SINGULAR VECTORS IN AFFINE SUBSPACES AND \( \Psi \)-DIRICHLET NUMBERS

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Abstract. We prove inheritance of measure zero property of the set of singular vectors for affine subspaces and submanifolds inside them in both \( \mathbb{R} \) and function field over finite fields. Another result of this paper shows that the only \( \psi \)-Dirichlet numbers in a function field over a finite field are rational functions, unlike \( \psi \)-Dirichlet numbers in \( \mathbb{R} \). We also prove that there are uncountably many totally irrational singular vectors with large uniform exponent in quadratic surfaces over a positive characteristic field.

1. Introduction

Diophantine approximation of real vectors by rational vectors starts with the famous Dirichlet’s theorem 1.1. One can also study Diophantine approximation of vectors in any local field \( K \) by vectors in a global field \( D \), where \( K \) is completion of \( D \). Over the years mathematicians tried to understand the same Diophantine approximation problems in different local fields. For instance, the much celebrated Oppenheim conjecture was proved in zero and positive characteristic; see [51, 11, 52]. The well known Sprindžuk’s conjecture was proved in local fields; see [39, 42, 28]. Recent developments in \( \nu \)-adic Diophantine approximation can be found in [42, 53, 6, 21, 5] and references therein. We refer the reader to the survey [24, 48] and [1, 24, 28, 27, 34, 4, 47, 56, 26, 25] for more recent results in Diophantine approximation in positive characteristic.

Suppose that \( \nu, p \) are primes, \( \mathbb{F}_q \) is a finite field of characteristic \( p \). In this paper, we are going to focus on \( D = \mathbb{Q} \) and \( D = \mathbb{F}_q(T) \). Let us recall Dirichlet theorem for \( K = \mathbb{R} \), \( \mathbb{F}_q((T^{-1})) \) and \( \mathbb{Q}_\nu \) (ref. [26, 42]). Here \( \mathbb{F}_q((T^{-1})) \) is the completion of \( \mathbb{F}_q(T) \), see §3.1. In this section, by \( | \cdot |_K \) we mean the standard norm in \( K \). For instance, \( | \cdot |_\mathbb{R} = | \cdot |_{\infty} \), \( | \cdot |_{\mathbb{Q}_\nu} \) is the \( \nu \)-adic norm \( | \cdot |_\nu \) in \( \mathbb{Q}_\nu \) and see §3.1 for \( | \cdot |_{F_q((T^{-1}))} \). We denote \( \| \cdot \|_{K,\infty} \) as the supremum norm in \( K^n \), and when \( K = \mathbb{R} \), we write \( \| \cdot \|_{\mathbb{R},\infty} \) as just \( \| \cdot \|_{\infty} \). In what follows, we denote \( x \cdot y = \sum_{i=1}^n x_i y_i \), where \( x = (x_i), y = (y_i) \in K^n \).

**Theorem 1.1** (Dirichlet theorem in \( \mathbb{R}^n \) and \( \mathbb{F}_q((T^{-1}))^n \)). Let \( K = \mathbb{R}, \mathbb{F}_q((T^{-1})) \). For every \( y \in \mathbb{R}^n \) (resp. \( \mathbb{F}_q((T^{-1}))^n \)), such that for all large \( Q > 0 \) there exist \( q \in \mathbb{Z}^n \) (resp. \( \mathbb{F}_q[T]^n \)) and \( q_0 \in \mathbb{Z} \) (resp. \( \mathbb{F}_q[T] \)) satisfying the following system,

\[
|q \cdot y + q_0|_K < \frac{1}{Q^n} \\
\|q\|_{K,\infty} \leq Q.
\]

Dirichlet’s theorem in \( \mathbb{Q}_\nu^n \) looks slightly different. The difference, that \( q_0 \) is occurring in the second inequality, is natural due to density of \( \mathbb{Z} \) in the unit ball in \( \mathbb{Q}_\nu \).

**Theorem 1.2** (Dirichlet theorem in \( \mathbb{Q}_\nu^n \)). For every \( y \in \mathbb{Q}_\nu^n \) there exists \( c(y) > 0 \) such that for all large \( Q > 0 \) there exist \( q \in \mathbb{Z}^n \) and \( q_0 \in \mathbb{Z} \) satisfying the following system of
inequalities,

\begin{align*}
|q \cdot y + q_0|_{Q^n} &< \frac{c(y)}{Q^{n+1}} \\
\|(q_0, q)\|_{R, \infty} &\leq Q.
\end{align*}

(1.2)

We are going to study singular vectors and \(\psi\)-Dirichlet vectors in this paper. These are the vectors for which the approximation in Dirichlet theorem can be improved.

1.1. Singular vectors. In recent years, the study of singular vectors in the classical setting i.e. in \(\mathbb{R}^n\), and in a real submanifold has drawn a lot of attention (see [14, 15, 19, 40, 45, 3, 32, 31] and references therein). Much less is known about singular vectors compared to its counterparts, very well approximable vectors or badly approximable vectors; see [7] for a survey. We recall the definition of singular vectors in \(\mathbb{R}^n\) and \(\mathbb{F}_q((T^{-1}))^n\). Singular vectors were originally introduced by A. Khintchine in the 1920s (see [33, 12]) for reals.

**Definition 1.1.** Let \(K = \mathbb{R}\) (resp. \(\mathbb{F}_q((T^{-1}))\), and \(\Lambda = \mathbb{Z}\) (resp. \(\mathbb{F}_q[T]\)). A vector \(x = (x_1, \cdots, x_n) \in \mathbb{K}^n\) is said to be singular if for every \(c > 0\), for all sufficiently large \(Q > 0\) there exist \(0 \neq q \in \Lambda^n, q_0 \in \Lambda\) satisfying the following system of inequalities,

\begin{align*}
|q \cdot x + q_0|_x &< \frac{c}{Q^n}, \\
\|q\|_{x, \infty} &\leq Q.
\end{align*}

(1.3)

Singular vectors in \(\mathbb{Q}_v^n\) can be defined accordingly using inequalities in (1.2). In [[12], Ch. V] Khintchine showed that set of real singular vectors has Lebesgue measure zero. In a similar manner, one can show that set of singular vectors in function field \(\mathbb{F}_q((T^{-1}))^n\) and \(\mathbb{Q}_v^n\) has measure zero. It also follows from Theorem 1.2 of [26] (for \(\mathbb{F}_q((T^{-1}))^n\)) and Theorem 2.3 (for \(\mathbb{Q}_v^n\)) in this paper.

In the late 1960s Davenport and Schmidt showed (see [23, 22]) that the set

\[ \{ t \in \mathbb{R} \mid (t, t^2) \text{ is singular} \} \]

has Haar measure zero. This problem is significantly difficult than studying singular vectors in Euclidean space due to the dependency of coordinates. This kind of problems boost a momentum after Kleinbock and Margulis proved the famous Sprindžuk conjecture in 1998 [39]. This conjecture states that for an analytic submanifold which is not contained in any affine subspace of \(\mathbb{R}^n\), the set of very well approximable vectors has measure zero. Kleinbock and Margulis proved Dani correspondence to translate the problem into a dynamical problem and they proved quantitative nondivergence to address the dynamical problem. Then onwards, using the similar philosophy there has been a major improvement in the study of diophantine approximation in manifolds; see [9, 8, 45, 42].

In the spirit of [39], it was shown in [45] (resp. [26]) that the set of singular vectors in a submanifold of \(\mathbb{R}^n\) (resp. \(\mathbb{F}_q((T^{-1}))^n\)), which is not contained inside any affine subspace has Haar measure zero. This shows that manifolds that are not contained inside any affine subspace obey the same rule for singular vectors as their ambient space \(\mathbb{R}^n\) (resp. \(\mathbb{F}_q((T^{-1}))^n\)). So naturally one would wonder what happens if a submanifold \(M\) is contained inside an affine subspace \(\mathcal{L}\) in \(\mathbb{K}^n\), where \(\mathbb{K}\) is a local field. This poses the following problem which seems very difficult to us with available technology.
Question 1.1. Find all submanifolds in \( \mathcal{K}^n \) such that Haar a.e. vector in the submanifold is not singular.

Kleinbock showed in [36] that the so-called diophantine exponents are the same for affine subspace \( L \subset \mathbb{R}^n \) and \( M \subset L \) if there is no proper subspace \( L \) that contains \( M \). Inspired by the works of Kleinbock in [36, 35] we ask the following question.

Question 1.2. Suppose that \( M \) is a submanifold in an affine subspace \( L \subset \mathcal{K}^n \). Let \( L \) be the smallest affine subspace in \( \mathcal{K}^n \) that contains \( M \). If the set of singular vectors in \( L \) has measure zero, will the set of singular vectors in \( M \) has measure zero? Moreover, is it an if and only if situation?

If the answer to the second part of Question 1.2 is affirmative then it confirms that the Question 1.1 can be solved if we can find all affine subspaces in \( \mathcal{K}^n \) such that the Haar measure of the set of singular vectors is zero. Finding all such affine subspaces is still a challenging question.

Another interesting question is as follows.

Question 1.3. Suppose \( L \) and \( M \) are as in Question 1.2. Is there a dichotomy regarding singular vectors, i.e. if \( M \) (resp. \( L \)) has one vector which is not singular, then Haar almost every vector in \( M \) (resp. \( L \)) is not singular?

We note that answering Question 1.3 does not answer Question 1.2.

1.1.1. State of the art in different fields \( \mathcal{K} \).

- Question 1.2 has not been studied for any \( \mathcal{K} \)-submanifold and proper affine subspace.
- Question 1.3 was addressed by Kleinbock for connected real analytic submanifolds; see [37].
- At present, the answers to Questions 1.2, 1.3 for submanifolds in \( \mathbb{Q}_p^n \) are not known. See Remark 2 for details.

In this paper, we answer Questions 1.2 and 1.3 affirmatively. Indeed we prove a stronger result, i.e. Theorems 2.1, 2.2 for submanifolds of \( \mathcal{K}^n \), when \( \mathcal{K} = \mathbb{R} \) or \( \mathcal{K} = \mathbb{F}_q((T^{-1})) \). As a special case of our Theorems 2.1 and 2.2, the following theorem follows.

Theorem 1.3. Suppose \( \mathcal{K} = \mathbb{R} \) or \( \mathbb{F}_q((T^{-1})) \). Let \( f : U \subset \mathcal{K}^d \to L \subset \mathcal{K}^n \) be an analytic map such that \( f(B) \) is not contained in any proper affine subspace of \( L \) for any \( B \) ball in \( U \). Let \( \lambda_U \) and \( \lambda_L \) are Haar measures on \( U \) and \( L \) respectively. Then the following are equivalent:

- There exists \( x \in U \) such that \( f(x) \) is not singular.
- There exists \( y \in L \) that is not singular.
- For \( \lambda_U \)-almost every \( x \), \( f(x) \) is not singular.
- For \( \lambda_L \)-almost every \( y \), \( y \) is singular.

The main tools to establishing Theorems 2.1 and 2.2 are quantitative nondivergence in homogeneous dynamics [39] and Dani correspondence [16]. We consider the space of unimodular lattices in \( \mathcal{K}^{n+1} \) i.e. \( \text{SL}_{n+1}(\mathcal{K})/\text{SL}_{n+1}(\Lambda) \), where \( \mathcal{K} = \mathbb{R} \) (resp. \( \mathbb{F}_q((T^{-1})) \)) and \( \Lambda = \mathbb{Z} \) (resp. \( \mathbb{F}_q[T] \)). Let us take the point \( u_x \Lambda^{n+1} \), where \( u_x \) is an unipotent element as defined in Section 3.8.1. Let \( g_k \) be a diagonal flow as defined in Section 3.8.1. The Dani correspondence links the singularity property of \( x \) with the divergence of \( u_x \Lambda^{n+1} \) under the diagonal flow \( g_k \), as \( k \to \infty \). Following ideas from [39, 36, 35], we used quantitative nondivergence to get measure estimates of the divergent orbits.
When $\mathcal{K} = \mathbb{Q}_\nu$, we have a weaker result. As a special case of our Theorem 2.3, the following theorem follows.

**Theorem 1.4.** Let $f : U \subset \mathbb{Q}_\nu^d \to \mathbb{Q}_\nu^n$ be an analytic map such that $f(B)$ is not contained inside any proper affine subspace of $\mathbb{Q}_\nu^n$, for any ball $B$ in $U$. Then for $\mu$ almost every $x \in U$, $f(x)$ is not singular.

In order to prove the above theorem we used quantitative nondivergence in the homogeneous space $G_1/\Gamma_1$, where $G_1 = (\mathbb{Q}_\nu \times \mathbb{R})^1 \rtimes \text{SL}_{n+1}(\mathbb{Q}_\nu \times \mathbb{R})$, $\Gamma_1 = \mathbb{Z}_{[\nu]}^1 \rtimes \text{SL}_{n+1}(\mathbb{Z}_{[\nu]})$ and $(\mathbb{Q}_\nu \times \mathbb{R})^1 = \{(x_\nu, x_{\infty}) : |x_\nu|_\nu |x_{\infty}| = 1\}$. In this setup Dani correspondence is not well studied. We proved one side of the correspondence, whereas for the proof of Theorem 1.3 both side of Dani correspondence played a crucial role.

### 1.2. $\psi$-Dirichlet numbers in $\mathcal{K}^n$.

Another way to improve (1.1) (resp. (1.2)) is by replacing the right side $\frac{1}{Q}$ (resp. $\frac{c(\psi)}{Q^{\psi}}$) by $\psi(Q)$, where $\psi$ is a positive function. Following [44], we define $\psi$-Dirichlet vectors in $\mathcal{K}^n$. We denote the set of $\psi$-Dirichlet vectors as $\mathcal{D}(\psi)$. Let $\mathcal{K}$ be $\mathbb{R}$ (resp. $\mathbb{F}_q((T^{-1}))$) and $\Lambda = \mathbb{Z}$ (resp. $\mathbb{F}_q[T]$).

**Definition 1.2.** Let $\psi : [t_0, +\infty) \to \mathbb{R}_+$ be a function. A vector $\mathbf{x} = (x_1, \cdots, x_n) \in \mathcal{K}^n$ is said to be $\psi$-Dirichlet if for all sufficiently large $Q > 0$ there exists $\mathbf{0} \neq \mathbf{q} \in \Lambda^n$, $q_0 \in \Lambda$ satisfying the following system

\begin{align}
&|\mathbf{q} \cdot \mathbf{x} + q_0|_{\mathcal{K}} < \psi(Q), \\
&\|\mathbf{q}\|_{\mathcal{K}, \infty} \leq Q.
\end{align}

(1.4)

When $\mathcal{K} = \mathbb{Q}_\nu$, we can similarly define $\psi$-Dirichlet vectors, with the only difference being the left side of the second inequality in (1.4), $\|\mathbf{q}\|_{\mathcal{K}, \infty}$ is replaced by $\|(q_0, \mathbf{q})\|_{\mathbb{R}, \infty}$.

