A subsystem-independent generalization of entanglement

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We introduce a generalization of entanglement based on the idea that entanglement is relative to a distinguished subspace of observables rather than a distinguished subsystem decomposition. A pure quantum state is entangled relative to such a subspace if its expectations are a proper mixture of those of other states. Many information-theoretic aspects of entanglement can be extended to the general setting, suggesting new ways of measuring and classifying entanglement in multipartite systems. By going beyond the distinguishable-subsystem framework, generalized entanglement also provides novel tools for probing quantum correlations in interacting many-body systems.

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Entanglement is a uniquely quantum phenomenon whereby a pure state of a composite quantum system may cease to be determined by the states of its constituent subsystems [1]. Entangled pure states are those that have mixed subsystem states. To determine an entangled state requires knowledge of the correlations between the subsystems. As no pure state of a classical system can be correlated, such correlations are intrinsically non-classical, as strikingly manifested by the violation of local realism and Bell’s inequalities [2]. In the science of quantum information processing (QIP), entanglement is regarded as the defining resource for quantum communication and an essential feature needed for unlocking the power of quantum computation. However, in spite of intensive investigation, a complete understanding of entanglement is far from being reached.

To unambiguously define entanglement requires a preferred partition of the overall system into subsystems. In conventional QIP scenarios, subsystems are associated with spatially separated “local” parties, which legitimates the distinguishability assumption implicit in standard entanglement theory. However, because quantum correlations are at the heart of many physical phenomena, it would be desirable for a notion of entanglement to be useful in contexts other than QIP. Strongly interacting quantum systems offer compelling examples of situations where the usual subsystem-based view is inadequate. Whenever indistinguishable particles are sufficiently close to each other, quantum statistics forces the accessible state space to be a proper subspace of the full tensor product space, and exchange correlations arise that are not a usable resource in the usual QIP sense. Thus, the natural identification of particles with preferred subsystems becomes problematic. Even if a distinguishable-subsystem structure may be associated to degrees of freedom different from the original particles (such as a set of modes [3]), inequivalent factorizations may occur on the same footing. Finally, the introduction of quasiparticles, or the purposeful transformation of the algebraic language used to analyze the system [4], may further complicate the choice of preferred subsystems.

While efforts are under way to obtain entanglement-like notions for bosons and fermions [3, 5] and to study entanglement in quantum critical phenomena [6–8], formulating a theory of entanglement applicable to the full variety of physical settings remains an important challenge.

In this Letter, we introduce a notion of generalized entanglement (GE) based on the relationship of a state to different sets of observables of the system of interest, without reference to a preferred subsystem decomposition. This is achieved by realizing that the salient features of entanglement are determined by the expectations of a distinguished subspace of observables. The latter may represent a limited means of manipulating and observing the system. For standard entanglement these means are limited to local observables acting on one subsystem only. The central idea is to generalize the observation that standard entangled pure states are those that look mixed to local observers. Each pure quantum state gives rise to a reduced state that only provides the expectations of the distinguished observables. The set of reduced states is convex and, like an ordinary quantum state space, it includes pure states (the extremal ones).

We say that a pure state is generalized unentangled relative to the distinguished observables, if its reduced state is pure, and generalized entangled otherwise. The definition extends to mixed states in a standard way: A mixed state is unentangled if it can be written as a mixture (or convex combination) of unentangled pure states. Because our definition depends only on convex properties of the distinguished spaces of observables and states we consider, it provides a notion of entanglement within a general convex framework suitable for investigating the foundations of quantum mechanics and related physical theories (cfr. [9] and references therein).

