BUNDLE 2-GERBES

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Abstract. We make the category $\mathbf{BGrb}_M$ of bundle gerbes on a manifold $M$ into a 2-category by providing 2-cells in the form of transformations of bundle gerbe morphisms. This description of $\mathbf{BGrb}_M$ as a 2-category is used to define the notion of a bundle 2-gerbe. To every bundle 2-gerbe on $M$ is associated a class in $H^4(M; \mathbb{Z})$. We define the notion of a bundle 2-gerbe connection and show how this leads to a closed, integral, differential 4-form on $M$ which represents the image in real cohomology of the class in $H^4(M; \mathbb{Z})$. Some examples of bundle 2-gerbes are discussed, including the bundle 2-gerbe associated to a principal $G$ bundle $P \rightarrow M$. It is shown that the class in $H^4(M; \mathbb{Z})$ associated to this bundle 2-gerbe coincides with the first Pontryagin class of $P$—this example was previously considered from the point of view of 2-gerbes by Brylinski and McLaughlin.

1. Introduction

Recently there has been interest in developing higher dimensional analogues of line bundles—so-called $p$-gerbes or $p$-line bundles—which realise classes in $H^{p+1}(M; \mathbb{Z})$ for a manifold $M$. Part of the motivation for this comes from physicists, who wish to interpret closed $p$-forms with integral periods on $M$ as a generalised curvature of a bundle-like object on $M$. A first step towards this goal was taken in the book [4] of Brylinski, who developed a theory of differential geometry for gerbes. Gerbes were originally introduced (in a very general setting) by Giraud in [12] for the purposes of developing a degree 2 non-abelian cohomology theory. The theory described by Brylinski allows one to realise classes in $H^3(M; \mathbb{Z})$ as equivalence classes of (abelian) gerbes. Murray in [16] invented the notion of a bundle gerbe. Bundle gerbes are simpler objects than gerbes but still provide a geometric realisation of $H^3(M; \mathbb{Z})$. The theory of gerbes and bundle gerbes has proved to be very useful tool: in [3] and [8] the authors studied anomalies in quantum field theory with the aid of bundle gerbes, Hitchin in [14] has used the theory of gerbes in his study of mirror symmetry, while Brylinski has made extensive applications of gerbes—one example is his use of gerbes in [5] to give an interpretation of Beilinson’s regulator maps in algebraic $K$-theory.

In [6] and [7] the authors constructed a canonical 2-gerbe associated to a principal $G$ bundle $P \rightarrow M$ where $G$ is a compact, simple, simply connected Lie group. 2-gerbes, introduced by Breen in [3], are higher dimensional analogues of gerbes. Breen used 2-gerbes to study three dimensional non-abelian sheaf cohomology, however there is a certain class of 2-gerbes—2-gerbes bound by the sheaf of abelian groups $\underline{\mathbb{C}}_M^\times$—that give rise to classes in $H^3(M; \mathbb{Z})$ via the exponential isomorphism $H^3(M; \underline{\mathbb{C}}_M^\times) = H^4(M; \mathbb{Z})$. This is the class of 2-gerbes studied by Brylinski.

1991 Mathematics Subject Classification. 18D05, 55R65.

The author acknowledges the support of the Australian Research Council.
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and McLaughlin. They show that the canonical 2-gerbe associated to the principal bundle $P$ has class in $H^4(M;\mathbb{Z})$ equal to $p_1$, the first Pontryagin class of $P$.

We shall consider here a related geometric object, the bundle 2-gerbe. Bundle 2-gerbes were originally introduced in [10] — we shall use a modification of the definition used there. A bundle 2-gerbe is a quadruple of manifolds $(Q,Y,X,M)$ where $(Q,Y,X^{[2]})$ is a bundle gerbe over the fibre product $X^{[2]}$. We also require that there is a bundle 2-gerbe product. In fact this requires two product structures, the first of which is a product on $Y$, which on the fibres takes the form $Y^{[2]}(x_1,x_2) = Y^{[2]}(x_3)$ for points $x_1$, $x_2$, and $x_3$ all lying in the same fibre. There is also a product in $Q$ covering this product on $Y$, and which commutes with the bundle gerbe product on $(Q,Y,X^{[2]})$. This product on $Q$ satisfies a certain associativity condition. One can associate to a bundle 2-gerbe $(Q,Y,X,M)$ a $\mathbb{C}^\times$ valued Čech 3-cocycle $g_{ijkl}$ representing a class in $H^4(M;\mathbb{Z})$. One can also develop the notion of a bundle 2-gerbe connection and a 2-curving for a bundle 2-gerbe connection in an analogous manner to [16] and show that a bundle 2-gerbe equipped with such structures has a 4-curvature. This is a closed, integral differential 4-form on $M$ which is a representative in $H^4(M;\mathbb{R})$ for the image, in real cohomology, of the class in $H^4(M;\mathbb{Z})$ defined by the cocycle $g_{ijkl}$.

There is a naturally arising bundle 2-gerbe $Q$ associated to a principal $G$ bundle $P$ on $M$ where $G$ is as above. If one calculates the Čech cocycle $g_{ijkl}$ associated to $Q$ then one recovers the results of [6] and [7] giving an explicit cocycle formula for the first Pontryagin class of $P$.

In outline then this paper is as follows. In Section 2 we review the theory of bundle gerbes from [10]. In Section 3 we discuss a gluing or ‘descent’ construction for line bundles from [4]. In Section 4 we explain how to make the category of bundle gerbes on a manifold $M$ into a 2-category by adding 2-cells in the form of transformations of bundle gerbe morphisms. This allows us in Section 5 to ‘categorify’ the definition of a bundle gerbe, so as to define a bundle 2-gerbe. The relationship of bundle 2-gerbes with bicategories is also examined here. This is also preparation for Section 6 where an example of a bundle 2-gerbe — the tautological bundle 2-gerbe — is introduced via the homotopy bigroupoid of a space. A Čech 3-class is associated to a bundle 2-gerbe in Section 7 and a de Rham representative for this class is defined in Section 8 via the notion of a bundle 2-gerbe connection. In Section 9 the example of a bundle 2-gerbe associated to a principal $G$-bundle is discussed and, using the work of Brylinski and McLaughlin, it is shown that the 4-class of this bundle 2-gerbe coincides with the first Pontryagin class of the bundle. In Sections 10 and 11 we discuss higher descent properties of bundle 2-gerbes and define the notion of a trivial bundle 2-gerbe. We finally show that a bundle 2-gerbe is trivial if and only if its 4-class vanishes. We will not discuss the relationship of bundle 2-gerbes with 2-gerbes, this will be done elsewhere. For some preliminary results in this direction one can consult [19].

This work is clearly influenced by the ideas presented in [6] and [7]. I am very grateful to Michael Murray for his supervision of my PhD thesis and for his help in the preparation of this paper.

2. Review of Bundle Gerbes

Let $\pi: X \to M$ be a surjection admitting local sections. Let $X^{[2]} = X \times_M X$ denote the fiber product of $X$ with itself over $M$ and let $X^{[p]} = X \times_M X \times_M \cdots \times_M X$
denote the $p$-fold such fiber product. We can form a simplicial manifold $X_* = \{X_p\}$ with $X_p = X^{[p+1]}$ and the face and degeneracy operators $d_i$ and $s_i$ given by omitting the $i^{th}$ factor and repeating the $j^{th}$ factor respectively. Thus the face operators $d_i : X^{[p+1]} \to X^{[p]}$ are given by $d_i = \pi_i$ where

$$
\pi_i(x_1, \ldots , x_{p+1}) = (x_1, \ldots , x_{i-1}, x_{i+1}, \ldots , x_{p+1})
$$

for $i = 1, \ldots , p+1$ and for $p = 1, 2, \ldots$. Recall from [16] that a bundle gerbe consists of a triple $(P, X, M)$ where $\pi : X \to M$ is a surjection admitting local sections and $P$ is a principal $\mathcal{C}^\infty$ bundle on $X^{[2]}$ with a product. This means that there is a $\mathcal{C}^\infty$ bundle isomorphism

$$
m_P : \pi_1^{-1}P \otimes \pi_3^{-1}P \to \pi_2^{-1}P
$$

covering the identity on $X^{[3]}$. Here $\pi_1^{-1}P \otimes \pi_3^{-1}P$ denotes the contracted product of the $\mathcal{C}^\infty$ bundles $\pi_1^{-1}P$ and $\pi_3^{-1}P$ — see [16]. Fiberwise the bundle gerbe product $m_P$ is a map

$$
m_P : P_{(x_2,x_3)} \otimes P_{(x_1,x_2)} \to P_{(x_1,x_3)}
$$

for $(x_1, x_2, x_3) \in X^{[3]}$ and we usually write $u_{23}u_{12}$ for $m_P(u_{23} \otimes u_{12})$ when $u_{23} \in P_{(x_2,x_3)}$ and $u_{12} \in P_{(x_1,x_2)}$. The bundle gerbe product $m_P$ is required to be associative in the following sense: whenever $u_{34} \in P_{(x_3,x_4)}$, $u_{23} \in P_{(x_2,x_3)}$ and $u_{12} \in P_{(x_1,x_2)}$ for $(x_1, x_2, x_3, x_4) \in X^{[4]}$ we have $u_{34}(u_{23}u_{12}) = (u_{34}u_{23})u_{12}$. When $M$ is understood we will frequently write $(P, X)$ or even $P$ for $(P, X, M)$.

Recall that a bundle gerbe also has an identity section; this is a section $e$ of $P$ over the diagonal $\Delta(X) = \{(x,x)\vert x \in X\} \subset X^{[2]}$ which behaves as an identity with respect to the bundle gerbe product. So if $u \in P_{(x_1,x_2)}$ then we have $ue(x_1) = u = e(x_2)u$. A bundle gerbe also has an inverse map $P \to inv^{-1}P$ where $inv : X^{[2]} \to X^{[2]}$ is the map which switches an ordered pair $(x_1, x_2)$, so $inv(x_1, x_2) = (x_2, x_1)$. We denote the image of $u \in P_{(x_1,x_2)}$ under $P \to inv^{-1}P$ by $u^{-1}$ — this has all the desired properties: $uu^{-1} = e(x_2), (uv)^{-1} = v^{-1}u^{-1}$ and so on. Note also that we can identify $inv^{-1}P$ with $P^*$, the $\mathcal{C}^\infty$ bundle $P$ with the action of $\mathcal{C}^\infty$ changed to its inverse. For more details we refer to [16].

Various operations can be performed on bundle gerbes; for example there is the notion of the pullback $(f^{-1}P, f^{-1}X, N)$ of a bundle gerbe $(P, X)$ on $M$ by a map $f : N \to M$. One can also form the product $(P \otimes Q, X \times_M Y)$ of two bundle gerbes $(P, X)$ and $(Q, Y)$ on $M$. Given a bundle gerbe $(P, X)$ we can also form its dual $(P^*, X)$. We refer to [16] for more details on these constructions.

Suppose $Q \to X$ is a principal $\mathcal{C}^\infty$ bundle on $X$ and $\pi : X \to M$ is a local-section-admitting surjection. Let $P$ be the $\mathcal{C}^\infty$ bundle on $X^{[2]}$ with fibre

$$
P_{(x,y)} = Aut_{\mathcal{C}^\infty}(Q_x, Q_y)
$$

(1)

at $(x, y) \in X^{[2]}$. $Q$ has an associative product via composition of isomorphisms. A bundle gerbe isomorphic to a bundle gerbe of the form (1) via an isomorphism preserving the bundle gerbe products is said to be trivial. The notation $\delta(Q) = \pi_1^{-1}Q \otimes \pi_2^{-1}Q^*$ is frequently used to denote the bundle gerbe (1).

In [16] the notion of a bundle gerbe connection on a bundle gerbe $(P, X)$ was introduced. Before we recall this notion it is useful to note (see [8]) that we can reformulate the definition of a bundle gerbe in terms of line bundles and line bundle isomorphisms by replacing the principal $\mathcal{C}^\infty$ bundle $P$ with its associated line bundle $L$. Then $L$ has an associative product $m_L : \pi_1^{-1}L \otimes \pi_2^{-1}L \to \pi_2^{-1}L$ described in the
Lemma 3.1 in \( H^\infty \) one can show that \( \pi \) of bundle gerbes and show that there is a bijection between stable isomorphism classes of bundle gerbes are in a bijective correspondence.

It is easy to see that the curvature \( F_{\nabla_L} \) of a bundle gerbe connection \( \nabla_L \) satisfies \( \delta(F_{\nabla_L}) = 0 \). Here \( \delta: \Omega^p(X[q]) \to \Omega^p(X[q+1]) \) is the map formed by adding the pullback maps \( \pi^* \) with an alternating sign: \( \delta = \sum (-1)^i \pi^*_i \). Therefore \( \delta \) commutes with the exterior derivative \( d \) and, since the \( \pi_i \) are face maps for a simplicial manifold, it follows that \( \delta^2 = 0 \). Hence we have a complex

\[
\Omega^p(M) \xrightarrow{\pi^*} \Omega^p(X) \xrightarrow{\delta} \Omega^p(X^{[2]}) \xrightarrow{\delta} \cdots \Omega^p(X^{[q]}_1) \xrightarrow{\delta} \cdots
\]

It is a fundamental result of \([16]\) that the complex \((2)\) has no cohomology as long as \( M \) supports partitions of unity. Hence we can solve the equation \( F_{\nabla_L} = \delta(f) \) for some two form \( f \) on \( X \). Following \([16]\) we call a choice of this two form \( f \) a curving for the bundle gerbe connection \( \nabla_L \). From the equation \( F_{\nabla_L} = \delta(f) \) we obtain \( \delta(df) = 0 \) and hence \( df = \pi^*(\omega) \) for some necessarily closed three form \( \omega \) on \( M \).

One can show that \( \omega \) has integral periods and hence is a representative of the image in \( H^3(M; \mathbb{R}) \) of a class in \( H^3(M; \mathbb{Z}) \). We call the three form \( \omega \) the 3-curvature of the bundle gerbe connection \( \nabla_L \) and curving \( f \).

One can associate to any bundle gerbe \( P \) on \( M \) a \( C^\infty \) valued Čech 2-cocycle \( g_{ijk} \) as described in \([16]\). \( g_{ijk} \) is a representative of a characteristic class \( DD(P) \) in \( H^3(M; \mathbb{Z}) \) — the Dixmier-Douady class of the bundle gerbe \( P \). The 3-curvature \( \omega \) of a bundle gerbe connection on \( P \) is a representative for the image, in real cohomology, of \( DD(P) \). The Dixmier-Douady class has the following properties.

**Proposition 2.1** ([16]). The Dixmier-Douady class \( DD(P) \) of a bundle gerbe \( P \) on \( M \) satisfies

1. \( DD(P \otimes Q) = DD(P) + DD(Q) \) for bundle gerbes \( P \) and \( Q \) on \( M \).
2. \( DD(P^*) = -DD(P) \) where \( P^* \) is the dual of the bundle gerbe \( P \).
3. \( DD(f^{-1}P) = f^*DD(P) \) where \( f^{-1}P \) denotes the pullback of the bundle gerbe \( P \) on \( M \) by a map \( f: N \to M \).

Recall from \([16]\) that a bundle gerbe morphism \( f: P \to Q \) between bundle gerbes \( P = (P, X) \) and \( Q = (Q, Y) \) is a triple of maps \( f = (\hat{f}, f, \phi) \) where \( \hat{f}: X \to Y \) is a map commuting with the projections \( \pi_X: X \to M, \pi_Y: Y \to M \) and covering \( \phi: M \to M \), while \( f: P \to Q \) is a \( C^\infty \) bundle morphism covering the induced map \( f^{[2]}: X^{[2]} \to Y^{[2]} \). We will only be interested in the case where \( \phi = \text{id}_M \). One could define an isomorphism of bundle gerbes \( P \) and \( Q \) to be a morphism of bundle gerbes \( (f, f, \phi): P \to Q \) in which each map was an isomorphism, however it is not true that isomorphism classes of bundle gerbes are in a bijective correspondence with \( H^3(M; \mathbb{Z}) \). Instead, one can consider the weaker notion of stable isomorphism \([17]\) of bundle gerbes and show that there is a bijection between stable isomorphism classes of bundle gerbes and \( H^3(M; \mathbb{Z}) \).

