HILBERTIAN REPULSIVE EFFECT AND DARK ENERGY

ANGELO LOINGER AND TIZIANA MARSICO

Abstract. A repulsive gravitational effect of general relativity (without cosmological term), which was pointed out by Hilbert many years ago, could play a decisive role in the explanation of the observational data concerning the accelerated expansion of the universe.

PACS 04.20 – General relativity.

Introduction. – In previous papers [1] we have illustrated some physical consequences of a repulsive gravitational effect, which was evidenced by Hilbert many years ago [2]. This effect comes forth – in particular regions – in the instance (e.g.) of the Einsteinian gravitational field (without cosmological term) of a mass concentrated in a very small volume. In accord with Einstein and Hilbert, we have considered the singularities (both “soft” and “hard”) of the metric tensor of the field of a mass point as unphysical loci, which however can give an idea of the behaviour of potential $g_{jk}$, $(j, k = 1, 2, 3, 4)$, generated by an extended body. Now, the minimal radius of an extended mass $M$ is equal to $(9/8)m$, where $m = GM/c^2$, as it was demonstrated by Schwarzschild [3] for a homogeneous sphere of an incompressible fluid, and by Weinberg [4] for a sphere of a generic fluid. In our papers [1] we have considered this case.

In the present Note we show that the above Hilbertian effect could play an important role in the problem of the dark energy. Our results can be regarded as a prolegomenon to a realistic cosmological model that explains physically the nature of the dark energy, by avoiding ad hoc assumptions, as (e.g.) the recourse to a cosmological term, or to a “quintessence” [5].

The behaviour of the geodesic lines of Schwarzschild manifold is interesting. For a gravitating mass point, test-particles and light-rays in radial motions arrive at the space surface $r = 2m$ (we employ here the customary standard coordinates) with a velocity $v = dr/dt$ and an acceleration $dv/dt$ equal to zero. (Accordingly, not even the tricking assumption that for the “internal” region $r < 2m$ the direction $\partial/\partial r$ acquires a temporal meaning – and $\partial/\partial t$ a radial meaning – can validate a physical extension to $r < 2m$ of the “external”, $r > 2m$, radial geodesics). On the contrary, test-particles and light-rays arrive at the surface of a gravitating sphere of radius $(9/8)2m$ with a velocity and an acceleration that can be relatively small, but non-zero.

This Hilbertian effect is an objective physical property, which is independent of the coordinate system. The same results are obtained, e.g., with the
original Schwarzschild’s coordinates ([3] and [6]), with Brillouin’s coordinates [7], with Fock’s coordinates [8] – and with any coordinate frame which does not “conceal” the “soft” singularity. Thus, the well-known coordinates \( u, v \) of Kruskal and Szekeres must be excluded, because the derivatives \( \partial u/\partial r \) and \( \partial v/\partial r \), \((r \text{ standard}),\) are singular at \( r = 2m \). An analogous exclusion holds for the celebrated coordinate systems of Eddington-Finkelstein, of Lemaître, of Synge [9], of Novikov.

The Hilbertian repulsion is particularly evident for the circular geodesics of Schwarzschild manifold. For the material particles these geodesics are restricted by the following inequalities (standard coordinates) [2]:

\[
1. \quad r > \frac{3}{2} \cdot 2m; \quad \frac{v}{c} < \frac{1}{\sqrt{3}},
\]

where \( v = c(m/r)^{1/2} \) is the (linear) velocity. For the circular trajectories of the light-rays, the coordinate radius \( r \) is equal to \((3/2) 2m\), and the velocity \( v \) is equal to \( c/\sqrt{3} \) [2]. For small \( r \)’s the Einsteinian gravity acts repulsively. Since \( 3/2 > 9/8 \), we see that no circular geodesic can touch the surface of a material sphere of physical radius \((9/8) 2m \). (With standard coordinates, this radius coincides with the coordinate radius, say \( r_c \), of the sphere).

