Stable Border Bases for Ideals of Points

John Abbott∗, Claudia Fassino†, Maria-Laure Torrente†

Abstract

Let $X$ be a set of points whose coordinates are known with limited accuracy; our aim is to give a characterization of the vanishing ideal $I(X)$ independent of the data uncertainty. We present a method to compute a polynomial basis $B$ of $I(X)$ which exhibits structural stability, that is, if $\tilde{X}$ is any set of points differing only slightly from $X$, there exists a polynomial set $\tilde{B}$ structurally similar to $B$, which is a basis of the perturbed ideal $I(\tilde{X})$.

Keywords: Empirical points, vanishing ideal, border bases.

MSC: 13P10, 65F20, 65G99.

1 Introduction

In this paper we present a method for computing “structurally stable” border bases of ideals of points whose coordinates are affected by errors.

If $X$ is a set of “empirical” points, representing real-world measurements, then typically the coordinates are known only imprecisely. Roughly speaking, if $\tilde{X}$ is another set of points, each differing by less than the uncertainty from the corresponding element of $X$, then the two sets can be considered as equivalent. Nevertheless, it can happen that their vanishing ideals have very different bases — this is a well known phenomenon in Gröbner basis theory. In order to emphasize the “numerical equivalence” of $X$ and its perturbation $\tilde{X}$, we look for a common characterization of the vanishing ideals $I(X)$ and $I(\tilde{X})$. More precisely our goal is to determine a polynomial basis $B$ of the vanishing ideal $I(X)$ which exhibits structural stability: namely, there is a basis $\tilde{B}$ for the perturbed ideal $I(\tilde{X})$, sharing the same structure as $B$, and whose coefficients differ only slightly, provided that $\tilde{X}$ differs from $X$ by only a small amount (up to some limit).

The decision to use border bases to describe vanishing ideals of sets of empirical points was due to two main reasons: border bases have always been considered a numerically stable tool (see [9], [11]); furthermore, it is easy to study their structure, i.e. the support of their polynomials, as it is completely determined once a suitable order ideal $\mathcal{O}$ has been chosen.

∗Dip. di Matematica, Università di Genova, via Dodecaneso 35, 16146 Genova, Italy - fassino@dima.unige.it
†Scuola Normale Superiore, piazza dei Cavalieri 7, 56126 Pisa, Italy - m.torrente@sns.it
We introduce the notion of stable quotient basis: given a set $X$ of empirical points and a permitted tolerance $\varepsilon$, a stable quotient basis $O$ guarantees the existence of an $O$-border basis $\tilde{B}$ for the vanishing ideal $I(\tilde{X})$ where $\tilde{X}$ is any set of points perturbed by amounts less than the tolerance $\varepsilon$. Once a stable quotient basis $O$ has been found, the corresponding stable border basis can be obtained by some simple combinatorical and linear algebra computations; so we focus our attention on determining $O$.

An alternative approach to the problem, presented in [7], is to use singular value decomposition of matrices to obtain a set of polynomials which are not required to vanish on $X$ but must nevertheless assume particularly small values there. In contrast, a stable border basis always comprises polynomials which vanish on $X$.

This paper is organized as follows. In Section 2 we introduce the concepts and tools we shall use. Section 3 provides a formal description of our problem. The main result, the SOI algorithm for computing a stable order ideal, is presented in Section 4. In Section 5 we give some numerical examples illustrating the functioning of the algorithm. Finally, Section 6 is an Appendix which contains the proof of a basic result about the first order approximation of rational functions, useful for the error analysis of the sensitivity of the border basis computation.

2 Basic definitions and notation

This section contains basic definitions and notation used later in the paper. To simplify the presentation, we shall implicitly suppose that each finite set of points or polynomials is in fact a tuple, so that the elements are ordered in some way, and we can refer to the $k$-th element using the index $k$.

Let $n \geq 1$; we recall (see [8, 9]) some basic concepts related to the polynomial ring $P = \mathbb{R}[x_1, \ldots, x_n]$.

**Definition 2.1** Let $X = \{p_1, \ldots, p_s\}$ be a non-empty finite set of points of $\mathbb{R}^n$ and let $G = \{g_1, \ldots, g_k\}$ be a non-empty finite set of polynomials.

(a) The ideal $I(X) = \{f \in P \mid f(p_i) = 0 \ \forall p_i \in X\}$ is called the vanishing ideal of $X$.

(b) The $\mathbb{R}$-linear map $\text{eval}_X : P \to \mathbb{R}^s$ defined by $\text{eval}_X(f) = (f(p_1), \ldots, f(p_s))$ is called the evaluation map associated to $X$. For brevity, we write $f(X)$ to mean $\text{eval}_X(f)$.

(c) The evaluation matrix of $G$ associated to $X$, written $M_G(X) \in \text{Mat}_{s \times k}(\mathbb{R})$, is defined as having entry $(i, j)$ equal to $g_j(p_i)$, i.e. whose columns are the images of the polynomials $g_j$ under the evaluation map.

**Definition 2.2** Let $T^n$ be the monoid of power products of $P$ and let $O$ be a non-empty subset of $T^n$. 
(a) The factor closure (abbr. closure) of \( \mathcal{O} \) is the set \( \overline{\mathcal{O}} \) of all power products in \( \mathbb{T}^n \) which divide some power product of \( \mathcal{O} \).

(b) The set \( \mathcal{O} \) is called an order ideal if \( \mathcal{O} = \overline{\mathcal{O}} \), i.e. if \( \mathcal{O} \) is factor closed.

(c) Let \( I \subseteq P \) be a zero-dimensional ideal, and \( s = \text{dim}(P/I) \); if \( \mathcal{O} \) is factor closed and the residue classes of its elements form a basis of \( P/I \) then we call it a quotient basis for \( I \).

(d) Let \( \mathcal{O} \) be an order ideal; the border \( \partial \mathcal{O} \) of \( \mathcal{O} \) is defined by
   \[
   \partial \mathcal{O} = (x_1 \mathcal{O} \cup \ldots \cup x_n \mathcal{O}) \setminus \mathcal{O}
   \]

(e) If \( \mathcal{O} \) is an order ideal then the elements of the minimal set of generators of the monomial ideal corresponding to \( \mathbb{T}^n \setminus \mathcal{O} \) are called the corners of \( \mathcal{O} \).

