TOPOLOGICAL QUANTUM FIELD THEORY FOR DORMANT OPERS

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ABSTRACT. The purpose of the present paper is to develop the enumerative geometry of dormant $G$-opers for a semisimple algebraic group $G$. In the present paper, we construct a compact moduli stack admitting a perfect obstruction theory by introducing the notion of a dormant faithful twisted $G$-oper (or a “$G$-do’per” for short). The resulting virtual fundamental class induces a semisimple 2d TQFT (= 2-dimensional topological quantum field theory) counting the number of $G$-do’pers. This 2d TQFT gives an analogue of the Witten-Kontsevich theorem describing the intersection numbers of psi classes on the moduli stack of $G$-do’pers.

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INTRODUCTION

The purpose of the present paper is to develop the enumerative geometry of the moduli of dormant $G$-opers (i.e., $G$-opers with vanishing $p$-curvature) for a semisimple algebraic group $G$ in characteristic $p > 0$. The formulations and background knowledge of dormant $G$-opers used in the present paper under the assumption that $G$ is of adjoint type were discussed in the author’s paper [57]. In the present paper, we generalize the previous work and construct a compact moduli stack admitting a perfect obstruction theory by introducing the notion of a dormant faithful twisted $G$-oper (or a “$G$-do’per” for short) defined on a stacky log curve. The resulting virtual fundamental class (in the cases of some classical types $G$) induces a semisimple 2d TQFT (= 2-dimensional topological quantum field theory), which arises from the nature of algebraic geometry in positive characteristic unlike many of the other examples of TQFTs constructed in geometry. This result may be thought of as an improvement of [57].
Proposition 7.33. In particular, an explicit description of (the characters of) the corresponding Frobenius algebra allows us to perform a computation for the counting problem of $G$-do’pers, i.e., the Verlinde formula for $G$-do’pers. This 2d TQFT gives also an analogue of the Witten-Kontsevich theorem, describing the intersection numbers of psi classes on the moduli stack of $G$-do’pers.

Thus, the results proved in the present paper show that the moduli stack of $G$-do’pers has many features similar to certain spaces dealt with in enumerative geometry related to high energy physics, say, the moduli stack of $r$-spin curves (denoted usually by $\mathcal{B}_{g,r}(\mathbb{G}_m, \omega^{1/r})$ or $\mathfrak{M}_g^{[1/r]}$) and the moduli stack of stable maps into a suitable variety $V$ (denoted by $\mathcal{M}_{g,r}(V)$). In the rest of this Introduction, we shall provide more detailed discussions, including the content of the present paper.

0.1. Recall that a dormant $G$-oper is, roughly speaking, a $G$-bundle on an algebraic curve in characteristic $p > 0$ equipped with a connection satisfying certain conditions, including the condition that its $p$-curvature vanishes identically. Various properties of dormant $\operatorname{PGL}_2$-opers and their moduli were discussed by S. Mochizuki (cf. [42], [43]) in the context of $p$-adic Teichmüller theory. If $G = \operatorname{PGL}_n$ or $\operatorname{SL}_n$ for a general $n$ (but the underlying curve is assumed to be unpointed and smooth over an algebraically closed field), then these objects has been studied by K. Joshi, S. Ramanan, E. Z. Xia, J. K. Yu, C. Pauly, T. H. Chen, X. Zhu et al. (cf. [13], [22], [23], and [24]).

As discussed in these references, dormant $G$-opers and their moduli have diverse aspects and occur naturally in mathematics. A detailed understanding of them from the viewpoint of enumerative geometry will be applied to the counting problems for various objects (e.g., lattice points inside a convex rational polytope, edge-colorings of a trivalent graph, and Frobenius-destabilized bundles, etc.) appearing in some areas of mathematics linked to the theory of dormant opers, see [23], [38], [50], and [55].

0.2. As an example, let us explain one aspect of the enumerative geometry of dormant $G$-opers concerning the algebraic-solution problem of linear differential equations in positive characteristic.

Let $X$ be a geometrically connected, proper, and smooth curve over a perfect field $k$ of characteristic $p$ with function field $K$. Consider a monic linear ordinary differential operator $D$ of order $n > 1$ with regular singularities defined on $X$; it may be expressed locally as follows:

\[
D := \frac{d^n}{dx^n} + q_1 \frac{d^{n-1}}{dx^{n-1}} + \cdots + q_{n-1} \frac{d}{dx} + q_n,
\]

where $q_1, \ldots, q_n \in K$ and $x$ denotes a local coordinate in $X$. Since the $p$-powers of elements of $K$ coincides with the constant field in $K$ (i.e., the kernel of the universal derivation $d : K \to \Omega_{K/k}$), the set of solutions to the equation $Dy = 0$ in $K$ forms a $K^p$-vector space.

We shall say that the differential equation $Dy = 0$ has a full set of solutions if it has $n$ solutions in $K$ linearly independent over $K^p$. The study of differential equations having many (algebraic) solutions was originally considered in the complex case, tackled and developed since the 1870s by many mathematicians: H. A. Schwarz (for the hypergeometric equations), L. I. Fuchs, P. Gordan, and C. F. Klein (for the second order equations), C. Jordan (for the $n$-th order) et al. We are interested in the positive characteristic analogue of this traditional study,
and, in particular, want to know how many differential equations $Dy = 0$ (associated with $D$ as in (1)) have a full set of solutions.

Let us consider a special case, i.e., the case of monic, linear, and second order differential operators on the projective line $\mathbb{P}^1$ having a full set of solutions. The classical theory of Riemann schemes shows that any such operator may be transformed (e.g., via pull-back by an automorphism of $\mathbb{P}^1$) into a Gauss’ hypergeometric differential operator

$$D_{a,b,c} := \frac{d^2}{dx^2} + \left(\frac{c}{x} + \frac{1-c+a+b}{x-1}\right) \frac{d}{dx} + \frac{ab}{x(x-1)}$$

determined by some triple $(a, b, c) \in k^{x3}$, where $x := s/t$ and $k^{x3}$ denotes the product of 3 copies of $k$.

We shall use the notation $(\sim)$ to denote the inverse of the bijective restriction $\{1, \ldots, p\} \rightarrow \mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ of the natural quotient $\mathbb{Z} \rightarrow \mathbb{F}_p$. If the equation $D_{a,b,c}y = 0$ has a full set of solutions, then the set $\{y_{a,b,c}(x), x^{-c}y_{a-c+1,b-c+1,2-c}(x)\} \subseteq k(x)$ forms a basis of the solutions. Here, $y_{a,b,c}(x)$ denotes a polynomial of $x$ defined by the following truncated hypergeometric series

$$y_{a,b,c}(x) := 1 + \frac{a \cdot b}{1 \cdot c} x + \frac{a \cdot (a+1) \cdot b \cdot (b+1)}{1 \cdot 2 \cdot c \cdot (c+1)} x^2 + \cdots$$

where we stop the series as soon as the numerator vanishes.

According to [18, §1.6] (or [27, §6.4]), the equation $D_{a,b,c}y = 0$ has a full set of solutions if and only if $(a, b, c)$ lies in $\mathbb{F}_p^{x3}$ and either $\tilde{b} \geq \tilde{c} > \tilde{a}$ or $\tilde{a} \geq \tilde{c} > \tilde{b}$ is satisfied. In particular, after a straightforward calculation, we see that there exists precisely $\frac{p^3-p}{3}$ hypergeometric equations having a full set of solutions.

0.3. Here, we shall recall the relationship between dormant opers and differential operators in terms of connections on vector bundles. To each differential operator $D$ as in (1), one can associate, in a well-known manner, a connection on a vector bundle expressed locally as follows:

$$\nabla = \frac{d}{dx} - \begin{pmatrix} -q_1 & -q_2 & -q_3 & \cdots & -q_{n-1} & -q_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$  

The assignment $y \mapsto t(\frac{d}{dx}y, \cdots, \frac{d^n}{dx^n}y)$ gives a bijective correspondence between the solutions of the equation $Dy = 0$ and the horizontal sections of this vector bundle with respect to $\nabla$.

A vector bundle equipped with a connection of the form (4) provides an oper. Indeed, any PGL$_n$-oper can be represented, via a gauge transformation, by such a connection (cf. [57, Theorem D]). Under a certain assumption on the subprincipal symbol of $D$ (i.e., the assumption that the 1-st order differential equation determined from the subprincipal symbol using the manner discussed in [57, Remark 4.30] has a nonzero solution), the differential equation $Dy = 0$
associated with $D$ as in (11) has a full set of solutions if and only if the associated $\text{PGL}_n$-oper is dormant (cf. [27, (6.0.5), Proposition]).

We go back to the hypergeometric case. Let $\mathcal{P}_k := (\mathbb{P}_k, \{[0], [1], [\infty]\})$ be the projective line with three marked points determined by 0, 1, and $\infty$. For a triple $(a, b, c) \in k^3$, denote by $\mathcal{E}_{a,b,c}^{\bullet}$ the dormant $\text{PGL}_2$-oper on $\mathcal{P}_k$ induced by $D_{a,b,c}$. If we set $\text{Ex}_{a,b,c} := (1-c, c-a-b, b-a)$ (i.e., the exponent differences at 0, 1, and $\infty$), then the isomorphism class of $\mathcal{E}_{a,b,c}^{\bullet}$ is determined by this data regarded as a triple of elements in $k/\{\pm 1\}$ ($:= \text{the quotient set of } k$ by the equivalence relation generated by $v \sim -v$ for every $v \in k$). On the other hand, it is verified that

$$\mathcal{E}_{a,b,c}^{\bullet} \cong \mathcal{E}_{a',b',c'}^{\bullet} \iff \text{Ex}_{a,b,c} = \text{Ex}_{a',b',c'} \in (k/\{\pm 1\})^3.$$  

(5)

Hence, since $\text{Ex}_{a,b,c} \neq (0, 0, 0)$ if $\mathcal{E}_{a,b,c}^{\bullet}$ is dormant, the assignment $D_{a,b,c} \mapsto \mathcal{E}_{a,b,c}^{\bullet}$ determines a $2^3$-to-1 correspondence

$$\left( \text{the set of hypergeometric operators } D_{a,b,c} \text{ such that the equation } D_{a,b,c}y = 0 \text{ has a full set of solutions} \right) \xrightarrow{2^3-1} \left( \text{the set of isomorphism classes of dormant } \text{PGL}_2\text{-opers on } \mathcal{P}_k \right).$$

(6)

In particular, the italicized assertion at the end of the previous subsection implies that the number of (isomorphism classes of) dormant $\text{PGL}_2$-opers on $\mathcal{P}_k$ is exactly equal to $\frac{2^3-1}{24}$.

The same computations were verified by S. Mochizuki (cf. [13, Chap. V, Corollary 3.7]), H. Lange-C. Pauly (cf. [55, Theorem 2]), and B. Osserman (cf. [51, Theorem 1.2]) by applying different methods. At any rate, the simplest case of the counting problem for dormant opers was completely resolved. When one sets about trying to compute the number of differential operators or the corresponding opers in more general cases (i.e., the case of general $n$ and $X$), the factorization property of their moduli stacks discussed in the present paper (or [57, Theorem F]) will provide an effective way to do this.

0.4. In the rest of this Introduction, we describe briefly the results obtained in the present paper.

Let $G$ be a split semisimple algebraic group over a perfect field $k$ of characteristic $p$. Then, we define a dormant faithful twisted $G$-oper (or a $G$-do’per for short) as a $G$-oper on a stacky log curve with vanishing $p$-curvature satisfying a certain representability condition (cf. Definitions 2.2.2 and 2.3.1). Under the assumption that $G$ is of adjoint type, the notion of a $G$-do’per coincides with the classical notion of a $\mathfrak{g}$-oper, where $\mathfrak{g}$ denotes the Lie algebra of $G$, in the sense of [57, Definitions 2.1 and 3.15] (cf. Remark 2.3.3).

Given a pair of nonnegative integers $(g, r)$ with $2g - 2 + r > 0$, we denote the usual moduli stack classifying $r$-pointed stable curves of genus $g$ over $k$ by $\mathcal{M}_{g,r}$. Then, one obtains the category fibered in groupoids

$$\mathcal{D}_{\mathcal{P}_{G,g,r}}^{2\mathfrak{m}_{g,r}}$$

(cf. [137]) classifying $G$-do’pers on stacky log curves whose coarse moduli spaces are classified by $\mathfrak{m}_{g,r}$. Here, let us consider two additional conditions $(*)_G$, $(**)_G$ on $G$ described as follows:

$(*)_G$: If $h$ denotes the integer $h_G$ defined in [57, Eq. (104)], where “$G$” is taken to be the adjoint group of $G$, then the inequality $p > 2h$ holds;

$(**)_G$: $G$ is of classical type $A_n$ (with $2n < p - 2$ or $(n, p) = (1, 3)$), $B_l$ (with $4l < p - 2$), or $C_m$ (with $4m < p$).
Notice that \((**)_G\) implies \((*)_G\). The following assertion is the first main result of the present paper, which concerns the structure of the moduli space \(\mathcal{O}p^{zzz...}_{G,g,r}\); this result generalizes [57, Theorems C and G].

**Theorem A** (cf. Theorems 3.3.1 and 12.1). (i) Assume that \(G\) satisfies the condition \((*)_G\). Then, \(\mathcal{O}p^{zzz...}_{G,g,r}\) may be represented by a nonempty proper Deligne-Mumford stack over \(k\) which is finite over \(\overline{M}_{g,r}\) and has an irreducible component that dominates \(\overline{M}_{g,r}\). Moreover, \(\mathcal{O}p^{zzz...}_{G,g,r}\) admits a perfect obstruction theory, and hence, has a virtual fundamental class \([\mathcal{O}p^{zzz...}_{G,g,r}]_{\text{vir}}\).

(ii) Assume further that \(G\) satisfies the condition \((**)_G\). Then, \(\mathcal{O}p^{zzz...}_{G,g,r}\) is generically étale over \(\overline{M}_{g,r}\) (i.e., any irreducible component of \(\mathcal{O}p^{zzz...}_{G,g,r}\) dominating \(\overline{M}_{g,r}\) has a dense open substack which is étale over \(\overline{M}_{g,r}\)), and moreover, has generic stabilizer isomorphic to the center \(Z\) of \(G\).

The above theorem asserts the existence of a canonical compact moduli space of dormant \(G\)-opers admitting a virtual fundamental class. Recall that virtual fundamental classes of moduli spaces play a central role in enumerative geometry as they represent a major ingredient in the construction of deformation invariants, e.g., Gromov-Witten and Donaldson-Thomas invariants. Virtual fundamental classes of Quot schemes were constructed in [39, Theorem 1]. Although there exists a connection between certain Quot schemes and the moduli space of PGL\(_n\)-opers on a sufficiently general curve (cf. [57, §9] or an unpublished version of [24, §11]), the virtual class \([\mathcal{O}p^{zzz...}_{G,g,r}]_{\text{vir}}\) does not seem to come, \(a\ priori\), from Quot schemes because of its construction.

0.5. Next, by means of the resulting virtual fundamental class, we construct a CohFT (= a cohomological field theory) associated with \(G\)-do’pers, which in fact forms a 2d TQFT (= a 2-dimensional topological quantum field theory).

Cohomological field theories were introduced by M. Kontsevich and Y. Manin in [33] to axiomatize the properties of Gromov-Witten classes of a given target variety over the field of complex numbers \(\mathbb{C}\). For instance, the trivial CohFT arises from the Gromov-Witten theory of one point. As it turns out, this notion is more general, in the sense that not all CohFTs come from Gromov-Witten theory. Indeed, one may find some examples of CohFTs, including Hodge CohFTs and CohFTs arising from Witten’s \(r\)-spin classes or Fan-Jarvis-Ruan-Witten (FJRW) theory.

On the other hand, in the axiomatic formulation due to Atiyah (cf. [7]), an \(N\)-dimensional topological quantum field theory (or just an \(N\)d TQFT) is a rule \(\Lambda\) which to each closed oriented manifold \(\Sigma\) of dimension \(N-1\) associates a vector space \(\Lambda_{\Sigma}\), and to each oriented manifold of dimension \(N\) whose boundary is \(\Sigma\) associates a vector in \(\Lambda_{\Sigma}\). This rule satisfies a collection of axioms which express that topologically equivalent manifolds have isomorphic associated vector spaces, and that disjoint unions of manifolds go to tensor products of vector spaces, etc. One may consider each 2d TQFT as a special kind of CohFT, i.e., a CohFT valued in the 0-th cohomology.

*This version is available at: https://arxiv.org/pdf/1311.4359v1.pdf*
Hereinafter, suppose that \( k \) is algebraically closed. In §5, we introduce (cf. Definition 5.2.1) the definition of a CohFT by means of the \( l \)-adic étale cohomology of \( \mathcal{M}_{g,r} \) and recall the notion of a 2d TQFT. One of key ingredients in the construction of the desired CohFT (or 2d TQFT) is the stack \( \mathcal{R}ad \) (cf. (141)) defined as follows. Let \( t \) be the Lie algebra of a fixed maximal torus of \( G \) (viewed as a \( k \)-scheme). Denote by \( t^{F \text{reg}} \) (cf. (139)) the subscheme of \( t \) consisting of the Frobenius-invariant regular elements. The natural action of the Weyl group \( W \) on \( t^{F \text{reg}} \) yields the quotient scheme \( t^{F \text{reg}}/W \). If \( Z \) denotes the center of \( G \), then the trivial \( Z \)-action gives rise to the quotient stack \( [(t^{F \text{reg}}/W)/Z] \). This stack induces the stack of cyclotomic gerbes \( \mathcal{I}_\mu \left([(t^{F \text{reg}}/W)/Z]\right) \), which we denote by \( \mathcal{R}ad \). The connected components of \( \mathcal{I}_\mu \left([(t^{F \text{reg}}/W)/Z]\right) \) are indexed by a certain finite set \( \Delta \) (cf. (142)). Each connected component \( \mathcal{R}ad_\rho \) (indexed by \( \rho \in \Delta \)) is canonically isomorphic to the classifying stack \( BZ \) of a certain quotient group \( Z \) (cf. Proposition 1.4.1).

Another key ingredient is called radius (cf. Definition 2.4.1); this is an invariant associated with each \( G \)-do'per (or more generally, each faithful twisted \( G \)-oper) and each marked point of the underlying curve. The notion of radius introduced in the present paper generalizes the classical definition given in [43, Definition 1.2] and [57, Definition 2.29]. The reason for the name of this notion is based on a certain analogy between the theory of “pants” in the classical Teichmüller theory and the structure theory of \( \text{PGL}_2 \)-opers on a Riemann sphere minus three points. In fact, we may make this analogy by identifying the radii in our sense with the size of the respective holes in the Riemann sphere.

The assignment from each \( G \)-do'per to its radius at the \( i \)-th marked point \( (i = 1, \cdots, r) \) may be realized geometrically as a morphism \( ev_i : \mathcal{O}p_{G,g,r}^{\text{zar}} \rightarrow \mathcal{R}ad \left( = \coprod_{\rho \in \Delta} \mathcal{R}ad_\rho \right) \). That is to say, the point classifying a \( G \)-do'per is mapped, via \( ev_i \), to a point in the connected component \( \mathcal{R}ad_\rho \) indexed by its radius \( \rho \) at the \( i \)-th marked point. Thus, we obtain a diagram of stacks

\[
\begin{array}{ccc}
\mathcal{O}p_{G,g,r}^{\text{zar}} & \xrightarrow{\pi_{g,r}} & \mathcal{M}_{g,r} \\
ev_i & \downarrow & \phantom{\coprod} \mathcal{R}ad, \\
& \pi_{g,r} & (8)
\end{array}
\]

where \( \pi_{g,r} \) denotes the natural projection.

By means of the various objects just discussed, the constituents in the desired CohFT are constructed. Given a prime \( l \) different from \( p \), let us consider the \( l \)-adic étale cohomology

\[
\mathcal{V} := \tilde{H}_{\text{et}}^*(\mathcal{R}ad, \overline{Q}_l)
\]

of the stack \( \mathcal{R}ad \), where \( \overline{Q}_l \) denotes the algebraic closure of \( Q_l \) and we write \( \tilde{H}_{\text{et}}^*(-, \overline{Q}_l) := \bigoplus_{i=0}^\infty H_{\text{et}}^i(-, \overline{Q}_l(\frac{i}{2})) \). We obtain a collection \( \{\Lambda_{G,g,r}^{g,r} \}_{g,r \geq 0, 2g-2+r>0} \) consisting of the \( \overline{Q}_l \)-linear morphisms

\[
\Lambda_{G,g,r} : \mathcal{V}^{\otimes r} \rightarrow \tilde{H}_{\text{et}}^*(\mathcal{M}_{g,r}, \overline{Q}_l)
\]

\[
\bigotimes_{i=1}^r v_i \mapsto \left( \pi_{g,r}^* \left( \bigotimes_{i=1}^r ev_i^* (v_i) \right) \right) \cap \text{cl}^{3g-3+r} \left( \left[ \mathcal{O}p_{G,g,r}^{\text{zar}} \right]_{\text{vir}} \right)
\]
(cf. (202) for its precise definition). Also, a \( \mathbb{T}_l \)-bilinear pairing \( \eta : \mathcal{V} \times \mathcal{V} \to \mathbb{T}_l \) (cf. (203)) and a distinguished element \( e_\mathcal{V} \) of \( \mathcal{V} \) (cf. (201)) are defined. Then, the second main result is as follows.

**Theorem B** (cf. Theorem 5.3.1 (ii)). Assume that \( G \) satisfies the condition \((**)_G\). Then, the collection of data

\[
\Lambda_G := (\mathcal{V}, \eta, e_\mathcal{V}, \{\Lambda_{G,g,r}\}_{g,r \geq 0, 2g-2+r > 0})
\]

(cf. (205)) forms a CohFT (with flat identity) valued in \( \widetilde{H}_0^0(\overline{\mathcal{M}}_{g,r}, \mathbb{T}_l) \), namely, forms a 2d TQFT over \( \mathbb{T}_l \). Moreover, the corresponding Frobenius algebra \((\mathcal{V}, \eta)\) is semisimple.

The 2d TQFT asserted above will play an important role in studying enumerative geometry of do’pers. In fact, each \( \mathbf{\rho} := (\rho_i)_{i=1}^r \in \Delta^r \) determines the closed substack \( \mathcal{D}_{PGL_2,g,r,\mathbf{\rho}} \) (cf. (146)) of \( \mathcal{D}_{PGL_2,g,r} \) classifying \( PGL_2 \)-do’pers of radii \( \mathbf{\rho} \). According to Theorem 5.3.1 (i), the value \( \Lambda_{G,g,r}(\bigotimes_{i=1}^r e_{\mathbf{\rho}_i}) \) coincides with the generic degree of \( \mathcal{D}_{PGL_2,g,r,\mathbf{\rho}}/\overline{\mathcal{M}}_{g,r} \). In particular, it is exactly equal to \( \frac{1}{|Z|} \) times the number of (isomorphism classes of) \( G \)-do’pers of radii \( \mathbf{\rho} \) on a sufficiently general curve classified by \( \overline{\mathcal{M}}_{g,r} \). An explicit description of the ring structure (e.g., the set of characters) of the Frobenius algebra \((\mathcal{V}, \eta)\) allows us to compute these values.

In [57, Proposition 7.33], it was already proved that the collection \( \{\Lambda_{G,g,r}\}_{g,r} \) together form a pseudo-fusion rule, i.e., a somewhat weaker variant of a fusion rule, as well as a 2d TQFT. The notion of a pseudo-fusion rule provides just enough framework to compute the generic degrees, but at the time of writing [57] some of the conditions for it to be 2d TQFT remained unknown. After proving Theorem 5.3.1 one can obtain the same calculations as obtained in loc. cit. by simply applying the general theory of 2d TQFTs.

For example, the structure of the Frobenius algebra \((\mathcal{V}, \eta)\) for \( G = PGL_2 \) can be completely understood by comparing with the fusion ring of the \( sl_2(\mathbb{C}) \) Wess-Zumino-Witten model (cf. §6.3). In fact, under the natural identification \( \Delta = \{0, 1, \cdots, \frac{p-3}{2}\} \) (cf. (233)), we obtain the following equality defined for each collection of radii \( (n_i)_{i=1}^r \in \Delta^r \):

\[
\Lambda_{PGL_2,g,r}(\bigotimes_{i=1}^r e_{n_i}) = \deg(\mathcal{D}_{PGL_2,g,r,(n_i)_{i=1}^r}/\overline{\mathcal{M}}_{g,r}) = \frac{p^{g-1}}{2^{g-1}} \sum_{j=1}^{p-1} \prod_{i=1}^r \sin \left( \frac{(2n_i+1)j\pi}{p} \right) \cdot \frac{1}{\sin^2 2g-2+r \left( \frac{j\pi}{p} \right)}.
\]

(cf. [57] Theorem 7.41]).

0.6. In the final section of the present paper, we discuss an analogue for \( \Lambda_G \) of the Witten-Kontsevich theorem (cf. [58], [32], [28], [29], [41], and [47]), which is one of the landmark results concerning the intersection theory on the moduli stacks of pointed stable curves over \( \mathbb{C} \). The well-known Witten-Kontsevich theorem asserts an equivalence of the intersection theory of psi classes on that space and the Hermitian matrix model of 2-dimensional gravity. This implies that the partition function of the trivial CohFT is a tau function of the KdV hierarchy, in other words, it satisfies a certain series of partial differential equations.
The partition function that we deal with is defined as follows (cf. Definition 6.2.1):

\[ Z_G := \exp \left( \sum_{g,r \geq 0} \frac{\hbar^{2g-2}}{r!} \sum_{d_1, \ldots, d_r \geq 0} \left( \int_{\text{Op}_{G,g,r}} \prod_{i=1}^r \text{ev}_i^*(c_{\rho_i}) \psi_i^{d_i} \right) \prod_{i=1}^r t_{d_i, \rho_i} \right) \]

\[
= \exp \left( \sum_{g,r \geq 0} \frac{\hbar^{2g-2}}{r!} \sum_{d_1, \ldots, d_r \geq 0} \left( \int_{\text{Mod}_{g,r}} \Lambda_{G,g,r} \left( \bigotimes_{i=1}^r c_{\rho_i} \right) \prod_{i=1}^r \psi_i^{d_i} \prod_{i=1}^r t_{d_i, \rho_i} \right) \right) \in \overline{Q}((h))[[\{t_{d, \rho}\}_{d \geq 0, \rho \in \Delta}]],
\]

where \( \psi_i \) and \( \widetilde{\psi}_i \) (\( i = 1, \ldots, r \)) denote the \( i \)-th psi classes on \( \text{Op}^{\text{Za}}_{G,g,r} \) and \( \text{Mod}_{g,r} \) respectively, and the equality in the parenthesis follows from Proposition 6.1.2. Also, let \( L_n \) (\( n \geq -1 \)) denote a differential operator

\[ L_n := -\frac{(2n+3)!!}{2^{n+1}} \frac{\partial}{\partial t_{n+1, \epsilon}} + \sum_{i=0}^{\infty} \frac{(2i+2n+1)!!}{(2i-1)!!} \frac{\partial}{\partial t_{i+n, \rho}} \left( \sum_{\rho \in \Delta} t_{i, \rho} \frac{\partial}{\partial t_{i+n, \rho}} \right) \]

\[ + \frac{|Z|\hbar^2}{2} \sum_{i=0}^{n-1} \frac{(2i+1)!!(2n-2i)!!}{2^{n+1}} \left( \sum_{\rho \in \Delta} \frac{\partial^2}{\partial t_{i, \rho} \partial t_{n-1-i, \rho}} \right) \]

\[ + \delta_{n,-1} \frac{\hbar^2}{2|Z|} \left( \sum_{\rho \in \Delta} \left| t_{0, \rho} t_{0, \rho} \right| \right) + \delta_{n,0} \frac{|\Delta|}{16} \cdot \]

Then, we prove the equality

\[ L_n Z_G = 0 \]

for every \( n \geq -1 \) (cf. Theorem 6.2.2). That is to say, the partition function of \( G \)-do'pers turns out to be a solution to infinitely many partial differential equations \( L_n y = 0 \) (\( n \geq -1 \)). In particular, this result gives nontrivial constraints on the psi classes \( \psi_i \) on \( \text{Op}^{\text{Za}}_{G,g,r} \).

Acknowledgements. The author cannot express enough his sincere and deep gratitude to Professors Shinichi Mochizuki and Kirti Joshi. Without their philosophical viewpoints, theoretical insights, and endless creativity, my study of mathematics would have remained “dormant”. Also, special thanks go to the moduli stack of dormant \( G \)-opers \( \text{Mod}_{G,g,r} \), who has guided him to the beautiful world of mathematics. The author was partially supported by the Grant-in-Aid for Scientific Research (KAKENHI No. 18K13385, 21K13770).

Notation and Conventions. Let us introduce some notation and conventions used in the present paper. Throughout the present paper, all schemes and algebraic stacks are assumed to be locally noetherian. We fix a perfect field \( k \), and denote by \( \mathcal{S}ch/k \) the category of schemes over \( k \).

We use the word stack to mean algebraic stack in the sense of the appendix in [54]. Let \( \mathfrak{X} \) be a Deligne-Mumford stack. By a sheaf on \( \mathfrak{X} \), we mean, unless otherwise stated, a sheaf on the small étale site of \( \mathfrak{X} \). In particular, one obtains the structure sheaf \( O_{\mathfrak{X}} \) of \( \mathfrak{X} \). Under the
assumption that $X$ is of finite type over $k$, we denote by $|X|$ the coarse moduli space of $X$, which has a natural projection $\text{coa}_X : X \to |X|$.

Each morphism $f : X \to Y$ of Deligne-Mumford stacks of finite type over $k$ induces naturally a morphism $|f| : |X| \to |Y|$ between their respective coarse moduli spaces. If, moreover, both $X$ and $Y$ are integral and $f$ is separated and dominant, then the degree $\deg(X/Y)$ of $X$ over $Y$ are defined (cf. [54, (1.15), Definition]). Notice that the value $\deg(X/Y)$ may not be an integer in general unless $f$ is representable. One may generalize naturally the notion of degree to the case where $X$ is a disjoint union of finite number of integral stacks. For each $n \in \mathbb{Z}_{\geq 0}$, denote by $A_n(X)_\mathbb{Q}$ (cf. [54, (3.4), Definition]) the rational Chow group of cycles of dimension $n$ on $X$ modulo rational equivalence tensored with $\mathbb{Q}$.

Basic references for the notion and properties of a log scheme (or more generally, a log stack) are [19], [25], and [26]. For a log stack indicated, say, by $Y_{log}$, we shall write $\mathcal{Y}$ for the underlying stack of $Y_{log}$. For a morphism $Y_{log} \to \mathcal{T}_{log}$ of fs log Deligne-Mumford stacks, let us write $\mathcal{T}_{\mathcal{Y}_{log}, \mathcal{T}_{log}}$ for the sheaf of logarithmic derivations of $Y_{log}$ over $\mathcal{T}_{log}$. Also, write $\Omega_{Y_{log}, \mathcal{T}_{log}}$ for its dual $\mathcal{T}_{\mathcal{Y}_{log}, \mathcal{T}_{log}}$, i.e., the sheaf of logarithmic 1-forms of $Y_{log}$ over $\mathcal{T}_{log}$.

For each positive integer $l$, we denote by $\mu_l$ the group of $l$-th roots of unity in an algebraic closure of $k$. If $G$ is an algebraic group over $k$, then let $BG$ denote the classifying stack of $G$, which is defined as the quotient stack $BG := [pt/G]$ for the trivial action of $G$ on $pt := \text{Spec}(k)$.

Given a right $G$-bundle $\mathcal{E}$ on a $k$-stack $Y$ in the étale topology and a $k$-vector space $\mathfrak{h}$ equipped with a left $G$-action, we shall write $\mathfrak{h}_G$ for the vector bundle on $Y$ associated with the relative affine space $\mathcal{E} \times^G \mathfrak{h} := (\mathcal{E} \times_k \mathfrak{h})/G$. Denote by $\mathbb{G}_m$ the multiplicative group over $k$.

Let $S$ be a scheme. By a nodal curve over $S$, we mean a flat morphism of schemes $f : X \to S$ whose geometric fibers are connected and reduced 1-dimensional schemes with at most nodal points as singularities. For simplicity, we often write $X$ instead of $f : X \to S$ when denoting a nodal curve over $S$.

Let $r$ be a nonnegative integer. An $r$-pointed nodal curve over $S$ is defined to be a collection $\mathcal{X} := (f : X \to S, \sigma_{X,i} : S \to X)_{i=1}^r$ consisting of a nodal curve $f : X \to S$ over $S$ and $r$ sections $\sigma_{X,i} : S \to X$ (i = 1, · · · , r) such that $\text{Im}(\sigma_{X,i})$ lies, for any $i$, in the smooth locus of $X$ (relative to $S$) and that $\text{Im}(\sigma_{X,i}) \cap \text{Im}(\sigma_{X,j}) = \emptyset$ for any pair $(i, j)$ with $i \neq j$.

1. Extended spin structures on twisted curves

This section deals with extended spin structures, which are some sort of natural generalization of spin structures (in the classical sense) defined on a twisted curve. The notion of an extended spin structure will be used to describe the gap between a faithful twisted $G$-oper and the associated $G_{ad}$-oper, where $G_{ad}$ denotes the adjoint group of $G$. In fact, we will see (cf. Theorem 2.3.3) that the moduli stack of faithful twisted $G$-opers is isomorphic to the product of the moduli stacks of $G_{ad}$-opers and of certain extended spin structures. In this section, we prove some properties of the moduli stack of extended spin structures (cf., e.g., Theorem 1.3.5, Propositions 1.5.2, 1.5.3, and 1.6.2).

1.1. The moduli stack of pointed stable curves. To begin with, we shall introduce some notation concerning pointed curves and the moduli stack classifying them. Given a pair of
nonnegative integers \((g, r)\) with \(2g - 2 + r > 0\), we shall write \(\mathcal{M}_{g, r, \mathbb{Z}}\) for the moduli stack of \(r\)-pointed stable curves of genus \(g\), and write
\begin{equation}
\mathcal{M}_{g, r} := \mathcal{M}_{g, r, \mathbb{Z}} \times \mathbb{Z}.
\end{equation}
Namely, \(\mathcal{M}_{g, r}\) classifies the pointed stable curves
\begin{equation}
\mathcal{X} := \{f : X \to S, \{\sigma_{X, i} : S \to X\}^r_{i=1}\}
\end{equation}
consisting of a proper nodal curve \(f : X \to S\) of genus \(g\) over a \(k\)-scheme \(S\) and \(r\) marked points \(\sigma_{X, i} : S \to X\) \((i = 1, \ldots, r)\) satisfying certain conditions (cf. [34, Definition 1.1]).