The mathematical foundation of GE is established in [10]. Here we highlight the significance of GE from a physics and information-physics perspective. For this purpose, we focus on the case where the observable subspace is a Lie algebra. A key result is then the identification of pure generalized unentangled states with the generalized coherent states (GCSs, a connection indepen-
dently noted by Klyachko [11]), which are well known for their applications in physics [12]. This encompasses the entanglement settings introduced to date in a unifying framework. Furthermore, it is now possible to extend information-theoretic notions to coherent state theory and beyond. We demonstrate that many concepts previously thought to be subsystem-specific are much more generic, define new measures of entanglement based on the general theory, and apply quantum information to condensed-matter problems. In particular, we introduce notions of Generalized Local Operations assisted by Classical Communication (GLOCC) under which the ordinary measures of standard entanglement do not increase, as well as measures of GE with the desired behavior under classes of GLOCC maps. New measures of standard entanglement are obtained for the multipartite case. In the Lie-algebraic setting, a simple GE measure obtained from the purity relative to a Lie algebra is a useful diagnostic tool for quantum many-body systems, playing the role of a disorder parameter for broken-symmetry quantum phase transitions.

**Generalized entanglement.**—We first revisit the standard setting for entanglement where we have two distinguishable subsystems forming a bipartite system. Let the $mn$-dimensional joint state space $\mathcal{H}$ factorize as $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$, with $\mathcal{H}_a$, $\mathcal{H}_b$, $m$, $n$-dimensional, respectively. In this setting, physical considerations distinguish a preferred set of observables, spanned by traceless Hermitian operators of the form $A \otimes 1$ and $1 \otimes B$, which are the local observables acting on system $a$ or $b$ alone. For each pure state $|\psi\rangle \in \mathcal{H}_a \otimes \mathcal{H}_b$, one may consider the reduced state describing the expectations of measurements of local observables. The reduced state is determined by the pair of reduced density operators, $\rho_a := \text{tr}_b |\psi\rangle \langle \psi|$ and $\rho_b := \text{tr}_a |\psi\rangle \langle \psi|$. Because pure product states are exactly those for which subsystem states are pure, our definition of GE relative to the local observable subspace coincides with the standard definition of entanglement. In this example, the distinguished observable space is a Lie algebra, $\mathfrak{h} = \mathfrak{su}(m) \oplus \mathfrak{su}(n)$, and $\mathfrak{h}$ is a subalgebra of the full Lie algebra $\mathfrak{g}$ of operators on $\mathcal{H}$. The connection with GCSs is established by associating the family of pure unentangled states with an orbit of the group of local unitary transformations acting on $\mathcal{H}$ (see below).

The extent to which our viewpoint extends the usual subsystem-based definition may be appreciated in situations where no subsystem partition exists and conventional entanglement is meaningless. Consider a single spin-1 system, whose three-dimensional state space $\mathcal{H}$ carries an irreducible representation of $\mathfrak{su}(2)$, with generators $J_x$, $J_y$, $J_z$ satisfying $[J_x, J_y] = i\varepsilon_{\alpha\beta\gamma} J_\gamma$, ($\varepsilon_{\alpha\beta\gamma}$ being the totally antisymmetric tensor). Suppose that the distinguished observables are linear in these generators so that they are the ones in the given representation of $\mathfrak{su}(2)$. The reduced states can be identified with vectors of expectation values of these three observables: They form a unit ball in $\mathbb{R}^3$, and the extremal points are those on the surface, which have maximal spin component 1 for some linear combination of $J_x$, $J_y$, $J_z$. These are the well-known “spin coherent states,” or GCSs for $\mathfrak{su}(2)$ [12]. For any choice of spin direction, $\mathcal{H}$ is spanned by the $|\uparrow\rangle$, $|\downarrow\rangle$, $|\uparrow\rangle$ eigenstates of that spin component; the first and last are GCSs, but $|\downarrow\rangle$ is not, characterizing $|\downarrow\rangle$ as a generalized entangled state relative to $\mathfrak{su}(2)$. All pure states appear unentangled if access to the full algebra $g = \mathfrak{su}(3)$ is available (that is, $\mathfrak{su}(3)$ is distinguished).

