THE $[46, 9, 20]_2$ CODE IS UNIQUE

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Abstract. The minimum distance of all binary linear codes with dimension at most eight is known. The smallest open case for dimension nine is length $n = 46$ with known bounds $19 \leq d \leq 20$. Here we present a $[46, 9, 20]_2$ code and show its uniqueness. Interestingly enough, this unique optimal code is asymmetric, i.e., it has a trivial automorphism group. Additionally, we show the non-existence of $[47, 10, 20]_2$ and $[85, 9, 40]_2$ codes.

Keywords: Binary linear codes, optimal codes

1. Introduction

An $[n, k, d]_q$-code is a $q$-ary linear code with length $n$, dimension $k$, and minimum Hamming distance $d$. Here we will only consider binary codes, so that we also speak of $[n, k, d]$-codes. Let $n(k, d)$ be the smallest integer $n$ for which an $[n, k, d]$-code exists. Due to Griesmer [7] we have

$$n(k, d) \geq g(k, d) := \sum_{i=0}^{k-1} \left\lceil \frac{d}{2^i} \right\rceil,$$

where $[x]$ denotes the smallest integer $\geq x$. As shown by Baumert and McEliece [1] for every fixed dimension $k$ there exists an integer $D(k)$ such that $n(k, d) = g(k, d)$ for all $d \geq D(k)$, i.e., the determination of $n(k, d)$ is a finite problem for every fixed dimension $k$. For $k \leq 7$, the function $n(k, d)$ has been completely determined by Baumert and McEliece [11] and van Tilborg [11]. After a lot of work of different authors, the determination of $n(8, d)$ has been completed by Bouyukliev, Jaffe, and Vavrek [4]. For results on $n(9, d)$ we refer e.g. to [5] and the references therein. The smallest open case for dimension nine is length $n = 46$ with known bounds $19 \leq d \leq 20$. Here we present a $[46, 9, 20]_2$ code and show its uniqueness. Interestingly enough, this unique optimal code is asymmetric, i.e., it has a trivial automorphism group. Speaking of a $\Delta$-divisible code for codes whose weights of codewords all are divisible by $\Delta$, we can state that the optimal code is $4$-divisible. $4$-divisible codes are also called doubly-even and $2$-divisible codes are called even. Additionally, we show the non-existence of $[47, 10, 20]_2$ and $[85, 9, 40]_2$ codes.

Our main tools – described in the next section – are the standard residual code argument (Proposition 2.2), the MacWilliams identities (Proposition 2.3), a result based on the weight distribution of Reed-Muller codes (Proposition 2.4), and the software package Q-Extension [2] to enumerate linear codes with a list of allowed weights. For an easy access to the known non-existence results for linear codes we have used the online database [6].

2. Basic tools

Definition 2.1. Let $C$ be an $[n, k, d]$-code and $c \in C$ be a codeword of weight $w$. The restriction to the support of $c$ is called the residual code $\text{Res}(C; c)$ of $C$ with respect to $c$. If only the weight $w$ is of importance, we will denote it by $\text{Res}(C; w)$.

Proposition 2.2. Let $C$ be an $[n, k, d]$-code. If $d > w/2$, then $\text{Res}(C; w)$ has the parameters

$$[n - w, k - 1, \geq d - \lfloor w/2 \rfloor].$$
Some authors call the result for the special case $w = d$ the one-step Griesmer bound.

**Proposition 2.3.** ([3] MacWilliams Identities) Let $C$ be an $[n, k, d]$-code and $C^\perp$ be the dual code of $C$. Let $A_i(C)$ and $B_i(C)$ be the number of codewords of weight $i$ in $C$ and $C^\perp$, respectively. With this, we have
\[ \sum_{j=0}^{n} K_i(j) A_j(C) = 2^k B_i(C), \quad 0 \leq i \leq n \] where
\[ K_i(j) = \sum_{s=0}^{n} (-1)^s \binom{n-j}{i-s} \binom{j}{s}, \quad 0 \leq i \leq n \] are the binary Krawtchouk polynomials. We will simplify the notation to $A_i$ and $B_i$ whenever $C$ is clear from the context.

