Study of the Potts Model on the Honeycomb and Triangular Lattices: Low-Temperature Series and Partition Function Zeros

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We present and analyze low-temperature series and complex-temperature partition function zeros for the \(q\)-state Potts model with \(q = 4\) on the honeycomb lattice and \(q = 3, 4\) on the triangular lattice. A discussion is given as to how the locations of the singularities obtained from the series analysis correlate with the complex-temperature phase boundary. Extending our earlier work, we include a similar discussion for the Potts model with \(q = 3\) on the honeycomb lattice and with \(q = 3, 4\) on the kagomé lattice.

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I. INTRODUCTION

The 2D $q$–state Potts models [1,2] for various $q$ have been of interest as examples of different universality classes for phase transitions and, for $q = 3, 4$, as models for the adsorption of gases on certain substrates. Unlike the $q = 2$ (Ising) case, however, for $q \geq 3$, the free energy has never been calculated in closed form for arbitrary temperature. It is thus of continuing value to obtain further information about the 2D Potts model. It has long been recognized that a very powerful method for doing this is via the calculation and analysis of series expansions for thermodynamic quantities such as the specific heat, magnetization, and susceptibility [3]. For $q = 2, 3$, and 4, the respective 2D $q$-state Potts ferromagnets have continuous phase transitions, and the critical singularities and associated exponents are known exactly [4,6,8]. Recently, two of us have calculated and analyzed long low-temperature series expansions for the Potts model with $q = 3$ on the honeycomb lattice and for the Potts model with $q = 3$ and $q = 4$ on the kagomé lattice [6]. These have been used to make very precise estimates of the respective critical points, to confirm a formula for the honeycomb lattice and to strengthen a previous refutation of an old conjecture for the kagomé lattice.

The other three authors have recently used a relation between complex-temperature (CT) properties of the Potts model on a given lattice and physical properties of the Potts antiferromagnet (AF) on the dual lattice to rule out other conjectures [7] and have calculated complex-temperature zeros of the partition function for these three cases of $q$ and lattice type [3]. The study of properties of spin models with the magnetic field and temperature generalized to complex values was pioneered by Yang and Lee [9] for the magnetic field and Fisher for the temperature [10]. Some of the earliest work on CT properties of spin models dealt with zeros of the partition function [10,12]. Another major reason for early interest in CT properties of spin models was the fact that unphysical, CT singularities complicated the analysis of low-temperature series expansions to get information about the location and critical exponents of the physical phase transition [13].

Here we shall present a unified study of the Potts model on the honeycomb lattice for $q = 4$ and on the triangular lattice for $q = 3$ and $q = 4$. For each $q$ value and lattice type, our results include (i) long, low-temperature series for the specific heat, spontaneous magnetization, and initial susceptibility derived using the finite-lattice method [14,15], extended by noting the structure of the correction terms [16]; (ii) a calculation of the complex-temperature zeros and, from these, an inference about the CT phase boundary; and (iii) a discussion of how the positions of the physical and unphysical singularities extracted from the series analysis correlate with the CT phase boundary. Since both the critical exponents and the location of the paramagnetic-to-ferromagnetic phase transition are known exactly for these models,
we shall focus mainly on getting new information on complex-temperature properties from the series and CT zeros. Using the results of Refs. [6–8], we shall also discuss subject (iii) for the Potts model with $q = 3$ model on the honeycomb lattice and with $q = 3$ and 4 on the kagomé lattice. It is useful to perform a unified analysis of this type because, aside from well-understood exceptions [17], the physical and complex-temperature singularities of the thermodynamic functions lie on the continuous locus of points $B$ which serves as the boundaries of the complex-temperature phases [19]; consequently, an approximate knowledge (or exact knowledge, if available) of where this boundary lies is of considerable help in checking which CT singularities that one extracts from a series analysis are trustworthy and which are not. This will be discussed further below. Note that low-temperature series on the honeycomb lattice correspond to high-temperature series on the triangular lattice, and vice versa.

II. MODEL

The (isotropic, nearest-neighbor) $q$-state Potts model at temperature $T$ on a lattice $\Lambda$ is defined by the partition function

$$Z = \sum_{\{\sigma_n\}} e^{-\beta\mathcal{H}} \tag{2.1}$$

with the Hamiltonian

$$\mathcal{H} = J \sum_{\langle nn' \rangle} (1 - \delta_{\sigma_n \sigma_{n'}}) + H \sum_n (1 - \delta_{0 \sigma_n}) \tag{2.2}$$

where $\sigma_n = 0, \ldots, q - 1$ are $\mathbb{Z}_q$-valued variables on each site $n \in \Lambda$, $\beta = (k_B T)^{-1}$, and $\langle nn' \rangle$ denotes pairs of nearest-neighbor sites. The symmetry group of the Potts Hamiltonian is the symmetric group on $q$ objects, $S_q$. We use the notation $K = \beta J$,

$$a = z^{-1} = e^K \tag{2.3}$$

$$x = \frac{e^K - 1}{\sqrt{q}} \tag{2.4}$$

and

$$\mu = e^{-\beta H} \tag{2.5}$$
The variable $z$ was denoted $u$ in Ref. [6]. The (reduced) free energy per site is denoted as $f = -\beta F = \lim_{N_s \to \infty} N_s^{-1} \ln Z$, where $N_s$ denotes the number of sites in the lattice. There are actually $q$ types of external fields which one may define, favoring the respective values $\sigma_n = 0, \ldots, q-1$; it suffices for our purposes to include only one. The order parameter (magnetization) is defined to be

$$m = \frac{qM - 1}{q - 1}$$

(2.6)

where $M = \langle \sigma \rangle = \lim_{h \to 0} \partial f / \partial h$. With this definition, $m = 0$ in the $S_q$-symmetric, disordered phase, and $m = 1$ in the limit of saturated ferromagnetic (FM) long-range order. Finally, the (reduced, initial) susceptibility is denoted as $\chi = \beta^{-1} \chi = \partial m / \partial h |_{h=0}$. We consider the zero-field model, $H = 0$, unless otherwise stated. For $J > 0$ and the dimensionality of interest here, $d = 2$, the $q$-state Potts model has a phase transition from the symmetric, high-temperature paramagnetic (PM) phase to a low-temperature phase involving spontaneous breaking of the $S_q$ symmetry and onset of ferromagnetic (FM) long-range order. This transition is continuous for $2 \leq q \leq 4$ and first-order for $q \geq 5$. As noted above, the model has the property of duality [1,4,21,22], which relates the partition function on a lattice $\Lambda$ with temperature parameter $x$ to another on the dual lattice with temperature parameter

$$x_d \equiv D(x) = \frac{1}{x}, \quad i.e. \quad a_d \equiv D(a) = \frac{a + q - 1}{a - 1}.$$

(2.7)

Other exact results include formulas for the PM-FM transition temperature on the square, triangular, and honeycomb lattices [1,22], and calculations of the free energy at the phase transition temperature, and of the related latent heat for $q \geq 5$ [23]. The case $J < 0$, i.e., the Potts antiferromagnet (AF), has also been of interest because of its connection with graph colorings. Depending on the type of lattice and the value of $q$, the antiferromagnetic model may have a low-temperature phase with AFM long-range order. Alternatively, it may not have any finite-temperature PM-AFM phase transition but instead may exhibit nonzero ground state entropy. For the Potts model on the honeycomb lattice, the well-known $q = 2$ (Ising) case [24,25] falls into the former category, while the model with $q \geq 3$ falls into the latter category [26,27] with nonzero ground state entropy [27–30]. Reviews of the model include Refs. [2,31].

