Truncated harmonic oscillator and Painlevé IV and V equations

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Dedicated to Professor Bogdan Mielnik for his 50 years of scientific career.

Abstract. Quantum systems described by second and third order polynomial Heisenberg algebras are obtained applying supersymmetric quantum mechanics to the harmonic oscillator with an infinite potential barrier. These systems are linked with the Painlevé IV and V equations, respectively, thus several solutions of these non-linear second-order differential equations will be found, along with a chain of Bäcklund transformations connecting such solutions.

1. Introduction
The methods and techniques of supersymmetric quantum mechanics [1–22] can be applied in a straightforward manner to the harmonic oscillator [9,23–25]. As a result, new systems are obtained, called supersymmetric (SUSY) partners of the harmonic oscillator. A well known fact about them is that they realize the polynomial Heisenberg algebras (PHA). These are deformations of the harmonic oscillator algebra, characterized by a commutator between the ladder operators which is a polynomial in the Hamiltonian.

Now, it has been shown that systems which realize a second-order PHA are connected to the Painlevé IV (PIV) equation while those ruled by a third-order PHA are linked with the Painlevé V (PV) equation [23,26–30]. Thus, once a family of potentials characterized by the required PHA is obtained, for example, through the techniques of supersymmetric quantum mechanics, said connections can be exploited to produce non-singular solutions to the PIV and PV equations [13,31,32].

In previous works we implemented this analysis for the harmonic oscillator with an infinite barrier at the origin, or truncated harmonic oscillator for short [6,33,34]. Solutions to the PIV and PV equations have been found, along with the required supersymmetric partners of the initial system. In this paper we shall study several connections among such solutions, in the form of Bäcklund transformations.

The organization of this work is the following: in Section 2 we review the results of applying supersymmetric quantum mechanics to the truncated harmonic oscillator. In Section 3 we discuss the connection between polynomial Heisenberg algebras and the Painlevé IV and V differential equations. In Section 4 we obtain several solutions to the PIV equation using realizations of second order PHA’s and relate them using Bäcklund transformations. In Section 5,
using realizations of third order PHA’s, we obtain solutions to the PV equation and relate them using Bäcklund transformations. Section 6 contains our conclusions and some final remarks.

2. Truncated harmonic oscillator
As was stated in the introduction, we will obtain solutions of the Painlevé IV and V equations using the supersymmetric partners of the truncated harmonic oscillator, the last being described by the Hamiltonian

\[
H = -\frac{1}{2} \frac{d^2}{dx^2} + V,
\]

with

\[
V = \begin{cases} 
\frac{x^2}{2} & \text{if } x > 0, \\
\infty & \text{if } x \leq 0.
\end{cases}
\]

The eigenfunctions of \( H \), corresponding to the eigenvalues \( \mathcal{E}_n = 2n + \frac{3}{2} \) are the odd eigenfunctions of the harmonic oscillator, i.e.,

\[
\psi_n(x) \propto x e^{-x^2/2} {}_1F_1\left(-n, \frac{3}{2}, x^2\right),
\]

since they have to satisfy the boundary conditions, particularly at the origin. The even eigenfunctions of the harmonic oscillator fail to do so, and this is the reason why the quantities \( \mathcal{E}_n = 2n + \frac{1}{2} \) do not belong to the spectrum of \( H \).

Anyway, they will be useful for the SUSY technique, thus we present them now:

\[
\chi_n(x) \propto e^{-x^2/2} {}_1F_1\left(-n, \frac{1}{2}, x^2\right), \quad \mathcal{E}_n = 2n + \frac{1}{2}.
\]

In order to obtain the supersymmetric partners of this system, we need to “intertwine” \( H \) with another Hamiltonian \( \tilde{H} \) through the operator relation

\[
\tilde{H} A^+ + A^+ H = E_n A^+ \psi_n(x),
\]

where \( A^+ \) is a differential operator of order \( k \).

If \( \psi(x) \) are eigenfunctions of \( H \) with eigenvalues \( E_n \), such a relation implies that \( \tilde{H} A^+ \psi_n(x) = E_n A^+ \psi_n(x) \), i.e.,

\[
\phi_n \propto A^+ \psi_n
\]

are eigenfunctions of \( \tilde{H} \) with eigenvalues \( E_n \).

