Graphical translating solitons for the mean curvature flow and isoparametric functions

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Abstract

In this paper, we consider a translating soliton for the mean curvature flow starting from a graph of a function on a domain in a unit sphere which is constant along each leaf of isoparametric foliation. First, we show that such a function is given as a composition of an isoparametric function on the sphere and a function which is given as a solution of a certain ordinary differential equation. Further, we analyze the shape of the graphs of the solutions of the ordinary differential equation. This analysis leads to the classification of the shape of such translating solitons. Finally, we investigate a domain of the function which is given as a composition of the isoparametric function and the solution of the ordinary differential equation in the case where the number of distinct principal curvatures of the isoparametric hypersurface defined by the regular level set for the isoparametric function is 1, 2, or 3.

1 Introduction

Let $N$ be a $n$-dimensional Riemannian manifold and $u : M \rightarrow \mathbb{R}$ be a function on a domain $M \subset N$. Define the immersion $f$ of $M$ into the product Riemannian manifold $N \times \mathbb{R}$ by $f(x) = (x, u(x))$, $x \in M$. Denote the graph of $u$ by $\Gamma$. If a $C^\infty$-family of $C^\infty$-immersions $\{f_t\}_{t \in I}$ of $M$ into $N \times \mathbb{R}$ ($I$ is an open interval including 0) satisfies

$$\begin{cases} (\frac{\partial f_t}{\partial t})^{\perp_{f_t}} = H_t \\ f_0 = f, \end{cases} \quad (1.1)$$

as $M_t = f_t(M)$, $\{M_t\}_{t \in I}$ is called the mean curvature flow starting from $\Gamma$. Here, $H_t$ is the mean curvature vector field of $f_t$ and $(\bullet)^{\perp_{f_t}}$ is the normal component of $(\bullet)$ with respect to $f_t$. Further, according to Hungerbühler and Smoczyk\cite{6}, we define a soliton of the mean curvature flow. Let $X$ be a Killing vector field on $N \times \mathbb{R}$ and $\{\phi_t\}_{t \in \mathbb{R}}$ be the
one-parameter transformation associated to $X$ on $N \times \mathbb{R}$, that is, $\phi_t$ satisfies
\[
\begin{cases}
\frac{\partial \phi_t}{\partial t} = X \circ \phi_t \\
\phi_0 = id_{N \times \mathbb{R}},
\end{cases}
\]
where $id_{N \times \mathbb{R}}$ is the identity map on $N \times \mathbb{R}$. Here, we note that $\phi_t$'s are isometries. Then, the mean curvature flow $\{M_t\}_{t \in I}$ is called a soliton of the mean curvature flow with respect to $X$, if $\tilde{f}_t = \phi_t^{-1} \circ f_t$ satisfies
\[
\left(\frac{\partial \tilde{f}_t}{\partial t}\right)^{1/2} = 0.
\]
(1.2)

In the sequel, we call such a soliton a $X$-soliton simply. In particular, when $X = (0, 1) \in T(N \times \mathbb{R}) = TN \oplus TR$, we call the $X$-soliton a translating soliton.

The translating soliton for $N = \mathbb{R}^n$ has been studied by several authors. When $n = 2$, Shahriyari[13] proved non-existence of complete translating graphs over bounded connected domains of $\mathbb{R}^2$ with smooth boundary. Also, she showed that if a complete translating soliton which is a graph over a domain in $\mathbb{R}^2$, then the domain is a strip, or a halfspace, or $\mathbb{R}^2$. Further, Hoffman, Ilmanen, Martín and White[5] showed that no complete translating soliton is the graph of a function over a halfspace in $\mathbb{R}^2$. For the function $u : \mathbb{R}^n \to \mathbb{R}$ defined by $u(x_1, \ldots, x_n) = -\log \cos x_n$, $(x_1, \ldots, x_n) \in \mathbb{R}^n$, the mean curvature flow starting from the graph of $u$ is the translating soliton. When $n = 1$, the curve of $u$ is called a grim reaper (Figure 1.1). When $n \geq 2$, the graph of $u$ is called a grim hyperplane (Figure 1.2). Further, Martin, Savas-Halilaj, and Smoczyk[7] gave the characterization of the grim hyperplane. Clutterbuck, Schnürer and Schulze[3] showed the existence of the complete rotationally symmetric graphical translating soliton which is called bowl soliton (Altschuler and Wu[1] had already showed the existence in the case $n = 2$) and a certain type of stability for the bowl soliton. Further, they showed that bowl solitons have the following asymptotic expansion as $r$ approaches infinity:
\[
\frac{r^2}{2(n-1)} - \log r + O(r^{-1}),
\]
where $r$ is the distance function in $\mathbb{R}^n$ because $u$ is the composition of $r$ and the solution of a certain ordinary differential equation. Wang[15] showd that the bowl soliton is the only convex translating soliton which is an entire graph. Further, Spruck and Xiao[14] showed that the bowl soliton is the only complete translating soliton which is an entire graph. In this paper, we consider the case where the symmetry of the graph is a little complicated.
In this paper, we consider the case where $N$ is the $n$-dimensional unit sphere $S^n$ and $u$ is a composition of an isoparametric function on $S^n$ and some function. The level sets of the isoparametric functions give compact isoparametric hypersurfaces of $S^n$. Münzner\cite{9} showed that the number $k$ of distinct principal curvatures of compact isoparametric hypersurfaces of $S^n$ is 1, 2, 3, 4 or 6 by a topological method. In cases $k = 1, 2, 3$, Cartan\cite{2} classified the isoparametric hypersurfaces. The hypersurfaces are $S^{n-1} \subset S^n$ in case $k = 1$, $S^k \times S^{n-k-1} \subset S^n$ in case $k = 2$ and the tubes over the Veronese surfaces $\mathbb{RP}^2 \subset S^4$, $\mathbb{CP}^2 \subset S^7$, $\mathbb{QP}^2 \subset S^{13}$, $\mathbb{OP}^2 \subset S^{25}$ (i.e., the principal orbits of the isotropy representations of the rank two symmetric spaces $SU(3)/SO(3)$, $(SU(3) \times SU(3))/SU(3)$, $SU(6)/Sp(3)$, $E_6/F_4$) in case $k = 3$. These hypersurfaces are homogeneous. In case $k = 6$, the hypersurfaces are homogeneous by the result of Dorfmeister and Neher\cite{4} and Miyaoka\cite{8}. The hypersurfaces are the principal orbits of the isotropy representations of $(G_2 \times G_2)/G_2$, $G_2/SO(4)$. In case $k = 4$, Ozeki and Takeuchi\cite{11, 12} found that non-homogeneous isoparametric hypersurfaces are constructed as the regular level sets of the restrictions of the Cartan-Münzner polynomial functions to the sphere.

