Abelian Yang-Baxter Deformations and TsT transformations

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Abstract

We prove that abelian Yang-Baxter deformations of superstring coset σ models are equivalent to sequences of commuting TsT transformations, meaning T dualities and coordinate shifts. Our results extend also to fermionic deformations and fermionic T duality, and naturally lead to a TsT subgroup of the superduality group OSp(\(d_b, d_\nu | 2d_f\)). In cases like AdS_5 x S^5, fermionic deformations necessarily lead to complex models. As an illustration of inequivalent deformations, we give all six abelian deformations of AdS_3. We comment on the possible dual field theory interpretation of these (super-)TsT models.
1 Introduction

Integrability is a key feature of the string σ model on AdS5 × S5 in the context of the AdS/CFT correspondence [1]. Progress in this field has led to substantial improvements in our understanding of both sides of this duality [2, 3, 4]. One way to further extend our understanding is to study deformations that extend beyond the maximally symmetric example of AdS5 × S5 and its lower dimensional cousins, while preserving integrability. The primary example of this is a string on the Lunin-Maldacena background [5, 6, 7], dual to real β deformed planar SYM. On the string side, this theory can be obtained by so-called TsT transformations – sequences of T dualities and shifts in commuting directions, also known as Melvin twists. More recently it was realised in the manifestly integrability preserving framework of Yang-Baxter deformations. The purpose of this paper is to elucidate the connection between these two approaches.

Yang-Baxter σ models were introduced as deformations of principal chiral models based on R operators solving the modified classical Yang-Baxter equation [8], preserving their integrability [9]. This notion was generalised to symmetric space coset σ models in [10] and then further to the supercoset σ model describing the AdS5 × S5 superstring [11]. By a simple limit this deformation procedure can be extended to solutions of the classical Yang-Baxter equation [24]. These equations admit many solutions, and correspondingly there are many different integrable deformations of the AdS5 × S5 string. In terms of general structure, at the level of symmetries, deformations based on the modified classical Yang-Baxter equation correspond to quantum deformations [25], while deformations based on the classical Yang-Baxter equation result in Drinfeld twists [26], see also [17]. At the level of string theory, the condition that the backgrounds of these models solve the supergravity equations of motion requires the associated R operator to be unimodular [27]. All Yang-Baxter deformations of the string preserve κ symmetry however [11, 27], meaning that their backgrounds necessarily solve a set of modified supergravity equations [28, 29], guaranteeing scale but not Weyl invariance.

1 These models are related to another type of integrable deformation known as the λ model [12, 13, 14] by analytic continuation and Poisson-Lie duality [15, 16, 17, 18, 19, 20], see also [21]. The λ-type models do correspond to solutions of supergravity [22, 23].
The structure described above matches with previously established results. Namely, the $\eta$ deformation of the string – based on the modified classical Yang-Baxter equation – was originally constructed using a non-unimodular $R$ operator, and indeed the associated background does not solve the supergravity equations [30], but rather the modified ones [28], see also [31]. Still, alternative $R$ operators exist [25, 32]. These appear to give inequivalent backgrounds, yet the same $S$ matrix [32]. None of the studied $R$ operators is unimodular, however, and it is not known whether a unimodular one exists.²

For classical Yang-Baxter deformations the situation is more diverse. $R$ operators of this type can be divided into abelian and non-abelian, depending on whether the associated generators all mutually commute or not. In the non-abelian class, bosonic jordanian $R$ operators are not unimodular, and indeed the associated backgrounds solve the modified supergravity equations [37], but not the regular ones [38, 37]. In fact, many jordanian deformations are closely related to the $\eta$ model, as they can be obtained from it by singular boosts [37]. Further bosonic jordanian examples were recently investigated in [39]. The conformal symmetry of AdS₅ is large enough, however, to admit other, unimodular non-abelian $R$ operators [27].

In contrast to non-abelian ones, abelian $R$ operators are always unimodular, meaning any such operator maps a solution of supergravity to a solution of supergravity. Various abelian deformations were studied at the bosonic level, see e.g. [40, 41, 42, 43], including the Lunin-Maldacena background mentioned above [44]. More recently some examples have been studied to quadratic order in fermions, both as singular boosts of the $\eta$ model [30, 37] and directly [38]. These individual examples all fit the proposal of one of the present authors [42], that abelian Yang-Baxter deformations are equivalent to sequences of commuting $TsT$ transformations.

The objective of this paper is to get closer to a complete understanding of this abelian class of Yang-Baxter deformations, by giving a general proof of the equivalence between abelian Yang-Baxter deformations and (sequences of commuting) $TsT$ transformations. This proof relies on always being able to find a group parameterisation such that the Maurer-Cartan forms manifest a set of chosen commuting isometries. For completeness, upon complexification we can extend our proof to include $R$ operators based on anticommuting supercharges. These are equivalent to a generalised fermionic version of $TsT$ transformations, which we introduce. Furthermore, in order to explore the various possible abelian deformations/$TsT$ transformations and to get a better idea of their general structure, we consider AdS₃ – the simplest nontrivial non-compact example – which admits six inequivalent abelian deformations.

This paper is organised as follows. In section 2 we establish our conventions for the type IIB superstring in AdS₅ × S⁵ and its integrable deformations based on the classical Yang-Baxter equation. Bosonic and fermionic $T$ duality is introduced in section 3, where we also briefly discuss the duality groups $O(d,d)$ and $OSp(d_0,d_0|2d_f)$ respectively. We prove equivalence between abelian deformations and $TsT$ transformations in section 4. In the last section we address the fact that there are different inequivalent commuting subalgebras in non-compact cosets, illustrating this with a discussion of all inequivalent abelian deformations of AdS₃. In the conclusions we indicate some open questions and comment on the possible dual field theory interpretation of these deformed models.

²Here it is interesting to recall that the bosonic part of the maximally deformed $\eta$ model can be completed to a solution of supergravity, giving the so-called mirror model [33, 34, 35]. Algebraically this maximal deformation limit corresponds to a contraction [36]. The mirror model is an integrable model itself, and is closely related to the direct contraction of the full $\eta$ model [30]. In particular the $S$ matrices of these models appear to match.
2 Yang-Baxter Deformations

The Undeformed AdS$_5 \times S^5$ Superstring Action

Let us briefly introduce the conventions for the supercoset $\sigma$ model with fields in

\[ \mathcal{M} = \frac{\text{PSU}(2,2|4)}{\text{SO}(1,4) \times \text{SO}(5)} \simeq \text{AdS}_5 \times S^5 \times \mathbb{C}^{0|16}, \]

which describes the Green-Schwarz type IIB superstring in AdS$_5 \times S^5$ [45], see [2] for an extensive review. The argumentation in the section 4 will also hold for general bosonic symmetric space $\sigma$ models and any supercoset $\sigma$ models which can be described similarly to the AdS$_5 \times S^5$ superstring.