The first-order supersymmetric partners of the truncated harmonic oscillator, or 1-SUSY partners for short, are obtained through a first-order operator \( A^+ \). Let us suppose now that we take a solution \( u(x) \) of the stationary Schrödinger equation for \( H \), i.e., \( H u = \epsilon u \), where \( u(x) \) does not necessarily satisfy the associated boundary conditions. Then, the expressions

\[
A^+ = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dx} + [\ln(u)]' \right],
\]

\[
\tilde{V} = V - [\ln(u)]''
\]

fulfill equation (1) for the first-order case and provides \( \tilde{V} \) once \( V \) is known. It is important to prevent that new singularities appear in \( \tilde{V} \). Such a prescription imposes restrictions on \( u(x) \), in particular, \( u(x) \) must not have zeros in the domain of interest.

In analogy to the previous case, the second-order supersymmetric partners of the truncated harmonic oscillator, or 2-SUSY partners, are obtained using a pair of Schrödinger solutions...
When a first order intertwining operator $\tilde{A}$ is given by

$$A^+ = \frac{d^2}{dx^2} - \ln W (u_1, u_2) + \frac{1}{2} \left( \ln W (u_1, u_2)'' + \ln W (u_1, u_2)'' \right) - 2\epsilon_1 + \epsilon_2,$$

$$\tilde{V} = V - \ln W (u_1, u_2)'',$$

where $W (u_1, u_2)$ is the Wronskian of $u_1$ and $u_2$. Once again, we need to consider those cases where the term $\ln W (u_1, u_2)'$ does not introduce new singularities to the supersymmetric partner, which translates in requiring that $W (u_1, u_2)$ does not have zeros in the domain of interest.

In this case, the energy levels of $\tilde{H}$ are $h_n = 2n + \frac{3}{2}$ and $\phi_n \propto A^+ \psi_n$ the corresponding eigenfunctions. The allowed values for $\epsilon$ are those which satisfy $\epsilon < \frac{3}{2}$, i.e., below the ground state of $H$. However, in the limit $\epsilon \to \frac{3}{2}$ the level $E_0 = \frac{3}{2}$ is expunged from the spectrum of $\tilde{H}$.

On the other hand, the use of an even transformation function $u$ produces new Hamiltonians $\tilde{H}$ with potentials of the form

$$\tilde{V} = V + 1 - \left( \ln \left[ \frac{1}{2} \frac{1}{2} x^2 \right] \right)''. \tag{5}$$

In this case, the energy levels of $\tilde{H}$ are given by $E_n = 2n + \frac{3}{2}$, with $\phi_n \propto A^+ \chi_n$ being the corresponding eigenfunctions. For $\epsilon < \frac{1}{2}$ no new singularities appear, i.e., these are the allowed values of $\epsilon$. In the limit $\epsilon = \frac{1}{2}$ the level $E_0 = \frac{1}{2}$ is expunged from the energy spectrum of $\tilde{H}$.

Notice that, for both cases, $\tilde{H}$ is isospectral to $H$, up to a shift in the ground state energy.

### 2.2. Second-order supersymmetric partners

For a second-order intertwining operator $A^+$ let us suppose, without lost of generality, that $\epsilon_2 < \epsilon_1$. There are four distinct parity combinations for $u_1(x)$ and $u_2(x)$.

Taking both $u_1$ and $u_2$ odd, the SUSY partner potential is

$$\tilde{V} = V + \frac{3}{x^2} + 2 - \ln \omega_1(x)''. \tag{6}$$
The absence of zeros of the function $\omega_1(x)$ sets the requirements on $\epsilon_1$, $\epsilon_2$. Then, no new singularities appear whenever $\epsilon_1, \epsilon_2 < \frac{3}{2}$ or $\frac{3+4n}{2} < \epsilon_1, \epsilon_2 < \frac{3+4(n+1)}{2}$. The energy spectrum of $\tilde{H}$ is given by $\{\mathcal{E}_n = 2n + \frac{3}{2}, \ n = 0, 1, \ldots\}$ with $\phi_n(x) \propto A^\dagger \psi_n(x)$ being the corresponding eigenfunctions. As a limiting case, when either $\epsilon_1$ or $\epsilon_2$ takes the value $\mathcal{E}_n$, then this level is expunged from the set of eigenvalues of $\tilde{H}$.

When $u_1$ and $u_2$ are both even we get

$$\tilde{V} = V + \frac{1}{x^2} + 2 - [\ln \omega_2(x)]''$$

where $\omega_2(x)$ has no zeros in $(0, \infty)$ as long as $\epsilon_1, \epsilon_2 < \frac{1}{2}$ or $\frac{1+4n}{2} < \epsilon_1, \epsilon_2 < \frac{1+4(n+1)}{2}$. The eigenvalues of $\tilde{H}$ are $\mathcal{E}_n = 2n + \frac{3}{2}$, $n = 0, 1, \ldots$, and $\phi_n(x) \propto A^\dagger \chi_n(x)$ are the corresponding eigenfunctions. One can erase one level $\mathcal{E}_n$ from the energy spectrum of $\tilde{H}$ by choosing $\epsilon_1$ or $\epsilon_2$ equal to said eigenvalue.

