Welded extensions and ribbon restrictions of diagrammatical moves

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Abstract

In this paper, we consider local moves on classical and welded diagrams of string links, and the notion of welded extension of a classical move. Such extensions being non-unique in general, the idea is to find a topological criterion which could isolate one extension from the others. To that end, we turn to the relation between welded string links and knotted surfaces in $\mathbb{R}^4$, and the ribbon subclass of these surfaces. This provides the topological interpretation of classical local moves as surgeries on surfaces, and of welded local moves as surgeries on ribbon surfaces. Comparing these surgeries leads to the notion of ribbon residue of a classical local move, and we show that up to some broad conditions there can be at most one welded extension which is a ribbon residue. The existence of such an extension is not guaranteed however, and we provide a counterexample.

Introduction

Knot theory aims at studying embeddings of circles in $\mathbb{R}^3$ up to ambient isotopies and, more generally, embeddings of codimension 2 submanifolds in $\mathbb{R}^n$. As shown by K. Reidemeister in dimension 3, and then extended to dimension 4 by D. Roseman, this topology–based study can be translated in a combinatorial way through the use of diagrams, which are generic projections of the submanifold onto $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$, up to local moves corresponding to some elementary local isotopies. Using this diagrammatic approach, one can extend the classical notion of knotted objects to the notion of welded objects, first defined for braids by R. Fenn, R. Rimányi and C. Rourke in [7]. This welded theory is a quotient of the virtual extension, defined independantly by L. Kauffman in [10] and M. Goussarov, M. Polyak and O. Viro in [8] by allowing a new type of, so-called virtual, crossings on diagrams and a new type of, so-called detour, moves which freely trade any piece of strand supporting only virtual crossings for any other such virtual strand with same extremities. For welded objects, strands are also allowed to pass above (but not under) virtual crossings. Whereas Kauffman/Goussarov–Polyak–Viro’s works are motivated by combinatorial aspects of link diagrams description, T. Brendle and A. Hatcher showed in [5] that Fenn–Rimányi–Rourke’s one is much more related to 4–dimensional topology, as their braid-permutation groups are closely related to paths of circles configurations, which are surfaces in $\mathbb{R}^4$ just like paths of points configurations are topological braids. As a matter of fact, welded link theory can be seen as an intermediary step between classical links and knotted surfaces. For general welded objects, the connection with knotted surfaces was made clear by S. Satoh [15] who extended the Tube map, first defined for classical objects by T. Yajima [17] by, roughly speaking, inflating strands into knotted tubes in $\mathbb{R}^4$, to any such welded object. In particular, as it is the codomain of the Tube map, it emphasized the important role played by the ribbon subclass of knotted surfaces, corresponding to embedded surfaces which are the boundary of immersed solid handlebodies with only ribbon singularities.

Besides ambient isotopies, other topological quotients were combinatorially modelized using additional local diagrammatical moves. In [2], B. Audoux, P. Bellingeri, J-B. Meilhan and E. Wagner started to study the question of potential welded extensions for such additional moves. Even though motivated by topological quotients in dimension 4, their study remained close to the classical knot theory side of the welded theory, a local move $M_w$ on welded diagrams being indeed said to extend a given local move $M_c$ on classical diagrams if two classical diagrams were related up to $M_w$ and welded Reidemeister moves if and only if they were related up to $M_c$ and classical Reidemeister moves. Surprisingly enough, it appeared that there exists classical local...
moves, e.g. the $\Delta$ move (see Figure 8), admitting multiple non-equivalent welded extensions.

The main goal of the present paper is to resolve such ambiguities by making the study of welded extensions closer to topology, using the knotted surface theory side of the welded theory. Indeed, another (actually equivalent) way to relate classical knots with knotted surfaces is to spin a 1–dimensional knotted object in $\mathbb{R}^3 \subset \mathbb{R}^4$ around a plane to obtain a surface. When similarly spinning a classical diagram, one obtains a broken surface diagram, the 4–dimensional counterpart of link diagrams, and when spinning a classical local move $M_c$, one obtains a surgery operation $\text{Spun}(M_c)$ which modifies in an explicit way broken surface diagrams inside some solid torus, and hence knotted surfaces inside some $S^1 \times B^3 \subset \mathbb{R}^4$. A welded local mode $M_w$ is then said to be a ribbon residue of $M_c$ if two ribbon surfaces $S^1 = \text{Tube}(L_1)$ and $S_2 = \text{Tube}(L_2)$ are related by $\text{Spun}(M_c)$ surgeries as knotted surfaces if and only if $L_1$ and $L_2$ are related by $M_w$ and welded Reidemeister moves.

As in [2], we focus on the string link case, which are embedded intervals with prescribed fixed ends, and consider specifically three local moves, namely $SC$ which modelizes link-homotopy, $\Delta$ which modelizes link-homology and $BP$ which modelizes band-passing (see Figure 8). More precisely:

- for $SC$, it was proven in [2] that the self-virtualization move $SV$, which turns any classical crossing involving portions of the same strand to a virtual crossing, is a welded extension. Without surprise, we prove that it is also a ribbon residue (Theorem 2.14);
- for $\Delta$, it was proven in [2] that both the fused move $F$, which allows any strand to pass under classical crossings, and the virtual conjugation move $VC$, which surrounds a classical crossing by two virtual ones, are welded extension. We prove that $F$ is a ribbon residue while $VC$ is not (Theorem 2.17). This provides a way to designate $F$ as a preferred welded extension, carrying more topological meaning;
- for $BP$, a welded extension was given in [2], but we prove that it is not a ribbon residue. More strikingly, there is no welded extension which is a ribbon residue. Indeed, we prove that the virtual band-passing move $BV$, which turns all four classical crossings of a band-passing patterns into virtual ones, is a ribbon residue (Theorem 2.20) even if, surprisingly enough, it is not a welded extension.

The paper is organized as follows. In Section 1, we set the global background: the general notation is set in Section 1.1, welded knot theory and its relationship with ribbon surfaces are presented in Section 1.2, and local moves, welded extensions and ribbon residues are defined in Section 1.3. Section 2 is devoted to the above-mentionned moves, $SC$ in Section 2.1, $\Delta$ in Section 2.2 and $BP$ in Section 2.3.

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1 Settings

1.1 Notation

We begin by introducing some notation. Let $n$ and $d$ be positive integers. We denote by $I := [0, 1]$ the unit interval, $B^d$ the closed unit ball in $\mathbb{R}^d$ and $S^d = \partial B^{d+1}$ the $d$-dimensional sphere. Let $B^{d,1} := B^d \times I$, $S^{d,1} := S^d \times I$, and $\partial_s B^{d,1} := B^d \times \{\varepsilon\}$, $\partial_s S^{d,1} := S^d \times \{\varepsilon\}$ for $\varepsilon = 0, 1$. The manifolds $I$ and $B^d$ are given their usual orientation (induced by the canonical orientation of $\mathbb{R}^d$), $S^d$ is oriented as the boundary of $B^{d+1}$, and $B^{d,1}$, $S^{d,1}$ are given the product orientation. Manifolds and maps are always in the smooth category.
We will work with submanifolds of $B^{d,1}$ which have a fixed cartesian product structure near $\partial_0 B^{d,1} \cup \partial_1 B^{d,1}$. More precisely, let $X$ be a manifold, and $b : X \to \hat{B}^d$ be an embedding. We will consider embeddings (resp. immersions) $f : X \times I \to B^{d,1}$ for which there exists a $\delta > 0$ such that:

- $f(x,t) = (b(x),t)$ for $t \in [0, \delta) \cup (1 - \delta, 1]$;
- $f(X \times [\delta, 1 - \delta]) \subset \hat{B}^{d,1}$.

