On a class of stochastic differential equations
used in quantum optics *

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Abstract

Stochastic differential equations for processes with values in Hilbert spaces are now largely
used in the quantum theory of open systems. In this work we present a class of such equations
and discuss their main properties; moreover, we explain how they are derived from purely
quantum mechanical models, where the dynamics is represented by a unitary evolution in
a Hilbert space, and how they are related to the theory of continual measurements. An
essential tool is an isomorphism between the bosonic Fock space and the Wiener space,
which allows to connect certain quantum objects with probabilistic ones.

1 Classical stochastic differential equations

Recently there was an increasing use of stochastic differential equations (SDEs) in quantum
optics and more generally in the theory of quantum open systems [1]–[6]. In particular SDEs
are used for theoretical purposes [2, 3, 6] in the theory of quantum continuous measurements
[4, 7], when a system is continually monitored in time, and for numerical simulations of quantum
master equations [4], which are mathematical models for the dynamics of quantum open systems.
In this work we want to describe a restricted class of such equations, to discuss their main
properties and to show how they can be derived from fully quantum mechanical models [8, 9].

We consider only SDEs of diffusive type, when the noises are Wiener processes; for the theory
of Itô’s stochastic integrals in Hilbert spaces we refer to [10]. Let \((\Omega, (\mathcal{F}_t), \mathcal{F}, P)\) be a stochastic
basis satisfying the usual hypotheses; in particular, \((\Omega, \mathcal{F}, P)\) is a probability space, which we
assume to be standard, \((\mathcal{F}_t)_{t \geq 0}\) is a filtration, i.e. \(\mathcal{F}_t\) is a sub–\(\sigma\)–algebra of \(\mathcal{F}\) and \(\mathcal{F}_s \subset \mathcal{F}_t\)
for \(s \leq t\); we also assume \(\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t\). Let \(W_k(\cdot), k = 0,1,\ldots,\) be a sequence of continuous
versions of adapted, standard, independent, Wiener processes with increments independent from
the past.

Let \(\mathcal{H}\) be a separable complex Hilbert space and let us consider the following linear SDE
with “multiplicative noise” for an \(\mathcal{H}\)–valued process \(\psi_t\):

\[
\begin{align*}
\frac{d\psi_t}{dt} &= \sum_{j=0}^{\infty} R_j(t) \psi_t \, dW_j(t) - iK(t)\psi_t \, dt \\
\psi_0 &= \psi
\end{align*}
\] (1.1)

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where the coefficients $R_j(t), K(t)$ are operators in $\mathcal{H}$ satisfying some assumptions to be discussed below and the initial condition $\psi$ is an $\mathcal{H}$-valued $\mathcal{F}_0$-measurable random variable. Let us state now some different sets of assumptions about the coefficients.

**Assumption 1.1** The coefficients are time dependent, bounded operators on $\mathcal{H}$ satisfying:

(i) $\sum_{j=0}^{\infty} R_j^*(t)R_j(t)$ is strongly convergent in $L(\mathcal{H})$ for all $t$;

(ii) the functions $t \mapsto R(t), t \mapsto K(t)$ are strongly measurable, where $R(t)$ is the operator from $\mathcal{H} \otimes l^2$ defined by $\langle x \otimes c | R(t)y \rangle = \sum_{j=0}^{\infty} \overline{c_j} \langle x | R_j(t)y \rangle$, $\forall x, y \in \mathcal{H}, \forall c \in l^2$;

(iii) $\forall T \in \mathbb{R}_+$, $\text{ess sup}_{t \leq T} \left( \left\| \sum_{j=0}^{\infty} R_j^*(t)R_j(t) \right\|^2 + \|K(t)\| \right) < \infty$.

**Assumption 1.2** The coefficients are time independent, (in general) unbounded operators, defined on a dense domain $D \subset \mathcal{H}$; we assume that

$$\sum_{j=0}^{\infty} \|R_j^*x\|^2_{\mathcal{H}} < \infty, \quad \forall x \in D^*,$$

where $D^*$ is a core for the adjoint operators $K^*$ and $R_j^*$. Moreover we assume the following dissipativity condition to hold:

$$\sum_{j=0}^{\infty} \|R_jx\|^2_{\mathcal{H}} + 2\text{Im} \langle x | Kx \rangle_{\mathcal{H}} \leq c\|x\|^2_{\mathcal{H}}, \quad \forall x \in D.$$

**Assumption 1.3** The coefficients are time independent, (in general) unbounded operators, defined on a dense domain $D \subset \mathcal{H}$, satisfying

$$\sum_{j=0}^{\infty} \|R_jx\|^2_{\mathcal{H}} < \infty, \quad \forall x \in D;$$

moreover, $-iK$ is the generator of an analytic operator semigroup $T(t)$.

Let us remark that, in the time independent case Assumption 1.1 is stronger than Assumption 1.2 and indeed Theorem 1.1 will show that, under Assumption 1.1, we can obtain stronger results than under Assumption 1.2. We note also that, for a linear equation, the so called “Lipschitz conditions” [10] are equivalent to the boundness of the coefficients.

**Definition 1.1** We consider three different kinds of solution:

(i) a (topological) strong solution of eq. (1.1) is an $\mathcal{H}$-valued process such that $\forall t \in \mathbb{R}_+$ the following equation holds almost surely (a.s.)

$$\psi_t = \psi + \sum_{j=0}^{\infty} \int_0^t R_j(s)\psi_s \, dW_j(s) - i \int_0^t K(s)\psi_s \, ds;$$  \hspace{1cm} (1.2)
(ii) a (topological) \textit{weak solution} of eq. (1.1) is an $\mathcal{H}$-valued process such that $\forall t \in \mathbb{R}_+$ and $\forall \chi \in \mathcal{D}^*$ the following equation holds a.s.

$$\langle \chi | \psi_t \rangle_\mathcal{H} = \langle \chi | \psi \rangle_\mathcal{H} + \sum_{j=0}^{\infty} \int_0^t \langle R^*_j(s) \chi | \psi_s \rangle_\mathcal{H} \, dW_j(s) - i \int_0^t \langle K^*(s) \chi | \psi_s \rangle_\mathcal{H} \, ds; \quad (1.3)$$

(iii) a (topological) \textit{mild solution} of eq. (1.1) with Assumption 1.3 is an $\mathcal{H}$-valued process such that $\forall t \in \mathbb{R}_+$ the following equation holds a.s.

$$\psi_t = T(t) \psi + \sum_{j=0}^{\infty} \int_0^t T(t - s) R_j \psi_s \, dW_j(s), \quad (1.4)$$

where $T(s)$ is the analytic semigroup operator generated by $-iK$.