This example illustrates that when the distinguished subspace forms an irreducibly represented Lie algebra, the set of unentangled states is the set of GCSs. Another, more physically motivated characterization is as the set of states that are unique ground states of a distinguished observable. To formally relate these characterizations of unentangled states we review the needed Lie representation theory [13]. A Cartan subalgebra (CSA) $c$ of a semisimple Lie algebra $\mathfrak{h}$ is a maximal commutative subalgebra. A vector space carrying a representation of $\mathfrak{h}$ decomposes into orthogonal joint eigenspaces $V_\lambda$ of the operators in $c$. That is, each $V_\lambda$ consists of the set of states $|\psi\rangle$ such that for $x \in c$, $x|\psi\rangle = \lambda(x)|\psi\rangle$. The label $\lambda$ is therefore a linear functional on $c$, called the weight of $V_\lambda$. In the above example, any spin component $J_z$ spans a (one-dimensional) CSA $c_z$. There are three weight spaces labeled by the angular momentum along $z$, and spanned by the states $|\uparrow\rangle$, $|\downarrow\rangle$, $|\uparrow\rangle$ of the previous paragraph. Note that any two CSAs are conjugate under elements of the Lie group, manifested in the spin example by the fact that $J_z$ transforms into any desired spin component via conjugation by a rotation in $\mathcal{SU}(2)$. The subspace of operators in $\mathfrak{h}$ orthogonal in the trace inner product to $c$ can be organized into orthogonal “raising and lowering” operators, which connect different weight spaces. In the example, choosing $J_x$ as the basis of our CSA, these are $J_\pm := (J_x \pm i J_y)/\sqrt{2}$. For a fixed CSA and irreducible representation, the weights generate a convex polytope; a lowest (or highest) weight is an extremal point of such a polytope, and the one-dimensional weight-spaces having those weights are known as lowest-weight states. The set of lowest-weight states for all CSAs is the orbit of any one such state under the Lie group generated by $c$. These are the group-theoretic GCSs [12]. Notably, the GCSs attain minimum uncertainty in an appropriate invariant sense [14].

A natural way to relate any state $|\psi\rangle \in \mathcal{H}$ to a Lie algebra $\mathfrak{h}$ of operators acting on $\mathcal{H}$ is to project $|\psi\rangle \langle \psi|$ onto $\mathfrak{h}$. This projection completely determines the expectations of operators in $\mathfrak{h}$ for $|\psi\rangle$. The generalized unentangled states are the ones for which the projection of $|\psi\rangle \langle \psi|$ onto $\mathfrak{h}$ is extremal. The intuition that these should be the states whose projection has largest distance from 0 turns out to be true. This motivates the following definition. Let $\{x_i\}$ be a Hermitian ($x_i = x_i^\dagger$) orthogonal ($x_i x_j \propto \delta_{ij}$) basis for $\mathfrak{h}$ [15]. The purity of $|\psi\rangle$ relative
to $\mathfrak{h}$ (or $h$-purity) is $P_\mathfrak{h}(|\psi\rangle) := \sum_i |\langle x_i | \psi \rangle|^2$, where the $x_i$ have a common, rescaled norm chosen to ensure that the maximal value is 1. $P_\mathfrak{h}(|\psi\rangle)$ is the square-distance from 0 of the projection of $|\psi\rangle$ for pure bipartite states, the $su(m) \oplus su(n)$-purity is (up to a constant) the conventional purity given by the trace of the square of states, the the maximal value is 1.

So far, $\mathfrak{h}$ has been assumed to be a real Lie algebra of Hermitian operators. These may be thought of as a preferred family of Hamiltonians, which generate (via $h \mapsto \exp(h)$) a Lie group of unitary operators. More generally, we want Lie-algebraically distinguished completely positive (CP) maps, $\rho \mapsto \sum_i A_i \rho A_i^\dagger$. A natural class is obtained by restricting the “Hellwig-Kraus” (HK) operators $A_i$ to lie in the topological closure $\overline{e^{\mathfrak{h} \oplus \mathbb{I}}}$ of the Lie group generated by the complex Lie algebra $\mathfrak{h} \oplus \mathbb{I}$ [16]. Having HK operators in a group ensures closure under composition. Using $\mathfrak{h} \oplus \mathbb{I}$ allows non-unitary HK operators. Topological closure introduces singular operators such as projectors. The following characterizations of unentangled states (proven in [10]) demonstrate the power of the Lie algebraic setting.

