Asymptotic behavior of the multiplicative counterpart of the Harish-Chandra integral and the $S$-transform

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In this note, we study the asymptotic of spherical integrals, which are analytical extension in index of the normalized Schur polynomials for $\beta = 2$, and of Jack symmetric polynomials otherwise. Such integrals are the multiplicative counterparts of the Harish-Chandra-Itzykson-Zuber (HCIZ) integrals, whose asymptotic are given by the so-called $R$-transform when one of the matrix is of rank one. We argue by a saddle-point analysis that a similar result holds for all $\beta > 0$ in the multiplicative case, where the asymptotic is governed by the logarithm of the $S$-transform. As a consequence of this result one can calculate the asymptotic behavior of complete homogeneous symmetric polynomials.

1 Introduction

1.1 Free probability transforms

Let $A_N$ (respectively $B_N$) be a random orthogonal/unitary/symplectic invariant matrices such that their empirical spectral distributions $\mu_{A_N}$ (resp. $\mu_{B_N}$) converge in the large $N$ limit towards a deterministic measure $\mu_A$ (resp. $\mu_B$) then it is well known from free probability that the empirical distribution of the free sum $C_N = A_N + GB_NG^*$, with $G$ taken uniformly distributed in the $N$-orthogonal/unitary/symplectic group, and $*$ denotes the conjugation operation, converges towards the free additive convolution of $\mu_A$ and $\mu_B$, denoted by $\mu_A \boxplus \mu_B$. The $R$-transform defined by:

$$\mathcal{R}_\mu(z) := \mathcal{G}_\mu^{-1}(z) - \frac{1}{z}$$

(1)

where $\mathcal{G}_\mu(z) := \int \frac{1}{z-x} \mu(dx)$ is the Stieltjes transform and $f^{-1}(.)$ denotes the functional inverse of the function $f(.)$, linearizes this convolution:

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$^*$we also assume that the minimum and maximum of the empirical spectral measure converge towards the edge of the limiting distribution
\[ R_{\mu_A \boxplus \mu_B}(z) = R_{\mu_A}(z) + R_{\mu_B}(z) \]  

Similarly, for two positive definite orthogonal/unitary/symplectic invariant matrices \( A_N \) and \( B_N \), the limiting spectral distribution of the free product \( C_N = \sqrt{A_N} G B_N G^* \sqrt{A_N} \) is given by the free multiplicative convolution \( \mu_A \boxplus \mu_B \) and we have:

\[ \tilde{S}_{\mu_A \boxplus \mu_B}(z) = \tilde{S}_{\mu_A}(z) \tilde{S}_{\mu_B}(z) \]  

where \( \tilde{S}(\cdot) \) is the (modified) \( S \)-transform defined by:

\[ \tilde{S}_\mu(z) = \frac{z}{z + 1} T_\mu^{(-1)}(z) \]

with \( T_\mu(z) := zG_\mu(z) - 1 \). In particular its logarithm plays the role of the \( R \)-transform since it linearizes the free multiplicative convolution.

1.2 Harish-Chandra-Itzykson-Zuber Integrals and free sum

The Harish-Chandra-Itzykson-Zuber (HCIZ in the following) integral [1] [2] [3] is defined in the RMT setting as:

\[ I^{(\beta)}(A_N, Z_N) := \int DG e^{\text{Tr} A_N G Z_N G^*} \]  

for \( \beta = 1, 2, 4 \) (5)

where the integral is over the orthogonal (respectively unitary, symplectic) group for \( \beta = 1 \) (respectively \( \beta = 2, 4 \)) and \( DG \) is the corresponding Haar measure normalized to unity \( \int DG = 1 \). Such integral has been proven to be of high interest in lattice gauge theory [4] [5] [2], algebraic geometry [6] and more generally in problems arising in random matrix theory [7] [8] [9] to cite few recent results. We refer the reader to [10] for other results concerning this integral. One can consistently extend the definition of the HCIZ integral to all \( \beta > 0 \) via the theory of rational Dunkl operator associated to generic root system [11], also known as the Bessel hypergeometric function in such setting. We list in the following interesting properties of the HCIZ integral:

From its definition (5), one may notice that the HCIZ integrals only depends on the eigenvalues of its entries. In the following, we denote by \( a = (a_1, \ldots, a_N) \) the vector of eigenvalues of the matrix \( A \), and \( q = \text{Diag}(a_1, \ldots, a_N) \) the corresponding diagonal matrix, in particular the identity matrix is denoted by \( 1 \). For a given vector of eigenvalues \( a \), we see the HCIZ integral as a function of the vector \( z \), and we denote it by \( I^{(\beta)}_a(z) \). It is also clear from its definition that the integral is invariant under the permutation of the vectors \( a \) and \( z \) and does not depend on the order of the entries of each vector. It also satisfies:

- For two diagonal matrices \( a \) and \( b \) and a matrix \( G \), if we denote by \( c(G) := \text{Eigen}(a + G b G^*) \), we have by Haar property:

\[ \int DG I^{(\beta)}_{c(G)}(z) = I^{(\beta)}_a(z) I^{(\beta)}_b(z) \]  

(6)

- The HCIZ integral is normalized so that:

\[ I^{(\beta)}_0(z) = 1 \]  

^2the usual \( S \)-transform is defined as \( \frac{1}{\tilde{S}} \)
For $\beta = 2$, the HCIZ integral admits a nice determinantal formula due to Itzykson and Zuber [2]:

$$I^{(2)}_a(z) = \left( \prod_{j=1}^{N-1} j! \right) \frac{\det \{ e^{a_j z_k} \}_{j,k=1}^N}{\text{Vand}(a) \text{Vand}(z)}$$

(8)

where Vand$(\cdot)$ denotes the Vandermonde product.

