Renormalization of masses of sterile neutrinos in the $\nu$MSM

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The quasi-degeneracy of heavier sterile neutrino masses in the Neutrino Minimal Standard Model ($\nu$MSM) facilitates the production of lepton asymmetry below the electroweak scale. The first order loop corrections to this mass-difference has been computed in this work along with a numerical estimate of the contribution.

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I. INTRODUCTION

A renormalizable extension of the Standard Model (SM), the $\nu$MSM extends the existing SM of particle physics by three sterile neutrinos, which are singlets under the SM gauge group. The $\nu$MSM attempts to address the various unresolved issues of the SM [1–5] (for a review see [3]), one of them being dark matter production.

The lepton asymmetry, produced at the same time as the generation of the dark matter sterile neutrinos (at temperature of $\sim 100$ MeV), affects their spectrum and number density (see [6–8]). Upon comparison of theoretical computation of the abundance of dark matter sterile neutrinos with cosmological and astrophysical observations, the lepton asymmetry ($\Delta L$) is required to be much larger than the baryon asymmetry ($\Delta B$): $\Delta L \geq 3 \times 10^3$, where $\Delta B \sim 10^{-10}$.

The high value of the lepton-asymmetry can be created provided the masses of the heavier sterile neutrinos in the $\nu$MSM are almost degenerate. Moreover, the mass-difference that leads to the requisite value of the lepton asymmetry must be much lower than the active neutrino mass-difference [3, 14]. In an accompanying paper [14], we discussed the naturalness of the fine-tuning required for satisfying the leptogenesis and active neutrino oscillation observations from the perspective of the renormalization group (RG) evolution of the $\nu$MSM parameters.

A complete analysis of the effect of mass-difference on leptogenesis, however, requires the computation of the physical mass-difference. This is the purpose of the present work, where we compute the radiative corrections to the mass-difference at the one loop level. For a review of the generalized on-shell renormalization procedure for Majorana neutrino theories, see [15].

The paper is organized as follows: in section II we review the Lagrangian of the $\nu$MSM, along with the definitions of the different parameters. In section III we describe our formalism of computation of loop corrections and compute the various contributions. In section IV we compute the mass difference with the loop corrections taken into account, while numerical estimates are given in V. Finally, in section VI we summarize our results.

II. THE $\nu$MSM LAGRANGIAN AND RELEVANT MASS-MATRIX

We use the Lagrangian of the $\nu$MSM in the parametrization [3, 14, 16]. In addition, considering the fact that the sterile neutrino Yukawa couplings are much smaller than the gauge couplings and neglecting the numerically much smaller charged lepton Yukawa couplings, we choose a basis for the leptonic doublets, in which the Lagrangian has the following simple form:

$$\mathcal{L}_{\nu MSM} = \mathcal{L}_0 + \Delta \mathcal{L},$$

$$\mathcal{L}_0 = \mathcal{L}_{SM} + \sum_{I=2,3} \overline{N_I}i\partial_\mu \gamma^\mu N_I + (f_2 \overline{N}_2 N_2$$

$$+ f_3 \overline{N}_3 N_3)\phi - M \overline{N}_2 N_3 + \text{h.c.},$$

$$\Delta \mathcal{L} = f_{23} \overline{N}_3 \Phi - \frac{\Delta M}{2} \sum_{I=2,3} \overline{N}_I N_I + \text{h.c.},$$

where $N_I$ are the right handed singlet leptons ($I = 2, 3$), $\phi$ and $l_{2,3}$ are the Higgs and the lepton doublets respectively, $M$ is the common mass of the two heavy neutral fermions, $\Delta M$ is the diagonal element of the Majorana mass matrix, $\phi = e^{i\frac{\pi}{2}}\phi$, $\delta = (\phi_1, \phi_2, \phi_3)$, and $M$ and $\Delta M$ are taken to be real. The Yukawa couplings $f_2, f_3$ can be chosen to be real by suitably defining the phases of $l_{2,3}$, while the $f_{23}$ is complex, with a phase of $\alpha$. The relation between $f_2, f_3$ and $h_{012}$ introduced in [3, 14] can be found by comparing eqs. (2) and (3) of [14].

We have omitted the dark matter sterile neutrino $N_1$ from the Lagrangian as its influence on the problem we are interested in is negligibly small [3].

