The tusk condition and Petrovskii criterion for the normalized $p$-parabolic equation

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Abstract

We study boundary regularity for the normalized $p$-parabolic equation in arbitrary bounded domains. Effros and Kazdan (Indiana Univ. Math. J. 20 (1970) 683–693) showed that the so-called tusk condition guarantees regularity for the heat equation. We generalize this result to the normalized $p$-parabolic equation, and also obtain Hölder continuity. The tusk condition is a parabolic version of the exterior cone condition. We also obtain a sharp Petrovskii criterion for the regularity of the latest moment of a domain. This criterion implies that the regularity of a boundary point is affected if one side of the equation is multiplied by a constant.

1. Introduction

Let $\Theta$ be a bounded open set in a Euclidean space and for every $f \in C(\partial \Theta)$ let $u_f$ be the solution of the Dirichlet problem for a given partial differential equation. Then a boundary point $\xi_0 \in \partial \Theta$ is regular if

$$\lim_{\Theta \ni \zeta \to \xi_0} u_f(\zeta) = f(\xi_0) \text{ for all } f \in C(\partial \Theta),$$

that is, if the solution to the Dirichlet problem attains the given boundary data continuously. In other words, the Dirichlet problem is solvable in the classical sense if and only if all boundary points are regular, in which case $\Theta$ is called regular. For solving the Dirichlet problem in this context we use Perron solutions.

In this paper, we consider boundary regularity for the normalized $p$-parabolic equation

$$u_t = \Delta^N_N u, \quad (1.1)$$

where $1 < p < \infty$ and, at least as long as $\nabla u \neq 0$,

$$\Delta^N_N u = \Delta u + (p - 2)\Delta^N_{\infty} u = |\nabla u|^{2-p} \Delta_p u, \quad$$

$$\Delta^N_{\infty} u = |\nabla u|^{-2} (D^2 u / \nabla u, \nabla u),$$

$$\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u).$$

For $p = 2$ both the normalized and the non-normalized $p$-parabolic equation

$$u_t = \Delta_p u \quad (1.2)$$

reduce to the heat equation.

The normalized $p$-parabolic equation (1.1) has applications in, for example, image processing (see Does [10]), and arises from tug-of-war games with noise (see Manfredi–Parviainen–Rossi [23]). Compared with (1.2), it has the advantage that solutions remain solutions when multiplied by constants. On the other hand, it is in non-divergence form and the solutions are

Received 20 December 2017; published online 20 May 2019.

2010 Mathematics Subject Classification 35K61 (primary), 35B30, 35B51, 35D40, 35K92 (secondary).

A. B. and J. B. were supported by the Swedish Research Council, grants 2016-63424 and 621-2014-3974, respectively. M. P. was partly supported by the Academy of Finland project #260791.
understood in the viscosity sense. Moreover, the normalized $p$-Laplacian $\Delta_p^N u$ is discontinuous at the zeros of the gradient $\nabla u$.

Boundary regularity for the normalized $p$-parabolic equation (1.1) was first studied by Banerjee–Garofalo [4] (see also their earlier paper [3]). More recently, Jin–Silvestre [17] established the interior $C^{1,\alpha}$-regularity for solutions of (1.1) (see also Imbert–Jin–Silvestre [15], Attouchi–Parviainen [1] and Parviainen–Ruosteenoja [24] for related regularity results).

The following is our main result.

**Theorem 1.1 (The tusk condition).** Assume that $\Theta$ satisfies the tusk condition at $\xi_0 = (0,0)$, that is, there are $\hat{x} \in \mathbb{R}^n$ and $R, T > 0$ such that the tusk
$$\{(x,t) \in \mathbb{R}^{n+1}: -T < t < 0 \text{ and } |x - (-t)^{1/2}\hat{x}|^2 < R^2(-t)\} \subset \mathbb{R}^{n+1} \setminus \Theta.$$ 
Then $\xi_0$ is regular. Moreover, if $f: \partial \Theta \to \mathbb{R}$ is bounded and Hölder continuous at $\xi_0$ then so is the upper Perron solution $Pf$.

For the heat equation, Effros–Kazdan [11] showed that the very same tusk condition implies boundary regularity (but without Hölder continuity part) and Lieberman [22] generalized this to linear uniformly parabolic equations.

Our proof is based on the strong minimum principle and the fact that the shape of the tusk is invariant under parabolic scaling. Compared with the proof in [11] we do not have their removability Lemma 1 at our disposal, and instead we use the parabolic comparison principle. We also need to first deduce the strong minimum principle, the parabolic comparison principle and show that the exterior ball condition guarantees boundary regularity. The exterior ball condition is much more restrictive than the tusk condition, but it is needed in our proof. We also improve on the earlier results (including the heat equation) by showing the Hölder continuity at boundary points with an external tusk.

In this paper, we also deduce the following generalization of Petrovskii’s criterion, which for the heat equation was proved in [25].

**Theorem 1.2 (The Petrovskii criterion).** Let
$$\Theta := \{(x,t) : |x|^2 < A|t| \log |t| \text{ and } -\frac{1}{3} < t < 0\},$$
where $A > 0$. Then $\xi_0 = (0,0)$ is regular if and only if $A \leq 4(p-1)$.

As in the case of the heat equation, it follows directly from the Petrovskii criterion that regularity is different for the normalized $p$-parabolic equation (1.1) and for its multiplied cousins
$$au_t = \Delta_p^N u, \quad a > 0, \quad a \neq 1.$$ 
This is in great contrast to the situation for the non-normalized $p$-parabolic equation (1.2), with $p \neq 2$, for which it was shown by Björn–Björn–Gianazza–Parviainen [7, Theorem 3.6] that it and all its cousins have the same regular points.

The natural parabolic scaling for (1.2) takes on a different form than for (1.1) and the heat equation (both of which are invariant under the same parabolic scaling) (see Björn–Björn–Gianazza [6, Section 4]). This is one reason for why the Petrovskii criterion for (1.2), obtained in [6, Theorem 1.1], looks quite different from Theorem 1.2. In particular, the constant $A$ in the Petrovskii criterion for (1.2) is unimportant and instead it is the power of $|x|$ that determines the regularity. Moreover, it follows from that result that the tusk condition (corresponding to the natural parabolic scaling) does not imply regularity for (1.2), at least not for $p > 2$.

A key tool for all these boundary regularity results is the barrier characterization saying that a boundary point is regular if and only if it has a barrier, or a family of barriers in the case
of (1.2). For (1.1), this characterization was obtained by Banerjee–Garofalo [4, Theorem 4.5], while for the heat equation it is due to Bauer [5, Theorems 30 and 31]. On the contrary, regularity for (1.2) is characterized by a family of barriers [7, Theorem 3.3], while one (traditional) barrier is not enough by [6, Proposition 1.2], at least not for $p < 2$.

Thus it seems that the boundary regularity theory for the normalized $p$-parabolic equation (1.1) is much more similar to the theory for the heat equation than for the non-normalized $p$-parabolic equation (1.2). However, this is not the complete picture. The main result in Banerjee–Garofalo [4, Theorem 1.1] says that a lateral point $(x_0, t_0)$ of a space–time cylinder $G_{t_1, t_2} := G \times (t_1, t_2) \subset \mathbb{R}^{n+1}$ (that is, $x_0 \in \partial G$ and $t_1 \leq t_0 \leq t_2$) is regular for (1.1) if and only if $x_0$ is regular for $p$-harmonic functions in $G$, when $p \geq 2$. The very same criterion was obtained for (1.2) by Kilpeläinen–Lindqvist [19] and Björn–Björn–Gianazza–Parviainen [7, Theorem 3.9], for all $1 < p < \infty$.

The boundary regularity theory for the heat equation has a long and colourful history (see, for example, Watson [29]). Since we have not been able to find a suitable reference containing all the details mentioned below, we state them here. Petrovskiǐ’s criterion [25] dates back to 1935. Soon afterwards, Tikhonov [26, Theorems 1–3] in 1938, showed that the parabolic boundary of a cylinder $G \times (t_1, t_2)$ is regular for the heat equation if and only if $G$ is regular for harmonic functions. That a specific point on the lateral boundary of a cylinder is regular if and only if the corresponding base point is regular for harmonic functions was shown by Babuška–Výborný [2] in 1962. They proved a barrier characterization in cylinders, and used it to prove their regularity result. The same year, Bauer [5, Theorems 30 and 31] obtained the general barrier characterization for the heat equation.

Evans–Gariepy [12] obtained the Wiener criterion for the heat equation, and Fabes–Garofalo–Lanconelli [13] generalized this to linear uniformly parabolic equations with $C^1$-Dini coefficients. A different type of Wiener criterion for the heat equation has also been obtained by Landis [20, 21]. However, contrary to the (linear and non-linear) elliptic case, where the Wiener criterion is really useful to deduce (ir)regularity, the parabolic Wiener criterion seems to be much harder to use in practice, and the preferred way of deducing (ir)regularity is using barriers, at least in most situations.

In a very recent preprint, Ubostad [27] independently studies the Perron method and the Petrovskiǐ’s criterion for the multiplied equation $u_t = p^{-1} \Delta_p^N u$.

2. Preliminaries

Let $\Theta \subset \mathbb{R}^{n+1}$ be an open set and $1 < p < \infty$. Points in $\mathbb{R}^{n+1}$ are written as $\xi = (x, t)$. We let $\nabla u$ denote the gradient of $u$ in the space directions, while $D^2 u$ is the matrix of all second derivatives in the space directions. Also the operators $\Delta_p$ and $\Delta_p^N$ are considered with respect to the space variable $x$.

Next we recall the definition of a viscosity (super/sub)solution to (1.1). If the gradient of a test function vanishes, then we need to specify how to interpret the equation. To this end, we use the standard way (see Giga [14] and Crandall–Ishii–Lions [9]) of replacing the operator by its lower/upper semicontinuous envelope.

