SIMPLE NORMAL CROSSING VARIETIES WITH
PRESCRIBED DUAL COMPLEX

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Let $W$ be a reducible variety with simple normal crossing singularities only. The combinatorial structure of $W$ is described by the dual complex $D(W)$ whose vertices correspond to the irreducible components $W_i \subset W$ and the positive dimensional cells correspond to the strata of $W$, that is, to irreducible components of intersections $\cap_{i \in J} W_i$ for some $J \subset I$; see Definition 7.

In the papers [Kol11a, KK11, Kol12] many interesting singularities were obtained by constructing a simple normal crossing variety $W$ whose dual complex is topologically complicated and then realizing $W$ as the exceptional divisor of a (partial) resolution of a singularity.

The aim of this note is to prove the following existence result for simplicial complexes.

**Theorem 1.** Let $C$ be a finite simplicial complex of dimension $\leq n$. Then there is a smooth projective variety $Y$ of dimension $n + 1$ and a simple normal crossing divisor $D \subset Y$ such that $D(D) \cong C$.

Moreover, we can achieve that either one (but not both) of the following conditions are also satisfied.

1. $Y$ and all the strata of $D$ are rational or
2. $K_Y + D$ is ample.

This is stronger than the results in [KK11] in three aspects. We realize $C$ up-to isomorphism (not just up-to homotopy equivalence) and the dimension estimate $\dim D \geq \dim C$ is optimal since $\dim W \geq \dim D(W)$ for every simple normal crossing variety $W$. Furthermore, by construction $D$ is naturally a hypersurface in a smooth projective variety, while in earlier versions this was achieved only after a detailed study of a resolution process in [Kol12, Sec.6]. The last property is very helpful in constructing singularities. It simplifies the method of [Kol11a] and also leads to isolated singularities.

**Theorem 2.** Let $C$ be a finite simplicial complex of dimension $\leq n$. Then there is an isolated rational singularity $(x \in X)$ of dimension $n + 1$ and a log resolution $\pi : Y \to X$ with exceptional divisor $E = \text{Supp} \pi^{-1}(x)$ such that $D(E) \cong C$.

Both of the alternatives in Theorem 1 are useful. The first case (1.1) can be utilized to complete the characterization of dual complexes of resolutions of isolated rational singularities started in [KK11, Kol12].

**Theorem 3.** For a finite simplicial complex $C$ the following are equivalent.

1. $C$ is connected, $\dim C \leq n$ and $H_i(C, \mathbb{Q}) = 0$ for $i > 0$.
2. There is an isolated rational singularity $(x \in X)$ of dimension $n + 1$ and a log resolution $\pi : Y \to X$ with exceptional divisor $E = \text{Supp} \pi^{-1}(x)$ such that $D(E) \cong C$. 
Let \((x \in X)\) be a singularity and \(g : Y \rightarrow X\) a resolution such that \(E := \text{Supp } g^{-1}(x) \subset Y\) is a simple normal crossing divisor. The dual complex \(D(E)\) depends on the choice of \(Y\) but its (simple) homotopy type does not; see Definition 10. This (simple) homotopy type is denoted by \(\mathcal{D}R(x \in X)\).

If \(K_X\) is Cartier (or at least \(\mathbb{Q}\)-Cartier) then \([dFK12]\) identifies a distinguished \(\text{PL-homeomorphism class } \mathcal{D}MR(x \in X)\) within the homotopy class \(\mathcal{D}R(x \in X)\); see Definition 11. Using the alternative (1.2), we see that the \(\text{PL-homeomorphism classes } \mathcal{D}MR(x \in X)\) can be arbitrary.

**Theorem 4.** For every finite, connected simplicial complex \(C\) of dimension \(\leq n\) there is a normal, isolated singularity \((x \in X)\) of dimension \(n + 1\) such that \(K_X\) is Cartier and \(\mathcal{D}MR(x \in X)\) is PL-homeomorphic to \(C\).

5 (Open problems). There are four main open problems in connection with dual complexes of varieties and of singularities.

5.1) It is interesting to understand the possible relation between the dual complex \(\mathcal{C}(X)\) and the canonical class \(K_X\).

If \(K_X\) is ample then \(\mathcal{C}(X)\) can be arbitrary by (1.2).

If \(-K_X\) is ample then \(\mathcal{C}(X)\) is a simplex by \([Fuj12a, Fuj12b]\). The really interesting case seems to be when \(K_X \sim \mathbb{Q} 0\). These lead to log canonical singularities and their structure is not yet understood.

By \([Kol11b]\) (see also \([Kol13, \text{Sec.4.4}]\)) in this case either \(\dim \mathcal{C}(X) = 1\) or \(\mathcal{C}(X)\) is a normal pseudomanifold. In particular, every codimension 1 cell is the face of exactly 2 maximal dimensional cells.

In all examples that I know of, \(\mathcal{C}(X)\) is an orbifold.

5.2) In many cases, the most economical realization of a PL-homeomorphism type is given not by a simplicial complex but by a \(\Delta\)-complex in the terminology of \([Hat02, \text{p.534}]\). Thus it would be quite useful to know that every \(\Delta\)-complex can be realized as the dual complex of a normal crossing variety. This would give the optimal answer along the lines proposed by \([Sim10]\). See \([Kap12]\) for very interesting related results.

