0-CYCLES ON SINGULAR SCHEMES AND CLASS FIELD THEORY

AMALENDU KRISHNA

Abstract. We show that the Chow group of 0-cycles on a singular projective scheme $X$ over a finite field $k$ describes the abelian extensions of its function field which are unramified over $X_{sm}$. As a consequence, we obtain the Bloch-Quillen formula for the Chow group of 0-cycles on such schemes. We deduce simple proofs of results of Kerz-Saito for a class of surfaces without any assumption on char$(k)$.

1. Introduction

The aim of the class field theory in the geometric case is to describe abelian extensions of a finitely generated field extension of a finite field in terms of motivic invariants. One knows that a certain class of the Galois extensions of the function field of a normal variety can be described in terms of the finite étale covers of the same normal variety. Hence the problem of class field theory for normal varieties reduces to describing the abelianized étale fundamental group of a normal variety in terms of its motivic invariants.

Let $k$ be a finite field of order $q = p^a$ for a prime number $p$. Let $X$ be a geometrically connected quasi-projective scheme over $k$ and let $\overline{X}$ denote its base change to a fixed algebraic closure $\overline{k}$. Then there is a short exact sequence of profinite groups:

$$0 \to \pi^0_1(X) \to \pi^0_1(\overline{X}) \to \hat{\mathbb{Z}} \to 0.$$ (1.1)

Let $X$ be a smooth projective connected scheme over $k$. Let $CH_0(X)$ denote the Chow group of 0-cycles and let $A_0(X)$ denote the kernel of the degree map $\deg_X : CH_0(X) \to \mathbb{Z}$.

The following is the main theorem of the geometric case of the class field theory for smooth projective schemes. The $d = 1$ case was earlier proven by Artin.

Theorem 1.1. (12, Theorem 1) Let $X$ be a smooth projective geometrically integral scheme over $k$. Then the map $\theta^0_X : A_0(X) \to \pi^0_1(X)$ is an isomorphism of finite groups.

1.1. Main results. Let us now assume that $X$ is a geometrically integral projective scheme over $k$ which is not necessarily smooth. Let $CH_0(X)$ denote the Chow group of 0-cycles in the sense of Levine-Weibel [20] (see below). Let $A_0(X)$ denote the kernel of the degree map $\deg_X : CH_0(X) \to \mathbb{Z}$. We shall say that $X$ has only isolated singularities if the singular locus $X_{sing}$ of $X$ is finite. Let $j : X_{reg} \hookrightarrow X$ denote the inclusion of the regular locus of $X$. We prove the following result in this note.

Theorem 1.2. Let $X$ be a geometrically integral projective scheme over $k$ of dimension $d \geq 2$ which is regular in codimension one. Then there exists a reciprocity map

$$\theta_X : CH_0(X) \to \pi^0_1(X_{reg})$$

which restricts to a map $\theta^0_X : A_0(X) \to \pi^0_1(X_{reg})$. The map $\theta_X$ is injective with dense image and $\theta^0_X$ is an isomorphism of finite groups, if $X$ has only isolated singularities.

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The above result extends Theorem 1.1 to schemes with isolated singularities and improves the main result of [27], where the tame quotient of $\pi^\text{ab}_1(X_{\text{reg}})$ was described using a quotient of $A_0(X)$. We shall show using an example that it is necessary to assume regularity in codimension one in order to construct the reciprocity map $\theta^0_X$ and prove its isomorphism. As an immediate consequence of (1.1) and Theorem 1.2, we get the following.

**Corollary 1.3.** Let $X$ be a geometrically integral projective scheme of dimension $d \geq 2$ over a finite field. Assume that $X$ has only isolated singularities. Assume further that $X_{\text{reg}}$ is simply connected. Then $\text{CH}_0(X) \cong \mathbb{Z}$.

Let $\text{Alb}_X$ denote the Albanese variety of $X$ (see [19, Chapter 2, §3]). As another consequence of Theorem 1.2, one obtains the following.

**Corollary 1.4.** Let $X$ be a geometrically integral projective scheme of dimension $d \geq 1$ over a finite field. Assume that $X$ has only isolated singularities. Then $A\!J_X : A_0(X) \rightarrow \text{Alb}_X(k)$ is a surjective map of finite groups.

1.2. **Results of Kerz-Saito.** As further application of Theorem 1.2, we obtain simple proofs of the main results (Theorems II and III) of [14] for the following class of smooth surfaces without any assumption on the characteristic of $k$.

Let $X$ be a normal projective scheme over $k$ and let $U \hookrightarrow X$ be an open subset which is smooth over $k$. Given an effective Cartier divisor $D \subseteq X$ with $|D| \subseteq X \setminus U$, let $\text{CH}_0(X, D)$ denote the Chow group of 0-cycles on $X$ with modulus $D$ in the sense of [14]. Let $C(U) = \lim \leftarrow D \, \text{CH}_0(X, D)$, where the limit is taken over all effective Cartier divisors which are disjoint from $U$. It is known that $C(U)$ depends only on $U$ and not on the choice of its compactification $X$ (see [14, Lemma 3.1]). Let $C_0(U)$ denote the kernel of the degree map $C(U) \rightarrow \mathbb{Z}$.

**Theorem 1.5.** Let $U$ be the regular locus of a normal projective surface over $k$. Then there is a reciprocity map $C(U) \rightarrow \pi^\text{ab}(U)$ which induces an isomorphism of finite groups $\rho_U : C_0(U) \cong \pi^\text{ab}_0(U)$.

This result was proven (without finiteness assertion) in [14] for any smooth surface $U$ over $k$. However, the proof given there has a high level of complexity and works under the assumption that $\text{char}(k) \neq 2$.

1.3. **Bloch-Quillen formula.** As another byproduct of the class field theory for singular projective schemes, we obtain the following Bloch-Quillen formula for the Chow group of 0-cycles on projective schemes over a finite field which have only isolated singularities.

**Theorem 1.6.** Let $X$ be a geometrically integral projective scheme of dimension $d$ over a finite field. Assume that $X$ has only isolated singularities. Then there are canonical isomorphisms

$$\text{CH}_0(X) \cong H^d(X, K^M_{d,X}) \cong H^d(X, K_{d,X}).$$

We remark here that the Bloch-Quillen type formula for the Chow group of 0-cycles in known to be false if we allow non-isolated singularities in dimension three or more (see [28, §3.2]).

1.4. **Outline.** Our strategy for proving Theorem 1.2 is to first prove it for surfaces. This case requires us to use a result of Kato and Saito [13] which describes the class field theory of regular open subsets of projective schemes in terms of a generalized ideal class group. The general case is deduced from surfaces using induction on dimension with the
aid of Lefschetz type theorems for fundamental groups and Bertini type theorems over finite fields. Theorem 1.5 is proven by combining Theorem 1.2 with some results of [17] and cycle class maps for Chow groups with modulus. We prove Theorems 1.6 using a combination of Theorem 1.2 and main results of [13].

2. Chow group of 0-cycles on singular schemes

In this section, we recall the definition of the Chow group of 0-cycles for singular schemes from [20]. Specializing to the schemes with isolated singularities, we show that there exist canonical maps from this Chow group to the top Zariski cohomologies of the Milnor and the Quillen $K$-sheaves.

2.1. Chow group of 0-cycles. We first define the Chow group of 0-cycles for curves. A curve will mean an equi-dimensional quasi-projective scheme over $k$ of dimension one.

Let $C$ be a reduced curve and let $D \subseteq C$ be a closed subscheme such that $C_{\text{sing}} \subseteq D$ and $D$ contains no irreducible component of $C$. Let $Z_0(C, D)$ denote the free abelian group on closed points in $C \setminus D$. Let $\{C_1, \ldots, C_r\}$ denote the set of irreducible components of $C$ with generic points $\{\eta_1, \ldots, \eta_r\}$, respectively. Let $k(C)$ denote the ring of total quotients on $C$. Since $C$ is a reduced curve, it is Cohen-Macaulay. This implies in particular that the map $k(C) \to \prod_{i=1}^r O_{C, \eta_i}$ is an isomorphism and hence the map $\iota_{(C,D)} : O_{C,D}^\times \to \prod_{i=1}^r O_{C, \eta_i}$ is injective.