In this paper, we obtain the following result.

\textbf{Theorem 1.1.} Let $r$ be an isoparametric function on $S^n$ $(n \geq 2)$ and $V$ be a $C^\infty$-function on an interval $J \subset r(S^n)$. If the mean curvature flow starting from the graph of the function $u = (V \circ r)|_{r^{-1}(J)}$ is a translating soliton, the shape of the graph of $V$ is like one of those defined in Figures 1.3–1.9 The real number $R \in (-1, 1)$ in Figures 1.3–1.9 is given by

$$R := \begin{cases} 0 & (k = 1, 3, 6) \\ -1 + \frac{km}{n-1} & (k = 2, 4), \end{cases}$$

where $k$ is the number of distinct principal curvatures of the compact isoparametric hypersurface defined by the regular level set for $r$ and $m$ is the multiplicity of the smallest principal curvature of the isoparametric hypersurface.
Figure 1.3: The graph of $V$ (Type I)

Figure 1.4: The graph of $V$ (Type II)

Figure 1.5: The graph of $V$ (Type III)

Figure 1.6: The graph of $V$ (Type IV)

Figure 1.7: The graph of $V$ (Type V)
The function $u = (V \circ r)|_{r=1(J)}$ in Theorem 1.1 is constant on the level set of $r$ and its behavior on the normal direction for the level set of $r$ is a little understood from the behavior of $V$ in Figures 1.3–1.9. In the last section, we investigate the domain of the function $u$ in Theorem 1.1.

2 Basic facts

Let $g$ be a Riemannian metric of a $n$-dimensional Riemannian manifold $N$ and $u : M \to \mathbb{R}$ be a function on a domain $M \subset N$. Define the immersion $f$ of $M$ into the product Riemannian manifold $N \times \mathbb{R}$ by $f(x) = (x, u(x))$, $x \in M$. Denote the graph of $u$ by $\Gamma$ and the mean curvature vector field of $f$ by $H$. Further, we assume that $X$ is a Killing vector field on $N \times \mathbb{R}$ and $\{\phi_t\}_{t \in \mathbb{R}}$ is the one-parameter transformation associated to $X$ on $N \times \mathbb{R}$. Then, we have the following lemma about the soliton of the mean curvature flow.

**Lemma 2.1.** If the mean curvature flow starting from $\Gamma$ is $X$-soliton, $f$ satisfies

\[(X \circ f)^{1/1} = H. \tag{2.1}\]

Conversely, if $f$ satisfies (2.1), the family of the images $\{M_t\}_{t \in \mathbb{R}}$ defined by $f_t = \phi_t \circ f$ and $M_t = f_t(M)$ is the $X$-soliton.

**Proof.** According to Hungerbühler and Smoczyk\[6\], we find the first half of the lemma. For the second half of the lemma, since $\phi_t$’s are isometries and $f$ satisfies (2.1), we find that $f_t = \phi_t \circ f$ satisfies

\[\left(\frac{\partial f_t}{\partial t}\right)^{1/1} - H_t = d\phi_t((X \circ f)^{1/1} - H) = 0,
\]

and $\{f_t\}_{t \in \mathbb{R}}$ satisfies (1.1). Therefore, $\{M_t\}_{t \in \mathbb{R}}$ is the mean curvature flow. Further, by $\phi^{-1}_t \circ f_t = f$, it turns out that $f_t$ satisfies (1.2). So, $\{M_t\}_{t \in \mathbb{R}}$ is the translating soliton. $\square$
Let $\nabla$ and div be the gradient and divergence with respect to $g$ respectively. For Lemma 2.1 considering the case where an $X$-soliton is a translating soliton, the following lemma is derived.