### 3. The Generalised Clutching Construction

Recall the following result from \([4]\).

**Lemma 3.1** ([4]). Suppose \( \pi: X \to M \) is a surjection admitting local sections and that \( P \) is a \( C^\infty \) bundle on \( X \) together with an isomorphism \( \phi: \pi_2^{-1}P \to \pi_1^{-1}P \) which
satisfies the descent cocycle condition
\[(3) \quad \pi_1^{-1} \phi \circ \pi_3^{-1} \phi = \pi_2^{-1} \phi\]
over \(X^3\). Then the \(\mathbb{C}^\times\) bundle \(P\) descends to \(M\), ie there is a \(\mathbb{C}^\times\) bundle \(Q = D(P)\) on \(M\) plus an isomorphism \(\psi: P \to \pi^{-1}Q\) which is compatible with \(\phi\). The converse is also true.

The \(\mathbb{C}^\times\) bundle isomorphism \(\phi\) above is called a descent isomorphism. Note that fiberwise \(\phi\) is a map \(P_{x_1} \to P_{x_2}\) and the descent cocycle condition (3) is simply that the diagram

\[
\begin{array}{ccc}
P_{x_1} & \xrightarrow{\phi} & P_{x_2} \\
\downarrow & & \downarrow \\
P_{x_3} & \xleftarrow{\phi} & P_{x_1}
\end{array}
\]
commutes. We give an example of this kind of formalism below.

**Example 3.1.** Suppose \((P, X)\) is a bundle gerbe on \(M\) and suppose that there are two trivialisations \(T_1\) and \(T_2\) of \(P\) on \(X\). Thus there exist isomorphisms \(P = \delta(T_1)\) and \(P = \delta(T_2)\) commuting with the respective bundle gerbe products. It is easy to see that there is a trivialisation of the bundle \(\delta(T_1 \otimes T_2^\ast)\) over \(X^2\). This corresponds to an isomorphism \(\phi: \pi_1^{-1}(T_1 \otimes T_2^\ast) \to \pi_2^{-1}(T_1 \otimes T_2^\ast)\) covering the identity on \(X^2\). Since the isomorphisms \(P = \delta(T_1)\) and \(P = \delta(T_2)\) commute with the bundle gerbe products on the respective bundle gerbes, one can show that \(\phi\) satisfies the descent cocycle condition. Hence the bundle \(T_1 \otimes T_2^\ast\) descends to a bundle \(D\) on \(M\), ie there is an isomorphism \(T_1 = T_2 \otimes \pi^{-1}D\) of bundles on \(X\), where \(\pi: X \to M\) denotes the projection.

There is the following strengthening of the above lemma [[3]]: there is an equivalence of categories \(D: \text{Desc}(X \xrightarrow{\pi} M) \to \text{Bund}_M\) between the so called descent category \(\text{Desc}(X \xrightarrow{\pi} M)\) and the category of principal \(\mathbb{C}^\times\) bundles \(\text{Bund}_M\) on \(M\). Here \(\text{Desc}(X \xrightarrow{\pi} M)\) is the category whose objects are pairs \((P, \phi)\) where \(\phi: \pi_2^{-1}P \to \pi_1^{-1}P\) is a descent isomorphism as above and whose arrows \((P, \phi) \to (Q, \psi)\) are \(\mathbb{C}^\times\) bundle isomorphisms \(f: P \to Q\) compatible with \(\phi\) and \(\psi\), so the following diagram commutes:

\[
\begin{array}{ccc}
\pi_2^{-1}P & \xrightarrow{\pi_2^{-1}f} & \pi_2^{-1}Q \\
\downarrow & & \downarrow \\
\pi_1^{-1}P & \xleftarrow{\pi_1^{-1}f} & \pi_1^{-1}Q.
\end{array}
\]

It is clear that the operation \(D\) which associates the \(\mathbb{C}^\times\) bundle \(D(P)\) on \(M\) to a bundle \(P\) on \(X\) with a descent isomorphism \(\phi\) extends to an operation on maps — if \(f: (P, \phi) \to (Q, \psi)\) then there is an induced map \(D(f): D(P) \to D(Q)\) — and this operation is functorial with respect to composition of maps.

One other point to note is that if we make \(\text{Desc}(X \xrightarrow{\pi} M)\) and \(\text{Bund}_M\) into monoidal categories via the contracted product \(\otimes\) of \(\mathbb{C}^\times\) bundles, then the equivalence of categories \(D: \text{Desc}(X \xrightarrow{\pi} M) \to \text{Bund}_M\) commutes with \(\otimes\) up to natural isomorphism. More specifically, we define a functor \(\otimes: \text{Desc}(X \xrightarrow{\pi} M) \times \text{Desc}(X \xrightarrow{\pi} M) \to \text{Desc}(X \xrightarrow{\pi} M)\) by a map on objects given by \(\otimes((P, \phi), (Q, \psi)) = (P \otimes Q, \phi \otimes \psi)\).
spectively. Therefore there is an induced isomorphism
\( D \) which commutes with the descent isomorphisms for \((g,h)\)
\( D \) is isomorphic to \( f,g \) and \( Q \). Then there is a natural isomorphism between the functors bounding the following diagram:

\[
\begin{array}{c}
\text{Desc}(X \xrightarrow{\pi} M) \times \text{Desc}(X \xrightarrow{\pi} M) \\
\downarrow \quad \downarrow
\end{array}
\xrightarrow{\mathbf{D} \times \mathbf{D}}
\begin{array}{c}
\text{Bund}_M \times \text{Bund}_M \\
\downarrow \quad \downarrow
\end{array}
\xrightarrow{\mathbf{D}}
\begin{array}{c}
\text{Desc}(X \xrightarrow{\pi} M) \\
\end{array}
\xrightarrow{\mathbf{D}^{\mathbf{M}}}
\begin{array}{c}
\text{Bund}_M.
\end{array}
\]

Note that such an isomorphism amounts to an isomorphism \( D(P) \otimes D(Q) \rightarrow D(P \otimes Q) \) which is natural with respect to maps.

4. The 2-Category of Bundle Gerbes

Given bundle gerbes \( P = (P,X) \) and \( Q = (Q,Y) \) together with a pair of bundle gerbe morphisms \( f,g: P \rightarrow Q \) with \( f = (\tilde{f},f) \) and \( g = (\tilde{g},g) \) let \( \tilde{D}_{f,g} \) denote the \( \mathbb{C}^\times \) bundle \((f,g)^{-1}Q\) on \( X \). Therefore \( \tilde{D}_{f,g} \) has fibre \( Q_{(f(x),g(x))} \) at \( x \in X \). We will construct a descent isomorphism \( \phi_{f,g}: \pi_2^{-1}\tilde{D}_{f,g} \rightarrow \pi_1^{-1}\tilde{D}_{f,g} \) for \( \tilde{D}_{f,g} \). Suppose \( v \in (\pi_2^{-1}\tilde{D}_{f,g})_{(x_1,x_2)} = (\tilde{D}_{f,g})_{x_1} \). Thus \( v \in Q_{(f(x_1),g(x_1))} \). Choose \( u \in P_{(x_1,x_2)} \) and put \( \phi_{f,g}(v) = \hat{g}(u)(v\hat{f}(u^{-1})) \). Notice that this is independent of the choice of \( u \in P_{(x_1,x_2)} \). \( \phi_{f,g} \) is a descent isomorphism — ie it satisfies

\[
\pi_1^{-1}\phi_{f,g} \circ \pi_3^{-1}\phi_{f,g} = \pi_2^{-1}\phi_{f,g}
\]

over \( X \)\(^3\). This is a consequence of the associativity of the bundle gerbe products on \( P \) and \( Q \). We have the following Lemma.

**Lemma 4.1** (*[19]*). 1. Suppose \((P,X)\) and \((Q,Y)\) are bundle gerbes on \( M \) and that there exist bundle gerbe morphisms \( f: P \rightarrow Q \) and \( g: P \rightarrow Q \). Then the \( \mathbb{C}^\times \) bundle \( \tilde{D}_{f,g} = (f,g)^{-1}Q \) on \( X \) descends to a \( \mathbb{C}^\times \) bundle \( D_{f,g} = D(\tilde{D}_{f,g}) \) on \( M \).

2. Suppose that \( P, Q, f \) and \( g \) are as above and that there is a third bundle gerbe morphism \( h: P \rightarrow Q \). Then there is an isomorphism

\[
D_{g,h} \otimes D_{f,g} \simeq D_{f,h}
\]

of \( \mathbb{C}^\times \) bundles on \( M \).

3. Suppose that \( P \) and \( Q \) are as above but now we have bundle gerbe morphisms \( f,g,h,k: P \rightarrow Q \). Then the following diagram of \( \mathbb{C}^\times \) bundle isomorphisms on \( M \) commutes:

\[
\begin{array}{ccc}
D_{h,k} \otimes D_{g,h} \otimes D_{f,g} & \xrightarrow{\sim} & D_{h,k} \otimes D_{f,h} \\
\downarrow & & \downarrow \\
D_{g,k} \otimes D_{f,g} & \xrightarrow{\sim} & D_{f,k}
\end{array}
\]

where the isomorphisms are those of (2) above.

(2) of this lemma is proved by noticing that the bundle gerbe product on \( Q \) gives an isomorphism \((g,h)^{-1}Q \otimes (f,g)^{-1}Q \rightarrow (f,h)^{-1}Q\) of \( \mathbb{C}^\times \) bundles on \( X \) which commutes with the descent isomorphisms for \((g,h)^{-1}Q \otimes (f,g)^{-1}Q\) and \((f,h)^{-1}Q\) respectively. Therefore there is an induced isomorphism \( D((g,h)^{-1}Q \otimes (f,g)^{-1}Q) \rightarrow D((f,h)^{-1}Q) \). (3) of the lemma is proved similarly, using the associativity of the
bundle gerbe product on \( Q \), the functoriality of the operation \( D \), and the fact that \( D \) commutes with \( \otimes \) up to natural isomorphism.

This Lemma suggests the following Definition.

**Definition 4.2** (\cite{14}). Let \((P, X)\) and \((Q, Y)\) be bundle gerbes on \( M \). A transformation \( \theta : f \Rightarrow g \) between two bundle gerbe morphisms \( f, g : P \rightarrow Q \) is a section of the \( \mathbb{C}^\times \) bundle \( D_{f,g} = D(\tilde{D}_{f,g}) \) on \( M \).

We would like to form a category \( \text{Hom}(P, Q) \) associated to bundle gerbes \((P, X)\) and \((Q, Y)\) with the bundle gerbe morphisms \( P \rightarrow Q \) as objects. Therefore we would like to be able to compose transformations between bundle gerbe morphisms. A way to do this is suggested by the previous lemma. Given bundle gerbe morphisms \( f, g, h : P \rightarrow Q \) together with transformations \( \theta : f \Rightarrow g \) and \( \lambda : g \Rightarrow h \) then we have the induced section \( \lambda \otimes \theta \) of \( D_{g,h} \otimes D_{f,g} \). We define the composed transformation \( \lambda \theta : f \Rightarrow h \) to be the image of this section \( \lambda \otimes \theta \) under the isomorphism \( D_{g,h} \otimes D_{f,g} \rightarrow D_{f,h} \). By the lemma above this operation of composition is associative. We can define an identity transformation \( 1_f : f \Rightarrow f \) by noticing that the identity section of the bundle gerbe \( Q \) pullback to define a section \( 1_f \) of \( (f, f)^{-1}Q \) which is compatible with the descent isomorphism for \( D_{f,f} = (f, f)^{-1}Q \). Therefore it descends to a section \( 1_f \) of \( D_{f,f} \) and it is straightforward to check that this acts as an identity.

The case where the manifold \( M \) is a point illuminates the preceding discussion. In this case a bundle gerbe over a point becomes a \( \mathbb{C}^\times \) groupoid—ie a groupoid such that the automorphism groups of each object of the groupoid are isomorphic to \( \mathbb{C}^\times \). Following \cite{14} we define the \( \mathbb{C}^\times \) groupoid \( \text{Gr}(P) \) associated to a bundle gerbe \((P, X, M)\) when the manifold \( M \) is restricted to a point \( m_0 \in M \) as follows. We let the objects of the groupoid \( \text{Gr}(P) \) be the points of \( X_{m_0} \) where \( X_{m_0} = \pi^{-1}(m_0) \).

Given two points of \( X_{m_0} \), \( x_1 \) and \( x_2 \), we define the set of arrows \( \text{Hom}(x_1, x_2) \) in \( \text{Gr}(P) \) to be the points of the fiber \( P(x_1, x_2) \). Composition of arrows in \( \text{Gr}(P) \) is then provided by the bundle gerbe product on \( P \) and the identity arrow from a point \( x \) to itself is provided by the identity section \( e(x) \) of \( P \) evaluated at the point \( x \). Since inverses exist in \( P \) every arrow is invertible and it is not hard to see that \( \text{Gr}(P) \) is a \( \mathbb{C}^\times \) groupoid. Thus we have a family of \( \mathbb{C}^\times \) groupoids, indexed by the points of \( M \). It is in this sense that a bundle gerbe is a ‘bundle of groupoids’.

It is not hard to see that in this case, when \( M \) is restricted to a point, a bundle gerbe morphism \( f : P \rightarrow Q \) induces a functor \( f : \text{Gr}(P) \rightarrow \text{Gr}(Q) \) (the important point here is that \( \tilde{f} \) preserves the bundle gerbe products on \( P \) and \( Q \)). Suppose that we are given a second bundle gerbe morphism \( g : P \rightarrow Q \) and a transformation \( \theta : f \Rightarrow g \).

So \( \theta \) is a section of the \( \mathbb{C}^\times \) bundle \( D_{f,g} = D(\tilde{D}_{f,g}) \) on \( M \) and hence lifts to a section \( \tilde{\theta} \) of the \( \mathbb{C}^\times \) bundle \( \tilde{D}_{f,g} = (f,g)^{-1}Q \) on \( X \). It follows from the definition of \( \tilde{D}_{f,g} \) that we have the following isomorphism of \( \mathbb{C}^\times \) bundles on \( X^{[2]} \):

\[
\psi : \pi_1^{-1} \tilde{D}_{f,g} \otimes (f^{[2]})^{-1}Q \xrightarrow{\sim} (g^{[2]})^{-1}Q \otimes \pi_2^{-1} \tilde{D}_{f,g}.
\]

It also follows that the section \( \tilde{\theta} \) of \( \tilde{D}_{f,g} \) is compatible with this isomorphism in the sense that \( \psi(\tilde{\theta}(x_2) \otimes \tilde{f}(u)) = \tilde{g}(u) \otimes \tilde{\theta}(x_1) \) where \( u \in P(x_1, x_2) \) and \( \tilde{f} : P \rightarrow (f^{[2]})^{-1}Q \) and \( \tilde{g} : P \rightarrow (g^{[2]})^{-1}Q \) are induced by \( \tilde{f} \) and \( \tilde{g} \) respectively. When we restrict \( M \) to a point \( m_0 \in M \), this is exactly the condition that \( \theta \) defines a natural transformation (in fact a natural isomorphism) between the functors \( f \) and \( g \).

