Z(N) BUBBLES AND THEIR DESCENDANTS IN HOT QCD: ALIVE AND BUBBLING

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Abstract

The spontaneous breaking of $Z(N)$ symmetry in hot QCD and the appearance of domain walls is reviewed.

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1. Introduction

In recent years much attention has been paid to the quark-gluon plasma, both numerically\(^1\) and analytically\(^2\). The advent of RHIC at Brookhaven and possibly heavy ion experiments at LHC and the relevance for early universe scenario’s amply justify its study.

The purpose of these notes is to provide some additional insights into the phenomenon of spontaneous Z(N) breaking\(^3\) at high temperatures, and to provide an up to date presentation of what has been achieved. I would like to apologise right from the start to the lattice people: they always asked the relevant questions and are not really concerned with the partly rather academic questions posed in this talk.

Z(N) is the center group of the SU(N) gauge group and as such does not appear in the gauge transformation law of the gauge potentials. The symmetry appears only at the quantum level: physical states do transform according to irreducible representations of the center group. When we compute the free energy of the system we find a low temperature phase where the free energy is non-degenerate. In the high temperature phase there appears a degeneracy.

Traditionally the phenomenon of broken Z(N) symmetry is described in close analogy with spontaneous magnetisation in spin systems\(^3\); in particular one expects the occurrence of ordered phases, and domain walls in between them. Already at this point the reader may raise his eyebrows: how can there be a close analogy between spontaneous magnetisation, a typical low temperature phenomenon, and the would be spontaneous breaking of Z(N) symmetry at \textit{high} temperatures? In other words, how can it possibly be, that a symmetry breaks at high temperatures? The answer is that the latter is perfectly possible, once we realise that we carry a prejudice on how the surface tension \(\alpha\) of a small bubble of say down spins behaves in a large sea of up spins. We usually think of it as being temperature independent at low temperatures. That means that the probability of having such a droplet of radius R at temperature T equals:

\[
\exp\left(-\frac{\alpha}{T} R^2\right)
\]

Given our prejudice on \(\alpha\) this probability disappears at small T, and an ordered phase prevails. However in QCD without quarks the only scale is T itself! Hence dimensional arguments tell us that \(\alpha \sim T^3\) and now the scenario is reversed: at \textit{high} temperatures order installs itself!* Actually the surface tension has been computed in perturbation theory\(^2\) with results that were in qualitative agreement with Montecarlo results\(^1\).

* I learnt this argument from R.D. Pisarski
There remains another, somewhat mystical element in the argument above: what are the degrees of freedom in QCD that do correspond to these spins? They are the expectation values of the Polyakov lines $P(A_0)$ and are related to the excess free energy of a single heavy test quark:

$$\frac{Tr\langle q|\exp{-\frac{H}{T}}|q\rangle}{Tr\exp{-\frac{H}{T}}} = \langle P(A_0) \rangle$$

In this formula the trace is over physical states, and $q$ represents the heavy quark. The average on the right hand side is the Euclidean path integral average. The point is now: make a gauge transformation periodic up to a $Z(N)$ phase in the time direction, and the Polyakov loop will pick up this phase (of course the action in the path integral stays invariant). On the other hand the free energy of a quark cannot acquire such a phase, and the way out is that both averages are zero. This does not imply, that $Z(N)$ symmetry never breaks. It just reflects the basic fact, that the order parameter can only become non-zero once we trigger it by appropriate boundary conditions, as is well known in spin systems. Incidentally, the fact that the Hamiltonian matrix element above is zero, can be deduced from the fact that the one quark state can never be made gauge invariant, since Gauss' law cannot be satisfied.

Of course, lattice people have been undaunted by this. They measured two point functions, and showed, that above a critical temperature the confining behaviour was replaced by a screened Coulomb potential. But to show the presence of a surface tension between coexisting phases, we have to be careful and specify boundary conditions. In section 3 we will do so. It will become clear, that the relative phase of Polyakov loops is perfectly physical. To illustrate the reasoning we have inserted a section on the paradigm: $\varphi^4$ theory in three dimensions. The analytic computations in QCD are discussed in section 4. Section 5 is a summary and an outlook.

