A Two-step Approach for Damage Identification in Axially Functionally Graded Beams

Z.R. Lu¹, R.Z. Yao, J.K. Liu, M. Huang
Department of Applied Mechanics, Sun Yat-sen University, Guangzhou, Guangdong Province, 510006, P.R. China
Email: lvzhr@mail.sysu.edu.cn(Z.R. Lu), huangm7@mail.sysu.edu.cn(M. Huang)

Abstract. This study presents a two-stage approach based on residual force vector and response sensitivity analysis for structural damage identification in axially functionally graded (AFG) beams. The local damage is simulated by a reduction in the elemental Young's modulus of the beam. The residual force vector is used to find the suspicious damaged elements in the beam at first. Then, a hybrid objective function is established and a sensitivity-based model updating method is adopted to identify the perturbation of the stiffness parameter from the measured acceleration responses. Two numerical examples are investigated to illustrate the correctness and efficiency of the proposed method. The effects of measurement noise on the identification results are investigated. Studies in this paper indicate that the proposed method is efficient and robust for identifying damages in the axially functionally graded beams. The advantage of the present approach lies in that only a few number of acceleration measurements and the first several natural frequencies of the beam are needed in the identification.

1. Introduction

A plenty of non-destructive techniques have been developed for health monitoring and structural damage identification in the last few decades. Doebling et al. [1] suggest a comprehensive review of the damage detection methods by testing changes in the dynamic parameters of a structure. Housner et al. [2] provided an extensive summary on the state-of-the-art in control and health monitoring in civil engineering structures. Zou et al. [3] reviewed the development on structural condition monitoring and damage identification for composite structures.

Usually, damage identification requires a finite element model of a structure in conjunction with experimental modal parameters of the structure. There are two categories of methods for damage identification, namely, the frequency domain methods and time domain methods. The frequency domain approaches are mainly based on the changes in the natural frequencies [4], mode shapes and their derivatives [5], measured dynamic flexibility [6] or frequency response function [7]. The natural frequency is easy to be measured with a high level of accuracy, and it is the most common method using dynamic parameter for damage detection. However, problems may occur in the symmetric structures if only natural frequencies are used, since the structural symmetry would lead to non-uniqueness in the solution of damage identification. This problem can be solved by incorporating the mode shapes in the inverse analysis.

The residual force vectors, derived from the natural frequencies and mode shapes of the intact and damaged structure, have been utilized to identify structural damages. Zimmerman and Kaouk [8]...
proposed a theory based on an original finite element model and a subset of measured eigenvalues and eigenvectors to identify structural damages. Damage locations are determined firstly and the damage extents are assessed by a minimum rank update theory. Li and Smith [9] used residual force vectors directly to solve for damaged element parameters, but again use experimentally determined undamped mode shapes and natural frequencies in the damage identification. Liu and Yang [10] made use of the mode shape matrix and the residual force matrix to determine the number of damaged elements and localized the damage elements by the damage localization matrix. And the damage extents can be easily obtained.

Recently, because of its excellent thermal and mechanical performance, FGM has great promising applications in many fields, such as space plane frames, spacecraft heat shield, etc. An intensive study on the vibration analysis of structures made of FGM has been conducted. Accurate condition assessment of the FGM structures is vital as most of them operate in severe conditions. However, the reported study on damage identification of structures made of FGM was very limited in the literature [11-17].

In this study, a two-step approach based on residual force vector and response sensitivity analysis is proposed to identify structural damage in functionally graded beams. A hybrid sensitivity matrix is constructed and used in the finite element model updating. The advantage of the present method lies in only the first several natural frequencies and a few number of acceleration responses are needed in the identification. A single span simply supported beam and a multi-span beam are studied as two numerical examples to illustrate the correctness and efficiency of the proposed method. Good identified results can be obtained from the short time histories of a few number of measurement points and the first several natural frequencies.

