with opposite sign in according to the symmetry replacement \( g \rightarrow -g \).

After elimination of all auxiliary fields \( a, a_5 \) and then \( \varphi, \varphi_5 \) we obtain the effective action

\[
W(A^{ex}) = -\frac{\hbar}{8} \int d^2 x A^{ex}_\mu D^{\mu\nu}_g A^{ex}_\nu,
\] (2.42)

where

\[
D^{\mu\nu}_g = \left( \epsilon^{\mu\rho} \epsilon^{\nu\sigma} \frac{\xi}{1 + \frac{g_{\epsilon h}}{\pi}} + \eta^{\rho\sigma} \eta^{\mu\nu} \frac{\eta}{1 - \frac{g_{\epsilon h}}{\pi}} \right) \partial_\rho \partial_\sigma \frac{1}{\sqrt{g}}.
\] (2.43)

Up to the normalization factor it is just relation (3.13) from the Hagen’s paper Ref. [12].

The expression for effective action is equivalent to the solution of the functional integral, [10],

\[
< 0 \mid 0 >_{A,g} = \int d\bar{\psi}d\psi e^{i\mathcal{L}^{Th}} = e^{iW(A^{ex})},
\] (2.44)

which corresponds to the Eq. (3.12) in the first Ref. [12]. Using these expressions it is easy to reproduce the other Hagen’s results.
3 Bosonization of Fermionic Fields

In this section we will apply canonical method of bosonization to the fermionic fields. Starting with PB of fermionic fields $\psi_\pm$ and corresponding momenta $\pi_\pm$ with currents $j_\pm$, fermionic fields will be expressed in terms of bosonic phase space coordinates $\varphi$ and $\pi$. After quantization, these classical fermionic fields become the operators. In order to show that these operators are really fermionic ones, we investigate their anticommutation relations. From the bosonic form of the operators $\Psi_\pm$, we easily obtain bosonic representation of scalar and pseudoscalar density $\hat{\Psi} \hat{\Psi}$ and $\Psi \gamma_5 \bar{\Psi}$, respectively. These results are consistent with ones obtained in Ref. [2].

3.1 Construction of the fermionic field operators

Poisson brackets of the fermionic fields $\psi_\pm$ and its conjugate momenta $\pi_\pm$ with the currents $j_\pm$ have the form

$$\{j_\pm(\sigma), \psi_\pm(\bar{\sigma})\} = i\sqrt{2}\psi_\pm(\sigma - \bar{\sigma}), \quad \{j_\pm(\sigma), \psi_\pm(\bar{\sigma})\} = 0,$$

$$\{j_\pm(\sigma), \pi_\pm(\bar{\sigma})\} = -i\sqrt{2}\pi_\pm(\sigma - \bar{\sigma}), \quad \{j_\pm(\sigma), \pi_\pm(\bar{\sigma})\} = 0.\tag{3.1}$$

Because the right hand side is linear in the fields, the anomaly is absent and after quantization, algebra of the operators $\hat{\psi}_\pm$, $\hat{\pi}_\pm$ and $\hat{j}_\pm$ preserves the original form

$$[\hat{j}_\pm(\sigma), \hat{\psi}_\pm(\bar{\sigma})] = -\hbar \sqrt{2}\hat{\psi}_\pm(\sigma - \bar{\sigma}), \quad [\hat{j}_\pm(\sigma), \hat{\psi}_\pm(\bar{\sigma})] = 0,$$

$$[\hat{j}_\pm(\sigma), \hat{\pi}_\pm(\bar{\sigma})] = \hbar \sqrt{2}\hat{\pi}_\pm(\sigma - \bar{\sigma}), \quad [\hat{j}_\pm(\sigma), \hat{\pi}_\pm(\bar{\sigma})] = 0.\tag{3.2}$$

