The normalized Laplacian spectra of the double corona based on $R$-graph

Ping-Kang Yu, Gui-Xian Tian∗

College of Mathematics, Physics and Information Engineering, Zhejiang Normal University, Jinhua, Zhejiang, 321004, P.R. China

Abstract

For simple graphs $G$, $G_1$ and $G_2$, we denote their double corona based on $R$-graph by $G^{(R)} \otimes \{G_1, G_2\}$. This paper determines the normalized Laplacian spectrum of $G^{(R)} \otimes \{G_1, G_2\}$ in terms of these of $G$, $G_1$ and $G_2$ whenever $G$, $G_1$ and $G_2$ are regular. The obtained result reduces to the normalized Laplacian spectra of the $R$-vertex corona $G^{(R)} \otimes G_1$ and $R$-edge corona $G^{(R)} \otimes G_2$ by choosing $G_2$ or $G_1$ as a null-graph, respectively. Finally, applying the results of the paper, we construct infinitely many pairs of normalized Laplacian cospectral graphs.

AMS classification: 05C50 05C90
Keywords : normalized Laplacian spectrum; double corona; $R$-graph; regular graph

1 Introduction

Throughout this paper, all graphs considered are finite simple graphs. Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The adjacency matrix $A(G)$ of $G$ is an $n \times n$ matrix whose $(i, j)$-entry is 1 if $v_i$ and $v_j$ are adjacent in $G$ and 0 otherwise. The degree of $v_i$ in $G$ is denoted by $d_i = d_G(v_i)$. Let $D(G)$ be the degree diagonal matrix of $G$ with diagonal entries $d_1, d_2, \ldots, d_n$. The normalized Laplacian matrix $\mathcal{L}(G)$ of $G$ is defined as $I_n - D(G)^{-1/2}A(G)D(G)^{-1/2}$, where $I_n$ denotes the identity matrix of order $n$. Denote the characteristic polynomial $\det(xI_n - \mathcal{L}(G))$ of $\mathcal{L}(G)$ by $\phi(G;x)$. Since $\mathcal{L}(G)$ is a symmetric and positive semi-definite matrix. Then its eigenvalues, denoted by $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$, are all real, non-negative and can be arranged in non-decreasing order $0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$. The set of all eigenvalues of $\mathcal{L}(G)$ is called the normalized Laplacian spectrum of $G$.

The normalized Laplacian matrix $\mathcal{L}(G)$, which is consistent with the transition probability matrix $P(G) = D(G)^{-1}A(G)$ in the random walk on $G$ and spectral geometry[8], has attracted people’s attention. For instance, Banerjee and Jost[1] studied how the normalized Laplacian spectrum is affected by operations such as motif doubling, graph splitting and joining. Huang and Li[13] studied the normalized Laplacian spectrum of some graph operations, such as subdivision graph, $Q$-graphs, $R$-graphs and so on. Butler and Grout[4] constructed many pairs of non-regular normalized Laplacian cospectral graphs. Chen et al.[5] gave an interlacing inequality on the normalized Laplacian eigenvalues of $G$. Chen and Zhang[7] obtained two formulae for the resistance distance and degree-Kirchhoff index in terms of the normalized Laplacian eigenvalues of $G$.

∗Corresponding author. E-mail address: gxtian@zjnu.cn or guixiantian@163.com.

*Corresponding author. E-mail address: gxtian@zjnu.cn or guixiantian@163.com.
eigenvalues and eigenvectors of $G$ and so on. For more review about the normalized Laplacian spectrum of graphs, readers may refer to [8]. Recently, Chen and Liao [6] determined the normalized Laplacian spectra of the (edge)corona for two graphs. Furthermore, they also obtained the degree-Kirchhoff index and the number of spanning trees of these graphs. In [11], the normalized Laplacian spectra of some subdivision-coronas for two regular graphs were computed by Das and Panigrahi. This paper considers the normalized Laplacian spectrum of double corona based on $R$-graph. We first recall that the $R$-graph [10] of a graph $G$, denoted by $G^{(R)}$, is the graph obtained from $G$ by adding a new vertex corresponding to each edge of $G$ and by joining each new vertex to the endpoints of the edge corresponding to it. The following graph operation based on $R$-graph comes from [2].

