Perfect codes in vertex-transitive graphs

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Abstract

Given a graph $\Gamma$, a perfect code in $\Gamma$ is an independent set $C$ of vertices of $\Gamma$ such that every vertex outside of $C$ is adjacent to a unique vertex in $C$, and a total perfect code in $\Gamma$ is a set $C$ of vertices of $\Gamma$ such that every vertex of $\Gamma$ is adjacent to a unique vertex in $C$. To study (total) perfect codes in vertex-transitive graphs, we generalize the concept of subgroup (total) perfect code of a finite group introduced in \cite{10} as follows: Given a finite group $G$ and a subgroup $H$ of $G$, a subgroup $A$ of $G$ containing $H$ is called a subgroup (total) perfect code of the pair $(G, H)$ if there exists a coset graph $\operatorname{Cos}(G, H, U)$ such that the set consisting of left cosets of $H$ in $A$ is a (total) perfect code in $\operatorname{Cos}(G, H, U)$. We give a necessary and sufficient condition for a subgroup $A$ of $G$ containing $H$ to be a (total) perfect code of the pair $(G, H)$ and generalize a few known results of subgroup (total) perfect codes of groups. We also construct some examples of subgroup perfect codes of the pair $(G, H)$ and propose a few problems for further research.

Keywords: vertex-transitive graph; perfect code; total perfect code; coset graph

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1 Introduction

In this paper, all groups considered are finite, and all graphs considered are finite, undirected and simple. Let $\Gamma$ be a graph with vertex set $V(\Gamma)$ and edge...
set $E(\Gamma)$, and let $t$ be a positive integer. A subset $C$ of $V(\Gamma)$ is called \[\text{a perfect } t\text{-code in } \Gamma\] if every vertex of $\Gamma$ is at distance no more than $t$ to exactly one vertex in $C$, where the distance in $\Gamma$ between two vertices is the length of a shortest path between the two vertices or $\infty$ if there is no path in $\Gamma$ joining them. A perfect 1-code is usually called a perfect code. Equivalently, a subset $C$ of $V(\Gamma)$ is a perfect code in $\Gamma$ if $C$ is an independent set of $\Gamma$ and every vertex in $V(\Gamma) \setminus C$ has exactly one neighbor in $C$. It is obvious that a total perfect code in $\Gamma$ induces a matching in $\Gamma$ and therefore has even cardinality. In graph theory, a perfect code in a graph is also called an efficient dominating set or independent perfect dominating set, and a total perfect code is called an efficient open dominating set.

The concept of $t$-perfect codes in graphs were firstly introduced by Biggs [1] as a generalization of the classical concept perfect $t$-error-correcting code in coding theory [8, 9, 15, 17]. For a set $A$ (usually with an algebraic structure such as group, ring, or field), we use $A^n$ to denote the $n$-fold Cartesian product of $A$. In coding theory, $A$ is called an alphabet and elements in $A^n$ are called words of length $n$ over $A$. A code $C$ over an alphabet $A$ is simply a subset of $A^n$, and every word in $C$ is called a codeword. The Hamming distance of two words in $A^n$ is the number of positions in which they differ. A code $C$ over $A$ is called a perfect $t$-error-correcting Hamming code if every word in $A^n$ is at Hamming distance no more than $t$ to exactly one codeword of $C$. The perfect $t$-error-correcting Lee code over $A$ is defined in a similar way if $A$ is the ring $\mathbb{Z}_m$ of integers (mod $m$), where the Lee distance of two words $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \mathbb{Z}_m^n$ is defined as follows: $d_L(x, y) = \sum_{i=1}^{n} \min(|x_i - y_i|, m - |x_i - y_i|)$. Recall that the Hamming graph $H(n, m)$ is the Cartesian product of $n$ copies of the complete graph $K_m$ and the grid-like graph $L(n, m)$ is the Cartesian product of $n$ copies of the $m$-cycle $C_m$. It is obvious that the perfect $t$-error-correcting Hamming codes over an alphabet of cardinality $m$ are precisely the perfect $t$-codes in $H(n, m)$. Similarly, the perfect $t$-error-correcting Lee codes over $\mathbb{Z}_m$ ($m \geq 3$) are precisely the perfect $t$-codes in $L(n, m)$.

A graph $\Gamma$ is called $G$-vertex-transitive if $G$ is a subgroup of $\text{Aut}(\Gamma)$ acting transitively on $V(\Gamma)$. In particular, a $G$-vertex-transitive graph is called a Cayley graph on $G$ if $G$ acts freely on the vertex set (nonidentity elements fix no vertex). It is well known and easy to check that both $K_m$ and $C_m$ are Cayley graphs on the cyclic group $\mathbb{Z}_m$. Therefore $H(n, q)$ and $L(n, m)$ are both Cayley graphs on the group $\mathbb{Z}_m^n$. Thus perfect $t$-codes in Cayley graphs are generalization of perfect
Perfect codes in Cayley graphs have received considerable attention in recent years; see [10, Section 1] for a brief survey and [3, 6, 18, 20, 21] for a few recent papers. In particular, perfect codes in Cayley graphs which are subgroups of the underlying groups are especially interesting since they are generalizations of perfect linear codes [15] in the classical setting. Another interesting avenue of research is to study when a given subset of a group is a perfect code in some Cayley graph of the group. In this regard the following concepts were introduced by Huang et al. in [10]: A subset \( C \) of a group \( G \) is called a (total) perfect code of \( G \) if there exists a Cayley graph of \( G \) which admits \( C \) as a (total) perfect code; a (total) perfect code of \( G \) which is also a subgroup of \( G \) is called a subgroup (total) perfect code of \( G \). Huang et al. [10] established a sufficient and necessary condition for the normal subgroups of a given group to be subgroup (total) perfect codes, and proved that every normal subgroup of a group of odd order or odd index is a subgroup perfect code. Ma et al. [16] proved that all subgroups of a group are subgroup perfect codes if and only if this group does not contain elements of order 4. Very recently, Zhang and Zhou [20] generalized several results about normal subgroups in [10] to general subgroups, and in particular they proved that every subgroup of a group of odd order or odd index is a subgroup perfect code.