**Definition 2.3** Let \( \mathcal{O} = \{t_1, \ldots, t_\mu\} \) be an order ideal, and let \( \partial \mathcal{O} = \{b_1, \ldots, b_\nu\} \) be the border of \( \mathcal{O} \). Let \( \mathcal{B} = \{g_1, \ldots, g_\nu\} \) be a set of polynomials having the form
   \[g_j = b_j - \sum_{i=1}^\mu \alpha_{ij} t_i\]
   where each \( \alpha_{ij} \in \mathbb{R} \). Let \( I \subseteq P \) be an ideal containing \( \mathcal{B} \). If the residue classes of the elements of \( \mathcal{O} \) form a \( \mathbb{R} \)-vector space basis of \( P/I \) then \( \mathcal{B} \) is called a border basis of \( I \) founded on \( \mathcal{O} \), or more briefly \( \mathcal{B} \) is an \( \mathcal{O} \)-border basis of \( I \).

**Proposition 2.4 (Existence and Uniqueness of Border Bases)**
Let \( I \subseteq P \) be a zero-dimensional ideal, and let \( \mathcal{O} = \{t_1, \ldots, t_\mu\} \) be a quotient basis for \( I \). Then there exists a unique \( \mathcal{O} \)-border basis \( \mathcal{B} \) of \( I \).

**Proof:** See Proposition 6.4.17 in [9].

Later on, in order to measure the distances between points of \( \mathbb{R}^n \), we will use the euclidean norm \( \| \cdot \| \). Additionally, given an \( n \times n \) positive diagonal matrix \( E \), we shall also use the weighted 2-norm \( \| \cdot \|_E \) as defined in [5]. For completeness, we recall here their definitions:

\[
\|v\| := \sqrt{\sum_{j=1}^n v_j^2} \quad \text{and} \quad \|v\|_E := \|Ev\|
\]

We recall the definition of empirical point (see [11], [2]).

**Definition 2.5** Let \( p \in \mathbb{R}^n \) be a point and let \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \), with each \( \varepsilon_i \in \mathbb{R}^+ \), be the vector of the componentwise tolerances. An empirical point \( p^\varepsilon \) is the pair \((p, \varepsilon)\), where we call \( p \) the specified value and \( \varepsilon \) the tolerance.

Let \( p^\varepsilon \) be an empirical point. We define its ellipsoid of perturbations:

\[
N(p^\varepsilon) = \{ \bar{p} \in \mathbb{R}^n : \|\bar{p} - p\|_E \leq 1 \}
\]

where the positive diagonal matrix \( E = \text{diag}(1/\varepsilon_1, \ldots, 1/\varepsilon_n) \). This set contains all the admissible perturbations of the specified value \( p \), i.e. all points differing from \( p \) by less than the tolerance.
We shall assume that all the empirical points share the same tolerance \( \varepsilon \), as is reasonable if they derive from real-world data measured with the same accuracy. In particular this assumption allows us to use the \( E \)-weighted norm on \( \mathbb{R}^n \) to measure the distance between empirical points.

Given a finite set \( \mathbb{X}^\varepsilon \) of empirical points all sharing the same tolerance \( \varepsilon \), we introduce the concept of a slightly perturbed set of points \( \tilde{\mathbb{X}} \) by means of the following definition.

**Definition 2.6** Let \( \mathbb{X}^\varepsilon = \{p_1^\varepsilon, \ldots, p_s^\varepsilon\} \) be a set of empirical points with uniform tolerance \( \varepsilon \) and with \( \mathbb{X} \subset \mathbb{R}^n \). Each set of points \( \tilde{\mathbb{X}} = \{\tilde{p}_1, \ldots, \tilde{p}_s\} \subset \mathbb{R}^n \) whose elements satisfy

\[
(\tilde{p}_1, \ldots, \tilde{p}_s) \in \prod_{i=1}^s N(p_i^\varepsilon)
\]

is called an **admissible perturbation** of \( \mathbb{X}^\varepsilon \).

Finally we introduce the definition of distinct empirical points.

**Definition 2.7** The empirical points \( p_1^\varepsilon \) and \( p_2^\varepsilon \), with specified values \( p_1, p_2 \in \mathbb{R}^n \), are said to be distinct if

\[
N(p_1^\varepsilon) \cap N(p_2^\varepsilon) = \emptyset
\]

### 3 The formal problem

We shall use the concept of empirical point to describe formally the given uncertain data: the input \( \mathbb{X} \) is viewed as the set of specified values of \( \mathbb{X}^\varepsilon \), which consists of \( s \) distinct empirical points all sharing the same fixed tolerance \( \varepsilon \).

Given the set \( \mathbb{X} \), we want to determine a numerically stable basis \( \mathcal{B} \) of the vanishing ideal \( \mathcal{I}(\mathbb{X}) \). Intuitively, a basis \( \mathcal{B} \) of \( \mathcal{I}(\mathbb{X}) \) is considered to be structurally stable if, for each admissible perturbation \( \tilde{\mathbb{X}} \) of \( \mathbb{X}^\varepsilon \), it is possible to produce a basis \( \tilde{\mathcal{B}} \) of \( \mathcal{I}(\tilde{\mathbb{X}}) \) only by means of a slight and continuous variation of the coefficients of the polynomials of \( \mathcal{B} \), that is if there exists a basis \( \tilde{\mathcal{B}} \) of \( \mathcal{I}(\tilde{\mathbb{X}}) \) whose polynomials have the same support as the corresponding polynomials of \( \mathcal{B} \). Given a polynomial basis \( \mathcal{B} \), we will call the union of the supports of its polynomials the **structure** of \( \mathcal{B} \).

A good starting point for us is the concept of border basis (see [9], [11]). In fact the structure of a border basis is easily computable and completely determined by the quotient basis \( \mathcal{O} \) upon which the border basis is founded (see Definition 2.3). Using border bases, the problem of computing a structurally stable representation of the vanishing ideal \( \mathcal{I}(\mathbb{X}) \) thus reduces to the problem of finding a quotient basis \( \mathcal{O} \) for \( \mathcal{I}(\tilde{\mathbb{X}}) \) valid for every admissible perturbation \( \tilde{\mathbb{X}} \). The following definition captures this notion and generalizes it to any order ideal.

**Definition 3.1** Let \( \mathcal{O} \) be an order ideal, then \( \mathcal{O} \) is **stable** w.r.t. \( \mathbb{X}^\varepsilon \) if the evaluation matrix \( M_\mathcal{O}(\tilde{\mathbb{X}}) \) has full rank for each admissible perturbation \( \tilde{\mathbb{X}} \) of \( \mathbb{X}^\varepsilon \).
The following proposition highlights the importance of stable quotient bases.

