A non-existence theorem for Morse type holomorphic foliations of codimension one transverse to spheres

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Abstract

We prove that a Morse type codimension one holomorphic foliation is not transverse to a sphere in the complex affine space. Also we characterize the variety of contacts of a linear foliation with concentric spheres.

1 Introduction

One of the main tools in the classical theory of codimension one real foliations is a theorem of A. Haefliger [3] which implies that an analytic codimension-one foliation admits no null (homotopic) transversals. The heart of the proof consists of a description of the dynamics of a real vector field in a neighborhood of the closed disc $D^2 \subset \mathbb{R}^2$ and transverse to the boundary $\partial D^2 \simeq S^1$. The use of Poincaré-Bendixson Theorem shows the existence of some one-sided hyperbolicity, for some closed orbit or graph $\gamma \subset D^2$, what is not compatible with the analytical behavior. Unfortunately, there is no feature like the classical Poincaré-Bendixson Theorem in the case of holomorphic vector fields. To overcome this difficult is one of the two basic motivations for the present work. The second comes from [4] where the authors study the topology of the variety of contacts of a holomorphic vector field $F = \sum_{j=1}^{n} F_j \frac{\partial}{\partial z_j}$ in an open neighborhood $U$ of the origin $0 \in \mathbb{C}^n$, having an isolated singularity at $0$, with the spheres around $0$. Let $M$ be this variety of contacts, i.e., $M = \{ z \in U | \langle F(z), z \rangle =: \sum_{j=1}^{n} F_j(z)z_j = 0 \}$. In case the singularity is in the Siegel domain we have $M \neq 0$ and results in [1], [2] and [5] imply that if $F$ is linear and generic then $M \setminus \{0\}$ is a smooth codimension two manifold in $\mathbb{C}^n$, each Siegel leaf of $F$ intersects $M \setminus \{0\}$ in exactly one point, which corresponds to the minimal distance of the leaf to the origin. The main result in [4] then generalizes some of these properties to a class of vector fields called of Morse type, corresponding to vector fields for which the distance function on each leaf has only nondegenerate singularities. Indeed, the class corresponds to those for which the distance flow, which is the real analytic vector field defined by $r_F := -\langle F(z), z \rangle F(z)$, has only nondegenerate singularities on each leaf. Using this notion the authors are able to prove that for a Morse type vector field $F$ either $M = 0$ or $M$ is a codimension two real analytic variety, singular only at $0$, indeed $M$ is a cone and $M \setminus \{0\}$ embeds into $\mathbb{C}^n$ with trivial normal bundle (it is transversal to the foliation of $F$), has only a finite number of connected components, and exhibits some stability with respect to the induced foliations on the small spheres around the origin.

In this paper we shall begin the study of the variety of contacts for a codimension one holomorphic foliation in complex dimension $n \geq 3$. This is also done by introducing a class of Morse type foliations and we prove that for such a foliation we never have $M = 0$. In other words, we give a negative answer for the following question in case of Morse type foliations:

...
Question 1. Is there a holomorphic codimension one foliation transverse to the unit sphere $S^{2n-1}(1) \subset \mathbb{C}^n$ if $n \geq 3$?

As mentioned above this question is related to the dynamics of holomorphic foliations of codimension one and it seems that the answer is no. In the last part we give a description of the variety of contacts for generic linear one-forms.

Let us make precise the notions that we use. Let $\mathcal{F}$ be a holomorphic foliation, possibly with singularities, on a complex manifold $V$. Given a smooth (real) submanifold $M \subset V$ we shall say that $\mathcal{F}$ is transverse to $M$ if $\text{sing}(\mathcal{F}) \cap M = \emptyset$ and for every $p \in M$ we have $T_p(L_p) + T_p(M) = T_p(V)$ as real spaces, where $L_p$ denotes the leaf of $\mathcal{F}$ that contains the point $p \in M$. In this paper we address Question 1. As it is known such foliation is defined by a holomorphic one-form $\Omega$, with singular set of codimension $\geq 2$, and satisfying the integrability condition $\Omega \wedge d\Omega = 0$. In previous papers \cite{7}, \cite{8} and \cite{9} we proved the non-existence of the foliation under some additional conditions: for instance, $\mathcal{F}$ is defined by an integrable homogeneous polynomial one-form, or $\mathcal{F}$ has a global separatrix transverse to each sphere $S^{2n-1}(r), 0 < r \leq 1$, or $n$ is odd. In the case of a non-integrable holomorphic one-form, if $n = 2m + 1$ is odd there is no holomorphic one-form $\Omega$ such that the distribution $\text{Ker}(\Omega) = \{ \Omega = 0 \}$ is transverse to the sphere $S^{2m+1}(1) \subset \mathbb{C}^{2m+1}$ (\cite{7}). If $n$ is even, we have a typical example (\cite{9}). In $\mathbb{C}^{2m}$ with coordinates $(z_1, ..., z_{2m})$ we define $\Omega = \sum_{j=1}^{2m} (z_{2j}dz_{2j-1} - z_{2j-1}dz_{2j})$. This holomorphic one-form satisfies the strong non-integrability condition $\Omega \wedge d\Omega \neq 0$, off the origin, and $\text{Ker}(\Omega)$ is transverse to the sphere $S^{4m-1}(1) \subset \mathbb{C}^{2m}$.

On the other hand, in the case of a holomorphic vector fields on $\mathbb{C}^n, n \geq 2$, a Poincaré-Bendixson type theorem has been proved in \cite{8} stating that if $Z$ is a holomorphic vector field in a neighborhood $U$ of the closed unit disk $\overline{D}^{2n} \subset \mathbb{C}^n, n \geq 2$, such that the corresponding foliation $\mathcal{F}(Z)$ the foliation defined by the solutions of $Z$ is transverse to $S^{2n-1}(1) = \partial \overline{D}^{2n}$, then $Z$ has only one singular point, say $p$, in $\overline{D}^{2n}$ and the index of $Z$ at $p$ is equal to one. Moreover, each solution $L$ of $Z$ which crosses $S^{2n-1}(1)$ tends to the unique singular point $p$ of $Z$ in the disk, $i.e.$, $p$ is in the closure of $L$. Furthermore, the restriction $\mathcal{F}(Z)|_{\overline{D}^{2n} \setminus \{p\}}$ is $C^w$-conjugate to the foliation $\mathcal{F}(Z)|_{S^{2n-1}(1) \times (0,1]} \mid_{\overline{D}^{2n} \setminus \{p\}}$.

This paper is dedicated to the proof of non-existence in case of generic foliations, where “generic” stands for a generic set tangencies with the spheres centered at the origin as follows. Let $\mathcal{F}(\Omega)$ be as above and denote by $\varphi$ the distance function with respect to the origin $0 \in \mathbb{C}^n$.

Definition 1 (Morse type). The foliation $\mathcal{F} = \mathcal{F}(\Omega)$ is of Morse type if for each leaf $L \in \mathcal{F}(\Omega)$, each critical point $p \in L$ of $\varphi|_L$ on $L$ is nondegenerate. $\Omega$ is of Morse type if $\mathcal{F}(\Omega)$ is.

Our main result is the following non-existence theorem:

Theorem 1. There is no Morse type holomorphic foliation of codimension one $\mathcal{F}$ in a neighborhood $U$ of the closed unit disk $\overline{D}^{2n} \subset \mathbb{C}^n, n \geq 3$ such that $\mathcal{F}$ is transverse to the boundary sphere $S^{2n-1}(1) = \partial \overline{D}^{2n}$.

A linear foliation is defined by a one-form $\Omega_A = \sum_{i,j=1}^{n} a_{ij}z_idz_j$, where $A = (a_{ij})_{i,j=1}^{n}$ is a symmetric matrix. Regarding transversality of linear foliations with spheres we prove in \cite{7} that a linear foliation $\mathcal{F}(\Omega_A)$ on $\mathbb{C}^n$ is not transverse to the sphere $S^{2n-1}(1)$. Moreover,
\[ \mathcal{F}(\Omega A) \text{ is transverse to the sphere } S^{2n-1}(1) \text{ off the singular set } \text{sing}(\mathcal{F}(\Omega A)) \cap S^{2n-1}(1) \text{ if and only if } \mathcal{F}(\Omega A) \text{ is a product } \mathcal{L}_\lambda \times \mathbb{C}^{n-2} \text{ for some linear foliation } \mathcal{L}_\lambda: x dy - \lambda y dx = 0, \text{ in the Poincaré domain on } \mathbb{C}^2. \]

The following result then describes the variety of contacts for a generic linear foliation.