Although every Cayley graph is vertex-transitive, there exist other vertex-transitive graphs that are not Cayley graphs. Somewhat surprisingly, there are very few known results on the perfect codes in vertex-transitive graphs in the literature. This motivates us to write the present paper. It is well known that a graph is \( G \)-vertex transitive if and only if it can be represented as a coset graph \( \text{Cos}(G, H, U) \) (see Section 2 for the details). To study the perfect codes in vertex-transitive graphs, we generalize the concept subgroup (total) perfect code of a finite group as follows: Given a finite group \( G \) and a subgroup \( H \) of \( G \), a subgroup \( A \) of \( G \) containing \( H \) is called a subgroup (total) perfect code of the pair \((G, H)\) if there exists a coset graph \( \text{Cos}(G, H, U) \) such that the set consisting of left cosets of \( H \) in \( A \) is a (total) perfect code in \( \text{Cos}(G, H, U) \). In this paper, we give a necessary and sufficient condition for a subgroup \( A \) of \( G \) containing \( H \) to be a (total) perfect code of the pair \((G, H)\) and generalize a few known results of subgroup (total) perfect codes of groups.

The rest of the paper is organized as follows. In Section 2, we recall the definition of coset graph and give a characterization of the relationship between subgroup perfect codes and subgroup total perfect codes of a pair \((G, H)\). In Section 3, we prove that \( A \) is a perfect code of a pair \((G, H)\) if and only if there exists a left transversal \( X \) of \( A \) in \( G \) such that \( XH = HX^{-1} \) (Theorem 3.1). Based
on Theorem 3.1, we generalize a few results about subgroup perfect codes in [21].
In Section 4, we deduce a few results on total perfect codes which are parallel to some results about perfect codes in Section 3. In Section 5, we construct several examples and propose a few problems for further research. In particular, we show that $S_{n-1}$ is a perfect code of $(S_n, S_3)$ for every positive integer $n \geq 5$.

2 Preliminaries

For a group $G$, we write $H \leq G$ to signify that $H$ is a subgroup of $G$ and we set $A/\ell H := \{aH \mid a \in A\}$ for all subset $A$ of $G$. For more group-theoretic terminology and notation used in the paper, please refer to [12]. The following proposition gives a nice way to represent vertex-transitive graphs. For the proof of this proposition, see [14].

Proposition 2.1. Let $G$ be a group and $H \leq G$. Let $U$ be a union of some double cosets of $H$ in $G$ such that $H \cap U = \emptyset$ and $U^{-1} = U$. Define a graph $\Gamma = \text{Cos}(G, H, U)$ as follows: the vertex set of $\Gamma$ is $G/\ell H$, and two vertices $g_1H$ and $g_2H$ are adjacent if and only if $g_1^{-1}g_2 \in U$. Then we have

(i) $\Gamma$ is a well defined graph and its valency is the number of left cosets of $H$ in $U$;

(ii) $G$ acts transitively on the vertex-set of $\Gamma$ by left multiplication and the kernel of this action is the core of $H$ in $G$;

(iii) every vertex-transitive graph can be represented as $\text{Cos}(G, H, U)$ for some $G$, $H$ and $U$.

The graph $\Gamma = \text{Cos}(G, H, U)$ defined in Proposition 2.1 is usually called a coset graph on $G/\ell H$. It is straightforward to check that the neighbourhood of $H$ in $\Gamma$ is $U/\ell H$ and $\Gamma$ is connected if and only if $G = \langle U \rangle$.

Definition 2.2. Let $G$ be a group, $H \leq G$ and $A \subseteq G$. If there exists a coset graph $\text{Cos}(G, H, U)$ on $G/\ell H$ admitting a (total) perfect code $A/\ell H$, then $A$ is called a (total) perfect code of the pair $(G, H)$. If further $H \leq A \leq G$, then $A$ is called a subgroup (total) perfect code of the pair $(G, H)$.

Remark 2.3. It is obvious that $\text{Cos}(G, 1, U)$ is a Cayley graph on $G$. Thus $A$ is a subgroup (total) perfect code of $(G, 1)$ if and only if it is a subgroup (total) perfect code of $G$. Therefore the concept of subgroup (total) perfect code of $(G, H)$ is a generalization of the concept of subgroup (total) perfect code of $G$. 

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The following lemma gives a characterization of the relationship between subgroup perfect codes and subgroup total perfect codes of a pair \((G, H)\).

**Lemma 2.4.** Let \(G\) be a group and \(H \leq A \leq G\). Then \(A\) is a total perfect code of \((G, H)\) if and only if \(A\) is a perfect code of \((G, H)\) and there exists an element \(x \in N_A(H) \setminus H\) such that \(x^2 \in H\).