We would like to define a 2-category \( \text{BGrb}_M \) whose objects are the bundle gerbes \( P \) on \( M \). We refer to \cite{15} for the definition of a 2-category (see also Section 3).
We take as the objects of $\text{BGrb}_M$ the bundle gerbes $P$ on $M$, and given two bundle gerbes $P$ and $Q$ on $M$, we define the category $\text{Hom}(P, Q)$ as above. Thus the objects of $\text{Hom}(P, Q)$ (1-arrows of $\text{BGrb}_M$) are the bundle gerbe morphisms $P \rightarrow Q$ and the arrows of $\text{Hom}(P, Q)$ (2-arrows of $\text{BGrb}_M$) are the transformations $\theta: f \Rightarrow g$. We need to define a composition functor

$$m: \text{Hom}(Q, R) \times \text{Hom}(P, Q) \rightarrow \text{Hom}(P, R).$$

It is clear how to define the action of $m$ on 1-arrows: if $g: Q \rightarrow R$ and $f: P \rightarrow Q$ are bundle gerbe morphisms, then we put $m(g, f) = g \circ f$. It is not so clear how to define the action of $m$ on 2-arrows. However we have the following result from [13].

**Lemma 4.3.** Suppose we are given three bundle gerbes $(P, X)$, $(Q, Y)$ and $(R, Z)$ on $M$ together with bundle gerbe morphisms $f_1, f_2: P \rightarrow Q$ and $g_1, g_2: Q \rightarrow R$. Then we have the following isomorphism of $\mathcal{C}^X$ bundles on $M$:

$$D_{g_1 \circ f_1, g_2 \circ f_2} \simeq D_{f_1, f_2} \otimes D_{g_1, g_2}.$$

This Lemma suggests a way to define the action of $m$ on 2-arrows. Suppose $\theta: f_1 \Rightarrow f_2$ is a transformation between bundle gerbe morphisms $(P, X) \rightarrow (Q, Y)$ and that $\lambda: g_1 \Rightarrow g_2$ is a transformation of bundle gerbe morphisms $(Q, Y) \rightarrow (R, Z)$. Then $\theta$ and $\lambda$ lift to sections $\hat{\theta}$ and $\hat{\lambda}$ of the $\mathcal{C}^X$ bundles $\hat{D}_{f_1, f_2} = (f_1, f_2)^{-1}Q$ and $\hat{D}_{g_1, g_2} = (g_1, g_2)^{-1}R$ on $X$ and $Y$ respectively. $\hat{g}_2$ induces an isomorphism $\hat{g}_2: Q \rightarrow (g_2^2)^{-1}R$, so if $x \in X$ then $\hat{g}_2(\hat{\theta}(x)) \in R_{(g_2 \circ f_1(x), g_2 \circ f_2(x))}$. Let $f_1^{-1}\hat{\lambda}$ denote the section of the pullback bundle $f_1^{-1}\hat{D}_{g_1, g_2} = (g_1 \circ f_1, g_2 \circ f_1)^{-1}R$ on $X$. Then if $x \in X$, $f_1^{-1}\hat{\lambda}(x) \in R_{(g_1 \circ f_1(x), g_2 \circ f_1(x))}$ and so $\hat{g}_2(\hat{\theta})f_1^{-1}\hat{\lambda}(x) \in R_{(g_1 \circ f_1(x), g_2 \circ f_2(x))}$. It is easy to check that $\hat{g}_2(\hat{\theta})f_1^{-1}\hat{\lambda}$ commutes with the descent isomorphism for $\hat{D}_{g_1 \circ f_1, g_2 \circ f_2} = (g_1 \circ f_1, g_2 \circ f_2)^{-1}R$ and therefore descends to a section $\theta \circ \lambda$ of $D_{g_1 \circ f_1, g_2 \circ f_2}$.

Note that fiberwise, i.e. regarding bundle gerbes as being bundles of groupoids, this is simply the operation of composing natural transformations in the 2-category $\text{Cat}$ with categories as objects, functors as 1-arrows and natural transformations as 2-arrows — recall that if we have categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ together with functors $F_1, F_2: \mathcal{C} \rightarrow \mathcal{D}$ and $G_1, G_2: \mathcal{D} \rightarrow \mathcal{E}$ plus natural transformations $\alpha: F_1 \Rightarrow F_2$ and $\beta: G_1 \Rightarrow G_2$ then one can define the composed natural transformation $\beta \circ \alpha: G_1 \circ F_1 \Rightarrow G_2 \circ F_2$.

We need to show that the action of $m$ on 2-arrows is functorial. Suppose that we have bundle gerbes $P$, $Q$ and $R$, bundle gerbe morphisms $f_1, f_2, f_3: P \rightarrow Q$, $g_1, g_2, g_3: Q \rightarrow R$ and transformations between them as pictured in the following diagram

$$
\begin{array}{ccc}
P & \overset{f_1}{\underset{f_3}{\downarrow}} & Q \\
\overset{f_2}{\underset{f_3}{\downarrow}} & \overset{g_1}{\underset{g_3}{\downarrow}} & R. \\
\end{array}
$$

To show that $m$ is a functor we need to show that the two different ways of composing 2-arrows coincide — i.e. $(\lambda_{23} \lambda_{12}) \circ (\theta_{23} \theta_{12}) = (\lambda_{23} \circ \theta_{23})(\lambda_{12} \circ \theta_{12})$. It is sufficient to show that

$$\hat{g}_3(\hat{\theta}_{23})f_2^{-1}\hat{\lambda}_{23}\hat{g}_2(\hat{\theta}_{12})f_1^{-1}\hat{\lambda}_{12} = \hat{g}_3(\hat{\theta}_{23}\hat{\theta}_{12})f_1^{-1}(\hat{\lambda}_{23}\hat{\lambda}_{12}).$$
Since $\hat{\lambda}_{23}$ is compatible with the descent isomorphisms for the bundle $\hat{D}_{\theta_2,\theta_3}$, we have $\lambda_{23}(y_2)\hat{g}_2(u) = \hat{g}_1(u)\lambda_{23}(y_1)$ for $(y_1, y_2) \in Y^{[2]}$ and $u \in Q_{(y_1, y_2)}$. Therefore $f_2^{-1}\hat{\lambda}_{23}\hat{g}_2(\theta_{12}) = \hat{g}_1(\theta_{12})f_1^{-1}\hat{\lambda}_{23}$, which establishes the equation above. One can also check that the functor $m$ is associative and that identity 1-arrows and identity 2-arrows behave as they should with respect to composition by $m$. Hence we have the following proposition.

**Proposition 4.4** ([12]). There is a 2-category $\mathbf{BGrb}_M$ whose objects are bundle gerbes $P$ on $M$, 1-arrows are bundle gerbe morphisms $P \to Q$ and whose 2-arrows are transformations between bundle gerbe morphisms with the composition laws given as above.

## 5. Simplicial Bundle Gerbes and Bundle 2-Gerbes

We use our description of $\mathbf{BGrb}_M$ as a 2-category to define the notion of a *simplicial bundle gerbe* on a simplicial manifold $X = \{X_p\}$. We are motivated by Brylinski and McLaughlin’s definitions of a *simplicial line bundle* ([8] Definition 5.1) and a *simplicial gerbe* ([3] page 617). We record here the definition of a simplicial line bundle.

**Definition 5.1** ([8]). A simplicial line bundle on a simplicial manifold $X_\bullet = \{X_p\}$ consists of the following data:

1. a line bundle $L \to X_1$
2. a non-vanishing section $s$ of the line bundle $\delta(L)$ on $X_2$ where
   \[
   \delta(L) = d_0^{-1}L \otimes d_1^{-1}L^* \otimes d_2^{-1}L,
   \]
   where $d_i: X_p \to X_{p-1}$ denote the face operators of the simplicial manifold $X_\bullet = \{X_p\}$.
3. $s$ induces a non-vanishing section $\delta(s)$ of the line bundle $\delta\delta(L)$ on $X_3$ where $\delta\delta(L)$ is defined by
   \[
   \delta\delta(L) = d_0^{-1}\delta(L) \otimes d_1^{-1}\delta(L)^* \otimes d_2^{-1}\delta(L) \otimes d_3^{-1}\delta(L)^*,
   \]
   and $\delta(s) = d_0^{-1}s \otimes d_1^{-1}s^* \otimes d_2^{-1}s \otimes d_3^{-1}s^*$. Notice that as a result of the simplicial identities satisfied by the face operators $d_i: X_p \to X_{p-1}$ the line bundle $\delta\delta(L)$ is canonically trivialised. We demand that $\delta(s)$ matches this canonical trivialisation.

Note the following consequences of this definition.

(i) The non-vanishing section $s$ of $\delta(L)$ defines a line bundle isomorphism $d_0^{-1}L \otimes d_2^{-1}L \to d_1^{-1}L$ covering the identity on $X_2$. The coherency condition on $s$ is equivalent to this line bundle isomorphism satisfying an ‘associativity’ condition on $X_3$.

(ii) In the special case where the simplicial manifold $X_\bullet = \{X_p\}$ is the simplicial manifold associated to a surjection $\pi: X \to M$ which locally admits sections, then a simplicial line bundle on $X_\bullet$ recovers the definition of a bundle gerbe.

(iii) Another important special case is when $X_\bullet = \{X_p\}$ is the simplicial manifold $NG$ associated to the classifying space of a Lie group $G$ (see [11]). Then a simplicial line bundle on $NG$ is the same thing as a central extension of $G$ by $\mathbb{C}^\times$ ([4]).
We use the notion of a simplicial line bundle to motivate our definition of a simplicial bundle gerbe. To avoid cluttered notation later on, it is convenient to restrict attention to the simplicial manifold $X_\bullet$ associated to a surjection $\pi: X \to M$ which admits local sections. This will not affect our results at all; everything we say will be true for an arbitrary simplicial manifold, however it is easier to state this for $X_\bullet$.

We start with a bundle gerbe $(Q,Y,X^2)$ on $X^2$. We suppose there is a bundle gerbe morphism $m: \pi_1^{-1}Q \otimes \pi_3^{-1}Q \to \pi_1^{-1}Q$. It is convenient to introduce some new notation (analogous to that used in [16]) to avoid large, complicated diagrams. Let us denote by $Y \to Y \to X^3$ the local-section-admitting surjection whose fiber at a point $(x_1,x_2,x_3) \in X^3$ is $Y_{(x_2,x_3)} \times Y_{(x_1,x_2)}$. So $Y \circ Y = \pi_1^{-1}Y \times_{X^3} \pi_3^{-1}Y$.

Another way of looking at this is that $Y \circ Y$ is the restriction of $X \times Y$ to $(Y \times Y)_{X^2 \circ X^2}$ where $X^2 \circ X^2 = \{(x,y),(y,z)\}(x,y),(y,z) \in X^2 = X^3$. A point of $Y \circ Y$ is of the form $(y_{32},y_{12})$ where $y_{32} \in Y_{(x_2,x_3)}$ and $y_{12} \in Y_{(x_1,x_2)}$ for some point $(x_1,x_2,x_3) \in X^3$. Similarly let $Q \circ Q$ denote the restriction of $Q \otimes Q \to Y^2 \times Y^2$ to $(Y \circ Y)^2 \subset Y^2 \times Y^2$. Thus $Q \circ Q = \pi_1^{-1}Q \otimes \pi_3^{-1}Q$ and has fiber $(Q \circ Q)((y_{32},y_{12}),y_{23},y_{12}))$ at a point $(y_{32},y_{12})$ of $(Y \circ Y)^2$ equal to $Q_{(y_{32},y_{12})} \otimes Q_{(y_{23},y_{12})}$.

By construction the triple $(Q\circ Q,Y\circ Y,X^3)$ is a bundle gerbe — the bundle gerbe $(\pi_1^{-1}Q \otimes \pi_3^{-1}Q,\pi_1^{-1}Y \times_{X^3} \pi_3^{-1}Y,X^3)$.

The bundle gerbe morphism $m: \pi_1^{-1}Q \otimes \pi_3^{-1}Q \to \pi_2^{-1}Q$ is then a bundle gerbe morphism (also denoted $m$) $Q \circ Q \to Q$ covering the map $\pi_2: X^3 \to X^2$ sending a point $(x_1,x_2,x_3)$ of $X^3$ to the point $(x_1,x_2)$ of $X^2$. Over $X^4$ we can define another bundle gerbe $(Q \circ Q,Y \to Y \circ Y,X^4)$ where $Y \circ Y \to X^4$ is the local-section-admitting surjection with fiber

$$Y_{(x_3,x_4)} \times Y_{(x_2,x_3)} \times Y_{(x_1,x_2)}$$

over a point $(x_1,x_2,x_3,x_4)$. $Q^{\circ^3} = Q \circ Q \circ Q$ is defined in an analogous fashion to $Q \circ Q$ above.

The bundle gerbe morphism $m$ gives rise to two bundle gerbe morphisms $m_1,m_2: Q^{\circ^3} \to Q$ which cover the map $X^4 \to X^2$ which sends $(x_1,x_2,x_3,x_4)$ to $(x_1,x_2)$. We have $m_1 = (\hat{m}_1,m_1)$, $m_2 = (\hat{m}_2,m_2)$ where $m_1,m_2: Q^{\circ^3} \to Q = Y \circ Y \circ Y$ are given by $m_1(y_{34},y_{23},y_{12}) = m(m(y_{34},y_{23},y_{12}))$ and $m_2(y_{34},y_{23},y_{12}) = m(y_{34},m(y_{23},y_{12}))$, and $\hat{m}_1,\hat{m}_2: Q^{\circ^3} \to Q$ are given by $\hat{m}_1(u_{34} \otimes u_{23} \otimes u_{12}) = \hat{m}(\hat{m}(u_{34} \otimes u_{23} \otimes u_{12}))$, and $\hat{m}_2(u_{34} \otimes u_{23} \otimes u_{12}) = \hat{m}(u_{34} \otimes u_{23} \otimes u_{12})$, for $u_{ij} \in Q_{(y_{ij},y_{ij})}$. We demand that there is a transformation of bundle gerbe morphisms $a: m_1 \Rightarrow m_2$. Recall that this means there is a section $\hat{a}$ of the $C^\infty$ bundle $(m_1,m_2)^{-1}Q$ on $Q^{\circ^3}$ which descends to a section $a$ of the $C^\infty$ bundle $A = D((m_1,m_2)^{-1}Q) = D_{m_1,m_2}Q$ on $X^4$.

Finally, over $X^5$ we can define a bundle gerbe $(Q^{\circ^i},Y^{\circ^i},X^5)$ where $Q^{\circ^i}$ and $Y^{\circ^i}$ are defined in the obvious way. So for example, $Y^{\circ^4}$ is the local-section-admitting surjection on $X^5$ with fiber

$$Y_{(x_4,x_5)} \times Y_{(x_3,x_4)} \times Y_{(x_2,x_3)} \times Y_{(x_1,x_2)}$$

at a point $(x_1,x_2,x_3,x_4,x_5) \in X^5$. Now the bundle gerbe morphism $m$ gives rise to five bundle gerbe morphisms $M_i: Q^{\circ^i} \to Q$, $i = 1,\ldots,5$ covering the map $X^5 \to X^2$ which sends $(x_1,x_2,x_3,x_4,x_5)$ to $(x_1,x_5)$. The bundle gerbe morphisms $M_i$ are given as follows: $M_1 = m(1 \circ m)(1 \circ m \circ 1)$, $M_2 = m(m(1 \circ 1)(1 \circ m \circ 1)$, $M_3 = m(1 \circ m)(1 \circ m \circ 1)$, $M_4 = m(1 \circ m)(1 \circ m \circ 1)$, $M_5 = m(1 \circ m)(1 \circ m \circ 1)$.
Here we have abused notation and denoted for example by $m \circ 1$ the bundle gerbe morphism $Q^3 \to Q^5$ which sends $u_{34} \otimes u_{23} \otimes u_{12}$ to $\hat{m}(u_{34} \otimes u_{23}) \otimes u_{12}$. Notice that $M_5$ can also be written as $M_5 = m(1 \circ m)(m \circ 1 \circ 1)$. It is not too hard to see that we have the following isomorphisms of $C^\times$ bundles on $X^{[5]}$. We have $D_{M_1,M_2} = \pi_1^{-1} A$, $D_{M_2,M_3} = \pi_3^{-1} A$, $D_{M_3,M_4} = \pi_5^{-1} A$, $D_{M_4,M_5} = \pi_2^{-1} A^*$ and $D_{M_5,M_1} = \pi_4^{-1} A^*$. From Lemma 4.1 there is an isomorphism

$$D_{M_1,M_2} \otimes D_{M_2,M_3} \otimes D_{M_3,M_4} \otimes D_{M_4,M_5} \otimes D_{M_5,M_1} = D_{M_1,M_1},$$

and therefore, since $D_{M_1,M_1}$ is canonically trivialised, the $C^\times$ bundle $\delta(A)$ on $X^{[5]}$ must be canonically trivialised. Here $\delta(A)$ is the $C^\times$ bundle given by

$$\delta(A) = \pi_1^{-1} A \otimes \pi_2^{-1} A^* \otimes \pi_3^{-1} A \otimes \pi_4^{-1} A^* \otimes \pi_5^{-1} A.$$ 

We finally require that the induced section $\delta(a) = \pi_1^{-1} a \otimes \pi_2^{-1} a^* \otimes \pi_3^{-1} a \otimes \pi_4^{-1} a^* \otimes \pi_5^{-1} a$ of $\delta(A)$ matches this canonical trivialisation. This coherency condition on the section $a$ should actually be viewed as an equality of transformations of bundle gerbe morphisms as indicated in Figure 1. Notice that this bit of theory is possible precisely because the $\pi_i$ are the face operators for a simplicial manifold. All that we have said applies equally well to an arbitrary simplicial manifold. Hence we make the following definition.

