A multi-step approximant for fixed point problem and convex optimization problem in Hadamard spaces

Muhammad Aqeel Ahmad Khan∗ and Hafiza Arham Maqbool
Department of Mathematics, COMSATS Institute of Information Technology, Lahore 54000, Pakistan
February 27, 2018

Abstract: The purpose of this paper is to propose and analyze a multi-step iterative algorithm to solve a convex optimization problem and a fixed point problem posed on a Hadamard space. The convergence properties of the proposed algorithm are analyzed by employing suitable conditions on the control sequences of parameters and the structural properties of the underlying space. We aim to establish strong and △-convergence results of the proposed iterative algorithm and compute an optimal solution for a minimizer of proper convex lower semicontinuous function and a common fixed point of a finite family of total asymptotically nonexpansive mappings in Hadamard spaces. Our results can be viewed as an extension and generalization of various corresponding results established in the current literature.

Keywords and Phrases: Convex optimization, Lower Semicontinuity, Proximal point algorithm, Total asymptotically nonexpansive mapping, Common fixed point, Asymptotic center.

2010 Mathematics Subject Classification: 47H09, 47H10, 65K10, 65K15.

1. Introduction

The theory of nonlinear analysis is mainly divided into three major areas, namely convex analysis, monotone operator theory and fixed point theory of nonlinear mappings. These theories have been largely developed in the abstract setting of spaces having linear structures such as Euclidean, Hilbert and Banach spaces. The theory of optimization, in particular, convex optimization is prominent in the theory of convex analysis which studies the properties of minimizers and maximizers of the under consideration functions. The analysis of such properties rely on various mathematical tools, topological notions and geometric ideas. Convex optimization not only provides a theoretical setting for the existence and uniqueness of a solution to a given optimization problem but also provides efficient iterative algorithms to construct the optimal solution for such an optimization problem. As a consequence, convex optimization solves a variety of problems arising in disciplines such as mathematical economics, approximation theory, game theory, optimal transport theory, probability and statistics, information theory, signal and image processing and partial differential equations, see, for example [1, 14, 15, 33, 34] and the references cited therein.

One of the major problems in optimization theory is to find a minimizer of a convex function. The class of proximal point algorithms (PPA) contributes significantly to the theory of convex optimization as to compute a minimizer of a convex lower semicontinuous (lsc) function. In 1970, Martinet [29] proposed and analyzed the initial draft of PPA as a sequence of successive approximation of resolvents. In
1976, Rockafellar [32] generally established, by the PPA, the convergence characteristics to a zero of a maximal monotone operator in Hilbert spaces. Brezis and Lions [7] improved the Rockafellar’s algorithm under a weaker condition on the parameters. The result established in [32] develops an interesting interplay between convex analysis, monotone operator theory and fixed point theory of nonlinear mappings. As a consequence, the PPA becomes an efficient tool for solving optimization problems, fixed point problems, variational inequality problems and zeros of maximal monotone operators. On the other hand, Rockafellar [32] posed an open question regarding the strong convergence characteristics of the PPA. The answer to the open question was settled in negative with a counterexample given by Güler [18]. In order to establish strong convergence of the PPA, one has to impose additional assumptions on the PPA, see for example [6, 9, 21, 36]. It is worth mentioning that the counterexamples for strong convergence of the PPA are still very rare and weak convergence is the best we can achieve without additional assumptions.

Since most of the results in the theory of optimization involving PPA and its various modifications are established within the spaces having linear structure such as Euclidean space, Hilbert space and Banach space. It is therefore natural to extend such beautiful and strong results from the linear domain to the corresponding nonlinear domain. Another motivation for this research direction is that various optimization problems, which are non-convex in nature, become convex with the introduction of an adequate metric defined on the under consideration spaces. Such metrics can also be used to define new algorithms for optimization. Moreover, computation of minimizers of the under consideration convex functions in such spaces plays a pivotal role in the fields of nonlinear analysis and geometry [19, 20]. It is worth to mention that some efforts have been made to generalize such results from the linear spaces to nonlinear spaces having non-positive sectional curvature, see, for example, [5, 12, 13, 16, 28, 31, 35] and the references cited therein. This research area is still open either to establish new convergence results for the class of PPA or to translate the existing linear version of a result into the corresponding nonlinear version in such spaces.

The outline of the paper is as follows: In Section 2, we first define the conventions to be held throughout the paper and then define the consequent notions, concepts and necessary results in the form of lemmas as required in the sequel. Section 3 is devoted for the convergence analysis of the proposed multi-step PPA to solve a convex optimization problem and a fixed point problem posed on a Hadamard space.

2. Preliminaries

This section is devoted to recall some fundamental definitions, properties and notations concerned with the fixed point problem and convex optimization problem in Hadamard spaces. We also list some useful results in the form of lemmas as required in the sequel. Throughout this paper, we write \( x_n \to x \) (resp. \( x_n \to x \)) to indicate the strong convergence (resp. the weak convergence) of a sequence \( \{x_n\}_{n=1}^{\infty} \). The set of fixed points of a self-mapping \( T \) on a nonempty subset \( C \) of a metric space \( (X, d) \) is defined and denoted as: \( F(T) = \{ x \in C : T(x) = x \} \).

Let \( (X, d) \) be a metric space and \( x, y \in X \) with \( l = d(x, y) \). A geodesic from \( x \) to \( y \) in \( X \) is a mapping \( \theta : [0, l] \to X \) such that

\[
\theta(0) = x, \quad \theta(l) = y \quad \text{and} \quad d(\theta(s), \theta(t)) = |s - t| \quad \text{for all} \quad s, t \in [0, l].
\]
The above characteristics show that $\theta$ is an isometry and $x = \theta(0)$ and $y = \theta(l)$ represent the end points of the geodesic segment. The metric space $(X, d)$ is called a geodesic space if for every pair of points $x, y \in X$, there is a geodesic segment from $x$ to $y$. Moreover, $(X, d)$ is uniquely geodesic if for all $x, y \in X$ there is exactly one geodesic from $x$ to $y$. A unique geodesic segment from $x$ to $y$ is denoted as $[x, y]$. A geodesic triangle $\triangle (x_1, x_2, x_3)$ in a geodesic metric space $(X, d)$ consists of three points $x_1, x_2, x_3$ in $X$ (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle (x_1, x_2, x_3)$ in $(X, d)$ is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle (\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean space $\mathbb{E}^2$ such that $d_{\mathbb{E}^2} (\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for each $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a $\text{CAT}(0)$ space if it is geodesically connected and if every geodesic triangle in $(X, d)$ is at least as thin as its comparison triangle in the Euclidean plane, that is $d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y})$. A complete $\text{CAT}(0)$ space is then called a Hadamard space. A nonempty subset $C$ of a $\text{CAT}(0)$ space is said to be convex if $[x, y] \subset C$. For a detailed discussion on this topic, we refer the reader to consult [8, 10].

