Motivated by the investigation of a black hole’s properties in the lab, some interesting subjects such as analogue gravity and transformation optics are generated. In this paper, we look for analogies between the geometry of a gravitating system and the optical medium. In addition, we recognize that appropriate two-dimensional metamaterials can be used to mimic the propagation of light in the curved spacetimes and behave like black holes which are incident with light rays in the equatorial plane. The resemblance of metamaterials with Kerr and Reissner–Nordström spacetimes is studied. Finally, we compare the results of two-dimensional simulation for light propagation in the corresponding two-dimensional metamaterials with those obtained from the geometrical optical limit.

1. Introduction

One of the interesting subjects in gravitational physics is analogue gravity in the optical frameworks, which is attributed to the pioneering work of Gordon [1]. Some of Gordon’s activities are devoted to describing dielectric media by an effective metric or, strictly speaking, mimicking a gravitational field with a dielectric medium. With the advent of transformation optics and analogue gravity, one can create systems that closely resemble general relativity objects such as black holes. Regarding the analogue gravity and transformation optics, one can investigate the resemblances of general relativistic gravitational fields to the equation of motion of other physical systems [2], such as mimicking some aspects of a black hole with a superfluid in the Bose–Einstein condensation [3]. Although such systems may not have exact analogues of any real spacetimes with corresponding metric tensors, transformation optics, where an optical material appears to perform a transformation of space, may use a formalism of general relativity to build a gradient index of static media in order to control the light paths in these media [4–11]. The theoretical concept of transformation optics is similar to equations of general relativity that describe how gravity warps space and time. However, instead of space and time, these equations show how light can be directed in a chosen manner, analogous to warp space [12]. In addition, transformation optics uses metamaterials to generate spatial variations derived from the coordinate transformation, which can control the chosen bandwidths of electromagnetic radiation. Therefore, it is possible to construct new artificial composite devices with a desired permittivity and permeability. Moreover, interactions of light and matter with spacetime, as predicted by the general relativity, can be studied by using new types of artificial optical materials that feature extraordinary abilities to bend light. This research creates an interesting link between...
the newly emerging field of artificial optical metamaterials and general relativity, and therefore it opens a new possibility to investigate various general relativity phenomena in a laboratory setting: chaotic motions observed in celestial objects that have been subjected to gravitational fields [4,13], mimicking an accurate laboratory model of a multi-dimensional universe landscape [4,14], optical analogues of black holes [15–19], Schwarzschild spacetime [20–22] and Hawking radiation [23].

The primary purpose of this paper is to generate an optical analogue of the known gravitational black hole and its characteristic phenomena, such as the existence of an event horizon (a one-way membrane) and the bending of light rays. From both theoretical and experimental points of view, optical metamaterials can mimic the geometry of light near black holes [12,15,19]. It has been shown that the permittivity and permeability tensors can be used instead of the refractive index in order to build the equivalent optical medium mimicking black holes [12,22]. In Ref. [22] the propagation of light waves outside the Schwarzschild black hole was simulated and the results were compared with ray paths obtained from the Hamiltonian method. Interestingly, the phenomenon of “photon sphere”, which is an essential feature of the black hole systems, was observed.

Here, as the first step, we extend the results of Ref. [22] to the case of a charged black hole. We consider a charged black hole and its optical simulation, and discuss the effect of electric charge on the photon sphere. Motivated by the lack of a definitive answer on the existence of astrophysical non-rotating black holes, we are encouraged to look at Kerr spacetime and look for its optical simulation as the second step. Also, we were interested in the effect of the spin parameter on the photon sphere.

The basic structure of this paper is twofold. First, we determine the permittivity and permeability mimicking the desired spacetime, then numerically solve the Maxwell equations using constitutive relations. Secondly, the Lagrangian for each spacetime is obtained; the ray paths in each spacetime can then be found using the equations of motion, and in order to check the accuracy of simulations, we compare the propagation of waves in optical materials with the ray paths derived from the equations of motion.

As an additional comment, we should note that analogue black holes and transformation optics are a bit different. Although these are very closely related and both take the form of the Plebanski equations in many circumstances, there is a subtle but critical distinction between them. In particular, transformation optics covariantly transforms solutions of Maxwell equations in a given spacetime. At the same time, analogue spacetimes are a non-covariant identification of electromagnetic solutions in curved spacetime with electromagnetic solutions in a medium in flat spacetime.

