Abstract

In this paper we find automorphic functions of coset manifolds with special Kähler geometry. We use ζ-functions to regularize an infinite product over integers which belong to a duality-invariant lattice, this product is known to produce duality-invariant functions. In turn these functions correspond to Eisenstein series which can be understood as string theory amplitudes that receive contributions from BPS states. The Ansatz is constructed using the coset manifold $\frac{SU(1,n)}{SU(n) \times U(1)}$ as an example but it can be generalized. Automorphic functions play an important role in the calculation of threshold corrections to gauge coupling and other stringy phenomena. We also find some connections between the theory of Abelian varieties and moduli spaces of Calabi-Yau manifolds.
1 Introduction

It is known that special Kähler manifolds \([1, 2]\) play an important role in the low-energy effective actions derived from superstrings. For example, the moduli space of of superstring theories compactified on Calabi-Yau manifolds have special Kähler geometry \([3, 4, 5]\); this is also the type of manifold parametrized by the scalar components of vector multiplets in \(N = 2\) supergravity theories in four dimensions \([6]\).

Supergravity models based on the special Kähler manifold \(SU(1, n) / SU(n) \times U(1)\) are known as “minimal models”. This is because the manifolds \(SU(1, n) / SU(n) \times U(1)\) are the ones one gets straightforwardly from superspace or superconformal tensor calculus \([7]\). In this way this coset provides a somehow general setting for supergravity models exhibiting special geometry. On the other hand, this cosets are a non-compact analogous to \(CP^n\) spaces with the advantage of being the simplest family of Special Kähler manifolds \([8]\).

One expects the low-energy effective theories derived from superstrings to be duality-invariant. Any supergravity can be written in terms of only two functions, namely \(W\) and \(K\), of the superfields. They are known as the superpotential and the Kähler potential, respectively (in turn this potential can be written as derivatives of another function called the pre-potential). These functions can actually be combined in one expression,

\[
\mathcal{G} = \exp(K) |W|^2. \tag{1}
\]

Duality transformations mix the moduli (the scalar field components of \(K\) and \(W\)) among themselves and leave the physics unchanged. This means that Kähler potential transforms like

\[
K \rightarrow K + \text{holomorphic} + \text{anti}\text{–holomorphic}
\]

under duality and therefore the superpotential should transform like \(W \rightarrow e^{\lambda} W\), where \(\lambda\) is some factor of automorphy, in order to keep \(\mathcal{G}\) invariant.

Duality transformation act on the Kähler potential. In general \(K\) can be written as the symplectic product of two vectors (for the class of manifolds under consideration); therefore, it is invariant under symplectic transformations –up to a Kähler factor which does not affect the Kähler metric. The duality group can be embedded into the symplectic group, \(\mathcal{D} \subset Sp(2n, \mathbb{Z})\) and, based on this fact, one can construct an Ansatz to obtain duality-invariant functions (in the case in which the moduli space has special Kähler geometry). In fact one can propose

\[
\mathcal{G} = \sum' \log |p_i t^i + q_i \mathcal{F}_i |^2 / t_i \mathcal{F}_i - t^i \mathcal{F}_i; \tag{2}
\]

with \(p_i = \{p_i\}, q_i = \{q_i\} \in \mathbb{Z}^n, \ p \neq 0 \neq \overline{0}, \) and the prime over sum indicating that it is to be carried out for “orbits” of some duality-invariant lattice \(\Gamma\) (we will see this later). The \(t^i\) are the moduli and, as we will see later in section \([\boxed{4}]\), \(\mathcal{F}_i\) are derivatives of the prepotential function \(\mathcal{F}, \mathcal{F}_i = \partial_i \mathcal{F}\). By construction this quantity has the required invariance.

Let us re-write \(\mathcal{G}\) of eq.\((1)\) as

\[
e^{\mathcal{G}} = e^K |\Delta(t^i)|^2, \tag{3}
\]

formally one has that \(\Delta\) is infinite and therefore needs to be regularized (the factor of 1/2
in front of $K$ is irrelevant given its logarithmic form). From this expression one obtains

$$\Delta(t^i) = \prod_{\vec{p}, \vec{q} \in \Gamma} (p_it^i + q^iF_i). \quad (4)$$

The suggested way for the regularization of this quantity is the so called zeta-function regularization \[9, 10\]; this will be the object of study in this paper.

We end this section by briefly discussing the threshold corrections to gauge couplings in low-energy effective field theories derived from superstrings \[11\]. First one should notice that the formula (3) has the same form of the threshold effect function \[12, 13, 14\], which describes the one-loop string corrections to the gauge coupling constant $g_a$ of the low energy effective actions. Indeed, this function has the form

$$T(\tau, \bar{\tau}) = \frac{1}{16\pi^2} C_a \log \left( \frac{2 \Im(\tau) |\eta(\tau)|^4}{|\Delta|^2} \right) \quad (5)$$

where $C_a$ are constants, that can be computed from the massless spectrum, and $\eta(\tau)$ is the Dedekind $\eta$-function. Here, $|\Delta|^2$ would be the analog of that function. Threshold corrections for orbifold cases are considered in reference \[15\].

It is understood that the one-loop correction to the gauge coupling constants have non-holomorphic parts (for a review on the subject, see references \[16, 17, 18\] and references therein). In the effective field theory Lagrangian of an $N = 1$ supersymmetric theory, the fermionic terms are determined by the bosonic part \[13\]. Apart from the condition that the bosonic fields span a Kähler manifold, the functions

$$f_{ab}(\phi) = \left( \frac{1}{g^2(\phi)} \right)_{ab} - \frac{i \Theta_{ab}(\phi)}{8\pi^2} \quad (6)$$

must be holomorphic functions of the moduli $\phi$. $g$ and $\Theta$ are scalar field-dependent gauge coupling and vacuum angle respectively.

At tree-level in string theory these functions can be written in terms of the dilaton-axion field $S$:

$$f_{ab}(\phi) = k_a \delta_{ab} S, \quad (7)$$

where $k$ is the level of the Kac-Moody algebra. One-loop corrections to this formula provide a useful mechanism for breaking supersymmetry. At one-loop the renormalized gauge couplings of the effective theory can be expressed as

$$g_a^{-2}(\mu) = k_a g_{\text{string}}^{-2} + \frac{1}{16\pi^2} b_a \log \frac{M_{\text{string}}^2}{\mu^2} + T(\tau, \bar{\tau}) \quad (8)$$

where $\mu$ is the renormalization scale, $g_{\text{string}}^{-2} \approx \Re S$, $b_a$ are coefficients of the one-loop $\beta$-function of the low-energy effective theory and $T$ is given in (5).

The non-holomorphicity of the renormalized gauge coupling is however not necessarily a problem since it can interpreted as an effective coupling, a purely low-energy effect \[16\].

More recently a lot of work in the computation of the threshold corrections in more realistic models has been done. In particular, there are some computations of the effective gauge couplings in orbifolds with Wilson lines done in reference \[19, 20\]. In these references the threshold effect is given in terms of Siegel modular forms.
Recent developments show the importance of automorphic functions in other aspects of string theory \[21, 22, 23, 24\] as well. In particular there has been a lot of progress in the search for functions with the right invariances. However, most of these has been done for toroidal compactifications of the string. In this paper we attempt to calculate functions with the right automorphy properties for the case of Calabi-Yau compactifications.

In section 3 we present how the regularization of the infinite product of eq.(4) can be done in terms of a $\zeta$-function and we also construct such function in section 29. After this we proceed to obtain, in sections 4 and 5, well known results of Dixon and Kaplunovski \[12, 13\] for the $SU(1,1)/U(1)$ manifold, by doing this we present our Ansatz for other (more complicated) cases. Section 6 contains the first of such cases, namely the $SU(1,2)/U(1) \times SU(2)$ manifold. We find that the automorphic function found has two different contributions, depending on the solutions to a Diophantine equation (90), we analyze them separately in the next subsections and provide the $\zeta$-regularized result in eq.(124), this has the asymptotic expansion (125). In section 7 we show the action of the symplectic group on the Riemann Theta functions found in previous sections, this provides the mechanism by which duality acts on the functions (in a way in which regularization should not be broken); in turn, this gives us the interpretation of (93) as the period matrix of some super-torus, with a map between its moduli space (an Abelian Variety) and the special Kähler manifold.

2 The Mass Formula.

We can explain equation (2) in a more consistent way; not just by constructing it by hand but by looking at the mass formula and the free energy of the particular low energy theory under consideration \[9, 25\]. In field theory one can obtain the tree-level free-energy for bosonic field using the path integral

\[ Z \equiv e^{F_{\phi}} = \int [D\phi] \exp(-\phi^\dagger M^2 \phi) + \cdots \]  

where the integral should be carried our over the massive states of the theory. Neglecting derivative terms one can use this formula as a topological free energy of the bosonic compactified string. In the bosonic case it will be interested in the contribution to the integral by those states in the spectrum which are due to string compactification (that carry momentum and winding number). The path integral (9) then gives in general

\[ F = \log(\det M^\dagger M). \]  

For $N = 1$ supergravity for instance \[26, 27\], the mass matrix has following form:

\[ M_{ij} = e^{S/2} \mathcal{G}^{-1/2}_{mn} \left[ \mathcal{G}_{mn} + 2 \mathcal{G}_{m n} - \mathcal{G}_{n k l} \mathcal{G}_k^{\dagger} m \mathcal{G}_{l n}^{-1/2} \mathcal{G}_k^{\dagger} \right] \mathcal{G}^{-1/2}_{n j}; \]  

with the subindex $i$ denoting

\[ \mathcal{G}_i = \frac{\partial S}{\partial \phi_i}; \quad \mathcal{G}^i = \frac{\partial S}{\partial \phi_i}; \quad \mathcal{G}_j^i = \frac{\partial S}{\partial \phi_j \partial \phi_i}; \]

and we lower and raise index with

\[ (\mathcal{G}^{-1})^j_k \mathcal{G}_k = \delta^j_i. \]
Supersymmetry is preserved when all the fields have vacuum expectation values that are invariant under supersymmetry transformations, in particular this means that we should have

\[ e^{\frac{G}{2}(S^{-1})_{ij}} G_i = 0 \]

for \( N = 1 \), since the expression in r.h.s is proportional to \( \delta \Psi \), the variation under supersymmetry of the fermionic fields.

Then we can write (11) as

\[ (M^\dagger M)_{\bar{i} \bar{j}} = e^{-K} (G^{-1})_{ik} \ W_{kl} (G^{-1})_{lm} \ W_{mn} (G^{-1})_{nj}. \tag{12} \]

Using the form \( K = -\log Y \), it one finds that

\[ \det(M^\dagger M) = \det \frac{|W|^2}{Y}. \tag{13} \]

In the Ansatz provided above it is therefore possible to read the superpotential from the mass formula.

Indeed, for the string theory case, the corresponding mass matrix has been found for many cases, if the moduli space \( S \) is

\[ \mathcal{M} = \frac{SO(2,2)}{SO(2) \times SO(2)} \tag{14} \]

corresponding to some orbifold compactifications, the lattice momenta \( \Gamma(2,2) \) is spanned by vectors \( p^{i_L}, p^{i_R} \) which satisfy

\[ (p^{i_L})^2 - (p^{i_R})^2 = 2 \mathbb{Z} \tag{15} \]

which is the conjugate to the condition satisfied by the moduli fields. The free energy is then given by

\[ F = \sum_{n,n',m,m'} \log \frac{|m + nTU + i(m'U + n'T)|^2}{(T + \bar{T})(U + \bar{U})} \tag{16} \]

where the integers \((0,0,0,0) \neq (n,n',m,m') \in \mathbb{Z}^4\) do not run over oscillator excitations and satisfy a stronger condition than that of equation (15), namely the belong to the orbit described by

\[ (p^{i_L})^2 - (p^{i_R})^2 = mn + m'n' = 0. \tag{17} \]

With this condition one has that (16) becomes the level-matching condition.