The string moving in a coset $\mathcal{M} = G/H$ is described by $G$ valued fields $g : \Sigma \to G$ defined on the worldsheet $\Sigma$. The theory can be formulated in terms of the Maurer-Cartan forms taking values in the Lie algebra $g$ of $G$

\[ A = -g^{-1}dg \in g. \]

Important for the integrability of the AdS$_5 \times S^5$ superstring is the existence of the $\mathbb{Z}_4$-grading of $g = su(2,2|4)$:

\[ g = g^{(0)} \oplus g^{(1)} \oplus g^{(2)} \oplus g^{(3)}, \]

with the properties

\[ [M^{(i)}, N^{(j)}] \in g^{(i+j \text{ mod } 4)} \quad \text{for } M^{(k)}, N^{(k)} \in g^{(k)}, \]

and for the supertrace of a matrix realisation of $g$

\[ \text{STr}(M^{(i)}N^{(j)}) = 0 \quad \text{for } m + n \neq 0 \text{ mod } 4. \]

$g^{(2)}$ denotes the bosonic coset algebra, $g^{(0)}$ the little group algebra of the bosonic coset, and $g^{(1)}$ and $g^{(3)}$ are the odd parts of the algebra.\(^3\)

The action of the superstring in AdS$_5 \times S^5$ in conformal gauge\(^4\) takes the form [45]

\[ S \propto \int d^2\sigma \mathcal{L} = \int d^2\sigma \text{STr}(A_+ d_-(A_-)), \]

with the worldsheets light-cone components of $A$

\[ A_{\pm} = A_M \partial_{\pm} \tilde{Z}^M, \]

and the linear combinations of projection operators on the $\mathbb{Z}_4$-components

\[ d_{\pm} = \mp \tilde{q}^{(1)} + 2\tilde{q}^{(2)} \pm \tilde{q}^{(3)}. \]

Key features of the $\sigma$ model (2.5) are $\kappa$ symmetry and integrability. The latter is associated to a spectral parameter dependent Lax pair

\[ L_{\pm}(\lambda) = A_{\pm}^{(0)} + \lambda A_{\pm}^{(1)} + \lambda^{-2} A_{\pm}^{(2)} + \lambda^{-1} A_{\pm}^{(3)}, \]

\(^3\)We choose our superalgebra conventions as in [2], where elements of the algebra may be represented as an even supermatrix

\[ \left( \begin{array}{cc} m & \eta \\ \bar{\theta} & n \end{array} \right) \quad \text{with } m, n : \text{matrices built from } c\text{-numbers, } \eta, \bar{\theta} \text{ Grassmann-valued matrices} \]

Let us note, that we work with bosonic generators $\{ h_i \}$ and fermionic generators $\{ Q_\alpha \}$ being even respectively odd supermatrices with only even entries, so that e. g.

\[ g = \exp(X^i h_i + \theta^\alpha Q_\alpha) \quad A = -g^{-1}dg \]

are even supermatrices for a Grassmann-valued fields $\theta^\alpha$.

\(^4\)This is purely a choice of convenience and does not affect our analysis.
where the flatness condition
\[ \partial_+ L_- - \partial_- L_+ - [L_+, L_-] = 0 \]  

is equivalent to the equations of motion.

Let us now introduce integrable deformations of (super)coset \( \sigma \) models such as (2.5), based on solutions of the classical Yang-Baxter equation.

**The Classical Yang-Baxter Equation and Linear \( R \) operators**

The standard form of the classical Yang-Baxter equation (CYBE) defined on tensor products of an algebra or superalgebra \( g \) is
\[
[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \quad \text{for } r \in g \otimes g.
\]

Deformations are formulated in terms of equivalent linear operators \( R : g \rightarrow g \). The transition from a graded skewsymmetric \( r \) matrix to an \( R \) operator is via the trace
\[
r = a \wedge b := \frac{1}{2} (a \otimes b - (-1)^{s(a)s(b)} b \otimes a)
\]
\[
\rightarrow \quad R(M) := \text{STr}_2 (r \cdot (1 \otimes M)) = \frac{1}{2} \left( a \text{STr} (bM) - (-1)^{s(a)s(b)} \text{STr} (aM) \right),
\]
extended by linearity, where we refer to the parity of a supermatrix \( a \) as \( s(a) \). The CYBE in terms of the \( R \) operator takes the form
\[
[R(M), R(N)] - R ([R(M), N] + [M, R(N)]) = 0. \tag{2.9}
\]

Note that for the parities of a \( r \) matrix \( r = a \wedge b \) and the associated \( R \) operator we have \( s(r) = s(R) = s(a)s(b) \) and \( s(R(M)) = s(R)s(M) \).

A simple solution of (2.9) over a given algebra \( g \) is the \( r \) matrix consisting of graded commuting generators. In the following we will call these \( r \) matrices abelian.

### Deformations based on Solutions of the Classical Yang-Baxter Equation

Yang-Baxter deformations of coset \( \sigma \) models of the form of eqn. (2.5) are generated by skew-symmetric\(^5\) linear \( R \) operators solving (2.9). A further ingredient is the “dressing” of the \( R \) operator \( R_g = A^{-1}_g \circ R \circ A_g \). The Yang-Baxter deformed action is given by [11, 24]
\[
S \propto \int d^2 \sigma \mathcal{L} = \int d^2 \sigma \text{STr} (A_+ d_- (J_-)), \tag{2.10}
\]

where we introduced the deformed currents \( J_\pm = \frac{1}{1 \pm R \not{\rightarrow} d} (A_\pm) \), and directly specified to the (unmodified) classical Yang-Baxter case. Note that we include deformation parameters already in the definition of \( R \). These can take any real respectively Grassmannian value depending on the parity of the generating \( R \) operator, as the CYBE (2.9) is invariant under rescalings of \( R \).

These deformations preserve the \( \kappa \) symmetry and integrability of the undeformed model (2.5). The associated deformed Lax pair is
\[
L_\pm = J_\pm^{(0)} + \lambda J_\pm^{(1)} + \lambda^{\mp 2} J_\pm^{(2)} + \lambda^{-1} J_\pm^{(3)}. \tag{2.11}
\]

These deformations break part of the global \( G \) symmetry \( g \mapsto g'g \) for \( g' \in G \) of the undeformed model. The unbroken symmetries are generated by the generators \( T \) for which [42]
\[
R ([T, M]) = [T, R(M)] \quad \forall M \in g. \tag{2.12}
\]

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\(^5\)This means \( \text{STr} (MR(N)) = -\text{STr} (R(M)N) \).
3 T Duality Groups and their TsT Subgroups

In this section we will briefly recall bosonic and fermionic T duality and the associated TsT transformations in the $\sigma$ model context.

3.1 The Notion of Bosonic and Fermionic T duality

Consider a generic (classical$^6$) string $\sigma$ model of the form

$$S \propto \int d^2\sigma \partial_+ Z^M \varepsilon_{MN}(Z) \partial_- Z^N \equiv \int d^2\sigma \mathcal{L}, \quad M, N = 1, ..., D,$$

where we work in conformal gauge for the sake of convenience, and understand $Z^M$ as

$$Z^M = (X^\mu(\sigma), \theta^A(\sigma))$$

with some bosonic fields $X^\mu$ and some fermionic Grassmann-valued fields $\theta^A$. We refer to the parity of the coordinate $Z^M$ as $s(M)$. $\varepsilon_{MN}(Z)$ is the background field describing the coupling between the fields with parity $s(\varepsilon_{MN}) = s(M) + s(N)$, so that $s(\mathcal{L}) = 0$.

Now we assume the model has a manifest isometry and choose the associated coordinate to be $Z^1$, meaning the symmetry is realised as a shift of $Z^1$. We write $Z^M = (Z^1, Z^D)$ with $M = 2, ..., D$, so that $\varepsilon_{MN} = \varepsilon_{MN}(Z^D)$, $Z^1$ can be either bosonic or fermionic$^5$. This allows us to rewrite the Lagrangian by introducing gauge fields $A_\pm$:

$$\partial_\pm Z^1 \rightarrow A_\pm \quad \mathcal{L} \rightarrow \mathcal{L} - \partial_1^1(\partial_+ A_- - \partial_- A_+),$$

where the Lagrange multiplier $\partial_1^1$ ensures $A_\pm = \partial_\pm Z^1$ by its equations of motion. Integrating out $A_\pm$ instead of $\partial_1^1$ yields the action of the dual model

$$S \propto \int d^2\sigma \partial_1^1 \tilde{X}^M \varepsilon_{MN} \partial_- \tilde{X}^N,$$

with the dual background $\tilde{\mathcal{E}}$ given by

$$\tilde{\mathcal{E}}_{11} = (-1)^{s(1)} \frac{1}{\mathcal{E}_{11}}, \quad \tilde{\mathcal{E}}_{1M} = (-1)^{s(1)} \frac{\mathcal{E}_{1M}}{\mathcal{E}_{11}}, \quad \tilde{\mathcal{E}}_{MN} = -\frac{\mathcal{E}_{MN}}{\mathcal{E}_{11}}$$

$$\tilde{\mathcal{E}}_{MN} = \frac{\mathcal{E}_{MN}}{\mathcal{E}_{11}} \quad \text{for } M, N = 2, ..., D.$$  \hfill (3.2)

For T duality along a bosonic isometry we reproduce Buscher’s T duality rules [46]. For details on topological considerations and fermionic T duality and its implications in general we refer to e.g. [47, 48].$^9$

$^6$A dilaton $\phi$ enters the string action at a higher order in the coupling $\alpha'$. At the classical level the dilaton has to be introduced in the corresponding supergravity (e.g. the RR-forms appear always as $e^{F_{1111}}$). As we will not do explicit field redefinitions, we neglect it and its behaviour under T duality from the start. Working at the classical level we also disregard any prefactors of the action and are only interested in its schematic form.

