Abstract

Let \((N, g)\) be a Riemannian manifold. For a compact, connected and oriented submanifold \(M\) of \(N\), we define the space of volume preserving embeddings \(\text{Emb}_\mu(M, N)\) as the set of smooth embeddings \(f : M \rightarrow N\) such that \(f^* \mu = \mu\), where \(\mu\) (resp. \(\mu\)) is the Riemannian volume form on \(f(M)\) (resp. \(M\)) induced by the ambient metric \(g\) (the orientation on \(f(M)\) being induced by \(f\)).

In this article, we use the Nash-Moser inverse function Theorem to show that the set of volume preserving embeddings in \(\text{Emb}_\mu(M, N)\) whose mean curvature is nowhere vanishing forms a tame Fréchet manifold, and determine explicitly the Euler-Lagrange equations of a natural class of Lagrangians.

As an application, we generalize the Euler equations of an incompressible fluid to the case of an “incompressible membrane” of arbitrary dimension moving in \(N\).

Introduction

Fluid mechanics and infinite dimensional geometry already share a long and common history. In 1966, Arnold [Arn66] suggested to regard the space of velocity fields of an incompressible fluid as the Lie algebra of the infinite dimensional Lie group of volume preserving diffeomorphisms:

\[
\text{SDiff}_\mu(M) := \{ \phi \in \text{Diff}(M) \mid \phi^* \mu = \mu \}. 
\]  

Here \(M\) is the oriented manifold on which the fluid is living, \(\mu\) is the volume form of \(M\) and \(\text{Diff}(M)\) is the group of all smooth diffeomorphisms of \(M\). In this setting, Arnold interpreted the Euler equations of an incompressible fluid as a geodesic equation on \(\text{SDiff}_\mu(M)\) for an appropriate right-invariant metric.

It was not until the 70’s that Arnold’s vision of fluid mechanics could be made partially rigorous with the development of Banach and Hilbert manifolds. In [EM70], Ebin and Marsden considered volume preserving diffeomorphisms on a compact manifold \(M\) which are not smooth, but of Sobolev classes. In doing so, they obtained topological groups locally modelled on Hilbert spaces, and were able, following Arnold’s ideas, to prove analytical results on the Euler equations. Their method is still an active research area (see for example [GB09, GBR05]).

On the geometrical side, volume preserving diffeomorphisms which are not smooth are problematic. For, the left-multiplication \(L_\phi : \text{Diff}(M) \rightarrow \text{Diff}(M), \psi \mapsto \phi \circ \psi\) consumes derivatives of \(\phi\), and thus, subgroups of the group of diffeomorphisms whose elements are not smooth cannot be turned into genuine infinite dimensional Lie groups (left multiplication is not smooth). Hence, from a Lie group theory point of view, one has to consider the group of smooth volume preserving diffeomorphisms of \((M, \mu)\), i.e., the group \(\text{SDiff}_\mu(M)\).

For technical reasons, \(\text{SDiff}_\mu(M)\) can only be given a Lie group structure modelled on topological vector spaces which are more general than Banach and Hilbert spaces, and an inverse function theorem, applicable beyond the usual Banach space category, is necessary. To our knowledge, only two authors succeeded in doing this. The first was Omori who showed and used an inverse function theorem in terms of ILB-spaces...
("inverse limit of Banach spaces", see [Omo97]), and later on, Hamilton with his category of tame Fréchet spaces together with the Nash-Moser inverse function Theorem (see [Ham82]). Nowadays, it is nevertheless not uncommon to find mistakes or big gaps in the literature when it comes to the differentiable structure of \( \text{SDiff}_\mu(M) \), even in some specialized textbooks in infinite dimensional geometry. The case of \( M \) being non-compact is even worse, and no proof that \( \text{SDiff}_\mu(M) \) is a “Lie group” is available in this case.

A natural generalization of \( \text{SDiff}_\mu(M) \), with which we shall be concerned in this paper, is the space of volume preserving embeddings \( \text{Emb}_\mu(M,N) \). This space is defined as follows. For a Riemannian manifold \((N,g)\) and a compact, connected and oriented submanifold \( M \) of \( N \),

\[
\text{Emb}_\mu(M,N) := \left\{ f \in \text{Emb}(M,N) \mid f^*\mu^f = \mu \right\},
\]

where \( \text{Emb}(M,N) \) is the space of smooth embeddings from \( M \) into \( N \), and where \( \mu^f \) (resp. \( \mu \)) is the Riemannian volume form on \( f(M) \) (resp. \( M \)) induced by the ambient metric \( g \) (the orientation on \( f(M) \) being induced by \( f \)).

When \( M \) is an open subset of \( \mathbb{R}^n \) with boundary\(^1\), then it is possible to extend Arnold’s method by introducing a \( L^2 \)-metric on \( \text{Emb}_\mu(M,N) \) and to show that the corresponding geodesics describe the dynamics of a liquid drop with free boundary. This has been discussed formally in [LMMR86], and rigorous results in this direction can be obtained using spaces of volume preserving embeddings of Sobolev classes, as pointed out to us by Sergiy Vasylkevych\(^2\).

In this paper, we focus on smooth volume preserving embeddings, i.e., on the space \( \text{Emb}_\mu(M,N) \) as defined above. To this end, we adopt a rigorous infinite dimensional point of view based on Hamilton’s category of tame Fréchet manifolds, and determine explicitly a natural class of Lagrangian equations on \( \text{Emb}_\mu(M,N) \). We allow \( M \) to be of arbitrary dimension, and we assume that it has no boundary.

More precisely, using the techniques developed by Hamilton in [Ham82], as well as a generalization of the Helmholtz-Hodge decomposition Theorem for vector fields supported on submanifolds (Proposition 2.1), we are able, in Theorem 1.6, to show the following result: the space \( \text{Emb}_\mu(M,N) \) of volume preserving embeddings whose mean curvature is nowhere vanishing forms a tame Fréchet submanifold of \( \text{Emb}(M,N) \). This result is a consequence of the Nash-Moser inverse function Theorem.