For the $q$-state Potts model, from duality and a star-triangle relation, together with a plausible assumption of a single transition, one can derive algebraic equations that yield the PM-FM critical points for the honeycomb (hc) and triangular (t) lattices [22]. The equation for the honeycomb lattice is

$$x^3 - 3x - \sqrt{q} = 0, \quad i.e., \quad a^3 - 3a^2 - 3(q - 1)a - q^2 + 3q - 1 = 0 \quad (\text{honeycomb}) \quad (2.8)$$
and, as follows from eq. (2.7), the corresponding formula for the triangular lattice is obtained by the replacement \( x \rightarrow 1/x \):

\[
\sqrt{q}x^3 + 3x^2 - 1 = 0, \quad \text{i.e.,} \quad a^3 - 3a + 2 - q = 0 \quad \text{(triangular)} .
\] (2.9)

It will be useful to have the explicit solutions for the cases studied here. For \( q = 4 \) on the honeycomb lattice, eq. (2.8) reduces to \( (a - 5)(a + 1)^2 = 0 \), yielding the PM-FM critical point

\[
a_{hc,\text{PM-FM},q=4} = z_{hc,\text{PM-FM},q=4}^{-1} = 5
\] (2.10)

together with a double root at the complex-temperature value

\[
a_{hc,2,q=4} = z_{hc,2,q=4}^{-1} = -1
\] (2.11)

For \( q = 3 \) on the triangular lattice, eq. (2.9) has the solutions

\[
a_{t,1,q=3} = a_{t,\text{PM-FM},q=3} = \cos(2\pi/9) + \sqrt{3}\sin(2\pi/9) = 1.879385...
\] (2.12)

i.e., \( z_{t,\text{PM-FM},q=3} = 0.5320889... \),

\[
a_{t,2,q=3} = \cos(2\pi/9) - \sqrt{3}\sin(2\pi/9) = -0.347296...
\] (2.13)

and

\[
a_{t,3,q=3} = -2\cos(2\pi/9) = -1.532089...
\] (2.14)

For \( q = 4 \), eq. (2.9) reduces to \( (a - 2)(a + 1)^2 = 0 \), so that the physical PM-FM critical point is given by

\[
a_{t,1,q=4} = a_{t,\text{PM-FM},q=4} = 2
\] (2.15)

and there is a double root at the CT value

\[
a_{t,2,q=4} = -1.
\] (2.16)

### III. SERIES EXPANSIONS

The low-temperature series expansion is based on perturbations from the fully aligned ground state and is expressed in terms of the low-temperature variable \( z \) and the field variable
\( y = 1 - \mu \). Details of the methods can be found in Ref. [6], so here it suffices to say that in order to derive series in \( z \) for the specific heat, magnetization and susceptibility one need only calculate the expansion in \( y \) to second order, i.e.,

\[
Z = Z_0(z) + yZ_1(z) + y^2Z_2(z),
\]

(3.1)

where \( Z_k(z) \) is a series in \( z \) formed by collecting all terms in the expansion of \( Z \) containing factors of \( y^k \). We use the finite-lattice method [15] to approximate the infinite-lattice partition function \( Z \) by a product of partition functions \( Z_{m,n} \) on finite \((m \times n)\) lattices, with each \( Z_{m,n} \) calculated by transfer matrix techniques. As explained in Ref. [6], this leads to a series in \( z \) correct to order \( w_s(m - 2) + m - 1 \), where \( w_s \) is the maximal number of sites contained within the largest width \( w \) of the rectangles, and \( m \) is the number of nearest neighbors of each site. The implementation of the algorithm on the honeycomb lattice [3] has \( w_s = 2w \) and \( m = 3 \). The triangular lattice is represented as a square lattice with additional interactions along one of the diagonals, and in this case \( w_s = w \) and \( m = 6 \). In addition, we make use of a recent extension procedure discussed in Ref. [16], which allows us to calculate additional series terms.

The extension procedure for the 4-state Potts model on the honeycomb lattice is the same as for the 3-state model [6]. For a given width the expansion is correct to order \( 2w + 2 \), and we calculated the series up to \( w = 12 \). Next we look at the integer sequences \( d_s(w) \) obtained by taking the difference between the expansions obtained from successive widths \( w \),

\[
Z_{w+1}(z) - Z_w(z) = z^{2w+3} \sum_{s \geq 0} d_s(w)z^s.
\]

(3.2)

In this case the formulae for the correction terms are simply given by polynomials of order \( 2s + k \). We managed to find formulæ for the first 4 correction terms, which enabled us to calculate the series for the specific heat \( C \), magnetization \( m \), and susceptibility \( \bar{\chi} \) to order 30. The resulting series for \( m, \bar{\chi} \), and the (reduced) specific heat \( \bar{C} = C/(k_B K^2) \) are given in Table 1.

The extension procedure for the triangular lattice is essentially the same as for the honeycomb lattice. The only difference is that the order of the polynomials is \( s + k \). For a given width the expansion is correct to order \( 4w + 5 \), and we calculated the series up to \( w = 14 \) for \( q = 3 \) and up to \( w = 12 \) for \( q = 4 \). We found formulæ for the first 7 or 8 correction terms in the case \( q = 3 \) and the first 6 or 7 correction terms for \( q = 4 \). The series were thus derived to order 69 (60) for the specific heat and magnetization and to order 68 (59) for the susceptibility in the case \( q = 3 \) \((q = 4)\). The resulting series for \( m, \bar{\chi} \), and the (reduced) specific heat \( \bar{C} = C/(k_B K^2) \) are given in Tables 2 and 3.
IV. ANALYSIS OF SERIES

A. Honeycomb Lattice, \( q = 4 \)

We have analyzed the series using dlog Padé approximants (PA’s) and differential approximants (DA’s); for a general review of these methods, see Ref. [3]. We first comment on the physical PM-FM phase transition. The series yield a value for the critical point in excellent agreement with the known value \( z_{hc,PM-FM,q=4} = 1/5 \). For example, the differential approximants of the type \([L/M_0, M_1]\) with \( L = 1 \) and \( L = 2 \) to the specific heat series yield \( z_{hc,PM-FM,q=4} = 0.19993(4) \) and \( 0.19991(5) \), while those for the magnetization yield \( 0.19999(3) \) and \( 0.20005(7) \), respectively, with similarly good agreement for other values of \( L \) and for the approximants to the susceptibility. Concerning the critical exponents at this transition, the value \( q = 4 \) is the borderline between the interval \( 2 \leq q \leq 4 \) where this transition is second-order and the interval \( q > 4 \) where it is first order. Related to this, the \( q = 4 \) 2D Potts model has the special feature that the thermodynamic functions have strong confluent logarithmic corrections to their usual algebraic scaling forms [4] at the PM-FM transition (on any lattice). For example, the singularities in the specific heat and magnetization are \( C_{\text{sing}} \sim |t|^{-2/3}(-\ln|t|)^{-1} \) for \( t \to 0 \), where \( t = (T - T_c)/T_c \), and \( M_{\text{sing}} \sim (-t)^{1/12}(-\ln|t|)^{-1/8} \) for \( t \to 0^- \). Consequently, simple fits of the series to an algebraic singularity without this confluent logarithmic correction are not expected to agree well with the known singularities. Indeed, this was the general experience in early series work, and the same is found for the longer series here. As an illustration, a naïve fit to a simple algebraic singularity for the specific heat would yield the value \( \alpha' \sim 0.5 \) rather than the known value \( \alpha' = 2/3 \). Since these confluent singularities may also affect singularities at complex-temperature points, it could be useful in future work, as was noted earlier for the square-lattice model [32], to carry out a more sophisticated analysis of the series including these confluent singularities. However, because our primary focus here is on obtaining new information on complex-temperature properties rather than reproducing exactly known results for the critical exponents of the physical PM-FM singularity, and because it is not known if the confluent logarithmic corrections do affect the CT singularities, we have not tried to include such logarithmic factors in fits to the CT singularities.

Proceeding to CT singularities, we find evidence for one on the negative real \( z \) axis at

\[
z_{hc,\ell,q=4} = -0.33(1), \quad \text{i.e.} \quad a_{hc,\ell,q=4} = -3.0(1)
\]

(4.1)

Here the subscript \( \ell \) stands for “leftmost” singularity on the negative real axis. We shall present below, as an application of the mapping discussed in Ref. [7], an analytic derivation
of the exact value \( a_{hc,\ell,q=4} = -3 \). Clearly, the value extracted from the series analysis is in excellent agreement with the exact determination. By the mapping of Ref. [7], it follows that the singularity in the specific heat at this point \( a_{hc,\ell,q=4} \), as approached from larger negative \( \alpha \), i.e. smaller negative \( z \), is the same as the singularity in the specific heat of the \( q = 4 \) Potts antiferromagnet on the triangular lattice at the \( T = 0 \) critical point as approached from finite temperature.

