HOMOLOGICAL PROPERTIES OF ORLIK-SOLOMON ALGEBRAS

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Abstract. The Orlik-Solomon algebra of a matroid can be considered as a quotient ring over the exterior algebra $E$. At first we study homological properties of $E$-modules as e.g. complexity, depth and regularity. In particular, we consider modules with linear injective resolutions. We apply our results to Orlik-Solomon algebras of matroids and give formulas for the complexity, depth and regularity of such rings in terms of invariants of the matroid. Moreover, we characterize those matroids whose Orlik-Solomon ideal has a linear projective resolution and compute in these cases the Betti numbers of the ideal.

1. Introduction

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an essential central affine hyperplane arrangement in $\mathbb{C}^m$, $X$ its complement and $K$ a field. We choose linear forms $\alpha_i \in (\mathbb{C}^m)^* \text{ such that } \text{Ker} \alpha_i = H_i$ for $i = 1, \ldots, n$. Let $E = K\langle e_1, \ldots, e_n \rangle$ be the standard graded exterior algebra over $K$ where $\deg e_i = 1$ for $i = 1, \ldots, n$ and $m = (e_1, \ldots, e_n)$. For $S = \{j_1, \ldots, j_t\} \subseteq [n] = \{1, \ldots, n\}$ we set $e_S = e_{j_1} \wedge \cdots \wedge e_{j_t}$. Usually we assume that $1 \leq j_1 < \cdots < j_t \leq n$. The elements $e_S$ are called monomials in $E$. It is well-known that the singular cohomology $H^*(X; K)$ of $X$ with coefficients in $K$ is isomorphic to $E/J$ where $J$ is the Orlik-Solomon ideal of $X$ which is generated by all

\[ \partial e_S = \sum_{i=1}^t (-1)^{i-1} e_{j_1} \wedge \cdots \wedge \hat{e}_{j_i} \wedge \cdots \wedge e_{j_t} \text{ for } S = \{j_1, \ldots, j_t\} \subseteq [n] \]

where $\{H_{j_1}, \ldots, H_{j_t}\}$ is a dependent set of hyperplanes of $\mathcal{A}$, i.e. $\alpha_{j_1}, \ldots, \alpha_{j_t}$ are linearly dependent. The algebra $E/J$ is also known as the Orlik-Solomon algebra of $X$. In the last decades many researchers have studied the relationship between ring properties of $E/J$ and properties of $\mathcal{A}$. See, e.g., the book of Orlik-Terao [15] and the survey of Yuzvinsky [21] for details.

Note that the definition of $E/J$ does only depend on the matroid of $\mathcal{A}$ on $[n]$. For an arbitrary matroid on $[n]$ the Orlik-Solomon algebra $E/J$ is defined as in the case of hyperplane arrangements, i.e. $J$ is the ideal generated by all $\partial e_S$ defined as in (1) where $S \subseteq [n]$ is a dependent set of the given matroid. We are in particular interested to investigate (co-)homological properties of Orlik-Solomon algebras as modules over $E$. See, e.g., [7, 9, 17, 18, 19] for related results.

In the first part of the paper we consider arbitrary graded modules over the exterior algebra and we study several algebraic and homological invariants of such modules. In the second part of the paper we apply these results to Orlik-Solomon algebras of matroids on $[n]$. 
Let \(\mathcal{M}\) be the category of finitely generated graded left and right \(E\)-modules \(M\) satisfying \(am = (-1)^{\deg a \deg m}ma\) for homogeneous elements \(a \in E\), \(m \in M\). For example if \(J \subseteq E\) is a graded ideal, then \(E/J\) belongs to \(\mathcal{M}\).

Let \(M \in \mathcal{M}\). Following \([1]\) we call an element \(v \in E_1\) regular on \(M\) (or \(M\)-regular) if the annihilator \(\mathfrak{a}_M\) of \(v\) in \(M\) is the smallest possible, that is, the submodule \(vM\). An \(M\)-regular sequence is a sequence \(v_1, \ldots, v_s\) in \(E_1\) such that \(v_i\) is \(M/(v_1, \ldots, v_{i-1})M\)-regular for \(i = 1, \ldots, s\) and \(M/(v_1, \ldots, v_s)M \neq 0\). Every \(M\)-regular sequence can be extended to a maximal one and all maximal regular sequences have the same length. This length is called the depth of \(M\) over \(E\) and is denoted by \(\text{depth} M\).

For \(i \in \mathbb{N}\) and \(j \in \mathbb{Z}\) we call \(\beta_i(M) = \dim_k \text{Tor}^E_j(K, M)\) the graded \(E\)-modules \(M\) and \(\mu_i,j(M) = \dim_k \text{Ext}^E_j(K, M)\) the graded \(E\)-modules \(M\). Recall that \(M \in \mathcal{M}\) has a \(d\)-linear (projective) resolution if \(\beta_{i+j}(M) = 0\) for all \(i\) and \(j \neq d\). We say that \(M \in \mathcal{M}\) has a \(d\)-linear injective resolution if \(\mu_{i+j}(M) = 0\) for all \(i\) and \(j \neq d\). (See Section 2 for reformulations of this definitions.) The complexity of \(M\) measures the growth rate of the Betti numbers of \(M\) and is defined as

\[
\text{cx} M = \inf \{ c \in \mathbb{N} : \beta_i(M) \leq \alpha i^{-1} \text{ for all } i \geq 1, \alpha \in \mathbb{R} \}
\]

where \(\beta_i(M) = \sum_{j \in \mathbb{Z}} \beta_{i,j}(M)\) is the \(i\)-th total Betti number of \(M\).

Aramova, Herzog and Hibi \([3]\) showed that analogously to the situation in a polynomial ring Gröbner basis theory can be developed over \(E\). Especially generic initial ideals can be constructed. In the following the monomial order considered on \(E\) is always the reverse lexicographic order induced by \(e_1 > \cdots > e_n\). Let \(\text{in} (J)\) denote the initial ideal and \(\text{gin} (J)\) denote the generic initial ideal of a graded ideal \(J \subseteq E\). For all results related to generic initial ideals we assume that \(|K| = \infty\). After some definitions and general remarks in Section 2 we consider in Section 3 the ideal \(\text{gin} (J)\) and study relations between \(E/J\) and \(E/\text{gin} (J)\). In \([3]\) it is observed that \(\beta_{i,j}(E/J) \leq \beta_{i,j}(E/\text{in} (J))\) for all \(i, j\). In Corollary 3.2 we show that also

\[
\mu_{i,j}(E/J) \leq \mu_{i,j}(E/\text{in} (J)) \text{ for all } i, j.
\]

Herzog and Terai proved in \([13]\) Proposition 2.3 that \(\text{depth} E/J = \text{depth} E/\text{gin} (J)\) and \(\text{cx} E/J = \text{cx} E/\text{gin} (J)\). These numbers can be computed in terms of combinatorial data associated to a generic initial ideal. More precisely, let \(\text{supp}(u) = \{ i \in [n] : e_i | u \} \) and \(\text{max}(u) = \text{max} \text{supp}(u)\) for a monomial \(u\) of \(E\). Similar we define \(\text{min}(u) = \text{min} \text{supp}(u)\). A direct consequence of Proposition 3.4 and Proposition 3.5 is that

\[
\begin{align*}
\text{cx} E/\text{gin} (J) & = \max \{ \text{max}(u) : u \in G(\text{gin} (J)) \}, \\
\text{d}(E/\text{gin} (J)) & = n - \max \{ \text{min}(u) : u \in G(\text{gin} (J)) \}
\end{align*}
\]

where \(G(\text{gin} (J))\) denotes the unique minimal set of monomial generators of \(\text{gin} (J)\) and \(d(M) = \max \{ i \in \mathbb{Z} : M_i \neq 0 \}\) for \(M \in \mathcal{M}\). Using the formula \(\text{cx} M + \text{depth} M = n\) (see \([1]\) Theorem 3.2) we get also an expression for \(\text{depth} E/\text{gin} (J)\).

In Section 4 we present some results related to \(\text{depth} M\). Let \(H(M,t) = \sum_{i \in \mathbb{Z}} \dim_k M_i t^i\) denote the \(Hilbert series\) of \(M\). Then depth of \(E/J\) where \(J \subseteq E\) is a graded ideal and \(E/J\) has a linear injective resolution can be computed as follows. We show in Theorem 4.1 that if \(|K| = \infty\), \(E/J\) has a linear injective resolution and \(\text{depth} E/J = s\), then there exists
a polynomial $Q(t) \in \mathbb{Z}[t]$ with non-negative coefficients such that
\[ H(E/J, t) = Q(t) \cdot (1 + t)^s \quad \text{and} \quad Q(-1) \neq 0. \]
Observe that it is not possible to generalize this equation in this form to the case of arbitrary quotient rings over $E$.

The $K$-algebra $E$ is injective and thus $(\cdot)^* = \text{Hom}_E(\cdot, E)$ is an exact functor on $\mathcal{M}$. By [3] Proposition 5.2 we know that $\mu_{i,j}(M) = \beta_{i,n-j}(M^*)$ for all $i, j$. In particular, we see that $M$ has a $d$-linear projective resolution if and only if $M^*$ has an $(n-d)$-linear injective resolution. In Theorem 4.3 we observe that additionally
\[ \text{depth} M = \text{depth} M^* \quad \text{and} \quad \text{cx} M = \text{cx} M^*. \]

Let $\Delta$ be a simplicial complex on $[n]$, i.e. $\Delta$ is a set of subsets of $[n]$ and if $F \subseteq G$ for some $G \in \Delta$, then we also have $F \in \Delta$. The exterior face ring of $\Delta$ is $E/I_\Delta$ where $I_\Delta = (e_F : F \subseteq [n], F \notin \Delta)$. Then it is easy to see that $E/I_\Delta$ has a linear injective resolution if and only if $\Delta$ is a Cohen-Macaulay complex. See Example 5.1 for details. A reformulation and generalization to the matroid case of [9, Theorem 1.1] is that the Orlik-Solomon algebra of a matroid has always a linear injective resolution. These examples motivate to study in general modules with linear injective resolutions, which is done in Section 5.

Recall that $\text{reg} M = \max \{ j - i : \beta_{i,j}(M) \neq 0 \}$ for $0 \neq M \in \mathcal{M}$ is the regularity of $M$. We prove in Theorem 5.3 that the regularity of a quotient ring $E/J$ with $d$-linear injective resolution satisfies
\[ \text{reg} E/J + \text{depth} E/J = d. \]

In the remainder of Section 5 we present several technical results related to modules with injective linear resolutions which we need in Section 6.

In Section 6 we investigate homological properties of Orlik-Solomon algebras of matroids. For convenience of the reader we start with all necessary matroid notions. At first we present a compact proof of the mentioned result of Eisenbud, Popescu and Yuzvinsky (see [9] Theorem 1.1) that Orlik-Solomon algebras have a linear injective resolution. We determine the depth and the regularity of an Orlik-Solomon algebra in Theorem 6.5 and Corollary 6.7. More precisely, if $|K| = \infty$ and $J \subseteq E$ is the Orlik-Solomon ideal of a loopless matroid on $[n]$ of rank $l$ with $k$ components, then
\[ \text{depth} E/J = k \quad \text{and} \quad \text{reg} E/J = l - k. \]

Finally we characterize in Theorem 6.10 those matroids whose Orlik-Solomon ideal has a linear projective resolution: The Orlik-Solomon ideal $J$ of a matroid has an $m$-linear projective resolution if and only if the matroid satisfies one of the following three conditions:

(i) The matroid has a loop and $m = 0$.
(ii) The matroid has no loops, but non-trivial parallel classes, $m = 1$ and the matroid is $U_{1,n_1} \oplus \cdots \oplus U_{1,n_k} \oplus U_{f,f}$ for some $k, f \geq 0$.
(iii) The matroid is simple and it is $U_{m,n-f} \oplus U_{f,f}$ for some $0 \leq f \leq n$.

