Local well-posedness of the coupled Yang–Mills and Dirac system in temporal gauge

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Abstract
We consider the classical Yang–Mills system coupled with a Dirac equation in 3+1 dimensions in temporal gauge. Using that most of the nonlinear terms fulfill a null condition we prove local well-posedness for small data with minimal regularity assumptions. This problem for smooth data was solved forty years ago by Y. Choquet-Bruhat and D. Christodoulou. The corresponding problem in Lorenz gauge was considered recently by the author in [14].

Keywords Yang–Mills · Dirac equation · Local well-posedness · Temporal gauge

Mathematics Subject Classification 35Q40 · 35L70

1 Introduction and the main theorem
Let $G$ be the Lie group $SU(n, \mathbb{C})$ (the group of unitary matrices of determinant 1) and $g$ its Lie algebra $su(n, \mathbb{C})$ (the algebra of trace-free skew hermitian matrices) with Lie bracket $[X, Y] = XY - YX$ (the matrix commutator). For given $A_\alpha : \mathbb{R}^{1+3} \to g$ we define the curvature $F = F[A]$ by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],$$

where $\mu, \nu \in \{0, 1, 2, 3\}$ and $D_\mu = \partial_\mu + [A_\mu, \cdot]$.

Then the Yang–Mills system is given by

$$D^\mu F_{\mu\nu} = 0$$

in Minkowski space $\mathbb{R}^{1+3} = \mathbb{R}_t \times \mathbb{R}_x^3$, with metric $diag(-1, 1, 1, 1)$. Greek indices run over $\{0, 1, 2, 3\}$, Latin indices over $\{1, 2, 3\}$, and the usual summation convention is used. We use the notation $\partial_\mu = \frac{\partial}{\partial x_\mu}$, where we write $(x^0, x^1, x^2, x^3) = (t, x^1, x^2, x^3)$ and also $\partial_0 = \partial_t$. 

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This system is coupled with a Dirac spinor field $\psi : \mathbb{R}^{1+3} \to \mathbb{C}^4$. Let $T_a$ be the set of generators of $SU(n, \mathbb{C})$ and $A_\mu = A_\mu^a T_a$, $F_{\mu\nu} = F_{\mu\nu}^a T_a$, $[T^\lambda, T^\beta]_a = f^{ab\lambda}$.

For the following considerations and also for the physical background we refer to the monograph by Schwartz [15]. We also refer to the pioneering work for the Yang–Mills, Higgs and spinor field equations by Choquet-Bruhat and Christodoulou [3], and Schwarz and Sniatycki [16].

The kinetic Lagrangian with $N$ Dirac fermions and the Yang–Mills Lagrangian are given by $\mathcal{L} = \sum_{i=1}^N \bar{\psi}_i (i\gamma^\mu \partial_\mu - m) \psi_i$ and $\mathcal{L}_{YM} = -\frac{1}{4} (F_{\mu\nu}^a)^2$, respectively. Here $\bar{\psi} = \psi^\dagger 0$, where $\psi^\dagger$ is the complex conjugate transpose of $\psi$.

Here $\gamma^\mu$ are the (4x4) Dirac matrices given by $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\gamma^j = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then we consider the following Lagrangian for the (minimally) coupled system

$$\mathcal{L} = -\frac{1}{4} (F_{\mu\nu}^a)^2 + \sum_{i,j=1}^N \bar{\psi}_i (\delta_{ij} i\gamma^\mu \partial_\mu + \gamma^\mu A_\mu^a T_{ij}^a - m \delta_{ij}) \psi_j$$

where $T_{ij}^a \in \mathbb{C}$ are the entries of the matrix $T_a$.

The corresponding equations of motion are given by the following coupled Yang–Mills–Dirac system (YMD)

$$\partial^\mu F_{\mu\nu}^a + f^{abc} A_\mu^a F_{\mu\nu}^c = -(\psi_i, \gamma^0 \gamma^\nu T_{ij}^a \psi_j)$$

$$i \gamma^\mu \partial_\mu - m) \psi_i = -A_\mu^a \gamma^\mu T_{ij}^a \psi_j.$$ 

Using $D_\mu F_{\mu\nu} = \partial_\mu F_{\mu\nu} + [A_\mu^a, F_{\mu\nu}^b]$ and 

$$[A_\mu^a, F_{\mu\nu}^b] = [A_\mu^a T_{\lambda}^c, F_{\mu\nu}^b T_{\beta}^d] = A_\mu^a T_{\lambda}^c F_{\mu\nu}^b T_{\beta}^d = A_\mu^a F_{\mu\nu}^b T_{\lambda}^c T_{\beta}^d = A_\mu^a F_{\mu\nu}^b f_{\lambda\beta\gamma}^c,$$

we obtain the following system which we intend to treat:

$$D_\mu F_{\mu\nu} = -(\psi_j, \alpha^\nu T_{ij} \psi_j T_{ij}) \psi_j,$$

$$i \alpha^\mu \partial_\mu \psi_i = -A_\mu^a \alpha^\mu T_{ij}^a \psi_j,$$

if we choose $m = 0$ just for simplicity and define the matrices $\alpha^\mu = \gamma^0 \gamma^\mu$, so that $\alpha^0 = I_{4x4}$ and $\alpha_j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}$. $\alpha^\mu$ are hermitian matrices with $(\alpha^\mu)^2 = I_{4x4}$, $\alpha^j \alpha^k + \alpha^k \alpha^j = 0$ for $j \neq k$.

Setting $\nu = 0$ in (3) we obtain the Gauss-law constraint

$$\partial^j F_{j0} = -[A^j, F_{j0}] + \langle \psi_j, T_{ij} \psi_j \rangle T_{ij}.$$

The system is gauge invariant. Given a sufficiently smooth function $U : \mathbb{R}^{1+3} \to \mathcal{G}$ we define the gauge transformation $T$ by $TA_\mu = A'_\mu$, $T(A_1, A_2, A_3) = (A'_1, A'_2, A'_3)$, $T\psi = \psi'$, where

$$A_\alpha \mapsto A'_\alpha = UA_\alpha U^{-1} - (\partial_\alpha U)U^{-1}$$

$$\psi \mapsto \psi' = U \psi.$$
Following [2] and [5] in order to rewrite the Dirac equation we define the projections
\[ \Pi(\xi) := \frac{1}{2} (I_{4\times 4} + \frac{\xi^j \alpha^j}{|\xi|} I_{4\times 4}) \]
and \( \Pi_{\pm}(\xi) := \Pi(\pm \xi) \), so that \( \Pi_{\pm}(\xi)^2 = \Pi_{\pm}(\xi) \), \( \Pi_+(\xi) \Pi_-(\xi) = 0 \), \( \Pi_+(\xi) + \Pi_-(\xi) = I_{4\times 4} \), \( \Pi_\mp(\xi) = \Pi_{\mp}(-\xi) \). We obtain
\[ \alpha^j \Pi(\xi) = \Pi(\pm \xi) \alpha^j + \frac{\xi^j I_{4\times 4}}{|\xi|} . \tag{5} \]
Using the notation \( \Pi_{\pm} = \Pi_{\pm}(\nabla) \) we obtain
\[ -i \alpha^j \partial_j = |\nabla| \Pi_+ - |\nabla| \Pi_- , \tag{6} \]
where \(|\nabla|\) has symbol \(|\xi|\). Moreover defining the modified Riesz transform by \( R^j_{\pm} = \mp (\frac{\partial^j}{|\xi|}) \) with symbol \( \mp \frac{\xi^j}{|\xi|} \) and \( R_0^j = -1 \) the identity (5) implies
\[ \alpha^j \Pi_\pm = (\alpha^j \Pi_\pm) \Pi_\pm = \Pi_\mp \alpha^j \Pi_\pm - R^j_\pm \Pi_\pm , \tag{7} \]
If we define \( \psi_{i, \pm} = \Pi_\pm \psi_i \) we obtain by applying the projection \( \Pi_\pm \) and (6) the Dirac type equation in the form
\[ (i \partial_t \pm |\nabla|) \psi_{i, \pm} = \Pi_\pm (A^a_{\mu} \alpha^\mu T^0_{ij} \psi^j) : = H_{i, \pm}(A, \psi) . \tag{8} \]
The Yang–Mills equation (3) may be written as
\[ \Box A = \partial_t \partial^j A_k - [\partial^j A_k, A_\nu] - [A_\mu, \partial^\mu A_\nu] - [A^\mu, F_{\mu\nu}] - \langle \psi_i, \alpha^i T^0_{ik} \psi^k \rangle T_a . \]
From now on we impose the temporal gauge
\[ A_0 = 0 . \]
This implies the wave equation
\[ \Box A_j = \partial_j \partial^i A_k - [\partial^i A_k, A_j] - [A_i, \partial^j A_k] - [A^i, F_{ij}] - \langle \psi_i, \alpha^i T^0_{ik} \psi^k \rangle T_a \\
= \partial_j \partial^j A_k - [\partial^j A_k, A_j] - [A^i, \partial^j A_i] - [A^j, [A_i, A_j]] \\
- \langle \psi^i, \alpha^i T^0_{ik} \psi^k \rangle T_a \tag{9} \]
and
\[ 0 = \partial_t \partial^i A_k - [A_i, \partial^i A_k] - \langle \psi^i, \alpha^i T^0_{ik} \psi^k \rangle T_a . \]
Now we use the Hodge decomposition of \( A = (A_1, A_2, A_3) \) into its divergence-free and curl-free parts:
\[ A = A^{df} + A^{cf} , \]
where
\[ PA := A^{df} = |\nabla|^{-2} \nabla \times (\nabla \times A) \Leftrightarrow A^{df}_j = R^k (R_j A_k - R_k A^j) \]
and
\[ A^{cf} = -|\nabla|^{-2} \nabla (div A) \Leftrightarrow A^{cf}_j = -R_j R_k A^k . \]
Here \( R_j = \frac{\partial_j}{|\nabla|} \) is the Riesz transform.
Then we obtain the following system:
\[ \partial_t A^c = |\nabla|^{-2} \nabla [A_i, \partial_t A^i] + |\nabla|^{-2} \nabla \langle \psi^i, \alpha_k T^a_{ik} \psi^k \rangle T_a, \]
\[ \Box A^d_j = -P[div A^c, A_j] - 2P[A_i, \partial^i A_j] + P[A^i, \partial_j A_i] - P[A^i, [A_i, A_j]] - P(\psi^i, \alpha_k T^a_{ik} \psi^k) T_a \]
\[ (i \partial_t \pm |\nabla|) \psi_{i, \pm} = \Pi_{\pm}(A)_{ij} k^a T^a_{ij} \psi^j.\]

We want to solve the system (10)–(12) simultaneously for \( A^c, A^d, \) and \( \psi_{\pm}. \) So to pose the Cauchy problem for this system, we consider initial data for \( (A^c, A^d, \psi) \) at \( t = 0: \)
\[ A^d(0) = A^d_0, \ (\partial_t A^d)(0) = a^d_i, \ A^c(0) = a^c_i, \]
\[ \psi_{i, \pm}(0) = \psi_{i, \pm}^0 = \Pi_{\pm} \psi_i^0.\]

Let us make some historical remarks. As is well-known we may impose a gauge condition. Convenient gauges are the Coulomb gauge \( \partial^i A_j = 0, \) the Lorenz gauge \( \partial^\alpha A_\alpha = 0 \) and the temporal gauge \( A_0 = 0. \) It is well-known that for the low regularity well-posedness problem for the Yang–Mills equation a null structure for some of the nonlinear terms plays a crucial role. This was first detected by Klainerman and Machedon [6], who proved global well-posedness in the case of three space dimensions in temporal and in Coulomb gauge in energy space. The corresponding result in Lorenz gauge, where the Yang–Mills equation can be formulated as a system of nonlinear wave equations, was shown by Selberg and Tesfahun [17], who discovered that also in this case some of the nonlinearities have a null structure. Tesfahun [20] improved the local well-posedness result to data without finite energy, namely for \( (A(0), (\partial_t A)(0)) \in H^s \times H^{s-1} \) and \( (F(0), (\partial_t F)(0)) \in H^r \times H^{r-1} \) with \( s > \frac{6}{7} \) and \( r > -\frac{1}{14}, \) by discovering an additional partial null structure. Local well-posedness in energy space was also shown by Oh [11] using a new gauge, namely the Yang–Mills heat flow. He was also able to show that this solution can be globally extended [12]. Tao [19] showed local well-posedness for small data in \( H^s \times H^{s-1} \) for \( s > \frac{3}{4} \) in temporal gauge.