Now, we will construct bosonic representation of the fermionic fields $\Psi_\pm$ and their conjugate momenta $\Pi_\pm$. We demand that Poisson brackets algebra of the fields $\Psi_\pm$, their conjugate momenta $\Pi_\pm$, and currents $J_\pm$, whose bosonic form is already known, is isomorphic to the algebra of the operators $\hat{\psi}_\pm$, $\hat{\pi}_\pm$ and $\hat{j}_\pm$, respectively. Therefore, we have

$$\{J_\pm(\sigma), \Psi_\pm(\bar{\sigma})\} = i\sqrt{2}\Psi_\pm(\sigma - \bar{\sigma}), \quad \{J_\pm(\sigma), \Psi_\pm(\bar{\sigma})\} = 0,$$

$$\{J_\pm(\sigma), \Pi_\pm(\bar{\sigma})\} = -i\sqrt{2}\Pi_\pm(\sigma - \bar{\sigma}), \quad \{J_\pm(\sigma), \Pi_\pm(\bar{\sigma})\} = 0.\tag{3.3}$$

In order to solve these equations in terms of $\Psi_\pm$ and $\Pi_\pm$, let us introduce the variables $I_\pm$ as follows

$$I_\pm(\sigma) \equiv \int_{-\infty}^{\sigma} d\sigma_1 J_\pm(\sigma_1).\tag{3.4}$$

With the help of bosonic representation of currents [2,3], we obtain the $I_\pm$ dependence of basic bosonic variables

$$I_\pm(\sigma) = \pm \int_{-\infty}^{\sigma} d\sigma_1 \pi(\sigma_1) + \kappa \varphi(\sigma).\tag{3.5}$$
Using Poisson brackets current algebra, given by Eq. (2.20), we find PB of the variables \( I_\pm \) with the currents \( J_\pm \)

\[
\{ J_\pm(\sigma), I_\pm(\bar{\sigma}) \} = \mp 2\kappa \delta(\sigma - \bar{\sigma}),
\]

(3.9)

\[
\{ J_\pm, I_\pm \} = 0.
\]

(3.10)

Let us suppose that fields \( \Psi_\pm \) and their conjugate momenta depend only on variables \( I_\pm \).

Under that assumption, from the algebra given by Eqs. (3.5) and (3.6), we get the following equations

\[
\{ J_\pm(\sigma), I_\pm(\bar{\sigma}) \} \frac{\partial \Psi_\pm}{\partial I_\pm} = i\sqrt{2} \Psi_\pm(\sigma) \delta(\sigma - \bar{\sigma}),
\]

(3.11)

\[
\{ J_\pm(\sigma), I_\pm(\bar{\sigma}) \} \frac{\partial \Pi_\pm}{\partial I_\pm} = -i\sqrt{2} \Pi_\pm(\sigma) \delta(\sigma - \bar{\sigma}).
\]

(3.12)

With the help of (3.9), we obtain bosonic representation of fermionic field \( \Psi_\pm \), and their conjugate momenta \( \Pi_\pm \)

\[
\Psi_\pm = C_\pm \exp (\mp \frac{i}{\kappa \sqrt{2}} I_\pm),
\]

(3.13)

\[
\Pi_\pm = D_\pm \exp (\pm \frac{i}{\kappa \sqrt{2}} I_\pm),
\]

(3.14)

where \( C_\pm \) and \( D_\pm \) are the constants which will be determined using regularization procedure. These constants are not independent. The relation \( D_\pm = i C_\pm^* \) follows from classical fermionic theory constraints \( \pi_\pm = i \psi_\pm^* \) and its bosonic analogue \( \Pi_\pm = i \Psi_\pm^* \).

After quantization the classical fields \( \Psi_\pm \) and their conjugate momenta \( \Pi_\pm \) become operators \( \hat{\Psi}_\pm \) and \( \hat{\Pi}_\pm \). So, normal ordering prescription has to be applied to the operators product in the right hand side in Eqs. (3.13) and (3.14)

\[
\hat{\Psi}_\pm = C_\pm : \exp (\mp \frac{i}{\kappa \sqrt{2}} \hat{I}_\pm) :,
\]

(3.15)

\[
\hat{\Pi}_\pm = i C_\pm^* : \exp (\pm \frac{i}{\kappa \sqrt{2}} \hat{I}_\pm) :.
\]

(3.16)

This means that after expansion of the exponent in the right hand side, annihilation operators are placed right to the creation ones (Appendix A).