**Definition 2.1.** A function $u : \Theta \to (-\infty, \infty]$ is a viscosity supersolution to (1.1) in $\Theta$ if

(i) $u$ is lower semicontinuous;
(ii) $u$ is finite in a dense subset of $\Theta$;
(iii) for all $(x_0, t_0) \in \Theta$ and $\varphi \in C^2(\Theta)$, such that $u(x_0, t_0) = \varphi(x_0, t_0)$ and $u(x, t) > \varphi(x, t)$ for $(x, t) \in \Theta \setminus \{(x_0, t_0)\}$, we have

\[
\varphi_t(x_0, t_0) - \Delta_p^N \varphi(x_0, t_0) \geq 0, \quad \text{if } \nabla \varphi(x_0, t_0) \neq 0, \\
\varphi_t(x_0, t_0) - \Delta_p \varphi(x_0, t_0) - (p-2)\lambda_{\text{eig}} \geq 0, \quad \text{if } \nabla \varphi(x_0, t_0) = 0,
\]
where $\lambda_{\text{eig}}$ denotes the smallest (if $p \geq 2$) or the largest (if $1 < p < 2$) eigenvalue of the matrix $D^2\varphi(x_0, t_0)$.

A function $u$ is a \textit{viscosity subsolution} if $-u$ is a viscosity supersolution, and a \textit{viscosity solution} if it is both a viscosity sub- and supersolution.

We sometimes briefly refer to the conditions above for the test function $\varphi$ by saying that $\varphi$ \textit{touches} $u$ at $(x_0, t_0)$ from below.

Since, for every $\eta \in \mathbb{R}^n$, the inner product $|\eta|^{-2}\langle D^2u(x_0, t_0)\eta, \eta \rangle$ lies between the smallest and the largest eigenvalue of $D^2u(x_0, t_0)$, this definition is equivalent to the ones in Jin–Silvestre [17, Definition 2.8] and Chen–Giga–Goto [8, Definition 2.1]. By Lemma 2 in Manfredi–Parviainen–Rossi [23], we may also reduce the number of test functions in the definition of the viscosity super/subsolutions. More precisely, condition (iii) in Definition 2.1 can equivalently be replaced by

(iii') for all $(x_0, t_0) \in \Theta$ and $\varphi \in \mathcal{C}^2(\Theta)$, such that $u(x_0, t_0) = \varphi(x_0, t_0)$ and $u(x, t) > \varphi(x, t)$ for $(x, t) \in \Theta \setminus \{(x_0, t_0)\}$, we have

$$\begin{align*}
&\varphi_t(x_0, t_0) - \Delta_p^N \varphi(x_0, t_0) \geq 0, 
&\quad \text{if } \nabla \varphi(x_0, t_0) \neq 0, 
&\varphi_t(x_0, t_0) \geq 0, 
&\quad \text{if } \nabla \varphi(x_0, t_0) = 0 \text{ and } D^2\varphi(x_0, t_0) = 0,
\end{align*}$$

while there is no requirement when $\nabla \varphi(x_0, t_0) = 0$ and $D^2\varphi(x_0, t_0) \neq 0$.

This definition is the same as Definition 2.2 in Banerjee–Garofalo [4], except that we do not require $u \in L^\infty$ and we allow for arbitrary domains $\Theta \subset \mathbb{R}^{n+1}$, not only cylinders. With this modification, it will be possible to obtain a full equivalence with superparabolic functions. This definition is also more common in the literature. Many of the specific supersolutions considered in this paper will be smooth enough to be checked by the following criterion, which may be of independent interest.

**Proposition 2.2.** Assume that $u \in \mathcal{C}^2(\Theta)$, and that for every $(x_0, t_0) \in \Theta$ we have

$$\begin{align*}
&u_t(x_0, t_0) - \Delta_p^N u(x_0, t_0) \geq 0, 
&\quad \text{if } \nabla u(x_0, t_0) \neq 0, 
&u_t(x_0, t_0) \geq 0, 
&\quad \text{if } \nabla u(x_0, t_0) = 0 \text{ and } D^2u(x_0, t_0) \geq 0,
\end{align*}$$

then $u$ is a viscosity supersolution in $\Theta$.

Note that there is no requirement if $\nabla u(x_0, t_0) = 0$ and $D^2u(x_0, t_0) \neq 0$. As usual, we say that $D^2u(x_0, t_0) \geq 0$ if it is a positive definite or positive semidefinite quadratic form.

**Proof.** Let $\varphi$ touch $u$ at $(x_0, t_0)$ from below as in (iii'). Then

$$\nabla \varphi(x_0, t_0) = \nabla u(x_0, t_0), \quad \varphi_t(x_0, t_0) = u_t(x_0, t_0) \text{ and } D^2(u - \varphi)(x_0, t_0) \geq 0. \quad (2.1)$$

Assume first that $\nabla \varphi(x_0, t_0) \neq 0$. Let

$$\eta = \frac{\nabla \varphi(x_0, t_0)}{||\nabla \varphi(x_0, t_0)||} = \frac{\nabla u(x_0, t_0)}{||\nabla u(x_0, t_0)||}$$

and let $0 \leq \lambda_1 \leq \ldots \leq \lambda_n$ be the eigenvalues of the matrix $D^2(u - \varphi)(x_0, t_0)$, which is positive (semi)definite by assumption. Then

$$\Delta_p^N u(x_0, t_0) - \Delta_p^N \varphi(x_0, t_0) = \Delta(u - \varphi)(x_0, t_0) + (p - 2)\langle (D^2(u - \varphi)(x_0, t_0)\eta, \eta \rangle$$

$$\geq \sum_{i=1}^{n} \lambda_i - \lambda_n + (p - 1)\langle D^2(u - \varphi)(x_0, t_0)\eta, \eta \rangle$$
\[\varphi_t(x_0, t_0) - \Delta^N_p \varphi(x_0, t_0) \geq u_t(x_0, t_0) - \Delta^N_p u(x_0, t_0) \geq 0.\]

Assume now, on the other hand, that \(\nabla \varphi(x_0, t_0) = 0\) and \(D^2 \varphi(x_0, t_0) = 0\). Then by (2.1), \(D^2 u(x_0, t_0) \geq 0\) and thus, using (2.1) again,

\[\varphi_t(x_0, t_0) = u_t(x_0, t_0) \geq 0,\]

by assumption. We see that (iii’) is fulfilled, and therefore \(u\) is a viscosity supersolution in \(\Theta\). □

Note that viscosity (super)solutions are more precisely defined than weak supersolutions used in connection with divergence-type operators, which can be changed arbitrarily on sets of measure zero. In fact, Proposition 2.4 implies that if we change a viscosity supersolution at a single point, it will never remain to be a viscosity supersolution.

Equation (1.1) satisfies some important invariance properties under multiplication, addition and parabolic scaling. More precisely, if \(u(x, t)\) is a viscosity (super/sub)solution and \(a, \lambda > 0\) and \(b \in \mathbb{R}\), then so is \(v(x, t) := au(\lambda x, \lambda^2 t) + b\).

The following weak Harnack inequality for viscosity supersolutions can be extracted from, for example, Wang [28, Corollary 4.14], Imbert–Silvestre [16, Theorem 4.15] or Jin–Silvestre [17, Theorem 2.1], upon noting that the normalized \(p\)-Laplace operator in (1.1) satisfies the assumptions therein. Indeed,

\[\Delta^N_p u = \sum_{i,j=1}^{n} a_{ij}(x) \partial_{ij} u(x),\]

where

\[a_{ij}(x) := \delta_{ij} + (p - 2) \frac{\partial_i u(x) \partial_j u(x)}{|\nabla u(x)|^2}\]

satisfy, for every \(\eta \in \mathbb{R}^n\) with \(|\eta| = 1\),

\[\sum_{i,j=1}^{n} a_{ij}(x) \eta_i \eta_j = \eta^2 + (p - 2) \frac{\langle \nabla u(x), \eta \rangle^2}{|\nabla u(x)|^2},\]

which clearly lies between \(\lambda = \min\{p - 1, 1\}\) and \(\Lambda = \max\{p - 1, 1\}\). This means that \(\Delta^N_p\) can be estimated from above and below by so-called Pucci operators, cf. [17, pp. 3–4] or [28, Lemmas 3.2, 3.9 and Proposition 3.10].

We state the weak Harnack inequality using the space–time cylinders

\[B_r \times (-r^2, 0) \quad \text{and} \quad B_r \times (-4r^2, -3r^2),\]

contained in \(B_{2r} \times (-4r^2, 0)\), where \(B_r = \{x \in \mathbb{R}^n : |x| < r\}\) are balls in \(\mathbb{R}^n\). These are easily obtained from the cylinders

\[B_{1/2} \times (\frac{1}{4}, 0), \quad B_{1/2} \times (-1, -\frac{3}{4}) \quad \text{and} \quad B_1 \times (-1, 0),\]
used in the weak Harnack inequality in [17, Theorem 2.1], by the solution-preserving dilation $(x, t) \mapsto (2x, (2r)^2t)$. When $p \geq 2$, the weak Harnack inequality also follows by a game-theoretic argument (see Parviainen–Ruosteenoja [24, Theorem 4.7]). Here and below, $f$ denotes the integral average, that is, $\int_A f \, d\mu = \int f \, d\mu / \mu(A)$.

**Theorem 2.3 (Weak Harnack inequality).** Let $u$ be a non-negative viscosity supersolution in the cylinder $B_{2r} \times (-4r^2, 0)$, for some $r > 0$. Then there are constants $q > 0$ and $C > 0$, only depending on $p$ and $n$, such that

$$\left( \int_{B_r \times (-4r^2, -3r^2)} u^q \, dx \, dt \right)^{1/q} \leq C \inf_{B_r \times (-r^2, 0)} u.$$ 

It follows from this and a covering argument that $u$ is finite a.e. Moreover, viscosity supersolutions are lower semicontinuously regularized in the following sense. Banerjee–Garofalo [4, Proposition 3.3] obtained a similar result stated in a slightly weaker form.