The link between the HCIZ integral and the $R$-transform of the previous paragraph is given by the following property, first derived by Parisi [12] and then proved rigorously by Guionnet and Maïda [13] (see [14] for an introduction):

If we denote by $I^{(\beta)}_a(z) := I^{(\beta)}_a(z, 0, \ldots, 0)$ the rank-one specialization of the HCIZ integral, and if the spectral distribution satisfies the conditions of the previous paragraph, then for $z$ small enough, we have:

$$\lim_{N \to \infty} \frac{2}{N \beta} \frac{d}{dz} \ln I^{(\beta)}_a \left( \frac{N \beta}{2} z \right) = R_{\mu_A}(z)$$

(9)

Note that Parisi and Guionnet and Maïda actually prove that this result can be extended for finite rank $k$ and arbitrary large $z$ in the following sense: if we denote by $H_{\mu_A}^R(\cdot)$ the unique function satisfying:

$$\frac{d}{dz} H_{\mu_A}^R(z) = \begin{cases} R_{\mu_A}(z) & \text{for } 0 \leq z \leq G_{\mu_A}(a_{\max}) \\ a_{\max} - \frac{1}{z} & \text{for } z \geq G_{\mu_A}(a_{\max}) \end{cases}$$

(10)

and with $H_{\mu_A}^R(0) = 0$. where $a_{\max}$ denotes the right edge of the limiting measure. Then we have:

$$\lim_{N \to \infty} \frac{2}{N \beta} \ln I^{(\beta)}_a \left( \frac{N \beta}{2} z_1, \ldots, \frac{N \beta}{2} z_k, 0, \ldots, 0 \right) = \sum_{i=1}^{k} H_{\mu_A}^R(z_i)$$

(11)

The goal of this note is to derive the multiplicative analogue of the theorem (9), which to the best knowledge of the authors, as not been yet established and is therefore not well known. In Section 2, we introduce the spherical functions on symmetric cones, which are the multiplicative counterparts of the HCIZ integral in the case $\beta = 1, 2, 4$ and then derived the asymptotic behavior of such functions by a saddle point analysis in this setting. In Section 3, we extend the derivation to all $\beta > 0$ using recent results concerning Macdonald polynomials and Heckman-Opdam hypergeometric functions.

2 Group integrals representations for $\beta = 1, 2, 4$ and asymptotic behavior

2.1 The spherical integral over symmetric cones

Unlike the HCIZ integral, its multiplicative counterpart has been far less studied in the random matrix community, exception made of [15] [16] [17] and references therein. We recall in this paragraph the construction of such function.

In the following the vector $a$ (and $b$) are assumed to be positive (that is for all $i$ we have $a_i > 0$). The idea is to replace in (6) and (7) the additive operation and its null element by the multiplicative operation and its null element. Namely we look a function $J_{a}^{(\beta)}(z)$ such that
\[
\int \mathcal{D} G J^{(\beta)}_{c(G)}(z) = J^{(\beta)}_a(z) J^{(\beta)}_b(z)
\]

It is well known from group theory, that if we take \( \beta = 2 \) and we specialize the vector \( z \) to be a partition, that is a vector of non-increasing non-negative integer, denoted by \( \lambda \), then the function \( J^{(2)}_a(\lambda) \) can be chosen to be the classical \textit{Schur polynomials} \cite{18} \( s_\lambda(a) \) normalized by \( s_\lambda(1, \ldots, 1) \). Following \cite{18}, the normalized Schur polynomials admits the following representation:

\[
\frac{s_\lambda(a)}{s_\lambda(1, \ldots, 1)} = \int \mathcal{D} U \Delta^\lambda U a U^*\]

where for a vector \( x \) (which is not necessarily a partition) \( \Delta^x(,.) \) is the \textit{multivariate power function} given by:

\[
\Delta^x(A) := (\det A_{(1)})^{x_1-x_2} \ldots (\det A_{(N-1)})^{x_{N-1}-x_N} (\det A)^{x_N}
\]

where \( A_{(j)} \) is the \( j \times j \) (top left) principal corner of the matrix \( A \).

Now for \( \beta = 1 \) (resp. \( \beta = 4 \)), with \( \lambda \) a partition, \( J^{(\beta)}_a(\lambda) \) can be chosen to be the normalized real (resp. quaternionic) \textit{zonal polynomial} \cite{18} which satisfies \cite{14} when one replaces the integral over the unitary group by an integral over the orthogonal (resp. symplectic) group. Since the multivariate power function \( \Delta^x(,.) \) can defined for a general complex vector \( x \), this leads us to the following natural definition for the so-called \textit{multiplicative spherical function}:

\[
J^{(\beta)}_a(z) := \int \mathcal{D} G \Delta^z(G a G^*)
\]

with \( \Delta^z(,.) \) defined in \cite{15}.