As direct computation shows (Appendix B), products of the operators \( \hat{\Psi}_\pm \hat{\Psi}_\pm \) at the same point of space are singular. In order to regularize these products, let us introduce the operators \( \hat{J}_\pm \), which are the product of field operators at the different points of space

\[
\hat{J}_\pm(\sigma_1, \sigma_2) \equiv \sqrt{2} \hat{\Psi}_\pm(\sigma_1) \hat{\Psi}_\pm(\sigma_2).
\]

(3.17)

After some calculations (Appendix B), we obtain

\[
\hat{J}_\pm(\sigma, \sigma + \eta)|_{\eta \to 0} \equiv \sqrt{2} \hat{\Psi}_\pm(\sigma) \hat{\Psi}_\pm(\sigma + \eta)|_{\eta \to 0} = \frac{F_\pm}{\eta \pm i\varepsilon} + Z_\pm \hat{J}_\pm(\sigma), \ (\varepsilon > 0),
\]

(3.18)
where $F_\pm$ and $Z_\pm$ are given by following expressions ($\Lambda$–cutoff parameter)

\[ F_\pm = \pm i \Lambda \sqrt{2} |C_\pm|^2, \quad Z_\pm = \frac{\Lambda}{\kappa} |C_\pm|^2. \quad (3.19) \]

Because $\Psi_\pm$ is a representation of the $\hat{\psi}_\pm$, we expect that bilinear combination in (3.18) produce $\hat{J}_\pm$, since it is the same combination as in (2.5). So, the natural choice for the constants is

\[ Z_\pm = 1. \]

From the last relation, we get the values of the constants $F_\pm$ and $C_\pm$

\[ C_\pm = \sqrt{\frac{\kappa}{\Lambda}}; \quad F_\pm = \pm i \kappa \sqrt{2}. \quad (3.20) \]

With that values of the constants $F_\pm$ and $C_\pm$, the operators $\hat{\Psi}_\pm$, $\hat{\Pi}_\pm$ and $\hat{J}_\pm$ take a form

\[ \hat{\Psi}_\pm = \sqrt{\frac{\kappa}{\Lambda}} : \exp \left( \mp \frac{i}{\kappa \sqrt{2}} \hat{I}_\pm \right) : , \quad (3.21) \]

\[ \hat{\Pi}_\pm = i \sqrt{\frac{\kappa}{\Lambda}} : \exp \left( \pm \frac{i}{\kappa \sqrt{2}} \hat{I}_\pm \right) : , \quad (3.22) \]

\[ \hat{J}_\pm(\sigma, \sigma + \eta)|_{\eta \to 0} = \pm \frac{i \kappa \sqrt{2}}{\eta \pm i \varepsilon} + \hat{J}_\pm(\sigma). \quad (3.23) \]

In order to compare these results with ones from the Ref. [2], we will derive Lagrangian form of the operators $\hat{\Psi}_\pm$. Note that up to this point in Sec. 3. we did not specify Hamiltonian of the theory. So, our canonical expression is valid for any theory satisfying PB algebra (3.1, 3.2). Passing to the Lagrangian formulation we must be more specific and we chose the example of the Thirring model, which we considered in the previous section. We will express the momentum $\pi$ in terms of corresponding velocity starting from equation of motion for the momentum, Eq. (2.29), and using equation of motion for auxiliary field $A_\mu$

\[ \pi = \frac{2}{\beta^2} \dot{\varphi}. \quad (3.24) \]

Substituting the last equation in Eq. (3.13), after rescaling $\varphi \rightarrow \frac{\sqrt{2}}{\beta} \varphi$, we obtain Lagrangian form of the fields $\Psi_\pm$

\[ \Psi_\pm(\sigma) = C_\pm \exp \left\{ - \frac{i}{\kappa \beta} \int_{-\infty}^{\sigma} d\bar{\sigma} \dot{\varphi}(\bar{\sigma}) \mp \frac{i \beta}{2} \varphi(\sigma) \right\}. \quad (3.25) \]