**Proposition 2.4.** Assume that $u$ is a viscosity supersolution in $\Theta$. Then for all $(x_0, t_0) \in \Theta$,

$$u(x_0, t_0) = \liminf_{(x, t) \to (x_0, t_0)} u(x, t) = \text{ess lim inf}_{(x, t) \to (x_0, t_0)} u(x, t) = \text{ess lim inf}_{t < t_0} u(x, t). \quad (2.2)$$

**Proof.** We can assume that $(x_0, t_0) = (0, 0)$. Since $u$, by definition, is lower semicontinuous we directly see that

$$u(0, 0) \leq \liminf_{(x, t) \to (0, 0)} u(x, t) \leq \text{ess lim inf}_{(x, t) \to (0, 0)} u(x, t) \leq \text{ess lim inf}_{t < 0} u(x, t).$$

If $u(0, 0) = \infty$, then there is nothing to prove. So, without loss of generality we may assume that $u(0, 0) = 0$. Let $a > 0$ be arbitrary and such that

$$a < \text{ess lim inf}_{t < 0} u(x, t). \quad (2.3)$$

If no such $a$ exists, then (2.2) clearly holds. By (2.3), there exists $r_0 > 0$ such that $u > a$ a.e. in $B_{2r_0} \times (-4r_0^2, 0)$.

As $u$ is lower semicontinuous and $u(0, 0) = 0$, we can for any $\varepsilon > 0$ find $r \in (0, r_0)$ and $\delta \in (0, r^2)$ such that $u > -\varepsilon$ everywhere in

$$B_{2r} \times (\delta - 4r^2, \delta) \subset \Theta.$$ 

Since $v := u + \varepsilon \geq 0$ is a viscosity supersolution and $(0, 0) \in B_r \times (\delta - r^2, \delta)$, it then follows from the weak Harnack inequality (Theorem 2.3) that

$$\left( \int_{B_r \times (\delta - 4r^2, \delta - 3r^2)} v^q \, dx \, dt \right)^{1/q} \leq Cv(0, 0) = C\varepsilon.$$ 

As $a < u < v$ a.e. in $B_{2r_0} \times (-4r_0^2, 0)$ we can conclude that $a \leq C\varepsilon$. Letting $\varepsilon \to 0$ shows that $a = 0$, and since $a$ in (2.3) was arbitrary, (2.2) follows. \qed

Next we recall the definition of superparabolic functions that frequently appears in the nonlinear parabolic potential theory (see Kilpeläinen–Lindqvist [19] and Banerjee–Garofalo [4]). Unless otherwise stated, $Q$ stands for the box $Q = (a_1, b_1) \times \ldots \times (a_n, b_n) \subset \mathbb{R}^n$, and the sets

$$Q_{t_1, t_2} = Q \times (t_1, t_2)$$
are called space–time boxes. The parabolic boundary of the space–time cylinder \( G_{t_1, t_2} := G \times (t_1, t_2) \subset \mathbb{R}^{n+1} \) is defined as

\[
\partial_p G_{t_1, t_2} = (G \times \{t_1\}) \cup (\partial G \times [t_1, t_2]).
\]

**Definition 2.5.** A function \( u : \Theta \to (-\infty, \infty] \) is superparabolic in \( \Theta \) with respect to (1.1) if

1. \( u \) is lower semicontinuous;
2. \( u \) is finite in a dense subset of \( \Theta \);
3. \( u \) satisfies the following comparison principle on each space–time box \( Q_{t_1, t_2} \subset \Theta \): If \( h \in C(Q_{t_1, t_2}) \) is a viscosity solution of (1.1) in \( Q_{t_1, t_2} \) satisfying \( h \leq u \) on \( \partial_p Q_{t_1, t_2} \), then \( h \leq u \) in \( Q_{t_1, t_2} \).

A function \( u : \Theta \to [-\infty, \infty) \) is subparabolic if \( -u \) is superparabolic.

In Banerjee–Garofalo [4, Definition 3.1] they use the name generalized super/subsolution rather than super/subparabolic function. They establish the following connection between viscosity supersolutions and superparabolic functions [4, Corollary 3.5 and Theorem 3.6]. In the case of the non-normalized \( p \)-parabolic equation, the corresponding equivalence was obtained in Juutinen–Lindqvist–Manfredi [18, Theorem 2.5].

**Theorem 2.6.** In a given domain, the viscosity supersolutions and superparabolic functions to (1.1) are the same.

**Corollary 2.7.** The weak Harnack inequality (Theorem 2.3) and the lower semicontinuous regularity (Proposition 2.4) hold also for superparabolic functions.

**Remark 2.8.** Observe that in [4, Definition 3.1], instead of our condition (iii) in Definition 2.5 they require the comparison principle on each open cylinder \( G_{t_1, t_2} \), not only on space–time boxes as here. Thus, their definition of superparabolicity is a priori more restrictive than ours.

On the other hand, Theorem 2.6 shows that our definition of superparabolicity is equivalent to viscosity supersolutions, whose definition differs from the one in [4] only in that we do not assume boundedness. Since boundedness is not needed to conclude that (possibly unbounded) viscosity supersolutions are superparabolic in the sense of [4, Definition 3.1], it also follows that (iii) is sufficient to define the same class of superparabolic functions as in [4, Definition 3.1].

**Proof of Theorem 2.6.** The only difference to [4, Corollary 3.5] is that here the viscosity supersolutions are not assumed to be bounded. However, the comparison principle for the viscosity super/subsolutions does not require this assumption (see Chen–Giga–Goto [8, Theorem 4.1] and Giga [14, Corollary 3.1.5]). It follows that viscosity supersolutions are superparabolic in the sense of [4, Definition 3.1] and thus also in the sense of our Definition 2.5. Note that [8, Theorem 4.1] is formulated in terms of lower semicontinuously regularized (viscosity) supersolutions, but this is provided by Proposition 2.4.

The converse direction is obtained through a counterassumption (see, for example, Juutinen–Lindqvist–Manfredi [18, p. 704]): Suppose that \( u \) is a superparabolic function but that there is \( \varphi \in C^2(\Theta) \) as in Definition 2.1 (iii′) which touches \( u \) at some \( (x_0, t_0) \in \Theta \) from below and either

\[
\varphi_t(x_0, t_0) - \Delta_N \varphi(x_0, t_0) < 0 \quad \text{and} \quad \nabla \varphi(x_0, t_0) \neq 0,
\]
Remark 2.8, it thus follows from Lemma 3.10 in Banerjee–Garofalo [4]. By pasting Lemma 2.12, which we however cannot yet deduce.

\[ \varphi_t(x, t) - \Delta_p^N \varphi(x, t) < 0 \quad \text{and} \quad \nabla \varphi(x, t) \neq 0, \]

or

\[ \varphi_t(x, t) < 0 \quad \text{and} \quad \nabla \varphi(x, t) = 0. \]

Thus, \( \varphi \) is a continuous viscosity subsolution in \( Q_{t_1, t_2} \), by Proposition 2.2. Since \( u \) is lower semicontinuous and \( \partial_p Q_{t_1, t_2} \) is compact, there is \( \delta > 0 \) such that \( \varphi + \delta \leq u \) on \( \partial_p Q_{t_1, t_2} \). As \( \varphi + \delta \in C^2(\Theta) \), using Theorem 2.6 in Banerjee–Garofalo [4] we can find a viscosity solution \( h \) with continuous boundary values \( \varphi + \delta \) on \( \partial_p Q_{t_1, t_2} \). By the first part of the proof, \( \varphi + \delta \) is subparabolic. Thus property (iii) in Definition 2.5 yields \( \varphi + \delta \leq h \leq u \) in \( Q_{t_1, t_2} \), which is a contradiction since \( \varphi(x_0, t_0) = u(x_0, t_0) \).

The following important elliptic-type comparison principle is a slight generalization of the one in Banerjee–Garofalo [4, Lemma 3.10], where it was proved for bounded functions. Note that if \( u \) and \( v \) are superparabolic, then it is easy to see that \( \min\{u, v\} \) is also superparabolic, and in particular \( u_k := \min\{u, k\} \) is superparabolic if \( k \in \mathbb{R} \). This fact is a special case of the pasting Lemma 2.12, which we however cannot yet deduce.

**Theorem 2.9 (Elliptic-type comparison principle).** Let \( \Theta \) be a bounded open subset of \( \mathbb{R}^{n+1} \). Suppose that \( u \) is superparabolic and \( v \) is subparabolic in \( \Theta \). If

\[ \infty \neq \limsup_{\Theta \ni (y, s) \to (x, t)} v(y, s) \leq \liminf_{\Theta \ni (y, s) \to (x, t)} u(y, s) \neq -\infty \]  \hspace{1cm} (2.4)

for all \((x, t) \in \partial \Theta\), then \( v \leq u \) in \( \Theta \).

**Proof.** By compactness, (2.4) and semicontinuity, \( u \) is bounded from below and \( v \) is bounded from above. Let \( M = \sup_{\Theta} v \), \( m = \inf_{\Theta} u \), \( u_M = \min\{u, M\} \) and \( v_m = \max\{v, m\} \). Then \( u_M \) and \( v_m \) satisfy a similar comparison on the boundary as in (2.4). In view of Theorem 2.6 and Remark 2.8, it thus follows from Lemma 3.10 in Banerjee–Garofalo [4] that \( v_m \leq u_M \), and hence \( v \leq u \) in \( \Theta \).

For evolution equations, a parabolic comparison principle is more natural since it avoids any requirements on the future boundary.

**Theorem 2.10 (Parabolic comparison principle).** Let \( \Theta \) be a bounded open subset of \( \mathbb{R}^{n+1} \). Suppose that \( u \) is superparabolic and \( v \) is subparabolic in \( \Theta \). Let \( T \in \mathbb{R} \) and assume that (2.4) holds for all \((x, t) \in \partial \Theta \) with \( t < T \). Then \( v \leq u \) in \( \Theta_- = \{(x, t) \in \Theta : t < T\} \).