$^7$The propagator $\varepsilon_{MN}$ could be decomposed into its graded symmetric (metric-like) and graded skew-symmetric part: $\varepsilon_{MN} = \eta_{MN} + B_{MN}$. But only the order $\theta^2$ terms in $\eta_{MN}$ respectively $B_{MN}$ would have a direct physical interpretation as the components of metric and $B$ field. We stick to the quite abstract ‘background’ $\varepsilon_{MN}$ as it is practical and sufficient for our further considerations.

$^8$In the fermionic case the generator $Q$ dual to the isometry coordinate has to anticommute with itself in order to correspond to a shift isometry. In other words, fermionic T duality requires a supercharge $Q$ with $Q^2 = 0$. We will come back to this point below.

$^9$Note that our conventions for the $\sigma$ model (3.1) differ from [47], leading to some different signs in (3.2). Furthermore note that, as defined, along a fermionic isometry coordinate only $T^2$, not $T^1$, is manifestly the identity operation. $T^2$ is a trivial and physically irrelevant coordinate redefinition of the background, $Z^1 \rightarrow (-1)^{(1)} Z^1$, however.
3.2 The O(d,d) Group of Bosonic T duality

Now we assume the model has d commuting bosonic isometries and choose the associated coordinates to be $X^i$ for $i = 1, ..., d$. We write $Z^M = (X^i, Z^i)$ with the $Z^i$-denoting the $D - d$ remaining non-isometry coordinates. In particular, $\tilde{E}_{MN} \equiv \tilde{E}_{MN}(Z^i)$. With the following fractional linear action of a $2D \times 2D$-matrix $G$ on $E$

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \rightarrow \quad \tilde{E} = (AE + B)(CE + D)^{-1},$$

(3.3)

a T duality transformation along $X^i$ can be represented for every $i \in \{1, ..., d\}$ as

$$G_{T_i} = \begin{pmatrix} 1_D - E_i & -E_i \\ -E_i & 1_D - E_i \end{pmatrix},$$

(3.4)

where $E_i$ is the $D \times D$-matrix with every element being zero, except for $(E_i)_{ii} = 1$. Other transformations, that even leave the Lagrangian invariant, are GL(d)-transformations of the isometry directions if we also transform $\tilde{E}$ accordingly. Let $A \in \text{GL}(d)$ and

$$X^i \rightarrow \tilde{X}^i = A^i_j X^j, \quad Z^i \rightarrow Z^i,$$

then the Lagrangian is invariant if

$$\tilde{E} = \begin{pmatrix} (A^T)^{-1} & \cdot \\ \cdot & 1_{D-d} \end{pmatrix} \cdot \tilde{E} \cdot \begin{pmatrix} A^{-1} \\ \cdot \end{pmatrix}.$$

This can be represented by fractional linear action (3.3) on $E$ of the group element

$$G_{\text{GL}} = \begin{pmatrix} (A^T)^{-1} & \cdot \\ \cdot & 1_{D-d} \end{pmatrix} \cdot \begin{pmatrix} A \\ \cdot \end{pmatrix},$$

(3.5)

Both $G_{T_i}$ and $G_{\text{GL}}$ are elements of $O(D,D)$, where we understand its elements as $2D \times 2D$-matrices $G$ fulfilling the pseudo-orthogonality relation

$$G J G^T = J, \quad J = \begin{pmatrix} 1_D & \cdot \\ \cdot & 1_{D-d} \end{pmatrix}.$$

(3.6)

The form of (3.4) and (3.5) suggests that we can write these as elements of $O(d,d)^{10}$ embedded in $O(D,D)$

$$G = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(d,d) \quad \rightarrow \quad G = \begin{pmatrix} a & 0 \\ \cdot & 1_{D-d} \end{pmatrix} \rightarrow \begin{pmatrix} b \\ \cdot \end{pmatrix} \rightarrow \begin{pmatrix} 1_D \\ 0_{D-d} \end{pmatrix} \in O(D,D).$$

(3.7)

Note that $\det G_{T_i} = -1$, so in fact bosonic T duality transformations itself are not in the component connected to the identity, in contrast to $G_{\text{GL}}$. But we can generate further elements of the component connected to the identity of $O(d,d)$ by a product of some general linear transformations and an even number of $T$ duality transformations.

\footnote{From discussions of the spectrum one can motivate the T duality group being the group of toroidal compactifications $O(d,d,Z)$. For example for closed strings, $O(d,d,Z)$ transformations correspond to "rotations" on the lattice describing winding numbers and Kaluza-Klein excitation numbers associated to the compact (toroidal) $(U(1))^2$-isometry, which leave the spectrum invariant. This is reviewed in e.g. [49]. In the above $c$ model, however, we consider theories that are equivalent modulo boundary conditions; $TST$ transformations can be absorbed in twisted boundary conditions [7, 50].}
Bosonic TsT Transformations

Now we introduce TsT transformations in the above framework. These gained some attention in the context of the AdS/CFT correspondence, as a particular TsT transformation of the AdS$_5 \times S^5$ background gives a supergravity background dual to $\beta$ deformed SYM [5]. To do TsT transformations we need at least two isometries, which we parameterise by $X^1$ and $X^2$ in the following. A single TsT transformation is generated by a $T$ duality transformation on the $X^1$, a shift

$$X_2 \rightarrow X_2 - \gamma X_1$$

and then a $T$ duality transformation on the $X^1$ direction back. In the above group language, in the minimal $d = 2$ setting this looks like

$$g_{\Gamma_1 \cdot \Gamma_2} = \left( \begin{array}{cc} 1 & \gamma \\ 0 & 1 \end{array} \right) \cdot g_{\Gamma_1} = \left( \begin{array}{cc} 1 & 1 \\ 0 & -\gamma & 1 \end{array} \right).$$

(3.12)

Generic TsT transformations can be understood as the straightforward generalisation to fractional linear transformations of the type (3.3) with the generating group element

$$g_{\Gamma} = \left( \begin{array}{c} 1_d \\ \Gamma \\ 1_d \end{array} \right) \in SO(d,d),$$

(3.13)

meaning we can construct generic TsT transformations by executing subsequent single TsT transformation. TsT transformations form an abelian subgroup of the component connected to the identity of O$(d,d)$.

3.3 OSp$(d_b, d_f | 2d_f)$ as the Superduality Group

Consider a background $\mathcal{E}$ with $d_b$ bosonic and $d_f$ fermionic isometries and $d = d_b + d_f$. Let us write our coordinates as

$$Z^M = (Z^a, Z^\theta) = (X^i, \theta^\alpha, Z^a),$$

with $i = 1, ..., d_b$ and $\alpha = 1, ..., d_f$.

