Quadratic BSDEs with $L^2$–terminal data Existence results, Krylov’s estimate and Itô–Krylov’s formula

K. Bahlali$^a$, M. Eddahbi$^b$ and Y. Ouknine$^c$

$^a$Université de Toulon, IMATH, EA 2134, 83957 La Garde, France
  e–mail: bahlali@univ-tln.fr

$^b$Université Cadi Ayyad, Faculté des Sciences et Techniques, Département de Mathématiques, B.P. 549, Marrakech, Maroc.
  e–mail: m.eddahbi@uca.ma

$^c$Université Cadi Ayyad, Faculté des Sciences Semlalia, Département de Mathématiques, B.P. 2390 Marrakech, Maroc.
  e–mail: ouknine@uca.ma

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Abstract

In a first step, we establish the existence (and sometimes the uniqueness) of solutions for a large class of quadratic backward stochastic differential equations (QBSDEs) with continuous generator and a merely square integrable terminal condition. Our approach is different from those existing in the literature. Although we are focused on QBSDEs, our existence result also covers the BSDEs with linear growth, keeping $\xi$ square integrable in both cases. As byproduct, the existence of viscosity solutions is established for a class of quadratic partial differential equations (QPDEs) with a square integrable terminal datum.

In a second step, we consider QBSDEs with measurable generator for which we establish a Krylov’s type a priori estimate for the solutions. We then deduce an Itô–Krylov’s change of variable formula. This allows us to establish various existence and uniqueness results for classes of QBSDEs with square integrable terminal condition and sometimes a merely measurable generator. Our results show, in particular, that neither the existence of exponential moments of the terminal datum nor the continuity of the generator are necessary to the existence and/or uniqueness of solutions for quadratic BSDEs. Some comparison theorems are also established for solutions of a class of QBSDEs.

1 Introduction

Let $(W_t)_{0 \leq t \leq T}$ be a $d$–dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $(\mathcal{F}_t)_{0 \leq t \leq T}$ the natural filtration of $W$ augmented with $\mathbb{P}$–negligible sets. Let $H(t, \omega, y, z)$ be a real valued $\mathcal{F}_t$–progressively measurable process defined on $[0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$. Let $\xi$ be an $\mathcal{F}_T$–measurable $\mathbb{R}$–valued random variable. In this paper, we consider a one dimensional BSDE of the form,

$$Y_t = \xi + \int_t^T H(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T \quad (eq(\xi, H))$$

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The data $\xi$ and $H$ are respectively called the terminal condition and the coefficient or the generator of the BSDE $eq(\xi, H)$.

A BSDE is called quadratic if its generator has at most a quadratic growth in the $z$ variable.

For given real numbers $a$ and $b$, we set $a \land b := \min(a, b)$, $a \lor b := \max(a, b)$, $a^- := \max(0, -a)$ and $a^+ := \max(0, a)$. We also define,

$\mathcal{W}^{2, \text{loc}} :=$ the Sobolev space of (classes) of functions $u$ defined on $\mathbb{R}$ such that both $u$ and its generalized derivatives $u'$ and $u''$ belong to $L^1_{\text{loc}}(\mathbb{R})$.

$S^2 :=$ the set of continuous, $\mathcal{F}_t$–adapted processes $\varphi$ such that

$$\mathbb{E} \sup_{0 \leq t \leq T} |\varphi_t|^2 < \infty.$$ 

$\mathcal{M}^2 :=$ the space of $\mathcal{F}_t$–adapted processes $\varphi$ satisfying $\mathbb{E} \int_0^T |\varphi_s|^2 ds < +\infty$.

$\mathcal{L}^2 :=$ the space of $\mathcal{F}_t$–adapted processes $\varphi$ satisfying $\mathbb{E} \int_0^T |\varphi_s|^2 ds < +\infty$ $\mathbb{P}$–a.s.

**Definition 1.1.** A solution to BSDE $eq(\xi, H)$ is an $\mathcal{F}_t$–adapted processes $(Y, Z)$ which satisfy the BSDE $eq(\xi, H)$ for each $t \in [0, T]$ and such that $Y$ is continuous and $\int_0^T |Z_s|^2 ds < \infty$ $\mathbb{P}$–a.s., that is $(Y, Z) \in C \times L^2$, where $C$ is the space of continuous processes.

The first results on the existence of solutions to QBSDEs were obtained independently in [16] and in [9] by two different methods. The approach developed in [16] is based on the monotone stability of QBSDEs and consists to find bounded solutions. Later, many authors have extended the result of [16] in many directions, see e.g. [6, 8, 13, 21, 24]. For instance, in [8], the existence of solutions was proved for QBSDEs in the case where the exponential moments of the terminal datum are finite. In [24], a fixed point method is used to directly show the existence and uniqueness of a bounded solution for QBSDEs with a bounded terminal datum and a (so–called) Lipschitz–quadratic generator. More recently, a monotone stability result for quadratic semimartingales was established in [6] then applied to derive the existence of solutions to QBSDEs in the framework of exponential integrability of the terminal data. The generalized stochastic QBSDEs were studied in [13] under more or less similar assumptions on the terminal datum. Applications of QBSDEs in financial mathematics are also given in [6] with a large bibliography in this subject.

It should be noted that all the previous papers in QBSDEs were developed in the framework of continuous generators and bounded terminal data or at least having finite exponential moments. It is natural to ask the following questions:

1) Are there quadratic BSDEs that have solutions without assuming the existence of exponential moments of the terminal datum? If yes, in what space these solutions lie?

2) Are there quadratic BSDEs with measurable generator that have solutions without assuming the existence of exponential moments of the terminal datum? If yes, in what space these solutions lie?

The present paper gives positive answers to these questions. It is a development and a continuation of our announced results [1]. We do not aim to generalize the previous papers on QBSDEs, but our goal is to give another point of view (on solving QBSDEs) which allows us to establish the existence of solutions, in the space $S^2 \times \mathcal{M}^2$, for a large class of QBSDEs with a square integrable terminal datum. Next, in order to deal with QBSDEs with measurable generator, we had to establish a Krylov’s type a priori estimate and an Itô–Krylov’s formula for the solutions of general QBSDEs.

To begin, let us give a simple example which is covered by the present paper but, to the best of our knowledge, is not covered by the previous results. This example shows that the existence
of exponential moments of the terminal datum is not necessary to the unique solvability of BSDEs in \( S^2 \times M^2 \). Assume that,

\[ \text{(H1)} \quad \xi \text{ is square integrable.} \]

Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a given continuous function with compact support, and set \( M := \sup_{y \in \mathbb{R}} |f(y)| \). The BSDE \( eq(\xi, f(y)|z|^2) \) is then of quadratic growth since \( |f(y)|z|^2| \leq M|z|^2 \). Let \( u(x) := \int_0^x \exp \frac{1}{2} \int_0^y f(y)dt \) dy. If \((Y,Z)\) is a solution to the BSDE \( eq(\xi, f(y)|z|^2) \), then Itô’s formula applied to \( u(Y_t) \) shows that,

\[ u(Y_t) = u(\xi) - \int_t^T u'(Y_s)Z_sdW_s \]

If we set \( \bar{Y}_t := u(Y_t) \) and \( \bar{Z}_t := u'(Y_t)Z_t \), then \((\bar{Y}, \bar{Z})\) solves the BSDE

\[ \bar{Y}_t = u(\xi) - \int_t^T \bar{Z}_s dW_s \]

Since both \( u \) and its inverse are \( C^2 \) smooth functions which are globally Lipschitz and one to one from \( \mathbb{R} \) onto \( \mathbb{R} \), we then deduce that the BSDE \( eq(\xi, f(y)|z|^2) \) admits a solution (resp. a unique solution) if and only if the BSDE \( eq(u(\xi), 0) \) admits a solution (resp. a unique solution). The BSDE \( eq(u(\xi), 0) \) has a unique solution in \( S^2 \times M^2 \) whenever \( u(\xi) \) is merely square integrable. According to the properties of \( u \) and its inverse, \( u(\xi) \) is square integrable if and only if \( \xi \) square integrable. Therefore, even when all the exponential moments are infinite the QBSDE \( eq(\xi, f(y)|z|^2) \) has a unique solution which lies in \( S^2 \times M^2 \). Note that, since the sign of \( f \) is not constant, our example also shows that the convexity of the generator is not necessary to the uniqueness. Assume now that \( \xi \) is merely \( \mathcal{F}_T \) measurable, but not necessarily integrable. According to Dudley’s representation theorem \([10]\), one can show as previously (by using the above transformation \( u \)) that when \( f \) is continuous and with compact support, the BSDE \( eq(\xi, f(y)|z|^2) \) has at least one solution \((Y, Z)\) which belongs to \( C \times \mathcal{L}^2 \).

In the first part of this paper, we establish the existence of solutions for a large class of QBSDEs having a continuous generator and a merely square integrable terminal datum. The generator \( H \) will satisfy

\[ |H(s, y, z)| \leq (a + b|y| + c|z| + f(|y|)|z|^2) \]

where \( f \) is some continuous and globally integrable function on \( \mathbb{R} \) (hence can not be a constant) and \( a, b, c \) are some positive constants.

Our approach consists to deduce the solvability of a BSDE (without barriers) from that of a suitable QBSDE with two Reflecting barriers whose solvability is ensured by \([13]\). This allows us to control the integrability we impose to the terminal datum. In other words, this idea can be summarized as follows: When \( |H(s, y, z)| \leq (a + b|y| + c|z| + f(|y|)|z|^2) \), the existence of solutions for the QBSDE \( eq(\xi, H) \) can be deduced from the existence of solutions to the BSDE driven by the dominating generator \( a + b|y| + c|z| + f(|y|)|z|^2 \). Using the transformation \( u \) (defined in the above first example), we show that the solvability of the QBSDE \( eq(\xi, a + b|y| + c|z| + f(|y|)|z|^2) \) is equivalent to the solvability of a BSDE without quadratic term which is more easily solvable.