Taking now $u_1$ odd and $u_2$ even leads to the new potential

$$\tilde{V} = V + 2 - [\ln \omega_3(x)]''$$

The energy spectrum is composed by the levels $\mathcal{E}_n = 2n + \frac{3}{2}$, the corresponding eigenfunctions are $\phi_n(x) \propto A^\dagger \psi_n(x)$, and the condition on $\epsilon_1, \epsilon_2$ is $\frac{1+4n}{2} < \epsilon_1, \epsilon_2 < \frac{3+4n}{2}$, with a limiting case $\epsilon_1 = \mathcal{E}_n$ for expunging the level $\mathcal{E}_n$. We can also add a level at $\epsilon_2$ by requesting that $\epsilon_2 \neq \mathcal{E}_n$, the corresponding eigenfunction being $\phi_2 \propto \frac{u_1}{W(u_1, u_2)}$.

As a final case, if $u_1$ is even and $u_2$ is odd, it turns out that

$$\tilde{V} = V + 2 - [\ln \omega_4(x)]''$$

where $\epsilon_1$ and $\epsilon_2$ are required now to satisfy $\epsilon_2 < \epsilon_1 < \frac{1}{2}$ or $\frac{3+4n}{2} < \epsilon_1 < \frac{5+4n}{2}$. The eigenfunctions of $\tilde{H}$ are $\phi_n(x) \propto A^\dagger \psi_n$ with eigenvalues $\mathcal{E}_n = 2n + \frac{3}{2}$. The choice $\epsilon_2 = \mathcal{E}_n$ expunges such a level, while for $\epsilon_1 \neq \mathcal{E}_n$ one adds a level at $\epsilon_1$ with eigenfunction $\phi_1 \propto \frac{u_2}{W(u_1, u_2)}$.

3. Polynomial Heisenberg algebras

The polynomial Heisenberg algebras (PHA) are generated by three operators $\{\mathbb{H}, L^+, L^\dagger\}$ satisfying the following commutation relations

$$[\mathbb{H}, L^\pm] = \pm L^\pm, \quad [L^-, L^+] = P_m(\mathbb{H}),$$

where $\mathbb{H}$ is a Hamiltonian with Schrödinger form, $P_m(\mathbb{H})$ is a polynomial of degree $m$ on the Hamiltonian and $L^\pm$ are $(m+1)$-th order differential ladder operators.

In particular, systems with third order ladder operators $L^\pm$ realize a second order PHA. By factorizing them as $L^+ = L_1^+ L_2^+$ and $L^- = L_2^- L_1^-$ where $L_{1,2}^- = (L_{1,2}^+)^\dagger$ and $L_1^+ = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dx} + f(x) \right]$, $L_2^+ = \frac{3}{2} \left[ \frac{d^2}{dx^2} + g(x) \frac{d}{dx} + h(x) \right]$, we find at the end the following set of equations [9]:

$$f(x) = x + g(x),$$

$$h = \frac{g'^2}{2} - \frac{g^2}{2} - 2xg - x^2 + \varepsilon_2 + \varepsilon_3 - 2\varepsilon_1 - 1,$$

$$\mathbb{V} = \frac{x^2}{2} - \frac{g'^2}{2} + \frac{g^2}{2} + xg + \varepsilon_1 - \frac{1}{2},$$

$$\frac{d^2g}{dx^2} = \frac{1}{2g} \left( \frac{dg}{dx} \right)^2 + \frac{3}{2}g^4 + 4xg^2 + 2(x^2 - a)g + \frac{b}{g}.$$
The last one is the Painlevé IV (PIV) equation with parameters \( a = \varepsilon_2 + \varepsilon_3 - 2\varepsilon_1 - 1 \) and \( b = -2(\varepsilon_2 - \varepsilon_3)^2 \). Thus, we have gotten a connection between second order PHA’s and PIV equation. We can either solve such a non-linear differential equation for realizing a second order PHA, or take a known realization of a second order PHA and obtain then solutions to the PIV equation.

