Giant acceleration in slow-fast space-periodic Hamiltonian systems

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Motion of an ensemble of particles in a space-periodic potential well with a weak wave-like perturbation imposed is considered. We found that slow oscillations of wavenumber of the perturbation lead to occurrence of directed particle current. This current is amplifying with time due to giant acceleration of some particles. It is shown that giant acceleration is linked with the existence of resonant channels in phase space.

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In the recent years considerable interest was devoted to the ratchet effect, generation of directed particle current in the absence of any biased forces. This phenomenon is relevant for wide range of applications, including controlled photocurrents in semiconductors [1, 2], motion of cold atoms in optical lattices [3, 4, 5], transport of passive tracers in meandering jet flows in the ocean [6], biological and chemical systems (see [7] for a comprehensive review).

The ratchet effect in space-periodic Hamiltonian systems is associated with the asymmetry of chaotic region in phase space [8, 9, 10]. Recently it was reported about new type of Hamiltonian ratchets, in which a periodic potential is subjected to a sum of external forces which are periodic in time and in coordinate [11]. Each of the forces induces strong but local chaotic diffusion in certain areas of phase space. This effect is achieved by means of resonance between temporal and spatial oscillations of the perturbation imposed. This resonance is asymmetric in momentum space, that provides asymmetry of crossing the separatrix and occurrence of directed transport. Combining such forces, we can make finite motion unstable for all range of the particle energy. By this way, even a weak perturbation of the potential can activate ballistic current of particles with the lowest initial energies. A similar effect was used in [12] in order to produce surfatron acceleration.

In the present Letter we demonstrate the mechanism providing simultaneously generation and giant acceleration of directed current by means of a weak external perturbation. Possibility of giant acceleration arises due to adiabatic variations of the perturbation.

Consider an ensemble of non-interacting unit-mass point particles driven by a wave-like external force. Motion of each particle is described by the Hamiltonian

\[ H = \frac{p^2}{2} - \cos x + \varepsilon \cos(\tilde{k}x + \nu t), \]  

(1)

where \( \varepsilon \ll 1 \), and wavenumber of the perturbation \( \tilde{k} \) is a slowly-varying parameter

\[ \tilde{k} = k(1 + a \cos \Omega t), \quad |a| < 1, \quad \Omega \ll 1. \]  

(2)

Physically this condition corresponds to slow libration of the external force with respect to axis \( x \). Particle trajectories obey the Hamiltonian equations

\[ \dot{x} = p, \quad \dot{p} = -\sin x + \varepsilon \tilde{k} \sin \phi, \]  

(3)

where we denote the perturbation phase \( \tilde{k}x + \nu t \) as \( \phi \).

Outside this region particle dynamics is close to integrable and can be described using the averaging technique [13]. According to (4), the resonant area in phase space is located along the line given by the following equation

\[ p_{\text{res}} = -\frac{\nu}{k(1 + a \cos \Psi)} + \frac{a \Omega \sin \Psi}{1 + a \cos \Psi} x, \]  

(5)

where \( \Psi = \Omega t \). In order to describe the motion inside the resonant region we derive, using (4) and (5), the “pendulum-like” equation for \( \phi \)

\[ \ddot{\phi} - \varepsilon \tilde{k}^2 \sin \phi + f(x, p, \Psi) = 0, \]  

(6)

where \( f(x, p, \Psi) \) is treated as a slowly-varying parameter given by the following equation

\[ f(x, p, t) = a k \Omega (2p \sin \Psi + \Omega x \cos \Psi) + \tilde{k} \sin x. \]

(7)

Equation (6) corresponds to the Hamiltonian of the following form:

\[ \tilde{H}(\phi, \psi) = \frac{1}{2} \dot{\phi}^2 + \phi f(x, p, \Psi) + \varepsilon \tilde{k}^2 \cos \phi. \]  

(8)

If the inequality

\[ |f(x, p, \Psi)| < \varepsilon \tilde{k}^2 \]

