Global wellposedness and scattering for 3D Schrödinger equations with harmonic potential and radial data

Zhang Xiaoyi\textsuperscript{1,2} *

\textsuperscript{1}Institute of Mathematics, Chinese Academy of Sciences,
\textsuperscript{2}Beijing Institute of Applied Physics and Computational Mathematics

P.O.Box 8009-11, Beijing 100088, China
Email:xiaoyizhangoo@yahoo.com.cn

Abstract

In this paper, we show that spherical bounded energy solution of the defocusing 3D energy critical Schrödinger equation with harmonic potential, \((i\partial_t + \frac{\Delta}{2} + \frac{|x|^2}{2})u = |u|^4u\), exits globally and scatters to free solution in the space \(\Sigma = H^1 \cap \mathcal{F}H^1\). We preclude the concentration of energy in finite time by combining the energy decay estimates and the ideas in the paper\([1, 7, 13]\).

Keywords: Schrödinger equation, harmonic potential, energy critical.
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1 Introduction

In this paper, we consider the Cauchy problem of defocusing energy critical equation with harmonic potential,

\[
(i\partial_t + \frac{\Delta}{2})u = -\frac{|x|^2}{2}u + |u|^4u, \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R},
\]

\[
u(x, 0) = u_0(x), \quad x \in \mathbb{R}^3,
\]

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where $u(t, x)$ is a complex function on $\mathbb{R}^3 \times \mathbb{R}$, $u_0(x)$ is a complex function on $\mathbb{R}^3$ satisfying

$$u_0(x) \in \Sigma = \{v; \|v\|_\Sigma = \|v\|_{H^1} + \|xv\|_2 < \infty\}.$$  

We will be interested in the global existence and long time behavior of the solution.

Schrödinger equation without potential has been extensively studied and we are mainly interested in the equation with power-like non-linearity:

$$(i\partial_t + \Delta)v = \mu |v|^p v, \quad \mu \neq 0,$$

where $\mu > 0$ and $\mu < 0$ correspond to the defocusing case and the focusing case, respectively. We concern such Cauchy problem as: there exists $t_0 \in \mathbb{R}$ such that the solution $v$ is in $H^1$ at this point. One thing that plays important role in the study of the Cauchy problem is the conservation of energy:

$$E(t) = \frac{1}{2} \|\nabla v(t)\|_2^2 + \frac{\mu}{p+2} \|v(t)\|^{p+2}_{p+2} = \text{const.}$$

Roughly speaking, (3) has local $H^1$ solution when $p$ is smaller than and equals to a certain exponent $p_c$ ($p_c = 4$ as $n = 3$) which is energy critical in the sense that the natural scale invariance

$$v(x, t) \longrightarrow \lambda^{-\frac{2}{p-2}} v(\lambda^{-1} x, \lambda^{-2} t)$$

of the equation leaves the $\dot{H}^1$ norm invariant. In the supercritical case $p > p_c$, (3) is locally illposed in the sense that the solution $v(x, t)$ does not continuously depend on the initial data $v_0(x)$ in $H^1$ space. To get more details, one refers to see [4], [5], [10], [6]. In the defocusing and subcritical case, the global existence is a direct consequence of the energy conservation. In the focusing case $\mu < 0$, blow up in finite time may appear, since the influence of kinetic energy $\|\nabla v\|_2^2$ may not always surpass the influence of the potential energy $\|v\|^{p+2}_{p+2}$. See [9], [4] for instance. The case of energy critical becomes rather difficult because the pure energy conservation is not enough to ensure the energy solution exist globally. In other words, the energy of the solution may concentrate somewhere, so that the solution may possibly blow up in finite time.

The first important work in this field is due to Bourgain [1] and Grillakis [12]. They pointed out that the solution will concentrate
somewhere in $\mathbb{R}^d$ unless the solution does not exist globally. To preclude this phenomenon, they use an apriori estimate which is called Morawetz estimate,

$$
\int_I \int_{|I|^{\frac{1}{2}}} \frac{|v|^{p_c+2}(x, t)}{|x|} dx dt \leq CAE(v)|I|^\frac{1}{2}, \quad A \geq 1. \quad (4)
$$

(4) is useful for preventing the concentration of $v(t, x)$ at origin $x = 0$. This is especially helpful when the solution is radially symmetric since in this case, it can be easily shown by bounded energy that $v$ will not concentrate at any other location than the origin. Their results are restricted to $d = 3, 4$. Recently in [13], T. Tao proved the energy critical Cauchy problem with radial data is globally wellposed and scatters to free solution in all dimensions by a different method.

To remove the radial assumption is not an easy thing. Recently in [7], J. Colliander and others proved this is feasible in the 3-dimensional case. Their proof involves many technical analysis especially in both physical space and frequency space. For details see [7].

Schrödinger equation with harmonic potential and power-like nonlinearity can be written in the form,

$$
(i\partial_t + \Delta)u = \omega |x|^2u + \mu |u|^p u. \quad (5)
$$

The Cauchy problem of it becomes much more complicated because of many-sided reasons. One of them lies in getting the estimate of the linear operator, another one may be that general structural conditions such as scaling invariance and spatially translation invariance that hold for the equation (4) will not hold for (5) any more. Even so, when we study Cauchy problem, we still define the energy critical exponent by omitting the potential just like we do for the equation (4). In both subcritical and critical case, it’s natural to seek for the solution with suitable decay in space, ie,

$$
u \in C_t(\Sigma). \quad (6)
$$

Indeed, recently in [2], [3], R. Carles systematically studied the Cauchy problem of (5). He found that when the nonlinearity is subcritical, defocusing and when a confining potential (ie. $\omega = -1$) is involved, the solution will be global and a scattering theory is available. Furthermore, when the nonlinearity is focusing and subcritical, a sufficient strong harmonic potential will prevent blow up in finite time. There
are also some results involving a general potential $V(x)$ which is a quadratic function.

There are two things that play important role in his proof. The first ingredient involved the Strichartz estimate for the linear operator $i\partial_t + \frac{x}{2} + \frac{|x|^2}{2}$. This may follow from Mehler’s formula at some special occasions. The next thing is that there exist two Galilean operators $J(t)$ and $H(t)$ which can commute with the linear operator and can be viewed as a substitute of $\nabla$ and $x$ in the nonpotential case.

The question remains open that what will happen when the nonlinearity is energy critical, that is, $p = p_c = \frac{4}{d-2}$, $d \geq 3$. In this paper, we restrict our attention to the case $d = 3$, $\omega = -1$, and $\mu = 1$. Enlightened by the work of Bourgain, Carles, Tao and others, we are expected to prove the global existence and scattering theory for the cauchy problem (1)-(2). In what follows, we sketch the proof.

The first thing we do is to find the local solution and small global solution of (1) and (2). They are available thanks to the Strichartz estimate for the linear operator. One interesting thing in the small solution theory is that: in order to get global solution, it suffices to require $\|\nabla u_0\|_2$ to be small enough. This result is fundamental in our proof. Another thing we must pay attention is that the maximal time interval of the local solution depends on the profile of the initial data, not depends on the $\Sigma$ norm of $u_0$ only. This is the main reason why the critical problem becomes much more complicated than the subcritical case.

Next, we show that: in order to extend the local solution to a global one and prove the scattering theory to the global solution, it suffices to prove an apriori space time bound for the solution. Being same with [1], this bound is $\|u\|_{L^{10}(R;L^{10})}$. So, it’s natural for us to see what the apriori bounds have been provided by the equation (1). First of all, we have two conservative quantities: mass and energy,

$$\|u(t_1)\|_2 = \|u(t_2)\|_2, \quad E(u(t_1)) = E(u(t_2)) = \frac{1}{2}\|\nabla u(t_2)\|_2^2 - \frac{1}{2}\|xu(t_2)\|_2^2 + \frac{1}{3}\|u(t_2)\|_6^6. \quad (7)$$

the energy (8) is non-positive, so we split it into two positive parts:

$$E_1(u(t)) = \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{3}\|u(t)\|_6^6, \quad E_2(u(t)) = \frac{1}{2}\|xu(t)\|_2^2, \quad (9)$$

and consider the Cauchy problem with $E_1(u(0)) = E$, $E_2(u(0)) = B$ for some fixed constant $E > 0$, $B > 0$. From now on, we call a
finite energy solution $u(x, t)$ on a time interval $I$ if it is such that $E_i(u(t)) < \infty$, $\forall t \in I$, $i = 1, 2$, and we may write the notations by omitting $u$ in some occasions for the sake of simplicity. Although $E_1(t)$ and $E_2(t)$ are nonnegative all the time, we are not clear about the evolution of them. So we introduce another way that is provided by R. Carles [2], [3]:

\[
\begin{align*}
\mathcal{E}_1(t) &= \frac{1}{2} \| J(t) u(t) \|_2^2 + \frac{1}{4} \cosh^2 t \| u(t) \|_6^6, \\
\mathcal{E}_2(t) &= \frac{1}{2} \| H(t) u(t) \|_2^2 + \frac{1}{3} \sinh^2 t \| u(t) \|_6^6,
\end{align*}
\]

$\mathcal{E}_1(t) - \mathcal{E}_2(t) = E(t)$ and they coincide with $E_1(t)$ and $E_2(t)$ only at $t = 0$. The benefits of this decomposition is that we know that $\mathcal{E}_1(t)$, $\mathcal{E}_2(t)$ all decay in time. (See Section2 for details). Using this facts combing the Strichartz estimate, it’s not difficult to get global solution in the subcritical case.

