Rényi entropy and Tsallis entropy associated with positive linear operators

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Abstract
This article is a continuation of my paper [arxiv: 1409.1015 v2]. Rényi and Tsallis entropies are associated to positive linear operators and properties of some functions related to these entropies are investigated.

1 Introduction
This paper is a continuation of [9]. In that article we considered discrete positive linear operators of the form

\[ Lf(x) = \sum_k f(x_k) a_k(x), \quad a_k(x) \geq 0, \quad \sum_k a_k(x) = 1 \]

for \( x \) in some interval \( I \subset \mathbb{R} \). In investigating the degree of non-multiplicativity of \( L \) an important role was played by the function \( S(x) = \sum_k a_k^2(x) \); see [1].

On the other hand, for each fixed \( x \in I \) the numbers \( (a_k(x))_k \) form a probability distribution. In this context \( -\log S(x) \) is a Rényi entropy [10] and \( 1 - S(x) \) is a Tsallis entropy [11]. So the properties of the function \( S(x) \) are relevant also in the study of these entropies.

Some properties of \( S(x) \) were investigated in [5]-[9]. In Section 2 we continue to study such properties in the case of discrete operators. Section 3 is devoted to some multivariate operators; see also [1]. In the last section we consider integral operators of the form \( Lf(x) = \int_I K(x, t)f(t)dt \) with \( K(x, t) \geq 0, \int_I K(x, t)dt = 1, x \in I \). Recall that in this case the associated Rényi entropy is defined by \( -\log \int_I K^2(x, t)dt \), and the Tsallis entropy by \( 1 - \int_I K^2(x, t)dt \).

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2 A conjecture from [9] and some of its consequences

We shall use the notation from [9]. In particular, we consider the function $S_{n,c}$ defined on the interval $I_c$. Conjecture 7.1 in [9] reads as follows:

\[(C) \text{ log } S_{n,c} \text{ is a convex function.}\]

Let us examine some consequences of this conjecture. With $X := x(1 + cx)$ and $X' = 1 + 2cx$, (3.10) in [9] can be written as

\[XX'\frac{S''_{n,c}(x)}{S_{n,c}(x)} + (4(n + c)X + 1)\frac{S'_{n,c}(x)}{S_{n,c}(x)} + 2nX' \leq 0. \tag{2.1}\]

\[(C) \text{ is equivalent to } S''_{n,c}S_{n,c} \geq (S'_{n,c})^2, \text{ and due to (2.1) both of them are equivalent to}\]

\[XX'\left(\frac{S'_{n,c}}{S_{n,c}}\right)^2 + (4(n + c)X + 1)\frac{S'_{n,c}}{S_{n,c}} + 2nX' \leq 0.\]

This leads immediately to

**Theorem 2.1** Conjecture (C) is equivalent to

\[\text{(C') } \frac{S'_{n,c}(x)}{S_{n,c}(x)} \text{ is between } z_1(x) \text{ and } z_2(x),\]

where

\[z_1(x) = \frac{-\sqrt{(1 + 4cX)^2 + (4nX)^2} - (1 + 4cX) - 4nX}{2XX'},\]

\[z_2(x) = \frac{\sqrt{(1 + 4cX)^2 + (4nX)^2} - (1 + 4cX) - 4nX}{2XX'}.\]

Let us consider the case $c = 0$. Then the function $K_n(x) := S_{n,0}(x)$ is defined for $x \in [0, +\infty)$. In [9 (6.4)] it was proved that

\[K_n(x) \leq \frac{1}{\sqrt{4nx + 1}}, \quad x \geq 0. \tag{2.2}\]

Recall also that $I_0$ is the modified Bessel function of first kind of order zero, and (see [9 (3.7)])

\[I_0(x) = e^xK_n\left(\frac{x}{2n}\right), \quad x \geq 0. \tag{2.3}\]

Combining (2.2) and (2.3) we get

\[I_0(x) \leq \frac{\exp x}{\sqrt{2x + 1}}, \quad x \geq 0. \tag{2.4}\]
Corollary 2.2 Under the hypothesis that \( \log S_{n,0} \) is convex, we have

\[
\frac{-\sqrt{1 + (4nt)^2} - 1 - 4nt}{2t} \leq \frac{K'_n(t)}{K_n(t)} \leq \frac{1}{2t}, \quad t > 0, \quad (2.5)
\]

\[
K^2_n(x) \leq \frac{2 \exp \left( \frac{\sqrt{1 + (4nx)^2} - 1 - 4nx}{\sqrt{1 + (4nx)^2} + 1} \right)}{x \geq 0}, \quad (2.6)
\]

\[
P'_0(x) \leq \frac{2 \exp \left( \frac{\sqrt{1 + 4x^2} - 1}{\sqrt{1 + 4x^2} + 1} \right)}{x \geq 0}, \quad (2.7)
\]

Proof. (2.5) is a direct consequence of Theorem 2.1. (2.6) can be obtained from the second inequality in (2.5) by integrating with respect to \( t \) between 0 and \( x \). (2.7) follows from (2.6) and (2.3).

