Co-competition numbers of graphs

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Abstract

In this paper, we introduce the notion of co-competition number \( p \) of a graph and show that the competition number \( k(G) \) and the co-competition number \( \tilde{k}(G) \) of a graph \( G \) are related as \( k(G) \geq \theta_e(G) - |V(G)| + \tilde{k}(G) \). As a matter of fact, this generalizes the inequality \( k(G) \geq \theta_e(G) - |V(G)| + 2 \) given by Opsut [12] as \( \tilde{k}(G) \geq 2 \) for a graph with at least one edge. In the process, we introduce a notion of effective competition cover, which is possessed by a large family of graphs, and show that our inequality actually becomes an equality for a graph having an effective competition cover and the competition number of a certain graph having an effective competition cover may be obtained by utilizing its co-competition number. Especially, we present a precise relationship between the competition number and the co-competition number of a diamond-free plane graph.

Keywords: competition graph, competition number, edge clique cover number, effective competition cover, co-competition number, plane graph

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1 Introduction

The competition graph of a digraph \( D \), denoted by \( C(D) \), is defined as a graph which has the same vertex set as \( D \) and has an edge \( xy \) between two distinct vertices \( x \) and \( y \) if and only if, for some vertex \( z \in V(D) \), the arcs \( (x, z) \) and \( (y, z) \) are in \( D \) (see [9][10][14][16][17] for reference). The notion of competition graphs is due to Cohen [8] and arose from ecology. Competition graphs also have applications in areas such as coding, radio transmission, and modeling of complex economic systems.

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For all undefined graph theoretical terms, see [1].

Roberts [15] observed that any graph $G$ together with $|E(G)|$ additional isolated vertices is the competition graph of an acyclic digraph. Then he defined the competition number of a graph $G$ to be the smallest number $k$ such that $G$ together with $k$ isolated vertices is the competition graph of an acyclic digraph, and denoted it by $k(G)$.

Computing the competition number of a graph is one of the important problems in the field of competition graphs. Yet, computing the competition number of a graph is usually not easy as Opsut has shown that computation of the competition number in general is NP-hard in 1982. While an upper bound $M$ of the competition number of a graph $G$ may be obtained by constructing an acyclic digraph whose competition graph is $G$ together with $M$ isolated vertices, getting a good lower bound is a very difficult task because there are usually so many cases to consider.

There has been much effort to compute the competition numbers of graphs (for some results on competition numbers, see [2, 4–8, 11–13, 18–20]).

The edge clique cover number of a graph is closely related to its competition number. For a clique $K$ and an edge $e$ of a graph $G$, we say that $K$ covers $e$ (or $e$ is covered by $K$) if and only if $K$ contains the two end points of $e$. An edge clique cover of a graph $G$ is a collection of cliques that cover all the edges of $G$. The edge clique cover number of a graph $G$, denoted by $\theta_e(G)$, is the smallest number of cliques in an edge clique cover of $G$. Opsut [12] showed that $k(G) \geq \theta_e(G) - |V(G)| + 2$ for any graph $G$.

In this paper, we introduce the notion of the co-competition number of a graph. Given a graph $G$, let $\tilde{D}(G)$ be the set of acyclic digraphs the competition graph of each of which is $G$ together with $k(G)$ isolated vertices, that is,

$$\tilde{D}(G) = \{ D \mid D \text{ is acyclic and } C(D) \text{ is } G \text{ together with } k(G) \text{ isolated vertices} \}.$$  

**Definition 1.1.** Let $G$ be a graph. Among the numbers of vertices of indegree 0 in the digraphs in $\tilde{D}(G)$, we call the maximum the co-competition number of $G$ and denote it by $\tilde{k}(G)$, that is,

$$\tilde{k}(G) = \max \{ i(D) \mid i(D) \text{ denotes the number of vertices of indegree 0 in } D \in \tilde{D}(G) \}.$$  

We say that $\tilde{k}(G)$ is attained by $D$ if $\tilde{k}(G) = i(D)$.

For a graph $G$, the number of vertices with indegree 0 in a digraph belonging to $\tilde{D}(G)$ is less than or equal to $|V(G)|$. Thus $\tilde{k}(G)$ is finite.

