RECENT DEVELOPMENTS IN STRING
THEORY

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Abstract

The purpose of this short review is to present progresses in string theory in the recent past. There have been very important developments in our understanding of string dynamics, especially the nonperturbative aspects. In this context, dualities play a cardinal role. The string theory provides a deeper understanding of the physics of special class of black holes from a microscopic point of view and has resolved several important questions. It is also recognized that M-theory provides a unified description of the five perturbatively consistent string theories. The article covers some of these aspects and highlights important progress made in string theory.

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1 Introduction

All along the progress in natural philosophy, curious minds have asked deep questions pertaining to the fundamental constituents of matter and creation and evolution of the cosmos. In the modern era, physicists have endeavored to comprehend natural phenomena in terms of a simple set of principles. Therefore, the search has continued to discover the elementary constituents of matter and identify the fundamental forces responsible for the natural phenomena. It is accepted that there are four fundamental forces: gravitation, the weak interaction, electromagnetism and the strong interaction. The unification of fundamental interactions has remained as one of the most outstanding challenge for generations of physicists. In the latter half of this century, some progress has taken place in this direction through the electroweak unification scheme. The electroweak theory together with quantum chromodynamics (QCD), referred to as the standard model, have been tested to a great degree of accuracy. Thus, the standard model provides a very good description of the ‘low energy physics’, comprising of the spectrum of elementary particles and their dynamics. The next step in fulfilling the dream of unification of forces were the schemes of grand unifications (GUT) which attempted to incorporate the three fundamental interactions, leaving aside gravitational interaction. The QED has been tested to a great degree of accuracy and two most important characteristics of that theory are the invariance under local gauge transformations and renormalizability. The electroweak theory and QCD respect the principle of gauge invariance and are renormalizable. Moreover, it is well known that the Einstein’s theory of general relativity respects a local symmetry: invariance under general coordinate transformations. However, the theory is not renormalizable since the Newton’s constant carries dimension of $\text{mass}^{-2}$, unlike the gauge coupling constants of the standard model which are dimensionless.

Although the standard model has successfully passed many stringent experimental tests, it is recognized that one must seek for a more fundamental theory. The standard model has many arbitrary parameters: the gauge coupling constants, the coupling constants of the scalars, Yukawa couplings of the Higgs bosons to fermions which are eventually responsible for generating fermion masses, just to mention a few. Furthermore, when one extrapolates the gauge coupling constants utilizing the renormalization group equations towards higher energy scale, there are evidences that the three coupling constants tend to converge to a point and it is natural to conclude that beyond that scale there might be a unified description of the standard model. Therefore, these observations lend support to the proposal of GUTS put forward in early seventies. As is well known, the existence of electroweak scale in the TeV region and another unification scale in the neighbourhood of $10^{16}$ to $10^{17}$ GeV leads to issues related to fine tuning of parameters, known as gauge hierarchy problem. The gauge hierarchy problem can be resolved in an elegant manner if one envisages supersymmetric version of the standard model (moreover, the convergence of gauge coupling constants in the unifying scale is more favourable in supersymmetric theories; see
Mohapatra’s article in this volume for details). The supersymmetric theories were constructed so that bosons and fermions can belong to a supermultiplet. The supersymmetry appeared in 2-dimensions in the construction of string theories. While attempts were being made to construct various types of grand unified theories, there were developments in incorporating gravity into supersymmetric theories which resulted in discovery of supergravity theories. However, it was not possible to construct renormalizable field theories which could unify the four fundamental forces. It was being perceived by many physicists, in the beginning of eighties, that new radical ideas were required to unify the fundamental interactions.

It is now accepted that string theory holds the promise of unifying all the fundamental interactions. The progress of the string theory in diverse directions, during the last fifteen years have been truly spectacular. The theory has not only has brought us nearer to the dream of unification, but also has influenced our understanding of various aspects of quantum gravity and has revealed many beautiful features relevant to the nonperturbative aspects of field theories.

The string theory was invented to describe the dynamics of strongly interacting particles. The vast amount of experimental data amassed from high energy accelerators, during fifties and sixties led to discovery of large number of hadronic resonances. One of the interesting characteristics of those resonances was that when one plots squared of mass vs spins of these particles, families of the resonances tend to lie on a straight line, known as the Chew-Frautschi plot. It was also evident from the high energy of scattering cross sections of hadrons that they follow a power law behaviour i.e. the crossed channel Regge poles controlled cross sections at high energies. The duality relation in strong interactions, that is sum over direct channel resonances (from low energy data) reproduces the Regge amplitude, was an important discovery for construction of dual models. Veneziano \[^2\] took crucial step step when he proposed a four point amplitude which satisfied requirements of duality and crossing symmetry.

\[
T(s,t) = B(s,t) + B(t,u) + B(u,s)
\] (1.1)

where

\[
B(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))\Gamma(-\alpha(s) - \alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))}
\] (1.2)

and \(\alpha(s) = \alpha_0 + \alpha's\) is the parameterization of the linear Regge trajectory. Here s,t and u refer to the Mandelstam variables; when we are in the center of mass frame, s is the squared of c.m. energy, t and u are related to the c.m. scattering angle. The B-function has the integral representation

\[
B(s,t) = \int_0^1 dw w^{-1-\alpha(s)}(1-w)^{-1-\alpha(t)}
\] (1.3)

Subsequently, generalized N-point amplitudes satisfying requirements of duality and crossing symmetry were proposed by several authors \[^2\] and one such amplitude is

\[
F(p_1,...,p_N) = |w_I - w_{II}| |w_{II} - w_{III}| |w_{III} - w_I| \int dw_1...dw_N \prod_{i<j} |w_i - w_j|^{2\alpha(p_i.p_j)}
\] (1.4)
$w_i$ are ordered cyclically, $w_I, w_{II}$ and $w_{III}$ are any three of the variables of the set \{w_i\}, but are held fixed. As in the case of 4-point Veneziano amplitude, the full N-point amplitude is sum of all cyclically inequivalent permutations. It was realized that it is possible to represent the N-point function in a path integral form 

$$F_N \sim \int \Pi_{\mu, \sigma} dX^\mu(\sigma) \int dw_1...dw_N \exp\left(-\frac{T}{2} \int_{\sigma_2 > 0} d^2\sigma \partial_\alpha X^\mu \partial^a X^\nu \eta_{\mu\nu}\right) \Pi^N e^{ip_I \cdot X(w_I)} (1.5)$$

where $\partial_a X^\mu = \frac{\partial X^\mu}{\partial \sigma^a}$, $\sigma^1$ and $\sigma^2$ are coordinates of a point in the upper plane, $X^\mu(\sigma)$ are integrated over all functions of $\sigma$. The boundary condition on $X^\mu$ is $\partial_2 X^\mu = 0$ for $\sigma^2 = 0$. The constant $T = \frac{1}{2\pi \alpha'}$ was later on identified as the tension of the string. Note the presence of $X^\mu(w_I)$; it is the value of $X^\mu(\sigma^1, \sigma^2)$ on the line $\sigma^1 = w, \sigma^2 = 0$.

The connection between dual amplitudes and dynamics of a relativistic string was recognized by several authors independently [4]. Now, of course we know that this amplitude is obtained from an open string theory and the action is that of a string, there are vertex operator in the path integral formula and the open string boundary conditions are to be specified. Virasoro had constructed another 4-point amplitude [5] fulfilling the requirement of duality and crossing symmetry in sequel to Veneziano’s paper and generalization of that amplitude for N-particle scattering was derived with a path integral representation [3]. It was realized that the Virasoro-Shapiro amplitude could be obtained from a closed string theory. Finally, Nambu proposed the action for the string so that one could start studying the dynamics of the string and proceed to examine the consequences of its quantization.

The string theory as a theory of strong interaction dynamics was not free from shortcomings. While attempts were going on to rectify the pitfalls of the theory and to construct new string theories as models of strong interactions; QCD was proposed as the fundamental theory of strongly interacting particle. The theory described interactions of the fundamental constituents, quarks, of the hadrons with gluon as the carrier of the force. Furthermore, the experimental data confirmed predictions of QCD steadily and consequently; string theory as a theory of strong interaction was no longer in the center stage.

In 1974, Joel Scherk and John Schwarz [7] made a bold proposition that string theory should be envisaged as a theory of gravity since the massless spin two particle appears naturally in the closed string spectrum and this theory might be a vehicle to achieve the goal of unification of the forces of Nature. If string theory were to incorporate the gravitational interaction, then the string tension should be order of the Planck scale in contrast to the the tension of the original string which was of the order of one GeV, the scale of hadronic interaction determined from the slope of the Regge trajectories. It was realized that one has to go up nineteen orders of magnitude in the energy scale if the Scherk-Schwarz proposal was to be realized. At that time, this radical idea did not receive wide spread acceptance amongst theoretical high energy physicists. The crucial work of Green and Schwarz [8] in the summer of 1984 led to conclusion that 10-dimensional super Yang-Mills theories coupled to supergravity
can be consistently constructed and are free from all anomalies \(^\[\]\) only for the gauge groups \(SO(32)\) and \(E_8 \times E_8\). The results of Green and Schwarz had profound impact on the field of high energy physics. It was recognized that string theory could fulfill the cherished dream of unifying fundamental forces. The construction of the heterotic string theory \([10]\) was a very important breakthrough towards realization of this goal since it had the desired gauge groups i.e. \(SO(32)\) or \(E_8 \times E_8\), depending on the construction one adopted. The ten dimensional theory had chiral fermions, \(N = 1\) supergravity coupled to supersymmetric Yang-Mills with appropriate gauge groups. Moreover, when the \(E_8 \times E_8\) heterotic string theory was compactified to four dimensions on a Calabi-Yau manifold, the resulting theory was shown to possess several desirable features that one expected from some of the grand unified theories. Furthermore, it was possible to demonstrate that the standard model gauge groups \(SU(3) \times SU(2) \times U(1)\) were contained in such four dimensional theories. Indeed, optimistically, one could feel that a unified theory was in sight and string theory was popularly named as the ‘Theory of Everything’.

Let us recapitulate some of the essential features of string theory. The string is a one dimensional object which executes motion in spacetime. There are, grossly speaking, two types of strings: open and closed strings. As the name suggests, the ends of open strings are free (there are special types whose ends might get stuck to some surfaces and they play very important roles too) and it is required to satisfy suitable boundary conditions for the end points. The closed string, by definition, has its both ends glued together, forming a loop. It is well known that when a point particle evolves in spacetime, it traces out a trajectory describing its history. In case of an open string, it sweeps a two dimensional surface and similarly the closed string sweeps a surface which is that of a cylinder. The natural question is why we do not observe these strings in high energy collisions. The answer to this question lies in the fact that the strings are much smaller in size than the present accelerators can probe. If we could have accelerators which have energies of the order of \(10^{19}\) GeV, then it will be possible to observe the dynamics of the strings directly and test the predictions of string theory at the Planckian energies. In contrast, the present day accelerators have energies of the order of TeV - almost 16 orders of magnitudes below the string scale.

The string has tension and it vibrates in an infinite number of modes. We identify each mode of the string with a particle. Of course, the string will have the lowest mode and we identify that with a particular particle. The next mode will correspond to an excited state and it is separated in energy from the lowest mode in suitable unit of string tension - separation between two neighboring levels is order of \(10^{19}\) GeV (recall that for the hadronic models they excitations were on Regge trajectories and there the tension was order of GeV). The string theories of interests to us contain massless particles in their lowest mode. For example, in 10-dimensional heterotic string theory, we have graviton, antisymmetric tensor and dilaton together with the super Yang-Mills multiplets corresponding to the gauge groups \(SO(32)\) or \(E_8 \times E_8\)
in its massless sector. Therefore, in the low energy limit, the string theory effectively reduces to a point particle field theory (this is when we want to describe physics at the present day accelerator energy scales). In other words, the zero slope limits of string theories correspond to known field theories - superstring theories go over to supergravity theories in this limit.

Now we give an outline of the rest of the article. Since it is to appear in a volume on ‘Field Theory’, we shall avoid involved technical details. The field has progressed in diverse directions and our strategy will be to adopt a course to high light important developments. We shall attempt to present different aspects of string theory in a pedagogical manner. In order to get across some issues, known examples from field theory will be presented. There has been intense activities in this field since 1984, when it was recognized that string theory is the most promising candidate for unification of forces of Nature. It is not possible to cover all the important literatures in a vibrant field like this within the frame work of this article. I apologize in advance to all the authors whose works have not been cited. There are two books which cover all the important aspects of string theory in detail besides several monographs and reprint collection volumes. The first one [11], in two volumes provides foundation for string theory and includes the developments up to 1986. The second one [12] has laid the emphasis on the progress made after the second superstring revolution. I have listed some of the review articles written in the first phase of the developments of string theory [13, 14, 15, 16, 17, 18]. There are a large number of review articles written in recent time [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. The next section deals with a brief review of the perturbative aspects of string theory to familiarize the reader with well known results. First, the string worldsheet action is introduced and the symmetries of the action are listed. A very quick exposition is given to the solutions of the equations of motion and mode expansions for the string coordinates and essentials of the Virasoro algebra are recalled. The evolution of the string in its massless background in the first quantized approach is discussed and the consequences of conformal invariance are noted. Section III deals with the symmetries of string theory. The theory is endowed with a rich symmetry structure in the target space besides the worldsheet symmetries. We introduce the duality symmetries since they play a very important role in our understanding of the string dynamics in various spacetime dimensions and they unravel the intimate connections between different string theories. The subsequent Section, IV, is devoted to to discuss the recent efforts to unify string theories. Besides duality symmetry, spatially extended objects, generically called p-branes, which appear as solutions to the effective action, are crucial to our understanding of string dynamics and to test some of the duality conjectures. We introduce some of the salient features of these objects and provide simple examples of the solutions. The raison de etre for M-theory is presented. We give an example how compactification of M-theory provides connections with the string theories and their various brane content. The fifth Section deals with issues related to black holes that appear in string theory. Since string theory describes gravity, it is expected that
the theory will be able to provide insights into deep questions in quantum gravity.
Indeed, some of the issues in the physics of the black holes have been resolved by string theory. It is known for more than two decades that a black hole is characterized by entropy and the Hawking temperature from the thermodynamic analogies. Moreover, the seminal work of Hawking demonstrated that the black holes radiate when quantum effects are taken into account. Recently, the black hole entropy has been computed as a microscopic derivation in the framework of string theory. Furthermore, the absorption cross sections for incident particles and the distribution of Hawking radiation emitted from special class of stringy black holes have been evaluated from a microscopic theory. Section VI contains a brief account of the M(atrix) model. The M(atrix) model proposal to describe M-theory has drawn considerable attention. Some of the calculations in this model give surprising agreements with results of supergravity theories. Moreover, when one considers compactification of the model on torii the resulting theory can be related to supersymmetric Yang-Mills theories through duality. We discuss some of the features of the Maldacena conjectures in Section VII. According to the conjecture, in a concrete form, if one considers N coincident 3-branes of type IIB theory on $AdS_5 \times S^5$ then the correlation functions of supergravity on the $AdS_5$ get related to correlation functions of the $N = 4$ super Yang-Mills theory living on the boundary of the $AdS_5$. This is a rapidly developing area and we shall be contented with some of the simple examples. In the last section we present an overview of the field. We make a few remarks to convey the reader how the work in string theory has influenced research in other branches of physics.

2 Perturbative Aspects of String Theory

We have outlined the historical backgrounds and the developments of string theory in its early phase in the previous section. In this section, we shall present some of the essential features of string theory such as its quantization, the perturbative spectrum of theory and the supersymmetric version of string theory.

Nambu had proposed the action for a string in analogy with the action for a relativistic point particle: the action for point particle in an integral over an line element; the string action is expected to be an integral over a surface. The Nambu-Goto action [34] was introduced almost three decades ago and has the form

$$S_{NG} = -T \int d^2 \sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2(\dot{X}')^2}$$

(2.1)

where $\sigma$ and $\tau$ are the coordinates on the surface swept out by the the string, called ‘worldsheet’; $\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}$ and $X'^\mu = \frac{\partial X^\mu}{\partial \sigma}$ and we shall follow this definition all along unless specified otherwise. The equations of motion can be derived after specifying boundary conditions for the types of string one is dealing with. One important point
to be noted is that the theory described by the above action satisfies two constraints

\[ \Pi X' = 0, \quad \Pi^2 + TX'^2 = 0 \quad (2.2) \]

where \( \Pi_\mu = \frac{\delta L}{\delta X'^\mu} \) is the canonical momentum of \( X^\mu \) obtained from this action. We reserve the notation \( P_\mu \) for the canonical momentum of the coordinate derived from the Polyakov action. We shall elaborate significance of these constraints later. However, this form of action was not very convenient to deal with the quantization of string and an alternative form of action was proposed by Polyakov \[35\]

\[ S = -\frac{T}{2} \int d^2 \sigma \sqrt{-\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \quad (2.3) \]

\( \gamma_{ab} \) is the worldsheet metric, \( \gamma^{ab} \) is its inverse, \( \gamma \) is determinant of worldsheet metric and \( \eta_{\mu\nu} \) is the flat space metric of the target space. The variation of the action with respect to \( \gamma_{ab} \) results in the worldsheet stress energy momentum tensor

\[ T_{ab} = \partial_a X^\mu \partial_b X^\nu - \frac{1}{2} \gamma_{ab} \gamma^{cd} \partial_c X^\mu \partial_d X^\nu \quad (2.4) \]

\( T_{ab} = 0 \), since there is no kinetic term i.e. as the analogue of Einstein-Hilbert piece, \( \int d^2 \sigma R^{(2)} \) is a topological term. We can solve for \( \gamma_{ab} \) from the above equation

\[ \gamma_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \quad (2.5) \]

If we insert the above expression for worldsheet metric into the Polyakov action, then we recover Nambu’s action.

The action (2.3) has following symmetry properties.

(a) Two dimensional reparameterization invariance

\[ \delta \gamma_{ab} = \xi^c \partial_c \gamma_{ab} + \partial_a \xi^c \gamma_{bc} + \partial_b \xi^c \gamma_{ac} \quad (2.6) \]

and hence \( \delta \sqrt{-\gamma} = \partial_a (\xi^a \sqrt{-\gamma}) \). The string coordinate transforms as

\[ \delta X^\mu = \xi^a \partial_a X^\mu \quad (2.7) \]

Weyl invariance

\[ \delta \gamma_{ab} = 2 \Omega \gamma_{ab}, \quad \delta X^\mu = 0 \quad (2.8) \]

Poincare invariance (in target space)

\[ \delta X^\mu = \omega^\mu_\nu X^\nu + a^\mu, \quad \delta \gamma_{ab} = 0 \quad (2.9) \]

where \( \omega^\mu_\nu \) are antisymmetric parameters associated with the Lorentz transformation and \( a^\mu \) are the parameters of translation.

Note that the Weyl invariance implies tracelessness of the two dimensional energy
momentum tensor for the classical theory. The quantum invariance of this symmetry has far reaching consequences in string theory.

If we make the orthonormal gauge choice for the worldsheet metric \( \gamma_{ab} = e^{2\Omega(\sigma, \tau)} \eta_{ab} \) with \( \eta_{ab} = \text{diag}(-1, +1) \) the, form of Polyakov action simplifies since \( \sqrt{-\gamma} \gamma^{ab} = \eta^{ab} \) in this gauge. The condition of the vanishing of \( T_{ab} \) reduces to two constraints

\[
(\dot{X} \pm X')^2 = 0 \tag{2.10}
\]

These are the Virasoro constraints. They take the following form in the Hamiltonian formalism

\[
P_\mu X'^\mu = 0, \quad H = \frac{1}{2}(P^2 + TX'^2) = 0 \tag{2.11}
\]

where \( P_\mu \) is momentum conjugate to \( X_\mu \) derived from Polyakov action. It is easy to see that the first constraint generates \( \sigma \) translation on the worldsheet, whereas latter being the canonical Hamiltonian generates \( \tau \) translation.

The equation of motion for the string coordinates, in the light-cone variables \( \xi_+ = \tau + \sigma \) and \( \xi_- = \tau - \sigma \), are given by

\[
\partial_+ \partial_- X'^\mu = 0 \tag{2.12}
\]

We note that the equation of motion is derived with following boundary conditions:

(i) \( X'^\mu(\tau, \sigma + 2\pi) = X'^\mu(\tau, \sigma) \) for the closed strings, and (ii) \( X'^\mu = 0 \) for \( \sigma = 0 \) and \( \sigma = 2\pi \) in the case of open strings, when we apply the variational method.

Let us illustrate the mode expansion for the closed string starting from the equation of motion with periodic boundary condition in \( \sigma \). We first note that the string coordinate can be decomposed as a sum of left-moving and right-moving coordinates.

\[
X'^\mu(\tau, \sigma) = X'^\mu_L(\tau + \sigma) + X'^\mu_R(\tau - \sigma) \tag{2.13}
\]

Then the two can be expanded as follows:

\[
X'^\mu_L(\tau + \sigma) = \frac{x'^\mu}{2} + \frac{p'^\mu}{4\pi T}(\tau + \sigma) + \frac{i}{\sqrt{4\pi T}} \sum m \frac{\alpha'^\mu m}{m} \ e^{-im(\tau + \sigma)} \tag{2.14}
\]

\[
X'^\mu_R(\tau - \sigma) = \frac{x'^\mu}{2} + \frac{p'^\mu}{4\pi T}(\tau - \sigma) + \frac{i}{\sqrt{4\pi T}} \sum m \frac{\bar{\alpha}'^\mu m}{m} \ e^{-im(\tau - \sigma)} \tag{2.15}
\]

The sum is over all integer values of \( m \) (\( m = 0 \) is excluded) in the above equations. \( \alpha'^\mu_m \) and \( \bar{\alpha}'^\mu_m \) are the Fourier modes. \( X'^\mu_{L,R} \) are real, so are \( x'^\mu \) and \( p'^\mu \); the Fourier modes satisfy

\[
(\alpha'^\mu_m)^* = \alpha'^\mu_{-m}, \quad (\bar{\alpha}'^\mu_m)^* = \bar{\alpha}'^\mu_{-m} \tag{2.16}
\]

\(^1\)We have adopted a shortcut route and taken the \( \alpha' \)’s to be time independent. The proper procedure would be, since we are yet to quantize, to allow them to be \( \tau \)-dependant and determine their equations of motion from the Poisson bracket with the Hamiltonian which in turn would determine the \( \tau \)-dependence of the \( \alpha' \)’s. Thus, the systematic steps would have been to take combination of \( X \) coordinates and \( P \)’s as one does in the case of harmonic oscillator go through the appropriate steps on this occasion also.
from the reality condition. For the closed string case, the classical Hamiltonian is given by

\[ H = \frac{1}{2} \left( \sum \alpha_m \alpha_{-m} + \sum \bar{\alpha}_m \bar{\alpha}_{-m} \right) \]  

(2.17)
in terms of the Fourier modes. The constraint, \( T_{ab} = 0 \), obtained from the Polyakov action, takes the form \( T_{--} = \frac{1}{2} (\partial_--X)^2 = 0 \) and \( T_{++} = \frac{1}{2} (\partial_{+} X)^2 = 0 \), in terms of the light-cone coordinates, after one has gone over to the ON gauge. It is more convenient to express these constraints in terms of the Fourier modes introduced above and define the Virasoro generators

\[ L_m = \frac{1}{2} \sum \alpha_{m-n} \alpha_n, \quad \bar{L}_m = \frac{1}{2} \sum \bar{\alpha}_{m-n} \bar{\alpha}_n \]  

(2.18)

And

\[ H = L_0 + \bar{L}_0 \]  

(2.19)

We can obtain classical Poisson bracket relations amongst \( L_m \), similarly for the set \( \bar{L}_m \), starting from the canonical Poisson bracket between \( X^\mu \) and \( P_\mu \).

\[ [L_m, L_n]_{PB} = -i (m-n) L_{m+n} \]  

(2.20)

\[ [L_m, \bar{L}_n]_{PB} = 0 \]  

(2.21)
the PB between \( \{ \bar{L}_m \} \) is same as for \( \{ L_m \} \). These are classical Virasoro algebra.

We have noted earlier that the string theory is endowed with local symmetries in the worldsheet and the action is that of D-scalar fields in 1 + 1 dimensions, since \( \mu = 0, 1 ... D-1 \) takes D values. When we proceed to quantize this theory, we encounter problems similar to the one faced in quantization of gauge theory. In other words we have to fix the gauge here too. One can choose to work in a noncovariant gauge which has the advantage of dealing with physical degrees of freedoms directly, but at the price of losing manifest Lorentz covariance. On the other hand one can adopt covariant quantization prescription with all its elegance and power. The light-cone quantization, although noncovariant, is very useful and gives us a physical picture. As the first step, the classical constraints are solved and one is left with less number of variables Recall that there were some remnant symmetries after choosing conformal (ON) gauge: \( \xi'_+ = \lambda_1 (\xi_+ \) and \( \xi'_- = \lambda_2 (\xi_- \). One can utilize this property to write

\[ X^+ = x^+ + \alpha' p^+ \tau \]  

(2.22)
Defining the light-cone string coordinated \( X^\pm = X^0 \pm X^1 \) one can impose the classical Virasoro constraints \( (\dot{X} \pm X')^2 = 0 \). Thus \( X^- \) is determined in terms of the rest of the (transverse) coordinates, \( X^i \); and in this process both \( X^+ \) and \( X^- \) are totally eliminated and we are left with \( \{ X^i \} \). Then the oscillators of these coordinates will create the states which could be identified with particles with physical degrees of freedom only. So it gives us a physical picture of the states. However, as mentioned earlier
and as is the case with noncovariant gauge fixing in QED or Yang-Mills theories, the Lorentz invariance must be checked explicitly. For the case of string theory, one is required to construct the generators of Lorentz transformations and ensure that the generators satisfy the algebra. It is well known that this requirement is not fulfilled unless the string propagates in 26-dimensional spacetime. On the other hand, if one adopts the covariant BRST procedure, it is necessary to add the ghost term to the action and construct the corresponding Virasoro generators for the ghosts. Thus the full Virasoro generator is a sum of the oscillators coming from string coordinates and those from the ghosts. When we compute the quantum Virasoro algebra, there is an anomaly of 26 from the ghost sector which gets precisely canceled if the spacetime dimension is 26 since each bosonic degrees of freedom contributes a factor of one to the anomaly with a sign just opposite to that coming from ghosts.