It is well known that a geodesic space is a $\text{CAT}(0)$ space if and only if
\[ d^2((1-t)x \oplus ty, z) \leq (1-t)d^2(x, z) + td^2(y, z) - t(1-t)d^2(x, y), \]
for all $x, y, z \in X$ and $t \in [0, 1]$. In particular, if $x, y$ and $z$ are points in a $\text{CAT}(0)$ space and $t \in [0, 1]$, then
\[ d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z). \]

A self-mapping $T : C \rightarrow C$ is said to be total asymptotically nonexpansive mapping [2] if there exists non-negative real sequences $\{k_n\}$ and $\{\varphi_n\}$ with $k_n \rightarrow 0$ and $\varphi_n \rightarrow 0$ as $n \rightarrow \infty$ and a strictly increasing continuous function $\xi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\xi(0) = 0$ such that
\[ d(T^n x, T^n y) \leq d(x, y) + k_n \xi(d(x, y)) + \varphi_n \text{ for all } x, y \in C, \ n \geq 1. \]

The class of total asymptotically nonexpansive mappings is the most general class of nonlinear mappings and contains properly various classes of mappings associated with the class of asymptotically nonexpansive mappings. These classes of nonlinear mappings have been studied extensively in the literature [17, 23, 24, 26] and the references cited therein. It is worth mentioning that the results established for total asymptotically nonexpansive mappings are applicable to the mappings associated with the class of asymptotically nonexpansive mappings and which are extensions of nonexpansive mappings.

It is well known that the concept of weak convergence in Hilbert spaces has been generalized to $\text{CAT}(0)$ spaces as $\triangle$-convergence. Moreover, many useful results from linear spaces involving weak convergence have precise analogue version of $\triangle$-convergence in geodesic spaces. The notion of asymptotic center of a sequence plays a key role to define the concept of $\triangle$-convergence in such spaces.

Let $\{x_n\}$ be a bounded sequence in a $\text{CAT}(0)$ $X$. For $x \in X$, define a continuous functional $r(\cdot, \{x_n\}) : X \rightarrow [0, \infty)$ by:
\[ r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n). \]
The asymptotic radius and asymptotic center of the bounded sequence \( \{x_n\} \) with respect to a subset \( C \) of \( X \) is defined and denoted as:

\[
r_C(\{x_n\}) = \inf \{ r(x, \{x_n\}) : x \in C \},
\]

and

\[
A_C(\{x_n\}) = \{ x \in C : r(x, \{x_n\}) \leq r(y, \{x_n\}) \text{ for all } y \in C \},
\]

respectively.

Recall that a sequence \( \{x_n\} \) in \( X \) is said to \( \Delta \)-converge to \( x \in X \) if \( x \) is the unique asymptotic center of \( \{u_n\} \) for every subsequence \( \{u_n\} \) of \( \{x_n\} \). In this case, we write \( \triangleq \lim_{n \to \infty} x_n = x \) and call \( x \) the \( \Delta \)-limit of \( \{x_n\} \).

A mapping \( T : C \to C \) is:

(i) semi-compact if every bounded sequence \( \{x_n\} \subseteq C \) satisfying \( d(x_n, Tx_n) \to 0 \), has a convergent subsequence; (ii) demiclosed at origin if for any sequence \( \{x_n\} \) in \( C \) with \( x_n \to x \) and \( \|x_n - Tx_n\| \to 0 \), we have \( x = Tx \). Let \( g \) be a nondecreasing self-mapping on \([0, \infty)\) with \( g(0) = 0 \) and \( g(t) > 0 \) for all \( t \in (0, \infty) \). Let \( \{T_i\}_{i=1}^m \) be a finite family of total asymptotically nonexpansive mappings on \( C \) with \( \cap_{i=1}^m F(T_i) \neq \emptyset \). Then the family of mappings is said to satisfy Condition (I) on \( C \) if:

\[
\|x - Tx\| \geq f(d(x, F)), \text{ for all } x \in C,
\]

holds for at least one \( T \in \{T_i\}_{i=1}^m \).

We now collect some basic concepts related to convex optimization in \( \text{CAT}(0) \) spaces:

Let \( C \) be a nonempty subset of a \( \text{CAT}(0) \) space \( X \), then a function \( f : C \to (-\infty, \infty] \) is said be convex if for any geodesic \( \theta : [a, b] \to C \) the function \( f \circ \theta \) is convex. Some important examples of convex function in \( \text{CAT}(0) \) spaces can be found in [8]. A function \( f \) defined on \( C \) is said to be lsc at a point \( x \in C \) if \( f(x) \leq \liminf_{n \to \infty} f(x_n) \), for each sequence \( x_n \to x \). A function \( f \) is said to be lsc on \( C \) if it is lsc at any point in \( C \). A convex minimization problem associated with a proper and convex function is to solve \( x \in C \) such that

\[
f(x) = \min_{y \in C} f(y).
\]

We denote by \( \arg \min_{y \in C} f(y) \) by the set of a minimizer of a convex function. For all \( k > 0 \), define the Moreau-Yosida resolvent of \( f \) in a complete \( \text{CAT}(0) \) space \( X \) as follows:

\[
J_k(x) = \arg \min_{y \in C} \left[ f(y) + \frac{1}{2k}d^2(y, x) \right],
\]

and put \( J_0(x) = x \) for all \( x \in X \). This definition in metric spaces with no linear structure first appeared in [13], see also [19]. The mapping \( J_k \) is well defined for all \( k \geq 0 \) (see [18] [19] [26]). For a proper, convex and lsc function, the set of fixed points of the resolvent \( J_k \) associated with \( f \) coincides with the set of minimizers of \( f \) [4]. Moreover, the resolvent \( J_k \) of \( f \) is nonexpansive for all \( k > 0 \) [19]. Some other relevant characteristics of the resolvent \( J_k \) of \( f \) are incorporated in the following couple of lemmas:

**Lemma 2.1** (Sub-differential Inequality) [3]. Let \( (X, d) \) be a complete \( \text{CAT}(0) \) space and \( f : X \to (-\infty, \infty] \) be a proper convex and lsc function. Then, for all \( x, y \in X \) and \( k > 0 \), we have:

\[
\frac{1}{2k}d^2(J_kx, y) - \frac{1}{2k}d^2(x, y) + \frac{1}{2k}d^2(J_kx, x) + f(J_kx) \leq f(y).
\]
Lemma 2.2 (The Resolvent Identity) [19, 26]. Let \((X, d)\) be a complete \(CAT(0)\) space and \(f : X \to (-\infty, \infty]\) be a proper convex and lsc function. Then, the following identity holds:

\[ J_k x = J_{\mu} \left( \frac{k - \eta}{k} J_k x \oplus \frac{\eta}{k} x \right), \]

for all \(x \in X\) and \(k > \eta > 0\).

We also require the following useful lemma for our main result.

Lemma 2.3 [36]. Let \(\{a_n\}, \{b_n\}\) and \(\{c_n\}\) be sequences of non-negative real numbers such that \(\sum_{n=1}^{\infty} b_n < \infty\) and \(\sum_{n=1}^{\infty} c_n < \infty\). If \(a_{n+1} \leq (1 + b_n)a_n + c_n, n \geq 1\), then \(\lim_{n \to \infty} a_n\) exists.

Lemma 2.4 [22]. Let \((X, d, W)\) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity \(\eta\). Let \(x \in X\) and \(\{\alpha_n\}\) be a sequence in \([a, b]\) for some \(a, b \in (0, 1)\). If \(\{x_n\}\) and \(\{y_n\}\) are sequences in \(X\) such that \(\limsup\limits_{n \to \infty} d(x_n, x) \leq c\), \(\limsup\limits_{n \to \infty} d(y_n, x) \leq c\) and \(\lim\limits_{n \to \infty} d(W(x_n, y_n, \alpha_n), x) = c\) for some \(c \geq 0\), then \(\lim\limits_{n \to \infty} d(x_n, y_n) = 0\).

Lemma 2.5 [22]. Let \(K\) be a nonempty closed convex subset of a uniformly convex hyperbolic space and \(\{x_n\}\) a bounded sequence in \(K\) such that \(A_K(\{x_n\}) = \{y\}\) and \(r_K(\{x_n\}) = \rho\). If \(\{y_m\}\) is another sequence in \(K\) such that \(\lim_{m \to \infty} r(y_m, \{x_n\}) = \rho\), then \(\lim_{m \to \infty} y_m = y\).

3. MAIN RESULTS

We now prove a result in the form of lemma which plays a critical role to establish strong and \(\Delta\)-convergence results of the proposed iterative algorithm and compute an optimal solution for a minimizer of proper convex lower semicontinuous function and a common fixed point of a finite family of total asymptotically nonexpansive mappings in Hadamard spaces.

Lemma 3.1. Let \(C\) be a nonempty closed convex subset of a Hadamard space \(X\). Let \(f : X \to (-\infty, \infty]\) be a proper convex and lsc function and let \(\{T_i\}_{i=1}^{m} : C \to C\) be a finite family of uniformly continuous total asymptotically quasi nonexpansive mappings with sequences \(\{\lambda_{in}\}\) and \(\{\mu_{in}\}, n \geq 1, i = 1, 2, \cdots, m\), such that

\begin{enumerate}
  \item[(C1)] \(\sum\limits_{n=1}^{\infty} \lambda_{in} < \infty\) and \(\sum\limits_{n=1}^{\infty} \mu_{in} < \infty\);
  \item[(C2)] there exists constants \(M_i, M_i^* > 0\) such that \(\xi_i(\theta_i) \leq M_i^* \theta_i\) for all \(\theta_i \geq M_i\).
\end{enumerate}
Let \( \{x_n\} \) be a sequence generated in the following manner:

\[
\begin{align*}
  x_1 &\in C, \\
x_{n+1} &= (1 - \alpha_n)y_{1n} \oplus \alpha_n T_1^n y_{1n}, \\
y_{1n} &= (1 - \alpha_n)y_{2n} \oplus \alpha_n T_2^n y_{2n}, \\
&\vdots \\
y_{in} &= (1 - \alpha_n)y_{(i+n)n} \oplus \alpha_n T_{i+n}^n y_{(i+n)n}, \\
y_{(m-1)n} &= (1 - \alpha_n)z_n \oplus \alpha_n T_m^n z_n, \\
z_n &= \arg \min_{y \in C} [f(y) + \frac{1}{2k}d(y, x_n)], \quad n \geq 1, \\
\end{align*}
\]

(3.1)

where \( \{\alpha_n\} \) is a sequence in \([0, 1]\) with \(0 < \alpha < \alpha_n \leq b < 1\) for all \(n \geq 1\) and for some constant \(a, b\) in \((0, 1)\). Assume that

\[
\mathbb{F} = \left( \bigcap_{i=1}^m F(T_i) \right) \cap \arg \min_{y \in C} f(y) \neq \emptyset,
\]

then, we have the following:

(i) \( \lim_{n \to \infty} d(x_n, p) \) exists for all \(p \in \mathbb{F}\);
(ii) \( \lim_{n \to \infty} d(x_n, z_n) = 0\);
(iii) \( \lim_{n \to \infty} d(T_i x_n, x_n) = 0\), for each \(i = 1, 2, \cdots, m\).

**Proof:** Let \(p \in \mathbb{F}\), then \(p = T_i(p)\) for each \(i = 1, 2, \cdots, m\) and \(f(p) \leq f(y)\) for all \(y \in C\). This implies that

\[
f(p) + \frac{1}{2k}d^2(p, p) \leq f(y) + \frac{1}{2k}d^2(y, p),
\]

for each \(y \in C\). Hence \(p = J_{k_n}(p)\) for each \(n \geq 1\).