We also find evidence from the series analyses for at least one complex conjugate (c. c.) pair of singularities. One such pair is observed at

\[
z_{hc,cc1,q=4}; \quad z_{hc,cc1,q=4}^* = 0.02(2) \pm 0.38(1)i
\]  

The central values correspond to \( a_{hc,cc1,q=4}; \quad a_{hc,cc1,q=4}^* = 0.14 \pm 2.6i \). As we shall show later, this pair of singularities is consistent with lying on the complex-temperature phase boundary \( B \).

**B. Triangular Lattice, \( q = 3 \)**

The series yield values for the PM-FM critical point in excellent agreement with the exactly known expression, eq. (2.12). For example, the first-order DA’s of the form \([L/M_0, M_1]\) with \( L = 1 \) for the free energy yield \( z_{t,PM-FM,q=3} = 0.532095(85) \), in complete agreement, to within the uncertainty, with the known value given by eq. (2.12). For reference, the thermal and field exponents for the 2D \( q = 3 \) Potts model are \( y_t = 6/5 \) and \( y_h = 28/15 \), so that the critical exponents for the specific heat, magnetization, and susceptibility are \( \alpha = \alpha' = 1/3, \beta = 1/9, \) and \( \gamma = \gamma' = 13/9 = 1.444... \) [2,5]. The above approximants yield the exponent \( \alpha' = 0.331(27) \), again in agreement with the known value. Similar statements apply to the magnetization and susceptibility.

Concerning complex-temperature singularities, the series for \( m \) and \( \bar{\chi} \) indicate a singularity on the negative real axis, at \( z_{t,-,q=3} \simeq -0.71 \) and \( z_{t,-,q=3} \simeq -0.65 \). If we assume that this is, as it should be, the same singularity, and average the positions, we get

\[
z_{t,-,q=3} \equiv z_{t,\ell,q=3} = -0.68(5)
\]  

or equivalently,

\[
a_{t,\ell,q=3} = -1.47(11)
\]  

where the numbers in parentheses are our estimates of the theoretical uncertainties. We observe that our numerical determination in eq. (4.4) is consistent, to within the uncertainty,
with being equal to the value given by the root in eq. (2.14), whence our use of the symbol \( a_{t,e,q=3} \) in eq. (4.4).

We find a complex-conjugate pair of singularities at

\[
z_{t,e,q=3}, \quad z^*_{t,e,q=3} = 0.0209(1) \pm 0.531(1)i
\]

(4.5)

From our analysis of the respective series, we infer the following values of singular exponents at the points (4.5):

\[
(\alpha', \beta, \gamma')_{t,e,q=3} = (1.19(1), -0.18(1), 1.17(1))
\]

(4.6)

The central values in eq. (4.5) correspond to

\[
a_{t,e,q=3}, \quad a^*_{t,e,q=3} = 0.0740 \pm 1.88i
\]

(4.7)

From the CT zeros to be discussed below, we see clearly that the c. c. members of this pair are endpoints of arcs of CT zeros of \( Z \), corresponding to continuous arcs of singularities of the free energy in the thermodynamic limit. (This type of correspondence with endpoints on \( B \) is indicated by the subscripts \( e \) here and in other cases below.) In passing, we observe that the exponent values in eq. (4.6) are not too different from the respective exponents obtained from the series analysis of Ref. [33], the singularities \( u_{s} = -0.301939(5) \pm 0.3787735(5)i \) in the 2D spin-1 Ising model on the square lattice, namely [33] \((\alpha', \beta, \gamma') = (1.1693(3), -0.1690(2), 1.1692(2))\). The c.c. pair of points \( u_{s} \) is analogous to the pair in eq. (4.7) because the members of this pair were shown [34] to be endpoints of arcs of CT zeros protruding into the complex-temperature extension of the FM phase of the spin-1 square-lattice Ising model. We also observe that the values of both these sets of exponents are consistent with the equality \( \alpha' = \gamma' \). However, we already know that such an equality, even if it were to hold for these cases, is not a general result for singular exponents at endpoints of arcs of a CT boundary \( B \) protruding into the complex-temperature extension of the FM phases for a spin model. A counterexample is provided by the (isotropic, spin 1/2) Ising model on the triangular lattice. In this case, one can determine the complex-temperature phase diagram exactly, and \( B \) consists of the union of the unit circle \(|u + 1/3| = 2/3\) and the semi-infinite line segment \(-\infty \leq u \leq -1/3\) [35], where \( u = z^2 \). Thus, in this case there is an exactly known analogue to the arc endpoints, viz., the endpoint at \( u_e = -1/3 \) (where the subscript \( e \) denotes “endpoint”) of the line segment protruding into the complex-temperature extension of the FM phase. An analysis of low-temperature series [36] had earlier yielded the inference that \( \gamma'_e = 5/4 \), while exact results [35] yielded \( \alpha'_e = 1 \) (and \( \beta_e = -1/8 \)), so that \( \alpha'_e \neq \gamma'_e \).
We find a second c. c. pair at
\[ z_{t,e',q=3}, z_{t,e',q=3}^* = -0.515(3) \pm 0.322(3)i \]  
(4.8)
with exponents \((\alpha', \beta, \gamma') = (1.2(1), -0.25(10), 1.2(1))\). The central values in eq. (4.8) correspond to
\[ a_{t,e',q=3}, a_{t,e',q=3}^* = -1.40 \pm 0.873i \]  
(4.9)
This pair is consistent with lying on the CT phase boundary, as will be discussed below. It should be noted that we would not expect the low-temperature series to be sensitive to the third root, \(a_{t,2,q=3}\), of eq. (2.9), since this root is masked by the nearer singularity \(a_{t,3,q=3}\) (that is, \(z_{t,3,q=3} = -0.652704...\) is closer to the origin in the \(z\) plane than \(z_{t,2,q=3} = -2.879385...\)).

C. Triangular Lattice, \(q = 4\)

For the physical PM-FM critical point of the \(q = 4\) Potts model on the triangular lattice, the discussion that we gave above for the honeycomb lattice applies; that is to say, the position of the physical singularity is well approximated, but the critical exponents are not, due to the presence of confluent logarithms. For complex-temperature properties, we first note that the series do not give a firm indication of a singularity on the negative real axis. We find a complex-conjugate pair of singularities at
\[ z_{t,e,q=4}, z_{t,e,q=4}^* = 0.0304(2) \pm 0.498(2)i \]  
(4.10)
We have also studied the exponents at this pair of singularities. If one assumes that there are no strong confluent singularities present, such as the logarithms that are present at the physical critical point, then from our series analysis we extract the following values, with their quoted uncertainties:
\[ (\alpha', \beta, \gamma')_{z_{t,e,q=4}} = (1.18(2), -0.17(2), 1.20(2)) \]  
(4.11)
However, we caution that it is not known whether strong confluent singularities are present at the points (4.10), and if they are, then the values in eq. (4.11) would have a lower degree of reliability. The central values in eq. (4.10) correspond to
\[ a_{t,e,q=4}, a_{t,e,q=4}^* = 0.122 \pm 2.00i \]  
(4.12)
As in the \(q = 3\) case, from the CT zeros to be presented below, we see clearly that the c. c. members of this pair of singularities are endpoints of arcs of CT zeros of \(Z\)
We find a second c. c. pair at

\[ z_{t,e,q=4}, \quad z^*_{t,e,q=4} = -0.461(5) \pm 0.281(5)i \]

The central values correspond to

\[ a_{t,e,q=4}, \quad a^*_{t,e,q=4} = -1.58 \pm 0.964i \]

Again, this pair can be associated with endpoints of arcs of zeros, as is especially clear from Fig. 7. There is also some sign of another pair of singularities in the vicinity of \( z \simeq -0.2 \pm 0.6i \), corresponding to \( a \simeq -0.5 \pm 1.5i \). The members of this c. c. pair are consistent with lying on the CT phase boundary. It is possible that there are also other c. c. pairs of singularities.