There are infinite tower of states in string theory. It is useful to arrange them according to their oscillator levels. Notice that the worldsheet degrees of freedom of the string are envisaged as a collection of infinite number of harmonic oscillators. If we consider creation operator of one of these oscillators, we could define a level such that the number of of units of worldsheet momenta created by this operator while acting on the vacuum. If we have a state, then the total oscillator level of that state is the sum of the levels of all the oscillators acting on the Fock vacuum to create this state. For the free string, the coordinates can be decomposed into left moving and right moving sectors. Therefore, one can define left and right moving oscillator oscillator levels (same decomposition is valid when we add fermionic degrees of freedom). Thus one can write \( L_0 = \frac{1}{2}(E + P) \) and \( \bar{L}_0 = \frac{1}{2}(E - P) \), where \( E \) and \( P \) are worldsheet energy and momentum respectively. Therefore, \( L_0 \) and \( \bar{L}_0 \) get contributions from the oscillators and from the Fock vacuum. We may remark in passing that the momenta of the spacetime D-dimensional theory (25 + 1 for bosonic string and 9 + 1 for superstring) are the ones conjugate to the zero modes of the bosonic/and/or fermionic worldsheet theory. Therefore, the ground state of the closed bosonic string is a tachyon satisfying the relation \( \alpha' m^2 = -4 \), with \( \alpha' = \frac{1}{2\pi T} \). The first excited (massless) states of closed string are:

- Spin 2 state, \( G_{\mu\nu} \), identified as graviton.
- An antisymmetric tensor field, \( B_{\mu\nu} \).
- A scalar, \( \phi \), called dilaton.

They belong to the irreducible representation of the \( SO(24) \) group. These states are created by action of a single creation operator from the left moving sector and another creation operator from the right moving sector. Therefore, they will have two target space Lorentz indices and one can decompose them according to irreducible representations of the corresponding rotation group.

Introduction worldsheet fermions has important consequences. In fact, if one demands worldsheet superconformal symmetry generalising from the bosonic string coordinates to include fermionic degrees of freedom, then resulting theory is the su-
perstring. First we need to construct two dimensional supergravity action. One needs to add to the action (2.3) the action

\[-\frac{T}{2} \int d^2 \sigma e \{ i \bar{\psi}^\mu \gamma^0 \gamma^a \partial_a \psi^\mu - i \bar{\lambda}_a \gamma^b \gamma^a \psi^\mu \partial_b \psi^\mu - \frac{1}{4} \psi^\mu \gamma^0 \psi^\mu \bar{\lambda}_a \gamma^b \gamma^a \lambda_b \} \tag{2.23}\]

The notations are as follows [37]: \( \psi^\mu \) are worldsheet two component Majorana fermions, \( e^a_i \) are the zweibeins associated with the worldsheet metric, \( e \) is its determinant. \( \lambda^a \) is the gravitino on worldsheet satisfying \( \lambda^*_a = \lambda_a \). The gamma matrices in the worldsheet have following representations: \( \gamma^0 = \sigma_2, \gamma^1 = i\sigma_1 \) and \( \gamma^5 = \gamma^0 \gamma^1 = \sigma_3 \), \( \sigma_i \) being the three Pauli matrices. We shall go over to the superorthonormal gauge, where the worldsheet metric is flat metric times a conformal factor (mentioned already) and gravitino is chosen to be \( \lambda_a = \gamma_a \zeta \) where \( \zeta \) is a constant Majorana spinor. Then the action (2.23) takes a simple form and is expressed in terms of the Weyl Majorana fermions (it is a free fermion theory now)

\[-\frac{iT}{2} \int d^2 \sigma \{ \psi^\mu_+(\partial_\tau - \partial_\sigma)\psi^\mu_+ + \psi^\mu_-(\partial_\tau + \partial_\sigma)\psi^\mu_- \} \tag{2.24}\]

with the definition of the chiral fermions: \( \psi^+_+ = \frac{1}{2}(1 - \gamma_5)\psi \) and \( \psi^+_--= \frac{1}{2}(1 + \gamma_5)\psi \), the spacetime index is suppressed. Now it is evident that the fermion equations of motion will separated according to the chiralities, as is expected for massless fermions. The worldsheet supersymmetry transformations are

\[\delta X^\mu = \bar{\epsilon} \psi^\mu \tag{2.25}\]
\[\delta \psi^\mu = -i \gamma^a \partial_a X^\mu \epsilon \tag{2.26}\]

For the two component Majorana fermions; \( \epsilon \) is the fermionic parameter associated with the supersymmetry transformation. The supercharge is the time component of supercurrent integrated over \( \sigma \) variable. The current is

\[ J^a = \gamma^b \partial_b X^\mu \gamma^a \psi^\mu \tag{2.27}\]

Next, one defines the super Virasoro generators and compute the quantum algebra and derive the condition for absence of anomaly. In case of the superstring the critical dimension is ten in contrast to bosonic string where it was 26.

Now we shall consider a few points before discussing how spacetime supersymmetry multiplets appear in the spectrum of the superstring. We had mentioned that the bosonic string has a tachyon in its lowest level which will render the theory unstable. Although, worldsheet supersymmetric theory moves in ten dimensional spacetime, the super Virasoro algebra does not impose sufficient constraints to remove the undesirable tachyon from the spectrum in general. Notice from the fermionic equations of motion (we suppress the bosonic part momentarily to focus attentions on fermions only) that there is some freedom in the choice
of the boundary condition as \( \sigma \) goes over a period of \( 2\pi \). The is due to the fact that the action remains invariant under \( \psi \rightarrow -\psi \) for fermions of either chirality. The boundary conditions are:

\[ \psi(\sigma + 2\pi) = -\psi(\sigma) \]

known as Neveu-Schwarz boundary condition is antiperiodic\[^{38}\]. The periodic boundary condition

\[ \psi(\sigma + 2\pi) = \psi(\sigma) \]

is the Ramond condition \(^{39}\); the indices are suppressed for notational convenience. The mode expansion for, say the holomorphic field, is

\[ \psi^\mu(\tau + \sigma) = \sum_n \psi^\mu_n e^{-n(\tau + \sigma)} \]

It is easy to see that for Ramond boundary condition, \( n \) must be integers. When we impose NS (Neveu-Schwarz) boundary condition and expand the fermions in Fourier modes, then \( n \) will take half integer values. We note that the NS fermions have no zero modes, whereas the Ramond fermions have zero modes in the Fourier expansions. Let us extend the arguments, we used for the bosonic string, for the superstring and examine their spectrum. The aim is to get rid of the tachyon and to construct states using bosonic and fermionic operators such that these states transform like fermions and the resulting theory be endowed with spacetime supersymmetry. We shall consider the light-cone gauge so that physical degrees of freedom become transparent. In addition to the condition \( X^+ = x^+ + p^+ \tau \), one imposes constraint

\[ \psi^+ = \bar{\psi}^+ = 0 \]

for the NS fermions, when we have Ramond fermions, they can be set to zero except for the zero modes. Now we look at the superconformal constraints and solve for \( X^-, \psi^- \), \( \bar{\psi}^- \) in terms of the rest of the coordinates. Thus we can use the (physical) transverse oscillators of both \( X \) and \( \psi \) to construct the physical states and keep in mind the presence of appropriate zero modes. It follows from straightforward calculation that the ground state in the NS sector is tachyon. The next level obtained by operating \( \psi^i \) contains massless states. Thus we need to remove the tachyon as well as some of the unwanted states, at the same time, keeping the massless spectrum intact. Notice that worldsheet fermions are anticommuting objects, although they create bosonic states while operating on a state of the theory. This feature is not very desirable as will be evident from the following example. Let us consider a specific bosonic state of a superstring and then operate on it a worldsheet spinor, \( \psi^i_+ \), obeying NS boundary condition. The resulting state will still be an integer spin object even if we have operated by anticommuting operator; this is rather unusual. We can think of a situation when odd number of NS operators act on a bosonic state and obviously same situation will continue to prevail, whereas for even number of such operators we face no problem since even number of anticommuting operators can be grouped to behave like bosonic operators. If we demand that all the states be even
under \((-1)^F\), then half of states which had above mentioned undesirable feature, are removed including the tachyon. This is the GSO projection. Moreover, after the unwanted states have been discarded from the spectrum, the remaining states of the theory belong to the representations of spacetime supersymmetry when we consider full spectrum of the superstring theory. Note that the operator \((-1)^F\) is defined up to a sign ambiguity. If we choose the sign convention that the first excited state has \((-1)^F = +1\) which arises due to action of \(\psi^i\), on the ground state, then we can fix \((-1)^F\) quantum number of the rest of the states. In this sign convention, tachyon will carry quantum number -1. There is another convention where tachyon has quantum number +1 and then the massless, first excited state, carries quantum number -1. The fermion numbers \(F_L\) and \(F_R\) can be introduced separately for the left and right moving sectors respectively. When one computes supercharge algebra with Ramond condition, the zero modes of the fermions in supercharge give an anomaly term besides the \(L_0\) term (that is Hamiltonian) and anomaly vanishes for \(D = 10\). Moreover, the anticommutation relation of the R-zero modes are like Dirac gamma matrices carrying target space indices. One finds that massless states appear in the R-sector and they satisfy Dirac equation. They transform as ten dimensional spinors (S) or conjugate spinors (C). Since we are considering the left moving sector here at the moment, S has +1 eigenvalue and C has -1 eigenvalue under the \((-1)^{F_L}\). When we construct other excited states on these states they turn out to be massive. In view of this one need not apply GSO projection, no tachyon is to be removed.

When we combine the left and right moving sectors four combinations will appear in the description of the closed string spectrum. NS-NS, NS-R, R-NS and R-R; where the first sector is from left movers and second is from right movers in the above four combinations. Let us look at them one by one.

(i) NS-NS: The states are created due to the action of the creation operators from the left and the right moving sector. They will transform as tensors under 10-dimensional Lorentz transformation. After GSO projection is implemented, the lowest lying states are massless and they can be decomposed into three groups, symmetric traceless, antisymmetric tensor and a scalar under the rotations.

(ii) NS-R: The GSO projection, as discussed is \((-1)^{F_L} = 1\) and one keeps the S representation of the R sector here. The the massless states consist of spacetime spinors.

(iii) R-NS: Here the GSO projection on NS sector from right side gives fermion number 1. We have the choice of keeping S spinor or the C spinor and obviously the states are spinorial.

(iv) R-R: The fermionic operators act from both sides and therefore, the resulting state will be bosonic in character. It will depend what combination we decide to keep. For example, if one keeps S from left side and \(\bar{C}\) from right side the product decomposes into a vector and a three index antisymmetric tensor (has to be antisymmetric - it arises from anticommuting objects). These belong to the bosonic sectors of type IIA theory. There is other combination which S from left and \(\bar{S}\) from right combine and their decomposition is a scalar, 2-form potential and 4-form (antisymmetric)
potential whose field strengths are self-dual in ten dimensions and these states are bosonic sector of type IIB theory.

We are in a position to classify string theories according to their important characteristics. There are two 10-dimensional theories which have $N = 2$ supersymmetry in target space. Their massless bosonic sectors are as follows: type IIA has graviton, $G_{\mu\nu}$, antisymmetric tensor $B_{\mu\nu}$ and dilaton, $\phi$, coming from the NS-NS sector and a gauge potential $A_\mu$ and three index antisymmetric tensor potential $C_{\mu\nu\lambda}$, coming from the R-R sector. These two theories have 32 generators of supersymmetry; type IIA is called non-chiral theory whereas type IIB is known as chiral theory. Although the bosonic fields coming from the RR sectors in these two string theories are tensors of different ranks, the total number of degrees of freedom of these tensors in each of the theories (A and B) are the same and this can be checked by counting the physical degrees of freedom RR gauge fields of type IIA and IIB.

Next, we introduce the heterotic string which is very attractive when one tries to establish connection of string theory with the gauge groups of the standard model. The heterotic string, in ten dimensions, contains $N = 1$ supergravity multiplet, super Yang-Mills gauge theory along with chiral fermions. There are two possible choices for the gauge groups: $SO(32)$ or $E_8 \times E_8$, in the construction of the heterotic string. Therefore, heterotic string theory fulfills Green-Schwarz anomaly cancellation condition. Moreover, when the theory is compactified to four dimensions on Calabi-Yau manifold, the resulting theory has many features of the standard model and the gauge group $SU(3) \times SU(2) \times U(1)$ can be embedded in the 4-dimensional theory. Let us briefly discuss how the heterotic string is constructed. We discussed the closed bosonic string and noted that the string coordinates can be decomposed to left movers and right movers and each can be expanded in Fourier modes. Moreover, the Virasoro generators are also separated into two groups, one group is expressed in terms of oscillators of one kind only (say left mover) and the other group of generators are expressed in terms of the oscillators of the other types (left movers). When one computes the quantum algebra, the anomaly free condition is imposed on each groups of Virasoro generators. In case of a closed string with worldsheet supersymmetry, same situation appears, because the fermion equations of motion is also written in terms of equations of motion of the Weyl Majorana fermions. If we were interested in constructing a string theory which satisfies requirements of conformal invariance, we could have a left moving closed bosonic string and a right moving superstring. The former will satisfy Virasoro algebra and latter the super Virasoro algebra.

The triumph of the Heterotic string is that, when we look at the massless spectrum of the theory, it has $N = 1$ spacetime supersymmetry, contains the appropriate gauge groups ($SO(32)$ or $E_8 \times E_8$) as is required for the consistency due to Green-Schwarz anomaly cancellation condition. Therefore, the closed bosonic string has 16 of its spatial coordinates compactified so that those coordinates themselves are periodic. Furthermore, using the standard techniques of 1 + 1 dimensional field theory, the compact bosonic coordinates could be fermionised to give 32 Weyl Majorana fermions.
which are left movers. Thus, we have 10 bosonic coordinates and their 10 super partners (in light-cone gauge 8 bosons and 8 fermions) in the right moving sector and 10 bosonic coordinates and 32 fermions (from compact coordinates) on the left moving sector. Whenever, we adopt NS boundary conditions for these fermions arising out of compactification, tachyon will appear in the spectrum. Of course, by introducing GSO projection on the right moving sector we shall have spacetime supersymmetry. So far as right moving part is concerned, bosonic states come from states with NS boundary condition and fermions arise due to the Ramond boundary conditions. The choice of boundary conditions on the left moving fermions (coming from compact directions), give rise to two different types of gauge groups. (i) All the left moving fermions can satisfy R-type (periodic) boundary condition or they can satisfy NS-type boundary conditions. Then there is GSO condition which ensures that there are only states which have even number of these fermions (only one type boundary condition). Thus the massless bosonic spectrum is given by symmetric second rank tensor field, antisymmetric tensor field and a scalar together with 496 gauge bosons belonging to the adjoint representation of $SO(32)$. (ii) The second possible choice of boundary condition for the left moving fermions is to divide them to two groups containing 16 fermions. Now there are four choices of boundary conditions (a) All satisfy R boundary conditions, (b) periodic (R) boundary condition is imposed on both the groups, (c) all the fermions in first group (call it I) have R boundary condition and the group II has NS antiperiodicity and finally (d) group I belong to NS boundary condition and II are in R. The GSO projection is such that it keeps even number of fermions from each group in the spectrum in every sector. When one works out the bosonic spectrum, it contains again second rank symmetric tensor, antisymmetric tensor of rank two, the scalar, dilaton and 496 gauge bosons in the adjoint representation of $E_8 \times E_8$.

There is another superstring theory, known as type I. A simple way to describe type I string is from the perspective of IIB theory. Consider the parity operation $\mathcal{P}$ on the worldsheet such that the ‘spatial’ coordinate $\sigma \rightarrow -\sigma$ under $\mathcal{P}$. In type IIB theory, $\mathcal{P}$ exchanges left and right moving sectors. Now, if we demand that we retain only those states which are invariant under $\mathcal{P}$, we get the type I string. In the NS-NS sector, graviton and the dilaton survive; the antisymmetric tensor is removed. From the RR sector, the only surviving field is the second rank antisymmetric tensor. Moreover, there are Weyl Majorana fermions and a gravitino surviving the operation giving rise to $N = 1$ supergravity multiplet. The open string states are also included in type I spectrum. In this case, the worldsheet degrees of freedom are same as in the closed string case. One imposes Neumann boundary conditions on the bosonic coordinates and suitable boundary conditions on worldsheet fermions. The gauge group that can get attached to the open string is $SO(32)$ and thus there is corresponding super Yang-Mills theory besides the states we mentioned above.

Thus there are five perturbatively consistent string theories. The scattering of particles belonging to spectrum of a string theory can be described by introducing vertex
operators. They are required to satisfy constraints due to conformal or superconformal transformations. They must transform as representations of Lorentz group, like a wave function. In the first quantized framework, one can calculate scattering of these particles in a well defined perturbation theory. It is one of the great virtues of the superstring theories that all these calculations are ultraviolet finite. Therefore, we have five different string theories in ten dimensions.

One of the most efficient ways to study properties of string theory is to investigate the evolution of a string in the background of its massless excitations and then explore the consequences of conformal invariance for such a situation. Let us consider closed bosonic string in the background of its massless excitations such as graviton, antisymmetric tensor and dilaton. The action (2.3) generalizes to

\[-\frac{T}{2} \left( \int d^2 \sigma \sqrt{-\gamma} \gamma^{ab} G_{\mu\nu}(X) + \epsilon^{ab} B_{\mu\nu}(X) \partial_\mu X^\rho \partial_\nu X^\sigma \right) + \frac{1}{2} \int d^2 \sigma \sqrt{-\gamma} R^{(2)}(X) \phi(X) \] (2.32)

Here $R^{(2)}$ is the scalar curvature of the worldsheet computed with $\gamma_{ab}$. The first two terms show the couplings of $G_{\mu\nu}$ and $B_{\mu\nu}$ to the string coordinates. In close string theory there is a massless state which transforms as symmetric second rank tensor and it is identified as graviton and there is an antisymmetric massless second rank tensor state. The above action describes motion of the string in the background of these massless states, $G_{\mu\nu}$ and $B_{\mu\nu}$; the last term is the coupling of the string to the massless scalar, the dilaton. This is an action for a two dimensional $\sigma$-model and we can interpret that $G_{\mu\nu}$ and $B_{\mu\nu}$ play the role of coupling constants. At the classical level the dilaton coupling breaks the conformal invariance explicitly. However, it is important to explore the consequences of the quantum invariance as we have seen that the quantum invariance principle imposes strong constraints on the theory. There is a well defined procedure to compute the conformal anomaly for such theories. One of the ways to ensure conformal invariance of the quantum theory is to demand that the two dimensional energy momentum stress tensor has vanishing trace. As is well known, the conformal anomaly is related to the corresponding $\beta$-function of the theory. Thus, vanishing of the $\beta$-functions will ensure conformal invariance. Moreover, the beta functions can be computed order by order in the $\sigma$-model perturbation theory; $\alpha'$ being the expansion parameter. The relevant $\beta$-functions are:

\[ \frac{\beta^G_{\mu\nu}}{\alpha'} = R_{\mu\nu} - \frac{1}{4} H_{\mu\rho\lambda} H^{\rho\lambda}_{\nu} + \nabla_\mu \nabla_\nu \phi \] (2.33)

\[ \frac{\beta^B_{\mu\nu}}{\alpha'} = \nabla^\rho [e^{-\phi} H_{\mu\nu\rho}] \] (2.34)

\[ \beta^\phi = \Lambda + 3\alpha' [(\nabla \phi)^2 - 2\nabla^\mu \nabla_\mu \phi - R + \frac{1}{12} H^2] \] (2.35)

The notations are as follows: $R_{\mu\nu}$ is the Ricci tensor for the target space computed from the string frame metric $G_{\mu\nu}$. $\Lambda = D - 26$ or $D - 10$ depending on whether we are dealing with a pure bosonic string or superstring (if we deal with superstring the
coupling of worldsheet fermions to the background has to be taken into account, $D$ being the spacetime dimension. $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \text{cycl.perm}$, is the field strength of two form potential $B_{\mu\nu}$. It might be worthwhile to point out that for the constant value of dilaton the last term in (2.32) is just the Euler character of the surface. When we write the path integral form with the action, we see that the factor $e^{-\chi \phi_0}$ comes out; where $\chi$ is the Euler character and $\phi_0$ is the constant value of the dilaton. In this light the string coupling constant is defined as

$$g_{\text{str}} = e^{\phi_0}/2$$  (2.36)

Let us look for an action in the target space such that the variation of that action with respect to the backgrounds $G_{\mu\nu}, B_{\mu\nu}$ and $\phi$ would reproduce the $\beta$-function equations we have obtained earlier. We also know that these $\beta$-functions must vanish (to the order in $\alpha'$ they are computed) in order to respect conformal invariance of the theory. The resulting action is

$$S = \int d^Dx \sqrt{-G} e^{-\phi} [R + (\partial \phi)^2 - \frac{1}{12} H^2]$$  (2.37)

This action is called the tree level string effective action. Solutions of the equation of motion of this action (same as solution to $\beta$-function equation) correspond to admissible background configurations with respect conformal invariance. In other words, every solution is an acceptable vacuum of the string theory to lowest order in $\alpha'$ since the effective action is obtained from the $\beta$-function equations keeping only lowest order terms in $\sigma$-model perturbation theory. Therefore, if we find solutions which correspond to cosmological situation with given $G$, $B$ and $\phi$, or a black hole solution, or a wormhole solution all these types of geometries with the appropriate matter content, consistent with the equations of motion, can be interpreted as string vacuum backgrounds.

So far we have been discussing the quantization of string theories and examining the consequences of conformal invariance. Note that all the consistent string theories are defined in spacetime dimensions higher than four i.e. $D = 10$. Therefore, one must answer the question what these theories have to do with the spacetime where we live. This issue has been taken up by Kaluza and Klein more than seven decades ago. The basic idea is rather simple. In order to construct a unified theory of gravity and electrodynamics, they considered an Einstein-Hilbert type action in 5-spacetime dimensions which is invariant under general coordinate transformations in five dimensions. Let us imagine that one of the dimensions, the 5th one, is a circle of very small radius which could not be probed today using any particle whose de Broglie wave length is comparable to the size of that circle. Then we shall not be aware of this scale. Let us assume, to a first approximation, that the metric does not depend on the 5th coordinate. Kaluza and Klein showed that the resulting theory looks like Einstein theory and Maxwell theory in four dimensions. What was general coordinate invariance in 5-dimensional theory, turned out to be general coordinate transformation
and Abelian gauge transformation (of Maxwell theory) in four dimensions. Although, the original Kaluza-Klein proposal had many shortcomings, the idea is very relevant for construction of four dimensional theories starting from the 10-dimensional string theories in the present context. We shall explore this aspect and we shall see how duality symmetries arise for compactified string theories. We shall set $T = 1$ from now on, whenever, we shall need to introduce the slope parameter/tension, we shall explicitly mention in that context.

3 Duality Symmetries in String Theory

One of the marvels of the string theory is its rich symmetry structure. We have noticed how the conformal invariance imposes strong constraints on the theory: when we consider flat spacetime the dimensionality is fixed by this symmetry. On the other hand if we consider strings in backgrounds, we get the equations of motion for them by demanding that the corresponding $\beta$-functions must vanish. Moreover, there are local symmetries like invariance associated with general coordinate transformation due to the presence of the graviton and an Abelian gauge symmetry since the antisymmetric tensor is also a part of the massless multiplet.