In the critical case, the decay estimates of $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$ are not sufficient to prevent blow up in finite time. However, it’s helpful in the sense that it provides strong decay of the potential energy $\| u(t) \|_6^6$. From this and some elementary analysis, we can fix a time $T$ only dependent with $E$ such that on $(-\infty, -T) \cup (T, \infty)$, $\| u \|_10$ has good control. Thus, we are left to do estimates on a finite time interval $[-T(E), T(E)].$

By the decay estimates of $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$ and the relation between $E_1(t)$ and $E_2(t)$, we know that on $[-T(E), T(E)]$, $E_1(t)$ and $E_2(t)$ are bounded uniformly by constants $\Lambda_1(E, B)$, $\Lambda_2(E, B)$ respectively. Now, we fix a small constant $\eta_1 = \eta_1(\Lambda_1, \Lambda_2 + \Lambda_1)$ and divide $[-T(E), T(E)]$ into finite intervals with fixed length $\eta_1$. If we can prove that on each subinterval, $u$ has finite $L^{10}$ estimate bounded only by $C(\Lambda_1, \Lambda_2)$, we can sum these intervals together and give the final result.

Now, let’s clarify again what is left to do. Let $t_0 \in \mathbb{R}$, and $v(t_0) \in \Sigma$ satisfy $E_1(v(t_0)) \leq \Lambda_1$, and $E_2(v(t_0)) \leq \Lambda_2$, then we are required to prove that there exists constant $\eta_1(\Lambda_1, \Lambda_2 + \Lambda_1)$ such that the Cauchy problem with the prescribed data $v(t_0)$ at time $t_0$ is at least solvable on $[t_0 - \eta_1^4, t_0 + \eta_1^4]$ and satisfies the estimate:

$$
\| v \|_{L^{10}([t_0 - \eta_1^4, t_0 + \eta_1^4]; L^{10})} \leq C(\Lambda_1, \Lambda_2).
$$

Thanks to the local solution theory, we need only to prove the above estimate by apriorily assuming that the solution has existed on interval $[t_0 - \eta_1^4, t_0 + \eta_1^4]$. Here, we adopt the ideas in [4] to get this estimate.
By time translation, we may assume \( t_0 = 0 \). Fix the small constant \( \eta_1 \) such that it satisfy all the conditions that will appear in the proof, we subdivide \([0, \eta_1^4]\) into \( J_1 \) intervals and \([-\eta_1^4, 0]\) into \( J_2 \) intervals such that on each subinterval, \( v \) has \( L^{10} \) norm comparable with \( \eta_1 \).

We do analysis forward in time and aim to estimate \( J_1 \) for simplicity. By some technical computation and the radial assumption, we get a sequence of bubbles located at the origin for a sequence of times in each subinterval. If the volume of every bubble is sizeable by the length of the corresponding time interval, then the solution is soliton like and \( J_1 \) can be estimated by using Morawetz estimate. Otherwise, there is concentration for \( E_1(v(t_*)) \) for some \( t_* \in (0, \eta_1^4) \). Our main task is to estimate \( J_1 \) in this case.

By removing the small bubble (because of the concentration), we get a new function \( w(t_*) \) for which \( E_1(w(t_*)) \leq E_1(v(t_*)) - c\eta_1^3 \). Here, we meet with a problem in making comparison between \( E_1(v(t_*)) \) and \( E_1(v(0)) \) because as we have mentioned before, we are not clear about the evolution of \( E_1(t) \) in time. However, thanks to the previous simplification to the initial problem, and by using the small length condition, we are able to roughly estimate the increment of \( E_1(v(t)) \) from 0 to \( t_* \) by \( C\eta_1^4 \). Therefore, we get the final estimate \( E_1(v(t_*)) \leq \Lambda_1 - c\eta_1^3 \).

This allows us to do induction on the size of energy \( E_1 \). Another difficulty comes from \( E_2 \) since there is no concentration property for it. However, we can deal with this trouble by noticing that the increment from \( E_2(v(0)) \) to \( E_2(w(t_*)) \) is also small, and any finite increment during the iteration is permitted by the small solution theory. This is the reason why we take \( \eta_1 = \eta_1(\Lambda_1, \Lambda_1 + \Lambda_2) \) instead of \( \eta_1(\Lambda_1, \Lambda_2) \).

By considering all the factors together, we may make an inductive assumption as follows:

Let \( v(t') \in \Sigma \) satisfy \( E_1(v(t')) \leq \Lambda_1 - C\eta_1^4 \) and \( E_2(v(t')) \leq \Lambda_2 + C\eta_1^4 \), then the Cauchy problem of \( v(t') \) with prescribed data \( v(t') \) at time \( t' \) is at least solvable on \([t' - \eta_1^4, t' + \eta_1^4]\), and there holds that \( \|v\|_{L^{10}([t' - \eta_1^4, t' + \eta_1^4]; L^1)} \leq C(\Lambda_1 - C\eta_1^4, \Lambda_2 + C\eta_1^4) \).

By using this assumption, \( J_1 \) and \( J_2 \) can be estimated by some technical arguments.

Finally, Let’s explain why we do induction on the size of \( E_1(t) \) and \( E_2(t) \) and not on the size of \( E_1(t) \) and \( E_2(t) \), since at first glance, the latter has good decay, thus is hopeful to be viewed as a substitute of Hamiltonian for Schrödinger equation without potential. Another reason supporting the idea is that, one can get small solution once for some \( t \in \mathbb{R} \), \( \|J(t)u(t)\|_2^2 \) is sufficiently small. However, we notice that,
not liking the quantity $E_i(t)$, the quantity $E_i(t)$ is not time-translation invariant, this will make essential trouble and is the key reason that one should not do induction on the size of $E_i(t)$.

The remaining part of this paper is arranged as follows: In Section 2, we give some notations and some basic estimates. They include: Littlewood-Paley decomposition, Galilean operator, Strichartz estimates for the linear operator with potential, basic properties of Galilean operator, etc. In the first part of Section 3, we give the local wellposedness and small solution theory. The small solution theory claims that the Cauchy problem of (1) is global wellposed and scatters to free solution if for some $t_0$, $E_1(t_0)$ is small enough. This is the fundamental theory which allows us to do induction. In the second part, we use the decay estimate to simplify the large data problem to an aproiri estimate on a finite time interval. Section 4 is devoted to Morawetz estimate of the solution of (1). In Section 5, we use Littlewood-Paley and paraproduct decomposition to prove the existence of a sequence of bubbles. In Section 6, we control $J_1$ and $J_2$ in the case of solitonlike solution. In Section 7, We control $J_1$ and $J_2$ if there is concentration by using the inductive assumption and close the induction by a perturbation analysis in Section 8.

## 2 Notations and basic estimates

Notations:

Let $\eta_1, \eta_2, \eta_3$ be small numbers satisfying $0 < \eta_3 < \eta_2 < \eta_1$ and to be defined in the proof, $c(\eta_1), c(\eta_2), c(\eta_3)$ be small numbers satisfying $0 < c(\eta_3) < c(\eta_2) < c(\eta_1) \ll 1$; $C(\eta_1), C(\eta_2), C(\eta_3)$ be large numbers such that $1 \ll C(\eta_1) \ll C(\eta_2) \ll C(\eta_3)$. $C, c$ are absolute numbers and may be different from one line to another.

For any time interval $I$, we use $\| \cdot \|_{L^q(I;L^r)}$ to denote the Lebesgue norm, where $1 \leq q, r \leq \infty$.