We show that the competition number $k(G)$ and the co-competition number $\tilde{k}(G)$ of a graph $G$ are related as $k(G) \geq \theta_e(G) - |V(G)| + \tilde{k}(G)$. As a matter of fact, this generalizes the inequality $k(G) \geq \theta_e(G) - |V(G)| + 2$ given by Opsut [12] as $\tilde{k}(G) \geq 2$ for a graph with at least one edge. By the way, Sano [18] gave a lower bound for the competition number, which generalizes the inequality given by Opsut in a viewpoint different from ours.
Our inequality becomes an equality and the competition number may be computed in terms of the co-competition number for the graphs having “effective competition covers” such as chordal graphs and diamond-free graphs with at least one edge.

Every graph in this paper is assumed to be simple unless otherwise stated.

2 Effective competition covers of graphs

In this section, we introduce the notion of effective competition covers of graphs.

Definition 2.1. Let $G$ be a graph with at least one edge. A minimum edge clique cover $C := \{C_1, \ldots, C_{\theta_e(G)}\}$ of $G$ is called an effective competition cover of $G$ if every clique in $C$ is maximal in $G$ and there exist an acyclic digraph $D \in \tilde{D}(G)$ satisfying the following property.

§ In $D$, there exist vertices $w_1, \ldots, w_{\theta_e(G)}$ such that $w_1, \ldots, w_{\theta_e(G)}$ are the only vertices of indegree nonzero in $D$ and $w_i$ is a common out-neighbor of all the vertices in $C_i$ for each $i = 1, \ldots, \theta_e(G)$.

For an effective competition cover $C$ and a digraph $D$ in Definition 2.1, we say that $D$ is a digraph accompanying $C$. By definition, $w_i$ in the property (§) is not an out-neighbor of any vertex not in $C_i$ for each $i = 1, \ldots, \theta_e(G)$. In this vein, we call $w_i$ in the property (§) of Definition 2.1 the sink of $C_i$ in $D$ for each $i = 1, \ldots, \theta_e(G)$.

The graph $G$ given in Figure 1 has an effective competition cover. The competition number of $G$ is one and the competition graph of $D$ in Figure 1 is $G \cup \{v_0\}$. Now consider the family $C = \{C_1 = \{v_8, v_9\}, C_2 = \{v_7, v_8\}, C_3 = \{v_6, v_7\}, C_4 = \{v_4, v_5, v_6, v_9\}, C_5 = \{v_3, v_4, v_5\}, C_6 = \{v_1, v_2, v_4, v_9\}\}$ of maximal cliques of $G$. It can easily be checked that $C$ is a minimum edge clique cover of $G$. Moreover, the $i$th term of $(v_7, v_6, v_5, v_3, v_2, v_0)$ is a common out-neighbor of all the vertices in $C_i$ for $i = 1, \ldots, 6$ and those terms are the only vertices of indegree nonzero in $D$.

Since an effective competition cover $C$ of a graph is a minimum edge clique cover, the sinks of cliques belonging to $C$ in a digraph accompanying $C$ are all distinct.

A perfect elimination ordering of a graph $G$ with $N$ vertices is an ordering $[v_1, v_2, \ldots, v_n]$ of the vertices of $G$ such that the neighborhood of $v_i$ is a clique in $G_i := G[v_1, v_i+1, \ldots, v_n]$ for each $i = 1, \ldots, n - 1$. It is well-known that every chordal graph has a perfect elimination ordering.

It is also well known that a digraph $D$ is acyclic if and only if there exists a bijection $\ell : V(D) \to \{1, 2, \ldots, |V(D)|\}$ such that whenever there is an arc from a vertex $u$ to a vertex $v$, $\ell(u) > \ell(v)$. We call such a function $\ell$ an acyclic labeling of $D$.

Given a graph $G$ and a minimum edge clique cover $C$, we may expand each clique in $C$ to a maximal clique. Therefore any graph has a minimum edge clique cover consisting of maximal cliques.
Given a graph $G$ and a vertex $v$ of $G$, we denote by $N_G[v]$ (resp. $N_G(v)$) the closed neighborhood (resp. open neighborhood) of $v$ in $G$.

Now we present some sufficient conditions under which a graph has an effective competition cover.