**Proof.** \(\Rightarrow\) Suppose that \(A\) is a total perfect code of \((G, H)\). Then there exists a coset graph \(\text{Cos}(G, H, U)\) on \(G/\ell H\) such that \(A/\ell H\) is a total perfect code of \(\text{Cos}(G, H, U)\). In particular, \(A/\ell H\) induces a matching in \(\text{Cos}(G, H, U)\). Thus there exists \(x \in A \setminus H\) such that \(xH\) is the unique vertex in \(A/\ell H\) which is adjacent to \(H\). By the definition of coset graph, \(U\) is a union of some double cosets of \(H, U^{-1} = U\) and \(x \in U\). Therefore \(hx, hx^{-1} \in U\) for every \(h \in H\). It follows that \(hxH\) and \(hx^{-1}H\) are both neighbours of \(H\) in \(\text{Cos}(G, H, U)\). Note that \(hxH, hx^{-1}H \in A/\ell H\). By the uniqueness of \(xH\), we have \(hxH = hx^{-1}H = xH\). Therefore \((hxH)^{-1} = HxH = xH\). Set \(W = U \setminus HxH\). Then \(W^{-1} = W\) and \(A/\ell H\) is a perfect code of the coset graph \(\text{Cos}(G, H, W)\). Therefore \(A\) is a perfect code of \((G, H)\). Since \(x^{-1}H = xH\), we have \(x^2 \in H\). Recall that \(x \in A \setminus H\). Since \(hxH = xH\) for every \(h \in H\), we get \(x^{-1}hx \in H\) and it follows that \(x \in N_A(H) \setminus H\).

\(\Leftarrow\) Suppose that \(A\) is a perfect code of \((G, H)\) and there exists an element \(x \in N_A(H) \setminus H\) such that \(x^2 \in H\). Then there exists a coset graph \(\text{Cos}(G, H, W)\) on \(G/\ell H\) such that \(A/\ell H\) is a perfect code of \(\text{Cos}(G, H, W)\). Since \(x \in N_A(H) \setminus H\) and \(x^2 \in H\), we have \((hxH)^{-1} = HxH = xH = x^{-1}H\). Set \(U = W \cup HxH\). Then \(U^{-1} = U\) and \(A/\ell H\) is a total perfect code of \(\text{Cos}(G, H, U)\). Therefore \(A\) is a total perfect code of \((G, H)\). \(\square\)

### 3 Subgroup perfect codes of \((G, H)\)

In this section, we deduce some general results about subgroup perfect codes of a pair \((G, H)\). Our first result below gives a necessary and sufficient condition for a subgroup \(A\) of \(G\) containing \(H\) to be a perfect code of \((G, H)\).

**Theorem 3.1.** Let \(G\) be a group and \(H \leq A \leq G\). Then \(A\) is a perfect code of \((G, H)\) if and only if there exists a left transversal \(X\) of \(A\) in \(G\) such that \(XH = HX^{-1}\).

**Proof.** \(\Rightarrow\) Let \(A\) be a perfect code of \((G, H)\). Then there exists a coset graph \(\Gamma := \text{Cos}(G, H, U)\) such that \(A/\ell H\) is a perfect code of \(\Gamma\). By the definition of coset graph, we get \(H \cap U = \emptyset\) and \(U^{-1} = U\). Assume \(|G : A| = n\) and let \(T = \{1, t_1, \ldots, t_{n-1}\}\) be a left transversal of \(A\) in \(G\). Then \(t_i \notin A\) for any...
holds. Actually, we have the following theorem.

Then $A/\ell H$ is a perfect code of $\Gamma$, $A/\ell H$ is adjacent to a unique vertex in $A/\ell H$. Therefore there is a unique $a \ell H \in A/\ell H$ such that $t_i a_i \in U$. Set $X = \{1, t_1 a_1, \ldots, t_{n-1} a_{n-1}\}$. Then $X$ is a left transversal of $A$ in $G$ and $X \setminus \{1\} \subseteq U$. Since $U$ is a union of some double cosets of $H$ in $G$, we have $H(X \setminus \{1\}) H \subseteq U$. We will further prove $U \subseteq (X \setminus \{1\}) H$. Take an arbitrary $u \in U$. Since $T$ is a left transversal of $A$ in $G$, $u$ can be uniquely written as $u = t a$ where $t \in T$ and $a \in A$. It follows that $ta \in U$ and therefore $t^{-1} H$ and $a H$ are adjacent in $\Gamma$. Since $A/\ell H$ is a perfect code of $\Gamma$, $A/\ell H$ is an independent set of $\Gamma$. Therefore $t \notin A$. Since $T = \{1, t_1, \ldots, t_{n-1}\}$, we have $t = t_i$ for some $i \in \{1, \ldots, n-1\}$. By the uniqueness of $a_i H$, we have $a H = a_i H$ and then $u = t_i a_i h$ for some $h \in H$. Therefore $u \in (X \setminus \{1\}) H$ and it follows that $U \subseteq (X \setminus \{1\}) H$. Now we have proved that $H(X \setminus \{1\}) H \subseteq U \subseteq (X \setminus \{1\}) H$. Therefore $U = H(X \setminus \{1\}) H = (X \setminus \{1\}) H$. Since $U^{-1} = U$, we have $H(X \setminus \{1\})^{-1} = U^{-1} = U = (X \setminus \{1\}) H$ and it follows that $X H = H X^{-1}$.