There has been an extensive study about $\psi$-approximable vectors in submanifolds of $\mathcal{K}^n$ [9, 8], but very little is known about the $\psi$-Dirichlet vectors (see [44, 43, 41]), even in the classical setting. In [43], $\mathcal{D}(\psi)$ was studied over $\mathbb{R}$ and the following theorem was proved.

**Theorem 1.5.** Suppose $\psi : [t_0, \infty) \to \mathbb{R}_+$ be non-increasing, and suppose the function $t \to t\psi(t)$ is non-decreasing and suppose

$$t\psi(t) < 1 \text{ for all } t \geq t_0.$$

Then if

$$\sum_{n} \frac{-\log(1 - n\psi(n))(1 - n\psi(n))}{n} = \infty \text{ (resp. } < \infty)$$

then the Lebesgue measure of $\mathcal{D}(\psi)$ (resp. $\mathcal{D}(\psi)^c$) in $\mathbb{R}$ is zero.

This immediately motivates the following question.

**Question 1.4.** What is the analogue of Theorem 1.5 for fields $\mathcal{K} = \mathbb{F}_q((T^{-1}))$ and $\mathcal{K} = \mathbb{Q}_\nu$?

#### 1.2.1. State of the art in different fields $\mathcal{K}$.

- The above Theorem 1.5 addresses Question 1.4 for $\mathcal{K} = \mathbb{R}$.

- At present, for $\mathcal{K} = \mathbb{Q}_\nu$, the answer to Question 1.4 is not known.

In Theorem 2.4 we addressed Question 1.4 for $\mathcal{K} = \mathbb{F}_q((T^{-1}))$. Surprisingly, it turns out that $\mathcal{D}(\psi)$ only contains rational functions, for any positive nonincreasing function $\psi$ such that $t\psi(t) < 1$.

**Remark 1.** In higher dimension and moreover for the space of matrices [41] gives sufficient conditions on $\psi$ such that the measure of the set $\mathcal{D}(\psi)$ is zero or full.
1.3. Plenty of totally irrational singular vectors. Suppose that \((x_1, \cdots, x_n) \in \mathbb{K}^n\) belongs to a rational affine hyperplane, then it must be singular. These are the most trivial singular vectors. In fact, the converse is also true when \(n = 1\) and \(\mathbb{K} = \mathbb{R}, \mathbb{F}_q((T^{-1}))\) (ref. [13], [27]).

**Definition 1.3.** We call a vector totally irrational vector if it is not inside a rational affine hyperplane of \(\mathbb{K}^n\).

For \(n > 1\), in [33] Khintchine showed the existence of infinitely many totally irrational singular vectors in \(\mathbb{R}^n\). Moreover, the authors in [40] showed that for real analytic submanifolds (of dimension greater than 2) which are not contained inside a rational affine subspace, there are uncountably many totally irrational singular vectors. This implies that even for many affine subspaces there are plenty of totally irrational singular vectors. It is natural to study the above situation when \(\mathbb{K} = \mathbb{F}_q((T^{-1}))\) and \(\mathbb{K} = \mathbb{Q}_\nu\).

**Question 1.5.** In a submanifold \(M\) of \(\mathbb{K}^n\), when are there uncountably many singular vectors that do not belong to a rational affine hyperplane?

1.3.1. **State of the art in different fields \(\mathbb{K}\).**

- When \(\mathbb{K} = \mathbb{R}\), the above question is studied for analytic submanifolds in [40, 45].
- The Question 1.5 has not been studied when \(\mathbb{K} = \mathbb{Q}_\nu\).

In this paper, we answer Question 1.5 for a proper subclass of analytic submanifolds in \(\mathbb{K}^n\), where \(\mathbb{K} = \mathbb{F}_q((T^{-1}))\); see Theorem 2.5. As a special case of our Theorem 2.5, we have the following theorem.

**Theorem 1.6.** Suppose \(\text{char}(\mathbb{F}_q((T^{-1}))) = p < \infty\). Let \(U\) be an open subset of \(\mathbb{F}_q((T^{-1}))\). We consider \(S\) of the following two types:

- \(S = \{(x, y, p_3(x, y), \cdots, p_n(x, y)) \mid x, y \in U\} \subset \mathbb{F}^n\), where and each \(p_i(x, y)\) is a degree 2 polynomial.
- \(S = \{(x, y, p(x, y)) \mid x, y \in U\} \subset \mathbb{F}^3\), where \(p(x, y) = \sum_{i=0}^m a_i x^i p_i + \sum_{j=0}^n b_j y^j p_j\).

Suppose that \(S\) is not contained inside any affine rational hyperplane, then there exist uncountably many totally irrational singular vectors in \(S\).

The main challenge comes from the lack of understanding about intersections of a surface and an affine subspace in the function field setting. Another difficulty comes due to total disconnectedness in this setting. For real submanifolds, intersection of a connected analytic surface and an affine subspace is well understood due to [10], §2. Both of these facts were used in [40] in a crucial manner. That is why we had to tackle case by case and we prove the theorem for a class of submanifolds which is smaller than the class of submanifolds that was taken in [40].

2. **Main Results**

We have three main theorems.

2.1. **Paucity of singular vectors.** The first two theorems deal with singular vectors in a submanifold of \(\mathbb{K}^n\), where \(\mathbb{K} = \mathbb{R}\) or \(\mathbb{F}_q((T^{-1}))\) and they are in the same flow as Theorem 0.3 in [36] and Theorem 1.3 in [35]. For definitions of Federer measure, Besicovitch space, good, nonplanar and nondegenerate the readers are referred to §3.
Theorem 2.1 (For \( \mathbb{R} \)). Let \( \mu \) be a Federer measure on a Besicovitch metric space \( X, \mathcal{L} \) an affine subspace of \( \mathbb{R}^n \), and let \( f : X \to \mathcal{L} \) be a continuous map such that \((f, \mu)\) is good and nonplanar in \( \mathcal{L} \). Let \( f_* \mu \) be the pushforward measure and \( \lambda_\mathcal{L} \) be the Haar measure on \( \mathcal{L} \). Then the following are equivalent:

1. There exists one \( x \in \text{supp}(\mu) \) such that \( f(x) \) is not singular.
2. There exists one \( y \in \mathcal{L} \) which is not singular.
3. For \( \lambda_\mathcal{L} \) almost every \( y \in \mathcal{L} \), \( y \) is not singular.
4. For \( \mu \) almost every \( x \in X \), \( f(x) \) is not singular.

Theorem 2.2 (For \( \mathbb{F}_q((T^{-1})) \)). Let \( \mu \) be a locally finite Borel measure on a Besicovitch ultrametric metric space \( X, \mathcal{L} \) an affine subspace of \( \mathbb{F}_q((T^{-1}))^n \), and let \( f : X \to \mathcal{L} \) be a continuous map such that \((f, \mu)\) is good and nonplanar in \( \mathcal{L} \). Let \( f_* \mu \) be the pushforward measure and \( \lambda_\mathcal{L} \) be the Haar measure on \( \mathcal{L} \). Then the following are equivalent:

1. There exists one \( x \in \text{supp}(\mu) \) such that \( f(x) \) is not singular.
2. There exists one \( y \in \mathcal{L} \) which is not singular.
3. For \( \lambda_\mathcal{L} \) almost every \( y \in \mathcal{L} \), \( y \) is not singular.
4. For \( \mu \) almost every \( x \in X \), \( f(x) \) is not singular.

The following theorem deals with singular vectors in \( \mathbb{Q}_\nu^n \) and it is much weaker than the previous two theorems.

Theorem 2.3 (For \( \mathbb{Q}_\nu \)). Let \( \mu \) be a Federer measure on a Besicovitch metric space \( X, \mathcal{L} \), and let \( f : X \to \mathcal{L} \) be a continuous map such that \((f, \mu)\) is good and nonplanar in \( \mathcal{L} \). Let \( f_* \mu \) be the pushforward measure on \( \mathcal{L} \). Then for \( \mu \) almost every \( x \in X \), \( f(x) \) is not singular.

Remark 2.

1. The proof of Theorems 2.1, 2.2 adopts the techniques from [36]. Previously it was only known that the set of singular vectors has measure zero in nondegenerate submanifolds in \( \mathbb{K}^n \), where \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{F}_q((T^{-1})) \) by [45, 26].
2. For \( \mathbb{Q}_\nu \), Theorem 2.1 is out of reach due to the fact that both sides of Dani correspondence are not known. In an upcoming preprint the first named author explores this direction.
3. In Theorem 2.2, we take the measure \( \mu \) to be locally finite, which is more general than Federer measure, introduced in [38] and considered in [36].
4. In a very recent paper [18] the authors proved that the set Dirichlet improvable vectors has measure zero for nonplanar manifolds whose coordinates are defined by polynomials, in the \( S \)-adic setting, where \( S \) contains all archimedean places. This, in particular, implies that the set of singular vectors in this setting will have measure zero.

2.2. \( \psi \)-Dirichlet numbers in \( \mathbb{K} \). The second theorem in this paper shows complementary results to the Theorem 2.5 in dimension 1, but for more general \( \psi \)-Dirichlet approximability in the classical setting. The following theorem was studied in real numbers, [46]. But when \( \mathbb{K} = \mathbb{F}_q((T^{-1})) \) as we showed below, \( \psi \)-Dirichlet numbers are precisely the rational functions:

Theorem 2.4. Let \( \psi : [t_0, +\infty) \to \mathbb{R}_+ \) be non-increasing. If \( \psi(t) < \frac{1}{t} \) for sufficiently large \( t \), then \( D(\psi) = \mathbb{F}_q(T) \).

Remark 3.
The above theorem shows that analogue of Theorem 1.5 in [46] over function field becomes drastically different than the real case.

The main tool in proving the above theorem is the use of continued fraction expansion.

Theorem 2.4 in [27] shows that only Dirichlet improvable numbers in function field are rational functions. Our Theorem 2.4 generalizes the above mentioned result and shows that even ψ-Dirichlet numbers are also only rational numbers. We note that the technique of [27] is different than ours.

When \( K = \mathbb{Q}_p \), there is no suitable continued fraction estimate that allows us to study the above theorem with the same techniques.

### 2.3. Abundance of singular vectors.

Our next theorem verifies that for a certain analytic submanifolds in \( \mathbb{F}^n \), there are plenty of totally irrational singular vectors, in fact their exponent \( \hat{\omega}(\cdot) \) is infinity. We define \( \hat{\omega}(\cdot) \) in §6.

**Theorem 2.5.** Let \( \text{char}(\mathbb{F}_q((T^{-1}))) = p < \infty \). Let \( U \) be an open subset of \( \mathbb{F}_q((T^{-1}))^d \). We consider \( S \) of the following two types:

- \( S = \{(x,y,p_3(x,y),\ldots,p_n(x,y)) \mid x,y \in U \} \subset F^n \), where and each \( p_i(x,y) \) is a degree 2 polynomial.
- \( S = \{(x,y,p(x,y)) \mid x,y \in U \} \subset F^3 \) where \( p(x,y) = \sum_{i=0}^{m} a_i x^i y^i + \sum_{j=0}^{n} b_j x^j y^j. \)

Suppose that \( S \) is not contained inside any rational affine hyperplane, then there exist uncountably many totally irrational \( y \) in \( S \) such that \( \hat{\omega}(y) = \infty \).

**Remark 4.**

1. By Lemma 6.7 we know that in order to solve the above theorem for nonplanar submanifolds, it is enough to prove the theorem for surfaces.

2. We believe that Theorem 2.5 should be true for any analytic submanifold of \( \mathbb{F}_q((T^{-1}))^n \) which is not contained inside an affine rational subspace. For real analytic submanifolds of dimension atleast 2, not contained inside an affine rational subspace, we know by [40] that there are uncountably many totally irrational singular vectors. The proof crucially relies on understanding how ‘semianalytic’ sets can spilt into connected analytic sets. This becomes difficult in \( \mathbb{F}_q((T^{-1}))^n \), as the notion of semianalyticity is not well defined due to the lack of order and the space \( \mathbb{F}_q((T^{-1})) \) is totally disconnected. Therefore, we examine case by case.

### 2.4. Corollaries of Theorems 2.1 and 2.2.

In Theorem 2.1 and 2.2, if we take \( X = U \subset \mathbb{R}^d \) (resp. \( \mathbb{F}_q((T^{-1}))^d \)) and \( f \) is a smooth nondegenerate map in \( L \), \( M = f(U) \) then the following corollary follows.

**Corollary 2.1.** Suppose that \( K = \mathbb{R} \) or \( \mathbb{F}_q((T^{-1})) \). Let \( L \) be an affine subspace of \( K^n \), and let \( M \) be a submanifold of \( L \) which is nondegenerate in \( L \). Let \( \lambda_L \) and \( \lambda_M \) be Haar measure in \( L \) and \( M \) respectively. Then the following are equivalent:

- There exists \( y \in M \) such that \( y \) is not singular.
- There exists \( y \in L \) such that \( y \) is not singular.
- For \( \lambda_L \) almost every \( y \in L \), \( y \) is not singular.
- For \( \lambda_M \) almost every \( y \in M \), \( y \) is not singular.

In [38] a class of measures called as ‘friendly measures’ was introduced, which contains Lebesgue measure, fractal measures, smooth measures on nondegenerate manifolds and many
measures naturally arising from geometric constructions. In particular, if \( \mu \) is friendly then \((\text{Id}, \mu)\) is good and nonplanar in \( \mathbb{K}^n \). Many natural measures coming from geometric constructions can be shown to have an even stronger property. These measures were referred to as ‘absolutely decaying and Federer’ in \([38]\) and as ‘absolutely friendly’ in \([54]\). If \( \mu \) is absolutely decaying, Federer and \( f \) is nondegenerate at \( \mu \)-a.e. point of \( \mathbb{R}^d \), then \((f, \mu)\) is good and nonplanar; see \([38]\) §7. Hence the following corollaries follow from Theorem 2.2 in the real case.

**Corollary 2.2.** Let \( \mathcal{L} \) be a \( d \)-dimensional affine subspace of \( \mathbb{R}^n \), and \( \mu \) be a (absolutely) friendly measure on \( \mathbb{R}^d \). Let \( f : \mathbb{R}^d \to \mathcal{L} \) be an affine isomorphism. Then the following are equivalent:

- There exists one \( y \in \text{supp}(\mu) \) such that \( f(y) \) is not singular.
- There exists one \( y \in \mathcal{L} \) which is not singular.
- For \( \lambda_\mathcal{L} \) almost every \( y \in \mathcal{L} \), \( y \) is not singular.
- For \( \mu \) almost every \( y \in \mathcal{L} \), \( f(y) \) is not singular.