(9)

is satisfied, then the phase portrait corresponding to the Hamiltonian (8) contains an oscillating resonant region
in $\phi$–$\dot{\phi}$ space, bounded by a separatrix. Falling into resonance (4) corresponds to crossing the separatrix and entering this region. Probability of entering depends on the area of the resonant region and increases with increasing the difference between the left- and right-hand sides of (9). A particle spends any time inside the resonant zone and then leaves it with strongly increased or decreased energy. Following [14, 15, 16], we derive an approximate formula for the energy jump

$$\Delta E = -\varepsilon k p^* \int_{-\infty}^{\phi^*} \frac{\sin \phi d\phi}{\sqrt{2(H - \phi f^* - \varepsilon k^2 \cos \phi)}}.$$  

(10)

where $0 < \phi < 2\pi$, $f^*$, $p^*$ and $\phi^*$ are values of $f$, $p$ and $\phi$, respectively, when the trajectory hits the resonant region. $\Delta E$ depends extremely on $\phi^*$, therefore multiple recurrences to the resonant area cause chaotic diffusion in phase space [17, 18, 19].

FIG. 1: Function $g(x, \Psi)$ with $x = -1000$, $x = -20000$, $x = -50000$, and $x = -100000$.

With fixed values of $k$ and $\nu$, the resonant condition (4) has the simplest form

$$kp + \nu \simeq 0.$$  

(11)

It should be emphasized, that inequality (9) is fulfilled only if a particle is not far from an extremum of the unperturbed potential. Hence the condition (11) should be replaced by the following one [18]:

$$p(E_{\text{res}}, x = \pi l) = p_{\text{res}} \simeq -\frac{\nu}{k},$$  

(12)

where $l$ is integer. Using (12) we find the resonant values of energy

$$E_{\text{res}} = \frac{\nu^2}{2k^2} \mp 1.$$  

(13)

Equation (13) determines locations of the chaotic layers in energy space [11, 17, 18]. If the chaotic layer induced by resonance (12) coalesces with the near-separatrix chaotic layer, then the chaotic sea formed has much larger width in the lower half-plane of phase space than in the upper one [11]. It follows from the asymmetry of the condition (12) in momentum space and implies the prevalence of chaos-induced particle flights towards $x = -\infty$.

It is natural to suggest that adiabatic variation of the resonant momentum (12) leads to a gradual displacing of areas of instability in phase space. If the timescale of diffusive mixing inside the chaotic areas is much smaller than the timescale of changing the resonant value of momentum (12), then these areas play the role of dynamical traps for particles, so-called stochastic layer traps (SLT) [20, 21]. Consequently displacement of a chaotic layer in

FIG. 2: (a) Mean coordinate, (b) mean momentum and (c) momentum variance as functions of time.
energy space can be followed by increasing or decreasing of mean energy of particles belonging to it. As it will be shown in this Letter, a rather complicated situation occurs if wavenumber of the perturbation varies according to the law (2).

Substituting (5) into (7), we obtain the expression for the criterion (9) on the resonant line

\[
\left|\frac{a\Omega^2(-a\cos^2\Psi + \cos \Psi + 2a)}{(1 + a \cos \Psi)^2}x + \sin x - \frac{2a\Omega \nu \sin \Psi}{k(1 + a \cos \Psi)^2}\right| \leq \tilde{\varepsilon}k. \tag{14}
\]

This inequality holds if \(\sin x \approx 0\) and, subsequently, \(x \approx \pi l\), where \(l\) is integer. When skipped the term \(\sim \sin x\), we can rewrite the criterion (14) as follows:

\[
g(x, \Psi) = \left|\frac{a\Omega^2(-a\cos^2\Psi + \cos \Psi + 2a)}{(1 + a \cos \Psi)^2}x - \frac{2a\Omega \nu \sin \Psi}{k(1 + a \cos \Psi)^2}\right| - \tilde{\varepsilon}k \leq 0. \tag{15}
\]