Whenever we speak of the first $l$ MacWilliams identities, we mean Equation (2) for $0 \leq i \leq l - 1$. Adding the non-negativity constraints $A_i, B_i \geq 0$ we obtain a linear program where we can maximize or minimize certain quantities, which is called the linear programming method for linear codes. Adding additional equations or inequalities strengthens the formulation.

**Proposition 2.4.** ([5] Proposition 5, cf. [9]) Let $C$ be an $[n, k, d]$-code with all weights divisible by $\Delta := 2^a$ and let $(A_i)_{i=0, 1, \ldots, n}$ be the weight distribution of $C$. Put
\[ \alpha := \min \{ k - a - 1, a + 1 \}, \]
\[ \beta := \lceil (k - a + 1)/2 \rceil, \text{ and} \]
\[ \delta := \min \{ 2\Delta i \mid A_{2\Delta i} \neq 0 \land i > 0 \}. \]

Then the integer
\[ T := \sum_{i=0}^{\lfloor n/(2\Delta) \rfloor} A_{2\Delta i} \] satisfies the following conditions.
1. $T$ is divisible by $2^{\lceil (k-1)/(a+1) \rceil}$.
2. If $T < 2^{k-a}$, then
\[ T = 2^{k-a} - 2^{k-a-t} \] for some integer $t$ satisfying $1 \leq t \leq \max \{ \alpha, \beta \}$. Moreover, if $t > \beta$, then $C$ has an $[n, k-a-2, \delta]$-subcode and if $t \leq \beta$, it has an $[n, k-a-t, \delta]$-subcode.
3. If $T > 2^k - 2^{k-a}$, then
\[ T = 2^k - 2^{k-a} + 2^{k-a-t} \] for some integer $t$ satisfying $0 \leq t \leq \max \{ \alpha, \beta \}$. Moreover, if $a = 1$, then $C$ has an $[n, k-t, \delta]$-subcode. If $a > 1$, then $C$ has an $[n, k-1, \delta]$-subcode unless $t = a + 1 \leq k-a-1$, in which case it has an $[n, k-2, \delta]$-subcode.

A special and well-known subcase is that the number of even weight codewords in a $[n, k]$ code is either $2^{k-1}$ or $2^k$.

### 3. Results

**Lemma 3.1.** Each $[ \leq 16, 4, 7]_2$ code contains a codeword of weight 8.

**Proof.** Let $C$ be an $[n, 4, 7]_2$ code with $n \leq 16$ and $A_8 = 0$. From the first two MacWilliams identities we conclude
\[ A_7 + A_9 + \sum_{i \geq 10} A_i = 2^4 - 1 = 15 \quad \text{and} \quad 7A_7 + 9A_9 + \sum_{i \geq 10} iA_i = 2^4n = 8n, \]
so that

$$2A_0 + 3A_{10} + \sum_{i \geq 11} (i - 7)A_i = 8n - 105.$$ 

Thus, the number of even weight codewords is at most $$8n/3 - 34$$. Since at least half the codewords have to be of even weight, we obtain $$n \geq \lceil 15.75 \rceil = 16$$. In the remaining case $$n$$ we use the linear programming method with the first four MacWilliams identities, $$B_1 = 0$$, and the fact that there are exactly 8 even weight codewords to conclude $$A_{11} + \sum_{i \geq 13} A_i < 1$$, i.e., $$A_{11} = 0$$ and $$A_i = 0$$ for all $$i \geq 13$$. With this and rounding to integers we obtain the bounds $$5 \leq B_2 \leq 6$$, which then gives the unique solution $$A_7 = 7$$, $$A_6 = 0$$, $$A_{10} = 6$$, and $$A_{12} = 1$$. Computing the full dual weight distribution unveils $$B_15 = -2$$, which is negative. $$\square$$