$\Leftarrow$ Let $X$ be a left transversal of $A$ in $G$ such that $X H = H X^{-1}$. Then $H X H = H H X^{-1} = H X^{-1}$, $(H X H)^{-1} = H X^{-1} H = X H$ and it follows that $(H X H)^{-1} = H X H = X H = H X^{-1}$.

Since $X$ is a left transversal of $A$ in $G$, $X \cap A$ contains a unique element, say $y$. Since $X H = H X^{-1}$, for every $h \in H$, there exists $w \in X$ such that $h y^{-1} \in w H$. Since $H \leq A$ and $h y^{-1} \in A$, we have $w \in A$. Therefore $w = y$ and it follows that $H y^{-1} = H y$. Set $U := H(X \setminus \{y\}) H$. Since $(H X H)^{-1} = H X H = X H = H X^{-1}$ and $H y^{-1} = y H$, we have $U^{-1} = U = (X \setminus \{y\}) H = H(X \setminus \{y\})^{-1}$. Furthermore, $A \cap U = H \cap U = \emptyset$. In particular, we obtain a coset graph $\Gamma := \text{Cos}(G, H, U)$ on $G/\ell H$. Since $A \cap U = \emptyset$, $a^{-1} b \notin U$ for any $a, b \in A$ and it follows that $A/\ell H$ is an independent set of $\Gamma$. Now consider an arbitrary vertex $g H \in G/\ell H$ with $g \notin A$. Since $X$ is a left transversal of $A$ in $G$, $g^{-1}$ can be uniquely written as $g^{-1} = x a^{-1}$ where $x \in X$ and $a \in A$. Since $g^{-1} \notin A$, we obtain $x \neq y$. Therefore $x \in U$, that is, $g^{-1} a \in U$. It follows that $a H$ is adjacent to $g H$ in $A/\ell H$. If there is $b \in A$ such that $b H$ is also adjacent to $g H$ in $A/\ell H$, then $g^{-1} b \in U$. Since $U = (X \setminus \{y\}) H$, we have $g^{-1} b = z h$ for some $z \in X$ and $h \in H$. Therefore $g^{-1} = z H b^{-1}$. Note that $h b^{-1} \in A$. By the uniqueness of the factorization $g^{-1} = x a^{-1}$, we have $z = x$ and $a^{-1} = h b^{-1}$. Thus $a H = b H$ and it follows that $a H$ is the unique vertex in $A/\ell H$ which is adjacent to $g H$. Therefore $A/\ell H$ is a perfect code of $\Gamma$, that is, $A$ is a perfect code of $(G, H)$.  \hfill $\Box$

If we replace the word ‘left’ with ‘right’ in Theorem 3.3, this theorem still holds. Actually, we have the following theorem.

}\hfill $\Box$
Theorem 3.2. Let $G$ be a group and $H \leq A \leq G$. Then $A$ is a perfect code of $(G, H)$ if and only if there exists a right transversal $Y$ of $A$ in $G$ such that $Y^{-1}H = HY$.

Proof. By Theorem 3.1, $A$ is a perfect code of $(G, H)$ if and only if there exists a left transversal $X$ of $A$ in $G$ such that $XH = HX^{-1}$. Replacing $X^{-1}$ by $Y$, we have that $A$ is a perfect code of $(G, H)$ if and only if there exists a right transversal $Y$ of $A$ in $G$ such that $Y^{-1}H = HY$. □

Theorem 3.1 has several interesting corollaries. The first one below is obvious and we omit its proof.

Corollary 3.3. Let $G$ be a group and $H \leq A \leq G$. Let $X$ be a left transversal of $A$ in $G$. If $XH = HX^{-1}$, then $A/\ell H$ is a perfect code of the coset graph $\text{Cos}(G, H, U)$ where $U = H(X \setminus A)H$.

Corollary 3.4. Let $G$ be a group and $H \leq A \leq G$. If $A$ is a perfect code of $(G, H)$, then for any $g \in G$, $g^{-1}Ag$ is a perfect code of $(G, g^{-1}Hg)$.

Proof. By the necessity of Theorem 3.1, there exists a left transversal $X$ of $A$ in $G$ such that $XH = HX^{-1}$. Set $Y = g^{-1}Xg$. Then $Y$ is a left transversal of $g^{-1}Ag$ in $G$ and $Yg^{-1}Hg = g^{-1}HgY^{-1}$. By the sufficiency of Theorem 3.1, $g^{-1}Ag$ is a perfect code of $(G, g^{-1}Hg)$. □

Corollary 3.5. Let $G$ be a group and $H \leq A \leq L \leq G$. If $A$ is a perfect code of $(G, H)$, then $A$ is a perfect code of $(L, H)$.

Proof. Let $A$ be a perfect code of $(G, H)$. By Theorem 3.1, $A$ has a left transversal $X$ in $G$ such that $XH = HX^{-1}$. Therefore $G = XA$ and $|X \cap A| = 1$. Set $Y = X \cap L$. Since $A \leq L \leq G$, we have $L = G \cap L = XA \cap L = (X \cap L)A = YA$ and $|Y \cap A| = |X \cap A| = 1$. Therefore $Y$ is a left transversal of $A$ in $L$. Since $H \leq L$, we get $YH = (X \cap L)H = XH \cap LH = XH \cap L$. Since $XH = HX^{-1}$, it follows that $YH = XH \cap L = HX^{-1} \cap HL^{-1} = H(X^{-1} \cap L^{-1}) = HY^{-1}$. Therefore, by Theorem 3.1, $A$ is a perfect code of $(L, H)$. □

It is natural to consider the opposite of Corollary 3.5. The following theorem is a preliminary exploration of that.