In order to compare (2.2) with (2.6), and (2.4) with (2.7), it is easy to check that

\[
\frac{2 \exp \left( \frac{\sqrt{1 + (4nx)^2} - 1 - 4nx}{\sqrt{1 + (4nx)^2} + 1} \right)}{x \geq 0}, \quad (2.8)
\]

This yields

\[
\frac{F'_n(x)}{F_n(x)} = \frac{F'_n(t)}{F_n(t)} - \frac{n}{\sqrt{t^2 - 1}} \frac{dt}{dx}.
\]

and consequently

\[
\frac{F'_n(t)}{F_n(t)} = -\frac{n}{\sqrt{t^2 - 1}} \frac{1}{4X} \frac{F'_n(x)}{F_n(x)}, \quad (2.9)
\]

Corollary 2.3 Under the hypothesis that \( \log S_{n,-1} \) is convex, we have

\[
\frac{F'_n(x)}{F_n(x)} \leq \frac{\sqrt{(1 - 4X)^2 + (4nX)^2} - (1 - 4X) - 4nX}{2XX'}, \quad x \in \left[ 0, \frac{1}{2} \right), \quad (2.10)
\]

\[
\frac{P'_n(t)}{P_n(t)} \leq \frac{\sqrt{4n^2(t^2 - 1)} + (t - \sqrt{t^2 - 1})^2 - (t - \sqrt{t^2 - 1})}{2(t^2 - 1)}, \quad t > 1. \quad (2.11)
\]
Proof. (2.10) is a consequence of Theorem 2.1. (2.11) follows from (2.10) and (2.9).

The following inequality was proved in [8, (1.2)]:
\[
\frac{n(n+1)}{2t + (n-1)\sqrt{t^2-1}} \leq \frac{P_n'(t)}{P_n(t)}, \quad t \geq 1.
\]
(2.12)

Using it and (2.9), we get
\[
-\frac{2nX'}{1 + (n-3)X} \leq \frac{F_n'(x)}{F_n(x)}, \quad x \in \left[0, \frac{1}{2}\right].
\]
(2.13)

Other lower and upper bounds for \(\frac{P_n'(t)}{P_n(t)}\) can be found in [8]. In particular, from [8, Theorems 2 and 3] we have
\[
\frac{P_n'(t)}{P_n(t)} \leq \frac{2n^2}{t + (2n - 1)\sqrt{t^2 - 1}}, \quad t \geq 1,
\]
(2.14)
and
\[
\frac{P_n'(t)}{P_n(t)} \leq \frac{n^2(2n+1)}{(n+1)t + (2n^2 - 1)\sqrt{t^2 - 1}}, \quad t \geq 1.
\]
(2.15)

(2.11) and (2.14) can be compared and we get
\[
\sqrt[4]{4n^2(t^2 - 1) + (t - \sqrt{t^2 - 1})^2 - (t - \sqrt{t^2 - 1})^2} \leq \frac{2n^2}{t + (2n - 1)\sqrt{t^2 - 1}}, \quad t > 1.
\]

The inequality
\[
\sqrt[4]{4n^2(t^2 - 1) + (t - \sqrt{t^2 - 1})^2 - (t - \sqrt{t^2 - 1})^2} \leq \frac{n^2(2n+1)}{(n+1)t + (2n^2 - 1)\sqrt{t^2 - 1}}, \quad t > 1,
\]
is equivalent to
\[
\frac{t}{t + \sqrt{t^2 - 1}} \geq \frac{3n + 2}{4n + 3}.
\]

This last inequality is true for \(t\) approaching 1, and false for \(t\) approaching \(+\infty\).

Let us remark that (2.16) yields by integration
\[
P_n(t) \leq (t + \sqrt{t^2 - 1})^{\frac{n(2n-1)}{2n^2 - n - 2}} \left( t + \frac{2n^2 - 1}{n + 1} \sqrt{t^2 - 1} \right)^{-\frac{n(n+1)}{2n^2 - n - 2}}, \quad t \geq 1.
\]
(2.16)

This inequality is stronger than
\[
P_n(t) \leq (t + \sqrt{t^2 - 1})^{\frac{n(2n-1)}{2n(n-1)}} \left( t + (2n - 1)\sqrt{t^2 - 1} \right)^{-\frac{n}{2n-1}}, \quad t \geq 1, n \geq 2.
\]
(2.17)
which can be obtained from (2.14).