After quantization, from the last relation, we get

\[ \hat{\Psi}_\pm(\sigma) = C_\pm : \exp \left\{ - \frac{i}{\kappa \beta} \int_{-\infty}^{\sigma} d\bar{\sigma} \dot{\varphi}(\bar{\sigma}) \mp \frac{i \beta}{2} \varphi(\sigma) \right\} : . \quad (3.26) \]
This form of the field operators is in agreement with one obtained in Ref. [2] from the requirement that operators $\hat{\Psi}_\pm$ have to be annihilation operators for solitons in sine-Gordon theory, as well as to anticommute with itself. Result given by Eq. (3.21) is more general, because it is in Hamiltonian form, so it can be applied to the other two-dimensional models. Additionally, this result is obtained from the less number of assumptions. Namely, we obtained this result demanding that fields $\Psi_\pm$ and currents $J_\pm$ have to obey Poisson brackets algebra which is isomorphic to algebra of the operators $\hat{j}_\pm$ and $\hat{\psi}_\pm$.

### 3.2 Anticommutation relations for the operators $\hat{\Psi}_\pm$ and $\hat{\Pi}_\pm$

In this subsection we will show that operators $\hat{\Psi}_\pm$ and $\hat{\Pi}_\pm$, given by (3.21) i (3.22), obey canonical anticommutation relations.

In order to justify interpretation of the operators $\hat{\Psi}_\pm$ and $\hat{\Pi}_\pm$ as the fermionic operators, we should show that they obey canonical anticommutation relations. Firstly, we will find anticommutation relations for the operators $\hat{\Psi}_\pm$. Using Eqs. (3.21) and (B.3), we find

$$
\hat{\Psi}_\pm(\sigma)\hat{\Psi}_\pm(\bar{\sigma}) = \frac{\kappa}{\Lambda} \exp \left\{ -\frac{1}{2\kappa^2} [\hat{I}_\pm(\sigma), \hat{I}_\pm(\bar{\sigma})] \right\} : \exp \left\{ \mp \frac{1}{2\kappa^2} [\hat{I}_\pm(\sigma) - \hat{I}_\pm(\bar{\sigma})] \right\} :. 
$$

(3.27)

With the help of Eq. (B.15), in the limit $\varepsilon \to 0$, we obtain that anticommutator for the fields $\Psi_\pm$ vanish

$$
[\hat{\Psi}_\pm(\sigma), \hat{\Psi}_\pm(\bar{\sigma})]_+ = 0.
$$

(3.28)

The calculation, which is very similar to the previous one, shows that the anticommutator for the momenta also vanish

$$
[\hat{\Pi}_\pm(\sigma), \hat{\Pi}_\pm(\bar{\sigma})]_+ = 0.
$$

(3.29)

Now we will find anticommutator for the fields $\Psi_\pm$ with their conjugate momenta $\Pi_\pm$. Using Eqs. (3.21), (3.22), and (B.3), we obtain

$$
\hat{\Psi}_\pm(\sigma)\hat{\Pi}_\pm(\bar{\sigma}) = \frac{\kappa}{\Lambda} \exp \left\{ \frac{1}{2\kappa^2} [\hat{I}_\pm(\sigma), \hat{I}_\pm(\bar{\sigma})] \right\} : \exp \left\{ \mp \frac{i}{\kappa\sqrt{2}} [\hat{I}_\pm(\sigma) - \hat{I}_\pm(\bar{\sigma})] \right\} :,
$$

and with the help of Eq. (B.14), we get

$$
\hat{\Psi}_\pm(\sigma)\hat{\Pi}_\pm(\bar{\sigma}) = i\hbar\delta(\pm)(\sigma - \bar{\sigma}) : \exp \left\{ \mp \frac{i}{\kappa\sqrt{2}} [\hat{I}_\pm(\sigma) - \hat{I}_\pm(\bar{\sigma})] \right\} :,
$$

which produces canonical anticommutation relations for the operators $\hat{\Psi}_\pm$ and $\hat{\Pi}_\pm$

$$
[\hat{\Psi}_\pm(\sigma), \hat{\Pi}_\pm(\bar{\sigma})]_+ = i\hbar\delta(\sigma - \bar{\sigma}) : \exp \left\{ \mp \frac{i}{\kappa\sqrt{2}} [\hat{I}_\pm(\sigma) - \hat{I}_\pm(\bar{\sigma})] \right\} : = i\hbar\delta(\sigma - \bar{\sigma}). 
$$