Given $f \in O_{C,D}^\times$, let $(f)_C$ denote the divisor of zeros and poles of $f$ on $C$ in the sense of [7]. Since $C_{\text{sing}} \subseteq D$, it is clear from the above definitions that $(f)_C = \sum_{i=1}^r (f_i)_C$. Let $R_0(C, D)$ denote the subgroup of $Z_0(C, D)$ generated by the set $\{(f)_{C}|f \in O_{C,D}^\times\}$. The group $R_0(C, D)$ is called the group of rational equivalences on $C$ relative to $D$. The Chow group of 0-cycles on $(C, D)$ is defined by

$$\text{CH}_0(C, D) := \frac{Z_0(C, D)}{R_0(C, D)}.$$

Proposition 2.1. ([20] Proposition 1.4) Let $(C, D)$ be as above. Then there is a cycle class map $\text{cycc} : \text{CH}_0(C, D) \to K_0(C)$ which induces an isomorphism $\text{cycc} : \text{CH}_0(C, D) \cong \text{Pic}(C)$. In particular, $\text{CH}_0(C, D)$ is independent of $D$.

Let $X$ now be a reduced and connected equi-dimensional quasi-projective scheme over $k$ of dimension $d \geq 2$. A reduced curve $C \hookrightarrow X$ is called Cartier if:

1. Each component of $C$ intersects $X_{\text{sing}}$ properly.
2. For every closed point $x \in X_{\text{sing}} \cap C$, the inclusion $C \hookrightarrow X$ is a local complete intersection at $x$.

Let $Z_0(X)$ denote the free abelian group on the set of closed points in $X_{\text{reg}}$. Given a Cartier curve $\iota : C \hookrightarrow X$, let $D = C \cap X_{\text{sing}}$ and let $O_{C,D}^\times \hookrightarrow k(C) \simeq \prod_{i=1}^r O_{C, \eta_i}$ be the inclusion as before. Here, $\{C_1, \ldots, C_r\}$ denotes the set of irreducible components of $C$ with generic points $\{\eta_1, \ldots, \eta_r\}$.

Given $f \in O_{C,D}^\times$, $1 \leq i \leq r$, let $f_i$ denote the $i$th component of $f$ in $k(C)$ and let $(f_i)_C$ denote the divisor of zeros and poles of $f_i$ on $C_i$ in the sense of [2]. We set $(f)_C := \sum_{i=1}^r \iota_*(f)_C$. Let $R_0(X)$ denote the subgroup of $Z_0(X)$ generated by the set $\{(f)_C|f \in O_{C,D}^\times, \text{ Cartier on } X\}$. The group $R_0(X)$ will be called the group of rational equivalences. The Chow group of 0-cycles on $X$ is defined by

$$\text{CH}_0(X) := \frac{Z_0(X)}{R_0(X)}.$$
The following result simplifies the definition of rational equivalence in special cases.

**Lemma 2.2.** (§ 2) In the definition of the group of rational equivalences of 0-cycles above, we can assume that a Cartier curve \( C \) is irreducible if \( X \) is so.

**Example 2.3.** Let \( C \) be the projective plane curve over \( k \) which has a simple cusp along the origin and is regular elsewhere. Its local ring at the singular point is analytically isomorphic to \( k[[t^2, t^3]] \) which is canonically a subring of its normalization \( k[[t]] \). Let \( \pi : \mathbb{P}^1 \to C \) denote the normalization map. Let \( S \simeq \text{Spec}(k[[t]]/(t^2, t^3)) \) denote the reduced conductor and let \( \tilde{S} \simeq \text{Spec}(k[t]/(t^2)) \) denote its scheme-theoretic inverse image in \( \mathbb{P}^1 \). We then have a commutative diagram with exact rows:

\[
\begin{array}{c}
0 \to \lim_{\longrightarrow} O^X(mS)/k^\times \to \lim_{\longrightarrow} K_0(C, mS) \to \text{Pic}(C) \to 0 \\
\downarrow \quad \downarrow \quad \downarrow \pi^* \\
0 \to \lim_{\longrightarrow} O^X(m\tilde{S})/k^\times \to \lim_{\longrightarrow} K_0(\mathbb{P}^1, m\tilde{S}) \to \text{Pic}(\mathbb{P}^1) \to 0.
\end{array}
\]

The isomorphism of the middle vertical map follows from the known result that the double relative \( K \)-groups \( K_0(C, \mathbb{P}^1, mS) \) and \( K_{-1}(C, \mathbb{P}^1, mS) \) vanish.

It is easy to check from the \( K \)-theory localization sequence that \( \text{Pic}(\mathbb{P}^1, m\tilde{S}) \cong K_0(\mathbb{P}^1, m\tilde{S}) \). On the other hand, the known class field theory for curves (with modulus) tells us that there is a canonical isomorphism \( \lim_{\longrightarrow} \text{Pic}^0(\mathbb{P}^1, m\tilde{S}) \cong \pi_{ab}(C_{\text{reg}})^0 \).

It follows that there is an isomorphism \( (1 + tk[[t]])^\times \cong \pi_{ab}(C_{\text{reg}})^0 \). On the other hand, \( A_0(C) \simeq \text{Pic}^0(C) \simeq k \) is finite which shows that there is no reciprocity map \( A_0(C) \to \pi_{ab}(C_{\text{reg}})^0 \) and the two can not be isomorphic.

### 2.2. 0-cycles and cohomology of \( K \)-theory sheaves

In this note, all cohomology groups will be with respect to the Zariski topology unless we explicitly mention otherwise. Given a \( k \)-scheme \( X \), let \( K^M_{m,X} \) denote the Zariski sheaf on \( X \) whose stalk at a point \( P \in X \) is the Milnor \( K \)-group \( K^M_{m,X}(\mathcal{O}_{X,P}) \) of the local ring \( \mathcal{O}_{X,P} \) (see [12] § 1.3)). The corresponding sheaf of Quillen \( K \)-theory will be denoted by \( K^M_{m,X} \). Let us now assume that \( X \) is a quasi-projective scheme of dimension \( n \geq 1 \) over \( k \) with only singularities and let \( S \) denote the singular locus of \( X \). For \( j \geq 0 \), let \( X^j \) denote the set of codimension \( j \) points on \( X \). Consider the maps of Zariski sheaves:

\[
(2.3) \quad K^M_{m,X} \xrightarrow{\epsilon} \left( \bigoplus_{x \in X^0} (i_x)_* K^M_m(k(x)) \right) \xrightarrow{d_0} \left( \bigoplus_{x \in X^1} (i_x)_* K^M_{m-1}(k(x)) \oplus \bigoplus_{P \in S} \bigoplus_{P \in x} (i_x)_* K^M_m(k(x)) \right) \xrightarrow{d_1} \ldots \xrightarrow{d_{n-1}} \left( \bigoplus_{x \in X^n} (i_x)_* K^M_{m-n+1}(k(x)) \oplus \bigoplus_{P \in S} \bigoplus_{P \in x} (i_x)_* K^M_{m-n}(k(x)) \right) \xrightarrow{d_n} \left( \bigoplus_{P \in S} (i_P)_* K^M_{m-n}(k(P)) \right) \to 0.
\]

Here, the map \( \epsilon \) is induced by the inclusion into both terms and the other maps are given by the matrices

\[
d_0 = \begin{pmatrix} \partial_1 & 0 \\ -\Delta & \epsilon \end{pmatrix}, \quad d_1 = \begin{pmatrix} \partial_1 & 0 \\ \Delta & \partial_2 \end{pmatrix}, \quad \ldots, \quad d_n = \begin{pmatrix} 0 & 0 \\ \Delta & \partial_2 \end{pmatrix}
\]
with \( \partial_1 \) and \( \partial_2 \) being the differentials of the Gersten-Quillen complex for Milnor \( K \)-theory sheaves as described in \([11]\) (see also \([6]\)) and \( \Delta \)'s being the diagonal maps.

**Lemma 2.4.** The above sequence of maps forms a complex which gives a flasque resolution of the sheaf \( \varepsilon(\mathcal{K}_{m,X}) \).

**Proof.** A similar complex for the Quillen \( K \)-theory sheaves is constructed in \([23] \ § 5\) and it is shown there that this complex is a flasque resolution of \( \varepsilon(\mathcal{K}_{m,X}) \). The same proof works here in verbatim. On all stalks except at \( S \), the exactness follows from \([6] \ Proposition 4.3\). The exactness at the points of \( S \) is an immediate consequence of the way the differentials are defined in \([2,3]\) (see \([23]\) for details). \( \square \)

For a scheme \( X \), let \( CH^F_0(X) \) denote the Chow group of 0-cycles on \( X \) in the sense of \([7]\). Our proof of the main results will be based on the following descriptions of the Chow groups of 0-cycles.

**Proposition 2.5.** Let \( X \) be an integral quasi-projective scheme of dimension \( d \geq 1 \) over \( k \) with only isolated singularities. Then there are canonical maps

\[
CH_0(X) \to H^d(X, \mathcal{K}^M_{d,X}) \to H^d(X, \mathcal{K}_{d,X}) \to CH^F_0(X),
\]

where the second map is an isomorphism.