Given nonnegative integers \(g_1, g_2, r_1, r_2\) with \(2g_i - 1 + r_i > 0\) \((i = 1, 2)\), we shall write
\begin{equation}
\Phi_{\text{tree}} : \mathcal{M}_{g_1, r_1 + 1} \times_k \mathcal{M}_{g_2, r_2 + 1} \to \mathcal{M}_{g_1 + g_2, r_1 + r_2}
\end{equation}
for the gluwing map obtained by attaching the respective last marked points of curves classified by \(\mathcal{M}_{g_1, r_1 + 1}\) and \(\mathcal{M}_{g_2, r_2 + 1}\) to form a node.

Similarly, given nonnegative integers \(g, r\) with \(2g + r > 0\), we shall write
\begin{equation}
\Phi_{\text{loop}} : \mathcal{M}_{g, r + 2} \to \mathcal{M}_{g+1, r}
\end{equation}
for the gluwing map obtained by attaching the last two marked points of each curve classified by \(\mathcal{M}_{g, r + 2}\) to form a node.

Finally, if \(2g - 2 + r > 0\), we define
\begin{equation}
\Phi_{\text{tail}} : \mathcal{M}_{g, r + 1} \to \mathcal{M}_{g, r}
\end{equation}
to be the morphism obtained by forgetting the last marked point and successively contracting any resulting unstable components of each curve classified by \(\mathcal{M}_{g, r + 1}\).

1.2. Twisted curves. Next, let us recall the notion of a pointed twisted curve and the log structure equipped with it. This notion will be used to define a twisted \(G\)-oper. Here, recall that a Deligne-Mumford stack \(\mathfrak{X}\) over \(k\) is called tame if, for any algebraically closed field \(\overline{k}\) over \(k\) and any morphism \(\overline{\mathfrak{X}} : \text{Spec}(\overline{k}) \to \mathfrak{X}\), the stabilizer group \(\text{Stab}_{\mathfrak{X}}(\overline{\mathfrak{X}})\) of \(\overline{\mathfrak{X}}\) has order invertible in \(k\).

**Definition 1.2.1.** Let \(T^\log\) be an fs log scheme over \(k\). A stacky log curve over \(T^\log\) is an fs log Deligne-Mumford stack \(\mathfrak{Y}\) together with a log smooth integrable morphism \(f^\log : \mathfrak{Y}^\log \to T^\log\) such that the geometric fibers of the underlying morphism \(f : \mathfrak{Y} \to T\) of stacks are reduced, connected, and 1-dimensional. (In particular, both \(T^\log_{\mathfrak{Y}^\log, T^\log}\) and \(\Omega^\log_{\mathfrak{Y}^\log, T^\log}\) are line bundles on \(\mathfrak{Y}\).)

**Definition 1.2.2** (cf. [48], Definition 1.2). Let \(S\) be a \(k\)-scheme.

(i) A (balanced) twisted curve over \(S\) is a proper flat morphism \(f : \mathfrak{X} \to S\) of tame Deligne-Mumford stacks over \(k\) satisfying the following conditions:

- The geometric fibers of \(f : \mathfrak{X} \to S\) are purely 1-dimensional and connected with at most nodal singularities.
\( |\mathcal{X}|^{\text{sm}} \) denotes the open subscheme of \(|\mathcal{X}|\) where \(|f| : |\mathcal{X}| \to S\) is smooth, then the inverse image \( \mathcal{X} \times_{|\mathcal{X}|} |\mathcal{X}|^{\text{sm}} \subseteq \mathcal{X} \) coincides with the open substack of \( \mathcal{X} \) where \( f \) is smooth.

- For any geometric point \( \mathfrak{p} \to S \), the map \( \text{coa}_{\mathcal{X}} \times \text{id}_{\mathfrak{p}} : \mathcal{X} \times_S \mathfrak{p} \to |\mathcal{X}| \times_S \mathfrak{p} \) is an isomorphism over some dense open subscheme of \( |\mathcal{X}| \times_S \mathfrak{p} \).

Consider a geometric point \( \mathfrak{p} \to |\mathcal{X}| \) mapping to a node. Note that there exist an affine open neighborhood \( T : = \text{Spec}(R) \subseteq S \) of \(|f|(\mathfrak{p})\), an affine étale neighborhood \( \text{Spec}(A) \to |f|^{-1}(T) \subseteq |\mathcal{X}| \) of \( \mathfrak{p} \), and an étale morphism

\[
\text{Spec}(A) \to \text{Spec}(R[s_0, t_0]/(s_0 t_0 - u_0))
\]

over \( R \) for some \( u_0 \in R \). Then, the pull-back \( \mathcal{X} \times_{|\mathcal{X}|} \text{Spec}(A) \) is isomorphic to the quotient stack

\[
[\text{Spec}(A)[s_1, t_1]/(s_1 t_1 - u_1, s_1^l - s_0, t_1^l - t_0))]/\mu_l]
\]

for some \( u_1 \in R \) and some positive integer \( l \) invertible in \( k \) such that \( \zeta \in \mu_l \) acts by \( (s_1, t_1) \mapsto (\zeta s_1, \zeta^{-1} t_1) \).

(ii) Let \( g \) be a nonnegative integer. We shall say that a twisted curve \( f : \mathcal{X} \to S \) is of genus \( g \) if the genus of every fiber of the proper nodal curve \( |f| : |\mathcal{X}| \to S \) coincides with \( g \).

**Definition 1.2.3** (cf. [5], Definition 4.1.2). Let \( g \) and \( r \) be nonnegative integers and \( S \) a \( k \)-scheme.

1. An \( r \)-pointed twisted curve (of genus \( g \)) over \( S \) is a collection of data

\[
\mathcal{X}^{\text{tw}} := (f : \mathcal{X} \to S, \{\sigma_{\mathcal{X}, i} : \mathcal{G}_i \to \mathcal{X}\}_{i=1}^r)
\]

consisting of a twisted curve \( f : \mathcal{X} \to S \) (of genus \( g \)) and disjoint closed substacks \( \sigma_{\mathcal{X}, i} : \mathcal{G}_i \to \mathcal{X} \) \( (i = 1, \ldots, r) \) of \( \mathcal{X} \) satisfying the following conditions:

- Each \( \text{Im}(\sigma_{\mathcal{X}, i}) \) is contained in the smooth locus in \( \mathcal{X} \) (relative to \( S \)).
- Each \( \mathcal{G}_i \) is étale gerbe over \( S \).
- If \( \mathcal{X}^{\text{gen}} \) denotes the complement of the union of \( \text{Im}(\sigma_{\mathcal{X}, i}) \) \( (i = 1, \ldots, r) \) in the smooth locus in \( \mathcal{X} \) (relative to \( S \)), then \( \mathcal{X}^{\text{gen}} \) may be represented by a scheme.

2. Let \( \mathcal{X}^{\text{tw}}_j := (f_j : \mathcal{X}_j \to S_j, \{\sigma_{\mathcal{X}_j, i} : \mathcal{G}_{j,i} \to \mathcal{X}_j\}_{i=1}^r) \) \( (j = 1, 2) \) be \( r \)-pointed twisted curves. A morphism of \( r \)-pointed twisted curves from \( \mathcal{X}^{\text{tw}}_1 \) to \( \mathcal{X}^{\text{tw}}_2 \) is a pair of morphisms

\[
(\alpha_S : S_1 \to S_2, \alpha_X : \mathcal{X}_1 \to \mathcal{X}_2)
\]

such that \( \alpha_X^{-1}(\mathcal{G}_{2,i}) = \mathcal{G}_{1,i} \) for any \( i = 1, \ldots, r \), and moreover, the square diagram

\[
\begin{array}{ccc}
\mathcal{X}_1 & \xrightarrow{f_1} & S_1 \\
\alpha_X \downarrow & & \downarrow \alpha_S \\
\mathcal{X}_2 & \xrightarrow{f_2} & S_2
\end{array}
\]

is commutative and cartesian.
(iii) Suppose that \(2g - 2 + r > 0\). A **twisted stable curve of type** \((g, r)\) over \(S\) is an \(r\)-pointed twisted curve \(\mathcal{X}^\text{tw} := (f : \mathcal{X} \to S, \{\sigma_{\mathcal{X}, i} : \mathcal{G}_i \to \mathcal{X}\}_{i=1}^r)\) over \(S\) whose coarse moduli space

\[
|\mathcal{X}^\text{tw}| := (|f| : |\mathcal{X}| \to S, \{|\sigma_{\mathcal{X}, i}| : |\mathcal{G}_i| \to |\mathcal{X}|\}_{i=1}^r)
\]

forms an \(r\)-pointed stable curve of genus \(g\).

**Definition 1.2.4.** Let \(r\) be a nonnegative integer and \(S\) a \(k\)-scheme.

(i) Let \(\mathcal{X} := (f : X \to S, \{\sigma_{X, i}\}_{i=1}^r)\) be an \(r\)-pointed nodal curve over \(S\). A **twistification** of \(\mathcal{X}\) is a pair

\[
(\mathcal{X}^\text{tw}, \gamma)
\]

consisting of an \(r\)-pointed twisted curve \(\mathcal{X}^\text{tw} := (f : \mathcal{X} \to S, \{\sigma_{\mathcal{X}, i}\}_{i=1}^r)\) and an \(S\)-morphism \(\gamma : \mathcal{X} \to X\) such that the induced morphism \(|\gamma| : |\mathcal{X}| \to X\) is an isomorphism and the equality \(|\gamma \circ \sigma_{\mathcal{X}, i}| = \sigma_{X, i}\) holds for every \(i = 1, \ldots, r\).

(ii) Let \(\mathcal{X}\) be an \(r\)-pointed nodal curve over \(S\) and let \((\mathcal{X}^\text{tw}_j, \gamma_j) (j = 1, 2)\) be twistifications of \(\mathcal{X}\). An **isomorphism of twistifications** from \((\mathcal{X}^\text{tw}_1, \gamma_1)\) to \((\mathcal{X}^\text{tw}_2, \gamma_2)\) is an isomorphism of \(r\)-pointed twisted curves from \(\mathcal{X}^\text{tw}_1\) to \(\mathcal{X}^\text{tw}_2\) compatible with \(\gamma_1\) and \(\gamma_2\).

Let \(r\) be a nonnegative integer and \(\mathcal{X}^\text{tw} := (f : \mathcal{X} \to S, \{\sigma_{\mathcal{X}, i}\}_{i=1}^r)\) an \(r\)-pointed twisted curve. There exist canonical log structures on \(\mathcal{X}\) and \(S\) obtained by the log structures described in [48, Theorem 3.6] and the closed substacks \(\sigma_{\mathcal{X}, i}\); we denote the resulting log stacks by \(\mathcal{X}^\text{log}\) and \(S^\text{log}\) respectively. The morphism \(f : \mathcal{X} \to S\) extends to a log smooth morphism between log stacks \(f^\text{log} : \mathcal{X}^\text{log} \to S^\text{log}\), by which \(\mathcal{X}^\text{log}\) specifies a stacky log curve over \(S^\text{log}\). If \(\mathcal{X}\) is smooth, then \(S^\text{log} = S\).

### 1.3. Extended spin structures

In the rest of this section, we assume that \(\text{char}(k) \neq 2\). Also, assume that we are given a pair \((Z, \delta)\) consisting of a finite abelian group \(Z\) (often regarded as a finite group scheme over \(k\)) which has order invertible in \(k\) and a morphism of groups \(\delta : (\{\pm 1\} =) \mu_2 \to Z\). Write

\[
\bar{\delta} : \mu_{\overline{2}} (:= \mu_2 / \text{Ker}(\delta)) \hookrightarrow Z,
\]

where \(\overline{2}\) is either 1 or 2, for the injection induced naturally by \(\delta\). Denote by

\[
\hat{Z}_\delta
\]

the cofiber product of \(\delta\) and the natural inclusion \(\mu_2 \hookrightarrow \mathbb{G}_m\), which fits into the following morphism of short exact sequences of algebraic groups:

\[
\begin{array}{ccccccc}
1 & \longrightarrow & \mu_2 & \xrightarrow{\text{incl}} & \mathbb{G}_m & \xrightarrow{\nu=\pi^2} & \mathbb{G}_m & \longrightarrow & 1 \\
\downarrow{\delta} & & \downarrow{\delta_{\mathbb{G}_m}} & & \downarrow{\text{id}_{\mathbb{G}_m}} & & \\
1 & \longrightarrow & Z & \longrightarrow & \hat{Z}_\delta & \longrightarrow & \mathbb{G}_m & \longrightarrow & 1.
\end{array}
\]
Definition 1.3.1. (i) Let $S^\log$ be an fs log scheme over $k$ and $U^\log$ a stacky log curve over $S^\log$. An extended $(Z, \delta)$-spin structure (or, a $(Z, \delta)$-structure, for short) on $U^\log/S^\log$ is a $\mathbb{Z}_\delta$-bundle

$$\pi_G : \mathcal{G} \to \mathfrak{U}$$

on $\mathfrak{U}$ in the étale topology whose classifying morphism $\mathfrak{U} \to B\mathbb{Z}_\delta$ is representable and fits into the following 1-commutative diagram:

$$\xymatrix{ \mathfrak{U} \ar[r]^{[\Omega_{\mathfrak{U}/S}^\log]} & B\mathbb{Z}_\delta \\
 & B\mathbb{G}_m \ar[ru]_{B\nu} }$$

where the upper horizontal morphism $[\Omega_{\mathfrak{U}/S}^\log]$ denotes the classifying morphism of the line bundle $\Omega_{\mathfrak{U}/S}^\log$ and $B\nu$ denotes the morphism induced naturally by $\nu$ (cf. (30)).

(ii) Let $r$ be a nonnegative integer, $S$ a $k$-scheme, and $X := (X, \{\sigma_{x,i}\}_{i=1}^r)$ an $r$-pointed nodal curve over $S$. A $(Z, \delta)$-structure on $X$ is a collection of data

$$(X^{\text{tw}}, \gamma, \pi_G : \mathcal{G} \to X),$$

consisting of a twistification $(X^{\text{tw}}, \gamma)$ of $X$ and a $(Z, \delta)$-structure $\pi_G : \mathcal{G} \to X$ on $X^{\log}/S^{\log}$.

Definition 1.3.2. In (i)-(iii) below, let $r$ be a nonnegative integer.

(i) Let $S$ be a $k$-scheme. An $r$-pointed $(Z, \delta)$-spin curve over $S$ is a collection of data

$$X := (X, X^{\text{tw}}, \gamma, \pi_G : \mathcal{G} \to X),$$

where $X$ denotes an $r$-pointed nodal curve over $S$ and $(X^{\text{tw}}, \gamma, \pi_G : \mathcal{G} \to X)$ denotes a $(Z, \delta)$-structure on $X$.

(ii) For each $j \in \{1, 2\}$, let $S_j$ be a $k$-scheme and $X_j := (X_j, X_j^{\text{tw}}, \gamma_j, \pi_{G,j} : G_j \to X_j)$ (where $X_j^{\text{tw}} := (f_j : X_j \to S_j, \{\sigma_{x,j,i}\}_{i=1}^r)$) an $r$-pointed $(Z, \delta)$-spin curve over $S_j$. A 1-morphism (or just a morphism) of $r$-pointed $(Z, \delta)$-spin curves from $X_1$ to $X_2$ is a triple of morphisms

$$\alpha := (\alpha_S, \alpha_X, \alpha_G)$$

which makes the following diagram 1-commutative:

$$\xymatrix{ \mathcal{G}_1 \ar[r]^{\pi_{G,1}} & X_1 \ar[r]^{f_1} & S_1 \\
\mathcal{G}_2 \ar[r]_{\pi_{G,2}} & X_2 \ar[r]_{f_2} & S_2, }$$

where

- the right-hand square forms a morphism of $r$-pointed twisted curves;
- the left-hand square is cartesian, and $\alpha_G$ is compatible with the respective $\mathbb{Z}_\delta$-actions on $\mathcal{G}_1$ and $\mathcal{G}_2$. 
In particular, by taking coarse moduli spaces, one may associate, to each such morphism \( \alpha : X_1 \to X_2 \), a morphism \( \alpha_X : \mathcal{X}_1 \to \mathcal{X}_2 \) between the underlying \( r \)-pointed nodal curves.

(iii) Let \( S_j, X_j \) \((j = 1, 2)\) be as in (ii) and \( \alpha_l := (\alpha_{S,l}, \alpha_{X,l}, \alpha_{G,l}) \) \((l = 1, 2)\) be morphisms \( X_1 \to X_2 \) of \( r \)-pointed \((Z, \delta)\)-spin curves. A \textbf{2-morphism} from \( \alpha_1 \) to \( \alpha_2 \) is a triple of natural transformations

\[
\alpha := (\alpha_{S,1} \Rightarrow \alpha_{S,2}, \alpha_{X,1} \Rightarrow \alpha_{X,2}, \alpha_{G,1} \Rightarrow \alpha_{G,2})
\]

compatible with each other (hence, \( \alpha_S \) coincides with the identity natural transformation).

(iv) Let \((g, r)\) be a pair of nonnegative integers with \(2g - 2 + r > 0\) and \( S \) a \( k \)-scheme. A \textbf{pointed stable \((Z, \delta)\)-spin curve of type \((g, r)\)} over \( S \) is an \( r \)-pointed \((Z, \delta)\)-spin curve over \( S \) whose underlying pointed nodal curve defines a pointed stable curve of type \((g, r)\).

By Definition 1.3.2 above, \( r \)-pointed \((Z, \delta)\)-spin curves form a \( 2 \)-category. It is verified that this \( 2 \)-category is equivalent to the \((1-)\)-category fibered in groupoids over \( \mathcal{G} \text{ch}/k \) whose fiber over \( S \in \text{Ob}(\mathcal{G} \text{ch}/k) \) forms the groupoid classifying \( r \)-pointed \((Z, \delta)\)-spin curves over \( S \) and \( 2 \)-isomorphism classes of \((1-)\)-morphisms between them. Indeed, since all \( 2 \)-morphisms are invertible, [1] Lemma 4.2.3 implies that any \( 1 \)-morphism in that \( 2 \)-category does not have nontrivial automorphisms. Thus, for each pair of nonnegative integers \((g, r)\) with \(2g - 2 + r > 0\), we obtain the \((1-)\)-category

\[
\mathcal{G} \text{p}_{Z, \delta, g, r}.
\]

consisting of pointed stable \((Z, \delta)\)-spin curves of type \((g, r)\) over \( k \)-schemes. The assignment \( X := (\mathcal{X}, \mathcal{X}^{tw}, \gamma, \pi_G : \mathcal{G} \to \mathcal{X}) \mapsto \mathcal{X} \) determines a functor

\[
\overline{\pi}_{g, r}^\text{gp} : \mathcal{G} \text{p}_{Z, \delta, g, r} \to \overline{\mathbf{M}}_{g, r}.
\]

**Remark 1.3.3.** Let \( r \) be a positive integer and \( \mathcal{X} \) (resp., \( \mathcal{X}^{tw} \)) an \( r \)-pointed nodal curve (resp., an \( r \)-pointed twisted curve) over a \( k \)-scheme \( S \). In the subsequent discussion, we shall refer to each \( 2 \)-isomorphism class of a \( 1 \)-isomorphism of pointed \((Z, \delta)\)-spin curves inducing the identity morphism of \( \mathcal{X} \) (resp., \( \mathcal{X}^{tw} \)) as an \textbf{isomorphism of \((Z, \delta)\)-structures} on \( \mathcal{X} \) (resp., \( \mathcal{X}^{tw} \)). In particular, we obtain the groupoid of \((Z, \delta)\)-structures on \( \mathcal{X} \) (resp., \( \mathcal{X}^{tw} \)).

**Remark 1.3.4.** Certain special cases of \( "{(Z, \delta)}" \) may be immediately understood or found in the previous works, as explained as follows.

(i) First, let us consider the case where \((Z, \delta) = (\mu_2, \text{id}_{\mu_2})\). The notion of a pointed stable \((\mu_2, \text{id}_{\mu_2})\)-spin curve is evidently equivalent to the notion of a \textbf{twisted 2-spin curve} in the sense of [1] §1.4. In particular, \( \mathcal{G} \text{p}_{\mu_2, \text{id}_{\mu_2}, g, r} \) is equivalent to the category \( \mathcal{B}_{g,r}(\mathbb{G}_m, \omega_{\log}^{1/2}) \) described in \textit{loc. cit.} For a general \((Z, \delta)\), changing the structure group of the underlying bundles via \( \delta_{\mathbb{G}_m} \) gives an assignment from each twisted 2-spin curve (i.e., a pointed stable \((\mu_2, \text{id}_{\mu_2})\)-spin curve) to a pointed stable \((Z, \delta)\)-spin curve. This assignment determines a functor

\[
\mathcal{G} \text{p}_{\mu_2, \text{id}_{\mu_2}, g, r} \to \mathcal{G} \text{p}_{Z, \delta, g, r}.
\]
over $\overline{M}_{g,r}$. Hence, since $\mathcal{S}p_{\mu_2, id_{\mu_2 g,r}}$ is nonempty (cf. [12] Corollary 4.11 and Proposition 4.19), $\mathcal{S}p_{Z,\delta,g,r}$ turns out to be nonempty.

(ii) Next, assume that $Z = \{1\}$ (hence $\delta$ is identical to the zero map 0 and $\hat{Z}_\delta \cong \mathbb{G}_m$). Then, there exists exactly one ($\{1\}, 0$)-structure on each pointed stable curve $\mathcal{X}/S$ given by the $\mathbb{G}_m$-bundle corresponding to $\Omega_{\mathcal{X}/S}^{\text{log}} \cong \mathcal{O}_{\mathcal{X}}^{\text{log}}$. In particular, $\pi_{\text{Sp}}^{\text{Rep}}$ is an equivalence of categories.

(iii) More generally, let us consider the case where $Z$ is arbitrary but $\delta$ coincides with the zero map. Then, there exists a natural isomorphism $\hat{Z}_\delta \cong G^{\times}k$ and the surjection $\nu$ (cf. (30)) may be identified (relative to this isomorphism) with the first projection $G^{\times}k \rightarrow G^{\times}k$. It follows immediately that $\mathcal{S}p_{Z,\delta,g,r}$ is naturally isomorphic to the moduli stack $M_{g,r}(B\mathbb{G}_m)$ of twisted stable maps into $B\mathbb{G}_m$, studied by D. Abramovich, T. Graber, A. Vistoli et al. (cf. [2], [3], [1], and [20]). According to [1] Theorems 2.1.7 and 3.0.2, $\overline{M}_{g,r}(BG) \cong \mathcal{S}p_{Z,\delta,g,r}$ may be represented by a proper smooth Deligne-Mumford stack with projective coarse moduli space. By applying an argument similar to the argument in the proof of this result, one may obtain Theorem 1.3.5 below, which may be thought of as its generalization to the case of an arbitrary $(Z,\delta)$.

**Theorem 1.3.5.** Let $(g,r)$ be a pair of nonnegative integers with $2g-2+r > 0$. Then, $\mathcal{S}p_{Z,\delta,g,r}$ may be represented by a nonempty proper smooth Deligne-Mumford stack over $k$ admitting a projective coarse moduli space. The forgetting morphism $\pi_{\text{Sp}}^{\text{Rep}} : \mathcal{S}p_{Z,\delta,g,r} \rightarrow \overline{M}_{g,r}$ is finite and flat. Moreover, its restriction $\mathcal{S}p_{Z,\delta,g,r} \times _{\overline{M}_{g,r}} \overline{M}_{g,r} \rightarrow \overline{M}_{g,r}$ is étale.

**Proof.** Since the relative cotangent complex of $B\nu : B\hat{Z}_\delta \rightarrow B\mathbb{G}_m$ is verified to be trivial, the first and second assertions follow from the arguments in [1] §1.5, §2.1, and §2.2 (or the argument in the proof of [1] Theorem 3.0.2), where $B_{g,n}(\mathbb{G}_m, \omega_1^{1/r})$ and $\kappa_r : B\mathbb{G}_m \rightarrow B\mathbb{G}_m$ in loc. cit. are replaced with $\mathcal{S}p_{Z,\delta,g,r}$ and $B\nu : B\hat{Z}_\delta \rightarrow B\mathbb{G}_m$ respectively. (The nonemptiness follows from the discussion in Remark 1.3.3(i) above.) Moreover, the last assertion follows from [48, Theorem 1.8] and the italicized assertion described above, which implies that deformations and obstructions of a pointed stable $(Z,\delta)$-spin curve are identical to those of the underlying twistification.

1.4. Radii of extended spin structures. Given a Deligne-Mumford stack $\mathcal{X}$ of finite type over $k$, we have the stack of cyclotomic gerbes

$$\mathcal{T}_\mu(\mathcal{X})$$

in $\mathcal{X}$, as described in [3] Definition 3.3.6. By definition, $\mathcal{T}_\mu(\mathcal{X})$ is the disjoint union $\bigsqcup_{l \geq 1} \mathcal{T}_\mu_l(\mathcal{X})$, where $\mathcal{T}_\mu_l(\mathcal{X})$ (for each positive integer $l$) denotes the category fibered in groupoids over $\text{Sch}_k$ whose fiber over $S \in \text{Ob}(\text{Sch}_k)$ is the groupoid classifying pairs $(\mathcal{G}, \phi)$ consisting of a gerbe $\mathcal{G}$ over $S$ banded by $\mu_l$ (hence $|\mathcal{G}| \cong S$) and a representable morphism $\phi : \mathcal{G} \rightarrow \mathcal{X}$ over $k$.

Let us consider the stack of cyclotomic gerbes $\mathcal{T}_\mu(B\mathbb{Z})$ in the classifying stack $B\mathbb{Z}$. Denote by

$$\text{Inj}(\mu, Z)$$


the set of injective morphisms of groups $\mu_l \to Z$, where $l$ is some positive integer. It is a finite set because $Z$ is finite. Given each element $\kappa : \mu_l \to Z$ of $\text{Inj}(\mu, Z)$, we obtain an open and closed substack

$$T_{\mu}(BZ)_\kappa$$

of $T_{\mu}(BZ)$ classifying representable morphisms $\phi : G \to BZ$ which arise, étale locally on $|G|$, identified with the composite $|G| \times_k B\mu_l \to BZ$ of the second projection $|G| \times_k B\mu_l \to B\mu_l$ and the morphism $B\kappa : B\mu_l \to BZ$ arising from $\kappa$.

The stack $T_{\mu}(BZ)$ decomposes into the disjoint union

$$T_{\mu}(BZ) = \bigcup_{\kappa \in \text{Inj}(\mu, Z)} T_{\mu}(BZ)_\kappa.$$

It gives rise to a decomposition

$$T_{\mu}(BZ)^{\times r} = \bigcup_{(\kappa_i)_{i=1}^{r} \in \text{Inj}(\mu, Z)^{\times r}} \prod_{i=1}^{r} T_{\mu}(BZ)_{\kappa_i}$$

(where $(-)^{\times r}$ denotes the product of $r$ copies of $(-)$).

**Proposition 1.4.1.** For each element $\kappa : \mu_l \to Z$ of $\text{Inj}(\mu, Z)$, there exists a canonical isomorphism of $k$-stacks

$$T_{\mu}(BZ)_\kappa \iso BC\text{oker}(\kappa).$$

**Proof.** Denote by $I_{\mu_l}(BZ)$ the $k$-stack classifying representable morphisms from $B\mu_l$ to $BZ$. It follows from [3, Proposition 3.4.1] that $T_{\mu_l}(BZ)$ is canonically isomorphic to the rigidification (cf. [1, Definition 5.1.9]) of $I_{\mu_l}(BZ)$ along $\mu_l$. The representable morphisms $B\mu_l \to BZ$ inducing, via rigidification, a morphism in $T_{\mu_l}(BZ)_\kappa$ ($\leq T_{\mu_l}(BZ)$) form a substack of $I_{\mu_l}(BZ)$, which is isomorphic to $BZ$. Hence, $T_{\mu_l}(BZ)_\kappa$ is isomorphic to the rigidification of $BZ$ along $\text{Im}(\kappa)(\subseteq Z)$, namely, isomorphic to $BC\text{oker}(\kappa)$. \hfill \Box

Now, let $(g, r)$ be a pair of nonnegative integers with $2g - 2 + r > 0$, $r > 0$, and $X^{tw} := (\mathcal{X}, \{\sigma_{x,i} : \mathcal{G}_i \to \mathcal{X}\}_{i=1}^{r})$ an $r$-pointed twisted curve of genus $g$ over a $k$-scheme $S$. Also, let $\mathcal{G}$ be a $(Z, \delta)$-structure on $\mathcal{X}^{log}/S^{log}$. For each $i = 1, \ldots, r$, the restriction $\sigma_{x,i}^*(\mathcal{G})$ of $\mathcal{G}$ to $\mathcal{G}_i$ corresponds to a representable morphism $\mathcal{G}_i \to B\mathcal{Z}_\delta$. Notice that the composite $\mathcal{G}_i \to B\mathcal{G}_m$ of this morphism and $B\nu : B\mathcal{Z}_\delta \to B\mathcal{G}_m$ classifies the line bundle $\sigma_{x,i}^*(\Omega_{\mathcal{X}^{log}/S^{log}})$, which is canonically identified with the trivial line bundle $\mathcal{O}_{\mathcal{G}_i}$ via the residue map. Hence, the morphism $\mathcal{G}_i \to B\mathcal{Z}_\delta$ factors through $BZ \to B\mathcal{Z}_\delta$. If $\phi_{\mathcal{G},i} : \mathcal{G}_i \to BZ$ denotes the resulting morphism, then we obtain an object

$$\tilde{\kappa}_{\mathcal{G},i} := (\mathcal{G}_i, \phi_{\mathcal{G},i}) \in \text{Ob}(T_{\mu}(BZ)(S)).$$

One may find a unique element

$$\kappa_{\mathcal{G},i} \in \text{Inj}(\mu, Z)$$

such that $\tilde{\kappa}_{\mathcal{G},i}$ lies in $\text{Ob}(T_{\mu}(BZ)_{\kappa_{\mathcal{G},i}}(S))$.

**Definition 1.4.2.** We shall refer to $\kappa_{\mathcal{G},i}$ as the **radius** of the $(Z, \delta)$-structure $\mathcal{G}$ at the marked point $\sigma_{x,i}$. 
For \( r \)-pointed \((Z, \delta)\)-spin curves \( X_j := (\mathcal{X}_j, \mathcal{X}_j^{\text{tw}}, \gamma_j, \mathcal{G}_j) \) \((j \in \{1, 2\})\), each morphism \( X_1 \to X_2 \) induces, in a natural way, a morphism \( \kappa_{G, i} \to \kappa_{G, i} \) in \( \mathcal{T}_\mu(BZ) \). Hence, the assignment \( X \mapsto \kappa_{G, i} \) gives rise to a morphism of \( k \)-stacks

\[
(49) \quad \text{ev}_i^{\text{Sp}} : \mathcal{G}p_{Z, \delta, g, r} \to \mathcal{T}_\mu(BZ).
\]

Thus, we obtain a morphism

\[
(50) \quad \text{ev}^{\text{Sp}} := (\text{ev}_1^{\text{Sp}}, \ldots, \text{ev}_r^{\text{Sp}}) : \mathcal{G}p_{Z, \delta, g, r} \to \mathcal{T}_\mu(BZ)^{xr}.
\]

Each \( \kappa := (\kappa_i)_{i=1}^r \in \text{Inj}(\mu, Z)^{xr} \) determines an open and closed substack

\[
(51) \quad \mathcal{G}p_{Z, \delta, g, r, \kappa} := (\text{ev}^{\text{Sp}})^{-1}(\prod_{i=1}^r \mathcal{T}_\mu(BZ)_{\kappa_i})
\]

of \( \mathcal{G}p_{Z, \delta, g, r, k} \). That is to say, \( \mathcal{G}p_{Z, \delta, g, r, \kappa} \) classifies pointed stable \((Z, \delta)\)-spin curves of type \((g, r)\) whose \((Z, \delta)\)-structure is of radii \( \kappa \).

1.5. **Gluing \((Z, \delta)\)-structures.** In this subsection, we observe certain factorization properties of the moduli stack of pointed stable \((Z, \delta)\)-spin curves according to the gluing maps \( \Phi_{\text{tree}}, \Phi_{\text{loop}} \) (cf. \((13), (19)\)).

Let \( \mathcal{G} \) be a gerbe over a \( k \)-scheme \( S \) banded by \( \mu_l \) \((l > 0)\). One may change the banding of the gerbe through the inversion automorphism \( \zeta \mapsto \zeta^{-1} \) of \( \mu_l \). Denote by \( \mathcal{G}^\vee \) the resulting gerbe and by \( \text{inv}^{\mathcal{G}} : \mathcal{G} \to \mathcal{G}^\vee \) the isomorphism over \( S \) arising from the inversion. The assignment \( (\mathcal{G}, \phi) \mapsto (\mathcal{G}^\vee, \phi^\vee) \) of \( \phi^\vee := \phi \circ \text{inv}^{\mathcal{G}} \) induces an involution of \( \mathcal{T}_\mu(BZ) \). By applying this argument to each piece \( \mathcal{T}_\mu(BZ) \) \((l \geq 1)\) of \( \mathcal{T}_\mu(BZ) \) separately, we obtain an involution

\[
(52) \quad (-)^\vee : \mathcal{T}_\mu(BZ) \to \mathcal{T}_\mu(BZ)
\]

of \( \mathcal{T}_\mu(BZ) \). Each injective morphism \( \kappa : \mu_l \to Z \) in \( \text{Inj}(\mu, Z) \) induces an injection \( \kappa^\vee : \mu_l \to Z \) given by \( \zeta \mapsto \kappa(\zeta^{-1}) \). The involution \( (-)^\vee \) obtained above restricts to an isomorphism

\[
(53) \quad \mathcal{T}_\mu(BZ)_\kappa \cong \mathcal{T}_\mu(BZ)_{\kappa^\vee}
\]

Let \( S \) be as above. For each \( j = 1, 2 \), let \((g_j, r_j)\) be a pair of nonnegative integers with \( 2g_j + r_j > 0 \), \( \mathcal{X}_j \) an \((r_j + 1)\)-pointed stable curve of genus \( g_j \) over \( S \), and \((\mathcal{X}_j^{\text{tw}}, \gamma_j)\) a twistification of \( \mathcal{X}_j \). Denote by \( \mathcal{X}_{\text{tree}} \) the \((r_1 + r_2)\)-pointed stable curve of genus \((g_1 + g_2)\) obtained by attaching the respective last marked points of \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) to form a node. Suppose that we are given an isomorphism \( \epsilon : \mathcal{G}_{1, r_1+1} \to \mathcal{G}_{2, r_2+1} \) over \( S \) compatible with the bands. Then, \((\mathcal{X}_1^{\text{tw}}, \gamma_1)\) and \((\mathcal{X}_2^{\text{tw}}, \gamma_2)\) may be glued together, by means of \( \epsilon \) (cf. the balanced-ness described in the fourth condition in Definition \( \square \) (i)), to a twistification

\[
(53) \quad (\mathcal{X}_{\text{tree}}^{\text{tw}}, \gamma_{\text{tree}})
\]

(\( \mathcal{X}_{\text{tree}}^{\text{tw}} := (\mathcal{X}_{\text{tree}}, \{\sigma_{\text{tree}, i}\}_{i=1}^{r_1+r_2}) \)) of \( \mathcal{X}_{\text{tree}} \). In particular, we obtain closed immersions

\[
(54) \quad \text{imm}_j : \mathcal{X}_j \to \mathcal{X}_{\text{tree}}
\]
\[ \text{(j = 1, 2), which fit into the following 1-commutative diagram:} \]
\[ (55) \]
\[ \begin{array}{ccc}
\mathcal{G}_{1,r_1+1} & \xrightarrow{\epsilon} & \mathcal{G}_{2,r_2+1} \\
\text{imm}_1 \circ \sigma_{1,r_1+1} & \downarrow & \text{imm}_2 \circ \sigma_{2,r_2+1} \circ \text{inv}_{1}^{-1} \\
\mathfrak{X}_{\text{tree}} & & \\
\end{array} \]

It is immediately verified that the morphism
\[ \text{(56)} \]
\[ \text{imm}_j^* (\Omega_{\mathcal{X}_{\text{tree}}^{\log}/S^{\log}}) \to \Omega_{\mathcal{X}_j^{\log}/S^{\log}} \]
induced by \text{imm}_j is an isomorphism. That is to say, there exists a canonical 2-isomorphism
\[ \text{(57)} \]
\[ [\Omega_{\mathcal{X}_{\text{tree}}^{\log}/S^{\log}}] \circ \text{imm}_j \cong [\Omega_{\mathcal{X}_j^{\log}/S^{\log}}] \]
(cf. (32)). The following lemma may be immediately verified.

