Common closed neighbourhood prime labeling

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Abstract. Let \( G = (V_G, E_G) \) be a connected graph of order \( n \). A bijection \( g: V_G \rightarrow \{1, 2, 3, \ldots, n\} \) is said to be prime labeling if for each two distinct vertices \( a, b \in V_G \) which \( a \) is adjacent to \( b \), \( \gcd(g(a), g(b)) = 1 \). A graph that satisfies the prime labeling is called a prime graph. Graph \( G \) is a neighbourhood prime graph if there is a bijection \( g: V_G \rightarrow \{1, 2, 3, \ldots, n\} \) so that for each vertex \( a \in V_G \) with \( \deg(a) > 1 \), \( \gcd(g(b): b \in N(a)) = 1 \). Graphs that have neighbourhood-prime labeling are called neighbourhood prime graphs. In this paper, we introduce a new variant of labeling as a development of prime labeling and neighbourhood prime labeling called common closed neighbourhood prime labeling. A bijection \( g: V_G \rightarrow \{1, 2, 3, \ldots, n\} \) is said to be common closed neighbourhood prime labeling if for each two distinct vertices \( a, b \in V_G \) which \( a \) is adjacent to \( b \), \( \gcd(g(c): c \in N[a, b]) = 1 \), where \( N[a, b] = N[a] \cap N[b] \). A common closed- neighbourhood prime graph is a graph that satisfies the common closed neighborhood prime labeling. Also, we studied an algorithm for some graphs which admits common closed neighbourhood prime labeling.

1. Introduction
The scope of graph theory is quite broad, one of which is graph labeling. Graph labeling was introduced in the mid-1960s. Graph labeling is growing very fast. In 50 years, over 200 techniques of graph labeling have been studied in more than 2800 papers. Generally, graph labeling could be implemented in numerous fields such as coding theory, radar, x-ray crystallography, astronomy, database management, circuit design, and so on [1].

One type of graph labeling is prime labeling. The idea of prime labeling was introduced by Tout et al. in 1982 [2]. One of the lastest of prime labeling is prime labeling of split graph of cycle \( C_n \) by S. Lavanya and Dr. V. Ganesan in October 2019 [3]. So far, there have been various expansion from prime labeling, one of them is neighbourhood prime labeling. Neighbourhood prime labeling was introduced by S.K. Patel and N.P. Shrimailin their paper in 2015. They examined Neighbourhood Prime Labeling on path graphs, cycle graphs, helmet graphs, closed helmet graphs, and floral graphs [4]. Furthermore, in this paper we develop a new type of neighbourhood prime labeling called common closed neighbourhood prime labeling. We also investigate several graphs that admit common closed neighbourhood prime labeling process.
2. Preliminary concepts

The definition of graph labeling and two preceding labelings (prime labeling and neighbourhood prime labeling) are shown in this section. These two earlier labelings encouraged a new type of graph labeling presented in this paper.

Chatrand et al. define graph labeling as in Definition 2.1.

Definition 2.1. [5] Graph labeling is a mapping from the elements of a graph $G$ (vertices and edges) to the label with special conditions.

One type of graph labeling is Prime Labeling which define by Tout et al. [1] as in Definition 2.2.

Definition 2.2. [2] Let $G = (V_G, E_G)$ be a connected graph of order $n$. A bijection function $g: V_G \rightarrow \{1,2,3, ..., n\}$ is named prime labeling if for any two distinct vertices $a, b \in V_G$ which $a$ is adjacent to $b$, $gcd(g(a), g(b)) = 1$. A prime graph is a graph which have prime labeling.

Neighbourhood prime labelings is defined as in Definition 2.3 was introduced by Patel and Shrimali.

Definition 2.3. [4] A connected graph $G = (V_G, E_G)$ of order $n$ is called a neighbourhood prime graph if there is a bijection $g: V_G \rightarrow \{1,2,3, ..., n\}$ so that for each vertex $a \in V_G$ with $\text{deg}(a) > 1$, $g(b): b \in N(a)$ are relatively prime. In shorts a neighbourhood prime graph is a graph which have neighbourhood prime labeling.