**Lemma 3.2.** Each even [46, 9, 20]_2 code $$C$$ is isomorphic to a code with generator matrix

$$
\begin{pmatrix}
1001010101110011010100111001100100100000000
1111001010100100011010100110010000000000
110011010000111101100110000101001000000
01101001101001101011001000111001000000
00111011110110011111100111001000000
0110011001101100111001101000111000000100
000111100001111111001000000000010
000000000000000000000000000000001
\end{pmatrix}.
$$

**Proof.** Applying Proposition 2.2 with $$w = 20$$ on a [45, 9, 20] code would give a [25, 8, 10] code, which does not exist. Thus, $$C$$ has full length $$n = 46$$, i.e., $$B_1 = 0$$. Since no [44, 8, 20] code exists, $$C$$ is projective, i.e., $$B_2 = 0$$. Since no [24, 8, 9] code exists, Proposition 2.2 yields that $$C$$ cannot contain a codeword of weight $$w = 22$$. Assume for a moment that $$C$$ contains a codeword $$c_{20}$$ of weight $$w = 26$$ and let $$R$$ be the corresponding residual [20, 8, 7] code. Let $$c' \neq c_{26}$$ be another codeword of $$C$$ and $$w'$$ and $$w''$$ be the weights of $$c'$$ and $$c' + c_{26}$$. Then the weight of the corresponding residual codeword is given by $$(w' + w'' - 26)/2$$, so that weight 8 is impossible in $$R$$ ($$C$$ does not contain a codeword of weight 22). Since $$R$$ has to contain a $$\lfloor 16, 4, 7 \rfloor_2$$ subcode, Lemma 3.1 shows the non-existence of $$R$$, so that $$A_{26} = 0$$.

With this, the first three MacWilliams Identities are given by

$$A_{20} + A_{24} + A_{28} + A_{30} + \sum_{i = 1}^{8} A_{2i+30} = 511$$

$$3A_{20} - A_{24} - 5A_{28} - 7A_{30} - \sum_{i = 1}^{8} (2i + 7) \cdot A_{2i+30} = -23$$

$$5A_{20} + 21A_{24} - 27A_{28} - 75A_{30} - \sum_{i = 1}^{8} (8i^2 + 56i + 75) \cdot A_{2i+30} = 1035.$$ 

Minimizing $$T = A_0 + A_{20} + A_{24} + A_{28} + A_{32} + A_{36} + A_{40} + A_{44}$$ gives $$T \geq \frac{6712}{15} > 384$$, so that Proposition 2.2 gives $$T = 512$$, i.e., all weights are divisible by 4. A further application of the linear programming method gives that $$A_{36} + A_{40} + A_{44} \leq \lfloor \frac{2}{4} \rfloor = 2$$, so that $$C$$ has to contain a $$\lfloor 44, 7, \{20, 24, 28, 32\} \rfloor_2$$ subcode.

Next, we have used Q-Extension to classify the $$[n, k, \{20, 24, 28, 32\}]_2$$ codes for $$k \leq 7$$ and $$n \leq 37 + k$$, see Table 7. Starting from the 537799 doubly-even $$\lfloor 44, 7, 20 \rfloor_2$$ codes, Q-Extension gives 424207 doubly-even $$\lfloor 45, 8, 20 \rfloor_2$$ codes and no doubly-even $$\lfloor 44, 8, 20 \rfloor_2$$ code (as the maximum minimum distance of a $$\lfloor 44, 8 \rfloor_2$$ code is 19). Indeed, a codeword of weight 36 or 40 can occur in a doubly-even $$\lfloor 45, 8, 20 \rfloor_2$$ code. We remark that largest occurring order of the automorphism group is 18. Finally, an
application of \(Q\)-Extension on the 424207 doubly-even \([45, 8, 20]_2\) codes results in the unique code as stated. (Note that there may be also doubly-even \([45, 8, 20]_2\) codes with two or more codewords of a weight \(w \geq 36\). However, these are not relevant for our conclusion.)

