Strings in a PP-wave background compactified on $T^8$ with twisted $S^1$

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Abstract

We study a torus-like compactification of type IIB maximally supersymmetric PP-wave background. As the most general case, we discuss a $T^8$ compactification of all the transverse directions. A nontrivial structure of the isometry group requires an additional light-like compactification. This additional $S^1$ fiber is twisted on the $T^8$. We determine the spectrum of closed strings in this twisted torus background and compute the thermal partition function.

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1 Summary and Conclusions

For some time, string theory has been recognized as a useful tool for investigation of Yang-Mills theories. String theories in various backgrounds are expected to be duals of gauge theories and we have obtained various information about gauge theories by studying strings. Recently, PP-waves have attracted great interest because of two facts that they are duals of certain subsectors of supersymmetric gauge theories and that string theories in PP-wave backgrounds can be exactly solved. The simplest, maximally supersymmetric example of PP-waves is obtained by taking a Penrose limit of the near horizon geometry of a number of coincident D3-branes.

In this paper, we discuss a torus-like compactification of this maximally supersymmetric PP-wave. Because the isometry of the PP-wave is non-Abelian, toroidal compactification is possible only for the directions corresponding to the Cartan part of the isometry group. Such compactifications have been already considered in the literature. Our purpose is to generalize these works to a torus-like compactification of non-commuting directions. As a general case, we consider a compactification of the eight transverse directions. In addition, such a compactification requires the light-like direction to be compactified because the commutator of two transverse translations gives a light-like translation. Therefore, the internal space we consider is nine-dimensional. Although uncompactified maximally supersymmetric PP-wave background is believed to be a dual description of the large R-charge sector of the $N=4$ supersymmetric $SU(N)$ Yang-Mills theory, it is not clear if our compactified background has anything to do with Yang-Mills theories.

Following the usual procedure, we obtain the worldsheet Hamiltonian $H$ and the worldsheet momentum $P$. (See (39) and (40).) Because the oscillator part is not affected by the compactification, we mainly focus on the zero-mode part. In addition to the occupation numbers for oscillators, string states are labeled by a set of the winding numbers and the Kaluza-Klein momenta. Because $H$ does not include the transverse momenta, the transverse momenta label degenerate states similar to the lowest Landau level states in $T^8$ as pointed out in the previous works. Due to the non-commutativity of the transverse space, only half of eight components of the transverse momentum $\vec{p}$ can be diagonalized simultaneously. This fact makes it subtle if we are able to diagonalize the worldsheet momentum $P$, which includes a term $\vec{R} \cdot \vec{p}$ where $\vec{R}$ is a vector representing the transverse winding of a string. We show that we can always choose a commutative set of four components of the momentum so that $\vec{R} \cdot \vec{p}$ is a linear combination of these. Therefore, we can use the Virasoro constraint in order to determine the spectrum without any trouble.

We also compute the thermal partition function of strings on the compactified background. Although we have to choose four-dimensional commutative subspace by hand to pick up commuting four components of the Kaluza-Klein momentum,
the final result does not depend on this choice. The result is quite similar to the \( S^1 \) compactified case given in [6], and the main change is that the sum over one winding number in the result for \( S^1 \) compactification [6] is replaced by the sum over the eight winding numbers.

This paper is organized as follows. In section 2, we briefly review the light-cone quantization of strings in the PP-wave background in a rotating coordinate system, which is convenient for discussions of the compactification. In section 3, we explain how we compactify the background in detail. In section 4, we determine the spectrum of strings in the compactified background and in section 5 we compute the thermal partition function of strings in the background.

2 The rotating coordinate of the PP-wave

In this section we briefly review the light-cone quantization of closed strings in the (uncompactified) maximally supersymmetric PP-wave background in the rotating coordinate. The PP-wave metric is given by

\[
ds^2 = 2dX^+dX^- + \sum_{i=1}^{8} (dX^i dX^i - \mu^2 X^i X^i (dX^+)^2).
\]

(1)

When we discuss compactifications of the PP-wave, it is convenient to use the rotating coordinate defined by [4]

\[
X^{r2a-1} = X^{2a-1} \cos(\mu X^+) - X^{2a} \sin(\mu X^+), \\
X^{r2a} = X^{2a-1} \sin(\mu X^+) + X^{2a} \cos(\mu X^+), \quad (a = 1, 2, 3, 4)
\]

(2)

Then, the potential term proportional to \((X^i)^2\) disappears and the metric becomes

\[
ds^2 = 2dX^+dX^- + dX^i dX^i - \frac{1}{2} F_{ij} X^i dX^j dX^+, \quad (3)
\]

where \(F_{ij}\) is the following skew-diagonal matrix

\[
F_{ij} = \begin{pmatrix}
2\mu & -2\mu \\
-2\mu & 2\mu \\
& & \ddots \\
& & & 2\mu & -2\mu \\
& & & -2\mu & 2\mu
\end{pmatrix}.
\]

(4)

Although the light-like coordinate \(X^-\) is often reparameterized as \(X^- \to X^- + f(X^+)\) so that some of the space-like isometries become manifest, we leave \(X^-\) unchanged here.