We also prove that the uniqueness of solutions holds for the class of QBSDEs \( eq(\xi, f(y)|z|^2) \) under the \( L^2 \)-integrability condition on the terminal data. It is worth to notice that the existence results of \([6, 8, 16, 20, 21]\) can be obtained by our method. We mention that, in contrast to the most previous papers on QBSDEs, our result also cover the BSDEs with linear growth (by putting \( f = 0 \)). It therefore provides a unified treatment for quadratic BSDEs and those of linear growth, keeping \( \xi \) square integrable in both cases.
In the second part of this paper, we begin by proving the Krylov inequality for the solutions of general QBSDEs from which we deduce the Itô–Krylov formula, i.e. we show that the Itô change of variable formula holds for \( u(Y_t) \) whenever \( Y \) is a solution of a QBSDE, \( u \) is of class \( C^1 \) and the second generalized derivative of \( u \) merely belongs to \( \mathbb{L}^1_{\text{loc}}(\mathbb{R}) \).

We then use this change of variable formula to establish the existence (and sometimes the uniqueness) of solutions. To explain more precisely how we get our second aim, let us consider the following assumption,

\[(H2) \text{ There exist a positive stochastic process } \eta \in \mathbb{L}^1([0, T] \times \Omega) \text{ and a locally integrable function } f \text{ such that for every } (t, \omega, y, z),\]

\[|H(t, y, z)| \leq \eta_t + |f(y)||z|^2 \quad \mathbb{P} \otimes dt \text{ a.e.}\]

We first use the occupation time formula to show that if assumption \((H2)\) holds, then for any solutions \((Y, Z)\) of the BSDE \( eq(\xi, H) \), the time spend by \( Y \) in a Lebesgue negligible set is negligible with respect to the measure \( |Z_t|^2 dt \). That is, the following Krylov’s type estimate holds for any positive measurable function \( \psi \),

\[\mathbb{E} \int_0^{T \wedge T_R} \psi(Y_s)|Z_s|^2 ds \leq C \|\psi\|_{L^1([0, T])} \cdot \|\xi\|_{L^1(\Omega)} \] (1.1)

where \( \tau_R \) is the first exit time of \( Y \) from the interval \([-R, R]\) and \( C \) is a constant depending on \( T \), \( \|\xi\|_{L^1(\Omega)} \) and \( \|f\|_{L^1([-R,R])} \).

We then deduce (by assuming \((H1)-(H2))\) that: if \((Y, Z)\) is a solution to the BSDE \( eq(\xi, H) \) which belongs to \( S^2 \times L^2 \), then for any function \( \varphi \in C^1(\mathbb{R}) \cap \mathcal{W}^2_{1, \text{loc}}(\mathbb{R}) \) the following change of variable formula holds true,

\[\varphi(Y_t) = \varphi(Y_0) + \int_0^t \varphi'(Y_s)dY_s + \frac{1}{2} \int_0^t \varphi''(Y_s)|Z_s|^2 ds\] (1.2)

Inequality (1.1) as well as formula (1.2) are interesting in their own and can have potential applications in BSDEs. They are established here with minimal conditions on the data \( \xi \) and \( H \). Indeed, it will be shown that these formulas hold for QBSDEs with a merely measurable generator. For formula (1.2), we require that the terminal datum is square integrable, while for inequality (1.1) we do not need any integrability condition on the terminal datum. Notice that, although the inequality (1.1) can be established by adapting the method developed by Krylov, which is based on partial differential equations [18] (see also [1, 2, 22, 19]), the proof we give here is purely probabilistic and more simple.

As application, we establish the existence of solutions in \( S^2 \times M^2 \) for the classes of QBSDEs \( eq(\xi, f(y)|z|^2) \) and \( eq(\xi, a + by + cz + f(y)|z|^2) \) assuming merely that \( f \) is globally integrable and \( \xi \) is square integrable. Remark that, when \( f \) is not continuous, the function \( u(x) := \int_0^x \exp \left( \int_0^y f(t)dt \right)dy \) is not of \( C^2 \)–class and the classical Itô’s formula can not be applied. Nevertheless, when \( f \) belongs to \( \mathbb{L}^1(\mathbb{R}) \), the function \( u \) belongs to the space \( C^1(\mathbb{R}) \cap \mathcal{W}^2_{1, \text{loc}}(\mathbb{R}) \) and hence formula (1.2) can be applied to \( u \). Our strategy consists then to use the idea we developed in the first part to show the existence of a minimal and a maximal solution for BSDE \( eq(\xi, a + by + cz + f(|y||z|^2)) \).

A comparison theorem is also proved for two BSDEs of type \( eq(\xi, f(|y||z|^2)) \) whenever we can compare their terminal data and \( a.e. \) their generators. We then deduce the uniqueness of solutions for the BSDEs \( eq(\xi, f(|y||z|^2)) \) when \( \xi \) is square integrable and \( f \) belongs to \( \mathbb{L}^1(\mathbb{R}) \). That is, even when \( f \) is defined merely \( a.e. \), the uniqueness holds. This gives a positive answer
to question 2. In particular, the QBSDE \(eq(\xi, H)\) has a unique solution \((Y, Z)\) which belongs to \(S^2 \times M^2\) when \(\xi\) is merely square integrable and \(H\) is one of the following generators:

\[
\begin{align*}
H_1(y, z) &= \sin(y)z^2 \text{ if } y \in [-\pi, \frac{\pi}{2}] \text{ and } H_1(y, z) := 0 \text{ otherwise,} \\
H_2(y, z) &= (1_{[a, b]}(y) - 1_{[c, d]}(y))z^2 \text{ for a given } a < b \text{ and } c < d, \\
H_3(y, z) &= \frac{1}{(1+y^2)^{\sqrt{|y|}}}z^2 \text{ if } y \neq 0 \text{ and } H_2(y, z) := 1 \text{ otherwise.}
\end{align*}
\]

It should be noted that the generator \(H_3(y, z)\) is neither continuous nor locally bounded and the QBSDE \(eq(\xi, H_3)\) has a unique solution in \(S^2 \times M^2\) when \(\xi\) is merely square integrable.

We finally consider the BSDE \(eq(\xi, H)\). We assume that \(\xi\) is square integrable and \(H\) is continuous in \((y, z)\) and \(|H(s, y, z)| \leq a + b|y| + c|z| + f(|y||z|^2)\), with \(f\) merely globally integrable and locally bounded but not necessarily continuous. Although one can argue as in the first part to obtain the existence of solutions from the solutions of a suitable Reflected BSDE, we use a different proof which is based on a classical comparison theorem and an appropriate localization by a suitable dominating process which is derived from the extremal solutions of the QBSDE \(eq(\xi, (a + b|y| + c|z| + f(|y||z|^2))\). This allows us to construct a suitable sequence of BSDEs \(eq(\xi_n, H_n)\) whose localized (i.e. stopped) solutions converge to a solution of the BSDE \(eq(\xi, H)\).

In the third part, we establish the existence of viscosity solutions for a class of non–divergence form semilinear PDEs with quadratic nonlinearity in the gradient variable. This is done with a continuous generator and an unbounded terminal datum. It surprisingly turns out that there is a gap between the BSDEs and the classical formulation of their associated semilinear PDEs (see Remark 5.2, section 5). Observe that the class of quadratic PDEs we study in this paper can be used as a simplified model in some incomplete financial markets, see e.g. [11].

The paper is organized as follows. In section 2, we study the QBSDEs with a continuous generator and a square integrable terminal datum. Krylov’s estimate and Itô-Krylov’s formula for QBSDEs are established in section 3. The solvability of a class of QBSDEs with measurable generator is studied in section 4. An application to the existence of viscosity solutions for Quadratic PDEs associated to the Markovian QBSDE \(eq(\xi, f(y)||z|^2)\) is given in section 5.

## 2 QBSDEs with \(L^2\) terminal data and continuous generators

We will establish the solvability in \(S^2 \times M^2\) for some BSDEs with a square integrable terminal data and a continuous generator. Our method consists to construct a solution of a BSDE without barriers from a solution of a suitable BSDE with two Reflecting barriers. More precisely:

Assuming that \(\xi\) is square integrable and \(f\) is continuous and globally integrable on \(\mathbb{R}\), we first establish the existence of a minimal and a maximal solution in \(S^2 \times M^2\) for the BSDE \(eq(\xi, a + b|y| + c|z| + f(|y||z|^2))\). An next, we consider the BSDE \(eq(\xi, H)\) with \(|H(s, y, z)| \leq (a + b|y| + c|z| + f(|y||z|^2))\). We then use the minimal solution of BSDE \(eq(-\xi^-, - (a + b|y| + c|z| + f(|y||z|^2)))\) and the maximal solution of BSDE \(eq(\xi^+, a+b|y|+c|z|+f(|y||z|^2))\) as barriers, and apply the result of [13] to get the existence of a solution which stays between these two barriers. We finally deduce the solvability of \(eq(\xi, H)\) by proving that the increasing stochastic processes, which force the solutions to stay between the barriers, are equal to zero.

The following lemma is needed for the sequel of the paper. It allows us to eliminate the additive quadratic term.
Lemma 2.1. Let $f$ be continuous and belongs to $L^1(\mathbb{R})$. The function

$$u(x) := \int_0^x \exp \left( 2 \int_0^y f(t) \, dt \right) \, dy$$

has the following properties,

(i) $u \in C^2(\mathbb{R})$ and satisfies the equation $\frac{1}{2} u''(x) - f(x)u'(x) = 0$, in $\mathbb{R}$.

(ii) $u$ is a one to one function from $\mathbb{R}$ onto $\mathbb{R}$.

(iii) The inverse function $u^{-1}$ belongs to $C^2(\mathbb{R})$.

(iv) $u$ is a quasi–isometry, that is there exist two positive constants $m$ and $M$ such that, for any $x, y \in \mathbb{R}$, $m|x-y| \leq |u(x) - u(y)| \leq M|x-y|$

2.1 The equation $eq(\xi, f(y)|z|^2)$

Remark 2.1. Let $\xi$ be an $\mathcal{F}_T$–measurable random variable. According to Dudley [11], there exists a (non necessary unique) $\mathcal{F}_t$–adapted process $(Z_t)_{0 \leq t \leq T}$ such that $\int_0^T |Z_s|^2 \, ds < \infty$ $\mathbb{P}$–a.s and $\xi = \int_0^T Z_s \, dW_s$. The process $(Y_t)_{0 \leq t \leq T}$ defined by $Y_t = \xi - \int_0^t Z_s \, dW_s$ is $\mathcal{F}_t$–adapted and satisfies the equation $eq(\xi,0)$. This solution $(Y_t, Z_t)_{0 \leq t \leq T}$ is not unique. However, if we assume $\xi \in L^2(\Omega)$ then the solution $(Y, Z)$ is unique and $Y_t = \mathbb{E}[\xi/\mathcal{F}_t]$.