It is commonly emphasized that if we adopt, e.g., the coordinates of Synge [9] or of Kruskal-Szekeres, or of Novikov, the geodesic lines penetrate the regions corresponding to the region \( r \leq 2m \): geodesic completeness. A physically insignificant result, because – as we have pointed out – these coordinate systems “hide” the “soft” singularity \( r = 2m \) – and are therefore unreliable. Moreover, in these global sets of coordinates the gravitational field is non-static. Synge, Kruskal, Szekeres, Novikov have merely camouflaged a stumbling-block.

Quite incomprehensibly, in the current literature Hilbert’s significant and complete treatment of the Schwarzschild manifold [2] is ignored; of special importance is his investigation of the relevant first integrals of the equations of motion of test-particles and light-rays, by means of which an intrinsic characterization of the geodesic lines can be given. (An English translation of Hilbertian memoirs on general relativity would be highly desirable).

1. – Recent observations tell us that: i) the mass density (of visible and dark matter) of the universe is \textit{circa} equal to \((1/3)\) of the critical density of Friedmann model; ii) the expansion of the universe is monotonically increasing – and in an \textit{accelerated} way; iii) the universe seems to be spatially flat [10]; iv) most of the universe mass is \textit{dark}; v) the present value \( H_0 \) of Hubble’s function \( H(t) \) is: \((65 \pm 10) \text{ km} \cdot \text{s}^{-1} \text{Mpc}^{-1} \); \((1 \text{ Mpc} = 3.26 \times 10^6 \text{ lyr})\).
2. — Many papers on dark energy contain improper considerations on Friedmann model, in which the mass density \( \rho \) is only composed of the “dust” of the “galaxy gas” \[1\]. If \( \rho_c := 3H(t)^2/(8\pi G) \) is the critical density, we have \( \Omega := \rho/\rho_c = 8\pi G\rho/(3H^2) \); the introduction of the cosmological term \( \Lambda_{ijk} \) in Friedmann equations is certainly allowed, but it is not allowed to add to \( \rho \) a \( \rho_v := c^2\Lambda/(8\pi G) \), interpreted – from the standpoint of quantum field theory – as the mass-energy density of the vacuum, and to consider an \( \Omega_v := 8\pi G\rho_v/(3H^2) \). Indeed, general relativity is a classical theory, and therefore it does not admit to be hybridized with quantum concepts. In particular, the concept of a vacuum energy is fully extraneous to GR.

A cosmological term can be introduced also in Newton theory, by substituting Poisson equation \( \nabla^2 \Phi = 4\pi G\rho \) with \( \nabla^2 \Phi - \Lambda \Phi = 4\pi G\rho \) \[12\]. There exists a perfect isomorphism between Friedmann model and the corresponding Newtonian model, both for \( \Lambda = 0 \) and for \( \Lambda \neq 0 \) \[11\].

We remark also that the consideration of a mass density proportional to space curvature is conceptually unjustified.

However, without the fictitious mass-energy density \( \rho_v \) it seems impossible to explain – with Friedmann model – the observational data: indeed, since \( \Omega \approx 1/3 \), the assumption \( \Omega_v \approx 2/3 \) appears as arithmetically natural in order to have an \( \Omega_{tot} = \Omega + \Omega_v = 1 \), which would ensure the monotonic expansion of a spatially flat universe; the accelerated motion would be the merit of the cosmological term.

Experience counsels us to try the construction of a more adequate cosmological model.

3. — A purely kinematic model of universe is based on the well-known Hubble’s relation:

\[ v = cz = H_0 r \; ; \]

where \( v = dr/dt \) is the recession velocity, \( z \) is the red-shift \( \Delta \lambda/\lambda_0 \), \( r \) is the distance of the considered galaxy. Eq.(2) can be written in a vectorial form:

\[ \mathbf{v} = H_0 \mathbf{r} \; ; \]

an observer in another galaxy at a distance \( r' \) and a velocity \( \mathbf{v}' \) relative to us \( (\mathbf{v}' = H_0 \mathbf{r}') \) would find

\[ \mathbf{v} - \mathbf{v}' = H_0 (\mathbf{r} - \mathbf{r}') \; , \]

and this proves that the kinematic model describes a homogeneous and isotropic universe. Under the condition \( \text{curl} \mathbf{v} = 0 \), eq. (3) gives the only velocity field which assures both homogeneity and isotropy.