For more details about these definitions see [10]. It is clear that a strong solution is also a weak solution and it is possible to show that a weak solution is also a mild solution (see [10], Propositions 6.2, 6.3, 6.4 and Theorem 6.5). In the following theorem we state some results about the existence, the uniqueness and the Markov properties of the solution of eq. (1.1) under different hypotheses.

**Theorem 1.1** (i) Under Assumption 1.1, eq. (1.1) admits a unique strong solution; (ii) under Assumption 1.2, eq. (1.1) admits a unique weak solution; (iii) under Assumption 1.3, eq. (1.1) admits a mild solution and this is also a weak solution. All these solutions are strong Markov processes.

**Proof.** The statement (i) is Theorem 1.1 of [11]; for the proof of (ii) see Theorem 1 of [12] and for the proof of (iii) see Theorems 6.19 and 6.22 of [10]. For the Markov property see [12] Theorems 9.8 and 9.14. $\square$

We are not interested in eq. (1.1) in general, but only when $\|\psi_t\|_\mathcal{H}^2$ is a martingale and can be interpreted as a probability density with respect to $P$. Indeed, if we have a positive martingale $\|\psi_t\|_\mathcal{H}^2$ with $E_P[\|\psi_t\|_\mathcal{H}^2] = 1$ ($E_P$ denotes the expectation with respect to $P$), then

$$\tilde{P}(A) := E_P[1_A \|\psi_t\|_\mathcal{H}^2], \quad \forall A \in \mathcal{F}_t, \forall t \in \mathbb{R}_+,$$

(1.5)

defines a new probability law $\tilde{P}$ on $(\Omega, \mathcal{F})$. To obtain this we need a further assumption.

**Assumption 1.4** For all $t \in \mathbb{R}_+$, the coefficients satisfy the following operator equality

$$-i (K^*(t) - K(t)) = \sum_{j=0}^{\infty} R^*_j(t) R_j(t),$$

(where the sum is convergent in the strong operator topology) and the initial condition is a square integrable, normalized random variable, i.e. $E_P[\|\psi\|_\mathcal{H}^2] = 1$.

**Theorem 1.2** Under Assumptions 1.3 and 1.4, the square norm $\|\psi_t\|_\mathcal{H}^2$ of the solution is a (positive) martingale and $E_P[\|\psi_t\|_\mathcal{H}^2] = 1$. Moreover, under the law $\tilde{P}$ defined by eq. (1.5), the processes

$$\tilde{W}_k(t) := W_k(t) - 2 \int_0^t \text{Re} \tilde{m}_k(s) \, ds, \quad k = 0, 1, \ldots,$$

(1.6)
where
\[
\hat{m}_k(t) := \frac{\langle \psi_t | R_k(t) \psi_t \rangle_H}{\| \psi_t \|^2_H},
\] (1.7)
are independent, adapted, standard Wiener processes.

For the proof of this theorem see [14], Theorem 1.2 and Proposition 1.1. The unbounded case
is more difficult and it is treated in [12] under an additional condition called of hyperdissipativity.

Let us introduce now the process
\[
\hat{\psi}_t := \frac{1}{\| \psi_t \|_H} \exp \left( -i \sum_{k=0}^{\infty} \int_0^t (\text{Im} \hat{m}_k(s)) (dW_k(s) - \text{Re} \hat{m}_k(s) ds) \right) \psi_t.
\] (1.8)
Note that \( \| \hat{\psi}_t \|_H^2 = 1 \); indeed, \( \exp(\cdots) \) is a stochastic phase with no particular meaning, which
we introduce only in order to simplify the form of the stochastic differential of \( \hat{\psi}_t \). Note also
that \( \hat{m}_k(t) = \langle \hat{\psi}_t | R_k(t) \hat{\psi}_t \rangle \).

Theorem 1.3 Under Assumptions 1.1 and 1.4, in the stochastic basis \((\Omega, (\mathcal{F}_t), \mathcal{F}, \hat{P})\), \( \hat{\psi}_t \) satisfies the non linear SDE
\[
d\hat{\psi}_t = -i \hat{K} \left( t, \hat{\psi}_t \right) dt + \sum_{k=0}^{\infty} \hat{R}_k \left( t, \hat{\psi}_t \right) d\hat{W}_k(t),
\] (1.9)
where, \( \forall f \in \mathcal{H} \),
\[
\hat{K}(t, f) := \frac{1}{2} \left( K(t) + K^*(t) \right) f - \frac{i}{2} \sum_{k=0}^{\infty} \left( \frac{\langle f | R_k(t) f \rangle_H}{\| f \|^2_H} R^*_k(t) - \frac{\langle f | R^*_k(t) f \rangle_H}{\| f \|^2_H} R_k(t) \right) f +
\]
\[-\frac{i}{2} \sum_{k=0}^{\infty} \left( R^*_k(t) f - \frac{\langle f | R^*_k(t) f \rangle_H}{\| f \|^2_H} \right) \left( R_k(t) - \frac{\langle f | R_k(t) f \rangle_H}{\| f \|^2_H} \right) f,
\] (1.10)
and
\[
\hat{R}_k(t, f) := \left( R_k(t) - \frac{\langle f | L_k(t) f \rangle_H}{\| f \|^2_H} \right) f.
\] (1.11)
Moreover \( \hat{\psi}_t \) is the unique strong solution of eq. (1.9) and it is a strong Markov process.

Proof. The first part of this theorem is essentially Theorem 1.3 of [1]. Moreover it is easy
to show that the coefficients \( \hat{K} \) and \( \hat{R}_k \) satisfy the local Lipschitz conditions (see [10], p. 198);
hence \( \hat{\psi}_t \) is the unique strong solution of eq. (1.9). Again the Markov property follows from [11]
Theorems 9.8 and 9.14.

Let us stress that the Markov property allows to associate to the process \( \hat{\psi}_t \) a transition
function satisfying a Chapman–Kolmogorov equation (see [10] Section 9.2.1 and, in particular,
Corollary 9.9); similar considerations apply to the solution of the linear SDE.

Examples of equations of the types (1.1) and (1.9) were introduced in the theory of quantum
continual measurement in Ref. [3].
In the statistical formulation of quantum mechanics the states of a quantum system are represented by statistical operators. Let $\mathcal{T}(\mathcal{H})$ be the trace class on $\mathcal{H}$ and $\mathcal{S}(\mathcal{H})$ denote the set of statistical operators, i.e.

$$
\mathcal{T}(\mathcal{H}) := \left\{ \rho \in \mathcal{L}(\mathcal{H}) : \text{Tr}_\mathcal{H}\left\{ \sqrt{\rho^*\rho} \right\} < \infty \right\},
$$

$$
\mathcal{S}(\mathcal{H}) := \left\{ \rho \in \mathcal{T}(\mathcal{H}) : \rho^* = \rho, \rho \geq 0, \text{Tr}_\mathcal{H}\{\rho\} = 1 \right\}.
$$

Equipped with the norm $\|\rho\|_1 := \text{Tr}_\mathcal{H}\{\sqrt{\rho^*\rho}\}$, $\mathcal{T}(\mathcal{H})$ becomes a Banach space whose dual is $\mathcal{L}(\mathcal{H})$.