5.3) Usually one can not satisfy both of the alternatives in Theorem 1 simultaneously. Indeed, if \(v_{n-1} \in \mathcal{D}(D)\) is an \((n - 1)\)-cell that is the face of 1 or 2 \(n\)-cells then the corresponding stratum is a smooth curve \(C \subset D\) with 1 or 2 smaller strata on it. Thus if \(C\) is rational then \((K_D \cdot C) = -1\) or \((K_D \cdot C) = 0\). This is, however, the only obstruction that I know.

It is possible that if \(C\) is a pseudomanifold then there is a simple normal crossing variety \(X\) such that all the strata of \(X\) are rational, \(K_X\) is nef and \(D(X) \cong C\).

5.4) A shortcoming of our construction is that it is not local. Let \(X = \cup_{i \in I} X_i\) be an snc variety. The link of a vertex \(v_i \in \mathcal{D}(X)\) is determined by those \(X_j\) for which \(X_j \cap X_i \neq \emptyset\). Conversely, give a complex \(\mathcal{C}\) one could hope to that there is a procedure to construct an snc variety \(X = \cup_{i \in I} X_i\) such that \(D(D) \cong \mathcal{C}\) and each \(X_i\) is determined by the link of \(v_i \in \mathcal{C}\). Such a procedure exists if \(\dim \mathcal{C} = 2\); see \([Kol11a]\). The study of parasitic intersections in \([KK11]\) suggests that this may be too much to hope, but leaves open the possibility that \(X_i\) should depend only on the \((\dim \mathcal{C} - 1)\)-times iterated link of \(v_i\).
One clearly needs a more local construction if one hopes to control the canonical class $K_X$.

1. Dual complexes

Since our main interest is in the topological aspects of these questions, we work with varieties over $\mathbb{C}$. However, everything applies to varieties over arbitrary fields. See [Kol13] for the more delicate aspects that appear when the base field is not algebraically closed and has positive characteristic.

**Definition 6** (Simple normal crossing varieties). Let $W$ be a variety with irreducible components $\{W_i : i \in I\}$. We say that $W$ is a simple normal crossing variety (abbreviated as nc) if the $W_i$ are smooth and every point $p \in W$ has an open (Euclidean) neighborhood $p \in U_p \subset W$ and an embedding $U_p \hookrightarrow \mathbb{C}^{n+1}$ such that the image of $U_p$ is an open subset of the union of coordinate hyperplanes $(z_1 \cdots z_{n+1} = 0)$.

A stratum of $W$ is any irreducible component of an intersection $\cap_{i \in J} W_i$ for some $J \subset I$.

The intersection of any two strata is also a union of lower dimensional strata, hence the union of all minimal strata is a smooth (reducible) subvariety of $W$.

**Definition 7** (Dual complex). The combinatorics of a simple normal crossing variety $W$ is encoded by a cell complex $D(W)$ whose vertices are labeled by the irreducible components of $W$ and for every stratum $Z \subset \cap_{i \in J} W_i$ we attach a $(|J| - 1)$-dimensional cell. Note that for any $j \in J$ there is a unique irreducible component of $\cap_{i \in J \backslash \{j\}} W_i$ that contains $Z$; this specifies the attaching map. $D(W)$ is called the dual complex or dual graph of $W$. (Although $D(W)$ is not a simplicial complex in general, it is a regular cell complex; cf. [Hat02, p.534].)

The minimal strata of $W$ correspond to the maximal cells of $D(W)$.

It is clear that $\dim D(W) \leq \dim W$ and $D(W)$ is connected iff $W$ is. More generally, the cohomology groups $H^i(D(W), \mathbb{Q})$ can be identified with the weight 0 part of the Hodge structure on $H^i(W, \mathbb{Q})$; see [ABW11]. One can think of $D(W)$ as the combinatorial part of the topology of $W$.

**Definition 8** (Simple normal crossing pairs). Let $X$ be a variety over $\mathbb{C}$ and $D \subset X$ a divisor. We say that $(X, D)$ is a simple normal crossing pair if

1. $X$ is smooth,
2. the irreducible components $D_i \subset D$ are smooth and
3. every point $x \in X$ has an Euclidean open neighborhood $x \in U_x \subset X$ with local coordinates $z_1, \ldots, z_n$ such that $D \cap U_x = (z_1 \cdots z_r = 0)$ for some $r$.

If $X$ is smooth then $(X, D)$ is a simple normal crossing pair iff $D$ is a simple normal crossing variety. (In Lemma 16 the key point is to understand what happens when $D$ is a simple normal crossing variety but $X$ is not smooth.)

One can think of a simple normal crossing variety $D = \cup_{i \in I} D_i$ as being glued together from the simple normal crossing pairs

$$(D_i, \sum_{j \neq i} D_j|_{D_i}).$$

We say that $(X, D)$ is a normal crossing pair if it satisfies conditions (1) and (3). The difference between the two variants is technical but important for several of our results.
9 (Blowing up strata). Let $W$ be a simple normal crossing variety and $Z \subset W$ a stratum. Then the blow-up $B_{Z}W$ is again a simple normal crossing variety. To see this, take a local chart where

$$(Z \subset W) = ((x_1 = \cdots = x_r = 0) \subset (x_1 \cdots x_n = 0)) \subset \mathbb{C}^{n+m}.$$ 