Definition 1.1 [2]. Let $G$ be a connected graph on $n$ vertices and $m$ edges. Let $G_1$ and $G_2$ be graphs on $n_1$ and $n_2$ vertices, respectively. The $R$-graph double corona of $G$, $G_1$ and $G_2$, denoted by $G^{(R)} \otimes \{G_1, G_2\}$, is the graph obtained by taking one copy of $G^{(R)}$, $n$ copies of $G_1$ and $m$ copies of $G_2$, and then by joining the $i$-th old-vertex of $G^{(R)}$ to every vertex of the $i$-th copy of $G_1$ and the $j$-th new-vertex of $G^{(R)}$ to every vertex of the $j$-th copy of $G_2$.

We remark that here $R$-graph double corona reduces to the $R$-vertex corona $G^{(R)} \circ G_1$ or $R$-edge corona $G^{(R)} \circ G_2$ (see [15] for more information) whenever we choose $G_2$ or $G_1$ as a null-graph in Definition 1.1, respectively.

In [2], Barik and and Sahoo determined the Laplacian spectra of $R$-graph double corona for regular graph $G$ and any two graphs $G_1$ and $G_2$. Song et al. [17] computed the spectra and Laplacian spectra of double corona based on subdivision graph. As applications, they determined the number of spanning trees of the double corona based on subdivision graph and constructed infinitely many pairs of cospectral(Laplacian cospectral) graphs. Recently, Lan and Zhou [15] characterized the spectra, Laplacian and signless Laplacian spectra of $R$-vertex corona and $R$-edge corona. At the same time, they also constructed infinitely many pairs of cospectral, Laplacian cospectral and signless Laplacian cospectral graphs.

Motivated by the works above, we focus on determining the normalized Laplacian spectrum of $R$-graph double corona $G^{(R)} \otimes \{G_1, G_2\}$ in terms of those of regular graphs $G$, $G_1$ and $G_2$ (see Theorem 2.3). As a special case, we give the normalized Laplacian spectra of the $R$-vertex corona $G^{(R)} \circ G_1$ and $R$-edge corona $G^{(R)} \circ G_2$ by choosing $G_2$ or $G_1$ as a null-graph, respectively (see Corollaries 2.4 and 2.5). Finally, applying these results, we construct infinitely many pairs of normalized Laplacian cospectral graphs.

2 Main results

In this section, we determine the normalized Laplacian spectrum of $G^{(R)} \otimes \{G_1, G_2\}$ in terms of those of regular graphs $G$, $G_1$ and $G_2$. To prove our results, we need some preliminaries. For two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of same size $m \times n$, the Hadamard product $A \circ B = (c_{ij})$ of $A$ and $B$ is a matrix of the same size $m \times n$ with entries $c_{ij} = a_{ij}b_{ij}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Similarly, the Kronecker product $A \otimes B$ of matrices $A = (a_{ij})$ of size $m \times n$ and $B$ of size $p \times q$ is the $mp \times nq$ partition matrix $a_{ij}B$. It is proved [12] that $AB \otimes CD = (A \otimes C)(B \otimes D)$, whenever the products $AB$ and $CD$ exist. Moreover, $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ for two nonsingular matrices $A$ and $B$. If $A$ and $B$ are two matrices of order $n$ and $p$ respectively, then $\det(A \otimes B) = (\det A)^p(\det B)^n$. For more review about the Kronecker product, see [12].