(i) Now, we first show that \(\lim_{n \to \infty} d(x_n, p)\) exists. Since \(z_n = J_{k_n} x_n\) and \(J_{k_n}\) is nonexpansive, therefore, we have

\[
d(z_n, p) = d(J_{k_n} x_n, J_{k_n} p) \leq d(x_n, p).
\]

(3.2)

It follows from (3.1) that

\[
d(y_{(m-1)n}, p) = d((1 - \alpha_n)z_n \oplus \alpha_n T_m^n z_n, p) \\
\leq (1 - \alpha_n)d(z_n, p) + \alpha_n d(T_m^n z_n, p) \\
\leq (1 - \alpha_n)d(z_n, p) + \alpha_n \{d(z_n, p) + \Lambda_{mn}(d(z_n, p)) + \mu_{mn}\}.
\]

Since \(\xi_m\) is an increasing function, therefore \(\xi_m (d(x_n, p)) \leq \xi_m (M_m)\) for \(d(x_n, p) \leq M_m\). Moreover \(\xi_m (d(x_n, p)) \leq d(x_n, p) M_m^*\) for \(d(x_n, p) \geq M_m\) (by C2). In either case, we have

\[
\xi_m (d(x_n, p)) \leq \xi_m (M_m) + d(x_n, p) M_m^*,
\]

where \(M_m, M_m^* > 0\). As a consequence, we get

\[
d(y_{(m-1)n}, p) \leq (1 + \alpha_n \Lambda_{mn} M_m^*)d(z_n, p) + \alpha_n \Lambda_{mn} \xi_m (M_{mn}) + \alpha_n \mu_{mn}. \quad (3.3)
\]

Let \(a_1 = \max \{\alpha_n, \alpha_n \xi_m (M_{mn}), \alpha_n M_m^*\} > 0\), the estimate (3.3) becomes

\[
d(y_{(m-1)n}, p) \leq (1 + a_1 \Lambda_{mn})d(x_n, p) + a_1 (\Lambda_{mn} + \mu_{mn}). \quad (3.4)
\]
Again, reasoning in the aforementioned manner, it follows from (3.1) that

\[ d(y_{(m-2)n}, p) = d\left( (1 - \alpha_n)y_{(m-1)n} \oplus \alpha_n T^m_{m-1} y_{(m-1)n}, p \right) \]

\[ \leq (1 - \alpha_n)d(y_{(m-1)n}, p) + \alpha_n d(T^m_{m-1} y_{(m-1)n}, p) \]

\[ \leq (1 - \alpha_n)d(y_{(m-1)n}, p) + \alpha_n \{ d(y_{(m-1)n}, p) + \lambda_{(m-1)n} \xi_{m-1}(d(y_{(m-1)n}, p)) + \mu_{(m-1)n} \} \]

\[ \leq \left( 1 + \alpha_n \lambda_{(m-1)n} M^*_{(m-1)n} \right) d(y_{(m-1)n}, p) + \alpha_n \lambda_{(m-1)n} \xi_{m-1}(M_{(m-1)n}) + \alpha_n \mu_{(m-1)n}. \]

Utilizing (3.3) in the above estimate and simplifying the terms, we have

\[ d(y_{(m-2)n}, p) \leq \left( 1 + \alpha_n \lambda_{mn} M^*_{mn} + \left( \alpha_n M^*_{(m-1)n} + \alpha_n^2 \lambda_{mn} M^*_{mn} M^*_{(m-1)n} \right) \lambda_{(m-1)n} \right) d(x_n, p) \]

\[ + \alpha_n \lambda_{mn} \xi_{m}(M_{mn}) + \alpha_n^2 \lambda_{mn} \lambda_{(m-1)n} \xi_{m}(M_{mn}) M^*_{(m-1)n} + \alpha_n \mu_{mn} \]

\[ + \alpha_n \mu_{(m-1)n} + \alpha_n^2 \lambda_{(m-1)n} M^*_{(m-1)n} \mu_{mn} + \alpha_n \lambda_{(m-1)n} \xi_{m-1}(M_{(m-1)n}). \] (3.5)

Let \( c_1, c_2 > 0 \) be such that \( \lambda_{mn} \leq c_1 \) and \( \lambda_{(m-1)n} \leq c_2 \) for all \( n \geq 1 \). Then for \( \alpha_n \leq b \), the estimate (3.5) simplifies as

\[ d(y_{(m-2)n}, p) \leq \left( 1 + b \lambda_{mn} M^*_{mn} + \left( b M^*_{(m-1)n} + b c_1 M^*_{mn} M^*_{(m-1)n} \right) \lambda_{(m-1)n} \right) d(x_n, p) \]

\[ + b \lambda_{mn} \xi_{m}(M_{mn}) + b c_2 \lambda_{mn} \xi_{m}(M_{mn}) M^*_{(m-1)n} + b \mu_{mn} + b \mu_{(m-1)n} \]

\[ + b c_2 M^*_{(m-1)n} \mu_{mn} + b \lambda_{(m-1)n} \xi_{m-1}(M_{(m-1)n}). \]

Similarly, let \( a_2 = \max\{ b, b M^*_{mn}, b M^*_{(m-1)n} + c_1 M^*_{mn} M^*_{(m-1)n}, b(c_2 \xi_{m}(M_{mn}) M^*_{(m-1)n} + \xi_{m}(M_{mn})), b(1 + c_2 M^*_{(m-1)n}), b \xi_{m-1}(M_{(m-1)n}) \} > 0 \). Then the above estimate becomes

\[ d(y_{(m-2)n}, p) \leq \left( 1 + a_2 \sum_{i=m-1}^{m} \lambda_{in} \right) d(x_n, p) + a_2 \sum_{i=m-1}^{m} (\lambda_{in} + \mu_{in}). \] (3.6)

Continuing in the similar fashion, for any \( m \geq 1 \), we have

\[ d(x_{n+1}, p) \leq \left( 1 + a_m \sum_{i=1}^{m} \lambda_{in} \right) d(x_n, p) + a_m \sum_{i=1}^{m} (\lambda_{in} + \mu_{in}), \] (3.7)

for some constant \( a_m > 0 \).

