NESTED COVARIANCE DETERMINANTS AND RESTRICTED TREK SEPARATION IN GAUSSIAN GRAPHICAL MODELS

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Abstract. Directed graphical models specify noisy functional relationships among a collection of random variables. In the Gaussian case, each such model corresponds to a semi-algebraic set of positive definite covariance matrices. The set is given via a parametrization, and much work has gone into obtaining an implicit description in terms of polynomial (in-)equalities. Implicit descriptions shed light on problems such as parameter identification, model equivalence, and constraint-based statistical inference. For models given by directed acyclic graphs, which represent settings where all relevant variables are observed, there is a complete theory: All conditional independence relations can be found via graphical d-separation and are sufficient for an implicit description. The situation is far more complicated, however, when some of the variables are hidden (or in other words, unobserved or latent). We consider models associated to mixed graphs that capture the effects of hidden variables through correlated error terms. The notion of trek separation explains when the covariance matrix in such a model has submatrices of low rank and generalizes d-separation. However, in many cases, such as the infamous Verma graph, the polynomials defining the graphical model are not determinantal, and hence cannot be explained by d-separation or trek-separation. In this paper, we show that these constraints often correspond to the vanishing of nested determinants and can be graphically explained by a notion of restricted trek separation.

1. Introduction

Let \( G = (V,E) \) be a directed graph with finite vertex set \( V \) and edge set \( E \subseteq V \times V \). The edge set is always assumed to be free of self-loops, so \( (i,i) \notin E \) for all \( i \in V \). For each vertex \( i \), define a set of parents \( \text{pa}(i) = \{ j \in V : (j,i) \in E \} \). The graph \( G \) induces a statistical model for the joint distribution of a collection of random variables \( X_i, i \in V \), indexed by the graph’s vertices. The model hypothesizes that each variable is a function of the parent variables and an independent noise term. In this paper we consider the Gaussian case, in which the functional relationships are linear so that

\[
X_i = \lambda_{0i} + \sum_{j \in \text{pa}(i)} \lambda_{ji} X_j + \epsilon_i, \quad i \in V,
\]

where the \( \epsilon_i, i \in V \), are independent and centered Gaussian random variables. The coefficients \( \lambda_{0i} \) and \( \lambda_{ji} \) are unknown real parameters that are assumed to be such that the system (1.1) admits a unique solution \( X = (X_i : i \in V) \). Typically termed a system of structural equations, (1.1) specifies cause-effect relations whose straightforward interpretability is behind the wide-spread use of the models [SGS00, Pea09].

The random vector \( X \) that solves (1.1) follows a Gaussian distribution whose mean vector may be arbitrary through the choice of the parameters \( \lambda_{0i} \) but whose covariance

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matrix is highly structured. The model obtained from (1.1) thus naturally corresponds to the set of covariance matrices, which we denote by $\mathcal{M}(G)$. This set is given parametrically with each covariance being a rational or even polynomial function of the parameters $\lambda_{ji}$ and the variances of the errors $\epsilon_i$, as we detail in Section 2.

While a parametrization is useful to specify a distribution and to optimize the likelihood function, many statistical problems can only be solved with some understanding of an implicit description. In our setting, an implicit description of the model amounts to a semi-algebraic description of the set of covariance matrices that belong to the model through polynomial equations and inequalities, and a combinatorial criterion on the graph which specifies how to obtain them. Specific problems that can be addressed through such an implicit description include model equivalence, parameter identification, and constraint-based statistical inference. We refer the reader to the recent work of [vOM17] and the reviews of [Drt18] and [DM17].

If the underlying graph $G$ is an acyclic digraph, also termed a directed acyclic graph (DAG), then probabilistic conditional independence yields an implicit description of $\mathcal{M}(G)$ [Lau96, Stu05]. For a Gaussian joint distribution, conditional independence corresponds to the vanishing of special subdeterminants of the covariance matrix, namely, subdeterminants that are almost principal in the sense that the row and the column index sets agree in all but one element [LM07, DSS09, Chap. 3.1]. The conditional independences holding in all distributions in the given model can be found graphically using the concept of $d$-separation. It follows in particular that two DAGs $G$ and $H$ give rise to the same model $\mathcal{M}(G) = \mathcal{M}(H)$ if and only if $G$ and $H$ have the same $d$-separation relations. This combinatorial criterion can be simplified to yield an efficient algorithm: $\mathcal{M}(G) = \mathcal{M}(H)$ if and only if $G$ and $H$ have the same skeleta and the same sets of unshielded colliders [Pry90, VP91].

While DAG models are well-understood, they only pertain to problems where all relevant variables are observed. A long-standing program in the fields of graphical modeling and causal inference seeks to develop combinatorial solutions to problems such as model equivalence in settings with hidden/latent variables. Mathematically, if only the variables indexed by a set $A \subset V$ are observed while those indexed by $V \setminus A$ are hidden, then the covariance matrices in the set $\mathcal{M}(G)$ are to be projected on their principal $A \times A$ sub-matrix. It is well known that conditional independence is no longer sufficient for implicit model description after such a projection.

**Example 1.1.** Let $G$ be the DAG in Figure 1.1 where vertex 5 indexes a hidden variable. Then no conditional independence involving only the observed $X_1$, $X_2$, $X_3$, and $X_4$ holds for all covariance matrices in $\mathcal{M}(G)$. Instead, a positive definite $4 \times 4$ matrix $\Sigma = (\sigma_{ij})$ is the projection of a matrix in $\mathcal{M}(G)$ if and only if

$$|\Sigma_{12,34}| := \det(\Sigma_{12,34}) = \sigma_{13}\sigma_{24} - \sigma_{14}\sigma_{23} = 0$$

and $\sigma_{j3} = 0$ implies $\sigma_{j4} = 0$ for $j = 1, 2$. 

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**Figure 1.1.** A DAG on five vertices. Vertex 5 indexes a hidden variable.
In the example just given the key constraint is a determinant of the covariance matrix that cannot be explained by $d$-separation. A major advance in this decade was the introduction of trek separation, which is a graphical criterion that can be used to decide the vanishing of any subdeterminant of the covariance matrix [DST13, STD10]. Although more work is required to fully exploit trek separation in model equivalence criteria, the notion has already seen application in parameter identification problems [WRD+18].

While greatly generalizing Gaussian conditional independence, determinantal constraints are again not sufficient to describe the sets $\mathcal{M}(G)$ after projection to the covariance matrix of observed variables. The following example is due to Thomas Verma.

**Example 1.2.** Let $G$ be the graph from Figure 1.2. Then as in the first example no conditional independence that holds for $\mathcal{M}(G)$ involves only the observed variables $X_1$, $X_2$, $X_3$, and $X_4$. Instead, a positive definite $4 \times 4$ matrix $\Sigma = (\sigma_{ij})$ is the projection of a matrix in $\mathcal{M}(G)$ if and only if

$$f_{\text{Verma}} = \sigma_{11}\sigma_{13}\sigma_{22}\sigma_{34} - \sigma_{11}\sigma_{13}\sigma_{23}\sigma_{24} - \sigma_{11}\sigma_{14}\sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{14}\sigma_{23}^2$$

$$-\sigma_{12}^2\sigma_{13}\sigma_{34} + \sigma_{12}^2\sigma_{14}\sigma_{33} + \sigma_{12}\sigma_{13}\sigma_{24}^2 - \sigma_{12}\sigma_{13}\sigma_{14}\sigma_{23} = 0;$$

(compare Example 3.3.14 in [DSS09]). The polynomial $f_{\text{Verma}}$ is not a subdeterminant of $\Sigma$ and, therefore, is neither explained by $d$-separation nor by trek-separation.

Another key advance in the area is a graph decomposition result of Jin Tian [TP02]; see also [Drt18, Sections 5-6]. This result allows one to derive constraints by applying $d$-separation in certain subgraphs. In particular, the vanishing polynomial $f_{\text{Verma}}$ from (1.2) can be shown to arise from the independence of variables $X_1$ and $X_4$ that holds for the subgraph obtained by removing the edges $1 \to 3$ and $2 \to 3$ from the Verma graph in Figure 1.2. For further details, we refer the reader to the review in [SERR14].

In the next example however, neither Tian’s graph decomposition nor trek separation provide any insight.

**Example 1.3.** Let $G$ be the graph from Figure 1.3. There are four observed variables, and projecting $\mathcal{M}(G)$ gives a set of codimension one. As discussed in [vOM17], any covariance $\Sigma = (\sigma_{ij}) \in \mathcal{M}(G)$ satisfies the constraint

$$f_{\text{VOM}} = \sigma_{22}\sigma_{34}\sigma_{13} - \sigma_{22}\sigma_{33}\sigma_{14} - \sigma_{23}\sigma_{24}\sigma_{13} + \sigma_{23}^2\sigma_{14} = 0.$$
The irreducible polynomial in (1.3) defines the hypersurface that contains the projection of \( \mathcal{M}(G) \).

A closer look at Examples 1.2 and 1.3 reveals some common structure. Both constraints are *nested determinants*, by which we mean determinants of a matrix whose entries are determinants themselves. This observation is the point of departure for our paper.

**Example 1.4.** The Verma polynomial from Example 1.2 admits a compact representation through nested determinants, namely,

\[
\begin{vmatrix}
\Sigma_{123} & \Sigma_{124} \\
\Sigma_{13} & \Sigma_{14}
\end{vmatrix}
\]

Such a representation is generally not unique. For instance,

\[
\begin{vmatrix}
\Sigma_{123} & \Sigma_{124} \\
\Sigma_{13} & \Sigma_{12}
\end{vmatrix}
= \begin{vmatrix}
\Sigma_{123} & \Sigma_{124} \\
\Sigma_{34} & \Sigma_{34,13}
\end{vmatrix}
\]

The polynomial from Example 1.3 is also a nested determinant, namely,

\[
\begin{vmatrix}
\Sigma_{23} & \Sigma_{23} \\
\Sigma_{1,3} & \Sigma_{1,4}
\end{vmatrix}
\]

We are not aware of any literature emphasizing these types of representations.

In this paper, we investigate combinatorial conditions on the graph \( G \) that entail the vanishing of nested determinants. We give a rigorous definition of the models we study in Section 2, where we also provide background on the current knowledge of their description. In particular, we introduce mixed graph models that play an important role in model selection [DM17, Section 5.2]. Section 3 shows how nested determinants arise under conditions of ancestrality. In Theorem 3.7 we show that such determinants completely describe the model \( \mathcal{M}(G) \) for a wide class of mixed graphs that are (nearly) ancestral. Section 4 describes our notion of *restricted trek separation* in the setting of arbitrary acyclic mixed graphs. In Section 5 we show how the vanishing of nested determinants can follow from restricted trek separation. The result we present also implies the vanishing of the constraints exhibited for (nearly) ancestral graphs in Section 3. In Section 6 we give examples that involve recursive nesting of determinants and are beyond the scope of our results. Nevertheless, we can relate these examples back to restricted trek separation. While our focus is on acyclic mixed graphs, our last example shows that a nested determinant may also arise for graphs containing directed cycles. We conclude with Section 7 where we discuss future work and open problems.

2. Background

2.1. Structural equation models. Let \( \epsilon = (\epsilon_i : i \in V) \) be the random error vector for the equation system in (1.1). As we are only concerned with the covariance structure, we disregard the offsets \( \lambda_{0i} \). The system can then be written as

\[
X = \Lambda^T X + \epsilon,
\]

where the matrix \( \Lambda = (\lambda_{ij}) \in \mathbb{R}^{V \times V} \) holds the unknown coefficients. Let \( \Omega = (\omega_{ij}) = \text{Var}[\epsilon] \in \mathbb{R}^{V \times V} \) be the covariance matrix of \( \epsilon \), which we assume positive definite. Assuming
further that $I - \Lambda$ is invertible, the random vector $X = (I - \Lambda)^{-1}\epsilon$ is the unique solution to the linear system in (2.1) and has covariance matrix

$$\text{Var}[X] = (I - \Lambda)^{-T}\Omega(I - \Lambda)^{-1}.\tag{2.2}$$

In the introduction we focused on the case where the individual error terms $\epsilon_i$ are independent. Their covariance matrix $\Omega$ is then diagonal. In this case, a model postulating that some of the coefficients in $\Lambda$ are zero is conveniently represented by a directed graph, as was our setup in Section 1. Going forward, we also allow for dependence among the $\epsilon_i$ and a possibly non-diagonal matrix $\Omega$. Nonzero off-diagonal terms of $\Omega$ are commonly supported over $X$, holding for the model in which random variable $V$ is hidden. Thus, the determinant $\det|I - \Lambda|$ indexed by $\sigma_{ij}$ can be written as $\det_{\sigma_{ij}}$, i.e.,

$$\det_{\sigma_{ij}} = \prod_{j \in \omega(i)} (\lambda_{ij} - \lambda_{ji}) - \lambda_{ij} \prod_{j \in \omega(i)} \lambda_{ij} = 0 \text{ if } j \in B.$$  

Let $\mathbb{R}^D$ be the set of $V \times V$ matrices $\Lambda$ with support $D$, i.e.,

$$\mathbb{R}^D = \{\Lambda \in \mathbb{R}^{V \times V} : \lambda_{ij} = 0 \text{ if } i \to j \notin D\}.$$  

Let $\mathbb{R}_{PD}^D$ be the subset of matrices $\Lambda \in \mathbb{R}^D$ for which $I - \Lambda$ is invertible. Let $PDV$ be the cone of positive definite $V \times V$ matrices, and define $PD(B)$ to be the subcone of matrices supported over $B$, i.e.,

$$PD(B) = \{\Omega = (\omega_{ij}) \in PDV : \omega_{ij} = 0 \text{ if } i \neq j \text{ and } i \leftrightarrow j \notin B\}.\tag{2.2}$$

The mixed graph $G$ is acyclic if its directed part $(V, D)$ does not contain any directed cycles. In this case, $V$ can be ordered such that all matrices $\Lambda \in \mathbb{R}^D$ are strictly upper-triangular. Thus, the determinant $|I - \Lambda| = 1$ and $\mathbb{R}^D = \mathbb{R}_{PD}^D$. By Cramer’s rule, the covariances in $\text{Var}[X]$ in (2.2) are then polynomial functions of the entries of $\Lambda$ and $\Omega$.