2. A useful finger exercise: $\varphi^4$ theory

The prime field theory example of a spin system with a $Z(2)$ symmetry is as we said the one with the three dimensional action:

$$S = \int d\vec{x}\left(\frac{1}{2}\sum_k (\partial_k \varphi)^2 + \frac{\lambda}{4!}(\varphi^2 - v^2)^2 \right)$$

The symmetry of the action is $\varphi \rightarrow -\varphi$. We have written the potential such that the $Z(2)$ symmetry is spontaneously broken (see fig. 1), the vacuum expectation value of the field being $\pm v$. The latter is related to the mass term $-m^2/\lambda \varphi^2$ by $v = \sqrt{6m^2/\lambda}$. 
The physics is identical in either of the two minima. One can think of a box with sizes $L \times L \times L_z$, with $L_z >> L >> m^{-1}$, in which we trigger the two phases by suitable boundary conditions (see fig. 2). The region in between the two coexisting phases is having a higher energy density due to the appearance of a domain wall

$$\varphi_c(z) \sim \tgh m z$$

as follows from minimising the action eq. 2.1 under the condition that $\varphi = \pm v$ at $z = \pm \infty$. The scale over which this wall develops is $m^{-1}$. The actual total energy stored in the wall per unit of surface is

$$\alpha_0 = 4\sqrt{2} \frac{m^3}{\lambda}$$

How can we see the wall, other than by its energy density? The obvious answer is: take the two point function $\langle \varphi(x)\varphi(y) \rangle$, with $y$ fixed in say the plus phase, and $x$ varying from the plus to the minus phase. In this trivial example there are no fluctuations so the correlation will trace out the profile eq. 2.2, and becomes for large enough separation negative (see fig. 2).

### 2.1 Quantum corrections

To compute the quantum corrections we have to resort to the path integral. We will take boundary conditions to be periodic in $x$ and $y$ directions and (anti)periodic in the $z$ direction and the corresponding partition functions will be called $(Z_{(0,0,1)}) Z_{(0,0,0)}$. To make sense of the path integrals we will put the action eq. 2.1 on the lattice, by discretising the derivatives in the kinetic term:

$$\partial_k \varphi(\vec{x}) \equiv \frac{\varphi(\vec{x} + \vec{i}_k a) - \varphi(\vec{x})}{a}$$

$a$ being the lattice length. This means that the kinetic term becomes

$$-a^{-2} \sum_k \varphi(\vec{x} + \vec{i}_k a)\varphi(\vec{x})$$

apart from local terms. The anti-periodicity in the $z$ direction can be reformulated simply by absorbing the - sign at $z=L_z$ into the coupling in eq. 2.4, and to consider the field to be periodic. The new action with this 2 d surface of anti ferromagnetic couplings is called “twisted”.

The ratio of the twisted partition function over the untwisted one will cancel out the volume effects and leaves us with the effect of the twist:

$$\frac{Z_{(0,0,1)}}{Z_{(0,0,0)}} = E(L) \exp -\alpha L^2$$
Here \( \alpha \) is the surface energy (the prefactor \( E \) contains powers of \( L \)). Classically it reduces to eq. 2.3, as will become explicit in the next section. It is amusing to note that eq. 2.5 can be seen as describing a disorder phenomenon, with a rapid fall-off. On the other hand the correlation function will signal an ordered behaviour: surface tension and correlation are dual aspects of the system. In fig. 3 we show how the correlation in the presence of the twist picks up a - sign when piercing the twisted links. This is seen by redefining the field variable with a minus sign where the correlation crosses, a manifestation of the algebra proposed in ref. 5.

### 2.2 Calculation of the surface energy

Let us introduce the constrained effective action \( U([\phi(z)]) \):

\[
\exp[-L^2U([\phi(z)])] \equiv \int D\varphi \prod_z \delta(\phi(z) - \bar{\varphi}(z)) \exp[-S(\varphi)]
\]

2.6

We wrote \( \bar{\varphi}(z) \equiv \frac{1}{L^2} \int dx dy \varphi(x, y, z) \), so up to normalisation the effective potential \( U \) gives us the probability that a configuration with profile \( \phi(z) \) appears. \( S \) is the untwisted action. The ratio in eq. 2.5 is related very simply to the effective potential:

\[
\frac{Z_{(0,0,1)}}{Z_{(0,0,0)}} = \int D\phi(z) \exp[-L^2U([\phi(z)])]
\]

2.7

The only way that the anti periodicity enters in eq. 2.7 is through the profile: \( \phi(0) = -\phi(L_z) \).

From eq. 2.5 and 2.7 it follows that knowledge of the effective action is enough to compute the surface energy. The former can be computed from eq. 2.6 in a loop expansion:

\[
U([\phi]) = S([\phi]) + U_1([\phi]) + \text{higher loops}
\]

2.8

S is the classical action, \( U_1 \) is the one loop contribution.