The coupled Yang–Mills and Dirac system in Lorenz gauge was considered from the physical point of view by M. D. Schwartz [15]. Local existence for smooth initial data, uniqueness in suitable gauges under appropriate conditions on the data and global existence for small and smooth data, i.e. \( (A(0), (\partial_t A)(0), F(0), (\partial_t F)(0), \psi(0)) \in H^s \times H^{s-1} \times H^{s-1} \times H^{s-2} \times H^{s} \) with \( s \geq 2 \) was proven by Y. Choquet-Bruhat and D. Christodoulou [3], and G. Schwarz and J. Sniatycki [16].

In [14] the author considered this problem in Lorenz gauge and obtained local well-posedness for \( s > \frac{3}{4}, r > -\frac{1}{8} \) and \( l > \frac{3}{8}, \) where existence holds in \( A \in C^0([0, T], H^s) \cap C^1([0, T], H^{s-1}), \) \( F \in C^0([0, T], H^r) \cap C^1([0, T], H^{r-1}), \) \( \psi \in C^0([0, T], H^l) \) and (existence and) uniqueness in a certain subspace of Bourgain–Klainerman–Machedon type \( X^{s,b}. \) We relied on Selberg-Tesfahun [17] and Tesfahun’s result [20], who detected the null structure in most—unfortunately not all—critical nonlinear terms. We also made use of the methods used by Huh and Oh [5] for the Chern–Simons–Dirac equation.

We now study the Yang–Mills–Dirac system in temporal gauge for low regularity data, which fulfill a smallness assumption, which reads as follows
\[ \|A(0)\|_{H^s} + \|\partial_t A(0)\|_{H^{s-1}} + \|\psi(0)\|_{H^l} < \epsilon \]
with a sufficiently small \( \epsilon > 0, \) under the assumption \( s > \frac{3}{4} \) and \( l > \frac{1}{4}. \) We obtain a solution which satisfies \( A \in C^0([0, 1], H^s) \cap C^1([0, 1], H^{s-1}), \) \( \psi \in C^0([0, 1], H^l). \) Uniqueness holds in spaces of Bourgain–Klainerman–Machedon type. Thus the parameter
can be weakened compared to the result in Lorenz gauge at the expense of a smallness assumption on the data. The basis for our results is Tao’s paper [19], who considered the corresponding result for the Yang–Mills equation. We carry over his result to the more general Yang–Mills–Dirac equation. The result relies on the null structure of all the critical bilinear terms. We review this null structure which was partly detected already by Klainerman–Machedon [7] in the situation of the Lorenz gauge. The necessary estimates for those nonlinear terms, which contain no terms depending on \( \psi \), in spaces of \( X^{s,b} \)-type then reduce essentially to Tao’s result [19]. One of these estimates is responsible for the small data assumption. Because these local well-posedness results can initially only be shown under the condition that the curl-free part \( A^{cf} \) of \( A \) (as defined below) vanishes for \( t = 0 \) we have to show that this assumption can be removed by a suitable gauge transformation (Lemma 4.1) which preserves the regularity of the solution. This uses an idea of Keel and Tao [19]. A proof for the Yang–Mills and Yang–Mills–Higgs case was given by [14].

Our main theorem reads as follows:

**Theorem 1.1** Let \( s > \frac{3}{4} \), \( l > \frac{1}{4} \), \( s \geq l \geq s - \frac{1}{2} \) and \( 2s - l > 1 \). Let \( a_0 \in H^s(\mathbb{R}^3) \), \( a_1 \in H^{s-1}(\mathbb{R}^3) \), \( \psi_0 \in H^l(\mathbb{R}^3) \) be given satisfying the Gauss law constraint \( \partial^j a^1_j = -\partial^j a^1_j + \langle \psi_0, T_{ij}^a \rangle \psi_0^j \rangle T_{ij} \). Assume

\[
\|a_0\|_{H^s} + \|a_1\|_{H^{s-1}} + \|\psi_0\|_{H^l} \leq \epsilon,
\]

where \( \epsilon > 0 \) is sufficiently small. Then the Yang–Mills–Dirac equations (3), (4) in temporal gauge \( A_0 = 0 \) with initial conditions

\[
A(0) = a_0, \quad (\partial_t A)(0) = a_1, \quad \psi(0) = \psi_0,
\]

where \( A = (A_1, A_2, A_3) \), has a unique local solution \( A = A^d_f + A^d_f + A^{cf} \) and \( \phi = \phi_+ + \phi_- \), where

\[
A^d_f \in X^{s+\frac{3}{2}, 0}_0 + [0, 1], \quad A^{cf} \in X^{s+\frac{1}{2}, 1}_0 + [0, 1], \quad \partial_t A^{cf} \in C^0([0, 1], H^{s-1}), \quad \psi_0 \in X^{s, 0}_0 + [0, 1],
\]

where these spaces are defined below. This solution fulfills

\[
A \in C^0([0, 1], H^s(\mathbb{R}^3)) \cap C^1([0, 1], H^{s-1}(\mathbb{R}^3)), \quad \psi \in C^0([0, 1], H^l(\mathbb{R}^3)).
\]

**Definition 1.1** The standard spaces \( X^{s,b}_{\pm} \) of Bourgain–Klainerman–Machedon type belonging to the half waves are the completion of the Schwarz space \( S(\mathbb{R}^d) \) with respect to the norm

\[
\|u\|_{X^{s,b}_\pm} = \|\langle \xi \rangle^{s} \langle \tau \mp |\xi| \rangle^b \hat{u}(\tau, \xi)\|_{L^{2}_{\tau \xi}}.
\]

Similarly we define the wave-Sobolev spaces \( X^{s,b}_{|\tau|=|\xi|} \) or for short \( H^{s,b} \) with norm

\[
\|u\|_{H^{s,b}} = \|u\|_{X^{s,b}_{|\tau|=|\xi|}} = \|\langle \xi \rangle^{s} \langle |\tau| - |\xi| \rangle^b \hat{u}(\tau, \xi)\|_{L^{2}_{\tau \xi}}
\]

and also \( X^{s,b}_{\tau=0} \) with norm

\[
\|u\|_{X^{s,b}_{\tau=0}} = \|\langle \xi \rangle^{s} \langle \tau \rangle^b \hat{u}(\tau, \xi)\|_{L^{2}_{\tau \xi}}.
\]

We also define \( X^{s,b}_{\pm}([0, T]) \) as the space of the restrictions of functions in \( X^{s,b}_{\pm} \) to \([0, T] \times \mathbb{R}^3\) and similarly \( X^{s,b}_{\tau=0}([0, T]) \) and \( X^{s,b}_{|\tau|=|\xi|}([0, T]) \). We frequently use the estimates \( \|u\|_{X^{s,b}_\pm} \leq \|u\|_{X^{s,b}_{|\tau|=|\xi|}} \) for \( b \leq 0 \) and the reverse estimate for \( b \geq 0 \).
We recall the fact that
\[ X^{s,b}_\pm[0, T] \hookrightarrow C^0([-T, T]; H^s) \quad \text{for } b > \frac{1}{2}. \]

We use the following notation: let \( \langle \nabla \rangle^\alpha, D^\alpha \rangle = |\nabla|^\alpha \) and \( D^\alpha - \) be the multipliers with symbols \( \langle \xi \rangle^\alpha, |\xi|^\alpha \) and \( ||\tau| - |\xi||^\alpha \), respectively, where \( \langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}} \). Finally \( a\pm \) and \( a\pm\pm \) is short for \( a\pm\epsilon \) and \( a\pm_2\epsilon \) for a sufficiently small \( \epsilon > 0 \).

## 2 Preliminaries

We recall that in the whole manuscript we consider functions in space dimension \( n = 3 \).

The following product estimates for wave-Sobolev spaces were proven in [1], Theorem 3.

**Proposition 2.1** For \( s_0, s_1, s_2, b_0, b_1, b_2 \in \mathbb{R} \) and \( u, v \in S(\mathbb{R}^{3+1}) \) the estimate
\[ \|uv\|_{H^{-s_0,-b_0}} \lesssim \|u\|_{H^{s_1,b_1}} \|v\|_{H^{s_2,b_2}} \]
holds, provided the following conditions are satisfied:
\[
\begin{align*}
    & b_0 + b_1 + b_2 > \frac{1}{2}, b_0 + b_1 \geq 0, \quad b_0 + b_2 \geq 0, b_1 + b_2 \geq 0 \\
    & s_0 + s_1 + s_2 > 2 - (b_0 + b_1 + b_2) \\
    & s_0 + s_1 + s_2 > \frac{3}{2} - \min(b_0 + b_1, b_0 + b_2, b_1 + b_2) \\
    & s_0 + s_1 + s_2 > 1 - \min(b_0, b_1, b_2) \\
    & s_0 + s_1 + s_2 > 1 \\
    & (s_0 + b_0) + 2s_1 + 2s_2 > \frac{3}{2} \\
    & 2s_0 + (s_1 + b_1) + 2s_2 > \frac{3}{2} \\
    & 2s_0 + 2s_1 + (s_2 + b_2) > \frac{3}{2} \\
    & s_1 + s_2 \geq \max(0, -b_0), \quad s_0 + s_2 \geq \max(0, -b_1), \quad s_0 + s_1 \geq \max(0, -b_2).
\end{align*}
\]

**Proposition 2.2** (Null form estimates, [17], Thm. 4) Let \( \sigma_0, \sigma_1, \sigma_2, \beta_0, \beta_1, \beta_2 \in \mathbb{R} \). Assume that
\[
\begin{align*}
    & 0 \leq \beta_0 < \frac{1}{2} < \beta_1, \beta_2 < 1, \\
    & \sum \sigma_i + \beta_0 > \frac{3}{2} - (\beta_0 + \sigma_1 + \sigma_2), \\
    & \sum \sigma_i > \frac{3}{2} - (\sigma_0 + \beta_1 + \sigma_2), \\
    & \sum \sigma_i > \frac{3}{2} - (\sigma_0 + \sigma_1 + \beta_2), \\
    & \sum \sigma_i + \beta_0 \geq 1, \\
    & \min(\sigma_0 + \sigma_1, \sigma_0 + \sigma_2, \beta_0 + \sigma_1 + \sigma_2) \geq 0.
\end{align*}
\]
and that the last two inequalities are not both equalities. Let
\[
\mathcal{F}(B_{\pm_1,\pm_2}(\psi_{1_1}, \psi_{2_2}))(\tau_0, \xi_0)
\]
\[
:= \int_{\tau_1+\tau_2=\tau_0} |\langle \pm_1 \xi_1, \pm_2 \xi_2 \rangle| \psi_{1_1}(\tau_1, \xi_1) \psi_{2_2}(\tau_2, \xi_2) d\tau_1 d\xi_1.
\]
(14)

Then we have the null form estimate
\[
\| B_{\pm_1,\pm_2}(\xi_1, \xi_2) (u, v) \|_{H^{-\alpha_0, -\rho_0}} \lesssim \| u \|_{X^{\alpha_1, \rho_1}_{\pm_1}^{s}} \| v \|_{X^{\alpha_2, \rho_2}_{\pm_2}}.
\]

The following multiplication law is well-known (cf. e.g. [1], Thm. 2.2).

**Proposition 2.3** (Sobolev multiplication law) Let \( s_0, s_1, s_2 \in \mathbb{R} \). Assume \( s_0 + s_1 + s_2 > \frac{3}{2} \), \( s_0 + s_1 \geq 0, s_0 + s_2 \geq 0, s_1 + s_2 \geq 0 \). Then the following product estimate holds:
\[
\| uv \|_{H^{-s_0}} \lesssim \| u \|_{H^{s_1}} \| v \|_{H^{s_2}}.
\]

**Proposition 2.4**

1. For \( 2 < q \leq \infty, 2 \leq r < \infty, \frac{1}{q} = \frac{1}{2} - \frac{1}{r}, \mu = 3(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q} \) the following estimate holds
\[
\| u \|_{L^q L^r} \lesssim \| u \|_{\dot{X}^{\mu, \frac{1}{q}+}_{|\tau|=|\xi|}}.
\]
(15)

2. For \( k \geq 0, p < \infty \) and \( \frac{1}{q} \geq \frac{1}{p} \geq \frac{1}{4} - \frac{k}{2} \) the following estimate holds:
\[
\| u \|_{L^p L^q} \lesssim \| u \|_{\dot{X}^{k+\frac{1}{q}+\frac{1}{2}+}_{|\tau|=|\xi|}}.
\]
(16)

**Proof** (15) is the Strichartz type estimate, which can be found for e.g. in [4], Prop. 2.1, combined with the transfer principle (cf. e.g. [10], Prop. 3.5).