(3.15)

Note that this a quite specific transformation. Generic coordinate transformations would also lead to contributions in the other blocks of an O$(d,d)$ element in comparison to (3.12). Shifts in the “other” direction like

$$\bar{X}_1 \rightarrow \bar{X}_1 - \theta \bar{X}_2$$

(3.8)

between two $T$ duality transformations would lead to

$$g_{\Theta} = \left( \begin{array}{ccc} 1 & 0 & -\theta \\ 0 & \theta & 0 \\ 1 & 1 \end{array} \right).$$

(3.9)

these are called $\Theta$ shifts and build an abelian subgroup of O$(d,d)$, created by skewsymmetric $d \times d$-matrices $\Theta$ in the upper right block:

$$g_{\Theta} = \left( \begin{array}{cc} 1_d & \Theta \\ 0 & 1_d \end{array} \right) \in SO(d,d).$$

(3.10)

The background is transformed with (3.7) and (3.3) only in the isometry components as

$$\tilde{E}_{ij} = E_{ij} + \Theta_{ij} \quad \leftrightarrow \quad \tilde{B}_{ij} = B_{ij} + \Theta_{ij},$$

where $B_{ij}$ are components corresponding to the isometry directions of the $B$-field. While these coordinate shifts (3.8) look quite similar to the ones of TsT transformations, $\Theta$ shifts act very differently on the background. $\Theta$ shifts clearly generate physically equivalent models up to boundary terms, as $H = dB$ remains invariant.
The matrix representation in the sense of (3.3) and (3.7) of a single T duality transformation (3.2) along the isometry coordinate \( Z^a \) is\(^{12} \)

\[
S_{T_a} = \begin{pmatrix}
1_d - E_a & -E_a \\
(-1)^{s(a)} E_a & 1_d - E_a
\end{pmatrix}.
\] (3.16)

We can further consider \( \text{GL}(d_b|d_f) \) coordinate transformations of the \( Z^a = (X^i, \theta^a) \)

\[
Z^a \rightarrow \bar{Z}^a = A^a_{\ b} Z^b
\]

with a supermatrix

\[
A = \begin{pmatrix} m & \eta \\ \theta & n \end{pmatrix} \in \text{GL}(d_b|d_f).
\]

With supertransposition defined as

\[
A^{ST} = \begin{pmatrix} m & \eta \\ \theta & n \end{pmatrix}^{ST} = \begin{pmatrix} m^T & \theta^T \\ -\eta^T & n^T \end{pmatrix},
\]

the “group element” of such a \( \text{GL}(d_b|d_f) \)-transformation with the action (3.3) on the background components \( E \) in the conventions of (3.1) is given similarly to (3.5) by

\[
S_{GL} = \begin{pmatrix} (A^{ST})^{-1} \\ A \end{pmatrix} \quad \text{for } A \in \text{GL}(d_b|d_f). \] (3.17)

It is easy to show that both (3.16) and (3.17) are elements of a group with elements

\[
S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{with } A, B, C, D \in \mathbb{R}^{(d_b|d_f)\times(d_b|d_f)}
\]

fulfilling a modified pseudoorthogonality relation (in comparison to (3.6))

\[
S S^{ST} = I \quad \text{with } \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{ST} = \begin{pmatrix} A^{ST} & C^{ST} \\ B^{ST} & D^{ST} \end{pmatrix} \quad \text{and } J = \begin{pmatrix} 1_{d_b} & 1_{d_f} \\ -1_{d_f} & 1_{d_b} \end{pmatrix}. \]

(3.18)

This is a representation\(^{13} \) of the orthosymplectic group \( \text{OSp}(d_b,d_b|2d_f) \) and nicely generalises the \( \text{O}(d_b,d_b) \) group of bosonic T duality. This group was previously introduced in [51], see also [52]. We will constrain further discussion of \( \text{OSp}(d_b,d_b|2d_f) \) to the generalisation of generic \( T\bar{S}\bar{T} \) transformations (3.13) of the bosonic case.

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\(^{12}\)Note that \( \det g_{T_a} = -(-1)^{s(a)}. \)

\(^{13}\)More commonly one defines \( \text{OSp}(m,m|2n) \) as the group consisting of \((2m|2n) \times (2m|2n)\)-supermatrices \( M \) preserving the supermetric \( J \)

\[
M J M^{ST} = J \quad \text{with } J = \begin{pmatrix}
1_n & -1_m \\
-1_m & 1_n
\end{pmatrix}. \]

(3.13)

\( J \) and \( J \) from (3.18) are connected via a similarity transformation

\[
J = O_2 J_1 J_0 O_2 \quad \text{with } O_1 = \begin{pmatrix} \frac{1}{1/2} & \frac{1}{1/2} & \frac{1}{1/2} & \frac{1}{1/2} \\ \frac{1}{1/2} & \frac{1}{1/2} & \frac{1}{1/2} & \frac{1}{1/2} \\ \frac{1}{1/2} & \frac{1}{1/2} & \frac{1}{1/2} & \frac{1}{1/2} \\ \frac{1}{1/2} & \frac{1}{1/2} & \frac{1}{1/2} & \frac{1}{1/2} \end{pmatrix} \quad \text{and } O_2 = \begin{pmatrix} 1_n & 0_n & 0_n & 1_n \\ 0_n & 1_{n\times n} & 1_{n\times n} & 0_n \\ 1_{n\times n} & 1_{n\times n} & 1_{n\times n} & 1_{n\times n} \\ 0_n & 1_{n\times n} & 1_{n\times n} & 1_{n\times n} \end{pmatrix}. \]
From here on, we therefore understand generic deformations (3.18) of the coset manifold with manifest shift invariance in \( d = d_b + d_f \) coordinates, we will prove that the (coordinate-dependent) \( TsT \) transformation behaviour (3.19) can be reproduced by an abelian \( K \) operator, and vice versa. As the Yang-Baxter deformed action (2.10) is independent of parameterisation this introduces a coordinate-independent notion of \( TsT \) transformations in the form of abelian Yang-Baxter deformations.

\[ \mathcal{S}_\Gamma = \begin{pmatrix} 1_d & \Gamma \\ \Gamma^\dagger & 1_d \end{pmatrix}. \]  

(3.19)

This lies in our representation (3.18) of \( \text{OSp}(d_b, d_b|2d_f) \) for

\[ \Gamma = \begin{pmatrix} \Lambda_b & \Omega \\ -\Omega^\dagger & \Lambda_f \end{pmatrix} \]

with a real skewsymmetric \( d_b \times d_b \) matrix \( \Lambda_b \), a Grassmann-valued \( d_b \times d_f \) matrix \( \Omega \) and a real symmetric \( d_f \times d_f \) matrix \( \Lambda_f \). Similarly to the bosonic case above, group elements of this type form an abelian subgroup of \( \text{OSp}(d_b, d_b|2d_f) \).

The group element (3.19) now corresponds to a sequence of \( Ts(T^{-1}) \) transformations, with shifts defined as in (3.11). Purely fermionic \( Ts(T^{-1}) \) transformations look like

\[ \mathcal{S}_{f_{1/2}} = \mathcal{S}_{f_1} \cdot \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \cdot \mathcal{S}_{f_1}^{-1} = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \]

(3.20)

and indeed schematically \( T_f \mathcal{S}_{ff} T_f^{-1} \) give rise to symmetric, but off-diagonal entries in \( \Lambda_f \) in (3.19). It turns out that the diagonal elements in \( \Lambda_f \) cannot be understood as a type \( \mathcal{S}_T \cdot \mathcal{S}_\Gamma \cdot \mathcal{S}_T^{-1} \) transformation. From here on, we therefore understand generic \( Ts(T^{-1}) \) transformations as group elements of \( \text{OSp}(d_b, d_b|2d_f) \) of the type (3.19) with generic symmetric, but off-diagonal \( \Lambda_f \).

Let us note that there is no ambiguity for \( Ts(T^{-1}) \) transformations “mixing” bosons and fermions: \( T_f \mathcal{S}_{ff} T_f^{-1} \) - and \( T_b \mathcal{S}_{sf} T_b^{-1} \)-type transformations are equivalent and both correspond to the (skewsymmetric) odd part of \( \Gamma \) in (3.19). Of course \( Ts(T^{-1}) \) transformations directly reduce to \( TsT \) transformations if the \( T \) duality is a bosonic one and so, for the sake of simplicity, we will refer to \( Ts(T^{-1}) \) transformations as \( TsT \) transformations from now on. Both only differ by a trivial coordinate redefinition in any case.