**Theorem 2.** Any linear foliation \( \mathcal{F} \) on \( \mathbb{C}^n \) can be arbitrarily approximated by Morse type linear foliations. In particular, \( \mathcal{F} \) is not transverse to \( S^{2n-1}(1) \) if \( n \geq 3 \). For a dense Zariski subset of linear foliations on \( \mathbb{C}^n \) the variety of contacts of its elements is a union of \( n \) complex lines through the origin.

## 2 Pugh’s generalization of Poincaré-Hopf theorem

In this section we recall Pugh’s generalization of Poincaré index formula \((\Pi)\) and we given an example which we use in §6 Lemma 2.

Let \( X \) be a \( C^\infty \) vector field on a manifold \( M \), such that:

1. \( X \) has no singularity on \( \partial M \).
2. The singularities of \( X \) in \( M \) are of Morse type.
3. \( X \) has a generic contact set with \( \partial M \).

Let \( n_i \) denote the number of zeros of \( X \) of Morse index \( i \), where the Morse index is the number of eigenvalues of \( DX(p) \) having negative real part. Let also \( \Sigma(X) := \sum_i (-1)^i n_i \).

Then we have

\[ \Sigma(X) = \mathcal{X}(M, \partial M) + \sum_{i \geq 1} \mathcal{X}(R^i_-, \Gamma^i) \]

where

1. \( R^i_- \) is the \( i \)-codimensional exit region.
2. \( \Gamma^i \) is the boundary of \( R^i_- \).
3. The Euler characteristic is defined as

\[ \chi(M, \partial M) = \sum_i \dim H_i(M, \partial M), \]

in terms of relative homology.

If \( M = D^2 \), the unit disk in \( \mathbb{C} \), we have \( \mathcal{X}(M, \partial M) = 1 \), \( R^i_- \) is the set of arcs on \( \partial D^2 = S^1 \) where \( X \) points out of \( D^2 \), \( \Gamma^1 \) is the set of tangency points, \( R^2_- \) is the set of points on \( \partial R^1_- = \Gamma^1 \) where \( X \) points out of \( R^1_- \). Because of the generic contact between \( X \) and \( \partial M \), there are no inflection points on \( S^1 \) and therefore, exit and entrance arcs alternate around \( S^1 \). Therefore \( \mathcal{X}(R^1_-) \) is half the number of tangency points, i.e., \( \mathcal{X}(\Gamma^1)/2 \). \( R^2_- \) is easily seen to be the set of interior tangencies and \( \Gamma^2 = \emptyset \). We have then

\[ \mathcal{X}(M, \partial M) + \sum_{i \geq 1} \mathcal{X}(R^i_-, \Gamma^i) = 1 + \mathcal{X}(R^1_-, \Gamma^1) + \mathcal{X}(R^2_-, \emptyset) \]

\[ = 1 + \mathcal{X}(\Gamma^1)/2 - \mathcal{X}(\Gamma^1) + i = 1 + \frac{2i - \mathcal{X}(\Gamma^i)}{2} = 1 + \frac{2i - (i + e)}{2} = 1 + \frac{i - e}{2}. \]
We obtain therefore the formula below quoted as Poincaré formula.

\[ I = 1 + \frac{i - e}{2} \]

where \( I \) is the topological index of the vector field, \( i \) is the number of interior tangencies and \( e \) is the number of exterior tangencies.

**Example 1.** Consider the case of an isolated Morse singularity of index \( i \geq 1 \). Let \( f(x) = -x_1^2 - \ldots - x_i^2 + x_{i+1}^2 + \ldots + x_n^2 \) be the Morse function of Morse index \( i \) at 0. We denote by \( \text{grad}(f) = -2x_1 \frac{\partial}{\partial x_1} - \ldots - 2x_i \frac{\partial}{\partial x_i} + 2x_{i+1} \frac{\partial}{\partial x_{i+1}} + \ldots + 2x_n \frac{\partial}{\partial x_n} \) the gradient vector field of \( f \) and by \( \vec{n} = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \) the radial vector field on \( \mathbb{R}^n \). Take \( r > 0 \), we investigate the gradient \( \text{grad}(f) \) in the disc \( D^n(r) \). Then we have \( R^1_+ = \{ x \in S^{n-1}(r) \mid (\text{grad}(f)(x), \vec{n}(x)) > 0 \} \) and \( \Gamma^1 = \{ x \in S^{n-1}(r) \mid (\text{grad}(f)(x), \vec{n}(x)) = 0 \} \). Therefore, \( R^1_+ \) is homotopic to \( D^i \times S^{n-i-1} \cong S^{n-i-1} \), where \( D^i \) is the \( i \)-dimensional disc, \( S^{n-i-1} \) the boundary of \( D^n \) and \( \cong \) means homotopic to. On the other hand \( \Gamma^1 \) is homotopic to \( S^{i-1} \times S^{n-i-1} \). Then we can calculate \( \chi(R^1_+, \Gamma^1) \) as follows:

\[ \chi(R^1_+, \Gamma^1) = \chi(R^1_+) - \chi(\Gamma^1) = \chi(S^{n-i-1}) - \chi(S^{i-1}) \cdot \chi(S^{n-i-1}). \]

We will apply Pugh’s generalization of Poincaré-Hopf theorem to the case of leaves of dimension \( n = 2(m - 1) \):

\[ (-1)^i \cdot 1 = 1 + \begin{cases} 2 - 2 \times 2, & \text{if } i \text{ is odd} \\ 0 - 0 \times 0, & \text{if } i \text{ is even.} \end{cases} \]

### 3 The variety of contacts

Let \( \Omega = \sum_{j=1}^{n} f_j(z) dz_j \) be a holomorphic one-form in a neighborhood \( U \) of the origin \( 0 \in \mathbb{C}^n, n \geq 2 \). We do not assume the integrability condition. We define the gradient of \( \Omega \) as the complex \( C^\infty \) vector field \( \text{grad}(\Omega) = \sum_{j=1}^{n} f_j(z) \frac{\partial}{\partial z_j} \). By construction, \( \text{grad}(\Omega) \) is orthogonal to the distribution \( \ker(\Omega) \), and also \( \Omega \cdot \text{grad}(\Omega) = \sum_{j=1}^{n} |f_j(z)|^2 \). Let also \( \varphi \) be a real analytic function \( \varphi : \mathbb{C}^n \rightarrow \mathbb{R} \) and \( \varphi \neq 0 \) in \( \mathbb{C}^n \setminus \{0\} \). The gradient vector field of \( \varphi \) is defined by

\[ \text{grad}(\varphi) = 2 \left[ \sum_{j=1}^{n} \left( \frac{\partial}{\partial z_j} \right) \frac{\partial}{\partial z_j} + \left( \frac{\partial}{\partial z_j} \right) \frac{\partial}{\partial z_j} \right] = 4 \text{Re} \left( \sum_{j=1}^{n} \left( \frac{\partial}{\partial z_j} \right) \frac{\partial}{\partial z_j} \right). \tag{3.1} \]

Given \( r > 0 \) let \( S^{2n-1}(0; r) \subset \mathbb{C}^n \) be the sphere given by \( \sum_{j=1}^{n} |z_j|^2 = r^2 \) and \( \vec{R} = \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} \) the (complex) radial vector field. If we write \( z_j = x_j + \sqrt{-1} y_j \) in standard euclidian coordinates then the real normal vector field to \( S^{2n-1}(0; r) \) is \( \vec{n} := \sum_{j=1}^{n} (x_j \partial/\partial x_j + y_j \partial/\partial y_j) \).