**Corollary 2.3.** Let \( \mu \) be an absolutely decaying and Federer measure on \( \mathbb{R}^d \), \( \mathcal{L} \) is an affine subspace of \( \mathbb{R}^n \), and let \( f : \mathbb{R}^d \to \mathcal{L} \) be a smooth map which is nondegenerate in \( \mathcal{L} \) at \( \mu \)-a.e. point of \( \mathbb{R}^d \). Then the following are equivalent:

- There exists one \( y \in \text{supp}(\mu) \) such that \( f(y) \) is not singular.
- There exists one \( y \in \mathcal{L} \) which is not singular.
- For \( \lambda_\mathcal{L} \) almost every \( y \in \mathcal{L} \), \( y \) is not singular.
- For \( \mu \) almost every \( y \in \mathcal{L} \), \( f(y) \) is not singular.

### 3. Several preliminaries

Let \( \mathcal{K} \) be a local field. We will require the definitions introduced in this section for \( \mathcal{K} = \mathbb{R}, \mathbb{Q}_p \) and \( \mathbb{F}_q((T^{-1})) \).

#### 3.1. Norms and topology

In this section and in the following sections, we will use \( | \cdot | \) (resp. \( \| \cdot \| \)) to denote norms in \( \mathbb{R} \) and \( \mathbb{F}_q((T^{-1})) \) (resp. \( \mathbb{R}^n \) and \( \mathbb{F}_q((T^{-1}))^n \)), unless otherwise mentioned. In \( \mathbb{Q}_p \), we denote the \( \nu \)-adic norm by \( | \cdot |_\nu \) and \( \| \cdot \|_\nu \) as the sup norm in \( \mathbb{Q}_p^\nu \).

Let \( p \) be a prime and \( q := p^r \), where \( r \in \mathbb{N} \) and consider the finite field \( \mathbb{F}_q \). We consider the integral domain \( \mathbb{F}_q[T] \), the set of polynomials with coefficients in \( \mathbb{F}_q \). Then we consider the function field \( \mathbb{F}_q(T) \). We define a norm \( | \cdot | \) on \( \mathbb{F}_q(T) \) as follow:

\[
|0| := 0; \quad \left| \frac{P}{Q} \right| := e^{\deg P - \deg Q}
\]

for all nonzero \( P, Q \in \mathbb{F}_q[T] \). Clearly \( | \cdot | \) is a nontrivial, non-archimedean and discrete absolute value in \( \mathbb{F}_q(T) \). The completion field of \( \mathbb{F}_q(T) \) with respect to this absolute value is \( \mathbb{F}_q((T^{-1})) \), i.e. the field of Laurent series over \( \mathbb{F}_q \). We will denote the absolute value of \( \mathbb{F}_q((T^{-1})) \) by the same notation \( | \cdot | \), is given as follows. Let \( a \in \mathbb{F}_q((T^{-1})) \),

\[
|a| := \begin{cases} 0 & \text{if } a = 0, \\ e^{k_0} & \text{if } a = \sum_{k \leq k_0} a_k T^k, k_0 \in \mathbb{Z}, a_k \in \mathbb{F}_q \text{ and } a_{k_0} \neq 0. \end{cases}
\]

This clearly extends the absolute value \( | \cdot | \) of \( \mathbb{F}_q(T) \to \mathbb{F}_q((T^{-1})) \) and moreover, the extension remains non-archimedean and discrete. In the above, we call \( k_0 \) as the degree of \( a \), \( \deg a \).
It is obvious that $F_q[T]$ is discrete in $F_q((T^{-1}))$. For any $n \in \mathbb{N}$, throughout $F_q((T^{-1}))^n$ is assumed to be equipped with the supremum norm which is defined as $\|x\| := \max_{1 \leq i \leq n} |x_i|$ for all $x = (x_1, x_2, ..., x_n) \in F_q((T^{-1}))^n$, and with the topology induced by this norm. Clearly $F_q[T]^n$ is discrete in $F_q((T^{-1}))^n$. Since the topology on $F_q((T^{-1}))^n$ considered here is the usual product topology on $F_q((T^{-1}))^n$, it follows that $F_q((T^{-1}))^n$ is locally compact as $F_q((T^{-1}))$ is locally compact. Note this construction $F_q[T] \subset F_q(T) \subset F_q((T^{-1}))$ is similar to $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. Let $\lambda$ be the Haar measure on $F_q((T^{-1}))^n$ which takes the value 1 on the closed unit ball $\|x\| = 1$.

3.2. **Besicovitch space.** A metric space $X$ is called Besicovitch [42] if there exists a constant $N_X$ such that the following holds: for any bounded subset $A$ of $X$ and for any family $B$ of nonempty open balls in $X$ such that every $x \in A$ is a center of some ball in $B$, there is a finite or countable subfamily $\{B_i\}$ of $B$ with

$$1_A \leq \sum_i 1_{B_i} \leq N_X.$$  

By [2], we know $\mathbb{R}, \mathbb{Q}_\nu$, and $F_q((T^{-1}))$ are Besicovitch spaces.

3.3. **Federer measure.** Let $X$ be a metric space. We define $D$-Federer measures following [38]. Let $\mu$ be a Radon measure on $X$, and $U$ an open subset of $X$ with $\mu(U) > 0$. We say that $\mu$ is $D$-Federer on $U$ if

$$\sup_{x \in \text{supp} \mu, r > 0} \frac{\mu(B(x, 3r))}{\mu(B(x, r))} < D.$$ 

We say that $\mu$ as above is **Federer** if, for $\mu$-a.e. $x \in X$, there exists a neighbourhood $U$ of $x$ and $D > 0$ such that $\mu$ is $D$-Federer on $U$. We refer the reader to [38] and [42] for examples of Federer measures.

3.4. **Nondegeneracy in an affine plane.** Let $U$ be an open subset of $\mathbb{K}^d$, and $\mathcal{L}$ be an affine subspace of $\mathbb{K}^n$. Following [36] we call a differentiable map $f : U \to \mathcal{L} \subset \mathbb{K}^n$ to be nondegenerate in $\mathcal{L}$ of $\mathbb{K}^n$ at $x \in U$ if the span of all the partial derivatives of $f$ at $x$ up to some order coincides with the linear part of $\mathcal{L}$. If $M$ is a $d$-dimensional submanifold of $\mathcal{L}$, we will say that $M$ is nondegenerate in $\mathcal{L}$ at $y \in M$ if any (equivalently, some) diffeomorphism $f$ between an open subset $U$ of $\mathbb{K}^d$ and a neighborhood of $y$ in $M$ is nondegenerate in $\mathcal{L}$ at $f^{-1}(y)$. We will say that $f : U \to \mathcal{L}$ (resp., $M \subset \mathcal{L}$) is nondegenerate in $\mathcal{L}$ if it is nondegenerate in $\mathcal{L}$ at $\lambda$-a.e. point of $U$, where $\lambda$ is the Haar measure on $U$ (resp., of $M$, in the sense of the smooth measure class on $M$).

3.5. **Nonplanar in an affine plane.** Let $X$ be a metric space, $\mu$ is a measure on $X$ and let $f : X \to \mathcal{L}$, where $\mathcal{L}$ is an affine subspace in $\mathbb{K}^n$. We will say $(f, \mu)$ is nonplanar in $\mathcal{L}$ if for any ball $B$ with $\mu(B) > 0$, $\mathcal{L}$ is the intersection of all affine subspaces that contain $f(B \cap \text{supp} \mu)$. If an analytic map $f : U \to \mathcal{L}$ is nondegenerate in $\mathcal{L}$ then $(f, \lambda)$ is nonplanar in $\mathcal{L}$. 
3.6. **Good maps.** Let $X$ be a normed space and $\mu$ be a measure in $X$. For $A \subset X$ with $\mu(A) > 0$ and $f$ a $K$-valued function on $X$, denote

$$\|f\|_{\mu,A} := \sup_{x \in A \cap \text{supp}\mu} |f(x)|.$$ 

A function $f : X \to K$ is called $(C, \alpha)$-good on $U \subset X$ with respect to $\mu$ if for any open ball $B \subset U$ centered in $\text{supp}(\mu)$ and $\varepsilon > 0$ one has

$$\mu(\{x \in B \mid |f(x)| < \varepsilon\}) \leq C \left( \frac{\varepsilon}{\|f\|_{\mu,B}} \right)^\alpha \mu(B).$$

Next, if $f = (f_1, \cdots, f_n) : X \to K^n$ be a map, and $\mu$ be a measure on $X$, we call $(f, \mu)$ is $(C, \alpha)$-good at $x \in X$ if there exists a neighborhood $U$ of $x$ such that any linear combination of $f_1, \cdots, f_n$ is $(C, \alpha)$-good on $U$ with respect to $\mu$. We will say $(f, \mu)$ is good if it is good at $\mu$-a.e point. When $\mu$ is the Haar measure, we omit ‘with respect to $\mu$’ and just say that the function $f$ is good at $x$. Polynomials are $(C, \alpha)$ good functions at every point, see Lemma 4.1 in [17]. In fact, there are many examples of good functions. We recall Lemma 2.5 of [39] and Proposition 4.2 from [42] which shows that non-degenerate functions are good.

**Proposition 3.1.** Let $K = \mathbb{R}$ or $\mathbb{Q}_\nu$ or $\mathbb{F}_q((T^{-1}))$. Let $f = (f_1, \cdots, f_n)$ be a $C^l$ map from an open subset $U \subset K^d$ to $K^n$ which is $l$-nondegenerate in $K^n$ at $x_0 \in U$. Then there is a neighborhood $V \subset U$ of $x_0$ such that any linear combination of $1, f_1, \cdots, f_n$ is $(C', \alpha)$-good on $V$, where $C', \alpha > 0$ only depends on $d, l$ and the field. In particular, the nondegeneracy of $f$ in $K^n$ at $x_0$ implies that $f$ is good at $x_0$.

Moreover, the following corollary is true. The proof is exactly the same as the proof of Corollary 3.2 in [35].

**Corollary 3.1.** Let $K = \mathbb{R}$ or $\mathbb{Q}_\nu$ or $\mathbb{F}_q((T^{-1}))$. Let $L$ be an affine subspace of $K^n$ and let $f = (f_1, \cdots, f_n)$ be a smooth map from an open subset $U$ of $K^d$ to $L$ which is nondegenerate in $L$ at $x_0 \in U$. Then $f$ is good at $x_0$.

3.7. **Quantitative Nondivergence.** Let $K = \mathbb{R}, \mathbb{F}_q((T^{-1}))$ and $\Lambda = \mathbb{Z}$ and $\mathbb{F}_q[T]$ respectively. If $\Delta$ is a $\Lambda$-submodule of $K^{n+1}$, we denote by $\mathcal{K}\Delta$ its $K$ linear span inside $K^{n+1}$, and define the rank of $\Delta$ by

$$\text{rank}(\Delta) := \dim_K(\mathcal{K}\Delta).$$

Similar to classification of $\mathbb{Z}$ modules in $\mathbb{R}^n$, by Lemma 4.1 of [28] we have a classification of $\mathbb{F}_q[T]$ modules in $\mathbb{F}_q((T^{-1}))^n$. Any discrete $\Lambda$-submodule of $K^n$ will be of the form $\Lambda x_1 + \cdots + \Lambda x_r$, where $x_1, \cdots, x_r$ are linearly independent over $K$. Following [42, 28] we say that $\Delta \subset \Lambda^n$, a $\Lambda$-submodule is primitive if

$$\Delta = \mathcal{K}\Delta \cap \Lambda^n.$$ 

Following notation from [42] §6.3, let

$$\mathcal{P}(\Lambda, n) := \text{the set of all nonzero primitive submodules of } \Lambda^n.$$ 

Let $e_0, e_1, \cdots, e_n \in K^n$ be the standard basis of $K^n$ over $K$. Then we can define the standard basis of $\bigwedge^j K^n$ over $K$ to be $\{e_{i_1} \wedge \cdots \wedge e_{i_j} \mid i \subset \{0, \cdots, n\} \text{ and } i_1 < i_2 < \cdots < i_j\}$. We can extend the norm in $K^n$ to a norm in the exterior algebra $\bigwedge^j K^n$. Namely, for an element
\( \mathbf{a} = \sum a_i \mathbf{e}_i \in \bigwedge^j \mathcal{K}^n \), we define \( \| \mathbf{a} \| := \max_I |a_I| \). For a \( \Lambda \)-submodule \( \Delta = \Lambda \mathbf{w}_1 + \cdots + \Lambda \mathbf{w}_r \) of \( \Lambda^n \), we can define

\[
\text{cov}(\Delta) := \| \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_r \|.
\]

We use the same notation \( \| \cdot \| \) for supremum norm in \( \mathcal{K}^n \), but it should be clear from the context.

Note that there is no lattice in \( \mathbb{Q}_\nu \). In order to deal with Diophantine problems in \( \mathbb{Q}_\nu^n \), we will consider the space \( \mathbb{Q}_\nu \times \mathbb{R}^n \), where \( \mathbb{Z}[\frac{1}{\nu}]^n \) is a lattice. This point of view was taken in [42]. Let us denote \( \mathbb{Q}_S := \mathbb{Q}_\nu \times \mathbb{R} \), and \( \mathbb{Z}_S := \mathbb{Z}[\frac{1}{\nu}]^n \). Proposition 7.2 in [42] gives that any discrete \( \mathbb{Z}_S \)-submodule of \( \mathbb{Q}_S \) is finitely generated and free. Given a vector \( \mathbf{x} = (x_\nu, x_\infty) \in \mathbb{Q}_S^n \), one can define content \( c(\mathbf{x}) = \max\{\|x_\nu\|, \|x_\infty\|_\infty\} \), where \( \| \cdot \|_\infty \) is the sup norm in \( \mathbb{R}^n \).

We will consider \( \wedge^j \mathbb{Q}_S^n = \wedge^j \mathbb{Q}_\nu^n \times \wedge^j \mathbb{R}^n \), which is a free \( \mathbb{Q}_S \)-module with standard basis \( \{\mathbf{e}_I = e_{i_1} \wedge \cdots \wedge e_{i_j} | I \subseteq \{0, \ldots, n+1\} \text{ and } i_1 < i_2 < \cdots < i_j \} \). Let \( \Delta = \mathbb{Z}_S \mathbf{a}_1 + \cdots + \mathbb{Z}_S \mathbf{a}_r \), where \( \mathbf{a}_1, \ldots, \mathbf{a}_r \in \mathbb{Q}_S^n \). Then \( \Delta \) is a lattice in \( \mathbb{Q}_S \Delta \), and by Lemma 7.4 in [42] we know that the covolume of \( \Delta \) is \( \text{cov}(\Delta) := c(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_r) \). Here \( c(\cdot) \) is the extension of content in \( \mathbb{Q}_S^n \) to \( \wedge^r \mathbb{Q}_S^n \). A submodule \( \Delta \) in \( \mathbb{Z}_S^n \) is called primitive if \( \Delta = \mathbb{Q}_S \Delta \cap \mathbb{Z}_S^n \). We denote \( \Psi(\mathbb{Z}_S, n) \) as the set of all nonzero primitive submodules of \( \mathbb{Z}_S^n \). The set of invertible matrices in \( \mathcal{K}_n \) is denoted as \( \text{GL}_n(\mathcal{K}) \).