Having a manifold structure on \( \text{Emb}_\mu(M,N) \), we then consider Lagrangian mechanics on it. The Lagrangians we consider are of the following form:

\[
\tilde{L}(X_f) := \int_M L \circ X_f \cdot \mu,
\]

where \( L : TN \to \mathbb{R} \) is a Lagrangian density and where \( X_f : M \to TN \) is a “divergence free vector field along \( f \)”, regarded as an element of \( T_f \text{Emb}_\mu(M,N) \). As it turns out, the resulting Euler-Lagrange equations are (pointwise) the usual finite dimensional Euler-Lagrange equations (written in a covariant form), twisted by a “Helmholtz-Hodge projection” (Proposition 2.3).

When \( L \) is the energy associated to the metric \( g \), then the corresponding Euler-Lagrange equations on \( \text{Emb}_\mu(M,N) \) are geodesic equations which generalize the Euler equations of an incompressible fluid to the case of an “incompressible membrane” of arbitrary dimension moving in \( N \) (Proposition 2.4).

It would be interesting to know if these equations have a physical meaning.

The paper is organized as follows. In 1.1 we review very briefly Hamilton’s category of tame Fréchet manifolds. In 1.2 we show that \( \text{Emb}_\mu(M,N) \) is a tame Fréchet submanifold of \( \text{Emb}(M,N) \); this requires a generalization of the Helmholtz-Hodge decomposition. In 2.1 we compute the Euler Lagrange equations on \( \text{Emb}_\mu(M,N) \) for a natural class of Lagrangians, and in 2.2 we identify the natural generalization of the Euler equations of an incompressible fluid.

\(^1\)In this paper, all manifolds have no boundary.

\(^2\)Private communication.
1 The differentiable structure of the space of volume preserving embeddings

Let $(N, g)$ be a Riemannian manifold and let $M$ be a compact, connected and oriented submanifold of $N$. We denote by $\text{Emb}(M,N)$ the space of smooth embeddings from $M$ into $N$.

For an embedding $f : M \hookrightarrow N$, we denote by $\mu^f$ the volume form on $f(M)$ induced by the restriction of the metric $g$ to the submanifold $f(M)$ (the orientation on $f(M)$ being induced by $f$). With this terminology, we define the space of volume preserving embeddings as

\[
\text{Emb}_\mu(M,N) := \left\{ f \in \text{Emb}(M,N) \mid f^*\mu = \mu \right\},
\]

where $\mu$ is the Riemannian volume form on $M$ induced by the metric $g$.

The aim of this section is to use the Nash-Moser inverse function Theorem (as formulated in [Ham82]) to define a differentiable structure on the open subset

\[
\text{Emb}_\mu(M,N) \times := \left\{ f \in \text{Emb}_\mu(M,N) \mid (\text{Tr} \Pi_f) x \neq 0 \text{ for all } x \in f(M) \right\},
\]

where $\text{Tr} \Pi_f$ denotes the trace of the second fundamental form of $f(M)$.

For the reader’s convenience, let us recall that the second fundamental form $\Pi_f$ of the submanifold $f(M)$ is defined, for $x \in f(M)$ and for two vector fields $X,Y$ on $f(M)$, by

\[
(\Pi_f)_x(X,Y) := \nabla_{\tilde{X}} \tilde{Y} - \nabla_{\tilde{X}} Y,
\]

where $\nabla$ (resp. $\nabla^f$) is the Levi-Civita connection on $N$ (resp. $f(M)$) induced by $g$ (resp. $g|_{f(M)}$), and where $\tilde{X}, \tilde{Y}$ are vector fields on $N$ extending $X$ and $Y$.

Let us also recall that the trace of the second fundamental form $\Pi_f$ is defined, for $x \in f(M)$, by

\[
(\text{Tr} \Pi_f)_x := \sum_{i=1}^k \Pi_f(e_i, e_i),
\]

where $k$ is the dimension of $M$ and where $\{e_1, ..., e_k\}$ is an orthonormal basis for $T_x f(M)$. In particular, $\text{Tr} \Pi_f$ is a section of the normal bundle $\text{Nor}_f$ of $f(M)$, the latter bundle being, by definition, the vector bundle over $f(M)$ whose fiber over $x \in f(M)$ is

\[
(\text{Nor}_f)_x := \left\{ u_x \in T_x N \mid g_x(u_x, v_x) = 0 \text{ for all } v_x \in T_x f(M) \right\}.
\]

Finally, recall that $\text{Tr} \Pi_f$ is, up to a multiplicative constant which depend on convention, the mean curvature of the submanifold $f(M)$.

1.1 Hamilton’s category of tame Fréchet manifolds

In this section, we review very briefly the category of tame Fréchet manifolds introduced by Hamilton in [Ham82].

**Definition 1.1.** (i) A graded Fréchet space $(F, \{\| \cdot \|_n\}_{n \in \mathbb{N}})$, is a Fréchet space $F$ whose topology is defined by a collection of seminorms $\{\| \cdot \|_n\}_{n \in \mathbb{N}}$ which are increasing in strength:

\[
\|x\|_0 \leq \|x\|_1 \leq \|x\|_2 \leq \cdots
\]

for all $x \in F$. 


3
1.2 The differentiable structure of \( \text{Emb}_\mu(M, N)^\times \)

Let \( \Sigma \) be an oriented submanifold of \( N \) endowed with the Riemannian volume form \( \mu^\Sigma \) induced by \( g \).

We shall use the following terminology:

- \( TN|\Sigma \) is the restriction of the bundle \( TN \) to \( \Sigma \) with associated space of sections \( \Gamma(TN|\Sigma) \).

\(^3\)By smooth we mean that \( f : U \subseteq F \rightarrow G \) is continuous and that for all \( k \in \mathbb{N} \), the \( k \)th derivative \( d^kf : U \times F \times \cdots \times F \rightarrow G \) exists and is jointly continuous on the product space, such as described in [Ham82].

\( \Sigma \) is the restriction of the bundle \( TN \) to \( \Sigma \) with associated space of sections \( \Gamma(TN|\Sigma) \)
• For a vector field $X \in \mathfrak{X}(\Sigma)$, $\text{div}_\Sigma(X)$ is the divergence of $X$ with respect to the volume form $\mu^\Sigma$, i.e., it is the only function which satisfies $\mathcal{L}_X \mu^\Sigma = \text{div}_\Sigma(X) \cdot \mu^\Sigma$, where $\mathcal{L}_X$ is the Lie derivative in direction $X$.

