Singular intersections of subgroups and character varieties

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Abstract
We prove a global local rigidity result for character varieties of 3-manifolds into $\text{SL}_2(\mathbb{C})$. Given a 3-manifold with toric boundary $M$ satisfying some technical hypotheses, we prove that all but a finite number of its Dehn fillings $M_{p/q}$ are globally locally rigid in the following sense: every irreducible representation $\rho : \pi_1(M_{p/q}) \to \text{SL}_2(\mathbb{C})$ is infinitesimally rigid, meaning that $H^1(M_{p/q}, \text{Ad}_\rho) = 0$. This question arose from the study of asymptotics problems in topological quantum field theory developed in Charles and Marché (Knot state asymptotics II. Irreducible representations and the Witten conjecture, 2011). The proof relies heavily on recent progress in diophantine geometry and raises new questions of Zilber–Pink type. The main step is to show that a generic curve lying in a plane multiplicative torus intersects transversally almost all subtori of codimension 1. We prove an effective result of this form, based mainly on a height upper bound of Habegger.

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1 Introduction
A simplified version of the main result of this article is the following elementary statement:

**Theorem 1.1.** Let $f(x, y) \in \mathbb{Q}[x, y]$ be a polynomial whose singular points are not pairs of roots of unity. Then, for all but a finite number of relatively prime integers $(p, q) \in \mathbb{N}$, the polynomial $f(t^p, t^q)$ has only simple roots.
This question arose while studying the Witten asymptotic expansion conjecture for quantum invariants of 3-manifolds and deals with the character variety of Dehn fillings of 3-manifolds. We provide here an introduction to these questions as a motivation; however, the article is indeed independent from quantum topology and character varieties. The reader can safely skip it and go directly to Definition 1.3.

Let $M$ be a compact connected oriented 3-manifold without boundary. For any integer $k$ called level, the quantum Chern-Simons theory associated to the group $SU_2$ and the level $k$ gives an invariant $Z_k(M) \in \mathbb{C}$ called Witten-Reshetikhin-Turaev invariant. This invariant was introduced in [24] as a path integral, and constructed rigorously by Reshetikhin and Turaev using the representation theory of the quantum group $U_q sl_2$, see [19]. Formally, one can write

$$Z_k(M) = \int e^{ikCS(A)} DA.$$ 

In this expression, $A$ is a 1-form on $M$ with values in the Lie algebra of $SU_2$ and

$$CS(A) = -\frac{1}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge [A \wedge A] \right).$$

The measure $DA$ is not rigorously defined; however, Witten understood its cut-and-paste properties from which Reshitikhin and Turaev constructed the invariant rigorously. Applying formally the stationary phase expansion to this path integral, it localizes around the critical points of the Chern-Simons functional which correspond to the flat connections, that is 1-forms $A$ satisfying $dA + \frac{1}{2}[A \wedge A] = 0$. The gauge equivalence classes of such connections correspond to conjugacy classes of representations $\rho : \pi_1(M) \to SU_2$. Witten formally obtained the following asymptotic expansion:

$$Z_k(M) = \sum_{\rho} e^{i \frac{\pi}{4} m(\rho) + ikCS(\rho)} \sqrt{T(M, \rho)} + O(k^{-1/2}).$$

In this formula, $\rho$ runs over the conjugacy classes of irreducible representations from $\pi_1(M)$ to $SU_2$, $m(\rho)$ is an element of $\mathbb{Z}/8\mathbb{Z}$ called spectral flow and $T(M, \rho)$ is the Reidemeister torsion of $M$ twisted by the representation $Ad_\rho$ of $\pi_1(M)$ on the Lie algebra of $SU_2$.

This formula is proved only in very few cases, one of the difficulties being that the Reidemeister torsion is defined only for those irreducible representations $\rho$ for which the space $H^1(M, Ad_\rho)$ vanishes. The space $H^1(M, Ad_\rho)$ can be identified to the Zariski tangent space of the character variety $X(M)$ at $\chi_\rho$ (see Sect. 2). Hence, a necessary condition for the Witten asymptotic formula to make sense is that the character variety is reduced of dimension 0.

If $M$ is a compact, connected and oriented 3-manifold with toric boundary, we call Dehn surgery the result of gluing back to $M$ a solid torus. Let $\phi$ be a homeomorphism from $\partial M$ to $\partial D^2 \times S^1$ reversing the orientation. It is well known that the homeomorphism type of the manifold $M \cup_\phi D^2 \times S^1$ depends only on the homotopy class of the
simple curve $\gamma = \phi^{-1}(S^1 \times \{1\}) \subset \partial M$. Hence we will denote by $M_\gamma$ this 3-manifold without boundary and call it the Dehn filling of $M$ with slope $\gamma$.

In [4], Charles and the first author proved that the Witten conjecture holds for $M_\gamma$ if $M = S^3 \setminus V(K)$ is the complement of a tubular neighborhood of the figure eight knot $K$, the linking number $\text{lk}(\gamma, K)$ is not divisible by 4 and the character variety $X(M_\gamma)$ is reduced of dimension 0. The strategy adopted in that article should generalize to any knot provided one has a strong version of the AJ-conjecture (detailed in [4]) and some information on the Reidemeister torsion.

The condition on the character variety to be reduced appeared as a technical point which was hard to check even in the case of the figure eight knot. However, in this article we prove that for a broad class of varieties $M$, this condition is satisfied for all but a finite number of slopes $\gamma$. More precisely, we show:

**Theorem 1.2** Suppose that $M$ is a compact connected oriented irreducible 3-manifold with toric boundary such that

(i) the map $r : X(M) \to X(\partial M)$ induced by the inclusion $\partial M \subset M$ is proper;
(ii) the character variety $X(M)$ is reduced;
(iii) the image by $r$ of the singular points of $X(M)$ are not torsion points of $X(\partial M)$ (see Sect. 2).

Then, for all but a finite number of slopes $\gamma$, the variety $X(M_\gamma)$ is reduced of dimension 0. Moreover, the number of exceptions can be effectively bounded.

It is well-known that $X(\partial M)$ is the quotient of a 2-dimensional torus $G^2_m$ by the involution $\sigma(x, y) = (x^{-1}, y^{-1})$. Denote by $\pi : G^2_m \to X(\partial M)$ the quotient. In the setting of the previous theorem, the variety $C = \pi^{-1}(r(X(M)))$ is a plane curve defined by the so-called A-polynomial, see [7]. The following notion will be crucial to the proof of Theorem 1.2.

**Definition 1.3** Let $C$ and $C'$ be two curves in $G^2_m$. We say that $C$ intersects $C'$ transversally at $P \in C \cap C'$ if the two curves are smooth at $P$ with distinct tangent lines. We define the singular intersection $C \cap_{\text{sing}} C'$ of $C$ and $C'$ as the set of all points $P \in C \cap C'$ where the two curves are smooth with equal tangent lines.

For any couple of relatively prime integers $(p, q)$, let $H_{p,q}$ be the subtorus of $G^2_m$ defined by the equation $x^p y^q = 1$. Through some standard argument in character varieties, we reduce the proof of the previous theorem to show that $C$ intersects transversally $H_{p,q}$ for almost all $(p, q)$. This fact is connected to recent questions in diophantine geometry surrounding the Zilber–Pink conjecture. Inexplicitly, it follows from the bounded height property proved in 1999 by Bombieri, Masser and Zannier (see Theorem 1 in [3]). An explicit version of this result was obtained later by Habegger in Appendix B1 of [10], which was also published as Theorem 1 of [11] (see also [12] for the generalisation to arbitrary dimension). Using these results, we give explicit upper bounds on the maximal size of a couple $(p, q)$ such that $C$ has non-empty singular intersection with a torsion translate of $H_{p,q}$, i.e a translate of $H_{p,q}$ by a torsion point.

These estimates might be of interest for the applications of Theorem 1.2 in topology. They only depend on quantities that can be computed from an irreducible equation...
\( f(x, y) = 0 \) defining \( C \) in \( \mathbb{G}^2_m \). The polynomial \( f \) is involved through its total and partial degrees and its logarithmic Weil height \( h(f) \) (see Sect. 3 for the definition).