\textbf{Remark:} Using relationships between determinants of blocks of a matrix and of blocks of its inverse, one can show that \( J^{(\beta)}_a(,) \) has the following symmetry:

\[
J^{(\beta)}_a(z) := J^{(\beta)}_a(-\sigma(z))
\]

where \( \frac{1}{z} := (\frac{1}{x_1}, \ldots, \frac{1}{x_N}) \) and where \( \sigma(.) \) reverses the order of a vector (that is the permutation which exchange the \( i^\text{th} \) argument with the \( (N + 1 - i)^\text{th} \) argument). When \( z \) is a partition, this property allows us to define the corresponding symmetric (Laurent) polynomials for a \textit{signature}, that is a vector of non-decreasing integers (not necessarily positive).

\footnote{the eigenvalues of this product are the same as the ones of the free symmetric product}
Starting from (16), it is possible to show that $J_a^{(\beta)}(z)$ satisfies (12) by use of the QR-decomposition and the properties of the multivariate power function [19]. Unlike the HCIZ integral, this spherical function does not admit a double matrix integral representation. Note also, to have similar property concerning symmetry and Dunkl/Calogero-Moser Operator, it is customary to look at a shifted version in $z$ of this function, this is the so-called Heckman-Opdam hypergeometric function of Section 3.1.

2.2 Asymptotic behavior of the rank one spherical integral for $\beta = 1, 2, 4$

As in the additive case, we introduce the rank one specification as $J_a(z) := J_a(z, 0, \ldots, 0)^4$ which is given by

$$J_a^{(\beta)}(z) = \int DG \det [(G^G^*)^{(1)}]^{\beta}$$

(18)

By first projecting along the first row $x$ of the group matrix $G$ and then using the usual inverse Laplace representation of the constraint $x^*x = 1$, we get:

$$J_a^{(\beta)}(z) \propto \int_{\mathbb{R}^N} dx_2 2\pi i \int_{\gamma^2} \int_{\gamma^1} e^{t(1-x^*x)} (x^*a^a)^{\beta}$$

(19)

where $\gamma'$ is a constant such that the complex integral along the vertical line is convergent and may change from line to line and $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ for $\beta = 1, 2, 4$ respectively. The constant of proportionality can be deduced using (13) and is equal to $\frac{\Gamma(\frac{\beta}{\pi})}{\pi^\beta}$. Using the inverse Laplace representation of the power function for $a > 0$:

$$a^z = \frac{\Gamma(z+1)}{2\pi i} \int_{\gamma^1} \int_{\gamma^2} ds s^{z-1} e^{sa}$$

(20)

we arrive at:

$$J_a^{(\beta)}(z) = \frac{\Gamma(z+1)\Gamma\left(\frac{N\beta}{2}\right)}{\pi^\frac{N\beta}{2}} \frac{1}{2\pi i} \int_{\gamma^1} \int_{\gamma^2} dt \int_{\mathbb{R}^N} dx e^{-t x^* x} \frac{1}{2\pi i} \int_{\gamma^1} \int_{\gamma^2} ds s^{z-1} e^{s(x^*a^a)}$$

(21)

by the change of variable $s = e^{-p}$ and deformation of the Bromwich contour, we get:

$$J_a^{(\beta)}(z) = \frac{\Gamma(z+1)\Gamma\left(\frac{N\beta}{2}\right)}{\pi^\frac{N\beta}{2}} \frac{1}{2\pi i} \int_{\gamma^1} \int_{\gamma^2} dt \int_{\mathbb{R}^N} dx e^{-t x^* x} \frac{1}{2\pi i} \int_{\gamma^1} \int_{\gamma^2} dp e^{pz} e^{s(p^* a^a)}$$

(22)

$$J_a^{(\beta)}(z) = \frac{\Gamma(z+1)\Gamma\left(\frac{N\beta}{2}\right)}{\pi^\frac{N\beta}{2}} \frac{1}{2\pi i} \int_{\gamma^1} \int_{\gamma^2} dt \int_{\mathbb{R}^N} dx e^{-t x^* x} \left(\int_{\gamma^1} \int_{\gamma^2} dp \left(\int_{\mathbb{R}^N} dx e^{-x^* (1-e^{-p} a^a)} \right) e^{pz} \right)$$

(23)

the term in bracket is a Gaussian integral so that after the change of variable $p \to p - \ln t$ and factorizing by $t^{-N}$
\[ J^{(\beta)}_a(z) = \frac{\Gamma(z+1)}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dt \, e^{t-\frac{\beta z}{2}} = \frac{\Gamma(z+1)}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dp \prod_{i=1}^{\infty} (1-e^{-p\alpha_i})^{-\frac{\beta}{2}} e^{pz} \] (24)

and finally we arrive at:

\[ J^{(\beta)}_a(z) = \frac{\Gamma(1+z)\Gamma\left(\frac{N\beta}{2}\right)}{\Gamma\left(\frac{N\beta}{2} + z\right)} \mathcal{L}^{-1} \left[ \prod_{i=1}^{\infty} (1-e^{-p\alpha_i})^{-\frac{\beta}{2}} \right] (z) \] (25)

