Non-Abelian Sine-Gordon Solitons:
Correspondence between $SU(N)$ Skyrmions and $\mathbb{C}P^{N-1}$ Lumps

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Abstract

Topologically stable non-Abelian sine-Gordon solitons have been found recently in the $U(N)$ chiral Lagrangian and a $U(N)$ gauge theory with two $N$ by $N$ complex scalar fields coupled to each other. We construct the effective theory on a non-Abelian sine-Gordon soliton that is a nonlinear sigma model with the target space $\mathbb{R} \times \mathbb{C}P^{N-1}$. We then show that $\mathbb{C}P^{N-1}$ lumps on it represent $SU(N)$ Skyrmions in the bulk point of view, providing a physical realization of the rational map Ansatz for Skyrmions of the translational (Donaldson) type. We find therefore that Skyrmions can exist stably without the Skyrme term.
I. INTRODUCTION

When a soliton equation is integrable, one can construct exact analytic solutions in principle. Among topological solitons and instantons, Yang-Mills instantons [1] and Bogomo’lnyi-Prasad-Sommerfield (BPS) monopoles [2, 3] are such examples studied in detail both in physics and mathematics, for which exact solutions are accessible from the Atiyah-Drinfeld-Hitchin-Manin [4] and Nahm [5] constructions, respectively. For BPS monopoles, Donaldson proposed a rational map construction [6], in which three dimensional space is decomposed into one particular direction and its orthogonal plane parametrized by a complex coordinate. Recently, a physical interpretation of the Donaldson’s rational map was provided in Ref. [7] by putting monopoles into the Higgs phase, in which vortices that confine monopoles extend to the above mentioned one particular direction. A spherical rational map construction was also proposed in Ref. [8] in which three dimensional space is decomposed into a sphere and a radial direction.

The Skyrme model that describes baryons as solitons known as Skyrmions [9] is not integrable, unlike its BPS version proposed recently whose Lagrangian is consisting of only a six derivative term and a potential term [10]. Since exact solutions are impossible to obtain for the original Skyrme model, approximate analytic solutions are the most useful if they exist, unless one obtains solutions numerically. One of such approximations is the Atiyah-Manton Ansatz [11, 12] in which an approximate Skyrme field is obtained from a holonomy of a Yang-Mills instanton configuration integrated along one particular direction. A physical realization of the Atiyah-Manton ansatz has been obtained recently [13] in which a Skyrmion is realized as a Yang-Mills instanton absorbed into a domain wall that is placed perpendicular to the above mentioned one particular direction. The other more useful approximation is the rational map Ansatz proposed in Ref. [14, 15], in which three dimensional space is decomposed into a sphere and a radial direction, as for the Jarvis’s spherical rational map Ansatz for BPS monopoles. This Ansatz was also generalized to $SU(N)$ Skyrmions [16]. As for a recent application to realistic situation, see Ref. [17]. While this Ansatz gives only an initial configuration for numerical relaxation, a physical realization of this Ansatz can be also given as a spherical domain wall [18, 19] which can be stabilized in a Skyrme model with a six derivative term. On the other hand, a Donaldson-type rational map Ansatz for Skyrmions has been found [20, 23] together with its physical realization in which $\mathbb{C}P^1$ lumps
inside a domain wall are Skyrmions in the bulk in the Skyrme model with the modified mass term admitting two discrete vacua [24]. However, a generalization to $SU(N)$ Skyrmions has a difficulty that such a potential term admitting a domain wall is not known.

The purpose of this paper is to give a physical realization of an $SU(N)$ rational map of the Donaldson type for Skyrmions. A key ingredient is a non-Abelian sine-Gordon soliton proposed recently [25], in which it has been found that a $U(N)$ chiral Lagrangian with usual pion mass term, instead of $SU(N)$, admits a topologically stable non-Abelian sine-Gordon soliton. The point is that the $U(N)$ group has the structure of $[SU(N) \times U(1)]/\mathbb{Z}_N$, and consequently there exists a topologically nontrivial closed path winding around the $U(1)$ group $1/N$ times together with an $SU(N)$ path from the unit element to an element in the center $\mathbb{Z}_N$. The diagonal $SU(N)$ symmetry in the vacuum is spontaneously broken into an $SU(N-1) \times U(1)$ subgroup in the presence of the non-Abelian sine-Gordon soliton, giving rise to localized $\mathbb{C}P^{N-1} \simeq SU(N)/[SU(N-1) \times U(1)]$ Nambu-Goldstone modes. Therefore, term “non-Abelian” is the same with that of non-Abelian vortices [26–31] carrying non-Abelian $\mathbb{C}P^{N-1}$ moduli, see Refs. [32–35] for a review. While a non-Abelian vortex can terminate on a non-Abelian monopole because of the matching of the moduli $\mathbb{C}P^{N-1}$ [36, 37], non-Abelian sine-Gordon soliton can terminate on a non-Abelian global vortex [35, 38–41]. Non-Abelian sine-Gordon solitons exist stably in the color-flavor locking (CFL) phase of dense quark matter [42] or confining phase of QCD as far as the axial anomaly term can be neglected at high density or high temperature [43].

In this paper, we construct the effective theory on the non-Abelian sine-Gordon soliton by using the moduli approximation [44, 45], that is a nonlinear sigma model with the target space $\mathbb{R} \times \mathbb{C}P^{N-1}$. We then show that $\mathbb{C}P^{N-1}$ lumps on it represent $SU(N)$ Skyrmions in the bulk point of view. This setting offers a physical realization of the rational map Ansatz for $SU(N)$ Skyrmions of the Donaldson type. One of interesting features is that Skyrmions can exist stably without the Skyrme term. This fact is consistent with the the Derrick’s scaling argument [46] that implies a three dimensional soliton in scalar field theories to shrink in the absence of the Skyrme term, because the sine-Gordon soliton has divergent energy proportional to the world-volume directions. This situation is similar to lumps inside a vortex representing a Yang-Mills instanton in the Higgs phase [29, 47].