**Theorem 1.4** Let $\psi_t$ be the solution of eq. (1.1) with Assumptions 1.1 and 1.2. Then, the formula

$$
\text{Tr}_\mathcal{H}\{\rho_t a\} = \mathbb{E}_\rho\left[ \left\langle \psi_t | a\psi_t \right\rangle_\mathcal{H} \right], \quad \forall a \in \mathcal{L}(\mathcal{H}),
$$

(1.12)

defines a family $\{\rho_t, t \geq 0\}$ of statistical operators on $\mathcal{H}$ with $\rho_0 = \rho$, where $\text{Tr}_\mathcal{H}\{\rho a\} := \mathbb{E}_\rho\left[ \left\langle \psi | a\psi \right\rangle_\mathcal{H} \right], \forall a \in \mathcal{L}(\mathcal{H})$; eq. (1.12) is equivalently written as

$$
\text{Tr}_\mathcal{H}\{\rho_t a\} = \mathbb{E}_\rho\left[ \left\langle \hat{\psi}_t | a\hat{\psi}_t \right\rangle_\mathcal{H} \right], \quad \forall a \in \mathcal{L}(\mathcal{H}).
$$

(1.13)

Moreover, $\rho_t$ is the unique solution of the integral equation

$$
\rho_t = \rho + \int_0^t \mathcal{L}(s)_*[\rho_s] \, ds,
$$

(1.14)

where $\mathcal{L}(s)_*$ is the preadjoint operator of $\mathcal{L}(s) \in \mathcal{L}(\mathcal{L}(\mathcal{H}))$ defined by

$$
\mathcal{L}(t)[a] := -\frac{i}{2} [K(t) + K^*(t), a] + \frac{1}{2} \sum_{i=0}^{\infty} \left( R_i^*(t) \left[ a, R_i(t) \right] + \left[ R_i^*(t), a \right] R_i(t) \right), \quad \forall a \in \mathcal{L}(\mathcal{H}),
$$

(1.15)

where $[a, b] := ab - ba$.

For the proof of this theorem see [1]. Theorem 1.2 and Section 3.

In the theory of open quantum systems eq. (1.14) is usually written as $d\rho_t/dt = \mathcal{L}(t)_*[\rho_t]$ (if $\mathcal{L}(t)$ is sufficiently smooth in time), it is called a *master equation* and describes the evolution of the statistical operator representing the state of the system.

Formula (1.13) connects the solution $\hat{\psi}_t$ of the SDE (1.9) with the solution $\rho_t$ of the master equation (1.14); this fact is at the basis of stochastic methods for numerical simulations of solutions of master equations [1].

For what concerns the theory of continual measurements, we shall see in Section 3 that the processes $W_k(t)$, or functionals of them, represent the output of the measurement and that the physical law of this output is given by the probability $\hat{P}$ defined in eq. (1.3). In particular, formulae for all the moments (under $\hat{P}$) of the processes $W_k(t)$ can be given. For instance, from Theorems 1.2 and 1.4 we get

$$
\mathbb{E}_\rho [W_k(t)] = 2 \int_0^t \text{Re} \mathbb{E}_\rho [\hat{m}_k(s)] \, ds \quad \text{and} \quad \mathbb{E}_\rho [\hat{m}_k(s)] = \text{Tr}_\mathcal{H}\{\rho_s R_k(s)\};
$$

therefore, the mean value of the output $W_k(t)$ is connected to the solution of the master equation by

$$
\mathbb{E}_\rho [W_k(t)] = \int_0^t \text{Tr}_\mathcal{H}\{\rho_s (R_k(s) + R_k^*(s))\} \, ds.
$$

(1.16)
The computation of the moments of the second order is a little bit more involved; one needs to go through the characteristic functional [4, 13] or to exploit the Markov properties of the solution of eq. (1.13). Let $\mathcal{U}(t,s)$ be the evolution operator (from time $s$ to $t$) associated with eq. (1.14), i.e. $\varrho_t = \mathcal{U}(t,0)[\varrho]$; then, the result is (cf [13] eq. (5.20))

$$
\mathbb{E}_{\varrho_t} [W_i(t_1)W_j(t_2)] = t_1 \wedge t_2 + \\
+ \int_0^{t_1} ds_1 \int_0^{t_2 \wedge s_1} ds_2 \frac{1}{\hbar} \text{Tr} \{ \left( R_i(s_1) + R_i^*(s_1) \right) \mathcal{U}(s_1, s_2) \left[ R_j(s_2) \varrho_{s_2} + \varrho_{s_2} R_j^*(s_2) \right] \} + \\
+ \int_0^{t_2} ds_2 \int_0^{t_1 \wedge s_2} ds_1 \frac{1}{\hbar} \text{Tr} \{ \left( R_j(s_2) + R_j^*(s_2) \right) \mathcal{U}(s_2, s_1) \left[ R_i(s_1) \varrho_{s_1} + \varrho_{s_1} R_i^*(s_1) \right] \}. \tag{1.17}
$$

Also the random vectors $\hat{\psi}_t$ have a physical interpretation [2, 3, 5, 6]. Indeed, it can be shown that the random orthogonal projection $|\hat{\psi}_t \rangle \langle \hat{\psi}_t|$ represents the state at time $t$ one has to attribute to the quantum system if the trajectory of all the processes $W_j$ is known up to time $t$; in this sense $|\hat{\psi}_t \rangle \langle \hat{\psi}_t|$ represents the a posteriori state of the system at time $t$. Then, $\varrho_t = \mathbb{E}_{\varrho_t} [|\hat{\psi}_t \rangle \langle \hat{\psi}_t|]$ is the state of the system when no information on the outputs is taken into account; therefore, $\varrho_t$ is called the a priori state at time $t$.

Note that no physical quantity, such as $\varrho_t, |\hat{\psi}_t \rangle \langle \hat{\psi}_t|$ and $\hat{P}$, depends on the arbitrary phase introduced in eq. (1.8).

## 2 The Fock and Wiener spaces

In order to be able to show how the stochastic equations of the previous section are related to the quantum dynamics, we need to recall some facts about Fock and Wiener spaces.