**Definition 5.2 (B).** A simplicial bundle gerbe on a simplicial manifold $X_* = \{X_p\}$ consists of the following data.

1. A bundle gerbe $(Q,Y,X_1)$ on $X_1$.
2. A bundle gerbe morphism $m: d_{i_0}^{-1} Q \otimes d_{i}^{-1} Q \to d_{i}^{-1} Q$ over $X_2$.
3. A transformation $a: m_1 \Rightarrow m_2$ between the two induced bundle gerbe morphisms $m_1$ and $m_2$ over $X_3$. $m_1$ and $m_2$ are defined as in the following...
diagram.
\[
d_0^{-1}(d_0^{-1}Q \otimes d_2^{-1}Q) \otimes d_2^{-1}d_2^{-1}Q \xrightarrow{m \otimes d_2^{-1}d_2^{-1}Q} d_1^{-1}d_0^{-1}Q \otimes d_3^{-1}(d_0^{-1}Q \otimes d_2^{-1}Q)
\]
\[
d_0^{-1}d_1^{-1}Q \otimes d_2^{-1}d_2^{-1}Q \xrightarrow{a} d_1^{-1}d_0^{-1}Q \otimes d_3^{-1}d_1^{-1}Q
\]
\[
d_2^{-1}(d_0^{-1}Q \otimes d_2^{-1}Q) \xrightarrow{d_2^{-1}m} d_1^{-1}(d_0^{-1}Q \otimes d_2^{-1}Q)
\]
\[
d_2^{-1}d_1^{-1}Q \xrightarrow{d_2^{-1}m} d_1^{-1}d_1^{-1}Q.
\]

So \( m_1 = d_2^{-1}m \circ (d_0^{-1}m \otimes d_2^{-1}d_2^{-1}1_Q) \) and \( m_2 = d_1^{-1}m \circ (d_1^{-1}d_0^{-1}1_Q \otimes d_3^{-1}m) \).

Thus \( a \) is a section of the \( \mathbb{C}^\times \) bundle \( A = D_{m_1,m_2} \) over \( X_3 \).

4. The transformation \( a \) satisfies the coherency condition
\[
d_0^{-1}a \otimes d_1^{-1}a^* \otimes d_2^{-1}a \otimes d_3^{-1}a^* \otimes d_4^{-1}a = 1,
\]
where 1 is the canonical section of the \( \mathbb{C}^\times \) bundle \( \delta(A) \) over \( X_4 \).

Note that the coherency condition on the transformation \( a \) can also be viewed as the commutativity of a diagram of the form Figure 3. Clearly the notion of a simplicial bundle gerbe is a special case of Brylinski and McLaughlin’s definition of a simplicial gerbe [3]. To recover the definition of simplicial gerbe from Definition 5.2 above, simply replace each occurrence of the word ‘bundle gerbe’ by the word ‘gerbe’, ‘bundle gerbe morphism’ by ‘gerbe morphism’ and so on (strictly speaking we should insert certain canonical equivalences of gerbes where we have equalities of bundle gerbes, but this is of no real importance). Note that the associator transformation of gerbe morphisms in the definition of a simplicial gerbe can be interpreted as a section of a certain line bundle on \( X_3 \), and the coherency condition on the transformation can be interpreted as a coherency condition on sections of line bundles on \( X_4 \), as above.

We define a bundle 2-gerbe to be a special case of the above definition.

**Definition 5.3** [3]. A **bundle 2-gerbe** consists of a quadruple of smooth manifolds \((Q,Y,X,M)\) where \( \pi: X \to M \) is a smooth surjection admitting local sections and where \((Q,Y,X^{[2]})\) is a simplicial bundle gerbe on the simplicial manifold \( X_* = \{X_p\} \) with \( X_p = X^{[p+1]} \) associated to \( \pi: X \to M \).

So given a bundle 2-gerbe \((Q,Y,X,M)\), we have a bundle gerbe \((Q,Y,X^{[2]})\) and a bundle gerbe morphism \( m: \pi_1^{-1}Q \otimes \pi_3^{-1}Q \to \pi_2^{-1}Q \). The bundle gerbe morphism \( m \) consists of a pair of maps \((\hat{m},m)\), where \( m: Y_{123} = \pi_1^{-1}Y \times_{X^{[3]}} \pi_3^{-1}Y \to Y_{13} = \pi_2^{-1}Y \) is a map commuting with the projections to \( X^{[3]} \) and \( \hat{m}: \pi_1^{-1}Q \otimes \pi_3^{-1}Q \to \pi_2^{-1}Q \) covers \( m^{[2]}: Y_{123}^{[2]} \to Y_{13}^{[2]} \) and commutes with the bundle gerbe products on \( \pi_1^{-1}Q \otimes \pi_3^{-1}Q \) and \( \pi_2^{-1}Q \). So fiberwise \( m \) is a map
\[
m: Y_{(x_1,x_2)} \times Y_{(x_1,x_2)} \to Y_{(x_1,x_2)}
\]
for \((x_1,x_2,x_3) \in X^{[3]}\) and \( \hat{m} \) is a map
\[
\hat{m}: Q_{(y_{23},y_{23}')} \otimes Q_{(y_{12},y_{12}')} \to Q_{(m(y_{23},y_{12}),m(y_{23},y_{12}'))}
\]
for \((y_{23}, y_{12})\), \((y'_{23}, y'_{12}) \in Y_{123}\). Thus for each pair of points \((x_1, x_2)\) lying in the same fiber of \(\pi: X \to M\), we obtain a \(\mathbb{C}^\times\) groupoid \(\text{Gr}(Q)_{(x_1, x_2)}\). Given a triple of points \((x_1, x_2, x_3)\) lying in the same fiber of \(\pi: X \to M\) the bundle gerbe morphism \(m\) gives rise to a functor \(m: \text{Gr}(Q)_{(x_2, x_3)} \times \text{Gr}(Q)_{(x_1, x_2)} \to \text{Gr}(Q)_{(x_1, x_3)}\) as explained in Section 3. Let us denote the action of the functor \(m\) on a pair of objects \((y_{23}, y_{12})\) of \(\text{Gr}(Q)_{(x_2, x_3)} \times \text{Gr}(Q)_{(x_1, x_2)}\) by \(y_{23} \circ y_{12}\). The transformation \(a\) gives rise to a natural transformation, also denoted \(a\), between the functors bounding the following diagram.

\[
\begin{array}{c}
\text{Gr}(Q)_{(x_3, x_4)} \times \text{Gr}(Q)_{(x_2, x_3)} \times \text{Gr}(Q)_{(x_1, x_2)}
\end{array}
\xrightarrow{\begin{array}{c}1 \times m \\ m \times 1 \end{array}}
\begin{array}{c}
\text{Gr}(Q)_{(x_2, x_3)} \times \text{Gr}(Q)_{(x_1, x_2)}
\end{array}
\xrightarrow{\begin{array}{c}m \end{array}}
\begin{array}{c}
\text{Gr}(Q)_{(x_1, x_3)}
\end{array}
\]

The coherency condition on the transformation \(a\) of bundle gerbe morphisms can be viewed as an associativity coherence condition on the natural transformation \(a\). Let us briefly recall the definition of a bicategory \([2]\). A bicategory \(\mathcal{B}\) consists of objects \(A, B, C, \ldots\) and for each pair of objects \(A\) and \(B\) a category \(\text{Hom}(A, B)\). The objects of \(\text{Hom}(A, B)\) are called 1-arrows or 1-cells of \(\mathcal{B}\) and the arrows of \(\text{Hom}(A, B)\) are called 2-arrows or 2-cells of \(\mathcal{B}\). A 2-cell \(\phi\) between 1-cells \(\alpha\) and \(\beta\) of \(\text{Hom}(A, B)\) is denoted \(\alpha \Rightarrow \beta\). Given three objects \(A, B\) and \(C\) of \(\mathcal{B}\) there is a composition functor \(\text{Hom}(B, C) \times \text{Hom}(A, B) \to \text{Hom}(A, C)\) whose action on a pair of objects \((\alpha, \beta)\) of \(\text{Hom}(B, C) \times \text{Hom}(A, B)\) (1-cells of \(\mathcal{B}\)) is denoted \(\alpha \circ \beta\) and similarly for 2-cells. The composition functor is associative up to a coherent isomorphism. This means that given objects \(A, B, C\) and \(D\) of \(\mathcal{B}\) with 1-cells \(\alpha \in \text{Hom}(A, B), \beta \in \text{Hom}(B, C)\) and \(\gamma \in \text{Hom}(C, D)\) then there is an isomorphism

\[
a(\gamma, \beta, \alpha): (\gamma \circ \beta) \circ \alpha \Rightarrow \gamma \circ (\beta \circ \alpha)
\]

in \(\text{Hom}(A, C)\) which is natural in \(A, B\) and \(C\). The natural isomorphism \(a\) is called the associator natural isomorphism. The associativity coherence condition means that the well known pentagonal diagram commutes. One also requires that for every object \(A\) of \(\mathcal{B}\) there is a 1-arrow \(1_A\) of \(\text{Hom}(A, A)\) and for every 1-arrow \(\alpha \in \text{Hom}(A, B)\) of \(\mathcal{B}\) there are left and right identity isomorphisms \(L_\alpha: \alpha \circ 1_A \Rightarrow \alpha\) and \(R_\alpha: 1_B \circ \alpha \Rightarrow \alpha\) which are natural in the 1-arrows \(\alpha\). These isomorphisms are finally required to satisfy the coherence condition that the following diagram commutes.

\[
\begin{array}{c}
\text{Gr}(Q)_{(x_3, x_4)} \times \text{Gr}(Q)_{(x_2, x_3)} \times \text{Gr}(Q)_{(x_1, x_2)}
\end{array}
\xrightarrow{\begin{array}{c}1 \times m \\ m \times 1 \end{array}}
\begin{array}{c}
\text{Gr}(Q)_{(x_2, x_3)} \times \text{Gr}(Q)_{(x_1, x_2)}
\end{array}
\xrightarrow{\begin{array}{c}m \end{array}}
\begin{array}{c}
\text{Gr}(Q)_{(x_1, x_3)}
\end{array}
\]

where \(\alpha\) is a 1-arrow of \(\text{Hom}(A, B)\) and \(\beta\) is a 1-arrow of \(\text{Hom}(B, C)\). A bicategory in which all of the natural isomorphisms \(a, L\) and \(R\) are the identities is a 2-category.

One can define the notion of a biequivalence between bicategories; we will refer to [2] for this. One can show [3] that every bicategory is biequivalent to a 2-category. We also have the notion of a bigroupoid.

**Definition 5.4** ([3]). A bigroupoid consists of a bicategory \(\mathcal{B}\) which satisfies the following two additional axioms.
1. 1-arrows are coherently invertible. This means that if \( \alpha \) is a 1-arrow of \( \text{Hom}(A, B) \) then there is a 1-arrow \( \beta \) of \( \text{Hom}(B, A) \) together with 2-arrows \( \phi: \beta \circ \alpha \Rightarrow 1_A \) of \( \text{Hom}(A, A) \) and \( \psi: \alpha \circ \beta \Rightarrow 1_B \) in \( \text{Hom}(B, B) \).

2. All 2-arrows are invertible.

By a \( C^X \) bigroupoid we mean a bigroupoid \( B \) in which the automorphism group of every 1-arrow is isomorphic to \( C^X \).

We have the following Proposition.

**Proposition 5.5** \((\text{[19]}))\). *For each point \( m \) of \( M \), the restriction of a bundle 2-gerbe \( (Q, Y, X, M) \) to the point \( m \) gives rise to a family of \( C^X \) bigroupoids \( Q_m \).*

We take as the objects of \( Q_m \) the points of \( X_m = \pi^{-1}(m) \). Given two such points \( x_1 \) and \( x_2 \) we define the category \( \text{Hom}(x_1, x_2) \) to be the category \( \text{Gr}(Q)(x_1, x_2) \) defined above. It is clear that the bundle gerbe morphism \( m \) provides the composition functor and that the transformation \( a \) plays the role of the associator natural isomorphism. All we have to do then is to define left and right identity morphisms and show that they are compatible with \( a \). We will not do this here and refer instead to \([19]\). Thus we can think of a bundle 2-gerbe as being a ‘bundle of bigroupoids’.

6. **The Homotopy Bigroupoid and the Tautological Bundle 2-Gerbe.**

An important example of a bigroupoid is the so-called *homotopy bigroupoid* or *fundamental bigroupoid* \( \Pi_2(X) \) associated to a topological space \( X \) (see \([\text{III}]\)). \( \Pi_2(X) \) is defined as follows. The objects of \( \Pi_2(X) \) are the points \( x \) of \( X \). Given two points \( x_1 \) and \( x_2 \) the category \( \text{Hom}(x_1, x_2) \) is defined to be the set of homotopy classes \([\mu]\) of maps \( I \times I \to X \) such that \( \mu(0, t) = \gamma_1(t) \), \( \mu(1, t) = \gamma_2(t) \), \( \mu(s, 0) = x_1 \) and \( \mu(s, 1) = x_2 \). Two such maps \( \mu \) and \( \mu' \) belong to the same homotopy class if there is a map \( H: I \times I \times I \to X \) such that \( H(0, s, t) = \mu(s, t), H(1, s, t) = \mu'(s, t), H(r, 0, t) = \gamma_1(t), H(r, 1, t) = \gamma_2(t) \), \( H(r, s, 0) = x_1 \) and \( H(r, s, 1) = x_2 \). To define the composite 2-cell \([\lambda][\mu]: \gamma_1 \Rightarrow \gamma_3 \) for 2-cells \([\mu]: \gamma_1 \Rightarrow \gamma_2 \) and \([\lambda]: \gamma_2 \Rightarrow \gamma_3 \) we choose representatives \( \mu \) and \( \lambda \) of \([\mu]\) and \([\lambda]\) respectively and define \([\lambda][\mu]\) to be the homotopy class of the map

\[
(\lambda \mu)(s, t) = \begin{cases} 
\mu(2s, t) & s \in [0, \frac{1}{2}], \ t \in [0, 1], \\
\lambda(2s - 1, t) & s \in [\frac{1}{2}, 1], \ t \in [0, 1].
\end{cases}
\]

It is straightforward to check that this law of composition is well defined and is associative. Notice that every 2-cell of \( \Pi_2(X) \) is invertible. We need to define the composition functor

\[
m: \text{Hom}(x_2, x_3) \times \text{Hom}(x_1, x_2) \to \text{Hom}(x_1, x_3).
\]

If \( \gamma_{23} \) is a 1-arrow of \( \text{Hom}(x_2, x_3) \) and \( \gamma_{12} \) is a 1-arrow of \( \text{Hom}(x_1, x_2) \) then we define \( m(\gamma_{23}, \gamma_{12}) \) to be the path \( \gamma_{23} \circ \gamma_{12}: I \to X \) given by

\[
(\gamma_{23} \circ \gamma_{12})(t) = \begin{cases} 
\gamma_{12}(2t), & t \in [0, \frac{1}{2}], \\
\gamma_{23}(2t - 1), & t \in [\frac{1}{2}, 1].
\end{cases}
\]
where \( \mu \) homotopy with endpoints fixed between \( \gamma \) identity 2-arrows. Given an object \( x \) of the homotopy class \( \mu \) path at \( x \).