**Proposition 3.2** Let $\mathbb{X}^e$ be a set of $s$ distinct empirical points, and let $\mathcal{O} = \{t_1, \ldots, t_s\}$ be a quotient basis for $\mathcal{I}(\mathbb{X})$ which is stable w.r.t. $\mathbb{X}^e$. Then, for each admissible perturbation $\bar{\mathbb{X}}$ of $\mathbb{X}^e$, the vanishing ideal $\mathcal{I}(\bar{\mathbb{X}})$ has an $\mathcal{O}$-border basis. Furthermore, if $\partial \mathcal{O} = \{b_1, \ldots, b_\nu\}$ is the border of $\mathcal{O}$ then $\bar{\mathcal{B}}$ consists of $\nu$ polynomials of the form

$$g_j = b_j - \sum_{i=1}^s \alpha_{ij} t_i \quad \text{for } j = 1 \ldots \nu$$

(1)

where the coefficients $\alpha_{ij} \in \mathbb{R}$ satisfy

$$b_j(\bar{\mathbb{X}}) = \sum_{i=1}^s \alpha_{ij} t_i(\bar{\mathbb{X}})$$

**Proof:** Let $\bar{\mathbb{X}}$ be an admissible perturbation of $\mathbb{X}^e$ and let $\text{eval}_{\bar{\mathbb{X}}} : P \rightarrow \mathbb{R}^s$ be the $\mathbb{R}$-linear evaluation map associated to the set $\bar{\mathbb{X}}$. It is easy to prove that $\mathcal{I}(\bar{\mathbb{X}}) = \ker(\text{eval}_{\bar{\mathbb{X}}})$ and consequently, that the quotient ring $P/\mathcal{I}(\bar{\mathbb{X}})$ is isomorphic to $\mathbb{R}^s$ as a vector space. Since $\mathcal{O}$ is stable w.r.t. the empirical set $\mathbb{X}^e$, it follows that $\{t_1(\bar{\mathbb{X}}), \ldots, t_s(\bar{\mathbb{X}})\}$ are linearly independent vectors. Moreover, $\# \bar{\mathbb{X}} = \# \mathcal{O}$, so the residue classes of the elements of $\mathcal{O}$ form a $\mathbb{R}$-vector space basis of $P/\mathcal{I}(\bar{\mathbb{X}})$.

Let $v_j = b_j(\bar{\mathbb{X}})$ be the evaluation vector associated to the power product $b_j$ lying in the border $\partial \mathcal{O}$; each $v_j$ can be expressed as

$$v_j = \sum_{i=1}^s \alpha_{ij} t_i(\bar{\mathbb{X}}) \quad \text{for some } \alpha_{ij} \in \mathbb{R}$$

For each $j$ we define the polynomial $g_j = b_j - \sum_{i=1}^s \alpha_{ij} t_i$; by construction $\text{eval}_{\bar{\mathbb{X}}}(g_j) = 0$, and so $\bar{\mathcal{B}} = \{g_1, \ldots, g_\nu\}$ is contained in $\mathcal{I}(\bar{\mathbb{X}})$; it follows that $\bar{\mathcal{B}}$ is the $\mathcal{O}$-border basis of the ideal $\mathcal{I}(\bar{\mathbb{X}})$.

We observe that the coefficients $\alpha_{ij}$ of each polynomial $g_j \in \bar{\mathcal{B}}$ are just the components of the solution $\alpha_j$ of the linear system $M_{\mathcal{O}}(\bar{\mathbb{X}}) \cdot \alpha_j = b_j(\bar{\mathbb{X}})$. It follows that $\alpha_{ij}$ are continuous functions of the points of the set $\bar{\mathbb{X}}$ and so, since the order ideal $\mathcal{O}$ is stable w.r.t. $\mathbb{X}^e$, they undergo only continuous variations as $\bar{\mathbb{X}}$ changes. Now, the definition of stable border basis follows naturally.

**Definition 3.3** Let $\mathbb{X}^e$ be a finite set of distinct empirical points, let $\mathcal{O}$ be a quotient basis for the vanishing ideal $\mathcal{I}(\mathbb{X})$. If $\mathcal{O}$ is stable w.r.t. $\mathbb{X}^e$ then the $\mathcal{O}$-border basis $\mathcal{B}$ for $\mathcal{I}(\mathbb{X})$ is said to be stable w.r.t. the set $\mathbb{X}^e$.

The problem of computing a stable border basis of the vanishing ideal of a set $\mathbb{X}^e$ of empirical points is therefore completely solved once we have found a
exists a tolerance $\delta$. Proposition 3.4 states that stability, as the following proposition shows.

Given tolerance $\varepsilon$ by the measurements, Proposition 3.4 does not solve our problem. If the determinant is a polynomial function in the matrix entries, and noting that the entries of $M(X)$ are polynomials in the points' coordinates, we can conclude that if there exists a tolerance $\delta = (\delta_1, \ldots, \delta_n)$, with each $\delta_i > 0$, such that $M(O(X))$ is not stable w.r.t. $X$ for any perturbation $\tilde{X}$ of $X$.

We end this section by observing that any $O$-border basis of the vanishing ideal $I(X)$ is stable w.r.t. $X$ for a sufficiently small value of the tolerance $\delta$. This is equivalent to saying that any quotient basis $O$ of $I(X)$ has a "region of stability", as the following proposition shows.

**Proposition 3.4** Let $X$ be a finite set of points of $\mathbb{R}^n$ and $I(X)$ be its vanishing ideal; let $O$ be a quotient basis for $I(X)$. Then there exists a tolerance $\delta = (\delta_1, \ldots, \delta_n)$, with $\delta_i > 0$, such that $O$ is stable w.r.t. $X$.

**Proof:** Let $M(O(X))$ be the evaluation matrix of $O$ associated to the set $X$; then $M(O(X))$ is a structured matrix whose coefficients depend continuously on the points in $X$. Since, by hypothesis, the $O$-border basis of the vanishing ideal $I(X)$ exists, it follows that $M(O(X))$ is invertible. Recalling that the determinant is a polynomial function in the matrix entries, and noting that the entries of $M(O(X))$ are polynomials in the points' coordinates, we can conclude that there exists a tolerance $\delta = (\delta_1, \ldots, \delta_n)$, with each $\delta_i > 0$, such that $\det(M(O(X))) \neq 0$ for any perturbation $\tilde{X}$ of $X$.