This paper is organized as follows. In Sec. [II] we give the $U(N)$ chiral Lagrangian and construct a non-Abelian sine-Gordon soliton. In Sec. [III] we construct the effective field
theory on a non-Abelian sine-Gordon soliton which is the $\mathbb{CP}^{N-1}$ model. In Sec. IV, we show that $\mathbb{CP}^{N-1}$ lumps on the non-Abelian sine-Gordon soliton are nothing but $SU(N)$ Skyrmions in the bulk point of view. Sec. V is devoted to summary and discussion. In Appendix, we summarize Abelian sine-Gordon soliton.

II. $U(N)$ CHIRAL LAGRANGIAN AND NON-ABELIAN SINE-GORDON SOLITON

In this section, we give the Lagrangian for a $U(N)$ principal chiral model (chiral Lagrangian) and its sine-Gordon solution. A $U(N)$-valued field $U(x)$ takes a value in the $U(N)$ group having a non-trivial first homotopy group:

$$U(x) \in U(N) \simeq \frac{U(1) \times SU(N)}{\mathbb{Z}_N}, \quad \pi_1[U(N)] = \mathbb{Z}. \quad (1)$$

The Lagrangian for a $U(N)$ chiral Lagrangian with the usual pion mass term is given by

$$\mathcal{L}/f_\pi^2 = \frac{1}{2} \text{tr} \left[ \partial_\mu U^\dagger \partial^\mu U - \frac{m^2}{2} \text{tr} \left( 2 \mathbf{1}_N - U - U^\dagger \right) \right]$$

$$= \frac{1}{2} \text{tr} (i U^\dagger \partial_\mu U)^2 - \frac{m^2}{2} \text{tr} (2 \mathbf{1}_N - U - U^\dagger), \quad (2)$$

with $f_\pi$ being a constant of the mass dimension one, and $\mu = 0, 1, \cdots, d - 1$. In the absence of the pion mass, $m = 0$, this Lagrangian is invariant under the chiral $SU(N)_L \times SU(N)_R$ symmetry

$$U(x) \rightarrow V_L U(x) V_R^\dagger, \quad V_L, R \in SU(N)_{L,R} \quad (3)$$

that is spontaneously broken to the vector-like symmetry

$$U(x) \rightarrow V U(x) V^\dagger, \quad V \in SU(N)_{L+R=V}. \quad (4)$$

In the presence of the pion mass, $m \neq 0$, the chiral symmetry is explicitly broken to the vector-like symmetry in Eq. (4) in the unique vacuum $U = \mathbf{1}_N$.

The energy density for static configuration and its Bogomol’nyi completion are given as

$$\mathcal{E}/f_\pi^2 = \frac{1}{2} \text{tr} (U^\dagger \partial_x U)^2 - \frac{m^2}{2} \text{tr} (2 \mathbf{1}_N - U - U^\dagger)$$

$$= \frac{1}{2} \text{tr} \left[ -\frac{i}{2} (U^\dagger \partial_x U - \partial_x U^\dagger U) \mp m \sqrt{2 \mathbf{1}_N - U - U^\dagger} \right]^2$$

$$\pm \frac{m}{2} \text{tr} \left[ -\frac{i}{2} (U^\dagger \partial_x U - \partial_x U^\dagger U) \sqrt{2 \mathbf{1}_N - U - U^\dagger} \right]$$

$$\geq |t_{U(N)}|, \quad (5)$$

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with the topological charge density, defined by
\[
t_{U(N)} \equiv -\frac{m}{2} \text{tr} \left[ i(U^\dagger \partial_x U - \partial_x U^\dagger U)\sqrt{21_N - U - U^\dagger} \right]. \tag{6}
\]
The BPS equation is obtained as
\[
-\frac{i}{2}(U^\dagger \partial_x U - \partial_x U^\dagger U) \mp m\sqrt{21_N - U - U^\dagger} = 0_N. \tag{7}
\]
This equation is invariant under the SU($N$)$_V$ symmetry in Eq. (4).

A non-Abelian sine-Gordon soliton solution is of the following form:
\[
U(x) = V \text{diag}(u(x), 1, \cdots, 1)V^\dagger, \quad V \in SU(N)_V, \tag{8}
\]
with \(u(x) \in U(1), \ |u|^2 = 1\) satisfying the Abelian sine-Gordon equation
\[
-\frac{i}{2}(u^* \partial_x u - (\partial_x u^*)u) \mp m\sqrt{2 - u - u^*} = 0 \tag{9}
\]
that allows for instance a single sine-Gordon soliton solution \[48\]
\[
u(x) = \exp \left(4i \arctan \exp[m(x - X)]\right) \tag{10}
\]
with the boundary condition \(u \to 1\) for \(x \to \pm \infty\) (see Appendix). Since there exists a redundancy in the action of \(V\) in Eq. \[8\], \(V\) in fact takes a value in the coset space
\[
V \in \frac{SU(N)_V}{SU(N - 1)_V \times U(1)_V} \simeq \mathbb{C}P^{N-1}. \tag{11}
\]
The single-soliton solution has the moduli
\[
\mathcal{M} = \mathbb{R} \times \mathbb{C}P^{N-1}, \tag{12}
\]
where the first and second factors are parameterized by \(X\) and \(V\), respectively. In terms of the group elements, the general solution can be rewritten as
\[
U(x) = \exp \left(\frac{i\theta(x)}{N}\right) \exp \left(i\theta(x)VT_0V^\dagger\right) = \exp \left(\frac{i\theta(x)}{N}\right) \exp i\frac{\theta(x)}{N}T = \exp \left(i\theta(x)\phi\phi^\dagger\right). \tag{13}
\]
with \(T_0 \equiv \frac{1}{N}\text{diag.}(N - 1, -1, \cdots, -1)\), where \(T \equiv VT_0V^\dagger\) can be any SU($N$) generator normalized as \(e^{i2\pi T} = \omega^{-1}1_N\ (\omega = \exp(2\pi/N))\). In the last line, we have introduced the
orientational vector $\phi \in \mathbb{C}^N$ that represents homogeneous coordinates of $\mathbb{C}P^{N-1}$ and satisfy