The blow-up $B_{Z} \mathbb{C}^{n+m}$ is defined by

$$B_{Z} \mathbb{C}^{n+m} = (x_is_j = x_js_i : 1 \leq i, j \leq r) \subset \mathbb{C}^{n+m} \times \mathbb{P}^{s-1}$$

and the blow-up $B_{Z}W$ is defined by

$$B_{Z}W = (s_1 \cdots s_r \cdot x_{r+1} \cdots x_n = 0) \subset B_{Z} \mathbb{C}^{n+m}.$$ 

In a typical local chart $U_r := (s_r \neq 0)$ we can use new coordinates

$$(x'_1 = s_1/s_r, \ldots, x'_{r-1} = s_{r-1}/s_r, s_r, x_{r+1}, \ldots, x_{n+m})$$

to get that

$$B_{Z}W \cap U_r = (x'_1 \cdots x'_{r-1} \cdot x_{r+1} \cdots x_n = 0).$$

Let $v_Z \subset D(W)$ be the cell corresponding to $Z$. We see from the above computations that $D_{\ast}(B_{Z}W)$ is obtained from $D(W)$ by removing all the cells that have $v_Z$ as a face. (That is, removing the star of $v_Z$.) In particular, if $Z$ is a minimal stratum then we remove the cell $v_Z$.

Although we do not need it, it is useful to know that there is another way to blow up a stratum. If $(X, D)$ is a simple normal crossing pair and $Z \subset D$ a stratum, one can consider the blow-up $\pi : B_{Z}X \to X$ with exceptional divisor $E_Z$. Then $B_{Z}D \cong \pi_{\ast}^{-1}D$ and $(B_{Z}X, E_Z + B_{Z}D)$ is again a simple normal crossing pair. It is not hard to see that $D(E_Z + B_{Z}D)$ is obtained as the stellar subdivision of $D(D)$ with vertex $v_Z$.

**Definition 10** (Dual complex of a singularity I). Let $X$ be a variety and $x \in X$ a point. Choose a resolution of singularities $\pi : Y \to X$ such that $E_x := \pi^{-1}(x) \subset Y$ is a simple normal crossing divisor. Thus it has a dual complex $D(E_x)$.

The dual graph of a normal surface singularity has a long history but systematic investigations in higher dimensions were started only recently; see [Thu07, Ste08, Pay11, ABW11]. It is proved in these papers that the homotopy type (even the simple homotopy type) of $D(E_x)$ is independent of the resolution $Y \to X$. We denote it by $\mathcal{D}(x \in X)$. ($\mathcal{D}$ stands for the Dual complex of a Resolution.)

It is clear that $\dim \mathcal{D}(x \in X) \leq \dim X - 1$ and $\mathcal{D}(x \in X)$ is connected if $X$ is normal.

**Definition 11** (Dual complex of a singularity II). It was observed in [dFKX12] that the dual complex of a divisor $D \subset Y$ is defined whenever $(Y, D)$ is divisorial log terminal (abbreviated as dlt); see [KM98, 2.37] for the definition of dlt. If $(x \in X)$ is an isolated singularity and $K_X$ is $\mathbb{Q}$-Cartier then there is a proper birational morphism $\pi : Y \to X$ such that $(Y, E_Y)$ is dlt and $K_Y + E_Y$ is $\pi$-nef where $E_Y = \text{Supp} \, \pi^{-1}(x)$. (Such a morphism is called a dlt modification.) It is proved in [dFKX12] that

1. $D(E_Y)$ is simple homotopy equivalent to $\mathcal{D}(x \in X)$ and
2. up-to PL-homeomorphism, $D(E_Y)$ does not depend on the choice of $Y$.

Thus, for such singularities, $D(E_Y)$ defines a distinguished PL-homeomorphism class, denoted by $\mathcal{D}(x \in X)$, within the simple homotopy class $\mathcal{D}(x \in X)$. ($\mathcal{D}$ stands for the Dual complex of a Minimal partial Resolution.)
The hard part of [dFKX12] is the proof of (11.1), but in our examples we check this directly in Lemma 16.

2. Construction of simple normal crossing varieties

In [KK11] we constructed snc varieties $W = \cup_{i \in I} W_i$ by first obtaining the irreducible components $W_i$ and the gluing them together to get $W$. This had two technical disadvantages. First, the projectivity of the resulting $W$ was hard to control and there was no natural embedding of $W$ into any smooth variety.

The process turns out to be much easier if we break the symmetry on the algebraic side but preserve it topologically. Given $C$, first we construct an snc variety $X_0$ such that $D(X_0)$ has the same dimension and the same vertex set as $C$ but the maximum possible number of positive dimensional cells. Then we remove the excess positive dimensional cells one dimension at a time.

A disadvantage is that we seem to lose information about finite group actions, while the construction of [KK11] preserved symmetries.

The key step is the following.

Lemma 12. Let $C$ be a finite simplicial complex of dimension $n$ with vertices $\{v_i : i \in I\}$. Fix $0 \leq r < n$. Let $X_r$ be an snc variety such that $D(X_r)$ is a simplicial complex and $\tau_r : C \hookrightarrow D(X_r)$ an embedding that is a bijection on the vertices and on cells of dimension $\geq n-r+1$.