The following proposition shows that the exponential moment of $\xi$ is not needed to obtain the existence and uniqueness of the solution to quadratic BSDEs.

Proposition 2.1. (i) Assume $(H1)$ be satisfied. Let $f$ be a continuous and integrable function. Then the BSDE $eq(\xi, f(y)|z|^2)$ has a unique solution in $S^2 \times L^2$ (resp. in $S^2 \times M^2$) if and only if the BSDE $eq(u(\xi),0)$ has a unique solution in $S^2 \times L^2$ (resp. in $S^2 \times M^2$).

(ii) In particular, the BSDE $eq(\xi, f(y)|z|^2)$ has a unique solution $(Y, Z)$ which belongs to $S^2 \times M^2$.

Proof. Let $u$ be the function defined in Lemma 3.1. Theorem 3.1 and Lemma 4.1 allow us to show that, $(Y_t, Z_t)$ is the unique solution of BSDE $eq(\xi, f(y)|z|^2)$ if and only if $(Y, Z) := (u(Y_t), u'(Y_t)Z_t)$ is the unique solution to BSDE $eq(u(\xi),0)$. We shall prove assertion (ii). Since $\xi$ is square integrable then $u(\xi)$ is square integrable too. Therefore $eq(u(\xi),0)$ has a unique solution in $S^2 \times M^2$. Assertion (ii) follows now from assertion (i). \hfill \blacksquare

2.2 The equation $eq(\xi, a + b|y| + c|z| + f(y)|z|^2)$

The BSDE under consideration in this subsection is,

$$Y_t = \xi + \int_t^T (a + b|Y_s| + c|Z_s| + f(Y_s)|Z_s|^2) \, ds - \int_t^T Z_s \, dW_s$$

(2.2)

where $a$, $b$, $c \in \mathbb{R}$ and $f : \mathbb{R} \mapsto \mathbb{R}$.

We refer to BSDE (2.2) as equation $eq(\xi, a + b|y| + c|z| + f(y)|z|^2)$.

Proposition 2.2. Assume that $(H1)$ holds. Assume also that $f$ is continuous and globally integrable on $\mathbb{R}$. Let $u$ be the function defined in Lemma 4.1. Then the BSDE $eq(\xi, a + b|y| + c|z| + f(y)|z|^2)$ has a minimal and a maximal solution. Moreover all solutions of $eq(\xi, a + b|y| + c|z| + f(y)|z|^2)$ are in $S^2 \times M^2$. 


Proof. Itô’s formula applied to the function $u$ (which is defined in Lemma 2.1) shows that $(Y_t, Z_t)$ is a (unique, resp. minimal, resp. maximal) solution to the BSDE $\text{eq}(\xi, a + b|y| + c|z| + f(y)|z|^2)$ if and only if $(\bar{Y}_t, \bar{Z}_t) := (u(Y_t), u'(Y_t)Z_t)$ is a (unique, resp. minimal, resp. maximal) solution to the BSDE $\text{eq}(u(\xi), (a + b|u^{-1}(\bar{y})|)|u'|u^{-1}(\bar{y})| + c|\bar{z}|)$. We shall prove the existence of solutions to BSDE $\text{eq}(u(\xi), (a + b|u^{-1}(\bar{y})|)|u'|u^{-1}(\bar{y})| + c|\bar{z}|)$ (c). By Theorem 3.1 we have

$$\bar{Y}_t = \bar{x} + \int_t^T G(\bar{Y}_s, \bar{Z}_s)ds - \int_t^T \bar{Z}_s dW_s \quad (2.3)$$

where $G(\bar{y}, \bar{z}) := (a + b|u^{-1}(\bar{y})|)|u'|u^{-1}(\bar{y})| + c|\bar{z}|$.

From Lemma 2.1, we deduce that the generator $G$ is continuous and with linear growth, and the terminal condition $\bar{x} := u(\xi)$ is square integrable (since Assumption (H1)). Hence, according to Lepeltier & San-Martin [20], the BSDE $(2.3)$ has at least one solution in $S^2 \times M^2$. To complete the proof, it is enough to observe that the function $u$ defined in Lemma 2.1 is strictly increasing.

Alternative proof to Proposition 2.2.

In the previous proof of Proposition 2.2 we had to use the Lepeltier & San-Martin result [20] in order to quickly deduce the existence of a minimal and a maximal solution to BSDE which belong to $S^2 \times M^2$ whenever the terminal datum $\xi$ is square integrable. This fact will be proved below by using an alternative proof which is in adequacy with the spirit of the present paper.

To this end, we use a result on two barriers Reflected QBSDEs obtained by Essaky & Hassani in [14] which establishes the existence of solutions for reflected QBSDEs without assuming any integrability condition on the terminal datum. For the self-contained, we state the result of [14] in the following theorem.

**Theorem 2.1.** ([14], Theorem 3.2). Let $L$ and $U$ be continuous processes and $\xi$ be a $\mathcal{F}_T$ measurable random variable. Assume that

1) for every $t \in [0, T]$, $L_t \leq U_t$

2) $L_T \leq \xi \leq U_T$.

3) there exists a continuous semimartingale which passes between the barriers $L$ and $U$.

4) $H$ is continuous in $(y, z)$ and satisfies for every $(s, \omega)$, every $y \in [L_s(\omega), U_s(\omega)]$ and every $z \in \mathbb{R}^d$.

$$|f(s, \omega, y, z)| \leq \eta_s(\omega) + C_s(\omega)|z|^2$$

where $\eta \in \mathbb{L}^1([0, T] \times \Omega)$ and $C$ is a continuous process.

Then, the following RBSDE has a minimal and a maximal solution.

$$
\begin{align*}
(i) & \quad Y_t = \xi + \int_t^T H(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s \\
& \quad \quad + \int_t^T dK^+_s - \int_t^T dK^-_s \text{ for all } t \leq T \\
(ii) & \quad \forall t \leq T, \; L_t \leq Y_t \leq U_t, \\
(iii) & \quad \int_t^T (Y_t - L_t)dK^+_t = \int_t^T (U_t - Y_t)dK^-_t = 0, \; a.s., \\
(iv) & \quad K^+_0 = K^-_0 = 0, \; K^+, K^- \text{ are continuous nondecreasing.} \\
(v) & \quad dK^+ \bot dK^-
\end{align*}
$$

We are now in the position to give our alternative proof to Proposition 2.2.

Note that, since $u$ is strictly increasing, we then only need to prove the existence of a minimal
Lemma 2.1, one can show that the generator and a maximal solutions for the BSDE (2.3). Since Lemma 2.1 the terminal condition \( u \) growth. Indeed, since \( u(0) = 0 \) and \( u' \) is bounded by \( M \) (Lemma 2.1 (iv)), we have

\[
G(y, z) = (a + b|u^{-1}(y)|)u'[u^{-1}(y)] + c|z| \\
\leq Ma + mMb|y| + c|z| := g(y, z)
\]

(2.5)

where \( m \) and \( M \) are the constants which appear in assertion (iv) of Lemma 2.1

Since the function \( g(y, z) := Ma + mMb|y| + c|z| \) is uniformly Lipschitz and with linear growth in \((y, z)\), then according to Pardoux & Peng result [23], the BSDEs \( eq(-\xi^-, -g) \) and \( eq(\xi^+, g) \) have unique solutions in \( S^2 \times M^2 \), which we respectively denote by \((Y^{-g}, Z^{-g})\) and \((Y^g, Z^g)\).

Note that \( Y^{-g} \) is negative and \( Y^g \) is positive. Using Theorem 2.1 (with \( L = Y^{-g}, U = Y^g, \eta_t = Ma + mMb(|Y^{-g}_t| + |Y^g_t|) + c^2, \) and \( C_t = 1 \)), we deduce the existence of a minimal solution \((Y^{min}_t, Z^{min}_t, K^{min+}_t, K^{min-}_t)\) and a maximal solution \((Y^{max}_t, Z^{max}_t, K^{max+}_t, K^{max-}_t)\) to the following Reflected BSDE,

\[
\begin{align*}
(i) \quad Y_t &= \xi + \int_t^T H(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \; t \leq T, \\
&\quad + \int_t^T dK^+_t - \int_t^T dK^-_t \quad \text{for all} \; t \leq T \\
(ii) \quad \forall \; t \leq T, \; Y^{t^-}_t \leq Y_t \leq Y^{t^+_t}, \\
(iii) \quad \int_0^T (Y_t - Y^{t^-}_t)dK^+_t = \int_0^T (Y^{t^+_t}_t - Y_t)dK^-_t = 0, \; a.s., \\
(iv) \quad K^+_t = K^-_t = 0, \; K^+, K^- \text{ are continuous nondecreasing,} \\
(v) \quad dK^+ \perp dK^-
\end{align*}
\]

and both \((Y^{min}_t, Z^{min}_t)\) and \((Y^{max}_t, Z^{max}_t)\) are in \( C \times \Delta^2 \).

We shall show that \( dK^+ = dK^- = 0 \). Let \((Y, Z, K^+, K^-)\) be a solution to equation (2.4). Since \( Y^g_t \) is a solution to the BSDE \( eq(\xi^+, g) \), then Tanaka’s formula applied to \((Y^g_t - Y_t)^+\) shows that

\[
(Y^g_t - Y_t)^+ = (Y^g_0 - Y_0)^+ + \int_0^t 1_{\{Y^g_s > Y_s\}}[H(s, Y_s, Z_s) - g(s, Y^g_s, Z^g_s)]ds \\
+ \int_0^t 1_{\{Y^g_s > Y_s\}}(dK^+_s - dK^-_s) + \int_0^t 1_{\{Y^g_s > Y_s\}}(Z^g_s - Z_s)dW_s \\
+ L^0_t(Y^g - Y)
\]

where \( L^0_t(Y^g - Y) \) is the local time at time \( t \) and level 0 of the semimartingale \((Y^g - Y)\).

