Skyrmions from Instantons inside Domain Walls

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Abstract

Some years ago, Atiyah and Manton described a method to construct approximate Skyrmion solutions from Yang-Mills instantons. Here we present a dynamical realization of this construction using domain walls in a five-dimensional gauge theory. The non-abelian gauge symmetry is broken in each vacuum but restored in the core of the domain wall, allowing instantons to nestle inside the wall. We show that the worldvolume dynamics of the wall is given by the Skyrme model, including the four-derivative term, and the instantons appear as domain wall Skyrmions.
Introduction

Consider an $SU(2)$ gauge potential $A_\mu$ over Euclidian $\mathbb{R}^4$. The path ordered holonomy along, say, the $x^4$ axis defines a group valued field $g(x)$ over $\mathbb{R}^3$.

\[ g(x) = \mathcal{P} \exp \left(i \int_{-\infty}^{+\infty} dx^4 A_4(x, x^4) \right). \tag{1} \]

We require that $A_\mu$ decays suitably rapidly as $x \to \infty$ so that $g(x) \to 1$. Then, regarding $g(x)$ as the Skyrme field, Yang-Mills configurations in $\mathbb{R}^4$ with instanton number $I$ map to Skyrme configurations in $\mathbb{R}^3$ with baryon number $B = I$.

Equation (1) does not map solutions of the Yang-Mills equations to solutions of the Skyrme equations. Indeed, while the explicit form of the instanton is well known, no analytic solution exists for the Skyrmion. Nevertheless, Atiyah and Manton showed that the Yang-Mills instanton gives a remarkably good approximation to the Skyrmion [1]. More precisely, the moduli space of $SU(2)$ instantons of charge $I$ has dimension $8I$. After losing the translational mode along $x^4$, the map (1) gives an $8I - 1$ dimensional space of Skyrme configurations. Minimizing the Skyrme energy functional over this space results in an approximate solution which, in the case of a single $B = I = 1$ Skyrmion, has energy about 1% above the true solution. This construction has been studied for higher $B$ and extended in other directions in [2–7].

The purpose of this paper is to provide a dynamical realization of this correspondence, with instantons appearing as “domain wall Skyrmions”. We construct a domain wall in a five dimensional Yang-Mills-Higgs theory whose low-energy dynamics is described by the Skyrme model, complete with four-derivative term. We show that the Skyrmions on the domain wall worldvolume are instanton particles in five dimensions, taking refuge inside the domain wall from the broken gauge symmetry of the bulk.

There has been some interest of late in realizing skyrmions as instantons in a higher dimensional space, including models using orbifold deconstruction [8, 9] and the AdS/CFT correspondence [10].

A Five Dimensional Theory

Our starting point is a five dimensional $U(2)$ gauge theory with a single real scalar field $\phi$ transforming in the Lie algebra valued adjoint representation of the gauge group, and four complex scalar fields $q_i, i = 1, \ldots, 4$, each transforming in the fundamental representation. The Lagrangian for our system is

\[ \mathcal{L} = \frac{1}{2e^2} \text{Tr}(-\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + (\mathcal{D}_\mu \phi)^2) + \sum_{i=1}^{4} (|\mathcal{D}_\mu q_i|^2 - q_i^\dagger (\phi - m_i)^2 q_i) - \frac{e^2}{2} \text{Tr}(\sum_{i=1}^{4} q_i \otimes q_i^\dagger - v^2)^2 \]
where $\mu = 0,1,\ldots,4$ as befits $d = 4 + 1$ Minkowski space. Both the masses $m_i$ and the Higgs vacuum expectation value $v^2$ are real parameters that are understood to come with an implicit $2 \times 2$ unit matrix. (e.g $v^2 = v^2 1_2$). If the masses $m_i$ are set to zero, then this Lagrangian enjoys an $SU(4)_F$ flavor symmetry, rotating the $q_i$. For generic $m_i$ this is broken to $U(1)^2_F$. In this paper we will be interested in an intermediate situation in which $m_1 = m_2 = m$, and $m_3 = m_4 = -m$. The surviving flavor symmetry in this case is $S[U(2)_{F_1} \times U(2)_{F_2}]$. For notational purposes it will prove useful to separate these fundamental fields into two pairs: $Q_1 = (q_1, q_2)$ and $Q_2 = (q_3, q_4)$. We will write each of these as $2 \times 2$ matrices, $(Q_1)^a_i$ and $(Q_2)^a_i$, with $a = 1,2$ the gauge index and $i = 1,2$ denoting flavor. Then the action of the $U(2)_G \times S[U(2)_{F_1} \times U(2)_{F_2}]$ gauge and flavor symmetries on the scalars is given by