We call the image $Y = f(X \times I)$ an embedded (resp. immersed) submanifold of $B^{d,1}$. We denote by $\partial_i Y := f(X \times \{ \varepsilon \})$ for $\varepsilon = 0, 1$ and $\partial_s Y := f(\partial X \times I)$ the lower, upper and lateral boundary of $Y$ respectively.

In what follows, we will consider sets of such submanifolds for a fixed oriented $X$ (typically a disjoint union of balls or spheres) and a fixed embedding $b$. Thanks to the boundary condition, we can define the stacking product $Y_1 \cdot Y_2$ for $Y_i = f_i(X \times I), i = 1, 2$, by:

$$Y_1 \cdot Y_2 = f(X \times I), \quad f(x,t) = \begin{cases} f_1(x, 2t) & \text{if } t \in [0, \frac{1}{2}], \\ f_2(x, 2t - 1) & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

To preserve the cartesian product structure of submanifolds near the lower and upper boundaries, we will only consider isotopies of $B^{d,1}$ which are the identity in a neighborhood of $\partial_0 B^{d,1} \cup \partial_1 B^{d,1}$.

Let $p_1 < \ldots < p_n$ be $n$ ordered points in the interval $(-1, 1)$, fixed once and for all (for example take $p_i = (2i - 1 - n)/n$). Moreover, let $b_D : B^2_1 \sqcup \ldots \sqcup B^2_n \to \hat{B}^3$ be an embedding of $n$ disjoint disks, and $b_C : S^1_1 \sqcup \ldots \sqcup S^1_n \to \hat{B}^3$ its restriction to the circles $S^1 := \partial B^2_i$. We will use these as our fixed $b$ in Definitions 1.6 and 1.7.

We will also make use of some algebraic notions: for a group $G$ normally generated by some elements $x_1, \ldots, x_n$, we denote by $RG$ the reduced group defined as the quotient of $G$ by the normal subgroup generated by the commutators $[x_i, g x_i g^{-1}]$ for $1 \leq i \leq n$ and $g \in G$. It is the biggest quotient of $G$ in which the $x_i$’s commute with their conjugates.

For a group $G$ normally generated by $x_1, \ldots, x_n$, we denote by $\text{End}_C(G)$ (resp. $\text{Aut}_C(G)$) the set of conjugating endomorphisms (resp. automorphisms) of $G$, i.e. the subset of $\text{End}(G)$ (resp. $\text{Aut}(G)$) whose elements send each $x_i$ to one of its conjugates. We also define $\text{Aut}_C^2(G)$ as the subset of $\text{Aut}_C(G)$ whose elements send the product $x_1 \cdots x_n$ to itself.

1.2 Welded theory

1.2.1 Definition

**Definition 1.1.** A string link is an embedding of $\sqcup_{1 \leq i \leq n} I_i \{1, \ldots, n\} \times I$ in $B^{2,1}$ with $b(i) = (0, p_i) \in \hat{B}^2$. The $I_i$ are called the strands of the string link, and are oriented from $\partial_0 I_i$ to $\partial_1 I_i$. We denote by $\mathcal{SL}_n$ the set of string links up to isotopy. It is given a monoid structure by the stacking product.

A string link can be represented in two dimensions by taking a generic projection on a plane. Up to isotopy, the string link can be arranged so that the singularities of its projection are transverse double points, called crossings. These crossings are represented by erasing part of the lower strand, and given a sign according to the orientation of the strands as indicated below:

- positive crossing
- negative crossing
Welded string links can be defined using this diagrammatic approach. First, we need to consider a third type of crossing, called virtual crossing:

\[
\begin{array}{c}
\text{X} \\
\end{array}
\]

**Definition 1.2.** A virtual string link diagram is an immersion of \(\bigcup_{1 \leq i \leq n} I_i\) in \(B^{1,1}\) such that:

- \(b(i) = p_i\), and \(I_i\) is oriented from \((p_i, 0)\) to \((p_i, 1)\);
- there is a finite number of singularities, which are transverse double points;
- each double point is labelled to indicate a positive, negative or virtual crossing.

We denote by \(vSLD_n\) the set of virtual string links up to isotopy and reparametrization. It is given a monoid structure by the stacking product. We denote by \(SLD_n\) the subset of \(vSLD_n\) composed of diagrams with no virtual crossing, which are called classical diagrams.

For a diagram \(D \in vSLD_n\), we call overstrands the portions of strands delimited by the undercrossings (i.e. the point on the lower strand at a classical crossing). If \(D\) is a classical diagram, the overstrands are simply the connected components obtained after erasing parts of the lower strands as described above. The overstrands connected to \(\partial_0 B^{1,1}\) are called the bottom overstrands, and the ones connected to \(\partial_1 B^{1,1}\) are called the top overstrands.

Using the construction given right after Definition 1.1, every string link can be represented by a classical string link diagram. In order to obtain a one-to-one correspondence between string links and diagrams, we need to identify certain diagrams.

As proven by Reidemeister (see [13] in the case of knots and links, which extends to string links), two classical diagrams represent the same string link if and only if one can be obtained from the other by applying some local moves (loosely speaking, the modification of a diagram inside a disk, see Definition 1.11 for more details). These are called Reidemeister moves, and are illustrated in Figure 1.

\[
\begin{array}{c}
\text{Figure 1: Reidemeister moves} \\
\end{array}
\]

We use this diagrammatical approach to define welded string links, by considering virtual string link diagrams up to some local moves. We keep the (classical) Reidemeister moves, but also add new moves involving virtual crossings, illustrated in Figure 2.

We denote by Reid (resp. vReid) the classical (resp. virtual) Reidemeister moves \(R_1, R_2, R_3\) (resp. \(vR_1, vR_2, vR_3\)), and by wReid the welded Reidemeister moves, consisting of Reid, vReid, Mixed and \(OC\).

**Definition 1.3.** We define by \(wSL_n := vSLD_n / \{wReid\}\) the monoid of welded string links.

Welded string links can be represented in a more combinatorial way by Gauss diagrams.
**Definition 1.4.** A *Gauss diagram* is a finite set of triplets \((t, h, \varepsilon) \in (\bigcup_{1 \leq i \leq n} I_i)^2 \times \{\pm 1\}\) such that the \(t\)'s and the \(h\)'s are all distinct. These triplets are called *arrows*, with a *tail* \(t\) and a *head* \(h\) on the \(n\) strands, and a *sign* \(\varepsilon\). We denote by \(GD_n\) the set of Gauss diagrams up to isotopy.

A virtual diagram can be described by a Gauss diagram by associating an arrow to each classical crossing, the tail (resp. the head) indicating the position of the preimage on the upper (resp. lower) strand, and the sign indicating the type of crossing. Virtual crossings are not represented.

Similarly to string link diagrams, we need to allow local moves on Gauss diagrams in order to obtain a one-to-one correspondence with welded string links. A local move on a Gauss diagram consists in modifying some arrows on some pieces of strands. The Gauss diagram equivalent of the welded Reidemeister moves are illustrated in Figure 4. Since the virtual Reidemeister moves only involve virtual crossings, they do not affect Gauss diagrams, and neither does the Mixed move. The \(R3\) move is labelled with a \((\ast)\) to indicate that it must satisfy some sign conditions: to apply \(R3\), we must have \(\delta_1 \varepsilon_1 = \delta_2 \varepsilon_2 = \delta_3 \varepsilon_3\), where \(\delta_i = 1\) or \(-1\) depending on whether \(i\)th potion of strand is oriented upward or downward.