Next, we present the definition of star graph and its development, which will be labeled by common closed neighbourhood prime labeling.

Definition 2.4. [6] An $n$-vertex star $S_n$ or also known as $K_{1,n-1}$ is a complete bipartite graph with vertexes set can be partitioned into two subset. One subset contains a vertex and the other subset contains $n−1$ vertices, where the vertex in one subset is joined to every $n−1$ vertices in the other subset.

Star graph which developed by joining each leaf vertex to form cycle is said to be wheel graph. General definition of wheel graph as follows:

Definition 2.5. [6] Wheel graph $W_n$ define as the join $C_n$ and $K_1$, and denoted by $C_n + K_1$. The vertex of $K_1$ is called apex vertex. Every vertex of $C_n$ is called rim vertex and every edge of $C_n$ is called rim edge.

A wheel graph where each rim vertex is connected by trivial graph $K_1$ is called helm graph. Then the vertex of $K_1$ is known as pendant vertex. The definition of helm graph is presented in Definition 2.6.

Definition 2.6. [7] Helm graph $H_n$ is obtained by connecting a pendant edge to each rim vertex of wheel graph $W_n$.

If every two different pendant vertex on the helm graph are connected to form cycle, then the formed graph is called closed helmet graph which defined as follows:

Definition 2.7. [7] Closed helm graph $CH_n$ is the graph obtained by connecting each pendant vertex of helm graph $H_n$ to form a cycle.

3. Main results

Based on the meaning of prime and neighbourhood prime labeling, we develop a new type of graph labeling define it as follows:

Definition 2.8. Let $G = (V_G, E_G)$ be a connected graph of order $n$. A bijection function $g: V_G \rightarrow \{1,2,3, ..., n\}$ is said common closed neighbourhood prime labeling if two arbitrary distinct vertices $a, b \in V_G$ where $a$ is adjacent to $b$, $gcd\{g(c): c \in N[a,b]\} = 1$, where $N[a, b] = N[a] \cap N[b]$. A
A graph that satisfy common closed neighbourhood-prime labeling is called common closed-neighbourhood-prime graph.

Where $N[a, b]$ is common closed neighbourhood of $a$ and $b$, which is the intersection between closed neighbourhood of $a$ and closed neighbourhood of $b$. It can be written as $N[a, b] = N[a] \cap N[b]$.

Furthermore, we showed star graph $S_n$ and its development (wheel $W_n$, helm $H_n$, and closed helm $CH_n$) are common closed neighbourhood prime graphs by providing steps in the form of an algorithm.

Algorithm
Common Closed Neighbourhood Prime of Star $S_n$.

1. Let $G$ be a star $S_n$.
   Then $V_G = \{a_1, a_2, ..., a_n\}$, $E_G = \{a_1a_2, a_1a_3, ..., a_1a_{n-1}\}$, $|V_G| = n$, and $|E_G| = n - 1$.

2. Investigate two adjacent vertices $a_i$ and $a_j$, where $i \in \{2, 3, ..., n\}$.
   Closed neighbourhood of $a_i$ is $N[a_i] = \{a_i, a_2, a_3, ..., a_n\}$, while closed neighbourhood of $a_i$ is
   $N[a_i] = \{a_i, a_1\}$. Then, we get $N[a_i, a_j] = \{a_i, a_j\}$.

3. Define a bijective $g: V_G \rightarrow \{1, 2, 3, ..., n\}$ with $g(a_i) = i$, $i \in \{1, 2, ..., n\}$.

4. Check $gcd\{g(a_1), g(a_2)\}$.
   If $gcd\{g(a_1), g(a_2)\} = 1$, then $G$ is a common closed neighbourhood prime graph. Since $g(a_1) = 1$, so $gcd\{g(a_1), g(a_2)\} = 1$ for any value of $g(a_2)$.

