Unwinding strings in semi-flatland

Dieter Lüst$^{1,2}$, Erik Plauschinn$^1$, Valentí Vall Camell$^{1,2}$

$^1$ Arnold Sommerfeld Center for Theoretical Physics
Theresienstraße 37, 80333 München, Germany

$^2$ Max-Planck-Institut für Physik
Föhringer Ring 6, 80805 München, Germany

Abstract

We study the dynamics of strings with non-zero winding number around T-duality defects. We deduce that the physics near the core of such non-geometric objects involves winding modes that are not captured by the supergravity approximation, and we argue that such corrections are T-dual to the modes responsible for quantum corrections of semi-flat elliptic metrics. We furthermore construct a solution of double field theory that captures part of such near-core physics.
1 Introduction

It has been known for quite some time that supersymmetric compactifications of string theory on elliptic Calabi-Yau manifolds admit non-geometric modifications, where the elliptic fiber develops monodromies in the U-duality group. In this note we consider perturbative solutions of string theory for which the monodromy is contained in the T-duality group (as part of the U-duality group), and which are generically referred to as T-folds [1].

A way to construct such spaces is to use an adiabatic fibration of the higher-dimensional toroidal theory, and to let all toroidal moduli vary over a base [2–4]. For example, one can reduce string theory on a $\mathbb{T}^2$ and then fiber the complex structure and Kähler modulus, that we will denote $\tau$ and $\rho$, respectively, meromorphically over a $\mathbb{P}^1$ base. If one only varies $\tau$ the familiar K3 compactifications are recovered, while if also $\rho$ is varied there will be degenerations that generically induce non-geometric monodromies. In analogy to the geometric case [2, 5], we will refer to this situation as a semi-flat approximation.

A semi-flat metric that respects the $U(1)^2$ isometries of the torus fiber can be written as follows (we omit additional transversal directions)

$$ds^2 = e^\phi d\tau d\bar{\tau} + \frac{\rho_2}{\tau_2} \left| d\xi^1 + \tau d\xi^2 \right|^2,$$

where $\xi^{1,2}$ are coordinates on the torus and $z$ is a local coordinate on $\mathbb{P}^1$. Furthermore, we set $\tau = \tau_1 + i\tau_2$ and $\rho = \rho_1 + i\rho_2$, and $e^\phi$ is a warp factor. An example which does not admit a global geometric interpretation is the local solution around a degeneration of the $\rho$-fibration with monodromy $1/\rho \to -1/\rho + 1$. In this case, the semi-flat metric (1.1) is usually referred to as a $5^3_2$-brane [6, 7]. More generally, as in the geometric case, one can have monodromies filling-in the conjugacy classes of $SL(2, \mathbb{Z})_\rho$ [8].

Close to a degeneration of the $\tau$- and $\rho$-fibration this semi-flat approximation breaks down, and the exact local description of the degeneration has to be glued-in. However, the latter in general breaks (some of) the isometries of the fiber. As we will review shortly, in the geometric situation we have a good understanding of such a local description, while in the non-geometric case the situation is more delicate. In fact, we can rely on a dual description of the local non-geometric solution, but we will point out that in all known cases such duality is strictly valid only in the semi-flat approximation. Given that we lack a conformal field theory description of the degeneration [2] and that supergravity is most certainly not valid for such stringy backgrounds, it is important to understand the physics of these degenerations.

By adapting the arguments of [10] to the torus case, in this note we argue that such physics is dominated by winding modes, and that the exotic brane solutions will receive stringy corrections that can be related to the modes correcting the semi-flat ansatz near geometric degenerations. It has been argued in the literature that modes of this kind can be captured

---

1. One can also consider the T-duality monodromy $(\tau, \rho) \to (\rho, \tau)$, which has been studied in [9] and has been shown to be an essential ingredient for the heterotic string away from the stable degeneration limit.

2. Except particular cases such as asymmetric orbifold points. However, we will be interested in parabolic monodromies which do not admit a CFT description.
in a doubled formalism such as double field theory (DFT) [11] (for reviews see [12–14]). We investigate this by constructing non-trivial solutions of the DFT equations of motion that reduce to the “semi-flat” exotic brane solution away from the degeneration. However, we did not find any additional justification within the doubled formalism, that this solution captures the correct T-duality of the symmetry-breaking modes.

Finally, in heterotic string theory we have an additional tool besides T-duality to study the physics of the $\tau$- and $\rho$-fibrations, as described in [15,16]. If we only consider monodromies that do not mix the moduli, the fibrations can be described algebraically with two elliptic fibrations that encode the solutions for the varying moduli. In this form, one can use the explicit heterotic/F-theory duality map [17] to construct the dual K3-fibered Calabi-Yau manifold. Since the moduli are mapped to geometric moduli of the fiber K3 (which is itself elliptically fibered), the dual F-theory model is geometric and can be used to read-off the physics of the non-geometric heterotic background. However, even in the simple case of a NS5-brane, this map misses precisely the information about the position of the brane on the fiber torus, and thus it cannot be used to understand the near-core physics of the $\rho$-degenerations. It would be interesting to understand how these corrections are seen in the F-theory dual and to compare such dual description with the double field theory solutions.

This note is organized as follows. In section 2 we review the exact metric for a Kodaira $I_1$ degeneration and for a dual NS5-brane localized on an elliptic curve, and discuss the dynamics of unwinding strings in the semi-flat limit of elliptic fibrations. We use the same conventions as in (1.1), for which the complex structure and Kähler modulus of a two-torus are expressed in terms of the metric and Kalb-Ramond field on the $T^2$ as

$$\tau = \frac{g_{12}}{g_{22}} + i \sqrt{\det g} \frac{g_{22}}{g_{22}}, \quad \rho = B_{12} + i \sqrt{\det g}.$$  (2.1)

2.1 $I_1$ degeneration

In the geometric setting, the simplest semi-flat solution corresponds to the Kodaira type $I_1$ singularity, which is uniquely determined by the monodromy acting on the fiber torus when encircling the singularity. The monodromy is a Dehn twist of fixed chirality around the shrinking cycle, sending $\tau \to \tau + 1$. In this case, we know that the exact metric is that of a Taub-NUT space with one transverse compact direction. The exact metric breaks one of
the \( U(1) \times U(1) \) isometries of the semi-flat metric, and the modes that break such isometry are the ones that localize the shrinking cycle on the orthogonal cycle of the torus. We thus see that specifying the type of degeneration is enough to capture the symmetries of the exact solution.

The local metric can be derived by starting from the Euclidean Taub-NUT solution and compactifying one base direction. To do so, let us consider the background

\[
ds^2 = h(\vec{x}) \, d\vec{x}^2 + \frac{1}{h(\vec{x})} \left( d\xi^2 + \omega^2 \right), \quad h(\vec{x}) = 1 + \frac{\tilde{R}_2}{2|\vec{x}|}, \tag{2.2}
\]

where \( \vec{x} \) denotes coordinates in \( \mathbb{R}^3 \) and \( \xi^2 \) denotes the coordinate on the \( S^1 \)-fiber. The background is regular if \( \xi^2 \) has a periodicity of \( 2\pi \tilde{R}_2 \), where \( \tilde{R}_2 \) is the radius of the fiber at infinity. Note furthermore that at the origin \( \vec{x} = 0 \) of the base, the cycle of the fiber shrinks to zero size. The one-form \( \omega \) is not closed and encodes the non-triviality of the fibration. It is determined up to shifts by exact forms through the relation \( d\omega = \star_3 dh \), where the latter ensures that the equations of motion with \( H = 0 \) and \( e^\phi = g_s = \text{const.} \) are satisfied.

The compactification of this background is achieved by considering an infinite array of sources on one of the base directions. The harmonic function \( h(\vec{x}) \) becomes

\[
h(\vec{r}, \xi^1) = 1 + \sum_{n \in \mathbb{Z}} \frac{\tilde{R}_2}{2\sqrt{r^2 + (\xi^1 - 2\pi \tilde{R}_1 n)^2}}, \tag{2.3}
\]

where we split the three-dimensional radial direction into \( |\vec{x}|^2 = r^2 + (\xi^1)^2 \). The sum in (2.3) does not converge but can be regularized. After Poisson resummation we obtain the Ooguri-Vafa metric \[18\] described by

\[
h(\vec{r}, \xi^1) = \frac{\tilde{R}_2}{2\pi \tilde{R}_1} \left[ \log(\mu/r) + \sum_{n \neq 0} e^{in\xi^1/\tilde{R}_1} K_0 \left( \frac{|n| r}{\tilde{R}_1} \right) \right], \tag{2.4}
\]

with \( \mu \) a constant that controls the regulator and absorbs also all other possible constants, for instance the first term in (2.3). \( K_0 \) is the zeroth-order modified Bessel function of the second kind, whose series expansion for large \( r \) reads

\[
K_0 \left( \frac{|n| r}{\tilde{R}_1} \right) = e^{-|n|r/\tilde{R}_1} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\frac{3}{2} + k)^2}{\sqrt{\pi} k!} \left( \frac{R_1}{2|n|r} \right)^{k+\frac{1}{2}}. \tag{2.5}
\]

Hence, the leading semi-flat term in (2.4) (i.e. the logarithm) is a good approximation of the exact metric far away from the degeneration point up to exponentially suppressed terms. In fact, the semi-flat approximation of a smooth K3 – repaired with the Ooguri-Vafa metric at the 24 \( I_1 \) points – gives a metric that is a good approximation of the exact Calabi-Yau metric \[19\].

The expression (2.4) can also be derived (in a simple way) field-theoretically \[20, 21\] and

\[3\] Here and in the following we omit the additional six space-time directions that make the background into a full ten-dimensional solution of string theory.
(in a complicated way) by solving explicitly the Riemann-Hilbert problem with wall-crossing technology \[22\].