$$\phi^\dagger \phi = 1,$$

$$T = VT_0V^\dagger = \phi \phi^\dagger - \frac{1}{N} 1_N. \quad (15)$$

This form of $T$ is known as the projector in the rational map Ansatz for Skyrmions [14,16], already implying a possibility of physical realization of the rational map.

### III. THE EFFECTIVE THEORY ON NON-ABELIAN SINE-GORDON SOLITON

In this section, we construct the low-energy effective theory, which is the $\mathbb{C}P^{N-1}$ model, by using the moduli approximation [44]. Let us place a single sine-Gordon soliton perpendicular to the $x^3$-coordinate, that we denote $x$ for simplicity. In the following, we will promote the moduli parameter $X$ and $\phi$ to be the fields on the $2 + 1$ dimensional soliton’s world-volume as

$$X \rightarrow X(x^\alpha), \quad \phi \rightarrow \phi(x^\alpha), \quad (\alpha = 0, 1, 2). \quad (16)$$

We will derive the effective theory including derivatives with respect to $x^\alpha$ up to the leading (second) order, by taking into account only the zero modes $X$ and $\phi$ and discarding massive modes. Therefore, what we will do in the rest of this section is integrating the kinetic term of the chiral Lagrangian over $x$

$$\mathcal{L}_{\text{eff}} = -f_x^2 \frac{1}{2} \int_{-\infty}^{\infty} dx \text{ tr } \left[ \left( U^\dagger \partial_\alpha U \right)^2 \right], \quad (17)$$

where $U$ is a non-Abelian sine-Gordon soliton solution in which the moduli parameters $X$ and $\phi$ are promoted to the fields on the world-volume. The effective Lagrangian correctly describe low energy physics with momenta sufficiently lower than the mass scale: $|p_\alpha| \ll m$.

#### A. The $U(2)$ case

As an exercise, we first consider the simplest case of $N = 2$. We start with specifying an inhomogeneous coordinate $\varphi$ of the $\mathbb{C}P^1$ manifold instead of the complex two vector $\phi$ defined in Eq. (15). Note that $T$ defined in Eq. (15) is invariant under the $U(1)_\varphi \in SU(2)_\varphi$ transformation

$$T \rightarrow VV_0(\eta)T_0V_0(\eta)^\dagger V^\dagger, \quad V_0(\eta) \equiv e^{i\eta T_0}, \quad (18)$$
with \( \eta \) being an arbitrary real number. Therefore, an \( SU(2)_V \) matrix \( V \) can be always transformed as \( V \rightarrow VV_0(\eta) \). By using this \( U(1)_V \) transformation, one can always cast the diagonal element of \( V \) to be real-valued. So, we will take the following concrete matrix

\[
V = \frac{1}{\sqrt{1 + |\varphi|^2}} \begin{pmatrix} 1 & -\varphi^* \\ \varphi & 1 \end{pmatrix}, \quad \varphi \in \mathbb{C}.
\]

Then, we have

\[
T = \frac{1}{2(1 + |\varphi|^2)} \begin{pmatrix} 1 - |\varphi|^2 & 2\varphi^* \\ 2\varphi & -(1 - |\varphi|^2) \end{pmatrix}.
\]

The relation between \( \phi \) and \( \varphi \) can be found through the equation \( T = \phi\phi^\dagger - 1_2/2 \) by

\[
\phi = V \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{1 + |\varphi|^2}} \begin{pmatrix} 1 \\ \varphi \end{pmatrix}.
\]

With these matrices, the concrete form of the matrix field \( U \) given in Eq. (8) is given by

\[
U(x; x^\alpha) = \frac{1}{1 + |\varphi(x^\alpha)|^2} \begin{pmatrix} u(x; X(x^\alpha)) + |\varphi(x^\alpha)|^2 & -(u(x; X(x^\alpha) - 1)\varphi^*(x^\alpha) \\ -(u(x; X(x^\alpha) - 1)\varphi(x^\alpha) & 1 + u(x; X(x^\alpha)|\varphi(x^\alpha)|^2 \end{pmatrix}.
\]