Next, we give the definition of Littlewood-Paley projection. Let $\{\phi_j(\xi)\}_{j=-\infty}^{\infty}$ be a sequence of smooth functions and each supported in an annuli $\{\xi; 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$, furthermore, for any $\xi \neq 0$,  
$$\sum_{j=-\infty}^{\infty} \phi_j(\xi) = 1.$$ 
For any $N = 2^j$, we define Littlewood-Paley projection as follows:  
$$P_N = P_{2^j} = F^{-1}(\phi_j)* \cdot,$$
\[ P_{\leq N} = P_{\leq 2j} = F^{-1}(\sum_{j' \leq j} \phi_{j'}) \ast, \]
\[ P_{> N} = I - P_{\leq N}. \]

We list some basic properties of the projector which will be used often:

- For any \( 1 \leq p \leq \infty \), and \( s \geq 0 \), we have:
  \[ \| \nabla^s P_N f \|_p \sim N^s \| P_N f \|_p, \]
  \[ \| \nabla^s P_{\leq N} \|_p \leq CN^s \| P_{\leq N} \|_p, \]

- Beinstein estimate: For any \( 1 \leq q \leq p \leq \infty \), we have
  \[ \| P_N f \|_p \leq CN^d \left( \frac{1}{q} - \frac{1}{p} \right) \| P_N f \|_q, \]
  \[ \| P_{\leq N} f \|_p \leq CN^d \left( \frac{1}{q} - \frac{1}{p} \right) \| P_{\leq N} f \|_q. \]

Let \( u(t, x) \) be the solution of 3-d linear Schrödinger equation with confining potential:

\[
(i \partial_t + \frac{\Delta}{2}) u = -|x|^2 u, \quad u|_{t=0}(x) = u_0(x), \tag{11}
\]

then it can be expressed by the Mehler’s formula (see [8]),

\[
u(t, x) = U(t)u_0 = e^{-\frac{it}{4}(-\Delta - |x|^2)} u_0
= e^{-\frac{3\pi}{4}sgnt} \left( \frac{1}{2\pi \sinh t} \right)^{\frac{3}{2}} \int_{\mathbb{R}^3} e\frac{3\pi}{4}sgnt \left( \frac{z^2 + y^2}{2} \cosh t - x \cdot y \right) u_0(y) dy, \tag{12}
\]

one sees from the above that the kernel of \( U(t) \) has the better dispersive estimate than the kernal of Schrödinger operator without potential. By using Mehler’s formula [12], and noting that \( U(\cdot) \) is unitary on \( L^2 \), one has the following decay estimate

\[
\| U(t)u_0 \|_\infty \leq C|t|^{-\frac{3}{2}} \| u_0 \|_1, \tag{13}
\]
\[
\| U(t)u_0 \|_p \leq C|t|^{\frac{3}{2} - \frac{1}{p}} \| u_0 \|_{p'}, \quad 2 \leq p \leq \infty. \tag{14}
\]

Using this decay estimate and by some standard arguments, one can get Strichartz estimates for the operator \( U(t) \).

**Definition 2.1** A pair \( (q, r) \) is admissible if \( 2 \leq r < 6 \) and \( \frac{2}{q} + \frac{3}{r} = \frac{3}{2} \).
Lemma 2.2 Strichartz estimates for \( U(t) \).

For any admissible pair \((q, r)\), there exists \( C_r > 0 \) such that
\[
\| U(\cdot) \phi \|_{L^q(R; L^r)} \leq C_r \| \phi \|_2.
\]

For any admissible pairs \((q_1, r_1), (q_2, r_2)\) and any time interval \( I \), there exists constant \( C_{r_1, r_2} \), such that
\[
\left\| \int_{I \cup \{ s < t \}} U(t-s) F(s) ds \right\|_{L^{q_1}(I; L^{r_1})} \leq C_{r_1, r_2} \| F \|_{L^{q_2'(I; L^{r_2'})}}.
\]

We omit the proof since it is exactly the same with linear Schrödinger operator \( e^{it\Delta} \).

Now, we introduce two Galilean operators, they are
\[
J(t) = x \sinh t + i \cosh t \nabla_x, \quad H(t) = x \cosh t + i \sinh t \nabla_x, \quad (15)
\]

conversely, \( x \) and \( \nabla_x \) can be expressed in terms of \( J(t) \) and \( H(t) \),
\[
x = \cosh t H(t) - \sinh t J(t), \quad i \nabla_x = \cosh t J(t) - \sinh t H(t). \quad (16)
\]

Furthermore, \( J(t) \) and \( H(t) \) enjoy the following property,

**Lemma 2.3:** The operators \( J \) and \( H \) satisfy

1. They are Heisenberg observables and consequently commute with the linear operator,
   \[
   J(t) = U(t) i \nabla U(-t), \quad H(t) = U(t) x U(-t),
   \]
   \[
   [i \partial_t + \frac{\Delta}{2} + \frac{|x|^2}{2}, J(t)] = [i \partial_t + \frac{\Delta}{2} + \frac{|x|^2}{2}, H(t)] = 0.
   \]

2. They can be factorized as follows, for \( t \neq 0 \),
   \[
   J(t) = i \cosh t e^{\frac{|x|^2}{2} \tanh t} \nabla_x (e^{-i \frac{|x|^2}{2} \tanh t}),
   \]
   \[
   H(t) = i \sinh t e^{\frac{|x|^2}{2} \coth t} \nabla_x (e^{-i \frac{|x|^2}{2} \coth t}).
   \]

3. Let \( F \in C^1(\mathbb{C}, \mathbb{C}) \) and \( F(z) = G(|z|^2)z \), then,
   \[
   J(t) F(u) = \partial_x F(u) J(t) u - \partial_x F(u) \overline{J(t) u},
   \]
   \[
   H(t) F(u) = \partial_x F(u) H(t) u - \partial_x F(u) \overline{H(t) u}.
   \]

4. There are embeddings (for instance),
   \[
   \| f \|_{10} \leq \| J(t) f \|_{30}^{13}, \quad \forall t \in \mathbb{R},
   \]
   \[
   \| f \|_{18} \leq \| J(t) f \|_{14}^{14}, \quad \forall t \in \mathbb{R}.
   \]

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Proof: The first point is easily checked thanks to (15). The second one holds by direct computation, and implies the last two one.

Formally, the solution of (1)-(2) satisfies the following two conservation laws,

Mass: \( M = \| u(t) \|_2 = \| u_0 \|_2 \),

Energy: \( E(t) = \frac{1}{2} \| \nabla u(t) \|_2^2 - \frac{1}{2} \| xu \|_2^2 + \frac{1}{3} \| u(t) \|_6^6 = \text{const.} \)

As mentioned in the introduction, we split \( E(t) \) by two ways. First, define

\[ E_1(u(t)) = \frac{1}{2} \| \nabla u(t) \|_2^2 + \frac{1}{3} \| u(t) \|_6^6, \quad E_2(u(t)) = \frac{1}{2} \| xu \|_2^2, \]

it follows easily that,

\[ E(u(t)) = E_1(u(t)) + E_2(u(t)). \]

Next, we define

\[ E_1(t) := \frac{1}{2} \| J(t)u(t) \|_{L^2}^2 + \frac{1}{3} \cosh^2 t \| u(t) \|_6^6, \]
\[ E_2(t) := \frac{1}{2} \| H(t)u \|_{L^2}^2 + \frac{1}{3} \sinh^2 t \| u(t) \|_6^6, \]  

we see that \( E_1(t) \) and \( E_2(t) \) coincide with \( E_1(t) \) and \( E_2(t) \) only at \( t = 0 \). Furthermore, we have,

**Lemma 2.4**: We can verify that:

1. \( E_1 \) and \( E_2 \) satisfy,

\[ E_1(t) - E_2(t) = E(t), \]
\[ \frac{dE_1(t)}{dt} = \frac{dE_2(t)}{dt} = -\frac{2}{3} \sinh(2t) \| u(t) \|_6^6. \]  

2. The potential energy \( \| u(t) \|_6^6 \) has exponentially decay in time:

\[ \| u(t) \|_6^6 \leq 3E_1(0) \cosh^{-6} t, \quad \forall \ t \in \mathbb{R}. \]

3. \( \forall t \in \mathbb{R}, \)

\[ E_1(t) \leq E_1(0) = E_1(0), \]
\[ \| H(t)u(t) \|_2 \leq \| H(0)u(0) \|_2 = \| xu_0 \|_2 = E_2(0). \]