Let $Z$ be a separable complex Hilbert space and let us denote by $\Gamma$ the symmetric Fock space over $\mathcal{X} := Z \otimes L^2(\mathbb{R}) \simeq L^2(\mathbb{R}; Z)$, which is defined by

$$
\Gamma := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} (\mathcal{X}^\otimes n)_S, \tag{2.1}
$$

where $(\mathcal{X}^\otimes n)_S$ denotes the symmetric part of the tensor product $\mathcal{X}^\otimes n$ and is called the $n$-particle space ([14] Sects. 17 and 19). The vectors $E(f) := \left( 1, f, \frac{1}{\sqrt{2!}} f \otimes f, \ldots, \frac{1}{\sqrt{n!}} f^\otimes n, \ldots \right), f \in \mathcal{X},$ are called exponential vectors ([14] p. 124); let us denote by $\mathcal{E}$ the linear span of the exponential vectors.

**Theorem 2.1** *The set of all the exponential vectors is linearly independent and total in $\Gamma$; moreover, for $f, g \in \mathcal{X}$ we have $(E(g)|E(f))_\Gamma = \exp(g|f).$ Let us fix $g \in \mathcal{X}$ and $V \in \mathcal{U}(\mathcal{X})$ (unitary operators on $\mathcal{X}$); there exists a unique unitary operator $W(g; V)$ (Weyl operator) such that

$$
W(g; V)E(f) := \exp \left\{ -\frac{1}{2} \|g\|^2 \right\}_\mathcal{X} \left( g|f \right)_\mathcal{X} E(V f + g), \quad \forall f \in \mathcal{X}. \tag{2.2}
$$

These operators satisfy the multiplication rules $(g_i \in \mathcal{X}, V_i \in \mathcal{U}(\mathcal{X}))$

$$
W(g_2; V_2)W(g_1; V_1) = \exp \left[ i \text{Im}(V_2 g_1|g_2)_\mathcal{X} \right] W(V_2 g_1 + g_2; V_2 V_1). \tag{2.3}
$$

|
Total means that \( \mathcal{E} \) is dense in \( \Gamma \); for the proof see [14] p. 124, Proposition 19.4 p. 126, pp. 134–135. Note that \( \|E(f)\|_\mathcal{E}^2 = \exp \|f\|_\mathcal{A}^2 \); by normalizing the exponential vectors one obtains the coherent vectors \( e(f) := \exp \left(-\frac{i}{2}\|f\|_\mathcal{A}^2\right) E(f) \). The vector \( e(0) \equiv E(0) \equiv (1, 0, 0, \ldots) \) is called the vacuum.

In the following we shall need particular Weyl operators. Let \( \mathcal{V} \in \mathcal{U}(\mathcal{X}) \) be such that \( (\mathcal{V} f)(t) = \mathcal{V}(t)f(t), \forall f \in \mathcal{X} \simeq L^2(\mathcal{R}; \mathcal{Z}) \); then \( \mathcal{V}(t) \in \mathcal{U}(\mathcal{Z}) \). Let \( g \in L^2_{\text{loc}}(\mathcal{R}; \mathcal{Z}) \), i.e. \( \int_\mathcal{A}^2 \|g(\tau)\|^2_\mathcal{Z} d\tau < +\infty, \forall s, t \in \mathcal{R}, s < t \). Then, we set

\[
\mathcal{W}_t(g; \mathcal{V}) := \mathcal{W}(\chi_{[0, t]}^{} g; \chi_{[0, t]}^{}(\mathcal{V} - \mathbb{1}) + \mathbb{1}).
\] (2.4)

The Fock space has a continuous tensor product structure, i.e. \( \forall s, t \in \mathcal{R}, s \leq t, \Gamma = \Gamma_{\mathcal{A}^\infty}^s \otimes \Gamma_{\mathcal{A}^\infty}^t \), where \( \Gamma_{\mathcal{A}^a}^b \) is the symmetric Fock space over \( L^2((a, b); \mathcal{Z}) \), \( -\infty \leq a < b \leq +\infty \). Correspondingly one has \( E(f) = E\left(\chi_{(-\infty, s)} f\right) \otimes E\left(\chi_{(s, t)} f\right) \otimes E\left(\chi_{(t, +\infty)} f\right) \); we also denote by \( \mathcal{E}_{\mathcal{A}^a}^b \subset \mathcal{A}^a_{\mathcal{A}^b} \) the linear span of the exponential vectors of the type \( E\left(\chi_{(a, b)}\right) \) ([14] Proposition 19.6 p. 127 and pp. 179–180). Note that \( \mathcal{W}_t(g; \mathcal{V}) \) leaves invariant \( \Gamma_{\mathcal{A}^a}^b \). By setting \( S(t) E(f) := E(f_t), f_t(s) := f(s - t) \), one defines a strongly continuous unitary group of shift operators on \( \Gamma \); moreover, one has \( S(t) \Gamma_{\mathcal{A}^a}^b = \Gamma_{\mathcal{A}^a}^{b+t} \).

Let \( \{ e_j, j = 0, 1, \ldots \} \) be a c.o.n.s. in \( \mathcal{Z}, g \in \mathcal{X} \) and let us define on \( \mathcal{E} \) the operators

\[
a(g)E(f) := \langle g|f \rangle_{\mathcal{X}} E(f), \quad \forall f \in \mathcal{X}, \quad A_j(t) := a\left(\chi_{[0, t]} \cdot e_j\right),
\]
\[
a(g)^t E(f) := \frac{d}{d\varepsilon} E\left(f + \varepsilon g\right)_{\varepsilon=0}, \quad \forall f \in \mathcal{X}, \quad A_j^t(t) := a^t\left(\chi_{[0, t]} \cdot e_j\right);
\]

let us note that one has \( \langle a^t(g) E(f_1)|E(f_2)\rangle_{\Gamma} = \langle E(f_1) a(g) E(f_2)\rangle_{\Gamma} \). The operator \( a^t(g) \) turns out to be linear in its argument and \( a(g) \) antilinear. The domains of these operators can be suitably extended; there exists a core \( \mathcal{D} \) for all these operators where all their products are well defined and such that \( \mathcal{D} \supset \mathcal{E}, \mathcal{D} \supset (\mathcal{X}^2)^n \) for every \( n \). Then, \( A_j(t) \) and \( a(g) \) send the \( n \)-particle space into the \((n - 1)\)–one and \( A_j(t) \) and \( a^t(g) \) send the \( n \)-particle space into the \((n + 1)\)–one; by this they are called annihilation and creation operators, respectively. On \( \mathcal{D} \) the annihilation and creation operators satisfy the canonical commutation rules \( [a(g), a(f)] = 0, \quad [a^t(g), a^t(f)] = 0, \quad [a(g), a^t(f)] = (g|f)_{\mathcal{X}} \), which characterize Bose fields ([14] pp. 136–146 and Example 24.1 p. 181). For the operators \( A_j(t) \) and \( A_j^t(t) \) these commutations relations become \( [A_j(t), A_i(s)] = 0, \quad [A_j^t(t), A_i^t(s)] = 0, \quad [A_j(t), A_i^t(s)] = \delta_{ij} \min(t, s) \); in particular

\[
[A_j(t) + A_j^t(t), A_i(s) + A_i^t(s)] = 0, \quad \forall i, j, t, s.
\] (2.7)

A connection between creation and annihilation operators and Weyl operators is given by the following theorem ([14] pp. 136, 139, 143–144).