One can check that the assignment of the 2-arrow \( a \) \((4)\) 2-arrow from \((\gamma_{45} \circ (\gamma_{34} \circ \gamma_{23})) \circ \gamma_{12} \) means that we have to check that the diagram of 2-arrows in Figure 2 is the identity constant path to itself.

There is a standard choice for \( \gamma_{34} \) in \( \text{Hom}(x_2, x_3) \) and \( \gamma_{12} \) in \( \text{Hom}(x_1, x_2) \) are 2-arrows, we define \( m([\mu_{23}], [\mu_{12}]) = [\mu_{23} \circ \mu_{12}] \) to be the homotopy class of the map

\[
(\mu_{23} \circ \mu_{12})(s, t) = \begin{cases} 
\mu_{12}(s, 2t), & s \in [0, 1], \ t \in [0, \frac{1}{2}], \\
\mu_{23}(s, 2t - 1), & s \in [0, 1], \ t \in [\frac{1}{2}, 1], 
\end{cases}
\]

where \( \mu_{23} \) is a representative of the homotopy class \( [\mu_{23}] \) and \( \mu_{12} \) is a representative of the homotopy class \( [\mu_{12}] \). Note that the map \( \mu_{23} \circ \mu_{12} : I \times I \to X \) defines a homotopy with endpoints fixed between \( \gamma_{23} \circ \gamma_{12} \) and \( \gamma_{23} \circ \gamma_{12}' \). It is straightforward to check that this defines a functor. We now need to define identity 1-arrows and identity 2-arrows. Given an object \( x \) of \( \Pi_2(X) \), we define \( 1_x \) to be the constant path at \( x \) and the identity 2-arrow \( 1_x \Rightarrow 1_x \) to be the constant homotopy from the constant path to itself.

Next we define the associator isomorphism. Given 1-arrows \( \gamma_{34} \) in \( \text{Hom}(x_2, x_3) \), \( \gamma_{23} \) in \( \text{Hom}(x_2, x_3) \) and \( \gamma_{12} \) in \( \text{Hom}(x_1, x_2) \) we need to define a 2-arrow

\[ a(\gamma_{34}, \gamma_{23}, \gamma_{12}) : (\gamma_{34} \circ \gamma_{23}) \circ \gamma_{12} \Rightarrow \gamma_{34} \circ (\gamma_{23} \circ \gamma_{12}) \].

There is a standard choice for \( a(\gamma_{34}, \gamma_{23}, \gamma_{12}) \) — see for example [13]. We set \( a(\gamma_{34}, \gamma_{23}, \gamma_{12}) \) equal to the homotopy class of the map \( \bar{a}(\gamma_{34}, \gamma_{23}, \gamma_{12}) : I \times I \to X \) given by

\[
a(\gamma_{34}, \gamma_{23}, \gamma_{12})(s, t) = \begin{cases} 
\gamma_{12}\left(\frac{2t}{s} \right), & s \in [0, 1], \ t \in [0, \frac{2s}{3}], \\
\gamma_{23}(4t - 2 + s), & s \in [0, 1], \ t \in [\frac{2s}{3}, \frac{3s}{4}], \\
\gamma_{34}\left(\frac{4t + 4s}{1 + s} \right), & s \in [0, 1], \ t \in [\frac{3s}{4}, 1]. 
\end{cases}
\]

One can check that the assignment of the 2-arrow \( a(\gamma_{34}, \gamma_{23}, \gamma_{12}) \) of \( \text{Hom}(x_1, x_4) \) to the 1-arrow \( \gamma_{12} \) of \( \text{Hom}(x_2, x_3) \) of \( \text{Hom}(x_3, x_4) \times \text{Hom}(x_2, x_3) \times \text{Hom}(x_1, x_2) \) is a natural transformation \( m \circ (m \times 1) \Rightarrow m \circ (1 \times m) \). We now have to check that the natural transformation \( a \) satisfies the associativity coherence condition. This means that we have to check that the diagram of 2-arrows in Figure 2 is the identity 2-arrow from \((\gamma_{45} \circ \gamma_{34}) \circ (\gamma_{34} \circ \gamma_{23}) \circ \gamma_{12} \) to itself. We will omit the proof of this fact and refer to [14] where an explicit homotopy between the composed 2-arrow from \((\gamma_{45} \circ \gamma_{34}) \circ (\gamma_{34} \circ \gamma_{23}) \circ \gamma_{12} \) to itself and the identity 2-arrow is given. To show that \( \Pi_2(X) \) is a bicategory, we need to produce left and right identity isomorphisms. If \( \gamma \in \text{Hom}(x_1, x_2) \) then \( L(\gamma) \) is a 2-arrow \( \gamma \Rightarrow \gamma \circ 1_{x_1} \). We define \( L(\gamma) \) to be the
homotopy class of the map
\[(s, t) \mapsto \begin{cases} x_1, & t \in [0, \frac{2}{3}], \\ \gamma(\frac{2t-1}{3}), & t \in [\frac{2}{3}, 1]. \end{cases}\]
Similarly if \( \gamma \in \text{Hom}(x_1, x_2) \) then \( R(\gamma) \) is a 2-arrow \( R(\gamma): \gamma \Rightarrow 1_{x_2} \circ \gamma \). We set \( R(\gamma) \) equal to the homotopy class of the map
\[(s, t) \mapsto \begin{cases} \gamma((s + 1)t), & t \in [0, \frac{1}{s+1}], \\ x_2, & t \in [\frac{1}{s+1}, 1]. \end{cases}\]

One can check (see [19]) that the assignments \( \gamma \mapsto L(\gamma) \) and \( \gamma \mapsto R(\gamma) \) define natural transformations and that moreover these natural transformations are compatible with \( a \). Hence \( \Pi_2(X) \) is an example of a bicategory. One can also show that the 1-arrows of \( \Pi_2(X) \) are coherently invertible and, as mentioned earlier, all 2-arrows of \( \Pi_2(X) \) are invertible. Therefore \( \Pi_2(X) \) is a bigroupoid - the homotopy bigroupoid of \( X \).

We will now use this description of the homotopy bigroupoid \( \Pi_2(X) \) of \( X \) to define the tautological bundle 2-gerbe of \([10]\) over a 3-connected manifold \( M \). Recall that we start with a closed four form \( \Theta \) on \( M \) with integral periods, representing a class in \( H^4(M; \mathbb{Z}) \). We then form the path fibration \( \pi: \mathcal{P}M \to M \), where \( \mathcal{P}M \) is the Frechet manifold consisting of piecewise smooth paths \( \gamma: [0, 1] \to M \), \( \gamma(0) = m_0 \) where \( m_0 \) is a basepoint of \( M \), and where \( \pi \) is the map sending such a path \( \gamma \) to its endpoint \( \gamma(1) \). The fibration \( \pi: \mathcal{P}M \to M \) has fiber \( F \) equal to the space of piecewise smooth loops in \( M, \Omega M \).

We will define a simplicial bundle gerbe on the simplicial manifold \( X_\bullet = \{ X_p \} \) with \( X_p = X^p \) and with face and degeneracy operators \( d_i: X^{p+1} \to X^p \), \( s_i: X^p \to X^{p+1} \) given respectively by
\[
d_i(x_1, \ldots, x_{p+1}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{p+1}) \\
s_i(x_1, \ldots, x_p) = (x_1, \ldots, x_i, x_{i+1}, \ldots, x_p).
\]
Performing the construction for the simplicial manifold \( X_\bullet \) above with \( X = \Omega M \) fiber by fiber on \( \mathcal{P}M \) will define the tautological bundle 2-gerbe. We start with a 2-connected manifold \( X \) and a closed 3-form \( \omega \) on \( X \) with integral periods. We construct a bundle gerbe on \( X^2 = X \times X \) in the usual way. We define a fibering \( Y \to X^2 \) with fiber \( Y_{(x_1, x_2)} \) at \( (x_1, x_2) \in X^2 \) equal to the space of piecewise smooth paths \( \alpha: I \to X \) with \( \alpha(0) = x_1 \) and \( \alpha(1) = x_2 \). Next we define a \( \mathbb{C}^\infty \) bundle \( Q \to Y^{[2]} \) whose fiber at \( (\alpha, \beta) \in Y^{[2]} \) is all equivalence classes \([\mu, z] \) where \( z \in \mathbb{C}^\infty \) and \( \mu: I^2 \to X \) is a homotopy with endpoints fixed between \( \alpha \) and \( \beta \), that is \( \mu(0, t) = \alpha(t), \mu(1, t) = \beta(t), \mu(s, 0) = x_1 \) and \( \mu(s, 1) = x_2 \). The equivalence relation \( \sim \) is defined by declaring \((\mu_1, z_1) \sim (\mu_2, z_2)\) if for any homotopy \( F: I^3 \to X \) with endpoints fixed between \( \mu_1 \) and \( \mu_2 \) we have
\[
z_2 = z_1 \exp\left( \int_{I^3} F^*(\omega) \right).
\]
Here we say that \( F \) is a homotopy with endpoints fixed between \( \mu_1 \) and \( \mu_2 \) if we have \( F(0, s, t) = \mu_1(s, t), F(1, s, t) = \mu_2(s, t), F(r, 0, t) = \alpha(t), F(r, 1, t) = \beta(t), F(r, s, 0) = x_1 \) and \( F(r, s, 1) = x_1 \). One can define an associative product \( m_Q \) on \( Q \to Y^{[2]} \) as in [3] by setting \( m_Q([\mu, z] \otimes [\nu, w]) = [\mu \nu, zw] \), where \( \mu \nu: I^2 \to X \) is
defined by
\[ (\mu \nu)(s, t) = \begin{cases} \nu(2s, t), & s \in [0, \frac{1}{2}], \ t \in [0, 1], \\ \mu(2s - 1, t), & s \in [\frac{1}{2}, 1], \ t \in [0, 1]. \end{cases} \]

One can check, see [3], that this is well defined and associative. Next we define a bundle gerbe morphism \( m: d_0^{-1}Q \otimes d_2^{-1}Q \to d_1^{-1}Q \) with \( m = (\hat{m}, m) \). So fiberwise \( m \) will be a map \( Y_{(x_2, x_3)} \times Y_{(x_1, x_2)} \to Y_{(x_1, x_3)} \). \( m \) is defined by the composition functor in the bigroupoid \( \Pi_2(X) \) so \( m(\alpha, \beta) = \alpha \circ \beta \) where \( \alpha \circ \beta: I \to X \) is the path from \( x_1 \) to \( x_3 \) given by
\[ (\alpha \circ \beta)(t) = \begin{cases} \beta(2t), & t \in [0, \frac{1}{2}], \\ \alpha(2t - 1), & t \in [\frac{1}{2}, 1]. \end{cases} \]

The map \( m: d_0^{-1}Q \otimes d_2^{-1}Q \to d_1^{-1}Q \) covering \( m^{[2]} \) is defined by \( m([\mu_{23}, z_{23}] \otimes [\mu_{12}, z_{12}]) = [\mu_{23} \circ \mu_{12}, z_{23} \circ z_{12}] \), where \( \mu_{23} \circ \mu_{12} \) is defined by the action of the composition functor \( m \) in the bigroupoid \( \Pi_2(X) \) on 2-arrows. Hence \( \mu_{23} \circ \mu_{12}: I^2 \to X \) is the homotopy given by
\[ (\mu_{23} \circ \mu_{12})(s, t) = \begin{cases} \mu_{12}(s, 2t), & s \in [0, 1], \ t \in [0, \frac{1}{2}], \\ \mu_{23}(s, 2t - 1), & s \in [0, 1], \ t \in [\frac{1}{2}, 1]. \end{cases} \]

Again, one can check (see [4]), that this is well defined and commutes with the bundle gerbe products. As usual, \( m \) defines two bundle gerbe morphisms \( m_1 = (\hat{m}_1, m_1), m_2 = (\hat{m}_2, m_2) \) between the appropriately defined bundle gerbes on \( X^4 \).

So fiberwise \( m_1 \) and \( m_2 \) are maps \( Y_{(x_3, x_4)} \times Y_{(x_2, x_3)} \times Y_{(x_1, x_2)} \to Y_{(x_1, x_4)} \) which are given by \( m_1(\alpha_{34}, \alpha_{23}, \alpha_{12}) = (\alpha_{34} \circ \alpha_{23}) \circ \alpha_{12}, m_2(\alpha_{34}, \alpha_{23}, \alpha_{12}) = \alpha_{34} \circ (\alpha_{23} \circ \alpha_{12}) \).

\( \hat{m}_1 \) and \( \hat{m}_2 \) are defined in an analogous fashion. As we have already seen, there is a homotopy \( m_1 \simeq m_2 \). We can use this homotopy to write down a section \( a \) which trivialises the \( \mathbb{C}^\times \) bundle \( (m_1, m_2)^{-1}Q \) on \( Y \circ Y \circ Y \). We have \( \hat{a}(\alpha_{34}, \alpha_{23}, \alpha_{12}) = [\alpha(\alpha_{34}, \alpha_{23}, \alpha_{12}), 1] \) where \( \alpha(\alpha_{34}, \alpha_{23}, \alpha_{12}): I^2 \to X \) is defined in equation [4]. Recall that the associator natural isomorphism for the bigroupoid \( \Pi_2(X) \) is defined via \( \alpha \). The fact that this is a natural isomorphism is exactly the requirement that \( \hat{a} \) descends to a section \( a \) of the bundle \( A \) on \( X^4 \). Finally, one needs to show that \( a \) satisfies the coherency condition over \( X^3 \) or, alternatively, that \( \hat{a} \) satisfies the analogous coherency condition. Let \( \delta(a) \) denote the 2-arrow in Figure 2 from \( (\gamma_{45} \circ (\gamma_{34} \circ \gamma_{23})) \circ \gamma_{12} \) to itself. In [4] an explicit homotopy from \( \delta(a) \) to the identity 2-arrow at \( (\gamma_{45} \circ (\gamma_{34} \circ \gamma_{23})) \circ \gamma_{12} \) was written down. One checks easily that the pullback of \( \omega \) by this homotopy is zero. This shows that \( a \) satisfies the required coherency condition.