Nevertheless, since the tolerance $\varepsilon$ of the empirical points in $X$ is given a priori by the measurements, Proposition 3.4 does not solve our problem. If the given tolerance $\varepsilon$ is larger than the "region of stability" of a chosen quotient basis $O$, the corresponding border basis will not be stable w.r.t. $X$; such a situation is shown in the following example.

**Example.** Let $X$ be the set of empirical points having

$$X = \{(-1, -5), (0, -2), (1, 1), (2, 4.1)\} \subset \mathbb{R}^2$$

as the set of specified values and $\varepsilon = (0.15, 0.15)$ as the tolerance; let

$$\tilde{X} = \{(-1 + e_1, -5 + e_2), (e_3, -2 + e_4), (1 + e_5, 1 + e_6), (2 + e_7, 4.1 + e_8)\}$$

be a generic admissible perturbation of $X$, where the parameters $e_i \in \mathbb{R}$ satisfy

$$\|(e_1, e_2)\|_E \leq 1 \quad \|(e_3, e_4)\|_E \leq 1 \quad \|(e_5, e_6)\|_E \leq 1 \quad \|(e_7, e_8)\|_E \leq 1$$

Consider first $O_1 = \{1, y, x, y^2\}$, which is a quotient basis for $I(X)$. The corresponding border basis $B_1$ of $I(X)$ is not stable w.r.t. $X$. Indeed, consider the perturbation $\tilde{X} = \{(-1, -5), (0, -2), (1, 1), (2, 4)\}$ of $X$. The evaluation matrix $M(O_1(\tilde{X}))$ is singular, so no $O_1$-border basis of $I(\tilde{X})$ exists. It follows that $O_1$ is not stable w.r.t. $X$ since its "region of stability" is too small w.r.t. the given tolerance $\varepsilon$.

Now consider the quotient basis $O_2 = \{1, y, y^2, y^3\}$, which is stable w.r.t. $X$. In fact, for each perturbation $\tilde{X}$ of $X$, we see that $M(O_2(\tilde{X}))$ is a Vandermonde
matrix whose determinant is equal to

\[(e_4 - e_2 + 3)(e_6 - e_2 + 6)(e_8 - e_2 + 9.1)(e_6 - e_4 + 3)(e_8 - e_4 + 6.1)(e_8 - e_6 + 3.1)\]

Since each \(|e_i| \leq 0.15\), it follows that, for each perturbation \(\tilde{X}\), the matrix \(M_{O_2}(\tilde{X})\) is invertible.

\[\diamond\]

4 A practical solution

In this section we address the problem of computing an order ideal \(O\) stable w.r.t. \(X^\varepsilon\), a finite set of distinct empirical points, and also the corresponding stable border basis when it exists.

Since in real-world measurements the tolerance \(\varepsilon\) present in the data is relatively small, our interest is focused on small perturbations \(\tilde{X}\) of the empirical set \(X^\varepsilon\). For this reason our approach is based on a first order error analysis of the problem. We present in Section 4.3 an algorithm which computes a stable order ideal \(O\) (up to first order). In order to investigate the stability of the order ideal \(O\) we use some results on the first order approximation of rational functions (see Section 4.1) and we introduce a parametric description of the admissible perturbations \(\tilde{X}\) of \(X^\varepsilon\) (see Section 4.2).

If the output of the algorithm is actually a quotient basis then the corresponding stable border basis \(B\) exists for \(I(X^\varepsilon)\). To determine \(B\) it suffices to find the border of \(O\) (a simple combinatorial computation), and then for each element of the border solve a linear system (as described in the proof of Proposition 3.2).

4.1 Remarks on first order approximation

Let \(n \in \mathbb{N}\); let \(F = \mathbb{R}(e_1, \ldots, e_n)\) be the field of rational functions on \(\mathbb{R}\) and let \(f \in F\). We use multi-index notation to give the Taylor expansion of \(f\) in a neighbourhood of 0

\[f(e_1, \ldots, e_n) = \sum_{|\alpha| \geq 0} \frac{D^\alpha f(0)}{\alpha!} e^\alpha\]

We recall that given \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\), we have \(|\alpha| = \alpha_1 + \ldots + \alpha_n\) and \(\alpha! = \alpha_1! \ldots \alpha_n!\). Similarly \(D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}\) (where \(D_i^j = \partial^j / \partial e_i^j\)) and \(e^\alpha = e_1^{\alpha_1} \ldots e_n^{\alpha_n}\).

Each \(f \in F\) can be decomposed into components of homogeneous degree in the following way:

\[f = \sum_{k \geq 0} f_k\]

where \(f_k = \sum_{|\alpha| = k} \frac{D^\alpha f(0)}{\alpha!} e^\alpha\)

and where, by convention, \(D^{(0 \ldots 0)} f = f\). Each polynomial \(f_k \in \mathbb{R}[e_1, \ldots, e_n]\) is called the homogeneous component of \(f\) of degree \(k\).

Analogously, we can decompose a matrix \(M \in \text{Mat}_{r,s}(F)\) into “homogeneous” parts in the following way.
Definition 4.1 Let $M = (m_{ij})$ be a matrix in $\text{Mat}_{r,s}(F)$; the matrix $M_k = ((m_{ij})_k)$, where $(m_{ij})_k = \sum_{|\alpha| = k} \frac{D^\alpha m_{ij}(0)}{\alpha!} e^\alpha \in \mathbb{R}[e_1, \ldots, e_n]$, is called the homogenous component of $M$ of degree $k$.

The following proposition characterizes the homogeneous components of degrees 0 and 1 of the solution and residual of a least squares problem.

Proposition 4.2 Let $r, s \in \mathbb{N}$ with $r > s$; let $M$ be a matrix in $\text{Mat}_{r,s}(F)$ and let $v$ be a vector in $\text{Mat}_{r,1}(F)$. Let $x \in \text{Mat}_{s,1}(F)$ and $\rho \in \text{Mat}_{r,1}(F)$ be respectively the solution and the residual of the least squares problem $Mx \approx v$.

The homogeneous components of degrees 0 and 1 of $x$ are

$$x_0 = (M_0^tM_0)^{-1}M_0^tv_0$$
$$x_1 = (M_0^tM_0)^{-1}(M_0^tv_1 + M_1^tv_0 - M_0^tM_1x_0 - M_1^tM_0x_0),$$

Moreover, the homogeneous components of degrees 0 and 1 of $\rho$ are

$$\rho_0 = v_0 - M_0x_0$$
$$\rho_1 = v_1 - M_0x_1 - M_1x_0$$

Proof: This lengthy proof has been deferred to an appendix.