Throughout this paper, $1_n$ denotes the column vector of size $n$ with all the entries equal to one. Let $G$ be a graph on $n$ vertices and $B$ be a matrix of order $n$. For any parameter $\lambda$, we
Lemma 2.2. Assume that the order of all four matrices $M_1, M_2, M_3$ and $M_4$ satisfy the rules of operations on matrices. If $M_1$ and $M_4$ are invertible, then

$$\det \left( \begin{array}{cc} M_1 & M_2 \\ M_3 & M_4 \end{array} \right) = \det M_4 \cdot \det (M_1 - M_2 M_4^{-1} M_3) = \det M_1 \cdot \det (M_4 - M_3 M_4^{-1} M_2).$$

Lemma 2.2[11]. If $G$ is an $r$-regular graph, then obviously

$$\mathcal{L}(G) = I_n - \frac{1}{r} A(G).$$

Theorem 2.3. Let $G$ be an $r$-regular graph with $n$ vertices and $m$ edges. Also let $G_1$ and $G_2$ be $r_1$-regular and $r_2$-regular with $n_1$ and $n_2$ vertices, respectively. Assume that $0 = \mu_1(G), \mu_2(G), \ldots, \mu_n(G); 0 = \eta_1(G_1), \eta_2(G_1), \ldots, \eta_{n_1}(G_1)$ and $0 = \delta_1(G_2), \delta_2(G_2), \ldots, \delta_{n_2}(G_2)$ be the normalized Laplacian spectra of $G, G_1$ and $G_2$, respectively. Then the normalized Laplacian spectrum of $G(R) \otimes \{G_1, G_2\}$ consists of:

- The eigenvalue $\frac{1 + r_1 \eta_j(G_1)}{r_1 + 1}$ with multiplicity $n$, for every eigenvalue $\eta_j(G_1)$ ($j = 2, 3, \ldots, n_1$) of $\mathcal{L}(G_1)$;

- The eigenvalue $\frac{1 + r_2 \delta_k(G_2)}{r_2 + 1}$ with multiplicity $m$, for every eigenvalue $\delta_k(G_2)$ ($k = 2, 3, \ldots, n_2$) of $\mathcal{L}(G_2)$;

- Four roots of equation

$$[(x - 1)(2 + n_2)(x r_2 + x - 1) - n_2][(x - 1)(2 r + n_1)(x r_1 + x - 1) + r(1 - \mu_i(G))(x r_1 + x - 1) - n_1] - r(\mu_i(G) - 2)(x r_1 + x - 1)(x r_2 + x - 1) = 0,$$

for each eigenvalue $\mu_i(G)$ ($i = 1, 2, \ldots, n$) of $\mathcal{L}(G)$;

- Two roots of equation $(x - 1)(2 + n_2)(x r_2 + x - 1) - n_2 = 0$ with multiplicity $m - n$ if $m > n$.

**Proof:** Let $M$ be the vertex-edge incidence matrix of $G$. Then one has

$$A(G(R) \otimes \{G_1, G_2\}) = \begin{pmatrix}
A(G) & M & 1_{n_1}^T \otimes I_n & 0 \\
M^T & 0 & 0 & 1_{n_2}^T \otimes I_m \\
1_{n_1} \otimes I_n & 0 & A(G_1) \otimes I_n & 0 \\
0 & 1_{n_2} \otimes I_m & 0 & A(G_2) \otimes I_m
\end{pmatrix}.$$
Now, we let $B = \alpha J_{n_1} + (1 - \alpha)I_{n_1}$ with $\alpha = r_1/(r_1 + 1)$; $C = \beta J_{n_2} + (1 - \beta)I_{n_2}$ with $\beta = r_2/(r_2 + 1)$ and

$$c_1 = \frac{1}{\sqrt{(2r + n_1)(2 + n_2)}}, \quad c_2 = \frac{1}{\sqrt{(2r + n_1)(r_1 + 1)}}, \quad c_3 = \frac{1}{\sqrt{(2 + n_2)(r_2 + 1)}}$$