As a result, we got that $G$, in this case $S_n$ is a common closed neighbourhood prime graph. The example of common closed neighbourhood prime labeling of star graph for $n = 9$ is showed in Figure 1. Labeling the vertices of $S_n$ by $g(a_i) = i$, for $i \in \{1, 2, ..., 9\}$.

![Figure 1. Common closed neighbourhood prime labelling of $S_9$.](image)

Algorithm
Common Closed Neighbourhood Prime of Wheel $W_n$.

1. Let $G$ be a wheel graph $W_n$.
   Then $V_G = \{a_0, a_1, a_2, ..., a_n\}$, $E_G = \{a_0a_1, a_0a_2, ..., a_0a_n, a_1a_2, a_2a_3, ..., a_{n-1}a_n, a_na_1\}$, $|V_G| = n + 1$, and $|E_G| = 2n$ with $a_0$ is a vertex of degree $n$ and $a_1, a_2, ..., a_n$ are vertices of degree $3$.

2. Investigate two adjacent vertices in $G$, divided into two cases:
   1. Two adjacent vertices $a_i$ and $a_{i+1}$, $i \in \{1, 2, ..., n\}$ which form rim edge.
      Closed neighbourhood of $a_i$ is $N[a_i] = \{a_0, a_i, a_{i-1}, a_{i+1}\}$, while closed neighbourhood of $a_{i+1}$ is $N[a_{i+1}] = \{a_0, a_i, a_{i+1}, a_{i+2}\}$. Then, we get $N[a_i, a_{i+1}] = \{a_0, a_i, a_{i+1}\}$. 
2. Two adjacent vertices $a_0$ and $a_i$, $i \in \{1, 2, ..., n\}$ which form apex edge.

Closed neighbourhood of $a_0$ is $N[a_0] = \{a_0, a_1, a_2, ..., a_n\}$, while closed neighbourhood of $a_i$ is $N[a_i] = \{a_0, a_i, a_{i-1}, a_{i+1}\}$. Then, we get $N[a_0, a_i] = \{a_0, a_i, a_{i-1}, a_{i+1}\}$.

**Step 3**

Define a bijection: $V_G \rightarrow \{1, 2, ..., n + 1\}$ with $g(a_i) = i + 1$, $i \in \{0, 1, 2, ..., n\}$.

**Step 4**

Check $gcd\{g(a_0), g(a_i), g(a_{i+1})\}$ and $gcd\{g(a_0), g(a_{i-1}), g(a_i), g(a_{i+1})\}$.

If obtained both value is 1, $i \in \{1, 2, ..., n\}$, then $G$ is a common closed neighbourhood prime graph. Notice that $g(a_0) = 1$ so $gcd\{g(a_0), g(a_{i-1}), g(a_i), g(a_{i+1})\} = 1$ and $gcd\{g(a_0), g(a_i), g(a_{i+1})\} = 1$, $i \in \{1, 2, ..., n\}$.

As a result, we got that $G$, in this case $W_n$ is a common closed neighbourhood prime graph. Figure 2 shows the example of common closed neighbourhood prime labeling of wheel graph for $n = 9$. Labeling the vertices of $W_9$ by $g(a_i) = i + 1$, for $i \in \{0, 1, 2, ..., 9\}$.

![Figure 2. Common closed neighbourhood prime labelling of $W_9$.](image)

**Algorithm**

Common Closed Neighbourhood Prime of Helm $H_n$

**Step 1**

Suppose $G$ be a helm $H_n$.

The vertices set of $G$ is $V_G = \{a_0, a_1, a_2, ..., a_n, b_1, b_2, ..., b_n\} = A \cup B$, with $A = \{a_0, a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$. The edges set of $G$ is $E_G = \{a_0a_1, a_0a_2, ..., a_0a_n, a_1a_2, a_2a_3, ..., a_{n-1}a_n, a_na_1, a_1b_1, a_2b_2, ..., a_nb_n\}$. Then, we get that $|V_G| = 2n + 1$ and $|E_G| = 3n$.