Concerning (16) we refer to [18], Prop. 4.1, or use [9], Thm. B.2:
\[
\| \mathcal{F}_t u \|_{L^2_x L^2_t} \lesssim \| u_0 \|_{H^{\frac{1}{4}}},
\]
if \( u = e^{it|\nabla|} u_0 \) and \( \mathcal{F}_t \) denotes the Fourier transform with respect to time. This immediately implies by Plancherel, Minkowski’s inequality and Sobolev’s embedding theorem
\[
\| u \|_{L^q_{x} L^r_t} = \| \mathcal{F}_t u \|_{L^q_{x} L^r_t} \leq \| \mathcal{F}_t u \|_{L^2_{x} L^2_t} \lesssim \| \mathcal{F}_t u \|_{L^2_{x} H^{k,4}_{t}} \lesssim \| u_0 \|_{H^{k+\frac{1}{2}}}.\]

Here \( H^{k,4}_x \) denotes the \( L^4 \)-based Sobolev space of order \( k \). The transfer principle implies (16). \( \square \)

### 3 Preliminary local well-posedness

Defining
\[
A^d_f = \frac{1}{2} (A^d_f \mp i \langle \nabla \rangle^{-1} \partial_t A^d_f ) \iff A^d_f = A^d_+ + A^d_- \quad \partial_t A^d_f = i \langle \nabla \rangle (A^d_+ - A^d_-)
\]
we can rewrite (11) as
\[
(i \partial_t \pm \langle \nabla \rangle) A^d_{f,\pm} = \mp 2^{-1} \langle \nabla \rangle^{-1} (R.H.S. \  of \ (11) \ - A^d_f).
\]
(17)
with initial data

\[ A^{df}_\pm (0) = \frac{1}{2}(A^{df}_\pm (0) + i^{-1}(\nabla)^{-1}(\partial_t A^{df}_\pm (0))). \]  

(18)

We now state and prove a preliminary local well-posedness result for (10), (11), (12), for which it is essential to have data for \( A \) with vanishing curl-free part.

**Proposition 3.1** Assume \( s > \frac{3}{4}, l > \frac{1}{4}, s \geq l \geq s - \frac{1}{2} \) and \( 2s - l > 1 \). Let \( a^{df}_0 \in H^s, a^{df}_1 \in H^{s-1}, \psi_0 \in H^l \) be given satisfying the Gauss law constraint \( \partial_j a^j_1 = -\partial^j a^j_1 + (\psi^j_0, T^j_{ij} \psi^k_0) T_a \) (necessary by (3) with \( v = 0 \)) with

\[ \|a^{df}_0\|_{H^s} + \|a^{df}_1\|_{H^{s-1}} + \|\psi_0\|_{H^l} \leq \epsilon_0 \]

where \( \epsilon_0 > 0 \) is sufficiently small. Then the system (10)–(12) with initial conditions

\[ A^{df}_+(0) = a^{df}_0, \quad (\partial_t A^{df}_+) (0) = a^{df}_1, \quad A^{cf}_+(0) = 0, \quad \psi(0) = \psi_0 \]

has a unique local solution

\[ A = A^{df}_+ + A^{df}_- + A^{cf}_-, \quad \psi = \psi_+ + \psi_-, \]

where

\[ A^{df}_+ \in X^{s+\frac{1}{2}}_+ [0, 1], \quad A^{cf}_+ \in X^{s+\frac{1}{2}+\frac{1}{2}} [0, 1], \quad \partial_t A^{cf}_+ \in C^0([0, 1], H^{s-1}), \quad \psi_+ \in X^{l+\frac{1}{2}}_+ [0, 1]. \]

Uniqueness holds (of course) for not necessarily vanishing initial data \( A^{cf}_+(0) = a^{cf}_+ \). The solution satisfies

\[ A \in C^0([0, 1], H^s) \cap C^1([0, 1], H^{s-1}), \quad \psi \in C^0([0, 1], H^l). \]

We want to use a contraction argument for \( A^{df}_+ \in X^{s+\frac{1}{2}+\epsilon}_+ [0, 1], \quad A^{cf}_+ \in X^{s+\frac{1}{2}+\frac{1}{2}+\epsilon}_+ [0, 1], \quad \partial_t A^{cf}_+ \in C^0([0, 1], H^{s-1}) \) and \( \psi_+ \in X^{l+\frac{1}{2}+\epsilon}_+ [0, 1] \). Provided that our small data assumption holds this can be reduced by well-known arguments to suitable multilinear estimates of the right hand sides of these equations. For (17) e.g. we make use of the following well-known estimate:

\[ \|A^{df}_\pm\|_{X^{s,b}_\pm [0,1]} \lesssim \|A^{df}_\pm(0)\|_{H^s} + \|R.H.S. of (17)\|_{X^{s,b-1}_\pm [0,1]}, \]

which holds for \( s \in \mathbb{R}, \quad \frac{1}{2} < b \leq 1 \). For (10) we make use of the estimate:

\[ \|A^{cf}\|_{X^{s+\frac{1}{2},b}_+ [0,1]} \lesssim \|R.H.S. of (10)\|_{X^{s+\frac{1}{2},b-1}_+ [0,1]}. \]

Here it is essential, that no term containing \( A^{cf}_+(0) \) appears on the right hand, because we do not want to assume \( A^{cf}_+(0) \in H^{s+\frac{1}{2}} \). Therefore in a first step we assume \( A^{cf}_+(0) = 0 \), which we later remove by an application of a suitable gauge transform. We are forced to admit the same parameter \( b \) on both sides of the latter inequality at one point, which prevents a large data result.

We now show that all the critical terms in (10)–(12), namely the quadratic terms which contain only \( A^{df} \) or \( \psi_\pm \) have null structure. Those quadratic terms which contain \( A^{cf} \) are less critical, because \( A^{cf} \) is shown to be more regular than \( A^{df} \), and the cubic terms are also less critical, because they contain no derivatives.

What we have to prove are estimates for the right hand sides of (10), (17) and (12).

First we consider the terms which do not contain \( A^{cf} \).
For the right hand side of (17) we have to prove the following estimates:

$$\|P[A^{df}_{i,\pm}, \partial^j A^{df}_{i,\pm}]\|_{H^{-1, -\frac{1}{2} + \epsilon}} \lesssim \|A^{df}_{i,\pm}\|_{H^{-1, \frac{3}{2} + \epsilon}} + \|A^{df}_{j,\pm}\|_{H^{-1, -\frac{1}{2} + \epsilon}}$$  \hspace{1cm} (19)

$$\|P[A^{df}_{\pm, i}, \partial^j A^{df}_{\pm, i}]\|_{H^{-1, -\frac{1}{2} + \epsilon}} \lesssim \|A^{df}_{i,\pm}\|_{H^{-1, \frac{3}{2} + \epsilon}} + \|A^{df}_{i,\pm}\|_{H^{-1, -\frac{1}{2} + \epsilon}}$$  \hspace{1cm} (20)

$$\|P(\psi_{1,\pm}, \alpha_j \psi_{2,\pm})\|_{H^{-1, -\frac{1}{2} + \epsilon}} \lesssim \|\psi_{1,\pm}\|_{H^{-1, \frac{3}{2} + \epsilon}} + \|\psi_{2,\pm}\|_{H^{-1, -\frac{1}{2} + \epsilon}}$$  \hspace{1cm} (21)

Concerning the right side of (12), ignoring the irrelevant term $T_{ij}^d$, we have to prove

$$\|\Pi_{\pm} (A^{\pm}_{k,d} \alpha^k \psi)\|_{H^{-1, -\frac{1}{2} + \epsilon}} \lesssim \|\psi\|_{H^{-1, \frac{3}{2} + \epsilon}} + \|A^{\pm}_{k,d}\|_{H^{-1, -\frac{1}{2} + \epsilon}}.$$  \hspace{1cm} (22)

In order to control $A^{cf}$ in (10) we use the Hodge decomposition $A = A^{cf} + A^{df}$, so that we need

$$\|\nabla^{-1}(\phi_1 \partial_\tau \phi_2)\|_{X^{s, \frac{3}{2} + \epsilon}} \lesssim \|\phi_1\|_{X^{s + \frac{1}{2} + \epsilon}} + \|\phi_2\|_{X^{s + \frac{1}{2} + \epsilon}}$$  \hspace{1cm} (23)

$$\|\nabla^{-1}(\phi_1 \partial_\tau \phi_2)\|_{X^{s + \frac{1}{2} + \epsilon}} \lesssim \|\phi_1\|_{X^{s + \frac{1}{2} + \epsilon}} + \|\phi_2\|_{X^{s + \frac{1}{2} + \epsilon}}$$  \hspace{1cm} (24)

$$\|\nabla^{-1}(\phi_1 \partial_\tau \phi_2)\|_{X^{s + \frac{1}{2} + \epsilon}} + \|\nabla^{-1}(\phi_2 \partial_\tau \phi_1)\|_{X^{s + \frac{1}{2} + \epsilon}} \lesssim \|\phi_1\|_{X^{s + \frac{1}{2} + \epsilon}} + \|\phi_2\|_{X^{s + \frac{1}{2} + \epsilon}}$$  \hspace{1cm} (25)

$$\|\nabla^{-1}(\psi_1 \psi_2)\|_{X^{s + \frac{1}{2} + \epsilon}} \lesssim \|\psi_1\|_{X^{s + \frac{1}{2} + \epsilon}} + \|\psi_2\|_{X^{s + \frac{1}{2} + \epsilon}}$$  \hspace{1cm} (26)

where $\phi_i$ represent components of $A^{cf}$ or $A^{df}$. In order to control $\partial_\tau A_{ij}^{cf}$ we need

$$\|\nabla^{-1}(A_1 \partial_\tau A_2)\|_{C^{0}(H^{-1})} \lesssim (\|A_1^{cf}\|_{X^{s, \frac{3}{2} + \epsilon}} + \sum_{\pm} \|A^{df}_{1,\pm}\|_{X^{s + \frac{1}{2} + \epsilon}}) \times (\|\partial_\tau A_2^{cf}\|_{C^{0}(H^{-1})} + \sum_{\pm} \|A^{df}_{2,\pm}\|_{X^{s, \frac{3}{2} + \epsilon}}).$$  \hspace{1cm} (27)

and

$$\|\nabla^{-2}\nabla(\psi_i, T_{ij} A^{df}_{ij})\|_{C^{0}(H^{-1})} \lesssim \sum_{\pm} \|\psi_i, \pm\|_{X^{s, \frac{1}{2} + \epsilon}} + \sum_{\pm} \|\psi_j, \pm\|_{X^{s, \frac{1}{2} + \epsilon}}.$$  \hspace{1cm} (28)

Concerning (17) it remains to consider the terms, which contain a factor $A^{cf}$. We need

$$\|\nabla A^{cf} A^{df}\|_{X^{s - 1, -\frac{1}{2} + \epsilon}} + \|A^{cf} \nabla A^{df}\|_{X^{s - 1, \frac{3}{2} + \epsilon}} \lesssim \|A^{cf}\|_{X^{s + \frac{1}{2} + \epsilon}} + \|A^{df}\|_{X^{s + \frac{3}{2} + \epsilon}}$$  \hspace{1cm} (29)

and

$$\|\nabla A^{cf} A^{cf}\|_{X^{s - 1, -\frac{1}{2} + \epsilon}} \lesssim \|A^{cf}\|_{X^{s + \frac{1}{2} + \epsilon}}^2.$$  \hspace{1cm} (30)

All the cubic terms are estimated by

$$\|A_1 A_2 A_3\|_{X^{s - 1, -\frac{1}{2} + \epsilon}} \lesssim \prod_{i=1}^3 \min(\|A_i\|_{X^{s, \frac{3}{2} + \epsilon}}, \|A_i\|_{X^{s + \frac{1}{2} + \epsilon}}).$$  \hspace{1cm} (31)
Concerning (8) it remains to prove the following estimate:
\[
\|A^{\text{df}} \psi\|_{X^{s,-\frac{1}{2}+}} \lesssim \|A^{\text{df}}\|_{X^{s+\frac{1}{2}+}} \|\psi\|_{X^{s+\frac{1}{2}+}}. \tag{32}
\]