We will use unitary gauge ($\Phi = \frac{1}{\sqrt{2}}(0 v + h)^T$) to reduce the computational complexity. Here $v$ is the vacuum expectation value of the Higgs boson and is taken to be 246...
corrections to this mass difference. Using the Euler-Lagrange’s equation of motion, we get the Dirac equation for the system of particles as follows:

\[ i\phi = \begin{pmatrix} N_{2R} \\ N_{3R} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix} - \begin{pmatrix} 0 & M_{\nu} \\ M_{\nu}^T & 0 \end{pmatrix} = 0, \quad (4) \]

where

\[
\begin{pmatrix} \Delta M & M \\ M & \Delta M \end{pmatrix} = \begin{pmatrix} \Delta M & M \\ M & \Delta M \end{pmatrix},
\]

At the tree level, the eigenvectors of \( M_\nu \) up to first order in Yukawa couplings with eigenvalues of \( M_\nu \) up to second order in the same are obtained by perturbative computation and are listed in the appendix B (see [7]).

The mass-difference between the sterile neutrino flavors were shown to be [2]:

\[ \delta m_{tree} = \frac{|m^2|}{M}, \quad (6) \]

The aim of the paper is to compute the first order loop corrections to this mass difference.

Also, the active neutrino mass-difference (\( \Delta m_{\nu} \)) may be solved at the tree-level to be [2]:

\[ \Delta m_{\nu} = \frac{f_2 |f_{23}| v^2}{M}. \quad (8) \]

### III. LOOP CORRECTIONS

#### A. Propagator

Consider the propagator for the system of active and sterile neutrinos:

\[ S_F(p) = \frac{i}{p - M - \Sigma}, \quad (9) \]

where \( \Sigma \) represents the first order loop correction to the mass matrix \( M \). It is useful to keep in mind that \( M \) and \( \Sigma \) represent the mass-matrix and the loop corrections of the complete system of 2 active and 2 sterile neutrinos, and hence are \( 8 \times 8 \) matrices. As usual, mass eigenvalues are given at the one loop level by the poles of the propagator.

Including only one particle irreducible diagrams for the computation, it is easy to see that only the following loop correction matrix elements are relevant: \( \Sigma_{ss}, \Sigma_{as}, \Sigma_{sa} \) and \( \Sigma_{aa} \), where

\[ \Sigma_{ss} = \begin{pmatrix} \langle N_{2R}|\Sigma|N_{2R} \rangle \\ \langle N_{3R}|\Sigma|N_{2R} \rangle \\ \langle N_{3R}|\Sigma|N_{3R} \rangle \end{pmatrix} \quad (10) \]

and so on. Here the subscripts \( s, a \) represent sterile and active flavors respectively.

#### B. Loop correction to active neutrino propagators: Computation of \( \Sigma_{aa} \)

Considering diagrams that give up to quadratic contribution in Yukawa coupling, we find that the only possible contribution comes from the internal W and Z boson loops. The following figure (Fig. 1) represents the loop-correction to the active neutrino propagator.

![FIG. 1. Contribution to active neutrino propagator from W, Z boson loop.](image)

Performing the one-loop computation using dimensional regularization, we find that

\[
\langle \nu_{2L}|\Sigma|\nu_{2L} \rangle = \frac{\frac{\mu^2}{v^2}(1 + \ln \frac{\mu^2}{M_W^2}) - \frac{\mu^2}{2v^2}(1 + \ln \frac{\mu^2}{M_Z^2})}{4\pi^2} + \frac{3g^2 + g'^2}{24}. \quad (11) \]

Define:

\[
C_1 = \frac{1}{4\pi^2} \left[ \frac{\mu^2}{v^2}(1 + \ln \frac{\mu^2}{M_W^2}) - \frac{\mu^2}{2v^2}(1 + \ln \frac{\mu^2}{M_Z^2}) + \frac{3g^2 + g'^2}{24} \right]. \quad (12) \]

Since the active neutrino masses are much smaller than the Higgs vacuum expectation value, we can assume that \( p^2 \ll v^2 \), implying

\[ C_1 = \frac{1}{4\pi^2} \frac{3g^2 + g'^2}{24}. \quad (13) \]

Thus,

\[ \langle \nu_{2L}|\Sigma|\nu_{2L} \rangle = pC_1. \quad (14) \]
Analogous computation shows:
\[ \langle \nu_{3L}|\Sigma|\nu_{3L}\rangle = pC_1. \]  
(15)