Write \( f_j(z) = g_j(x, y) + \sqrt{-1} h_j(x, y), 1 \leq j \leq n \). We have the real representations of these gradient vector fields:
\[
\text{grad}(\Omega) = \frac{1}{2} \left\{ \sum_{j=1}^{n} \left( (g_j \partial/\partial x_j - h_j \partial/\partial y_j) - \sqrt{-1} \sum_{j=1}^{n} \left( (h_j \partial/\partial x_j + g_j \partial/\partial y_j) \right) \right) \right\} \quad (3.2)
\]

\[
= \frac{1}{2} \left\{ X - \sqrt{-1} J X \right\} \quad (3.3)
\]

where \( X = \sum_{j=1}^{n} (g_j \partial/\partial x_j - h_j \partial/\partial y_j) \), and \( J \) is the canonical complex structure of \( \mathbb{C}^n \). Also we have

\[
\text{grad}(\varphi) = 4 \text{Re} \left\{ \frac{1}{4} \left( \sum_{j=1}^{n} \left( \frac{\partial \varphi}{\partial x_j} \frac{\partial}{\partial x_j} + \frac{\partial \varphi}{\partial y_j} \frac{\partial}{\partial y_j} \right) - \sqrt{-1} \left( \sum_{j=1}^{n} \left( -\frac{\partial \varphi}{\partial y_j} \frac{\partial}{\partial x_j} + \frac{\partial \varphi}{\partial x_j} \frac{\partial}{\partial y_j} \right) \right) \right) \right\} \quad (3.4)
\]

\[
= \text{Re}(\tilde{n}_\varphi - \sqrt{-1} J \tilde{n}_\varphi) = \tilde{n}_\varphi \quad (3.5)
\]

where by definition we have

\[
\tilde{n}_\varphi := \sum_{j=1}^{n} \left( \frac{\partial \varphi}{\partial x_j} \frac{\partial}{\partial x_j} + \frac{\partial \varphi}{\partial y_j} \frac{\partial}{\partial y_j} \right) \quad (3.6)
\]

**Definition 2** (Variety of contacts). The variety of contacts \( \Sigma = \Sigma(\Omega, \varphi) \) of the one-form \( \Omega \) and the function \( \varphi \), i.e., the variety of contacts \( \Sigma \) of the two foliations \( \mathcal{F}(\Omega) \) and \( \mathcal{F}(d\varphi) \), is defined by the real analytic equations: \( \text{grad}(\varphi)(p) \in \text{grad}(\Omega)(p) \) at \( p \in \Sigma \), that is, \( \tilde{n}_\varphi = a X + b J X \), for some \( a, b \in \mathbb{R} \) at \( p \in \Sigma \).

**Remark 1.** With the above definitions we have:

1. Consider the distributions \( \mathcal{F}(\Omega) \) defined by \( \Omega \) and \( \mathcal{F}(d\varphi) \) defined by the real one-form \( d\varphi \). The relation between two tangent spaces of these distributions is \( T_p(\mathcal{F}(\Omega)) \subset T_p(\mathcal{F}(d\varphi)) \) at \( p \in \Sigma \).

2. Put \( \tilde{R}_\varphi = \frac{1}{2}(\tilde{n}_\varphi - \sqrt{-1} J \tilde{n}_\varphi) = 2 \sum_{j=1}^{n} \left( \frac{\partial \varphi}{\partial z_j} \right) \frac{\partial}{\partial z_j} \). Then, \( \Sigma(\Omega, \varphi) \) is defined by the equations:

\[
\frac{f_1(z)}{\frac{\partial z_1}{\partial z_1}} = \frac{f_2(z)}{\frac{\partial z_2}{\partial z_2}} = ... = \frac{f_n(z)}{\frac{\partial z_n}{\partial z_n}} \quad (3.7)
\]

**Proof.** (1) Given \( v \in T_p(\mathcal{F}(\Omega)) \) we have \( v, X = 0 \) and \( v, JX = 0 \). This implies \( \langle v, \tilde{n}_\varphi \rangle = a v, X \rangle + b \langle v, JX \rangle = 0 \).

(2) By definition and the above equations we have

\[
\tilde{R}_\varphi = \frac{1}{2}(a X + b J X - \sqrt{-1}(a J X - b X)) = (a + \sqrt{-1}b)v \quad \frac{1}{2}(X - \sqrt{-1} J X) = (a + \sqrt{-1}b) \text{grad}(\Omega) \quad (3.8)
\]

Thus \( \tilde{R}_\varphi = \lambda \text{grad}(\Omega) \) for some \( \lambda \in \mathbb{C} \), i.e., \( (3.7) \).
3.1 The distance function

Let now \( \varphi(z) = \sum_{j=1}^{n} |z_j|^2 \) be the distance function from the origin to the point \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \). Then

\[
\text{grad}(\varphi) = \vec{n}_\varphi = 2\vec{n}. \tag{3.9}
\]

where

\[
\vec{n} := \sum_{j=1}^{n} \left( x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j} \right) \tag{3.10}
\]

In particular, the variety of contacts \( \Sigma \) is given by

\[
\frac{f_1(z)}{\overline{z_1}} = \frac{f_2(z)}{\overline{z_2}} = \ldots = \frac{f_n(z)}{\overline{z_n}} \tag{3.11}
\]

3.2 Homogeneous case

Proposition 1. Let \( \Omega = \sum_{j=1}^{n} f_j(z)dz_j \) be an integrable homogeneous polynomial one-form of degree \( k \geq 1 \), in a neighborhood \( U \) of the closed unit disk \( D^{2n} \subset \mathbb{C}^n \), \( n \geq 3 \). Assume that the singularity of \( \Omega \) inside the disk is the origin. Let \( \Sigma \) be the set of contact points between spheres \( S^{2n-1}(r) \), \( r > 0 \) and the foliation \( \mathcal{F}(\Omega) \) defined by \( \Omega \). We denote by \( \vec{R} = \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} \) the complex radial vector field. Then \( \Sigma \) is a non-empty \( \vec{R} \)-invariant set.

Proof. The assumption on the singular set of \( \Omega \) implies that \( \text{cod}_{\mathbb{C}}(\text{Sing}(\Omega))_0 = n \geq 3 \). According to Malgrange’s theorem \([10]\), there exists a holomorphic first integral \( f: V \to \mathbb{C} \) of \( \Omega \) in a neighborhood \( V \) of the origin; \( \Omega|_V = gdf \), where \( g \) is a non-zero holomorphic function on \( V \). Since a point of \( S^{2n-1}(\epsilon) \subset V \) where a maximal value of \( |f|_{S^{2n-1}(\epsilon)} \) is attained is a contact point, \( \Sigma \) is non-empty. Take a point \( p = (p_1, \ldots, p_n) \in \Sigma \), the orbit of \( \vec{R} \) passing through \( p \) is \( p.e^T = (p_1.e^T, \ldots, p_n.e^T), T \in \mathbb{C} \). We will prove that \( f_j(p.e^T) = \lambda(T).p_j.e^T, \lambda(T) \in \mathbb{C}, 1 \leq j \leq n \). Since \( p \in \Sigma \) we have \( f_j(p) = \lambda.p_j, \lambda \in \mathbb{C}, 1 \leq j \leq n \). For any \( j, f_j(p.e^T) = (\lambda.e^T.e^{-T}).p_j.e^T \). Put \( \lambda(T) = \lambda.e^{kT}.e^{-T} \), then \( p.e^T \in \Sigma \).

\[\square\]

4 The projected gradient vector field

Let \( \Omega \) be an integrable holomorphic one-form with \( \text{sing}(\Omega) = \{0\} \) and \( \varphi \) be the distance function as above. Under the same notation of \([3]\) we define the projection \( t_{\mathcal{F}(\Omega)}(\vec{n}) \) of \( \vec{n} \) onto \( T(\mathcal{F}(\Omega)) \) along \( \text{grad}(\Omega) \) as:

\[
t_{\mathcal{F}(\Omega)}(\vec{n}) = \vec{n} - \left[ < \vec{n}, \frac{X}{||X||} > \frac{X}{||X||} + < \vec{n}, \frac{JX}{||JX||} > \frac{JX}{||JX||} \right] \tag{4.1}
\]

where \(<,>\) is the natural inner product of \( \mathbb{R}^{2n} \simeq \mathbb{C}^n \) and \( ||X|| \) is the norm of \( X \).

