Plebański–Demiański-like solutions in metric–affine gravity

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Abstract. We consider a (non-Riemannian) metric–affine gravity theory, in particular its non-metricity–torsion sector ‘isomorphic’ to Einstein–Maxwell theory. We map certain Einstein–Maxwell electrovacuum solutions to it, namely the Plebański–Demiański class of Petrov type D metrics.

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1. Introduction

There is currently a revival of interest in metric–affine gravity (MAG) theories. It has been demonstrated that they contain the axi–dilatonic sector of low-energy string theory [1] as a special case. Moreover, the gravitational interactions involving the axion and dilaton may be derived from a geometrical action principle involving the curvature scalar with a non-Riemannian connection. In other words, the axi–dilatonic sector of low-energy string theory can be expressed in terms of a geometry with torsion and non-metricity [2]. This formulation emphasizes the geometrical nature of the axion and dilaton fields and raises questions about the most appropriate geometry for the discussion of physical phenomena involving these fields.

Recently, it has been proposed that certain MAG models can be reduced to an effective Einstein–Proca system [3, 4]. Indeed, we have in these kinds of models, beside the orthonormal coframe of spacetime, effectively only one extra 1-form (co-vector) field as an additional degree of freedom.

Very important classes of Petrov type D solutions of the Einstein–Maxwell equations are the Plebański classes. The most general of these is the so-called Plebański–Demiański
solution [5], which, as is well known, contains as special cases, among others, the Plebański–Carter, the Kerr–Newman, and the Kerr solutions [6]. In this paper we are going to map this complete space of general electrovacuum solutions to a metric–affine gravity model, which generalizes Einstein’s general relativity. Thus, we are able to present solutions to this MAG model and give these solutions a physical interpretation.

One arrives at the metric–affine gauge theory of gravity if one gauges the affine group $A(4, R)$ and additionally allows for a metric $g$ [7]. The four-dimensional affine group $A(4, R)$ is the semidirect product of the translation group $R^4$ and the linear group $GL(4, R)$, that is, $GL(4, R) = R^4 \triangleright \triangleleft GL(4, R)$. The spacetime of MAG encompasses two different post-Riemannian structures: the non-metricity 1-form $Q_{\alpha\beta} = Q_{i\alpha\beta} \, dx^i$ and the torsion 2-form $T^a = \frac{1}{2} T^{ij}_a \, dx^i \wedge dx^j$. According to Yang–Mills theory, gauge Lagrangians of MAG are quadratic in curvature, torsion and non-metricity. One way to investigate the potential of such models is to look for exact solutions.

The search for exact solutions of MAG began with the work of Tresguerres [8, 9], Tucker and Wang [10], Obukhov et al [11], Vlashinsky et al [12] and Puntigam et al [13]. Macías et al [14] and Socorro et al [15] mapped the Einstein–Maxwell sector of dilaton–gravity, emerging from low-energy string theory, and found new soliton and multipole solutions of MAG. However, it is important to note that in order to incorporate the scalar dilaton field, one could, for instance, generalize the torsion kink of Baekler et al [16], an exact solution with an external massless scalar field, or one could turn to the axi–dilaton sector of MAG [1]. Moreover, solutions implying the existence of torsion shock waves have already been found by García et al [17]. In this spirit, we are going to look for a wide class of solutions of the vacuum field equations of MAG. Note that such a solution with the additional electromagnetic field of a point charge has been presented in [13]. There it was confirmed that the electromagnetic field is not directly influenced by the post-Riemannian structures of torsion and non-metricity.