**Theorem 2.2.** Every chordal graph with at least one edge has an effective competition cover.

**Proof.** Take a chordal graph $G$ with at least one edge. If a graph $H$ has an effective competition cover without having isolated vertices, then $H$ together with isolated vertices still has an effective competition cover. In this context, we may assume that $G$ has no isolated vertices. Since $G$ is chordal, there exists a perfect elimination ordering $[v_1, v_2, \ldots, v_n]$ of $G$. Let $G_i = G[v_1, v_{i+1}, \ldots, v_n]$, and $N_i = N_{G_i}[v_i]$ for each $i = 1, 2, \ldots, n - 1$. In addition, let $\theta = \theta_e(G)$ and $\mathcal{C} = \{C_1, \ldots, C_\theta\}$ be a minimum edge clique cover of $G$ consisting of maximal cliques.

Given a subset $X = \{v_{r_1}, v_{r_2}, \ldots, v_{r_j}\}$ of $V(G)$ with $r_1 < r_2 < \cdots < r_j$ for some positive integer $j$, we may correspond the ordered $j$-tuple $\mathbf{a}(X) := (r_1, r_2, \ldots, r_j)$. We rearrange $C_1, \ldots, C_\theta$ so that $\mathbf{a}(C_1) \prec \mathbf{a}(C_2) \cdots \prec \mathbf{a}(C_\theta)$ where $\prec$ is the lexicographic order.

To show that $\mathcal{C}$ consists of some elements in $\{N_1, N_2, \ldots, N_{n-1}\}$, fix $l \in \{1, \ldots, \theta\}$ and let $n_l = \min\{j \mid v_j \in C_l\}$. By the choice of $n_l$,

$$C_l \subset V(G_{n_l}).$$

(1)

Thus $C_l$ is a clique in $G_{n_l}$ containing $v_{n_l}$. By the definition of perfect elimination ordering, $v_{n_l}$ is a simplicial vertex of $G_{n_l}$ and so $C_l \subset N_{n_l}$. However, since $C_l$ is a maximal clique in $G$, it is also a maximal clique in $G_{n_l}$. Therefore $C_l = N_{n_l}$ as $N_{n_l}$ is a clique in $G_{n_l}$.
and so $C$ consists of some elements in \{$N_1, N_2, \ldots, N_{n-1}$\}. Thus $C = \{N_{n_1}, \ldots, N_{n_\theta}\}$. Note that $n_1 = 1$. To see why, recall our assumption that $G$ has no isolated vertices. Since $C_1, \ldots, C_\theta$ are arranged based upon the lexicographic order of $a(C_1), \ldots, a(C_\theta)$, $v_1$ belongs to $C_1$.

Let $D$ be a digraph with $V(D) = V(G) \cup \{v_0\}$ and

$$A(D) = \bigcup_{l=1}^{\theta}\{(v, v_{n_l-1}) \mid v \in N_{n_l}\}.$$ 

Now the following are true:

$$v_iv_j \in E(G) \iff v_iv_j \in N_{n_l} \text{ for some } l \in \{1, \ldots, \theta\}$$

$$\iff (v_i, v_{n_l-1}) \in A(D) \text{ and } (v_j, v_{n_l-1}) \in A(D) \text{ for some } l \in \{1, \ldots, \theta\}$$

$$\iff v_iv_j \in E(C(D)).$$

Thus the competition graph of $D$ is $G$ together with the isolated vertex $v_0$. To show that $D$ is acyclic, take $(v_i, v_j) \in A(D)$. Then $v_i \in N_{n_l}$ and $j = n_l - 1$ for some $l \in \{1, \ldots, \theta\}$. Now, since $N_{n_l} = C_l$, by (1), $N_{n_l} \subset V(G_{n_l}) = \{v_{n_l}, v_{n_l+1}, \ldots, v_n\}$, which implies $i \geq n_l$. Thus $i > j$ and so $D$ is acyclic. By the definition of $D$, $v_{n_1-1}, \ldots, v_{n_\theta-1}$ are the only vertices of indegree nonzero in $D$ and $C$ satisfies (§) of Definition 2.1. This completes the proof.

Remark 2.3. In the proof given above, we have actually shown a stronger statement that every minimum edge clique cover of a chordal graph consisting of maximal cliques is an effective competition cover.

Given a maximal clique $C$ of a graph $G$, an edge of $G$ is said to be occupied by $C$ if $C$ is the only maximal clique that covers it.

Theorem 2.4. Suppose that a graph $G$ with at least one edge satisfies the property that, for each maximal clique $C$, each vertex of $C$ is an end vertex of an edge occupied by $C$. Then $G$ has an effective competition cover.

