Monopoles in non-Abelian Einstein-Born-Infeld Theory

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Abstract

We study static spherically symmetric monopole solutions in non-Abelian Einstein-Born-Infeld-Higgs model with normal trace structure. These monopoles are similar to the corresponding solution with symmetrised trace structure and are existing only up to some critical value of the strength of the gravitational interaction. In addition, similar to their flat space counterpart, they also admit a critical value of the Born-Infeld parameter $\beta$.

The Dirac-Born-Infeld (DBI) model has recently received special consideration in connection with string theory and D-brane dynamics after the discovery that the low energy effective action for D-branes is precisely described by the DBI action (for a review see ). BPS solutions to DBI theory for gauge fields and scalars were then studied and interpreted in terms of branes pulled by strings.

Different non-Abelian generalization of the DBI action have been proposed. They differ in the way a scalar action is defined from objects carrying group indices. The use of a symmetrised trace proposed by Tesytlin in the context of superstring theory seems to be the natural one in connection with supersymmetry and leads to a linearized Lagrangian with BPS equations identical to Yang-Mills ones. The use of the ordinary trace and other recipes for defining the DBI action have been also investigated.
Vortex, monopole and other soliton-like solutions to theories containing a DBI action have been studied for Abelian and non-Abelian gauge symmetry [19]-[23], in this last case both using the symmetrised and the normal trace operation to define a scalar DBI action. A distinctive feature was discovered for DBI vortices and monopoles: there exists a critical value $\beta_c$ of the Born-Infeld $\beta$-parameter below which regular solutions cease to exist [20]-[22]. In the non-Abelian case, this critical value can be tested only when the normal trace is adopted since, for the symmetrised trace, the Lagrangian is known only as a perturbative series in $1/\beta$ and the results are valid outside the domain where $\beta$-criticality takes place.

The existence of $\beta_c$ is much resemblant to the phenomenon occurring for self-gravitating monopoles [24]-[32], that exhibit a critical behavior with respect to the strength of the gravitational interaction: above some maximum value of a parameter $\alpha$, related to the strength of gravitational interaction, regular monopole solutions collapse. This behavior, originally described in Einstein-Yang-Mills-Higgs theory, has been shown to take place also when the DBI action governs the dynamics of the gauge field [33, 34]. Since the symmetric trace has been adopted in these last references, the existence of $\beta_c$ in DBI theories coupled to gravity could not be analysed; it is the purpose of the present work to address this issue, namely, to study the Einstein-DBI-Higgs model using the normal trace for defining the DBI action, find self-gravitating monopole solutions, determine whether they cease to exist below some critical value $\beta_c$ as in the flat space case and, in the affirmative, study the interplay between $\alpha$ and $\beta_c$.

We consider the following Einstein-Born-Infeld-Higgs action for $SU(2)$ fields with the Higgs field in the adjoint representation

$$S = \int d^4x \sqrt{-g} \left[ L_G + L_{BI} + L_H \right]$$ (1)

with

$$L_G = \frac{1}{16\pi G} R,$$

$$L_H = - \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - \frac{e^2 g^2}{4} \left( \phi^a \phi^a - v^2 \right)^2$$

and the non Abelian Born-Infeld Lagrangian,

$$L_{BI} = \beta^2 tr \left( 1 - \sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{8\beta^4} \left( F_{\mu\nu} F^{\mu\nu} \right)^2} \right)$$ (2)

where

$$D_\mu \phi^a = \partial_\mu \phi^a + e e^{abc} A^b_\mu \phi^c,$$

$$F_{\mu\nu} = F_{\mu\nu}^a t^a = \left( \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e e^{abc} A^b_\mu A^c_\nu \right) t^a$$
and the trace $tr$ in Lagrangian (2) is defined so that $tr(t^at^b) = \delta^{ab}$.