$$\phi \rightarrow U\phi U^+, \quad Q_1 \rightarrow UQ_1V_1^+, \quad Q_2 \rightarrow UQ_2V_2^+ \quad (2)$$

where $U \in U(2)_G$ is a gauge symmetry and $V_i \in U(2)_{F_i}$ are flavor symmetries.

Domain walls in this theory with degenerate masses were studied previously by Shifman and Yung [11] and further in [12]. We will be interested in walls interpolating between two specific vacua of the theory, each of which is isolated with a mass gap. They are:

- Vacuum 1: $\phi = m 1_2$, $Q_1 = v 1_2$ and $Q_2 = 0$. In this vacuum the $U(2)_G \times SU(2)_{F_1}$ is spontaneously broken to the diagonal $SU(2)_{\text{diag}}$, while the $U(2)_{F_2}$ flavor symmetry survives.

- Vacuum 2: $\phi = -m 1_2$, $Q_1 = 0$ and $Q_2 = v 1_2$. In this vacuum the $U(2)_G \times SU(2)_{F_2}$ is spontaneously broken to the diagonal $SU(2)_{\text{diag}}$, while the $U(2)_{F_1}$ flavor symmetry survives.

Notice that each of these vacua lies in a ”color-flavor” locked phase, where a flavor rotation requires a matching color rotation. However, a different flavor symmetry is locked with color in each vacuum. This will prove important in the following.

There is also a third branch of vacua in which $Q_1$ and $Q_2$ are both non-zero and there is no mass gap. Although these vacua play an interesting role in the dynamics of domain walls [11], they will not be of immediate interest for our story and will enter our discussion only in passing.

**The Domain Wall**

Various parameters in our Lagrangian have been finely tuned so that the theory admits a completion to one with eight supercharges. However, supersymmetry will play no crucial role in this paper and all arguments below depend only on the flavor and gauge symmetries
of the model. Nevertheless, we stick with the supersymmetric theory since domain walls are somewhat simpler in this context.

We will examine the properties of the domain wall interpolating between the two vacua described above in, say, the $x^4$ direction. The worldvolume of these domain walls is $\mathbb{R}^{1,3}$ Minkowski space, spanned by $x^\alpha$ with $\alpha = 0, 1, 2, 3$ and to find static configurations we set $\partial_\alpha = A_\alpha = 0$. One of the advantages of working with the supersymmetric theory is that the second order equations of motion can be integrated once to yield first order Bogomolnyi equations,

$$
\mathcal{D}_4 \phi = e^2 (Q_1 Q_1^\dagger + Q_2 Q_2^\dagger - v^2 1_2) , \quad \mathcal{D}_4 Q_1 = (\phi - m) Q_1 , \quad \mathcal{D}_4 Q_2 = (\phi + m) Q_2 \quad (3)
$$

with boundary conditions imposed so that the fields asymptote to Vacuum 1 as $x^4 \to -\infty$ and Vacuum 2 as $x^4 \to +\infty$. Solutions to these equations are domain walls with tension given by $T_{\text{wall}} = 4v^2m$.