Taking the error $\epsilon$ to be Gaussian, a given mixed graph induces the following statistical model for the joint distribution of $X$.

**Definition 2.1.** The linear structural equation model given by a mixed graph $G = (V, D, B)$ is the family of all multivariate normal distributions on $\mathbb{R}^V$ with covariance matrix in the set

$$\mathcal{M}(G) = \{(I - \Lambda)^{-T}\Omega(I - \Lambda)^{-1} : \Lambda \in \mathbb{R}^D_{\text{reg}}, \Omega \in PD(B)\}.$$  

The set $\mathbb{R}_{\text{reg}}^D \times PD(B)$ is semialgebraic. Since $\mathcal{M}(G)$ is the image of this set under a rational map, the Tarski-Seidenberg theorem yields that $\mathcal{M}(G)$ itself is a semialgebraic set and, thus, admits a polynomial description. In this paper, we are interested in studying polynomial equations that are satisfied by the matrices in $\mathcal{M}(G)$. With $\Sigma = (\sigma_{ij})$ interpreted as a symmetric $V \times V$ matrix of indeterminates, define $\mathbb{R}[\Sigma]$ to be the ring of polynomials in the $\sigma_{ij}$. Then the polynomial relations we seek to understand make up the vanishing ideal

$$\mathcal{I}(G) := \{f \in \mathbb{R}[\Sigma] : f(\Sigma) = 0 \text{ for all } \Sigma \in \mathcal{M}(G)\}.$$  

Suppose a variable $X_j$, $j \in V$, is hidden. Then the remaining variables $(X_i : i \neq j)$ have their covariance matrix in the set obtained by projecting each matrix in $\mathcal{M}(G)$ onto its $(V \setminus \{j\}) \times (V \setminus \{j\})$ submatrix. Two comments are in order. First, we emphasize that for a fixed $j \in V$, the polynomials $f \in \mathcal{I}(G)$ that do not involve any of the indeterminates indexed by $j$, i.e., $f$ is free of $\sigma_{jk}$ for $k \in V$, give precisely the polynomial constraints holding for the model in which random variable $X_j$ is hidden. Second, the paradigm of mixed graphs allows one to directly capture relations after projection. Indeed, a graphical
operation known as “latent projection” creates a new mixed graph $G'$ over the observed variables that represents key relations among covariances of observed variables; see [Pea09, Section 2.6], [Kos02] or [Wer11]. For instance, the three examples from our introduction would be represented by the three mixed graphs in Figure 2.1. In these examples, the ideal $\mathcal{I}(G')$ of the given mixed graph coincides with the ideal of polynomial relations among the observed covariances in the hidden variable model given by the original DAG $G$.

2.2. Trek rule. Again let $G = (V, D, B)$ be any mixed graph, possibly cyclic. The starting point for any combinatorial understanding of polynomials in the vanishing ideal $\mathcal{I}(G)$ is the trek rule. This rule specifies each entry of the covariance matrix in (2.2) as a sum of monomials associated with certain paths in the graph.

**Definition 2.2.** A trek is a path $\tau$ of the form

(a) $i_\ell \leftarrow \cdots \leftarrow i_1 \leftrightarrow j_1 \rightarrow \cdots \rightarrow j_r$, or

(b) $i_\ell \leftarrow \cdots \leftarrow i_1 = j_1 \rightarrow \cdots \rightarrow j_r$,

for integers $\ell, r \geq 0$ with $\ell + r \geq 1$. Here, a path may visit a vertex more than once. If $\ell = 0$, the trek is simply the directed path $j_1 \rightarrow \cdots \rightarrow j_r$. Similarly, it is $i_\ell \leftarrow \cdots \leftarrow i_1$ if $r = 0$.

We call $\tau$ a trek from $i_\ell$ to $j_r$ or also a trek between $i_\ell$ and $j_r$. The sets $\{i_k : k = 1, \ldots, \ell\}$ and $\{j_k : k = 1, \ldots, r\}$ are the left side and the right side of $\tau$, respectively.

Let $\Lambda = (\lambda_{ij}) \in \mathbb{R}^D_{\text{reg}}$ and $\Omega = (\omega_{ij}) \in \text{PD}(B)$. To any trek $\tau$, specified as in Definition 2.2, associate a trek monomial

(2.3) $\sigma(\tau) = \omega_{i_1 j_1} \prod_{k=1}^{\ell-1} \lambda_{i_k i_{k+1}} \prod_{k=1}^{r-1} \lambda_{j_k j_{k+1}}$.

The trek rule now states that the covariance matrix $\Sigma = (\sigma_{ij}) = (I - \Lambda)^{-T} \Omega (I - \Lambda)^{-1}$ has its entries given by

(2.4) $\sigma_{ij} = \sum_{\text{treks } \tau \text{ from } i \text{ to } j} \sigma(\tau)$.

The rule, which originates in the work of [Wri34], is obtained by observing that $(I - \Lambda)^{-1} = I + \Lambda + \Lambda^2 + \ldots$. The right-hand side of (2.4) is a polynomial when $G$ is acyclic and a (formal) power series otherwise.
2.3. Conditional independence and subdeterminants. The notion of $d$-separation allows one to decide by inspection of paths in a mixed graph $G$ whether a conditional independence relation holds for all distributions in the model given by $G$; see e.g. [Drt18 Section 10]. In algebraic terms, for a Gaussian joint distribution, variables $X_i$ and $X_j$ are conditionally independent given a subvector $X_S$ with $i, j \notin S$ if and only if the subdeterminant $|\Sigma_{iS,jS}|$ is zero. Here, $iS$ denotes the union of a singleton set $\{i\}$ and the set $S$. Thus, $d$-separation gives a combinatorial characterization of when a subdeterminant of the form $|\Sigma_{iS,jS}|$ belongs to the ideal $\mathcal{I}(G)$. If $G$ is a DAG, then the covariance model $\mathcal{M}(G)$ admits a semi-algebraic description by conditional independence. Indeed, $\mathcal{M}(G)$ is the set of positive definite matrices $\Sigma$ for which all conditional independence determinants $|\Sigma_{iS,jS}|$ vanish. This is also true for mixed graphs that are maximal ancestral [RS02], but false more generally as the examples in the introduction show.

In seminal work, Sullivant, Talaska and Draisma [STD10] move beyond conditional independence determinants and give a combinatorial characterization of when an arbitrary subdeterminant $|\Sigma_{A,B}|$ is in $\mathcal{I}(G)$. We briefly review their concept of trek-separation; see also [Drt18 Section 11].

**Definition 2.3.** Two sets $A, B \subseteq V$ are trek-separated by the pair $(S_L, S_R)$, where $S_L, S_R \subseteq V$, if every trek between a vertex from $A$ and a vertex from $B$ intersects either $S_L$ on its left side or $S_R$ on its right side.

In the case $|A| = |B| = m$ the following theorem shows that $|\Sigma_{A,B}| \in \mathcal{I}(G)$ if and only if the sets of vertices $A$ and $B$ are trek-separated by a pair $(S_L, S_R)$ with $|S_L| + |S_R| < |A|$.

**Theorem 2.4** (Thm. 2.17, [STD10]). Let $A, B \subseteq V$. The submatrix $\Sigma_{A,B}$ has rank at most $r$ for all covariance matrices $\Sigma \in \mathcal{M}(G)$ if and only if there exist subsets $S_L, S_R \subseteq V$ such that $|S_L| + |S_R| \leq r$ and $(S_L, S_R)$ trek-separates $A$ from $B$. For a generic choice of $\Sigma \in \mathcal{M}(G)$,

$$\text{rank}(\Sigma_{A,B}) = \min\{|S_L| + |S_R| : (S_L, S_R) \text{ trek-separates } A \text{ from } B\}.$$ 

While trek-separation greatly generalizes $d$-separation and can yield a generating set of $\mathcal{I}(G)$ for some mixed graphs [FRS16], it is in general not sufficient to understand the vanishing ideal $\mathcal{I}(G)$ as we demonstrated in Examples 1.2 and 1.3.

3. Ancestral vertices and overdetermined linear systems

We now proceed to a first combinatorial condition (see Proposition 3.4) for the vanishing of very special nested determinants (Definition 3.1). Fix a mixed graph $G = (V, \mathcal{D}, \mathcal{B})$, and let $\Sigma \in PD_V$. For a pair of matrices $\Lambda \in \mathbb{R}_{reg}^D$ and $\Omega \in PD(\mathcal{B})$, it holds that

$$\Sigma = (I - \Lambda)^{-T} \Omega (I - \Lambda)^{-1} \iff (I - \Lambda)^T \Sigma (I - \Lambda) = \Omega.$$ 

In turn, $\Sigma \in \mathcal{M}(G)$ if and only if

$$[(I - \Lambda)^T \Sigma (I - \Lambda)]_{ij} = 0 \quad \forall i,j \in V \text{ with } i \neq j, \ i \leftrightarrow j \notin \mathcal{B}. \quad (3.1)$$

For some graphs it is known that all entries of $\Lambda$ can be recovered as rational expressions of $\Sigma$, at least for generic choices of positive definite $\Sigma$. For instance, the half-trek criterion [FDD12] and its extensions [Che16 DW16 WRD18] can be used to certify graphically that such rational identification of $\Lambda$ from $\Sigma$ is possible and to find rational expressions. If now both the $i$th and the $j$th column of $\Lambda$ are rationally identifiable from $\Sigma$, then the left-hand side of the equation in (3.1) can be expressed as a rational function of $\Sigma$. If $i \neq j$
and $i \leftrightarrow j \notin \mathcal{B}$, then one finds a rational constraint on $\Sigma$ that after clearing denominators yields a polynomial in $\mathcal{I}(G)$. This approach is used, for instance, in [VOM17].

In this section we follow a similar approach in which we substitute solutions for some of the entries of $\Lambda$ that appear in (3.1). However, we only linearize the equations and then observe that nested determinantal constraints arise from overdetermined linear equation systems. Specifically, we study the following constraints.

**Definition 3.1.** Let $i$ be a vertex of the mixed graph $G$, and let $J$ be a subset of vertices in $G$. Define a matrix of polynomials of size $(|\text{pa}(i)| + |J|) \times (|\text{pa}(i)| + 1)$ as

$$F_{i,J} = \left(\left|\Sigma_{\text{pa}(r) \cup \{r\}, \text{pa}(r) \cup \{c\}}\right|_{r \in \text{pa}(i) \cup J, c \in \text{pa}(i) \cup \{i}\}}\right).$$

The *parentally nested determinants* for the pair $(i, J)$ are the minors of order $|\text{pa}(i)| + 1$ of the matrix $F_{i,J}$. When $J = \{j\}$ is a singleton, there is only one parentally nested determinant

$$f_{ij} = \left|\left|\Sigma_{\text{pa}(r) \cup \{r\}, \text{pa}(r) \cup \{c\}}\right|_{r \in \text{pa}(i) \cup \{j\}, c \in \text{pa}(i) \cup \{i}\}}\right|.$$

Here, index sets are treated as multisets with possibly repeated elements, and the determinants are formed according to a prespecified linear order for the vertex set $V$. The symbol $\cup$ stands for the sum (or disjoint union) of multisets; e.g., $\{1, 1, 2\} \cup \{1, 3\} = \{1, 1, 1, 2, 3\}$.

Suppose $j \in J \cap \text{pa}(i)$. Then $j$ is repeated in the row index set $\text{pa}(i) \cup J$ for the matrix $F_{i,J}$. In this case $j$ indexes two rows for a minor, which is then zero. In particular, if $j \in \text{pa}(i)$ then $f_{ij} = 0$. We may therefore always restrict the set $J$ to satisfy $J \cap \text{pa}(i) = \emptyset$.

A repeated index may also arise for the column index sets of the matrices whose determinants yield the entries of $F_{i,J}$. Indeed, if $c \in \text{pa}(i) \cup \{i\}$ is also in $\text{pa}(r)$ for $r \in \text{pa}(i) \cup \{j\}$, then the $(r, c)$ entry of $F_{i,J}$ is zero.

**Example 3.2.** It holds that $f_{\text{Verma}} = f_{41}$ in Example 1.2, and $f_{\text{VOM}} = f_{41}$ in Example 1.3.

In the remainder of this section, we identify conditions under which parentally nested determinants vanish.

**Definition 3.3.** A vertex $j$ in the mixed graph $G = (V, \mathcal{D}, \mathcal{B})$ is *ancestral* if (i) $j$ is not on any directed cycle, and (ii) no vertex $k \neq j$ has both $k \leftrightarrow j \in \mathcal{B}$ and a directed path from $k$ to $j$.

Let $\Lambda \in \mathbb{R}_{\text{reg}}^\mathcal{D}$ and $\Omega \in \mathcal{P}D(\mathcal{B})$, and define $\Sigma = (I - \Lambda)^{-T}\Omega(I - \Lambda)^{-1} \in \mathcal{M}(G)$. If $j$ is ancestral, then all treks from a vertex $r \in \text{pa}(j)$ to $j$ end with a directed edge pointing to $j$. The trek rule from (2.4) then implies that

$$\Sigma_{\text{pa}(j), \text{pa}(j)} \Lambda_{\text{pa}(j), j} = \Sigma_{\text{pa}(j), j}.$$

For our next result it is convenient to introduce the set of *siblings* of a vertex $j$, which is $\text{sib}(j) = \{k \in V : k \leftrightarrow i \in \mathcal{B}\}$, the set of neighbors of $j$ in the bidirected part of the graph.

**Proposition 3.4.** Let $i$ be a vertex of a mixed graph $G = (V, \mathcal{D}, \mathcal{B})$ such that

(i) $\text{pa}(i) \cap \text{sib}(i) = \emptyset$,
(ii) all vertices in $\text{pa}(i)$ are ancestral, and
(iii) the set $J$ of all ancestral vertices in $V \setminus (\text{pa}(i) \cup \text{sib}(i) \cup \{i\})$ is non-empty.