The semi-flat approximation of the above background is a flat two-torus fibration parameterized by the coordinates \((\xi^1, \xi^2)\) over a two-dimensional base \(\mathbb{R}^2\). For the latter, we introduce polar coordinates \((r, \theta)\), and the one-form \(\omega\) mentioned above can be brought into the form \(\omega^{sf} = f d\xi^1\) with \(df = \star_2 dh\). Note that in later calculations we take a gauge where

\[
\omega^{sf} = \frac{\tilde{R}_2}{2\pi R_1} \theta d\xi^1. \tag{2.6}
\]

Furthermore, we observe that after encircling the defect in the base as \(\theta \to \theta + 2\pi\), the shift \(\omega^{sf} \to \omega^{sf} + \frac{\tilde{R}_2}{R_1} d\xi^1\) should be compensated by the shift \(\xi^2 \to \xi^2 - \frac{\tilde{R}_2}{R_1} \xi^1\) which, as expected, corresponds to the action of a Dehn twist on the torus cycles. The corrections to the semi-flat term in \((2.4)\) explicitly break one of the \(U(1)\) isometries of the torus fiber. This affects also the one-form \((2.6)\), which is corrected (up to gauge transformations) by modified Bessel functions of the second kind as

\[
\omega = \omega^{sf} - \frac{\tilde{R}_2}{\pi R_1} r \sum_{k>0} K_1 \left( \frac{kr}{R_1} \right) \sin \left( \frac{k\xi^1}{R_1} \right) d\theta. \tag{2.7}
\]

The above analysis can be easily extended to a \(I_n\) degeneration. The solution is given by coalescing \(n\) Taub-NUT centers, and the Ooguri-Vafa corrections \((2.4)\), which completely smooth out the semi-flat metric for \(n = 1\), now replace the semi-flat singularity with an \(A_{n-1}\) singularity, as expected.

**Monodromy and unwinding strings**

An important point is that even within the semi-flat approximation, some “remnant” of the corrections \((2.4)\) survives. In fact, it is useful to look at the action of the monodromy on the momentum and winding of strings propagating on the torus fiber. Recall that for a \(SL(2, \mathbb{Z})_\tau \times SL(2, \mathbb{Z})_\rho\) monodromy \((M_\tau, M_\rho)\), acting on the moduli as

\[
\tau \to M_\tau[\tau] \equiv \frac{a\tau + b}{c\tau + d}, \quad \rho \to M_\rho[\rho] \equiv \frac{\tilde{a}\rho + \tilde{b}}{\tilde{c}\rho + \tilde{d}}, \tag{2.8}
\]

the corresponding \(O(2, 2, \mathbb{Z})\) transformation on the combined momentum\((n)/winding(m)\) vector \((n, m)\) is given by

\[
n \to \tilde{a} \begin{pmatrix} a & b \\ c & d \end{pmatrix} n + \tilde{b} \begin{pmatrix} -b & a \\ -d & c \end{pmatrix} m, \tag{2.9}
\]

\[
m \to \tilde{c} \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} n + \tilde{d} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} m.
\]

\(^4\)We neglect a constant shift which is not captured by the action on the homology \[22\].
In the simple case of constant $\rho$, that is say $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = (+1, 0, 0, +1)$, the above transformation reduces to

$$n \rightarrow M_\tau n, \quad m \rightarrow (M_\tau^t)^{-1} m. \quad (2.10)$$

In our case of interest, namely $\tau \rightarrow \tau + 1$, we have

$$M_\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (2.11)$$

giving the transformation

$$(n_1, n_2) \rightarrow (n_1 + n_2, n_2), \quad (m^1, m^2) \rightarrow (m^1, m^2 - m^1). \quad (2.12)$$

We see that momentum along the $(1, 0)$-cycle ($\xi^1$-direction) is not conserved for a string that moves around a degeneration point at which the $(0, 1)$-cycle ($\xi^2$-direction) shrinks. This is in contrast with the translation invariance of the semi-flat metric along both directions of the torus. The exact metric cures this problem by breaking the $U(1)$ isometry along the $(1, 0)$-cycle, as we discussed above.

Note however that winding along the $(0, 1)$-direction is also not conserved. This is easy to see by taking a string wrapped along the cycle $(1, 1)$. Denoting the world-sheet coordinates by $(\hat{\tau}, \hat{\sigma})$, we consider the trajectory

$$\xi^1 = 2\pi R_1 \hat{\sigma}, \quad \theta = 2\pi \hat{\tau},$$
$$\xi^2 = 2\pi \tilde{R}_2 \hat{\sigma}, \quad r = r_0, \quad (2.13)$$

where $(r, \theta)$ are again polar coordinates on $\mathbb{R}^2$ and $(\xi^1, \xi^2)$ are flat coordinates on $\mathbb{T}^2$ with periodicity $(\xi^1, \xi^2) = (\xi^1 + 2\pi R_1, \xi^2 + 2\pi \tilde{R}_2)$. The monodromy around the defect is a Dehn twist, which corresponds to cutting the torus along $(1, 0)$, rotating by $2\pi$ and gluing it back. The process of unwinding the string along the $(1, 0)$-cycle corresponds to the patching $\xi^2_{(2\pi)} = \xi^2_{(0)} - (\tilde{R}_2/R_1)\xi^1_{(0)}$, where $\xi^a_{(0)}$ and $\xi^a_{(2\pi)}$ are the torus coordinates at $\theta = 0$ and $\theta = 2\pi$, respectively. With this transformation the trajectory $\text{(2.13)}$ unwinds the direction $\xi^2$ at $\theta = 2\pi$, which is the semi-flat version of the unwinding trajectory in the Taub-NUT space considered in $[10]$. In the latter case, the string can be unwound by taking it arbitrarily far-away from the core of the monopole because the $S^1$ circle is non-trivially fibered over $S^2$ at spatial infinity in the $\mathbb{R}^3$ base. Such a fibration is in fact the Hopf fibration of a three-sphere. The trajectory in this case takes a string wrapping the fiber and a $S^1 \subset S^2$ from the north pole to the south pole, where a rotation of the fiber effectively unwinds the string. Our case is a compactified version of this process. Although far away from the degeneration the space is locally $\mathbb{T}^2 \times S^1$, the global twist gives it the topology of a nilmanifold, which can be seen as a non-trivial fibration of the $(0, 1)$-cycle over the remaining torus $\tilde{T}^2 = (1, 0) \times S^1$. The non-triviality of such fibration gives the unwinding in our case.

There is yet another way to understand equation $\text{(2.12)}$. In the semi-flat limit we can quantize the string on the $\mathbb{T}^2$-fiber, and find for the left- and right-moving momenta the expressions

$$p_{L,R}^I = \pi_I \pm (G \mp B)_{IJ}L^J, \quad I, J = 1, 2, \quad (2.14)$$
where \( \pi_I \) denotes the canonical momentum and \( L^I \) is the winding vector. In the present case, these are given by

\[
\pi_I = \left( \frac{n_1}{R_1}, \frac{n_2}{\tilde{R}_2} \right), \quad L^I = \left( \frac{R_1 m^1}{\tilde{R}_2 m^2} \right). \tag{2.15}
\]

Furthermore, when encircling the defect as \( \theta \to \theta + 2\pi \) the coordinates change as \( (\hat{\xi}^1, \hat{\xi}^2) = (\xi^1, \xi^2 - \tilde{R}_2 \xi^1) \), as discussed below equation (2.6). This gives rise to the diffeomorphism

\[
\Omega^I_{\ J} = \frac{\partial \hat{\xi}^I}{\partial \xi^J} = \left( \begin{array}{cc} 1 & 0 \\ -\frac{\tilde{R}_2}{R_1} & 1 \end{array} \right). \tag{2.16}
\]

If we require the spectrum to be invariant under \( \theta \to \theta + 2\pi \), we see that the momenta \( p_{L,R} \) appearing in the mass formula have to be invariant. Recalling then that in the present situation \( B_{IJ} = 0 \) and \( G(\theta + 2\pi) = \Omega^{-T} G(\theta) \Omega^{-1} \), we find

\[
0 = \Delta(p_{L,R})_I = \left[ (\Omega^T)_I^J \left( p_{L,R}(\theta + 2\pi) \right)_J - (p_{L,R}(\theta))_I \right] = \left[ (\Omega^T)_I^J \pi_J(\theta + 2\pi) - \pi_I(\theta) \right] \pm G_{IJ}(\theta) \left[ (\Omega^{-1})^J_K L^K(\theta + 2\pi) - L^K(\theta) \right], \tag{2.17}
\]

which leads to the identifications shown in equation (2.12).

**Charge inflow**

As in [10], the non-conservation of the winding charge along the \( (1,0) \)-cycle is compensated by a radial inflow of charge towards the degeneration point. This arises from a coupling between the string and a collective coordinate excitation of the monopole.