**Proof.** The \( d = 1 \) case is already dealt with in Proposition \([2,4]\). We can thus assume that \( d \geq 2 \). Let \( S \) denote the singular locus of \( X \) and let \( X^j_S \) denote the set of points on \( X \) of codimension \( j \) such that \( S \cap \{x\} = \emptyset \). We first observe that the map of sheaves \( \mathcal{K}^M_{d,X} \to \varepsilon(\mathcal{K}^M_{d,X}) \) is generically an isomorphism and the same holds for the Quillen \( K \)-theory sheaves. It follows that (see \([10] \ Exer. II.1.19, Lemma III.2.10\)) that the map \( H^d(X, \mathcal{K}^M_{d,X}) \to H^d(X, \varepsilon(\mathcal{K}^M_{d,X})) \) is an isomorphism and ditto for the Quillen \( K \)-theory sheaves. It follows from Lemma \([2,4]\) that both \( H^d(X, \mathcal{K}^M_{d,X}) \) and \( H^d(X, \mathcal{K}_{d,X}) \) are given by the middle homology of the complex \( \mathcal{C}_X \):

\[
\left( \begin{array}{c}
\prod_{x \in X^{d-1}} K_1(k(x)) \\
\prod_{P \in S} \prod_{P \in \{x\}} K_2(k(x))
\end{array} \right) \xrightarrow{d_1} \left( \begin{array}{c}
\prod_{x \in X^d} K_0(k(x)) \\
\prod_{P \in S} \prod_{P \in \{x\}} K_1(k(x))
\end{array} \right) \xrightarrow{d_2} \left( \begin{array}{c}
0 \\
\prod_{P \in S} K_0(k(P))
\end{array} \right).
\]

On the other hand, letting \( \mathcal{C}^0_X \) and \( \mathcal{C}^{F,0}_X \) denote the complexes

\[
\prod_{x \in X^{d-1}} K_1(k(x)) \xrightarrow{\text{div}} \prod_{x \in X^d} K_0(k(x)) \to 0 \quad \text{and}
\prod_{x \in X^d} K_1(k(x)) \xrightarrow{\text{div}} \prod_{x \in X^{d-1}} K_0(k(x)) \to 0,
\]

respectively, we see that there are canonical maps of chain complexes \( \mathcal{C}^0_X \to \mathcal{C}_X \to \mathcal{C}^{F,0}_X \). This yields canonical maps \( H_1(\mathcal{C}^0_X) \to H^d(X, \mathcal{K}^M_{d,X}) \xrightarrow{\sim} H^d(X, \mathcal{K}_{d,X}) \to H_1(\mathcal{C}^{F,0}_X) \). It follows however by using the definition of \( CH_0(X) \) and Lemma \([2,2]\) that \( H_1(\mathcal{C}^0_X) \cong CH_0(X) \). This finishes the proof. \( \square \)
2.3. **Cycle class map for singular schemes.** For a scheme $X$ of dimension $d \geq 2$ and a closed subscheme $D \subseteq X$, let $K(X,D)$ denote the relative $K$-theory spectrum given by the homotopy fiber of the restriction map of spectra $K(X) \to K(D)$. Given a closed point $x \in X_{\text{reg}} \setminus D$, the map of pairs $(\text{Spec}(k(x)),\emptyset) \to (X,D)$ is of finite tor-dimension. Hence, it yields a map $K_0(k(x)) \to K_0(X,D)$ which allows us to define a cycle class $\text{cyc}(x) \in K_0(X,D)$ as the image of $1 \in K_0(k(x))$. Let $F^dK_0(X,D)$ denote the subgroup of $K_0(X,D)$ generated by the cycle classes of all closed points in $X_{\text{reg}} \setminus D$. Let $F^dK_0(X)$ denote the subgroup of $K_0(X)$ generated by cycle classes of closed points of $X_{\text{reg}}$. It follows from [13, Proposition 2] that the cycle class map $\text{cyc}_X: Z_0(X) \to F^dK_0(X)$ kills the group of rational equivalences to yield a surjective map

\[(2.4) \quad \text{cyc}_X : \text{CH}_0(X) \to F^dK_0(X).\]

**Lemma 2.6.** Suppose that $X$ has only isolated singularities and let $S \hookrightarrow X$ be a closed subscheme supported on $X_{\text{sing}}$. Then the maps $Z_0(X) \to K_0(X,S) \to K_0(X)$ induce surjective maps

\[(2.5) \quad \text{CH}_0(X) \to F^dK_0(X,S) \to F^dK_0(X)\]

such that the second map is an isomorphism.

**Proof.** In any case, we have the maps $Z_0(X) \to F^dK_0(X,S) \to F^dK_0(X)$ and it follows from (2.4) that the composite map kills rational equivalences. So it suffices to show that the map $F^dK_0(X,S) \to F^dK_0(X)$ is an isomorphism. But this follows from [15, Lemma 3.1].

3. **Class field theory for surfaces**

In this section, we prove our main results for surfaces. Theorem 1.2 is proven by combining the results of [13] and the isomorphism of the canonical map $\text{CH}_0(X) \to H^2(X, K^M_{2,X})$. Theorem 1.3 is proven with the aid of Theorem 1.2 and an explicit formula for the Chow group of 0-cycles on normal surfaces.

For a $k$-scheme $X$, let $\tilde{K}_0(X) := \text{Ker}(\text{rk} : K_0(X) \to H^0(X,\mathbb{Z}))$ and let $SK_0(X) = \text{Ker}(\text{det} : \tilde{K}_0(X) \to \text{Pic}(X))$. Let us now assume that $X$ is a surface with isolated singularities. If $x \in X_{\text{reg}}$, then $O_{\{x\}}$ is locally defined by a regular sequence of length two. It follows that $[O_{\{x\}}] \in K_0(X)$ lies in $SK_0(X)$. We conclude that the cycle class map $\text{cyc}_X : \text{CH}_0(X) \to K_0(X)$ factors through $\text{CH}_0(X) \to F^2K_0(X) \hookrightarrow SK_0(X)$.

**Proposition 3.1.** Let $X$ be a quasi-projective surface over $k$ with isolated singularities and let $S \hookrightarrow X$ be a closed subscheme supported on $X_{\text{sing}}$. Then the following hold.

1. The maps $\text{CH}_0(X) \to F^2K_0(X,S) \to F^2K_0(X) \hookrightarrow SK_0(X)$ are all isomorphisms.
2. There are isomorphisms

\[F^2K_0(X,S) \simeq F^2K_0(X) \simeq H^3_s(X, K_2) \simeq H^3_s(X, K_{2,\text{reg}})\]

for $\mathcal{C} = \text{Zar}$ or $\text{Nis}$.

3. If $\pi : \tilde{X} \to X$ is the normalization, then the maps $\pi^* : \text{CH}_0(X) \to \text{CH}_0(\tilde{X})$ and $\pi_{1\text{ab}}^*(\text{reg}) \to \pi_{1\text{ab}}^*(\tilde{X}_{\text{reg}})$ are isomorphisms.

**Proof.** Since the first two maps in (1) are anyway surjective, it suffices to show that the composite map $\text{CH}_0(X) \to SK_0(X)$ is an isomorphism. Since $X$ is a surface with isolated singularities, it follows from the above definition of Cartier curves and Lemma 2.3 that we can assume that the group of rational equivalences in $Z_0(X)$ is generated by the
divisors of rational functions on integral curves on $X$ which do not meet $X_{\text{sing}}$. It follows now from [23 Theorem 2.2] that there is an exact sequence

$$SK_1(X_{\text{sing}}) \to \text{CH}_0(X) \xrightarrow{\text{deg}} SK_0(X) \to 0.$$  

As $X_{\text{sing}}$ is finite, the first term in this sequence is zero and we get (1).

We now show (2). The isomorphism $SK_0(X) \simeq H^2_C(X, K_2)$ follows directly from the Thomason-Trobaugh spectral sequence $E_2^{p,q} = H^p_C(X, K_q) \Rightarrow K_{q-p}(X)$ with differential $d_r : E_2^{p,q} \to E_2^{p+r,q+r-1}$ and the fact that the map $K_1(X) \to H^0_C(X, K_1) \simeq O^\times(X)$ is split surjective (see [17 pg. 162]).

We are left with showing that the map $H^2_C(X, K_2(X,S)) \simeq H^2_C(X, K_2)$ is an isomorphism for $C = \text{Zar}$ or $\text{Nis}$. We have an exact sequence of sheaves

$$0 \to K_{3, S}/K_{3, X} \to K_{2, (X,S)} \to K_{2, X} \to K_{2, S} \to 0.$$  

This yields two short exact sequences $0 \to K_{3, S}/K_{3, X} \to K_{2, (X,S)} \to F \to 0$ and $0 \to F \to K_{2, X} \to K_{2, S} \to 0$.

