Quantum-mechanical correlations and Tsirelson bound from geometric algebra

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Received: 25 February 2021 / Accepted: 17 July 2021 / Published online: 12 September 2021
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Abstract The Bell–Clauser–Horne–Shimony–Holt (Bell–CHSH) inequality [1] is a well-known generalization of the Bell inequality [2,3] and, like its famous predecessor, is designed to prove that no hidden-variable theory can reproduce the correlations predicted by quantum mechanics (QM). It can be shown that certain QM correlations lead to a violation of the classical bound established by the inequality, while all correlations, QM and classical, respect a QM bound (the Tsirelson bound). Here, we show that these well-known results depend crucially on the assumption that the values of physical magnitudes are scalars. More specifically, the assumption that these values are not scalars, but vectors that are elements of the geometric algebra $G^3$ over $\mathbb{R}^3$, makes it possible that the classical bound is violated and the QM bound respected, even given a locality assumption. The result implies, first, that the origin of the Tsirelson bound is geometrical, not physical; and, second, that a local hidden-variable theory does not contradict QM if the values of physical magnitudes are vectors in the geometric algebra $G^3$.

Keywords No-hidden-variables theorems · Bell–CHSH inequality · Tsirelson bound · Geometric algebra

1 Introduction

The Bell–Clauser–Horne–Shimony–Holt (Bell–CHSH) inequality [1] is a well-known generalization of the Bell inequality [2,3] and, like its famous predecessor, is designed to prove that no hidden-variable theory can reproduce the correlations predicted by quantum mechanics (QM) if it respects an assumption of locality [4,5]. It can be shown that some QM correlations lead to a violation of the classical bound established by the inequality and it can also be shown that no correlations, QM or classical, exceed a specific QM bound, i.e. the Tsirelson bound. Here, we show that these well-known results depend crucially on the assumption that the values of physical magnitudes are scalars. More specifically, the assumption that these values are not scalars, but vectors that are elements of the geometric algebra $G^3$ over $\mathbb{R}^3$, makes it possible that the classical bound is violated and the QM bound respected, even given a locality assumption. The result implies, first, that the origin of the Tsirelson bound is geometrical, not physical; and, second, that a local hidden-variable theory does not contradict QM if the values of physical magnitudes are vectors in the geometric algebra $G^3$.

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2 The Bell–CHSH argument

The Bell–CHSH inequality and the argument based on it are well known (see, e.g. [6]), so a brief introduction of both will suffice here. Consider a large collection of pairs of spin-1/2 particles measured for their spin components in certain directions of \( \mathbb{R}^3 \). Consider four unit vectors \( \mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}' \in \mathbb{R}^3 \), such that \( \mathbf{a} \) and \( \mathbf{a}' \) define two measurement directions for the left particle and analogously for \( \mathbf{b}, \mathbf{b}' \) and the right particle. Let the oriented axis defined, e.g. by vector \( \mathbf{a} \) be identical with the physical magnitude \( A \) (which later in QM becomes the observable \( A \)) and let the magnitude’s value found upon measurement be either 1 or \(-1\), and analogously for \( \mathbf{a}', \mathbf{b}, \mathbf{b}' \). Consequently, the expectation value \( E(\mathbf{ab}) \) of the product of values of \( A \) and \( B \), revealed in a measurement of spin in the \( \mathbf{a} \)-direction on the left and the \( \mathbf{b} \)-direction on the right, is an element of \([-1, 1]\), and similarly for other combinations of the four vectors. We define the classical correlation \( c(\mathbf{a}, \mathbf{b}) \) as: \( c(\mathbf{a}, \mathbf{b}) = \int_\lambda (E(\mathbf{a} \mathbf{b}) | \lambda) \, dP(\lambda) \), where \( \lambda \) is a variable describing an eventual joint influence on the values of \( A \) and \( B \) and \( E(\mathbf{ab} | \lambda) \) is the expected value of the product \( \mathbf{ab} \) with respect to the probability measure \( P_{\mathbf{ab}}(\cdot | \lambda) \). To derive the desired inequality, we make three assumptions. First, we assume locality, i.e. we assume that an expectation value of the form \( E(\mathbf{ab} | \lambda) \) factorizes: \( E(\mathbf{ab} | \lambda) = E(\mathbf{a} | \lambda) E(\mathbf{b} | \lambda) \). Second, we assume that \(|E(\mathbf{x} | \lambda)| \leq 1\), for \( \mathbf{x} = \mathbf{a}, \mathbf{a}', \mathbf{b} \) or \( \mathbf{b}' \), something we will refer to as boundedness. Third, we note the simple algebraic fact that \(|x + y| + |x - y| \leq 2\) for real numbers \( x, y \in [-1, 1] \). Consider now the four classical correlations \( c(\mathbf{a}, \mathbf{b}), c(\mathbf{a}, \mathbf{b}'), c(\mathbf{a}', \mathbf{b}), \) and \( c(\mathbf{a}', \mathbf{b}') \). Using the above definition, we have