Then the parentally nested determinants for the pair $(i, J)$ are in the vanishing ideal $\mathcal{I}(G)$. 

Given \( f \), in fact the ideal \( \langle f \rangle \) and \( \langle \Sigma X \rangle \) correspond to conditional independence of \( f \) and \( \Sigma X \). As there are no bidirected edges, \( \sigma_{ri} \) implies the vanishing of its minors. Note that in the case that \( \sigma_{ri} \) differs from \( \sigma_{rc} \), neither \( \text{pa}(\Sigma) \) has rank at most \( |\text{pa}(i)| \). This in turn implies the vanishing of its minors. Note that in the case that \( i \) is not trek reachable from \( r \), the last equation is trivial and corresponds to a row of zeros in \( F_{i,j} \).

\[
\sigma_{ri} - \Sigma_{r,pa(r)}^{-1} \Sigma_{pa(r),i} - \Sigma_{r,pa(i)} \Lambda_{pa(i),i} + \Lambda_{r,pa(r)} \Sigma_{pa(r),pa(i)} \Lambda_{pa(i),i} = 0.
\]

Now observe that for any vertex \( c \),

\[
\left( \sigma_{rc} - \Sigma_{r,pa(r)}^{-1} \Sigma_{pa(r),c} \right) \left| \Sigma_{pa(r),pa(r)} \right| = \left| \Sigma_{pa(r),\cup \{r\},pa(r),\cup \{c\}} \right|.
\]

Hence, multiplying the equation in (3.7) by \( \left| \Sigma_{pa(r),pa(r)} \right| \) gives

\[
\left| \Sigma_{pa(r),\cup \{r\},pa(r),\cup \{i\}} \right| - \sum_{e \in \text{pa}(i)} \left| \Sigma_{pa(r),\cup \{r\},pa(r),\cup \{e\}} \right| \cdot \lambda_{ei} = 0.
\]

With one equation for every \( r \in \text{pa}(i) \cup J \), the system is overdetermined and admits a solution only if the matrix \( F_{i,j} \) from Definition 3.1 has rank at most \( |\text{pa}(i)| \). This, in turn, implies the vanishing of its minors. Note that in the case that \( i \) is not trek reachable from \( r \), the last equation is trivial and corresponds to a row of zeros in \( F_{i,j} \). \( \square \)

**Example 3.5.** The graph \( G \) from Figure 3.1 is a DAG, and thus all its vertices are ancestral. As there are no bidirected edges, \( f_{ij} \in I(G) \) for all \( i \neq j \). As previously noted, for any graph \( f_{ij} = 0 \) if \( j \in \text{pa}(i) \). Here, \( f_{21} = f_{32} = f_{42} = f_{43} = 0 \). Moreover, \( f_{12} = f_{34} = 0 \). The nonzero polynomials are

\[
\begin{align*}
f_{13} &= |\Sigma_{12,23}|, & f_{31} &= \sigma_{11} \cdot |\Sigma_{12,23}|, & f_{23} &= -\sigma_{12} \cdot |\Sigma_{12,23}| \\
f_{14} &= |\Sigma_{12,234}|, & f_{41} &= \sigma_{11} \sigma_{22} \cdot |\Sigma_{12,234}|, & f_{24} &= -\sigma_{12} \cdot |\Sigma_{12,234}|.
\end{align*}
\]

The irredicible polynomial \( f_{13} \) corresponds to conditional independence of \( X_1 \) and \( X_3 \) given \( X_2 \). The second irredicible polynomial \( f_{14} \) encodes conditional independence of \( X_1 \) and \( X_4 \) given \( X_2, X_3 \). It turns out that

\[
\mathcal{M}(G) = \{ \Sigma \in PD_{\{1,2,3,4\}} : f_{13}(\Sigma) = f_{14}(\Sigma) = 0 \}
\]

In fact, the ideal \( \langle f_{ij} : i \neq j \rangle = \langle f_{13}, f_{14} \rangle \) differs from \( I(G) \) only through components that do not vanish at positive definite matrices \([RL14\text{ Example 2}]. Specifically,

\[
\langle f_{13}, f_{14} \rangle = I(G) \cap \langle |\Sigma_{23,23}|, |\Sigma_{13,23}|, |\Sigma_{12,23}| \rangle.
\]

![Figure 3.1. DAG on 4 vertices used to illustrate the nested determinants \( f_{ij} \).](image-url)
Figure 3.2. An ancestral graph that is not maximal.

Here,
\[ I(G) = \langle |\Sigma_{12,23}|, |\Sigma_{12,24}|, |\Sigma_{12,25}| \rangle \]
is generated by three subdeterminants, two of which are conditional independences.

Example 3.6. The mixed graph in Figure 3.2 is an ancestral graph, that is, all vertices are ancestral \[ \text{RS02}. \] It is not maximal, that is, there are non-adjacent vertices, namely, 3 and 5, that cannot be \( d \)-separated. There is then no conditional independence constraint associated to the non-adjacency. Precisely two of the \( f_{ij} \) are nonzero, namely,

\[
\begin{align*}
f_{35} &= \begin{vmatrix} |\Sigma_{1,1}| & |\Sigma_{1,2}| & |\Sigma_{1,3}| \\ |\Sigma_{1,2}| & |\Sigma_{2,2}| & |\Sigma_{2,3}| \\ |\Sigma_{124,245}| & |\Sigma_{224,245}| & |\Sigma_{324,245}| \end{vmatrix} = \begin{vmatrix} |\Sigma_{1,1}| & |\Sigma_{1,2}| & |\Sigma_{1,3}| \\ |\Sigma_{1,2}| & |\Sigma_{2,2}| & |\Sigma_{2,3}| \\ |\Sigma_{124,245}| & 0 & |\Sigma_{234,245}| \end{vmatrix}, \\
f_{53} &= \begin{vmatrix} |\Sigma_{2,2}| & |\Sigma_{2,4}| & |\Sigma_{2,5}| \\ |\Sigma_{4,2}| & |\Sigma_{4,4}| & |\Sigma_{4,5}| \\ |\Sigma_{123,124}| & |\Sigma_{123,124}| & |\Sigma_{123,125}| \end{vmatrix} = \begin{vmatrix} |\Sigma_{2,2}| & |\Sigma_{2,4}| & |\Sigma_{2,5}| \\ |\Sigma_{4,2}| & |\Sigma_{4,4}| & |\Sigma_{4,5}| \\ 0 & |\Sigma_{123,124}| & |\Sigma_{123,125}| \end{vmatrix}.
\end{align*}
\]

In fact, \( f_{35} = f_{53} \), and \( I(G) = \langle f_{35} \rangle = \langle f_{53} \rangle \). We note that there is also the alternative representation of

\[
f_{35} = \begin{vmatrix} |\Sigma_{12,12}| & |\Sigma_{12,23}| \\ |\Sigma_{124,245}| & |\Sigma_{234,245}| \end{vmatrix} = \begin{vmatrix} |\Sigma_{24,24}| & |\Sigma_{24,25}| \\ |\Sigma_{123,124}| & |\Sigma_{123,125}| \end{vmatrix} = f_{53}.
\]

We now give a model description for a class of graphs that includes all ancestral graphs. It also covers the two graphs from Figure 2.1(b)(c). Recall that a sink of a mixed graph \( G = (V, D, B) \) is any vertex that is not a parent of any other vertex. A subgraph of \( G \) is a mixed graph \( G' = (V', D', B') \) with \( V' \subseteq V \), \( D' \subseteq D \), and \( B' \subseteq B \). We call \( G \) globally identifiable if \( G \) is acyclic and none of its subgraphs \( G'' = (V', D', B') \) has both a connected bidirected part \( (V', B') \) and a unique sink vertex in its directed part \( (V', D') \).

For any set of polynomials \( F \subset \mathbb{R}[\Sigma] \), we let \( V_F = \{ \Sigma : f(\Sigma) = 0 \ \forall f \in F \} \) be the algebraic subset it defines in the space of symmetric matrices.

Theorem 3.7. Let \( G = (V, D, B) \) be a globally identifiable mixed graph with vertex set \( V = [p] \equiv \{1, \ldots, p\} \) enumerated in a topological order. Suppose all vertices in \([p - 1]\) are ancestral. Let \( F(G) \) be the set of all parentally nested determinants obtained from the pairs \( (i, [i - 1] \setminus (pa(i) \cup sib(i))) \) for \( i \in V \). Then

\[ \mathcal{M}(G) = PD_V \cap V_{F(G)}. \]

Proof. The inclusion \( \mathcal{M}(G) \subseteq PD_V \cap V_{F(G)} \) follows from application of Proposition 3.4.

To show the reverse inclusion, we proceed by induction on the number of vertices \( p \). The statement is trivial for \( p = 1 \). In the induction step, let \( \Sigma \in PD_V \cap V_{F(G)} \). Let
\( \Sigma_{[p-1],[p-1]} \) be the submatrix obtained by removing the \( p \)th row and column. Let \( G[p-1] = ([p-1], \mathcal{D}[p-1], \mathcal{B}[p-1]) \) be the subgraph induced by \([p-1]\). Now, \( \Sigma_{[p-1],[p-1]} \in PD_{[p-1]} \cap \mathcal{V}_G(G[p-1]) \). The induction hypothesis yields that \( \Sigma_{[p-1],[p-1]} \in \mathcal{M}(G[p-1]) \). Let 

\[
\Sigma_{[p-1],[p-1]} = (I - \Lambda')^{-T} \Omega' (I - \Lambda')^{-1}
\]

for \( \Lambda' \in \mathbb{R}^{[p-1]} \) and \( \Omega' \in PD(\mathcal{B}[p-1]) \).

Consider the matrix \( F_{p,[p-1]\setminus\{pa(p),\text{sib}(p)\}} \) from Definition 3.1 evaluated at the given matrix \( \Sigma \). For each \( r \in [p-1] \setminus \text{sib}(p) \), divide the corresponding row by \( |\Sigma_{\text{pa}(r),\text{pa}(r)}| > 0 \). The resulting matrix \( \tilde{F} \) has entries

\[
\sigma_{rc} - \Sigma_{r,pa(r)} \Sigma^{-1}_{pa(r),pa(r)} \Sigma_{pa(r),c}
\]

for \( r \in [p-1] \setminus \text{sib}(p) \) and \( c \in \text{pa}(p) \cup \{p\} \); recall (3.3). Using (3.3), we obtain that

\[
\tilde{F}_{[p-1]\setminus\text{sib}(p),\text{pa}(p)} = [(I - \Lambda')^{-T} \Sigma_{[p-1],V_{[p-1]\setminus\text{sib}(p),\text{pa}(p)\cup\{p\}}}].
\]

Form the submatrix \( \tilde{F}_{[p-1]\setminus\text{sib}(p),\text{pa}(p)} \), that is, we omit the column indexed by \( p \). Then

\[
\tilde{F}_{[p-1]\setminus\text{sib}(p),\text{pa}(p)} = [\Omega' (I - \Lambda')^{-1}]_{[p-1]\setminus\text{sib}(p),\text{pa}(p)}.
\]

Lemma 2 in \cite{DFS11} yields that \( \tilde{F}_{[p-1]\setminus\text{sib}(p),\text{pa}(p)} \) has full column rank.

Since \( \Sigma \in \mathcal{V}_F(G) \), the matrix \( F_{p,[p-1]\setminus\{\text{pa}(p),\text{sib}(p)\}} \) and thus also \( \tilde{F} \) do not have full column rank. We conclude that the kernel of \( \tilde{F} \) contains a vector \( x \in \mathbb{R}^{\text{pa}(p)\cup\{p\}} \) for which the last coordinate \( x_p \neq 0 \). Dividing \( x_{\text{pa}(p)} \) by \(-x_p\) gives a vector \( \lambda_{\text{pa}(p),p} \in \mathbb{R}^{\text{pa}(p)} \) that solves the equation system in (3.5). Define a \( p \times p \) matrix \( \Lambda \in \mathbb{R}^D \) by using \( \lambda_{\text{pa}(p),p} \) to define its last column. Then \( \Lambda \) solves (3.5) for \( i = p \) and all \( r \in [p-1] \setminus \text{sib}(p) \) and thus also (3.1). Therefore, \( \Sigma \in \mathcal{M}(G) \).

The above facts leverage existence of ancestral vertices. In the next sections, we seek to give a more general condition for the vanishing of nested determinants. The results on vanishing nested determinants from this section can be recovered as a special case; see Proposition 5.7.

4. Restricted Trek Separation

As we reviewed in Section 2.3, the notion of trek separation \cite{STD10} provides a combinatorial characterization of when a subdeterminant of the covariance matrix \( \Sigma \) vanishes over a model \( \mathcal{M}(G) \). Underlying the trek separation result we stated in Theorem 2.4 is the observation that determinants correspond to sums of certain products of trek monomials. In this section, we recall this observation and then introduce a notion of restricted trek separation, in which separation only needs to occur with respect to treks that avoid certain vertices on their left or right sides. This notion will be used in Section 5 to obtain conditions that imply the vanishing of nested determinants.

Let \( A \) and \( B \) be two subsets of the vertex set of a mixed graph \( G \), with \(|A| = |B|\). A system of treks from \( A \) to \( B \) is a set of treks that each are between a vertex in \( A \) and a vertex in \( B \). Let \( \mathcal{T} \) be such a system. Then \( \mathcal{T} \) has no sided intersection if any two distinct treks in \( \mathcal{T} \) have disjoint left sides and disjoint right sides. In particular, each vertex in \( A \) and each vertex in \( B \) is on precisely one trek, so that \( \mathcal{T} \) induces a bijection between \( A \) and \( B \). Fixing an ordering of the elements of \( A \) and \( B \), the trek system induces a permutation of \( B \) in which the \( i \)th element of \( B \) is mapped to the end point of the trek that starts at
the $i$th element of $A$. Write $(-1)^T$ for the sign of this permutation. Now define
\begin{equation}
\mathcal{P}_{A,B} = \sum (-1)^T \prod_{\tau \in T} \sigma(\tau)
\end{equation}
with the summation being over all systems of treks $T$ from $A$ to $B$ with no sided intersection; recall the definition of trek monomials from (2.3).