**Lemma 2.2.** If the mean curvature flow starting from $\Gamma$ is a translating soliton, $u$ satisfies

$$\sqrt{1 + \|\nabla u\|^2} \text{ div} \left( \frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}} \right) = 1. \quad (2.2)$$

Conversely, if $u$ satisfies $(2.2)$, the family of the images $\{M_t\}_{t \in \mathbb{R}}$ defined by $f_t(x) = (x, u(x) + t)$, $x \in M$ and $M_t = f_t(M)$ is a translating soliton.

**Proof.** Let $(x^1, \ldots, x^n, s)$ be local coordinates of $N \times \mathbb{R}$. Define the Killing vector $X = (0, 1) \in T(N \times \mathbb{R}) = TN \oplus T\mathbb{R}$. By \( f(x) = (x, u(x)) \), $x \in M$ and $X = \frac{\partial}{\partial s}$, we find

$$(X \circ f)^{-1} \frac{\partial}{\partial s} - \frac{1}{1 + \|\nabla u\|^2} df(\nabla u)$$

$$H = \sqrt{1 + \|\nabla u\|^2} \text{ div} \left( \frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}} \right) \left( \frac{\partial}{\partial s} - \frac{1}{1 + \|\nabla u\|^2} df(\nabla u) \right).$$

Therefore, we obtain that $(2.1)$ and $(2.2)$ are equivalent in this case. \qed

Next, we consider the case where $u$ is a composition of an isoparametric function and some function. Let $\Delta$ be the Laplacian with respect to $g$. A non-constant $C^\infty$-function $r : N \rightarrow \mathbb{R}$ is called an isoparametric function if there exist $C^\infty$-functions $\alpha, \beta$ such that

$$\begin{cases} \|\nabla r\|^2 = \alpha \circ r \\ \Delta r = \beta \circ r. \end{cases}$$

Further, the level set of $r$ with respect to a regular value is called an isoparametric hypersurface.

In case where $N$ is the $n$-dimensional unit sphere $S^n$, Münzner showed the following theorem for an isoparametric function on $S^n$.

**Theorem 2.3.** (Münzner) \( i \) An isoparametric function $r$ on $S^n$ is a restriction to $S^n$ of a homogeneous polynomial $h : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ which satisfies

$$\begin{cases} \|\nabla^R h\|^2_x = k^2 |x|^{2k-2} \\ (\Delta^R h)_x = \frac{m_2 - m_1}{2} k^2 |x|^{k-2} \quad (x \in \mathbb{R}^{n+1}), \end{cases} \quad (2.3)$$

where $| \cdot |$ is the Euclidean norm and $\nabla^R$ and $\Delta^R$ are the gradient and Laplacian for the Euclidean space $\mathbb{R}^n$. Here, we assume that the isoparametric hypersurface defined by the level set of $r$ has $k$ distinct principal curvatures $\lambda_1 > \cdots > \lambda_k$ with respective multiplicities $m_1, \ldots, m_k$.

\( ii \) The above natural number $k$ is 1, 2, 3, 4 or 6.
Remark 2.4. According to Münzner\cite{9,10}, we find the following two facts.

(i) If \( k = 1, 3, 6 \), then the multiplicities are equal. If \( k = 2, 4 \), then there are at most two distinct multiplicities \( m_1, m_2 \).

(ii) By (2.3), we obtain

\[
\left\{
\begin{array}{l}
\|\nabla r\|^2 = k^2(1 - r^2) \\
\Delta r = \frac{m_2 - m_1}{2}k^2 - k(n + k - 1)r.
\end{array}
\right.
\tag{2.4}
\]

From the first equation of (2.4), we find that \( r(\mathbb{S}^n) = [-1, 1] \).

For Lemma 2.2, considering the case where \( u \) is the composition of the isoparametric function and some function, the following lemma is derived.

Lemma 2.5. Let \( r : N \to \mathbb{R} \) be an isoparametric function on \( N \). If the mean curvature flow starting from \( \Gamma \) is a translating soliton and there is exists a \( C^\infty \)-function \( V \) on \( r(M) \) such that \( u = (V \circ r)|_M \), the function \( V \) satisfies

\[
2\alpha V'' - \alpha(\alpha' - 2\beta)V'^2 - 2\alpha V'^2 + 2\beta V' - 2 = 0,
\tag{2.5}
\]

where \( ' \) denotes derivative on \( r(M) \) and \( \alpha, \beta \) are \( C^\infty \)-functions which satisfy \( \|\nabla r\|^2 = \alpha \circ r, \Delta r = \beta \circ r \). Conversely, if \( V \) satisfies (2.5), the family of the images \( \{M_t\}_{t \in \mathbb{R}} \) defined by \( f_t(x) = (x, (V \circ r)(x) + t) \), \( x \in M \) and \( M_t = f_t(M) \) is the translating soliton.