\[
c(\mathbf{a}, \mathbf{b}) + c(\mathbf{a}, \mathbf{b}') + c(\mathbf{a}', \mathbf{b}) - c(\mathbf{a}', \mathbf{b}') \\
\leq \int_\lambda (E(\mathbf{ab} | \lambda) + E(\mathbf{ab}' | \lambda) + E(\mathbf{a} \mathbf{b} | \lambda) - (E(\mathbf{a}' \mathbf{b}' | \lambda))) \, dP(\lambda). \tag{1}
\]

We abbreviate the RHS of (1) as \( \int_\lambda S(\lambda) \, dP(\lambda) \) and consider the integrand \( S(\lambda) \) separately, suppressing the \( \lambda \)-dependence. Then, using our three assumptions in turn, we get:

\[
S = E(\mathbf{ab}) + E(\mathbf{ab}') + E(\mathbf{a} \mathbf{b}') - E(\mathbf{a}' \mathbf{b}') \\
= E(\mathbf{a})(E(\mathbf{b}) + E(\mathbf{b}')) + E(\mathbf{a}')(E(\mathbf{b}) - E(\mathbf{b}')) \quad \text{(locality)} \\
\leq |E(\mathbf{b}) + E(\mathbf{b}')| + |E(\mathbf{b}) - E(\mathbf{b}')| \quad \text{(boundedness)} \\
\leq 2. \quad \text{(algebraic fact).} \tag{2}
\]

From (1) and (2), we now have the Bell–CHSH inequality:

\[
c(\mathbf{a}, \mathbf{b}) + c(\mathbf{a}, \mathbf{b}') + c(\mathbf{a}', \mathbf{b}) - c(\mathbf{a}', \mathbf{b}') \leq 2. \tag{3}
\]

Inequality (3) is violated by suitable QM systems, as is readily shown. Define an observable for the spin component in the direction of vector \( \mathbf{a} \) as a self-adjoint operator \( A = \sigma^1 \mathbf{a} \) on a two-dimensional Hilbert space \( \mathbb{H}^A \) (where \( \sigma^1 \) is the Pauli spin operator on \( \mathbb{H}^A \), acting on the state vector representing system \( A \)), with \( \| A \| \leq 1 \) (where ‘\( \| ... \| \)’ is the operator norm). Similarly, for vector \( \mathbf{a}' \) define operator \( A' = \sigma^1 \mathbf{a}' \) on \( \mathbb{H}^A \), and analogously, for vectors \( \mathbf{b}, \mathbf{b}' \), define operators \( B = \sigma^2 \mathbf{b} \) and \( B' = \sigma^2 \mathbf{b}' \) on another two-dimensional space \( \mathbb{H}^B \) for system \( B \). Define the QM correlation \( \langle A, B \rangle := \langle \sigma^1 \mathbf{a} \otimes \sigma^2 \mathbf{b} \rangle_\psi \) for QM state \( \psi \in \mathbb{H}^A \otimes \mathbb{H}^B \). Assuming that \( \psi \) is the singlet state, we calculate \( \langle A, B \rangle = -\cos \theta_{ab} \), and analogously for \( \langle A, B' \rangle, \langle A', B \rangle, \langle A', B' \rangle \). It is easy to show that the classical and QM correlations do not match. Suppose, for reductio, that correlations of both kinds each lead to an inequality exhibiting the same bound, i.e. that (3) becomes:

\[
\langle A, B \rangle + \langle A, B' \rangle + \langle A', B \rangle - \langle A', B' \rangle \leq 2. \tag{4}
\]