7. The Čech 3-class associated to a Bundle 2-gerbe.

Let \( (Q, Y, X, M) \) be a bundle 2-gerbe. We will explain how to construct a \( \mathbb{C}^\times \) valued Čech 3-cocycle associated to \( Q \). Choose an open covering \( \{U_{ij}\}_{i \in I} \) of \( M \) all of whose finite intersections are empty or contractible and such that there exist local sections \( s_i: U_i \to X \) of \( \pi: X \to M \). Form maps \( (s_i, s_j): U_{ij} \to X \) by sending a point \( m \) of \( U_{ij} \) to the point \( (s_i(m), s_j(m)) \) of \( X \). Let \( (Q_{ij}, Y_{ij}, U_{ij}) \) denote the pullback of the bundle gerbe \( (Q, Y, X, M) \) to \( U_{ij} \) via \( (s_i, s_j) \). Therefore \( Y_{ij} \to U_{ij} \) is a local-section-admitting surjection and the fiber \( (Y_{ij})_m \) of \( Y_{ij} \) at \( m \in U_{ij} \) is \( Y(s_i(m), s_j(m)) \).
Since $U_{ij}$ is contractible, the bundle gerbe $(Q_{ij}, Y_{ij}, U_{ij})$ is trivial. Hence there is a $\mathbb{C}^\infty$ bundle $P_{ij}$ on $Y_{ij}$ and an isomorphism $Q_{ij} \rightarrow \delta(P_{ij})$ over $Y_{ij}^{[2]}$ which commutes with the bundle gerbe products on $Q_{ij}$ and the trivial bundle gerbe $\delta(P_{ij})$. The bundle gerbe morphism $m: \pi_1^{-1}Q \otimes \pi_3^{-1}Q \rightarrow \pi_2^{-1}Q$ pulls back to define a bundle gerbe morphism $Q_{jk} \otimes Q_{ij} \rightarrow Q_{ik}$, also denoted $m$. In particular there is a map $m: Y_{jk} \times_M Y_{ij} \rightarrow Y_{ik}$ covering the identity on $U_{ijk}$. Let $\hat{P}_{ijk} = P_{jk} \otimes m^{-1}P_{ik} \otimes P_{ij}$. Thus $\hat{P}_{ijk}$ is a $\mathbb{C}^\infty$ bundle on $Y_{jk} \times_M Y_{ij}$. Note that there is an isomorphism $\delta(\hat{P}_{ijk}) \rightarrow Q_{jk} \otimes (m^{-1})^{-1}Q_{ik} \otimes Q_{ij}$ which commutes with the respective bundle gerbe products. Moreover, the $\mathbb{C}^\infty$ bundle $Q_{jk} \otimes (m^{-1})^{-1}Q_{ik} \otimes Q_{ij}$ has a canonical trivialisation provided by the bundle gerbe morphism $m$. The following Lemma follows easily from Example 3.1.

**Lemma 7.1.** Suppose $(P, X, M)$ and $(Q, Y, M)$ are bundle gerbes with a bundle gerbe morphism $f: P \rightarrow Q$. If $P$ and $Q$ are both trivial, so there exist $\mathbb{C}^\infty$ bundles $T_P$ and $T_Q$ on $X$ and $Y$ respectively, with $\delta(T_P) = P$ and $\delta(T_Q) = Q$, then the bundle $T_P \otimes f^{-1}T_Q$ descends to $M$.

Applying this result we see that $\hat{P}_{ijk}$ descends to a $\mathbb{C}^\infty$ bundle $P_{ijk}$ on $U_{ijk}$. Next, over $U_{ijkl}$ we have two induced bundle gerbe morphisms $m_1, m_2: Q_{kl} \otimes Q_{jk} \otimes Q_{ij} \rightarrow Q_{il}$. By Lemma 7.1 the $\mathbb{C}^\infty$ bundle $(m_1, m_2)^{-1}Q_{il}$ on $Y_{ijkl} = Y_{kl} \times_M Y_{jk} \times_M Y_{ij}$ descends to a $\mathbb{C}^\infty$ bundle $A_{ijkl}$ on $U_{ijkl}$, and it is clear that $A_{ijkl} = (s_i, s_j, s_k, s_l)^{-1}A$. We will show that there is an isomorphism

$$A_{ijkl} = P_{kl} \otimes P_{kl}^* \otimes P_{ij} \otimes P_{ij}^*$$

of $\mathbb{C}^\infty$ bundles on $U_{ijkl}$. Recall that the map $m_1: Y_{ijkl} \rightarrow Y_{il}$ is defined by composition: $Y_{ijkl} \xrightarrow{m_1} Y_{ij} \xrightarrow{m_3} Y_{il}$, where $Y_{ij} = Y_{jl} \times_M Y_{ij}$. It is not hard to show that $P_{kl} \otimes P_{jk} \otimes P_{ij} \otimes m_1^{-1}P_{il}^* \simeq \pi_{ijkl}^{-1}(P_{ij} \otimes P_{jl})$. Similarly we get another isomorphism $P_{kl} \otimes P_{jk} \otimes P_{ij} \otimes m_2^{-1}P_{il}^* \simeq \pi_{ijkl}^{-1}(P_{jk} \otimes P_{kl})$. Since we have an isomorphism $m_1^{-1}P_{il}^* \otimes m_2^{-1}P_{il} \simeq \pi_{ijkl}^{-1}(P_{jk} \otimes P_{kl})$, we get the required isomorphism $A_{ijkl} \simeq P_{kl} \otimes P_{kl}^* \otimes P_{ij} \otimes P_{ij}^*$. Now choose sections $\sigma_{ijk}$ of $P_{ijk}$ over $U_{ijk}$ and define $g_{ijkl}: U_{ijkl} \rightarrow \mathbb{C}^\infty$ by

$$\sigma_{ijkl} \otimes \sigma_{ijkl}^* \otimes \sigma_{ijkl} \otimes \sigma_{ijkl}^* : g_{ijkl} = a_{ijkl}.$$

One can show that $g_{ijkl}$ is a Čech $3$-cocycle. We have the following Proposition.

**Proposition 7.2.** $g_{ijkl}$ satisfies the Čech $3$-cocycle condition

$$g_{ijkl}g_{iklm}g_{ijlm}g_{ijkl}^{-1}g_{ijkl} = 1,$$

and hence is a representative of a class in $\check{H}^3(M; \mathbb{C}^\infty_M) = H^4(M, \mathbb{Z})$.

There is another method of calculating the Čech $3$-cocycle $g_{ijkl}$ which is similar in spirit to the method used to calculate the Čech representative of the Dixmier-Douady class of a bundle gerbe. Let $(Q, Y, X, M)$ be a bundle $2$-gerbe. Choose an open cover $\{U_i\}_{i \in I}$ of $M$ all of whose finite non-empty intersections are contractible and such that there exist local sections $s_i: U_i \rightarrow X$ of $\pi$. Form the maps $(s_i, s_j): U_{ij} \rightarrow X^{[2]}$ as above and again denote the pullback of the bundle gerbe $(Q, Y, X^{[2]})$ to $U_{ij}$ via $(s_i, s_j)$ by $(Q_{ij}, Y_{ij}, U_{ij})$. In certain circumstances, for instance if $\pi_Y: Y \rightarrow X^{[2]}$ is a fibration, one can choose sections $\sigma_{ij}: U_{ij} \rightarrow Y_{ij}$ of $\pi_{Y_{ij}}: Y_{ij} \rightarrow U_{ij}$. Note that in general one would only be able to choose an open cover
\( \{ U^a_{ij} \}_{a \in \Sigma_{ij}} \) of \( U_{ij} \) such that there were local sections \( \sigma^a_{ij} : U^a_{ij} \to Y_{ij} \) of \( \pi Y_{ij} \). We will assume here that we are in the former situation described above. For ease of notation denote \( m(\sigma_{jk}, \sigma_{ij}) \) by \( \sigma_{jk} \circ \sigma_{ij} \). Then we have a map \( (\sigma_{ik}, \sigma_{jk} \circ \sigma_{ij}) : U_{ij} \to Y^{[2]}_{ik} \) which sends \( m \in U_{ij} \to (\sigma_{ik}(m), (\sigma_{jk} \circ \sigma_{ij})(m)) \in Y^{[2]}_{ik} \). Let \( Q_{ijk} \) denote the pullback bundle \( (\sigma_{ik}, \sigma_{jk} \circ \sigma_{ij})^{-1}Q_{ik} \) on \( U_{ijk} \). We then have

\[
(\sigma_{kl} \circ (\sigma_{jk} \circ \sigma_{ij}), \sigma_{il})^{-1}Q = (\sigma_{kl} \circ (\sigma_{jk} \circ \sigma_{ij}), \sigma_{kl} \circ \sigma_{ik})^{-1}Q \otimes Q_{ikl} = (\sigma_{kl}, \sigma_{kl})^{-1}Q \otimes Q_{ijkl} = Q_{ijkl} \otimes Q_{ikl}.
\]

Similarly we have \((\sigma_{kl} \circ \sigma_{jk} \circ \sigma_{ij}, \sigma_{il})^{-1}Q = Q_{jkl} \otimes Q_{ijl}\). Also it is clear that

\[
((\sigma_{kl} \circ \sigma_{jk} \circ \sigma_{ij}, \sigma_{kl} \circ \sigma_{jk} \circ \sigma_{ij})^{-1}Q = A_{ijkl},
\]

where we denote the pullback bundle \( (s, s), s), s) = A_{ijkl} \) on \( U_{ijkl} \) by \( A_{ijkl} \). It follows as above that there is an isomorphism

\[
A_{ijkl} = Q_{jkl} \otimes Q_{ikl} \otimes Q_{ijl} \otimes Q_{ijkl}
\]

of \( \mathbb{C}^\times \) bundles on \( U_{ijkl} \). Choose a section \( \rho_{ijkl} \) of \( Q_{ijkl} \) over \( U_{ijkl} \) and define a map \( \epsilon_{ijkl} : U_{ijkl} \to \mathbb{C}^\times \) by \( \rho_{ijkl} \otimes \hat{\rho}_{ijkl} \otimes \rho_{ijkl} \otimes \hat{\rho}_{ijkl} = a(s, s), s, s) \epsilon_{ijkl} \). As above \( \epsilon_{ijkl} \) satisfies the Čech 3-cocycle condition \( \delta(\epsilon)_{ijkl} = 1 \) and hence is a representative of a class in \( H^3(M; \mathbb{C}^\times_M) = H^4(M; \mathbb{Z}) \). It is straightforward to check that these two methods of assigning a Čech 3-cocycle to a bundle 2-gerbe give rise to the same class in \( H^4(M; \mathbb{Z}) \).

It is also a straightforward exercise to define such notions as the pullback of a bundle 2-gerbe and the product of two bundle 2-gerbes and prove that the four classes behave as one would expect under these operations.

8. Bundle 2-gerbe Connections and 2-curvings

Just as there is a notion of a bundle gerbe connection on a bundle gerbe, there is also a notion of a bundle 2-gerbe connection on a bundle 2-gerbe \((Q, Y, X, M)\). This requires a choice of both a bundle gerbe connection \( \nabla \) on the bundle gerbe \((Q, Y, X^{[2]}\) and a curving \( f \) for \( \nabla \).

**Definition 8.1** (\([10]\)). Let \((Q, Y, X, M)\) be a bundle 2-gerbe. A **bundle 2-gerbe connection** on \( Q \) is a pair \((\nabla, f)\) where \( \nabla \) is a bundle gerbe connection on the bundle gerbe \( Q \) and \( f \) is a curving for \( \nabla \) such that the associated 3-curvature \( \omega \) on \( X^{[2]} \) satisfies \( \delta(\omega) = 0 \).

For a proof that bundle gerbe connections always exist, see \([10]\). Note that this is a non-trivial fact to prove, as one has to deal with two complexes \((\Omega^p(X^{[\ast]}), \delta_X)\) and \((\Omega^p(Y^{[\ast]}), \delta_Y)\) associated to the two local-section-admitting surjections \( \pi_X : X \to M \) and \( \pi_Y : Y \to X^{[2]} \). The idea of the proof is to first choose any bundle gerbe connection \( \nabla \) on \( Q \) and any curving \( f \) for \( \nabla \). Then one can show that there is a two form \( \mu \in \Omega^2(X^{[4]}) \) such that \( \delta(\omega) = d\mu \), where \( \omega \) is the 3-curvature associated to the bundle gerbe connection \( \nabla \) and curving \( f \). Similarly one can show that there is a one form \( \alpha \in \Omega^1(X^{[4]}) \) such that \( \delta(\alpha) = d\alpha \) and moreover \( \delta(\alpha) = 0 \). Hence, using the exactness of the complex \([\mathbb{R}]\), one can solve the equation \( \alpha = \delta(\beta) \) for some one form \( \beta \in \Omega^1(X^{[3]}) \). Continuing in this way one can show that it is possible to adjust the curving \( f \) by the pullback of a two form on \( X^{[2]} \) so that the 3-curvature \( \omega' \) associated to \( \nabla \) and the new curving \( f' \) satisfies \( \delta(\omega') = 0 \).
Given a bundle 2-gerbe connection \((\nabla, f_1)\) on a bundle 2-gerbe \((Q, Y, X, M)\) we can solve the equation \(\omega = \delta(f_2)\) for some three form \(f_2\). A choice of \(f_2\) is called a 2-curving for the bundle 2-gerbe connection \((\nabla, f_1)\). Given a choice of 2-curving \(f_2\), we have \(\delta(df_2) = 0\) and hence \(df_2 = \pi^*(\Theta)\) for some necessarily closed four form \(\Theta\) on \(M\). We call \(\Theta\) the four curvature of the bundle 2-gerbe connection \((\nabla, f_1)\) and 2-curving \(f_2\). We have the following Proposition.

**Proposition 8.2** ([12]). The four curvature \(\Theta\) is a closed, integral four form on \(M\) which represents the image in \(H^4(M; \mathbb{R})\) of the class in \(H^4(M; \mathbb{Z})\) represented by the Čech cocycle \(g_{ijkl}\).

As an example of this structure, consider the tautological bundle 2-gerbe on a 3-connected manifold \(M\) associated to a closed, integral four form \(\Theta\) on \(M\). Recall that the tautological bundle 2-gerbe \((Q, Y, \mathcal{P}M, M)\) was defined by constructing the tautological bundle gerbe on each fiber of \(\pi: \mathcal{P}M \to M\). Another way of viewing this construction is to first pull back the four form \(\Theta\) on \(M\) to \(\mathcal{P}M\). Since \(\mathcal{P}M\) is contractible we can solve \(\pi^*\Theta = df_2\) for some three form \(f_2\) on \(\mathcal{P}M\). Then it is easy to see that the three form \(\delta(f_2)\) on \(\mathcal{P}M[^2]\) is closed. Since \(M\) is 3-connected, \(\mathcal{P}M[^2]\) is 2-connected and we can construct the tautological bundle gerbe \((Q, Y, \mathcal{P}M[^2])\) on \(\mathcal{P}M[^2]\) from the three form \(\delta(f_2)\) using the methods of [6] and [10]. \((Q, Y, \mathcal{P}M, M)\) is then the tautological bundle 2-gerbe. In [13] it is shown how to construct a bundle gerbe connection on the tautological bundle gerbe over \(\mathcal{P}M[^2]\) and a curving such that the associated 3-curvature is \(\delta(f_2)\). This choice of bundle gerbe connection and curving therefore defines a bundle 2-gerbe connection on the tautological bundle 2-gerbe and \(f_2\) provides a 2-curving for this bundle 2-gerbe connection. \(\Theta\) is then the associated 4-curvature.

It can be shown [13] that given a bundle 2-gerbe \((Q, Y, X, M)\) with bundle 2-gerbe connection \((\nabla, f_1)\) and 2-curving \(f_2\) there is a class \(D(Q, \nabla, f_1, f_2)\) in the Deligne hypercohomology group \(H^3(M; \mathbb{Z}_M^3) \to \Omega^3_M \to \Omega^3_M \to \Omega^3_M\) associated to \(Q\). As a consequence of this one can show that the class in \(H^4(M)\) defined by the 4-curvature \(\Theta\) equals the image in \(H^4(M)\) of the class in \(H^3(M; \mathbb{Z}_M^3)\) defined by the Čech 3-cocycle \(g_{ijkl}\).

9. **Bundle 2-Gerbes and the First Pontraygin Class**

Suppose we are given a principal \(G\) bundle \(P \to M\), where \(G\) is a compact, simply connected, simple Lie group. Then it is well known that \(\pi_2(G) = 0\) and \(H^3(G; \mathbb{Z}) = \mathbb{Z}\). It is shown in [3] that there is a closed, bi-invariant three form \(\nu\) on \(G\) with integral periods which represents the canonical generator of \(H^3(G; \mathbb{Z}) = \mathbb{Z}\). If \(G = SU(N)\), then \(\nu\) is the three form \(\frac{1}{24\pi i} \text{tr}(dgg^{-1})^3\).