**Lemma 1.5.1.** Let us keep the above notation.

(i) Let \( \pi_G : \mathcal{G} \to \mathfrak{X}_{\text{tree}} \) be a \((Z, \delta)\)-structure on \( \mathfrak{X}_{\text{tree}}^{\log}/S^{\log} \). Then, for each \( j = 1, 2 \), the pull-back \( \pi_G \mid \mathcal{G}_j : \mathcal{G}_j \to \mathcal{X}_j \) of \( \mathcal{G} \) by \( \text{imm}_j \) forms, via (57), a \((Z, \delta)\)-structure on \( \mathfrak{X}_j^{\log}/S^{\log} \). Moreover, by restricting \( \mathcal{G}_j \) to \( \sigma_{j,r_1+1} \) and applying the commutativity of diagram (55), we obtain an isomorphism
\[ \text{(58)} \]
\[ \epsilon_\mathcal{G} := (\epsilon, \epsilon_\phi) : \kappa_{g,r_1+1} \left( = (\mathcal{G}_{1,r_1+1}, \phi_{g,r_1+1}) \right) \cong \kappa_{G_2,r_2+1} \left( = (\mathcal{G}_{2,r_2+1}, \phi_{G_2,r_2+1}) \right) \]
in \( \mathcal{I}_\mu(\mathcal{B}Z) \), where \( \epsilon \) is as in (55) and \( \epsilon_\phi \) denotes some 2-isomorphism \( \phi_{g,r_1+1} \cong \phi_{G_2,r_2+1} \circ \epsilon \).

(ii) Conversely, for each \( j \in \{1, 2\} \), let \( \pi_G : \mathcal{G}_j \to \mathcal{X}_j \) be a \((Z, \delta)\)-structure on \( \mathfrak{X}_j^{\log}/S^{\log} \).

Also, assume that we are given an isomorphism of the form \( (\epsilon, \epsilon_\phi) : \kappa_{g,r_1+1} \cong \kappa_{G_2,r_2+1} \).

Then, there exists a unique (up to isomorphism) \((Z, \delta)\)-structure \( \pi_{G_{\text{tree}}} : \mathcal{G}_{\text{tree}} \to \mathfrak{X}_{\text{tree}} \)
on \( \mathfrak{X}_{\text{tree}}^{\log}/S^{\log} \) together with isomorphisms of \((Z, \delta)\)-structures \( \text{imm}_j^*(\mathcal{G}_{\text{tree}}) \cong \mathcal{G}_j \) \( (j = 1, 2) \) via which the equality \( (\epsilon, \epsilon_\phi) = \epsilon_{G_{\text{tree}}} \) (cf. (58) above) holds.

By applying the above lemma, we obtain the following proposition.

**Proposition 1.5.2.** Let \( g_1, g_2, r_1, \) and \( r_2 \) be nonnegative integers with \( 2g_j - 1 + r_j > 0 \) \( (j = 1, 2) \). Also, let \( \kappa_1 \in \text{Inj}(\mu, Z)^{x_1} \) and \( \kappa_2 \in \text{Inj}(\mu, Z)^{x_2} \). We shall write \( g := g_1 + g_2, r := r_1 + r_2. \)

(i) Let \( \kappa : \mu_1 \to Z \) \((l \geq 1)\) be an element of \( \text{Inj}(\mu, Z) \). We shall write
\[ \text{(59)} \]
\[ \mathfrak{Sp}_{\text{tree}, \kappa} := \mathfrak{Sp}_{Z, \delta, g, r_1+1, (\kappa_1, \kappa)} \times_{\text{ev}_{r_1+1}} \mathfrak{I}_{\mu}(\mathcal{B}Z)_{\kappa, (-)} \times_{\text{ev}_{r_2+1}} \mathfrak{Sp}_{Z, \delta, g, r_2+1, (\kappa_2, \kappa^\vee)}. \]

Then, the assignment
\[ \text{(60)} \]
\[ ((\mathcal{X}_1, \mathcal{X}_1^{\text{tw}}, \gamma_1, \mathcal{G}_1), (\mathcal{X}_2, \mathcal{X}_2^{\text{tw}}, \gamma_2, \mathcal{G}_2)) \mapsto (\mathcal{X}_{\text{tree}}, \mathcal{X}_{\text{tree}}^{\text{tw}}, \gamma_{\text{tree}}, \mathcal{G}_{\text{tree}}) \]
resulting from Lemma 1.5.1 determines a morphism
\[ \text{(61)} \]
\[ \Phi_{\text{tree}, \kappa}^{\mathfrak{Sp}} : \mathfrak{Sp}_{\text{tree}, \kappa} \to \mathfrak{Sp}_{Z, \delta, g, r, (\kappa_1, \kappa_2)}. \]
which makes the following square diagram commute:

\[
\begin{array}{ccc}
\mathcal{M}_{g,1,r_1} \times \mathcal{M}_{g,2,r_2} & \xrightarrow{\Phi_{\text{tree},\kappa}} & \mathcal{M}_{Z,\delta,g,r,1}(\kappa_1,\kappa_2) \\
\mathcal{M}_{g,r+1} \times_k \mathcal{M}_{g,r+2} & \xrightarrow{\Phi_{\text{tree}}} & \mathcal{M}_{g,r}.
\end{array}
\]

Moreover, the ramification index of \( \pi_{g,r} \) at the image of \( \Phi_{\text{tree},\kappa} \) coincides with \( l \).

(ii) We shall write

\[
\mathcal{M}_{\text{tree}} := (\mathcal{M}_{g,1,r_1} \times_k \mathcal{M}_{g,2,r_2+1}) \times_{\Phi_{\text{tree}}} \mathcal{M}_{g,r} \times_{\pi_{g,r}} \mathcal{G}_{Z,\delta,\kappa}(\kappa_1,\kappa_2)
\]

and write \( \mathcal{M}_{\text{tree}}^{\text{red}} \) for the reduced stack associated to \( \mathcal{M}_{\text{tree}} \). Then, the morphism

\[
\prod_{\kappa \in \text{Inj}(\mu,Z)} \mathcal{G}_{\text{tree},\kappa} \to \mathcal{M}_{\text{tree}}^{\text{red}}
\]

induced by \( \prod_{\kappa \in \text{Inj}(\mu,Z)} \Phi_{\text{tree},\kappa} \) (because of the commutativity of (62)) is an isomorphism.

(iii) The following equality holds:

\[
\deg(\mathcal{G}_{Z,\delta,g,r,1}(\kappa_1,\kappa_2)/\mathcal{M}_{g,r}) = |Z| \cdot \sum_{\kappa \in \text{Inj}(\mu,Z)} \prod_{j=1}^2 \deg(\mathcal{G}_{Z,\delta,g,r,j+1,1}(\kappa_j,\kappa)/\mathcal{M}_{g,r,j+1}).
\]

Proof. First, let us prove assertion (i). The nontrivial portion is the last statement. Let \( q \) be a geometric point in \( \mathcal{G}_{Z,\delta,g,r,1}(\kappa_1,\kappa_2) \) determined by the image via \( \Phi_{\text{tree},\kappa} \) of an irreducible component of \( \mathcal{G}_{\text{tree},\kappa} \). Denote by \( \overline{7} \in \mathcal{G}_{g,r} \) the image of \( q \) via \( \pi_{g,r} \). Also, denote by \( \mathcal{X} \) the pointed stable curve classified by \( \overline{7} \) and by \( (\mathcal{X}^{tw},\gamma,G) \) the \((Z,\delta)\)-structure on \( \mathcal{X} \) classified by \( q \). Since the relative cotangent complex of \( B\mathcal{V} : B\mathcal{Z}_k \to B\mathcal{G}_m \) is trivial, the deformations of the \((Z,\delta)\)-structure \((\mathcal{X}^{tw},\gamma,G_{Z,\delta})\) are identical to those of the underlying twistorification \((\mathcal{X}^{tw},\gamma)\) (cf. the proof in Theorem 1.3.3). Let \( D := \text{Spec}(R) \) (for some \( k \)-algebra \( R \)) be the versal deformation space of \( \overline{7} \in \mathcal{M}_{g,r} \). Then, by [12, Theorem 4.5] (or the discussion in the proof of Theorem 4.4 in loc. cit.), the deformation space of the twistorification \((\mathcal{X}^{tw},\gamma)\) may be represented by the quotient stack \( \tilde{D} := [\text{Spec}(R[z]/(z^l-w))/\mu_l] \) over \( D \), where \( w \in R \) denotes a function defining \( \overline{7} \) and the \( \mu_l \)-action on \( \text{Spec}(R[z]/(z^l-w)) \) is given by \( (\zeta,z) \mapsto \zeta \cdot z \) (for each \( \zeta \in \mu_l \)). (To be precise, we apply the pointed curve version of the result in loc. cit., which may be formulated and proved similarly.) In particular, the ramification index of \( \tilde{D}/D \) at the point classifying \((\mathcal{X}^{tw},\gamma)\) coincides with \( l \). This completes the proof of assertion (i).

Next, assertion (ii) follows from the local description [22] of a twisted curve at a node. Indeed, if \( R \) is a reduced ring, then the étale stack over \( \text{Spec}(R[s_0,t_0]/(s_0t_0)) \) of the form [22] must satisfy the equality \( u_1 = 0 \). This stack may be obtained by gluing together the two stacks \([\text{Spec}(A[s_1]/(s_1^2-s_0))/\mu_l]\) and \([\text{Spec}(A[t_1]/(t_1^2-t_0))/\mu_l]\). This implies that the pointed \((Z,\delta)\)-spin curves classified by \( \mathcal{M}_{\text{tree}}^{\text{red}} \) come from \( \prod_{\kappa \in \text{Inj}(\mu,Z)} \mathcal{G}_{\text{tree},\kappa} \). Hence, (64) turns out to be an isomorphism.
Finally, let us consider assertion (iii). Observe that the following equalities hold:
\begin{equation}
\deg(\mathcal{G}_r^{\mu,Z}/(\mathbb{M}_{g_1,r_1+1} \times_k \mathbb{M}_{g_2,r_2+1})) = |\text{Coker}(\kappa)| \cdot \prod_{j=1}^2 \deg(\mathcal{G}_{\delta,g_j,r_j+1,\kappa}/\mathbb{M}_{g_j,r_j+1})
\end{equation}
\begin{equation}
= \frac{|Z|}{l} \cdot \prod_{j=1}^2 \deg(\mathcal{G}_{\delta,g_j,r_j+1,\kappa}/\mathbb{M}_{g_j,r_j+1}).
\end{equation}

Given an element \( \kappa: \mu_t \hookrightarrow Z \) of \( \text{Inj}(\mu,Z) \), we shall write \( \mathbb{M}_{\text{tree},\kappa} \) for the open and closed substack of \( \mathbb{M}_{\text{tree}} \) determined by the image via (64) of \( \mathcal{G}_{\text{tree},\kappa} \). Also, write \( \mathbb{M}_{\text{tree},\kappa}^{\text{red}} \) for the reduced stack associated to \( \mathbb{M}_{\text{tree},\kappa} \) (hence, \( \mathbb{M}_{\text{tree}}^{\text{red}} = \bigsqcup_{\kappa \in \text{Inj}(\mu,Z)} \mathbb{M}_{\text{tree},\kappa}^{\text{red}} \)). By the last assertion of (ii), we have
\begin{equation}
\deg(\mathbb{M}_{\text{tree},\kappa}/(\mathbb{M}_{g_1,r_1+1} \times_k \mathbb{M}_{g_2,r_2+1})) = l \cdot \deg(\mathbb{M}_{\text{tree},\kappa}^{\text{red}}/(\mathbb{M}_{g_1,r_1+1} \times_k \mathbb{M}_{g_2,r_2+1})).
\end{equation}
Thus, the following sequence of equalities holds:
\begin{equation}
\deg(\mathcal{G}_{\delta,g,r,(\kappa_1,\kappa_2)}/\mathbb{M}_{g,r}) = \deg(\mathbb{M}_{\text{tree}}/(\mathbb{M}_{g_1,r_1+1} \times_k \mathbb{M}_{g_2,r_2+1}))
\end{equation}
\begin{equation}
= \sum_{\kappa \in \text{Inj}(\mu,Z)} \deg(\mathbb{M}_{\text{tree},\kappa}/(\mathbb{M}_{g_1,r_1+1} \times_k \mathbb{M}_{g_2,r_2+1}))
\end{equation}
\begin{equation}
\overset{(66)}{=} \sum_{\kappa \in \text{Inj}(\mu,Z)} l \cdot \deg(\mathbb{M}_{\text{tree},\kappa}^{\text{red}}/(\mathbb{M}_{g_1,r_1+1} \times_k \mathbb{M}_{g_2,r_2+1}))
\end{equation}
\begin{equation}
\overset{(ii)}{=} \sum_{\kappa \in \text{Inj}(\mu,Z)} l \cdot \deg(\mathcal{G}_{\text{tree},\kappa}/(\mathbb{M}_{g_1,r_1+1} \times_k \mathbb{M}_{g_2,r_2+1}))
\end{equation}
\begin{equation}
= \frac{|Z|}{l} \cdot \prod_{\kappa \in \text{Inj}(\mu,Z)} \prod_{j=1}^2 \deg(\mathcal{G}_{\delta,g_j,r_j+1,\kappa}/\mathbb{M}_{g_j,r_j+1}),
\end{equation}
where the first equality follows from the fact that \( \pi_{g,r}^{\mathbb{G}} \) is finite and flat (cf. Theorem 1.3.5). This completes the proof of assertion (iii). \( \square \)

Next, let \((g,r)\) be a pair of nonnegative integers with \(2g+r > 0\), \( \mathcal{X} \) an \((r+2)\)-pointed stable curve of genus \(g\) over \(S\), and \((\mathcal{X}^{\text{tw}},\gamma)\) (where \( \mathcal{X}^{\text{tw}} := (\mathcal{X},\{\sigma_{x,i}: \mathcal{G}_i \to \mathcal{X}\}_{i=1}^{r+2})\)) a twifistation of \(\mathcal{X}\). Denote by \(\mathcal{X}_{\text{loop}}\) the \(r\)-pointed stable curve of genus \(g\) obtained by attaching the last two marked points of \(\mathcal{X}\) to form a node. Suppose that we are given an isomorphism \(\epsilon: \mathcal{G}_{r+1} \leftrightarrows \mathcal{G}_{r+2}^{\gamma}\) over \(S\) compatible with the bands. By attaching the last two marked points of \(\mathcal{X}_{\text{loop}}^{\text{tw}}\) along \(\epsilon\), we obtain a twifistation \((\mathcal{X}_{\text{loop}}^{\text{tw}},\gamma_{\text{loop}})\) of \(\mathcal{X}_{\text{loop}}\).

There exists a natural bijective correspondence \(\mathcal{G} \mapsto \mathcal{G}_{\text{loop}}\) between the set of isomorphism classes of \((Z,\delta)\)-structures \(\mathcal{G}\) on \(\mathcal{X}_{\text{loop}}^{\text{log}}/S_{\text{log}}\) admitting an isomorphism \(\bar{\alpha}_{g,r+1} \to \bar{\alpha}_{g,r+2}^{\gamma}\) compatible with \(\epsilon\) and the set of isomorphism classes of \((Z,\delta)\)-structures \(\mathcal{G}_{\text{loop}}\) on \(\mathcal{X}_{\text{loop}}^{\text{log}}/S_{\text{log}}\). Thus, we obtain the following propositions.
Proposition 1.5.3. Let \((g, r)\) be a pair nonnegative integers with \(2g + r > 0\), and let \(\bar{\kappa} \in \text{Inj}(\mu, Z)^{\times r}\).

(i) Let \(\kappa : \mu_1 \hookrightarrow Z\) \((l \geq 1)\) be an element of \(\text{Inj}(\mu, Z)\). We shall write

\[
\mathfrak{S} p_{\text{loop}, \kappa} := \text{Ker} \left( \mathfrak{S} p_{Z, \delta, g, r+2, (\bar{\kappa}, \kappa, \kappa^\vee)} \Rightarrow \overline{I}_\mu(\mathcal{B}Z)_\kappa \right). \tag{69}
\]

Then, the assignment

\[
(\mathcal{X}, \mathcal{X}^{\text{tw}}, \gamma, \mathcal{G}) \mapsto (\mathcal{X}_{\text{loop}}, \mathcal{X}^{\text{tw}}_{\text{loop}}, \gamma_{\text{loop}}, \mathcal{G}_{\text{loop}})
\]

resulting from the above discussion determines a morphism

\[
\Phi_{\text{loop}, \kappa}^{\mathfrak{S} p} : \mathfrak{S} p_{\text{loop}, \kappa} \rightarrow \mathfrak{S} p_{Z, \delta, g+1, r, \bar{\kappa}}, \tag{70}
\]

and the following square diagram is commutative:

\[
\begin{array}{ccc}
\mathfrak{S} p_{\text{loop}, \kappa} & \xrightarrow{\Phi_{\text{loop}, \kappa}^{\mathfrak{S} p}} & \mathfrak{S} p_{Z, \delta, g+1, r, \bar{\kappa}} \\
\mathfrak{M}_{g, r+2} & \xrightarrow{\Phi_{\text{loop}}} & \mathfrak{M}_{g+1, r}. \\
\end{array}
\]  

Moreover, the ramification index of \(\mathfrak{S} p_{Z, \delta, g+1, r, \bar{\kappa}}^{\mathfrak{S} p}\) at the image of \(\Phi_{\text{loop}, \kappa}^{\mathfrak{S} p}\) coincides with \(l\).

(ii) We shall write

\[
\mathfrak{W}_{\text{loop}} := \overline{\mathfrak{M}}_{g, r+2} \times_{\Phi_{\text{loop}} \mathfrak{M}_{g+1, r, \mathfrak{S} p_{Z, \delta, g, r+2, (\bar{\kappa}, \kappa, \kappa^\vee)}}} \mathfrak{S} p_{Z, \delta, g+1, r, \bar{\kappa}}, \tag{73}
\]

and write \(\mathfrak{W}_{\text{loop}}^{\text{red}}\) for the reduced stack associated to \(\mathfrak{W}_{\text{loop}}\). Then, the morphism

\[
\bigoplus_{\kappa \in \text{Inj}(\mu, Z)} \Phi_{\text{loop}, \kappa}^{\mathfrak{S} p} : \bigoplus_{\kappa \in \text{Inj}(\mu, Z)} \mathfrak{S} p_{\text{loop}, \kappa} \rightarrow \mathfrak{W}_{\text{loop}}^{\text{red}} \tag{74}
\]

induced by \(\bigoplus_{\kappa \in \text{Inj}(\mu, Z)} \Phi_{\text{loop}, \kappa}^{\mathfrak{S} p}\) (because of the commutativity of (72)) is an isomorphism.

(iii) The following equality holds:

\[
deg(\mathfrak{S} p_{Z, \delta, g+1, r, \bar{\kappa}} / \overline{\mathfrak{M}}_{g+1, r}) = |Z| \cdot \sum_{\kappa \in \text{Inj}(\mu, Z)} \deg(\mathfrak{S} p_{Z, \delta, g, r+2, (\bar{\kappa}, \kappa, \kappa^\vee)} / \overline{\mathfrak{M}}_{g, r+2}). \tag{75}
\]

Proof. The assertions follow from arguments similar to the arguments in the proof of Proposition 1.5.2.

\[\square\]

1.6. Forgetting tails. Next, we observe the factorization of \(\mathfrak{S} p_{Z, \delta, g, r, \bar{\kappa}}\) according to the forgetting-tails map \(\Phi_{\text{tail}}\) (cf. (20)). Let \(r\) be a nonnegative integer, \(S\) a \(k\)-scheme, and \(\mathcal{X} := (X, \{\sigma_{X,i}\}_{i=1}^{r+1})\) an \((r+1)\)-pointed smooth curve over \(S\). Write \(\mathcal{X}_{\text{tail}} := (X, \{\sigma_{X,i}\}_{i=1}^r)\), i.e., the \(r\)-pointed curve obtained from \(\mathcal{X}\) by forgetting the last marked point. Moreover, let \((\mathcal{X}^{\text{tw}}_{\text{tail}}, \gamma_{\text{tail}})\) (where \(\mathcal{X}^{\text{tw}}_{\text{tail}} := (X_{\text{tail}}, \{\sigma_{X_{\text{tail}},i}\}_{i=1}^r)\)) be a twistification of \(\mathcal{X}_{\text{tail}}\).

Lemma 1.6.1. There exists a unique (up to isomorphism) twistification \((\mathcal{X}^{\text{tw}}_{\text{tail}}; \gamma_{\text{tail}})\) of \(\mathcal{X}\) which is isomorphic to \((\mathcal{X}^{\text{tw}}_{\text{tail}}; \gamma_{\text{tail}})\) when restricted to \(X \setminus \text{Im}(\sigma_{X,r+1})\) and such that the stabilizer at any geometric point in \(\mathcal{X}^{\text{tw}}_{\text{tail}}(\mathcal{X}_{\text{tail}}; \{\sigma_{X_{\text{tail}},i}\}_{i=1}^r)\) has order 2.
Proof. Let us construct the desired twistification. Consider the category \( X[\text{Im}(\sigma_{X,r+1})/2] \) (cf. [12, Definition 2.2]) consisting of collections \((S,M,j,s)\), where

- \( S \) is an \( X \)-scheme \( S \to X \);
- \( M \) is a line bundle on \( S \);
- \( j \) is an isomorphism between \( M^{\otimes 2} \) and the pull-back of \( O_X(\text{Im}(\sigma_{X,r+1})) \) on \( S \);
- \( s \) is a global section of \( M \) such that \( j(s^{\otimes 2}) \) equals the tautological section of \( O_X(\text{Im}(\sigma_{X,r+1})) \) vanishing along \( \sigma_{X,r+1} \).

The morphisms in this category are defined in an obvious way. \( X[\text{Im}(\sigma_{X,r+1})/2] \) may be represented by a Deligne-Mumford stack over \( k \) with coarse moduli space \( X \). The projection \( \text{coa} : X[\text{Im}(\sigma_{X,r+1})/2] \to X \) (i.e., \( \text{coa} := \text{coa}_X[\text{Im}(\sigma_{X,r+1})/2] \)) is an isomorphism over \( X \setminus \text{Im}(\sigma_{X,r+1}) \). Also, \( X[\text{Im}(\sigma_{X,r+1})/2] \) is equipped with a tautological line bundle \( \mathcal{M} \) and an isomorphism \( \mathcal{M}^{\otimes 2} \cong \text{coa}^*(O_X(\text{Im}(\sigma_{X,r+1}))) \). Let us write \( \mathcal{X}_{+\delta} := \mathcal{X}_{\text{tail}} \times_X X[\text{Im}(\sigma_{X,r+1})/2] \) and write \( \xi : \mathcal{X}_{+\delta} \to \mathcal{X}_{\text{tail}} \) for the projection to the first factor. There exists a unique closed immersion \( \mathcal{X}_{+\delta} \times_{\mathcal{X}_{\text{tail}}} \mathcal{G}_{r+1} \to \mathcal{X}_{+\delta} \) (where \( \mathcal{G}_{r+1} \) is an \( \text{étale} \) gerbe over \( S \) banded by \( \mu_2 \)) inducing the closed immersion \( \sigma_{X,+\delta} \) between their respective coarse moduli spaces. Since \( \xi \) restricts to an isomorphism \( \mathcal{X}_{+\delta} \setminus \text{Im}(\sigma_{X,+\delta,r+1}) \cong \mathcal{X}_{\text{tail}} \setminus \text{Im}(\xi \circ \sigma_{X,+\delta,r+1}) \), each \( \sigma_{X,i} \) \((i = 1, \ldots, r)\) may be regarded as a marked point \( \sigma_{X,+\delta,i} \) of \( \mathcal{X}_{+\delta} \). By setting \( \mathcal{X}_{+\delta}^{\text{tw}} := (\mathcal{X}_{+\delta}, \{\sigma_{X,+\delta,i}\}_{i=1}^{r+1}) \) and \( \gamma_+ := \gamma_{\text{tail}} \circ \xi \), we obtain a pair \((\mathcal{X}_{+\delta}^{\text{tw}}, \gamma_+)\) forming the desired twistification of \( \mathcal{X} \).

Next, let us consider the uniqueness assertion. Let \((\mathcal{X}^{\text{tw}}, \gamma)\) (where \( \mathcal{X}^{\text{tw}} := (\mathcal{X}, \{\sigma_{X,i}\}_{i=1}^{r+1}) \)) be a twistification of \( \mathcal{X} \) satisfying the required properties. The line bundle \( O_X(\text{Im}(\sigma_{X,r+1}))^{\otimes 2} \) on \( \mathcal{X} \) descends to \( O_X(\text{Im}(\sigma_{X,r+1})) \). Hence, the line bundle \( O_X(\text{Im}(\sigma_{X,r+1}))^{\otimes 2} \) and its tautological section vanishing along \( \sigma_{X,r+1} \) induce a morphism \( \mathcal{X} \to X[\text{Im}(\sigma_{X,r+1})/2] \). This morphism extends naturally to a morphism \( \mathcal{X} \to \mathcal{X}_{+\delta}^{\text{tw}} \) \( (= \mathcal{X}_{\text{tail}} \times_X X[\text{Im}(\sigma_{X,r+1})/2]) \); it specifies, by construction, an isomorphism of twistifications \((\mathcal{X}^{\text{tw}}, \gamma) \cong (\mathcal{X}_{+\delta}^{\text{tw}}, \gamma_+) \). This completes the proof of the uniqueness assertion.

Next, let \( \mathcal{G}_{\text{tail}} \) be a \((Z, \delta)\)-structure on \( \mathcal{X}_{\text{tail}}^{\log} / S \). By using this, we shall construct a \((Z, \delta)\)-structure on \( \mathcal{X}_{+\delta}^{\log} / S \). Observe that

\[
\Omega_{\mathcal{X}_{+\delta}^{\log} / S} \cong \xi^*(\Omega_{\mathcal{X}_{\text{tail}}^{\log} / S^{\log}} \otimes (\text{coa}_{\mathcal{X}_{\text{tail}}} \circ \xi)^*(O_X(\text{Im}(\sigma_{X,r+1}))))
\]

\[
\cong \xi^*(\Omega_{\mathcal{X}_{\text{tail}}^{\log} / S} \otimes (\text{coa} \circ \text{coa})^*(O_X(\text{Im}(\sigma_{X,r+1}))))
\]

\[
\cong \xi^*(\Omega_{\mathcal{X}_{\text{tail}}^{\log} / S} \otimes \text{coa}^*(\mathcal{M})^{\otimes 2},
\]

where \( \text{coa} \) denotes the projection \( \mathcal{X}_{+\delta} \to X[\text{Im}(\sigma_{X,r+1})/2] \). Hence, by twisting \( \mathcal{G}_{\text{tail}} \) by the \( \mathcal{G}_m \)-bundle corresponding to \( \text{coa}^*(\mathcal{M}) \), we obtain a \( \mathcal{G}_\delta \)-bundle \( \mathcal{G}_{+\delta} \), which forms a \((Z, \delta)\)-structure on \( \mathcal{X}_{+\delta}^{\log} / S \). Since the nontrivial automorphism of \( \mathcal{G}_{r+1} \) over \( S \) arises from the automorphism of \( \mathcal{M} \) given by multiplication by \(-1\) \( \in \mu_2 \), the commutativity of the left-hand square in \([30]\) implies that the radius of \( \mathcal{G}_{+\delta} \) at \( \sigma_{X,+\delta,r+1} \) coincides with \( \delta : \mu_2 \hookrightarrow Z \subseteq \tilde{\mathbb{Z}}_\delta \). Thus, we have obtained a \((Z, \delta)\)-structure

\[
(\mathcal{X}_{+\delta}^{\text{tw}}, \gamma_+ ; \mathcal{G}_{+\delta})
\]

on \( \mathcal{X} \) with \( \kappa_{\mathcal{X}_{+\delta,r+1}} = \delta \).

Conversely, suppose that we are given a \((Z, \delta)\)-structure \((\mathcal{X}^{\text{tw}}, \gamma, \mathcal{G})\) on \( \mathcal{X} \) with \( \kappa_{\mathcal{X},r+1} = \delta \). One may find a unique (up to isomorphism) twistification \((\mathcal{X}_{-\delta}^{\text{tw}}, \gamma_- ; \mathcal{G}_{-\delta})\) (where \( \mathcal{X}_{-\delta}^{\text{tw}} := (\mathcal{X}_{-\delta}, \{\sigma_{X,-\delta,i}\}_{i=1}^{r+1}) \)) of \( \mathcal{X}_{\text{tail}} \) such that there exists an isomorphism \( \mathcal{X} \to \mathcal{X}_{-\delta} \times_X X[\text{Im}(\sigma_{X,r+1})/2] \)
whose composite with $\gamma_{-\delta} \times \cos : \mathcal{X}_{-\delta} \times_X X[\text{Im}(\sigma_{X,r+1})/2] \to X$ coincides with $\gamma$. If $G \cdot M$ denotes the twist of $1.7$.\(\text{ }(80)\)

$\tilde{\text{admitting an isomorphism}}\) Proposition 1.6.2.

(i) The assignments $(\mathcal{X}_{\text{tail}}, \gamma_{\text{tail}}, G) \mapsto (\mathcal{X}^\text{tw}_{\text{tail}}, \gamma^\text{tw}_{\text{tail}}, G^\text{tw}_{\text{tail}})$ and $(\mathcal{X}^\text{tw}, \gamma, G) \mapsto (\mathcal{X}^\text{tw}, \gamma^\text{tw}, G^\text{tw})$ constructed above determine an equivalence of categories between the groupoid of $(Z, \delta)$-structures on $\mathcal{X}_{\text{tail}}$ and the groupoid of $(Z, \delta)$-structures on $\mathcal{X}$ whose radius at $\sigma_{X,r+1}$ coincides with $\delta$.

(ii) Let $(g, r)$ be a pair of nonnegative integers with $2g - 1 + r > 0$, and let $\vec{v} \in \text{Inj}(\mu, Z)^{g,r}$. Then, the assignment

$$\text{(78)} \quad (\mathcal{X}, (\mathcal{X}_{\text{tail}}, \mathcal{X}^\text{tw}_{\text{tail}}, \gamma_{\text{tail}}, G)) \mapsto (\mathcal{X}, (\mathcal{X}^\text{tw}, \gamma, G^\text{tw}))$$

determines an isomorphism

$$\text{(79)} \quad \mathcal{M}_{g,r+1} \times_{\mathcal{M}_{g,r}} \mathcal{G}p_{Z,\delta,g,r,\vec{v}} \sim \mathcal{M}_{g,r+1} \times_{\mathcal{M}_{g,r+1}} \mathcal{G}p_{Z,\delta,g,r+1,\vec{v}}$$

over $\mathcal{M}_{g,r+1}$.

(iii) The following equality holds:

$$\text{(80)} \quad \text{deg}(\mathcal{G}p_{Z,\delta,g,r+1,\vec{v}}/\mathcal{M}_{g,r+1}) = \text{deg}(\mathcal{G}p_{Z,\delta,g,r,\vec{v}}/\mathcal{M}_{g,r}).$$

1.7. Automorphisms of $(Z, \delta)$-structures. In this subsection, suppose that $k$ is algebraically closed. Let $V$ be a finite set and $\{\mathcal{X}_v\}_{v \in V}$ a collection of pointed irreducible stable curves over $k$ indexed by $V$, where each $\mathcal{X}_v$ is $r_v$-pointed and of genus $g_v$ with $2g_v - 2 + r_v > 0$. Also, let $\mathcal{X} := (X, \{\sigma_{X_v}\}_{v=1}^r)$ be a pointed stable curve over $k$ obtained from the $\mathcal{X}_v$’s by attaching some of their marked points to form nodes (cf. [57 §7.3]). In particular, $V$ may be identified with the set of vertices of the dual graph $\Gamma$ of $\mathcal{X}$ (cf. [21 §1.5]). Denote by $E$ the set of edges of $\Gamma$. Given an element $e$ of $E$, we denote by $e_1, e_2$ the two half-edges belonging to $e$ (i.e., the branches of the node corresponding to $e$) and by $v_{e,j}$ (for each $j \in \{1, 2\}$) the element of $V$ at which $e_j$ is attached. Moreover, for any $e \in E$ and $j \in \{1, 2\}$, denote by $i_{e,j}$ the unique element of $\{1, \cdots, r_{v_{e,j}}\}$ such that the $i_{e,j}$-th marked point of $\mathcal{X}_{v_{e,j}}$ corresponds to $e_j$.

Suppose further that, for each $v \in V$, we are given a $(Z, \delta)$-structure $(\mathcal{X}_v^{\text{tw}}, \gamma_v, G_v)$ on $\mathcal{X}_v$ admitting an isomorphism $\tilde{\kappa}_{G_{v_{e,j}}^{e_1}} \sim \tilde{\kappa}_{G_{v_{e,j}}^{e_2}}$ for any $e \in E$. In particular, we have the equality

$$\text{(81)} \quad \tilde{\kappa}_e := \tilde{\kappa}_{G_{v_{e,j}}^{e_1}} \sim \tilde{\kappa}_{G_{v_{e,j}}^{e_2}} \in \text{Inj}(\mu, Z)$$

for every $e \in E$. Then, one may apply Lemma 1.5.1 (iii) successively to obtain a $(Z, \delta)$-structure $(\mathcal{X}^{\text{tw}}, \gamma, G)$ on $\mathcal{X}$. In particular, $(\mathcal{X}^{\text{tw}}, \gamma, G)$ becomes $(\mathcal{X}_v^{\text{tw}}, \gamma_v, G_v)$ ($v \in V$) after pull-back to $\mathcal{X}_v$. Let us write

$$\text{(82)} \quad \mathcal{X} := (\mathcal{X}, \mathcal{X}^{\text{tw}}, \gamma, G), \quad \mathcal{X}_v := (\mathcal{X}_v, \mathcal{X}_v^{\text{tw}}, \gamma_v, G_v)$$

($v \in V$). Then, we obtain the following proposition, which may be found in [21 Proposition 1.18] (and the comment following that proposition) when $(Z, \delta) = (\mu_2, \text{id}_{\mu_2})$. 