Proof. Let $C$ be the set of all maximal cliques of $G$. We first show that $C$ is a minimum edge clique cover. Obviously $C$ is an edge clique cover of $G$. Let $C^*$ be a minimum edge clique cover of $G$ consisting of maximal cliques. Then clearly $C^* \subset C$. By the hypothesis, every clique in $C$ has an edge occupied by it. Since an edge occupied by a maximal clique cannot be covered by any other maximal cliques, $C$ with an element omitted no longer covers the edges of $G$. Therefore $C^* = C$ and so $C$ is a minimum edge clique cover.

Let $D$ be a digraph in $\overline{D}(G)$ and $\ell$ be an acyclic labeling of $D$.

Take a maximal clique $C$ in $C$. Then the end vertices of each of the edges occupied by $C$ has a common out-neighbor in $D$. We consider such common out-neighbors and take one, say $x_C$, with the smallest $\ell$-value among them. Since $x_C$ is a common out-neighbor
of the end vertices of an edge occupied by \( C \) in \( D \), \( x_C \neq x_{C'} \) if \( C \neq C' \) for \( C, C' \in \mathcal{C} \). We consider the digraph \( D^* \) with the vertex set \( V(D) \) and the arc set

\[
\bigcup_{C \in \mathcal{C}} \{(v, x_C) \mid v \in C\}.
\]

Since \( x_C \neq x_{C'} \) for distinct \( C, C' \in \mathcal{C} \), the competition graph of \( D^* \) is \( G \) together with \( k(G) \) isolated vertices. Therefore it remains to show that \( D^* \) is acyclic in order to prove \( D^* \in \overline{D}(G) \).

Take an arc in \( D^* \). Then it is in the form of \((v, x_C)\) for some \( C \in \mathcal{C} \) and \( v \in C \). By the hypothesis, \( v \) is an end vertex of an edge occupied by \( C \). Then \( v \) and the other end of that edge have a common out-neighbor \( x \) in \( D \). Since \( \ell \) is an acyclic labeling of \( D \), \( \ell(x) < \ell(v) \). By the choice of \( x_C \), \( \ell(x_C) \leq \ell(x) \), so \( \ell(x_C) < \ell(v) \). Thus \( \ell \) is still an acyclic labeling of \( D^* \) and so \( D^* \) is acyclic. Hence \( D^* \in \overline{D}(G) \). By the definition of \( D^* \), it is obvious that \( \mathcal{C} \) satisfies (5) of Definition 2.1 and the theorem statement holds.

A diamond is a graph obtained from \( K_4 \) by deleting an edge. A graph is called diamond-free if it does not contain a diamond as an induced subgraph. It is easy to see that a graph is diamond-free if and only if no two of its maximal cliques cover a common edge.

**Corollary 2.5.** Every diamond-free graph with at least one edge has an effective competition cover.

**Proof.** If \( G \) is a diamond-free graph with at least one edge, then each edge of \( G \) is occupied by a maximal clique. Hence the corollary immediately follows from Theorem 2.4. \( \square \)

**Proposition 2.6.** Let \( G \) be a graph with at least one edge having no isolated vertices and let \( \mathcal{C} \) be a minimum edge clique cover of \( G \) consisting of maximal cliques. Suppose that \( \mathcal{C} \) contains a subset \( \mathcal{F} \) having \( \theta_e(G) - k(G) + 1 \) elements such that the subgraph \( H \) induced by the edges covered by \( \mathcal{F} \) has at least \( \theta_e(G) - k(G) \) simplicial vertices. Then \( \mathcal{C} \) is an effective competition cover of \( G \).

**Proof.** We label the elements in \( \mathcal{F} \) as \( C_{k(G)}, C_{k(G)+1}, \ldots, C_{\theta_e(G)} \) so that the number of simplicial vertices in \( H \) belonging to \( C_i \) is greater than or equal to the number of simplicial vertices in \( H \) belonging to \( C_j \) if \( i \geq j \). Then we label the elements in \( \mathcal{C} \setminus \mathcal{F} \) as \( C_1, \ldots, C_{k(G)-1} \). We construct a digraph \( D \) whose vertex set is \( V(G) \) together with added isolated vertices \( a_1, \ldots, a_{k(G)} \) as follows. For each \( i = 1, \ldots, k(G) \), we add arcs from every vertex in \( C_i \) to \( a_i \). By the hypothesis, \( \theta_e(G) \geq k(G) \). If \( \theta_e(G) = k(G) \), then \( D \) is obviously a digraph accompanying \( \mathcal{C} \). Therefore we may assume \( \theta_e(G) > k(G) \). Then, by the way in which the cliques in \( \mathcal{F} \) are labeled, \( C_{k(G)} \) has at least one simplicial vertex in \( H \). We take one of them and add arcs from every vertex in \( C_{k(G)+1} \) to that vertex. Again, we may take a simplicial vertex in \( H \) from \( C_{k(G)} \cup C_{k(G)+1} \) that is distinct from the one just