We denote by \( C^{(1)} \) the subset of \( C \) defined by

\[
C^{(1)} = \bigcup_{\gcd(p, q) = 1} C(\bar{\mathbb{Q}}) \cap_{\text{sing}} H_{p,q}(\bar{\mathbb{Q}}).
\]

We first prove the following result.

**Theorem 1.4** Let \( C \) be an irreducible curve in \( A \) with defining equation \( f(x, y) = 0 \) for an irreducible polynomial \( f \in \bar{\mathbb{Q}}[x, y] \). Let \( \delta = \deg(f) \), \( \delta_x = \deg_x(f) \) and \( \delta_y = \deg_y(f) \). Assume that \( C \) is not a translate of a subtorus. Then, for any torsion translate \( \xi H_{p,q} \) with non-empty singular intersection with \( C \),

\[
\max(|p|, |q|) \leq \delta^3 \exp \left( (6 \cdot 10^5 + 1)\delta^4 \max(\delta_x \delta_y, h(f)) \right).
\]

In particular, \( C^{(1)} \) is a finite set.

Using a different approach inspired by Lemma 5.2 of [1], we prove a sharper bound under an additional hypothesis on \((p, q)\). It is based on the combination of a degree upper bound (see Lemma 3.3) and more advanced techniques in diophantine approximation, such as lower bounds for the height of Dobrowolski type. Denoting by \( \Delta \) the degree of a field of definition for \( C \), we obtain

\[
\max(|p|, |q|) \leq 3 \cdot 2^4 10^5 \Delta^9 \log(3\delta^2 \Delta)^3 \max(\delta_x \delta_y, h(f)),
\]

under the stronger assumption that at least one singular intersection of \( C \) with a torsion translate of \( H_{p,q} \) is non-empty and not entirely torsion (see Theorem 3.6). Let us also point out that we can take \( \Delta = 1 \) in the applications to character varieties, because the curves \( C \) that we need to consider are all defined over \( \mathbb{Q} \).

Unfortunately, on the contrary to Theorem 1.4, this sharper estimate is not sufficient on its own to determine effectively all the exceptional slopes \( \gamma \) of Theorem 1.2. In fact, because of the additional assumption on \((p, q)\), it only gives a partial result. However, together with an upper bound of Ruppert for the number of torsion points of \( C \), it implies an explicit upper bound for the number of all \( \gamma \)'s which is much sharper than the one coming from Theorem 1.4 (see Corollary 3.8).

We conclude the article by proving a strengthening of these finiteness results that looks like a perfect analogue of the Zilber–Pink conjecture in the context of plane singular intersections.

When the ambient space is a multiplicative torus \( T = \mathbb{G}^n_m \) over \( \mathbb{C} \), Zilber–Pink conjecture predicts what happens to a subvariety \( X \) when intersected to the union of all algebraic subgroups of \( T \) of fixed codimension \( m \) (see [18, 26] for the original conjectures and [25] for a recent panorama of the subject). More precisely, denote by
$X^{[m]}$ the subset of $X$ given by

$$X^{[m]} = \bigcup_{\text{codim } H = m} X(C) \cap \zeta H(C), \quad (1.4.2)$$

where the union runs over all subtori $H$ of codimension $m$ and all torsion points $\zeta$ of $T$. Then, the conjecture asserts that, for any $X$ that is not contained in a proper algebraic subgroup of $T$, the subset $X^{[m]}$ is not Zariski-dense in $X$ for $m \geq \dim X + 1$. The motivation is that for such values of $m$, the $H$’s are so small that most of their intersections with $X$ are empty, giving rise to the notion of *unlikely intersections* introduced by Bombieri, Masser and Zannier.

In the particular case of an irreducible curve $C$ lying in $G^2_m$, the assumption on $C$ means precisely that $C$ is not a translate of a subtorus by a torsion point. In comparison with the hypotheses of Theorem 1.4, this is weaker, yet it turns out to be sufficient for the finiteness of $C^{[1]}$. Indeed, the only new case that appears with this more general hypothesis is the case where $C$ is a translate of a subtorus by a non-torsion point, but it turns out to be trivial, because then $C^{[1]} = \emptyset$ (see the proof of theorem 1.5).

Finally, under this weaker assumption, we can even prove the finiteness of a slightly larger subset of $C$. Its definition is derived from formula (1.4.2) with $C^{[1]}$ by changing all intersections for singular intersections.

**Theorem 1.5** If $C$ is an irreducible curve in $G^2_m$ that is not a translate of a subtorus by a torsion point, then

$$C^{[1, \text{tor}]} = \bigcup_{\gcd(p, q) = 1} C(\bar{Q}) \cap_{\text{sing}} \zeta H_{p, q}(\bar{Q})$$

is a finite subset of $C$.

It is well known that, in the Zilber–Pink conjecture, the codimension value $m = \dim X + 1$ is optimal for Zariski non-density. If $m$ is decreased further, then $X^{[m]}$ contains $X^{[\dim X]}$ that is dense in $X$ for all $X$ in the case of $G^n_m$. In this respect, the main feature of Theorem 1.5 is to show that positive multiplicity of intersection can make up for a codimension drop among the $H$’s: going from $C^{[2]}$ to the larger subset $C^{[1]}$ generates infinitely many new points, but we recover a finite set by restricting to the case of positive multiplicity. This line of thought goes further than the case of plane curves and makes sense in a more general framework, opening the way to new conjectures of Zilber–Pink type.

To conclude the article, we study the relation between subsets of the form $C^{[1, \text{tor}]}$ and $C^{[2]}$, showing that the first can be seen as a subset of the second type for a curve that lies a slightly different ambient space (see Theorem 3.12 and Remark 3.13).
2 Character variety and a reduction

Let \( \Gamma \) be a finitely generated group. We denote by \( R(\Gamma) \) the algebraic variety of all representations \( \rho : \Gamma \to \text{SL}_2(\mathbb{Q}). \) This variety is generally used over \( \mathbb{C} \) by topologists whereas it is actually defined over \( \mathbb{Z} \). We adopt here the field \( \mathbb{Q} \) which is more convenient for our purposes. The group \( \text{SL}_2(\mathbb{Q}) \) acts on \( R(\Gamma) \) by \( g.\rho = g \rho g^{-1} \): we denote the algebraic quotient by \( X(\Gamma) = R(\Gamma) // \text{SL}_2(\mathbb{Q}) \). We refer to [6, 16] for the general theory and recall here some facts.

(i) Given a representation \( \rho \in R(\Gamma) \), we define its character \( \chi_\rho : \Gamma \to \hat{\mathbb{Q}} \) by the formula \( \chi_\rho(\gamma) = \text{Tr} \rho(\gamma) \). As a set, \( X(\Gamma) \) is the quotient of \( R(\Gamma) \) by the relation \( \rho \sim \rho' \) if and only if \( \chi_\rho = \chi_{\rho'} \). This justifies the name character variety.

(ii) If \( \rho, \rho' \) are two elements of \( R(\Gamma) \) with \( \chi_\rho = \chi_{\rho'} \) and \( \rho \) irreducible, then \( \rho \) and \( \rho' \) are conjugated.

(iii) The algebra of regular functions on \( X(\Gamma) \) is generated by the so-called trace functions defined for any \( \gamma \in \Gamma \) by \( f_\gamma(\rho) = \text{Tr} \rho(\gamma) \).

(iv) A representation \( \rho \in R(\Gamma) \) is reducible if and only if for all \( \alpha, \beta \in \Gamma \) one has \( f_{[\alpha, \beta]}(\rho) = 2 \). In particular the set of reducible characters is Zariski-closed in \( X(\Gamma) \) and is denoted by \( X^{\text{red}}(\Gamma) \) whereas its complement is denoted by \( X^{\text{irr}}(\Gamma) \).