In the case of a holomorphic vector field \( Z \) in \( \mathbb{C}^n, n \geq 2 \), X. Gomez-Mont, J. Seade and A. Verjovsky \([3]\) defined the real analytic vector field \( r_Z = -1 < Z, \vec{R} >_C.Z \in Z \) and investigated the properties of \( r_Z \), where \( < Z, \vec{R} >_C \) is the canonical hermitian inner product of \( \mathbb{C}^n \) between \( Z \) and \( \vec{R} \). Our definition is a version for holomorphic foliations of codimension one. In the following, we explain some properties of \( t_{\mathcal{F}(\Omega)}(\vec{n}) \).
Proposition 2. We have:

1. \( t_{\mathcal{F}(\Omega)}(\vec{n}) \in T(\mathcal{F}(\Omega)) \).
2. \( \text{Sing}(t_{\mathcal{F}(\Omega)}(\vec{n})) = \Sigma \).
3. \( \text{grad}(\varphi|_L) = 2t_{\mathcal{F}(\Omega)}(\vec{n}) \).
4. Away from the variety of contacts \( \Sigma \), the distributions \( \mathcal{F}(\Omega) \) and \( \mathcal{F}(d\varphi) \) meet transversally.
5. Let \( L \) be an integral submanifold of \( \mathcal{F}(\Omega) \). The vector field \( t_{\mathcal{F}(\Omega)}(\vec{n}) \) is transversal to each level surface of \( \varphi|_L \) on \( L \setminus (L \cap \Sigma) \).

Proof. (1) \( < t_{\mathcal{F}(\Omega)}(\vec{n}), X > = < \vec{n}, X > - \{ < \vec{n}, \frac{X}{||X||} >, X > + < \vec{n}, \frac{JX}{||JX||} >, JX \} = 0 \) Similarly, \( < t_{\mathcal{F}(\Omega)}(\vec{n}), JX > = 0 \).

(2) Given \( p \in \text{Sing}(t_{\mathcal{F}(\Omega)}(\vec{n})) \) we have \( \vec{n} \in \text{grad}(\Omega) \) and then \( p \in \Sigma \). By its turn, \( p \in \Sigma \) implies \( \vec{n} = aX + bJX, a, b, \in \mathbb{R} \) and then

\[
\begin{align*}
< \vec{n}, \frac{X}{||X||} > &= aX \\
< \vec{n}, \frac{JX}{||JX||} > &= bJX
\end{align*}
\]

This finally implies \( t_{\mathcal{F}(\Omega)}(\vec{n}) = 0 \).

(3) Let \( p \) be a point of \( \mathcal{F}(\Omega) \setminus \{0\} \). Since \( \Omega \) is non-singular at \( p \), there exists a neighborhood \( U \) of \( p \) such that \( \Omega|_U \) writes \( \Omega|_U = GdF \) on \( U \) where \( G \) and \( F \) are holomorphic functions on \( U \) and \( G \) is a non-zero holomorphic function on \( U \). The leaf \( L \) through \( p \) is given by \( L = \{ z \in U | F(z) = F(p) = c \} \). We can assume that \( f_n(z) \neq 0 \) on \( U \) because \( \Omega \) is non-singular on \( U \). By the implicit function theorem, we can take a local coordinates \( (z_1, ..., z_{n-1}, z_n(z_1, ..., z_{n-1}, c)) \) to represent the leaf \( L \). Therefore, we have the relations:

\[
f_j(z) + f_n(z) \frac{\partial z_n}{\partial z_j} = 0, \quad 1 \leq j \leq n - 1,
\]

i.e.,

\[
\begin{align*}
g_j + g_n \frac{\partial x_n}{\partial x_j} - h_n \frac{\partial y_n}{\partial x_j} &= 0 \\
h_j + h_n \frac{\partial x_n}{\partial x_j} + g_n \frac{\partial y_n}{\partial x_j} &= 0, \quad 1 \leq j \leq n - 1
\end{align*}
\]

Then we have the gradient \( \text{grad}(\varphi|_L) \) of \( \varphi|_L(z) = \sum_{j=1}^{n-1} (x_j^2 + y_j^2) + (x_n^2 + y_n^2) \) and then

\[
\text{grad}(\varphi|_L) = 2\sum_{j=1}^{n-1} (x_j + x_n \frac{\partial x_n}{\partial x_j} + y_n \frac{\partial y_n}{\partial x_j}) \frac{\partial}{\partial x_j} + \sum_{j=1}^{n-1} (y_j + x_n \frac{\partial x_n}{\partial y_j} + y_n \frac{\partial y_n}{\partial y_j}) \frac{\partial}{\partial y_j}
\]

On the other hand, directly calculating, we get \( (t_{\mathcal{F}(\Omega)}(\vec{n})) = \frac{1}{2} \text{grad}(d\varphi|_L) :\)

\[
(t_{\mathcal{F}(\Omega)}(\vec{n})) = \vec{n} - \frac{\langle \vec{n}, X \rangle}{||X||^2} \left[ \sum_{j=1}^{n-1} (g_j \frac{\partial}{\partial x_j} - h_j \frac{\partial}{\partial y_j}) \right] - \frac{\langle \vec{n}, JX \rangle}{||JX||^2} \left[ \sum_{j=1}^{n-1} (h_j \frac{\partial}{\partial x_j} + g_j \frac{\partial}{\partial x_j}) \right]
\]
the equations:

\[-\langle n, X \rangle = \frac{g}{|X|^2} \{ gn \left[ \sum_{j=1}^{n-1} \left( \frac{\partial x_n}{\partial x_j} \frac{\partial}{\partial x_j} + \frac{\partial x_n}{\partial y_j} \frac{\partial}{\partial y_j} \right) + h \left[ \sum_{j=1}^{n-1} \left( \frac{\partial y_n}{\partial x_j} \frac{\partial}{\partial x_j} + \frac{\partial y_n}{\partial y_j} \frac{\partial}{\partial y_j} \right) \right] \}
\]

\[= \sum_{j=1}^{n-1} (x_j + x_n \frac{\partial x_n}{\partial x_j} + y_n \frac{\partial y_n}{\partial x_j}) \frac{\partial}{\partial x_j} + \sum_{j=1}^{n-1} (y_j + x_n \frac{\partial x_n}{\partial y_j} + y_n \frac{\partial y_n}{\partial y_j}) \frac{\partial}{\partial y_j} = \frac{1}{2} \text{grad}(\varphi) \]

Item (4) follows immediately from (2). (5) follows from (3).

4.1 The complex projected tangential vector field

The complex projected tangential vector field \( T_\varphi(\Omega) \in T(\varphi(\Omega)) \) is defined as

\[ T_\varphi(\Omega) := \bar{\varphi} - \mu \text{ grad}(\varphi) = \frac{1}{2} [t_\varphi(\Omega) - \sqrt{-1} J_t\varphi(\Omega)] \]

where

\[ \mu = \frac{\langle \bar{\varphi}, \text{grad}(\varphi) \rangle}{||\text{grad}(\varphi)||^2} \]

Then we have

\[ \Sigma = \text{sing}(T_\varphi(\Omega)) = \{ z \in \mathbb{C}^n : \bar{\varphi} = \mu \text{ grad}(\varphi) \} \]

5 The variety of contacts of a Morse foliation

Let \( \Omega \) be an integrable holomorphic one-form with Sing(\( \Omega \)) = \{0\}. Under the same notation of \( \S \) we give a characterization of Morse type foliations.

**Proposition 3.** Assume that each critical point \( p \in L \cap \Sigma \) of \( \varphi \) is non-degenerate. Then the real dimension of \( \Sigma \setminus \{0\} \) is two and \( \Sigma \setminus \{0\} \) is transverse to the foliation \( \varphi(\Omega) \).

**Proof.** First we note the existence of a critical point of \( \varphi \) for any leaf \( L' \) close enough to \( L \) at \( p \), i.e., passing through a neighborhood of \( p \in L \cap \Sigma \), where \( p \) is a critical point of \( \varphi \), is proved by Poincaré-Hopf theorem (or Pugh’s generalization of Poincaré-Hopf theorem). Take a distinguished coordinate neighborhood \( (w_1, ..., w_{n-1}, w_n) \in U \) for \( \varphi(\Omega) \) where \( p \in U \) corresponds to the origin and \( L' \cap U = \{ w_n = c \} \). Then \( \Sigma \cap U \) is defined by the equations:

\[ \frac{\partial \varphi}{\partial w_1} = 0, ..., \frac{\partial \varphi}{\partial w_{n-1}} = 0 \]

i.e., \( \frac{\partial w_j}{\partial w_n} = 0 \), \( j = 1, ..., 2n-2 \), where we write \( w_j = u_{2j-1} + \sqrt{-1} u_{2j} \), \( 1 \leq j \leq n \), by the real coordinate. Since \( p \) is non-degenerate on \( L \),

\[ \text{det}(\frac{\partial^2 \varphi}{\partial u_i \partial u_j})_{1 \leq i,j \leq 2n-2} \]

is different from 0. By the Implicit function theorem, \( \Sigma \) is parametrized by \( u_{2n-1} \) and \( u_{2n} : (u_1(u_{2n-1}, u_{2n}), ..., u_2(u_{2n-1}, u_{2n}), u_{2n-1}, u_{2n}) \). Then the dimension of \( \Sigma \setminus \{0\} \) is two and \( \Sigma \setminus \{0\} \) is transverse to \( \varphi(\Omega) \).
Proposition 4. If the real dimension of $\Sigma \setminus \{0\}$ is two and $\Sigma \setminus \{0\}$ is transverse to the foliation $\mathcal{F}(\Omega)$, then each critical point $p \in \Sigma \cap L$ of $\varphi|_L$ on the leaf $L$ passing through $p$ is non-degenerate.