Hence the characteristic polynomial of $G^{(R)} \otimes \{G_1, G_2\}$ is $\Phi_{G^{(R)} \otimes \{G_1, G_2\}}(x) = \det B_0$ where

$$B_0 = xI - L(G^{(R)} \otimes \{G_1, G_2\}) = \begin{pmatrix}
(x - 1)I_n & c_1M & c_21_{n_1}^T \otimes I_n & 0 \\
\frac{A(G)}{2r + n_1} & (x - 1)I_m & 0 & 0 \\
c_21_{n_1} \otimes I_n & 0 & (xI_{n_1} - L(G_1) \otimes B) \otimes I_n & 0 \\
0 & c_31_{n_2} \otimes I_m & 0 & (xI_{n_2} - L(G_2) \otimes C) \otimes I_m
\end{pmatrix}.
$$

Denoted by $M_0$ the elementary block matrix below,

$$M_0 = \begin{pmatrix}
I_n & 0 & -c_21_{n_1}^T(xI_{n_1} - L(G_1) \otimes B)^{-1} \otimes I_n & 0 \\
0 & I_m & 0 & -c_31_{n_2}^T(xI_{n_2} - L(G_2) \otimes C)^{-1} \otimes I_m \\
0 & 0 & I_{n_1} \otimes I_n & 0 \\
0 & 0 & 0 & I_{n_2} \otimes I_m
\end{pmatrix}.
$$

Now, we let $B_1 = M_0B_0$. It follows from $\det M_0 = 1$ that

$$\Phi_{G^{(R)} \otimes \{G_1, G_2\}}(x) = \det B_1 = \det(xI_{n_2} - L(G_2) \otimes C)^m \cdot \det(xI_{n_1} - L(G_1) \otimes B)^n \cdot \det S,$$  \hspace{1cm} (1)

where

$$S = \begin{pmatrix}
[x - 1 - c_2^21_{n_1}^T(xI_{n_1} - L(G_1) \otimes B)^{-1}1_{n_1}]I_n & \frac{A(G)}{2r + n_1} & c_1M \\
\frac{A(G)}{2r + n_1} & [x - 1 - c_3^21_{n_2}^T(xI_{n_2} - L(G_2) \otimes C)^{-1}1_{n_2}]I_m
\end{pmatrix}.
$$

Let $\chi_{G_1}(B) = 1_{n_1}^T(xI_{n_1} - L(G_1) \otimes B)^{-1}1_{n_1}$ and $\chi_{G_2}(C) = 1_{n_2}^T(xI_{n_2} - L(G_2) \otimes C)^{-1}1_{n_2}$. From Lemma 2.1, one obtains

$$\det S = \det[(x - 1 - c_3^2\chi_{G_2}(C))I_m] \cdot \det P,$$  \hspace{1cm} (2)
where
\[
P = (x - 1 - c_2^2\chi(G_1(B)))I_n + \frac{A(G)}{2r + n_1} - \frac{c_2^2}{x - 1 - c_2^2\chi(G_2)(rI_n + A(G)).}
\] (3)

Next we shall compute \(\chi_1(G_1(B)\) and \(\chi_2(G_2(C). From Lemma 2.2, we get
\[
\mathcal{L}(G_1) \circ B = I_n - \frac{A(G_1)}{r_1} = \frac{1}{r_1 + 1} (I_n + r_1 \mathcal{L}(G_1)).
\] (4)

Similarly,
\[
\mathcal{L}(G_2) \circ C = \frac{1}{r_2 + 1} (I_{n_2} + r_2 \mathcal{L}(G_2)).
\] (5)

Observe that \(\mathcal{L}(G_1)1_{n_1} = 0\) and \(\mathcal{L}(G_2)1_{n_2} = 0\). Then we get
\[
(xI_{n_1} - \mathcal{L}(G_1) \circ B)1_{n_1} = (x - \frac{1}{r_1 + 1})1_{n_1};
\]
\[
(xI_{n_2} - \mathcal{L}(G_2) \circ C)1_{n_2} = (x - \frac{1}{r_2 + 1})1_{n_2}.
\]