Proposition 1.7.1. Let us keep the above notation. Denote by Aut_{\mathcal{Y}}(X) (resp., Aut_{\mathcal{Y}_v}(X_v)) for each v \in V) the automorphism group of X (resp., X_v) inducing the identity of \mathcal{Y} (resp., \mathcal{Y}_v).

(i) For each v \in V, Aut_{\mathcal{Y}_v}(X_v) is canonically isomorphic to Z.

(ii) The group Aut_{\mathcal{Y}}(X) fits into the following exact sequence:

\begin{equation}
1 \to \text{Aut}_{\mathcal{Y}}(X) \xrightarrow{\alpha} \prod_{v \in V} \text{Aut}_{\mathcal{Y}_v}(X_v) \xrightarrow{\beta} \prod_{e \in E_{nl}} \text{Coker}(\kappa_e),
\end{equation}

where

- \alpha denotes the morphism induced by pull-back to \mathcal{Y}_v (v \in V);
- E_{nl} denotes the set of edges which do not start and end at the same vertex (i.e., non-loops);
- \beta maps each element (\zeta_v)_{v \in V} \in \prod_{v \in V} \text{Aut}_{\mathcal{Y}_v}(X_v) (= \prod_{v \in V} Z) to (\zeta_{v,1} \zeta_{v,2}^{-1} \mod \text{Im}(\kappa_e))_{e \in E_{nl}}.

Proof. Assertion (i) follows from the fact that any element of Aut_{\mathcal{Y}_v}(X_v) may be expressed uniquely as the automorphism given by translation by an element of Z. Assertion (ii) follows immediately from Proposition 1.4.1 and Lemma 1.5.1. \qed

1.8. \((Z, \delta)-\text{structures on the 2-pointed projective line.}

Let S be a k-scheme, and let \mathbb{P}_S := \text{Proj}(\mathcal{O}_S[x, y]) denote the projective line over S, i.e., the moduli space classifying ratios [x : y] (for x, y \in \mathcal{O}_S with (x, y) \neq (0, 0)). Denote by \sigma_{\mathbb{P}, 1} and \sigma_{\mathbb{P}, 2} the marked points of \mathbb{P}_S determined by the values 0 (= 0 : 1) and \infty (= 1 : 0) respectively. In particular, we have a 2-pointed curve \mathcal{P}_S := (\mathbb{P}_S, \{\sigma_{\mathbb{P}, 1}, \sigma_{\mathbb{P}, 2}\}) over S. The S-scheme \mathbb{P}_S has two open subschemes U_1 := \mathbb{P}_S \setminus \text{Im}(\sigma_{\mathbb{P}, 2}) = \text{Spec}(\mathcal{O}_S[s_1]) (where s_1 := x/y) and U_2 := \mathbb{P}_S \setminus \text{Im}(\sigma_{\mathbb{P}, 1}) = \text{Spec}(\mathcal{O}_S[s_2]) (where s_2 := y/x).

Proposition 1.8.1. (i) Let (\mathcal{P}^{\text{tw}}, \gamma, \mathcal{G}) be a \((Z, \delta)-\text{structure on } \mathcal{P}_S. Then, the equality \kappa_{\mathcal{G}, 1} = \kappa_{\mathcal{G}, 2}^\vee \in \text{Inj}(\mu, Z) holds.

(ii) For each \kappa \in \text{Inj}(\mu, Z), there exists a unique (up to isomorphism) 2-pointed \((Z, \delta)-\text{spin curve } (\mathcal{P}^{\text{tw}}, \gamma, \mathcal{G}) \text{ over } S \text{ whose underlying pointed curve is } \mathcal{P}_S^l \text{ and which satisfies the equalities } \kappa = \kappa_{\mathcal{G}, 1} = \kappa_{\mathcal{G}, 2}^\vee.

Proof. Since assertions (i) and (ii) are of local nature with respect to the \(\math{\acute{e}}\text{tale} topology of S, one may assume that \mu_l is contained in \Gamma(S, \mathcal{O}_S) for any l dividing |Z|.

Before proceeding, let us introduce some notation. Given a k-scheme V, we denote by Z_{V}^{\text{triv}} the trivial Z-bundle on V. For each j \in \{1, 2\} and a positive integer l, we shall define \mathcal{U}_{j,l} to be the \mathbb{U}_j \text{-scheme } Spec(\mathcal{O}_S[s_j^{1/l}]), equipped with the } \mu_l \text{-action given by } s_j^{1/l} \mapsto \zeta s_j^{1/l} (\zeta \in \mu_l); given each \zeta \in \mu_l, we denote the corresponding automorphism of \mathcal{U}_{j,l} by \alpha_{\zeta, j, l}. Denote by \pi_{j,l} : \mathcal{U}_{j,l} \to [\mu_{j,l}] \text{ the natural projection. Finally, given } \xi \in Z, \text{ we denote by } \beta_\xi : Z \to Z \text{ the translation by } \xi.

Now, we shall prove assertion (i). To this end, one may assume, without loss of generality, that S = Spec(k) and k is algebraically closed. Let (\mathcal{P}^{\text{tw}}, \gamma, \mathcal{G}) \text{ (where } \mathcal{P}^{\text{tw}} := (\mathcal{P}, \{\sigma_{\mathcal{P}, 1}, \sigma_{\mathcal{P}, 2}\})\text{) be a } (Z, \delta)-\text{structure on } \mathcal{P}_k := \mathcal{P}_S. Since

\begin{equation}
\Omega_{\mathcal{P}^{\text{log}, k}} \cong \gamma^*(\Omega_{\mathcal{P}^{\text{log}, k}}) \cong \gamma^*(\mathcal{O}_{\mathcal{P}_k}) \cong \mathcal{O}_{\mathcal{P}},
\end{equation}

\begin{itemize}
\item[(ii)] The group Aut_{\mathcal{Y}}(X) fits into the following exact sequence:
\end{itemize}
the classifying morphism $\Psi \to BZ_\delta$ of $G$ factors through $BZ \to B\hat{Z}_\delta$. That is to say, $G$ may be regarded as a $Z$-bundle on $\Psi$. Denote by $l_j$ (for each $j = 1, 2$) the order of the stabilizer in $\Psi$ at $\sigma_{\Psi,j}$. Then, there exists an isomorphism $\iota_{l_j,j} : [U_{j,l_j}/\mu_{l_j}] \cong \Psi \setminus \text{Im}({\sigma_{\Psi,j}})$ of stacks over $U_j$. Since we have assumed that $Z$ has order invertible in $k$ and $k$ is algebraically closed, $(\iota_{l_j,j} \circ \pi_{j,l_j})^*(G)$ turns out to be trivial. Let us identify $(\iota_{l_j,j} \circ \pi_{j,l_j})^*(G)$ with $Z^{\text{triv}}_{l_j,j}$ by a fixed isomorphism. For each $\zeta \in \mu_{l_j}$, each $\alpha$ induces the surjection given by $\zeta$ of $\kappa$ and $\alpha$ of $\zeta$ denotes the surjection given by $\zeta$ of $\kappa$ of $\alpha$ of $\zeta$. Since we have assumed that $k \in \mu_{l_j}$ and $\beta_{l_j,j}$, consider the trivial $Z^{\text{triv}}$ of $G$ whose restriction to $l_j \in \mu_{l_j}$ and $\beta_{l_j,j}$ is $Z^{\text{triv}}$ of $G$ over $U_{l_j,j}$. It follows from the definition of $\kappa_{G,j}$ that this automorphism may be expressed as $\alpha_{l_j,j} \times \beta_{l_j,j}$ under the identification $(\iota_{l_j,j} \circ \pi_{j,l_j})^*(G) = Z^{\text{triv}}_{U_{l_j,j}} (= U_{l_j,j} \times_k Z)$.

Next, let us consider the commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{v_1} & U_{1,l_1} \\
\downarrow & & \downarrow \\
U_{2,l_2} & \xleftarrow{v_2} & \Psi,
\end{array}
$$

where $U := \text{Spec}(k[s_1^{1/L}]) = \text{Spec}(k[s_2^{1/L}])$ ($L := \text{lcm}(l_1, l_2)$, $s_1^{1/L} = s_2^{1/L}$). If $\tau_j : \mu_L \to \mu_{l_j}$ denotes the surjection given by $\zeta \mapsto \zeta^{l_j}$, then, for each $\zeta \in \mu_L$, the following equality of automorphisms of $Z^{\text{triv}}_U (= v_j^*(Z^{\text{triv}}_{U_{l_j,j}}))$ holds:

$$
v_j^*(\alpha_{l_j,j} \circ \pi_{j,l_j}) = (\alpha_{l_j,j})_{l_j}(U) \times (\beta_{l_j,j \circ \tau_j})_{l_j}(U).
$$

The commutativity of (85) gives rise to an isomorphism

$$
(Z^{\text{triv}}_U =) v_j^*(Z^{\text{triv}}_{U_{l_j,j}}) \cong v_j^*(Z^{\text{triv}}_{U_{l_j,j}}) (= Z^{\text{triv}}_U)
$$

over $U$ (given by $\beta_\zeta$ for some $\zeta \in Z$). On the other hand, the equality $s_1^{1/L} = s_2^{1/L}$ implies $\alpha_{l_j,j}(U) = \alpha_{l_j,j}(U)$. By passing to (87) and taking account of (86), we obtain the equalities

$$
\beta_{l_j,j \circ \tau_1}(U) = \left(\beta_{l_j,j \circ \tau_2}(U) \circ \beta_\zeta \right) = \left(\beta_{l_j,j \circ \tau_2 \circ \tau_1}(U) \circ \beta_\zeta \right) = \beta_{l_j,j \circ \tau_1}(U) \circ \beta_\zeta = \beta_{l_j,j \circ \tau_1}(U) \circ \beta_\zeta = \beta_{l_j,j \circ \tau_1}(U) \circ \beta_\zeta
$$

(88)

for all $\zeta \in \mu_L$. Thus, the equality $\kappa_{G,l_j} \circ \tau_1 = (\kappa_{G,l_j} \circ \tau_2)^\vee$ of morphisms $\mu_L \to Z$ holds. Since both $\kappa_{G,l_j}$ and $\kappa_{G,l_j}$ are surjective, we have $l_1 = l_2$ and hence, $\kappa_{G,l_1} = \kappa_{G,l_2}$. This completes the proof of assertion (i).

Next, let us consider assertion (ii). Let $\kappa : \mu_1 \to Z$ be an element of $\text{Inj}(\mu, Z)$. Denote by $\tilde{\kappa} : S \times_k B\mu_1 \to BZ$ the object of $\mathcal{F}_\mu(BZ)_\kappa$ defined as the composite of the second projection $S \times_k B\mu_1 \to B\mu_1$ and $B\kappa : B\mu_1 \to BZ$. Since

$$
[U_{1,l_1}/\mu_1] \times_{FS} (U_1 \cap U_2) = U_1 \cap U_2 \cong [U_{2,l_1}/\mu_1] \times_{FS} (U_1 \cap U_2),
$$

the two stacks $[U_{1,l_1}/\mu_1]$ and $[U_{2,l_1}/\mu_1]$ may be glued together to obtain a twisted curve $\Psi_\kappa$ over $S$ equipped with two marked points $\sigma_{\Psi,1}$, $\sigma_{\Psi,2}$ over $\sigma_{\Psi,1}$, $\sigma_{\Psi,2}$ respectively. Denote by $\mathcal{P}^{\text{triv}}_\kappa := (\Psi_\kappa, (\sigma_{\Psi,1}, \sigma_{\Psi,2}))$ the resulting pointed twisted curve and by $\gamma_\kappa : \Psi_\kappa \to \Psi_S$ the natural projection, i.e., $\gamma_\kappa := \text{coa}_{\Psi_\kappa}$. One may find a unique (up to isomorphism) $Z (= \tilde{Z}_\delta)$-bundle on $\Psi_\kappa$ whose restriction to $\sigma_{\Psi,1}$ and $\sigma_{\Psi,2}$ are classified by $\tilde{\kappa}$ and $\tilde{\kappa}^\vee$ respectively. Indeed, for each $j = 1, 2$, consider the trivial $Z$-bundle $Z^{\text{triv}}_{U_{l_j,j}}$ on $U_{l_j,j}$. Let us equip $Z^{\text{triv}}_{U_{l_1,j}}$ (resp., $Z^{\text{triv}}_{U_{l_2,j}}$) with
the $\mu_t$-action given by $\alpha_{1,t,\zeta} \times \beta_{\kappa(\zeta)}$ (resp., $\alpha_{2,t,\zeta}^{-1} \times \beta_{\kappa(\zeta)}$) for each $\zeta \in \mu_t$. These $\mu_t$-actions on $Z^{\text{triv}}_{U,1,t}$ and $Z^{\text{triv}}_{U,2,t}$ are compatible when restricted to $U := U_{1,t} \times_{\text{PS}} U_{2,t}$. By means of these actions and the identity morphism $(Z^{\text{triv}}_U = Z^{\text{triv}}_{U,1,t}|_U \simeq Z^{\text{triv}}_{U,2,t}|_U (= Z^{\text{triv}}_U)$ of $Z^{\text{triv}}_U$, the bundles $Z^{\text{triv}}_{U,1,t}$ and $Z^{\text{triv}}_{U,2,t}$ may be glued together to obtain the desired $Z$-bundle $\mathcal{G}_{\tilde{\kappa}}$. The injectivity of $\kappa$ implies that the classifying morphism $\mathcal{P}_{\tilde{\kappa}} \rightarrow \mathcal{B}Z$ of $\mathcal{G}_{\tilde{\kappa}}$ is representable. Thus, we obtain a $(Z,\delta)$-structure $(\mathcal{P}_{\tilde{\kappa}}, \gamma_{\tilde{\kappa}}, \mathcal{G}_{\tilde{\kappa}})$ on $\mathcal{P}_S$ satisfying the required conditions. This implies the validity of the existence portion. The uniqueness portion follows immediately from the above construction of $(\mathcal{P}_{\tilde{\kappa}}^{\text{tw}}, \gamma_{\tilde{\kappa}}, \mathcal{G}_{\tilde{\kappa}})$. This completes the proof of assertion (ii).

\[\Box\]

**Corollary 1.8.2.** We have

\[
\mathcal{G}_{\mathcal{P}Z,\delta,0,3,(\kappa_1,\kappa_2,\delta)} \simeq \begin{cases} 
\mathcal{B}Z & \text{if } \kappa_1 = \kappa_2; \\
\emptyset & \text{if otherwise}.
\end{cases}
\]

**Proof.** The assertion follows from Propositions 1.6.2 (i), 1.7.1 (i), and 1.8.1. \[\Box\]

## 2. Twisted opers on pointed stable curves

In this section, we define the notion of a \textit{(faithful) twisted $G$-oper} on a pointed stable curve. We construct the moduli space, denoted by $\mathcal{D}p_{G,G,r,\rho,\mathfrak{g}}$, classifying pointed stable curves together with a faithful twisted $G$-oper (of prescribed radii $\rho$). The main result of this section asserts (cf. Theorems 2.3.5 and 2.4.2) that this moduli space may be represented by a smooth Deligne-Mumford stack which is flat over $\mathcal{M}_{g,r}$ of constant relative dimension. Also, we study the faithful twisted $G$-opers on the projective line with two or three marked points (cf. Proposition 2.6.1 and Corollary 2.6.2).

### 2.1. Algebraic groups and Lie algebras

First, we introduce some notation concerning algebraic groups and Lie algebras (cf. [57, §§ 2.1-2.2]). Let $(G, T)$ be a split semisimple algebraic group over $k$, where $T$ denotes a maximal torus $T$ of $G$. In this section, we shall assume that $\text{char}(k) = 0$ or $\text{char}(k) = p$ for some prime $p$ satisfying the condition $(*)_G$ described in Introduction. Let us fix a Borel subgroup $B$ defined over $k$ containing $T$. In particular, we obtain a natural surjection $B \twoheadrightarrow (B/[B, B] \cong T)$. Denote by $\mathfrak{g}$, $\mathfrak{t}$, and $\mathfrak{b}$ the Lie algebras of $G$, $T$, and $B$ respectively (hence, $\mathfrak{t} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$).

For each character $\beta$ of $T$, we write

\[g^\beta := \{ x \in \mathfrak{g} \mid \text{ad}(t)(x) = \beta(t) \cdot x \text{ for all } t \in T \}. \tag{91}\]

Let $\Gamma$ denote the set of simple roots in $B$ with respect to $T$. For each $\alpha \in \Gamma$, we fix a generator $x_\alpha$ of $g^\alpha$. Write $p_\Gamma := \sum_{\alpha \in \Gamma} x_\alpha \ (\in \mathfrak{g})$ and $\hat{\rho} := \sum_{\alpha \in \Gamma} \hat{\omega}_\alpha \ (\in \mathfrak{t})$, where $\hat{\omega}_\alpha$ (for each $\alpha \in \Gamma$) denotes the fundamental coweight of $\alpha$, regarded as an element of $\mathfrak{t}$ via differentiation. There exists a unique collection $(y_\alpha)_{\alpha \in \Gamma}$, where $y_\alpha$ is a generator of $\mathfrak{g}^{-\alpha}$, such that if we write $p_{-1} := \sum_{\alpha \in \Gamma} y_\alpha$, then the set \{ $p_{-1}, 2\hat{\rho}, p_\Gamma$ \} forms an $\mathfrak{sl}_2$-triple.

Finally, recall a canonical decreasing filtration \{ $\mathfrak{g}^j_{j \in \mathbb{Z}}$ \} on $\mathfrak{g}$ such that $\mathfrak{g}^0 = \mathfrak{b}$, $\mathfrak{g}^0 / \mathfrak{g}^1 = \bigoplus_{\alpha \in \Gamma} g^\alpha$, and $[g^{j_1}, g^{j_2}] \subseteq g^{j_1+j_2}$ for $j_1, j_2 \in \mathbb{Z}$. 


2.2. **Twisted $G$-opers on stacky log curves.** Let $S^{\text{log}}$ be an $\text{fs}$ log scheme (or, more generally, an $\text{fs}$ log stack) over $k$, $\mathcal{U}^{\text{log}}$ a stacky log curve over $S^{\text{log}}$, and $\pi : \mathcal{E} \to \mathcal{U}$ a right $G$-bundle on $\mathcal{U}$. By pulling-back the log structure of $\mathcal{U}^{\text{log}}$ via $\pi$, one may obtain a log structure on $\mathcal{E}$; we denote the resulting log stack by $\mathcal{E}^{\text{log}}$. The $G$-action on $\mathcal{E}$ carries a $G$-action on the direct image $\pi_*(T_{\mathcal{E}^{\text{log}}/S^{\text{log}}})$ of $T_{\mathcal{E}^{\text{log}}/S^{\text{log}}}$. Denote by $\widetilde{T}_{\mathcal{E}^{\text{log}}/S^{\text{log}}}$ the subsheaf of $G$-invariant sections of $\pi_*(T_{\mathcal{E}^{\text{log}}/S^{\text{log}}})$. The differential of $\pi$ gives rises to a short exact sequence

$$0 \to \mathfrak{g}_{\mathcal{E}} \to \widetilde{T}_{\mathcal{E}^{\text{log}}/S^{\text{log}}} \xrightarrow{d_{\mathfrak{E}}^{\text{log}}} \mathcal{T}_{\mathcal{U}^{\text{log}}/S^{\text{log}}} \to 0$$

of $\mathcal{O}_{\mathcal{U}}$-modules.

**Definition 2.2.1.** An $S^{\text{log}}$-connection on $\mathcal{E}$ is an $\mathcal{O}_{\mathcal{U}}$-linear morphism $\nabla : \mathcal{T}_{\mathcal{U}^{\text{log}}/S^{\text{log}}} \to \widetilde{T}_{\mathcal{E}^{\text{log}}/S^{\text{log}}}$ with $d_{\mathfrak{E}}^{\text{log}} \circ \nabla = \text{id}_{\mathcal{T}_{\mathcal{U}^{\text{log}}/S^{\text{log}}}}$.

Now, suppose that we are given a right $B$-bundle $\pi_B : \mathcal{E}_B \to \mathcal{U}$ on $\mathcal{U}$. Denote by $\pi_G : (\mathcal{E}_B \times^B G =: \mathcal{E}_G) \to \mathcal{U}$ the $G$-bundle on $\mathcal{U}$ obtained by change of structure group via the inclusion $B \hookrightarrow G$. The natural morphism $\mathcal{E}_B \to \mathcal{E}_G$ yields a canonical isomorphism $\mathfrak{g}_{\mathcal{E}_B} \simeq \mathfrak{g}_{\mathcal{E}_G}$ and moreover a morphism between short exact sequences:

$$0 \longrightarrow \mathfrak{b}_{\mathcal{E}_B} \longrightarrow \widetilde{T}_{\mathcal{E}_B^{\text{log}}/S^{\text{log}}} \xrightarrow{d_{\mathfrak{E}_B}^{\text{log}}} \mathcal{T}_{\mathcal{U}^{\text{log}}/S^{\text{log}}} \longrightarrow 0$$

and

$$0 \longrightarrow \mathfrak{g}_{\mathcal{E}_G} \longrightarrow \widetilde{T}_{\mathcal{E}_G^{\text{log}}/S^{\text{log}}} \xrightarrow{d_{\mathfrak{E}_G}^{\text{log}}} \mathcal{T}_{\mathcal{U}^{\text{log}}/S^{\text{log}}} \longrightarrow 0,$$

where the upper and lower horizontal sequences are (92) applied to $\mathcal{E}_B$ and $\mathcal{E}_G$ respectively. Since $\mathfrak{g}^j (\subseteq \mathfrak{g})$ is closed under the adjoint action of $B$, one obtains vector bundles $\mathfrak{g}_{\mathcal{E}_B}^j (j \in \mathbb{Z})$ associated with $\mathcal{E}_B \times^B \mathfrak{g}^j$. The collection $\{\mathfrak{g}_{\mathcal{E}_B}^j\}_{j \leq 0}$ defines a decreasing filtration on $\mathfrak{g}_{\mathcal{E}_B} (\simeq \mathfrak{g}_{\mathcal{E}_G})$. On the other hand, diagram (93) induces a composite isomorphism

$$\mathfrak{g}_{\mathcal{E}_B} / \mathfrak{g}_{\mathcal{E}_B}^0 \simeq \mathfrak{g}_{\mathcal{E}_G} / \mathfrak{g}_{\mathcal{E}_G}^0 \xrightarrow{t_{\mathfrak{g}/b}} \mathfrak{g}_{\mathcal{E}_B} / \mathfrak{g}_{\mathcal{E}_B}^0 \xrightarrow{t_{\mathfrak{g}/b}} \widetilde{T}_{\mathcal{E}_G^{\text{log}}/S^{\text{log}}} / t_{\mathfrak{g}/b}(\mathfrak{g}_{\mathcal{E}_B} / \mathfrak{g}_{\mathcal{E}_B}^0).$$

The filtration $\{\mathfrak{g}_{\mathcal{E}_B}^j\}_{j \leq 0}$ carries, via this composite isomorphism, a decreasing filtration $\{\widetilde{T}_{\mathcal{E}_G^{\text{log}}/S^{\text{log}}}^j\}_{j \leq 0}$ on $\widetilde{T}_{\mathcal{E}_G^{\text{log}}/S^{\text{log}}}$ in such a way that $\widetilde{T}_{\mathcal{E}_G^{\text{log}}/S^{\text{log}}}^j = t_{\mathfrak{g}/b}(\mathfrak{g}_{\mathcal{E}_B} / \mathfrak{g}_{\mathcal{E}_B}^0)$ and the resulting morphism $\mathfrak{g}_{\mathcal{E}_B}^j / \mathfrak{g}_{\mathcal{E}_B}^{j-1} \to \widetilde{T}_{\mathcal{E}_G^{\text{log}}/S^{\text{log}}}^j / \mathfrak{g}_{\mathcal{E}_B}^j$ is an isomorphism. Since each $\mathfrak{g}^{-\alpha} (\alpha \in \Gamma)$ is closed under the $B$-action defined by the composite $B \to T \overset{\text{adj. rep.}}{\longrightarrow} \text{Aut}(\mathfrak{g}^{-\alpha})$, the canonical decomposition $\mathfrak{g}^{-1} / \mathfrak{g}^0 = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}^{-\alpha}$ gives rise to a decomposition

$$\widetilde{T}_{\mathcal{E}_G^{\text{log}}/S^{\text{log}}}^{-1} / \mathfrak{g}_{\mathcal{E}_B}^0 = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_{\mathcal{E}_B}^{-\alpha}.$$

**Definition 2.2.2.**

(i) A **twisted $G$-oper** on $\mathcal{U}^{\text{log}}/S^{\text{log}}$ is a pair

$$\mathfrak{E}^{\bullet} := (\pi_B : \mathcal{E}_B \to \mathcal{U}, \nabla : \mathcal{T}_{\mathcal{U}^{\text{log}}/S^{\text{log}}} \to \widetilde{T}_{\mathcal{E}_B^{\text{log}}/S^{\text{log}}})$$

consisting of a $B$-bundle $\mathcal{E}_B$ on $\mathcal{U}$ and an $S^{\text{log}}$-connection $\nabla$ on the $G$-bundle $\pi_G : \mathcal{E}_G \to \mathcal{U}$ induced by $\mathcal{E}_B$ satisfying the following two conditions:
\( \nabla (T_{\log} / S_{\log}) \subseteq \tilde{T}^{-1}_{G_{\log} / S_{\log}} \)

- For any \( \alpha \in \Gamma \), the composite

\[
\xymatrix{T_{\log} / S_{\log} \ar[r]^\nabla & \tilde{T}^{-1}_{G_{\log} / S_{\log}} \ar[r] & \tilde{T}^{-1}_{E_{2,0} / S_{\log}} / \tilde{T}^0_{G_{\log} / S_{\log}} \ar[r] & g_{E_B}^\alpha}
\]

is an isomorphism, where the third arrow denotes the natural projection with respect to decomposition \( \ref{decomposition} \).

If \( U_{\log} / S_{\log} = X_{\log} / S_{\log} \) for a pointed twisted curve \( X_{\log} := (X / S, \{\sigma_{x,i}\}_i) \), then we shall refer to any twisted \( G \)-oper on \( X_{\log} / S_{\log} \) as a twisted \( G \)-oper on \( X_{\log} \).

If \( X \) is a pointed nodal curve, then we define a twisted \( G \)-oper on \( X \) to be a collection of data

\[
E^\bullet := (X_{\log} \times G, \gamma, E^\bullet),
\]

consisting of a twistification \( (X_{\log}, \gamma) \) of \( X \) and a twisted \( G \)-oper \( E^\bullet \) on \( X_{\log} \).

(ii) For each \( j \in \{1, 2\} \), let \( S_j \) be a \( k \)-scheme and \( X^\bullet_j := (X_j, X_{\log}^{\text{tw}}, \gamma_j, E^\bullet_j) \) a collection consisting of a pointed nodal curve \( X_j \) and a twisted \( G \)-oper \( (X_{\log}^{\text{tw}}, \gamma_j, E^\bullet_j) \) on it, where \( X_{\log}^{\text{tw}} := (f_j : X_j \to S_j, \{\sigma_{x,i}\}_i) \) and \( E^\bullet_j := (\pi_{B,j} : \mathcal{E}_{B,j} \to X_j, \nabla_j) \). A 1-morphism (or just a morphism) from \( X^\bullet_1 \) to \( X^\bullet_2 \) is a triple

\[
\alpha^\bullet := (\alpha^\bullet_S, \alpha^\bullet_X, \alpha^\bullet_E)
\]

consisting of morphisms of \( k \)-stacks which make the following diagram 1-commutative:

\[
\xymatrix{\mathcal{E}_{B,1} \ar[r]^{\pi_{B,1}} & X_1 \ar[r]^{f_1} & S_1 \ar[d]^{\alpha^\bullet_X} \ar[d]^{\alpha^\bullet_S} \ar[d]^{\alpha^\bullet_E} \\
\mathcal{E}_{B,2} \ar[r]_{\pi_{B,2}} & X_2 \ar[r]_{f_2} & S_2,}
\]

where

- the right-hand square diagram forms a morphism of pointed twisted curves (cf. Definition \ref{definition} (ii));
- the left-hand square is cartesian, and \( \alpha^\bullet_E \) is compatible with the respective \( B \)-actions of \( \mathcal{E}_{B,1} \) and \( \mathcal{E}_{B,2} \);
- the morphism \( \mathcal{E}_{G,1} := \mathcal{E}_{B,1} \times B G \to \mathcal{E}_{G,2} := \mathcal{E}_{B,2} \times B G \) induced by \( \alpha^\bullet_E \) is compatible with the respective connections \( \nabla_1, \nabla_2 \).

In particular, one may associate, to such a morphism \( \alpha^\bullet \), a morphism \( \alpha^{\text{loc}} : \mathcal{X}_1 \to \mathcal{X}_2 \) between the underlying pointed nodal curves.

(iii) Let \( X^\bullet_j \ (j = 1, 2) \) be as in (ii) and \( \alpha^\bullet_l := (\alpha^\bullet_{S,l}, \alpha^\bullet_{X,l}, \alpha^\bullet_{E,l}) \) \((l = 1, 2)\) morphisms from \( X^\bullet_1 \) to \( X^\bullet_2 \). A 2-morphism from \( \alpha^\bullet_1 \) to \( \alpha^\bullet_2 \) is a triple of natural transformations

\[
a^\bullet := (\alpha^\bullet_{S,1} \Rightarrow \alpha^\bullet_{S,2}, \alpha^\bullet_{X,1} \Rightarrow \alpha^\bullet_{X,2}, \alpha^\bullet_{E,1} \Rightarrow \alpha^\bullet_{E,2})
\]

compatible with each other (hence, \( a^\bullet_S \) coincides with the identity natural transformation).
Definition 2.3.1. We shall say that a twisted $G$-oper $\mathcal{E}^\bullet := (\mathcal{E}_B, \nabla)$ on $\mathfrak{U}^{\log}/S^{\log}$ is faithful if the classifying morphism $\mathfrak{U} \to \mathfrak{BT}$ of the $T$-bundle $\mathcal{E}_B \times^B T$ is representable. Also, let $\mathfrak{E}^\bullet := (\mathcal{E}^\text{tw}, \gamma, \mathcal{E}^\bullet)$ be a twisted $G$-oper on a pointed nodal curve. Then, we shall say that $\mathfrak{E}^\bullet$ is faithful if $\mathcal{E}^\bullet$ is faithful.

We shall examine the relationship with the notion of an extended spin structure discussed in §1. Denote by $Z$ the center of $G$. Because of the assumption imposed at the beginning of §2.1, $G$ is finite and has order invertible in $k$. Write $G_{\text{ad}} := G/Z$ (i.e., the adjoint group of $G$) and $T_{\text{ad}} := T/Z$. Let $\lambda : \mathbb{G}_m \to T_{\text{ad}}$ (cf. [11, §3.4.1]) be the morphism determined by the condition that for any $\alpha \in \Gamma$, $\lambda(t)$ acts on $g^\alpha$ (via the adjoint representation $(T_{\text{ad}} \subseteq) G_{\text{ad}} \to \text{GL}(\mathfrak{g})$) as multiplication by $t$. Then, one may find a unique morphism $\lambda^\sharp : \mathbb{G}_m \to T$ (cf. [11, Eq. (54)]) such that $\lambda(t)^2 = \lambda^\flat(t)^2 \mod Z$ for any $t \in \mathbb{G}_m$. The morphism $\lambda^\sharp$ restricts to a morphism $\delta^\sharp : \mu_2 \to Z (\subseteq T)$. The following square diagram is verified to be commutative:

\[
\begin{array}{ccc}
\mu_2 & \xrightarrow{\text{incl.}} & \mathbb{G}_m \\
\delta^\sharp \downarrow & & \downarrow \xrightarrow{\text{incl.}} \\
Z & \xrightarrow{z \mapsto (z,e)} & T \times_{T_{\text{ad}} \lambda} \mathbb{G}_m,
\end{array}
\]

where $e$ denotes the unit of $\mathbb{G}_m$. Hence, this diagram determines a morphism

\[
\hat{Z}_{\delta^\sharp} (\cong Z \times_{\delta^\sharp, \mu_2} \mathbb{G}_m) \to T \times_{T_{\text{ad}} \lambda} \mathbb{G}_m.
\]

This morphism is an isomorphism since it fits into the following morphism of short exact sequences:

\[
\begin{array}{cccccc}
0 & \longrightarrow & Z & \longrightarrow & \hat{Z}_{\delta^\sharp} & \longrightarrow & \mathbb{G}_m & \longrightarrow & 0 \\
\downarrow \text{id}_Z & & \downarrow \text{Id}_{\hat{Z}_{\delta^\sharp}} & & \downarrow \text{id}_{\mathbb{G}_m} \quad \text{(103)} & & \quad \text{(104)}\end{array}
\]

where the upper horizontal sequence is the lower horizontal sequence in (103).

Now, let $\mathfrak{E}^\bullet := (\mathcal{E}_B, \nabla)$ be a twisted $G$-oper on $\mathfrak{U}^{\log}/S^{\log}$. If $(\Omega_{\mathfrak{U}^{\log}/S^{\log}})^\times$ denotes the $\mathbb{G}_m$-bundle on $\mathfrak{U}$ corresponding to the line bundle $\Omega_{\mathfrak{U}^{\log}/S^{\log}}^{\log}$, then the $T_{\text{ad}}$-bundle $\mathcal{E}_B \times^B T_{\text{ad}}$ induced from $\mathcal{E}_B$ via change of structure group by the composite $B \to T \to T_{\text{ad}}$ is isomorphic to $(\Omega_{\mathfrak{U}^{\log}/S^{\log}})^\times \times_{\mathbb{G}_m, \lambda} T_{\text{ad}}$ (cf. the discussion in [11, §3.4.1]). Hence, by passing to (103), one may use the $T$-bundle $\mathcal{E}_B \times^B T$ and the $\mathbb{G}_m$-bundle $(\Omega_{\mathfrak{U}^{\log}/S^{\log}})^\times$ to obtain a $\hat{Z}_{\delta^\sharp}$-bundle

\[
\pi_{\mathfrak{G}_\mathfrak{E}^\bullet} := \mathfrak{G}_\mathfrak{E}^\bullet \to \mathfrak{U}.
\]

By definition, $\mathfrak{G}_\mathfrak{E}^\bullet \times_{\hat{Z}_{\delta^\sharp}, \mu_2} \mathbb{G}_m$ is isomorphic to $(\Omega_{\mathfrak{U}^{\log}/S^{\log}})^\times$.