Plugging this into Eq. (17), we have

\[
\mathcal{L}_{\text{eff}} = C_X \partial_\alpha X \partial^\alpha X + C_\varphi \frac{\partial_\alpha \varphi \partial^\alpha \varphi^*}{(1 + |\varphi|^2)^2},
\]

with

\[
C_X = \frac{f_\pi^2}{2} \int_{-\infty}^{\infty} dx \left( \frac{\partial \theta(x; X(x^\alpha))}{\partial x} \right)^2 = \frac{f_\pi^2 T_{\text{sG}}}{2},
\]

\[
C_\varphi = 4f_\pi^2 \int_{-\infty}^{\infty} dx \sin^2 \frac{\theta(x; X(x^\alpha))}{2} = \frac{f_\pi^2 T_{\text{sG}}}{m^2},
\]

where \( \theta \) is an ordinary sine-Gordon field which is related with \( u \) by \( u = e^{i\theta} \), see Appendix A. In the calculation above, we have used the BPS equation \( \partial_x \theta = \pm 2m \sin \theta/2 \), and the tension of the sine-Gordon domain wall is given by

\[
T_{\text{sG}} = 8m.
\]

Some comments are in order: First, the coefficient \( C_X = T_{\text{sG}}/2 \) of the translational zero mode \( X \) is consistent with the Nambu-Goto action of the order \( O(\partial_\alpha^2) \). Second, it is
remarkable that the coefficient $C_\varphi$, called the Kähler class, has been exactly obtained. The situation is similar to the BPS non-Abelian local vortex [26, 28]. Note that the Kähler class of the non-Abelian orientational zero modes cannot always been obtained. For example, the one for the non-BPS non-Abelian vortex in the dense QCD [31] is only numerically determined.

B. The $U(N)$ case

Now we generalize the results in the previous subsection for $N = 2$ to the generic $N$. Let us first specify the orientational zero modes as in the previous subsection. Let $V_{ij}$ be an $(i,j)$ element of an $SU(N)_V$ matrix. Since the $SU(N)$ generator $T_0$ is expressed as $(T_0)_{ij} = \delta_{i1}\delta_{j1} - \delta_{ij}/N$, Eq. (15) can be written as

$$T_{il} = V_{ij} \left( \delta_{jk} - \frac{1}{N} \right) V_{lk}^* = V_{i1} V_{l1}^* - \delta_{il} \frac{1}{N} = \phi_i \phi_l^* - \delta_{il} \frac{1}{N}. \quad (27)$$

We thus can identify $\phi$ as the first column vector of $V$, namely $\phi_i \equiv V_{i1}$. Of course, the condition Eq. (14) is automatically satisfied: $\phi^\dagger \phi = \phi_i^* \phi_i = V_{i1}^* V_{i1} = \delta_{i1} = 1$. Similarly, we can explicitly write down the matrix $U$ in Eq. (8) as

$$U_{il} = (V U_0 V^\dagger)_{il} = V_{ij} \left( \delta_{jk} + (u - 1) \delta_{1j} \delta_{1k} \right) V_{lk}^* = \delta_{il} + (u - 1) \phi_i \phi_l^*, \quad (28)$$

where we have introduced $U_0 = \text{diag}(u, 1, \cdots, 1) \in U(N)$. In the matrix notation, this can be simply expressed as

$$U = 1_N + (u - 1) \phi \phi^\dagger. \quad (29)$$

Note that this can be also derived from Eq. (13) as

$$\exp \left( i \theta \phi \phi^\dagger \right) = 1_N + i \theta \phi \phi^\dagger + \frac{1}{2!} \left( i \theta \phi \phi^\dagger \right)^2 + \frac{1}{3!} \left( i \theta \phi \phi^\dagger \right)^3 + \cdots$$

$$= 1_N + \left( i \theta + \frac{1}{2!} (i \theta)^2 + \frac{1}{3!} (i \theta)^3 + \cdots \right) \phi \phi^\dagger$$

$$= 1_N + (e^{i \theta} - 1) \phi \phi^\dagger. \quad (30)$$

Thus, we have

$$\partial_\alpha U = \phi \phi^\dagger \partial_\alpha u + (u - 1) \left( \partial_\alpha \phi \phi^\dagger + \phi \partial_\alpha \phi^\dagger \right). \quad (31)$$
By plugging this into the integrand of Eq. (17), we find
\[
\text{tr} \left[ \partial_\alpha U \partial^\alpha U^\dagger \right] = \partial_\alpha u \partial^\alpha u + 2|1 - u|^2 \left[ \partial_\alpha \phi^\dagger \partial^\alpha \phi + (\phi^\dagger \partial_\alpha \phi)(\partial^\alpha \phi) \right].
\] (32)

In order to compute this, let us recall the following equations
\[
\partial_\alpha u = \partial_\alpha e^{i\theta(x; x''')} = i\partial_\alpha X \frac{\partial \theta}{\partial x} u = -i\partial_\alpha X \frac{\partial \theta}{\partial x} u,
\] (33)
\[
2|1 - u|^2 = 2(2 - u - u^*) = 8 \sin^2 \frac{\theta}{2}.
\] (34)

By plugging these into Eq. (17) and performing the integral over $x$, we again find the same integrals in Eqs. (24) and (25). Thus, we eventually reach at the following Lagrangian for the generic $N$
\[
\mathcal{L}_{\text{eff}} = \frac{f_2^2 T_{\pi G}}{2} \partial_\alpha X \partial^\alpha X + \frac{f_2^2 T_{\pi G}}{m^2} \left[ \partial_\alpha \phi^\dagger \partial^\alpha \phi + (\phi^\dagger \partial_\alpha \phi)(\partial^\alpha \phi) \right].
\] (35)

The first term corresponds to the translational zero modes while the second term is the well-known Lagrangian for the $\mathbb{C}P^{N-1}$ nonlinear sigma model.