**Theorem.** The following are equivalent for an irreducible representation of $\mathfrak{h}$ on $\mathcal{H}$:

1. $\rho$ is generalized unentangled relative to $\mathfrak{h}$.
2. $\rho = |\psi\rangle\langle \psi|$ with $|\psi\rangle$ the unique ground state of some $H$ in $\mathfrak{h}$.
3. $\rho = |\psi\rangle\langle \psi|$ with $|\psi\rangle$ a lowest-weight vector of $\mathfrak{h}$.
4. $\rho$ has maximum $h$-purity.
5. $\rho$ is a one-dimensional projector in $\overline{e^{\mathfrak{h} \oplus \mathbb{I}}}$.

**Generalized LOCC.**— The semigroup of LOCC maps [17] and the preordering it induces on states according to whether or not a given state can be transformed to another by an LOCC operation are at the core of entanglement theory. Given an HK representation $\{A_i\}$ of a CP map $M$, we can view each $A_i$ as being associated with measurement outcome $i$, obtained with probability $tr A_i \rho$, and leading to the state $A_i \rho A_i^\dagger$. The set $\{A_i\}$ and a list of maps $M_i$, with HK operators $\{B_{ij}\}$, specify a new map with representation $\{B_{ij} A_i\}$. This map can be implemented by first applying $M$ and then, given measurement outcome $i$, applying $M_i$. We call this conditional composition of maps. Closing the set of one-party maps (for all parties) under conditional composition gives the LOCC maps. When the distinguished observables form a semisimple Lie algebra $\mathfrak{h}$, a natural multipartite structure can be exploited to generalize LOCC. $\mathfrak{h}$ can be uniquely expressed as a direct sum of simple Lie algebras, $\mathfrak{h} = \oplus_i \mathfrak{h}_i$. A Hilbert space irreducibly representing $\mathfrak{h}$ factorizes as $\mathcal{H} = \oplus_i \mathcal{H}_i$, with $\mathcal{H}_i$ acting non-trivially on $\mathcal{H}_i$ only. This resembles ordinary entanglement, except that the “local” systems $\mathcal{H}_i$ may not be physically local, and actions on them are restricted to involve operators in the topological closure of a “local” Lie group representation which need not be $GL(\dim(\mathcal{H}_i))$ as in standard entanglement. For each simple algebra $\mathfrak{h}_i$, a natural restriction is to CP maps with HK operators in $e^{\mathfrak{h}_i \oplus \mathbb{I}}$. GLOCC, generalized LOCC, is the closure under conditional composition of the set of operations each of which is representable with HK operators in the topological closure of $e^{\mathfrak{h}_i \oplus \mathbb{I}}$ for some $i$.

In conventional entanglement, there is also interest in separable maps (SLOCC, [18–20]), which are those representable with HK operators that are tensor products. The generalization of these maps is obtained by considering the semigroup of maps whose HK operators are in $e^{\mathfrak{h} \oplus \mathbb{I}}$. Another potential generalization of LOCC involves using spectra of operators to classify them as analogues of single-party operators. Yet another begins from maps that induce well-defined maps on the set of reduced states, as single-party maps do in the standard setting. These alternative proposals are discussed further in [10].

