DOA Estimation using a Broadband Steering Vector

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Abstract—The steering vector has traditionally been useful only for narrowband signals because it is based on a specific wavelength. It is possible to construct a more general steering vector that is based on the sample period. This allows a variety of array processing techniques to be used for broadband signals.

INTRODUCTION

Direction-of-arrival estimation of signals is an important function of array processing. This information can be used for localization applications or used for beamforming applications. A number of approaches have been devised to solve this problem. It is essentially a timing problem where the signal arrival times at the array elements can be used to determine the direction of arrival.

For narrowband signals time delays are equivalent to phase shifts with respect to the wavelength of the signal. This fact has been exploited to construct useful steering vectors that are based on phase shifts. Although these steering vectors work well with narrowband signals, they are not very useful for broadband signals which have a range of wavelengths. Another approach is needed for broadband signals.

This paper develops a novel steering vector that is purely based on time delays. The natural unit of measurement for these time delays is the sample period. This broadband steering vector approach can essentially be used just like the narrowband steering vector. This allows standard array processing methods to be easily adapted for broadband direction-of-arrival and beamforming applications.

THE NARROWBAND STEERING VECTOR

The crucial pieces of information in array processing are the delays associated with a signal’s arrival at the sensors of the array. This is based on the sensor spacing, \( a \), of the array and the incident angle, \( \theta \), of the signal. Fig. 1 shows an example of a signal delay due to the extra distance, \( \Delta \), caused by the relative geometry of the signal and the array.

The signal, \( s \), has to travel the extra distance, \( \Delta \), to the second array element, \( y_2 \). This extra distance is simply

\[
\Delta = a \sin(\theta) .
\]  

(1)

For narrowband signals it is convenient to express this distance as the radian measure of the fraction of a wavelength extra distance that the signal travels.

\[
\varphi = \frac{2 \pi}{\lambda} a \sin(\theta) .
\]  

(2)

This phase angle can then be used to effectively advance or delay the timing of incoming signals onto the array elements by multiplication of a phase term and the signal term.

\[
e^{i\varphi} e^{i\omega t} = e^{i(\omega t + \varphi)}
\]  

(3)

The narrowband steering vector, \( D \), uses this relationship to map the signal onto the array elements. This vector is given below for the case of a uniform linear array.

\[
D = [1 \ e^{i\varphi} \ e^{i2\varphi} \ldots \ e^{i(n-1)\varphi}]^T
\]  

(4)

The narrowband steering vector is used in a wide variety of signal and array processing applications.

THE BROADBAND STEERING VECTOR

Broadband signals require a slightly different approach. The fundamental problem with broadband signals is that they include a range of wavelengths. Hence, any method based on a specific wavelength will not work.

The solution for broadband signals requires a new approach. Instead of measuring the extra distance traveled as a radian fraction of a wavelength, it is proposed to measure this distance as a radian fraction of the sample period, \( T \), times the speed of the signal, \( c \).

\[
\tau = \frac{2 \pi}{c T} a \sin(\theta)
\]  

(5)

This broadband approach is actually very similar to the narrowband case. The difference is only in the unit of measurement. For broadband signals the sample period is a natural unit of measurement.

The functional utility of this representation is demonstrated by the following multiplication:

\[
e^{i\tau}e^{if(t)} = e^{i(f(t)+\tau)}
\]  

(6)

Assuming the broadband signal can be expressed as \( e^{i\varphi(t)} \), which is a reasonable assumption for most signals, then multiplying the signal by \( e^{i\tau} \) results in the desired time shift.
The steering vector for a broadband signal with \( n \) uniformly spaced elements can then be constructed as

\[
D = \begin{pmatrix}
1 & e^{i\tau_1} & \cdots & e^{i(n-1)\tau_1} \\
1 & e^{i\tau_2} & \cdots & e^{i(n-1)\tau_2} \\
\vdots & \vdots & \ddots & \vdots \\
e^{i(n-1)\tau_1} & e^{i(n-1)\tau_2} & \cdots & e^{i(n-1)\tau_n}
\end{pmatrix}^T \tag{7}
\]

This steering vector can be used in a similar manner as its narrowband sibling. However, the broadband steering vector is applicable to essentially all signals.