To establish the Guionnet-Maïda theorem, we make the usual rescaling \( z \rightarrow \frac{\beta N}{2} z \) and take the log derivative:

\[
\frac{2}{N\beta} \frac{d}{dz} \ln J^{(\beta)}_a\left(\frac{N\beta}{2} z\right) = \frac{2}{\beta} \frac{d}{dz} \ln \left( \frac{\Gamma\left(\frac{N\beta}{2}\right) \Gamma\left(1 + \frac{N\beta}{2} z\right)}{\Gamma\left(\frac{N\beta}{2} + \frac{N\beta}{2} z\right)} \right) + \frac{2}{N\beta} \frac{d}{dz} \ln \mathcal{L}^{-1} \left[ \prod_{i=1}^{\infty} (1-a_i e^{-p})^{-\frac{\beta}{2}} \right] \] (26)

Taking the \( N \) limit and by property of the digamma function \( \psi(z) := \frac{d}{dz} \ln \Gamma(z) \) at infinity \( \psi(z) \sim z \rightarrow \infty \ln z \), we have for the first term:

\[
\lim_{N \rightarrow \infty} \frac{2}{N\beta} \frac{d}{dz} \ln \left( \frac{\Gamma\left(\frac{N\beta}{2}\right) \Gamma\left(1 + \frac{N\beta}{2} z\right)}{\Gamma\left(\frac{N\beta}{2} + \frac{N\beta}{2} z\right)} \right) = \ln \frac{z}{z+1} \] (27)

while the second term is obtained by a saddle point method since:

\[
\mathcal{L}^{-1} \left[ \prod_{i=1}^{\infty} (1-a_i e^{-p})^{-\frac{\beta}{2}} \right] \left(\frac{N\beta}{2} z\right) = \int dp \, e^{\frac{\beta}{2} N \mathcal{H}(z,p)} \] (28)

with:

\[
\mathcal{H}(z,p) := zp - \int \mu_a(dx) \ln (1-x e^{-p}) \] (29)

from which we deduce that integral is dominated by the critical point \( p^* \) solution of:

\[
z = \int \mu_a(dx) \frac{x}{e^{p^*} - x} =: T_{\mu_a}(e^{p^*}) \] (30)

where the second equality is the definition of the \( T \)-transform. We then have:

\[ p^* = \ln T_{\mu_a}^{(-1)}(z) \] (31)

Since \( p^* \) is an critical point, by total derivative of \( \mathcal{H}(z,p) \) with respect to \( z \), we conclude using (4):

\[
\lim_{N \rightarrow \infty} \frac{2}{N\beta} \frac{d}{dz} \ln J^{(\beta)}_a\left(\frac{N\beta}{2} z\right) = \ln \tilde{S}_{\mu_A}(z) \quad \text{(for } \beta = 1, 2, 4\text{)} \] (32)
The above argument is only valid for positive $z$ small enough so that the limiting $T$-transform is invertible. For larger $z$, one need to work with the discrete $T$-transform, in this case we find that $p^* = \ln a_{\text{max}}$ where $a_{\text{max}}$ is the largest element of $a$. If we denote by $H^{S}_{\mu_A}(\cdot)$ the solution of:

$$
\frac{d}{dz} H^{S}_{\mu_A}(z) = \begin{cases} 
\ln S_{\mu_A}(z) & \text{for } 0 \leq z \leq T_{\mu_A}(a_{\text{max}}) \\
\ln a_{\text{max}} + \ln \left(\frac{z}{z+1}\right) & \text{for } z \geq T_{\mu_A}(a_{\text{max}})
\end{cases}
$$

with $H^{S}_{\mu_A}(0) = 0$, then we have:

$$
\lim_{N \to \infty} \frac{2}{N\beta} \ln J^{(\beta)}_{\alpha} \left( \frac{N\beta}{2} z, 0, \ldots, 0 \right) = H^{S}_{\mu_A}(z_i)
$$

Note that from [33] and the power series expansion of the $S$-transform [14], one can compute explicitly the first coefficients of the power series of $H^{S}_{\mu_A}(\cdot)$ near the origin, in terms of the first moments of the distribution $\mu_A$, which we denote by $m_k$. One has:

$$
H^{S}_{\mu_A}(z) = (\ln m_1) z + \left( \frac{m_2}{m_1^2} - 1 \right) \frac{z^2}{2} + \left( \frac{2m_3 m_1 - 3m_2^2}{m_1^4} + 1 \right) \frac{z^3}{6} + O(z^4)
$$

Remark: Using non-rigorous arguments that we expect to be valid in the large $N$ limit, we convinced ourselves that this result should also hold in the finite $k$ rank setting then one should have:

$$
\lim_{N \to \infty} \frac{2}{N\beta} \ln J^{(\beta)}_{\alpha} \left( \frac{N\beta}{2} z_1, \ldots, \frac{N\beta}{2} z_k, 0, \ldots, 0 \right) = \sum_{i=1}^{k} H^{S}_{\mu_A}(z_i)
$$

Note that the since spherical integral [16] is not permutation invariant on the elements of $z$, it matters which elements of $z$ are non-zero in our low-rank limit. As long as the non-zero elements are all below some finite position $n$ as $N$ goes to infinity, we expect our permutation invariant conjecture [36] to hold true.