4.2 A parametric description of $X^\varepsilon$

Let $X^\varepsilon = \{p_1^\varepsilon, \ldots, p_s^\varepsilon\}$ be a finite set of distinct empirical points with specified values $X \subset \mathbb{R}^n$; we represent an admissible perturbation $\tilde{X} = \{\tilde{p}_1, \ldots, \tilde{p}_s\}$ of $X^\varepsilon$ by using first order infinitesimals for the perturbation in each coordinate. In particular we express $\tilde{X}$ as a function of $ns$ variables

$$\varepsilon = (e_{11}, \ldots, e_{s1}, e_{12}, \ldots, e_{s2}, \ldots, e_{1n}, \ldots, e_{sn})$$

called error variables; specifically, we have

$$\tilde{p}_k = (p_k + e_{k1}, p_k + e_{k2}, \ldots, p_k + e_{kn})$$

The condition that each $\tilde{p}_k$ is an admissible perturbation of the point $p_k$ is equivalent to the following:

$$\|(e_{k1}, \ldots, e_{kn})\|_E \leq 1$$
We denote by \( \tilde{X}(e) = (\tilde{p}_1(e), \ldots, \tilde{p}_s(e)) \) a generic admissible perturbation of \( X^e \). We observe that the coordinates of each perturbed point \( \tilde{p}_k(e) \) are elements of the polynomial ring \( R = \mathbb{R}[e] \) and that each variable \( e_{kj} \) represents the perturbation in the \( j \)-th coordinate of the original point \( p_k \). The domain of the perturbed set \( \tilde{X}(e) \), viewed as a function of \( ns \) variables, is denoted by \( D_e \).

Obviously, if \( e \in D_e \) we have

\[
\|e\|^2 = \sum_{j=1}^n \sum_{k=1}^s \varepsilon_{kj}^2 \leq \sum_{j=1}^n s \epsilon_j^2,
\]

and consequently

\[
\|e\| \leq \sqrt{s} \|\varepsilon\| \tag{5}
\]

To keep evident the dependence on the error variables \( e \), we extend the concepts of evaluation map of a polynomial \( f \in P \) and evaluation matrix of a set of polynomials \( G = \{g_1, \ldots, g_k\} \subset P \) (see Definition 2.1) to a generic perturbed set \( \tilde{X}(e) \), using the following notation:

\[
\text{eval}_e X(e)(f) = (f(\tilde{p}_1(e)), \ldots, f(\tilde{p}_s(e))) \in R \times \ldots \times R = R^s
\]

for brevity denoted by \( f(\tilde{X}(e)) \); similarly we write the evaluation matrix

\[
M_G(\tilde{X}(e)) = \left( g_1(\tilde{X}(e)), \ldots, g_k(\tilde{X}(e)) \right)
\]

### 4.3 The SOI Algorithm

In this section we present the SOI algorithm which computes, up to first order, an order ideal \( \mathcal{O} \) stable w.r.t the empirical set \( X^e \).

The strategy for computing a stable order ideal \( \mathcal{O} \) is the following. As in the Buchberger-Möller algorithm (\[3\], \[1\]) the order ideal \( \mathcal{O} \) is built stepwise: initially \( \mathcal{O} \) comprises just the power product 1; then at each iteration, a new power product \( t \) is considered. If the evaluation matrix \( M_{\mathcal{O} \cup \{t\}}(\tilde{X}(e)) \) has full rank for all \( e \in D_e \) then \( t \) is added to \( \mathcal{O} \); otherwise \( t \) is added to the corner set of the order ideal.

A first observation concerns the choice of the power product \( t \) to analyze at each iteration: any strategy that chooses a term \( t \) such that the set \( \mathcal{O} \cup \{t\} \) preserves the property of being an order ideal can be applied. A possible technique is the one used in the Buchberger-Möller algorithm, where the power product \( t \) is chosen according to a fixed term ordering \( \sigma \). The version of the SOI Algorithm presented below employs this latter strategy. Note that \( \sigma \) is only used as a computational tool for choosing \( t \); in fact the final computed set \( \mathcal{O} \) is not, in general, the same as that which would be obtained processing the set \( X \) by the Buchberger-Möller algorithm with the same term ordering.

Another observation concerns the main check of the algorithm: note that the rank condition is equivalent to checking whether \( \rho(e) \), the component of
the evaluation vector \( t(\widetilde{X}(e)) \) orthogonal to the column space of the matrix \( M_O(\widetilde{X}(e)) \), vanishes for any \( e ∈ D_ε \). This check is greatly simplified by our restriction to first order error terms.

**Algorithm 4.3** (Stable Order Ideal Algorithm)

Let \( \sigma \) be a term ordering on \( T^n \) and let \( X^ε = \{p_1^ε, \ldots, p_s^ε\} \) be a finite set of distinct empirical points, with \( X ⊂ R^n \) and a common tolerance \( ε = (ε_1, \ldots, ε_n) \). Let \( e = (ε_{11}, \ldots, ε_{sn}) \) be the error variables whose constraints are given in (4). Consider the following sequence of instructions.

**S1** Start with the lists \( O = [1] \), \( L = [x_1, \ldots, x_n] \), the empty list \( C = [] \), and the matrices \( M_0 ∈ \text{Mat}_{s,1}(R) \) with all the elements equal to 1, and \( M_1 ∈ \text{Mat}_{s,1}(R) \) with all the elements equal to 0.

**S2** If \( L = [] \) then return the set \( O \) and stop. Otherwise let \( t = \min_σ(L) \) and delete it from \( L \).

**S3** Let \( v_0 \) and \( v_1 \) be the homogeneous components of degrees 0 and 1 of the evaluation vector \( v = t(\widetilde{X}(e)) \). Solve up to first order the least squares problem

\[
M_O(\widetilde{X}(e)) \alpha(e) ≈ v,
\]

by computing the vectors

\[
\rho_0 = v_0 - M_0α_0
\]

\[
ρ_1 = v_1 - M_0α_1 - M_1α_0
\]

where

\[
α_0 = (M_0^tM_0)^{-1}M_0^tv_0
\]

\[
α_1 = (M_0^tM_0)^{-1}(M_0^tv_1 + M_1^tv_0 - M_0^tM_1α_0 - M_1^tM_0α_0).
\]

**S4** Let \( C_t ∈ \text{Mat}_{s,n}(R) \) be such that \( ρ_1 = C_t e \). Compute the minimal 2-norm solution \( \hat{e} \) of the underdetermined system \( C_t e = -ρ_0 \).