Hence,
\[
\chi_1(G_1(B) = \frac{n_1}{x - \frac{1}{r_1 + 1}}; \chi_2(G_2(C) = \frac{n_2}{x - \frac{1}{r_2 + 1}}.
\] (6)

Now plugging (6) into (3), again from Lemma 2.2, we have
\[
\det P = \det \left[ (x - 1 - \frac{c_2^2 n_1}{x - \frac{1}{r_1 + 1}} + \frac{r}{2r + n_1} - \frac{c_2^2 2r}{x - 1 - c_2^2 \frac{n_2}{x - \frac{1}{r_2 + 1}}}) I_n + \frac{c_2^2 r \mathcal{L}(G)}{x - 1 - c_2^2 \frac{n_2}{x - \frac{1}{r_2 + 1}}} - \frac{r \mathcal{L}(G)}{2r + n_1} \right].
\] (7)

Let \(\mu_i(G), \eta_j(G_1)\) and \(\delta_k(G_2)\) be the eigenvalues of \(\mathcal{L}(G), \mathcal{L}(G_1)\) and \(\mathcal{L}(G_2)\), respectively, for \(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n_1, \) and \(k = 1, 2, \ldots, n_2\). Then, by (1), (2), (7), along with (4) and (5), we obtain
\[
\Phi_{\mathcal{L}(G_1(B) \circ G_2)}(x) = \prod_{k=1}^{n_2} \left( x - \frac{1 + r_2 \delta_k(G_2)}{r_2 + 1} \right)^{m} \cdot \prod_{j=1}^{n_1} \left( x - \frac{1 + r_1 \eta_j(G_1)}{r_1 + 1} \right)^{n} \cdot \left( x - \frac{1}{2r + n_2} \right)^{m-n} \cdot \prod_{i=1}^{n} \left[ \left( x - 1 - \frac{r_1 \eta_j(G_1)}{2r + n_1}(x r_1 + x - 1) + \frac{r(1 - \mu_i(G))}{2r + n_1} \right) + \frac{r(\mu_i(G) - 2)}{2r + n_1} \right].
\]

From the above characteristic polynomial, we have
- The eigenvalue \(\frac{1 + r_2 \delta_k(G_2)}{r_2 + 1}\) with multiplicity \(m\), for every eigenvalue \(\delta_k(G_2) (k = 2, 3, \ldots, n_2)\) of \(\mathcal{L}(G_2)\);
- The eigenvalue \(\frac{1 + r_1 \eta_j(G_1)}{r_1 + 1}\) with multiplicity \(n\), for every eigenvalue \(\eta_j(G_1) (j = 2, 3, \ldots, n_1)\) of \(\mathcal{L}(G_1)\).
• Four roots of equation
\[
\left( x - 1 - \frac{n_2}{2 + n_2}(xr_2 + x - 1) \right) \left( x - 1 - \frac{n_1}{2r + n_1}(xr_1 + x - 1) + \frac{r(1 - \mu_i(G))}{2r + n_1} \right)
\]
\[+ \frac{r(\mu_i(G) - 2)}{(2r + n_1)(2 + n_2)} = 0\]
for each eigenvalue \(\mu_i(G)\) \((i = 1, 2, \ldots, n)\) of \(\mathcal{L}(G)\);

• Two roots of equation
\[
x - 1 - \frac{n_2}{2 + n_2}(xr_2 + x - 1) = 0
\]
with multiplicity \((m - n)\) whenever \(m > n\).