**BEAMFORMING BROADBAND SIGNALS**

The general \( n \)-array element and \( m \)-signal model may be written as a mapping of the signals onto the array plus noise.

\[
\begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{pmatrix} = D
\begin{pmatrix}
s_1 \\
s_2 \\
\vdots \\
s_m
\end{pmatrix} + \text{noise} \tag{8}
\]

The multiple-signal steering matrix, \( D \), can be expressed in terms of the time delays associated with the various direction of arrivals and has the following form for a uniform linear array.

\[
D = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
e^{i\tau_1} & e^{i\tau_2} & \cdots & e^{i(n-1)\tau_1} \\
\vdots & \vdots & \ddots & \vdots \\
e^{i(n-1)\tau_1} & e^{i(n-1)\tau_2} & \cdots & e^{i(n-1)\tau_n}
\end{pmatrix} \tag{9}
\]

It is often desirable to process the array data such that the individual signals can be resolved. Historically, this process has been called beamforming. Mathematically, this process requires inverting (8). The optimal way to do this in a least-squares sense is to use the Moore-Penrose inverse [1].

\[
\begin{pmatrix}
s_1 \\
s_2 \\
\vdots \\
s_m
\end{pmatrix} = (D^TD)^{-1}D^T
\begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{pmatrix} \tag{10}
\]

This inverse is effectively a very powerful beamformer, especially for multiple signals. In this representation the signals are completely decoupled or separated.

For the case of one signal the term \( D^TD \) is a scalar equal to \( n \). This allows the optimal reception of the signal to be obtained by simply steering the array in the direction of the signal.

Fig. 2 shows a generic example of a broadband chirp signal. It ranges from 5 to 10 kHz and is sampled at 100 kHz. This sample rate is unnecessarily high but does produce recognizable signals. Two sensors were used to beamform the signal. A nominal sensor spacing of half a wavelength at 20 kHz was used. A signal-to-noise ratio of 10 dB was also used.

The signal begins at time sample 50 and lasts for 100 time samples. The top plot shows the signal and noise measured at the first sensor. The bottom plot shows the signal beamformed with the broadband steering vector. It can be seen that the beamformed signal is less noisy than the signal measured at sensor 1. This is simply due to combining the random noise from two sensors while coherently summing the signal. As can be seen, the broadband vector method works very well in this case.

**Fig 2. Two-sensor array with a single chirp signal at \(-3^\circ\).**

Things are a lot more interesting for multiple signals. In this case the matrix \( D^TD \) needs to be inverted. This matrix is only \( m \times m \) and can be easily inverted. However, the off-diagonal terms of this matrix are critical in order to decouple or separate the signals.

An example is shown in Fig. 3 for two overlapping chirp signals arriving on a two element array at \( \pm 3^\circ \). This is similar to the example in Fig. 2, except two signals with time delays of 50 and 96 are used. The top plot shows the signal plus noise measured at the first sensor. The next two plots show the outputs of the broadband beamerformer. The first and second chirp signals can be clearly seen. These are completely separated and the noise level is reduced.

**Fig 3. Two-sensor array with two chirp signals at \( \pm 3^\circ \).**
A more challenging example is shown in Fig. 4 for three overlapping chirp signals arriving on a four-sensor array at +3°, 0°, and −3°. Time delays of 50, 96, and 127 are used. The top plot shows the signal plus noise measured at the first sensor. The next three plots show the outputs of the broadband beamformer. All the chirp signals can be clearly seen and are completely separated.

Fig 4. Four-sensor array with three chirp signals at ±3° and 0°.

The noise levels are reduced for the first and third signals. However, the second signal is fairly noisy. This problem can be traced back to the matrix $D^\dagger D$. Since the steering vectors are so close, the inverse starts to become poorly conditioned. This effectively allows more noise into the result. In this case the middle signal suffers most by being spatially close to the two other chirp signals. More array elements would help alleviate this problem by increasing the dimensionality of $D$.

It should be noted that the above broadband examples can also be processed purely in the time domain. This is the topic of a companion paper in this conference [2].