(v) Given an irreducible representation \( \rho \), we denote by \( \text{Ad}_\rho \) the Lie algebra \( \text{sl}_2(\mathbb{Q}) \) viewed as a \( \Gamma \)-module through the formula \( \gamma : \xi = \rho(\gamma) \xi \rho(\gamma)^{-1} \). There is a natural isomorphism

\[
T_{\chi_\rho}X(\Gamma) \simeq H^1(\Gamma, \text{Ad}_\rho).
\]

(vi) If \( \Gamma = \mathbb{Z}^2 \), we consider the morphism \( \pi : \mathbb{G}_m^2 \to X(\Gamma) \) mapping \( (x, y) \) to the character of the representation \( \rho_{x,y} \) defined by

\[
\rho_{x,y}(a, b) = \begin{pmatrix} x^ay^b & 0 \\ 0 & x^{-a}y^{-b} \end{pmatrix}.
\]

It is well-known that \( \pi \) induces an isomorphism between the quotient of \( \mathbb{G}_m^2 \) by the involution \( \sigma(x, y) = (x^{-1}, y^{-1}) \) and \( X(\Gamma) \). In particular, we will denote by \( X(\Gamma)_{\text{tor}} \) the image by \( \pi \) of the torsion points of \( \mathbb{G}_m^2 \).

(vii) If \( \phi : \Gamma \to \Gamma' \) is a group homomorphism, it induces an algebraic morphism \( \phi^* : X(\Gamma') \to X(\Gamma) \).

If \( M \) is a connected compact oriented manifold, we set \( X(M) = X(\pi_1(M)) \). If \( M \) is a surface or a 3-manifold as in Theorem 1.2, then it is an Eilenberg-Maclane space, which means that there is a natural isomorphism \( H^1(\pi_1(M), \text{Ad}_\rho) \simeq H^1(M, \text{Ad}_\rho) \). Let \( i : \partial M \to M \) be the inclusion morphism: it induces a map \( i_* : \pi_1(\partial M) \to \pi_1(M) \). We denote by \( r \) the map \( (i_*)^* : X(M) \to X(\partial M) \) induced by the inclusion and call it the restriction map.

If \( M \) is a connected compact oriented 3-manifold with toric boundary, one can study representations of \( M_\gamma \) for a given slope \( \gamma \subset \partial M \) in the following way: by Van-Kampen
theorem, the fundamental group of $M$ is the amalgamated product $\pi_1(M) \ast \pi_1(D^2 \times S^1)$. Moreover, the map $\pi_1(\partial M) \to \pi_1(D^2 \times S^1)$ is surjective with kernel generated by $\gamma$. This subgroup injects as a non-normal subgroup of $\pi_1(M)$ hence one has $\pi_1(M_\gamma) = \pi_1(M)/\langle \gamma \rangle$ where $\langle \gamma \rangle$ is the normal closure of $\gamma$.

In particular, a representation $\rho : \pi_1(M_\gamma) \to \text{SL}_2(\bar{\mathbb{Q}})$ is the same as a representation of $\rho : \pi_1(M) \to \text{SL}_2(\bar{\mathbb{Q}})$ such that $\rho(\gamma) = 1$. In terms of character varieties, $X(M_\gamma)$ fits in the following diagram (which may not be cartesian):

$$
\begin{array}{ccc}
X(M) & \xrightarrow{r} & X(D^2 \times S^1) \\
\downarrow & & \downarrow \quad r' \\
X(\partial M) & & 
\end{array}
$$

The image of $r'$ is the projection of a subtorus of $\mathbb{G}_m^2$ by the map $\pi$. We will reduce Theorem 1.2 to a statement about the intersection of $\pi^{-1} r(X(M))$ with subtori of $\mathbb{G}_m^2$.

We start with a technical lemma.

**Lemma 2.1** Let $M$ be a manifold satisfying the assumptions of Theorem 1.2, then every irreducible component of $X(M)$ has dimension 1.

**Proof** From now on, we denote the local system $\text{Ad}_\rho$ with a subscript $\rho$. Let $Y$ be an irreducible (reduced) component of $X(M)$ and $x_\rho$ be a smooth point of it. A standard argument involving Poincaré duality (see [14] p. 42) shows that the rank of the map $i^* : H^1_\rho(M) \to H^1_\rho(\partial M)$ is half the dimension of $H^1_\rho(\partial M)$. By Poincaré duality again, $\text{rk } i^* = \dim H^0_\rho(\partial M) \in \{1, 3\}$. As $r$ is proper, $r(Y)$ is a subvariety of the 2-dimensional variety $X(\partial M)$. This shows that $r(Y)$ has dimension 1 and, because $r$ is proper, also $Y$ has dimension 1. \hfill \Box

Let $M$ be a manifold satisfying the assumptions of Theorem 1.2 and fix a homeomorphism between $\partial M$ and $S^1 \times S^1$. A slope $\gamma$ corresponds to a pair $(\rho, q)$ of relatively prime integers.

Given $x_\rho$, a character of $X(M_{p/q})$, we denote by the same letter its restriction to $X(M)$. By the above remarks, $r(x_\rho) = \pi(x, \gamma)$ for some $(x, y) \in \bar{\mathbb{Q}}^2$ with $x^p y^q = 1$.

We relate the vanishing of $H^1(M_{p/q}, \text{Ad}_\rho)$ and the transversality of the intersection of $H_{p,q}$ and $\pi^{-1} r(X(M))$ in the following technical proposition.

**Proposition 2.2** Let $(x, y)$ be the boundary parameters of a representation $\rho : \pi_1(M_{p/q}) \to \text{SL}_2(\bar{\mathbb{Q}})$ as above.

**Case 1:** $x \neq \pm 1$ or $y \neq \pm 1$.

In that case, one can suppose that, up to conjugation $\rho \circ i^* = \rho_{x,y}$. Then, one has $H^1(M_{p/q}, \text{Ad}_\rho) = (i^*)^{-1} T_{x,y} H_{p,q}$ where $i^* : H^1(M, \text{Ad}_\rho) \to H^1(\partial M, \text{Ad}_{\rho_{x,y}})$ is induced by the restriction map.
In particular, if $\chi_\rho$ is a smooth point of $X(M)$ and $\pi^{-1} r(X(M))$ intersects $H_{p,q}$ transversally, then $H^1(M_{p/q}, \mathbb{A}_\rho) = 0$.

**Case 2:** $x = \pm 1$ and $y = \pm 1$.

If $\chi_\rho$ is a smooth point of $X^{irr}(M)$, then $\rho$ factors through at most 1 Dehn filling $M_{p/q}$.

**Proof** The main point is a consequence of the Mayer-Vietoris sequence applied to the decomposition $M_{p/q} = M \cup D^2 \times S^1$ given below.

$$
\begin{align*}
  H^0_\rho(M) &\oplus H^0_\rho(D^2 \times S^1) \to H^0_\rho(S^1 \times S^1) \to \\
  H^1_\rho(M_{p/q}) &\to H^1_\rho(M) \oplus H^1_\rho(D^2 \times S^1) \to H^1_\rho(\partial M)
\end{align*}
$$

By the assumption $(x, y) \neq (\pm 1, \pm 1)$, the first map is onto and the result follows from the fact that the map $H^1_\rho(D^2 \times S^1) \to H^1_\rho(S^1 \times S^1)$ is the differential of the inclusion $H_{p,q} \subset \mathbb{G}_m^2$.

In the second case, if $\chi_\rho$ is a smooth point of $X^{irr}(M)$, then Poincaré duality implies that $H^0_\rho(\partial M)$ has dimension 1 and hence $\rho \circ \iota^*$ is a parabolic non-central representation. More explicitly, one has $\rho(a, b) = \pm \left( \begin{array}{cc} 1 & au + bv \\ 0 & 1 \end{array} \right)$ for some $(u, v) \neq (0, 0)$. Hence we can have $\rho(p, q) = 1$ for at most one slope $[p : q] \in \mathbb{P}_1(\mathbb{Q})$ and the result follows.