Let $J_{n-r+1} \subset \binom{I}{n-r+1}$ be those $(n-r+1)$-element subsets that do not span an $(n-r)$-cell in $C$ and $Z_r \subset X_r$ the union of the strata that correspond to elements of $\tau_r(J_{n-r+1})$. Then

1. $Z_r$ is smooth,
2. $X_{r+1} := B_{Z_r}X_r$, the blow-up of $Z_r \subset X_r$, is an snc variety,
3. $D(X_{r+1}) \subset D(X_r)$ and
4. the restriction of $\tau_r$ gives an embedding $\tau_{r+1} : C \hookrightarrow D(X_{r+1})$ which is a bijection on the vertices and on cells of dimension $\geq n-r$.

Proof. If $s_{n-r+1} \subset I$ does not span an $(n-r)$-cell in $C$ then no subset of $I$ containing $s_{n-r+1}$ spans a cell in $C$. Thus either $s_{n-r+1}$ does not span a cell in $D(X_r)$ or it spans a maximal cell in $D(X_r)$. Since maximal cells correspond to minimal strata, $Z_r$ is a union of minimal strata. The minimal strata are disjoint from each other, hence $Z_r$ is smooth and of pure dimension $r$.

Thus, by Paragraph [13] $X_{r+1} = B_{Z_r}X_r$ is an snc variety and $D(X_{r+1})$ is obtained from $D(X_r)$ by removing all $(n-r)$-cells that do not correspond to a cell in $C$. □

The following yields a proof of (11.1).

Proposition 13. Let $C$ be a finite simplicial complex of dimension $n$. Then there is an snc variety $X$ such that

1. $D(X) \cong C$,
2. $\dim X = \dim C$,
3. $X$ is a hypersurface in a smooth, projective variety $Y$,
4. $X$ is defined over $\mathbb{Q}$ and
5. the strata of $X$ and also $Y$ are all rational varieties.

Proof. Let $\{v_i : i \in I\}$ be the vertices of $C$. In $\mathbb{P}^{n+1}$ let $\{H_i : i \in I\}$ be hyperplanes in general position.
One can get concrete examples by fixing distinct numbers \(a_i\) and setting
\[
H_i = (x_0 + a_1 x_1 + a_2^2 x_2 + \cdots + a_i^{n+1} x_{n+1} = 0) \subset \mathbb{P}^{n+1}.
\]
No \(n+2\) of the \(H_i\) have a point in common since the coefficients form a Vandermonde matrix. If \(a_i \in \mathbb{Q}\), then \(H_i\) is defined over \(\mathbb{Q}\). If the \(a_i\) are conjugate over \(\mathbb{Q}\), their union is defined over \(\mathbb{Q}\) but has a nontrivial Galois action on its geometric irreducible components.

Set \(X_0 := \cup_{i \in I} H_i\) and \(Y_0 := \mathbb{P}^{n+1}\).

Note that \(X_0\) is an snc variety, all of its strata are linear spaces and for \(0 \leq r \leq n\) the \((n-r)\)-dimensional strata of \(X_0\) are in a natural bijection with \(\binom{I}{r}\), the set of \(r\)-element subsets of \(I\).

Thus every subset \(J \subset I\) with at most \(n + 1\) elements spans a cell in \(\mathcal{D}(X_0)\) hence there is a natural embedding \(\tau_0 : \mathcal{C} \hookrightarrow \mathcal{D}(X_0)\) that is a bijection on vertices. Thus \(X_0\) and \(\tau_0\) satisfy the assumptions of Lemma \[12\] with \(r = 0\) since neither \(\mathcal{C}\) nor \(\mathcal{D}(X_0)\) contain any cells of dimension \(> n\).

Using Lemma \[12\] inductively for \(0 \leq r < n\), we obtain embedded snc varieties \(X_r \subset Y_r\) and smooth subvarieties \(Z_r \subset X_r\) such that
\[
X_{r+1} = B_{Z_r} X_r \quad \text{and} \quad Y_{r+1} = B_{Z_r} Y_r.
\]
Furthermore, \(\tau_0\) restricts to embeddings \(\tau_r : \mathcal{C} \hookrightarrow \mathcal{D}(X_r)\) which are bijections on the vertices and on cells of dimension \(\geq n - r + 1\).

Finally we get \(X_n\) such that \(\mathcal{C} \hookrightarrow \mathcal{D}(X_n)\) is a bijection on cells in every dimension.

The strata of \(X_r\) are obtained from linear subspaces in \(Y_0 := \mathbb{P}^{n+1}\) by blowing up lower dimensional linear subspaces, hence the strata are rational.

Therefore \(X := X_n\) satisfies all the requirements. \(\square\)

Using the pair \(X \subset Y\) obtained in Proposition \[13\] the following shows \(\{12\}\).

**Lemma 14.** Let \(Y\) be a smooth, projective variety of dimension \(n\), \(D \subset Y\) an snc divisor and \(\{p_j : j \in J\}\) the \(0\)-dimensional strata. Let \(\tau : Y_1 \to Y\) be a double cover ramified along a general, sufficiently ample divisor \(H' \sim 2H\). For each \(p_j\) pick a preimage \(q_j \in \tau^{-1}_1(p_j)\) and let \(\pi : Y_2 \to Y_1\) be the blow-up of the points \(q_j \in \{q_j : j \in J\}\). Set \(D_1 := \tau^* D\) and \(D_2 := \pi^{-1}_2 D_1\). Then
\[
\begin{align*}
(1) & \text{ } Y_2 \text{ is a smooth, projective variety and } D_2 \subset Y_2 \text{ is an snc divisor,} \\
(2) & \text{ } K_{Y_2} + D_2 \text{ is ample and} \\
(3) & \text{ the composite } \tau \circ \pi \text{ induces an isomorphism } \mathcal{D}(D_2) \cong \mathcal{D}(D).
\end{align*}
\]