6
taken. Then we add arcs from every vertex in $C_{k(G)+2}$ to that vertex. We may continue in this way until we add arcs from every vertex in $C_{\theta_e(G)}$ to a simplicial vertex in $H$ that belongs to $\bigcup_{i=k(G)}^{\theta_e(G)-1} C_i$. By construction, each arc goes either from a vertex in $C_j$ to a newly added isolated vertex or from a vertex in $C_j$ to a simplicial vertex of $H$ in $C_i$ for some $i, j \in \{k(G), \ldots, \theta_e(G)\}$ with $j > i$. As a simplicial vertex of $H$ belonging to $C_i$ does not belong to $C_j$ if $j > i$ for $i, j \in \{k(G), k(G)+1, \ldots, \theta_e(G)\}$, $D$ is acyclic. Furthermore, $D$ is obviously a digraph accompanying $C$, so $C$ is an effective competition cover of $G$. □

3 Relationship between the competition number and the co-competition number of a graph

In this section, we give a new lower bound for the competition number of a graph in terms of its co-competition number. In addition, we show that the graphs having effective competition covers have the lower bound as their competition numbers.

**Proposition 3.1.** For any graph $G$ with at least two vertices, $\tilde{k}(G) \geq 2$

*Proof.* Let $G$ be a graph with at least two vertices. In $\tilde{D}(G)$, we take a digraph $D$ with arcs as few as possible. Since $G$ has at least two vertices, $n := |V(D)| \geq 2$. Let $\ell$ be an acyclic labeling of $D$. We denote by $v_1$ the vertex $v$ satisfying $\ell(v) = 1$. By definition of acyclic labeling, $v_n$ is of indegree 0. Suppose that the indegree of $v_{n-1}$ is nonzero. By definition of acyclic labeling, $v_n$ is the only in-neighbor of $v_{n-1}$, which implies that $v_{n-1}$ does not induce any edge in $G$ as a common out-neighbor of two vertices. Therefore we may delete the arc $(v_n, v_{n-1})$ to obtain an acyclic digraph in $\tilde{D}(G)$, which contradicts the choice of $D$. Thus $v_{n-1}$ is of indegree 0. Hence $\tilde{k}(G) \geq 2$. □

**Theorem 3.2.** Let $G$ be a graph with an effective competition cover $C$. Then the cliques in $C$ can be labeled as $C_1, \ldots, C_{\theta_e(G)}$ so that $\tilde{k}(G) \geq \left| \bigcup_{i=k}^{\theta_e(G)} C_i \right| - \theta_e(G) + k$ for any $1 \leq k \leq \theta_e(G)$.

*Proof.* Let $D$ be a digraph accompanying $C$. We consider the subdigraph $D'$ of $D$ induced by the sinks of the cliques belonging to $C$ in $D$. Obviously $D'$ is acyclic, so we may label the vertices of $D'$ as $w_1, \ldots, w_{\theta_e(G)}$ so that

$$(w_j, w_i) \text{ is an arc only if } i < j.$$  \hfill (⋆)

Now we label the cliques in $C$ as $C_1, \ldots, C_{\theta_e(G)}$ so that $w_i$ is the sink of $C_i$ for each $i = 1, \ldots, \theta_e(G)$.