The duality symmetries play a crucial role in understanding various features of string theory. Since string is an extended object, there are symmetries special to string theory. Consider a particle whose motion is on a circular path, the momentum is quantized in suitable units of the inverse radius in order that the wave function maintains single valuedness. However, in case of a string, one of whose coordinate has geometry of a circle, offers more interesting possibilities. In fact a string theory with one spatial direction compactified as $S^1$ of radius $R$ cannot be distinguished from another theory whose coordinate is compactified on a circle of radius $\frac{1}{R}$. Let the compactified coordinate be denoted by $Y(\sigma, \tau)$ with the periodicity condition

$$Y(\sigma, \tau) + 2\pi R = Y(\sigma, \tau)$$

(3.1)

Furthermore, the string coordinate is also periodic when $\sigma$ goes over $2\pi$ for the closed string. Since, the coordinate is compact, zero momentum mode must be quantized to maintain single valuedness of the wave function just as the case in field theory. In case of the string, the string can wind around the compact direction. It will cost more energy if the string winds $m$-number time, because it will have to stretch more. Therefore, the effect due to windings has to be taken into account too while estimating energy levels [45]. Thus the mode expansions for left and right moving sectors are:

$$Y_R = y_R + \sqrt{\frac{1}{2}p_R(\tau - \sigma)} + \text{oscillators}$$

(3.2)

$$Y_L = y_L + \sqrt{\frac{1}{2}p_L(\tau + \sigma)} + \text{oscillators}$$

(3.3)
The momentum zero modes $p_{R,L}$ will have the following form to be consistent with what we said earlier

$$p_R = \frac{1}{\sqrt{2}}(n - Rm), \quad p_L = \frac{1}{\sqrt{2}}(n + Rm)$$

The above equation states that in general the contribution of the Kaluza-Klein mode is $\frac{1}{R}$ times an integer and the winding mode is an integer times the radius. The total momentum is just $P = \frac{1}{\sqrt{2}}(p_R + p_L)$, which is integral of momentum density over $\sigma$. The total Hamiltonian is

$$H = L_0 + \bar{L}_0 = \frac{1}{2}(p_L^2 + p_R^2) + \text{oscillators}$$

Now we consider the general case of toroidal compactification and present the derivation as was done in reference [49]. Let $G_{\alpha\beta}$ and $B_{\alpha\beta}$ be constant backgrounds, $\alpha, \beta = 1, \ldots, d$, and $Y^\alpha(\sigma, \tau)$ are the string coordinates. The two-dimensional $\sigma$-model action containing these coordinates is

$$I_{\text{compact}} = \frac{1}{2} \int d^2 \sigma \left[ G_{\alpha\beta} \eta^{ab} \partial_a Y^\alpha \partial_b Y^\beta + \epsilon^{ab} B_{\alpha\beta} \partial_a Y^\alpha \partial_b Y^\beta \right]$$

where $G_{\alpha\beta}$ and $B_{\alpha\beta}$ are constant backgrounds. The coordinates are taken to satisfy the periodicity conditions $Y^\alpha(2\pi, \tau) = Y^\alpha(0, \tau) + 2\pi m^\alpha$. Here we take the compactification radius to be unity for simplicity in calculations. For closed strings it is necessary that

$$Y^\alpha(2\pi, \tau) = Y^\alpha(0, \tau) + 2\pi m^\alpha$$

where the integers $m^\alpha$ are called winding numbers. It follows from the single-valuedness of the wave function on the torus that the zero modes of the canonical momentum, $P_\alpha = G_{\alpha\beta} \partial_\tau Y^\beta + B_{\alpha\beta} \partial_\sigma Y^\beta$, are also integers $n_\alpha$. Therefore the zero modes of $Y^\alpha$ are given by

$$Y_0^\alpha = y^\alpha + m^\alpha \sigma + G^{\alpha\beta}(n_\beta - B_{\beta\gamma} n^\gamma) \tau$$

where $G^{\alpha\beta}$ is the inverse of $G_{\alpha\beta}$. The Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2} G_{\alpha\beta}(\dot{Y}^\alpha \dot{Y}^\beta + Y^\alpha Y^{\prime\beta})$$

where $\dot{Y}^\alpha$ and $Y^{\prime\beta}$ are derivatives with respect to $\tau$ and $\sigma$, respectively. Let us elaborate a little bit on the significance of what we have done with respect to the compact coordinates. Since the coordinates $Y^\alpha$, are compact, they satisfy eq.(3.7). Moreover, these coordinates can be expanded as usual in terms of their zero modes and the oscillators. However, for the discussion of T-duality, we focus our attentions on the zero mode parts and the contribution of these parts to the Hamiltonian, given above.
Since \( Y^\alpha(\sigma, \tau) \) satisfies the free wave equation, we can decompose it as the sum of left- and right-moving pieces. The zero mode of \( P^\alpha = G^{\alpha\beta} P_\beta \) is given by \( p^\alpha_L + p^\alpha_R \) where

\[
p^\alpha_L = \frac{1}{2} \left[ m^\alpha + G^{\alpha\beta} (n_\beta - B_\beta m^\gamma) \right] \quad (3.10)
\]

\[
p^\alpha_R = \frac{1}{2} \left[ -m^\alpha + G^{\alpha\beta} (n_\beta - B_\beta m^\gamma) \right] \quad (3.11)
\]

The mass-squared operator, which corresponds to the zero mode of \( \mathcal{H} \), is given (aside from a constant) by

\[
(mass)^2 = G_{\alpha\beta} (p^\alpha_L p^\beta_L + p^\alpha_R p^\beta_R) + \sum_{m=1}^{\infty} \sum_{i=1}^{d} (\alpha_m^i \alpha_m^i + \bar{\alpha}_m^i \bar{\alpha}_m^i) \quad (3.12)
\]

As usual, \( \{\alpha_m\} \) and \( \{\bar{\alpha}_m\} \) denote oscillators associated with right- and left-moving coordinates, respectively. Substituting the expressions for \( p_L \) and \( p_R \), the mass squared can be rewritten as

\[
(mass)^2 = \frac{1}{2} G_{\alpha\beta} m^\alpha m^\beta + \frac{1}{2} G^{\alpha\beta} (n_\alpha - B_\alpha m^\gamma)(n_\beta - B_\beta m^\delta) + \sum (\alpha_m^i \alpha_m^i + \bar{\alpha}_m^i \bar{\alpha}_m^i) \quad (3.13)
\]

It is significant that the zero mode portion of (3.13) can be expressed in the form

\[
(M_0)^2 = \frac{1}{2} (m \cdot n) M^{-1} \begin{pmatrix} m \\ n \end{pmatrix}, \quad (3.14)
\]

where \( M \) is the \( 2d \times 2d \) symmetric matrix expressed in terms of constant backgrounds \( G \) and \( B \)

\[
M = \begin{pmatrix} G^{-1} & \frac{-G^{-1}B}{BG^{-1}} \\ 
\frac{BG^{-1}}{G^{-1}B} & G - BG^{-1}B \end{pmatrix} \quad (3.15)
\]

In order to satisfy \( \sigma \)-translation symmetry, the contributions of left- and right-moving sectors to the mass squared must agree; \( L_0 = \bar{L}_0 \). The zero mode contribution to their difference is

\[
G_{\alpha\beta} (p^\alpha_L p^\beta_L - p^\alpha_R p^\beta_R) = m^\alpha n_\alpha \quad (3.16)
\]

Since this is an integer, it always can be compensated by oscillator contributions, which are also integers.

Equation (3.16) is invariant under interchange of the winding numbers \( m^\alpha \) and the discrete momenta \( n_\alpha \). Indeed, the entire spectrum remains invariant if we interchange \( m^\alpha \leftrightarrow n_\alpha \) simultaneously let

\[
(G - BG^{-1}B) \leftrightarrow G^{-1} \quad \text{and} \quad BG^{-1} \leftrightarrow -G^{-1}B \quad (3.17)
\]

These interchanges precisely correspond to inverting the \( 2d \times 2d \) matrix \( M \). This is the spacetime duality transformation generalizing the well-known duality \( R \leftrightarrow \frac{1}{R} \) in
the $d = 1$ case discussed earlier. The general duality symmetry implies that the $2d$-dimensional Lorentzian lattice spanned by the vectors $\sqrt{2}(p_L^\alpha, p_R^\alpha)$ with inner product

$$\sqrt{2}(p_L, p_R) \cdot \sqrt{2}(p_L', p_R') \equiv 2G_{\alpha\beta}(p_L^\alpha p_L'^\beta - p_R^\alpha p_R'^\beta) = (m^\alpha n'_\alpha + m'^\alpha n_\alpha)$$

is even and self-dual \((\text{[47]})\). For toroidally compactified string theory, the coordinates satisfy periodicity condition and the conjugate momenta belong to the dual space and are quantized in suitable units. Furthermore, one can define corresponding metric to introduce the norm for the coordinates and their dual momentum vectors and define an inner product also. For a class of lattices the space of the coordinates (since the coordinates satisfy periodicity condition it is like crystals) is the same as the dual space, then the lattice is called self-dual. Of special significance, are the spaces where the length of the vector is even (with the definition of norm). In that case we have even self-dual lattice. These types of lattices are very important in construction of string theories with nonabelian gauge groups and to satisfy consistency requirements of the theory.

The moduli space parametrized by $G_{\alpha\beta}$ and $B_{\alpha\beta}$ is locally the coset $O(d, d)/O(d) \times O(d)$. The global geometry requires also modding out the group of discrete symmetries generated by $B_{\alpha\beta} \rightarrow B_{\alpha\beta} + N_{\alpha\beta}$ and $G + B \rightarrow (G + B)^{-1}$. These symmetries generate the $O(d, d, Z)$ subgroup of $O(d, d)$. An $O(d, d, Z)$ transformation is given by a $2d \times 2d$ matrix $A$ having integral entries and satisfying $A^T \eta A = \eta$, where $\eta$ consists of off-diagonal unit matrices defined below. Under an $O(d, d, Z)$ transformation

$$\begin{pmatrix} m \\ n \end{pmatrix} \rightarrow \begin{pmatrix} m' \\ n' \end{pmatrix} = A \begin{pmatrix} m \\ n \end{pmatrix} \quad \text{and} \quad M \rightarrow AMA^T$$

It is evident that

$$m \cdot n = \frac{1}{2}(m \, n)\eta \begin{pmatrix} m \\ n \end{pmatrix}$$

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which appears in eq.\((3.16)\), and $M_0^2$ in eq.\((3.13)\) are preserved under these transformations. Note that $\eta$ is symmetric $2d \times 2d$ matrix with off diagonal elements which are d-dimensional unit matrices. The crucial fact, already evident from the spectrum, is that toroidally compactified string theory certainly does not share the full $O(d, d)$ symmetry of the low energy effective theory. It is at most invariant under the discrete $O(d, d, Z)$ subgroup.

So far, in discussing issue compactifications, we have considered situations when all the coordinates are compact. However, one can envisage the scenario, when some of the string string coordinates are compactified and the rest are noncompact. Furthermore, we treated the backgrounds to be constant; however, in more realistic situations the backgrounds should be allowed to depend on noncompact coordinates. This is the more interesting situation where we have a ten dimensional string theory and six of
its spatial coordinates are compactified on a torus $T^6$ so that the resulting theory is reduced to a four dimensional effective theory. We shall adopt the general prescription of dimensional reduction \[18, 43, 50\] so that we can compactify an arbitrary number of dimensions so that the effective theory is defined in a lower spacetime dimension, not necessarily four. This will be useful, since the duality conjectures are in various spacetime dimensions and string theories are related by the web of dualities in diverse dimensions.

The starting point is to consider the string effective action in $D$ spacetime dimensions. The coordinates, metric and all other tensors in the $D$ dimensional space are specified with a ‘hat’. The coordinates in $D$-dimensional spacetime are denoted by $x^\mu, \mu, \nu, \text{etc}$ are spacetime indices. Therefore, $D = D + d$. The theory is compactified on a d-dimensional torus, $T^d$, to $D-d$ dimension spacetime. The coordinates on the torus, sometimes referred to coordinates of internal dimensions, are denoted as $y^\alpha, \alpha = 1, ..., d$. The bosonic part of the action is given by

$$
\hat{S} = \int d^D x \sqrt{-\hat{G}} e^{-\hat{\phi}} \left[ \hat{R}(\hat{G}) + \hat{G}_{\hat{\mu}\hat{\nu}} \partial_{\hat{\mu}} \hat{\phi} \partial_{\hat{\nu}} \hat{\phi} - \frac{1}{12} \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} \hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}} \right].
$$

Note that $\hat{S}$ is the bosonic part of the string effective action with backgrounds coming from NS-NS sector. $\hat{H}$ is the field strength of antisymmetric tensor and $\hat{\phi}$ is the dilaton. The backgrounds are taken to be independent of the internal coordinates, $y^\alpha$ of the torus. Consequently, any transformations of the coordinated $y^\alpha, \alpha = 1, 2, ... d$ does not affect the background fields and we recognize that there are $d$ isometries. Furthermore, associated with these isometries, there will be $d$ Abelian gauge fields since the $\hat{D}$-dimensional metric will have components carrying a $D$-dimensional spacetime index and an internal index $\alpha$. There will be components of the $\hat{D}$-dimensional metric which will carry indices of the toroidal coordinates, say $\alpha, \beta$ and these will transform as scalars, often refer to as moduli. Similarly, if we consider the components of the $D$-dimensional antisymmetric tensor field it will have $D \times D$ component antisymmetric tensor, $d$ Abelian gauge fields coming from spacetime and internal component and $d \times d$ dimensional moduli (antisymmetric) when considered from $D$-dimensional point of view.

The metric $\hat{G}_{\hat{\mu}\hat{\nu}}$ can be decomposed as

$$
\hat{G}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} G_{\mu\nu} + A^{(1)}_{\mu} A^{(1)}_{\nu} & A^{(1)}_{\mu} \\ A^{(1)}_{\nu} & G_{\alpha\beta} \end{pmatrix},
$$

where $G_{\alpha\beta}$ is the internal metric and $G_{\mu\nu}$, the $D$-dimensional space-time metric, depend on the coordinates $x^\mu$. Note the appearance of Abelian gauge fields $A^{(1)}_\alpha$ due to the presence of the isometries. We also expect same number of gauge fields from the antisymmetric tensor $\hat{B}_{\mu\nu}$. Thus The dimensionally reduced action is,

$$
S_D = \int d^D x \sqrt{-g} e^{-\phi} \left\{ R + g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right\}
$$
Here \( \phi = \hat{\phi} - \frac{1}{2} \log \det G \) is the shifted dilaton.

\[
H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2} A_\mu^i \eta_{ij} F^j_{\nu\rho} + (\text{cyc. perms.}),
\]

\( F^i_{\mu\nu} \) is the 2d-component vector of field strengths

\[
F^i_{\mu\nu} = \begin{pmatrix} F^{(1)\alpha}_{\mu\nu} \\ F^{(2)\mu\alpha}_{\mu\alpha} \end{pmatrix} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu,
\]

\( A^{(2)}_{\mu\alpha} = \hat{B}_{\mu\alpha} + B_{\alpha\beta} A^{(1)\beta}_{\mu} \) (recall \( B_{\alpha\beta} = \hat{B}_{\alpha\beta} \)), and the 2d \times 2d matrices \( M \) and \( \eta \) are defined as

\[
M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Note that the elements of the matrix \( M \), \( G_{\alpha\beta} \) and \( B_{\alpha\beta} \) depend on spacetime coordinates \( x^\mu \) in contrast to the earlier case (3.15) where those backgrounds were taken to be constant. The action (3) is invariant under a global \( O(d, d) \) transformation,

\[
M \rightarrow \Omega^T M \Omega, \quad \Omega \eta \Omega^T = \eta, \quad A^i_\mu \rightarrow \Omega^i_j A^j_\mu, \quad \text{where} \quad \Omega \in O(d, d).
\]

and the shifted dilaton, \( \phi \), remains invariant under the \( O(d, d) \) transformations. Moreover, \( M \in O(d, d) \) and \( M^T \eta M = \eta \). Thus if we solve for a set of backgrounds, \( M, F \) and \( \phi \), satisfying the equations of motion they correspond to a vacuum configuration of the string theory. The \( O(d, d) \) symmetry is known as the target space duality (or T-duality) symmetry, it is a stringy symmetry and there is no analogue of winding modes in ordinary field theory. The symmetry holds good order by order in string perturbation theory. Therefore, predictions of T-duality can be tested within the frame work of perturbation theory. We remark in passing that, if we had considered an effective action in \( \hat{D} \) dimensions with \( n \) Abelian gauge fields, the reduced action in \( D \) dimensions will be invariant under \( O(d, d+n) \) symmetry. This is of importance, since in case of the heterotic string, the ten dimensional action with 16 Abelian gauge fields corresponding to the Cartan subalgebra of the nonabelian gauge groups of the theory, when reduced to lower dimensions with exhibit the symmetry \( O(d, d+n) \) we mentioned.

Thus if we have a set of background configurations it is possible to generate another set of gauge inequivalent backgrounds by implementing suitable \( O(d, d) \) transformations. The new backgrounds will also satisfy the equations of motion and they will be acceptable vacuum configurations. In fact the \( O(d, d) \) symmetry was discovered for
non-constant backgrounds in the context of cosmological solutions in string theory \cite{51, 52}, when the backgrounds carried only time dependence. One could generate new cosmological solutions through $O(d, d)$ transformations \cite{53, 54}. The applications of $O(d, d)$ transformations in the context of black holes was to generate new black hole solutions was initiated by Sen \cite{55} and there is a vast literature in this subject \cite{19, 31}.

Next, we discuss S-duality in string theory. This symmetry relates a theory in the weak coupling regime to a theory in the strong coupling domain. In some it is the same theory which gets related to itself, like the type IIB theory. In some other situations one theory gets related to another one: a familiar example is that heterotic string compactified on $T^4$ is related to type IIA theory compactified on $K_3$. A simple example is the Maxwell electrodynamics. The equations are invariant under $E \rightarrow B$ and $B \rightarrow -E$. However, in the presence of sources, one has to be careful. The usual Maxwell equations have only sources carrying electric charges and then the equations are not symmetric under the above duality transformations. Thus it is necessary to add sources carrying magnetic charges to maintain electric-magnetic duality. This led Dirac to formulate the theory of magnetic monopoles. As is well known, the existence of magnetic monopole in the theory leads to the famous charge quantization condition: $e \cdot g = 2\pi n$, where $e$ is the electric charge and $g$ is the magnetic charge. This relation has profound implications; if the theory of electrically charged particles is described by a small coupling constant (indeed fine structure constant $\alpha = \frac{1}{137}$), then the theory describing magnetic monopoles will have large value for such charges corresponding to strong coupling constant. In the case of gauge theories with spontaneous symmetry breaking, magnetic monopoles appear as classical solutions of nonlinear field equations \cite{56, 57}. Note that the electric charge in such theories are obtained from the Noether currents whereas, the magnetic charge of 't Hooft-Polyakov monopoles are of topological nature. The charges respect the Dirac quantization condition. Furthermore, the massive gauge bosons (acquiring mass through Higgs's mechanism) have masses proportional to the gauge coupling constant, whereas the monopole masses are inversely proportional to the gauge coupling constant (electric charge). Consequently, if the gauge bosons are light in a SSB theory, the monopoles are heavy; indeed the monopoles have the interpretation of being the solitons of the theory. One of the most fundamental contributions to developments in S-duality came from the work of Montonen and Olive \cite{58}. According to them, we might envisage a dual formulation of fundamental physics where the role of Noether charges and topological charges are interchanged. One can visualize that monopoles will appear as elementary particles and the W-bosons will be solitonic counter parts. In fact one could check their mass formula $m^2 = C(e^2 + g^2)$; where $C$ is related to VEV of Higgs in SSB theories. In fact W boson and photon satisfy this formula. If a particle had been discovered with magnetic charge this relation could be verified. Since it is symmetric under the interchange of $e$ and $g$ and Dirac’s rule tells us that $e$ and $g$ are related, one could formulate the theory in the dual picture. However, the
monopole mass obtained in SSB theory is a classical one and it is subject to quantum corrections. Thus, Montonen-Olive idea could not be consistently checked in usual field theories. There are special types of supersymmetric field theories where there is no quantum correction to the mass and furthermore, the W-bosons and monopoles belong to the same multiplet. In such cases there is the possibility of checking this conjecture.

We recall that the Yang-Mills theory also admits the introduction of the $\theta$ term in its action. Thus, gauge theories have two parameters, the Yang-Mills coupling constant $e$ and the $\theta$ parameter. The latter couples to the field strengths as follows:

$$\frac{\theta e^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a,$$

(3.29)

where $\tilde{F}^{a\mu\nu} = \epsilon^{\rho\lambda\mu\nu} F^a_{\rho\lambda}$. Note that this term is a surface term and does not contribute to classical equations of motion and presence of this term does not affect renormalizability in the perturbation theory. It was noted by Witten [59] that in the presence of monopoles, this term shifts the allowed values of the electric charge in the monopole sector. Thus we can have electrically charged, magnetically charged particles and a third kind of particles carrying both the charges. The Yang-Mills Lagrangian can be written in the following form after taking into account the effect of the $\theta$ term and introducing a complex coupling constant $\tau = \frac{\theta}{2\pi} + \frac{4i\pi}{e}$

$$\mathcal{L} = -\frac{1}{32\pi} Im(\tau[F^a_{\mu\nu} + i\tilde{F}^a_{\mu\nu}][F^a_{\mu\nu} + i\tilde{F}^a_{\mu\nu}])$$

(3.30)

Following qualitative argument tells us about the strong-weak duality group. (i) When $\theta$ goes over its period $2\pi$ physics is the same. Thus, we expect that the theory be invariant when $\tau \rightarrow \tau + 1$. (ii) We also know that, under electric magnetic duality, $\tau \rightarrow -\frac{1}{\tau}$. One can argue that, when $\theta$ is arbitrary, the duality group is generated by these transformations. Thus, the duality group is identified to be $SL(2, \mathbb{Z})$. Therefore, in a theory with $SL(2, \mathbb{Z})$ symmetry one could check the spectrum with charged particles, monopoles and dyons. The complex coupling constant $\tau$ is often referred to as modular parameter or moduli. Moreover, when we discuss strong-weak duality in the context of string theory, dilaton and axion will be combined to define the moduli field. As mentioned earlier, string theory does not admit any arbitrary parameters as coupling constants. All the coupling constants appear as VEV of some scalar fields, i.e. moduli. Therefore, very often, the term coupling constants and moduli are used interchangeably in string theory.

As mentioned earlier, the mass formulas are protected from quantum corrections in supersymmetric theory. Moreover, some of the solitonic solutions in the supersymmetric theories satisfy special properties: (i) They saturate the BPS bound and (ii) these solutions preserve a part of the supersymmetry of the original theory. These attributes play a very important part in testing duality conjectures in field theory and in string theory. In order to illustrate the basic point, let us consider a two
dimensional example due to Witten and Olive \[60\], where the field content is a scalar field and Majorana fermion. The Lagrangian density is

$$ \mathcal{L} = \frac{1}{2} [ (\partial_\mu \Phi)^2 + i \bar{\Psi} \gamma^\mu \partial_\mu \Psi - V^2(\Phi) - V'(\Phi) \bar{\Psi} \Psi ] $$

(3.31)

The potential is arbitrary function of \( \Phi \) and ‘prime’ denotes derivative with respect to \( \Phi \). As was the case in worldsheet supersymmetry, we can work in terms of chiral components of fermions and the two super charges are

$$ Q_+ = \int dx [ (\partial_0 + \partial_1) \Phi \Psi_+ - V(\Phi) \Psi_- ] $$

(3.32)

$$ Q_- = \int dx [ (\partial_0 - \partial_1) \Phi \Psi_- + V(\Phi) \Psi_+ ] $$

(3.33)

In light-cone variables \( Q^2_\pm = P_\pm \), with \( P_\pm = P_0 \pm P_1 \) and it turns out that \( \{ Q_+, Q_- \} = 0 \), in most of the case. However, careful analysis shows that the anticommutator, is proportional to a surface integral

$$ \{ Q_+, Q_- \} = 2 \int dx \frac{\partial}{\partial x} H(\Phi) $$

(3.34)

and \( H'(\Phi) = V(\Phi) \). This surface integral does not necessarily vanish when one considers solitonic states. If we denote the R.H.S. of (3.34) by the operator \( T \), then it can be evaluated for the case at hand. Now the algebra of charges are different from usual case and one can write

$$ P_+ + P_- = T + (Q_+ - Q_-)^2 $$

(3.35)

$$ P_+ + P_- = -T + (Q_+ + Q_-)^2 $$

(3.36)

The R.H.S. of each equation above has a piece which is a complete square and we have \( P_+ + P_- \geq |T| \). If we consider single particle of mass \( M \) and go to its rest frame \( P_\pm = M \); we arrive at

$$ M \geq |T| $$

(3.37)

The bound will be equality when we have states, \( |s\rangle \) such that \( (Q_+ + Q_-)|s\rangle = 0 \) or \( (Q_+ - Q_-)|s\rangle = 0 \). The bound on \( M \) is the Bogomolny bound. The state which saturates it is called a BPS state. This bound also can be derived in a Lorentz covariant manner. We note that, for the states saturating the BPS bound, only half of the supersymmetries are preserved. In string theory or field theories with large number of supersymmetries, the algebra of the charges for a set of charges \( \{ Q_\alpha \}, \alpha = 1, ... N \), can be brought to the form

$$ \{ Q_\alpha, Q_\beta \} = \delta_{\alpha\beta} $$

(3.38)
This will be possible if there are no states which are annihilated by some of these charges and in that case, we shall get supermultiplets as usual. However, just like the soliton case considered earlier, if there are states which will be annihilated by some charges then we shall have a situation where

\[ \{Q_a, Q_b\} = \delta_{ab}, \quad \text{for } a, b = 1, \ldots M \] (3.39)

\[ \{Q_\alpha, Q_\beta\} = 0, \quad \alpha, \beta = M + 1, \ldots N \] (3.40)

So we see that these states will be lower dimensional representations since \( M < N \). Again, citing the example of two dimensional case, we can state the general result that when there are soliton like states getting annihilated by some of the supercharges, then the symmetric matrix \( \{Q_\alpha, Q_\beta\} \) will have some zero eigen values. The charges (analog of \( T \)) and masses get related in the process. This is true for monopoles in 4-dimensional theories. The string effective action is defined in 10 dimensions and one can seek solutions for extended objects in space and there are BPS states in this regime too.

Let us compactify the heterotic string effective action on \( T^6 \) to come to a four dimensional theory. As mentioned earlier, the T-duality group is \( O(6,22) \) with scalars parameterizing the moduli \( O(6) \times O(22) \), 28 gauge bosons, graviton \( G_{\mu\nu} \) and antisymmetric tensor \( B_{\mu\nu} \). The four dimensional effective action for the heterotic string, following the prescriptions of [49], can be obtained in a straightforward manner. The T-duality invariance is manifest when we are in the string frame metric with shifted dilaton \( \phi^{-1/2} \ln \det G_{\alpha\beta} \). However, when one considers the S-duality properties of the theory, it is convenient to go over to the Einstein frame metric, \( g_{\mu\nu} \) through the conformal transformation, \( g_{\mu\nu} = e^{-\phi} G_{\mu\nu} \). In string theory, all the coupling constants are related to the VEV of the dilaton and therefore, in order to identify the parameters of S-duality group, we have to choose the field whose VEV will coincide with the \( \theta \) parameter. Notice that the field strength of antisymmetric tensor, \( H_{\mu\nu\rho} \) has only one degree of freedom in four dimensions when we fix all gauge freedoms. In fact, if we take dual of this field, it is a pseudoscalar particle and that is what we need, an axion.

The starting point is the four dimensional effective action [64] with Einstein frame spacetime metric

\[
S^{(4)} = \int_M dx \sqrt{-g} \left\{ R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \mathcal{L}_2 + e^{-\phi} \mathcal{L}_3 + e^{-2\phi} \mathcal{L}_4 \right\}
\] (3.41)

with \( \mathcal{L}_2, \mathcal{L}_3, \) and \( \mathcal{L}_4 \) defined as follows

\[
\mathcal{L}_2 = \frac{1}{8} \text{tr} (\partial_\mu M^{-1} \partial^\mu M).
\] (3.42)

\[
\mathcal{L}_3 = -\frac{1}{4} F_{\mu\nu} (M^{-1})_{ij} F^{\mu\nu ij}.
\] (3.43)
Here we closely follow the notation of [49] and [64]. The next step is to perform a duality transformation, which replaces the field $B_{\mu\nu}$ by a scalar field $\chi$. This is achieved by first forming the $B_{\mu\nu}$ equation of motion

$$\partial_{\mu}(\sqrt{-g} e^{-2\phi} H^{\mu\nu}) = 0$$

(3.45)

and solving it by setting

$$\sqrt{-g} e^{-2\phi} H^{\mu\nu} = \gamma \epsilon^{\mu\nu\rho\lambda} \partial_{\rho} \chi$$

(3.46)

where $\chi$ is the “axion” and $\gamma$ is a constant to be fixed later. In the language of differential forms,

$$H = \gamma e^{2\phi} * d\chi$$

(3.47)

or, using $H = dB - \frac{1}{2} \eta_{ij} A^i_\lambda \tilde{F}^j$, 

$$dB = \frac{1}{2} \eta_{ij} A^i_\lambda \tilde{F}^j + \gamma e^{2\phi} * d\chi$$

(3.48)

The Bianchi identity ($d^2 B = 0$) now turns into the $\chi$ field equation

$$\frac{1}{2} \eta_{ij} \tilde{F}^i_\lambda \tilde{F}^j + \gamma d(e^{2\phi} * d\chi) = 0$$

(3.49)

or, in terms of components, (choosing a convenient value for $\gamma$)

$$\partial_{\mu}(e^{2\phi} \sqrt{-g} g^{\mu\nu} \partial_{\nu} \chi) - \frac{1}{8} \eta_{ij} \epsilon^{\mu\nu\rho\lambda} \tilde{F}^{i}_{\mu\nu} \tilde{F}^{j}_{\rho\lambda} = 0,$$

(3.50)

This is an equation of motion if we replace the $L_4$ term in $S^{(4)}$ by

$$S_{\chi} = - \int dx \sqrt{-g} \left( \frac{1}{2} e^{2\phi} g^{\mu\nu} \partial_{\mu} \chi \partial_{\nu} \chi + \frac{1}{4} \chi \tilde{F} \cdot \tilde{F} \right),$$

(3.51)

where

$$\tilde{F} \cdot \tilde{F} = \frac{1}{2} \sqrt{-g} e^{\mu\nu\rho\lambda} \tilde{F}^{i}_{\mu\nu} \eta_{ij} \tilde{F}^{j}_{\rho\lambda}.$$  

(3.52)

Let us briefly recapitulate the steps we have taken to modify the four dimensional action in the Einstein frame. The field strength $H_{\mu\nu\lambda}$ appearing in $L_4$ is traded for the pseudoscalar axion, $\chi$. The resulting action (3.51) contains not only the kinetic energy term of the axion, but also the topological term which is like the $\theta$ dependant term of the Yang-Mills action if the VEV of $\chi$ is identified with that parameter.