\[
\begin{array}{cccccccccccc}
  k / n & 20 & 24 & 28 & 30 & 32 & 34 & 35 & 36 & 37 & 38 & 39 & 40 & 41 & 42 & 43 & 44 \\
 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 2 & 1 & 1 & 2 & 0 & 3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 3 & 1 & 1 & 2 & 4 & 6 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 4 & 1 & 3 & 10 & 13 & 26 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 5 & 3 & 15 & 163 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 6 & 24 & 3649 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Table 1. Number of \([n, k, \{20, 24, 28, 32\}]_2\) codes.

We remark that the code of Lemma 3.2 has a trivial automorphism group and weight enumerator \(1x^0 + 235x^{20} + 171x^{24} + 97x^{28} + 8x^{32}\), i.e., all weights are divisible by four. The dual minimum distance is \(3 (A_2 = 1, A_4 = 276)\), i.e., the code is projective. Since the Griesmer bound, see Inequality (1), gives a lower bound of \(47\) for the length of a binary linear code with dimension \(k = 9\) and minimum distance \(d \geq 21\), the code has the optimum minimum distance. The linear programming method could also be used to exclude the weights \(w = 40\) and \(w = 44\) directly (and to show \(A_{30} \leq 2\)). While the maximum distance \(d = 20\) was proven using the Griesmer bound directly, the \([46, 9, 20]_2\) code is not a Griesmer code, i.e., where Inequality (1) is satisfied with equality. For the latter codes the \(2^d\)-divisibility would follow from [12, Theorem 9] stating that for Griesmer codes over \(\mathbb{F}_p\), where \(p\) is a divisor of the minimum distance, all weights are divisible by \(p^d\).

**Theorem 3.3.** Each \([46, 9, 20]_2\) code \(C\) is isomorphic to a code with the generator matrix given in Lemma 3.2.

**Proof.** Let \(C\) be a \([46, 9, 20]_2\) with generator matrix \(G\) which is not even. Removing a column from \(G\) and adding a parity check bit gives an even \([46, 9, 20]_2\) code. So, we start from the generator matrix of Lemma 3.2 and replace a column by all \(2^9 - 1\) possible column vectors. Checking all \(46 \cdot 511\) cases gives either linear codes with a codeword of weight 19 or the generator matrix of Lemma 3.2 again.

**Lemma 3.4.** No \([47, 10, 20]_2\) code exists.

**Proof.** Assume that \(C\) is a \([47, 10, 20]_2\) code. Since no \([46, 10, 20]_2\) and no \([45, 9, 20]_2\) code exists, we have \(B_1 = 0\) and \(B_2 = 0\), respectively. Let \(G\) be a systematic generator matrix of \(C\). Since removing the \(i\)th unit vector and the corresponding column (with the 1-entry) from \(G\) gives a \([46, 9, 20]_2\) code, there are at least 1023 codewords in \(C\) whose weight is divisible by 4. Thus, Proposition 2.4 yields that \(C\) is doubly-even. By Theorem 3.2 we have \(A_{32} \geq 8\). Adding this extra inequality to the linear inequality system of the first four MacWilliams identities gives, after rounding down to integers, \(A_{14} = 0, A_{40} = 0, A_{36} = 0, B_2 = 0\). (We could also conclude \(B_2 = 0\) directly from the non-existence of a \([44, 8, 20]_2\)-code.) The unique remaining weight enumerator is given by \(1x^0 + 418x^{20} + 318x^{24} + 278x^{28} + 9x^{32}\). Let \(C\) be such a code and \(C'\) be the code generated by the nine codewords of weight 32. We eventually add codewords from \(C\) to \(C'\) till \(C'\) has dimension exactly nine and denote the corresponding code by \(C''\). Now the existence of \(C''\) contradicts Theorem 3.3.

