THE SUPERIORITY OF STOCHASTIC SYMPLECTIC METHODS FOR A LINEAR STOCHASTIC OSCILLATOR VIA LARGE DEVIATIONS PRINCIPLES

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Abstract. It is well known that symplectic methods have been rigorously shown to be superior to non-symplectic ones especially in long-time computation, when applied to deterministic Hamiltonian systems. In this paper, we focus on the superiority of stochastic symplectic methods applied to a linear stochastic oscillator, from the perspective of large deviations principle (LDP). Based on the Gärtner–Ellis theorem, we first study the LDPs of the mean position and the mean velocity for both the exact solution of the stochastic oscillator and its numerical approximations. Then, by giving the conditions which make numerical methods have first order convergence in mean-square sense, we prove that symplectic methods asymptotically preserve these two LDPs, in the sense that the modified rate functions of symplectic methods converge to the rate functions of exact solution. This indicates that stochastic symplectic methods are able to approximate well the exponential decay speed of the “hitting probability” of the mean position and mean velocity of the original system. However, it is shown that non-symplectic methods do not asymptotically preserve these two LDPs by using the tail estimation of Gaussian random variables. To the best of our knowledge, this is the first result about using LDP to show the superiority of stochastic symplectic methods compared with non-symplectic ones.

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1. Introduction

Many stochastic differential equations, such as the stochastic oscillator, have stochastic Hamiltonian formulation and an associated stochastic symplectic structure. Concerning their numerical integration, stochastic symplectic methods have received extensive attentions (see e.g., [1, 2, 4, 5, 8, 17, 18, 19, 27, 25, 26] and references therein), for their superiority in numerical computations compared with non-symplectic ones. The motivation of this paper is to explain the superiority of stochastic symplectic methods, by studying the LDPs of numerical methods for a linear stochastic oscillator $\dot{X}_t + X_t = \alpha \dot{W}_t$ with $\alpha > 0$, and $W_t$ being a 1-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$. By focusing on this system, we are able to obtain precise results about the rate functions of LDPs for both exact and numerical solutions. The linear stochastic oscillator can be rewritten as a 2-dimensional stochastic Hamiltonian system

$$
\begin{align*}
\begin{array}{c}
d \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} X_t \\ Y_t \end{pmatrix} dt + \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} dW_t, \\
\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},
\end{array}
\end{align*}
$$

(1.1)

Key words and phrases. symplectic methods; large deviations principle; stochastic oscillator; mean position; mean velocity; rate function; logarithmic moment generating function.

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whose phase flow preserves symplectic structure. Namely, the oriented areas of the projection of the phase flow are invariant:
\[ dX_t \wedge dY_t = dx_0 \wedge dy_0, \quad \forall \quad t \geq 0, \]
where the exact solution \((X_t, Y_t)\) of (1.1) (see [16, Chapter 8]) is
\[
X_t = x_0 \cos(t) + y_0 \sin(t) + \alpha \int_0^t \sin(t - s) dW_s, \\
Y_t = -x_0 \sin(t) + y_0 \cos(t) + \alpha \int_0^t \cos(t - s) dW_s.
\]
To inherit the symplecticity of this stochastic oscillator, different kinds of symplectic methods have been constructed (see [7, 9, 14, 22, 23, 24, 28] and references therein).

For SDE (1.1), we introduce the so-called mean position
\[
A_T = \frac{1}{T} \int_0^T X_t \, dt, \quad \forall \quad T > 0,
\]
and the mean velocity:
\[
B_T = \frac{X_T}{T}, \quad \forall \quad T > 0.
\]
Both \(A_T\) and \(B_T\) are important observations, and they have many applications in physics. For example, the Ornstein–Uhlenbeck process is often used to describe the velocity of a particle moving in a random environment ([21]). In this case, \(A_T\) can be interpreted as the mean value of the displacement process \(\int_0^T X_t \, dt\), and \(B_T\) as the mean value of velocity \(X_t\) on the time interval \([0, T]\) (see also [13]). Based on the Gärtner–Ellis theorem, we show that both \(A_T\) and \(B_T\) of the exact solution (1.2) satisfy the LDPs with the good rate functions
\[
I(y) = \frac{y^2}{3\alpha^2} \quad \text{and} \quad J(y) = \frac{y^2}{\alpha^2},
\]
respectively. Generally speaking, the theory of large deviations is concerned with the exponential decay of probabilities of very rare events, which can be regarded as an extension or refinement of the law of large numbers and central limit theorem. The theory of large deviations has been widely applied to many other branches of sciences (see e.g., [12]). For a numerical approximation \(\{x_n, y_n\}\) of the linear stochastic oscillator (1.1), two natural questions are: Do its discrete mean position \(A_N = \frac{1}{N} \sum_{n=0}^{N-1} x_n\) and discrete mean velocity \(B_N = \frac{X_N}{N}\) satisfy similar LDPs as continuous case? Is the method able to preserve or asymptotically preserve the LDPs of \(A_T\) and \(B_T\) in the sense that the modified rate functions converge to the rate functions of exact solution? The main purpose of this paper is to answer the above questions.

For the numerical approximations of (1.1), we consider the general numerical methods in form of
\[
\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \alpha b \Delta W_n := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \alpha \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \Delta W_n, \tag{1.5}
\]
where \(\Delta W_n = W_{t_{n+1}} - W_{t_n}\), and the real matrix \(A\) and the real vector \(b\) depend on both the method and the constant step-size \(h\) (see [22]). When \(\det(A) = 1\), the method (1.5) preserves the discrete symplectic structure, and vice versa. We are interested in whether the symplectic methods are able to preserve the LDPs of \(A_T\) and \(B_T\) better than the non-symplectic ones.
For the case of the fixed step-size $h$, we find that the LDPs hold for $A_N$ of numerical solution of (1.5) with the rate function

$$I^h(y) = \frac{(2 + \text{tr}(A))(2 - \text{tr}(A))^2 y^2}{2a^2 h [(b_1 + a_{12}b_2 - a_{22}b_1)^2(4 + \text{tr}(A)) - 2b_1(a_{12}b_2 - a_{22}b_1)(2 - \text{tr}(A))]^2}$$

for symplectic methods, and with

$$I^h(y) = \frac{y^2}{2a^2 h} \left( \frac{1 - \text{tr}(A) + \det(A)}{b_1 + a_{12}b_2 - a_{22}b_1} \right)^2$$

for non-symplectic ones. In order to compare the rate functions of numerical methods with the one of continuous case, we introduce the concept of asymptotical preservation for LDP of the original system, in the sense that the modified rate functions $I^h_{\text{mod}}(\cdot) = I^h(\cdot)/h$ of numerical methods converge to the rate function $I(\cdot)$ of the original system. Since the matrix $A$ and vector $b$ are some functions of step-size $h$, which can be determined for a specific numerical method, extra requirements are needed to investigate the limit behaviour of the modified rate functions. To this end, we restrict the numerical approximation to the one which is mean-square convergent to the original system. By establishing the conditions which make numerical methods have first order convergence in mean-square sense, we prove that symplectic methods asymptotically preserve the LDP for $A_T$, but non-symplectic ones do not. We would like to mention that the conclusion is valid for general stochastic symplectic methods, not only for some specific ones. As for the discrete mean velocity $B_N$ of symplectic methods, we show that they satisfy the LDP based on the Gärtner–Ellis theorem, and asymptotically preserve the LDP for $B_T$ under some given conditions. However, the Gärtner–Ellis theorem is not valid for non-symplectic methods, since the logarithmic moment generating function of $B_N$ is not essentially smooth. Based on the tail estimation of Gaussian random variables, we prove that $B_N$ of non-symplectic methods satisfy the LDP, and that non-symplectic methods do not asymptotically preserve the LDP for $B_T$. Hence, we conclude that symplectic methods have better asymptotical behavior, compared with non-symplectic ones, in aspect of preserving the LDPs for $A_T$ and $B_T$.

The paper is organized as follows. In Section 2 we give some basic concepts about LDP and establish the LDPs for both $A_T$ and $B_T$. Sections 3 and 4 study the LDP for $A_N$ of general numerical methods, and show that symplectic methods asymptotically preserve the LDP for $A_T$. In Section 5 by following the ideas of dealing with $A_N$, we investigate the LDP for $B_N$ and show that symplectic methods asymptotically preserve the LDP for $B_T$. In Section 6 we verify our theoretical results by discussing about some concrete numerical methods, and construct some methods preserving exactly the LDPs for $A_T$ or $B_T$. These imply the superiority of symplectic methods in preserving the LDPs for $A_T$ and $B_T$ of the linear stochastic oscillator.

2. LDPs for $A_T$ and $B_T$

In this section, we aim to prove that both the mean position $A_T$ and mean velocity $B_T$ of the exact solution of stochastic oscillator (1.1) satisfy the LDPs. Before showing the LDPs of $A_T$ and $B_T$, we introduce some preliminaries upon LDP theory, which can be found in [10] [15].

**Definition 2.1.** $I : E \to [0, \infty]$ is called a rate function, if it is lower semicontinuous, where $E$ is a Polish space, i.e., complete and separable metric space. If all level sets $I^{-1}([-\infty, a])$, $a \in [0, \infty)$, are compact, then $I$ is called a good rate function.
Definition 2.2. Let $I$ be a rate function and $(\mu_\epsilon)_{\epsilon>0}$ be a family of probability measures on $E$. We say that $(\mu_\epsilon)_{\epsilon>0}$ satisfies a large deviations principle (LDP) with rate function $I$ if

(LDP1) $\liminf_{\epsilon \to 0} \epsilon \log(\mu_\epsilon(U)) \geq -\inf U \in E$,

(LDP2) $\limsup_{\epsilon \to 0} \epsilon \log(\mu_\epsilon(C)) \leq -\inf C \in E$.

Based on Definition 2.2, one can give the definition of LDP for a family of random variables similarly. Namely, let $\{X_\epsilon\}_{\epsilon>0}$ (resp. $\{X_n\}_{n \in \mathbb{N}}$) be a family of random variables from $\Omega, \mathcal{F}, \mathbb{P}$ to $(E, \mathcal{B}(E))$. $\{X_\epsilon\}_{\epsilon>0}$ (resp. $\{X_n\}_{n \in \mathbb{N}}$) is said to satisfy an LDP with the rate function $I$, if its distribution law $(\mathbb{P} \circ X_\epsilon^{-1})_{\epsilon>0}$ (resp. $(\mathbb{P} \circ X_n^{-1})_{n \in \mathbb{N}}$) satisfies (LDP1) and (LDP2) in Definition 2.2 (see e.g., [10, 6]).

The Gärtner–Ellis theorem plays an important role in dealing with the LDPs for a family of not independent random variables. When utilizing this theorem, one needs to examine whether the logarithmic moment generating function is essentially smooth.

Definition 2.3. A convex function $\Lambda: \mathbb{R}^d \to (-\infty, \infty]$ is essentially smooth if:

1. $\mathcal{D}_\Lambda^\circ$ is non-empty, where $\mathcal{D}_\Lambda^\circ$ is the interior of $\mathcal{D}_\Lambda := \{x \in \mathbb{R}^d : \Lambda(x) < \infty\}$;
2. $\Lambda(\cdot)$ is differentiable throughout $\mathcal{D}_\Lambda^\circ$;
3. $\Lambda(\cdot)$ is steep, namely, $\lim_{n \to \infty} |\nabla \Lambda(\lambda_n)| = \infty$ whenever $\{\lambda_n\}$ is a sequence in $\mathcal{D}_\Lambda^\circ$ converging to a boundary point of $\mathcal{D}_\Lambda^\circ$.

Theorem 2.4 (Gärtner–Ellis). Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of random vectors taking values in $\mathbb{R}^d$. Assume that for each $\lambda \in \mathbb{R}^d$, the logarithmic moment generating function, defined as the limit

$$\Lambda(\lambda) \equiv \lim_{n \to \infty} \frac{1}{n} \log \left( \mathbb{E} e^{n\langle \lambda, X_n \rangle} \right)$$

exists as an extended real number. Further, assume that the origin belongs to $\mathcal{D}_\Lambda^\circ$. If $\Lambda$ is an essentially smooth and lower semicontinuous function, then the LDP holds for $\{X_n\}_{n \in \mathbb{N}}$ with the good rate function $\Lambda^*(\cdot)$. Here $\Lambda^*(x) = \sup_{\lambda \in \mathbb{R}^d} \{\langle \lambda, x \rangle - \Lambda(\lambda)\}, x \in \mathbb{R}^d$, is the Fenchel–Legendre transform of $\Lambda(\cdot)$.