Here $U_{m,n}$ is the uniform matroid, whose independent sets are all subsets of $[n]$ with $m$ or less elements. In Theorem 6.12 we give formulas for the total Betti numbers of these Orlik-Solomon ideals.

We conclude the paper with examples of matroids with small rank or small number of elements to which we apply our results.
2. Preliminaries

In this section we recall some definitions and facts about the exterior algebra. Let \( M \in \mathcal{M} \) with minimal graded free resolution
\[
\ldots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow 0.
\]
To distinguish it from the injective resolution of \( M \), we call this resolution also the projective one. Because of the minimality the \( i \)-th free module in this resolution is \( F_i = \bigoplus_{j \in \mathbb{Z}} E(-j)^{\beta_i(M)} \). We see that the resolution is \( d \)-linear (as defined in Section I) for some \( d \in \mathbb{Z} \) if and only if it is of the form
\[
\ldots \rightarrow E(-d-2)^{\beta_2(M)} \rightarrow E(-d-1)^{\beta_1(M)} \rightarrow E(-d)^{\beta_0(M)} \rightarrow 0.
\]
This is equivalent to say that if we choose matrices for the maps in the resolution, then all entries in these matrices are elements in \( m = (e_1, \ldots, e_n) \) of degree 1.

Next we consider for \( M \) its minimal graded injective resolution
\[
0 \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \ldots
\]
Since \( E \) is injective, we have \( I^i = \bigoplus_{j \in \mathbb{Z}} E(n-j)^{\mu_i,j(M)} \). Computing \( \text{Ext}^E_*(K,M) \) via the latter resolution shows that this resolution is \( d \)-linear if and only if it is of the form
\[
0 \rightarrow E(n-d)^{\mu_0(M)} \rightarrow E(n-d+1)^{\mu_1(M)} \rightarrow E(n-d)^{\mu_2(M)} \rightarrow \ldots
\]
In particular, in this case \( d(M) = \max \{ i : M_i \neq 0 \} = d \) because the socle 0 :\( _M m \) of \( M \) is isomorphic to the socle of \( E(n-d)^{\mu_0(M)} \) which lives in degree \( d \).

Let \( M^* = \text{Hom}_K(M,E) \). We call \( M^* \) the dual of \( M \). Note that the dual of a (minimal) graded projective resolution of \( M \) is a (minimal) graded injective resolution of \( M^* \).

For a \( K \)-vector space \( W \) let \( W^\vee = \text{Hom}_K(W,K) \) be the \( K \)-dual of \( W \). In [2 Proposition 5.1] it was observed that \( (M^*)_i \cong (M_{n-i})^\vee \) as \( K \)-vector spaces.

A very useful complex over \( E \) is the Cartan complex which plays a similar role as the Koszul complex for the polynomial ring. It is defined as follows. For a sequence \( v = v_1, \ldots, v_m \in E_1 \) let \( C_i(v;E) = C_i(v_1, \ldots, v_m;E) \) be the free divided power algebra \( E(x_1, \ldots, x_m) \). It is generated by the divided powers \( x_i^{(j)} \) for \( i = 1, \ldots, m \) and \( j \geq 0 \) which satisfy the relations \( x_i^{(j)} x_k^{(l)} = ((j+k)!(j!k!)) x_{i+k}^{(j+l)} \). Thus \( C_i(v;E) \) is a free \( E \)-module with basis \( x^{(a)} = x_1^{(a_1)} \cdots x_m^{(a_m)} \), \( a \in \mathbb{N}^m \), \( |a| = i \). The \( E \)-linear differential on \( C_i(v_1, \ldots, v_m;E) \) is
\[
\partial_i : C_i(v_1, \ldots, v_m;E) \rightarrow C_{i-1}(v_1, \ldots, v_m;E), \quad x^{(a)} \mapsto \sum_{a_j > 0} v_j x_1^{(a_1)} \cdots x_j^{(a_j-1)} \cdots x_m^{(a_m)}.
\]
One easily sees that \( \partial \circ \partial = 0 \) so this is indeed a complex.

**Definition 2.1.** Let \( M \in \mathcal{M} \). The complexes
\[
C_i(v;M) = C_i(v;E) \otimes E M \quad \text{and} \quad C^*(v;M) = \text{Hom}_E(C_i(v;E),M)
\]
are called the **Cartan complex** and **Cartan cocomplex** of \( v \) with values in \( M \). The corresponding homology modules
\[
H_i(v;M) = H_i(C_i(v;M)) \quad \text{and} \quad H^i(v;M) = H^i(C^*(v;M))
\]
are called the **Cartan homology** and **Cartan cohomology** of \( v \) with values in \( M \).
The elements of $C'(v;M)$ can be identified with homogeneous polynomials $\sum m_a y^a$ with $m_a \in M$, $a \in \mathbb{N}^m$, $|a| = i$ and

$$\sum m_a y^a(x^b) = \begin{cases} m_a & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases}$$

In particular, $C'(v;E)$ is just the polynomial ring $S = K[y_1, \ldots, y_m]$. After this identifications the differential on $C'(v;M)$ is simply the multiplication with $\sum_{i=1}^m v_i y_i$.

Setting $\deg x_i = 1$ and $\deg y_i = -1$ induces a grading on the complexes and their homologies. Cartan homology and Cartan cohomology are related as follows:

**Proposition 2.2.** [2, Proposition 4.2] Let $M \in \mathcal{M}$ and $v = v_1, \ldots, v_m \in E_1$. Then

$$H_i(v;M)^* \cong H^i(v;M^*)$$

as graded $E$-modules for all $i \in \mathbb{N}$.

Cartan (co)homology can be used inductively as there are long exact sequences connecting the (co)homologies of $v_1, \ldots, v_j$ and $v_1, \ldots, v_{j+1}$.

**Proposition 2.3.** [2, Propositions 4.1, 4.3] Let $M \in \mathcal{M}$ and $v = v_1, \ldots, v_m \in E_1$. For all $j = 1, \ldots, m$ there exist long exact sequences of graded $E$-modules

$$\cdots \rightarrow H_i(v_1, \ldots, v_{j}; M) \rightarrow H_i(v_1, \ldots, v_{j+1}; M) \rightarrow H_{i-1}(v_1, \ldots, v_{j+1}; M)(-1) \rightarrow \cdots$$

and

$$\cdots \rightarrow H_{i-1}(v_1, \ldots, v_{j}; M) \rightarrow H_{i-1}(v_1, \ldots, v_{j+1}; M) \rightarrow H_i(v_1, \ldots, v_{j+1}; M) \rightarrow \cdots$$

It is well-known that the Cartan complex $C(v_1, \ldots, v_m; E)$ with values in $E$ is exact and hence it is the minimal graded free resolution of $E/(v_1, \ldots, v_m)$ over $E$. Thus it can be used to compute $\text{Tor}_i^E(E/(v_1, \ldots, v_m), \cdot)$ and $\text{Ext}_i^E(E/(v_1, \ldots, v_m), \cdot)$:

**Proposition 2.4.** [3, Theorem 2.2] Let $M \in \mathcal{M}$ and $v = v_1, \ldots, v_m \in E_1$. There are isomorphisms of graded $E$-modules

$$\text{Tor}_i^E(E/(v_1, \ldots, v_m), M) \cong H_i(v;M), \quad \text{Ext}_i^E(E/(v_1, \ldots, v_m), M) \cong H^i(v;M).$$

Regularity of a sequence can be detected by its Cartan complex:

**Proposition 2.5.** [1, Remark 3.4] Let $M \in \mathcal{M}$ and $v = v_1, \ldots, v_m \in E_1$. The following statements are equivalent:

(i) $v$ is $M$-regular;
(ii) $H_1(v;M) = 0$;
(iii) $H_i(v;M) = 0$ for $i \geq 1$.

In particular, permutations of regular sequences are regular sequences because the vanishing of the first Cartan homology does not depend on the order of the elements as one easily sees using Proposition 2.3.
3. INITIAL AND GENERIC INITIAL IDEALS

In this section we describe some properties of generic initial ideals and stable ideals. The existence of the generic initial ideal \( \text{gin}(J) \) of a graded ideal \( J \) in the exterior algebra over an infinite field is proved by Aramova, Herzog and Hibi in [3, Theorem 1.6], analogously to the case of ideals in the polynomial ring. (See, e.g., also [11, Chapter 5] or [14] for related results.)

A monomial ideal \( J \subseteq E \) is called stable if \( e_j \frac{u}{e_{\max(u)}} \in J \) for every monomial \( u \in J \) and \( j < \max(u) \). The ideal \( J \) is called strongly stable if \( e_j \frac{u}{e_i} \in J \) for every monomial \( u \in J \), \( i \in \text{supp}(u) \) and \( j < i \).

The generic initial ideal \( \text{gin}(J) \) of a graded ideal \( J \) is strongly stable if it exists (see, e.g., [3, Proposition 1.7]). This is independent of the characteristic of \( K \) in contrast to ideals in a polynomial ring. By (the proof of) [13, Lemma 1.1] we have:

**Lemma 3.1.** Let \( |K| = \infty \) and \( J \subseteq E \) be a graded ideal in \( E \). Then

\[
\text{in}((E/J)^*) \cong (E/\text{in}(J))^*
\]

as graded \( E \)-modules, where \((E/J)^*\) is identified with the ideal \( 0 :_E J \). In particular,

\[
\text{gin}((E/J)^*) \cong (E/\text{gin}(J))^*.
\]

With this result we can compare the Bass numbers of a graded ideal with the Bass numbers of its initial ideal because we already know \( \beta_{i,j}(E/J) \leq \beta_{i,j}(E/\text{in}(J)) \) for all \( i, j \) by [3 Proposition 1.8].

**Corollary 3.2.** Let \( J \subseteq E \) be a graded ideal. Then

\[
\mu_{i,j}(E/J) \leq \mu_{i,j}(E/\text{in}(J)) \text{ for all } i, j.
\]

**Proof.** It follows from the inequalities

\[
\beta_{i,j}(E/J) \leq \beta_{i,j}(E/\text{in}(J))
\]

and Lemma 3.1 that

\[
\mu_{i,j}(E/J) = \beta_{i,n-j}((E/J)^*) \leq \beta_{i,n-j}(\text{in}((E/J)^*)) = \beta_{i,n-j}((E/\text{in}(J))^*) = \mu_{i,j}(E/\text{in}(J)).
\]

\( \square \)

In the following we collect some results on (strongly) stable ideals. They are inspired by the chapter on squarefree strongly stable ideals in the polynomial ring in [12]. Let \( G(J) \) be the unique minimal system of monomials generators of a monomial ideal \( J \).

Aramova, Herzog and Hibi [3] computed a formula for the graded Betti numbers of stable ideals:

**Lemma 3.3.** [3 Corollary 3.3] Let \( 0 \neq J \subseteq E \) be a stable ideal. Then

\[
\beta_{i,i+j}(J) = \sum_{u \in G(J)_j} \binom{\max(u) + i - 1}{\max(u) - 1} \text{ for all } i \geq 0, j \in \mathbb{Z}.
\]
In particular, if $J$ is stable and generated in one degree, it has a linear projective resolution. An example for such an ideal is the maximal ideal $m$ of $E$ and all its powers.

The complexity of a stable ideal $J$ can be interpreted in terms of $G(J)$.

**Proposition 3.4.** Let $0 \neq J \subset E$ be a stable ideal. Then

\[
\text{cx} E/J = \max \{ \max(u) : u \in G(J) \}.
\]

**Proof.** This is evident from the formula for the Betti numbers of stable ideals since

\[
\beta_i(J) = \sum_{k=1}^{n} m_k(J) \binom{k + i - 1}{k - 1}
\]

where $m_k(J) = \{ u \in G(J) : \max(u) = k \}$. The binomial coefficient in this sum is a polynomial in $i$ of degree $k - 1$ and the number $\max \{ \max(u) : u \in G(J) \}$ is exactly the maximal $k$ for which $m_k(J) \neq 0$.