V. COMPLEX-TEMPERATURE ZEROS

A. General

The (zero-field) Potts model partition function \( Z \) for a finite lattice is, up to a possible prefactor, a polynomial in the Boltzmann weight \( a \). We calculate this polynomial by standard transfer matrix methods. From this, we then compute the zeros. In the thermodynamic limit, via a coalescence of zeros, there forms a continuous locus \( B \) of points where the free energy is nonanalytic. As was noted, this locus serves as the union of boundaries of the various complex-temperature phases [19] and, aside from well-understood exceptions [17], the CT singularities of thermodynamic functions occur on the continuous locus of points \( B \) where the free energy is nonanalytic, since it is analytic in the interior of physical phases and their complex-temperature extensions. Thus, calculations of CT zeros on sufficiently large finite lattices yield useful information on the CT phase diagram in the thermodynamic limit. Hence, when investigating CT singularities, it is useful to do so in conjunction with a calculation of the CT zeros of the partition function to infer the approximate location of the CT phase boundary \( B \).

To illustrate this, let us return briefly to the \( q = 2 \) Ising special case of the Potts model, for which both the free energy [24] and the magnetization [25] are known exactly. We recall that the expression for the spontaneous magnetization is

\[ M = \frac{(1 + u)^{1/4}(1 - 6u + u^2)^{1/8}}{(1 - u)^{1/2}} \]
where $u = z^2$ in the FM phase and the complex-temperature extension of it (with $M = 0$ elsewhere). Let us pretend that we did not know the exact free energy or magnetization, but that we had a low-temperature (small-$u$) series for $M$ and analyzed it using dlog Padé approximants. We would find singularities at the following four points: (i) $u_{PM-FM} = 3 - 2\sqrt{2}$, the physical PM-FM phase transition point; (ii) $u_{PM-AFM} = u_c^{-1} = 3 + 2\sqrt{2}$, the PM-AFM transition point; (iii) $u = -1$; and (iv) $u = 1$. This shows the value and importance of analyzing CT singularities of thermodynamic functions from series expansions; these can give one deeper knowledge of the exact functions. Indeed, in this illustrative example, a dlog Padé analysis of the series for $M$ would enable one to reconstruct the exact analytic expression for this quantity. The knowledge of the CT zeros and corresponding CT phase diagram give complementary information, in particular, information on which of the singularities found from the series analysis occur in the true thermodynamic function. Thus, from calculations of CT zeros for finite lattices, we could determine the approximate CT phase boundaries, and, in particular, the CT extension of the FM phase. We would then infer that this (CT) FM phase does not include the point $3 + 2\sqrt{2}$, so that the second apparent singularity extracted from the series does not occur in the true function $M$, since the low-temperature series only apply in the physical FM phase and its complex-temperature extension. Even without the CT zeros, in this case, we would also know that the apparent singularity at $u = 1$ does not occur in the true $M$, since $M$ is certainly zero, and all thermodynamic functions are analytic, at the infinite-temperature point $u = 1$ (and since this point is clearly in the PM phase, the low-temperature series are again not applicable in its vicinity). This example thus illustrates the value of both the study of CT singularities from series expansions and CT zeros of the partition function. Here, of the four apparent singularities extracted from the series, only two, namely the physical critical point (i) and the CT singularity (iii) are true singularities of $M$, since the others occur in regions outside the CT extension of the FM phase where the series applies. In this exactly solved case, these results are obvious, but the lesson holds more generally and illustrates the usefulness of having at least approximate knowledge of the CT phase boundary of a given model when analyzing series expansions to obtain locations of CT singularities. Note that all of the CT singularities occur on the locus of points $\mathcal{B}$ where the free energy is nonanalytic, which in this case is a limaçon of Pascal (given by eqs. (2.17) and (2.18) in Ref. [20], the image in the $u$ plane of the circles $|z \pm 1| = \sqrt{2}$ in the $z$ plane.) This is also a general feature (with the exception noted in Theorem 6 of Ref. [18]) and constitutes another reason for the value that a knowledge of CT zeros and the corresponding locus $\mathcal{B}$ have for series analyses and vice versa.

As a technical remark, we note that the problem of calculating the zeros of the partition
function for large lattices is a challenging one, since the degree of the polynomial is equal to the number of bonds, \( N_b = (\Delta/2)N_s \), where \( \Delta \) is the coordination number, \( N_s \) is the number of sites, and there is a very large range in the sizes of the coefficients, from \( q \) for the highest-degree term \( a^{N_b} \) to exponentially large values for intermediate terms. The latter property is obvious from the fact that for \( K = 0 \), i.e., \( a = 1 \), the sum of the coefficients in \( Z \) is \( q^N_s \). A general property of the CT phase boundary for any lattice and \( q \) value is invariance under complex-conjugation: \( \mathcal{B} \rightarrow \mathcal{B} \) as \( a \rightarrow a^* \). For previous calculations of CT zeros for the Potts model on the triangular and square lattices, see [37,38,32,31].

The complex-temperature zeros are equivalently plotted in the complex \( a \) or \( z \) plane; we shall plot them in the \( a \) plane because the resultant figures are somewhat more compact and because this maintains conformity with the plots for the square lattice, where the \( \text{Re}(a) > 0 \) part of the phase boundary is very simple (either exactly or approximately part of a circle, depending on boundary conditions).

In making inferences about the CT phase boundary \( \mathcal{B} \) in the thermodynamic limit from calculations on CT zeros on finite lattices, it is important to get an idea of the effects of different boundary conditions and lattice sizes. Accordingly, in Ref. [8], the authors performed calculations with three different boundary conditions and compared the resultant CT zeros with the exactly known CT phase boundary for the \( q = 2 \) Ising model on the honeycomb and kagomé lattices. This also served as a check on the computer programs used. We shall use the same three types of boundary conditions here, and we identify them next, following the notation of Ref. [8].

### B. Honeycomb Lattice, \( q = 4 \)

Before we start to present our results, we have to introduce our notation for the sizes and orientations of the lattices. To indicate the size of a given lattice, we count the number of hexagons. As an illustration, the size of the honeycomb lattice in Fig. [1] is \( 4 \times 3 \) hexagons. The number of sites in a lattice is also dependent on the boundary conditions: with periodic boundary conditions in the horizontal direction for example, the sites on the left and right are identified, while with free boundary conditions they are counted independently from each other.
Since we make use of duality in this work, we use lattices that have a corresponding dual lattice. This excludes lattices that are periodic in both directions, for the following reason: duality relies on the fact that every closed polygon divides the lattice into at least two regions. However, a lattice with periodic boundary conditions in both directions, and hence with toroidal geometry, has the property that one can easily draw a closed contour that does not divide the surface into two disjoint regions. Since boundary effects are, in general, best suppressed if one uses periodic boundary conditions in as many directions as possible, we use boundary conditions that are periodic in one direction and free in the other. Our notation for the boundary conditions (BC’s) is \((fbc,pbc)\) for free and periodic BC’s in the horizontal \((x)\) and vertical \((y)\) directions, respectively (see Fig. 1), and \((pbc,fbc)\) for periodic and free BC’s in the \(x\) and \(y\) directions. Note that for the \((fbc, pbc)\) choice, there is one site per hexagon at the boundary with only two instead of the usual \(\Delta = 3\) bonds. For the \((pbc, fbc)\) BC’s, there are two of these sites per boundary hexagon. This motivated the formulation of a third kind of boundary condition \([8]\): starting from the \((pbc, fbc)\) BC’s, one adds bonds connecting the boundary sites with fewer than three bounds so that all sites on the lattice have the same coordination number \(\Delta = 3\). This type of boundary conditions is denoted as \((pbc,fbc)_{\Delta}\). In Ref. \([8]\), a comparison was made with the Ising case \(q = 2\) where the CT phase boundary is exactly known, and it was found that (for the same lattice sizes as are used here) the CT zeros calculated with all three types of boundary conditions tracked the exactly known CT phase boundary reasonably well. In particular, the \((pbc,fbc)_{\Delta}\) choice
produced CT zeros with, in general, the least scatter. The (fbc,pbc) choice also exhibited
the special feature that a subset of zeros lay exactly on a certain circular arc comprising part
of $\mathcal{B}$.

FIG. 2. CT zeros of $Z$ in the $a$ plane for the $q = 4$ Potts model on a honeycomb lattice of size
$7 \times 6$ hexagons and boundary conditions of type (fbc,pbc).