It is known that the key point of the Gärtner–Ellis theorem is to study the logarithmic moment generating function. Moreover, we would like to mention that the Gärtner–Ellis theorem is valid in the case of continuous parameter family $\{X_\epsilon\}_{\epsilon>0}$ (see the remarks of [10, Theorem 2.3.6]). By means of the Gärtner–Ellis theorem, we show that both the mean position $A_T$ and mean velocity $B_T$ of the exact solution satisfy the LDPs.

Theorem 2.5. $\{A_T\}_{T>0}$ satisfies an LDP with the good rate function $I(y) = \frac{y^2}{2\alpha^2}$, i.e.,

$$\liminf_{T \to \infty} \frac{1}{T} \log \left( \mathbb{P}(A_T \in U) \right) \geq -\inf_{y \in U} I(y) \quad \text{for every open } U \subset \mathbb{R},$$

$$\limsup_{T \to \infty} \frac{1}{T} \log \left( \mathbb{P}(A_T \in C) \right) \leq -\inf_{y \in C} I(y) \quad \text{for every closed } C \subset \mathbb{R}.$$

Proof. It follows from (1.2) and (1.3) that

$$TA_T = \int_0^T X_t dt = x_0 \sin(T) + y_0(1 - \cos(T)) + \alpha \int_0^T \int_0^t \sin(t-s) dW_s dt$$


\[= x_0 \sin(T) + y_0(1 \cos(T)) + \alpha \int_0^T \int_s^T \sin(t - s) dt \, dW_s \]
\[= x_0 \sin(T) + y_0(1 - \cos(T)) + \alpha \int_0^T \int_s^T [1 - \cos(T - s)] \, dW_s, \]
where the stochastic Fubini theorem is used in the third equality. Thus, we have
\[E[T_A_T] = x_0 \sin(T) + y_0(1 - \cos(T)), \]
and
\[\text{Var}[T_A_T] = \alpha^2 \int_0^T [1 - \cos(T - s)]^2 \, ds = \alpha^2 \left[ \frac{3T}{2} - 2 \sin(T) + \frac{\sin(2T)}{4} \right]. \]
Hence \(\lambda T A_T \sim \mathcal{N}(\lambda E[T_A_T], \lambda^2 \text{Var}[T_A_T])\) for every \(\lambda \in \mathbb{R}\). It follows from the characteristic function of \(\lambda T A_T\) that
\[Ee^{\lambda T A_T} = e^{\lambda E[T_A_T] + \frac{\lambda^2}{2} \text{Var}[T_A_T]}, \]
In this way, we obtain the logarithmic moment generating function
\[\Lambda(\lambda) = \lim_{T \to \infty} \frac{1}{T} \log Ee^{\lambda T A_T} \]
\[= \frac{3\lambda^2}{4}, \]
which means that \(\Lambda(\cdot)\) is an essentially smooth, lower semicontinuous function. Moreover, we have that the origin 0 belongs to \(D^+_0 = \mathbb{R}\). By the Theorem 2.4, we obtain that \(\{A_T\}_{T > 0}\) satisfies an LDP with the good rate function \(I(y) = \Lambda^*(y) = \sup_{\lambda \in \mathbb{R}} \{y \lambda - \Lambda(\lambda)\} = \frac{y^2}{2\alpha^2}. \)

Notice that the LDP for \(\{A_T\}_{T > 0}\) is independent of the initial value \((x_0, y_0)\) of the stochastic oscillator (\ref{eq:1.1}).\ Theorem 2.5 indicates that, for any initial value \((x_0, y_0)\), the probability that the mean position \(A_T\) hits the interval \([a, a + da]\) decays exponentially and formally satisfies
\[P(A_T \in [a, a + da]) \approx e^{-TI(a) da} = e^{-T \frac{\sin^2(\alpha t)}{2\alpha^2} da}, \quad \text{for sufficiently large } T. \quad (2.1)\]
Similarly, we give the result of the LDP for \(\{B_T\}_{T > 0}\) in the following theorem.

**Theorem 2.6.** \(\{B_T\}_{T > 0}\) satisfies an LDP with the good rate function \(J(y) = \frac{y^2}{2\alpha^2}\), i.e,
\[
\liminf_{T \to \infty} \frac{1}{T} \log \mathbb{P}(B_T \in U) \geq -\inf_{y \in U} J(y) \quad \text{for every open } U \subset \mathbb{R},
\]
\[
\limsup_{T \to \infty} \frac{1}{T} \log \mathbb{P}(B_T \in C) \leq -\inf_{y \in C} J(y) \quad \text{for every closed } C \subset \mathbb{R}.
\]

**Proof.** This proof is analogous to that of Theorem 2.5. According to (\ref{eq:1.2}), the distribution of \(X_T\) is
\[\mathcal{N}\left( x_0 \cos(T) + y_0 \sin(T), \alpha^2 \left( \frac{T}{2} - \frac{\sin(2T)}{4} \right) \right). \]
Then, we get the logarithmic moment generating function
\[\Lambda(\lambda) = \lim_{T \to \infty} \frac{1}{T} \log Ee^{\lambda T B_T} \]
\[= \lim_{T \to \infty} \frac{1}{T} \log Ee^{\lambda X_T}. \]
\[
\lim_{T \to \infty} \frac{1}{T} \left[ \lambda \mathbb{E}(X_T) + \frac{1}{2} \lambda^2 \text{Var}(X_T) \right] = \frac{\alpha^2 \lambda^2}{4}.
\]

Accordingly, it follows from Theorem 2.4 that \( \{B_T\}_{T > 0} \) satisfies an LDP with the good rate function

\[
J(y) = \Lambda^*(y) = \frac{y^2}{\alpha^2},
\]

which proves this theorem. \( \square \)

3. LDP for discrete mean position \( A_N \)

In this section, we study the LDP for the discrete mean position of general numerical methods. We show that symplectic methods and non-symplectic ones satisfy different types of LDPs.

Let \( \{(x_n, y_n)\}_{n \geq 1} \) be the discrete approximations at \( t_n = nh \) with \( x_n \approx X_{t_n}, \ y_n \approx Y_{t_n}, \) where \( h > 0 \) is the given step-size. Following [22], we consider the general numerical methods in form of

\[
\begin{pmatrix}
x_{n+1} \\ y_{n+1}
\end{pmatrix} = A \begin{pmatrix}
x_n \\ y_n
\end{pmatrix} + \alpha b \Delta W_n = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \alpha \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \Delta W_n, \tag{3.1}
\]

with \( \Delta W_n = W_{t_{n+1}} - W_{t_n} \). In fact, the real matrix \( A \) and the real vector \( b \) depend on both the method and the constant step-size \( h \). In the previous section, we derive the LDP for the mean position \( A_T \) of the continuous system \( \text{[11]} \). In what follows, we consider the LDP for discrete mean position \( A_N \) of the method \( \text{[3.1]} \) and study how closely the LDP for \( A_N \) approximates the LDP for \( A_T \). We recall that \( A_N \) is defined as

\[
A_N = \frac{1}{N} \sum_{n=0}^{N-1} x_n, \quad N = 1, 2, \ldots \tag{3.2}
\]

Aiming at giving the general formula of \( \{x_n\} \), we denote \( M_n = \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} \) for \( n \geq 1 \). It follows from recurrence \( \text{[3.1]} \) that

\[
M_n = BM_{n-1} + r_n, \quad n \geq 1,
\]

with

\[
B = \begin{pmatrix}
\text{tr}(A) & -\text{det}(A) \\ 1 & 0
\end{pmatrix}, \quad r_n = \begin{pmatrix}
\alpha (b_1 \Delta W_n + (a_{12} b_2 - a_{22} b_1) \Delta W_{n-1}) \\ 0
\end{pmatrix},
\]

where \( \text{tr}(A) \) and \( \text{det}(A) \) denote the trace and the determinant of \( A \), respectively. In this way, we have

\[
M_n = B^n M_0 + \sum_{j=1}^{n} B^{n-j} r_j, \quad n \geq 1. \tag{3.3}
\]

Suppose that the coefficients in matrix \( B \) satisfy

\[
(A1) \quad 4 \text{det}(A) - (\text{tr}(A))^2 > 0,
\]

which guarantees that the eigenvalues of \( B \) are

\[
\lambda_{\pm} = \frac{\text{tr}(A)}{2} \pm i \frac{\sqrt{4 \text{det}(A) - (\text{tr}(A))^2}}{2} = \sqrt{\text{det}(A)} e^{\pm i \theta}, \quad i^2 = -1,
\]

where \( \theta = \tan^{-1} \left( \frac{2 \text{tr}(A) \text{det}(A)}{(\text{tr}(A))^2 - 4 \text{det}(A)} \right) \). The matrix \( B \) is then diagonalized as

\[
B = \begin{pmatrix}
\lambda_+ & 0 \\ 0 & \lambda_-
\end{pmatrix}.
\]
for some $\theta \in (0, \pi)$ satisfying
\[
\cos(\theta) = \frac{\text{tr}(A)}{2\sqrt{\det(A)}}, \quad \sin(\theta) = \frac{\sqrt{4\det(A) - (\text{tr}(A))^2}}{2\sqrt{\det(A)}}.
\] (3.4)

Let $\hat{\alpha}_n = (\det(A))^{n/2} \frac{\sin((n+1)\theta)}{\sin(\theta)}$ and $\hat{\beta}_n = -(\det(A))^{(n+1)/2} \frac{\sin(n\theta)}{\sin(\theta)}$, for any integer $n$. It follows from (3.3) (one can refer to [22]) that

\[
x_{n+1} = \hat{\alpha}_n x_1 + \hat{\beta}_n x_0 + \alpha \sum_{j=1}^{n} \hat{\alpha}_{n-j} [b_1 \Delta W_j + (a_{12} b_2 - a_{22} b_1) \Delta W_{j-1}], \quad n \geq 0.
\]

Noting that $x_1 = a_{11} x_0 + a_{12} y_0 + \alpha b_1 \Delta W_0$, $\hat{\alpha}_{-1} = 0$ and $\hat{\alpha}_0 = 1$, we have that, for $n \geq 1$,

\[
x_n = \hat{\alpha}_{n-1} (a_{11} x_0 + a_{12} y_0 + \alpha b_1 \Delta W_0) + \hat{\beta}_{n-1} x_0 + \alpha \sum_{j=1}^{n-1} \hat{\alpha}_{n-1-j} [b_1 \Delta W_j + (a_{12} b_2 - a_{22} b_1) \Delta W_{j-1}]
\]

\[
= \left( a_{11} \hat{\alpha}_{n-1} + \hat{\beta}_{n-1} \right) x_0 + a_{12} \hat{\alpha}_{n-1} y_0 + \alpha b_1 \hat{\alpha}_{n-1} \Delta W_0 + \alpha \sum_{j=1}^{n-2} [b_1 \hat{\alpha}_{n-1-j} + (a_{12} b_2 - a_{22} b_1) \hat{\alpha}_{n-2-j}] \Delta W_j + \alpha b_1 \Delta W_{n-1}
\]

\[
= \left( a_{11} \hat{\alpha}_{n-1} + \hat{\beta}_{n-1} \right) x_0 + a_{12} \hat{\alpha}_{n-1} y_0 + \alpha \sum_{j=0}^{n-1} [b_1 \hat{\alpha}_{n-1-j} + (a_{12} b_2 - a_{22} b_1) \hat{\alpha}_{n-2-j}] \Delta W_j.
\] (3.5)

By (3.2) and (3.5), we have

\[
NA_N = x_0 + \sum_{n=1}^{N-1} x_n
\]

\[
= x_0 + \left( a_{11} \sum_{n=1}^{N-1} \hat{\alpha}_{n-1} + \sum_{n=1}^{N-1} \hat{\beta}_{n-1} \right) x_0 + a_{12} \sum_{n=1}^{N-1} \hat{\alpha}_{n-1} y_0
\]

\[
+ \alpha \sum_{n=1}^{N-1} \sum_{j=0}^{n-1} [b_1 \hat{\alpha}_{n-1-j} + (a_{12} b_2 - a_{22} b_1) \hat{\alpha}_{n-2-j}] \Delta W_j
\]

\[
= \left( 1 + a_{11} \sum_{n=0}^{N-2} \hat{\alpha}_n + \sum_{n=0}^{N-2} \hat{\beta}_n \right) x_0 + a_{12} \sum_{n=0}^{N-2} \hat{\alpha}_n y_0
\]

\[
+ \alpha \sum_{j=0}^{N-2} \sum_{n=j+1}^{N-1} [b_1 \hat{\alpha}_{n-1-j} + (a_{12} b_2 - a_{22} b_1) \hat{\alpha}_{n-2-j}] \Delta W_j
\]

\[
= \left( 1 + a_{11} S_N + S_N \right) x_0 + a_{12} S_N y_0 + \alpha \sum_{j=0}^{N-2} c_j \Delta W_j,
\] (3.6)
where \( S^\alpha_N = \sum_{n=0}^{N-2} \hat{\alpha}_n, \) \( S^\beta_N = \sum_{n=0}^{N-2} \hat{\beta}_n \) and
\[
c_j := \sum_{n=j+1}^{N-1} \left[ b_1 \hat{\alpha}_{n-1-j} + (a_{12} b_2 - a_{22} b_1) \hat{\alpha}_{n-2-j} \right]
\]
\[
= b_1 \sum_{n=0}^{N-2-j} \hat{\alpha}_n + (a_{12} b_2 - a_{22} b_1) \sum_{n=-1}^{N-3-j} \hat{\alpha}_n
\]
\[
= b_1 \hat{\alpha}_{N-2-j} + (b_1 + a_{12} b_2 - a_{22} b_1) S^\alpha_{N-1-j}. \tag{3.7}
\]

To give precise results of (3.6), we need to compute \( S^\alpha_N \) and \( S^\beta_N \) respectively, and then the following lemma is required.