If one wants to express the $\mathbb{C}P^{N-1}$ Lagrangian in terms of the inhomogeneous coordinate $\varphi^a (a = 1, 2, \ldots, N - 1)$, as in the previous subsection, we take the $SU(N)_V$ matrix
\[
v = \frac{1}{\sqrt{1 + |\bar{\varphi}|^2}} \begin{pmatrix}
1 & \varphi_1^* & -\varphi_2^* & \cdots & -\varphi_{N-1}^* \\
\varphi_1 & 1 + i |\bar{\varphi}|^2 & -\varphi_2 & \cdots & -i \varphi_{N-1}^* \\
\varphi_2 & -i \varphi_1 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\varphi_{N-1} & -i \varphi_{N-2} & \cdots & -i \varphi_{N-2}^* & 1 + i |\bar{\varphi}|^2
\end{pmatrix}.
\] (36)

This $V$ includes $N - 1$ complex parameters $\bar{\varphi}^T = (\varphi_1, \varphi_2, \ldots, \varphi_{N-1})$. A compact form of the elements of $V$ is given by
\[
v_{ij} = \frac{1}{\sqrt{1 + |\bar{\varphi}|^2}} \left( \delta_{ij} + i \frac{|\bar{\varphi}|^2 \delta_{ij} - \varphi_{i-1}^* \varphi_{j-1}^*}{|\bar{\varphi}|} \right), \quad \varphi_0 = -i |\bar{\varphi}|.
\] (37)

By making use of the identity $\sum_{i=1}^N |\varphi_{i-1}|^2 = 2|\varphi|^2$, it is straightforward to check that the condition $(VV^\dagger)_{ik} = V_{ij} V_{kj} = \delta_{ik}$ is satisfied. The effective Lagrangian can be rewritten as
\[
\mathcal{L}_{\text{eff}} = \frac{f_2^2 T_{\pi G}}{2} \partial_\alpha X \partial^\alpha X + \frac{f_2^2 T_{\pi G}}{m^2} g_{ab}^* \partial_\alpha \varphi^a \partial^\alpha \varphi^{b*},
\]
\[
g_{ab}^* = \delta_{ab} \left( 1 + |\bar{\varphi}|^2 \right) - \varphi^b \varphi^{a*} = \partial_\alpha \partial_{b*} \log(1 + |\bar{\varphi}|^2).
\] (38)
IV. $\mathbb{C}P^{N-1}$ LUMPS ON SINE-GORDON SOLITON AS $SU(N)$ SKYRMIONS

In this section, we construct $\mathbb{C}P^{N-1}$ lumps in the effective theory on the sine-Gordon soliton in $d = 3 + 1$ dimensions, and then show that they represent $SU(N)$ Skyrmions. A similar case was found before in the $SU(2)$ model principal chiral model and the Skyrme model with a potential term admitting two discrete vacua [20–22].

By placing a single sine-Gordon soliton perpendicular to the $x^3$-coordinate, the effective theory on it is defined in the $x^0, x^1, x^2$ coordinates as in the last section. Apart from the translational modulus $X$, the energy of static configuration and this Bogomol’nyi completion are given by

$$E = \frac{f^2 T_{SG}}{m^2} \int d^2 x g_{ab^*} (\partial_1 \varphi^a \partial_1 \varphi^{*b} + \partial_2 \varphi^a \partial_2 \varphi^{*b})$$

$$= \frac{f^2 T_{SG}}{m^2} \int d^2 x g_{ab^*} (\partial_1 \varphi^a \pm i \partial_2 \varphi^{a})(\partial_1 \varphi^{*b} \mp i \partial_2 \varphi^{*b}) \pm \frac{f^2 T_{SG}}{m^2} \int d^2 x \epsilon^{mn} i g_{ab} \partial_m \varphi^a \partial_n \varphi^{*b}$$

$$\geq |Q| \quad (39)$$

with spatial indices $m, n = 1, 2$ on the world-volume. Here, $Q$ is the topological lump charge defined by

$$Q \equiv \frac{f^2 T_{SG}}{m^2} \int d^2 x \epsilon^{mn} i g_{ab} \partial_m \varphi^a \partial_n \varphi^{*b} = \frac{f^2 T_{SG}}{m^2} 2 \pi k = \frac{16 \pi f^2}{m} k \quad (40)$$

with $k \in \pi_2(\mathbb{C}P^{N-1})$ being the topological lump number. The topological lump charge is the pullback of the Kähler form on $\mathbb{C}P^{N-1}$. In terms of homogeneous coordinates $\phi$, the lump charge $k$ can be also expressed by [49]:

$$k = \frac{i}{2 \pi} \int dz d\bar{z} \text{tr} \left( [\partial_z \mathcal{P}, \partial_{\bar{z}} \mathcal{P}] \mathcal{P} \right), \quad \mathcal{P} \equiv \phi \phi^\dagger. \quad (41)$$

Note that $\mathcal{P}$ is an projection operator $\mathcal{P}^2 = \mathcal{P}$.

The inequality of the Bogomol’nyi energy bound in Eq. (39) is saturated if and only if the BPS or anti-BPS lump equation

$$\partial_z \varphi^a = 0, \quad \partial_{\bar{z}} \varphi^a = 0, \quad (a = 1, 2, \cdots, N - 1), \quad (42)$$

is satisfied, where we have defined a complex coordinate by $z \equiv x^1 + i x^2$. Generic BPS solutions in terms of $\phi$ are given by a set of holomorphic function $\{P_i(z)\}$

$$\phi^T = (\phi_1, \cdots, \phi_N) = \frac{1}{\sqrt{\sum_{i=1}^{N} |P_i(z)|^2}} (P_1(z), P_2(z), \cdots, P_N(z)). \quad (43)$$
The lump charge \( k \) in Eq. (41) corresponds to the degree of the highest-order polynomial \( P_i(z) \). For instance, a single BPS lump solution in the \( \mathbb{C}P^{N-1} \) model is given by

\[
k = 1: \quad P_1 = z - z_0, \quad P_2 = a, \quad P_{i \geq 3} = 0,
\]

up to symmetry, where \( a \) is a complex modulus representing the size (|\( a \)|) and the phase (\( \text{arg} \, a \)), and \( z_0 \) is the position moduli which we will set to be zero in the following.