If we take the light-cone gauge

\[
X^+ = \frac{p_+}{2\pi T} \tau,
\]

(5)
the bosonic part of the worldsheet Lagrangian is given by
\[ L = \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[ p_- \partial_\tau X^- - \frac{2\pi T}{2} \partial_\alpha X^i \partial^\alpha X^i + \frac{p_-}{2} F_{ij} X^i \partial_\tau X^j \right]. \]

The worldsheet Hamiltonian and the worldsheet momentum are straightforwardly obtained from this Lagrangian as
\[ H = 2\pi T \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[ p_- \partial_\tau X^- + \frac{1}{2} (\partial_\tau X^i \partial_\tau X^i + \partial_\sigma X^i \partial_\sigma X^i) \right], \]
\[ P = -\int \frac{d\sigma}{2\pi} \left[ p_- \partial_\sigma X^- + 2\pi T \partial_\sigma X^i \partial_\tau X^i \right]. \]

The mode expansion of \( X^i \) is
\[ X^{2a-1} + iX^{2a} = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{(2\pi T)\omega_n}} \left( a_{n,a} e^{-i(\omega_n + \omega)\tau + in\sigma} + b_{n,a}^\dagger e^{i(\omega_n - \omega)\tau - in\sigma} \right) \]
\[ (a = 1, 2, 3, 4), \]
where \( \omega \) and \( \omega_n \) are defined by
\[ \omega = \frac{\mu p_-}{2\pi T}, \quad \omega_n = \sqrt{\omega^2 + n^2}. \]

The shift of the frequency by \( \omega \) is due to the rotation of the coordinate system.

The action of the fermionic sector is obtained straightforwardly by use of an appropriate covariant derivative. The transformation changes the spin connection, and induces an additional mass term \(-i \frac{\theta \gamma}{2} \gamma \gamma^{-1} \gamma^2 \theta \). This term shifts fermion masses by \( \pm \omega/2 \). In the case of a \( T^8 \) compactification, there are contributions from four rotating planes, and the mass eigenvalues of the eight fermions are one \( \omega_n + 2\omega \), six \( \omega_n \)'s, and one \( \omega_n - 2\omega \). Although this mass shift breaks the boson-fermion degeneracy, it does not affect the zero-point energy because the shift of the zero-point energy is proportional to the sum of the mass eigenvalues, which is kept invariant under the coordinate transformation.

Since the fermionic sector is not involved by compactifications, we will ignore the fermionic sector in the rest of this paper.

The bosonic oscillators satisfy the commutation relations
\[ [a_{m,a}, a_{n,b}^\dagger] = [b_{m,a}, b_{n,b}^\dagger] = \delta_{mn} \delta_{ab}, \quad \text{others} = 0. \]

The Hamiltonian and the momentum are represented by these oscillators as
\[ H = \frac{1}{2\pi T} p_- p_+ + \sum_n \left( \omega_n + \omega \right) a_{n,a}^\dagger a_{n,a} + \sum_n \left( \omega_n - \omega \right) b_{n,a}^\dagger b_{n,a}, \]
\[ P = \sum_{n,a} n(a_{n,a}^\dagger a_{n,a} + b_{n,a}^\dagger b_{n,a}). \]
We can now determine the mass spectrum of strings by the Virasoro constraint $H = P = 0$.

As investigated in Refs. [9, 10] for NS-backgrounds and mentioned in [5] for the RR case, this system is regarded as a system of charged string $s$ moving in the constant gauge flux $F_{ij}$ given by (4). To make this fact manifest, we rewrite the Lagrangian (6) in the following form.

$$L = \int_{0}^{2\pi} \frac{d\sigma}{2\pi} \left[ p_\tau X_\tau - \frac{2\pi T}{2} \partial_\alpha X^i \partial^\alpha X^i + p_\tau (A_i \partial_\tau X^i) \right]. \quad (14)$$

We introduced $A_i$ by

$$A_{2a-1} = \mu X^{2a}, \quad A_{2a} = -\mu X^{2a-1}, \quad (a = 1, 2, 3, 4) \quad (15)$$

and these are related to $F_{ij}$ by $F_{ij} = \partial_i A_j - \partial_j A_i$. From this viewpoint, the light-cone momentum $p_\tau$ is identified with the total charge of a closed string. The Lagrangian (14) is invariant under the gauge transformation

$$X_\tau \to X_\tau - \lambda, \quad A_i \to A_i + \frac{\partial}{\partial X^i} \lambda, \quad (16)$$

where $\lambda = \lambda(X^i)$ is a parameter depending on the transverse coordinates $X^i$.

In the expansion (9), we have two $\sigma$-independent modes $a_{0,a}$ and $b_{0,a}$. These describe motion of the center of mass of a closed string. The $a_{0,a}$ term has a factor $e^{-2i\omega \tau}$ and represents the cyclotron motion of the string. The eigenvalues of the operator $a_{0,a}^\dagger a_{0,a}$ determine the radius $r_a$ of the cyclotron motion on the $X^{2a-1}X^{2a}$ plane by the relation

$$r_a^2 \sim \frac{1}{(2\pi T)\omega} a_{0,a}^\dagger a_{0,a}, \quad (a = 1, 2, 3, 4). \quad (17)$$

On the other hand, we identify $b_{0,a}^\dagger$ term with a coordinate $x^i$ of the center of the cyclotron motion by the relation

$$x^{2a-1} + ix^{2a} = \frac{1}{\sqrt{(2\pi T)\omega}} b_{0,a}^\dagger, \quad (a = 1, 2, 3, 4). \quad (18)$$

From the commutation relation (11), the $x^i$-space possesses the following non-commutativity.