**Lemma 3.1.** For arbitrary \( \theta \in (0, \pi), \) \( N \in \mathbb{N}^+ \) and \( a \in \mathbb{R}, \) it holds that
\[
\sum_{n=1}^{N} \sin(n\theta)a^n = \frac{a \sin(\theta) - a^{N+1} \sin((N+1)\theta) + a^{N+2} \sin(N\theta)}{1 - 2a \cos(\theta) + a^2}. \tag{3.8}
\]
In particular, if \( a = 1, \) then
\[
\sum_{n=1}^{N} \sin(n\theta) = \frac{\cos(\frac{\theta}{2}) - \cos((N + \frac{1}{2})\theta)}{2 \sin(\frac{\theta}{2})}. \tag{3.9}
\]

**Proof.** Since \( \sin(n\theta) = \frac{1}{2i} (e^{in\theta} - e^{-in\theta}), \)
\[
\sum_{n=1}^{N} \sin(n\theta)a^n = \frac{1}{2i} \left[ \sum_{n=1}^{N} (ae^{i\theta})^n - \sum_{n=1}^{N} (ae^{-i\theta})^n \right]
\]
\[
= \frac{1}{2i} \left[ \frac{ae^{i\theta} - (ae^{i\theta})^{N+1}}{1 - ae^{i\theta}} - \frac{ae^{-i\theta} - (ae^{-i\theta})^{N+1}}{1 - ae^{-i\theta}} \right]
\]
\[
= Im(\frac{ae^{i\theta} - (ae^{i\theta})^{N+1}}{1 - ae^{i\theta}})
\]
\[
= \frac{a \sin(\theta) - a^{N+1} \sin((N+1)\theta) + a^{N+2} \sin(N\theta)}{1 - 2a \cos(\theta) + a^2}.
\]
Here, \( Im(\frac{ae^{i\theta} - (ae^{i\theta})^{N+1}}{1 - ae^{i\theta}}) \) is the imaginary part of \( \frac{ae^{i\theta} - (ae^{i\theta})^{N+1}}{1 - ae^{i\theta}}. \)

For \( a = 1, \) utilizing the formula \( \sin(\alpha) - \sin(\beta) = 2 \cos(\frac{\alpha + \beta}{2}) \sin(\frac{\alpha - \beta}{2}) \) gives
\[
\sum_{n=1}^{N} \sin(n\theta) = \frac{\sin(\theta) - \sin((N+1)\theta) + \sin(N\theta)}{2(1 - \cos(\theta))}
\]
\[
= \frac{2 \sin(\frac{\theta}{2}) \cos(\frac{\theta}{2}) - 2 \cos((N + \frac{1}{2})\theta) \sin(\frac{\theta}{2})}{4 \sin^2(\frac{\theta}{2})}
\]
\[
= \frac{\cos(\frac{\theta}{2}) - \cos((N + \frac{1}{2})\theta)}{2 \sin(\frac{\theta}{2})},
\]
which completes the proof. \( \square \)
It follows from \([3.8]\) that, for every \(N \geq 1\),

\[
S_N^\alpha = \sum_{n=0}^{N-2} \frac{\sin((n+1)\theta)}{\sin(\theta)} \left( \sqrt{\det(A)} \right)^{n+1}
\]

\[
= \frac{1}{\sin(\theta) \sqrt{\det(A)}} \sum_{n=1}^{N-1} \sin(n\theta) \left( \sqrt{\det(A)} \right)^n \sin(\theta) - \left( \sqrt{\det(A)} \right)^{N-1} \sin((N-1)\theta) + \left( \sqrt{\det(A)} \right)^N \sin((N-2)\theta) \sin(\theta) \left( 1 - 2\sqrt{\det(A)} \cos(\theta) + \det(A) \right) .
\]

Further, because \(\hat{\beta}_n = - \det(A)\hat{\alpha}_{n-1}\) and \(\alpha_{-1} = 0\), we have

\[
S_N^\beta = - \det(A) \sum_{n=0}^{N-2} \hat{\alpha}_{n-1}
\]

\[
= - \det(A) S_N^\alpha - \det(A) \sin(\theta) - \left( \sqrt{\det(A)} \right)^N \sin((N-1)\theta) + \left( \sqrt{\det(A)} \right)^{N+1} \sin((N-2)\theta) \sin(\theta) \left( 1 - 2\sqrt{\det(A)} \cos(\theta) + \det(A) \right) ,
\]

and

\[
c_j = \frac{b_1}{\sin(\theta)} \sin((N-1-j)\theta) \left( \sqrt{\det(A)} \right)^{N-2-j} + \frac{b_1 + a_{12}b_2 - a_{22}b_1}{\sin(\theta)} \sin(\theta) - \left( \sqrt{\det(A)} \right)^{N-2-j} \sin((N-1-j)\theta) + \left( \sqrt{\det(A)} \right)^{N-1-j} \sin((N-2-j)\theta) \sin(\theta) \left( 1 - 2\sqrt{\det(A)} \cos(\theta) + \det(A) \right) .
\]

Based on \([3.10]-[3.12]\), we obtain the expression of \(NA_N\). To obtain the LDP for \(A_N\), we exclude the case \(\det(A) > 1\) which makes \(S_N^\alpha, S_N^\beta\) and \(c_N\) exponentially increase as \(N\) increases. This is to say, we need to deal with the case \(\det(A) = 1\) and the case \(\det(A) < 1\) separately. It is known that the method \([3.1]\) preserves the symplectic structure, i.e., \(dx_{n+1} \wedge dy_{n+1} = dx_n \wedge dy_n\), if and only if \(\det(A) = 1\) (see e.g., \([22]\)). Thus, we investigate the LDPs of \(A_N\) for the symplectic methods and non-symplectic methods separately.

### 3.1. LDP of \(A_N\) for symplectic methods

In this part, we derive the LDP for \(A_N\) of the method \([3.1]\) in the case of preserving the symplecticity. Hereafter we use the notation \(K(a_1, \ldots, a_m)\) to denote some constant dependent on the parameters \(a_1, \ldots, a_m\) but independent of \(N\), which may vary from one line to another.

We assume that

\[
(A2) \quad \det(A) = 1.
\]

Under \((A2)\), we have \(\hat{\alpha}_n = \frac{\sin((n+1)\theta)}{\sin(\theta)}\), \(\hat{\beta}_n = - \frac{\sin(n\theta)}{\sin(\theta)}\). Then by \([3.7]\) and \([3.9]\), we obtain

\[
S_N^\alpha = \frac{\cos \left( \frac{\theta}{2} \right) - \cos((N-\frac{1}{2})\theta)}{2 \sin(\theta) \sin \left( \frac{\theta}{2} \right)} ,
\]

\[
(3.13)
\]
According to the increment independence of Brownian motions, it follows from (3.6) that

\[ S_N^\beta = -\frac{\cos \left( \frac{\theta}{2} \right) - \cos \left( \frac{N - \frac{3}{2} \theta}{2} \right)}{2 \sin(\theta) \sin \left( \frac{\theta}{2} \right)}, \]

\[ c_j = \frac{(b_1 + a_1b_2 - a_2b_1) \cos \left( \frac{\theta}{2} \right) - b_1 \cos (N - \frac{1}{2} - j)\theta - (a_1b_2 - a_2b_1) \cos (N - \frac{3}{2} - j)\theta}{2 \sin(\theta) \sin \left( \frac{\theta}{2} \right)}. \]

By (3.13) and (3.14), it holds that

\[ |S_N^\alpha| + |S_N^\beta| \leq K(\theta), \quad \forall \ N \geq 2. \]

According to the increment independence of Brownian motions, it follows from (3.6) that NA_N is Gaussian. Further, it follows from (3.6) and (3.15)

\[ |E[N A_N]| = \left| (1 + a_1S_N^\alpha + S_N^\beta)x_0 + a_1S_N^\alpha y_0 \right| \leq K(x_0, y_0, \theta), \]

and

\[ \Var[N A_N] = \alpha^2 h \sum_{j=0}^{N-2} c_j^2 = \frac{\alpha^2 h}{4 \sin^2(\theta) \sin^2 \left( \frac{\theta}{2} \right)} \sum_{j=0}^{N-2} c_j^2, \]

with

\[ c_j^2 = (b_1 + a_1b_2 - a_2b_1)^2 \cos^2 \left( \frac{\theta}{2} \right) + b_1^2 \left[ \frac{1}{2} + \frac{1}{2} \cos ((2N - 1 - 2j)\theta) \right] \]

\[ + 2(b_1 + a_1b_2 - a_2b_1)(a_1b_2 - a_2b_1) \cos \left( \frac{\theta}{2} \right) \cos \left( \frac{(2N - 1 - 2j)\theta}{2} \right) \]

\[ - 2(b_1 + a_1b_2 - a_2b_1)(a_1b_2 - a_2b_1) \cos \left( \frac{\theta}{2} \right) \cos \left( \frac{(2N - 3 - 2j)\theta}{2} \right) \]

\[ + b_1(a_1b_2 - a_2b_1) \cos \left( \frac{\theta}{2} \right) \cos \left( \frac{(2N - 2 - 2j)\theta}{2} \right) \]

\[ = (b_1 + a_1b_2 - a_2b_1)^2 \cos^2 \left( \frac{\theta}{2} \right) + \frac{1}{2} b_1^2 \left[ \frac{1}{2} + \frac{1}{2} \cos ((2N - 2 - 2j)\theta) \right] \]

\[ + b_1(a_1b_2 - a_2b_1) \cos \left( \frac{\theta}{2} \right) \cos \left( \frac{(2N - 2 - 2j)\theta}{2} \right) \]

\[ + b_1(a_1b_2 - a_2b_1) \cos \left( \frac{\theta}{2} \right) \cos \left( \frac{(2N - 3 - 2j)\theta}{2} \right) \]

\[ + b_1 \cos \left( \frac{\theta}{2} \right) \cos \left( \frac{(2N - 1 - 2j)\theta}{2} \right). \]

We claim

\[ \sum_{j=0}^{N-2} R_j \leq K(\theta). \]

In detail, by \( \sum_{n=1}^{N} \cos \left( 2n + 1 \right) \theta = \frac{\sin(2N\theta) - \sin(2\theta)}{2 \sin(\theta)} \), we have

\[ \sum_{j=0}^{N-2} \cos \left( 2N - 1 - 2j \right) \theta = \sum_{n=1}^{N-1} \cos \left( 2n + 1 \right) \theta = \frac{\sin(2N\theta) - \sin(2\theta)}{2 \sin(\theta)} \leq K(\theta). \]
Analogously, we obtain
\[
\begin{align*}
\left| \sum_{j=0}^{N-2} \cos((2N - 3 - 2j)\theta) \right| &+ \left| \sum_{j=0}^{N-2} \cos \left( \frac{(2N - 1 - 2j)\theta}{2} \right) \right| \\
+ \left| \sum_{j=0}^{N-2} \cos \left( \frac{(2N - 3 - 2j)\theta}{2} \right) \right| &+ \left| \sum_{j=0}^{N-2} \cos((2N - 2 - 2j)\theta) \right| \leq K(\theta),
\end{align*}
\]
which proves the claim \((3.19)\).