4 Equivalence of Abelian Yang-Baxter Deformations and TsT Transformations

In this section we prove that any Yang-Baxter deformation generated by an abelian solution to the CYBE is equivalent to a \( TsT \) transformation at the level of the corresponding \( \sigma \) model.

This equivalence was previously proposed in [42], and is supported by many examples, see e.g. [44, 40, 41], but a general proof is still missing. We will also extend this claim by considering \( r \) matrices built out of anticommuting supercharges. Using a parameterisation of the coset manifold with manifest shift invariance in \( d = d_b + d_f \) coordinates, we will prove that the (coordinate-dependent) \( TsT \) transformation behaviour (3.19) can be reproduced by an abelian \( K \) operator, and vice versa. As the Yang-Baxter deformed action (2.10) is independent of parameterisation this introduces a coordinate-independent notion of \( TsT \) transformations in the form of abelian Yang-Baxter deformations.

\(^{14}\) Up to \( T \) duality transformations, the effect of diagonal elements of \( \Lambda_f \) on the background is equivalent to a shift of \( \mathcal{E} \). Namely \( \mathcal{S}_{f_{1/2}} = T^{-1} \circ (\mathcal{E}_a \rightarrow \mathcal{E}_a + \Lambda_{f,aa}) \circ T, \quad a = 1, \ldots, d_f \).
4.1 Natural Parameterisation with Manifest Shift Isometries

The starting point of our proof is to choose a natural parameterisation of the coset manifold where we have shift isometries in the coordinates associated to (anti)commuting generators \( t_a \), namely

\[
\hat{g} = \exp(Z^i t_i) \hat{g}(Z^\bar{i}).
\]  

(4.1)

There the \( Z^i \) are the \( d = d_p + d_f \) isometry coordinates and \( Z^\bar{i} \) are the remaining coordinates, \( Z^M = (Z^i, Z^\bar{i}) = (X^i, \bar{\theta}^i, Z^\bar{i}) \). \( \hat{g} \) is assumed to be chosen in a way that the metric is non-degenerate, so we can consider (4.1) to be a valid parameterisation of the coset manifold. This is motivated for instance by the group parameterisations of AdS in Poincaré coordinates as

\[
\hat{g}_{\text{AdS}} = e^{X^\mu p_\mu} z^{-D}, \quad \text{with } \mu = 0, 1, 2, ..., N-2
\]

where \( p^\mu \) respectively \( D \) are the momentum respectively dilatation generators of the conformal algebra \( \text{so}(2, N-1) \). There we have \( N-1 \) isometries parameterised by \( X^\mu \), as \( [p^\mu, p^\nu] = 0 \) by means of the conformal algebra. This type of group parameterisation should always be possible for general group and coset manifolds and any choice of (anti)commuting generators \( t_a \) in the symmetry algebra. Let us sketch a proof for the bosonic case.

We assume that we have a geometry with \( d \) commuting Killing vector fields. Then there are coordinates \( Z^M = (X^i, Y^i) \) in which these vector fields are \( \frac{\partial}{\partial X^i} \), thus the commuting isometries are parameterised by \( X^i \). In particular, the background and a choice of a local frame \( e^a_i \) with a corresponding spin connection \( \omega^{ab}_i \) are independent of the \( X^i \).

The Maurer-Cartan form on a coset manifold (see e.g. [45]) decomposes into

\[
A = -\hat{g}^{-1} d\hat{g} = e^a_i P_a dX^\mu + \omega^{ab}_i j_{ab} dX^\mu
\]

with coset generators \( P_a \) and isotropy generators \( j_{ab} \), so in our case

\[
A = A_i(Y) dX^i + A_2(Y) dY^2,
\]

The flatness of \( A \) implies that

\[
[A_i(Y), A_j(Y)] = 0 \quad \text{due to } \partial_i A_j = 0 \quad \forall i, j = 1, ..., d.
\]

For every \( Y \) these span a \( d \)-dimensional commuting algebra. It follows there is similarity transformation with a group valued function \( g_2(Y) \)

\[
A_i(Y) = g_2^{-1}(Y) h_i g_2(Y) \quad \forall i = 1, ..., d,
\]

(4.3)

where the \( h_i \) are the constant commuting generators of the algebra corresponding to the Lie algebra of the commuting Killing vector fields. Note that we use the notation \( h_i \) for a general set of commuting generators, which in the non-compact case will generically not be the Cartan generators.

Now consider a group parameterisation \( \hat{g} = \exp(X^h h_i) g_2(Y) \) with \( \hat{A} = -\hat{g}^{-1} d\hat{g} \). It follows that

\[
\hat{A}_i = A_i \quad \Rightarrow \quad \hat{g} = \hat{g}_1(Y) \exp(X^h h_i) g_2(Y) \quad \text{for some } \hat{g}_1(Y).
\]

Again from the flatness of \( \hat{A} \) follows that

\[
\partial_i \hat{A}_j = \partial_j \hat{A}_i + [A_i, A_j] = 0 \quad \Rightarrow \quad [A_i, A_j] = [A_i, \hat{A}_j]
\]

\[
\Rightarrow \quad [\Ad_{\hat{g}}^{-1}(-\hat{g}_1^{-1} \partial \hat{g}_1), A_i] = \Ad_{\hat{g}}^{-1}\left([-\hat{g}_1^{-1} \partial \hat{g}_1, h_i]\right) = 0,
\]

so that \( \hat{g}_1 \) is generated by the \( h_i \). It follows that a group parameterisation of the form

\[
\hat{g} = \exp(X^h h_i) \hat{g}_1(Y) g_2(Y) \equiv \exp(X^h h_i) \hat{g}(Y)
\]

(4.4)

exists for any choice of commuting generators \( h_i \).

---

15 In the non-compact case there are inequivalent choices of commuting subalgebras/isometries. These inequivalent choices would correspond to different choices of our Killing vector fields at the beginning of the proof.
4.2 Bosonic Abelian Yang-Baxter Deformations

Now consider a generic abelian matrix that consists some bosonic commuting generators \( h_i \) of the global symmetry algebra of the coset model

\[
r = -\Gamma^{ij} h_i \wedge h_j,
\]

(4.5)

with a (real) antisymmetric \( d \times d \) parameter matrix \( \tilde{\Gamma}^{ij} \). Consider a parameterisation of the form (4.1),

\[
g = \exp(X^i h_i) \tilde{g}(Y).
\]

(4.6)

Due to the fact that the \( h_i \) commute, the Maurer-Cartan form becomes

\[
A = -g^{-1} d g = -\text{Ad}_{g}^{-1}(dX^i h_i) + \bar{A}(Y) = -\text{Ad}_{g}^{-1}(h_i) dX^i + \bar{A}(Y) \equiv A_i(Y) dX^i + \bar{A}(Y),
\]

(4.7)

and the Lagrangian is manifestly shift-invariant in the \( X^i \)-coordinates. With this we see that the abelian \( r \) matrix (4.5) is actually built from some components of the conserved currents with respect to the global symmetry of the coset \( \sigma \) model, \( A^R = \text{Ad}_{g}(A) = -d g g^{-1} \). The corresponding dressed \( r \) matrix then is

\[
r_s = (\text{Ad}_{g}^{-1} \otimes \text{Ad}_{g}^{-1}) \cdot r
\]

(4.8)

and the associated linear \( R \) operator can be expressed nicely in terms of the Maurer-Cartan form components

\[
r_s = -\tilde{\Gamma}^{ij} A_i \wedge A_j \quad \Rightarrow \quad R_s(M) = \text{STr}_2 (r_s \cdot (1 \otimes M)) = -\tilde{\Gamma}^{ij} A_i \text{STr}(A_j M).
\]

(4.9)

Writing

\[
\Gamma = \begin{pmatrix}
\tilde{\Gamma} \\
0_{D-d}
\end{pmatrix},
\]

it follows that

\[
R_s \circ d_-(A_N) = -\Gamma^{ij} A_i \text{STr}(A_j d_-(A_M)) = A_M(-\Gamma \varepsilon)^M_N
\]

\[
(R_s \circ d_-)^n(A_N) = A_M((-\Gamma \varepsilon)^n)^M_N.
\]