The equations (3) have an interesting space of solutions but there are a particularly simple subset of solutions that will interest us here [11]. We can satisfy the $U(2)$ group structure of these equations by setting all fields proportional to the $2 \times 2$ unit matrix,

$$
\phi = \phi(x^4) 1_2 , \quad Q_1 = Q_1(x^4) 1_2 , \quad Q_2 = Q_2(x^4) 1_2 \quad \text{and} \quad A_4 = 0 \quad (4)
$$

in which case the equations (3) reduce to those of an abelian gauge theory studied previously in [13–16]. The spatial profile of this domain wall is rather interesting and depends on the dimensionless\(^1\) ratio $m^2/e^2v^2$. When $e^2v^2 \ll m^2$, the wall exhibits a three layer structure shown in the figure. The fundamental fields $Q_1$ and $Q_2$ drop to zero over a

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\(^1\)Recall that in $d = 4 + 1$, the engineering dimensions of the various quantities are $[e^2] = -1$, $[v^2] = 3$ and $[m] = 1$. 

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thin shell of width 1/ev. The scalar $\phi$ takes a more leisurely path, interpolating between its expectation values $\pm m$ over a length $L_{\text{wall}} \sim 2m/e^2v^2$. Note that in the middle of the domain wall, the $U(2)$ gauge symmetry is naively restored. Of course, this isn’t true over distances larger than $L_{\text{wall}}$ for which long range non-abelian fields are affected by the boundary conditions imposed by the bulk. In contrast, in the other limit $e^2v^2 \gg m^2$, the three layer structure is lost and all fields vary over a length $1/m$.

Having found one solution using the ansatz (4), we may now act with the $U(2)_G \times S[U(2)_{F_1} \times U(2)_{F_2}]$ symmetry action to sweep out further solutions. The transformation of the scalar fields is given in (2) while the requirement that the vacua at left and right infinity are untouched translates into the boundary condition on the gauge transformation $U \in U(2)_G$,

$$U(x^4) \rightarrow \begin{cases} V_1 & x^4 \rightarrow -\infty \\ V_2 & x^4 \rightarrow +\infty \end{cases}.$$  \hfill (5)

Under such a gauge transformation, the $A_4$ component of the gauge field also picks up a contribution,

$$A_4(x^4) = -i(\partial_4 U)U^\dagger.$$  \hfill (6)

Because the gauge symmetry is locked with a different flavor symmetry at left and right infinity, a fact reflected in the boundary condition (5), the holonomy of $A_4$ along the $x^4$ direction transverse to the wall is non-vanishing. We have,

$$g = \mathcal{P} \exp \left(i \int_{-\infty}^{+\infty} dx^4 A_4(x^4) \right) = V_2 V_1^{-1}.$$  \hfill (7)

This $U(2)$ valued field $g$ plays the role of the gauge invariant coordinate [11] on the moduli space $\mathcal{M}$ of the domain wall,

$$\mathcal{M} \cong \frac{U(2)_{F_1} \times U(2)_{F_2}}{U(2)_G} \cong U(2).$$  \hfill (8)

The solution of the form (4) corresponds to the point $g = 1$. All other solutions on $\mathcal{M}$ are related by a symmetry transformation.

The domain wall has further moduli. There is, of course, the overall center of mass of the wall. There are also zero modes corresponding to the domain wall splitting into two walls, each with tension $\frac{1}{2}T_{\text{wall}}$, and with the third branch of vacua mentioned above lying between them. We may consistently ignore these moduli for the purpose of our story. In particular, the moduli corresponding to splitting into two walls would not exist in a non-supersymmetric theory.

Note the similarity between the coordinate on the moduli space of domain walls (7) and the Atiyah-Manton equation (1). The difference is that the collective coordinate $g$ of
the domain wall defined in (7) does not yet depend on the wall worldvolume coordinates \( x^\alpha \) with \( \alpha = 0, 1, 2, 3 \). We shall now rectify this.