**Proof of (19)** We conclude
\[
[A_i^{\text{df}}, \partial^i A^{\text{df}}] = [R^k (R_i A_k - R_k A_i), \partial^i A^{\text{df}}]
\]
\[
= \frac{1}{2} \left( [R^k (R_i A_k - R_k A_i), \partial^i A^{\text{df}}] + [R_i^k (R_k A_i - R_i A_k), \partial^k A^{\text{df}}] \right)
\]
\[
= \frac{1}{2} \left( [R^k (R_i A_k - R_k A_i), \partial^i A^{\text{df}}] - [R_i^k (R_k A_i - R_i A_k), \partial^k A^{\text{df}}] \right)
\]
\[
= \frac{1}{2} Q^{jk} \mathcal{A}_{ij}^{-1} (R_i A_k - R_k A_i), A^{\text{df}},
\]
where
\[
Q_{ij} [u, v] := [\partial_i u, \partial_j v] - [\partial_j u, \partial_i v] = Q_{ij} (u, v) + Q_{ji} (v, u)
\]
with the standard null form
\[
Q_{ij} (u, v) := \partial_i u \partial_j v - \partial_j u \partial_i v.
\]

Thus, ignoring \(P\), which is a bounded operator, we obtain
\[
P[A_i^{\text{df}}, \partial^i A^{\text{df}}] \sim \sum Q_{ik} \mathcal{A}_{ij}^{-1} A^{\text{df}}, A^{\text{df}}. \tag{34}
\]
It is well-known (cf. e.g. [17]) that the bilinear form \(Q_{\pm 1, \pm 2}^{jk}\), defined by
\[
Q_{\pm 1, \pm 2}^{jk} (\phi_{\pm 1}, \phi_{\pm 2}) := R_{\pm 1}^j \phi_{\pm 1} R_{\pm 2}^k \phi_{\pm 2} - R_{\pm 2}^k \phi_{\pm 1} R_{\pm 1}^j \phi_{\pm 2},
\]
similarly to the standard null form \(Q_{ik}\), which is defined by replacing the modified Riesz transforms \(R_{\pm}^k\) by \(\partial^k\), fulfills the following estimate:
\[
Q_{\pm 1, \pm 2}^{jk} (\phi_{\pm 1}, \phi_{\pm 2}) \lesssim B_{\pm 1, \pm 2} (\psi_{\pm 1}, \psi_{\pm 2}).
\]

Let \(u \lesssim v\) be defined by \(|\vec{u}| \lesssim |\vec{v}|\). We have to prove
\[
\|B_{\pm 1, \pm 2} (u, v)\|_{H^{\sigma-\frac{1}{2}+}} \lesssim \|u\|_{X^{\frac{s}{4} + \frac{1}{4} +}} \|v\|_{X^{\frac{s}{4} - \frac{1}{4} +}}.
\]

This is implied by Proposition 2.2 with parameters \(\sigma_0 = 1 - s, \sigma_1 = s, \sigma_2 = s - 1, \beta_0 = \frac{1}{4} -\), \(\beta_1 = \beta_2 = \frac{3}{4} +\), provided \(s > \frac{3}{4}\). \(\square\)

**Proof of (20)**
\[
(P (A_i^{\text{df}} \nabla A_i^{\text{df}})) j = R^k (R_j (A_i^{\text{df}} \partial_k A_i^{\text{df}}) - R_k (A_i^{\text{df}} \partial_j A_i^{\text{df}}))
\]
\[
= |\nabla|^{-2} \partial^k (\partial_j (A_i^{\text{df}} \partial_k A_i^{\text{df}}) - \partial_k (A_i^{\text{df}} \partial_j A_i^{\text{df}}))
\]
\[
= |\nabla|^{-2} \partial^k (\partial_j A_i^{\text{df}} \partial_k A_i^{\text{df}} - \partial_k A_i^{\text{df}} \partial_j A_i^{\text{df}})
\]
so that
\[
P[A_i^{\text{df}}, \nabla A_i^{\text{df}}] \sim \sum |\nabla|^{-1} Q_{jk} [A^{\text{df}}, A^{\text{df}}]. \tag{35}
\]

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We have to prove
\[ \|Q_{jk}(u, v)\|_{H^{s-2, -\frac{1}{4}} +} \lesssim \|u\|_{H^{s+\frac{1}{4}}} + \|v\|_{H^{s+\frac{1}{4}}}. \]

We use the estimate
\[ Q_{jk}(u, v) \lesssim D^\frac{1}{2} D^\frac{1}{2} (D^\frac{1}{2} u D^\frac{1}{2} v) + D^\frac{1}{2} (D^\frac{1}{2} D^\frac{1}{2} u D^\frac{1}{2} v) + D^\frac{1}{2} (D^\frac{1}{2} u D^\frac{1}{2} D^\frac{1}{2} v), \]
which was proven by [8], Prop. 1. This reduces the proof to the estimates
\[ \|uv\|_{H^{s-\frac{1}{2}, \frac{1}{4}}} \lesssim \|u\|_{H^{s+\frac{1}{4}}} + \|v\|_{H^{s+\frac{1}{4}}}, \]
\[ \|uv\|_{H^{s-\frac{1}{2}, \frac{1}{4}}} \lesssim \|u\|_{H^{s+\frac{1}{4}}} + \|v\|_{H^{s+\frac{1}{4}}}. \]

Both are implied by Proposition 2.1 with parameters \( s_0 = \frac{3}{2} - s, s_1 = s - \frac{1}{2}, \) and \( b_0 = -\frac{1}{4}, b_1 = b_2 = \frac{3}{4} + \) for the first one and \( b_0 = \frac{1}{4}, b_1 = \frac{1}{4} +, b_2 = \frac{3}{4} + \) for the second one, which both require the assumption \( s > \frac{3}{4}. \)

**Proof of (21)** Using the definition of \( P \) and ignoring the irrelevant term \( T^a \) we have to prove
\[ \|R^k(R_j(\psi_1, \alpha_k \psi_2) - R_k(\psi_1, \alpha_j \psi_2))\|_{\xi^{s-1, -\frac{1}{4} +}} \lesssim \|\psi_1\|_{\overline{X}_{\alpha_0}^{\frac{1}{2} +}} \|\psi_2\|_{\overline{X}_{\alpha_2}^{\frac{1}{2} +}}. \]

We obtain by (7):
\[
\begin{align*}
R^k(R_j(\psi_1, \alpha_k \psi_2) - R_k(\psi_1, \alpha_j \psi_2)) & = \sum_{\pm_1, \pm_2} R^k(R_j(\psi_{1\pm_1}, \alpha_k \Pi_{\pm_2} \psi_{2\pm_2}) - R_k(\psi_{1\pm_1}, \alpha_j \Pi_{\pm_2} \psi_{2\pm_2})) \\
& = \sum_{\pm_1, \pm_2} R^k(R_j(\psi_{1\pm_1}, \Pi_{\mp_2} (\alpha_k \psi_{2\pm_2})) - R_k(\psi_{1\pm_1}, \Pi_{\mp_2} (\alpha_j \psi_{2\pm_2}))) \\
& - \sum_{\pm_1, \pm_2} R^k(R_j(\psi_{1\pm_1}, R^k_{\pm_2} \psi_{2\pm_2}) - R_k(\psi_{1\pm_1}, R^k_{\pm_2} \psi_{2\pm_2})) \\
& = I + II.
\end{align*}
\]
Both terms are null forms.

Concerning I we consider each term separately and remark that at this point \( R_k \) and \( R_j \) are irrelevant. We obtain
\[
\mathcal{F}((\Pi_{\pm_1} \psi_{1\pm_1}, \Pi_{\mp_2} \alpha_k \psi_{2\pm_2}))(\tau_0, \xi_0)
= \int_{\tau_1 + \tau_2 = \tau_0, \xi_1 + \xi_2 = \xi_0} \langle \Pi_{\pm_1} \psi_{1\pm_1}(\tau_1, \xi_1), \Pi_{\mp_2} \alpha_k \Pi_{\pm_2} \psi_{2\pm_2}(\tau_2, \xi_2) \rangle d\tau_1 d\xi_1
= \int_{\tau_1 + \tau_2 = \tau_0, \xi_1 + \xi_2 = \xi_0} \langle \psi_{1\pm_1}(\tau_1, \xi_1), \Pi(\pm_1 \xi_1) \Pi(\mp_2 \xi_2) \alpha_k \Pi(\pm_2 \xi_2) \psi_{2\pm_2}(\tau_2, \xi_2) \rangle d\tau_1 d\xi_1.
\]

(36)

Now we use the estimate ("spinorial null structure")
\[ |\Pi(\pm_1 \xi_1)\Pi(\mp_2 \xi_2)z| \lesssim |z|\angle(\pm_1 \xi_1, \pm_2 \xi_2) \]
proven by [1], Lemma 2. This implies
\[ I \lesssim B_{\pm_1, \pm_2}(\psi_{1\pm_1}, \psi_{2\pm_2}), \]
where \( B_{\pm_1, \pm_2} \) is defined by (14).
We have to prove
\[ \| B_{\pm1, \pm2} (\psi_{1\pm1}, \psi_{2\pm2}) \|_{X_{\pm0}^{-r, -\frac{1}{4} +}} \lesssim \| \psi_{1\pm1} \|_{X_{\pm1}^{l, \frac{1}{4} +}} \| \psi_{2\pm2} \|_{X_{\pm2}^{l, \frac{1}{4} +}}. \]  
(37)

We apply Proposition 2.2 with parameters \( \sigma_0 = 1 - s, \sigma_1 = \sigma_2 = l, \beta_0 = l, \beta_1 = \beta_2 = \frac{1}{2} + \). This requires \( 2l - s > -\frac{1}{4}, 4l - s > 0 \) and \( 3l - 2s > -1 \), which follows from our assumptions.

Next we obtain
\[ II \lesssim \sum_{\pm1, \pm2} (R_j \pm0 (\psi_{1\pm1}, R_{\pm2}^k \psi_{2\pm2}) - R_k \pm0 (\psi_{1\pm1}, R_{\pm2}^l \psi_{2\pm2})). \]

By duality we have to prove
\[ \left| \int (\langle \psi_{1\pm1}, R_{\pm2}^k \psi_{2\pm2} \rangle R_{\pm0} \psi_{0\pm0} - \langle \psi_{1\pm1}, R_{\pm2}^l \psi_{2\pm2} \rangle R_{\pm0} \psi_{0\pm0} \rangle dx dt \right| \lesssim \| \psi_{1\pm1} \|_{X_{\pm1}^{l, \frac{1}{4} +}} \| \psi_{2\pm2} \|_{X_{\pm2}^{l, \frac{1}{4} +}} \| \psi_{0\pm0} \|_{X_{\pm0}^{-l, -\frac{1}{4} -}}. \]

We remark that the left hand side possesses a \( Q^{jk} \)-type null form between \( \psi_{2\pm2} \) and \( \psi_{0\pm0} \).

We have to prove
\[ \| B_{\pm2, \pm0} (\psi_{2\pm2}, \psi_{0\pm0}) \|_{X_{\pm1}^{-l, -\frac{1}{4} -}} \lesssim \| \psi_{2\pm2} \|_{X_{\pm2}^{l, \frac{1}{4} +}} \| \psi_{0\pm0} \|_{X_{\pm0}^{-l, -\frac{1}{4} -}}. \]  
(38)

We apply Proposition 2.2 with \( \sigma_0 = \sigma_1 = l, \sigma_2 = 1 - s, \beta_0 = \frac{1}{2} - \), \( \beta_1 = \frac{1}{2} + \), \( \beta_2 = \frac{3}{4} - \), which requires \( 3l - 2s > -1 \) and \( 4l - s > 0 \) as before. This implies (38), because we may trivially replace \( \beta_0 = \frac{1}{2} - \) by \( \beta_0 = \frac{1}{2} + \).