Based on \((3.16)\), \((3.17)\), \((3.18)\) and \((3.19)\), we have
\[
\Lambda^h(\lambda) := \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} e^{\Lambda^h AN}
\]
\[
= \lim_{N \to \infty} \frac{1}{N} \left[ \lambda \mathbb{E}(NA_N) + \frac{\lambda^2}{2} \mathbb{Var}(NA_N) \right]
\]
\[
= \frac{\lambda^2}{2} \lim_{N \to \infty} \frac{1}{N} \mathbb{Var}(NA_N)
\]
\[
= \frac{\alpha^2h\lambda^2}{8 \sin^2(\theta) \sin^2(\frac{\theta}{2})} \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-2} \varepsilon_j^2
\]
\[
= \frac{\alpha^2h\lambda^2}{8 \sin^2(\theta) \sin^2(\frac{\theta}{2})} \left[ \left( b_1 + a_{12}b_2 - a_{22}b_1 \right)^2 \cos^2 \left( \frac{\theta}{2} \right) + \frac{1}{2} b_1^2 \\
+ \frac{1}{2} \left( a_{12}b_2 - a_{22}b_1 \right)^2 + b_1 \left( a_{12}b_2 - a_{22}b_1 \right) \cos(\theta) \right].
\]

As a result of \((3.4)\) with \(\det(A) = 1\), it holds that
\[
\cos(\theta) = \frac{\text{tr}(A)}{2}, \quad \sin(\theta) = \frac{\sqrt{4 - \left( \text{tr}(A) \right)^2}}{2},
\]
\[
\sin^2 \left( \frac{\theta}{2} \right) = \frac{1 - \cos(\theta)}{2} = \frac{2 - \text{tr}(A)}{4}, \quad \cos^2 \left( \frac{\theta}{2} \right) = \frac{1 + \cos(\theta)}{2} = \frac{2 + \text{tr}(A)}{4}.
\]

Substituting \((3.22)\) into \((3.21)\) yields that
\[
\Lambda^h(\lambda) = \frac{\alpha^2h\lambda^2}{2(2 + \text{tr}(A))(2 - \text{tr}(A))^2} \left[ \left( b_1 + a_{12}b_2 - a_{22}b_1 \right)^2 \left( 2 + \text{tr}(A) \right) + 2b_1^2 \\
+ 2(a_{12}b_2 - a_{22}b_1)^2 + 2b_1(a_{12}b_2 - a_{22}b_1)\text{tr}(A) \right]
\]
\[
= \frac{\alpha^2h\lambda^2}{2(2 + \text{tr}(A))(2 - \text{tr}(A))^2} \left[ \left( b_1 + a_{12}b_2 - a_{22}b_1 \right)^2 \left( 4 + \text{tr}(A) \right) \\
- 2b_1(a_{12}b_2 - a_{22}b_1)(2 - \text{tr}(A)) \right].
\]

In order to assure that \(\Lambda^h(\cdot)\) is essentially smooth, we further assume
\((A3)\) \(\left( b_1 + a_{12}b_2 - a_{22}b_1 \right)^2 \left( 4 + \text{tr}(A) \right) - 2b_1(a_{12}b_2 - a_{22}b_1)(2 - \text{tr}(A)) > 0.\)

Under assumptions \((A1)\), \((A2)\) and \((A3)\), it follows from Theorem \((2.4)\) that \(\{A_N\}_{N \geq 1}\) satisfies an LDP with the good rate function
\[
I^h(y) = \sup_{\lambda \in \mathbb{R}} \{ y \lambda - \Lambda^h(\lambda) \}.
\]
where it follows from (3.6) and (3.12) that

\[
A_4 = \exp(2 + \text{tr}(A))(2 - \text{tr}(A))^2 y^2 \frac{2\alpha_2}{1 + \sigma_2(\sigma_2 - 2\beta_1)}.
\]  

Finally, we acquire the following theorem:

**Theorem 3.2.** If the numerical method \([3.1]\) for approximating the SDE \([1.1]\) satisfies the assumptions \((A1), (A2)\) and \((A3)\), then its mean position \(\{A_N\}_{N \geq 1}\) satisfies an LDP with the good rate function given by \([3.21]\).

**Remark 3.3.** Theorem \([3.2]\) indicates that to make the LDP hold for \(A_N\), the step-size \(h\) need to be restricted such that conditions \((A1), (A2)\) and \((A3)\) hold. Moreover, the rate function \(I^h(y)\) does not depend on the initial \((x_0, y_0)\). That is to say, for appropriate step-size \(h\) and arbitrary initial value, \(A_N\) formally satisfies \(P(A_N \in [a, a + da]) \approx e^{-N I^h(a)} da\) for sufficiently large \(N\).

### 3.2. LDP of \(A_N\) for non-symplectic methods.

In this part, we show the LDP for \(A_N\) of method \([3.1]\) when it does not preserve the symplecticity. To this end, we firstly suppose that

\[ (A4) \quad 0 < \det(A) < 1. \]

Under condition \((A4)\), one immediately concludes from \([3.10]\) and \([3.11]\) that

\[
|S_N^\delta| + |\hat{S}_N^\delta| \leq K(\theta), \quad \forall \quad N \geq 2,
\]

which gives

\[
|\mathbb{E}[N A_N]| \leq K(x_0, y_0, \theta).
\]  

(3.25)

It follows from \((3.6)\) and \((3.12)\) that

\[
\text{Var}(N A_N) = \alpha^2 h \sum_{j=0}^{N-2} c_j^2,
\]

where

\[
c_j^2 = \left( \frac{b_1 + a_{12} b_2 - a_{22} b_1}{1 - 2 \sqrt{\det(A) \cos(\theta) + \det(A)}} \right)^2 + \tilde{R}_j,
\]

\[ (3.27) \]

with

\[
\tilde{R}_j = \frac{b_2^2 (N - 1 - j)^2 \sin^2(\theta)(\det(A))^N - 2 - j}{\sin^2(\theta)} + \frac{(b_1 + a_{12} b_2 - a_{22} b_1)^2}{\sin^2(\theta) \left(1 - 2 \sqrt{\det(A) \cos(\theta) + \det(A)}\right)^2}.
\]

\[
\text{Var}(N A_N) \approx e^{-N h \theta(a)} \text{ for sufficiently large } N.
\]

(3.28)

Since

\[
|\tilde{R}_j| \leq K(\theta) \left( (\det(A))^{N-j} + \left( 2 \sqrt{\det(A)} \right)^{N-j} \right) \leq 2 \tilde{K}(\theta) \left( \sqrt{\det(A)} \right)^{N-j},
\]

we have

\[
\sum_{j=0}^{N-2} |\tilde{R}_j| \leq K(\theta) \sum_{j=0}^{N-j} \left( \sqrt{\det(A)} \right)^{j} \leq K(\theta).
\]  

(3.28)
Combining (3.25), (3.26), (3.27) and (3.28) leads to
\[
\bar{\Lambda}^h(\lambda) = \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} e^{NA_N}
\]
\[
= \lim_{N \to \infty} \frac{1}{N} \left[ \lambda \mathbb{E}(NA_N) + \frac{\lambda^2}{2} \text{Var}(NA_N) \right]
\]
\[
= \frac{\alpha^2 h \lambda^2}{2} \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-2} c_j^2
\]
\[
= \frac{\alpha^2 h \lambda^2}{2} \lim_{N \to \infty} \frac{1}{N} \left[ \frac{b_1 + a_{12} b_2 - a_{22} b_1}{1 - 2 \sqrt{\det(A)} \cos(\theta) + \det(A)} \right]^2 (N - 1) + \sum_{j=0}^{N-2} \tilde{R}_j
\]
\[
= \frac{\alpha^2 h \lambda^2}{2} \left( \frac{b_1 + a_{12} b_2 - a_{22} b_1}{1 - 2 \sqrt{\det(A)} \cos(\theta) + \det(A)} \right)^2.
\]

If we assume that
\[(A5) \quad b_1 + a_{12} b_2 - a_{22} b_1 \neq 0,\]
then it follows from Theorem 2.4 that \( \{A_N\}_{N \geq 1} \) satisfies an LDP with the good rate function
\[
\bar{I}^h(y) = \frac{y^2}{2 \alpha^2 h} \left( \frac{1 - 2 \sqrt{\det(A)} \cos(\theta) + \det(A)}{b_1 + a_{12} b_2 - a_{22} b_1} \right)^2
\]
\[
= \frac{y^2}{2 \alpha^2 h} \left( \frac{1 - \text{tr}(A) + \det(A)}{b_1 + a_{12} b_2 - a_{22} b_1} \right)^2,
\]
where we have used (3.3) in the second equality. Finally, we obtain the following theorem:

**Theorem 3.4.** If the numerical method (3.1) for approximating the SDE (1.1) satisfies the assumptions (A1), (A4) and (A5), then its mean position \( \{A_N\}_{N \geq 1} \) satisfies an LDP with the good rate function
\[
\bar{I}^h(y) = \frac{y^2}{2 \alpha^2 h} \left( \frac{1 - \text{tr}(A) + \det(A)}{b_1 + a_{12} b_2 - a_{22} b_1} \right)^2.
\]

4. **ASYMPTOTICAL PRESERVATION FOR THE LDP OF \( A_T \)**

In Section 3, we acquire the LDP for mean position \( A_N \) when the method (3.1) is symplectic or non-symplectic separately, for given appropriate step-size. In this section, we study their asymptotical preservation for the LDP of \( A_T \) as step-size tends to 0. In fact, as are shown in (2.1) and Remark 3.3, if the LDP holds for \( A_T \) with the rate function \( I(y) \) and for \( A_N \) with the rate function \( I^h(y) \), then
\[
\mathbb{P}(A_T \in [a, a + da]) \approx e^{-T I(a)} da, \quad \text{for sufficiently large } T,
\]
\[
\mathbb{P}(A_N \in [a, a + da]) \approx e^{-N I^h(a)} da, \quad \text{for sufficiently large } N.
\]
This shows that the probability in (4.1) decays exponentially as the “time” \( T \) tends to infinity, while in (4.2) the observation changes into the “number of steps” \( N \). In order to unify the observation scale, we rewrite (4.2) as
\[
\mathbb{P}(A_N \in [a, a + da]) \approx e^{-N I^h(a)} da = e^{-N h \frac{I^h(a)}{h}} da = e^{-t N I^h_{\text{mod}}(a)} da,
\]
where $I_{\text{mod}}^h(a) = \frac{I^h(a)}{h}$. To illustrate how the LDP of $A_N$ approximates the LDP of $A_T$, we need to introduce the so-called modified rate function.

**Definition 4.1.** If discrete mean position $\{A_N\}_{N \geq 1}$ of method (3.1) satisfies an LDP with the rate function $I^h(\cdot)$, then we call $I_{\text{mod}}^h(\cdot) = I^h(\cdot)/h$ the modified rate function of $I^h(\cdot)$. Furthermore, method (3.1) is said to asymptotically preserve the LDP of $A_T$ if

$$\lim_{h \to 0} I_{\text{mod}}^h(y) = I(y), \quad \forall \ y \in \mathbb{R},$$

where $I(\cdot)$ is the rate function of the LDP for $A_T$. In particular, method (3.1) is said to exactly preserve the LDP of $A_T$ if for all sufficiently small step-size $h$, $I_{\text{mod}}^h(\cdot) = I(\cdot)$.

By Definition 4.1, we obtain the modified rate functions of the rate functions appearing in Theorems 3.2 and 3.4, respectively, as follows:

$$I_{\text{mod}}^h(y) = \frac{(2 + \text{tr}(A))(2 - \text{tr}(A))^2 y^2}{2\alpha^2 h^2 [(b_1 + a_{12} b_2 - a_{22} b_1)^2 (4 + \text{tr}(A)) - 2b_1 (a_{12} b_2 - a_{22} b_1)(2 - \text{tr}(A))]},$$  \hspace{1cm} (4.3)

$$\tilde{I}_{\text{mod}}^h(y) = \frac{y^2}{2\alpha^2 h^2} \left( \frac{1 - \text{tr}(A) + \text{det}(A)}{b_1 + a_{12} b_2 - a_{22} b_1} \right)^2.$$  \hspace{1cm} (4.4)

It would fail to get the asymptotically convergence for $I_{\text{mod}}^h(y)$ and $\tilde{I}_{\text{mod}}^h(y)$ only by means of conditions (A1) – (A5) in two aspects: one is that both $A$ and $b$ are some functions of step-size $h$, which are unknown unless a specific method is applied; the other is that for some $A$ and $b$, the numerical approximation may not be convergent to the original system. A solution to this problem is studying the convergence on finite interval of numerical methods. In what follows, we consider the mean-square convergence of the method (3.1).

For the sake of simplicity, we first give some notations. Let $R = O(h^p)$ stand for $|R| \leq Ch^p$, for all sufficiently small step-size $h$, where $C$ is independent of $h$ and may vary from one line to another. $f(h) \sim h^p$ means that $f(h)$ and $h^p$ are equivalent infinitesimal. Furthermore, $\| \cdot \|_2$ denotes 2-norm of a vector or matrix and $\| \cdot \|_F$ denotes Frobenius norm of a matrix.

To give the conditions about the mean-square convergence of the method (3.1), we introduce the Euler-Maruyama method of form (3.1) with

$$A^{EM} = \begin{pmatrix} 1 & h \\ -h & 1 \end{pmatrix}, \quad b^{EM} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  \hspace{1cm} (4.5)

Based on the fundamental convergence theorem, we acquire the sufficient conditions which make numerical method (3.1) converge with mean-square order 1 as follows.