So, the unique \([46, 9, 20]_2\) code is strongly optimal in the sense of [10, Definition 1], i.e., no \([n - 1, k, d]_2\) and no \([n + 1, k + 1, d]_2\) code exists. The strongly optimal binary linear codes with dimension at most seven have been completely classified, except the \([56, 7, 26]_2\) codes, in [3]. The next open case is the existence question for a \([65, 9, 29]_2\) code, which is equivalent to the existence of a \([66, 9, 30]_2\) code.
The technique of Lemma 3.2 to conclude the 4-divisibility of an optimal even code can also be applied in further cases and we give an example for [78, 9, 36]_2 codes, whose existence is unknown.

**Lemma 3.5.** Each $[\leq 33, 5, 15]_2$ code contains a codeword of weight 16.

**Proof.** We verify this statement computationally using Q-Extension. \( \square \)

We remark that a direct proof is possible too. However, the one that we found is too involved to be presented here. Moreover, there are exactly 3 $[\leq 32, 4, 15]_2$ codes without a codeword of weight 16.

**Lemma 3.6.** If an even $[78, 9, 36]_2$ code $C$ exists, then it has to be doubly-even.

**Proof.** Since no $[77, 9, 36]_2$ and no $[76, 8, 36]_2$ code exists, we have $B_1 = 0$ and $B_2 = 0$. Proposition 2.2 yields that $C$ does not contain a codeword of weight 38. Assume for a moment that $C$ contains a codeword with weight 42 and let $R$ be the corresponding residual $[36, 8, 15]_2$ code. Let $c' \neq c_{42}$ be another codeword of $C$ and $w'$ and $w''$ be the weights of $c'$ and $c' + c_{42}$. Then the weight of the corresponding residual codeword is given by $(w' + w'' - 42)/2$, so that weight 16 is impossible in $R$ ($C$ does not contain a codeword of weight 38). Since $R$ has to contain a $[\leq 33, 5, 15]_2$ subcode, Lemma 3.5 shows the non-existence of $R$, so that $A_{12} = 0$.

We use the linear programming method with the first four MacWilliams identities. Minimizing the number $T$ of doubly-even codewords gives $T \geq 1076 > 384$, so that Proposition 4 gives $T = 512$, i.e., all weights are divisible by 4. \( \square \)

Two cases where 8-divisibility can be concluded for optimal even codes are given below.

**Theorem 3.7.** No $[85, 9, 40]_2$ code exists.

**Proof.** Assume that $C$ is a $[85, 9, 40]_2$ code. Since no $[84, 9, 40]_2$ and no $[83, 8, 40]_2$ code exists, we have $B_1 = 0$ and $B_2 = 0$, respectively. Considering the residual code, Proposition 2.2 yields that $C$ contains no codewords with weight $w \in \{42, 44, 46\}$. With this, we use the first four MacWilliams identities and minimize $T = A_0 + \sum_{i=10}^{21} A_{i}$. Since $T \geq 416 > 384$, so that Proposition 2.2 gives $T = 512$, all weights are divisible by 4. Minimizing $T = A_0 + \sum_{i=5}^{10} A_{i}$ gives $T \geq 472 > 384$, so that Proposition 2.2 gives $T = 512$, i.e., all weights are divisible by 8. The residual code of each codeword of $C$ is a projective 4-divisible code of length $85 - w$. Since no such codes of lengths 5 and 13 exist, $C$ does not contain codewords of weight 80 or 72, respectively.