For large \( L \) we only have to minimise \( U \), as follows from eq. 2.7. The classical term has been minimised in the previous sub-section. The one loop contribution is computed\(^4\) from the propagator with inverse \( S''(\varphi_c) \). This inverse has the correct spectrum (i.e. no negative eigenvalues). When letting the lattice length go to zero, we have to reexpress the parameters in the action in terms of their renormalised counterparts.

At small coupling, \( \frac{\lambda}{m} \ll 1 \), the vacuum expectation value \( v \) provides a much smaller scale, over which the wall, eq. 2.2, develops adiabatically, like \( \sqrt{\lambda v z} \). Nevertheless, we are not allowed to make a derivative expansion, since
$S''(\phi)$ for a constant profile $\phi$ develops negative eigenvalues, as the potential in fig. 1 shows, when it is convex.

This is the prime distinction with the interface in hot gauge theory (see section 4).

3. The free energy in QCD, string tension and surface tension

In this section we will give a definition of the surface tension using the Hamiltonian approach. The use of Euclidean path integrals will be completely avoided, until the actual evaluation. The method we use is essentially an adaptation of ’t Hooft’s method $^5$ and a short version was published some time ago $^6$ by us.

3.1 Surface tension from the Hamiltonian

The Hamiltonian is in a familiar notation:

$$H = \int d\vec{x} Tr(g^2 E^2 + \frac{1}{g^2} B^2)$$

with $g$ the bare coupling. The electric and magnetic field strengths are written in NxN matrix form:

$$E_k = E^a_k \lambda^a$$

and

$$Tr\lambda^a \lambda^b = \frac{1}{2} \delta^{a,b}$$

The indices $a$, $b$ are the $N^2 - 1$ colour indices. As before we imagine the system in a box of macroscopic size $L_x L_y L_z$, with the size $L_z$ in the $z$ direction much larger than the transverse ones. The gauge potentials have periodic boundary conditions. The system is put in a heatbath at temperature $T$.

We will now introduce symmetry operators $\Omega_{\vec{k}}$, commuting with $H$. They are gauge transformations in the box which are periodic up to a centergroup element $exp i k \cdot \vec{e} \frac{2\pi}{N}$ in the $x$ direction and analogous for $y$ and $z$ direction. Thus they will leave gauge potentials periodic, since the $Z(N)$ phase does not appear in the gauge transform of $A_\mu \equiv A^a_\mu \lambda^a$:

$$A_\mu^\Omega = \Omega A_\mu \Omega^{-1} + i \Omega \partial_\mu \Omega^{-1}$$

If we define physical states as states obtained by averaging over all truly periodic gauge transformations, then these physical states fall into $N^3$ different Hilbert spaces, labeled by electric flux $\vec{e}$. Each space is invariant under the Hamiltonian, and any state vector $|\vec{e}\rangle$ is eigenvector of $\Omega_{\vec{k}}$ with eigenvalue $exp i \vec{k} \cdot \vec{e} \frac{2\pi}{N}$. It is then obvious that we can obtain any electric flux state from
the no flux state $|\vec{e} = 0\rangle$ by acting on it with the Wilson line* winding the appropriate number of times in the spatial directions, because a Wilson line $U_x$ in the $x$-direction will pick up a phase $\exp(i k_x \frac{2\pi}{N})$:

$$\Omega_k U_x \Omega_k^{-1} = \exp(i k_x \frac{2\pi}{N} U_x)$$

We are now in a position to write down the free energy of a given electric flux $\vec{e}$:

$$\exp\left(-\frac{F_{\vec{e}}}{T}\right) \equiv \text{Tr}_{\text{phys}} \langle \vec{e} | \exp\left(-\frac{H}{T}\right) | \vec{e}\rangle$$  \hspace{1cm} 3.1

The trace in eq. 3.1 is over all physical states with electric flux $\vec{e}$ and it is not hard to see that any such state with a given electric flux can be written as

$$|\vec{e}\rangle = P_{\vec{e}} |\text{phys}\rangle$$

where the projector is related to the gauge transformations $\Omega_k$ by

$$P_{\vec{e}} = \frac{1}{N^3} \sum_k \exp(i \vec{k} \cdot \vec{e} \frac{2\pi}{N} \Omega_k)$$  \hspace{1cm} 3.2

For large spatial volume volume and low enough temperature we find for the difference (we want to get rid of volume effects, and only study effects due to the Wilson line):

$$F_{(0,0,1)} - F_{(0,0,0)} = \rho(T) L_z$$  \hspace{1cm} 3.3

whereas for high enough temperatures we will find in perturbation theory:

$$F_{(0,0,1)} - F_{(0,0,0)} = D(T, L) \exp\left(-\frac{\alpha(T)}{T} L^2\right)$$  \hspace{1cm} 3.4

The prefactor $D$ consists of powers of $L$ and $T$.

