Multi-Martingale Optimal Transport

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ADVANCES IN FINANCIAL MATHEMATICS
Martingale Optimal Transport (MOT) Problem in One dimension

- Borel probability measures $\mu, \nu$ on $\mathbb{R}$ in convex order: $\mu \leq_c \nu$
- (continuous) cost function $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$\text{MT}(\mu, \nu)$: probability measures $\pi$ on $\mathbb{R} \times \mathbb{R}$ which not only project to the marginals $\mu, \nu$, but also its disintegration $(\pi_x)_{x \in \mathbb{R}}$ has barycenter at $x$ (martingale constraint):

$$f(x) \leq \int \limits_{\mathbb{R}} f(y) \, d\pi_x(y) \quad \forall f \text{ convex}.$$ 

- Disintegration = Conditional probability: $\pi_x(A) = \mathbb{P}(Y \in A | X = x)$.

Study the optimal solutions of the minimization problem

$$\min_{\pi \in \text{MT}(\mu, \nu)} \int \limits_{\mathbb{R} \times \mathbb{R}} c(x, y) \, d\pi(x, y).$$
Probabilistic statement of MOT

- $(\Omega, \mathcal{F}, \mathbb{P})$: probability space
- $X : \Omega \to \mathbb{R}$, $Y : \Omega \to \mathbb{R}$: random variables
- cost function $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$
- $\text{Law}(X) = \mu$, $\text{Law}(Y) = \nu$
- $E(Y|X) = X$.

Study the one-step martingales (stocks) $(X, Y)$ with prescribed marginals, which minimize the expected cost (option price)

$$\min_{X \sim \mu, Y \sim \nu, E(Y|X)=X} E_{\mathbb{P}} c(X, Y).$$

Motivation:
- [Model-free Finance] find the minimum price of option $c(x, y)$ given market information $\mu, \nu$, that is, given the prices of call / put options.
A structure result in 1-dimension

Theorem (Hobson-Neuberger-Klimmek, Beiglböck-Juillet ’13)

Let $c(x, y) = \pm |x - y|$ and $d = 1$ (in financial term, this means that the option $|X - Y|$ depends only on one stock process), and assume $\mu$ is dispersed ($\mu \ll L^1$). Then the optimal martingale transport $\pi$ is unique for any given $\nu$, and it exhibits an extremal property: for each $x \in \mathbb{R}$, the conditional probability $\pi_x$ is concentrated at two boundary points of an interval.

Question: What is a right generalization of this theorem in higher dimension?
Multi-Martingale Optimal Transport (MMOT) Problem [L. ’16]

- probability measures $\mu_i, \nu_i$ on $\mathbb{R}$ in convex order, $i = 1, 2, ..., d$
- cost function (option) $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$
- $(X_i, Y_i)$: one-step martingales ($\mathbb{E}(Y_i | X_i) = X_i$) with the prescribed marginal laws $X_i \sim \mu_i$ and $Y_i \sim \nu_i$
- $\mu := (\mu_1, ..., \mu_d), \quad \nu := (\nu_1, ..., \nu_d)$
- $\text{MMT}(\mu, \nu)$: the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ such that each $\pi \in \text{MMT}(\mu, \nu)$ is the joint law of martingales $(X_i, Y_i)_{i \leq d}$ having $(\mu_i, \nu_i)_{i \leq d}$ as its marginals, respectively.

Study the optimal solutions of the minimization problem

Minimize $\text{Cost}[\pi] = \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) \, d\pi(x, y)$ over $\pi \in \text{MMT}(\mu, \nu)$.

Motivation:

- [Finance] find the minimum price of the option whose value depends on many stocks $(X_i, Y_i), i = 1, ..., d$, given the information that can be observed from the market.
Probabilistic description of MMOT

- $(\Omega, \mathcal{F}, \mathbb{P})$: probability space
- $X_i: \Omega \to \mathbb{R}, \ Y_i: \Omega \to \mathbb{R}$: random variables, $i = 1, 2, \ldots, d$
- cost function $c: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$
- $\text{Law}(X_i) = \mu_i, \text{Law}(Y_i) = \nu_i$
- $E(Y|X) = X$, where $X = (X_1, \ldots, X_d), \ Y = (Y_1, \ldots, Y_d)$

Study the one-step martingales (stocks) $(X, Y)$ with prescribed marginals, which minimize the expected cost (option price)

$$\min_{X_i \sim \mu_i, Y_i \sim \nu_i, E(Y|X) = X} E_\mathbb{P} c(X, Y).$$
Extremal structure of MMOT holds true in every dimension

**Theorem [L. ’16]** Assume:

- \( \mu_i \leq c \nu_i \) (not necessarily irreducible)
- \( \mu_i \ll \mathcal{L}^1 \)
- \( c(x, y) = \pm ||x - y|| \) where \( || \cdot || \) is any strictly convex norm on \( \mathbb{R}^d \)
- \( \pi = \text{Law}(X, Y) \) is any minimizer of MMOT with copula \( \pi^1 = \text{Law}(X) \)

**Then:** for any disintegration \((\pi_x)_x\) of \( \pi \) with respect to \( \pi^1 \), the support of \( \pi_x \) coincides with the extreme points of the closed convex hull of itself:

\[
\text{supp } \pi_x = \text{Ext} \left( \overline{\text{conv}}(\text{supp } \pi_x) \right), \quad \pi^1 - a.e. \ x.
\]