2. Spacetime geometry and media

Using the mathematical machinery of differential geometry, one can write Maxwell equations in arbitrary coordinates and geometry. It has been shown that the source-free Maxwell equations in arbitrary right-handed spacetime coordinates can be written as the macroscopic Maxwell equations in the right-handed Cartesian coordinates with Plebanski’s constitutive equations [12]. It means that Maxwell equations in curved coordinates are equivalent to their standard form for the flat space but in the presence of an effective medium. The Plebanski’s constitutive relations have been found in the form [24]:

\[ D^i = \varepsilon_0 \varepsilon^{ij} E_j + \frac{1}{c} [ijk] \Gamma_j H_k, \quad B^i = \mu_0 \mu^{ij} H_j - \frac{1}{c} [ijk] \Gamma_j E_k, \]  

(1)

where permittivity and permeability tensors and vector \( \Gamma \) are given by:

\[ \varepsilon^{ij} = \mu^{ij} = -\frac{\sqrt{-g}}{g_{00}} g^{ij}, \quad \Gamma_i = \frac{g_{0i}}{g_{00}}. \]  

(2)
Here, $g^{ij}$ is the inverse of $g_{ij}$, the spatial part of full spacetime metric $g_{\mu\nu}$, and $g$ is the determinant of $g_{\mu\nu}$. Designing artificial materials with the permittivity and permeability introduced in Eq. (2) enables us to mimic different spacetimes. It is notable that all the information about the gravitational field is embedded in the material properties of the effective medium.

Anisotropic or isotropic materials in which the constitutive relations can be described by Eq. (1) are called bi-anisotropic or bi-isotropic materials [25–29]. The vector $\Gamma$ is the magnetoelectric, coupling parameter coupling the electric and magnetic fields. Metamaterials are artificial materials made of sub-wavelength metallic constituents, which are randomly or periodically distributed in a dielectric background. In some metamaterials, the cross-polarization effect (an electric polarization which results from an applied magnetic field and vice versa) occurs; they are called bi-anisotropic or isotropic metamaterials [25]. By choosing a suitable distribution of the metallic inclusions in these materials, it is possible to obtain the desired electric permittivity, magnetic permeability and magnetoelectric coupling parameter following Eq. (2). Fabrication of bi-anisotropic and bi-isotropic metamaterials operating in the visible wavelengths is possible using different techniques such as lithography and laser writing [25]. To understand how light waves propagate near a black hole in different spacetimes, we can experimentally and theoretically study the propagation of electromagnetic waves in the corresponding metamaterials. Then, to prove the correctness of this equivalence, we compare our results with trajectories obtained using Lagrangian formalism. In other words, since the trajectories near black holes in different spacetimes are well studied, we compare them with the trajectories obtained from the equivalence optical medium to prove the correctness of our method. It should be noted that the electromagnetic (optical) black hole was fabricated for the first time in 2010 based on metamaterials composed of non-resonant I-shaped inclusions and electric field coupled resonators [16]. This structure can absorb the electromagnetic wave incident from every direction, like a black hole or a black body.

In other words, the mixing of electric and magnetic fields is addressed by $\Gamma$ with the physical dimension of velocity. It is also shown that for a slow-moving medium, $u/c \ll 1$, $\Gamma$ is proportional to the velocity of the medium [30].

3. Maxwell equations in medium

Regarding the first approach, Maxwell equations written in a source-free form [31]

\[
\nabla \cdot \vec{D} = 0, \quad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}, \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \nabla \cdot \vec{B} = 0, \quad (3)
\]

should be supplemented by the constitutive relations (1) and medium tensors which express space-time, in order to be solved. Since the constitutive equations couple the magnetic and electric fields and cause Eq. (3) to be complicated to solve, we have to seek for a method to decouple equations. In this regard, we apply the time-harmonic Maxwell equations, assuming an exponential time-dependence of the form $e^{-i\omega t}$, with $t$ and $\omega$ being the time variable and angular frequency, respectively:

\[
\nabla \cdot \vec{D} = 0, \quad \nabla \times \vec{H} = -i\omega \vec{D}, \quad \nabla \times \vec{E} = i\omega \vec{B}, \quad \nabla \cdot \vec{B} = 0, \quad (4)
\]

where $\vec{E}$, $\vec{H}$, $\vec{D}$ and $\vec{B}$ are, respectively, the electric field, magnetic field, electric displacement and magnetic flux density. We have considered that the constitutive equations obey Eq. (1), and therefore, substituting this equation to Eq. (4) we can find the following Maxwell–Tellegen equations [32]:

\[
\nabla \times \vec{H} = -i\omega \epsilon \vec{E} + i\omega \Gamma \times \vec{H}, \quad \nabla \times \vec{E} = i\omega \mu \vec{H} + i\omega \Gamma \times \vec{E}. \quad (5)
\]
We have assumed that the material parameters $\epsilon$, $\mu$ and $\Gamma$ and the electromagnetic fields are invariant in one direction; here the $z$-direction is considered. Since $\epsilon$, $\mu$ and $\Gamma$ are $z$-anisotropic tensors, in general, we can write

$$
\epsilon^{ij} = \begin{bmatrix}
\epsilon_{11} & \epsilon_{12} & 0 \\
\epsilon_{21} & \epsilon_{22} & 0 \\
0 & 0 & \epsilon_{33}
\end{bmatrix}, \quad 
\mu^{ij} = \begin{bmatrix}
\mu_{11} & \mu_{12} & 0 \\
\mu_{21} & \mu_{22} & 0 \\
0 & 0 & \mu_{33}
\end{bmatrix}, \quad \Gamma = \begin{bmatrix}
\Gamma_1 \\
\Gamma_2 \\
0
\end{bmatrix}. \tag{6}
$$