$$i[x^i, x^j] = \frac{1}{p_\tau} (F^{-1})^{ij}. \quad (19)$$

Because of this commutation relation, we can represent the momentum $p_i$ conjugate to $x^i$ by the coordinate $x^i$ itself as

$$p_i = p_\tau F_{ij} x^j. \quad (20)$$
3 Compactification

Let us discuss a $T^8$ compactification of the transverse space $R^8$. Let $\vec{b}_A = (b_1^A, \ldots, b_8^A)$ ($A = 1, \ldots, 8$) form a basis of the compactification lattice on $R^8$. In many works studying compactifications of PP-waves, each vector $\vec{b}_A$ is taken in one of $X^{2a-1}-X^{2a}$ planes. Now, however, $\{\vec{b}_A\}$ can be an arbitrary basis in $R^8$ except that they must satisfy the following flux quantization condition.

$$p_-(b_A^i F_{ij} b_B^j) \in 2\pi Z. \quad (21)$$

This is necessary for the string wave function on the torus to be well-defined. If the dimension of the basis was equal to or less than 4, we could take a basis satisfying $b_A^i F_{ij} b_B^j = 0$. Such a basis gives a commutative toroidal compactification, which has been already studied in the literature\[4, 7\]. In the case of the $T^8$ compactification, however, it is impossible to take such a basis, and we necessarily have to assume the quantization of the charge $p_-$. This implies that the light-like direction $X^-$ have to be compactified. Once $X^-$ direction is compactified with radius $R^-$, the charge $p_-$ is quantized as

$$p_- = \frac{k}{R^-}, \quad k \in Z, \quad (22)$$

and the condition (21) is rewritten as

$$\Phi_{AB} \equiv \frac{1}{2\pi R} (b_A^i F_{ij} b_B^j) \in Z. \quad (23)$$

The necessity for the light-like compactification and the flux quantization condition (23) is also explained from the structure of the isometry group of the PP-wave\[4\]. Let $P_i$ and $P_-$ denote shift operators along $x^i$ and $x^-$, respectively. These are related to the notation in [4] as

$$P_{2a-1} = -k_{s_{2a-1,2a}}, \quad P_{2a} = -k_{s_{2a,2a-1}}, \quad (a = 1, 2, 3, 4), \quad P_- = -k e_- \quad (24)$$

In the rotating coordinate, these isometries are represented as

$$P_i = \partial_i + \frac{1}{2} F_{ij} X^j \partial_-, \quad P_- = \partial_-, \quad (25)$$

and satisfy the commutation relation

$$[P_i, P_j] = -F_{ij} P_-. \quad (26)$$

Because of the non-vanishing commutation relation (26), the commutator of two space-like translations $\exp(b_A^i P_i)$ and $\exp(b_B^i P_i)$ gives the light-like translation:

$$e^{b_A^i P_i} e^{b_B^i P_i} = e^{-b_A^i P_i} e^{-b_B^i P_i} = \exp(-b_A^i F_{ij} b_B^j P_-). \quad (27)$$
This implies that the light-like direction $X^-$ must be compactified by a certain radius $R^-$ and $b^i_A F_{ij} b^j_B$ must be a multiple of the period $2\pi R^-$ for any pair of $b^i_A$ and $b^j_B$. This reproduces the constraint (23).

Let $G$ be an orbifold group generated by elements $\exp(b^i_A P_i)$ and $\exp(2\pi R^- P_-)$. An arbitrary element of $G$ is represented as

$$g(w^A, n) = e^{w^A b^i_A P_i} e^{2\pi(n + \sigma(w^A)/2)R^- P_-},$$

where $\sigma(w^A)$ is an integral function satisfying

$$\sigma(w^A + w'^A) = \sigma(w^A) + \sigma(w'^A) - w^A \Phi_{AB} w'^B \mod 2.$$ (29)

For example, we can define $\sigma(w^A)$ by

$$\sigma(w^A) = \sum_{A < B} w^A \Phi_{AB} w^B.$$ (30)

The multiplication rule is $g(w^A_1, n_1) g(w^A_2, n_2) = g(w^A_3, n_3)$ with $w^A_3$ and $n_3$ defined by

$$w^A_3 = w^A_1 + w^A_2,$$
$$n_3 = n_1 + n_2 + \frac{1}{2}(-w^A_1 \Phi_{AB} w^B_2 + \sigma(w^A_1) + \sigma(w^A_2) - \sigma(w^A_3)).$$ (31)

In general, compactifications break supersymmetries which are changed by compactification isometries. On a rotating plane $X^{2a-1} - X^{2a}$, isometries used by the compactification are $k_{S_{2a-1,2a}}^+$ and $k_{S_{2a,2a-1}}^-$. Both the isometries preserve the same 24 supersymmetries; for instance $k_{S_{12}}^+$ or $k_{S_{21}}^-$ preserves supersymmetries corresponding to the killing spinors which vanish by the action of $\gamma^+(1 + i\gamma^3 \gamma^A)[4]$. So, when we choose general directions of the compactification, the number of preserved supersymmetries depends only on the number of rotating planes which are involved by the compactification (See Table 1).

| The number of planes involved by compactification | The number of preserved supersymmetries | Possible maximal compactification |
|--------------------------------------------------|---------------------------------------|---------------------------------|
| 1                                                | 24                                    | $T^2$                           |
| 2                                                | 20                                    | $T^4$                           |
| 3                                                | 18                                    | $T^6$                           |
| 4                                                | 16                                    | $T^8$                           |