The residual code $\hat{C}$ of a codeword of weight 64 is a projective 4-divisible 8-dimensional code of length 21. Note that $\hat{C}$ cannot contain a codeword of weight 20 since no even code of length 1 exists. Thus we have $A_{64} \leq 1$. Now we look at the two-dimensional subcodes of the unique codeword of weight 64 and two other codewords. Denoting their weights by $a$, $b$, $c$ and the weight of the corresponding codeword in $\hat{C}$ by $w$ we use the notation $(a, b, c; w)$. W.l.o.g. we assume $a = 64$, $b \leq c$ and obtain the following possibilities: $(64, 40, 40; 8)$, $(64, 40, 48; 12)$, $(64, 40, 56; 16)$, and $(64, 48, 48; 16)$. Note that $(64, 48, 56; 20)$ and $(64, 56, 56; 24)$ are impossible. By $x_8, x_{12}, x'_{16}$, and $x'^{16}_0$ we denote the corresponding counts. Setting $x_{16} = x'_{16} + x'^{16}_0$, we have that $x_i$ is the number of codewords of weight $i$ in $\hat{C}$. Assuming $A_{64} = 1$ the unique (theoretically) possible weight enumerator is $1x^0 + 360x^{40} + 138x^{48} + 12x^{56} + 1x^{64}$. Double-counting gives $A_{40} = 360 = 2x_8 + x_{12} + x'_{16}$, $A_{48} = 138 = x_{12} + 2x'_{16}$, and $A_{56} = 12 = x'_{16}$. Solving this equation system gives $x_{12} = 348 - 2x_8$ and $x_{16} = 348 - 2x_8$. Using the first four MacWilliams identities for $\hat{C}$ we obtain the unique solution $x_8 = 102, x_{12} = 144$, and $x_{16} = 9$, so that $x'^{16}_0 = 9 - 12 = -3$ is negative – contradiction. Thus, $A_{64} \neq 0$ and the unique (theoretically) possible weight enumerator is given by $1x^0 + 361x^{40} + 135x^{48} + 15x^{56}$ ($B_3 = 60$).

Using Q-Extension we classify all $[n, k, \{40, 48, 56\}]_2$ codes for $k \leq 7$ and $n \leq 76 + k$, see Table 2. For dimension $k = 8$, there is no $[83, 8, \{40, 48, 56\}]_2$ code and exactly 106322 $[84, 8, \{40, 48, 56\}]_2$ codes exist.

\[ \text{We remark that a 4-divisible non-projective binary linear code of length 13 exists.} \]
codes. The latter codes have weight enumerators
\[ 1x^0 + (186 + l)x^{40} + (69 - 2l)x^{48} + lx^{56} \]
\((B_2 = l - 3)\), where \(3 \leq l \leq 9\). The corresponding counts are given in Table 3. Since the next step would need a huge amount of computation time we derive some extra information on a \([84, 8, \{40, 48, 56\}]_2\)-subcode of \(C\). Each of the 15 codewords of weight 56 of \(C\) hits 56 of the columns of a generator matrix of \(C\), so that there exists a column which is hit by at most \(\lfloor 56 \cdot 15/85 \rfloor = 9\) such codewords. Thus, by shortening of \(C\) we obtain a \([84, 8, \{40, 48, 56\}]_2\)-subcode with at least \(15 - 9 = 6\) codewords of weight 56. Extending the corresponding 5666 cases with \(Q\)-Extension results in no \([85, 9, \{40, 48, 56\}]_2\) code. (Each extension took between a few minutes and a few hours.)

\[\begin{array}{ccccccccccccccc}
  k & n & 40 & 48 & 56 & 60 & 64 & 68 & 70 & 72 & 74 & 75 & 76 & 77 & 78 & 79 & 80 & 81 & 82 & 83 \\
  \hline
  1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  2 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  3 & 1 & 1 & 2 & 0 & 3 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  4 & 1 & 1 & 2 & 3 & 6 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  5 & 1 & 3 & 11 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  6 & 2 & 8 & 106 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  7 & 7 & 56 & 13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}\]

**Table 2.** Number of \([n, k, \{40, 48, 56\}]_2\) codes.

\[\begin{array}{cccccccc}
  A_{56} & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
  \hline
  25773 & 48792 & 26091 & 5198 & 450 & 17 & 1 \\
\end{array}\]

**Table 3.** Number of \([84, 8, \{40, 48, 56\}]_2\) codes per \(A_{56}\).

**Lemma 3.8.** Each \([\leq 47, 4, 23]_2\) code satisfies \(A_{24} + A_{25} + A_{26} \geq 1\).

**Proof.** We verify this statement computationally using \(Q\)-Extension. \(\square\)

We remark that there are 1 \([44, 3, 23]_2\), 3 \([45, 3, 23]_2\), and 9 \([46, 3, 23]_2\) codes without codewords of a weight in \([24, 25, 26]\).