**Proof.** By the Hurwitz formula,
\[
K_{Y_1} + D_1 \cong \mathbb{Q} \tau^* (K_Y + D + H)
\]
thus if \(H\) is sufficiently ample then \(K_{Y_1} + D_1\) is the pull-back of a very ample line bundle from \(Y\). (As for hyperelliptic curves, usually \(K_{Y_1} + D_1\) is not very ample.) We may even arrange that
\[
\mathcal{O}_Y (K_Y + D + H)(-\sum_j p_j) \quad \text{(14.4)}
\]
is very ample on \(Y \setminus \{p_j : j \in J\}\).

Let \(E \subset Y_2\) be the exceptional divisor of \(\pi\); it is a union of \(|J|\) disjoint copies of \(\mathbb{P}^{n-1}\). Then
\[
K_{Y_2} \sim \pi^* K_{Y_1} + (n-1) E \quad \text{and} \quad D_2 \sim \pi^* D_1 - nE.
\]
Therefore

\[ K_{Y_2} + D_2 \sim \pi^*(K_{Y_1} + D_1)(-E). \]

This shows that \( K_{Y_2} + D_2 \) is ample on \( E \) and \( 14.4 \) implies that it is ample on \( Y_2 \setminus E \cong Y_1 \setminus \{ q_j : j \in J \} \). Thus \( K_{Y_2} + D_2 \) is ample.

Finally, every positive dimensional stratum \( W_1 \subset D_1 \) is a double cover of a stratum \( W \subset D \) ramified along \( W \cap H' \neq \emptyset \). Thus \( \pi_* : \mathcal{D}(D_1) \to \mathcal{D}(D) \) is a bijection on the cells of dimension \( < n \) but every \( n \)-dimensional cell of \( \mathcal{D}(D) \) has 2 preimages in \( \mathcal{D}(D_1) \). The blow-up of the points \( q_j \) removes one of these cells, giving the isomorphism \( \mathcal{D}(D_2) \cong \mathcal{D}(D) \). \( \square \)

3. Construction of singularities

Starting with a simplicial complex \( \mathcal{C} \), in Theorem 11 we obtained an embedded snc variety \( D \subset Y \) such that \( \mathcal{D}(D) \cong \mathcal{C} \). If \( D \subset Y \) can be contracted to a point then we get an isolated singularity \( (x \in X) \) one of whose log resolutions is \( (D \subset Y) \).

In general \( D \) is not contractible; we have to perform some blow-ups first.

Lemma 15. Let \( Y \) be a smooth variety, \( D \subset Y \) an snc divisor and \( H \subset Y \) another divisor intersecting \( D \) transversally. Set \( Z := D \cap H \) and let \( \pi : Y_1 := B_2Y \to Y \) be the blow-up with exceptional divisor \( E \). Let \( D_1 := \pi^{-1}_*D \) denote the birational transform of \( D \). Then

(1) \( D_1 \sim \pi^*D - E \) is a Cartier divisor,

(2) \( Y_1 \setminus D_1 \) is smooth,

(3) \( Y_1 \setminus D_1 \) is an snc variety and at every singular point \( \text{Sing} \) is disjoint from \( D_1 \).

(5) \( \mathcal{D}(D_1) \) is dlt and

(6) \( N_{D_1,Y_1} \cong \mathcal{O}_{Y_1}(D_1)|_{D_1} \cong \mathcal{O}_Y(D - H)|_D \).

Proof. The first 5 assertions are local on \( Y \), hence we may assume that there are local coordinates \( y_1, \ldots, y_n \) such that \( D = (y_1 \cdots y_r = 0) \) and \( H = (y_n = 0) \) for some \( r < n \). The blow-up is given by an equation

\[ (y_1, \ldots, y_r, s = y_n \cdot t) \subset Y \times \mathbb{P}_\mathbb{A}_{st}^1 \quad \text{and} \quad D_1 = (y_1 \cdots y_r - t = 0) = (t = 0). \]

In the affine chart \( t \neq 0 \), we get the equation \( y_1 \cdots y_r \cdot (s/t) = y_n \) which shows that \( Y_1 \setminus D_1 \) is smooth.

The singularities of \( (Y_1, D_1) \) are locally of the form

\[ (y_1 \cdots y_r = v = 0) \subset (y_1 \cdots y_r = y_n \cdot v) \subset Y \times \mathbb{A}_v^1. \]

This shows (4) and (5) is proved in Lemma 16.