The infinite sum (16), subject to the condition (17), can be found using zeta-functions following the procedure of references [9, 28]. Ferrara et al. found that

\[ F|_{\text{reg}} = \log \left( |\eta(T)|^4 |\eta(T)|^4 (T + \bar{T})(U + \bar{U}) \right) \tag{18} \]

where \( \eta(T) \) is the Dedekind function, defined as

\[ \eta(T) \equiv e^{\frac{1}{12} \pi T} \prod_{n>0} [1 - e^{2\pi nT}]. \tag{19} \]
However, the form of the actual calculation, is difficult to generalize to other cases. The condition (17) is resolved in two different orbits: \( n = 0, \ n' = 0 \) and \( n = 0, \ m' = 0 \). This decouples (16) into two separate sums \( (T^1 = T, \ T^2 = U) \),

\[
F(T_i) = \sum_{i=1}^{2} \sum_{(n,m)\neq(0,0)} \log \frac{m_i + in_i T^i}{(T^i + T^i)}
\]

(20)

whose solution is (18). We will do this calculation later in this paper using a different approach to that of reference [9].

The mass formula for other moduli spaces is also known to be of the form (2). For example, in references [29, 25] it is shown that, for a moduli space isomorphic to \( SU(1,2) \times SU(2) \times U(1) \), the chiral mass formula is of the form

\[
M^2 = 2 \left| mA - n\tau + p \right|^2 \frac{1}{1 - \tau \bar{\tau} - AA}
\]

(21)

where \( \tau, A \) are the two complex moduli and the Gaussian integers \( m, n, p \) (integers of the form \( a + ib \) with \( a, b \in \mathbb{Z} \)) satisfy the level matching condition

\[
|m|^2 - |n|^2 - |p|^2 = 0.
\]

(22)

This constraint is not easy to solve and therefore makes it difficult to find a regularization for (21). In [29] a solution is proposed for the automorphic function of \( SU(1,2) \times SU(2) \times U(1) \) by truncating a \( SO(2) \times SO(4) \) coset. The sums are carried out in orbits of the sublattice \( \Gamma_p \) of the \( E_8 \times E_8 \) lattice \( \Gamma_{16} \). The orbits considered not only obey the level matching condition but also severely restrict the numbers over which the sum takes place.

### 3 The Zeta Function.

We briefly introduce the powerful mechanism of zeta-function regularization [30]. First consider a simple example [31, 32] in field theory in which \( \zeta \)-functions are used to find determinants.

Let us consider a quantum operator \( \Lambda \) with eigenvalues \( \lambda_n \), it obeys the eigenvalue equation \( \Lambda f_n(x) = \lambda_n f_n(x) \). We define the \( \zeta \)-function associated with \( \Lambda \) as

\[
\zeta_\Lambda(s) = \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} \right)^s
\]

(23)

if \( \Lambda = 1 \), this function would coincide with the Riemann function. From this we can see that

\[
\det \Lambda \equiv \prod_n \lambda_n = e^{-\zeta'(0)}.
\]

(24)

To find \( \zeta_\Lambda(s) \), we try to solve the heat equation

\[
\Lambda_x G(x,y,t) = -\frac{\partial}{\partial t} G(x,y,t),
\]

(25)
with the boundary condition \( G(x, y, 0) = \delta(x - y) \). We construct the heat function \( G \) given as
\[
G(x, y, t) \equiv \sum_n e^{-\lambda_n t} f_n(x) f_n^*(y).
\]
(26)

It can easily be verified that \( G \) solves (25). The associated \( \zeta \)-function can expressed as
\[
\zeta_\Lambda(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \int d^p x \, G(x, x, t).
\]
(27)

This analytic form of the \( \zeta \)-function can then be used to compute a regularized determinant of the operator \( \Lambda \).

### 3.1 Regularization of the Automorphic Function.

With all this in mind, we can propose as an Ansatz for the regularization of the infinite product (4), via the limit formula
\[
\mathcal{G} = \log (|\Delta|^2 e^K) = -\sum_{(p_i, q_i) \in L_\Gamma} \log i \frac{|p_i t^i + q_i \mathcal{F}_i|^2}{(t^i \mathcal{F}_i - t^i \mathcal{F}_i)}
\]
\[
= - \left[ \sum_{(p_i, q_i) \in L_\Gamma} \left[ \log (p_i t^i + q_i \mathcal{F}_i) + \text{c.c.} \right] - \sum_{(p_i, q_i) \in L_\Gamma} \log (t^i \mathcal{F}_i - t^i \mathcal{F}_i) \right]
\]
\[
= - \lim_{s \to 0} \frac{d}{ds} \zeta(s).
\]
(28)

With the associated \( \zeta \)-function defined as
\[
\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \, \tau^{s-1} \sum_{(p, q) \in L_\Gamma} \left[ \exp \left( -i \tau \pi (p_i t^i + q_i \mathcal{F}_i) \right) \right.
\]
\[
\left. + \exp \left( -i \tau \pi (\bar{p}_i \bar{t}^i + \bar{q}_i \bar{\mathcal{F}}_i) \right) - \frac{1}{2} \exp \left( -i \tau \pi (t^i \mathcal{F}_i - \bar{t}^i \bar{\mathcal{F}}_i) \right) \right];
\]
(29)

the sums are taken over the orbits \( L_\Gamma \) in the sublattice spanned by the momentum and winding numbers, not considering oscillator excitations.

The integers \((q, p)\) obey constraints similar to those of equations (17,22), specialized for the particular example of the special manifold under consideration. They transform as a vector under the symplectic group of which the duality group is a subgroup. This is easily seen since we can write the numerator of \( \mathcal{G} \) as
\[
\left( \begin{array}{c} \bar{q} \\ \bar{p} \end{array} \right) \theta \left( \begin{array}{cr} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{array} \right) \left( \begin{array}{c} \bar{t} \\ i\mathcal{F} \end{array} \right);
\]
(30)
since \((t, \mathcal{F})\) is a vector under \( Sp(2n, \mathbb{R}) \) we must have that, for \( \mathcal{G} \) to be invariant under duality transformations, \((\bar{p}, \bar{q})\) must transform as a symplectic vector as well. Moreover, if \( \mathcal{G}(\mathbb{R}) \) is the duality group then, given the embedding of the duality group into the symplectic group, \((\bar{p}, \bar{q})\) should transform as a vector of \( \mathcal{G}(\mathbb{Z}) \).
4 The $SU(1, 1)/U(1)$ Case.

We should now try to solve the first and most simple case of special Kähler manifold. As mentioned earlier, this example is solved in reference [9]. However, we are going to consider a different approach here which will have the advantage of being more general and therefore possible to apply to other cases.

The particular form of $G$ for this special manifold depends on the coset representatives chosen. Indeed, if one chooses an unbounded realization of the coset, the representatives obey the constraint

$$x_0^\dagger x_0 - X^\dagger X = 1. \quad (31)$$

in homogeneous coordinates. The prepotential $F$ is given by

$$F(X) = \frac{i}{2} \eta^{I J} X_I X_J, \quad (32)$$

we have

$$(p_i t^i + q_i f_i) = (p_0 x_0 + p_1 x_1 + i q_0 x_0 + i q_1 x_1)$$

$$= P_0 x_0 + P_1 x_1 \quad \text{for} \quad P_i = p_i + i q_i \quad (33)$$

just as we expected from the form of the chiral mass formula (21). The integers are subject to certain conditions. In fact they satisfy the constraint

$$|P_0|^2 - |P_1|^2 = 0, \quad (34)$$

the sum (29) over $(p, q)$ is carried out only over these $SU(1, 1) \equiv L_\Gamma$ orbits.

The constraint (34) can be generally solved. However, we first should impose an extra condition to avoid over-counting. Indeed, when one looks at the relation between the parameters of the coset $SU(1, n)/U(1) \times SU(n)$, given by equation (31), it is obvious that the parameter $x_0$ is real by definition. Therefore, since the $\zeta$-function defined so far includes $x_0$ as a complex variable and is treated as any other parameter $x_i$, we need to impose some condition to avoid counting $x_0$ twice. The obvious choice is to impose a similar constraint over the Gaussian integer associated with this parameter, we then set $q_0 = 0$, to set $P_0 \in \mathbb{R}$.

The solution of the resulting constraint is an old number theory problem, which has been considered by many mathematicians [33]. The proposed solution is

$$q_1 = 2 n m$$

$$p_1 = n^2 - m^2$$

$$p_0 = n^2 + m^2 \quad \text{for} \quad (n, m) \in \mathbb{Z}^2. \quad (35)$$

The first term of the $\zeta$-function (29) is therefore given by

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \sum_{(m,n) \in \mathbb{Z}^2} e^{-\tau \pi (n^2 (1+t) + m^2 (1+t) + 2 inmt)} \quad (36)$$
where we have used the parametrization of $SU(1,1)/U(1)$ given by the inhomogeneous physical coordinates $t = x_1$ and $x_0 = 1$, and with the condition $(n,m) \neq (0,0)$. After this, our $\zeta$-function can be written as

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \sum_{(m,n) \in \mathbb{Z}^2} \exp[-\tau \pi Q(n,m,t)]$$  \hspace{1cm} (37)$$

where

$$Q(n,m,t) = \left( \begin{array}{c} n \\ m \end{array} \right) \cdot \left( \begin{array}{cc} 1 + t & it \\ it & 1 - t \end{array} \right) \cdot \left( \begin{array}{c} n \\ m \end{array} \right)$$

$$\equiv \vec{n}^t \Omega \vec{n}. \hspace{1cm} (38)$$

The matrix $\Omega$ satisfies

$$\det \Omega = 1$$

$$\Omega^{-1} = \left( \begin{array}{cc} 1 - t & -it \\ -it & 1 + t \end{array} \right). \hspace{1cm} (39)$$

The sum over integers in (37) is an example of the Riemann Theta function (34), which we will briefly review in the next section. But finally we should note that we could have constructed a solution to (34) without setting $q_0 = 0$. However, it turns out that the matrix $\Omega$ in such solution in not invertible (always has an eigenvalue zero). It seems that this condition is justified in every sense.

4.1 Riemann Theta Functions.

Riemann Theta function is defined in general as

$$\theta(z, \Omega) \equiv \sum_{n \in \mathbb{Z}^g} \exp \left( \pi i n^t \cdot \Omega \cdot n + 2\pi i n^t \cdot z \right), \hspace{1cm} (40)$$

where $z \in \mathbb{C}^g$, $\Omega$ is a $p \times p$ complex matrix which satisfies

$$\Omega^t = \Omega \quad \text{and} \quad \text{Im} \Omega > 0. \hspace{1cm} (41)$$

These conditions define the Siegel upper half space $\mathcal{H}_g$. It is a generalization to the usual upper complex plane $\mathcal{H}_1$ defined by $\text{Im} \tau > 0$, these conditions guarantees the convergence of (40). These conditions are just a particular form of the Riemann Bilinear Conditions.