(3.30)
3.3 Bosonization of the scalar densities

Products of the fermionic field operators $\hat{\Psi} \hat{\Psi}$ and $\hat{\Psi} \gamma^5 \hat{\Psi}$ we will express in terms of bosonic variables using Eq. (3.21). Note that the relation $[\hat{I}_+, \hat{I}_-] = 0$ simplifies the calculations. With the help of Eq. (3.3), we obtain

$$\hat{\Psi}^\pm \hat{\Psi} = \frac{\kappa}{\Lambda} : \exp \left\{ \pm \frac{i}{\kappa \sqrt{2}} (\hat{I}_+ + \hat{I}_-) \right\} : .$$

(3.31)

Substituting Eq. (3.8) in the last equation, we find

$$\hat{\Psi}^\pm \hat{\Psi} = \frac{\kappa}{\Lambda} : \exp (\pm i \sqrt{2} \varphi) : .$$

(3.32)

So, the bosonic representations for scalar density $\hat{\Psi} \hat{\Psi}$ and pseudoscalar density $\hat{\Psi} \gamma^5 \hat{\Psi}$ are

$$\hat{\Psi} \hat{\Psi} = \hat{\Psi}^*_+ \hat{\Psi}_+ + \hat{\Psi}^*_- \hat{\Psi}_- = \frac{\hbar}{\pi \Lambda} : \cos (\sqrt{2} \varphi) : ,$$

(3.33)

$$\hat{\Psi} \gamma^5 \hat{\Psi} = -\hat{\Psi}^*_+ \hat{\Psi}_+ + \hat{\Psi}^*_- \hat{\Psi}_- = \frac{i \hbar}{\pi \Lambda} : \sin (\sqrt{2} \varphi) : .$$

(3.34)

These results are consistent with ones obtained by direct applying of the method to the scalar densities.

4 Conclusion

In this paper we presented a complete and independent derivation of the Thirring sine-Gordon relationship, using the Hamiltonian methods. We also obtained Hamiltonian and Lagrangian representation for the Mandelstam’s fermionic operators .

We started with canonical analysis of the theory where fermions are coupled to the auxiliary external gauge field. The massive Thirring model can be easily obtained from this Lagrangian by adding square of the auxiliary field and eliminating it on its equation of motion. We found that there exists FCC $j_{\pm}$ in our theory, whose PB are equal to zero. In the quantum theory, the central term appears in the commutation relations of the operators $\hat{j}_{\pm}$. This changes the nature of constraints because they become SCC.

We define the new effective theory, postulating the PB of the constraints and Hamiltonian density, following the method developed in [10]. We require that the classical PB algebra of the bosonic theory is isomorphic to the quantum commutator algebra of the fermionic theory. Then we found the representation for the currents and Hamiltonian density in terms of phase space coordinates. Finally, we derived effective action using general canonical formalism and obtained the equivalent bosonized model. Together with auxiliary field term this is just the sine-Gordon action, up to some parameters identifications which agrees with Refs. [1, 2]. For the massless Thirring model it is shown that its
quantum effective action has one-parameter which does not exist in the classical one. We determined the quantum action using formal invariance of the massless Thirring model under replacements given by Eq. (2.40) and already established Thirring sine-Gordon relationship.

The algebra of the currents \( J_{\pm} \) is the basic PB algebra. Knowing the representation of the currents \( J_{\pm} \) in terms of \( \varphi \) and \( \pi \) we can find the representation for all other quantities from their PB algebra with the currents. In Sec. 2. we found the bosonization rules for the chiral densities. The main result of Sec. 3. is the bosonic representation for the fermions, which has been obtained in the same way. Beside usual bosonization rules and usual Mandelstam’s fermionic representations, we also got the Hamiltonian ones, expressing the currents \( J_{\pm} \) in terms of both coordinate \( \varphi \) and momentum \( \pi \). These rules are more general, because they are valid for arbitrary Hamiltonian and they are consequence of the commutation relations. After momenta elimination on their equations of motion, we came back to the conventional bosonization rules and to the conventional Mandelstam’s fermionic representations. The Schwinger term, and consequently the sine-Gordon action have the correct dependence on Planck’s constant \( \hbar \), because \( \kappa \) is proportional to \( \hbar \). The fact that \( \hbar \) arises in the classical effective theory and in the coupling constant relation, shows the quantum origin of the established equivalence.