Any reasonable cosmological model must have as a consequence eqs. (2), (3), (1).
4. – Let us assume that the core of a future cosmological model is a material sphere \( S \) of mass \( M \), whose radius is \((9/8)m\) (with \( m \equiv GM/c^2 \)), and consider its Schwarzschild manifold \([3], [6]\). The “galaxy gas” be composed of a spherically symmetric swarm of test-particles, which at a given time abandon abruptly the periphery of \( S \) with a suitable radial velocity – why? because “Im Anfang war die Tat”! – as in Friedmann model.

According to Hilbert’s treatment \([2], [1]\), the equation of the radial geodesics can be written, in standard coordinates:

\[
\frac{1}{c^2} \frac{d^2 r}{dt^2} - \frac{3}{2} \frac{2m}{r(r-2m)} \left( \frac{dr}{cdt} \right)^2 + \frac{m(r-2m)}{r^3} = 0 ;
\]

with the first integral:

\[
\left( \frac{dr}{cdt} \right)^2 = \left( \frac{r-2m}{r} \right)^2 + A \left( \frac{r-2m}{r} \right)^3 ,
\]

the constant \( A \), which is equal to zero for the light-rays, is negative for the material particles; it is: \( \varepsilon \leq |A| \leq 1 \), with \( \varepsilon > 0 \) and \( ad lhibitum \) small.

Eqs. \((5)\) and \((6)\) tell us that the acceleration is negative (attractive gravity) or positive (repulsive gravity) where, respectively:

\[
\left| \frac{dr}{cdt} \right| < \frac{1}{\sqrt{3}} \frac{r-2m}{r} ,
\]

\[
\left| \frac{dr}{cdt} \right| > \frac{1}{\sqrt{3}} \frac{r-2m}{r} .
\]

Putting \( x := r/(2m) \) and \( y := (dr/cdt)^2 \), eq. \((6)\) can be rewritten as follows:

\[
y(x) = \left( \frac{x-1}{x} \right)^2 \left( 1 - |A| \frac{x-1}{x} \right) ;
\]

the following nine figures give the diagrams of \( y(x) \) for the following values of \( |A| \): 1; 0.9; 0.8; 0.7; 2/3; 0.5; \( 10^{-1} \); \( 10^{-3} \); \( 10^{-6} \). The last five diagrams represent motions through everywhere-repulsive regions. (Fig.10 gives the \( y(x) \) for light-rays, \( A = 0 \)).
Figure 1. Diagram of $y(x) = [(x - 1)/x]^2[1 - (x - 1)/x]$ for some values of $x$; $(9/8) \leq x < +\infty$; max(3.0; 4/27); $[y(9/8)]^{1/2} = 2\sqrt{2}/27$.

Figure 2. Diagram of $y(x) = [(x - 1)/x]^2[1 - 0.9*(x - 1)/x]$ for some values of $x$; $(9/8) \leq x < +\infty$; max(3.75; 0.182844); $[y(9/8)]^{1/2} = 0.105409$. 
Figure 3. Diagram of $y(x) = [(x-1)/x]^2[1 - 0.8*(x-1)/x]$ for some values of $x$: $(9/8) \leq x < +\infty$; max(6.0; 0.231481); $[y(9/8)]^{1/2} = 0.106058$.

Figure 4. Diagram of $y(x) = [(x-1)/x]^2[1 - 0.7*(x-1)/x]$ for some values of $x$: $(9/8) \leq x < +\infty$; max(21.0; 0.302343); $[y(9/8)]^{1/2} = 0.106703$. 
Figure 5. Diagram of $y(x) = [(x - 1)/x]^2[1 - (2/3) \times (x - 1)/x]$ for some values of $x$; $(9/8) \leq x < +\infty$; $\max(+\infty, 1/3)$; $[y(9/8)]^{1/2} = 0.106917$.