**Theorem 4.1** (DST13). Suppose the underlying graph $G$ is acyclic. Then the determinant of $\Sigma_{A,B}$ equals $\mathcal{P}_{A,B}$.

This result admits a generalization to the case where the graph $G$ contains directed cycles. Indeed, DST13 give a rational expression for the determinant of $\Sigma_{A,B}$ in terms of self-avoiding trek flows, which reduce to trek systems without sided intersection in the acyclic case. As this generalization is more involved, we will not give any details here and focus instead on acyclic graphs only.

We now extend the combinatorial characterization of determinants and the trek separation result from Theorem 2.4 to allow for restricted treks.

**Definition 4.2.** Let $A$, $B$, $P$, and $Q$ be subsets of vertices of a mixed graph $G$. A $(P, Q)$-restricted trek between $A$ and $B$ is a trek between a vertex in $A$ and a vertex in $B$ that has its left side in $P$ and its right side in $Q$. Let $S_L$ and $S_R$ be two further subsets of vertices. Then $A$ and $B$ are $(P, Q)$-restricted trek-separated by $(S_L, S_R)$ if every $(P, Q)$-restricted trek between $A$ and $B$ intersects $S_L$ on its left side or $S_R$ on its right side.

**Example 4.3.** Consider the Verma graph from Figure 2.1(b). Take $A = \{2, 4\}$, $B = \{2, 3\}$, $P = \{2, 4\}$, and $Q = \{2, 3, 4\}$. Then $A$ and $B$ are $(P, Q)$-restricted trek-separated by $\{(\{\}, \{2\})\}$. Indeed, every trek between $A$ and $B$ that only uses $P$ on the left and only uses $Q$ on the right has to go through $2$ on the right. Note, however, that this is not true if, for example, $P = Q = V$ or if $3 \in P$.

The main observation of this section is that restricted trek separation is equivalent to a rank constraint on a special matrix. Note also that part (ii) of the theorem is a direct generalization of Theorem 4.1 to the restricted case.

**Theorem 4.4.** Let $G = (V, D, B)$ be an acyclic mixed graph, and let $\Lambda \in \mathbb{R}^D$ and $\Omega \in PD(B)$. For $P, Q \subseteq V$, consider the matrix
\[ \Sigma^{(P,Q)} = [(I - \Lambda)_{P,P}]^{-T} \Omega_{P,Q} [(I - \Lambda)_{Q,Q}]^{-1}, \]
and its submatrix $\Sigma_{A,B}^{(P,Q)}$ for a choice of row indices $A \subseteq P$ and column indices $B \subseteq Q$.

(i) The rank of $\Sigma_{A,B}^{(P,Q)}$ is at most
\[ \min\{|S_L| + |S_R| : A \text{ and } B \text{ are } (P, Q)\text{-restricted trek-separated by } (S_L, S_R)\}, \]
and equal to this minimum generically.

(ii) If $|A| = |B|$, then the determinant of $\Sigma_{A,B}^{(P,Q)}$ is equal to
\[ \mathcal{P}_{A,B,(P,Q)} = \sum (-1)^T \prod_{\tau \in T} \sigma(\tau) \]
where the summation runs over all systems of treks $T$ that comprise only $(P, Q)$-restricted treks from $A$ to $B$ and have no sided intersection.
The proof of Theorem 4.4 is located in Appendix A. It is analogous to the proofs of Theorems 2.4 and 4.1 as developed in STDI, DSTI.

Example 4.5. Consider once more the Verma graph from Figure 2.1(b). Let $A = \{2, 4\}$, $B = \{2, 3\}$, $P = \{2, 4\}$, and $Q = \{2, 3, 4\}$. We saw in Example 4.3 that $A$ and $B$ are $(P, Q)$-restricted trek-separated by $(\{\}, \{2\})$. Now consider the matrix

$$
\Sigma^{(P, Q)} = [(I - \Lambda)_{24, 24}]^{-T} \Omega_{24, 234} [(I - \Lambda)_{234, 234}]^{-1}
$$

$$
= \begin{pmatrix}
\omega_{22} & \omega_{22} \lambda_{23} & \omega_{22} \lambda_{23} \lambda_{34} + \omega_{24} \\
\omega_{24} & \omega_{24} \lambda_{23} & \omega_{24} \lambda_{23} \lambda_{34} + \omega_{14}
\end{pmatrix}.
$$

As predicted by Theorem 4.4(i), the submatrix $\Sigma_{A, B}^{(P, Q)} = \Sigma_{24, 23}^{(P, Q)}$ has rank 1.

5. Nested determinants

In this section we demonstrate how restricted trek separation may lead to polynomial equations in the vanishing ideal $\mathcal{I}(G)$ of the model $\mathcal{M}(G)$ of an acyclic mixed graph $G$. These equations are in general not determinantal, instead they are given by specific types of nested determinants. In Section 5.1 we introduce a swapping property based on which in Theorem 5.3 we show how restricted trek separation gives rise to the vanishing of such nested determinants. In Section 5.2 we show how the swapping property and Theorem 5.3 are sufficient to explain the vanishing of the parenally nested determinants from Proposition 3.4 and Theorem 3.7 in terms of restricted trek separation.

5.1. Restricted trek separation and nested determinants. We begin by defining a swapping property that allows us to factor certain subdeterminants of $\Sigma$. Recall that $\uplus$ denotes disjoint union (of multisets).

Definition 5.1. Let $A_1, \ldots, A_k, B_1, \ldots, B_k$ be sets of vertices of an acyclic mixed graph $G$ with $|A_i| = |B_i|$ for every $i = 1, \ldots, k$. Suppose every system of treks without sided intersection between $A_1 \uplus \cdots \uplus A_k$ and $B_1 \uplus \cdots \uplus B_k$ connects $A_i$ to $B_i$ for every $i$. Moreover, suppose that for any two trek systems $T_1$ and $T_2$ without sided intersection between $A_1 \uplus \cdots \uplus A_k$ and $B_1 \uplus \cdots \uplus B_k$, if we swap the treks between $A_i$ and $B_i$ from $T_1$ with those from $T_2$, then we obtain another two trek systems with no sided intersection between $A_1 \uplus \cdots \uplus A_k$ and $B_1 \uplus \cdots \uplus B_k$. Then, we say that $(A_1, B_1), \ldots, (A_k, B_k)$ satisfy the swapping property.

The next lemma, which is proven in Appendix B.1, shows how the swapping property gives factorizations of subdeterminants of $\Sigma$ into different systems of trek monominal sums.

Lemma 5.2. Assume that $(A_1, B_1), \ldots, (A_k, B_k)$ satisfy the swapping property. Then,

$$
|\Sigma_{A_1 \uplus \cdots \uplus A_k, B_1 \uplus \cdots \uplus B_k}| = \prod_{i=1}^{k} P_{A_i, B_i, (C_i, D_i)},
$$

where $C_1, \ldots, C_k, D_1, \ldots, D_k$ are sets of vertices determined by the trek systems without sided intersection between $A_1 \uplus \cdots \uplus A_k$ and $B_1 \uplus \cdots \uplus B_k$.

We now proceed to the main result of this section, Theorem 5.3, which shows that the swapping property for suitable sets of vertices implies that certain nested determinants can be factored into sums of restricted trek monomial systems. Later in the section we will see that though the conditions of this theorem appear to be quite special, they are very
natural. In particular, they apply to a multitude of examples, and moreover, the theorem generalizes our results from Proposition 3.4 and Theorem 3.7.

**Theorem 5.3.** Let \( a_1, \ldots, a_n, b_1, \ldots, b_n \in V \), and let \( A_1, \ldots, A_n, B_1, \ldots, B_n, C_1, \ldots, C_n, D_1, \ldots, D_n \subseteq V \) be such that \( |A_i| = |B_i| = |C_j| = |D_j| \) for all \( i, j \). Assume further that for each \( i, j \), the sets \((A_i, B_i), (C_j, D_j)\), and \((\{a_i\}, \{b_j\})\) satisfy the swapping property such that
\[
|\Sigma_{A_i \cup B_j \cup \{a_i\}, B_i \cup D_j \cup \{b_j\}}| = \mathcal{P}_{A_i, B_i}(P_i, Q_i) \mathcal{P}_{C_j, D_j}(R_j, S_j) \mathcal{P}_{a_i, b_j}(E_{ij}, F_{ij})
\]
for sets of vertices \( P_i, Q_i, R_j, S_j, E_{ij}, F_{ij} \) and every \( i \) and \( j \). Assume also that
\[
\left| \left( \mathcal{P}_{a_i, b_j}(E_{ij}, F_{ij}) \right) \right|_{1 \leq i, j \leq n} = \mathcal{P}_{\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_n\}, (E, F)}
\]
for some \( E, F \). Then,
\[
\left| \left( \left| \Sigma_{A_i \cup B_j \cup \{a_i\}, B_i \cup D_j \cup \{b_j\}} \right| \right|_{1 \leq i, j \leq n} \right| = \\
= \left( \prod_i \mathcal{P}_{A_i, B_i}(P_i, Q_i) \right) \left( \prod_j \mathcal{P}_{C_j, D_j}(R_j, S_j) \right) \mathcal{P}_{\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_n\}, (E, F)}.
\]

**Proof.** Let \( M \) be the matrix with \( i, j \)-th entry equal to \( |\Sigma_{A_i \cup B_j \cup \{a_i\}, B_i \cup D_j \cup \{b_j\}}| \) for \( 1 \leq i, j \leq n \). Since the entries in the \( i \)-th row of \( M \) are divisible by \( \mathcal{P}_{A_i, B_i}(P_i, Q_i) \), we can factor the determinant of \( M \) as
\[
|M| = \left( \prod_{i=1}^{n} \mathcal{P}_{A_i, B_i}(P_i, Q_i) \right) \det((\mathcal{P}_{C_j, D_j}(R_j, S_j) \mathcal{P}_{a_i, b_j}(E_{ij}, F_{ij}))_{1 \leq i, j \leq n}).
\]
Since the \( j \)-th column is divisible by \( \mathcal{P}_{C_j, D_j}(R_j, S_j) \), we can further factor as
\[
|M| = \left( \prod_i \mathcal{P}_{A_i, B_i}(P_i, Q_i) \right) \left( \prod_j \mathcal{P}_{C_j, D_j}(R_j, S_j) \right) \det(\mathcal{P}_{a_i, b_j}(E_{ij}, F_{ij}))
\]
\[
= \left( \prod_i \mathcal{P}_{A_i, B_i}(P_i, Q_i) \right) \left( \prod_j \mathcal{P}_{C_j, D_j}(R_j, S_j) \right) \mathcal{P}_{\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_n\}, (E, F)}
\]
as required. \( \square \)

Condition (5.1) deserves further discussion. By Theorem 4.4(ii),
\[
\left| \left( \mathcal{P}_{a_i, b_j}(E, F) \right) \right|_{i, j} = \mathcal{P}_{\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_n\}, (E, F)}
\]
for every \( a_1, \ldots, a_n, b_1, \ldots, b_n \in V \) and \( E, F \subseteq V \). However, for this equality, it is not necessary to have the exact sets \( E \) and \( F \) for all matrix entries on the left-hand side. Instead, slightly different sets \( E_{ij} \) and \( F_{ij} \) can be used as the following lemma indicates.

**Lemma 5.4.** Let \( a_1, \ldots, a_n, b_1, \ldots, b_n \in V \) and \( E, F \subseteq V \). Then,
\[
\left| \left( \mathcal{P}_{a_i, b_j}(E_{ij}, F_{ij}) \right) \right|_{i, j} = \mathcal{P}_{\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_n\}, (E, F)}
\]
if \( E \setminus \{a_k : k \neq i\} \subseteq E_{ij} \subseteq E \cup (V \setminus \text{an}(a_i)) \) and \( F \setminus \{b_\ell : \ell \neq j\} \subseteq F_{ij} \subseteq F \cup (V \setminus \text{an}(b_j)) \).
The proof can be found in Appendix [3.2]. We remark that additional choices of $E_{ij}$ and $F_{ij}$ are certainly possible depending on the graph structure at hand. Nevertheless, this lemma gives us a wide variety of sets $E_{ij}$ and $F_{ij}$ for which equality (5.3) holds.

Example 5.5. Recall the Verma graph from Figure 2.1(b). In the context of Theorem 5.3 let

$$a_1 = 2, \quad a_2 = 3, \quad b_1 = 2, \quad b_2 = 4,$$

$$C_1 = \{1\}, \quad C_2 = \{1\}, \quad D_1 = \{1\}, \quad D_2 = \{3\}, \quad A_1 = A_2 = B_1 = B_2 = \emptyset.$$  

Then $(C_1, D_1), \{(a_1), (b_1)\}$ satisfy the swapping property for $i = 1, 2$. The same is true for $(C_2, D_2), \{(a_2), (b_2)\}$ for $i = 1, 2$. Moreover,

$$\Sigma_{12,12} = P_{1,1} P_{2,2}, ((234)\{234\}), \quad \Sigma_{12,34} = P_{1,3} P_{2,4}, ((234)\{24\}),$$

$$\Sigma_{13,12} = P_{1,1} P_{3,2}, ((234)\{24\}), \quad \Sigma_{13,34} = P_{1,3} P_{3,4}, ((234)\{24\}).$$

By Lemma 5.4 we have that

$$\begin{bmatrix} P_{2,2}, ((234)\{234\}) & P_{2,4}, ((234)\{24\}) \\ P_{3,2}, ((234)\{24\}) & P_{3,4}, ((234)\{24\}) \end{bmatrix} = P_{(2,3), (2,4)}, ((234)\{24\}).$$

Therefore, the conditions of Theorem 5.3 are satisfied, and

$$f_{\text{Verma}} = \frac{\Sigma_{12,12}}{\Sigma_{13,12}} \frac{\Sigma_{12,34}}{\Sigma_{13,34}} = P_{1,1} P_{1,3} P_{(2,3), (2,4)}, ((234)\{24\}).$$

Now $P_{(2,3), (2,4)}, ((2,3,4), (2,4))$ is zero because $\{2, 3\}$ and $\{2, 4\}$ are $\{(234), (24)\}$-restricted trek separated by $\{(2, \emptyset)\}$. That is, the nested determinant giving $f_{\text{Verma}}$ vanishes because treks between $\{2, 3\}$ and $\{2, 4\}$ that only use $\{2, 3, 4\}$ on the left and $\{2, 4\}$ on the right must all pass through 2 on the left.