A Normal Ordering and Central Term

In this appendix we will derive commutation relations of the current operators \( \hat{j}_{\pm} \equiv -i\sqrt{2} : \hat{n}_{\pm} \hat{\psi}_{\pm} : \)

\[
[\hat{j}_{\pm}(\sigma), \hat{j}_{\pm}(\bar{\sigma})] = \pm 2i\hbar \kappa \delta'(\sigma - \bar{\sigma}), \quad [\hat{j}_{\pm}(\sigma), \hat{j}_{\mp}(\bar{\sigma})] = 0, \quad (A.1)
\]

where \( \kappa \equiv \frac{\hbar}{2\pi} \).

The current operators \( \hat{j}_{\pm} \) we define using normal ordering prescription. In order to decompose these operators in positive and negative frequencies in position space, let us introduce two parts of the delta function

\[
\delta^{(+)}(\sigma) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \theta(\mp 2\pi k) e^{i(\sigma \pm \epsilon)}, \quad (\epsilon > 0), \quad (A.2)
\]

where \( \theta \) is unit step function. They obviously obey the relation \( \delta(\sigma) = \delta^{(+)}(\sigma) + \delta^{(-)}(-\sigma) \) and have the following properties

\[
\delta^{(+)}(\sigma) = \delta^{(-)}(-\sigma), \quad (A.3)
\]

\[
[\delta^{(+)}(\sigma)]^2 - [\delta^{(-)}(\sigma)]^2 = \frac{i}{2\pi} \delta'(\sigma). \quad (A.4)
\]
Any operator $\hat{O}$ can also be decomposed in two parts
\begin{equation}
\hat{O}^{(\pm)}(\tau, \sigma) = \int_{-\infty}^{\infty} d\bar{\sigma} \delta^{(\pm)}(\sigma - \bar{\sigma}) \hat{O}(\tau, \sigma),
\end{equation}
so that $\hat{O} = \hat{O}^{(+)} + \hat{O}^{(-)}$. We promote the operators $\hat{\pi}^{(-)}$ and $\hat{\psi}^{(-)}$ as annihilation ones, and the operators $\hat{\pi}^{(+)}$ and $\hat{\psi}^{(+)}$ as creation ones
\begin{equation}
\hat{\pi}^{(-)}(0) = \hat{\psi}^{(-)}(0) = 0, \quad 0|\hat{\psi}^{(+)} = \langle 0| \hat{\pi}^{(+)} = 0.
\end{equation}

In order to preserve the symmetry under parity transformations, we define creation and annihilation operators for $\hat{\pi} - i \hat{\psi}$ in an opposite way [operators with index ($-$) are creation and ones with index ($+$) are annihilation]. Normal order for product of the operators means that creation operators are placed to the left from the annihilation ones.

From the basic commutation relations
\begin{equation}
[\hat{\psi}^{(\pm)}(\sigma), \hat{\pi}^{(\pm)}(\bar{\sigma})] = i\hbar \delta^{(\pm)}(\sigma - \bar{\sigma}),
\end{equation}
we have
\begin{equation}
[\hat{\psi}^{(\pm)}(\sigma), \hat{\pi}^{(\mp)}(\bar{\sigma})] = i\hbar \delta^{(\pm)}(\sigma - \bar{\sigma}),
\end{equation}
\begin{equation}
[\hat{\psi}^{(\mp)}(\sigma), \hat{\pi}^{(\pm)}(\bar{\sigma})] = i\hbar \delta^{(\pm)}(\sigma - \bar{\sigma}),
\end{equation}
and other commutation relations are trivial. Since the Poisson brackets for the currents $j^{\pm}$ vanish, only possible difference between classical and quantum algebra is appearance of central term at the quantum level. Because of that, we find the form of the current algebra taking vacuum expectation value of the commutators
\begin{equation}
[\hat{j}^{\pm}(\sigma), \hat{j}^{\pm}(\bar{\sigma})] = \Delta^{\pm}(\sigma, \bar{\sigma}) - \Delta^{\pm}(\bar{\sigma}, \sigma),
\end{equation}
where $\Delta^{\pm}(\sigma, \bar{\sigma}) \equiv \langle 0| \hat{j}^{\pm}(\sigma) \hat{j}^{\pm}(\bar{\sigma}) | 0 \rangle$.