Since $K_{3, S}/K_{3, X}$ and $K_{2, S}$ are supported on $S$ which is $0$-dimensional, the isomorphism $H^2_{\text{Zar}}(X, K_{2, (X,S)}) \simeq H^2_{\text{Zar}}(X, K_2)$ follows at once by considering the long cohomology exact sequences. To prove this for the Nisnevich site, we apply the same argument using the following inputs: if $G$ is a sheaf supported on $i : S \hookrightarrow X$, then $G \xrightarrow{\sim} i_*(G|_S)$. In particular, $H^i_{\text{Nis}}(X, G) = 0$ for $i > 0$ because $i_*$ is exact in Nisnevich topology and the Nisnevich cohomological dimension of $S$ is zero. This finishes the proof of (2).

The isomorphism $\text{CH}_0(X) \xrightarrow{\sim} \text{CH}_0(\bar{X})$ follows directly from (1), (2) and [16 Proposition 2.3]. It should be observed here that the relevant part of the cited result holds over any field even though the result there is stated over $\mathbb{C}$. The isomorphism $\pi^\text{ab}_1(X_{\text{reg}}) \xrightarrow{\sim} \pi^\text{ab}_1(\bar{X}_{\text{reg}})$ follows from the Zariski-Nagata purity theorem (see [9 Exposé X, Théorème 3.1]) because $X_{\text{reg}} \hookrightarrow \bar{X}_{\text{reg}}$ is an open immersion of smooth schemes whose complement has codimension two. This proves (3).  

**Corollary 3.2.** Let $X$ be as in Proposition 2.3 and let $\mathcal{I}_S$ denote the sheaf of ideals defining a closed subscheme $i : S \hookrightarrow X$ supported on the singular points. Let $K^M_{2,(X,S)}$ denote the Zariski sheaf defined by the exact sequence

$$0 \to K^M_2(X,S) \to K^M_2(X) \to i_*(K^M_{2,S}) \to 0.$$  

Then the maps $\text{CH}_0(X) \to H^2(X, K^M_2)$ and $H^2(X, K^M_{2,(X,S)}) \to H^2(X, K^M_{2,X})$ are isomorphisms.

**Proof.** The first isomorphism follows by combining Propositions 2.3 and 3.1. For the second isomorphism, observe that the map $K^M_{2,X} \to K_{2,X}$ is an isomorphism. In particular, the map $K_{2,(X,S)} \to K_{2,(X,S)}$ is surjective and an isomorphism away from $S$. It follows that the map $H^2(X, K_{2,(X,S)}) \to H^2(X, K^M_{2,(X,S)})$ is an isomorphism. We now apply Proposition 3.1 again.

Let $X$ be a connected and projective scheme over $k$ of dimension $d \geq 2$ which is regular in codimension one. Let $\text{deg} : \text{CH}_0(X) \to \mathbb{Z}$ denote the degree map. Observe that this degree map is surjective. The reason is that we can find a smooth curve $C \hookrightarrow X$ which does not meet $X_{\text{sing}}$ and it is a consequence of Lang’s density theorem [18] that the map $\text{deg} : \text{CH}_0(C) \to \mathbb{Z}$ is surjective. Let $A_0(X)$ denote the kernel of the degree map.

**Theorem 3.3.** Let $X$ be a geometrically integral and projective surface over $k$ with isolated singularities. Let $U \hookrightarrow X$ denote the regular locus of $X$. Then there exists a reciprocity map $\theta_X : \text{CH}_0(X) \xrightarrow{\sim} \pi^\text{ab}_1(U)^0$.
Proof. To construct the reciprocity map $\theta_X : \mathrm{CH}_0(X) \to \pi_1^{ab}(U)$, we let $x \in U$ be a closed point and consider the inclusion $i^x : \text{Spec}(k(x)) \hookrightarrow U$. This induces a natural map $i^x_* : \pi_1^{ab}(\text{Spec}(k(x))) \to \pi_1^{ab}(U)$. We set $\theta_X([x]) = i^x_*(F_x)$, where $F_x$ is the Frobenius element of $\pi_1^{ab}(\text{Spec}(k(x))) \simeq \text{Gal}(\overline{k(x)}/k(x))$. Extending linearly, we get the reciprocity map

$$\theta_X : Z_0(X) \to \pi_1^{ab}(U).$$

To show that $\theta_X$ kills rational equivalences, recall from [13, § 3] that there is a reciprocity map $\rho_X : \lim_{S \subseteq X, \text{sing}} H^2(X, K_{M, 2, (X, S)}) \to \pi_1^{ab}(U)$. It follows from Corollary 3.2 that this map factors through a reciprocity map $\rho_X : H^2(X, K_{M, 2, X}) \to \pi_1^{ab}(U)$.

We now consider the maps $Z_0(X) \xrightarrow{\rho_X} H^2(X, K_{M, 2, X}) \xrightarrow{\rho_X} \pi_1^{ab}(U)$, where the first map is the composite $Z_0(X) \to \mathrm{CH}_0(X) \to H^2(X, K_{M, 2, X})$ given by Proposition 2.5. To prove that $\theta_X$ kills rational equivalences, it suffices therefore to show that $\theta_X = \rho_X \circ \eta_X$. For this, it suffices to show that given any closed point $x \in U$ and any finite separable field extension $k(X) \to L$ in which $U$ is unramified, the composite $K_0(k(x)) \simeq H^2_x(X, K_{2, 2, X}) \to H^2(X, K_{M, 2, X}) \to \pi_1^{ab}(U) = \text{Gal}(L/k(X))$ takes the element $1 \in K_0(k(x))$ to the Frobenius substitution $F_x$ over $x$. But this follows from [13, Proposition 3.6].

It is clear from the above construction that $\theta_X$ induces a map $\theta_X^* : A_0(X) \to \pi_1^{ab}(U)^0$. The assertion that this map is an isomorphism follows from Corollary 3.2 and [26 Theorem 6.2].

3.1. Proof of Theorem 1.5. Let $X$ be an integral normal projective surface over $k$ and let $S$ denote the singular locus of $X$ with reduced induced closed subscheme structure. Set $U = X_{\text{reg}}$. Let $f : \tilde{X} \to X$ denote a resolution of singularities of $X$. Recall that such a resolution of singularities exists for surfaces over any field. Let $E \subseteq \tilde{X}$ denote the reduced exceptional divisor. We identify $f^{-1}(U)$ with $U$ in what follows. We begin by recalling the Chow group of 0-cycles with modulus from [14].

Let $D \subseteq \tilde{X}$ be an effective Cartier divisor supported on $E$. Let $Z_0(\tilde{X}, D)$ denote the free abelian group on closed points in $\tilde{X} \setminus D$. Let $C \hookrightarrow \tilde{X} \times \mathbb{P}_k^1$ be a closed irreducible curve satisfying

1. $C$ is not contained in $\tilde{X} \times \{0, 1, \infty\}$.

2. If $\nu : C^N \to \tilde{X} \times \mathbb{P}_k^1$ denotes the composite map from the normalization of $C$, then one has an inequality of Weil divisors on $C^N$:

$$\nu^*(D \times \mathbb{P}_k^1) \leq \nu^*(\tilde{X} \times \{1\}).$$

We call such curves admissible. Let $Z_1(\tilde{X}, D)$ denote the free abelian group on admissible curves and let $R_0(\tilde{X}, D)$ denote the image of the boundary map $(\partial_0 - \partial_\infty) : Z_1(\tilde{X}, D) \to Z_0(\tilde{X}, D)$. The Chow group of 0-cycles on $\tilde{X}$ with modulus $D$ is defined as the quotient

$$\mathrm{CH}_0(\tilde{X}, D) := \frac{Z_0(\tilde{X}, D)}{R_0(X, D)}.$$

Proposition 3.4. There is a cycle class map $\mathrm{cyc}_{(\tilde{X}, D)} : \mathrm{CH}_0(\tilde{X}, D) \to K_0(X, D)$.

Proof. This is a special case of the more general construction of the cycle class map by Bindu and Krishna [2]. They give a functorial construction of the cycle class map $\mathrm{cyc}_{(\tilde{X}, D)} : \mathrm{CH}_0(\tilde{X}|D, n) \to K_n(X, D)$ from the higher Chow groups with modulus to the higher relative $K$-groups. A completely different construction of this cycle class map is
also given in [1]. We reproduce the construction of [2] in the present special case for the sake of completeness.