To go back from a Gauss diagram to a virtual string link diagram, we start by drawing the classical crossings that are indicated by the arrows, and then join the pieces of strands together according to the
order of the arrow extremities, potentially creating virtual crossings if needed. The purpose of the virtual Reidemeister and Mixed moves is to remove the ambiguity between the different ways of joining the strands. In particular, they enable what is called the detour move: if a portion of a strand only involves virtual crossings, it can be changed for any other portion of strand involving only virtual crossings and having the same extremities.

It is well known and straightforwardly checked that up to these local moves, diagrams and Gauss diagrams are faithful representations of string links:

**Proposition 1.5.** The following monoid isomorphisms hold:

- $\mathcal{S}L_n \simeq \mathcal{SL}_D_n / \{\text{Reid}\}$;
- $v\mathcal{SL}_D_n / \{v\text{Reid}, \text{Mixed}\} \simeq \mathcal{GD}_n$;
- $w\mathcal{S}L_n := v\mathcal{SL}_D_n / \{w\text{Reid}\} \simeq \mathcal{GD}_n / \{\text{Reid}, \text{OC}\}$.

### 1.2.2 Relation with knotted surfaces

String links can be related to knotted surfaces through two maps, called Spun and Tube. The Spun map consists in spinning a classical string link around a plane in 4 dimensions to obtain a surface, while the Tube map, first defined for classical knots by T. Vajima in [17] and then extended to the welded case by S. Satoh in [15], consists in inflating a welded string link and taking the boundary to obtain “tubes”. One important fact is that the Tube map sends welded string links to the ribbon subclass (see Definition 1.8 below) of the surfaces considered here.

A knotted surface is an embedding of a surface in $\mathbb{R}^4$. As in the case of knots, such surfaces can be projected on a hyperplane, in order to obtain a surface in $\mathbb{R}^3$ with three types of singularities (see [14] or [6]): lines of double points (where it is locally the intersection of two planes), isolated triple points (locally the intersection of three planes in a point) and isolated branch points (where the projection is not an immersion).

Such a projection, together with the information of upper/lower parts of the surface at singularities, is a called *broken surface diagram* of the knotted surface. As in the 1–dimensional case, a broken surface diagram can be represented by deleting thin bands around the lines of double points on the lower part of the diagram. This is illustrated below, where the upper and lower parts of the diagrams have been chosen arbitrarily:
As in the case of knots, there are local moves on broken surface diagrams which identify different diagrams associated to the same knotted surface. These are called Roseman moves (see [14] for a detailed description).

We now define an equivalent of string links in the case of surfaces.

**Definition 1.6.** A string 2–link is an embedding of \( \bigsqcup_{1 \leq i \leq n} S^1_i \times I \) in \( B^{3,1} \) with \( b = b_C \) as our fixed boundary embedding. We denote by \( 2\mathcal{SL}_n \) the set of string 2–links up to isotopy. It is given a monoid structure by the stacking product.

We will also consider the subclass of ribbon string 2–links, which is the string 2–link equivalent of the ribbon subclass of knots.

**Definition 1.7.** A 3–ribbon is an immersion of \( \bigsqcup_{1 \leq i \leq n} B^2_{i,1} = \left( \bigsqcup_{1 \leq i \leq n} B^2_i \right) \times I \) in \( B^{3,1} \) with \( b = b_D \), and a singular set composed of a finite number of ribbon singularities, which are defined as follows: a connected singularity \( \delta \) is ribbon if it is a disk given by a transverse intersection of the images of two components \( B^2_{i,1} \) and \( B^2_{j,1} \) (with possibly \( i = j \)), with preimages \( \delta_c \subset B^2_{i,1} \) and \( \delta_{ess} \subset B^2_{j,1} \) satisfying the following conditions:

- \( \delta_c \subset \tilde{B}^{2,1}_{i}; \)
- \( \delta_{ess} \subset \tilde{B}^{2,1}_{j} \text{ and } \partial \delta_{ess} \subset \partial B^2_{j} \times I \) is non-trivial in \( H_1(\partial B^2_{j} \times I) \).

We call \( \delta_c \) the contractible preimage and \( \delta_{ess} \) the essential preimage.

By considering the images of tangent vectors at the preimages \( x_c \in \delta_c \) and \( x_{ess} \in \delta_{ess} \) of a point \( x \in \delta \), we can associate a sign to a ribbon singularity. See [1, §3.2.1] for more details.

**Definition 1.8.** A ribbon string 2–link is a string 2–link \( L \) which is the lateral boundary of a 3–ribbon \( R: L = \partial_s R \). We say that \( R \) is a ribbon filling of \( L \). We denote by \( 2\mathcal{rSL}_n \) the monoid of ribbon string 2–links.

As described in Section 3.2 of [1], we can associate a Gauss diagram to each 3–ribbon, with arrows corresponding to ribbon singularities. This induces a one-to-one monoid morphism between 3–ribbons up to isotopy and Gauss diagrams up to the OC move. The inverse of this morphism becomes invariant under Reidemeister moves when composed with the “lateral boundary” map \( \partial_s : \{3–ribbon\}/\{isotopy\} \to 2\mathcal{rSL}_n \), and induces a surjective morphism Tube : \( GD_n/\{OC\} \to 2\mathcal{rSL}_n \).

Figure 6 illustrates what a broken surface diagram of a ribbon string 2–link looks like at a ribbon singularity. We can give a more geometric definition of this Tube map in terms of broken surface diagrams:
Definition 1.9. For a welded string link \( L \in wSL_n \), let \( D \in vSLD_n \) be a diagram of \( L \), which we place in \( \{0\} \times B^{2,1} \subset B^{2,1} \). Let \( N \) be a tubular neighborhood of \( D \), and \( \partial N \) its lateral boundary. At each crossing of \( D \), we modify \( \partial^* N \) as indicated on Figure 7: a positive (resp. negative) crossing gives a broken surface diagram of a positive (resp. negative) ribbon singularity, and a virtual crossing gives two disjoint tubes. The Tube map is then defined as sending \( L \) to the element of \( 2-rSL_n \) represented by this broken surface diagram.

We now define the Spun map, which gives another way to obtain a knotted surface from a string link. Note however that this map is only defined on classical string links, while the Tube map is defined on welded objects.

Definition 1.10. Let \( L \in SL_n \) be given by a parametrization \( f(i, t) = (x_i(t), y_i(t), z_i(t)) \in B^{2,1} \) for \( 1 \leq i \leq n \) and \( t \in [0, 1] \). Then \( \text{Spun}(L) \) is defined to be the string 2-link parametrized by:

\[
(i, t, \theta) \in \{1, \ldots, n\} \times [0, 1] \times [0, 2\pi] \mapsto \left( \frac{x_i(t)}{2}, \frac{y_i(t) - 1}{2} \cos(\theta), \frac{y_i(t) - 1}{2} \sin(\theta), z_i(t) \right) \in B^{3,1}.
\]

In other words, we start by placing \( L \) in \( B^2((0, -\frac{1}{2}), \frac{1}{2}) \times \{0\} \times [0, 1] \subset B^{3,1} \subset \mathbb{R}^4 \) by applying the map \( (x, y, z) \mapsto (x/2, (y-1)/2, 0, z) \), then we take its trace under a complete rotation around the plane \( \mathbb{R} \times \{0\}^2 \times \mathbb{R} \). This defines a map \( \text{Spun} : SL_n \to 2\mathcal{S}L_n \). Indeed, an isotopy of \( B^{2,1} \) induces an isotopy of \( B^{3,1} \) by the same construction.