Riemann Theta functions satisfy beautiful periodicity conditions:

$$\theta(z + m, \Omega) = \theta(z, \Omega) \hspace{1cm} (42)$$

$$\theta(z + \Omega \cdot m, \Omega) = e^{-(\pi im^t \cdot \Omega \cdot m + 2\pi im^t \cdot z)} \theta(z, \Omega), \hspace{1cm} (43)$$

9
for $z \in \mathbb{C}^g$, $m \in \mathbb{Z}^g$. A generalized version of the Riemann theta functions is the theta function with characteristics $\Theta_{a,b}$, defined as

$$\Theta_{a,b}(z, \wp) = \sum_{n \in \mathbb{Z}^g} \exp \left[ \pi i (n + a) \cdot \wp \cdot (n + a) + 2\pi i (n + a) \cdot (z + b) \right]$$

for $a, b \in \mathbb{Q}$.

With this notation we now realize that (37) can be written in terms of the Riemann theta function $\theta(0, \Omega)$ as

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \left( \theta_\tau(0, \Omega') - 1 \right)$$

where $\theta_\tau(0, \Omega') \equiv \theta(0, i\tau \Omega)$. In other words our $\zeta$-function is nothing but a Mellin transformation of a Riemann theta function. The factor of 1 correspond to the 0-modes that are not present in the original (37). How (45) transforms under duality will be discussed later.

The limit (28) can be taken simply by considering the behavior of $\Gamma(s)$ as $s$ goes to zero. If we denote (45) as $\zeta(s) = (\Gamma(s))^{-1} I(s)$, we have

$$\lim_{s \to 0} \frac{d}{ds} \zeta(s) = \lim_{s \to 0} \left[ I(s) \frac{d}{ds} (\Gamma(s))^{-1} + \frac{1}{\Gamma(s)} \frac{d}{ds} I(s) \right].$$

(46)

The first term is easily solved, we make use of

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \mathcal{O}(\epsilon^2)$$

$$\psi(\epsilon) = -\gamma - \frac{1}{\epsilon} + \mathcal{O}(\epsilon^2) \quad \text{with} \quad \psi(s) \equiv \frac{d \ln \Gamma(s)}{ds} = \frac{\Gamma'(s)}{\Gamma(s)}$$

where $\gamma = 0.5772157...$ is the Euler-Mascheroni constant. Therefore we have that

$$\lim_{s \to 0} \frac{d}{ds} \frac{1}{\Gamma(s)} \approx -1.$$

The second term in (46) is more involved. We will prove that $I(s)$ can be expanded in a Laurent series around $s \to 0$ with simple pole at $s = 0$. We will also calculate the residue and therefore we should be able to write

$$I(s) \approx \frac{C_{-1}}{s} + C_0 + \mathcal{O}(s)$$

(48)

taking the derivative and then multiplying by $\frac{1}{\Gamma(s)} \approx s$ for $s$ small, we can write

$$\Psi (\Omega(t)) \equiv \lim_{s \to 0} \frac{d}{ds} \zeta(s)$$

$$= -\lim_{s \to 0} \left[ \int_0^\infty d\tau \tau^{s-1} \left( \theta_\tau(0, \Omega') - 1 \right) + \frac{C_{-1}}{s} \right];$$

(49)

hence the function $\Psi(\Omega)$ is well behaved and finite once the pole is subtracted.

Invariances of (49) are not difficult to establish. One just has to look at the periodicity conditions (42 and 43) of the Riemann theta function for $z = 0$. We will discuss this below, first we need to establish some properties of (49).
4.2 Mellin Transformation.

The function $\Psi(\Omega(t), s)$ of equation (49) is known as a Mellin Transformation, we briefly review a few facts about this transform since they are important in our study of the automorphic functions.

We define the Mellin Transformation of a function $f(x)$ as

$$M[f(u)] \equiv \int_0^\infty dx x^{u-1} f(x);$$

for instance the Gamma function, $\Gamma(s)$ is the Mellin Transformation of the exponential function, $f(x) = e^{-x}$. The condition for $M[f(u)]$ to be properly defined is that, for $x > 0$, the function $f(x)$ decays rapidly at infinity, and at the origin it diverges at most as a polynomial $f(1/x) \sim O(x^{-K})$ as $x \to 0$. With these conditions $M[f(u)]$ converges for $u \in \mathbb{C}$ and Re $s > K$.

With the conditions described above we can study (49). Let's assume that a function $\phi(\tau)$, convergent as $\tau \to \infty$, satisfies the functional equation

$$\phi\left(\frac{1}{\tau}\right) = \sum_{k=1}^{K} b_k \tau^{\lambda_k} + \alpha \tau^h \phi(\tau)$$

for $\tau > 0$ and $\alpha \neq 0$, $\lambda_k$, $h$, $b_k$ complex numbers. Then the Mellin transformation of $\phi(\tau)$ can be written as a sum $\int_0^\infty = \int_0^1 + \int_1^\infty$, and we have

$$M[\phi(\alpha, s)] = \int_1^\infty \left[ \sum_{k} b_k \tau^{\lambda_k} + \alpha \tau^{h} \phi(\tau) \right] \tau^{-s-1} d\tau + \int_1^\infty \phi(\tau) \tau^{s-1} d\tau$$

where we have used the functional equation in the first term.

After we perform the first of these integrals we are left with

$$M[\phi(\alpha, s)] = \sum_{k} \frac{b_k}{\alpha \tau^{h+\lambda_k}} + \int_1^\infty \phi(\tau) \tau^{s-1} d\tau + \alpha \int_1^\infty \phi(\tau) \tau^{h-s-1} d\tau.$$  

The first term gives the poles of the Mellin transformation at $\lambda_k$ with residues $b_k$. The second term is convergent for all values of $s$. If we now use the functional equation twice, we obtain a second version of it

$$\phi\left(\frac{1}{\tau}\right) = \frac{1}{\alpha} \tau^h \phi(\tau) - \frac{1}{\alpha} \sum_{k=1}^{K} b_k \tau^{h-\lambda_k}.$$ 

Integrating as before, we obtain

$$M[\phi(\alpha, s)] = \sum_{k} \frac{-b_k/\alpha}{s-h+\lambda_k} + \int_1^\infty \phi(\tau) \tau^{s-1} d\tau + \frac{1}{\alpha} \int_1^\infty \phi(\tau) \tau^{h-s-1} d\tau.$$ 

Therefore we arrive at the conclusion that the Mellin transformation of $\phi$ as defined, is invariant (up to a factor of $\alpha$) under $s \to h - s$, $\alpha \to \frac{1}{\alpha}$,

$$M[\phi(\alpha, s)] = \alpha M[\phi\left(\frac{1}{\alpha}, h - s\right)];$$
with poles at \( \lambda_k \) and residues \( b_k \).

In the Mellin transformation under consideration \((49)\), the functional equation can be obtained using the Poisson’s summation formula:

\[
\sum_{\vec{n} \in \mathbb{Z}^p} \exp[-\pi \vec{n}^t \cdot A \cdot \vec{n}] = (\det A)^{-\frac{1}{2}} \sum_{\vec{m} \in \mathbb{Z}^p} \exp[-\pi \vec{m}^t \cdot A^{-1} \cdot \vec{m}].
\] (57)

This amounts to obtaining the transformations properties of \( \theta(0, \tau \Omega) \), defined by equation \((45)\), when \( \tau \rightarrow \frac{1}{\tau} \).

We define \( \phi(\tau) \equiv \theta_{\tau}(0, \Omega) - 1 \equiv \theta(\tau \Omega) - 1 \) and make use of \((57)\) to get

\[
\phi\left(\frac{1}{\tau}\right) = \theta(0, \frac{1}{\tau} \Omega) - 1
\]
\[
= \frac{1}{(\det \tau^{-1} \Omega)^{\frac{1}{2}}} \theta(0, \tau \Omega^{-1}) - 1
\]
\[
= \frac{\tau}{\alpha} \theta(\tau \Omega^{-1}) - 1
\] (58)

where we have set \( \alpha = (\det \Omega)^{1/2} \). However we also have that

\[
\theta(\tau \Omega) = \theta(\tau \Omega^{-1})
\] (59)

this is because the sum implied in the right hand side of the equation can be obtained by re-arranging the sum in the left hand side. In fact, just by looking at the form of \( \Omega \) and \( \Omega^{-1} \) in \((39)\) and \((38)\) one can see that, by redefining the integers in the sum \((36)\) as

\[
n \rightarrow -m \quad \text{and} \quad m \rightarrow n,
\] (60)

we go from the l.h.s. to the r.h.s. of \((59)\).

Therefore we have the following functional equation:

\[
\phi\left(\frac{1}{\tau}\right) = \alpha \tau \phi(\tau) - 1 + \alpha \tau;
\] (61)

where we just have to use what we have already learned about the Mellin transformation to learn about \( \Psi(\Omega) \) in \((49)\). We have, in \((51)\),

\[
h = 1 \quad \text{and} \quad \lambda_1 = 0 \quad b_0 = -1
\]
\[
\lambda_2 = 1 \quad b_2 = \alpha.
\] (62)

We conclude that \( \Psi(s, \Omega) \) extends to a holomorphic function of \( s \), except for simple poles located at \( s = 0 \) and \( s = 1 \), with residues \(-1\) and \( \alpha = (\det \Omega)^{1/2} \) respectively. Moreover, up to factors of \( \alpha \), the function \( \Psi(s) \) is invariant under \( s \rightarrow 1 - s \).

Thus we have completed the expression \((49)\) which, after a mild abuse of our notation, we write as

\[
\lim_{s \rightarrow 0} (\Psi(s, \Omega) + \frac{1}{s})
\] (63)

the desired \( \zeta \)-function is finite and well defined. It is however difficult to give an general solution to the Mellin transformation, we will try to give an answer to that later.
4.3 Putting It All Together.

Having found the form of the first term in the expression (29) it is now easy to compute the automorphic function of \( SU(1,n) \). The second term is nothing but \( \Psi(\Omega(t)) \), therefore the first two terms pick the real part of this function. The third term can be computed using Riemann \( \zeta \)-functions as defined before in section 3.

We recall here the \( \zeta \)-function related to the third term, it is given by

\[
\zeta_3(s) = -\frac{1}{2} \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \sum_{\substack{m,n \in \mathbb{Z} \setminus \{0\}}} \exp \left[ -\tau \pi (1 - \bar{t} t) \right]. \tag{64}
\]

The contribution of this term to the function (3) is given by its derivative and in the limit \( s \to 0 \):

\[
-\lim_{s \to 0} \frac{d}{ds} \zeta_3(s) = \frac{1}{2} \lim_{s \to 0} \frac{d}{ds} \left[ \frac{1}{\Gamma(s)} \int_0^\infty d\tau' \tau'^{s-1} \sum_{\substack{0 \neq m,n \in \mathbb{Z}}} \frac{1}{[1 - t \bar{t} \tau']^s} \right] \tag{65}
\]

the integral is the Gamma function, so we get

\[
-\lim_{s \to 0} \frac{d}{ds} \zeta_3(s) = \frac{1}{2} \lim_{s \to 0} \frac{d}{ds} \left[ (1 - t \bar{t}) \right]^{-s} \left( 4 \sum_{n>0, m>0} 1 \right); \tag{66}
\]

here we use \( \zeta(0) = -1/2 \), to finally get

\[
-\lim_{s \to 0} \frac{d}{ds} \zeta_3(s) = \ln |1 - t \bar{t}| \tag{67}
\]

Collecting all the results of this section we can write

\[
e^G = |\Delta|^2 e^K = e^{4 \Re \Psi(\Omega(t))} |1 - t \bar{t}| \tag{68}
\]

Finding a form for this expression in terms of known functions of mathematical physics is rather complicated. We will make some attempts in that direction later. It is also important to verify that the expression found is invariant under duality transformation. The original expression (28) obviously is, but we should discuss this later with more detail.