Applying Eq. (6) to Eq. (5) leads to

$$
-\partial_x E_3 = i\omega (\mu_{21} H_1 + \mu_{22} H_2) - i\omega \Gamma_1 E_3, \tag{7}
$$

$$
\partial_y E_3 = i\omega (\mu_{11} H_1 + \mu_{12} H_2) + i\omega \Gamma_2 E_3, \tag{8}
$$

$$
\partial_x E_2 - \partial_y E_1 = i\omega \mu_{33} H_3 + i\omega (\Gamma_1 E_2 - \Gamma_2 E_1), \tag{9}
$$

$$
\partial_x H_3 = i\omega (\epsilon_{21} E_1 + \epsilon_{22} E_2) + i\omega \Gamma_1 H_3, \tag{10}
$$

$$
\partial_y H_3 = -i\omega (\epsilon_{11} E_1 + \epsilon_{12} E_2) + i\omega \Gamma_2 H_3, \tag{11}
$$

$$
\partial_x H_2 - \partial_y H_1 = -i\omega \epsilon_{33} E_3 + i\omega (\Gamma_1 H_2 - \Gamma_2 H_1). \tag{12}
$$

It is evident from Eqs. (7)–(12) that electric and magnetic fields are coupled because of the constitutive equations. In order to decouple these equations, we follow the approach introduced in Ref. [32]. As the first step, we rewrite Eqs. (7) and (8) as the unified relation

$$
AE_3 = i\omega \mu_T H + i\omega \Gamma_T E_3, \tag{13}
$$

and also Eqs. (10) and (11) as

$$
AH_3 = i\omega \epsilon_T E + i\omega \Gamma_T H_3, \tag{14}
$$

where

$$
A = \begin{bmatrix}
\partial_x \\
\partial_y
\end{bmatrix}, \quad E = \begin{bmatrix}
E_2 \\
-E_1
\end{bmatrix}, \quad H = \begin{bmatrix}
H_2 \\
-H_1
\end{bmatrix}, \quad \Gamma_T = \begin{bmatrix}
\Gamma_1 \\
\Gamma_2
\end{bmatrix},
$$

$$
\epsilon_T = \begin{bmatrix}
\epsilon_{22} & -\epsilon_{21} \\
-\epsilon_{12} & \epsilon_{11}
\end{bmatrix}, \quad \mu_T = \begin{bmatrix}
-\mu_{22} & \mu_{21} \\
\mu_{12} & -\mu_{11}
\end{bmatrix}. \tag{15}
$$

Next, one can derive $H$ and $E$ from Eqs. (13) and (14), respectively:

$$
H = (i\omega \mu_T)^{-1} (A - i\omega \Gamma_T) E_3, \tag{16}
$$

and

$$
E = (i\omega \epsilon_T)^{-1} (A - i\omega \Gamma_T) H_3. \tag{17}
$$

We can also rewrite Eqs. (9) and (12) in the following forms:

$$
\begin{bmatrix}
\partial_x & \partial_y
\end{bmatrix} E = i\omega \mu_{33} H_3 + i\omega \begin{bmatrix}
\Gamma_1 & \Gamma_2
\end{bmatrix} E, \tag{18}
$$

and

$$
\begin{bmatrix}
\partial_x & \partial_y
\end{bmatrix} H = -i\omega \epsilon_{33} E_3 + i\omega \begin{bmatrix}
\Gamma_1 & \Gamma_2
\end{bmatrix} H. \tag{19}
$$
Substituting Eqs. (17) and (16) into Eqs. (18) and (19), one finds

$$\left[ \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right] (i \omega \epsilon T)^{-1} (A - i \omega \Gamma T) H_3$$

$$= i\omega \mu_{33} H_3 + i\omega \left[ \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right] (i \omega \epsilon T)^{-1} (A - i \omega \Gamma T) H_3,$$

(20)

and

$$\left[ \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right] (i \omega \mu T)^{-1} (A - i \omega \Gamma T) E_3$$

$$= -i\omega \epsilon_{33} E_3 + i\omega \left[ \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right] (i \omega \mu T)^{-1} (A - i \omega \Gamma T) E_3.$$  (21)

It is notable that all Eqs. (7–12) are resolved to two equations, (20) and (21), which are decoupled equations with the longitudinal fields $E_3$ and $H_3$ as unknowns.

We have considered Transverse Electric (TE) polarization for an electromagnetic wave for which $E$ is perpendicular to the $xy$ plane, so the only non-zero component of $E$ is $E_3$. Because the electric field is perpendicular to the magnetic field, $H_3$ is zero. Thus, one concludes that only Eq. (21) is required to solve for TE polarization, since Eq. (20) is always satisfied.