Since \( Y^g \geq Y \), then \((Y^g_t - Y_t)^+ = (Y^g_t - Y_t)\). Therefore, identifying the terms of \((Y^g_t - Y_t)^+\) with those of \((Y^g_t - Y_t)\) and using the fact that:

\[
1 - 1_{\{Y^g_s > Y_s\}} = 1_{\{Y^g_s \leq Y_s\}} = 1_{\{Y^g_s = Y_s\}},
\]

we prove that

\[
Z = Z^g \quad a.e.
\]

Using the previous equalities, one can show that

\[
\int_0^t 1_{\{Y^g_s = Y_s\}}(dK^+_s - dK^-_s) = L^0_t(Y^g - Y) \\
+ \int_0^t 1_{\{Y^g_s = Y_s\}}[g(s, Y^g_s, Z^g_s) - H(s, Y_s, Z_s)]ds
\]

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Since $\int_0^t 1_{\{Y^g_s = Y_s\}} dK^+_s = 0$, it holds that
\[ 0 \leq L_0^0(Y^g - Y) + \int_0^t 1_{\{Y^g_s = Y_s\}} [g(s, Y^g_s, Z^g_s) - H(s, Y_s, Z_s)] ds \]
\[ = - \int_0^t 1_{\{Y^g_s = Y_s\}} dK^-_s \leq 0 \]
Hence, $\int_0^t 1_{\{Y^g_s = Y_s\}} dK^-_s = 0$, which implies that $dK^- = 0$. Arguing symmetrically, one can show that $dK^+ = 0$. Therefore $(Y, Z)$ is a solution to the (non reflected) BSDE $eq(\xi, H)$. Note that since both $Y^g$ and $Y^{-g}$ belong to $S^2$, then $Y$ also belongs to $S^2$. Since $G$ is with linear growth and $Y$ belongs to $S^2$, then using standard arguments of BSDEs, one can show that $Z$ belongs to $M^2$. We now show that $(Y^{\min}, Z^{\min})$ and $(Y^{\max}, Z^{\max})$ we just construct are respectively a minimal and a maximal solution to BSDE (2.3). Let $(Y, Z)$ be another solution to BSDE (2.3). Since $g$ and $-g$ are Lipschitz, then by the classical comparison theorem we have $Y^{\min}_t \leq Y_t \leq Y^{\max}_t$ for each $t$, $\mathbb{P}$-a.s. Hence, $(Y, Z, 0, 0)$ is a solution to the previous Reflected BSDE such that $(Y, Z)$ lives in $S^2 \times M^2$. Therefore, $Y^{\min}_t \leq Y_t \leq Y^{\max}_t$ for each $t$, $\mathbb{P}$-a.s. This completes the “alternative proof of Proposition 2.2).

Remark 2.2. The previous proof also constitute an alternative proof to the result of Lepeltier & San-Martin for the existence of a minimal and a maximal solution to BSDEs with continuous and at most of linear growth generator. It is worth noting that the idea consists to construct a solution of a BSDE with linear growth from a solution of a Reflected Quadratic BSDE.

2.3 The BSDE $eq(\xi, H)$, with $|H(s, y, z)| \leq a + b|y| + c|z| + f(|y|)|z|^2$

Consider the assumptions,

(H4) For a.e. $(s, \omega)$, $H$ is continuous in $(y, z)$

(H5) There exist positive real numbers $a, b, c$ such that for every $s, y, z$

\[ |H(s, y, z)| \leq a + b|y| + c|z| + f(|y|)|z|^2 \equiv g(y, z), \]

where $f$ is some positive continuous and integrable function.

Theorem 2.2. Assume that (H1), (H4) and (H5) are fulfilled. Then, the BSDE $eq(\xi, H)$ has at least one solution $(Y, Z)$ in $S^2 \times M^2$.

Proof of Theorem 2.2. The idea is close to the above “Alternative proof of proposition 2.2” and consists to derive the existence of solution for the BSDE without reflection from solutions of a suitable 2-barriers Reflected BSDE. Put $g(y, z) := a + b|y| + c|z| + f(|y|)|z|^2$. According to Proposition 2.2 let $(Y^g, Z^g)$ be the maximal solution of BSDE $eq(\xi^+, g)$ and $(Y^{-g}, Z^{-g})$ be the minimal solution of BSDE $eq(-\xi^-, -g)$. We know by Proposition 2.2 that $(Y^g$ and $Y^{-g})$ belong to $S^2$. Using Theorem 2.1 (with $L = Y^{-g}$, $U = Y^g$, $\eta_t = a + b(|Y^{-g}_t| + |Y^g_t|) + c^2$, and $C_t = 1 + \sup_{s \leq t} \sup_{a \in [0, 1]} |f(aY^{-g}_s + (1 - a)Y^g_s)|$), we deduce the existence of solution $(Y, Z, K^+, K^-)$ to the following Reflected BSDE, such that $(Y, Z)$ belongs to $C \times \mathcal{L}^2$. 

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Lemma 2.2. Let $f$ be continuous and integrable function on $\mathbb{R}$. Set

$$K(y) := \int_0^y \exp \left( -2 \int_0^x f(r) \, dr \right) \, dx.$$ 

The function

$$u(x) := \int_0^x K(y) \exp \left( 2 \int_0^y f(t) \, dt \right) \, dy$$

We shall show that $dK^+ = dK^- = 0$. Since $Y^g_t$ is a solution for the BSDE $eq(\xi, g)$, then Tanaka's formula applied to $(Y^g_t - Y_t)^+$ shows that

$$
(Y^g_t - Y_t)^+ = (Y^g_0 - Y_0)^+ + \int_0^t \mathbf{1}_{\{Y^g > Y_s\}} [H(s, Y_s, Z_s) - g(s, Y^g_s, Z^g_s)] \, ds
$$

$$+ \int_0^t \mathbf{1}_{\{Y^g > Y_s\}} (dK^+_s - dK^-_s) + \int_0^t \mathbf{1}_{\{Y^g > Y_s\}} (Z^g_s - Z_s) \, dW_s
$$

$$+ L^0_t (Y^g - Y)$$

where $L^0_t (Y^g - Y)$ denotes the local time at time $t$ and level $0$ of the semimartingale $(Y^g - Y)$. Since $Y^g \geq Y$, then $(Y^g_t - Y_t)^+ = (Y^g_t - Y_t)$. Therefore, identifying the terms of $(Y^g_t - Y_t)^+$ with those of $(Y^g_t - Y_t)$ and using the fact that $1 - \mathbf{1}_{\{Y^g \leq Y_s\}} = \mathbf{1}_{\{Y^g = Y_s\}}$, we show that $Z = Z^g$ a.e.

Using the previous equalities, we deduce that

$$\int_0^t \mathbf{1}_{\{Y^g = Y_s\}} (dK^+_s - dK^-_s) = L^0_t (Y^g - Y)
$$

$$+ \int_0^t \mathbf{1}_{\{Y^g = Y_s\}} [g(s, Y_s, Z_s) - H(s, Y_s, Z_s)] \, ds$$

Since $\int_0^t \mathbf{1}_{\{Y^g = Y_s\}} dK^+_s = 0$, it holds that

$$0 \leq L^0_t (Y^g - Y) + \int_0^t \mathbf{1}_{\{Y^g = Y_s\}} [g(s, Y_s, Z_s) - H(s, Y_s, Z_s)] \, ds
$$

$$= - \int_0^t \mathbf{1}_{\{Y^g = Y_s\}} dK^-_s \leq 0$$

Hence, $\int_0^t \mathbf{1}_{\{Y^g = Y_s\}} dK^-_s = 0$, which implies that $dK^- = 0$. Arguing symmetrically, one can show that $dK^+ = 0$. Therefore $(Y, Z)$ satisfies the (non reflected) BSDE $eq(\xi, H)$. Note that since both $Y^g$ and $Y^{-g}$ belong to $S^2$, then $Y \in S^2$ belongs to $S^2$ too.

In order to complete the proof of Theorem 2.2, it remains to show that $Z$ belongs to $M^2$. To this end, we need the following lemma.
We now prove that $N > 0$ and $\text{sgn}(x) = 1$ if $x \geq 0$ and $\text{sgn}(x) = 0$ if $x < 0$. Let $u$ be the function defined in Lemma 2.2 and $v := u(|y|)$. Since $v$ belongs to $C^3(\mathbb{R})$, then using Itô’s formula it holds that for every $t \in [0, T]$,

$$u(|Y_{t \vee \tau_N}|) = u(|Y_0|) + \int_0^{t \wedge \tau_N} \left[ \frac{1}{2} u''(|Y_s|)|Z_s|^2 - \text{sgn}(Y_s) u'(|Y_s|) H(s, Y_s, Z_s) \right] ds$$

$$+ \int_0^{t \wedge \tau_N} \text{sgn}(Y_s) u'(|Y_s|) Z_s dW_s$$

Passing to expectation and using successively assumption (H5) and Lemma 2.2, we get for any $N > 0$

$$u(|Y_0|) = u(|Y_{t \vee \tau_N}|)$$

$$+ \int_0^{t \wedge \tau_N} \left[ \text{sgn}(Y_s) u'(|Y_s|) H(s, Y_s, Z_s) - \frac{1}{2} u''(|Y_s|)|Z_s|^2 \right] ds$$

$$\leq u(|Y_{t \vee \tau_N}|)$$

$$+ \int_0^{t \wedge \tau_N} \left[ u'(|Y_s|)(a + b|Y_s| + c|Z_s| + f(|Y_s|)|Z_s|^2) - \frac{1}{2} u''(|Y_s|)|Z_s|^2 \right] ds$$

$$\leq u(|Y_{t \vee \tau_N}|) + \int_0^{t \wedge \tau_N} \left[ u'(|Y_s|)(a + b|Y_s| + c|Z_s|) - \frac{1}{2} |Z_s|^2 \right] ds$$

$$\leq u(|Y_{t \vee \tau_N}|) + \int_0^{t \wedge \tau_N} \left[ u'(|Y_s|)(a + b|Y_s|) + c u'(|Y_s|)|Z_s| - \frac{1}{2} |Z_s|^2 \right] ds$$

$$\leq u(|Y_{t \vee \tau_N}|)$$

$$+ \int_0^{t \wedge \tau_N} \left[ u'(|Y_s|)(a + b|Y_s|) + 4c^2 u'(|Y_s|)^2 + \frac{1}{4} |Z_s|^2 - \frac{1}{2} |Z_s|^2 \right] ds$$

Hence,

$$\frac{1}{4} \mathbb{E} \int_0^{t \wedge \tau_N} |Z_s|^2 ds \leq u(|Y_0|) + \mathbb{E} \int_0^T [(a + b|Y_s|) u'(|Y_s|) + 4c^2 (u'(|Y_s|))^2] ds$$

We successively use Lemma 2.2- (iii), the fact that the process $Y$ belongs to $\mathcal{S}^2$ and Fattou’s lemma, to show that $\mathbb{E} \int_0^T |Z_s|^2 ds < \infty$. Theorem 2.2 is proved.