We will deduce the parabolic comparison principle from the elliptic-type comparison principle. In order to do so we will need the following simple pasting lemma, which may be of independent interest.
LEMMA 2.11. Assume that \( u \) is superparabolic in \( \Theta_\frac{1}{2} := \{ (x, t) \in \Theta : t < T \} \). Let \( k \in \mathbb{R} \). Then the function

\[
v(x, t) = \begin{cases} 
\min\{u(x, t), k\}, & \text{if } (x, t) \in \Theta \text{ and } t < T, \\
\min \left\{ \liminf_{\theta \rightarrow (x, t)} u(\zeta), k \right\}, & \text{if } (x, t) \in \Theta \text{ and } t = T, \\
k, & \text{if } (x, t) \in \Theta \text{ and } t > T,
\end{cases}
\]

is superparabolic in \( \Theta \).

Note that the complicated definition for \( t = T \) is needed for \( v \) to be lower semicontinuous.

Proof. By construction, \( v \) is lower semicontinuous and bounded from above. It remains to show that \( v \) satisfies the comparison principle. We therefore let \( Q_{t_1, t_2} \subseteq \Theta \) be a space–time box and \( h \in C(\overline{Q_{t_1, t_2}}) \) be a a viscosity solution in \( Q_{t_1, t_2} \) satisfying

\[
h \leq v \quad \text{on } \partial_p Q_{t_1, t_2}.
\]

We first note that since \( h \leq k \) on \( \partial_p Q_{t_1, t_2} \) and the constant function \( k \) is superparabolic, it is true that \( h \leq k \) in \( Q_{t_1, t_2} \). To verify that \( h \leq v \) in \( Q_{t_1, t_2} \) we let \( (x, t) \in Q_{t_1, t_2} \), and consider the three cases: \( t > T \), \( t < T \) and \( t = T \) separately.

If \( t > T \), then \( h(x, t) \leq k = v(x, t) \). On the other hand if \( t < T \), we let \( t' = \frac{1}{2}(t + T) < T \). Then \( Q_{t_1, t'} \subseteq \Theta \) and \( h \leq u \) on \( \partial_p Q_{t_1, t'} \). Together with the superparabolicity of \( u \) in \( \Theta_\frac{1}{2} \), this yields \( h(x, t) \leq u(x, t) \). As \( h \leq k \), we conclude that \( h(x, t) \leq v(x, t) \) if \( t < T \).

Finally, if \( \xi = (x, T) \in Q_{t_1, t_2} \) then, since \( h \leq u \) in \( \Theta_\frac{1}{2} \), we conclude from the continuity of \( h \) that

\[
h(\xi) = \lim_{\Theta_\frac{1}{2} \ni \zeta \to \xi} h(\zeta) \leq \liminf_{\Theta_\frac{1}{2} \ni \zeta \to \xi} u(\zeta).
\]

As \( h \leq k \), it follows that \( h(x, T) \leq v(x, T) \). \( \square \)

Proof of Theorem 2.10. Let \( (x_0, t_0) \in \Theta \) with \( t_0 < T \), and set \( T' = \frac{1}{2}(t_0 + T) \). Then the lower semicontinuity of \( u \) and upper semicontinuity of \( v \), together with (2.4) show that \( u \) is bounded from below and \( v \) is bounded from above in

\[
\tilde{\Theta}_\frac{1}{2} := \{ (x, t) \in \Theta : t \leq T' \}.
\]

Let \( m = \inf_{\tilde{\Theta}_\frac{1}{2}} u \), \( M = \sup_{\tilde{\Theta}_\frac{1}{2}} v \),

\[
\tilde{u} = \begin{cases} 
\min\{v, M\} & \text{in } \tilde{\Theta}_\frac{1}{2}, \\
M & \text{in } \Theta \setminus \tilde{\Theta}_\frac{1}{2},
\end{cases} \quad \text{and} \quad \tilde{v} = \begin{cases} 
\max\{v, m\} & \text{in } \tilde{\Theta}_\frac{1}{2}, \\
m & \text{in } \Theta \setminus \tilde{\Theta}_\frac{1}{2}.
\end{cases}
\]

By Lemma 2.11, \( \tilde{u} \) is superparabolic and \( \tilde{v} \) is subparabolic in \( \Theta \). Now \( \tilde{u} \) and \( \tilde{v} \) satisfy the assumptions for the elliptic comparison principle (Theorem 2.9) in \( \Theta \), and thus \( \tilde{v} \leq \tilde{u} \) in \( \Theta \). Hence \( v(x_0, t_0) \leq \tilde{v}(x_0, t_0) \leq \tilde{u}(x_0, t_0) \leq u(x_0, t_0) \). \( \square \)

Having established the parabolic comparison principle (Theorem 2.10), we can obtain the following generalization of Lemma 2.11, which is useful when constructing new superparabolic functions.
Lemma 2.12 (Pasting lemma). Let $U \subset \Theta$ be open. Also let $u$ and $v$ be superparabolic in $\Theta$ and $U$, respectively, and let

$$w = \begin{cases} \min\{u,v\} & \text{in } U, \\ u & \text{in } \Theta \setminus U. \end{cases}$$

If $w$ is lower semicontinuous, then $w$ is superparabolic in $\Theta$.

Proof. Since $-\infty < w \leq u$, we see that $w$ is finite in a dense subset of $\Theta$, and we only have to obtain the comparison principle. Therefore, let $Q_{t_1,t_2} \Subset \Theta$ be a space–time box, and $h \in C(\overline{Q_{t_1,t_2}})$ be a viscosity solution in $Q_{t_1,t_2}$ such that

$$h \leq w \quad \text{on } \partial_p Q_{t_1,t_2}. \quad (2.5)$$

Since $h \leq u$ on $\partial_p Q_{t_1,t_2}$ and $u$ is superparabolic, we directly have that

$$h \leq u \quad \text{in } Q_{t_1,t_2}. \quad (2.6)$$

To complete the proof we show that

$$h \leq v \quad \text{in } Q_{t_1,t_2} \cap U.$$

To this end, we intend to use the parabolic comparison principle for $Q_{t_1,t_2} \cap U$ after verifying that $h(x,t) \leq v(x,t)$ for $(x,t) \in \partial(Q_{t_1,t_2} \cap U)$ with $t < t_2$. There are two cases: either $(x,t) \in U$ or $(x,t) \notin U$.

First, suppose that $(x,t) \in U$, then $(x,t) \in \partial_p Q_{t_1,t_2}$ and thus by the lower semicontinuity of $v$,

$$\liminf_{Q_{t_1,t_2} \cap U \ni (y,s) \to (x,t)} v(y,s) \geq v(x,t) \geq w(x,t) \geq h(x,t),$$

where the last inequality follows from (2.5). On the other hand, if $(x,t) \notin U$, then by the lower semicontinuity of $w$,

$$\liminf_{Q_{t_1,t_2} \cap U \ni (y,s) \to (x,t)} v(y,s) \geq \liminf_{Q_{t_1,t_2} \cap U \ni (y,s) \to (x,t)} w(y,s) \geq w(x,t) = u(x,t) \geq h(x,t),$$

where the last inequality follows from (2.5) or (2.6), depending on whether $(x,t) \in \partial_p Q_{t_1,t_2}$ or $(x,t) \in Q_{t_1,t_2}$. Hence, the parabolic comparison principle (Theorem 2.10) shows that $h \leq v$ in $Q_{t_1,t_2} \cap U$. Together with (2.6) this shows that $h \leq w$ in $Q_{t_1,t_2}$. \qed

The strong minimum principle for superparabolic functions will be an important tool for us. In the statement, we will use polygonal paths. A polygonal path is a continuous and piecewise linear function $\gamma : [0, 1] \to \Theta$.

Theorem 2.13 (Strong minimum principle). Let $u \geq 0$ be superparabolic in $\Theta$, $\xi_0 \in \Theta$ and let $\Lambda$ be the set of all points $\xi \in \Theta$ such that there is a polygonal path $\gamma : [0, 1] \to \Theta$ from $\xi = \gamma(0)$ to $\xi_0 = \gamma(1)$ along which the time variable is strictly increasing. If $u(\xi_0) = 0$, then $u \equiv 0$ in $\Lambda \cap \Theta$.

Proof. First, let $\xi \in \Lambda$, and let $\gamma : [0, 1] \to \Theta$ be a polygonal path from $\xi = \gamma(0)$ to $\xi_0 = \gamma(1)$ along which the time variable is strictly increasing. Also let

$$\sigma = \inf\{s \in [0, 1] : u(\gamma(s)) = 0\}.$$

By the lower semicontinuity of $u$, we see that $u(\gamma(\sigma)) = 0$. For simplicity, we assume that $\gamma(\sigma) = (0, 0)$.  

If $\sigma > 0$, then there is $0 < s < \sigma$ and $r > 0$ such that

$$B_{2r} \times (-4r^2, 0) \subset \Theta \quad \text{and} \quad \gamma(s) \in B_r \times (-4r^2, -3r^2).$$

It then follows from the weak Harnack inequality (Theorem 2.3) together with Proposition 2.4 that

$$\left( \int_{B_r \times (-4r^2, -3r^2)} u^q \, dx \, dt \right)^{1/q} \leq C \inf_{B_r \times (-r^2, 0)} u \leq C \ess \liminf_{t < 0} u(x, t) = C u(0, 0) = 0.$$

Thus $u = 0$ a.e. in $B_r \times (-4r^2, -3r^2)$. Since $u \geq 0$ is lower semicontinuous it must be identically 0 therein. In particular $u(\gamma(s)) = 0$, but this contradicts the fact that $s < \sigma$. Hence $\sigma = 0$ and $u(\xi) = u(\gamma(\sigma)) = 0$.

Finally, as $u \geq 0$ is lower semicontinuous it follows that $u \equiv 0$ in $\Lambda \cap \Theta$. \hfill $\square$

3. Perron solutions and boundary regularity

In this section we assume that $\Theta \subset \mathbb{R}^{n+1}$ is a bounded open set.

Perhaps the most general method to solve the Dirichlet problem in arbitrary bounded domains is the Perron method. For us it will be enough to consider Perron solutions for bounded functions, so for simplicity we restrict ourselves to this case throughout the rest of this paper.