Theorem 3.6. Let $G$ be a group admitting a normal subgroup $K$ and a subgroup $L$ such that $G = KL$ and $K \cap L = \{1\}$. Let $H \leq A \leq L$. Then $A$ is a perfect code of $(G, H)$ if and only if $A$ is a perfect code of $(L, H)$. 

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Proof. The necessity follows Corollary 3.5. Now we prove the sufficiency. Suppose that $A$ is a perfect code of $(L, H)$. By Theorem 3.1 there exists a left transversal $Y$ of $A$ in $L$ such that $YH = HY^{-1}$. Since $Y$ is a left transversal of $A$ in $L$, we have $L = YA$ and $|L| = |Y||A|$. Set $X = KY$. Since $G = KL$ and $K \cap L = \{1\}$, we have $G = KYA = XA$ and

$$|G| = |KL| = |K||L| = |K||YA| = |K||Y||A| = |KY||A| = |X||A|.$$  

Therefore $X$ is a left transversal of $A$ in $G$. Since $K$ is a normal subgroup of $G$ and $YH = HY^{-1}$, we have $XH = KYH = YHK = HY^{-1}K = HX^{-1}$. By Theorem 3.1, $A$ is a perfect code of $(G, H)$. \qed

We use $S_1 \cup S_2$ to denote the union of two disjoint sets $S_1$ and $S_2$, and $\bigcup_{i=1}^{m} S_i$ the union of pairwise disjoint sets $S_1, \ldots, S_m$. The following theorem generalizes the necessary part of [21] Theorem 3.1.

Theorem 3.7. Let $G$ be a group and $H \leq A \leq G$. If $A$ is a perfect code of $(G, H)$, then for any $g \in G$ either the left coset $gA$ contains an element $x$ such that $x^2 \in b^{-1}Hb$ for some $b \in A$ or $A\{g, g^{-1}\}A = \bigcup_{i=1}^{m} g_iA$ for some $g_1, \ldots, g_m \in G$ where $m$ is an even integer.

Proof. Suppose that $A$ is a perfect code of $(G, H)$. By Theorem 3.1, $A$ has a left transversal $T$ in $G$ such that $TH = HT^{-1}$.

Take an arbitrary $g \in G$. If $g \in A$, then $x \in gA$ and $x^2 \in H$ for each $x \in H$. Now assume $g \in G \setminus A$. It is obvious that $A\{g, g^{-1}\}A$ is a disjoint union of some left cosets of $A$ in $G$. Suppose that $A\{g, g^{-1}\}A = \bigcup_{i=1}^{m} g_iA$ for some $g_1, \ldots, g_m \in G$ where $m$ is an odd integer. It suffices to prove that $gA$ contains an element $x$ such that $x^2 \in b^{-1}Hb$ for some $b \in A$. Since $T$ is a left transversal of $A$ in $G$, there is a unique $x_i \in T$ such that $g_iA = x_iA$ for each $1 \leq i \leq m$. Set $X = \{x_1, x_2, \ldots, x_m\}$. Then $A\{g, g^{-1}\}A = \bigcup_{i=1}^{m} x_iA = XA$. Since $X \subseteq T$ and $H \leq A$, we get $XH = TH \cap XA$. Therefore $HX^{-1} = HT^{-1} \cap AX^{-1}$. Since $XH = AX^{-1}$ and it follows that $(HXH)^{-1} = HXH = XH$.

Let $Y$ be a subset of $X$ of minimal cardinality such that $HXH = HYH$. Then $HgY \cap Hg'Y = \emptyset$ for any pair of distinct elements $y, y' \in Y$. Since $(HYH)^{-1} = HYH$, we can set $Y = \{v_1, \ldots, v_k, w_1, \ldots, w_k, z_1, \ldots, z_l\}$ such that $(Hv_iH)^{-1} = Hw_iH$ and $(Hz_jH)^{-1} = Hz_jH$ for all $1 \leq i \leq k$ and $1 \leq j \leq \ell$. Set $V_i = Hv_iH \cap X$, $W_i = Hw_iH \cap X$ and $Z_j = Hz_jH \cap X$ for all $1 \leq i \leq k$ and $1 \leq j \leq \ell$. Then $X = (\bigcup_{i=1}^{k} V_i) \cup (\bigcup_{i=1}^{k} W_i) \cup (\bigcup_{j=1}^{\ell} Z_j)$. Since $HXH = XH$, we have $HXH = (\bigcup_{i=1}^{k} V_iH) \cup (\bigcup_{i=1}^{k} W_iH) \cup (\bigcup_{j=1}^{\ell} Z_jH)$. Then, since $V_i \subseteq Hv_iH$
and \((X \setminus V_i) \cap Hv_i = \emptyset\), we have \(Hv_i = V_i H\). Similarly, \(Hv_i = V_i H\), \(Hv_i = W_i H\) and \(HZ_j = Z_j H\) for all \(1 \leq i \leq k\) and \(1 \leq j \leq \ell\). Since \((V_i H)^{-1} = (Hv_i H)^{-1} = Hw_i H = W_i H\), we have \(|V_i H| = |W_i H|\) and it follows that \(|V_i| = |W_i|\). Therefore \(m = |X| = 2(\sum_{i=1}^k |V_i|) + \sum_{j=1}^\ell |Z_j|\). Since \(m\) is an odd integer, we have \(\ell \neq 0\). Note that the inequality \(\ell \neq 0\) ensure the existence of \(z_1\). Since \(Hz_1^{-1} = (Hz_1 H)^{-1} = Hz_1 H\), we have \(z_1^{-1} H = hz_1 H\) for some \(h \in H\). It follows that \((z_1 h)^2 \in H\). Since \(Az_1 H = Az_1^{-1} A\) and \(z_1 \in A\{g, g^{-1}\} A\), we have \(Az_1 A = AgA = Ag^{-1} A\). Therefore \(z_1 = b g c\) for some \(b, c \in A\). Set \(x = g c h b\). Then \(x \in g A\) and \(x^2 = gc h b g c h b c h b = b^{-1} h c b g c h b c h b = b^{-1} (z_1 h)^2 b \in b^{-1} H b\).

