Properties Analysis of Fuzzy Variational Inequality Problems

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Abstract. The fuzzy variable was introduced in the variational inequality, the model of the fuzzy variational inequality problem was presented in possibility space. And some properties were investigated. This paper was organized as follow, firstly, we introduced fuzzy variable in variational inequality. Furthermore, the expected residual minimization of the regularized gap function was established. Then, we concentrated on a subclass of fuzzy variable in variational inequality in which the function involved was assumed to be affine and the properties of the variational inequality problem with fuzzy variable were discussed.

Introduction

The variational inequality problem (VIP for short) is very important in mathematics. Researchers mainly studies three aspects of VIP including solution to the existence and uniqueness, effective solution algorithms and application such as optimization theory, economics, engineering, game theory and networks. Recently, Chang and Zhu [1] introduced the concepts of variational inequalities for fuzzy mappings which was later studied by Noor [2], Chang and Huang [2], and Lee et al. [3]. Our paper first considers the variational inequality problem under possibility space (FVIP for short). Finding a vector \( \bar{x} \in s \subset \mathbb{R}^n \) such that

\[
\left( x - \bar{x} \right)^T F(x^*, \xi) \geq 0 \quad \forall x, \xi \in \Xi \text{ a.s.,}
\]

where \( \xi \) is fuzzy variable in the possibility space, \( F: \mathbb{R}^n \times \Xi \rightarrow \mathbb{R}^n \) is a mapping, and a.s. is the abbreviation for almost surely under the given possibility measure. There is no doubt that both of them are natural extensions of deterministic VIP. Therefore, In this paper, our focus is talking about the property of FVIP. The remainder of this paper is organized as follows. Some basic concepts will be reviewed in Section 2. In Section 3, we establish the expected model based on possibility measure. In Section 4, some properties were investigated.

Preliminaries

In this section, we will introduce some fundamental concepts and properties concerning possibility measure, membership function and so on.

Definition 1. [4] Let \( \Theta \) is a nonempty set. \( F \) is the power set of \( \Theta \). The element in \( F \) is called an event. If the function \( Pos : F \rightarrow [0, 1] \) satisfies

(1) \( Pos (\emptyset) = 1 \),
(2) \( Pos (A) \leq Pos (B), A \subset B, A, B \in F \)
(3) \( Pos (\bigcup_{i=1}^{n} A_i) \supset \sup_{i=1}^{n} Pos (A_i) \), for \( i = 1, 2, \ldots, A_i \in F \).

Then, the function is possibility measure. \( (\Theta, F(\Theta), Pos) \) is possibility measure space.

Definition 2 Let \( \tilde{x} \) is a fuzzy variable defined in the possibility space. If
\[ \mu_\xi^p(t) = \text{Pos}\{\theta \in \Theta | \bar{x}(\theta) = t\}, \ t \in R \]

\( \mu_\xi^p(t) \) is the membership function of \( \bar{x} \), denoted as \( \mu(t) \).

When we study the uncertainty problem, we naturally think of whether we can find a reasonable average value of this fuzzy variable, and then we use this average value instead of the fuzzy variable to simplify the problem. Liu Y K and Liu B [5] defined the upper expectations and the lower expected value of the fuzzy variable based on fuzzy measure. Then the expected model is presented based on the expectation they proposed.

**Definition 3** Let \( \bar{x} \) be a fuzzy variable in the possibility space. The expected value of \( \bar{x} \) is

\[ E(\xi) = \int_0^\infty \text{Pos}\{\xi \geq x\}dx - \int_0^0 (1 - \text{Pos}\{\xi \geq x\})dx \]

provided that the integral exist.

**Theorem 1** Let \( \Phi(\xi)_+ = \text{Pos}\{\xi \geq t\} \), and \( \Phi(\xi)_- = 1 - \text{Pos}\{\xi \geq t\} \). Then

\[ E(\xi) = \int_\xi^\infty \xi d \Phi(\xi)_- - \int_\xi^\infty \xi d \Phi(\xi)_+ \]

**Proof** From Definition 3,

\[ E(\xi) = \int_0^\infty \text{Pos}\{\xi \geq x\}dx - \int_0^0 (1 - \text{Pos}\{\xi \geq x\})dx \]

we get

\[ E(\xi) = \int_0^\infty \Phi(\xi)_+d\xi - \int_\xi^\infty \xi d \Phi(\xi)_- - \int_\xi^\infty \xi d \Phi(\xi)_+ + \int_\xi^\infty \xi d \Phi(\xi)_- \]

Establishment of the Model

We considered a expected residual minimization variational inequality problem on the basis of possibility theory. So we define a regularized gap function.