**Theorem 2.2** For every \( k \in \mathcal{X} \) the operator \( a(k) + a^t(k) \) is essentially selfadjoint on \( \mathcal{E} \). Moreover, if we denote again by \( a(k) + a^t(k) \) its selfadjoint extension, we have \( \mathcal{W}(ik; \mathbb{1}) = \exp\{i[a(k) + a^t(k)]\} \).

From now on, when \( v \in \mathcal{Z} \), we shall set \( v_j := \langle e_j|v\rangle_{\mathcal{Z}} \). Moreover, we shall confuse essentially selfadjoint operators with their selfadjoint extensions.

Let us set \( \mathcal{R} := \{k \in L^2_{\text{loc}}(\mathcal{R}^+; \mathcal{Z}) : k_j(t) \in \mathcal{R}, \forall j, t\} \). From eq. (2.3) we have that

\[
[\mathcal{W}_{t_1}(ik_1; \mathbb{1}), \mathcal{W}_{t_2}(ik_2; \mathbb{1})] = 0, \quad \forall t_1, t_2 \in \mathcal{R}^+ , \quad \forall k_1, k_2 \in \mathcal{R},
\] (2.8)
or, which is the same, from the canonical commutation rules we have

\[
\left[ a \left( \chi_{[0,t_1]} k_1 \right), a^\dagger \left( \chi_{[0,t_1]} k_1 \right), a \left( \chi_{[0,t_2]} k_2 \right), a^\dagger \left( \chi_{[0,t_2]} k_2 \right) \right] = 0.
\]  

(2.9)

Then, we set \( \mathcal{M}(t) := \{ \mathcal{W}_t(ik; \mathbb{I}), k \in \mathcal{R} \}'' \) (double commutant in \( \mathcal{L}(\Gamma) \)); \( \mathcal{M}(t) \) is the von Neumann subalgebra of \( \mathcal{L}(\Gamma) \) generated by the Weyl operators \( \{ \mathcal{W}_t(ik; \mathbb{I}), k \in \mathcal{R} \} \) and it turns out to be a commutative algebra. Moreover, by the spectral theorem, the previous one and the definition of the canonical commutation rules we have

\[ \text{Theorem 2.3} \]

There exists a unique unitary isomorphism \( \mathcal{I} : \Gamma_0 \rightarrow L^2_w \) satisfying

\[
\mathcal{I} E(f) = E_w(f) := \exp \left\{ \sum_{j=0}^{\infty} \int_0^{+\infty} \left[ f_j(t) \text{d}W_j(t) - \frac{1}{2} (f_j(t))^2 \text{d}t \right] \right\}, \quad \forall f \in L^2(\mathbb{R}_+; \mathcal{Z}).
\]  

(2.11)

Moreover, we have

\[
\mathcal{I} \mathcal{W}_t(ik; \mathbb{I}) \mathcal{I}^{-1} = \exp \left\{ \sum_{j=0}^{\infty} \int_0^t k_j(s) \text{d}W_j(s) \right\}, \quad \forall k \in \mathcal{R}, \quad \forall t > 0,
\]  

(2.12)

and, \( \forall j = 0, 1, \ldots, \forall t > 0,

\[
\mathcal{I} \left( A_j(t) + A_j^\dagger(t) \right) \mathcal{I}^{-1} = W_j(t), \quad \mathcal{I} \mathcal{M}(t) \mathcal{I}^{-1} = L^\infty(\Omega, \mathcal{F}_t, P); \quad (2.13)
\]

here \( W_j(t) \) is a multiplication operator in \( L^2_w \).

Proof. The isomorphism \( \mathcal{I} \) is constructed in [14] Example 19.9 pp. 130–131.

By eqs. (2.2), (2.4), (2.11), we have \( \forall k \in L^2_{\text{loc}}(\mathbb{R}_+; \mathcal{Z}) \) and \( \forall f \in L^2(\mathbb{R}_+; \mathcal{Z}) \)

\[
\mathcal{I} \mathcal{W}_t(ik; \mathbb{I}) \mathcal{I}^{-1} E_w(f) = \exp \left\{ \sum_{j=0}^{\infty} \int_0^t \left[ k_j(s) \text{d}W_j(s) + (2f_j(s) + ik_j(s)) \text{Im} k_j(s) \text{d}s \right] \right\} E_w(f).
\]  

(2.14)

Then, Theorems 2.1, 2.2 and eq. (2.14) give eq. (2.13) and the first of eqs. (2.13). By eq. (2.12) and the definition of \( \mathcal{M}(t) \) we have also the second of eqs. (2.13). \( \square \)

It is possible to extend the isomorphism \( \mathcal{I} \) to the whole \( \Gamma \equiv \Gamma_{-\infty}^0 \otimes \Gamma_0 \) by mapping it into \( L^2_w \otimes L^2_w \). Let us stress that \( \mathcal{I} \) is the unitary transformation which simultaneously diagonalizes all the commuting selfadjoint operators \( \{ A_j(t) + A_j^\dagger(t), j = 0, 1, \ldots, t \in \mathbb{R}_+ \} \).
3 Quantum stochastic differential equations

Let $\mathcal{H}$ be a separable complex Hilbert space as in Section 1. Hudson and Parthasarathy \[3] developed a quantum stochastic calculus which gives meaning to integrals with respect to $dA_j(t)$, $dA_j^\dagger(t)$, $dt$; we give only a rough idea of the definition, just what we shall need in the following. Let $G(t)$ be an operator with domain including $\mathcal{H} \otimes \mathcal{E}$ (the linear span of the vectors of the type $\varphi \otimes E(f)$) and such that $G(t) \mathcal{H} \otimes \mathcal{E}_{\mathbb{C}^\infty} \subset \mathcal{H} \otimes \Gamma_{\mathbb{C}^\infty}$. Then, the integral $\int_0^t G(s) dA_j(s)$ is defined as a suitable limit (if it exists) of a sum of terms of the type $G(t_i) (A_j(t_{i+1}) - A_j(t_i))$; this kind of definition is analogous to the definition of classical Itô’s stochastic integral (\[14\] Sects. 24 and 25). For instance, one has

$$a(k) = \sum_{j=0}^\infty \int_0^{+\infty} k_j(t) dA_j(t), \quad a^\dagger(k) = \sum_{j=0}^\infty \int_0^{+\infty} k_j(t) dA_j^\dagger(t), \quad k \in L^2(\mathbb{R}^+; \mathcal{Z}).$$