It now follows from (C1) and Lemma 2.3 that \( \lim_{n \to \infty} d(x_n, p) \) exists for all \( p \in \mathbb{F} \). This completes the proof of part (i).

(ii). In order to proceed for part (ii), we assume, without loss of any generality, that

\[ \lim_{n \to \infty} d(x_n, p) = r \geq 0. \] (3.8)

Taking \( \limsup \) on both sides of the estimate (3.2), we have

\[ \limsup_{n \to \infty} d(z_n, p) \leq r. \] (3.9)

Consider the following variant of the estimate (3.7)

\[ d(x_{n+1}, p) \leq \left( 1 + a_m \sum_{i=1}^{m} \lambda_{in} \right) d(z_n, p) + a_m \sum_{i=1}^{m} (\lambda_{in} + \mu_{in}). \]

Applying \( \liminf \) on both sides of the above estimate, we get

\[ \liminf_{n \to \infty} d(z_n, p) \geq r. \] (3.10)
The estimates (3.9) and (3.10) collectively imply that
\[ \lim_{n \to \infty} d(z_n, p) = r. \] (3.11)

Now, from Lemma 2.1, we have
\[ \frac{1}{2k_n}[d^2(z_n, p) - d^2(x_n, p) + d^2(x_n, z_n)] \leq f(p) - f(z_n). \]

Since \( f(p) \leq f(z_n) \) for each \( n \geq 1 \), it follows that
\[ d^2(x_n, z_n) \leq d^2(x_n, p) - d^2(z_n, p). \]

Utilizing (3.8) and (3.11), the above estimate implies that
\[ \lim_{n \to \infty} d(x_n, z_n) = 0. \] (3.12)

This completes the proof of part (ii).

(iii). We now establish asymptotic regularity of the sequence \( \{x_n\} \) involving a finite family of uniformly continuous total asymptotically quasi nonexpansive mappings.

Consider the following another variant of the estimate (3.7)
\[ d(x_{n+1}, p) \leq \left(1 + a_m \sum_{i=1}^{m-1} \lambda_{in}\right) d(y_{(m-1)n}, p) + a_m \sum_{i=1}^{m-1} (\lambda_{in} + \mu_{in}). \]

Taking \( \lim \inf \) on both sides of the above estimate, we get
\[ \lim \inf_{n \to \infty} d(y_{(m-1)n}, p) \geq r. \] (3.13)

Moreover, taking \( \lim \sup \) on both sides of (3.4), we have
\[ \lim \sup_{n \to \infty} d(y_{(m-1)n}, p) \leq r. \] (3.14)

Hence, by (3.13) and (3.14), we obtain
\[ \lim_{n \to \infty} d(y_{(m-1)n}, p) = \lim_{n \to \infty} d((1 - \alpha_n)z_n \oplus \alpha_n T_m^n z_n, p) = r. \] (3.15)

It follows from the definition of \( T_m \) that \( \lim \sup_{n \to \infty} d(T_m^n z_n, p) \leq r \). Utilizing this fact together with (3.9) and (3.15), it then follows from Lemma 2.3 that
\[ \lim_{n \to \infty} d(z_n, T_m^n z_n) = 0. \] (3.16)

Now, observe the following variant of (3.7)
\[ d(x_{n+1}, p) \leq \left(1 + a_m \sum_{i=1}^{m-2} \lambda_{in}\right) d(y_{(m-2)n}, p) + a_m \sum_{i=1}^{m-2} (\lambda_{in} + \mu_{in}). \]

Taking \( \lim \inf \) on both sides of the above estimate, we get
\[ \lim \inf_{n \to \infty} d(y_{(m-2)n}, p) \geq r. \] (3.17)

Also, taking \( \lim \sup \) on both sides of the estimate (3.6), we have
\[ \lim \sup_{n \to \infty} d(y_{(m-2)n}, p) \leq r. \] (3.18)

Hence, by (3.17) and (3.18), we obtain
\[ \lim_{n \to \infty} d(y_{(m-2)n}, p) = \lim_{n \to \infty} d((1 - \alpha_n)y_{(m-1)n} \oplus \alpha_n T_{(m-1)}^n y_{(m-1)n}, p) = r. \] (3.19)
Again, it follows from the definition of $T_{m-1}$ that $\limsup_{n \to \infty} d(T_{m-1}^n y_{(m-1)n}, p, p) \leq r$. Utilizing this fact together with (3.15) and (3.19), it then follows from Lemma 2.3 that

$$\lim_{n \to \infty} d(y_{(m-1)n}, T_{m-1}^n y_{(m-1)n}) = 0.$$ 

Continuing in the similar fashion, we have

$$\lim_{n \to \infty} d(y_{in}, T_{in}^n y_{in}) = 0, \text{ for } i = 1, 2, \ldots, m - 1. \quad (3.20)$$

Note that $d(x_{n+1}, y_{in}) \leq b \cdot d(y_{in}, T_{in}^n y_{in})$. Therefore, letting $n \to \infty$ and utilizing (3.20), we get

$$\lim_{n \to \infty} d(x_{n+1}, y_{in}) = 0. \quad (3.21)$$

Moreover, $d(y_{in}, y_{(i+1)n}) \leq b \cdot d(y_{(i+1)n}, T_{(i+1)n}^n y_{(i+1)n})$, for $i = 1, 2, \ldots, m - 2$. Again, letting $n \to \infty$ and utilizing (3.20), we get

$$\lim_{n \to \infty} d(y_{in}, y_{(i+1)n}) = 0, \text{ for } i = 1, 2, \ldots, m - 2. \quad (3.22)$$

As a consequence of the estimates (3.21) and (3.22), we have

$$\lim_{n \to \infty} d(x_{n}, y_{in}) = 0 \text{ for } i = 1, 2, \ldots, m - 1. \quad (3.23)$$

Now, observe that

$$d(T_{m}^{n} x_{n}, x_{n}) \leq d(T_{m}^{n} x_{n}, T_{m}^{n} x_{n} z_{n}) + d(T_{m}^{n} z_{n}, z_{n}) + d(z_{n}, x_{n}) \leq L d(x_{n}, z_{n}) + d(T_{m}^{n} z_{n}, z_{n}) + d(z_{n}, x_{n}).$$