Let us take the non-Abelian sine-Gordon solution \( U \) whose moduli parameter \( \phi \) is replaced by the lump solution

\[
U(z, \bar{z}, x^3) = \exp \left( i\theta(x^3)\phi(z, \bar{z})\phi^\dagger(z, \bar{z}) \right)
\]

\[
= 1_N + \left( u(x^3) - 1 \right) \phi(z, \bar{z})\phi^\dagger(z, \bar{z}).
\]

As long as the condition \( \partial_1, \partial_2 \ll \partial_3 \sim m \) holds, this is an approximate solution of the full equations of motion in 3+1 dimensions. So we should keep the size moduli of the lump \( |a|^{-1} \) to be smaller than \( m \). For a configuration with \( |a|^{-1} \gg m \), one should take into account higher derivative corrections to the effective action or solve the full equations of motion in 3+1 dimensions without using the effective theory at all, which we do not work out in this paper.

By using the Maurer-Cartan one form \( R_i \equiv U^\dagger \partial_i U \), the baryon (Skyrmion) number \( B \) taking a value in \( \pi_3[SU(N)] \simeq \mathbb{Z} \) in the bulk can be calculated as

\[
B = \frac{1}{24\pi^2} \int d^3x \epsilon_{ijk} \text{tr} \left( R_i R_j R_k \right)
= -\frac{1}{8\pi^2} \int d^3x \text{ tr} \left[ \left( \partial_1 U^\dagger \partial_2 U - \partial_2 U^\dagger \partial_1 U \right) U^\dagger \partial_3 U \right]
= -\frac{1}{8\pi^2} \int d^3x dx^2 \text{ tr} \left( [\partial_1 \mathcal{P}, \partial_2 \mathcal{P}] \mathcal{P} \right) \int dx^3 |u - 1|^2 u^* \partial_3 u
= \frac{1}{8\pi^2} \int dz d\bar{z} \text{ tr} \left( [\partial_2 \mathcal{P}, \partial_3 \mathcal{P}] \mathcal{P} \right) \int dx^3 4i \sin^2 \frac{\theta}{2} \partial_3 \theta
= \frac{i}{2\pi} \int dz d\bar{z} \text{ tr} \left( [\partial_2 \mathcal{P}, \partial_3 \mathcal{P}] \mathcal{P} \right) \times \frac{1}{2\pi} \int dx^3 (1 - \cos \theta) \partial_3 \theta
= k \times \frac{\Delta}{2\pi}.
\]

where we have defined the sine-Gordon soliton charge by

\[
\Delta \equiv \theta(x^3 = +\infty) - \theta(x^3 = -\infty) = 2\pi(n_+ - n_-), \quad (n_+ \in \mathbb{Z}).
\]
FIG. 1: An $SU(N)$ Skyrmion as a $\mathbb{C}P^{N-1}$ lump inside a $U(N)$ non-Abelian sine-Gordon soliton.

FIG. 2: An isosurface of the baryon density $(1/4\pi^2)\text{tr} \left[ R_1 R_2 R_3 \right]$. We take $m = 1$ and $|a| = 2$. The left panel shows the top view and the right panel shows the front view.

The single non-Abelian sine-Gordon soliton has $\Delta = 2\pi$. Therefore, we have found that $\mathbb{C}P^{N-1}$ lumps on the non-Abelian sine-Gordon soliton represents $SU(N)$ Skyrmions as illustrated in Fig. 1. Note that the Skyrmion confined in the non-Abelian sine-Gordon soliton is not spherical but looks like a pancake, Go stone or m&M’s, see Fig. 2. Although this calculation is valid only for $a$ satisfying $|a| \gg m^{-1}$, the result is independent of the size moduli $a$. Therefore, we expect that this is true for any $a$.

Configurations with higher baryon charges are also easy to be constructed in terms of the effective theory. For example, $B = 2$ configurations are described by

$$k = 2 : \quad P_1 = (z - d)(z + d), \quad P_2 = a, \quad P_{k \geq 3} = 0. \quad (48)$$

Two Skyrmions sit at $z = \pm d$. Some configurations are shown in Fig. 3. When the two
Skyrmions have an overlap region, the baryon density isosurface becomes a donut shape as usual $B = 2$ Skyrmions in the Skryme model. A difference appears for $k \geq 3$. It is known that when usual $B = 3$ Skyrmions coincide on top of each other, the baryon density exhibits a tetrahedral structure. On the other hand, since the Skyrmions are confined inside a soliton in our model, such three dimensional structure does not appear. For instance, a $\mathbb{Z}_3$ symmetric $B = 3$ configuration can be given by

$$k = 3 : \quad P_1 = (z - d)(z - d\omega)(z - d\omega^2), \quad P_2 = a, \quad P_{i \geq 3} = 0,$$

with $z = 1, \omega, \omega^2$ being roots of $z^3 = 1$. As can be seen in Fig. 4, instead of having a tetrahedron, a torus structure again appear when multiple Skyrmions are coincide ($d = 0$) inside the sine-Gordon soliton.