Proposition 2.3.2. The twisted $G$-oper $\mathfrak{E}^\bullet$ is faithful if and only if $\mathfrak{G}_\mathfrak{E}^\bullet$ forms a $(Z, \delta^\sharp)$-structure on $\mathfrak{U}^{\log}/S^{\log}$.
Proof. Let us consider the composite
\[ \tilde{Z}_{\delta} \xrightarrow{(106)} T \times_{T_{ad, X}} G_m \to T, \]
where the second arrow denotes the first projection. The composite of the classifying morphism \([G_E]: \mathcal{U} \to \tilde{Z}_{\delta}\) of \(G_E\) and the morphism \(B\tilde{Z}_{\delta} \to BT\) induced by \((106)\) coincides with the classifying morphism \([E_B \times_T T]: \mathcal{U} \to BT\) of \(E_B \times_T T\). Since \((106)\) is a closed immersion, it follows from [5, Lemma 4.4.3] that \([G_E]\) is representable if and only if \([E_B \times_T T]\) is representable. This completes the proof of Proposition 2.3.2. □

Remark 2.3.3. Let us consider the case where \(G\) is of adjoint type, i.e., \(G = G_{ad}\). The definition of a faithful twisted \(G\)-oper introduced above may be identified with the classical definition of a \(g\)-oper in the sense of [57, Definition 2.1, (i)]. Indeed, let \(\mathcal{X}\) be a pointed nodal curve and \(E^{\bullet} := (\mathcal{X}^{tw}, \gamma, E^{\bullet})\) a twisted \(G\)-oper on \(\mathcal{X}\). The representability of the classifying morphism \(\mathcal{X} \to \tilde{Z}_{\delta}\) of \(G_E^{\bullet}\) together with the assumption \(Z = \{1\}\) imply (cf. [5, Lemma 4.4.3]) that the stabilizers of the nodes and the marked points of \(\mathcal{X}^{tw}\) are trivial. Hence, the morphism \(\gamma: \mathcal{X} \to X\) must be an isomorphism and \(E^{\bullet}\) specifies a \(g\)-oper on \(\mathcal{X}\) in the classical sense. Conversely, given a \(g\)-oper \(E^{\bullet}\) on \(\mathcal{X}\), we obtain a faithful twisted \(G\)-oper \((\mathcal{X}, id_X, E^{\bullet})\) on \(\mathcal{X}\). In this way, we shall not distinguish between faithful twisted \(G\)-opers (in the case of \(G = G_{ad}\)) and \(g\)-opers.

By Definition 2.2.2, (i)-(iii), the collections \((\mathcal{X}, E^{\bullet})\) of a pointed nodal curve \(\mathcal{X}\) and a twisted \(G\)-oper \(E^{\bullet}\) on \(\mathcal{X}\) form a 2-category. Just as in the case of \(\mathfrak{S}p_{Z,\delta,g,r}\) (cf. the discussion following Definition 1.3.2), this 2-category specifies a category fibered in groupoids over \(\mathfrak{S}ch_{/k}\).

Given a pair of nonnegative integers \((g, r)\) with \(2g - 2 + r > 0\), we shall denote by
\[ \mathcal{O}p_{G,g,r} \]
the category of pairs \((\mathcal{X}, E^{\bullet})\) consisting of an \(r\)-pointed stable curve \(\mathcal{X}\) of genus \(g\) over a \(k\)-scheme and a faithful twisted \(G\)-oper \(E^{\bullet}\) on \(\mathcal{X}\). The assignments \((\mathcal{X}, E^{\bullet}) \mapsto \mathcal{X}\) and \((\mathcal{X}, E^{\bullet}) \mapsto (\mathcal{X}, G_E^{\bullet})\) (where \(E^{\bullet} := (\mathcal{X}^{tw}, \gamma, E^{\bullet})\)) determine functors
\[ \mathcal{O}p_{G,g,r} \to \overline{M}_{g,r} \quad \text{and} \quad \mathcal{O}p_{G,g,r} \to \mathfrak{S}p_{Z,\delta,g,r} \]
respectively. Also, by change of structure group via \(G \to G_{ad}\), we obtain a functor
\[ \alpha_{\text{op}}: \mathcal{O}p_{G,g,r} \to \mathcal{O}p_{G_{ad},g,r} \]
over \(\overline{M}_{g,r}\).

Remark 2.3.4. Let \(\mathcal{X}\) be a pointed stable curve. We shall refer to each 2-isomorphism class of a 1-isomorphism defined in Definition 2.2.2 (ii) inducing the identity morphism of \(\mathcal{X}\) as an isomorphism of faithful twisted \(G\)-opers on \(\mathcal{X}\). In this way, we obtain the groupoid of faithful twisted \(G\)-opers on \(\mathcal{X}\).

Theorem 2.3.5. (i) The functor
\[ \mathcal{O}p_{G,g,r} \xrightarrow{(107)} \mathcal{O}p_{G_{ad},g,r} \times_{\overline{M}_{g,r}} \mathfrak{S}p_{Z,\delta,g,r} \]
induced by \((109)\) and the second morphism in \((108)\) is an equivalence of categories over \(\overline{M}_{g,r}\).
(ii) $\mathcal{D}p_{G,g,r}$ may be represented by a nonempty smooth Deligne-Mumford stack over $k$ which is flat over $\mathcal{M}_{g,r}$ of relative dimension
\begin{equation}
(g - 1) \cdot \dim(G) + \frac{r}{2} \cdot (\dim(G) + \text{rk}(G)),
\end{equation}
where $\dim(G)$ and $\text{rk}(G)$ denote the dimension and rank of $G$ respectively. Also, the morphism $\mathcal{O}_{\mathcal{D}p_{G,g,r}}$ is finite, flat, and generically étale.

**Proof.** We shall consider assertion (i). Let $\mathcal{X}$ be an $r$-pointed stable curve of genus $g$ over a $k$-scheme $S$. Suppose that we are given a $G_{ad}$-oper $E_{ad} := (E_{B_{ad}}, \nabla)$ on $\mathcal{X}$ and a $(Z, \delta^2)$-structure $(\mathcal{X}^{tw}, \gamma, G)$ on $\mathcal{X}$. In particular, $\mathcal{X} := (\mathcal{X}, \mathcal{X}^{tw}, \gamma, G)$ forms a pointed stable $(Z, \delta^2)$-spin curve. Both the $B_{ad}$-bundle $\gamma^*(E_{B_{ad}})$ on $\mathcal{X}$ and the $T$-bundle $G \times_{Z^{tw}} T$ obtained from $G$ via change of structure group by the composite of (103) and the first projection $T \times_{T_{ad}} G \rightarrow T$ induce the same $T$-bundle $(\Omega_{\mathcal{X}^{log}} / S^{log}) \times \mathbb{G}_m \rightarrow T$ obtained from $G$ via change of structure group by the composite of (103) and the first projection $T \times_{T_{ad}} G \rightarrow T$.

Since $B \cong B_{ad} \times_{T_{ad}} T$, these bundles give rise to a $B$-bundle $E_B$ on $\mathcal{X}$. The natural morphism $E_G := E_B \times_B G \rightarrow E_{G_{ad}}$ induces an isomorphism $\mathbb{I}_{E_{G_{ad}}}^{log} / S^{log} \rightarrow \mathbb{I}_{E_B}^{log} / S^{log}$. By this isomorphism, $\nabla$ may be regarded as an $S^{log}$-connection on $E_G$. Hence, the collection $E^{\bullet} := (E^{tw}, \gamma, (E_B, \nabla))$ forms a faithful twisted $G$-oper on $\mathcal{X}$. One may verify that the resulting assignment $(((\mathcal{X}, E^{\bullet}), X) \mapsto ((\mathcal{X}, E^{\bullet}))$ determines a functor
\begin{equation}
\mathcal{D}p_{G_{ad}, g, r} \times_{\mathcal{D}p_{Z, \delta^2, g, r}} \mathcal{D}p_{Z, \delta^2, g, r} \rightarrow \mathcal{D}p_{G, g, r},
\end{equation}
and that it specifies the inverse of (110). This complete the proof of assertion (i).

Assertion (ii) follows from Theorem [1.3.5] and [57, Theorem A].

2.4. **Radii of twisted $G$-opers.** Let us write $c := g / G$, i.e., the GIT quotient of $g$ by the adjoint action of $G$. Also, write $\chi : g \rightarrow c$ for the natural projection. The $k$-scheme $c$ has the involution $\rho \mapsto \rho^\vee$ arising from the automorphism of $g$ given by multiplication by $(-1)$. This involution induces an involution $(\rho, \kappa) \mapsto (\rho^\vee, \kappa^\vee)$. Let us write
\begin{equation}
\varepsilon := \chi(\bar{\rho}) \in c(k), \quad \xi := (\varepsilon, \delta^\vee) \in c(k) \times \text{Inj}(\mu, Z).
\end{equation}
Then, the equalities $\varepsilon^\vee = \varepsilon$ and $\xi^\vee = \xi$ hold.

The quotient stack $[c / Z]$ induced by the trivial $Z$-action on $c$ is canonically isomorphic to $c \times_k BZ$. We identify the stack of cyclotomic gerbes $\mathcal{T}_\mu([c / Z])$ in $[c / Z]$ with $c \times_k \mathcal{T}_\mu(BZ)$ by the composite of natural isomorphisms
\begin{equation}
\mathcal{T}_\mu([c / Z]) \rightarrow \mathcal{T}_\mu(c \times_k BZ) \rightarrow c \times_k \mathcal{T}_\mu(BZ).
\end{equation}
Moreover, according to decomposition (114), $\mathcal{T}_\mu([c / Z])$ may be identified with $\coprod_{\kappa \in \text{Inj}(\mu, Z)} c \times_k \mathcal{T}_\mu(BZ)^\kappa$. Given $\rho := (\rho, \kappa) \in c(k) \times \text{Inj}(\mu, Z)$, we obtain a closed substack
\begin{equation}
\mathcal{T}_\mu([c / Z])_\rho
\end{equation}
of $\mathcal{T}_\mu([c / Z])$ corresponding, via (114), to the close immersion $\mathcal{T}_\mu(BZ)^{\kappa} (= \text{Spec}(k) \times_k \mathcal{T}_\mu(BZ)^{\kappa}) \hookrightarrow c \times_k \mathcal{T}_\mu(BZ)$ defined as the product of $\rho$ and the inclusion $\mathcal{T}_\mu(BZ)^{\kappa} \hookrightarrow \mathcal{T}_\mu(BZ)$.

Let $r$ be a positive integer, $\mathcal{X} := (X, \{\sigma_X, i\}_{i=1}^r)$ an $r$-pointed nodal curve over a $k$-scheme $S$, and $E^{\bullet} := (\mathcal{X}^{tw}, \gamma, \delta^{\bullet})$ a faithful twisted $G$-oper on $\mathcal{X}$. Let $(E^{\bullet}_{ad}, G)$ be the pair of a faithful
twisted \( G_{\text{ad}} \)-oper on \( \mathcal{X} \) and a \((Z, \delta')\)-structure on \( \mathcal{X}^\text{tw} \) corresponding to \( E^\bullet \) via equivalence of categories (110).

Recall from \cite{w1} Definition 2.32 that the radius of \( \mathcal{E}^\bullet_{\text{ad}} = (\mathcal{E}_{B, \text{ad}}, \nabla) \) at each \( \sigma_{X,i} \) \((i = 1, \cdots, r)\) is defined as a certain element of \( \mathfrak{c}(S) \), which we shall denote by \( \rho^\bullet_{\text{ad}, i} \). That is to say, if \( \mathcal{E}_{G, \text{ad}} \) denotes the \( \mathcal{G}_{\text{ad}}\)-bundle \( \mathcal{E}_{B, \text{ad}} \times_{B, \text{ad}} G_{\text{ad}} \) and \( \mu_i^\nabla \) denotes the monodromy operator of \( \nabla \) at \( \sigma_{X,i} \) in the sense of \cite{w1} Definition 1.46, then \( \rho^\bullet_{\text{ad}, i} \) is defined as the image of the pair \((\sigma^\bullet_{X,i}((\mathcal{E}_{G, \text{ad}}), \mu_i^\nabla)) \) via the natural projection \([g/G] \to \mathfrak{c}\). Here, the quotient stack \([g/G]\) represents the functor which, to any \( k\)-scheme \( T \), assigns the groupoid of pairs \((\mathcal{F}, \mu)\) consisting of a \( G\)-bundle on \( T \) and \( \mu \in \Gamma(T, g_T) \). Let us write \( \rho^\bullet_{\text{ad}, i} \) as the radius of \( E^\bullet \) at \( \sigma_{X,i} \). In particular, if \( G \) is of adjoint type, then the notion of radius coincides with the classical definition discussed in loc. cit.

**Definition 2.4.1.** Let \( \mathcal{X} \) be as above and \( \tilde{\rho} : = (\rho_i)^r_{i=1} \) an element of \( (\mathfrak{c}(S) \times \text{Inj}(\mu, Z))^r \). Then, we shall say that a faithful twisted \( G\)-oper \( E^\bullet \) on \( \mathcal{X} \) is of radii \( \tilde{\rho} \) if \( \rho^\bullet_{E, i} = \rho_i \) for every \( i = 1, \cdots, r \).

For each \( i = 1, \cdots, r \), the assignment \( E^\bullet \mapsto \tilde{\rho}^E_{i, i} \) determines a morphism

\[
ev_{i}^{\text{Op}} : \mathcal{D}p_{G, g, r} \to \mathcal{T}_{\mu}(\mathfrak{c}/Z).
\]

Moreover, the morphisms \( \ev^{\text{Op}} \) determine a morphism

\[
ev^{\text{Op}} : = (\ev_{1}^{\text{Op}}, \cdots, \ev_{r}^{\text{Op}}) : \mathcal{D}p_{G, g, r} \to \mathcal{T}_{\mu}(\mathfrak{c}/Z)^r.
\]

Given an element \( \tilde{\rho} : = (\rho_i)^r_{i=1} \in (\mathfrak{c}(k) \times \text{Inj}(\mu, Z))^r \), we obtain the closed substack

\[
\mathcal{D}p_{G, g, r, \tilde{\rho}} : = (\ev^{\text{Op}})^{-1}(\prod_{i=1}^{r} \mathcal{T}_{\mu}(\mathfrak{c}/Z)_{\rho_i}),
\]

i.e., the substack classifying faithful twisted \( G\)-opers of radii \( \tilde{\rho} \).

**Theorem 2.4.2.** Let \( \tilde{\rho} : = ((\rho_i, \kappa_i))^{r}_{i=1} \in (\mathfrak{c}(k) \times \text{Inj}(\mu, Z))^r \). Then, isomorphism (110) restricts to an isomorphism

\[
\mathcal{D}p_{G, g, r, \tilde{\rho}} \sim \mathcal{D}p_{G_{\text{ad}}, g, r, (\rho_i)^r_{i=1} \times_{\mathcal{M}_{g,r}} \mathcal{I}_g, \delta_{g, r, (\kappa_i)^r}_{i=1}}.
\]

In particular, \( \mathcal{D}p_{G, g, r, \tilde{\rho}} \) may be represented by a smooth Deligne-Mumford stack over \( k \) which is flat over \( \mathcal{M}_{g, r} \) of relative dimension

\[
(g - 1) \cdot \dim(G) + \frac{r}{2} \cdot (\dim(G) - \text{rk}(G))
\]

**Proof.** The assertion follows from the various definitions involved together with Theorem 2.3.5 and \cite{w1} Theorem A. \( \square \)
2.5. Forgetting tails. Let \( r \) be a nonnegative integer, \( \bar{\rho} \) an element of \((\epsilon(S) \times \mathrm{Inj}(\mu, Z))^r\), and \( X' := (X, \{\sigma_{X,i}^r\}) \) an \((r + 1)\)-pointed smooth curve over a \( k\)-scheme \( S \) (hence \( S^{\log} = S \)). We shall write \( X' = (X, \{\sigma_{X,i}^r\}) \) i.e., the \( r \)-pointed curve obtained from \( X' \) by forgetting the last marked point. In particular, \( X_{\text{tail}} = X \) and \( \sigma_{X,i}^r = \sigma_{X,i} \) for every \( i \in \{1, \ldots, r\} \). Also, write

\[
\mathcal{D}_p X_{\text{tail}} \hat{\rho} \quad \text{(resp.,} \quad \mathcal{D}_p X \hat{\rho} \varepsilon) \tag{122}
\]

for the groupoid of faithful twisted \( G \)-opers on \( X'_{\text{tail}} \) of radii \( \hat{\rho} \) (resp., on \( X' \) of radii \( \hat{\rho}, \varepsilon \)). In what follows, we shall construct a canonical functor \( \mathcal{D}_p X_{\text{tail}} \hat{\rho} \to \mathcal{D}_p X \hat{\rho} \varepsilon \).

First, let us consider the case where \( G \) is of adjoint type (hence, \( \mathrm{Inj}(\mu, Z) = \{1\} \) and \( \varepsilon = \varepsilon \)). Denote by \( \mathcal{E}_{B,X} \hat{\rho} \) (resp., \( \mathcal{E}_{B,X_{\text{tail}}} \hat{\rho} \)) the \( B \)-bundle \( \mathcal{E}_{B,h,U^{\log}/S^{\log}} \) introduced in [57, Eq. (210)], where the triple \( "(B, h, U^{\log}/S^{\log})" \) is taken to be \((B, 1, X_{\text{tail}}^{\log}/S)\) (resp., \((B, 1, X_{\text{tail}}^{\log}/S)\)). Write \( \mathcal{E}_{B,X} := \mathcal{E}_{B,X_{\text{tail}}} \times^B G \) and \( \mathcal{E}_{G,X_{\text{tail}}} := \mathcal{E}_{B,X_{\text{tail}}} \times^B G \).

Now, let \( \mathcal{E}_{\alpha}^\bullet := (\mathcal{E}_{B,X_{\text{tail}}}, \nabla_{\text{tail}}) \) be a faithful twisted \( G \)-oper (i.e., a \( \mathfrak{g} \)-oper in the sense of [57, Definition 2.1]) on \( X'_{\text{tail}} \) of radii \( \hat{\rho} \in (S)^r \). According to [57, Proposition 2.19], there exists a unique pair \( (\mathcal{E}_{\alpha}^\bullet, \text{nor}_{\alpha}) \) consisting of a faithful twisted \( G \)-oper \( \mathcal{E}_{\alpha}^\bullet := (\mathcal{E}_{B,X_{\text{tail}}}, \nabla_{\text{tail}}) \) on \( X'_{\text{tail}} \) which is \( \{x_\alpha\}_{\alpha} \)-normal (cf. [57, Definition 2.14]) and an isomorphism of \( G \)-opers \( \text{nor}_{\alpha} : \mathcal{E}_{\alpha}^\bullet \to \mathcal{E}_{\alpha}^\bullet \). By using \( \text{nor}_{\alpha} \), we shall identify \( \mathcal{E}_{\alpha}^\bullet \) with \( \mathcal{E}_{\alpha}^\bullet \), i.e., assume that \( \mathcal{E}_{\alpha}^\bullet \) is \( \{x_\alpha\}_{\alpha} \)-normal.

We shall construct from \( \nabla_{\text{tail}} \) an \( S^{\log} (= S) \)-connection on \( \mathcal{E}_{G,X} \). Let us take a pair \( U = (U, t) \) consisting of an open sub.scheme \( U \) of \( X \) with \( U \cap \mathrm{Im}(\sigma_{X,i+1}) \neq \emptyset, U \cap \mathrm{Im}(\sigma_{X,i}) = \emptyset \) (\( i = 1, \ldots, r \)) and an element \( t \in \Gamma(U, O_X) \) defining the closed sub.scheme \( \sigma_{X,i+1} \) such that \( d\log(t) \) generates \( \Omega_{X^{\log}/S}|_U \). Hence, \( t \cdot d\log(t) (= dt) \) generates \( \Omega_{X^{\log}/S}|_U \). According to [57, Eq. (211)], there exists a canonical trivialization \( \text{triv}_{U} : \mathcal{E}_{B,X_{\text{tail}}}|_U \to U \times_k B \) of the \( B \)-bundle \( \mathcal{E}_{B,X_{\text{tail}}}|_U \) arising from the pair \( t \) \( U := (U, dt) \). After a change of structure group by \( B \to G \), the differential of this trivialization specifies an \( \mathcal{O}_U \)-linear isomorphism

\[
\widetilde{\mathcal{T}}_{\mathcal{E}_{G,X_{\text{tail}}}}|_U \simeq \mathcal{T}_{U/S} \oplus (\mathcal{O}_U \otimes \mathfrak{g}). \tag{123}
\]

For any \( h \in T \) and \( \alpha \in \Gamma \), denote by \( h_\alpha \) the automorphism of \( \mathfrak{g}^\alpha \) given by multiplication by \( \alpha(h) \in \mathfrak{g}_m \). The assignment \( h \mapsto (h_\alpha)_{\alpha \in \Gamma} \) determines an isomorphism

\[
T \simeq \prod_{\alpha \in \Gamma} \text{GL}(\mathfrak{g}^\alpha). \tag{124}
\]

Write \( U^\circ := U \setminus \mathrm{Im}(\sigma_{X,r+1}) \). Also, write \( \iota_U : U^\circ \to U \) for the natural open immersion and \( \text{mult}_{\alpha,U} \) (where \( \alpha \in \Gamma \)) for the automorphism of \( \mathcal{O}_{U^\circ} \otimes \mathfrak{g}^\alpha \) given by multiplication by \( \iota_U^\ast(t) \in \Gamma(U^\circ, O_X) \). The translation by the element of \( T(U^\circ) \) corresponding to \( \text{mult}_{\alpha,U} \) \( \alpha \in \Gamma \in \prod_{\alpha \in \Gamma} \text{GL}(\mathfrak{g}^\alpha) \) via (124) determines an automorphism \( \text{mult}_U \) of the trivial \( B \)-bundle \( U^\circ \times_k B \) on \( U^\circ \). The composite

\[
\mathcal{T}_{U^\circ/S} \xrightarrow{\nabla_{\text{tail}}|_{U^\circ}} \mathcal{T}_{\mathcal{E}_{G,X_{\text{tail}}}}|_{U^\circ} \xrightarrow{\iota_U^\ast|_{U^\circ}} \mathcal{T}_{U^\circ/S} \oplus (\mathcal{O}_{U^\circ} \otimes \mathfrak{g}) \xrightarrow{d(\text{mult}_U)} \mathcal{T}_{U^\circ/S} \oplus (\mathcal{O}_{U^\circ} \otimes \mathfrak{g}) \tag{125}
\]

specifies an \( S^{\log} \)-connection on \( U^\circ \times_k G \), and it extends uniquely to an \( S^{\log} \)-connection

\[
\nabla_U : \mathcal{T}_{X^{\log}/S}|_U \to \mathcal{T}_{X^{\log}/S}|_U \oplus (\mathcal{O}_U \otimes \mathfrak{g}) \tag{126}
\]

on \( X^{\log}|_U \times_k G \). Moreover, the monodromy operator \( \mu_{r+1}^U \in \mathfrak{g}(U \times \sigma_{X,r+1}) \) of \( \nabla_U \) at \( \sigma_{X,r+1} \) coincides with \( -\bar{\rho} \in \mathfrak{g} \) (cf. [15, Proposition 9.2.1]). The pair \( (U \times_k B, \nabla_U) \) forms a faithful twisted
G-oper on $X^\log|U/S$ whose radius at $\sigma_{X,r+1}$ coincides with $\varepsilon = \chi(-\hat{\rho})$. Hence, by \cite[Proposition 2.19]{[57]}, there exists uniquely an $\{\alpha\}$-normal faithful twisted G-oper $^\dagger E_\mathcal{U} := (^\dagger E_{B,\mathcal{X}}|U, ^\dagger \nabla_U)$ on $X^\log|U|$ isomorphic to $(U \times_k B, \nabla_U)$. On the other hand, since there exists a natural identification $^\dagger\mathcal{E}_{G,\mathcal{X}}|\mathcal{X}\setminus \text{Im}(\sigma_{X,r+1}) = (^\dagger\mathcal{E}_{G,\mathcal{X}}|X \setminus \text{Im}(\sigma_{X,r+1}))$, the restriction $\nabla_{\text{tail}}|\mathcal{X}\setminus \text{Im}(\sigma_{X,r+1})$ of $\nabla_{\text{tail}}$ to $\mathcal{X}\setminus \text{Im}(\sigma_{X,r+1})$ may be regarded as an $S^\log$-connection on $(^\dagger\mathcal{E}_{B,\mathcal{X}}|\mathcal{X}\setminus \text{Im}(\sigma_{X,r+1}))$. Because of the equality $^\dagger \nabla_U|U^\op = \nabla_{\text{tail}}|U^\op$, the $S^\log$-connection $\nabla_{\text{tail}}|\mathcal{X}\setminus \text{Im}(\sigma_{X,r+1})$ and the various $S^\log$-connections $^\dagger \nabla_U$ (where $U$ ranges over the pairs $U = (U, t)$ as above) may be glued together to obtain an $S^\log$-connection $\nabla_{+\varepsilon}$ on $^\dagger \mathcal{E}_{G,\mathcal{X}}$. The resulting pair
\begin{equation}
(127)
E_{+\varepsilon} := (^\dagger \mathcal{E}_{B,\mathcal{X}}, \nabla_{+\varepsilon})
\end{equation}
forms a faithful twisted G-oper on $\mathcal{X}$ of radii $(\hat{\rho}, \varepsilon) \in \mathfrak{c}(S)^{(r+1)}$. The assignment $E_{\text{tail}} \mapsto E_{+\varepsilon}$ determines a functor $\mathfrak{D}p\mathcal{X}_{\text{tail}}\hat{\rho} \rightarrow \mathfrak{D}p\mathcal{X}((\hat{\rho}, \varepsilon))$.

Next, let us remove the assumption that $\mathfrak{g}$ is of adjoint type. Let $E_{\text{tail}}$ be a faithful twisted G-oper on $\mathcal{X}_{\text{tail}}$; it corresponds, via \cite{[110]}, to a pair $(E_{\text{tail}}^\circ, (\mathcal{X}_{\text{tail}}, \gamma_{\text{tail}}, \mathcal{G}_{\text{tail}}))$ consisting of a faithful twisted $G_{\text{ad}}$-oper $E_{\text{tail}}^\circ$ on $\mathcal{X}_{\text{tail}}$ and a $(Z, \delta^\circ)$-structure $(\mathcal{X}_{\text{tail}}, \gamma_{\text{tail}}, \mathcal{G}_{\text{tail}})$ on $\mathcal{X}_{\text{tail}}$. According to the above discussion and the discussion in \cite{[110]}, these data induce a faithful twisted $G_{\text{ad}}$-oper $E_{+\varepsilon}^\circ$ on $\mathcal{X}$ and a $(Z, \delta^\circ)$-structure $(\mathcal{X}^\circ_{+\varepsilon}, \gamma_{+\varepsilon}, \mathcal{G}_{+\varepsilon})$ on $\mathcal{X}$ respectively. The pair $(E_{+\varepsilon}^\circ, (\mathcal{X}^\circ_{+\varepsilon}, \gamma_{+\varepsilon}, \mathcal{G}_{+\varepsilon}))$ corresponds, via \cite{[110]} again, to a faithful twisted G-oper
\begin{equation}
(128)
E_{+\varepsilon}^\circ
\end{equation}
on $\mathcal{X}$. The assignment $E_{\text{tail}} \mapsto E_{+\varepsilon}^\circ$ determines a functor $\mathfrak{D}p\mathcal{X}_{\text{tail}}\hat{\rho} \rightarrow \mathfrak{D}p\mathcal{X}((\hat{\rho}, \varepsilon))$. We shall write
\begin{equation}
(129)
\mathfrak{D}p\mathcal{X}_{\text{tail}}\hat{\rho} \rightarrow \mathfrak{D}p\mathcal{X}((\hat{\rho}, \varepsilon))
\end{equation}
for the essential image of this functor. The above discussion and Proposition 1.6.2 imply the following proposition.

**Proposition 2.5.1.**

(i) Let $\mathcal{X}$ and $\mathcal{X}_{\text{tail}}$ be as above. Then, the assignment $E_{\text{tail}} \mapsto E_{+\varepsilon}^\circ$ determines an equivalence of categories
\begin{equation}
(130)
\mathfrak{D}p\mathcal{X}_{\text{tail}}\hat{\rho} \sim \mathfrak{D}p\mathcal{X}((\hat{\rho}, \varepsilon)).
\end{equation}

(ii) Let $(g, r)$ be a pair of nonnegative integers with $2g - 1 + r > 0$, and let $\hat{\rho} \in (\mathfrak{c}(k) \times \text{Inj}(\mu, Z))^r$. Then, the assignment
\begin{equation}
(131)
(\mathcal{X}, (\mathcal{X}_{\text{tail}}, E_{\text{tail}})) \mapsto (\mathcal{X}, (\mathcal{X}, E_{+\varepsilon}^\circ))
\end{equation}
determines a fully faithful functor
\begin{equation}
(132)
\mathfrak{M}_{g, r+1} \times \mathfrak{M}_{g, r} \mathfrak{D}p_{G,g,r}\hat{\rho} \rightarrow \mathfrak{M}_{g, r+1} \times \mathfrak{M}_{g, r+1} \mathfrak{D}p_{G,g,r+1}(\hat{\rho}, \varepsilon)
\end{equation}
over $\mathfrak{M}_{g, r+1}$.

2.6. Faithful twisted G-operators on the 2-pointed projective line. In this subsection, we shall study the faithful twisted G-operators on the 2-pointed projective line. Let us keep the notation in §1.5.

**Proposition 2.6.1.**

(i) Let $E^\circ$ be a faithful twisted G-operator on $\mathcal{P}_S$. Then, the equality $\rho_{E^\circ, 1} = \rho_{E^\circ, 2}$ holds.
(ii) For each \( \rho \in \mathfrak{c}(S) \times \text{Inj}(\mu, Z) \), there exists a unique (up to isomorphism) faithful twisted \( G \)-operator \( \mathbb{P}^\bullet \) on \( \mathcal{P}'_S \) with \( \rho = \rho_{S,1}^\vee = \rho_{S,2}^\vee \).

Proof. By Proposition 1.8.1 and Theorem 2.4.2 one may assume that \( G = G_{ad} \) (hence \( T_p([\ell/Z]) = \mathfrak{c} \)). First, we shall consider assertion (i). Let \( \mathcal{E}^\bullet := (\mathfrak{E}_B, \nabla) \) be an \( \{x_\alpha\}_\alpha \)-normal faithful twisted \( G \)-operator on \( \mathcal{P}'_S \). The global section \( ds_1/s_1 = s_2/s_2 \in \Gamma(P_S, \Omega_{P_{S}}) \) generates globally \( \Omega_{P_{S}} \). It follows that the pair \( (P_S, ds_1/s_1) \) gives a trivialization \( \mathfrak{E}_{G, \mathcal{P}_S} \approx P_S \times_k G \) (cf. [57, Eq. (211)]). By this trivialization, we have

\[
\text{Hom}(\mathcal{T}_{P_{S}/S}, \mathcal{T}_{G, \mathcal{P}_{S}/S}) \cong \text{Hom}(\mathcal{T}_{P_{S}/S}, \mathcal{T}_{(P_{S} \times_k G)/S})
\]

\[
\cong \Gamma(P_S, \Omega_{P_{S}} \otimes (\mathcal{T}_{P_{S}/S} \oplus (\mathcal{O}_{P_{S}} \otimes \mathfrak{g})))
\]

\[
\cong \Gamma(P_S, \mathcal{O}_{P_{S}} \oplus (\Omega_{P_{S}} \otimes \mathfrak{g})).
\]

The \( \nabla' \)-connection \( \nabla' \) on \( P_S \times_k G \) corresponding to \( \nabla \) via (133) may be expressed as

\[
(1, \frac{ds_1}{s_1} \otimes v) = (1, -\frac{ds_2}{s_2} \otimes v) \in \Gamma(P_S, \mathcal{O}_{P_{S}} \oplus (\Omega_{P_{S}} \otimes \mathfrak{g}))
\]

for some \( v \in \mathfrak{g}(S) \). Hence, we obtain the following sequence of equalities:

\[
\rho(P_S \times_k G, \nabla, 1) = \chi \left( \sigma^*_{P,1} \left( \frac{ds_1}{s_1} \otimes v \right) \right)
\]

\[
= \chi \left( \sigma^*_{P,2} \left( -\frac{ds_2}{s_2} \otimes v \right) \right)
\]

\[
= \chi \left( \sigma^*_{P,2} \left( \frac{ds_2}{s_2} \otimes v \right) \right)
\]

\[
= \rho(P_S \times_k G, \nabla', 2).
\]

This completes the proof of assertion (i).

Next, we shall consider assertion (ii). Write \( \mathfrak{g}^{ad(p_1)} := \{ x \in \mathfrak{g} \mid \text{ad}(p_1)(x) = 0 \} \). Since we have assumed that \( \text{char}(k) = 0 \) or \( \text{char}(k) = p \) for some prime \( p \) satisfying the condition \( (\ast)_G \), the morphism \( \text{kos} : \mathfrak{g}^{ad(p_1)} \to \mathfrak{c} \) given by \( s \mapsto \chi(p_{-1} + s) \) (for each \( s \in \mathfrak{g}^{ad(p_1)} \)) is an isomorphism (cf. [44, Lemma 1.2.1]). Hence, for each \( \rho \in \mathfrak{c}(S) \), there exists a unique \( v_0 \in \mathfrak{g}^{ad(p_1)}(S) \) with \( \text{kos}(v_0) = \rho \). Let \( \nabla_\rho \) denote the \( S_{log}' \)-connection on \( P_S \times_k G \) of the form (134), where \( v \) is taken to be \( v := p_{-1} + v_0 \). Then, the pair \( \mathcal{E}^\bullet : (P_S \times_k B, \nabla_\rho) \) forms a faithful twisted \( G \)-operator on \( \mathcal{P}'_S \) satisfying the required condition. This proves the existence portion of (ii). The uniqueness portion follows immediately from the construction of \( \mathcal{E}^\bullet \). Thus, the proof of assertion (ii) is completed.

**Corollary 2.6.2.** Let \( \mathcal{X} \) be a 3-pointed projective line over \( k \). Then, we have

\[
\mathbb{P}_{\mathcal{X},(p_1, p_2, e)} = \begin{cases} 
\mathcal{BZ} & \text{if } p_1 = p_2^\vee; \\
\emptyset & \text{if otherwise}.
\end{cases}
\]

**Proof.** The assertion follows from Propositions 2.5.1 and 2.6.1. 

\( \square \)
3. Do’pers and their moduli

In this section, we study G-do’pers (= dormant faithful twisted G-opers) and their moduli, which are central objects in the present paper. We prove that the moduli stack of G-do’pers is a proper Deligne-Mumford stack (cf. Theorem 3.3.1) and satisfies certain factorization properties according to clutching morphisms of the stacks \( \overline{M}^{\gamma}_{g,r} \) for various pairs \((g,r)\) (cf. Propositions 3.3.2, 3.3.3, and 3.3.4).