**S5** If \( \|\hat{e}\| > \sqrt{s} \|ε\| \) then adjoin the vector \( v_0 \) as a new column of \( M_0 \) and the vector \( v_1 \) as a new column of \( M_1 \). Append the power product \( t \) to \( O \), and add to \( L \) those elements of \( \{x_1t, \ldots, x_n t\} \) which are not multiples of an element of \( L \) or \( C \). Continue with step S2.

**S6** Otherwise append \( t \) to the list \( C \), and remove from \( L \) all multiples of \( t \). Continue with step S2.

**Theorem 4.4** Algorithm 4.3 stops after finitely many steps and returns a set \( O ⊂ T^n \) which is an order ideal stable (up to first order) w.r.t. the empirical set \( X^ε \). Furthermore, if \( |O| = s \) then \( I(X) \) has a corresponding stable border basis w.r.t. \( X^ε \).

**Proof:** First we claim that the vectors \( ρ_0 \), \( ρ_1 \), \( α_0 \), \( α_1 \) computed in step S3 are the homogeneous components of degrees 0 and 1 of the residual \( ρ(e) \) and of the solution \( α(e) \) to the least squares problem

\[
M_O(\widetilde{X}(e)) α(e) ≈ t(\widetilde{X}(e)) \quad (6)
\]

10
where $t$ is the power product being considered at the current iteration, and $\mathcal{O}$ is the order ideal computed so far. To prove this claim it is sufficient to apply Proposition 4.2 to (6) and to observe that the matrices $M_0$ and $M_1$ coincide with the homogeneous components of degrees 0 and 1 of $M_{\mathcal{O}}(\tilde{X}(e))$. Clearly, this is true at the first iteration, since $M_{\mathcal{O}}(\tilde{X}(e)) = (1 \ldots 1)^t$. We apply induction on the number of iterations. Assume that $M_0$ and $M_1$ are the components of degrees 0 and 1 of $M_{\mathcal{O}}(\tilde{X}(e))$ and suppose that the power product $r$ is added to $\mathcal{O}$. Since the last column of $M_{\mathcal{O} \cup \{r\}}(\tilde{X}(e))$ is given by $r(\tilde{X}(e))$, whose components of degrees 0 and 1 are $r_0$ and $r_1$ respectively, the new matrices $[M_0, r_0]$ and $[M_1, r_1]$ are the components of degrees 0 and 1 of $M_{\mathcal{O} \cup \{r\}}(\tilde{X}(e))$. We conclude that the vectors $\rho_0 + \rho_1(e)$ and $\alpha_0 + \alpha_1(e)$ coincide with $\rho(e)$ and $\alpha(e)$, up to first order.

Now we prove the finiteness and the correctness of Algorithm 4.3. First we show finiteness. At each iteration the algorithm performs either step S5 or step S6. We observe that step S5 can be executed at most $s - 1$ times; in fact, when $M_0$ becomes a square matrix, i.e. after $s - 1$ iterations of step S5, the residual vector $\rho_0$ is zero, and consequently the minimal 2-norm solution $\hat{e}$ of the linear system $C_t e = -\rho_0$ is also zero. Moreover, step S5 is the only place where the set $L$ is enlarged with a finite number of terms, while each iteration removes from $L$ at least one element; we conclude that the algorithm reaches the condition $L = \{\}$ after finitely many iterations.

In order to show correctness we prove, by induction on the number of iterations and using a first order error analysis, that the output set $\mathcal{O}$ is an order ideal stable w.r.t. $X^e$. This is clearly true after zero iterations, i.e. after step S1 has been executed. By induction assume that a number of iterations has already been performed and that the set $\mathcal{O}$ satisfies the given requirements; let us follow the steps of the new iteration, in which a power product $t$ is considered. If step S6 is performed the claim is true because $\mathcal{O}$ does not change. Otherwise, if step S5 is performed, the set $\mathcal{O}^* = \mathcal{O} \cup \{t\}$ is an order ideal by construction. Further, since the minimal 2-norm solution $\tilde{e}$ of the linear system $C_t e = -\rho_0$ satisfies condition $\|\tilde{e}\| > \sqrt{s} \|\varepsilon\|$ it follows that $\tilde{e}$ does not belong to $D_\varepsilon$ and that the vector $\rho_0 + \rho_1(e)$ does not vanish as $e$ varies in $D_\varepsilon$. Therefore, up to first order, we can consider $\rho(e)$ as a non-vanishing vector for each perturbation $\tilde{X}(e)$, i.e. the matrix $M_{\mathcal{O}^*}(\tilde{X}(e))$ has full rank for each $e \in D_\varepsilon$.

For the last part of the theorem we simply observe that when $\# \mathcal{O} = s$ then $\mathcal{O}$ is a quotient basis; the rest is immediate.

5 Numerical examples

In this section we present some numerical examples to show the effectiveness of the SOI algorithm. Our algorithm is implemented using the C++ language and the CoCoALib, see [4], and all computations have been performed on an Intel Pentium M735 processor (at 1.7 GHz) running GNU/Linux. In all the examples, the SOI algorithm is performed using a fixed precision of 1024 bits.
for the RingTwinFloat implemented in CoCoALib, and the degree lexicographic term ordering σ; in addition, the coefficients of the polynomials are displayed as truncated decimals.

The first two examples show how the SOI algorithm detects the simplest geometrical configuration almost satisfied by the empirical set $X^\varepsilon$.