We leave it as an open problem whether the converse of Theorem 3.7 holds. Now we give two corollaries of Theorem 3.7.

**Corollary 3.8.** Let \(G\) be a group and \(H \leq A \leq G\). If there exists an element \(x \in G \setminus A\) such that \(x^2 \in A\), \(x A\) contains no element whose square is contained in a conjugate of \(H\) in \(A\) and \(|A : A \cap x A x^{-1}|\) is an odd integer, then \(A\) is not a perfect code of \((G, H)\).

**Proof.** Let \(x\) be an element in \(G \setminus A\) such that \(x^2 \in A\), \(x A\) contains no element whose square is contained in a conjugate of \(H\) in \(A\) and \(|A : A \cap x A x^{-1}|\) is an odd integer. Set \(|A : A \cap x A x^{-1}| = m\). Since \(ax A = bx A\) if and only if \(a^{-1} b \in x A x^{-1}\) for any \(a, b \in A\), we have that \(A x A = \bigcup_{i=1}^m g_i A\) for some \(g_1, \ldots, g_m \in G\). Since \(x^2 \in A\), we have \(A\{x, x^{-1}\} A = A x A = \bigcup_{i=1}^m g_i A\). Since \(m\) is an odd integer and \(x A\) contains no element whose square is contained in a conjugate of \(H\) in \(A\), it follows from Theorem 3.7 that \(A\) is not a perfect code of \((G, H)\).

**Corollary 3.9.** Let \(G\) be a group, \(H\) a subgroup of \(G\) and \(A\) a normal subgroup of \(G\) such that \(H \leq A \leq G\). If \(A\) is a perfect code of \((G, H)\), then for any \(x \in G\) with \(x^2 \in A\) there exists \(b \in A\) such that \((xb)^2 \in H\).

**Proof.** Let \(A\) be a normal subgroup of \(G\) and a perfect code of \((G, H)\). If \(x \in A\), then \((xb)^2 = 1 \in H\) where \(b = x^{-1} \in A\). Now consider an arbitrary element \(x \in G \setminus A\) with \(x^2 \in A\). Since \(A\) is normal in \(G\), we have \(A\{x, x^{-1}\} A = A x A = A x^{-1} A = x A\). By Theorem 3.7 \(x A\) contains an element \(y\) satisfying \(y^2 \in b_1^{-1} H b_1\) for some \(b_1 \in A\). Set \(y = xa\) where \(a \in A\). Since \(A\) is normal in \(G\), we have \(x^{-1} b_1 x \in A\). Set \(b = x^{-1} b_1 x a b_1^{-1}\). Then \(b \in A\). Since \((xb)^2 = (b_1 x a b_1^{-1})^2 = (b_1 y b_1^{-1})^2 = b_1 y^2 b_1^{-1}\) and \(y^2 \in b_1^{-1} H b_1\), we have \((xb)^2 \in H\).

The following result is a generalization of [21, Theorem 3.7 (i)].
Theorem 3.10. Let $G$ be a group and $H \leq A \leq G$. Let $N$ be a normal subgroup of $G$ which is contained in $A$. If $A$ is a perfect code of $(G, H)$, then $A/N$ is a perfect code of $(G/N, H/N)$.

Proof. Suppose that $A$ is a perfect code of $(G, H)$. By Theorem 3.1, there exists a left transversal $X$ of $A$ in $G$ such that $XH = HX^{-1}$. Since $N$ is a normal subgroup of $G$ and $N \leq A$, $X/N$ is a left transversal of $A/N$ in $G/N$. Since $XH = HX^{-1}$, we have $(X/N)(H/N) = XH/N = HX^{-1}/N = (H/N)(X^{-1}/N)$. By Theorem 3.1, $A/N$ is a perfect code of $(G/N, H/N)$.

4 Subgroup total perfect codes of $(G, H)$

In this section, we deduce a few results on total perfect codes which are parallel to some results about perfect codes we obtained in Section 3.

Theorem 4.1. Let $G$ be a group and $H \leq A \leq G$. Then $A$ is a total perfect code of $(G, H)$ if and only if there exists a left transversal $X$ of $A$ in $G$ such that $XH = HX^{-1}$ and $X$ contains an element in $A \setminus H$.