Table 1: The number of supersymmetries preserved by compactifications
4 Quantization of winding sectors

Let us consider a winding sector with the boundary condition

\[ X^\mu (\sigma + 2\pi) = g(w^A, n)X^\mu (\sigma)g^{-1}(w^A, n), \]

(32)

where \( w^A \) and \( n \) are the transverse and light-like winding numbers, respectively. For the space-like and light-like coordinates, (32) represents the following boundary conditions, respectively.

\[ X^i (\sigma + 2\pi) = X^i (\sigma) + 2\pi R^i, \]

(33)

\[ X^- (\sigma + 2\pi) = X^- (\sigma) + 2\pi nR^- + \pi F_{ij}R^iX^j, \]

(34)

\[ X^+ (\sigma + 2\pi) = X^+ (\sigma), \]

(35)

where we defined \( R^i = \frac{w^A b^i A}{2\pi} \).

First we have to represent the Hamiltonian \( H \) and the momentum \( P \) by the creation and annihilation operators, the zero-mode momenta and the winding numbers. This is straightforward and almost parallel to the uncompactified case. We just mention several points we should be careful of.

We decompose \( X^i (\sigma) \) into the periodic part \( X^i_0 \) and the winding part \( R^i \sigma \) by

\[ X^i (\sigma) = X^i_0 + R^i \sigma, \]

(36)

and the periodic part is expanded by (9). We can obtain \( H \) and \( P \) in terms of the oscillators, the winding numbers and the momenta by substituting the expansion into (7) and (8). Then, we should bear in mind that some quantities are not periodic in \( \sigma \). For example, substituting (36) into the worldsheet momentum (8), we obtain

\[ P = \int_{\sigma_0}^{\sigma_0+2\pi} d\sigma \left[ -p_- \partial_\sigma X^- - 2\pi T \partial_\sigma X^i_0 \partial_\tau X^i_0 - \frac{p_-}{2} F_{ij}X^i_0 \partial_\sigma X^j_0 \right. \]

\[ -2\pi TR^i \partial_\tau X^i_0 + \left. \frac{p_-}{2} F_{ij}R^iX^j_0 - \frac{p_-}{2} \sigma F_{ij}R^i \partial_\sigma X^j \right]. \]

(37)

Because \( X^- \) is not periodic, we cannot drop the first term. Similarly, the last term in this expression is not periodic. In fact, the non-periodicity of these two terms cancels out each other. To see this, we integrate the last term by part, and obtain

\[ P = \int_{\sigma_0}^{\sigma_0+2\pi} d\sigma \left[ -2\pi T \partial_\sigma X^i_0 \partial_\tau X^i_0 - \frac{p_-}{2} F_{ij}X^i_0 \partial_\sigma X^j_0 \right. \]

\[ -2\pi TR^i \partial_\tau X^i_0 + \left. \frac{p_-}{2} \sigma F_{ij}R^i X^j \right] \sigma_0+2\pi \]

(38)
The first line in (38) does not depend on $R^i$ and is equal to the worldsheet momentum for the uncompactified PP-wave (13). The second line includes only the time independent part ($b_0$-term) of $X^i$, and represents the winding number contribution to the level matching condition. The third line comes from the integration by part. Although each term in the bracket in the third line depends on a choice of the interval $[\sigma_0, \sigma_0 + 2\pi]$, the sum of them is independent of $\sigma_0$ due to the boundary conditions (33) and (34). As a result, we obtain

$$P = -p_- R^i + \sum_{n,a} n(a^\dagger_{n,a} a_{n,a} + b^\dagger_{n,a} b_{n,a}) - p_i R^i. \quad (39)$$

We have used the equivalence (20) between the coordinates and the momenta to obtain the last term in (39).

The worldsheet Hamiltonian is obtained in a similar way as

$$H = \frac{p_+ p_-}{2\pi T} + \sum_{n,a} (\omega_n + \omega) a^\dagger_{n,a} a_{n,a} + \sum_{n,a} (\omega_n - \omega) b^\dagger_{n,a} b_{n,a} + \frac{2\pi T}{2} R^i R^i. \quad (40)$$

In deriving this Hamiltonian, we defined the momentum $p_+$ as a constant part of $\Pi_+$, where $\Pi_+$ is the canonical momentum for the field $X^+$:

$$\Pi_+ = \partial_\tau X^- + \frac{p_-}{2(2\pi T)} F_{ij} X^i \partial_\tau X^j. \quad (41)$$

Because the periodicity of $\Pi_+$ is guaranteed by the boundary conditions (33) and (34), we can define $p_+$ unambiguously.

We can make an arbitrary string state by acting the bosonic creation oscillators $a^\dagger_{n,a}$ ($n \in \mathbb{Z}$), $b^\dagger_{n,a}$ ($n \in \mathbb{Z} - \{0\}$) and the fermionic ones on a ground state

$$|n, w^A, k; \vec{p}_V\rangle, \quad (42)$$

in each winding and momentum sector. $n$, $w^A$ and $k$ are the light-like winding number, the transverse winding numbers and the quantum number of the light-like Kaluza-Klein momentum (22), respectively. The vector $\vec{p}_V$ is the transverse momentum in $T^8$.