**Proof**: The first point can be verified by (15) and the equation (1), see[2] for details. Now let us prove the second point. Integrating in time from 0 to \( t \), we see from (16) that,

\[ E_1(t) = E_1(0) - \frac{2}{3} \int_0^t \sinh(2s) \| u(s) \|_6^6 ds. \]
By (17), we have
\[
\cosh^2 t \|u(t)\|_6^6 \leq 3 \mathcal{E}_1(0) - 2 \int_0^t \sinh(2s) \|u(s)\|_6^6 \, ds
\]
\[
= 3 \mathcal{E}_1(0) - 2 \int_0^t \frac{\sinh(2s)}{\cosh^2 s} \cosh^2 s \|u(s)\|_6^6 \, ds.
\]
Applying the Gronwall inequality yields:
\[
\cosh^2 t \|u(t)\|_6^6 \leq 3 \mathcal{E}_1(0) \exp \left[ -2 \int_0^t \frac{\sinh(2s)}{\cosh^2 s} \, ds \right].
\]
Noting by direct computation,
\[
\int_0^t \frac{\sinh(2s)}{\cosh^2 s} \, ds = \ln \cosh t,
\]
thus, we have
\[
\cosh^2 t \|u(t)\|_6^6 \leq 3 \mathcal{E}_1(0) \cosh^{-4} t,
\]
and,
\[
\|u(t)\|_6^6 \leq 3 \mathcal{E}_1(0) \cosh^{-6} t.
\]
Now, let’s prove the third point. First, (20) is easily verified by using (19). Next, noting (17), (16) and energy conservation, we see that,
\[
\frac{1}{2} \|H(t)u(t)\|_2^2 + \frac{1}{3} \sinh^2 t \|u(t)\|_6^6 = \mathcal{E}_1(t) - E(t)
\leq \mathcal{E}_1(0) - E(0)
= \frac{1}{2} \|\nabla u_0\|_2^2 + \frac{1}{3} \|u_0\|_6^6 - \left( \frac{1}{2} \|\nabla u_0\|_2^2 - \frac{1}{2} \|xu_0\|_2^2 + \frac{1}{3} \|u_0\|_6^6 \right)
= \frac{1}{2} \|xu_0\|_2^2.
\]
Thus we get
\[
\|H(t)u(t)\|_2 \leq \|xu_0\|_2,
\]
which is exactly (21).

Before ending this Section, we give the main theorems of this paper.

**Theorem 1:** Let $u_0 \in \Sigma$ be radial, then the Cauchy problem (11)-(12) has a unique global solution in $C(\mathbb{R}; \Sigma) \cap L_{xt}^{10}(\mathbb{R} \times \mathbb{R}^3)$ and satisfies
\[
\|u\|_{L_{xt}^{10}(\mathbb{R}; L^w)} \leq C(\|\nabla u_0\|_2, \|xu_0\|_2), \tag{22}
\]
\[
\max_{A \in \{J,H,I\}} \|A(\cdot)u\|_{L^q(\mathbb{R}; L^r)} \leq C(\|u_0\|_\Sigma). \tag{23}
\]
Furthermore, there exits a unique $u_+ \in \Sigma$ such that
\[ \|U(-t)u(t) - u_+\|_{\Sigma} \to 0, \text{ as } t \to \infty; \]
there exists a unique $u_- \in \Sigma$ such that
\[ \|U(-t)u(t) - u_-\|_{\Sigma} \to 0, \text{ as } t \to -\infty. \]

**Theorem 2:** (Existence of wave operator)
Let $u_+ \in \Sigma$ be radial, then there exists a unique solution $u(x,t)$ of equation (1) satisfying
\[ \max_{A \in \{J,H,I\}} \left\| u \right\|_{L^{q}(\mathbb{R}; L^r)} \leq C(\| u_+ \|_{\Sigma}), \quad (q, r) \text{ admissible}, \]
and
\[ \|U(-t)u(t) - u_+\|_{\Sigma} \to 0, \quad \text{as } t \to \infty; \]
Let $u_- \in \Sigma$ be radial, then there exists a unique solution of equation (1) satisfying
\[ \max_{A \in \{J,H,I\}} \left\| u \right\|_{L^{q}(\mathbb{R}; L^r)} \leq C(\| u_- \|_{\Sigma}), \quad (q, r) \text{ admissible}, \]
and
\[ \|U(-t)u(t) - u_-\|_{\Sigma} \to 0, \quad \text{as } t \to -\infty. \]

3 Local wellposedness and global small solution

In this section, we aim to get local solution and global small solution to the energy critical Schrödinger equation with harmonic potential. By Duhamel’s formula, it’s enough to find solutions to the integral equation
\[ u(t) = U(t)u_0 - i \int_0^t U(t-s)|u|^4u(s)ds. \]
Define solution map by $\Phi(u)(x,t) = U(t)u_0 - i \int_0^t U(t-s)|u|^4u(s)ds$, one is required to find fixed point of the map $\Phi$.

**Proposition 3.1** (Local wellposedness)
For any $u_0 \in \Sigma$, there exists maximal time interval $(T^-_*, T^+_*)$, such that (1)-(2) has a unique solution

$$u(x, t) \in C((T^-_*, T^+_*); \Sigma) \cap L^{10}_{\text{loc}}((T^-_*, T^+_*); L^{10}).$$

Furthermore, for any admissible pair $(q, r)$, one has

$$\|A(\cdot)\|_{L^q_{\text{loc}}((T^-_*, T^+_*); L^r)} < \infty, \quad A \in \{J, H, I\}.$$ 

**Proof:** Let

$$R = \max_{A \in \{J, H, I\}} \|A(\cdot)U(\cdot)u_0\|_{L^{10}_{\text{loc}}((T^-_*, T^+_*); L^{10}) \cap L^{10}_{\text{loc}}((T^-_*, T^+_*); L^{10})},$$

with $T^-, T^+$ being specified later. By Strichartz estimate Lemma 2.2 and using the facts

$$A(t)U(t) = U(t)B, \quad B \in \{i \nabla_x, x, I\},$$

we see that $R$ is uniformly bounded w.r.t $T^-, T^+$, and in particular,

$$R \leq C\|u_0\|_{\Sigma},$$

this in turn shows that $R$ is small when $T^+$ and $T^-$ are small.

Define a set

$$X = \left\{ u(x, t) \bigg| \max_{A \in \{J, H, I\}} \|Au\|_{L^{10}_{\text{loc}}((T^-_*, T^+_*); L^{10}) \cap L^{10}_{\text{loc}}((T^-_*, T^+_*); L^{10})} \leq 2R \right\},$$

and the norm $\| \cdot \|_X$ is taken as the same as the one in the capital bracket. First, we show that $X$ is stable under the solution map $\Phi$. Choosing $u \in X$, and using Lemma 2.2, Lemma 2.3, one computes that

$$A(t)\Phi(u)(t) = A(t)U(t)u_0 - i \int_0^t U(t-s)A(s)|u|^4u(s)ds,$$

and,

$$\|A\Phi(u)\|_{L^{10}_{\text{loc}}((T^-_*, T^+_*); L^{10}) \cap L^{10}_{\text{loc}}((T^-_*, T^+_*); L^{10})} = R + \| \int_0^t U(t-s)A(s)|u|^4u(s)ds\|_{L^{10}_{\text{loc}}((T^-_*, T^+_*); L^{10}) \cap L^{10}_{\text{loc}}((T^-_*, T^+_*); L^{10})}. $$

By Strichartz estimate and Lemma 2.3, the second term can be controlled by

$$C\|A(\cdot)|u|^4u\|_{L^{10}_{\text{loc}}((T^-_*, T^+_*); L^{10})} \leq C\|u\|_{10}^4\|A(\cdot)u\|_{L^{10}_{\text{loc}}((T^-_*, T^+_*); L^{10})},$$

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by embedding, this is smaller than $C\|u\|^5_X$. Consequently, we obtain

$$\|A(\cdot)\Phi(u)\|_X \leq R + C(2R)^5.$$ 

Thus, $X$ is stable if $R$ is such that $C2^5R^4 < \frac{1}{4}$. This is available by choosing $T^-$ and $T^+$ small enough. Donate the metric on $X$ by

$$d(u_1, u_2) = \|u_1 - u_2\|_{L^{10}_X(T^-, T^+); L^{10}_X},$$

we need only to prove the contraction under this weak metric. Taking $u_1, u_2 \in X$, we have

$$d(\Phi(u_1), \Phi(u_2)) = \left\| \int_0^t U(t-s)(|u_1|^4u_1(s) - |u_2|^4u_2(s))ds \right\|_{L^{10}_X(T^-, T^+); L^{10}_X} \leq C\|u_1|^4u_1 - |u_2|^4u_2\|_{L^{10}_X(T^-, T^+); L^{10}_X} \leq C\|u_1\|^4_X + \|u_2\|^4_X \leq 2C(2R)^4d(u_1, u_2) \leq \frac{1}{2}d(u_1, u_2).$$

Now, we now first fix $R$ such that $C2^5R^4 < \frac{1}{4}$, then choose $T^-, T^+$ such that

$$\max_{A \in \{J, H, I\}} \left\| A(\cdot)U(\cdot)u_0 \right\|_{L^{10}_X(T^-, T^+); L^{10}_X(T^-, T^+)} < R,$$

by the fixed point theorem, we get a solution on $[T^-, T^+]$. Once this is done, we extend this solution to the maximal time interval $(T^-_*, T^+_*)$. The regularity property of the solution follows from the Strichartz estimate. Thus, we conclude the proof of Proposition 3.1.