(3.1)

Let us introduce now the operators $H, L_j \in \mathcal{L}(\mathcal{H})$, such that $H = H^*$ and $\sum_{j=0}^\infty L_j^* L_j$ strongly convergent in $\mathcal{L}(\mathcal{H})$; let us set $\tilde{K} := H - \frac{1}{2} \sum_{j=0}^\infty L_j^* L_j$. We consider the quantum stochastic Schrödinger equation

$$dU(t, s) = \left\{ \sum_{j=0}^\infty \left[ L_j \, dA_j^\dagger(t) - L_j^* \, dA_j(t) \right] - i\tilde{K} \, dt \right\} U(t, s),$$

(3.2)

$$U(s, s) = \mathbb{I};$$

(3.3)

here $s \leq t$, $U(t, s)$ is an operator on $\mathcal{H} \otimes \Gamma$ and $L_j$ is identified with $L_j \otimes \mathbb{I}$, $A_j(t)$ with $\mathbb{I} \otimes A_j(t)$ and so on.

**Theorem 3.1** (Hudson, Parthasarathy, Frigerio) Equations (3.2), (3.3) have a unique solution $U(t, s), t \geq s$; we have also $U(t, s) \in \mathcal{U}(\mathcal{H} \otimes \Gamma_\mathbb{C}_j) \subset \mathcal{U}(\mathcal{H} \otimes \Gamma)$ and $U(t, r) U(r, s) = U(t, s)$ for $s \leq r \leq t$. If we fix $f \in \mathcal{X}$ and set $\forall a \in \mathcal{L}(\mathcal{H})$

$$\tilde{E}_t[a] := \mathsf{i} \left[ H + \sum_{j=0}^\infty \left( f_j(t) L_j - f_j(t) L_j^\dagger \right), \quad a \right] + \frac{1}{2} \sum_{j=0}^\infty \left( [L_j^*, a] L_j + L_j^* [a, L_j] \right),$$

(3.4)

then we have, $\forall \xi \in \mathcal{H}, \forall a \in \mathcal{L}(\mathcal{H}), \forall t, s \in \mathbb{R}, t \geq s$,

$$\langle U(t, s) \xi \otimes e(f) | a U(t, s) \xi \otimes e(f) \rangle_{\mathcal{H} \otimes \mathcal{X}} = \langle \xi | a \xi \rangle_{\mathcal{H}} +$$

$$+ \int_s^t \langle U(\tau, s) \xi \otimes e(f) | \tilde{E}_r[a] U(\tau, s) \xi \otimes e(f) \rangle_{\mathcal{H} \otimes \mathcal{X}} d\tau.$$  (3.5)

Moreover, the quantity $S^*(t + s) U(t + s, s) S(s)$ does not depend on $s$ and $\{\tilde{U}(t), t \in \mathbb{R}\}$, where $\tilde{U}(t) := S^*(t) U(t, 0)$ for $t \geq 0$ and $\tilde{U}(t) := U^*(-t, 0) S(-t)$ for $t < 0$, is a strongly continuous one-parameter group of unitary operators on $\mathcal{H} \otimes \Gamma$.

For the proof of the first part of the theorem see \[14\] Theorem 27.8 p. 228, Proposition 26.7 p. 216, Corollaries 27.9 and 27.10 p. 230; for the statements about $\tilde{U}(t)$ see \[16\]. Let us stress that from the strong continuity we have that $\tilde{U}(t)$ is strongly differentiable on some dense domain and the same is true for $S(t)$. But the two infinitesimal generators are unbounded and mutually related in such a way that $U(t, s) = S(t) \tilde{U}(t - s) S^*(s)$ is not differentiable in the usual sense; however, although the usual derivative of $U(t, s)$ does not exist, the quantum stochastic
calculation gives meaning to its stochastic differential (3.2). Another interesting point is that, from
the connection between \( \hat{U}(t) \) and the master equation (3.2), it is possible to deduce that the
Stone generator of \( \hat{U}(t) \) is a selfadjoint operator, necessarily unbounded from above and from
below.

In the standard formulation of quantum mechanics it is assumed that the evolution of an
isolated system is represented by a strongly continuous group of unitary operators. So, by
the previous theorem, we can think we have a quantum system \( \mathcal{A} \), described in the Hilbert
space \( \mathcal{H} \), interacting with a second system \( \mathcal{B} \), a bosonic field described in \( \Gamma \). The dynamics of
\( \mathcal{A} + \mathcal{B} \) is given by \( \hat{U}(t) \); if we interpret \( S(t) \) as the free dynamics of the field, then \( U(t,s) \)
is the evolution operator of the compound system in the interaction picture with respect to the
free-field dynamics. It is possible to show that the evolutions such as \( U(t,s) \) are a sensible
approximation to the dynamics of a photo-emissive source \( \mathcal{A} \), such as an atom, an optical system
in a cavity, \ldots, interacting with the electromagnetic field \( \mathcal{B} \).

Let us fix now \( s = 0 \) as initial time and \( \xi \otimes e(f) \) as initial state, \( \xi \in \mathcal{H}, \|\xi\|_\mathcal{H} = 1, 
f \in L^2_{\text{loc}}(\mathbb{R}; \mathbb{Z}) \); we have not defined coherent vectors for such an \( f \), but we shall see in the
following that in our applications the queue of \( f \) does not matter and that it is often useful to
consider periodic functions. Indeed the choice \( f(t) \sim \exp(-i\omega_0 t) \) could represent a stimulating
monochromatic laser. Moreover, we set \( U_t := U(t,0) \), so that \( U_t \xi \otimes e(f) \) is the state of our
compound system at time \( t \).