Because the fermionic oscillators and the bosonic ones except $b_{0,a}$ are not affected by the compactification, we discuss only the structure of Fock space associated with the zero-mode oscillator $b_{0,a}$, which is related to $\vec{p}_V$ via (18) and (20).

The transverse space is non-commutative and the eight components of $p_i$ do not commute:

$$[p_i, p_j] = -ip_- F_{ij}. \quad (43)$$

Thus we can diagonalize only four of the eight components of the transverse momentum. To choose four linearly independent momenta commutative among
them, we define a four-dimensional commutative subspace $V$ of the transverse space $\mathbb{R}^8$, and we use four components of a momentum $\vec{p}_V$ on $V$ as quantum numbers of ground states. We assume the subspace $V$ satisfies the following conditions.

- The sublattice $\Gamma_V \equiv V \cap \Gamma_8$ of the $\mathbb{T}^8$ compactification lattice $\Gamma_8$ is four-dimensional. This is equivalent to the statement that we can take a basis of $V$ in the lattice $\Gamma_8$.

- Any two momenta along $V$ commute with each other. Namely, arbitrary two vectors $\vec{v}_1 \in V$ and $\vec{v}_2 \in V$ satisfy

$$\vec{v}_1 \cdot J \cdot \vec{v}_2 = 0,$$

where $J$ is a skew-diagonal matrix proportional to $F$ and satisfying $J^2 = -1$. Thanks to this condition, we can use the four components of $\vec{p}_V$ as independent quantum numbers specifying ground states.

- The sublattice $\Gamma_V$ includes the winding vector $w^A \vec{b}_A$. This guarantees that the term $R^i p_i$ appearing in the worldsheet momentum (39) is automatically diagonalized on the ground states (42). This condition implies that the subspace $V$ has to be chosen after the winding numbers $w^A$ are specified.

We can always take such a subspace $V$. Indeed, we can construct a basis of $V$ in the following way. First, we adopt the winding vector $2\pi \vec{R} = w^A \vec{b}_A$ as the first vector $\vec{v}_1$ in the basis. If we assume that we have already determined $k$ vectors $\vec{v}_i \in \Gamma_8$ ($1 \leq k < 4$) in the basis, we can choose one solution of the commutativity condition

$$\vec{v}_{k+1} \cdot F \cdot \vec{v}_i = 0, \quad \text{for } i = 1, \ldots, k$$

as the $k+1$-th vector in the basis of $V$. Thanks to the flux quantization condition (21), all the linearly independent $8 - k$ solutions can be taken in the lattice $\Gamma_8$. Because this equation has $8 - k$ linearly independent solutions, we can choose $\vec{v}_{k+1}$ which is independent of $k$ vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$, whenever $k$ is less than 4. This procedure can be repeated before $k$ reaches 4, in which case all the solutions of (45) are linear combinations of $\{\vec{v}_1, \ldots, \vec{v}_4\}$. In this way, we obtain four vectors spanning four-dimensional sublattice of $\Gamma_8$, and we obtain $V$ as a four-dimensional subspace of $\mathbb{R}^8$ containing the sublattice. The above procedure does not guarantee that $\vec{v}_i$ are the basis of $V \cap \Gamma_8$. The lattice spanned by $\vec{v}_i$ may be a sublattice of $V \cap \Gamma_8$. However, once we have obtained $V$, we can always choose $\vec{v}_i$ as a basis of $V \cap \Gamma_8$. We will assume $\vec{v}_i$ to be defined in this way below.

The complement subspace of $V$ in $\mathbb{R}^8$ is denoted by $W$. It is given by

$$W = \{ \vec{w} | \vec{w} = J \vec{v}, \vec{v} \in V \}.$$  

Indeed, $\vec{w} \in W$ and $\vec{v} \in V$ are always orthogonal because $\vec{w}$ can be represented as $J \vec{v}'$ for some vector $\vec{v}' \in V$, and the inner product $\vec{v} \cdot \vec{w} = \vec{v} J \vec{v}'$ is 0 due
to the assumption (44). In addition, the subspace $W$ is commutative as well as $V$ because arbitrary pair of vectors $\vec{w}_1 = J\vec{v}_1$ and $\vec{w}_2 = J\vec{v}_2$ in $W$ satisfies $\vec{w}_1 J\vec{w}_2 = \vec{v}_1 J\vec{v}_2 = 0$.

Now, we have obtained a set of quantum numbers specifying string states. To obtain physical string states, we need to pick up states invariant under the orbifold group $G$. This is realized by the identification $g(s^A, t)|s\rangle \sim |s\rangle$ for arbitrary elements $g(s^A, t) \in G$. The invariance under elements in the form $g(0, t)$ requires the quantization of the light-like momentum. This is already taken into account by (22). The invariance under

$$g(s^A, 0) = \exp(\vec{s} \cdot \vec{P}), \quad \vec{s} \equiv s^A b_A,$$

should also be considered. If $\vec{s}$ is an element of $V$, $\vec{P}$ in (47) can be replaced by a c-number $i\vec{p}_V$ when it acts on (42), and we obtain

$$g(s^A, 0)|n, w^A, k, \vec{p}_V\rangle = e^{i\vec{s} \cdot \vec{p}_V}|n, w^A, k, \vec{p}_V\rangle, \quad \vec{s} \in \Gamma_V.$$