We should note that the actual waves in the black hole background may not localize on the equatorial plane and they may vary in the $z$-direction. However, in the general case, one can obtain both permittivity and permeability tensors from Eq. (2), which depends on the spacetime metric, which is the solution of Einstein’s equation. In the following sections, we first write the exact solution of Einstein’s equation, and then we set $z = 0$ in order to simplify the calculations. If we set $z = 0$, the permittivity and permeability components will be independent of $z$. Thus it can be assumed that the material parameters, and then the magnetic and electric fields, are invariant in the $z$-direction, and as a result Eq. (21) can be derived.

For the sake of completeness, we take the opportunity to comment that one can ignore the mentioned simplification ($z = 0$) and investigate wave propagation in the general case. For this general case, one needs to solve the coupled differential equations for electric and magnetic fields in three dimensions. This work is very time-consuming and requires supercomputers, but it is possible.

4. A brief review of Lagrangian formalism

In the previous section, we used wave optics to determine the propagation of light in optical materials mimicking spacetime. In this section, we apply geometrical optics in order to determine the propagation of rays in optical materials to show that our model works well. In this regard, an appropriate Lagrangian is introduced and then, based on the equation of motion, the ray path is obtained.

It is worth mentioning that the thin light ray is used in the geometrical optics to show the path of wave propagation in the cases in which the wavelength of light is much smaller than the size of structures with which light waves interact. As shown in Figs. 1–5, the simulation regions for the proposed metamaterials are squares with dimension larger than $5 \times 5$, and the incident wavelength is 0.3. Thus, the structure size is 16-fold larger than the incident wavelength and geometrical optics can be applied for these structures.

It has been shown [33] that the equations governing the geodesics in a spacetime with the line element $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ can be derived from the Lagrangian

$$L = g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = g_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta,$$  (22)
Fig. 1. Metamaterial mimicking the Reissner–Nordström spacetime for $Q = 0$, $m = 1$ and different impact parameters $D$.

where $\tau$ is the affine parameter along the geodesics. From the calculus of variations, one can write the following Euler–Lagrange equation:

$$\frac{\partial L}{\partial q^\alpha} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}^\alpha} = 0,$$

(23)

where $q^\alpha = (t, r, \theta, \phi)$ is the generalized coordinate, and the generalized momentum $p_\alpha$ is defined as

$$p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha},$$

(24)

It is easy to determine $\dot{p}_i$ from the Euler–Lagrange equation, Eq. (23), as below

$$\dot{p}_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha},$$

(25)

where a dot denotes the derivative with respect to the affine parameter, $\tau$. Based on Eqs. (22) and (24), we can rewrite the Lagrangian as a function of generalized momentum, $p_i$. Since we are seeking the ray path, the Lagrangian, Eq. (22), is zero.

5. Reissner–Nordström spacetime and its related medium tensors

First, we apply the analogy to the Reissner–Nordström spacetime. The Reissner–Nordström metric is a static solution to the Einstein–Maxwell field equations, which corresponds to the gravitational
field of a charged, non-rotating, spherically symmetric body of mass \( M \) \cite{34}. This metric in the Cartesian coordinates can be written as

\[
ds^2 = \left( \frac{2m}{r} - 1 - \frac{Q^2}{r^2} \right) dt^2 + \left( \frac{r^2-2mr+Q^2}{r^2} + \frac{x^2}{r^2(r^2-z^2)} + \frac{y^2}{r^2(r^2-z^2)} \right) dx^2 \\
+ \left( \frac{r^2-2mr+Q^2}{r^2} + \frac{y^2}{r^2(r^2-z^2)} + \frac{z^2}{r^2(r^2-y^2)} \right) dy^2 \\
+ \left( \frac{r^2-2mr+Q^2}{r^2} + \frac{x^2}{r^2(r^2-y^2)} + \frac{z^2}{r^2(r^2-x^2)} \right) dz^2 \\
+ 2 \left( \frac{x^2 y^2}{r^2(r^2-z^2)} - \frac{x^2 y^2}{r^2(r^2-y^2)} \right) dxdy \\
+ 2 \left( \frac{x^2 z^2}{r^2(r^2-z^2)} - \frac{x^2 z^2}{r^2(r^2-x^2)} \right) dxdz.
\]

Here \( Q \) is the charge of the black hole, \( m \) is the geometrical mass of the source of gravitation and \( r = \sqrt{x^2 + y^2 + z^2} \).

The anisotropic equivalent medium tensors can be obtained from Eq. (2). It is notable that \( \Gamma_i = 0 \) because the Reissner–Nordström metric is static. Therefore, permittivity and permeability tensors in
Metamaterial mimicking the Reissner–Nordström spacetime for $Q = 0.5$, $m = 1$ and different impact parameters $D$.