2. Theory

2.1. Finite element model of intact and damaged AFG beam

As shown in Figure 1, a straight uniform AFG beam has length $L$, width $b$, thickness $h$, with a Cartesian coordinate system O-xyz, where $x$ axis is taken along the central axis, the $y$ axis is in the width direction and the $z$ axis is in the depth direction.

In this study, it is assumed that the material properties of beam such as, Young’s modulus $E$, mass density $\rho$ vary continuously along the transversal direction. Both $E$ and $\rho$ are assumed to vary according to the following power-law functions:

$$E(x) = (E_L - E_R)(1 - x/L)^g + E_R$$

$$\rho(x) = (\rho_L - \rho_R)(1 - x/L)^g + \rho_R$$

where $E_L$ and $E_R$ are the Young’s modulus of the left and the right side of the beam, $\rho_L$ and $\rho_R$ are the mass densities of the left and the right side of the beam, $g$ is the non-negative power-law component which dictates the material variation profile through the axis of the beam.

The equation of motion of the intact AFG beam structure under the external force can be expressed as

$$M\ddot{d} + C\dot{d} + Kd = P(t),$$

where $M$, $K$ and $C$ are the system mass, stiffness and damping matrices respectively, $\ddot{d}$, $\dot{d}$ and $d$ are the acceleration, velocity and displacement vectors of the structure, $P(t)$ is the vector of external force. The well-known Rayleigh damping model is assumed in the study, i.e.,

$$C = \eta_1 M + \eta_2 K,$$

where $\eta_1$ and $\eta_2$ are constants to be determined from two given damping ratios that corresponding to two unequal modal frequencies of vibration. The forced vibration response of the beam under external force can be calculated using Newmark direct integration method.

When the beam is damaged, it is assumed that the local damage only leads to the change in the stiffness parameter (for instance, Young’s modulus $E$) and the mass property remains unchanged.
[19]. Instead of the absolute value of elemental Young’s modulus $E$, its relative perturbation to the initial value $E_0$ is chosen as dimensionless updating parameter $\alpha$ in this study, it is expressed as

$$E = E_0(1 + \alpha), \quad \text{with} \quad -1 \leq \alpha \leq 0$$  \hspace{1cm} (4)

Supposing the $i$th element is damaged, the elemental stiffness matrix $k_i'$ is stated as

$$k_i' = k_i(1 + \alpha^i)$$  \hspace{1cm} (5)

And the global stiffness matrix $K_d$ for the damaged structure can be expressed as

$$K_d = \sum_{i}^{NE} k_i'(1 + \alpha^i)$$  \hspace{1cm} (6)

The equation of motion for free and forced vibration of the damaged beam can be written as

$$\ddot{r}_i + K_d r_i = 0$$  \hspace{1cm} (7)

$$\ddot{d} + C_d d + K_d d = P(t)$$  \hspace{1cm} (8)

The natural frequencies and mode shapes of the damaged beam can be calculated from Eq. (9).

$$(K_d - \omega_{di}^2I)V_{di} = 0$$  \hspace{1cm} (9)

where $\omega_{di}$ is the $i$th natural frequency of the damaged beam and $V_{di}$ is the associated $i$th modeshape.

2.2. Localization of damage from residual force vector method

The change of structural stiffness matrix $\Delta K$ introduced by the structural damage can be modeled by a perturbation in the stiffness parameter of the structure in this study. So the global stiffness matrix $K_d$ can be expressed as

$$K_d = K - \Delta K.$$  \hspace{1cm} (10)

Substituting equation (10) into equation (9), we have

$$(K - \omega_{dq}^2M)V_{dq} = \Delta KV_{dq}.$$  \hspace{1cm} (11)

The residual force vector (RFV) (Zimmerman and Kaouk [9], Liu and Yang [10]) $f_j$ from the $j$th mode of the damaged structure is defined as

$$f_j = \Delta K \phi_{dq} = (K - \omega_{dq}^2M)V_{dq}.$$  \hspace{1cm} (12)

The residual force vector $f_j$ from equation (12) will have nonzero values only in the damaged elements. Furthermore, we can identify the damaged elements according to the relation between the element number and the DOF number. To avoid miss some damage information when a single mode is used, the normalized average value of the first $p$ RFV is adopted, i.e.

$$\bar{f} = \sum_{j=1}^{p} \frac{f_j}{\max |f_j|} \cdot p^{-1}$$  \hspace{1cm} (13)

where $p$ denotes the number of mode used to calculate.