Up to here we have studied the evolution of the system. Let us consider now a measurement
process in which some observables of the system are monitored with continuity in time; the
mathematical model of the detection scheme will be given again in terms of operators on the
Fock space and quantum stochastic integrals. Quantum stochastic calculus has been introduced
into the theory of continual measurements in [18]

Let us recall that, in the usual formulation of quantum mechanics, observables are represented
by selfadjoint operators. Let us consider a quantum system represented in a Hilbert space \( \mathcal{H} \) and let \( \psi_t \) be
the state of the system at time \( t \) (\( \psi_t \in \mathcal{H}, \|\psi_t\|_\mathcal{H} = 1 \)). Let \( X_1, \ldots, X_n \) be
commuting selfadjoint operators with joint spectral projections \( \Pi_{X_i}(\mathcal{B}), B \) Borel subset of \( \mathbb{R}^n \).
Then, \( \langle \psi_t | \Pi_{\hat{X}}(\mathcal{B}) | \psi_t \rangle_{\mathcal{H}} \) is interpreted as the probability that the measurement of the “observable”
\( \hat{X} \) gives a result in \( \mathcal{B} \) at time \( t \); indeed, \( \langle \psi_t | \Pi_{\hat{X}}(\cdot) | \psi_t \rangle_{\mathcal{H}} \) is a probability measure on \( \mathbb{R}^n \). Obviously,
the whole probability measure is known if we know its Fourier transform (the characteristic
function), which turns out to be given by \( \langle \psi_t | \exp\{i\hat{k} \cdot \hat{X}\} | \psi_t \rangle_{\mathcal{H}}, \hat{k} \in \mathbb{R}^n \). We can also say
that we know completely the probability law of \( \hat{X} \) if we know all the quantities \( \langle \psi_t | a | \psi_t \rangle_{\mathcal{H}} \) with
\( a \) belonging to the commutative von Neumann sub-algebra of \( \mathcal{L}(\mathcal{H}) \) generated by the set of
operators \( \{ \exp[i\hat{X} \cdot \hat{X}], \hat{k} \in \mathbb{R}^n \} \). This can be generalized even to families of infinitely many
commuting selfadjoint operators.

Now, let us consider again our system \( \mathcal{A} + \mathcal{B} \) and the family of commuting selfadjoint operators
\begin{equation}
Z_j(t) := \sum_{i=0}^{\infty} \int_0^t \left[ V_{ij}(s) \ dA_i(s) + V_{ij}(s) \ dA_i^1(s) \right], \quad t \geq 0, \ j = 0, 1, \ldots ; \quad (3.6)
\end{equation}

here \( V, V(t) \) are unitary operators as in eq. (2.4) and \( V_{ij}(t) := \langle e_j | V(t) e_i \rangle x \). We consider \{\( Z_j(t) \}\)
as a set of continuously monitored observables. Indeed, in the case of electromagnetic field
emitted by some source, the monitoring of observables of the type (3.6) can be concretely realized
by a measurement scheme called balanced heterodyne detection [13]. Let us denote by \( \mathcal{M}_Z(t) \) the
von Neumann algebra generated by the Weyl operators \( \{ \exp \left[ i \sum_{j=0}^{\infty} \int_0^t k_j(s) dZ_j(s) \right], k \in \mathcal{R} \} \).

By the previous discussion, the law of the measurement of \( \mathcal{Z}(s) \) for \( s \) from 0 to \( t \) is known if we
know the quantities \( (U_t \xi \otimes e(f) | Y U_t \xi \otimes e(f))_{\mathcal{H} \otimes \mathcal{F}_t}, \forall Y \in \mathcal{M}_Z(t) \). The next step is to transform these quantities and to link them to the stochastic processes introduced in Section 1.

From now on \( V \) is the unitary operator appearing in the observables (3.3), \( f \) is the function characterizing the initial state of \( B \), \( \xi \) is the initial state of \( A \) (\( \xi \in \mathcal{H} ||\xi||_\mathcal{H} = 1 \)) and \( H_0 = H_0^* \in \mathcal{L}(\mathcal{H}) \) is a selfadjoint operator which can be chosen arbitrarily and is introduced for future convenience.

**Theorem 3.2** For every \( t \in \mathbb{R}_+ \) and \( Y \in \mathcal{M}_Z(t) \), we set
\[
\tilde{U}_t := \exp[iH_0 t] \mathcal{W}_t(f; V) U_t \mathcal{W}_t(f; V),
\]
\[
\psi_t := \mathcal{I} \tilde{U}_t \xi \otimes e(0),
\]
\[
\tilde{Y} := \mathcal{W}_t^*(f; V) Y \mathcal{W}_t(f; V).
\]

Then, we have \( \tilde{U}_0 = \mathbb{I}, \tilde{Y} \in \mathcal{M}(t), \mathcal{I} \tilde{Y} \mathcal{I}^{-1} \in L^\infty(\Omega, \mathcal{F}_t, P) \),
\[
\langle U_t \xi \otimes e(f) | Y U_t \xi \otimes e(f) \rangle_{\mathcal{H} \otimes \mathcal{F}_t} = \langle \tilde{U}_t \xi \otimes e(0) | \tilde{Y} \tilde{U}_t \xi \otimes e(0) \rangle_{\mathcal{H} \otimes \mathcal{F}_t} = \langle \psi_t | \mathcal{I} \tilde{Y} \mathcal{I}^{-1} \psi_t \rangle_{\mathcal{H} \otimes \mathcal{L}^2_\mathcal{W}},
\]
\[
d\tilde{U}_t = \left\{ \sum_{j=0}^\infty \left[ R_j(t) dA_j^\dagger(t) - R_j^\dagger(t) dA_j(t) \right] - iK(t) dt \right\} \tilde{U}_t,
\]
\[
K(t) := \exp[iH_0 t \left\{ \hat{K} - H_0 + i \sum_{j=0}^\infty \left[ f_j(t) L_j - f_j^* (t) L_j^* \right] \right\} e^{-iH_0 t},
\]
\[
R_j(t) := \sum_{i=0}^\infty (V^* \chi^i(t) ) \chi_i(t) e^{iH_0 t} L_i e^{-iH_0 t};
\]

finally, \( \psi_t \) satisfies the SDE (1.1) with initial condition \( \psi_0(\omega) = \xi, \forall \omega \in \Omega \). Moreover, if we assume \( \text{ess sup} \| f(t) \|_\mathcal{F} < +\infty \) for all \( t \in \mathbb{R}_+ \), then the coefficients (3.12) and (3.13) satisfy Assumptions (1.4) and (1.4).