5 Known Results.

We should now focus in the problem of finding an analytic form of the function we have defined. Fortunately this is possible for the case of the coset we have just seen in the previous section and we should show that it is possible in general. In particular, we shall find the results of Ferrara et. al. in [9].

To proceed we need to make a direct integration of the \( \zeta \)-function, as defined in (29). The first term renders the Eisenstein series

\[
\frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \left( \theta_\tau(0, \Omega') - 1 \right) = \sum_{m,n \neq 0} \left( \pi Q(m, n) \right)^{-s-1} \tag{69}
\]

where \( Q \) is the quadratic form defined (38) and where we have also identified it with the Mellin transformation studied in the preceding section. This Eisenstein series is convergent
for Res > 1 but can be analytically continued to the whole complex plane. The derivative of the resulting function – just as we saw – has poles at s = 0, 1. Taking into account the three terms of the ζ-function and once the derivative against s is taken, one can easily see that final Eisenstein series to calculate becomes

\[ G(t, s) = \sum_{m,n \neq (0,0)} \frac{(1 - tt)^s}{\pi Q(t, m, n) |2s|}. \quad (70) \]

This form is important because it allows us to study duality transformations on the automorphic function in a convenient way. But, for the effects of the calculation, we will work with the form broken up in three pieces and will concentrate our attention in (69).

First we will change our variables as follows:

\[ t \rightarrow T = \frac{1 - t}{t + 1} \quad \text{and} \quad t = \frac{1 - T}{T + 1} \quad (71) \]

which means, for the automorphic function,

\[ \Omega \rightarrow \Omega = \left( \frac{i}{2} \frac{1}{(1 - T)} \frac{i}{2} \frac{1}{T} \right) (1 - tT) \rightarrow \frac{1}{2} (T + T). \quad (72) \]

Using the so-called Chowla-Selberg formula, we can find an analytic form of the automorphic function, in terms of known functions of mathematical physics. This process is similar to the demonstration of Kronecker’s First Limit Formula. To do this we need to use the general form of the Poisson summation formula, namely,

\[ \sum_{n \in \mathbb{Z}} g(x + n) = \sum_{m \in \mathbb{Z}} \left[ \int_{\mathbb{R}} g(\tau) e^{-2i\pi m\tau} d\tau \right] e^{2i\pi m x}. \quad (73) \]

This amounts to converting the sum in the l.h.s. for \( g(x) \) into a sum over its Fourier transformation \( \tilde{g}(\tau) \) in the r.h.s. Here \( g(x) \) is a continuous function of \( x \) which decreases rapidly for \( x \to \infty \).

In our case (69), using the quadratic form defined in (72), we can write

\[ \sum_{m,n \neq (0,0)} \frac{1}{(m^2 + mni(1 - T) + n^2T)^s} = \sum_{m+n \neq (0,0)} \frac{1}{(m + n(z + i\alpha))^s (m + n(z - i\alpha))^s} \]

where we have defined

\[ z = \frac{i}{2} (1 - T) \quad \text{and} \quad \alpha = \frac{1}{2} (T + 1) \quad (75) \]

we also can split the sum as \( \sum_{m,n \neq (0,0)} = (2 \sum_{n=0} \sum_{m>0} + 2 \sum_{n>0} \sum_{m \in \mathbb{Z}}) \).
5.1 The Fourier Transformation.

After some algebra, which we will just sketch here, one can find the Fourier transformation (73), the expression

\[ \sum_m g(x + m) = \sum_m (m + x + iy + i\delta)^{-s} (m + x + iy - i\delta)^{-s} \]  

leads to

\[ \sum_m g(x + m) = \sum_r \left( \int_R \frac{1}{(w + iy + i\delta)^s(w + iy - i\delta)^s} e^{-2\pi rw} \, dw \right) e^{2\pi rx}. \]  

Using the integral (37)

\[ \int_{-\infty}^{\infty} e^{-ivu} (u^2 + 1)^{-s} \, du = \begin{cases} \frac{1}{\Gamma(s)} \frac{2\pi^{1/2} \left(\frac{|v|}{2}\right)^{s-3/2}} K_{s-3/2}(|v|) & \text{for } v \neq 0 \\ \frac{1}{\Gamma(s)} \frac{\pi^{1/2} \Gamma(s - 1/2)} & \text{for } v = 0 \end{cases} \]

where \( K_{\alpha}(b) = \int_0^\infty e^{-\beta \cosh(b)} \cos(a\beta) \, d\beta \) is the modified Bessel function, and, using the notation \( \sigma = x + iy \) in (76), one obtains

\[ \sum_m g(x + m) = 2 \left[ \frac{1}{\Gamma(s)} \pi^{1/2} \Gamma(s - 1/2) \delta^{-2s+1} \right. \]

\[ + \left. \frac{2}{\Gamma(s)} \pi^s \sum_{r \neq 0} e^{2\pi i r \sigma} \delta^{-s+\frac{1}{2}} |r|^{-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|\delta|) \right]. \]

The last expression can now be used to calculate the sum (75) with \( \sigma = nz \) and \( \delta = n\alpha \). Splitting the sum as announced before, one obtains for (80)

\[ \sum_{m,n \neq 0} (\pi Q(T, m, n))^{-s} = 2\pi^{-s} \zeta(2s) + \frac{2\pi^{1/2-s} \Gamma(s - 1/2)}{\Gamma(s)} \alpha^{1-2s} \zeta(2s - 1) \]

\[ + \frac{4}{\Gamma(s)} \alpha^{1/2-s} \sum_{n > 0, r \neq 0} e^{2\pi inz} n^{1/2-s} |r|^{-s-1/2} K_{s-1/2}(2\pi|r|n\alpha). \]

in this formula \( \zeta(s) \) denotes the ordinary Riemann \( \zeta \)-function.

From previous sections we learned that the function

\[ 2G^*(s) \equiv \Gamma(s) \sum_{(m,n) \neq (0,0)} (\pi Q(m, n, z))^{-s} \]

extends to a meromorphic function in the complex plane. We also learned that it has a pole at \( s = 0 \) and that, once the pole is subtracted, the resulting function is regular in the limit. We know too that (49)

\[ \lim_{s \to 0} \frac{d}{ds} \zeta(s) = \lim_{s \to 0} \left( 2G^*(s) + \frac{1}{s} \right) \]

1The absolute value here only means that the sign of the argument of the Bessel function is set so that integral is convergent.
where $\zeta(s)$ now denotes the $\zeta$-function defined in (63).

Collecting all our findings so far, we can write

$$2G^*(s) = \pi^{-s} \Gamma(s) \zeta(2s) + \pi^{1/2-s} \Gamma(s-1/2) \alpha^{1-2s}\zeta(2s-1)$$

$$+ 2\alpha^{1/2-s} \sum_{n>0,r\neq0} e^{2i\pi r n z} n^{1/2-s} |r|^{s-1/2} K_{s-1/2}(2\pi |r| n\alpha).$$

(83)

The function $\zeta^*(a)\pi^{-a/2} \Gamma(a/2) \zeta(a)$ is holomorphic in the complex plane $\mathbb{C}$, it has a simple pole at $s = 0$ with residue $-1$.

Using the identity

$$K_{1/2}(x) = \sqrt{\pi/2x} \ e^{-x}$$

(84)

we find

$$\lim_{s \to 0} \left( 2G^*(s) + \frac{1}{s} \right) = C_0 + \frac{1}{6} \pi\alpha +$$

$$\sum_{n,r>0} \frac{1}{r} \left( e^{2i\pi r n(z+i\alpha)} + e^{-2i\pi r n(z-i\alpha)} \right),$$

(85)

where $C_0$ is a constant of no importance to us and we have make use of the value $\zeta(-1) = -1/12$. The sums implied here can be performed with the help of the identity $[38],$

$$\ln \eta(z) = \frac{i\pi z}{12} - \sum_{m,k>0} \frac{1}{k} e^{2i\pi kmz};$$

(86)

where $\eta(z)$ is the Dedekind function. Thus we end with the expression

$$\lim_{s \to 0} \left( 2G^*(s) + \frac{1}{s} \right) = \ln \eta(z + i\alpha) + \ln \eta(-z + i\alpha) + C_0$$

(87)

This equation resembles the First Kronecker Limit Formula of the theory of elliptic functions $[38, 32]$. Both terms are convergent.

### 5.2 Putting It All Together Again.

We recall here that we need to add the complex conjugate to the formula (87), corresponding to the second term of the $\zeta$-function expansion (29) to obtain the automorphic function. A third term should be added as well, provided in the equation (68), with the Kähler potential written as in (72) in the transformed variables $T$.

In short one finds, with the values of $z$ and $\alpha$ given by (75), that the automorphic function of $SU(1,1)/U(1)$ can be written as

$$e^G = |\Delta|^2 e^K = [ |\eta(T)|^4 (T - \bar{T}) ] + C$$

(88)

where we have included a factor of $i$ into the variable $T$ and the constant term $C$ is moduli independent. What we have obtained here is the classical result of reference $[3]$ (up to constant moduli-independent factors).
The automorphic function is invariant under $T \rightarrow T + 1$ and $T \rightarrow -\frac{1}{T}$, therefore it is $SL(2, \mathbb{Z})$ invariant (up to roots of unity).

The question now is why go through all this lengthy process to find what we already knew? The reason for this is that the procedure described here can be generalized to cosets beyond $SU(1,1)/U(1)$ and to any $SU(1,n)/U(1)\times SU(n)$. We will describe the next case in the following section.

6 The $SU(1,2)/U(1) \times SU(2)$ Coset and the General Ansatz.

The form of the automorphic function in this case, taken from (28), is going to depend on

$$\left(p_i x^i + q^i F_i\right) = p_0 x_0 + p_1 x_1 + i q_1 x_1 + p_2 x_2 + i q_2 x_2.$$  (89)

As before the integers are subject to the $SU(1,n,\mathbb{Z})$-invariant constraint

$$(p_0)^2 = (p_1)^2 + (q_1)^2 + (p_2)^2 + (q_2)^2$$  (90)

where we have also set the to zero the partner of the imaginary part of $x_0$.

The constraint (90) is also possible to solve [33], in the same spirit of the solutions provided for the previous example. Indeed, one can write

$$p_0 = (n_1)^2 + (n_2)^2 + (n_3)^2 + (n_4)^2$$
$$p_1 = -(n_1)^2 + (n_2)^2 + (n_3)^2 + (n_4)^2$$
$$q_1 = 2n_1 n_2$$
$$q_2 = 2n_1 n_3$$
$$p_2 = 2n_1 n_4$$  (91)

for $\vec{n} \equiv (n_1, n_2, n_3, n_4) \in \mathbb{Z}^4$. This provides a general solution to (90). However, there is one set of numbers that are not produced by this array. Nevertheless, such solution can be constructed using it.