FIG. 3. CT zeros of $Z$ for the $q = 4$ Potts model on a honeycomb lattice of size $8 \times 6$ hexagons
and boundary conditions of type (pbc,fbc).
FIG. 4. CT zeros of $Z$ for the $q = 4$ Potts model on a honeycomb lattice of size $8 \times 6$ hexagons and boundary conditions of type (pbc,fbc)$_{\Delta}$.

We show our calculations of the CT zeros of the $q = 4$ Potts model on the honeycomb lattice, using the above three types of boundary conditions, in Figs. 2, 3, and 4. The zeros cross the positive real $a$ axis only at one point, which is the PM-FM transition point; this value is in good agreement with the exact result $a_c = 5$ of eq. (2.10). The zeros thus exclude a PM-AFM transition and associated low-temperature phase with antiferromagnetic long-range order, since such a transition would be represented by a curve of CT zeros crossing the real $a$ axis at some point in the interval $0 \leq a < 1$. Concerning earlier work that bears on this, we note that a recent Monte Carlo study of the $q = 3$ Potts antiferromagnet on the honeycomb lattice [27] yielded evidence that that model has no symmetry-breaking phase transition and thus is disordered at all temperatures, including $T = 0$, where it exhibits nonzero ground state entropy measured to be $S_0/k_B = 0.957$. The latter value is close to an estimate [27] from earlier large-$q$ series [28] and is bracketed closely by rigorous upper and lower bounds [30]. Because increasing $q$ makes the spins “floppier”, one expects that the Potts antiferromagnet on the honeycomb lattice for $q \geq 4$ is similarly disordered at all temperatures, and, indeed, this has been rigorously proved [26].

A second remark is that the zeros are also consistent with the inference that a curve on the CT phase boundary $B$ crosses the real axis at the value in eq. (2.11), $a = -1$. This crossing is clearest with the (pbc,fbc) boundary conditions, shown in Fig. 3. Two other possible crossings occur at $a = -2.0(2)$ and $a = -0.47(5)$. For the latter point we have another source of information, using duality; if a crossing did occur at this point, it would
be the closest, on the left, to the origin of the $a$ plane and consequently its dual image would be the leftmost crossing of the CT phase boundary of the $q = 4$ Potts model on the triangular lattice, in the $a_d$ plane. In the other cases of $q$ value and lattice type where such Potts model series have been calculated and analyzed [6], they have been able to locate the leftmost singularity on the real $a$ axis (corresponding to the nearest singularity left of the origin in the $z$ plane) with good accuracy. However, the analysis of the low-temperature series for the $q = 4$ Potts model on the triangular lattice does not yield very strong evidence for such a singularity.

The leftmost crossing, in the $a$ plane, of the CT zeros for the $q = 4$ Potts model on the honeycomb lattice, and hence of $B$ in the thermodynamic limit, is related by duality to physical properties of the $q = 4$ Potts antiferromagnet on the dual, i.e. triangular, lattice; as discussed in Ref. [7], the full temperature interval $0 \leq a_d \leq 1$ of the $q$-state Potts AF on this dual lattice $\Lambda_d$ maps in a 1-1 manner, under duality, to the complex-temperature interval $-\infty \leq a \leq -(q-1)$ of the Potts model on the lattice $\Lambda$. Now it has been argued [39] that the Potts AF on the triangular lattice has a zero-temperature critical point (see also Ref. [40], where a study of the closely related $T = 0$ critical point of the $q = 3$ Potts AF on the kagomé lattice is given). Using the duality connection [7], one then deduces that the leftmost crossing of $B$ for the $q = 4$ Potts model on the honeycomb lattice is at

$$a_\ell = -3$$  \hspace{1cm} (5.2)

The CT zeros that we have calculated are consistent with this. The slight flaring out of the zeros to the left of this point appears as a finite lattice-size effect.

As was the case with the $q = 2$ Ising case and with $q = 3$, we again observe complex-conjugate arcs of zeros protruding into, and terminating in, the CT extension of the PM phase, ending at

$$a_{hc,e,q=4}, \quad a_{hc,e,q=4}^* = 0.27(3) \pm 1.68(4)i$$  \hspace{1cm} (5.3)

We recall that in the exactly known $q = 2$ case, these arc endpoints occur at $a = e^{\pm \pi i/3}$; as discussed in [8], as $q$ increases, these arc endpoints in the CT PM phase move to larger magnitudes $|a_e|$ and larger values of the angle $\theta = \arg(a_e)$. There also appears to be another c. c. pair of arcs protruding into the CT PM phase, with endpoints at

$$a_{hc,e',q=4}, \quad a_{hc,e',q=4}^* = -0.34(3) \pm 0.45(7)i$$  \hspace{1cm} (5.4)

A general observation is that all of the complex-temperature singularities obtained from the analysis of the low-temperature series are consistent with lying on the CT phase boundary.
This clearly includes the physical PM-FM critical point $a_{hc,PM-FM,q=4} = 5$, the leftmost crossing at $a_{hc,\ell,q=4} = -3$, and also the complex-conjugate pair given in eq. (4.1). We note that (i) the crossing at $a = -1$, and the c. c. pairs of arc endpoints in the PM phase in eqs. (ii) (5.3), and (iii) (5.4) are not expected to be seen with the low-temperature series because they are not contiguous with the complex-temperature extension of the FM phase but rather are within a presumed O phase [19] for (i) and the CT PM phase for (ii) and (iii).

C. Triangular Lattice, $q = 3$

In Figs. 5 and 6, we present calculations of CT zeros for the $q = 3$ Potts model on the triangular lattice. For consistency, we have plotted all of our zeros in the $a$ plane; however, we note that when relating zeros of $Z$ for the Potts model on one lattice $\Lambda$ to those on the dual lattice $\Lambda_d$, the connection is simplest if one plots the zeros in the $x$ plane, where $x$ was given in eq. (2.4) since in this case the duality transformation (2.7) just amounts to the inversion map $x \to 1/x$. A comparison of these figures gives a quantitative indication of the effects of different boundary conditions. These effects are somewhat stronger for $Re(a) < 0$ than $Re(a) > 0$.

FIG. 5. CT zeros of $Z$ for the $q = 3$ Potts model on the triangular lattice, obtained via duality from a honeycomb lattice of size $8 \times 6$ hexagons and (pbc,fbc) boundary conditions.
FIG. 6. CT zeros of \( Z \) for the \( q = 3 \) Potts model on the triangular lattice, obtained via duality from results for a honeycomb lattice of size \( 8 \times 6 \) hexagons and (pbc,fbc)\( _\Delta \) boundary conditions.

One observes complex-conjugate arc endpoints protruding into, and ending in, the (complex-temperature extension of the) FM phase at

\[
a_e, a_e^* = 0.072(10) \pm 1.85(2)i
\]

(5.5)

The points at which one can infer crossings of the zeros, and hence the CT phase boundary \( B \) are (in order, proceeding from right to left along the real \( a \) axis) (1) the PM-FM critical point \( a_{PM-FM,q=3} = 1.879... \) given by eq. (2.12); (2) the PM-AFM critical point \( a_{PM-AFM,q=3} \), which has recently been measured by a Monte Carlo simulation to high precision, \( a_{PM-AFM,q=3} = 0.20309(3) \) \( \text{[13]} \); (3) \( a_{2,q=3} = -0.3473... \) in eq. (2.13); (4) \( a = -0.82(3) \), and (5) \( a_{3,q=3} = -1.532... \) in eq. (2.14). In Ref. \( \text{[37]} \), it was suggested that points (2) and (4) were given by two of the roots of the equation \( a^3 + 6a^2 + 3a - 1 = 0 \), viz., \( a = 0.226682... \), \( a = -0.8152075... \), while the third root, \( a = -5.411474... \) would be associated with the completion of the complex-conjugate arcs of zeros (labelled as branches 6 in Ref. \( \text{[37]} \)) to form a closed curve crossing the negative real \( a \) axis at this root. However, neither the early \( \text{[11,12]} \) determinations nor the recent high-precision determination \( \text{[13]} \) of \( a_{PM-AFM} \) agrees with the value \( a = 0.226682 \), and the suggestion about the closing of the arcs to form a closed curve crossing the negative real axis at \( a = -5.411474 \) has been refuted \( \text{[7]} \) since, by duality, it is equivalent to a finite-temperature phase transition in the \( q = 3 \) Potts antiferromagnet on the honeycomb lattice, which is known not to occur \( \text{[27]} \). The CT phase diagram \( \text{[19]} \) in the \( a \) plane for the \( q = 3 \) Potts model on the triangular lattice thus consists of a PM phase and an FM phase, with
indications of at least one $O$ phase.