Let us fix a split semisimple algebraic group \((G,T)\) over \(k\), where \(T\) denotes a maximal torus \(T\) of \(G\). Also, fix a Borel subgroup of \(G\) defined over \(k\) containing \(T\). Denote by \(\mathfrak{g},\mathfrak{t}\), and \(\mathfrak{b}\) the Lie algebras of \(G\), \(T\), and \(B\) respectively. We shall suppose that \(\text{char}(k) = p > 0\) and \(G\) satisfies the condition \((\ast)_G\).

3.1. G-do’pers. To begin with, we recall the definition of the \(p\)-curvature of a logarithmic connection (cf., e.g., [57, §3.3]). Let \(S^{\log}\) be an fs log scheme over \(k\), \(U^{\log}\) a stacky log curve over \(S^{\log}\), and \((\pi : \mathcal{E} \to U, \nabla)\) a \(G\)-bundle \(\pi : \mathcal{E} \to U\) on \(U\) paired with an \(S^{\log}\)-connection \(\nabla\) on \(\mathcal{E}\). If \(\partial\) is a logarithmic derivation corresponding to a local section of \(T^{\log/S^{\log}}\) (resp., \(\nabla^{\log/S^{\log}} := (\pi^*(T^{\log/S^{\log}}))^G\)), then we shall denote by \(\partial^{[p]}\) the \(p\)-th symbolic power of \(\partial\) (i.e., “\(\partial \mapsto \partial^{[p]}\)” asserted in [16, Proposition 1.2.1]), which is also a logarithmic derivation corresponding to a local section of \(T^{\log/S^{\log}}\) (resp., \(\nabla^{\log/S^{\log}}\)). Then, there exists a unique \(\mathcal{O}_U\)-linear morphism \(\psi(\xi,\nabla) : T^{\log/S^{\log}} \to \mathfrak{g}\mathfrak{c} \subseteq \nabla^{\log/S^{\log}}\) determined by \(\partial^{\log} \mapsto \nabla(\partial)^{[p]} - \nabla(\partial^{[p]})\). We shall refer \(\psi(\xi,\nabla)\) to as the \(p\)-curvature map of \((\xi,\nabla)\).

**Definition 3.1.1.** (i) We shall say that a twisted \(G\)-oper \(\mathcal{E}^{\bullet} := (\mathcal{E}_B, \nabla)\) on \(U^{\log}/S^{\log}\) is \begin{**}dormant if \(\psi(\xi_B \times_{\mathcal{E}_B} \mathfrak{g},\nabla)\) vanishes identically on \(U\).

(ii) Let \(\mathcal{X}\) be a pointed nodal curve. A \begin{**}G-do’per on \(\mathcal{X}\) is defined to be a faithful twisted \(G\)-oper \(\mathcal{E}^{\bullet} := (\mathcal{E}^{\text{tw}}, \gamma, \mathcal{E}^{\bullet})\) on \(\mathcal{X}\) such that \(\mathcal{E}^{\bullet}\) is dormant.

Let \((g,r)\) be a pair of nonnegative integers with \(2g - 2 + r > 0\). Write

\[
\mathcal{O} \mathcal{P}^{\text{zar}}_{G,g,r}
\]

for the closed substack of \(\mathcal{O} \mathcal{P}_{G,g,r}\) classifying G-do’pers. Also, write

\[
\pi_{g,r} : \mathcal{O} \mathcal{P}^{\text{zar}}_{G,g,r} \to \overline{M}^{\gamma}_{g,r}, \quad r_{g,r} : \mathcal{O} \mathcal{P}^{\text{zar}}_{G,g,r} \to \mathcal{O} \mathcal{P}_{G,g,r}
\]

for the forgetting morphism and the natural closed immersion respectively.

3.2. Radii of G-do’pers. Let \(t_{\text{reg}}\) denote the open subscheme of \(t\) classifying regular elements. Since \(G\) is a Chevalley group, all the groups \(G, T,\) and \(B\) can be defined over the prime field \(\mathbb{F}_p\). It follows that the \(k\)-scheme \(t_{\text{reg}}\) comes from a certain \(\mathbb{F}_p\)-scheme \((t_{\text{reg}})_{\mathbb{F}_p}\) via base-change. In particular, the Frobenius twist \(t_{\text{reg}}^{(1)}\) is isomorphic to \(t_{\text{reg}}\) itself, and the relative Frobenius morphism \(F_{\text{reg}/k}\) may be regarded as a \(k\)-endomorphism of \(t_{\text{reg}}\). We shall write

\[
t_{\text{reg}}^{F} := \text{Ker} \left( t_{\text{reg}}^{F_{\text{reg}/k}, \text{id}} \right).
\]
If $W$ denotes the Weyl group of $(G, T)$, then the natural $W$-action on $t$ restricts to a $W$-action on $t_{\text{reg}}^F$. The composite $t \hookrightarrow g \xrightarrow{\chi} c := g//G$ induces a closed immersion from the resulting quotient $t_{\text{reg}}^F/W$ to $c$. The finite set

$$\tag{140} (t_{\text{reg}}^F/W)(k) \simeq t_{\text{reg}}^F(k)/W$$

may be considered as a subset of $c(k)$. The $k$-scheme $t_{\text{reg}}^F/W$ decomposes into the disjoint union $\coprod_{\rho \in (t_{\text{reg}}^F/W)(k)} \operatorname{Spec}(k)_\rho$, where $\operatorname{Spec}(k)_\rho := \operatorname{Spec}(k)$. So we occasionally identify $t_{\text{reg}}^F/W$ with the set of its $k$-rational points in a natural manner. The trivial $Z$-action on $t_{\text{reg}}^F/W$ gives rise to a closed substack

$$\tag{141} \mathfrak{Rad} := \mathcal{I}_\mu((t_{\text{reg}}^F/W)/Z)$$

of $\mathcal{I}_\mu([t/Z])$. Let us define a finite set $\Delta$ to be

$$\tag{142} \Delta := (t_{\text{reg}}^F/W) \times \text{Inj}(\mu, Z).$$

Decomposition (142) determines a decomposition

$$\tag{143} \mathfrak{Rad} = \mathcal{I}_\mu((t_{\text{reg}}^F/W) \times_k BZ)$$

$$= (t_{\text{reg}}^F/W) \times_k \mathcal{I}_\mu(BZ)$$

$$= \coprod_{\rho \in t_{\text{reg}}^F/W} \operatorname{Spec}(k)_\rho \times \coprod_{\kappa \in \text{Inj}(\mu, Z)} \mathcal{I}_\mu(BZ)_\kappa$$

$$= \coprod_{\rho \in \Delta} \mathfrak{Rad}_\rho,$$

where $\mathfrak{Rad}_{(\rho, \kappa)} := \operatorname{Spec}(k)_\rho \times_k \mathcal{I}_\mu(BZ)_\kappa$.

**Proposition 3.2.1.** Let $r$ be a positive integer, $S$ a $k$-scheme, $\mathcal{X} := (X, \{\sigma_{X,i}\}_{i=1}^r)$ an $r$-pointed nodal curve over $S$, and $\mathbb{E}^\bullet$ a $G$-doper on $\mathcal{X}$. Then, for each $i \in \{1, \ldots, r\}$, the radius $\rho_{\mathbb{E}^\bullet, i} \in c(S) \times \text{Inj}(\mu, Z)$ of $\mathbb{E}^\bullet$ at $\sigma_{X,i}$ lies in $\Delta$.

**Proof.** To complete the proof, it suffices to consider the case where $G = G_{\text{ad}}$ and $\mathbb{E}^\bullet$ represents a faithful twisted $G(= G_{\text{ad}})$-oper $\mathcal{E}^\bullet := (\mathcal{E}_B, \nabla)$ which is normal with respect to a fixed choice of generators $\{x_\alpha\}_\alpha$ in $\mathfrak{g}$'s (cf. [57, Definition 2.14]). By [57, Proposition 3.13, (i)], $\rho_{\mathbb{E}^\bullet, i}$ belongs to $c(k)$. Hence, one may assume, without loss of generality, that $S = \operatorname{Spec}(k)$ and $k$ is algebraic closed. Denote by $\mu_i^\nabla \in \mathfrak{g}(k)$ the monodromy operator of $\nabla$ at $\sigma_{X,i}$. Let us consider a Jordan decomposition $\mu_i^\nabla = \mu_s + \mu_n$ with $\mu_s$ semisimple and $\mu_n$ nilpotent. Denote by $\text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ the adjoint representation of $\mathfrak{g}$, which is injective and compatible with the respective restricted structures, i.e., $p$-power maps. One may find an isomorphism of restricted Lie algebras $\alpha : \text{End}(\mathfrak{g}) \overset{\sim}{\rightarrow} \mathfrak{gl}_{\text{lin}}(\mathfrak{g})$ which sends $\alpha(\text{ad}(\mu_i^\nabla)) := \alpha(\text{ad}(\mu_s)) + \alpha(\text{ad}(\mu_n))$ to a Jordan normal form. Hence, $\alpha(\text{ad}(\mu_s))$ is diagonal and every entry of $\alpha(\text{ad}(\mu_n))$ except the superdiagonal is 0. Let us observe the following sequence of equalities:

$$\tag{144} \alpha(\text{ad}(\mu_s)) + \alpha(\text{ad}(\mu_n)) = \alpha(\text{ad}(\mu_s + \mu_n)) = \alpha(\text{ad}(\mu_s + \mu_n)^{[p]})$$

$$= \alpha(\text{ad}(\mu_s + \mu_n)^p) = (\alpha(\text{ad}(\mu_s)) + \alpha(\text{ad}(\mu_n)))^p,$$

where the second equality follows from the assumption that $\mathcal{E}^\bullet$ has vanishing $p$-curvature (and [57, Eq. (345)]). An explicit computation of $(\alpha(\text{ad}(\mu_s)) + \alpha(\text{ad}(\mu_n)))^p$ shows (by (144)) that $\alpha(\text{ad}(\mu_n)) = 0 \iff \mu_n = 0$, namely, $\mu_i^\nabla$ is conjugate to some $v \in t$. Since the map
Let us consider the former assertion. To complete the proof, it suffices to consider
\[\mu_\nabla \in \mathfrak{g}_{\text{ad}(p_1)}(k)\] (cf. \[\S 2.1\] for the definitions of \(p_{-1}\) and \(p_1\)). Hence, \(\mu_\nabla\) is regular (cf. the comment preceding 144 Lemma 1.2.3). It follows that \(v \in t_{\text{reg}}^F(k)\). Consequently, we have \(\rho_{\mathbf{x},v} = (\chi(\mu_\nabla) = \chi(v)) \in (t_{\text{reg}}^F/W)(k)\), which completes the proof of the assertion.

Because of Proposition 3.2.1 above, the morphism \(\text{ev}^\text{Op}_i (i = 1, \cdots, r)\) and \(\text{ev}^\text{Op}\) restrict to morphisms
\[
eq \prod_{\rho \in \Delta} \text{Rad}_\rho \rightleftharpoons \prod_{\rho \in \Delta} \text{Rad}_\rho,
\]
respectively. Given \(\tilde{\rho} := (\rho_i)^r_{i=1} \in \Delta^\times r\), we obtain an open and closed substack
\[
\mathcal{O}_{\mathbf{p}_{G,g,r}}^{\text{zzz}}(\tilde{\rho}) := \text{ev}^{-1}\left(\prod_{i=1}^r \text{Rad}_{\rho_i}\right) = \mathcal{O}_{\mathbf{p}_{G,g,r}}^{\text{zzz}} \cap \mathcal{O}_{\mathbf{p}_{G,g,r}}^{\text{zzz}}
\]
of \(\mathcal{O}_{\mathbf{p}_{G,g,r}}^{\text{zzz}}\). The stack \(\mathcal{O}_{\mathbf{p}_{G,g,r}}^{\text{zzz}}\) decomposes into the disjoint union
\[
\mathcal{O}_{\mathbf{p}_{G,g,r}}^{\text{zzz}} = \bigsqcup_{\tilde{\rho} \in \Delta^\times r} \mathcal{O}_{\mathbf{p}_{G,g,r}}^{\text{zzz}}(\tilde{\rho}).
\]
Depending on the choice of \(\tilde{\rho}\), \(\mathcal{O}_{\mathbf{p}_{G,g,r}}^{\text{zzz}}(\tilde{\rho})\) may be empty. The following two lemmas will be applied to conclude Theorem 3.3.1 (iv), described later.

**Lemma 3.2.2.** Let \(S, \mathcal{X}, \text{ and } \mathcal{X}_{\text{tail}}\) be as in \[\S 2.3\] and let \(\tilde{\rho}\) be an element of \(\Delta^\times r\). Denote by \(\mathcal{O}_{\mathcal{X}_{\text{tail}}(\tilde{\rho}, \varepsilon)}\) (resp., \(\mathcal{O}_{\mathcal{X}_{\text{tail}}(\tilde{\rho}, \varepsilon)}\)) the full subcategory of \(\mathcal{D}_{\mathcal{X}_{\text{tail}}(\tilde{\rho}, \varepsilon)}\) consisting of \(G\)-do’pers. Then, \(\mathcal{O}_{\mathcal{X}_{\text{tail}}(\tilde{\rho}, \varepsilon)}\) is contained in \(\mathcal{O}_{\mathcal{X}_{\text{tail}}(\tilde{\rho}, \varepsilon)}\). In particular, the fully faithful functor \(\mathcal{O}_{\mathcal{X}_{\text{tail}}(\tilde{\rho}, \varepsilon)} \to \mathcal{O}_{\mathcal{X}_{\text{tail}}(\tilde{\rho}, \varepsilon)}\) constructed in \[\S 2.5\] restricts to an equivalence of categories \(\mathcal{O}_{\mathcal{X}_{\text{tail}}(\tilde{\rho}, \varepsilon)} \cong \mathcal{O}_{\mathcal{X}_{\text{tail}}(\tilde{\rho}, \varepsilon)}\).

**Proof.** Let us consider the former assertion. To complete the proof, it suffices to consider the case where \(G\) is of adjoint type and \(S = \text{Spec}(R)\) for some \(k\)-algebra \(R\). Moreover, by considering the formal neighborhood \(D := \text{Spec}(R[[t]])\) of \(\sigma_{X,r+1}\) in \(X\), we can reduce the problem to proving the following assertion: let \(\nabla\) be a dormant \(G\)-oper on \(D\) having regular singularity along the divisor \(t = 0\) with residue \(\varepsilon\) (in the sense of [15 §9.1]), then \(\nabla\) becomes, after the gauge transformation by some element of \(B(R((t)))\), a \(G\)-oper on \(D\). In what follows, we try to prove this assertion.

We shall write \(N := [B, B]\), i.e., the unipotent radical of \(B\), and write \(n\) for its Lie algebra. Also, let us fix a collection of generators \(\{x_\alpha\}_\alpha\) of \(\mathfrak{n}_{\alpha}\)'s, which induces an element \(p_{-1} \in \mathfrak{g}\) as mentioned in \[\S 2.1\]. Since we have assumed that \(G\) satisfies the condition \((*)_G\), the exponential map \(\text{Exp} : n \to N\) resulting from [57 Proposition 1.31] can be defined. By means of this, we can apply an argument similar to the argument in the proof of [15 Proposition 9.2.1] (for \(\lambda = 0\)) to our discussion even though we are working in positive characteristic (see also [57 Proposition 2.19]). It follows that, by the gauge transformation by some element of \(B(R((t)))\),
\[ \nabla \] may be brought to a connection of the form \[ \frac{dt}{dt} + p_{-1} + v(t) + \psi \] for some \( v(t) \in \mathfrak{b}(R[[t]]) \) and \( u \in \mathfrak{n} \). The mod \( t \) reduction of the \( p \)-curvature of \( \nabla \) with respect to this expression is given by \( u^{[p]} - u \), which must be equal to 0 because \( \nabla \) has vanishing \( p \)-curvature. But, since \( u \in \mathfrak{n} \) (which implies \( u^{[p]} = 0 \)), the equality \( u = 0 \) holds. Therefore, the connection \( \nabla \) is gauge equivalent to \[ \frac{dt}{dt} + p_{-1} + v(t) \] and forms a \( G \)-oper on \( D \). This completes the proof of the former assertion. The latter assertion follows from the former assertion and Proposition \ref{prop:2.5.1} (i).

**Lemma 3.2.3.** The unique (up to isomorphism) faithful twisted \( G \)-oper \( E^\bullet \) on \( \mathcal{P}'_k := \mathcal{P}'_{\text{Spec}(k)} \)

with \( \rho_{E^\bullet,1} = \rho^\vee_{E^\bullet,2} \in \Delta \) (cf. Proposition \ref{prop:2.6.1} (ii)) is dormant.

**Proof.** To complete the proof, it suffices to consider the case where \( G = G_{\text{ad}} \), \( k \) is algebraically closed, and \( E^\bullet \) is normal with respect to a fixed collection of generators \( \{ x_\alpha \}_\alpha \) of \( \mathfrak{g}^\alpha \) (cf. \ref{def:2.14}). Denote by \( \rho \in t^F_{\text{reg}} / W \) the radius of \( E^\bullet \) at \( \sigma_p,1 \). Let \( p_1 \) and \( p_{-1} \) be as in §2.1 Then, the connection \( \nabla \) defining \( E^\bullet \) may be expressed as \( \ref{eq:134} \), where the element \( v \) is taken to be \( v = p_{-1} + v_0 \) for a unique \( v_0 \in \mathfrak{g}^{\text{ad}(p_1)} \) with \( \chi(v) = \rho \) (cf. \ref{lem:1.2.3}). It follows from \ref{lem:1.2.3} that \( \nabla \) becomes, after the gauge transformation by some element of \( G \), a connection of the form \( \ref{eq:134} \) with \( v = \tilde{\rho} \) for some lifting \( \tilde{\rho} \in t^F_{\text{reg}} \) of \( \rho \). The \( p \)-curvature of \( \nabla \) with respect to this form is given by \( \left( s_1 \frac{d}{ds_1} \right)^{\otimes p} \mapsto 1 \otimes (\tilde{\rho}^{[p]} - \tilde{\rho}) \). But, the equality \( \tilde{\rho}^{[p]} = \tilde{\rho} \) holds since \( \tilde{\rho} \) belongs to \( t^F_{\text{reg}} \), so \( \nabla \) has vanishing \( p \)-curvature. That is to say, \( E^\bullet \) turns out to be dormant. \( \square \)

3.3. **The moduli space of \( G \)-do'pers.** In what follows, let us describe an assertion concerning the structure of the moduli stack of \( G \)-do'pers. This assertion contains a part of Theorem \ref{thm:A}

**Theorem 3.3.1.** (i) Isomorphism \( \ref{eq:110} \) restricts to an isomorphism

\[
\mathcal{D}p_{G,g,r} \xrightarrow{Z_{\text{ax}}} \mathcal{D}p_{G_{\text{ad}},g,r} \times_{\mathfrak{g}_{\text{reg}}} \mathcal{S}p_{Z,\delta,g,r}.
\]

In particular, the morphism \( \text{op}_{\text{ad}} \) (cf. \( \ref{eq:109} \)) restricts to a finite, flat, and generically étale morphism

\[
\text{op}_{\text{ad}} : \mathcal{D}p_{G_{\text{ad}},g,r} \to \mathcal{D}p_{G_{\text{ad},g,r}},
\]

and the following commutative diagram is cartesian:

\[
\begin{array}{ccc}
\mathcal{D}p_{G,g,r} & \xrightarrow{Z_{\text{ax}}} & \mathcal{D}p_{G_{\text{ad}},g,r} \\
\downarrow{}^{t_{g,r}} & & \downarrow{}^{t_{g,r}} \\
\mathcal{D}p_{G_{\text{ad}},g,r} & \xrightarrow{\text{op}_{\text{ad}}} & \mathcal{D}p_{G_{\text{ad}},g,r}.
\end{array}
\]

Moreover, for each \( \tilde{\rho} := (\rho_i, \kappa_i)_{i=1}^m \in \Delta^x_r \), isomorphism \( \ref{eq:148} \) restricts to an isomorphism

\[
\mathcal{D}p_{G,g,r,\tilde{\rho}} \xrightarrow{Z_{\text{ax}}} \mathcal{D}p_{G_{\text{ad}},g,r,(\rho_i)_{i=1}^m} \times_{\mathfrak{g}_{\text{reg}}} \mathcal{S}p_{Z,\delta^2,g,r,(\kappa_i)_{i=1}^m}.
\]
(ii) Both \( \mathcal{O}_{p_{G,g,r}} \) and \( \mathcal{O}_{p_{G,g,r,\bar{p}}} \) (for every \( \bar{p} \in \Delta^{x|r} \)) may be represented by (possibly empty) proper Deligne-Mumford stacks over \( k \) which are finite over \( \overline{\mathcal{M}}_{g,r} \). Moreover, \( \mathcal{O}_{p_{G,g,r}} \) is nonempty and has an irreducible component that dominates \( \overline{\mathcal{M}}_{g,r} \).

(iii) Let us assume further that \( G \) satisfies the condition (**)_G described in Introduction. Then, \( \mathcal{O}_{p_{G,g,r,\bar{p}}} \) is (finite and) flat over the points of \( \overline{\mathcal{M}}_{g,r} \) classifying pointed totally degenerate curves (cf. [57, Definition 7.15] for the definition of a pointed totally degenerate curve). Moreover, \( \mathcal{O}_{p_{G,g,r,\bar{p}}} \) admits a dense open subscheme which is étale over \( \overline{\mathcal{M}}_{g,r} \) and has generic stabilizer isomorphic to the center \( Z \) of \( G \).

(iv) The following assertion holds (without the assumption imposed in (iii)):

\[
\mathcal{O}_{p_{G,0,3},(\rho_1,\rho_2,\varepsilon)} \simeq \begin{cases} BZ & \text{if } \rho_1 \in \Delta \text{ and } \rho_1 = \rho_2^\vee; \\ \emptyset & \text{if otherwise.} \end{cases}
\]

Proof. Assertion (i) follows from the various definitions involved together with Theorem 1.3.5. Assertion (ii) follows from assertion (i) and [57, Theorem C]. Assertion (iii) follows from assertion (i), Theorem 1.3.5 (in the present paper), Theorem G in loc. cit., and [43, Chap. II, Theorem 2.8]. Finally, assertion (iv) follows from Corollary 2.6.2, Proposition 3.2.1 and Lemmas 3.2.2 and 3.2.3.

By the above theorem, it makes sense to speak of the \textit{generic degree}

\[
\deg_{\text{gen}}(\mathcal{O}_{p_{G,g,r,\bar{p}}}/\overline{\mathcal{M}}_{g,r}) (\in \mathbb{Q})
\]

of \( \mathcal{O}_{p_{G,g,r,\bar{p}}} \) (for each \( \bar{p} := ((\rho_i, \kappa_i))_{i=1}^r \in \Delta^{x|r} \)) over \( \overline{\mathcal{M}}_{g,r} \). Moreover, isomorphism (151) yields the equality

\[
\deg_{\text{gen}}(\mathcal{O}_{p_{G,g,r,\bar{p}}}/\overline{\mathcal{M}}_{g,r}) = \deg_{\text{gen}}(\mathcal{O}_{p_{G,\ad,g,r,\bar{p}},(\rho_i)_{i=1}^r}/\overline{\mathcal{M}}_{g,r}) \cdot \deg(\mathcal{O}_{p_{Z,\ad^*,g,r,(\kappa_i)_{i=1}^r}}/\overline{\mathcal{M}}_{g,r}).
\]

Then, we obtain the following three propositions.

**Proposition 3.3.2.** Assume that \( G \) satisfies the condition (**)_G. Let \( g_1, g_2, r_1, \) and \( r_2 \) be nonnegative integers with \( 2g_j + 1 + r_j > 0 \) \( (j = 1, 2) \), and let \( \bar{p}_j \in \Delta^{x|r_j} \). Write \( g := g_1 + g_2, \)

\( r = r_1 + r_2 \).

(i) For each \( \rho \in \Delta \), there exists a morphism

\[
\Phi_{\text{tree},\rho} : \mathcal{O}_{p_{G,g_1,r_1+1},(\bar{p}_1,\rho)} \times_{\ev_{r_1+1},\mathcal{M}_{g_0}} \mathcal{O}_{p_{G,g_2,r_2+1},(\bar{p}_2,\rho)} \to \mathcal{O}_{p_{G,g,r,\bar{p}_1,\rho}}
\]

obtained by gluing together two \( G \)-do’pers along the fibers over the respective last marked points of the underlying curves. Moreover, this morphism makes the following square diagram commute:

\[
\begin{array}{ccc}
\mathcal{O}_{p_{G,g_1,r_1+1},(\bar{p}_1,\rho)} \times_{\mathcal{M}_{g_0}} \mathcal{O}_{p_{G,g_2,r_2+1},(\bar{p}_2,\rho)} & \xrightarrow{\Phi_{\text{tree},\rho}} & \mathcal{O}_{p_{G,g,r,\bar{p}_1,\rho}} \\
\pi_{g_1,r_1} \times \pi_{g_2,r_2} & & \pi_{g,r} \\
\mathcal{M}_{g_1,r_1+1} \times_k \mathcal{M}_{g_2,r_2+1} & & \mathcal{M}_{g,r} \\
\end{array}
\]
(ii) The following equality holds:

\[
\deg_{\text{gen}}(\Omega_{\mathcal{P}_{G,g,r},(\vec{\rho},\vec{\epsilon})}/\overline{\mathcal{M}_{g,r}}) = |Z| \cdot \sum_{\rho \in \Delta} \prod_{j=1}^{2} \deg_{\text{gen}}(\mathcal{O}_{\mathcal{P}_{G,g,j+1,\rho_{j}},\vec{\rho},\vec{\epsilon}}/\overline{\mathcal{M}_{g,j+1}}).
\]

**Proof.** The assertions of the case where \( G = G_{\text{ad}} \) follow from [57, Theorem 7.13]. The general case can be reduced to that case because of Proposition 1.5.2, Theorem 3.3.1 (i), and equality (154). \( \square \)

**Proposition 3.3.3.** Assume that \( G \) satisfies the condition \((**)_G\). Let \((g, r)\) be a pair of nonnegative integers with \(2g + r > 0\), and let \( \vec{\rho} \in \Delta^r \).

(i) For each \( \rho \in \Delta \), there exists a morphism

\[
\Phi_{\text{loop}, \rho}^{Z_{\text{g}}}: \text{Ker} \left( \mathcal{O}_{\mathcal{P}_{G,g,r+2,\rho},(\vec{\rho},\vec{\epsilon})}/\mathcal{M}_{g,r} \right) \rightarrow \mathcal{O}_{\mathcal{P}_{G,g+1,\rho},(\vec{\rho},\vec{\epsilon})}/\mathcal{M}_{g+1,r}
\]

obtained by gluing each \( G \)-do'per along the fibers over the last two marked points of the underlying curve. Moreover, this morphism makes the following square diagram commute:

\[
\begin{array}{ccc}
\text{Ker} \left( \mathcal{O}_{\mathcal{P}_{G,g,r+2,\rho},(\vec{\rho},\vec{\epsilon})}/\mathcal{M}_{g,r} \right) & \xrightarrow{\Phi_{\text{loop}, \rho}^{Z_{\text{g}}}} & \mathcal{O}_{\mathcal{P}_{G,g+1,\rho},(\vec{\rho},\vec{\epsilon})}/\mathcal{M}_{g+1,r} \\
\pi_{g,r+2} & & \pi_{g+1,r} \\
\mathcal{M}_{g,r+2} & \xrightarrow{\Phi_{\text{loop}}} & \mathcal{M}_{g+1,r}
\end{array}
\]

(ii) The following equality holds:

\[
\deg_{\text{gen}}(\Omega_{\mathcal{P}_{G,g+1,\rho},(\vec{\rho},\vec{\epsilon})}/\overline{\mathcal{M}_{g+1,r}}) = \deg_{\text{gen}}(\Omega_{\mathcal{P}_{G,g,r},(\vec{\rho},\vec{\epsilon})}/\overline{\mathcal{M}_{g,r}}).
\]

**Proof.** The assertions of the case where \( G = G_{\text{ad}} \) follow from [57, Theorem 7.13]. The general case can be reduced to that case because of Proposition 1.5.3, Theorem 3.3.1 (i), and equality (154). \( \square \)

**Proposition 3.3.4.** Assume that \( G \) satisfies the condition \((**)_G\). Let \((g, r)\) be a pair of nonnegative integers with \(2g - 1 + r > 0\), and let \( \vec{\rho} \in \Delta^r \).

(i) There exists an isomorphism

\[
\mathcal{M}_{g,r+1} \times \overline{\mathcal{M}_{g,r}} \overset{\sim}{\rightarrow} \mathcal{M}_{g,r+1} \times \overline{\mathcal{M}_{g,r+1}} \overset{\sim}{\rightarrow} \mathcal{M}_{g,r+1} \times \overline{\mathcal{M}_{g,r+1}}
\]

over \( \mathcal{M}_{g,r+1} \).

(ii) The following equality holds:

\[
\deg_{\text{gen}}(\Omega_{\mathcal{P}_{G,g,r+1,\rho},(\vec{\rho},\vec{\epsilon})}/\overline{\mathcal{M}_{g,r+1}}) = \deg_{\text{gen}}(\Omega_{\mathcal{P}_{G,g,r,\rho},(\vec{\rho},\vec{\epsilon})}/\overline{\mathcal{M}_{g,r}}).
\]

**Proof.** The assertions follow from Proposition 1.6.2, Theorem 3.3.1 (i), and equality (154). \( \square \)
Remark 3.3.5. At the time of writing the present paper, the author does not know to what extent one can weaken the condition (***)$_G$ imposed in Theorem 3.3.1 (iii), and Propositions 3.3.2, 3.3.3, and 3.3.4.

4. The virtual fundamental class on the moduli of do'pers

In this section, we construct a perfect obstruction theory for the moduli stack $\mathcal{D}^{zar}_{G,g,r}$ (cf. Theorem 4.2.1). As a result, we obtain a virtual fundamental class on that moduli (cf. (179)). This virtual class will be used to construct a CohFT (forming, in fact, a 2d TQFT) of $G$-do'pers.

Let us keep the notation at the beginning of §3.

4.1. General definition of a perfect obstruction theory. First, we shall recall from [10] the notions of a perfect obstruction theory and the virtual fundamental class associated with it.

Let $\mathfrak{X}$ be a separated Deligne-Mumford stack, locally of finite type over $k$. Denote by $D(O_{\mathfrak{X}})$ the derived category of the category $\text{Mod}(O_{\mathfrak{X}})$ of $O_{\mathfrak{X}}$-modules and by $L^\bullet_{\mathfrak{X}} \in \text{Ob}(D(O_{\mathfrak{X}}))$ the cotangent complex of $\mathfrak{X}$ relative to $k$.

For a morphism $E^0_{\mathfrak{X}} \to E^1_{\mathfrak{X}}$ of abelian sheaves on the big fppf site $\mathfrak{X}_{fl}$ of $\mathfrak{X}$, one obtains the quotient stack $[E^1_{\mathfrak{X}}/E^0_{\mathfrak{X}}]$. That is to say, for an object $T \in \text{Ob}(\mathfrak{X}_{fl})$ the groupoid $[E^1_{\mathfrak{X}}/E^0_{\mathfrak{X}}](T)$ of sections over $T$ is the category of pairs $(P,f)$, where $P$ is an $E^0_{\mathfrak{X}}$-torsor over $T$ and $f$ is an $E^0_{\mathfrak{X}}$-equivariant morphism $P \to E^1_{\mathfrak{X}}|_T$ of sheaves on $\mathfrak{X}_{fl}$. If $E^\bullet_{\mathfrak{X}}$ is a complex of arbitrary length of abelian sheaves on $\mathfrak{X}_{fl}$, we shall write

$$h^1/h^0(E^\bullet_{\mathfrak{X}}) := [Z^1/C^0],$$

where $Z^1 := \text{Ker}(E^1_{\mathfrak{X}} \to E^2_{\mathfrak{X}})$, $C^0 := \text{Coker}(E^{-1}_{\mathfrak{X}} \to E^0_{\mathfrak{X}})$. Denote by $\mathfrak{N}_{\mathfrak{X}}$ the stack defined to be $\mathfrak{N}_{\mathfrak{X}} := h^1/h^0(((L^\bullet_{\mathfrak{X}})_{\mathfrak{X}})^\vee)$, where for each complex $E^\bullet$ on (the small étale site of) $\mathfrak{X}$, we shall write $L^\bullet_{\mathfrak{X}}$ for the complex on $\mathfrak{X}_{fl}$ associated with $E^\bullet$.

Let us take a diagram

$$\begin{array}{ccc}
U & \overset{\iota}{\longrightarrow} & M \\
\downarrow{\pi} & & \downarrow{\pi} \\
\mathfrak{X}, & & \\
\end{array}$$

where $U$ denotes an affine scheme of finite type over $k$, $M$ denotes a smooth affine scheme of finite type over $k$, $\iota$ is a closed immersion, and $\pi$ is an étale morphism. Let $I$ denote the ideal sheaf on $M$ defining the closed subscheme $U$, and we consider the natural morphism $I/I^2 \to \iota^*(\Omega_{M/k})$ as a complex $[I/I^2 \to \iota^*(\Omega_{M/k})]$ concentrated in degrees $-1$ and $0$. Then, there exists a natural quasi-isomorphism

$$\phi : L^\bullet_{\mathfrak{X}}|_U \cong [I/I^2 \to \iota^*(\Omega_{M/k})].$$

Write $T_M := \text{Spec}(\text{Sym}_{\mathfrak{O}_M}(\Omega_{M/k}))$ (i.e., the total space of the tangent bundle $T_{M/k}$) and $N_{U/M} := \text{Spec}(\text{Sym}_{\mathfrak{O}_U}(I/I^2))$. Note that $N_{U/M}$ admits an $\iota^*(T_M)$-action induced from the
morphism $I/I^2 \to \iota^*(\Omega_{M/k})$. Quasi-isomorphism (165) gives rise to an isomorphism
\begin{equation}
[N_{U/M}/\iota^*(T_M)] \xrightarrow{\sim} \mathcal{N}_\mathcal{X}|_U.
\end{equation}

The intrinsic normal cone of $\mathcal{X}$ (cf. [10, Definition 3.10]) is defined as a unique closed substack $\mathcal{C}_X$ of $\mathcal{N}_\mathcal{X}$ determined by the condition that if we are given a diagram as in (164), then $\mathcal{C}_X|_U$ may be identified, via (166), with the closed substack $[C_{U/M}/\iota^*(T_M)]$, where $C_{U/M}$ denotes the normal cone $\text{Spec}(\bigoplus_{n \geq 0} I^n/I^{n+1})$ of $U$ in $M$. A perfect obstruction theory for $\mathcal{X}$ (cf. [10, Definitions 4.4 and 5.1]) is a morphism $\phi : E^\bullet \to L^\bullet_{\mathcal{X}}$ in $D(\mathcal{O}_{\mathcal{X}})$ satisfying the following conditions:

- $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective.
- $E^\bullet$ is of perfect amplitude contained in $[-1, 0]$, i.e., is locally isomorphic (in $D(\mathcal{O}_{\mathcal{X}})$) to a complex $[E^{-1} \to E^0]$ of locally free sheaves of finite rank.

