On unfolded off-shell formulation for higher-spin theory

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Abstract
We present an unfolded off-shell formulation for free massless higher-spin fields in 4d Minkowski space in terms of spinorial variables. This system arises from the on-shell one by the addition of external higher-spin currents, for which we find an unfolded description. Also we show that this off-shell system can be interpreted as Schwinger–Dyson equations and restore two-point functions of higher-spin fields this way.

1 Introduction
Higher-spin (HS) gauge theory (for a review see e.g. [1, 2]), describing interactions of massless fields of all spins and possessing infinite gauge symmetry, is of great interest for high-energy physics. Firstly, it is regarded as possible symmetric high-energy phase of string theory [3], secondly, it provides an example of weak-weak AdS/CFT correspondence, being dual to different conformal vectorial models according to Klebanov–Polyakov conjecture [4]. Available formulation of HS theory represents a full nonlinear system of classical equations of motion written in the so-called unfolded form – Vasiliev equations [5, 6]. Unfolded dynamics approach is a first-order formalism, possessing a manifest coordinate-independence (due to exterior forms language) and allowing an easy control of gauge symmetries of the theory. However, obtaining conventional field-theoretical information from the unfolded formulation turns out to be a very nontrivial problem. Up to date only cubic HS vertices were managed to be extracted from Vasiliev equations and compared with AdS/CFT predictions [7–12].

One of the main problems of HS theory is the absence of a full nonlinear action (see, however, [13]), that does not allow the use of standard AdS/CFT techniques and the study of quantization issue. A general systematic method of classifying all gauge-invariant functionals of unfolded system was proposed in [14]. It consists in studying cohomologies of certain operator determined by unfolded equations of the theory. In the case of an on-shell unfolded system (i.e. when it contains dynamical equations on primary fields) such functionals correspond to conserved charges. In the case of an off-shell unfolded system (when unfolded equations represent
a set of constraints expressing descendants via primaries with prescribed gauge symmetries) these functionals can be considered as potential actions.

Thus, in order to proceed in the search for HS action, one should find an off-shell completion of Vasiliev equations – an unfolded system with the same spectrum of primary fields, proper gauge transformations laws, but without any dynamical equations. Such off-shell construction, resulting from relaxing tracelessness condition, for Vasiliev theory of totally symmetric bosonic HS fields in arbitrary dimension [15] was presented in [16]. However, the most elaborated is another version of Vasiliev system, namely, 4d theory formulated in terms of spinors, which in most cases is the only one where practical computations are feasible. Here traces are absent by construction, so the method of [16] is inapplicable. In this paper we propose an alternative way to look for off-shell completion for this theory, by coupling it to external currents. Concretely, we present an off-shell completion of unfolded spinorial system of free HS in 4d flat spacetime. One of the advantage of the proposed approach is that it allows one to look for quantum correlation functions via interpreting off-shell equations as the Schwinger–Dyson ones. We illustrate this by finding two-point functions of HS fields from the off-shell system we built.

The paper is organized as follows. In Section 2 we recall some facts about unfolded dynamics approach and HS equations. In Section 3 we construct unfolded description of Fronsdal currents and use it to find an off-shell completion of free HS equations. In 4 two-point correlation functions of HS fields are evaluated from the off-shell system of Section 3. In 5 we present our conclusions.

2 Unfolded formulation and free on-shell HS equations

Unfolding of the theory means reformulating it via equations of the form

\[ dW^A(x) = G^A(W), \]

(2.1)

where \( d \) is spacetime de Rham differential and \( W^A(x) \) are differential forms representing unfolded fields with \( A \) denoting all indices they carry. The identity \( d^2 = 0 \) imposes the following consistency condition on the system (2.1)

\[ G^B \frac{\delta G^A}{\delta W^B} \equiv 0. \]

(2.2)

A consistent unfolded system is manifestly invariant under a set of gauge transformations

\[ \delta W^A = d\varepsilon^A(x) - \varepsilon^B \frac{\delta G^A(W)}{\delta W^B}. \]

(2.3)

A simple example of unfolded system is provided by equations describing Minkowski space. In this case unfolded fields are 1-forms of vielbein \( e^a = e^a_m dx^m \) and Lorentz spin-connection \( \omega^{a,b}_L = \omega^{a,b}_m dx^m = -\omega^{b,a}_L \) which obey

\[ de^a + \omega^{a,b}_L e_b = 0, \]

(2.4)

\[ d\omega^{a,b}_L + \omega^{a,c}_L \omega^{c,b}_L e_b = 0. \]