Proof. For the left side of (2.2), we have

\[
\sqrt{1 + \|\nabla u\|^2} \text{ div} \left( \frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}} \right) = \Delta u - \frac{1}{2(1 + \|\nabla u\|^2)} \nabla u(\|\nabla u\|^2).
\]

By \( u = V \circ r \), we find

\[
\|\nabla u\|^2 = (\alpha V'^2) \circ r, \\
\nabla u(\|\nabla u\|^2) = (\alpha V'^2 (2\alpha V'' + \alpha' V')) \circ r, \\
\Delta u = (\alpha V'' + \beta V') \circ r.
\]

Therefore, (2.2) is reduced to the following equation

\[
\frac{\alpha V''}{1 + \alpha V'^2} + \beta V' - \frac{\alpha \alpha' V'^3}{2(1 + \alpha V'^2)} = 1.
\]

By this equation, we obtain (2.5). \( \square \)

3 Proof of Theorem 1.1

In this section, we assume that \( N \) is the \( n \)-dimensional unit sphere \( \mathbb{S}^n (n \geq 2) \) and \( u = (V \circ r)|_{r^{-1}(J)} \) with an isoparametric function \( r \) on \( \mathbb{S}^n \) and a \( C^\infty \)-function \( V \) on
interval $J \subset r(S^n) = [-1, 1]$. By (2.3), substituting $\alpha(r) = k^2(1-r^2)$ and $\beta(r) = \frac{m_2-m_1}{2}k^2 - k(n+k-1)r$ for (2.5), we obtain
\[
V''(r) = k((n-1)r - \frac{m_2-m_1}{2})V'(r)^3 + V'(r)^2 + \frac{(n+k-1)r - (n-1)r}{k(1-r^2)}V'(r) + \frac{1}{k^2(1-r^2)}, \quad r \in (-1, 1).
\]

The local existence of the solution $V$ of (3.1) is clear. By Remark 2.4 (i), we find
\[
m_2 - m_1 = \begin{cases} 0 & (k = 1, 3, 6) \\ 2(m_2 - \frac{n-1}{k}) & (k = 2, 4). \end{cases}
\]

Therefore, (3.1) is reduced to
\[
V''(r) = k((n-1)(r-R))V'(r)^3 + V'(r)^2 + \frac{(n+k-1)r - (n-1)r}{k(1-r^2)}V'(r) + \frac{1}{k^2(1-r^2)}, \quad r \in (-1, 1).
\]

Here, $R \in (-1, 1)$ is the constant defined by
\[
R := \begin{cases} 0 & (k = 1, 3, 6) \\ -1 + \frac{km_2}{n-1} & (k = 2, 4), \end{cases}
\]
and when $k = 2, 4$, $m_2$ is equal to the multiplicity of the smallest principal curvature of the isoparametric hypersurface defined by the level set of $r$. To prove Theorem 1.1, we consider the graph of the solution $V$ of (3.2). Define $\psi(r) = k\sqrt{1-r^2}V'(r)$. The equation (3.2) is reduced to
\[
\psi'(r) = \frac{1}{k(1-r^2)} \left( \psi(r)^2 + 1 \right) \left( (n-1)(r-R)\psi(r) + \sqrt{1-r^2} \right).
\]

Therefore, we consider the behavior of the solution $\psi$ of (3.3). Define $\eta(r) = -\frac{\sqrt{1-r^2}}{(n-1)(r-R)}$. Then, the following lemma holds clearly.

**Lemma 3.1.**

(i) When $r \in (R, 1)$:
(a) if $\psi(r) > \eta(r)$, then $\psi'(r) > 0$,
(b) if $\psi(r) = \eta(r)$, then $\psi'(r) = 0$,
(c) if $\psi(r) < \eta(r)$, then $\psi'(r) < 0$.

(ii) When $r \in (-1, R)$:
(a) if $\psi(r) < \eta(r)$, then $\psi'(r) > 0$.
(b) if \( \psi(r) = \eta(r) \), then \( \psi'(r) = 0 \),
(c) if \( \psi(r) > \eta(r) \), then \( \psi'(r) < 0 \).

(iii) When \( r = R \) or \( \psi(r) = 0 \) : \( \psi'(r) > 0 \).

![Figure 3.1: The graph of \( \eta \)](image)

For the shape of \( \psi \) in the case where \( \psi > 0 \), we obtain the following lemmata.