**Proposition 3.2** (Global small solution)

There exists an absolute constant $\varepsilon > 0$ such that when $u_0 \in \Sigma$ and

$$\|\nabla_x u_0\|_2 \leq \varepsilon,$$

has a unique global solution satisfying

$$\|u\|_{L^{10}(R; L^{10})} \leq 2C\varepsilon, \quad (24)$$

$$A(t)u(t, x) \in L^g(R; L^r), \quad (q, r) \text{ admissible}, A \in \{J, H, I\}. \quad (25)$$

Furthermore, there exists a unique function $u_+ \in \Sigma$ such that

$$\|U(-t)u(t) - u_+\|_{\Sigma} \to 0, \quad \text{as } t \to \infty,$$
and there exists a unique function \( u_- \in \Sigma \) such that
\[
\| U(-t)u(t) - u_- \|_{\Sigma} \to 0, \quad \text{as } t \to -\infty.
\]

**Proof:** Being slightly different from the proof of Proposition 3.1, we define

\[
R := \| \nabla u_0 \|_2, \\
X := \left\{ u(x,t) \left| \begin{array}{c}
u \in L^{\frac{10}{3}}(R;L^\frac{30}{1}) \\
\| u \|_{L^{\frac{10}{3}}(R;L^\frac{30}{1})} \leq 2C \| u_0 \|_2,
\| J(\cdot)u \|_{L^{\frac{10}{3}}(R;L^\frac{30}{1}) \cap L^{\frac{10}{30}}(R;L^\frac{30}{13})} \leq 2CR \right. \right\},
\]

where, \( C \) in the bracket is the Strichartz constant. First, we show \( \Phi \) is onto from \( X \) to \( X \). Taking \( u \in X \), we verify that

\[
\| J(\cdot)\Phi(u) \|_{L^{\frac{10}{3}}(R;L^{\frac{30}{1}} \cap L^{\frac{10}{30}}(R;L^\frac{30}{13}))} \leq C\| \nabla u_0 \|_2 + \| J(\cdot)u \|_{L^{\frac{10}{3}}(R;L^\frac{30}{13})} \leq CR.
\]

Here, we have used Strichartz estimate and embedding. If \( R \) is taken such that
\[
C(2CR)^5 < \frac{1}{2} CR,
\]
then we obtain
\[
\| J(\cdot)\Phi(u) \|_{L^{\frac{10}{3}}(R;L^{\frac{30}{1}} \cap L^{\frac{10}{30}}(R;L^\frac{30}{13}))} \leq 2CR.
\]

Now, we verify the first two properties in the bracket. Taking \( u \in X \), we have

\[
\| \Phi(u) \|_{L^{\frac{10}{3}}(R;L^\frac{30}{13})} \leq C\| u_0 \|_2 + \| J(\cdot)u \|_{L^{\frac{10}{3}}(R;L^\frac{30}{13})} \leq CR.
\]
By the same way, one gets
\[ \|H(\cdot)\Phi u\|_{L^{10}(\mathbb{R}; L^{10})} \leq 2C \|x u_0\|_2. \]

Hence, \( \Phi \) is a map from \( X \) to \( X \). To complete the proof of the existence part, we donate \( X \) with the weak metric
\[ d(u_1, u_2) = \|u_1 - u_2\|_{L^{10}(\mathbb{R}; L^{10})}, \]
and plan to prove the contraction under this metric. Choosing \( u_1, u_2 \in X \) with same data, we have
\[ \Phi(u_1)(t) - \Phi(u_2)(t) = -i \int_0^t U(t-s)(|u_1|^4 u_1 - |u_2|^4 u_2)(s)ds. \]

Taking Lebesgues norm on each sides of the equation and using Strichartz estimate and embedding, we easily get
\[ d(\Phi(u_1), \Phi(u_2)) \leq \frac{1}{2} d(u_1, u_2). \]

By the fixed point theorem, we obtain a unique solution \( u \in X \). The properties (25) and (24) follows directly from the Strichar tz estimate and embedding. Now, Let’s turn to the proof of the second part. Let \( u_+(x, t) = i \int_0^\infty U(-s)|u|^4 u(x, s)ds \), one sees that
\[ U(-t)u(t) - u_+(t) = i \int_t^\infty U(-s)|u|^4 u(s)ds. \]

Noting that by Lemma2.3, there holds
\[ i\nabla_x(U(-t)u(t) - u_+(t)) = i \int_t^\infty U(-s)J(s)|u|^4 u(s)ds, \]
\[ x(U(-t)u(t) - u_+(t)) = i \int_t^\infty U(-s)H(s)|u|^4 u(s)ds, \]

We see that
\[ \|U(-t)u(t) - u_+(t)\|_\Sigma \leq \max_{A \in \{J, H, I\}} \| \int_t^\infty U(-s)A(s)|u|^4 u(s)ds\|_2. \]

Noting that the operator \( U(\cdot) \) is unitary on \( L^2 \), we can bound the right side of the above equation by
\[ \|u\|_{L^{10}((t, \infty); L^{10})} \|A(\cdot)u\|_{L^{10}((t, \infty); L^{10})} \]
which tends to 0 as $t$ tends to $\infty$. The scattering in the negative direction follows from the way. This finally gives Proposition 3.2.

The above two proposition provide no answer about whether the solution with large data is global. Assume $u$ is the local solution on the maximal time interval $(T^-, T^+_*)$, our first purpose is to show that $T^- = -\infty$, and $T^*_+ = \infty$ once we prove an apriori estimate on $u$ as follows:

**Lemma 3.3** Assume $u$ be a maximal solution on $(T^-, T^+_*)$ with finite energy. Then if for any $I \in (T^-, T^+_*)$, $u$ satisfies

$$\|u\|_{L^{10}(I; L^{10})} < C(|I|, E_1(u_0), E_2(u_0)),$$

(27)

then $T^- = -\infty$, $T^*_+ = \infty$. Here, $|I|$ denotes the length of $I$.

**Proof:** Let’s discuss in the positive time direction. Assume otherwise that $T^*_+ < \infty$, we will get a contradiction by showing the solution can be extended beyond $T^*_+$. Our strategy is as follows: we first take $t_0 \in [0, T^*_+]$ that is close enough to $T^*_+$, then aim to solve the same Cauchy problem from $t_0$ forward. Once we have shown that there exists $\delta > 0$ such that

$$\max_{A \in \{J,H,I\}} \|A(\cdot - t_0)u(t_0)\|_{L^{10}(l_0, T^*_+ + \delta); L^{10}(l_0, T^*_+ + \delta)} < R,$$

(28)

where $R$ is a same constant in Proposition 3.1, we will establish a contraction mapping. This allows us to extend the solution at least beyond $T^*_+ + \delta$ and contradicts the maximum property of $T^*_+$. So, let’s prove (28). First of all, by Strichartz estimate and some routine arguments, we see that (27) implies that

$$\max_{A \in \{J,H,I\}} \|A(\cdot)u\|_{L^q([0,T^*_+]; L^r)} \leq C(|T^*_+|, \|u_0\|_\Sigma), \ (q, r) \text{ admissible}.$$

(29)

By Duhamel’s formula, we see that $u$ solves the equation

$$u(t) = U(t - t_0)u(t_0) - i \int_{t_0}^t U(t - s)|u|^4u(s)ds, \text{ for } t \in [t_0, T^*_+),$$

and hence,

$$A(t)U(t - t_0)u(t_0) = A(t)u(t) + i \int_{t_0}^t A(t)U(t - s)|u|^4u(s)ds$$

$$= A(t)u(t) + i \int_{t_0}^t U(t - s)A(s)|u|^4u(s)ds.$$
Taking Lebesgue’s norm on each side to the equation, one gets
\[
\|A(\cdot)U(\cdot - t_0)\|_{L^{10}(\langle t_0, T^+_*\rangle; L^{30/13}\mathbb{R}))} \\
\leq \|A(\cdot)u\|_{L^{10}(\langle t_0, T^+_*\rangle; L^{30/13}\mathbb{R}))} + C\|A(\cdot)|u|^4 u\|_{L^{10}(\langle t_0, T^+_*\rangle; L^{10}\mathbb{R}))}.
\]
(30)

Having (29) in mind and applying Hölder, we see (30) is smaller than \(\frac{R}{2}\) if \(t_0\) is close enough to \(T^+_*\).