Of course the low temperature relation is telling us that the string tension $\rho$ will cause a linearly rising energy difference between the various free energies and that $F_{(0,0,0)}$ is the lowest energy state.

The high temperature relation renders the free energies all degenerate in the infinite volume limit, and the quantity $\alpha$ that controls the exponential decay of the difference is the surface tension, as we will see at the end of this section. We have here a definition of the surface tension without introducing path integrals or even Polyakov loops. We will only use those when computing $\alpha$ in perturbation theory.

* In our terminology a Wilson line is a Polyakov line, eq. 3.6, in one of the space directions
3.2 From Hamiltonian to path integral

The transition from the Hamiltonian expression for the free energy, eq. 3.1, to a path integral expression is provided by eq. 3.2. Indeed, after substitution of eq. 3.2 into eq. 3.1, we are left with a linear combination of terms of the type:

\[\langle \text{phys} | \exp\left(-\frac{H}{T}\Omega_{\vec{k}}\right)|\text{phys} \rangle\]

Were it not for the presence of the gauge transformation \(\Omega_{\vec{k}}\) this matrix element would equal the four dimensional path integral with periodic boundary conditions in all directions, including the Euclidean time direction \(\tau\), that runs from 0 to \(\frac{1}{T}\). The presence of \(\Omega_{\vec{k}}\) in eq. 3.5 renders the periodicity twisted. To illustrate this we take a specific \(\vec{k} = (0,0,1)\). At any point with \(\tau = 0, z = 0\) we follow first the z-direction, and pick up the non-periodic gauge transformation at \(\tau = 0, z = L_z\). Then we go from time \(\tau = 0\) to \(\tau = \frac{1}{T}\) and pick up a periodic transform. Thus we find in the corner \(\tau = \frac{1}{T}, z = L_z\) a gauge potential, which is up to a non periodic gauge transformation the same as we stated from in the opposite corner. Following the path first along the time and then along the z-direction leaves us with a periodic gauge transformation. Either path leaves us with one and the same gauge potential in the corner, because it does not feel the centergroup.

What is the physical meaning of such a twisted matrix element? First we note that the Polyakov loops \(P(\vec{x})\) defined by

\[P(\vec{x}) \equiv \frac{1}{N} Tr P \exp i \int_0^{\frac{1}{T}} d\tau A_0(\vec{x}, \tau)\]

are periodic modulo \(exp i \frac{2\pi}{N}\) in the z-direction. Thus the Polyakov loops have the same periodicity due to the twist, as the spin variables in the previous section, and the twisted matrix element has to be considered as the analogue of the anti-periodic partition function in the spin case. Its lattice version has twisted plaquettes (instead of links as in section 2.1.)

Let us study the ratio

\[Z_{(0,0,1)}/Z_{(0,0,0)}\]

As in eq. 3.3, this ratio contains no volume effects.

In the low temperature phase it is related to the string tension in eq. 3.3 by strong coupling methods. In the high temperature phase one can use continuum perturbation theory methods to evaluate the ratio and the outcome is

\[\frac{Z_{(0,0,1)}}{Z_{(0,0,0)}} = \exp\left(-\frac{\alpha}{T}L^2\right)\]
Like in section 2 for the spin system, the surface tension is dual to the correlation of two Polyakov loops, and an algebra like in ref. 5 can be established.

We have computed the surface tension to two loop order, as will be explained in the next subsection. Accepting eq. 3.8 for the moment, it is not hard to show from eq. 3.2, 3.4 and 3.8, that

$$F_{(0,0,1)} - F_{(0,0,0)} = 2T(1 - \cos \frac{2\pi}{N}) \frac{Z_{(0,0,1)}}{Z_{(0,0,0)}}$$

up to exponentially smaller terms.

### 3.3 Computing the surface tension

The ratio of the two path integrals can be written as a path integral over all possible Polyakov-loops $p(z)$, with as integrand the constrained effective potential $U([p])$.

$$\frac{Z_{(0,0,1)}}{Z_{(0,0,0)}} \equiv \int Dp(z) \exp -L^2 \frac{U([p])}{T}$$

The latter is defined as

$$\exp -L^2 \frac{U([p])}{T} \equiv \int DA_0 D\vec{A} \delta(p - \bar{p}(A_0)) \exp -\frac{1}{g^2} S(A)$$

We dropped for notational convenience the $z$-dependence in the delta function constraint on the averaged Polyakov loop:

$$\bar{p}(A_0(z)) \equiv \frac{1}{L^2} \int dx dy P(A_0(\vec{x}))$$

The boundary conditions on the loop in eq. 3.10 are of course anti periodic as mentioned in the discussion of the twisted matrix element. The analogy with the spin case in section 2.2 is obvious. The constrained effective potential is independent of the boundary conditions, and is readily accessible in MonteCarlo simulations⁸,¹.

