A method for research on behaviour of a di-mesoatomic wave function at small distances

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Abstract

The Schrödinger equation describing the local potential model of a strong $\pi^+\pi^-$-interaction was studied. The influence of the strong $\pi^+\pi^-$-interaction on the behaviour of pionium $nS$-state wave functions at small distances is studied both analytically (perturbatively) and numerically. It is shown that in the whole the accounting of strong interaction results in multiplying pure Coulomb pionium wave functions by some function (2) independent of the value of the principal quantum number $n$. Due to this reason, the $n$-independence of probability of $\pi^+\pi^-$-atom production in $nS$-state remains the same as in the case of a pure Coulomb $\pi^+\pi^-$-interaction.

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1. Introduction

The method for the measurement of the pionium ($\pi^+\pi^-$-atom) lifetime proposed in ref. [1], is essentially based on the assumption that the $n$-dependence of probability $w_n$ on creation of $\pi^+\pi^-$-atom in $nS$-state is well known or, at least, may be calculated with a high degree accuracy. The first consideration of this problem has been done in L. Nemenov’s paper [2], where the following relation has been derived from a more general result of the author [2]

\[ w_n \sim n^{-3} \]  \hspace{1cm} (1)

has been derived from a more general result of the author [2]

\[ w_n \sim \left| \int M(\vec{r})\psi_n(\vec{r})d^3r \right|^2 , \] \hspace{1cm} (2)

\[ M(\vec{p}) = \frac{1}{(2\pi)^3} \int M(\vec{p})e^{-i\vec{p}\vec{r}}d^3p , \] \hspace{1cm} (3)

where $M(p)$ is the amplitude of production of free $\pi^+\pi^-$-pairs with relative momentum $\vec{p}$ in hadron – hadron or hadron – nucleus collisions and $\psi_n(r)$ is the wave function of $nS$-state of pionium.

In line with accounting the short range nature of amplitude $M(r)$ in his original derivation of (1) from (2) L. Nemenov also has made an assumption that the pure Coulomb wave functions describe quite well a distribution of pions in the pionium not only at large distances, but at small ones also. However, as it has been shown recently by E. Kuraev, this assumption is unjustified due to some noticeable influence of strong a $\pi^+\pi^-$-interaction on the behaviour of the pionium wave functions in the nearest of origin. Due to this reason a more careful analysis of this problem is needed.

Below we shall represent some preliminary results of the analysis based on the local potential model of the strong $\pi^+\pi^-$-interaction. In this model the "reduced" pionium wave functions

\[ \Phi_n(r) = \sqrt{4\pi}\psi_n(r), \quad \int |\Phi_n(r)|^2dr = 1 , \] \hspace{1cm} (4)

obey the Schrödinger equation

\[ \Phi''_n(r) + m[U_c(r) + U_s(r)]\Phi_n(r) = m\varepsilon_n\Phi_n(r) , \] \hspace{1cm} (5)

where \( m \) is the pion mass, $\varepsilon_n$ — binding energy, $U_c = \alpha/r$, $U_s$ — Coulomb and strong potentials, respectively.
2. The scheme and results of perturbative analysis

First of all, let us apply the methods of the perturbation theory to the Schrödinger equation (5), treating the strong interaction potential \( U_s \) as perturbation, in order to obtain some qualitative estimations. Putting

\[
\Phi_n(r) \approx \Phi_n^{(0)}(r) + \Phi_n^{(1)}(r),
\]

\[
\varepsilon_n = \varepsilon_n^{(0)} + \varepsilon_n^{(1)}, \quad \text{where}
\]

\[
\varepsilon_n^{(0)} = \frac{m\alpha^2}{4n^2}, \quad \varepsilon_n^{(1)} = \int_0^\infty U_s(r)\Phi_n^{(0)2}(r)dr;
\]

\[
\Phi_n^{(0)\nu}(r) + m[U_c(r) - \varepsilon_n^{(0)}]\Phi_n^{(0)}(r) = 0,
\]

