Anticoncentration versus the Number of Subset Sums

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Abstract: Let $\vec{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$. We show that for any $n^{-2} \leq \varepsilon \leq 1$, if

$$\# \{ \xi \in \{0, 1\}^n : \langle \vec{\xi}, \vec{w} \rangle = \tau \} \geq 2^{-en} \cdot 2^n$$

for some $\tau \in \mathbb{R}$, then

$$\# \{ \langle \vec{\xi}, \vec{w} \rangle : \vec{\xi} \in \{0, 1\}^n \} \leq 2^{O(\sqrt{n})}.$$

This exponentially improves the $\varepsilon$ dependence in a recent result of Nederlof, Pawlewicz, Swennenhuis, and Węgrzycki and leads to a similar improvement in the parameterized (by the number of bins) runtime of bin packing.

Key words and phrases: anticoncentration

1 Introduction

For $\vec{w} := (w_1, \ldots, w_n) \in \mathbb{R}^n$ and a real random variable $\xi$, recall that the Lévy concentration function of $\vec{w}$ with respect to $\xi$ is defined for all $r \geq 0$ by

$$\mathcal{L}_\xi(\vec{w}, r) = \sup_{\tau \in \mathbb{R}} \mathbb{P}[|w_1\xi_1 + \cdots + w_n\xi_n - \tau| \leq r],$$

where $\xi_1, \ldots, \xi_n$ are i.i.d. copies of $\xi$. In combinatorial settings (where $\vec{w} \in \mathbb{Z}^n$) a particularly natural and interesting case is when $r = 0$ and $\xi$ is a Bernoulli random variable, i.e., $\xi = 0$ with probability $1/2$ and $\xi = 1$ with probability $1/2$. For lightness of notation, we will denote this special case by

$$\rho(\vec{w}) = \mathcal{L}_{\text{Ber}(1/2)}(\vec{w}, 0) = \sup_{\tau \in \mathbb{R}} \mathbb{P}[\langle \vec{w}, \vec{\xi} \rangle = \tau].$$

In this note, we study the following question.

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**Question 1.1.** For a vector $\vec{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$ with $\rho(\vec{w}) \geq \rho$, how large can the range

$$\mathcal{R}(\vec{w}) = \{w_1\xi_1 + \cdots + w_n\xi_n : \xi_i \in \{0, 1\}\}$$

be?

The two extremal examples here are $\vec{w} = (0, 0, \ldots, 0)$, which corresponds to $\rho(\vec{w}) = 1$, $|\mathcal{R}(\vec{w})| = 1$ and $\vec{w} = (1, 10, \ldots, 10^{n-1})$, which corresponds to $\rho(\vec{w}) = 2^{-n}$, $|\mathcal{R}(\vec{w})| = 2^n$. Motivated by these examples, one may ask if there is a smooth trade-off between $\rho(\vec{w})$ and $|\mathcal{R}(\vec{w})|$. This turns out not to be the case. Indeed, for any $\epsilon > 0$, Wiman [6] gave an example of a $\vec{w} \in \mathbb{Z}^n$ for which $|\mathcal{R}(\vec{w})| \geq 2^{(1-\epsilon)n}$ and $\rho(\vec{w}) \geq 2^{-0.7447n}$. At the other end of the spectrum, when $\rho(\vec{w}) \geq 2^{-\epsilon n}$, the so-called inverse Littlewood–Offord theory [3–5] heuristically suggests that $\vec{w}$ is essentially contained in a low-rank generalized arithmetic progression of ‘small’ volume so that $|\mathcal{R}(\vec{w})|$ is also ‘small’. However, the number of ‘exceptional elements’ in the inverse Littlewood–Offord theorems [3–5] is unfortunately too large to be able to rigorously establish such a statement.