The latter is done by using the factorization

\[ N \equiv L^+ L^- = (\mathbb{H} - \varepsilon_1)(\mathbb{H} - \varepsilon_2)(\mathbb{H} - \varepsilon_3), \tag{7} \]

to see that there are three extremal states \( \phi_\lambda \) such that \( L^- \phi_\lambda = L^- L^+ \phi_\lambda = 0 \) satisfying also \( \mathbb{H} \phi_\lambda = \lambda \phi_\lambda \), with \( \lambda = \varepsilon_1, \varepsilon_2, \varepsilon_3 \). In particular, a \( \phi_\lambda \) such that

\[ L^- \phi_\lambda = \frac{1}{\sqrt{2}} \left[ \frac{d}{dx} + f(x) \right] \phi_\lambda = 0 \]

satisfies the extremal state condition. Here we have fixed \( \lambda \), but keep in mind that there are three possible identifications for such value. Let us recall also that \( f(x) = x + g(x) \) to obtain

\[ g_\lambda(x) = -x - [\ln \phi_\lambda]' \tag{8} \]

which is a solution of the PIV equation with parameters \( a, b \).

On the other hand, systems with fourth order ladder operators \( L^+ = L^+_1 L^+_2 \) and \( L^- = L^-_2 L^-_1 \) realize a third order PHA [9, 30]. Choosing \( L^+_1 = \frac{1}{2} \left[ \frac{d^2}{dx^2} + g_1(x) \frac{d}{dx} + h_1(x) \right] \), \( L^-_2 = \frac{1}{2} \left[ \frac{d^2}{dx^2} + g(x) \frac{d}{dx} + h(x) \right] \) and \( L^- = (L^+_1)_2 \) we find another set of equations:

\[ g_1(x) = -g(x) - x, \quad g(x) = \frac{x}{w - 1} \tag{9} \]

\[ \frac{d^2 w}{dz^2} = \left( \frac{1}{2w} + \frac{1}{w - 1} \right) \left( \frac{dw}{dz} \right)^2 - \frac{1}{2z} \frac{dw}{dz} + \left( \frac{(w - 1)^2}{z^2} \right) \left( \frac{aw + b}{w} \right) + \left( \frac{cw}{z} + d \right) \frac{w(w + 1)}{w - 1}, \]

where \( z = x^2 \). Expression (9) is the Painlevé V (PV) equation, with \( a = \frac{(\varepsilon_1 - \varepsilon_2)^2}{2}, b = -\frac{(\varepsilon_3 - \varepsilon_4)^2}{2}, \]
\( c = \varepsilon_1 + \varepsilon_2 - \varepsilon_4 - \varepsilon_3 - 1 \), \( d = -\frac{1}{8} \) being fixed parameters.

Now we are concerned with the factorization

\[ N \equiv L^+ L^- = (\mathbb{H} - \varepsilon_1)(\mathbb{H} - \varepsilon_2)(\mathbb{H} - \varepsilon_3)(\mathbb{H} - \varepsilon_4) \tag{10} \]

which indicates that there are four extremal states such that \( L^- \phi_\lambda = L^- L^- \phi_\lambda = 0 \) and \( \mathbb{H} \phi_\lambda = \lambda \phi_\lambda \), \( \lambda = \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \).

In a similar way as in the previous case, but with lengthier calculations, we can write

\[ g_1 = (\ln |W(\phi_\lambda, \phi_\sigma)|)' \]

for a fixed choice of a pair of distinct extremal states \( \phi_\lambda, \phi_\sigma \), thus obtaining

\[ g(x) = -x - (\ln |W(\phi_\lambda, \phi_\sigma)|)' \]

The use of two extremal states in this construction allows us to choose among six different identifications for \( \phi_\lambda \) and \( \phi_\sigma \) leading to six different functions \( g \).

Finally, after the change of variable \( x = \sqrt{z} \), it turns out that

\[ w(z) = 1 + \frac{\sqrt{z}}{g(\sqrt{z})} \]

is the solution to the PV equation we are looking for, with fixed parameters \( a, b, c \) and \( d \) [27, 35].
4. Solutions to the Painlevé IV equation

Now we want to connect the SUSY partners of $H$ obtained in Section 2 with the PIV equation. This is done by first noting that, if $a^\pm$ are the standard raising and lowering operators for the harmonic oscillator, then $[H, a^\pm] = \pm a^\pm$.

On the other hand, if we define a pair of new operators for $\hat{H}$ as $L^\pm \equiv A^+ a^\pm A$, where $A = (A^+)^l$, then $[\hat{H}, L^\pm] = \pm L^\pm$, which is a commutator characteristic for ladder operators. Moreover, the commutator between $L^+$ and $L^-$ is a polynomial of degree $2k$ in $\hat{H}$, i.e.,

$$[L^-, L^+] = P_{2k}(\hat{H}),$$

which defines a PHA of $2k$-th order.