Figure 3 represents the function \(g(x, \Psi)\) with different fixed values of \(x\). According to this figure, the criterion (14) is satisfied with \(|x| < 50000\) for the large intervals of \(\Psi\), centered at \(2\pi m\), where \(m = 0, 1, 2, \ldots\). This implies existence of those trajectories which, being passed the resonance area at once, will visit the resonant area repeatedly on the subsequent cycles of pendulum, till the slowly-varying phase \(\Psi\) remains close to \(2\pi m\). Such particles move along the lines described by (5), towards the point \(x = 0\) when \(\sin \Psi < 0\) and from it when \(\sin \Psi > 0\). The latter ones are capable to perform ballistic flights with increasing velocity.

Occurrence of such ballistic flights is confirmed by numerical simulation. We computed evolution of the ensemble of particles, initially distributed with gaussian probability density

\[
\rho(x, p, t = 0) = \frac{1}{2\pi \sigma_{0x} \sigma_{0p}} \exp \left( -\frac{x^2}{\sigma_{0x}^2} - \frac{p^2}{\sigma_{0p}^2} \right), \tag{16}
\]
where $\sigma_{0x} = \sigma_{0y} = 0.1$. The parameters of the perturbation we used are the following: $\varepsilon = 0.04$, $k_0 = 12$, $\nu = 4$, $\alpha = 0.75$, $\Omega = 2\pi/1000$. Figure 2 represents the temporal dependence of mean coordinate, mean momentum and variance of momentum. It is shown that there occurs a particle flux directed towards $t \to -\infty$. The mean momentum grows nonmonotonically and abrupt accelerations are alternating with abrupt slowdowns down. Acceleration takes place when the slow phase $\Psi$ is close to $2\pi m$, that agrees with our analysis. Each act of acceleration is followed by step-like increasing of momentum variance. It should be emphasized that momentum variance is much larger than mean momentum, that indicates variance. It is demonstrated in Fig. 4, where instantaneous particle distributions at $t = 3200$, $t = 5200$ and $t = 9200$ are presented. Evolution of the particle cloud is also presented in the media files, which are available at [22].

We can distinguish three stages of evolution of the particle ensemble. At the first stage directed current is activating. Until a particle is not far from $x = 0$, the location of the resonant zone is determined by the formula $p_{\text{res}} = -\nu/k(t)$. That infers that the initial particle cloud is placed inside the chaotic layer caused by resonance with $p_{\text{res}} = -\nu/k(t = 0) = -4/21$. Adiabatic decreasing of $k$ displaces this layer to the separatrix. The time of diffusive mixing inside the chaotic layer is much smaller than $2\pi/\Omega$, so that the particle cloud follows the chaotic layer. At $\Psi = \pi$ chaotic layer caused by resonance [11] merges into the near-separatrix chaotic layer, that leads to occurrence of ballistic particle current towards $x = -\infty$.

The second stage starts when the particle cloud becomes enough wide and some particles are capable to fall into the resonant channels described by [9]. This stage is characterized by fast growth of momentum variance due to events of giant acceleration. Note that some particles turn around and then perform ballistic flights in the opposite direction. Nevertheless, number of particles accelerating in the direction $x \to -\infty$ is much larger, therefore the turned particles give negligible contribution into the resulting particle flux. Since the resonant channels have finite length, one can call the zone where they exist as the accelerating zone.

The third stage is not presented in the figures. It starts when the particle cloud becomes very wide and only negligible fraction of a particle ensemble remains within the accelerating zone. At this stage momentum variance achieves saturation and stops increasing.

In conclusion, in this work we demonstrated the effect of giant particle acceleration in the simple space-periodic Hamiltonian system subjected to a slowly-modulated external force. The effect arises from the specific topology of resonance [11] in phase space, which permits capturing of a particle into the accelerating channel.

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