Recall that

\[
d(M) = \max \{ i \in \mathbb{Z} : M_i \neq 0 \} = n - \min \{ i \in \mathbb{Z} : (M^*)_i \neq 0 \}.
\]

Here the second equality results from the isomorphism $(M^*)_i \cong (M_{n-i})^\vee$. In the case of strongly stable ideals $J$ this number has a meaning in terms of $G(J)$.

**Proposition 3.5.** Let $0 \neq J \subset E$ be a strongly stable ideal. Then

\[
d(E/J) = n - \max \{ \min(u) : u \in G(J) \}.
\]

Observe that the right hand side of the equation does not change when replacing $G(J)$ by $J$ is strongly stable.

**Proof.** Set $s = \max \{ \min(u) : u \in G(J) \}$. We want to show that

\[
s = \min \{ i : (E/J)^s_i \neq 0 \}.
\]

As $J \subseteq (e_1, \ldots, e_s)$ we obtain “$\geq$” immediately from the equivalence

\[
J \subseteq (e_{i_1}, \ldots, e_{i_r}) \iff e_{i_1} \cdots e_{i_r} \in 0 :_E J \cong (E/J)^s
\]

for $i_1, \ldots, i_r \in [n]$.

The other inequality “$\leq$” follows if we show that $J \subseteq (e_{i_1}, \ldots, e_{i_r})$ implies $r \geq s$.

First consider the case that $J \subseteq (e_s)$. As $J$ is strongly stable, $e_i, e_{i}^u / e_s \in J$ for all monomials $u \in J$ and all $i < s$. But $e_i, e_{i}^u / e_s \not\in (e_s)$ for $i \not\in \text{supp}(u)$ and thus $i \in \text{supp}(u)$ for all $i < s$. By the definition of $s$ this implies $s = 1$ and hence $r \geq s$.

Now assume $J \not\subseteq (e_s)$ and consider the ideal $\overline{J} = J + (e_s) / (e_s)$. This ideal is again strongly stable in the exterior algebra $E$ in $n - 1$ variables $e_1, \ldots, e_{s-1}, e_{s+1}, \ldots, e_n$. The position of $e_i$ diminishes by one for one for $i > s$. We see that

\[
\min(\overline{u}) = \begin{cases} 
\min(u) & \text{if } \min(u) < s \\
\min(u) - 1 & \text{if } \min(u) > s
\end{cases}
\]

for the residue class of a monomial $u$ of $E$ with $e_s / u$. By the choice of $s$ we see immediately that $\max \{ \min(\overline{u}) : \overline{u} \in \overline{J} \} = s$. On the other hand $\overline{J} \subseteq (e_{i_1}, \ldots, e_{i_r}) + (e_s) / (e_s)$. By an appropriate induction on $n$ we get that $s \leq r$ if $s \not\in \{i_1, \ldots, i_r\}$ and $s \leq r - 1$ otherwise. \qed
4. Depth of Graded \( E \)-modules

The purpose of this section is to present further results on regular sequences over the exterior algebra.

Recall that \( H(M,t) = \sum_{i \in \mathbb{Z}} \dim K M_i t^i \) denotes the Hilbert series of a graded \( E \)-module \( M \). Analogously to the well-known Hilbert-Serre theorem (see, e.g., [5, Proposition 4.4.1]) we have the following result.

**Theorem 4.1.** Let \( |K| = \infty \) and \( 0 \neq J \subset E \) be a graded ideal with \( \text{depth} E/J = s \). Let \( E/J \) have a linear injective resolution. Then there exists a polynomial \( Q(t) \in \mathbb{Z}[t] \) with non-negative coefficients such that the Hilbert series of \( E/J \) has the form

\[
H(E/J, t) = Q(t) \cdot (1 + t)^s \quad \text{with} \quad Q(-1) \neq 0.
\]

Note that it is not possible to generalize the equation in this form to the case of arbitrary quotient rings. The ideal \( (e_1 e_2, e_1 e_3, e_1 e_4, e_2 e_3 e_4) \) provides a counterexample.

**Proof.** Let \( M = E/J \). First of all we show that if \( v \) is \( M \)-regular, then

\[
H(M, t) = (1 + t)H(M/vM, t).
\]

We have the exact sequence

\[
0 \to vM \to M \to M/vM \to 0
\]

which implies

\[
H(vM, t) = H(M, t) - H(M/vM, t).
\]

As \( v \) is \( M \)-regular the sequence

\[
0 \to vM(-1) \to M(-1) \xrightarrow{v} M \to M/vM \to 0
\]

is exact and gives

\[
(1 - t)H(M, t) = H(M/vM, t) - tH(vM, t).
\]

Equations (3) and (4) together show (2).

Thus if \( v_1, \ldots, v_s \) is a maximal \( E/J \)-regular sequence, we obtain inductively

\[
H(E/J, t) = (1 + t)^s H(E/J + (v_1, \ldots, v_s), t).
\]

The Hilbert series of \( E/(J + (v_1, \ldots, v_s)) \) is a polynomial with nonnegative coefficients and \( \text{depth} E/(J + (v_1, \ldots, v_s)) = 0 \). We claim that the polynomial \( 1 + t \) does not divide \( H(E/(J + (v_1, \ldots, v_s)), t) \).

To this end we may assume that \( \text{depth} E/J = 0 \). The Hilbert series and the depth of \( E/J \) and \( E/\text{gin}(J) \) coincide, so we may assume in addition that \( J \) is strongly stable. Then we know the Betti numbers of \( J \). Proving that \( 1 + t \) does not divide the Hilbert series of \( E/J \) is the same as showing this for \( J \) as the Hilbert series of \( E \) is \( (1 + t)^n \).
Let \( m_{kj}(J) = |\{u \in G(J) : \max(u) = k, \deg(u) = j\}|. \) Computing the Hilbert series of \( J \) via the minimal graded free resolution of \( J \) gives

\[
H(J,t) = \sum_{i,j} \frac{(-1)^i}{i!} \sum_{j \in \mathbb{Z}} t^j \beta_{ij}(J)(1+t)^n
\]

\[
= \sum_{i,j} (-1)^i \sum_{j \in \mathbb{Z}} t^j \beta_{ij}(J)(1+t)^n \sum_{u \in G(J)} \left( \max(u) + i - 1 \right)
\]

\[
= \sum_{i \in \mathbb{Z}} (-1)^i \sum_{j \in \mathbb{Z}} t^{ij} \beta_{ij}(J)(1+t)^n \sum_{u \in G(J)} \left( \max(u) + i - 1 \right)
\]

\[
= \sum_{i \in \mathbb{Z}} (-1)^i \sum_{j \in \mathbb{Z}} t^{ij} \beta_{ij}(J)(1+t)^n \sum_{u \in G(J)} \left( \max(u) + i - 1 \right)
\]

\[
= \sum_{i \in \mathbb{Z}} (-1)^i \sum_{j \in \mathbb{Z}} t^{ij} \beta_{ij}(J)(1+t)^n \sum_{u \in G(J)} \left( \max(u) + i - 1 \right)
\]

\[
= \sum_{i \in \mathbb{Z}} (-1)^i \sum_{j \in \mathbb{Z}} t^{ij} \beta_{ij}(J)(1+t)^n \sum_{u \in G(J)} \left( \max(u) + i - 1 \right)
\]

\[
= \sum_{i \in \mathbb{Z}} (-1)^i \sum_{j \in \mathbb{Z}} t^{ij} \beta_{ij}(J)(1+t)^n \sum_{u \in G(J)} \left( \max(u) + i - 1 \right)
\]

\[
= \sum_{i \in \mathbb{Z}} (-1)^i \sum_{j \in \mathbb{Z}} t^{ij} \beta_{ij}(J)(1+t)^n \sum_{u \in G(J)} \left( \max(u) + i - 1 \right)
\]

\[
= \sum_{i \in \mathbb{Z}} (-1)^i \sum_{j \in \mathbb{Z}} t^{ij} \beta_{ij}(J)(1+t)^n \sum_{u \in G(J)} \left( \max(u) + i - 1 \right)
\]

\[
= \sum_{i \in \mathbb{Z}} (-1)^i \sum_{j \in \mathbb{Z}} t^{ij} \beta_{ij}(J)(1+t)^n \sum_{u \in G(J)} \left( \max(u) + i - 1 \right)
\]

All coefficients appearing in the last sum are non-negative hence no term can be canceled by another. \( n - cx E/J = \text{depth} E/J = 0 \) and Proposition [3,4] imply that \( m_{nj}(J) \neq 0 \) for some \( j \). Let \( u = e_F e_n \in G(J) \). We have \( e_F e_i \in J \) for all \( i = 1, \ldots, n \) because \( J \) is stable. The dual of \( E/J \) is \( (E/J)^* \cong 0 : E J \), which is generated by all monomials \( e_F \) with \( e_F \notin J \) (cf. Example [5.1]). But then \( e_F (\cup i)^* = e_F \setminus \{i\} \notin (E/J)^* \) for all \( i \notin F \). As \( e_F \notin J \) (otherwise \( e_F e_n \) would not be a minimal generator), the complement \( e_F \) is in \( (E/J)^* \) and even a minimal generator. The ideal \( (E/J)^* \) has an \((n - d)\)-linear projective resolution, in particular it is generated in degree \( n - d \), so \( |F| = n - |F^c| = n - (n - d) = d \). Thus we have seen that every minimal generator \( u \in G(J) \) with \( n \in \text{supp}(u) \) has degree \( d + 1 \). Hence \( m_{nj}(J) = 0 \) for \( j \neq d + 1 \) and \( m_{n,d+1} \neq 0 \). Thus there is exactly one summand in \( H(J,t) \) that is not divisible by \( 1 + t \) and we see that \( 1 + t \) does not divide \( H(J,t) \). \( \square \)

Next we want to compare regular sequences on a module and its dual. To this end we need the following lemma. Since \( v^2 = 0 \) for \( v \in E_1 \) the multiplication map on a graded \( E\)-module \( M \) induces a complex

\[
(M, v) : \quad \ldots \rightarrow M_{i-1} \xrightarrow{\cdot v} M_i \xrightarrow{\cdot v} M_{i+1} \rightarrow \ldots
\]

The homology of this complex is denoted by \( H_i(M,v) \). Then \( v \) is regular on \( M \) if and only if \( H_i(M,v) = 0 \) for all \( i \).

**Lemma 4.2.** Let \( 0 \rightarrow U \rightarrow M \rightarrow N \rightarrow 0 \) be an exact sequence of modules in \( \mathcal{M} \). If \( v \in E_1 \) is regular on two of the three modules, then it is regular on the third.
Proof. The short exact sequence induces a short exact sequence of complexes
\[ 0 \longrightarrow (U, v) \longrightarrow (M, v) \longrightarrow (N, v) \longrightarrow 0 \]
which induces a long exact sequence of homology modules
\[ \ldots \longrightarrow H_{i-1}(N, v) \longrightarrow H_i(U, v) \longrightarrow H_i(M, v) \longrightarrow H_i(N, v) \longrightarrow H_{i+1}(U, v) \longrightarrow \ldots \]
Then the observation that \( v \) is regular on one of these modules, say \( M \), if and only if the corresponding homology \( H_i(M, v) \) is zero for all \( i \) concludes the proof \( \square \)

Let \( v_1, \ldots, v_s \in E_1 \) and \( M \in \mathcal{M} \). To simplify notation we define
\[ H_i(k) = H_i(v_1, \ldots, v_k; M) \text{ for } i > 0, \ k = 1, \ldots, s \]
and
\[ \tilde{H}_0(k) = \frac{0 : M/(v_1, \ldots, v_{k-1})M v_k}{v_k(M/(v_1, \ldots, v_{k-1})M)}. \]

Analogously
\[ H^i(k) = H^i(v_1, \ldots, v_k; M) \text{ for } i > 0, \ k = 1, \ldots, s \]
and
\[ \tilde{H}^0(k) = \frac{0 : \mathfrak{m}(v_1, \ldots, v_{k-1}) v_k}{v_k(0 : \mathfrak{m}(v_1, \ldots, v_{k-1}))}. \]

Finally we set \( H_i(0) = H^i(0) = 0 \) for \( i > 0 \). The modules \( \tilde{H}_0(k) \) and \( \tilde{H}^0(k) \) are not the 0-th Cartan homology and cohomology but defined such that the long exact sequences of Cartan homology and cohomology modules of Proposition 2.3 induces exact sequences
\[ \ldots \longrightarrow H_2(k) \longrightarrow H_1(k)(-1) \longrightarrow H_1(k-1) \longrightarrow H_1(k) \longrightarrow \tilde{H}_0(k)(-1) \longrightarrow 0 \]
and
\[ 0 \longrightarrow \tilde{H}^0(k)(+1) \longrightarrow H^1(k) \longrightarrow H^1(k-1) \longrightarrow H^1(k)(+1) \longrightarrow H^2(k) \longrightarrow \ldots \]

Theorem 4.3. Let \( 0 \neq M \in \mathcal{M} \). A sequence \( \mathbf{v} = v_1, \ldots, v_s \in E_1 \) is an \( M \)-regular sequence if and only if it is an \( M^* \)-regular sequence. In particular,
\[ \text{depth } M = \text{depth } M^* \text{ and } \text{cx } M = \text{cx } M^*. \]

Proof. We may assume that \( |K| = \infty \). It is enough to prove \( \text{depth } M = \text{depth } M^* \). Then \( \text{cx } M = \text{cx } M^* \) follows from the formula \( \text{cx } M + \text{depth } M = n \).