(2.5)

Another example is an unfolded description of free massless scalar field. Appropriate system look as follows

\[ D^L C_{a(n)} = e^b C_{ba(n)}, \]

(2.6)
where $D^L = d + \omega_L$ is Lorentz-covariant derivative and $C_{a(n)}(x)$ are symmetric rank-$n$ Lorentz tensors. In Cartesian coordinates

$$e^a_m = \delta^a_m, \quad \omega^{a,b}_L = 0$$

(2.7)

one can solve (2.6) as

$$C_{a(n)}(x) = \partial_{a_1}...\partial_{a_n} C(x),$$

(2.8)

which shows that $C_{a(n)}(x), n > 0$ are descendant fields, forming the tower of all derivatives of the primary scalar $C(x)$. If there is no further constraints on the unfolded fields, the system (2.6) is off shell in the sense that there is no any differential constraints on the primary field $C(x)$. However if one requires $C_{a(n)}(x)$ to be traceless, then from (2.8) Klein-Gordon equation $\Box C(x) = 0$ follows. So the same unfolded system (2.6) for traceless tensors describes on-shell scalar field.

A spectrum of unfolded fields of HS theory includes master 1-form $\omega$ and master 0-form $C$ that depend on a pair of auxiliary Lorentz vectors $Y_1^a$ and $Y_2^a$.

$$\omega(Y|x) = \sum_{n \geq m} \omega_{a(n),b(m)} Y_1^a_1...Y_1^a_{n} Y_2^b_1...Y_2^b_{m}, \quad C(Y|x) = \sum_{n \geq m} C_{a(n),b(m)} Y_1^a_1...Y_1^a_{n} Y_2^b_1...Y_2^b_{m}.$$

(2.9)

where tensors possess symmetry of two-row Young diagrams. Submodule describing spin-$s$ field is formed by $\omega_{a(s-1),b(t)}$ (Fronsdal gauge spin-$s$ field and its first $(s - 1)$ derivatives) and $C_{a(s+t),b(s)}$ (gauge-invariant HS curvatures and infinite towers of their descendants). To write down on-shell equations, one requires all tensors to be traceless.

The most elaborated is 4d Vasiliev system due to the isomorphism $so(3,1) \approx sl(2,C)$ that allows one to replace auxiliary vectors $Y_i^a$ with auxiliary spinor variables $Y^A = (y^\alpha, \bar{y}^{\dot{\alpha}})$, $\alpha, \dot{\alpha} = \{1,2\}$. In this case requirement of tracelessness for Lorentz tensors gets replaced with a simple condition of symmetricity of $Y^A$. 4d unfolded master-fields now are

$$\omega(Y|x) = \sum_{n,m} \frac{1}{n!m!} \omega_{\alpha(n),\beta(m)} y_1^{\alpha_1}...y_1^{\alpha_n} \bar{y}_1^{\beta_1}...\bar{y}_1^{\beta_m}, \quad C(Y|x) = \sum_{n,m} \frac{1}{n!m!} C_{\alpha(n),\beta(m)} y_1^{\alpha_1}...y_1^{\alpha_n} \bar{y}_1^{\beta_1}...\bar{y}_1^{\beta_m}.$$

(2.10)

In terms of them one can write down an unfolded system describing propagation of free massless HS fields (so-called Central On-mass-shell Theorem) [6]. For Minkowski background it is

$$D^L \omega(Y|x) + e^{\alpha\beta} y_\alpha \partial_\beta \Pi^- \omega(Y|x) + e^{\alpha\beta} \partial_\alpha \bar{y}_\beta \Pi^+ \omega(Y|x) =$$

$$= \frac{i}{4} \tilde{H}^{\alpha\beta} \partial_\alpha \partial_\beta C(0,\bar{y}|x) + \frac{i}{4} \bar{H}^{\alpha\beta} \partial_\alpha \partial_\beta C(y,0|x),$$

(2.11)

$$D^L C(Y|x) + i e^{\alpha\beta} \partial_\alpha \partial_\beta C(Y|x) = 0.$$  

(2.12)