Proof. Take a distinguished coordinate neighborhood $U$, $(w_1, ..., w_{n-1}, w_n)$, at $p$ such that the leaf $L \cap U$ of $\mathcal{F}(\Omega)$ passing through $p$ is defined by $\{w_n = 0\}$. Using the real coordinate $(u_1, ..., u_{2n-1}, u_{2n-1}, u_{2n})$: $w_j = u_{j-1} + \sqrt{-1}u_{2j}$, $1 \leq j \leq n$, $\Sigma$ is parametrized in $U$ by $u_{2n-1}$ and $u_{2n}$:

$$\Sigma \cap U = \{(u_1(u_{2n-1}, u_{2n}), ..., u_{2n-2}(u_{2n-1}, u_{2n}), u_{2n-1}, u_{2n})| u_{2n-1}, u_{2n}\}.$$

Then the tangent space of $T\Sigma|_U$ of $\Sigma \cap U$ is generated by $\vec{v}_{2n-1}$ and $\vec{v}_{2n}$:

$$\vec{v}_{2n-1} = \sum_{j=1}^{2n-2} \frac{\partial u_j}{\partial u_{2j-1}} \frac{\partial}{\partial u_j} + \frac{\partial}{\partial u_{2n-1}}$$

$$\vec{v}_{2n} = \sum_{j=1}^{2n-2} \frac{\partial u_j}{\partial u_{2n}} \frac{\partial}{\partial u_j} + \frac{\partial}{\partial u_{2n}}$$

On the other hand, $\Sigma$ is defined by the critical points of the distance function $\varphi$ as follows:

$$\Sigma \cap U \{w \in U| \frac{\partial \varphi}{\partial u_1} = 0, ..., \frac{\partial \varphi}{\partial u_{2n-2}} = 0\}$$

Then the tangent space $T\Sigma|_U$ is defined by $\{d(\frac{\partial \varphi}{\partial u_1}) = 0, ..., d(\frac{\partial \varphi}{\partial u_{2n-2}}) = 0\}$:

$$T\Sigma|_U = \{\vec{v} \in T\mathbb{R}^{2n}| d(\frac{\partial \varphi}{\partial u_i})(\vec{v}) = 0, 1 \leq i \leq 2n-2\}$$

Since $d(\frac{\partial \varphi}{\partial u_i})(\vec{v}_k) = 0$ for $k = 2n-1, 2n$, $1 \leq i \leq 2n-2$, we get the following relations:

$$\begin{pmatrix}
\frac{\partial^2 \varphi}{\partial u_1 \partial u_1} & ... & \frac{\partial^2 \varphi}{\partial u_{2n-1} \partial u_1} \\
... & ... & ...
\end{pmatrix}
\begin{pmatrix}
\frac{\partial u_1}{\partial u_1} \\
... \\
\frac{\partial u_{2n-2}}{\partial u_{2n-2}}
\end{pmatrix}
+ 
\begin{pmatrix}
\frac{\partial^2 \varphi}{\partial u_i \partial u_1} & ... & \frac{\partial^2 \varphi}{\partial u_i \partial u_{2n-1}} \\
... & ... & ...
\end{pmatrix}
\begin{pmatrix}
\frac{\partial u_i}{\partial u_1} \\
... \\
\frac{\partial u_i}{\partial u_{2n-2}}
\end{pmatrix}
= \begin{pmatrix} 0 \\
... \\
0 \end{pmatrix}$$

Then the fact that the rank of $\{d(\frac{\partial \varphi}{\partial u_1}), ..., d(\frac{\partial \varphi}{\partial u_{2n-2}})\}$ is $2n-2$ means that the rank of the $(2n-2) \times (2n-2)$ matrix

$$\begin{pmatrix}
\frac{\partial^2 \varphi}{\partial u_1 \partial u_1} & ... & \frac{\partial^2 \varphi}{\partial u_{2n-1} \partial u_1} \\
... & ... & ...
\end{pmatrix}
\begin{pmatrix}
\frac{\partial u_1}{\partial u_1} \\
... \\
\frac{\partial u_{2n-2}}{\partial u_{2n-2}}
\end{pmatrix}$$

is $2n-2$. Therefore the critical point $p$ is non-degenerate.

\[\square\]

Summarizing Proposition\ref{prop:2}(3) and Propositions\ref{prop:3} and\ref{prop:4} we have the following theorem:

**Theorem 3** (Characterization of Morse type foliations). Given a holomorphic integrable one-form $\Omega$ with $\text{Sing}(\Omega) = \{0\}$, in a neighborhood of the disk $D^{2n}$, the following conditions are equivalent:
(i) $\mathcal{F}(\Omega)$ is of Morse type, i.e., each critical point $p \in \Sigma \cap L$ of $\varphi|_L$ on each leaf $L$ is nondegenerate.

(ii) $\Sigma - \{0\}$ has real dimension two and is transverse to the foliation $\mathcal{F}(\Omega)$.

(iii) The singularities of $t_{\mathcal{F}(\Omega)}(\vec{n})$ on each leaf are nondegenerate.

**Corollary 1.** If the real dimension of $\Sigma \setminus \{0\}$ is two and $\Sigma \setminus \{0\}$ is transverse to the foliation $\mathcal{F}(\Omega)$, then $\Sigma$ has a cone structure.

**Proof.** We denote by $t_{\Sigma}(\vec{n})$ the projection of $\vec{n}$ onto $T\Sigma$ along $T(\mathcal{F}(\Omega))$. We note that $t_{\Sigma}(\vec{n})$ is different from zero. Hence, the orbits of $t_{\Sigma}(\vec{n})$ define a cone structure for $\Sigma$. 

\[ \square \]

6 Proof of the non-existence theorem

Let $\Omega$ be an integrable holomorphic one-form of Morse type and $\Sigma(\Omega)$ of dimension two. Let $L \in \mathcal{F}(\Omega)$ be a leaf of $\mathcal{F}(\Omega)$ such that $\Sigma \cap L \neq \emptyset$, indeed we assume that there is a point $p \in L$ such that $\varphi|_L$ has a nondegenerate critical point of index 0 at $p$ (in other words, $p$ is a local minimum point for $\varphi|_L$).

**Lemma 1.** There is a neighborhood $U$ of $p$ in $\overline{D^{2n}}$ such given a leaf $L'$ intersecting $U$, the restriction $\varphi|_{L'}$ exhibits a critical point $p' \in U$ of index 0.

**Proof.** Indeed, we choose a distinguished coordinate neighborhood $U$ for $\mathcal{F}(\Omega)$, such that $\mathcal{F}(\Omega)|_U$ is equivalent to a foliation by $(2n - 2)$-discs. The variety of contacts $\Sigma$ has a cone structure and is a 2-dimensional analytic manifold in a neighborhood of $p$ transverse to $\mathcal{F}(\Omega)$. The level surfaces of $\varphi|_{L' \cap U}$ are $(2n - 3)$-dimensional spheres on $L$ centered at $p$. Moreover, for small $\epsilon > 0$ a connected component $\Sigma_0$ of the intersection $\Sigma \cap S^{2n-1}(\epsilon)$ is a circle $S^1$. Hence, $\Sigma_0$ is a cone over $S^1$ and the gradient vector field $t_{\mathcal{F}(\Omega)}(\vec{n})$ of $\varphi|_L$ is tangent to $L$ and has a Morse type singularity at $p$, of index zero. Given a leaf $L'$ such that $L'$ intersects $U$ at a point close enough to $p$, then by Poincaré-Hopf theorem, applied to a disc $D' \subset L' \cap U$, obtained as lifting of a sufficiently small disc $D \subset L \cap U$ centered at $p$ whose boundary is a level curve of $\varphi|_{L'}$, we conclude that $t_{\mathcal{F}(\Omega)}(\vec{n})|_{L'}$ on $L'$ has a Morse type singularity at some point $p' \in D'$. This singularity of Morse index zero and corresponding to a point of minimum for the restriction $\varphi|_{L'}$. 

\[ \square \]

Let now $\Sigma$ be of dimension two and transverse to $\mathcal{F}(\Omega)$. Let $L \in \mathcal{F}(\Omega)$ be a leaf of $\mathcal{F}(\Omega)$ such that $\Sigma \cap L \neq \emptyset$, indeed we assume that there is a point $p \in L$ such that $\varphi|_L$ has a non-degenerate critical point of index $k > 0$ at $p$.