To prove the assertion it is enough to show that if \( \mathbf{v} \) is an \( M^* \)-regular sequence, then it is an \( M \)-regular sequence as well.

First of all we state two observations which will be used several times in the proof. Let \( N, N' \in \mathcal{M} \) and \( v \in E_1 \).

(*) \quad \text{If } v \text{ is } N' \text{-regular and } vN' \subseteq vN, \text{ then } vN \cap N' = vN'.

This is obvious since \( x \in vN \cap N' \) implies \( x \in 0 :_{N'} v = vN' \).

(**) \quad \text{If } v \text{ is regular on } N, N' \text{ and } N \cap N', \text{ then } v \text{ is regular on } N + N'.

This follows from the short exact sequence
\[ 0 \to N \cap N' \to N \oplus N' \to N + N' \to 0 \]
and Lemma 4.2.

The main task is to show by an induction on \( t \) that \( v_k \) is regular on each module of the form \( v_1 \cdots v_r (v_{j_1}, \ldots, v_{j_r}) M \) for \( \{i_1, \ldots, i_r, j_1, \ldots, j_r\} \subseteq \{1, \ldots, s\} \setminus \{k\} \) and all \( k = 1, \ldots, s \).

Then with \( r = 0, t = k - 1 \) this means that \( v_k \) is \((v_1, \ldots, v_{k-1})M\)-regular, with \( r = t = 0 \) that \( v_k \) is \( M \)-regular. Hence the exact sequence
\[ 0 \to (v_1, \ldots, v_{k-1})M \to M \to M/(v_1, \ldots, v_{k-1})M \to 0 \]
implies by Lemma 4.2 that \( v_k \) is \( (v_1, \ldots, v_{k-1})M \)-regular for all \( k = 1, \ldots, s \).

For the induction on \( t \) let \( t = 0 \). For simplicity we show that \( v_k \) is \( v_1 \cdots v_{k-1}M \)-regular. But as permutations of regular sequences are regular sequences, the proof works for arbitrary elements of the sequence as well.

The Cartan homology of \( \mathfrak{v} \) with values in \( M^* \) vanishes (see Proposition 2.5) which implies by Proposition 2.2 that the Cartan cohomology of \( \mathfrak{v} \) with values in \( M \) vanishes. In particular
\[ 0 = \tilde{H}^0(k; M) = \frac{0 :_{0, M}(v_1, \ldots, v_{k-1}) v_k}{v_k(0 :_{M}(v_1, \ldots, v_{k-1}))} \]
for all \( k = 1, \ldots, s \). We show by a second induction on \( k \) that \( v_k \) is \( v_{k-1} \cdots v_1 M \)-regular.

If \( k = 1 \) we have
\[ 0 = \tilde{H}^0(1; M) = \frac{0 :_M v_1 v_1}{v_1 M} \]
Hence \( v_1 \) is \( M \)-regular. Now suppose that the assertion is known for \( k - 1 \).

The module \( 0 :_M (v_1, \ldots, v_{k-1}) \) contains all elements of \( M \) that are annihilated by all \( v_i, i = 1, \ldots, k - 1 \). Since every \( v_i \) is \( M \)-regular (where we use the same argument as for \( v_1 \), since permutations of regular sequences are regular sequences), \( 0 :_M (v_1, \ldots, v_{k-1}) = v_{k-1} M \cap \ldots \cap v_1 M \). We show \( v_{l-1} M \cap \ldots \cap v_1 M = v_{l-1} \cdots v_1 M \) by another induction on \( l \), \( 2 \leq l \leq k \).

If \( l = 2 \) this is obvious. Now if \( l > 2 \) we have
\[ v_{l-1} M \cap \ldots \cap v_1 M = v_{l-1} M \cap (v_{l-2} M \cap \ldots \cap v_1 M) = v_{l-1} M \cap v_{l-2} \cdots v_1 M = v_{l-1} \cdots v_1 M \]
where the induction hypothesis of the induction on \( k \) is used, i.e. that \( v_{l-1} \) is regular on \( v_{l-2} \cdots v_1 M \) since \( l - 1 \leq k - 1 \).

Then
\[ 0 = \tilde{H}^0(k; M) = \frac{0 :_{0, M}(v_1, \ldots, v_{k-1}) v_k}{v_k(0 :_{M}(v_1, \ldots, v_{k-1}))} = \frac{0 :_{v_{k-1} \cdots v_1 M} v_k}{v_k(v_{k-1} \cdots v_1 M)} \]
implies that \( v_k \) is \( v_{k-1} \cdots v_1 M \)-regular. Thus we proved the basis for the induction on \( t \).

Now suppose \( t > 0 \). We decompose \( v_i \cdots v_r (v_{j_1}, \ldots, v_{j_r}) M \) in two parts. By induction hypothesis \( v_k \) is regular on \( v_i \cdots v_r (v_{j_1}, \ldots, v_{j_r}) M \) and on \( v_i \cdots v_r v_{j_i} M \). Furthermore the induction hypothesis gives that \( v_{j_i} \) is \( v_i \cdots v_r (v_{j_1}, \ldots, v_{j_r}) M \)-regular. Hence it follows from (*) that the intersection of the two parts is
\[ v_i \cdots v_r (v_{j_1}, \ldots, v_{j_i-1}) M \cap v_i \cdots v_r v_{j_i} M = v_{j_i} v_i \cdots v_r (v_{j_1}, \ldots, v_{j_i-1}) M. \]
Again by induction hypothesis \( v_k \) is regular on this intersection. So \((**)\) implies that
\( v_k \) is regular on \( v_i \cdots v_r(v_{j_1}, \ldots, v_{j_{r-1}})M + v_i \cdots v_rv_{j_i}M = v_i \cdots v_r(v_{j_1}, \ldots, v_{j_r})M. \) This
concludes the proof of our induction on \( t. \)

Finally it remains to show that \( M/(v_1, \ldots, v_s)M \neq 0 \) for \( v_1, \ldots, v_s \) being an \( M \)-regular
sequence. To this end we prove by (a new) induction on \( s \) that \( M/(v_1, \ldots, v_s)M^* \cong v_s \cdots v_1M^*. \) If \( s = 1 \) this follows from the exact sequence
\[
0 \longrightarrow 0 :_M v_1 \longrightarrow M \overset{\cdot v_1}{\longrightarrow} M \longrightarrow M/v_1M \longrightarrow 0
\]
and the corresponding exact dual sequence
\[
0 \longrightarrow (M/v_1M)^* \longrightarrow M^* \overset{\cdot v_1}{\longrightarrow} (0 :_M v_1)^* \longrightarrow 0
\]
because here \( (M/v_1M)^* \) is the kernel of the multiplication with \( v_1 \) which is \( v_1M^* \) as \( v_1 \) is
\( M^* \)-regular.

Now suppose \( s > 1 \) and the assertion is proved for sequences of length \( < s. \)

An induction on \( r \) similar as in the first part of the proof shows that \( v_k \) is regular on
\( v_i \cdots v_r(v_{j_1}, \ldots, v_{j_r})M^* \) for \( \{i_1, \ldots, i_r, j_1, \ldots, j_r\} \subseteq [s] \setminus \{k\} \) for \( k = 1, \ldots, s, \) this time using
the decomposition
\[
v_i \cdots v_r(v_{j_1}, \ldots, v_{j_r})M^* = v_i \cdots v_{i_{r-1}}(v_{j_1}, \ldots, v_{j_r})M^* \cap v_i \cdots v_{i_{r-2}}v_{i_r}(v_{j_1}, \ldots, v_{j_r})M^*.
\]
In particular, \( v_s \) is regular on \( v_{s-1} \cdots v_1M^*. \) By the induction hypothesis (of the induction on \( s \)) we have
\[
v_s \cdots v_1M^* = v_s(v_{s-1} \cdots v_1M^*) \cong v_s(M/(v_{s-1}, \ldots, v_1)M)^*.
\]
We have already seen that \( v_s \) is regular on \( M/(v_{s-1}, \ldots, v_1)M \) and hence also on its dual \( (M/(v_{s-1}, \ldots, v_1)M)^*. \) Thus a second application of the induction hypothesis gives
\[
v_s(M/(v_{s-1}, \ldots, v_1)M)^* \cong (M/(v_{s-1}, \ldots, v_1)M)/(v_s(M/(v_{s-1}, \ldots, v_1)M))^* \cong (M/(v_s, v_{s-1}, \ldots, v_1)M)^*.
\]

The module \( M/(v_s, v_{s-1}, \ldots, v_1)M \) is zero if and only if \( (M/(v_s, v_{s-1}, \ldots, v_1)M)^* \) is
zero. As we have just seen the latter is isomorphic to \( v_1 \cdots v_sM^* \). If \( v_1 \cdots v_sM^* \) were zero, this would imply \( 0 = v_1 \cdots v_sM^* = 0 :_{v_1 \cdots v_{s-1}M^*} v_s = v_1 \cdots v_{s-1}M^* \). Inductively we would obtain \( M^* = 0 \), a contradiction. This concludes the proof. \( \square \)

We state a corollary which has been proved by the way in the proof of Theorem 4.3.

**Corollary 4.4.** Let \( M \in \mathcal{M} \) and \( v_1, \ldots, v_s \) be an \( M^* \)-regular sequence. Then
\[
(M/(v_1, \ldots, v_s)M)^* \cong v_1 \cdots v_sM^*
\]
as graded \( E \)-modules.

The relation between Cartan homology and Cartan cohomology in Proposition 2.2 provides
the following corollary.

**Corollary 4.5.** Let \( M \in \mathcal{M} \) and \( v = v_1, \ldots, v_s \in E_1 \). Then the following statements are equivalent:

(i) \( v_1, \ldots, v_s \) is \( M \)-regular;
(ii) \( H_1(\mathcal{F};M) = 0; \)
(iii) \( H_i(\mathcal{F};M) = 0 \) for all \( i > 0; \)
(iv) $H^1(v; M) = 0$;
(v) $H^i(v; M) = 0$ for all $i > 0$.

**Proof.** The equivalence of the first three conditions is stated in Proposition 2.5. An $E$-module is zero if and only if its dual is zero. Thus the equality of condition (ii) and (iv) resp. (iii) and (v) follows from $H_i(v; M^*) ≃ H^i(v; M)^*$ as seen in Proposition 2.2. □

5. Modules with Linear Injective Resolutions

In this section we focus on $E$-modules having linear injective resolutions. We begin with an example.