\[
\Phi_n^{(1)\nu}(r) + m[U_c(r)\varepsilon_n^{(1)}]\Phi_n^{(1)}(r) = m[\varepsilon_n^{(1)} - U_s(r)]\Phi_n^{(0)}(r);
\]

and applying the general methods of solving the linear inhomogeneous equations \( \Phi_n^{(0)} \) one can obtain

\[
\Phi_n^{(1)} = \Phi_n^{(0)}[c_n - \chi_n(r)], \quad \text{where}
\]

\[
\chi_n(r) = \int_0^r \frac{dr_1}{|\Phi_n^{(0)}(r_1)|^2} \int_0^{r_1} |\Phi_n^{(0)}(r_1)|^2 \times [-\varepsilon_n^{(1)} + U_s(r_2)]dr_2 \quad \text{and}
\]

\[
c_n = \int_0^\infty dr |\Phi_n^{(0)}(r)|^2 \chi_n(r).
\]

If we define the ratio

\[
R_n(r) = \frac{\Phi_n(r)}{\Phi_n^{(0)}(r)} = \frac{\psi_n(r)}{\psi_n^{(e)}(r)}
\]

as a measure of the influence of strong interactions on the values of the pionium wave functions, then in the first order of the perturbation theory

\[
R_n(0) \approx 1 + c_n.
\]

With explicit expressions for the pure Coulomb wave functions \( \Phi_n^{(0)} \) we have proceeded in the calculation of \( c_n \) with \( n = 1, 2, 3 \).

The result looks like as follows:

\[
c_n = \int_0^\infty mU_s(r)dr + \sum_{n=1} d_k^{(n)} \times
\]

\[
\times \int_0^\infty mU_s(r)\left(\frac{r}{r_B}\right)^k \ln\left(\frac{r}{r_B}\right)dr,
\]

where \( d_k^{(n)} \sim 1, \epsilon_k^{(n)} \sim r_B \approx 400 \, fm \).

Taking into account the relation

\[
\int_0^\infty mU_s(r)^2dr \approx a \approx 0.15 \, fm,
\]

where \( a \) is the \( \pi^+\pi^- \) scattering length, it is easy to see that the \( n \)-dependent contributions to \( c_n \) are numerically small (of order \( 10^{-3} \)) and may be neglected.

Putting \( U_s = g/r \exp(-br) \) with the values of parameters \( g \approx 3, b = m_\rho \approx 3.8 \, fm^{-1}, \) that corresponds to applying the \( \rho \)-exchange model for a description of the strong \( \pi^+\pi^- \)-interaction, one can obtain for \( n \)-independent part of \( c_n \) (see also \( 3 \))

\[
\int_0^\infty mU_s(r)^2dr = \frac{gm}{m_\rho} \approx 0.55.
\]

These estimations show that the strong \( \pi^+\pi^- \)-interaction can change noticeably the value of the pionium wave functions at origin and this effect cannot be ignored at evaluating the probabilities \( w_n \) (2). On the other hand, the large correction to the values of the wave functions, obtained in the first order of perturbation theory, means that the higher order corrections are not small and must be taken into account. Their calculation does not look a simple problem.

3. Numerical investigation

Due to this reason we have applied numerical methods for an accurate investigation of the pionium wave functions behaviour at small distances, using the following equation with boundary conditions and normalization:

\[
\Phi_1 = \frac{d^2\chi}{d\rho^2} + U(\rho)\chi(\rho) - \varepsilon\chi(\rho) = 0,
\]

\[
\Phi_2 = \chi(0) = 0, \quad \Phi_3 = \chi(\infty) = 0,
\]

\[
\Phi_4 = \int_0^\infty \chi^2(\rho)d\rho - 1 = 0, \quad \text{where}
\]
So we have an operator equation

\[ \Phi(z) = 0 \]

where

\[ \Phi(z) = \begin{pmatrix} \Phi_1(z) \\ \Phi_2(z) \\ \Phi_3(z) \\ \Phi_4(z) \end{pmatrix}, \quad z = (\chi(\rho), \varepsilon) \quad (24) \]

In order to calculate the normalized pionium wave functions, we have used an improved version of code \cite{4}, based on applying the continuous analogy of Newton method, developed in ref. \cite{5}. In this way we change the primary equation by the following differential equation with parameter \( t \)

\[ \frac{d}{dt} \Phi(z(t)) = -\Phi(z(t)), \quad \Phi(0) = \Phi_0, \quad 0 < t < \infty \quad (25) \]

When \( t \to \infty \), the solution of this equation tends to the solution of primary equation (23) \( z(t) \to z \).