Here we are interested in purely magnetic configurations, hence we have $F_{\mu\nu} \tilde{F}^{\mu\nu} = 0$. The elementary excitations are a massless photon, two massive charged vectors bosons with mass $M_W = ev$, and a massive neutral Higgs scalar with mass $M_H = \sqrt{2} \text{gev}$. Varying the action with respect to the matrix we obtain the following expression for the energy-momentum tensor

$$T_{\lambda\rho} = -\frac{g^{\mu\nu} F^a_{\mu\lambda} F^a_{\nu\rho}}{\sqrt{1 + \frac{1}{16\beta^2} F^a_{\mu\nu} F^a_{\mu\nu}}} - 2\beta^2 g_{\lambda\rho} \left( 1 - \sqrt{1 + \frac{1}{4\beta^2} F^a_{\mu\nu} F^a_{\mu\nu}} \right)$$  \hspace{1cm} (3)

For static spherical symmetric solutions, the metric can be parametrized as \cite{26, 35}

$$ds^2 = -e^{2\nu(R)} dt^2 + e^{2\lambda(R)} dR^2 + r^2(R)(d\theta^2 + \sin^2\theta d\phi^2) \hspace{1cm} (4)$$

We consider the ’t Hooft-Polyakov ansatz for the gauge and scalar fields

$$A^a_R(R) = 0 = A^a_\theta, \quad A^a_\phi = e^{\phi^a R} \left[ \frac{W(R)}{e} - 1 \right], \quad A^a_\phi = -e^{\phi^a R} \left[ \frac{W(R) - 1}{e} \right] \sin \theta, \hspace{1cm} (5)$$

and

$$\phi^a = e^{\phi R} vH(R). \hspace{1cm} (6)$$

Putting the above ansatz in Eq.1, defining $\alpha^2 = 4\pi Gv^2$ and rescaling $R \to R/ev$, $r(R) \to r(R)/ev$ and $\beta \to \beta ev^2$ we get the following expression for the Lagrangian

$$\int dRe^{\nu+\lambda} \left[ \frac{1}{2} \left( 1 + e^{-2\lambda} \left( (r')^2 + \nu'(r^2) \right) \right) - \alpha^2 r^2 \left( 1 - \sqrt{1 + \frac{V_1}{\beta^2}} \right) - V_2 \right] \hspace{1cm} (7)$$

where

$$V_1 = \frac{1}{r^2} e^{-2\lambda} \left( W' \right)^2 + \frac{1}{2r^4} \left( W^2 - 1 \right)^2 \hspace{1cm} (8)$$

and

$$V_2 = \frac{1}{2} e^{-2\lambda} \left( H' \right)^2 + \frac{1}{r^2} \left( HW \right)^2 + \frac{1}{4} g^2 \left( H^2 - 1 \right)^2 \hspace{1cm} (9)$$

Here the prime denotes differentiation with respect to $R$. The dimensionless parameter $\alpha$ can be expressed as the mass ratio

$$\alpha = \sqrt{4\pi} \frac{M_W}{eM_{Pl}} \hspace{1cm} (10)$$

where $M_{Pl} = 1/\sqrt{G}$ is the Planck mass. As expected in the limit of $\beta \to \infty$ the above action reduces to that of the Einstein-Yang-Mills-Higgs model. Also, the limit $\alpha = 0,$
for which we must have \( \nu(R) = 0 = \lambda(R) \) corresponds to the flat space Born-Infeld-Higgs theory\[21\]. Note that the use of the normal trace allowed to reaccomodate the Lagrangian in terms of a square root without reference to the gauge group generators; this is not possible for the case of the symmetrized trace.