To be more precise, the square of an integer is always 0 or 1 modulo 4. That is, the l.h.s. of (90) equals 0, 1 (mod 4). Since the sum in the r.h.s is a sum of four squares we can have that the r.h.s can be any integer. (In fact, it was Lagrange who proved that any integer can be written as the sum of four squares.) Since there is the equality in the middle, only some combinations of numbers can satisfy the condition. The correct combinations are given in Table I.

| Solution | $p_0$ | $p_1$ | $q_1$ | $p_2$ | $q_2$ |
|----------|------|------|------|------|------|
| I        | 0 (mod 4) | e | e | e | e |
| II       | 1 (mod 4) | o | e | e | e |
| III      | 0 (mod 4) | o | o | o | o |

Table 1: Solutions to the constraint (90) according to whether $p_i, q_i$ are even or odd.
6.1 First Contribution to the Automorphic Function.

It is not difficult to see that solutions I and II can be generated by (91). Solution III is generated as follows: take all numbers in \( \vec{n} \) to be odd, this means that all the expressions in the l.h.s are even. We now divide by two all the expressions of the r.h.s of (91). The resulting solution corresponds to case III. We will concentrate first in the contribution of the cases I and II, the third will be discussed at the end.

Implementing the solution given above leads to the \( \zeta \)-function

\[
\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \sum_{\vec{n} \in \mathbb{Z}^4} \left( \exp[-\tau \pi Q(\vec{n}, t, A)] - 1 \right)
\]

\[
= \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} (\Theta(\tau, t, A) - 1)
\] (92)

where we have the quadratic form in terms of the inhomogenous (physical) coordinates \( x_0 = 1, \ x_1 = t, \ x_2 = A \). The quadratic form \( Q \) is given by

\[
Q(\vec{n}, t, A) = \vec{n}^T \cdot \Omega_2 \cdot \vec{n}
\]

with

\[
\Omega_2 = \begin{pmatrix}
1 - t & it & iA & A \\
it & 1 + t & 0 & 0 \\
iA & 0 & 1 + t & 0 \\
A & 0 & 0 & 1 + t
\end{pmatrix};
\] (93)

which satisfies \( \det \Omega_2 = (1 + t)^2 \), i.e. it is invertible. (Using a solution with \( q_0 \neq 0 \) leads to a non-invertible matrix.)

The resulting Mellin transformation is not easy to analyze in this case. However, we can still work out some form of the Fourier transformation in the other half of the procedure. A general form of such expansion is provided by the recursive use of the Poisson summation formula (73) as shown in reference [39]. If \( \zeta_{\Omega_2}(s) = \sum_{\vec{n} \neq n \in \mathbb{Z}^d} (Q)^{-s} \) this procedure, in general, renders the \( \zeta \)-function expansion

\[
\zeta_{\Omega_2}(s) = 2a^{-s} \zeta(s) + \frac{\pi}{a} \Gamma(s - 1/2) \zeta_{\Omega_{\text{Red}}}(s - 1/2) +
\]

\[
\frac{4\pi^s}{a^{s/2+1/4} \Gamma(s)} \sum_{\vec{0} \neq \vec{n} \in \mathbb{Z}^{d-1}} \sum_{m=1}^{\infty} \left[ \cos\left(\frac{\pi}{a} m \vec{b} \cdot \vec{n} \right) m^{s-1/2} (\vec{n}^T \cdot \Omega_{\text{Red}} \cdot \vec{n}^T)^{1/4-s/2} \right.
\]

\[
\left. \times K_{s-1/2}\left(\frac{2\pi}{\sqrt{a}} m \sqrt{\vec{n}^T \cdot \Omega_{\text{Red}} \cdot \vec{n}^T} \right) \right];
\] (94)

where \( d \) is the dimensionality of \( \Omega_2 \), \( a = \Omega_{2a}, \ \vec{b} = (\Omega_{2x,1}, \Omega_{2x,2}, \ldots, \Omega_{2x,d}) \) and

\[
\Omega_{\text{Red}} = \Omega_{2d-1 \times d-1} - (1/4a) \vec{b} \otimes \vec{b},
\]

with \( \Omega_{2d-1 \times d-1} \) being the reduced matrix \( \Omega_2 \) without the first column and row.

This is a recursive formula which leads to an analytic—although complicated—Fourier expansion of the automorphic function. One can, however, use the fact that \( \Omega_2 \) has some...
nice symmetries to obtain the final formula. Before this, it is better to write the quadratic form as we did in (72). In this case we change variables as follows

\[ t \to T = \frac{1-t}{1+t}, \quad A \to A' = \frac{\sqrt{2}A}{1+t}. \]  

Under which the quadratic \( Q(n, t, A) \to Q(n, T, A') \) has the matrix

\[
\Omega'_2 = \begin{pmatrix}
T & \frac{1}{2}i(1-T) & \frac{1}{\sqrt{2}}iA' & \frac{1}{\sqrt{2}}A' \\
\frac{1}{2}i(1-T) & 1 & 0 & 0 \\
\frac{1}{\sqrt{2}}iA' & 0 & 1 & 0 \\
\frac{1}{\sqrt{2}}A' & 0 & 0 & 1
\end{pmatrix}
\]

and the Kähler potential transforms as

\[ 1 - t\bar{t} - AA \to 2(T + \bar{T} - A'A'). \]

Correspondingly, the \( \zeta \)-function is given by

\[
\zeta_{\Omega'_2}(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \sum_{\vec{n} \in \mathbb{Z}^4} \exp[-\tau \pi Q(\vec{n}, T, A')]
\]

\[ = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \sum_{\vec{n} \in \mathbb{Z}^4} e^{-\tau \pi [T(n_1)^2 + \sum_{j=2}^4 (n_j)^2 + n_1n_2(1-T) + n_1n_3\sqrt{2}A' + n_1n_4\sqrt{2}A']]. \]

Completing squares for each \( n_j \) one obtains

\[ \zeta_{\Omega'_2}(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \sum_{\vec{n} \in \mathbb{Z}^4} e^{-\tau \pi \left[\frac{1}{4}(1+T)^2(n_1)^2 + \left(\frac{n_1}{\sqrt{2}}(1-T)\right)^2 + \left(n_3 + \frac{n_4}{\sqrt{2}}A'\right)^2 + (n_4 + \frac{n_4}{\sqrt{2}}A')^2\right]}, \]

where we can split the sum into \( \sum_{\vec{n}} = \sum_{\vec{n} \in \mathbb{Z}^3, n_1=0} + \sum_{\vec{n} \neq \vec{0}, \vec{n} \in \mathbb{Z}^3} \equiv \zeta_I + \zeta_{II}, \) and define \( \vec{N} \equiv (n_2, n_3, n_4). \) The second term in this sum can be analyzed using the properties of Mellin transformations, for that term we obtain

\[ \zeta_{II} = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \sum_{\vec{N} \neq \vec{0}} e^{-\tau \pi [n_2^2 + (n_3)^2 + (n_4)^2]} \]

which can be written

\[ M[\phi(s)] = \int_0^\infty d\tau \tau^{s-1} \phi(\tau) \]

with

\[ \phi(\tau) = [\theta_3(\tau)]^3 - 1 \quad \text{and} \quad \theta_3(\tau) = \sum_n e^{-\pi \tau n^2}. \]

Jacobi’s \( \theta_3(\tau) \) function satisfies \( [\theta_3(\tau)]^3 = \tau^{-3/2} [\theta_3(\tau^{-1})]^3. \) Therefore, we have the functional equation

\[ \phi(\tau^{-1}) = \tau^{3/2} \phi(\tau) + \tau^{3/2} - 1. \]
Thus we have simple poles at $s = 0$ and $s = 3/2$ with residues $-1$ and $+1$ respectively.

The first part of the sum is more involved. First we make use of the Poisson formula [40],

$$\sum_{n \in \mathbb{Z}} e^{-\pi \tau (n + x m)^2} = \frac{1}{\sqrt{\tau}} \sum_{n \in \mathbb{Z}} e^{2\pi i x n} e^{-\pi n^2 / \tau}$$

over every component of $\vec{N}$. Then we can integrate with the help of the identity

$$\int_0^\infty dx \frac{1}{x} e^{-\beta x - \alpha x} = 2 \frac{(\beta/\alpha)^{\nu/2} K_{\nu}(2 \sqrt{\beta \alpha})}{\beta \alpha},$$

for $\beta, \alpha \neq 0$.

Doing all this, we get

$$\zeta_I = \frac{1}{\Gamma(s)} \left[ \sum_{n_1 \neq 0, \vec{N} \in \mathbb{Z}^3} e^{i\pi n_1 [n_2 i (1-T) + n_3 \sqrt{2} A' + n_4 \sqrt{2} A']} \right.
\left. \times \int_0^\infty d\tau \tau^{(s-3/2) - 1} e^{-\pi \tau (n_1^2 + (n_2)^2 + (n_3)^2 + (n_4)^2)} \right].$$

Considering the cases $\vec{N} = 0$ and $\vec{N} \neq 0$ separately (and integrating the second part over $\tau$) we obtain

$$\zeta_{\Omega'_2}(s) = \zeta_{II} + \left( \frac{1 + T}{2} \right)^{-2s+3} \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{(s-3/2) - 1} \left[ \theta_3(\tau) - 1 \right] +
\frac{1}{\Gamma(s)} \sum_{0 \neq n_1 \in \mathbb{Z}, \vec{N} \neq 0} \left[ e^{i\pi n_1 [n_2 i (1-T) + n_3 \sqrt{2} A' + n_4 \sqrt{2} A']} \right.
\left. \times 2^{s-1/2} |\vec{N}|^{-s/2} n_1^{-s+3/2} \left( 1 + T \right)^{-s+3/2} K_{s-3/2} \left( \pi n_1 |\vec{N}| (1 + T) \right) \right].$$

we have to notice here that this expression can not be brought to a form similar to the case of $SU(1,1)/U(1)$, where the indexes and exponentials were $1/2$ instead of $3/2$. That is, the presence of the other moduli cannot be switched off simply by setting the moduli and the winding and momentum numbers to zero. If one is to consider that case, the approximation should be made in the previous step of the calculation and not implemented as truncations in the Fourier expansion [108].

The pole structure of the above formula is not difficult to study. However, according to what we have already learned, we are interested in the function $(\Gamma(s)\zeta_{\Omega'_2}(s))$ which extends to a meromorphic function in the complex plane, except for poles at certain values of $s$.

Since the third term is convergent as long as the exponentials are in the upper half plane, the poles are given by the first two terms. We find that –after the use of the properties of the Mellin transformation– the function $2G(s) \equiv \Gamma(s) s^{-s} \sum_{n \in \mathbb{Z}^4} (Q(n,t,A'))^{-s}$ (defined as in [31]) has simple poles located at $s = 0$ and $s = 2$ with residues $-1$ and $+1$. Hence, after making a Laurent expansion and subtracting the pole, we can proceed to study the limit $s \to 0$ in the expression [108].
From (108), we therefore obtain

$$\lim_{s \to 0} (2G(s) + \frac{1}{s}) = C_0 + \frac{\pi^2}{45} \left( \frac{1 + T}{2} \right)^3 +$$

$$\left( \frac{1 + T}{2} \right)^{3/2} \sum_{0 \neq n_1 \in \mathbb{Z}, \tilde{N} \neq 0} \left[ e^{i\pi n_1|n_2i(1-T)+n_3\sqrt{2}iA'+n_4\sqrt{2}A'|} \times |\tilde{N}|^{-3/2} n_1^{3/2} K_{-3/2}(\pi n_1|\tilde{N}|(1 + T)) \right]$$

(108)

where $C_0$ is a $T$-independent constant. The recurrence relations [37] for the Bessel function $K_{3/2}(z) = K_{-3/2}(z)$ allow us to write $K_{3/2}(z) = z^{-1} K_{1/2}(z) + K_{1/2}(z)$, where we can also make use of [84]. So, we can write

$$K_{3/2}(\pi n_1|\tilde{N}|(1 + T)) = e^{-\pi n_1|\tilde{N}|(1 + T)}$$

(109)

Introducing this expression into (108), we find

$$\lim_{s \to 0} (2G(s) + \frac{1}{s}) = C_0 + \frac{\pi^2}{45} \left( \frac{1 + T}{2} \right)^3 +$$

$$\frac{1}{4} \sum_{0 \neq n_1 \in \mathbb{Z}, \tilde{N} \neq 0} \left[ e^{i\pi n_1|n_2i(1-T)+n_3\sqrt{2}iA'+n_4\sqrt{2}A'|+i|\tilde{N}|(1 + T)} \right]$$

$$\times \left( \pi|\tilde{N}|^{-2} + |\tilde{N}|^{-2} n_1(1 + T) \right)$$

(110)

This is essentially the last formula of this section. Presumably there could be a way of writing the sums in (110) in terms of known functions but that looks like a difficult task.