Let us recall Theorem 2.2 from [36], which is an improvement to one of the main theorems in [39] on quantitative nondivergence.

**Theorem 3.1.** Let \( m \in \mathbb{N} \), \( C, \alpha > 0 \), \( 0 < \rho < 1 \) and \( X \) is a Besicovitch space. Let \( B = B(x_0, r_0) \subset X \) and a continuous map \( h : B := B(x_0, 3^m r_0) \to \text{GL}_m(\mathbb{R}) \) be given and \( \mu \) be a \( D \)-Federer measure on \( B \). Suppose that the following conditions are satisfied by the function \( h \) and the measure \( \mu \):

(i) for every \( \Delta \in \Psi(\Lambda, m) \), the function \( \text{cov}(h(\cdot)\Delta) \) is \((C, \alpha)\)-good on \( B \) with respect to \( \mu \);

(ii) for every \( \Delta \in \Psi(\Lambda, m) \),

\[
\sup_{x \in B \cap \text{supp}\mu} \text{cov}(h(x)\Delta) \geq \rho^{\text{rank}(\Delta)}.
\]

Then for any positive \( \varepsilon \leq \rho \), one has

\[
\mu \left( \left\{ x \in B \mid \delta(h(x)\Lambda^m) < \varepsilon \right\} \right) \leq mC(ND^2)^m \left( \frac{\varepsilon}{\rho} \right)^\alpha \mu(B).
\]

Theorem 5.3 in [20] gives the \( S \)-adic version of the above theorem. Next, we state a general version of Theorem 4.4 of [28]. The following theorem can be proved by improving Theorem 4.2 of [28], adopting the same proof of Theorem 2.2 from [36].

**Theorem 3.2.** Let \( m \in \mathbb{N} \), \( C, \alpha > 0 \), \( 0 < \rho < 1 \) and \( X \) is ultrametric Besicovitch space. Let \( B = B(x_0, r_0) \subset X \) and a continuous map \( h : B \to \text{GL}_m(\mathbb{F}_q((T^{-1}))) \) be given and \( \mu \) be a locally finite Borel measure on \( X \). Suppose that the following conditions are satisfied by the function \( h \) and the measure \( \mu \):

(i) for every \( \Delta \in \Psi(\Lambda, m) \), the function \( \text{cov}(h(\cdot)\Delta) \) is \((C, \alpha)\)-good on \( B \) with respect to \( \mu \);

(ii) for every \( \Delta \in \Psi(\Lambda, m) \),

\[
\sup_{x \in B \cap \text{supp}\mu} \text{cov}(h(x)\Delta) \geq \rho^{\text{rank}(\Delta)}.
\]

Then for any positive \( \varepsilon \leq \rho \), one has

\[
\mu \left( \left\{ x \in B \mid \delta(h(x)\Lambda^m) < \varepsilon \right\} \right) \leq mC \left( \frac{\varepsilon}{\rho} \right)^\alpha \mu(B).
\]
Remark 5. Note that the main difference between Theorem 3.1 and Theorem 3.2 is that in Theorem 3.2 we can remove the Federer condition on the measure $\mu$, and it is fine to take $\mu$ to be just locally finite.

3.8. Dynamics.

3.8.1. Homogeneous space when $\mathcal{K} = \mathbb{R}, \mathbb{F}_q((T^{-1}))$. Let $\mathcal{K} = \mathbb{R}$ (resp. $\mathbb{F}_q((T^{-1}))$) and $\Lambda = \mathbb{Z}$ (resp. $\mathbb{F}_q[T]$). We consider $\Omega_{n+1} := \text{SL}_{n+1}(\mathcal{K})/\text{SL}_{n+1}(\Lambda)$ which is noncompact and it has finite volume (c.f. [57]). Moreover, $\Omega_{n+1}$ can be indentified with the space of covolume 1 lattices in $\mathcal{K}^{n+1}$.

3.8.2. Homogeneous space when $\mathcal{K} = Q_{\nu}$. We define $\text{GL}_m^1(Q_{\mathbb{S}}) := \{ g \in \text{GL}_m(Q_{\mathbb{S}}) \mid c(\det(g)) = 1 \}$, and we consider $\Omega^1_{S,n+1} := \text{GL}_n^1(Q_{\mathbb{S}})/\text{GL}_{n+1}(\mathbb{Z}_{\mathbb{S}})$, which is the space of lattices in $Q_{S}^{n+1}$ with covolume 1. This homogeneous space is same as $G_1/\Gamma_1$, defined in 1.1.1.

3.8.3. The smallest vector. Let us take $\mathcal{K} = \mathbb{R}, \mathbb{F}_q((T^{-1}))$. Let us define the length of smallest vector in a lattice $\Delta$ in $\mathbb{K}^{n+1}$,

$$\delta(\Delta) := \inf\{ \|x\| \mid x \in \Delta \setminus \{0\} \}.$$ 

Next we take a lattice $\Delta$ in $Q_{S}^{n+1}$, and we define

$$\delta_S(\Delta) := \inf\{ c(x) \mid x \in \Delta \setminus \{0\} \}.$$ 

We recall the following theorem which is Corollary 2.4 from [28], which describes all the compact sets of the space $\Omega_n$.

**Theorem 3.3.** The set $Q_{\varepsilon} = \{ \Delta \in \Omega_n \mid \delta(\Delta) \geq \varepsilon \}$ is compact for all $\varepsilon > 0$.

Similar to the above theorem, we state Theorem 7.10 of [42] below. This describes all the bounded sets, in particular compact sets in the S-adic setting.

**Theorem 3.4.** A subset $\mathcal{F} \subset \Omega^1_{S,n+1}$ is bounded if and only if it is seperated from $0$, i.e. there exists a nonempty neighborhood $U \subset Q_{S}^{n+1}$ such that $U \cap \Delta = \{0\}$ for all $\Delta \in \mathcal{F}$.

3.8.4. Flows in the homogeneous spaces. Let $\mathcal{K} = \mathbb{R}$ (resp. $\mathbb{F}_q((T^{-1}))$) and $\Lambda = \mathbb{Z}$ (resp. $\mathbb{F}_q[T]$). For $y \in \mathcal{K}^n$, let us denote $\Lambda_y = \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}$ and $u_y = \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}$. We also consider the diagonal flows as follows. When $\mathcal{K} = \mathbb{R}$, we take a diagonal flow

$$g_k = \begin{bmatrix} e^{nk} & 0 & \ldots & 0 \\ 0 & e^{-k} & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & e^{-k} \end{bmatrix}, \text{ where } k \in \mathbb{N}.$$ 

When $\mathcal{K} = \mathbb{F}_q((T^{-1}))$, we take a diagonal flow

$$g_k = \begin{bmatrix} T^{nk} & 0 & \ldots & 0 \\ 0 & T^{-k} & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & T^{-k} \end{bmatrix}, \text{ where } k \in \mathbb{N}.$$
Let $\mathcal{K} = \mathbb{Q}_\nu$. For $\mathbf{y} \in \mathbb{Q}_\nu^n$, we define $\mathbf{u}_\mathbf{y} := (u_{\mathbf{y}, \nu}, u_{\mathbf{y}, \infty})$, where $u_{\mathbf{y}, \nu} = \begin{bmatrix} 1 & \mathbf{y} \\ 0 & \mathbf{I}_n \end{bmatrix}$ and $u_{\mathbf{y}, \infty} = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{I}_n \end{bmatrix}$. Here $\mathbf{I}_n$ is the identity matrix of size $n \times n$. We consider the diagonal flow $g_k = ((g_k), (g_k))$ as follows:

$$
(g_k)^{(n+1)k} := \begin{bmatrix} \nu^{-n}k & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \text{ where } k \in \mathbb{N},
$$

and

$$
(g_k)^{(n+1)k} := \begin{bmatrix} \nu^{-k} & 0 & \cdots & 0 \\ 0 & \nu^{-k} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \nu^{-k} \end{bmatrix}, \text{ where } k \in \mathbb{N}.
$$

4. Proof of theorems 2.1 and 2.2

4.1. Connecting number theory and dynamics. The following connection, often referred to as Dani correspondence, between number theory and dynamics has been explored in several breakthrough works; see [39, 19]. For the sake of completeness we provide a proof.

**Lemma 4.1** (Dani Correspondence when $\mathcal{K} = \mathbb{R}, \mathbb{F}_q((T^{-1})))$. A vector $\mathbf{x} \in \mathcal{K}^n$ is singular if and only if the corresponding trajectory $\{g_ku_{\mathbf{x}}\Lambda^{n+1} | k \geq 0\}$ is divergent in $\Omega_{n+1}$. Here $\mathcal{K} = \mathbb{R}$ (resp. $\mathcal{F}_q((T^{-1}))$) and $\Lambda = \mathbb{Z}$ (resp. $\mathbb{F}_q[T]$).

**Proof.** Let $\mathcal{K} = \mathbb{F}_q((T^{-1}))$ and $\Lambda = \mathbb{F}_q[T]$. Suppose that $\{g_ku_{\mathbf{x}}\Lambda^{n+1}\}$ is not divergent in $\Omega_{n+1}$. By Lemma 3.3 this means that, there exist an $\varepsilon > 0$ and a sequence $k_r \to \infty$ as $r \to \infty$ such that, $\|g_ku_{\mathbf{x}}(q_0^{(r)}, \ldots, q_m^{(r)})\| \geq \varepsilon$ for some nonzero $(q_0^{(r)}, \ldots, q_m^{(r)}) \in \Lambda^{n+1}$. The inequality above gives

$$
\|T^{nk_r}(q_0^{(r)} + q_1^{(r)}x_1 + \cdots + q_n^{(r)}x_n, T^{-k_r}q_1^{(r)}, \ldots, T^{-k_r}q_n^{(r)})\| \geq \varepsilon.
$$

Therefore, for $Q_r = e^{k_r}$ the system (1.3) has no nonzero solution in $\Lambda^{n+1}$ for $c = e^{n+1}$. Hence $\mathbf{x}$ is not singular, conversely, suppose that $\mathbf{x}$ is not singular. Then, there exists an $\varepsilon > 0$ such that, for $Q_r \to \infty$ the system (1.3) has no nonzero solution in $\Lambda^{n+1}$. Choosing $k_r \in \mathbb{N}$ such that $k_r = [\log(Q_r)] + 1$ yields $g_ku_{\mathbf{x}}\Lambda^{n+1}$ avoids an open $\varepsilon$-neighbourhood of $0$ as $k_r \to \infty$, i.e., $g_ku_{\mathbf{x}}\Lambda^{n+1}$ does not diverge as $k \to \infty$.

The exact same proof will work in $\mathbb{R}$; see Proposition 2.1 in [45].

**Lemma 4.2** (One side of Dani correspondence when $\mathcal{K} = \mathbb{Q}_\nu$). If a vector $\mathbf{x} \in \mathbb{Q}_\nu^n$ is singular then the corresponding trajectory $\{g_ku_{\mathbf{x}}\Lambda^{n+1} | k \geq 0\}$ is divergent in $\Omega_{1, n+1}^{1}$. \hfill \Box
Proof. Suppose \( x \in \mathbb{Q}_\nu^n \) is singular, i.e. for every \( c > 0 \) and for all large enough \( Q > 0 \), there are integer \( (q_0, q) \in \mathbb{Z} \setminus \{0\} \times \mathbb{Z}^n \) solutions to the following system of inequalities,

\[
\begin{align*}
|q \cdot x + q_0|_\nu < \frac{c}{Q^{n+1}}, \\
\|(q_0, q)\|_\infty \leq Q,
\end{align*}
\]

where \( \| \cdot \|_\infty \) is the sup norm in \( \mathbb{R}^n \). We can see

\[
\delta_S(g_k u_x \mathbb{Z}^n_{S}) = \min_{(q_0, q) \in \mathbb{Z}^n_{S} \setminus \{0\}} c \left( g_k u_x \left[ \begin{array}{c} q_0 \\ q \end{array} \right] \right)
= \min_{(q_0, q) \in \mathbb{Z}^n_{S} \setminus \{0\}} \max_{q} (\nu^{(n+1)k} |q_0 + q \cdot x|_\nu, |\|q\|_\nu^{n-k})\|(q_0, q)\|_\infty.
\]

Let us take \( Q = c^{-\frac{1}{n+1}} \nu^k \), and any integer \( (q_0, q) \in \mathbb{Z}^n_{S} \) solution to (4.1) will give \( \nu^{(n+1)k} |q_0 + q \cdot x|_\nu < 1, \|q\|_\nu \leq 1 \). From the second inequality of (4.1) we have \( \nu^k \|(q_0, q)\|_\infty \leq c^{-\frac{1}{n+1}} \) for all large \( k \in \mathbb{N} \). Hence by Theorem 3.4 this lemma follows.

**Remark 6.** The other side of the previous lemma is not clear and we are not sure if that is true. The reason is that for singular vectors in \( \mathbb{Q}_\nu^1 \), we are seeking integer solutions to the system (4.1), and in the homogeneous space we consider the lattice \( \mathbb{Z}^n_{S} \) in \( \mathbb{Q}_\nu^n \). This issue did not occur in Lemma 4.1 because in (1.3) the solutions are in \( \mathbb{Z} \) (resp. \( \mathbb{F}_q[T] \)) and the corresponding lattices are the same.

The following theorem follows from Theorem 3.2.

**Theorem 4.1.** Suppose that \( \mathcal{K} = \mathbb{F}_q((T^{-1})) \) and \( \Lambda = \mathbb{F}_q[T] \). Let \( X \) be a Besicovitch ultrametric space, \( B = B(x, r) \subset X \) a ball, \( \mu \) be a locally finite Borel measure on \( X \), and suppose that \( f : B \to \mathcal{K}^n \) is a continuous map. We assume that the following two properties are satisfied.