2.3. The objective function for model updating problem

In finite element model updating, an optimization problem is usually set-up in which the differences between the experimental and numerical modal parameters or dynamic responses have to be minimized by modifying some parameters of the analytical model. During the practice, the natural frequencies of the structures can be measured easily and accurately but the mode shapes cannot, especially the higher ones. On the other hand, the acceleration responses of the structure can be measured easily. In this study, we establish the objective function making use of the natural frequency and acceleration measurements. The objective functions are expressed as follows:
\[ g_1(\alpha) = \frac{1}{2} \sum_{p=1}^{nf} (\hat{\lambda}_p - \lambda_p(\alpha))^T W_1 (\hat{\lambda}_p - \lambda_p(\alpha)), \text{ with } \lambda_p = \omega_p^2 \] (14a)

\[ g_2(\alpha) = \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{nf} (\hat{R}_{ij} - R_{ij}(\alpha))^T W_2 (\hat{R}_{ij} - R_{ij}(\alpha)) \] (14b)

where \( l \) is the number of measurement locations, \( nt \) is the number of time instances of the measured data, \( nf \) is the number of eigenvalue. \( R(\alpha) \) is the vector of calculated response of the structure from a known set of \( \alpha \) and \( \hat{R} \) is the vector of measured response. \( W_1 \) and \( W_2 \) are the weighting matrices for the eigenvalues and dynamic responses respectively, which are taken as unit matrices in this study. \( \alpha \) is the vector of unknown damage parameters \( (\alpha^1, \alpha^2, ..., \alpha^n)^T \) to be identified.

2.4. Hybrid sensitivity for model updating

Based on Eq. (14a) and Eq. (14b), the increment vector of the damage parameter in the \( j \)th iteration can be calculated from the following equation using the penalty function method [20]

\[ S^l_\lambda \Delta \lambda^j = \Delta \lambda^j \] (15a)

\[ S^l_\alpha \Delta \alpha^j = \Delta R^j \] (15b)

where

\[ \Delta \lambda^j = \hat{\lambda}^j - \lambda^j(\alpha) \] (16a)

\[ \Delta R^j = \hat{R}^j - R^j(\alpha) \] (16b)

is the difference between the measured and calculated responses at the \( j \)th iteration. \( S^l_\lambda \) is the sensitivity matrix of natural frequency with respect to damage parameter, which can be calculated using formulas of Fox and Kapoor [21]. \( S^l_\alpha \) is the dynamic response sensitivity matrix, which can be obtained using formulas of Lu and Law [22].

Combining Eq. (15a) and Eq. (15b), we have

\[ \overline{S}^j \Delta \lambda^j = \Delta H^j \] (17)

where \( \overline{S}^j = \begin{bmatrix} S^l_\lambda^j \\ S^l_\alpha^j \end{bmatrix} \) is the hybrid sensitivity matrix, \( \Delta H^j = \begin{bmatrix} \Delta \lambda^j \\ \Delta R^j \end{bmatrix} \)

Eq. (12) can be solved by the damped least-squares method (DLS) [23] and the incremental in the damage parameter is identified as

\[ \Delta \alpha^j = (\overline{S}^{jT} \overline{S}^{j} + \gamma A) \overline{S}^{jT} \Delta H^j \] (18)

where \( \gamma \) is the non-negative damping (regularization) coefficient governing the participation of least-squares error in the solution. The solution of Eq. (17) equals to the minimizing the function,

\[ J(\Delta \alpha^j, \gamma) = \left\| \overline{S}^j \Delta \alpha^j - \Delta H^j \right\|^2 + \gamma \left\| \Delta \alpha^j \right\|^2 \] (19)

with the second term in Eq. (18) providing bounds to the solution. In this paper, Tikhonov regularization approach is used to obtain the optimal regularization parameter \( \gamma \) [24].