Here $H^{\alpha\beta} = e^\alpha_\gamma e^\beta_\gamma, \tilde{H}^{\alpha\beta} = e^\dot{\alpha}_\gamma e^\gamma_\beta$ and we introduced projectors $\Pi^+ (\Pi^-)$ to components with $N > \bar{N}$ ($N < \bar{N}$, respectively), where $N = y^\alpha \partial_\alpha$ and $\bar{N} = \bar{y}^{\dot{\alpha}} \partial_{\dot{\alpha}}$ count the number of $y$ and $\bar{y}$. The system (2.11)-(2.12) splits into independent subsystems with $(N + \bar{N}) \omega = (2s - 2) \omega$, $|N - \bar{N}| C = 2s C$ that describe spin-$s$ field. Our goal is to build and off-shell completion of (2.11)-(2.12).
As was mentioned, for Vasiliev system formulated in terms of Lorentz tensors one can formulate an off-shell generalization via relaxing the tracelessness projections [16]. Another example of off-shell HS system is presented in [14], where it is shown that unfolded system consisting of covariant flatness and covariant constancy equations for 1- and 0-forms taking values in \(d\)-dimensional oscillator algebra can be treated for special vacuum solution as nonlinear off-shell HS theory in Minkowski space. Corresponding HS fields however turn out to be traceful, differing from standard Fronsdal fields. How to perform proper reduction of this system that would drive out traces remains unclear.

The situation with 4\(d\) spinorial formulation of Vasiliev system is peculiar, because in this case it is the commutativity of auxiliary spinors \(Y^A\) that puts the system on-shell and there is no trace projections that could be relaxed. To proceed in this case we choose a following strategy. Consider a massless scalar field obeying Klein-Gordon equation \(\Box \phi(x) = 0\). This system is on shell. Now deform it writing \(\Box \phi(x) = J(x)\). If the source \(J(x)\) is some given function, say, field-dependent correction describing interactions or fixed external field, then the theory remains on shell. However if we treat \(\Box \phi(x) = J(x)\) as equation determining \(J(x)\), then our theory becomes off-shell one. Now it contains two scalar fields, primary \(\phi(x)\) and its descendant \(J(x)\), without any differential restrictions on \(\phi(x)\).

Thus we can build an off-shell completion for (2.11)-(2.12) by coupling it to sources that obeys no constraints other than required by consistency condition. In standard formulation Fronsdal equation for double-traceless spin-s field \(\phi_a(s)(x)\) coupled to double-traceless external current \(J_a(s)(x)\) has the form

\[
\Box \phi_a(s) - s \partial_a \partial^b \phi_{ba(s-1)} + \frac{s(s-1)}{2} \partial_a \partial_a \phi_{ba(s-2)} = J_a(s),
\]

(3.1)

with consistency condition for the current

\[
\partial^b J_{ba(s-1)} = \frac{(s-1)}{2} \partial_a J_{ba(s-2)}.
\]

(3.2)

First let us consider an example of off-shell completion for unfolded scalar field. This field is described on-shell by a subsystem of (2.12) with \(NC(Y|x) = \bar{N}C(Y|x)\)

\[
D^L C + ie^{\alpha\bar{\beta}} \partial_\alpha \bar{\partial}_{\bar{\beta}} C = 0.
\]

(3.3)

(which in fact is (2.1) rewritten in terms of spinors). In this case the source is just another (unconstrained) scalar field. We can describe it by the 0-form master-field \(J(Y|x)\), \(NJ = \bar{N}J\) and put it to the r.h.s. of (3.3). Next we should write down an unfolded system for \(J(Y|x)\). To this end we can use the same equation (3.3) with \(C\) replaced with \(J\), but now to avoid imposing \(\Box J = 0\) we once again should introduce one more “source for source” to its r.h.s. Obviously this process is infinitely repeating. So it is convenient to introduce a new parameter \(b\) and organize the whole sequence of sources as expansion in it

\[
J(Y|b|x) = \sum_{k=0}^{\infty} \frac{b^k}{k!} J^{(k)}(Y|x).
\]

(3.4)
Then an unfolded system describing off-shell scalar field takes the form

\[
D^L C + i e^{\alpha\beta} \partial_\alpha \bar{\partial}_\beta C = i e^{\alpha\beta} y_\alpha \bar{y}_\beta \frac{1}{(N + 1)(N + 2)} J (b = 0), \tag{3.5}
\]

\[
D^L J + i e^{\alpha\beta} \partial_\alpha \bar{\partial}_\beta J = i e^{\alpha\beta} y_\alpha \bar{y}_\beta \frac{1}{(N + 1)(N + 2)} \partial_b J, \tag{3.6}
\]

with condition \( NC = \bar{N} C, NJ = \bar{N} J \). \( N \)-dependent coefficients in (3.5)-(3.6) are fixed by consistency condition (2.2). By analyzing this system in Cartesian coordinates it is easy to see that \( b \)-expansion (3.4) is in fact an expansion in boxes of off-shell primary scalar field \( C(x) \):

\[
J_{\alpha(n),\dot{\alpha}(n)}^{(k)} (x) \sim \left( \frac{\partial}{\partial x^{a\alpha}} \right)^n \square^{k+1} C (x). \tag{3.7}
\]

Let us note, that this construction also proposes a simple way of imposing higher-order equations. For instance, to get an unfolded description of scalar field subjected to equation

\[
\square^n C (x) = 0, \tag{3.8}
\]

one should simply restrict the limit of summation in (3.4) to \((n - 1)\).