For the Kaluza-Klein monopole this comes from a gauge transformation of the \( B \)-field in terms of the unique (up to a constant) self-dual two-form [23]

\[
\mathcal{B} = \alpha d\Lambda, \quad \Lambda = \frac{C}{\hbar} \left( d\xi^2 + \omega \right), \tag{2.18}
\]

where \( \alpha \) is a parameter that becomes dynamical at the quantum level. The normalization constant \( C \) can be fixed by demanding that \( \alpha \) has periodicity of \( 2\pi/\tilde{R}_2 \). After compactification, it is possible to derive an exact expression for \( \Lambda \) [24]. Here, we will only need the semi-flat limit where \( \Lambda \) reduces to

\[
\Lambda = \frac{C}{\hbar} \left[ d\xi^2 + \frac{\tilde{R}_2}{2\pi R_1} \theta d\xi^1 \right]. \tag{2.19}
\]

which is indeed a self-dual form for the semi-flat \( I_1 \) degeneration. Let us now investigate the coupling of the string trajectory with \( \alpha \). For this, we embed the semi-flat supergravity configuration into \( 4 + 1 \) dimensions and study the dynamics of a string moving in this background,
described by the action
\[ S = S_{\text{sugra}} - \frac{1}{4\pi} \int d^5x \int d\hat{\rho} d\hat{\tau} \delta(x^a - X^a) \left[ \sqrt{\gamma^{AB}} \partial_A X^a \partial_B X^b G_{ab} + \epsilon^{AB} \partial_A X^a \partial_B X^b B_{ab} \right], \]
where \( G \) and \( B \) are the five-dimensional background fields, \( S_{\text{sugra}} \) is the usual NS-NS supergravity action and we have set \( \alpha' = 1 \). Promoting \( \alpha \) to \( \alpha(t) \) and considering the unwinding trajectory \( \hat{\tau} \), but leaving the motion along the base directions as arbitrary functions of \( \hat{\tau} \), we find that the dynamics of \( \alpha(t) \) can be described in terms of the Lagrangian density
\[ L_{\alpha} = \frac{1}{2} \dot{\alpha}^2 + K\alpha \left( \hat{h}^{-1} \frac{d\theta}{dt} + (2\pi - \theta) \frac{h'}{h^2} \frac{dr}{dt} \right), \]
where \( K \) is a constant. The corresponding equations of motion are solved by
\[ \dot{\alpha}(t) = K \frac{\theta - 2\pi}{h} + \alpha_0, \]
with \( \alpha_0 \) an integration constant. For trajectories with \( r = \text{const.} \), we see that after encircling the defect \( \dot{\alpha} \) increases by \( 2\pi K/h \). We have therefore checked that a string configuration with initial winding charge \( m_2 = 1 \) following the unwinding trajectory \( \hat{\tau} \), couples non-trivially with the background fields via the zero mode. Along this trajectory the string loses its winding charge but this is compensated by an increase of the kinetic energy of the zero mode. From the point of view of the theory reduced on the unwinding cycle, the winding charge is an electric-type charge associated to the gauge field obtained from the reduction of the \( B \)-field. With the discussed non-trivial coupling the unwinding trajectory generates an inflow of “winding” current which is eventually absorbed by the brane configuration \( \text{[10]} \). We will come back to this point below when discussing the T-dual configurations.

Finally, generalizing our above discussion, there exist configurations in which an arbitrary \((p,q)\)-cycle shrinks when encircling the singularity. A Dehn twist around a \((p,q)\)-cycle is represented by the monodromy
\[ M_{\tau}^{(p,q)} = \begin{pmatrix} 1 + pq & p^2 \\ -q^2 & 1 - pq \end{pmatrix}, \]
which is the generic element of the parabolic conjugacy class of \( SL(2,\mathbb{Z}) \). This corresponds to a compactification of a Taub-NUT space where the coordinates \((\xi^1, \xi^2)\) have been rotated by an angle \( \phi = \arctan(q/p) \) compared to the previous example. To obtain a space which is asymptotically flat one needs at least 12 mutually non-local \((p,q)\)-degenerations, which follows from the minimal Dehn twist decomposition of the identity
\[ \left( M_{\tau}^{(1,0)} M_{\tau}^{(0,1)} \right)^6 = 1. \]
If the number of singular fibers is 24, the base space becomes compact and the total space is a K3 surface.
2.2 NS5-branes on $\mathbb{R}^2 \times T^2$

In the semi-flat limit, a fiber-wise T-duality on the shrinking cycle relates the $I_1$ degeneration to a NS5-brane with monodromy $\rho \to \rho + 1$ [25][26] (see also [27]). In this case, however, the exact solution breaks all the isometries of the fiber torus.

**Compactification**

To be more concrete, let us start from the uncompactified NS5-brane background. Omitting (as in the last section) the six longitudinal space-time directions, it takes the general form

$$ds^2 = h(\vec{x}) \, d\vec{x}^2,$$

$$e^\phi = g_s \, h(\vec{x}),$$

$$H_3 = \star_4 dh(\vec{x}),$$

where $\vec{x} \in \mathbb{R}^4$. Next, we compactify two of the transversal directions on a two-torus. To this end, we split $\mathbb{R}^4 \to \mathbb{R}^2 \times T^2$ and introduce polar coordinates $(r, \theta)$ on $\mathbb{R}^2$ and coordinates $(\xi^1, \xi^2)$ on $T^2$. The above solution can then be expressed in the following way

$$ds^2 = h(r, \xi^1, \xi^2) \left[ dr^2 + r^2 d\theta^2 + (d\xi^1)^2 + (d\xi^2)^2 \right],$$

$$e^\phi = g_s \, h(r, \xi^1, \xi^2),$$

$$H_3 = \star_4 dh(r, \xi^1, \xi^2),$$

and the function $h$ can be determined by considering a rectangular lattice of NS5-branes as

$$h(r, \xi^1, \xi^2) = 1 + \sum_{\vec{n} \in \mathbb{Z}^2} \frac{1}{r^2 + (\xi^1 - 2\pi R_1 n_1)^2 + (\xi^2 - 2\pi R_2 n_2)^2}.$$  \hspace{1cm} (2.27)

This sum is not convergent, but can be regulated with a regulator of the form [21][24]

$$\frac{1}{2\pi R_1 R_2} \sum_{n \in \mathbb{Z}^*} \frac{1}{|n|^2}. \hspace{1cm} (2.28)$$

By subtracting this term from the original function and performing a Poisson resummation we find

$$h(r, \xi^1, \xi^2) = \frac{1}{2\pi R_1 R_2} \log \left( \frac{\mu}{r} \right) + \sum_{\vec{k} \in (\mathbb{Z}^2)^*} K_0(\lambda r) \, e^{-i \left( \frac{k_1 \xi^1}{R_1} + \frac{k_2 \xi^2}{R_2} \right)}, \hspace{1cm} (2.29)$$

where $(\mathbb{Z}^2)^* = \mathbb{Z}^2 - \{(0,0)\}$ and $\lambda = \sqrt{(k_1/R_1)^2 + (k_2/R_2)^2}$. The same result can be determined in purely field-theoretic terms [28][21] from one-loop corrections to the gauge coupling of the non-linear sigma model, obtained by reducing a $\mathcal{N} = 2$ theory on the torus. We emphasize that expression (2.29) makes evident the origin of the symmetry-breaking corrections to the semi-flat metric

$$h(r) = \frac{1}{2\pi R_1 R_2} \log \left( \frac{\mu}{r} \right), \hspace{1cm} (2.30)$$
that is, the terms involving $K_0(\lambda r)$ break the $U(1)^2$ isometry of the background. Note also that taking the decompactification limit $r, \xi^1, \xi^2 \ll R_1, R_2$ in (2.27), one recovers the non-compact harmonic function shown in (2.25), that is

$$h(r, \xi^1, \xi^2) = 1 + \frac{1}{r^2 + (\xi^1)^2 + (\xi^2)^2}. \quad (2.31)$$

If we de-compactify only one of the cycles of the torus, say the one corresponding to $\xi^1$, and define as before $|\vec{x}|^2 = r^2 + (\xi^1)^2$, we obtain the familiar result for the H-monopole compactified along one direction [29,10]

$$h(|\vec{x}|, \xi^2) = 1 + \frac{1}{2 R_2 |\vec{x}|} \frac{\sinh(|\vec{x}|/R_2)}{\cosh(|\vec{x}|/R_2) - \cos(\xi^2/R_2)}. \quad (2.32)$$

This solution encodes the breaking of the $U(1)$ isometry along the cycle which is dual to the shrinking one in the Taub-NUT space.

**Monodromies**

As in the previous case, we can understand the breaking of the $U(1)$ isometries from the action of the monodromies (2.9). Since now $\rho$ is varying, there will be a mixing between momentum and winding states. Using the general expression shown in (2.9), we can deduce the action of $\rho \to \rho + 1$ on the momentum and winding modes as

$$(n_1, n_2) \to (n_1 + m^2, n_2 - m^1), \quad (m^1, m^2) \to (m^1, m^2), \quad (2.33)$$

as expected from T-duality. Momentum can now unwind from the T-dual of the (0,1)-cycle, with a trajectory dual to (2.13)

$$\xi^1 = 2\pi R_1 \hat{\sigma}, \quad \theta = 2\pi \hat{\tau},$$

$$\xi^2 = \frac{2\pi}{\tilde{R}_2} \hat{\tau}, \quad r = r_0, \quad (2.34)$$

where we used that $\tilde{R}_2 = 1/R_2$. Similarly, for the (1,0)-cycle we can write

$$\xi^1 = \frac{2\pi}{R_1} \hat{\tau}, \quad \theta = 2\pi \hat{\tau},$$

$$\xi^2 = -2\pi R_2 \hat{\sigma}, \quad r = r_0. \quad (2.35)$$

The canonical momenta that generate translations along the fiber directions are

$$\pi_a = \int d\hat{\sigma} \left[ g_{ab} \partial_{\hat{\tau}} X^b + B_{ab} \partial_{\hat{\sigma}} X^b \right]. \quad (2.36)$$

In order to compute such quantities we can set $h = 1$, effectively putting the brane in an asymptotically flat background. For the above trajectories we find, respectively,

$$\pi_2 = \frac{1}{R_2} (2\pi - \theta), \quad \pi_1 = \frac{1}{R_1} (2\pi - \theta), \quad (2.37)$$
which indeed vanish after encircling the defect. Accordingly, the exact metric breaks translational invariance in both directions.