Assume, second, that \( \mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}' \) are coplanar with angles:

\[
\langle (\mathbf{a}, \mathbf{a}') \rangle = \langle (\mathbf{b}', \mathbf{b}) \rangle = \pi/2 \text{ and } \langle (\mathbf{a}', \mathbf{b}') \rangle = \pi/4. \tag{5}
\]

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Then, the QM correlations are calculated to be \( \langle A, B \rangle = \langle A, B' \rangle = \langle A', B \rangle = - \langle A', B' \rangle = + 1/\sqrt{2} \), such that the LHS of (4) equals \( 4/\sqrt{2} = 2\sqrt{2} \). Thus, for the angles in (5), (4) implies \( 2\sqrt{2} \leq 2 \), a contradiction. Instead of the naively assumed (4), QM yields, again for the angles in (5)

\[
\langle A, B \rangle + \langle A, B' \rangle + \langle A', B \rangle - \langle A', B' \rangle = 2\sqrt{2}.
\] (6)

Accordingly, QM correlations in certain cases violate the Bell–CHSH inequality. (Henceforth, the argument leading to the contradiction of (4) and (6) is referred to as the Bell–CHSH argument; and the situation containing many pairs of spin-1/2 particles in the singlet state as the Bell–CHSH situation.)

3 The Tsirelson bound

However, the violation of (4) by QM cannot be arbitrarily large. It was shown by Tsirelson [7] that, for any choice of unit vectors \( a, a', b, b' \),

\[
\langle A, B \rangle + \langle A, B' \rangle + \langle A', B \rangle - \langle A', B' \rangle \leq 2\sqrt{2}.
\] (7)

The RHS of (7) is the quantum-mechanical (or Tsirelson) bound for the violation of the Bell–CHSH inequality. Again, we just sketch a proof of the result. Define an operator \( B = AB + AB' + A'B' - A'B' \). To prove (7) it is sufficient to show that \( \|B\| \leq 2\sqrt{2} \). We first recall that \( \|A\| \leq 1 \) and similarly for \( \|A'\|, \|B\|, \|B'\| \). Moreover, we recall the commutation relations \( [A, A'] \neq 0 \neq [B, B'] \) and \( [A, B] = [A, B'] = [A', B] = [A', B'] = 0 \). Using these norms and relations, we have, by an elementary calculation [8],

\[
B^2 \leq 4 \cdot 1 - [A, A'][B, B'],
\] (8)

where \( 1 \) is the unit operator. Since \( \| [A, A'] \| \leq 2 \), and analogously for \( B, B' \), we immediately have

\[
\|B\| = \sqrt{\|\{4 \cdot 1 - [A, A'][B, B']\}\|} \leq \sqrt{(4 + \|[A, A']\|\|[B, B']\|)} \leq \sqrt{(4 + 4)} = 2\sqrt{2},
\] (9)

From (8) and (9), we get

\[
\|B\| = \sqrt{\|\{4 \cdot 1 - [A, A'][B, B']\}\|} \leq \sqrt{(4 + \|[A, A']\|\|[B, B']\|)} \leq \sqrt{(4 + 4)} = 2\sqrt{2},
\] (10)

such that (7) is proved.

4 Quantum-mechanical correlations and Tsirelson bound from geometric algebra

The derivations of (6) and (7) make essential use of two basic concepts of QM: the definition of QM correlations from Hilbert space operators and the non-commutation of some of these operators (i.e. \( [A, A'] \neq 0 \neq [B, B'] \)). However, there is a way to obtain these very results without making any quantum-mechanical assumptions and instead employing elementary Geometric Algebra (GA). This alternative derivation is of interest as it uses GA elements (vectors in \( R^3 \) and constructions from them) but no QM machinery (no operators acting on vectors in \( H^A \otimes H^B \)) and thus opens a way to a new, geometric understanding of (6) and (7).