Let us now regroup the terms in the dual action in the following way:

$$\tilde{S}^{(4)} = \int_{M} dx \sqrt{-g} (R + L_2) + S_D + S_F,$$

(3.53)
where

\[ S_D = -\frac{1}{2} \int_M dx \sqrt{-g} g^{\mu \nu} \left( \partial_\mu \phi \partial_\nu \phi + e^{2\phi} \partial_\mu \chi \partial_\nu \chi \right) \]  (3.54)

\[ S_F = -\frac{1}{4} \int_M dx \sqrt{-g} \left( e^{-\phi} F^2 + \chi \mathcal{F} \cdot \bar{\mathcal{F}} \right) \]  (3.55)

and \( \mathcal{F}^2 \equiv g^{\mu \rho} g^{\nu \lambda} F^i_{\mu \lambda} (M^{-1})_{ij} F^j_{\rho \lambda} \). Note that \( S^{(4)} \) contains the usual Einstein-Hilbert action and the part coming from kinetic energy term of the M-matrix. We have rearranged the actions coming from dilaton kinetic energy, gauge field part and the axionic part (together with the ‘topological’ term) to define \( S_D \) and \( S_F \) so that dilaton and axion are put together and the gauge field kinetic energy along with the topological term are clubbed together. This is very useful to study the S-duality properties of the action. In order to describe the \( SL(2, R) \) symmetry of the dilaton and axion kinetic terms, let us introduce a complex modular parameter (recall the case of Yang-Mills)

\[ \tau = \chi + ie^{-\phi} , \]  (3.56)

which has the nice property that under a linear fractional transformation

\[ \tau \rightarrow \frac{a \tau + b}{c \tau + d} \]  (3.57)

the combination

\[ \frac{g^{\mu \nu} \partial_\mu \tau \partial_\nu \bar{\tau}}{(\text{Im} \, \tau)^2} = g^{\mu \nu} \left( \partial_\mu \phi \partial_\nu \phi + e^{2\phi} \partial_\mu \chi \partial_\nu \chi \right) \]  (3.58)

is invariant. It immediately follows that

\[ S_D = -\frac{1}{2} \int_M dx \sqrt{-g} g^{\mu \nu} \frac{\partial_\mu \tau \partial_\nu \bar{\tau}}{(\text{Im} \, \tau)^2} \]  (3.59)

Now we consider the gauge field action, \( S_F \). Notice that the \( SL(2, R) \) transformations give rise to an electric-magnetic duality rotation. Let us define

\[ F^\pm_{\mu \nu} = M \eta F_{\mu \nu} \pm i \bar{F}_{\mu \nu} \]  (3.60)

Then, using the identity \( F^{+\mu \nu} M^{-1} F^{-\mu \nu} = 0 \), we can express \( S_F \) as

\[ S_F = -\frac{1}{16 i} \int_M dx \sqrt{-g} \left( \tau F^{+\mu \nu} M^{-1} F^+_{\mu \nu} - \bar{\tau} F^{-\mu \nu} M^{-1} F^-_{\mu \nu} \right) \]  (3.61)

The \( A_\mu \) equation of motion is

\[ \nabla^\mu (\tau F^+_{\mu \nu} - \bar{\tau} F^-_{\mu \nu}) = 0 \]  (3.62)

and the Bianchi identity is

\[ \nabla^\mu (F^+_{\mu \nu} - F^-_{\mu \nu}) = 0 \]  (3.63)
To exhibit $SL(2, R)$ symmetry it is necessary to have $A_\mu$ transform at the same time as $\tau$. The appropriate choice is to require that $F_{\mu\nu}^\pm$ transform as modular forms as follows
\[
F_{\mu\nu}^+ \rightarrow (c\tau + d)F_{\mu\nu}^+, \quad F_{\mu\nu}^- \rightarrow (c\bar{\tau} + d)F_{\mu\nu}^-.
\]
This implies that
\[
\tau F_{\mu\nu}^+ \rightarrow (a\tau + b)F_{\mu\nu}^+, \quad \bar{\tau} F_{\mu\nu}^- \rightarrow (a\bar{\tau} + b)F_{\mu\nu}^-.
\]
Thus the equation of motion (3.62) and the Bianchi identity (3.63) transform into linear combinations of one another and are preserved. In particular, the negative of the unit matrix sends $F_{\mu\nu}^\pm \rightarrow -F_{\mu\nu}^\pm$. This result is acceptable if we identify the symmetry as $SL(2, R)$. Note that $SL(2, R)$ is not a symmetry of the action. The transformation in (3.64) is a nonlocal transformation of $A_\mu$, and such transformations can do strange things to the action. For example, the total derivative $F \cdot \tilde{F}$ transforms into an expression that is not a total derivative.

Thus far we have focused the attention to dilaton-axion system and the gauge field part of the action. The explicit checks show that the rest of the equations of motion are invariant under S-duality transformation. While checking the invariance of the Einstein equation we must ensure that that the contribution of $S_F$ to the energy–momentum tensor is $SL(2, R)$ invariant. After a short calculation one finds that only terms of the structure $e^{-\phi}F^+ F^-$ survive, and these are invariant since $e^{-\phi} \rightarrow |c\tau + d|^{-2}e^{-\phi}$. The symmetry of the equations motion is $SL(2, R)$. Notice that the axion couples to the topological density term, product of $F$ and its dual. We can argue qualitatively that the part of the $SL(2, R)$ group which gives rise to the translation symmetry of the axion (VEV of $\chi$ is the $\theta$ angle) should break down to discrete group of translations due the instanton effects. A more careful analysis is necessary \[12\] to show that $SL(2, R)$ breaks to $SL(2, Z)$.

The low energy string effective action, in four dimensions, contains graviton, antisymmetric tensor, dilaton and nonabelian gauge bosons. Furthermore, the Poincare dual of the three form field strength is a pseudoscalar and this field can be identified as the axion. One can combine dilaton and axion to form a doublet of the S-duality group $SL(2, R)$. It was argued \[51\] \[52\] that S-duality is an exact symmetry of the string theory. Schwarz and Sen \[63\] provided a general formulation of S-duality in string theory. Indeed the heterotic string compactified on $T^6$ has the effective action of $N = 4$ supersymmetric theory. How one can test S-duality in this case. One of the important results in this direction was first derived by Sen \[96\] when he showed that there are certain dyonic states in the theory whose existence can be demonstrated using S-duality transformations on heterotic string actions. These states precisely coincide with the ones we expect from Montonen-Olive conjecture. The theory has electrically charged states and magnetically charged states and each is 28-dimensional vector for the heterotic string. Due to nonrenormalization theorem of $N = 4$ supersymmetric theory, the electric charges are not renormalized. Moreover, the spectrum of the magnetic charges are fixed by the generalized Dirac quantization condition; the
magnetic charges are not renormalized either. Thus, spectrum of theses charges will be same as in the tree level theory. Indeed, the multimonopole moduli could be computed for the heterotic string [96]. In fact, the study of nonperturbative aspects of supersymmetric Yang-Mills theories took new directions through the works of Seiberg and Witten [66] in sequel to Sen’s work.

It is interesting to look for extended objects which appear as solution to equations of motion of string effective action. Simplest extended object is a string which is one dimensional. Let us denote the worldsheet coordinates of this string as $\xi^0$ and $\xi^1$ and the spacetime coordinates as $\{x^\mu\}$. This should appear as solution to string effective action. Suppose, we consider a frame where $(\xi^0, \xi^1)$ lie along the spacetime coordinates $(x^0, x^1)$ respectively. We look for a ‘spherically symmetric’ solution such that the solution is static and it depends only on the magnitude of the transverse distance, $r = \sqrt{y_1^2 + \ldots + y_8^2}$ where $x^2 ... x^9$ are denoted as $y_i$’s. The effective action has graviton, dilaton and antisymmetric tensor fields. In the Einstein frame the action has the form

$$S_E = \frac{1}{\kappa^2} \int d^{10}x \sqrt{-g} [R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{12} e^{-\phi} H^2] \quad (3.66)$$

The macroscopic string solution which was identified with the heterotic string [67] is obtained for following background configurations

$$ds^2 = f^{-\frac{3}{4}} (-dt^2 + (dx^1)^2) + f^{\frac{1}{4}} dy^i dy^i \quad (3.67)$$

$$B_{01} = \frac{1}{f} \quad (3.68)$$

The rest of the components of $B_{\mu \nu}$ are set to zero and

$$f = 1 + \frac{q}{3 r^6} \quad (3.69)$$

Here $Q$ is the charge carried by the string and it is associated with antisymmetric tensor field. The field equations one needs to satisfy are: Einstein equation, dilaton field equation and axionic charge conservation which follow from field equation of $H$. If we look at field equation carefully there is a delta-function singularity at $r = 0$ in the Laplace equation $\nabla^2 f$. Therefore, it was proposed [67] to resolve this problem by introducing a source for the string which will be the $\sigma$-model action

$$S_\sigma = \frac{-T}{2} \int d^2\xi [\partial^a X^\mu \partial_a X^\nu G_{\mu \nu} + e^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu \nu}] \quad (3.70)$$

Here of course the metric $G_{\mu \nu}$ is the string frame metric. This is the string solution carrying ‘electric’ charge and this charge can be obtained from the conservation law. Indeed, $q = \kappa^2 T / \omega_7$ where $\omega_7$ refers to the volume of $S^7$. In the supersymmetric case, there are BPS saturating solutions and here mass per unit length is equal to the charge.

In four dimensions the dual of electromagnetic field tensor is also a two form, thus if
we have point particles, the dual objects are point-like (‘t Hooft-Polyakov monopoles look point-like at large distances). However, if we have a string in ten dimensions it couples to 3-form field strength the dual of that field strength is 7-form. Therefore, the solitonic object for the string is a 5-brane, extended in five spatial dimensions. In fact the p-brane solutions were found in sequel to the string solutions [68]. As in case of monopole solution, we do not have magnetic source term while looking for field equations (W-bosons carry electric charge), the solitonic five-branes solutions are derived without adding a source term. Moreover, if $e_2$ is ‘electric’ charge of the string and $g_6$ is ‘magnetic’ charge of soliton, the Dirac quantization condition is

$$e_2 g_6 = 2\pi n$$  \hspace{1cm} (3.71)

One has to be careful in deriving strong-weak duality relation here. The coupling constant is determined in terms of dilaton expectation value. The relations are $e_2 = e^{\phi_0/2}$ and $g_6 = e^{-\phi_0/6}$.

There are special type of extended objects, the Dp-brane (D-branes), which carry R-R charges [69]. The type II theories admit gauge fields from the RR sector. The corresponding effective contain these fields. If one look for p-brane solutions with these gauge fields: strings, membranes and so on, they have interesting properties. These are hypersurfaces or spacetime defects on which the open strings can end. In D-dimensions, if there is a Dp-brane, there are Neumann boundary conditions satisfied in (p+1)-directions, these are the directions of the worldvolume coordinates of Dp-brane and we have Dirichlet boundary conditions along the remaining transverse directions that is $(D-p-1)$ coordinates. Written explicitly,

$$\partial_\sigma X^\mu = 0, \quad for \quad \mu = 0, \ldots p$$  \hspace{1cm} (3.72)

$$X^\mu (\sigma = 0, \pi) = a_0^\mu, \quad for \quad \mu = p + 1, \ldots 9$$  \hspace{1cm} (3.73)

A Dp-brane will couple to $(p+2)$-form RR field strength; therefore, D0-brane is a particle, D1-brane is a string and so on. The corresponding fermions satisfy boundary conditions in accordance with the bosonic fields in order to maintain the worldsheet supersymmetry. The BPS saturated solutions, then preserve half of the supersymmetry. From our earlier discussions, we note that type IIA admits D0-brane and D2-brane (their dual objects too) and IIB string has D-string, D3-brane and D-instantons, along with the duals. Thus, we conclude that IIA has even D-branes and odd D-branes belong to IIB theory. Of course, we are discussing the 10 dimensional case. The D-branes are dynamical objects and there are excitation of such extended objects since open string ends are attached to them.

Consider a situation when two D-branes are separated from each other. Since open string ends can get attached to this surface, they will be connected by open string/strings. The farther apart the two branes, it will cost more energy to stretch the open string. More interesting is the configuration when D-branes lie on top of each other. Then we can visualize an open string starting from a brane and ending on it, open string

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starting from one brane and ending on another coincident brane. In this situation we have massless states since there is no stretching of strings. Open strings contain massless vector state in their spectrum. One can incorporate nonabelian gauge symmetry for such a theory by introducing the Chan-Paton factors. We can imagine a scenario where a quark belonging to representation $i$ of $U(n)$ is attached to one end of the string and an antiquark in representation $\bar{j}$ attached to the other end. Thus the gauge field will carry index $i$ and $j$ like usual Yang-Mills fields and these are called Chan-Paton factors. This characteristic of open string turned out to be useful when we consider coincident D-branes. Therefore, if there are $N$ coincident branes, we get $U(N)$ Yang-Mills action, in fact we get supersymmetric gauge theory on the worldvolume of the brane.

Let us discuss some of the implications of dualities in the context of the branes we just introduced. The experience from monopole solution is that the charged particle couples to the field strength tensor and the soliton couples to the dual tensor in four dimensions. In ten dimensions, the solitonic counter part of string is five brane and we saw that couplings are not really reciprocals of each other. If we consider six spacetime dimensions, then we note that dual of 3-form field strength is also another 3-form tensor and string couples to this tensor. Therefore, the conjecture is that in six dimensions there is string/string duality. If there is a fundamental string the solitonic counter part is a string too and their coupling constants satisfy the reciprocal relation. For simplicity, consider a six dimensional reduced action, with only metric, antisymmetric tensor field and the dilaton $\Phi$.

$$I_6 = \frac{1}{2\kappa^2} \int d^6x \sqrt{-G} e^{-\phi} [R_G + (\partial \phi)^2 - \frac{1}{12} H^2]$$ (3.74)

Where $G_{MN}$ is six dimensional metric in string frame and $H_{NMP}$ is the 3-form field strength associated with $B_{MN}$ and it is understood that $H$ is defined up to Chern-Simons terms. We can go over to Einstein metric by the relation $G_{MN} = e^{\phi/2} g_{MN}$; $\phi$ being the dilaton in six dimensions. Let us consider the dual six dimensional action

$$\tilde{I}_6 = \frac{1}{2\kappa^2} \int d^6x \sqrt{-\tilde{G}} e^{-\tilde{\phi}} [R_{\tilde{G}} + (\partial \tilde{\phi})^2 - \frac{1}{12} \tilde{H}^2]$$ (3.75)

Here $\tilde{\phi}$ is the corresponding dilaton and $\tilde{H}$ is the field strength of the $\tilde{B}$, 2-form potential of the dual theory. The two actions (3.74) and (3.75) are related if we identify

$$\phi = -\tilde{\phi} \quad \text{and} \quad \tilde{H} = e^{-\phi} \ast H$$ (3.76)

The two metric being identified to be equal. Here $\ast$ stands for Hodge dual. Note that just as in case of gauge field kinetic energy term in four dimensions is conformally invariant, the $H^2$ term is also conformally invariant in six dimensions and it is immaterial which metric we use while taking Hodge dual. As noted earlier, the fundamental string solution with action (3.74) can be obtained by adding a $\sigma$-model
source term with coupling of the G and B backgrounds. The solution is given by

\[ ds^2 = (1 - \frac{q^2}{r^2})[-dt^2 + (x^1)^2 + (1 - \frac{q^2}{r^2})^{-2}dr^2 + r^2d\Omega_3^2] \] (3.77)

\[ e^\phi = 1 - \frac{q^2}{r^2} \] (3.78)

\[ e^{-\phi} \ast H_3 = 2q^2\epsilon_3 \] (3.79)

with

\[ q^2 = \frac{\kappa^2 T}{\Omega_3} \] (3.80)

Of course we have the BPS saturated mass relation

\[ M = T <e^\frac{\phi}{2}> \] (3.81)

Therefore, the mass density gets heavier as string coupling proceeds towards strong coupling domain. The source free action \((3.74)\) also admits solitonic string which is nonsingular and the solution is

\[ ds^2 = -dt^2 + (dx^1)^2 + (1 - \frac{q^2}{r^2})^{-2}dr^2 + r^2d\Omega_3^2 \] (3.82)

\[ e^{-\phi} = 1 - \frac{\tilde{q}^2}{r^2} \] (3.83)

\[ H_3 = 2\tilde{q}^2\epsilon_3 \] (3.84)

Where \(\tilde{q}^2 = \frac{q^2\tilde{T}}{\Omega_3}\). The mass density is

\[ \tilde{M} = \tilde{T} <e^{-\frac{\phi}{2}}/> \] (3.85)

In the weak coupling regime this string is heavier as one expects of a solitonic string. Notice that the solitonic string differs from the fundamental string by the replacement \(\phi \rightarrow -\phi, G_{MN} \rightarrow \bar{G}_{MN}, H \rightarrow \bar{H} = e^{-\phi} \ast H, \alpha' \rightarrow \bar{\alpha}'\). The Noether charge and the topological ‘magnetic’ charge are respectively given by

\[ e_2 = \frac{1}{\sqrt{2\kappa}} \int_{S^3} \ast H_3, \text{ and } g_2 = \frac{1}{\sqrt{2\kappa}} \int_{S^3} H_3 \] (3.86)

The Dirac quantization rule for charges: \(e_2g_2 = 2\pi n\) gets translated to relation between tensions. Moreover, the fundamental string and dual string saturate Bogomolnyi bound for mass densities and break half of the supersymmetry as expected. These solutions have the interpretation of being limiting cases of more general solutions. They can be viewed as extreme mass equals charge limit of two-parameter black string solutions.
Again the question arises where can we test the string/string duality? It has been conjectured that \([71, 72, 73]\) heterotic string compactified on \(T^4\) is S-dual to type IIA theory compactified on \(K_3\). When heterotic string is compactified on \(T^4\), the theory has charged states saturating Bogomolnyi bound. On the IIA side, elementary string states are neutral since the gauge fields arise from RR sector. Moreover, for type IIA, the analysis of the Bogomolnyi formula tells us the charged states (under gauge fields) have their masses as \(\frac{1}{g_{str}}\), implying that these are solitonic states. The duality between heterotic and type IIA is understood in the following sense \([74, 75]\): In type IIA theory, there are nonsingular soliton solutions and these carry quantum numbers of fundamental heterotic string. The properties of those strings are consistent with those of the heterotic string. On the other hand the heterotic string admits solitonic solutions carrying the quantum numbers of type IIA string. Moreover, we know that the moduli of heterotic string compactified on \(T^4\) parameterize the coset \(O(4) \times O(20) \rightarrow O(4)\). When type IIA is compactified on \(K_3\), the moduli also turns out to be exactly the same. Therefore, there is a very good evidence for this heterotic - type IIA duality conjecture. Another duality relation, that has been verified, is toroidal compactification of IIA and IIB theory via T-duality. Again the simplest one being compactification on \(S^1\). If one theory is compactified on circle of radius \(R\), it is equivalent to the other theory compactified on circle of reciprocal radius \(\frac{1}{R}\), although in ten dimensions these are two different theories. Some of the important consequences of S-duality can be examined in type IIB theory. It is conjectured that type IIB theory is self-dual and the effective action can be cast in a manifestly \(SL(2, \mathbb{Z})\) invariant form. We shall study this aspect in the next section. The two heterotic strings i.e. \(SO(32)\) and \(E_8 \times E_8\) when compactified on \(S^1\) are T-dual to each other in the reciprocal radius sense that one theory compactified on a circle of radius \(R\) is equivalent to the other which is compactified on a circle of radius \(\frac{1}{R}\). Finally, we comment that heterotic string with \(SO(32)\) gauge group is S-dual to type I theory with \(SO(32)\) group. The heterotic string effective action, with \(SO(32)\) gauge group has the following form

\[
S_{het} = \int d^{10}x \sqrt{-g}[R - \frac{1}{8}(\partial \phi)^2 - \frac{1}{4}e^{-\frac{\phi}{4}}\text{Tr}(F_{\mu\nu})^2 - \frac{1}{12}e^{-\frac{\phi}{4}}H^2]
\]  

(3.87)

Here \(F_{\mu\nu}\) is the nonabelian field strength and \(H = dB\). We work in the Einstein frame as it is the most convenient frame to study S-duality properties, since this metric remains invariant under S-duality. This action is obtained after rescaling the backgrounds and the slope parameters. The type I string has graviton and dilaton coming from the closed string NS sectors and closed string RR sector gives the antisymmetric tensor. The gauge fields come from NS sector of the open string and they have to be in the adjoint representation of \(SO(32)\). Again with appropriate scalings the effective action can be brought to the following form

\[
S_I = \int d^{10} \sqrt{-g}[R - \frac{1}{8}(\partial \bar{\phi})^2 - \frac{1}{4}e^{\frac{\phi}{4}}\text{Tr}(F_{\mu\nu})^2 - \frac{1}{12}e^{\frac{\phi}{4}}H^2]
\]  

(3.88)
Here all the fields of type I theory are defined with ‘bar’ to distinguish from those of heterotic string theory and the metric is in Einstein frame. Now, the comparison between the two actions shows that they will be identical if

\[ \phi = -\bar{\phi}, \quad g_{\mu\nu} = \bar{g}_{\mu\nu}, \quad H_{\mu\nu\rho} = \bar{H}_{\mu\nu\rho}, \quad A_{\mu} = \bar{A}_{\mu} \quad (3.89) \]

Thus, if we compare the two actions, (3.87) and (3.88), we see that the two theories are related to each other by strong-weak duality in 10-dimensions, since \( g_{\text{str}}^2 = e^\phi \). There are host of duality relations among various string theories in diverse dimensions; we refer the interested reader to large number of review articles in this area.

4 M-theory and Unified String Dynamics

We have briefly introduced some of the essential features of string theory and their symmetry properties. There are five perturbatively consistent string theories and one of their most attractive attributes is that they describe quantum gravity which is perturbatively finite and unitary. The dualities are powerful symmetry properties which provide important information about intimate connections between string theories. We have seen that one string theory, in a spacetime dimension, is related to another string theory either through T-duality or by the S-duality. When two theories are S-dual to each other, we can study strong coupling regime of one theory by going over to the weak, perturbative domain of its dual theory. Therefore, the nonperturbative aspects of some of the string theories could be investigated by these powerful tools. However, we still have five string theories. Therefore, the natural goal is to search for a theory which will provide a unified description of all the five string theories. The zero slope limits of the string theories yield all the known 10-dimensional supergravity theories. However, there is the \( D = 11 \) supergravity theory consisting of graviton and 3-form potential, endowed with total 128 bosonic degrees of freedom, and the 128 fermionic degrees of freedom. It was shown several years ago \cite{77} that compactification of 11-dimensional theory on a circle gives rise to \( N = 2 \) supergravity theory in 10-dimensions. It was not possible to establish any relation between the 11-dimensional theory and any string string theory for a long time. The connection of \( N = 2, 10 \)-dimensional supergravity with string theory is rather transparent since the supergravity actions can be obtained in the zero slope limit of corresponding type I string theories. There was no string theory that could be related in some such limit to 11-dimensional supergravity. Therefore, if 11-dimensional supergravity were to have any connection with one of the string theories, then only the nonperturbative regime of a theory will show the inter-relation. Moreover, when one views from the 11-dimensional perspective, the supergravity theory does not have any small parameter, like \( e^\phi \), in string theory, which can be chosen to take small value as an expansion parameter.

The connection between type IIA string theory and 11-dimensional supergravity were...
recognised by Witten [73] and Townsend [78] following the developments in string dualities. The massless bosonic sector of the type IIA theory, we might recall from our discussions of Section II, consists of dilaton, $\phi$, graviton, $G_{\mu\nu}$ and gauge field, $A_\mu$, antisymmetric tensor, $C_{\mu\nu\lambda}$ coming from the NS and Ramond sectors respectively. The effective action of type IIA theory

$$S_{IIA} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left[ e^{-\phi}(R + (\partial\phi)^2 - \frac{1}{12}H^2) - \left( \frac{1}{4}F^2 - \frac{1}{48}F'^2 \right) \right]$$

We have suppressed the Lorentz indices of the field strengths and we shall define them now: R is the scalar curvature, $H_{\mu\nu\rho}$ is the field strength of $B_{\mu\nu}$ from the NS sector, $F_{\mu\nu}$ is the field strength of RR gauge potential $A_\mu$ and in form notations, 4-form field strength, $F' = dC_3 + A \wedge dB$; $C_3$ being the 3-form potential coming from the RR sector and B is the 2-form potential whose field strength is $H$. Last term in (4.1) is the Chern-Simons term. and $F_4 = dC_3$ is the antisymmetric 4-form field strength of potential $C_3$ A few remarks are in order at this point: the metric used in action (4.1) is the string frame metric. Note that the factor $e^\phi$ multiplies only R and $H^2$ piece; fields coming from the NS sector. The reason is that in the worldsheet supersymmetric formulation of NSR type II theories the R-R sector fields through local worldsheet interactions (in NS sector the worldsheet fields couple to potentials), couple via bilinears of spin fields (in fact to field strengths). As a consequence, there are cuts and the usual arguments that tree level term starts with $\frac{1}{g_{str}}$ does not go through. Thus we see this mismatch of $e^\phi$ between NS and RR fields in the effective action. Now, it is easy to see that this theory will admit D0-brane and D2-brane and their duals will be D6-brane and D4-brane from RR sector and a string and its dual five brane from the NS sector.