The non-conservation of momentum shown in (2.33) can also be understood in a fashion similar to the \( I_1 \)-degeneration discussed in the previous section. Quantizing the string on the \( \mathbb{T}^2 \)-fiber in the semi-flat limit, the left- and right-moving momenta are again given by the general expression (2.14), where

\[
\pi_I = \left( \frac{n_1}{R_1}, \frac{n_2}{R_2} \right), \quad L^I = \left( \frac{R_1 m^1}{n_2 R_2}, \frac{R_2 m^2}{n_1 R_1} \right). \tag{2.38}
\]

When encircling the defect as \( \theta \to \theta + 2\pi \), the coordinates change as \( (\hat{\xi}^1, \hat{\xi}^2) = (\xi^1, \xi^2) \) and hence the diffeomorphism is trivial, however, now the \( B \)-field depends non-trivially on \( \theta \). Demanding again that the spectrum is invariant, we are led to requiring

\[
0 = \Delta (p_{L,R})_I = (p_{L,R}(\theta + 2\pi))_I - (p_{L,R}(\theta))_I \tag{2.39}
= \left[ \pi_I(\theta + 2\pi) + (B_{IJ} L^J)(\theta + 2\pi) - \pi_I(\theta) - (B_{IJ} L^J)(\theta) \right] \pm G_{IJ} \left[ L^J(\theta + 2\pi) - L^J(\theta) \right],
\]

which gives \( L^I(\theta + 2\pi) = L^I(\theta) \) and \( \pi_I(\theta + 2\pi) = \pi_I(\theta) \). Hence, we find the identifications of momentum and winding numbers shown in equation (2.33).

Charge inflow

As we did for the KK-monopole case, we can compute the coupling of the string with the background collective coordinates. In this case, the zero mode dual to (2.19) is a shift along the toroidal coordinate, \( \xi^2 \to \xi^2 + \alpha \). In analogy to the above situation, we embed the semi-flat configuration into 4 + 1 dimensions and study the dynamics of a string moving in this background using the action (2.20). Promoting \( \alpha \) to \( \alpha(t) \) and considering the trajectory (2.34) but letting the motion along the angular coordinate on the base be an arbitrary function \( \theta(\hat{\tau}) \), the dynamics of the zero modes is described by the effective Lagrangian density

\[
L_\alpha = \frac{1}{2} \dot{\alpha}^2 + \hat{K} \dot{\alpha} (4\pi h - \theta), \tag{2.40}
\]

where \( \hat{K} \) is a constant. The corresponding equations of motion are solved by

\[
\alpha = \hat{K} \int^t dt \theta. \tag{2.41}
\]

As for the KK-monopole, after going around the defect \( \dot{\alpha} \) increases by \( 2\pi \hat{K} \). In this case, the non-conserved charge along the trajectory is momentum, which couples to the background fields via the zero mode associated to the position of the brane along the fiber. Again, one can also perform an analysis from the point of view of the dimensionally reduced theory. In this case, momentum charge is associated to the KK gauge field coming from the reduction of the metric. The trajectory (2.34) will then produce an equivalent current inflow that is absorbed by the background via the discussed mechanism.
T-duality of exact metrics

It is interesting to ask what happens to the T-duality transformation between the NS5-brane and the $I_1$ degeneration, once corrections to the semi-flat approximation are taken into account and thus the Buscher rules cannot be applied.

Corrections to the $U(1)$ isometry of the $(1,0)$-cycle have a physical interpretation related to the non-conservation of momentum along that cycle. For the $I_1$ degeneration the corrections are captured by the Ooguri-Vafa metric related to (2.4) and have the form of a sum of non-perturbative terms

$$C_n \sim e^{-|n|r_1} e^{-i{n_1} \xi_1} \frac{1}{R_1},$$  \hspace{1cm} (2.42)

where each of these contains also the perturbative sum (2.5). The NS5-brane metric related to (2.29) on the other hand contains a double-sum of terms

$$\tilde{C}_{n_1,n_2} \sim e^{-\lambda r} e^{-i{n_1} \xi_1} e^{-i{n_2} \xi_2} \text{ with } \lambda = \sqrt{(n_1/R_1)^2 + (n_2/R_2)^2}.$$  \hspace{1cm} (2.43)

The corrections depending on $\xi^2$ break the isometry along the $(0,1)$-cycle, along which we dualize to arrive at the $I_1$ degeneration. The problem of dualizing these higher Fourier modes is in fact similar to the problem considered in [10], where it has been suggested (and to some extend checked in [30, 31]) that the modes in $\tilde{C}_{n_1,n_2}$ map to stringy modes of the Taub-NUT space. In our case we see that $\tilde{C}_{n,0} = C_n$ – including numerical factors – and it is plausible to conjecture that T-duality of the full NS5-background sends each mode $\tilde{C}_{n,m}$ for $m \neq 0$ to a mixed momentum-winding mode $(n, m)$ on the Taub-NUT side. Note that this is a very specific rule for the massive modes, and it might be valid only in the regime where the semi-flat approximation is broken only mildly.

A similar conclusion is found by considering elements of the T-duality group that are merely changes of basis, belonging to the geometric $SL(2, \mathbb{Z})_\tau$ subgroup. An example is the rotation that sends $\tau \rightarrow -1/\tau$, exchanging a $(1,0)$ $I_1$ degeneration with a $(0,1)$ one. The T-duality of exact metrics can now be derived by noticing that the two configurations are obtained by a compactification of Taub-NUT spaces along two orthogonal directions, and they are therefore related by a $\pi/2$ rotation of the toroidal coordinates. This results in a specific map for the massive modes (2.42) that sends $C_n \rightarrow e^{-|n|r_{R_2}} e^{-i n \xi^2/R_2'}$, with $R_2' = R_1$ and $R_1' = R_2$.

3 T-folds

Starting from the semi-flat metric of a compactified $I_1$ degeneration – the Kaluza-Klein monopole smeared on a $S^1$ discussed in section 2.1 – we want to perform a T-duality along the circle parametrized by $\xi^1$. At first this seems problematic because the monodromy around the $I_1$ degeneration acts non-trivially on this $S^1$ and the corresponding Killing vector $\partial/\partial \xi^1$
is not globally defined. However, the semi-flat NS5-solution has two $U(1)$ isometries and we can perform a collective T-duality transformation or a duality rotation in $O(2,2,\mathbb{Z})$ that corresponds to a fiberwise transformation $(\rho \to -1/\rho, \tau \to -1/\tau)$. An independent argument showing the appearance of such solutions uses heterotic/F-theory duality\cite{15,18}. The result is the line element

$$\begin{align*}
    ds^2 &= h(r) \left( dr^2 + r^2 d\theta^2 \right) + \frac{4\pi^2 h(r)}{4\pi^2 h(r)^2 + \tilde{R}_1^2 \tilde{R}_2^2 \theta^2} \left[ (d\xi^1)^2 + (d\xi^2)^2 \right], \\
    B &= -\frac{2\pi \tilde{R}_1 \tilde{R}_2 \theta}{4\pi^2 h(r)^2 + \tilde{R}_1^2 \tilde{R}_2^2 \theta^2} d\xi^1 \wedge d\xi^2, \\
    e^{2\phi} &= \frac{4\pi^2 h(r)}{4\pi^2 h(r)^2 + \tilde{R}_1^2 \tilde{R}_2^2 \theta^2},
\end{align*}$$

where $\tilde{R}_a = 1/R_a$, $(r,\theta)$ parametrize $\mathbb{R}^2$, and $h(r)$ is the semi-flat harmonic function (2.30). This solution – usually denoted as $5_2^2$-brane\cite{7} or called a Q-brane\cite{34} – clearly induces a monodromy $-1/\rho \to -1/\rho + 1$, corresponding to $\beta$-transformations in $O(2,2,\mathbb{Z})$. It is therefore a globally non-geometric background.

Note that the metric in (3.1) is translational invariant along both the fiber directions. Correspondingly, from (2.9) we deduce the non-geometric monodromy action on the momentum and winding states as

$$(n_1, n_2) \to (n_1, n_2), \quad (m^1, m^2) \to (m^1 + n_2, m^2 - n_1).$$

This can also be obtained from the NS5-monodromy by the action of the transformation $\rho \to -1/\rho$ and $\tau \to -1/\tau$, which interchanges $n_a \leftrightarrow m^a$ for all $a$. This suggests that the $U(1)^2$ isometries of the metric (3.1) will not receive quantum corrections, as the momenta are conserved on both the torus directions. However, in analogy with the duality between the Taub-NUT space and the NS5-brane, there exist now trajectories along which strings initially wrapped along the $(1,0)$- and $(0,1)$-cycle of the torus unwind. For example, for the trajectory

$$\begin{align*}
    \xi^1 &= 2\pi \tilde{R}_1 \hat{\sigma}, \\
    \xi^2 &= -\frac{2\pi}{\tilde{R}_2} \hat{\tau}, \\
    \theta &= 2\pi \hat{\tau}, \\
    r &= r_0,
\end{align*}$$

a string with winding along the $(1,0)$-cycle and momentum along the $(0,1)$-cycle will unwind after encircling the defect. Note that the monodromy action is similar to the NS5-monodromy (2.33), up to an interchange of momenta and windings.

Let us again derive the change in momentum and winding numbers using the invariance of the left- and right-moving momenta (2.14) when encircling the defect. More concretely, we quantize the string on the $\mathbb{T}^2$-fiber in the semi-flat approximation, and find for the canonical momentum and winding vector the expressions

$$\begin{align*}
    \pi_I &= \begin{pmatrix} n_1/\tilde{R}_1 \\ n_2/\tilde{R}_2 \end{pmatrix}, \\
    L' &= \begin{pmatrix} \tilde{R}_1 m^1 \\ \tilde{R}_2 m^2 \end{pmatrix}.
\end{align*}$$

\footnote{See for example\cite{32} for a clear discussion of these issues.}
Under $\theta \to \theta + 2\pi$, the background can be made globally-defined by identifying $\mathbb{T}^2(\theta + 2\pi)$ and $\mathbb{T}^2(\theta)$ using an $O(2,2)$ transformation. In particular, we have

\[(G \mp B)(\theta + 2\pi) = O^\beta_1 \left[ (G \mp B)(\theta) \right], \quad (3.5)\]

where $O_\beta$ is a so-called $\beta$-transformation. On the combined momentum-winding vector $(L', \pi_I)^T$ this transformation acts by matrix multiplication as

\[O_\beta = \begin{pmatrix}
1 & 0 & -\tilde{R}_1 \tilde{R}_2 \\
0 & +\tilde{R}_1 \tilde{R}_2 & 0 \\
0 & 0 & 1
\end{pmatrix}. \quad (3.6)\]

We now demand that the spectrum does not change when encircling the defect, which means that the left- and right-moving momenta have to be invariant under $\theta \to \theta + 2\pi$. We then compute

\[0 \equiv \Delta(p_{L,R})_I \quad (3.7)\]

leading to the relation

\[0 \equiv O_\beta \cdot \left( \frac{L^I(\theta + 2\pi)}{\pi_I(\theta + 2\pi)} \right) - \left( \frac{L^I(\theta)}{\pi_I(\theta)} \right), \quad (3.8)\]

which is solved by (3.2).