On the other hand, since by the Strichartz estimates
\[
\|A(\cdot)U(\cdot - t_0)\|_{L^{10}(\langle t_0, T^+_*\rangle; L^{30/13}\mathbb{R}))} = \|U(\cdot - t_0)A(t_0)u(t_0)\|_{L^{10}(\langle t_0, T^+_*\rangle; L^{30/13}\mathbb{R}))} \\
\leq C\|A(t_0)u(t_0)\|_2 \leq C(\|u_0\|_\Sigma),
\]
thus, once \(t_0\) is fixed by (30) \(\leq \frac{R}{2}\), one is allowed to choose a \(\delta > 0\) sufficiently small such that
\[
\|A(\cdot)U(\cdot - t_0)\|_{L^{10}(\langle T^+_* + \delta, T^+_* + \delta, T^+_* + \delta\rangle; L^{30/13}\mathbb{R}))} \leq \frac{R}{2}.
\]
(31)

(28) then follows by collecting (30) \(\leq \frac{R}{2}\) and (31), also implies Lemma3.3.

Lemma3.3 says the solution is global in the sense that it exists on arbitrary finite time interval \((-T, T)\). In particular, it doesn’t imply that the solution enjoy certain global space-time estimate which is the usual requirement in the scattering theory. However, we can complement this by the decay estimate Lemma2.4.

**Lemma3.4:** Assume \(u\) be the global solution in the sense above, then \(u\) satisfies
\[
\max_{A \in \{J, H, I\}} \|A(\cdot)u\|_{L^q(R; L^r)} \leq C(\|u_0\|_\Sigma), \quad \forall (q, r) \text{ admissible.} \tag{32}
\]
and there is scattering.

**Proof:** Fixing a small number \(\varepsilon\) and taking \(T \geq T_0 = (\frac{3E_1(0)}{\varepsilon^6})^\frac{1}{6}\), we have
\[3E_1(0) \cosh^{-6} T \leq \varepsilon^6,
\]
thus by the decay estimate Lemma2.4, one has
\[
\|u\|_{L^\infty([T, \infty); L^6)} \leq \varepsilon. \tag{33}
\]
By Duhamel’s formula, on \([T, \infty)\), \(u\) satisfies the equation
\[
u(t) = U(t - T)u(T) - i \int_T^t U(t - s)|u|^4u(s)ds,
\]
taking a special admissible pair \((6, \frac{18}{7})\) and applying Strichartz estimate gives,
\[
\|J(\cdot)u\|_{L^6([T, \infty); L^{\frac{18}{7}})} \\
\leq \|J(\cdot)U(\cdot - T)u(T)\|_{L^6([T, \infty); L^{\frac{18}{7}})} + \| \int_T^t J(\cdot)U(\cdot - s)|u|^4u(s)ds\|_{L^6([T, \infty); L^{\frac{18}{7}})}
\]
\[
\leq C\|J(T)u(T)\|_2 + C\|J(\cdot)|u|^4u\|_{L^\frac{9}{2}([T, \infty); L^{\frac{18}{7}})}
\]
The first term is smaller than \(CE_1^\frac{1}{2}(u_0)\), and by Hölder, the second term is controlled by
\[
C\|u\|_{L^\infty([T, \infty); L^6)}\|u\|_{L^6([T, \infty); L^{18})}^3\|J(\cdot)u\|_{L^6([T, \infty); L^{\frac{18}{7}})},
\]
in view of (33) and embedding, we further estimate it by
\[
C\|J(\cdot)u\|_{L^6([T, \infty); L^{\frac{18}{7}})}^4.
\]
Hence we get an estimate for \(J(\cdot)u\) as follows,
\[
\|J(\cdot)u\|_{L^6([T, \infty); L^{\frac{18}{7}})} \leq CE_1(u_0)^\frac{1}{2} + C\|J(\cdot)u\|_{L^6([T, \infty); L^{\frac{18}{7}})}^4.
\]
(The more rigorous way is to do estimate on \([T, R)\), \(R < \infty\), then take supreme w.r.t \(R\).) This implies that \(\|J(\cdot)u\|_{L^6([T, \infty); L^{\frac{18}{7}})}\) is bounded if \(\varepsilon\) is smaller than a constant which depends only on \(E_1(u_0)\). Once this has been obtained, one can get
\[
\|J(\cdot)u\|_{L^6([0, \infty); L^{\frac{18}{7}})} \leq \|J(\cdot)u\|_{L^6([0, T_0); L^{\frac{18}{7}})} + \|J(\cdot)u\|_{L^6([T_0, \infty); L^{\frac{18}{7}})} \\
\leq C(E_1(u_0), E_2(u_0)).
\]
By time reversing and Strichartz estimate, we obtain (32).

Having Lemma 3.3 and Lemma 3.4 in mind, in order to prove Theorem 1.1, we need only to show
\[
\|u\|_{L^{10}([-T_0, T_0]; L^{10})} \leq C(E_1(u_0), E_2(u_0)),
\]
where \(T_0\) is defined in Lemma 3.4 and depends only on \(E_1(u_0)\).
Now, we fix two constants $E$ and $B$ such that
\[ E_1(u_0) = E, \quad E_2(u_0) = B, \]
then our task becomes to prove
\[ \|u\|_{L^{10}([-T_0(E), T_0(E)]; L^{10})} \leq C(E, B), \tag{34} \]
if $u$ is a finite energy solution on $[-T_0(E), T_0(E)]$ with $u(0) = u_0$.

From (16) and Lemma 2.4, we compute that,
\[
\begin{align*}
\|\nabla u(t)\|_2 & \leq \cosh t \|H(t)u(t)\|_2 + \sinh t \|J(t)u(t)\|_2 \\
\|xu(t)\|_2 & \leq \sinh t \|H(t)u(t)\|_2 + \cosh t \|J(t)u(t)\|_2
\end{align*}
\]
\[ \leq C(E, B), \quad \forall t \in [-T_0(E), T_0(E)]. \]

Thus, there exists $\Lambda_1(E, B)$ and $\Lambda_2(E, B)$ such that
\[ E_1(u(t)) \leq \Lambda_1, \quad E_2(u(t)) \leq \Lambda_2, \quad \forall t \in [-T_0(E), T_0(E)]. \]

If there exists $\eta_1 = \eta_1(\Lambda_1, \Lambda_2)$ such that on every time interval $I$ with length $2\eta_1^4$, one has
\[ \|u\|_{L^{10}(I; L^{10})} \leq C(\Lambda_1, \Lambda_2), \]
then we can divide $[-T(E), T(E)]$ into $O\left(\frac{T(E)}{\eta_1^4}\right)$ subintervals, and get \cite{14} by summing the estimates on each subinterval. Indeed, we plan to prove the following proposition.

**Proposition 3.5:** Let $t' \in \mathbb{R}$ be arbitrarily fixed and $u(t') \in \Sigma$ satisfying
\[ E_1(u(t')) \leq \Lambda_1, \quad E_2(u(t')) \leq \Lambda_2, \]
then we have a small constant $\eta_1$ which depends only on $(\Lambda_1, \Lambda_2 + \Lambda_1)$ such that the Cauchy problem of (11) with prescribed data $u(t')$ at time $t'$ is at least solvable on $[t' - \eta_1^4, t' + \eta_1^4]$ and the solution $u$ satisfy
\[ \|u\|_{L^{10}([t' - \eta_1^4, t' + \eta_1^4]; L^{10})} \leq C(\Lambda_1, \Lambda_2). \]

Assuming Proposition 3.5 hold true, let’s give a remark about the proof of Theorem 1, Theorem 2. First of all, in Theorem 1, we are left to prove the regularity part and the scattering part (22), (23) of the global solution which can be deduced from the Strichartz estimate and some
routine arguments. See [1] and the proof of Proposition 3.2 for details. The proof Theorem 2 is a bit different, so we sketch it below.