Let us consider the bosonic part of the eleven dimensional supergravity action

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-G} \left[ \tilde{R} - \frac{1}{48} \tilde{F}_4^2 \right] - \frac{1}{12\kappa_{11}^2} \int \tilde{C}_3 \wedge \tilde{F}_4 \wedge \tilde{F}_4$$

Here the field with tilde belong to bosonic components of 11-dimensional supergravity. Let us compactify one of the spatial dimensions on $S^1$, following the procedure outlined in the last section. There will be a gauge field and a scalar field, when the metric is expressed in terms of the metric of the 10-dimensional theory. The 3-form potential will decompose into a 3-form potential but with additional piece according to the procedure of [48, 49] and a two form potential will appear as well. It is most convenient to express the 11-dimensional metric in the following form

$$\tilde{G}_{MN} = e^{-\frac{1}{2}\phi} \begin{pmatrix} G_{\mu\nu} + e^{\phi} A_\mu A_\nu & e^{\phi} A_\mu \\ e^{\phi} A_\mu & e^{\phi} \end{pmatrix}$$

The dimensional reduction of (1.2) goes over exactly to the type IIA action (4.1). Note that if we had not adopted this form of the decomposition of the 11-dimensional
metric with the overall factor of $e^{-\frac{1}{3}\phi}$ and the factors of $e^\phi$ in various places inside the matrix; but had compactified on a circle of radius, say, $R$; we would have obtained a reduced action with 10-dimensional metric, the moduli $R$ and the antisymmetric tensor potentials (2-form and 3-form) with appropriately modified C-S terms. The resulting action in ten dimensions would need some field redefinitions to match with the type IIA action. Let us see how the radius of compactification $R_{11}$ is related to type IIA string coupling constant $g_s^{(A)}$. Note from (4.3) that $R^2_{11} = (e^{(\frac{2}{3})\phi})^2$ and by definition $e^\phi = (g_{str})^2$. Therefore, we conclude that

$$R_{11} = (g_{str}^{(A)})^\frac{2}{3}$$

(4.4)

Therefore, in the perturbative regime of the type IIA theory, the radius of compactification of the 11-dimensional theory is very small. When we want to go over to the decompactification regime i.e. large radius limit of 11-dimensional theory, we can’t realise that domain since it is the strong coupling phase of the type IIA theory and perturbation theory does not provide any clue for the existence of the 11th dimension in the ten dimensional theory. The correspondence established between type IIA theory and 11-dimensional theory is at the level of the effective action. The 11-dimensional supergravity has a 3-form potential in the bosonic sector and the natural extended object is a membrane. The 10-dimensional theory admits a string as a fundamental object and supergravity action is zero slope limit of the string theory. How can one establish the relation between membrane and the string? The idea of double dimensional reduction provides an important clue. One can envisage a situation where we start from a membrane in eleven dimensions and compactify 11th dimension on a circle. Then, according to the prescription of double dimensional reduction [79], the membrane wraps around the compact direction so that the end result is the ten dimensional string.

We have described in the previous section how one can establish connections among the five string theories various dimensions through duality transformations in different spacetime dimensions; although there are five distinct ten dimensional theories when viewed in the perturbative frame work. The 11-dimensional theory is also recognised to play an important role in string dynamics. It is believed that there is an underlying fundamental theory, yet to be discovered, so that the manifestations of the theory in its various phases are realized through the string theories. It is postulated that in the low energy limit, we should derive the 11-dimensional supergravity action as an effective theory. The unknown fundamental theory is named U-theory. Since the 11-dimensional theory naturally admits membrane as a fundamental extended solution, it has been argued that the underlying fundamental theory is a theory of membranes. The M-theory is taken to be the underlying theory. We shall illustrate, with a few examples, that starting with an eleven dimensional theory with membrane, how one can obtain a host of relations about the structure of branes in various string theories. Since the BPS states do not get any quantum corrections, it is interesting to look for BPS states and then propose tests for the theory. When we compactify M-theory on
a circle, the momenta in that direction will be quantized and we shall get towers of KK massive states. These states will fall into representations of the 11-dimensional supergravity. In fact they are BPS states. In the KK reduction, the charge of a state (in the lower dimensions) is related to the momentum along the compact direction (thus automatically quantized) and in some suitable units the charge is proportional to \( \frac{m}{R_{11}} \), \( m \) being an integer. This is the charge associated with the gauge field \( A_\mu \) as a result of \( S^1 \) compactification (4.3). From the type IIA point of view this charge is that of gauge field coming from RR sector and the whole tower should exist as BPS state. We know already that elementary string states are RR-charge neutral and those massive towers belong to RR sector. We can identify the state with unit charge, \( m = 1 \) as a D0-brane of type IIA theory. The open string ends can get attached to D0-brane and act as the collective coordinates to give excitations. One can show that IIA theory has those BPS states belonging to the ultra-short multiplets and these also correspond to the states counting done from M-theory side. Therefore, we notice that duality between type IIA theory and M-theory is established for such states. In case of \( m > 1 \), the test is not so simple. One of the properties of BPS states is that the binding energy for composite BPS states is zero. That means, if we have a single D0-brane, a BPS state with \( m \) units of charge, we can’t distinguish it from collection of \( m \) BPS particles each carrying unit charge. Thus a test for the general case is rather difficult.

The relation between M-theory and type II theories can be established by exploiting the duality relations. Note that type IIA and type IIB theories are T-dual to each other when one of the directions is compactified. Since M-theory with one compact direction, \( S^1 \), is related to type IIA, therefore, M-theory with two compact dimensions, compactified on \( T^2 \), is expected to be intimately connected to type IIB with one direction compactified to \( S^1 \). We shall see that one needs to exploit the \( SL(2,\mathbb{Z}) \) S-duality symmetry of type IIB theory in this context (81). The type IIB theory has graviton, 2-form antisymmetric potential, \( B^{(1)}_{\mu\nu} \) and dilaton, \( \phi \) in the NS sector and 2-form potential, \( B^{(2)}_{\mu\nu} \), axion, \( \chi \) and 4-form potential, \( D_{\mu\nu\rho\lambda} \) in the RR sector; the field strength of D-field is self dual. For our purpose, it suffices to drop the D-field from considerations presently. The action is

\[
S_{str} = \frac{1}{\kappa^2} \int d^{10}x \sqrt{-G} [e^{-\phi}(R + (\partial\phi)^2) - \frac{1}{12} H^{(1)2} - \frac{1}{2}(\partial\chi)^2 - \frac{1}{12} \chi^2 H^{(1)2} - \frac{1}{6} \chi H^{(1)} \cdot H^{(2)} - \frac{1}{12} H^{(2)2}] 
\]

(4.5)

This action is written in the string frame metric. It is useful to go over to the Einstein frame by the conformal transformation. Furthermore, to write the Einstein frame action in a manifestly \( SL(2,\mathbb{Z}) \) invariant form, let us define

\[
\mathcal{M} = \begin{pmatrix} \chi^2 + e^{-\phi} & \chi e^\phi \\ \chi e^\phi & e^\phi \end{pmatrix}, \quad H = \begin{pmatrix} H^{(1)} \\ H^{(2)} \end{pmatrix}
\]

(4.6)
Then the action,

\[ S_E = \frac{1}{2\kappa^2} \int d^{10}x [R_g + \frac{1}{4} \text{Tr}(\partial_\mu \mathcal{M} \partial^\mu \mathcal{M}) - \frac{1}{12} H^T \mathcal{M} H] \]  

(4.7)

This action is invariant under the transformations

\[ \mathcal{M} \rightarrow \Lambda \mathcal{M} \Lambda^T, \quad H \rightarrow (\Lambda^T)^{-1} H \quad \text{and} \quad g_{\mu\nu} \rightarrow g_{\mu\nu} \]  

(4.8)

If one looks for a string solution in this theory then the solutions will be of three kinds: strings carrying NS charge, strings with RR charge and ones with both NS and RR charge. The procedure adopted in [81] is as follows: first look for a string solution with NS charge such that asymptotic values of axion, \( \chi_0 = 0 \) and that of dilaton \( \phi_0 = 0 \). In the language of complex moduli introduced earlier, asymptotic value of \( \tau_0 = i \). Moreover, one starts with \( H^{(2)} = 0 \), since one is looking for a string carrying NS charge only. Next introduce a specific \( SL(2, \mathbb{Z}) \) transformation such that the resulting string carries both types of charges; the relevant matrix is

\[ \Lambda = \frac{1}{\sqrt{q_1^2 + q_2^2}} \begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix} \]  

(4.9)

Although the string carries both types, NS and RR charges, still the modulus preserves the asymptotic value, \( \tau_0 = i \). Finally, introduce a general \( SL(2, \mathbb{Z}) \) transformation so that \( \tau_0 \) will take arbitrary value as a result of the duality transformation. The matrix is \( \Lambda = \begin{pmatrix} e^{-\phi_0/2} & \chi_0 e^{\phi_0/2} \\ 0 & e^{\phi_0/2} \end{pmatrix} \). As a consequence of the \( SL(2, \mathbb{Z}) \) transformation, not only we have strings which carry charges \( (q_1, q_2) \), but also the tensions of the strings depend on these charges; after all these are BPS strings. The formula for the tension of string with \( (q_1, q_2) \) charges is

\[ T_q = \left[ e^{\phi_0} (q_2 \chi_0 - q_1)^2 + e^{-\phi_0} q_2^2 \right]^\frac{1}{2} T \]  

(4.10)

where \( T \) is the tension of the NS string one started with for which \( \phi_0 = \chi_0 = 0 \) i.e. \( \tau_0 = i \). Since we consider \( SL(2, \mathbb{Z}) \) transformations, \( q_1 \) and \( q_2 \) should be integers. For stable strings, \( (q_1, q_2) \) should be relatively prime; otherwise these string will decay into multiple strings. If we compactify this theory on \( S^1 \), more interesting results follow. The spectrum of the nine dimensional theory is governed by the mass formula

\[ M_B^2 = \left( \frac{m}{R} \right)^2 + (2\pi R T_q n)^2 + 4\pi (N_L + N_R) \]  

(4.11)

Here we have explicitly kept the tension term showing that when the string winds ‘n’ times it stretches by its perimeter and energy is obtained by multiplying the tension \( T_q \). The last term is sum of contributions from left and right oscillators. The level matching condition tells us \( N_R - N_L = mn \). The BPS saturating multiplets have
either \( N_L = 0 \) or \( N_R = 0 \); ultrashort corresponds to both being zero. If we choose \( N_L = 0 \), then mass formula is (also level matching relation is used)

\[
M_B^2 = (2\pi nRT_a + \frac{m}{R})^2
\]

(4.12)

We have a rich spectrum and these masses should remain protected from any quantum corrections.

We can describe the same phenomena by compactifying the M-theory on \( T^2 \). There is membrane in M-theory with tension \( T_{11} \) and if it wraps \( m \) times on a torus of area \( A_{11} \), then the contribution to mass will be of the form \( mA_{11}T_{11} \). But the area is \( A_{11} = (2\pi R_{11})^2 \rho_2 \), \( \rho_2 \) being the modular parameter of the torus and \( \rho = \rho_1 + i\rho_2 \) the area is computed using 11-dimensional metric. Since we are considering compactification of the M-theory on \( T^2 \), the wave function of the two dimensional Laplacian (corresponding to two coordinates on the torus) must satisfy periodicity property appropriate to the torus and the mass formula should be suitably generalised with respect to mass formula for a string compactified on a circle.

\[
M_{11}^2 = [m(2\pi R_{11})^2 \rho_2 T_{11}]^2 + \frac{1}{R_{11}} [l_2^2 + \frac{1}{\rho_2^2} (l_1 - l_2\rho_1)^2] \]

(4.13)

\( l_1, l_2 \) are integers which enter the mass formula as the contribution to the KK part since the two dimensional Laplacian \(-\partial^2_x - \partial^2_y \) acting on the wave function

\[
\psi_{l_1,l_2}(x,y) = exp\{\frac{i}{R_{11}}[xl_2 + \frac{1}{\rho_2}y(l_1 - l_2\rho_1)]\}
\]

(4.14)

It is easy to see the periodicity property of the wave function by defining \( z = (x + iy)/2\pi R_{11} \), since the invariance now translates to \( z \rightarrow z + 1 \) and \( z \rightarrow z + \rho \) In order to compare the above mass formula, obtained from M-theory with \( T^2 \) compactification, with the corresponding one (4.12) from type IIB in 9-dimensions, we should recognize that (4.12) is derived using 11-dimensional metric. Therefore they could differ from each other by a multiplicative constant: \( M_{11} = CM_B \). Now the exact matching of the mass formula implies that the the modular parameters of \( T^2 \), denoted as \( \rho \), should be identical to the parameters of \( SL(2,Z) \), \( \tau \). Thus the modular group appearing in \( T^2 \) compactification of M-theory , \( SL(2,Z) \) is identical to the duality group, \( SL(2,Z) \), of type IIB theory. The following relations should be satisfied for matching of (4.12) and (4.14)

\[
R^{-2} = TT_{11}A_{11}^3, \quad \text{and} \quad C^2 = \frac{2\pi R_{11}e^{-\phi_0/2}T_{11}}{T}
\]

(4.15)

Since type IIA theory, in 9-dimensions, is related to IIB theory by T-duality, we can also get some insight into D0-branes in IIA theory. The mass spectrum of these (point) particles can viewed from two perspectives. One way is to identify the winding modes of the family of type IIB strings on the circle with the KK modes of the torus; and the other way of looking is to identify KK modes of the circle with wrapping of
the membrane on the torus. Again the mass formula matching relations can be used to relate parameters on both sides. There is also a five brane, the soliton counter part of fundamental membrane, in the M-theory and these are the only two extended objects in the 11-dimensional theory. Therefore, one expects that M-theory should be able to give description of NS branes and Dp-branes in the lower dimensional string theories. Moreover, the membrane tension of 11-dimensional theory, \( T_2^{(M)} \), is the only parameter since the 5-brane tension, \( T_5^{(M)} \) is determined from the Dirac quantization relation in terms of \( T_2^{(M)} \). In order to study the branes in 9-dimensional theory, we should see how different branes arise from M-theory and from type IIB theory. The simplest is the 2-brane. In case of M-theory, the membrane remains a membrane; but type IIB in 10-dimensions has no membrane (due to the absence of even RR field strength), therefore, the D3-brane of ten dimensional IIB theory wraps around \( S^1 \) to produce a membrane. But the type IIB string tension also is related to \( T_2^{(M)} \) since membrane wraps around torus to produce the string. So the simplest result is D3-brane tension and string tension of type IIB are related: \( T_3^{(B)} = \frac{1}{2\pi}(T_2^{(B)})^2 \); this result involves only IIB theory tensions derived through M-theory route. The 9-dimensional IIB theory will have D3-branes too. They will arise, from M-theory view point, as wrapping of the 5-brane around \( T^2 \). There are 4-branes in 9-dimensional IIB theory. Since 10-dimensional IIB theory has \( SL(2, Z) \) pair of strings, their solitonic partners 5-branes will come in same multiplets too. These 5-branes, compactified on \( S^1 \), will give the 4-branes in 9-dimensions. Similarly, one can discuss type IIA theory from both the perspectives in ten dimensions. If we define \( L = 2\pi R_{11} \) as the perimeter of the compact circle, then tension of IIA string gets related to \( T_2^{(M)} \) since IIA string, in 10-dimensions, will arise due to wrapping of membrane around the circle. The relation is \( T_1^{(A)} = g_A^{-\frac{1}{2}} LT_2^{(M)} \). For type IIA membrane we have \( T_2^{(A)} = g_A^{-1} T_3^{(A)} \). The 4-brane in IIA theory will come from wrapping of M-theory 5-brane around the circle and the relation becomes \( T_4^{(A)} = g_A^{-\frac{5}{2}} LT_5^{(M)} \) and we must also have \( T_5^{(A)} = g_A^{-2} T_5^{(M)} \). Then using relation between \( T_2^{(M)} \) and \( T_5^{(M)} \) together with the relation between \( T_4^{(A)} \) and \( T_5^{(M)} \), one can get an expression: \( T_5^{(A)} = \frac{1}{2\pi}(T_2^{(A)})^2 \).

The purpose of above examples was to illustrate how one can derive a large number relations using the M-theory. In lower spacetime dimensions, the theory provides a rich basis to understand branes coming from various theories. We would like to record one important fact for our future considerations. Note that the formula (4.10) of general \((q_1, q_2)\) string is for a string carrying NS and RR charges. For a string with only NS charge the \((1, 0)\) the tension scales as \( T \sim g_B^{\frac{1}{2}} \) and for the one carrying only unit RR charge it is \( T \sim g_B^{-\frac{1}{2}} \). In the string frame, after rescaling the metric, we find that a string with one unit of NS charge has tension of order 1 and the string with one unit of RR charge has tension \( g_B^{-1} \). Therefore, the mass density also has same dependence on coupling constant.

There are duality relations which relate compactified M-theory to other string theo-
ries. One of the interesting cases is $E_8 \times E_8$ heterotic string in ten dimensions \[84\]. There is no other string theory which can be related to this one. So it is expected that $E_8 \times E_8$ is connected to the 11-dimensional theory. But it cannot be $S^1$ compactified M-theory, because that compactified theory is type IIA as we have seen. Moreover, 11-dimensional theory as such is free from anomalies. However, if one considers compactification on $\frac{S^1}{Z_2}$ of M-theory it gets related to $E_8 \times E_8$ ten dimensional theory. The orientation of $S^1$ is reversed under $Z_2$ and it flips the sign of 3-form potential $C$. As a consequence of this projection, we are left with the metric in ten dimension, the dilaton and 2-form potential. The gauge boson and 3-form potential $C$ are projected out. The surviving fermions are Majorana-Weyl gravitino and Majorana-Weyl fermion. This is the supergravity in the bulk. Actually, $\frac{S^1}{Z_2}$ is a line segment with fixed points at the boundary. These are two copies of 10-dimensional flat space. The states from twisted sector should be localized on these planes. It was shown that half of the anomalous variation is localized in one plane and the other half on the other plane. The possible gauge groups that can cancel the anomaly are $E_8 \times E_8$ or $U(1)^{496}$. It is obvious the string theory to be identified is the $E_8 \times E_8$ heterotic string. There are other duality conjectures \[85, 86, 87\] between M-theory and other string theories in lower dimensions: M-theory on $K_3 \leftrightarrow$ heterotic/Type I on $T^3$. Compactification of M-theory on $\frac{T^9}{Z_2}$ is dual to type IIB on $K_3$. The lower dimensional compactifications $\frac{T^8}{Z_2}$ and $\frac{T^9}{Z_2}$ are related to Type I/ heterotic on $T^7$ and type IIB on $\frac{T^8}{Z_2}$ respectively. There are attempts to construct gauge supersymmetric gauge theories by choosing suitable combinations of intersecting branes and establish Seiberg-Witten dualities from this M-theory point of view of SUSY Yang-Mills gauge theories. Undoubtedly, the dualities together with the proposal for M-theory has brought us nearer to the goal of unified description of string theories. However, the underlying fundamental theory is yet to be discovered although we have seen many facets of that theory.

5 Black holes and String Theory

The physics of the black holes has many fascinating aspects. The classical black hole is the final stage of a collapsing heavy star. As the name suggests, matter falls into it and nothing comes out; there is an event horizon. However, deeper investigations have revealed, almost a quarter of a century ago, that there are strong similarities between thermodynamics and black hole mechanics \[88, 89\]. If $M$ is mass of the black hole,

$$dM = \frac{1}{8\pi G} k dA, \quad \delta A \geq 0$$

Here $G$ is the Newton’s constant, $A$ is the area of the event horizon and $k$ is the surface gravity. This is to be compared with thermodynamical relation,

$$dE = T dS, \quad \delta S \geq 0$$
Hawking’s startling discovery \cite{90} that black holes radiate with a black body spectrum of temperature \( T = \frac{\hbar}{2\pi} \), when quantum effects are accounted for, raised several important issues in black hole physics. One can also associate entropy with a black hole

\[
S_{\text{BH}} = \frac{A}{4\bar{g}\hbar}
\]

The thermodynamical relations used to describe macroscopic phenomena can be derived from statistical mechanics starting with microscopic fundamental laws of physics. Since \( \hbar \) appears in the black hole entropy formula, it is expected that the microscopic derivation of black hole entropy requires quantum gravity calculations. Moreover, entropy of a system, when interpreted from statistical mechanical point of view, counts the total number of degrees of freedom in the system. How do we count the number of degrees of freedom in a black hole and obtain the expression for entropy? There are more fundamental issues related to quantum mechanics when we carefully examine the implications of Hawking radiation. We can think of allowing some matter to go into the black hole, prepare the initial state as a pure quantum state to be the incident wave. However, the emitted Hawking radiation has a black body distribution and thus these are mixed states. Therefore, the S-matrix that will describe the above process will lose its unitarity property.

In the perturbative regime, string theory can provide reliable results for computations of processes involving graviton. The resulting S-matrix elements respect the required unitarity and analyticity properties. Thus, it is pertinent to ask what string theory has to offer in resolving the issues alluded to earlier. Recently, one of the important achievements of the string theory has been the microscopic derivation of the black hole entropy, for a special class of black holes that arise in string theory. We shall, initially, not set \( G = 1 \), to bring out a few salient points in discussions of stringy black holes and some times we shall display presence of \( \hbar \) in formulas. Recall, that the Newton’s constant is related to string coupling and tension as \( G \sim g_{\text{str}}^2 / T \), in four spacetime dimensions. If we have a massive string state, the gravitational field is \( GM_s \), where \( M_s \) is mass of a string state measured in units of \( T \); also some times we shall denote it as \( M \). Thus, the field increases as string coupling increases. String states are given my the mass formula \( M^2 = NT \) and it is well known that at a given mass there are a lot of states and the degeneracy \cite{92} grows exponentially with mass, i.e. \( e^M \). Thus one might think that the excited states, if treated as black holes, will reproduce the entropy formula; however, this simple argument in not adequate since black hole entropy grows like \( M^2 \), whereas the naive argument will give \( S_{\text{BH}} \sim M \). There have been attempts to explain this discrepancy saying that the mass that would appear in microscopic derivation of \( S_{\text{BH}} \) is not the same as the one appearing in Bekenstein-Hawking formula and there might be renormalization effects to be accounted for \cite{91}.

The perturbative string states appear in infinite levels and thus, for high enough mass, the massive elementary string state will lie inside the Schwarzschild radius associated with it. Consequently, they will require black hole descriptions. One of the ways
to derive black hole entropy microscopically is to consider such BPS states, so that when string coupling gets strong, the state is unchanged. In this approach, first step is to pick up appropriate BPS state and compute the microscopic entropy. Next, compute the Bekenstein-Hawking entropy of the BPS state, it is also an extremal black hole, and verify whether the two ways of calculating entropy are in agreement. This is the first clue that string theory might explain black hole entropy in microscopic way. However, the black holes constructed from the elementary string states had some short comings while computing the entropy. The area of the event horizon, for such black holes, tends to zero as one approaches the extremal limit; moreover, the dilaton also diverges at the horizon in this limit. This problem was encountered for string states in the NS sector.

The D-brane in RR sector can come as elementary states and there are corresponding solitonic states contained in the full spectrum. We had argued in the context of type IIB $SL(2,\mathbb{Z})$ strings that in string frame metric, NS states have tensions of order 1, whereas, D-strings had mass density of the order of $\frac{1}{g_{str}}$. For the solitons of NS sector the mass goes as $\frac{1}{g_{str}}$; but the solitons for RR sector still have mass order $\frac{1}{g_{str}}$. In the weak coupling regime NS solitons and RR ones are heavy. We should account for the gravitational fields they produce, which is $GM$. In view of above discussions, (i) NS elementary states produce very low field and (ii) RR states also produce low field in weak coupling limit; field tends to 0 as $g_{str} \to 0$. We may argue that in this regime, flat spacetime is a good description of the geometry. Since we are dealing with BPS states, as string coupling increases the mass remains unchanged, but the gravitational field keeps increasing and after some critical coupling, the spacetime is not flat any more; we must employ general theory of relativity. If these states describe black holes, then we should be able to compute the degrees of freedoms associated with them. It is possible to construct black hole configuration such that the area of the horizon is not zero nor the dilaton diverges at the horizon, when we take the extremal limit. For five dimensional black holes, we need at least three charges to have nonzero area for the horizon together with constant value for the dilaton at the horizon. In case of the four dimensional black hole needs four charges in order to satisfy the requirement of nonzero horizon area and finite value of dilaton (at the horizon).

The black holes which we shall consider now have some special characteristics. They can be thought of as composites of many D-branes carrying Ramond charges. We have mentioned before that the BPS states have the property that mass of composite BPS state is the sum of the masses of the constituents. One starts in the weak string coupling phase with such D-branes and proceeds towards strong coupling domain when gravity becomes strong. In weak coupling regime, the degeneracy of the level can be estimated reliably and microscopic entropy can be computed. In the strong coupling domain, the D-brane is inside the horizon and one can treat this like a black hole and compute the ratio $\frac{A}{\sigma}$, which is independent of string coupling $g_{str}$ since both area and Newton’s constant grow like $g_{str}^2$.

Let us discuss how the five dimensional black hole configuration is constructed with
D-branes [24]. We start with type IIB theory in 10-dimensions. We know that it will admit D1-string and D5-brane. We want to make the composite object heavy; therefore, we put \(Q_5\) number of D5-branes and \(Q_1\) number of D1-strings together. Let us compactify this theory on \(T^5\) such that the \(Q_5\) number of D5-branes are wrapped around \(T^5\), the \(Q_1\) D1-strings wrap along one of the directions of the torus. Then put some momentum along the direction in which the D-string wrapped; this momentum will be quantized in units of inverse radius of \(S^1\). The aim is to evaluate the microscopic entropy by counting number of degrees of freedom for this system and it involves some detail technical steps [95, 96, 97, 98]; but we shall outline only essential points. We expect to have a \(U(Q_5)\) supersymmetric Yang-Mills theory on the D5-brane worldvolume. This will be a gauge theory in 5 + 1 dimensions which is derived by dimensional reduction of \(N = 1\) supersymmetric Yang-Mills theory from ten dimensions [95]. The D-string is inside this pack of D5-branes (\(Q_5\) of them). The D-string can be viewed as an instanton in this six dimensional spacetime, since an instanton in 6-dimensional theory with no time dependence and extension in one direction is a string. There are \(Q_1\) such strings in the D5-brane configuration. Their low energy dynamics is described by two dimensional supersymmetric sigma model in \(4Q_1Q_5\) dimensional hyper Kahler manifold. Every boson contributes factor 1 and every fermion contributes \(\frac{1}{2}\) to the central charge as we noted in Sec. II. Thus, total central charge is

\[
c = 6Q_1Q_5
\]  

(5.4)

Since we are dealing with BPS states, for these states \(L_0 = 0\) and the momentum given along \(S^1\) is related to the difference \(L_0 - \bar{L}_0\). If we take momentum to be large i.e. \(P_s = -\frac{n}{R}, n\) large; then using Cardy’s result (relating degeneracy to central charge), one gets

\[
d(Q_1, Q_5, n) = \exp(2\pi \sqrt{Q_1Q_5n})
\]

(5.5)

The black hole entropy computed from the microscopic viewpoint is given by

\[
S_{\text{microscopic}} = 2\pi \sqrt{Q_1Q_5n}
\]

(5.6)

In order to derive the black hole entropy, \(S_{BH}\), from Bekenstein-Hawking formula, we have to specify the metric, the charges and then compute the area of the event horizon in the extremal limit.