**Definition 3.1.** Given a bounded function $f : \partial \Theta \to \mathbb{R}$, let the upper class $U_f$ be the set of all superparabolic functions $u$ on $\Theta$ such that

$$\liminf_{\Theta \ni \eta \to \xi} u(\eta) \geq f(\xi) \quad \text{for all } \xi \in \partial \Theta.$$ 

Define the **upper Perron solution** of $f$ by

$$\overline{P}_\Theta f(\xi) = \inf_{u \in U_f} u(\xi), \quad \xi \in \Theta.$$ 

Similarly, let the lower class $L_f$ be the set of all subparabolic functions $v$ on $\Theta$ such that

$$\limsup_{\Theta \ni \eta \to \xi} v(\eta) \leq f(\xi) \quad \text{for all } \xi \in \partial \Theta$$

and define the **lower Perron solution** of $f$ by

$$\underline{P}_\Theta f(\xi) = \sup_{v \in L_f} v(\xi), \quad \xi \in \Theta.$$ 

It follows directly from the elliptic comparison principle (Theorem 2.9) that we always have $P f \leq \overline{P} f$. Moreover, $P f$ and $\overline{P} f$ are viscosity solutions (see Banerjee–Garofalo [4, Theorem 3.12] and also Giga [14, Section 2.4]). When the Perron solution is taken with respect to $\Theta$ we often drop $\Theta$ from the notation.

**Definition 3.2.** A boundary point $\xi_0 \in \partial \Theta$ is **regular** with respect to $\Theta$ if

$$\lim_{\Theta \ni \xi \to \xi_0} \overline{P} f(\xi) = f(\xi_0) \quad \text{whenever } f \in C(\partial \Theta).$$

Since $\overline{P} f = -\overline{P}(-f)$, boundary regularity can equivalently be formulated using lower Perron solutions.
DEFINITION 3.3. A function $w$ is a barrier in $\Theta$ at the point $\xi_0 \in \partial \Theta$ if

(i) $w$ is a positive superparabolic function in $\Theta$;
(ii) $\lim_{\Theta \ni \zeta \to \xi_0} w(\zeta) = 0$;
(iii) $\liminf_{\Theta \ni \zeta \to \xi} w(\zeta) > 0$ for every $\xi \in \partial \Theta \setminus \{\xi_0\}$.

Banerjee–Garofalo [4, Section 4] obtained a number of results about boundary regularity which we summarize as follows. (Part (c) follows from (b) together with [4, Proposition 4.7 and Theorem 4.8].)

THEOREM 3.4. Let $\xi_0 = (x_0, t_0) \in \partial \Theta$ and let $\Theta_- = \{(x, t) \in \Theta : t < t_0\}$.

(a) $\xi_0$ is regular if and only if there is a barrier at $\xi_0$.
(b) Regularity is a local property, that is, if $U$ is an open neighbourhood of $\xi_0$, then $\xi_0$ is regular with respect to $\Theta$ if and only if it is regular with respect to $\Theta \cap U$.
(c) $\xi_0$ is regular with respect to $\Theta$ if and only if either $\xi_0 \notin \partial \Theta_-$ or $\xi_0$ is regular with respect to $\Theta_-$.
(d) If $\xi_0$ is regular and $f : \partial \Theta \to \mathbb{R}$ is a bounded function which is continuous at $\xi_0$, then

$$\lim_{\Theta \ni \zeta \to \xi_0} Pf(\zeta) = \lim_{\Theta \ni \zeta \to \xi_0} \overline{P}f(\zeta) = f(\xi_0).$$

In particular, part (c) implies that a first point is always regular because in this case $\Theta_- = \emptyset$. Another important consequence of the barrier characterization is the following restriction property.

PROPOSITION 3.5. Let $\xi_0 \in \partial \Theta$ and let $U \subset \Theta$ be open and such that $\xi_0 \in \partial U$. If $\xi_0$ is regular with respect to $\Theta$, then $\xi_0$ is regular with respect to $U$.

Proof. By Theorem 3.4(a) there is a barrier $w$ in $\Theta$ at $\xi_0$. As $w$ is lower semicontinuous and positive it follows directly that $w|_U$ is a barrier with respect to $U$. Thus Theorem 3.4(a) implies that $\xi_0$ is a regular boundary point with respect to $U$. $\square$

4. The tusk condition

DEFINITION 4.1. A tusk at $\xi_0 = (0, 0) \in \partial \Theta$ is a set in $\mathbb{R}^{n+1}$ of the form

$$V := \{(x, t) \in \mathbb{R}^{n+1} : -T < t < 0 \text{ and } |x - (-t)^{1/2} \hat{x}|^2 < R^2(-t)\},$$

for some $\hat{x} \in \mathbb{R}^n$ and with positive constants $R$ and $T$ (see Figure 1). We say that $\Theta$ satisfies the tusk condition at $\xi_0 = (0, 0) \in \partial \Theta$ if there is a tusk $V$ at $\xi_0$ with $V \cap \Theta = \emptyset$.

At points $(0, 0) \neq \xi_0 \in \partial \Theta$, the definition is analogous except that we use translations of $V$.

It is well known that if $\xi_0$ satisfies the tusk condition, then $\xi_0$ is regular for the heat equation (see Effros–Kazdan [11], which refers to $\xi_0$ as being parabolically touchable, and Lieberman [22]). We extend this result to the normalized $p$-parabolic equation. Compared to [11], we do not establish a counterpart of their Lemma 1, but use the iterative argument directly together with the parabolic comparison principle. We also improve on their result (also for the heat equation) by showing Hölder continuity.

To start with, we prove an auxiliary exterior ball condition. We let $B(\zeta, R) = \{\xi \in \mathbb{R}^{n+1} : |\zeta - \xi| < R\}$ denote a ball in $\mathbb{R}^{n+1}$. 
LEMMA 4.2 (Exterior ball condition, preliminary version). Let $\xi_0 = (x_0, t_0) \in \partial \Theta$. Suppose that there exists a ball $B = B(\xi_1, R_1)$, $\xi_1 = (x_1, t_1)$, such that $B \cap \Theta = \emptyset$ and $\xi_0 \in \partial B \cap \partial \Theta$. If $x_1 \neq x_0$, or if $\xi_0$ is the north pole of $B$ (that is, $\xi_1 = (x_0, t_0 - R_1)$) and the additional radius condition $R_1 > n + p - 2$ is satisfied, then $\xi_0$ is regular with respect to $\Theta$.

In Proposition 4.7 we will remove the above restriction on the radius in the north pole case.

Proof. For simplicity, we assume that $\xi_0 = (0, 0)$. By choosing a smaller ball, if necessary, we may without loss of generality assume that $\partial B \cap \partial \Theta = \{\xi_0\}$. For $\xi = (x, t)$ define

$$w(\xi) = e^{-jR_1^2} - e^{-jR^2},$$

where $R = |\xi - \xi_1|$ and $R_1 = |\xi_1|$, while $j$ will be chosen later. Then $w > 0$ in $\overline{\Theta} \setminus \{\xi_0\}$ and $\lim_{\xi \to \xi_0} w(\xi) = 0$. Elementary calculations show that

$$w_t(x, t) = 2je^{-jR_1^2}(t - t_1),$$
$$\nabla w(x, t) = 2je^{-jR_1^2}(x - x_1),$$
$$\Delta_p w(x, t) = (2j)^{p-1}|x - x_1|^{p-2}e^{-j(p-1)R_1^2}[n + p - 2 - 2j(p - 1)|x - x_1|^2],$$

and, provided that $\nabla w(x, t) \neq 0$,

$$\Delta_p^N w(x, t) = 2je^{-jR_1^2}[n + p - 2 - 2j(p - 1)|x - x_1|^2].$$

Thus, still assuming that $\nabla w(x, t) \neq 0$,

$$\Delta_p^N w(x, t) - w_t(x, t) = 2je^{-jR_1^2}[n + p - 2 - 2j(p - 1)|x - x_1|^2 - (t - t_1)]. \quad (4.1)$$

To show that $w$ is superparabolic, we need to show that the last bracket is non-positive. Since regularity is a local property by Theorem 3.4(b), we may restrict our considerations to a small neighbourhood of $\xi_0$.

If $x_1 \neq 0$ then, in view of Theorem 3.4(b), we can assume that $(x, t) \in \Theta$ satisfy $|x|, |t| < \delta := \frac{1}{2}|x_1|$. In particular, $|x - x_1| > \delta$, $\nabla w(x, t) \neq 0$ and $t_1 - t < t_1 + \delta$. Hence we can choose $j$ so that the bracket in (4.1) is non-positive and thus $\Delta_p^N w(x, t) - w_t(x, t) \leq 0$ for all such $x$ and $t$. By Proposition 2.2, we get that $w$ is superparabolic.
If, on the other hand, \( x_1 = 0 \) and \( t_1 = -R_1 \), then we can assume that \( t > n + p - 2 - R_1 \) (which is negative by assumption) whenever \( (x, t) \in \Theta \). In particular, \( w_1(x, t) > 0 \). Moreover, if \( \nabla w(x, t) \neq 0 \), then
\[
\Delta^N w(x, t) - w_1(x, t) \leq 2je^{-jR^2}|n + p - 2 - R_1 - t| < 0.
\]
Hence \( w \) is superparabolic, by Proposition 2.2.

**Lemma 4.3.** Let \( V \) be a tusk at \( \xi_0 = (0, 0) \), determined by \( T = 1, R \) and \( \hat{x} \). Then \( \xi_0 \) is regular with respect to the domain \( \Theta_0 = \hat{\Theta} \setminus \overline{V} \), where
\[
\hat{\Theta} = (B_{R_0} \times (-1, 0]) \cup \{(x, t) \in \mathbb{R}^{n+1} : |x| < R_0(1 - t) \text{ and } 0 \leq t < 1 \}
\]
for some \( R_0 > |\hat{x}| + R \) (see Figure 1). Moreover, the viscosity solution \( u := \overline{T}_{\Theta_0} f \), with
\[
f(x, t) = \begin{cases} -t, & \text{if } (x, t) \in \partial \Theta_0 \cap \partial V, \\ 1, & \text{if } (x, t) \in \partial \Theta_0 \setminus \partial V, \end{cases}
\]
is a positive continuous barrier in \( \Theta_0 \), which is Hölder continuous at \( \xi_0 \) and continuously attains its boundary values \( f \) everywhere on \( \partial \Theta_0 \).