**Proof of (22)** Using (7) we obtain
\[ \Pi \pm0 (A_{k, df}^\pm \alpha^k \psi) = \sum_{\pm} \Pi \pm0 (A_{k, df}^\pm \alpha^k \Pi \pm \psi) \]
\[ = \sum_{\pm} \Pi \pm0 (A_{k, df}^\pm \Pi \mp (\alpha^k \Pi \pm \psi)) - \sum_{\pm} \Pi \pm0 (A_{k, df}^\pm R_{\pm}^k \psi_{\pm}) = I + II. \]

Concerning I we have to prove by duality
\[ \left| \int \int \langle \Pi \pm0 (A_{k, df}^\pm \Pi \mp (\alpha^k \psi_{\pm})), \psi_{0\pm0} \rangle dx dt \right| \lesssim \| A_{k, df}^\pm \|_{X_{\pm2}^{l, \frac{1}{4} +}} \| \psi_{1\pm1} \|_{X_{\pm1}^{l, \frac{1}{4} +}} \| \Pi \pm0 \psi_{0\pm0} \|_{X_{\pm0}^{-l, -\frac{1}{4} -}}. \]

The left hand side equals
\[ \left| \int \int A_{k, df}^\pm (\Pi \mp (\alpha^k \psi_{\pm})), \Pi \pm0 \psi_{0\pm0} \rangle dx dt \right| \]
\[ = \left| \int \int A_{k, df}^\pm (\Pi \mp (\alpha^k \psi_{\pm})), \psi_{0\pm0} \rangle dx dt \right|. \]

It contains a spinorial null form between \( \psi_{\pm1} \) and \( \psi_{0\pm0} \) as in (36), so that it remains to prove
\[ \| B_{\pm1, \pm0} (\psi_{\pm1}, \psi_{0\pm0}) \|_{X_{\pm2}^{-l, -\frac{1}{4} -}} \lesssim \| \psi_{1\pm1} \|_{X_{\pm1}^{l, \frac{1}{4} +}} \| \psi_{0\pm0} \|_{X_{\pm0}^{-l, -\frac{1}{4} -}}. \]  
(39)
First we prove the estimate
\[
\|B_{\pm 1, \pm 0}(\psi_{\pm 1}, \psi_{0, \pm 0})\|_{X_{\pm 2}^{-\frac{1}{2} + \epsilon}} \lesssim \|\psi_{\pm 1}\|_{X_{\pm 1}^{\frac{1}{2} + \epsilon}} \|\psi_{0, \pm 0}\|_{X_{\pm 0}^{-\frac{1}{2} + \epsilon}},
\]
which follows from Proposition 2.2 with parameters \(\sigma_0 = s, \sigma_1 = l, \sigma_2 = -l, \beta_0 = \frac{1}{2} -, \beta_1 = \frac{1}{2} +, \beta_2 = \frac{1}{2} +\), where we need \(2s - l > 1\) and \(s > \frac{3}{2}\). Interpolation with the estimate
\[
\|B_{\pm 1, \pm 0}(\psi_{\pm 1}, \psi_{0, \pm 0})\|_{X_{\pm 2}^{s - \frac{3}{2} - \frac{1}{2} + \epsilon}} \lesssim \|\psi_{\pm 1}\|_{X_{\pm 1}^{l, \frac{1}{2} + \epsilon}} \|\psi_{0, \pm 0}\|_{X_{\pm 0}^{-\frac{s}{2} - \frac{1}{2} + \epsilon}}
\]
which is easily implied by Sobolev (or use Proposition 2.1), gives (39) with \(s\) replaced by \(s +\). This however does not change the assumptions \(2s - l > 1\) and \(s > \frac{3}{4}\).

Concerning II we remark that
\[
A^{df} = |\nabla|^{-2} \nabla \times (\nabla \times A) = |\nabla|^{-2} \nabla \times (\nabla \times A^{df}) + |\nabla|^{-2} \nabla \times (\nabla \times A^{cf}) = |\nabla|^{-2} \nabla \times (\nabla \times A^{df}).
\]
This implies
\[
A^{df}_l R_{\pm 1} \psi_{\pm 1} = \epsilon^{km} \partial_k w_m R_{\pm 1} \psi_{\pm 1} = (\nabla w_m \times \frac{\nabla}{|\nabla|} \psi_{\pm 1})^m,
\]
where \(\epsilon^{km}\) denotes the Levi-Civita symbol with \(\epsilon^{123} = 1\) and \(w = |\nabla|^{-2} \nabla \times A^{df}\), so that \(\partial_j w_m = |\nabla|^{-2} \partial_j \partial_k A^{df}_l \epsilon^{km}\). This is a \(Q_{ij}\)-type null form between \(w_m\) and \(|\nabla|^{-1} \psi_{\pm 1}\), so that we have to prove
\[
\|B_{\pm 1, \pm 2}(A^{df}_l, \psi_{\pm 1})\|_{X_{\pm 2}^{1, \frac{1}{2} + \epsilon}} \lesssim \|A^{df}_l\|_{X_{\pm 2}^{1, \frac{1}{2} + \epsilon}} \|\psi_{\pm 1}\|_{X_{\pm 1}^{\frac{1}{2} + \epsilon} + \epsilon}.
\]
This is implied by Proposition 2.2 with parameters \(\sigma_0 = -l, \sigma_1 = l, \sigma_2 = s, \beta_0 = \frac{1}{2} -\), \(\beta_1 = \frac{1}{2} +, \beta_2 = \frac{3}{4} +\), if \(2s - l > 1\) and \(s > \frac{3}{4}\).

The estimates (23)-(25) have been essentially given by Tao [19]. For the sake of completeness we give the details. We remark that it is especially (25) which prevents a large data result, because it seems to be difficult to replace \(X_{\tau=0}^{s + \frac{1}{2} - \frac{1}{2} + \epsilon}\) by \(X_{\tau=0}^{s + \frac{1}{2} - \frac{1}{2} + \epsilon}\) on the left hand side.

**Proof of (24)** As usual the singularity of \(|\nabla|^{-1}\) is harmless in dimension 3 ([18], Cor. 8.2) and it can be replaced by \((\nabla)^{-1}\). Taking care of the time derivative we reduce to
\[
| \int \int u_1 u_2 u_3 dx dt | \lesssim \|u_1\|_{X_{\tau=0}^{s + \frac{1}{2} + \epsilon}} \|u_2\|_{X_{\tau=0}^{s + \frac{1}{2} + \epsilon}} \|u_3\|_{X_{\tau=0}^{s + \frac{1}{2} - 2\epsilon}}
\]
which follows from Sobolev’s multiplication rule, because \(s > \frac{1}{2}\).

**Proof of (25)** a. If \(\tilde{\phi}\) is supported in \(||\tau| - |\xi|| \gtrsim |\xi||\), we obtain
\[
\|\phi\|_{X_{\tau=0}^{s + \frac{1}{2} + \epsilon}} \lesssim \|\phi\|_{X_{|\tau|=|\xi|}^{s + \frac{3}{2} + \epsilon}}.
\]
Thus (25) follows from (24).
b. It remains to show

$$\left| \int \int (uv, w + uvw) \, dx \, dt \right| \lesssim \|u\|_{X_{t=0}^{3-s}} \|v\|_{X_{\tau=0}^{1-\epsilon, 1+\epsilon}} \|w\|_{X_{\tau=0}^{s, \frac{1}{2} + \epsilon}}$$

whenever \( \hat{w} \) is supported in \( ||\tau| - |\xi|| \ll |\xi| \). This is equivalent to

$$\int m(\xi_1, \xi_2, \xi_3, \tau_1, \tau_2, \tau_3) \prod_{i=1}^{3} \hat{u}_i(\xi_i, \tau_i) d\xi d\tau \lesssim \prod_{i=1}^{3} \|u_i\|_{L^2_{ut}}$$

where \( d\xi = d\xi_1 d\xi_2 d\xi_3 \), \( d\tau = d\tau_1 d\tau_2 d\tau_3 \) and * denotes integration over \( \sum_{i=1}^{3} \xi_i = \sum_{i=1}^{3} \tau_i = 0 \). The Fourier transforms are nonnegative without loss of generality. Here

$$m = \frac{(|\tau_2| + |\tau_3|) \chi_{||\tau_3|| - |\xi_3|| \ll |\xi_3|}}{(\xi_1)_{1-s}^{2} (\tau_1)_{1-s}^{2} (\xi_2)_{1-s}^{2} (\xi_3)_{1-s}^{2}} \lesssim \frac{1}{|\tau_3|}$$

Since \( \langle \tau_3 \rangle \sim \langle \xi_3 \rangle \) and \( \tau_1 + \tau_2 + \tau_3 = 0 \) we have

$$|\tau_2| + |\tau_3| \lesssim \langle \tau_1 \rangle_{1-s, \tau_2}^{2} + \langle \tau_1 \rangle_{1-s, \tau_2}^{2} + \langle \tau_3 \rangle_{1-s, \tau_2}^{2} + \langle \tau_3 \rangle_{1-s, \tau_2}^{2},$$

so that concerning the first term on the right hand side of (40) we have to show

$$\left| \int \int uvw \, dx \, dt \right| \lesssim \|u\|_{X_{t=0}^{3-s}} \|v\|_{X_{\tau=0}^{1-\epsilon, 1+\epsilon}} \|w\|_{X_{\tau=0}^{s, \frac{1}{2} + \epsilon}},$$

which follows from Sobolev’s multiplication rule, because \( s > \frac{1}{2} \). This is sharp with respect to the time derivative. As a consequence we need the smallness assumption on the data for local existence.

Concerning the second term on the right hand side of (40) we use \( \langle \xi_1 \rangle_{s-\frac{3}{4}}^{2} \lesssim \langle \xi_2 \rangle_{s-\frac{3}{4}}^{2} + \langle \xi_3 \rangle_{s-\frac{3}{4}}^{2} \), so that we reduce to

$$\left| \int \int uvw \, dx \, dt \right| \lesssim \|u\|_{X_{t=0}^{0, 0}} \|v\|_{X_{\tau=0}^{s-\epsilon, \frac{1}{2} + \epsilon}} \|w\|_{X_{\tau=0}^{s-\epsilon, \frac{1}{2} + \epsilon}}$$

and

$$\left| \int \int uvw \, dx \, dt \right| \lesssim \|u\|_{X_{t=0}^{0, 0}} \|v\|_{X_{\tau=0}^{s+\epsilon, \frac{1}{2} + \epsilon}} \|w\|_{X_{\tau=0}^{s-\epsilon, \frac{1}{2} + \epsilon}}.$$

(41) is implied by Sobolev and (16) as follows:

$$\left| \int \int uvw \, dx \, dt \right| \lesssim \|u\|_{L^2_{t} L^t_{1}} \|v\|_{L^t_{t} L^\infty_{t}} \|w\|_{L^t_{t} L^2_{t}} \lesssim \|u\|_{X_{t=0}^{0, 0}} \|v\|_{X_{\tau=0}^{s-\epsilon, \frac{1}{2} + \epsilon}} \|w\|_{X_{\tau=0}^{s+\epsilon, \frac{1}{2} + \epsilon}}.$$

For (42) we obtain

$$\left| \int \int uvw \, dx \, dt \right| \lesssim \|u\|_{L^2_{t} L^t_{1}} \|v\|_{L^t_{t} L^\infty_{t}} \|w\|_{L^t_{t} L^2_{t}}$$

where \( \frac{1}{q} = \frac{1}{2} - \epsilon \) and \( \frac{1}{p} = \frac{1}{2} + \epsilon \). For \( s > \frac{3}{4} \) we obtain by Sobolev

$$\|v\|_{L^q_{t} L^\infty_{t}} \lesssim \|v\|_{X_{t=0}^{s+\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon}}.$$
Interpolation between (16) \( \|w\|_{L^q_t L^r_x} \lesssim \|w\|_{\dot{X}^{\frac{1}{2}, \frac{1}{2} + \epsilon}_{t=0}^{\frac{1}{2}, \frac{1}{2} + \epsilon}} \) and the trivial identity \( \|w\|_{L^2_t L^2_x} = \|w\|_{\dot{X}_t^{0,0}} \) implies

\[
\|w\|_{L^q_t L^r_x} \lesssim \|w\|_{\dot{X}^{\frac{1}{2}, \frac{1}{2} + \epsilon}_{t=0}^{\frac{1}{2}, \frac{1}{2} + \epsilon}} \]

so that (42) follows.