**Proof of Theorem 1.2 from Theorem 1.4:** Set $C = \pi^{-1}(r(X(M))) \subset \mathbb{G}_m^2$. In the setting of Theorem 1.2, this is a curve defined by the so-called $A$-polynomial introduced in [7]. Applying Theorem 1.4 to $C$, we obtain that $C$ is transverse to $H_{p,q}$ at smooth points of $C$ for all but a finite number of slopes $(p, q)$. Moreover by assumption, singular points of $X(M)$ do not map to torsion points of $X(\partial M)$ and hence belong to at most one subtorus. The only remaining case is a singular point $(x, y)$ of $C$ which is not a singular value of $r$. In the neighborhood of $(x, y)$, $C$ is a union of branches with non-trivial tangents. Removing these tangents from the list of admissible $(p, q)$, we finally proved Theorem 1.2.

We would like to end this section with some comments on the topological meaning of the assumptions of Theorem 1.2.

(i) By Culler-Shalen theory (see [21]), the properness assumption of $r : X(M) \to X(\partial M)$ is implied by the assumption that $M$ is “small”, meaning that it does not contain any closed incompressible surfaces (not boundary parallel). This assumption holds for a large family of knots such as 2-bridge knots. Such a hypothesis is necessary as the global local rigidity does not hold for instance for Whitehead doubles of knots.

(ii) The reducibility of the character variety is a notoriously hard question. There is no reason to believe that the character variety of a knot complement in $S^3$ is reduced, however, we do not know any counter-example.

(iii) The last assumption on singular points seems hard to check without knowing explicitly the character variety of $M$. However it is well-known that the singular points of $X(M)$ belonging to $X^{red}(M)$ are encoded in the roots of the Alexander...
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polynomial of $M$. Hence, the assumption contains in particular the fact that the Alexander polynomial of $M$ does not vanish at roots of unity.

3 Plane curves

This section is devoted to the proof of Theorem 1.4. It relies heavily on the notions of degree and height.

Recall that the degree $\deg(P)$ of a point $P = (x_0 : \ldots : x_n)$ of $\mathbb{P}_n(\overline{\mathbb{Q}})$ is defined as the minimal degree of a number field containing a system of homogeneous coordinates of $P$. Equivalently, assuming for simplicity that $x_0 \neq 0$,

$$\deg(P) = \left[ \mathbb{Q} \left( \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0} \right) : \mathbb{Q} \right].$$

The logarithmic Weil height $h(P)$ of $P$ is defined as follows. Let $K$ be a number field containing a system of homogeneous coordinates $(x_0, \ldots, x_n)$ of $P$. At any place $v$ of $K$, there is a unique absolute value $|\cdot|_v$ associated to $v$ such that $|p|_v \in \{1/p, 1, p\}$, for any prime number $p$. Let $K_v$ (respectively $\overline{\mathbb{Q}}_v$) be the completion of $K$ (resp. $\overline{\mathbb{Q}}$) with respect to this absolute value (resp. the absolute value induced by $|\cdot|_v$). Then, $h(P)$ is given by the formula:

$$h(P) = \sum_v \frac{[K_v : \overline{\mathbb{Q}}_v]}{[K : \overline{\mathbb{Q}}]} \log \left( \max_{0 \leq i \leq n} |x_i|_v \right),$$

where the sum runs over all places of $K$. Because of the normalization factors $[K_v : \overline{\mathbb{Q}}_v]/[K : \overline{\mathbb{Q}}]$, the right-hand side neither depends on $K$, nor on the system of homogenous coordinates $(x_0, \ldots, x_n)$ that is chosen for $P$. Therefore, $h(P)$ is well-defined.

Then, for any non-zero vector $a$ of $\overline{\mathbb{Q}}^{n+1}$, we define the projective height of $a$ as the quantity $h(P_a)$, where $P_a$ is the point of $\mathbb{P}_n(\overline{\mathbb{Q}})$ with homogenous coordinates $a$. Finally, we define the height $h(f)$ of a polynomial $f$ with coefficients in $\overline{\mathbb{Q}}$ as the projective height of its vector of coefficients.

Roughly speaking, the height of a point measures its arithmetic complexity. For example, if $P \in \mathbb{P}_1(\overline{\mathbb{Q}})$ is a rational point of coordinates $(p : q)$, where $p$ and $q$ are coprime integers, the definition above yields

$$h(p : q) = \log \{\max(|p|, |q|)\}.$$

In particular, for any positive real number $M$, there are only finitely many rational points $P$ of the projective line such that $h(P) \leq M$.

More generally, Northcott’s theorem asserts that if the degree and height are bounded over a subset $S$ of $\mathbb{P}_n(\overline{\mathbb{Q}})$, then $S$ is a finite set. This crucial fact is one of the main properties of the height.

As we will see, this can be used to obtain a second proof of Theorems 1.4 and 1.5, except for the upper bound for $\max(|p|, |q|)$ of the former which follows from
a different approach. This estimate will be derived from a height upper bound of Habegger through the following lemma.

**Lemma 3.1** Let $f_1, f_2 \in \overline{\mathbb{Q}}[x, y]$ be polynomials of total degree $N_1$ and $N_2$ and let $N = \max(N_1, N_2)$. Let $\mathcal{F}$ be the family of coefficients appearing in either $f_1$ or $f_2$ and let $h(\mathcal{F})$ be the projective height of $\mathcal{F}$. Then, for any point $P = (x_P, y_P)$ of $\mathbb{A}^2(\overline{\mathbb{Q}})$,

$$h(f_1(P) : f_2(P)) \leq Nh(x_P : y_P : 1) + h(\mathcal{F}) + \log \left( \frac{N + 2}{2} \right).$$

**Proof** Let $f_1(x, y) = \sum_{k+\ell \leq N_1} a_{k,\ell} x^k y^\ell$ and let $v$ be a place of a number field $K$ containing both the coordinates of $P$ and the family $\mathcal{F}$. Then,

$$|f_1(P)|_v \leq \varepsilon(v) \max_{k,\ell} (|a_{k,\ell}|_v) \max(1, |x_P|_v, |y_P|_v)^{N_1},$$

(3.1.1)

where

$$\varepsilon(v) = \begin{cases} \left( \frac{N_1 + 2}{2} \right) & \text{if } v \mid \infty, \\ 1 & \text{otherwise.} \end{cases}$$

Assume for simplicity that $N_1 \geq N_2$, so $f_2(x, y)$ is again of the form $\sum_{k+\ell \leq N_1} b_{k,\ell} x^k y^\ell$, for some $b_{k,\ell} \in \overline{\mathbb{Q}}$. Then, an upper bound similar to (3.1.1) holds for $|f_2(P)|_v$, with the only difference that the $a_{k,\ell}$’s are replaced by the $b_{k,\ell}$’s. Therefore,

$$\max(|f_1(P)|_v, |f_2(P)|_v) \leq \varepsilon(v) \max_{f \in \mathcal{F}} \max(1, |x_P|_v, |y_P|_v)^{N_1},$$

and it follows that

$$h(f_1(P) : f_2(P)) \leq N_1h(x_P : y_P : 1) + h(\mathcal{F}) + \sum_{v \mid \infty} \log(\varepsilon(v))$$

$$= N_1h(x_P : y_P : 1) + h(\mathcal{F}) + \log \left( \frac{N_1 + 2}{2} \right),$$

which concludes the proof as $N_1 = \max(N_1, N_2)$. \qed

We are now ready to prove Theorem 1.4. From now on, we let $A = \mathbb{G}_m^2$ be the ambient multiplicative torus. Recall that, for any curve $C$ lying in $A$, we denote by $C^{[1]}$ the union of all singular intersections of the form $C \cap_{\text{sing}} H_{p,q}$, where $(p, q)$ varies among all couples of relatively prime integers. Finally, recall that a translate of a subtorus $\gamma H_{p,q}$ is a torsion translate if $\gamma$ is a torsion point of $A$.

**Theorem 1.4** Let $C$ be an irreducible curve in $A$ with defining equation $f(x, y) = 0$ for an irreducible polynomial $f \in \overline{\mathbb{Q}}[x, y]$. Let $\delta = \deg(f)$, $\delta_x = \deg_x(f)$ and...
\[ \delta_y = \deg_y(f). \] Assume \( C \) is not a translate of a subtorus. Then, for any torsion translate \( \zeta H_{p,q} \) with non-empty singular intersection with \( C \), the quantity \( \max(|p|, |q|) \) is at most

\[ \delta^3 \exp \left( (6 \cdot 10^5 + 1)\delta^4 \max(\delta_x, \delta_y, h(f)) \right). \]

Moreover, \( C^{[1]} \) is a finite set of effectively bounded degree and height.