The following corollary gives a combinatorial interpretation of the vanishing of a nested determinant like the ones specified in Theorem 5.3.

Corollary 5.6. Suppose the conditions in Theorem 5.3 are satisfied. Define matrix entries $M_{ij} = |\Sigma_{A_i \cup C_j \cup \{a_i\}, B_i \cup D_j \cup \{b_j\}}|$. Then, $|M| = 0$ if and only if at least one of the following holds:

(i) The sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are (E, F)-restricted trek separated by some sets $(X, Y)$ with $|X| + |Y| < n$; or

(ii) For some $i = 1, \ldots, n$, the sets $A_i$ and $B_i$ are $P_i, Q_i$-restricted trek separated by some sets $(X, Y)$ with $|X| + |Y| < |A_i|$; or

(iii) For some $j = 1, \ldots, n$, the sets $C_j$ and $D_j$ are $R_j, S_j$-restricted trek separated by some sets $(X, Y)$ with $|X| + |Y| < |C_j|$.

Proof. By Theorem 5.3 $|M|$ factors as in (5.2). Therefore, $|M| = 0$ if and only if one of the factors in this expression vanishes. By Theorem 4.4 each of these factors vanishes if and only if the corresponding restricted trek separation is satisfied.

5.2. Restricted trek separation and ancestral vertices. We return to the parentally nested determinants and specifically the nested determinant $f_{ij}$ defined in (5.3). In Proposition 3.4 we gave conditions that entailed the vanishing of $f_{ij}$. We now see how this result is also implied by restricted trek separation.

Proposition 5.7. Consider the conditions from Proposition 3.4, i.e., $i$ is a vertex in $G = (V, D, B)$ satisfying
Figure 5.1. Ancestral graph examples.

(i) $\text{pa}(i) \cap \text{sib}(i) = \emptyset$,
(ii) all vertices in $\text{pa}(i)$ are ancestral, and
(iii) the set $J$ of all ancestral vertices in $V \setminus (\text{pa}(i) \cup \text{sib}(i) \cup \{i\})$ is non-empty.

Then for every $j \in J$ the sets $\text{pa}(i) \cup \{j\}$ and $\text{pa}(i) \cup \{i\}$ are $(\text{pa}(v), V)$-restricted trek separated by $(\emptyset, \text{pa}(v))$. This restricted trek separation implies that $f_{ij} \in I(G)$ for all $j \in J$, i.e., all parentally nested determinants for $(i, J)$ lie in $I(G)$.

The proof can be found in Appendix B.3.

Example 5.8. Again consider the Verma graph from Figure 2.1(b). We have that

$$f_{\text{Verma}} = \begin{vmatrix} \sigma_{13} & \sigma_{14} \\ |\Sigma_{123,123}| & |\Sigma_{123,124}| \end{vmatrix}.$$ 

As mentioned in Section 3, $f_{\text{Verma}} = f_{41}$. Indeed, $(\{3,1\}, \{3,4\})$ are $(\{3,1\}, V)$-restricted trek separated by $(\emptyset, \{3\})$.

Example 5.9. Consider the ancestral graph from Figure 5.1(a), which was studied in more detail in [RS02]. Applying Proposition 5.7 we choose $i = 3$ and $j = 4$ to obtain that the corresponding polynomial $f_{34}$ vanishes. Indeed, by Lemma 5.2 we have that

$$f_{34} = \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ |\Sigma_{24,21}| & |\Sigma_{24,23}| \end{vmatrix} = \frac{\mathcal{P}_{1,1,4,1,2,3}(1,3,4,\{1,2,3,4\})}{\mathcal{P}_{2,2,2,2,4,1,2,3}(1,3,4,\{1,2,3,4\})} \cdot \frac{\mathcal{P}_{1,3,4,1,2,3}(1,3,4,\{1,2,3,4\})}{\mathcal{P}_{2,2,2,2,4,1,2,3}(1,3,4,\{1,2,3,4\})} = \mathcal{P}_{2,2,2,2,4,1,2,3}(1,3,4,\{1,2,3,4\}) \cdot \mathcal{P}_{2,2,2,2,4,1,2,3}(1,3,4,\{1,2,3,4\}).$$

As all treks between $\{1,4\}$ and $\{1,3\}$ which avoid 2 on the left must intersect 1 on the right we have that $\mathcal{P}_{1,4,1,3,4,1,2,3}(1,3,4,\{1,2,3,4\}) = 0$ and thus the above nested determinant is an element of the vanishing ideal for the graph. It can be checked by computational algebra that the above determinant generates the ideal of the model.
Example 5.10. For the ancestral graph in Figure 5.1(b), we choose $i = 6, j = 3$. Then, using Lemma 5.2, we have that

$$f_{63} = \begin{vmatrix} \sigma_{123,124} & \sigma_{123,125} & \sigma_{123,126} \\ \sigma_{44} & \sigma_{45} & \sigma_{46} \\ \sigma_{54} & \sigma_{55} & \sigma_{56} \end{vmatrix} = \mathcal{P}_{\{1,2\},\{1,2\}} \mathcal{P}_{\{3,4,5\},\{3,4,5\},\{3,4,5\},\{3,4,5\},\{3,4,5\}}.$$

As all treks between $\{3, 4, 5\}$ and $\{4, 5, 6\}$ which are restricted to only use $\{3, 4, 5, 6\}$ on their left or right sides must use 4 or 5 on their right side it follows that

$$\mathcal{P}_{\{3,4,5\},\{4,5,6\},\{3,4,5\},\{3,4,5\},\{3,4,5\}} = 0$$

and thus the above determinant is an element of the vanishing ideal.

Example 5.11. Consider the graph from Figure 5.1(c). Lemma 5.2 implies that

$$f_{53} = \begin{vmatrix} \Sigma_{123,122} & \Sigma_{123,124} & \Sigma_{123,125} \\ \sigma_{22} & \sigma_{24} & \sigma_{25} \\ \sigma_{42} & \sigma_{44} & \sigma_{45} \end{vmatrix} = \mathcal{P}_{\{1,2\},\{1,2\}} \mathcal{P}_{\{3,2,4\},\{2,4,5\},\{2,3,4,5\},\{1,2,3,4,5\}}.$$

As any trek from $\{3, 2, 4\}$ to $\{2, 4, 5\}$ which avoids 1 on the left must use 2 or 4 on the right it follows that $\mathcal{P}_{\{3,2,4\},\{2,4,5\},\{2,3,4,5\},\{1,2,3,4,5\}} = 0$. Hence $f_{53}$ is in (and generates) the vanishing ideal.

6. BEYOND SWAPPING: RECURSIVE NESTING, DIRECTED CYCLES, AND THE PENTAD

In this section we explore examples that are not covered by Theorem 5.3 and Corollary 5.6 but whose constraints are still nested determinants. In Section 6.1 we consider two mixed graphs for which the constraints could be presented as nested determinants but for which—we argue—a recursive nesting of determinants is more natural and more directly tied to restricted trek separation. In Section 6.2 we discuss an example of a directed graph with a directed cycle for which restricted trek separation also implies a nested determinant constraint. Finally, in Section 6.3 we turn to the pentad from factor analysis [DSS07], and show that it is also defined by a nested determinant.

6.1. Restricted trek separation and determinants of recursively nested matrices.

It is apparent from Theorem 5.3 and Corollary 5.6 that (singly) nested determinants give a way to express restricted trek systems as factors as follows. Recall the Verma graph and
Moreover, using Lemma 5.2, we get the following factorization:

$$\det A = \det B$$

and

$$\det C = \det D$$

Thus, we can express the determinant of the matrix whose entries are themselves nested determinants, namely,

$$f = \begin{vmatrix} 0 & |\Sigma_{12,12}| & |\Sigma_{12,14}| \\ |\Sigma_{12,12}| & |\Sigma_{12,14}| & |\Sigma_{12,12}| \\ |\Sigma_{13,12}| & |\Sigma_{13,14}| & |\Sigma_{14,12}| \\ |\Sigma_{13,12}| & |\Sigma_{13,14}| & |\Sigma_{14,12}| \\ |\Sigma_{13,12}| & |\Sigma_{13,14}| & |\Sigma_{14,12}| \\ |\Sigma_{13,12}| & |\Sigma_{13,14}| & |\Sigma_{14,12}| \\ |\Sigma_{13,12}| & |\Sigma_{13,14}| & |\Sigma_{14,12}| \end{vmatrix}.$$
This implies that
\[ f = \mathcal{P}_1 \mathcal{P}_2 \left| \begin{array}{ccc}
| \Sigma_{12,12} | & | \Sigma_{13,13} | & | \Sigma_{12,12} | \\
| \Sigma_{13,12} | & \sigma_{33} & \sigma_{32} \\
| \Sigma_{14,12} | & \sigma_{43} & \sigma_{42}
\end{array} \right| \]

where we have used Lemma 5.4 in the second step. Indeed, this is the case, and they are separated by \((\emptyset, \{2\})\).

**Example 6.2.** Now consider the graph in Figure 6.1(b). Its vanishing ideal is generated by the polynomial
\[ f = \left| \begin{array}{ccc}
| \Sigma_{12,12} | & | \Sigma_{12,13} | & | \Sigma_{12,12} | \\
| \Sigma_{13,12} | & \Sigma_{2,3} & \Sigma_{3,3} \\
| \Sigma_{14,12} | & \Sigma_{12,12} & \Sigma_{13,12}
\end{array} \right| . \]

While the above representation of \( f \) suggests applying Theorem 5.3, this, unfortunately, does not seem to satisfy the conditions of the theorem either. On the other hand, we can express \( f \) as the determinant of a matrix whose entries are themselves nested determinants:
\[ f = \left| \begin{array}{ccc}
| \Sigma_{12,13} | & | \Sigma_{12,14} | \\
| \Sigma_{2,3} & \Sigma_{3,3} & \Sigma_{4,3} \\
| \Sigma_{12,12} & | \Sigma_{13,12} | & | \Sigma_{13,12} |
\end{array} \right| . \]

Moreover, by Lemma 5.2,
\[ | \Sigma_{12,12} | = \mathcal{P}_{1,1} \mathcal{P}_{2,2}(\{2,3,4\},\{2,3,4\}), \quad | \Sigma_{12,13} | = \mathcal{P}_{1,1} \mathcal{P}_{2,3}(\{2,3,4\},\{2,3,4\}), \quad | \Sigma_{12,14} | = \mathcal{P}_{1,1} \mathcal{P}_{2,4}(\{2,3,4\},\{2,3,4\}). \]

Consequently,
\[ \left| \begin{array}{cc}
\Sigma_{2,3} & \Sigma_{3,3} \\
| \Sigma_{12,12} | & | \Sigma_{13,12} |
\end{array} \right| = \mathcal{P}_{1,1} \mathcal{P}_{2,2}(\{2,3,4\},\{2,3,4\}) \mathcal{P}_{3,3}(\{3,4\},\{3,4\}), \]
\[ \left| \begin{array}{cc}
\Sigma_{2,4} & \Sigma_{3,4} \\
| \Sigma_{12,12} | & | \Sigma_{13,12} |
\end{array} \right| = \mathcal{P}_{1,1} \mathcal{P}_{2,2}(\{2,3,4\},\{2,3,4\}) \mathcal{P}_{3,4}(\{3,4\},\{3,4\}). \]

Thus, we can write the full determinant as
\[ f = \mathcal{P}_{1,1}^2 \mathcal{P}_{2,2}(\{2,3,4\},\{2,3,4\}) \left| \begin{array}{ccc}
| \Sigma_{12,13} | & | \Sigma_{12,14} | \\
| \Sigma_{2,3} & \Sigma_{3,3} & \Sigma_{4,3} \\
| \Sigma_{12,12} & | \Sigma_{13,12} | & | \Sigma_{13,12} |
\end{array} \right| . \]

By Lemma 5.4,
\[ f = \mathcal{P}_{1,1}^2 \mathcal{P}_{2,2}(\{2,3,4\},\{2,3,4\}) \mathcal{P}_{2,3}(\{3,4\},\{2,3,4\}) \mathcal{P}_{3,4}(\{3,4\},\{2,3,4\}). \]

The last term suggests that \( \{2,3\} \) and \( \{3,4\} \) are \((\emptyset, \{3\})\)-restricted trek separated. Indeed, this is the case, and they are separated by \((\emptyset, \{3\})\).

**Remark 6.3.** Appendix B of [vOM17] explicitly lists the (minimal) generators of all vanishing ideals of acyclic mixed graphs on 4 nodes. Of these graphs, only those from Figure 6.1 cannot be immediately recognized as being determinantal constraints on the covariance matrix. From what we have shown above we now see that all generators of vanishing ideals of acyclic mixed graphs on four nodes can be written as nested determinants with, at most, a single level of nesting (i.e., as determinants of determinants), and moreover can be explained via restricted trek separation.
As we also record in Section 7, we believe that restricted trek separation can always be formed as a factor of a vanishing recursively nested determinant. Moreover, we deem it possible that such determinants define all acyclic linear structural equation models. We defer further exploration of these questions to a future study.

6.2. Graphs with cycles. Although we believe our results from the previous two sections can be extended to graphs containing cycles, the situation there is a bit more complicated. Even extending Theorem 4.1 to the cyclic case is not a simple task. It was accomplished (along with other results) in a separate article [DST13].