The Yang-Baxter deformed Lagrangian (2.10) then becomes

\[
\mathcal{L} \propto \partial_+ X^M \tilde{\varepsilon}_{MN} \partial_- X^N
\]

(4.10)

with the general coordinates \( X^M = (X^i, Y^2) \) and the deformed background

\[
\tilde{\varepsilon}_{MN} = \text{STr} \left( A_M d_- \circ \frac{1}{1 - R_s \circ d_-(A_N)} \right)
\]

\[
= \sum_{n=0}^{\infty} \text{STr} \left( A_M d_- \circ (R_s \circ d_-)^n(A_N) \right) = \sum_{n=0}^{\infty} \text{STr} \left( A_M d_- (A_K) \right) ((-\Gamma \varepsilon)^n)^K_N
\]

\[
= \tilde{\varepsilon}_{MK} (1 + \Gamma \varepsilon)^{-1}_N.
\]

(4.11)

This directly corresponds to the \( \text{O}(d,d) \) group element (3.13) describing a generic bosonic \( T \tau T \) transformation.
4.3 Inclusion of Fermions

A generic abelian graded skewsymmetric $r$ matrix over a Lie superalgebra in our conventions is built out of (anti)commuting even (odd) generators $\{ h_i, Q_\alpha \}$ with
\[
[h_i, h_j] = 0, \quad [h_i, Q_\alpha] = 0 \quad \{ Q_\alpha, Q_\beta \} = 0 \quad \text{for } i, j = 1, \ldots, d_b \text{ and } \alpha, \beta = 1, \ldots, d_f,
\]
as
\[
r = -\Lambda_b^i h_i \wedge h_j - \Omega^{ia} h_i \wedge Q_\alpha - \Omega^{ai} Q_\alpha \wedge h_j - \Lambda_f^\beta Q_\beta \wedge Q_\beta \equiv -\Omega^{ab} t_a \wedge t_b, \quad \text{(4.12)}
\]
with $t_a = (h_i, Q_\alpha)$ and a graded skewsymmetric $(d_b \vert d_f) \times (d_b \vert d_f)$-matrix
\[
\Gamma = \begin{pmatrix} \Lambda_b & \Omega \\ -\Omega^T & -\Lambda_f \end{pmatrix}.
\]

Here $\Lambda_f$ is a symmetric, but off-diagonal real $d_f \times d_f$-matrix, $\Omega$ is an arbitrary Grassmann-valued $d_b \times d_f$-matrix and $\Lambda_b$ is a skewsymmetric real $d_b \times d_b$-matrix. We should emphasize that $su(2|2)$ and $psu(2|2)$ do not contain real supercharges that anticommute with themselves, so these fermionic extensions of abelian $r$ matrices do not exist for the real $AdS_5 \times S^5$ superstring, or its $AdS_3$ and $AdS_2$ cousins. To consider them we need to work with the complexified model. The $r$ matrices are then complex and break reality of the action, but are otherwise admissible.

With some care\textsuperscript{16} regarding the Grassmann-valued fields $\theta$ the proof works in the same way as in the bosonic case. First we choose a group parameterisation with manifest isometries corresponding to the (anti)commuting generators and express the $R_\xi$ operator corresponding to (4.12) by some components of the Maurer-Cartan form.

\[
\xi = \exp(\theta^a Q_\alpha) \xi(Z^\alpha)
\]

\[
A = -\text{Ad}^{-1}_g (dX^i h_i + d\theta^a Q_\alpha) + A(Z^\alpha)
\]

\[
\equiv -A_i dX^i - A_\alpha d\theta^\alpha + A(Z^\alpha) = -A_i dX^i - d\theta^a A_\alpha + A(Z^\alpha)
\]

\[
R_\xi(M) = -A_\alpha^T \xi \text{STr}(A_\alpha^T M)
\]

The undeformed background $\mathcal{E}_{MN}$ is given terms of the components of the Maurer-Cartan form in the conventions of (3.1) and (2.5) by
\[
\mathcal{E}_{MN} = \text{STr}(A_M^T d_- (A_N^T)),
\]
so we get
\[
(R_\xi \circ d_-)^n(A_N^T) = A_M^T (-\Gamma \xi)^n M_N \quad \text{with } \Gamma = \begin{pmatrix} \tilde{\Gamma} & 0_{D-d_b-d_f} \\ 0_{d_b-d_f} & -\xi \end{pmatrix}.
\]

In the same way as in the bosonic case the abelian Yang-Baxter deformation results in a deformed background
\[
\tilde{\xi} = \xi(1 + \Gamma \xi)^{-1}.
\]

In other words, we directly reproduce the generic $TsT$ transformation behaviour (3.19) of the superduality group $\text{OSp}(d_b \vert d_f)$, and vice versa.

The direct approach via a natural parameterisation with manifest isometries like (4.1) is useful to see the $TsT$ behaviour of abelian Yang-Baxter deformations as in (3.13), in particular to determine its effect on the concrete background. The abelian Yang-Baxter deformation in the form (2.10) on the other hand, gives a coordinate-independent representation of

16This is rather tedious with our conventions, as for the fermionic Maurer-Cartan components
\[
A^{\alpha} := A_\alpha^T d\theta^\alpha = d\theta^a A_\alpha^T \quad \text{with e.g. } A_\alpha^T = -g^{-1} Q_\alpha (g^{ST})^{ST}.
\]
It is important to pay attention to some subtleties of the graded tensor product in the definition of $r_\xi = (\text{Ad}_\xi^{-1} \otimes \text{Ad}_\xi^{-1}) \cdot r$ which match the above ambiguity and lead to the desired $R_\xi$ operator in (4.14).
TsT transformations (in contrast to the OSp(d_f,d_f|2d_f)-approach). Moreover this manifestly shows that every TsT transformation of such a (super)coset gives an integrable model with (2.11) as the associated Lax pair.

Abelian Yang-Baxter deformed models correspond to supergravity solutions by construction, as T duality and thus TsT transformations map two supergravity solutions to each other [53], also in the fermionic case [47]. This matches the analysis of [27], as any abelian r matrix is unimodular.

5 On Inequivalent TsT Transformations

In this section we want to illustrate the fact that there are different inequivalent sets of commuting shift isometries and thus TsT transformations on non-compact backgrounds. For completeness we start with TsT transformation of S^3.

5.1 Sphere S^3

We have seen in the previous section that a natural parameterisation of the background with d commuting isometries is \( g = \exp(X^i h_i) \) with a choice of d commuting generators \( \{ h_i \} \). As \( S^N \) and its isometry group \( O(N+1) \) is compact, any other choice of the commuting generators \( \{ k_i \} \) is connected via a similarity transformation with a group element S related to the \( \{ h_i \} \) as \( k_i = Sh_iS^{-1} \). Exactly as in (4.3) the corresponding group element

\[
g_k = \exp(X^i k_i) \quad \Rightarrow \quad A_k = -g_k^{-1}d g_k = A
\]

yields the same background as g because S is constant.