**Example 5.1.** Let $\Delta$ be a simplicial complex on $[n]$. Then $\Delta$ is Cohen-Macaulay if and only if the face ideal $J_{\Delta}^* = (e_F : F \not\subseteq \Delta^*)$ of the Alexander dual $\Delta^* = \{ F \subseteq [n] : F^c \not\subseteq \Delta \}$ (here $F^c$ denotes the complement of $F$ in $[n]$) has a linear projective resolution as was shown in [2] Corollary 7.6.

This is equivalent to say that the face ring $K\{\Delta\} = E/J_\Delta$ has a linear injective resolution as it is the dual $(J_{\Delta^*})^* ≃ E/(E/J_{\Delta^*})^* ≃ E/0 : E J_{\Delta^*} ≃ E/J_\Delta$ of $J_{\Delta^*}$.

If $v \in E_1$ is $M$-regular then $M$ has a $t$-linear projective resolution over $E$ if and only if $M/vM$ has a $t$-linear resolution over $E/(v)$. Linear injective resolutions behave more complicated under reduction modulo regular elements.

**Lemma 5.2.** Let $M \in \mathcal{M}$ and $v \in E_1$ be an $M$-regular element. Then $M$ has a $d$-linear injective resolution over $E$ if and only if $vM$ has a $d$-linear injective resolution over $E/(v)$.

In particular, if $v$ is $E/J$-regular for some graded ideal $J \subseteq E$, then we have that $E/J$ has a $d$-linear injective resolution over $E$ if and only if $E/(J + (v))$ has a $(d-1)$-linear injective resolution over $E/(v)$.

**Proof.** Let

$I^* : \quad 0 \longrightarrow \bigoplus_{j \in \mathbb{Z}} E(n - j)\mu_0,j(M) \longrightarrow \bigoplus_{j \in \mathbb{Z}} E(n - j)\mu_1,j(M) \longrightarrow \ldots$

be the minimal graded injective resolution of $M$ over $E$. We claim that $\text{Hom}_E(E/(v), I^*)$ is the minimal graded injective resolution of $\text{Hom}_E(E/(v), M) \cong 0 : M v = vM$ over $E/(v)$ with the same ranks and degree shifts, i.e. $\mu_{E/(v)}(vM) = \mu_{E/(v)}^j(M)$. From this the claim follows.

The homology of $\text{Hom}_E(E/(v), I^*)$ is isomorphic to the Cartan cohomology $H^i(v; M)$ of $M$ with respect to $v$ by Proposition 2.4. As $v$ is $M$-regular, Corollary 4.5 implies that $H^i(v; M) = 0$ for $i > 0$. So this is indeed a resolution of $\text{Hom}_E(E/(v), M) \cong vM$.

The modules in this resolution are

$$\text{Hom}_E(E/(v), \bigoplus_{j \in \mathbb{Z}} E(n - j)\mu_{i,j}(M)) \cong \bigoplus_{j \in \mathbb{Z}} \text{Hom}_E(E/(v), E(n - j)\mu_{i,j}(M))$$

$$\cong \bigoplus_{j \in \mathbb{Z}} (0 : E v)(n - j)\mu_{i,j}(M)$$

$$\cong \bigoplus_{j \in \mathbb{Z}} (v)(n - j)\mu_{i,j}(M)$$

$$\cong \bigoplus_{j \in \mathbb{Z}} E/(v)(n - 1 - j)\mu_{i,j}(M).$$
\[ \text{Hom}_E(E/(v), I^*) \] is an injective resolution of \( vM \) with \( \mu_{i,j}^{E/(v)}(vM) = \mu_{i,j}^E(M) \) (bear in mind that \( E/(v) \) is an exterior algebra with \( n-1 \) variables). The minimality is preserved because an injective resolution over \( E \) is minimal if and only if all entries in the matrices of the maps are in the maximal ideal. This property is not touched by applying \( \text{Hom}_E(E/(v), \cdot) \).

Now suppose \( M = E/J \) for some graded ideal \( J \). As just proved \( E/J \) has a \( d \)-linear injective resolution over \( E \) if and only if \( v(E/J) \) has one over \( E/(v) \). The latter module is isomorphic to the \( E/(v) \)-module
\[ v(E/J) = (J + (v))/J \cong (E/(v))/(J + (v)/(v))(-1) \]
where the isomorphism is induced by the homomorphism
\[ E/(v) \to (J + (v))/J(1), \ a + (v) \mapsto av + J. \]
Thus \( v(E/J) \) has a \( d \)-linear injective resolution if and only if \( (E/(v))/(J + (v)/(v)) \) has a \((d-1)\)-linear injective resolution.

Recall that for \( 0 \neq M \in \mathcal{M} \) the number
\[ \text{reg}M = \max\{ j - i : \text{Tor}_i^E(M, K)_J \neq 0 \} \]
is the regularity of \( M \). The regularity of \( M \) is bounded by \( d(M) = \max\{ i : M_i \neq 0 \} \) which can be seen when computing \( \text{Tor}_i^E(M, K) \) via the Cartan complex. For a graded ideal \( 0 \neq J \subset E \) there is the relationship \( \text{reg}J = \text{reg}E/J + 1 \). Reducing modulo a regular element \( v \) does not change the regularity because the minimal graded free resolution of \( M/\nu M \) over \( E/(v) \) has the same ranks and shifts as the minimal graded free resolution of \( M \) over \( E \). For quotient rings with linear injective resolution there is a nice formula for the regularity.

**Theorem 5.3.** Let \( |K| = \infty \) and \( E/J \) have a \( d \)-linear injective resolution. Then
\[ \text{reg}E/J + \text{depth}E/J = d. \]

**Proof.** At first assume \( \text{depth}E/J = 0 \). Then \( d = d(E/J) \) is an upper bound for \( \text{reg}E/J \) and we want to show that both numbers are equal. In [2] Theorem 5.3 it is proved that \( \text{reg}E/J = \text{reg}E/\text{gin}(J) \). Thus we may assume in addition that \( J \) is strongly stable. Then by [3 Corollary 3.2] the regularity of \( J \) is
\[ \text{reg}J = \max\{ \deg(u) : u \in G(J) \}. \]

In particular, \( J \) is a monomial ideal such that it can be seen as the face ideal of a simplicial complex \( \Delta \), i.e. \( J = J_\Delta = (e_F : F \notin \Delta) \). Then we have already seen that \( (E/J)^* \cong 0 : E J = J_\Delta^* \) is generated by all monomials \( e_F \) with \( e_F \notin J \) (cf. Example 5.1).

From \( n = \text{cx} E/J + \text{depth} E/J = \text{cx} E/J = \text{cx} J \) and Proposition 3.4 follows the existence of a monomial \( e_F e_n \in G(J) \). We have \( e_F e_i \in J \) for all \( i = 1, \ldots, n \) because \( J \) is stable. But then \( e_{(F \cup \{i\})} = e_{F \setminus \{i\}} \notin (E/J)^* \) for all \( i \notin F \). As \( e_F \notin J \) (otherwise \( e_F e_n \) would not be a minimal generator), the complement \( e_{F^\complement} \) is in \( (E/J)^* \) and even a minimal generator. The ideal \( (E/J)^* \) has an \((n-d)\)-linear projective resolution and is thus generated in degree \( n - d \), so \( |F| = n - |F^\complement| = n - (n - d) = d. \)

This means that there exists a minimal generator of \( J \) of degree \( d + 1 \) which implies \( \text{reg}E/J = \text{reg}J - 1 = d + 1 - 1 = d. \)
Now suppose depth \( E/J = s \). Reducing modulo a maximal regular sequence \( v_1, \ldots, v_s \) does not change the regularity, but \( E/J + (v_1, \ldots, v_s) \) has a \((d-s)\)-linear injective resolution over \( E/(v_1, \ldots, v_s) \) by Lemma 5.2. Then \( \text{reg}(E/J + (v_1, \ldots, v_s)) = d-s \) and so \[
\text{reg} E/J = \text{reg}(E/J + (v_1, \ldots, v_s)) = d-s = d - \text{depth} E/J.
\]

\[\square\]

**Remark 5.4.** Let \( |K| = \infty \) and \( 0 \neq J \subset E \) with \( d \)-linear injective resolution. By [1] Theorem 3.2 we have \( \text{cx} E/J = n - \text{depth} E/J \). As \( d \leq n \) this proves that \[
\text{reg} E/J \leq \text{cx} E/J.
\]

This inequality is even true for general quotient rings \( E/J \). For arbitrary graded \( E \)-modules there is no such relation between the regularity and the complexity since the first one is changed by shifting while the other is invariant.

For a graded ideal \( J \subset E \) Eisenbud, Popescu and Yuzvinsky characterize in [9] the case when both \( J \) has a linear projective and \( E/J \) a linear injective resolution over \( E \). In their proof they use the Bernstein-Gel’fand-Gel’fand-correspondence between resolutions over \( E \) and resolutions over the polynomial ring in \( n \) variables. We present a (partly) more direct proof using generic initial ideals.

**Theorem 5.5.** [9] Theorem 3.4] Let \( |K| = \infty \) and \( 0 \neq J \subset E \) be a graded ideal. Then \( J \) and \((E/J)^*\) have linear projective resolutions if and only if \( J \) reduces to a power of the maximal ideal modulo some (respectively any) maximal \( E/J \)-regular sequence of linear forms of \( E \).

**Proof.** At first we show that it is enough to consider the case \( \text{depth} E/J = 0 \). Note that the ideal \( J \) has a linear projective resolution over \( E \) if and only if \( J + (v_1, \ldots, v_s)/({v_1, \ldots, v_s}) \) has a linear projective resolution over \( E/({v_1, \ldots, v_s}) \). Furthermore Lemma 5.2 says that \( E/J \) has a linear injective resolution over \( E \) if and only if the \( E/({v}) \)-module \( E/J + (v) \) has a linear injective resolution for some \( E/J \)-regular element \( v \). Thus inductively \( E/J \) has a linear injective resolution over \( E \) if and only if \( E/(J + (v_1, \ldots, v_s)) \) has one over \( E/({v_1, \ldots, v_s}) \). All in all we may indeed assume that \( \text{depth} E/J = 0 \).

The \( t \)-th power of the maximal ideal \( m = (e_1, \ldots, e_n) \) has a \( t \)-linear projective resolution because it is strongly stable and generated in one degree (cf. Lemma 5.3). For the same reason \( (E/m^t)^* \cong 0 : E m^t \cong m^{n-t+1} \) has a linear projective resolution. Hence the “if” direction is proved.

Now it remains to show that if \( J \) has a \( t \)-linear projective resolution, \( E/J \) has a \( d \)-linear injective resolution and \( \text{depth} E/J = 0 \), then \( J = m^t \).

In a first step we will see that \( J \) may be replaced by its generic initial ideal. If \( J \) has a \( t \)-linear projective resolution, its regularity is obviously \( t \). Then by [2] Theorem 5.3] the regularity of \( \text{gin}(J) \) is also \( t \). As \( \text{gin}(J) \) is generated in degree \( \geq t \) this implies that \( \text{gin}(J) \) has a \( t \)-linear resolution as well.

Generic initial ideals and duality commute by Lemma 3.1 i.e.

\[
\text{gin}((E/J)^*) \cong (E/\text{gin}(J))^*.
\]
Then a similar argument shows that $E / \gin(J)$ has a $d$-linear injective resolution as well. Finally
\[
\text{depth} E / \gin(J) = \text{depth} E / J = 0
\]
by \cite[Proposition 2.3]{13}. Altogether $\gin(J)$ satisfies the same conditions as $J$. Assume that $\gin(J) = m'$. The Hilbert series of $J$ and $\gin(J)$ are the same which implies that in this case $J = m'$ as well because $J \subseteq m' = \gin(J)$.

This allows us to replace $J$ by $\gin(J)$ so in the following we assume that $J$ is strongly stable.