We note that configurations in Eq. (45) show a physical realization of the rational map Ansatz of the Donaldson type. For conventional Skyrmions, rational maps give merely initial configurations for numerical relaxations, although a spherical version of the rational map eventually gives a good approximation to the final configurations [14-17]. On the other hand, we would like to emphasize that, in our case, the rational map of the Donaldson type solves the equation of motion in the moduli approximation, that is, as far as the condition $\partial_1, \partial_2 \ll \partial_3 \sim m$ holds.

We have seen that the Skyrmion can exist stably even in the absence of the Skyrme term, while it shrinks from the Derrick’s scaling argument [46] in the bulk. This is because the sine-Gordon soliton has divergent energy proportional to the world-volume directions. This
situation is parallel to lumps inside a vortex corresponding to Yang-Mills instantons in the Higgs phase \[29, 47\].

The sine-Gordon soliton is BPS saturating the Bogomol'nyi bound and lumps are also BPS saturating the Bogomol'nyi bound in the world-volume theory. However the Skyrmion as the composite soliton itself is not BPS.

V. SUMMARY AND DISCUSSION

We have constructed the effective theory on a non-Abelian sine-Gordon soliton in the \(U(N)\) chiral Lagrangian to obtain the nonlinear sigma model with the target space \(\mathbb{R} \times \mathbb{C}P^{N-1}\). We have shown that \(\mathbb{C}P^{N-1}\) lumps on the non-Abelian sine-Gordon soliton are nothing but \(SU(N)\) Skyrmions in the bulk point of view. This setting offers a physical realization of the rational map Ansatz for Skyrmions of the Donaldson type that solves the equations of motion in the moduli approximation. Skyrmions can exist stably inside the soliton without the Skyrme term.

Several discussions are addressed here. Since we have not considered the Skyrme term in this paper, the \(\mathbb{C}P^{N-1}\) Lagrangian on the soliton admits \(\mathbb{C}P^{N-1}\) lumps with arbitrary sizes and there is no force between Skyrmions. If we add the Skyrme term in the original Lagrangian, it will induces a baby-Skyrme term (as well as enhancement of kinetic term) in the \(\mathbb{C}P^{N-1}\) model on the soliton, that was the case of the \(SU(2)\) chiral Lagrangian with two discrete vacua \[20, 23\]. In this case, the lumps are unstable to expand. In order to
stabilize them, one has to further introduce a mass term that explicitly breaks the $SU(N)_V$ symmetry in the bulk, resulting in $\mathbb{C}P^{N-1}$ baby Skyrmions \[50\] on the soliton corresponding to $SU(N)$ Skyrmions in the bulk.

In this paper, we have constructed the effective theory on a single soliton. It is well known that the sine-Gordon equation admits more general solutions such as breather of two solitons and a static multiple soliton lattice. Constructing the effective theory on these cases will be interesting with paying attentions to the relation between orientational zero modes localized on different solitons, in particular for a soliton lattice. The orientational modes on a non-Abelian $U(N)$ vortex lattice has been discussed recently, and found to give an inhomogeneous $\mathbb{C}P^{N-1}$ model \[51\]. Therefore, a similar mechanism may work here.

In the Skyrme model, Skyrmions are identified with baryons. In the quark model, baryons are composite states of constituent quarks. Hence, one would naively expect that Skyrmions are also composite state of constituent solitons in the Skyrme model. Unfortunately, no fractional solitons that could be identified with quarks have been found in the original Skyrme model of hadrons. On the other hand, there can exist fractional solitons in our model with some modifications. It is known that one $\mathbb{C}P^{N-1}$ lump can be decomposed into $N$ fractional lumps (merons) with $1/N$ lump charges in certain situations such as a twisted boundary condition \[29, 52, 53\], an introduction of a suitable potential \[54, 55\] or a deformation of the target space metric \[56, 57\]. Since the lump is identified with the Skyrmion (baryon) in our model, the merons with $1/N$ baryon charge might be identified with quarks. The “quarks” are confined to baryons and cannot be observed in our model as it is. In order to obtain deconfined quarks, we might need to break the chiral symmetry explicitly. We will report this interesting problem elsewhere.

The CFL phase of dense quark matter \[42\] admits a non-Abelian $U(3)$ sine-Gordon soliton. Therefore, we can construct $SU(3)$ Skyrmions stably inside a non-Abelian sine-Gordon soliton. It was conjectured that Skyrmions in the CFL phase are quarks (called qualitons) rather than baryons as in the usual Skyrme model \[35, 58\]. In the CFL phase, however, it was a problem that Skyrmions cannot exist stably in the absence of the Skyrme term. In our case, they exist stably inside a non-Abelian sine-Gordon soliton. Physical implications of this remain a future problem.

While non-Abelian sine-Gordon soliton is stable in the framework of chiral Lagrangian, it can be unstable or metastable in a linear sigma model because it can be terminated by
a non-Abelian global vortex \[25\]. Consequently, a soliton is bounded by a closed loop of a non-Abelian vortex. The effective theory is therefore the $\mathbb{C}P^{N-1}$ model with the boundary, that will be interesting to explore.

We have considered the group $U(N)$ for sine-Gordon solitons. In the case of non-Abelian vortices, $U(N)$ were extended to arbitrary gauge groups $G$ in the form of $G \times U(1)/Z_r$ with the center $Z_r$ of $G$ of rank $r$ \[59\] such as $SO(N)$ and $USp(2N)$ groups \[60\]. In the same way, non-Abelian $U(N)$ sine-Gordon solitons can be also extended to such cases. The effective theory on such a $G$ sine-Gordon soliton can be constructed to obtain a nonlinear sigma model with the target space $\mathbb{R} \times G/H$ with a suitable subgroup $H$. In this case, $G/H$ lumps on the sine-Gordon soliton will represent $G$ Skyrmions.