1. There exists \( C, \alpha > 0 \) such that all the functions \( x \to \text{cov}(g_k u_{f(x)} \Gamma), \Gamma \in \mathfrak{P}(\Lambda, n + 1) \), are \((C, \alpha)\) good on \( B \) w.r.t. \( \mu \);
2. there exists \( c > 0 \) and \( k_i \to \infty \) such that for any \( \Gamma \in \mathfrak{P}(\Lambda, n + 1) \) one has

\[
\sup_{x \in B \cap \text{supp} \mu} \text{cov}(g_k u_{f(x)} \Gamma) \geq c^{\text{rank}(\Gamma)}.
\]

Then

\[
\mu \{ x \in B \mid f(x) \text{ is singular} \} = 0.
\]

**Proof.** We will check that the map \( h = g_k u_{f} \) satisfies the assumptions of Theorem 3.2 with respect to the measure \( \nu = f_* \mu |_{B} \) where, condition (i) of this proposition is same as the condition (i) of Theorem 3.2. Then condition (ii) of this proposition gives (ii) of Theorem 3.2, for all integers \( k_i \). Therefore by Theorem 3.2 for any \( 1 > \varepsilon > 0 \) we have that

\[
\mu \left( \left\{ x \in B \mid \delta(g_k u_{f(x)} \Lambda^{n+1}) < \varepsilon c \right\} \right)
\leq (n + 1)C \varepsilon^\alpha \mu(B)
\]

\[
= E \varepsilon^\alpha.
\]
From Lemma 4.1 we have
\[ \mu \{ x \in B \mid f(x) \text{ is singular} \} \subset \mu \{ x \in B \mid g_ku_f(x)\Gamma, \ k \in \mathbb{N} \text{ is divergent} \}. \]
We want to show that for any \( \varepsilon > 0 \), we have \( \mu \{ x \in B \mid f(x) \text{ is singular} \} \leq \varepsilon^\alpha \). Let \( \varepsilon > 0 \) be given. Then for any \( j \in \mathbb{N} \), there exists a set \( B_j \subset \{ x \in B \mid f(x) \text{ is singular} \} \) such that \( \mu(\{ x \in B \mid f(x) \text{ is singular} \}) \leq \mu(B_j) + \frac{1}{j} \) and there exists a \( k_i \) such that \( B_j \subset \{ x \in B \mid \delta(g_k u_f(x)\Lambda^{n+1}) < \varepsilon c \} \). Hence, for every \( k \in \mathbb{N} \) and \( \varepsilon > 0 \) we have
\[ \mu(\{ x \in B \mid f(x) \text{ is singular} \}) \leq E \varepsilon^\alpha + \frac{1}{j}, \]
and we conclude. \( \square \)

**Remark 7.** Note that only one side of Dani correspondence 4.1 is needed in the above theorem.

Similar to the previous theorem, using Theorem 3.1 in the case of \( \mathbb{R} \) and using Theorem 5.3 of [20] in the case of \( \mathbb{Q}_\nu \) we have

**Theorem 4.2.** Suppose that \( \mathcal{K} = \mathbb{R} \) (resp. \( \mathbb{Q}_\nu \)) and \( \Lambda = \mathbb{Z} \) (resp. \( \mathbb{Z}_S \)). Let \( X \) be a Besicovitch space, \( B = B(x,r) \subset X \) a ball, \( \mu \) a Federer measure on \( X \), and suppose that \( f : B \to \mathcal{K}^N \) is a continuous map. We assume that the following two properties are satisfied.

1. There exists \( C, \alpha > 0 \) such that all the functions \( x \to \text{cov}(g_k u_f(x)\Gamma), \Gamma \in \mathfrak{P}(\Lambda, n + 1) \), are \((C, \alpha)\) good on \( \tilde{B} \) w.r.t. \( \mu \);
2. there exists \( c > 0 \) and \( k_i \to \infty \) such that for any \( \Gamma \in \mathfrak{P}(\Lambda, n + 1) \) one has
\[ (4.6) \sup_{B \cap \text{supp } \mu} \text{cov}(g_k u_f(x)\Gamma) \geq c^{\text{rank } \Gamma}. \]

Then
\[ \mu \{ x \in B \mid f(x) \text{ is singular} \} = 0. \]

**Proof.** The proof is exactly same as the proof of Theorem 4.1. \( \square \)

**Remark 8.** In order to prove Theorem 2.3 one does not need the full strength of the above theorem. In particular, \( c^{\text{rank } \Gamma} \) in the second condition can be replaced by \( c \), making the condition stronger and the theorem weaker.

The following lemma follows from Minkowski’s convex body theorem in \( \mathbb{R} \) and \( \mathbb{F}_q((T^{-1})) \), [49].

**Lemma 4.3.** Let \( \mathcal{K} = \mathbb{R} \) (resp. \( \mathbb{F}_q((T^{-1})) \)) and \( \Lambda = \mathbb{Z} \) (resp. \( \mathbb{F}_q[T] \)). For any \( m > 0, g \in \text{SL}(m, \Lambda), \Delta \in \mathfrak{P}(\mathcal{K}, m) \), we have that
\[ \delta(g\Lambda^m) \leq C_{\mathcal{K}} \text{cov}(g\Delta)_{\text{rank } \Delta}^{\frac{1}{\text{rank } \Delta}}, \]
\( C_{\mathbb{R}} = 2 \) and \( C_{\mathbb{F}_q((T^{-1}))} = 1 \).

**Proposition 4.1.** Let \( \mathcal{K} = \mathbb{R}, \mathbb{F}_q((T^{-1})) \). The second Condition 4.6 in Theorem 4.2 for \( \mathbb{R} \), the second Condition 4.2 in the Theorem 4.1 are necessary. In fact, if the Condition does not hold, then \( f(\text{supp } \mu \cap B) \) is contained in the set of singular vectors.
Proof. If the second condition does not hold then for every $c > 0$ and for all large enough $k$, there exists $\Gamma_k \in \Psi(\Lambda, n + 1)$ such that

$$\sup_{B \supset \supp \mu} \cov(g_k u_f(x) \Gamma_k) < e^{\rank(\Gamma_k)}.$$ 

Hence by Proposition 4.1 for $\mu$-a.e. every $x \in B$, $\delta(g_k u_f(x)\Lambda^{n+1}) \leq \cov(g_k u_f(x) \Gamma_k)^{\frac{1}{\rank(\Gamma_k)}} \Rightarrow \delta(g_k u_f(x)\mathbb{Z}^{n+1}) < c$ for all sufficiently large $k$. Now using Lemma 4.1, we can conclude that for $\mu$-a.e. $x \in B$ we have $f(x)$ to be singular. \qed

Remark 9. Note that in the previous proposition, the other side of Dani correspondence 4.1, i.e. divergent orbits give singular vectors, play a crucial role.

4.2. Covolume calculation for $\mathbb{R}$ and $\mathbb{F}_q((T^{-1}))$. Suppose $\mathcal{K} = \mathbb{R}$ (resp. $\mathbb{F}_q((T^{-1}))$) and $\Lambda = \mathbb{Z}$ (resp. $\mathbb{F}_q[T]$). Let us denote the set of rank $j$ submodules of $\Lambda^{n+1}$ as $S_{n+1,j}$. Let $Q_{\mathcal{K}} = \left\{ \begin{array}{ll} e & \text{when } \mathcal{K} = \mathbb{R} \\ T & \text{when } \mathcal{K} = \mathbb{F}_q((T^{-1})) \end{array} \right.$. In this subsection it is necessary to distinguish the norms in different fields. So we will use $| \cdot |_{\mathcal{K}}$ for the standard norm, and $\| \cdot \|_{\mathcal{K}}$ for the sup norm in $\mathcal{K}^n$.

Lemma 4.4. Let $w \in \bigwedge^j(\Lambda^{n+1})$, $w = \sum w_I e_I$. Then

$$g_k u_x w = Q_{\mathcal{K}}^{-jk} \sum_{\{I \mid 0 \not\in I\}} w_I e_I + Q_{\mathcal{K}}^{(n-j+1)k} \sum_{\{I \mid 0 \in I\}} \left( w_I + \left( \sum_{i \not\in I} \pm w_{I \setminus \{0\} \cup \{i\}} x_i \right) e_I \right).$$

Proof. Note that $u_x$ leaves $e_0$ invariant and sends $e_i$ to $x_i e_0 + e_i$ for $i \geq 1$. Therefore

$$u_x(e_I) = \begin{cases} e_I & \text{if } 0 \in I \\ e_I + \sum_{i \in I} \pm x_i e_{I \setminus \{0\} \cup \{i\}} & \text{if } 0 \not\in I. \end{cases}$$

Also, under the diagonal action of $g_k$, the vectors $e_i$ are eigenvectors with eigenvalue $Q_{\mathcal{K}}^{-jk}$ for $i \geq 1$ and $e_0$ is an eigenvector with eigenvalue $Q_{\mathcal{K}}^{nk}$. Therefore,

$$g_k u_x(e_I) = \begin{cases} Q_{\mathcal{K}}^{(n-j+1)k} e_I & \text{if } 0 \in I \\ Q_{\mathcal{K}}^{-jk} e_I \pm Q_{\mathcal{K}}^{(n-j+1)k} \sum_{i \in I} x_i e_{I \setminus \{0\} \cup \{i\}} & \text{if } 0 \not\in I. \end{cases}$$

Thus, for $w \in \bigwedge^j(\Lambda^{n+1})$, $w = \sum w_I e_I$ with $w_I \in \Lambda$ the conclusion follows. \qed

Let $V_0$ be the $\Lambda$ submodule of $\mathcal{K}^{n+1}$ generated by $e_1, \ldots, e_n$. Note we can write

$$c(w) = \begin{bmatrix} c(w)_0 \\ c(w)_1 \\ \vdots \\ c(w)_n \end{bmatrix},$$

where $c(w)_i = \sum_{J \subset \{1, \ldots, n\} \atop \# J = j-1} w_{J \cup \{i\}} e_J \in \bigwedge^{j-1}(V_0)$ for $i = 0, \ldots, n$. Let $\bar{x} = (1, x)$ and $\pi$ be the orthogonal projection from $\bigwedge^j(\mathcal{K}^{n+1}) \to \bigwedge^j V_0$. 

Proposition 4.2. The second Condition (2) in Theorem 4.1 for $\mathbb{R}$, and the second Condition (2) in Theorem 4.1 are equivalent to the following condition.

\[ (2) \]

- There exists $c > 0$ and $k_i \to \infty$ such that $\forall \ j, \cdots, n$ and $\forall \ w \in \bigwedge^j \Lambda^{n+1}$, one has

\[ \max \left( e^{(n-j+1)k} \left\| R_c(w) \right\|_{\mathcal{X}}, e^{-jk} \left\| \pi(w) \right\|_{\mathcal{X}} \right) \geq c^j, \]

where $R$ is a $(s+1) \times (n+1)$ matrix that depends on the ball $B$, the function $f$ and the measure $\mu$.

Proof. By Lemma 4.4, we can write

\[ g_k u_x w \]

\[ = Q_{\mathcal{X}}^{(n-j+1)k} \left( e_0 \wedge \sum_{i=0}^n x_i \mathbf{c}(w)_i \right) + Q_{\mathcal{X}}^{-jk} \sum_{\{I \mid 0 \notin I\}} w_I e_I \]

\[ = Q_{\mathcal{X}}^{(n-j+1)k} (e_0 \wedge \bar{x} \cdot \mathbf{c}(w)) + Q_{\mathcal{X}}^{-jk} \pi(w). \]

Hence, we have

\[ \text{cov}(g_k u_x \Delta) = \left\| g_k u_x w \right\|_{\mathcal{X}} \]

\[ = \max \left( e^{(n-j+1)k} \left\| \sum_{i=0}^n x_i \mathbf{c}(w)_i \right\|_{\mathcal{X}}, e^{-jk} \left\| \pi(w) \right\|_{\mathcal{X}} \right). \]

Thus for $x = f(x)$,

\[ \sup_{x \in B \cap \text{supp} \mu} \text{cov}(g_k u_{f(x)} \Delta) \]

\[ = \max \left( e^{(n-j+1)k} \sup_{x \in B \cap \text{supp} \mu} \left\| \tilde{f}(x) \cdot \mathbf{c}(w) \right\|_{\mathcal{X}}, e^{-jk} \left\| \pi(w) \right\|_{\mathcal{X}} \right), \]

where $\tilde{f} = (1, f_1, \cdots, f_n)$ and $B$ is a ball in $\mathcal{K}$. Now suppose that the $\mathcal{K}$-span of the restrictions of $1, f_1, \cdots, f_n$ to $B \cap \text{supp} \mu$ has dimension $s + 1$ and choose $g_1, \cdots, g_s : B \cap \text{supp} \mu \to \mathcal{K}$ such that $1, g_1, \cdots, g_s$ form a basis of the space. Therefore there exists a matrix $R = (r_{ij})_{(s+1) \times (n+1)}$ such that $\tilde{f}(x) = \tilde{g}(x)R \forall \ x \in B \cap \text{supp} \mu$ where $\tilde{g} = (1, g_1, \cdots, g_s)$. We can rewrite

\[ \sup_{x \in B \cap \text{supp} \mu} \left\| \tilde{f}(x) \mathbf{c}(w) \right\|_{\mathcal{X}} = \sup_{x \in B \cap \text{supp} \mu} \left\| \tilde{g}(x) R_c(w) \right\|_{\mathcal{X}}. \]

Hence by (4.12) and (4.13) the second Conditions (2) for $\mathbb{R}$ and (2) are equivalent to the Condition (2)' (4.2) by equivalence of norms since $1, g_1, \cdots, g_s$ are linearly independent. \hfill $\Box$

Lemma 4.5. For all $(f, \mu)$ nonplanar in $\mathcal{L}$, the matrix $R$ in Condition (2) can be chosen uniformly for all ball $B$ with $\mu(B) > 0$.

Proof. Let $\dim \mathcal{L} = s$ and let

\[ h : \mathcal{K}^s \to \mathcal{L} \subset \mathcal{K}^n \]

be an affine isomorphism, and $\tilde{h}(x) = \bar{x} R, x \in \mathcal{K}^s$, where $\tilde{h} := (1, h_1, \cdots, h_n)$. Then $\tilde{g} = h^\perp \circ f$ spans the space of $F$-span of the restrictions of $1, f_1, \cdots, f_n$ to $B \cap \text{supp} \mu$ and satisfies $\tilde{f}(x) = \tilde{g}(x) R \forall \ x \in B \cap \text{supp} \mu$. Moreover, since $(f, \mu)$ is nonplanar in $\mathcal{L}$, we have $1, g_1, \cdots, g_s$ are linearly independent. Since we chose $R$
such as it only depends on the affine subspace $L$, Condition (2)′ (4.2) only depends on the subspace $L$ as long as $(f, \mu)$ is nonplanar in $L$. \hfill \square

4.3. Completing the proof of Theorems 2.1 and 2.2. Suppose $K = \mathbb{R}$ or $\mathbb{F}_q((T^{-1}))$.

4.3.1. (3) $\iff$ (4) in Theorems 2.1 and 2.2. By Proposition 4.1 if (3) is true, then Condition (2) $\iff$ Condition (2)′, (4.2), should be satisfied. By the hypothesis of Theorem 2.1 and 2.2 $(f, \mu)$ is good and therefore by Theorems 4.2 and 4.1, (4) is true. Since $(I_d, \lambda_L)$ is nonplanar in $L$, same argument as above shows (4) implies (3) due to Lemma 4.5.

4.3.2. (1) $\implies$ (2) in Theorems 2.1 and 2.2. Since $f(x) \in L$ for all $x \in B \cap \text{supp}(\mu)$, (1) $\implies$ (2).