In absence of terms contributing to the mixing between \(\nu_2\) and \(\nu_3\), the lowest order contribution to the loop correction matrix may be written as follows:
\[ \Sigma_{aa} = pC_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]  
(16)

C. Loop Correction to the Sterile Neutrino propagators:

Computation of \(\Sigma_{ss}\)

The sterile neutrino propagator receives contributions from the internal Higgs loop in addition to W and Z boson loops. From here on, we use the intuitive notation that \(\Sigma_{w,z,h,t}\) respectively represent the contribution to the loop-correction \(\Sigma\) from W, Z, Higgs-boson loop and tadpole graphs.

1. Contribution from the W boson loop

![Fig. 2. Contribution to sterile neutrino propagator from W boson loop; N, \(\nu\) represent the different possible flavors.](image)

The contribution coming from the W boson loop is represented in Fig. 2 and is given by
\[ \langle N_2R|\Sigma_w|N_2R\rangle = \frac{f_2^2}{(4\pi)^2} p \left[ -1 \left( \ln \frac{\mu^2}{m_\nu^2} + 1 \right) + \frac{(g_2^2 + g_1^2)v^2}{24p^2} \right]. \]  
(17)

2. Contribution from Z boson loop

![Fig. 3. Contribution to sterile neutrino propagator from Z boson loop.](image)

Fig. 3 represents the contribution from the Z-boson loop. Performing the computation, we get
\[ \langle N_2R|\Sigma_z|N_2R\rangle = \frac{f_2^2}{(4\pi)^2} p \left[ -1 \left( \ln \frac{\mu^2}{M_Z^2} + 1 \right) + \frac{(g_2^2 + g_1^2)v^2}{48p^2} \right]. \]  
(18)

3. Contribution from the Higgs boson loop

![Fig. 4. Contribution to sterile neutrino propagator from Higgs boson loop.](image)

The above figure (Fig. 4) shows the contribution from the Higgs loop, which leads to
\[ \langle N_2R|\Sigma_h|N_2R\rangle = -\frac{f_2^2}{2(4\pi)^2} \left[ \frac{1}{2} \ln \frac{\mu^2}{m_H^2} + \frac{1}{4} \right]. \]  
(19)

4. Matrix Element representing the loop contribution

The total loop-correction will be the sum of the aforementioned contributions. Thus,
\[ \Sigma_{ss} = p(F_1 + F_2) \left( f_2^2 f_{23}^2 + |f_{23}|^2 \right), \]  
(20)

where
\[ F_1 = -\frac{1}{2(4\pi)^2} \left[ \frac{1}{2} \ln \frac{\mu^2}{m_H^2} + \frac{1}{4} \right], \]  
(21)
\[ F_2 = \frac{1}{(4\pi)^2} \left[ -1 \left( \ln \frac{\mu^2}{M_Z^2} + 1 \right) + \frac{(g_2^2 + g_1^2)v^2}{48p^2} \right] \]
\[ + \frac{1}{(4\pi)^2} \left[ -1 \left( \ln \frac{\mu^2}{M_W^2} + 1 \right) + \frac{(g_2^2 + g_1^2)v^2}{24p^2} \right]. \]  
(22)

D. Loop Correction to the Active-Sterile Neutrino propagator: Computation of \(\Sigma_{as}\) and \(\Sigma_{sa}\)

The active-sterile neutrino propagator receives similar contribution from the W and Z boson loops on the external active neutrino leg and in addition receives contributions from tadpoles. We consider only the top quark loop among the possible fermion loops in the tadpole graphs.
1. Contribution from W boson

The W-boson loop contribution is represented by Fig. 5 and it is given by
\[
\langle \nu_2^L | \Sigma_w | N_{2R} \rangle = -\frac{f_2 v}{\sqrt{2}} \left[ \frac{1}{(4\pi)^2} \left( -\frac{p^2}{v^2} (1 + \ln \frac{\mu^2}{M_W^2}) + \frac{g^2}{12} \right) + \frac{3m_H^2}{2\sqrt{2}v} \ln \frac{\mu^2}{m_H^2} - \frac{6M_W^4}{\sqrt{2}vm_H^2} \ln \frac{\mu^2}{M_W^2} \right].
\] (23)