Proof of Theorem 2: We need only to show the integral equation
\[ u(t) = U(t)u_+ + i \int_{t}^{\infty} U(t-s)|u|^4u(s)ds, \] (35)
has a unique global solution with global spacetime estimates. First of all, we seek for local solution. Define the solution map by \( \Phi(u)(t) = U(t)u_+ + i \int_{t}^{\infty} U(t-s)|u|^4u(s)ds \), and denote \( R = \|\nabla u_+\|_2 \). By choosing \( T = T(R) \) large enough, say, \( \cosh T \geq CR \), we see that \( \Phi \) is a contraction map on the set
\[ X = \left\{ (x,t); \begin{align*}
\|u\|_{L^\infty([T,\infty);L^\frac{10}{3})} &\leq 2C\|u_+\|_2, \\
\|H(\cdot)u\|_{L^\frac{10}{3}([T,\infty);L^\frac{10}{3})} &\leq 2C\|xu_+\|_2, \\
\|J(\cdot)u\|_{L^\frac{10}{3}([T,\infty);L^\frac{10}{3}) \cap L^{10}([T,\infty);L^{\frac{30}{13}})} &\leq 2CR
\end{align*} \right\}, \]
donated with the metric \( d(u_1, u_2) = \|u_1 - u_2\|_{L^\frac{10}{3}([T,\infty);L^\frac{10}{3})} \). The proof is routine, except needing to notifying that the gain \( \cosh^{-1} T \) from the embedding \( \|u\|_{L^{10}([T,\infty);L^{10})} \leq C \cosh^{-1} T \|J(\cdot)u\|_{L^{10}([T,\infty);L^{\frac{30}{13}})} \) gives the dependence of \( T \) on \( R \). Once we get the local solution, we can find a finite time \( T = T(E_1(u_+)) \) such that \( u(T) \in \Sigma \) and has the bound only depending on \( \|u_+\|_\Sigma \). At that moment, by solving a finite time Cauchy problem, we are allowed to get global solution of (35) by Theorem 1 and the uniqueness. The scattering part of Theorem 2 follows easily from the global spacetime bound of \( u(x,t) \), thus we end the proof of Theorem 2.

The remaining part of the paper is devoted to the proof of Proposition 3.5, by time translation, we may assume \( t' = 0 \). We begin the proof by giving the Morawetz estimate of the Schrödinger equation with potential in the following section.

4 Morawetz estimate for solutions of Schrödinger equations with potential

We first give the local mass conservation of \( u \). Taking a smooth function \( \chi(x) \in C_0^\infty(\mathbb{R}^3) \) such that \( \chi(x) = 1 \) if \( |x| \leq \frac{1}{2} \) and \( \chi(x) = 0 \) if \( |x| \geq 1 \). Then we have,
Proposition 4.1: Let $u$ be the smooth solution of (11), and define local mass of $u$ to be

$$\text{Mass}(u(t), B(x_0, R)) = \left( \int \chi^2 \left( \frac{x-x_0}{R} \right) |u(t,x)|^2 dx \right)^\frac{1}{2},$$

then,

$$\partial_t \text{Mass}(u(t), B(x_0, R)) \leq \frac{\|u(t)\|_{H^1}}{R},$$

and

$$\text{Mass}(u(t), B(x_0, R)) \leq R\|u(t)\|_{H^1}.\quad (37)$$

For the self-containedness of the paper, we give the proof of this proposition.

Proof: Noting that $u$ satisfies the equation (11), we have

$$\partial_t \text{Mass}(u(t), B(x_0, R))^2 = \int \chi^2 \left( \frac{x-x_0}{R} \right) 2Re \left[ \bar{u} \left( \frac{i}{2} \Delta u + \frac{i}{2} |x|^2 u - i |u|^4 u \right) \right](x)dx,$$

which equals to

$$-\int \chi^2 \left( \frac{x-x_0}{R} \right) \text{Im} \left( \bar{u} \Delta u \right)(x)dx,$$

by a simple computation. Using integrating by parts, we finally get

$$\partial_t \text{Mass}(u(t), B(x_0, R))^2 = \frac{2}{R} \int \chi \left( \frac{x-x_0}{R} \right) (\nabla \chi) \left( \frac{x-x_0}{R} \right) \text{Im} (u \nabla \bar{u})(x,t)dx.$$

By Hölder inequality, the right hand side can be controlled by

$$\frac{2}{R} \text{Mass}(u(t), B(x_0, R)) \| \nabla u(t) \|_2,$$

from which (36) follows. Now, let’s prove (37). Using Hardy’s inequality, one has

$$\text{Mass}(u(t), B(x_0, R))^2 = \int \chi^2 \left( \frac{x-x_0}{R} \right) |u(t,x)|^2 dx \leq \sup_{x \in \mathbb{R}^3} \chi^2 \left( \frac{x-x_0}{R} \right) |x-x_0|^2 \int \frac{|u(t,x)|^2}{|x-x_0|^2} dx \leq R^2 \|u(t)\|^2_{H^1}.$$

Thus, we get (37).
Proposition 4.2: (Morawetz inequality) Let $u$ be the solution of (1) with finite energy. Then we have
\[
\int_I \int_{|x| \leq A} \frac{|u(t,x)|^6}{|x|} \, dx \, dt \\
\leq CA^2 \left( \| \nabla u \|^2_{L^2(I;L^2)} + \| xu \|^2_{L^2(I;L^2)} + \| u \|^6_{L^2(I;L^2)} \right) \tag{38}
\]
for all $A \geq 1$.

Proof: We prove this result by following the idea in [13]. Assume that $u$ is a smooth solution of (1). First, by direct computation, we get
\[
\frac{\partial}{\partial t} Im(\partial_k u \bar{u}) = \frac{1}{4} \partial_k \Delta (|u|^2) - Re \partial_j (\bar{u}_k u_j) - \frac{2}{3} \partial_k (|u|^6) + x_k |u|^2, \tag{39}
\]
where we use $\partial_k f$ or $f_k$ to denote $\frac{\partial f}{\partial x_k}$. Let $a(x)$ be a smooth radial solution to be choose later. Multiplying (39) by $a_k(x)$ and integrating on $\mathbb{R}^3$, we get
\[
\frac{\partial}{\partial t} \int_{\mathbb{R}^3} \text{Im}(\partial_k u \bar{u})(x) a_k(x) \, dx = \int_{\mathbb{R}^3} a_j k(x) \text{Re}(\bar{u}_k u_j)(x) \, dx \\
- \frac{1}{4} \int_{\mathbb{R}^3} \Delta a(x) |u|^2(x) \, dx + \frac{2}{3} \int_{\mathbb{R}^3} \Delta a(x) |u|^6(x) \, dx + \int_{\mathbb{R}^3} a_k(x) x_k |u|^2(x) \, dx. \tag{40}
\]
Taking $\chi(x) \in C_0^\infty(\mathbb{R}^3)$ satisfying $\chi(x) = 1$ as $|x| \leq 1$ and $\chi(x) = 0$ as $|x| \leq 2$. Letting $a(x) = (\varepsilon^2 + |x|^2)^{\frac{1}{2}} \chi(\frac{x}{\varepsilon})$, we claim that, on $|x| \leq R$,
\[
a(x) = (\varepsilon^2 + |x|^2)^{\frac{1}{2}}, \quad a_k(x) = \frac{x_k}{(\varepsilon^2 + |x|^2)^{\frac{1}{2}}}, \quad \Delta a(x) = \frac{2}{(\varepsilon^2 + |x|^2)^{\frac{1}{2}}} + \frac{\varepsilon^2}{(\varepsilon^2 + |x|^2)^{\frac{1}{2}}}, \quad \Delta \Delta a(x) = \frac{15\varepsilon^2}{(\varepsilon^2 + |x|^2)^{\frac{1}{2}}},
\]
\[
a_j k(x) \text{Re}(\bar{u}_k u_j)(x) \geq 0, \quad a_k(x) x_k = \frac{|x|^2}{(\varepsilon^2 + |x|^2)^{\frac{1}{2}}} \geq 0.
\]
The first four points follow by directly differentiating $a(x)$ on $|x| \leq R$. To see the fifth point, we do further computation,
\[
a_j k(x) \text{Re}(\bar{u}_k u_j)(x) = \left( \frac{\delta_{jk}}{\varepsilon^2 + |x|^2} - \frac{x_j x_k}{\varepsilon^2 + |x|^2} \right) \text{Re}(\bar{u}_k u_j)(x) \\
= \frac{|\nabla u|^2}{\varepsilon^2 + |x|^2} - \frac{\text{Re}(x_j u_j x_k \bar{u}_k)}{\varepsilon^2 + |x|^2} \\
= \frac{|\nabla u|^2}{\varepsilon^2 + |x|^2} - \frac{|x|^2 |u_r|^2}{\varepsilon^2 + |x|^2}. \tag{41}
\]
where \( u_r \) denotes the radial derivative. Since \( |u_r| \leq |\nabla u| \), one sees that (41) is greater than 
\[
\frac{|\nabla u|^2 \varepsilon^2}{(\varepsilon^2 + |x|^2)^{\frac{3}{2}}} \geq 0,
\]
from which the positivity follows. The last point is an easy consequence of the second one. Keeping the above claims in mind, we get from (40) that
\[
\frac{4}{3} \int_{|x| \leq R} \frac{|u|^6}{(\varepsilon^2 + |x|^2)^{\frac{3}{2}}} \, dx \leq \partial_t \int_{\mathbb{R}^3} \text{Im}(\partial_k u \bar{u})(x) a_k(x) \, dx + \int_{R \leq |x| \leq 2R} |a_{jk}(x) \text{Re}(\bar{u}_k u_j)(x)| \\
+ \frac{1}{4} |\Delta \Delta a(x)||u|^2(x) + \frac{2}{3} |\Delta a(x)||u|^6(x) + |a_k(x)x_k||u|^2(x) \, dx.
\]
Integrating in time on \( I \), we get
\[
\frac{4}{3} \int_I \int_{|x| \leq R} \frac{|u|^6}{(\varepsilon^2 + |x|^2)^{\frac{3}{2}}} \, dx \leq \sup_{t \in I} \left| \int_{\mathbb{R}^3} \text{Im}(\bar{u}_k u)(x) a_k(x) \, dx \right| \\
+ |I| \sup_{t \in I} \left( \int_{R \leq |x| \leq 2R} |a_{jk}(x) u_k u_j(x, t)| + \frac{1}{4} |\Delta \Delta a(x)||u|^2(x, t)| \\
+ \frac{2}{3} |\Delta a(x)||u|^6(x, t) + |a_k(x)x_k||u|^2(x, t)| \, dx \right).
\]
(42)