**Lemma 3.2.** *If there exists \( r_0 \in (R, 1) \) with \( \psi(r_0) > 0 \), there exists \( r_1 \in (r_0, 1) \) such that*
\[
\lim_{r \uparrow r_1} \psi(r) = +\infty.
\]

*Proof.* For all \( r \in (r_0, 1) \), we find \( \psi'(r) > 0 \) and \( \psi(r) > 0 \). Also, we have
\[
\psi'(r) = \frac{1}{k(1-r^2)} \left( \psi(r)^2 + 1 \right) \left( (n-1)(r-R)\psi(r) + \sqrt{1-r^2} \right)
\]
\[
> \frac{(n-1)(r-R)}{k(1-r^2)} \psi(r)^3.
\]

Therefore, we find
\[
\frac{\psi'(r)}{\psi(r)^3} > \frac{(n-1)(r-R)}{k(1-r^2)}.
\]

By integrating from \( r_0 \) to \( r \), we have
\[
\frac{1}{\psi(r)^2} < \frac{(n-1)}{k} \log (1-r^2) + \frac{(n-1)R}{k} \log \frac{1+r}{1-r}
\]
\[
- \frac{(n-1)}{k} \log (1-r_0^2) - \frac{(n-1)R}{k} \log \frac{1+r_0}{1-r_0} + \frac{1}{\psi(r_0)^2} =: h_1(r).
\]

Here, \( h_1 \) is decreasing on \( (r_0, 1) \) and
\[
h_1(r_0) = \frac{1}{\psi(r_0)^2} > 0, \quad \lim_{r \uparrow 1} h_1(r) = -\infty.
\]
Therefore, there exists \( r_1 \in (r_0, 1) \) with \( h_1(r_1) = 0 \) and
\[
\psi(r) > \frac{1}{\sqrt{h_1(r)}} \to +\infty \quad (r \uparrow r_1).
\]
Then, we obtain the statement of this lemma.

Figure 3.2: The behavior of the graph of \( \psi \) in Lemma 3.2

**Lemma 3.3.** If there exists \( r_0 \in (-1, R) \) with \( 0 < \psi(r_0) < \eta(r_0) \), there exists \( C \in (\psi(r_0), +\infty) \) such that
\[
\lim_{r \to R} \psi(r) = C.
\]

**Proof.** First, we consider the case \( k = 1 \). For all \( r \in (r_0, 0) \), we find \( \psi'(r) > 0 \) and \( 0 < \psi(r) < \eta(r) \). Also, we have
\[
\psi'(r) = \frac{1}{1 - r^2} \left( \psi(r)^2 + 1 \right) \left( (n - 1) r \psi(r) + \sqrt{1 - r^2} \right) < \frac{1}{\sqrt{1 - r^2}} \left( \psi(r)^2 + 1 \right).
\]
Therefore, we find
\[
\frac{\psi'(r)}{\psi(r)^2 + 1} < \frac{1}{\sqrt{1 - r^2}}.
\]
By integrating from \( r_0 \) to \( r \), we have
\[
\arctan \psi(r) < \arcsin r - \arcsin r_0 + \arctan \psi(r_0) =: h_2(r).
\]
Here, \( h_2 \) is increasing on \((r_0, 0)\) and
\[
h_2(r_0) = \arctan \psi(r_0), \quad h_2(0) = \arctan \psi(r_0) - \arcsin r_0.
\]
Since we find
\[
\psi(r_0) < \eta(r_0) = -\frac{\sqrt{1 - r_0^2}}{(n - 1)r_0} \leq -\frac{\sqrt{1 - r_0^2}}{r_0} = \tan \left( \arcsin r_0 + \frac{\pi}{2} \right),
\]
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we have
\[ h_2(0) = \arctan \psi(r_0) - \arcsin r_0 < \frac{\pi}{2}. \]

Therefore, \( \tan (h_2(r)) \) is defined on \((r_0, 0]\) and
\[ \psi(r) < \tan (h_2(r)). \]

Then, we obtain the statement of this lemma for \( k = 1 \).

Next, we consider the case \( k = 2, 3, 4 \) or \( 6 \). For all \( r \in (r_0, R) \), we find \( \psi(r) > 0 \) and \( 0 < \psi(r) < \eta(r) \). Also, we have
\[ \psi'(r) = \frac{1}{k(1 - r^2)} \left( \psi(r)^2 + 1 \right) \left( (n - 1)(r - R)\psi(r) + \sqrt{1 - r^2} \right) \]
\[ < \frac{1}{2\sqrt{1 - r^2}} (\psi(r)^2 + 1). \]

Therefore, we find
\[ \frac{\psi'(r)}{\psi(r)^2 + 1} < \frac{1}{2\sqrt{1 - r^2}}. \]

By integrating from \( r_0 \) to \( r \), we have
\[ \arctan \psi(r) < \frac{1}{2} \arcsin r - \frac{1}{2} \arcsin r_0 + \arctan \psi(r_0) =: \hat{h}_2(r). \]

Here, \( \hat{h}_2 \) is increasing on \((r_0, R)\) and
\[ \hat{h}_2(r_0) = \arctan \psi(r_0), \quad \hat{h}_2(R) = \arctan \psi(r_0) + \frac{1}{2} \arcsin R - \frac{1}{2} \arcsin r_0. \]

Since we find
\[ \psi(r_0) < \eta(r_0) = -\frac{\sqrt{1 - r_0^2}}{(n - 1)(r_0 - R)} \]
\[ < -\frac{\sqrt{1 - r_0^2} + \sqrt{1 - R^2}}{r_0 - R} \]
\[ = \tan \left( \frac{1}{2} \arcsin r_0 - \frac{1}{2} \arcsin R + \frac{\pi}{2} \right), \]
we have
\[ \hat{h}_2(R) = \arctan \psi(r_0) - \arcsin r_0 < \frac{\pi}{2}. \]