A general quadratic Lagrangian in MAG reads [4, 7]:

$$V_{\text{MAG}} = \frac{1}{2\kappa} \left[ - a_0 R^{\alpha\beta} \wedge \eta_{\alpha\beta} - 2\lambda_{\text{cosm}} \eta + T^a \wedge \left( \sum_{l=1}^{3} a_l \, T^a \right) \right. \\
+ 2 \left( \sum_{l=2}^{4} c_l \right) \wedge \left( \sum_{l=1}^{8} c_l \, \eta_{\alpha\beta} \wedge \left( \sum_{l=1}^{4} b_l \, Q_{\alpha\beta} \right) \right) \\
- \frac{1}{2} R^{\alpha\beta} \wedge \left( \sum_{l=1}^{6} w_l \, W^{\alpha\beta} + \sum_{l=1}^{5} z_l \right) \right]^{(1.1)}$$

The signature of spacetime is $(-+++)$, the volume 4-form $\eta := *1$, the 2-form $\eta_{\alpha\beta} := *\left( \partial_\alpha \wedge \partial_\beta \right)$, and the dimensionless coupling constants are

$$a_0, \ldots, a_3, b_1, \ldots, b_4, c_2, c_3, c_4, w_1, \ldots, w_6, z_1, \ldots, z_5 \, . \, (1.2)$$

Moreover, $\kappa$ is the gravitational constant and $\lambda_{\text{cosm}}$ the cosmological constant. In suitable units $\kappa = 1$, which will be assumed hereafter. In the curvature squared term we introduced the antisymmetric part $W^{\alpha\beta} := R^{(\alpha\beta)}$ and the symmetric part $Z_{\alpha\beta} := R_{(\alpha\beta)}$ of the curvature 2-form. In $Z_{\alpha\beta}$, we meet a purely post-Riemannian term. Weyl’s segmental curvature $^{(4)}Z_{\alpha\beta} := \frac{1}{4} R^{\gamma\delta} g_{\alpha\beta} = g_{\alpha\beta} \, dQ$, with the Weyl covector $Q := \frac{1}{4} Q_{\gamma\delta}$, has formally a similar structure to the electromagnetic field strength $F = dA$, but is physically quite different since it is related to Weyl rescalings.

For the torsion and non-metricity field configurations, we concentrate on the simplest non-trivial case with shear. According to its irreducible decomposition [7], the non-metricity...
contains two covector terms, namely

\[ Q_{\alpha\beta}^{(4)} = \frac{1}{H} \left( \partial_{(\alpha} \epsilon_{\beta)} \Lambda - \frac{1}{2} \mathcal{R}_{\alpha\beta} \Lambda \right), \]

a proper shear piece. Accordingly, our ansatz for the non-metricity reads

\[ Q_{\alpha\beta} = Q_{\alpha\beta}^{(3)} + Q_{\alpha\beta}^{(4)} . \]  

(1.4)

The torsion, in addition to its tensor term, contains covector and axial covector terms. Let us choose only the covector term as non-vanishing:

\[ T^a = \frac{2}{3} \theta^a \wedge T, \quad \text{with} \quad T := e_a \gamma T^a. \]  

(1.5)

Thus we are left with the three non-trivial 1-forms \( Q, \Lambda \) and \( T \). We shall assume that this triplet of 1-forms shares the spacetime symmetries, that is, its members are proportional to each other [11–14].

With propagating non-metricity \( Q_{\alpha\beta} \) two types of charge are expected to arise: a dilation charge related by Noether’s theorem to the trace of the non-metricity, the Weyl covector \( Q = Q_i \, dx^i \). This represents the connection associated with gauging the scale transformations (instead of the \( U(1) \) connection in the case of Maxwell’s field). Furthermore, nine shear charges are expected to arise, that are related to the remaining traceless term of the non-metricity \( Q_{\alpha\beta} := Q_{\alpha\beta} - Q g_{\alpha\beta} \).

The Lagrangian (1.1) is very complicated, in particular, on account of its curvature squared terms. Therefore we have to restrict its generality in order to stay within manageable limits. Our ansatz for the non-metricity is expected to require a non-vanishing post-Riemannian term quadratic in the segmental curvature. Accordingly, in (1.1) we choose

\[ w_1 = \cdots = w_6 = 0, \quad z_1 = z_2 = z_3 = z_5 = 0, \]  

(1.6)

that is, only \( z_4 \) is allowed to survive.