Using the fact that the operators $\hat{j}^{\pm}$ are normal ordered, the only nontrivial contributions have the form
\begin{equation}
\Delta^{\pm}(\sigma, \bar{\sigma}) = -2\langle 0| \hat{\pi}^{(\mp)}(\sigma) \hat{\psi}^{(\mp)}(\bar{\sigma}) \hat{\psi}^{(\mp)}(\sigma) \hat{\pi}^{(\mp)}(\bar{\sigma}) | 0 \rangle = -2\hbar^2 [\delta^{(\mp)}(\sigma - \bar{\sigma})]^2.
\end{equation}

With the help of the $\delta^{(\pm)}$ functions properties (A.4) we obtain Eq. \( \text{A.1} \). Commutator for the currents with different lower indices do not have central term, as well as the commutators of the currents with the operators $\hat{t}^{\pm}$ and $\hat{\rho}^{\pm}$.

## B Regularization of the Field Products

In this Appendix, we will show, using regularization procedure, that following relations hold
\begin{equation}
\sqrt{2} \hat{\psi}^{(\pm)}(\sigma) \hat{\psi}^{(\pm)}(\sigma + \eta)|_{\eta \to 0} \equiv \hat{J}^{\pm}(\sigma, \sigma + \eta)|_{\eta \to 0} = \frac{F^{\pm}}{\eta \pm i\varepsilon} + Z^{\pm} \hat{J}^{\pm}, \quad (\varepsilon > 0)
\end{equation}
where $F_{\pm} i Z_{\pm}$ are given by ($\Lambda$–cutoff parametar)

$$F_{\pm} = \pm i \Lambda \sqrt{2} |C_{\pm}|^2, \quad Z_{\pm} = \frac{\Lambda}{\kappa} |C_{\pm}|^2. \quad (B.2)$$

Starting with the definition of the operator $\hat{J}_{\pm}$ and using formula (Eq. (3.5) in Ref. [2])

$$e^{\hat{A}} = \frac{e^{[\hat{A}(+)\hat{B}(-)]}}{e^{\hat{A}+\hat{B}}}; \quad (B.3)$$

$([A^{(+)}, B^{(-)}] \sim - \text{c} \sim \text{number})$, we get

$$\hat{J}_{\pm}(\sigma, \bar{\sigma}) = \sqrt{2} |C_{\pm}|^2 \exp \left\{ \frac{1}{2\kappa^2} X_{\pm}(\sigma, \bar{\sigma}) \right\} \exp \left\{ \pm \frac{i}{\kappa \sqrt{2}} [\hat{I}_{\pm}(\sigma) - \hat{I}_{\pm}(\bar{\sigma})] \right\}; \quad (B.4)$$

where

$$X_{\pm}(\sigma, \bar{\sigma}) \equiv [\hat{I}_{\pm}^{(\pm)}(\sigma), \hat{I}_{\pm}^{(\mp)}(\bar{\sigma})]. \quad (B.5)$$

We compute the commutators $X_{\pm}$ using the algebra of the operators $\hat{I}_{\pm}$

$$[\hat{I}_{\pm}(\sigma), \hat{I}_{\pm}(\bar{\sigma})] = \int_{-\infty}^{\sigma} d\sigma_1 \int_{-\infty}^{\sigma} d\sigma_2 [\hat{J}_{\pm}(\sigma_1), J_{\pm}(\sigma_2)]. \quad (B.6)$$