For a first order $A^+$, the ladder operators $L^\pm$ are of third order. In addition, for the 1-SUSY partner of the truncated harmonic oscillator we recognize that the extremal state energies for $\hat{H}$ are given by

$$L^+ L^- = (\hat{H} - \epsilon)(\hat{H} - \epsilon - 1) \left(\hat{H} - \frac{1}{2}\right),$$

where the identification $\epsilon_1 = \epsilon$, $\epsilon_2 = \epsilon + 1$, $\epsilon_3 = \frac{1}{2}$ has been done.

As a result, when one uses an odd transformation function $u(x)$, the solution of the PIV equation obtained from the first extremal state is

$$g_{\epsilon_1} = \frac{1}{x} - 2x + \left(1 - \frac{2}{3}x\right) x \frac{1}{1} \frac{F_1(\frac{-2x}{3}; \frac{1}{2}; x^2)}{F_1(\frac{-2x}{3}; \frac{1}{2}; x^2)}.$$

Meanwhile, for an even function $u(x)$ the solution of the PIV equation obtained from the first extremal state is

$$g_{\epsilon_1} = -2x + (1 - 2x) x \frac{1}{1} \frac{F_1(\frac{-2x}{3}; \frac{1}{2}; x^2)}{F_1(\frac{-2x}{3}; \frac{1}{2}; x^2)}.$$

In both cases the other two PIV solutions can be calculated from the first one through the following expressions

$$g_{\epsilon_2} = -g_e - 2x - 2 \left(\frac{x + (2\epsilon - x^2)g_e + x + (g_e + x)^3}{x^2 - 2\epsilon - 1} \right),$$

$$g_{\epsilon_3} = -\frac{g_e + 2}{g_e} + 2x.$$

For a second order $A^+$, the ladder operators $L^\pm$ are of fifth order. So, in order to repeat the procedure carried out for the 1-SUSY partners of $H$ we need to reduce the order of $L^\pm$. For this purpose we use a reduction theorem [13] which states that under the condition that $u_2(x) = a^- u_1(x)$, $\epsilon_2 = \epsilon_1 - 1$, there exists a pair of third order ladder operators $l^\pm$ for $\hat{H}$ which product can be factorized into

$$l^+ l^- = (\hat{H} - \epsilon_1 + 1)(\hat{H} - \epsilon_1 - 1)(\hat{H} - 1/2).$$

Then, the three extremal states are those associated with $\epsilon_1 = \epsilon - 1$, $\epsilon_2 = \epsilon + 1$, $\epsilon_3 = \frac{1}{2}$.

Recall that the reduction theorem required that $\epsilon_2$ depended on $\epsilon_1$, and $u_2(x)$ on $u_1(x)$. Thus, we can write $u_1(x) = u(x)$ and $\epsilon_1 = \epsilon$ for simplicity. So, after repeating the method used for the 1-SUSY partners of $H$, in the reduced case for the 2-SUSY partners the three different solutions
of the PIV equation are given by

\[ G_{\varepsilon_1} = -x - \alpha + 2 \left[ \frac{x + \alpha}{x^2 + 1 - 2\epsilon - \alpha^2} \right], \]
\[ G_{\varepsilon_2} = G_{\varepsilon_1} + \frac{2\alpha^2 - 2x^2 + 2(2\epsilon + 1)}{\alpha - G_{\varepsilon_1} - x}, \]
\[ G_{\varepsilon_3} = \frac{(x + \alpha)G_{\varepsilon_1}^2 + [2\epsilon - 1 + (x + \alpha)^2]}{(x + \alpha)^2 + (x + \alpha)G_{\varepsilon_1} + 2\epsilon - 1}, \]

where \( \alpha = [\ln(u)]' \) and the solutions \( G_{\varepsilon_2} \) and \( G_{\varepsilon_3} \) are given in terms of \( G_{\varepsilon_1} \). We must remark that in this case we have used a capital \( g \) for the PIV solutions in order to distinguish them from those obtained through the first-order SUSY partners.

4.1. Bäcklund transformations

Now we would like to connect the six solutions of PIV equation we have just found by transformations converting one into another. With this purpose let us recall that heuristically a Bäcklund transformation relates one solution of a given equation either to another solution of the same equation, possibly with different values of the parameters, or to a solution of another equation [36]; for a more formal definition you can see [37].