The $xy$ plane can be obtained as

$$
\epsilon_{ij} = \mu_{ij} = \begin{pmatrix}
\frac{r^4 + Q^2 - 2mrx^2}{r^2(r^2 + Q^2 - 2mr)} & \frac{x y(Q^2 - 2mr)}{r^2(r^2 + Q^2 - 2mr)} & 0 \\
\frac{x y(Q^2 - 2mr)}{r^2(r^2 + Q^2 - 2mr)} & \frac{r^4 + Q^2 - 2mry^2}{r^2(r^2 + Q^2 - 2mr)} & 0 \\
0 & 0 & \frac{r^2}{r^2 + Q^2 - 2mr}
\end{pmatrix}.
$$

As we are working in the $xy$ plane, here $r = \sqrt{x^2 + y^2}$. Having permittivity and permeability tensors, Eq. (21) can be solved. We have solved the wave equation Eq. (21) numerically for a beam with the appropriate boundary conditions.

It is known that having a non-trivial, spherically symmetric electromagnetic wave is not possible. The reason for this comes from the fact that the polarization vectors of such a field configuration will introduce a continuous nowhere-vanishing vector field tangent to the 2-sphere, which contradicts the well-known fact that the 2-sphere is not parallelizable. This fact motivated us to use other coordinate...
systems. The optical parameters of the material equivalent to the black hole depend on the coordinate system. Because working with the Cartesian coordinate in COMSOL is more comfortable than other coordinate systems, we use it in this paper. It is also notable that although the (non-physical) coordinate singularity can be removed in the Cartesian coordinate, its one-way membrane property is being kept for the equivalent metamaterials.

The Reissner–Nordström metric in spherical coordinates reads [34]

\[ ds^2 = -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \] (28)

Substituting this metric into Eq. (22) and considering the xy plane, we find

\[ L = -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)\dot{t}^2 + \frac{\dot{r}^2}{1 - \frac{2m}{r}} + \frac{\dot{\phi}^2}{\frac{Q^2}{r^2}}. \] (29)

Considering the obtained Lagrangian and using Eqs. (24) and (25), \( p_i \) and \( \dot{p}_i \) can be easily calculated as

\[ \dot{\hat{r}} = \frac{dp_i}{d\tau} = 0, \] (30)
Fig. 5. Metamaterial mimicking the Kerr spacetime for $m = 0.5$ and different impact parameters $D$ and spin parameters $a$.

\[
p_\phi = \frac{dp_\phi}{d\tau} = 0, \quad (31)
\]

\[
p_t = -2 \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) \dot{r}, \quad (32)
\]

\[
p_\phi = 2r^2 \dot{\phi}. \quad (33)
\]

We define $p_t = 2E$ and $p_\phi = 2L$ as constants, since $p_t = 0$ and $p_\phi = 0$, which just state the conservation of energy and angular momentum for the static spherically symmetric system. Hence, the Lagrangian Eq. (29) can be rewritten as

\[
\mathcal{L} = -\frac{E^2}{\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)} + \frac{\dot{r}^2}{\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)} + \frac{L^2}{r^2}. \quad (34)
\]

Considering $\mathcal{L} = 0$, one finds

\[
\dot{r}^2 = \left(\frac{dr}{d\tau}\right)^2 = E^2 - \frac{L^2}{r^2} \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) = f(r), \quad (35)
\]
and $\phi$ can be determined from Eq. (33), therefore
\[
\left( \frac{dr}{d\phi} \right)^2 = \frac{r^4}{D^2} - r^2 + 2mr - Q^2.
\] (36)

Because photons are massless, the ray trajectories can be determined by only one character, which is called the impact parameter $D = \frac{L}{E}$ [33]. The ray trajectories can be obtained by solving Eq. (36) [33,35]. The results for Reissner–Nordström spacetime, both for wave optics and ray optics, are shown in Figs. 1–4.

Interestingly, depending on the roots of $f(r)$, we have three types of trajectories: “terminating orbit”, “bound orbit” and “flyby orbit”. Those roots only depend on the impact parameter, $D$, since both $m$ and $Q$ are constants. Therefore, for $D = D_c$ we have a bound orbit, if $D > D_c$ there is a flyby orbit, and if $D < D_c$ there is a terminating orbit [35]. The critical value of impact parameter, $D_c$, can be derived by considering $f''(r_c) = 0$ and $f(r_c) = 0$:
\[
r_c = \frac{3}{2} m \left[ 1 + \frac{1}{2} \sqrt{\frac{8Q^2}{9m^2}} \right], \quad D_c = \frac{r_c^2}{\sqrt{r_c^2 - 2mr_c + Q^2}}. \] (37)

It is evident that for different values of charge, $Q$, the critical value of the impact parameter differs.