Fix $1 \leq k \leq \theta_e(G)$ and suppose that $\bigcup_{i=k}^{\theta_e(G)} C_i \setminus \{w_{k+1}, \ldots, w_{\theta_e(G)}\}$ contains a sink $w_j$ for some $j \in \{1, \ldots, k\}$. Then $w_j \in C_i$ for some $i \in \{k, k+1, \ldots, \theta_e(G)\}$. Since $w_i$ is the sink of $C_i$ and $w_j \in C_i$, there is an arc from $w_j$ to $w_i$ and so, by (⋆), $i < j$, which is a contradiction. Therefore $\bigcup_{i=k}^{\theta_e(G)} C_i \setminus \{w_{k+1}, \ldots, w_{\theta_e(G)}\}$ does not contain a sink, that
is, every vertex in $\bigcup_{i=k}^{\theta_e(G)} C_i \setminus \{w_{k+1}, \ldots, w_{\theta_e(G)}\}$ has indegree 0 in $D$. By the definition of $\tilde{k}(G)$,

$$\tilde{k}(G) \geq \bigcup_{i=k}^{\theta_e(G)} C_i \setminus \{w_{k+1}, \ldots, w_{\theta_e(G)}\} \geq \bigcup_{i=k}^{\theta_e(G)} C_i - \theta_e(G) + k.$$

and the theorem follows. \hfill \Box

Opsut [12] showed that, for any graph $G$, $k(G) \geq \theta_e(G) - |V(G)| + 2$. We generalize this inequality. To do so, we need the following theorem.

**Theorem 3.3.** Let $G$ be a graph with at least one edge and $D$ be a digraph in $\tilde{D}(G)$. Then $D$ has at least $\theta_e(G)$ vertices of indegree nonzero.

**Proof.** Let $C$ be a minimum edge clique cover of $G$ consisting of maximal cliques $C_1, \ldots, C_{\theta_e(G)}$. We define

$$A_i := \{v \in V(D) \mid v \text{ is a common out-neighbor of two vertices in } C_i\}$$

for each $i = 1, \ldots, \theta_e(G)$. Since $C$ is an edge clique cover of $G$, $G[C_i]$ contains at least one edge whose two end vertices, therefore, has a common out-neighbor in $D$ and so $A_i \neq \emptyset$ for each $i = 1, \ldots, \theta_e(G)$.

Let $B = (X, Y)$ be a bipartite graph, where $X = \{A_1, A_2, \ldots, A_{\theta_e(G)}\}$ and $Y = V(D)$, such that, for $A_i \in X$ and $v \in Y$, $\{A_i, v\}$ is an edge of $B$ if and only if $v \in A_i$. By definition, $A_i = N_B(A_i)$ for each $i = 1, \ldots, \theta_e(G)$. To show that $B$ satisfies Hall’s condition for Hall’s marriage theorem, suppose, to the contrary, that there exists $S \subseteq X$ such that $|S| > |N_B(S)|$. We denote $S = \{A_{i_1}, A_{i_2}, \ldots, A_{i_l}\}$ and $N_B(S) = \{z_1, z_2, \ldots, z_l\}$. Then $k > l$ by our assumption.

To show that $N^-(z_1), N^-(z_2), \ldots, N^-(z_l)$ in $D$ cover all the edges covered by $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$, take an edge $e$ of $G$ covered by $C_{i_j}$ for some $j \in \{1, \ldots, k\}$. Then there exists a vertex $z \in A_{i_j}$ such that $z$ is a common out-neighbor of the end vertices of $e$. Therefore $e$ is covered by $N^-(z)$. Since $A_{i_j} \subseteq X$, $A_{i_j} = N_B(A_{i_j})$. Since $A_{i_j} \subseteq S$, $N_B(A_{i_j}) \subseteq N_B(S)$. Therefore $z \in N_B(S)$ since $z \in A_{i_j}$ and $A_{i_j} = N_B(A_{i_j})$. As the in-neighborhood of each vertex of $N_B(S)$ clearly forms a clique in $G$, $\{N^-(z_1), N^-(z_2), \ldots, N^-(z_l)\}$ is a family of cliques of $G$ covering all the edges covered by $\{C_{i_1}, C_{i_2}, \ldots, C_{i_k}\}$. Since $k > l$, we replace $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$ with $N^-(z_1), N^-(z_2), \ldots, N^-(z_l)$ in $C$ to obtain a new edge clique cover of $G$ consisting of fewer cliques than $C$, a contradiction. Thus $B$ satisfies Hall’s condition and so, by Hall’s marriage theorem, $B$ has a matching $M = \{\{A_i, w_i\} \mid i = 1, \ldots, \theta_e(G)\}$ that saturates $X$. By the definition of $A_i$, $w_1, w_2, \ldots, w_{\theta_e(G)}$ are vertices of indegree at least two. Since each of $w_1, w_2, \ldots, w_{\theta_e(G)}$ is saturated by the matching $M$, $w_1, w_2, \ldots, w_{\theta_e(G)}$ are all distinct and the theorem statement follows. \hfill \Box