In the proof of Theorem \cite{5,3} was proved in the same situation that there exists a minimal generator of $J$ of degree $d + 1$. As $J$ is generated in degree $t$, this implies $d = t - 1$.

Finally, we will see that this equality implies $J = m'$. As $E / J$ has a $d$-linear injective resolution, the number $d(E / J) = \max\{i : (E / J)_i \neq 0\}$ equals $d$. Then, by Proposition \cite{5,5}
\[
\max\{\min(u) : u \in G(J)\} = n - d = n - t + 1.
\]

Thus there exists a monomial $u \in G(J)$ of degree $t$ with $\min(u) = n - t + 1$. The only possibility for $u$ is $u = e_{n-t+1} \cdots e_n$. Then every monomial of degree $t$ is in $J$ because $J$ is strongly stable and this implies $J = m'$ since $J$ is generated in degree $t$.

Let $v \in E_1$. Recall that $H_i(M, v)$ is the homology of the complex
\[
(M, v) : \quad \ldots \longrightarrow M_{i-1} \xrightarrow{v} M_i \xrightarrow{v} M_{i+1} \longrightarrow \ldots
\]
In Section \cite{6} we need the following technical result from \cite{9}.

**Theorem 5.6.** \cite[Theorem 4.1(b)]{9} Let $M \in \mathcal{M}$ have a $d$-linear injective resolution. Then $H_i(M, v) = 0$ for all $i \in \mathbb{Z}$ if and only if $H_d(M, v) = 0$.

### 6. Orlik-Solomon Algebras

In this section we investigate homological properties of the Orlik-Solomon algebra of a matroid. It is one example for $E$-modules with linear injective resolutions. We determine the depth and the regularity of the Orlik-Solomon algebra and characterize the matroids whose Orlik-Solomon ideal has a linear resolution. In the following the letter “$M$” denotes always a matroid and never a module.

For the convenience of the reader we first collect all necessary matroid notions that will be used in this section. They can be found in introductory books on matroids, as for example \cite{16} or \cite{20}.

Let $M$ be a non-empty matroid over $[n] = \{1, \ldots, n\}$, i.e. $M$ is a collection $\mathcal{I}$ of subsets of $[n]$, called independent sets, satisfying the following conditions:

(i) $\emptyset \in \mathcal{I}$.
(ii) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
(iii) If $A, B \in \mathcal{I}$ and $|A| < |B|$, then there exists an element $i \in B \setminus A$ such that $A \cup \{i\} \in \mathcal{I}$.

The subsets of $[n]$ that are not in $\mathcal{I}$ are called dependent, minimal dependent sets are called circuits. The cardinality of maximal independent sets (called bases) is constant and denoted by $r(M)$, the rank of $M$. 
On $E$ exists a derivation $\partial : E \to E$ of degree $-1$ which maps $e_i$ to 1 and obeys the Leibniz rule

$$\partial(ab) = (\partial a)b + (-1)^{\deg a}(\partial b)$$

for homogeneous $a \in E$ and all $b \in E$. One easily checks

$$\partial e_S = (e_{i_1} - e_{i_0}) \cdots (e_{i_m} - e_{i_0}) = \sum_{j=0}^{m} (-1)^j e_{S \setminus \{i_j\}}$$

for $S = \{i_0, \ldots, i_m\}$. The Orlik-Solomon ideal of $M$ is the ideal

$$J(M) = (\partial e_S : S \text{ is dependent}) = (\partial e_C : C \text{ is a circuit}).$$

If there is no danger of confusion we simply write $J$ for $J(M)$. The quotient ring $E/J$ is called the Orlik-Solomon algebra of $M$.

A circuit whose minimal element (with respect to a chosen order on $[n]$) is deleted is called a broken circuit. A set that does not contain any broken circuit is called nbc. Björner proves in [4] Theorem 7.10.2] that the set of all nbc-sets is a $K$-linear basis of $E/J$.

A loop is a subset $\{i\}$ that is dependent. If $M$ has a loop $\{i\}$, then $\partial e_i = 1$ in $J$ and thus $E/J$ is zero. Quite often it is enough to consider the case that $M$ is simple, i.e. $M$ has no loops and no non-trivial parallel classes. A parallel class is a maximal subset such that any two distinct members $i, j$ are parallel, i.e. $\{i, j\}$ is a circuit.

Note that if $M$ has no loops, a monomial $e_S$ is contained in $J$ if and only if the set $S$ is dependent (see for example [4] Lemma 7.10.1]).

**Example 6.1.** The simplest matroids are the uniform matroids $U_{m,n}$ with $m \leq n$. They are matroids on $[n]$ such that all subsets of $[n]$ of cardinality $\leq m$ are independent. The rank of $U_{m,n}$ is obviously $m$ and the circuits of $U_{m,n}$ are all subsets of $[n]$ of cardinality $m+1$. Thus the Orlik-Solomon ideal $J_{m,n} := J(U_{m,n})$ of $U_{m,n}$ is the ideal $J_{m,n} = (\partial e_A : A \subseteq [n], |A| = m+1)$. The relation

$$\partial e_S = \sum_{j=0}^{k} (-1)^j \partial e_{S \setminus \{i_j\} \cup \{1\}}$$

for $S = \{i_0, \ldots, i_k\} \subseteq [n]$ with $1 \notin S$ is easily verified by a simple computation. Then we can rewrite the Orlik-Solomon ideal as

$$J_{m,n} = (\partial e_A : A \subseteq [n], |A| = m+1, 1 \in A).$$

The rank of a subset $X \subseteq [n]$ is the rank of the matroid $M|X$ which results from restricting $M$ on $X$. Then the closure operator $\cl$ is defined as

$$\cl(X) = \{i \in [n] : r(X \cup \{i\}) = r(X)\}$$

for $X \subseteq [n]$. If $\cl(X) = X$, then $X$ is called a flat (or a closed set). The by inclusion partially ordered set $L$ of all flats of $M$ is a graded lattice. On $L$ we consider the Möbius function which can be defined recursively by

$$\mu(X, X) = 1 \quad \text{and} \quad \mu(X, Z) = - \sum_{X \leq Y < Z} \mu(X, Y) \text{ if } X < Z$$
and the characteristic polynomial
\[ p(L; t) = \sum_{X \in L} \mu(\emptyset, X) t^{r(M) - r(X)}. \]

The beta-invariant \( \beta(M) \) of a matroid \( M \) was introduced by Crapo in \( [6] \) as
\[ \beta(M) = (-1)^{r(M)} \sum_{S \subseteq [n]} (-1)^{|S|} r(S) = (-1)^{r(M)} \sum_{X \in L} \mu(\emptyset, X) r(X). \]

The Möbius function, the characteristic polynomial and the beta-invariant are considered in detail, e.g., in \( [22] \).

The direct sum of two matroids \( M_1 \) and \( M_2 \) on disjoint ground sets \( E_1 \) and \( E_2 \) is the matroid \( M_1 \oplus M_2 \) on the ground set \( E_1 \cup E_2 \) whose independent sets are the unions of an independent set of \( M_1 \) and an independent set of \( M_2 \). The circuits of \( M_1 \oplus M_2 \) are those of \( M_1 \) and those of \( M_2 \). The Hilbert series of the Orlik-Solomon algebra is multiplicative on direct sums, i.e.
\[ H(E/J(M_1 \oplus M_2), t) = H(E/J(M_1), t) \cdot H(E/J(M_2), t). \]
This can be proved using the fact that the set of all nbc-sets of cardinality \( k \) is a \( K \)-basis of \( (E/J)_{k} \) and that the nbc-sets of \( M_1 \oplus M_2 \) are the unions of an nbc-set of \( M_1 \) and an nbc-set of \( M_2 \).

On a matroid \( M \) exists the equivalence relation
\[ x \sim y \Leftrightarrow x = y \text{ or there is a circuit which contains both } x \text{ and } y. \]

The equivalence classes of this relation are called the connected components or, more briefly, components of \( M \). They are disjoint subsets of the ground set and each circuit contains only elements of one component. If \( T_1, \ldots, T_k \) are the components of \( M \) then \( M = M|T_1 \oplus \cdots \oplus M|T_k \). The matroid \( M \) is called connected if it has only one connected component.

The Orlik-Solomon algebra has a linear injective resolution, which was first observed by Eisenbud, Popescu and Yuzvinsky in \( [9] \) for Orlik-Solomon algebras defined by hyperplane arrangements, although their proof works for arbitrary Orlik-Solomon algebras as well. For the convenience of the reader we present a compact proof.

**Theorem 6.2.** \( [9, \text{Theorem } 1.1] \) Let \( l = r(M) \) be the rank of the matroid \( M \). Then the Orlik-Solomon algebra \( E/J \) of \( M \) has an \( l \)-linear injective resolution.

**Proof.** Let \( \Gamma \) be the simplicial complex whose faces are the nbc-sets of \( M \). The face ideal of \( \Gamma \) is the ideal
\[ J_{\Gamma} = (e_A : A \not\subseteq \Gamma) = (e_A : A \text{ is a broken circuit}). \]
This ideal is just the initial ideal \( \in(J) \) of \( J \) (this is implicitly contained in the proof of \([7, \text{Theorem } 3.3]\)).

By \([4, \text{Theorem } 7.4.3]\) the complex \( \Gamma \) is shellable and hence Cohen-Macaulay. So as in Example \([5,1]\) it follows that \( E/J_{\Gamma} = E/\in(J) \) has a linear injective resolution. Then Corollary \([3,2]\) implies that \( E/J \) has a linear injective resolution, too.

Every subset of \([n]\) of cardinality greater than \( l \) is dependent and thus every monomial of degree greater than \( l \) is contained in \( J \). Hence \( d(E/J) = \max\{i : (E/J)_i \neq 0 \} \leq l \). There
exists an independent subset $A \subseteq [n]$ of cardinality $l$. Then $e_A \not\in J$ and $d(E/J) = l$. So $E/J$ has an $l$-linear injective resolution as was observed in Section 2. \qed

Next we want to determine the depth of the Orlik-Solomon algebra. We are able to find at least one $E/J$-regular element if $M$ has no loops.

**Proposition 6.3.** If the matroid $M$ has no loops, then the variable $e_i$ is $E/J$-regular for all $i \in [n]$. In particular, $\text{depth} E/J \geq 1$.

**Proof.** By Theorem 5.6 and Theorem 6.2 it is enough to show that the annihilator of $e_i$ in $E/J$ and the ideal $(\bar{e}_i) = e_i(E/J)$ in $E/J$ coincide in degree $l$.

Every set of cardinality $l+1$ is dependent and therefore every monomial of degree $l+1$ is contained in $J$ whence $(E/J)_{l+1} = 0$. So every element in $E/J$ of degree $l$ is annihilated by $e_i$.

Now let $T$ be an independent set of cardinality $l$ that does not contain $i$. Then $T \cup \{i\}$ is dependent and thus $\partial e_{T \cup \{i\}} \in J$. Arrange $T \cup \{i\}$ such that $i$ is the first element. Then in $E/J$ there is the relation

$$\bar{e}_T = \bar{e}_T - \partial e_{T \cup \{i\}} = \bar{e}_T - e_T + (\ldots)e_i = (\ldots)e_i.$$  

So the residue class of every monomial of degree $l$ is in the ideal generated by $\bar{e}_i$, which shows that the annihilator and the ideal $(\bar{e}_i)$ coincide in degree $l$. This shows that $e_i$ is $E/J$-regular and thus the depth of $E/J$ is at least 1. \qed

The matroids $M$ whose corresponding depth is exactly 1 can be characterized by their beta-invariant $\beta(M)$.

**Theorem 6.4.** If $|K| = \infty$ and $M$ has no loops, then the depth of the Orlik-Solomon algebra $E/J$ equals 1 if and only if $\beta(M) \neq 0$.

**Proof.** Theorem 4.1 shows that the depth of $E/J$ is the maximal number $s$ such that the Hilbert series can be written as $H(E/J,t) = (1 + t)^sQ(t)$ for some $Q(t) \in \mathbb{Z}[t]$.