2. Contribution from Z boson loop

Fig. 6 represents the contribution from the Z boson loop, which leads to
\[
\langle \nu_2^L | \Sigma_z | N_{2R} \rangle = \frac{f_2 v}{\sqrt{2}} \left[ \frac{1}{(4\pi)^2} \left( -\frac{p^2}{v^2} (1 + \ln \frac{\mu^2}{M_Z^2}) + \frac{g^2}{12} \right) - \frac{3M_Z^4}{\sqrt{2}vm_H^2} \ln \frac{\mu^2}{M_Z^2} + \frac{12m_f^4}{\sqrt{2}m_H^2} \ln \frac{\mu^2}{m_f^2} \right].
\] (24)

3. Contribution from tadpole graphs

The tadpole contribution to the active-sterile neutrino propagator is represented by Fig. 7 which leads to
\[
\langle \nu_2^L | \Sigma_t | N_{2R} \rangle = \frac{f_2 v}{\sqrt{2}} \left[ \frac{1}{(4\pi)^2} \left( -\frac{3m_H^2}{2\sqrt{2}v} \ln \frac{\mu^2}{m_H^2} - \frac{6M_W^4}{\sqrt{2}vm_H^2} \ln \frac{\mu^2}{M_W^2} \right) - \frac{3M_Z^4}{\sqrt{2}vm_H^2} \ln \frac{\mu^2}{M_Z^2} + \frac{12m_f^4}{\sqrt{2}m_H^2} \ln \frac{\mu^2}{m_f^2} \right].
\] (25)

4. Matrix Element representing the loop contribution

Thus,
\[
\Sigma = (K_1 + K_2) \left( \begin{array}{cc} f_2 & f_2^* \\ 0 & f_3 \end{array} \right) = (K_1 + K_2)C^\dagger,
\] (26)
where
\[
C = \left( \begin{array}{cc} f_2 & 0 \\ f_3 & f_3 \end{array} \right),
\] (27)
\[
K_1 = \frac{1}{(4\pi)^2} \left[ -\frac{3m_H^2}{2\sqrt{2}v} \ln \frac{\mu^2}{m_H^2} - \frac{6M_W^4}{\sqrt{2}vm_H^2} \ln \frac{\mu^2}{M_W^2} \right]
\] (29)
and
\[
K_2 = \frac{v}{\sqrt{2}} \left[ -\frac{p^2}{v^2} (1 + \ln \frac{\mu^2}{M_Z^2}) + \frac{g^2}{12} \right]
\] (30)
\[
-\frac{3m_Z^4}{\sqrt{2}vm_H^2} \ln \frac{\mu^2}{M_Z^2} + \frac{12m_f^4}{\sqrt{2}m_H^2} \ln \frac{\mu^2}{m_f^2} \right].
\] (30)

Reasoning as in III.B we arrive at \( p^2 \ll v^2 \). This implies
\[
K_2 = -\frac{v}{\sqrt{2}} \left[ \frac{1}{(4\pi)^2} \left( 3g^2 + g'^2 \right) \right].
\] (31)

IV. COMPUTATION OF THE MASS-DIFFERENCE

Having obtained the one loop corrections to the propagator of the system of active and sterile neutrinos, we proceed to compute the mass difference between the two heavy sterile neutrinos.

The loop correction can be written in the matrix form as follows:
In order to obtain the mass-eigenvalues, we find the poles of the propagator given in (9) i.e. the roots of

\[ \phi - \mathcal{M} - \Sigma = 0, \]  

where

\[ \mathcal{M} = \left( \begin{array}{ccc} 0 & 0 & M_R - \frac{\sqrt{2}}{2} C^* \\ 0 & 0 & -\frac{\sqrt{2}}{2} C^\dagger \\ M_R & -\frac{\sqrt{2}}{2} C^\dagger & 0 \end{array} \right). \]  

Considering only terms up to second order in gauge and Yukawa couplings, we arrive at equation of the form:

\[ \phi \left( \begin{array}{cc} 1_F & 0 \\ 0 & 1_F \end{array} \right) - \mathcal{M} - \left( \begin{array}{cc} 0 & Q_1 \\ Q_2 & 0 \end{array} \right) = 0, \]  

where \( \left( \begin{array}{cc} 1_F & 0 \\ 0 & 1_F \end{array} \right) \) is the unit flavor matrix and \( Q_1, Q_2 \) are obtained by multiplying equation (33) with appropriate matrices.