From now on we consider the gauge \( r'(R) = R \), corresponding to the Schwarzschild-like coordinates and rename \( R = r \). We define \( A = e^{\nu+\lambda} \) and \( N = e^{-2\lambda} \). Integrating the \( tt \) component of the energy-momentum we get the mass of the monopole equal to \( M/evG \) where

\[
M = \alpha^2 \int_0^\infty dr r^2 \left\{ V_2 - 2\beta^2 \left( 1 - \sqrt{1 + \frac{V_1}{\beta^2}} \right) \right\} \tag{11}
\]

Following ’t Hooft the electromagnetic \( U(1) \) field strength \( F_{\mu\nu} \) can be defined as

\[
F_{\mu\nu} = \frac{\phi^a F^a_{\mu\nu}}{|\phi|} - \frac{1}{e|\phi|} \epsilon^{abc} \phi^a D_{\mu} \phi^b D_{\nu} \phi^c.
\]

Then using the ansatz(3) the magnetic field

\[
B_i = \frac{1}{2} \epsilon^{ijk} F_{jk}
\]

is equal to \( e_i^i/er^2 \) with a total flux \( 4\pi/e \) and unit magnetic charge.

The \( tt \) and \( rr \) components of Einstein’s equations are

\[
\frac{1}{2} (1 - (rN)') = \alpha^2 r^2 \left\{ V_2 - 2\beta^2 \left( 1 - \sqrt{1 + \frac{V_1}{\beta^2}} \right) \right\} \tag{12}
\]

\[
\frac{A'}{A} = \alpha^2 r \left[ (H')^2 + \frac{2(W')^2}{r^2 \sqrt{1 + \frac{V_1}{\beta^2}}} \right] \tag{13}
\]

and the matter field equations are

\[
\left( AN \frac{W'}{\sqrt{1 + \frac{V_1}{\beta^2}}} \right)' = WA \left( H^2 + \frac{W^2 - 1}{r^2 \sqrt{1 + \frac{V_1}{\beta^2}}} \right) \tag{14}
\]

\[
(ANr^2H')' = AH \left( 2W^2 + g^2 r^2 (H^2 - 1) \right) \tag{15}
\]

Note that the field \( A \) can be eliminated from the matter field equations using Eq.(12).

Now we consider the globally regular solution to the above equations. Expanding the fields in powers of \( r \) and keeping the leading order terms we obtain the following expressions near the origin

\[
H = ar + O(r^3), \tag{16}
\]

\[
W = 1 - br^2 + O(r^4), \tag{17}
\]

\[
N = 1 - cr^2 + O(r^4) \tag{18}
\]
where $c$ is expressed in terms of the free parameters $a$ and $b$ as

$$c = \alpha^2 \left[ a^2 + \frac{g^2}{6} + \frac{4}{3} \beta^2 \left( \sqrt{1 + \frac{6b^2}{\beta^2}} - 1 \right) \right]$$

We are looking for asymptotically flat solution and hence we impose

$$N = 1 - \frac{2M}{r}.$$  

Then the gauge and the Higgs fields has the following behaviour in the asymptotically far region:

$$W = Cr^{-M} e^{-r} \left( 1 + O \left( \frac{1}{r} \right) \right)$$

$$H = \begin{cases} 
1 - Br^{-\sqrt{2}gM-1} e^{-\sqrt{2}gr}, & \text{for } 0 < g \leq \sqrt{2} \\
1 - \frac{c^2}{g^2-2} e^{-2M-2e^{-2gr}}, & \text{otherwise.}
\end{cases}$$

We solved the equations of motion numerically using the above boundary conditions. The solutions are pretty much the same as those for the case of the symmetrised trace\[33\] for $\beta \sim 1$ and they agree with the corresponding solution for the Yang-Mills-Higgs case for large $\beta$\[27, 26\]. For a definite value of $\beta$, the solution exists up to some critical value $\alpha_{\text{max}}$ of the parameter $\alpha$. The minima of the metric function decreases as we increase the value of $\alpha$ for $\alpha < \alpha_{\text{max}}$ and it approaches zero for $\alpha \to \alpha_{\text{max}}$. The solution does not exist for $\alpha > \alpha_{\text{max}}$. It is observed that when we decrease the value of $\beta$ the $\alpha_{\text{max}}$ goes on increasing. The values of $\alpha_{\text{max}}$ for different $\beta$ are given in Table 1. However it is also found that, similar to the flat space monopoles\[21\] there is a critical value $\beta_c$ for finite $\alpha$, and the solutions does not exist for $\beta < \beta_c$. For $\alpha = 1$ and $g = 0$ we find $\beta_c \sim 0.1$ which is much smaller than the corresponding value for flat space which is approximately 0.5. The profile for different values of $\alpha, \beta$ for the case of $g = 0$ are given in the Figs.1, 2&3.