Björner proves in [4, Corollary 7.10.3] that

$$H(E/J,t) = (-t)^{r(M)} p(L; -\frac{1}{t}).$$

Replacing the characteristic polynomial $p(L; -\frac{1}{t})$ by its definition gives

$$H(E/J,t) = \sum_{X \in L} \mu(\emptyset, X)(-1)^{r(X)}t^{r(X)}.$$  

Thus the Taylor expansion of $H(E/J,t)$ at $-1$ is

$$H(E/J,t) = \sum_{X \in L} \mu(\emptyset, X)(-1)^{r(X)}r(X)(-1)^{r(X)-1}(1+t) + (1+t)^2(\ldots)$$

$$= - \sum_{X \in L} \mu(\emptyset, X)r(X)(1+t) + (1+t)^2(\ldots)$$

$$= (-1)^{r(M)-1}\beta(M)(1+t) + (1+t)^2(\ldots).$$
Now one sees that $H(E/J,t)$ can be divided twice by $1+t$ if and only if $\beta(M) = 0$. Observe that $H(E/J,-1) = 0$ because $1+t$ divides $H(E/J,t)$ at least once since $e_i$ is regular on $E/J$ by the preceding lemma.\hfill \Box

Crapo [6, Theorem II] proved that $M$ is connected if and only if $\beta(M) \neq 0$ (see also Welsh [20, Chapter 5.2]). Thus the above result says that if $M$ is connected, the depth of $E/J$ equals the number of components of $M$. This is true in general.

**Theorem 6.5.** Let $|K| = \infty$ and $M$ be a loopless matroid with $k$ components and $J$ its Orlik-Solomon ideal. Then depth $E/J = k$.

**Proof.** Let $M_1, \ldots, M_k$ be the matroids on the components of $M$, i.e. $M = M_1 \oplus \ldots \oplus M_k$ and let $J_i = J(M_i)$ be the corresponding Orlik-Solomon ideals. Theorem 4.1 and Theorem 6.4 imply that their Hilbert series can be written as $$H(E/J_i,t) = Q_i(t) \cdot (1+t)$$ such that $Q_i(-1) \neq 0$. The Hilbert series is multiplicative on direct sums, thus

$$H(E/J,t) = \prod_{i=1}^{k} (Q_i(t) \cdot (1+t)) = Q(t) \cdot (1+t)^k$$

with $Q(-1) \neq 0$ and so depth $E/J = k$.\hfill \Box

For Orlik-Solomon algebras of hyperplane arrangements this result was already proved by Eisenbud, Popescu and Yuzvinsky. In [9, Corollary 2.3] they state that the codimension of the singular variety (i.e. the set of all non-regular elements on the Orlik-Solomon algebra) of the arrangement is the number of central factors in an irreducible decomposition of the arrangement. This codimension is exactly the depth of the Orlik-Solomon algebra as Aramova, Avramov and Herzog showed in [11, Theorem 3.1].

**Remark 6.6.** Let $M$ be a loopless matroid with components $T_1, \ldots, T_k$ and $M_i = M|T_i$. A “canonical” maximal regular sequence on $E/J$ can be found as follows. For every component $T_j$ choose an element $i_j \in T_j$. Then $e_{i_1}, \ldots, e_{i_k}$ is an $E/J$-regular sequence. As $E/J + (e_{i_1}, \ldots, e_{i_{j-1}})$ has an $(l-j+1)$-linear injective resolution over $E/(e_{i_1}, \ldots, e_{i_{j-1}})$ by Lemma 5.2, it is enough to prove that $e_{i_j}$ is regular on $E/J + (e_{i_1}, \ldots, e_{i_{j-1}})$ in degree $l-j+1$. Let $A$ be an independent subset of $[n] \setminus \{i_1, \ldots, i_{j-1}\}$ with $|A| = l-j+1$. Then $A = S_1 \cup \ldots \cup S_k$ with $S_i \subseteq T_i$. The rank of $M$ is the sum of the ranks of the $M_i$, i.e. $l = r(M_1) + \ldots + r(M_k)$. So at most $j-1$ of the $S_i$ are not bases of their matroid, which means that there exists a $t \in \{1, \ldots, j\}$ such that $S_t \cup \{i_t\}$ is dependent in $M_t$. Then $A \cup \{i_t\}$ is dependent in $M$. The same trick as in the proof of Proposition 6.3 shows that $e_A \in J + (e_{i_1}, \ldots, e_{i_j})$.

As we know now the depth, we can compute the regularity of the Orlik-Solomon algebra as well.

**Corollary 6.7.** Let $|K| = \infty$ and $M$ be a loopless matroid of rank $l$ with $k$ components. The regularity of its Orlik-Solomon algebra is $$\text{reg} E/J = l-k.$$ **Proof.** This is just an application of Theorem 5.3.\hfill \Box
Example 6.8. We consider the uniform matroids $U_{m,n}$ and their Orlik-Solomon ideals $J_{m,n}$.

If $m = 0$ then every set is dependent. The circuits are all sets with one element, in particular they are loops. Thus $U_{0,n}$ has rank 0 and $n$ components $U_{0,1}$. The Orlik-Solomon ideal is $J_{0,n} = E$.

If $m = n$ then every set is independent. There are no circuits hence $J_{n,n} = 0$. The rank of $U_{n,n}$ is $n$ and it has $n$ components $U_{1,1}$. Thus depth $E/J = n$ and $c x E/J = 0$. The regularity is $\text{reg} E/J = n - n = 0$.

If $m \neq 0, n$ then $U_{m,n}$ is connected. Thus $\text{depth} E/J = 1$ and $c x E/J = n - 1$. The rank is $m$ hence the regularity is $\text{reg} E/J = m - 1$.

We say that an $E$-module has linear relations if it is generated in one degree and the first syzygy module is generated in degree one. Thus a linear projective resolution implies linear relations.

Theorem 6.9. Let $M$ be a simple matroid and have no singleton components. If the Orlik-Solomon ideal $J$ has linear relations then $M$ is connected.

Proof. As $M$ is simple there exists no circuits with one or two elements, so $J$ is generated in degree $m \geq 2$. Suppose $J = (\mathcal{P} e_{C_i} : i = 1, \ldots, r)$ where $C_1, \ldots, C_r$ are circuits of $M$ of cardinality $m + 1$. Let $f_1, \ldots, f_r$ be the free generators of $\bigoplus_{i=1}^r E(-m)$ such that $f_i$ is mapped to $\mathcal{P} e_{C_i}$ in the minimal graded free resolution of $J$. Then the assumption says that the kernel of this map,

$$U = \{ \sum_{i=1}^r a_i f_i : a_i \in E, \sum_{i=1}^r a_i \partial e_{C_i} = 0 \},$$

is generated by elements $r_k = \sum_{i=1}^r v_{ik} f_i$ with $v_{ik} \in E_1$. We may assume that the generators $r_k$ are minimal, i.e. no sum $\sum_{i \in I'} v_{ik} f_i$ with $I' \subseteq \{ 1, \ldots, r \}$ is in $U$. The support of a linear form $v = \sum_{j=1}^n \alpha_j e_j$ with $\alpha_j \in \mathbb{K}$ is the set $\text{supp}(v) = \{ j : \alpha_j \neq 0 \}$.

Under this conditions we claim that for each $k$ the elements of the circuits $C_i$ with $v_{ik} \neq 0$ are in the same component of $M$, which we call the component of $C_i$, and consequently the support of $v_{ik}$ is in this component, too.

The monomials in $\sum_{l=1}^r v_{ik} \partial e_{C_l}$ have the form $e_j e_{C_i \setminus \{ l \}}$ with $l \in C_i$ and $j \in \text{supp}(v_{ik})$. Because of the structure of $\partial e_{C_i}$ the monomials $e_j e_{C_i \setminus \{ l \}}$ cannot be zero for all $l \in C_i$. If it is not zero, then there exists $C_p, q \in C_p$ and $t \in \text{supp}(v_{pk})$ such that

$$\{ j \} \cup C_i \setminus \{ l \} = \{ t \} \cup C_p \setminus \{ q \}.$$ 

As $C_i$ and $C_p$ have at least three elements, it follows that their intersection is not empty. This means that their elements are both in the same component of $M$. Then the minimality of $r_k$ implies that all elements of circuits $C_i$ with $v_{ik} \neq 0$ belong to the same component.

Every $j \in \text{supp}(v_{ik})$ must belong to some circuit $C_p$ with $v_{pk} \neq 0$, otherwise we see that

$$e_j \sum_{i \in \text{supp}(v_{ik})} \alpha_{ij} \partial e_{C_i} = 0$$

when $v_{ik} = \sum_{j=1}^n \alpha_{ijk} e_j$. This implies $\sum_{i \in \text{supp}(v_{ik})} \alpha_{ijk} \partial e_{C_i} \in (e_j)$ and thus this sum equals zero. But this is not possible by our assumption on $U$. Hence all indices of the support of $v_{ik}$ belong to the same component of $M$ as the elements of the circuits $C_i$. 

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If $M$ is not connected and has no singleton components, there exists at least two components and thus two circuits $C_i$ and $C_j$ whose intersection is empty. There is a trivial relation of degree $2m$ between the generators corresponding to these two circuits, namely $\partial e_{C_i} f_j \pm \partial e_{C_j} f_i$. This relation has a representation

$$\partial e_{C_i} f_j \pm \partial e_{C_j} f_i = \sum_k g_k r_k = \sum_k \sum_{i=1}^r g_k v_{ik} f_i$$

where $g_k \in E_{m-1}$. Then

$$\partial e_{C_i} = \sum_k g_k v_{jk}$$

since the $f_i$ are free generators. Each monomial in the sum on the right side has a variable whose index is in the support of $v_{jk}$. As shown above this support is contained in the component of $C_j$. Thus $C_i$ contains elements of the component of $C_j$ which implies that both circuits belong to the same component, a contradiction to the choice of $C_i$ and $C_j$. □

Finally we classify all Orlik-Solomon ideals with linear projective resolutions. Only joining or removing “superfluous” variables has no effect on the linearity of $J$. This operation can be expressed using the direct sum of matroids. A singleton component and thus two circuits belong to the same component, a contradiction to the choice of $C_i$ and $C_j$. □

**Theorem 6.10.** Let $|K| = \infty$ and $M$ be a matroid on $[n]$. The Orlik-Solomon ideal $J$ of $M$ has an $m$-linear projective resolution if and only if $M$ satisfies one of the following three conditions:

(i) $M$ has a loop and $m = 0$.

(ii) $M$ has no loops, but non-trivial parallel classes, $m = 1$ and $M = U_{1,n_1} \oplus \cdots \oplus U_{1,n_k} \oplus U_{f,f}$ for some $k, f \geq 0$.

(iii) $M$ is simple and $M = U_{m,n-f} \oplus U_{f,f}$ for some $0 \leq f \leq n$.

**Proof.** First of all we will see that if $M$ satisfies one of the three conditions then $J$ has a linear projective resolution:

(i) If $M$ has a loop $\{i\}$, then $\partial e_i = 1 \in J$ so $J$ is the whole ring $E$ which has a linear resolution.

If $M$ satisfies (ii) then the circuits of $M$ are the circuits of the $U_{1,n_1}$. Thus all circuits of $M$ have cardinality two which means that $J$ is generated by linear forms $v_1, \ldots, v_s$ and has the Cartan complex $C(v_1, \ldots, v_s; E)$ as a linear resolution.

(iii) Following the remark preceding this theorem we may assume that $M$ has no singleton components, so we have $M = U_{m,n}$. If $m = 0$ or $m = n$ then the Orlik-Solomon ideal is $E$ or zero and has a linear resolution. By Example 6.8 the matroid $U_{m,n}$ is connected if $m \neq 0, n$. Hence it follows from Theorem 6.4 that depth $E/J = 1$. By Proposition 6.3 the
variable $e_1$ is $E/J$-regular. The Orlik-Solomon ideal $J = J_{m,n} = (\partial e_A : |A| = m + 1, 1 \in A)$ of $U_{m,n}$ was computed in Example 6.1. Then $J + (e_1) = (e_A : |A| = m) + (e_1)$ and thus $J$ reduces modulo $e_1$ to the $m$-th power of the maximal ideal in the exterior algebra $E/(e_1)$ and hence has a linear projective resolution by Theorem 5.5.