We conclude this section with a remark concerning the function \( A_{n,c} := S'_{n,c} \).

By using \([9, (3.10)]\), or (2.1), we deduce easily that \( A_{n,c} \) satisfies the Riccati equation

\[
x(1 + cx)(1 + 2cx)(A'_{n,c} + A^2_{n,c}) + (4(n + c)x(1 + cx) + 1)A_{n,c} + 2n(1 + 2cx) = 0.
\]

### 3 Multivariate operators

First, consider the classical Bernstein operators on the canonical simplex of \( \mathbb{R}^2 \).

The sum of the squared fundamental Bernstein polynomials is in this case

\[
R_n(x, y) := \sum_{i+j \leq n} \left( \frac{n!}{i!j!(n-i-j)!} \right)^2 x^i y^j (1-x)^{2(n-i-j)}
\]

\[
= \sum_{j=0}^n \sum_{i=0}^{n-j} \left( \frac{n!}{i!j!(n-i-j)!} \right)^2 \left( \frac{n-j}{i} \right)^2 x^i y^j (1-x)^{2(n-i-j)},
\]

for \( x \geq 0, \ y \geq 0, \ x + y \leq 1 \); see \([2, (6.3.6)]\), \([3, Sect. 3.1.2]\).

Let \( y \in [0, 1) \) be fixed. Then for \( x \in [0, 1 - y] \) we have

\[
R_n(x, y) = \sum_{j=0}^n \left( \frac{n!}{i!j!(n-i-j)!} \right)^2 \left( \frac{n-j}{i} \right)^2 x^i y^j (1-x)^{2(n-i-j-i)}.
\]

For each \( j \in \{0, 1, \ldots, n\} \),

\[
\sum_{i=0}^{n-j} \left( \frac{n-j}{i} \right)^2 \left( \frac{x}{1-y} \right)^{2i} \left( 1 - \frac{x}{1-y} \right)^{2(n-j-i)} = F_{n-j} \left( \frac{x}{1-y} \right),
\]

where \( F_{n-j} = S_{n-j-1} \). It is known (see \([3, 6, 8, 9]\)) that \( F_{n-j} \) is convex on \([0, 1]\). It follows that for each fixed \( y \in [0, 1] \), the function \( R_n(\cdot, y) \) is convex on \([0, 1 - y]\). In other words, \( R_n \) is convex on each segment parallel to \( Ox \).

Similarly we see that \( R_n \) is convex on each segment parallel to a side of the canonical triangle of \( \mathbb{R}^2 \). This means that \( R_n \) is axially-convex; concerning this terminology see \([2, p. 407]\), \([3, Sect. 3.5]\).

Now consider the classical Bernstein operators on the square \([0, 1]^2\): see \([2, (6.3.101)]\), \([3, Sect. 3.1.5]\). The sum of the squared fundamental Bernstein polynomials is in this case

\[
Q_n(x, y) = \sum_{i=0}^n \sum_{j=0}^n \left( \frac{n!}{i!j!(n-i-j)!} \right)^2 x^i y^j (1-x)^{2n-2i-y} (1-y)^{2n-2j} = F_n(x)F_n(y).
\]

It is easy to verify that the following three statements are equivalent:

i) \( \log F_n \) is convex on \([0, 1]\);

ii) \( Q_n \) is convex on \([0, 1]^2\);

iii) \( \log Q_n \) is convex on \([0, 1]^2\).
4 \hspace{1em} \textbf{Entropy and variance. Integral operators}

Let $I$ be an interval and $L$ a positive linear operator on a space of functions defined on $I$, containing the functions $e_i(x) = x^i$, $i = 0, 1, 2$. Suppose that $Le_0 = e_0$.

The \textit{variance} associated with $L$ is the function

$$V(x) := Le_2(x) - (Le_1(x))^2, \quad x \in I.$$ 

If $L$ is a discrete operator of the form $Lf(x) = \sum_k a_k f(x_k)$, let $S(x) := \sum_k a_k^2(x)$. If $L$ is an integral operator of the form $Lf(x) = \int_I K(x, t)f(t)dt$, let $S(x) := \int_I K^2(x, t)dt$, $x \in I$.

In both cases the Rényi entropy associated with $L$ is $-\log S(x)$, and the Tsallis entropy is $1 - S(x)$, $x \in I$.