**Theorem 4.2.** If the numerical method (3.1) satisfies

$$\|A - A^{EM}\|_F = O(h^2) \quad \text{and} \quad \|b - b^{EM}\|_2 = O(h),$$  \hspace{1cm} (4.6)

then its convergence order is 1 in mean-square sense on any finite interval $[0, T_0]$, i.e.,

$$\sup_{n \geq 0, nh \leq T_0} \left[ \mathbb{E} \left( (x_n - X(t_n))^2 + (y_n - Y(t_n))^2 \right) \right]^{1/2} \leq K(T_0)h.$$

**Proof.** The one-step approximations based on the Euler-Maruyama method (4.5) and the method (3.1) are

$$\tilde{Z} = A^{EM} z + ab^{EM} \Delta W$$

and

$$\tilde{Z} = Az + ab \Delta W,$$
respectively, where \( \Delta W = W_{t+h} - W_t \). Based on the above two equations, we obtain
\[
\left\| E(\tilde{Z} - \bar{Z}) \right\|_2 \leq C \left\| A - A^{EM} \right\|_2 \| z \|_2 \leq C \left\| A - A^{EM} \right\|_F \| z \|_2, \tag{4.7}
\]
where the second equality uses the equivalence of norms in finite-dimensional normed linear spaces. In addition, it holds that
\[
E \left\| \tilde{Z} - \bar{Z} \right\|_2^2 = \left\langle (A - A^{EM}) z, z \right\rangle + \alpha^2 \left\| b - b^{EM} \right\|_2^2 E(\Delta W^2) 
\leq C \left\| A - A^{EM} \right\|_F^2 \| z \|_2^2 + \alpha^2 \left\| b - b^{EM} \right\|_2^2 \Delta t. \tag{4.8}
\]
Since SDE (1.1) is driven by additive noise, the Euler-Maruyama method (4.5) satisfies the Theorem 2.1 in [17] with \( p_1 = 2, p_2 = \frac{3}{2} \). It follows from Lemma 2.1 in [17] (4.6), (4.7) and (4.8) that method (3.1) is of mean-square order 1.

By the definitions of 2-norm and Frobenius norm, (4.6) is equivalent to
\[
(B) \quad |a_{11} - 1| + |a_{22} - 1| + |a_{12} - h| + |a_{21} + h| = O(h^2), \quad \text{and} \quad |b_1| + |b_2 - 1| = O(h). \tag{4.9}
\]

Using this condition (B), we have the following lemma, which is used to study whether method (3.1) asymptotically preserves the LDPS for \( A_T \) or \( B_T \) of exact solution.

**Lemma 4.3.** Under the condition (B), the following properties hold:

1. \( \text{tr}(A) \to 2 \) as \( h \to 0 \);
2. \( 1 - \text{tr}(A) + \text{det}(A) \sim h^2; \)
3. \( b_1 + a_{12}b_2 - a_{22}b_1 \sim h. \)

**Proof.** If (B) holds, then \( a_{11} = 1 + O(h^2), a_{22} = 1 + O(h^2) \). Thus, \( \text{tr}(A) = 2 + O(h^2) \), which leads to the assertion (1). Further,
\[
1 - \text{tr}(A) + \text{det}(A) = 1 - (a_{11} + a_{22}) + a_{11}a_{22} - a_{12}a_{21}
= (a_{11} - 1)(a_{22} - 1) - a_{12}a_{21}.
\]

Noting that \( a_{12} \sim h \) and \( a_{21} \sim -h \), one has \( (1 - \text{tr}(A) + \text{det}(A)) \sim h^2 \). Finally, since
\[
\lim_{h \to 0} \frac{a_{12}b_2}{h} = \lim_{h \to 0} \frac{(a_{12} - h)(b_2 - 1) + h(b_2 - 1) + a_{12}}{h} = 1,
\]

it holds that
\[
\lim_{h \to 0} \frac{b_1 + a_{12}b_2 - a_{22}b_1}{h} = \lim_{h \to 0} \frac{a_{12}b_2}{h} + \lim_{h \to 0} \frac{b_1(1 - a_{22})}{h} = 1,
\]

which is nothing but the assertion (3). □

By Lemma 4.3, we obtain the convergence of the modified rate functions in (4.3) and (4.4).

**Case 1:** Let (A1), (A2), (A3) and (B) hold. Noting \( \text{det}(A) = 1 \) in this case, Lemma 4.3(2) yields \( (2 - \text{tr}(A)) \sim h^2 \). Hence,
\[
\lim_{h \to 0} \frac{b_1(a_{12}b_2 - a_{22}b_1)(2 - \text{tr}(A))}{h^2} = 0. \tag{4.9}
\]

It follows from Lemma 4.3, 4.3 and (4.9) that
\[
\lim_{h \to 0} h^{-1} \int_{mod(A)} \frac{y^2}{2\alpha^2} \left| b_1 + a_{12}b_2 - a_{22}b_1 \right|^2 \frac{(2 + \text{tr}(A))}{h^2} (4 + \text{tr}(A)) - 2b_1(a_{12}b_2 - a_{22}b_1)(2 - \text{tr}(A))
= \frac{y^2}{2\alpha^2} \lim_{h \to 0} (2 + \text{tr}(A)) (4 + \text{tr}(A))(b_1 + a_{12}b_2 - a_{22}b_1)^2/h^2 - 2 \lim_{h \to 0} b_1(a_{12}b_2 - a_{22}b_1)(2 - \text{tr}(A))/h^2.
\]
\[
\frac{y^2}{2\alpha^2} \frac{4}{6-0} = \frac{y^2}{3\alpha^2}.
\]

(4.10)

Case 2: Let \((A1), (A4), (A5)\) and \((B)\) hold. According to (4.4) and Lemma 4.3 we have
\[
\lim_{h \to 0} \tilde{I}_h = \frac{y^2}{2\alpha^2} \lim_{h \to 0} \frac{(h^2)^2}{h^2} = \frac{y^2}{2\alpha^2}.
\]

(4.11)

Therefore, we get the following two theorems.

**Theorem 4.4.** For the numerical method (3.1) approximating the stochastic oscillator (1.1), if the assumptions \((A1), (A2)\) and \((A3)\) hold, then we have
1. The method (3.1) is symplectic;
2. The discrete mean position \(\{A_N\}_{N \geq 1}\) of method (3.1) satisfies an LDP with the good rate function
   \[
   I^h(y) = \frac{(2 + \text{tr}(A))(2 - \text{tr}(A))^2 y^2}{2\alpha^2 h [(b_1 + a_{12}b_2 - a_{22}b_1)^2 (4 + \text{tr}(A)) - 2b_1(a_{12}b_2 - a_{22}b_1)(2 - \text{tr}(A))]};
   \]
   (4.12)
3. Moreover, if assumption \((B)\) holds, then method (3.1) asymptotically preserves the LDP of \(A_T\), i.e., the modified rate function \(\tilde{I}^h(y) = I^h(y)/h\) satisfies:
   \[
   \lim_{h \to 0} \tilde{I}^h(y) = I(y), \quad \forall \ y \in \mathbb{R},
   \]
   where \(I(\cdot)\) is the rate function of LDP for \(A_T\).

**Theorem 4.5.** For the numerical method (3.1) approximating the stochastic oscillator (1.1), if the assumptions \((A1), (A4)\) and \((A5)\) hold, then we have
1. The method (3.1) is non-symplectic;
2. The discrete mean position \(\{A_N\}_{N \geq 1}\) of method (3.1) satisfies an LDP with the good rate function
   \[
   \tilde{I}^h(y) = \frac{y^2}{2\alpha^2 h} \left(1 - \text{tr}(A) + \text{det}(A)\right)^2;
   \]
   (4.13)
3. Moreover, if assumption \((B)\) holds, then method (3.1) does not asymptotically preserve the LDP of \(A_T\), i.e., for \(y \neq 0\),
   \[
   \lim_{h \to 0} \tilde{I}^h(y) \neq I(y),
   \]
   where \(\tilde{I}^h(y) = \tilde{I}^h(y)/h\), and \(I(\cdot)\) is the rate function of LDP for \(A_T\).

**Remark 4.6.** Theorems 4.4 and 4.5 indicate that under appropriate conditions, the symplectic methods asymptotically preserve the LDP for the mean position \(A_T\) of original system (1.1), while the non-symplectic methods do not. This implies that, in comparison with non-symplectic methods, symplectic methods have long-time stability in the aspect of LDP for the mean position.
5. LDP for discrete mean velocity $B_N$

In Section 2, we obtain the LDP for time-average current $B_T$ of original system (1.1). In this section, following the ideas of dealing with discrete mean position, we investigate the LDP for discrete mean velocity.

We consider the numerical approximation of $B_T = \frac{X_T}{t_N}$ at $t_N = Nh$. Noting that $x_N$ is used to approximate $X_{t_N}$ in terms of numerical method (3.1), we define discrete mean velocity as

$$B_N = \frac{x_N}{Nh}, \quad N = 1, 2, \ldots \quad (5.1)$$

In what follows, we study the LDP for $B_N$ of method (3.1) and its asymptotical preservation for LDP of $B_T$. Similar to the arguments on $A_N$, we introduce the modified rate function to characterize how the LDP for $B_N$ approximates the LDP for $B_T$.

**Definition 5.1.** If the discrete mean velocity $\{B_N\}_{N \geq 1}$ of method (3.1) satisfies an LDP with the rate function $J^h(\cdot)$, then we call $J^h_{\text{mod}}(\cdot) = \frac{J^h(\cdot)}{h}$ the modified rate function of $J^h(\cdot)$. Furthermore, method (3.1) is said to asymptotically preserve the LDP of $B_T$ if

$$\lim_{h \to 0} J^h_{\text{mod}}(y) = J(y), \quad \forall \ y \in \mathbb{R},$$

where $J(\cdot)$ is the rate function of the LDP for $B_T$. In particular, method (3.1) is said to exactly preserve the LDP of $B_T$ if for all sufficiently small step-size $h$, $J^h_{\text{mod}}(\cdot) = J(\cdot)$.

We still assume that (A1) holds. In this case, the equality (3.5) holds. Then, we have

$$x_N = (a_{11} \hat{\alpha}_{N-1} + \hat{\beta}_{N-1})x_0 + a_{12} \hat{\alpha}_{N-1}y_0 + \alpha \sum_{n=0}^{N-1} [b_1 \hat{\alpha}_{N-1-n} + (a_{12}b_2 - a_{22}b_1)\hat{\alpha}_{N-2-n}] \Delta W_n$$

with

$$\hat{\alpha}_n = \frac{(\det(A))^{n/2} \sin((n + 1)\theta)}{\sin(\theta)}, \quad \hat{\beta}_n = -\frac{(\det(A))^{n+1} \sin(n\theta)}{\sin(\theta)}.$$  

According to (5.2), $x_N$ is Gaussian whose expectation is

$$E(x_N) = (a_{11} \hat{\alpha}_{N-1} + \hat{\beta}_{N-1})x_0 + a_{12} \hat{\alpha}_{N-1}y_0$$

$$= \left(a_{11} (\det(A))^{\frac{N-1}{2}} \frac{\sin(N\theta)}{\sin(\theta)} - (\det(A))^{\frac{N}{2}} \frac{\sin((N-1)\theta)}{\sin(\theta)}\right)x_0$$

$$+ a_{12} (\det(A))^{\frac{N-1}{2}} \frac{\sin(N\theta)}{\sin(\theta)}y_0.$$

If $0 < \det(A) \leq 1$, then $|E(x_N)| \leq K(\theta)$ which leads to

$$\lim_{N \to \infty} \frac{E(x_N)}{N} = 0. \quad (5.3)$$

From (5.2) and the fact $\hat{\alpha}_{-1} = 0$, we get

$$\text{Var}(x_N) = \alpha^2 h \sum_{n=0}^{N-1} [b_1 \hat{\alpha}_{N-1-n} + (a_{12}b_2 - a_{22}b_1)\hat{\alpha}_{N-2-n}]^2$$

$$= \alpha^2 h \left[b_1^2 \sum_{n=0}^{N-1} \hat{\alpha}_{N-1-n} + (a_{12}b_2 - a_{22}b_1)^2 \sum_{n=0}^{N-1} \hat{\alpha}_{N-2-n}^2\right]$$
symplectic methods, so we assume that (LDP of 5.1. have non-symplectic ones (0 < 2). As is analogous to the treatment of CHUCHU CHEN, JIALIN HONG, DIANCONG JIN, AND LIYING SUN

\[ + 2b_1(a_{12}b_2 - a_{22}b_1) \sum_{n=0}^{N-1} \hat{a}_{n-1-n} \hat{a}_{N-n-1} \]

\[ = \alpha^2 h \left[ \sum_{n=0}^{N-1} \hat{a}_n^2 + (a_{12}b_2 - a_{22}b_1)^2 \sum_{n=1}^{N-2} \hat{a}_n^2 + 2b_1(a_{12}b_2 - a_{22}b_1) \sum_{n=0}^{N-1} \hat{a}_n \hat{a}_{n-1} \right] \]

\[ = \alpha^2 h \left[ (b_1^2 + (a_{12}b_2 - a_{22}b_1)^2) \sum_{n=0}^{N-2} \hat{a}_n^2 + b_1^2 \hat{a}_{N-1} + 2b_1(a_{12}b_2 - a_{22}b_1) \sum_{n=1}^{N-1} \hat{a}_n \hat{a}_{n-1} \right]. \] (5.4)

Further, we have

\[ \sum_{n=0}^{N-2} \hat{a}_n^2 = \sum_{n=0}^{N-2} (\det(A))^n \sin^2((n + 1)\theta) \] (5.5)

\[ 2 \sum_{n=1}^{N-1} \hat{a}_n \hat{a}_{n-1} = 2 \sum_{n=1}^{N-1} (\det(A))^{2n-1} \sin((n + 1)\theta) \sin(n\theta) \]

\[ = \frac{1}{\sin^2(\theta)} \sum_{n=1}^{N-1} (\det(A))^{2n-1} (\cos(\theta) - \cos((2n + 1)\theta)). \] (5.6)

As is analogous to the treatment of $A_N$, we deal with symplectic methods (det($A$) = 1) and non-symplectic ones (0 < det($A$) < 1), respectively.