Recall from [4] that we can define a bundle gerbe \((Q, \mathcal{P}G, G)\) on \(G\) with three curvature equal to \(\nu\). The fibre of \(Q \to \mathcal{P}G[^2]\) at a point \((\alpha, \beta) \in \mathcal{P}G[^2]\) is the set of all equivalence classes \([\phi, z]\) where \(z \in \mathbb{C}^x\) and \(\phi: I^3 \to G\) is a homotopy with end points fixed between \(\alpha\) and \(\beta\). Two pairs \((\phi_1, z_1)\) and \((\phi_2, z_2)\) are declared equivalent if for all homotopies \(F: I^3 \to G\) with end points fixed between \(\phi_1\) and \(\phi_2\) we have \(z_2 = z_1 \exp(\int_{f_2} F^*\nu)\). The bundle gerbe product is defined by

\[
[\phi_1, z_1] \otimes [\phi_2, z_2] \mapsto [\phi_1\phi_2, z_1z_2],
\]
where $\phi_1\phi_2$ denotes the homotopy defined by

$$
(\phi_1\phi_2)(s, t) = \begin{cases} 
\phi_1(2s, t) & \text{for } 0 \leq s \leq 1/2 \\
\phi_2(2s - 1, t) & \text{for } 1/2 \leq s \leq 1 
\end{cases}
$$

It is shown in [10] that this is well defined, associative, etc.

**Proposition 9.1 ([10]).** The bundle gerbe $(Q, PG, G)$ is a simplicial bundle gerbe on the simplicial manifold $NG$.

**Proof.** We first need to define the bundle gerbe morphism $m = (\hat{m}, m, id)$ which maps

$$
m : d_0^{-1}Q \otimes d_2^{-1}Q \to d_1^{-1}Q.
$$

Define $m : d_0^{-1}PG \times_{G^2} d_2^{-1}PG \to d_1^{-1}PG$ covering the identity on $G^2 = G \times G$ by sending $(\alpha, \beta)$ to the piecewise smooth path $\alpha \circ \alpha(1)\beta$ given by

$$(\alpha \circ \alpha(1)\beta)(t) = \begin{cases} 
\alpha(2t), & 0 \leq t \leq 1/2 \\
\alpha(1)\beta(2t - 1), & 1/2 \leq t \leq 1.
\end{cases}
$$

Next, we need to define a $C^\infty$ equivariant map $\hat{m} : d_0^{-1}Q \otimes d_2^{-1}Q \to d_1^{-1}Q$ covering $m^{[2]} : (d_0^{-1}PG \times_{G^2} d_2^{-1}PG)^{[2]} \to d_1^{-1}PG^{[2]}$ and check that it commutes with the bundle gerbe product. So take pairs $(\phi, z)$ and $(\psi, w)$ where $z, w \in C^\infty$ and $\phi : I^2 \to G$ and $\psi : I^2 \to G$ are homotopies with endpoints fixed between paths $\alpha_1, \alpha_2$ and $\beta_1, \beta_2$ respectively. Then we put

$$
\hat{m}((\phi, z), (\psi, w)) = (\phi \circ \phi(0, 1)\psi, zw)
$$

where $\phi \circ \phi(0, 1)\psi : I^2 \to G$ is the homotopy with endpoints fixed between $\alpha_1 \circ \alpha_1(1)\beta_1$ and $\alpha_2 \circ \alpha_2(1)\beta_2$ given by

$$(\phi \circ \phi(0, 1)\psi)(s, t) = \begin{cases} 
\phi(s, 2t), & 0 \leq t \leq 1/2 \\
\phi(0, 1)\psi(s, 2t - 1), & 1/2 \leq t \leq 1.
\end{cases}
$$

We need to check firstly that this map is well defined — that is it respects the equivalence relation $\sim$ — and secondly that $\hat{m}$ commutes with the bundle gerbe products. So suppose $(\phi, z) \sim (\phi', z')$ and $(\psi, w) \sim (\psi', w')$, where $\phi$ and $\phi'$ are homotopies with endpoints fixed between paths $\alpha_1$ and $\alpha_2$ and where $\psi$ and $\psi'$ are homotopies with endpoints fixed between paths $\beta_1$ and $\beta_2$. We want to show that

$$
(\phi \circ \phi(0, 1)\psi) \sim (\phi' \circ \phi'(0, 1)\psi', zw').
$$

Therefore we want to show that for all homotopies $H : I^3 \to G$ with endpoints fixed between $\phi \circ \phi(0, 1)\psi$ and $\phi' \circ \phi'(0, 1)\psi'$ we have

$$
z w' = zw \exp(\int_{I^3} H^*\nu).
$$

Note that if $\Phi : I^3 \to G$ is a homotopy with endpoints fixed between $\phi$ and $\phi'$ and $\Psi : I^3 \to G$ is a homotopy with endpoints fixed between $\psi$ and $\psi'$, then by integrality of $\nu$ we have

$$
\exp(\int_{I^3} H^*\nu) = \exp(\int_{I^3} (\Phi \circ \Phi(0, 0, 1)\Psi)^*\nu).
$$
Therefore we are reduced to showing that
\[ z'w' = zw \exp(\int_{I^3} (\Phi \circ \Phi(0,0,1)\Psi)^*\nu). \]
We have
\[ \exp(\int_{I^3} (\Phi \circ \Phi(0,0,1)\Psi)^*\nu) = \exp(\int_{I^3} \Phi^*\nu) \exp(\int_{I^3} (\Phi(0,0,1)\Psi)^*\nu). \]
By the bi-invariantness of \( \nu \), we get \( (\Phi(0,0,1)\Psi)^*\nu = \Phi^*\nu \), hence
\[ \exp(\int_{I^3} (\Phi \circ \Phi(0,0,1)\Psi)^*\nu) = \exp(\int_{I^3} \Phi^*\nu) \exp(\int_{I^3} \Psi^*\nu), \]
which implies the result. Hence \( \tilde{m} \) is well defined. It is a straightforward matter to verify that \( \tilde{m} \) respects the bundle gerbe products.

It remains to show that there is a transformation of the bundle gerbe morphisms \( m_1 \) and \( m_2 \) over \( G \times G \times G \) which satisfies the compatibility criterion over \( G \times G \times \tilde{G} \times \tilde{G} \). This has already been done above for the tautological bundle 2-gerbe and the proof given there carries over to this case.

Suppose that we have a principal \( G \) bundle \( \pi : P \to M \). Form the canonical map \( \tau : P^{[2]} \to G \) defined by \( p_2 = p_1 \tau(p_1, p_2) \) for \( p_1 \) and \( p_2 \) in the same fiber. We can extend \( \tau \) to define maps \( \tau : P^{[q]} \to G^{q-1} \) for any \( q \geq 2 \) by
\[ \tau(p_1, p_2, \ldots, p_q) = (\tau(p_1, p_2), \ldots, \tau(p_{q-1}, p_q)). \]
Notice that \( \tau \) defines a simplicial map \( P^{[\bullet]} \to NG_{\bullet} \) between the simplicial manifolds \( P^{[\bullet]} \) and \( NG_{\bullet} \). Clearly the pullback bundle gerbe \( \tau^{-1}Q \) is a bundle gerbe on the simplicial manifold \( P^{[\bullet]} \). We have the following Proposition.

**Proposition 9.2.** The quadruple of manifolds \( (\tilde{Q}, \tilde{P}, P, M) \) is a bundle 2-gerbe.

Brylinski and McLaughlin \[3, 4\] defined a canonical 2-gerbe associated to a principal \( G \) bundle \( P \) on \( M \) where \( G \) was a compact, simple, simply connected Lie group. They showed that the class in \( H^4(M; \mathbb{Z}) \) associated to this 2-gerbe was equal to the first Pontryagin class \( p_1 \) of the principal bundle \( P \). If we calculate the four class associated to the bundle-2-gerbe \( Q \) of Proposition 9.2 then we recapture the result of Brylinski and McLaughlin.

**Proposition 9.3 \[3, 4\].** The class in \( H^4(M; \mathbb{Z}) \) associated to the bundle 2-gerbe \( \tilde{Q} \) is the transgression of \([\nu]\), that is the first Pontryagin class \( p_1 \) of \( P \).

**Proof.** We will calculate the Čech four class of the bundle 2-gerbe \( \tilde{Q} \) and show that it is exactly equal to the Čech cocycle obtained by Brylinski and McLaughlin in \[3\] and \[4\]. We then apply Theorem 6.2 of \[3\] to conclude that this Čech four class is \( p_1 \). We calculate the Čech cocycle \( g_{ijkl} \) as follows. First choose an open cover \( \{U_i\}_{i \in I} \) of \( M \) relative to which \( \pi : P \to M \) has local sections \( s_i \). Since \( P \to P^{[2]} \) is a fibration, we can choose sections \( s_{ij} \) of the pullback fibration \( \tilde{P} = \tilde{P}_{ij} \to U_{ij} \). This is equivalent to choosing maps \( \gamma_{ij} : U_{ij} \times I \to G \) such that \( \gamma_{ij}(m,0) = 1 \) and \( \gamma_{ij}(m,1) = g_{ij}(m) \). Next we choose sections \( \rho_{ijk} : U_{ijk} \to \sigma_{ik} \circ \sigma_{jk} \circ \sigma_{ij}^{-1}Q_{jk} \).

This amounts to choosing maps \( \gamma_{ijk} : U_{ijk} \times I \to G \) such that \( \gamma_{ijk}(m,0,t) = \gamma_{ijk}(m,t), \gamma_{ijk}(m,1,t) = (\gamma_{ij} \circ g_{ijk} \gamma_{jk})(m,t), \gamma_{ijk}(m,s,0) = 1 \) and \( \gamma_{ijk}(m,s,1) = g_{ij}(m)g_{jk}(m) \). Such maps \( \gamma_{ijk} \) exist because \( G \) is simply connected. Define a section \( t_{ijkl} \) of the bundle \( (\sigma_{il} \circ (\sigma_{jk} \circ \sigma_{ij}^{-1}Q_{kl}) \] by \( t_{ijkl} = (c(\sigma_{kl}) \circ \rho_{ijkl}) \rho_{ijkl} \).

In a similar manner construct a section \( s_{ijkl} \) of the bundle
over each triple intersection \(U_i\), \(U_j\), \(U_k\) there exist bundle gerbe morphisms \(g_{ijk}\). We can get an explicit formula for \(g_{ijk}\) as follows: we choose a homotopy with endpoints fixed \(H_{ijk}: U_{ijk} \times I \times I \times I \to G\) such that \(H_{ijk}(m, 0, s, t) = (\varphi_{ij} \circ g_{ijk})\gamma_{ijk}(s, t)\), \(H_{ijk}(m, 1, s, t) = a(\gamma_{ij}, \gamma_{jk}, \gamma_{ij})\gamma_{ijk} \circ g_{ik}\gamma_{ik}(s, t)\), \(H_{ijk}(m, r, 0, t) = \gamma_{ij}(m, t)\), \(H_{ijk}(m, r, 1, t) = (\gamma_{ij} \circ g_{jk}\gamma_{jk}) \circ g_{ik}\gamma_{ik}, \) \(H_{ijk}(m, r, s, 0) = 1\) and \(H_{ijk}(m, r, s, 1) = g_{il}\) and we set \(g_{ijk} = \exp(\int^1_0 H_{ijk}^* \nu)\). This is just the integral of \(\nu\) over the tetrahedron shown in the following diagram,

\[
\begin{array}{ccc}
1 & \rightarrow & g_{il} \\
\downarrow & & \downarrow \\
g_{jk} & \rightarrow & g_{ik}
\end{array}
\]

as described in [3] and [6]. Thus our cocycle agrees with the cocycle defined by Brylinski and McLaughlin. 

10. Higher Gluing Laws

As a prelude to the discussion of trivial bundle 2-gerbes in the next section, we will discuss some features of 2-descent (see [7]). We have already seen that if we are given a family of \(\mathbb{C}^\times\) bundles \(P_i\) defined on an open cover \(\{U_i\}_{i \in I}\) of a manifold \(M\) such that there exist isomorphisms \(\phi_{ij}: P_i \rightarrow P_j\) satisfying the cocycle condition \(\phi_{jk} \circ \phi_{ij} = \phi_{ik}\) then we can construct a \(\mathbb{C}^\times\) bundle \(P\) defined on \(M\) which is locally isomorphic to \(P_i\) over each open set \(U_i\). If we replace \(\mathbb{C}^\times\) bundles by bundle gerbes then new complications arise. Rather than demanding that the equation \(\phi_{jk} \circ \phi_{ij} = \phi_{ik}\) is satisfied on the nose, we can settle for the weaker condition that there is a transformation of bundle gerbe morphisms \(\psi_{ijk}: \phi_{jk} \circ \phi_{ij} \Rightarrow \phi_{ik}\) which satisfies a certain cocycle condition — the non-abelian 2-cocycle condition. We shall see that it is still possible to ‘glue’ the various bundle gerbes \(P_i\) together to form a bundle gerbe \(P\) on \(M\).

Suppose we are given an open cover \(\{U_i\}_{i \in I}\) of a manifold \(M\) such that there exist bundle gerbes \(Q_i\) over \(U_i\). Suppose also that over each intersection \(U_{ij}\) there exist bundle gerbe morphisms \(\phi_{ij}: Q_i|_{U_{ij}} \rightarrow Q_j|_{U_{ij}}\). Suppose as well, that over each triple intersection \(U_{ijk}\) there exist transformations of bundle gerbe morphisms \(\psi_{ijk}: \phi_{jk}|_{U_{ijk}} \circ \phi_{ij}|_{U_{ij}} \Rightarrow \phi_{ik}|_{U_{ijk}}\). Finally, suppose that the diagram of transformations of bundle gerbe morphisms in Figure 3 commutes. If we let \(L_{ijk}\) denote the \(\mathbb{C}^\times\) bundle \(D_{\phi_{jk} \circ \phi_{ij}, \phi_{ik}}\) on \(U_{ijk}\) then we have an isomorphism \(L_{ijk} \otimes L_{ikl} = L_{jkl} \otimes L_{jkl}\) of \(\mathbb{C}^\times\) bundles on \(U_{ijk}\) or, put another way, a canonical trivialisation \(L_{ijk} \otimes L_{ikl} \otimes L_{jkl} = 1\). The condition that the diagram of bundle gerbe transformations in Figure 3 commutes translates into the requirement that the induced section \(\psi_{jkl} \otimes \psi_{ikl} \otimes \psi_{jkl} \otimes \psi_{jkl}^*\) matches this canonical trivialisation.

We have the following proposition.