**Proof** Let \( P = (x_P, y_P) \) be a point belonging to the singular intersection of \( C \) and a torsion translate \( \zeta H \), with \( H = H_{p,q} \). First, because of the relation \( x_P y_P = 1 \), the tangent line \( T_P(\zeta H) \) is defined by the following equation, as a subspace of \( T_P A \):

\[ (T_P(\zeta H)) : \frac{p}{x_P} dx_P + \frac{q}{y_P} dy_P = 0, \]

where \( dx_P \) and \( dy_P \) denote the germs of the global 1-forms \( dx \) and \( dy \) at \( P \). Then, we have

\[ (T_{PC}) : \frac{\partial f}{\partial x}(P) dx_P + \frac{\partial f}{\partial y}(P) dy_P = 0. \]

Hence, the assumption \( T_P C = T_P(\zeta H) \) implies that the two vectors of partial derivatives

\[ \left( \frac{p}{x_P}, \frac{q}{y_P} \right) \quad \text{and} \quad \left( \frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P) \right) \]

are collinear.

Using height theory, we now derive that \( \max(|p|, |q|) \) is bounded from above, thereby showing that the possible \( H \)'s are finitely many. Indeed, \( p \) and \( q \) are coprime, so \( \log(\max(|p|, |q|)) = h(p : q) \) is the logarithmic Weil height of the point \( (p : q) \in \mathbb{P}_1(\overline{\mathbb{Q}}) \). Then, collinearity of the vectors of partial derivatives means that the corresponding points of \( \mathbb{P}_1(\overline{\mathbb{Q}}) \) are equal, which implies

\[ (p : q) = \left( x_P \frac{\partial f}{\partial x}(P) : y_P \frac{\partial f}{\partial y}(P) \right). \]

Hence, using Lemma 3.1 with \( f_1 = x \partial f/\partial x \) and \( f_2 = y \partial f/\partial y \), we obtain the following upper bound:

\[ h(p : q) \leq \delta h(x_P : y_P : 1) + h(\mathcal{F}) + \log \left( \frac{(\delta + 1)(\delta + 2)}{2} \right), \]

where \( \mathcal{F} \) is the family of coefficients of the partial derivatives polynomials \( \partial f/\partial x \) and \( \partial f/\partial y \).

Here, we can obtain a slightly better log-term, with numerator \( \delta(\delta + 1) \) instead of \( (\delta + 1)(\delta + 2) \), by using the relation \( h(p : q) = h_m(p/q) \) where the right-hand side
is the height of the point \( p/q \) of \( \mathbb{G}_m(\overline{\mathbb{Q}}) \). This height satisfies a triangle inequality relative to multiplication, so

\[
 h(p : q) = h\left( x_p \frac{\partial f}{\partial x}(P) : y_p \frac{\partial f}{\partial y}(P) \right) = h_m\left( \frac{x_p}{y_p} \frac{\partial f}{\partial x}(P) \right)
\]

\[
 \leq h_m\left( \frac{x_p}{y_p} \right) + h_m\left( \frac{\partial f}{\partial x}(P) \right) + h_m\left( \frac{\partial f}{\partial y}(P) \right)
\]

\[
 \leq h(P) + h\left( \frac{\partial f}{\partial x}(P) : \frac{\partial f}{\partial y}(P) \right),
\]

and then, we apply our lemma to the partial derivatives of \( f \) instead of \( x \frac{\partial f}{\partial x} \) and \( y \frac{\partial f}{\partial y} \), which gives

\[
 h(p : q) \leq \delta h(x_p : y_p : 1) + h(F) + \log \left( \frac{\delta(\delta + 1)}{2} \right). \tag{3.1.2}
\]

We will use this sharper bound in the sequel. Finally, any element of \( F \) is a product \( ka_0 \) of a coefficient of \( f(x, y) = \sum_{a \in \mathbb{A}} a_{a_1} x^{a_1} y^{a_2} \) and a positive integer \( k \) equal to \( \alpha_1 \) or \( \alpha_2 \), depending on which partial derivative of \( f \) is considered. In any case, we have \( k \leq \max(\delta_x, \delta_y) \leq \delta \), so

\[
 h(F) \leq h(f) + \log \delta.
\]

and therefore

\[
 h(p : q) \leq \delta h(x_p : y_p : 1) + h(f) + \log \left( \frac{\delta^2(\delta + 1)}{2} \right). \tag{3.1.3}
\]

To conclude the proof, we use an explicit height upper bound for the points of \( C^{[1]} \) due to Habegger (see Theorem B.1 in [10] and Theorem 1 in [11]). As \( P = (x_p, y_p) \) belongs to \( \zeta H_{p,q} \) and \( \zeta \) is a torsion point, \( P \) belongs to \( C^{[1]} \), so Habegger’s bound implies

\[
 \max(h_m(x_p), h_m(y_p)) \leq 3 \cdot 10^5 \delta^3 \max(\delta_x \delta_y, h(f)). \tag{3.1.4}
\]

Combining this result with inequality (3.1.2) and using the elementary fact that \( h(x : y : 1) \leq h_m(x) + h_m(y) \), we obtain

\[
 h(p : q) \leq 6 \cdot 10^5 \delta^4 \max(\delta_x \delta_y, h(f)) + h(f) + \log \left( \frac{\delta^2(\delta + 1)}{2} \right)
\]

\[
 \leq (6 \cdot 10^5 + 1)\delta^4 \max(\delta_x \delta_y, h(f)) + \log(\delta^3)
\]

and taking exponentials of both sides gives the upper bound of the theorem.
Finiteness of $C^{[1]}$ follows easily: the bound for $\max(|p|, |q|)$ implies there are only finitely many subtori $H_{p,q}$ with non-empty singular intersection with $C$, and the assumption on $C$ guarantees that its intersection with any $H_{p,q}$ is finite.

Finally, the effective upper bound for $h(P)$ mentioned in our theorem is Habegger's bound (3.1.4). Whereas the effective upper bound for $\deg(P)$ follows from our bound for $\max(|p|, |q|)$ by applying Bézout’s theorem to the intersection of $C$ and $H_{p,q}$, which gives

$$\deg(P) \ll \max(|p|, |q|)\deg(C)[K_0 : \mathbb{Q}],$$

where $K_0$ is any number field such that $C$ is defined over $K_0$. So, we also get an effective upper bound for $\deg(P)$. We don’t go any further and postpone explicit degree bounds until Lemma 3.3 below, where we obtain a much stronger result than the one we would get here, from a qualitative and quantitative standpoint. \hfill \Box

We point out that the first part of Theorem 1.4, giving the upper bound for $\max(|p|, |q|)$, holds for all torsion translates of dimension 1, not just subtori. It suggests that more generally, this result should imply finiteness of the larger subset $C^{[1,\text{tor}]}$, instead of $C^{[1]}$.

This turns out to be true (see Corollary 3.2 below), but the deduction is not trivial:

- Again, the bound for $\max(|p|, |q|)$ implies that we only need to consider finitely many $H_{p,q}$’s. But then, taking all possible torsion translates of these subtori, we obtain infinitely many possible intersections to look for, so the argument given for $C^{[1]}$ fails for $C^{[1,\text{tor}]}$.

- One could also try to generalize the last part of the proof of Theorem 1.4 to show that the degree of the points in $C^{[1,\text{tor}]}$ is bounded. Indeed, using Northcott’s theorem, finiteness would follow by applying Habegger height bound (3.1.4). Yet again, although the bound for $\max(|p|, |q|)$ is enough to bound the degree of $H_{p,q}$, it doesn’t yield any bound for the degrees of the torsion translates $\zeta H_{p,q}$, because $\deg(\zeta H_{p,q})$ also depends on $\deg(\zeta)$, which is unbounded.