**Definition 4** \( g : R^n \times \Xi \rightarrow R \) is a regularized gap function of fuzzy variational inequality problem, if

\[ g(x, \xi) := \max\{(x - y)^T F(x, \xi) - \frac{\alpha}{2} \|x - y\|^2 \} \quad (2) \]

where \( \xi \) is fuzzy variable in the possibility space \( \Xi \).

**Theorem 2** \( \max\{(x - y)^T F(x, \xi) - \frac{\alpha}{2} \|x - y\|^2 \} \) is essentially equivalent to the below

\[ \min_{y \in S} \|y - (x - \alpha^{-1}G^{-1}F(x, \xi))\|^2_{G} \]

**Proof** Let \( \text{Pr}_{aj_{s,G}}(x - \alpha^{-1}G^{-1}F(x, \xi)) \) be the optimal solution of (2) and the projection of the point \( x - \alpha^{-1}G^{-1}F(x, \xi) \) onto the set \( S \) with respect to the norm \( \|\cdot\|_{G} \). Let \( H : R^n \times \Xi \rightarrow R^n \) be

\[ H(x, \xi) = \text{Pr}_{aj_{s,G}}(x - \alpha^{-1}G^{-1}F(x, \xi)) \]

So we have, for any \( x \in R^n \) and any \( \xi \in \Xi \),

\[ g(x, \xi) = (x - H(x, \xi))^T F(x, \xi) - \frac{\alpha}{2} \|x - H(x, \xi)\|^2_{G}, \forall x \in S, \xi \in \Xi \]

(3)

where \( \xi \) is fuzzy variable, \( F : R^n \times \Xi \rightarrow R^n \) is a convex function.

**Theorem 3** Let \( g(x, \xi) : R^n \times \Xi \rightarrow R^n \) be defined by (3). Then

(1) \( g(x, \xi) \geq 0 \), for every \( x \in S, \xi \in \Xi \) and
(2) \( g(x, \xi) = 0 \) if and only if \( x \) solves VI(f, S).

(3) \( x \) solves (3) if and only if it solves (1).

**Proof** the proof of the theorem is easy to get.

Let \( g(x, \star) \) be integrable on \( \Xi \), the vector \( x^* \in S \) that minimizes an expected residual,

\[
\min_{\theta(x)} := E[g(x, \xi)], \text{s.t.} x \in S
\]

Here, \( E \) stands for the expectation with respect to the fuzzy variable \( \xi \in \Xi \).

In the rest of this paper, we assume that \( F(x, \xi) \) is affine, that is,

\[
F(x, \xi) := M(\xi)x + q(\xi)
\]

where \( M : \Xi \rightarrow \mathbb{R}^{n \times n} \) and \( q : \Xi \rightarrow \mathbb{R}^n \) are continuous and integrable on \( \Xi \). We further suppose that

\[
E[\|M(\xi) + q(\xi)\|^2] < +\infty
\]

where \( \|\cdot\| \) means the Euclidean norm. From the Cauchy-Schwarz inequality, we have

\[
E[\|M(\xi) + q(\xi)\|] \leq (E[\|M(\xi) + q(\xi)\|^2])^{1/2}
\]

Moreover, we denote \( f(x) := \bar{M}x + \bar{q} \) with \( \bar{M} := E[M(\xi)] \) and \( \bar{q} := E[q(\xi)] \). Note that

\[
\sqrt{\lambda_{\min}} ||x|| \leq ||f(x)|| \leq \sqrt{\lambda_{\max}} ||x||
\]

where \( \lambda_{\min} \) and \( \lambda_{\max} \) indicate the smallest and largest eigenvalues of \( G \), respectively. Therefore, if \( \bar{M} \) is positive definite, then \( f \) is strongly monotone in the sense that

\[
(x - y)^T (f(x) - f(y)) \geq \mu_{\min} \lambda_{\max}^2 ||x - y||_G^2
\]

where \( \mu_{\min} \) is the smallest eigenvalue of \( \bar{M} + \frac{1}{2} \bar{M}^T \bar{M} \)

**Properties of the Function \( \theta \)**

We first discuss the differentiability of the function \( \theta \)

**Lemma 1.**

Let \( \xi \) be a fuzzy variable defined in the possibility space and \( f(\xi, t) \) be a measurable and integrable function with \( \xi \) for each \( t \) in \((a, b)\). Let \( \Psi(t) := E[f(\xi, t)] \). For any \( \xi \in \Xi \), \( f(\xi, t) \) has a derivative in \((a, b)\), and for \( f(\xi, t) \) and \( g(\xi) \), let \( |f'(\xi, t)| \leq g(\xi) \) for \( \xi \in A \) and \( t \in (a, b) \), where \( g \) is integrable. Then \( \Psi(t) \) has derivative \( \Psi'(t) := E[f(\xi, t)] \) on \((a, b)\).