**Lemma 2.** There is a neighborhood $U$ of $p$ in $\overline{D^{2n}}$ such given a leaf $L'$ intersecting $U$, the distance function $\varphi|_{L'}$ on $L'$ exhibits a critical point $p' \in U \cap L'$ of index $k$.

**Proof.** The proof is similar to the above. Choose a distinguished coordinate neighborhood $U$ for $\mathcal{F}(\Omega)$, such that $\mathcal{F}(\Omega)|_U$ is equivalent to a foliation by $(2n - 2)$-discs. By Morse’s lemma, we have a sufficiently small disc $D \subset L \cap U$ centered at $p$ whose boundary is in

\[ \text{1}\] We can therefore consider the situation as $\Sigma \cap U$ being a (cylinder) product of the intersection a circle $\gamma \subset (\Sigma \cap S^{2n-1}(r) \cap U)$ by an interval $(-\delta, \delta)$, with $p$ corresponding to the level zero. The level surfaces of $\varphi|_{L' \cap U}$ are $(2n - 3)$-spheres centered at $p$ and therefore level surfaces of $\varphi|_{L'}$ can be thought as cylinders transverse to $\mathcal{F}(\Omega)$ off $\Sigma \cap U$. 

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generic position for $t_{\mathcal{F}(\Omega)}(\vec{n})|_{L}$. Arguing as in the proof of Lemma 1 we obtain a disc $D' \subset L' \cap U$ as lift of $D \subset L \cap U$, and we may deform $D'$ to a disc $\tilde{D}'$ whose boundary is in generic position for $t_{\mathcal{F}(\Omega)}(\vec{n})|_{L'}$. By Pugh’s generalization of Poincaré-Hopf theorem we conclude that $t_{\mathcal{F}(\Omega)}(\vec{n})|_{L'}$ has a Morse type singularity at some point $p' \in \tilde{D}'$, of Morse index $k$.

Proof of Theorem 1. Assume by contradiction that $\mathcal{F}$ is transverse to the boundary sphere $S^{2n-1}(1)$. We note that $\mathcal{F}$ is defined by an integrable holomorphic one-form $\Omega$ in $U \supset \overline{D}^{2n}$ and according to [9] there is a unique singular point $p$ of $\Omega$ inside the disc $\overline{D}^{2n}$ and this is a non-degenerate singular point: the determinant of the matrix $D(\Omega)(p)$ of the coefficients of the linear part of $\Omega$ is different from 0. By a Möbius transformation we can assume that the origin is this singular point. Since $\Omega$ is integrable and $n \geq 3$, by Malgrange’s theorem (10) $\Omega$ admits a local holomorphic first integral $f: V \to \mathbb{C}$, $f|_{V} = gdf$ in a neighborhood $V$ of 0 in $\mathbb{C}^{n}$, where $g$ is a non-zero holomorphic function in $V$. Then $\mathcal{F}(\Omega)|_{V}$ is defined by the level hypersurfaces of $f$ and only finitely many leaves accumulate on the origin. Let $\varphi$ be the distance function with respect to the origin 0 $\in \mathbb{C}^{n}$. We consider the variety of contacts $\Sigma = \Sigma(\Omega, \varphi)$. Since by hypothesis $\mathcal{F}$ is of Morse type there is a minimal point (that is, a critical point of Morse index 0) for the restricted distance function $\varphi|_{L}$ to any leaf $L$ intersecting $V$ and these points belong to $\Sigma$. Let $\Sigma^*_0$ be a connected component of $\Sigma \setminus \{0\}$ which contains such minimal points in a neighborhood of the origin. According to Theorem 2 and Lemma 1, $\Sigma^*_0$ in $U \supset \overline{D}^{2n}$ is of real dimension two and has an unbounded cone structure. Then $\Sigma^*_0$ intersects $S^{2n-1}(1)$. This is in contradiction to the assumption of transversality.

Remark 2. Though Lemma 2 has not been used in a direct way in the proof of Theorem 1 (it is enough to begin with a minimal (index zero) singular point for $\varphi|_{L}$ for a leaf $L$ close to the origin) we have stated it because we think it will be useful in a more general setting.

7 Examples

Example 2. Consider the function $f(z) = \sum_{j=1}^{n} z_{j}^{2}$. The foliation $\mathcal{F}(df)$ is defined by the level surfaces of $f$. The variety of contacts $\Sigma$ between $\mathcal{F}(df)$ and $\mathcal{F}(d\varphi)$ is defined by the equations: $2\pi j = \lambda z_{j}, 1 \leq j \leq n, \lambda \in \mathbb{C}$.

We will write complex numbers $z_{j} = r_{j}e^{\sqrt{-1}\theta_{j}}$ and $c = re^{\sqrt{-1}\theta}$ in polar coordinates.

We will denote by $L_{c}$ the leaf of $\mathcal{F}(df)$ defined by $\{f = c\}$. To explain $\Sigma$ we prepare some notations:

\[ \begin{align*}
\Sigma_{1}^{1} \cap L_{c} &= \{(0, \ldots, 0, r_{i}e^{\sqrt{-1}\theta_{i}}, o, \ldots, 0)|r_{i}^{2} = r, \theta_{i} = \frac{\theta}{2} \text{ or } \frac{\theta}{2} + \pi\} \quad (7.1) \\
\Sigma_{1,2}^{2} \cap L_{c} &= \{(0, \ldots, 0, r_{i}e^{\sqrt{-1}\theta_{i}}, 0, \ldots, r_{j}e^{\sqrt{-1}\theta_{j}}, 0, \ldots, 0)|r_{i}^{2} + r_{j}^{2} = r, \theta_{i}, \theta_{j} = \frac{\theta}{2} \text{ or } \frac{\theta}{2} + \pi\} \quad (7.2) \\
\Sigma_{1,2,\ldots,n}^{n} \cap L_{c} &= \{(r_{1}e^{\sqrt{-1}\theta_{1}}, \ldots, r_{n}e^{\sqrt{-1}\theta_{n}})|r_{1}^{2} + \ldots + r_{n}^{2} = r, \theta_{i} = \frac{\theta}{2} \text{ or } \frac{\theta}{2} + \pi\} \quad (7.3)
\end{align*} \]
Each dimension of $\Sigma_{i_1,\ldots,i_\ell} \cap L_c$ is $\ell - 1$ and $\dim(\Sigma_{i_1,\ldots,i_\ell})$ is equal to $\ell + 1$. Then we can explain $\Sigma$ as follows:

$$\Sigma = \left( \bigcup_{i=1}^{n} \Sigma^1_i \right) \cup \left( \bigcup_{i,j} \Sigma^2_{i,j} \right) \cup \ldots \cup (\Sigma^n_{1,2,\ldots,n}). \quad (7.4)$$

In this degree two homogeneous case $\Sigma$ is a $\bar{R}$-invariant set.