There is way to visualize the physical processes that lead to microscopic [24] derivation of the entropy formula. The D-string is inside D5-brane and the low level excitations are the lowest lying modes of the open strings attached to this one. If we think of the physical degrees of freedom, these are 8 transverse vectors and their super partners. Since these have to satisfy the Dirichlet boundary condition, they are constrained to move along the D-string. We are dealing with BPS state, therefore, these move only in one direction (say left). Since the D-string is wrapped around one circle of \(T^5\), we choose \(x_1\), then length is winding number times the radius of the circle. But the momenta of individual open strings moving on this unidirectional path on
the circle is quantized. Moreover, sum of their momentum is constrained too by the total momentum we have put on that direction. Therefore, this is analogous to solving statistical mechanics of a one dimensional system on a circle where total energy (momenta are same as energy) is fixed.

The next step is to define the metric for the above configuration of the branes and obtain the harmonic functions that are necessary to satisfy the equations of motion for the brane configurations [100, 101].

\[
\begin{aligned}
    ds^2 &= H_1^{1/2} H_5^{1/2} \left\{ [H_1^{-1} H_5^{-1} (-K^{-1} dt^2 + K (dx_1 - (K^{-1} - 1) dt)^2) + H_5^{-1} (dx_2^2 + \cdots + dx_9^2)] \right. \\
    &\quad\left. + dx_6^2 + \cdots + dx_9^2 \right\} 
\end{aligned}
\]  

(5.7)

We specify the compact directions as follows: the \(Q_5\) number of D5-branes are wrapped in \(x_1, \ldots, x_5\) directions, D-string is wrapped in \(x_1\) and the momentum is along \(x_1\) too. Since we toroidally compactify to five dimensions \(x_i, i = 1, \ldots, 5\) are periodic and the radius of compactification is \(R_i\) along \(i\)th direction. and

\[
e^{-2\phi} = H_1^{-1} H_5, \quad B_{01} = H_1^{-1} - 1
\]

(5.8)

\[
H_{ijk} = \frac{1}{2} \epsilon_{ijkl} \partial_l H_5, \quad i, j, k, l = 6, \ldots, 9
\]

(5.9)

\[
r^2 = x_6^2 + \cdots + x_9^2
\]

(5.10)

The harmonic functions are equal to

\[
H_1 = 1 + C_1 \frac{Q_1}{r^2}, \quad C_1 = \frac{g_{str} \alpha'^3}{V}
\]

(5.11)

\[
H_5 = 1 + C_5 \frac{Q_5}{r^2}, \quad C_5 = g_{str} \alpha'
\]

(5.12)

\[
K = 1 + C_K \frac{Q_K}{r^2}, \quad C_K = \frac{ng_{str}^2 \alpha'^4}{R_1^2 V}
\]

(5.13)

where \(V = R_2 R_3 R_4 R_5\), we displayed the \(\alpha'\) dependence to show how the dimensionality of the charges appear, but now on we set the slope to unity as usual. Let us briefly note how the charges arise in this black hole. There is electric charge \(Q_1\) coming from \(B_{01}\) which is a gauge field now, after compactification of \(x_1\) coordinate. The \(Q_5\) charge is magnetic type originally attributed to D5-brane in 10-dimensions. After compactification the Poincare dual of that 3-form RR field strength is two form field strength and it becomes an electric charge counting D-brane charges. Of course, the third charge comes from momentum given along \(x_1\) direction and is quantized. When any one of these charges vanishes, the area of the event horizon vanishes too.
5-dimensional effective action. The metric in the five dimensional space takes the following form

\[ ds^2 = \lambda^{-2/3} dt^2 + \lambda^{1/3} (dr^2 + r^2 \Omega_3^2) \]  

(5.14)

where

\[ \lambda = H_1 H_5 K = (1 + C_1 Q_1/r^2)(1 + C_5 Q_5/r^2)(1 + C_K Q_K/r^2) \]  

(5.15)

This corresponds to an extremal charged black hole and the horizon is located at \( r = 0 \). However, the area of the horizon is nonzero and it is proportional to the product of the charges. The expression for the area is

\[ A_5 = (r^2 \lambda^{1/3})^{3/2} \bigg|_{r=0} = \sqrt{C_1 Q_1 C_5 Q_5 C_K Q_K} (2\pi^2) = \frac{g_{str}^2}{R_1 V} \]  

(5.16)

The Newton’s constant in five dimensions gets related to the ten dimensional Newton’s constant after we compactify on \( T^5 \) and the relation is

\[ G_N^{(5)} = \frac{G_N^{(10)}}{(2\pi)^5 R_1 V} = \frac{1}{4} \frac{g_{str}^2}{R_1 V} \]  

(5.17)

Therefore, the entropy is equal to

\[ S_{BH} = \frac{A_5}{4G_5} = 2\pi \sqrt{Q_1 Q_5 n} \]  

(5.18)

This expression exactly agrees with the expression for \( S_{microscopic} \). A few comments are in order to discuss the constraints on the parameters for the above relation to be valid. The string effective action adopted to obtain the brane solutions is valid when string loop corrections and \( \alpha' \) corrections are nonleading. The string loop corrections are small when \( g_{str} \to 0 \) with the values of the charges held fixed. The charges correspond to characteristic scales of the system. If we want ignore \( \alpha' \) correction terms then the charges should be larger than string scale i.e. \( Q_1, Q_5 \) and \( n \) are much larger than \( \alpha' \). If the compactification radii of the torii be taken as order of string length scale, then we should have \( g_{str} Q_1, g_{str} Q_5, g_{str}^2 n \gg 1 \). This tells us that \( n \gg Q_1 \sim Q_5 \gg 1 \).

The entropy of nonextremal black holes can be considered in a similar manner; however, we must keep several points in mind. First of all, the extremal black holes are BPS stated and they get no quantum corrections. Therefore, whereas the microscopic entropy is computed in the weak coupling phase, the Bekenstein-Hawking entropy is obtained after we go over to the strong coupling domain so that the composite D-brane configuration lies inside the horizon. In case of nonextremal black holes, we have no theorem against quantum corrections and therefore, passage to strong coupling limit is not so simple. It is argued, that a black hole which is slightly away from extremality might allow smooth increase of the coupling constant as one starts from weak coupling limit. This type of black holes configuration can be achieved by
allowing some low level right moving oscillators compared to the high left moving levels (note that for extremal case $N_R = 0$). We shall not discuss the properties of these black hole in detail here.

The BPS extremal black holes are stable and they have zero temperature; therefore, they will not emit Hawking radiation. If we intend to understand the Hawking radiations from black holes in string theory, we have to look for those ones which are excited states and can decay into lower energy state. The starting point is to consider a nonextremal black hole. Since there will be left and right movers, the open string states will be going in opposite directions on the D-string. Again, it is a one dimensional problem where one can imagine that two oppositely moving open string states collide to give a closed string state. If we were to calculate the S-matrix element for such a process, we shall consider initial state, final state and a suitable interaction Hamiltonian for our computational purpose. In order to get the emission rate, one will take modulus square of this amplitude, average over initial states, sum over final states and divide by usual phase space factor. The state of the initial nonextremal black hole is given by occupation numbers $N_L$ and $N_R$ and the amount of momentum we give on the compact circle which are going in opposite directions. The momenta are quantized as $\frac{n}{N}$ in either direction and thus the closed string state will carry momentum $\frac{2n}{N}$. As we have seen there are $4Q_1Q_5$ bosonic and fermionic oscillators. The string theory calculation gives the amplitude for emission of a closed string state from these initial state $[100]$. The sum over final state and averaging over initial states leads to a factor $\rho_L \rho_R$, where for example

$$\rho_R = \frac{1}{N_i} \sum_i \langle i | N_R | i \rangle$$

(5.19)

where $N_i$ is the total number of initial states and $N_R$ is the number operator of right movers. We might carry out the averaging over all possible initial states with a given value of $N_R$ by adopting the statistical mechanical prescription. The problem actually maps to the case of one dimensional gas and the microcanonical ensemble can be used since we are holding $N_R$ fixed; energy is held constant. The configuration of the black hole is such that $N_L \gg N_R > 1$. If $k_0$ is the momentum of outgoing massless closed string the final calculation give the decay rate as

$$d\Gamma \sim (\text{Area}) \frac{e^{-\frac{k_0}{T_R}}}{1 - e^{-\frac{k_0}{T_R}}} d^4k$$

(5.20)

A more careful calculation $[102]$ reveals a surprising result that not only the form of thermal distribution is recovered, but also the numerical coefficients match with semiclassical results of Hawking. The result has been derived for four dimensional black holes as well $[103]$. It is an interesting question to ask whether one can calculate the absorption cross section of an extremal black hole for a closed string massless scalar and then relate that cross section to the decay rate of a nonextremal black hole by
using the principle of detailed balance in quantum mechanics taking into account all
the subtleties. Indeed explicit verification shows that such a check yields the correct
result [104].

6 M-theory and the M(atrix) model

Our present understanding of string dynamics together with duality symmetries
strengthen the belief that there is a fundamental theory and the five perturbatively
consistent theories are different phases of that underlying theory. However, we do
not know what this theory is except the conjecture that the low energy limit of this
theory is the 11-dimensional supergravity action. There are deep questions about
the structure of this theory. We shall call it the M-theory. We recall that strong
coupling limit of the type IIA theory is identified with 11-dimensional supergravity.
When viewed from type IIA perspective, the existence of D0-branes as nonpertur-
bative RR point like objects is quite important for our discussion. They are BPS
states and their mass is of the order $\frac{1}{g_{str}}$ and scaled by 10-dimensional length scale
$l_s$. These being BPS states, one could assume that there are threshold bound states
of many, say N, D0-branes which satisfy the properties of bound BPS states. Now
if we take the strong coupling limit, then it is found that the low energy spectrum
is same as the spectrum of the 11-dimensional supergravity. This is an important
evidence. Furthermore, the 11-dimensional theory is known to admit membrane and
five brane and we have argued how one can study properties of various brane con-
fignurations in string theories after compactifying the M-theory. The M(atrix) model
[105, 32, 33] can describe perturbation expansions of various string theories. There
is a limit in which the theory provides connection with 11-dimensional supergravity
theory. However, one would like to seek answers to several questions from this the-
ory. For example, the general prescription for the compactification of the theory is
not known. Similarly, the complete set of degrees of freedom of this theory is to be
obtained. The M(atrix) theory, nevertheless, provides insight into nonperturbative
definition of string theory and it also exhibits string dualities [106]. One can also go
over to various string theories by adopting different limiting prescriptions.
The model resorts to infinite momentum frame (IMF) technique boosted along a
compact direction. The momenta along compact direction is quantized; and one
starts with N units of these momenta and then $N \rightarrow \infty$ limit is taken. Since one
is working in the light-cone frame while constructing M(atrix) theory, the theory is
not manifestly Lorentz invariance. Thus Lorentz invariance might be recovered in the
large N limit. In the M(atrix) model formulations one encounters parameters which
have the interpretation of being expectation values of scalars when viewed from the
string theory side. But in the M(atrix) model when we have IMF formulation, these
constant modes have infinite frequency and they are frozen into fixed configuration.
The theory in its present formulation is not background independent. Moreover, one
encounters problems while compactifying the theory on an arbitrary d-dimensional
torus. We may remind the reader that the M(atrix) theory provides a rich structure to study various aspects of string theory from M-theory standpoint.

The infinite momentum frame (IMF) technique played a very useful role in current algebra \[107\]. In field theoretic calculations it simplifies perturbation theory calculations \[108, 109\]. When we have to deal with a collection of particles, we can define IMF to be a frame where the total momentum is taken to be very large. If we designate particles by index I, J... then

\[
P_I = \eta I P + P_{TI}
\]

(6.1)

where T stands for 'transverse' and \( P \cdot P_{TI} = 0 \), \( \sum P_{TI} = 0 \) and \( \eta I \leq 1 \). For a highly boosted coordinate system we could have all \( \eta I \) positive. Particularly, for the case at hand, we deal with massive particles and we can choose an appropriate frame to satisfy our requirement. Energy of any particle satisfies relativistic relation

\[
E_I = \sqrt{P^2_I + m^2_I} = \eta I P + \frac{P^2_{TI} + m^2_I}{2\eta I P} + ..
\]

(6.2)

it is understood that there are terms higher order in \( \frac{1}{P} \) denoted by dots. The expression for energy is similar to that of a nonrelativistic particle in a lower dimension with mass term taking a modified form. When we use a light-cone (LC) frame, a spatial direction is identified and designated as longitudinal. The longitudinal momentum is \( P_{LI} = \eta I P \) and one defines \( P_{\pm I} = E_I \pm P_{LI} = E_I \pm \eta I P \). The mass shell condition translates to \( P_{+I}P_{-I} - P^2_{TI} = m^2_I \) and we can rewrite this relation as

\[
E_I - \eta I P = \frac{P^2_{TI} + m^2_I}{P_{+I}}
\]

(6.3)

In the limit of large \( P \), we have \( \eta I P \) large and therefore, \( E_I \to \eta I P \) with \( P_{+I} \sim 2\eta I P \). When M-theory is envisaged in IMF, let us designate the momenta as \( p_0, p_i, i = 1,..9 \) and \( p_{11} \). One compactifies 11th direction with and this is also boosted, therefore \( \{p_i\} \) are collectively denoted as \( p_T \). Thus for collection of the D0 particles

\[
E - p_{11}^{total} = \sum_I \frac{p_{TI}^2}{2p_{11}}
\]

(6.4)

We note that there are 32 real supercharges in the theory. When one adopts IMF description, it is convenient to split them into two groups each having 16 of them. The charges in every group transform as spinors of \( SO(9) \). Let us denote charges as \( Q_\alpha, \alpha = 1,..16 \), and \( q_A, A = 1,..16 \). The algebra of these charges are

\[
\{Q_\alpha, Q_\beta\} = \delta_{\alpha\beta} H, \quad \{q_A, q_B\} = \delta_{AB} P_{11}
\]

(6.5)

\[
\{Q_\alpha, q_A\} = \gamma^i_{A\alpha} P_i
\]

(6.6)
Here $H$ is the Hamiltonian operator, P’s are the corresponding momentum operators and $\gamma_i$ are 16 dimensional gamma matrices.

We have discussed earlier, how D0-brane has a natural interpretation from the 11-dimensional theory with a compact coordinate and the RR charge is related to quantized momenta along this direction. The relation between mass and charge is satisfied since these are BPS states. There exists a sector with $N$ units of D0-brane charge, carrying Kaluza-Klein momentum $\frac{N}{R_{11}}$. If we hold $N$ fixed and take a limit $R_{11} \to 0$, we go over to the weak coupling phase of string theory; however, in the passage to this limit, the string scale is not held fixed. The aim is to study the phenomena in the 11-dimensional theory and thus $l_{11}$ is to be kept fixed. We recall that

$$R_{11}^3 = g_{str}^2 l_{11}^3$$

and the string length scale, $l_s^2 = \frac{l_{11}^3}{R_{11}}$. Thus as the compactification radius tends to zero string scale diverges. We have also seen earlier, as the radius shrinks, the mass of D0-brane tends to infinity, when measured in 11-dimensional Planck units in ten dimensions. In other words the mass of D0-brane is

$$\frac{1}{g_s l_s} = \frac{1}{R_{11}}$$

and therefore, it is appropriate to identify them as the KK modes. Thus, when we consider mass of the these particles in 10-dimensions, in scales of eleven dimensional theory, the particles become very heavy and a nonrelativistic description is quite adequate. If we were to describe M-theory in terms of type IIA zero branes, then we have a scenario where M-theory is equivalent to $N \to \infty$ limit of the nonrelativistic quantum mechanics of N D0-branes which are in weak coupling phase of type IIA theory. Furthermore, as Witten has argued [95] the physics of $N$ coincident D0-branes is described by dimensionally reducing ten dimensional $U(N)$ supersymmetric Yang-Mills theory to 0 + 1 dimensions [110]. Let us consider supersymmetric quantum mechanics of a single D0 particle. The starting point is the action

$$\int dt \text{Tr} \left( \frac{1}{2g_{str}} (D_0 X^i)^2 - i \theta^T D_0 \theta + \frac{1}{4g_{str}} ([X^i, X^j])^2 + \theta^T \gamma_i [X^i, \theta] \right)$$

This is the action obtained from 10-dimensional super Yang-Mills theory reduced to one dimension. Here $i = 1, \ldots, 9$ stands for transverse directions and $\theta$ are real spinors with 16 components. Since $X^i$ and $\theta$ come from the gauge groups, they are in the adjoint representations of $U(N)$. Since they carry only time dependence, these are $N \times N$ matrices. $D_0 = \partial_t + [A_0,]$ is the covariant derivative and this can be converted to ordinary derivative with the gauge choice $A_0 = 0$. The mass of D0-brane is order $\frac{1}{g_{str}}$, thus the first term in (6.8) can be written as $\int dt \frac{M}{2} \left( \frac{dX^i}{dt} \right)^2$. Note that the action (6.9) contains parameters of type IIA theory. It is convenient to scale $X^i = g_{str}^{\frac{1}{2}} Y^i$ which amounts to rescaling of the metric to that of 11-dimensional theory. Moreover,
one scales the time variable as \( t = g_{str} \frac{1}{4} \tau \) and denotes the \( \tau \) derivative by a dot. The action is rewritten as

\[
S = \int d\tau \text{Tr} \left( \frac{1}{2R_{11}} (\dot{Y}^i)^2 - i\theta^T \dot{\theta} + \frac{R_{11}}{4} ([Y^i, Y^j])^2 + R\theta^T \gamma_i [Y^i, \theta] \right) \tag{6.10}
\]

If \( \Pi_i = \frac{\dot{Y}^i}{R_{11}} \) and \( \pi = -i\theta^T \) are conjugate momenta of \( Y^i \) and \( \theta \) respectively, the corresponding Hamiltonian is given by

\[
H = R_{11} \text{Tr} \left( \frac{1}{2} \Pi_i^2 - \frac{1}{4} ([Y^i, Y^j])^2 - \theta^T \gamma_i [Y^i, \theta] \right) \tag{6.11}
\]

One can define \( H \equiv R_{11} \tilde{H} \) for convenience factoring out over all \( R_{11} \). Notice also that the potential energy term \( \frac{1}{4} R_{11} \text{Tr}([Y^i, Y^j])^2 \) is non-negative. When \( R_{11} \to \infty \), we are in decompactification phase of M-theory. Thus, the finite energy states of \( H \) are those for which the Hamiltonian \( \tilde{H} \) has vanishing eigenvalues. One seeks those states for which \( \tilde{H}|\psi\rangle = \frac{\epsilon}{N} |\psi\rangle \), which is equivalent to seeking a solution \( H|\psi\rangle = \frac{\epsilon}{N} |\psi\rangle \) where \( \epsilon \) is finite. We know that, for collection of \( N \) number of D0-branes, the total momentum \( p_{11} = \frac{N}{R_{11}} \) and therefore, the energy is given by \( E = \frac{\epsilon}{p_{11}} \). We have to identify \( \epsilon \) with \( \frac{1}{2} P_T^2 \) if we recall (6.4). The \( N \times N \) matrices \( X^i \) can be interpreted as the location of \( N \) D0-branes. When we consider the potential term in \( Y^i \) variables (6.10), we notice that there are flat directions when \( [Y^i, Y^j] = 0 \). Here we deal with a quantum mechanical system and \( Y^i \) have are the collective coordinates. In such situation as mutually commuting \( Y^i \), we can diagonalize \( Y^i = \text{diag} \left( y_1^i, y_2^i, ..., y_N^i \right) \). Thus \( y_n^i \) is the \( i \)th coordinate of the \( n \)th D0-brane. It is easy to see that there is invariance under Galilean translation, \( Y^i \to Y^i + d^i1 \) and the Galilean boost \( Y^i \to Y^i + v^i t 1 \) as is expected of a nonrelativistic system, here \( 1 \) is the unit matrix. The boost will affect the center of mass momentum; but neither the relative momenta nor interaction term are affected by these transformations.

We can consider two clusters separated from one another. This is familiar in composite model of hadrons where quarks are the basic constituents. In the parton picture, the proton is made of large number of partons with very small binding energy and one could describe photon-hadron deep inelastic scattering in IMF [111]. In this case we can think of configurations where the \( N \times N \) matrices \( Y^i \) can be decomposed to block diagonal form of say \( n \) blocks of \( N_1 \times N_1, N_2 \times N_2, ..., N_n \times N_n \) such that \( \sum_m N_m = N \). This decomposition can be interpreted as if we have \( n \) separated clusters of D0-branes where each of the clusters has \( N_1, N_2, ..., N_n \) number of particles. The distance between two clusters can be defined as

\[
r_{ab} = \frac{1}{N_a} \text{Tr} Y_a - \frac{1}{N_b} \text{Tr} Y_b \tag{6.12}
\]

where \( a \) and \( b \) are the two clusters. Now we can visualize how the potential will arise. It comes from \( \text{Tr}([Y^i, Y^j])^2 \) and this goes like modulus squared of the off diagonal block elements multiplied by the minimum of the \( r_{ab}^2 \) and an appropriate
numerical constant. Thus, if we consider well separated cluster of D particles, the off diagonal elements are required to be small; otherwise, the potential will grow like \( r_{ab}^2 \).
We should keep in mind that the system is supersymmetric and having a harmonic oscillator type potential does not imply ground state energy is that of the oscillator. The supersymmetric quantum mechanical system has a very rich structure. This becomes transparent if we consider a single D0-brane, i.e. \( N = 1 \).

\[
H = \frac{R_{11}}{2} \Pi_i^2 = \frac{R_{11}}{2} P_T^2 = \frac{P_T}{p_{11}}
\]  

(6.13)

When we look at this equation from 11-dimensional point of view, this corresponds to the relation between energy and momentum of a massless particle in IMF. When we take into account the 16 component fermions, \( \theta \) we eventually get the supermultiplet with 256 total degrees of freedom and this agrees with the massless degrees of freedom of \( N = 1 \) supergravity in eleven dimensions. In fact the bosonic components are 128 equal to fermionic degrees of freedom. As is well known, there 44 components from graviton and 84 from the antisymmetric tensor field in 11-dimensions. When we have \( N > 1 \), it is necessary to separate the center of mass motion and define the relative coordinates and the decomposition is as follows:

\[
Y^i = Y^i_r + Y^i_{cm} \mathbf{1}, \quad Y^i_{cm} = \frac{1}{N} \text{Tr} \ Y^i
\]

(6.14)

\[
\Pi_i = \Pi_i^r + \frac{1}{N} P_{cm} \ i \ \mathbf{1}, \quad P_{cm} \ i = \text{Tr} \ \Pi_i
\]

(6.15)

and \( \text{Tr} \ Y_r^i = \text{Tr} \ \Pi_r^i = 0 \). Now the total Hamiltonian will be written as a sum of two terms

\[
H = H_{cm} + H_r
\]

(6.16)

with

\[
H_{cm} = \frac{R_{11}}{2N}(P_{cm} \ i)^2 = \frac{1}{p_{11}}(P_{cm} \ i)^2
\]

(6.17)

Note the appearance of the factor \( \frac{R}{N} = \frac{1}{p_{11}} \) as expected. We have defined the center of mass coordinate, canonical momentum and the Hamiltonian by taking trace over \( U(N) \) matrices. Therefore, the relative Hamiltonian is a function of \( \{ Y_r^i, \ \Pi_i^r \} \). Thus \( H_r \) is quite similar to the original Hamiltonian; however, all the variables are \( SU(N) \) matrices, they are traceless since the trace part is separated out. It has been shown that the relative Hamiltonian has zero energy bound states due to the presence of supersymmetry [9, 12, 13]. The total energy is due to the center of mass energy: \( E = E_{cm} = \frac{1}{2p_{11}}(P_{cm}^r)^2 \). In this case one also gets the supergravity multiple which has 256 states. Therefore, for any \( N \), we see that the spectrum contains supergravitons. Suppose we decompose \( Y^i \) to various blocks which describe clusters of D-particles. In the simplest case, if the submatrices are exactly block diagonal so that off diagonal elements are zero, then the total Hamiltonian will be given by sum
of \( n \) separate Hamiltonians without any interactions amongst them. If we let the off diagonal elements appear (give them small values), that will amount to switching on interactions between the clusters. The physical picture is that we have several clusters, each cluster will have its supergraviton in the spectrum. There could be arbitrary number of them and therefore, we let \( N \) go to infinity. Thus the matrix model contains the full Fock space of supergravitons. The interaction among the supergravitons is described due to the presence of off diagonal elements and one should be able to describe various processes involving supergravitons in this picture. In order to compute S-matrix element for scattering of two supergravitons when their transverse velocities are small, we have to determine potential between them. One starts by considering the classical configurations and the fluctuations over them to compute the effective action \[114\]. Suppose we give transverse velocity \( v \) and define the impact parameter as \( b \) and expand the coordinates around their backgrounds as follows:

\[
X^9 = \frac{1}{2} b \sigma_3 + \sqrt{g_{str}} \delta X^9, \quad X^8 = \frac{1}{2} v t \sigma_3 + \sqrt{g_{str}} \delta X^8
\]

\[
X^i = \sqrt{g_{str}} \delta X^i, \quad i \neq 8, 9
\]

(6.18)

(6.19)

Here \( \delta X^i \) denotes the fluctuations and \( \sigma_3 \) is the Pauli matrix. When we have vanishing fluctuation, the classical configuration is such that total transverse center of mass momentum and position vanish. The \( 2 \times 2 \) matrices are block diagonal which describes two clusters of D0-branes and in this case we have \( N_1 = N_2 = 1 \). Now the separation between the two particles is given by \( r_{ab} = \sqrt{v^2 t^2 + b^2} \). The effective action can be computed using the standard techniques and the order \( \hbar \) term will contain determinant of (basically) propagators when we restrict to one loop level. Thus

\[
S_{eff} = S_0 + \int d\tau V_{eff}(r(\tau)) \equiv \int d\tau V_{eff}(\sqrt{v^2 \tau^2 + b^2})
\]

(6.20)

For large impact parameter, the long range part of the potential in the leading order is given by \[114\]

\[
V_{eff}(r) = -\frac{15}{16} \frac{v^4}{r^4} + \text{higher orders}
\]

(6.21)

The result is striking in the sense that this form of the potential can be derived from the supergravity action at the tree level i.e. considering graviton exchange. Thus starting from a simple M(atrix) model description, one could extract a result of 11-dimensional supergravity.