We work with generators \( n_{ij} \) of \( so(N+1) \), satisfying

\[
n_{ij} n_{kl} = \delta_{il} n_{jk} - \delta_{jl} n_{ik} - \delta_{jk} n_{il} + \delta_{jk} n_{il} \quad i,j,k,l = 1,\ldots,N+1.
\]

S^3 is the minimal example for the study of TsT transformations on spheres, with the rank of \( so(4) \) being two. We choose \( n_{12}, n_{34} \) as the Cartan basis, \( r = -\gamma n_{12} \wedge n_{34} \) and the corresponding group parameterisation with manifest isometries to be

\[
\exp(\phi_1 n_{12} + \phi_2 n_{34}) \exp(\theta n_{24}).
\]

This corresponds to the metric

\[
(ds)^2 = \sin^2 \theta(d\phi_1)^2 + \cos^2 \theta(d\phi_2)^2 + (d\theta)^2.
\]

The TsT deformed three-sphere looks like

\[
(ds)^2_{\text{def}} = \frac{1}{1 + \frac{2}{3}(1 - \cos(4\theta))} \left( \sin^2 \theta(d\phi_1)^2 + \cos^2 \theta(d\phi_2)^2 \right) + (d\theta)^2
\]

\[
B_{\text{def}} = \frac{2}{1 + \frac{2}{3}(1 - \cos(4\theta))}d\phi_1 \wedge d\phi_2.
\]

\[17\]In terms of the action on the background fields, the standard treatment of T duality for a supergravity background coupling to a Green-Schwarz superstring [54, 55] does not admit an immediate O(d,d)-like formulation of TsT transformations. However, an appropriate extension to the Ramond-Ramond forms exists [56, 57, 58]. The action of the superduality group OSp(d_f,d_f|2d_f) on the supergravity fields has not been investigated yet to our knowledge. For fermionic T duality transformations themselves some progress was made in [59] in the canonical formulation. TsT transformations including fermions were studied previously in [50] for deformations of S^3 in the r model approach.
5.2 Anti-de Sitter Space AdS₃

In the non-compact case there are inequivalent choices of commuting generators. We will only explicitly discuss the inequivalent deformations of AdS₃, where this undertaking is greatly simplified due to the structure of so(2,2). This gives some insight in the various possible abelian Yang-Baxter deformations of AdS₃.

The symmetry algebra of AdS₃ is so(2,2), which has the nice decomposition\(^{18}\)
\[ so(2,2) \simeq sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R}). \] (5.4)

From here we can immediately read off all possible commuting isometries, namely one arbitrary element of each factor. We work with the following representation of \(sl(2,\mathbb{R})\)
\[ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]
\[ [h,a] = 2a, \quad [h,b] = -2b, \quad [a,b] = h \]
and so(2,2) generators \(m_{ij}\) resp. conformal generators \(p_\mu, k_\mu, D, m_{01}\)
\[ [m_{ij}, m_{kl}] = \eta_{ij}m_{jk} - \eta_{ij}m_{kj} - \eta_{ik}m_{jl} + \eta_{ik}m_{lj} \quad i,j,k,l = 0, \ldots, 3 \]
\[ \eta = \text{diag}(-1,1,1,-1) \]
\[ p_\mu = m_{\mu2} + m_{\mu3}, \quad k_\mu = m_{\mu2} - m_{\mu3} \quad \text{and} \quad D = m_{23} \quad \mu = 1,2. \]

Then we see that the two copies of \(sl(2,\mathbb{R})\) in so(2,2) are spanned by
\[ h_1 = m_{01} - D \quad a_1 = p_+ \quad b_1 = k_- \]
respectively
\[ h_2 = m_{01} + D \quad a_2 = k_+ \quad b_2 = p_- \]
with \(v_{\pm} := \frac{1}{2}(v_0 \pm v_1)\). Explicitly, generic abelian \(r\) matrices are of the form
\[ r = s_1 \wedge s_2 \quad \text{with} \quad (s_1, s_2) \in sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R}) \simeq so(2,2). \] (5.5)

From the point of view of the Yang-Baxter deformations the overall scaling of the \(r\) matrix only contributes to the deformation parameter, so for each factor in (5.5) we only need to consider \(\det s < 0, \det s > 0\) or \(\det s = 0\). These three classes of generators are clearly inequivalent to each other under similarity transformations \(S = ScS^{-1}\) with \(S \in SL(2,\mathbb{R})\). SL(2,\mathbb{R}) moreover acts transitively on each class (up to rescaling). Convenient representants are
1. \(\det s = -1: s \sim h\)
2. \(\det s = 0: s \sim a\)
3. \(\det s = 1: s \sim a - b\).

We can now combine these \(sl(2,\mathbb{R})\) generators of both copies in so(2,2) to a generic \(r\) matrix. Exchanging the two copies of \(sl(2,\mathbb{R})\) is an outer automorphism of so(2,2)
\[ h_1 \leftrightarrow h_2 \quad a_1 \leftrightarrow a_2 \quad b_1 \leftrightarrow b_2 \]
The physical interpretation is either
\[ D \leftrightarrow -D, \quad p \leftrightarrow k \quad \text{or} \quad D \leftrightarrow -D, \quad + \leftrightarrow -. \] (5.6)

With use of (5.6) we are left with six types of abelian \(r\) matrices, namely:

\(^{18}\)This structure essentially makes it possible to independently deform the two factors also for quantum deformations [60].
• \( h_1 \wedge h_2 \) corresponds to the (non-compact) Cartan \( r \) matrix \( r = -\gamma m_{01} \wedge D \). A convenient parameterisation is given by \( g = \exp(\theta m_{01} + \ln(z)D) \exp((uz)p_0) \), corresponding to the metric

\[
(ds)^2 = -(zd\rho)^2 + (uz)^2(d\theta)^2 + (d\ln(z))^2
\]

of hyperpolar Poincaré coordinates. A coordinate change \( u \to x/z \) yields \( \ln(z) \) and the boost-angle \( \theta \) as isometry coordinates. The associated Yang-Baxter deformed background reads

\[
(ds)_{\text{def}}^2 = \frac{1}{1 + \gamma^2(uz)^2} \left[ -(1 + \gamma^2(uz)^2)z^2(d\rho)^2 + (uz)^2(d\theta)^2 - 2\gamma^2 u^2 z^4 d\rho \ln(z) + \left(1 - \gamma^2(uz)^4\right)(d\ln(z))^2 \right],
\]

\[
B_{\text{def}} = \frac{2\gamma(uz)^2(z^2 u^2 d\rho + d\ln(z))}{1 + \gamma^4(uz)^2 - \gamma^2(uz)^4} \wedge d\theta,
\]

in terms of the original hyperpolar Poincaré coordinates.

• \( (a_1 - b_1) \wedge (a_2 - b_2) \) translates to the (compact) Cartan \( r \) matrix \( r = -\gamma m_{03} \wedge m_{12} \) leading to a TST transformation corresponding to time shifts and spatial rotations. These are natural in global coordinates, where both isometries are manifest. With a group parameterisation \( g = \exp(\phi m_{03} + \theta m_{12}) \exp(\rho m_{23}) \) the undeformed and deformed backgrounds are

\[
(ds)^2 = -\cosh^2 \rho(d\phi)^2 + \sinh^2 \rho(d\theta)^2 + (d\rho)^2,
\]

\[
(ds)_{\text{def}}^2 = \frac{1}{1 + \frac{1}{4}(1 - \cosh(4\rho))} \left[ -\cosh^2 \rho(d\phi)^2 + \sinh^2 \rho(d\theta)^2 \right],
\]

\[
B_{\text{def}} = \frac{\frac{1}{2}\sinh^2(2\rho)}{1 + \frac{1}{4}(1 - \cosh(4\rho))} d\phi \wedge d\theta.
\]

• \( a_1 \wedge a_2 \) corresponds to \( \tilde{r} = -\gamma p_+ \wedge p_- \sim r = -\gamma p_0 \sim p_1 \). With group parameterisation \( g = \exp(-x_0 p_0 + x_1 p_1) z^D \) the undeformed and deformed backgrounds are

\[
(ds)^2 = z^2 \left[ -(d\rho)^2 + (d\rho_1)^2 \right] + (d\ln(z))^2,
\]

\[
(ds)_{\text{def}}^2 = \frac{z^2}{1 - \gamma^2 z^2} \left[ -(d\rho)^2 + (d\rho_1)^2 \right] + (d\ln(z))^2,
\]

\[
B_{\text{def}} = \frac{2\gamma z^4}{1 - \Gamma^2 z^4} d\rho_0 \wedge d\rho_1.
\]

The manifest isometry coordinates for the remaining three \( r \) matrices are not very intuitive as the \( r \) matrices mix the generators corresponding to costumary choices of coordinates (like global or Poincaré coordinates). We therefore give the deformed backgrounds in light-cone Poincaré coordinates (group parameterisation \( g = \exp(x_+ p_+ + x_- p_-) z^D \))

\[
(ds)_{\text{def}}^2 = -z^2 dx_+ dx_- + (d\ln(z))^2.
\]