Concerning the last term on the right hand side of (40) we use \( (\xi_1)^{s - \frac{3}{4}} \lesssim (\xi_2)^{s - \frac{3}{4}} + (\xi_3)^{s - \frac{3}{4}} \) so that we reduce to

\[
\left| \int \int uvw dxdt \right| \lesssim \|u\|_{\dot{X}^{0, \frac{1}{2} - \epsilon}_{t=0}^{1, \frac{1}{2} - \epsilon}} \|v\|_{\dot{X}^{\frac{1}{2}, -\epsilon, 0}_{t=0}^{\frac{1}{2}, -\epsilon, 0}} \|w\|_{\dot{X}^{\frac{1}{2}, \frac{1}{2} + \epsilon}_{t=0}^{\frac{1}{2}, \frac{1}{2} + \epsilon}} \quad (43)
\]

and

\[
\left| \int \int uvw dxdt \right| \lesssim \|u\|_{\dot{X}^{0, \frac{1}{2} - \epsilon}_{t=0}^{1, \frac{1}{2} - \epsilon}} \|v\|_{\dot{X}^{\frac{1}{2}, -\epsilon, 0}_{t=0}^{\frac{1}{2}, -\epsilon, 0}} \|w\|_{\dot{X}^{\frac{1}{2}, \frac{1}{2} + \epsilon}_{t=0}^{\frac{1}{2}, \frac{1}{2} + \epsilon}}. \quad (44)
\]

We estimate as follows:

\[
\left| \int \int uvw dxdt \right| \lesssim \|u\|_{L^q_t L^r_x} \|v\|_{L^q_t L^r_x} \|w\|_{L^q_t L^r_x} \quad (45)
\]

which would be sufficient for (43) and (44) under our assumption \( s > \frac{3}{4} \). For the proof of (45) we choose \( \frac{1}{q} = \frac{1}{2} - \epsilon + \epsilon \), \( \frac{1}{r} = \frac{1}{2} + \frac{\epsilon}{3} \) and \( \frac{1}{r} = \frac{1}{2} - \frac{\epsilon}{6} \) so that \( \|u\|_{L^q_t L^r_x} \lesssim \|u\|_{\dot{X}^{\frac{1}{2}, \frac{1}{2} - \epsilon}_{t=0}^{\frac{1}{2}, \frac{1}{2} - \epsilon}} \) and by Sobolev \( \|v\|_{L^q_t L^r_x} \lesssim \|v\|_{\dot{X}^{\frac{1}{2}, -\epsilon, 0}_{t=0}^{\frac{1}{2}, -\epsilon, 0}} \). Moreover interpolation between (16) \( \|w\|_{L^q_t L^r_x} \lesssim \|w\|_{\dot{X}^{0, \frac{1}{2} - \epsilon}_{t=0}^{1, \frac{1}{2} - \epsilon}} \) and the trivial identity \( \|w\|_{L^2_t L^2_x} = \|w\|_{\dot{X}_t^{0,0}} \) implies

\[
\|w\|_{L^q_t L^r_x} \lesssim \|w\|_{\dot{X}^{\frac{1}{2}, \frac{1}{2} + \epsilon}_{t=0}^{\frac{1}{2}, \frac{1}{2} + \epsilon}},
\]

Interpolation between Strichartz’ inequality (15) \( \|w\|_{L^4_t X^4} \lesssim \|w\|_{\dot{X}^{\frac{1}{2}, \frac{1}{2} + \epsilon}_{t=0}^{\frac{1}{2}, \frac{1}{2} + \epsilon}} \) and the trivial identity gives

\[
\|w\|_{L^q_t L^r_x} \lesssim \|w\|_{\dot{X}^{\frac{1}{2}, \frac{1}{2} + \epsilon}_{t=0}^{\frac{1}{2}, \frac{1}{2} + \epsilon}},
\]

so that another interpolation between these estimates implies the following bound for the last factor

\[
\|w\|_{L^q_t L^r_x} \lesssim \|w\|_{\dot{X}^{\frac{1}{2}, \frac{1}{2} + \epsilon}_{t=0}^{\frac{1}{2}, \frac{1}{2} + \epsilon}},
\]

as one easily checks. This implies (45).

\[\square\]

**Proof of (23)** If \( \tilde{\phi} \) is supported in \( ||\tau|| - ||\xi|| \gtrsim ||\xi|| \) we obtain

\[
\|\tilde{\phi}\|_{\dot{X}^{\frac{1}{2} + 3 \epsilon, \frac{1}{2} + 3 \epsilon}_{t=0}^{\frac{1}{2} + 3 \epsilon, \frac{1}{2} + 3 \epsilon}} \lesssim \|\tilde{\phi}\|_{\dot{X}^{\frac{1}{2} + 3 \epsilon, \frac{1}{2} + 3 \epsilon}_{t=0}^{\frac{1}{2} + 3 \epsilon, \frac{1}{2} + 3 \epsilon}}.
\]

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which implies that (23) follows from (25), if \( \tilde{\varphi}_1 \) or \( \tilde{\varphi}_2 \) have this support property. So we may assume that both functions are supported in \(||\tau| - |\xi|| \ll |\xi|\). This means that it suffices to show
\[
\int m(\xi_1, \xi_2, \xi_3, \tau_1, \tau_2, \tau_3) \prod_{i=1}^{3} a_i(\xi_i, \tau_i) d\xi d\tau \lesssim \prod_{i=1}^{3} \|u_i\|_{L^2_{\tau_i}},
\]
where
\[
m = \frac{|\tau_3|^p |\tau_2 - |\xi_2|| \ll |\xi_2| |\tau_2 - |\xi_2|| \ll |\xi_2| |\tau_3 - |\xi_3|| \ll |\xi_3| |\tau_3 - |\xi_3|| \ll |\xi_3|}{\langle \xi_1 \rangle^{\frac{3}{2} - \varepsilon} (\langle \xi_1 \rangle^{\frac{1}{2} - \varepsilon} - \langle \xi_2 \rangle^\varepsilon (\langle |\tau_2| - |\xi_2|| \ll |\xi_2| |\tau_3 - |\xi_3|| \ll |\xi_3|)^{\frac{3}{2} + \varepsilon}}.
\]
Since \( \langle \tau_3 \rangle \sim \langle \xi_3 \rangle \), \( \langle \tau_2 \rangle \sim \langle \xi_2 \rangle \) and \( \tau_1 + \tau_2 + \tau_3 = 0 \) we have
\[
|\tau_3| \lesssim |\tau_1|^{\frac{1}{2} - \varepsilon} - \langle \xi_3 \rangle^{\frac{1}{2} + \varepsilon} + \langle \xi_2 \rangle^{\frac{1}{2} - \varepsilon} - \langle \xi_3 \rangle^{\frac{1}{2} + \varepsilon} + 3
\]
(46)
Concerning the first term on the right hand side we have to show
\[
|\int \int uvwdxdt| \lesssim \|u\|_{X^{\frac{3}{2},0}_{\tau}} \|v\|_{X^{\frac{3}{2},0}_{|\tau| = |\xi|}} \|w\|_{X^{s - \frac{3}{2},-\frac{3}{2} + \varepsilon}_{|\tau| = |\xi|}}.
\]
We use Proposition 2.2, which shows
\[
\|vw\|_{L^2_{\tau}H^{s - \frac{3}{2}}_{\xi}} \lesssim \|v\|_{X^{\frac{1}{2},\frac{1}{2}}_{|\tau| = |\xi|}} \|w\|_{X^{s - \frac{3}{2} - \varepsilon, -\frac{3}{2} + \varepsilon}_{|\tau| = |\xi|}}
\]
under the assumption \( s > \frac{3}{4} \).

Concerning the second term on the right hand side we use \( \langle \xi_1 \rangle^{s - \frac{3}{2}} \lesssim \langle \xi_2 \rangle^{s - \frac{3}{2} + \varepsilon} + \langle \xi_3 \rangle^{s - \frac{3}{2}} \), so that we reduce to
\[
|\int \int uvwdxdt| \lesssim \|u\|_{X^{\frac{1}{2},\frac{1}{2}}_{|\tau| = |\xi|}} \|v\|_{X^{s - \frac{3}{2} - \varepsilon, \frac{3}{2} + \varepsilon}_{|\tau| = |\xi|}} \|w\|_{X^{s - \frac{3}{2} - \varepsilon, -\frac{3}{2} + \varepsilon}_{|\tau| = |\xi|}}
\]
and
\[
|\int \int uvwdxdt| \lesssim \|u\|_{X^{\frac{1}{2},\frac{1}{2}}_{|\tau| = |\xi|}} \|v\|_{X^{\frac{1}{2},\frac{1}{2}}_{|\tau| = |\xi|}} \|w\|_{X^{\frac{1}{2},\frac{1}{2}}_{|\tau| = |\xi|}}.
\]
We obtain
\[
|\int \int uvwdxdt| \lesssim \|u\|_{L^2_{\tau}} \|v\|_{L^{\frac{1}{2},\varepsilon}_{\tau}} \|w\|_{L^{\frac{1}{2},\varepsilon}_{\tau}} \lesssim \|u\|_{X^{\frac{1}{2},\frac{1}{2}}_{|\tau| = |\xi|}} \|v\|_{X^{\frac{1}{2},\frac{1}{2}}_{|\tau| = |\xi|}} \|w\|_{X^{\frac{1}{2},\frac{1}{2}}_{|\tau| = |\xi|}}.
\]
which implies both estimates for \( s > \frac{3}{4} \) and \( \varepsilon > 0 \) sufficiently small. Then we choose \( \frac{1}{r} = \frac{1}{4} - \varepsilon, \frac{1}{q} = \frac{1}{2} + 2\varepsilon \) and \( \frac{1}{p} = \frac{1}{4} + \varepsilon \). Here for the first factor we interpolated between \( \|u\|_{L^2_{\tau}L^{\infty}_{x}} \lesssim \|u\|_{X^{\frac{1}{2},\frac{1}{2}}_{|\tau| = |\xi|}} \) and the trivial identity \( \|u\|_{L^2_{\tau}} = \|u\|_{X^{\frac{1}{2},0}_{|\tau| = |\xi|}} \), for the second factor between (16) \( \|v\|_{L^{\frac{1}{2}_{\tau}}L^{\frac{1}{4}_{x}}_{\tau}} \lesssim \|v\|_{X^{\frac{1}{2},\frac{1}{2}}_{|\tau| = |\xi|}} \) and the Sobolev type inequality \( \|v\|_{L^2_{\tau}} \lesssim \|v\|_{X^{\frac{3}{2},\frac{1}{2}}_{|\tau| = |\xi|}} \), and for the last factor between (16) and the trivial identity, both with interpolation parameter \( \theta = 1 - 4\varepsilon - \varepsilon \).
**Proof of (26)** By the fractional Leibniz rule we have to prove
\[
\| u v \|_{X^{l, \frac{1}{2}+, \epsilon, \pm}_{t = 0^+}} \lesssim \| u \|_{X^{l - s, \frac{1}{2}+, \frac{1}{2}, \epsilon, \pm}_{t = |\tau| = |\xi|}} \| v \|_{X^{l, \frac{1}{2}+, \epsilon, \pm}_{t = 0^+}},
\]
which is equivalent to
\[
\left| \int \int u v w \, dx \, dt \right| \lesssim \| u \|_{X^{l - s, \frac{1}{2}+, \frac{1}{2}, \epsilon, \pm}_{t = |\tau| = |\xi|}} \| v \|_{X^{l, \frac{1}{2}+, \epsilon, \pm}_{t = 0^+}} \| w \|_{X^{l, \frac{1}{2}+, \epsilon, \pm}_{t = 0^+}}.
\]
By Hölder’s inequality we obtain:
\[
\left| \int \int u v w \, dx \, dt \right| \lesssim \| u \|_{L^4_{t} L^2_{x}} \| v \|_{L^2_{t} L^4_{x}} \| w \|_{L^2_{t} L^\infty_{x}}.
\]
The first factor is estimated by (16) using the assumption \( l - s \geq -\frac{1}{2} \) and the last factor by Sobolev. For the second factor we interpolate between (16) \( \| v \|_{L^1_{t} L^2_{x}} \lesssim \| v \|_{X^{l, \frac{1}{2}+, \epsilon, \pm}_{t = 0^+}} \) and Strichartz’ estimate \( \| v \|_{L^1_{t} L^4_{x}} \lesssim \| v \|_{X^{l, \frac{1}{2}+, \epsilon, \pm}_{t = 0^+}} \), which implies
\[
\| v \|_{L^1_{t} L^2_{x}} \lesssim \| v \|_{X^{l, \frac{1}{2}+, \epsilon, \pm}_{t = 0^+}} \quad \text{for } l > \frac{1}{4},
\]
so that (26) is proven. \( \square \)