**Proof.** Note that from eqs. (2.2)–(2.4) we have
\[
\mathcal{W}_t(f; V) e(0) = \mathcal{W} \left( (1 - \chi_{[0,t]} f) ; \mathbb{I} \right) \mathcal{W}_t(f; V) e(0) = e(f).
\]
The operator \( U_t \) acts not trivially only on \( \mathcal{H} \otimes \mathcal{F}_t^0 \), while \( \mathcal{W} \left( (1 - \chi_{[0,t]} f) ; \mathbb{I} \right) \) only on \( \mathcal{F}_t^0 \otimes \mathcal{F}_t \); therefore, these two operators commute. Then, by eqs. (3.14) and (3.7) one has
\[
U_t \xi \otimes e(f) = \mathcal{W} \left( (1 - \chi_{[0,t]} f) ; \mathbb{I} \right) \mathcal{W}_t(f; V) e^{-iH_0 t} \tilde{U}_t \xi \otimes e(0).
\]
The first of eqs. (3.10) follows from eqs. (3.15), (3.1) and the second one from eq. (3.8). The fact that \( \tilde{Y} \) belongs to \( \mathcal{M}(t) \) follows from the definitions of \( \mathcal{M}(t) \) and \( \mathcal{M}_Z(t) \) and from eqs. (2.3) and (2.4); the fact that one has \( \mathcal{I} \tilde{Y} \mathcal{I}^{-1} \in L^\infty(\Omega, \mathcal{F}_t, P) \) follows from the second of eqs. (2.13).  

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From eqs. (2.3), (2.4), (2.5), (2.6) one obtains how the operation $W^*_t(f;V) \cdot W_t(f;V)$ transforms the operators $A_j(t), A_j^*(t)$; then, eqs. (3.11)–(3.13) follow from eqs. (3.7), (3.8).

By eq. (2.5), the operators $A_j(t)$ annihilate the vacuum. Moreover, by the factorization properties of $\Gamma$ and the definition of quantum stochastic integrals, we have that $dA_j(t)$, acting on $\Gamma^{t+dt}$, commutes with $\tilde{U}_t$, acting on $\mathcal{H} \otimes \Gamma^0$, or, better, $\int_0^t dA_j(s)\tilde{U}_s = \int_0^t \tilde{U}_s dA_j(s)$. Therefore, when eq. (3.11) is applied to $\xi \otimes e(0)$, the integral with respect to $A_j$ gives a vanishing contribution and the integrand can be changed at will; so we can write

$$d\tilde{U}_t \xi \otimes e(0) = \left\{ \sum_{j=0}^{\infty} R_j(t) \left[ dA_j^*(t) + dA_j(t) \right] - iK(t)dt \right\} \tilde{U}_t \xi \otimes e(0).$$

(3.16)

By the first of eqs. (2.13) we have that $\psi_t$ satisfies the SDE (1.1) with initial condition $\psi_0 = \xi \otimes E_w(0)$; but $E_w(0) \equiv 1$, so that we can write $\psi_0(\omega) = \xi$, $\forall \omega \in \Omega$.

Finally, it is possible to prove that $\left\| \sum_{j=0}^{\infty} R_j^*(t)R_j(t) \right\| = \left\| \sum_{j=0}^{\infty} L_j^*L_j \right\|$ and $\|K(t)\| \leq \|H - H_0\| + \frac{1}{2} \left\| \sum_{j=0}^{\infty} L_j^*L_j \right\| + 2\|f(t)\| \left\| \sum_{j=0}^{\infty} L_j^*L_j \right\|^{1/2}$; then, one can check that our coefficients satisfy Assumptions (1.3) and (1.4).

Let us comment the content of Theorem 3.2. All the physical quantities (probabilities, characteristic functional, moments) are given by the “quantum expectations” $\langle U_t \xi \otimes e(f)|YU_t \xi \otimes e(f) \rangle_{\mathcal{H} \otimes \Gamma_0}$ with $Y \in \mathcal{M}_Z(t)$. If we set

$$\tilde{Y} := \mathcal{I} \tilde{Y} \mathcal{I}^{-1} \equiv \mathcal{I} W_t(f;V) Y W^*_t(f;V) \mathcal{I}^{-1},$$

(3.17)

we have that $\tilde{Y}$ is a random variable in $(\Omega, \mathcal{F}_t, P)$ or $(\Omega, \mathcal{F}_t, \tilde{P})$; moreover, by the definitions of $L^2_W$ and $\tilde{P}$, we can write

$$\langle \psi_t | \tilde{Y} \psi_t \rangle_{\mathcal{H} \otimes L^2_W} = \mathbb{E}_P \left[ \| \psi_t \|_{\mathcal{H}}^2 \tilde{Y} \right] = \mathbb{E}_{\tilde{P}} \left[ \tilde{Y} \right].$$

(3.18)

Finally, by eq. (3.14) we obtain

$$\langle U_t \xi \otimes e(f)|YU_t \xi \otimes e(f) \rangle_{\mathcal{H} \otimes \Gamma_0} = \mathbb{E}_{\tilde{P}} \left[ \tilde{Y} \right],$$

(3.19)

where $\tilde{Y}$ is given by eq. (3.17), $\tilde{P}$ by (1.3), $\psi_t$ is the solution of (1.1) with non random initial condition $\xi$, $R_j(t)$ and $K(t)$ are given by eqs. (3.12) and (3.13). Therefore, $\tilde{P}$ is the physical law of the output, as stated at the end of Section 1.

To discuss concrete physical applications of the previous formalism would be too long. However, let us stress that the first non trivial example can be realized in the Hilbert space $\mathcal{H} = \mathbb{C}^2$. Indeed, in this space one can construct a model describing a two-level atom of resonance frequency $\omega$ stimulated by a laser of frequency $\omega_0$ and emitting fluorescence light; we are interested in the spectrum of the emitted light. Let us just give the list of the choices that characterize the model: $\mathcal{H} = \mathbb{C}^2$, $H = \omega \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ with $\omega > 0$, $L_j = \alpha_j \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ with $\alpha_j \in \mathbb{C}$ and $0 < \sum_j |\alpha_j|^2 < +\infty$, $f_j(t) = \lambda_j e^{-i\omega_0 t}$ with $\omega_0 > 0$, $\lambda_j \in \mathbb{C}$ and $\lambda_0 = 0$; the arbitrary selfadjoint operator $H_0$ is chosen to be $H_0 = \omega_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (this eliminates some time dependence from the resulting equations). The fluorescence light is made to interfere with a strong laser of frequency $\nu$ and the intensity of the resulting light is measured with a photodetector. It can be shown
that this scheme corresponds to a continual measurement of one of the components of \( \vec{Z} \) (say \( Z_0 \)) given by eq. (3.6) with \( V_{ij}(s) = e^{-isu} \delta_{ij} \), \( \nu > 0 \). The intensity of the emitted light can be obtained from the second moments \( \mathbb{E}_p [W_0(t_1)W_0(t_2)] \) of \( W_0(t) = \mathcal{I}W_0(f;V)Z_0(t)\mathcal{I}^{-1} \); as a function of \( \nu \), this intensity is the spectrum of the atom. Computations are given in [20] where only quantum stochastic calculus is used (classical SDEs are not explicitly introduced); the result is a three–peaked spectrum, already known in the quantum–optical literature as the Mollow spectrum.
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