Therefore, \( \tan \left( \hat{h}_2(r) \right) \) is defined on \((r_0, R]\) and
\[ \psi(r) < \tan \left( \hat{h}_2(r) \right). \]

Then, we obtain the statement of this lemma. \( \square \)
Lemma 3.4. If there exists \( r_0 \in (-1, R) \) with \( \psi(r_0) > \eta(r_0) \), there exists \( r_1 \in (-1, r_0) \) such that
\[
\lim_{r \to r_1} \psi(r) = +\infty.
\]

Proof. For all \( r \in (-1, r_0) \), we find \( \psi'(r) < 0 \) and \( \psi(r) > \eta(r) \). Also, we have
\[
\psi'(r) = \frac{1}{k(1 - r^2)} \left( \psi(r)^2 + 1 \right) \left( (n - 1)(r - R)\psi(r) + \sqrt{1 - r^2} \right) < \frac{1}{k(1 - r^2)} \left( (n - 1)\psi(r_0)(r - R) + \sqrt{1 - r^2} \right) \psi(r)^2.
\]
Therefore, we find
\[
\frac{\psi'(r)}{\psi(r)^2} < \frac{(n - 1)}{k} \psi(r_0) \frac{r - R}{1 - r^2} + \frac{1}{k \sqrt{1 - r^2}}.
\]
By integrating from \( r_0 \) to \( r \), we have
\[
\frac{1}{\psi(r)} < \frac{(n - 1)\psi(r_0)}{2k} \log (1 - r^2) - \frac{1}{k} \arcsin r
+ \frac{(n - 1)R\psi(r_0)}{2k} \log \frac{1 + r}{1 - r} - \frac{(n - 1)\psi(r_0)}{2k} \log (1 - r_0^2)
+ \frac{1}{k} \arcsin r_0 - \frac{(n - 1)R\psi(r_0)}{2k} \log \frac{1 + r_0}{1 - r_0} + \frac{1}{\psi(r_0)} =: h_3(r).
\]
Here, \( h_3 \) is increasing on \((-1, r_0)\) and
\[
h_3(r_0) = \frac{1}{\psi(r_0)} > 0, \quad \lim_{r \to r_1} h_3(r) = -\infty.
\]
Therefore, there exists \( r_1 \in (-1, r_0) \) with \( h_3(r_1) = 0 \) and
\[
\psi(r) > \frac{1}{h_3(r)} \to +\infty \quad (r \downarrow r_1)
\]
Then, we obtain the statement of this lemma. \(\square\)
Since the existence of the solution $\psi$ of (3.3) which is defined to $r = -1$ could not be excluded, we consider that case.

**Lemma 3.5.** If $\psi$ is defined to $r = -1$, $\psi(-1) = 0$ and $V'(-1) = \frac{1}{k(k+n-1)(1+R)}$.

**Proof.** It is clear that $\psi(-1) = 0$. By $V'(r) = \frac{1}{k\sqrt{1-r^2}}\psi(r)$, we find

$$\frac{\psi'(r)}{(k\sqrt{1-r^2})'} = -\frac{1}{k^2r} \left( k^2(1-r^2)V'(r)^2 + 1 \right) \left( k(n-1)(r-R)V'(r) + 1 \right)$$

$$\rightarrow -\frac{1}{k^2} (k(n-1)(1+R)V'(-1) - 1) \quad (r \downarrow -1)$$

By the l’Hôpital’s rule, we find $V'(-1) = \frac{1}{k(k+n-1)(1+R)}$.

Also, in the case where $\psi < 0$, by proofs similar to Lemmas 3.2, 3.3, 3.5, we obtain following lemmas.
**Lemma 3.6.** If there exists \( r_0 \in (-1, R) \) with \( \psi(r_0) < 0 \), there exists \( r_1 \in (-1, r_0) \) such that
\[
\lim_{r \to r_1} \psi(r) = -\infty.
\]

![Figure 3.6: The behavior of the graph of \( \psi \) in Lemma 3.6](image)

**Lemma 3.7.** If there exists \( r_0 \in (R, 1) \) with \( 0 > \psi(r_0) > \eta(r_0) \), there exists \( C \in (-\infty, \psi(r_0)) \) such that
\[
\lim_{r \downarrow R} \psi(r) = C.
\]

![Figure 3.7: The behavior of the graph of \( \psi \) in Lemma 3.7](image)

**Lemma 3.8.** If there exists \( r_0 \in (R, 1) \) with \( \psi(r_0) < \eta(r_0) \), there exists \( r_1 \in (r_0, 1) \) such that
\[
\lim_{r \uparrow r_1} \psi(r) = -\infty.
\]
Lemma 3.9. If $\psi$ is defined to $r = 1$, $\psi(1) = 0$ and $V'(1) = -\frac{1}{k(n-1)(1-R)}$. 

By Lemmas 3.1-3.9, we obtain the following proposition for the behavior of the graph of $\psi$.