This should be identified with the state (42) and it requires the Kaluza-Klein momentum $\vec{p}_V$ to be quantized as

$$\vec{p}_V \in 2\pi \Gamma_V^{-1}. \quad (49)$$

For the case of $\vec{s} \not\in V$, it is convenient to decompose $\vec{s}$ into the sum of two vectors $\vec{s}_V \in V$ and $\vec{s}_W \in W$. If we define $\Gamma_W$ as a projection of $\Gamma_8$ to $W$, $\vec{s}_W$ is an element of $\Gamma_W$. Due to the commutation relation (43), $\exp(\vec{s}_W \cdot \vec{P})$ shifts the momentum $\vec{p}_V$ by $p_\perp \cdot \vec{s}_W$ while $\exp(\vec{s}_V \cdot \vec{P})$ just gives a phase factor when it acts on the ground states. For the shift of the transverse vector $\vec{p}_V$ and the quantization (49) of $\vec{p}_V$ to be consistent to each other, the vector $p_\perp \cdot \vec{s}_W$ must be an element of the lattice $2\pi \Gamma_V^{-1}$ as well as $\vec{p}_V$. This is actually guaranteed by the flux quantization condition (23).

The action of the orbifold group $G$ also changes the light-like winding number. This is because the shift replaces the $X$ in the boundary condition (32) by $g(s^A, t)Xg^{-1}(s^A, t)$ and it gives new boundary condition (32) with $g(w^A, n)$ replaced by $g(w^A, n')$ defined by

$$g^{-1}(s^A, t)g(w^A, n)g(s^A, t) = g(w^A, n'). \quad (50)$$

The explicit form of $n'$ can be determined by the multiplication rule (31) as

$$n' = n + s^A \Phi_{AB} w_B. \quad (51)$$

Taking account of these facts, $g(s^A, 0)$ changes the ground state $|n, w^A, k, \vec{p}_V\rangle$ to

$$e^{\vec{s} \cdot \vec{P}}|n, w^A, k, \vec{p}_V\rangle = (\text{phase factor}) \times |n', w^A, k, \vec{p}_V + p_\perp \cdot \vec{s}_W\rangle. \quad (52)$$
The change of the winding number by the action of the orbifold group is a general feature of non-Abelian orbifolds. Topologically inequivalent sectors of a non-Abelian orbifold are labeled not by elements of an orbifold group but by its conjugacy classes. In our case, there are the following conjugacy classes.

\[ \tilde{g}(0, n) = \{ g(0, n) \}, \quad (n \in \mathbb{Z}) , \]
\[ \tilde{g}(w^A, n) = \{ g(w^A, n + p(w^A)l) : l \in \mathbb{Z} \}, \quad (n = 0, 1, \ldots, p(w^A) - 1), \] (53)

where \( p(w^A) \) is an integer defined by

\[ p(w^A) = \gcd(\Phi_{1A}w^A, \Phi_{2A}w^A, \ldots, \Phi_{8A}w^A). \] (54)

Although the topologically inequivalent sectors are labeled by conjugacy classes \( \tilde{g}(w^A, n) \), we use elements \( g(w^A, n) \) of the orbifold group \( G \) to label sectors and treat these sectors independently. Instead, we restrict the transverse momentum \( \vec{p}_V \) inside a fundamental region of the lattice \( p_- F \Gamma W \) to take the identification (52) into account. In other words, we treat \( \vec{p}_V \) as a vector in the following compactified lattice.

\[ \vec{p}_V \in 2\pi \Gamma^\perp / (p_- F \Gamma W). \] (55)

Now we have a complete set of quantum numbers and the restriction on them to avoid multiple counting of states. We can obtain the string spectrum by the Virasoro constraint \( H = P = 0 \).

As a consistency check, let us confirm the spectrum reduces to that of ordinary toroidal compactification in the vanishing flux limit \( \mu \rightarrow 0 \). For the limit to be taken smoothly, we assume large transverse compactification radii and small light-like one. In this situation, the components of the field strength \( F_{ij} \) satisfying the quantization condition (23) can be treated as continuous quantities. To show the coincidence of the spectrum, let us compute the light-cone partition function \( Z_{lc} \) for fixed light-like winding \( n \) and light-like momentum \( k \) defined by

\[ Z_{lc} = \text{Tr} e^{2\pi (-\tau_2 H + i\tau_1 P)}, \] (56)

where the trace is taken over states with fixed \( k \) and \( n \). Because of the large transverse compactification radii, we can focus only on the \( w^A = 0 \) sector. For the \( w^A = 0 \) sector, the partition function is factorized into

\[ Z_{lc} = Z_{0} N_{LLL} \prod_{n} Z_{n,+}^4 \prod_{n \neq 0} Z_{n,-}^4, \] (57)

where \( Z_0 = e^{2\pi (-\tau_2 p_+/p_-/(2\pi T) - i\tau_1 p_- R^-)} \) represents the factor depending on the light-like components of momentum and the light-like winding number. \( Z_{n,+} \) and \( Z_{n,-} \) are the factors coming from the sum over the occupation numbers for the oscillators \( a_{n,a} \) and \( b_{n,a} \), respectively, and are defined as

\[ Z_{n,\pm} = \sum_{N=0}^{\infty} e^{2\pi (-\tau_2 (\omega_n \pm \omega) + i\tau_1 n) N}. \] (58)
\( N_{\text{LLL}} \) is the number of states in the lowest Landau level labeled by \( \vec{p}_V \). It is obtained as the ratio between volumes of the fundamental regions of lattices in the numerator and the denominator in (55).

\[
N_{\text{LLL}} = \frac{(2p_\mu)^4 \text{vol}[\Gamma_W]}{(2\pi)^4 (\text{vol}[\Gamma_V])^{-1}} = \frac{(2p_\mu)^4}{(2\pi)^4} \text{vol}[\Gamma_8] = k^4 \text{Pf}(\Phi_{AB}).
\]  

(59)

\text{vol}[\Gamma] \) represents the volume of a fundamental region of a lattice \( \Gamma \). The final expression shows that this quantity is obviously integer.