Letting $n \to \infty$ in the above estimate and utilizing (3.12) and (3.16), we have

$$\lim_{n \to \infty} d(T_{m}^{n} x_{n}, x_{n}) = 0. \quad (3.24)$$

Similarly

$$d(T_{m-1}^{n-1} x_{n}, x_{n}) \leq d(T_{m-1}^{n} x_{n}, T_{m-1}^{n-1} y_{(m-1)n}) + d(T_{m-1}^{n} y_{(m-1)n}, y_{(m-1)n}) + d(y_{(m-1)n}, x_{n}) \leq L d(x_{n}, y_{(m-1)n}) + d(T_{m-1}^{n} y_{(m-1)n}, y_{(m-1)n}) + d(y_{(m-1)n}, x_{n}).$$

Letting $n \to \infty$ in the above estimate and utilizing (3.20) and (3.23), we have

$$\lim_{n \to \infty} d(T_{m-1}^{n} x_{n}, x_{n}) = 0. \quad (3.25)$$

Continuing in the similar fashion, we get

$$\lim_{n \to \infty} d(T_{i}^{n} x_{n}, x_{n}) = 0 \text{ for } i = 1, 2, \ldots, m. \quad (3.26)$$

Now, utilizing the uniform continuity of $T_{i}$, the following estimate:

$$d(x_{n}, T_{i} x_{n}) \leq d(x_{n}, T_{i}^{n} x_{n}) + d(T_{i}^{n} x_{n}, T_{i} x_{n})$$

implies that

$$\lim_{n \to \infty} d(T_{i} x_{n}, x_{n}) = 0 \text{ for } i = 1, 2, \ldots, m. \quad (3.26)$$

This completes the proof.

**Theorem 3.2.** Let $C$ be a nonempty closed convex subset of a Hadamard space $X$. Let $f : X \to (-\infty, \infty]$ be a proper convex and lsc function and let $\{T_{i}\}_{i=1}^{m} : C \to C$ be a finite family of uniformly continuous total asymptotically quasi nonexpansive mappings with sequences $\{\lambda_{in}\}$ and $\{\mu_{in}\}$, $n \geq 1$, $i = 1, 2, \ldots, m$, such that

(C1) $\sum_{n=1}^{\infty} \lambda_{in} < \infty$ and $\sum_{n=1}^{\infty} \mu_{in} < \infty$;
Proof: In fact, it follows from (3.12) and Lemma 2.2, that

\[ \lim_{n \to \infty} u_n \text{ exists and is unique.} \]

Then the sequence \( \{x_n\} \) \( \Delta \)-converges to a common element of \( F \).

**Proof:** In fact, it follows from (3.12) and Lemma 2.2, that

\[
d(J_k x_n, x_n) \leq d(J_k x_n, z_n) + d(z_n, x_n)
\]

\[
\leq d(J_k x_n, J_k x_n) + d(z_n, x_n)
\]

\[
\leq d(J_k x_n, J_k \left( \frac{k_n - k}{k_n} J_k x_n + \frac{k}{k_n} x_n \right)) + d(z_n, x_n)
\]

\[
\leq d(x_n, (1 - \frac{k}{k_n}) J_k x_n) + d(z_n, x_n)
\]

\[
= \left( 1 - \frac{k}{k_n} \right) d(x_n, J_k x_n) + d(z_n, x_n)
\]

\[
\leq \left( 1 - \frac{k}{k_n} \right) d(x_n, z_n) + d(z_n, x_n)
\]

\[
\to 0 \text{ as } n \to \infty.
\]

Moreover, it follows from Lemma 3.1(i) that \( \lim_{n \to \infty} d(x_n, p) \) exists for all \( p \in F \), hence \( \{x_n\} \) is bounded and has a unique asymptotic center, that is, \( A_C(\{x_n\}) = \{x\} \). Let \( \{u_n\} \) be any subsequence of \( \{x_n\} \) such that \( A_C(\{u_n\}) = \{u\} \) and by Lemma 3.1(iii), we have \( \lim_{n \to \infty} d(T_i u_n, u_n) = 0 \) for \( i = 1, 2, \ldots, m \). Next, we show that \( u \in F \). For each \( i \in \{1, 2, 3, \ldots, m\} \), we define a sequence \( \{z_n\} \) in \( K \) by \( z_j = T_i^j u \). In the presence of increasing function \( \xi_i \) and (C2), we calculate

\[
d(z_j, u_n) \leq d(T_i^j u, T_i^j u_n) + d(T_i^j u_n, T_i^{j-1} u_n) + \cdots + d(T_i u_n, u_n)
\]

\[
\leq d(u, u_n) + \lambda_{i_n} \xi_i (d(u, u_n)) + \mu_{i_n} + \sum_{r=0}^{j-1} d(T_i^r u_n, T_i^{r+1} u_n)
\]

\[
\leq (1 + \lambda_{i_n} M^*_i) d(u, u_n) + \lambda_{i_n} \xi_i (M_i) + \mu_{i_n} + \sum_{r=0}^{j-1} d(T_i^r u_n, T_i^{r+1} u_n).
\]

Taking \( \limsup \) on both sides of the above estimate and utilizing (3.9) and the fact that each \( T_i \) is uniformly continuous, we have

\[
r(z_j, \{u_n\}) = \limsup_{n \to \infty} d(z_j, u_n) \leq \limsup_{n \to \infty} d(u, u_n) = r(u, \{u_n\}).
\]

This implies that \( |r(z_j, \{u_n\}) - r(u, \{u_n\})| \to 0 \) as \( j \to \infty \). It follows from Lemma 2.5 that \( \lim_{j \to \infty} T_i^j u = u \). Again, utilizing the uniform continuity of \( T_i \), we have that \( T_i(u) = T_i(\lim_{j \to \infty} T_i^j u) = \lim_{j \to \infty} T_i^{j+1} u = u \). From the arbitrariness of \( i \), we conclude that \( u \) is the common fixed point of \( \{T_i\}_{i=1}^m \). It remains to show that
Theorem 3.4
for the strong convergence of the sequence \((3.1)\).

in \((3.1)\) in a Hadamard space \(X\).