We know that if \( |e\rangle \) is an eigenvector of \( M^\dagger \mu M_\nu \), then \( \frac{1}{\sqrt{2}} \left( |e\rangle |e\rangle \right) \) is eigenvector for the matrix \( \left( \begin{array}{cc} 0 & M_\nu \\ M_\nu & 0 \end{array} \right) \). Let \( \lambda_i, i = 1 \cdots 4 \) be the eigenvalues for \( \phi \) at the tree level. Then, applying first order perturbation theory, we obtain

\[ \delta \lambda_i = \frac{1}{2} \left( \langle e_i^* | (e_i) \right) \left( \begin{array}{cc} 0 & Q_1 \\ Q_2 & 0 \end{array} \right) \left( \langle e_i^* | (e_i) \right). \]  

Thus,

\[ \delta \lambda_i = \lambda_i + \delta \lambda_i, \]

where, \( \delta \lambda_i \) represents the change in \( \lambda_i \) due to the perturbation and is a function of \( p^2 \). For the on-shell computation of the mass-difference, we can safely replace \( p^2 \) by \( M^2 \) up to second order in Yukawa couplings.

Denoting the physical mass-difference as \( \delta m_{\text{phys}} \), we arrive at a relation:

\[ \delta m_{\text{phys}} = \delta m_{\text{tree}} + \delta m_{\text{loop}}, \]  

\[ \Rightarrow \delta m_{\text{phy}} = \left| 2 \Delta M + \frac{f_2 f_{23} v^2}{M} \right| + \delta m_{\text{loop}}. \]  

where neglecting terms of order \( C_1^2 \),

\[ \delta m_{\text{loop}} = \left[ \Delta M \cos a (f_2^2 + f_3^2 + |f_{23}|^2) + 2 f_2 f_{23} M \cos (a + n) \right] (\tilde{F}_1 + \tilde{F}_2) \]

\[ + \frac{f_2 f_{23} v^2}{M} \cos (a - n) C_1 - \frac{2 \sqrt{2} f_2 f_{23} v}{M} \cos n \cos a (K_1 + K_2). \]  

This is central result of this work. As expected, the physical mass-difference \( \delta m_{\text{phys}} \) is RG invariant, which can be verified using the RG equations for the parameters involved. The necessary RG equations are given, for example, in [14] 2 2

2 The conventions for the scalar quartic coupling are different in the two papers, which must be kept in mind while performing the verification.

V. NUMERICAL ESTIMATES

In this section, we provide a numerical estimate of the loop contribution to the mass-difference. Evaluating equation (40) at \( \mu \sim M_W \), we obtain that

\[ \delta m_{\text{loop}}(M_W) \approx \frac{2 f_2 f_{23} v^2}{(4\pi)^2 M}. \]  

Comparing with our result with (1), we see that

\[ \delta m_{\text{loop}}(M_W) \sim \Delta m_\nu, \]

where \( \Delta m_\nu \) is the active neutrino mass-difference. Thus, the loop correction may be absorbed into the Higgs con-
densate contribution to the tree-level mass-difference. The numerical estimate of the loop correction is taken into consideration for the fine-tunings made on the parameters of the $\nu$MSM to satisfy the leptogenesis conditions, as is described in an accompanying paper \cite{14}.

VI. CONCLUSIONS

The generation of low temperature lepton-asymmetry has considerable impact on the production of dark matter sterile neutrinos. Oscillations or decays of the singlet neutrinos in the $\nu$MSM can give rise to the requisite lepton asymmetry provided their masses are sufficiently degenerate. A complete analysis of the problem requires computation of the physical mass-difference between the heavier neutrinos. In this paper, we computed the loop corrections to this mass-difference. On performing the computation, we see that the loop correction is of the same order as the active neutrino mass-difference. The loop-correction is then incorporated into the considerations for satisfying the leptogenesis conditions, which is done in an accompanying paper \cite{14}.