The updated vector of the \((j+1)\)th iteration of damage parameter \( \alpha^{j+1} \), can be obtained in the next iteration as follows:

\[ \alpha^{j+1} = \alpha^j + \Delta \alpha^j \] (20)

The convergence is recognized as accomplished when the following criteria is met

\[ \max(\Delta \alpha^j) \leq \text{Tol.} \] (21)
In this study, the tolerance $T_{ol}$ is taken as $1 \times 10^{-3}$.

Starting with an initial damage parameter vector $\alpha^0$, which is taken as a null vector, the flowchart of iteration is shown in Figure 1.

![Flowchart of iteration](image)

Figure 1. Flowchart of iteration

3. **Numerical simulation**

3.1. *A simply supported AFG beam*

As the first example, an AFG beam made of steel and aluminum is examined using the proposed method as shown Figure 2. The length of the beam under study is 2 m, the width $b$ 0.1m and height $h$ 0.05m. Table 1 shows the material properties of the beam. The right side of beam is pure steel and left side is pure aluminum. The non-negative power-law $\vartheta$ is taken as 1.5 in this study.
MATLAB software package was employed to build the finite element model of the beam, in which the beam was discretized into twelve 2-node Euler-Bernoulli beam elements. The first five natural frequencies of the intact beam are 37.20, 151.75, 342.69, 609.54, and 804.93 Hz, respectively. To obtain the forced vibration response of the beam, an impulsive force is supposed to act at the 7th node of the beam in the global z direction with

\[
F(t) = \begin{cases} 
10^5 (t - 0.02) \text{ N} & (0.02 \text{ s} < t \leq 0.04 \text{ s}) \\
10^5 (0.06 - t) \text{ N} & (0.04 \text{ s} < t \leq 0.06 \text{ s}) 
\end{cases}
\]

In calculating the dynamic response, the time increment is 0.0002 second and the time duration for the response calculation is 2.0 seconds unless otherwise specified. The two damping coefficients used for calculating Rayleigh damping matrix are both assumed to be 0.01. In this numerical example the first 6 natural frequencies and five acceleration measurements locating at nodes 3, 5, 7, 9, and 10 are used in damage identification.

In this case, it is assumed that the natural frequency and the acceleration are contaminated with measurement noise. The noise polluted eigenvalues and mode shapes are expressed as

\[
\hat{\lambda}_j = \lambda_j (1 + \chi_i \varsigma)
\]

\[
\hat{\phi}_j = \phi_j (1 + \chi_i \varsigma |_{\phi_{\text{max}}})
\]

where \(\hat{\lambda}_j\) and \(\hat{\phi}_j\) are the \(j\)th simulated measured noisy eigenvalue and mode shape, \(\lambda_j\) and \(\phi_j\) are the calculated ones, \(\chi_i\) is random number with a mean equal to zero and a variance equal to one, \(\varsigma\) is noise level in percentage. Messina et al. [25] suggested a standard error of \(\pm 0.15\%\) as a benchmark figure for natural frequencies in the laboratory with the impulse hammer technique.

Two different noise levels are utilized to study the robustness of the proposed method: 1) low level noise: 0.15% noise in the natural frequencies, 5% noise in the mode shapes and 2% noise in the acceleration responses; 2) high level noise 1% noise in the natural frequencies, 15% noise in the mode shapes and 15% noise in the acceleration responses. Figure 3 shows the identified results with noisy measurements. One can find that all the four damages have been identified successfully. The largest identification error is only 0.15% at the 3rd element for low noise level. In this case the number of iteration required is 22 and the optimal regularization parameter \(\lambda_{\text{opt}}\) is found to be 143.48 at the last iteration. The largest identification error is only 1.73% at the 1st element for high level noise, the number of iteration required is 26 and the optimal regularization parameter \(\lambda_{\text{opt}}\) is found to be 443.14 at the last iteration.
3.2. A four span continuous AFG beam

An AFG beam made of Zirconia (ZrO2)/Titanium (Ti-6Al-4V) is studied as the second numerical example as shown in Figure 4. The length of the beam is 12 m, the width $b$ 0.2 m and height $h$ 0.2 m. Table 2 shows the material properties of the beam. The right side of beam is pure ZrO2 and left side is pure Ti-6Al-4V.