6. Kerr spacetime and its related medium tensors

In this section, we are going to generalize the Schwarzschild static solution to the case of a stationary one, and thereby discuss the Kerr spacetime. The Kerr metric describes the geometry of empty spacetime around a rotating, uncharged, axially symmetric black hole with an appropriate event horizon. The Kerr metric is an exact solution of the Einstein field equations of general relativity with a stationary constraint [34]. This metric in the Cartesian coordinates reads
\[
ds^2 = d\tau^2 - dx^2 - dy^2 - dz^2 - \frac{2mr^3}{a^2 + r^2} \left( \frac{dx}{a^2 + r^2} + \frac{ydy}{a^2 + r^2} \right)^2 \quad \text{(38)}
\]
where $a$ and $m$ are spin parameter and geometrical mass, respectively, the radial coordinate is $r = \sqrt{x^2 + y^2 + z^2}$ and also $d\tau = dt + \frac{(2m(\varepsilon x + \eta y + \zeta z))/(r^2 - 2mr + a^2))}$.

Based on Eq. (2), permittivity and permeability tensors and $\Gamma_i$ in the $xy$ plane can be obtained as
\[
\epsilon^{xx} = \mu^{xx} = -A_z \left[ \frac{-r^8 + 2mr^7 - 2r^6a^2 + 4ma^2 + (a^2 - \frac{r^2}{m})r^2 + (-a^4 + 4mxya)4 + 2ma^2(a^2 - x^2)r}{(1 - \frac{2mr}{m})(a^2 + r^2)(r^6 - 2mr^5 + r^4a^2 - 2m(a^2 - r^2)r^3)} \right],
\]
\[
\epsilon^{xy} = \epsilon^{yx} = \mu^{xy} = \mu^{yx} = 2A_z \left[ \frac{(ay + bx)(ax - by)r}{(1 - \frac{2mr}{m})(a^2 + r^2)(r^6 - 2mr^5 + r^4a^2 - 2m(a^2 - r^2)r^3)} \right],
\]
\[
\epsilon^{xz} = \epsilon^{zx} = \mu^{xz} = \mu^{zx} = 0,
\]
\[
\epsilon^{yy} = \mu^{yy} = -A_z \left[ \frac{-r^8 + 2mr^7 - 2r^6a^2 + 4ma^2 + (a^2 - \frac{r^2}{m})r^2 + (-a^4 + 4mxya)4 + 2ma^2(a^2 - y^2)r}{(1 - \frac{2mr}{m})(a^2 + r^2)(r^6 - 2mr^5 + r^4a^2 - 2m(a^2 - r^2)r^3)} \right],
\]
\[
\epsilon^{yz} = \epsilon^{zy} = \mu^{yz} = \mu^{zy} = 0,
\]
\[
\epsilon^{zz} = \mu^{zz} = -A_z \left[ \frac{-r^8 + 2mr^7 - 2r^6a^2 + 2ma^2(a^2 - y^2)r^3}{(1 - \frac{2mr}{m})(a^2 + r^2)(r^6 - 2mr^5 + r^4a^2 - 2m(a^2 - r^2)r^3)} \right],
\]
where \( A_z = \sqrt{(r^4a^2 - 2mr^3a^2 + r^6)/[r^4(a^2 + r^2)]} \). We also find that

\[
\Gamma_x = \frac{2ma(-2mr^3xa-r^5y+2r^4ym-r^3a^2y)}{r^4(r^2-2mr+a^2)(a^2+r^2)(1-\frac{2mr}{r^2})},
\]

\[
\Gamma_y = \frac{2ma(-2mr^3ya+r^5x-2r^4xm+r^3a^2x)}{r^4(r^2-2mr+a^2)(a^2+r^2)(1-\frac{2mr}{r^2})},
\]

\[
\Gamma_z = 0.
\]

Interestingly, the vector \( \vec{\Gamma} \) is non-zero because the Kerr metric is not static. It can be said that the spin parameter, \( a \), makes the equivalent material bi-anisotropic. Besides, Eq. (21) can be solved numerically by considering Eqs. (39) and (40).

Regarding the Lagrangian formalism, first, we write the Kerr metric in the spherical coordinates [34]

\[
ds^2 = \frac{\Delta}{\rho^2} \left( dt - \sin^2 \theta d\phi \right)^2 - \frac{\sin^2 \theta}{\rho^2} \left[ (r^2 + a^2) d\phi - adt \right]^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2,
\]

where \( \rho^2 = r^2 + a^2 \cos^2 \theta \) and \( \Delta = r^2 - 2mr + a^2 \). As we are working in the \( xy \) plane, we set \( \theta = \frac{\pi}{2} \).