• $\Gamma_\mu(TN|_{\Sigma}) := \{ X \in \Gamma(TN|_{\Sigma}) \mid \text{div}_\Sigma(X^\top) - g(X^{\perp}, \text{Tr} \Pi_\Sigma) = 0 \}$, where $X^\top$ and $X^{\perp}$ are respectively the tangential and orthogonal projections of $X$ on the tangent and normal bundles of $\Sigma$.

If $\Sigma = f(M)$ for some embedding $f \in \text{Emb}(M, N)$, then we shall replace “$\Sigma$” by “$f$” in the above notation. For example, $\text{div}_f$ instead of $\text{div}_\Sigma$, etc.

**Proposition 1.4.** Let $\Sigma$ be a compact, connected, oriented submanifold of $N$ whose mean curvature is not identically zero. Then, for every section $X$ of $TN|_{\Sigma}$, there exist a unique $X_\mu \in \Gamma_\mu(TN|_{\Sigma})$ and a unique function $p : \Sigma \to \mathbb{R}$ such that

$$X = X_\mu + \text{grad}(p) + p \cdot \text{Tr} \Pi_\Sigma,$$

(13)

where $\text{grad}(p) \in \mathfrak{X}(\Sigma)$ is the Riemannian gradient of $p$ taken with respect to $g|_{\Sigma}$.

**Proof.** Let $X$ be an element of $\Gamma(TN|_{\Sigma})$. If $X$ could be written $X = X_\mu + \text{grad}(p) + p \cdot \text{Tr} \Pi_\Sigma$ with $X_\mu \in \Gamma_\mu(TN|_{\Sigma})$ and $p \in C^\infty(\Sigma, \mathbb{R})$, then $p$ would be a solution of the following partial differential equation:

$$\text{div}_\Sigma(X^\top) - g(X^{\perp}, \text{Tr} \Pi_\Sigma) = \Delta p - ||\text{Tr} \Pi_\Sigma||^2 \cdot p.$$  

(14)

The differential operator $\Delta p - ||\text{Tr} \Pi_\Sigma||^2 \cdot p$ acting on functions $p : \Sigma \to \mathbb{R}$ is an operator of the form $\Delta - c$, where $c$ is a smooth function, and, being an elliptic operator, it is well known that this operator is Fredholm, and that its analytical index is a topological invariant (see [Pal65]). Hence, the index of $\Delta - c$ equals the index of $\Delta$, which is zero on the space $C^\infty(\Sigma, \mathbb{R})$. Moreover, as $c$ is nonnegative, and since $\Sigma$ connected, we can use the maximum principle (see for example [Amb98], p.96) or [Jos05], Thm 24.10, p.355), to deduce that the kernel of $\Delta - c : C^\infty(\Sigma, \mathbb{R}) \to C^\infty(\Sigma, \mathbb{R})$ is included in the space of constant functions, and as $c$ is not identically zero in our case, this kernel has to be trivial. Hence, $\Delta - c$ is bijective, and (14) possesses a unique solution $p \in C^\infty(\Sigma, \mathbb{R})$.

Now, if we take a function $p$ solution to (14) and set $X_\mu := X - \text{grad}(p) - p \cdot \text{Tr} \Pi_\Sigma$, then it is straightforward to check that $X = X_\mu + \text{grad}(p) + p \cdot \text{Tr} \Pi_\Sigma$ is the desired decomposition.

Proposition 1.4 yields a topological decomposition

$$\Gamma(TN|_{\Sigma}) = \Gamma_\mu(TN|_{\Sigma}) \oplus \Gamma_\mu(TN|_{\Sigma})^{\perp},$$

(15)

where

$$\Gamma_\mu(TN|_{\Sigma})^{\perp} := \{ \text{grad}(p) + p \cdot \text{Tr} \Pi_\Sigma \in \Gamma(\text{Nor}_\Sigma) \mid p \in C^\infty(\Sigma, \mathbb{R}) \}.$$  

(16)

Moreover, by application of Stokes’ Theorem, one easily sees that (16) is an orthogonal decomposition (whence the notation “$\Gamma_\mu(TN|_{\Sigma})^{\perp,n}$”) with respect to the following weak scalar product:

$$\Gamma(TN|_{\Sigma}) \times \Gamma(TN|_{\Sigma}) \to \mathbb{R}, \ (X, Y) \mapsto \int_{\Sigma} g(X, Y) \cdot \mu^\Sigma.$$  

(17)

**Remark 1.5.**

(i) Proposition 1.4 also holds for $\Sigma = N$. In this case, (15) reduces to the well known Helmholtz-Hodge decomposition for vector fields (see for example [Arn98], p.341) :

$$\mathfrak{X}(N) = \mathfrak{X}_\mu(N) \oplus \text{grad}(C^\infty(N, \mathbb{R})),$$

(18)

where $\mathfrak{X}_\mu(N) := \{ X \in \mathfrak{X}(N) \mid \text{div}(X) = 0 \}$ is the space of divergence free vector fields on $N$ (here $\mu$ denotes the Riemannian volume form of $(N, g)$).
(ii) A direct consequence of the existence of a topological direct summand for \( \Gamma_\mu(TN|_\Sigma) \) in the tame Fréchet space \( \Gamma(TN|_\Sigma) \), is that the space \( \Gamma_\mu(TN|_\Sigma) \) is also a tame Fréchet space (see [Ham82]).

(iii) We shall denote by \( P_\Sigma : \Gamma(TN|_\Sigma) \to \Gamma_\mu(TN|_\Sigma) \) the continuous projection given by Proposition 1.4.