Nevertheless, in a recent work on the parameterized complexity of the bin packing problem (see Section 1.1), Nederlof, Pawlewics, Swennenhuis and Węgrzycki [2] showed that for any $\epsilon > 0$,

$$\rho(\vec{w}) \geq 2^{-\epsilon n} \implies |\mathcal{R}(\vec{w})| \leq 2^{\delta(\epsilon)n},$$

where

$$\delta(\epsilon) = O\left(\frac{\log \log (\epsilon^{-1})}{\sqrt{\log (\epsilon^{-1})}}\right). \quad (1.1)$$

In particular, $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. Moreover, we must have $\delta(\epsilon) \geq (2 - o(1))\epsilon$, as can be seen by considering

$$\vec{w} = (C_1, \ldots, C_1, C_2, \ldots, C_2, \ldots, C_{n/k}, \ldots, C_{n/k}) \in \mathbb{R}^n,$$

where each $C_i$ is repeated $k$ times, and $C_i$ is sufficiently small compared to $C_{i+1}$ for all $i$. Indeed, for such $\vec{w}$, we have $\rho(\vec{w}) = 2^{-\frac{1}{2} + o(1)} \frac{1}{k} \log_2 k$ while $|\mathcal{R}(\vec{w})| \leq 2^{(1+o(1))\frac{1}{k} \log_2 k}$.

We conjecture that this example is essentially the worst possible, so that $\delta(\epsilon) \leq 2\epsilon$. We are able to show that

$$\delta(\epsilon) = O(\sqrt{\epsilon}), \quad (1.2)$$

d thereby obtaining an exponential improvement over (1.1). More precisely,

**Theorem 1.2.** Let $\epsilon > 0$. For any $n \geq \epsilon^{-1/2}$ and any $\vec{w} \in \mathbb{R}^n$ satisfying $\rho(\vec{w}) \geq \exp(-\epsilon n)$, we have

$$|\mathcal{R}(\vec{w})| \leq \exp(C_{1.2} \epsilon^{1/2} n),$$

where $C_{1.2}$ is an absolute constant.

We prove this theorem in Section 2.
1.1 Application to bin packing

The bin packing problem is a classic NP-complete problem whose decision version may be stated as follows: given \( n \) items with weights \( w_1, \ldots, w_n \in [0, 1] \) and \( m \) bins, each of capacity 1, is there a way to assign the items to the bins without violating the capacity constraints? Formally, is there a map \( f : [n] \to [m] \) such that \( \sum_{i \in f^{-1}(j)} w_i \leq 1 \) for all \( j \in [m] \)?

Björklund, Husfeldt, and Koivisto [1] provided an algorithm for solving bin packing in time \( \tilde{O}(2^n) \) where the tilde hides polynomial factors in \( n \). It is natural to ask whether the base of the exponent may be improved at all i.e. is there a (possibly randomized) algorithm to solve bin packing in time \( \tilde{O}(2^{1-\varepsilon}n) \) for some absolute constant \( \varepsilon > 0 \)?

In recent work, Nederlof, Pawlewics, Swennehuis and Węgrzycki [2] showed that this is true provided that the number of bins \( m \) is fixed. More precisely, they showed that there exists a function \( \sigma : \mathbb{N} \to \mathbb{R}^+ \) and a randomized algorithm for solving bin packing which, on instances with \( m \) bins, runs in time \( \tilde{O}(2^{1-\sigma(m)}n) \), where \( \tilde{O} \) hides polynomials in \( n \) as well as exponential factors in \( m \). Their analysis, which crucially relies on (1.1), gives a very small value of \( \sigma(m) \) satisfying

\[
\sigma(m) \leq 2^{-m^9}. \tag{1.3}
\]

Using Theorem 1.2 instead of (1.1) in a black-box manner in the analysis of [2], we exponentially improve the bound on \( \sigma(m) \).

**Corollary 1.3.** With notation as above, the randomized algorithm of [2] solves bin packing instances with \( m \) bins in time \( \tilde{O}(2^{1-\sigma(m)}n) \) with high probability, for \( \sigma : \mathbb{N} \to \mathbb{R}^+ \) satisfying

\[
\sigma(m) = \tilde{\Omega}(m^{-12}), \tag{1.4}
\]

where \( \tilde{\Omega} \) hides logarithmic factors in \( m \).