On the other hand, in the case of ordinary toroidal compactification, the light-cone partition function defined by (56) is factorized into

\[
Z_{lc} = Z_0 \sum_p e^{-\tau_2 \frac{1}{4\pi} p^2} \prod_{n \neq 0} Z_{n,0}^8, \tag{60}
\]

where \( Z_0 \) is the same with that in (57), and \( Z_{n,0} \) is the factor coming from the bosonic oscillators and is given by

\[
Z_{n,0} = \sum_{N=0}^{\infty} e^{2\pi i (\tau_2 |n| + \tau_1 n) N}. \tag{61}
\]

Thanks to the large transverse compactification, the summation over the transverse momentum in (60) is replaced by an integration as

\[
\sum_p e^{-\tau_2 \frac{1}{4\pi} p^2} = \frac{\text{vol}[\Gamma_8]}{(2\pi)^8} \int d^8 p e^{-\tau_2 \frac{1}{4\pi} p^2} = \text{vol}[\Gamma_8] \frac{T^4}{(2\pi \tau_2)^4}. \tag{62}
\]

In the \( \mu \to 0 \) limit, both \( Z_{n,+} \) and \( Z_{n,-} \) reduce into \( Z_{n,0} \). Therefore, for the two partition functions (57) and (60) to coincide in the limit, we need the relation

\[
N_{\text{LLL}} Z_{0,+} \stackrel{\mu \to 0}{\to} \sum_p e^{-\tau_2 \frac{1}{4\pi} p^2}. \tag{63}
\]

Indeed, this relation can be easily shown to hold.

5 Thermal Partition Function

In this section, we calculate the thermal partition function (TPF) of strings on our background. The computation of this quantity is well done for DLCQ strings. It is considered that TPF has a property of the modular invariance. Although this fact is made clear by the path integral calculation of TPF, we will calculate it by the operator method, because these two methods are considered to give the same results [11, 12, 13]. Our procedure is almost parallel to that of [6], in which \( S^1 \) compactification is treated.
The TPF is defined by

\[ F(\beta) = -\sum_{l=1}^{\infty} \frac{1}{\beta l} \text{Tr} \left[ (-1)^{(l+1)} F e^{-\beta p^0} \right], \]  

(64)

where \( \beta \) is the inverse temperature, \( F \) is the space-time fermion number and \( p^0 = \frac{1}{\sqrt{2}} (p^+ - p^-) \). The trace runs only physical states satisfying the level matching condition.

In the case of an \( S^1 \) compactification, TPF is calculated in [6]. The result is

\[ F_{S^1}(\beta) = -\sum_{k=1}^{\infty} \sum_{l=1}^{k-1} \sum_{q=0}^{w(k/d)} e^{-\frac{\beta^2 R^2}{4\pi \alpha'} \left( 1 + \frac{R^{-2} \mu_0^2}{2\pi^2 \alpha'} \right)} \times \text{tr} \left[ (-1)^{(l+1)} F e^{-2\pi \tau_2 H_{osc} + 2\pi \tau_1 P_{osc}} \right], \]  

(65)

where \( R_T \) is the transverse compactification radius, and \( \mathcal{N} \) represents the number of the lowest Landau level states on non-commutative plane compactified on \( S^1 \). In [6], a cut-off length \( L \) for the direction transverse to the \( S^1 \) is introduced to make \( \mathcal{N} \) finite. In our case, as we will see below, \( \mathcal{N} \) is replaced by \( \mathcal{N}_{LLL} \) in (59), the number of the lowest Landau level states in \( T^8 \), which is finite due to the compactness of the \( T^8 \). The trace is over the oscillator modes and \( d = \gcd(k, q) \).

\( \tau \) is defined by

\[ \tau = \tau_1 + i \tau_2 \equiv \frac{q + il\nu}{k}, \quad \nu = \frac{\sqrt{2} \beta R^{-}}{4\pi \alpha'}, \]  

(66)

If \( k \) is very large, the sum over \( l \) and \( q \) can be regarded as an integration over the complex variable \( \tau \).