Let \( \text{Emb}(M,N)^x \) be the open subset of \( \text{Emb}(M,N) \) defined by

\[
\text{Emb}(M,N)^x := \left\{ f \in \text{Emb}(M,N) \mid (\text{Tr} \Pi_f)_x \neq 0 \text{ for all } x \in f(M) \right\}.
\]

The global version of Proposition 1.4 is:

**Theorem 1.6.** The space \( \text{Emb}_\mu(M,N)^x \) is a tame Fréchet submanifold of the Fréchet manifold \( \text{Emb}(M,N)^x \), and for \( f \in \text{Emb}_\mu(M,N)^x \), we have the following natural isomorphism

\[
T_f \text{Emb}_\mu(M,N)^x \cong \Gamma_\mu(f^*TN),
\]

where \( \Gamma_\mu(f^*TN) := \left\{ X \in \Gamma(f^*TN) \mid X \circ f^{-1} \in \Gamma_\mu(TN|_{f(M)}) \right\} \).

In order to show Theorem 1.6 let us recall the construction of the “standard” chart \((U_f, \varphi_f)\) of \( \text{Emb}(M,N) \) centered at a point \( f \in \text{Emb}(M,N) \). For this, we need

- a sufficiently small neighborhood \( \Theta_f \subseteq f^*TN \) of the zero section of \( f^*TN \) such that the map \( \Theta_f \to N \times N, \nu \in \Theta_f \cap T_xN \mapsto (x, \exp_x(\nu)) \) (here the exponential map \( \exp \) is taken with respect to the metric \( g \)) is a diffeomorphism onto an open subset of \( N \times N \).
- \( \varphi_f(U_f) := \{ X \in \Gamma(f^*TN) \mid X(M) \subseteq \Theta_f \} \).
- \( \varphi_f^{-1} : \varphi_f(U_f) \to U_f \) is defined for \( X \in \varphi_f(U_f) \) and \( x \in M \), by \( (\varphi_f^{-1}(X))(x) := \exp_{(2)}(x)X_x \). For brevity’s sake, we shall write \( f_X := \varphi_f^{-1}(X) \in \text{Emb}(M,N) \) (\( f_X \) can be seen as a “perturbation of the embedding \( f \) by the vector field \( X^\mu \)).

It is well known that \( \text{Emb}(M,N) \) endowed with these charts is a tame Fréchet manifold (see for example [Ham82] [KM97]), and it is clear, restricting the open sets \( \Theta_f \) if necessary, that we also get an atlas for \( \text{Emb}(M,N)^x \).

We will also need the following map

\[
\text{Emb}(M,N) \xrightarrow{\rho} C^\infty(M,\mathbb{R}), \ f \mapsto f^*\mu^f/\mu,
\]

i.e., for \( f \in \text{Emb}(M,N) \), \( \rho(f) \) is the unique function satisfying \( f^*\mu^f = \rho(f) \cdot \mu \) on \( M \). Observe that \( \rho(f) > 0 \).

Finally, for \( f \in \text{Emb}(M,N)^x \), we define

- \( P_f : \varphi_f(U_f) \to C^\infty(M,\mathbb{R}), \ X \mapsto (\rho \circ \varphi_f^{-1})(X) \) \( (P_f \) is nothing but the local expression of \( \rho \) in the chart \((U_f, \varphi_f)) \).
- \( Q_f : \varphi_f(U_f) \to \Gamma_\mu(TN|_{f(M)}) \oplus C^\infty(M,\mathbb{R}), \ X \mapsto (X, P_f(X) - 1) \), where we use Proposition 1.5 to write \( X = (X_\mu + \text{grad}(\rho) + \rho \cdot \text{Tr} \Pi(f(M)) \circ f \).

Observe that \( f_X = \varphi_f^{-1}(X) \in U_f \) is volume preserving if and only if \( P_f(X) \equiv 1 \).

Following Hamilton in [Ham82 Thm.2.5.3], if we prove that \( Q_f \) is a local diffeomorphism near the zero section, then it would be possible, using \( Q_f \circ \varphi_f \), to define splitting charts for \( \text{Emb}(M,N)^x \), and thus to prove that \( \text{Emb}_\mu(M,N)^x \) is a tame submanifold of \( \text{Emb}(M,N)^x \). We will do this with two lemmas, the main point being the use of the inverse function theorem of Nash-Moser.
The map $P_f : \varphi_f(U_f) \to C^\infty(M, \mathbb{R})$ is a smooth tame map, and its derivative $(P_f)_* X$ is a family of linear partial differential operators of degree $1$ in $Y$ with coefficients which are nonlinear partial differential operators of degree $1$ in $X$. Moreover,

$$(P_f)_* Y = \left[ \text{div}_f(Y^\top \circ f^{-1}) - g(Y^\top \circ f^{-1}, \text{Tr} \, P_f) \right] \circ f \cdot P_f(0),$$

where $Y \in \Gamma(fn^T N)$.

Proof. Let $(U, \phi = (x_1, \ldots, x_m))$ be a positively oriented chart for $M$ and $(V, \psi = (y_1, \ldots, y_n))$ a chart for $N$ such that $f_X(U) \subseteq V$ for $X$ sufficiently small.

For $x \in U$, a direct calculation shows that

$$(P_f(X))(x) = \frac{\det(g_{ij}(x)((f_x)_*(\partial_{x_i} \circ f_x)_*(\partial_{x_j} \circ f_x)))^{1/2}}{\det(g_{ij}(\partial_{x_i} \circ f_x, \partial_{x_j} \circ f_x))^{1/2}}.$$  

(23)

As $(f_X)_* \partial_{x_i} = \exp X_* \partial_{x_i}$, we see that (24) is a nonlinear differential operator of degree $1$ in $X \in \Gamma(f^* T N)$, and it is well known that a nonlinear differential operator is a smooth tame map (see [Ham82 Cor. 2.27]).