Our GA approach to (6) and (7) proceeds in two steps. First, we interpret the unit vectors \( a, a', b, b' \in R^3 \) as elements of a geometric algebra. Second, we allow the values of physical magnitudes to be identical with these vectors. We begin with the introduction of a geometric algebra. Let \( \{ e_1, e_2, e_3 \} \) be an orthonormal basis of \( R^3 \) generating \( G^3 \), the geometric algebra over \( R^3 \), by means of the geometric product [9–11]. The unit vectors \( a, a', b, b' \in R^3 \) now can be written in terms of the basis vectors as \( a = a_1 e_1 + a_2 e_2 + a_3 e_3 \), etc. The basis vectors instantiate a characteristic non-commutative structure, given by \( e_i e_j + e_j e_i = 2 \delta_{ij} (i, j = 1, 2, 3) \), that carries over to other vectors: generally, neither one of the vectors \( a, a', b, b' \) commutes with any other. Consider, on the other hand, that we are interested in a geometric-algebraic account of the Bell–CHSH situation, where the Pauli operators describing the spin in QM exhibit the same non-commutative structure as the vectors \( e_1, e_2, e_3 \) in \( G^3 \). We would thus expect this structure to be the critical factor in our GA approach—but it turns out that this is not
the case. Although the structure arguably is crucial in GA accounts of other (non-statistical) no-hidden-variable arguments [12], it does not play a prominent role in a GA description of the Bell–CHSH situation. Here, as we will see presently, the reinterpretation of values of magnitudes as vectors is all-important.

We thus assume that the values of the physical magnitudes referred to in the Bell–CHSH argument (i.e. spin components) are vectors. In particular, we assume that the unit vectors \( \mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}' \) introduced above are possible values of such magnitudes and thus that functions of these values are functions of vectors. This assumption immediately suggests a revised notion of expectations. According to the standard definitions of the expectations ‘\( E(\cdot) \)’ and ‘\( E(\cdot;\cdot) \)’ in (1) and (2), they are functions from sets of possible outcomes, or sets of pairs of them, into \( \mathbb{R} \). Since these functions’ range is \( \mathbb{R} \), they are \textit{real-valued} random variables. Analogously, we can now introduce new functions: \textit{vector-valued} random variables. We choose unary functions ‘\( F(\cdot) \)’ from sets of possible outcomes into \( \mathbb{R}^3 \) and binary functions ‘\( F(\cdot, \cdot) \)’ from sets of pairs of possible outcomes into \( \mathbb{R} \). Hence, possible outcomes are mapped into vectors and pairs of them are mapped into scalars.

More explicitly, we define a set \( \mathcal{O}_A \) of possible outcomes of a measurement of magnitude \( A \) simply as \( \mathcal{O}_A = \{ \pm \mathbf{a} \} \), where \( \pm \mathbf{a} \) are two antiparallel unit vectors in \( \mathbb{R}^3 \) given by \( A \); and we define the unary function \( F(\cdot \cdot) \) from \( \mathcal{O}_A \) into \( \mathbb{R}^3 \) as: \( F(\mathbf{a}) = \alpha \mathbf{a} \), where \( \alpha \in [-1, 1] \). (Analogously for \( \mathcal{O}_B \).) We define the binary function \( F(\cdot, \cdot) \) from \( \mathcal{O}_A \times \mathcal{O}_B \) into \( \mathbb{R} \) as: \( F(\mathbf{a}, \mathbf{b}) = F(\mathbf{a}) \cdot F(\mathbf{b}) = \alpha \beta \mathbf{a} \cdot \mathbf{b} \), where \( \alpha, \beta \in [-1, 1] \) and the product ‘\( \cdot \)’ is the inner product induced by the geometric product on \( \mathbb{G}^3 \). We write \( F(\mathbf{a}, \mathbf{b}) = F(\mathbf{ab}) \).

Now, consider a new magnitude \( S' \), structurally similar to \( S \) in (2), and define it as

\[
S' = F(ab) + F(ab') + F(a'b) - F(a'b').
\]

Due to our definition of the \( F \)-functions, \( S' \) is a scalar. Moreover, the \( F \)-functions automatically satisfy the first two steps of (2) above, i.e. locality (factorizability) for \( F(ab) \), etc. and boundedness for \( |F(ab)|, |F(a'b)| \). Hence, starting from (11) we can, without further ado, repeat these first two steps for \( S' \). Thus, we get

\[
S' \leq |F(b) + F(b')| + |F(b) - F(b')| = |\alpha \mathbf{b} + \beta \mathbf{b}'| + |\alpha \mathbf{b} - \beta \mathbf{b}'|,
\]

where we have chosen \( \alpha, \beta \in [-1, 1] \) such that \( F(b) = \alpha \mathbf{b} \) and \( F(b') = \beta \mathbf{b}' \). Suppose first that \( \alpha = \beta = 1 \) such that \( F(b) = \mathbf{b} \) and \( F(b') = \mathbf{b}' \) and thus

\[
S' \leq |\mathbf{b} + \mathbf{b}'| + |\mathbf{b} - \mathbf{b}'|.
\]