For numerical calculations the grid on \( t \) was used \( \{ t_i, i = 1, 2, \ldots; t_0; t_{i+1} - t_i = \tau \} \). For every \( t_i \) we obtained the solution \( dz(t = t_i)/dt \) to equation (25)

\[ \frac{dz(t_i)}{dt} = -\phi'(z(t_i))^{-1} \Phi(z(t_i)) \quad (27) \]

where \( \phi' \) is Frechet derivative of function \( \Phi \), \( \phi'^{-1} \) is an inverse operator.

For approximation of \( dz(t_i)/dt \) we use a finite difference

\[ dz(t_i) \approx \Delta z_i = \tau_i^{-1} [z(t_{i+1}) - z(t_i)] \quad (28) \]

Thus,

\[ z(t_{i+1}) \approx \tau_i \Delta z_i + z(t_i) \quad (29) \]

If \( z(t_i) \) is known, then as a result of this iterative procedure with parameter \( \tau_i \) we obtained a sequence of the approximate values of the solution to equation (25) \( \{ z_i \} \to z \). As a starting approximation \( z_0 \) we choose an analytical solution of the corresponding Coulomb problem. The iterative procedure was continued until residual || \( \Phi(z(t)) || < 10^{-5} \).

The problem (27) was solved numerically in the interval \((a, b)\), where \( a = 10^{-8}, b \) is a sufficiently large number, depending on the principal quantum number \( n \). The solution at the point \( b \) was adjusted logarithmically to the appropriate Coulomb eigenfunction

\[ \frac{\chi'(b)}{\chi(b)} = \frac{\chi^{(c)'}(b)}{\chi^{(c)}(b)} \quad (30) \]

To calculate, we used a nonuniform grid on \( \rho \) as follows: in the interval \((a, 0.02) \) — 301 nodes, in the interval \((0.02, 2) \) — 301 nodes and in the interval \((2, b) \) — 200 nodes.

The accuracy of the method was tested on Coulomb problem. Fig. 1 shows the numerically obtained solution of Coulomb problem for \( n = 1 \) for potential \( U_c = 2/\rho \). Fig. 2a gives the corresponding value of the wave function \( \psi^{(c)}(\rho) \). The same functions for \( n = 2, 3, 4 \) are given in Figs. 2b-2d. The difference between the numerical and analytical values of function \( \psi^{(c)}(\rho), n = 1, 2, 3, 4 \), is represented in Figs. 3a-3d.
Fig. 2. Results of numerical calculations of Coulomb pionium $nS$-state wave functions $\psi_n^{(c)}(\rho)$ for $n = 1$ (a), $n = 2$ (b), $n = 3$ (c), $n = 4$ (d).

Fig. 3. Difference between numerical and analytical values of functions $\psi_n^{(c)}(\rho)$ for $n = 1$ (a), $n = 2$ (b), $n = 3$ (c), $n = 4$ (d).
The input of parameters of the code has been chosen in this way to guarantee an absolute accuracy of the calculations higher than $10^{-4}$. To check up this accuracy, we have compared the numerical solution of Shrödinger equation with the pure Coulomb interaction with the analytical one.