**Remark.** This follows by noting that the function \( f_\epsilon(m) \) in [2, Section 3.6] is \( \tilde{O}(m^{-2}) \) so that \( \delta \) in [2, Section 3.6] is \( \tilde{\Omega}(m^{-3}) \). With Theorem 1.2, the function \( \epsilon(\delta) \) in the runtime analysis of [2, Section 3.4] satisfies \( \epsilon(\delta) = O(\delta^2) \). Therefore, the function \( f_B(\delta) \) in the same section is \( \tilde{O}(\delta^4) \), which is \( \tilde{\Omega}(m^{-12}) \). Note that if one were able to establish the conjecturally optimal bound \( \delta = O(\epsilon) \), this would lead to \( f_B(\delta) = \tilde{O}(\delta^2) \), thereby giving the quadratically better \( \sigma(m) = \tilde{\Omega}(m^{-6}) \).

1.2 Notation

We use big-\( O \) notation to mean that an absolute multiplicative constant is being hidden. We use \( \text{Ber}(1/2) \) to denote the balanced \( \{0, 1\} \) Bernoulli distribution and \( \text{Bin}(k) \) to denote the binomial distribution on \( k \) trials with parameter 1/2. Recall that \( \text{Bin}(k) \) is the sum of \( k \) independent \( \text{Ber}(1/2) \) random variables. Given a distribution \( \mu \), we let \( \mu^\otimes n \) denote the distribution of a random vector with \( n \) independent samples from \( \mu \) as its coordinates. We also use the following standard additive combinatorics notation: \( C + D = \{c + d : c \in C, d \in D\} \) is the sumset (if \( C, D \) are subsets of the same abelian group), and for a positive integer \( k \), we let \( k \cdot C = C + \cdots + C \) (\( k \) times) be the iterated sumset. Finally, in some cases we will use the notation \( \Sigma \cdot \) or \( f \cdot \) to denote that the expression in the sum or integral is the same as in the previous line to simplify the presentation of long expressions.
1.3 Outline of the proof

As in [2], the starting point of our proof is the following observation: let $A$ denote a fixed (but otherwise arbitrary) set of unique preimages for points in $\mathcal{R}(\bar{w})$ (hence, $|A| = |\mathcal{R}(\bar{w})|$) and let $B$ denote the the set of preimages of a value $\tau \in \mathbb{R}$ realising $\rho(\bar{w})$. Then (Lemma 2.2) for any $k \geq 1$, the map $A \times (k \cdot B) \rightarrow A + k \cdot B$ is a bijection. In particular, if $\bar{a}$ is sampled from the uniform distribution on $A$ and $\bar{b}_1, \ldots, \bar{b}_k$ are independently sampled from the uniform distribution on $B$, then

$$|A| = |A| \cdot \mathbb{P}[\bar{a} + \bar{b}_1 + \cdots + \bar{b}_k \in \{0, \ldots, k+1\}^n]$$

$$= |A| \cdot \sum_{\bar{x} \in \{0, \ldots, k+1\}^n} \mathbb{P}[\bar{a} + \bar{b}_1 + \cdots + \bar{b}_k = \bar{x}]$$

$$\leq |A| \cdot \sum_{\bar{x} \in \{0, \ldots, k+1\}^n} \mathbb{P}[\bar{a} = \bar{a}(\bar{x})] \cdot \mathbb{P}[\bar{b}_1 + \cdots + \bar{b}_k = \bar{x} - \bar{a}(\bar{x})]$$

$$\leq \sum_{\bar{x} \in \{0, \ldots, k+1\}^n} \mathbb{P}[\bar{b}_1 + \cdots + \bar{b}_k = \bar{x} - \bar{a}(\bar{x})]$$

In [2], the largeness of $B$ is exploited by finding, for every $a \in A$, a large subset of $B$ which is ‘balanced’ (in a certain sense) with respect to $a$. Instead, we exploit the largeness of $B$ directly by using the observation that the density of the uniform measure on $B$ with respect to the uniform measure on $\{0, 1\}^n$ is at most $2^n/|B| \leq 2^n$. In particular, if we let $\mu_k$ denote the measure on $k \cdot B$ induced by the product measure on $B \times \cdots \times B$ via the map $(b_1, \ldots, b_k) \mapsto b_1 + \cdots + b_k$ and if we let $\text{Bin}(k)^n$ denote the $n$-fold product of the Binomial$(k, 1/2)$ distribution, then the density of $\mu_k$ with respect to $\text{Bin}(k)^n$ is at most $2^{k\varepsilon n}$. This allows us to replace the measure $\mu_k$ appearing in the last line of the above equation by $\text{Bin}(k)^n$, at the cost of a factor of $2^{k\varepsilon n}$. Thus,

$$|A| \leq 2^{k\varepsilon n} \cdot \sum_{\bar{x} \in \{0, \ldots, k+1\}^n} \mathbb{P}_{\bar{x} \sim \text{Bin}(k)^n} [\bar{x} - \bar{a}(\bar{x})]$$