Hence the required result follows. \(\square\)

Next we consider two special situations of \(G^{(R)} \otimes \{G_1, G_2\}\). By choosing \(G_2\) as a null-graph, we can reduce \(G^{(R)} \otimes \{G_1, G_2\}\) to \(R\)-vertex corona \(G^{(R)} \otimes G_1\). Thus, from Theorem 2.3, we obtain

**Corollary 2.4.** Let \(G\) be an \(r\)-regular graph with \(n\) vertices and \(m\) edges, \(G_1\) be an \(r_1\)-regular graph with \(n_1\) vertices. Also let \(\mu_1(G), \mu_2(G), \ldots, \mu_n(G)\) and \(\eta_1(G_1), \eta_2(G_1), \ldots, \eta_{n_1}(G_1)\) be the normalized Laplacian spectra of \(G\) and \(G_1\), respectively. Then the normalized Laplacian spectrum of \(G^{(R)} \otimes G_1\) consists of:

- The eigenvalue \(\frac{1 + r_1 \eta_j(G_1)}{r_1 + 1}\) with multiplicity \(n\), for every eigenvalue \(\eta_j(G_1)\) \((j = 2, 3, \ldots, n_1)\) of \(\mathcal{L}(G_1)\);

- The roots of equation
\[
2(x - 1)[(x - 1)(2r + n_1)(xr_1 + x - 1) - n_1 + r(1 - \mu_i(G))(xr_1 + x - 1)] + r(\mu_i(G) - 2)(xr_1 + x - 1) = 0
\]
for each eigenvalue \(\mu_i(G)\) \((i = 1, 2, \ldots, n)\) of \(\mathcal{L}(G)\), and

- The eigenvalue \(1\) with multiplicity \(m - n\), if \(m > n\).

Instead of choosing \(G_2\) as a null-graph, if we choose \(G_1\) as a null-graph, then \(G^{(R)} \otimes \{G_1, G_2\}\) reduces to \(R\)-edge corona \(G^{(R)} \otimes G_2\). Thus we arrive at

**Corollary 2.5.** Let \(G\) be an \(r\)-regular graph with \(n\) vertices and \(m\) edges, \(G_2\) be an \(r_2\)-regular graph with \(n_2\) vertices. Also let \(\mu_1(G), \mu_2(G), \ldots, \mu_n(G)\) and \(\delta_1(G_2), \delta_2(G_2), \ldots, \delta_{n_2}(G_2)\) be the normalized Laplacian spectra of \(G\) and \(G_2\), respectively. Then the normalized Laplacian spectrum of \(G^{(R)} \otimes G_2\) consists of:

- The eigenvalue \(\frac{1 + r_2 \delta_k(G_2)}{r_2 + 1}\) with multiplicity \(n\), for every eigenvalue \(\delta_k(G_2)\) \((k = 2, 3, \ldots, n_2)\) of \(\mathcal{L}(G_2)\);

- Three roots of equation
\[
(2x - 1 - \mu_i(G))[x - 1](2 + n_2)(xr_2 + x - 1) - n_2] + (\mu_i(G) - 2)(xr_2 + x - 1) = 0
\]
for each eigenvalue \(\mu_i(G)\) \((i = 1, 2, \ldots, n)\) of \(\mathcal{L}(G)\), and
Two roots of equation
\[ (x - 1)(2 + n_2)(x^2 + x - 1) - n_2 = 0 \]
with multiplicity \( m - n \), if \( m > n \).

Next we shall present an example to explain our Theorem 2.3.

Example 2.6. Let us consider three graphs \( G = K_3 \), \( G_1 = P_2 \), and \( G_2 = P_2 \). Then the normalized Laplacian eigenvalues of \( G \) are \( \left( \frac{3}{2} \right)^{(2)} \) and \( 0^{(1)} \), where \( a^{(0)} \) indicates that \( a \) is repeated \( b \) times. The normalized Laplacian eigenvalues of \( G_1 \) and \( G_2 \) are \( 2^{(1)} \) and \( 0^{(1)} \). Applying Theorem 2.3, the normalized Laplacian spectrum of \( G(R) \otimes \{ G_1, G_2 \} \) consists of:

- \( \left( \frac{3}{2} \right)^{(3)} \) for the normalized Laplacian eigenvalue \( 2 \) of \( G_1 \);
- \( \left( \frac{3}{2} \right)^{(3)} \) for the normalized Laplacian eigenvalue \( 2 \) of \( G_2 \);
- For the normalized Laplacian eigenvalue \( \frac{3}{2} \) of \( G \), the roots of \( 24x^4 - 76x^3 + 75x^2 - 24x + \frac{9}{4} = 0 \) with multiplicity 2 each, that is, \( \left( \frac{3}{2} \right)^{(2)}, \left( \frac{3 + \sqrt{3}}{4} \right)^{(2)}, \left( \frac{3 - \sqrt{3}}{4} \right)^{(2)} \) and \( \left( \frac{1}{6} \right)^{(2)} \);
- For the normalized Laplacian eigenvalue 0 of \( G \), the roots of \( 24x^4 - 64x^3 + 48x^2 - 9x = 0 \) with multiplicity 1 each, that is, \( 0^{(1)}, \left( \frac{7 - \sqrt{13}}{12} \right)^{(1)}, \left( \frac{7 + \sqrt{13}}{12} \right)^{(1)}, \left( \frac{3}{2} \right)^{(1)} \).

On the other hand, according to the computation of Matlab, we get directly the normalized Laplacian eigenvalues of \( G(R) \otimes \{ G_1, G_2 \} \) are \( 0^{(1)}, \left( \frac{1}{6} \right)^{(2)}, \left( \frac{3}{2} \right)^{(9)}, \left( \frac{3 - \sqrt{3}}{4} \right)^{(2)}, \left( \frac{3 + \sqrt{3}}{4} \right)^{(2)}, \left( \frac{7 - \sqrt{13}}{12} \right)^{(1)}, \left( \frac{7 + \sqrt{13}}{12} \right)^{(1)} \). This example also shows that Theorem 2.3 is valid.

Similarly, applying corollary 2.4, the normalized spectrum of \( G(R) \otimes G_1 \) consists of: (1) \( \left( \frac{3}{2} \right)^{(3)} \); (2) the roots of \( 12x^3 - 32x^2 + 24x - \frac{9}{2} = 0 \) with multiplicity 2 each; (3) the roots of \( 6x^3 - 13x^2 + 6x = 0 \) with multiplicity 1 each. Applying corollary 2.5, the normalized spectrum of \( G(R) \otimes G_2 \) consists of: (1) \( \left( \frac{3}{2} \right)^{(3)} \); (2) the roots of \( 16x^3 - 44x^2 + 33x - \frac{9}{2} = 0 \) with multiplicity 2 each; and the roots of \( 4x^3 - 8x^2 + 3x = 0 \) with multiplicity 1 each.

From above theorem and corollaries, we find that the normalized spectrum of \( R \)-graph double corona depends on the degree of regularities, number of vertices, number of edges and normalized Laplacian eigenvalues of \( G, G_1 \) and \( G_2 \). Thus, we can construct infinitely many pairs of normalized Laplacian cospectral graphs.

Lemma 2.7 \[11\]. Two regular graphs are normalized Laplacian cospectral if and only if they are cospectral.

Theorem 2.8. If \( G \) and \( H \) are cospectral regular graphs (not necessarily distinct), so as to \( G_i \) and \( H_i \) (for \( i = 1, 2 \)) (not necessarily distinct), then \( G(R) \otimes \{ G_1, G_2 \} \) (respectively \( G(R) \otimes G_1 \), \( G(R) \otimes G_2 \)) is normalized Laplacian cospectral to \( H(R) \otimes \{ H_1, H_2 \} \) (respectively \( H(R) \otimes H_1 \), \( H(R) \otimes H_2 \)).
Proof: From Theorem 2.3 and Lemma 2.7, the result follows. □

Remark 2.9. The graphs $G$ and $H$, along with $G_i$ and $H_i$ (for $i = 1, 2$) are regular in Theorem 2.8, but $G^{(R)} \otimes \{G_1, G_2\}$ and $H^{(R)} \otimes \{H_1, H_2\}$ are non-regular in the general case. Hence we can construct infinitely many pairs of non-regular normalized Laplacian cospectral graphs by using double corona operations based on $R$-graphs. In addition, we remark that the degree Kirchhoff index and the number of spanning trees of some graph operations have been studied extensively (for example, see [2, 3, 6, 7, 13, 14, 18]). Our results can also help us to compute the number of spanning trees and degree Kirchhoff index for $R$-graph double corona operations of graphs, omitted.