• \( h_1 \wedge a_2; r = -\gamma (m_{01} - D) \wedge p_- \)

\[
(ds)_{\text{def}}^2 = -C \left( \frac{7}{4} z^4 (dx_+)^2 + z^2 dx_+ dx_- + \gamma^2 x_- z^3 dz dx_- \right) + (d\ln(z))^2,
\]

\[
B_{\text{def}} = \gamma C \left( x_- z^4 dx_- \wedge dx_+ + z dx_- \wedge dz \right).
\]

with \( C^{-1} = 1 - \gamma^2 x_- z^4 \).
• \( h_1 \wedge (a_2 - b_2): r = -\gamma (m_{01} - D) \wedge (p_- - k_+) \)

\[
(d\sigma)^2_{\text{def}} = -C \left( \frac{r^2}{4} (1 + x_+^2)^2 z^4 \left( dx_- \right)^2 + z^2 \left( 1 - \frac{r^2}{2} (2x_- x_+ (1 + x_+^2)(x_+^2 - x_-^2) - 1) \right) dx_- dx_+ \\
+ \gamma^2 x_- (1 + x_+^2)^2 z^3 dx_- dz + \frac{r}{4} (1 - 2x_- x_+ z^2)^2 \left( dx_+ \right)^2 \\
- \gamma^2 x_- (1 + x_+^2) z (1 - 2x_- x_+ z^2) dx_+ dz - \frac{1 - \gamma^2 x_+^2 (1 + x_+^2)^2 z^4}{z^2} (dz)^2 \right),
\]

\[
B_{\text{def}} = -\gamma \ C \left( x_- (1 + x_+^2) z^4 dx_- \wedge dx_+ + (1 + x_+^2) z dx_- \wedge dz + (1 - 2x_- x_+ z^2) dx_+ \wedge dz \right). 
\]

(5.11)

with \( C^{-1} = 1 - \gamma^2 (1 + (x_+ - x_- (1 + x_+^2)z^2)^2) \).

• \( (a_1 - b_1) \wedge a_2: r = -\gamma (p_+ - k_-) \wedge p_- \)

\[
(d\sigma)^2_{\text{def}} = -C \left( \frac{r^2}{4} x_+^2 z^4 \left( dx_- \right)^2 + z^2 dx_-,dx_+ + \frac{r^2}{2} x_- (1 + x_+^2) z^3 dz dx_- \right) + (d \ln(z))^2,
\]

\[
B_{\text{def}} = -\frac{1}{2} \gamma \ C \left( (1 + x_+^2) z^4 dx_- \wedge dx_+ + x_- z dx_- \wedge dz \right). 
\]

(5.12)

with \( C^{-1} = 1 - \frac{r^2}{4} (1 + x_+^2)^2 z^4 \).

**AdS**

The conformal symmetry of AdS does not decompose nicely as in the AdS\(_3\) case, and we will not give an extensive list of inequivalent \(TsT\) transformations here. To illustrate the extent of the full list, note that we could for instance consider abelian Yang-Baxter deformations based on the subalgebras

\[
\text{so}(2,4) \supset \text{so}(2,2) \oplus \text{so}(2)_{\text{space}} \simeq \text{sl}(2, \mathbb{R}) \oplus \text{sl}(2, \mathbb{R}) \oplus \text{so}(2)_{\text{space}}, \\
\text{so}(2,4) \supset \text{so}(2)_{\text{time}} \oplus \text{so}(4) \simeq \text{so}(2)_{\text{time}} \oplus \text{su}(2) \oplus \text{su}(2), \\
\text{or so}(2,4) \simeq \text{conf}(1,3) \supset \text{span}(p_+) \text{ or } \text{span}(k_+),
\]

(5.13)

leading to many tens of inequivalent deformations already. A method to obtain and classify all inequivalent commuting subalgebras of \(\text{so}(2,4)\) and thus also abelian Yang-Baxter deformations was proposed in principle in [61]. In addition to pure AdS\(_5\) deformations we could of course mix AdS\(_5\) and S\(^3\) directions.

### 6 Conclusion and Outlook

In this paper we proved that abelian Yang-Baxter deformations are equivalent to sequences of commuting \(TsT\) transformations. This proof is completely generic and holds for any group or (semi-)symmetric coset \(\sigma\) model, including fermions to all orders. We included the fermionic generalisation of these transformations, which however typically requires complexification. Including fermionic transformations naturally leads to a \(TsT\) subgroup of the superduality group \(\text{OSp}(2d_0, d_0|2d_f)\) generalising the bosonic \(T\) duality group \(O(d_h, d_h)\). For illustrative purposes we moreover presented all six possible inequivalent abelian deformations of AdS\(_5\). In terms of the \(\text{so}(2,2)\)-generators the associated \(r\) matrices are given by

\[
m_{01} \wedge D, \quad (m_{01} - D) \wedge p_-, \quad m_{03} \wedge m_{12}, \quad (m_{01} - D) \wedge (p_- - k_+), \quad p_0 \wedge p_1, \quad (p_+ - k_-) \wedge p_-.
\]
One natural question to ask is what the dual field theory interpretation of Yang-Baxter deformations is. For \( r \) matrices solving the regular classical Yang-Baxter equation – which includes the present abelian ones – these duals are generically conjectured to be noncommutative versions of supersymmetric Yang-Mills theory [26], provided they exist. This conjecture relies on the twisted symmetry structure of the gravitational models, whose realisation on the hypothetical field theory side requires a nontrivial star product. Several abelian deformed theories are known to fit this description, notably the gravity duals of \( \beta \) deformed SYM [5] and canonical spacelike noncommutative SYM [62, 63]. As discussed in [26], the situation is less clear for the naive time-like noncommutative version of SYM and the related abelian deformation of AdS\(_5\) × S\(_5\) for example. The generalisation from the \( \beta \) to the \( \gamma_i \) deformation [7] shows subtleties as well, though at least in the spectrum a notion of duality appears to remain, see e.g. [64, 65, 66]. It is important to understand in which (isolated) cases, and how, the general dual field theory picture breaks down.

In principle we can formally extend the conjecture of [26] to our fermionic TsT transformations, replacing field products in the SYM Lagrangian by star products built on the twist \( e^{rT} \), where \( r \) is associated \( r \) matrix. As such \( r \) matrices are not real, however, this would be a complex deformation of SYM. Moreover, manifest conformal invariance would be broken, cf. eqn. (2.12).\(^{19}\) In particular such star products introduce new, possibly dimensionful, couplings in the theory. On the gravity side it would be useful to gain a better understanding of the action of fermionic TsT transformations on the supergravity fields (and their reality). Duals of mixed bosonic-fermionic deformations could be defined similarly, though the nature of their deformation parameter is slightly odd.

There are a number of further open questions. First, it would be interesting to consider classical solutions and associated integrable classical mechanical models for these abelian deformed models, as well as non-abelian ones, as done for the \( \beta \) deformation [6], and the \( \eta \) model in e.g. [67, 68, 69, 70, 71]. Second, given the classical equivalence between the \( \eta \) and \( \lambda \) models via Poisson-Lie duality (cf. footnote 1), we might wonder whether similar dual theories exist for CYBE-based deformations. Third, non-Cartan abelian deformations (and non-abelian ones) invariably break the isometries required to fix the standard BMN light cone gauge of the exact S matrix approach to the quantum string \( \sigma \) model [2]. In other words, the effect of these deformations at the quantum level is mysterious, in contrast to the \( \beta \) deformation for example [65].

Recently, hints of generalised TsT structures have been found also in non-abelian cases [39, 27]. It would be interesting to try and extend our approach here, especially to the uni-modal (supergravity) cases described in [27].

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\(^{19}\)Suitably choosing an anticommuting supercharge \( Q \) and superconformal \( S \), it is possible to preserve scale invariance, at least classically. Fermionic abelian deformations always break Lorentz invariance however.
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