Figure 6. Diagram of $y(x) = [(x - 1)/x]^2[1 - 0.5 \times (x - 1)/x]$ for some values of $x$; $(9/8) \leq x < +\infty$; $\max(+\infty, 0.5)$; $[y(9/8)]^{1/2} = 0.107981$. 
Figure 7.
Diagram of $y(x) = [(x - 1)/x]^2 [1 - 10^{-1} \cdot (x - 1)/x]$ for some values of $x; (9/8) \leq x < +\infty$; max$(+\infty; 0.9); [y(9/8)]^{1/2} = 0.110492.$

Figure 8.
Diagram of $y(x) = [(x - 1)/x]^2 [1 - 10^{-3} \cdot (x - 1)/x]$ for some values of $x; (9/8) \leq x < +\infty$; max$(+\infty; 1 - 10^{-3}); [y(9/8)]^{1/2} = 0.111105.$
Figure 9.
Diagram of \( y(x) = \left(\frac{x - 1}{x}\right)^2 [1 - 10^{-6} \times (x - 1)/x] \) for some values of \( x \); \( (9/8) \leq x < +\infty \); max(\(+\infty; 1 - 10^{-6}\))
\[ y(9/8) \]^{1/2} = 0.111111.

Figure 10. Diagram of \( y(x) = \left(\frac{x - 1}{x}\right)^2 \) for some values of \( x \); \( (9/8) \leq x < +\infty \); max(\(+\infty; 1.0\))
\[ y(9/8) \]^{1/2} = 1/9.
In Friedmann model with $\Lambda = 0$, there are relations among the scale factor $F(t)$, its derivatives $\dot{F}(t)$ and $\ddot{F}(t)$, the mass density $\rho(t)$, the constant $\zeta (=-1, 0, +1)$ of space curvature. Experience determines $\rho(t_0)$ and $H_0 = H(t_0)$, where $t_0$ is the present time, and $H(t) = \dot{F}(t)/F(t)$ is Hubble’s function.

In our skeleton of cosmological model there are relations among the radial coordinate $r(t)$, its derivatives $\dot{r}(t)$ and $\ddot{r}(t)$, the mass $M$ of the gravitating centre of radius $(9/8)2m$, the integration constant $A$; for us the function $\rho(t)$ is theoretically free. Of course, $H(t) = \dot{r}(t)/r(t)$, $A$ and $m$ are free parameters. Experience determines $H(t_0) = \dot{r}(t_0)/r(t_0)$ and $\rho(t_0)$.

If $d(t)$ is the distance between two test-particles (two galaxies), we have

$$d(t) = \text{const} \cdot r(t) \quad ,$$

for which

$$\dot{d}(t) = d(t) \frac{\dot{r}(t)}{r(t)} = d(t) H(t) \quad ,$$

and we see that the fundamental relation of the kinematic model is satisfied.

The galaxy gas and the gravitating body, hard core of the model, can be composed, partially or even entirely, of dark matter.

The main result of paper [10] is the following: the temperature anisotropies $\Delta T/T$ of CMB (cosmic microwave background) tell us that the universe is spatially flat ($\zeta = 0$ in Friedmann model). However, this assertion is not purely empirical, since depends heavily on the assumed adequacy of the inflationary model. In our scheme the universe has the spacetime curvature of Schwarzschild manifold, which decreases gradually going away from the gravitating centre.

Conclusion. The present paper intends simply to draw the attention on the possible role of Hilbertian repulsive effect ([2], [1]) for the explanation of the accelerated expansion of the universe.

A final remark. Our insistence on the field generated by an extended gravitating centre, with a minimal radius $(9/8)2m$, has the aim to show how useless are all the attempts to give a physical meaning to singular geometrical loci.
Hilbert’s eqs. (41), (42), (43) – see [2] – give immediately (p is an affine evolution-parameter):

\[
\left( \frac{dr(p)}{dp} \right)^2 = C^2 - \left( 1 - \frac{2m}{r(p)} \right) \left( |A| + \frac{B^2}{r^2(p)} \right),
\]

where \( A = -|A|, B, C \) are integration constants; \( A \) and \( C \) are numbers, \([B] = [\text{LENGTH}].\) Since \( r^2 \frac{d\varphi(p)}{dp} = B, \) we see that \( B \) gives the value of the angular momentum in the plane of motion \( \vartheta = \pi/2. \) The meaning of constant \( C \) is given by Hilbert’s eq. (43), (H. puts \( c = 1):\)

\[
\frac{r(p) - 2m}{r(p)} \frac{cdt(p)}{dp} = C;
\]

with a trivial change of the parameter \( p, \) the constant \( C \) can be normalized to 1.