5.1. LDP of $B_N$ for symplectic methods. In this part, we study the LDP for $B_N$ of symplectic methods, so we assume that (A2) holds. Based on det($A$) = 1, (5.5) and (5.6), we have

\[ \sum_{n=0}^{N-2} \hat{a}_n^2 = \frac{1}{\sin^2(\theta)} \sum_{n=1}^{N-1} \sin^2(n\theta) \]

\[ = \frac{1}{\sin^2(\theta)} \sum_{n=1}^{N-1} \left( \frac{1}{2} - \frac{1}{2} \cos(n\theta) \right) \]

\[ = \frac{1}{\sin^2(\theta)} \left( \frac{N-1}{2} - \frac{\sin((2N-1)\theta) - \sin(\theta)}{4 \sin(\theta)} \right), \] (5.7)

and

\[ 2 \sum_{n=1}^{N-1} \hat{a}_n \hat{a}_{n-1} = \frac{1}{\sin^2(\theta)} \sum_{n=1}^{N-1} (\cos(\theta) - \cos((2n + 1)\theta)) \]

\[ = \frac{1}{\sin^2(\theta)} \left[ (N-1) \cos(\theta) - \frac{\sin(2N\theta) - \sin(2\theta)}{2 \sin(\theta)} \right]. \] (5.8)

Substituting (5.7) and (5.8) into (5.4) yields

\[ \text{Var}(x_N) = \alpha^2 h \left[ \frac{b_1^2 + (a_{12}b_2 - a_{22}b_1)^2 + 2b_1(a_{12}b_2 - a_{22}b_1) \cos(\theta)}{2 \sin^2(\theta)} (N-1) \right. \]

\[ - \left. \frac{b_1^2 + (a_{12}b_2 - a_{22}b_1)^2 \sin((2N-1)\theta) - \sin(\theta)}{4 \sin^3(\theta)} + \frac{b_2^2 \sin^2(N\theta)}{\sin^2(\theta)} \right]. \]
\[
- \frac{b_1(a_{12}b_2 - a_{22}b_1)(\sin(2N\theta) - \sin(\theta))}{2 \sin^2(\theta)}.
\] (5.9)

Using (5.3), (5.9) and (3.4), we have

\[
\Lambda^h(\lambda) = \lim_{N \to \infty} \frac{1}{N} \log E e^{\lambda N B_N} = \lim_{N \to \infty} \frac{1}{N} \left[ \frac{\lambda}{h} E(x_N) + \frac{\lambda^2}{2h^2} \text{Var}(x_N) \right] = \frac{\lambda^2}{2h^2} \lim_{N \to \infty} \frac{1}{N} \text{Var}(x_N)
\]

\[
= \frac{\lambda^2}{2h^2} \alpha^2 h^2 + (a_{12}b_2 - a_{22}b_1)^2 + b_1(a_{12}b_2 - a_{22}b_1) \cos(\theta)
\]

\[
= \frac{\lambda^2}{4h} \left[ (b_1 + a_{12}b_2 - a_{22}b_1)^2 - b_1(a_{12}b_2 - a_{22}b_1)(1 - \cos(\theta)) \right]
\]

\[
= \frac{\lambda^2}{4h} \frac{(b_1 + a_{12}b_2 - a_{22}b_1)^2 - b_1(a_{12}b_2 - a_{22}b_1)(2 - \text{tr}(A))}{(4 - (\text{tr}(A))^2)h},
\] (5.10)

where we have used (3.4) with \(\det(A) = 1\) in the last equality.

Further, we assume

\[(A6) \quad (b_1 + a_{12}b_2 - a_{22}b_1)^2 - b_1(a_{12}b_2 - a_{22}b_1)(2 - \text{tr}(A)) > 0.\]

Under the assumption (A6), \(\Lambda^h(\cdot)\) is essentially smooth and lower semicontinuous. Then, using Theorem [2.4] we conclude that \(B_N\) satisfies an LDP with the good rate function

\[
J^h(y) = \sup_{\lambda \in \mathbb{R}} \{\lambda y - \Lambda^h(\lambda)\}
\]

\[
= h \left[ 4 - (\text{tr}(A))^2 \right] y^2
\]

\[
= \frac{4\alpha^2 [(b_1 + a_{12}b_2 - a_{22}b_1)^2 - b_1(a_{12}b_2 - a_{22}b_1)(2 - \text{tr}(A))] y^2}{(4 - (\text{tr}(A))^2)h}.
\] (5.11)

By Definition 5.1, the modified rate function is

\[
J_{\text{mod}}^h(y) = \frac{(4 - (\text{tr}(A))^2) y^2}{4\alpha^2 [(b_1 + a_{12}b_2 - a_{22}b_1)^2 - b_1(a_{12}b_2 - a_{22}b_1)(2 - \text{tr}(A))]},
\] (5.12)

In what follows, we study the asymptotical convergence of \(J_{\text{mod}}^h(\cdot)\) as step-size \(h\) tends to 0 based on mean-square convergence condition. To this end, let condition (B) hold. Then it follows from Lemma [4.3] that

\[(2 - \text{tr}(A)) \sim h^2, \quad (b_1 + a_{12}b_2 - a_{22}b_1) \sim h.\]

In addition, (B) implies that \(b_1(a_{12}b_2 - a_{22}b_1) \to 0\) as \(h \to 0\). In this way, we have

\[
\lim_{h \to 0} J_{\text{mod}}^h(y) = \lim_{h \to 0} \frac{2 + \text{tr}(A)}{4\alpha^2 \left[ \frac{(b_1 + a_{12}b_2 - a_{22}b_1)^2}{2 - \text{tr}(A)} - b_1(a_{12}b_2 - a_{22}b_1) \right] y^2}
\]

\[
= \frac{2 + \lim_{h \to 0} \text{tr}(A)}{4\alpha^2 [1 - 0]} y^2
\]

\[
= \frac{4}{4\alpha^2 [1 - 0]} y^2
\]
According to the above results, we write them into the following theorem.

**Theorem 5.2.** For the numerical method (3.1) approximating the stochastic oscillator (1.1), if the assumptions (A1), (A2), and (A6) hold, then we have

1. The method (3.1) is symplectic;
2. The discrete mean velocity \( \{B_N\}_{N \geq 1} \) of method (3.1) satisfies an LDP with the good rate function
   \[
   J^h(y) = \frac{h}{4\alpha^2} \left[ 4 - (\text{tr}(A))^2 \right] \frac{y^2}{(b_1 + a_1b_2 - a_2b_1)^2 - b_1(a_1b_2 - a_2b_1)(2 - \text{tr}(A))};
   \]  
   (5.13)

3. Moreover, if assumption (B) holds, then method (3.1) asymptotically preserves the LDP of \( B_T \), i.e., the modified rate function \( J^h_{\text{mod}}(y) = J^h(y)/h \) satisfies:
   \[
   \lim_{h \to 0} J^h_{\text{mod}}(y) = J(y), \quad \forall \ y \in \mathbb{R},
   \]
   where \( J(\cdot) \) is the rate function of the LDP for \( B_T \).

5.2. **LDP of \( B_N \) for non-symplectic methods.** In this part, we consider the discrete mean velocity \( B_N \) of general non-symplectic methods. We study whether the LDP holds for \( B_N \). Let conditions (A1) and (A4) hold. Then, (5.3) and (5.6) satisfy, respectively,

\[
\sum_{n=0}^{N-2} \hat{a}_n^2 \leq K(\theta) \sum_{n=0}^{N-2} (\det(A))^n \leq K(\theta), \quad \bigg| \sum_{n=1}^{N-1} \hat{a}_n \hat{a}_{n-1} \bigg| \leq K(\theta) \sum_{n=1}^{N-1} (\det(A))^{2n-1} \leq K(\theta).
\]

Additionally, it holds that \( |\hat{a}_{N-1}| = |(\det(A))^{N-1} \sin^2(N\theta)| \leq K(\theta) \). Thus, (5.4) satisfies

\[
|\text{Var}(x_N)| \leq \alpha^2 h K(\theta).
\]  
(5.14)

It follows from (5.3) and (5.14) that the logarithmic moment generating function is

\[
\tilde{\Lambda}^h(\lambda) = \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} e^{\lambda N B_N} = \lim_{N \to \infty} \left[ \frac{\lambda}{N} \mathbb{E} (x_N) + \frac{\lambda^2}{2h^2} \text{Var}(x_N) \right] = 0.
\]  
(5.15)

We note that \( \tilde{\Lambda}^h(\cdot) \) is not essentially smooth, for which Theorem 2.4 is not valid. In our case, we can directly prove that the LDP holds for \( B_N \) of non-symplectic methods by the definition of LDP. We claim that \( B_N \) of non-symplectic methods satisfy the LDP with the good rate function:

\[
\tilde{J}^h(y) = \begin{cases} 
0, & y = 0, \\
+\infty, & y \neq 0.
\end{cases}
\]  
(5.16)

We divide the proof of this claim into three steps.

**Step 1:** We show the limit behavior of \( P(B_N \geq x_0) \) and \( P(B_N \leq x_0) \) for non-symplectic methods.

We need to use the following fact: if \( X \sim N(\mu, \sigma^2) \), then it follows from [15] Lemma 22.2 that, for any \( x > \mu \),

\[
P(X \geq x) = P \left( \frac{X - \mu}{\sigma} \geq \frac{x - \mu}{\sigma} \right) \leq \frac{1}{\sqrt{2\pi}} \frac{\sigma}{x - \mu} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.
\]  
(5.17)
In addition, for any \( x < \mu \),

\[
\mathbb{P}(X \leq x) = \mathbb{P}\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) = \mathbb{P}\left(\frac{X - \mu}{\sigma} \geq -\frac{x - \mu}{\sigma}\right) \leq \frac{1}{\sqrt{2\pi}} \frac{\sigma}{x - \mu} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{5.18}
\]

Since \( B_N = \frac{x_N}{N} \), we have \( B_N \sim N\left(\frac{E(x_N)}{N}, \frac{\text{Var}(x_N)}{N^2}\right) \) with \( |E(x_N)| \leq K(\theta) \) and \( |\text{Var}(x_N)| \leq K(\theta) \). Noting that \( \lim\limits_{N \to \infty} E(B_N) = 0 \), one has that for the given \( x_0 > 0 \), there exists some \( N_0 \) such that \( E(B_N) < x_0 \) for every \( N > N_0 \). Accordingly, it follows from (5.17) that

\[
\mathbb{P}(B_N \geq x_0) \leq \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\text{Var}(x_N)}}{Nh_x_0 - E(x_N)} \exp\left\{-\frac{(Nh_x_0 - E(x_N))^2}{2\text{Var}(x_N)}\right\}, \quad \forall \ N > N_0.
\]

In this way, for every \( x_0 > 0 \),

\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{P}(B_N \geq x_0) = -\infty. \tag{5.19}
\]

Analogously, using (5.18), one has that for the given \( x_0 < 0 \),

\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{P}(B_N \leq x_0) = -\infty. \tag{5.20}
\]

Step 2: We prove the upper bound LDP (LDP2): For every closed \( C \subseteq \mathbb{R} \),

\[
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}(B_N \in C) \leq -\inf \bar{h}(C). \tag{5.21}
\]

If \( 0 \in C \), then it follows from (5.16) that \( \inf \bar{h}(C) = 0 \). Since \( \mathbb{P}(B_N \in C) \leq 1 \), (5.21) naturally holds.