Summarizing, we have that $\Psi(\Omega_2(T, A')) = \lim_{s \to 0} (2G(s) + \frac{1}{s})$ and therefore the automorphic function (88), considering only the contribution from the solutions I and II of table [1], is given by $e^G = e^{4 \text{Re} \Psi(\Omega_2(T, A'))} |T + \bar{T} + A' \bar{A}'|$. A general Ansatz for finding the automorphic functions of coset manifolds of the form $SU(1, n)/SU(n) \times U(1)$ can be deduced from the above discussion. Indeed, the recursive use of the formulas (88) is not difficult to implement and the rest of the equations are also easy to generalize. The inclusion of more moduli $A_i$ into our formulas would only increase the dimension of the matrix $\Omega_2$ but its general structure remains the same.

$$\Omega'_n = \begin{pmatrix}
T & \frac{1}{\sqrt{2}} i(1 - T) & \frac{1}{\sqrt{2}} iA' & \frac{1}{\sqrt{2}} iA' & \cdots & \frac{1}{\sqrt{2}} iA'_n & \frac{1}{\sqrt{2}} A'_n \\
\frac{1}{\sqrt{2}} i(1 - T) & 1 & 0 & \cdots & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} iA' & 0 & 1 & 0 & \cdots & 0 & 0 \\
\frac{1}{\sqrt{2}} iA' & 0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{\sqrt{2}} A'_n & 0 & \cdots & \cdots & \cdots & 1 & 0
\end{pmatrix}$$

(111)
So the \( \zeta \)-function to be calculated is the Mellin transformation

\[
\zeta_{\text{gen}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \, \tau^{s-1} \left[ \Theta_{\Omega_n}(\tau) - 1 \right].
\] (112)

It is also easy to see that the positions of the poles of the Fourier expansion of \( \Gamma(s)\zeta_{\text{gen}}(s) \) are affected; nonetheless, one can see that there is always a pole at the position \( s = 0 \) (due to the particular points \( n_1 \neq 0, \vec{N} = 0 \) for example). Once this pole is subtracted we obtain a regular expression for the automorphic function. The final form for a general coset will depend on the recursion formulas for the Bessel functions, but the structure of the solution is similar in form to (106).

### 6.2 Second Contribution to the Automorphic Superpotential and Final Expression.

We saw in the previous section that the \( SU(1,n) \)-orbits described by the constraint (90) can be divided in three types given in table 1. However, we have only considered the solutions provided by the cases I and II. We will now see what happens to the extra piece, coming from the solution III. In short we just find that the Riemann theta function (92) gets extra terms and gets promoted to a \( \theta \)-function whose general expression is given by the definition (44).

The solution to constraint, following the procedure explained after the equation (91), is given by

\[
\begin{align*}
p_0 &= 2 \left[ (n_1)^2 + (n_2)^2 + (n_3)^2 + (n_4)^2 \right] + 2 \left[ n_1 + n_2 + n_3 + n_4 \right] + 2 \\
p_1 &= 2 \left[ - (n_1)^2 + (n_2)^2 + (n_3)^2 + (n_4)^2 \right] + 2 \left[ - n_1 + n_2 + n_3 + n_4 \right] + 1 \\
q_1 &= 4 n_1 n_2 + 2(n_1 + n_2) + 1 \\
q_2 &= 4 n_1 n_4 + 2(n_1 + n_4) + 1 \\
p_2 &= 4 n_1 n_3 + 2(n_1 + n_3) + 1
\end{align*}
\] (113)

following the same steps as before, we find that the theta function involved is

\[
\Theta'[\Omega_2(t,A)] = \sum_{\vec{n} \in \mathbb{Z}^4} e^{-2\pi\tau(\vec{n} \cdot \Omega_2 \cdot \vec{n} + 2\vec{n} \cdot \vec{Z} + \vec{B})},
\] (114)

where \( \Omega_2 \) is the same as in (93).

Here the product \( \cdot \) is taken with an Euclidean metric and we have introduced

\[
\vec{Z} \equiv \begin{bmatrix} 1 - (1 - i)t + (1 + i)A \\ 1 + (1 + i)t \\ 1 + t + iA \\ 1 + t + A \end{bmatrix} \quad \text{and} \quad \vec{B} \equiv 2 + (1 + i)(t + A)
\] (115)

The \( \zeta \)-function, in the lines seen many times before here can be expanded in a Fourier series. This is possible because of the properties (42) and (43). This Fourier expansion
of (114) can be found using the results from reference [39]. There they use the Poisson summation (57), generalized as follows

$$\sum_{\vec{n} \in \mathbb{Z}^d} \exp \left( -\frac{1}{2} \vec{n} \cdot \vec{A} \cdot \vec{n} + \vec{b} \cdot \vec{n} \right) = \frac{(2\pi)^{d/2}}{\sqrt{\det \vec{A}}} \sum_{\vec{m} \in \mathbb{Z}^d} \exp \left( \frac{1}{2} \vec{b} \cdot \vec{m} + \vec{b} \cdot \vec{m} \right) (116)$$

where \(d\) is the dimensionality of the complex-valued matrix \(\vec{A}\).

The \(\zeta\)-function \(\sum_{\vec{n}} \Omega(\vec{n}, t, A)^{-s}\) that we defined above is promoted in this case, by the means of a Poisson summation, to \(\sum_{\vec{n}} \Omega(\vec{n} + \vec{c}, t, A) + B^{-s}\) where \(\vec{c} \in \mathbb{C}^d\) and \(B\) is a term which does not contribute to the sum as in (115). Formally one can use (116) to obtain, using the formula provided by Elizalde in reference [39],

$$\zeta_{\Theta'}(s) = \frac{B^{2-s} \Gamma(s - 2)}{\sqrt{\det \Omega^2(s)}} + \frac{2^{s/2+1} \pi s - 2 B^{-s/2+1}}{\sqrt{\det \Omega^2(s)}}$$

$$+ (2\pi)^{s/2+1} \sum_{\vec{n} \in \mathbb{Z}^d} \cos(2\vec{n} \cdot \vec{Z}) (\vec{n} \cdot \Omega^2 \cdot \vec{n})^{s/2-1} K_{s-2} \left( 2 \sqrt{\pi B \vec{n} \cdot \Omega^2 \cdot \vec{n}} \right) (117)$$

in which \(\mathbb{Z}^d_+\) indicates that only half of the integers contribute to the sum, for example those with positive first entry will contribute and not those with a negative one.

The last expression is quite formal; indeed, we could have written something similar for the case we studied rigorously in the previous section. It is reassuring to see the same structure that we found before for the automorphic function emerging here as well.

The transformation rule for Jacobi’s \(\theta\)-function

$$\sum_{n \in \mathbb{Z}} e^{\pi i n^2 T + 2\pi i n v} = \sqrt{\frac{T}{i}} \sum_{n \in \mathbb{Z}} e^{\pi i T (n+v)^2} (118)$$

allow us to repeat the same steps of the previous calculation and perform a Fourier expansion in the same lines that we have already tried. Hence, we split the sum –and the \(\zeta\)-function– in (114) as

$$\sum_{(n_1, n_2, n_3, n_4) \neq (0,0,0,0)} = \sum_{n_1 = 0, \vec{N} \neq \vec{0}} + \sum_{n_1 \neq 0, \vec{N} \neq \vec{0}} + \sum_{n_1 \neq 0, \vec{N} = \vec{0}}. (119)$$

Using the same kind of manipulation of the previous section for each of the components of the \(\zeta\)-function \(\zeta_{\Theta'}(s) \equiv \zeta_I + \zeta_{II} + \zeta_{III}\) (labeled according to the sums in [119]), we obtain

$$\zeta_I = \sum_{\vec{N} \neq \vec{0}} \frac{\exp[\pi i (\sum_{j=2}^{4} n_j Z_j)]}{2^{s-1} \Gamma(s)} |\vec{N}|^{s-3/2}$$

$$\times \left( B - \frac{1}{4} \sum_{j=2}^{4} Z_j^2 \right)^{-s/2+3/4} K_{s-3/2} \left( 2\pi |\vec{N}| \left[ B - \frac{1}{4} \sum_{j=2}^{4} Z_j^2 \right]^{1/2} \right); (120)$$

where we have obviously used the notation established in (113) duly transformed from \(B(t, A)\) to \(B(T, A')\) and from \(Z_i(t, A)\) to \(Z_i(T, A')\).
As for the second and third parts we find

\[ \zeta_{II} = \sum_{n_1 \neq 0, \bar{N} \neq \bar{0}} \frac{\exp[\pi i \sum_{j=2}^{4} n_j (n_1 \rho_j + Z_j)]}{2^{s-1} \Gamma(s)} |\bar{N}|^{s-3/2} \gamma_{n_1}^{s/2+3/4} K_{s-3/2} \left( 2\pi |\bar{N}| \sqrt{\gamma_{n_1}} \right) \]  

(121)

and

\[ \zeta_{III} = 2 \sum_{n_1 \neq 0} \frac{e^{\pi n_1 Z_1/T}}{(2T)^s \Gamma(s)} |n_1|^{-s/2} \lambda^{-s/2+1/4} K_{s-1/2} \left( 2|n_1| \pi \sqrt{\lambda} \right) \]  

(122)

where \( \rho_j \equiv (\Omega_2)_{1j} \) for \( j = 2, 3, 4 \) and

\[ \gamma_{n_1} = (n_1)^2 T + B - \frac{1}{4} \sum_{j=2,3,4} (n_1 \rho_j + Z_j), \]

\[ \lambda = \frac{4TB + Z_1^2}{4T^2}. \]  

(123)

Here it is possible to take the limit \( s \to 0 \) of \( \left( \Gamma(s) \zeta_{s'}(s) \right) \) and use the recurrence relations for the Bessel functions to provide a final expression in terms of \( T \) and \( A' \). Each of the functions \( \Gamma(s) \zeta_1, \Gamma(s) \zeta_{II} \) and \( \Gamma(s) \zeta_{III} \) is regular in that limit.