### 3 Krylov’s estimates and Itô–Krylov’s formula in QBSDEs

**Remark 3.1.** (i) The Krylov estimate for QBSDEs is obtained with minimal conditions. Indeed, the generator $H$ will be assumed merely measurable and the terminal condition $\xi$ merely integrable.
(ii) It is worth noting that the change of variable formula we will establish here for the solutions of QBSDEs is valid although the martingale part of $Y$ can degenerate. Actually, the martingale part of $Y$ can degenerate with respect to the Lebesgue measure but remains nondegenerate with respect to the measure $|Z_t|^2 dt$.

(iii) The Krylov estimate for QBSDE we state in the next proposition can be established by using Krylov's method \cite{18} (see also \cite{12, 13, 22}), which is based on partial differential equations. The proof we give here is probabilistic and very simple. It is based on the time occupation formula.

3.1 Krylov’s estimates in QBSDEs.

Proposition 3.1. (Local estimate) Assume (H2) holds. Let $(Y,Z)$ be a solution of the BSDE eq$(\xi,H)$ and assume that $\int_0^T |H(s,Y_s,Z_s)|ds < \infty \ \mathbb{P}$–a.s. Then, there exists a positive constant $C$ depending on $T,R$ and $\|f\|_{L^1([-R,R])}$ such that for any nonnegative measurable function $\psi$,

$$
\mathbb{E} \int_0^{\tau_R} \psi(Y_s)|Z_s|^2 ds \leq C \|\psi\|_{L^1([-R,R])},
$$

where $\tau_R := \inf\{t > 0 : |Y_t| \geq R\}$.

Proof. Without loss of generality, we can and assume that $\eta = 0$ in assumption (H2). Set $\tau'_N := \inf\{t > 0, \int_0^t |Z_s|^2 ds \geq N\}$, $\tau''_M := \inf\{t > 0, \int_0^t |H(s,Y_s,Z_s)|ds \geq M\}$, and put $\tau := \tau_R \wedge \tau'_N \wedge \tau''_M$. Let $a$ be a real number such that $a \leq R$. By Tanaka’s formula, we have

$$(Y_{t \wedge \tau} - a)^- = (Y_0 - a)^- - \int_0^{t \wedge \tau} 1_{\{Y_s < a\}}dY_s + \frac{1}{2} L^a_{t \wedge \tau}(Y)$$

$$= (Y_0 - a)^- - \int_0^{t \wedge \tau} 1_{\{Y_s < a\}}H(s,Y_s,Z_s)ds$$

$$+ \int_0^{t \wedge \tau} 1_{\{Y_s < a\}}Z_s dW_s + \frac{1}{2} L^a_{t \wedge \tau}(Y)$$

Since the map $y \mapsto (y-a)^-$ is Lipschitz, we obtain

$$\frac{1}{2} L^a_{t \wedge \tau}(Y) \leq |Y_{t \wedge \tau} - Y_0| + \int_0^{t \wedge \tau} 1_{\{Y_s < a\}}H(s,Y_s,Z_s)ds$$

$$- \int_0^{t \wedge \tau} 1_{\{Y_s < a\}}Z_s dW_s$$

(3.1)

Passing to expectation, we obtain

$$\sup_a \mathbb{E} [L^a_{t \wedge \tau}(Y)] \leq 4R + 2M \quad (3.2)$$

Since $-R \leq Y_{t \wedge \tau} \leq R$ for each $t$, then $\text{Support}(L^2(Y_{t \wedge \tau})) \subset [-R,R]$. Therefore, using inequality (3.1), assumption (H2) and the time occupation formula, we get

$$\frac{1}{2} L^a_{t \wedge \tau}(Y) \leq |Y_{t \wedge \tau} - Y_0| + \int_0^{t \wedge \tau} 1_{\{Y_s < a\}}|f(Y_s)||Z_s|^2 ds - \int_0^{t \wedge \tau} 1_{\{Y_s < a\}}Z_s dW_s$$

$$\leq |Y_{t \wedge \tau} - Y_0| + \int_0^{t \wedge \tau} 1_{\{Y_s < a\}}|f(Y_s)|dY_s - \int_0^{t \wedge \tau} 1_{\{Y_s < a\}}Z_s dW_s$$

$$\leq |Y_{t \wedge \tau} - Y_0| + \int_{-R}^{a} |f(x)|L^a_{t \wedge \tau R}(Y)dx - \int_0^{t \wedge \tau} 1_{\{Y_s < a\}}Z_s dW_s$$

$$\leq |Y_{t \wedge \tau} - Y_0| + \int_{-R}^{a} |f(x)|L^a_{t \wedge \tau R}(Y)dx - \int_0^{t \wedge \tau} 1_{\{Y_s < a\}}Z_s dW_s$$

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Passing to expectation, we obtain
\[
\frac{1}{2} \mathbb{E}[L^a_{t\wedge \tau}(Y)] \leq \mathbb{E}|Y_{t\wedge \tau} - Y_0| + \int_{-R}^a |f(x)| \mathbb{E}[L^a_{t\wedge \tau}(Y)] \, dx < \infty
\]

Hence, by inequality (3.2) and Gronwall lemma we get
\[
\mathbb{E}[L^a_{t\wedge \tau}(Y)] \leq 2 \mathbb{E}(|Y_{t\wedge \tau} - Y_0|) \exp \left(2 \int_{-R}^a |f(x)| \, dx\right)
\leq 2 \mathbb{E}(|Y_{t\wedge \tau} - Y_0|) \exp \left(2 \|f\|_{L^1([-R,R])}\right)
\leq 4R \exp \left(2 \|f\|_{L^1([-R,R])}\right) \|\psi\|_{L^1([-R,R])}
\]

Passing to the limit on $N$ and $M$ (having in mind that $\tau := \tau_{R\wedge \tau'N\wedge \tau'M}$) and using Beppo–Levi theorem we get
\[
\mathbb{E}[L^a_{t\wedge \tau}(Y)] \leq 4R \exp \left(2 \|f\|_{L^1([-R,R])}\right) \|\psi\|_{L^1([-R,R])}
\]

Let $\psi$ be an arbitrary positive function. We use the previous inequality to show that
\[
\mathbb{E} \int_0^{T\wedge \tau_R} \psi(Y_s)|Z_s|^2 \, ds = \mathbb{E} \int_0^{T\wedge \tau_R} \psi(Y_s)d(Y_s)
\leq \mathbb{E} \int_{-R}^R \psi(a)L^a_{t\wedge \tau_R}(Y) \, da
\leq \int_{-R}^R \psi(a) \mathbb{E}L^a_{t\wedge \tau_R}(Y) \, da
\leq 4R \exp \left(2 \|f\|_{L^1([-R,R])}\right) \|\psi\|_{L^1([-R,R])}
\]

Proposition 3.1 is proved. \hfill \blacksquare

We now consider the following assumption.

(H3) The function $f$, defined in assumption (H2), is globally integrable on $\mathbb{R}$. 

Arguing as previously, one can prove the following global estimate.

Corollary 3.1. Assume that (H1), (H2) and (H3) are satisfied. 

Let $(Y, Z) \in S^2 \times L^2$ be a solution of BSDE $\text{eq}(\xi, H)$. Assume moreover that \( \int_0^T |H(s, Y_s, Z_s)| \, ds < \infty \) $\mathbb{P}$-a.s. Then, there exists a positive constant $C$ depending on $T$, $\|\xi\|_{L^1(\Omega)}$, $\|f\|_{L^1(\mathbb{R})}$ and $\mathbb{E}(\sup_{t \leq T} |Y_t|)$ such that, for any nonnegative measurable function $\psi$,

\[
\mathbb{E} \int_0^T \psi(Y_s)|Z_s|^2 \, ds \leq C \|\psi\|_{L^1(\mathbb{R})}
\]

In particular,

\[
\mathbb{E} \int_0^{T\wedge \tau_R} \psi(Y_s)|Z_s|^2 \, ds \leq C \|\psi\|_{L^1([-R,R])}
\]

where $\tau_R := \inf\{t > 0 : |Y_t| \geq R\}$. 

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3.2 An Itô–Krylov’s change of variable formula in BSDEs

In this subsection we shall establish an Itô–Krylov’s change of variable formula for the solutions of one dimensional BSDEs. This will allows us to treat some QBSDEs with measurable generator. Let’s give a summarized explanation on Itô–Krylov’s formula. The Itô change of variable formula expresses that the image of a semimartingale, by a $C^2$–class function, is a semimartingale. When

$$ X_t := X_0 + \int_0^t \sigma(s, \omega) dW_s + \int_0^t b(s, \omega) ds $$

is an Itô’s semimartingale, the so-called Itô–Krylov’s formula (established by N.V. Krylov) expresses that if $\sigma^*$ is uniformly elliptic, then Itô’s formula also remains valid when $u$ belongs to $W^{2,p}_{p,loc}$ with $p$ strictly more large than the dimension of the process $X$. Here $W^{2,p}_{p,loc}$ denotes the Sobolev space of (classes) of functions $u$ defined on $\mathbb{R}$ such that both $u$ and its generalized derivatives $u', u''$ belong to $L^p_{loc}(\mathbb{R})$. The Itô–Krylov formula was extended in [2] to continuous semimartingales $X_t := X_0 + M_t + V_t$ with a non degenerate martingale part and some additional conditions. The non degeneracy means that the matrix of the increasing processes $\langle M^i, M^j \rangle$ is uniformly elliptic.