It follows from Lemma 3.1(i) that the sequence

This is a contradiction. Hence \(x = u\). This implies that \(u\) is the unique asymptotic center of \(\{x_n\}\) for every subsequence \(\{u_n\}\) of \(\{x_n\}\). This completes the proof.

Remark 3.3. It is worth mentioning that the analogous weak convergence result in Hilbert spaces for the sequence \(\{x_n\}\) defined in (3.1) can easily be obtained as a corollary of Theorem 3.2.

We now establish strong convergence characteristics of the sequence \(\{x_n\}\) defined in (3.1) in a Hadamard space \(X\). We first give a necessary and sufficient condition for the strong convergence of the sequence (3.1).

Theorem 3.4 Let \(C\) be a nonempty closed convex subset of a Hadamard space \(X\). Let \(f : X \to (-\infty, \infty]\) be a proper convex and lsc function and let \(\{T_i\}_{i=1}^m : C \longrightarrow C\) be a finite family of uniformly continuous total asymptotically quasi nonexpansive mappings with sequences \(\{\lambda_{in}\}\) and \(\{\mu_{in}\}\), \(n \geq 1, i = 1, 2, \ldots, m\), such that

\[
\sum_{n=1}^{\infty} \lambda_{in} < \infty \text{ and } \sum_{n=1}^{\infty} \mu_{in} < \infty;
\]

(C2) there exists constants \(M_i, M_i^* > 0\) such that \(\xi_i (\lambda_i) \leq M_i^* \lambda_i\) for all \(\lambda_i \geq M_i\).

Let \(\{x_n\}\) be the sequence generated in (3.1) such that

\[
F = \left( \bigcap_{i=1}^{m} F(T_i) \right) \cap \arg\min_{y \in C} f(y) \neq \emptyset.
\]

Then the sequence \(\{x_n\}\) converges strongly to a point in \(F\) if and only if \(\liminf_{n \to \infty} d(x_n, F) = 0\), where \(d(x, F) = \inf \{d(x, p) : p \in F\}\).

Proof: The necessity of the conditions is obvious. Thus, we only prove the sufficiency. It follows from Lemma 3.1(i) that the sequence \(\{d(x_n, p)\}_{n=1}^{\infty}\) converges. Moreover, \(\liminf_{n \to \infty} d(x_n, F) = 0\) implies that \(\lim_{n \to \infty} d(x_n, F) = 0\). This completes the proof.

Theorem 3.5 Let \(C\) be a nonempty closed convex subset of a Hadamard space \(X\). Let \(f : X \to (-\infty, \infty]\) be a proper convex and lsc function and let \(\{T_i\}_{i=1}^m : C \longrightarrow C\) be a finite family of uniformly continuous total asymptotically quasi nonexpansive mappings with sequences \(\{\lambda_{in}\}\) and \(\{\mu_{in}\}\), \(n \geq 1, i = 1, 2, \ldots, m\), such that

\[
\sum_{n=1}^{\infty} \lambda_{in} < \infty \text{ and } \sum_{n=1}^{\infty} \mu_{in} < \infty;
\]

(C2) there exists constants \(M_i, M_i^* > 0\) such that \(\xi_i (\lambda_i) \leq M_i^* \lambda_i\) for all \(\lambda_i \geq M_i\).

Let \(\{x_n\}\) be the sequence generated in (3.1) such that

\[
F = \left( \bigcap_{i=1}^{m} F(T_i) \right) \cap \arg\min_{y \in C} f(y) \neq \emptyset.
\]

Assume that \(\{T_i, J_k\}\) satisfies Condition (I), then the sequence \(\{x_n\}\) converges strongly to a point in \(F\).
Proof: It follows from Lemma 3.1(iii) that
\[ \lim_{n \to \infty} d(T_i x_n, x_n) = 0 \text{ for } i = 1, 2, \cdots, m. \]
Moreover, from Theorem 3.2, we have
\[ \lim_{n \to \infty} d(J_k x_n, x_n) = 0. \]
Since \( \{T_i, J_k\} \) satisfies Condition (I), so we have, either
\[ \lim_{n \to \infty} g(d(x_n, F)) \leq \lim_{n \to \infty} d(T_i x_n, x_n) = 0, \]
or
\[ \lim_{n \to \infty} g(d(x_n, F)) \leq \lim_{n \to \infty} d(J_k x_n, x_n) = 0, \]
In both cases, it imply that \( \lim_{n \to \infty} g(d(x_n, F)) = 0 \). Since \( g \) is nondecreasing and \( g(0) = 0 \), we have \( \lim_{n \to \infty} d(x_n, F) = 0 \). Rest of the proof follows from Theorem 3.4 and is, therefore, omitted.

Remark 3.6. It is remarked that the strong convergence characteristics of the sequence \( \{x_n\} \) defined in (3.1) in a Hadamard space \( X \) can also be established by utilizing the compactness condition of \( C \) or \( T(C) \). Moreover, one utilize the modified version of the semi-compactness condition satisfied by a family of mappings. We further remark that our results can be viewed as an extension and generalization of various corresponding results established in the current literature. In particular: (i). Theorems 3.2 generalizes the corresponding results in [27, Theorem 3], [13, Theorem 3.2] and [30, Theorem 3.2]; (ii). Theorem 3.4 generalizes the corresponding results in [27, Theorem 5] and [13, Theorem 3.5] and (iii). Theorem 3.5 generalizes the corresponding results in [13, Theorem 3.6] and [30, Theorem 3.4].

Open Questions: (i). Can we modify the sequence (3.1) involving nonself-mapping in a Hadamard space \( X \). (ii). Can we modify the sequence (3.1) in the form of a shrinking projection method for the strong convergence results in a Hadamard space \( X \).