**Proof of (27)** Sobolev’s multiplication law shows the estimate
\[
\| |\nabla|^{-1} (A_1 \partial_t A_2) \|_{C^0_0(H^s - 1)} \lesssim \| A_1 \|_{C^0_0(H^s)} \| \partial_t A_2 \|_{C^0_0(H^s - 1)}
\]
for \( s > \frac{1}{2} \). Use now
\[
A = A^c + \sum_{\pm} A^d_{\pm}, \quad \partial_t A = \partial_t A^c + i \langle \nabla \rangle (A^d_+ - A^d_-)
\]
from which the estimate (27) easily follows. \( \square \)

**Proof of (28)** This reduces to the estimate
\[
\| \psi_1 \psi_2 \|_{C^0_0(H^{s - 2})} \lesssim \| \psi_1 \|_{C^0_0(H^s)} \| \psi_2 \|_{C^0_0(H^s)},
\]
which by the Sobolev multiplication law requires \( 2 - s + 2l > \frac{3}{2} \). This is implied by our assumption \( l - s \geq -\frac{1}{2} \) and \( l > \frac{1}{4} \). \( \square \)

**Proof of (29)** This a variant of a proof given by Tao ([19]) for the Yang–Mills case. We have to show
\[
\int \star m(\xi, \tau) \prod_{i = 1}^{3} \tilde{u}_i(\xi_i, \tau_i) \, d\xi \, d\tau \lesssim \prod_{i = 1}^{3} \| u_i \|_{L^2_{3t}},
\]
where \( \xi = (\xi_1, \xi_2, \xi_3), \tau = (\tau_1, \tau_2, \tau_3), \star \) denotes integration over \( \sum_{i = 1}^{3} \xi_i = \sum_{i = 1}^{3} \tau_i = 0 \), and
\[
m = \frac{((\xi_2 | + | \xi_3 |)(\xi_1)^{s - 1}(|\tau_1| - |\xi_1|)^{-\frac{1}{2} + 2\epsilon})}{(\xi_2)^{s}(|\tau_2| - |\xi_2|)^{\frac{3}{2} + \epsilon} (\xi_3)^{s + \frac{1}{2}} (\tau_3)^{\frac{1}{2} + \epsilon}}.
\]
Case 1: \( |\xi_2| \leq |\xi_1| \) \( (\Rightarrow |\xi_2| + |\xi_3| \lesssim |\xi_1|) \).
By two applications of the averaging principle ([18], Prop. 5.1) we may replace $m$ by

$$m' = \frac{\langle \xi_1 \rangle^s \chi_{|\tau_2| - |\xi_2| \sim |1 \chi_{|\tau_3| - 1}}}{\langle \xi_2 \rangle^s \langle \xi_3 \rangle^{s+\frac{1}{2}}}.$$ 

Let now $\tau_2$ be restricted to the region $\tau_2 = T + O(1)$ for some integer $T$. Then $\tau_1$ is restricted to $\tau_1 = -T + O(1)$, because $\tau_1 + \tau_2 + \tau_3 = 0$, and $\xi_2$ is restricted to $|\xi_2| = |T| + O(1)$. The $\tau_1$-regions are essentially disjoint for $T \in \mathbb{Z}$ and similarly the $\tau_2$-regions. Thus by Schur’s test ([18], Lemma 3.11) we only have to show

$$\sup_{T \in \mathbb{Z} - \ast} \left| \int_{\tau_1} \frac{\langle \xi_1 \rangle^s \chi_{|\tau_2| - |\xi_2| \sim |1 \chi_{|\tau_3| - 1}}}{\langle \xi_2 \rangle^s \langle \xi_3 \rangle^{s+\frac{1}{2}}} \int \tilde{u}_i(\xi_i, \tau_i) d\xi d\tau \right| \lesssim \prod_{i=1}^{3} \| u_i \|_{L^2_{\xi_i}}.$$

The $\tau$-behaviour of the integral is now trivial, thus we reduce to

$$\sup_{T \in \mathbb{N}, \sum_{i=1}^{3} \xi_i = 0} \left| \int \frac{\langle \xi_1 \rangle^s \chi_{|\xi_2| = |T + O(1)| \chi_{|\xi_3| = |T|} \chi_{|\tau_3| - 1}}}{\langle T \rangle^s \langle \xi_3 \rangle^{s+\frac{1}{2}}} \int \tilde{f}_1(\xi_1) \tilde{f}_2(\xi_2) \tilde{f}_3(\xi_3) d\xi \lesssim \prod_{i=1}^{3} \| f_i \|_{L^2_{\xi_i}}. \quad (47)$$

Assuming now $|\xi_3| \leq |\xi_1|$ (the other case being simpler) it only remains to consider the following two cases:

Case 1.1: $|\xi_1| \sim |\xi_3| \geq T$. We obtain in this case

$$L.H.S. of \ (47) \lesssim \sup_{T \in \mathbb{N}} \frac{1}{T^{s+\frac{1}{2}}} \| f_1 \|_{L^2} \| f_3 \|_{L^2} \| F^{-1}(\chi_{|\xi_1| = T + O(1)} \tilde{f}_2) \|_{L^\infty(\mathbb{R}^3)}$$

$$\lesssim \sup_{T \in \mathbb{N}} \frac{1}{T^{s+\frac{1}{2}}} \| f_1 \|_{L^2} \| f_3 \|_{L^2} \| \chi_{|\xi_1| = T + O(1)} \tilde{f}_2 \|_{L^1(\mathbb{R}^3)}$$

$$\lesssim \sup_{T \in \mathbb{N}} \frac{T}{T^{s+\frac{1}{2}}} \prod_{i=1}^{3} \| f_i \|_{L^2} \lesssim \prod_{i=1}^{3} \| f_i \|_{L^2},$$

provided $s \geq \frac{3}{4}$.

Case 1.2: $|\xi_1| \sim T \geq |\xi_3|$. An elementary calculation shows that

$$L.H.S. of \ (47) \lesssim \sup_{T \in \mathbb{N}} \| \chi_{|\xi_1| = T + O(1)} \langle \xi \rangle^{-2(s+\frac{1}{2})} \|_{L^\infty(\mathbb{R}^2)} \prod_{i=1}^{3} \| f_i \|_{L^2} \lesssim \prod_{i=1}^{3} \| f_i \|_{L^2}$$

for $s > \frac{3}{4}$, so that the desired estimate follows.

Case 2. $|\xi_1| \leq |\xi_2|$ ($|\xi_2| + |\xi_3| \lesssim |\xi_2|$).

Exactly as in case 1 we reduce to

$$\sup_{T \in \mathbb{N}, \sum_{i=1}^{3} \xi_i = 0} \left| \int \frac{\langle \xi_1 \rangle^{s-1} \chi_{|\xi_2| = |T + O(1)| \chi_{|\xi_3| = |T|} \chi_{|\tau_3| - 1}}}{\langle T \rangle^{s-1} \langle \xi_3 \rangle^{s+\frac{1}{2}}} \int \tilde{f}_1(\xi_1) \tilde{f}_2(\xi_2) \tilde{f}_3(\xi_3) d\xi \lesssim \prod_{i=1}^{3} \| f_i \|_{L^2_{\xi_i}}.$$

This can be treated as in case 1.

**Proof of (30)** By the Sobolev multiplication law we obtain

$$| \int \int fgh dx dt | \lesssim \| f \|_{X_{\tau=0}^{s+\frac{1}{2}}} \| g \|_{X_{\tau=0}^{s-\frac{1}{2}}} \| h \|_{X_{\tau=0}^{\frac{3}{2}-\frac{3}{2}}} \| f \|_{X_{\tau=0}^{\frac{3}{2}-2\epsilon}} \| h \|_{X_{\tau=0}^{\frac{3}{2}-2\epsilon}}.$$
for \( s > \frac{3}{4} \). The elementary estimate \( \langle \xi \rangle^{\frac{1}{2} - 2\epsilon} \langle \tau \rangle^{-(\frac{1}{2} - 2\epsilon)} \lesssim \langle |\tau| - |\xi| \rangle^{\frac{1}{2} - 2\epsilon} \) implies
\[
\|h\|_{X_{\tau=0}^s}^{\frac{3}{2} - s - 2\epsilon, -\frac{1}{2}} \lesssim \|h\|_{X_{\tau=|\xi|}^{1-s, \frac{1}{2} - 2\epsilon}},
\]
thus the claimed estimate.

**Proof of (31)** We may assume \( \frac{3}{4} < s \leq 1 \), because the case \( s > 1 \) follows easily by the fractional Leibniz rule. We obtain
\[
\|A\|_{L_t^4 L_x^2} \lesssim \|A\|_{X_{\tau=0}^{s, \frac{1}{2}-2\epsilon}} \lesssim \|A\|_{X_{\tau=|\xi|}^{1-s, \frac{1}{2}-2\epsilon}},
\]
so that by duality
\[
\|A_1 A_2 A_3\|_{X_{\tau=|\xi|}^{s, \frac{1}{2}+}} \lesssim \|A_1 A_2 A_3\|_{L_t^4 L_x^6} \lesssim \prod_{i=1}^3 \|A_i\|_{L_t^{4+s} L_x^6},
\]
Now by Sobolev
\[
\|A_i\|_{L_t^{4+s} L_x^6} \lesssim \|A_i\|_{L_t^{4+s} L_x^6} \lesssim \|A_i\|_{X_{\tau=|\xi|}^{\frac{1}{2}+}},
\]
and by Strichartz and Sobolev as well
\[
\|A_i\|_{L_t^{4+s} L_x^6} \lesssim \|A_i\|_{L_t^{4+s} H_x^\frac{3}{2}+} \lesssim \|A_i\|_{X_{\tau=|\xi|}^{\frac{3}{2}+}},
\]
which implies the claim.

**Proof of (32)** We apply again Tao’s method [19]. We have to show
\[
\int \sum_{i=1}^3 \bar{u}_i(\xi_j, \tau_i) d\xi_i d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_t^4 L_x^2},
\]
where
\[
m = \frac{\langle \xi_1 \rangle^s (\langle |\tau_1| - |\xi_1| \rangle)^{-\frac{1}{2}+\frac{1}{s}}} {\langle \xi_2 \rangle^s (\langle |\tau_2| - |\xi_2| \rangle)^{\frac{1}{2}+\frac{1}{s}} + \langle \xi_3 \rangle^{\frac{3}{2}+\frac{1}{s}} (\langle |\tau_3| \rangle)^{\frac{1}{2}+\frac{1}{s}}}.
\]
By two applications of the averaging principle ([18], Prop. 5.1) we may replace \( m \) by
\[
m' = \frac{\langle \xi_1 \rangle^s (\langle |\tau_1| - |\xi_1| \rangle)^{-\frac{1}{2}+\frac{1}{s}}}{\langle \xi_2 \rangle^s (\langle |\tau_2| \rangle)^{\frac{1}{2}+\frac{1}{s}} + \langle \xi_3 \rangle^{\frac{3}{2}+\frac{1}{s}}}.
\]
Let now \( \tau_2 \) be restricted to the region \( \tau_2 = T + O(1) \) for some integer \( T \). Then \( \tau_1 \) is restricted to \( \tau_1 = -T + O(1) \), because \( \tau_1 + \tau_2 + \tau_3 = 0 \), and \( \xi_2 \) is restricted to \( |\xi_2| = |T| + O(1) \). The \( \tau_1 \)-regions are essentially disjoint for \( T \in \mathbb{Z} \) and similarly the \( \tau_2 \)-regions. Thus by Schur’s test ([18], Lemma 3.11) we only have to show
\[
\sup_{T \in \mathbb{Z}} \int \frac{\langle \xi_1 \rangle^s (\langle |\tau_1| \rangle)^{-\frac{1}{2}+\frac{1}{s}}}{\langle \xi_2 \rangle^s (\langle |\tau_2| \rangle)^{\frac{1}{2}+\frac{1}{s}} + \langle \xi_3 \rangle^{\frac{3}{2}+\frac{1}{s}}} \sum_{i=1}^3 \bar{u}_i(\xi_i, \tau_i) d\xi_i d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_t^4 L_x^2}.
\]
The $\tau$-behaviour of the integral is now trivial, thus we reduce to
\[
\sup_{T \in \mathbb{N}} \sum_{i=1}^{3} \frac{\langle \xi \rangle^{j} \left|\mathcal{X}^{\xi} \right| \left|\mathcal{X}^{\xi} \right| = |T| + O(1)}{(T)^{j}(\langle \xi \rangle)^{s + \frac{1}{4}}} \mathcal{T}_{1}(\langle \xi \rangle) \mathcal{T}_{2}(\langle \xi \rangle) \mathcal{T}_{3}(\langle \xi \rangle) d\xi \lesssim \prod_{i=1}^{3} \| f_i \|_{L^2}. \tag{48}
\]
Assuming now $|\xi| \leq |\xi|_1$ (the other case being simpler) it only remains to consider the following two cases:

Case 1.1: $|\xi|_1 \sim |\xi|_3 \gtrsim T$. We obtain in this case
\[
L.H.S. of (49) \lesssim \sup_{T \in \mathbb{N}} \frac{1}{T^{s + \frac{1}{4}}} \| f_1 \|_{L^2} \| f_3 \|_{L^2} \| f_5 \|_{L^2} \mathcal{F}^{-1}(\mathcal{X}^{T + O(1)} \mathcal{T}_2) \|_{L^\infty(\mathbb{R}^3)} \lesssim \sup_{T \in \mathbb{N}} \frac{1}{T^{s + \frac{1}{4}}} \| f_1 \|_{L^2} \| f_3 \|_{L^2} \| \mathcal{X}^{T + O(1)} \mathcal{T}_2 \|_{L^1(\mathbb{R}^3)}
\]
\[
\lesssim \sup_{T \in \mathbb{N}} \frac{T}{T^{s + \frac{1}{4}}} \prod_{i=1}^{3} \| f_i \|_{L^2} \lesssim \prod_{i=1}^{3} \| f_i \|_{L^2},
\]
because $s \geq \frac{3}{4}$.

Case 1.2: $|\xi|_1 \sim T \gtrsim |\xi|_3$. An elementary calculation shows that
\[
L.H.S. of (48) \lesssim \sup_{T \in \mathbb{N}} \| \mathcal{X}^{T + O(1)} * (\langle \xi \rangle)^{-2(s + \frac{1}{4})} \|_{L^\infty(\mathbb{R}^3)} \prod_{i=1}^{3} \| f_i \|_{L^2} \lesssim \prod_{i=1}^{3} \| f_i \|_{L^2},
\]
using that $2(s + \frac{1}{4}) > 2$, so that the desired estimate follows. \(\square\)

### 4 Removal of the assumption $A^{df}(0) = 0$

Applying an idea of Keel and Tao [19] we use the gauge invariance of the Yang–Mills–Dirac system to show that the condition $A^{df}(0) = 0$, which had to be assumed in Proposition 3.1, can be removed.

**Lemma 4.1** Let $s > \frac{3}{4}$ and $0 < \epsilon \ll 1$. Assume $(A, \psi) \in (C^0([0, 1], H^s) \cap C^1([0, 1], H^{s-1})) \times (C^0([0, 1], H^s), A_0 = 0$ and
\[
|A^{df}(0)|_{H^s} + |(\partial_t A)^{df}(0)|_{H^{s-1}} + |A^{cf}(0)|_{H^s} + |\psi(0)|_{H^s} \lesssim \epsilon. \tag{49}
\]
Then there exists a gauge transformation $T$ preserving the temporal gauge such that $(TA)^{df}(0) = 0$ and
\[
|T(A)^{df}(0)|_{H^s} + |(\partial_t T A)^{df}(0)|_{H^{s-1}} + |(T \psi(0))|_{H^s} \lesssim \epsilon. \tag{50}
\]
T preserves also the regularity, i.e. $T A \in C^0([0, 1], H^s) \cap C^1([0, 1], H^{s-1})$. $T \psi \in C^0([0, 1], H^s)$. If $A \in X_{+, s}^{\frac{3}{4}}([0, 1] + X_{-, s}^{\frac{3}{4}}([0, 1]) + X_{+, s+\frac{1}{2}}([0, 1], \partial_t A^{cf} \in C^0([0, 1], H^{s-1})$ and $\psi \in X_{+, \frac{1}{2}}([0, 1] + X_{-, \frac{1}{2}}([0, 1])$, then $T A, T \psi$ belong to the same spaces. Its inverse $T^{-1}$ has the same properties.

**Proof of Lemma 4.1** For details of the proof we refer to a similar result for the Yang–Mills and Yang–Mills–Higgs equation in in [13], Lemma 4.1 (cf. also the sketch of proof in [19]).

It is achieved by an iteration argument. We use the Hodge decomposition of $A$:
\[
A = A^{cf} + A^{df} = -|\nabla|^{-2} \nabla div A + A^{df}.
\]
We define \( V_1 := -|\nabla|^{-2} \text{div} A(0) \), so that \( \nabla V_1 = A^{cf}(0) \). Thus
\[
\| V_1 \|_X := \| \nabla V_1 \|_{H^s} = \| A^{cf}(0) \|_{H^s} \leq \epsilon.
\]

We define \( U_1 := \exp(V_1) \) and consider the gauge transformation \( T_1 \) with
\[
\begin{align*}
A_0 &\mapsto U_1 A_0 U_1^{-1} - (\partial_t U_1) U_1^{-1} \\
\psi\pm &\mapsto U_1 \psi\pm, \\
A &\mapsto U_1 A U_1^{-1} - (\nabla U_1) U_1^{-1}
\end{align*}
\]

Then \( T_1 \) preserves the temporal gauge, because \( U_1 \) is independent of \( t \). We obtain by Sobolev
\[
\| (T_1 \psi\pm)(0) \|_{H^t} \lesssim \| \exp V_1 \|_X \| \psi\pm(0) \|_{H^t} \lesssim (1 + \epsilon) \| \psi\pm(0) \|_{H^t}.
\]

Iteratively we define for \( k \geq 2 : \nabla V_k := (T_{k-1} A)^{cf}(0) \) and \( U_k := \prod_{l=k}^{1} \exp V_l \), so that as in \([13], \text{Lemma 4.1}\) we obtain
\[
\| V_k \|_X = \| (T_{k-1} A)^{cf}(0) \|_{H^t} \lesssim \epsilon^{\frac{k+1}{2}} \forall k \geq 2
\]
and \( \| U_k - I \|_X \leq \epsilon \) (thus \( \| U_k \|_X \leq 1 \)). Let the gauge transformation \( T_k \) be defined by
\[
\begin{align*}
A_0 &\mapsto U_k A_0 U_k^{-1} - (\partial_t U_k) U_k^{-1} \\
\psi\pm &\mapsto U_k \psi\pm.
\end{align*}
\]

This implies
\[
\| (T_k \psi\pm)(0) \|_{H^t} \lesssim \| \exp V_k \|_X \| \psi\pm(0) \|_{H^t} \lesssim (1 + \epsilon) \| \psi\pm(0) \|_{H^t}
\]

independently of \( k \). As in \([13], \text{Lemma 4.1}\) this allows to define a gauge transformation \( T \) by
\( TA := \lim_{k \to \infty} T_k A \) in \( C^0([0, 1]; H^s) \), \( \partial_t TA := \lim_{k \to \infty} \partial_t T_k A \) in \( C^0([0, 1]; H^{s-1}) \) and
\( T \psi\pm := \lim_{k \to \infty} T_k \psi\pm \) in \( C^0([0, 1]; H^t) \), which fulfills \((TA)^{cf}(0) = 0\). We also deduce
\[
\| (TA)^{cf}(0) \|_{H^s} + \| (\partial_t TA)^{cf}(0) \|_{H^{s-1}} + \| (T \psi\pm)(0) \|_{H^t} \lesssim \epsilon
\]
and
\[
TA = UAU^{-1} - \nabla UU^{-1}, \ T \psi\pm = U \psi\pm,
\]
where \( U = \prod_{l=\infty}^{1} \exp V_l, U^{-1} = \prod_{l=\infty}^{1} \exp(-V_l) \) and the limits are taken with respect to
\( \| \cdot \|_X \). It has the property \( \| U \|_X = \| \nabla U \|_{H^s} \lesssim 1 \).

We want to show that \( T \) preserves the regularity. That \( TA \) has the same regularity was shown in \([13], \text{Lemma 4.1}\). Let now \( \chi = \chi(t) \) be a smooth function with \( \chi(t) = 1 \) for \( 0 \leq t \leq 1 \) and \( \chi(t) = 0 \) for \( t \geq 2 \). We obtain :
\[
\begin{align*}
\| U \psi\pm \|_{X_{\pm}^\frac{1}{2} + \epsilon, [0, 1]} &\lesssim \| U \psi\pm \chi \|_{X_{\pm}^\frac{1}{2} + \epsilon} \lesssim \| \nabla U \chi \|_{X_{\pm}^{-\frac{1}{2}}} \| \psi\pm \|_{X_{\pm}^{\frac{1}{2} + \epsilon}} \lesssim \| \nabla U \|_{H^s} \| \psi\pm \|_{X_{\pm}^{\frac{1}{2} + \epsilon}} \\
&\lesssim \| \psi\pm \|_{X_{\pm}^{\frac{1}{2} + \epsilon}} < \infty
\end{align*}
\]

Here we applied the estimate
\[
\| uv \|_{X_{\pm}^{\frac{1}{2} + \epsilon}} \lesssim \| \nabla u \|_{X_{\pm}^{-\frac{1}{2}}} \| v \|_{X_{\pm}^{\frac{1}{2} + \epsilon}},
\]
provided \( s > \frac{1}{2} \), for the second step, which is proved as \([13], \text{Lemma 4.2}\). Thus the regularity of \( \psi\pm \) is also preserved.
The inverse $T^{-1}$, defined by

$$T^{-1}B = U^{-1}BU + U^{-1}\nabla U, \quad T^{-1}\psi_{\pm} = U^{-1}\psi_{\pm},$$

has the same properties as $T$. \hfill \Box

5 Proof of Theorem 1.1

Proof It suffices to construct a unique local solution of (10)–(12) with initial conditions

$$A^{df}(0) = a^{df}, \quad (\partial_t A^{df})(0) = a^{\prime df}, \quad A^{cf}(0) = a^{cf}, \quad \psi(0) = \psi_0,$$

which fulfill

$$\|A^{df}(0)\|_{H^s} + \|(\partial_t A)^{df}(0)\|_{H^{s-1}} + \|A^{cf}(0)\|_{H^s} + \|\psi(0)\|_{H^s} \leq \epsilon$$

for a sufficiently small $\epsilon > 0$. By Lemma 4.1 there exists a gauge transformation $T$ which fulfills the smallness condition (50) and $(TA)^{cf}(0) = 0$. We use Proposition 3.1 to construct a unique solution $(\tilde{A}, \tilde{\psi})$ of (10)–(12), where $\tilde{A} = \tilde{A}^+_\pm + \tilde{A}^-_\pm + \tilde{A}^{cf}$ and $\tilde{\psi} = \tilde{\psi}_+ + \tilde{\psi}_-$, with data

$$\tilde{A}^{df}(0) = (TA)^{df}(0), \quad (\partial_t \tilde{A})^{df}(0) = (\partial_t (TA)^{df})(0), \quad \tilde{A}^{cf}(0) = (TA)^{cf}(0) = 0,$$

$$\tilde{\psi}(0) = (T\psi)(0),$$

with the regularity

$$\tilde{A}^{df}_\pm \in X^{s,\frac{1}{2}}_\pm[0, 1], \quad \tilde{A}^{cf} \in X^{s+\frac{1}{2}, \frac{1}{2}+}[0, 1], \quad \partial_t \tilde{A}^{cf} \in C^0([0, 1], H^{s-1}), \quad \tilde{\psi}_\pm \in X^{l, \frac{1}{2}}_\pm[0, 1].$$

This solution satisfies also $\tilde{A} \in C^0([0, 1], H^s) \cap C^1[0, 1], H^{s-1})$, $\tilde{\psi} \in C^0([0, 1], H^l)$. Applying the inverse gauge transformation $T^{-1}$ according to Lemma 4.1 we obtain a unique solution of (10)–(12) with the required initial data and also the same regularity. \hfill \Box

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