![Figure 4. A four span functionally graded beam under study](image)

### Table 2. Properties of beam in the numerical study

| Properties | ZrO2   | Ti-6Al-4V |
|------------|--------|-----------|
| $E$        | 168GPa | 105GPa    |
| $\rho$    | 3000 kg/m$^3$ | 4429 kg/m$^3$ |

In the finite element, the beam was discretized into sixty 2-node Euler-Bernoulli beam elements. The first six natural frequencies of the beam are 60.29, 76.11, 99.23, 129.42, 236.44 and 282.64 Hz, respectively. To obtain the forced vibration response of the beam, four impulsive forces are supposed to act at the $8^{th}$, $17^{th}$, $34^{th}$, $46^{th}$ nodes of the beam in the global $z$ direction with the same expression:

$$F(t) = 10^6 (t - 0.02) \text{ N} \quad (0.02 \text{ s} < t \leq 0.04 \text{ s})$$

In computing the dynamic response, the time increment is $0.0005$ second and the time duration for the response calculation is $2.0$ seconds unless otherwise specified. The two damping coefficients used for calculating Rayleigh damping matrix are both assumed to be 0.01.

In this study case, we try to use the displacement measurements instead of acceleration for damage identification. The last study case is re-examined. The same number of natural frequencies...
and the displacement measurements at the same nodes as the last case is used. The noise level is assumed to be 0.5% in the natural frequency and 10% in the displacement time histories respectively. After 32 iterations, the identification results converge to the seven assumed damages in the beam as shown in Figure 5. The max identified error is only 1.5% at element 8. The optimal regularization parameter $\lambda_{opt}$ is found to be 14.73 at the last iteration. This case further illustrates the correctness and robustness of the proposed method.

![Figure 5. Damage identification using displacement measurements](image)

4. Conclusion

In this study, a hybrid sensitivity matrix is constructed for finite element model updating. It is found to be effective and robust for structural damage identification in AFG beam structures. The proposed method only needs several natural frequencies and a few number of dynamic responses of the structure. Both acceleration and displacement measurements under impulsive or sinusoidal excitation force can be used in the identification. Damage identification results are obtained iteratively with the penalty function method with Tikhonov regularization from the measured natural frequencies and structural dynamic responses. Two numerical examples studied in this contribution show the proposed method effective in identifying multiple damages in the AFG beams. Study shows that the proposed method own excellent robustness to the artificial measurement noise.

5. Acknowledgements

The work described in this paper is supported by the National Natural Science Foundation of China (11572356), Doctoral Program Foundation of Ministry of Education of China (20130171110039), Guangdong Province Natural Science Foundation (2015A030313126), and the Guangdong Province Science and Technology Program (2016A020223006). Such financial aids are gratefully acknowledged.

References

[1] S.W. Doebling, C.R. Farrar, M.B. Prime, A review of damage identification methods that examine changes in dynamic properties, Shock Vibr. Dig. 30(1998) 91-105.

[2] G.W. Housner, L.A. Bergman, T.K. Caughey, et al., Structural control: Past, present, and future, J. Eng. Mech. 123(1997) 897-971.

[3] T. Zou, L. Tong, G.P. Steve, Vibration based model-dependent damage (delamination) identification and health monitoring for composite structures-a review, J. Sound Vib. 230(2000) 357-378.