**Proof** By Theorem 1, we can get \( E[\xi] = \int_0^\infty \xi d\Phi(\xi) - \int_0^\infty \xi d\Phi(\xi)_+ \). From the definition of \( \Psi(t) \),

\[
\Psi(t) = E[f(\xi, t)] = \int_0^\infty f(\xi, t) d\Phi(\xi) - \int_0^\infty f(\xi, t) d\Phi(\xi)_+.
\]

Consider a fixed \( \xi \in \Xi \), for \( t \in (a, b) \), we obtain

\[
\frac{f(\xi, t + h) - f(\xi, t)}{h} = f'(\xi, s)
\]

where \( s \) lies between \( t \) and \( t + h \). The ratio on the left goes to \( f'(\xi, t) \) as \( h \to 0 \), by hypothesis it is Dominated by the integrable function \( g(\xi) \). Therefore,
\[
\frac{\Psi(t+h)-\Psi(t)}{h} = \int_{-\infty}^{0} \frac{f(\xi,t+h)-f(\xi,t)}{h} d\Phi(\xi) - \int_{0}^{+\infty} \frac{f(\xi,t+h)-f(\xi,t)}{h} d\Phi(\xi)
\]

The condition involving \( g \) in this part can be weakened. It suffices to assume that for each \( t \) there is an integrable \( g(\xi,t) \) such that \( \int f(\xi,t) d\Phi(\xi) \leq g(\xi,t), \) for \( \xi \in A \).

**Theorem 4** Let \( g \) and \( \theta \) are differentiable with respect to \( x \). In particular, for any \( x \in S \), we have

\[
\nabla \theta(x) = E[\nabla_x g(x,\xi)]
\]

**Proof** Since \( F(x,\xi) \) and \( g \) are differentiable with respect to \( x \),

\[
\nabla_x g(x,\xi) = F(x,\xi) - (\nabla_x F(x,\xi) - \alpha G)(H(x,\xi) - x)
\]

On the other hand, note that \( g(\xi,t) \geq 0 \) for any \( x \in S \) and \( \xi \in \Xi \), from (4) and (8) that

\[
\frac{\alpha}{2} \left\| F(x,\xi) \right\|_G^2 \leq (x - H(x,\xi))^T F(x,\xi)
\]

\[
\leq (x - H(x,\xi))^T F(x,\xi)
\]

\[
\leq \left\| F(x,\xi) \right\|_G \left\| F(x,\xi) \right\|
\]

\[
\leq \frac{1}{\sqrt{\lambda_{\min}}} \left\| F(x,\xi) \right\|_G \left\| F(x,\xi) \right\|
\]

It follows that

\[
\left\| F(x,\xi) \right\|_G \leq \frac{2}{\alpha \sqrt{\lambda_{\min}}} \left\| F(x,\xi) \right\|
\]

and

\[
\left\| F(x,\xi) \right\|_G \leq \frac{2}{\alpha \sqrt{\lambda_{\min}}} \left\| F(x,\xi) \right\|
\]

Then, from (11) and (12), we have

\[
\left\| \nabla_x g(x,\xi) \right\| = \left\| F(x,\xi) + (\nabla_x F(x,\xi) - \alpha G)(H(x,\xi) - x) \right\|
\]

\[
\leq \left\| F(x,\xi) \right\| + \left\| \nabla_x F(x,\xi) - \alpha G \right\| \left\| H(x,\xi) - x \right\|
\]

\[
\leq \left\| F(x,\xi) \right\| + \left\| \nabla_x F(x,\xi) + \alpha G \right\| \left\| H(x,\xi) - x \right\|
\]

\[
\leq (1 + \frac{2}{\alpha \lambda_{\min}}) (\left\| M(\xi) \right\| + \alpha \left\| G \right\|) \left\| F(x,\xi) \right\|
\]

\[
\leq (1 + \frac{2}{\alpha \lambda_{\min}}) (\left\| M(\xi) \right\| + \alpha \left\| G \right\|) \left\| M(\xi) x + q(\xi) \right\|
\]

\[
\leq (1 + \frac{2}{\alpha \lambda_{\min}}) (\left\| M(\xi) \right\| + \alpha \left\| G \right\|) (\left\| x \right\| + 1) \left( \left\| M(\xi) x + q(\xi) \right\| \right)
\]

By (6) and (7) and Lemma 1. the function \( \theta \) is differentiable and (10) holds.