**Theorem 3.1.** Assume that (H1) and (H2) are satisfied. Let $(Y, Z)$ be a solution of BSDE $eq(\xi, H)$ in $S^2 \times \mathcal{L}^2$. Then, for any function $u$ belonging to the space $C^1(\mathbb{R}) \cap W^{2,1}_{1,loc}(\mathbb{R})$, we have

$$ u(Y_t) = u(Y_0) + \int_0^t u'(Y_s) dY_s + \frac{1}{2} \int_0^t u''(Y_s)|Z_s|^2 ds \quad (3.4) $$

Using Sobolev’s embedding theorem and Lemma 3.1, we get,

**Corollary 3.2.** Assume (H1) and (H2) be satisfied. Let $(Y, Z)$ be a solution of BSDE $eq(\xi, H)$ in $S^2 \times \mathcal{L}^2$. Then, for any function $u \in W^{2,1}_{p,loc}(\mathbb{R})$ with $p > 1$, the formula (3.4) remains valid.

**Proof of Theorem 3.1**

For $R > 0$, let $\tau_R := \inf\{t > 0 : |Y_t| \geq R\}$. Since $\tau_R$ tends to infinity as $R$ tends to infinity, it then suffices to establish the formula for $u(Y_{t \wedge \tau_R})$. Using Proposition 3.1, the term $\int_0^{t \wedge \tau_R} u''(Y_s) Z_s^2 ds$ is well defined.

Let $u_n$ be a sequence of $C^2$–class functions satisfying

(i) $u_n$ converges uniformly to $u$ in the interval $[-R, R]$.
(ii) $u'_n$ converges uniformly to $u'$ in the interval $[-R, R]$
(iii) $u''_n$ converges in $\mathbb{L}^1([-R, R])$ to $u''$.

We use Itô’s formula to show that,

$$ u_n(Y_{t \wedge \tau_R}) = u_n(Y_0) + \int_0^{t \wedge \tau_R} u'_n(Y_s) dY_s + \frac{1}{2} \int_0^{t \wedge \tau_R} u''_n(Y_s)|Z_s|^2 ds $$

 Passing to the limit (on $n$) in the previous identity and using the above properties (i), (ii), (iii) and Proposition 3.1 we get

$$ u(Y_{t \wedge \tau_R}) = u(Y_0) + \int_0^{t \wedge \tau_R} u'(Y_s) dY_s + \frac{1}{2} \int_0^{t \wedge \tau_R} u''(Y_s)|Z_s|^2 ds $$

Indeed, the limit for the left hand side term, as well as those of the first and the second right hand side terms can be obtained by using properties (i) and (ii). The limit for the third right hand side term follows from property (iii) and Proposition 3.1.
4 QBSDEs with $L^2$ terminal data and measurable generators

The present section will be developed in the same spirit of section 3. The Itô–Krylov formula (established in section 4) will replace the Itô formula in all proofs. Thanks to Itô–Krylov’s formula, the following lemma, will play the same role as Lemma 2.1 when $f$ is merely measurable. In particular, it allows us to eliminate the additive quadratic term from the simple QBSDEs $eq(\xi, f(y)|z|^2)$ and $eq(\xi, a + b|y| + c|z| + f(y)|z|^2)$.

Lemma 4.1. Let $f$ belongs to $L^1(\mathbb{R})$. The function

$$u(x) := \int_0^x \exp \left( 2 \int_0^y f(t)dt \right) dy$$

satisfies the following properties,

(i) $u \in C^1(\mathbb{R}) \cap W^2_{1,loc}(\mathbb{R})$ and satisfies, for a.e. $x$, $\frac{1}{2}u''(x) - f(x)u'(x) = 0$.

(ii) $u$ is a one to one function from $\mathbb{R}$ onto $\mathbb{R}$.

(iii) The inverse function $u^{-1}$ belongs to $C^1(\mathbb{R}) \cap W^2_{1,loc}(\mathbb{R})$.

(iv) There exist two positive constants $m$ and $M$ such that, for any $x, y \in \mathbb{R}$, $m |x - y| \leq |u(x) - u(y)| \leq M |x - y|$

4.1 The equation $eq(\xi, f(y)|z|^2)$

The following proposition shows that neither the exponential moment of $\xi$ nor the continuity of the generator are needed to obtain the existence and uniqueness of the solution to quadratic BSDEs.

Proposition 4.1. Assume (H1) be satisfied. Let $f$ be a globally integrable function on $\mathbb{R}$. Then, the BSDE $eq(\xi, f(y)|z|^2)$ has a unique solution $(Y, Z)$ which belongs to $S^2 \times M^2$.

Proof. Let $u$ be the function defined in Lemma 4.1. Since $\xi$ is square integrable then $u(\xi)$ is square integrable too. Therefore $eq(u(\xi), 0)$ has a unique solution in $S^2 \times M^2$. The proposition follows now by applying Itô–Krylov’s formula to the function $u^{-1}$ which belongs to the space $C^1(\mathbb{R}) \cap W^2_{1,loc}(\mathbb{R})$.

The following proposition allows us to compare the solutions for QBSDEs of type $eq(\xi, f(y)|z|^2)$. The novelty is that the comparison holds whenever we can only compare the generators for a.e. $y$. Moreover, both the generators can be non–Lipschitz.

Proposition 4.2. (Comparison) Let $\xi_1, \xi_2$ be $F_T$–measurable and satisfy assumption (H1). Let $f, g$ be in $L^1(\mathbb{R})$. Let $(Y^f, Z^f)$, $(Y^g, Z^g)$ be respectively the solution of the BSDEs $eq(\xi_1, f(y)|z|^2)$ and $eq(\xi_2, g(y)|z|^2)$. Assume that $\xi_1 \leq \xi_2$ a.s. and $f \leq g$ a.e. Then $Y_t^f \leq Y_t^g$ for all $t \in \mathbb{R}$.

Proof. According to Proposition 4.1 the solutions $(Y^f, Z^f)$ and $(Y^g, Z^g)$ belong to $S^2 \times M^2$. For a given function $h$, we put

$$u_h(x) := \int_0^x \exp \left( 2 \int_0^y h(t)dt \right) dy$$
Since $Y$ is positive, then $u_f(Y_T^g) = u_f(Y_t^g) + \int_t^T u_f'(Y_s^g) dY_s^g + \frac{1}{2} \int_t^T u_f''(Y_s^g) d\langle Y_s^g \rangle_s$

$$= u_f(Y_t^g) + M_T - M_t - \int_t^T u_f'(Y_s^g) |g(Y_s^g)| Z_s^g ds + \frac{1}{2} \int_t^T u_f''(Y_s^g) |Z_s^g|^2 ds$$

Since $u_f'(x) - 2g(x)u_f(x) = 0$, $u_f'(x) - 2f(x)u'_f(x) = 0$ and $u_f'(x) \geq 0$, then

$$u_f(Y_T^g) = u_f(Y_t^g) + M_T - M_t - \int_t^T u_f'(Y_s^g) [g(Y_s^g) - f(Y_s^g)] |Z_s^g|^2 ds$$

where $(M_t)_{0 \leq t \leq T}$ is a martingale.

Since the term

$$\int_t^T u_f'(Y_s^g) [g(Y_s^g) - f(Y_s^g)] |Z_s^g|^2 ds$$

is positive, then

$$u_f(Y_t^g) \geq u_f(Y_T^g) + M_T - M_t$$

Since $Y_t^f$ and $Y_t^g$ is $\mathcal{F}_t$-adapted, then passing to conditional expectation and using the fact that $u_f$ is an increasing function and $\xi_2 \geq \xi_1$, we get

$$u_f(Y_t^g) \geq \mathbb{E} \left[ u_f(Y_T^g) / \mathcal{F}_t \right]$$

$$= \mathbb{E} \left[ u_f(\xi_2) / \mathcal{F}_t \right]$$

$$\geq \mathbb{E} \left[ u_f(\xi_1) / \mathcal{F}_t \right]$$

$$= u_f \left( Y_t^f \right)$$

Passing to $u_f^{-1}$, we get $Y_t^g \geq Y_t^f$. Proposition 4.2 is proved.

The following uniqueness result is a consequence of the previous proposition.

**Corollary 4.1.** Let $\xi$ satisfies (H1) and $f$, $g$ be integrable functions. Let $(Y^f, Z^f)$ and $(Y^g, Z^g)$ respectively denote the (unique) solutions of the BSDE $eq(\xi, f(y)|z|^2)$ and $eq(\xi, g(y)|z|^2)$. If $f = g$–a.e., then $(Y^f, Z^f) = (Y^g, Z^g)$ in $S^2 \times M^2$.

**Remark 4.1.** Proposition 4.2 and Corollary 4.1 will be used in the PDEs part, to show the existence of a gap in the classical relation between the BSDEs and their corresponding PDEs.

### 4.2 The equation $eq(\xi, a + b|y| + c|z| + f(y)|z|^2)$

The BSDE under consideration in this subsection is,

$$Y_t = \xi + \int_t^T (a + b|Y_s| + c|Z_s| + f(Y_s)|Z_s|^2) ds - \int_t^T Z_s dW_s$$

where $a$, $b$, $c \in \mathbb{R}$ and $f : \mathbb{R} \mapsto \mathbb{R}$.

We refer to BSDE (4.2) as equation $eq(\xi, a + b|y| + c|z| + f(y)|z|^2)$.
Proposition 4.3. Assume that \((H1)\) is satisfied. Assume moreover that \(f\) is globally integrable on \(\mathbb{R}\). Then, the BSDE \(\text{eq}(\xi, a + b|y| + c|z| + f(y)|z|^2)\) has a minimal and a maximal solution. Moreover all solutions are in \(S^2 \times M^2\).

Proof. Let \(u\) be the function defined in Lemma 4.1 Consider the BSDE

\[
\dot{Y}_t = \xi + \int_t^T G(\bar{Y}_s, \bar{Z}_s)ds - \int_t^T \bar{Z}_s dW_s
\]

where \(G(\bar{y}, \bar{z}) := (a + b|u^{-1}(\bar{y})|)|u'|^{2}(\bar{y}) + c|\bar{z}|\).