4.3.3. (2) $\implies$ (4) in Theorems 2.1 and 2.2. Let (2) be true, that is there is one $y \in L$ such that $y$ is not singular. Then by Proposition 4.1, the second Condition 2 in Theorems 4.2, 4.1 is true. Since $(f, \mu)$ is good, we have that $\mu$-a.e. $x \in X$ has a neighbourhood $V$ such that $(f, \mu)$ is good. If $K = \mathbb{R}$, then we have that $\mu$ is Federer, and we can choose $V$ such that $\mu$ is $D$-Federer on $V$ for some $D > 0$. If $K = \mathbb{F}_q((T^{-1}))$, then $\mu$ is locally finite which is the condition that is required in Theorem 4.1. If $K = \mathbb{R}$, choose a ball $B = B(x, r)$ of positive measure such that the dilated ball $\tilde{B} = B(x, 3^{n+1}r)$ is contained in $V$. We have already noted in (4.12) that $\text{cov}(g_k u_{f(x)} \Gamma)$ is the maximum of the norms of linear combinations of $1, f_1, \cdots, f_n$. Hence Condition 1 of Theorems 4.2, 4.1 is satisfied. Thus we can conclude (4).

4.3.4. (4) $\implies$ (1) in Theorems 2.1 and 2.2. It is straightforward that (4) $\implies$ (1).

4.4. Covolume calculation for $Q_\nu$. In this subsection we are going to denote the norm on $\mathbb{R}^n$ as $\| \cdot \|_\infty$.

Lemma 4.6. Let $w \in \bigwedge^j (\Lambda^{n+1}), \ w = \sum w_I e_I$. Then

\begin{equation} (g_k u_{\chi} w)_{\nu} = \sum_{\{I, 0 \notin I\}} w_I e_I^\nu + \nu^{-(n+1)k} \sum_{\{I, 0 \in I\}} \left( w_I + \left( \sum_{i \notin I} \pm w_{I \setminus \{0\} \cup \{i\}} x_i \right) \right) e_I^\nu \end{equation}

and

\begin{equation} (g_k u_{\chi} w)_\infty = \nu^{-kj} \sum w_I e_I^\infty. \end{equation}

Proof. Note that $u_{\chi, \nu}$ leaves $e_I^\nu$ invariant and sends $e_I^\nu$ to $x_i e_I^\nu + e_I^\nu$ for $i \geq 1$. Therefore

\[ u_{\chi, \nu}(e_I^\nu) = \begin{cases} e_I^\nu & \text{if } 0 \in I \\ e_I^\nu + \sum_{i \notin I} \pm x_i e_{I \setminus \{0\} \cup \{i\}} \end{cases} \text{ if } 0 \notin I. \]

Since $u_{\chi, \infty} = \text{Id}_{n+1}$, everything is invariant under this. Under the diagonal flow $(g_k)_{\nu}$, the vectors $e_I^\nu$ are invariant for $i \geq 1$, and $e_I^\nu$ is an eigenvector with eigenvalue $\nu^{-k(n+1)}$. Therefore,

\[ (g_k)_{\nu} u_{y, \nu}(e_I^\nu) = \begin{cases} \nu^{-(n+1)k} e_I^\nu & \text{if } 0 \in I \\ e_I^\nu + \nu^{-(n+1)k} \sum_{i \in I} \pm y_i e_{I \setminus \{0\} \cup \{i\}} \end{cases} \text{ if } 0 \notin I. \]

Each $e_I^\infty$ is an eigenvector of $(g_k)_{\infty}$ of eigenvalue $\nu^{-k}$. Thus

\[ (g_k)_{\infty} u_{y, \infty} e_I^\infty = \nu^{-kj} e_I^\infty. \]
Combining all yields this lemma. □

Now let us recall \(c(w)\) as in (4.9) and \(\pi(w)\) as before.

**Proposition 4.3.** The second Condition (2) in Theorem 4.2 in the case of \(Q_\nu\) is equivalent to the following condition.

\(\bullet\) (2)′ There exists \(c > 0\) and \(k_i \to \infty\) such that \(\forall j = 1, \cdots, n\) and \(\forall w \in \bigwedge^j \Lambda^{n+1}\), one has

\[
\max \left( \nu^{(n-j+1)k} \|Rc(w)\|_\nu, \nu^{-jk} \|\pi(w)\|_\nu \right) \geq c^j,
\]

where \(R\) is a \((s+1) \times (n+1)\) matrix that depends on the ball \(B\), the function \(f\) and the measure \(\mu\).

**Proof.** By Lemma 4.6, we can write

\[
(g_{k_u}w)_\nu = \nu^{-(n+1)k} \left( e_0 \land \sum_{i=0}^n x_i c(w)_i \right) + \sum_{\{I \neq I\}} w_I e_I = \nu^{-(n+1)k} (e_0 \land x \cdot c(w)) + \pi(w).
\]

Hence, we have

\[
\text{cov}(g_{k_u}x) = c(g_{k_u}w) = \|\nu^{(n+1)k} \sum_{i=0}^n x_i c(w)_i \|_\nu \nu^{-kj} \|\pi(w)\|_\nu \|w\|_\infty.
\]

Thus for \(x = f(x)\),

\[
\text{sup}_{x \in B_{\text{supp}} \mu} \text{cov}(g_{k_{uf}}x) = \max \left( \nu^{(n+1)k} \sup_{x \in B_{\text{supp}} \mu} \|f(x) \cdot c(w)\|_\nu \|w\|_\infty, \nu^{-kj} \|\pi(w)\|_\nu \|w\|_\infty \right).
\]

Now the proof will progress as same as Proposition 4.2. □

**4.5. Completing the proof of Theorem 2.3.** It is given that \((f, \mu)\) is nonplanar. This means that the restriction of \(1, f_1, \cdots, f_n\) to \(B \cap \text{supp} \mu\) are linearly independent. Hence by Proposition 4.3, Condition (2) in Theorem 4.2 in the case of \(Q_\nu\) is satisfied. We have \((f, \mu)\) is good in \(Q^n\) by one of the hypotheses, which guarantees Condition (1) of Theorem 4.2. Hence the conclusion follows from Theorem 4.2.

**5. \(\psi\)-Dirichlet numbers in function field**

In this section and Section 6 we are going to deal with the field \(F_q((T^{-1}))\) only. So we refer \(F_q((T^{-1}))\) as \(F\), in order to reduce notational cumbersome.
5.1. Continued fraction over function field. Suppose \( a = \sum_{k \leq k_0} a_k T^k \in F \) where \( a_{k_0} \neq 0 \), we call \([a] := \sum_{k \leq k_0} a_k T^k\) as the integer part of \( a \) and \( \langle a \rangle = \sum_{k < 0} a_k T^k\) as the fractional part of \( a \). Note that \( \|a\| = e^{k_0} \geq 1 \) if \( a_{k_0} \neq 0 \), and otherwise we have \( \|a\| = 0 \). Also, note that \( \|\langle a \rangle\| \leq 1 \). This observation leads us to construct continued fraction expansion of \( a \). An expression of the form \( a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}} \) where \( a_0, a_1, a_2 \in F_q[T] \) is called a simple continued fraction; see §1 in [55]. An expression of the form \( \frac{p_n}{q_n} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} \), where \( p_n, q_n, a_0, a_1, \ldots, a_n \in F_q[T] \) is called a finite continued fraction. An element of \( F_q(T) \), can be represented as an unique finite continued fraction. An \( \alpha \in F \setminus F_q[T] \) can be represented as a simple continued fraction in the form of \([a_0, a_1, a_2, \ldots]\) and we call the numbers \( \frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]\) the convergents of \( \alpha \). Note that \( |q_n| \) is increasing as \( n \to \infty \). The relation between two consecutive convergents is given by the following equation.

\[
p_i q_{i+1} - p_{i+1} q_i = (-1)^{i+1} \quad \text{for } i \in \mathbb{Z}, i \geq -2.
\]

Hence we have,

\[
\left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| = \left| \frac{p_{n+1}q_n - p_n q_{n+1}}{q_n q_{n+1}} \right| = \left| \frac{\pm 1}{q_n q_{n+1}} \right| = \frac{1}{|q_n| |q_{n+1}|} \leq \frac{1}{|q_n|^2}.
\]

In fact, by Equation 1.12 in [55] we have

\[
(5.1) \quad \left| \alpha - \frac{p_n}{q_n} \right| = \frac{1}{|q_n q_{n+1}|},
\]

where \( \frac{p_n}{q_n} \) is a convergent of \( \alpha \in F \setminus F_q(T) \). We recall the following definition and theorem of best approximation [50], §1.2.

**Definition 5.1.** We say a rational \( \frac{a}{b} \) is the best approximation to some \( \alpha \in F \) if for all \( \frac{c}{d} \) such that \( |d| \leq |b| \) we have \( |b \alpha - a| \leq |d \alpha - c| \).

**Theorem 5.1.** Let \( \alpha \in F \) and let \( \left( \frac{p_n}{q_n} \right)_n \) be its convergents. Let \( p, q \in F_q[T] \) with \( q \neq 0 \) be two relatively prime polynomials. Then \( \frac{p}{q} \) is a best approximation to \( \alpha \) if and only if it is a convergent to \( \alpha \).

We want to recall the following Lemma 2.1 from [43] which was stated for real numbers. The verbatim proof will give the following lemma for function field. The proof uses the fact that convergents are best approximations, which we have by Theorem 5.1. In what follows \( \frac{p_n}{q_n} \) are convergents of \( x \).

**Lemma 5.1.** Let \( \psi : [t_0, +\infty) \to \mathbb{R}_+ \) be non-increasing. Then \( x \in F \setminus F_q(T) \) is \( \psi \)-Dirichlet if and only if \( |\langle q_{n-1}x \rangle| < \psi(|q_n|) \) for sufficiently large \( n \).

5.2. Proof of Theorem 2.4. It is easy to see that \( F_q(T) \subset D(\psi) \). We want to show that \( D(\psi) \subset F_q(T) \). By Equation (5.1) for \( x \in F \setminus F_q(T) \) we have \( |\langle q_{n-1}x \rangle| = \frac{1}{|q_n|}, \forall n \). Since \( \psi(t) < \frac{1}{t} \) for all large enough \( t \), by Lemma 5.1 we conclude that there is no \( x \in F \setminus F_q(T) \) such that \( x \) is \( \psi \)-Dirichlet.

6. Too many vectors with high uniform exponent

In this section we study totally irrational singular vectors in submanifolds of \( F^n \). In dimension \( n = 1 \), the following theorem follows from Theorem 2.4 in [27].
**Theorem 6.1.** The set of numbers $y$ in $F$ that are singular is $\mathbb{F}_q(T)$. 

Going one step ahead one can define $\omega(\cdot)$, as follows, which quantifies singularity of a vector.

(6.1) 
$$\hat{\omega}(y) := \sup \left\{ \omega \mid \text{for all large enough } Q > 0, \exists (q_0, q) \in \mathbb{F}_q[T]^n \text{ s.t. } \|q \cdot y + q_0\| \leq \frac{1}{Q^\omega}, \|q\| \leq Q \right\}$$

Dirichlet’s Theorem 1.1 gives that $\hat{\omega}(y) \geq n$ for all $y \in F^n$.

In order to state the main theorem of this section, we need to define the *irrationality measure* function as follows. We follow the definition in [40].

**Definition 6.1.** We define $\Phi : \mathbb{F}_q[T] \setminus \{0\} \to \mathbb{R}_+$ to be a proper function if the set $\{q \in \mathbb{F}_q[T] : \Phi(q) \leq C\}$ being finite for any $C > 0$. For any arbitrary and any $y \in F^n$, we define the irrationality measure function $\psi_{a,y}(t) := \min_{(q_0, q) \in \mathbb{F}_q[T] \times \mathbb{F}_q[T] \setminus \{0\}, \Phi(q) \leq t} |q \cdot y + q_0|$.

We can now state one of the main theorems of this section.

**Theorem 6.2.** Let $\text{char}(F) = p < \infty$. Let $U$ be an open subset of $F$. We consider $S$ of the following two types:

- $S = \{(x, y, p_3(x, y), \ldots, p_n(x, y)) \mid x, y \in U\} \subset F^n$, where and each $p_i(x, y)$ is a degree 2 polynomial.
- $S = \{(x, y, p(x, y)) \mid x, y \in U\} \subset F^3$ where $p(x, y) = \sum_i a_i x^i y^i + \sum_j b_j y^j$.

Suppose that $S$ is not contained inside any rational affine hyperplane. Then for any proper function $\Phi : \mathbb{F}_q[T] \setminus \{0\} \to \mathbb{R}_+$ and for any non-increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, there exists uncountably many totally irrationals $y \in S$ such that $\psi_{a,y}(t) \leq \phi(t)$ for all large enough $t$.

As an application of the previous Theorem we get Theorem 2.5.

**Proof of Theorem 2.5.** By taking $\Phi(q) = \|q\|$ Theorem 2.5 follows from Theorem 6.2. \qed

Let us recall Theorem 1.1 from [40], which was proved for locally closed subsets of $\mathbb{R}^n$. The same proof verbatim will work for locally closed subsets in $F^n$. It is noteworthy that the proof follows Khintchine’s argument in [33]. We define $|A| := \max_{i=0}^n |a_i|$, where $A = a_1 x_1 + \cdots + a_n x_n = a_{n+1}$, and $(a_1, \ldots, a_{n+1})$ is a primitive vector in $\mathbb{F}_q[T]^{n+1}$.

**Theorem 6.3.** Let $S \subset F^n$ be a nonempty locally closed subset. Let $\{L_1, L_2, \ldots\}$ and $\{L_1', L_2', \ldots\}$ be disjoint collections of distinct closed subsets of $S$, each of which is contained in a rational affine hyperplane in $F^n$, and for each $i$ let $A_i$ be a rational affine hyperplane containing $L_i$, assume the following hold:

(a) 
$$\bigcup_i L_i \cup \bigcup_j L_j' = \{x \in S : x \text{ is contained in a rational affine hyperplane}\};$$

(b) For each $i$ and each $\alpha > 0$, 
$$L_i = \bigcup_{|A_i| > \alpha} L_i \cap L_j;$$
(c) For each $i$, and for any finite subsets of indices $F, F'$ with $i \notin F$, we have
\[ L_i = L_i - \left( \bigcup_{k \in F} L_k \cup \bigcup_{k' \in F'} L'_{k'} \right); \]

(d) $\bigcup_i L_i$ is dense in $S$.

Then for arbitrary $\Phi : \mathbb{F}_q[T]^n \setminus \{0\} \to \mathbb{R}_+$ proper function and for any non-increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, there exists uncountably many totally irrationals $y \in S$ such that $\psi_{\Phi,y}(t) \leq \phi(t)$ for all large enough $t$.

We will call the property (a), (b), (c), and (d) defined above as “property A”. Let us recall the following Theorem 2.1.1 in [30] which we are going to use again and again in the rest of the section.

**Theorem 6.4.** Let $K$ be an arbitrary field and assume for some $m$, $n$ that every $F_i(x,y)$ in $F(x,y) = (F_1(x,y), \ldots, F_m(x,y))$ is in $K[[X,Y]] = K[[x_1, \ldots, x_n, y_1, \ldots, y_m]]$ satisfying $F_i(0,0) = 0$ and further $\frac{\partial(F_1, \ldots, F_m)}{\partial(y_1, \ldots, y_m)} |_{(0,0)} \neq 0$, in which $\frac{\partial(F_1, \ldots, F_m)}{\partial(y_1, \ldots, y_m)}$ is the Jacobian. Then there exists a unique $f(x) = (f_1(x), \ldots, f_m(x))$ with every $f_i(x)$ in $K[[x]] = K[[x_1, \ldots, x_m]]$ satisfying $f_i(0) = 0$ and further $F(x, f(x)) = 0$.