Let us now compute its derivative in local coordinates. For this purpose, let us take a smooth curve of sections $X_t$ in $\varphi_f(U_f)$. We shall denote $Y_t := \partial_t X_t$, and $f_t := f_X(X_t)$ the corresponding smooth curve of embeddings in $\text{Emb}(M, N)$ (in the following, we may forget the subscript “$t$”). After elementary differential calculus, one finds,

$$(P_f)_* X = \partial_t (P_f(X_t)) = \frac{d}{dt} \frac{\det(g_{ij}(x)((f_t)_*(\partial_{x_i} \circ f_t)_*(\partial_{x_j} \circ f_t)))^{1/2}}{\det(g_{ij}(\partial_{x_i} \circ f_t, \partial_{x_j} \circ f_t))^{1/2}}$$

(24)

where $A$ is the matrix whose entries are $A_{ij} := g_{ij}^{(M)} := g_{ij}((f_t)_*(\partial_{x_i} \circ f_t)_*(\partial_{x_j} \circ f_t))$. To carry out the calculation of $\partial_t A$ in local coordinates, we will also denote $g_{ij}^{\perp} := g(\partial_{y_i}, \partial_{y_j})$, $Z_t := \partial_t f_t \in \Gamma(f^* T N)$, and $\Gamma_{\alpha \beta}^k \in \mathcal{C}^\infty(V, \mathbb{R})$ the Christoffel symbols associated to the metric $g$ on $V$. Using Einstein summation convention and the formula $\partial_{\alpha}, g_{ab} = \Gamma_{\alpha \beta}^k g_{kb} + \Gamma_{\alpha \beta}^k g_{k a}$, it is then easy to see that

$$\partial_t A_{ij} = \Gamma_{\alpha \beta}^k \partial_{\alpha} f \cdot g_{kj}^{\perp} f \cdot Z^\alpha \cdot \partial_{x_i} f^a \cdot \partial_{x_j} f^b$$

$$+ \Gamma_{\alpha \beta}^k \partial_{\alpha} g_{\perp} \cdot g_{kj}^{\perp} f \cdot Z^\alpha \cdot \partial_{x_i} f^a \cdot \partial_{x_j} f^b$$

$$+ g_{\perp}^{kj} \partial_{x_i} Z^\alpha \cdot \partial_{x_j} f^b + g_{\perp}^{kj} \partial_{x_i} f^a \cdot \partial_{x_j} Z^\alpha.$$  

(25)

Since $Z_t = \partial_t f_t = \partial_t f_{X_t} = \exp X_t Y_t$ and $\partial_{x_i} f^a = \partial_{x_i} f_{X_t}$, and $X_t$ can be considered respectively as a partial differential operator of order $0$ (nonlinear) in $X$ and (linear) in $Y$, and a nonlinear partial differential operator of order $1$ in $X_t$, it follows easily, in view of (24) and (25), that $(P_f)_* X$ is a family of nonlinear partial differential operators of degree $1$ in $Y$ with coefficients which are nonlinear partial differential operators of degree $1$ in $X$.

Formula (23) can be obtained after direct calculations, splitting $Z$ into its tangential and normal parts $Z^\top, Z^\perp$, and using, among others, equation (24), $\partial_{\alpha}, g_{ab} = \Gamma_{\alpha \beta}^k g_{kb} + \Gamma_{\alpha \beta}^k g_{k a}$ as well as $g_{\perp}^{kj} f \cdot (Z^\alpha)^\perp \partial_{x_j} f^b = 0$. One finds

$$(P_f)_* Y = 1/2 \cdot P_f(X) \cdot \text{Tr} (A^{-1} \partial_t A) = P_f(X) \cdot \left[ (g_{ij}^{(M)} f \cdot g_{\perp}^{kj} f \cdot \partial_{x_i} f^a \cdot \partial_{x_j} f^b) (\partial_{x_i} f^b - \partial_{x_j} f^a \cdot \partial_{x_j} f^b \cdot \Gamma_{\alpha \beta}^k \partial_{\alpha} f) \right.$$  

$$- (g_{ij}^{(M)} f \cdot g_{\perp}^{kj} f \cdot (Z^\alpha)^\perp \partial_{x_j} f^b) \left( \Gamma_{\alpha \beta}^k \partial_{\alpha} f \right] \right.$$  

$$= (g_{\perp}^{kj} f \cdot (Z^\alpha)^\perp \partial_{x_j} f^b) \left( \Gamma_{\alpha \beta}^k \partial_{\alpha} f \right].$$  

(26)

(27)
One recognizes \[26\] as being $\text{div}(Z^\top \circ f^{-1}) \circ f$ and \[27\] to be the negative of $g_f(Z^\top \circ f^{-1}, \text{Tr} \Pi_f)$. Taking $t = 0$ and $X_0 = 0$, then $Z_0 = \exp_{x_0} Y = Y$ and formula \[22\] follows.

**Remark 1.8.**

(i) Throughout the last proof, we have actually proved that the “density map” $\rho : \text{Emb}(M, N) \rightarrow C^\infty(M, \mathbb{R})$, $f \mapsto f^*\mu / \mu$ is a smooth tame map, and that its derivative at a point $f \in \text{Emb}(M, N)$ in direction $X \in \Gamma(f^*TN)$, is

$$\rho_{*,X} Y = \left[\text{div}_f(X^\top \circ f^{-1}) - g(X^\top \circ f^{-1}, \text{Tr} \Pi_f)\right] \circ f \cdot \rho(f).$$

Equation \[28\] is a classical formula in differential geometry (see for example [Jos82, p.158] or [AM82]).

(ii) It may seem weird, in view of \[27\] where there are second order partial differentials, that $(P_f)_{*,X} Y$ is only a nonlinear partial differential operator of order 1 in $X$. This comes from the “artificial” splitting $Z = Z^\top + Z^\perp$ (recall that $Z = \exp_{x_0} Y$, see the proof above) which introduces a first order nonlinear partial differential operator in $X$, since projecting a tangent vector on the tangent space of $f_X(M)$ consumes the first derivatives of $f_X$.

From Lemma \[1.7\] it follows that $Q_f : \varphi_f(U_f) \rightarrow \Gamma_f(TN|_{f(M)}) \oplus C^\infty(M, \mathbb{R})$ is a smooth tame map, and one may try to invert it on a neighborhood of the zero section.

**Lemma 1.9.** For $f \in \text{Emb}_x(M, N)^\times$, the smooth tame map $Q_f : \varphi_f(U_f) \rightarrow \Gamma_f(TN|_{f(M)}) \oplus C^\infty(M, \mathbb{R})$ is invertible on an open neighborhood of the zero section, and its local inverse is also a smooth tame map.