Additionally, the non-conservation of the winding charge should be compensated by an inflow current, and we expect the winding modes to couple to two dyonic coordinates arising from the flux. It is hard to make this concrete because of the non-geometric nature of the local metric (3.1), but we can make the following argument based on T-duality. Let us start from the solution of NS5-branes smeared on the $\mathbb{T}^2$, and consider the coordinate shifts $\xi^1 \to \xi^1 + f_1(r, \theta)$ and $\xi^2 \to \xi^2 + f_2(r, \theta)$. If one applies T-duality along the $\xi^2$-direction to the transformed solution, we see that $f_1$ remains as a coordinate shift of the Taub-NUT solution, while $f_2$ is mapped to a gauge transformation of the $B$-field. This is consistent with the analysis of the monodromy action in section 2.1. On the other hand, if we start from the NS5-brane configuration and perform two T-dualities, both transformations become gauge transformations of the $B$-field and the metric is not affected. This suggests that the Q-brane has two dyonic zero modes, as expected from T-duality.

These transformations are coordinate transformations, which result in another supergravity solution. The zero-mode is a particular case thereof.
Beyond semi-flat approximation

As for the duality between A-type singularities and NS5-branes, we should ask what is the transformation of the modes (2.29) that localize the NS5-branes on the fiber torus under the conjectured T-duality that leads to the solution (3.1). The answer is roughly a T-dual version of the transformation between a \((1, 0)\) and a \((0, 1)\) type \(I_1\) degeneration. A naive guess is that the NS5 Fourier modes are mapped to

\[
\tilde{C}_{n_1, n_2} \sim e^{-\tilde{\lambda} r} e^{-in_1\tilde{\xi}_1} e^{-in_2\tilde{\xi}_2} \quad \text{with} \quad \tilde{\lambda} = \sqrt{(n_1\tilde{R}_1)^2 + (n_2\tilde{R}_2)^2}, \tag{3.9}
\]

where we define \(\tilde{R}_i = \frac{1}{R_i}\). As in [10], the modes \(\tilde{\xi}_i\) should be identified with dyonic degrees of freedom of the non-geometric solution, as follows from a particular effective action describing the type of couplings between winding and dyonic modes described above. The rationale for such transformations is that both the geometrical coordinates \(\xi^i = \xi^i_L + \xi^i_R\) and the dual ones \(\tilde{\xi}^i = \xi^i_L - \xi^i_R\) play a non-trivial role. The semi-flat solution for a NS5-brane is written in terms of a trivial fibration of the \((\xi^1, \xi^2)\) fiber coordinates, and the excited Kaluza-Klein momentum states break both the U(1) symmetries associated to shifts in such coordinates. The dual stringy coordinates are instead exact. Note that from the previous discussion it seems that such stringy coordinates are associated with a non-geometric fibration structure, so that the present situation is substantially more complicate than the usual duality between \(H\)-monopoles and Taub-NUT spaces. In this latter situation, the stringy coordinate is associated with a topologically non-trivial circle fibration, which is traded by T-duality with a \(B\)-field in the dual, trivially fibered solution. In fact there is a well-known geometrical construction that unifies both fibrations [37]. Starting from an oriented \(S^1\) bundle over a compact connected manifold \(M\): \(S^1 \to E \xrightarrow{\pi} M\), one constructs the correspondence space \(C = E \times_M \hat{E}\), where \(\hat{E}\) is the T-dual fibration. \(C\) is both a circle bundle over \(E\) and a circle bundle over \(\hat{E}\), and if \(\hat{E}\) is a trivial fibration, as in the \(H\)-monopole case, we have that \(C = E \times S^1\). For the present case of elliptic fibrations, this geometric construction cannot be easily generalized [32], in line with the above discussion. The breaking of both \(U(1)^2\) isometries of the NS5-background poses in fact additional challenges for a geometric description in a extended space, as we will discuss in the next section.

4 Description in extended space

In the previous sections we have seen evidence for a “generalized T-duality” acting on higher Fourier modes of the string fields. We now want to discuss to what extend this physics can be captured by a T-duality covariant formalism such as the doubled formalism of [1].

\[\]
4.1 Doubled torus fibrations

In the semi-flat limit, the defect solutions are fully characterized by associating at each base point a string state $\Psi = \sum \Psi_{n,m}(n,m)$ together with a monodromy, where the latter is an $O(2,2,\mathbb{Z})$ transformation acting on the momentum and winding numbers. We can then Fourier transform the basis $|n,m\rangle$ to position space as

$$|\xi,\bar{\xi}\rangle = \sum_{n_i,m_i} e^{in_1\xi_1/R_1+in_2\xi_2/R_2}e^{im_1\bar{\xi}_1\bar{R}_1+im_2\bar{\xi}_2\bar{R}_2}|n,m\rangle,$$

where we introduced coordinates $(\bar{\xi}_1,\bar{\xi}_2)$ conjugate to the winding numbers. These additional coordinates can be thought of as defining an extended compact space, a four-torus $T^4$, which leads to the doubled formalism of [1]. In our case, the parabolic monodromies act as generalized Dehn twist on such a four-torus, and define a (non-principal) fibration over the two-dimensional base. For the local model of a Kodaira $I_1$ degeneration the monodromy (2.12) obviously defines the $SO(2,2,\mathbb{Z}) \subset SL(4,\mathbb{Z})$ monodromy

$$A_\tau = \begin{pmatrix} M^{(1,0)} & 0 \\ 0 & M^{(0,1)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

All the monodromies for the $(p,q)_\tau$- and $(p,q)_\rho$-defects can be obtained by a simple change of basis for the $T^4$, which corresponds to different embeddings of two-tori $T^2 \times \bar{T}^2$. The $(0,1)$ degeneration, that we will denote by $B_\tau$, is obtained by exchanging $M^{(1,0)}_\tau \leftrightarrow M^{(0,1)}_\tau$. By exchanging $(\xi_2 \leftrightarrow \bar{\xi}_2)$ or $(\xi_1 \leftrightarrow \bar{\xi}_1)$ we obtain respectively the NS5- and Q-brane monodromies

$$A_\rho = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_\rho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

corresponding to a B-shift and $\beta$-transform as expected. Restricting these $T^4$ fibrations over the boundary of a disk around the degeneration, $S^1 = \partial D^2$, one recovers the fibrations of [38]. In our case these fibrations extend over the punctured base and we need to ask if there exists a degeneration of the four-torus giving rise to such monodromies, and if the local models can be glued together to form a global space.

Singular fibers

We expect the type of singular fiber in the $T^4$ fibration to be determined by the conjugacy class of the monodromy around the boundary of a small disk encircling the degeneration. In the doubled description all the parabolic $\tau$- and $\rho$-monodromies are related by the change of basis described above, and so all singular fibers should have the same topology. If we assume that monodromies of the type (4.3) arise as a Picard-Lefschetz type monodromy around a singular
fiber, where two of the cycles of the $\mathbb{T}^4$ are pinched, we obtain a topology of type $I_1 \times I_1$. Higher Fourier modes will correct the semi-flat approximation by localizing the shrinking cycles in a similar way as for the $I_1$ degenerations. The change of $SL(4,\mathbb{Z})$ duality frames gives then a precise generalized T-duality between higher Fourier modes, of the kind discussed in the previous sections. It would be interesting to study in more details this quantum corrected metric for the $I_1 \times I_1$ degeneration.

**Global construction**

The global issues appear quite subtly, and we would like to make the following observation. A global, supersymmetric, non-geometric model can be obtained by pairing 12 non-local $\tau$-degenerations with 12 non-local $\rho$-degenerations [3]. In the doubled space this is described by the factorization of the identity in terms of the $(A,B)$ twists:

$$ (A_\tau B_\tau)^6 (A_\rho B_\rho)^6 = 1. \quad (4.4) $$

However, as a $\mathbb{T}^4$ fibration (including monodromies in the full mapping class group $SL(4,\mathbb{Z})$), each degeneration can be seen as a collision of two elementary degenerations in which one cycle shrinks. Locally, these correspond to a singular fiber of type $I_1 \times T^2$. We see then that the global doubled fibration is specified by 48 elementary degenerations, which appears to be incompatible with a holomorphic fibration of the $\mathbb{T}^4$ moduli.

This can be seen already in the geometric setting. Let us consider a smooth K3 surface, described by 24 mutually non-local $I_1$ degenerations, corresponding to the monodromy decomposition $(M^{(1,0)} M^{(0,1)})^{12} = 1$. The doubled torus fibration will then be described by the decomposition $(A_\tau B_\tau)^{12} = 1$. Now, there exists a global polarization that identifies the physical fiber with the $(\xi_1, \xi_2)$ directions, and the fibration reduces to a $T^2 \times \tilde{T}^2$ fibration over $\mathbb{P}^1$. If we try to fiber the two complex structure moduli $\tau$ and $\tilde{\tau}$ of the two tori we can write the metric [39,40]

$$ ds^2 = e^\varphi \tau_2 \tilde{\tau}_2 d\bar{z}d\bar{z} + \frac{1}{\tau_2} |d\xi^1 + \tau d\xi^2|^2 + \frac{1}{\tilde{\tau}_2} |d\tilde{\xi}_1 + \tilde{\tau} d\tilde{\xi}_2|^2. \quad (4.5) $$

Each $I_1$ degeneration of $\tau$ or $\tilde{\tau}$ would give the same deficit angle as the physical $I_1$ singularity we started with, and a compact model seems to require a total of 24 degenerations, precisely half of the degenerations required to build the 24 $I_1 \times I_1$ degenerations of the doubled model. We leave this issue for future investigation.