The above expression is still complicated by the presence of the shift $\bar{a}(\bar{x})$, about which we have no information except that it lies in the set $A$. The key technical lemma in the proof is Lemma 2.1, which essentially allows us to remove this shift after paying a factor which depends on $|A|$. Ultimately, this gives an upper bound on the sum in terms of $|A|$ and $k$, which amounts to an upper bound on $|A|$ in terms of $k, \varepsilon$, and $|A|$. Optimizing the value of the free parameter $k$ now gives the desired conclusion.

2 Proof of Theorem 1.2

We begin by recording the following key comparison bound, which will be proved at the end of this section.

**Lemma 2.1.** Let $n \geq k \geq C_{2.1}$, where $C_{2.1}$ is a sufficiently large absolute constant and let $\delta > 0$. For any $A \subseteq \{0, 1\}^n$ with $|A| \leq \exp(\delta n)$, the following holds. Let $\bar{x}, \bar{b} \sim \text{Bin}(k)^n$ be independent $n$-dimensional random vectors. Then,

$$\mathbb{E}_{\bar{x}} \left[ \sup_{\bar{a} \in A} \frac{\mathbb{P}_{\bar{b}}(\bar{b} = \bar{x} - \bar{a})}{\mathbb{P}_{\bar{b}}(\bar{b} = \bar{x})} \right] \leq \exp \left( C_{2.1} \left( \frac{1}{k} + \sqrt{\frac{\delta}{k}} \right)^n \right).$$
Let $n$, $\varepsilon$, and $\vec{w}$ be as in Theorem 1.2. Let $\tau$ be such that $\mathbb{P}[(\vec{w}, \vec{\xi}) = \tau] = \rho(\vec{w})$, where $\vec{\xi}$ is a random vector with i.i.d. Ber$(1/2)$ components. Let

$$B = \{\vec{\xi} \in \{0, 1\}^n : \langle \vec{w}, \vec{\xi} \rangle = \tau \}.$$ 

In particular, $|B| \geq \exp(-\varepsilon n) \cdot 2^n$. Let $|\mathcal{R}(\vec{w})| = \exp(\delta n)$. For each $r \in \mathcal{R}(\vec{w})$, let $\vec{\xi}(r)$ be a fixed (but otherwise arbitrary) element of $\{0, 1\}^n$ such that $\langle \vec{w}, \vec{\xi}(r) \rangle = r$. Let

$$A = \{\vec{\xi}(r) \in \{0, 1\}^n : r \in \mathcal{R}(\vec{w}) \}.$$ 

Note that, by definition, for any distinct $\vec{a}_1, \vec{a}_2 \in A$, we have that $\langle \vec{w}, \vec{a}_1 \rangle \neq \langle \vec{w}, \vec{a}_2 \rangle$ and that $|A| = |\mathcal{R}(\vec{w})| = \exp(\delta n)$.

We will make use of the simple, but crucial, observation from [2] that $A$ and $k \cdot B$ have a full sumset for all $k \geq 1$.

**Lemma 2.2** ([2, Lemma 4.2]). The map $(\vec{a}, \vec{c}) \mapsto \vec{a} + \vec{c}$ from $A \times (k \cdot B)$ to $A + k \cdot B$ is injective.