Here and below, we mean Hölder continuity with respect to parabolic scaling, that is, \( g \) is Hölder continuous at \( (0, 0) \) with Hölder exponent \( \beta \) if
\[
|g(x, t) - g(0, 0)| \leq C(|x| + |t|^{1/2})^\beta.
\]

**Proof.** Continuity of \( u \) within \( \Theta_0 \) is clear since it is a viscosity solution therein. By the exterior ball condition (Lemma 4.2), all \( (x, t) \in \partial \Theta_0 \setminus \{\xi_0\} \) are regular and hence
\[
\lim_{\xi \to (x, t)} u(\xi) = f(x, t) > 0 \quad \text{for all } (x, t) \in \partial \Theta_0 \setminus \{\xi_0\}.
\]
(4.2)

From the strong minimum principle (Theorem 2.13), together with (4.2) and the fact that viscosity solutions are preserved under multiplication and addition by constants, we conclude that \( 0 < u < 1 \) in \( \Theta_0 \).

To show that \( u \) is a barrier, it suffices to show that \( \lim_{\Theta_0 \ni \xi \to \xi_0} u(\xi) = 0 \). To this end, let \( v(x, t) = u(2x, 4t) \) and \( \Theta_k = \hat{\Theta}_k \setminus \overline{V} \), where
\[
\hat{\Theta}_k = \{(x, t) \in \mathbb{R}^{n+1} : (2^k x, 4^k t) \in \hat{\Theta}\}, \quad k = 0, 1, \ldots,
\]
see Figure 1. Note that \( \Theta_{k+1} \subset \Theta_k \), \( k = 0, 1, \ldots \), with identical boundaries near \( \xi_0 \), and that
\[
K := \partial \Theta_1 \setminus \partial V \subset \overline{\Theta}_0
\]
is compact. Hence, by continuity and (4.2), we see that \( \alpha_1 := \sup_K u < 1 \). At the same time,
\[
v = 1 \quad \text{on } K
\]
and \( v \) attains the boundary values
\[
v(x, t) = -4t = 4u(x, t) \quad \text{on } \partial \Theta_1 \cap \partial V \setminus \{\xi_0\}.
\]
The parabolic comparison principle (Theorem 2.10), applied to \( \Theta_1^- := \{(x, t) \in \Theta_1 : t < 0\} \), implies that
\[
u \leq \alpha v \quad \text{in } \Theta_1^-,
\]
(4.4)
where \( \alpha = \max\{\alpha_1, \frac{1}{4}\} \). Thus, if
\[
A := \limsup_{\Theta_1^- \ni \xi \to \xi_0} u(\xi),
\]
\[
0 < A < 1.
\]
then $0 \leq A \leq \alpha A$, from which it follows that $A = 0$. At the same time, since $0 < u < 1$ in $\Theta_0$, we conclude from the continuity of $u$ in $\Theta_0$ that

$$\liminf_{\Theta_0 \ni \xi \to (x,0)} u(\xi) = u(x,0) > 0$$

whenever $x \neq 0$. Thus, $u$ is a barrier for $\Theta_0$ at $\xi_0$ and $\xi_0$ is regular for $\Theta_0$. Theorem 3.4 then implies that $\xi_0$ is regular for $\Theta_0$ as well. In particular, this means that $\lim_{\Theta_0 \ni \xi \to \xi_0} u(\xi) = 0$.

We shall now show that $u$ is H"older continuous at $\xi_0$. From the first part of the proof we see that

$$\limsup_{\Theta_0 \ni \xi \to (x,t)} u(\xi) \leq \liminf_{\Theta_0 \ni \xi \to (x,t)} \alpha v(\xi)$$

for all $(x,t) \in \partial \Theta$. The elliptic comparison principle (Theorem 2.9) then implies that $u \leq \alpha v$ in $\Theta_1$. An iteration of this inequality then gives for $(x,t) \in \Theta_k \setminus \Theta_{k+1}$ that

$$u(x,t) \leq \alpha u(2^{2k}x,4^{2k}t) \leq \ldots \leq \alpha^k u(2^{k}x,4^{k}t) \leq \alpha^k \leq C(|x| + |t|^{1/2})^\beta,$$  \hspace{1cm} (4.5)

where $\beta = -\log \alpha/\log 2 > 0$. Since this holds for all $k = 1,2,\ldots$, we see that $u$ is H"older continuous at $\xi_0$.

\textbf{Theorem 4.4 (The tusk condition).} If $\Theta$ satisfies the tusk condition at $\xi_0$ then $\xi_0$ is regular. If moreover, $f : \partial \Theta \to \mathbb{R}$ is bounded and H"older continuous at $\xi_0$ then so is $\overline{P}f$.

It follows from the proof below that if $f$ is bounded and H"older continuous at $\xi_0$ with H"older exponent $\gamma > 0$, and $\gamma$ is small enough, then $f$ is H"older continuous at $\xi_0$ with H"older exponent $\frac{1}{2}\gamma$. How small $\gamma$ has to be depends on the tusk.

In fact, replacing the scaling $(2^{k}x,4^{k}t)$ in (4.3) by $(b^{k}x,b^{k}t)$ with any $b > 1$, and $l = k, k + 1$ by $l = k, l_k + 1,\ldots,l_{k+1}$, where $l_k = [-\gamma k \log b/\log \alpha]$, in the proof below, makes it possible to obtain H"older continuity at $\xi_0$ with any exponent $\beta < \gamma$, at the cost of an increasing constant $C''$ in (4.6). We leave the details to the interested reader.

\textbf{Proof.} We can assume that $\xi_0 = (0,0)$. The regularity of $\xi_0$ follows from Lemma 4.3 by means of the restriction property (Proposition 3.5) and the fact that regularity is a local property, by Theorem 3.4.

To prove the H"older regularity, assume that $f : \partial \Theta \to \mathbb{R}$ is H"older continuous near $\xi_0$ with H"older exponent $\gamma$ and that $|f| \leq M$ on $\partial \Theta$. We can also assume that $f(0,0) = 0$. Let $k \geq 0$ be arbitrary, but such that $\Theta \cap \hat{\Theta}_k \subset \Theta_k$ and

$$|f(x,t)| \leq C(|x| + |t|^{1/2})^\gamma \quad \text{on} \ \partial \Theta \cap \hat{\Theta}_k,$$

where $\hat{\Theta}_k$ and $\Theta_k$ are as in (4.3). Let $u$ be the barrier from Lemma 4.3 and $\alpha$ be the corresponding constant from (4.4). Set $u_k(x,t) = u(2^{k}x,4^{k}t)$ in $\Theta_k$. Extend $u_k$ by 1 to $\Theta \setminus \Theta_k$ and then by continuity to $\partial \Theta$. By the pasting Lemma 2.12, $u_k$ is superparabolic in $\Theta$, and provides us with a H"older continuous barrier therein, in view of Lemma 4.3.

Since $u_k = 1$ on $\partial \Theta \setminus \hat{\Theta}_k$, we have $f \leq C'2^{-\gamma k} + M u_k$ everywhere on $\partial \Theta$. Hence, by the definition of Perron solutions, $\overline{P}f \leq C'2^{-\gamma k} + M u_k$ in $\Theta$. Thus, for $l \geq 1$ and $(2^{k}x,4^{k}t) \in \Theta \cap (\hat{\Theta}_l \setminus \hat{\Theta}_{l+1})$ we conclude from (4.5) that

$$\overline{P}f(x,t) \leq C'2^{-\gamma k} + M u(2^{k}x,4^{k}t) \leq C'2^{-\gamma k} + M \alpha^l.$$

In particular, with $l = k$ and $l = k + 1$, that is, for $(x,t) \in \Theta \cap (\hat{\Theta}_{2k} \setminus \hat{\Theta}_{2(k+1)})$,

$$\overline{P}f(x,t) \leq C''(|x| + |t|^{1/2})^\beta,$$  \hspace{1cm} (4.6)
where $\beta = \min\{\gamma/2, -\log \alpha/2 \log 2\} > 0$. Applying the same argument to $-f$ and by considering all sufficiently large $k$ shows that $\overline{P}f$ is Hölder continuous at $\xi_0$. \hfill $\Box$

As a direct consequence of the tusk condition we can now deduce the following ‘wedge’ condition for cylinders.

**Corollary 4.5.** Let $G \subset \mathbb{R}^n$ be open and $\Theta = G_{t_1,t_2}$. Let $(x_0, t_0) \in \partial G \times [t_1, t_2]$ be a point on the lateral boundary. Assume that there is $\alpha > 0$ and a vector $y \in \mathbb{R}^n$ such that the cone

$$\{x \in \mathbb{R}^n : (x - x_0) \cdot y > \alpha|x - x_0|\}$$

belongs to the complement $\mathbb{R}^n \setminus G$ of $G$. Then $(x_0, t_0)$ is a regular boundary point for $\Theta$.

**Remark 4.6.** The proofs of Lemma 4.3 and Theorem 4.4 reveal that the tusk $V$ therein can be replaced by the following union of geometrically spaced ellipses,

$$E_k = \left\{(x, t) \in \mathbb{R}^{n+1} : \left(\frac{|x - x_k|}{a_k}\right)^2 + \left(\frac{t - t_k}{b_k}\right)^2 < 1\right\},$$

where $x_k = q^k \hat{x}$, $a_k = a q^k$, $b_k = b q^k$ and $t_k = -a \hat{q}^k$ for some $a, b, c > 0$, $\hat{x} \in \mathbb{R}^n$ and $0 < q < 1$. More precisely, assuming that $\Theta \cap E_k = \emptyset$, $k = 1, 2, \ldots$, we have that $\xi_0 = (0, 0)$ is regular for $\Theta$, with a Hölder continuous barrier. Moreover, Hölder continuity of the boundary data $f$ at $\xi_0$ implies Hölder continuity of $\overline{P}f$ at $\xi_0$.