Substituting this metric into Eq. (22), one can show that

\[
\mathcal{L} = \left( 1 - \frac{2ma}{r} \right) i^2 + \left( \frac{4ma}{r} \right) i \dot{\phi} - \frac{r^2}{r^2 - 2mr + a^2} \dot{\phi}^2 - \left[ r^2 + a^2 + \frac{2ma^2}{r} \right] \dot{\phi}^2.
\]

Using Eqs. (24) and (25), the following relations are obtained

\[
\dot{p}_t = \frac{dp_t}{d\tau} = 0,
\]

\[
\dot{p}_\phi = \frac{dp_\phi}{d\tau} = 0
\]

\[
p_t = 2 \left( 1 - \frac{2m}{r} \right) i + \frac{4ma^2}{r} \dot{\phi},
\]

\[
p_\phi = \frac{4ma}{r} i - \dot{\phi} \left[ \frac{r^2 + a^2 + \frac{2ma^2}{r}}{r} \right].
\]

where we call \( p_t = 2E \) and \( p_\phi = -2L \), as before. By solving Eqs. (45) and (46) simultaneously, \( i \) and \( \dot{\phi} \) can be calculated as

\[
i = \frac{1}{r^2 - 2mr + a^2} \left[ \left( 1 - \frac{2m}{r} \right) L + \frac{2ma}{r} E \right],
\]

\[
\dot{\phi} = \frac{1}{r^2 - 2mr + a^2} \left[ \left( r^2 + a^2 + \frac{2ma^2}{r} \right) E - \frac{2ma}{r} L \right].
\]

Moreover, substituting these parameters into the Lagrangian Eq. (42), one finds

\[
\mathcal{L} = \left( a^2 + r^2 + \frac{2ma^2}{r} \right) E^2 + \left( \frac{2m}{r} \right) L^2 - \frac{4ma}{r} LE - r^2 \dot{i}^2.
\]

Since \( \mathcal{L} = 0 \) for light beam, we have

\[
\dot{i}^2 = E^2 + \frac{2m}{r^2} (aE - L)^2 + \frac{1}{r^2} (a^2E^2 - L^2) = f(r).
\]
Table 1. Reissner–Nordström solutions: the numerical results of $D_c$ and $r_c$ for different values of $Q$ in the case of $m = 1$.

| $Q$ | $D_c$  | $r_c$ |
|-----|--------|-------|
| 0   | 5.196  | 3     |
| 0.1 | 5.187  | 2.99  |
| 0.5 | 4.97   | 2.82  |
| 1   | 4      | 2     |

It is noteworthy to mention that only the impact parameter affects the ray trajectories, and its effects lead to the three types of orbit mentioned in the previous section. At this point, since the equations related to Kerr spacetime are too complicated, we only focus on the critical value of the impact parameter. In this regard, by calculating $f'(r_c) = 0$ and $f(r_c) = 0$, one finds

$$r_c = \frac{3m}{E} = \frac{3mD_c - a}{D_c + a}, \quad (D_c + a)^3 = 27m^2(D_c - a). \tag{51}$$

It is clear that the spin parameter, $a$, controls the critical value of the impact parameter. By considering $u = 1/r$ with Eqs. (50) and (51), we can rewrite Eq. (50) as

$$\dot{u}^2 = E^2u^4m(D_c - a)^2(u - u_c)^2(2u + u_c). \tag{52}$$

Taking into account $\dot{\phi}$ in Eq. (48), we can obtain

$$\frac{du}{d\phi} = \frac{(1 - 2mu + a^2u^2)(u - u_c)(D_c + a)\sqrt{2u + u_c}}{\sqrt{m[3u_cD_c - 2u(D_c + a)]}}. \tag{53}$$

Furthermore, solving Eq. (53) gives the ray trajectories [33]. The results for Kerr spacetime are shown in Fig. 5.

7. Results and discussion

Here, we are in a position to solve the equations of two approaches and simulate the results for comparison. It is notable that we use the geometric unit system with unit mass, as $c = G = m = 1$. Equation (21) is solved in the Cartesian coordinates and a rectangular 2D geometry of the space for a TE polarized wave of frequency $\omega = 6.3 \times 10^9$ injected from the right and the results are compared with the ray path (red line) calculated from the equations of motion. The computational domain is surrounded by a perfectly matched layer that absorbs the outward waves to ensure that there are no unwanted reflections. The simulations are done using a standard software solver (COMSOL, with the following fixed parameters: maximum element size: 0.02, minimum element size: $3.2 \times 10^{-4}$, maximum element growth rate: 1.1, curvature factor: 0.2 and resolution of the narrow region: 1).

For the Reissner–Nordström spacetime, the medium parameters are given by Eq. (27) and the results are depicted in Table 1 and Figs. 1–4. In order to obtain dimensionless parameters, we normalize all parameters to $r_0 = r_s/2$ in which $r_s = 2m$ is the event horizon of a Schwarzschild black hole where the gravity is so strong that light cannot escape [33].