Recall the elementary algebraic fact, used in (2), that, if \( b, b' \in [-1, 1] \), then \( |b + b'| + |b - b'| \leq 2 \). Unsurprisingly, this fact does not carry over to our GA approach. It is not the case that generally \( |\mathbf{b} + \mathbf{b}'| + |\mathbf{b} - \mathbf{b}'| \leq 2 \); instead, it is the case that \( |\mathbf{b} + \mathbf{b}'| + |\mathbf{b} - \mathbf{b}'| \leq 2 \) if \( \mathbf{b} = \pm \mathbf{b}' \).

Due to the fact that \( \mathbf{b} \) and \( \mathbf{b}' \) are vectors, the term on the RHS of (13) can be larger than 2. Assume again that \( \mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}' \) form the angles, by now familiar, from (5). Choosing \( \mathbf{a} = \mathbf{e}_1 \) and \( \mathbf{a}' = \mathbf{e}_2 \), these angles imply that

\[
\mathbf{b} = -(\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{2}
\]

and

\[
\mathbf{b}' = -(\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{2}.
\]

Thus, \( \mathbf{b} + \mathbf{b}' = -\sqrt{2} \mathbf{e}_1 \) and \( \mathbf{b} - \mathbf{b}' = -\sqrt{2} \mathbf{e}_2 \). Thus, we have

\[
|\mathbf{b} + \mathbf{b}'| = |\mathbf{b} - \mathbf{b}'| = \sqrt{2}
\]

and (13) becomes

\[
S' \leq |\mathbf{b} + \mathbf{b}'| + |\mathbf{b} - \mathbf{b}'| = 2\sqrt{2}.
\]

Comparing (2) and (14), we see that \( S \) and \( S' \) both have been evaluated using a locality assumption—but (2) sets a stricter bound for \( S \) than (14) does for \( S' \). In other words, \( S' \) is not in a logical conflict with QM predictions, although it is classical in the sense that it respects a locality assumption.

(14) has been derived for particular unit vectors \( \mathbf{b}, \mathbf{b}' \), specified as in (5), i.e. such that \( \mathbf{b} \perp \mathbf{b}' \), but it extends to arbitrary unit vectors. Clearly, the term \( |\mathbf{b} + \mathbf{b}'| + |\mathbf{b} - \mathbf{b}'| \) is maximal if \( |\mathbf{b} + \mathbf{b}'| = |\mathbf{b} - \mathbf{b}'| = \sqrt{2} \), which is the case if \( \mathbf{b} \perp \mathbf{b}' \). Thus, for arbitrary unit vectors \( \mathbf{b}, \mathbf{b}' \), it is true that

\[
S' \leq |\mathbf{b} + \mathbf{b}'| + |\mathbf{b} - \mathbf{b}'| \leq 2\sqrt{2}.
\]

Finally, we consider the general case: \( \alpha, \beta \in [-1, 1] \) and arbitrary unit vectors \( \mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}' \). We distinguish three special cases. First, assume \( \alpha \beta > 0 \). Then, \( \alpha \mathbf{b} + \beta \mathbf{b}' \leq |\mathbf{b} + \mathbf{b}'| \) and \( \alpha \mathbf{b} - \beta \mathbf{b}' \leq |\mathbf{b} - \mathbf{b}'| \). Second, assume \( \alpha \beta < 0 \). Then, \( \alpha \mathbf{b} + \beta \mathbf{b}' \leq |\mathbf{b} - \mathbf{b}'| \) and \( \alpha \mathbf{b} - \beta \mathbf{b}' \leq |\mathbf{b} + \mathbf{b}'| \). Thus, in both cases

\[
|\alpha \mathbf{b} + \beta \mathbf{b}'| + |\alpha \mathbf{b} - \beta \mathbf{b}'| \leq |\mathbf{b} + \mathbf{b}'| + |\mathbf{b} - \mathbf{b}'|.
\]
From (15) and (16),
\[
|\alpha b + \beta b'| + |\alpha b - \beta b'| \leq 2\sqrt{2}.
\]