Suppose \( L \in S\mathcal{L}_n \) is represented by a diagram \( D \in SLD_n \), obtained by projecting \( L \) onto the \((yOz)\) plane. Then \( \text{Spun}(L) \) is represented by the broken surface diagram obtained by rotating \( D \), parametrized...
by:

\[(i, t, \theta) \in \{1, \ldots, n\} \times [0, 1] \times [0, 2\pi] \mapsto \left( \frac{y_i(t) - 1}{2} \cos(\theta), \frac{y_i(t) - 1}{2} \sin(\theta), z_i(t) \right) \in B^{2,1},\]

with the upper/lower information being given by the value of \(x_i(t)\). Note that the only singularities of this broken surface diagram are lines of double points, obtained by rotating the crossings of \(D\).

### 1.3 Local moves

We will now consider local moves in a general way.

#### 1.3.1 Definition

**Definition 1.11.** A *local move* on a virtual string link diagram is given by two pieces of diagram inside a disk, which are identical near the boundary. Two diagrams are related by this move if they differ only inside a disk, where each of them coincides with one of the piece of diagram given by the local move.

![Diagram](image)

Figure 8: Some local moves on virtual string link diagrams

A few examples are illustrated on Figure 8, where a dotted line indicates that the crossing occurs between two portions of the same strand. Some local moves on classical diagrams are derived from topological operations on links. For example, the \(CC\) (for “crossing change”) move corresponds to homotopy, which allows strands to cross each other, while \(SC\) (for “self-crossing change”) corresponds to link homotopy, only allowing each strand to cross itself. The \(BP\) (for “band pass”) move represents the crossing of two “bands”, delimited by parallel strands.

In an effort to extend the effect of some classical local moves to welded string links, certain local moves on virtual diagrams are introduced which closely resemble their classical counterpart. For example, \(V\) (for “virtualization”) and \(SV\) (for “self-virtualization”) are derived from \(CC\) and \(SC\), while \(F\) (for “fused”) is derived from \(\Delta\), and \(BV\) (for “band virtualization”) is derived from \(BP\). This notion of extension will be discussed in the next section.
Finally, some virtual local moves come naturally from Gauss diagrams. For example, $VC$ (for “virtual conjugation”) reverses the orientation of the arrow representing the classical crossing, while $SR$ (for “sign reversal”) changes its sign.

We also give their version on Gauss diagrams in Figure 9. The dotted lines in $SC$ and $SV$ indicate that the extremities of the arrows belong to the same strand, but are not necessarily adjacent on this strand. These are called self-arrows. As before, the $(\ast)$ indicates the presence of some sign conditions: the $\Delta$ move must verify the same conditions as the $R3$ move, while the $BP$, $wBP$ and $BV$ moves must verify $\varepsilon_{ij}\varepsilon_{kl} = \delta_i\delta_j\delta_k\delta_l$, where the $\delta$'s are defined as in Section 1.2.1.

**Definition 1.12.** Let $M_1$ and $M_2$ be local moves on classical (resp. virtual) string link diagrams. We say that $M_2$ $c$-generates (resp. $w$-generates) $M_1$ if $M_1$ can be realised using $M_2$ and classical (resp. welded) Reidemeister moves. We denote it by $M_2 \xrightarrow{c} M_1$ (resp. $M_2 \xrightarrow{w} M_1$). If $M_1 \xrightarrow{c} M_2$ and $M_2 \xrightarrow{c} M_1$ (resp. $M_1 \xrightarrow{w} M_2$ and $M_2 \xrightarrow{w} M_1$), we say that $M_1$ and $M_2$ are $c$-equivalent (resp. $w$-equivalent).

For a classical (resp. virtual) local move $M$, we denote by $\mathcal{SL}_n^M$ (resp. $w\mathcal{SL}_n^M$) the quotient of classical (resp. welded) string links by the equivalence relation induced by this move. We then have $M_2 \xrightarrow{c} M_1$ (resp. $M_2 \xrightarrow{w} M_1$) if and only if the identity map of $\mathcal{SL}_n$ (resp. $w\mathcal{SL}_n$) induces a well defined map $\mathcal{SL}_n^M \rightarrow \mathcal{SL}_n^{M_2}$ (resp. $w\mathcal{SL}_n^M \rightarrow w\mathcal{SL}_n^{M_2}$).
Examples: As proven in [2], we have the following relations:

\[ V \cong CC, \quad SV \cong SC, \quad F \cong UC, \quad VC \cong F \cong \Delta, \quad wBP \cong SR,F,BP. \]

1.3.2 Welded extension

If \( M_c \) is a classical local move and \( M_w \) is a local move such that \( M_w \Rightarrow M_c \), then the inclusion \( SLD_n \rightarrow vSLD_n \) induces a map \( \mathcal{S}L_n^{M_c} \rightarrow w\mathcal{S}L_n^{M_w} \).

**Definition 1.13.** When a local move \( M_w \) w-generates a classical move \( M_c \), we say that \( M_w \) is a welded extension of \( M_c \) if the induced map \( \mathcal{S}L_n^{M_c} \rightarrow w\mathcal{S}L_n^{M_w} \) is injective.

The local move \( M_w \) extends \( M_c \) in the sense that if two classical diagrams are related by welded Reidemeister moves and \( M_w \), then they are also related by classical Reidemeister moves and \( M_c \).

**Definition 1.14.** Let \( M \) be a classical (resp. welded) local move, \( A \) a monoid and \( \phi : SLD_n \rightarrow A \) (resp. \( \phi : vSLD_n \rightarrow A \)) a monoid morphism. We say that \( \phi \) c-classifies (resp. w-classifies) \( M \) if it is preserved by \( M \) and classical (resp. welded) Reidemeister moves, and the induced map \( \bar{\phi} : \mathcal{S}L_n^M \rightarrow A \) (resp. \( \bar{\phi} : w\mathcal{S}L_n^M \rightarrow A \)) is an isomorphism.

**Definition 1.15.** For \( i,j \in \{1,\ldots,n\}, i \neq j \), we define the virtual linking number \( vlk_{ij} : vSLD_n \rightarrow \mathbb{Z} \) by counting, with signs, the number of crossings where \( I_i \) passes over \( I_j \). If \( D \in vSLD_n \) is represented by a Gauss diagram, \( vlk_{ij}(D) \) is the number of arrows from \( I_i \) to \( I_j \), counted with their sign. We note \( lk_{ij} : SLD_n \rightarrow \mathbb{Z} \) the restriction of \( vlk_{ij} \) to classical string link diagrams.

The (virtual) linking numbers are preserved by classical and welded Reidemeister moves, hence they are well-defined on classical and welded string links. By taking combinations of the linking numbers, we obtain classifying invariants for certain local moves. We will make use of \( vlk_{ij} := \sum_{j \neq i} vlk_{ij}, vlk_{si} := \sum_{j \neq i} vlk_{ji} \) and \( vlk_{ij}^{\mod} := vlk_{ij} \mod 2 \in \mathbb{Z}_2 \). The same notation is used on classical linking numbers.

**Proposition 1.16.** We have the following classification:

- \([2] \quad (lk_{ij})_{1 \leq i < j \leq n} : SLD_n \rightarrow \mathbb{Z}^{(n-1)/2} \) c-classifies \( \Delta \);
- \([2] \quad (vlk_{ij} - vlk_{ji})_{1 \leq i < j \leq n} : vSLD_n \rightarrow \mathbb{Z}^{(n-1)/2} \) w-classifies \( CC \);
- \([2] \quad (vlk_{ij})_{1 \leq i \neq j \leq n} : vSLD_n \rightarrow \mathbb{Z}^{(n-1)} \) w-classifies \( F \);
- \([2] \quad (vlk_{ij} + vlk_{ji})_{1 \leq i < j \leq n} : vSLD_n \rightarrow \mathbb{Z}^{(n-1)/2} \) w-classifies \( VC \);
- \([11] \quad \text{and} \quad [12] \quad (lk_{is}^{\mod})_{1 \leq i \leq n-1} : SLD_n \rightarrow \mathbb{Z}_2^{(n-1)} \) c-classifies \( BP \);
- \([2] \quad (vlk_{ij}^{\mod} + vlk_{ji}^{\mod})_{1 \leq i < j \leq n} + (vlk_{ij}^{\mod})_{1 \leq i < n-1} : vSLD_n \rightarrow \mathbb{Z}^{(n-1)/2} \oplus \mathbb{Z}_2 \) w-classifies \( wBP \).