For HS sources the situation is more complicated, because possible tensor symmetries of descendants are richer – we can antisymmetrize derivatives with spin indices of the primary source field or take divergences of it. Moreover, HS sources are double-traceless and obey the generalized conservation law (3.2). These two properties can be treated as follows: spin-\(s\) source represents a combination of two traceless tensors of rank \(s\) and \((s - 2)\) with the only condition that rank-\(s\) tensor has a divergence proportional to the first traceless symmetrized derivative of the rank-\((s - 2)\) one; apart from that tensors are unrestricted. So it is easier to start with considering the following problem: what is the unfolded description of unconstrained traceless rank-\(n\) Lorentz tensor field \( T_{a(n)} (x) \).

As the first step consider the case of conserved field obeying Klein-Gordon equation

\[
\square T_{a(n)} = 0, \quad \partial^b T_{ba(n-1)} = 0. \tag{3.9}
\]

Let us analyze the spectrum of descendants it generates. In the language of Young diagrams, we start with the one-row diagram of length \(n\) and then successively add one cell step by step (successively differentiate primary \( T_{a(n)} \)). Then, taking into account that all derivatives are automatically symmetrized, we see that the space of descendants gets parameterized by the set of all one- and two-row traceless Young diagrams with the first row of no less than \(n\) cells and the second row (if presented) with no more than \(n\) cells. In the language of multispinors, this is equivalent to the set of \( T_{a(n),\dot{a}(m)} (x) \) with \(n + m \geq 2s\), \(|n - m| \leq 2s\). (Let us remind that traceless tensor \( T_{a(n),b(m)} \) with Young symmetry corresponds to a pair of symmetric multispinors \( T_{a(n+m),\dot{a}(n-m)} \) and \( T_{a(n-m),\dot{a}(n+m)} \)).

Corresponding unfolded system must present some generalization of (3.3). An operator \( ie^{\alpha\beta} \partial_\alpha \bar{\partial}_\beta \) corresponds to the adding cell to the upper (for scalar field – the only) row. This must be complemented by operators of the form \( e^{\alpha\beta} y_\alpha \bar{y}_\beta \Pi^- \) and \( e^{\alpha\beta} \partial_\alpha \bar{x}_\beta \Pi^+ \) which account for the possibility of adding cell to the bottom row. Fixing the coefficients by consistency, one arrives at the following unfolded system, corresponding to (3.9),

\[
D^L T + ie^{\alpha\beta} \partial_\alpha \bar{\partial}_\beta T + e^{\alpha\beta} y_\alpha \bar{y}_\beta \frac{1}{(N + 1)(N + 2)} \Pi^- T + e^{\alpha\beta} \partial_\alpha \bar{x}_\beta \frac{1}{(N + 1)(N + 2)} \Pi^+ T = 0. \tag{3.10}
\]
At the second step let us get rid of the constraint $\square T_{a(n)} = 0$. To this end, as in the scalar field example, we introduce a dependence on the additional parameter $b$ (“power of boxes”) and add corresponding $b$-dependent terms to the r.h.s. of the unfolded equations. As taking box means removing one cell from the Young diagram of descendant, these terms must be of the form $ie^{\alpha\beta} y_\alpha \bar{y}_\beta$ (removing cell from the upper row, presented already in the scalar case), $e^{\alpha\beta} y_\alpha \partial_\beta \Pi^{+0}$ and $e^{\alpha\beta} \partial_\alpha \bar{y}_\beta \Pi^{-0}$ (removing from the bottom one), where $\Pi^{+0}$ ($\Pi^{-0}$) are projectors to $N \geq \bar{N}$ ($N \leq \bar{N}$). After determining coefficients from consistency condition one gets