**Proof.** Indeed, if $\vec{a}_1 + (\vec{b}_1^{(1)} + \cdots + \vec{b}_1^{(1)}) = \vec{a}_2 + (\vec{b}_2^{(1)} + \cdots + \vec{b}_2^{(1)})$, where $\vec{a}_i \in A$ and $\vec{b}_j^{(i)} \in B$, then taking the inner product of both sides with $\vec{w}$ and using $\langle \vec{w}, \vec{b} \rangle = \tau$ for all $\vec{b} \in B$, we see that $\langle \vec{w}, \vec{a}_1 \rangle = \langle \vec{w}, \vec{a}_2 \rangle$, which implies that $\vec{a}_1 = \vec{a}_2$ by the definition of $A$. \hfill \square

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let $k \geq 2$ be a parameter which will be chosen later depending on $\varepsilon$. We may assume $\varepsilon \in (0, (2C_{2.1})^{-2})$ by adjusting $C_{1.2}$ appropriately at the end to make larger values trivial. By Lemma 2.2, for each $\vec{x} \in \{0, \ldots, k+1\}^n$ for which there exist $\vec{a} \in A$ and $\vec{c} \in k \cdot B$ with $\vec{a} + \vec{c} = \vec{x}$, there exists a unique such choice $\vec{a} = \vec{a}(\vec{x}) \in A$. (For $\vec{x} \notin A + k \cdot B$, we let $\vec{a}(\vec{x})$ be an arbitrary element of $A$.)

Now, let $\vec{a}$ be uniform on $A$, let $\vec{b}_1, \ldots, \vec{b}_k$ be uniform on $B$, and let $\vec{v}_1, \ldots, \vec{v}_k$ be uniform on $\{0, 1\}^n$. Let $C_i \subseteq \{0, \ldots, k+1\}^n$ be the set of vectors of size $i$ with coordinates equal to $k+1$. For $\vec{x} \in \{0, \ldots, k+1\}^n$, we let $\vec{x}^i \in \{0, \ldots, k\}^n$ denote the vector obtained by setting every occurrence of $k+1$ in $\vec{x}$ to $k$. We have

$$1 = \mathbb{P}[\vec{a} + \vec{b}_1 + \cdots + \vec{b}_k \in \{0, \ldots, k+1\}^n]$$
$$= \sum_{i=0}^{n} \sum_{\vec{x} \in C_i} \mathbb{P}[\vec{a} + \vec{b}_1 + \cdots + \vec{b}_k = \vec{x}]$$
$$\leq \sum_{i=0}^{n} \sum_{\vec{x} \in C_i} \mathbb{P}[\vec{a} = \vec{a}(\vec{x})] \mathbb{P}[\vec{b}_1 + \cdots + \vec{b}_k = \vec{x} - \vec{a}(\vec{x})]$$
$$\leq \frac{1}{|A|} \sum_{i=0}^{n} \sum_{\vec{x} \in C_i} \left( \frac{2^n}{|B|} \right)^k \mathbb{P}[\vec{v}_1 + \cdots + \vec{v}_k = \vec{x} - \vec{a}(\vec{x})]$$
$$\leq \frac{e^{\varepsilon B}}{|A|} \sum_{i=0}^{n} \sum_{\vec{x} \in C_i} \mathbb{P}[\vec{v}_1 + \cdots + \vec{v}_k = \vec{x}^i] \sup_{\vec{a} \in A} \frac{\mathbb{P}[\vec{v}_1 + \cdots + \vec{v}_k = \vec{x} - \vec{a}]}{\mathbb{P}[\vec{v}_1 + \cdots + \vec{v}_k = \vec{x}^i]}$$
$$= \frac{e^{\varepsilon B}}{|A|} \sum_{i=0}^{n} \left( \frac{1}{2k} \right)^j \sum_{S \subseteq \binom{\{0, \ldots, n\}}{i}} \mathbb{P}_{\vec{x} \sim \text{Bin}(k) \cap [\{0\}^n \times \{k+1\}]} \left[ \sup_{\vec{a} \in A} \frac{\mathbb{P}[\vec{v}_1 + \cdots + \vec{v}_k = \vec{x} - \vec{a}]}{\mathbb{P}[\vec{v}_1 + \cdots + \vec{v}_k = \vec{x}^i]} \right].$$
Let \( A_S \) be the set of elements in \( A \subseteq \{0,1\}^n \) whose support contains \( S \). Let
\[
A'_S = \{ \vec{a}' \in \{0,1\}^{[n]\setminus S} : \exists \vec{a} \in A_S \text{ with } \vec{a}|_{[n]\setminus S} = \vec{a}' \}.
\]