Putting together the expressions (111) + (120) + (121) + (122) + c.c. at \( s = 0 \), we obtain the final expression for the automorphic function that we are trying to find. We obtain \( e^S = e^\delta \Re \Psi(\Omega_2(T,A')) \) \( |T + \bar{T} + A' \bar{A}'| \) with \( \Psi(\Omega_2(T,A')) \) given in this case by

\[ \Psi(\Omega_2(T,A')) = C_0 + \frac{\pi^2}{45} \left( \frac{1 + T}{2} \right)^3 + \left( \frac{1 + T}{2} \right)^{3/2} \sum_{0 \neq n_1 \in \mathbb{Z}, \bar{N} \neq \bar{0}} \exp \left[ 2\pi n_1 \left( \sum_{j=2}^{4} n_j \rho_j \right) \right] \]

\[ \times |\bar{N}|^{-3/2} n_1^{3/2} K_{-3/2} \left( \pi n_1 |\bar{N}| (1 + T) \right) \]

+ \[ \sum_{\bar{N} \neq \bar{0}} \exp \left[ \pi i \left( \sum_{j=2}^{4} n_j Z_j \right) \right] |\bar{N}|^{-3/2} \]

\[ \times \left( B - \frac{1}{4} \sum_{j=2}^{4} Z_j^2 \right)^{3/4} K_{-3/2} \left( 2\pi |\bar{N}| \left[ B - \frac{1}{4} \sum_{j=2}^{4} Z_j^2 \right]^{1/2} \right) \]

+ \[ \sum_{n_1 \neq 0, \bar{N} \neq \bar{0}} \exp \left[ \pi i \sum_{j=2}^{4} n_j (n_1 \rho_j + Z_j) \right] |\bar{N}|^{-3/2} \gamma_{n_1}^{3/4} K_{-3/2} \left( 2\pi |\bar{N}| \sqrt{\gamma_{n_1}} \right) \]

+ \[ \sum_{n_1 \neq 0} \exp \left[ \pi n_1 i Z_1/T \right] |n_1|^{-1/2} \lambda^{1/4} K_{-1/2} \left( 2|n_1| \pi \sqrt{\lambda} \right). \]  

(124)
$K_{3/2}(x)$ can be expressed in terms of the exponential functions and it is also possible to find an asymptotic form for this formula, for both the $x \to 0$ and the $x \gg 15/8$ limits. For instance, taking the first of these two limits renders

$$
\Psi(\Omega_{2}(T, A'))_{\text{asymp}} = C_0 + \frac{\pi^2}{45} \left( \frac{1+T}{2} \right) + \frac{1}{2} \sum_{n \neq 0} \frac{1}{|n|} e^{i \pi n Z_1 / T + 2 \pi i \sqrt{\lambda}}
$$

$$
\left( \frac{1+T}{2} \right) \sum_{0 \neq n_1 \in \mathbb{Z}, \bar{N} \neq 0} \frac{n_1}{|\bar{N}|^2} e^{2 \pi i n_1 \bar{N} \cdot \bar{\rho} + \pi i |\bar{N}|(1+T)} + 
$$

$$
\frac{(4B - |\bar{Z}|^2)^{1/2}}{4} \sum_{\bar{N} \neq 0} \frac{1}{|\bar{N}|^2} e^{\pi i \bar{N} \cdot \bar{Z} + \pi i |\bar{N}| \sqrt{4B - |\bar{Z}|^2}} + 
$$

$$
\sum_{0 \neq n_1 \in \mathbb{Z}, \bar{N} \neq 0} \frac{\sqrt{m_1}}{\sqrt{2} |\bar{N}|^2} e^{\pi i n_1 (\bar{N} \cdot \bar{Z} + \bar{N} \cdot \bar{\rho}) + 2 \pi i |\bar{N}| \sqrt{\gamma_n}} ;
$$

where we have used the convention $\bar{\rho} = \rho_{j=2,3,4}$ and $\bar{Z} = Z_{j=2,3,4}$. Here we can note that the truncation $\bar{N} = (m, 0, 0)$ and $A' = 0$ does not render the results of the previous section, unless it is implemented before the integrations and re-summations are carried out. Otherwise, one is just visiting a particular point, but not a special one, of the function $\Psi(\Omega_{2}(T, A'))$.

7 The Action of the Symplectic Group.

There is some similarity between the matrices $\Omega(T, A_i)$ of the previous section and period matrices of compact Riemann surfaces [3, 4, 5]. For instance, it has been noticed that manifolds with Special Kähler geometry (like the moduli space of Calabi-Yau manifolds) give rise to matrices which transform in the same way as period matrices. Here we will see another example of such relation, the profound meaning of which is yet to be clarified.

The moduli of Calabi-Yau manifolds are of two different classes, those parametrizing the deformations of the Complex and those parametrizing the deformations the Kähler structures. The dimensionality of each of these pieces is given by the elements of the Hodge diamond which contains the information about the $(2,1)$ and $(1,1)$-forms that parametrize the deformations.

The holomorphic three-form $\Psi$ that describes the variations to the complex structure can be expanded in terms of a canonical homology basis of $H_3(M, \mathbb{Z})$, where $M$ is the Calabi-Yau manifold in which the compactification of the string is taking place. If we denote this basis by $(\mathfrak{A}^a, \mathfrak{B}^b)$, with $a, b = 0, 1 \ldots 21$, and the dual cohomology basis by $(\alpha_a, \beta^b)$, so we have the symplectic relations

$$
\int_{\mathfrak{A}^a} \alpha_a = \oint_{\mathcal{M}} \alpha_a \wedge \beta^b = \delta^b_a \quad \int_{\mathfrak{B}^b} \beta^b = \int_{\mathcal{M}} \beta^b \wedge \alpha_a = -\delta^b_a.
$$

The periods of $\Psi$ are given by $z^a \equiv \int_{\mathfrak{A}^a} \Psi$ and $\partial_a \mathcal{F} = \mathcal{F}_a \equiv \int_{\mathfrak{B}^a} \Psi$. Where we recognize the moduli and prepotential of special Kähler geometry.
A symplectic transformation over the basis \((\alpha_a, \beta_b)\) leads to the transformation rule
\[
\begin{pmatrix}
\tilde{z}_a \\
\tilde{\mathcal{F}}_a
\end{pmatrix} = \begin{pmatrix} U & V \\
W & Z
\end{pmatrix}\begin{pmatrix} z_a \\
\mathcal{F}_a
\end{pmatrix}
\] (127)
for the symplectic vector.

The symplectic group has a natural action over the periods matrix \(\Omega (t, A_i)\) of the previous sections (to be precise a period matrix of a compact Riemann surface has dimension \(g \times 2g\) usually of the form \(\Pi = (1, \Omega)\), they also obey the Riemann bilinear conditions, which are equivalent to being in the Siegel upper half space \(\mathcal{H}_g\) that we discussed earlier). Indeed, a symplectic transformation on the basis of \(H_3(\mathcal{M}, \mathbb{Z})\), induces a transformation similar to the transformation rule for period matrices of compact Riemann surfaces, i.e.
\[
\tilde{\Omega} = (U \Omega + V) (W \Omega + Z)^{-1}.
\] (128)
This is also very similar to the transformation rules that one finds in toroidal and orbifold compactifications, see \([1, 1]\) for examples. Since the duality group can be embedded into the symplectic group then \((128)\) represents the transformation rule of \(\Omega (t, A_i)\) under T-duality.

There are also some more striking similarities between period matrices of the three-form of Calabi-Yau manifolds and period matrices of Riemann surfaces. These are given by the appearance of the Theta functions that we came across during the computations of the automorphic function. In fact Riemann’s theta functions, as defined before in \((40)\), are defined within the context of Riemann surfaces.

In the context of a compact Riemann surface \(C\), the \(2g\) column-vectors of the period matrix \(\Pi = (1, \Omega)\) generate a lattice \(\Lambda\) in \(\mathbb{C}^g\). The quotient space \(\mathcal{A} \equiv \mathbb{C}^g/\Lambda = \mathbb{C}^g/(\mathbb{Z}^g \Omega + \mathbb{Z}^g)\) defines a complex torus. Such torus is called the Jacobian variety of \(C\), denoted \(J(C)\). Moreover, since the period matrix satisfies,
\[
\Pi \begin{pmatrix} 0 & 1 \\
-1 & 0
\end{pmatrix} \Pi^t = 0
\] (129)
the Jacobian variety \(\mathcal{A}\) is in fact a principally polarized Abelian variety (rigorously, a principally polarized Abelian variety is a pair consisting of a complex torus \(\mathcal{A}\) together with the Chern class \(\chi\) of an ample line bundle on \(\mathcal{A}\) such that \(1/g! \int_\mathcal{A} \chi = 1\), see references \([12, 22, 4]\).

A Divisor in \(C\) is a linear combination of points, i.e. \(D \equiv \sum_i n_i p_i\) for \(p_i \in C\), \(n_i \in \mathbb{Z}\). Define a meromorphic function \(\phi\) in \(C\); then, in a local holomorphic coordinate \(z\), \(h(z)\) defined by \(\phi = h(z)\) is a meromorphic function. If \(P \in C\) corresponds to the origin of the \(z\)-complex plane, we can write \(h(z) = z^\mu g(z)\). We define the order of \(\phi\) at \(P\) as \(\mu_P(\phi) = \mu\). The divisor \((\phi)\) associated with the meromorphic function \(\phi\) is defined as \((\phi) = \sum_{P \in C} \mu_P(\phi) [P]\). Due to the periodicity conditions of Riemann’s theta function, defined by \((12)\) and \((13)\), over the lattice \(\Lambda\), the divisor \((\theta)\) induces a divisor on \(\mathcal{A} = \mathbb{C}^g/\Lambda\).

Two Abelian varieties are isomorphic if their period matrices are related by the symplectic transformation \((128)\). The moduli space of the complex torus is therefore identified as the quotient space \(\mathbb{A}_g = \mathcal{H}_g/Sp(2g, \mathbb{Z})\). Moreover, duality transformations are induced by a particular subgroup of \(Sp(2g, \mathbb{Z})\), that given by the embedding of \(SU(1, n)\) in the symplectic group as seen in \([43]\).
The action of the discrete group $Sp(2g, \mathbb{Z})$ on $A_g$ is holomorphic and well defined. Thus $A_g$ is endowed with the structure of a complex analytic space (a complex manifold away from some points fixed under the action of subgroups of $Sp(2g, \mathbb{Z})$). One can define for this space a projective embedding $A_g \to \mathbb{P}^N \mathbb{C}$ for some $N \in \mathbb{Z}$. To define this embedding one needs modular forms, in this case the so-called Siegel modular forms, i.e. functions of $\Omega$ which transform as

$$f \left[ (U\Omega + V) (W\Omega + Z)^{-1} \right] = \det((W\Omega + Z)^{-1})^k f(\Omega)$$

where $k \in \mathbb{Z}$ and

$$\begin{pmatrix} U & V \\ W & Z \end{pmatrix} \in \Gamma \subset Sp(2g, \mathbb{Z}).$$

Riemann’s theta function, as defined in (40), for example, obeys the relation

$$\theta(0, -\Omega^{-1})^2 = \det(-i\Omega) \theta(0, \Omega)^2.$$ (131)

Hence, $\theta(0, -\Omega^{-1})^2$ is a modular form of weight 1. In the case $g = 1$ that we addressed when we studied the space $SU(1, 1)/U(1)$ the automorphic function could be given in terms of a well knows modular function (the Dedekind eta function) obtained from the Mellin transformation of a theta function (via the Kronecker Limit formula). However, showing that the automorphic function, in the general case, is a Siegel modular function seems like a formidable task, at least from the explicit expression (124).