$$D^L T (Y|b|f|x) + ie^{\alpha\beta} \partial_\alpha \bar{y}_\beta T + e^{\alpha\beta} y_\alpha \bar{y}_\beta \frac{1}{(N + 1)(N + 2)} \Pi^{-} T + e^{\alpha\beta} \partial_\alpha \bar{y}_\beta \frac{1}{(N + 1)(N + 2)} \Pi^{+} T = \nonumber$$

$$= i e^{\alpha\beta} y_\alpha \bar{y}_\beta \frac{1}{(N + 1)(N + 2)} \frac{\partial}{\partial b} T - e^{\alpha\beta} \partial_\alpha \bar{y}_\beta \frac{1}{(N + 1)(N + 2)} \Pi^{+0} \frac{\partial}{\partial b} T - \nonumber$$

$$- e^{\alpha\beta} y_\alpha \partial_\beta \frac{1}{(N + 1)(N + 2)} \Pi^{+0} \left( \frac{\partial}{\partial b} + \frac{\partial}{\partial f} \right) T - e^{\alpha\beta} \partial_\alpha \bar{y}_\beta \frac{1}{(N + 1)(N + 2)} \Pi^{-0} \left( \frac{\partial}{\partial b} + \frac{\partial}{\partial f} \right) T. \quad (3.11)$$

Let us note that this system is of interest by itself, because it provides a description of traceless conserved tensor fields of arbitrary ranks, i.e. $4d$ conformal HS currents, which are the subject of the study in the literature [17–19].

The last step is to remove conservation condition $\partial^b T_{ba(n-1)} = 0$. This requires the introduction of one more parameter $f$ and corresponding expansion (“power of divergences”), analogous to $b$ (of course, an expansion in $f$ is finite, as one can take no more than $n$ divergences of rank-$n$ tensor)

$$T (Y|b, f|x) = \sum_{p=0}^{\infty} \sum_{q=0}^{n} \frac{f^q b^p}{q! p!} T^{(p,q)} (Y|x). \quad (3.12)$$

From the standpoint of Young diagrams taking divergencies reveals in the same way as taking box – both remove one cell. The difference however is that modules corresponding to higher powers of $f$ (higher divergences) describes tensor fields of less ranks, so for them one has $N + \bar{N} \geq 2 \left( n - f \frac{\partial}{\partial f} \right)$, $|N - \bar{N}| \leq 2 \left( n - f \frac{\partial}{\partial f} \right)$. Thus we have

$$D^L T + ie^{\alpha\beta} \partial_\alpha \bar{y}_\beta T + e^{\alpha\beta} y_\alpha \partial_\beta \left( \frac{\partial}{\partial b} + \frac{\partial}{\partial f} \right) T = \nonumber$$

$$= i e^{\alpha\beta} y_\alpha \bar{y}_\beta \left( \frac{\partial}{\partial b} + \frac{\partial}{\partial f} \right) T - e^{\alpha\beta} \partial_\alpha \bar{y}_\beta \left( \frac{\partial}{\partial b} + \frac{\partial}{\partial f} \right) T. \quad (3.13)$$

This system solves our auxiliary problem, describing totally unconstrained traceless rank-$n$ tensor field $T_{a(n)} (x)$ in terms of master 0-form $T (Y|b, f|x)$ such that $(N + \bar{N}) T \geq 2 \left( n - f \frac{\partial}{\partial f} \right) T$, $|N - \bar{N}|T \leq 2 \left( n - f \frac{\partial}{\partial f} \right) T$.

Now, using this, we can write down a full system describing a space of HS sources. To avoid degeneracy in spins we need to introduce last additional parameter $m$ that encodes spin of the
current

\[ T(Y|b, f, m|x) = \sum_{2s-4=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{s-2} \frac{m^{2s} f^q b^p}{(2s)! q! p!} T^{(p,q,2s-4)}(Y|x). \]  

(3.14)

This parameter allows one to distinguish between tensors of the same Young symmetry but corresponding to different spins (e.g. \( T_{a(k)} \) may equally describe primary source of spin-\( k \), or \( k \)-th derivative of scalar source, or divergence of spin-(\( k + 1 \)) source etc.) The role of parameters \( b, f \) and \( m \) (and need for them) can be illustrated by the following relation, giving a manifest expression for particular unfolded descendant in terms of derivatives of primary field:

\[ T^{(p,q,2s-4)}_{\alpha(n),\bar{\alpha}(\bar{n})}(x) \sim \Box^p (\partial^q)^m (\partial_{\alpha})^{n-s+2q} T_{b(q)a(s-2-q)}(x). \]  

(3.15)