From this classification, we obtain the following extensions:

**Corollary 1.17.** The \( F \) and \( VC \) moves both extend \( \Delta \), and the \( wBP \) move extends \( BP \).

It can be noted that, as in the case of \( \Delta \), a classical move can have several welded extensions. When so, we will try to use the relation between string links and knotted surfaces to isolate one extension among the others.

1.3.3 Ribbon residue

In this section, we provide two ways to extend the action of the local moves on (welded) string links to (ribbon) string 2-links using the Tube and Spun maps. The final goal is to compare the action of a move \( Tube(M_w) \) to the restriction on ribbon string 2-links of the action of \( Spun(M_c) \) when \( M_w \) is a welded extension of \( M_c \).

**Definition 1.18.** Let \( M \) be a local move. Let us consider the binary relation on \( 2r\mathcal{S}L_n \) which identifies two elements having \( M \)-equivalent preimages by \( Tube \) in \( w\mathcal{S}L_n \). We define \( Tube(M) \) as the transitive closure of this relation, which is an equivalence relation on \( 2r\mathcal{S}L_n \).
Remark: If $M$ w-generates the $SV$ move, we can show (see the last remark in section 2.1.3) that the binary relation defined above is already an equivalence relation, so there is no need to take its transitive closure. In this case, the induced map $\text{Tube} : w\mathcal{SL}_n^M \to 2^{-r}\mathcal{SL}_n/\text{Tube}(M)$ is bijective. Hence for two local moves $M_1$ and $M_2$, each w-generating $SV$, we have $\text{Tube}(M_1) = \text{Tube}(M_2)$ if and only if $M_1$ and $M_2$ are w-equivalent.

Definition 1.19. Let $L$ and $L'$ be string 2–links represented by broken surface diagrams $D$ and $D'$. For a classical local move $M$, we say that $L$ and $L'$ are related by a $\text{Spun}(M)$ move if there exists a solid torus $T \subset B^{2,1}$ such that $D = D'$ outside of $T$, and $D$ and $D'$ differ in $T$ by the spun of the move $M$. We denote by $\text{Spun}(M)$ the equivalence relation on $2^{-r}\mathcal{SL}_n$ which identifies two such string 2–links $L$ and $L'$.

For a classical local move $M$ (i.e. involving only classical crossings), we can consider both $\text{Spun}(M)$ and $\text{Tube}(M)$. In general, the restriction of $\text{Spun}(M)$ to ribbon string 2–links is not equal to $\text{Tube}(M)$. For example, for $M = CC$, every ribbon string 2–links is trivial up to $\text{Spun}(CC)$ (see the example below), while $2^{-r}\mathcal{SL}_n/\text{Tube}(CC)$ is not trivial. This can be proved using the classification of $CC$ on welded string links and the generalization of linking numbers for string 2–links developed in section 2.2.

Definition 1.20. Let $M_c$ be a classical local move. We say that a local move $M_w$ is a ribbon residue of $M_c$ if $\text{Tube}(M_w)$ is the restriction to $2^{-r}\mathcal{SL}_n$ of the equivalence relation $\text{Spun}(M_c)$ on $2^{-r}\mathcal{SL}_n$. This amounts to say that two ribbon string 2–links are equivalent under $\text{Tube}(M_w)$ if and only if they are equivalent under $\text{Spun}(M_c)$ in the set of string 2–links.

Notation: Considering an equivalence relation $\mathcal{R}$ on a set $X$ as its defining subset $\{(x, x') \mid x \mathcal{R} x'\}$ of $X \times X$, for $Y \subset X$ we denote by $\mathcal{R}|_Y := \mathcal{R} \cap (Y \times Y)$ the restriction of $\mathcal{R}$ to $Y$. If $\mathcal{R}$ and $\mathcal{R}'$ are two equivalence relations on $X$, $\mathcal{R} \subset \mathcal{R}'$ means that $x \mathcal{R} x'$ implies $x \mathcal{R}' x'$ for $x, x' \in X$.

With this notation, a local move $M_w$ is a ribbon residue of $M_c$ if and only if $\text{Spun}(M_c)|_{2^{-r}\mathcal{SL}_n} = \text{Tube}(M_w)$.

From the remark above, it follows that a classical move can have at most one ribbon residue which w-generates the $SV$ move. As we will see in section 2.2, the $F$ and $VC$ moves both w-generate $SV$, and by their classification they are not w-equivalent, so only one of them (if any) can be a residue of $\Delta$.

Example: It is not difficult to see that $V$ is a ribbon residue of $CC$: since $w\mathcal{SL}_n^V$ is trivial, so is $2^{-r}\mathcal{SL}_n/\text{Tube}(V)$, so it is enough to verify that $\text{Tube}(V)$ can be performed using a $\text{Spun}(CC)$ move. Figure 10 illustrates how this can be done, with the $\text{Spun}(CC)$ move being used in a torus neighborhood of the top right line of double points. We can then use Roseman moves to separate the two tubes.

In what follows, we focus on three different cases. In order to prove that a local move $M_w$ is a residue of a classical move $M_c$, we will use the following strategy: first we find a w-classifying invariant $\varphi : w\mathcal{SL}_n \to A$ of $M_w$, and a morphism $\psi : 2^{-r}\mathcal{SL}_n \to A$ which is invariant under $\text{Spun}(M_c)$ and such that $\psi \circ \text{Tube} = \varphi$. This gives $\text{Spun}(M_c)|_{2^{-r}\mathcal{SL}_n} \subset \text{Tube}(M_w)$. We then check on broken surface diagrams that $\text{Tube}(M_w)$ can be performed using a $\text{Spun}(M_c)$ move, so that $\text{Spun}(M_c)|_{2^{-r}\mathcal{SL}_n} \supset \text{Tube}(M_w)$.

2 Three examples of residues

2.1 The SC and SV moves

2.1.1 Classification of the SC move

We begin by introducing a c-classifying invariant of the $SC$ move, which was established by Habegger and Lin in [9] as a classification of string links up to homotopy.
For a string link \( L \), let \( X_L \) denote the complement of an open tubular neighborhood of \( L \) in \( B^2 \). For \( \varepsilon = 0, 1 \), \( \partial_\varepsilon X_L \) is a disk with \( n \) smaller and disjoint open disks removed. Hence \( \pi_1(\partial_\varepsilon X_L) \simeq F_n \), with generators \( m^{(\varepsilon)}_i \), called meridians, given by the positively oriented boundaries of these small disks, up to some given fixed path joining them to the basepoint.

A quick calculation of the homology of \( X_L \) shows that the inclusion maps \( \iota_\varepsilon : \partial_\varepsilon X_L \to X_L \) induce isomorphisms at the \( H_1 \) and \( H_2 \)-level. We can then apply a theorem of J. Stallings:

**Theorem 2.1.** [16, Thm. 3.4] If a map \( f : X \to Y \) induces an isomorphism at the \( H_1 \)-level and an epimorphism at the \( H_2 \)-level, then for all \( k \geq 1 \), it induces an isomorphism between \( \pi_1(X)/\Gamma_k \pi_1(X) \) and \( \pi_1(Y)/\Gamma_k \pi_1(Y) \).

