A Note on ”Global solutions for nonlinear fuzzy fractional integral and integrodifferential equations”

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Abstract

The authors in [1] have given two examples to illustrate their results in which they have been eliminated the technical details. However, the authors in [4] claimed that the examples are incorrect. In fact they conjectured that the authors in [1] have employed the incorrect statement
\(x - x = 0\) for \(x \in \mathbb{R}_F \setminus \mathbb{R}\) to construct the examples. Here we intend to observe that the basic method used in [1] to prove the validity of the examples is the well-known L-U representation of a fuzzy-number valued function. In this sense, we will make use of the \(\alpha\)-level sets and show that the examples are correct.

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1 Introduction

The authors in [1] have considered the following fuzzy fractional integrodifferential equations (FFIDEs) involving Riemann-Liouville derivatives of order \(0 < q < 1\),

\[
D^q u(t) = f(t, u, Tu), \quad \forall \, t \in J,
\]

\[
\lim_{t \to 0^+} t^{1-q}u(t) = u_0,
\]

where \(J = (0, b], u_0 \in \mathbb{R}_F, f \in C(J \times \mathbb{R}_F^2, \mathbb{R}_F)\), \(D^q u\) denotes fuzzy derivative of fractional order \(q\) introduced in [1], and

\[
(Tu)(t) = \int_0^t k(t, s)u(s)ds,
\]

where \(k \in C(I, \mathbb{R}_+), I = \{(t, s) \in J \times J : t \geq s\}, \mathbb{R}_+ = [0, +\infty)\) and \(0 < q < 1\). To prove the existence theorems, they applied concepts of the upper and lower solutions for more generic-following fuzzy fractional integral equation

\[
u(t) = g(t) + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, u(s), (Tu)(s))ds, \quad t \in J,
\]

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where \( J = (0, b] \), \( g \in C(J, \mathbb{R}_F) \), \( f \in C(J \times \mathbb{R}^2_F, \mathbb{R}_F) \). They combined these concepts with the monotone iterative technique and proved the existence of solutions for Problem (1.1). They have given two examples to illustrate the method in which they gave the upper and lower solutions and the exact solution corresponded with examples. However, while the proof of details was almost clear, then they eliminated them. On the other hand the authors in [2] have asserted that there exist some errors in the given examples in [1], and the given exact solutions are incorrect. In fact they supposed the authors in [1] have employed the incorrect expression \( x - x = \hat{0} \) for \( x \in \mathbb{R}_F \setminus \mathbb{R} \) to prove the details of their assertion. In this study we recall the definition of the solution for Problem (1.1) in [1], and we prove the exact solutions by verify the corresponding problems using \( \alpha \)-level sets not by applying the incorrect expression \( x - x = \hat{0} \). Also the authors in [3] have asserted these solutions do not lie between the lower and upper solutions and have given some examples to prove it. In this paper we will state that according to Theorem 4.5-4.6 in [1], it is not necessary that all of the solutions lie between the lower and upper solutions, it is true if this happens just for some solutions not for all.

2 Preliminaries

In this section we recall a few known results that are needed in our discussion.

\( \mathbb{R}_F \) denotes the space of fuzzy numbers on \( \mathbb{R} \). For \( 0 < \alpha \leq 1 \), \( \alpha \)-level set of \( x \in \mathbb{R}_F \) is defined by \( [x]^{\alpha} = \{ t \in \mathbb{R} \mid x(t) \geq \alpha \} \) and \( [x]^0 = \{ t \in \mathbb{R} \mid x(t) > 0 \} \). For any \( \alpha \in (0, 1) \), \( [x]^{\alpha} \) is a bounded closed interval, we denote \( [x]^{\alpha} = [x_{l_\alpha}, x_{r_\alpha}] \). Also, we define \( \hat{0} \in \mathbb{R}_F \) as \( \hat{0}(t) = 1 \) if \( t = 0 \) and \( \hat{0}(t) = 0 \) if \( t \neq 0 \). We denote a triangular fuzzy number as \( x = (a, b, c) \), where \( a \) and \( c \) are endpoints of the 0-level set and 1-level set = \{\}(b)\).

For the proof of the validity of the examples in [1], we need to utilize \( \alpha \)-level sets concepts. In this sense, we will make use of the following well-known lemma.

\begin{lemma}
(See e.g. [2].) Assume \( x_l : [0, 1] \to \mathbb{R} \) and \( x_r : [0, 1] \to \mathbb{R} \) satisfy the conditions:

(i) \( x_l \) is a bounded increasing function.