The following model, although it has cycles, is defined by the vanishing of a nested determinant, which can be explained by restricted trek separation. However, we wish to point out that a more sophisticated example might need the definition of further notions, like those that appear in [DST13].

Example 6.4. Consider the graph \( G \) in Figure 6.2, which was treated in [Drt09] where a degree 6 polynomial \( f \) generating \( \mathcal{I}(G) \) was displayed. This polynomial can be written as the following doubly nested determinant:

\[
f = \begin{vmatrix}
|\Sigma_{34,12}| & |\Sigma_{34,13}| \\
|\Sigma_{12,12}| & |\Sigma_{12,34}| & |\Sigma_{12,13}| & |\Sigma_{12,34}| \\
|\Sigma_{14,12}| & |\Sigma_{14,34}| & |\Sigma_{14,13}| & |\Sigma_{14,34}|
\end{vmatrix}.
\]

We will now show that the vanishing of this determinant corresponds to the fact that \( \{4,2\} \) and \( \{2,3\} \) are \( (\{1,2,4\},\{2,3,4\}) \)-restricted-trek separated by \((\emptyset,\{2\})\). For our derivations we use results from [DST13], where subdeterminants of \( \Sigma \) corresponding to graphs with cycles are given by rational expressions. The entries of the above matrix are:

\[
|\Sigma_{34,12}| = \frac{(\lambda_{13} + \lambda_{12} \lambda_{23}) \omega_{11} \omega_{44} \lambda_{42}}{(1 - \lambda_{23} \lambda_{34} \lambda_{42})^2} = \frac{P_{3,1,((1,2,3),(1,2,3))} P_{4,2,((1,2,4),(2,3,4))}}{(1 - \lambda_{23} \lambda_{34} \lambda_{42})},
\]

\[
|\Sigma_{34,13}| = \frac{(\lambda_{13} + \lambda_{12} \lambda_{23}) \omega_{11} \omega_{44} \lambda_{42} \lambda_{23}}{(1 - \lambda_{23} \lambda_{34} \lambda_{42})^2} = \frac{P_{3,1,((1,2,3),(1,2,3))} P_{4,3,((1,2,4),(2,3,4))}}{(1 - \lambda_{23} \lambda_{34} \lambda_{42})},
\]

\[
|\Sigma_{12,12}| = \frac{1}{1 - \lambda_{23} \lambda_{34} \lambda_{42}} = \frac{P_{3,1,((1,2,3),(1,2,3))} P_{4,2,((1,2,4),(2,3,4))}}{(1 - \lambda_{23} \lambda_{34} \lambda_{42})},
\]

\[
|\Sigma_{14,12}| = \frac{1}{1 - \lambda_{23} \lambda_{34} \lambda_{42}} = \frac{P_{3,1,((1,2,3),(1,2,3))} P_{4,2,((1,2,4),(2,3,4))}}{(1 - \lambda_{23} \lambda_{34} \lambda_{42})},
\]

It follows that

\[
f = \frac{\Sigma_{14,34} \omega_{11} P_{3,1,((1,2,3),(1,2,3))} P_{4,2,((1,2,4),(2,3,4))} P_{4,3,((1,2,4),(2,3,4))} P_{2,3,((1,2),(2,3,4))}}{(1 - \lambda_{23} \lambda_{34} \lambda_{42})},
\]
where the last equality follows by Lemma 5.4. The last term in the numerator vanishes due to the above mentioned restricted trek separation. We remark that the extra term \((1 - \lambda_{23}\lambda_{34}\lambda_{42})\) in the denominator cannot be obtained via the formula given in Theorem 5.3 (even with the usage of geometric series).

6.3. Nested determinants with no restricted trek separation. We have not found any examples of acyclic mixed graphs \(G\) for which defining equations of the model \(\mathcal{M}(G)\) cannot be explained using restricted trek separation. However, there are other, closely related models, for which restricted trek separation does not seem to provide the same combinatorial explanation.

**Example 6.5** (The pentad). Consider a factor analysis model with five normally distributed observed variables and two latent factors as in Figure 6.3. Its defining equation is a degree 5 polynomial in the covariance matrix entries:

\[
f_{\text{pentad}} = \sigma_{12}\sigma_{13}\sigma_{24}\sigma_{35}\sigma_{45} - \sigma_{12}\sigma_{13}\sigma_{25}\sigma_{34}\sigma_{45} - \sigma_{12}\sigma_{14}\sigma_{23}\sigma_{35}\sigma_{45} + \sigma_{12}\sigma_{14}\sigma_{25}\sigma_{34}\sigma_{45} + \sigma_{12}\sigma_{15}\sigma_{23}\sigma_{34}\sigma_{45} - \sigma_{12}\sigma_{15}\sigma_{24}\sigma_{35}\sigma_{45} + \sigma_{13}\sigma_{14}\sigma_{23}\sigma_{25}\sigma_{34} - \sigma_{13}\sigma_{14}\sigma_{25}\sigma_{23}\sigma_{34} + \sigma_{14}\sigma_{15}\sigma_{23}\sigma_{24}\sigma_{35}.
\]

This polynomial can be expressed in nested determinantal form as

\[
f_{\text{pentad}} = \begin{vmatrix} \Sigma_{23,45} & | & \Sigma_{25,34} \\ \Sigma_{123,145} & | & \Sigma_{125,134} \end{vmatrix}.
\]

Combinatorially, we can see that all trek systems stemming from the second row of the matrix are in one-to-one correspondence with the trek systems from the first row of the matrix, and are obtained by just adding the trek \(1 - 1\). However, we have not been able to interpret this nested determinant via restricted trek separation. Note that the mixed graph obtained by latent projection would be a complete graph with a bidirected edge between each \(i, j \in \{1, \ldots, 5\}\).

7. Discussion

We conclude by giving a brief review of the results presented in this paper and then discussing problems for future work.

**Contributions.** This paper demonstrates the importance of nested determinants as constraints on covariance matrices in graphical causal/structural equation models associated to mixed graphs. Nested determinants are determinants of matrices whose entries are determinants themselves. Theorem 3.7 shows that a special class of parentally nested determinants is sufficient for a semialgebraic description of a class of models that is slightly more general than the class of ancestral graph models. Theorem 5.3 provides a framework
for explaining the vanishing of more general nested determinants via trek separation under restrictions on the vertices that treks may visit on their left and their right sides.

The examples from Section 6 depict graphs for which the conditions of Theorem 5.3 do not apply. While it is often possible to present the defining equations of such models in terms of (singly) nested determinants, we suggest to instead view the equations as recursively nested determinants. In other words, we consider determinants of smaller matrices whose entries are (recursively) nested determinants. As we exemplified, such recursively nested determinants may admit an explanation by restricted trek separation. We further exhibit an example of a graph with a cycle in which the model is also described by a recursively nested determinant that admits a restricted trek separation interpretation.

*Definition of nested and recursively nested determinants.* Theorem 5.3 is concerned with a particular type of nested determinants where rows and columns of the considered matrix correspond to vertices of the graph/the given random variables. This setup contains as a special case the parentally nested determinants from Section 3. We anticipate that the nested determinants considered in Theorem 5.3 are sufficiently general to describe mixed graph models as long as we allow for a suitable notion of recursive nesting as encountered in the Examples in Section 6.

In a general definition of recursively nested determinants, the subdeterminants of the original covariance matrix would be recursively nested determinants with depth of recursion zero. At depth $k$, we would take determinants of matrices whose entries are recursively nested determinants of depth at most $k - 1$. However, it would be desirable to constrain this construction such that for any recursively nested determinant the rows and columns of the considered matrix can be put in correspondence with two sets of vertices. These sets of vertices may then admit a restricted trek separation.

*Problem 7.1.* Develop a notion of recursively nested determinants for which row and column indices are in correspondence with graph vertices. The depth of recursion should be such that the subdeterminants of the original matrix are the only recursively nested determinants of depth 0. The recursively nested determinants of depth 1 should be of the type encountered in Theorem 5.3.

*Tian decomposition.* In the introduction, after Example 1.2, we mentioned Tian’s graph decomposition, which may yield subgraphs whose covariance matrix can be rationally identified from the covariance matrix for the original graph $G$. Trek separation in the subgraph then gives a rational constraint. Clearing denominators yields a polynomial in $\mathcal{I}(G)$.

*Conjecture 7.2.* Trek separation relations in subgraphs obtained from Tian’s graph decomposition correspond to recursively nested determinants.

*Vanishing nested determinants.* The results we have given so far are sufficient conditions for the vanishing of nested determinants.

*Problem 7.3.* Using restricted trek separation, obtain graphical conditions that are necessary for the vanishing of the nested determinants from Theorem 5.3.

If a characterization of the vanishing of nested determinants is established, it can be used to decide model equivalence questions. More generally, it would be desirable to obtain conditions, sufficient and necessary, for the vanishing of recursively nested determinants. We formulate a “hopeful” conjecture for acyclic mixed graphs.
Conjecture 7.4. The equality of two models $M(G)$ and $M(H)$ can be decided by comparing restricted trek separation relations in $G$ and $H$.

In all examples of graphs $G$ we inspected, the vanishing ideal $I(G)$ is in fact generated by nested or recursively nested determinants.

Conjecture 7.5. The vanishing ideal $I(G)$ can always be generated by recursively nested determinants.

Computing restricted trek separation. Assuming that restricted trek separation is what characterizes equivalence classes of models, as suggested by Conjecture 7.4, we may need to either use the graphical conditions from Problem 7.3 or to be able to compute restricted trek separation in order to find equivalent graphs.

Problem 7.6. Design computationally efficient algorithms for checking/finding restricted trek separations.

Feedback cycles. Our focus was on acyclic mixed graphs, for which determinants of the covariance matrix have expansions in terms of systems of treks without sided intersection. However, as the example of Figure 6.2 shows, (recursively) nested determinants are also relevant for cyclic graphs.

Problem 7.7. Generalize Theorem 5.3 to the general possibly cyclic case.

Of course, all problems mentioned above also pertain to graphs with cycles.

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Appendix A. Proofs for Section 4: restricted trek separation

This section is devoted to proving Theorem 4.4. The proof proceeds through rather minor modifications of the ideas of STD10.

A.1. Proof of Theorem 4.4(i) for directed acyclic graphs. We begin by proving Theorem 4.4 in the case when $G$ is a directed acyclic graph (DAG). We extend it to acyclic mixed graphs in the next section. We first record the following combinatorial interpretation of the entries of $(I - \Lambda_{C,C})^{-1}$ for a subset of vertices $C$.

Proposition A.1. Let $P(i,j,C)$ be the set of directed paths from $i \in C$ to $j \in C$ that only use vertices from a subset $C \subseteq V$ in the directed graph $G$. For each path $P$, define $\lambda_P = \prod_{i \to j \in P} \lambda_{ij}$. Then

$$[(I - \Lambda_{C,C})^{-1}]_{ij} = \sum_{P \in P(i,j,C)} \lambda_P.$$

Proof. The claim follows from Proposition 3.1 in STD10 if we consider the induced subgraph of $G$ with vertex set $C$. □
When $G$ is a directed graph, the error covariance matrix $\Omega$ is diagonal. This allows us to show the following lemma. We emphasize that in our discussion a determinant is zero if it is identically zero as a polynomial/function.

**Lemma A.2.** In a directed graph consider sets of vertices $A \subseteq P, B \subseteq Q$ with $|A| = |B|$. Then $\det \Sigma_{A,B}^{(P,Q)} = 0$ if and only if for every set $S \subseteq P \cap Q$ with $|S| = |A| = |B|$ either $\det \left( (I - \Lambda)_{P,P}^{-1} \right)_{S,A} = 0$ or $\det \left( (I - \Lambda)_{Q,Q}^{-1} \right)_{S,B} = 0$.

**Proof.** Since $\Sigma^{(P,Q)} = ((I - \Lambda)_{P,P})^{-T} \Omega_{P,Q} ((I - \Lambda)_{Q,Q})^{-1}$, we have

$$\Sigma_{A,B}^{(P,Q)} = \left( (I - \Lambda)_{P,P}^{-1} \right)_{A,P} \Omega_{P,Q} \left( (I - \Lambda)_{Q,Q}^{-1} \right)_{Q,B}.$$ 

By the Cauchy-Binet theorem,

$$\det \Sigma_{A,B}^{(P,Q)} = \sum_{S \subseteq P, R \subseteq Q} \det \left( (I - \Lambda)_{P,P}^{-1} \right)_{A,S} \det (\Omega_{S,R}) \det \left( (I - \Lambda)_{Q,Q}^{-1} \right)_{R,B},$$

where the sum runs over $S$ and $R$ of cardinality $|A| = |B|$. As $\Omega$ is diagonal, we obtain that

$$\det \Sigma_{A,B}^{(P,Q)} = \sum_{S \subseteq P \cap Q} \det \left( (I - \Lambda)_{P,P}^{-1} \right)_{A,S} \det (\Omega_{S,S}) \det \left( (I - \Lambda)_{Q,Q}^{-1} \right)_{S,B}$$

$$= \sum_{S \subseteq P \cap Q} \det \left( (I - \Lambda)_{P,P}^{-1} \right)_{S,A} \det \left( (I - \Lambda)_{Q,Q}^{-1} \right)_{S,B} \prod_{s \in S} \omega_{s,s}.$$ 

Since each monomial $\prod_{s \in S} \omega_{s,s}$ appears only in one term in this expansion, the result follows. □

We now recall the Gessel-Viennot-Lindström lemma.

**Lemma A.3** (Gessel-Viennot-Lindström lemma). Suppose $G$ is a DAG with vertex set $\{1, \ldots, m\}$. Let $A, B \subseteq \{1, \ldots, m\}$ be such that $|A| = |B| = \ell$. Then

$$\det ((I - \Lambda)^{-1})_{A,B} = \sum_{S \in \mathcal{N}(A,B)} (-1)^{S} \lambda^{S},$$

where $\mathcal{N}(A,B)$ is the set of all nonintersecting systems of $\ell$ directed paths in $G$ from $A$ to $B$, and $(-1)^{S}$ is the sign of the induced permutation of elements from $A$ to $B$. In particular, $\det ((I - \Lambda)^{-1})_{A,B} = 0$ if and only if every system of $\ell$ directed paths from $A$ to $B$ has two paths which share a vertex.