We next investigate the conditions for the boundedness of the level set defined by

\[
(L_g(\xi))^\delta := \{ x \in S | \theta(x) \leq c \}
\]
where \( c \) is a nonnegative number. In the rest of this section, we suppose that \( \overline{M} \) is positive definite and \( \mu_{\min} > 0 \) is the smallest eigenvalue of \( \frac{\overline{M}^T + \overline{M}}{2} \). The function \( f \) is strongly monotone and (9) holds. From the strong monotonicity of \( f \), it follows that VI(f,S) has a unique solution of (11).

Therefore, we have

\[
(x - x^*)^T f(x^*) \geq 0, \forall x \in S. \tag{11}
\]

**Theorem 5** Let \( \xi \) be a fuzzy variable defined in the possibility space, \( f \) be a convex function. Then

\[
f(E(\xi)) \leq E[f(\xi)].
\]

**Proof** Since \( f \) is a convex function, for each \( y \), there exists a \( k \) such that \( f(x) - f(y) \geq k(x - y) \).

Replacing \( x \) with \( \xi \) and \( y \) with \( E(\xi) \), we obtain

\[f(\xi) - f(E(\xi)) \geq k(\xi - E(\xi)).\]

We have further that

\[E[f(\xi)] - f(E(\xi)) \geq k(E(\xi) - E(\xi)) = 0.\]

The inequality is proved.

**Theorem 6** Let \( \alpha \in (0, 2\mu_{\min} \lambda_{\max}^{-1}) \). The level set \((L_{\alpha}(c))^S\) is bounded for any \( c \geq 0 \).

**Proof** Suppose that there is a nonnegative number \( c \) such that \((L_{\alpha}(c))^S\) is unbounded. This implies that there exists a sequence \( x^k \subseteq (L_{\alpha}(c))^S \) such that \( \lim_{k \to \infty} \|x^k\| = +\infty \),

we have

\[
\bar{c} \geq \theta(x^k) = E[\max\{(x^k - y)^T F(x^k, \xi) - \frac{\alpha}{2} \|x^k - y\|_G^2\}]
\geq \max\{(x^k - y)^T E[F(x^k, \xi)] - \frac{\alpha}{2} \|x^k - y\|_G^2\}
\geq (x^k - x^*)^T E[F(x^k, \xi)] - \frac{\alpha}{2} \|x^k - x^*\|_G^2
\geq (x^k - x^*)^T f(x^k) - \frac{\alpha}{2} \|x^k - x^*\|_G^2
\geq (x^k - x^*)^T f(x^k) + (\mu_{\min} \lambda_{\max}^{-1}) \frac{\alpha}{2} \|x^k - x^*\|_G^2
\geq (\mu_{\min} \lambda_{\max}^{-1}) \frac{\alpha}{2} \|x^k - x^*\|_G^2 \to +\infty
\]

where the second inequality follows from Theorem 5, the fourth inequality follows from (9), and the last inequality follows from (11). This is a contradiction and hence \((L_{\alpha}(c))^S\) is bounded for any number \( c \geq 0 \). We further have the following result.

\[
(y - x^*)^T f(x^*) \geq 0, \forall y \in S.
\]

**Theorem 6** Let \( \alpha \in (0, 2\mu_{\min} \lambda_{\max}^{-1}) \) and \( x^* \) be the unique solution of VI(f,S). We have

\[
\|x - x^*\|_G \leq \sqrt{(\mu_{\min} \lambda_{\max}^{-1} - \frac{\alpha}{2})^{-1} \theta(x)}
\]

**Proof** Let \( x \in S \), we can obtain

\[
(\mu_{\min} \lambda_{\max}^{-1} - \frac{\alpha}{2}) \|x - x^*\|_G^2 \leq (x - x^*)^T f(x) - \frac{\alpha}{2} \|x - x^*\|_G^2
\]
\[
\leq \max_{y \in S} \{(x-x^*)^T f(x) - \frac{\alpha}{2} \|x-x^*\|^2_G \}
\]
\[
\leq \max_{y \in S} \{(x-x^*)^T E[F(x, \xi)] - \frac{\alpha}{2} \|x-x^*\|^2_G \}
\]
\[
\leq E[\max_{y \in S} \{(x-x^*)^T F(x, \xi)\}] - \frac{\alpha}{2} \|x-x^*\|^2_G 
= \theta(x)
\]

where the last inequality follows from Theorem 5. The Theorem is proofed.

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