To describe Fronsdal currents, we treat \( T \) as a space of totally unconstrained traces of currents (this is why we started sum with \( s = 2 \) in (3.14)) and complement it with analogous system for \( J(Y|b, f, m|x) \)

\[ J(Y|b, f, m|x) = \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{s} \frac{m^{2s} f^q b^p}{(2s)! q! p!} J^{(p,q,2s)}(Y|x) \]  

(3.16)

- traceless components of Fronsdal currents, that have divergences proportional to first symmetrized derivatives of \( T \):

\[
D^L J + i e^{\alpha\beta} \partial_{\alpha} \bar{\partial}_{\beta} J + e^{\alpha\beta} y_{\alpha} \bar{y}_{\beta} \frac{1}{(N + 1)(N + 2)} \Pi^{-} J + e^{\alpha\beta} \partial_{\alpha} \bar{y}_{\beta} \frac{1}{(N + 1)(N + 2)} \Pi^{+} J =
\]

\[
= i e^{\alpha\beta} y_{\alpha} \bar{y}_{\beta} \frac{1}{(N + 1)(N + 2)} \frac{1}{(N + 1)(N + 2)} \left( \frac{\partial}{\partial b} J + T \right) -
\]

\[
- e^{\alpha\beta} y_{\alpha} \bar{y}_{\beta} \frac{1}{(N + 1)(N + 2)} \Pi^{+0} \left( \frac{\partial}{\partial b} J + T \right) - e^{\alpha\beta} \partial_{\alpha} \bar{y}_{\beta} \frac{1}{(N + 1)(N + 2)} \Pi^{0} \left( \frac{\partial}{\partial b} J + T \right).
\]  

(3.17)

Conditions on \( Y \)-powers for \( J \) and \( T \) now can be formulated as follows

\[
(N + \bar{N}) J \geq \left( m \frac{\partial}{\partial m} - 2f \frac{\partial}{\partial f} \right) J, \quad |N - \bar{N}| J \leq \left( m \frac{\partial}{\partial m} - 2f \frac{\partial}{\partial f} \right) J, \tag{3.18}
\]

\[
(N + \bar{N}) T \geq \left( m \frac{\partial}{\partial m} - 4 - 2f \frac{\partial}{\partial f} \right) T, \quad |N - \bar{N}| T \leq \left( m \frac{\partial}{\partial m} - 4 - 2f \frac{\partial}{\partial f} \right) T. \tag{3.19}
\]

Finally, to get an off-shell completion of unfolded HS system (2.11)-(2.12) we must couple it to unfolded HS currents (3.13), (3.17) we found. The result is

\[
D^L \omega (Y|x) + e^{\alpha\beta} y_{\alpha} \bar{y}_{\beta} \Pi^{-} \omega (Y|x) + e^{\alpha\beta} \partial_{\alpha} \bar{y}_{\beta} \Pi^{+} \omega (Y|x) =
\]

\[
= \frac{i}{4} H^{\alpha\beta} \partial_{\alpha} \bar{y}_{\beta} C(0, \bar{y}|x) + \frac{i}{4} H^{\alpha\beta} \partial_{\alpha} C(y, 0|x) +
\]

\[ + \left( \frac{i}{4} H^{\alpha\beta} \partial_{\alpha} \bar{y}_{\beta} \Pi^{+0} \frac{N! (\bar{N} - 2)!}{N + \bar{N}} \int_{m=0}^{\infty} \frac{dm}{2\pi i m} J \left( \frac{1}{m}, \frac{1}{\bar{m}} \right) \right) \]

\[ + \frac{i}{4} H^{\alpha\beta} y_{\alpha} y_{\beta} \frac{N! (\bar{N} + 1)! N!}{(N + 3)! (N + \bar{N} + 4)} \Pi^{+0} \int_{m=0}^{\infty} \frac{dm}{2\pi i m^5} T \left( \frac{1}{m}, \frac{1}{\bar{m}} \right) + h.c. \right|_{b=f=0}(3.20).
\]
\[ D^L C (Y|x) + ie^{\alpha \beta} \partial_{\alpha} \bar{\partial}_{\beta} C (Y|x) = -ie^{\alpha \beta} y_{\alpha} \bar{y}_{\beta} \frac{1}{(N+1)(N+2)} J - \]
\[ - \left( e^{\alpha \beta} y_{\alpha} \bar{y}_{\beta} \frac{1}{(N+1)(N+2)(N-N+2)} \Pi^{+0} \int_{m=0}^\infty \frac{dm}{2\pi i m^3} J \left( \frac{1}{m} y, \bar{m}y \right) + \right. \]
\[ + ie^{\alpha \beta} y_{\alpha} \bar{y}_{\beta} \frac{1}{(N+1)(N+2)(N-N)} \Pi^{+} \int_{m=0}^\infty \frac{dm}{2\pi i m} J \left( \frac{1}{m} y, \bar{m}y \right) + h.c. \right) _{b=f=0}. \tag{3.21} \]