For the PIV equation, [36] presents a list of Bäcklund transformations relating PIV solutions among them. In particular, using the notation of this reference, we found that the solutions of the PIV equation \( g_{\sigma} \), \( G_{\sigma} \) with \( \sigma = \varepsilon_1, \varepsilon_2, \varepsilon_3 \) are related by Bäcklund transformations according to the following diagram:

1-SUSY

\[
\begin{align*}
g_{\varepsilon_1} & \rightarrow g_{\varepsilon_2} \\
W^+ W^+ & \rightarrow W^- W^- \\
g_{\varepsilon_3} & \rightarrow W^-
\end{align*}
\]

2-SUSY

\[
\begin{align*}
G_{\varepsilon_1} & \rightarrow G_{\varepsilon_3} \\
W^+ W^+ & \rightarrow W^- W^- \\
G_{\varepsilon_2} & \rightarrow W^-
\end{align*}
\]

where \( g_{\varepsilon_2} \) is obtained by applying two transformations \( W^+ W^+ \) and \( W^- W^- \) in this order to \( g_{\varepsilon_1} \), i.e., \( g_{\varepsilon_2} = W^+ W^+ W^- w_{\varepsilon_1} \) and similarly for the rest of the diagram.

Notice that we could relate not only the set of solutions obtained through 1-SUSY, and those obtained by 2-SUSY, but we could also connect both sets of solutions through the expression \( G_{\varepsilon_1} = W^+ g_{\varepsilon_3} \).

5. Solutions to the Painlevé V equation

To establish the connection between the PV equation and \( \tilde{H} \) we need to take into account that, although \( L^\pm = A^+ a^\pm A \) (again \( A = (A^+)^{\dagger} \)) have the right commutators with \( \tilde{H} \) to be called ladder operators, it is \( \mathcal{L}^\pm = A^+(a^\pm)^2 A \) which defines genuine ladder operators, which move a physical state of \( \tilde{H} \) to another physical state of \( \tilde{H} \). This makes \( \mathcal{L}^\pm \) fourth-order ladder operators and thus the set \( \{\tilde{H}, \mathcal{L}^+, \mathcal{L}^-\} \) supplies a realization of a third-order PHA, whenever \( A^+ \) is of first-order.

This time the factorization

\[ \mathcal{L}^+ \mathcal{L}^- = \left( \tilde{H} - \epsilon \right) \left( \tilde{H} - \epsilon - 2 \right) \left( \tilde{H} - \frac{1}{2} \right) \left( \tilde{H} - \frac{3}{2} \right) \]

indicates that there are four extremal states satisfying \( \mathcal{L}^- \phi_\lambda = 0 \) with \( \lambda = \epsilon, \epsilon + 2, \frac{1}{2}, \frac{3}{2} \), then we have six inequivalent choices of two of them for a given \( \tilde{H} \). The six solutions of the PV equation are:

\[ w_{1\alpha} = 1 + \frac{2\sqrt{2z}(1 + 2\epsilon - z + \sqrt{2z\alpha})}{\sqrt{2z(2 + z)} - 4(1 + z)\alpha + 2\sqrt{2z\alpha}^2}, \]
for \( \lambda = \epsilon, \sigma = \epsilon + 2 \).

\[
\nu_{1b} = \frac{(2\alpha + \sqrt{2\alpha}) (4\alpha - 4\epsilon \sqrt{2\alpha} + \sqrt{2\alpha} z^{3/2} - 2\sqrt{2\alpha} \sqrt{z} - 4\sqrt{2\alpha})}{(2\alpha - \sqrt{2\alpha})(4\alpha - 4\epsilon \sqrt{2\alpha} + \sqrt{2\alpha} z^{3/2} - 2\sqrt{2\alpha} \sqrt{z} + 4\sqrt{2\alpha})},
\]

for \( \lambda = \epsilon + 2, \sigma = \frac{1}{2} \).

\[
\nu_{1c} = 1 + \frac{\sqrt{2} \zeta \left( 8\epsilon \zeta - 2\epsilon^2 + 4\alpha^2 z + 4\sqrt{2} \alpha \zeta - 16 \right)}{8\alpha - 2\sqrt{2}\zeta^{3/2} \left( \alpha^2 + 2\epsilon - 3 \right) + 4\alpha z \left( \alpha^2 + 2\epsilon - 1 \right) - 8\sqrt{2} \zeta \left( \alpha^2 - \epsilon - 1 \right) + \sqrt{2}\zeta^{5/2} - 2\alpha^2 z^2},
\]

for \( \lambda = \epsilon + 2, \sigma = \frac{3}{2} \).

\[
\nu_{1d} = \frac{2\sqrt{2} - 2\alpha \sqrt{\zeta} - \sqrt{2} \zeta}{2\sqrt{2} - 2\alpha \sqrt{\zeta} + \sqrt{2} \zeta},
\]

for \( \lambda = \epsilon, \sigma = \frac{3}{2} \).