Recall that \( |A| = \exp(\delta n) \). Abusing notation so that the supremum of an empty set is 0, we can continue the above chain of inequalities to get that
\[
1 \leq \frac{e^{\kappa n}}{|A|} \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{i} \cdot 2^{\gamma - k} \cdot 2^n \cdot \left( \max_{\ell} \left\{ \binom{k}{\ell - 1}, \binom{k}{\ell} \right\} \right)^{n-i} \leq \frac{e^{\kappa n}}{|A|} \left( \sum_{i=0}^{n/2} \frac{n}{i} \cdot 2^{\gamma - k} \cdot 2^n \cdot k^n \right) \leq \frac{e^{\kappa n}}{|A|} \left( \sum_{i=0}^{n/2} \binom{n}{i} \cdot 2^{-k} \cdot 2^n \right) \leq\exp(-\delta n) \exp \left( O(k \varepsilon + k^{-1} + \delta^{1/2} k^{-1/2} n) \right)
\]
by Lemma 2.1 applied to \( A_S \), as long as \( n/2 \geq k \geq C_{2.1} \geq 20 \). To deduce the last line, note that \( \binom{n}{i} 2^{-k} \leq (2^{-k/2})^{i} \), so for \( i \geq \lceil en/2^{k-1} \rceil \) the sum of weighted binomials is bounded by a geometric series. Additionally, for \( 1 \leq i \leq \lfloor en/2^{k} \rfloor \), if this interval is nonempty, the sum of binomials is certainly bounded by \( \exp(O(k^{-1}n)) \).

Hence, the above inequality yields
\[
\delta \leq C(k \varepsilon + k^{-1} + \delta^{1/2} k^{-1/2})
\]
for some absolute constant \( C > 0 \). Now letting \( k = \varepsilon^{-1/2}/2 \) (note that this satisfies \( 2C_{2.1} \leq 2 k = \varepsilon^{-1/2} \leq n \)), we find that
\[
\delta = O(\varepsilon^{1/2}),
\]
as desired. \(\square\)

The proof of Lemma 2.1 relies on the following preliminary estimate.

**Lemma 2.3.** If \( 1 \leq s \leq k/(16\pi) \), then
\[
\mathbb{E}_{x \sim \text{Bin}(k)} \left( \frac{x}{k+1-x} \right)^s \leq \exp(10\pi s^2/k) + 2k^s (4/5)^k.
\]
**Proof.** We let \( x \sim \text{Bin}(k) \) and \( y = x - k/2 \sim \text{Bin}(k) - k/2 \) throughout. We let \( z \sim \mathcal{N}(0, k\pi/8) \). We have

\[
\mathbb{E}_{x \sim \text{Bin}(k)} \left[ \left( \frac{x}{k+1-x} \right)^s \right] = \mathbb{E}_y \left[ \left( 1 + \frac{2y-1}{k/2+1-y} \right)^s \right] \\
\leq \mathbb{E}_y \left[ \left( 1 + \frac{2y}{k/2+1-y} \right)^s \right] \mathbb{1}_{|y| \leq k/3} + k^s \mathbb{P}[|y| \geq k/3] \\
\leq \mathbb{E}_y \left[ \left( 1 + \frac{2y}{k/2+1-y} \right)^s \right] \mathbb{1}_{|y| \leq k/3} + 2k^s (4/5)^k.
\]

Note that the probability estimate for \( \mathbb{P}[|y| \geq k/3] \) follows from the sharp (entropy) version of the Chernoff-Hoeffding theorem. Since for \( |y| \leq k/3 \),

\[
\frac{2y}{(k/2+1-y)} \leq \frac{2y}{k/2+1} + \frac{8y^2}{(k/2+1)^2},
\]

and using \((1+x)^s \leq \exp(x)\), we can continue the previous inequality as

\[
\mathbb{E}_{x \sim \text{Bin}(k)} \left[ \left( \frac{x}{k+1-x} \right)^s \right] \leq \mathbb{E}_y \left[ \left( 1 + \frac{2y}{k/2+1} + \frac{8y^2}{(k/2+1)^2} \right)^s \mathbb{1}_{|y| \leq k/3} \right] + 2k^s (4/5)^k \\
\leq \mathbb{E}_y \left[ \exp \left( \frac{4sy}{k+2} + \frac{32sy^2}{k^2} \right) \right] + 2k^s (4/5)^k.
\]