Nevertheless, the integral from which we originate our results, namely (28), is invariant under duality transformations. The integral is regular up to a pole that we subtract, independent of the moduli and, therefore, the final result should keep these invariances. In this sense one should be able to express the resulting function (124), in terms of Siegel modular forms. The correct functional dependency is still work in progress. Siegel forms are presented in [44]. It is interesting to note that the threshold corrections to gauge couplings in $(0, 2)$ compactifications have been constructed in terms of the these modular forms [19]. This type of compactification arise with the introduction of Wilson lines.

Another possible use of regularization procedure is the calculation of integrals like those appearing in [19], the integral proposed in this reference is very similar to the integrals that we regularized here, namely

$$\Delta = \int \frac{d\tau}{\tau_2} \sum_{k_1, k_2 \in \mathbb{Z}} Z^{4d}(\tau, T, \mathcal{A}, k_1, k_2) \mathcal{C}_a(\tau, k_1, k_2)$$ (132)

where $\mathcal{C}_a$ is a holomorphic moduli-independent function and

$$Z^{4d} = \sum_{n_1, n_2, m_1, m_2 \in \mathbb{Z}} \exp \left[ \frac{-\pi \tau_2}{\Im(T - i\sqrt{3}/8|\mathcal{A}|^2) \Im(U)} \left| TU n_2 + T n_1 - U (m_1 + \mathcal{A} k_1 - 1/2 \mathcal{A} k_2) + m_2 - 1/2 \mathcal{A} k_1 |^2 \\
+ 2\pi i \tau (m_1 n_1 + m_2 n_2) \right] \right].$$ (133)

Although not exactly the same problem, one can easily see how to use what we have learned in this paper to compute this type of integral.
Going back a bit in our discussion one can conclude that there is a correspondence between the moduli space of a Calabi-Yau manifold and the moduli space of a Complex Abelian variety. Moving on the moduli space of the Calabi-Yau would correspond to moving on the moduli space of the associated complex torus (associated through the period matrix $\Omega$). Exactly how far this similarity between these two spaces can go is difficult to establish. The structure of $A_g$ is still the matter of much research by mathematicians, this is known as the “Schottky problem” (see the references given above in this subsection).

The map $(\tilde{z}_a, \tilde{F}_a) \rightarrow \Omega(\tilde{z}_a, \tilde{F}_a)$ of (127) and (128), provides an isomorphism (at least locally)

$$\frac{SU(1,n)}{U(1) \times SU(n)} \rightarrow \mathbb{C}^g \frac{\Omega + \mathbb{Z}^g}{\mathbb{Z}^g}$$

where $2n = g = 2b_{21}$ ($n$, the number of complex moduli, $g$ the genus of the associated Riemann surface and $b_{21}$ the dimension of the homology group respectively). Duality transformations in the compactified string correspond to usual $SL(2, \mathbb{Z})$ transformations of the torus, only generalized to the torus of given by the previous expression. The automorphic functions therefore are nothing but generalizations of the usual elliptic functions usually associated with the torus. However, the role of this underlying compact Riemann surface is not clear at the moment.

## 8 Conclusions

The main result of this paper is the construction of automorphic functions of manifolds with special Kähler geometry. This kind of geometry appears in manifolds which are the moduli space of Calabi-Yau spaces. Our method starts with the duality-invariant formula and using $\zeta$-functions we first obtain a well known result for the simple case of $SU(1,1)/U(1)$, then we proceed to solve the case of $SU(1,2)/[SU(2) \times U(1)]$.

It is also interesting to note that the construction of such solution is one comes across Riemann theta functions which have a period matrix. Given this plus the action of the symplectic group over this functions (which can be seen as duality-transformations) one can interpret this matrix as the period matrix of some Abelian Variety. Unfortunately, at this point, one can only speculate about the nature of this underlying variety. This situations is somehow similar to the Seiberg-Witten solution of $N = 2$ supergravity and some research in this line could provide some interesting insights into the structure of Calabi-Yau compactifications of string theory. Of course this connection has been noticed before by Candelas but we are able to provide particular examples of such varieties. Generally speaking our method can be generalized to other manifolds to which the formula applies. In many ways therefore one should be able to reproduce the results of $SO(2,2n)/[SO(2) \times SO(2n)]$ coset manifolds.

### Acknowledgements.

I have benefited from many useful discussions with Dr. S. Thomas and Dr. José Figueroa-O’Farrill for which am deeply grateful; they not only gave me useful input but also read the manuscript. I should acknowledge talks and encouragement from the people of the Theory Group at Queen Mary & Westfield College, London, as well.
References

[1] A. Strominger. Special Geometry. *Commun. Math. Phys.*, 133:163, 1990.

[2] B. De Wit and A. Van Proyen. Special Geometry and Symplectic Transformations. hepth/9510186, 1995.

[3] P. Candelas and X. de Ossa. A Pair of Calabi-Yau Manifolds as an Exactly Soluble Superconformal Theory. *Nucl. Phys. B*, 359:21, 1991.

[4] P. Candelas and X. de la Ossa. Moduli Space of Calabi-Yau Manifolds. *Nucl. Phys. B*, 355:455, 1991.

[5] P. Candelas, P.S. Green, and T. Hubsch. Rolling Among Calabi-Yau Vacua. *Nucl. Phys. B*, 330:49, 1990.

[6] B. Zumino. Supersymmetry and Kähler Manifolds. *Phys. Lett. B*, 87:203, 1979.

[7] N. Seiberg. Observations on the Moduli Space of Superconformal Field Theories. *Nucl. Phys. B*, 303:286, 1988.

[8] S. Cecotti, S. Ferrara, and L. Girardello. Geometry of the Type II Superstrings and the Moduli of the Superconformal Field Theories. *Int. J. of Mod. Phys. A*, 4(10):2475, 1989.

[9] S. Ferrara, C. Kounnas, D. Lüst, and F. Zwirner. Duality-Invariant Partition Functions and Automorphic Superpotentials for (2,2) String Compactifications. *Nucl. Phys. B*, 365:431, 1991.

[10] S. Ferrara, P. Fré, and P. Soriani. On the Moduli Space of the $T^6/Z_3$ Orbifold and Its Modular Group. *Class. Quant. Grav.*, 9:1649, 1992.

[11] A. Giveon, M. Porrati, and E. Ravinovici. Target Space Duality in String Theory. *Physics Reports*, 244:72, 1994.

[12] L.J. Dixon, V.S. Kaplunovsky, and J. Louis. On Effective Field Theories Describing (2,2) Vacua of the Heterotic String. *Nucl. Phys. B*, 329:27, 1990.

[13] L.J. Dixon, V.S. Kaplunovsky, and J. Louis. Moduli-Dependence of String Loop Corrections to Gauge Coupling Constants. *Nucl. Phys. B*, 355:649, 1991.

[14] I. Antoniadis, E. Gava, K.S. Narain, and T.R. Taylor. Superstring Threshold Corrections to Yukawa Couplings. *Nucl. Phys. B*, 407:706, 1993.

[15] P. Mayr and S. Stieberger. Threshold Corrections to Gauge Couplings in Orbifold Compactifications. *Nucl. Phys. B*, 407:725, 1993.

[16] V. Kaplunovsky and J. Louis. On Gauge Couplings in String Theory. *Nucl. Phys. B*, 444:191, 1995.

[17] V. Kaplunovsky and J. Louis. Field Dependent Gauge Couplings in Locally Supersymmetric Effective Quantum Field Theories. *Nucl. Phys. B*, 422:57, 1994.

[18] J.P. Derendinger, S. Ferrara, C. Kounnas, and F. Zwirner. On Loop Corrections to String Effective Field Theories: Field-Independent Gauge Couplings and $\sigma$-Model Anomalies. *Nucl. Phys. B*, 372:145, 1992.
19. P. Mayr and S. Stieberger. Moduli Dependence of One-Loop Gauge Couplings in (0,2) Compactifications. *Phys. Lett. B*, 355:107, 1995.
20. P. Mayr and S. Stieberger. Low-Energy Properties of (0,2) Compactifications. hep-th/9412144, 1994.
21. N.A. Obers and B. Pioline. Eisenstein Series and String Thresholds. 1999.
22. E. Kiritsis and B. Pioline. On $R^4$ Threshold Corrections in IIB String Theory and $(p, q)$ String Instantons. *Nucl. Phys. B*, 508:509, 1997.
23. A. Gregori, E. Kiritsis, C. Kounnas, N.A. Obers, P.M. Petropoulos, and B. Pioline. $R^2$ Corrections and Non-perturbative Dualities of $N = 4$ Strings Ground States. hep-th/9708062.
24. M.B. Green and M. Gutperle. Effects of D-instantons. *Nucl. Phys. B*, 498:195, 1997.
25. W. Sabra. Space-Time Duality and $SU(n, 1)/SU(n) \times U(1)$ Cosets of Orbifold Compactifications. *Mod. Phys. Lett. A*, 11:1497, 1996.
26. E. Cremmer, S. Ferrara, L. Girardello, and A. Van Proyen. Yang-Mills Theories with Local Supersymmetry. *Nucl. Phys. B*, 212:413, 1983.
27. D. Bailin and A. Love. Supersymmetric Gauge Field Theory and String Theory. Institute of Physics Publishing, 1st. edition, 1994.
28. H. Ooguri and C. Vafa. Geometry of $N = 2$ Strings. *Nucl. Phys. B*, 361:469, 1991.
29. G. Lopes-Cardoso, D. Lüst, and T. Mohaupt. Threshold Corrections and Symmetry Enhancement in String Compactifications. *Nucl. Phys. B*, 450:115, 1995.
30. J.R. Quine, S.H. Heydari, and R. Song. Zeta Regularized Products. *Trans. Am. Math. Soc.*, 338(1):213, 1993.
31. P. Ramond. *Field Theory: a Modern Primer*. Addison-Wesley, 2nd. edition, 1990.
32. M. Waldschmidt, P. Moussa, J.-M. Luck, and C. Itzykson. *From Number Theory to Physics*. Springer-Verlag, 1992.
33. L. E. Dickson. *History of the Theory of Numbers. Diophantine Analysis*. Number 256. Carnegie Institution of Washington, 1920.
34. H. Lange and Ch. Birkenhake. *Complex Abelian Varieties*. Springer-Verlag, 1992.
35. L.H. Ryder. *Quantum Field Theory*. Cambridge University Press., 1985.
36. S. Chowla and A. Selberg. On Epstein Zeta Function (I). *Proceedings of the National Academy of Sciences U.S.A.*, 35:371, 1949.
37. I.S. Gradshteyn and I.M. Ryzhik. *Table of Integrals, Series and Products*. Academic Press, 1980.
38. K. Chandrasekharan. *Elliptic Functions*. Springer-Verlag, 1985.
39. E. Elizalde. Multidimensional Extension of the Generalized Chowla-Selberg Formula. hep-th/9707257, July 1997.
40. S. Lange. *Elliptic Functions*. Springer-Verlag, 1987.
41. A. Giveon, E. Rabinovici, and G. Veneziano. Duality in String Background Space. *Nucl. Phys. B*, 322:167, 1989.
[42] E. Arbarello, M. Cornalba, P.A. Griffiths, and J. Harris. *Geometry of Algebraic Curves*. Springer-Verlag, 1985.

[43] W. Sabra, S. Thomas, and N. Vanegas. Symplectic Embeddings, the Prepotentials and Automorphic Functions of $SU(1, n)/(SU(n) \times U(1))$. hep-th/9608075.

[44] H.P. Nilles and S. Stieberger. String Unification, Universal One-Loop and Strongly Coupled Heterotic String Theory. *Nucl. Phys. B*, 499:3, 1997.