The normal bundle of \( D_1 \) is \( \mathcal{O}_{Y_1}(D_1)|_{D_1} \). Note that \( D_1 \sim \pi^*D - E \) and \( \pi^{-1}_*H \sim \pi^*H - E \) is disjoint from \( D_1 \). Thus

\[ \mathcal{O}_{Y_1}(D_1) \cong \mathcal{O}_{Y_1}(\pi^*D - \pi^*H) \cong \pi^*\mathcal{O}_Y(D - H). \]

Restricting to \( D_1 \cong D \) gives (6). \( \square \)

Lemma 16. Let \( Y \) be a normal variety and \( D \subset Y \) a codimension 1 subvariety. Assume that \( D \) is an snc variety and at every singular point \( x \in \text{Sing} \) there are
local analytic coordinates such that the pair \((Y, D)\) is given as a product of an snc pair with one of the form
\[
((y_1 \cdots y_r = y_{n+1} = 0) \subset (y_1 \cdots y_r = y_n y_{n+1})) \subset \mathbb{A}^{n+1}
\]
for some \(0 \leq r < n \leq \dim Y\). Then \((Y, D)\) has a small log resolution \(\pi : (Y^*, D^*) \to (Y, D)\) and \(\mathcal{D}(D^*) = \mathcal{D}(D)\). In particular, \((Y, D)\) is dlt.

Proof. Let \(D_0 \subset D\) be an irreducible component. Let \(\pi_0 : Y' \to Y\) denote the blow-up of \(D_0\). Then \(\pi_0\) is an isomorphism at the points where \(D_0\) is Cartier. At other points, in suitable local coordinates we have
\[(D_0 \subset D \subset Y) \cong \left((y_r = y_{n+1} = 0) \subset (y_1 \cdots y_r = y_{n+1} = 0) \subset (y_1 \cdots y_r = y_n y_{n+1})\right).
\]
The blow-up is covered by 2 coordinate charts. In one of them we have new coordinate \(y'_r = y_r/y_{n+1}\) and the local equations are
\[(D' \subset Y') \cong \left((y_{n+1} = 0) \subset (y_1 \cdots y_{r-1} \cdot y'_r = y_n)\right).
\]
Thus, in this chart, \(D'\) has only 1 irreducible component and \((Y', D')\) is an snc pair.

In the other chart the new coordinate is \(y'_{n+1} = y_{n+1}/y_r\) and the local equations are
\[
((y_r = 0) \subset (y_r = 0) \cup (y_1 \cdots y_{r-1} = y'_{n+1} = 0) \subset (y_1 \cdots y_{r-1} = y_n y'_{n+1}))
\]
Thus the \(y_r\)-coordinate splits off as a direct factor and the remaining equation is
\[
((y_1 \cdots y_{r-1} = y'_{n+1} = 0) \subset (y_1 \cdots y_{r-1} = y_n y'_{n+1})).
\]
We also see that the dual complex of \(D\) is unchanged.

Performing these blow-ups for every irreducible component of \(D\) we end up with \(\pi : Y^* \to Y\) such that \((Y^*, D^* := \pi_\ast^{-1} D)\) is an snc pair and \(\mathcal{D}(D^*) = \mathcal{D}(D)\).

Since \(\pi\) is small, the discrepancies are unchanged as we go from \((Y, D)\) to \((Y^*, D^*)\) by [KM98, 2.30]. Since \(\pi\) does not contract any stratum of \(D^*\), we see that \((Y, D)\) is dlt. \(\square\)

Remark 17. Note that the above resolution process requires an ordering of the irreducible components of \(D\). Thus it does not apply if the simple normal crossing variety \(D\) is replaced by a normal crossing variety. This makes the study of the dual complex of resolutions with normal crossing exceptional divisors quite a bit harder.

18 (Proof of Theorem\textsuperscript{2}). Start with a simplicial complex \(C\) and use Theorem\textsuperscript{1} to obtain an snc pair \(D \subset Y\) such that \(\mathcal{D}(D) \cong C\).

Let \(H \subset Y\) be a smooth divisor intersecting \(D\) transversally such that \(H - D\) is ample. As in Lemma\textsuperscript{15} and let \(\pi : Y_1 \to Y\) be the blow-up of \(Z := D \cap H\) and set \(D_1 := \pi^{-1}_\ast D\). By (15\textsuperscript{2}), \(D_1 \cong D\) and by (15\textsuperscript{6}) its normal bundle is \(N_{D_1, Y_1} \cong \mathcal{O}_Y(D - H)|_Y\). Thus \(N_{D_1, Y_1}^{-1}\) is ample. Therefore, by [Art70], \(D_1 \subset Y_1\) can be contracted to a point, at least analytically or étale locally. That is, there is an analytic space (resp. an algebraic variety) \(U_1\) containing \(D_1\) with an open embedding (resp. an étale morphism) \(g : (D_1 \subset U_1) \to (D_1 \subset Y_1)\) and a contraction morphism
\[
\begin{array}{ccc}
D_1 & \subset & U_1 \\
\downarrow & & \downarrow \pi \\
x & \in & X
\end{array}
\]
such that $X$ is normal, $U_1 \setminus D_1 \cong X \setminus \{x\}$ and $D_1 = \text{Supp} \pi^{-1}(x)$.

By (15.4) $U_1$ is not smooth, but Lemma 16 shows that there is a log resolution
\[ \tau : (D_2 \subset U_2) \rightarrow (D_1 \subset U_1) \]
such that $\mathcal{D}(D_2) = \mathcal{D}(D_1)$. Thus $\pi \circ \tau$ gives a resolution
\[
\begin{array}{ccc}
D_2 & \subset & U_2 \\
\downarrow & \downarrow & \pi \circ \tau \\
x & \in & X
\end{array}
\]  
(15.2)

This proves Theorem 2. \hfill \Box

**Remark 19.** Although, as we noted, the above construction does not seem to be compatible with symmetries of $C$, it is usually possible to realize symmetries of $C$ as Galois group actions on $D_1$ (though not on $D_2$ by Remark 17). This may help in understanding the algebraic fundamental groups of links.