Using the Wirtinger presentation of \( \pi_1(X_L) \) and the fact that the reduced quotient of a group \( G \) normally generated by \( k \) elements is nilpotent of class \( \leq k \) [9, Lem. 1.3], by taking the reduced groups for \( k \) sufficiently large, we have:

**Lemma 2.2.** [9, Cor. 1.4] The induced maps \( \iota_\varepsilon : \partial_\varepsilon X_L \to X_L \) are isomorphisms.

We can then define \( \varphi_L := \iota_0^{-1} \circ \iota_1 \in \text{Aut}(RF_n) \). As seen in the Wirtinger presentation of \( L \), for each \( i \) the meridians \( m^{(0)}_i \) and \( m^{(1)}_i \) are conjugates of each other, so \( \varphi_L \in \text{Aut}_C(RF_n) \). Moreover, the product \( m^{(0)}_1 \cdots m^{(0)}_n \) is homotopic to \( m^{(1)}_1 \cdots m^{(1)}_n \) in \( X_L \), so \( \varphi_L \in \text{Aut}_C(RF_n) \).

**Proposition 2.3.** [9] Lem. 1.6] The map \( \Phi_{S\ell_n} : L \in S\ell_n \mapsto \varphi_L \in \text{Aut}_C(RF_n) \) is a monoid homomorphism which is invariant under the \( SC \) move.

We obtain the following classification result:

**Theorem 2.4.** [9, Thm. 1.7] The homomorphism \( \Phi_{S\ell_n} \) \( c \)-classifies \( SC \).

### 2.1.2 Classification of the SV move

We now give a \( w \)-classifying invariant of the \( SV \) move, which was established in [3] by Audoux–Bellingeri–Meilhan–Wagner using the notion of coloring on Gauss diagrams. In order to facilitate the transition to string 2-links, we use the equivalent approach of virtual string link diagrams, rather than Gauss diagrams. The correspondence between the two consists in identifying overstrands of welded diagrams with what is referred to as “tail intervals” of Gauss diagrams in [3].

We can generalize the Wirtinger presentation to virtual string link diagrams by associating a generator to each overstrand, and the usual conjugating relation at each classical crossing (and no relation at virtual crossings). For \( D \in vSLD_n \), we denote by \( \pi_1(D) \) the group given by this presentation.
Definition 2.5. If \( y_i \in RF_n \) is a conjugate of \( x_i \) for each \( i \), a \((y_1, \ldots, y_n)\)-coloring of a virtual string link diagram \( D \) is a map from the overstrands of \( D \) to \( RF_n \), which sends the \( i^{th} \) bottom overstrand to \( y_i \), and which satisfies the Wirtinger relation at each classical crossing. Equivalently, this last condition can be replaced by stating that the coloring induces a homomorphism from \( \pi_1(D) \) to \( RF_n \).

Proposition 2.6. [3] Lem. 4.20 If \( D, D' \in vSLD_n \) are related by a move \( M \in \text{wReid} \cup \{ \text{SV} \} \), then there exists a one-to-one correspondence between the \((y_1, \ldots, y_n)\)-colorings of \( D \) and \( D' \), which preserves the image of the top overstrands.

It is clear that a virtual pure braid diagram (i.e. a diagram with monotone strands) admits a unique \((y_1, \ldots, y_n)\)-coloring. Since up to \( SV \), any virtual string link diagram is equivalent to a virtual pure braid diagram (see [3] Thm. 4.12), it follows from Proposition 2.6 that it is also the case for any virtual string link diagram. Hence for \( L \in wSL_n \) represented by a diagram \( D \), we can define \( \varphi_L \in \text{End}_C(RF_n) \) by \( \varphi_L(x_i) = z_i \), where \( z_i \in RF_n \) is the image of the \( i^{th} \) top overstrand in the unique \((x_1, \ldots, x_n)\)-coloring of \( D \).

Proposition 2.7. [3] Lem. 4.20 The map \( \Phi_{wSL_n} : wSL_n \rightarrow \text{End}_C(RF_n) \) is a monoid homomorphism which is invariant under the \( SV \) move. Moreover, we have \( \text{End}_C(RF_n) = \text{Aut}_C(RF_n) \).

We obtain the following classification result:

Theorem 2.8. [3] Thm. 4.17 The homomorphism \( \Phi_{wSL_n} : wSL_n \rightarrow \text{Aut}_C(RF_n) \) \( w \)-classifies \( SV \).

Remark: If \( L \in SL_n \) and \( \pi_1(X_L) \) is given by the Wirtinger presentation associated to a diagram \( D \) of \( L \), then \( \iota_0^{-1} : R\pi_1(X_L) \simeq R\pi_1(D) \rightarrow R\pi_1(\partial_0X_L) \simeq RF_n \) gives a \((x_1, \ldots, x_n)\)-coloring of \( D \), and since \( \iota_1 \) sends \( x_i \) to the generator of \( \pi_1(X_L) \) associated to the top overstrand of \( L_i \), we have \( \Phi_{wSL_n}(L)(x_i) = \iota_0^{-1}(\iota_1(x_i)) = \Phi_{wSL_n}(L) = \Phi_{wSL_n}(L) \).

In particular, we obtain that the inclusion \( SLD_n \rightarrow vSLD_n \) induces an injection \( SL_n^{SC} \rightarrow wSL_n^{SV} \), so \( SV \) is a welded extension of \( SC \). We will now see that it is also a ribbon residue of \( SC \).

2.1.3 Extension to string 2–links

Let \( L \in 2–SL_n \) be a string 2–link, \( X_L \) the complement of an open tubular neighborhood of \( L \), and \( D \) a broken surface diagram of \( L \). As described in [6], we get a Wirtinger presentation of \( \pi_1(X_L) \) from \( D \), with one generator for each connected component, called oversheet, and Wirtinger relations at lines of double points:

![Diagram](image)

\[ g_+ = g_0^{-1}g_0. \]

Definition 2.9. If \( y_i \in RF_n \) is a conjugate of \( x_i \) for each \( i \), a \((y_1, \ldots, y_n)\)-coloring of a broken surface diagram \( D \) is a map from the connected components of \( D \) to \( RF_n \), which sends the \( i^{th} \) bottom component to \( y_i \), and which satisfies the Wirtinger relation at each line of double points.

As in the case of string links, we have:
Proposition 2.10. [4 §4.1] For a string 2–link $L$, the inclusion maps $\iota_c : \partial_c X_L \rightarrow X_L$ induce isomorphisms $\iota_c : R\pi_1(\partial_c X_L) \simeq RF_n \rightarrow R\pi_1(X_L)$.

Proposition 2.11. For $L \in \mathcal{S}\mathcal{L}_n$ (resp. $L \in w\mathcal{S}\mathcal{L}_n$), there exists a one-to-one correspondence between colorings of $L$ and colorings of $\text{Spun}(L)$ (resp. of $\text{Tube}(L)$), which preserves the images of the top and bottom components.

Proof: In the case of the Spun map, it follows directly from the fact that a broken surface diagram of $\text{Spun}(L)$ can be obtained from a diagram of $L$ by a rotation, which sends overstrands to oversheets and crossings to lines of double points with the same Wirtinger relations.

In the case of the Tube map, we begin by taking a broken surface diagram $D$ of $\text{Tube}(L)$, obtained by the construction described after definition 1.9. There is a correspondence between overstrands of $L$ and oversheets of $D$, except for one additional small disk in $D$ at each ribbon singularity. However it is easily seen that given the images of the other oversheets of $D$, the Wirtinger relations give a unique value for these disks, hence removing all ambiguity.