As usual, all coefficients before HS sources get fixed by consistency requirement (2.2). Contour integrals in (3.20)-(3.21) ensure that only a current of spin-s (determined by power of \( m \)) sources gauge field of spin-s (determined by powers of \( y \) and \( \bar{y} \)): only when their spins coincide a pole arises. Coupling of a current of some spin to a gauge field of another spin is forbidden by consistency. Because new master-fields \( J \) and \( T \), introduced to build off-shell completion, are 0-forms, it means, according to (2.3), that the gauge symmetries did not changed under this completion, preserving their correct form. An on-shell reduction of the system (3.20)-(3.21), (3.13), (3.17) is trivially achieved by putting \( J = T = 0 \).

4 Two-point functions

One of the advantage of the proposed approach to building an off-shell extension in comparison with the 'traceful' one is that introducing auxiliary HS sources allows the direct looking for the partition function of the corresponding quantum theory. The idea is that one can treat a proper sector of the off-shell unfolded equations as the functional Schwinger–Dyson equations that determine the partition function. In standard QFT if the classical equations of motion for fields \( \{ \varphi_n \} \) are
\[ \frac{\delta S}{\delta \varphi_k} (\varphi) = 0, \tag{4.1} \]
then corresponding Schwinger–Dyson equations for the partition function \( Z [J] = \int \mathcal{D} \varphi e^{iS[\varphi] + iJ\varphi^m} \) are
\[ \frac{\delta S}{\delta \varphi_n} \left[ -i \frac{\delta}{\delta J_k} (x) \right] Z = -J_n (x) Z. \tag{4.2} \]
Introducing generating functional of connected correlators \( W = -i \log Z \) and considering a free theory with linear e.o.m. one can reformulate (4.2) as
\[ \frac{\delta S_{\text{free}}}{\delta \varphi_n} \left[ \frac{\delta W}{\delta J_k} (x) \right] = -J_n (x). \tag{4.3} \]
(For construction of Schwinger–Dyson equations in a non-Lagrangian case see [20].) A transition from (4.1) to (4.3) is similar to the off-shell extension we considered.

Let us illustrate this scheme by evaluating two-point functions of free HS fields from the off-shell system we found. One can solve corresponding equations from (3.20)-(3.21) for primary HS fields in terms of \( J \) and \( T \) and then treat them as derivatives of \( W \) with respect to the corresponding sources. In Cartesian coordinates (2.7) equations of the form
\[ \left( D^L + ie^{\alpha \beta} \partial_{\alpha} \bar{\partial}_{\beta} \right) C (Y|x) = F (Y|x), \tag{4.4} \]
\[ \left( D^L + e^{\alpha \beta} y_{\alpha} \bar{\partial}_{\beta} \Pi^+ + e^{\alpha \beta} \partial_{\alpha} \bar{y}_{\beta} \Pi^+ \right) \omega (Y|x) = G (Y|x) \tag{4.5} \]
for 0-form $C$ and 1-form $\omega = e^{\alpha \hat{\omega}} \omega_{\alpha \hat{\alpha}}$ sourced by 1-form $F = e^{\alpha \hat{\alpha}} F_{\alpha \hat{\alpha}}$ and 2-form $G = \frac{1}{2} H^\alpha \beta G_{\alpha \beta} + \frac{1}{2} \hat{H}^\alpha \beta \hat{G}_{\alpha \beta}$ can be solved in the following manner

$$C (Y|x) = - \frac{i}{2 (2\pi)^8} \int d^4 p \int d^4 z \frac{e^{ip(x-z)}}{p^2} (p_{\alpha \hat{\alpha}} - \partial_{\alpha} \partial_{\hat{\alpha}}) F^{\alpha \hat{\alpha}} (Y|z) ,$$

$$\omega_{\alpha \hat{\alpha}} (Y|x) = - \frac{1}{2 (2\pi)^8} \int d^4 p \int d^4 z \frac{e^{ip(x-z)}}{p^2} \left[ i p^\gamma \alpha G_{\alpha \gamma} (Y|z) + i p^\gamma \alpha \hat{G}_{\alpha \gamma} (Y|z) - \hat{y} \partial_{\alpha} \Pi^+ \hat{G}_{\alpha \gamma} (Y|z) - y \partial_{\alpha} \Pi^- G_{\alpha \gamma} (Y|z) - y_{\alpha} \partial_{\beta} \Pi^- \hat{G}_{\alpha \gamma} (Y|z) \right] .$$

In checking that these formulas indeed solve (4.4)-(4.5) one has to use consistency conditions (2.2) for $F$ and $G$.