Let $\phi : E^\bullet \to L^\bullet_{\mathcal{X}}$ be a perfect obstruction theory for $\mathcal{X}$. The virtual dimension of $\mathcal{X}$ with respect to $\phi : E^\bullet \to L^\bullet_{\mathcal{X}}$ is a well-defined locally constant function on $\mathcal{X}$, denoted by $\text{rk}(E^\bullet)$, defined in such a way that if $E^\bullet$ is locally written as a complex of vector bundles $[E^{-1} \to E^0]$, then $\text{rk}(E^\bullet) := \dim(E^0) - \dim(E^{-1})$. Let us suppose further that $\text{rk}(E^\bullet)$ is constant and that $E^\bullet$ has a global resolution, i.e., has a morphism of vector bundles $F^\bullet := [F^{-1} \to F^0]$ considered as a complex concentrated in degrees $-1$ and $0$ together with an isomorphism $F^\bullet \xrightarrow{\sim} E^\bullet$ in $D(\mathcal{O}_{\mathcal{X}})$. Since $h^1/h^0((E^\bullet)^\vee)$ is isomorphic to $[(F^{-1}_n)^\vee/(F^0_n)^\vee]$, the relative affine space $\mathcal{W}$ associated with $(F^{-1}_n)^\vee$ specifies a global presentation $\mathcal{W} \to h^1/h^0((E^\bullet)^\vee)$. Let $\mathcal{W}$ be the fiber product
\begin{equation}
\begin{aligned}
\mathcal{W} & \longrightarrow \mathcal{W} \\
\downarrow & \quad \quad \downarrow \\
\mathcal{C}_\mathcal{X} & \longrightarrow h^1/h^0((E^\bullet)^\vee),
\end{aligned}
\end{equation}
where the lower horizontal arrow denotes the composite of the closed immersion $\mathcal{C}_\mathcal{X} \to \mathcal{N}_\mathcal{X}$ and the morphism $\mathcal{N}_\mathcal{X} \to h^1/h^0((E^\bullet)^\vee)$ induced by $\phi$, being a closed immersion (cf. [10, Theorem 4.5]). We define the virtual fundamental class $[\mathcal{X}, E^\bullet]^\text{virt}$ to be the intersection of $\mathcal{W}$ with the zero section $0_\mathcal{W} : \mathcal{X} \to \mathcal{W}$, i.e.,
\begin{equation}
[\mathcal{X}, E^\bullet]^\text{virt} := 0_\mathcal{W} [\mathcal{W}],
\end{equation}
which is an element of $A_{\text{rk}(E^\bullet)}(\mathcal{X})\mathbb{Q}$. This class is independent of the global resolution $F^\bullet$ used to construct it. If $\mathcal{X}$ is smooth, then the virtual fundamental class $[\mathcal{X}, L^\bullet_{\mathcal{X}}]^\text{virt}$ is equal to the usual fundamental class $[\mathcal{X}]$.

4.2. The perfect obstruction theory for the moduli space of $G$-do’pers. Let $(g, r)$ be a pair of nonnegative integers with $2g - 2 + r > 0$. The goal of this subsection is to construct a perfect obstruction theory for $\mathcal{O}_{G,g,r}$.

Denote by $\mathcal{C}\text{on}_{G,g,r}$ the category classifying triples $(\mathcal{X}, \mathcal{E}, \nabla)$ consisting of an $r$-pointed stable curve $\mathcal{X} := (X/S, \{x_i\}_{i=1}^r)$ of genus $g$ over a $k$-scheme $S$, a $G$-bundle $\mathcal{E}$ on $X$, and an $S^\text{log}$-connection $\nabla$ on $\mathcal{E}$. This category may be represented by a smooth algebraic stack over $k$. Also, denote by $\mathcal{C}\text{on}_{G,g,r}^{\text{virt}}$ the closed substack of $\mathcal{C}\text{on}_{G,g,r}$ classifying triples $(\mathcal{X}, \mathcal{E}, \nabla)$ with $\psi(\mathcal{E}, \nabla) = 0$. The assignment from each pair $(\mathcal{X}, \mathcal{E}^\bullet)$ (where $\mathcal{X}$ denotes a pointed stable curve and $\mathcal{E}^\bullet$ denotes a faithful twisted $G$-oper on $\mathcal{X}$) to the triple $(\mathcal{X}, \mathcal{E}_G, \mathcal{E}^\bullet)$ (where $\mathcal{E}_G$ denotes a smooth stabilization of $\mathcal{E}$) is a perfect obstruction theory for $\mathcal{X}$ on $\mathcal{X}$.
denotes the $S^{\log}$-connection defining the $\{x_\alpha\}_{\alpha}$-normalization of $E^\bullet$ asserted in [57, Proposition 2.19]) determines morphisms

$$
\xi : \mathcal{O}p_{G,g,r} \to \text{Conn}_{G,g,r}, \quad \xi^{2g} : \mathcal{O}p_{G,g,r} \to \text{Conn}^{2g}_{G,g,r}
$$

over $\mathfrak{M}_{g,r}$. The following square diagram is commutative and cartesian:

$$
\begin{array}{ccc}
\text{Conn}^{2g}_{G,g,r} & \xrightarrow{\xi^{2g}} & \text{Conn}_{G,g,r} \\
\uparrow{\xi} & & \uparrow{\xi} \\
\mathcal{O}p_{G,g,r} & \xrightarrow{\xi^{2g}} & \mathcal{O}p_{G,g,r},
\end{array}
$$

(169)

where the upper horizontal arrow $\xi^{2g}$ denotes the natural closed immersion. Denote by $\mathcal{I}_G$ (resp., $\hat{\mathcal{I}}_G$) the ideal sheaf on $\mathcal{O}p_{G,g,r}$ (resp., $\text{Conn}_{G,g,r}$) defining the closed substack $\mathcal{O}p^{2g}_{G,g,r}$ (resp., $\text{Conn}^{2g}_{G,g,r}$). Then, diagram (170) induces a commutative diagram of coherent sheaves on $\mathcal{O}p^{2g}_{G,g,r}$.

$$
\begin{array}{ccc}
\xi^{2g} : (\mathcal{I}_G/\mathcal{I}_G^2) & \xrightarrow{\xi^{2g}} & ((\xi^{2g}) \circ \xi^{2g})^*(\Omega_{\text{Conn}_{G,g,r}/k}) \\
\downarrow{\xi} & & \downarrow{\xi} \\
\mathcal{I}_G/\mathcal{I}_G^2 & \xrightarrow{\iota_{g,r}} & \iota_{g,r}^*(\Omega_{\mathcal{O}p_{G,g,r}/k}).
\end{array}
$$

(170)

Since (170) is cartesian, $\xi^{2g}$ is verified to be surjective. Let us consider the composite $\xi^{2g} \circ \xi^{2g}$ (= $\xi \circ \xi^{2g}$) as a complex

$$
E^\bullet := [\xi^{2g} : (\mathcal{I}_G/\mathcal{I}_G^2) \to \iota_{g,r}^*(\Omega_{\mathcal{O}p_{G,g,r}/k})]
$$

(172)

concentrated in degrees $-1$ and $0$. The Deligne-Mumford stack $\mathcal{O}p_{G,g,r}$ is smooth over $k$ (cf. Theorem 2.3.5 (ii)), so the cotangent complex $L^\bullet_{\mathcal{O}p^{2g}_{G,g,r}}$ is naturally isomorphic (in $D(\mathcal{O}_{\mathcal{O}p^{2g}_{G,g,r}})$) to the complex $[\mathcal{I}_G/\mathcal{I}_G^2 \to \iota_{g,r}^*(\Omega_{\mathcal{O}p_{G,g,r}/k})]$. Thus, the pair of $\xi^{2g}$ and the identity morphism of $\iota_{g,r}^*(\Omega_{\mathcal{O}p_{G,g,r}/k})$ specifies a morphism

$$
\phi : E^\bullet \to L^\bullet_{\mathcal{O}p^{2g}_{G,g,r}}
$$

(173)

in $D(\mathcal{O}_{\mathcal{O}p^{2g}_{G,g,r}})$.

Theorem 4.2.1. Suppose that $G$ is of adjoint type. Then, the morphism $\phi$ just obtained forms a perfect obstruction theory for $\mathcal{O}p^{2g}_{G,g,r}$ of constant virtual dimension $3g-3+r$ with $E^\bullet$ perfect.

Proof. It follows from Theorem 2.3.5 (ii), that $\Omega_{\mathcal{O}p_{G,g,r}/k}$ is locally free of rank $3g-3+r+N$, where $N := (g-1) \cdot \dim(G) + \frac{r}{2} \cdot (\dim(G) + \text{rk}(G))$. Hence, the problem is reduced to proving that $\xi^{2g} : (\mathcal{I}_G/\mathcal{I}_G^2)$ is locally free of rank $N$.

For simplicity, let us denote by $\mathcal{E} := (f : X \to S, \{\sigma_X,i\}_{i=1}^r)$ the tautological family of pointed stable curves over $S := \mathcal{O}p_{G,g,r}$. Also, denote by $\mathcal{E}^\bullet := (\mathcal{E}_B, \nabla)$ the tautological dormant $G$-do’per on $\mathcal{E}$. Let $\nabla^{\text{ad}} : \mathfrak{g} \to \Omega_{X^{\log}/\mathcal{E}^{\log}} \otimes \mathfrak{g}_G$ be the $S^{\log}$-connection on the
adjoint vector bundle $g_{CE}$ induced by $\nabla$; we regard it as a complex $\mathcal{K}^\bullet[\nabla^{ad}]$ concentrated in degrees 0 and 1. Note that there exists a short exact sequence of $\mathcal{O}_S$-modules
\[(174) \quad 0 \to \mathbb{R}^1 f_*(\text{Ker}(\nabla^{ad})) \to \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla^{ad}]) \to f_*(\text{Coker}(\nabla^{ad})) \to 0\]
arising from the conjugate spectral sequence of $\mathcal{K}^\bullet[\nabla^{ad}]$ (cf. [57, Proposition 6.5, (iii)]). It follows from well-known generalities of deformation theory that one may construct a canonical isomorphism
\[(175) \quad (i_{g,r} \circ \xi^{2a})^*(\mathcal{T}_{\text{conn}G,g,r/\overline{\mathbb{M}}_{g,r}}) \sim \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla^{ad}]).\]
This isomorphism restricts to an isomorphism
\[(176) \quad \xi^{2a}_*(\mathcal{T}_{\text{conn}G,g,r/\overline{\mathbb{M}}_{g,r}}) \sim \mathbb{R}^1 f_*(\text{Ker}(\nabla^{ad}))\]
(cf. [57, Proposition 6.11]). According to [57, Proposition 6.18], $f_*(\text{Coker}(\nabla^{ad}))$ is locally free of rank $N$. Hence, the duals of (175) and (176) induce a commutative diagram
\[
\begin{array}{c}
0 \\
\downarrow \\
\xi^{2a}_*(\mathcal{T}_{\mathcal{I}_G/\mathcal{I}_G}) \\
\downarrow (i_{g,r} \circ \xi^{2a})_*(\Omega_{\text{conn}G,g,r/\overline{\mathbb{M}}_{g,r}}) \\
\downarrow \\
\xi^{2a}_*(\Omega_{\text{conn}G,g,r/\overline{\mathbb{M}}_{g,r}}) \\
\downarrow (176) \\
0
\end{array}
\begin{array}{c}
\longrightarrow \\
\longrightarrow
\end{array}
\begin{array}{c}
f_*(\text{Coker}(\nabla^{ad}))^\vee \\
R^1 f_*(\mathcal{K}^\bullet[\nabla^{ad}])^\vee \\
R^1 f_*(\text{Ker}(\nabla^{ad}))^\vee \\
R^1 f_*(\text{Ker}(\nabla^{ad}))^\vee \\
0
\end{array}
\]
where the both sides of vertical sequences are exact. Since we have the equalities $\mathbb{R}^2 f_*(\mathcal{K}^\bullet[\nabla^{ad}]) = 0$ (cf. [57, Proposition 6.5, (iii)]) and $\mathbb{R}^2 f_*(\text{Ker}(\nabla^{ad})) = 0$ (which follows from $\text{dim}(X/S) = 1$), the constructions of (175) and (176) implies that both $\mathcal{O}_{\text{conn}G,g,r}$ and $\mathcal{O}_{\text{conn}G,g,r}$ are smooth over $\overline{\mathbb{M}}_{g,r}$ at the points lying over $S$. Hence, the left-hand top vertical arrow in (177) is injective. It follows that the top horizontal arrow $\xi^{2a}_*(\mathcal{T}_{\mathcal{I}_G/\mathcal{I}_G}) \to f_*(\text{Coker}(\nabla^{ad}))^\vee$ is an isomorphism. Consequently, $\xi^{2a}_*(\mathcal{T}_{\mathcal{I}_G/\mathcal{I}_G})$ is verified to be locally free of rank $N$. This completes the proof of the theorem. \hfill \Box

By the above result, we obtain the virtual fundamental class
\[(178) \quad [\mathcal{O}_{\mathcal{P}_{\mathcal{G}_{ad},g,r}}]^\text{vir} := [\mathcal{O}_{\mathcal{P}_{\mathcal{G}_{ad},g,r}}]^{2a} \in A_{3g-3+r}(\mathcal{O}_{\mathcal{P}_{\mathcal{G}_{ad},g,r}})_Q.\]

Moreover, since the morphism $o_{\mathcal{P}_{\mathcal{G}_{ad},g,r}}^{2a}$ (cf. (149)) is finite and flat (cf. Theorem 3.3.1 (i)), the pull-back of this cycle class by $o_{\mathcal{P}_{\mathcal{G}_{ad},g,r}}$ can be defined; we shall denote it by
\[(179) \quad [\mathcal{O}_{\mathcal{P}_{\mathcal{G},g,r}}]^\text{vir} := (o_{\mathcal{P}_{\mathcal{G}_{ad},g,r}})^*[\mathcal{O}_{\mathcal{P}_{\mathcal{G}_{ad},g,r}}]^\text{vir} \in A_{3g-3+r}(\mathcal{O}_{\mathcal{P}_{\mathcal{G},g,r}})_Q.\]

**Definition 4.2.2.** We shall refer to $[\mathcal{O}_{\mathcal{P}_{\mathcal{G},g,r}}]^\text{vir}$ as the virtual fundamental class on $\mathcal{O}_{\mathcal{P}_{\mathcal{G},g,r}}$. 
Remark 4.2.3. In the case of $G = \text{PGL}_2$, the moduli stack $\mathcal{O}_{\text{PGL}_2,g,r}^{\text{Zas}}$ is smooth over $k$ (cf. [43 Chap. II, Theorem 2.8]). In particular, the virtual fundamental class $[\mathcal{O}_{\text{PGL}_2,g,r}^{\text{Zas}}]_{\text{vir}}$ coincides with the usual fundamental class $[\mathcal{O}_{\text{PGL}_2,g,r}]$.

5. 2d TQFTs for do’pers

In this section, we define the notion of a CohFT ($= \text{a cohomological field theory}$) mapped into the $l$-adic étale cohomology groups of $\mathcal{M}_{g,r}$’s. After that, a CohFT for $G$-do’pers is constructed by using the virtual fundamental class obtained in the previous section and the factorization properties resulting from Propositions 3.3.2, 3.3.3, and 3.3.4. It also forms a 2d TQFT ($= \text{a 2-dimensional topological quantum field theory}$) such that the corresponding Frobenius algebra is semisimple.

Let us keep the notation at the beginning of §3. Moreover, we suppose that $k$ is algebraically closed. Also, let us fix a prime $l$ different from $p = \text{char}(k)$.

5.1. $l$-adic cohomology and Borel-Moore homology. Let $\mathcal{M}$ be a separated Deligne-Mumford stack of finite type over $k$. Denote by $D^b_c(\mathcal{M}, \overline{\mathbb{Q}}_l)$ the derived category of constructible $\overline{\mathbb{Q}}_l$-modules on $\mathcal{M}$. Hence, for each complex $\mathcal{L}$ in $D^b_c(\mathcal{M}, \overline{\mathbb{Q}}_l)$, we obtain its étale cohomology $H^i_{\text{ét}}(\mathcal{M}, \mathcal{L})$ ($i \in \mathbb{Z}$). Also, we write

\[
\widetilde{H}^i_{\text{ét}}(\mathcal{M}, \mathcal{L}) := H^i_{\text{ét}}(\mathcal{M}, \mathcal{L}([\frac{i}{2}))), \quad \overline{H}^i_{\text{ét}}(\mathcal{M}, \mathcal{L}) := \bigoplus_{i \in \mathbb{Z}} \widetilde{H}^i_{\text{ét}}(\mathcal{M}, \mathcal{L}).
\]

For each $i \in \mathbb{Z}$, write $H^i_{\text{BM}}(\mathcal{M}, \overline{\mathbb{Q}}_l)$ for the $i$-th Borel-Moore homology of $\mathcal{M}$ defined in [49 Definition 2.2]. That is to say, we set

\[
H^i_{\text{BM}}(\mathcal{M}, \overline{\mathbb{Q}}_l) := H^{-i}_{\text{ét}}(\mathcal{M}, \omega_{\mathcal{M}}),
\]

where $\omega_{\mathcal{M}} \in D^b_c(\mathcal{M}, \overline{\mathbb{Q}}_l)$ denotes the $l$-adic dualizing complex of $\mathcal{M}$. Also, set

\[
\widetilde{H}^i_{\text{BM}}(\mathcal{M}, \overline{\mathbb{Q}}_l) := H^i_{\text{BM}}(\mathcal{M}, \overline{\mathbb{Q}}_l)(-[\frac{i}{2}]), \quad \overline{H}^i_{\text{BM}}(\mathcal{M}, \overline{\mathbb{Q}}_l) := \bigoplus_{i \in \mathbb{Z}} \widetilde{H}^i_{\text{BM}}(\mathcal{M}, \overline{\mathbb{Q}}_l).
\]

The $i$-th cycle map is the $\mathbb{Q}$-linear map

\[
\text{cl}^i : A_i(\mathcal{M}) \otimes \mathbb{Q} \to \widetilde{H}^i_{\text{BM}}(\mathcal{M}, \overline{\mathbb{Q}}_l)
\]

mentioned in [49 §2.10]. If $\mathcal{M}$ is smooth of dimension $d$, then $\omega_{\mathcal{M}} \cong \overline{\mathbb{Q}}_l(d)[2d]$ and hence, we obtain a composite of natural isomorphisms

\[
(-)^{\vee} : \overline{H}^i_{\text{BM}}(\mathcal{M}, \overline{\mathbb{Q}}_l) \cong \bigoplus_{i \geq 0} H^{-i}_{\text{ét}}(\mathcal{M}, \overline{\mathbb{Q}}_l(d)[2d])(-[\frac{i}{2}])
\]

\[
\cong \bigoplus_{i \geq 0} H^{2d-i}_{\text{ét}}(\mathcal{M}, \overline{\mathbb{Q}}_l([\frac{2d-i}{2}]))
\]

\[
\cong \widetilde{H}^{2d-i}_{\text{ét}}(\mathcal{M}, \overline{\mathbb{Q}}_l).
\]

Then, the cycle map $\text{cl}^i$ coincides, via (183), with the classical definition of the cycle map $A_i(\mathcal{M}) \otimes \mathbb{Q} \to \widetilde{H}^{2d-2i}_{\text{ét}}(\mathcal{M}, \overline{\mathbb{Q}}_l)$.  


By the definition of $\tilde{H}^*_{BM}(-, \overline{Q}_l)$, the cup product in $l$-adic cohomology induces a natural pairing
\begin{equation}
(-) \cap (-) : \tilde{H}^{2j}_{\text{ét}}(M, \overline{Q}_l) \otimes_{\overline{Q}_l} \tilde{H}^{2i}_{BM}(M, \overline{Q}_l) \to \tilde{H}^{2i}_{BM}(\overline{Q}_l, \overline{Q}_l).
\end{equation}

If $M$ is another separated Deligne-Mumford stack of finite type over $k$ and $f : M \to N$ is a proper morphism over $k$, then there exists the pushforward map
\begin{equation}
f_*^{\text{hom}} : \tilde{H}^*_{BM}(M, \overline{Q}_l) \to \tilde{H}^*_{BM}(N, \overline{Q}_l)
\end{equation}
along $f$ described in [49, §2.10]. The following projection formula may be immediately verified:
\begin{equation}
f_*^{\text{hom}}(f^*(\alpha) \cap \beta) = \alpha \cap f_*^{\text{hom}}(\beta) \in \tilde{H}^{2l(\text{dim } N)}_{BM}(N, \overline{Q}_l),
\end{equation}
where $\alpha \in \tilde{H}^{2j}_{\text{ét}}(M, \overline{Q}_l)$ and $\beta \in \tilde{H}^{2i}_{BM}(N, \overline{Q}_l)$ (cf. [37, Proposition 5.2]). If, moreover, $M = \text{Spec}(k)$, then by composing $f_*^{\text{hom}}$ and (185), we obtain, for each class $\alpha \in A_i(M)_{\mathbb{Q}}$, a $\overline{Q}_l$-linear morphism
\begin{equation}
\int : \tilde{H}^{2j}_{\text{ét}}(M, \overline{Q}_l) \to \tilde{H}^{2j}_{BM}(\overline{Q}_l, \overline{Q}_l) \to \tilde{H}^0_{BM}(\text{Spec}(k), \overline{Q}_l) \to \overline{Q}_l.
\end{equation}

By assigning $\nu \mapsto 0$ for any $\nu \in \tilde{H}_j^{2j}(M, \overline{Q}_l)$ with $j \neq 2i$, we shall regard $\int$ as a morphism $\tilde{H}^{2j}_{\text{ét}}(M, \overline{Q}_l) \to \overline{Q}_l$.

5.2. $l$-adic CohFTs and 2d TQFTs. We shall describe the definition of a cohomological field theory by means of the $l$-adic étale cohomologies $\tilde{H}^*_{BM}(\overline{M}_{g,r}, \overline{Q}_l)$ of the stacks $\overline{M}_{g,r}$. Also, we recall the notion of a 2-dimensional topological quantum field theory as a special kind of cohomological field theory.

**Definition 5.2.1.** (i) A(n) ($l$-adic) cohomological field theory (with flat identity), abbreviated CohFT, is a collection of data
\begin{equation}
\Lambda := (\mathcal{H}, \eta, 1, \{\Lambda_{g,r}\}_{g,r \geq 0, 2g-2+r > 0})
\end{equation}
consisting of
- a finite dimensional $\overline{Q}_l$-vector space $\mathcal{H}$ (called the state space) with basis $e := \{e_1, \ldots, e_{\text{dim}(\mathcal{H})}\}$;
- a symmetric nondegenerate pairing $\eta : \mathcal{H} \times \mathcal{H} \to \overline{Q}_l$ (called the metric), where we shall write $(\eta^{\text{act}_{b}})_{a,b}$ for the inverse of the matrix corresponding to the metric $\eta$ with respect to the basis $e$;
- an element $1$ of $\mathcal{H}$;
- $\overline{Q}_l$-linear morphisms $\Lambda_{g,r} : \mathcal{H}^{\otimes r} \to \tilde{H}^*_{BM}(\overline{M}_{g,r}, \overline{Q}_l)$ (called the correlators), where $\mathcal{H}^{\otimes 0} := \overline{Q}_l$,
and satisfying the following conditions:
- Each $\Lambda_{g,r}$ is compatible with the respective actions of the symmetric group $S_r$ on $\mathcal{H}^{\otimes r}$ and $\tilde{H}^*_{BM}(\overline{M}_{g,r}, \overline{Q}_l)$ arising from permutations of the $r$ factors in $\mathcal{H}^{\otimes r}$ and the $r$ punctures in the tautological family of curves over $\overline{M}_{g,r}$ respectively.
• For any \( v_1, v_2 \in \mathcal{H} \), the following equality holds:

\[
\eta(v_1, v_2) = \int_{\mathcal{M}_{0,3}} \Lambda_{0,3}(v_1 \otimes v_2 \otimes 1).
\]

• For any \( v_1, \ldots, v_{r+2} \in \mathcal{H} \), the following equality holds:

\[
\Phi^*_\text{tree}(\Lambda_{g_1+g_2, r+1}(v_1 \otimes \cdots \otimes v_{r+2})) = \sum_{\epsilon_1, \epsilon_2 \in \mathbb{S}} \Lambda_{g_1,r+1}(v_1 \otimes \cdots \otimes v_r \otimes \epsilon_1 \otimes \epsilon_2) \eta^{\epsilon_1 \epsilon_2} \otimes \Lambda_{g_2,r+1}(e_2 \otimes v_{r+1} \otimes \cdots \otimes v_{r+2}),
\]

where \( \Phi^*_\text{tree} \) denotes the morphism

\[
\tilde{H}^*_\text{et}(\overline{\mathbb{M}}_{g_1+g_2, r+2}, \overline{\mathbb{Q}}_l) \to \tilde{H}^*_\text{et}(\overline{\mathbb{M}}_{g_1,r+1}, \overline{\mathbb{Q}}_l) \otimes_{\overline{\mathbb{Q}}_l} \tilde{H}^*_\text{et}(\overline{\mathbb{M}}_{g_2,r+1}, \overline{\mathbb{Q}}_l)
\]

induced by \( \Phi^*_\text{tree} \) (cf. \[18\]).

• For any \( v_1, \ldots, v_r \in \mathcal{H} \), the following equality holds:

\[
\Phi^*_\text{loop}(\Lambda_{g+1,r}(v_1 \otimes \cdots \otimes v_r)) = \sum_{\epsilon_1, \epsilon_2 \in \mathbb{S}} \Lambda_{g,r+1}(v_1 \otimes \cdots \otimes v_r \otimes \epsilon_1 \otimes \epsilon_2) \eta^{\epsilon_1 \epsilon_2},
\]

where \( \Phi^*_\text{loop} \) denotes the morphism \( \tilde{H}^*_\text{et}(\overline{\mathbb{M}}_{g+1,r}, \overline{\mathbb{Q}}_l) \to \tilde{H}^*_\text{et}(\overline{\mathbb{M}}_{g,r+2}, \overline{\mathbb{Q}}_l) \) induced by \( \Phi^*_\text{loop} \) (cf. \[19\]).

• For any \( v_1, \ldots, v_r \in \mathcal{H} \), the following equality holds:

\[
\Phi^*_\text{tail}(\Lambda_{g,r}(v_1 \otimes \cdots \otimes v_r)) = \Lambda_{g,r+1}(v_1 \otimes \cdots \otimes v_r \otimes 1),
\]

where \( \Phi^*_\text{tail} \) denotes the morphism \( \tilde{H}^*_\text{et}(\overline{\mathbb{M}}_{g,r}, \overline{\mathbb{Q}}_l) \to \tilde{H}^*_\text{et}(\overline{\mathbb{M}}_{g,r+1}, \overline{\mathbb{Q}}_l) \) induced by \( \Phi^*_\text{tail} \) (cf. \[20\]).

(ii) A 2-dimensional topological quantum field theory (over \( \overline{\mathbb{Q}}_l \)), abbreviated 2d TQFT, is an \( l \)-adic cohomological field theory whose correlators \( \Lambda_{g,r} \) are all valued in \( \tilde{H}^0_\text{et}(\overline{\mathbb{M}}_{g,r}, \overline{\mathbb{Q}}_l) = \overline{\mathbb{Q}}_l \) (cf. \[30\] §1.3.32) for the naive definition of an \( n \)-dimensional topological quantum field theory).

It is well-known that 2d TQFTs correspond to Frobenius algebras. Here, by a Frobenius algebra over \( \overline{\mathbb{Q}}_l \), we mean (cf. \[30\] §2.2.5) a pair \((\mathcal{H}, \eta)\) consisting of a unital, associative, and commutative \( \overline{\mathbb{Q}}_l \)-algebra \( \mathcal{H} \) of finite dimension and a nondegenerate \( \overline{\mathbb{Q}}_l \)-bilinear pairing \( \eta: \mathcal{H} \times \mathcal{H} \to \overline{\mathbb{Q}}_l \) such that

\[
\eta(v_1, (v_2 \times v_3)) = \eta((v_1 \times v_2), v_3)
\]

for any \( v_1, v_2, v_3 \in \mathcal{H} \), where \( \times \) denotes the multiplication in \( \mathcal{H} \). We shall say that a Frobenius algebra \((\mathcal{H}, \eta)\) is semisimple if there exists a basis \( e^i := \{ e^i_a \}_{a \in I} \) of \( \mathcal{H} \) such that

\[
e^i_a \times e^j_b = \delta_{ab} e^j_a \quad \text{and} \quad \eta(e^i_a, e^j_b) = \delta_{ij} \nu_a
\]

for any \( a, b \in I \), where each \( \nu_a \) is some nonzero element of \( \overline{\mathbb{Q}}_l \). We shall refer to \( e^i \) as a canonical basis of \((\mathcal{H}, \eta)\).

Now, let \( \Lambda := (\mathcal{H}, \eta, 1, \{ \Lambda_{g,r} \}_{g,r}) \) be a 2d TQFT over \( \overline{\mathbb{Q}}_l \) (where let \( \mathfrak{e} := \{ e^i_a \}_{a \in I} \) denote the distinguished basis of \( \mathcal{H} \)). There exists a multiplication \( \times: \mathcal{H} \times \mathcal{H} \to \mathcal{H} \) given by

\[
u_a \times e^j_b := \sum_{a, b \in I} \Lambda_{0,3}(u \otimes v \otimes e^i_a) \eta^{ab} e^j_b.
\]
The \( \mathbb{Q}_l \)-vector space \( \mathcal{H} \) together with this multiplication forms a unital, associative, and commutative \( \mathbb{Q}_l \)-algebra, in which \( 1 \) is the unit. Moreover, the pair \( (\mathcal{H}, \eta) \) forms a Frobenius algebra over \( \mathbb{Q}_l \), which we shall refer to as the Frobenius algebra associated with \( \Lambda \).

5.3. The 2d TQFT associated with the moduli space of do\'pers. As described in Introduction, we set

\[
V := \tilde{\mathcal{H}}^*_\text{et}(\mathfrak{Rad}, \mathbb{Q}_l).
\]

Decomposition (143) gives rise to a composite isomorphism

\[
\mathcal{V} \simeq \tilde{\mathcal{H}}^*_\text{et}(\mathfrak{Rad}, \mathbb{Q}_l) \simeq \bigoplus_{\rho \in \Delta} \tilde{\mathcal{H}}^*_\text{et}(\mathfrak{Rad}_\rho, \mathbb{Q}_l).
\]

For each \( \rho \in \Delta \), denote by \( e_\rho \) the element of \( V \) corresponding, via (199), to \( 1 \in \tilde{\mathcal{H}}^0_\text{et}(\mathfrak{Rad}_\rho, \mathbb{Q}_l) \). In particular, we obtain

\[
e_\varepsilon \in V.
\]

Next, let \( (g, r) \) be a pair of nonnegative integers with \( 2g - 2 + r > 0 \). We define a \( \mathbb{Q}_l \)-linear morphism

\[
\Lambda_{G,g,r} : V^\otimes r \to \tilde{\mathcal{H}}^*_\text{et}(\mathfrak{M}_{g,r}, \mathbb{Q}_l)
\]

to be the morphism determined uniquely by

\[
\Lambda_{G,g,r}(\bigotimes_{i=1}^r v_i) := \left( \pi^\hom_{g,r}( \left( \prod_{i=1}^r \text{ev}_i^*(v_i) \right) \cap \text{cl}^{3g-3+r}(\mathcal{D}_{G,g,r}^{zaz\cdots}) \right)^\circ
\]

(cf. (138) and (145) for the definitions of \( \pi_{g,r} \) and \( \text{ev}_i \) respectively) for every \( v_1, \ldots, v_r \in V \).

Finally, let us write

\[
\eta : \mathcal{V} \times \mathcal{V} \to \mathbb{Q}_l
\]

for the \( \mathbb{Q}_l \)-bilinear pairing determined by

\[
\eta(v_1, v_2) = \left( \pi^\hom_{0,3} \left( \left( \text{ev}_1^*(v_1)\text{ev}_2^*(v_2)\text{ev}_3^*(e_\varepsilon) \right) \cap \text{cl}^0(\mathcal{D}_{G,0,3}^{zaz\cdots}) \right) \right)^\circ.
\]

Thus, we have obtained a collection of data

\[
\Lambda_G := (\mathcal{V}, \eta, e_\varepsilon; \{\Lambda_{G,g,r}\}_{g,r \geq 0, 2g - 2 + r > 0}).
\]

**Theorem 5.3.1.** Suppose that \( G \) satisfies the condition \((**)_G\).

(i) The set of elements \( e_\Delta := \{e_\rho\}_{\rho \in \Delta} \) forms a basis of \( \mathcal{V} \). Also, for each \( \tilde{\rho} := (\rho_i)_{i=1}^r \in \Delta^{\times r} \),

\[
\Lambda_{G,g,r}(\bigotimes_{i=1}^r e_{\rho_i}) \text{ lies in } \tilde{\mathcal{H}}^0_{\text{et}}(\mathfrak{M}_{g,r}, \mathbb{Q}_l) \text{ and the following equality holds:}
\]

\[
\Lambda_{G,g,r}(\bigotimes_{i=1}^r e_{\rho_i}) = \deg^{\text{gen}}(\mathcal{D}_{G,g,r,\tilde{\rho}}^{zaz\cdots}/\mathfrak{M}_{g,r})
\]
Let us consider assertion (i). The first assertion follows from the following sequence of isomorphisms:

\[
\tilde{H}_c^r(\mathcal{M}, \overline{\mathbb{Q}}) \xrightarrow{\cong} \tilde{H}_c^r(\mathcal{M}^0, \overline{\mathbb{Q}}) \xrightarrow{\cong} \tilde{H}_c^r(\mathcal{M}^0, \overline{\mathbb{Q}}) = \overline{\mathbb{Q}},
\]

where \( \rho := (\rho, \kappa) \in \Delta \) and the first isomorphism follows from Proposition 1.4.1.