**4.2 Double field theory and generalized duality**

We now discuss solutions of double field theory [11] that capture part of the physics discussed in the previous sections. From a slightly different perspective, DFT (and EFT) configurations describing the NS5-brane and its dual backgrounds have been studied also in [41,43]. We denote the coordinates on the “doubled” manifold as $X^N = (x^\mu, \tilde{x}_\mu)$, where the capital indices
\( N \) are raised and lowered by the \( O(d,d) \) metric

\[
\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\] (4.6)

The metric \( g \) of the manifold \( M \) and the anti-symmetric tensor field \( B \) are reorganized into the \( O(d,d) \) matrix

\[
\mathcal{H} = \begin{pmatrix} g - B g^{-1} B & B g^{-1} \\ -g^{-1} B & g^{-1} \end{pmatrix},
\] (4.7)

also called a generalized metric, and the dilaton \( \phi \) is expressed using the generalized dilaton \( \tilde{\phi} \) defined by \( e^{-2\tilde{\phi}} = \sqrt{g} e^{-2\phi} \). In order for the algebra of infinitesimal diffeomorphisms to close, one has to impose the so-called strong constraint

\[
\eta^{MN} \partial_M \partial_N = 0,
\] (4.8)

which implies that the fields only depend on half of the generalized coordinates. Taking this field content, one can write down a manifestly \( O(d,d) \)-covariant theory \[44\]

\[
S \sim \int d^N X e^{-2\tilde{\phi}} \mathcal{R}(\mathcal{H},\tilde{\phi}),
\] (4.9)

where \( \mathcal{R}(\mathcal{H},\tilde{\phi}) \) is the generalized curvature scalar (see for instance equation (4.24) in \[44\]). Solving the strong constraint by demanding no winding-coordinate dependence of the fields, the NS-NS supergravity action is recovered.

Turning to the symmetry transformations, under a global \( O(d,d) \)-transformations of the form

\[
h = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O(d,d),
\] (4.10)

the generalized metric \( \mathcal{H} \) transforms as \( \mathcal{H}'(X') = h \mathcal{H}(X) h^t \), where \( X' = h^{-t} X \). In terms of the combined background field \( E = g + B \), the \( O(d,d) \) transformation \( h \) acts as

\[
E' = h[E] = (AE + B)(CE + D)^{-1}.
\] (4.11)

This is the extension of the Buscher rules to arbitrary elements of the duality group \[45\]. \( \Theta \) shifts and \( GL(d) \) transformations are exact, while other elements generically receive corrections from the path-integral measure. An example of the latter situation is a T-duality along \( x^a \) direction given by the factorized duality

\[
h = \begin{pmatrix} 1 - e_a & e_a \\ e_a & 1 - e_a \end{pmatrix},
\] (4.12)

where \( 1 \) is the \( d \times d \) identity matrix and \( e_a \) is a \( d \times d \) matrix with all entries equal zero except for the \( a \)'th diagonal element. The action on fields give the familiar Buscher rules, and in double field theory this corresponds to the interchange \( x^a \leftrightarrow \tilde{x}_a \). In this formalism, one can
still apply these transformations in the case the generalized metric depends on the coordinate $x^a$, and the resulting expression will still be a solution of the double field theory equations of motion. Furthermore, the solution will be compatible with the strong constraint provided one chooses $(x^1, \ldots, \tilde{x}_a, \ldots, x^d)$ to be the physical coordinates.

One can now ask the question, to what extent these transformations give rise to equivalent string theory backgrounds? The fact that such dualization of massive modes could correct supergravity Buscher rules was emphasized and discussed in [11] by using closed-string field theory. It should be emphasized that such possibility, if true, is closely tied to the particular toroidal background. We will discuss this possibility in connection of the fibered structure of the semi-flat metrics and their quantum corrections.

Dual backgrounds

In the semi-flat limit, one can easily construct solutions of the equations of motion derived from the action (4.9) that roughly correspond to the semi-flat limit of the doubled torus fibrations discussed in the previous section. The semi-flat NS5-brane solution is lifted to

$$\begin{align}
\text{\text{ds}}^2_{\text{DFT}} &= h(r) \left[ dr^2 + r^2 d\theta^2 + (d\xi^1)^2 + (d\tilde{\xi}^2)^2 \right] \\
&\quad + \frac{1}{h(r)} \left[ \left( \tilde{\xi}^1 - \frac{\theta}{2\pi R_1 R_2} d\xi^2 \right)^2 + \left( \tilde{\xi}^2 + \frac{\theta}{2\pi R_1 R_2} d\tilde{\xi}^1 \right)^2 \right].
\end{align}
$$

(4.13)

This has a similar structure to the doubled torus fibration (4.5) with

$$\begin{align}
\tau &= \frac{i}{2\pi R_1 R_2} \log(z^{-1}), \\
\tilde{\tau} &= \frac{2\pi i R_1 R_2}{\log(z^{-1})},
\end{align}
$$

(4.14)

where $z = re^{i\theta}$, giving the expected monodromy $A_{\rho}$ (4.3). By the simple basis change discussed above, one recovers the different semi-flat backgrounds discussed in the previous section.

An interesting question is to what extent we can construct solutions that incorporate the higher Fourier modes that localize the NS5-brane on the torus fiber. In fact, this is possible by applying the generalized dualization discussed in the previous subsection. This essentially corresponds to the particular T-duality transformation (4.12) on the massive fields. Starting from the NS5 solution on $\mathbb{R}^2 \times \mathbb{T}^2$, we obtain the following configuration

$$\begin{align}
\text{ds}^2 &= \tilde{h} \left[ dr^2 + r^2 d\theta^2 + (d\xi^1)^2 \right] + \frac{1}{\tilde{h}} \left[ d\xi^2 + \frac{\theta}{2\pi R_1} d\xi^1 + \tilde{\Pi}_2 d\theta \right]^2, \\
B &= \tilde{\Pi}_1 d\theta \wedge d\xi^1, \\
\text{e}^{2\Phi} &= \text{const.},
\end{align}
$$

(4.15)

where $\tilde{\Pi}_{1,2} \equiv \Pi_{1,2}(r, \xi^1, \tilde{\xi}_2)$ and $\Pi_{1,2}$ the functions collected in equation (B.7) in appendix B. Furthermore, $\tilde{h} \equiv h(r, \xi^1, \tilde{\xi}_2)$, with $h$ the localized harmonic function (2.29), and we write

We use $\text{ds}^2_{\text{DFT}}$ as a short-hand notation to encode the form of the generalized metric as $\text{ds}^2_{\text{DFT}} = \mathcal{H}_{MN} dx^M dx^N$.}
the result in terms of $\tilde{R}_2 = 1/R_2$. This is a compactification of the solution presented in [30] and it is compatible with the naive T-duality discussed in the previous sections. A second dualization leads to the background described by

$$
\begin{align*}
\text{ds}^2 &= \tilde{h}[dr^2 + r^2d\theta^2] + \frac{4\pi^2\tilde{h}}{4\pi^2\tilde{h}^2 + \tilde{R}_1^2\tilde{R}_2^2\theta^2}\left[(d\xi^1 + \tilde{\Pi}_1 d\theta)^2 + (d\xi^2 + \tilde{\Pi}_2 d\theta)^2\right], \\
B &= -\frac{2\pi \tilde{R}_1 \tilde{R}_2 \theta}{4\pi^2\tilde{h}^2 + \tilde{R}_1^2\tilde{R}_2^2\theta^2}(d\xi^1 + \tilde{\Pi}_1 d\theta) \wedge (d\xi^2 + \tilde{\Pi}_2 d\theta), \\
e^{2\Phi} &= \frac{4\pi^2\tilde{h}}{4\pi^2\tilde{h}^2 + \tilde{R}_1^2\tilde{R}_2^2\theta^2},
\end{align*}
$$

where now $\tilde{\Pi}_{1,2} \equiv \Pi_{1,2}(r, \tilde{\xi}_1, \tilde{\xi}_2)$ and $\tilde{h} \equiv h(r, \tilde{\xi}_1, \tilde{\xi}_2)$. We also substitute $R_{1,2} \rightarrow \tilde{R}_{1,2} = 1/R_{1,2}$. Configurations (4.15) and (4.16) depend explicitly on the winding coordinates, and should be understood as DFT configurations by inserting the fields into (4.7) to obtain the corresponding generalized metric. In the DFT language the above localized configurations can be obtained from the localized NS5-brane generalized metric by simple transformations of the type $\xi^a \leftrightarrow \tilde{\xi}_a$. All these configurations have vanishing generalized curvature $R_{MN}$ and therefore are solutions of the equations of motion of the DFT action (4.9).

Before closing this section, let us mention that in principle one can consider defects with more general $\tau$ and $\rho$ monodromies, of the form of $(ADE, ADE)$ type. The physics of such non-geometric defects has been recently studied in [16]. Analogous puzzles related to generalized T-duality will arise. However, in this situation both winding and momentum might not be conserved along both fiber directions, and a solution of the strong constraint is a priori not guaranteed.

5 Discussion

In this note we argued that winding modes are crucial for understanding the near-core physics of T-duality defects. This is already evident in the T-duality between $I_1$ singularities and NS5-branes, where our analysis becomes essentially a compactified version of [10]. We argued that a similar physics describes non-geometric defects, where an essential role is played by two dyonic coordinates dual to the isometric directions of the fiber torus. In particular, the picture that emerges is that winding modes correspond to the localization of the Q-monopole compactified on a two-torus similarly as winding modes localize the KK-monopole compactified on a circle. It would be important to search for an explicit CFT description of such winding mode physics, or a dual formulation in terms of more conventional dynamics.