We choose a commutative space \( V \) as to contain the transverse winding vector \( w^A \vec{b}_A \). We can change the basis to \( \{ \vec{\tilde{b}}_A \} \) whose first four are the four basis vectors of \( \Gamma_Y \). We denote these four vectors as \( \vec{\tilde{v}}_i (i = 1, 2, 3, 4) \). The other four vectors in \( \{ \vec{b}_A \} \) are denoted by \( \vec{w}_i \). In this basis, \( \Phi_{AB} \) is written in the following form.

\[ \Phi_{AB} = \frac{1}{2\pi R^{-}} (\vec{\tilde{b}}_A \cdot \vec{F} \cdot \vec{\tilde{b}}_B) = \begin{pmatrix} 0 & B \\ -B & C \end{pmatrix}, \]  

(67)

It is always possible by basis transformation of \( \{ \vec{\tilde{v}}_i \}, \{ \vec{w}_i \} \) in each subspace to make \( B \) diagonal\(^3\). In this choice of basis, the matrix \( B \) becomes

\[ B = \begin{pmatrix} N_1 \\ \vdots \\ N_A \end{pmatrix}. \]  

(68)

\(^3\text{Basis change } \vec{v}'s \text{ and } \vec{w}'s \text{ is each represented by unimodular matrices. It is known that any integer valued matrix is made integer valued diagonal matrix by multiplication of two unimodular matrices on both sides.}\)
The winding vector is represented as a linear combination of $\vec{v}_i$'s and we denote the coefficients by $u_i$:

$$\sum_{A=1}^{8} w^A \vec{b}_A = \sum_{i=1}^{4} u^i \vec{v}_i.$$  \hspace{1cm} (69)

We use $R_{\{w^A\}}$ to represent the length of $\sum_{A=1}^{8} w^A \vec{b}_A / 2\pi$.

Because of the projection by the orbifold group, the ground states are identified as

$$|n, k, w_A, m_i\rangle \sim |n + \sum_{i=1}^{4} \alpha_i u_i N_i, k, w_A, m_i + \alpha_i k N_i\rangle,$$  \hspace{1cm} (70)

where $\alpha_i$ are arbitrary integers. To avoid multiple counting of states, we restrict $m_i$ by $0 \leq m_i < k N_i$.

In this setting, the physical states are labeled by the zero-mode quantum numbers in Table 2. The worldsheet momentum takes the following form.

| quantum numbers                      | variables | values |
|--------------------------------------|-----------|--------|
| light-cone winding number:           | $n$       | $\mathbb{Z}$ |
| light-cone momentum:                 | $k$       | $\mathbb{N}$ |
| transverse winding number:           | $w_A$     | $\mathbb{Z}$ |
| transverse momentum: $m_i (i = 1, \ldots, 4)$ | $\mathbb{Z}$, $0 \leq m_i < k N_i$ |

Table 2: The zero-mode quantum numbers

$$P = P_{osc} - kn - \sum_{i=1}^{4} m_i u_i$$ \hspace{1cm} (71)

We first insert

$$\delta_{P,0} = \int_0^1 dt e^{2\pi i t P}$$ \hspace{1cm} (72)

into the trace in (64) in order to pick up states satisfying the level matching condition, and then sum over all states of Hilbert space. The partition function becomes

$$F(\beta) = -\sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \sum_{n \in \mathbb{Z}} \sum_{w^A \in \mathbb{Z}^8} \sum_{0 \leq m_i < N_i k} \frac{1}{\beta l} \text{tr} \left[ (-1)^{(l+1)f} \int_0^1 dt e^{2\pi i t(P_{osc} - m_i u_i - kn)} \right.$$  

$$\times e^{\frac{\beta L}{2k}(H_{osc} + |R_{\{w^A\}}|^2 + \frac{m_i}{2\alpha_i})} \bigg] \bigg].$$ \hspace{1cm} (73)

We decomposed the trace into the zero-mode part and the oscillator-mode part, and explicitly represent the summation over the zero-modes by $\sum$. So, the trace
in (73) means the summation only over the oscillator modes. By computing the level matching part and the summation over $n$, we obtain

$$\sum_{n \in \mathbb{Z}} \int_0^1 dt e^{2 \pi i t P} = \sum_{q \in \mathbb{Z}_k} e^{2 \pi i \frac{q}{k} (P_{osc} - \sum_i m_i u_i)}.$$  \hfill (74)

The factor $e^{-2 \pi i \frac{q}{k} \sum m_i u_i}$ together with the summation over $m_i$'s becomes

$$\sum_{m_i \in (\mathbb{Z}_k N_i)^4} e^{-2 \pi i \frac{q}{k} \sum m_i u_i} = \prod_i (k N_i \sum_{p_i \in \mathbb{Z}} \delta_{q u_i, k p_i}).$$  \hfill (75)

The Kronecker’s delta from the $i$-th direction $\sum_{p_i} \delta_{q u_i, k p_i}$ restricts values of $u_i$ to $\frac{k}{d} \mathbb{Z}$, where $d$ is gcd$(k, q)$. This is equivalent to the restriction of $w^A$ to $(k/d) \mathbb{Z}$.

Finally, we obtain

$$F(\beta) = -\sum_{k=1}^\infty \sum_{l=1}^\infty \sum_{q=0}^{k-1} \sum_{w^A \in ([k/d] \mathbb{Z})^8} \frac{N_{LLL}}{\beta l k} e^{\frac{-q^2 \omega^2}{4 \alpha'^2}} \left(1 + \frac{q^2}{2 \omega^2 k^2}\right) \times (\text{oscillator part}),$$  \hfill (76)

where $N_{LLL} = k^4 \prod_i N_i$. Namely, the result for $S^1$ compactification (65) is generalized to the $T^8$ compactification by replacing the summation over one winding number $w$ by the summation over the eight winding numbers $w^A$ and $N$ by $N_{LLL}$. The oscillator part is not affected by the compactification and is written by massive theta function according to the oscillator spectrum of the $T^8$ case (6).

While we chose the basis $\{ \vec{b}_A \}$ depending on the winding direction during the calculation, the final expression of the TPF does not depend on this choice.

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