\[
\nu_{1e} = \frac{2\alpha + \sqrt{2} \zeta}{2\alpha - \sqrt{2} \zeta},
\]

for \( \lambda = \epsilon, \sigma = \frac{1}{2} \).

\[
\nu_{1f} = \frac{-4\alpha + \sqrt{2} \zeta^{3/2} + 2\sqrt{2} \left( \alpha^2 - 1 \right) \sqrt{\zeta} + 4\alpha z}{4\alpha - 2\sqrt{2} \sqrt{\zeta} \left( \alpha^2 + 2\epsilon - 2 \right) + \sqrt{2}\zeta^{3/2}},
\]

for \( \lambda = \frac{1}{2}, \sigma = \frac{3}{2} \). We just need to remember that \( \alpha = [\ln(u)]' \).

A straightforward generalization of the reduction theorem used in the case of the PIV equation requires that \( u_2 = (a^{-})^2 u_1 \), such that \( c_2 = \epsilon_1 - 2 \), thus supplying us with ladder operators \( l^\pm \) such that

\[
l^+ l^- = \left( H_2 - \frac{1}{2} \right) \left( H_2 - \frac{3}{2} \right) (H_2 - \epsilon - 2) (H_2 - \epsilon - 2) .
\]

Notice that the reduction theorem leaves only one free parameter out of the two original \( \epsilon_1, \epsilon_2 \). Thus we have chosen to set \( \epsilon_1 = \epsilon \) and \( \epsilon_2 = \epsilon - 2 \).

The notation we use for these solutions of the PV equation is the following: \( w_{2a} \) if \( \lambda = \epsilon - 2, \sigma = \epsilon + 2 \), \( w_{2b} \) if \( \lambda = \epsilon + 2, \sigma = \frac{1}{2} \), \( w_{2c} \) if \( \lambda = \epsilon + 2, \sigma = \frac{3}{2} \), \( w_{2d} \) if \( \lambda = \epsilon - 2, \sigma = \frac{3}{2} \), \( w_{2e} \) if \( \lambda = \epsilon - 2, \sigma = \frac{1}{2} \), and \( w_{2f} \) if \( \lambda = \frac{1}{2}, \sigma = \frac{3}{2} \).

Explicit expressions for the solutions obtained through 2-SUSY applied to the truncated oscillator are too long to be put in here, but let us give some simple examples.

For the choice \( E_1 = \epsilon - 2, E_2 = \epsilon + 2, E_3 = \frac{1}{2}, E_4 = \frac{3}{2} \), where \( \epsilon = \frac{5}{2} \) and \( u(x) \) has even parity:

\[
w_{2f} = \frac{4(z - 1)}{z^2 + 2z - 1}.
\]

For the permutation \( E_1 = \epsilon - 2, E_2 = \epsilon + 2, E_3 = \frac{1}{2}, E_4 = \frac{3}{2} \), where \( \epsilon = \frac{1}{2} \) and \( u(x) \) has odd parity:

\[
w_{2f} = \frac{3\sqrt{2} \sqrt{\zeta} - 2(z + 3) F \left( \frac{\sqrt{2}}{\sqrt{2}} \right)}{\sqrt{2}\sqrt{\zeta}(z + 3) - 2(z^2 + 2z + 3) F \left( \frac{\sqrt{2}}{\sqrt{2}} \right)}.
\]

In the last equation \( F(x) \) is the Dawson function, also called Dawson integral.
5.1. Bäcklund transformations

In [38] the authors studied the Bäcklund transformations for the fifth Painlevé equation. As presented in this paper, they considered a solution of the PV equation $w = w(z; \alpha, \beta, \gamma, \delta)$ with parameters $\alpha, \beta, \gamma, \delta \neq 0$, such that

$$F_1(w) = zw' - k_1cw^2 + (k_1c - k_2a + k_3hz)w + k_2a \neq 0,$$

where $c^2 = 2\alpha$, $a^2 = -2\beta$, $h^2 = -2\delta$. Then they found that the transformation

$$T_{k_1,k_2,k_3} : w(z; \alpha, \beta, \gamma, \delta) \rightarrow w_{1}(z; \alpha_1, \beta_1, \gamma_1, \delta_1) = 1 - \frac{2k_3hzw}{F_1}$$

defines another solution $w_1(z; \alpha_1, \beta_1, \gamma_1, \delta_1)$ of the PV equation with parameters

$$\alpha_1 = -\frac{1}{16\delta} [\gamma + k_3h(1 - k_2a - k_1c)]^2, \quad \beta_1 = \frac{1}{16\delta} [\gamma - k_3h(1 - k_2a - k_1c)]^2,$$

$$\gamma_1 = k_3h(k_2a - k_1c), \quad \delta_1 = \delta,$$

where $k_i^2 = 1, i \in \{1, 2, 3\}$.