We see that Hilbert utilizes five first integrals: those characterized by \( A, B, C, \) plus the two remaining components of angular momentum, that are implicitly determined by the choice \( \vartheta = \pi/2. \) The physical first integrals are obviously four: the energy and the components of the angular momentum. The energy constant is \( A. \) The constant \( C \) owes its existence to the introduction of the auxiliary parameter \( p. \)

Hilbert employed an affine \( p, \) in lieu of the proper time \( \tau \) or of \( s = c\tau \), in order that his equations hold both for material particles and light-rays. Most authors employ \( s = c\tau \) as an evolution parameter for the motions of the test-particles. Then, the constant \( A \) has the unique value \(-1, \) the constant \( B \) maintains its meaning, and the meaning of \( C \) is given by the following equation:

\[
\frac{r(s) - 2m}{r(s)} \frac{cdt(s)}{ds} = C,
\]

from which we see that \( C \) gives the value of \( g_{44}(s) = -(r(s) - 2m)/r(s) \) for \( s = s_0, \) when the particle begins its motion and \( [dct(s)/ds]_{s=s_0} = 1. \)

When the evolution parameter is \( p, \) the constant \( A \) is dynamically important, and \( C \) plays only a limited role. Vice-versa, when the evolution parameter is \( s, \) the constant \( C \) is dynamically important, while \( A \) plays only a limited role.

The description of the motions by means of coordinate time \( t \) is the most adequate from the standpoint of the physical meaning.
References

[1] A. Loinger and T. Marsico: a) arXiv:0706.3891 v3 [physics.gen-ph] 16 Jul 2007; b) ibid.: 0710.3927 v1 [physics.gen-ph] 21 Oct 2007; c) ibid.: 0711.4997 v3 [physics.gen-ph] 22 Dec 2007.

[2] D. Hilbert, Mathem. Annalen, 92 (1924) 1; also in Gesammelte Abhandlungen, Dritter Band (Springer-Verlag, Berlin) 1935, p.258.

[3] K. Schwarzschild, Berl. Ber., (1916) 424; an English version in arXiv:physics/9912033 (December 16th, 1999).

[4] S. Weinberg, Gravitation and Cosmology etc. (Wiley and Sons, New York, etc.) 1972, Chapt.11, sect. 6.

[5] See, e.g.: Y. Wang and P. Mukherjee, arXiv:astro-ph/0703780 – 21 March 2007; A. Balbi and C. Quercellini, arXiv:0704.2350 v3 [astro-ph] 12 Nov 2007. And references therein.

[6] K. Schwarzschild, Berl. Ber., (1916) 189; an English translation in: i) arXiv:physics/9905030 (May 12th, 1999); ii) Gen. Rel. Grav., 35 (2003) 951.

[7] M. Brillouin, Journ. Phys. Rad., 23 (1923) 45; an English version in arXiv:physics/0002009 (February 3rd, 2000).

[8] V. Fock, The Theory of Space, Time and Gravitation, (Pergamon Press, Oxford, etc.) 1964, sect. 57.

[9] J.L. Synge, Proc. Roy. Ir. Acad., 53A (1950) 83.

[10] W. Hu and S. Dodelson, arXiv:astro-ph/0110414 v1 – 18 Oct 2001, and in Annu. Rev. Astron. and Astrophys. 2002. And references therein.

[11] A. Unsöld and B. Baschek, The New Cosmos, Fourth Completely Revised Edition (Springer-Verlag, Berlin, etc.) 1991, sect. 5.9; A. Loinger, arXiv:physics/0504018 v1 – 3 April 2005. And references therein.

[12] W. Pauli, Teoria della Relatività (Boringhieri, Torino) 1958, p.271; and the reference therein.

A.L. – Dipartimento di Fisica, Università di Milano, Via Celoria, 16 - 20133 Milano (Italy)

T.M. – Liceo Classico “G. Berchet”, Via della Commenda, 26 - 20122 Milano (Italy)

E-mail address: angelo.loinger@mi.infn.it
E-mail address: martiz64@libero.it