*The Skyrme Model on the Domain Wall*

In the spirit of the moduli space approximation, we promote the collective coordinates \( g \) to fields on the domain wall worldvolume: \( g \to g(x^\alpha) \), with \( \alpha = 0, 1, 2, 3 \). The low-energy dynamics on the domain wall is derived by inserting this varying collective coordinate back into the action, together with a suitable ansatz for the gauge fields \( A_\alpha \), parallel to the domain wall, which are sourced by \( \partial_\alpha g \). We now describe this procedure.

Given the symmetries of our model, the low-energy dynamics of the collective coordinate \( g \) at the two-derivative level is fixed to be the \( U(2) \) chiral Lagrangian. We need only determine the overall coefficient \( "f_2^2/16" \). This can be achieved by studying the fluctuations about the simplest solution (4), corresponding to the point \( g = 1 \) in moduli space. The zero modes may be written as

\[
\delta \phi = i[\Gamma, \phi] \quad , \quad \delta Q_i = i\Gamma Q_i - iQ_i\Omega_i \quad \text{and} \quad \delta A_4 = \mathcal{D}_4\Gamma
\]

where \( U(x^4) = e^{i\Gamma(x^4)} \) is the gauge transformation and \( V_i = e^{i\Omega_i} \) are the two flavor transformations. The gauge transformation is restricted to obey the boundary conditions (5) which, in infinitesimal form, read,

\[
\Gamma(x^4) \to \begin{cases} 
\Omega_1 & x^4 \to -\infty \\
\Omega_2 & x^4 \to +\infty 
\end{cases}.
\]

The full \( x^4 \) dependence of the gauge transformation \( \Gamma(x^4) \) is determined by a suitable background gauge fixing condition which, for the fluctuations (9), reads

\[
\mathcal{D}_4^2\Gamma - [\phi, [\phi, \Gamma]] = e^2 \sum_{i=1}^2 \left( \{Q_iQ_i^\dagger, \Gamma\} + 2Q_i\Omega_iQ_i^\dagger \right).
\]

Since a rotation of the form \( V_1 = V_2 \) can always be undone by a gauge transformation, we may choose to work in a gauge in which \( V_1 = V_2^\dagger \), so that \( \Omega_1 = -\Omega_2 \equiv \Omega \) and \( g \approx 1 - 2i\Omega \). Then the above equation has the simple solution \( \Gamma(x^4) = -(\phi(x^4)/m)\Omega \), and the zero modes read,

\[
\delta \phi = 0 \quad , \quad \delta Q_1 = i\frac{q_1}{m}(\phi - m)\Omega \quad , \quad \delta Q_2 = i\frac{q_2}{m}(\phi + m)\Omega \quad \text{and} \quad \delta A_4 = -\frac{\partial_4 \phi}{m}\Omega.
\]

The coefficient in front of the two-derivative terms of the \( U(2) \) chiral Lagrangian is determined by the overlap of these zero modes, yielding

\[
\mathcal{L}_2 = -\frac{v^2}{4m} \text{Tr} (g^{-1}\partial_\alpha g)(g^{-1}\partial^\alpha g).
\]
This result for the low-energy dynamics of the domain wall was previously derived in [11].

For solitons with worldvolume dimension greater than one, there is an extra term in the moduli space approximation that is usually ignored since it is subleading in the derivative expansion. It comes from the field strength $F_{\alpha\beta}$, with $\alpha$ and $\beta$ indices along the soliton worldvolume. To see this, we first need the equation of motion for the gauge fields $A_{\alpha}$

$$
\mathcal{D}_\mu F_{\alpha\mu} - i[\phi, \mathcal{D}_\alpha \phi] = ie^2 (Q_i \mathcal{D}_\alpha Q_i^\dagger - (\mathcal{D}_\alpha Q_i)Q_i^\dagger) .
$$