To understand these results, let us recall that the existence of an upper bound $\alpha_{\text{max}}$ for the gravitation strength can be interpreted by observing that monopole mass/radius ratio increases as $\alpha$ increases so that $\alpha_{\text{max}}$ can be seen as the value at which the monopole becomes gravitationally unstable and collapses \[24\]-\[31\]. This for Yang-Mills action, which corresponds to the $\beta \to \infty$ limit of DBI action. Now, as $\beta$ decreases from its limiting value, the mass of the monopole decreases (see Table 2) so that the collapse should occur for a value $\alpha_{\text{max}}^\beta > \alpha_{\text{max}}^{\beta=\infty}$, as observed in our solutions. The same kind of analysis can be performed regarding the lowering of $\beta_c$ as $\alpha$ increases: as shown in \[20\]-\[24\], the singular behavior occurring at $\beta_c$ manifests as an abrupt descent in the soliton mass, a phenomenon which is in concurrence with the enhancing of the mass as $\alpha$ grows. That is the reason why $\beta_c^\alpha < \beta_c^{\alpha=0}$.

In summary, we analysed the gravitating monopole solutions in non-Abelian Born-Infeld-Higgs system with a normal trace structure. The solutions exist up to some critical value $\beta_c$ of the Born-Infeld parameter $\beta$ below which there is no solution. It was not possible to study this feature in the corresponding model with symmetrised trace structure since the perturbative expansion implicit in this case is not valid for
small $\beta$. We also found that the parameter $\alpha$ has some maximum allowed value $\alpha_{\text{max}}$ for a definite $\beta$. This $\alpha_{\text{max}}$ increases as we decrease $\beta$. It would be worth studying if the similar behaviour occurs in case of non-Abelian black holes. Also it should be possible to study the dyon solutions in the non-Abelian model with normal trace structure.

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| $\beta$ | $\alpha^2_{max}$ |
|--------|----------------|
| 0.10   | 8.1            |
| 0.15   | 7.2            |
| 0.20   | 6.5            |
| 0.25   | 6.1            |
| 0.30   | 5.9            |
| 0.50   | 5.7            |
| 1.00   | 5.6            |

Table 1 $\alpha^2_{max}$ for different $\beta$ ($g = 0$). $\alpha_{max}$ decreases as we increase $\beta$.

| $\beta$ | $M/evG$ |
|--------|---------|
| 1.0    | 1.22097 |
| 0.5    | 1.19034 |
| 0.27   | 1.11625 |
| 0.2    | 1.05062 |
| 0.15   | 0.98329 |

Table 2 $M/evG$ for different $\beta$ ($g = 0$ and $\alpha^2 = 2.5$). Mass decreases as we decrease $\beta$. 
Figure 1: Plot for the metric function $N$ as a function of $r$ for $g = 0$, $\alpha^2 = 2.5$ for different values of $\beta$. Curve I is for $\beta = 1.0$ and curve II for $\beta = 3.0$.

Figure 2: Plot for the gauge field $W$ as a function of $r$ for $g = 0$, $\alpha^2 = 2.5$ for different values of $\beta$. Curve I for $\beta = 1.0$ and curve II for $\beta = 3.0$. 
Figure 3: Plot for the Higgs field $H$ as a function of $r$ for $g = 0$, $\alpha^2 = 2.5$ for different $\beta$. Curve I is for $\beta = 1.0$ and curve II for $\beta = 3.0$.