Let \( x \in \tilde{X} \setminus D \) be a closed point with residue field \( k(x) \) and let \( \iota^x : \text{Spec}(k(x)) \hookrightarrow \tilde{X} \) denote the inclusion. Then the composite map \( K(\text{Spec}(k(x))) \xrightarrow{\iota^x_*} K(\tilde{X}) \to K(D) \) is null-homotopic and hence there is a unique factorization \( \iota^x_* : K(\text{Spec}(k(x))) \to K(\tilde{X}, D) \to K(\tilde{X}) \). We set \( [x] = \iota^x_*(1) \in K_0(\tilde{X}, D) \). Extending this linearly, we get a cycle class map \( \text{cyc}_{(\tilde{X}, D)} : Z_0(\tilde{X}, D) \to K_0(\tilde{X}, D) \).

Our next task is to show that this map has the desired factorization. So let \( C \subsetneq \tilde{X} \times \mathbb{P}_k^1 \) be an irreducible admissible curve. Let \( C^N \) denote the normalization of \( C \) and let \( g : C^N \to \tilde{X} \) be the projection map. If \( C \) lies over a closed point of \( \tilde{X} \), then it is immediate that \( \text{cyc}_{(\tilde{X}, D)}(\partial([C])) = 0 \). So we can assume that \( C \) does not lie over a closed point. In that case, the map \( g : C^N \to \tilde{X} \) is finite. This gives rise to a Cartesian square

\[
\begin{array}{ccc}
C^N & \xrightarrow{\nu} & N \\
\downarrow{\phi} & & \downarrow{p} \\
C^N \times \mathbb{P}_k^1 & \xrightarrow{\iota} & \tilde{X} \times \mathbb{P}_k^1 \\
\downarrow{id} & & \downarrow{p} \\
C^N & \xrightarrow{g} & \tilde{X}.
\end{array}
\]

The finiteness of \( g \) and admissibility of \( C \) imply that \( C^N = \phi(C^N) \) is an admissible cycle on \( C^N \times \mathbb{P}_k^1 \). We set \( D' = g^*(D) \). Notice that a consequence of the admissibility condition is that \( D' \) is a proper Cartier divisor on \( C^N \). Our first claim is the following:

**Claim:** The finite map \( g \) induces a push-forward map between \( K \)-theory spectra \( g_* : K(C^N, D') \to K(\tilde{X}, D) \).

**Proof of the claim:** Since \( g \) is a morphism between non-singular schemes, it is of finite tor-dimension and hence there is a push-forward map of \( K \)-theory spectra \( g_* : K(C^N) \to K(\tilde{X}) \). To prove the claim, it is enough to show that the map \( D' \to D \) is a also of finite tor-dimension. To show this, all we need to know is that \( C^N \) and \( D \) are tor-independent over \( \tilde{X} \). Since \( D \) is an effective Cartier divisor, only possible tor term will be \( \text{Tor}^1_{\mathcal{O}_{\tilde{X}}} (\mathcal{O}_{C^N}, \mathcal{O}_D) \) which is same as \( \mathcal{I}_D \)-torsion subsheaf of \( \mathcal{O}_{C^N} \). Since \( C^N \) is integral, this torsion subsheaf is non-zero if and only if the ideal \( \mathcal{I}_D \) is zero. But this can not happen as \( D' \) is a proper divisor on \( C^N \). This proves the claim.

It is easy to check from the above construction that

\[
\partial^\epsilon([C]) = g_*(\partial^\epsilon([C^N])) \quad \text{for} \quad \epsilon = 0, \infty \quad \text{and}
\]

\[
\text{cyc}_{(\tilde{X}, D)} \circ g_*(\alpha) = g_* \circ \text{cyc}_{(C^N, D')} (\alpha) \quad \text{for} \quad \alpha \in Z_0(C^N, D').
\]

This reduces the problem of constructing the cycle class map to the case when \( \tilde{X} \) is replaced by a smooth curve \( C \) and \( D \) is an effective Cartier divisor on \( C \). This case follows from Lemma 3.5 below. \( \square \)

**Lemma 3.5.** Let \( C \) be a smooth curve and let \( D \subseteq C \) be an effective Cartier divisor. Then there is a cycle class map \( \text{cyc}_{(C, D)} : CH_0(C, D) \to K_0(C, D) \) which induces an isomorphism \( CH_0(C, D) \xrightarrow{\sim} \text{Pic}(C, D) \).
Proof. We let \( R'_0(C,D) \) denote the subgroup of \( \mathbb{Z}_0(C,D) \) generated by the cycles \( \text{div}(f) \), where \( f \in \mathcal{O}_{C,D}^\times \mapsto k(C)^\times \) is such that \( f \equiv 1 \) modulo \( I_D \). Using (3.2), it is not difficult to check that \( R'_0(C,D) = R_0(C,D) \). So we shall use this new definition of rational equivalences.

Let \( A \) denote the semi-local ring \( \mathcal{O}_{C,D} \) and let \( I \) denote the ideal of \( D \) in \( A \) giving an exact sequence

\[
0 \to K_1(A, I) \to K_1(A) \to K_1(A/I) \to 0.
\]

We now consider the commutative diagram of homotopy fiber sequences:

\[
\begin{array}{cccccc}
\prod_{x \notin D} K(k(x)) & \longrightarrow & K(C) & \longrightarrow & K(A) \\
\downarrow & & \downarrow & & \downarrow \\
K(A/I) & \longrightarrow & K(A/I).
\end{array}
\]

This yields a homotopy fiber sequence

\[
\prod_{x \notin D} K(k(x)) \to K(C, D) \to K(A, I)
\]

and in particular, an exact sequence

\[
K_1(A, I) \xrightarrow{\partial} Z_0(C, D) \to K_0(C, D) \to 0
\]

and we conclude from this that

\[
\text{Coker}(\partial) = \text{CH}_0(C, D) \xrightarrow{\sim} K_0(C, D).
\]

Finally, it is easy to check that \( \text{Pic}(C, D) \simeq K_0(C, D) \).

Proposition 3.6. Let \( f : \tilde{X} \to X \) be a resolution of singularities of a normal projective surface \( X \) as above. For any \( m \geq 1 \), there exists a commutative diagram

\[
\begin{array}{cccc}
\text{CH}_0(X) & \xrightarrow{\text{cyc}_{(X,mS)}} & F^2K_0(X, mS) & \\
f^* & & f^* \xrightarrow{\sim} & F^2K_0(\tilde{X}, mE) \\
\text{CH}_0(\tilde{X}, mE) & \xrightarrow{\text{cyc}_{(\tilde{X}, mE)}} & F^2K_0(\tilde{X}, mE) & \xrightarrow{f^*} \\
\text{CH}_0(\tilde{X}) & \xrightarrow{\text{cyc}_{\tilde{X}}} & F^2K_0(\tilde{X}).
\end{array}
\]

such that all arrows are surjective. Moreover, all arrows in the top square are isomorphisms for all \( m \gg 1 \). In particular, the cycle class map \( \text{CH}_0(\tilde{X}, mE) \to K_0(\tilde{X}, mE) \) is injective for all \( m \gg 1 \).

Proof. Let \( F^2K_0(\tilde{X}, mE) \) denote the image of \( \text{CH}_0(\tilde{X}, mE) \) under the cycle class map of Proposition 3.4. The maps on the \( K \)-theory side are obvious maps induced by the pull-back \( f^* : K(X, mS) \to K(\tilde{X}, mE) \). The surjectivity of arrows in the triangle on the right is immediate from the construction of the underlying groups. The map \( \text{cyc}_{(X,mS)} \) exists and is surjective by Lemma 2.6. The maps \( \text{cyc}_{(\tilde{X}, mE)} \) and \( \text{cyc}_{\tilde{X}} \) clearly are surjective. The map \( \text{CH}_0(\tilde{X}, mE) \to \text{CH}_0(\tilde{X}) \) is the forgetful map which is surjective by the moving lemma for \( \text{CH}_0(\tilde{X}) \).

It is clear that there is a pull-back map \( f^* : Z_0(X) \to Z_0(\tilde{X}, mE) \) which is surjective. To show that it preserves rational equivalences, let \( C \hookrightarrow X \) be an integral curve not meeting \( X_{\text{sing}} \) and let \( h \in k(C)^\times \). Let \( \Gamma_h \to C \times \mathbb{P}^1 \to X \times \mathbb{P}^1 \) be the graph of
the function $h : C \to \mathbb{P}^1$. It is then clear that $\Gamma_h \cap (X_{\text{sing}} \times \mathbb{P}^1) = \emptyset$. In particular, $f^{-1}(\Gamma_h) \cap (E \times \mathbb{P}^1) = \emptyset$. This shows that $[\Gamma_h] \in Z_1(\widetilde{X}, mE)$ is an admissible 1-cycle such that 

$$f^*(\text{div}(h)) = f^*((h^*(0) - h^*(\infty))) = f^*(\partial_0([\Gamma_h]) - \partial_\infty([\Gamma_h])) = (\partial_0 - \partial_\infty([\Gamma_h])).$$

This shows that $f^*(\text{div}(h)) \subseteq \mathcal{R}_0(\widetilde{X}, mE)$ and yields the pull-back $f^* : CH_0(X) \to CH_0(\widetilde{X}, mE)$. It is clear from the constructions that the diagram 3.4 commutes. This proves the first part of the proposition.