**Proof.** The conditions required by the inverse function theorem of Nash-Moser are that $(Q_f)_{*,X}$ is invertible for all $X$ in a neighborhood of the zero section, and also that the family of inverses forms a smooth tame map (see [Ham82, Thm 1.1.1]).

So, let us take $X \in \varphi_f(U_f)$, $Y \in \Gamma(f^*TN)$, $Z_\mu \in \Gamma_M(TN|_{f(M)})$ and $p_Z \in C^\infty(M, \mathbb{R})$. Denoting $Y = (Y_\mu + \text{grad}(p) + p \cdot \text{Tr} \Pi_f) \circ f$, we have:

$$\begin{align*}
(Q_f)_{*,X} Y &= (Z_\mu, p_Z) \quad \Leftrightarrow \\
(Y_\mu, (P_f)_{*,X} Y) &= (Z_\mu, p_Z) \quad \Leftrightarrow \\
Y_\mu &= Z_\mu \quad \text{and} \quad (P_f)_{*,X} Y = p_Z \quad \Leftrightarrow \\
&\left\{ Y_\mu = Z_\mu, \quad (P_f)_{*,X} (\text{grad}(p) + p \cdot \text{Tr} \Pi_f) \circ f = p_Z - (P_f)_{*,X} (Z_\mu) \right\}.
\end{align*}$$

From the equivalence between equation \[29\] and equation \[30\], we see that $(Q_f)_{*,X}$ is invertible if and only if the operator $(P_f)_{*,X} (\text{grad}(p) + p \cdot \text{Tr} \Pi_f) \circ f$ acting on $p$ is invertible.

Now, according to Lemma \[1.7\] $(P_f)_{*,X} (\text{grad}(p) + p \cdot \text{Tr} \Pi_f) \circ f$ is a family of linear partial differential operators of degree 2 in $p$ with coefficients which are nonlinear partial differential operators of degree 1 in $X$. Moreover, as

$$(P_f)_{*,0}(\text{grad}(p) + p \cdot \text{Tr} \Pi_f) \circ f = (\Delta p - \|\text{Tr} \Pi_f\|^2 \cdot p) \circ f$$

is an elliptic operator in $p$ with analytical index zero (see the proof of Proposition \[1.4\]), it follows by the topological invariance of the analytical index that $(P_f)_{*,X} (\text{grad}(p) + p \cdot \text{Tr} \Pi_f) \circ f$ is a also elliptic in $p$ with analytical index zero for $X$ sufficiently small. This family is also a family of injective operators by the maximum principle (see [Amb98, p.96]). The maximum principle can be applied here, because $(P_f)_{*,0}(\text{grad}(1) + 1 \cdot \text{Tr} \Pi_f) \circ f = -\|\text{Tr} \Pi_f\|^2 \circ f$ is strictly negative (recall that we assume $\text{Tr} \Pi_f \not= 0$ for all $x \in M$), and thus, the term of order zero $(P_f)_{*,X} (\text{grad}(1) + 1 \cdot \text{Tr} \Pi_f) \circ f$ will remain strictly negative for small $X$. It follows that this family is actually a family of invertible elliptic operators, and one can apply [Ham82, Thm 3.3.1] to deduces that this family of inverses forms a smooth tame family of linear maps. The same conclusion being obviously true for the family of inverses $((Q_f)_{*,X})^{-1}$, the lemma follows. \[\square\]
2 Mechanics on the space of volume preserving embeddings

2.1 Euler-Lagrange equations on Emb$_\mu$(M, N)$^\times$

Let M be a compact, connected and oriented submanifold of a Riemannian manifold (N, g). We denote by $\mu$ the Riemannian volume form on M induced by the ambient metric g.

In this section, we consider Lagrangian mechanics on Emb$_\mu$(M, N)$^\times$ for Lagrangians of the following type:

$$\tilde{L}(X_f) := \int_M L \circ X_f \cdot \mu = \int_{f(M)} (L \circ X_f \circ f^{-1}) \cdot \mu^f,$$

where $L : T_N \rightarrow \mathbb{R}$ is a Lagrangian density and where $X_f \in T_f$Emb$_\mu$(M, N)$^\times$ $\cong \Gamma_M(f^*TN)$.

Observe that the last equality in (32) comes from a change of variables together with the formula $f^*\mu^f = \mu$.

In order to formulate the Euler-Lagrange equations on Emb$_\mu$(M, N)$^\times$, we have to introduce some terminology. Recall that the metric $g$ induces a connector $K : T(TN) \rightarrow TN$ (see [Lam02], chapter 10, page 284), and that for $v_x \in T_xN$, there is an isomorphism

$$T_{v_x}TN \xrightarrow{\cong} T_xN \oplus T_xN, \quad \xi \mapsto (\pi_{v_x}\xi, K\xi),$$

where $\pi : TN \rightarrow N$ is the canonical projection. Hence, for $\xi \in T_{v_x}TN$, we have the decomposition $\xi = \xi^h + \xi^v$ which is characterized by $K\xi^h = 0$ and $\pi_{v_x}\xi^v = 0$. This decomposition defines a splitting of the bundle $T(TN)$ into a direct sum $T(TN) = HN \oplus VN$, where $HN$ is the horizontal vector bundle and $VN$ the vertical vector bundle (see [Lam02]).

With this notation, for $v_x \in T_xN$, we have:

$$L_{sv_x}|(HN)_{v_x} \in (HN)_{v_x} \cong T^*_xN \cong T_xN.$$  \hspace{1cm} (34)

Thus, there exists $(\nabla^hL)_{v_x} \in T_xN$ such that

$$L_{sv_x}\xi^h = g((\nabla^hL)_{v_x}, \pi_{v_x}\xi^h),$$

for all $\xi^h \in (HN)_{v_x}$. Similarly, there exists $(\nabla^vL)_{v_x} \in T_xN$ such that

$$L_{sv_x}\xi^v = g((\nabla^vL)_{v_x}, K\xi^v),$$

for all $\xi^v \in (VN)_{v_x}$. In this way, we define two maps $\nabla^hL : TN \rightarrow TN$ and $\nabla^vL : TN \rightarrow TN$ which are smooth and fiber preserving. For practical calculations, let “↓”: $TN \rightarrow T^*N$ be the canonical isomorphism induced by the metric $g$ and “↑”: $T^*N \rightarrow TN$ its inverse. For $v_x \in T_xN$, it is not hard to see that:

- $(\nabla^vL)_{v_x}$
- $(\nabla^hL)_{v_x}$

The fact that Emb$_\mu$(M, N)$^\times$ is a submanifold of Emb(M, N)$^\times$ is now a simple consequence of Lemma 1.2 as we already remarked.
Example 2.1. If \( L := \frac{1}{2} g(\ldots) - V \circ \pi \), where \( V \) is a function on \( N \), then \((\nabla^v L)_{v_x} = v_x \) and \((\nabla^h L)_{v_x} = -(\text{grad}(V))_x \) for all \( v_x \in T_x N \).