Now let $J$ have an $m$-linear projective resolution. If $M$ has a loop, then this is a circuit of cardinality one whence $m = 0$. Thus $M$ satisfies (i).

Now we consider the case that $M$ is simple. As above we assume that $M$ has no singleton components. So we have to show that $M = U_{m,n}$. Theorem 6.9 implies that $M$ is connected. Then $\text{depth } E/J = 1$ by Theorem 6.4 and $e_1$ is a maximal regular sequence on $E/J$ by Proposition 6.3. Reducing $J$ modulo $(e_1)$ gives the $m$-th power of the maximal ideal of the exterior algebra $E/(e_1)$ by Theorem 5.5.

Let $A \subseteq [n]$ with $1 \in A$, $|A| = m + 1$ and let $A' = A \setminus \{1\}$. The degree of the residue class of $e_{A'}$ in $E/(e_1)$ is $m$ and so $e_{A'} \in J + (e_1)/(e_1)$. Thus there exists a representation

$$e_{A'} = f + ge_1 \quad f \in J, g \in E.$$ 

Then

$$e_A = \pm e_{A'} e_1 = \pm fe_1 \in J$$

which is the case if and only if $A$ is dependent. So every subset of cardinality $m + 1$ containing 1 is dependent. An analogous argument for $i > 1$ shows that every subset of cardinality $m + 1$ is dependent. No subset of cardinality $\leq m$ is dependent because $J_j = 0$ for $j < m$. Thus we conclude $M = U_{m,n}$.

Finally we assume that $M$ has no loops or singleton components, but non-trivial parallel classes. Then there exists at least one circuit with two elements. As $J$ is generated in degree $m$ this implies $m = 1$. Let $J_1, \ldots, J_k$ be the Orlik-Solomon ideals of the components $M_1, \ldots, M_k$ of $M$, i.e. $J = J_1 + \ldots + J_k$. Each $J_j$ is generated by linear forms, because no $\partial e_C$ with $C$ of one component can be represented by elements $\partial e_{C_i}$ with $C_i$ of other components. Ideals generated by linear forms have the Cartan complex with respect to these linear forms as minimal graded free resolution and this is a linear resolution. Thus $J_j$ has a linear resolution. It is the Orlik-Solomon ideal of the connected loopless matroid $M_j$. Following the argumentation in the preceding paragraph for simple matroids this implies $M_j = U_{1,n_j}$ with $n_j$ the cardinality of the $j$-th component of $M$ and $M = \bigoplus_{j=1}^k U_{1,n_j}$. □

Since the powers of the maximal ideal of $E$ are strongly stable, their minimal resolution and especially their Betti numbers are known from [3]. Also Eisenbud, Floystad and Schreyer give in [10, Section 5] an explicit description of the minimal graded free resolution of the power of the maximal ideal using Schur functors. Their result gave the hint how a “nicer” formula of the Betti numbers could look like.

**Proposition 6.11.** The graded Betti numbers of $m'$ are

$$\beta_{i,t}(m') = \binom{n+i}{t+i} \binom{t+i-1}{i} \text{ and } \beta_{i,j}(m') = 0 \text{ for } j \neq t.$$
Proof. There are \( \binom{k-1}{t-1} \) monomials of degree \( t \) whose highest supporting variable is \( e_k \), i.e. \( m_k(m') = |\{ u \in G(m') : \max(u) = k \}| = \binom{k-1}{t-1} \). Hence by Lemma 3.3 we obtain

\[
\beta_{i,i+t}(m') = \sum_{k=t}^{n} m_k(m') \binom{k+i-1}{k-1} = \sum_{k=t}^{n} \left( k-1 \right) \binom{k+i-1}{k-1}.
\]

That this sums equals \( \binom{n+i}{t+i} \binom{t+i-1}{i} \) can be seen by an induction on \( n \), where the induction step from \( n \) to \( n+1 \) is the following:

\[
\sum_{k=t}^{n+1} \binom{k-1}{k-1} \binom{k+i-1}{k-1} = \binom{n+i}{t+i} \binom{t+i-1}{i} + \binom{n}{t-1} \binom{n+i}{n} = \binom{n+i+1}{t+i} \binom{t+i-1}{i} + \binom{n}{t-1} \binom{n+i}{n}.
\]

□

Now we obtain:

Theorem 6.12. Let \( M \) be a matroid and \( J = J(M) \) be its Orlik-Solomon ideal.

(i) If \( M = U_{m,n-f} \oplus U_{f,f} \) for some \( f \geq 0 \), then

\[
\beta_{i}^{E}(J) = \binom{n-f-1+i}{m+i} \binom{m+i-1}{i}.
\]

(ii) If \( M = U_{1,n_1} \oplus \cdots \oplus U_{1,n_k} \oplus U_{f,f} \) for some \( k, f \geq 0, n = f + \sum_{i=1}^{k} n_i \), then

\[
\beta_{i}^{E}(J) = \binom{n-f-k+i}{i+1}.
\]

Proof. Observe that reducing modulo a regular sequence does not change the Betti numbers so the Betti numbers of \( m^m \) give the Betti numbers of \( J_{m,n} \).

(i) The Betti numbers of \( J \) are the same as the Betti numbers of the \( m \)-th power of the maximal ideal in the exterior algebra \( E \) on \( n-f-1 \) variables:

\[
\beta_{i}^{E}(J) = \beta_{i}^{E}(m^m) = \binom{n-f-1+i}{m+i} \binom{m+i-1}{i}.
\]

(ii) In this case \( J \) reduces to the maximal ideal in the exterior algebra on \( n-f-k \) variables because for each component \( U_{1,n_i} \) one reduces modulo one variable as in Remark 6.6. □
7. Examples

In this section we study some examples of matroids with small rank or small number of elements.

Oxley enumerates in [16, Table 1.1] all non-isomorphic matroids with three or fewer elements. The only loopless matroids among them are the uniform matroids $U_{1,1}$, $U_{1,2}$, $U_{2,2}$, $U_{1,3}$, $U_{2,3}$ and $U_{3,3}$. Their depth, complexity and regularity were already computed in Example 6.8.

Now we turn to matroids defined by central hyperplane arrangements in $\mathbb{C}^l$ with $l \leq 3$. The arrangement is called central if the common intersection of all hyperplanes is not empty. A set of $t$ hyperplanes defines an independent set if and only if their intersection has codimension $t$. Thus every two hyperplanes in a central arrangement define an independent set and so the matroids defined by central hyperplane arrangements are simple.

In $\mathbb{C}^1$ the only central hyperplane arrangement consists of a single point, thus the underlying matroid is $U_{1,1}$.

In $\mathbb{C}^2$ a central hyperplane arrangement consists of $n$ lines through the origin. The underlying matroid is $U_{2,n}$ if $n \geq 2$ and $U_{1,1}$ if $n = 1$.

In $\mathbb{C}^3$ central hyperplane arrangement define various matroids. One single hyperplane defines a $U_{1,1}$, two hyperplanes a $U_{2,2}$. Three hyperplanes intersecting in a point give a $U_{3,3}$, if their intersection is a line then the underlying matroid is $U_{2,3}$. More generally $n$ hyperplanes through a line define the matroid $U_{2,n}$. Such an arrangement is called a pencil. For the first time one obtains a matroid that is not uniform with four hyperplanes taking three hyperplanes intersecting in a line and a fourth in general position, i.e. the intersection of the fourth with every two others is a point. The underlying matroid has two components, one containing the first three hyperplanes and one singleton component for the fourth hyperplane. It is the matroid $U_{2,3} \oplus U_{1,1}$. Such an arrangement is an example for a near pencil. For simplicity we define the notions of pencil and near pencil in terms of their underlying matroid.

**Definition 7.1.** A central arrangement of $n \geq 3$ hyperplanes is called

(i) a pencil if its underlying matroid is $U_{2,n}$.
(ii) a near pencil if its underlying matroid is $U_{2,n-1} \oplus U_{1,1}$.

In abuse of notation we also call the matroid $U_{2,n}$ a pencil and $U_{2,n-1} \oplus U_{1,1}$ a near pencil.

A matroid defined by $n$ hyperplanes in $\mathbb{C}^3$ is a simple matroid of rank 3 unless it is not a pencil which has rank 2. We classify all simple rank 3 matroids by their connectedness. Then we determine their homological invariants depth, complexity and regularity.

It is well-known that a near pencil is the unique reducible central hyperplane arrangement in $\mathbb{C}^3$; we present a homological proof for this fact.

**Theorem 7.2.** Let $M$ be a simple matroid of rank 3. Then $M$ is connected if and only if it is not a near pencil.

**Proof.** Note that $n \geq 3$ since $M$ has rank 3. If $M = U_{2,n-1} \oplus U_{1,1}$ is a near pencil, it has two components if $n > 3$ and three components if $n = 3$. Thus is it not connected in any case.
Suppose that $M$ has $k$ components with $k > 1$ and let $J$ be its Orlik-Solomon ideal. It is zero if and only if all subsets are independent. Then $r(M) = 3$ implies that $M = U_{3,3}$ is a near pencil. So from now on we assume $J \neq 0$. Since $M$ is simple, $J$ is generated in degree $\geq 2$ and thus $\text{reg} J \geq 2$. Theorem 6.5 and Corollary 6.7 imply that $\text{reg} J = \text{reg} E/J + 1 = 3 - k + 1 = 4 - k \leq 2$.

Thus the regularity of $J$ is exactly 2 and $k = 2$. Then $J$ has a 2-linear resolution and we may apply Theorem 6.10 which says that $M = U_{m,n-1} \oplus U_{1,1}$ for some $0 \leq m, i \leq n$. We may assume $m < n - i$ otherwise $M$ is $U_{3,3}$ and has three components. Since $M$ is simple, $m$ must be at least 2. Then $3 = r(M) = m + i$ so $i$ can only take the values 0 or 1. If $i = 0$ then $M = U_{3,n}$ has one or three (if $n = 3$) components, so this case cannot occur. Hence $i = 1$ and $M = U_{2,n-1} \oplus U_{1,1}$ is a near pencil. □

In the following table we have collected the homological invariants investigated in this paper of all simple matroids of rank 3 which are given by the above Theorem 7.2, using [1, Theorem 3.2], Theorem 6.5 and Corollary 6.7. It is a generalization of Proposition 4.6 of Schenck and Suciu in [19], even including the special case $n = 3$.

| Case                        | depth $E/J$ | $\text{cx} E/J$ | reg $E/J$ |
|-----------------------------|-------------|-----------------|-----------|
| no near pencil              | 1           | $n - 1$         | 2         |
| near pencil, $n > 3$        | 2           | $n - 2$         | 1         |
| near pencil, $n = 3$        | 3           | 0               | 0         |

The number of simple rank 3 matroids is e.g. determined in [8]. If $n = 4$ there exist only two simple rank 3 matroids, namely $U_{3,4}$ and $U_{2,3} \oplus U_{1,1}$. If $n = 5$ there exist 4 simple rank 3 matroids, $U_{3,5}$, $U_{2,4} \oplus U_{1,1}$ and two further which cannot be expressed as sum of uniform matroids since they must be connected by Theorem 7.2. One is the underlying matroid of an arrangement of five hyperplanes, three intersecting in a line and two in general position to each other and to the first three hyperplanes. The matroid has only one circuit with three elements corresponding to the first three hyperplanes and three circuits with four elements. The arrangement of five hyperplanes defining the second matroid has twice three hyperplanes intersecting in a line. The matroid has two circuits with three elements corresponding to these triples, and one circuit with four elements, not containing the element in the intersection of the other circuits.

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