The composite Skyrmions constructed in this paper are not BPS although their constituents, sine-Gordon solitons and lumps, are all BPS. On the other hand, the Skyrmions are BPS in the BPS Skyrme model consisting of a six derivative term and a potential term \[10\]. A corresponding model to our $U(N)$ case may admit BPS Skyrmions as BPS lumps inside a BPS soliton.

We have not discussed supersymmetry in this paper. If we promote the target space $U(N)$ to $T^*U(N) \simeq GL(N, \mathbb{C})$, the model can be made supersymmetric \[61\]. For that case, sine-Gordon solitons and lumps may preserve a half supersymmetry, while the total configuration breaks all supersymmetry because it is non-BPS.

As a lower dimensional analogue, a lump (baby Skyrmion) can be constructed \[62\] as a sine-Gordon soliton on a $\mathbb{C}P^1$ kink \[63\]. This relation can be generalized to $\mathbb{C}P^{N-1}$ lumps as sine-Gordon solitons on $\mathbb{C}P^{N-1}$ kinks \[64\]. Combining with the result in this paper, $SU(N)$ Skyrmions can be constructed only from domain walls, as was so for $SU(2)$ Skyrmions \[21\].

As shown in this paper, the target space of the effective theory on a single non-Abelian sine-Gordon soliton is $\mathbb{C}P^{N-1}$ having the nontrivial second homotopy group $\pi_2(\mathbb{C}P^{N-1}) \simeq \mathbb{Z}$. In the latter part of the paper, we have constructed the lump solutions as topological textures characterized by this homotopy group. On the other hand, the same homotopy group admits a monopole as a topological defect if the soliton world-volume is 3+1 dimensional, that is, the bulk is 4+1 dimensional. This gives a D-brane soliton, that is, a Skyrmion string ending on a domain wall, as has been recently shown in Ref. \[65\] for the $SU(2)$ Skyrme model with two vacua, as a higher dimensional generalization of lump strings ending on a domain wall in the $\mathbb{C}P^1$ model \[66\], $\mathbb{C}P^N$ or Grassmann sigma model \[67\]. One advantage of our model
is the existence of parallel solitons as many as possible without anti-solitons, in contrast to previous works (the $\mathbb{C}P^N$ model admits at most $N$ parallel walls).

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**Appendix A: The sine-Gordon model**

Here, we summarize the conventional sine-Gordon soliton to fix notations. The Lagrangian density of conventional sine-Gordon model is

$$\mathcal{L} = \frac{1}{2}(\partial_{\mu}\theta)^2 - m^2(1 - \cos \theta)$$  \hspace{1cm} (A1)

with $\mu = 0, 1, \cdots, d - 1$ and $0 \leq \theta < 2\pi$. The sine-Gordon soliton is characterized by the first homotopy group $\pi_1[U(1)] \simeq \mathbb{Z}$. The static energy density of static configurations depending on one spatial direction $x$ and its Bogomol’nyi completion are given by

$$\mathcal{E} = \frac{1}{2}(\partial_x\theta)^2 + m^2(1 - \cos \theta)$$

$$= \frac{1}{2}(\partial_x\theta)^2 + 2m^2 \sin^2 \frac{\theta}{2}$$

$$= \frac{1}{2}\left( \partial_x\theta \mp 2m \sin \frac{\theta}{2} \right)^2 \pm 2m \partial_x\theta \sin \frac{\theta}{2}$$

$$\geq \left| 2m \partial_x\theta \sin \frac{\theta}{2} \right| = |t_{sG}|$$ \hspace{1cm} (A2)

with the topological charge density defined by

$$t_{sG} \equiv 2m \partial_x\theta \sin \frac{\theta}{2} = -4m \partial_x \left( \cos \frac{\theta}{2} \right).$$ \hspace{1cm} (A3)

The inequality is saturated by the BPS equation

$$\partial_x\theta \mp 2m \sin \frac{\theta}{2} = 0.$$ \hspace{1cm} (A4)
A single-soliton solution interpolating between $\theta = 0$ at $x \to -\infty$ to $\theta = 2\pi$ at $x \to +\infty$ and its topological charge are

$$\theta(x) = 4 \arctan \exp m(x - X), \quad (A5)$$

$$T_{sG} = \int dx t_{sG} = -4m \left[ \cos \frac{\theta}{2} \right]_{x=-\infty}^{x=+\infty} = -4m(-1 - 1) = 8m, \quad (A6)$$

respectively. Here, $X$ is the sine-Gordon soliton position and the width of the soliton is $1/m$.

By using the field $u \equiv e^{i\theta}$ taking a value in the $U(1)$ group, the BPS equation, the topological charge density and the single-soliton solution can be rewritten as

$$-\frac{i}{2} (u^* \partial_x u - (\partial_x u^*) u) \mp m\sqrt{2 - u - u^*} = 0, \quad (A7)$$

$$t_{U(1)} = \frac{im}{2} (u^* \partial_x u - (\partial_x u^*) u) \sqrt{2 - u - u^*} = -2m \partial_x \left( \sqrt{2 + u + u^*} \right), \quad (A8)$$

$$u(x) = \exp \left( 4i \arctan \exp[m(x - X)] \right), \quad (A9)$$

respectively, with the boundary condition $u \to 1$ for $x \to \pm \infty$.

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