Similarly, the ‘wedge’ condition (4.7) in Corollary 4.5 can be replaced by the requirement that

$$G \cap B_k = \emptyset, \quad k = 1, 2, \ldots,$$

where $x_k$ and $a_k$ are as above, and $B_k = \{x \in \mathbb{R}^n : |x - x_k| < a_k\}$.

For the non-normalized $p$-parabolic equation (with $p > 1$) it was shown in Kilpeläinen–Lindqvist [19] and Björn–Björn–Gianazza–Parviainen [7, Theorem 3.9] that a point $(x_0, t_0)$ on the lateral boundary of a cylinder $G_{t_1,t_2} \subset \mathbb{R}^{n+1}$ is regular if and only if $x_0$ is regular for $p$-harmonic functions with respect to $G \subset \mathbb{R}^n$. The main result in Banerjee–Garofalo [4, Theorem 1.1] says that the same equivalence holds for the normalized $p$-parabolic equation provided that $p \geq 2$. However, due to the power $2 - p$ in $\Delta_p u = |\nabla u|^{2-p} \Delta_p u$, which leads to the singular right-hand side in $\Delta_p u = -|\nabla u|^{p-2} \Delta_p u$, they did not cover the case $p < 2$. Corollary 4.5 and Remark 4.6 are currently the best known results about boundary regularity for cylinders when $p < 2$. Note, however, that the necessity part of the proof of [4, Theorem 1.1] (that is, from the regularity of $(x_0, t_0)$ to the regularity of $x_0$) holds true also for $p < 2$.

We end this section by deducing the full exterior ball condition.

**Proposition 4.7 (Exterior ball condition).** Let $\xi_0 = (x_0, t_0) \in \partial \Theta$. Suppose that there exists a ball $B = B(\xi_1, R_1)$, $\xi_1 = (x_1, t_1)$, such that $B \cap \Theta = \emptyset$ and $\xi_0 \in \partial B \cap \partial \Theta$. If $x_1 \neq x_0$, or if $\xi_0$ is the north pole of $B$ (that is, $\xi_1 = (x_0, t_0 - R_1)$), then $\xi_0$ is regular with respect to $\Theta$.

Note that this result is a direct corollary of the tusk condition (Theorem 4.4), since the exterior ball condition is always a stronger requirement than the tusk condition. Nevertheless, we only need to directly appeal to the tusk condition for the north pole case.

**Proof.** Apart from the north pole case this follows from Lemma 4.2, while the north pole case follows directly from the tusk condition (Theorem 4.4). \hfill $\Box$
Note that the well-known irregularity of non-lateral last points in cylinders shows that an exterior ball touching at the south pole does not guarantee regularity, which leads us directly into the topic of the next section.

5. The Petrovskii criterion

In this section, we consider the regularity of the last point of a domain. Non-lateral last points in cylinders are known to be irregular. On the other hand, last points of paraboloids are regular by, for example, the tusk condition. The idea in the Petrovskii criterion is to find a sharper regularity condition for the shape of the domain near a last point. This condition has also interesting consequences. Just as for the heat equation, Theorem 5.1, together with a simple scaling argument, shows that regularity of a boundary point for the multiplied equation

\[ au_t - \Delta_p^N u = 0, \quad \text{with } a > 0, \]

depends on \( a \).

**Theorem 5.1.** Let

\[ \Theta := \{ (x, t) : |x|^2 < A|t| \log |t| \text{ and } -\frac{1}{3} < t < 0 \}, \]

where \( A > 0 \). Then \( \xi_0 = (0, 0) \) is regular if and only if \( A \leq 4(p - 1) \).

In view of Theorem 3.4(b) the constant \(-\frac{1}{3}\) can be replaced by any other negative constant, but here it has been chosen so that \( \log |t| > 0 \) for all such \( t \).

**Proof.** We first consider the case when \( 0 < A \leq 4(p - 1) \), in which case we shall show regularity by constructing a barrier. Let

\[ k = 4(p - 1), \quad f(t) = |\log |t||^{-a - 1}, \]

\[ a = \frac{n + p - 2}{k} > 0, \quad h(t) = 2|\log |t||^{-a}, \]

where we only consider \( t \in \left[ -\frac{1}{3}, 0 \right) \) throughout the proof. We see that

\[ f'(t) = -\frac{a + 1}{|t||\log |t||^{a+2}} < 0 \quad \text{and} \quad h'(t) = -\frac{2a}{|t||\log |t||^{a+1}} = -\frac{2a}{|t|} f(t) < 0. \tag{5.1} \]

We want to show that

\[ u(\xi) = -f(t)e^{\frac{|x|^2}{k|t|}} + h(t) \]

is a barrier at \( \xi_0 \), where we write \( \xi = (x, t) \) from now on.

First, note that \( u \in C^2(\overline{\Theta} \setminus \{\xi_0\}) \) and it is positive if and only if

\[ 2|\log |t|| > e^{\frac{|x|^2}{k|t|}}, \]

that is, if and only if

\[ |x|^2 < k|t| \log |\log |t|| + k|t| \log 2, \]

which holds in \( \overline{\Theta} \setminus \{\xi_0\} \) since \( A \leq k \). Moreover,

\[ \lim_{\Theta \ni \xi \to \xi_0} u(\xi) = 0. \]
It remains to show that \( u \) is superparabolic in \( \Theta \), to conclude that \( u \) is a barrier. Since \( u \in C^2(\Theta) \) we will show this using Proposition 2.2. To this end, we see that

\[
\nabla u(\xi) = -\frac{2f(t)}{k|t|} e^{x^2/k|t|} x,
\]

\[
|\nabla u(\xi)|^{p-2} \nabla u(\xi) = -\left(\frac{2f(t)}{k|t|}\right)^{p-1} e^{(p-1)\frac{|x|^2}{k|t|}} |x|^{p-2} x,
\]

\[
\Delta_p u(\xi) = -\left(\frac{2f(t)}{k|t|}\right)^{p-1} e^{(p-1)\frac{|x|^2}{k|t|}} \left(\frac{2(p-1)}{k|t|}|x|^p + (n + 2)|x|^{p-2}\right).
\]

Thus, if in addition \( \nabla u(\xi) \neq 0 \), we have

\[
\Delta_p^N u(\xi) = -\frac{2f(t)}{k|t|} e^{x^2/k|t|} \left(\frac{2(p-1)}{k|t|}|x|^2 + n + p - 2\right).
\]

Also

\[
u_t(\xi) = e^{x^2/k|t|} \left(-f'(t) - f(t) \frac{|x|^2}{kt^2} + h'(t)e^{-|x|^2/k|t|}\right), \tag{5.2}
\]

which together yields, still provided that \( \nabla u(\xi) \neq 0 \),

\[
u_t(\xi) - \Delta_p^N u(\xi) = e^{x^2/k|t|} \left(-f'(t) - f(t) \frac{|x|^2}{kt^2} (k - 4(p - 1)) + \frac{2f(t)(n + p - 2)}{k|t|} + h'(t)e^{-|x|^2/k|t|}\right).
\]

Using (5.1) and that \( k = 4(p - 1) \), we then obtain that

\[
u_t(\xi) - \Delta_p^N u(\xi) \geq e^{x^2/k|t|} f(t) \left(\frac{2(n + p - 2)}{k|t|} - \frac{2a}{|t|}\right) = 0.
\]

When \( \nabla u(x, t) = 0 \), we see that \( x = 0 \). Moreover,

\[
\partial_i \partial_j u(x, t) = -\frac{2f(t)}{k|t|} e^{x^2/k|t|} \left(\delta_{ij} + \frac{2x_i x_j}{k|t|}\right),
\]

so \( D^2 u(0, t) \) is negative definite, as \( f(t) \) and \( k \) are positive. The requirements in Proposition 2.2 are thus met (even though \( u_t(0, t) < 0 \) for \( t \) close to 0), so \( u \) is superparabolic in \( \Theta \). Hence it is a barrier, and Theorem 3.4 shows that \( \xi_0 \) is regular if \( A \leq 4(p - 1) \).

Now we turn to the case \( A > 4(p - 1) \), in which case we shall show irregularity by producing a so-called ‘irregularity barrier’. To be more precise, \( u \) is an ‘irregularity barrier’ if it can be used as a comparison function to show that the upper Perron solution does not attain its boundary values continuously at \( \xi_0 = (0, 0) \). We will construct \( u \) such that

(i) \( u \) is subparabolic in \( \Theta' = U \cap \Theta \) for some open neighbourhood \( U \) of \( \xi_0 \);

(ii) \( u \) has an extension to \( \overline{\Theta} \) so that both \( u|_{\overline{\Theta} \setminus \{\xi_0\}} \) and \( u|_{\partial \Theta'} \) are continuous;

(iii) \( \lim_{t \to 0^-} u(0, t) = 0 > u(\xi_0) \).