From Lemma 4.1 we deduce that the terminal condition \(\bar{\xi} := u(\xi)\) is square integrable (since Assumption \((H1)\)) and the generator \(G\) is continuous and with linear growth. Arguing then as in the "alternative proof of Proposition 2.2", one can prove that the BSDE \((4.3)\) has a maximal and a minimal solutions in \(S^2 \times M^2\). Applying now the Itô–Krylov formula to the function \(u^{-1}(\bar{Y}_t)\), we show that the BSDE \(\text{eq}(\xi, a + b|y| + c|z| + f(y)|z|^2)\) has a solution. Since \(u\) is a strictly increasing function, we then deduce the existence of a minimal and a maximal solutions for the initial equation \(\text{eq}(\xi, a + b|y| + c|z| + f(y)|z|^2)\).

4.3 The BSDE \(\text{eq}(\xi, H)\) with \(|H(s, y, z)| \leq a + b|y| + c|z| + f(|y|)|z|^2\)

Consider the assumption,

\((H6)\) There exist positive real numbers \(a, b, c\) such that for every \(s, y, z\)

\[
|H(s, y, z)| \leq a + b|y| + c|z| + f(|y|)|z|^2 := g(y, z),
\]

where \(f\) is some positive locally bounded integrable function, but not necessarily continuous.

Theorem 4.1. Assume that \((H1), (H4)\) and \((H6)\) are fulfilled. Then, the BSDE \(\text{eq}(\xi, H)\) has at least one solution \((Y, Z)\) which belongs to \(S^2 \times M^2\).

Remark 4.2. Although the proof of Theorem 4.1 may be performed as that of Theorem 2.2, we will give another proof which consists to use a comparison theorem and an appropriate localization by a suitable dominating process which is derived from the extremal solutions of the two QBSDEs \(\text{eq}(-\xi^-, -(a + b|y| + c|z| + f(|y|)|z|^2))\) and \(\text{eq}(\xi^+, (a + b|y| + c|z| + f(|y|)|z|^2))\).

To prove Theorem 4.1 we need the following two lemmas. The first one allows us to show that \(Z\) belongs to \(M^2\) while the second is a comparison theorem for our context.

Lemma 4.2. Let \(f\) belongs to \(L^1(\mathbb{R})\) and put \(K(y) := \int_0^y \exp(-2\int_0^x f(r)dr)dx\). The function

\[
u(x) := \int_0^x K(y) \exp\left(2\int_0^y f(t)dt\right)dy\]

satisfies following properties:

(i) \(u\) belongs to \(C^1(\mathbb{R}) \cap W^2_{1,\text{loc}}(\mathbb{R})\), and, \(u(x) \geq 0\) and \(u'(x) \geq 0\) for \(x \geq 0\).

Moreover, \(u\) satisfies for a.e. \(x\), \(\frac{1}{2}u''(x) - f(x)u'(x) = \frac{1}{2}\).

(ii) The map \(x \mapsto \nu(x) := u(|x|)\) belongs to \(C^1(\mathbb{R}) \cap W^2_{1,\text{loc}}(\mathbb{R})\), and \(\nu'(0) = 0\).

(iii) There exist \(c_1 > 0\), \(c_2 > 0\) such that for every \(x \in \mathbb{R}, u(|x|) \leq c_1|x|^2\) and \(u'(|x|) \leq c_2|x|\).
Lemma 4.3. (Comparison) Let $h_1(t, \omega, y, z)$ be uniformly Lipschitz in $(y, z)$ uniformly with respect to $(t, \omega)$. Let $h_2(t, \omega, y, z)$ be $\mathcal{F}_t$-progressively measurable and such that for every process $U, V \in \mathcal{S}^2 \times \mathcal{M}^2$, $\mathbb{E}\int_0^T |h_2(s, U_s, V_s)| ds < \infty$. Let $\xi_i \in L^2(\Omega), (i = 1, 2)$, be an $\mathcal{F}_T$-measurable random variables. Let $(Y^1, Z^1) \in \mathcal{S}^2 \times \mathcal{M}^2$ be the unique solution of BSDE $eq(\xi_1, h_1)$ and $(Y^2, Z^2) \in \mathcal{S}^2 \times \mathcal{M}^2$ be a solution of BSDE $eq(\xi_2, h_2)$. Assume that, $\xi_1 \leq \xi_2$ for a.s. $\omega$ and $h_1(s, Y^1_s, Z^1_s) \leq h_2(s, Y^2_s, Z^2_s)$ for a.e. $s, \omega$. Then, $Y^1_t \leq Y^2_t$ for every $t$ and a.s. $\omega$.

Proof. Applying Itô’s formula to $(Y^1_t - Y^2_t)^+$, and since $(\xi_1 - \xi_2)^+ = 0$,

$$
\left( (Y^1_t - Y^2_t)^+ \right)^2 + \int_t^T 1_{\{Y^1_s > Y^2_s\}} \|Z^1_s - Z^2_s\|^2 ds
$$

$$
= 2 \int_t^T (Y^1_s - Y^2_s)^+ \left[ h_1(s, Y^1_s, Z^1_s) - h_1(s, Y^2_s, Z^2_s) \right] ds
$$

$$
- 2 \int_t^T (Y^1_s - Y^2_s)^+ \left[ Z^1_s - Z^2_s \right] dW_s
$$

Passing to expectation and using the fact that $h_1(s, Y^2_s, Z^2_s) \leq h_2(s, Y^2_s, Z^2_s)$, we obtain

$$
\mathbb{E} \left[ (Y^1_t - Y^2_t)^+ \right]^2 + \mathbb{E} \int_t^T 1_{\{Y^1_s > Y^2_s\}} \|Z^1_s - Z^2_s\|^2 ds
$$

$$
= 2\mathbb{E} \int_t^T (Y^1_s - Y^2_s)^+ \left[ h_1(s, Y^1_s, Z^1_s) - h_1(s, Y^2_s, Z^2_s) \right] ds
$$

$$
+ 2\mathbb{E} \int_t^T (Y^1_s - Y^2_s)^+ \left[ h_1(s, Y^2_s, Z^2_s) - h_2(s, Y^2_s, Z^2_s) \right] ds
$$

$$
\leq 2\mathbb{E} \int_t^T (Y^1_s - Y^2_s)^+ \left[ h_1(s, Y^1_s, Z^1_s) - h_1(s, Y^2_s, Z^2_s) \right] ds
$$

$$
\leq 2\mathbb{E} \int_t^T (Y^1_s - Y^2_s)^+ \left[ 1_{\{Y^1_s > Y^2_s\}} + 1_{\{Y^1_s \leq Y^2_s\}} \right]
$$

$$
\times \left[ h_1(s, Y^1_s, Z^1_s) - h_1(s, Y^2_s, Z^2_s) \right] ds
$$

Since $1_{\{Y^1_s \leq Y^2_s\}} (Y^1_s - Y^2_s)^+ = 0$, we then have

$$
\mathbb{E} \left[ (Y^1_t - Y^2_t)^+ \right]^2 + \mathbb{E} \int_t^T 1_{\{Y^1_s > Y^2_s\}} \|Z^1_s - Z^2_s\|^2 ds
$$

$$
\leq 2\mathbb{E} \int_t^T (Y^1_s - Y^2_s)^+ \left[ 1_{\{Y^1_s > Y^2_s\}} + 1_{\{Y^1_s \leq Y^2_s\}} \right]
$$

$$
\times \left[ h_1(s, Y^1_s, Z^1_s) - h_1(s, Y^2_s, Z^2_s) \right] ds
$$

Using the fact that $h_1$ is Lipschitz and the inequality $ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$, we obtain

$$
E \left[ (Y^1_t - Y^2_t)^+ \right]^2 \leq CE \int_t^T \left[ (Y^1_s - Y^2_s)^+ \right]^2 ds.
$$

Using Gronwall’s lemma, we get $(Y^1_t - Y^2_t)^+ = 0$ for every $t$ and a.s. $\omega$, which implies that $Y^1_t \leq Y^2_t$ for every $t$, and a.s. $\omega$. \[\square\]
Proof of Theorem 4.1. We assume for simplicity that \( \xi \) is positive. Let \( Y^g \) be the maximal solution of the BSDE \( eq(\xi, g) \) and \( Y^{-g} \) be the minimal solution of the BSDE \( eq(-\xi, -g) \). By Proposition 4.3, \( Y^{-g} \) and \( Y^g \) exist and belong to \( S^2 \times M^2 \). Let \( \xi_n := \xi \land n \). Let \( (H_n) \) be an increasing sequence of Lipschitz functions which converges to \( H \) uniformly on compact sets. For each \( n \), we denote by \( (Y^n, Z^n) \) the unique solution of the BSDE \( eq(\xi_n, H_n) \). We know that for every \( n \), \( (Y^n, Z^n) \) belongs to \( S^2 \times M^2 \). Since \( \xi_n \leq \xi \) and \( H_n \leq g \) for each \( n \), then Lemma 4.3 (comparison) shows that for every \( n \), \( t \) and a.s. \( \omega \),

\[
|Y_t^n| \leq |Y_t^{-g}| + |Y_t^g| := S_t
\]

For \( R > 0 \), we define a stopping time \( \tau_R \) by

\[
\tau_R := \inf\{ t \geq 0 \mid S_t \geq R \} \land T
\]

The process \( (Y_t^{n,R}, Z_t^{n,R}) := (Y_t^{n,R}, 1_{\{t \leq \tau_R\}}Z_t^n)_{0 \leq t \leq T} \) satisfies then the BSDE \( eq(Y_t^{n,R}, H_n, R) \)

\[
Y_t^{n,R} = Y_t^{n,R} + \int_t^T 1_{\{s \leq \tau_R\}} H_n(s, Y_s^{n,R}, Z_s^{n,R}) ds - \int_t^T Z_s^{n,R} dW_s
\]

From inequality (4.4) and the definition of \( \tau_R \), we deduce that for every \( n \) and every \( t \in [0, T] \),

\[
|Y_t^{n,R}| \leq S_{t \land \tau_R} \leq R
\]

As in [3] (see also [21]), we define a function \( \rho \) by,

\[
\rho(y) := -R1_{\{y < -R\}} + y1_{\{-R \leq y \leq R\}} + R1_{\{y > R\}}
\]

It is not difficult to prove that \( (Y^{n,R}, Z^{n,R}) \) solves the BSDE \( eq(Y_t^{n,R}, H_n(t, \rho(y), z)) \).