[4] P. Cawley, R.D. Adams, The location of defects in structures from measurements of natural frequencies, J. Strain Anal. 14(1979) 49-57.

[5] A. K. Pandey, M. Biswas, M. M. Samman, Damage detection from change in curvature mode shapes, J. Sound Vib. 145(1991) 321-332.
A. K. Pandey, M. Biswas, Damage detection in structures using change in flexibility, J. Sound Vib. 169(1994) 3-17.

X. Liu, N.A.J. Lieven, P.J. Escamilla-Ambrosio, Frequency response function shape-based methods for structural damage localization, Mech. Syst. Signal Proc., 23(2009) 1243-1259.

D.C. Zimmerman and M. Kaouk, Structural damage detection using a minimum rank update theory, J. Vib. Acoust. 116(2) (1994) 222-231.

C. Li and W. Smith, Hybrid approach for damage detection in flexible structures, J. Guid. Control Dynam. 18(3) (1995) 419-425.

J.K. Liu and Q.W.Yang, A new structural damage identification method, J. Sound Vib. 297(3)(2006)694-703.

A. Khatir, M. Tehami, S. Khatir and M. Abdel Wahab, Multiple damage detection and localization in beam-like and complex structures using co-ordinate modal assurance criterion combined with firefly and genetic algorithms Journal of Vibroengineering 18(8) (2016) 5063-5073

Y.L. Zhou, NMM. Maia, R. Sampaio and MA. Wahab, Structural damage detection using transmissibility together with hierarchical clustering analysis and similarity Structural health monitoring, 2016, doi: 10.1177/1475921716680849

Y.L. Zhou, N. Maia and M. Abdel Wahab, Damage detection using transmissibility compressed by principal component analysis enhanced with distance measure Journal of Vibration and Control, 2016, doi: 10.1177/1077546316674544

Y.L. Zhou and M. Abdel Wahab, Rapid early damage detection using transmissibility with distance measure analysis under unknown excitation in long-term health monitoring Journal of Vibroengineering 18(7) (2016) 4491-4499

S. Khatir, I. Belaidi, R. Serra, M. Abdel Wahab and T. Khatir, Numerical study for single and multiple damage detection and localization in beam-like structures using BAT algorithm Journal of Vibroengineering 18(1)(2016) 202-213

S. Khatir, I. Belaidi, R. Serra, M. Abdel Wahab and T. Khatir, Damage detection and localization in composite beam structures based on vibration analysis Mechanika 21(6)(2015) 472-479

G.R. Gillich, Z.I. Praisach, M. Abdel Wahab, N. Gillich, IC. Mituletu and C. Nitescu, Free vibration of a perfectly clamped-free beam with stepwise eccentric distributed masses Shock and Vibration 2016(Article ID 2086274) 10 pages

K. Wakashima, T. Hirano, M. Niino. Space applications of advanced structural materials. ESA 1990. SP 303-97.

A. Messina, E. J. Williams, T. Contursi, Structural damage detection by a sensitivity and statistical-based method, J. Sound Vib. 216(1998) 791-808.

M.I. Friswell, J.E. Mottershead, Finite Element Model Updating in Structural Dynamics, Dordrecht: Kluwer Academic Publisher, 1995.

R. Fox, M. Kapoor, Rate of change of eigenvalues and eigenvectors, AIAA J. 6(1968) 2426-2429.

Z.R. Lu and S.S. Law, Features of dynamic response sensitivity and its application in damage detection, J. Sound Vib. 303(1-2) (2007) 305-329.

A.M. Tikhonov, On the solution of ill-posed problems and the method of regularization, Soviet Math. 4(1963)1035-1038.

P.C. Hansen, D.P.O. Leary, The use of the L-curve in the regularization of discrete ill-posed problem. SIAM J. Sci. Comput. 14(1993) 1487-503.

A. Messina, I.A. Jones, E.J. Williams, Damage detection and localization using natural frequency changes, in: Proceedings of Conference on Identification in Engineering Systems, Swansea, U.K., 1996, pp. 67-76.