6.1. **When each $p_i(x, y)$ is a degree 2 polynomial and $F$ is of any positive characteristic.** Let us consider

\[ S = \{(x, y, p_3(x, y), \ldots, p_n(x, y)) \mid x, y \in U\}, \]

where $U$ is an open subset of $F$, and

\[ p_i(x, y) = b_{i,1}x^2 + b_{i,2}xy + b_{i,3}y^2 + b_{i,4}x + b_{i,5}y + b_{i,6} \]

with $b_{i,1}, b_{i,2}, \ldots, b_{i,6} \in F$ and $b_{i,1}, b_{i,2}, b_{i,3}$ not being zero simultaneously for $i = 3, \ldots, n$. Let us take $A$ to be a rational affine hyperplane in $F^n$ and we assume that $S$ is not contained inside $A$. We can define $A$ by the linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = a_{n+1}$, where $(a_1, a_2, \cdots, a_{n+1}) \in \mathbb{F}_q[T]^{n+1}$ is primitive. Note that $S \cap A$ is given by the solutions to the equation;

\[ f(x,y) := 0, \]

where

\[ f(x,y) = \sum_{i=3}^{n} a_ip_i(x,y) + a_1x + a_2y - a_{n+1}. \]

We see that $f$ is a polynomial of degree less than or equal to 2. Now note that

\[ \frac{\partial f}{\partial x} = \sum_{i=3}^{n} (2a_ib_{i,1}x + a_ib_{i,2}y) + (a_1 + \sum_{i=3}^{n} a_ib_{i,4}) \]

and

\[ \frac{\partial f}{\partial y} = \sum_{i=3}^{n} (a_ib_{i,2}x + 2a_ib_{i,3}y) + (a_2 + \sum_{i=3}^{n} a_ib_{i,5}). \]

If $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$, then by Theorem 6.4 we get a neighborhood of $(x_0, y_0)$, where $y$ is a $F$-analytic function of $x$. If $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$ then locally we can write $x$ as a $F$-analytic function.
of \( y \). Hence in order to find out all possible \((x_0, y_0)\) such that there is no neighborhood of \((x_0, y_0, p_3(x_0, y_0), \ldots, p_n(x_0, y_0)) \in S \cap A\) that is analytic curve in \( S \cap A \), we consider the linear system

\[
\begin{cases}
\frac{\partial f}{\partial x} = 0; \\
\frac{\partial f}{\partial y} = 0.
\end{cases}
\]

The corresponding coefficient matrix \( M \in \text{Mat}_{2\times2}(F) \) of the system is

\[
\begin{bmatrix}
\sum_{i=3}^{n} 2a_i b_{i,1} & \sum_{i=3}^{n} a_i b_{i,2} \\
\sum_{i=3}^{n} a_i b_{i,2} & \sum_{i=3}^{n} 2a_i b_{i,3}
\end{bmatrix}
\]

First note that if \( a_3, \ldots, a_n \) are zero, then \( S \cap A \) is an analytic curve as the equation of \( A \) would be \( a_1 x + a_2 x^2 = a_{n+1} \). Therefore one of \( a_3, \ldots, a_n \) must be nonzero, and without loss of generality we assume that \( a_3 \neq 0 \). Next denote

\[
b_k = \sum_{i=3}^{n} a_i b_{i,k}
\]

for \( k = 1, \ldots, 6 \). With this setting, we have the following lemma.

**Lemma 6.1.** If \( b_2^2 \neq 4b_1 b_3 \), then there are at most finitely many points of \( A \cap S \) such that their neighborhood is not an \( F \)-analytic curve in \( F^n \).

**Proof.** Note that \( \det(M) = 4b_1 b_3 - b_2^2 \). Hence by the hypothesis, we know that the system has only one solution. This completes the proof.

From the proof above we know that the key is to solve the following equations;

\[
f(x, y) = \sum_{i=3}^{n} a_i p_i(x, y) + a_1 x + a_2 y - a_{n+1} = b_1 x^2 + b_2 xy + b_3 y^2 + (a_1 + b_1) x + (a_2 + b_5) y + (b_6 - a_{n+1}) = 0,
\]

\[
\frac{\partial f}{\partial x}(x, y) = 2b_1 x + b_2 y + (a_1 + b_1) = 0,
\]

\[
\frac{\partial f}{\partial y}(x, y) = b_2 x + 2b_3 y + (a_2 + b_5) = 0.
\]

We will omit the setting up in the proof of Lemma 6.2 and Lemma 6.3 and dive directly into the system.

**Lemma 6.2.** If

\[
b_2^2 = 4b_1 b_3
\]

and \( b_2 \neq 0 \), then there are at most finitely many points of \( A \cap S \) that its neighborhood is not an \( F \)-analytic curve in \( F^n \).

**Proof.** Suppose that there exists no point that satisfies System (6.5), then conclusion of the lemma holds trivially.
Now suppose that there exists a point \((x_0, y_0)\) that satisfies System (6.5). Using Equation (6.9) we have,

\[
(6.10) \quad b_2(a_2 + b_5) = -b_2^2 x_0 - 2b_3(b_2 y_0) \quad (6.7) \quad -b_2^2 x_0 + 2b_3(2b_1 x_0 + (a_1 + b_4)) \quad (6.9) \quad \frac{2b_3}{b_2}(a_1 + b_4).
\]

For any \(x, y \in U\),

\[
(6.11) \quad b_2^2 f(x, y) = b_2^2(b_1 x^2 + b_2 xy + b_3 y^2) + b_2^2((a_1 + b_4)x + (a_2 + b_5)y) + b_2^2(b_6 - a_{n+1})
\]

\[
= b_3(2b_1 x + b_2 y)^2 + b_2^2((a_1 + b_4)x + (a_2 + b_5)y) + b_2^2(b_6 - a_{n+1})
\]

\[
= b_3(2b_1 x + b_2 y)^2 + b_2(a_1 + b_4)(b_2 x + 2b_3 y) + b_2^2(b_6 - a_{n+1}).
\]

For any point that satisfies System (6.5),

\[
(6.12) \quad b_3(2b_1 x + b_2 y)^2 + b_2(a_1 + b_4)(b_2 x + 2b_3 y) + b_2^2(b_6 - a_{n+1})
\]

\[
= b_3(a_1 + b_4)^2 - b_2(a_1 + b_4)(a_2 + b_5) + b_2^2(b_6 - a_{n+1})
\]

\[
= b_2^2(b_6 - a_{n+1}) - b_3(a_1 + b_4)^2.
\]

Suppose that \(b_2^2(b_6 - a_{n+1}) \neq b_3(a_1 + b_4)^2\). We know that Equation (6.6) would never be satisfied for any points satisfying the System (6.5). Therefore there is no point in \(S \cap A\) such that its neighborhood is not an \(F\)-analytic curve.

Now suppose that

\[
(6.13) \quad b_2^2(b_6 - a_{n+1}) = b_3(a_1 + b_4)^2.
\]

Let \(g(x) := -\frac{2b_1 x + (a_1 + b_4)}{b_2}\) and \(\gamma\) be \(\{(x, g(x), p_3(x, g(x)), \ldots, p_n(x, g(x))) \mid x \in U\}\). Then clearly any point in \(\gamma\) satisfies (6.7), (6.8) and using (6.13) one can see that any point in \(\gamma\) also satisfies (6.6). Hence we have \(\gamma \subseteq S \cap A\). Note here Equations (6.7) and (6.8) are essentially the same.

Then for any point in \(S \cap A\), we have

\[
f(x, y) = 0 \quad (6.11)
\]

\[
\Rightarrow b_3(2b_1 x + b_2 y)^2 + b_2(a_1 + b_4)(b_2 x + 2b_3 y) + b_2^2(b_6 - a_{n+1}) = 0
\]

\[
\Rightarrow b_3(2b_1 x + b_2 y + a_1 + b_4)^2 = 0
\]

\[
\Rightarrow y = -\frac{2b_1 x + (a_1 + b_4)}{b_2}.
\]

The last equality holds because \(b_3 \neq 0\) as \(b_2 \neq 0\). Since \(\gamma\) is already an analytic curve, this completes the proof of the lemma.

\[\square\]

**Lemma 6.3.** If \(b_2^2 = 4b_1 b_3\) and \(b_2 = 0\), then there are at most finitely many points of \(A \cap S\) that its neighbourhood is not an \(F\)-analytic curve in \(F^n\).

**Proof.** If \(b_1 = 0, b_2 = 0, b_3 = 0\) and there exists one point whose neighborhood is not an \(F\)-analytic curve, then \(S \cap A\) being nonempty implies \(S \cap A = A\), because \(f(x, y) = b_6 - a_{n+1} = 0\). This contradicts the standing assumption that \(S\) is not contained inside \(A\).

If \(\text{char}(F) \neq 2\), then \(b_1 b_3 = 0\). Since \(b_1, b_2\) and \(b_3\) cannot be zero simultaneously, assume without loss of generality that \(b_1 \neq 0\) and \(b_3 = 0\).
Suppose that there exists no point that satisfies System (6.5), then conclusion of the lemma holds. So let us assume that there exists a point \((x_0, y_0)\) that satisfies System (6.5), then Equation (6.8) gives us \(a_2 + b_5 = 0\). By Equation (6.6) we have
\[
b_1 x^2 + (a_1 + b_4) x + (b_6 - a_{n+1}) = 0.
\]
At most two \(x\) can satisfy the above equation, say they are \(x_1\) and \(x_2\) respectively. Then \(\gamma_1 = \{(x_1, y, p_3(x_1, y), \ldots, p_n(x_1, y)) \mid y \in U\}\) and \(\gamma_2 = \{(x_2, y, p_3(x_2, y), \ldots, p_n(x_2, y)) \mid y \in U\}\) are both analytic curves and they are inside \(S \cap A\).

In addition, since
\[
f(x, y) = b_1 x^2 + (a_1 + b_4) x + (b_6 - a_{n+1}),
\]
we know that any point on \(S \cap A\) must have an \(x\)-coordinate equal to \(x_1\) or \(x_2\), which means that it is in \(\gamma_1\) or \(\gamma_2\). In other words, \(S \cap A = \gamma_1 \cup \gamma_2\). Thus the proof is complete for \(\text{char}(F) \neq 2\).

If \(\text{char}(F) = 2\), then \(\frac{\partial f}{\partial x} = a_1 + b_4\) and \(\frac{\partial f}{\partial y} = a_2 + b_5\). If either of the two is nonzero then the conclusion of the lemma holds. Otherwise, for every \(x, y\) we have
\[
f(x, y) = b_1 x^2 + b_3 y^2 + (b_6 - a_{n+1}).
\]
Since \(b_1, b_2\) and \(b_3\) cannot be zero simultaneously, assume without loss of generality that \(b_3 \neq 0\). Suppose \(\frac{b_2}{b_3}\) is not square. Now if we have two points \((x_0, y_0, p_3(x_0, y_0), \ldots, p_n(x_0, y_0))\) and \((x_1, y_1, p_3(x_1, y_1), \ldots, p_n(x_1, y_1))\) in \(S \cap A\), then
\[
b_1 x_0^2 + b_3 y_0^2 = b_1 x_1^2 + b_3 y_1^2
\]
\[
\implies b_1(x_1 - x_0)^2 = b_3(y_1 - y_0)^2.
\]
The above gives a contradiction to the assumption that \(\frac{b_2}{b_3}\) is not a square. Hence in this case there could be at most one point in \(S \cap A\).

Now assume \(\frac{b_2}{b_3} = \alpha^2\) for some \(\alpha \in F\). Any point in \(S \cap A\) satisfies the equation \(b_1 x^2 + b_3 y^2 = (a_{n+1} - b_6)\). This is equivalent to \(\alpha^2 x^2 + y^2 = \beta\), where \(\beta = \frac{a_{n+1} - b_6}{b_3}\). Suppose \((x_0, y_0, p_3(x_0, y_0), \ldots, p_n(x_0, y_0))\) is a point in \(S \cap A\), which implies \(\alpha^2 x_0^2 + y_0^2 = \beta\). Suppose \((x(t), y(t), p_3(x(t), y(t)), \ldots, p_n(x(t), y(t)))\) is in \(S \cap A\). For any \(t \in F\), let \(x(t) = x_0 + t\). Then the corresponding \(y(t)\) can be given as \((y(t))^2 = \beta - \alpha^2 (x(t))^2) = (\beta - \alpha^2 x_0^2 - \alpha^2 t^2) = y_0^2 + \alpha^2 t^2\). Therefore \(y(t) = y_0 + \alpha t\), and this shows that \(S \cap A\) gives an analytic curve.

\[\Box\]

Combining Lemma 6.1, Lemma 6.2 and Lemma 6.3 we get the following proposition.

**Proposition 6.1.** Let \(S = \{(x, y, p_3(x, y), \ldots, p_n(x, y)) \mid x, y \in U\}\), where \(U\) is an open set of \(F\), and each \(p_i(x, y)\) is a degree 2 polynomial and \(A\) be an affine rational hyperplane in \(F^n\). Suppose that \(S\) is not contained inside \(A\) then there are at most finitely many points of \(S \cap A\) such that its neighbourhood is not an \(F\)-analytic curve in \(F^n\).

By the above proposition we have that \(S \cap A \setminus J = \bigcup_{\gamma \in \gamma_j} \gamma(\mathbf{z})\), where \(J\) is a finite set of points which do not have an \(F\)-analytic curve, and \(\gamma(\mathbf{z})\) is an \(F\)-analytic curve which is a neighborhood of \(\mathbf{z}\) in \(S \cap A\). Also note that \(\gamma(\mathbf{z})\) is open and closed. Since \(S \cap A \setminus J\) is a second countable space, we know that there exists a countable subcovering \(\gamma_j\), i.e. \(S \cap A \setminus J = \bigcup_{i} \gamma_i\).

**Theorem 6.5.** Let \(S\) be as in the previous proposition. There exist \(\{L_i\}, \{L'_i\}, \{A_j\}\) as mentioned in Theorem 6.3, that satisfy property \(A\).
Proof. Let \( \{A_i\} \) be the set of affine rational hyperplanes normal to one of \( x \)-axis or \( y \)-axis in \( F^n \). By possibly replacing \( S \) with a smaller restriction on \( x \) and \( y \), we can ensure that for any \( \zeta \in S \), \( T_\zeta S \) is normal to \( x \) and \( y \)-axis. Now let us define \( L_i = S \cap A_i \), which are closed subsets and curves of \( S \).