**Proposition 10.1.** Suppose we are given an open cover \(\{U_i\}\) of \(M\) and a triple \((Q_i, \phi_{ij}, \psi_{ijk})\) as described above. Then there is a bundle gerbe \(Q\) on \(M\) and bundle gerbe morphisms \(\chi_i: Q|_{U_i} \rightarrow Q_i\) over \(U_i\) together with transformations \(\xi_j: \phi_{ij} \circ \chi_i \Rightarrow \chi_j\) which are compatible with the transformations \(\psi_{ijk}\).
Proof. Suppose the bundle gerbes $Q_i$ are given by triples $(Q_i, X_i, U_i)$. We first construct the bundle gerbe $(Q, X, M)$. Let $X = \prod_{i,j \in I} X_i$. Then the fibre product of $X$ with itself over $M$ is $X^{[2]} = \prod_{i,j \in I} X_i \times_M X_j$. Suppose the bundle gerbe morphisms $\phi_{ij}$ are given by $\phi_{ij} = (\phi_{ij}, \phi_{ij}).$ Define a map $f_{ij} : X_i \times_M X_j \to X^{[2]}_j$ by sending $(x_i, x_j) \in X_i \times_M X_j$ to $(\phi_{ij}(x_i), x_j)$. Let $Q_{ij} = f_{ij}^{-1}Q_i$. Define a $\mathbb{C}^\times$ bundle $Q$ on $X^{[2]}$ by setting $Q = \prod_{i,j \in I} Q_{ij}$ with projection map $Q \to X^{[2]}$ induced by the various projections $Q_{ij} \to X_i \times_M X_j$. We want to show that the triple $(Q, X, M)$ is a bundle gerbe. We first define the product in $Q$. This is a $\mathbb{C}^\times$ bundle isomorphism $\pi^{-1}_1Q \otimes \pi^{-1}_3Q \to \pi^{-1}_2Q$ covering the identity on $X^{[3]}$. Since $X^{[3]} = X_1 \times_M X_j \times_M X_k$ this amounts to finding a $\mathbb{C}^\times$ bundle map $Q_{ijk} \otimes Q_{ij} \to Q_{i}$ satisfying an associativity condition over $X_i \times_M X_j \times_M X_k \times_M X_l$.

Let $u_{jk} \in (Q_{jk}(x_j, x_k), u_{ij} \in (Q_{ij}(x_i, x_j))$ for $(x_i, x_j, x_k) \in X_i \times_M X_j \times_M X_k$. Then $u_{jk} \in (Q_{j})_{(\phi_{ij}(x_j), x_k)}$ and $u_{ij} \in (Q_{k})_{(\phi_{jk}(x_k), x_k)}$. Apply $\phi_{ij}$ to $u_{ij}$. Then $\phi_{jk}(u_{ij}) \in (Q_{k})_{(\phi_{jk}(x_k), x_k)}$. Using the bundle gerbe product in $Q_k$ we have that

$$u_{jk} \phi_{jk}(u_{ij}) \in (Q_{k})_{(\phi_{jk}(x_k), x_k)}.$$

Let $\hat{\psi}_{ijk}$ denote the section of the $\mathbb{C}^\times$ bundle $(\phi_{jk} \circ \phi_{ij}, \phi_{ik})^{-1}Q_k$ on $X_i|_{U_{ijk}}$ which descends to $\psi_{ijk}$. Using the bundle gerbe product in $Q_k$ again, we have that

$$u_{jk} \hat{\psi}_{jk}(u_{ij}) \hat{\psi}_{ijk}^{-1}(x_i) \in (Q_{k})_{(\phi_{ik}(x_k), x_k)}.$$

We define a product in $Q$ by sending $u_{jk} \otimes u_{ij}$ to $u_{jk} \cdot u_{ij} = u_{jk} \hat{\psi}_{jk}(u_{ij}) \hat{\psi}_{ijk}^{-1}(x_i)$. We have to check that this product is associative. This follows easily from the following equation satisfied by $\psi_{ijk}$:

$$\hat{\psi}_{ijk}(x_i) \hat{\psi}_{kl}(\hat{\psi}_{ijkl}(x_i)) = \hat{\psi}_{ijkl}(x_i) \hat{\psi}_{ijkl}(\phi_{ij}(x_i)).$$

This equation is a consequence of the coherency condition satisfied by $\psi_{ijk}$. Therefore $(Q, X, M)$ is a bundle gerbe. Now we need to define the bundle gerbe morphism $Q|_{U_i} \to Q_i$. First of all we define a map $X|_{U_i} \to X_i$ covering the identity on $U_i$. If $x_j \in X_j$ and $\pi_{X_i}(x_j) \in U_i$, then $\phi_{ji}(x_j) \in X_i$. Since $X|_{U_i} = \prod_{j \in J} X_j|_{U_i}$ this defines a map $X|_{U_i} \to X_i$. Now suppose $(x_j, x_j') \in X^{[2]}_{U_i}$ and $u_{jj'} \in Q_{(x_j, x_j')}$. So $u_{jj'} \in (Q_{j'})_{(\phi_{j'}(x_j), x_j')}$ - Hence applying $\phi_{jj'}$ to $u_{ij}$ means that $\phi_{jj'}(u_{jj'}) \in (Q_{j'})_{(\phi_{j'}(x_j), x_j')}$. Therefore

$$\phi_{jj'}(u_{jj'}) \hat{\psi}_{jj'}^{-1}(x_j) \in (Q_{j'})_{(\phi_{j'}(x_j), x_j')}.$$
This defines a $C^\infty$ bundle map $Q|_{U_i} \to Q_i$. It is not hard to check that this map commutes with the bundle gerbe products on $Q$ and $Q_i$ and hence defines a bundle gerbe morphism $\chi_i: Q|_{U_i} \to Q$. Similarly, one can define a transformation of bundle gerbe morphisms $\xi_i: \phi_{ij} \circ \chi_i \Rightarrow \chi_j$ which is compatible with $\psi_{ijk}$.

The triple $(Q_i, \phi_{ij}, \psi_{ijk})$ is called 2-descent data relative to the open covering $\mathcal{U} = \{U_i\}_{i \in I}$. One can think of these 2-descent data as being objects of a 2-category $\mathbf{2-Desc}(\mathcal{U})$. Let $(Q_i, \phi_{ij}, \psi_{ijk})$ and $(P_i, \hat{\phi}_{ij}, \hat{\psi}_{ijk})$ be two sets of 2-descent data. A 1-arrow from $(Q_i, \phi_{ij}, \psi_{ijk})$ to $(P_i, \hat{\phi}_{ij}, \hat{\psi}_{ijk})$ is a pair $(f_i, \tau_{ij})$ where $f_i: Q_i \to P_i$ is a bundle gerbe morphism and $\tau_{ij}$ is a transformation of bundle gerbe morphisms as pictured in the following diagram

\[
\begin{array}{ccc}
Q_i & \xrightarrow{f_i} & P_i \\
\phi_{ij} & \searrow \tau_{ij} & \hat{\phi}_{ij} \\
| & & |
\phi_{ij} & \downarrow f_j & \downarrow \hat{\phi}_{ij} \\
Q_j & \xrightarrow{f_j} & P_j
\end{array}
\]

which is compatible with $\psi_{ijk}$ and $\hat{\psi}_{ijk}$. Given two 1-arrows $(f_i, \tau_{ij})$ and $(g_i, \rho_{ij})$ a 2-arrow $(f_i, \tau_{ij}) \Rightarrow (g_i, \rho_{ij})$ is a transformation of bundle gerbe morphisms $\lambda_i: f_i \Rightarrow g_i$ which is compatible with $\tau_{ij}$ and $\rho_{ij}$. Horizontal and vertical composition in $\mathbf{2-Desc}(\mathcal{U})$ is defined in the obvious manner.

The gluing procedure of Proposition [10.4] above allows us to define a 2-functor $\mathbf{2-Desc}(\mathcal{U}) \to \mathbf{BGrb}_M$. The action of this functor on objects of $\mathbf{2-Desc}(\mathcal{U})$ is clear: a triple $(Q_i, \phi_{ij}, \psi_{ijk})$ of 2-descent data is mapped to the bundle gerbe $Q$ of Proposition [10.1]. With a little work one can show that a 1-arrow $(f_i, \tau_{ij})$ from $(Q_i, \phi_{ij}, \psi_{ijk})$ to $(P_i, \hat{\phi}_{ij}, \hat{\psi}_{ijk})$ induces a bundle gerbe morphism $f: Q \to P$ and that a 2-arrow $\lambda_i: (f_i, \tau_{ij}) \Rightarrow (g_i, \rho_{ij})$ between two 1-arrows $(f_i, \tau_{ij})$ and $(g_i, \rho_{ij})$ induces a transformation of bundle gerbe morphisms $\lambda: f \Rightarrow g$. Both of these constructions are functorial.

Note that bundle gerbe morphisms do not glue together in the fashion that one would like. One would like to say that given bundle gerbes $P$ and $Q$ such that relative to some open cover $\{U_i\}_{i \in I}$ of $M$ there exist local bundle gerbe morphisms $f_i: P|_{U_i} \to Q|_{U_i}$ together with transformations of bundle gerbe morphisms $\tau_{ij}: f_i \Rightarrow f_j$ which satisfy the cocycle condition $\tau_{jk} \tau_{ij} = \tau_{ik}$, there exists a bundle gerbe morphism $f: P \to Q$ locally isomorphic to $f_i$. Unfortunately this is not true; it is however true for gerbes.

11. Trivial Bundle 2-Gerbes

In [16] it was shown that a bundle gerbe $P$ had vanishing Dixmier-Douady class precisely when the bundle gerbe was trivial — i.e. $P$ was of the form $\delta(T)$ for some $C^\infty$ bundle $T$. We would like to know under what conditions the four class of a bundle 2-gerbe is zero. We will define a certain class of bundle 2-gerbes, trivial bundle 2-gerbes and show that the four class associated to a bundle 2-gerbe belonging to this class vanishes. We will then prove that the converse is true.

**Definition 11.1.** Let $(Q, Y, X, M)$ be a bundle 2-gerbe. We say that $Q$ is *trivial* if there exists a bundle gerbe $(L, Z, X)$ on $X$ together with a bundle gerbe morphism $\eta: \pi_1^{-1}L \otimes Q \to \pi_2^{-1}L$ over $X^{[2]}$ and a transformation of bundle gerbe morphisms...
Let us agree to call \( \eta_1 = \pi_2^{-1} \eta \circ (1 \otimes m) \) and \( \eta_2 = \pi_3^{-1} \eta \circ (\pi_1^{-1} \eta \otimes 1) \). Then \( \theta \) is a section trivialising the \( C^\times \) bundle \( B = D_{\eta_1, \eta_2} \) on \( X^{[2]} \). Moreover there is a canonical isomorphism \( \delta(B) = \pi_1^{-1} B \otimes \pi_2^{-1} B^* \otimes \pi_3^{-1} B \otimes \pi_4^{-1} B^* = A \) of \( C^\times \) bundles over \( X^{[2]} \). As a final condition we demand that the induced section \( \delta(\theta) = \pi_1^{-1} \theta \otimes \pi_2^{-1} \theta^* \otimes \pi_3^{-1} \theta \otimes \pi_4^{-1} \theta^* \) of \( \delta(B) \) is mapped to \( a \) under this isomorphism.

Suppose we are now given a bundle 2-gerbe \( (Q,Y,X,M) \) with vanishing four class. We will make the additional assumption that \( \pi_Y: Y \to X^{[2]} \) is a fibration. Let \( \{U_i\}_{i \in I} \) be an open covering of \( M \) all of whose finite intersections \( U_{i_0} \cap \cdots \cap U_{i_p} \) are empty or contractible and such that there exist local sections \( s_i: U_i \to X \) of the surjection \( \pi: X \to M \) over \( U_i \). Define maps \( \hat{s}_i: X_i \to X^{[2]} \) where \( X_i = \pi^{-1}(U_i) \) by \( \hat{s}_i(x) = (x, s_i(\pi(x))) \). Let \( (L_i, Z_i, X_i) \) denote the pullback of the bundle gerbe \( (Q,Y,X^{[2]}) \) to \( X_i \) via the map \( \hat{s}_i \). Then \( Z_i \to X_i \) is a fibering with fibre \( (Z_i)_x \) at \( x \in X_i \) equal to \( Y_{(x,s_i(\pi(x)))} \). One can also define maps \( (s_i, s_j): U_{ij} \to X^{[2]} \) in the usual fashion by sending \( m \in U_{ij} \) to \( (s_i(m), s_j(m)) \) \( \in X^{[2]} \). Let \( Y_{ij} \) denote the pullback of the fibration \( Y \to X^{[2]} \) via this map. Choose sections \( \sigma_{ij} \) of the pullback fibering \( Y_{ij} \to U_{ij} \). Now we can define maps \( \phi_{ij}: Z_i \to Z_j \) by sending \( y_i \in Z_i \) to \( m(\sigma_{ij}, y_i) \in Z_j \). The \( \phi_{ij} \) extend to define bundle gerbe morphisms \( \phi_{ij} = (\phi_{ij}, \phi_{ij}) : L_i \to L_j \) with \( \phi_{ij}(u_i) = \hat{m}(e(\sigma_{ij}) \otimes u_i) \) where \( e \) denotes the identity section of the bundle gerbe \( (Q,Y,X^{[2]}) \).

We now wish to define transformations \( \psi_{ijk}: \phi_{jk} \circ \phi_{ij} \Rightarrow \phi_{ik} \) satisfying the non-abelian 2-cocycle condition over \( X_{ijkl} \). To do this, first note that \( \hat{a}(\sigma_{jk}, \sigma_{ij}, y_i) \in Q_{m(m(\sigma_{jk}, \sigma_{ij}), y_i), m(\sigma_{jk}, m(\sigma_{ij}, y_i))} \) where \( \hat{a} \) denotes the lift of the associator section \( a \) to \( Y \circ Y \circ Y \). Also, as in Section \( 6 \) let \( \rho_{ijk} \) denote a section of the pullback bundle \( (m(\sigma_{jk}, \sigma_{ij}), \sigma_{ik})^{-1} Q \) over \( U_{ijk} \). Then \( \hat{m}(\rho_{ijk} \otimes e(y_i)) \in Q_{m(\rho_{ijk}, e(y_i), m(\sigma_{ijk}, y_i))}. \)

Therefore
\[
\hat{\psi}_{ijk}(y_i) = \hat{m}(\rho_{ijk} \otimes e(y_i))\hat{a}(\sigma_{jk}, \sigma_{ij}, y_i)^{-1} \in Q_{\rho_{ijk}(\sigma_{ijk}(y_i), \phi_{ik}(y_i))}.
\]

Since the Čech 3-cocycle \( g_{ijkl} \) representing the four class of \( Q \) is trivial, one can show that it is possible to choose \( \rho_{ijk} \) so that the sections \( \psi_{ijk} \) defined above satisfy the non-abelian 2-cocycle condition. Therefore, using Proposition \( 10.1 \), one can form a bundle gerbe \( (L, Z, X) \) on \( X \) which is locally isomorphic to each \( (L_i, Z_i, X_i) \).

However, more is true. The bundle gerbes \( (L_i, Z_i, X_i) \) provide local trivialisations of the bundle 2-gerbe \( Q \). To see this, note that the bundle gerbe morphism \( m: \pi_1^{-1} Q \otimes \pi_3^{-1} Q \to \pi_2^{-1} Q \) provides a bundle gerbe morphism \( \eta_i: \pi_1^{-1} L_i \otimes Q \to \pi_2^{-1} L_i \otimes Q \).
π_{−1}^{−1}L_i by sending a point \((y_i, y)\) of \(\pi_{−1}^{-1}Z_i \times X\) to \(m(y_i, y) \in \pi_2^{-1}Z_i\) and a point \(u_i \otimes u\) of \(\pi_1^{-1}L_i \otimes Q\) to \(\tilde{m}(u_i \otimes u)\). One can also define transformations \(\theta_i\) as in Definition 11.1 above. It is possible to show \(\phi\), although it is very tedious, that \(\eta_i\) and \(\theta_i\) are compatible with the 2-descent data \((L_i, \phi_{ij, \psi_{ijk}})\) relative to the open covering \(\{X_i\}_{i \in I}\) of \(X\). It follows that \(\eta_i\) and \(\theta_i\) glue together to form a bundle gerbe morphism \(\eta_i: \pi_1^{-1}L \otimes Q \to \pi_2^{-1}L\) and a transformation \(\theta_i\) as in Definition 11.1. Thus the bundle 2-gerbe \(Q\) is trivial.

One can show \(\psi\) that it is possible to remove the restriction that \(\pi_Y: Y \to X\) be a fibration.

**Proposition 11.2** \([19]\). The four class of a bundle 2-gerbe \((Q, Y, X, M)\) vanishes if and only if the bundle 2-gerbe \(Q\) is trivial.

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