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Appendix A: Notations and Conventions

1. Majorana Fermions

\[ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  
\[ \sigma^\mu = (1, \sigma), \overline{\sigma}^\mu = (1, -\sigma). \]

where the Pauli matrices are given by the standard representation

\[ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

The left and right projectors are given by:

\[ P_L = \frac{1}{2}(1 + \gamma_5), \quad P_R = \frac{1}{2}(1 - \gamma_5). \]

2. Higgs sector

We use the following form of the Higgs potential:

\[ V(\phi) = -\mu^2(\phi^\dagger \phi) + \lambda(\phi^\dagger \phi)^2. \]  
\[ (A9) \]

The vacuum expectation value is taken to be $v = 246$ GeV. From the above two relations, it can be shown that the Higgs mass may be written as:

\[ m_H^2 = 2\mu^2 = 2\lambda v^2. \]  
\[ (A10) \]

Appendix B: Mass Eigenstates and Mass Eigenvalues of 2 active and 2 sterile neutrino system

Define the following parametrization:

\[ f_2 = -A; f_3 = -Af; f_{23} = -\epsilon Af e^{in}; \] \[ v' = \frac{v}{\sqrt{2}}. \]

where

\[ \epsilon = \frac{|f_{23}|}{f_3}. \]  
\[ (B2) \]

This leads to the following form of the mass-matrix:

\[ M_\nu = \begin{pmatrix} \Delta M & M & \epsilon' \epsilon' v' & 0 \\ M & \Delta M & A\epsilon e^{-in} v' & A\epsilon v' \\ \epsilon' \epsilon' v' & A\epsilon e^{-in} v' & 0 & 0 \\ 0 & A\epsilon v' & 0 & 0 \end{pmatrix}. \]  
\[ (B3) \]

Furthermore, define the parameters:

\[ \sigma = \left\{ 1 + (1 + \epsilon^2)f \right\} v'^2; \sin a = \frac{f_{23}v'^2 \sin n}{\rho}, \]  
\[ (B4) \]
where
\[ \rho = \epsilon \nu^2 \left[ f^2 (1 + f^2 \kappa^2 + 2 f \kappa \cos n) + \kappa^2 + 2 f \kappa \cos n + 2 f^2 \kappa^2 \right]^\frac{1}{2}, \]

and
\[ \Delta M = \frac{A^2 (1 + f^2) \nu^2 \epsilon \kappa}{M}. \]

Then the eigenstates and eigenvalues may be written as follows:

\[ |e_1\rangle = \frac{1}{\sqrt{2}} \left( e^{\frac{i \pi}{4}} \sqrt{\frac{M}{A}} e^{\frac{i \pi}{4} (2n - a)} + e^{\frac{i \pi}{4} (2n - a)} \right), \]

\[ |e_2\rangle = \frac{1}{\sqrt{2}} \left( -i e^{-\frac{i \pi}{2}} \sqrt{\frac{i A}{M}} e^{-\frac{i \pi}{2} (2n - a)} \right), \]

\[ m_1 = M + \frac{\sigma + 2 \rho}{2M} A^2. \]

\[ |e_3\rangle = \frac{1}{\sqrt{2}} \left( e^{\frac{i \pi}{4}} \sqrt{\frac{A}{M}} e^{\frac{i \pi}{4} (2n - a)} \right), \]

\[ |e_4\rangle = \frac{1}{\sqrt{2}} \left( -i e^{-\frac{i \pi}{2}} \sqrt{\frac{1}{M}} e^{-\frac{i \pi}{2} (2n - a)} \right), \]

\[ m_2 = M + \frac{\sigma - 2 \rho}{2M} A^2. \]

\[ \rho = \epsilon \nu^2 \left[ f^2 (1 + f^2 \kappa^2 + 2 f \kappa \cos n) + \kappa^2 \right]^\frac{1}{2}, \]

\[ |e_3\rangle = \frac{1}{\sqrt{2}} \left( e^{\frac{i \pi}{4}} \sqrt{\frac{A}{M}} e^{\frac{i \pi}{4} (2n - a)} \right), \]

\[ |e_4\rangle = \frac{1}{\sqrt{2}} \left( -i e^{-\frac{i \pi}{2}} \sqrt{\frac{1}{M}} e^{-\frac{i \pi}{2} (2n - a)} \right), \]

\[ m_3 = \left( \sqrt{1 + \epsilon^2 + \epsilon} \right) \frac{f \nu^2}{M} A^2. \]

\[ m_4 = \left( \sqrt{1 - \epsilon^2} - \epsilon \right) \frac{f \nu^2}{M} A. \]

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