Proof: Let $\varphi_L \in \text{End}_C(RF_n) = \text{Aut}_C(RF_n)$ in the same way as for welded string links, and the map $\Phi_{2,\mathcal{S}\mathcal{L}_n} : L \in 2–\mathcal{S}\mathcal{L}_n \rightarrow \varphi_L \in \text{Aut}_C(RF_n)$ is a monoid morphism.

Proposition 2.12. The morphism $\Phi_{2,\mathcal{S}\mathcal{L}_n}$ is invariant under the $\text{Spun}(SC)$ move.

Proof: Let $L \in 2–\mathcal{S}\mathcal{L}_n$ be a string 2–link, represented by a broken surface diagram $D$. Executing a $\text{Spun}(SC)$ move on $D$ yields a broken surface diagram $D'$, equal to $D$ outside of a solid torus $T$ in which the move occurs. Inside this torus, $D$ is subdivided into three annuli, which are all coming from the same connected component $S^{1,1}_n$ of $\text{Spun}(L)$. As such, their images in the unique $(x_1, \ldots, x_n)$–coloring of $D$ are conjugates of $x_i$, which commute in $RF_n$, so the Wirtinger relation on the line of double points in $T$ identifies the images of the two lower oversheets. This also holds true for $D'$. From this, it follows that we obtain the unique $(x_1, \ldots, x_n)$–coloring of $D'$ by taking the coloring of $D$ outside of $T$, where $D = D'$, and extending it inside of $T$ according to its values on $\partial T$. In particular, we have $\varphi_L(x_j) = \varphi_L'(x_j)$ for all $j$, where $L' \in 2–\mathcal{S}\mathcal{L}_n$ is represented by $D'$, and thus $\Phi_{2,\mathcal{S}\mathcal{L}_n}(L) = \Phi_{2,\mathcal{S}\mathcal{L}_n}(L')$.

As a direct corollary of proposition 2.11 we obtain:

Proposition 2.13. With the notations introduced in the previous sections, $\Phi_{2,\mathcal{S}\mathcal{L}_n} \circ \text{Tube} = \Phi_{w\mathcal{S}\mathcal{L}_n}$ and $\Phi_{2,\mathcal{S}\mathcal{L}_n} \circ \text{Spun} = \Phi_{w\mathcal{S}\mathcal{L}_n}$.

We can now prove the main result of this section, i.e. the relation between $\text{Spun}(SC)$ and $\text{Tube}(SV)$.

Theorem 2.14. The $SV$ move is a ribbon move of the $SC$ move.

Proof: As indicated earlier, we first prove that $\text{Spun}(SC)|_{2–\mathcal{S}\mathcal{L}_n} \subset \text{Tube}(SV)$. Let $R_1, R_2 \in 2–\mathcal{S}\mathcal{L}_n$ be equivalent under $\text{Spun}(SC)$, and $L_1, L_2 \in w\mathcal{S}\mathcal{L}_n$ be preimages of $R_1$ and $R_2$ by the Tube map. By propositions 2.12 and 2.13 we have:

$$\Phi_{w\mathcal{S}\mathcal{L}_n}(R) = \Phi_{2,\mathcal{S}\mathcal{L}_n}(L) = \Phi_{2,\mathcal{S}\mathcal{L}_n}(L') = \Phi_{w\mathcal{S}\mathcal{L}_n}(R'),$$

and since $\Phi_{w\mathcal{S}\mathcal{L}_n}$ is a w-classifying invariant of $SV$, $R$ and $R'$ are equivalent under $SV$. By definition, this implies that $L$ and $L'$ are equivalent under $\text{Tube}(SV)$.

The other inclusion $\text{Spun}(SC)|_{2–\mathcal{S}\mathcal{L}_n} \supset \text{Tube}(SV)$ follows from the fact that we can perform a $\text{Tube}(SV)$ move using $\text{Spun}(SC)$ on a broken surface diagram. This was illustrated on Figure 10 for the $CC$ and $V$ moves, which transposes directly to the $SC$ and $SV$ case by restricting ourselves to ribbon singularities of a component with itself.

\[15\]
Remark: Another consequence of proposition 2.13 is that if $L, L' \in \omega \mathcal{SL}_n$ have the same image by the Tube map, then $\Phi_{w \mathcal{SL}_n}(L) = \Phi_{w \mathcal{SL}_n}(L')$, so $L$ and $L'$ are equivalent under the $SV$ move. This is also true for any local move $M$ $w$-generating $SV$, which proves the remark made in section 1.3.3.

2.2 The $\Delta$ and $F$ moves

Since the $\Delta$ and $F$ moves are classified by the linking numbers, we begin by defining an equivalent of the linking numbers for string 2–links. This will be done using the $\Phi_{2,\mathcal{SL}_n}$ map defined in the previous section, so let us first show how the virtual linking numbers on welded string links can be obtained via $\Phi_{w \mathcal{SL}_n}$.

For $1 \leq j \leq n$, let $RF_{n-1}^{(j)}$ denote the subgroup of $RF_n$ generated by the $x_k$’s for $k \neq j$. For a welded string link $L$, let $\lambda_j \in RF_n$ be such that $\varphi_L(x_j) = \lambda_j^{-1} x_j \lambda_j$. Since $x_j$ commutes with its conjugates, $\lambda_j$ can be taken in $RF_{n-1}^{(j)}$. As proven in [3, Lem. 4.25], such a $\lambda_j \in RF_{n-1}^{(j)}$ is unique.

Let $x_j^*: RF_n \to \mathbb{Z}$ be the group homomorphism defined by $x_i^*(x_i) = 1$ and $x_k^*(x_k) = 0$ for $k \neq i$. Keeping track of the undercrossings of $L$ on the $j^{th}$ strand, it is not difficult to check that $\text{vlk}_{ij}(L) = x_j^*(\lambda_j)$.

Since the unicity of the $\lambda_j$’s is true for any element of $\text{Aut}_C(RF_n)$, it holds for $\Phi_{2,\mathcal{SL}_n}(L)$, $L \in 2-\mathcal{SL}_n$. This gives us a way of extending the notion of linking numbers to string 2–links.

Definition 2.15. For $i \neq j \in \{1, \ldots, n\}$, we define the linking number $LK_{ij} : 2-\mathcal{SL}_n \to \mathbb{Z}$ by $LK_{ij}(L) := x_i^* \lambda_j$, where $\lambda_j$ is the unique element of $RF_{n-1}^{(j)}$ such that $\Phi_{2,\mathcal{SL}_n}(L)(x_j) = \lambda_j^{-1} x_j \lambda_j$.

We can give a more geometric interpretation of the linking numbers for string 2–links as follows: for $L \in 2-\mathcal{SL}_n$ represented by a broken surface diagram $D$, let $\gamma_j$ be a path on $L_j \subset D$ from $\partial_0 L_j$ to $\partial_1 L_j$, having transverse intersections with lines of double points. Then $LK_{ij}(L)$ is the number of times (counted with a sign given by the orientation) $\gamma_j$ crosses a line of double points where $L_j$ passes behind $L_i$.

From this interpretation, it follows that $LK_{ij}(L)$ only depends on the components $L_i$ and $L_j$ of $L$. Thus the linking numbers are invariant under any move which does not change the relative position of any pair of components.

Proposition 2.16. For $i, j \in \{1, \ldots, n\}$, $i \neq j$, we have $LK_{ij} \circ \text{Tube} = \text{vlk}_{ij}$ and $LK_{ij} \circ \text{Spun} = \text{lk}_{ij}$.

Proof: For the Tube map, it follows directly from proposition 2.13 and definition 2.15. For the Spun map, we also use the fact that $\Phi_{w \mathcal{SL}_n}|_{\mathcal{SL}_n} = \Phi_{\mathcal{SL}_n}$ and $\text{vlk}_{ij}|_{\mathcal{SL}_n} = \text{lk}_{ij}$.