Remark 2.2. Using the Legendre transform \( \mathbb{F}L : TN \to T^*N \) (see for example [AM78]), one observes that \((\nabla^v L)_{v_x} = (\mathbb{F}L(v_x))^\flat \).

Finally, let us introduce, for a given \( f \in \text{Emb}(M, N) \), the following operator:

\[
\mathbb{P}_f : \left\{ \begin{array}{l}
\Gamma(f^*TN) \to \Gamma(f^*TN), \\
X \mapsto \mathbb{P}_f(M)(X \circ f^{-1}) \circ f,
\end{array} \right.
\] (37)

(see Remark [AM78] (iii) for the definition of \( \mathbb{P}_f(M) \)).

Proposition 2.3. The Euler-Lagrange equations on \( \text{Emb}_\mu(M, N)^\times \) associated to a Lagrangian density \( L : TN \to \mathbb{R} \) are:

\[
\mathbb{P}_f[\nabla_{\partial_t} f(\nabla^v L)_{\partial_t f} - (\nabla^h L)_{\partial_t f}] = 0,
\] (38)

where \( f = f_t \) is a smooth curve in \( \text{Emb}_\mu(M, N)^\times \), and where \( \nabla \) is the Levi-Civita connection associated to \( g \).

Proof. Let \( f_t \) be a smooth curve in \( \text{Emb}_\mu(M, N)^\times \) and let \( \tilde{f}_s \) be a proper variation in \( \text{Emb}_\mu(M, N)^\times \) of the curve \( f_t \), i.e. a variation with fixed ends (see [AM78]). We have:

\[
\frac{d}{ds}\bigg|_0 \int_a^b L(\partial_t \tilde{f}) \, dt = \frac{d}{ds}\bigg|_0 \int_a^b \int_M (L \circ \partial_t \tilde{f}) \cdot \mu \, dt
\]
\[
= \int_a^b \int_M \partial_s |_0 (L \circ \partial_t \tilde{f}) \cdot \mu \, dt - \int_a^b \int_M (L \circ \partial_s |_0 \partial_t \tilde{f}) \cdot \mu \, dt
\]
\[
= \int_a^b \int_M \left[ g_f((\nabla^h L)_{\partial_t f}, \partial_s |_0 \tilde{f}) + g_f((\nabla^v L)_{\partial_t f}, K \partial_s |_0 \partial_t \tilde{f}) \right] \mu \, dt. \quad (39)
\]

As \( K \partial_s |_0 \partial_t \tilde{f} = \nabla_{\partial_t f} \partial_s |_0 \tilde{f} \), the second term in (38) can be rewritten

\[
\int_a^b \int_M g_f((\nabla^v L)_{\partial_t f}, \nabla_{\partial_t f} \partial_s |_0 \tilde{f}) \cdot \mu \, dt
\]
\[
= \int_a^b \int_M \left[ \partial_t g_f((\nabla^v L)_{\partial_t f}, \partial_s |_0 \tilde{f}) - g_f((\nabla_{\partial_t f} (\nabla^v L)_{\partial_t f}, \partial_s |_0 \tilde{f}) \right] \mu \, dt
\]
\[
= - \int_a^b \int_M g_f(\nabla_{\partial_t f} (\nabla^v L)_{\partial_t f}, \partial_s |_0 \tilde{f}) \cdot \mu \, dt. \quad (40)
\]

The proposition follows from (39), (40) and the fact that (11) is an orthogonal decomposition.

Remark 2.4.

(i) If the submanifold \( M \subseteq N \) is a point, then (39) reduces to a coordinate-free formulation of the classical Euler-Lagrange equations on \( N \):

\[
\nabla_{\dot{a}(t)} (\nabla^v L)_{\dot{a}(t)} - (\nabla^h L)_{\dot{a}(t)} = 0, \quad (41)
\]

where \( \alpha \) is a smooth curve in \( N \). Equation (41) is a particular case of a more general free-coordinate formulation of the Euler-Lagrange equations using connections (see [GSS03]).
(ii) According to the above remark, the Euler-Lagrange equations (38) on $\text{Emb}_\mu(M,N)^\times$ are simply the “pointwise” classical Euler-Lagrange equations twisted by the “Helmholtz-Hodge projector” $\mathbb{P}_f$.

An alternative description of the Euler-Lagrange equations on $\text{Emb}_\mu(M,N)^\times$, which is straightforward and maybe more explicit than the one given in Proposition 2.3 is as follows.

**Proposition 2.5.** The Euler-Lagrange equations on $\text{Emb}_\mu(M,N)^\times$ associated to a Lagrangian density $L: TN \to \mathbb{R}$ are:

\[
\nabla_{\partial_t f}(\nabla^* L)_{\partial_t f} - (\nabla^h L)_{\partial_t f} = \text{grad}(p \circ f^{-1}) + (p \circ f^{-1}) \cdot \text{Tr} \Pi_f \\
\text{div}_f(\partial_t f^\top) = g(\partial_t f^\perp, \text{Tr} \Pi_f),
\]

where $f = f_t$ is a smooth curve in $\text{Emb}_\mu(M,N)^\times$, $p = p_t : M \to \mathbb{R}$ is a time-dependent function, $\text{grad}(p \circ f^{-1})$ is the Riemannian gradient of $p \circ f^{-1}$ taken with respect to the induced metric on $f(M)$ and where $\nabla$ is the Levi-Civita connection associated to $g$.