3 Extension to all $\beta > 0$: Heckman-Opdam hypergeometric function

3.1 Heckman-Opdam and the spherical integral

The Harish-Chandra integral can be extended to all $\beta > 0$, by abstracting out its group integral representation thanks to the theory of rational Dunkl operators associated to root system of type $A_{N-1}$ [11] [20]. A similar construction can be performed in the multiplicative case for the so-called Heckman-Opdam hypergeometric function [21] [22] by use of the trigonometric Dunkl theory [11], such function is the permutation-invariant symmetric version (in its argument $z$) of the spherical integral of the previous section. We first give its group integral representation in the $\beta = 1, 2, 4$ and show how one can naturally extend it to all $\beta > 0$, without involving the theory of root systems.

For $\beta = 1, 2, 4$ the Heckman-Opdam is a shifted version of the spherical function of Section 2.1, namely:

$$
F_{\alpha}^{(\beta)}(z) := J_{-\alpha}^{(\beta)}(-z - \rho) \quad \text{for } \beta = 1, 2, 4
$$
\[ \rho_i := \frac{\beta}{2} (N - i) \quad \text{for } i = 1, \ldots, N \]  
(38)

This operation is equivalent to put an additional \( \frac{\beta}{2} \) term in the exponent of each determinant in (15), except for the last one.

**Remark:** Using the symmetry relation (17) and the permutation invariance of the Heckman-Opdam hypergeometric, this can also be written as:

\[ F_{\alpha}^{(\beta)} (z) := F_{\epsilon \alpha}^{(\beta)} (z + \sigma (\rho)) \]  
(39)

with \( \sigma (\rho) = \frac{\beta}{2} (i - 1) \).

In particular for \( \beta = 2 \), the Heckman-Opdam hypergeometric function admits a multiplicative counterpart of the Itzykson-Zuber determinantal formula (8), known as the Gelfand-Naimark formula:

\[
\mathcal{F}_{\alpha}^{(2)} (z) = \prod_{j=1}^{N-1} j! \frac{\det [e^{a_{j} z_{k}}]_{j,k=1}^{N}}{\text{Vand} (e^{-a}) \text{Vand} (-z)} = \prod_{j=1}^{N-1} j! \frac{\det (e^{a})^{N-1} \det [e^{a_{j} z_{k}}]_{j,k=1}^{N}}{\text{Vand} (e^{a}) \text{Vand} (z)}
\]  
(40)

where \( \text{Vand}(.) \) is the usual Vandermonde determinant.

The HCIZ integral can be obtained as a limit of the Heckman-Opdam hypergeometric function \( F_{\alpha}^{(2)} (z) \) (41) namely we have:

\[
\mathcal{I}_{\alpha}^{(\beta)} (z) = \lim_{\epsilon \to 0^+} \mathcal{F}_{\epsilon \alpha}^{(\beta)} (\epsilon^{-1} z)
\]  
(41)

### 3.2 The spherical integral for arbitrary beta

The spherical integral and the Heckman-Opdam function are defined using the Haar measure on the orthogonal/unitary/simplectic group. We will show how to generalize their definition to arbitrary beta. The goal is to write the joint distribution of eigenvalues of all principal minors of randomly rotated fixed matrix. Consider first the principal minor \( \mathbf{M} \) of size \( N - 1 \) of a rotated matrix \( \mathbf{A} \) of size \( N \), its eigenvalues are equivalent to the non-zero eigenvalues of \( \mathbf{M} = \Pi \mathbf{A} \Pi \mathbf{A}^{*} \) (42)

where \( \mathbf{1} \) is the identity matrix and \( \mathbf{z} \) is a normalized Gaussian vector whose statistics can easily be generalized to arbitrary \( \beta \). The non-zero eigenvalues \( \{ \lambda_i \} \) of \( \mathbf{M} \) satisfy the interlacing condition \( a_i \geq \lambda_i \geq a_{i+1} \) where we have assumed that the eigenvalues of \( \mathbf{A} \) are in decreasing order. The joint law of \( \{ \lambda_i \} \) is given by the Dixon-Anderson integral (26) (see chapter 4 of (28) for a derivation)

\[ P_{\alpha}^{(\beta)} (\{ \lambda_i \}) = \frac{\Gamma (\frac{N \beta}{2})}{\Gamma (\frac{\beta}{2} N)} \left( \prod_{1 \leq i < j \leq N-1} |\lambda_i - \lambda_j| \right)^{\frac{\beta}{2}} \left( \prod_{1 \leq i < j \leq N} |a_i - a_j|^{1-\beta} \right)^{N-1} \prod_{i=1}^{N-1} \prod_{j=1}^{N} |\lambda_i - a_j|^{\frac{\beta}{2} - 1} \]  
(43)

one can then iterate the procedure to obtain the joint law of eigenvalues of all principal minors, known as the *beta corner process* (24). When applied to (37), where we have reordered each
vector $a$ such that it is decreasing, we obtain the following expression for $\mathcal{F}_a^{(\beta)}(z)$:

$$
\mathcal{F}_a^{(\beta)}(z) = \frac{1}{Z_{N,\beta,e^{-a}}} \int \prod_{j=1}^{N} e^{a_j z^N} \prod_{k=1}^{N-1} \prod_{i=1}^{k} (\lambda_i^{(k)})^{-(z_k - z_{k+1})} \prod_{k=1}^{N-1} \prod_{i=1}^{k} \left( \prod_{u=1}^{k+1} (\lambda_u^{(k)} - \lambda_v^{(k+1)}) \prod_{v=1}^{k} \right) \prod_{i=1}^{N-1} d\lambda_i^{(k)}
$$