\textbf{Example 4.1}

Let $L_{n,c}f(x) = \sum_{j=0}^{\infty} f\left(\frac{x}{n}\right) p_{n,j}^c(x)$, see [9, Sect. 2]. Then $S(x) = S_{n,c}(x)$ and $V(x) = V_{n,c}(x) = \frac{x^{1+cx}}{1+cx}$. According to [9, (3.5), (3.8)],

$$S_{n,c}(x) = \frac{1}{\pi} \int_0^\pi \left(1 + 4ncV_{n,c}(x)\sin^2\frac{\varphi}{2}\right)^{-n/c} d\varphi, \quad c \neq 0, \quad (4.1)$$

$$S_{n,0}(x) = \frac{1}{\pi} \int_0^\pi \exp\left(-4n^2V_{n,0}(x)\sin^2\frac{\varphi}{2}\right) d\varphi. \quad (4.2)$$

\textbf{Example 4.2}

For the Kantorovich operators [2, p. 333] we have $S_n(x) = (n + 1)F_n(x)$ and $V_n(x) = (n + 1)^{-2}(nx(1-x) + \frac{1}{12})$.

\textbf{Example 4.3}

Consider the Gauss-Weierstrass operators [2 p. 310], [4 p. 114]:

$$W_rf(x) = \int_R (4\pi r)^{-1/2} \exp\left(-\frac{(t-x)^2}{4r}\right)f(t)dt, \quad r > 0.$$ 

Then $V_r(x) = 2r$ and $S_r(x) = (8\pi r)^{-1/2}, x \in R$.

Generally speaking, for a convolution operator

$$Lf(x) = \int_R \varphi(x-t)f(t)dt$$

we have $V(x) = \int_R s^2\varphi(s)ds - \left(\int_R s\varphi(s)ds\right)^2$

and $S(x) = \int_R \varphi^2(s)ds$, so that $V$ and $S$ are constant functions.

\textbf{Example 4.4}
For the Post-Widder operators [1 p. 114],

\[ V_n(x) = \frac{x^2}{n} \quad \text{and} \quad S_n(x) = \left(\frac{2n - 2}{n - 1}\right)^{2^{1-2n} \frac{n}{x}}, \quad x > 0. \]

**Example 4.5**

Consider the Durrmeyer operators [2 p. 335].

In this case

\[ V_n(x) = \frac{n + 1}{(n + 2)^2(n + 3)}(2nx(1 - x) + 1), \]

\[ S_n(x) = \sum_{k=0}^{2n} c_{n,k} \left(\frac{2n}{k}\right)x^k(1 - x)^{2n-k}, \]

where

\[ c_{n,k} := \frac{(n + 1)^2}{2n + 1} \left(\frac{2n}{k}\right)^{-2} \sum_{j=0}^{k} \left(\frac{n}{j}\right)^{2} \left(\frac{n}{k-j}\right)^{2}, \quad k = 0, 1, ..., 2n, \]

where, as usual, \( \binom{n}{m} = 0 \) if \( m > n \).

It is easy to see that \( c_{n,2n-k} = c_{n,k}, k = 0, 1, ..., 2n \).

**Conjecture 4.6** The sequence \( (c_{n,k})_{k=0,1,...,2n} \) is convex and, consequently, the function \( S_n \) is convex on \([0, 1]\).

**Example 4.7**

For the genuine Bernstein-Durrmeyer operators, defined by

\[ U_n f(x) = f(0)b_{n,0}(x) + f(1)b_{n,n}(x) + (n - 1) \sum_{k=1}^{n-1} b_{n,k}(x) \int_0^1 b_{n-2,k-1}(t)f(t)dt, \]

with \( b_{n,k}(x) = \binom{n}{k}x^k(1 - x)^{n-k} \), we have

\[ V_n(x) = \frac{2x(1 - x)}{n + 1} \]

and

\[ S_n(x) = (1 - x)^{2n} + x^{2n} + \frac{(n-1)^2}{2n-3} \sum_{k,j=1}^{n-1} \binom{n-2}{k-1} \binom{n-2}{j-1} \binom{n}{k} \binom{n}{j} \binom{2n-4}{k+j-2}^{-1} x^{k+j}(1 - x)^{2n-k-j}. \]

**Remark 4.8** In Examples 4.1-4.4, and also in Example 4.5 under Conjecture 4.6, the functions \( V(x) \), \( 1 - S(x) \) and \( -\log S(x) \) are all increasing or all decreasing on suitable subintervals of \( I \). In other words, the variance, the Tsallis entropy and the Rényi entropy are synchronous functions.
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