References

[1] R. Adler, J. P. Dedieu, J. Y. Margulies, M. Martens and M. Shub, Newton’s method on Riemannian manifolds and a geometric model for human spine, IMA J. Numer. Anal., 22(2002), 359-390.
[2] Ya. I. Alber, C. E. Chidume and H. Zegeye, Approximating fixed points of total asymptotically nonexpansive mappings, Fixed Point Theory Appl., (2006), 2006:10673.
[3] L. Ambrosio, N. Gigli, and G. Savare, Gradient flows in metric spaces and in the space of probability measures, 2nd ed., Lectures in Mathematics ETH Zurich, Birkhauser Verlag, Basel, 2008.
[4] D. Ariza-Ruiz, L. Leustean, and G. Lopez, Firmly nonexpansive mappings in classes of geodesic spaces, Trans. Amer. Math. Soc., 366 (2014), 4299-4322.
[5] M. Bacak, The proximal point algorithm in metric spaces, Israel. J. Math., 194(2013), 689-701.
[6] O. A. Boikanyo and G. Morosanu, A proximal point algorithm converging strongly for general errors, Optim. Lett., 4(2010), 635-641.
[7] H. Brezis and P. L. Lions, Produits infinis de résolvantes, Israel J Math., 29(1978), 329-345.
[8] M.R. Bridson and A. Haefliger, Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften 319, Springer, Berlin, 1999.
[9] R. E. Bruck and S. Reich, Nonexpansive projections and resolvents of accretive operators in Banach spaces, Houston J. Math., 3(1977), 459-470.
[10] F. Bruhat and J. Tits, Groupes réductifs sur un corps local. I. Données radicielles valuées, Inst. Hautes Études Sci. Publ. Math., 41(1972), 5-251.
[11] C. E. Chidume and E. U. Ofoedu, Approximation of common fixed points for finite families of total asymptotically nonexpansive mappings, J. Math. Anal. Appl., 333(2007), 128-141.

[12] P. Cholamjiak, The modified proximal point algorithm in CAT(0) spaces, Optim. Lett., 9(2015), 1401-1410.

[13] P. Cholamjiak, A. A. Abdou and Y. J. Cho, Proximal point algorithms involving fixed points of nonexpansive mappings in CAT(0) spaces, Fixed Point Theory Appl., (2015), 2015:227.

[14] P. L. Combettes, J. C. Pesquet, Proximal splitting methods in signal processing, in: H.H. Bauschke, R. Burachik, P. L. Combettes, V. Elser, D. R. Luke, H. Wolkowicz (eds.) Fixed-Point Algorithms for Inverse Problems in Science and Engineering, Springer, New York (2010).

[15] P. L. Combettes and V. R. Wajs, Signal recovery by proximal forward-backward splitting, Multiscale Model. Simul., 4(2005) 1168-1200.

[16] O.P. Ferreira and R. P. Oliveira, Proximal point algorithm on Riemannian manifolds, Optim., 51 (2002), 257-270.

[17] H. Fukhar-ud-din, A. R. Khan and M. A. A. Khan, A new implicit algorithm of asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces, IAENG Int. J. Appl. Math., 42(3) (2012), 5 pages.

[18] O. Guler, On the convergence of the proximal point algorithm for convex minimization, SIAM J. Control Optim., 29(1991), 403-419.

[19] J. Jost, Convex functionals and generalized harmonic maps into spaces of non positive curvature, Comment. Math. Helvetici, 70(1995), 659-673.

[20] J. Jost, Nonpositive Curvature: Geometric and Analytic Aspects, Lectures in Mathematics ETH Zurich, Birkhauser Verlag, Basel, 1997.

[21] S. Kamimura and W. Takahashi, Approximating solutions of maximal monotone operators in Hilbert spaces, J. Approx. Theory, 106(2000), 226-240.

[22] A. R. Khan, H. Fukhar-ud-din and M. A. A. Khan, An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces, Fixed Point Theory Appl., (54) 2012, doi:10.1186/1687-1812-2012-54.

[23] M. A. A. Khan, Convergence analysis of a multi-step iteration for a finite family of asymptotically quasi-nonexpansive mappings, J. Inequal. Appl., 2013:423, (doi:10.1186/1029-242X-2013-423) 10 pp.

[24] M. A. A. Khan and H. Fukhar-ud-din, Convergence analysis of a general iteration schema of nonlinear mappings in hyperbolic spaces, Fixed Point Theory Appl., 2013: 238, (doi: 10.1186/1687-1812-2013-238) 18 pp.

[25] M. A. A. Khan, H. Fukhar-ud-din and A. Kalsoom, Existence and higher arity iteration for total asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces, Fixed Point Theory Appl., (2016) 2016:3, 18 pages.

[26] U. F. Mayer, Gradient flows on nonpositively curved metric spaces and harmonic maps, Commun. Anal. Geom., 6(1998), 199-253.

[27] K. Lerkchailaphum, and W. Phuengrattana, Iterative approaches to solving convex minimization problems and fixed point problems in complete CAT(0) spaces, Numer. Algor., (2017), pp.1-14, DOI:10.1007/s11075-017-0337-6.

[28] C. Li and J.C. Yao, Variational inequalities for set-valued vector fields on Riemannian manifolds: convexity of the solution set and the proximal point algorithm, SIAM J. Control Optim., 50(2012), 2486-2514.

[29] B. Martinet, Regularisation des équations variationnelles par approximations successives, Rev. Fr. Inform. Rech. Oper., 4(1970), 154-158.

[30] N. Pakkaranang, P. Kumam and Y. J. Cho, Proximal point algorithms for solving convex minimization problem and common fixed points problem of asymptotically quasi-nonexpansive mappings in CAT(0) spaces with convergence analysis, Numer. Algor., (2017), pp.1-19, DOI:10.1007/s11075-017-0402-1.

[31] E.A.P. Quiroz, An extension of the proximal point algorithm with Bregman distances on Hadamard manifolds, J. Global Optim., 56(2013), 43-59.

[32] R.T. Rockafellar, Monotone operators and the proximal point algorithm, SIAM J. Control Optim., 14(1976), 877-898.

[33] S.T. Smith, Optimization techniques on Riemannian manifolds, in: Fields Institute Communications, Amer. Math. Soc., Providence, RI. 3(1994), 113-146.
[34] C. Udriste, Convex Functions and Optimization Methods on Riemannian Manifolds, Mathematics and its Applications, Vol. 297, Kluwer Academic, Dordrecht, 1994.

[35] J.H. Wang and G. Lopez, Modified proximal point algorithms on Hadamard manifolds, Optimization, 60(2011), 697-708.

[36] H.-K. Xu, A regularization method for the proximal point algorithm, J. Glob. Optim., 36(2006), 115-125.