The plan of this paper is as follows. In section 2 a class of solutions, which is related to the Plebaniński–Demiański solution of the Einstein–Maxwell system, is presented. In section 3 we discuss the results and further prospects of the theory.

2. Plebaniński–Demiański-like solution in MAG

We start from the coframe of Plebaniński and Demiański [5], which is specified in terms of the coordinates \( (\tau, y, x, \sigma) \):

\[ \theta^0 = \frac{1}{H} \sqrt{\frac{Y}{\Delta}} \left( d\tau - z^2 \, d\sigma \right), \]  

(2.1)

\[ \theta^1 = \frac{1}{H} \sqrt{\frac{\Delta}{Y}} \, dy, \]  

(2.2)

\[ \theta^2 = \frac{1}{H} \sqrt{\frac{\Delta}{X}} \, dx, \]  

(2.3)

\[ \theta^3 = \frac{1}{H} \sqrt{\frac{X}{\Delta}} \left( d\tau + y^2 \, d\sigma \right). \]  

(2.4)

Here \( H = H(x, y), X = X(x), Y = Y(y), \) and \( \Delta = \Delta(x, y) \) are unknown functions. The coframe is orthonormal:

\[ g = o_{\alpha\beta} \theta^\alpha \otimes \theta^\beta, \quad \text{with} \quad o_{\alpha\beta} = \text{diag}(-1, +1, +1, +1). \]  

(2.5)
Thus we find the following explicit expression for the metric:

\[
g = \frac{1}{H^2} \left[ -\frac{Y}{\Delta} (dx - x^2 \, d\sigma)^2 + \frac{\tilde{\Delta}}{Y} \, dy^2 + \frac{\Delta}{X} \, dx^2 + \frac{X}{\Delta} (dy + y^2 \, d\sigma)^2 \right]. \tag{2.6}
\]

For the non-metricity and torsion we assume that they are represented by a *triplet of 1-forms*, the Weyl covector \( Q \), the covector \( \Lambda \) corresponding to the third irreducible non-metricity piece, and the torsion \( T \).

We substitute the local metric \( o_{\alpha \beta} \), the coframe (2.1)–(2.4), the non-metricity (1.4), and the torsion (1.5) into the two field equations following from the Lagrangian (1.1) with (1.6) by variation with respect to metric and connection. Then, provided the (rather weak) constraint (2.7) into the two field equations following from the Lagrangian (1.1) with (1.6) by variation with respect to metric and connection. Then, provided the (rather weak) constraint

\[
32a_0^2b_4 - 4a_0a_2b_4 + 64a_0b_3b_4 - 32a_2b_3b_4 + 48a_0b_4c_3 + 24b_4c_3^2 + 24b_3c_4^2 + 12a_0a_2b_3 \\
+ 48a_0b_3c_4 - 9a_0c_3^2 + 18a_0c_3c_4 + 3a_0c_4^2 + 6a_0^2a_2 + 24a_0^2c_4 = 0, \tag{2.7}
\]
on the coupling constants (1.2) is satisfied, we find a general exact solution for the following expressions:

\[
\begin{aligned}
Q &= \frac{\Lambda}{k_0} = \frac{T}{k_2} = H \frac{N_e y}{\sqrt{\Delta}} \, \theta^0 + H \frac{N_g x}{\sqrt{\Delta}} \, \theta^1, \\
H(x, y) &= 1 - \mu xy, \\
X(x) &= (b - g^2) + 2nx - \epsilon x^2 + 2m \mu x^3 - \left( \frac{\lambda_{\text{cosm}}}{3a_0} + \mu^2 (b + e^2) \right) x^4, \\
Y(y) &= (b + e^2) - 2my + \epsilon y^2 - 2n \mu y^3 - \left( \frac{\lambda_{\text{cosm}}}{3a_0} + \mu^2 (b - g^2) \right) y^4, \\
\tilde{\Delta}(x, y) &= x^2 + y^2. \tag{2.8}
\end{aligned}
\]