**D. Triangular Lattice, $q = 4$**

![Diagram](image)

**FIG. 7.** CT zeros of $Z$ for the $q = 4$ Potts model on the triangular lattice, obtained via duality from a honeycomb lattice of size $8 \times 6$ hexagons and (pbc,fbc) boundary conditions.
In Figs. 7 and 8 we show CT zeros of the partition function for the $q = 4$ Potts model on the triangular lattice with two different sets of boundary conditions. Qualitatively, the patterns of CT zeros are similar to those for $q = 3$. The points at which the zeros, and hence the CT phase boundary $B$ inferred in the thermodynamic limit, cross the real $a$ axis can be obtained via duality from those for the $q = 4$ model on the honeycomb lattice. The rightmost crossing is consistent with the known exact value $a_{t,PM} = 2$ in eq. (2.15). This is equivalent, by duality, to the known value $a_{hc,PM} = 5$ for the $q = 4$ Potts model on the honeycomb lattice. Assuming the correctness of the suggested zero-temperature critical point in the model [39], we deduce that the CT zeros and the CT phase boundary $B$ have no further crossings on the positive real axis but cross this axis at $a = 0$. This would imply that in the thermodynamic limit, the innermost complex-conjugate arcs of zeros pinch together at this point. As can be seen from Figs. 7 and 8, this requires that the two complex-conjugate arcs of zeros nearest to the origin in the $a$ plane must pull back slightly to the left as the lattice size goes to infinity. By duality, this crossing of the CT zeros at $a = 0$ on the triangular lattice is equivalent to the crossing of the CT zeros at $a = -3$ for the $q = 4$ Potts model on the honeycomb lattice. In the present case, the CT zeros are also consistent with the conclusion that another crossing is at $a = a_{t,2,q=4} = -1$, the multiple root of eq. (2.9) given above in eq. (2.16); the dual equivalent is that the CT zeros in the $q = 4$ Potts model on the honeycomb lattice also cross the real $a$ axis at $a = -1$. This is, of course, consistent.
with our calculations of CT zeros on the honeycomb lattice. There are also suggestions of other possible crossings of CT zeros for the present $q = 4$ case on the triangular lattice. These include a possible crossing at $a = -0.33(3)$, corresponding to the observed crossing of zeros for the honeycomb lattice at $a = -2.0(2)$. We have noted above that an analysis of the low-temperature series for the $q = 4$ Potts model on the triangular lattice does not yield a firm indication of a singularity on the negative real $z$ (equivalently $a$) axis. Such a singularity would occur at the leftmost crossing of the zeros, $z_\ell = a_\ell^{-1}$. It is possible that the reason for this is that, if, indeed, $a_{t,\ell,q=4} = -1$, then the effect of this singularity is shielded by the effects of singularities lying to the left in the complex $a$ plane (perhaps associated with the arcs of zeros in Figs. 7 and 8), i.e., closer to the origin of the $z$ plane.

We also observe clear c. c. arcs of zeros protruding into the FM phase, with endpoints at

$$a_{t,e,q=4} = 0.12(1) \pm 1.97(3)i \quad (5.6)$$

As indicated in the notation, this c. c. pair of points is in very good agreement with the c. c. pair of singularities given in eq. (4.12), identified from the analysis of the low-temperature series. Thus, in both of these cases, the $q = 3$ and $q = 4$ Potts models on the triangular lattice, we find excellent agreement between such c. c. pairs of singularities extracted from low-temperature series analyses and endpoints of arcs of zeros obtained from the calculation of CT zeros on finite lattices.

**E. Comparison of CT Singularities with Phase Boundary for Triangular Lattice**

Evidently, the respective values of the physical PM-FM critical point $a_{PM-FM,q}$ and the leftmost point where $B$ crosses the real $a$ axis, $a_{t,q}$, as obtained from the analysis of the low-temperature series are in excellent agreement with the roots of the general formula (2.9) and also with the crossing points seen with the CT zeros for both $q = 3$ and $q = 4$. The respective complex-conjugate pair of singularities at $z_{e,q}, z_{e,q}^*$ from the analysis of the low-temperature series are seen to be the ends of c. c. arcs of zeros (in the thermodynamic limit, continuous arcs of singularities) protruding into and ending in the CT FM phase. For $q = 3$ and $q = 4$, from our analysis of the series we have also obtained evidence for a complex-conjugate pair of singularities at the respective values $a_{e',q}, a_{e',q}^*$ as given in eqs. (4.13) and (4.14). Comparing these respective pairs of singularities with the CT phase boundary $B$ inferred from the CT zeros for $q = 3$ and $q = 4$, we observe that each pair is consistent with lying on the respective boundary $B$, in the “northwest” and “southwest” quadrants of the $a$ plane.
F. Triangular Lattice, \( q = 5 \)

It is also of interest to present an example of CT zeros calculated for the \( q \)-state Potts model on the triangular lattice with \( q \) in the range where the Potts antiferromagnet is disordered at all temperatures including \( T = 0 \). Accordingly, we show in Fig. 9 the CT zeros for the case \( q = 5 \). For \( q > 4 \), eq. (2.9) has only one real root, together with a conjugate pair of complex roots. The real root for this case is \( a_{PM-FM,q=5} = 2.103803 \ldots \). This is in agreement with the rightmost crossing point of the zeros on the real \( a \) axis. As is evident from Fig. 9, the curves of zeros that we had inferred to pass through the origin for the \( q = 4 \) model have moved further to the left, consistent with the conclusion that no branch of \( B \) passes through the interval \( 0 \leq a < 1 \), i.e., that the \( q = 5 \) Potts antiferromagnet is disordered for all temperatures including \( T = 0 \). The other features of the CT phase diagram are similar to those that we have observed before, including the prominent c. c. arcs of zeros protruding into the FM phase near to the vertical axis in the \( a \) plane.

![CT zeros of Z for the q = 5 Potts model on the triangular lattice, obtained via duality from a honeycomb lattice of size 7x6 hexagons and (fbc,pbc) boundary conditions.](image)

G. Comparison with \( q = 2 \) Ising Case for the Triangular Lattice

One can gain some further insight into these results from a comparison with the exactly solved \( q = 2 \) Ising case. The CT phase diagram in the \( z \) plane is shown as Fig. 1(b) of Ref.
in the $a$ plane, the CT phase boundary maps to the union of (i) an oval with its longer side along the real $a$ axis, crossing this axis at $\pm a_{PM-FM,q=2} = \pm \sqrt{3}$, and (ii) a vertical line segment along the imaginary $a$ axis extending from $-\sqrt{3}i$ upward to $\sqrt{3}i$. This line segment bisects the oval and divides the interior into the PM phase to the right and an O phase to the left; outside of the oval is the FM phase. The two components (i) and (ii) of $\mathcal{B}$ intersect at the multiple points $a = \pm i$.

Now taking the broader perspective of general $q$, one sees that as $q$ increases from 2 to 4, the PM-FM phase transition point $a_{PM-FM,q}$, which is the largest root of eq. (2.9), moves monotonically to the right, as dictated by general inequalities (as $q$ increases, the spins become “floppier”, and one must go to lower temperature to attain FM long-range order). A qualitative change occurs as one increases $q$ above 2, in that the middle root of eq. (2.9) moves to negative values, as does this portion of the CT phase boundary. Thus, while the $T = 0$ critical point of the Ising antiferromagnet corresponds to the middle root of eq. (2.9), the $T = 0$ critical point of the $q = 4$ model that has been argued for does not correspond to any root of this eq. (2.9).