Acknowledgements This work was supported in part by NNSFC (No. 11671053) and the Natural Science Foundation of Zhejiang Province, China (No. LY15A010011).

References

[1] A. Banerjee, J. Jost, On the spectrum of the normalized graph Laplacian, Linear Algebra Appl. 428 (2008) 3015-3022.
[2] S. Barik, G. Sahoo, On the Laplacian spectra of some variants of corona, Linear Algebra Appl., 512 (2017) 32-47.
[3] C.-J. Bu, B. Yan, X.-Q. Zhou, J. Zhou, Resistance distance in subdivision-vertex join and subdivision-edge join of graphs, Linear Algebra Appl. 458 (2014) 454-462.
[4] S. Butler, J. Grout, A construction of cospectral graphs for the normalized Laplacian, Electron. J. Combin. 18 (1) (2011) #P231.
[5] G.T. Chen, G. Davis, F. Hall, Z.S. Li, K. Patel, M. Stewart, An interlacing result on normalized Laplacians, SIAM J. Discrete Math. 18(2) (2004) 353-361.
[6] H.Y. Chen, L.W. Liao, The normalized Laplacian spectra of the corona and edge corona of two graphs, Linear Multilinear Algebra, 65 (2017) 582-592.
[7] H.Y. Chen, F.J. Zhang, Resistance distance and the normalized Laplacian spectrum, Discrete Appl. Math. 155 (2007) 654-661.
[8] F.R.K. Chung, Spectral Graph Theory, CBMS Regional Conference Series in Mathematics, Amer. Math. Soc., Providence, 1997.
[9] S.-Y. Cui, G.-X. Tian, The spectrum and the signless Laplacian spectrum of coronae, Linear Algebra Appl., 437 (2012) 1692-1703.
[10] D. M. Cvetković, P. Rowinson, H. Simić, An introduction to the Theory of Graph Spectra, Cambridge University Press, Cambridge, 2009.
[11] A. Das, P. Panigrahi, Normalized Laplacian spectrum of some subdivision-coronas of two regular graphs, Linear Multilinear Algebra 65 (2017) 962-972.
[12] R. A. Horn, C. R. Johnson, Topics in matrix analysis, Cambridge University Press, 1991.
[13] J. Huang, S. Li, On the normalized Laplacian spectrum, degree-Kirchhoff index and spanning trees of graphs, Bull. Aust. Math. Soc. 91 (2015) 353-367.

[14] J. Huang, S. Li, The normalized Laplacians, degree-Kirchhoff index and the spanning trees of linear hexagonal chains, Discrete Appl. Math. 207 (2016) 67-79.

[15] J. Lan, B. Zhou, Spectra of graph operations based on \( R \)-graph, Linear and Multilinear Algebra, 63(2014) 1401-1422.

[16] C. McLeman, E. McNicholas, Spectra of coronae, Linear Algebra Appl. 435 (2011) 998-1007.

[17] C.-X. Song, Q.-X. Huang, X.-Y. Huang, Spectra of Subdivision Vertex-edge Corona for Graphs, Advances in Mathematics(China), 45(2016) 37-47.

[18] G.-X. Tian, The asymptotic behavior of (degree-)Kirchhoff indices of iterated total graphs of regular graphs, to appear in Discrete Appl. Math., (2017).

[19] F.-Z. Zhang, The Schur complement and its applications, Springer, 2005.