Then from (3.20) and (3.17) one finds an expression for the (traceless components of) Fronsdal spin-$s > 1$ field $\phi_{\alpha(s), \hat{\alpha}(s)} = ((s-1)!^2 \omega_{\alpha(s-1), \hat{\alpha}(s-1)} e^{2 s_{\alpha \hat{\alpha}}}$ as a function of currents

$$\phi_{\alpha(s), \hat{\alpha}(s)} (x) = \frac{i ((s-1)!^2}{4 (2\pi)^8 (2s)!} \int d^4 p \int d^4 z \frac{e^{ip(x-z)}}{p^2} \left( (1 + s + s^2) J_{\alpha(s), \hat{\alpha}(s)} (z) - s^2 p_{\alpha \hat{\alpha}} p^{\beta \hat{\beta}} J_{\beta(s-1), \hat{\beta}(s-1)} (z) \right) .$$

The second $p$-proportional term in fact can be chosen arbitrarily due to the gauge symmetry (2.3), which for $\phi_{\alpha(s), \hat{\alpha}(s)}$ has the form

$$\delta \phi_{\alpha(s), \hat{\alpha}(s)} (x) = i \int d^4 p e^{ip x} p_{\alpha \hat{\alpha}} \epsilon_{\alpha(s-1), \hat{\alpha}(s-1)} (p) ,$$

for arbitrary $\epsilon_{\alpha(s-1), \hat{\alpha}(s-1)} (p)$. So for simplicity we gauge away the second term in (4.8) (impose the Feynman gauge)

$$\phi_{\alpha(s), \hat{\alpha}(s)} (x) = \frac{i ((s-1)!^2}{4 (2\pi)^8 (2s)!} (1 + s + s^2) \int d^4 p \int d^4 z \frac{e^{ip(x-z)}}{p^2} J_{\alpha(s), \hat{\alpha}(s)} (z) .$$

Analogously one can find expressions for electromagnetic and scalar fields, using (3.21).

Now treating (4.10) in the spirit of (4.3) as the $J$-derivative of $W$

$$\phi_{\alpha(s), \hat{\alpha}(s)} (x) = - \frac{\delta W}{\delta J_{\alpha(s), \hat{\alpha}(s)}} (x)$$

and considering that connected correlators are given by $G_{\alpha(s)}^\alpha (x_1, ... x_n) = \left. \frac{\delta}{\iota \delta J (x_1)} ... \frac{\delta}{\iota \delta J (x_n)} iW \right|_{J=0}$ one finds two-point correlation functions for traceless HS fields

$$\left< \phi_{\alpha(s), \hat{\alpha}(s)} (x_1) \phi_{\beta(s), \hat{\beta}(s)} (x_2) \right> = k_s \int d^4 p \frac{e^{ip(x_1-x_2)}}{p^2} (\epsilon_{\alpha \beta})^s (\epsilon_{\hat{\alpha} \hat{\beta}})^s ,$$

with $k_s$ being $s$-dependent constants that can be properly normalized by rescaling of $J$. 

9
5 Conclusion

In this paper we found an off-shell completion for unfolded system of free HS fields in 4d flat spacetime in spinorial formalism. In order to do this we derived an unfolded description of HS currents. These currents, when coupled to on-shell unfolded HS equations, remove differential constraints on HS fields and get an interpretation of descendants of these fields. As byproduct we found an unfolded spinorial description of unconstrained traceless Lorentz tensor field of arbitrary rank, whose reduction, in particular, corresponds to conformal HS currents, which may help in the study of this subject. We also showed that such an off-shell unfolded system can be reinterpreted as Schwinger–Dyson system, and restored two-point functions of HS fields this way.

The next issue to be studied is the deformation of the presented system to AdS space, where the full-fledged nonlinear HS theory lives. One may hope that at the end of the day this will allow one to build an off-shell completion for the full 4d Vasiliev theory, making possible the systematic study of HS action principle problem as well as the quantization issue.

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