Now, let \( z_1, \ldots, z_k \) be i.i.d. \( \mathcal{N}(0, 1) \) random variables. Then,

\[
y \sim \frac{1}{2} (\text{sgn} z_1 + \cdots + \text{sgn} z_k).
\]

Moreover, for any \(-k \leq \ell \leq k\),

\[
\mathbb{E}[z_1 + \cdots + z_k \mid \text{sgn}(z_1) + \cdots + \text{sgn}(z_k) = \ell] = \sqrt{\frac{2}{\pi}} \ell.
\]

In particular, under this coupling of \( y, z_1, \ldots, z_k \), we have

\[
\mathbb{E}[z_1 + \cdots + z_k \mid y] = \sqrt{\frac{8}{\pi}} y.
\]

Let \( z = z_1 + \cdots + z_k \), so that \( z \sim \mathcal{N}(0, k) \). Then, by the convexity of

\[
f(y) = \exp \left( \frac{4sy}{k+2} + \frac{32sy^2}{k^2} \right)
\]
and using Jensen’s inequality, we have

\[
\mathbb{E}_y f(y) = \mathbb{E}_{y,z_1,...,z_k} f(y) \\
= \mathbb{E}_{y,z_1,...,z_k} f\left(\sqrt{\frac{\pi}{8}}\mathbb{E}[z | y]\right) \\
\leq \mathbb{E}_z f\left(\sqrt{\pi}z/\sqrt{8}\right) \\
= \mathbb{E}_{w \sim N(0,1)} \exp\left(\frac{4s\pi}{k}w^2\right) \\
= \left(1 - \frac{8\pi s}{k}\right)^{-1/2} \exp\left(\frac{\pi s^2 k^2}{(k+2)^2(k-8\pi s)}\right) \\
\leq \exp\left(\frac{8\pi s}{k} + \frac{2\pi s^2}{k}\right) \\
\leq \exp(10\pi s^2/k).
\]

Finaly, we can prove Lemma 2.1

**Proof of Lemma 2.1.** We may assume that \( \delta \geq 2000/k \) since the statement for \( \delta < 2000/k \) follows from the statement for \( \delta = 2000/k \). Also, note that we may assume that \( \delta \leq \log 2 \). For any \( t \in \mathbb{R} \), we have

\[
\mathbb{P}_x\left[\sup_{\bar{a} \in A} \mathbb{P}_{\bar{b}}[\bar{b} = \bar{x} - \bar{d}] \geq e^{tn}\right] \leq |A| \sup_{\bar{b}} \mathbb{P}_{\bar{b}}[\bar{b} = \bar{x}] \geq e^{tn} \\
\leq |A| \sup_{\bar{a} \in A, s \geq 2} \exp(-stn) \mathbb{E}_x\left[\left(\frac{\mathbb{P}_x[\bar{b} = \bar{x} - \bar{d}]}{\mathbb{P}_{\bar{b}}[\bar{b} = \bar{x}]}\right)^s\right] \\
= |A| \sup_{\bar{a} \in A, s \geq 2} \exp(-stn) \prod_{i=1}^n \mathbb{E}_{x \sim \text{Bin}(k)}\left[\left(\frac{\mathbb{P}[\text{Bin}(k) = x - a]}{\mathbb{P}[\text{Bin}(k) = x]}\right)^s\right] \\
\leq |A| \inf_{s \geq 2} \exp(-stn) \left(\mathbb{E}_{x \sim \text{Bin}(k)}\left(\frac{x}{k+1-x}\right)^s\right)^n.
\]