(44)

where the integral is over the set of $\{\lambda_i^{(k)}\}_{1 \leq i \leq k \leq N}$ satisfying the interlacing constraints $\lambda_i^{(k)} \geq \lambda_i^{(k+1)}$ and $\lambda_i^{(N)} = e^{-a}$, and

$$
Z_{N,\beta,e^{-a}} := (-1)^{\frac{N(N+1)}{2}} \prod_{k=1}^{N} \frac{\Gamma(\beta/2)}{\Gamma(k\beta/2)} \prod_{1 \leq i < j \leq N} (e^{-a_j} - e^{-a_i})^{\beta-1}
$$

(45)

If we make the change of variable $\lambda_i^{(k)} = e^{-l_i^{(k)}}$, with $\{l_i^{(k)}\}$ satisfying the interlacing constraints with $l^{(N)} = a$, this introduces a constant $(-1)^{\frac{N(N+1)}{2}}$ which exactly cancel out with the one in (45).

We have:

$$
\mathcal{F}_a^{(\beta)}(z) = \frac{\prod_{k=1}^{N} \Gamma(k\beta/2)}{\Gamma(\beta/2)^{\frac{N(N+1)}{2}}} \prod_{1 \leq i < j \leq N} (e^{-a_j} - e^{-a_i})^{\beta-1} \int e^{\sum_{k=1}^{N} z_k \left( \sum_{i=1}^{k} l_i^{(k)} - \sum_{i=1}^{k-1} l_i^{(k-1)} \right)} \prod_{k=1}^{N-1} \prod_{1 \leq i < j \leq k} (e^{-l_i^{(k)}} - e^{-l_j^{(k)}})^{2-\beta} \prod_{\alpha=1}^{k+1} \prod_{\nu=1}^{k} \left( e^{-l_\nu^{(k)}} - e^{-l_\nu^{(k+1)}} \right)^{\beta-1} \prod_{i=1}^{N} e^{\left( \frac{\beta}{2} - 1 \right) l_i^{(k)}} d l_i^{(k)}
$$

(46)

From this formula it is possible to establish a link between the Heckman-Opdam hypergeometric function and the normalized Macdonald polynomials and we refer the reader to [29] [30] [31] for a derivation of the result. The Macdonald polynomials $P_\lambda(x,q,t)$ are $q,t$-deformation of the Schur polynomials and we refer to [18] for an introduction on this subject. We have:

$$
\mathcal{F}_a^{(\beta)}(z) = \lim_{\epsilon \to 0^+} \frac{P_{\left\lfloor \frac{\epsilon}{2} \right\rfloor} \left( e^{\epsilon} | e^{-\epsilon}, e^{-\frac{\beta}{2}} \right)}{P_{\left\lfloor \frac{\epsilon}{2} \right\rfloor} \left( 1, e^{-\epsilon}, e^{-\epsilon}, e^{-\frac{\beta}{2}} \right)}
$$

(47)

where $\epsilon = (\epsilon^1, \ldots, \epsilon^N)$ and $| x | := ([x_1], \ldots, [x_N])$ with $[\cdot]$ the integer part function. Our interest for this expression lies in the fact that the normalized Macdonald polynomials admits a simple integral representation when all its arguments except one are fixed.

### 3.3 Rank one formula and asymptotic behavior

We recall that we want to extend the asymptotic behavior of the rank one function $J_a^{(\beta)}(z,0,\ldots,0)$ of Section 2.1 to all $\beta > 0$, which is equivalent to the study of $\mathcal{F}^{(\beta)}_{-\infty a}(-z - (N - 1)\frac{\beta}{2}, \ldots, -\frac{\beta}{2})$ by [37]. The negative-value partition, also known as a signature, Macdonald polynomials are defined by (23):

$$
P_{-\lambda}(x|q,t) = P_{\lambda}(x^{-1}|q,t)
$$

(48)

and by homogeneity of the Macdonald polynomials and (47) we are left with the study of $^6$.
\[ \mathcal{J}_{\alpha}^{(\beta)} (z) := \lim_{\epsilon \to 0^+} \frac{P_{\left[ \frac{\ln q}{q} \right]} \left( e^{\epsilon z}, e^{-\epsilon \frac{q}{2}}, \ldots, e^{-\epsilon \frac{q}{2}(N-1)} \right) e^{-\epsilon}, e^{-\epsilon \frac{q}{2}}}{P_{\left[ \frac{\ln q}{q} \right]} \left( 1, e^{-\epsilon \frac{q}{2}}, \ldots, e^{-\epsilon \frac{q}{2}(N-1)} \right) e^{-\epsilon}, e^{-\epsilon \frac{q}{2}}} \quad \text{(for } \beta > 0) \]  