**Step 2**

Investigate two adjacent vertices in $G$, divided into three cases:

1. Two adjacent vertices $a_i$ and $a_{i+1}$, $i \in \{1, 2, ..., n\}$ which form rim edge.

Closed neighbourhood of $a_i$ is $N[a_i] = \{a_0, a_{i-1}, a_i, a_{i+1}, b_1\}$, while closed neighbourhood of $a_{i+1}$ is $N[a_{i+1}] = \{a_0, a_i, a_{i+1}, a_{i+2}, b_{i+1}\}$. Then, we get $N[a_i, a_{i+1}] = \{a_0, a_i, a_{i+1}\}$.

2. Two adjacent vertices $a_0$ and $a_i$, $i \in \{1, 2, ..., n\}$ which form apex edge.

Closed neighbourhood of $a_0$ is $N[a_0] = \{a_0, a_1, a_2, ..., a_n\}$, while closed neighbourhood of $a_i$ is $N[a_i] = \{a_0, a_{i-1}, a_i, a_{i+1}, b_1\}$. Then, we get $N[a_0, a_i] = \{a_0, a_i, a_{i-1}, a_{i+1}\}$.

3. Two adjacent vertices $a_i$ and $b_i$, $i \in \{1, 2, ..., n\}$ which form the pedant edge.

Closed neighbourhood of $a_i$ is $N[a_i] = \{a_0, a_{i-1}, a_i, a_{i+1}, b_1\}$, while closed neighbourhood of $b_i$ is $N[b_i] = \{a_i, b_1\}$. Then, we get $N[a_i, b_i] = \{a_i, b_i\}$.

**Step 3**

Define a bijection: $g: V_G \rightarrow \{1, 2, ..., 2n + 1\}$ which divided into: $g: A \rightarrow \{1, 3, 5, ..., 2n + 1\}$ with:

$g(a_i) = 2i + 1, i \in \{0, 1, 2, ..., n\}$. 


\( g : B \to \{2, 4, 6, ..., 2n\} \) with:
\[
g(b_i) = 2i, i \in \{1, 2, ..., n\}.
\]

**Step 4**

Check \( gcd\{g(a_0), g(a_i), g(a_{i+1})\} \), \( gcd\{g(a_0), g(a_i), g(a_{i-1}), g(a_{i+1})\} \), and \( gcd\{g(a_i), g(b_i)\} \).

If obtained all three value is 1, \( i \in \{1, 2, ..., n\} \) then \( G \) is a common closed neighbourhood prime graph. Since \( g(a_0) = 1 \), so \( gcd\{g(a_0), g(a_i), g(a_{i-1}), g(a_{i+1})\} \) and \( gcd\{g(a_0), g(a_i), g(a_{i+1})\} \) must be 1 with \( i \in \{1, 2, ..., n\} \). Then notice that \( g(a_i) = 2i + 1 \) and \( g(a_i) = 2i \), so \( gcd\{g(a_i), g(b_i)\} = 1, i \in \{1, 2, ..., n\} \).

As a result, we got that \( G \), in this case \( H_n \), is a common closed neighbourhood prime graph. Figure 3 is the illustration of common closed neighbourhood prime labeling of helm graph for \( n = 9 \). Labeling the vertices of \( H_9 \) by using:
\[
\begin{align*}
g(a_i) &= 2i + 1, i \in \{0, 1, 2, ..., 9\}.
g(b_i) &= 2i, i \in \{1, 2, ..., 9\}.
\end{align*}
\]

**Algorithm**

Common Closed Neighbourhood Prime of Closed Helm \( CH_n \)

**Step 1**
Suppose \( G \) be a closed helm \( CH_n \).

The vertices set of \( G \) is \( V_G = \{a_0, a_1, a_2, ..., a_n, b_1, b_2, ..., b_n\} = A \cup B \), with \( A = \{a_0, a_1, a_2, ..., a_n\} \), and \( B = \{b_1, b_2, ..., b_n\} \). The edges set of \( G \) is \( E_G = \{a_0a_1, a_0a_2, ..., a_0a_n, a_1a_2, a_2a_3, ..., a_{n-1}a_n, a_0a_3, a_1b_1, a_2b_2, ..., a_nb_n, b_1b_2, b_2b_3, ..., b_{n-1}b_n, b_nb_1\} \).