We are going to use this lemma by restricting the original directed acyclic graph $G$ to the induced subgraphs on the subsets $P$ and $Q$. The lemma applies to all these subgraphs because they themselves are directed acyclic graphs.

Let $A \subseteq P, B \subseteq Q$ with $|A| = |B| = \ell$. Consider a system $\mathcal{T} = \{\tau_{1}, \ldots, \tau_{\ell}\}$ of $\ell$ $(P, Q)$-restricted treks from $A \subseteq P$ to $B \subseteq Q$, connecting the $\ell$ distinct vertices in $A$ to the $\ell$ distinct vertices in $B$. Let $\text{top}(\mathcal{T})$ denote the multiset $\{\text{top}(\tau_{1}), \ldots, \text{top}(\tau_{\ell})\}$. Here $\text{top}(\tau)$ is the unique source of the trek $\tau$, i.e., the vertex contained in both the left side and the right side of the trek. Note that the trek system $\mathcal{T}$ consists of two systems of directed paths, a path system $S_A$ from $\text{top}(\mathcal{T})$ to $A$ which only uses vertices in $P$, and a path system $S_B$ from $\text{top}(\mathcal{T})$ to $B$ which only uses vertices in $Q$. We say that $\mathcal{T}$ has a sided intersection if two paths in $S_A$ share a vertex or if two paths in $S_B$ share a vertex.
Proposition A.4. In a DAG consider sets of vertices $A \subseteq P$ and $B \subseteq Q$ with $|A| = |B|$. Then,

$$\det(\Sigma^{(P,Q)}_{A,B}) = 0$$

if and only if every system of (simple) $(P,Q)$-restricted treks from $A$ to $B$ has a sided intersection.

Proof. Suppose that $\det(\Sigma^{(P,Q)}_{A,B}) = 0$, and let $T$ be a $(P,Q)$-restricted trek system from $A$ to $B$. If all elements of the multiset $\top(T)$ are distinct, then Lemma A.2 implies that either $\det(((I - \Lambda)^{-1})_{\top(T),A}) = 0$ or $\det(((I - \Lambda)^{-1})_{\top(T),B}) = 0$. If $\top(T)$ has repeated elements, then these determinants are also zero since there are repeated rows. Thus, in both cases, Lemma A.3 implies that there is an intersection in the path system from $\top(T)$ to $A$ or in the path system from $\top(T)$ to $B$. Hence, $T$ has a sided intersection.

Conversely, suppose that every $(P,Q)$-restricted trek system from $A$ to $B$ has a sided intersection, and let $S \subseteq P \cap Q$. If $\hat{R} = \top(T)$ for some $(P,Q)$-restricted trek system $T$ from $A$ to $B$, then either the path system from $\top(T)$ to $A$ or the path system from $\top(T)$ to $B$ has an intersection. If $\hat{R}$ is not the set of top elements for some $(P,Q)$-restricted trek system $T$ from $A$ to $B$, then there is no $P$-restricted path system connecting $R$ to $A$ or there is no $Q$-restricted path system from $R$ to $B$. In both cases, Lemma A.3 implies that either $\det(((I - \Lambda)^{-1})_{R,A}) = 0$ or $\det(((I - \Lambda)^{-1})_{R,B}) = 0$. Then, Lemma A.2 implies that $\det(\Sigma^{(P,Q)}_{A,B}) = 0$.

Note that it is sufficient to check the systems of simple treks only. Here, simple indicates that a trek has no repeated vertices.

We now define a new DAG associated to $G$, denoted $\hat{G}_{P,Q}$ in order to be able to invoke the Max-Flow-Min-Cut Theorem (see Theorem A.6). Let $P' = \{i' : i \in P\}$ be a set of new vertices, each being the copy of a corresponding vertex in $P$. The vertex set of graph $\hat{G}_{P,Q}$ is $P' \cup Q$. The edge set of $\hat{G}_{P,Q}$ includes the edge $i \to j$ for all $i,j \in Q$ such that $i \to j$ is an edge in $G$. Moreover, it includes the edge $j' \to i'$ for all $i,j \in P$ such that $i \to j$ is an edge in $G$, and the edge $i' \to i$ for all $i \in P \cap Q$.

**Proposition A.5.** The $(P,Q)$-restricted treks in $G$ from $i \in P$ to $j \in Q$ are in bijective correspondence with directed paths from $i'$ to $j$ in $\hat{G}_{P,Q}$. Simple $(P,Q)$-restricted treks in $G$ from $i$ to $j$ are in bijective correspondence with directed paths from $i'$ to $j$ in $\hat{G}_{P,Q}$ that use at most one edge from any pair $a \to b$ and $b' \to c'$ where $a,b \in Q$, $b,c \in P$.

Proof. Every trek from $i$ to $j$ is the union of two paths with a common top, the left path in $P$, the right path in $Q$. The part of the trek from the top to $i$ corresponds to the subpath with only vertices in $P'$, and the part of the trek from the top to $j$ corresponds to the subpath with only vertices in $Q$. The unique edge of the form $k' \to k$ corresponds to the top of the trek. Excluding $a \to b$ and $b' \to c'$ implies that a trek never visits the same vertex $b$ twice.

Menger’s theorem, also known as the Max-Flow-Min-Cut theorem, now allows us to turn the sided crossing result on $G$ into a blocking characterization on $\hat{G}_{P,Q}$.

**Theorem A.6** (Vertex version of Menger’s theorem). The cardinality of the largest set of vertex disjoint directed paths between two nonadjacent vertices $u$ and $v$ in a DAG is equal to the cardinality of the smallest blocking set, where a blocking set is a set of vertices whose removal from the graph ensures there is no directed path from $u$ to $v$. 


Proof of Theorem 4.4 for DAGs. We first focus on the case where \( \det \Sigma_{A,B}^{(P,Q)} = 0 \) so that the rank is at most \( k - 1 \), where \( k = |A| = |B| \). According to Proposition A.4, every system of \( k \) \((P,Q)\)-restricted treks from \( A \) to \( B \) must have a sided intersection. That is, the number of vertex disjoint paths from \( A' \) to \( B \) is at most \( k - 1 \) in the graph \( \tilde{G}_{P,Q} \). We add two new vertices to \( \tilde{G}_{P,Q} \), one vertex \( u \) that points to each vertex in \( A' \) and one vertex \( v \) that each vertex in \( B \) points to \( v \). Thus, there are at most \( k - 1 \) vertex disjoint paths from \( u \) to \( v \). Applying Menger’s theorem, there is a blocking set \( W \) in \( \tilde{G}_{P,Q} \) of cardinality \( |W| \leq k - 1 \). Set \( J_A = \{ i \in P : i' \in W \} \) and \( J_B = \{ i \in Q : i \in W \} \). Then, we have that \( |J_A| + |J_B| \leq k - 1 \), and these two sets \((P,Q)\)-restricted trek-separates \( A \) from \( B \).

Conversely, suppose there exist sets \( J_A \subseteq P \) and \( J_B \subseteq Q \) with \( |J_A| + |J_B| \leq k - 1 \) which \((P,Q)\)-restricted trek-separate \( A \) from \( B \). Then \( W = \{ i : i \in J_B \} \cup \{ i' : i \in J_A \} \) is a blocking set between \( u \) and \( v \) as above. By Menger’s theorem, since \( |W| \leq k - 1 \), there is no vertex disjoint system of \( k \) paths from \( A' \) to \( B \) in \( \tilde{G}_{P,Q} \). Thus, every \((P,Q)\)-restricted trek system from \( A \) to \( B \) has a sided intersection so that \( \det \Sigma_{A,B}^{(P,Q)} = 0 \) by Proposition A.4. From the special case of determinants, we deduce the general result, because if the smallest blocking set has size \( r \), there exists a collection of \( r \) disjoint paths between any subset of \( A' \) and any subset of \( B \), and this is the largest possible number of paths in such a collection. This means that all \( (r + 1) \times (r + 1) \) minors of \( \Sigma_{A,B}^{(P,Q)} \) are zero, but at least one \( r \times r \) minor is not zero. Hence, \( \Sigma_{A,B}^{(P,Q)} \) has rank \( r \) for generic choices of the parameters. \( \square \)

A.2. Proof of Theorem 4.4(i) for mixed graphs. A standard argument allows us to reduce to the case where there are no bidirected edges in the graph. This can be achieved by subdividing the bidirected edges; that is, for each bidirected edge \( i \leftrightarrow j \) in the graph, where \( i \leq j \), we replace \( i \leftrightarrow j \) with a vertex \( v_{i,j} \), directed edges \( v_{i,j} \rightarrow i \) and \( v_{i,j} \rightarrow j \). If \( i \) or \( j \) lie in \( P \) or \( Q \), then we add \( v_{i,j} \) to \( P \) or \( Q \) respectively. Call the enhanced sets \( \overline{P} \) and \( \overline{Q} \). The graph \( \overline{G} \) obtained from \( G \) by subdividing all of its bidirected edges is called the bidirected subdivision, or canonical DAG associated to \( G \).

Proposition A.7. Let \( A \subseteq P \), \( B \subseteq Q \) be sets of vertices of a mixed graph with \( |A| = |B| \).

(i) The matrix \( \Sigma_{A,B}^{(P,Q)} \) associated to \( G \) has the same generic rank as the matrix \( \Sigma_{A,B}^{(\overline{P},\overline{Q})} \) associated to \( \overline{G} \).

(ii) There exist \( J_L \subseteq P \), \( J_R \subseteq Q \) with \( |J_L| + |J_R| = r \) that \((J_L,J_R)\) \((P,Q)\)-restricted trek-separates \( A \) from \( B \) in \( G \) if and only if there exist \( J_L \subseteq \overline{P} \), \( J_R \subseteq \overline{Q} \) with \( |J_L| + |J_R| = r \) that \((J_L,J_R)\) \((\overline{P},\overline{Q})\)-restricted trek-separates \( A \) from \( B \) in \( \overline{G} \).

Proof. (i) Let \( \Lambda = (\lambda_{k,l}) \) and \( \Omega = (\omega_{k,l}) \) be parameters for \( G \). Define parameters for \( G = (V,\mathcal{D},\mathcal{B}) \) as follows. For any directed edge \( i \rightarrow j \) in \( G \), set \( \lambda_{i,j} = \lambda_{i,j} \). For any bidirected edge \( i \leftrightarrow j \) in \( G \), set

\[
\omega_{i,j} = \overline{w}_{v_{i,j}, v_{i,j}} \overline{\lambda}_{v_{i,j}, v_{i,j}}.
\]

Finally, for each vertex \( i \) in \( G \), set

\[
\omega_{i,i} = \overline{w}_{i,i} + \sum_{j \leftrightarrow i \in G} \overline{w}_{v_{i,j}, v_{i,j}} \overline{\lambda}_{v_{i,j}, v_{i,j}}.
\]

Clearly, \( \Lambda = (\lambda_{i,j}) \in \mathbb{R}^{\mathcal{D}} \). Since all \( \overline{w}_{i,i} > 0 \), the matrix \( \Omega = (\omega_{i,j}) \) is positive definite and, thus, in PD(\( \mathcal{B} \)). Let \( \Sigma_{A,B}^{(P,Q)} \) be the matrix defined by \( (\Lambda, \Omega) \), and let \( \Sigma_{A,B}^{(\overline{P},\overline{Q})} \) be the matrix
defined by \((\Lambda, \Omega)\). Applying the \((P,Q)\)-restricted trek rule to \(G\) and \(\bar{G}\), respectively, we see that \(\Sigma_{A,B}^{(P,Q)} = \Sigma_{A,B}^{(\bar{P}, \bar{Q})}\). We conclude that the set of matrices \(\Sigma_{A,B}^{(P,Q)}\) associated to \(G\) is contained in the set of matrices \(\Sigma_{A,B}^{(P,Q)}\) associated to \(\bar{G}\).

In general the reverse inclusion does not hold [DY10]. Nevertheless, the set of matrices \(\Sigma_{A,B}^{(P,Q)}\) for \(G\) has the same Zariski closure as the set of \(\Sigma_{A,B}^{(\bar{P}, \bar{Q})}\) for \(\bar{G}\). Let \(U \subset \mathbb{R}^P \times PD(B)\) be a neighborhood of \((0, I)\), i.e., we consider matrices \(\Lambda\) with entries of small magnitude and \(\Omega\) near the identity matrix. To prove equality of the Zariski closures, it suffices to show that every matrix \(\Sigma_{A,B}^{(P,Q)}\) given by a choice of \((\Lambda, \Omega) \in U\) is equal to a matrix \(\Sigma_{A,B}^{(\bar{P}, \bar{Q})}\) associated to a choice of \(\Lambda\) and \(\Omega\) for \(\bar{G}\). This in turn will follow from the trek rule if we can find \((\Lambda, \Omega)\) such that (A.1) and (A.2) hold. However, this is possible because near the identity matrix, each off-diagonal entry \(\omega_{i,j}\) is small. Specifically, we choose \(\overline{\omega}_{v_{i,j},v_{i,j}} = 1\), and set \(\overline{\omega}_{v_{i,j},i} = \sqrt{|\omega_{i,j}|}\), and \(\overline{\omega}_{v_{i,j},j} = \text{sign}(\omega_{i,j}) \sqrt{|\omega_{i,j}|}\). When the \(\omega_{i,j}\) are small enough, the sum on the right-hand side of (A.2) is smaller than \(\omega_{i,i}\), which is near one. Hence, we can find a positive \(\overline{\omega}_{i,i}\) satisfying (A.2), which ensures that \(\Omega\) is a diagonal matrix with positive diagonal entries as required.