**20 (Proof of Theorem 3).** The proof is essentially in \[KK11\]; see also \[Kol12\] Sec.8.

The implication (3.2) ⇒ (3.1) was established in the cited papers but the key ingredients are contained already in \[GS75\] pp.68–72 and \[Ste83\] 2.14.

The converse statements are not fully proved in \[Kol12\] Sec.8, but the reason is that instead of our Theorem 1, weaker existence results were used. Using Paragraph 18 the proof is even simpler.

Let $C$ be a $\mathbb{Q}$-acyclic simplicial complex. First, \[KK11\] 3.63 shows that with $D$ as in (11.1) we have $H^i(D, \mathcal{O}_D) = 0$ for $i > 0$. Then \[Kol12\] 3.54 implies that for $H$ sufficiently ample, the singularity $(x \in X)$ obtained in (18.1–2) is rational. \hfill \Box

**21 (Proof of Theorem 4).** Start with a simplicial complex $C$ and use Theorem 1 to obtain an embedded snc variety $D \subset Y$ such that $\mathcal{D}(D) \cong C$.

As before, we blow up a subvariety of the form $D \cap H$ and then contract the birational transform of $D$ to obtain the required singularity $(x \in X)$. However, we have to choose $H$ with some care.

By (11.2) we may assume that $K_Y + D$ is ample. Choose $m \gg 1$ such that $m(K_Y + D) + D$ is very ample and let $H \sim m(K_Y + D) + D$ be a smooth divisor that intersects $D$ transversally. Let $\pi : Y_1 \rightarrow Y$ be the blow-up of $Z := D \cap H$ with exceptional divisor $E$. Then

\[ K_{Y_1} \sim \pi^* K_Y + E, \quad D_1 := \pi_*^{-1} D \sim \pi^* D - E \quad \text{and} \quad \pi_*^{-1} H \sim \pi^* H - E. \]

Therefore

\[ mK_{Y_1} + (m + 1)D_1 \sim \pi^* (mK_Y + (m + 1)D) - E \sim \pi_*^{-1} H. \]  
(21.1)

Let $(D_1 \subset U_1) \rightarrow (D_1 \subset Y_1)$ be any analytic or étale neighborhood of $D_1$ whose image is disjoint from $\pi_*^{-1} H$. Then (21.1) implies that

\[ \mathcal{O}_{U_1}(mK_{U_1} + (m + 1)D_1) \cong \mathcal{O}_{U_1}. \]  
(21.2)

Let $g : (D_1 \subset U_1) \rightarrow (x_1 \in X_1)$ be the contraction of $D_1$ to a normal singularity. Then the isomorphism (21.2) pushes forward to an isomorphism

\[ \mathcal{O}_{X_1}(mK_{X_1}) \cong g_* \mathcal{O}_{U_1}(mK_{U_1} + (m + 1)D_1) \cong g_* \mathcal{O}_{U_1} \cong \mathcal{O}_{X_1}. \]  
(21.3)

Thus $mK_{X_1}$ is Cartier. Furthermore, $(U_1, D_1)$ is dlt by (15.4) and

\[ \mathcal{O}_{U_1}(K_{U_1} + D_1)|_{D_1} \cong \mathcal{O}_{D_1}(KD_1) \cong \mathcal{O}_D(KD) \cong \mathcal{O}_Y(K_Y + D)|_D \]
is ample by assumption, hence \( g : U_1 \to X_1 \) is a dlt modification. Therefore, as noted in Definition \[11\] \( \text{DMR}(x_1 \in X_1) \) is PL-homeomorphic to \( \mathcal{D}(D_1) \cong \mathcal{D}(D) \cong \mathcal{C} \). This gives a requisite example where the canonical class is \( \mathbb{Q} \)-Cartier. From this we get an example where the canonical class is Cartier as follows.

The isomorphism \( \mathcal{O}_{X_1}(mK_{X_1}) \cong \mathcal{O}_{X_1} \) determines a degree \( m \) cyclic cover
\[
\tau : (x \in X) \to (x_1 \in X_1).
\]
(See [KM98, 2.49–53] for the construction and basic properties of cyclic covers.) Let \( \pi : (U_2, D_2) \to (U_1, D_1) \) be the log resolution constructed in Lemma \[16\] such that \( \mathcal{D}(D_2) \cong \mathcal{D}(D_1) \). Corresponding to \( X \to X_1 \) we have a cyclic cover \( \tau_U : U \to U_2 \) that comes from the isomorphism
\[
\mathcal{O}_{U_2}(K_{U_2} + D_2)^{\otimes m} \cong \mathcal{O}_{U_2}(-D_2).
\]
This shows that \( D := \text{Supp} \tau_U^{-1}(D_2) \to D_2 \) is an isomorphism. In particular, \( U \) and \( X \) are irreducible. \( K_U + D \) is not relatively ample any more, but \( (K_U + D)|_D \sim K_D \sim K_{D_2} \) is the pull-back of \( K_{D_1} \sim (K_U + D_1)|_{D_1} \), hence \( K_U + D \) is nef over \( X \).