**Lemma 3.9.** Each even \([\leq 46, 5, 22]_2\) code contains a codeword of weight 24.

**Proof.** We verify this statement computationally using \(Q\)-Extension. \(\square\)

We remark that there are 2 \([44, 4, 22]_2\) and 6 \([45, 4, 22]_2\) codes that are even and do not contain a codeword of weight 24.

**Lemma 3.10.** If an even \([117, 9, 56]_2\) code exist, then the weights of all codewords are divisible by 8.

**Proof.** From the known non-existence results we conclude \(B_1 = 0\) and \(C\) does not contain codewords with a weight in \([58, 60, 62]\). If \(C\) would contain a codeword of weight 66 then its corresponding residual code \(R\) is a \([51, 8, 23]_2\) code without codewords with a weight in \([24, 25, 26]\), which contradicts Lemma 3.8. Thus, \(A_{66} = 0\). Minimizing the number \(T_4\) of doubly-even codewords using the first four MacWilliams identities gives \(T_4 \geq \frac{2048}{64} > 384\), so that Proposition 2.4 gives \(T_4 = 512\), i.e., all weights are divisible by 4.

If \(C\) contains no codeword of weight 68, then the number \(T_8\) of codewords whose weight is divisible by 8 is at least \(475.86 > 448\), so that Proposition 2.4 gives \(T_8 = 512\), i.e., all weights are divisible.
by 8. So, let us assume that \( C \) contains a codeword of weight 68 and consider the corresponding residual \([49, 8, 22]_2\) code \( R \). Note that \( R \) is even and does not contain a codeword of weight 24, which contradicts Lemma 3.9. Thus, all weights are divisible by 8. □

**Lemma 3.11.** If an even \([118, 10, 56]_2\) code exist, then its weight enumerator is either 
\[
1x^0 + 719x^{56} + 218x^{64} + 85x^{72} + 1x^{80} \text{ or } 1x^0 + 720x^{56} + 215x^{64} + 88x^{72}.
\]

**Proof.** Assume that \( C \) is an even \([118, 10, 56]_2\) code. Since no \([117, 10, 56]_2\) and no \([116, 9, 56]_2\) code exists we have \( B_1 = 0 \) and \( B_2 = 0 \), respectively. Using the known upper bounds on the minimum distance for 9-dimensional codes we can conclude that no codeword as a weight \( w \in \{58, 60, 62, 66, 68, 70\} \). Maximizing \( T = \sum_i A_{4i} \) gives \( T \geq 1011.2 > 768 \), so that \( C \) is 4-divisible, see Proposition 2.4(3). Maximizing \( T = \sum_i A_{8i} \) gives \( T \geq 1019.2 > 768 \), so that \( C \) is 8-divisible, Proposition 2.4(3). Maximizing \( A_i \), for \( i \in \{88, 96, 104, 112\} \) gives a value strictly less than 1, so that the only non-zero weights can be 56, 64, 72, and 80. Maximizing \( A_{80} \) gives an upper bound of \( \frac{3}{2} \), so that \( A_{80} = 1 \) or \( A_{80} = 0 \). The remaining values are then uniquely determined by the first four MacWilliams identities. □

The exhaustive enumeration of all \([117, 9, \{56, 64, 72\}]_2\) codes remains a computational challenge. We remark that it is not known whether a \([117, 9, 56]_2\) code exists.

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