**Remark 2.6.** The “pressure term” $p$ in (42) is uniquely determined by the equation

\[
\Delta (p \circ f^{-1}) - (p \circ f^{-1}) \cdot \|\text{Tr} \Pi_f\|^2 = \text{div}_f \left( [\nabla_{\partial_t f}(\nabla^* L)_{\partial_t f} - (\nabla^h L)_{\partial_t f}]^\top \right) \\
- g \left( [\nabla_{\partial_t f}(\nabla^* L)_{\partial_t f} - (\nabla^h L)_{\partial_t f}]^\perp, \text{Tr} \Pi_f \right),
\]

where $\Delta$ is the Laplacian operator on $f(M)$ for the induced metric (see the proof of Proposition 1.4).

### 2.2 Application: generalization of the Euler equations

By “Euler equations”, we are referring to the equations of an incompressible fluid on an oriented Riemannian manifold $(M,g)$ with Riemannian volume form $\mu$:

\[
\partial_t X_t + \nabla_{X_t} X_t = \text{grad}(p_t) \\
\text{div}_\mu(X_t) = 0,
\]

where $X_t$ is a time-dependent vector field describing the velocity of the fluid, $p_t : M \to \mathbb{R}$ is the pressure of the fluid, $\nabla_{X_t} X_t$ is the Riemannian covariant derivative of $X_t$ in direction $X_t$ and where $\text{grad}(p_t)$ is the Riemannian gradient of $p_t$. The condition $\text{div}_\mu(X_t) = 0$ guarantees that the fluid is incompressible.

It is known, since Arnold’s paper [Arn66], that the above equations can be interpreted as geodesic equations on the Fréchet Lie group $\text{SDiff}_\mu(M)$ for the right invariant $L^2$-metric which is defined, at the identity diffeomorphism, as $\langle X, Y \rangle := \int_M g(X, Y) \cdot \mu$, where $X, Y \in \mathfrak{X}(M) := \{ Z \in \mathfrak{X}(M) \mid \text{div}_\mu(Z) = 0 \}$. Observe that the latter space is identified with the Lie algebra of $\text{SDiff}_\mu(M)$.

Briefly, the fact that the Euler equations (44) are equivalent to the geodesic equations of $\text{SDiff}_\mu(M)$ comes from the right-invariance of the metric $\langle \cdot, \cdot \rangle$, together with the following general fact: geodesic equations on a Lie group for a right (or left) invariant metric are equivalent to an evolution equation on the Lie algebra called *Euler equation* [Arn66], which, in the particular case $\text{SDiff}_\mu(M)$ yields the Euler equations of an incompressible fluid; this is Arnold’s remarkable observation (see [Arn66, AK95, EM70]).

For us, the important point is that the Euler equations (44) are equivalent to the geodesic equations for the metric $\langle \cdot, \cdot \rangle$, and that this metric can be naturally generalized to $\text{Emb}_\mu(M,N)^\times$, as follows:

\[
\langle X_f, Y_f \rangle = \int_M g(X_f, Y_f) \cdot \mu,
\]

Their are several formulations of the Euler equation. One of them is as follows. If $G$ is a Lie group with Lie algebra $\mathfrak{g}$, endowed with a right or left invariant metric $\langle \cdot, \cdot \rangle$, then its associated Euler equation is $\dot{a}(t) = \text{ad}^*(\alpha(t)) (\alpha(a(t)))$, where $\text{ad}^*$ is the coadjoint representation of $G$, $\alpha(t)$ is a smooth curve in $\mathfrak{g}^*$ and where $\alpha(t)$ is the unique curve in $\mathfrak{g}$ which satisfies $\alpha(0)(\xi) = \langle \alpha(t), \xi \rangle$ for all $\xi \in \mathfrak{g}$. The sign in front of $\text{ad}^*$ depends on convention and whether the metric is right or left invariant. One can show that the Euler equation is equivalent to the geodesic equations for the metric $\langle \cdot, \cdot \rangle$, see [AK95].
where \( f \in \text{Emb}_\mu (M, N)^\times \) and where \( X^f_t, Y^f_t \in T^f \text{Emb}_\mu (M, N)^\times \cong \Gamma_\mu (f^*TN) \).

It is well known in the context of Riemannian geometry that geodesics are solutions of the Euler-Lagrange equations associated to the energy, which in our case reads

\[
T\text{Emb}_\mu (M, N)^\times \to \mathbb{R}, \quad X^f_t \mapsto \frac{1}{2} \int_M g(X^f_t, X^f_t) \cdot \mu.
\]  

(46)

According to Proposition 2.5 and taking into account Example 2.1, we thus get

**Proposition 2.7.** The geodesic equations on \( \text{Emb}_\mu (M, N)^\times \) for the metric \( \langle \cdot , \cdot \rangle \) defined in (45) are

\[
\nabla_{\partial_t f} \partial_t f = \text{grad}(p \circ f^{-1}) + (p \circ f^{-1}) \cdot \text{Tr} \Pi f, \\
\text{div}_f (\partial_t f^{-1}) = g(\partial_t f^\perp, \text{Tr} \Pi f),
\]

(47)

where \( f = f_t \) is a smooth curve in \( \text{Emb}_\mu (M, N)^\times \), and where \( p = p_t : M \to \mathbb{R} \) is a time-dependant function.

In the special case \( M = N \), then \( \text{Emb}_\mu (M, N) = \text{SDiff}_\mu (M) \) and the term \( \text{Tr} \Pi f_t \) in (47) vanishes. Moreover, If \( f = f_t \) is a smooth curve in \( \text{Diff}(M) \) and if \( X_t := (\partial_t f_t) \circ f_t^{-1} \), then one has the formula

\[
\nabla_{\partial_t f} \partial_t f = (\partial_t X_t + \nabla_{X_t} X_t) \circ f,
\]

(48)

from which, together with \( \text{Tr} \Pi f_t \equiv 0 \), one easily sees that (47) reduces to the usual Euler equations of an incompressible fluid (14). Hence, (47) is indeed a generalization of the Euler equations; we call it the **Euler equations of an incompressible membrane**.

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