We now prove the second part. Proposition 3.1 says that $\text{cyc}_{(X, mS)}$ is an isomorphism and so is the map $F^2K_0(X, mS) \to F^2K_0(X)$. On the other hand, it follows from Theorem 1.1 that the map $F^2K_0(X) \to F^2K_0(\widetilde{X})$ factors through $F^2K_0(X) \to F^2K_0(\widetilde{X}, mE) \to F^2K_0(\widetilde{X})$ and the map $F^2K_0(X) \to F^2K_0(\widetilde{X}, mE)$ is an isomorphism for all $m \gg 1$. It follows that the right vertical arrow in the top square is an isomorphism for all $m \gg 1$. A simple diagram chase now shows that all arrows in the top square are isomorphisms for all $m \gg 1$. \hfill \Box

**Proposition 3.7.** Let $X$ be a connected projective surface over $k$ with isolated singularities. Then $A_0(X)$ is finite.

**Proof.** Using Proposition 3.1 we can assume that $X$ is normal and hence integral. Let $f : \widetilde{X} \to X$ be a resolution of singularities such that the reduced exceptional divisor $E$ is strict normal crossing. Using Proposition 3.6 and the finiteness of $A_0(\widetilde{X})$ (see Theorem 1.1), we reduce to showing that $\text{Ker}(F^2K_0(\widetilde{X}, mE) \to F^2K_0(\widetilde{X}))$ is finite. By Lemma 2.2, this kernel is contained in the image of the boundary map $\partial : SK_1(mE) \to K_0(\widetilde{X}, mE)$. Hence, it suffices to prove inductively that $SK_1(mE)$ is finite for $m \geq 1$.

Let $U \hookrightarrow E$ be a dense open subscheme of $E$ containing the singular locus of $E$ and set $Z = (E \setminus U)_{\text{reg}}$. The localization sequence of Thomason-Trobaugh yields an exact sequence $K_1(E\text{ on } Z) \to SK_1(E) \to SK_1(U)$. Since $U$ is affine, $SK_1(U)$ vanishes by Proposition 2.1. Since $Z \hookrightarrow E_{\text{reg}}$, the excision isomorphism $K_1(E\text{ on } Z) \cong K_1(E_{\text{reg}}\text{ on } Z)$ (see Proposition 3.19) and the localization fiber sequence $K(Z) \to K(E_{\text{reg}}\to Z)$ show that the map $K_1(E\text{ on } Z) \to K_1(Z)$ is an isomorphism. Since $K_1(Z) \cong O_Z^{\times}$ is finite, the surjection $K_1(E\text{ on } Z) \twoheadrightarrow SK_1(E)$ shows that $SK_1(E)$ is finite.

Assume now that $SK_1(mE)$ is finite for some $m \geq 1$. Let $S = \{E_1, \ldots, E_r\}$ denote the set of irreducible components of $E$ and let $S_{ij} = E_i \cap E_j$. By [21 § 6.4], there is an exact sequence of Zariski sheaves

$$\mathcal{K}_{2,(m+1)E} \to \bigoplus_{E_i \in S} \mathcal{K}_{2,(m+1)E_i} \to \bigoplus_{E_i \in S} \bigoplus_{P \in S_{ij}} \mathcal{K}_{2,(m+1)E_i} \to 0$$

on $(m+1)E$, where $P_{(m+1)}$ is the closed subscheme of $\widetilde{X}$ with support $P \in S_{ij}$ whose local ring is analytically isomorphic to $l[[x, y]]/(x^{m+1}, y^{m+1})$ for some finite field extension $k \hookrightarrow l$. The surjectivity of the last map can be easily checked locally. This yields a long cohomology exact sequence

$$\bigoplus_{E_i \in S} \bigoplus_{P \in S_{ij}} K_2(P_{(m+1)}) \to H^1((m+1)E, \mathcal{K}_{2,(m+1)E}) \to \bigoplus_{E_i \in S} H^1((m+1)E_i, \mathcal{K}_{2,(m+1)E_i}) \to 0.$$
Using the isomorphism $SK_1((m+1)E) \xrightarrow{\sim} H^1((m+1)E, \mathcal{K}_2,(m+1)E)$ (see [17 Lemma 2.3]), it suffices to show that each $SK_1((m + 1)E_i)$ is finite. We can thus assume that $E$ is smooth.

The short exact sequence of sheaves

$$
\mathcal{K}_{2,(m+1)E,mE} \rightarrow \mathcal{K}_{2,(m+1)E} \rightarrow \mathcal{K}_{2,mE} \rightarrow 0
$$

yields an exact sequence

$$
H^1((m+1)E, \mathcal{K}_{2,(m+1)E,mE}) \rightarrow SK_1((m+1)E) \rightarrow SK_1(mE) \rightarrow 0.
$$

Since $k$ is finite, it follows from [17 Theorem 7.1] that $\mathcal{K}_{2,(m+1)E,mE}$ is a quotient of a coherent sheaf on $E$. In particular, $H^1((m + 1)E, \mathcal{K}_{2,(m+1)E,mE})$ is finite. We conclude by induction that $SK_1((m + 1)E)$ is finite and this finishes the proof of the proposition.

**Proof of Theorem 1.5:** Let $U$ be the regular locus of a normal projective surface $X$ over $k$. Since $C(U)$ does not depend on the choice of a compactification of $U$, we take this compactification to be $\tilde{X}$, where $f : \tilde{X} \rightarrow X$ is a resolution of singularities of $X$. Note that such a resolution always exists for surfaces.

If $D \subseteq \tilde{X}$ is an effective Cartier divisor with support $|D| \subseteq E$, then $mE - D$ must be an effective Cartier divisor some $m \gg 1$. This implies that the canonical map $C(U) \rightarrow \lim_{m} CH_{0}(\tilde{X},mE)$ is an isomorphism. Moreover, it follows from Proposition 3.6 that the maps $CH_{0}(X) \rightarrow C(U) \rightarrow CH_{0}(\tilde{X},mE)$ are isomorphisms for all $m \gg 1$.

Since the map $\mathbb{Z}_{0}(X) \xrightarrow{f^*} \mathbb{Z}_{0}(\tilde{X},mE)$ is identity, the reciprocity map for $X$ defines a similar map $\mathbb{Z}_{0}(\tilde{X},mE) \rightarrow \pi_{1}^{ab}(U)$. It follows now from Theorem 3.3 that this induces a reciprocity map $\rho_U : C(U) \rightarrow \pi_{1}^{ab}(U)$ and an isomorphism $C_{0}(U) \xrightarrow{\sim} \pi_{1}^{ab}(U)$. Moreover, it follows from Propositions 3.6 and 3.7 that $C_{0}(U)$ is finite. This proves Theorem 1.5.

### 4. Class Field Theory in Higher Dimensions

In this section, we shall prove Theorem 1.2 using an inductive argument on the dimension of the scheme. Let $X$ be a geometrically integral projective scheme over $k$ of dimension $d \geq 2$ which is regular in codimension one and let $j : U \subseteq X$ denote the inclusion of the regular locus of $X$. We can define the reciprocity map $\theta_{X} : \mathbb{Z}_{0}(X) \rightarrow \pi_{1}^{ab}(U)$ exactly as we did for surfaces in Theorem 3.3. We shall prove that this map kills rational equivalences using the following steps.