If \( 0 \notin C \). Define \( x_+ = \inf(C \cap (0, +\infty)) \) and \( x_- = \sup(C \cap (-\infty, 0)) \). Then, \( \mathbb{P}(B_N \in C) \leq \mathbb{P}(B_N \geq x_+) + \mathbb{P}(B_N \leq x_-) \). In order to prove (5.21), we need to use the following lemma (see [15] Lemma 23.9]).

Lemma 5.3. Let \( N \in \mathbb{N} \) and let \( a_i^\epsilon, i = 1, \ldots, N, \epsilon > 0 \), be nonnegative numbers. Then

\[
\lim_{\epsilon \to 0} \epsilon \log \sum_{i=1}^{N} a_i^\epsilon = \max_{i=1, \ldots, N} \limsup_{\epsilon \to 0} \epsilon \log(a_i^\epsilon).
\]

Using (5.19), (5.20) and Lemma 5.3 yields

\[
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}(B_N \in C) \leq \max\left\{ \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}(B_N \geq x_+), \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}(B_N \leq x_-) \right\} = -\infty.
\]

Noting that \( 0 \notin C \), one obtains \( \inf \bar{h}(C) = +\infty \). Thus, (5.21) also holds for this case.

Step 3: We prove the lower bound LDP (LDP1): For every open \( U \subseteq \mathbb{R} \),

\[
\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}(B_N \in U) \geq -\inf \bar{h}(U). \tag{5.22}
\]

If \( 0 \notin U \), then \( \inf \bar{h}(U) = +\infty \). Since \( \mathbb{P}(B_N \in C) \geq 0 \), (5.22) naturally holds.

If \( 0 \in U \), then there exists some \( \delta > 0 \) such that \( (-\delta, \delta) \subseteq U \). Accordingly,

\[
\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}(B_N \in U) \geq \liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}(|B_N| < \delta). \tag{5.23}
\]
It follows from (5.19) that for arbitrary given $M \in (-\infty, 0)$, there exists some $N_1$ such that for every $N > N_1$, $\frac{1}{N} \log \mathbb{P}(B_N \geq \delta) < M$. Thus,

$$\mathbb{P}(B_N \geq \delta) \leq e^{NM}, \quad \forall \ N > N_1,$$

which leads to $\lim_{N \to \infty} \mathbb{P}(B_N \geq \delta) = 0$. Similarly, utilizing (5.20) gives $\lim_{N \to \infty} \mathbb{P}(B_N \leq -\delta) = 0$. Hence, $\lim_{N \to \infty} \mathbb{P}(|B_N| < \delta) = 1$, which implies

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{P}(|B_N| < \delta) = 0. \quad (5.24)$$

Combining (5.23) and (5.24), we have

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}(B_N \in U) \geq 0.$$

Further, since $0 \in U$, $\inf \mathcal{J}^h(U) = 0$. Hence, we prove (5.22).

Combining the above discussion, we deduce that $B_N$ of non-symplectic methods satisfy the LDP with the good rate function $\mathcal{J}^h$ given by (5.10) and the modified rate function $\mathcal{J}^h_{\text{mod}} = \mathcal{J}^h/h = \mathcal{J}^h$. Finally, we get the following theorem.

**Theorem 5.4.** For the numerical method (3.1) approximating the stochastic oscillator (1.1), if the assumptions (A1) and (A4) hold, then we have

1. The method (3.1) is non-symplectic;
2. The discrete mean velocity $\{B_N\}_{N \geq 1}$ of method (3.1) satisfies an LDP with the good rate function

   $$\mathcal{J}^h(y) = \begin{cases} 
   0, & y = 0, \\
   +\infty, & y \neq 0;
   \end{cases}$$

3. Method (3.1) does not asymptotically preserve the LDP of $B_T$, i.e., for $y \neq 0$,

   $$\lim_{h \to 0} \mathcal{J}^h_{\text{mod}}(y) \neq J(y),$$

where $\mathcal{J}^h_{\text{mod}}(y) = \mathcal{J}^h(y)/h$, and $J(y) = \frac{y^2}{\alpha^2}$ is the rate function of LDP for $B_T$. 

6. **Concrete numerical methods**

In this section, we show and compare the LDPs of some concrete numerical methods to verify the theoretical results obtained in previous sections. For symplectic methods, we consider symplectic $\beta$-method, Exponential method, INT method and OPT method. For non-symplectic ones, we examine $\theta$-method, PC (PEM-MR) method and PC (EM-BEM) method. All of the methods can be found in [22], except symplectic $\beta$-method (see (3.6) in [19]). Furthermore, we construct some symplectic methods which preserve the LDP for $A_T$ or $B_T$ exactly.

6.1. **Symplectic methods.**

- Symplectic $\beta$-method ($\beta \in [0, 1]$):

  $$A^\beta = \frac{1}{1 + \beta(1 - \beta)h^2} \begin{pmatrix} 1 - (1 - \beta)^2h^2 & h \\ -h & 1 - \beta^2h^2 \end{pmatrix}, \quad b^\beta = \frac{1}{1 + \beta(1 - \beta)h^2} \begin{pmatrix} (1 - \beta)h \\ 1 \end{pmatrix}.$$
The straightforward calculation leads to
\[
\det(A^\beta) = 1, \quad \text{tr}(A^\beta) = \frac{2 - (2\beta^2 - 2\beta + 1)h^2}{1 + \beta(1 - \beta)h^2}, \quad (6.1)
\]
\[
a_{12}b_2 - a_{22}b_1 = \frac{\beta h}{1 + \beta(1 - \beta)h^2}, \quad b_1 + a_{12}b_2 - a_{22}b_1 = \frac{h}{1 + \beta(1 - \beta)h^2}. \quad (6.2)
\]

It can be verified that condition (B) holds, and if \( h \in (0, 2) \), then for every \( \beta \in [0, 1] \), conditions (A1), (A2) and (A3) hold. Substituting (6.1) and (6.2) into (4.12), we have
\[
I^h(y) = \frac{hy^2}{3\alpha^2} \left[ \frac{3}{2} - \frac{3}{2 - 6(2\beta - 1)^2h^2} \right],
\]
which is the good rate function of LDP for \( A_N \) of symplectic \( \beta \)-method by Theorem 4.4. Furthermore, we get the modified rate function
\[
I^h_{mod}(y) = I^h(y) / h = \frac{y^2}{3\alpha^2} \left[ \frac{3}{2} - \frac{3}{6 - (2\beta - 1)^2h^2} \right].
\]

Further, we have that \( \lim_{h \to 0} I^h_{mod}(y) = I(y) = \frac{y^2}{3\alpha^2} \), for every \( y \in \mathbb{R} \), which is consistent with the third conclusion of Theorem 4.4. Moreover, for every \( h > 0 \), the modified rate function of the mean position for the midpoint method with \( \beta = \frac{1}{2} \) is same as that for the exact solution. These indicate that midpoint method exactly preserves the LDP for \( A_T \). In case of \( \beta \neq \frac{1}{2} \), \( I^h_{mod}(y) < I(y) \) provided \( y \neq 0 \). That is, as the time \( T \) and \( t_N \) tend to infinity simultaneously, the exponential decay speed of \( P(A_N \in [a, a + da]) \) provided \( a \neq 0 \).

On the other hand, if \( h \in (0, 2) \) and \( \beta \in (0, 1) \), conditions (A1), (A2) and (A6) hold. By Theorem 5.2 \( B_N \) of symplectic \( \beta \)-method satisfies an LDP with the good rate function
\[
J^h(y) = \frac{h[4 - (2\beta - 1)^2h^2][1 + \beta(1 - \beta)h^2]y^2}{4a^2}. \quad (6.3)
\]
This means that the modified rate function \( J^h_{mod}(\cdot) \) satisfies \( \lim_{h \to 0} J^h_{mod}(y) = \frac{y^2}{\alpha^2} = J(y) \), which verifies the third conclusion of Theorem 5.2.

- Exponential method (EX):
  \[
  A^{EX} = \begin{pmatrix}
  \cos(h) & \sin(h) \\
  -\sin(h) & \cos(h)
  \end{pmatrix}, \quad b^{EX} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
  \]
  For this method, it holds that
  \[
  \det(A^{EX}) = 1, \quad \text{tr}(A^{EX}) = 2\cos(h), \quad a_{12}b_2 - a_{22}b_1 = \sin(h), \quad b_1 + a_{12}b_2 - a_{22}b_1 = \sin(h).
  \]
  If \( h \in (0, \pi) \), then the conditions (A1), (A2) and (A3) hold. Then, we obtain that \( A_N \) satisfies an LDP with the modified rate function
  \[
  I^h_{mod}(y) = \frac{2y^2}{\alpha^2} \frac{1 - \cos(h)}{h^2(2 + \cos(h))}.
  \]
  Hence, we have \( \lim_{h \to 0} I^h_{mod}(y) = \frac{y^2}{3\alpha^2} = I(y) \).

  For the sake of simplicity, we denote \( f(h) = \frac{1-\cos(h)}{h^2(2+\cos(h))} \) such that \( I^h_{mod}(y) = \frac{2y^2}{\alpha^2} f(h) \).
  Denote \( g(h) = 3h \sin(h) + 2\cos(h) + 2\cos^2(h) - 4 \). Then
  \[
  f'(h) = \frac{g(h)}{h^2(2+\cos(h))^2}, \quad g'(h) = \sin(h) + 3h \cos(h) - 4\cos(h) \sin(h) \quad \text{and} \quad g''(h) = 4\cos(h) - 4\cos^2(h) + 4\sin(h)(\sin(h) - \frac{3h}{2})
  \]
  If \( h \in (0, \pi/6) \), then \( \sin(h) - \frac{3h}{2} > 0 \). In this way, \( g'(h) > g'(0) = 0 \), which leads to \( g(h) > g(0) = 0 \) for \( h \in (0, \pi/6) \), and then \( f(h) > f(0) := \lim_{h \to 0} f(h) = \frac{1}{6} \). Finally, we have
  \[
  I^h_{mod}(y) > I(y) \quad \text{provided} \quad h \in (0, \pi/6) \quad \text{and} \quad y \neq 0.
  \]
According to the discussions above, if \( h \in (0, \pi/6) \), then the mean position \( A_N \) of exponential method satisfies an LDP, which asymptotically preserves the LDP for \( A_T \). In addition, as the time \( T \) and \( t_N \) tend to infinity simultaneously, the exponential decay speed of \( \Pi(A_N \in [a, a + da]) \) is faster than that of \( \Pi(A_T \in [a, a + da]) \) provided \( a \neq 0 \).

Analogously, we have that conditions (A1), (A2) and (A6) hold for \( h \in (0, \pi) \). Hence, for \( h \in (0, \pi) \), \( B_N \) of exponential method satisfies an LDP with the modified rate function \( J_{mod}^h(y) = \frac{y^2}{\alpha^2} = J(y) \). In this way, exponential method exactly preserves the LDP for \( B_T \).

- Integral method (INT):

\[
A^{INT} = \begin{pmatrix}
\cos(h) & \sin(h) \\
-\sin(h) & \cos(h)
\end{pmatrix}, \quad b^{INT} = \begin{pmatrix}
\sin(h) \\
\cos(h)
\end{pmatrix}.
\]

For this method, \( \det(A^{INT}) = 1 \), \( \text{tr}(A^{INT}) = 2 \cos(h) \), \( a_{12}b_2 - a_{22}b_1 = 0 \) and \( b_1 + a_{12}b_2 - a_{22}b_1 = \sin(h) \). It is shown that its modified rate functions of \( A_N \) and \( B_N \) are \( I_{mod}^h(y) = \frac{2y^2}{\alpha^2} \), and \( J_{mod}^h(y) = \frac{y^2}{\alpha^2} = J(y) \), respectively. This case is exactly the same as that of exponential method.

- Optimal method (OPT):

\[
A^{INT} = \begin{pmatrix}
\cos(h) & \sin(h) \\
-\sin(h) & \cos(h)
\end{pmatrix}, \quad b^{INT} = \frac{1}{h} \begin{pmatrix}
2 \sin^2 \left(\frac{h}{y}\right) \\
\sin(h)
\end{pmatrix}.
\]

Based on the above two formulas, one has
\[
\det(A^{INT}) = 1, \quad \text{tr}(A^{INT}) = 2 \cos(h), \quad a_{12}b_2 - a_{22}b_1 = b_1 = \frac{1 - \cos(h)}{h}.
\]

If \( h \in (0, \pi) \), then assumptions (A1), (A2) and (A3) hold such that \( A_N \) of optimal method satisfies an LDP with the modified rate function \( I_{mod}^h(y) = \frac{y^2}{\alpha^2} = I(y) \). Thus, we conclude that the LDP for mean position \( A_N \) of optimal method exactly preserves the LDP for \( A_T \).