Here \( N_e \) and \( N_g \) are the quasi-electric and quasi-magnetic nonmetricity–torsion charges of the source, which satisfy

\[
\frac{z_4 k_0^2}{2a_0} (N_e^2 + N_g^2) = g^2 + e^2. \tag{2.10}
\]

The coefficients \( k_0, k_1, k_2 \) in (2.8) are determined by the dimensionless coupling constants (1.2) of the Lagrangian:

\[
\begin{aligned}
k_0 &= \left( \frac{1}{2} a_2 - a_0 \right) (8b_3 + a_0) - 3(c_3 + a_0)^2, \\
k_1 &= -9[a_0 \left( \frac{1}{2} a_2 - a_0 \right) + (c_3 + a_0)(c_4 + a_0)], \\
k_2 &= \frac{3}{4} \left[ 3a_0(c_3 + a_0) + (8b_3 + a_0)(c_4 + a_0) \right].
\end{aligned} \tag{2.11-2.13}
\]

Then the constraint (2.7) can be put into the following more compact form:

\[
b_4 = \frac{a_0 k + 2c_4 k_2}{8k_0}, \quad \text{with} \quad k := 3k_0 - k_1 + 2k_2. \tag{2.14}
\]

The constants \( \mu, b, g, e, n, \) and \( m \) are free parameters. The parameter \( \epsilon \) is related to the two-dimensional spacelike \( xy \) surface, it has the value \( \epsilon = 1 \) for spherical, \( \epsilon = 0 \) for flat, and \( \epsilon = -1 \) for hyperbolic geometry.
If we collect our results, then the non-metricity and the torsion read as follows:

\[
Q^{αβ} = \left[ k_0 o^{αβ} + \frac{3}{4} k_1 \left( \vartheta^{α(ε_β)} - \frac{1}{2} η^{αβ} \right) \right] \frac{H}{\sqrt{Δ}} \left( N_{eY} \frac{η^{0}}{\sqrt{Y}} + N_{eX} \frac{η^{3}}{\sqrt{X}} \right), \tag{2.15}
\]

\[
T^α = \frac{1}{4} k_2 θ^α ∧ \frac{H}{\sqrt{Δ}} \left( N_{eY} \frac{η^{0}}{\sqrt{Y}} + N_{eX} \frac{η^{3}}{\sqrt{X}} \right). \tag{2.16}
\]

We recognize, see also (2.8), that the members \( Q, Λ, T \) of the triplet are proportional to each other. Therefore, we have in our model, besides the spacetime metric, effectively only one extra 1-form as an additional degree of freedom. This makes it clear why a mapping of our MAG model to the Einstein–Maxwell system and, accordingly, the use of the Plebański–Demiański ansatz is possible, i.e. both models have the same number of degrees of freedom. Indeed, using this ansatz of Plebański and Demiański for a stationary metric and a corresponding ansatz for non-metricity and torsion, where we additionally assumed that only co-vector parts of these post-Riemannian structures are non-vanishing, we arrived at a general class of solutions for a MAG model.

The physical interpretation of the post-Riemannian parameters of the solution, as described above, is clear: the dilation (‘Weyl’) charges (related to \( (4)Q_{αβ} \)) are described by \( k_0 N_e \) and \( k_0 N_g \), the shear charges (related to \( (3)Q_{αβ} \)) by \( k_1 N_e \) and \( k_1 N_g \), and, eventually, the spin charges (related to \( (2)T^α \)) by \( k_2 N_e \) and \( k_2 N_g \).

The solution (2.1)–(2.5), (2.9), (2.15), (2.16) found above, was checked with the help of the computer algebra system Reduce [18, 19], using its Excalc package [20] for handling exterior differential forms, and by means of the Reduce-based GRG computer algebra system [21].