**Theorem 3.4.** For any graph $G$, $k(G) \geq \theta_e(G) - |V(G)| + \tilde{k}(G)$. 

8
Proof. If $G$ is an edgeless graph, then obviously $k(G) = 0$, $\theta_e(G) = 0$, $\tilde{k}(G) = |V(G)|$, and the inequality holds. Let $G$ be a graph with at least one edge and $D$ be a digraph by which $\tilde{k}(G)$ is attained. By definition,

$$|V(D)| = |V(G)| + k(G).$$

(2)

On the other hand, by Theorem 3.3 there are at least $\theta_e(G)$ vertices of indegree nonzero in $D$. Since any vertex in $D$ has indegree 0 or indegree nonzero,

$$|V(D)| \geq \tilde{k}(G) + \theta_e(G).$$

(3)

Then (2) and (3) yield the desired inequality.

Opsut’s inequality for competition numbers immediately follows from Proposition 3.4 and Theorem 3.3.

**Proposition 3.5.** Let $G$ be a graph with an effective competition cover $C$. Then $\tilde{k}(G)$ is attained by any digraph accompanying $C$. Furthermore,

$$|V(D)| = \tilde{k}(G) + \theta_e(G)$$

for any digraph $D$ accompanying $C$.

Proof. Let $D$ be a digraph accompanying $C$. By definition, $D$ has exactly $\theta_e(G)$ vertices of indegree nonzero. Therefore, by Theorem 3.3 $D$ has the smallest number of vertices of indegree nonzero. This implies that $D$ has the largest number of vertices of indegree 0. By the definition of $\tilde{k}(G)$, $\tilde{k}(G)$ is attained by $D$. Now, since the number of vertices of indegree nonzero is $\theta_e(G)$ and the number of vertices of indegree 0 is $\tilde{k}(G)$ in $D$, we have $|V(D)| = \tilde{k}(G) + \theta_e(G)$.

In the following, we present some sufficient conditions for a graph to make the inequality given in Theorem 3.4 sharp.

**Proposition 3.6.** Let $G$ be a graph with an effective competition cover $C$. Then $k(G) = \theta_e(G) - |V(G)| + \tilde{k}(G)$.

Proof. Let $D$ be a digraph accompanying $C$. By Proposition 3.5,

$$|V(D)| = \tilde{k}(G) + \theta_e(G).$$

By the definition of competition number,

$$|V(D)| = |V(G)| + k(G).$$

Then the above equalities yield the desired equality.
Now we present an example showing how Theorem 3.2 and Proposition 3.6 can be applied to obtain a lower bound for the competition number of a graph having an effective competition cover. The graph $G$ given in Figure 2 is a diamond-free graph with at least one edge, so it has an effective competition cover by Corollary 2.5. Note that $\theta_e(G) = 9$ and the union of any three maximal cliques of $G$ contains at least five vertices.

By applying Theorem 3.2 for $k = 7$, we have $\tilde{k}(G) \geq 3$. Thus, by Proposition 3.6, $k(G) = \theta_e(G) - |V(G)| + \tilde{k}(G) \geq 3$. This bound is sharper than Opsut’s bound in [12] that $k(G) \geq \theta_e(G) - |V(G)| + 2 = 2$. In fact, we can show $k(G) \leq 3$ by constructing an acyclic digraph whose competition graph is $G$ together with three isolated vertices, and therefore $k(G) = 3$.

The following proposition guarantees that the equality $k(G) = \theta_e(G) - |V(G)| + \tilde{k}(G)$ given in Proposition 3.6 is still true under the following condition.

**Proposition 3.7.** Let $G$ be a graph with at least one edge and $\mathcal{C}$ be a set of maximal cliques of $G$. Suppose that every clique in $\mathcal{C}$ has an edge that is occupied by it. Then $k(G) = \theta_e(G) - |V(G)| + \tilde{k}(G)$.

**Proof.** Since $\mathcal{C}$ is a set of maximal cliques of $G$, $\mathcal{C}$ is an edge clique cover of $G$. Moreover, by the hypothesis that every clique in $\mathcal{C}$ has an edge that is occupied by it, it is a minimum edge clique cover.