(ii) Any pair of sets \(S_L\) and \(S_R\) that are \((P,Q)\)-restricted trek-separating in \(G\) are also clearly \((\bar{P}, \bar{Q})\)-restricted trek-separating in \(\bar{G}\). Conversely, suppose that \((\bar{J}_L, \bar{J}_R)\) is a minimal \((\bar{P}, \bar{Q})\)-restricted trek-separating set in \(\bar{G}\); that is, if any vertex is deleted from \((\bar{J}_L, \bar{J}_R)\), we no longer have a \((\bar{P}, \bar{Q})\)-restricted trek-separating set. We show that such a minimal \((\bar{P}, \bar{Q})\)-restricted trek-separating set in \(\bar{G}\) corresponds to a \((P,Q)\)-restricted trek-separating set in \(G\). Define

\[ J_L = (\bar{J}_L \cap P) \cup \{ i \in P : v_{i,j} \in \bar{J}_L \}, \]

\[ J_R = (\bar{J}_R \cap Q) \cup \{ j \in Q : v_{i,j} \in \bar{J}_R \}. \]

If \(\bar{J}_L\) and \(\bar{J}_R\) contain none of the vertices \(v_{i,j}\), then \(J_L\) and \(J_R\) clearly \((P,Q)\)-restricted trek-separate \(A\) and \(B\) in \(G\). Otherwise, the way that \(\{ i \in P : v_{i,j} \in \bar{J}_L \}\) and \(\{ j \in Q : v_{i,j} \in \bar{J}_R \}\) are chosen is important. Given a vertex \(v_{i,j} \in \bar{J}_L \cup \bar{J}_R\), let \(T(v_{i,j})\) denote the set of \((P,Q)\)-restricted treks \(\tau = (\tau_L, \tau_R)\) from \(A\) to \(B\) such that \(\tau_L \cap \bar{J}_L = \{v_{i,j}\}\) or \(\tau_R \cap \bar{J}_R = \{v_{i,j}\}\). Since \((\bar{J}_L, \bar{J}_R)\) is minimal, then \(T(v_{i,j})\) must be nonempty. This implies that in every \((P,Q)\)-restricted trek \(\tau = (\tau_L, \tau_R) \in T(v_{i,j})\), up to relabeling, \(i\) occurs in \(\tau_L\) (whose sink lies in \(A\)) and \(j\) occurs in \(\tau_R\) (whose sink lies in \(B\)). For if there were also a trek \(\tau = (\tau_L, \tau_R)\) in \(T(v_{i,j})\) which has \(j\) in \(\tau_L\) or \(i\) in \(\tau_R\), we could patch two halves of these treks together to find a \((P,Q)\)-restricted trek from \(A\) to \(B\) that does not have a sided intersection with \((\bar{J}_L, \bar{J}_R)\). So, assume \(i\) lies in \(\tau_L\), and \(j\) lies in \(\tau_R\) for all \((P,Q)\)-restricted treks in \(T(v_{i,j})\). In this case, add \(i\) to \(J_L\) whenever \(v_{i,j} \in \bar{J}_L\), and add \(j\) to \(J_R\) whenever \(v_{i,j} \in \bar{J}_R\). Then, \(|J_L| + |J_R| \leq |\bar{J}_L| + |\bar{J}_R|\), and \((J_L, J_R)\) \((P,Q)\)-restricted trek-separates \(A\) from \(B\) in \(G\).

To finish the proof of Theorem 4.4(ii), note that Proposition A.7 immediately reduces the statement to the case of directed acyclic graphs, which was given in the previous subsection.

A.3. Proof of Theorem 4.4(ii). Using first the Cauchy-Binet Theorem and then the Gessel-Viennot-Lindström Lemma A.3, we have that

\[\det(\Sigma_{A,B}^{(P,Q)}) = \det(((I - \Lambda)_{P,P})^{-T})_{A,P}\Omega_{P,Q} (((I - \Lambda)_{Q,Q})^{-1})_{Q,B}\]
The latter sum goes over all trek systems between $A$ and $B$ whose left directed parts have no sided intersection and only use vertices from $P$, whose right directed parts have no sided intersection and only use vertices from $Q$, and use left and right sides are joined via “middle vertices” in $S$ and $R$. Each summand is the product of the trek monomials of the treks in each such system times the sign of the permutation induced by each such trek system. Moreover, note that each trek system with no sided intersection between $A$ and $B$ appears in this sum.

APPENDIX B. PROOFS FOR NESTED DETERMINANTS

B.1. Proof of Lemma 5.2

Proof. Suppose first that there exist $A_i$ and $A_j$ such that $A_i \cap A_j \neq \emptyset$ for $i \neq j$. The case where two of the $B_i$’s intersect is analogous. Then, $|\Sigma_{A_1 \cup \cdots \cup A_k, B_1 \cup \cdots \cup B_k}| = 0$ since this matrix has a repeated row. On the other hand, we can select $C_i = \emptyset$, which makes $P_{A_i, B_i, (C_i, D_i)} = 0$, so that for any choice of the rest of the $C_j$ and $D_j$, we have that $\prod_{j=1}^{k} P_{A_j, B_j, (C_j, D_j)} = 0$. Thus, both sides are equal to 0, which establishes the statement.

Now, assume that $A_i \cap A_j = B_i \cap B_j = \emptyset$ for all $i \neq j$. We know by Theorem 4.1 that

$$|\Sigma_{A_1 \cup \cdots \cup A_k, B_1 \cup \cdots \cup B_k}| = P_{A_1 \cup \cdots \cup A_k, B_1 \cup \cdots \cup B_k}.$$

For every $i = 1, \ldots, k$ let $C_i^c$, the complement of $C_i$, be the union over all trek systems with no sided intersection between $A_1 \cup \cdots \cup A_k$ and $B_1 \cup \cdots \cup B_k$ of the vertices that take part in the left side of the treks that start at $A_1 \cup \cdots \cup A_{i-1} \cup A_{i+1} \cup \cdots \cup A_k$. Let $D_i^c$ be the union over all trek systems with no sided intersection between $A_1 \cup \cdots \cup A_k$ and $B_1 \cup \cdots \cup B_k$ of the vertices that take part in the right side of the treks that start at $A_1 \cup \cdots \cup A_{i-1} \cup A_{i+1} \cup \cdots \cup A_k$ (and end at $B_1 \cup \cdots \cup B_{i-1} \cup B_{i+1} \cup \cdots \cup B_k$).

By assumption, if we are given two trek systems $T_1$ and $T_2$ with no sided intersection between $A_1 \cup \cdots \cup A_k$ and $B_1 \cup \cdots \cup B_k$, then we can swap the treks from $A_i$ to $B_i$ from the first system $T_1$ with those from the second system $T_2$ and obtain two other trek systems between $A_1 \cup \cdots \cup A_k$ and $B_1 \cup \cdots \cup B_k$ with no sided intersection. Hence, each summand in $P_{A_1 \cup \cdots \cup A_k, B_1 \cup \cdots \cup B_k}$ can be factored uniquely as a product of one element from each of $P_{A_i, B_i, (C_i, D_i)}$. Conversely, the product of one element from each of $P_{A_i, B_i, (C_i, D_i)}$ gives an element from $P_{A_1 \cup \cdots \cup A_k, B_1 \cup \cdots \cup B_k}$. Thus,

$$|\Sigma_{A_1 \cup \cdots \cup A_k, B_1 \cup \cdots \cup B_k}| = \prod_{i=1}^{k} P_{A_i, B_i, (C_i, D_i)},$$

as required. \qed
B.2. Proof of Lemma 5.4

Proof. Recall that \( P_{\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_n\}, \{E, F\}} \) is the sum of the trek monomials of all trek systems with no sided intersection between \(\{a_1, \ldots, a_n\}\) and \(\{b_1, \ldots, b_n\}\) that only use \(E\) on the left and \(F\) on the right. For each such trek system, the trek starting at \(a_i\) only uses \(E_{ij}\) on the left, and the trek ending at \(b_j\) only uses \(F_{ij}\) on the right. On the other hand, the determinant of \( (P_{a_i, b_j, (E_{ij}, F_{ij})})_{i,j} \) is the sum of the trek monomials of all trek systems with no sided intersection between \(\{a_1, \ldots, a_n\}\) and \(\{b_1, \ldots, b_n\}\) for which the trek starting at \(a_i\) only uses \(E_{ij}\) on the left, and the trek ending at \(b_j\) only uses \(F_{ij}\) on the right. Therefore, the two quantities are equal.

\[ \square \]

B.3. Proof of Proposition 5.7

Proof. We will show that the determinant of the matrix with entries
\[
(\prod_{x \in \text{pa}(u) \cup \{x\}} P_{\text{pa}(u) \cup \{x\}, \text{pa}(u) \cup \{x\}})_{u \in \text{pa}(i) \cup \{j\}, x \in \text{pa}(i) \cup \{i\}}
\]
is divisible by \( P_{\text{pa}(u) \cup \{j\}, \text{pa}(u) \cup \{i\}, \text{pa}(i) \cup \{j\}, V} \). Combinatorially, this means that there is a \((\text{pa}(i) \cup \{j\}, V)\)-restricted trek separation between the sets \(\text{pa}(i) \cup \{j\}\) and \(\text{pa}(i) \cup \{j\}\). Indeed, they are \((\text{pa}(i) \cup \{j\}, V)\)-restricted trek separated by \((\emptyset, \text{pa}(i))\).

We begin by showing that the sets \((\text{pa}(u), \text{pa}(u)), (u, x)\) for \(u \in \text{pa}(i) \cup \{j\}\) and \(x \in \text{pa}(i) \cup \{i\}\) satisfy the swapping property. Firstly, consider a system of treks with no sided intersection between \(\text{pa}(u) \cup \{u\}\) and \(\text{pa}(u) \cup \{x\}\). Suppose that in this system it is not the case that \(\text{pa}(u)\) is connected to \(\text{pa}(u)\) and \(u\) is connected to \(x\). Then, there must exist a trek between \(u\) and an element from \(\text{pa}(u)\). Since \(u\) is ancestral, the left side of this trek has to end in a directed edge. That means that the left side of this trek contains an element from \(\text{pa}(u)\), which is impossible since this creates a sided intersection on the left side of this system. Therefore, we have a contradiction, and any such trek system connects \(\text{pa}(u)\) to \(\text{pa}(u)\) and \(u\) to \(x\).

Now, suppose that we have two systems of treks with no sided intersection between \(\text{pa}(u) \cup \{u\}\) and \(\text{pa}(u) \cup \{i\}\). Call them \(T_1\) and \(T_2\). In both systems, the treks connecting \(u\) and \(x\) need to start with a bidirected edge at \(u\) or with a directed edge away from \(u\) in order to avoid intersections on the left. We need to show that we can exchange the part connecting \(\text{pa}(u)\) to \(\text{pa}(u)\) in \(T_1\) with the corresponding part of \(T_2\), thereby obtaining two new trek systems with no sided intersection. Suppose for contradiction that once we make such an exchange, we get a sided intersection. Then, one of the treks from \(u\) to \(x\) gets a sided intersection with a trek from \(\text{pa}(u)\) to \(\text{pa}(u)\). Since the former trek has the form \(u \leftrightarrow \cdots \rightarrow x\), the created intersection has to be on its right side. Switch the tails of the two intersecting treks. We get a trek of the form \(u \leftrightarrow \cdots \rightarrow z \in \text{pa}(u)\). But this is a contradiction to \(u\) being ancestral.

We have shown that the sets \((\text{pa}(u), \text{pa}(u)), (u, x)\) for \(u \in \text{pa}(i) \cup \{j\}\) and \(x \in \text{pa}(i) \cup \{i\}\) satisfy the swapping property. We now show that
\[
|\prod_{x \in \text{pa}(u) \cup \{x\}} P_{\text{pa}(u) \cup \{x\}, \text{pa}(u) \cup \{x\}}| = P_{\text{pa}(u), \text{pa}(u), \text{pa}(u) \cup \{u\}, V}.
\]

Note that \(|\prod_{x \in \text{pa}(u) \cup \{x\}} P_{\text{pa}(u) \cup \{x\}, \text{pa}(u) \cup \{x\}}| = P_{\text{pa}(u) \cup \{u\}, \text{pa}(u) \cup \{x\}}\). Since the sets \((\text{pa}(u), \text{pa}(u)), (u, x)\) for \(u \in \text{pa}(i) \cup \{j\}\) and \(x \in \text{pa}(i) \cup \{i\}\) satisfy the swapping property, every trek system with no sided intersection between \(\text{pa}(u) \cup \{u\}\) and \(\text{pa}(u) \cup \{x\}\) splits into a trek system connecting \(\text{pa}(u)\) and \(\text{pa}(u)\) and a single trek connecting \(u\) and \(x\). The latter trek only has the vertex \(u\) on its left side. In other words, it starts either with a bidirected edge at \(u\) or with a directed edge pointing away from \(u\).
On the other hand we claim that every trek system connecting $pa(u)$ to $pa(u)$ with no sided intersection, and every trek from $u$ to $x$ that only has $u$ on the left can be combined into a trek system connecting $pa(u) \cup \{u\}$ and $pa(u) \cup \{x\}$ with no sided intersection. Suppose for contradiction that the combination gives a sided intersection. So, there is a trek from $a \in pa(u)$ to $b \in pa(u)$ that has sided intersection with the considered trek from $u$ to $x$. The intersection cannot be on the left since otherwise we would have a loop $u \to \cdots \to a \to u$ which is not allowed. Thus, there is intersection on the right. Swapping the right tails then gives a trek $u(\leftrightarrow) \to \cdots \to b$. But since $u$ is ancestral, we know that every trek between $u$ and its parents has to end with a directed edge at $u$. We have arrived at a contradiction and, thus, the claimed combination into a trek system connecting $pa(u) \cup \{u\}$ and $pa(u) \cup \{x\}$ with no sided intersection is possible. This proves (B.1).

Finally, it remains to show that

$$\det(P_{u,x,(u,V)})_{u \in pa(i) \cup \{j\}, x \in pa(i) \cup \{i\}} = P_{pa(i) \cup \{j\}, pa(i) \cup \{i\}, (pa(i) \cup \{j\}, V)}.$$

But this equality follows directly from Lemma 5.3. \qed

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