This equation is solved by $A_{\alpha} = -[1 + (\phi/m)] \partial_\alpha \Omega$, where the constant piece has been chosen so that $A_{\alpha}$ is pure gauge at $x^4 = \pm\infty$. With this ansatz, we have $\mathcal{D}_\alpha Q_i = \delta Q_i \partial_\alpha \Omega$, with the zero mode given by (12) and one can check that the equation above reduces to that of (11). While this form of $A_{\alpha}$ holds for small fluctuations about $g = 1$, the generalization to arbitrary $g$ is simply [11]

$$
A_{\alpha} = \frac{i}{2} \left[ 1 + \frac{\phi(x^4)}{m} \right] g^{-1} \partial_\alpha g
$$

which, indeed, becomes pure gauge as $x^4 \to -\infty$ and vanishes as $x^4 \to +\infty$, ensuring that any field strength is localized on the wall. To see how it leads to a four-derivative term, we compute the field strength

$$
F_{\alpha\beta} = \frac{i}{4} \left( 1 - \frac{\phi^2}{m^2} \right) [g^{-1} \partial_\alpha g, g^{-1} \partial_\beta g] .
$$

Inserting this into the original five-dimensional Lagrangian and integrating over $x^4$ gives rise to the promised four-derivative term,

$$
\mathcal{L}_{4-\phi} = c \text{ Tr} [(g^{-1} \partial_\alpha g, (g^{-1} \partial_\beta g)]^2
$$

where the coefficient $c$ is given by the integral,

$$
c = \frac{1}{64e^2} \int_{-\infty}^{+\infty} \left( 1 - \frac{\phi(x^4)^2}{m^2} \right)^2 dx^4 .
$$

2 A similar discussion holds for other solitons in gauge theories — for example, vortices, monopoles or instantons — when their worldvolume dimension is greater than one. In general, if a soliton has collective coordinates $X^a$ then the zero modes are a combination of variations with respect to $X^a$ together with an infinitesimal gauge transformation, e.g. $\delta_a A_\mu = \partial A_\mu / \partial X^a + \mathcal{D}_\mu \epsilon_a$ where $A_\mu$ are the gauge fields transverse to the soliton worldvolume. The equations of motion for the gauge fields $A_{\alpha}$ parallel to the soliton worldvolume are solved to leading order by $A_{\alpha} = \epsilon_a \partial_\alpha X^a$, resulting in the field strength

$$
F_{\alpha\beta} = (\partial \epsilon_a / \partial X^b - \partial \epsilon_b / \partial X^a - i[\epsilon_a, \epsilon_b]) \partial_\alpha X^a \partial_\beta X^b .
$$

The curvature term in brackets is part of the "universal bundle" defined in [17]. Some properties of this object are detailed in [18]. Inserting this expression into the action will lead to four-derivative terms.
Parametrically, we see that $c \sim L_{\text{wall}}/e^2$. The numerical prefactor can be computed in the limit $e^2 v^2 \ll m^2$ where the domain wall has the three layer structure shown in figure 1. Up to corrections of order $e v/m$, arising from the two outer layers, the profile of $\phi$ can be taken to be piecewise linear,

$$\phi(x^4) \approx \begin{cases} 
  m & x^4 < -\frac{m}{e^2 v^2} \\
  -e^2 v^2 x^4 & -\frac{m}{e^2 v^2} < x^4 < \frac{m}{e^2 v^2} \\
  -m & x^4 > \frac{m}{e^2 v^2} 
\end{cases}$$

(19)

which gives us

$$c \approx \frac{1}{60} \frac{m}{e^4 v^2}.$$ 

(20)

Note that terms with four derivatives and higher will also typically appear if we go beyond the moduli space approximation, taking into account the backreaction of the motion $\partial_\alpha X^a$ on the fields themselves. The computation of these is a much harder problem that we do not attempt here.