**Lemma 4.1.** Let $f : X' \rightarrow X$ be a morphism of projective schemes such that $f^{-1}(X_{\text{sing}}) \subseteq X'_{\text{sing}}$. Then there is a commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z}_{0}(X') & \xrightarrow{\theta_{X'}} & \pi_{1}^{ab}(X'_{\text{reg}}) \\
\downarrow f_* & & \downarrow f_* \\
\mathbb{Z}_{0}(X) & \xrightarrow{\theta_{X}} & \pi_{1}^{ab}(X_{\text{reg}}). \\
\end{array}
$$

**Proof.** It is clear from our assumption that $f(X'_{\text{reg}}) \subseteq X_{\text{reg}}$. In particular, the vertical arrows in (4.1) are defined. Let $x' \in X'_{\text{reg}}$ be a closed point with residue field $k(x')$ and
let $f(x') = x$ with residue field $k(x)$. We then have a diagram

\[
\begin{array}{ccc}
Z_0(k(x')) & \rightarrow & \pi_1^{ab}(k(x')) \\
\downarrow^{\iota'_*} & & \downarrow \pi_1^{ab}(X') \\
Z_0(X') & \rightarrow & \pi_1^{ab}(X') \\
\downarrow & & \downarrow \\
Z_0(k(x)) & \rightarrow & \pi_1^{ab}(k(x)) \\
\downarrow & & \downarrow \\
Z_0(X) & \rightarrow & \pi_1^{ab}(X).
\end{array}
\]

It suffices to show that $f_* \circ \theta_{X'}(\iota'_*(1)) = \theta_X \circ f_*(\iota'_*(1))$ in order to prove the lemma. In other words, we need to show that all faces of the cube commute.

The left and the right faces commute by the covariant functoriality of the fundamental group and the group of 0-cycles. The top and the bottom faces commute by the definition of the reciprocity map. It thus suffices to show that the back face commutes. In other words, we need to show that all faces of the cube commute.

Lemma 4.2. Let $f : X' \to X$ be a finite morphism of irreducible quasi-projective schemes over $k$ of dimension $d \geq 2$ with isolated singularities. Assume that $f^{-1}(X'_{\text{sing}}) \subseteq X'_{\text{sing}}$. Then there is a push-forward map $f_* : \text{CH}_0(X') \to \text{CH}_0(X)$.

Proof. Our assumption implies that there is a push-forward map $f_* : Z_0(X') \to Z_0(X)$. Let $C' \subseteq X'$ be an irreducible curve let $h \in k(C')^\times$. As $\dim(X') \geq 2$ and it has only isolated singularities, we must have $C' \cap X'_{\text{sing}} = \emptyset$. Setting $C = f(C')$, we see that $C' \subseteq X_{\text{reg}}$ is an irreducible curve. Let $N : k(C') \to k(C)$ denote the norm map. It follows now from [7] that $f_*(\text{div}(h)) = \text{div}(N(h)) \in \mathcal{R}_0(X)$. In particular, $f_*$ preserves the groups of rational equivalences and yields the desired map on the Chow groups. □

Proposition 4.3. Let $X$ be a integral projective scheme over $k$ of dimension $d \geq 2$ which is regular in codimension one and let $j : U \to X$ denote the inclusion of the regular locus of $X$. Then the map $\theta_X : Z_0(X) \to \pi_1^{ab}(U)$ kills the group of rational equivalences to induce a map

$$\theta_X : \text{CH}_0(X) \to \pi_1^{ab}(U).$$

which restricts to a map $\theta_X^0 : A_0(X) \to \pi_1^{ab}(U)^0$.

Proof. If $d = 2$, then the proposition follows Theorem 3.3. We now let $d \geq 3$ and assume that $\theta_X$ kills rational equivalences if $2 \leq \dim(X) \leq d - 1$.

Let $\iota : C \to X$ be a closed irreducible curve such that $C \subseteq U$ and let $f \in k(C)^\times$ be a rational function. Since $X$ is irreducible and regular in codimension one, it suffices to
show (see the definition of Cartier curves and Lemma 4.1) that \( \theta_X \circ \iota_s(\text{div}(f)) = 0 \) in order to show that \( \theta_X \) kills rational equivalences.

We assume that \( C \) is singular. The other case is easier as we will see. We claim that there exists a finite sequence of point blow-ups \( \pi : X' \to X \) with centers in \( U \) such that the strict transform \( C' \) of \( C \) is the normalization of \( C \).

To see this, we first observe that if \( x \in C \) is a closed point, then the blow-up of \( C \) at \( x \) is same as the strict transform of \( C \) in the blow-up of \( X \) at \( x \). Using this observation and using the inclusion \( C \to U \), it suffices to show that the normalization \( C^N \to C \) is a composite of point blow-ups of \( C \) with centers in \( C_{\text{sing}} \). Let \( D \) denote the singular locus of \( C \).

Let us choose a point \( x \in D \) and let \( \pi_x : C^x \to C \) be the blow-up of \( C \) at \( x \). We now notice that \( \pi_x \) is a finite birational map and dominated by the normalization \( C^N \to C^x \to C \). We next observe that \( \pi_x \) can not be an isomorphism. This is because of the fact (which one can easily check) that \( \pi_x \) will be an isomorphism if and only if \( x \in C \) is a Cartier divisor. But as \( C \) is a curve, this will happen if and only if \( C \) is regular at \( x \), which we have assumed it is not. Since all such blow-ups are dominated by \( C^N \) which is finite over \( C \), it follows that this process will end at \( C^N \) (otherwise, \( \mathcal{O}_{CN} \) will not be a coherent \( \mathcal{O}_C \)-module). This proves our claim.

Using the above claim, we choose a composite of point blow-ups \( \pi : X' \to X \) with centers in \( U \) such that the strict transform \( C' \) of \( C \) is the normalization of \( C \). Notice that \( \pi : X' \to X \) is an isomorphism over a neighbourhood of \( X_{\text{sing}} \). In particular, the following hold.

1. \( X' \) is irreducible and regular in codimension one.
2. \( U' := X'_{\text{reg}} = \pi^{-1}(U) \).
3. \( C' \) is smooth and \( C' \subseteq U' \).
4. If we let \( \nu : C' \to C \) denote the restriction of \( \pi \), then we can think of \( f \) as a rational function on \( C' \) such that \( \text{div}(f|_{C'}) = \nu_*(\text{div}(f|_{C'})) \) (see [7]).

We embed \( X' \hookrightarrow \mathbb{P}^N_k \) as a closed subscheme for some \( N \gg 1 \). By [25, Theorem 1.1], we can find a hypersurface section \( H \subseteq X' \) containing \( C' \) such that \( H \cap U' \) is smooth and \( X'_{\text{sing}} \not\subseteq H \). By [5, Corollary 1.4], we can also assume that \( H \) is irreducible.

Since \( H \) is a hypersurface section of \( X' \), it must be singular along \( X'_{\text{sing}} \). In particular, \( H_{\text{sing}} = X'_{\text{sing}} \cap H \). As \( X'_{\text{sing}} \not\subseteq H \), we see that \( \dim(H \cap X'_{\text{sing}}) = \dim(X'_{\text{sing}}) - 1 \leq d - 3 \). We conclude that \( H \) is a hypersurface section of \( X' \) such that \( H_{\text{sing}} \) has dimension at most \( d - 3 \), \( H_{\text{reg}} = H \cap U' \) and \( C' \not\subseteq H_{\text{reg}} \). In particular, \( H \) is regular in codimension one. Let \( \pi' : H \hookrightarrow X' \to X \) denote the composite map. We now apply Lemma 4.1 to \( \pi' : H \to X \) to get
\[
\theta_X(\text{div}(f|_{C'})) = \theta_X \circ \pi'_*(\text{div}(f|_{C'})) = \pi'_* \circ \theta_H(\text{div}(f|_{C'})) = 0,
\]
where \( = \) follows from the induction hypothesis. This shows that \( \theta_X \) kills rational equivalences. Finally, it is clear from the construction of \( \theta_X \) that it induces a map \( \theta_X^0 : A_0(X) \to \pi_{1,0}^\text{ab}(U)^0 \). This finishes the proof. \( \square \)

4.1. Conclusion of the Proof of Theorem 1.2. We now complete the proof of Theorem 1.2. We are only left with showing that \( \theta_X^0 \) is an isomorphism of finite groups if \( X \) has only isolated singularities. It will then follow from [4.3] that \( \theta_X \) is injective with dense image. We shall prove the isomorphism of \( \theta_X^0 \) with the aid of the following Bertini type theorem over finite fields, which is our third main step.

Theorem 4.4. ([5, Theorem 1.3]) Let \( U \hookrightarrow \mathbb{P}^N_k \) be a smooth and geometrically integral locally closed subscheme of dimension at least three. Let \( T_1, T_2 \subseteq \mathbb{P}^N_k \) be two disjoint
finite sets of closed points. Then for all sufficiently large \( d \gg 1 \), there exist hypersurfaces \( H \subseteq \mathbb{P}^N_k \) of degree \( d \) such that \( H \cap T_1 = \emptyset, T_2 \subseteq H \) and \( H \cap U \) is a smooth and geometrically integral hypersurface section of \( U \).