The last relation can be rewritten in a form

$$[\hat{I}_{\pm}(\sigma), \hat{I}_{\pm}(\bar{\sigma})] = \pm i \hbar \kappa \int_{-\infty}^{\sigma} d\sigma_1 \int_{-\infty}^{\sigma} d\sigma_2 [\partial_{\sigma_1} \delta(\sigma_1 - \sigma_2) - \partial_{\sigma_2} \delta(\sigma_1 - \sigma_2)]; \quad (B.7)$$

which is obtained from algebra of the currents $\hat{J}_{\pm}$. Performing integration we get

$$[\hat{I}_{\pm}(\sigma), \hat{I}_{\pm}(\bar{\sigma})] = \mp i \hbar \kappa \epsilon(\sigma - \bar{\sigma}), \quad (B.8)$$

where $\epsilon(\sigma) = \theta(\sigma) - \theta(-\sigma)$ is a sign function. From the expression

$$[\hat{J}_{\pm}^{(\pm)}, \hat{I}_{\pm}] = [\hat{J}_{\pm}^{(\pm)}, \hat{I}_{\pm}^{(\mp)}], \quad (B.9)$$

follows that

$$X_{\pm}(\sigma, \bar{\sigma}) \equiv [\hat{I}_{\pm}^{(\pm)}(\sigma), \hat{I}_{\pm}^{(\mp)}(\bar{\sigma})] = \mp i \hbar \kappa \int_{-\infty}^{\infty} d\sigma_1 \delta^{(\pm)}(\sigma - \sigma_1) \epsilon(\sigma_1 - \bar{\sigma}). \quad (B.10)$$

Using Eq. [B.2], the last relation obtains the form

$$X_{\pm} = \mp i \hbar \kappa \int_{-\infty}^{\infty} d\sigma_1 \frac{\int_{-\infty}^{\infty} d\sigma_1 \frac{1}{\sigma - \sigma_1 + i\epsilon} \frac{1}{\sigma - \sigma_1 + i\epsilon}}{2\pi(\sigma - \sigma_1 + i\epsilon)^2} \epsilon(\sigma_1 - \bar{\sigma}). \quad (B.11)$$

After regularization ($\Lambda$–cutoff parameter) integral on the right hand side takes a form

$$X_{\pm} = -\kappa^2 \left\{ \int_{\Lambda}^{\lambda} d\sigma_1 \frac{1}{\sigma - \sigma_1 + i\epsilon} - \int_{-\Lambda}^{\lambda} d\sigma_1 \frac{1}{\sigma - \sigma_1 + i\epsilon} \right\}. \quad (B.12)$$

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Computing the integral and taking $\Lambda \gg \sigma$, we get

$$X_\pm = \kappa^2 \left\{ \ln \left( \frac{\Lambda^2}{(\sigma - \sigma \pm i\varepsilon)^2} \right) \pm i\pi \right\}. \quad (B.13)$$

The last equation implies

$$e^{\frac{1}{2\kappa^2} X_\pm(\sigma, \bar{\sigma})} = \frac{\mp i\Lambda}{\sigma - \bar{\sigma} \mp i\varepsilon} = 2\pi \Lambda \delta(\pm)(\sigma - \bar{\sigma}), \quad (B.14)$$

$$e^{-\frac{1}{2\kappa^2} X_\pm(\sigma, \bar{\sigma})} = \pm i \frac{\sigma - \bar{\sigma} \mp i\varepsilon}{\Lambda}. \quad (B.15)$$

Substituting Eq. (B.14) in Eq. (B.4), we get the regularized form of the operator $\hat{J}_\pm$

$$\hat{J}_\pm(\sigma, \bar{\sigma}) = \sqrt{2}|C_\pm|^2 \frac{\mp i\Lambda}{\sigma - \bar{\sigma} \mp i\varepsilon} : \exp \{ \pm \frac{i}{\kappa \sqrt{2}} [\hat{I}_\pm(\sigma) - \hat{I}_\pm(\bar{\sigma})] \} ::, \quad (\varepsilon > 0). \quad (B.16)$$

For the infinitesimal $\eta = \bar{\sigma} - \sigma$ we get

$$\hat{J}_\pm(\sigma, \sigma + \eta)_{|\eta \to 0} = \sqrt{2}|C_\pm|^2 \frac{\mp i\Lambda}{\eta \pm i\varepsilon} + \frac{|C_\pm|^2 \Lambda}{\kappa} \hat{J}_\pm, \quad (\varepsilon > 0), \quad (B.17)$$

which is exactly the relation (B.1), and $F_\pm$ and $C_\pm$ are given by Eq. (B.2).

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