(ii) \( x_r \) is a bounded decreasing function.

(iii) \( x_l(1) \leq x_r(1) \).

(iv) For \( 0 < k \leq 1 \), \( \lim_{\alpha \to k^+} x_l(\alpha) = x_l(k) \) and \( \lim_{\alpha \to k^-} x_r(\alpha) = x_r(k) \).

(v) \( \lim_{\alpha \to 0^+} x_l(\alpha) = x_l(0) \) and \( \lim_{\alpha \to 0^+} x_r(\alpha) = x_r(0) \).

Then \([x_l(\alpha), x_r(\alpha)]\) is the parametric form of a fuzzy number.
\end{lemma}

\begin{lemma}
(See e.g. [2].)

(i) If we denote \( \hat{0} = \chi_{\{0\}} \), then \( \hat{0} \in \mathbb{R}_F \) is neutral element with respect to \( + \), i.e. \( u + \hat{0} = \hat{0} + u = u \), for all \( u \in \mathbb{R}_F \).

(ii) With respect to \( \hat{0} \), none of \( u \in \mathbb{R}_F \setminus \mathbb{R} \), has opposite in \( \mathbb{R}_F \) (with respect to \( + \)).

(iii) For any \( a, b \in \mathbb{R} \) with \( a, b \geq 0 \) or \( a, b \leq 0 \) and any \( u \in \mathbb{R}_F \), we have \( (a+b) \cdot u = a \cdot u + b \cdot u \). For general \( a, b \in \mathbb{R} \), the above property does not hold.

(iv) For any \( \lambda \in \mathbb{R} \) and any \( u, v \in \mathbb{R}_F \), we have \( \lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v \).

(v) For any \( \lambda, \mu \in \mathbb{R} \) and any \( u \in \mathbb{R}_F \), we have \( \lambda \cdot (\mu \cdot u) = (\lambda \mu) \cdot u \).
\end{lemma}
The space of continuous fuzzy functions is denoted by $C(\bar{J}, \mathbb{R}_F)$. Moreover, let $0 < r < 1$. We consider the following space

$$C_r(\bar{J}, \mathbb{R}_F) = \{ g \in C(J, \mathbb{R}_F) : t^r g \in C(J, \mathbb{R}_F) \}.$$ 

The definition of solution for the problem (1.1) has been recalled from [1] as follows.

**Definition 2.3.** We say that $u \in C_{1-q}(\bar{J}, \mathbb{R}_F)$ is a solution for the problem (1.1), if it satisfies Problem (1.1).

In the following examples we intend to state the details of Examples 1 and 2 in [1] and also to show what is wrong in Example 1.1-1.2 [4], respectively.

**Example 2.4.** Consider the linear fuzzy initial value problem

$$D^q u(t) = \left( \frac{t^{-q}}{\Gamma(1-q)} - t \right) u(t) + \int_0^t u(s) ds, \quad t \in (0, b],$$

$$\lim_{t \to 0^+} t^{1-q} u(t) = 0,$$  \hspace{1cm} (2.1)

where $0 < q < 1$ and $b = \frac{1+q}{\sqrt{\Gamma(1-q)}}$. $u_1 = \hat{0}$ and $\overline{u}_1 = t^q$ are the lower and upper solutions of Problem (2.1), respectively, and $u(t) = c$ for $c \in \mathbb{R}_F$ is an exact solution as stated in [1]. Here in details we prove $u(t) = c$ for all $c \in \mathbb{R}_F$ is a solution for Problem (2.1) based on Definition 2.3. To this end we have to prove

$$D^q c = \left( \frac{c^{-q}}{\Gamma(1-q)} - t \right) c + \int_0^t c ds, \quad t \in (0, b],$$

i.e.

$$t^{-q} \frac{c}{\Gamma(1-q)} = \left( \frac{t^{-q}}{\Gamma(1-q)} - t \right) c + tc, \quad t \in (0, b].$$ \hspace{1cm} (2.2)