Hence, we need a new argument to deduce finiteness of $C^{[1,\text{tor}]}$ from the upper bound for $\max(|p|, |q|)$.

A crucial tool in this context is the notion of ramification points for finite morphisms between curves. Indeed, for any point $P$ of $C^{[1,\text{tor}]}$, there exists coprime integers $p, q$ such that $P$ is a ramification point for the finite morphism $C \to A/H_{p,q}$ induced by the quotient $A \to A/H_{p,q}$.

**Corollary 3.2** Let $C$ be an irreducible curve in $A$ that is not a translate of a subtorus. Then $C^{[1,\text{tor}]}$ is a finite set of effectively bounded height.

**Proof** Again, as in Theorem 1.4, the effective height bound is already known from the inclusion of $C^{[1,\text{tor}]}$ in $C^{[1]}$ and we only need to prove finiteness. We use the upper bound of Theorem 1.4. It shows that the points of $C^{[1,\text{tor}]}$ are all coming from a finite number of subtori, translating each one by all torsion points of $A$ and taking singular intersections with $C$. Now, fix one of these subtorus $H = H_{p,q}$ and consider the points of $C^{[1,\text{tor}]}$ coming from singular intersections with torsion translates of $H$. Let
\( \pi_H : A \to \mathbb{G}_m \) be the morphism given in coordinates by \( \pi_H(x, y) = x^p y^q \). Then, \( \text{Ker}(\pi_H) = H \) and, more generally, for any \( \gamma \in A(\hat{\mathbb{Q}}) \) we have \( \pi_H^{-1}(\pi_H(\gamma)) = \gamma H \).

Hence, denoting by \( \iota : C \hookrightarrow A \) the inclusion of \( C \) in \( A \), a point \( P \) lies in a singular intersection of the form \( C \cap \text{sing} \gamma H \) for some \( \gamma \) if and only if the curve \( C \) and the fiber of \( \pi_H \) passing through \( P \) have the same tangent line at \( P \). In other words, the composition \( \phi_H = \pi_H \circ \iota \) is ramified at \( P \).

It follows that the points of \( C^{[1,\text{tor}]} \) coming from singular intersections with arbitrary translates of \( H \) are precisely the ramification points of \( \phi_H \). Finally, because of the assumption on \( C \), \( \phi_H \) is a finite separable morphism of curves, so it only has finitely many ramification points, which concludes the proof (see for example point (a) in Proposition 2.2, Chapter IV in [13]). \( \square \)

Theorem 1.4 and Corollary 3.2 are non trivial in the sense that both \( C^{[1]} \) and

\[
C^{[1,A(\hat{\mathbb{Q}})]} = \bigcup_{\gcd(p,q) = 1} \{ P \in C_{\text{reg}}(\hat{\mathbb{Q}}) \mid TP_C = TP(PH_{p,q}) \}
\]

are infinite for all \( C \). The first is obtained from \( C^{[1,\text{tor}]} \) by removing the tangency condition, while the second is the set of smooth points \( P \in C(\hat{\mathbb{Q}}) \) at which \( TP_C \) coincides with the tangent line of a translate of an \( H_{p,q} \), for some \((p, q)\). Indeed, this translate is then the unique translate of \( H_{p,q} \) passing through \( P \), which is \( PH_{p,q} \) for obvious reasons, so we recover the formula above.

The case of \( C^{[1]} \) is well known. One of the two projections \( \pi \) induces a dominant morphism over \( C \), so \( \pi(C) \) contains a dense open subset of \( \mathbb{G}_m \). In particular, it contains infinitely many torsion points and their inverse images form an infinite subset of \( C^{[1]} \).

For \( C^{[1,A(\hat{\mathbb{Q}})]} \), the argument is completely similar, up to a different choice of the dominant morphism. Assuming \( f(x, y) = 0 \) is a defining equation for \( C \), with \( f \) irreducible, let \( \sigma_C \) be the rational map from \( C \) to the dual projective line which sends any smooth point \( P = (x_P, y_P) \) of \( C(\hat{\mathbb{Q}}) \) to the point

\[
\sigma_C(P) = \left( x_P \frac{\partial f}{\partial x}(P) : y_P \frac{\partial f}{\partial y}(P) \right)
\]

(3.2.1) of \( \mathbb{P}^1 \). Note that it doesn’t depend on the irreducible polynomial \( f \) chosen initially. Moreover, \( \sigma_C \) has the following property:

\[
\sigma_C(P) = (\alpha : \beta) \iff (TP_C) : \alpha \frac{dx_P}{x_P} + \beta \frac{dy_P}{y_P} = 0,
\]

(3.2.2)

where \( dx_P \) and \( dy_P \) denote respectively the germs of \( dx \) and \( dy \) at \( P \). It follows that \( \sigma_C \) is constant if and only if \( C \) is a translate of a subtorus, in which case \( \sigma_C(C) \) is a rational point and \( C^{[1,A(\hat{\mathbb{Q}})]} = C \). Otherwise, \( \sigma_C \) is dominant, so its image contains a dense open subset of the line, hence infinitely many rational points. And by considering the inverse images of these points in \( C \), we obtain an infinite subset of \( C^{[1,A(\hat{\mathbb{Q}})]} \).

Hence, \( C^{[1,A(\hat{\mathbb{Q}})]} \) is always Zariski-dense in \( C \). However, under the assumption of Theorem 1.4, it turns out to be a sparse subset of \( C \) in the following sense.
Lemma 3.3 Let $C$ be an irreducible curve in $A$ defined over a number field $K_0$ and assume $C$ is not a translate of a subtorus. Then, for all $P \in C^{1, A(\bar{\mathbb{Q}})}$, 

$$\deg(P) \leq \deg(C)^2 [K_0 : \mathbb{Q}].$$

Proof By hypothesis, $C$ can be defined by an equation $f(x, y) = 0$ with coefficients in $K_0$. Moreover, we can assume $f$ is irreducible over $\bar{\mathbb{Q}}$, so $\deg(f) = \deg(C)$.

Then, by (3.2.1), the rational map $\sigma_C$ is also defined over $K_0$, and so is the fiber of $\sigma_C$ over any rational point $(p : q)$ of $\mathbb{P}^n$. Therefore, if $P = (x_P, y_P)$ belongs to $\sigma^{-1}_C(p : q)$, then so does any Galois conjugate of $P$ over $K_0$. Moreover, the number of distinct conjugates of $P$ is precisely the number of embeddings of $K_0(x_P, y_P)$ in $\bar{\mathbb{Q}}$ over $K_0$, that is, $[K_0(x_P, y_P) : K_0]$. Hence, $[K_0(x_P, y_P) : K_0] \leq \deg(C)$ and we obtain

$$\deg(P) = [\mathbb{Q}(x_P, y_P) : \mathbb{Q}] \leq [K_0(x_P, y_P) : \mathbb{Q}] \leq [K_0(x_P, y_P) : K_0][K_0 : \mathbb{Q}] \leq \deg(C).$$

The lemma thus follows from the estimate

$$|\sigma^{-1}_C(p : q)| \leq \deg(C)^2. \quad (3.3.1)$$

To prove (3.3.1), consider the irreducible components $C_1, \ldots, C_r$ of the algebraic subset of $A$ defined by the vanishing of

$$g_{p,q}(x, y) = qx \frac{\partial f}{\partial x}(x, y) - py \frac{\partial f}{\partial y}(x, y).$$

Then, $\sigma^{-1}_C(p : q)$ is the union of the intersections $C_{\text{reg}} \cap C_i$ and

$$\sum_{i=1}^{r} \deg(C_i) \leq \deg(g) \leq \deg(f) = \deg(C).$$

Using Bézout’s theorem, it follows that

$$|\sigma^{-1}_C(p : q)| \leq \sum_{i=1}^{r} \deg(C) \deg(C_i) \leq \deg(C)^2,$$

which concludes the proof. \qed

Remark 3.4 In comparison with Theorem 1.4, this lemma provides a much sharper bound for the degrees of the points of $C^{[1]}$ and extends it to the much larger subset $C^{1, A(\bar{\mathbb{Q}})}$. Recall that the one from the theorem was derived from the fact that

$$\deg(P) \ll \max(|p|, |q|) \deg(C)[K_0 : \mathbb{Q}]$$
combined with our estimate on $\max(|p|, |q|)$, thereby leading to a large upper bound depending on the height of $C$.