Finally, assume \(\alpha\beta = 0\), e.g. \(\alpha = 0\); then the LHS of (16) equals \(2|\beta| \leq 2\). Thus, (17) follows for all three cases and we have the unconditional inequality
\[
S' \leq 2\sqrt{2}.
\]

All in all, we have shown that, for arbitrary \(\alpha, \beta \in [-1, 1]\) and unit vectors \(a, a', b, b' \in \mathbb{G}^3\), the sum \(S'\) is bounded by \(2\sqrt{2}\). As we see from (14), this bound is attained when \(\alpha = \beta = 1\) and the unit vectors form angles as prescribed in (5) above. Since \(S' \leq 2\sqrt{2}\)—in contrast with \(S \leq 2\)—the Bell–CHSH inequality (3) cannot be derived from it and thus the Bell–CHSH argument cannot be repeated for \(S'\).

The \(F\)-functions have been introduced above to produce an expression \(S'\) that is the classical \(S\), but with scalars replaced by vectors. These functions, it turns out, are closely related to the quantum-mechanical expectations. To see this, we make one further assumption, namely, we assume that \(\alpha = -\beta = 1\). (This is well motivated from the physical situation (the singlet state), which dictates that the correlations in question are anti-correlations.) As a result, we get \(F(a, b) = -a \cdot b = -\cos \theta_{ab}\), which is just the QM correlation \(<A, B>\). (Analogously for the other \(F\)-functions.) The QM correlations thus appear as special cases of the \(F\)-functions, such that whatever holds for the latter also holds for the former. Accordingly, it is no wonder that from these functions we can derive the counterparts (14) and (18) of the QM results (6) and (7)—effectively: that we can derive these QM results not from QM, but from GA. The crucial background assumption of these derivations is that the unary-place \(F\)-functions are functions of vectors and are vectors themselves.

5 Review

The derivations in Sect. 4 invite critical questions. To answer them it is helpful to review the argument as a whole. We began with the Bell–CHSH argument, which rests on three assumptions, two of them being trivial statistics or algebra, the third one an equivalent of Bell's original locality assumption of classical physics [13]. From these three assumptions, we derived inequality (2): \(S = E(ab) + E(ab') + E(a'b) - E(a'b') \leq 2\). The expectations here are classical (i.e. are defined in the usual statistical way) and locality is taken to be a tenet of classical physics, as emphasized. A counterexample (wherein \(a, a', b, b'\) are chosen as in (5)) was introduced, exemplifying that quantum-mechanical expectations in some cases violate the inequality, i.e. giving us (6) above, \(<A, B> + <A, B'> + <A', B> - <A', B'> = 2\sqrt{2}\). The QM expectations \(<A, B>\), etc. are, of course, calculated quantum-mechanically. The juxtaposition of both derivations exhibits a conflict between classical physics, containing classical locality, and QM. Accordingly, no second type of locality is in play or has to be defined. Arguably, locality can be defined in QM as commutativity of operators (see [14]) but this is inessential for the Bell–CHSH argument that yields a logical conflict between classical locality and QM. Note also that this conflict's empirical consequences are indirect. Assume that apart from classical locality there are no additional premises in play that can reasonably be doubted or disputed; if this is the case and QM is verified experimentally (see, e.g. [15]), then this is evidence that the empirical world is nonlocal, i.e. locality is not universal.

In the next step, an additional premise was exposed. The Bell–CHSH argument was shown to rest on a hidden assumption that is not undisputable or trivially true. This assumption is the implicit definition of expectation values as products of scalars. An alternative definition—defining expectations as products of vectors—is not entirely implausible and thus had to be considered. We built a mini-theory wherein expectations (the binary \(F\)-functions) are constructed from vectors, the unary \(F\)-functions. In the presence of locality (the same classical notion as before!), we derived (14) \(S' = F(ab) + F(ab') + F(a'b) - F(a'b') \leq 2\sqrt{2}\) (where again \(a, a', b, b'\) make angles as in (5)). This derivation is not quantum-mechanical but classical in the sense that no Hilbert space formalism is employed. Finally, for arbitrary unit vectors \(a, a', b, b'\), we derived (18): \(S' \leq 2\sqrt{2}\). The calculation again is classical, by the criterion just mentioned. But the result matches the quantum-mechanical one—in the presence of classical locality!
By contrast, the standard derivation of the Tsirelson bound (in Sect. 4) was quantum-mechanical in the sense that it concerned Hilbert space operators. But we have now given an alternative derivation of the bound using only $\mathbb{R}^3$ geometry and no Hilbert space formalism.