**Example 3.** Let $\lambda_j$ (1 ≤ $j$ ≤ $n$) be complex numbers such that $|\lambda_i| \neq |\lambda_j|$ for $i \neq j$. Consider the function $f_\lambda(z) = \sum_{j=1}^{n} \lambda_j z_j^2$. We define the foliation $F(d\lambda)$. The variety of contacts $\Sigma$ between $F(d\lambda)$ and $F(d\varphi)$ is defined by the equations:

$$2\lambda_jz_j = \xi z_j, \ 1 \leq j \leq n, \ \xi \in \mathbb{C}. \quad (7.5)$$

$\Sigma \setminus \{0\}$ consists of $n$-connected components $\Sigma_1 \cup \Sigma_2 \cup \ldots \cup \Sigma_n$ where $\Sigma_i = \{0, \ldots, 0, r_i e^{\sqrt{-1} \theta_i}, 0, \ldots, 0 | r_i > 0, 0 \leq \theta_i < 2\pi\}$. Let $L_c = \{f_\lambda = c\}$ be a leaf of $F(d\lambda)$. Take $p = (\omega_1, 0, \ldots, 0) \in \Sigma_1 \cap L_c$. From now on, let us calculate the real hessian matrix of $\varphi|_{L_c}$ represented by the equations $(z_1(z_2, \ldots, z_n,c), z_2, \ldots, z_n)$, $z_i = x_i + \sqrt{-1} y_i$, in a neighborhood of $p \in \Sigma_1 \cap L_c$. First we note that $(z_1(z_2, \ldots, z_n,c), z_2, \ldots, z_n)$ is the holomorphic function of complex variables $z_2, \ldots, z_n$. By Cauchy-Riemann equations, we get

$$\frac{\partial z_1}{\partial z_j} = \frac{\partial x_1}{\partial x_j} + \sqrt{-1} \frac{\partial y_1}{\partial y_j} = \frac{\partial y_1}{\partial y_j} - \sqrt{-1} \frac{\partial x_1}{\partial y_j}. \quad (7.6)$$

Differentiating $c = f_\lambda(z)$ by $\frac{\partial}{\partial z_j}$, we have

$$0 = 2\lambda_1 z_1 \frac{\partial z_1}{\partial z_j} + 2\lambda_j z_j. \quad (7.7)$$

Since $0 = 2\lambda_1 \omega_1 (\frac{\partial x_1}{\partial z_j}(p) + \sqrt{-1} \frac{\partial y_1}{\partial z_j}(p))$, we have $\frac{\partial x_1}{\partial z_j}(p) = \frac{\partial y_1}{\partial y_j}(p) = 0$ and $\frac{\partial y_1}{\partial x_j}(p) = -\frac{\partial x_1}{\partial y_j}(p) = 0$. Then we obtain

$$\frac{\partial \varphi}{\partial x_j}(p) = 2u_1 \frac{\partial x_1}{\partial x_j}(p) + 2v_1 \frac{\partial y_1}{\partial x_j}(p) + 2.0 = 0 \quad (7.8)$$

and

$$\frac{\partial \varphi}{\partial y_j}(p) = 2u_1 \frac{\partial x_1}{\partial y_j}(p) + 2v_1 \frac{\partial y_1}{\partial y_j}(p) + 2.0 = 0 \quad (7.9)$$

where $w_1 = u_1 + \sqrt{-1} v_1$.

To obtain the real Hessian matrix $H(\varphi|_{L_c})(p)$ at $p$ we calculate the following equations:

$$\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = 2\frac{\partial x_1}{\partial x_i} \frac{\partial x_1}{\partial x_j} + 2x_1 \frac{\partial^2 x_1}{\partial x_i \partial x_j} + 2\frac{\partial y_1}{\partial x_i} \frac{\partial y_1}{\partial x_j} + 2y_1 \frac{\partial^2 y_1}{\partial x_i \partial x_j} + 2\delta_{ij} \quad (7.10)$$

$$\frac{\partial^2 \varphi}{\partial y_i \partial x_j} = 2\frac{\partial x_1}{\partial y_i} \frac{\partial x_1}{\partial x_j} + 2x_1 \frac{\partial^2 x_1}{\partial y_i \partial x_j} + 2\frac{\partial y_1}{\partial y_i} \frac{\partial y_1}{\partial x_j} + 2y_1 \frac{\partial^2 y_1}{\partial y_i \partial x_j} + 2\delta_{ij} \quad (7.11)$$

$$\frac{\partial^2 \varphi}{\partial y_i \partial y_j} = 2\frac{\partial x_1}{\partial y_i} \frac{\partial x_1}{\partial y_j} + 2x_1 \frac{\partial^2 x_1}{\partial y_i \partial y_j} + 2\frac{\partial y_1}{\partial y_i} \frac{\partial y_1}{\partial y_j} + 2y_1 \frac{\partial^2 y_1}{\partial y_i \partial y_j} + 2\delta_{ij} \quad (7.12)$$
Differentiating $0 = 2\lambda_1 z_1 \frac{\partial f}{\partial z_j} + 2\lambda_j z_j$ by $\frac{\partial}{\partial z_j}$, we have

$$\frac{\partial^2 z_1}{\partial z_i \partial z_j}(p) = \begin{cases} -\frac{\lambda_1}{\lambda_1 - \lambda_i} & (i \neq j) \\ 0 & (i \neq j) \end{cases}$$ (7.13)

Put $\frac{\lambda_i}{\lambda_1} = a_i + \sqrt{-1}b_i$. Then we get each component of $H(\varphi|_{L_c})(p)$:

$$\frac{\partial^2 \varphi}{\partial x_i \partial x_j}(p) = \begin{cases} 2\left[\frac{-a_i(u_j^2 - v_j^2) + 2b_iu_j v_j}{|w_i|^2} - 1\right] & (i = j) \\ 0 & (i \neq j) \end{cases}$$ (7.14)

$$\frac{\partial^2 \varphi}{\partial y_i \partial x_j}(p) = \begin{cases} 2\left[\frac{b_i(u_j^2 - v_j^2) - a_i u_j v_j}{|w_i|^2}\right] & (i = j) \\ 0 & (i \neq j) \end{cases}$$ (7.15)

$$\frac{\partial^2 \varphi}{\partial y_i \partial y_j}(p) = \begin{cases} 2\left[\frac{a_i(u_j^2 - v_j^2) + 2b_i u_j v_j}{|w_i|^2} - 1\right] & (i = j) \\ 0 & (i \neq j) \end{cases}$$ (7.16)

Then the characteristic equation of $H(\varphi|_{L_c})(p)$ is

$$\Pi_{i=2}^n [(1 - \mu)^2 - (a_i^2 + b_i^2)] = 0$$ (7.17)

that is, $\mu = 1 \pm \sqrt{a_i^2 + b_i^2}$, $2 \leq i \leq n$. The sign of the eigenvalues $\mu$ is defined by $|\lambda_i| > 1$ or $|\lambda_i| < 1$. Furthermore, we assume that $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$, each $\Sigma_i$ consists of critical points of Morse index $i - 1$ of $\varphi|_{L_c}$.

**Example 4.** Let $\epsilon_j$ ($1 \leq j \leq n$) be sufficiently small complex numbers such that $|1 + \epsilon_i| \neq |1 + \epsilon_j|$ for $i \neq j$. Consider $f_\epsilon(z) = \sum_{j=1}^n (1 + \epsilon_j)z_j^2$. We mean that $f_\epsilon(z)$ is a deformation of $f(z) = \sum_{j=1}^n z_j^2$. We can check that the dimension of the variety of contacts $\Sigma(df)$ is $n + 1$ and the dimension of $\Sigma(df_\epsilon)$ is two.