Proof. $\Rightarrow$ Suppose that $A$ is a total perfect code of $(G, H)$. By Lemma 2.4, $A$ is a perfect code of $(G, H)$ and there exists an element $x \in N_A(H) \setminus H$ such that $x^2 \in H$. Then, by Theorem 3.1, there exists a left transversal $Y$ of $A$ in $G$ such that $YH = HY^{-1}$. Since $x \in N_A(H) \setminus H$ and $x^2 \in H$, we have $xH = Hx^{-1}$. Since $Y$ a left transversal of $A$, $Y \cap A$ contains a unique element, say $y$. If $y \in A \setminus H$, then we set $X = Y$. If $y \in H$, then we set $X = (Y \setminus \{y\}) \cup \{x\}$. In both cases, we have $XH = HX^{-1}$ and $X$ contains an element in $A \setminus H$.

$\Leftarrow$ Suppose that there exists a left transversal $X$ of $A$ in $G$ such that $XH = HX^{-1}$ and $X$ contains an element in $A \setminus H$. By Theorem 3.1, $A$ is a perfect code of $(G, H)$. Since $X$ is a left transversal of $A$, $X \cap A$ contains a unique element. Set $X^{'} \cap A = \{x\}$ and $T = X \setminus \{x\}$. Since $H \leq A$, we have $xH$ and $Hx^{-1}$ are both contained in $A$. Therefore $TH \cap xH = TH \cap Hx^{-1} = \emptyset$. Then, since $XH = HX^{-1}$, we get $xH = Hx^{-1}$. Thus $x^2 \in H$ and $x \in N_A(H)$. By Lemma 2.4, $A$ is a total perfect code of $(G, H)$.

Similar to Theorem 3.1, the word ‘left’ in Theorem 4.1 can be replaced by ‘right’. The proof of the following theorem is the same as that of Theorem 3.2 and therefore omitted.

Theorem 4.2. Let $G$ be a group and $H \leq A \leq G$. Then $A$ is a total perfect code of $(G, H)$ if and only if there exists a right transversal $Y$ of $A$ in $G$ such that $Y^{-1}H = HY$ and $Y$ contains an element in $A \setminus H$. 

The following theorem is a parallel result to Corollary 3.5.

**Theorem 4.3.** Let $G$ be a group and $H \leq A \leq L \leq G$. If $A$ is a total perfect code of $(G, H)$, then $A$ is a total perfect code of $(L, H)$.

**Proof.** Let $A$ be a total perfect code of $(G, H)$. By Lemma 2.4, $A$ is a perfect code of $(G, H)$ and there exists an element $x \in N_A(H) \setminus H$ such that $x^2 \in H$. Since $H \leq A \leq L \leq G$, it follows from Corollary 3.5 that $A$ is a perfect code of $(L, H)$. Then, since there exists an element $x \in N_A(H) \setminus H$ such that $x^2 \in H$, Lemma 2.4 ensures that $A$ is a total perfect code of $(L, H)$.

The following theorem is the counterpart of Theorem 3.6 for total perfect codes. We omit its proof as it can be proceed by using a similar approach to the proof of Theorem 4.3.

**Theorem 4.4.** Let $G$ be a group admitting a normal subgroup $K$ and a subgroup $L$ such that $G = KL$ and $K \cap L = \{1\}$. Let $H \leq A \leq L$. Then $A$ is a total perfect code of $(G, H)$ if and only if $A$ is a total perfect code of $(L, H)$.

## 5 Examples and problems

In this section, we construct some examples and propose a few open problems.

**Problem 5.1.** Whether the converse of Theorem 3.7 holds? If it does not hold, then what conditions should we add to make it true?

For a normal subgroup $N$ of a given group $G$, it is straightforward to check that every coset graph $\text{Cos}(G, N, U)$ is a Cayley graph on the quotient group $G/N$. Therefore for every subgroup $A$ of $G$ containing $N$, $A$ is a perfect code of $(G, N)$ if any only if $A/N$ is perfect code of $G/N$. By using this fact and Theorem 3.6 we construct an infinite family of perfect codes as follows.

**Example 5.2.** Let $G$ be the one dimensional affine group over a finite field $F$, and $L$ be the subgroup of $G$ consisting of elements in $G$ fixing the additive identity of $F$. Let $H \leq A \leq L$. Suppose that either the index $|L : A|$ of $A$ in $L$ is odd or the index $|A : H|$ of $H$ in $A$ is odd. Then $A$ is perfect code of $(G, H)$.

**Proof.** By [5, Example 3.4.1], $G$ is a Frobenius group, $L$ is a Frobenius complement of $G$ and $L$ is isomorphic to the cyclic group of order $|F| - 1$. In particular, $H$ is normal in $L$ and the quotient group $L/H$ is cyclic. Since either $|L : A|$ or $|A : H|$ is odd, either $A/H$ is of odd order or $A/H$ is of odd index in $L/H$. By
Corollary 2.8, \( A/H \) is a perfect code in \( L/H \). Therefore \( A \) is perfect code of \((L, H)\). Let \( K \) be the Frobenius kernel of \( G \). Then \( G = KL, K \cap L = \{1\} \) and \( K \) is normal in \( G \). By Theorem 3.6, \( A \) is perfect code of \((G, H)\).

Let \( G \) be a group and \( H \leq A \leq G \). If there exists a left transversal \( X \) of \( A \) in \( G \) such that \( XH = HX^{-1} \), then we call \((A, H, X)\) a perfect triple of \( G \). By Theorem 3.1, \( A \) is a perfect code of \((G, H)\) if and only if there exists a subset \( X \) of \( G \) such that \((A, H, X)\) is a perfect triple of \( G \). We propose the following problems for further research.