The assumptions (A1), (A2) and (A6) hold provided that \( h \in (0, \pi) \). Thus, for \( h \in (0, \pi) \), \( B_N \) of optimal method satisfies an LDP with the modified rate function \( J_{mod}^h(y) = \frac{y^2}{\alpha^2} \). Further, we have that \( \lim_{h \to 0} J_{mod}^h(y) = \frac{y^2}{\alpha^2} = J(y) \) and \( J_{mod}^h(y) > J(y) \). Hence, optimal method asymptotically preserves the LDP for \( B_T \). When the time \( T \) and \( t_N \) tend to infinity simultaneously, the exponential decay speed of \( \Pi(B_N \in [a, a + da]) \) is faster than that of \( \Pi(B_T \in [a, a + da]) \) provided \( a \neq 0 \).

6.2. Non-symplectic methods.

- stochastic \( \theta \)-method (\( \theta \in [0, 1/2) \cup (1/2, 1] \)):

\[
A^{\theta} = \frac{1}{1 + \theta^2 h^2} \begin{pmatrix}
1 - (1 - \theta)h^2 & h \\
h & 1 - (1 - \theta)h^2
\end{pmatrix}, \quad b^{\theta} = \frac{1}{1 + \theta^2 h^2} \begin{pmatrix}
\theta h \\
1
\end{pmatrix}.
\]

For this method, we have
\[
\det(A^{\theta}) = \frac{1 + (1 - \theta)^2 h^2}{1 + \theta^2 h^2}, \quad 1 - \text{tr}(A^{\theta}) + \det(A^{\theta}) = \frac{h^2}{1 + \theta^2 h^2}, \quad b_1 + a_{12}b_2 - a_{22}b_1 = \frac{h}{1 + \theta^2 h^2}.
\]

Notice that \( 0 < \det(A^{\theta}) < 1 \) is equivalent to \( \theta \in (1/2, 1] \). One can show that, with \( \theta \in (1/2, 1] \), (A1), (A4) and (A5) hold for every \( h > 0 \). Hence, for every \( \theta \in (1/2, 1] \) and \( h > 0 \), the mean position \( A_N \) satisfies an LDP with the modified rate function \( J_{mod}^h(y) = \frac{y^2}{2\alpha^2} \), which verifies
the third conclusion of Theorem 4.5.

- PC (PEM-MR):
  \[
  A^1 = \begin{pmatrix}
  1 - h^2/2 & h(1 - h^2/2) \\
  -h & 1 - h^2/2
  \end{pmatrix}, \quad b^1 = \begin{pmatrix}
  h/2 \\
  1
  \end{pmatrix}.
  \]

One has that \(1 - \text{tr}(A^1) + \det(A^1) = h^2 - \frac{h^4}{4}\) and \(b_1 + a_{12}b_2 - a_{22}b_1 = h - \frac{h^3}{4}\). We obtain that (A1), (A4) and (A5) hold, provided \(h \in (0, \sqrt{2})\). Thus, by Theorem 4.5, \(A_N\) of this method satisfies an LDP with the modified rate function \(\tilde{J}^h_{\text{mod}}(y) = \frac{y^2}{2\sigma^2}\).

- PC (EM-BEM):
  \[
  A^2 = \begin{pmatrix}
  1 - h^2/2 & h \\
  -h & 1 - h^2
  \end{pmatrix}, \quad b^2 = \begin{pmatrix}
  h/2 \\
  1
  \end{pmatrix},
  \]
  which means that \(1 - \text{tr}(A^2) + \det(A^2) = h^2 + h^4\) and \(b_1 + a_{12}b_2 - a_{22}b_1 = h + h^3\). In this case, (A1), (A4) and (A5) hold, provided \(h \in (0, 1)\). Thus, by Theorem 4.5, \(A_N\) of this method satisfies an LDP with the modified rate function \(\tilde{J}^h_{\text{mod}}(y) = \frac{y^2}{2\sigma^2}\).

We observe that all methods shown in Sections 6.1 and 6.2 satisfy the condition (B). When the step-size \(h\) is sufficiently small, the symplectic methods in Section 6.1 satisfy the conditions (A1) (A2), (A3) and (A6), and the non-symplectic methods in Section 6.2 satisfy the conditions (A1), (A4) and (A5). By studying these methods, we verify the theoretical results in Theorems 4.3, 4.5 and 5.2. It is shown that symplectic methods are superior to non-symplectic methods in terms of preservation of the LDP for both \(A_T\) and \(B_T\).

### 6.3. Construction for methods exactly preserving the LDP for \(A_T\) or \(B_T\).

In this part, we construct several symplectic methods exactly preserving the LDP for \(A_T\) (resp. \(B_T\)) based on Theorem 4.3 (resp. Theorem 5.2).

- Methods exactly preserving the LDP for \(A_T\):

  Motivated by assumption (B), we consider the method (6.1) with
  \[
  A = \begin{pmatrix}
  1 + c_{11}h^2/2 & h + c_{12}h^2 \\
  -h + c_{21}h^2/2 & 1 + c_{22}h^2
  \end{pmatrix}, \quad b = \begin{pmatrix}
  D_1h \\
  1 + D_2h
  \end{pmatrix}
  \]
  (6.3)
  with real constants \(c_{ij}\) and \(D_i, i, j = 1, 2\), independent of \(h\). In order to make the condition \(\det(A) = 1\) hold, we have
  \[
  (1 + c_{11}h^2)(1 + c_{22}h^2) = 1 + (h + c_{12}h^2)(-h + c_{21}h^2), \quad \forall \quad h > 0.
  \]

  Comparing the coefficients and we obtain
  \[
  c_{11} + c_{22} = -1, \quad c_{11}c_{22} = c_{12}c_{21}, \quad c_{12} = c_{21}.
  \]
  Let \(c_{12} = c_{21} = \sigma\), then \(c_{11}\) and \(c_{22}\) are the roots of equation \(x^2 + x + \sigma^2 = 0\). To assure that \(c_{11}\) and \(c_{22}\) are real numbers, we assume \(\sigma \in [-1/2, 1/2]\). Solving the equation \(x^2 + x + \sigma^2 = 0\) yields
  \[
  c_{11} = \frac{-1 - \sqrt{1 - 4\sigma^2}}{2}, \quad c_{22} = \frac{-1 + \sqrt{1 - 4\sigma^2}}{2}
  \]
  or
  \[
  c_{11} = \frac{-1 + \sqrt{1 - 4\sigma^2}}{2}, \quad c_{22} = \frac{-1 - \sqrt{1 - 4\sigma^2}}{2}.
  \]
where the case $c_{11} = c_{22} = -1/2$, $\sigma = \pm 1/2$ is included in the above two cases. In order to acquire the methods exactly preserving the LDP for $A_T$, a necessary condition is that the modified rate function (4.3) satisfies

$$ I^h_{mod}(y) = \frac{(2 + \text{tr}(A))(2 - \text{tr}(A))^2 y^2}{2\alpha^2 h^2 [(b_1 + a_{12}b_2 - a_{22}b_1)^2(4 + \text{tr}(A)) - 2b_1(a_{12}b_2 - a_{22}b_1)(2 - \text{tr}(A))]} = \frac{y^2}{3\alpha^2}. $$

(6.4)

According to (6.3), it is known that

$$ \text{tr}(A) = 2 - h^2, \ a_{12}b_2 - a_{22}b_1 = h \left[(1 - D_1) + (D_2 + \sigma)h + (D_2\sigma - D_1c_{22})h^2]\right]. $$

Substituting the above equation into (6.4), we have

$$ 6 - \frac{3h^2}{2} = (1 + (D_2 + \sigma)h + (D_2\sigma - D_1c_{22})h^2)^2 (6 - h^2) $$

$$ - 2D_1h^2 [1 - D_1 + (D_2 + \sigma)h + (D_2\sigma - D_1c_{22})h^2]. $$

(6.5)

Comparing the coefficients of $h^6$ and $h^4$ in (6.5), we obtain

$$ D_2\sigma - D_1c_{22} = 0, \quad D_2 + \sigma = 0, $$

(6.6)

which leads to

$$ 6 - \frac{3h^2}{2} = 6 - h^2 - 2D_1h^2(1 - D_1). $$

The above formula implies that

$$ D_1 = 1/2. $$

(6.7)

Combining (6.6) and (6.7) produces

$$ 2\sigma^2 + c_{22} = 0. $$

(6.8)

Note that $c_{22} = \frac{-1 + \sqrt{1 - 4\sigma^2}}{2}$ or $c_{22} = \frac{-1 - \sqrt{1 - 4\sigma^2}}{2}$. Thus, we have

$$ \sigma = 0, \pm \frac{1}{2}, \ c_{22} = \frac{-1 + \sqrt{1 - 4\sigma^2}}{2}, \ c_{11} = \frac{-1 - \sqrt{1 - 4\sigma^2}}{2}. $$

Finally, we acquire the following three methods exactly preserving the LDP for $A_T$ and their coefficients are separately

$$ A^{[1]} = \begin{pmatrix} 1 - h^2 & h \\ -h & 1 \end{pmatrix}, \quad b^{[1]} = \begin{pmatrix} h/2 \\ 1 \end{pmatrix}; $$

(6.9)

$$ A^{[2]} = \begin{pmatrix} 1 - h^2/2 & h + h^2/2 \\ -h + h^2/2 & 1 - h^2/2 \end{pmatrix}, \quad b^{[2]} = \begin{pmatrix} h/2 \\ 1 - h/2 \end{pmatrix}; $$

(6.10)

$$ A^{[3]} = \begin{pmatrix} 1 - h^2/2 & h - h^2/2 \\ -h - h^2/2 & 1 - h^2/2 \end{pmatrix}, \quad b^{[3]} = \begin{pmatrix} h/2 \\ 1 + h/2 \end{pmatrix}. $$

(6.11)

Moreover, if $h \in (0, 2)$, methods based on (6.9), (6.10) and (6.11) satisfy the assumptions (A1), (A2) and (A3) and have the same modified rate function $I^h_{mod}(y) = \frac{y^2}{3\alpha^2} = I(y)$.

- Methods exactly preserving the LDP for $B_T$:

We still consider the method with coefficients satisfying (6.3). By the straightforward computation, we get the following methods exactly preserving the LDP for $B_T$, whose coefficients are

$$ A = \begin{pmatrix} 1 - \frac{1 + \sqrt{1 - 4\sigma^2}}{2}h^2 & h + \sigma h^2 \\ -h + \sigma h^2 & 1 - \frac{1 - \sqrt{1 - 4\sigma^2}}{2}h^2 \end{pmatrix}, \quad b = \begin{pmatrix} h/2 \\ 1 - \sigma h \end{pmatrix}, $$

(6.12)
with $\sigma = 0, \pm \frac{1}{2}$, or
\[
A = \begin{pmatrix}
1 - \frac{1 - \sqrt{1 - 4\sigma^2}}{2} h^2 & h + \sigma h^2 \\
-h + \sigma h^2 & 1 - \frac{1 + \sqrt{1 - 4\sigma^2}}{2} h^2
\end{pmatrix}, \quad b = \begin{pmatrix}
-h/2 \\
1 - \sigma h
\end{pmatrix},
\]
with $\sigma = 0, \pm \frac{1}{2}$. Finally, besides methods based on (6.9), (6.10) and (6.11), we obtain three more methods exactly preserving the LDP for $B_T$ with coefficients given by
\[
A[4] = \begin{pmatrix}
1 & h \\
-h & 1 - h^2
\end{pmatrix}, \quad b[4] = \begin{pmatrix}
-h/2 \\
1
\end{pmatrix};
\]
\[
A[5] = \begin{pmatrix}
1 - h^2/2 & h + h^2/2 \\
-h + h^2/2 & 1 - h^2/2
\end{pmatrix}, \quad b[5] = \begin{pmatrix}
-h/2 \\
1 - h/2
\end{pmatrix};
\]
\[
A[6] = \begin{pmatrix}
1 - h^2/2 & h - h^2/2 \\
-h - h^2/2 & 1 - h^2/2
\end{pmatrix}, \quad b[6] = \begin{pmatrix}
-h/2 \\
1 + h/2
\end{pmatrix}.
\]

In fact, it is verified that methods based on (6.9), (6.10), (6.11), (6.12), (6.13) and (6.14) satisfy the assumptions (A1), (A2) and (A6) for $h \in (0, 2)$ and have the same modified rate function $J_{h_{\text{mod}}}(y) = \frac{y^2}{2\sigma} = J(y)$.

**Remark 6.1.** Note that three symplectic methods constructed based on (6.12) (6.13) and (6.14) preserve exactly the LDP for $A_T$ and $B_T$ at the same time.

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