### 8 Linear foliations of Morse type

Let $\Omega = \sum_{i=1}^n \alpha_i(z)dz_i$ be a linear one-form on $\mathbb{C}^n$, $n \geq 3$, defined by a quadratical polynomial $f(z) = \frac{1}{2} \sum_{i,j=1}^n a_{ij}z_i z_j$, with $a_{ij} = a_{ji}$: $\Omega = df$. We denote by $A = (a_{ij})$ the $n \times n$ complex invertible matrix with $(i,j)$ component $a_{ij}$. The corresponding gradient vector field is $\text{grad}(\Omega) = \sum_{i=1}^n \alpha_i(z) \frac{\partial}{\partial z_i}$, and the radial vector field of the distance function $\varphi(z) = \sum_{j=1}^n |z_j|^2$ in $\mathbb{C}^n$ is $\vec{R} = \sum_{i=1}^n z_i \frac{\partial}{\partial z_i}$. The corresponding variety of contacts $\Sigma$ between $\mathcal{F}(\Omega)$ and $\{d\varphi = 0\}$ is defined by the following equation:

$$\Sigma = \{z \in \mathbb{C}^n : \vec{R}(z) = \mu(z) \cdot \text{grad}(\Omega)(z)\}$$ (8.1)

where

$$\mu(z) = \frac{(\vec{R}(z), \text{grad}(\Omega)(z))}{\|	ext{grad}(\Omega)(z)\|^2}$$ (8.2)
By its turn \( \tilde{R}(z) = \mu(z) \). grad(\( \Omega \))(z) is equivalent to
\[
\begin{align*}
  z_j = \mu(z). \alpha_j(z), \forall j = 1, ..., n \\
  \text{i.e.,}
  \begin{pmatrix}
    z_1 \\
    \vdots \\
    z_n
  \end{pmatrix} = \frac{1}{\mu(z)} \begin{pmatrix}
    z_1 \\
    \vdots \\
    z_n
  \end{pmatrix} \quad (8.3)
\end{align*}
\]

Thus we obtain
\[
\begin{align*}
  z = \frac{1}{\mu(z)} A^{-1}(z) = \frac{1}{\mu(z)} A^{-1} \begin{pmatrix}
    \frac{1}{\mu(z)} A^{-1} \Sigma
  \end{pmatrix} = \frac{1}{|\mu(z)|^2} A^{-1}.A^{-1} z \quad (8.5)
\end{align*}
\]

Thus we obtain

**Lemma 3.** Equation (8.3) implies equation equation (8.6) below:
\[
A^{-1}.A^{-1}(z) = \frac{1}{|\mu(z)|^2} z \quad (8.6)
\]

Notice that the matrix \( B := A^{-1}.A^{-1} \) is an hermitian matrix because \( A \) is symmetric. Moreover we have:

**Lemma 4.** The eigenvalues of \( B = A^{-1}.A^{-1} \) are all positive.

**Proof.** Given an eigenvector \( \vec{u} \in \mathbb{C}^n \) with eigenvalue \( \lambda \) we have \( B \vec{u} = \lambda \vec{u} \) and then \( \langle A^{-1}A^{-1} \vec{u}, \vec{u} \rangle = \langle \lambda \vec{u}, \vec{u} \rangle = \lambda \langle \vec{u}, \vec{u} \rangle \). On the other hand, \( \langle A^{-1}A^{-1} \vec{u}, \vec{u} \rangle = \langle A^{-1} \vec{u}, (A^{-1})^t \vec{u} \rangle = ||A^{-1}\vec{u}||^2 \). Thus we have \( \lambda ||\vec{u}||^2 = ||A^{-1}\vec{u}||^2 \) what implies \( \lambda > 0 \).

Let now \( w^{(1)} \in \Sigma \) be a tangency point, \( w^{(1)} \in S^{n-1}(r) \). We introduce the complex line
\[
\Sigma(w^{(1)}) = \{ T.w^{(1)} : T \in \mathbb{C} \}.
\]

**Lemma 5.** The complex line \( \Sigma(w^{(1)}) \) is contained in the variety of contacts \( \Sigma \).

**Proof.** Given a point \( T.w^{(1)} \in \Sigma(w^{(1)}) \) we have \( A(T.w^{(1)}) = T.A.w^{(1)} = T.\frac{1}{\mu(w^{(1)})}.w^{(1)} = T.\frac{1}{\mu(w^{(1)})}.T.w^{(1)} \). On the other hand, since \( \Omega = \Omega A \) is linear we have \( \mu(T.w^{(1)}) = \| \tilde{R}(T.w^{(1)}), \text{grad}(\Omega)(T.w^{(1)}) \| \| \text{grad}(\Omega)(T.w^{(1)}) \| \) = \( T.\tilde{R}(w^{(1)}), \text{grad}(\Omega)(w^{(1)}) \| \) = \( T.\mu(w^{(1)}) \). Hence we obtain \( A(T.w^{(1)}) = \frac{1}{\mu(w^{(1)})} T.w^{(1)} \), i.e., \( T.w^{(1)} \in \Sigma(w^{(1)}) \).

Let now \( w^{(2)} \in \Sigma \) be another contact point, say \( w^{(2)} \in S^{2n-1}(r') \). Suppose that the contact points \( w^{(1)} \) and \( w^{(2)} \) are linearly independent. Given a linear combination \( w = T_1.w^{(1)} + T_2.w^{(2)} \) of the contact points we investigate whether this is also a contact point. Suppose therefore that \( T_1 \neq 0 \neq T_2 \) and that \( w \in \Sigma \). By equation (8.4) we obtain
\[
A(T_1.w^{(1)} + T_2.w^{(2)}) = \frac{1}{\mu(T_1.w^{(1)} + T_2.w^{(2)})}.(T_1.w^{(1)} + T_2.w^{(2)}) \quad (8.4)
\]

On the other hand, \( A(T_1.w^{(1)} + T_2.w^{(2)}) = A(T_1.w^{(1)}) + A(T_2.w^{(2)}) = \frac{1}{\mu(T_1.w^{(1)})}.T_1.w^{(1)} + \frac{1}{\mu(T_2.w^{(2)})}.T_2.w^{(2)} \). Therefore, by linear independence of \( w^{(1)} \) and \( w^{(2)} \) we have \( \mu((T_1.w^{(1)} = \mu(T_1.w^{(1)} + T_2.w^{(2)}) = \mu(T_2.w^{(2)}) \). Since \( \mu(T_1.w^{(1)}) = \mu(T_2.w^{(2)}) \) we conclude that \( |\mu(w^{(1)})| = |\mu(w^{(2)})| \). Thus
Lemma 6. If \(|\mu(w^{(1)})| \neq |\mu(w^{(2)})|\) then the point \(T_1w^{(1)} + T_2w^{(2)}, T_1 \neq 0, T_2 \neq 0\) is not a contact point.

Then we have the following:

**Proposition 5.** Let \(A\) be a \(n \times n\) nonsingular complex symmetric matrix. If the eigenvalues of \(A^{-1} \cdot A^{-1}\) are pairwise distinct with eigenvectors say \(w^{(1)}, ..., w^{(n)}\) then the variety of contacts \(\Sigma(\Omega_A, \varphi)\), is the union of the \(n\) lines \(\Sigma(w^{(j)}), j = 1, ..., n\).

Denote by:

- \(\text{Sim}(n)\) the subspace of symmetric \(n \times n\) complex matrices.
- \(\text{Sim}(n)^* \subset \text{Sim}(n)\) the open subset of invertible symmetric matrices.
- \(\mathcal{M}(n) \subset \text{Sim}(n)^*\) the set of Morse type symmetric invertible matrices.
- \(\mathcal{MRS}(n) \subset \text{Sim}(n)^*\) the set of all invertible symmetric real matrices having eigenvalues \(\lambda_i (1 \leq i \leq n)\) such that \(\lambda_i^2 \neq \lambda_j^2, \forall i \neq j\).

**Proposition 6.** We have \(\mathcal{M}(n) \subset \text{Sim}(n)^* \subset \text{Sim}(n)\) and the inclusions are dense.

**Proof.** First we observe that \(\text{Sim}(n)^*\) is a Zariski open subset of the affine manifold (affine vector space) \(\text{Sim}(n)\), it is the complement of the proper Zariski closed set (algebraic submanifold) \(\{A \in \text{Sim}(n) : \det A = 0\}\). We note that \(\mathcal{M}(n)\) contains \(\mathcal{MRS}(n)\). The complement \(\text{Sim}(n)^* \setminus \mathcal{M}(n)\) is characterized by the fact that for \(A \in \text{Sim}(n)^* \setminus \mathcal{M}(n)\) the hermitian matrix \(A^{-1} \cdot A^{-1}\) has some multiple eigenvalue if and only if its characteristic polynomial \(m_{A^{-1} \cdot A^{-1}}(\lambda) \in \mathbb{C}[\lambda]\) has some multiple zero. This is an algebraic condition on its coefficients. This argumentation shows that \(\text{Sim}(n)^* \setminus \mathcal{M}(n)\) is a closed Zariski (algebraic) subset of \(\text{Sim}(n)^*\). Thus \(\mathcal{M}(n)\) is dense in \(\text{Sim}(n)^*\) which is dense in \(\text{Sim}(n)\).

**Definition 3.** A \(n \times n\) symmetric complex matrix \(A\) will be called of Morse type if the corresponding foliation \(\Omega_A\) is of Morse type.

**Proof of Theorem 2.** Theorem 2 follows from Proposition 5 and 6.

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