**Literature in OT:**
Sudakov, Evans, Gangbo, McCann, Ambrosio, Kirchheim, Pratelli, Caffarelli, Feldman, Otto, Kinderlehrer, Jordan, Bianchini, Cavalletti, Ma, Trudinger, Wang, Champion, De Pascale, and others...
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We say that a triple of functions \((\phi, \psi, h)\) is a dual maximizer of the MOT problem, if for every minimizer \(\pi\) of MOT we have

\[
\phi(x) + \psi(y) + h(x) \cdot (y - x) \leq c(x, y) \quad \forall x \in \mathbb{R}, \forall y \in \mathbb{R},
\]

\[
\phi(x) + \psi(y) + h(x) \cdot (y - x) = c(x, y) \quad \pi - a.e. (x, y).
\]  

\(\phi(x) + \psi(y) + h(x) \cdot (y - x)\) can be interpreted as an optimal subhedging strategy for the option \(c(x, y)\).
Irreducibility of \((\mu, \nu)\) is essential to achieve duality in MOT

- Beiglböck-Juillet, Beiglböck-Nutz-Touzi showed that in dimension one \((d = 1)\), duality is attained if the marginals \((\mu, \nu)\) are **irreducible**.

- The irreducibility of \((\mu, \nu)\) is characterized by their potential functions
  \[
  u_{\mu}(x) := \int |x - y| \, d\mu(y), \quad u_{\nu}(x) := \int |x - y| \, d\nu(y).
  \]

- This is also where the OT and MOT are divergent: in OT theory essentially no relation between \(\mu, \nu\) is required for duality.

- The seemingly harmless linear term \(h(x) \cdot (y - x)\) drastically changes the picture.
Duality in MMOT (is also possible!)

Theorem [L. ’16] Assume:

- \((\mu_i, \nu_i)\) is irreducible, \(\forall i = 1, \ldots, d\)
- \(\pi\) is any minimizer of MMOT

Then: there exist a bunch of functions \(\phi_i, \psi_i : \mathbb{R} \to \mathbb{R}\), \(i=1,\ldots,d\), \(h : \mathbb{R}^d \to \mathbb{R}^d\) which is a dual maximizer:

\[
\begin{align*}
\sum_{i=1}^{d} \phi_i(x_i) + \sum_{i=1}^{d} \psi_i(y_i) + h(x) \cdot (y - x) &\leq c(x, y) \quad \forall x \in \mathbb{R}^d, \forall y \in \mathbb{R}^d, \\
\sum_{i=1}^{d} \phi_i(x_i) + \sum_{i=1}^{d} \psi(y_i) + h(x) \cdot (y - x) &= c(x, y) \quad \pi - a.e. (x, y).
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But not only this, we find that \(\text{Law}(X)\) and \(\text{Law}(Y)\) also solve a classical dual optimal transport problem:
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- But not only this, we find that \(\text{Law}(X)\) and \(\text{Law}(Y)\) also solve a classical dual optimal transport problem:
Law$(X)$, Law$(Y)$ are also optimizers for OT

**Theorem [L. ’16]** Assume:

- $(\phi_i, \psi_i, h_i)_{i \leq d}$ is a dual maximizer
- $\pi = \text{Law}(X,Y)$ is any minimizer of MMOT

Then: its first and second copulas $\pi^1, \pi^2$ (i.e. $\pi^1 = \text{Law}(X)$, $\pi^2 = \text{Law}(Y)$) solve the dual optimal transport problem with respect to the costs $\alpha, \beta$ respectively:

\[
\sum_i \phi_i(x_i) \leq \alpha(x) \quad \mu_i - \text{a.e. } x_i \quad \forall i \in (d), \quad \text{and} \quad \sum_i \phi_i(x_i) = \alpha(x) \quad \pi^1 - \text{a.e. } x,
\]
\[
\sum_i \psi_i(y_i) \geq \beta(y) \quad \nu_i - \text{a.e. } y_i \quad \forall i \in (d), \quad \text{and} \quad \sum_i \psi_i(y_i) = \beta(y) \quad \pi^2 - \text{a.e. } y.
\]

Here the functions $\alpha : \mathbb{R}^d \rightarrow \mathbb{R}$, $\beta : \mathbb{R}^d \rightarrow \mathbb{R}$ are naturally defined in terms of the function $y \mapsto \sum_{i=1}^d \psi_i(y_i)$ and are called the martingale Legendre transform. (Ghoussoub-Kim-L. ’15)

OT theory can enter for the study of the structure of Law$(X)$, Law$(Y)$. 
Law($X$), Law($Y$) are also optimizers for OT

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- OT theory can enter for the study of the structure of Law($X$), Law($Y$).
Law(\(X\)), Law(\(Y\)) are also optimizers for OT

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- OT theory can enter for the study of the structure of \(\text{Law}(X), \text{Law}(Y)\).
Conclusion:

- The duality attainment results presented so far shall serve as the cornerstones for further development of the MOT / MMOT theory, as it did so in the classical OT theory.

- As the classical optimal transport theory (in higher dimensions) has made important contributions to many areas of mathematics and economics, I believe that this new development of probabilistic optimal embedding theory in higher dimensions will have far-reaching consequences as well.
Thank You Very Much!