Now let us go back to the solutions obtained in the previous subsection. Then for fixed value of $\epsilon$ one can show that $T_{k_1,k_2,k_3}$ is a BT for the solutions previously found in the following cases:

- $T_{k_1,k_2,k_3} : w_{1b} \rightarrow w_{2a}$ whenever $k_1 = -1, k_2 = 1, k_3 = 1$ if $\epsilon < -\frac{3}{2}$; or $k_1 = -1, k_2 = -1, k_3 = 1$ if $\epsilon < -\frac{3}{2}$.
- $T_{k_1,k_2,k_3} : w_{1b} \rightarrow w_{2c}$ whenever $k_1 = -1, k_2 = -1, k_3 = 1$ if $\epsilon < -\frac{3}{2}$; or $k_1 = -1, k_2 = 1, k_3 = 1$ if $\epsilon < -\frac{3}{2}$.
- $T_{k_1,k_2,k_3} : w_1c \rightarrow w_{2a}$ whenever $k_1 = 1, k_2 = -1, k_3 = 1$ and $\frac{1}{2} < \epsilon$.
- $T_{k_1,k_2,k_3} : w_1c \rightarrow w_{2d}$ whenever $k_1 = 1, k_2 = 1, k_3 = 1$ if $\frac{1}{2} < \epsilon$; $k_1 = -1, k_2 = 1, k_3 = 1$ if $\frac{1}{2} < \epsilon$; $k_1 = -1, k_2 = -1, k_3 = 1$ if $\frac{1}{2} < \epsilon$.
- $T_{k_1,k_2,k_3} : w_{1f} \rightarrow w_{2d}$ whenever $k_1 = -1, k_2 = -1, k_3 = 1$ and for all values of $\epsilon$.
- $T_{k_1,k_2,k_3} : w_{1f} \rightarrow w_{2c}$ whenever $k_1 = -1, k_2 = 1, k_3 = 1$ and for all values of $\epsilon$.
- $T_{k_1,k_2,k_3} : w_{2d} \rightarrow w_1c$ whenever $k_1 = 1, k_2 = 1, k_3 = -1$ if $\epsilon < -\frac{3}{2}$; or $k_1 = -1, k_2 = -1, k_3 = -1$ if $\epsilon < -\frac{3}{2}$.
- $T_{k_1,k_2,k_3} : w_{2d} \rightarrow w_{1f}$ whenever $k_1 = -1, k_2 = 1, k_3 = -1$ if $\epsilon < -\frac{3}{2}$; $k_1 = 1, k_2 = 1, k_3 = -1$ if $\epsilon < -\frac{3}{2}$; $k_1 = 1, k_2 = -1, k_3 = -1$ if $\frac{\epsilon}{2} < \epsilon$.
- $T_{k_1,k_2,k_3} : w_{2c} \rightarrow w_{1b}$ whenever $k_1 = 1, k_2 = 1, k_3 = -1$ if $\epsilon \leq -\frac{1}{2}$; $k_1 = -1, k_2 = 1, k_3 = -1$ if $\epsilon \leq -\frac{1}{2}$; or $k_1 = -1, k_2 = -1, k_3 = 1$ if $\frac{5}{2} < \epsilon$.
- $T_{k_1,k_2,k_3} : w_{2c} \rightarrow w_{1f}$ whenever $k_1 = -1, k_2 = 1, k_3 = -1$ if $\epsilon \leq -\frac{1}{2}$; or $k_1 = 1, k_2 = 1, k_3 = -1$ if $\epsilon \leq -\frac{1}{2}$.

6. Conclusions

In this work we implemented a simple and direct procedure to obtain several explicit solutions of the Painlevé IV and V equations. We used the SUSY partners of the harmonic oscillator with an infinite potential barrier at the origin to construct realizations of the polynomial Heisenberg algebras of second and third order, which are connected to the Painlevé IV and V equations, respectively. Such connections established a simple dependence of the particular solutions of both differential equations on the extremal states of the SUSY partners of the truncated oscillator.

Finally, we found some Bäcklund transformations relating the obtained solutions of the Painlevé IV and V equations. This was done in order to connect the results of distinct instances, such as different order of the SUSY transformation involved, or inequivalent permutations of the extremal states used.
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