Domain Wall Skyrmions

One can see at a glance from the figure that, inside the domain wall, the $U(2)$ gauge symmetry is restored. One may suspect that this allows us to place an instanton, which in five-dimensions is a particle-like soliton, inside the domain wall as long as its size is smaller than $L_{\text{wall}} \sim 2m/e^2 v^2$. In a supersymmetric theory, the instanton embedded within the domain wall is not BPS [19]. A related fact is that such a configuration is not necessarily a stable solution because the long range fields of the instanton are subject to the boundary conditions imposed by the bulk. Since the vacuum is in the Higgs phase, these boundary conditions are those of a five-dimensional superconductor, ensuring that the magnetic field $F_{\mu\nu}$ with $\mu, \nu = 1, 2, 3, 4$ must lie parallel to the surface of the wall: $\partial_4 F_{\mu\nu}$ at the boundary. We can ask how such a configuration is viewed from the domain wall theory.

Far from the core of the instanton, up to a power-law tail, the gauge field configuration is pure gauge. We may write $A_\mu = -i(\partial_\mu U)U^\dagger$, $\mu = 1, 2, 3, 4$. The instanton number $I$ is given by the integral

$$I = \frac{1}{24\pi^2} \int_{R^3_3 - R^3_1} d^3 x \text{ Tr } \left[ (\partial_\nu U)U^{-1}(\partial_\rho U)U^{-1}(\partial_\sigma U)U^{-1} \right] \epsilon^\mu\nu\rho\sigma$$

(21)

where we evaluate the integral over the two left and right boundaries $R^3_{L/R}$ of the domain wall. (These actually have width equal to the penetration depth $1/ev$). But we’ve seen above that such a large gauge transformation must be compensated by a flavor transformation outside the domain wall in order to leave the vacuum invariant. Thus $U \to V_1$
on $\mathbb{R}^3_L$ and $U \to V_2$ on $\mathbb{R}^3_R$. We may work in gauge in which $V_1 = 1$, while $V_2$ varies, in which case the integral above does not pick up a contribution from $\mathbb{R}^3_L$ while, on $\mathbb{R}^3_R$, $U = V_2 = g$, giving us

$$I = \frac{1}{24\pi^2} \text{Tr} \int d^3x \left[ (\partial_\alpha g)g^{-1}(\partial_\beta g)g^{-1}(\partial_\gamma g)g^{-1} \right] \epsilon^{\alpha\beta\gamma} = B \quad (22)$$

where $\alpha, \beta, \gamma$ run over 1, 2, 3, the spatial indices of the domain wall. The above expression measures the baryon number $B$ of the Skyrme model, reiterating the result of Atiyah and Manton [1]: a Yang-Mills configuration of instanton charge $I$ maps into a Skyrme configuration with baryon number $B = I$.

So much for topological charges. The real surprise of Atiyah and Manton is that the instanton solution gives such a good approximation to the Skyrmion. How can we understand this from our set-up? The Skyrmion solution balances the two-derivative (13) and four derivative (17) terms against each other, so that they are comparable. This means that the fields of the Skyrmion vary with wavelength,

$$L_{\text{Skyrmin}} \sim (\partial g)^{-1} \sim m/e^2 v^2 \sim L_{\text{wall}} \quad (23)$$

suggesting that an instanton much smaller than $L_{\text{wall}}$ will expand until it just touches the wall. Indeed, Atiyah and Manton showed the Skyrme functional is minimized by an instanton of a particular size which, from our perspective, is the width of the wall $L_{\text{wall}}$.