3. Discussion

The physical motivation to go beyond classical Einstein gravity by means of MAG models is fairly clear and well founded, see the discussion in [22]. One may suspect that the spin-3 modes of the linear connection in the framework of MAG lead to acausalities. However, no detailed investigation has been done into this question so far. Also, in view of the problems of other theories, like supergravity and even string field theory [26] in this respect, it appears unfair to ask questions about such things as the renormalizability of MAG.

Due to the fact that torsion couples to the spin of matter, a discussion of those experiments which may lead to restrictions on torsion also leads, due to (2.8), to restrictions on the two covector parts of the non-metricity. Therefore, under our triplet ansatz—which certainly describes a highly idealized situation—it is not necessary to devise separate experiments testing the coupling of non-metricity to the shear current of some matter model.

However, spin-\( \frac{1}{2} \) matter fields couple to the axial vector piece \( (3)T^α \) of the torsion alone, (massless) gauge fields carry a helicity of 1 and do not couple to torsion at all, see e.g. [13]. For massive fields with spin \( s = \frac{1}{2}, 1, \frac{3}{2} \), we can extract the following formula for the torsion \( T^α \) from the literature, see [23] and [24]:

\[
T^α_{\text{as seen by spin } s} = \left( 1 - \frac{1}{2s} \right) T^α + \frac{3}{2s} (3)T^α = \left( 1 - \frac{1}{2s} \right) (1)T^α + (2)T^α + \left( 1 + \frac{1}{s} \right) (3)T^α. \tag{3.1}
\]

Thereby we recognize that for massive higher-spin fields the trace part \( (2)T^α \) of the torsion couples to the spin of these matter fields in the same way as the axial part \( (3)T^α \), modulo
numerical factors of the order of unity. Accordingly, we can assume that restrictions on axial torsion also restrict the trace part in a similar way. Analysing known experiments, we find, with \[25\], \(t_i \leq 1.5 \times 10^{-15} \text{ m}^{-1}\) and, consequently, \((k_2/k_0)Q_i \leq 1.5 \times 10^{-15} \text{ m}^{-1}\) and \((k_2/k_1)\Lambda_i \leq 1.5 \times 10^{-15} \text{ m}^{-1}\). Here \(T = t_i \, dx^i\), \(Q = Q_i \, dx^i\), and \(\Lambda = \Lambda_i \, dx^i\).

On the other hand, we presented here a complete class of solutions of MAG. The physical interpretation of the parameters involved in (2.9) can be given as follows: \(\mu\) is the acceleration parameter, \(b\) is related to the angular momentum of the solution, \(m\) is the mass and \(n\) the NUT parameter [6]. The quasi-electric and quasi-magnetic charges \(e\) and \(g\), via (2.10), are related to the torsion and non-metricity charges \(N_e\) and \(N_g\), respectively.

It is important to point out that the generalization of these results to the whole electrovacuum sector of MAG, i.e. including an electromagnetic field as source, is straightforward, and these results will be reported elsewhere.

We want to conclude with two remarks. First, one would like to know at which energy scale such a MAG framework can be regarded as an effective gravitational model. According to [7], the motivation for MAG came mainly from particle physics and the manifold description of an infinite tower of fermions. One may regard such a gauge theory of gravity with Weyl invariance as a small but decisive step towards quantum gravity. Circumstances under which spacetime might become non-Riemannian near Planck energies occur in string theory or in the inflationary model during the early epoch of our universe. The simplest such geometry is metric–affine geometry, in which non-metricity appears as a field strength, side by side with torsion and curvature.

Secondly, on the one hand, the axion–dilaton theory emerges at the low-energy limit of string models. On the other hand, such models represent one sector of the MAG models. Since these two models have one important sector in common, we should consider the MAG models in a new perspective, as an effective low-energy theory of quantum gravity.

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