**Example. Almost aligned points**

We consider the empirical set $X^\varepsilon$ given in Example we recall here the points in $X$ 

$$X = \{(-1, -5), (0, -2), (1, 1), (2, 4.1)\} \subset \mathbb{R}^2$$

and the tolerance $\varepsilon = (0.15, 0.15)$. Applying algorithm SOI to $X^\varepsilon$ we obtain the quotient basis $O = \{1, y, y^2, y^3\}$ which is stable w.r.t. $X^\varepsilon$, as we proved in Example As $O$ is a quotient basis we can compute the border basis founded on it:

$$B = \begin{cases} 
\text{ord} = x + 0.0002y^3 + 0.0012y^2 - 0.3328y - 0.6686 \\
xy + 0.0008y^3 - 0.3286y^2 - 0.6643y - 0.0079 \\
x^2y - 0.3301y^3 - 0.6471y^2 + 0.0098y - 0.0326 \\
x^3y - 0.0199y^3 - 7.1199y^2 - 7.3933y + 13.533 \\
y^4 + 1.9y^3 - 21.6y^2 - 22.3y + 41 
\end{cases}$$

Note that the lowest degree polynomial of $B$, $f = x + 0.0002y^3 + 0.0012y^2 - 0.3328y - 0.6686$, highlights the fact that $X$ contains “almost aligned” points. In fact, if we neglect the terms with smallest coefficients, $f$ simplifies to $x - 0.3328y - 0.6686$. Since the coefficients of a polynomial are continuous functions of its zeros and the quotient basis $O$ is stable w.r.t. $X^\varepsilon$, we can conclude that there exists a small perturbation $\tilde{X}$ of $X$ containing aligned points and for which the associated evaluation matrix $M_O(\tilde{X})$ is invertible. A simple example of such a set is given by $\tilde{X} = \{(-1, -5), (0, -2), (1, 1), (2, 4)\}$. A completely different result is obtained by applying to the set $X$ the Buchberger-Möller algorithm w.r.t. the same term ordering $\sigma$. The $\sigma$-Gröbner basis $G$ of $I(X)$ is:

$$G = \begin{cases} 
x^2 - 1/9y^2 - 121/30x + 9/10y + 101/45 \\
xy - 1/3y^2 - 41/10x + 7/10y + 41/15 \\
y^3 + 6y^2 + 516243/100x - 171781/100y - 172581/50 
\end{cases}$$

and the associated quotient basis is $O_\sigma(I(X)) = T^2 \setminus LT_\sigma(I(X)) = \{1, y, x, y^2\}$. We observe that $O_\sigma(I(X))$ is not stable (see Example because the evaluation matrix $M_{O_\sigma}(\tilde{X})$ is singular for some admissible perturbations of $X$. In particular, the information that the points of $X$ are “almost aligned” is not at all evident from $G$.

**Example. Empirical points close to an ellipse**

Let $X \subset \mathbb{R}^2$ be a set of points created by perturbing by less than 0.1 the
coordinates of 10 points lying on the ellipse $x^2 + 0.25y^2 - 25 = 0$,
\[
\mathbf{X} = \{(-5.07, 0.02), (4.98, 0), (3.05, 8.07), (3.01, -8.02), (-3.02, 7.99),
\quad (-2.98, -8), (4.01, 5.94), (3.98, -6.06), (-3.92, 6.03), (-4.01, -6)\}
\]
Let $\mathbf{X}^\varepsilon$ be the set of empirical points whose set of specified values is $\mathbf{X}$ and whose common tolerance is $\varepsilon = (0.1, 0.1)$.
Applying SOI on $\mathbf{X}^\varepsilon$ we obtain, after 11 iterations, the stable quotient basis
\[
\mathcal{O} = \{1, y, x, y^2, xy, y^3, xy^2, y^4, xy^3, xy^4\}
\]
We use linear algebra to compute the corresponding stable border basis $\mathcal{B}$ of $\mathcal{I}(\mathbf{X})$. We can identify the “almost elliptic” configuration of the points of $\mathbf{X}$ by looking at $f$ the lowest degree polynomial contained in $\mathcal{B}$:
\[
f = x^2 + 0.273y^2 - 25.250 + 10^{-2}(0.004xy^4 + 0.020xy^3 - 0.034y^4 - 0.489xy^2
\quad -0.177y^3 - 1.371xy + 9.035x + 9.810y)
\]
We observe that $f$ highlights the fact that $\mathbf{X}$ contains points close to an ellipse. In fact, if we neglect the terms with smallest coefficients, $f$ simplifies to $x^2 + 0.273y^2 - 25.250$. Since the coefficients of a polynomial are continuous functions of its zeros and the quotient basis $\mathcal{O}$ is stable w.r.t. $\mathbf{X}^\varepsilon$, we can conclude that there exists a small perturbation $\tilde{\mathbf{X}}$ of $\mathbf{X}$ containing points lying on an ellipse and such that the associated evaluation matrix $M_\mathcal{O}(\tilde{\mathbf{X}})$ is invertible. A simple example of such a set is given by
\[
\tilde{\mathbf{X}} = \{(-5, 0), (5, 0), (3, 8), (3, -8), (-3, 8),
\quad (-3, -8), (4, 6), (4, -6), (-4, 6), (-4, -6)\}
\]

Example. Empirical points close to a circle
In this example we show the behaviour of the SOI algorithm when applied to several sets of points with similar geometrical configuration but with different cardinality.
Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4 \subset \mathbb{R}^2$ be sets of points created by perturbing by less than 0.01 the coordinates of 8, 16, 32 and 64 points lying on the circumference $x^2 + y^2 - 1 = 0$, and let $\varepsilon = (0.01, 0.01)$ be the tolerance. We summarize in Table I the numerical tests performed by applying the SOI algorithm to the empirical set $\mathbf{X}_i^\varepsilon$, for $i = 1 \ldots 4$. The first two columns of the table contain the name of the processed set and the value of its cardinality. The column labelled with “Corners” refers to the set of corners of the stable order ideal computed by the algorithm; the column labelled with “Time” contains the time taken to compute the quotient bases.

Note that the set of corners of the stable quotient bases computed by the SOI algorithm always contain the power product $x^2$: this means that there is an “almost linear dependence” among the power products $\{1, y, x, y^2, xy, x^2\}$ and that some useful information on the geometrical configuration of the points could be found.

13
Example. Empirical points close to an hyperbola, a circle and a cubic

The numerical tests suggest that in most cases the SOI algorithm computes a stable quotient basis, allowing us to determine a stable border basis of $\mathcal{I}(X)$. Nevertheless, this is not true in general, as the following example illustrates.

Let $X$ be the set of distinct empirical points having

$$X = \{(1, 6), (2, 3), (2.449, 2.449), (3, 2), (6, 1)\} \subset \mathbb{R}^2$$

as the set of specified values and $\varepsilon = (0.25, 0.25)$ as the tolerance.

Applying the algorithm SOI to $X$, we obtain the stable order ideal $O = \{1, y, x, y^2\}$; however, this is not a quotient basis, so we cannot obtain a corresponding stable border basis. This is due to the fact that the points of $X$ lie close to the hyperbola $xy - 6 = 0$, the circle $(x - 6)^2 + (y - 6)^2 - 25 = 0$ and the cubic $y^3 - 12y^2 + 6x + 47y - 73 = 0$. So, if the tolerance $\varepsilon$ is too big, they “almost satisfy” all of them.