VI. COMPARISON OF CT SINGULARITIES WITH PHASE BOUNDARIES FOR OTHER CASES

In Ref. [8], it was noted that for the Potts model with $q = 3$ on the honeycomb lattice and with $q = 3, 4$ on the kagomé lattice, the positions of the physical PM-FM transition points, as obtained from eq. (2.8) for the honeycomb lattice and from series analyses for the kagomé lattice, agreed nicely with the maximal real points at which the CT zeros crossed the real $a$ axis. It was also noted that the lefmost crossing point of the zeros at the respective points $a_\ell$ were in good agreement (i) for the $q = 3$ triangular case with a prediction from duality [7] and a precise Monte Carlo measurement of the PM-AFM transition temperature of the $q = 3$ Potts AF on the triangular lattice [43]; and (ii) for the $q = 3, 4$ kagomé case with the values obtained from low-temperature series analyses [6]. Here, we extend this comparison to the complex-$a$ singularities.

A. Honeycomb Lattice, $q = 3$

The low-temperature series analysis of the $q = 3$ Potts model on the honeycomb lattice in Ref. [8] yielded evidence for a c. c. pair of singularities in the thermodynamic functions at
$z_\pm = -0.06(2) \pm 0.47(3)i$. The central values correspond to $a_\pm = -0.27 \pm 2.1i$. Comparing with the results of Ref. \cite{8}, one sees that these points lie slightly to the upper left and lower left of the curve of CT zeros in the “northwest” and “southwest” quadrants of the $a$ plane. This c. c. pair may be associated with possible c. c. cusp-like structures in this vicinity, as is hinted at in Fig. 7 of Ref. \cite{8}.

B. Kagomé Lattice, $q = 3$

In addition to the PM-FM critical point and the leftmost crossing point, the low-temperature series analysis of Ref. \cite{3} found evidence for CT singularities at four c. c. pairs of points in the $z$ plane, viz., $z_{1,\pm} = 0.38(2) \pm 0.24(2)i$, $z_{2,\pm} = 0.278(10) \pm 0.38(1)i$, $z_{3,\pm} = -0.113(6) \pm 0.515(10)i$, and $z_{4,\pm} = -0.37(2) \pm 0.30(5)i$. The central values correspond approximately to the points $a_{1,\pm} = 1.9 \pm 1.2i$, $a_{2,\pm} = 1.25 \pm 1.7i$, $a_{3,\pm} = -0.41 \pm 1.85i$, and $a_{4,\pm} = -1.6 \pm 1.3i$. As discussed above, one expects the true singularities of the thermodynamic functions to lie on the CT phase boundaries, since these quantities are analytic functions of complex temperature in the interiors of the CT phases. However, as our illustration with the exactly calculated magnetization of the $q = 2$ Ising case showed, low-temperature series may, in general, indicate singularities which lie off the CT phase boundary in regions where these series do not apply; the corresponding factors are presumably present in the true function (e.g. the factor $(1 - u)^{-1/2}$ in (5.1)). The poles labeled $a_{2,\pm}$ and $a_{4,\pm}$ are definitely consistent with lying on CT phase boundaries which may be plausibly inferred in the thermodynamic limit from the zeros calculated in Ref. \cite{8} for finite lattices. For some, but not all, types of boundary conditions, there is an indication of c. c. arcs of zeros protruding into the CT FM phase and ending therein at points near to $a_{4,\pm}$. The previous experience with exactly known CT phase diagrams \cite{36,35,18} and with a comparison between CT zeros and low-temperature series expansions for the higher-spin Ising model \cite{33,34} showed that, in the cases studied, the magnetization diverges at the ends of arcs or line segments of singularities of the free energy which protrude into the CT FM phase. Thus, if the c. c. singularities at $a_{4,\pm}$ do lie at the ends of such arcs, this would be in accord with the divergence found in the magnetization in Ref. \cite{3}. The points $a_{3,\pm}$ lie somewhat outside the curves of zeros, in the interior of the FM phase, while the points $a_{1,\pm}$ lie slightly inside of the CT phase boundary $B$ in the interior of the CT PM phase.
C. Kagomé Lattice, $q = 4$

In this case, Ref. [6] obtained the two c. c. pairs of CT singularities at $z_{1,±} = 0.275(10) ± 0.305(10)i$ and $z_{2,±} = -0.345(10) ± 0.235(1)i$. The central values correspond to the points $a_{1,±} = 1.63 ± 1.81i$, and $a_{2,±} = -1.98 ± 1.35i$. scattered set of poles in the region of the second pair.) The c. c. pair $a_{1,±}$ lie on the inferred phase boundary $B$ in the upper and lower right quadrants of the $a$ plane. Similarly, the c. c. pair $a_{2,±}$ lie on $B$ in the upper and lower quadrants of the left-hand half plane $Re(a) < 0$, in the region of arc-like structures on this boundary.

VII. CONCLUSIONS

We have carried out a unified study of the $q$-state Potts model with $q = 4$ on the honeycomb lattice and with $q = 3, 4$ on the triangular lattice, including the calculation and analysis of long low-temperature series and the calculation of complex-temperature zeros of the partition function which allow one to make reasonable inferences about the CT phase boundary $B$ in the thermodynamic limit. In all cases, the series are in excellent agreement with the known values of the respective PM-FM critical points. For the $q = 4$ Potts model on the honeycomb lattice, there is no PM-AFM critical point and, concerning CT properties, we find that the series analysis and CT zeros yield a value of the leftmost crossing $a_ℓ$ in good agreement with the inference from duality and the zero-temperature critical point of the $q = 4$ Potts model on the triangular lattice, viz., $a_ℓ = -3$. For the triangular lattice, the CT zeros agree well with the known PM-AFM transition of the $q = 3$ model and are also consistent with the property that the $q = 4$ model has a $T = 0$ critical point. The singularities seen in the series at the largest negative values of $a$ are seen to be the leftmost points where the CT phase boundary crosses the negative real $a$ axis. For both $q = 3$ and $q = 4$ the series also yield clear indications of a complex-conjugate pair of singularities which are seen to lie at the ends of arcs of CT zeros protruding into the CT FM phase. In each case, there are indications of another c. c. pair lying on the respective CT phase boundaries. We have also discussed how the positions of various CT singularities lying at complex values of $a$ in this model and also in the $q = 3, 4$ model on the kagomé lattice correlate with the respective CT phase boundaries.

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TABLE I. Low-temperature series for the 4-state honeycomb lattice Potts model magnetization \((m(z) = \sum n m_n z^n)\), susceptibility \((\bar{\chi}(z) = \sum n x_n z^n)\), and specific heat \((\bar{C}(z) = \sum n c_n z^n)\).