At the next step we solved our problem for the sum of strong and Coulomb potentials as follows:

$$U(\rho) = \frac{2}{\rho} \left(1 - e^{-b\rho}\right) + \frac{a}{\rho} e^{-b\rho}, \quad \text{where} \quad (31)$$

$$b = \frac{m_\rho}{\alpha \mu} \approx 1.5 \cdot 10^3, \quad a = \frac{2\alpha_{\rho\pi\pi}}{\alpha} \approx 8 \cdot 10^2, \quad (32)$$

$$\alpha_{\rho\pi\pi} = \frac{g_{\rho\pi\pi}^2}{4\pi} \approx 3.$$  

The results of the numerical calculations of the wave function $\psi(\rho)$ for $n = 1$ are represented in Fig. 4. At a long interval of $\rho$ we cannot see a difference between it and the corresponding wave function of Coulomb problem, but when $\rho$ is compatible with Fermi radius, then this difference is significant. This fact is shown for $n = 1$ in Fig. 5a. The difference also takes place for $n = 2, 3, 4$ (see Figs. 5b-5d).
The most important conclusion from our numerical calculations is as follows: the functions
\[ R_n(\rho) = \frac{\psi_n(\rho)}{\psi_n^{(c)}(\rho)} \]  
are practically independent of \( n \) \((n = 1, 2, 3, 4)\). This is illustrated by Fig. 6.

Fig. 7a shows the differences between functions \( R_n(\rho) \) for the considered \( n \). One can see from the plots that the differences between functions \( R_n \) for \( n = 1 - 4 \) are of the order less or equal \( 10^{-3} \). Thus, we can replace functions \( R_n(\rho), (n = 1 - 4) \) with accuracy \( 10^{-3} \) with \( R_1(\rho) \)
\[ R_n(\rho) = R_1(\rho) + O(10^{-3}) \, . \]  

The same results were obtained for potential
\[ \tilde{U}(\rho) = \frac{2}{\rho} (1 - e^{-b\rho}) + \tilde{a} e^{-b\rho}, \]  
where
\[ b = \frac{m_\rho}{\alpha \mu} \approx 1.5 \cdot 10^3, \quad \tilde{a} = \frac{1}{2} a \cdot b \approx 6 \cdot 10^5. \]  

As one can see from Fig. 7b, the estimation (34) is valid for potential (35).

So, we can make the following conclusions from our calculations: for the considered potentials (31) and (35) for the principal quantum numbers \( n = 1, 2, 3, 4 \) the following estimation is valid
\[ \frac{\psi_n(\rho)}{\psi_n^{(c)}(\rho)} = \frac{\psi_1(\rho)}{\psi_1^{(c)}(\rho)} + O_n(10^{-3}) \, , \]  
i.e. with accuracy \( 10^{-3} \) function \( \psi_n(\rho) \) can be represented as
\[ \psi_n(\rho) \approx \alpha(\rho) \psi_n^{(c)}(\rho) \, , \]  
where \( \alpha(\rho) \) is independent of \( n, \rho \) is compatible with Bohr radius.

4. Conclusion

Thus, the results of the numerical solution to equation (5) with Yukawa-type strong potential have confirmed the main conclusions of the perturbative consideration, namely: the ratios
\[ R_n(\rho) = \frac{\psi_n(\rho)}{\psi_n^{(c)}(\rho)} \equiv R_n(\rho) \]  
being numerically large (see Fig. 6) in the region \( r \leq r_s \sim 1 \, fm \) and essential for the problem under consideration \((n\)-dependence of values \( w_n \)), are practically \( n \)-independent (their \( n \)-independence is illustrated by Fig. 7).
This means that with a high degree accuracy one can substitute
\[ \psi_n(r) = R(r)\psi_n^{(c)}(r) \] (40)
in eq. (2) and, replacing
\[ M(\vec{r}) \Rightarrow \tilde{M}(\vec{r}) = M(\vec{r})R(r), \]
obtain
\[ w_n \sim \left| \int \tilde{M}(\vec{r})\psi_n^{(c)} \, dr \right|^2 \sim \] (41)
\[ \sim \left| \psi_n^{(c)}(0) \right|^2 \int \tilde{M}(\vec{r}) \, dr \sim n^{-3}. \]

Therefore, we can conclude that strong interaction corrections to the pionium wave functions at small distances, being sufficiently large, do not change the \( n^{-3} \)-law (1) primarily derived in paper \(^3\) assuming that the pionium wave functions are pure Coulomb.

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