Observe how the problem does not arise if we use a smaller tolerance, e.g. $\delta = (0.2, 0.2)$. Applying SOI to $X^\delta$ we obtain the stable quotient basis $O' = \{1, y, x, y^2, y^3\}$, and its corresponding border basis:

$$B' = \begin{bmatrix}
xy & + & 0.0047y^3 - 0.0560y^2 + 0.0280x + 0.2194y - 6.336 \\
x^2 & - & 0.4265y^3 + 6.118y^2 - 14.559x - 32.047y + 77.711 \\
xy^2 & + & 0.0114y^3 - 0.1372y^2 + 0.0686x - 5.463y - 0.8231 \\
y^4 & - & 14.477y^3 + 76.724y^2 - 14.862x - 188.419y + 214.345 \\
xy^3 & + & 0.0280y^3 - 6.336y^2 + 0.1680x + 1.316y - 2.016
\end{bmatrix}$$

### Table 1: SOI on sets of points close to a circle

| Input | $\#X_i$ | Corners          | Time  |
|-------|---------|------------------|-------|
| $X_1$ | 8       | $\{x^2, xy^3, y^3\}$ | 0.5 s |
| $X_2$ | 16      | $\{x^2, xy^7, y^3\}$ | 8.5 s |
| $X_3$ | 32      | $\{x^2, xy^{15}, y^{17}\}$ | 79 s  |
| $X_4$ | 64      | $\{x^2, xy^{31}, y^{33}\}$ | 2320 s|

6 Appendix

In this section we present the proof of Proposition 4.2. **Proof:** Let $n \in \mathbb{N}$; first we prove a result on the homogeneous components of degrees 0 and 1 of the inverse of a square matrix $A \in \text{Mat}_{n,n}(F)$.

Let $A$ be a non singular matrix in $\text{Mat}_{n,n}(F)$ and let $B$ be the inverse of $A$.

The homogeneous components $B_0$ and $B_1$ of $B$ are given by

$$B_0 = A_0^{-1} \quad B_1 = -A_0^{-1} A_1 A_0^{-1}$$

Define $\Delta A = \sum_{i \geq 2} A_i$ and $\Delta B = \sum_{i \geq 2} B_i$, so we have

$$A = A_0 + A_1 + \Delta A \quad \text{and} \quad B = B_0 + B_1 + \Delta B$$
Since $AB = I$, where $I$ is the $n \times n$ identity matrix, we have

$$(A_0 + A_1 + \Delta A)(B_0 + B_1 + \Delta B) = I$$

and our claim is immediate.

Now we prove the result of the proposition. Applying the classical least squares method to the linear system $Mx \approx v$ we obtain

$$x = (M^tM)^{-1}M^tv$$

$$\rho = v - Mx$$

Applying to (9) the homogeneous degree decomposition up to degree 1 we have

$$\rho_0 + \rho_1 = (v_0 - M_0x_0) + (v_1 - M_0x_1 - M_1x_0)$$

thus (3) follows.

Since $(M^tM)_0 = M_0^tM_0$ and $(M^tM)_1 = M_0^tM_1 + M_1^tM_0$, from formula (7) we have to first order,

$$(M^tM)^{-1} \cong (M_0^tM_0)^{-1} - (M_0^tM_0)^{-1}(M_0^tM_1 + M_1^tM_0)(M_0^tM_0)^{-1}$$

Up to degree 1, formula (8) becomes

$$x_0 + x_1 = (M^tM)_0^{-1}(M_0^tM_0 + M_1^tM_0) - (M^tM)_1^{-1}M_0^tv_0$$

$$= (M_0^tM_0)^{-1} \left( M_0^tM_0 + M_0^tM_0 + M_1^tM_0 - (M_0^tM_1 + M_1^tM_0)(M_0^tM_0)^{-1}M_0^tv_0 \right)$$

and so

$$x_0 = (M_0^tM_0)^{-1}M_0^tv_0$$

$$x_1 = (M_0^tM_0)^{-1}(M_0^tM_0 + M_1^tM_0 - M_0^tM_1x_0 - M_1^tM_0x_0)$$

thus the proof is concluded.

Acknowledgements. The authors would like to thank Prof. L. Robbiano for his useful and constructive remarks.

During the development of this work John Abbott was a member of a project financially supported by the Shell Research Foundation.

This work was partly conducted during the Special Semester on Gröbner Bases (from 1st February to 31st July 2006) organized by RICAM (Radon Institute for Computational and Applied Mathematics) of the Austrian Academy of Sciences and RISC (Research Institute for Symbolic Computation) of the Johannes Kepler University, Linz, Austria, under the direction of Professor Bruno Buchberger.

References

[1] Abbott, J., Bigatti, A., Kreuzer, M., Robbiano, L. (2000). Computing ideals of points. J. Symb. Comput., 30:341-356.
[2] Abbott, J., Fassino, C., Torrente, M. (2007). Thinning Out Redundant Empirical Data. *Mathematics in Computer Science* - To appear.

[3] Buchberger, B., Möller, H. M. (1982). The construction of multivariate polynomials with preassigned zeros. *Proc. EUROCAM ’82, Lecture Notes in Comp.Sci.*, 144:24-31.

[4] CoCoA Team. CoCoA: a system for doing computations in Commutative Algebra. Available at [http://cocoa.dima.unige.it/](http://cocoa.dima.unige.it/)

[5] Dahlquist, G., Björck, Å., Anderson, N. (1974). *Numerical Methods*. Englewood Cliffs, New Jersey.

[6] Demmel, J. W., Higham, N. J. (1993). Improved error bounds for underdetermined system solvers. *SIAM J. Matrix Anal. Appl.*, 14:1-14.

[7] Heldt, D., Kreuzer, M., Pokutta, S., Poulisse, H. (2006). Approximate computation of zero-dimensional polynomial ideals. Available at [http://www.mathematik.uni-dortmund.de/algebraic-oil/](http://www.mathematik.uni-dortmund.de/algebraic-oil/).

[8] Kreuzer, M., Robbiano, L. (2000). *Computational Commutative Algebra 1*. Springer, Berlin.

[9] Kreuzer, M., Robbiano, L. (2005). *Computational Commutative Algebra 2*. Springer, Berlin.

[10] Sauer, T. (2007). Approximate varieties, approximate ideals and dimension reductions. *Numerical Algorithms*. To appear.

[11] Stetter, H. (2004) *Numerical Polynomial Algebra*. SIAM, Philadelphia.