Next we define \( \{L'_i\} \). For any affine rational hyperplane \( A \) that has a nonempty intersection with \( S \), by proposition 6.1, we have that \( S \cap A \) is union of \( \gamma_j \), excluding finitely many points. Let \( \{L'_i\} = \{\gamma_j : \forall i, \gamma_j \not\subset L_i\} \). Now we want to verify that these collections satisfy four hypotheses of Theorem 6.3.

Property (a) of Theorem 6.3 follows directly from how we defined these sets.

First let us consider those \( L_i = S \cap A_i \), where \( A_i : x = a, a \in \mathbb{F}_q(T) \). Now let us consider \( A_{i_k} : y = \frac{b}{a^k} \), where \( b \in \mathbb{F}_q[T] \). Since \( L_i = \{(a, y, p_3(a, y), \ldots, p_n(a, y)|y \in U\} \) or \( \{(x, a, p_3(x, a), \ldots, p_n(x, a)|x \in U\} \) and \( \{\frac{b}{a^k}|k \geq m\} \) for some \( m \in \mathbb{N} \) is dense in \( U \), property (b) follows.

Let \( F \) and \( F' \) as in hypothesis (c) of Theorem 6.3. Let
\[
L_i = \{(a, y, p_3(a, y), \ldots, p_n(a, y)|x, y \in U\},
\]
where \( a \in \mathbb{F}_q(T) \). Note that each \( L_i \cap L_{i_k} \) with \( k \in F \) is either empty or consists of only a single point. For any \( k' \in F' \), by the equation given above, \( L_i \cap L_{k'} = L_i \cap \gamma_{k'} \). We can write \( \gamma_{k'} \) is a subset of \( \frac{a_1x + a_2y + \sum_{i=3}^n a_ip_i(x, y) - a_{n+1} = 0} {a_1a + a_2y + \sum_{i=3}^n a_ip_i(a, y) - a_{n+1} = 0} \). Hence only finitely many solution is possible and \( L_i \cap L_{k'} \) has no interior. The same proof will work when \( L_i = A_i \cap S \) and \( A_i \) is normal to \( y \)-axis.

To verify property (d) of Theorem 6.3 it is enough to observe that \( S \cap A_i \) looks like \( \{(x, a, p_3(x, y), \ldots, p_n(x, y)|x, y \in U\} \) or \( \{(b, y, p_3(x, y), \ldots, p_n(x, y)|x, y \in U\} \), where \( a, b \in \mathbb{F}_q(T) \). Clearly they form a dense set in \( S \).\( \square \)

6.2. A special case in higher degree \( p(x, y) \).

**Proposition 6.2.** Let \( char(F) = p < +\infty \). Let \( S = \{(x, y, p(x, y))|x, y \in U\} \), where \( p(x, y) = \sum_{i=0}^m b_i^p x^{p^i} + \sum_{j=0}^n c_j^{p^j} y^{p^j} \) and \( U \) is an open subset in \( F \). There are at most finitely many points of \( S \cap A \) such that its neighbourhood is not an \( F \)-analytic curve in \( F^3 \).

**Proof.** Without loss of generality let us assume that \( S \cap A \) is nonempty, and that there exists at least one point of \( S \cap A \) whose neighbourhood is not an \( F \)-analytic curve in \( F^n \). This implies \( \alpha_3 \neq 0 \), and we can also assume \( \alpha_3 = 1 \) after normalization. This implies that \( a_1 + b_0 = a_2 + c_0 = 0 \). Suppose also without loss of generality that \( m \geq n, b_m \neq 0, \) and \( c_n \neq 0 \). The intersection \( S \cap A \) is given by \( b(x, y) = 0 \), where
\[
f(x, y) = \sum_{i=1}^m b_i^p x^{p^i} + \sum_{j=1}^n c_j^{p^j} y^{p^j} - a_4
\]
\[
= \left( \sum_{i=1}^m b_i^p x^{p^{i-1}} + \sum_{j=1}^n c_j^{p^j} y^{p^{j-1}} \right)^p - a_4
\]
Since \( S \cap A \) is nonempty \( a_4 \) must have a \( p \)-th root, say it is \( a_{4,0}^p = a_4 \). Therefore, we have that \( S \cap A \) is given by
\[
f_0(x, y) := \sum_{i=1}^m b_i^p x^{p^{i-1}} + \sum_{j=1}^n c_j^{p^j} y^{p^{j-1}} - a_{4,0}.
\]
If \( S \cap A \) has one point which has no neighbourhood that is an \( F \)-analytic curve, we have \( b_1 = c_1 = 0 \). Hence we can write \( f_0(x, y) = 0 \) as,

\[
f_0(x, y) = \left( \sum_{i=2}^{m} b_i y^{p_{i-2}} + \sum_{j=2}^{n} c_j y^{p_{j-2}} \right)^p - a_{4,0}.
\]

Since \( S \cap A \) is nonempty \( a_{4,0} \) must have a \( p \)-th root, say it is \( a_{4,1}^p = a_{4,0} \). Since \( f_0(x, y) = (f_1(x, y))^p \), where

\[
f_1(x, y) := \sum_{i=2}^{m} b_i y^{p_{i-2}} + \sum_{j=2}^{n} c_j y^{p_{j-2}} - a_{4,1},
\]

we have that \( S \cap A \) is defined by \( f_1(x, y) = 0 \).

We use an induction to derive the desired results. For any \( k \in \mathbb{N} \) satisfying \( 1 \leq k \leq n-1 \), \( b_1 = \cdots = b_{k-1} = 0 \), and \( c_1 = \cdots = c_{k-1} = 0 \), assume that \( S \cap A \) is given by \( f_k(x, y) = 0 \), where

\[
f_k(x, y) = \sum_{i=k+1}^{m} b_i y^{p_{i-1}} + \sum_{j=k+1}^{n} c_j y^{p_{j-1}} - a_{4,k}
\]

for some \( a_{4,k} \in F \). We have \( \partial f_k / \partial x = b_{k+1} \) and \( \partial f_k / \partial y = c_{k+1} \). By System (6.5), we derive that \( b_{k+1} = c_{k+1} = 0 \). Now since

\[
f_k(x, y) = \sum_{i=k+2}^{m} b_i y^{p_{i-1}} + \sum_{j=k+2}^{n} c_j y^{p_{j-1}} - a_{4,k}
\]

and by our assumption that \( S \cap A \) is nonempty, we know that \( a_{4,k} \) must have a \( p \)-th root, say it is \( a_{4,(k+1)} \in F \), i.e., \( a_{4,(k+1)}^p = a_{4,k} \). Define

\[
f_{k+1}(x, y) := \sum_{i=k+2}^{m} b_i y^{p_{i-1}} + \sum_{j=k+2}^{n} c_j y^{p_{j-1}} - a_{4,(k+1)}.
\]

Then

\[
f_k(x, y) = (f_{k+1}(x, y))^p.
\]

This implies that the intersection \( S \cap A \) is given by the equation \( f_{k+1}(x, y) = 0 \).

Repeat the steps above, and we see that the intersection \( S \cap A \) is given by \( f_{n-2}(x, y) = 0 \). Also from the induction above, we see that

\[
f_{n-2}(x, y) = c_{n} y^{p_n} - a_{4,n-2} + \sum_{i=n}^{m} b_i y^{p_{i-n+1}}.
\]

This tells us that \( S \cap A \) is completely given by the curve

\[
\{(x, g(x), p(x, g(x))) \mid x \in U\},
\]

where

\[
g(x) := \frac{1}{c_n^{p_n}} \left( a_{4,n-2} - \sum_{i=n}^{m} b_i y^{p_{i-n+1}} \right),
\]

which is certainly analytic.
The exact same proof as Theorem 6.5 with suitable changes gives the following theorem.

**Theorem 6.6.** Let \( S = \{(x, y, \sum_{i=0}^{m} b_i^i x^i + \sum_{j=0}^{n} c_j^j y^j)\} \), where \( x, y, b_i, c_j \in F \), and not all \( b_i \)'s or \( c_j \)'s are being zero. Then there exists \( \{L_i\}, \{L_j\}, \{A_j\} \) that satisfies property A.

We now have everything we need to prove the main theorem of this section:

**Proof of Theorem 6.2.** Theorem 6.5 and Theorem 6.6 guarantee that the hypotheses of Theorem 6.3 are met by surfaces considered in Theorem 6.2. Therefore by Theorem 6.3, Theorem 6.2 follows. \( \square \)

### 6.3. Surface to higher dimensional manifold.

The following theorem is somewhat an analogue to Lemma 3.5 in [40].

**Theorem 6.7.** Suppose that for \( k \geq 3 \) and \( B(0,1) \) is the open and closed ball in \( F \) of radius 1. Let \( M \) be a \( k \)-dimensional submanifold of \( F^n \) which is the image of \( B(0,1)^k \) under an immersion \( f : B(0,1)^k \to F^n \). Suppose that \( (f, \lambda_k) \) is nonplanar, where \( \lambda_k \) is the Lebesgue measure in \( B(0,1)^k \). Then there exists \( y \in B(0,1)^{d-2} \) such that the surface \( M_y := f_y(B(0,1)^2) \) is not contained inside an rational affine hyperplane, where \( f_y : B(0,1)^2 \to F^n \), \( f_y(x_1, x_2) = f(x_1, x_2, y) \).

**Proof.** We will prove by contradiction. Suppose for every \( y \in B(0,1)^{k-2} \) there exists a rational affine subspace \( A_y \) such that \( B(0,1)^2 \times \{y\} \subset f^{-1}(A_y \cap M) \implies B(0,1)^k = \bigcup A \) is a rational affine subspace \( f^{-1}(A \cap M) \). By Baire category theorem there is one \( A \) such that \( f^{-1}(A \cap M) \) contains an open ball inside \( B(0,1)^k \). This contradicts the fact that \( (f, \lambda_k) \) is nonplanar. \( \square \)

**Remark 10.**

1. In the above theorem we don’t need to consider the manifold to be analytic but we need a stronger assumption that \( (f, \lambda) \) is nonplanar as compared to the manifold being not inside an affine hyperplane.
2. The above theorem shows it is enough to prove Theorem 2.5 for surfaces.

### 6.4. Product of perfect sets.

Let us recall that a subset of \( F \) is called perfect if it is compact and has no isolated points. If \( L = \{y \in F^n \mid \sum_{i=1}^{n} a_i y_i = a_0\} \), we can define \( |L| := \max_{i=1}^{n} |a_i| \). The proof of the next proposition will be exactly the same as the proof of Theorem 1.6 in [40].

**Proposition 6.3.** Let \( n \geq 2 \) and let \( S_1, \ldots, S_n \) be perfect subsets of \( F \) such that \( F_q(T) \cap S_1 \) is dense in \( S_1 \) and \( F_q(T) \cap S_2 \) is dense in \( S_2 \). Let \( S = \prod_{j=1}^{n} S_j \). Then there exist a collection of \( \{L_i\}, \{L_j\}, \{A_i\} \) of \( S \) that satisfy property A.

Thus we have the following theorem combining Theorem 6.3 and Proposition 6.3.

**Theorem 6.8.** Let \( n \geq 2 \) and let \( S_1, \ldots, S_n \) be perfect subsets of \( F \) such that \( (F_q(T) \cap S_1) \) is dense in \( S_1 \) and \( (F_q(T) \cap S_2) \) is dense in \( S_2 \). Let \( S = \prod_{j=1}^{n} S_j \). Then there exists uncountably many totally irrational singular vectors in \( S \).
7. What’s next?

7.1. For a $n \times m$ matrix $A$ over $\mathcal{K}$, we define singularity as follows:

**Definition 7.1.** Suppose $\mathcal{K} = \mathbb{R}$ (resp. $\mathbb{F}_q((T^{-1}))$) and $\Lambda = \mathbb{Z}$ (resp. $\mathbb{F}_q[T]$). Let $A$ be a $n \times m$ matrix over $\mathcal{K}$. We define $A$ to be singular if for every $c > 0$, for all sufficiently large $Q > 0$ there exist $0 \neq q \in \Lambda^n$, $q_0 \in \Lambda^m$ satisfying the following system of inequalities,

\[
\begin{align*}
|qA + q_0|_\mathcal{K}^n &< cQ^n, \\
\|q\|_{\mathcal{K},\infty} &\leq Q.
\end{align*}
\]  

(7.1)

If $\mathcal{L}$ is an $s$-dimensional affine subspace of $\mathcal{K}^n$, one can choose a parametrization $x \rightarrow (x, xA_0 + a_0)$, where $A_0$ is a $n \times (n - s)$ matrix and $a_0 \in \mathcal{K}^{n-s}$. Here both $x$ and $a_0$ are row vectors. We rewrite the parametrization above as $x \rightarrow (x, x\tilde{A})$, where $\tilde{A} = \begin{bmatrix} a_0 \\ A_0 \end{bmatrix}$ and $\tilde{x} = (1, x)$. Following the [35, 36] it is easy to show that if $A$ is singular then all vectors in $\mathcal{L}$ are singular. This springs the following question:

**Question 7.1.** If $A$ is not singular then does the set of singular vectors in $\mathcal{L}$ have measure zero?

Following methods in [36] it should be possible to answer the above question for affine hyperplanes. But for any affine subspace the question remains very interesting. The authors expect techniques from [29] can be useful in answering the above question.

7.2. We propose the following conjecture which is a function field analogue of Corollary 2.3.

**Conjecture 7.1.** Let $\mu$ be an absolutely decaying and Federer (more generally locally finite) on $\mathbb{F}_q((T^{-1}))^d$, $\mathcal{L}$ be an affine subspace of $\mathbb{F}_q((T^{-1}))^n$, and let $f : \mathbb{F}_q((T^{-1}))^d \rightarrow \mathcal{L}$ be a smooth map which is nondegenerate in $\mathcal{L}$ at $\mu$-a.e. point of $\mathbb{F}_q((T^{-1}))^d$. Suppose that $\lambda_\mathcal{L}$ is the Haar measure on $\mathcal{L}$ Then the following are equivalent:

- There exists one $y \in \text{supp}(\mu)$ such that $f(y)$ is not singular.
- There exists one $y \in \mathcal{L}$ which is not singular.
- For $\lambda_\mathcal{L}$ almost every $y \in \mathcal{L}$, $y$ is not singular.
- For $\mu$ almost every $y$, $f(y)$ is not singular.

One of the crucial fact that is used to deduce Corollary 2.3 from Theorem 2.2 is that if $\mu$ is absolutely decaying, Federer and $f$ is nondegenerate at $\mu$-a.e. point of $\mathbb{R}^d$, then $(f, \mu)$ is good, [38] §7. The proof of the above-mentioned fact uses mean value theorem extensively. So one needs to come up with an alternative to prove §7 of [38] over function field. Then that can be used to deduce the Conjecture 7.1 from Theorem 2.2.

7.3. When $\mathcal{K} = \mathbb{Q}_\nu$ or finite extensions of $\mathbb{Q}_\nu$, it would be interesting to prove analogue of Theorem 2.1, improving Theorem 2.3. The techniques developed in this paper can be used to prove analogue results in number fields.

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