**Remark 3.5** For the connoisseur of unlikely intersections, the situation here is in striking contrast with the classical case described in [3], where one derive that the degree is bounded over $C^{[2]}$ from the fact that the height is bounded over $C^{[1]}$. The proof of this implication relies usually on advanced diophantine tools, such as estimates on the Lehmer problem or Bogomolov-type properties.

Here, on the contrary, the bounded degree property is elementary and unconditional, holding true over a larger subset than $C^{[1,\text{tor}]}$ of unbounded height, without any multiplicative dependence assumption. It shed some light on the novelty of our problem in comparison with the classical Zilber–Pink framework.

Applying Northcott’s theorem, it follows from boundedness of the degree that any subset of $C^{[1,A(\bar{Q})]}$ of bounded height is finite. In particular, it implies that

$$C^{[1,\text{tor}]} \subset C^{[1]} \cap C^{[1,A(\bar{Q})]}$$

is a finite set, which gives a second proof of Theorem 1.4 and Corollary 3.2, except for the upper bound for $\max(|p|, |q|)$.

As we will see, such a bound can still be recovered partially from the upper bounds on the degree and height. It even yields a sharper estimate, but it doesn’t give any result in the case where all singular intersections of torsion translates of $H_{p,q}$ with $C$ are entirely torsion.

The method is coming from a lemma of Barroero and Capuano (see Lemma 5.2 of [1] and also [2] where the same ideas were already used in a different context). It relies primarily on Masser’s Theorem $G_{m}$ (see [17]). Using a lower bound for the height of Dobrowolski type, we first prove an explicit version of Barroero and Capuano’s lemma that follows readily from their proof. In combination with our Lemma 3.3 above, it gives the following upper bound for $\max(|p|, |q|)$.

**Theorem 3.6** Let $K_0$ be a number field of degree $\Delta$ and let $C$ be an irreducible curve in $A$ with defining equation $f(x, y) = 0$ for a polynomial $f \in K_0[x, y]$ of degree $\delta$ irreducible over $\bar{Q}$. Assume $C$ is not a translate of a subtorus and let $(p, q)$ be a couple of coprime integers such that there is a torsion translate of $H_{p,q}$ whose singular intersection with $C$ is non-empty and not entirely torsion. Then

$$\max(|p|, |q|) \leq 3 \cdot 2^4 5^9 3^3 \log(3^2 \Delta)^3 \max \left( \delta_x \delta_y, h(f) \right).$$

**Proof** Following the proof of Lemma 5.2 in [1], let $K$ be a number field of degree $\kappa$, let $\omega$ be the number of roots of unity in $K$ and let $\eta$ be a lower bound for the height of elements of $K^*$ which are not roots of unity. Then, consider a point $P$ of $A^{[1]}$, i.e. a point of $A(\bar{Q})$ whose coordinates satisfy at least one non-trivial multiplicative dependence relation $x^\alpha y^\beta = 1$ for $\alpha, \beta \in \mathbb{Z}$. Let $L(P)$ be the subgroup of $\mathbb{Z}^\omega$ formed by the vectors of exponents $(\alpha, \beta)$ of all multiplicative dependence relations satisfied at $P$. Assuming $P$ has coordinates in $K$ and height bounded from above by a real
number \( h \geq \eta \), Masser’s Theorem \( \mathbb{G}_m \) from [17] implies that \( L(P) \) can be generated by vectors \((a, b)\) such that

\[
\max(|a|, |b|) \leq 2\omega h / \eta. \tag{3.6.1}
\]

Finally, as \( \kappa \geq \phi(\omega) \geq \sqrt{\omega} / 2 \), where \( \phi \) is Euler’s function, we have

\[
\omega \leq 2\kappa^2. \tag{3.6.2}
\]

Also, it follows from Corollary 2 of [23] that a suitable \( \eta \) can be defined as follows:

\[
\eta = \frac{1}{2\kappa \log(3\kappa)^3}. \tag{3.6.3}
\]

Here, in comparison with Voutier, we take a slightly coarser bound to include the case of \( K = \mathbb{Q} \) as well, where \( \kappa = 1 \) and \( \log(2) \) is the minimal value of the height over \( K^* \). Notice in particular that \( \eta \leq 1 \).

In conclusion, by inequalities (3.6.1), (3.6.2) and definition (3.6.3) above, we obtain

\[
\max(|a|, |b|) \leq 2^3 \kappa^3 \log(3\kappa)^3 h. \tag{3.6.4}
\]

Finally, we consider a couple \((p, q)\) such that there exists a torsion point \( \zeta \) and a non-torsion point \( P \) with \( P \in C \cap \text{sing} \zeta H_{p,q} \). We apply the previous estimate with \( K = \mathbb{Q}(P) \), so \( \kappa = \deg(P) \leq \delta^2 \Delta \) by Lemma 3.3. Moreover, we take \( h \) to be Habegger’s upper bound for the height \( h(x : y : 1) \) over \( C^{[1]} \), so

\[
h = 6 \cdot 10^5 \delta^3 \max(\delta_x, \delta_y, h(f)),
\]

\(i.e.\) twice the upper bound for \( \max(h_m(x), h_m(x)) \) of (3.1.4). In particular, as we assume \( C \) is not a translate of a subtorus, we have \( \delta_x, \delta_y > 0 \) and it follows that \( h \geq 1 \geq \eta \). Also, the condition \( h(P) \leq h \) is satisfied, so (3.6.4) implies

\[
\max(|a|, |b|) \leq 2^3 \delta^6 \Delta^3 \log(3\delta^2 \Delta)^3 h, \tag{3.6.5}
\]

for certain generators \((a, b)\) of \( L(P) \), which gives the desired upper bound by substituting the definition of \( h \) above. Then, as \( P \) is non torsion, \( L(P) \) has rank 1, so this bound holds true for any generator \((a, b)\) of \( L(P) \). Moreover, for any such \((a, b)\), we have \((p, q) = \pm(a/d, b/d)\), where \( d \) is the greatest common divisor of \( a \) and \( b \). Therefore, \( \max(|p|, |q|) \) is also bounded as in (3.6.5), proving the claim. \( \square \)

**Remark 3.7** This new bound is sharper than the one from Theorem 1.4. In particular, it depends linearly on the height upper bound \( h \), as in (3.6.5), while the bound coming from Theorem 1.4 was exponential. Indeed, in comparison with (3.6.5), the proof of Theorem 1.4 only gives

\[
\max(|p|, |q|) \leq \frac{\delta^2(\delta + 1)}{2} \exp(\delta h + h(f)),
\]
with the same definition of $h$ (see inequality (3.1.2)).

We then derive an effective upper bound for the number of couples $(p, q)$ such that some torsion translate of $H_{p,q}$ has non-empty singular intersection with $C$, without any further assumption. In addition to Theorem 3.6, we thus need a bound for the number of “degenerate” cases, that is, couples $(p, q)$ such that at least one singular intersection of a torsion translate of $H_{p,q}$ with $C$ is non-empty, but all such singular intersections are entirely torsion.

As we will see, these degenerate cases are controlled by the torsion points of $C$, whose number is at most $22\delta_x\delta_y$ by a theorem of Ruppert (see Corollary 5 in Section 2 of [20]). It gives the following counting result.

**Corollary 3.8** Let $K_0$ be a number field of degree $\Delta$ and let $C$ be an irreducible curve in $A$ with defining equation $f(x, y) = 0$ for a polynomial $f \in K_0[x, y]$ of degree $\delta$ irreducible over $\overline{\mathbb{Q}}$. If $C$ is not a translate of a subtorus, then the number of couples $(p, q)$ of coprime integers such that there is a torsion translate of $H_{p,q}$ with non-empty singular intersection with $C$ is at most

$$c \cdot \delta^{18} \Delta^6 \log(3\delta^2 \Delta)^6 \max (\delta_x \delta_y, h(f))^2, \quad c = 20^{11}.$$  

**Proof** First, for any real number $T \geq 1$, the number of couples of coprime integers $(p, q)$ with $\max(|p|, |q|) \leq T$ is at most $4 + 4T^2 \leq 8T^2$. Hence, the number of couples $(p, q)$ as in Theorem 3.6 is bounded from above by a quantity of the desired form, with constant $c' = 9 \cdot 2^{11}10^{10}$ instead of $c$, which proves the result in the generic case.