We consider $\alpha$-level set of $c$ as $[c]_\alpha = [c_{\alpha}, c_{\alpha}]$. Since $0 < t \leq b$, the sign of $t^{-q} \frac{c}{\Gamma(1-q)} - t$ is positive. Thus we can write the right-hand as follows:

$$\left( \frac{t^{-q}}{\Gamma(1-q)} - t \right) [c_{\alpha}, c_{\alpha}] + t [c_{\alpha}, c_{\alpha}] = \left( \frac{t^{-q}}{\Gamma(1-q)} - t \right) c_{\alpha} + t c_{\alpha},$$

$$= \left( \frac{t^{-q}}{\Gamma(1-q)} c_{\alpha} + t^{-q} \frac{c_{\alpha}}{\Gamma(1-q)} c_{\alpha} \right)$$

$$= t^{-q} \frac{[c_{\alpha}, c_{\alpha}]}{\Gamma(1-q)}$$

This proves the equality of (2.2). It then follows that $c$ is an exact solution of Problem (2.1). The authors in [3] have guessed that in the proof of equality (2.2) in Example 1 in [1], $tc - tc = 0$ has been used. It sounds like the authors in [3] have employed the following idea to prove Eq. (2.2)

$$\left( \frac{t^{-q}}{\Gamma(1-q)} - t \right) c + tc = \frac{t^{-q}}{\Gamma(1-q)} c - tc + tc = \frac{t^{-q}}{\Gamma(1-q)} c.$$

By virtue of Lemma (ii)-(iii), it is not possible. Since $u_1 = \hat{0}$ and $\overline{u}_1 = t^q$, all the conditions in Theorem 4.5 in [1] are fulfilled. Then there exists a solution of the problem (2.1), $u$, so that
\( 0 \leq u(t) \leq t^q \) for \( t \in (0,b] \), \( q \in (0,1) \). Here this happens for \( u(t) = c = \hat{0} \). The authors in [4] have misunderstood that this should be happen for all \( c \in \mathbb{R}_F \), and they have given an example as a contraction example.

**Example 2.5.** Consider the linear fuzzy initial value problem

\[
D^q u(t) = \frac{c}{\Gamma(1-q)}(t^{-q} - 1 - t^{q-1}) + \frac{1}{\Gamma(1-q)}u(t), \quad t \in (0,0.32],
\]

\[
\lim_{t \to 0^+} t^{1-q}u(t) = c,
\]

(2.3)

where \( 0.58 < q \leq 0.88 \) and \( c \in \mathbb{R}_F \). \( u_1 = ct^{q-1} \) and \( \bar{u}_1 = 10ct^{q-1} \) are the lower and upper solutions of Problem (2.3), respectively, and \( u(t) = c + ct^{q-1} \) is an exact solution as stated in [4]. Here we prove in details that \( u(t) = c + ct^{q-1} \), for all \( c \in \mathbb{R}_F \) is a solution for Problem (2.3) based on Definition 2.3. To this end, we have to prove that

\[
D^q(c + ct^{q-1}) = \left( \frac{c}{\Gamma(1-q)}(t^{-q} - 1 - t^{q-1}) + \frac{c + ct^{q-1}}{\Gamma(1-q)} \right), \quad t \in (0,0.32],
\]

i.e.

\[
\frac{ct^{-q}}{\Gamma(1-q)} = \frac{c}{\Gamma(1-q)}\left(t^{-q} - 1 - t^{q-1}\right) + \frac{c + ct^{q-1}}{\Gamma(1-q)}, \quad t \in (0,0.32].
\]

(2.4)

We consider \( \alpha \)-level set of \( c \) as \( [c]_\alpha = [c_{l_\alpha},c_{r_\alpha}] \). Since \( t \in (0,0.32] \), the sign of \( t^{-q} - 1 - t^{q-1} \) is positive. Thus, we can prove the equality below in a similar way to Example 2.1.

\[
\frac{[c_{l_\alpha},c_{r_\alpha}]t^{-q}}{\Gamma(1-q)} = \frac{c_{l_\alpha}}{\Gamma(1-q)}\left(t^{-q} - 1 - t^{q-1}\right) + \frac{c_{l_\alpha} + [c_{l_\alpha},c_{r_\alpha}]t^{q-1}}{\Gamma(1-q)}, \quad t \in (0,0.32].
\]

This proves Eq. (2.3), and then \( u(t) = c + ct^{q-1} \) is the exact solution of Problem (2.3). Since all the conditions of Theorem 4.5 in [4] are fulfilled, there exists a solution \( u \) for this problem so that \( ct^{q-1} \leq u(t) \leq 10ct^{q-1} \) for all \( t \in (0,0.32] \) and \( 0.58 < q \leq 0.88 \). Here this happens for \( u(t) = c + ct^{q-1} \) where \( c \geq \hat{0} \) not for all \( c \in \mathbb{R}_F \).