We can now determine the relation between $\text{Spun}(\Delta)$ and $\text{Tube}(F)$.

Theorem 2.17. The $F$ move is a ribbon residue of the $\Delta$ move.

Proof: Let $R_1, R_2 \in 2-r\mathcal{SL}_n$ be equivalent under $\text{Spun}(\Delta)$, and $L_1, L_2 \in w\mathcal{SL}_n$ be preimages of $R_1$ and $R_2$ by the Tube map. A $\text{Spun}(\Delta)$ move modifies the relative position of three components of a string 2–links, but not the relative position of any pair of components, so it does not affect the linking numbers of the string 2–link. Hence we have $LK_{ij}(R_1) = LK_{ij}(R_2)$. By proposition 2.16, we have $\text{vlk}_{ij}(L_1) = LK_{ij}(R_1) = LK_{ij}(R_2) = \text{vlk}_{ij}(L_2)$, and since $F$ is $w$-classified by the virtual linking numbers, $L_1 \sim_F L_2$. Hence $R_1$ and $R_2$ are equivalent under $\text{Tube}(F)$, and we have $\text{Spun}(\Delta)|_{2-r\mathcal{SL}_n} \subset \text{Tube}(F)$.

Since $F$ is $w$-equivalent to $UC$, we have $\text{Tube}(F) = \text{Tube}(UC)$, so to prove $\text{Spun}(\Delta)|_{2-r\mathcal{SL}_n} \subset \text{Tube}(F)$ it is enough to verify that we can perform a $\text{Tube}(UC)$ move using $\text{Spun}(\Delta)$ on a broken surface diagram. This is illustrated on Figure 11 at the end of the article, with the following conventions:
2.3 The BP, wBP and BV moves

As noted in Corollary 1.17, the wBP move is a welded extension of the BP move. However, we will see that it is not a ribbon residue of BP, and neither is any SV-generating welded extension of BP. First, let us take a closer look at the BV move.

Lemma 2.18. The BV move w-generates the F, SR and VC moves.

Proof: First, we show on virtual diagrams that BV w-generates F, as illustrated below:

\[ \text{R1, R2} \quad \text{BV} \quad \text{Detour move} \]

\[ \text{R1, R2} \quad \text{BV} \quad \text{Detour move} \]
Since $F \cong UC$, we can use $UC$ to show that $BV$ w-generates $SR$. It is easily checked that, up to one $R2$ move, the move described below is w-equivalent to $SR$.

Hence the sign restriction on the $BV$ move on Gauss diagrams can be omitted. We can now show that $BV$ w-generates $VC$ on Gauss diagrams, without worrying about signs:

We have the following classification of the $BV$ move:

**Proposition 2.19.** The $BV$ move is w-classified by $(\text{vlk}_{i*}^{\text{mod}} + \text{vlk}_{si}^{\text{mod}})_{1 \leq i \leq n-1} : vS\text{LD}_n \rightarrow \mathbb{Z}_2^{n-1}$.

**Proof:** This combination of linking numbers counts for each strand $L_i$ (except for the $n^{th}$ one) the parity of the number of classical overcrossings and undercrossings involving $L_i$. Note that we can include the self-crossings of $L_i$, since these count as two crossings on $L_i$. Since the $BV$ move deletes or adds an even number of classical crossings on each strand, the homomorphism defined above is invariant under $BV$.

For a given element $z \in \mathbb{Z}_2^{n-1}$, we define a Gauss diagram $G_z$ as follows: for $1 \leq i \leq n-1$, we put a positive arrow from the $i^{th}$ strand to the $n^{th}$ strand if $z_i = 1$, and no arrow if $z_i = 0$. These arrows are taken to be horizontal, with the heads on the $n^{th}$ strand arranged in the order given by the index $i$. By construction, we have $\text{vlk}_{i*}^{\text{mod}}(G_z) + \text{vlk}_{si}^{\text{mod}}(G_z) = z_i$.

Using Lemma 2.18 we can show that any Gauss diagram representing a welded string link $L$ with $\text{vlk}_{i*}^{\text{mod}}(L) + \text{vlk}_{si}^{\text{mod}}(L) = z_i$ for $1 \leq i \leq n-1$ is equivalent up to $BV$ to the diagram $G_z$ defined above. To achieve this, we first use $VC$ moves to transform arrows from $L_i$ to $L_1$ into arrows from $L_1$ to $L_i$. Self-arrows of $L_1$ are deleted using $SV$ moves, which are allowed since $BV \Rightarrow F \Rightarrow SV$. The following move allows us to turn arrows from $L_1$ to $L_i$ for $1 < i < n$ into arrows from $L_1$ to $L_n$, by creating an additional arrow between $L_i$ and $L_n$. Since we have access to $SR$, signs are irrelevant, and are not displayed:
Using $F$, $UC$ and $OC$ moves, we can regroup the arrows from $L_1$ to $L_n$, then assign them alternating signs using $SR$ in order to delete them by pairs using $R2$. After these steps, there is at most one arrow attached to $L_1$, which points to $L_n$, and can be made positive by $SR$. Moreover, its head can be positioned at the bottom of $L_n$.

We then iterate this process on the diagram obtained by ignoring the first strand. In the end, we are left with a diagram of the form $G_{z'}$. Since the process described above does not change the invariant $(\text{vlk}_{s_i}^{\text{mod}} + \text{vlk}_{s_1}^{\text{mod}})_{1 \leq i \leq n-1}$, we have $z = z'$, which completes the proof.

Because of the symmetry of linking numbers on $SLD_n$, the restriction of this invariant on classical diagrams vanishes. This shows that $BV$ is not a welded extension of $BP$, as the induced map $SL_n^{BP} \rightarrow wSL_n^{BV}$ is trivial. However, we have:

**Theorem 2.20.** The $BV$ move is a ribbon residue of the $BP$ move.

**Proof:** Since $\text{Spun}(BP)$ merely changes the over/under information on some lines of double points, it does not modify $\text{LK}_{s_i}^{\text{mod}} + \text{LK}_{s_1}^{\text{mod}}$. We then conclude that $\text{Spun}(BP)|_{2-\tau SL_n \subset \text{Tube}(BV)}$ with the same reasoning as in Theorem 2.17.

For the other inclusion, we only need to check that the $\text{Tube}(BV)$ move can be performed using $\text{Spun}(BP)$, which is illustrated on Figure 12, where we use the same convention as in Theorem 2.17. This figure illustrates the easier case where the overstrands have opposite orientations. If they have the same orientation, an extra step is needed, where we slide one of the black tubes into the other. This brings us to a configuration where we can apply $\text{Spun}(BP)$.

Finally, the orientation of the understrands is irrelevant. Indeed, changing the orientation of an understrand has no effect on the broken surface diagram of the image under the $\text{Tube}$ map, as the thin tube has its co-orientation reversed, which compensates the change of sign of the ribbon singularity.

As a result, the $BP$ move does not have any $SV$-generating welded extension which is also a ribbon residue. Indeed, if such a welded extension $M$ existed, we would have $\text{Tube}(M) = \text{Spun}(BP)|_{2-\tau SL_n} = \text{Tube}(BV)$, so $M \leftrightarrow BV$ by the remark in section 1.3.3. But then the induced map $SL_n^{BP} \rightarrow wSL_n^{BV}$ would be trivial, contradicting the fact that $M$ is a welded extension of $BP$. In this situation, the notion of ribbon residue does not provide a distinguished ($SV$-generating) welded extension, as was the case for the $\Delta$ move.
Figure 11: Performing Tube($UC$) using Spun($\Delta$)
Figure 12: Performing Tube(BV) using Spun(BP)
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