**Measures of generalized entanglement.**— Because GE relative to $\mathfrak{h}$ reflects incoherence relative to $\mathfrak{h}$, and incoherence amounts to mixing from the point of view of $\mathfrak{h}$, a natural Lie-algebraic entanglement measure for mixed $\rho$ is obtained by minimizing the expected difference from 1 of the $\mathfrak{h}$-purity over pure state ensembles for $\rho$: $\min \{1 - \sum_i \pi_i P_\mathfrak{h}(\pi_i)\}$, where the minimum is over all $\pi_i > 0$ and pure $\pi_i$ such that $\sum_i \pi_i \rho_i = \rho$. A different approach is suggested by the convex structure of reduced states and uses natural mixedness measures $\sigma$ on finite probability distributions $p = (p_1, \ldots, p_k)$. Such measures are concave and permutation-invariant (Schur concave). Examples are entropy, $\sigma_H(p) := -\sum_i p_i \ln p_i$, and Renyi entropy, $\sigma_1(p) := 1 - \sum_i p_i^2$. For a reduced state $\rho$, define $\sigma(\mu)$ by minimizing $\sigma(p)$ over ways of writing $\mu = \sum_i p_i \mu_i$ with $p_i$ probabilities and $\mu_i$ pure reduced states. For an unreduced pure state $\rho$ with reduction $\nu$, define $\sigma(\rho) := \sigma(\nu)$. For general unreduced $\rho$, define $\sigma(\rho) = \min \{\pi_i \sigma(\pi_i)\}$ where the minimum is over all $\pi_i > 0$ and pure $\pi_i$ such that $\sum_i \pi_i \rho_i = \rho$. This measure will be convex as all measures of GE should be. It is also desirable that it is non-increasing under GLOCC. In [10], we have established that the above measures are non-increasing under those GLOCC operations implementable via conditional composition of operations with unitary HK operators in the Lie group. Generalizations of these results beyond the Lie-algebraic setting are discussed in [10]. As with standard entanglement, no single measure can capture the complexity of GE.

**Generalized multipartite entanglement.**— Multipartite systems are examples where GE contributes to the study of conventional entanglement. For $N$ qubits, the relevant algebra for conventional entanglement is $\mathfrak{h} = \oplus_{i=1}^N su(2)_i$, generated by the Pauli matrices for each qubit. The pure product states have maximal purity $P_\mathfrak{h} = 1$ (unentangled), whereas the states $|GHZ_N\rangle := 2^{-1/2}(|\uparrow\cdots\uparrow\rangle + |\downarrow\cdots\downarrow\rangle)$ have minimal purity 0 (maximally entangled). States of the form...
\[ |W_N\rangle := N^{-1/2} \sum_{\sigma} N^{\lambda(\sigma)} \prod_{i=1}^{N} \uparrow \uparrow \cdots \uparrow \downarrow \cdots \uparrow \] have an intermediate purity \( (N^{-1/2})^2 \). In the \( N \to \infty \) limit, \( P_b(|W_N\rangle) \to 1 \), whereas \( |\text{GHZ}_N\rangle \) remains maximally entangled. Interestingly, \( 1 - P_b \) (for this \( \lambda \)) coincides with the global entanglement measure introduced in [21]. Different choices of observable algebras, constructed for instance by using the full \( su(2N) \) or collective \( su(2) \) subalgebras for clusters of \( k \) qubits, may be considered to further refine the study of GE. Algebras such as these can be seen to be partially ordered by the subalgebra relationship. In this case, one can further relativize entanglement by considering any pair of relevant Lie algebras \( \mathfrak{h}_1 \supseteq \mathfrak{h}_2 \). By starting with a state reduced to \( \mathfrak{h}_1 \) and considering its further reduction to \( \mathfrak{h}_2 \), contributions to entanglement due to this pair of subalgebras can be extracted. The measures of entanglement \( \sigma(\rho) \) can provide additional information on the fine-structure of multipartite quantum correlations.

Another example consists of two spin-1 particles in the total spin representation of \( su(2) \). Suppose that the two spins can only be accessed collectively, e.g. using a global external field. Then the distinguished observable subspace is spanned by operators \( J_{\alpha} := J_{\alpha}^{(1)} \otimes \mathbf{1} + \mathbf{1} \otimes J_{\alpha}^{(2)} \), \( J_{\alpha}^{(1)} \), \( J_{\alpha}^{(2)} \) being spin-1 generators for each \( su(2) \). The (unentangled) GCSs here are states of maximal total spin projection in some direction \( \alpha \) (states of the form \( |1_\alpha\rangle |1_\alpha\rangle \)), whereas product states, like \( |0_\alpha\rangle |0_\alpha\rangle \) with zero spin projection, are generalized (maximally) entangled relative to this algebra. This reflects the fact that no SU(2) spin rotation can connect \( |0_\alpha\rangle |0_\alpha\rangle \) to the unentangled state \( |1_\alpha\rangle |1_\alpha\rangle \).