Consider also the fact that QM expectations in special cases have the form of inner products of $\mathbb{R}^3$ vectors. (Consider two-valued operators $A$ and $B$ on $\mathbb{C}^2$ and their respective associated Bloch vectors $a$ and $b$, with $a, b \in \mathbb{R}^3$. Then for a pure, maximally entangled state, we have $\langle A, B \rangle = a \cdot b$, where $\langle A, B \rangle$ is the QM expectation value.) Thus, we find expectations in the form of inner products of vectors within ordinary QM. Accordingly, it seems not surprising that expectations of this form can be used to derive the Tsirelson bound. But of course, even such an unsurprising derivation deserves to be carried out explicitly and the result is indeed of theoretical interest.

While the standard derivation, presented in Sect. 3 above, makes crucial use of Hilbert space operators and their properties, the alternative derivation in Sect. 4 uses entities not exclusive to the Hilbert space formalism: simple $\mathbb{R}^3$ unit vectors.

6 Implications

The significance of our result for the interpretation of QM is twofold. First, it is often suggested that the existence of an upper bound $2\sqrt{2}$ for the Bell–CHSH expression (the LHS of (7)), is a specific characteristic of QM that must be explained from physical principles [16]. But the derivations in Sect. 4 imply something else, i.e. that this bound is of purely geometric origin, being due solely to the particular arrangement of the four $\mathbb{R}^3$ unit vectors $a$, $a'$, $b$, $b'$. To be sure, the crucial QM assumption made in the derivation of (7) is that $[A, A'] \neq 0 \neq [B, B']$, but this is not a QM-specific, or even specifically physical, assumption, as witnessed by the fact that its GA counterpart $[a, a'] \neq 0 \neq [b, b']$ follows directly from (a) writing $a, a', b, b'$ in terms of the basis vectors $e_1, e_2, e_3$, (b) the assumptions $a \neq \pm a'$ and $b \neq \pm b'$, and (c) the non-commutativity of $e_1, e_2, e_3$.

The second, much weightier, implication concerns the Bell–CHSH argument for a disproof of local hidden variable theories for QM. A locality assumption is explicitly present in (2) and thus implicitly so in (4), and it apparently is the only assumption within the argument that can reasonably be doubted. This has led to the oft-repeated claim that the argument constitutes a general proof of the nonlocality of QM (see, e.g. [17]). However, it is clear now that this claim requires careful qualification.

As we saw, the first two steps of the three-step derivation of (2) and the two steps summarized in (12) include the same locality (factorizability) assumption. This assumption is employed before the classical and GA derivations diverge. Hence, there is reason to infer that the GA account is just as local as the classical one. This, however, implies that the GA account respects the locality assumption and nevertheless matches the QM predictions summarized in (6) and (7). Hence, this account casts the unqualified opposition between locality and QM predictions into doubt. An explicit contradiction of both has been exhibited above, but it arises only given the tacit premise that the values of physical magnitudes are scalars; without that premise it can be avoided.

Our derivations of QM correlations (in (14)) and the Tsirelson bound (in (18)) show that there is no general, entirely unconditional, conflict between QM correlations and locality; the contradiction we have is conditional on the premise that the values of physical magnitudes are scalars, not vectors. Certainly, at some point we will have to ask whether the idea of vectors as values is at all plausible. However, what is at issue here is not plausibility, but the mere compossibility of QM correlations and locality—which is demonstrated by means of the new idea. Moreover, at first glance the idea is plausible. The Bell–CHSH situation involves vectorial physical magnitudes (spin components) and in this context the assumption that these magnitudes have vectorial values, unfamiliar as it may be, makes good sense.
An obvious next step in the discussion would be to consider whether the GA approach to the Bell–CHSH situation is adaptable to other (non-statistical) no-hidden-variable arguments like the Greenberger–Horne–Zeilinger argument [18,19]. This question needs a separate treatment and a recent proposal [12] has not yet answered it completely. Further research is required.

Declarations

Conflict of interest The author states that there is no conflict of interest.

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