A pseudo dispersive, nonlinear equation and its corresponding linear envelope

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Abstract. A general dispersive and nonlinear equation is considered. The equation is given in the form of pseudo differential operators. Some examples of dispersion and nonlinearity types are discussed. Taking a special case of the dispersion and nonlinear terms, the equation leads to a well-known Korteweg - de Vries (KdV) equation. A derivation of the linear wave packet evolution is also discussed. This will put a fundamental step for deriving a Nonlinear Schrödinger (NLS) equation for more general dispersive and nonlinear terms. NLS equations are in the interest of applied mathematics, theoretical physics as well as engineering.

1. Introduction  
Nonlinear Schrödinger (NLS) equation has been in the interest of applied mathematics, theoretical physics as well as engineering. The nonlinearity of the equation is still a challenging research in (partial) differential equation, both from theoretical point of view and applications. NLS equation has been discussed in broader fields, for example to study optical wave packet in nonlinear media, [1,2], to model slowly varying dispersive wave envelope propagating in nonlinear medium [3]. The evolution of surface wave group modeled by an NLS equation might be related to the envelope change due to the dynamics of third order side band [4, 5]. The relation of envelope with its original wave can be traced from Korteweg- de Vries (KdV)equation and NLS equation[1, 6].

NLS type of equations have also been applied in fundamental study on quantum mechanics [7, 8] and to study Bose-Einstein condensate theory [9, 10]. NLS equations have also appeared in photonics, a study of light or photon generation, and manipulation, see [11] or more recently[12]. It was also applied to study plasma, among others [13, 14]. Plasmains one of the four fundamental states of matter (solid, liquid, gas and plasma). Plasma was first described by chemist Irving Langmuir in the 1920s [15]. NLS equation is also appeared in the study of molecular dynamics, a study of the motion of atoms and molecules, including for particle accelerator [16, 17]. Especially, for bio-molecule dynamics can found in [18, 19].

Brezjiet al., and Lopez et al. [20, 21] used NLS equation to study semiconductor electronics. Semiconductor is among materials applied for solar cells,see among others [22, 23]. Understanding this material is one of the objectives to optimize solar energy transfer into electric energy, together by applying signaling transfer discussed in [24].

In a basic form, NLS equation can be viewed as follows. Let $x, t$ be real numbers, and $x$ be a spatial variable, while $t > 0$ be a temporal variable. In the standard form, NLS equation is given by
\[ A_t + iβA_ξξ + iy|A|^2A = 0. \]  

(1)

A is from
\[ u(x,t) = A(x,t)\exp(i(kx - ω(k)t)) + c.c. \]

(2)

for \( u \) satisfies a standard KdV equation
\[ u_t + u_x + \frac{1}{6}u_{xxx} + \frac{3}{2}uux = 0. \]

(3)

The relation among \( x, t, ξ, τ, k \) and \( ω \) will be discussed in the next section. This paper is a generalization on the dispersive and nonlinear terms, which is more general than [25].

2. General dispersive and nonlinear equation

We consider a general dispersive and nonlinear equation in the form
\[ u_t + D(u) + N(u) = 0 \]

(4)

where \( D \) and \( N \) are dispersive and nonlinear operators, respectively. The operators \( D \) and \( N \) should be interpreted as
\[ D(e^{i(kx−ωt)}) = iω(k) e^{i(kx−ωt)} \]

(5)

and
\[ N(e^{i(kx−ωt)}) = ln(k)e^{im(kx−ωt)}. \]

(6)

Without the presence of the nonlinear term \( N(u) \), (1) becomes
\[ u_t + D(u) = 0. \]

(7)

Equation (7) gives the dispersion relation
\[ ω = ω(k). \]

For the case of standard KdV (3), \( D(u) = u_x + \frac{1}{6}u_{xxx} \) and \( N(u) = \frac{3}{2}uu_x \). This yields a dispersion relation \( ω(k) = k - \frac{1}{6}k^3 \), \( n(k) = \frac{3}{2}k \) and \( m(k) = 2 \), which is applied for surface waves over shallow water. For a more general case, in [26, 4] the dispersion was modified to the exact dispersion relation \( ω(k) = k \sqrt{\frac{\tanh k}{k}} \) which was for surface waves over a moderate depth. For a standard KdV equation power series expansion of \( ω(k) \) is considered. The third order approximation of the dispersion relation is
\[ ω(k) = k - \frac{1}{6}k^3. \]

Figure 1 shows the dispersion relation for moderate depth and shallow water waves.

On the other hand, the nonlinearity \( n(k) \) for KdV equation is given by \( n(k) = \frac{3}{2}k \), and in [4] it is a pseudo operator for second order nonlinearity in the form
\[ n(k) = kR(2k) \left( \frac{1}{R(k)R(2k)} + \frac{1}{2R^2(k)} \right), \]

where \( R(k) = \sqrt{\frac{\tanh k}{k}} \). Figure 2 shows these nonlinearities. Observe that for small \( k \), both are in a good agreement. Moreover, for small \( k \) the series expansion of
\[ kR(2k) \left( \frac{1}{R(k)R(2k)} + \frac{1}{2R^2(k)} \right) \approx \frac{3}{2}k. \]
Figure 1. The dispersion relation for moderate depth and shallow water waves.

Figure 2. The nonlinearity for moderate depth and KdV approximation.
3. Wave packet of general dispersive equation

We consider a solution of general dispersive equation in the form

\[ u(x, t) = \int a(k) \exp(i(kx - \omega(k)t)) dk + c.c. \quad (8) \]

where c.c. stands for complex conjugate. Equation (8) is written in the form

\[ u(x, t) = A(x, t) \exp(i(\bar{k}x - \omega(\bar{k})t)) + c.c. \]

\( A(x, t) \) represents the envelope of varying amplitude ‘wave’ in the form \( \exp(i(\bar{k}x - \omega(\bar{k})t)) \).

Observe that

\[ A(x, t) = \int a(\bar{k} + K) \exp\left(i\left((K)x - (\omega(\bar{k} + K) - \omega(\bar{k}))t\right)\right) dK \]

Hence

\[ A_t = -i \int (\omega(\bar{k} + K) - \omega(\bar{k})) a(\bar{k} + K) \exp\left(i\left((K)x - (\omega(\bar{k} + K) - \omega(\bar{k}))t\right)\right) dK. \]

Let \( D_1 \) be a linear operator in the sense

\[ D_1(\exp iKx) = i\left(\omega(\bar{k} + K) - \omega(\bar{k})\right) \exp iKx. \]

This results in

\[ A_t + D_1(A) = 0. \quad (9) \]

Introducing a transformation \( t = \tau \) and \( x = \omega(\bar{k})\tau \), meaning that we follow the wave by the speed \( \omega(\bar{k}) \), equation (9) becomes

\[ A_\tau + D_2(A) = 0, \quad (10) \]

where operator \( D_2 \) is given by

\[ D_2(\exp iKx) = i\left(\omega(\bar{k} + K) - \omega(\bar{k}) - \omega(\bar{k})\right) \exp iKx. \quad (11) \]

For the case \( \omega(k) = k \sqrt{\frac{\tanh k}{k}} \), the first order Taylor expansion of (11) around \( k = 0 \)is given by

\[ \omega(k) = k - \frac{1}{6} k^3. \quad (12) \]

Remark:
The derivation of equations (9) and (10) will be a fundamental step for deriving anNLS equation for more general dispersive and nonlinear terms.

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