(49)

It turns out that the corresponding specification of the normalized Macdonald polynomials appearing in [19] admits a simple formula in this setting [32]. For a signature \( \lambda, q \in (0, 1) \) and a complex \( |x| > 1 \), we have:

\[
\frac{P_{\lambda} \left( x, q^\beta, \ldots, q^\beta(N-2), q, q^\beta \right)}{P_{\lambda} \left( 1, q^\beta, \ldots, q^\beta(N-1), q, q^\beta \right)} = \ln q \frac{q - 1}{q} \Gamma_q \left( \frac{\beta N}{2} - \frac{q}{2}; q \right) \left( \frac{q}{2}; q \right)_\infty \times \frac{1}{2\pi i} \int_C dp e^{pz} \prod_{i=1}^{N} \frac{\Gamma_q \left( p - \left( \lambda_i - \frac{\beta}{2} i + \frac{\beta}{2} \right) \right)}{\Gamma_q \left( p - \left( \lambda_i - \frac{\beta}{2} i \right) \right)}
\]

(50)

where \( \Gamma_q (z) := (1 - q)^{1-z} \frac{\left( \frac{q}{2}; q \right)_\infty}{\left( \frac{q^z}{2}; q \right)_\infty} \) is the q-gamma function and \( (.; q)_\infty \) the q-Pochhammer of q-calculus [33] and C is a usual complex Bromwich contour which left all the poles to the left of the integration line. Without loss of generality we fix \( \Re z > 0 \), the other case can be obtained using the symmetry relation [17] and the behavior of the S-transform of the inverse of a matrix [14]. By doing the change of variables \( p \to \epsilon p \), we have:

\[
\mathcal{J}_{\alpha}^{(\beta)} (z) = \lim_{\epsilon \to 0^+} -\epsilon \frac{1}{1 - e^{-\epsilon}} \Gamma q \left( \frac{\beta N}{2} - \frac{e^{-\epsilon (\frac{\lambda N}{2} + \beta i)}; e^{-\epsilon}}{e^{-\epsilon (1+z)}; e^{-\epsilon}} \right) \left( 1 - e^{-\epsilon} \right)^{\frac{N-1}{2}} \frac{1}{\epsilon} \\
\times \frac{1}{2\pi i} \int C dp e^{pz} \prod_{i=1}^{N} \frac{e^{\epsilon \left( \frac{\ln a_i}{x} \right) e^{-\epsilon \frac{q}{2} (i-1); e^{-\epsilon}}} \left( e^{\epsilon \left( \frac{\ln a_i}{x} \right) e^{-\epsilon \frac{q}{2} (i-1); e^{-\epsilon}}} \right)_\infty}{e^{\epsilon \left( \frac{\ln a_i}{x} \right) e^{-\epsilon \frac{q}{2} (i-1); e^{-\epsilon}}} \left( e^{\epsilon \left( \frac{\ln a_i}{x} \right) e^{-\epsilon \frac{q}{2} (i-1); e^{-\epsilon}}} \right)_\infty}
\]

(51)

Next taking the limit \( \epsilon \to 0^+ \) together with the following limit relation of q-calculus [33]:

1. \( \lim_{q \to 1^-} \Gamma_q (x) = \Gamma (x) \)
2. \( \lim_{q \to 1^-} \frac{(q^a q^b)_\infty}{(q^a q^b)_\infty} (1 - q)^{a-b} = \frac{\Gamma (b)}{\Gamma (a)} \)
3. \( \lim_{q \to 1^-} \frac{(u q^a q^b)_\infty}{(u q^a q^b)_\infty} = (1 - u)^{b-a} \quad \text{for } |u| < 1 \)

and \( \lim_{\epsilon \to 0^+} e^{\epsilon \left( \frac{\ln a_i}{x} \right) e^{-\epsilon \frac{q}{2} (i-1); e^{-\epsilon}}} = a_i \), we have:

\[
\mathcal{J}_{\alpha}^{(\beta)} (z) = \frac{\Gamma \left( \frac{\beta N}{2} \right) \Gamma \left( 1 + z \right)}{\Gamma \left( \frac{N \beta}{2} + \frac{z}{2} \right)} \frac{1}{2\pi i} \int_{C^+} dp e^{pz} \prod_{i=1}^{N} \left( 1 - a_i e^{-p} \right)^{-\frac{\beta}{2}}
\]

(52)

which is the generalization to all \( \beta > 0 \) of the previous formula:

\[
\mathcal{J}_{\alpha}^{(\beta)} (z) = \frac{\Gamma \left( \frac{\beta N}{2} \right) \Gamma \left( 1 + z \right)}{\Gamma \left( \frac{N \beta}{2} + \frac{z}{2} \right)} \mathcal{L}^{-1} \left[ \prod_{i=1}^{N} \left( 1 - a_i e^{-p} \right)^{-\frac{\beta}{2}} \right] (z)
\]