In the case of duality between the $I_1$ singularity and NS5-branes, there is an interesting observation [46] (see also [47]). Consider a stack of $N$ NS5-branes on the Coulomb branch, symmetrically distributed on a contractible circle. There exists a particular large $N$ limit of such configuration in which it looks like a periodic array of NS5-branes on a line. In such limit the harmonic function is precisely of the form (2.32). Namely, the solution reduces to the $H$-monopole. Since we have an explicit CFT description of such a NS5-ring, one could understand
the fate of the modes that localize the NS5-branes after a T-duality. Unfortunately, to study T-duality to a non-geometric configuration we would need to start with a configuration of NS5-branes arranged on $S^1 \times S^1$, for which an explicit CFT description is lost.

A different approach to study torus fibrations is to fiber over a common base $B$ the eight dimensional duality between the heterotic string compactified on a torus and F-theory compactified on a elliptic K3-surface. If we keep the gauge group unbroken the eight-dimensional moduli are $\tau$ and $\sigma$, and a fibration over $B = \mathbb{P}^1$ reproduces our models. The $\tau$- and $\sigma$-fibrations are described by two Weierstrass forms

$$\tau: y^2 = x^3 + f_\tau x + g_\tau, \quad \rho: \tilde{y}^2 + \tilde{x}^3 + f_\rho \tilde{x} + g_\rho,$$

from which we can reconstruct the dual elliptic K3. Note that the NS5-brane, as well as the Q-brane and the general $(p, q)_\rho$-branes, are identified with a $I_1$ degeneration of the $\rho$-fibration. In fact, the explicit map to F-theory is completely symmetric between $\tau$ and $\rho$, as expected from T-duality. This however leads to a puzzle if we consider the leading symmetries of the exact solutions, namely the corrections to the semi-flat approximation. The $I_1$ degeneration of the $\tau$-fibration correctly captures the breaking of the $U(1)$ isometry of the cycle which is orthogonal to the shrinking one. This is described by the metric (2.4). On the contrary, the $I_1$ degeneration of the auxiliary $\rho$-fibration misses the symmetries of the NS5-solution. Additionally, the $(p, q)_\rho$ conjugacy class interpolates between the NS5-brane, where both the $U(1)^2$ isometries of the torus are broken, and the Q-brane, where they are expected to be exact. How these facts are reconciled with T-duality? In fact, the moduli corresponding to the position of the NS5-branes on the fiber are missed by the heterotic/F-theory duality. The explicit solutions on the doubled space that we obtained in section 4 (assuming they are not substantially modified for the heterotic string) give a precise prediction for the corrections to the semi-flat $\rho$-fibration. Hence, it should be possible to check whether they capture the essential physics by extending the duality to F-theory to include these extra moduli. It would also be interesting to see if some of these results can be obtained by using D-brane probes, generalizing the analysis of [48].

**Acknowledgments**

We are grateful to Stefano Massai for collaboration on this project. We furthermore thank Ismail Achmed-Zade, Daniel Junghans and Felix Rudolph for valuable discussions. This work was partially supported by the ERC Advanced Grant “Strings and Gravity” (Grant. No. 320045) and by the DFG cluster of excellence “Origin and Structure of the Universe”.

21
In this appendix, we briefly consider the case in which the space transversal to the monopoles is $\mathbb{R} \times \mathbb{T}^3$. Let us begin with the NS5-brane, for which the exact metric can be determined as before by considering a three-dimensional array of harmonic sources

$$h(y, \xi^1, \xi^2, \xi^3) = \frac{1}{4\pi R_1 R_2 R_3} \left[ |y| + \sum_{a=1}^{3} \sum_{\vec{k} \in \mathbb{Z}^3} \frac{1}{\lambda} e^{-\left(\frac{\vec{k} \cdot \vec{n}}{R_a} + \lambda |y|\right)} \right],$$

where $y$ is the coordinate on $\mathbb{R}$ and $\xi^i$ are coordinates on the $\mathbb{T}^3$ with periodicities $\xi^i \sim \xi^i + 2\pi R_i$. The sum does not converge, but can be regularized. After a Poisson resummation it becomes

$$h(y, \xi^1, \xi^2, \xi^3) = \frac{1}{4\pi R_1 R_2 R_3} \left[ |y| + \sum_{a=1}^{3} \sum_{\vec{k} \in \mathbb{Z}^3} \frac{1}{\lambda} e^{-\left(\frac{\vec{k} \cdot \vec{n}}{R_a} + \lambda |y|\right)} \right],$$

where

$$\lambda = \sqrt{\left(\frac{k_1}{R_1}\right)^2 + \left(\frac{k_2}{R_2}\right)^2 + \left(\frac{k_3}{R_3}\right)^2}. \quad (A.3)$$

In the limit $|y| \gg R_1, R_2, R_3$ the branes are smeared on the torus and the solution is simply

$$ds^2 = h(y) \left[ dy^2 + (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2 \right],
$$

$$H = -\frac{\text{sgn}(y)}{4\pi R_1 R_2 R_3} d\xi^1 \wedge d\xi^2 \wedge d\xi^3, \quad e^{2\Phi} = h(y), \quad (A.4)$$

with $h(y) = |y|/(4\pi R_1 R_2 R_3)$. A T-duality along say the $\xi^2$-direction gives a background of the form $\mathbb{R} \times \text{Nil}_3$, which is in fact a Taub-NUT space $[2,2]$ with special circle $\xi^2$, where two of the base directions have been compactified. The smeared limit is given by the metric

$$ds^2 = h(y) \left[ dy^2 + (d\xi^1)^2 + (d\xi^3)^2 \right] + \frac{1}{h(y)} \left[ d\xi^2 - \frac{\text{sgn}(y)}{4\pi R_1 R_2 R_3} \xi^3 d\xi^1 \right]^2. \quad (A.5)$$

At fixed $y$, there is a torus parametrized by $(\xi^1, \xi^2)$ which is fibered over the cycle corresponding to $\xi^3$, with monodromy given by a Dehn twist around $\xi^1$ resulting in a nilmanifold as a total space. For a detailed discussion of the domain-wall metric $[A.5]$ see for example $[49,8]$. As for the elliptic cases discussed in the main text, an additional T-duality along say the $\xi^1$-direction gives a solution in which, at fixed $y$, there is a non-geometric T-fold given by the $(\xi^1, \xi^2)$-torus fibration over $\xi^3$ with monodromy $1/\rho \rightarrow 1/\rho - 1$. This T-fold solution has no additional isometry, but it has been speculated from considerations in the effective reduced theory that a last T-duality is possible, giving a so-called R-space $[50,34]$. From the present perspective it seems that their physics in the closed-string sector would be essentially determined by the T-dual of the massive modes in $[A.4]$. 

22
B  \(B\)-fields of the localized NS5-brane

For completeness, in this appendix we give the full expressions of the \(B\)-field for the localized NS5-brane solutions.

\(\text{NS5 on } \mathbb{R}^3 \times S^1\)

Consider an NS5-brane on \(\mathbb{R}^3 \times S^1\). Using spherical coordinates for the non-compact transversal space of the form

\[
(\vec{x}, \xi^2) = (|\vec{x}| \sin \theta \cos \varphi, |\vec{x}| \sin \theta \sin \varphi, |\vec{x}| \cos \theta, \xi^2),
\]

the \(B\)-field up to gauge transformations reads

\[
B = \frac{1 - \cos \theta}{2R_2} d\varphi \wedge d\xi^2 + B_{\theta \varphi} d\theta \wedge d\varphi,
\]

with

\[
B_{\theta \varphi} = \sin \theta \left[ \tan^{-1} \left( \frac{\coth |\vec{x}|}{2R_2} \tan \frac{\xi^2}{2R_2} \right) - \frac{\xi^2}{2R_2} + \frac{|\vec{x}| \sin \left( \frac{\xi^2}{R_2} \right)}{2R_2 \cosh \left( \frac{|\vec{x}|}{R_2} \right) - 2R_2 \cos \left( \frac{\xi^2}{R_2} \right)} \right].
\]

In the semi-flat limit, \(B_{\theta \varphi} \to 0\) and the \(B\)-field simplifies to

\[
B^{sf} = \frac{1 - \cos \theta}{2R_2} d\varphi \wedge d\xi^2.
\]

\(\text{NS5 on } \mathbb{R}^2 \times \mathbb{T}^2\)

For an NS5-brane on \(\mathbb{R}^2 \times \mathbb{T}^2\) we have the following expression for the \(B\)-field

\[
B = \frac{\theta}{2\pi R_1 R_2} d\xi^1 \wedge d\xi^2 + \Pi_1 d\theta \wedge d\xi^1 + \Pi_2 d\theta \wedge d\xi^2,
\]

where \(\Pi_{1,2}\) satisfy the equations

\[
\partial_r \Pi_1 = r \partial_\varphi h, \quad \partial_r \Pi_2 = -r \partial_\varphi h, \quad (\partial_\varphi \Pi_2 - \partial_\varphi \Pi_1) = r \partial_\varphi h + (2\pi R_1 R_2)^{-1},
\]

where \(h\) is the localized harmonic function (2.29). These equations are solved by

\[
\Pi_1(r, \xi^1, \xi^2) = + \sum_{k_1, k_2 \geq 0} \frac{(2 - \delta_{k_1,0} - \delta_{k_2,0})}{\pi R_1 R_2} \frac{k_2}{\lambda} r K_1(\lambda r) \cos \left( \frac{k_1 \xi^1}{R_1} \right) \sin \left( \frac{k_2 \xi^2}{R_2} \right),
\]

\[
\Pi_2(r, \xi^1, \xi^2) = - \sum_{k_1, k_2 \geq 0} \frac{(2 - \delta_{k_1,0} - \delta_{k_2,0})}{\pi R_1 R_2} \frac{k_1}{\lambda} r K_1(\lambda r) \sin \left( \frac{k_1 \xi^1}{R_1} \right) \cos \left( \frac{k_2 \xi^2}{R_2} \right),
\]

with \(K_1\) the first order modified Bessel function of second kind and \(\lambda = \sqrt{(k_1/R_1)^2 + (k_2/R_2)^2}\).
References

[1] C. Hull, “A Geometry for non-geometric string backgrounds,” JHEP 0510 (2005) 065, hep-th/0406102.