The 11-dimensional supergravity admits supermembrane. It is worthwhile to ask how much the M(atrix) model can tell us about the underlying membrane theory. The membrane is extended object in two spatial directions as the name suggests. Moreover, the dimension of spacetime in which the supermembrane can exist is quite restricted \[113, 23\]. The reason for such constraints lies in the fact that the action contains Wess-Zumino-Witten term and the supersymmetry invariance of the full action restricts the spacetime dimensions to 4,5,7 or 11. The membrane is described
by $Z^\mu(\sigma, \xi, \tau)$, where $\sigma, \xi$ and $\tau$ are the worldvolume coordinates. When one adopts a Hamiltonian formalism, a fixed $\tau$-slice is chosen and thus the explicit $\tau$ dependence in $Z^\mu$ does not appear and the derivatives with respect to worldvolume time are traded for canonical momenta $P_\mu$. The light-cone gauge is a convenient description to see the physical degrees of freedom and in this gauge the membrane Hamiltonian takes the following form \[116\]

$$H_M = \frac{1}{2p_{11}} \int d\sigma d\xi \left( \frac{(2\pi T_2)^2}{4p_{11}} \int d\sigma d\xi \{ \{ Z^i(\sigma, \xi), Z^j(\sigma, \xi) \} \}^2 + \text{fermionic terms} \right)$$

(6.22)

where the brackets appearing is the second term are defined as

$$\{ A, B \} = \partial_\sigma A \partial_\xi B - \partial_\xi A \partial_\sigma B$$

(6.23)

and $T_2$ is the membrane tension. Let us assume that the worldvolume of the membrane can be written as $\Sigma \times \mathbb{R}$, where $\Sigma$ has the topology of a torus. For this topology, $Z^i(\sigma, \xi)$ is a double periodic function and we can expand $Z^i$ in double Fourier series with $Z_{mn}^i$ as the Fourier coefficients. Thus we have nine $\infty \times \infty$ matrices and same would be the case if we had considered nine $Y^i$’s in the $N \to \infty$ limit. In order to establish relation with the membrane Hamiltonian (6.22), we have show how the commutator $[Y^i, Y^j]$ goes over to the bracket $\{ Z^i, Z^j \}$. For arbitrary finite N, introduce two $N \times N$ matrices $U$ and $V$, satisfying the properties

$$U^N = V^N = 1, \quad \text{and} \quad UV = e^{\frac{2\pi i}{N}} VU$$

(6.24)

This can be realized if U and V have the following special form $U_{j,j+1} = U_{N,1} = 1$ and $V_{j,j} = e^{\frac{2\pi i(j-1)}{N}}$ and all other matrix elements set to zero. A more abstract, ‘t Hooft, representation is

$$U = e^{ip}, \quad V = e^{iq}, \quad [p, q] = \frac{2\pi}{N} i$$

(6.25)

This is the canonical commutation relation between position and momentum when the space is taken to be compact and discrete. It is worthwhile to point out that the above commutation relation will not hold good for finite dimensional matrices. However, acting on states with low wave number, the error on the r.h.s of the commutator $[p, q]$ is further down by power of N and therefore, $\frac{2\pi}{N}$ is the leading term. Thus as N assumes higher and higher values, the error gets smaller and smaller. From $U^N = V^N = 1$ we can conclude that $p$ and $q$ take eigenvalues $m \frac{2\pi}{N}$, where $m$ takes values 0, 1, 2...($N-1$). Moreover $\text{Tr} U^n V^m = N \delta_{n,0} \delta_{m,0}$, where 0 in both the Kronecker delta are to be understood as mod N. Now we can expand any $N \times N$ matrix in terms of Fourier modes.

$$A = \sum_{n,m=N/2-1}^{N/2} A_{nm} U^n V^m = \sum_{n,m=N/2-1}^{N/2} A_{nm} e^{inp} e^{imq}$$

(6.26)
Since commutator of $p$ and $q$ is order $\frac{1}{N}$, in the $N \to \infty$ limit, they will commute. The eigenvalues of these two operators will fill the interval $[0, 2\pi]$ and 0 is to be identified with $2\pi$ since we have toric geometry. The double Fourier expansion (6.26) takes the form

$$A(p, q) = \sum_{n, m=-\infty}^{\infty} A_{nm} e^{inp} e^{imq}$$

(6.27)

and the Fourier coefficients with the double index are defined as

$$A_{nm} = \int_{0}^{2\pi} \int_{0}^{2\pi} \frac{dp}{2\pi} \frac{dq}{2\pi} A(p, q) e^{-inp} e^{-imq}$$

(6.28)

Also $\text{Tr} \ A = NA_{00}$, when we take $N \to \infty$ limit, $\text{Tr} \ A \to N \int_{0}^{2\pi} \int_{0}^{2\pi} \frac{dp}{2\pi} \frac{dq}{2\pi} A(p, q)$. One can show with some algebra that the commutator of two matrices in the infinite $N$ limit goes over to the $\{, \}$. Finally bosonic part of the M(atrix) model Lagrangian goes over to a form (identify $\frac{dp}{2\pi} = d\sigma$ and $\frac{dq}{2\pi} = d\xi$)

$$L_m \to \frac{N}{2R} \int d\sigma d\xi (\dot{Z}^i(\sigma, \xi))^2 - \frac{R}{4N} \int d\sigma d\xi (Z^i(\sigma, \xi), Z^j(\sigma, \xi))^2$$

(6.29)

Note that $\frac{N}{R} = p_{11}$, therefore conjugate momentum of $Z^i$ is $p_{11} \dot{Z}^i$. Thus passage to the Hamiltonian (in light-cone gauge) gives the membrane Hamiltonian (6.22). This is indeed a remarkable result that a simple supersymmetric quantum mechanical system encodes the dynamics of the supermembrane.

It is natural to ask whether one obtain a string starting from the M(atrix) model. First, one compactifies the theory to ten dimension. When the compactification radius is small, the theory contains the Fock space of the type IIA string. As the radius tends to zero the string becomes free [117] and correct leading order string interactions could be reproduced. In order to carry out compactification, we replace the matrices by infinite dimensional operators. The compact coordinate is represented as $X^a \to -i \frac{\partial}{\partial \sigma^a} I_{N \times N} - A_a(\sigma)$. Here A is a $U(N)$ gauge potential. The rest of the variables are taken to be matrix valued function of $\sigma$. If we use this ansatz, the resulting Hamiltonian is that of maximally supersymmetric 1 + 1 dimensional Yang-Mills theory. In the limit when radius goes to zero and $N$ is taken to be infinity, the moduli space of this model coincides with the Fock space of type IIA theory.

Indeed, the M(atrix) model has opened up new avenues to study dualities between compactified model on torus and Yang-Mills theory on dual space. Moreover, there are applications of the M(atrix) model to study black holes we refer the interested reader to the review on the subject [33]. Another interesting development has been to understand type IIB theory and its dualities from a matrix model formulation. In this approach one adopts procedure of Eguchi and Kawai to consider reduced 10-dimensional super Yang-Mills theory and it is a theory of $N \times N$ matrices which even carry no time dependence [118]. We refer the reader to the review article of Makeenko [119].
7 Anti-de Sitter Space and Boundary field Theory Correspondence

Recently, attentions have been focused in constructing supersymmetric gauge theories by considering various configurations of branes in string theories as well as in M-theory. When we have N coincident Dp-branes, a supersymmetric $U(N)$ gauge theory lives in worldvolume of the branes. The $\frac{1}{N}$ expansion proposed by ’t Hooft [121] revealed several aspects of $SU(N)$ Yang-Mills theory. According to ’t Hooft, one should consider large N limit of the theory keeping $g_{YM}^2 N$ fixed, $g_{YM}$ being the gauge coupling constant. Then a Feynman diagram is designated by the topological factor $N^{\chi}$, $\chi$ being the Euler characteristic of the Feynman diagram. When we consider, expansion in $\frac{1}{N}$, rather than in coupling constant, each order in $\frac{1}{N}$, contains diagrams to all orders in coupling constant and the leading order corresponds to the planar diagrams. Maldacena [120] has made remarkable conjecture regarding large N conformal gauge theories. The proposal states that large N limit of a conformally invariant theory in $d$ dimensions is determined by supergravity theory on $d+1$ dimensional Anti-de Sitter space times a compact space (for a sphere it is maximally supersymmetric). The AdS/CFT connection has led to the generalization of the holography principle in this context [123, 124] which was first introduced in black hole physics [125, 126] in order to understand the Bekenstein entropy bound and the area law for black hole entropy. Thus the conjectures of Maldacena led to reveal deeper connections between string theory and superconformal gauge theories.

We have emphasized earlier that gravity is an integral part of string theory since graviton is a part of the spectrum. Moreover, gauge fields also invariably appear in string theories. Let us recapitulate a few points in order to get a perspective of AdS/CFT connections. We have seen that the heterotic strings, through their constructions, contain nonabelian gauge groups and graviton in their massless spectrum. The type II theories have graviton, coming from NS sector, in their perturbative spectrum. However, with the discovery of Dp-branes, we know that supersymmetric gauge theories can arise if we consider coincident Dp-branes in type II theories. Type I string theory admit nonabelian gauge field since Chan-Paton factors can be attached to the end points as was discussed earlier. Furthermore, consistency of the theory requires that we have to incorporate closed string sector in order to account for nonplanar loop corrections; therefore there is gravity coming from the closed string spectrum. For this theory, when we take $\alpha' \rightarrow 0$ limit Yang-Mills theory appears automatically and since consistency requires inclusion of closed string states, gravity also will appear in the zero slope limit. In view of preceding remarks, one might conclude that, in string theory, gravity and gauge theory invariably appear simultaneously. Thus the important question to answer is that how the string theory can describe the strong interaction among quarks and gluons. The recent developments [120, 122, 123, 127] have provided connections between string theory and gauge theories.

The configuration under consideration is N coincident Dp-branes and open strings
can end on these hypersurfaces. When we look into the dynamics in the worldvolume we have collection of these open strings and their excitations. Moreover, the worldvolume fields have their interactions and also there exists interaction with the bulk. An interesting limit to consider is when dilaton remains at a fixed value and the slope parameter tends to zero value. Then, at low energies, the gravity decouples; but to keep the interactions in the worldvolume in tact, we should have gauge coupling finite, for the $U(N)$ gauge theory. In fact, if we ignore the center of mass part, then we need to consider the $SU(N)$ gauge theory. It is necessary to go near the horizon, $r \rightarrow 0$, to see the connection between AdS and CFT. In the near horizon limit, recall eq.(5.11) and eq.(5.12), that the factor 1 appearing in the definition of the harmonic function of the Dp-brane can be neglected. To be specific let us first consider the metric in the case of N coincident branes.

$$\text{d}s^2 = H_p^{-\frac{1}{2}}(r)\eta_{\mu\nu}dx^\mu dx^\nu + H_p^{\frac{1}{2}}dy_i dy_i$$

(7.1)

where, $\{y_i\}$ are the transverse coordinates and $r = \sqrt{y_i y^i}$. The indices $\mu, \nu,...$ are for tensors on the worldvolume. The dilaton and the $(p+1)$-form potential, coming from the RR sector, are given by

$$e^{-(\phi - \phi_0)} = H_p(r)^{\frac{(p-3)}{4}}, \text{ and } A = [H_p(r)]^{-1}$$

(7.2)

and

$$H_p(r) = 1 + \frac{C_p N}{r^{7-p}}, \quad C_p = \frac{(2\pi \sqrt{\alpha'})^{7-p}}{(7-p)\Omega_{8-p} g_{str}}$$

(7.3)

Here we have suppressed the indices of the $(p+1)$-form gauge potential and $\Omega_r = \frac{2\pi^{(n+1)}}{\Gamma\left[\frac{n+1}{2}\right]}$ and $\phi_0$ is the asymptotic constant value of the dilaton. When we have N coincident D-branes, the worldvolume action is the generalised Born-Infeld action proposed by Tseytline [128]

$$S_{BI} = -\tau_p^{(0)} \int d^{p+1} \xi e^{-\phi} \text{STr} \sqrt{-\text{det}[G_{\mu\nu} + 2\pi \alpha' F_{\mu\nu}]}$$

(7.4)

Here $G_{\mu\nu}$ is the pullback of the metric $G_{MN}$ to the world volume and $F_{\mu\nu}$ is the gauge field strength on the brane. The tension of the brane is

$$T_p = \frac{(2\pi \sqrt{\alpha'})^{(1-p)}}{2\pi \alpha' g_{str}} = \frac{\tau_p^{(0)}}{g_s}$$

(7.5)

and $g_{str}$ is the string coupling constant. The action (7.4) under the square root can be expended and keeping the second order term in gauge field strength one can write the action in more familiar form

$$S_{gauge} = -\frac{1}{4g_{YM}^2} \int d^{p+1} \xi \text{Tr} F_{\mu\nu} F^{\mu\nu}$$

(7.6)
where $\text{Tr}$ is taken over the gauge group matrices and the gauge coupling constant is identified as $g_{YM}^2 = 2g_{\text{str}}(2\pi)^{(p-2)}(\alpha')^{(p-3)}$. We know from the solutions discussed in previous section (recall eq.(5.12) and eq.(5.13)) that, in the limit, when $r \to \infty$, the metric is flat. Here one is looking for the behaviour of the solution in the $r \to 0$ limit and one chooses a brane for which the dilaton is constant at the horizon. If we consider D3-branes, then we find that not only the dilaton is independent of $r$, but also the Yang-Mills coupling constant is dimensionless. As mentioned above, one examines the configuration of $N$ coincident branes in the following limit

$$r \to 0, \quad \alpha' \to 0, \quad \text{and } U \equiv \frac{r}{\alpha'} = \text{fixed} \quad (7.7)$$

Therefore, we can neglect 1 appearing in the harmonic function. and the D3-brane metric goes over to

$$\frac{ds^2}{\alpha'} \to \frac{U^2}{\sqrt{4\pi N g_{\text{str}}}}(dx_{3+1})^2 + \sqrt{\frac{4\pi N g_{\text{str}}}{U^2}}dU^2 + \sqrt{4\pi N g_{\text{str}}}d\Omega_5^2 \quad (7.8)$$

The last term is the line element of five sphere and the metric describes the manifold $AdS \times S_5$. The radius of AdS is the same as that of $S_5$ and the radius is given by $R_{AdS} = (\alpha'\sqrt{4\pi N g_{\text{str}}})^{\frac{1}{2}}$. Since the Yang-Mills coupling constant satisfies the relation $g_{YM}^2 = 4\pi g_{\text{str}}$, the radius of the AdS gets related to the Yang-Mills coupling constant as

$$\frac{R_{AdS}^2}{\alpha'} = \sqrt{Ng_{YM}^2} \quad (7.9)$$

We know that the worldvolume theory of $N$ coincident Dp-branes is supersymmetric Yang-Mills theory in $p + 1$ dimensions and therefore, in this case the $N = 4$ SUSY gauge theory will appear. This is known to be a conformally invariant theory. From the supergravity side, we could describe the theory even for large radius; but that will amount to taking $Ng_{YM}^2$ to large values. Maldacena’s conjecture states that strongly coupled $N = 4$ super Yang-Mills theory is equivalent to 10-dimensional supergravity compactified on $AdS_5 \times S_5$. However, the consistency of the supergravity theory requires string theory at a deeper level. Thus supersymmetric four dimensional Yang-Mills theory is equivalent to type IIB theory compactified on $AdS_5 \times S_5$. The relations among the parameters are

$$g_{YM}^2 \equiv \frac{\lambda}{N} = 4\pi g_{\text{str}}, \quad \text{and } R_{AdS}^2 = \alpha'\sqrt{\lambda} \quad (7.10)$$

Let us very briefly recall some essential features of the Anti-de Sitter space. The Einstein-Hilbert action in the presence of cosmological constant term is

$$S_{EH} = \frac{1}{16\pi G_D} \int d^Dx \sqrt{|g|}[R + \Lambda] \quad (7.11)$$
We consider D-dimensional spacetime with Minkowski metric. The field equations are

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} g_{\mu\nu} \Lambda \]  

(7.12)

Taking the trace of this equation, we can determine curvature scalar \( R \) in terms of \( \Lambda \), and then derive the relation

\[ R_{\mu\nu} = \frac{\Lambda}{2 - D} g_{\mu\nu} \]  

(7.13)

In this case the Ricci tensor is proportional to the metric and these are Einstein spaces. This is also maximally symmetric space with the property that

\[ R_{\mu\nu\rho\lambda} = \frac{R}{D(D-1)} (g_{\nu\lambda} g_{\mu\rho} - g_{\nu\rho} g_{\mu\lambda}) \]  

(7.14)

The example of such space, with nonzero curvature, are de Sitter, Anti-de Sitter and D-spheres. In this sign convention, AdS space has positive cosmological constant. The AdS space is best described by an embedding. We start with \( D + 1 \) dimensional pseudo-Euclidean embedding space with coordinates \( \{y^a = y^0, y^1, \ldots y^{D-1}, y^D\} \) and metric \( \eta = \text{diag}(+, -,-,\ldots+) \) and the distance squared is

\[ y^2 \equiv (y^0)^2 + (y^D)^2 - \sum_{n=1}^{D-1} (y^n)^2 \]  

(7.15)

Note the appearance of two time coordinates from the form of the metric. The length remains invariant under \( SO(D-1,2) \) global transformations

\[ y^n \rightarrow y'^n = L^n_m y^m \]  

(7.16)

where \( L^n_m \) is an \( SO(D-1,2) \) matrix. If we consider the locus of

\[ y^2 = b^2 = \text{constant} \]  

(7.17)

and that defines \( AdS_D \). It is worth noting that the invariance group for theories defined on \( AdS_D \) is same as that of the D-dimensional flat space that is D generators corresponding to translations and \( \frac{1}{2} D(D-1) \), generators from Lorentz rotations.

Next, let us consider what is the conformal group in D-dimensional Euclidean space \( E^n \). In this case the Poincare group has altogether \( \frac{1}{2} D(D+1) \) generators (D translations and rest from Lorentz group). Then we have following extra generators:

\[ \vec{x} \rightarrow \lambda \vec{x} \]  

(7.18)

this is \textit{dilation} and \( \lambda \) is a real number. Furthermore, there is special conformal transformation

\[ \frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} + \alpha^\mu \]  

(7.19)
This transformation involves \( n \) parameters \( \alpha^\mu \). The transformation (7.19) can be rewritten as

\[
x'^\mu = \frac{x^\mu + \alpha^\mu x^2}{1 + 2x^\mu \alpha_\mu + \alpha^2 x^2}
\]  

(7.20)

Thus we see that the total number of generators are:

\[
\frac{1}{2} D(D + 1) + 1 + D = \frac{1}{2} (D + 1)(D + 2).
\]

This is the same number of generators that \( AdS_{D+1} \) space has. Indeed, in view of the recent developments, one can establish the connection that the isometry group of \( AdS_{D+1} \), \( SO(2, D) \) acts on the boundary as the conformal group acting on Minkowski/Euclidean space. We list below the generators of conformal group and their algebra

\[
[M_{\mu\nu}, P_\lambda] = i(g_{\nu\lambda}P_\mu - g_{\mu\lambda}P_\nu) \quad (7.21)
\]

\[
[M_{\mu\nu}, M_{\lambda\rho}] = i(g_{\nu\rho}M_{\mu\lambda} - g_{\mu\lambda}M_{\nu\rho} + g_{\nu\lambda}M_{\mu\rho} - g_{\mu\rho}M_{\nu\lambda}), \quad (7.22)
\]

\[
[M_{\mu\nu}, K_\lambda] = i(g_{\nu\lambda}K_\mu - g_{\mu\lambda}K_\nu) \quad (7.23)
\]

\[
[D, P_\mu] = iP_\mu \quad (7.24)
\]

\[
[D, K_\mu] = iK_\mu \quad (7.25)
\]

\[
[P_\mu, K_\nu] = 2i(g_{\mu\nu} + M_{\mu\nu}) \quad (7.26)
\]

The generators of conformal transformation have the following representations, when we choose Cartesian coordinate system and consider transformation properties of a real scalar field: \( P_\mu = -i\partial_\mu, \quad M_{\mu\nu} = x_\nu P_\mu - x_\mu P_\nu, \quad D = x^\mu P_\mu \) and \( K_\mu = x^2 P_\mu - 2x_\mu D \), corresponding to translation, Lorentz transformation, dilation and special conformal transformations respectively.

Let us discuss the evidences in support of Maldacena’s conjecture. When we consider collections of D3-branes of the type IIB theory we note that D3-branes couple to the 5-form field strength and \( N \) units of this flux will pass through the five sphere of the \( AdS_5 \times S_5 \) manifold. The isometry group of \( S_5 \) is \( SO(6) \) and the \( AdS_5 \) is endowed with isometry group \( SO(4, 2) \) as we have just mentioned. The IIB theory has fermions and therefore, it is more relevant to consider the covering groups \( SU(4) \) and \( SU(2, 2) \) of \( SO(6) \) and \( S(4, 2) \) respectively. We also know that type IIB theory has 32 Majorana supercharges. These supersymmetries are preserved by the background under consideration. The invariance group is the super Lie group \( SU(2, 2|4) \) for this theory. On the super Yang-Mills part, one has to examine how the above symmetry appears on the boundary theory. We have mentioned how the conformal group, for the case at hand, is to be identified as \( SO(4, 2) \) or \( SU(2, 2) \). It is well known that \( N = 4 \) super Yang-Mills theory is conformally invariant in four dimensions, since the theory has vanishing \( \beta \)-function \([30]\), and thus the origin of the conformal group is well understood. Let us now focus our attention on the other symmetries present in type IIB theory. The ten dimensional super Yang-Mills has gauge bosons, \( A_\mu^a, \quad \mu = 0, 1, \ldots, 9, \) a being \( U(N) \) group index and thus there are 8 physical states corresponding to each gauge field. The superpartners are Majorana Weyl gauginos having matching numbers. The theory has 16 Majorana supercharges in \( D = 10 \). When we consider the
4-dimensional action, dimensionally reduced from ten dimensions \[ R, 49 \] physical degrees of freedom of each of the ten dimensional gauge field decomposes into 2 (corresponding to physical degrees of freedom of gauge field in \( D = 4 \)) and six scalars, \( \phi^i, i = 1, 2...6 \), \( a \) is group index suppressed from now on. The number of, gauginos are given by the Weyl spinors, \( \lambda^A_\alpha \), \( A = 1, 2, 3, 4 \), \( \alpha = 1, 2 \). One of these fermions, together with the gauge field can be grouped define a vector superfield. The rest of the three spinors can be grouped with the scalars (which appeared after dimensional reduction) to define 3 chiral superfields. The 16 supercharges can be grouped into 4 sets of complex Majorana charges \( Q^A_\alpha, \bar{Q}^A_\alpha \), \( A = 1, 2, 3, 4 \) and \( \alpha = 1, 2 \) These two supercharges transform as \( \{4\} \) and \( \{\bar{4}\} \) of the R-symmetry group \( SU(4) \). The scalars \( \phi_i \) transform as \( \{6\} \) of the \( SO(6) \), since we deal with the covering group \( SU(4) \), the scalars transform in the antisymmetric, rank 2 representation of the \( SU(4) \). We see that type IIB theory has 32 supercharges, but the super Yang-Mills has only 16 of them. We know from discussions in Sec.IV that in the presence of the coincident D3-branes, half of the supersymmetries are preserved. When we consider the super conformal algebra the rest also appear as the extension of the superconformal group \[ 131 \].

Another important nonperturbative symmetry of type IIB theory is the \( SL(2, Z) \) symmetry where dilaton and axion define the moduli. In the Yang-Mills sector the S-duality symmetry is robust and is known to be, again, \( SL(2, Z) \). In this case, the modular parameter \( \tau = \frac{g_2}{2\pi} + \frac{4ie^{\phi}}{g_4} \) whereas in the former case it is \( \tau = \chi + ie^{-\phi} \).