Proof. For a closed point \( P \in \mathbb{P}^N_k \), let \( \hat{O}_P \) denote the completion of the local ring of \( \mathbb{P}^N_k \) at \( P \). For \( P \in T_1 \), let \( U_P = (\hat{O}_P)^\times \) and for \( Q \in T_2 \), let \( U_Q = m_Q \), the maximal ideal of \( \hat{O}_Q \). Then for each \( P \in T_1 \cup T_2 \), the set \( U_P \subseteq \hat{O}_P \) is a union of cosets of \( m_P \). Moreover, \( m_P \) has finite index in \( \hat{O}_P \). We can now apply [5, Theorem 1.3] (see its proof) to conclude that for all sufficiently large \( d \gg 1 \), there exist hypersurfaces \( H = H(f) \subseteq \mathbb{P}^N_k \) which have the property that \( x_{ijP}^{-1} f \in \mathcal{O}_P \) for each \( P \in T_1 \cup T_2 \) and \( H \cap U \) is smooth. Here, \( x_{ijP} \) is a non-vanishing coordinate of \( \mathbb{P}^N_k \) at \( P \) so that \( x_{ijP}^{-1} f \in \mathcal{O}_P \). Our conditions at the points of \( T_1 \cup T_2 \) imply that \( T_1 \cap H = \emptyset \) and \( T_2 \subseteq H \). Using [5, Corollary 1.4], we can also assume that \( H \cap U \) is geometrically integral. This finishes the proof.

Proof of Theorem 1.2 Let \( X \) be a geometrically integral projective scheme over \( k \) of dimension \( \geq 2 \) with isolated singularities. If \( d = 2 \), Theorem 1.2 follows from Theorem 3.3 and Proposition 3.7. We now let \( d \geq 3 \) and assume that \( \theta_X^0 \) is an isomorphism of finite groups if \( 2 \leq \dim(X) \leq d - 1 \).

Let \( p : X' \to X \) denote the normalization map and set \( S' = p^{-1}(X_{\text{sing}}) \), and \( U = X_{\text{reg}} \). We can then identify \( U \) as the open subset \( X' \setminus S' \). Notice that \( S' \) is a finite set since \( X \) has only isolated singularities.

We now embed \( X' \to \mathbb{P}^N_k \) as a closed subscheme for some \( N \gg 1 \). To show that \( \theta_X^0 \) is injective, let \( \alpha \in A_0(X) \) be a 0-cycle such that \( \theta_X^0(\alpha) = 0 \). Since the support of \( p^*(\alpha) \) and \( S' \) are two disjoint finite subsets of closed points in \( X' \), we can apply Theorem 4.4 to get a geometrically integral hypersurface section \( Y = H \cap X' \) of \( X' \) which is disjoint from \( S' \), contains the support of \( p^*(\alpha) \) and \( H \cap U \) is smooth. In particular, \( Y \subseteq U \) is a closed subscheme of \( X \) and \( \alpha \) is a 0-cycle of degree zero on \( Y \).

Applying Lemmas 4.1 and 4.2 to the inclusion \( \pi : Y \to U \to X \), we get a commutative diagram:

\[
\begin{array}{ccc}
A_0(Y) & \xrightarrow{\theta_X^0} & \pi_1^{ab}(Y)^0 \\
\downarrow \pi_* & & \downarrow \pi_* \\
A_0(X) & \xrightarrow{\theta_X} & \pi_1^{ab}(U)^0.
\end{array}
\]

We claim that the map \( \pi_* : \pi_1^{ab}(Y)^0 \to \pi_1^{ab}(U)^0 \) is an isomorphism. Since \( X' \) is normal of dimension at least two, it has depth at least two at each of its closed points. Since \( Y \) is a hypersurface section of \( X' \) which is contained in \( U \) (which is an open subset of \( X' \)) and since \( d \geq 3 \), we see that \( X' \) has depth at least three at each closed point of \( Y \). We conclude from [8, Exposé XII, Corollaire 3.5] that the map \( \pi_1(Y) \to \lim_{W} \pi_1(W) \) is an isomorphism, where the limit is taken over all open neighborhoods of \( Y \) contained in \( U \).

It therefore suffices to show that the map \( \pi_1(W) \to \pi_1(U) \) is an isomorphism for each such \( W \).

Set \( Z' = U \setminus W \) and let \( Z \) denote its closure in \( X' \). Then \( Z \subseteq \mathbb{P}^N_k \) is a closed subscheme disjoint from \( Y = X' \cap H \) and hence from \( H \). But this can happen only if \( \dim(Z) \leq 0 \). If \( Z = \emptyset \), we are done. If \( \dim(Z) = 0 \), then \( W \to U \) is an open immersion of smooth schemes such that \( U \setminus W \) has codimension \( d \geq 2 \). It follows from the Zariski-Nagata theorem (see [9, Exposé X, Théorème 3.1]) that the map \( \pi_1(W) \to \pi_1(U) \) must be an isomorphism and this proves the claim.
Since $Y$ is smooth projective and geometrically integral, it follows from [12, Theorem 1] that $\theta^0_{\mathcal{Y}}$ is an isomorphism. Combining this with the above claim, it follows that the map $\pi_* \circ \theta^0_{\mathcal{Y}}$ is an isomorphism. It follows that $\alpha$ dies as a class in $A_0(Y)$ and hence it dies in $A_0(X)$. This shows that $\theta^0_X$ is injective. Furthermore, the isomorphism of $\pi_* \circ \theta^0_{\mathcal{Y}} = \theta^0_X \circ \pi_*$ also shows that $\theta^0_X$ is surjective as well.

Lastly, the finiteness of $A_0(X)$ and $\pi_1^{ab}(U)$ follows because we have shown that all arrows in (4.2) are isomorphisms and the groups on the top are finite by induction. This completes the proof of Theorem 1.2.

Proof of Corollary 1.4: The case of curve is easy because the maps $A_0(X) \to A_0(X^N) \to \text{Alb}_X(k)$ are surjective maps of finite groups. If $X$ is a surface with isolated singularities, we can assume it to be normal by Proposition 3.1. If $\tilde{X} \to X$ is a resolution of singularities, we have the maps $A_0(X) \to A_0(\tilde{X}) \to \text{Alb}_X(k)$. The first map is clearly surjective and the second map is surjective by [12, Proposition 9]. The finiteness of these groups is already shown before.

If $d \geq 3$, we can find a general hypersurface section $Y = H \cap X$ as in the proof of Theorem 1.2. It is shown in (4.2) that the map $A_0(Y) \to A_0(X)$ is an isomorphism. On the other hand, it follows from [19, Chapter 8, §2, Theorem 5] that the map $\text{Alb}_Y \to \text{Alb}_X$ is an isomorphism. The induction hypothesis now finishes the proof.

Proof of Theorem 1.6: Let $X$ be as in Theorem 1.6. The $d \leq 1$ case follows from Proposition 2.1. So we assume $d \geq 2$. Proposition 2.5 says that the map $H^d(X, \mathcal{K}^{M}_{d,X}) \to H^d(X, \mathcal{K}^{M}_{d,X})$ is an isomorphism. It also says that there is a commutative diagram with exact rows:

\[
\begin{array}{cccc}
0 & \to & A_0(X) & \to & \text{CH}_0(X) & \to & \mathbb{Z} & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^d(X, \mathcal{K}^{M}_{d,X})^0 & \to & H^d(X, \mathcal{K}^{M}_{d,X}) & \to & \mathbb{Z} & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \pi_1^{ab}(U)^0 & \to & \pi_1^{ab}(U) & \to & \mathbb{Z} & \to & 0.
\end{array}
\]

Note that the maps $\text{CH}_0(X) \to H^d(X, \mathcal{K}^{M}_{d,X}) \to \mathbb{Z}$ are the composite $\text{CH}_0(X) \to H^d(X, \mathcal{K}^{M}_{d,X}) \to \text{CH}_0^F(X) \to \mathbb{Z}$ (see Proposition 2.5), where the last map is the degree map induced by the projective push-forward $\text{CH}_0^F(X) \to \text{CH}_0^F(\text{Spec}(k))$.

Theorem 1.2 says that the the composite vertical arrow on the left is an isomorphism. It follows from Corollary 3.2 and its obvious higher dimensional analogue that the map $H^d(X, \mathcal{K}^{M}_{d,X,S}) \to H^d(X, \mathcal{K}^{M}_{d,X})$ is an isomorphism for all closed subschemes $S \hookrightarrow X$ supported on $X_{\text{sing}}$. Combining this with [26, Theorem 6.2], we see that the bottom vertical arrow on the left is also an isomorphism. It follows that the left vertical arrow on the top is an isomorphism and this implies that the middle vertical arrow on the top must also be an isomorphism.

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School of Mathematics, Tata Institute of Fundamental Research, 1 Homi Bhabha Road, Colaba, Mumbai, India
E-mail address: amal@math.tifr.res.in