Proposition 3.10. For the solution $\psi$ of the equation (3.3), the behavior of the graph of $\psi$ is like one of Figures 3.10-3.16.
Figure 3.10: The graph of $\psi$ (Type I$''$)

Figure 3.11: The graph of $\psi$ (Type II$''$)

Figure 3.12: The graph of $\psi$ (Type III$''$)

Figure 3.13: The graph of $\psi$ (Type IV$''$)

Figure 3.14: The graph of $\psi$ (Type V$''$)
For the graph of $\phi$ in Proposition 3.10, we have not yet show whether $\psi$ in the case of Figures 3.10 and 3.16 exists or not. By the following lemma, we obtain the existence.

**Lemma 3.11.** The solutions $\psi$ of the equation (3.3) in Figures 3.15 and 3.16 exist.

**Proof.** For the set $S$ of all solutions of the equation (3.3), we define sets $S_1, S_2, S_3 \subset S$ by

\[
S_1 := \{\psi \in S | \exists r_0 \in (-1, 1) : \psi(r_0) = 0\}
\]

\[
S_2 := \{\psi \in S | \exists r_0 \in (-1, 1) : \psi(r_0) = \eta(r_0)\}
\]

\[
S_3 := \{\psi \in S | \psi(1) = 0 \text{ or } \psi(-1) = 0\}
\]

Then, we have

\[
(-1, 1) \times \mathbb{R} = \cup_{\psi \in S_1 \cup S_2 \cup S_3} \text{Im}(\psi).
\]

Since $\cup_{\psi \in S_1} \text{Im}(\psi)$ and $\cup_{\psi \in S_2} \text{Im}(\psi)$ are open sets and $(-1, 1) \times \mathbb{R}$ is connected, we find $S_3$ is not empty set. Therefore, we obtain the statement of this lemma.

Define $\zeta(r) = -\frac{1}{k(n-1)(r-R)}$. By $V'(r) = \frac{k}{k\sqrt{1-r^2}} \psi(r)$ and Proposition 3.10, we have the following proposition for the behavior of the graph of $V'$. Besides, by Proposition 3.12, we obtain Theorem 1.1.

**Proposition 3.12.** For the solution $V$ of the equation (3.2), the behavior of the graph of $V'$ is like one of Figures 3.17-3.23. Here, the dotted curve in Figures 3.17-3.23 is the graph of $\zeta$. 

Figure 3.15: The graph of $\psi$ (Type VI$''$)  Figure 3.16: The graph of $\psi$ (Type VII$''$)
Figure 3.17: The graph of $V'$ (Type I')

Figure 3.18: The graph of $V'$ (Type II')

Figure 3.19: The graph of $V'$ (Type III')

Figure 3.20: The graph of $V'$ (Type IV')

Figure 3.21: The graph of $V'$ (Type V')
Figure 3.22: The graph of $V'$ (Type VI')

Figure 3.23: The graph of $V'$ (Type VII')

4 The domain of the function $u$ in Theorem 1.1

In this section, we investigate the domain of the function $u = V \circ r$ over $M \subset S^n$ in Theorem 1.1 in the case where the number $k$ of distinct principal curvatures of the isoparametric hypersurface for $r$ is 1, 2 or 3. From the result of Theorem 1.1, we find that $M$ does not contain some tubular neighborhoods of the focal submanifolds $r^{-1}(1)$ and $r^{-1}(-1)$ in case that the type of $V$ in Theorem 1.1 is I-V. Also, we find that $M$ contains $r^{-1}(-1)$ and does not contain a tubular neighborhood of the focal submanifold $r^{-1}(1)$ in case that the type of $V$ is VI and $M$ contains $r^{-1}(1)$ and does not contain a tubular neighborhood of the focal submanifold $r^{-1}(-1)$ in case that the type of $V$ is VII.

When $k = 1$, the isoparametric function $r$ is defined by

\[ r(x_1, \ldots, x_{n+1}) = x_{n+1} (x_1, \ldots, x_{n+1}) \in S^n. \]

Therefore, from the result of Theorem 1.1 the domain $M$ of $u$ is an open set of $S^n$ including the set $\{(x_1, \ldots, x_n, 0) \in \mathbb{R}^{n+1} | x_1^2 + \cdots + x_n^2 = 1\} \subset S^n$. Also, as $p = (0, \ldots, 0, 1), q = (0, \ldots, 0, -1)$, we find that $p, q \notin M$ in case that the type of $V$ in Theorem 1.1 is I-V, $p \notin M, q \in M$ in case that the type of $V$ is VI and $p \in M, q \notin M$ in case that the type of $V$ is VII.