The preceding discussions were focused to show that the symmetry properties of the type IIB theory and those of \( N = 4 \) super Yang-Mills are the same. It is important to investigate which physical properties are common to both the theories. Indeed, if the two theories are equivalent, it should be possible to identify a physical field \( \Psi \) in the bulk theory and find the corresponding object on the boundary theory. Then, one of the tests will be to compute the correlators involving relevant objects in each of the theories and check the consistencies. Thus it is important to identify the physical quantities (operators) in both the theories. In the case of the boundary theory, one obvious criterion will be to choose gauge invariant operators while computing the correlators. One could formally express the equivalence between the theories through the relation among the generating functionals.

\[
e^{-S_{II}[\Phi(J)]} = \int D Ae^{-\left(S_{YM}[A]+O_{\Delta}[A]J\right)} \tag{7.27}
\]

The l.h.s. of the above equation is to be identified as the generating function for the supergravity theory (rather low energy limit of IIB theory). The action \( S_{II} \) is determined in terms of the massless states of the supergravity and the Kaluza-Klein towers and these are collectively denoted as \( \Phi(z, \omega) \). Here the coordinates \( z^N \equiv (x^\mu, r) \) and \( \mu \) taking values 0,1,2,3 are to be identified as the AdS coordinates and \( \omega \) is the coordinate on five sphere. Moreover, it is implied due to the presence of \( J(x) \) that it also depends on the boundary data of the bulk fields. The r.h.s. defines the generating function for \( N = 4 \) super Yang-Mills theory; however, one only
computes the correlation functions of gauge invariant composite operators denoted by $O(A)$ with couplings to $J(x)$. In this general setting [122, 123, 132], one will be able to compute the correlation functions from both the theories and establish the correspondence between the two theories. Let us consider a simple example as illustration for the case of minimally coupled scalar in the bulk theory which could be identified with the dilaton. The action on the bulk for the dilaton on $AdS_5 \times S_5$ is

$$S = \frac{\pi^3 b^3}{4G_{10}} \int d^5 x \sqrt{|g|} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

(7.28)

The factor $\pi^3 b^3$ comes from the volume of $S_5$, through implicit assumption that $\phi$ has no dependence on coordinates of five sphere. The metric is $g_{\mu\nu} = \frac{b^2}{x_0^2} \delta_{\mu\nu}$, is metric on $AdS_5$, now in the Poincare coordinates. For large $\lambda >> 1$, the classical supergravity can be taken to be a good approximation (7.10). The dilaton equation of motion is given by

$$\partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \phi) = 0$$

(7.29)

Of course, this equation can be solved by the standard Green’s function method. The purpose is to determine the generating function with value of dilaton computed on the boundary, call it $\phi_0$ which is value of $\phi$ as $x_0 \rightarrow 0$. Thus we can write

$$\phi(x_0, \vec{x}) = \int d^4 \vec{z} K(x_0, \vec{x}, \vec{z}) \phi_0(\vec{z})$$

(7.30)

the vectors refer to four dimensional vectors on the boundary space and the Green’s function is defined as,

$$K(x_0, \vec{x}, \vec{z}) \sim \frac{x_0^4}{[x_0^2 + (\vec{x} - \vec{z})^2]^4}$$

(7.31)

Now, one can insert the solution for $\phi$ into the action to determine it at the classical value of dilaton

$$S = \frac{\pi^3 b^8}{4G_{10}} \int \frac{d^4 \vec{x}}{x_0^3} \phi \partial_0 \phi|_{\epsilon}$$

(7.32)

$\epsilon$ is the cut off for the lower limit of integration. Once expression for $\phi$ is inserted into the action, then it is possible to take cut off to zero and everything is finite. The action is given by

$$S \sim -\frac{\pi^3 b^8}{4G_{10}} \int d^4 \vec{x} \int d^4 z \frac{\phi_0(\vec{x})\phi_0(\vec{z})}{(\vec{x} - \vec{z})^8}$$

(7.33)

Then the generating function can be obtained by exponentiating this action. On the super Yang-Mills side, since it is a conformal field theory in four dimensions, the quadratic of Yang-Mills field strength has dimension 4 and product of two of the $F^2$ terms behave as

$$\langle F^2(\vec{x})F^2(\vec{z}) \rangle \sim \frac{N^2}{(\vec{x} - \vec{z})^8}$$

(7.34)

If we want to determine the dilaton correlation function on boundary, we compute
\[
\frac{\delta^2 Z_{II}(\phi_0)}{\delta \phi_0(\vec{x}) \delta \phi_0(\vec{z})} \sim \frac{N^2}{(\vec{x} - \vec{z})^8}
\]

(7.35)

Now comparing (7.34) and (7.35) we find that they are in agreement. If one considers, metric perturbation of the form \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) and then computes the two point correlation of this perturbation on the brane taking the boundary limit; this correlation is identical to the correlation of stress energy momentum tensors (product of a pair of them; just as we took correlation of two \( F^2 \) terms while identifying the dilaton two point functions).

Let us recall that the 't Hooft coupling \( \lambda \equiv N g_{YM}^2 \) and the length parameter \( b^4 = l_s^4 \lambda = 4\pi l_s^4 N g_{str} \) are related. If we hold \( \lambda \) fixed and let \( N \to \infty \), then the string coupling tends to zero. Therefore, string perturbation theory can give reliable result in this limit. Thus, one can get a full quantum theoretic description of the Yang-Mills theory in the \( N \to \infty \) limit. Instead of holding \( \lambda \) fixed, if we allow it to take large values, then in the domain, where AdS radius is kept constant, the relevant limit is \( \alpha' \to 0 \). We know that in the zero slope limit the string theory goes over to supergravity theory. We saw the matching of AdS/CFT in this limit. But the consequences of Maldacena conjecture is very interesting in this regime, it tells us how the superconformal gauge theory in the \( N \to \infty \) limit behaves in strong coupling domain. Of course, the example we have been considering is the one where the \( \beta \)-function of the theory vanishes identically and therefore, it is not a realistic theory if we want to establish connection with supersymmetric gauge theories which have running coupling constants leading to asymptotic freedom. There are attempts to construct field theories which will have broken SUSY and conformal invariance (for example classical SQCD is scale invariant, but in the quantized theory scale invariance is broken). Witten [133] has proposed that one should consider \( AdS_7 \times S_4 \). The resulting boundary theory corresponds to 6-dimensional theory whose action is yet to be explicitly constructed. Then one compactifies the theory on \( T^2 \) and require that fermions satisfy anti-periodic boundary condition around a cycle of the two-torus. Then the boundary theory is a four dimensional one. Conformal invariance and supersymmetry are broken in this 4-dimensional theory and we have a pure gauge theory with large \( N \).

There has been rapid developments in studying the interconnection between supergravity (rather type IIB) theory on AdS space and boundary gauge theory. Several important issues pertaining to string theory and gauge theories have been addressed in this context. We refer to some of the recent review articles in this subject [134, 135, 136].

8 Cosmology and String Theory

The remarkable attribute of string theory is that it provides a unified description of laws of Nature. Although, a connection with the phenomenological aspects of elementary particle physics is not firmly established so far, there are several indications
that we are pursuing the right path. We have discussed, in Action V, how string theory has provided an adequate description of the physics of the black holes from a microscopic point of view as is expected from a theory describing gravity.

It is natural to address questions intimately related to evolution of the Universe and its creation in the framework of string theory. Einstein's theory provides a very good description of classical gravity and has been tested with precision measurements. The principles of equivalence, cosmological principle and the big bang hypothesis are fundamental ingredients of the standard cosmological model [137]. The experimental data have verified the predictions of the standard cosmological model to a great degree of accuracy. However, in order to understand some of the salient features of our Universe such as its flatness, isotropy and homogeneity, the horizon problem and the large scale structure; the paradigm of inflation has been accepted as an integral part of the theory of the cosmos. In simple words, our Universe underwent rapid superluminary expansion after the big bang so that we can understand some of the cosmological observations alluded to above. Indeed, considerable efforts have been made, in recent times, to understand mechanisms of inflation and to explore consequences of various inflationary models. It is necessary to introduce a scalar field, in generic inflationary model, to explain the mechanism; however, the inflaton field is introduced in an ad hoc manner.

We expect that string theory should provide answers to the questions related to the evolution of our Universe. In the cosmological scenario, in this approach, one considers Einstein and matter field equations obtained from the string effective action, (2.37), when the metric and background fields (corresponding to dimensionally reduced 4-dimensional action) depend only on the time coordinate; usually identified with the cosmic time. In string theory, the scalar dilaton, appears naturally in the massless spectrum of theory and it is tempting to identify this field as the one responsible for inflation in early epochs [138]. It is well known that a dilatonic potential cannot be generated in a superstring theory perturbatively. Furthermore, the VEV of dilaton determines the Newton’s gravitational constant, gauge coupling constant, Yukawa couplings of the fermions amongst other parameters of string theory. In a cosmological context, the dilaton acquires time dependence and it will roll with evolution of the Universe. However, the dilaton must decouple at some appropriate time in the history of the Universe [139], otherwise, the nice (tested) predictions of late time cosmology will be seriously affected due to the fact that a time dependent dilaton controls masses and coupling constants. Notice that, eventually, the dilaton potential becomes important and the dilaton settle down at the bottom of the well. There are important consequences for a massless dilaton: it violates equivalence principle [140]. Moreover, the dilaton mass is bounded as $M_\phi > 10^{-4}$ eV, in order to fulfill the observational constraints [141].

There is a novel approach to describe inflation phenomena in string theory [142, 143], known as the pre-big bang (PBB) proposal. Since, considerable attention has been focused to investigate the consequences of the PBB scenario, we shall discuss impor-
tant features of this proposal and refer the interested reader to some of the recent reviews in the subject [144, 143, 146, 147, 159]. The target space duality, often called T-duality, is a key ingredient leading to a new mechanism for inflation in string cosmology. In this approach, one does not need potential for the dilaton for accelerated expansion of the Universe unlike the case of standard mechanism for inflation. It is the coupled dilaton and metric evolution equations which are responsible for inflation. It follows from the properties of these equations, as discussed below, that there are two branches of solution, denoted as ± and in each branch there are two sets of solutions. One of the solutions in the (+)-branch is such that the corresponding Hubble parameter and the second derivative of the scale factor are positive. Therefore, the Universe has accelerated expansion for this case. This branch gets related, by combined operation of duality and time reversal to a solution in the (−)-branch which has the features of the FRW metric in the sense that it is expanding but decelerating. If there is a mechanism for a smooth transition from the aforementioned (+)-branch to the (−)-branch, then following interesting scenario emerges. In the pre-big bang scenario the time begins somewhere in the infinite past, in contrast with the big bang model in which we identify beginning of time with the big bang singularity. Thus the Universe evolves from a low curvature regime proceeding towards strong coupling, high curvature domain with accelerated expansion. Then the Universe emerges into the FRW like post big bang phase in which standard cosmological model applies. Of course, it is essential to understand the mechanism for transition from one phase to another known as the problem of graceful exit. Let us recall, very briefly, how T-duality and time reversal transformations relate different solutions of the cosmological evolution equations of string effective action.

The string effective action, in the cosmological scenario, is given by

$$S = -\frac{1}{2\lambda_s} \int dt \sqrt{|g|} e^{-\phi}(R + \dot{\phi}^2) \tag{8.1}$$

where $\lambda_s$ is the inverse of the string scale. The $d+1$ dimensional metric has the form $ds^2 = dt^2 - g_{ij} dx^i dx^j$, $g_{ij}$ is the spatial part of the string frame metric since we can always choose $g_{00} = 1$ and $g_{0i} = 0$ in the cosmological context. We have omitted the field strength of the antisymmetric tensor field. In the presence of graviton, antisymmetric tensor and dilaton the string effective action can be brought to manifestly $O(d,d)$ invariant form, $d$ being the number of spatial dimensions [143]. As we have mentioned above the duality symmetry of string theory plays a very important role in relating the two epochs i.e. $t > 0$ and $t < 0$, in the evolution of the Universe. Let us consider a homogeneous, isotropic Universe so that we have only one scale factor $a(t)$ and the spatial metric is diagonal: $g_{ij} = diag (a(t)^2, a(t)^2, a(t)^2, ...)$.

The resulting field equations take the following form:

$$2\ddot{\phi} + 2d\dot{\phi}H - \dot{\phi}^2 - 2dH - dH^2 - (dH)^2 = 0 \tag{8.2}$$

$$\dot{\phi}^2 - 2d\dot{\phi}H - dH^2 + (dH)^2 = 0 \tag{8.3}$$
\[
\dot{H}^2 + dH^2 - H\dot{\phi} - d\dot{H} + \frac{1}{2}dH^2 - \frac{1}{2}(dH)^2 - \frac{1}{2}\dot{\phi}^2 + \ddot{\phi} + d\dot{\phi}H = 0
\] (8.4)

The first of the above three equation comes from variation of the dilaton \(\phi\), the second follows from the \(0-0\) component of the Einstein’s equation which is the Hamiltonian constraint on this occasion. Here, \(H = \frac{\dot{a}}{a}\) is the Hubble parameter. The last equation, in an expanded form, is the \((i,i)\) component of the Einstein’s equation and the off-diagonal space-space components are found to be trivially satisfied. If we stare at the dilaton equation and the last equation, the \((i,i)\) part, at the first sight, we note that both have second derivative of time for dilaton and it might be a formidable task to solve the graviton dilaton equation. However, the last five terms of the third equation coincide with dilaton equation of motion and we are left with a simple equation

\[
\dot{H} - H\dot{\phi} - dH^2 = 0
\] (8.5)

Notice that (8.3) is quadratic in \(\dot{\phi}\) and therefore, the solution has two roots. Moreover, (8.5) is an evolution equation for the Hubble parameter with a term involving \(\dot{\phi}\). Thus \(\dot{H}\) will have two equations corresponding to each root of \(\dot{\phi}\). Therefore, there are altogether four branches when we inspect the solutions of coupled graviton-dilaton equations. Let us introduce the shifted dilaton

\[
\bar{\phi}(t) = \phi(t) - d\ln a(t)
\] (8.6)

It is obvious from the evolution equations that if \(\{a(t), \phi(t)\}\) satisfy the equations of motion, then the new set \(\{a(t)^{-1}, \phi - 2d\ln a(t)\}\) are also solutions to the equations of motion; indeed, this is a part of the \(O(d,d)\) group and the shifted dilaton, \(\bar{\phi}\), is invariant under this T-duality. Another property of the time evolution equations for dilaton and that for the scale factor is that they are invariant under time inversion \(t \to -t\) with \(H \to -H\) and \(\dot{\phi} \to -\dot{\phi}\). Let us consider a simple illustrative example to demonstrate how one can generate solutions in the four branches. We begin with a specific isotropic solution

\[
a(t) = t^{1/\sqrt{d}} \quad \bar{\phi}(t) = -\ln t
\] (8.7)

for \(t > 0\). We can generate new set of solutions by implementing duality and time reversal transformations. In fact we can get the four solutions, mentioned above, starting from the (seed) solution of the previous equation. The solutions are

\[
a_{\pm}(t) = t^{\pm 1/\sqrt{d}}, \quad \bar{\phi}(t) = -\ln t
\] (8.8)

\[
a_{\pm}(-t) = (-t)^{\pm 1/\sqrt{d}}, \quad \bar{\phi}(-t) = -\ln(-t)
\] (8.9)

Let us examine the characteristics of the four solutions in some detail. \(a_{+}(t)\) corresponds to decelerated expansion, whereas \(a_{-}(t)\) is identified for decelerated contraction. These two solutions lie in the positive \(t\) branch. In the negative time branch,
$a_+(t)$ and $a_-(t)$ are identified to be accelerating, contraction, and accelerating expansion respectively. If $\dot{a} > 0$ ($\dot{a} < 0$) the solution describes expansion (contraction). Similarly, a solution is called an accelerated one (decelerated) if $\dot{a}$ and $\ddot{a}$ have same sign (opposite sign). Since the dilaton is the coupling constant in the theory, it is important to extract time dependence of the dilaton in the four branches.

$$\phi_{\pm}(\pm t) = (\pm \sqrt{d - 1}) \ln(\pm t)$$

which can be obtained from definition of $\tilde{\phi}$. Note that $a_+(t)$ has the characteristics of FRW solution in the sense that it corresponds to expanding Universe with deceleration and the singularity lies in its past. On the other hand, $a_-(t)$ is the one which is expanding with an acceleration and the singularity is in its the future. Furthermore, these two solutions are related to each other under simultaneous duality and time reversal transformations as we mentioned earlier. If one introduces a dilatonic potential, the full set of solutions continues to exhibit same characteristics: there are two branches and each branch having two solutions. In view of remarks at the beginning of the section, a dilatonic potential is very much desirable. According to the PBB proposal, the Universe initially is flat, cold and is in the weak coupling regime and therefore, the tree level string effective action is a good starting point. Moreover, we can trust the perturbative vacuum. Subsequently, the Universe evolves towards curved, hot and strong coupling domain under going accelerated expansion. If the Universe could smoothly pass over to the FRW like regime, then we would have not only resolved the mechanism of inflation, but also the initial singularity problem could be circumvented. It is well known that a smooth transition from the $(+)$-branch to the $(-)$-branch is not possible if we consider the tree level string effective action due to the no-go theorems [149, 150]. Therefore, graceful exit is an important issue when we envisage pre-big bang scenario. One of the possibilities is to appeal to quantum string cosmology in order to resolve the graceful exit problem [151] and the other approach is to consider the string effective action with higher derivative terms due to string look and/or $\alpha'$ corrections [152].

The examples, illustrating mechanisms for inflation in the pre-big bang scenario, correspond to spatially flat, homogeneous solutions of the equations of motion. It is not desirable to start with a homogeneous solution from the onset, if we want to solve the homogeneity and flatness problem in cosmology. Therefore, a more appropriate approach will be to consider generic initial condition in the vicinity of the perturbative vacuum. One could consider a scenario where, long before the big bang, the Universe was inhomogeneous so that the fields (dilaton in this case) were spacetime dependent and moreover, the time derivative and spatial gradients were comparable. Furthermore, if we want to follow the PBB approach, these derivatives were small, to begin with, so that initially we are in the perturbative regime. If one looks at the evolution, it is noted that certain patches develop where time derivatives dominate over spatial gradients. If the kinetic energy of the dilaton contributes a good part to the critical density, the dilaton driven inflation sets in so that the patch gets blown up
and it becomes homogeneous, isotropic and spatially flat. Another important aspect is to investigate situations when the spatial curvature is nonzero and examine how various cosmological criteria are affected [153]. In view of above discussions, there have been considerable activities to study the evolution of the Universe taking into account effects of spatial curvature, inhomogeneity and examine various aspects of pre-big bang string cosmology.

Recently, the principle of holography has attracted considerable attentions in cosmological context and especially in string cosmology. The Bekenstein-Hawking entropy formula states that the entropy associated with a black hole is proportional to the area of the horizon. According to holography principle, when we are dealing with a system with gravity, the degrees of freedom of that system is bounded by the surface area of the volume, $V$, in which the system resides. Recently, Fischler and Susskind [154] examined the issue of holography in the cosmological framework. If we consider the Universe as a whole and explore the consequences of holography as applied to the black holes, we encounter difficulty in the following way. Let us consider the FRW metric as an example. The entropy per unit comoving volume is constant as can be inferred from the covariant conservation of stress energy momentum tensor. Thus, if we take a large enough value for the scale factor, the holographic bound $\frac{S}{A} \leq 1$ will be violated. It was proposed in [154] that one should consider the following situation while applying holography in the cosmological scenario. Let us consider a four dimensional spacetime manifold, $M$. Suppose $B$ is the two dimensional boundary of a spatial region $R$. We define the light surface $L$ to be the one bounded by $B$ and generated by the past light rays from $B$ towards the center of $R$. The cosmological holography principle, expounded by Fischler and Susskind, states that the entropy passing through $L$ never exceeds the area of the bounding surface $B$. In the special case of adiabatic evolution of the Universe, the total entropy of the matter in the horizon should be smaller than area of the horizon. The issue of holography, in cosmology, has been addressed in a general framework recently [155]. Moreover, the consequences of the principle has been studied in the cosmological context [156].

Let us consider a PBB scenario in the Einstein frame with graviton and dilaton [157]. The entropy per comoving volume remains constant when the Universe undergoes adiabatic expansion (or contraction) and the location of the horizon is determined from the condition $ds^2 = 0$. It follows from the covariant conservation of the stress energy tensor that

$$\sqrt{g} \frac{dp}{dt} = \frac{d}{dt} (\sqrt{g} (\rho + p))$$

(8.11)

where $\rho$ and $p$ are defined from the diagonal of $T^\mu_\nu$ in the cosmological context, $g$ being the determinant of the spatial part of the metric. It is easy to see that the comoving entropy density remains unchanged with time through out PBB and is given by

$$S^c = \frac{(\rho + p) \sqrt{g}}{T}$$

(8.12)
where $T$ is identified as the temperature of the fluid. For the dilatonic matter $\rho = p$ and then

$$S^c = \text{Const. } (\rho g)^{1/2} \tag{8.13}$$

where the constant is related to the Stefan’s constant of the dilaton. Now, consider a simple homogeneous PBB cosmological case where

$$ds^2 = -dt^2 + \sum \left(\frac{t}{t_0} - 1\right)^{2\lambda_a^2} (dx^a)^2 \tag{8.14}$$

and

$$\phi(x, t) = \phi_0 - \sqrt{2} \sqrt{(1 - \sum \lambda_a^2) \ln \left(\frac{t}{t_0} - 1\right)} \tag{8.15}$$

Here $\{\lambda_a\}$ are independent of $x$ and they satisfy Kasner condition

$$\sum \lambda_a = 1, \quad \sum \lambda_a^2 = \rho^2, \quad \frac{1}{3} \leq \rho^2 \leq 1 \tag{8.16}$$

Now the volume is given by

$$V_H^c = \Pi_a (X_H^a) \tag{8.17}$$

and $X_H^a$ is given by

$$X_H^a = t_0 \left(\frac{t}{t_0} - 1\right)^{1-\lambda_a} \left(1 - \lambda_a\right) \tag{8.18}$$

The area of the surface bounding the volume enclosed by the horizon is given by

$$A_H = \{\Pi_a (X_H^a) \left(\frac{t}{t_0} - 1\right)^{\lambda_a}\}^{\frac{2}{\lambda_a}} \tag{8.19}$$

Having obtained the expressions for the volume factor and the area, we compute the holographic ratio to be

$$\frac{S}{A} = \frac{\sigma^{\frac{1}{2}}}{l_p^2} \frac{1}{2\sqrt{\pi}} \frac{(1 - \rho^2)^{\frac{1}{2}}}{\Pi(1 - \lambda_a)^{\frac{1}{4}}} \tag{8.20}$$

We have now introduced the Stefan’s constant, $\sigma$, for dilaton explicitly and the Planck length makes its appearance in the above equation so that $\sigma$ is dimensionless. Since the exponents $\{\lambda_a\}$ appearing in the definition of the line element satisfy the two constraints, the holographic ratio $\frac{S}{A}$ is function of only one exponent. Therefore, by eliminating two of the $\lambda$’s we can write the ratio as an expression involving $\sigma, \rho$ and one of the $\lambda_a$’s, call it $Y$.

$$\frac{S}{A} = \frac{\sigma^{\frac{1}{2}}}{l_p^2} \frac{1}{2\sqrt{\pi}} \frac{(1 - \rho^2)^{\frac{1}{2}}}{\left((1 - Y)Y^2 + \frac{(1 - \rho^2)}{2}\right)^{\frac{1}{4}}} \tag{8.21}$$
The minimization and maximization of the denominator as a function of $Y$ will give us an upper bound and lower bound on the holographic ratio, respectively as given below,

$$\frac{S}{A} \leq \sqrt{\frac{\sigma}{\pi}} \frac{1}{2l_p^2} \frac{\sqrt{1 - \rho^2}}{\left[(\frac{11}{3} - 3\rho^2 - \frac{(3\rho^2 - 1)^{3/2}}{3\sqrt{2}})/9\right]^{1/3}}$$ (8.22)

and

$$\frac{S}{A} \geq \sqrt{\frac{\sigma}{\pi}} \frac{1}{2l_p^2} \frac{\sqrt{1 - \rho^2}}{\left[(\frac{11}{3} - 3\rho^2 + \frac{(3\rho^2 - 1)^{3/2}}{3\sqrt{2}})/9\right]^{1/3}}$$ (8.23)

Note the appearance of $\sigma$, the Stefan constant for the dilaton, the holographic ratio which could be determined in principle from string theory. It is quite interesting that the ratio could be bounded from above as well as from below in this case.

The field of string cosmology and especially PBB string cosmology is still developing and there are important issues to be resolved. We would like to make a few comments before closing this section. One would like to study phenomenological aspects of PBB cosmology and compare and contrast the results of PBB scenario with standard inflationary models. A lot of work has been done in this direction and we refer the reader to the review article for more detailed references. It might be worth while to point out a few features of PBB cosmology. In the standard inflationary scenario, the Universe goes through de Sitter type phase where the curvature remains constant; however, in the PBB inflation the curvature changes with time. Thus the quantum fluctuation of background fields are amplified in different modes with different spectra. Therefore, some of the distinct features of PBB cosmology could be experimentally observed in gravitational wave detectors in future. Similarly, the axion spectrum has been computed in PBB \[158\] and if detected, it will be another test of this scenario. There have been attempts to provide an understanding of galactic and intergalactic magnetic fields from the point of string cosmology. Since dilaton couples to the gauge field the mechanism for amplification of the magnetic field is related to the growth of the dilaton. It is quite possible that some of the predictions of PBB cosmology will be tested by on going experiments and/or by the experiments planned in the near future.

9 Summary and Conclusion

We have made some efforts to convey to the reader some of the interesting and important developments in string theory through this article. It is not possible to include all developments in the field in diverse directions in an article of this nature. A global perspective of string theory is contained in the article of John Schwarz \[160\] in this volume. We may recall that the research in string theory has stimulated progress in
other fields such as mathematics, quantum field theory and statistical mechanics of lower dimensional systems to mention a few areas. We have seen that string theory has made very important contributions to our understanding of the physics of the black holes. As we have mentioned, for a special class of black holes, the Bekenstein-Hawking entropy formula could be derived from an underlying microscopic theory. Similarly, the nature of the Hawking radiation from a stringy black hole, slightly away from extremality, could derived from the theory.

We have noted that, there are intimate connections between the five string theories. Some of them are inter related through dualities in ten dimensions and some are related in lower dimensions. Thus it is recognized that dualities have a special role in our understanding of string dynamics. Moreover, there are increasing evidence that there is a unique, fundamental theory and the five perturbatively consistent string theories are various phases of the fundamental theory. It is argued that M-theory might be that theory and the low energy effective action of M-theory is to be identified with the eleven dimensional supergravity theory. In this context, we discussed the M(atrix) model proposal to show that the model captures many important features of M-theory.

Recently, the conjecture due to Maldacena has attracted considerable attention since it provides an important connection between supergravity on the bulk and the supersymmetric gauge theories living on the boundary. The connection between type IIB theory on $\text{AdS}_5 \times S_5$ and $N = 4$ supersymmetric gauge theory on the boundary has been at the center of attention. Furthermore, there are interesting developments in the study of theories on $\text{AdS}_3$ and corresponding two dimensional conformal field theories.

One of the most important achievements of string theories has been to address important issues in quantum gravity and provide answers to some of the puzzles. However, the theory is yet to provide a satisfactory answer to the cosmological constant problem. The cosmological constant is a parameter in physics which is measured to be closest to zero. It plays a dual role. When we look at it from the point of view of macroscopic physics, the smallness of the constant conveys to us that the Universe is very large and it is flat. On the other hand, it is expected that, the cosmological constant, like other parameters in Nature, should be explained from a microscopic theory and the short distance physics, i.e. quantum gravity, will explain the smallness of the cosmological constant. Therefore, one expects that string theory will be able to resolve this outstanding problem [161, 162]. The author along with his collaborators had made an attempt in this direction [163]. It is expected that string theory will provide us clues to understand the creation of the Universe and the evolution of the Universe in early epochs. Indeed, string cosmology has attracted considerable attention is recent years; however, we have not included discussions on this topic in this article due to limitations of space. Indeed, string cosmology makes several predictions which might be subjected to experimental tests in next few years [164].
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