Since \( f \) is locally bounded, then for every \( y \) satisfying \( |y| \leq R \) we have,

\[
|H_n(t, \rho(y), z)| \leq a + b|\rho(y)| + c|z| + \sup_{|y| \leq R} f(|\rho(y)|)|z|^2
\]

\[
\leq a_1 + bR + C(R)|z|^2
\]

where \( a_1 := a + c^2 \) and \( C(R) = 1 + \sup_{|y| \leq R} f(|\rho(y)|) \).

Therefore, passing to the limit on \( n \) and using the Kobylanski monotone stability result [16], one can show that for any \( R > 0 \), the sequence \( (Y^{n,R}, Z^{n,R}) \) converges to a process \( (Y^R, Z^R) \) which satisfies the following BSDE on \([0, T]\),

\[
Y_t^R = \xi^R + \int_t^T 1_{\{s \leq \tau_R\}} H(s, Y_s^R, Z_s^R) ds - \int_t^T Z_s^R dW_s
\]

where \( Y_t^R = \sup_n Y_t^{n,R} \) and \( \xi^R := \sup_n Y_{\tau_R}^n \).

Arguing as in the proof of Theorem 2.2 but we use Lemma 4.2 (in place of Lemma 2.2) and Itô–Krylov’s formula (in place of Itô’s formula), one can show that \( (Y^R, Z^R) \) belongs to \( S^2 \times M^2 \).

We define

\[
Y_t := \lim_{R \to \infty} Y_{t \land \tau_R} \quad \text{for} \quad t \in [0, T]
\]
and 
\[ Z_t := Z^R_t \text{ for } t \in (0, \tau_R) \]

Passing to the limit on \( R \) in the previous BSDE eq\((\xi, H, R)\), one can show that the process \((Y, Z) := (Y_t, Z_t)_{t \leq T}\) satisfies the BSDE eq\((\xi, H)\).

Since \( Y^{-g} \) and \( Y^g \) belong to \( S^2 \), we deduce that \( Y \) belongs to \( S^2 \) also.

Arguing as in the proof of Theorem 2.2, but use Lemma 4.2 (in place of Lemma 2.2) and Itô–Krylov’s formula (in place of Itô’s formula) one can prove that \( Z \in M^2 \).

We shall prove that \( Y_T = \xi \). Let \( Y' \) be the minimal solution of BSDE eq\((\xi, -g)\) and \( Y'' \) be the maximal solution of BSDE eq\((\xi, g)\). Using Lemma 4.3 we obtain, for any \( n, R \) and \( t \in [0, T] \)
\[ Y'_{t \wedge \tau_R} \leq Y^n_{t \wedge \tau_R} \leq Y''_{t \wedge \tau_R} \]

Passing to the limit on \( n \), we obtain,
\[ Y'_{t \wedge \tau_R} \leq \sup_n Y^n_{t \wedge \tau_R} := Y_{t \wedge \tau_R} \leq Y''_{t \wedge \tau_R} \]

Putting \( t = T \), we get for any \( R \)
\[ Y'_{\tau_R} \leq Y_{\tau_R} \leq Y''_{\tau_R} \]

Since \( Y' \) and \( Y'' \) are continuous and \( Y_T = Y''_T = \xi \), then letting \( R \) tends to infinity, we get
\[ \xi \leq \liminf_{R \to \infty} Y_{\tau_R} \leq \limsup_{R \to \infty} Y_{\tau_R} \leq \xi \]

Theorem 4.1 is proved.

Remark 4.3. We are currently working to drop the global integrability condition on the function \( f \). The situation becomes more delicate under this (local integrability) condition. It requires some localization arguments and supplementary assumptions on the terminal condition.

5 Application to Quadratic Partial Differential Equations

Let \( \sigma, b \) be measurable functions defined on \( \mathbb{R}^d \) with values in \( \mathbb{R}^{d \times d} \) and \( \mathbb{R}^d \) respectively. Let \( a := \sigma \sigma^* \) and define the operator \( L \) by

\[ L := \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i} \]

Let \( \psi \) be a measurable function from \( \mathbb{R}^d \) to \( \mathbb{R} \). Consider the following semi-linear PDE

\[ \begin{cases} \frac{\partial v}{\partial s}(s, x) = L v(s, x) + f(v(s, x)) |\nabla_x v(s, x)|^2, \text{ on } [0, T) \times \mathbb{R}^d \\ v(T, x) = \psi(x) \end{cases} \] (5.1)

Assumptions.

(H7) \( \sigma, b \) are uniformly Lipschitz.

(H8) \( \sigma, b \) are of linear growth and \( f \) is continuous and integrable.

(H9) The terminal condition \( \psi \) is continuous and with polynomial growth.
Theorem 5.1. Assume (H7), (H8) and (H9) hold. Then, \( v(t, x) := Y_t^{t,x} \) is a viscosity solution for the PDE (5.1).

Remark 5.1. The conclusion of Theorem 5.1 remains valid if the assumption (H7) is replaced by: “the martingale problem is well–posed for \( a := \sigma \sigma^* \) and \( b \).”

To prove the existence of viscosity solution, we will follow the idea of [16]. To this end, we need the following touching property. This allows to avoid the comparison theorem. The proof of the touching property can be found for instance in [16].

Lemma 5.1. Let \((\xi_t)_{0 \leq t \leq T}\) be a continuous adapted process such that

\[
d\xi_t = \beta(t)dt + \alpha(t)dW_t,
\]

where \( \beta \) and \( \alpha \) are continuous adapted processes such that \( b, |\sigma|^2 \) are integrable. If \( \xi_t \geq 0 \) a.s. for all \( t \), then for all \( t \),

\[
1_{\{\xi_t = 0\}} \alpha(t) = 0 \quad \text{a.s.},
\]

\[
1_{\{\xi_t = 0\}} \beta(t) \geq 0 \quad \text{a.s.},
\]

Proof of Theorem 5.1. We first prove the continuity of \( v(t, x) := Y_t^{t,x} \). Let \( u \) be the transformation defined in Lemma 4.1. Let \((\bar{Y}_t^{t,x}, \bar{Z}_t^{t,x})\) be the unique \( \mathcal{M}^2 \) solution of the BSDE eq\((u(\psi(X_t^{t,x}), 0). \) Using assumption (H7), one can show that the map \((t, x) \mapsto \bar{Y}_t^{t,x}\). Using Lemma 4.1 we deduce that \( v(t, x) := Y_t^{t,x} \) is continuous in \((t, x)\). We now show that \( v \) is a viscosity subsolution for PDE (5.1). We denote \((X_t, Y_t, Z_t) := (X_t^{t,x}, Y_t^{t,x}, Z_t^{t,x})\). Since \( v(t, x) = Y_t^{t,x} \), then the Markov property of \( X \) and the uniqueness of \( Y \) show that for every \( s \in [0, T] \)

\[
v(s, X_s) = Y_s
\]  

(5.2)

Let \( \phi \in C^{1,2} \) and \((t, x)\) be a local maximum of \( v - \phi \) which we suppose global and equal to 0, that is:

\[
\phi(t, x) = v(t, x) \quad \text{and} \quad \phi(t, \bar{x}) \geq v(t, \bar{x}) \quad \text{for each} \quad (t, \bar{x}).
\]

This and equality (5.2) imply that

\[
\phi(s, X_s) \geq Y_s
\]  

(5.3)

By Itô’s formula we have

\[
\phi(s, X_s) = \phi(t, X_t) + \int_t^s \left( \frac{\partial \phi}{\partial r} + L\phi \right)(r, X_r)dr + \int_t^s \sigma \nabla_x \phi(r, X_r)dW_r
\]

Since \( \phi(s, X_s) \geq Y_s \), and \( Y \) satisfies the equation

\[
Y_t = Y_s + \int_t^s f(Y_r)|Z_r|^2 dr - \int_t^s Z_r dW_r,
\]

then the touching property shows that for each \( s \),

\[
1_{\{\phi(s, X_s) = Y_s\}} \left( \frac{\partial \phi}{\partial t} + L\phi \right)(s, X_s) + f(Y_s)|Z_s|^2 \geq 0 \quad \text{a.s.,}
\]

and

\[
1_{\{\phi(s, X_s) = Y_s\}} |\sigma^T \nabla_x \phi(s, X_s) - Z_s| \geq 0 \quad \text{a.s.}
\]

Since for \( s = t \), \( \phi(t, x) := \phi(t, X_t) = Y_t := v(t, x) \), then the second equation gives \( Z_t = \sigma \nabla_x \phi(t, X_t) := \sigma \nabla_x \phi(t, x) \), and the first inequality gives the desired result. \( \blacksquare \)
Remark 5.2. From Theorem 5.1, one can see that there is a gap between the solution of a BSDE and the viscosity solutions of its associated PDE. That is, the existence of a unique solution to a BSDE (even when the comparison theorem holds) does not systematically allow to define a viscosity solution to the associated PDE. Indeed, the Corollary 4.1 shows that the QBSDE \( \xi, f_1(y)|z|^2 \) and \( \xi, f_2(y)|z|^2 \) generate the same solution when \( f_1 \) and \( f_2 \) are equal almost surely. Thereby, for a square integrable \( \xi \) and \( f \) belonging to \( L^1(\mathbb{R}) \), the QBSDE \( \xi, f(y)|z|^2 \) has a unique solution in \( S^2 \times M^2 \), but how define the associated PDE (5.1) when \( f \) is defined merely a.e. ? What meaning to give to \( f(v(t, x)) \) when \( v(t, x) \) stays to the set where \( f \) is not defined ? We think that, when \( f \) is defined merely a.e., the associated PDE associated to BSDE \( \xi, f_2(y)|z|^2 \) would has the form

\[
\begin{cases}
\frac{\partial v}{\partial s}(s, x) = Lv(s, x) + f(v(s, x))|\nabla_x v(s, x)|^2, & \text{on } [0, T) \times \mathbb{R}^d \\
v(T, x) = \psi(x) \\
\nabla v(t, x) = 0 \quad \text{if } v(t, x) \in N_f
\end{cases}
\] (5.4)

where \( N_f \) denotes the negligible set of all real numbers \( y \) for which \( f \) is not defined.

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