Let $D$ be a digraph with the most arcs among which acyclic digraphs by which $\tilde{k}(G)$ are attained. We show that the number $r$ of vertices with at least one in-neighbor in $D$ equals $\theta_e(G)$. By Theorem 3.3 $r \geq \theta_e(G)$. To reach a contradiction, we assume $r > \theta_e(G)$. Let $w_1, \ldots, w_r$ be the vertices with at least one in-neighbor in $D$. Since the co-competition number of $G$ is attained by $D$, $w_i$ has at least two in-neighbors in $D$ and so $N_D^{-}(w_i)$ forms a clique of size at least two for each $i = 1, \ldots, r$. Let $C_i$ be a maximal clique including $N_D^{-}(w_i)$ for each $i = 1, \ldots, r$. Then $C_i$ belongs to $\mathcal{C}$. Since $|\mathcal{C}| = \theta_e(G)$ and $r > \theta_e(G)$, $C_p = C_q$ for some distinct $p, q \in \{1, \ldots, r\}$ by the Pigeonhole principle. Without loss of generality, we may assume $w_p$ has a lower label than $w_q$ in an acyclic labeling of $D$. Now we detour the arcs from $N_D^{-}(w_i)$ to $w_q$ so that their heads change from $w_q$ to $w_p$ in this way, we obtain a new acyclic digraph $D^*$. Since $N_D^{-}(w_p)$ and $N_D^{-}(w_q)$ are included in the same clique in $G$, $C(D^*) = C(D)$. However, $w_q$ is a vertex of indegree 0 in $D^*$, so the
number of vertices of indegree 0 in $D^*$ is greater than that of vertices of indegree 0 in $D$, which contradicts the choice of $D$. Therefore $r = \theta_e(G)$.  

By using Proposition 3.6 we may represent the competition number of a certain plane graph in terms of co-competition number, the number of faces, and the number of maximal cliques. Let $G$ be a plane graph. Kuratowski’s theorem tells us that $G$ contains no $K_5$ as a subgraph, so any maximal cliques in $G$ consists of at most four vertices. For $i = 2, 3, 4$, we denote by $c_i(G)$ the number of maximal cliques of size $i$ in $G$.

**Proposition 3.8.** Let $G$ be a connected diamond-free plane graph. Then

$$k(G) = f(G) + \tilde{k}(G) - 2c_3(G) - 5c_4(G) - 2$$

where $f(G)$ denotes the number of faces in $G$.

**Proof.** If $G$ has only one vertex, then $k(G) = 0$, $f(G) = 1$, $\tilde{k}(G) = 1$, $c_3(G) = c_4(G) = 0$ and so (4) holds.

Suppose that $G$ has at least two vertices. Since $G$ is connected, $G$ has at least one edge. Let $C$ be a minimum edge clique cover of $G$ consisting of maximal cliques. Since $G$ is diamond-free, each edge of $G$ belongs to exactly one maximal clique. Thus $C$ consists of all the maximal cliques of $G$. On the other hand, since $G$ is connected and planar, any maximal clique of $G$ has size 2 or 3 or 4. Thus

$$\theta_e(G) = c_2(G) + c_3(G) + c_4(G).$$

Since $G$ is diamond-free, any maximal cliques of $G$ are edge-disjoint. Therefore $|E(G)| = \binom{2}{2}c_2(G) + \binom{3}{2}c_3(G) + \binom{4}{2}c_4(G)$ and so $c_2(G) = |E(G)| - 3c_3(G) - 6c_4(G)$. Substituting this into (5), we have

$$\theta_e(G) = |E(G)| - 2c_3(G) - 5c_4(G).$$

Since $G$ is a diamond-free graph having at least one edge, it has an effective competition cover by Corollary 2.5 and so $k(G) = \theta_e(G) - |V(G)| + \tilde{k}(G)$ by Proposition 3.6. This equality together with (6) give $k(G) = |E(G)| - |V(G)| - 2c_3(G) - 5c_4(G) + \tilde{k}(G)$. Since $G$ is connected and planar, by Euler’s formula, (4) follows.  

4 Concluding remarks

We conjecture that every graph $G$ satisfies the equality $k(G) = \theta_e(G) - |V(G)| + \tilde{k}(G)$. In addition, we have a strong belief that every graph with at least one edge has an effective competition cover, which implies the above conjecture by Proposition 3.6.
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