(53)
and since the saddle-point analysis of Section 2.2 still holds for $\beta > 0$, we deduce that for $z$ small enough (with positive real value), we have:

$$\lim_{N \to \infty} \frac{2}{N \beta} \frac{d}{dz} \ln \mathcal{J}^{(\beta)}_{\frac{N\beta}{2}}(\frac{N\beta}{2} z) = \ln \tilde{S}(z)$$

(for $\beta > 0$) (54)

4 Asymptotics of symmetric polynomials

Just like the spherical function of Section 2.1 can be seen as an extension in index of the Schur polynomials ($\beta = 2$) and zonal polynomials ($\beta = 1, 4$), the Heckman-Opdam hypergeometric function can be seen as an extension of (a $\rho$-shifted version of) the normalized Jack polynomials $j^{(\frac{\beta}{2})}_{\lambda}(a)$ [31] for all $\beta > 0$. For $\lambda$ a partition, we have:

$$\mathcal{F}^{(\beta)}_{\alpha}(\lambda - \rho) = \frac{j^{(\frac{\beta}{2})}_{\lambda}(e^{-\alpha})}{j^{(\frac{\beta}{2})}_{\lambda}(1, \ldots, 1)}$$

(55)

For $\beta = 2$, the asymptotic behavior of the HCIZ integral can be translated as an asymptotic behavior over normalized Schur polynomial: if $\frac{1}{N} \sum_{i=1}^{N} \delta_{N-1}(\lambda_i + N - i)$ converge toward a deterministic measure $\mu$, than the corresponding normalized Schur polynomial with index $\lambda$ and with all its arguments except one fixed, converges (up to an integration term) exponentially towards the integral of the $R$-transform [10] see [13]. Since the multiplicative spherical function $\mathcal{J}^{(\frac{\beta}{2})}_{\alpha}(z)$ of this note is nothing else than the analytical extension of the Jack polynomials $j^{(\frac{\beta}{2})}_{\lambda}(z)$ we have a similar interpretation, except that now it is the vector in argument of the Jack polynomial which converges towards a deterministic measure while the index is the trivial partition $\lambda = ([\frac{N\beta}{2} z], 0, \ldots, 0) =: [\frac{N\beta}{2} z]$

$$\lim_{N \to \infty} \frac{2}{N \beta} \ln \frac{j^{(\frac{\beta}{2})}_{\frac{N\beta}{2} z}(a)}{j^{(\frac{\beta}{2})}_{\frac{N\beta}{2} z}(1, \ldots, 1)} = H^S_{\mu,A}(z)$$

(56)

with $H^S_{\mu,A}(\cdot)$ defined by [33]. In particular, for $\beta = 2$, the Jack polynomials become Schur polynomials and Schur polynomials of a trivial partition degenerate into complete homogeneous polynomials defined by:

$$h_k(a) = \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq N} a_{i_1} \ldots a_{i_k}$$

(57)

so that the LHS of (56) has a simple explicit expression in terms of the $a_i$ in this case.

As an illustration of this example, we take $\mu_A$ to be the uniform distribution between 0 and 2, then after some calculation one has:

$$H^S_{\mu,A}(z) = z \left( \ln \left| \frac{2z}{z + 1 + W(\frac{2z}{-z + 1 + W(\frac{2z}{1 - (z + 1)e^{-(z+1)})}) - 1)} \right| - \ln \left| W\left(\frac{2z}{-z + 1 + W(\frac{2z}{1 - (z + 1)e^{-(z+1)})}) - 1\right) \right| \right)$$

(58)
Figure 1: Value of the logarithm of the normalized complete homogeneous symmetric polynomials (59) for equidistributed entries between 0 and 2 for different $N$ and different $k = Nz$, compared with the limiting behavior (58), represented by a dashed line. The inset graph represents the convergence at $k = N$ ($z = 1$) for more values of $N$, represented as a function of $\frac{1}{N}$.

where $W(.)$ is the Lambert W function, which we compare with:

$$\frac{1}{N} \ln \frac{h_k(a)}{h_k(1, \ldots, 1)} = \frac{1}{N} \ln \left[ \frac{k!(N - 1)!}{(k + N - 1)!} \right]$$

(59)

for different $N$ and $z = k/N$, where the $a_i$ are the $N$ equidistributed points between 0 and 2. The results are shown in Fig. 1.

Note that the conjecture (36) implies that Jack polynomials with few non-trivial entries in the index partition should completely decoupled in this regime. In particular “low-rank” Schur polynomials should converge to a product of complete homogeneous symmetric polynomials.

Conclusion

In this note we have studied the asymptotic behavior of the multiplicative spherical function and its link with the $S$-transform, establishing the multiplicative counterpart of the Parisi-Guionnet-Maïda theorem. Our result is expected to be true in the large $N$ limit when the fraction of non-zero entries in the vector $z$ goes to zero. For the HCIZ integral there exists an asymptotic regime where both matrices are full rank and their eigenvalues converge to well determined measures [34]. It would be quite interesting to find out if the Heckman-Opdam function (or equivalently the spherical integral) can be computed asymptotically in the regime where both vectors $z$ and $a$ converge to full measures.
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