The singularities of \( D \subset U \) are locally of the form
\[
\left( y_1 \cdots y_r = y_n = 0 \right) \subset \left( y_1 \cdots y_r = y_n^m \right) \subset \mathbb{A}^{n+1}
\]
These pairs are not dlt, but they are quotients of dlt pairs. Indeed, they can be written as
\[
\left( u_1 \cdots u_r = 0 \right) \subset \mathbb{A}^n / G_m
\]
where \( G_m \) is the subgroup of diagonal matrices
\[
\left\{ \text{diag}(a_1, \ldots, a_r, 1, \ldots, 1) : a_1^m = \cdots = a_r^m = a_1 \cdots a_r = 1 \right\}.
\]
Thus, using [dFKX12 Cor.38], \( \mathcal{C} \cong \mathcal{D}(D_1) \cong \mathcal{D}(D_2) \cong \mathcal{D}(D) \) is PL-homeomorphic to \( \text{DMR}(x \in X) \). \( \square \)

4. Fundamental groups of rational links

A group \( G \) is called \( \mathbb{Q} \)-superperfect if \( H_1(G, \mathbb{Q}) = H_2(G, \mathbb{Q}) = 0 \), see [KK11 Def.40]. By [KK11 Thm.42], for every finitely presented, \( \mathbb{Q} \)-superperfect group \( G \) there is a 6-dimensional rational singularity \( (x \in X) \) such that \( \pi_1(\text{Lk}(x \in X)) \), the fundamental group of the link of \( (x \in X) \), is isomorphic to \( G \). Our results yield such examples also in dimensions 4 and 5. In addition, we get isolated singularities.

**Theorem 22.** Let \( G \) be a finitely presented, \( \mathbb{Q} \)-superperfect group. For \( n \geq 4 \), there is an \( n \)-dimensional rational singularity \( (x \in X) \) such that \( \pi_1(\text{Lk}(x \in X)) \cong G \).

**Proof.** By Lemma \[24\] there is a 3-dimensional, \( \mathbb{Q} \)-acyclic, simplicial complex \( \mathcal{C} \) such that \( \pi_1(\mathcal{C}) \cong G \). Thus, for \( n \geq 4 \), Theorem \[5\] constructs an \( n \)-dimensional rational singularity \( (x \in X) \) such that \( \pi_1(\text{DR}(x \in X)) \cong G \).

By [KK11 Thm.35] there is a surjection
\[
\pi_1(\text{Lk}(x \in X)) \to \pi_1(\text{DR}(x \in X)) \tag{22.1}
\]
whose kernel is finite and cyclic. The discussion at the end of [KK11 Sec.6] shows how to modify the construction to ensure that (22.1) is an isomorphism. \( \square \)
Remark 23. As discussed in [KK11, Sec.7], if \( x \in X \) is a 3-dimensional rational singularity then \( \pi_1(\mathcal{D}R(x \in X)) \) has a balanced presentation; that is, the number of relations equals the number of generators. For arbitrary rational singularities \((22.1)\) can have a very large kernel which is very hard to control in general. It is not known which groups can occur as fundamental groups of links of rational singularities in dimension 3 or higher.

The following is well known.

Lemma 24. Let \( G \) be a finitely presented group.

1. There is a 2-dimensional simplicial complex \( C_2 \) such that \( \pi_1(C_2) \cong G \).
2. There is a 3-dimensional simplicial complex \( C_3 \) such that \( \pi_1(C_3) \cong G \), \( H_1(C_3, \mathbb{Z}) \cong H_1(G, \mathbb{Z}) \) for \( i = 1, 2 \) and \( H_3(C_3, \mathbb{Z}) \cong 0 \).
3. If \( G \) is \( \mathbb{Q} \)-superperfect then \( C_3 \) is \( \mathbb{Q} \)-acyclic.

Proof. Let \((a_1, \ldots, a_r, \omega_1, \ldots, \omega_s)\) be a presentation of \( G \).

Start with \( C_1 \), a bouquet of \( r \) circles. If \( \omega_i \) has word length \( l_i \), we can kill it in \( \pi_1(C_1) \) by attaching an \( l_i \)-gon to \( C_1 \) appropriately. We can think of the \( l_i \)-gon as made up of \( l_i \) triangles. This way we have a 2-dimensional \( \Delta \)-complex \( C_2 \) whose fundamental group is \( G \). Note that \( H_3(C_2, \mathbb{Z}) \) is free.

Let \( \beta_1, \ldots, \beta_t \in H_2(C_2, \mathbb{Z}) \) be a basis of the image of the Hurewicz map \( \pi_2(C_2) \to H_2(C_2, \mathbb{Z}) \). We attach 3-balls \( B_i \) to the \( \beta_i \) to obtain a 3-dimensional \( \Delta \)-complex \( C_3 \). By construction \( \pi_1(C_3) \cong G \) (hence also \( H_1(C_3, \mathbb{Z}) \cong H_1(G, \mathbb{Z}) \)) and \( H_3(C_3, \mathbb{Z}) = 0 \).

By Hopf’s theorem,

\[
H_2(G, \mathbb{Z}) \cong \text{coker}(\pi_2(C_2) \to H_2(C_2, \mathbb{Z})) \cong H_2(C_3, \mathbb{Z})
\]

which completes the proof of (2). If \( G \) is \( \mathbb{Q} \)-superperfect then \( H_i(C_3, \mathbb{Q}) = 0 \) for \( i = 1, 2 \), hence then \( C_3 \) is \( \mathbb{Q} \)-acyclic. \( \square \)

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