Taking the plotted figures into account, we find that the ray path shows the same behavior as the beam. It is notable that although the medium is highly anisotropic, suitable boundary conditions restrict the direction of the beam and the numerical results confirm an acceptable agreement between the ray path and the beam. Moreover, the ray trajectories are consistent with the behavior expected...
Table 2. Kerr solutions: the numerical results of $D_c$ and $r_c$ for different values of $a$ in the case of $m = 1$.

| $a$  | $D_c$ | $r_c$ |
|------|-------|-------|
| 0.1  | 2.48  | 1.38  |
| 0    | 2.60  | 1.5   |
| −0.1 | 2.74  | 1.61  |
| −0.5 | 3.5   | 2     |

from theory. Speaking more precisely, the case of $D = D_c$ represents the photon sphere [36], which is a spherical region of space where the gravitational effect is strong enough that photons are forced to travel in an orbit (in the $xy$ plane). This area is patently obvious in Figs. 1, 2, 3 and 4. The photon sphere is located farther from the center of a black hole than the event horizon. The radius of a photon sphere can be determined as $r_c$ in Eq. (37), and some numerical results of $r_c$ can be found in Table 1. In addition, for $D < D_c$, we expect that the photons are drowned in the event horizon, which can be seen in Figs. 1, 2, 3 and 4. Moreover, in the case of $D > D_c$, a deflection of the ray trajectory can be observed in Figs. 1, 2, 3 and 4. Furthermore, it is notable that the charge of black hole, $Q$, affects the critical value of the impact parameter, $D_c$, and as a result the radius of the photon sphere, $r_c$, is changed. It can be seen from Table 1 that when $Q$ increases, the critical impact parameter, $D_c$, and the radius of a photon sphere, $r_c$, decrease.

It can be observed from all figures that the beam splits into a set of rays or sub-beams; one part falls into the black hole due to having $D < D_c$, another part escapes from the black hole because of $D > D_c$, and another part bends around the photon sphere of the black hole and interferes with the primary beam.

It is expected from the theory that the case of $Q = 0$ represents Schwarzschild spacetime. The radius of a photon sphere is determined to be $r_c = 3m$ for $Q = 0$, which is similar to our expectation; also, Fig. 1 matches the results reported in Ref. [22].

For the Kerr spacetime, the medium parameters are given by Eqs. (39) and (40), and the numerical and simulated results are shown in Table 2 and Fig. 5. It is notable that in order to work with dimensionless parameters we can normalize them to $r_s$, as before.

As we mentioned before, computational analysis of rotating spacetime is complicated, and therefore we only focus on the critical value of the impact parameter which leads to the photon sphere. In contrast to the Reissner–Nordström solution, which is a static, spherically symmetric spacetime, the Kerr solution describes a stationary spacetime with the axially symmetry, and therefore it has profound consequences for the photon orbits. A circular orbit can only exist in the equatorial plane [36]. Fortunately, both methods of simulations—numeric solutions to Maxwell equations and the ray trajectory arising from the Hamiltonian formalism—are in agreement with each other. As is depicted in Table 2, by decreasing the spin parameter ($a$), the critical impact parameter ($D_c$) and the radius of the photon sphere ($r_c$) increase. Furthermore, for the static case ($a = 0$), the results of the Kerr black hole match with those being derived from the Schwarzschild black hole, as expected.

As a final comment, we should note that because we set $z = 0$ in the spacetime metric, the propagation of a wave near the black hole and in corresponding designed two-dimensional optical metamaterial are the same only in the cases where the beam is incident on to the black hole in the $xy$ plane. If one tends to obtain the path of the beam incident on to the black hole in any $z = constant$ plans, it is sufficient to find the corresponding three-dimensional metamaterial. However, the simulation of light wave propagation in these materials is very time-consuming and needs strong computational
equipment, which we leave for the future. It is worth mentioning that if these three-dimensional metamaterials can be realized using the lithography technique, one can directly study the wave propagation through them in the laboratory.

8. Concluding remarks

Applying various methods of the analogue gravity and transformation optics, one can ask how we can simulate different properties of Einstein’s equations and black hole solutions in the laboratory. In this paper, we compared the geometrical description of gravitational systems with optical media in flat space through two approaches. We observed a good correspondence between the two methods.

We considered two supplementary factors of the Schwarzschild black hole; the electric charge and the rotation parameters. More precisely, we investigated the effects of electric charge and rotation factors on the trajectory of light with various impact parameters. We found that the medium parameters given for Reissner–Nordström and Kerr spacetime can mimic the ray path near the black hole in the flat spacetime. Interestingly, some theoretical phenomena like the photon sphere are observed, and the effects of charge, $Q$, and spin parameter, $a$, are investigated. We showed that the critical impact parameter is a decreasing function of both the electric charge and spin parameters. This behavior is expected since increasing the electric charge and rotation leads to decreasing the event horizon radius (weakening the gravitational effect) in the classical black hole scenario.

It will be interesting to apply these calculations to the Kerr–Newman black hole and other attractive nontrivial solutions of Einstein gravity. Moreover, one may consider the effect of thermal (quantum) fluctuations [37–42] on the geodesic motion of particles near different black holes.

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