Entanglement in condensed matter. GE can be applied to the study of interacting quantum systems, where the characterization of quantum correlations is essential to a complete understanding of quantum phase transitions. Consider the case of an anisotropic one-dimensional spin-1/2 XY model in a transverse field, described by the Hamiltonian acting on the \( N \)-spin space:

\[
H = -g \sum_{i=1}^{N} \left( (1 + \eta) J_{i+}^{(1)} J_{i+}^{(1)} + (1 - \eta) J_{i-}^{(1)} J_{i-}^{(1)} \right) + \sum_{i=1}^{N} J_{i}^{(2)} \tag{1}
\]

where \( \eta \in [0, 1] \) is the anisotropy, \( g \in [0, \infty) \) is a tunable parameter and \( J_{i}^{(2)} = J_{i}^{(1)} \). \( H \) can be diagonalized by performing a Jordan-Wigner mapping to spinless fermions. The resulting ground state is BCS-like. A transition between a paramagnetic state (disorder) and a ferromagnetic state (order) occurs for all \( \eta \) at the critical value \( g_c = 1 \), in the thermodynamic limit. Relevant algebras, generated by subsets of bilinear products of spinless-fermion operators [12], include \( u(N) = \{ c_i^c \mid 1 \leq i \leq N \} \), \( c_i^c \mid 1 \leq i < j \leq N \). A BCS state is a GCS of \( so(2N) \), thus it is generalized-unentangled relative to \( so(2N) \), capturing the fact that quasiparticles are non-interacting in this description. However, GE may be present relative to the smaller algebra \( u(N) \subset so(2N) \) [22]. Remarkably, \( P_{\text{u}(N)} \) as a function of \( g \) plays the role of a disorder parameter (Fig. 1). The fact that the purity relative to an appropriate algebra succeeds at detecting a quantum phase transition and characterizing its universality class appears to be a generic feature of broken-symmetry (here \( Z_2 \)) phase transitions. The purity, a sum of squared expectations of observables, is a natural measure of fluctuations. Changes in the nature of the fluctuations identify those transitions. In some cases [6, 7], nearest-neighbor lattice-site entanglement or other standard entanglement measures may suffice, but in general highly non-local correlations or fluctuations, whose nature depends on the physics and symmetries of the problem, may be required. An extended analysis of these issues will be presented elsewhere [23].

Conclusion. We have introduced a generalization of entanglement which goes beyond the standard subsystem-based approach by considering entanglement as a quantum feature of states with respect to any physically relevant, distinguished subspace of observables. These subspaces arise naturally from the algebraic languages [4] used to describe quantum systems. In addition to tying together the theory of entanglement and the theory of coherent states, our results carry the potential for a number of conceptual and practical advances. From a condensed-matter perspective, GE might naturally provide measures of correlation strength useful for establishing, for example, whether interactions within a given quasiparticle description are sufficiently weak for a mean-field theory to be meaningful. Conversely, one might use a typology of GE to better understand situations where mean-field theory is not easily applied. For QIP, our formalism can give additional insight into standard entanglement theory. It suggests novel entanglement measures for the multipartite case. By scaling system sizes, asymptotic measures can be obtained to...

![FIG. 1: Purity \( P_{u(N)} \) for the BCS state as a function of \( g \). \( P_{u(N)} \) scales with an exponent \( \nu = 1 \) near \( g_c \). Thus the correlation length diverges as \( (g_c - g)^{-\nu} \) (Ising universality class).]
help investigate information-theoretic or thermodynamic limits, with possible uses in renormalization group analyses. Finally, because the occurrence of a superselected structure in a quantum system provides an important avenue for effectively restricting the set of physically accessible operations, and can be formally associated to the reducible action of an appropriate operator set, our framework may provide further insight on the issue of entanglement in the presence of superselection rules as recently addressed in [24, 25].

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