First choose \( k \) such that

\[
4(p - 1) < k < A
\]
and let

\[ a = \frac{A}{k} - 1 > 0 \quad \text{and} \quad b = \frac{4(n + p - 2)}{k} > 0. \]

Note that the parameters \( a \) and \( k \), as well as the function \( h \) below, are not the same as in the first part of the proof. The functions

\[ f(t) = |\log |t||^{-a} \quad \text{and} \quad h(t) = \frac{2b}{a|\log |t||^{a/2}}, \]

are positive for \(-\frac{1}{3} \leq t < 0\), considered in this proof. Moreover,

\[ f'(t) = -\frac{f(t)}{|t|} \frac{a+1}{|\log |t||} \quad \text{and} \quad h'(t) = -\frac{b}{|t||\log |t||^{1+a/2}} = -\frac{bf(t)}{|t||\log |t||^{a/2}}. \]

We want to show that

\[ u(\xi) = -f(t)e^{t^{2}/k|t|} + h(t) \]

is an ‘irregularity barrier’ for small enough \( t \). We first observe that

\[ \lim_{t \to 0^-} u(0, t) = 0, \]

while if \( \xi \in \partial \Theta \) and \(-\frac{1}{3} < t < 0\), then since \(-a - 1 + A/k = 0\),

\[ u(\xi) = -|\log |t||^{-a} e^{(A/k)\log |t|} + h(t) \]

\[ = -|\log |t||^{-a} e^{A/k} + h(t) = h(t) - 1 \to -1, \quad \text{as} \ t \to 0. \]

We will show that \( u \) is subparabolic in

\[ \Theta' := \{(x, t) \in \Theta : t > -\tau\}, \]

for some \( 0 < \tau < \frac{1}{3} \) which will be determined later. For now, we take this fact for granted and show how it implies that \( u \) is an ‘irregularity barrier’ in \( \Theta' \), and how this yields the irregularity of \( \xi_0 \). Let

\[ \tilde{u}(\xi) = \begin{cases} 
    u(\xi), & \text{if } \xi \in \partial \Theta' \setminus \{\xi_0\}, \\
    -1, & \text{if } \xi = \xi_0.
\end{cases} \]

Observe that \( \tilde{u} \in C(\partial \Theta') \), and let \( v \in \mathcal{U}_a \). Then,

\[ \limsup_{\Theta' \ni \xi \to \xi_0} u(\xi) = \liminf_{\Theta' \ni \xi \to \xi_0} v(\xi) \quad \text{for all } \xi \in \partial \Theta' \setminus \{\xi_0\}. \]

Hence, by Theorem 2.10, \( u \leq v \) in \( \Theta' \), and thus \( u \leq \overline{\mathcal{P}}_{\Theta'} \tilde{u} \) in \( \Theta' \). Therefore

\[ \limsup_{\Theta' \ni \xi \to \xi_0} \overline{\mathcal{P}}_{\Theta'} \tilde{u}(\xi) \geq \lim_{t \to 0^-} u(0, t) = 0 > \tilde{u}(\xi_0), \]

showing that \( \xi_0 \) is irregular with respect to \( \Theta' \). By Theorem 3.4(b), \( \xi_0 \) is irregular also with respect to \( \Theta \).

It remains to verify that \( u \) is subparabolic in \( \Theta' \), if \( \tau \) is small enough. As in the first part, we get, provided that \( \nabla u(\xi) \neq 0 \),

\[ u_t(\xi) - \Delta_p^N u(\xi) = e^{t^{2}/k|t|} \left( -f'(t) - \frac{f(t)}{k|t|^2} (k - 4(p - 1)) + \frac{2f(t)(n + p - 2)}{k|t|} + h'(t)e^{-t^{2}/k|t|} \right) \]
\[
e^{\frac{c|x|^2}{k^2|t|}} \frac{f(t)}{|t|} \left( \frac{a + 1}{|\log |t||} - \frac{c|x|^2}{k^2|t|} + \frac{b}{2} - be^{-\frac{|x|^2}{k|t|}|\log |t||^{a/2}} \right),
\]
where \( c = k - 4(p - 1) > 0 \). We will need three conditions on \( \tau \). The first is that it is so small that
\[
a + 1 \leq b \quad \text{for} \quad -\tau < t < 0,
\]
which we assume from now on. To show that \( u_t - \Delta_p u \leq 0 \), it therefore suffices to verify that
\[
c|x|^2 \leq b| \log |t||^{a/2}
\]
for
\[
|\log |t||^{a/2} \leq b,
\]
while if \( |x|^2 > \frac{1}{2} ak|t| \log |t| \), then
\[
be^{-\frac{|x|^2}{k|t||}} |\log |t||^{a/2} \geq be^{-\frac{a}{\log |t|}} |\log |t||^{a/2} = b,
\]
provided that \( \tau \) is small enough. Hence, (5.3) holds in both cases.
Moreover, if \( \nabla u(x,t) = 0 \), then \( x = 0 \), and from (5.2) we see that
\[
w_t(0,t) = h(t) - f(t) = \frac{f(t)}{|t|} \left( \frac{a + 1}{|\log |t||} - b|\log |t||^{a/2} \right) < 0 \quad \text{for} \quad -\tau < t < 0,
\]
provided that \( \tau \) is small enough.
Hence \( u \) is subparabolic in \( \Theta' \), by Proposition 2.2, if \( \tau \) is chosen small enough, which concludes the proof.

Acknowledgements. This research started in the fall of 2015, and was in particular conducted during several visits of the authors to Jyväskylä and Linköping, respectively, in 2015–2016. We thank these institutions for their kind hospitality.

References
1. A. Attouchi and M. Parviainen, ‘Hölder regularity for the gradient of the inhomogeneous parabolic normalized \( p \)-Laplacian’, Commun. Contemp. Math. 20 (2018) 1750035, https://doi.org/10.1142/S0219199717500353.
2. I. Babuška and R. Výborný, ‘Reguläre und stabile Randpunkte für das Problem der Wärmeleitungsgleichung’, Ann. Polon. Math. 12 (1962) 91–104.
3. A. Banerjee and N. Garofalo, ‘Gradient bounds and monotonicity of the energy for some nonlinear singular diffusion equations’, Indiana Univ. Math. J. 62 (2013) 699–736.
4. A. Banerjee and N. Garofalo, ‘On the Dirichlet boundary value problem for the normalized \( p \)-Laplacian evolution’, Commun. Pure Appl. Anal. 14 (2015) 1–21.
5. H. Bauer, ‘Axiomatische Behandlung des Dirichletschen Problems für elliptische und parabolische Differentialgleichungen’, Math. Ann. 146 (1962) 1–59.
6. A. Björn, J. Björn and U. Gianazza, ‘The Petrovskii criterion and barriers for degenerate and singular \( p \)-parabolic equations’, Math. Ann. 368 (2017) 885–904.
7. A. Björn, J. Björn, U. Gianazza and M. Parviainen, ‘Boundary regularity for degenerate and singular parabolic equations’, Calc. Var. Partial Differential Equations 52 (2015) 797–827.
8. Y. G. Chen, Y. Giga and S. Goto, ‘Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations’, J. Differential Geom. 33 (1991) 749–786.
9. M. G. Crandall, H. Ishii and P.-L. Lions, ‘User’s guide to viscosity solutions of second order partial differential equations’, Bull. Amer. Math. Soc. (N.S.) 27 (1992) 1–67.
10. K. Does, ‘An evolution equation involving the normalized \( p \)-Laplacian’, Commun. Pure Appl. Anal. 10 (2011) 361–396.
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11. E. Effros and J. L. Kazdan, ‘On the Dirichlet problem for the heat equation’, Indiana Univ. Math. J. 20 (1970) 683–693.
12. L. C. Evans and R. F. Gariepy, ‘Wiener’s criterion for the heat equation’, Arch. Ration. Mech. Anal. 78 (1982) 293–314.
13. E. B. Fabes, N. Garofalo and E. Lanconelli, ‘Wiener’s criterion for divergence form parabolic operators with $C^1$–Dini continuous coefficients’, Duke Math. J. 59 (1989) 191–232.
14. Y. Giga, Surface evolution equations, Monographs in Mathematics 99 (Birkhäuser, Basel, 2006).
15. C. Imbert, T. Jin and L. Silvestre, ‘Hölder gradient estimates for a class of singular or degenerate parabolic equations’, Adv. Nonlinear Anal. 8 (2019) 845–867.
16. C. Imbert and L. Silvestre, ‘An introduction to fully nonlinear parabolic equations’, An introduction to the Kähler–Ricci flow, Lecture Notes in Mathematics 2086 (Springer, Berlin–Heidelberg, 2013) 7–88.
17. T. Jin and L. Silvestre, ‘Hölder gradient estimates for parabolic homogeneous $p$-Laplacian equations’, J. Math. Pures Appl. 108 (2017) 63–87.
18. P. Juutinen, P. Lindqvist and J. J. Manfredi, ‘On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation’, SIAM J. Math. Anal. 33 (2001) 699–717.
19. T. Kilpeläinen and P. Lindqvist, ‘On the Dirichlet boundary value problem for a degenerate parabolic equation’, SIAM J. Math. Anal. 27 (1996) 661–683.
20. E. M. Landis, ‘Necessary and sufficient conditions for the regularity of a boundary point for the Dirichlet problem for the heat equation’, Dokl. Akad. Nauk SSSR 185 (1969) 517–520 (Russian). English translation: Soviet Math. Dokl. 10 (1969) 380–384.
21. E. M. Landis, ‘Regularity of a boundary point for the heat equation’, Qualitative theory of boundary value problems of mathematical physics (eds V. K. Kalantarov and I. T. Mamedov; Ehlm, Baku, 1991) 69–96 (Russian).
22. G. M. Lieberman, ‘Intermediate Schauder theory for second order parabolic equations. III. The tusk conditions’, Appl. Anal. 33 (1989) 25–43.
23. J. J. Manfredi, M. Parviainen and J. D. Rossi, ‘An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games’, SIAM J. Math. Anal. 42 (2010) 2058–2081.
24. M. Parviainen and E. Ruosteenoja, ‘Local regularity for time-dependent tug-of-war games with varying probabilities’, J. Differential Equations 261 (2016) 1357–1398.
25. I. Petrovskii, ‘Zur ersten Randwertaufgabe der Wärmeleitungsgleichung’, Compos. Math. 1 (1935) 383–419.
26. A. Tikhonov, ‘Sur l’équation de la chaleur a plusieurs variables’, Bull. Univ. Moscow Sect. A Math. Mech. (Ser. Int.) 9 (1938) 1–44. Russian simultaneous version: Bull. Univ. Moscow Sect. A Math. Mech. (Ser. Rus.) 9 (1938) 1–45 (Russian).
27. N. Uboñad, ‘On the normalized $p$-parabolic equation in arbitrary domains’, Preprint, 2017, arXiv:1711.11369.
28. L. Wang, ‘On the regularity theory of fully nonlinear parabolic equations. I’, Comm. Pure Appl. Math. 45 (1992) 27–76.
29. N. A. Watson, Introduction to heat potential theory, Mathematical Surveys and Monographs 182 (American Mathematical Society, Providence, RI, 2012).

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