Let’s look at this in more detail. There is a Bogomolnyi type argument bounding the mass of the Skyrmion: $M_{\text{Skyrme}} \geq M_{\text{bound}}$. However, unlike BPS solitons, the bound is never saturated. Numerical study reveals that for a single $B = 1$ Skyrmion, $M_{\text{Skyrme}} \sim 1.2 \times M_{\text{bound}}$. Using the scaling above, and working in the $e^2 v^2 \ll m^2$ limit, the bound on the Skyrmion mass is

$$M_{\text{bound}} \approx 48\pi^2 \sqrt{\frac{v^2}{4m} \frac{m}{60e^4 v^2}} \approx \frac{4.4\pi^2}{e^2} I \approx 1.1 \times M_{\text{instanton}} \quad (24)$$

where, in our conventions, the mass of a single instanton is $M_{\text{inst}} = 4\pi^2/e^2$. We see that the true mass of the Skyrmion in the domain wall is about 30% higher than that of the instanton in Coulomb phase. Nevertheless, the fact that the Atiyah-Manton construction (1) results in a good approximation to the Skyrmion at the 1% level suggests that the instanton solution is not greatly deformed by its residence inside the domain wall. It would be interesting to use this perspective to quantify the Atiyah-Manton approximation.

Note that the above analysis is from the perspective of the domain wall and its validity becomes dubious for instantons much smaller than $L_{\text{wall}}$ since, for $(\partial g)^{-1} \sim L_{\text{wall}}$, one might worry about higher derivative terms in the effective Lagrangian. This is the standard drawback of the Skyrme model: if the two and four derivative terms are of
equal parametric importance, then so are all higher derivatives. In the present context, our small dimensionless parameter \( e^2 v^2 / m^2 \) may mitigate matters. Dimensional analysis requires higher derivative terms to be of the form \( b_n m v^2 (\partial / m)^{2n} \), where \( b_n \) is a dimensionless coefficient. However, to compete with the two and four derivative terms, one would need \( b_n \sim (m^2 / e^2 v^2)^{2n-2} \) and it is not clear that such large coefficients arise. Further study of this issue would be worthwhile.

**Generalizations**

There exists an obvious generalization of our analysis to higher gauge groups \( SU(N) \). One starts with a five dimensional \( U(N) \) gauge theory with \( 2N \) flavors. As above, the first \( N \) flavors are assigned mass \( (\phi - m) \), with the remaining assigned mass \( (\phi + m) \). There exists a domain wall interpolating between the two vacua \( \phi = \pm m 1_N \), whose low-energy dynamics is described by the \( SU(N) \) Skyrme model, complete with four derivative term. As above, the instantons of the gauge theory descend to Skyrmions on the domain wall.

For \( SU(N) \) gauge theories with \( N \geq 3 \), there exists a further term that one can add to the five dimensional action: a Chern-Simons coupling, with gauge invariance enforcing an integer valued coefficient \( k \) [20]. How does such a term appear in the domain wall worldvolume theory? The answer is that induces a Wess-Zumino term for the Skyrme model. As is well known, a Lagrangian for such a term usually involves introducing a five-manifold \( M \) with boundary \( \partial M \cong 1^3 \) [21]. From our perspective, the five dimensional manifold appears naturally as the bulk, with the domain wall acting as the boundary. Such a term determines the spin properties of the Skyrmion: fermion for \( k \) odd, boson for \( k \) even. It should be possible to repeat this soliton quantization from the perspective of the instanton in five dimensions.

Finally, let us mention that a Chern-Simons term is also induced in five dimensional gauge theories by integrating out massive Dirac fermions. If these also couple to the scalar field \( \phi \), then two effects occur: firstly, there are chiral, fermionic zero modes localized on the domain wall. Secondly, they induce a term of the form \( \phi F_{\mu \nu} F^{\mu \nu} \), which is related to the Chern-Simons term by supersymmetry. It would be interesting to understand the interplay of these two effects in the dynamics of the domain wall.
Acknowledgement

We would like to thank Sean Hartnoll, Matt Headrick, Youichi Isozumi, Nick Manton and Norisuke Sakai for useful discussions. The work of M. N. and K. O. (M. E.) is supported by Japan Society for the Promotion of Science under the Post-doctoral (Predoctoral) Research Program. D. T. is supported by the Royal Society and thanks the Benasque and Aspen centers for physics for hospitality while this work was completed.

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