**Problem 5.3.** Given a group \( G \) and its subgroup \( H \), construct or classify the perfect triples \((A, H, X)\) of \( G \).

**Problem 5.4.** Given a group \( G \) and its subgroup \( A \), classify the perfect triples \((A, H, X)\) of \( G \).

Let \( S_n \) be the symmetric group on \( \{1, 2, \ldots, n\} \). For two positive integers \( m \) and \( n \) with \( m < n \), we treat \( S_m \) as the subgroup of \( S_n \) consisting of elements in \( S_n \) fixing every number in \( \{m + 1, \ldots, n\} \).

**Example 5.5.** Set \( X = \{1, (2, 5, 3), (1, 3, 5), (1, 5, 2, 3), (4, 5)(1, 3, 2)\} \). Then one can check that \((S_4, S_3, X)\) is a perfect triple of \( S_5 \). Therefore \( S_4 \) is perfect code of \((S_5, S_3)\).

**Proposition 5.6.** Let \( n \) be a positive integer at least 5. Then \( S_{n-1} \) is a perfect code of \((S_n, S_3)\).

**Proof.** We proceed the proof by induction on \( n \). By Example 5.5, the proposition is true for \( n = 5 \). Suppose that the proposition is true for \( n = k \) \( (k \geq 5) \). It suffices to prove that the proposition is true for \( n = k + 1 \). Let \( G \) be the subgroup of \( S_{k+1} \) consisting of elements in \( S_{k+1} \) fixing \( k \). Since \( k \geq 5 \), we have \( S_3 < S_{k-1} < G \). Let \( z \) be the involution \((k, k + 1)\) in \( S_{k+1} \). Then \( z^{-1}S_kz = G \), \( z^{-1}S_3z = S_3 \) and \( z^{-1}S_{k-1}z = S_{k-1} \). By induction hypothesis, \( S_{k-1} \) is a perfect code of \((S_k, S_3)\). Therefore, by Corollary 3.3, \( S_{k-1} \) is a perfect code of \((G, S_3)\). By Theorem 3.1 there exists a left transversal \( X \) of \( S_{k-1} \) in \( G \) such that \( XS_3 = S_3X^{-1} \). Set \( Y = X \cup \{z\} \). Since \( XS_3 = S_3X^{-1} \) and \( z^{-1}S_3z = S_3 \), we obtain \( YS_3 = S_3Y^{-1} \). We will further prove that \( Y \) is a left transversal of \( S_k \) in \( S_{k+1} \).

Since \( X \) is a left transversal of \( S_{k-1} \) in \( G \), \( x^{-1}y \notin S_{k-1} \) for each pair of distinct elements \( x, y \in X \). Therefore \( x^{-1}y \) does not fix \( k+1 \) and it follows that \( x^{-1}y \notin S_k \). Thus \( xS_k \neq yS_k \). Since \( z = (k, k + 1) \) and \( x^{-1}z \) fixes \( k \) for every \( x \in X \), \( x^{-1}z \) takes \( k+1 \) to \( k \). Therefore \( x^{-1}z \) does not fix \( k+1 \) and it follows that \( x^{-1}z \notin S_k \). Thus \( xS_k \neq zS_k \).
From the above discussion, we have that $xS_k \neq yS_k$ for each pair of distinct elements $x, y \in Y$. Therefore $|YS_k| = |Y||S_k| = (k + 1)! = |S_{k+1}|$. Thus $YS_k = S_{k+1}$ and it follows that $Y$ is a left transversal of $S_k$ in $S_{k+1}$.

Now we have proved that $Y$ is a left transversal of $S_k$ in $S_{k+1}$ and $YS_3 = S_3Y^{-1}$. By Theorem 3.1 $S_k$ is a perfect code of $(S_{k+1}, S_3)$. In other word, the proposition is true for $k + 1$. Therefore the proposition is true for all positive integer $n \geq 5$.

By the proof of Proposition 5.6 if $(S_{k-1}, S_3, X)$ is a perfect triple of $S_k$, then $(S_k, S_3, z^{-1}Xz \cup \{z\})$ is a perfect triple of $S_{k+1}$ where $z = (k, k+1)$. This provides us with a way to construct perfect triples $(S_{n-1}, S_3, X)$ of $S_n$ for every $n \geq 5$. Based on Example 5.5 the following two examples are constructed through this approach.

**Example 5.7.** Set

$$X = \{1, (2, 6, 3), (1, 3, 6), (1, 6, 2, 3), (4, 6)(1, 3, 2), (5, 6)\}.$$  

Then $(S_5, S_3, X)$ is a perfect triple of $S_6$.

**Example 5.8.** Set

$$X = \{1, (2, 7, 3), (1, 3, 7), (1, 7, 2, 3), (4, 7)(1, 3, 2), (5, 7), (6, 7)\}.$$  

Then $(S_6, S_3, X)$ is a perfect triple of $S_7$.

More generally, we have the following proposition.

**Proposition 5.9.** Let $n \geq 3$ be a positive integer. Set

$$X = \{1, (2, n, 3), (1, 3, n), (1, n, 2, 3), (4, n)(1, 3, 2), (5, n), \ldots, (n - 1, n)\}.$$  

Then $(S_{n-1}, S_3, X)$ is a perfect triple of $S_n$.

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