Then, we get that \( |V_G| = 2n + 1 \) and \( |E_G| = 4n \).

**Step 2**

Investigate two adjacent vertices in \( G \), divided into four cases:

1. Two adjacent vertices \( a_i \) and \( a_{i+1} \), \( i \in \{1, 2, ..., n\} \) which form rim edge.

   Closed neighbourhood of \( a_i \) is \( N[a_i] = \{a_0, a_{i-1}, a_i, a_{i+1}, b_i\} \), while closed neighbourhood of \( a_{i+1} \) is \( N[a_{i+1}] = \{a_0, a_i, a_{i+1}, a_{i+2}, b_{i+1}\} \). Then, we get \( N[a_i, a_{i+1}] = \{a_0, a_i, a_{i+1}\} \).

2. Two adjacent vertices \( a_0 \) and \( a_i \), \( i \in \{1, 2, ..., n\} \) which form apex edge.

   Closed neighbourhood of \( a_0 \) is \( N[a_0] = \{a_0, a_3, a_2, ..., a_n\} \), while closed neighbourhood of \( a_i \) is \( N[a_i] = \{a_0, a_{i-1}, a_i, a_{i+1}, b_i\} \). Then, we get \( N[a_0, a_i] = \{a_0, a_{i-1}, a_i, a_{i+1}\} \).

3. Two adjacent vertices \( a_i \) and \( b_i \), \( i \in \{1, 2, ..., n\} \) which form pedant edge.
Closed neighbourhood of $a_i$ is $N[a_i] = \{a_0, a_{i-1}, a_i, a_{i+1}, b_i\}$, while closed neighbourhood of $b_i$ is $N[b_i] = \{a_i, b_{i-1}, b_i, b_{i+1}\}$. Then, we get $N[a_i, b_i] = \{a_i, b_i\}$.

4. Two adjacent vertices $b_i$ and $b_{i+1}$, $i \in \{1, 2, ..., n\}$ which form outer rim edge.

Closed neighbourhood of $b_i$ is $N[b_i] = \{a_i, b_{i-1}, b_i, b_{i+1}\}$, while closed neighbourhood of $b_{i+1}$ is $N[b_{i+1}] = \{a_{i+1}, b_i, b_{i+1}, b_{i+2}\}$. Then, we get $N[b_i, b_{i+1}] = \{b_i, b_{i+1}\}$.

**Step 3**
Define a bijective $g: V_G \rightarrow \{1, 2, 3, ..., 2n + 1\}$ which divided into:

$g: A \rightarrow \{1, 2, 4, 6, ..., 2n\}$ as follows:

\[
g(a_i) = \begin{cases} 1 & \text{for } i = 0 \\ 2i & \text{for } i \in \{1, 2, ..., n\} \end{cases}
\]

$g: B \rightarrow \{3, 5, ..., 2n + 1\}$ as follows:

\[
g(b_i) = 2i + 1 \text{ for } i \in \{1, 2, ..., n\}.
\]

**Step 4**
Check $gcd\{g(a_0), g(a_i), g(a_{i+1})\}$, $gcd\{g(a_0), g(a_{i-1}), g(a_i), g(a_{i+1})\}$, $gcd\{g(a_i), g(b_i)\}$, and $gcd\{g(b_i), g(b_{i+1})\}$.

If the value obtained from all four is 1 with $i \in \{1, 2, ..., n\}$, then $G$ is a common closed neighbourhood prime graph. Since $g(a_0) = 1$, so $gcd\{g(a_0), g(a_{i-1}), g(a_i), g(a_{i+1})\}$ and $gcd\{g(a_0), g(a_i), g(a_{i+1})\}$ must be 1, $i \in \{1, 2, ..., n\}$. Then notice that $g(b_i) = 2i + 1$ and $g(a_i) = 2i$, so $gcd\{g(a_i), g(b_i)\} = 1, i \in \{1, 2, ..., n\}$.