When $k = 2$, the isoparametric function $r$ is defined by

\[ r(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{l} x_i^2 - \sum_{i=l+1}^{n+1} x_i^2 (x_1, \ldots, x_{n+1}) \in S^n. \]

Here, $l \in \{1, \ldots, n\}$. Since $r^{-1}(t) = \{(x, y) \in \mathbb{R}^l \times \mathbb{R}^{n-l+1} | \|x\|^2 = \frac{1+t}{2}, \|y\|^2 = \frac{1-t}{2}\}$ for $t \in (-1, 1)$, as $S_\theta := \{((\cos \theta, 0, \ldots, 0)A, (\sin \theta, 0, \ldots, 0)B) \in \mathbb{R}^l \times \mathbb{R}^{n-l+1} | A \in SO(l-1), B \in SO(n-l)\}$, we obtain that $r^{-1}(t) = S_{\theta_t}$ for $\theta_t \in (0, \frac{\pi}{2})$ with $\cos \theta_t = \sqrt{\frac{1-t}{2}}$ and $\sin \theta_t = \sqrt{\frac{1+t}{2}}$. Therefore, from the result of Theorem 1.1 we find that the domain $M$ is the open set of $S^n$ including $S_{\theta_t}$. Also, we find that $M = \cup_{\theta \in I} S_\theta$ for an interval
\( I \subset (0, \frac{\pi}{2}) \) in case that the type of \( V \) in Theorem 1.1 is I-V, \( M = \cup_{\theta \in (a, \frac{\pi}{2})} S_\theta \) for some \( a \in (0, \frac{\pi}{2}) \) in case that the type of \( V \) is VI and \( M = \cup_{\theta \in (0, a)} S_\theta \) for \( a \in (0, \frac{\pi}{2}) \) in case that the type of \( V \) is VII.

When \( k = 3 \), an isoparametric hypersurface is a principal orbit of the isotropy representation of the rank two symmetric space \( G/K = SU(3)/SO(3), (SU(3) \times SU(3))/SU(3), SU(6)/Sp(3) \) or \( E_6/F_4 \). Since the principal orbit of the isotropy representation intersects with the Weyl domain \( C \) at only one point, we find that there exists an open subset \( U \subset C \cap S^n \) such that \( K \cdot U = M \). Here, \( T_e(G/K) \) for \( e \in G/K \) is identified with \( \mathbb{R}^{n+1} \). Also, we find that \( M \subset K \cdot C \) in case that the type of \( V \) in Theorem 1.1 is I-V and \( M \cap (C \setminus C) \neq \emptyset \) in case that the type of \( V \) is VI or VII.

In the rest of this paper, we shall give explicit descriptions of Weyl domains for the symmetric space \( G/K = SU(3)/SO(3), (SU(3) \times SU(3))/SU(3) \) or \( SU(6)/Sp(3) \). Define \( g, k \) and \( p \) by

\[
\mathfrak{g} = \text{Lie} G, \quad \mathfrak{k} = \text{Lie} K, \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.
\]

Denote by \( \mathfrak{a} \) the maximal abelian subspace of \( \mathfrak{p} \). When \( G/K = SU(3)/SO(3), \) we have that

\[
\mathfrak{p} = \{ A : 3 \times 3 \text{ symmetric purely imaginary matrix such that the trace of } A = 0 \}.
\]

Define \( e_i (1 \leq i \leq 3) \) as \( e_i(A) \) is the diagonal element of \( A \). Then, we find that for the basis \( \{ e_1, e_2 \} \) the positive restricted root system \( \Delta_+ = \{ \sqrt{-1}(e_1 - e_2), \sqrt{-1}(e_1 - e_3), \sqrt{-1}(e_2 - e_3) \} \). Since the Killing form \( B \) is defined by \( B(X, Y) = 6Tr(XY) \), as

\[
A_1 = \begin{pmatrix}
\sqrt{-1} & 0 & 0 \\
0 & -\sqrt{-1} & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
\sqrt{-1} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -\sqrt{-1}
\end{pmatrix}, \quad A_3 = \begin{pmatrix}
0 & 0 & 0 \\
0 & \sqrt{-1} & 0 \\
0 & 0 & \sqrt{-1}
\end{pmatrix},
\]

we obtain the Weyl domain \( C \) in Figure 4.1. Here, for the angle \( \theta_{ij} \) with respect to \( A_1 \) and \( A_j \), we find \( \theta_{12} = \theta_{23} = \frac{\pi}{3} \) and \( \theta_{13} = \frac{2\pi}{3} \).

![Figure 4.1: The Weyl domain C](image-url)
When $G/K = (SU(3) \times SU(3))/SU(3)$, we have that
\[
p = \left\{ \begin{pmatrix} A & O \\ O & -A \end{pmatrix} \bigg| \text{A : } 3 \times 3 \text{ skew Hermitian matrix, the trace of A = 0} \right\}
\]
and the diagonal matrices in $p$ form $a$.

Also, when $G/K = SU(6)/Sp(3)$, we have that
\[
p = \left\{ \begin{pmatrix} A & B \\ \overline{B} & -A \end{pmatrix} \bigg| \text{A : } 3 \times 3 \text{ skew Hermitian matrix, the trace of A = 0, } \text{B : } 3 \times 3 \text{ skew symmetric matrix} \right\}
\]
and the diagonal matrices in $p$ form $a$.

In a similar way we obtain in the case that $G/K = SU(3)/SO(3)$, that the Weyl domains $C$ for $(SU(3) \times SU(3))/SU(3)$ and $SU(6)/Sp(3)$ are as in Figure 4.1.

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