The remaining couples $(p, q)$ are the degenerate ones, those such that at least one singular intersection of a torsion translate of $H_{p,q}$ with $C$ is non-empty, but all such singular intersections are entirely torsion. Then, there exists a non singular torsion point $\xi_{p,q}$ of the curve $C$ such that $\xi_{p,q}H_{p,q}$ and $C$ have the same tangent line at $\xi_{p,q}$. Finally, the map which sends $(p, q)$ to $\xi_{p,q}$ is one-to-one, because of the tangency condition; indeed, $(p, q)$ can be recovered from $\xi_{p,q}$ by considering the tangent line of $C$ at $\xi_{p,q}$. Therefore, the number of degenerate couples $(p, q)$ is bounded from above by the number of torsion points of $C$, which is at most $22\delta_x\delta_y$ by [20].

Finally, the desired result follows by adding up our upper bounds for the number of couples of the first and second types and using the trivial inequalities

$$22 \leq 2^{11}10^{10}\delta^{18} \Delta^6 \log(3\delta^2 \Delta)^6 \quad \text{and} \quad \delta_x \delta_y \leq \max (\delta_x \delta_y, h(f))^2.$$

**Remark 3.9** The proof shows that, for counting estimates, the generic upper bound that follows from Theorem 3.6 is still true in the degenerate case. The situation seems to be different for effective height estimates, i.e. upper bounds on $\max(|p|, |q|)$. Indeed, for a torsion point $P \in C(\overline{\mathbb{Q}})$, the condition that $P$ belongs to a torsion translate $\xi H_{p,q}$ of $H_{p,q}$ is always satisfied with $\xi = P$, for any $(p, q)$. Therefore, $P$ belongs to a singular intersection of the form $C \cap_{\text{sing}} \xi H_{p,q}$ for some $\xi \in A_{\text{tor}}$, if and only if
$T_P(C) = T_P(PH_{p,q})$, i.e. $\sigma_C(P) = (p : q)$. Hence, for degenerate couples $(p, q)$, the only approach to effective bounds on $\max(|p|, |q|)$ seems to be as in the proof of Theorem 1.4. Finally, notice that in the particular case of a degenerate couple $(p, q)$, the condition $h(P) = 0$ can be used to simplify the upper bound of this Theorem, starting from (3.1.2). It gives

$$\max(|p|, |q|) \leq \frac{\delta^2(\delta + 1)}{2} \exp(h(f))$$

(see Remark 3.7), which is still exponential in the height of $f$, so not as sharp as Theorem 3.6.

We finally turn to the completion of the proof of Theorem 1.5. Recall that it gives the same conclusion as Corollary 3.2 under the weaker assumption that $C$ is not a torsion translate. The proof turns out to be very easy: the corollary implies the theorem, because the result is almost trivial in the remaining case.

**Proof of Theorem 1.5** Assume first that $C$ is not a translate of a subtorus. Then, finiteness follows from Corollary 3.2. We may therefore assume that $C$ is a translate $\gamma H$ of a subtorus $H$ of $A$. Then, because of the assumption on $C$, $\gamma$ is non-torsion. Hence $C \neq \zeta H$ for all $\zeta \in A_{\text{tor}}$, so $C \cap \zeta H = \emptyset$ because the two are distinct translates of the same subtorus. Finally, if $H' \neq H$ is a second subtorus, then $T_P(PH') \cap T_P(PH) = \{0\}$ at any point $P \in A(\overline{Q})$. Therefore, $C \cap_{\text{sing}} \zeta H'$ is again empty, which gives $C_{[1, \text{tor}]} = \emptyset$ and completes the proof of the theorem. □

We conclude this section with an alternative formulation of Theorem 1.5, showing the equivalence of this statement to a tangential Zilber–Pink problem. It also clarifies the relation between subsets of the form $C_{[1, \text{tor}]}$ and $C_{[2]}$: the first one can be seen as a subset of the second type by passing from $A$ to the dual projectivized tangent bundle

$${\mathcal A} = \mathbb{P}^s(TA),$$

replacing curves in $A$ by their **tangent sections** in $\mathcal A$ (see definition below).

First, recall that the fiber of $\mathcal A$ over a point $P \in A(\overline{Q})$ parametrizes lines in $T_P A$. For any such line $\Delta$, we denote by $(P, [\Delta])$ the point of the fiber $A_P$ corresponding to $\Delta$.

**Definition 3.10** Let $C$ be an irreducible curve lying in $A$. The **tangent section** of $\mathcal A$ over $C$ is the curve $\tilde{C} \subset \mathcal A$ defined as follows:

- If $C$ is smooth, then $\tilde{\sigma}_C(P) = (P, [TP C])$ defines a global section of $\mathcal A$ over $C$, and we define $\tilde{C}$ as its image in $\mathcal A$, i.e.

$$\tilde{C} = \{(P, [TP C]) \mid P \in C(\overline{Q})\} = \mathbb{P}^s(TC).$$

- If $C$ is singular, then $\tilde{\sigma}_C(P) = (P, [TP C])$ still defines a rational section of $\mathcal A$ over $C$, and we define $\tilde{C}$ as the Zariski closure of its image in $\mathcal A$. 

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Remark 3.11 The rational section $\tilde{\sigma}_C$ of this definition is similar to the rational map $\sigma_C$ previously defined. The connection lies in the trivialization of the cotangent bundle $T^*A$ associated to the global 1-forms $dx/x$ and $dy/y$. Using this trivialization, the map $\pi : \mathbb{P}^*(TA) \to A$ can be identified to the first projection $A \times \mathbb{P}^*_1 \to A$ and, by (3.2.2), we have $\tilde{\sigma}_C(P) = (P, \sigma_C(P))$ at any smooth point $P$ of $C$.

For any couple of relatively prime integers $(p, q)$ and any torsion point $\zeta$ of $A$, let $\mathcal{H}_{p, q}^\zeta$ be the tangent section of $A$ over the torsion translate $\zeta H_{p, q}$. Then $\mathcal{H}_{p, q}^\zeta$ is of codimension 2 in $A$.

Theorem 3.12 Let $C$ be an irreducible curve in $A$ and let $C$ be the tangent section of $A$ over $C$. If $C$ is not of the form $\mathcal{H}_{p, q}^\zeta$, then

$$C^{[2]} = \bigcup_{\gcd(p, q) = 1} \mathcal{C}(\bar{Q}) \cap \mathcal{H}_{p, q}^\zeta$$

is a finite set.

Remark 3.13 In the formula above, intersections are usual intersections, not singular ones. Moreover $\operatorname{codim}(\mathcal{H}_{p, q}^\zeta, A) = 2$, so the union of the theorem is indeed a subset of the type $C^{[2]}$ for a Zilber–Pink-like problem in $A$.

Proof It suffices to show that there are finitely many smooth points $P$ of $C$ such that $\tilde{\sigma}_C(P) = (P, [TP C])$ lies in the union of the $\mathcal{H}_{p, q}^\zeta$. Notice that assuming $\tilde{\sigma}_C(P) \in \mathcal{H}_{p, q}^\zeta$ means precisely that there is a point $Q$ in $C' = \zeta H_{p, q}$ such that

$$\tilde{\sigma}_C(P) = \tilde{\sigma}_{C'}(Q).$$

This implies that $P = Q$ and $C$ and $C'$ have equal tangent lines at $P$, so $P \in C \cap \text{sing } C'$ and finiteness thus follows from Theorem 1.5. $\square$

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Declarations

Conflict of interest  On behalf of all authors, the corresponding author states that there is no conflict of interest.

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