There’s a two cases to determine the value of $gcd\{g(b_i), g(b_{i+1})\}$.

1. If $2n + 1$ isn’t a multiple of 3.

Because it is known that $g(b_i) = 2i + 1$ and $g(b_{i+1}) = 2(i + 1) + 1 = 2i + 3$, so $gcd\{g(b_i), g(b_{i+1})\} = 1, i \in \{1, 2, ..., n\}$.

2. If $2n + 1$ is a multiple of 3.

Since $g(b_1) = 2(1) + 1 = 3$ and $g(b_n) = 2n + 1$, so $gcd\{g(b_n), g(b_1)\} \neq 1$. Then we have to do the following step 5.

**Step 5**
Assumed that $gcd\{g(b_n), g(b_1)\} \neq 1$, so exchange the label of $g(b_n)$ and $g(a_n)$.

As a result, we got that $G$, in this case $CH_n$ is a common closed neighbourhood prime graph. The example of common closed neighbourhood prime labeling of closed helm graph for $n = 7$ can be showed below. Labeling the vertices of $CH_7$ by using:

\[
g(a_i) = \begin{cases} 1 & \text{for } i = 0 \\ 2i & \text{for } i \in \{1, 2, ..., 7\} \end{cases}
\]

\[
g(b_i) = 2i + 1 \text{ for } i \in \{1, 2, ..., 7\}.
\]

So, we get the following labeled graph:

![Figure 4. Vertices labelling of $CH_7$.](image-url)
Checking the labels, then we get that $\gcd\{g(b_1), g(b_7)\} \neq 1$. So we should exchange the label of $b_7$ and $a_7$.

After we exchange the label of $b_7$ and $a_7$, we got the common closed neighbourhood prime labeling of $CH_7$.

Based on the definitions of prime labeling and common closed neighbourhood prime labeling, we have got the relation between prime graphs and common closed neighbourhood prime graphs which stated in the following theorem:

**Theorem 3.1.** Given a connected graph $G$ of order $n \geq 2$. If $G$ is a prime graph then $G$ is a common-closed-neighbourhood prime graph.

**Proof.** We have a prime graph $G$. Let $a$ and $b$ be two adjacent vertices in $V_G$. Since $G$ is a prime graph, $N[a]$ must be relatively prime. Likewise, $N[b]$ must also be relatively prime. So $N[a, b] = N[a] \cap N[b]$ must also be relatively prime. As a result, $G$ also admit common closed neighbourhood prime labelling.

From the proof above, we know that the prime graphs are a subset of common closed neighbourhood prime graph. In other words, a graph that satisfies prime labeling must admit common closed neighbourhood prime labeling, but it does not apply otherwise. An example of a common closed neighbourhood prime graph which is not a prime graph is wheel graph $W_n$ where $n$ is odd.

**4. Conclusion**

From the description above, it can be seen that the process of common closed neighbourhood prime labeling of star graph can be done in four steps. However, the process of common closed neighbourhood prime labeling of the development of star graph (wheel graph, helm graph, and closed helm graph) can’t always be done in four steps. For example in the process common closed neighbourhood prime labeling of $CH_7$ graph, we need the fifth step which is an exchange label between two vertices so that the common closed neighbourhood prime is achieved.
In this paper, we have derived algorithms which admit common closed neighbourhood prime labeling to star graph $S_n$ and its development which is wheel graph $W_n$, helm graph $H_n$, and closed helm graph $CH_n$.

Based on the conclusions above, we suggest the following:
1. For further research, can be discussed about common closed neighbourhood prime labeling of other operation graph.
2. Common closed neighbourhood prime labeling can be developed into total common closed neighbourhood prime labeling by labeling the graph’s vertices and edges.

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