[2] B. R. Greene, A. D. Shapere, C. Vafa, and S.-T. Yau, “Stringy Cosmic Strings and Noncompact Calabi-Yau Manifolds,” Nucl. Phys. B337 (1990) 1.

[3] S. Hellerman, J. McGreevy, and B. Williams, “Geometric constructions of nongeometric string theories,” JHEP 0401 (2004) 024, hep-th/0208174.

[4] S. Hellerman and J. Walcher, “Worldsheet CFTs for Flat Monodrofolds,” hep-th/0604191.

[5] A. Strominger, S.-T. Yau, and E. Zaslow, “Mirror symmetry is T duality,” Nucl. Phys. B479 (1996) 243–259, hep-th/9606040.

[6] N. Obers and B. Pioline, “U duality and M theory,” Phys. Rept. 318 (1999) 113–225, hep-th/9809039.

[7] J. de Boer and M. Shigemori, “Exotic Branes in String Theory,” Phys. Rept. 532 (2013) 65–118, 1209.6056.

[8] D. Lüst, S. Massai, and V. Vall Camell, “The monodromy of T-folds and T-fects,” JHEP 09 (2016) 127, 1508.01193.

[9] I. García-Etxebarria, D. Lüst, S. Massai, and C. Mayrhofer, “Ubiquity of non-geometry in heterotic compactifications,” 1611.10291.

[10] R. Gregory, J. A. Harvey, and G. W. Moore, “Unwinding strings and t duality of Kaluza-Klein and h monopoles,” Adv. Theor. Math. Phys. 1 (1997) 283–297, hep-th/9708086.

[11] C. Hull and B. Zwiebach, “Double Field Theory,” JHEP 0909 (2009) 099, 0904.4664.

[12] G. Aldazabal, D. Marques, and C. Nunez, “Double Field Theory: A Pedagogical Review,” Class. Quant. Grav. 30 (2013) 163001, 1305.1907.

[13] D. S. Berman and D. C. Thompson, “Duality Symmetric String and M-Theory,” Phys. Rept. 566 (2014) 1–60, 1306.2643.

[14] O. Hohm, D. Lüst, and B. Zwiebach, “The Spacetime of Double Field Theory: Review, Remarks, and Outlook,” Fortsch. Phys. 61 (2013) 926–966, 1309.2977.

[15] J. McOrist, D. R. Morrison, and S. Sethi, “Geometries, Non-Geometries, and Fluxes,” Adv. Theor. Math. Phys. 14 (2010) 0404.5447.
A. Font, I. García-Etxebarria, D. Lüst, S. Massai, and C. Mayrhofer, “Heterotic T-fects, 6D SCFTs, and F-Theory,” *JHEP* **08** (2016) 175, [1603.09361](http://arxiv.org/abs/1603.09361).

G. Lopes Cardoso, G. Curio, D. Lüst, and T. Mohaupt, “On the duality between the heterotic string and F theory in eight-dimensions,” *Phys. Lett. B* **389** (1996) 479–484, [hep-th/9609111](http://arxiv.org/abs/hep-th/9609111).

H. Ooguri and C. Vafa, “Summing up D instantons,” *Phys.Rev.Lett.* **77** (1996) 3296–3298, [hep-th/9608079](http://arxiv.org/abs/hep-th/9608079).

M. Gross and P. M. H. Wilson, “Large Complex Structure Limits of K3 Surfaces,” *J. Differential Geom.* **55** (07, 2000) 475–546, [math/0008018](http://arxiv.org/abs/math/0008018).

N. Seiberg and S. H. Shenker, “Hypermultiplet moduli space and string compactification to three-dimensions,” *Phys. Lett. B* **388** (1996) 521–523, [hep-th/9608086](http://arxiv.org/abs/hep-th/9608086).

K. Becker and S. Sethi, “Torsional Heterotic Geometries,” *Nucl.Phys. B* **820** (2009) 1–31, [0903.3769](http://arxiv.org/abs/0903.3769).

D. Gaiotto, G. W. Moore, and A. Neitzke, “Four-dimensional wall-crossing via three-dimensional field theory,” *Commun. Math. Phys.* **299** (2010) 163–224, [0807.4723](http://arxiv.org/abs/0807.4723).

A. Sen, “Kaluza-Klein dyons in string theory,” *Phys. Rev. Lett.* **79** (1997) 1619–1621, [hep-th/9705212](http://arxiv.org/abs/hep-th/9705212).

T. W. Grimm, D. Klevers, and M. Poretschkin, “Fluxes and Warping for Gauge Couplings in F-theory,” *JHEP* **01** (2013) 023, [1202.0285](http://arxiv.org/abs/1202.0285).

H. Ooguri and C. Vafa, “Two-dimensional black hole and singularities of CY manifolds,” *Nucl.Phys. B* **463** (1996) 55–72, [hep-th/9511164](http://arxiv.org/abs/hep-th/9511164).

D. Kutasov, “Orbifolds and solitons,” *Phys. Lett. B* **383** (1996) 48–53, [hep-th/9512145](http://arxiv.org/abs/hep-th/9512145).

T. Kimura, S. Sasaki, and M. Yata, “Hyper-Kähler with torsion, T-duality, and defect (p, q) five-branes,” *JHEP* **03** (2015) 076, [1411.3457](http://arxiv.org/abs/1411.3457).

D.-E. Diaconescu and N. Seiberg, “The Coulomb branch of (4,4) supersymmetric field theories in two-dimensions,” *JHEP* **07** (1997) 001, [hep-th/9707158](http://arxiv.org/abs/hep-th/9707158).

J. P. Gauntlett, J. A. Harvey, and J. T. Liu, “Magnetic monopoles in string theory,” *Nucl. Phys. B* **409** (1993) 363–381, [hep-th/9211056](http://arxiv.org/abs/hep-th/9211056).

J. A. Harvey and S. Jensen, “Worldsheet instanton corrections to the Kaluza-Klein monopole,” *JHEP* **10** (2005) 028, [hep-th/0507204](http://arxiv.org/abs/hep-th/0507204).

S. Jensen, “The KK-Monopole/NS5-Brane in Doubled Geometry,” *JHEP* **07** (2011) 088, [1106.1174](http://arxiv.org/abs/1106.1174).
[32] D. M. Belov, C. M. Hull, and R. Minasian, “T-duality, gerbes and loop spaces,” 0710.5151

[33] E. Plauschinn, “On T-duality transformations for the three-sphere,” Nucl.Phys. B893 (2015) 257–286, 1408.1715

[34] F. Hassler and D. Lüst, “Non-commutative/non-associative IIA (IIB) Q- and R-branes and their intersections,” JHEP 1307 (2013) 048, 1303.1413.

[35] T. Kimura and S. Sasaki, “Gauged Linear Sigma Model for Exotic Five-brane,” Nucl. Phys. B876 (2013) 493–508, 1304.4061.

[36] T. Kimura and S. Sasaki, “Worldsheet instanton corrections to 52-brane geometry,” JHEP 08 (2013) 126, 1305.4439.

[37] P. Bouwknegt, J. Evslin, and V. Mathai, “T duality: Topology change from H flux,” Commun. Math. Phys. 249 (2004) 383–415, hep-th/0306062.

[38] C. M. Hull and R. A. Reid-Edwards, “Non-geometric backgrounds, doubled geometry and generalised T-duality,” JHEP 09 (2009) 014, 0902.4032.

[39] P. Candelas, A. Constantin, C. Damian, M. Larfors, and J. F. Morales, “Type IIB flux vacua from G-theory I,” JHEP 02 (2015) 187, 1411.4785.

[40] P. Candelas, A. Constantin, C. Damian, M. Larfors, and J. F. Morales, “Type IIB flux vacua from G-theory II,” JHEP 02 (2015) 188, 1411.4786.

[41] D. S. Berman and F. J. Rudolph, “Branes are Waves and Monopoles,” JHEP 05 (2015) 015, 1409.6314.

[42] D. S. Berman and F. J. Rudolph, “Strings, Branes and the Self-dual Solutions of Exceptional Field Theory,” JHEP 05 (2015) 130, 1412.2768.

[43] I. Bakhmatov, A. Kleinschmidt, and E. T. Musaev, “Non-geometric branes are DFT monopoles,” JHEP 10 (2016) 076, 1607.05450.

[44] O. Hohm, C. Hull, and B. Zwiebach, “Generalized metric formulation of double field theory,” JHEP 08 (2010) 008, 1006.4823.

[45] A. Giveon and M. Rocek, “Generalized duality in curved string backgrounds,” Nucl. Phys. B380 (1992) 128–146, hep-th/9112070.

[46] D. Israel, C. Kounnas, A. Pakman, and J. Troost, “The Partition function of the supersymmetric two-dimensional black hole and little string theory,” JHEP 06 (2004) 033, hep-th/0403237.

[47] E. J. Martinec and S. Massai, “String Theory of Supertubes,” 1705.10844.
[48] E. Witten, “Branes, Instantons, And Taub-NUT Spaces,” *JHEP* 06 (2009) 067, 0902.0948

[49] G. W. Gibbons and P. Rychenkova, “Single sided domain walls in M theory,” *J. Geom. Phys.* 32 (2000) 311–340, hep-th/9811045

[50] J. Shelton, W. Taylor, and B. Wecht, “Nongeometric flux compactifications,” *JHEP* 0510 (2005) 085, hep-th/0508133