DIRECTION-OF-ARRIVAL ESTIMATION

The least-squares method offers a robust high resolution approach to direction-of-arrival estimation. The metric or likelihood function, $L$, of interest can be written as the least-squares difference between the measured array data, $y$, and the parametric model, $Ds$.

$$L = \|y - Ds\|^2$$  \hspace{1cm} (11)

It is useful to insert the Moore-Penrose representation for $s$ into (11). This yields the following representation for $L$.

$$L = y^\dagger y - y^\dagger D(D^\dagger D)^{-1} D^\dagger y$$  \hspace{1cm} (12)

It is usually convenient to drop the constant term $y^\dagger y$ and flip the sign to yield a maximization problem.

$$L = y^\dagger D(D^\dagger D)^{-1} D^\dagger y$$  \hspace{1cm} (13)

The likelihood function can also be expressed using a projection operator and a sample covariance matrix representation. Taking the trace of (13) and rotating the vector $y^\dagger$ to the right side of the equation yields

$$L = tr(D(D^\dagger D)^{-1} D^\dagger y y^\dagger)$$

$$= tr(D(D^\dagger D)^{-1} D^\dagger R)$$  \hspace{1cm} (14)

The least-squares metric, $L$, for the one signal case can alternatively be written as a periodogram representation or sample covariance matrix representation.

$$L = \|D^\dagger y\|^2 = y^\dagger DD^\dagger y = tr (DD^\dagger R)$$  \hspace{1cm} (15)

An example of the likelihood function for one signal is plotted with respect to the incident angle in Fig. 5. The broadband signal used arrives at the array from −3° and is exactly the one seen in Fig. 2. A variety of uniform linear arrays were used. Doubling the number of sensors decreases the width of the likelihood function by a factor of two. However, it is the maximum of the likelihood function that yields the best estimate, and these examples all peak at −3°.

Fig 5. Likelihood function for multiple arrays with signal at −3°.
Several alias signals can be seen in Fig. 5 at +20º and −26º. These are due to the high sample rate which increases the phase rate of \( \tau \). The alias signals will disappear with lower sampling rates or with a smaller value for the sensor spacing.

Fig. 6 shows the results of sampling at a 10 kHz rate. Although this is below the Nyquist rate, the likelihood function can still be calculated. More information from more sensors and more snapshots is better, since this allows the likelihood function to more accurately estimate the direction of arrival. An important consideration from a design viewpoint is that the sample frequency is a parameter that can be optimized for a given application.

Fig. 7 shows the results of sampling at a 10 kHz rate. Although this is below the Nyquist rate, the likelihood function can still be calculated. More information from more sensors and more snapshots is better, since this allows the likelihood function to more accurately estimate the direction of arrival. An important consideration from a design viewpoint is that the sample frequency is a parameter that can be optimized for a given application.

Fig. 6. Likelihood function for signal at −3º with sub-Nyquist sampling.

The multiple signal case requires a likelihood function with multiple parameters for the multiple directions. The maximum of this function yields the best estimate for the directions of arrival. An example is shown in Fig. 7 for two overlapping chirp signals. These are exactly the same +10 dB signals seen in Fig. 3.

Two maxima can be seen. This is due to the likelihood function being symmetric with respect to interchanging the two signals. The first signal might be +3º and the second −3º, or the first signal might be −3º and the second +3º. The diagonal line separates these two redundant areas. The diagonal line also represents where the two signals have the same angle and where the inverse of \( D^T D \) does not exist. This presents numerical problems, although in the analytical limit of the angles approaching each other a solution does exist.

The maxima of the likelihood function yield the best estimates for the directions of arrival. More sensors, of course, contain more information and yield likelihood functions that are more sharply peaked.

Fig. 8 shows the same example as Fig. 7 except the SNR has been reduced to 0 dB. At this limit it is difficult to recognize the signal from the noise. However, the likelihood function can still extract high resolution estimates of the directions of arrival. In general the noise mostly just increases the background levels and has little effect on the accuracy of the estimates for the 4, 6, and 8 sensor cases.
COMMENTS

- **Steering vectors are essentially based on time delays.** For narrowband signals this time delay can be represented as a fraction of the wavelength. For broadband steering vectors this time delay can be effectively represented as a fraction of the sample period.

- **A few broadband beamforming examples are shown.** These use the somewhat unappreciated but very powerful Moore-Penrose inverse as the beamformer. This approach is simple to use and allows multiple signals to be easily and naturally decoupled or separated.

- **Direction-of-arrival estimates based on a least-squares approach** were obtained for single and multiple broadband signals. High resolution estimates were obtained using the broadband steering vector even at a 0 dB SNR.

- **The sample period is a natural time unit to use for broadband processing.** This can be a design parameter that can be optimized for a particular application.

ACKNOWLEDGMENT

This work was supported in part by the In-Laboratory Independent Research Program funded by the Office of Naval Research.

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