Next, let us consider the second assertion. The definition of \( \Lambda_{G, g, r} \) implies the equality

\[
\Lambda_{G, g, r} \bigotimes_{i=1}^r e_{\rho_i} = \left( \frac{\text{cl}^{3g-3+r}}{\pi_{g, r}^*} \left( \frac{[\mathcal{O}_{G, g, r}]_{\text{vir}}}{[\mathcal{O}_{G, g, r}]_{\text{vir}}} \right) \right)^{\circ},
\]

where \(-\bigotimes_{i=1}^r e_{\rho_i} \) denotes the restriction of the class \(-\) to the component \( \mathcal{O}_{G, g, r, \rho} \). Also, since the square diagram

\[
\begin{array}{ccc}
A_{3g-3+r}(\mathcal{O}_{G, g, r})_{\overline{\mathbb{Q}}} & \xrightarrow{\cl^{3g-3+r}} & \tilde{H}^{BM}_{6g+6+2r}(\mathcal{O}_{G, g, r}, \overline{\mathbb{Q}}) \\
\pi_{g, r}^* & & \downarrow \\pi_{g, r}^*
\end{array}
\]

is commutative (cf. [49, §2.10]), we have

\[
\frac{\text{cl}^{3g-3+r}}{\pi_{g, r}^*} \left( \frac{[\mathcal{O}_{G, g, r}]_{\text{vir}}}{[\mathcal{O}_{G, g, r}]_{\text{vir}}} \right) = \text{cl}^{3g-3+r} \left( \frac{[\mathcal{O}_{G, g, r}]_{\text{vir}}}{[\mathcal{O}_{G, g, r}]_{\text{vir}}} \right).
\]

Since \( \mathcal{O}_{G, g, r, \rho} \) is finite over \( \mathcal{M}_{g, r} \) (cf. Theorem 3.3.1 (ii)) and \( \mathcal{M}_{g, r} \) is an irreducible stack of dimension \( 3g - 3 + r \), any prime cycle in \( A_{3g-3+r}(\mathcal{O}_{G, g, r})_{\overline{\mathbb{Q}}} \) dominates \( \mathcal{M}_{g, r} \). On the other hand, the generic étaleness of \( \mathcal{O}_{G, g, r, \rho} \) (where we set \(( \rho_i, \kappa_i) \)) \( \subset \rho \) over \( \mathcal{M}_{g, r} \) (cf. [57, Theorem G]) implies the generic étaleness of \( \mathcal{O}_{G, g, r, \rho} \) \( \cong \mathcal{O}_{G, g, r, \rho} \times \mathcal{M}_{g, r} \) \( \mathcal{O}_{G, g, r, \rho} \) by Theorem 3.3.1 (i) over \( \mathcal{M}_{g, r} \). Hence, the smoothness of \( \mathcal{O}_{G, g, r, \rho} \) (cf. Theorem 1.3.5) implies that \( \mathcal{O}_{G, g, r, \rho} \) is generically smooth over \( k \). To be precise, any irreducible component \( \mathcal{N} \) of \( \mathcal{O}_{G, g, r, \rho} \) dominating \( \mathcal{M}_{g, r} \) has a dense open substack \( \mathcal{N} \) which (does not intersect any other irreducible components and) is smooth over \( k \). The restriction of \( \frac{[\mathcal{O}_{G, g, r}]_{\text{vir}}}{[\mathcal{O}_{G, g, r}]_{\text{vir}}} \) to \( \mathcal{N} \) coincides with the usual fundamental class \([\mathcal{N}]\) in the usual sense. By the observations made so far and the definition of the pushforward map \( \pi_{g, r*} \) between rational Chow groups, the following equality turns out to be satisfied:

\[
\pi_{g, r*} \left( \frac{[\mathcal{O}_{G, g, r}]_{\text{vir}}}{[\mathcal{O}_{G, g, r}]_{\text{vir}}} \right) = \deg \left( \frac{[\mathcal{O}_{G, g, r, \rho}]_{\text{vir}}}{[\mathcal{O}_{G, g, r, \rho}]_{\text{vir}}} \right),
\]
Remark 5.3.2.\(\) Thus, (209), (211), and (212) give the following sequence of equalities:

\[
\Lambda_{G,g,r}^r \left( \bigotimes_{i=1}^r e_{\rho_i} \right) = \sum_{\gamma \in \mathbb{Z}} \left( \text{cl}^{3g-3+r} \left( \left[ \mathcal{D}_{\rho,\gamma,\mathcal{Q}_g} \right]^{\text{vir}} \bigotimes_{\rho \in \Delta} \mathcal{M}_{\rho,\gamma} \right) \right)
\]

This completes the proof of the second assertion. The third assertion follows from Theorem 3.3.1 (iv), and hence, the proof of assertion (i) is completed.

Finally, let us consider assertion (ii). The former assertion follows from assertion (i) and Propositions 3.3.2 (ii), 3.3.3 (ii), and 3.3.4 (ii). In what follows, we shall consider the latter assertion, i.e., the semisimplicity of the Frobenius algebra \(\mathcal{V}, \eta\). To this end, it suffices to prove that the \(\mathcal{Q}_l\)-algebra \(\mathcal{V}\) is reduced. Let us fix an isomorphism \(\mathcal{Q}_l \cong \mathbb{C}\). Denote by \(\gamma\) the involution on \(\mathcal{V}\) (viewed as an \(\mathbb{R}\)-algebra) given by \(\sum_{\rho \in \Delta} (\eta e_{\rho} e_{\rho}^* = \sum_{\rho \in \Delta} \gamma e_{\rho}^* e_{\rho}\), where each \(e_{\rho}\) is an element of \(\mathbb{C}\) \(\cong \mathcal{Q}_l\) and \((-)\) denotes the complex conjugation. Note that the equality \(\eta(x, \gamma(x)) = \frac{1}{|z|} \sum_{\rho \in \Delta} |\eta e_{\rho}|^2\) holds for each \(x := \sum_{\rho \in \Delta} \eta e_{\rho} e_{\rho}^* \in \mathcal{V}\). Hence, \(\eta(x, \gamma(x)) = 0\) implies \(x = 0\). Now, let us take an element \(x := \sum_{\rho \in \Delta} \eta e_{\rho} e_{\rho}^* \in \mathcal{V}\) with \(x = 0\). Then,

\[
\eta(x \times \gamma(x), \gamma(x \times \gamma(x))) = \eta(x \times \gamma(x), \gamma(x \times x)) = \eta(x \times x, \gamma(x \times x)) = 0,
\]

which implies \(x \times \gamma(x) = 0\). It follows that \(\eta(x, \gamma(x)) = 0\), and hence, that \(x = 0\). Consequently, \(\mathcal{V}\) turns out to be reduced. This completes the proof of the latter assertion of (ii). \(\square\)

**Remark 5.3.2.** (i) Since the Frobenius algebra \((\mathcal{V}, \eta)\) associated with \(\Lambda_G\) is semisimple, there exists a canonical basis \(\{e_{\rho}^1\}_{\rho \in \Delta}\) of \(\mathcal{V}\). Let us write \(e_{\rho}^{1/2} := \left(\eta e_{\rho}^1\right)^{-1} e_{\rho}^1\) for each \(\rho \in \Delta\) with \(\left(\eta e_{\rho}^{1/2}\right)^2 = \eta e_{\rho}\). The \(S\)-matrix \(S := (S_{\rho\lambda})_{\rho,\lambda} \in \text{GL}(\mathcal{V})\) is defined in such a way that \(e_{\rho} = \sum_{\lambda \in \Delta} S_{\rho\lambda} e_{\lambda}^1\) (i.e., \(S_{\rho\lambda} := S_{\lambda\rho}^{-1}\) in the sense of \([\text{[6]} \text{[3.3.5, Eq. (20)]}) for any \(\rho \in \Delta\). Hence, \(e_{\rho} = S_{\rho\rho}^{1/2}\) \(\rho \in \Delta\) \(\text{(cf. \([6] \text{[3.3.8])}\)}\). By applying the discussion in \([6] \text{[4.3]}, we obtain the Verlinde formula for G-do’pers. That is to say, for each \(\{\rho_i\}_{i=1}^r \in \Delta^r\), the following equality holds:

\[
\Lambda_{G,g,r}^r \left( \bigotimes_{i=1}^r e_{\rho_i} \right) = \sum_{\lambda \in \Delta} \left( \text{deg} \left( \mathcal{D}_{\rho,\lambda,\mathcal{Q}_g} \bigotimes_{\rho \in \Delta} \mathcal{M}_{\rho,\lambda} \right) \right) = \sum_{\lambda \in \Delta} \prod_{i=1}^r S_{\rho_i\lambda}.
\]

This formula contains the case of \(g = 0\), i.e.,

\[
\Lambda_{G,g,r}(1) = \sum_{\rho \in \Delta} \nu_{\rho}^{1-g} = \sum_{\rho \in \Delta} S_{\rho\rho}^{2-2g}.
\]
(ii) Assume further that $G$ is of adjoint type, i.e., $|Z| = 1$. Then, $\Lambda_G$ defines (under the fixed isomorphism $\mathbb{Q}_l \cong \mathbb{C}$) a fusion ring in the sense of [52, Definition 11.10]. Formula (215) may be expressed as

\begin{equation}
\Lambda_{G,g,r} \left( \bigotimes_{i=1}^{r} e_{\rho_i} \right) = \sum_{\chi \in \mathcal{C}} \chi(CasG)^{g-1} \prod_{i=1}^{r} \chi(e_{\rho_i})
\end{equation}

(cf. [57, Theorem F]), where $CasG := \sum_{\rho \in \Delta} \rho \times \rho^\vee$ and $\mathcal{C}$ denotes the set of characters (i.e., morphisms of $\mathbb{Q}_l$-algebras) $\mathcal{V} \to \mathbb{Q}_l$. Recall that the forgetting morphism $\mathcal{D}_{\rho_{G,g,r}} \to \mathcal{M}_{g,r}$ is representable, finite, and generically étale. Hence, for a sufficiently general curve $X$ in $\mathcal{M}_{g,r}$, the number of $G$-do’pers of radii $(\rho_i)_{i=1}^{r}$ on $X$ is exactly equal to the value $\Lambda_{G,g,r} \left( \bigotimes_{i=1}^{r} e_{\rho_i} \right)$.

If, moreover, $G = PGL_n$ for a small $n$ (relative to $g$ and $p$), then the result of [57, Theorem H] allows us to compute the value $\Lambda_{PGL_n,g,0}(1)$ without explicit knowledge of the characters $\mathcal{V} \to \mathbb{Q}_l$. Indeed, under the assumption that $p > n \cdot \max\{g - 1, 2\}$, the following formula holds:

\begin{equation}
\Lambda_{PGL_n,g,0}(1) \left( = \deg_{\text{gen}} \left( \mathcal{D}_{\rho_{PGL_n,g,0}} / \mathcal{M}_{g,0} \right) \right) = \frac{p^{(n-1)(g-1)-1}}{n!} \sum_{(\zeta_1, \ldots, \zeta_n) \in \mathbb{Q}_l^{\times n}} \frac{\prod_{i=1}^{n} \zeta_i^{(n-1)(g-1)}}{\prod_{i \neq j} (\zeta_i - \zeta_j)^{g-1}}.
\end{equation}

6. THE WITTEN-KONTSEVICH THEOREM FOR DO’PERS

In this final section, we introduce the partition function for do’pers and apply the discussion in [20, § 4] in order to obtain a result analogous to the Witten-Kontsevich theorem; this result gives nontrivial relationships among the intersection numbers of the psi classes on $\mathcal{D}_{\rho_{G,g,r}}$ (cf. Theorem 6.2.2).

Let us keep the notation at the beginning of §3. Also, suppose that $G$ satisfies the condition $(**)_G$.

6.1. Correlator functions. Denote by $\psi_i \in \tilde{H}_{\text{et}}^2(\mathcal{M}_{g,r}, \mathcal{Q}_l)$ $(i = 1, \ldots, r)$ the $i$-th psi class on $\mathcal{M}_{g,r}$. Given a pair of nonnegative integers $(g, r)$ and an $r$-tuple of nonnegative integers $d_1, \ldots, d_r$, we shall recall (cf., e.g., [41, Introduction]) the invariants

\begin{equation}
\langle \tau_{d_1} \cdots \tau_{d_r} \rangle_g := \left( \prod_{i=1}^{r} \tau_{d_i} \right)_g := \int_{\mathcal{M}_{g,r}} \prod_{i=1}^{r} \psi_i^{d_i} \in \mathcal{Q}_l,
\end{equation}

where $\langle - \rangle_g := 0$ if either $r = 0$ or $2g - 2 + r \leq 0$ is satisfied.

Remark 6.1.1. In most cases, the intersection theory of psi classes is discussed in terms of the orbifold $\mathcal{M}^{\text{top}}_{g,r,C}$ (or the corresponding topological stack in the sense of [13]) associated with $\mathcal{M}_{g,r,C} := \mathcal{M}_{g,r,Z} \times \mathbb{Z} \mathbb{C}$ and the usual complex cohomology $H^\ast(\mathcal{M}^{\text{top}}_{g,r,C}, \mathbb{C})$ of $\mathcal{M}^{\text{top}}_{g,r,C}$. However,
after fixing an isomorphism $\mathcal{Q}_l \rightarrow \mathbb{C}$, the argument in [10] Chap. III, Theorem 3.12] together with Riemann’s existence theorem for stacks (cf. [45 Theorem 20.4]) yields an isomorphism $H^*_\text{et}(\mathcal{M}_{g,r,c}, \mathcal{Q}_l) \sim H^*(\mathcal{M}_{g,r,c}^{\text{top}}, \mathbb{C})$ preserving the cup product and the Chern class maps. On the other hand, if we set $\mathcal{M}_{g,r,z,p} := \mathcal{M}_{g,r} \times \mathbb{Z}_p$, then the natural morphism $\mathbb{Z}_p \rightarrow k$ and a fixed inclusion $\mathbb{Z}_p \rightarrow \mathbb{C}$ induce the base change maps $H^*_\text{et}(\mathcal{M}_{g,r,z,p}, \mathcal{Q}_l) \rightarrow H^*_\text{et}(\mathcal{M}_{g,r}, \mathcal{Q}_l)$ and $H^*_\text{et}(\mathcal{M}_{g,r,z,p}, \mathcal{Q}_l) \rightarrow H^*_\text{et}(\mathcal{M}_{g,r,c}, \mathcal{Q}_l)$ respectively. These maps preserve the cup product and moreover preserve the 1-st Chern class map $c_1(\cdot)$ because of the construction using the Kummer sequence. Thus, we conclude that the invariants $\left\langle \tau_{d_1} \cdots \tau_{d_r} \right\rangle_g$ (which in fact belong to $\mathcal{Q}$) defined in the $l$-adic étale cohomology of $\mathcal{M}_{g,r}$ as above coincide with the usual intersection numbers of the corresponding psi classes on $\mathcal{M}_{g,r,c}^{\text{top}}$.

Next, let us define the class $\hat{\psi}_i \in \widetilde{H}^2(\mathcal{D}_{G,g,r}^{\text{pair}}, \mathcal{Q}_l)$ on $\mathcal{D}_{G,g,r}^{\text{pair}}$ to be the pull-back of $\psi_i$. Given a pair of nonnegative integers $(g, r)$, an $r$-tuple of nonnegative integers $(d_1, \ldots, d_r)$, and an $r$-tuple of elements $(v_1, \ldots, v_r)$ of $\mathcal{V}$, we write

$$\left\langle \tau_{d_1}(v_1) \cdots \tau_{d_r}(v_r) \right\rangle_{G,g} = \left( \prod_{i=1}^{r} \tau_{d_i}(v_i) \right)_{G,g} := \int_{\mathcal{D}_{G,g,r}^{\text{pair}}} \prod_{i=1}^{r} ev_i^*(v_i) \hat{\psi}_i^{d_i} \in \mathcal{Q}_l,$$

where $\left\langle \cdot \right\rangle_{G,g} := 0$ if either $r = 0$ or $2g - 2 + r \leq 0$ is satisfied. The invariants $\left\langle \tau_{d_1}(v_1) \cdots \tau_{d_r}(v_r) \right\rangle_{G,g}$ are called the $r$-point correlators.

**Proposition 6.1.2.** Let $(g, r), (d_1, \ldots, d_r),$ and $(v_1, \ldots, v_r)$ be as above. Then, the following equality holds:

$$\left\langle \tau_{d_1}(v_1), \ldots, \tau_{d_r}(v_r) \right\rangle_{G,g} = \Lambda_{G,g,r}(\bigotimes_{i=1}^{r} v_i) \left\langle \tau_{d_1} \cdots \tau_{d_r} \right\rangle_g.$$

**Proof.** The assertion follows from the following sequence of equalities:

$$\begin{align*}
\left\langle \tau_{d_1}(v_1) \cdots \tau_{d_r}(v_r) \right\rangle_{G,g} & \quad = (f \circ \pi_{g,r})^\text{hom} \left( \prod_{i=1}^{r} ev_i^*(v_i) \hat{\psi}_i^{d_i} \right) \cap \text{cl}^{3g-3+r}(\mathcal{D}_{G,g,r}^{\text{pair}}) \\
& \quad = f_{\text{hom}} \circ \pi_{g,r}^\text{hom} \left( \prod_{i=1}^{r} \psi_i^{d_i} \right) \cap \left( \prod_{i=1}^{r} ev_i^*(v_i) \right) \cap \text{cl}^{3g-3+r}(\mathcal{D}_{G,g,r}^{\text{pair}}) \\
& \quad = f_{\text{hom}} \left( \prod_{i=1}^{r} \psi_i^{d_i} \right) \cap \pi_{g,r}^\text{hom} \left( \prod_{i=1}^{r} ev_i^*(v_i) \right) \cap \text{cl}^{3g-3+r}(\mathcal{D}_{G,g,r}^{\text{pair}}) \\
& \quad = f_{\text{hom}} \left( \prod_{i=1}^{r} \psi_i^{d_i} \right) \cap \Lambda_{G,g,r}(\bigotimes_{i=1}^{r} v_i) \left[ \mathcal{M}_{g,r} \right] \\
& \quad = \Lambda_{G,g,r}(\bigotimes_{i=1}^{r} v_i) f_{\text{hom}} \left( \prod_{i=1}^{r} \psi_i^{d_i} \right) \cap \text{cl}^{3g-3+r}(\mathcal{M}_{g,r}) \\
& \quad = \Lambda_{G,g,r}(\bigotimes_{i=1}^{r} v_i) \left\langle \tau_{d_1} \cdots \tau_{d_r} \right\rangle_g.
\end{align*}$$


where \( f \) denotes the structure morphism \( \mathfrak{M}_{g,r} \to \text{Spec}(k) \) of \( \mathfrak{M}_{g,r} \) and the third equality follows from the projection formula (cf. \([187]\)). \( \square \)

6.2. The partition function of \( G \)-do'pers. Let \( h \) and \( t_{d,\rho} \) \((d \in \mathbb{Z}_{\geq 0}, \rho \in \Delta)\) be formal parameters. Given a basis \( \mathcal{E} := \{ e'_\rho \}_{\rho \in \Delta} \) of \( \mathcal{V} \) and an integer \( g \geq 0 \), we set

\[
\Phi_{G,\mathcal{E}} := \langle \exp \left( \sum_{d \in \mathbb{Z}_{\geq 0}} \tau_d \langle e'_\rho | t_{d,\rho} \rangle \right) \rangle_{G,g}
\]

\[
= \sum_{r \geq 0} \frac{1}{r!} \sum_{d_1, \ldots, d_r \geq 0} \langle \prod_{i=1}^r \tau_{d_i} \langle e'_\rho | \rangle \rangle_{G,g} \prod_{i=1}^r t_{d_i, \rho_i}
\]

\[
= \sum_{(s_d, \rho_d) \geq 0} \langle \prod_{d, \rho} \tau_d \langle e'_\rho | \rangle \rangle_{G,g} \prod_{d, \rho \in \Delta} s_d \rho
\]

where the sum in the rightmost of this sequence runs over the set of sequences of nonnegative integers \((s_d, \rho_d)\) indexed by the elements of \( \mathbb{Z}_{\geq 0} \times \Delta \) with finitely many nonzero integers. Also, write

\[
\Phi_{G,\mathcal{E}} := \sum_{g \geq 0} \Phi_{G,\mathcal{E}} h^{2g-2} \left( \in \mathcal{M}_\partial[[(h)]][[\{(t_{d,\rho})_{d \in \mathbb{Z}_{\geq 0}, \rho \in \Delta}\}]] \right),
\]

\[
Z_{G,\mathcal{E}} := \exp (\Phi_{G,\mathcal{E}}) \left( \in \mathcal{M}_\partial(((h)]][[\{(t_{d,\rho})_{d \in \mathbb{Z}_{\geq 0}, \rho \in \Delta}\}]] \right).
\]

If \( \mathcal{E}' \) is another basis of \( \mathcal{V} \), then the change of basis from \( \mathcal{E} \) to \( \mathcal{E}' \) induces naturally an automorphism of the \( \mathcal{M}_\partial(((h)]][[\{(t_{d,\rho})_{d \in \mathbb{Z}_{\geq 0}, \rho \in \Delta}\}]] \), by which \( Z_{G,\mathcal{E}} \) is mapped to \( Z_{G,\mathcal{E}'} \). Notice that \( Z_{G,\mathcal{E} \Delta} \) coincides with \( Z_G \) described in Introduction.

**Definition 6.2.1.** We shall refer to \( Z_G (= Z_{G,\mathcal{E} \Delta}) \) as the **partition function of \( G \)-do'pers**.

Next, let us fix a canonical base \( \mathcal{E} \) := \{ \mathcal{E}_{\rho} \}_{\rho \in \Delta} \) with \( \mathcal{E}_{\rho} = \eta(\mathcal{E}_{\rho}, \mathcal{E}_{\rho}) = \nu_{\rho} \) of the Frobenius algebra corresponding to the 2d TQFT \( \Lambda_G := (\mathcal{V}, \eta, \epsilon, \{ \Lambda_{G,g,r} \}_{g,r}) \). Also, fix elements \( \nu_{\rho}^{1/3} \) of \( \mathcal{M}_\partial \) with \( \nu_{\rho}^{1/3} = \nu_{\rho} \). For each \( \rho \in \Delta \) and \( n \geq -1 \), we shall write

\[
L_n^{(\rho)} := -\frac{(2n+3)!!}{2^{n+1}} (\nu_{\rho}^{1/3})^n \frac{\partial}{\partial t_{\rho,n+1}} + \sum_{i=0}^{\infty} \frac{(2i+2n+1)!!}{(2i-1)!!2^{n+1}} (\nu_{\rho}^{1/3})^{n-3} \frac{\partial^2}{\partial t_{\rho,i} \partial t_{\rho,n-i}}
\]

\[
+ \frac{h^2}{2} \sum_{i=0}^{n-1} (2i+1)!!(2n-2i-1)!! (\nu_{\rho}^{1/3})^{n-3} \frac{\partial}{\partial t_{\rho,i}}
\]

\[
+ \delta_{n,1} \frac{h}{2} (\nu_{\rho}^{1/3})^{1/2} t_{\rho,0}^2 + \delta_{n,0} \frac{1}{16}.
\]

There operators satisfy \([L_n^{(\rho_1)}, L_m^{(\rho_2)}] = (n-m)\delta_{\rho_1,\rho_2} L_{n+m}^{(\rho_1)}\) for any \( n, m \geq -1 \) and any \( \rho_1, \rho_2 \in \Delta \) (cf. \([20]\) §4.3)). Similarly, if \( L_n (n \geq -1) \) are the differential operators defined in Introduction, then the equality \([L_n, L_m] = (n-m)L_{n+m}\) holds for any \( n, m \geq -1 \).
Theorem 6.2.2. \(\text{(i)}\) Given \(\rho \in \Delta, \ n \geq -1\), we obtain the following equality:

\[
L_n^{(\rho)} Z_{G,t^1} = 0.
\]

Moreover, these equalities for various \((\rho, n)\)'s completely determine \(Z_G\) (cf. the discussion preceding Definition 6.2.7).

\(\text{(ii)}\) For any \(n \geq -1\), the following equality holds:

\[
L_n Z_G = 0.
\]

\textbf{Proof.} The assertions follow from Proposition 6.1.2 and [20, Proposition 4.4] applied to \(\Lambda_G\). Indeed, by a fixed isomorphism \(\overline{\mathbb{Q}}_l \cong \mathbb{C}\), Proposition 4.4 in \textit{loc. cit.} is available because the invariants \(\langle \tau_1, \cdots, \tau_d \rangle\) defined in \(219\) coincide with the usual intersection numbers of the corresponding psi classes on the module space of complex curves (cf. Remark 6.1.1). In particular, the Witten-Kontsevich theorem, which is used in the proof of that proposition, also holds for the psi classes in the \(l\)-adic étale cohomology groups of \(\mathfrak{M}_{g,r}\)'s. \(\square\)

Let \((g, r)\) be a pair of nonnegative integers, \((d_1, \cdots, d_r)\) an \(r\)-tuple of nonnegative integers, and \((v_1, \cdots, v_r)\) an \(r\)-tuple of elements of \(\mathcal{V}\). Then, we shall write

\[
\langle \tau_{d_1} (v_1) \cdots \tau_{d_r} (v_r) \rangle_{G}\ := \langle \tau_{d_1} (v_1) \cdots \tau_{d_r} (v_r) \rangle_{G,G} \exp \left( \sum_{d \in \mathbb{Z}_{\geq 0}, \rho \in \Delta} \tau_{d} (e_{\rho}) t_{d, \rho} \right)_{G,g},
\]

where \(\langle \langle - \rangle \rangle \) is available because the \((g, r)\)-WZW conformal field theory at \(g, r\) is available because the unique root in a \(\mathfrak{sl}_2\)-WZW conformal field theory at level \(p - 2\) (i.e., \(\mathcal{R}_{p-2}(\mathfrak{sl}_2)\) in the notation of [8]), and \(Z_{\mathfrak{sl}_2}^{\text{WZW}, p-2}\) for the partition function associated with the corresponding 2d TQFT.

Then, by [20, Proposition 4.6], the following proposition holds.

\textbf{Theorem 6.2.3.} For any \(d \in \mathbb{Z}_{\geq 0}, v \in \mathcal{V}\), and \(\rho_1, \rho_2, \rho_3, \rho_4 \in \Delta\), the following equality holds:

\[
\frac{2d + 1}{h^2} \langle \tau_{d} (v) \tau_0 (e_{\rho_1}) \tau_0 (e_{\rho_2}) \rangle_{G} \eta^{\rho_1, \rho_2}^d
\]

\[
= \langle \tau_{d-1} (v) \tau_0 (e_{\rho_1}) \rangle_{G} \eta^{\rho_1, \rho_2}^{d-1} \langle \tau_0 (e_{\rho_2}) \tau_0 (e_{\rho_1}) \rangle_{G} \eta^{\rho_1, \rho_2} + 2 \langle \tau_{d-1} (v) \tau_0 (e_{\rho_1}) \tau_0 (e_{\rho_2}) \rangle_{G} \eta^{\rho_1, \rho_2} \eta^{\rho_1, \rho_2} \langle \tau_0 (e_{\rho_2}) \tau_0 (e_{\rho_1}) \rangle_{G} \eta^{\rho_1, \rho_2}
\]

\[
+ \frac{1}{4} \langle \tau_{d-1} (v) \tau_0 (e_{\rho_1}) \tau_0 (e_{\rho_2}) \tau_0 (e_{\rho_3}) \tau_0 (e_{\rho_4}) \rangle_{G} \eta^{\rho_1, \rho_2} \eta^{\rho_1, \rho_2} \eta^{\rho_1, \rho_2}.
\]

Moreover, equation \(230\) and the equalities \(L_{-1}^{(\rho)} Z_{G,t^1} = 0 \ (\rho \in \Delta)\) resulting from Theorem 6.2.2 completely determine \(\Phi_{G,t^1}\).

6.3. The 2d TQFT for PGL₂-dopers. In this subsection, we compare the 2d TQFT for PGL₂-dopers with the \(\mathfrak{sl}_2(\mathbb{C})\)-WZW (= Wess-Zumino-Witten) conformal field theory.

Write \((\mathcal{U}, \eta^\mu)\) for the fusion ring associated with the \(\mathfrak{sl}_2(\mathbb{C})\)-WZW conformal field theory at level \(p - 2\) (i.e., \(\mathcal{R}_{p-2}(\mathfrak{sl}_2)\) in the notation of [8]), and \(Z_{\mathfrak{sl}_2}^{\text{WZW}, p-2}\) for the partition function associated with the corresponding 2d TQFT.

First, let us recall the structure of the \(C\)-algebra \(\mathcal{U}\). Denote by \(\alpha\) the unique root in a Borel subgroup of PGL₂ and by \(\langle -, - \rangle\) the Killing form on \(\mathfrak{sl}_2(\mathbb{C})\) normalized as \(\langle H_{\alpha}, H_{\alpha} \rangle = 2\), where \(H_{\alpha}\) denotes the coroot (considered as an element of \(t\) via differentiation) associated to
α. Also, denote by $P_{p-2}$ the set of dominant weights $λ$ of $\mathfrak{sl}_2(\mathbb{C})$ with $0 ≤ \lambda(H_\alpha) ≤ p - 2$. That is to say, we have $P_{p-2} = \{0, \frac{1}{2}α, \alpha, \cdots, \frac{p-2}{2}α\}$. We shall identify $P_{p-2}$ with the set $\{0, \frac{1}{2}, \cdots, \frac{p-2}{2}\} (\subseteq \frac{1}{2}\mathbb{Z})$ via the correspondence $j\alpha \leftrightarrow j$. Then, the underlying $\mathbb{C}$-vector space of $U$ is isomorphic to the direct sum $\bigoplus_{j\in P_{p-2}} \mathbb{C}e_j^U$ with basis $\{e^U_j\}_{j\in P_{p-2}}$ indexed by $P_{p-2}$.

(In particular, $Z_{\mathfrak{sl}_2}^{\text{WZW,}p-2}$ may be considered as an element of $\mathbb{C}((h))[\{(t_{d,m})_{d\geq 0,m\in P_{p-2}}\}]$.) The multiplication “$\times$” in $U$ is determined by the following conditions (cf. [8 Lemma 4.2]):

- If $\{N_{a,b,c}\in \mathbb{C} | a, b, c \in P_{p-2}\}$ denotes a collection defined in such a way that $e_a^U \times e_b^U = \sum_{c \in P_{p-2}} N_{a,b,c} e_c^U$, then $N_{a,b,c} \in \{0, 1\}$ for any triple $(a, b, c)$.
- The equality $N_{a,b,c} = 1$ holds if and only if $(a, b, c)$ satisfies:

\begin{equation}
 a + b + c \in \mathbb{Z}, \quad a + b + c ≤ p - 2, \quad \text{and} \quad |b - c| ≤ a ≤ b + c.
\end{equation}

On the other hand, let $(\mathcal{V}, \eta^\mathcal{V})$ be the Frobenius algebra associated with $\Lambda_{\text{PGL}_2}$. We regard $(\mathcal{V}, \eta^\mathcal{V})$ as a $\mathbb{C}$-algebra by fixing an isomorphism $\mathbb{C}_l \cong \mathbb{C}$. We shall write

\begin{equation}
 P_{p-2}^\mathcal{V} := \left\{ m \in \mathbb{Z} \mid 0 ≤ j ≤ \frac{p - 2}{2} \right\} (= P_{p-2} \cap \mathbb{Z}).
\end{equation}

Then, the finite set $\Delta$ (cf. (141)) in the case of $G = \text{PGL}_2$ (where the maximal torus $T$ is taken to be the subgroup consisting of the images of invertible diagonal matrices via the quotient $\text{GL}_2 \rightarrow \text{PGL}_2$) may be identified with $P_{p-2}^\mathcal{V}$ via the bijection

\begin{equation}
P_{p-2}^\mathcal{V} \rightarrow \Delta
\end{equation}

given by assigning, to each $a \in P_{p-2}^\mathcal{V}$, the element of $t_{\text{reg}}^F/W$ represented by the diagonal matrix

\begin{equation}
\begin{pmatrix}
\frac{(2a + 1)}{2} & 0 \\
0 & -\frac{(2a + 1)}{2}
\end{pmatrix}.
\end{equation}

Then, it follows from [13 Introduction, Theorem 1.3]) that, under the identification $P_{p-2}^\mathcal{V} = \Delta$ given by (233), $\mathcal{V}$ is isomorphic to the $\mathbb{C}$-vector space $\bigoplus_{m\in P_{p-2}^\mathcal{V}} \mathbb{C}e^\mathcal{V}_m$ (where $\{e^\mathcal{V}_m\}_{m\in P_{p-2}^\mathcal{V}}$ is a basis indexed by $P_{p-2}^\mathcal{V}$) with multiplication characterized uniquely by the condition that the assignment $e^\mathcal{V}_m \mapsto e^\mathcal{V}_m$ defines a $\mathbb{C}$-algebra homomorphism

\begin{equation}
\text{incl} : \mathcal{V} \rightarrow U.
\end{equation}

Just as in the case of the pseudo-fusion ring for dormant $\text{PGL}_2$-opers discussed in [57 § 7.8.2], we can describe the characters $\mathcal{V} \rightarrow \mathbb{C}$ of $\mathcal{V}$ from those of $U$ via this homomorphism incl. In particular, by [8 Proposition 6.3], the explicit knowledge of the characters allows us to perform some computations that we need in the ring $U$, e.g., the formula displayed in (12). Also, we obtain the following assertion regarding the partition function $Z_{\text{PGL}_2}$:

**Theorem 6.3.1.** Let us consider the surjective morphism of $\mathbb{C}$-algebras

\begin{equation}
\theta : \mathbb{C}((h))[[\{(t_{d,m})_{d\geq 0,m\in P_{p-2}}\} \rightarrow \mathbb{C}((h))[[\{(t_{d,m})_{d\geq 0,m\in P_{p-2}^\mathcal{V}}\}]
\end{equation}

given by $h \mapsto \frac{h}{2}$, $t_{d,m} \mapsto t_{d,m}$ for any $m \in P_{p-2}^\mathcal{V}$, and $t_{d,m} \mapsto 0$ for any $m \in P_{p-2} \setminus P_{p-2}^\mathcal{V}$. Then, the following equality holds:

\begin{equation}
Z_{\text{PGL}_2} = \theta(Z_{\mathfrak{sl}_2}^{\text{WZW,}p-2}).
\end{equation}

\[\text{In [56 Eq. (1032)]}, \text{this condition is incorrectly described.}\]
Proof. The assertion follows from the fact that if \( \Lambda_{g,r}^{WZW,p-2} \) denote the correlators of the 2d TQFT corresponding to the Frobenius algebra \((U, \eta_U)\), then the equality

\[
\Lambda_{g,r}^{WZW,p-2}(\bigotimes_{i=1}^r e^U_{n_i}) = 2^{g-1} \cdot \Lambda_{PGL_2,g,r}^{U,V}(\bigotimes_{i=1}^r e^V_{n_i})
\]

holds for any \( n_1, \ldots, n_r \in P_{p-2}^Z \) (cf. (12) and [8, Corollary 9.8]).

\[\square\]

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