If we are not interested in finite size effects due to $L$, the ratio in eq. 3.10 is determined by the saddle point of $U([p])$. The value of $U$ in the saddle point $\hat{p}$ is the surface tension $\alpha$, according to eq. 3.8.

So we are left with the task of calculating $U$ from eq. 3.11. This is done in section 4.

### 4. Results of the perturbative calculation

We have calculated the effective potential eq. 3.11 up to two loop order²,⁹. The method is well known; determine the saddlepoint of the integrand in
eq. 3.11, and expand around it, using a background gauge fixing. The important difference with $\varphi^4$ theory is that we have to use the gradient expansion method!

The reason for this is the following. In the classical action there is no scale. There is only a gradient term, no potential term like in $\varphi^4$ theory. The way the only scale in the theory, the temperature, comes into the effective potential is through the 1-loop result $V_1$, which is order $g^2$ smaller than the gradient term. To make both of the same order, we have to admit \textit{gradients of order} $g$.

The effective potential, to this order, takes the form:

$$U(p) = \frac{1}{g^2} \int dz (1 + g^2 K_1(q)) \left( \frac{\partial}{\partial z} q \right)^2 + \int dz (V_1(q) + g^2 V_2(q))$$

The variable $q$ is defined by

$$A_0 = \text{diag} \left( \frac{q}{N}, \frac{q}{N}, ..., -\frac{(N-1)}{N} q \right)$$

In fig. 4 we have depicted the 1-loop effective potential $V_1$ for the case $N=3$, with global minima at the $Z(3)$ values of the Polyakov-loop. In fact the 2 loop contribution $V_2$ leaves these minima invariant. The very first check is the independence of the gauge fixing procedure, and this has been done.

The formula 4.1 depends on one parameter $q$ and describes only the rim of the figure. The full expression for the inside of the base triangle in fig. 4 is given in ref.9, including two loops.

In triangles neighbouring the one in fig. 3 one obtains the potential by reflection. Hence the potential has a local minimum along this rim, and the minimum configuration that gives us the surface tension is precisely developing along this rim, from $q=0 (p=1)$ to $q=1 (p=\exp i2\pi/3)$. Strikingly, this is what is found in MonteCarlo simulations. The result for the surface tension including the two loop effects in eq. 4.1 is then:

$$\alpha = \frac{4(N-1)}{3\sqrt{3N}} \pi^2 T^3 \left(1 - (15.27853...) \frac{g^2 N}{16\pi^2} \right)$$

Note the overall factor $\frac{1}{g}$, not $\frac{1}{g^2}$, for the lowest order result. This follows immediately from eq. 4.1, and that gradients are of order $g$. One just has to rescale the $z$ variable into $gz$.

5. Conclusions

We have shown, that $Z(N)$ bubbles are well alive, with the thickness of the wall $O(1/\overline{\nu}gT)$. They are not an artifact of the Euclidean path integral,
since the surface tension can be defined without mentioning the pathintegral. Including quarks lifts the degeneracy of the vacua in fig. 4, see fig. 5. But there stays a metastable state. This metastable state has received a lot of attention, because of its strange thermodynamical properties\textsuperscript{11}.

With quarks, there is no surface tension tension in the vein of section 3. Generally speaking, the introduction of a field, that feels the centergroup of the gauge group (quarks in QCD, electrons in QED) will render the definition of states with a given flux as in section 3.1 impossible, since the field will become multivalued.

It has been noticed\textsuperscript{12}, that the 1-loop kinetic term contains a pole at $q=0$. This means that for $q = O(g)$ perturbation theory for the kinetic term gets upset. But those values of $q$ contribute only to order $g^3$ to the surface tension eq. 4.2 as is readily seen, when minimising eq. 4.1.

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Fig. 1
Classical Potential in $\lambda \varphi^4$ with spontaneous symmetry breaking.

Fig. 2
System with fixed boundary conditions $\varphi = \pm v$. The correlation function has endpoints $x$ and $y$.

Fig. 3
Deformation of the dislocation line (dotted) in the presence of the correlation (the crosses). Horizontal bars are twisted links.

Fig. 4
One loop pure glue potential for $N = 3$.

Fig. 5
As in fig. 4, but with two quark flavours.