In the last line, we have used that

\[
\mathbb{E}_{x \sim \text{Bin}(k)}\left(\frac{x}{k+1-x}\right)^s \geq \left(\mathbb{E}_{x \sim \text{Bin}(k)}\left(\frac{x^2}{(k+1-x)^2}\right)\right)^{s/2} \\
= \left(\sum_{\ell=0}^{k-1} \frac{\ell + 1}{k-\ell} \left(\frac{k}{\ell}\right)^{2-\ell}\right)^{s/2} \\
= \left(\sum_{\ell=0}^{k-1} \left(\frac{k+2}{k} + \frac{4(k+1)(\ell-k/2)}{k^2} + \frac{(k+1)(k-2\ell)^2}{k^2(k-\ell)}\right)\left(\frac{k}{\ell}\right)^{2-\ell}\right)^{s/2} \\
\geq \left(\sum_{\ell=0}^{k-1} \left(\frac{k+2}{k} + \frac{4(k+1)(\ell-k/2)}{k^2}\right)\left(\frac{k}{\ell}\right)^{2-\ell}\right)^{s/2}.
\]
\[ \left( \frac{k+2}{k} - \frac{3k+4}{k}e^{-k} \right)^{s/2} \geq 1 \]

if \( k \geq 3 \). Therefore, by Lemma 2.3, we have

\[ \mathbb{P}_{\tilde{x}} \left[ \sup_{\vec{a} \in \mathcal{A}} \mathbb{P}_{\tilde{b}} [ \tilde{b} = \tilde{x} - \vec{a} ] \geq e^n \right] \leq |A| \inf_{s \geq 2} \exp(-stn) \left( \mathbb{E}_{x \sim \text{Bin}(k)} \left( \frac{x}{k+1-x} \right)^s \right)^n \]

\[ \leq |A| \inf_{2 \leq s \leq k/(16\pi)} \exp(-stn) \left( \exp(10\pi s^2/k) + 2k^{s}(4/5)^k \right)^n \]

\[ \leq |A| \inf_{2 \leq s \leq k/(10\log k)} \exp(-stn) \left( \exp(12\pi s^2/k) \right)^n \]

\[ \leq \begin{cases} 
|A| \exp \left( -\frac{k^2n}{48\pi} \right) & \text{if } \frac{96\pi \delta}{k} \leq t \leq (\log k)^{-1} \leq t \leq \log k.
\end{cases} \]

Here, the second case follows by plugging in \( s = k/(24\pi \log k) \) and simplifying (assuming \( C_{2.1} \) is large enough so \( s \geq 2 \)), and the first case follows from plugging in \( s = kt/(24\pi) \) which satisfies \( 2 \leq s \leq k/(10\log k) \) by the restriction on \( t \) and \( \delta \). Finally, since

\[ 0 \leq \sup_{\vec{a} \in \mathcal{A}} \mathbb{P}_{\tilde{b}} [ \tilde{b} = \tilde{x} - \vec{a} ] \leq \left( \max_{\ell} \frac{\max \left\{ \frac{k}{k-1}, \frac{k}{\ell} \right\}}{\left( \frac{k}{\ell} \right)_{\ell}} \right)^n \leq k^n, \]

we have

\[ \mathbb{E}_{\tilde{x}} \left[ \sup_{\vec{a} \in \mathcal{A}} \mathbb{P}_{\tilde{b}} [ \tilde{b} = \tilde{x} - \vec{a} ] \right] = \int_{-\infty}^{\log k} \mathbb{P}_{\tilde{b}} [ \tilde{b} = \tilde{x} - \vec{a} ] \geq e^n \right] \leq k^n \]

\[ \leq \int_{-\infty}^{\log k} \int_{1/\log k}^{1/\log k} \frac{1}{\sqrt{96\pi \delta/k}} e^{\log k} dt \]

\[ \leq e^{\sqrt{96\pi \delta/k}n} + \int_{1/\log k}^{1/\log k} |A| \exp \left( -\frac{k^2n}{48\pi} \right) e^{\log k} dt \]

\[ + \int_{1/\log k}^{\log k} |A| \exp \left( -\frac{kn}{48\pi (\log k)^2} \right) e^{\log k} dt \]

\[ \leq \exp \left( O(\sqrt{\delta/k})n \right) + \int_{1/\log k}^{1/\log k} ne^{-tn} dt + 1 \]

\[ \leq \exp \left( O(\sqrt{\delta/k})n \right). \]
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