| \(n\) | \(m_n\) | \(x_n\) | \(c_n\) |
|------|--------|--------|--------|
| 0    | 1      | 0      | 0      |
| 1    | 0      | 0      | 0      |
| 2    | 0      | 0      | 0      |
| 3    | -4     | 6      | 54     |
| 4    | -12    | 36     | 144    |
| 5    | -60    | 234    | 900    |
| 6    | -220   | 1284   | 2916   |
| 7    | -936   | 6804   | 14112  |
| 8    | -4092  | 38160  | 59616  |
| 9    | -17840 | 198912 | 280908 |
| 10   | -80868 | 1070316| 1304100|
| 11   | -356172| 5499054| 5974254|
| 12   | -1640872| 29005692| 28501416|
| 13   | -7433604| 149318838| 133160508|
| 14   | -34541160| 776570508| 641771424|
| 15   | -159080304| 3987307152| 3037720320|
| 16   | -743832276| 20560750344| 14671207872|
| 17   | -3469487112| 105345948384| 70242548778|
| 18   | -16321682424| 540305120844| 340125653664|
| 19   | -76796957940| 2761471319562| 1640652533460|
| 20   | -363235185312| 14111436147228| 7963315328520|
| 21   | -1720415299660| 71964766006350| 38614602921930|
| 22   | -8176521038556| 366780011157360| 187903674109404|
| 23   | -38925659520072| 1866864944056032| 914552556040350|
| 24   | -185771131129720| 9495487987576116| 44607344444147344|
| 25   | -888069677637192| 4825104682543824| 2177182349435750|
| 26   | -4253549708242236| 24502903414632628| 10641506076772476|
| 27   | -2040430261163396| 1243326018023082990| 520535130747734844|
| 28   | -98033976216116940| 6305270741929760652| 2548904536404499392|
| 29   | -471655884252852348| 3195659934515563546| 12490681376369529306|
| 30   | -2272238036173908576| 161878582502746522164| 61260473924462872080|
| \( n \) | \( m_{nz} \) | \( z_n \) | \( x_n \) | \( c_n \) |
|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 |
| 6 | -3 | 2 | 72 | 0 |
| 7 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 |
| 10 | -18 | 24 | 60 | 0 |
| 11 | -18 | 24 | 72 | 0 |
| 12 | -18 | 24 | 72 | 0 |
| 13 | 0 | 0 | 0 | 0 |
| 14 | -123 | 171 | 306 | 0 |
| 15 | 0 | 0 | 0 | 0 |
| 16 | -18 | 24 | 60 | 0 |
| 17 | 0 | 0 | 0 | 0 |
| 18 | 0 | 0 | 0 | 0 |
| 19 | -2106 | 324 | 8100 | 0 |
| 20 | 1998 | -1356 | 10000 | 0 |
| 21 | 2586 | -1820 | 20000 | 0 |
| 22 | 1364 | -324 | 30000 | 0 |
| 23 | -28098 | 99252 | 200000 | 0 |
| 24 | 19008 | -3024 | 300000 | 0 |
| 25 | 43020 | -5352 | 500000 | 0 |
| 26 | -147024 | 68988 | 1000000 | 0 |
| 27 | -317304 | 1382336 | 2000000 | 0 |
| 28 | 125775 | 285870 | 3000000 | 0 |
| 29 | 612954 | -926172 | 4000000 | 0 |
| 30 | -573087 | 10897804 | 5000000 | 0 |
| 31 | 178570 | 53068062 | 6000000 | 0 |
| 32 | -115684132 | -2347019916 | 7000000 | 0 |
| 33 | -101751129 | 1023192774 | 8000000 | 0 |
| 34 | -543365058 | 3619881892 | 9000000 | 0 |
| 35 | -122314741 | 2436809622 | 10000000 | 0 |
| 36 | 1155684132 | -2347019916 | 11000000 | 0 |
| 37 | -703522320 | 10960701972 | 12000000 | 0 |
| 38 | -6365950592 | 47574029772 | 13000000 | 0 |
| 39 | -2758467240 | 39732903192 | 14000000 | 0 |
| 40 | 13464222858 | -27776348840 | 15000000 | 0 |
| 41 | -1676208686 | 2735689898 | 16000000 | 0 |
| 42 | -14576455058 | 9376180586412 | 17000000 | 0 |
| 43 | -101751129 | 189786712276 | 18000000 | 0 |
| 44 | -573087 | 53068062 | 19000000 | 0 |
| 45 | -115684132 | -2347019916 | 20000000 | 0 |
| 46 | -78925126236 | -310099907604 | 21000000 | 0 |
| 47 | -27843500676 | 1118867211276 | 22000000 | 0 |
| 48 | -824251279038 | 6680614520384 | 23000000 | 0 |
| 49 | -709717601998 | 9376180586412 | 24000000 | 0 |
| 50 | 171260965746 | -2931391777128 | 25000000 | 0 |
| 51 | 102936045820 | 10000000034056 | 26000000 | 0 |
| 52 | -9179822752182 | 9552726710760 | 27000000 | 0 |
| 53 | -11186857401165 | 13625863972976 | 28000000 | 0 |
| 54 | 18287963891184 | -1998221659216 | 29000000 | 0 |
| 55 | 1978784095701 | 79741546755022 | 30000000 | 0 |
| 56 | -9179822752182 | 9552726710760 | 31000000 | 0 |
| 57 | -11186857401165 | 13625863972976 | 32000000 | 0 |
| 58 | 18287963891184 | -1998221659216 | 33000000 | 0 |
| 59 | 1978784095701 | 79741546755022 | 34000000 | 0 |
| 60 | -9179822752182 | 9552726710760 | 35000000 | 0 |
| 61 | -11186857401165 | 13625863972976 | 36000000 | 0 |
| 62 | 18287963891184 | -1998221659216 | 37000000 | 0 |
| 63 | 1978784095701 | 79741546755022 | 38000000 | 0 |

TABLE II. Low-temperature series for the 3-state triangular lattice Potts model magnetization \( m(z) = \sum_n m_n z^n \), susceptibility \( \chi(z) = \sum_n x_n z^n \), and specific heat \( C(z) = \sum_n c_n z^n \).
TABLE III. Low-temperature series for the 4-state triangular lattice Potts model magnetization \( (m(z) = \sum_n m_n z^n) \), susceptibility \( (\chi(z) = \sum_n \chi_n z^n) \), and specific heat \( (C(z) = \sum_n c_n z^n) \).

| n  | \( m_n \) | \( \chi_n \) | \( c_n \) |
|----|---------|---------|---------|
| 0  | 1       | 0       | 0       |
| 1  | 0       | 0       | 0       |
| 2  | 0       | 0       | 0       |
| 3  | 0       | 0       | 0       |
| 4  | 0       | 0       | 0       |
| 5  | 0       | 0       | 0       |
| 6  | -4      | 3       | 108     |
| 7  | 0       | 0       | 0       |
| 8  | 0       | 0       | 0       |
| 9  | 0       | 0       | 0       |
| 10 | -24     | 36      | 900     |
| 11 | -48     | 72      | 2178    |
| 12 | -60     | -72     | -3672   |
| 13 | 0       | 0       | 0       |
| 14 | -300    | 711     | 14112   |
| 15 | -480    | 1080    | 27000   |
| 16 | 144     | 144     | -18432  |
| 17 | 1392    | -2556   | -114444 |
| 18 | -4392   | 12852   | 290628  |
| 19 | -7248   | 23004   | 467856  |
| 20 | 2904    | -504    | -354600 |
| 21 | 13280   | -21192  | -1479114|
| 22 | -27348  | 122877  | 1768536 |
| 23 | -142512 | 525996  | 12073896|
| 24 | 29948   | 69366   | -5861808|
| 25 | 241872  | -531576 | -29193750|
| 26 | -336072 | 1970154 | 22900176|
| 27 | -1711936| 7833756 | 165214728|
| 28 | -990208 | 6613164 | 64153152|
| 29 | 4759680 | -12953124| -654007014|
| 30 | -2790212| 21423621| 1635354054|
| 31 | -2559960| 137623572| 2795890388|
| 32 | -1764872| -130318974| 1579935744|
| 33 | 53777216| -134509232| -8670448116|
| 34 | 2451472| 115953372| 65670776728|
| 35 | -385317888| 2338653528| 48371018850|
| 36 | -379526360| 2854976280| 42623145144|
| 37 | 757341312| -2039815332| -13619436566|
| 38 | 678358892| 433835991| -156222316800|
| 39 | -460520100| 3321142320| 62040507776|
| 40 | -8295782520| 62346454416| 1141035505200|
| 41 | 9858368640| -23201806140| -2037399414364|
| 42 | 16985972056| -32015102700| -3676310701364|
| 43 | -5955502524| 499277914144| 8580503647908|
| 44 | -136699873260| 11189201518738| 210381881323904|
| 45 | 8105525424| 49161610540| -21085184576700|
| 46 | 365702065748| -118105194617| -8127290605524|
| 47 | -702809980704| 6918782749164| 105969879599940|
| 48 | -2310592067252| 20279205891438| 39153381047712|
| 49 | 494570429328| 605152630600| -19975090321224|
| 50 | 5980182924792| -22190276278656| -141875008839000|
| 51 | -6198365886016| 84541391331996| 823212895965966|
| 52 | -37899130736064| 354875036323788| 704465404212800|
| 53 | -6090856346112| 205839844753932| -40924404630198|
| 54 | 97187254024404| -40744863461367| -2449066568902484|
| 55 | -39928634332608| 98450245280336| 1572414807677440|
| 56 | 56208134061556| 5775418442187300| 112350099816114364|
| 57 | -34066821277956| 59395305173556| 50948011956261300|
| 58 | 148201224840712| -654261994118038| -40108047490250964|
| 59 | 28354474049040| 889439649832452| -181847403757220400|
| 60 | -8231813619904556| 1752211155816298440|
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