To estimate each term on the right hand side of the inequality, we need rough bounds on the derivatives of \( a \) as \( R \leq |x| \leq 2R \), they are
\[
\text{ when } R \leq |x| \leq 2R: \quad |a_k(x)| \leq C \frac{(\varepsilon^2 + R^2)^{\frac{1}{2}}}{R}, \quad |a_{jk}(x)| \leq C \frac{(\varepsilon^2 + R^2)^{\frac{1}{2}}}{R^2}, \quad |\Delta \Delta a(x)| \leq C \frac{(\varepsilon^2 + R^2)^{\frac{1}{2}}}{R^4}.
\]
Using these bounds, we can further control (42) by
\[
C(\varepsilon^2 + R^2)^{\frac{1}{2}} \| \nabla u \|_{L^\infty(I;L^2)} + C |I| \frac{(\varepsilon^2 + R^2)^{\frac{1}{2}}}{R^2} \| \nabla u \|_{L^\infty(I;L^2)}^2 \\
+ C |I| \frac{(\varepsilon^2 + R^2)^{\frac{1}{2}}}{R^2} \| u \|_{L^\infty(I;L^5)}^6 + C |I| \frac{(\varepsilon^2 + R^2)^{\frac{1}{2}}}{R^2} \| xu \|_{L^\infty(I;L^2)}^2,
\]
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which is smaller than
\[
C \left( \frac{\epsilon^2 + R^2 \frac{1}{2} |I|}{R^2} \right) \left( \| \nabla u \|_{L^\infty(I; L^2)}^2 + \| xu \|_{L^\infty(I; L^2)}^2 + \| u \|_{L^6(I; L^6)}^6 \right).
\]

Choosing \( R = A |I|^{\frac{1}{2}} \) and letting \( \epsilon \to 0 \), (42) becomes
\[
\int_I \int_{|x| \leq A |I|^{\frac{1}{2}}} \frac{|u|^6}{|x|} dx dt \leq C \left( A |I|^{\frac{1}{2}} + A^{-1} |I|^{\frac{1}{2}} \right) \left( \| \nabla u \|_{L^\infty(I; L^2)}^2 + \| xu \|_{L^\infty(I; L^2)}^2 + \| u \|_{L^6(I; L^6)}^6 \right)
\]
\[
\leq C A |I|^{\frac{1}{2}} \left( \| \nabla u \|_{L^\infty(I; L^2)}^2 + \| xu \|_{L^\infty(I; L^2)}^2 + \| u \|_{L^6(I; L^6)}^6 \right),
\]
since \( A \geq 1 \). This is exactly (38).

Being different from the Morawetz estimate for equations without potential, the term
\[
\| \nabla u \|_{L^\infty(I; L^2)}^2 + \| xu \|_{L^\infty(I; L^2)}^2 + \| u \|_{L^6(I; L^6)}^6
\]
can’t be substituted by a quantity independent on \( I \) since it is not conserved in time. However, because we have restricted this problem in a finite time interval, we are allowed to control this term. Indeed, we have

**Corollary 4.3:** Let \( u \) be a finite energy solution on \( I \) and satisfy
\[
E_1(u(t)) \leq C_1; \quad E_2(u(t)) \leq C_2, \forall t \in I,
\]
then we have
\[
\int_I \int_{|x| \leq A |I|^{\frac{1}{2}}} \frac{|u(t, x)|^6}{|x|} dx dt \leq C(C_1, C_2) A |I|^{\frac{1}{2}} \quad \text{for all } A \geq 1. \quad (43)
\]

### 5 Paraproduct decomposition and Littlewood-Paley

Keeping Corollary 4.3 in mind, we begin to prove Proposition 3.5 from this Section to the end. Thanks to the local solution theory, we may assume the solution has been existed on \([ -\eta_1^4, \eta_1^4 \]) and only aim to show the spacetime bound on it. Let \( \eta_1 \) be a small number that meets all the conditions that will appear in the proof, then dividing \([0, \eta_1^4]\) into \( J_1 \) subintervals and \([ -\eta_1^4, 0 \]) into \( J_2 \) subintervals such that on each subinterval \( I_j \), we have \( \eta_1 \leq \| u \|_{L^{10}(I_j, L^{10})} \leq 2 \eta_1 \). So we are left to control \( J_1, J_2 \) by constant \( C(\Lambda_1, \Lambda_2) \). Without loss of generality, we
only do analysis in the positive time direction. Following Bourgain [1], we classify the subintervals into three components $I^{(1)}, I^{(2)}, I^{(3)}$, and each contains $\frac{\lambda}{3}$ consecutive subintervals. It’s on the middle component that we do most analysis. Our first aim is to show the existence of a sequence of bubbles somewhere in space at a sequence of times $t_j$ which belongs to the subinterval $I_j$. We realize this by doing analysis on one specified subinterval. At first, we show some regularity property. (From this section to the end, the constant $C$ may depend on $\Lambda_1, \Lambda_2$.)

Proposition 5.1: Let $I_j$ be one of the subintervals, that is $I_j \subset [0, \eta^4]$ and

$$\eta_1 \leq \|u\|_{L^{10}(I_j; L^{10})} \leq 2\eta_1.$$  

Then $u$ satisfies

$$\|\nabla u\|_{L^\frac{10}{7}(I_j; L^{\frac{10}{7}})} + \|xu\|_{L^\frac{10}{7}(I_j; L^{\frac{10}{7}})} \leq C(\Lambda_1, \Lambda_2).$$

Proof: Noting that $I_j \subset [0, \eta^4]$, it’s suffices to prove the same space time bound for $J(t)u$ and $H(t)u$. By Duhamel, on $I_j = [t_j, t_{j+1}]$, $u$ satisfies

$$u(t) = U(t - t_j)u(t_j) - i \int_{t_j}^t U(t - s)|u|^4u(s)ds.$$  

Let $A(t) \in \{J(t), H(t)\}$, then we have

$$A(t)u(t) = U(t - t_j)A(t_j)u(t_j) - i \int_{t_j}^t U(t - s)A(s)|u|^4u(s)ds.$$  

Using Strichartz estimate, we get

$$\|A(\cdot)u\|_{L^\frac{10}{7}(I_j; L^{\frac{10}{7}})} \leq C\|A(t_j)u(t_j)\|_2 + C\|u\|_{L^{10}(I_j; L^{10})}^4\|A(\cdot)u\|_{L^\frac{10}{7}(I_j; L^{\frac{10}{7}})}$$

$$\leq C\|A(t_j)u(t_j)\|_2 + C\eta_1^4\|A(\cdot)u\|_{L^\frac{10}{7}(I_j; L^{\frac{10}{7}})}.$$  

Noting that $\|A(t_j)u(t_j)\|_2$ is bounded by $C(\Lambda_1, \Lambda_2)$, we get

$$\|A(\cdot)u\|_{L^\frac{10}{7}(I_j; L^{\frac{10}{7}})} \leq C(\Lambda_1, \Lambda_2),$$  

by $C\eta_1^4 < \frac{1}{4}$. Thus we end this Proposition.

Proposition 5.2 Let $I_j$ be one of the subintervals, that is $I_j \in [0, \eta^4]$ and $\eta_1 \leq \|u\|_{L^{10}(I_j; L^{10})} \leq 2\eta_1$. Then there exists $t_j \in I_j$, $x_j \in$
\( \mathbb{R}^3 \) and \( N \geq N_{j0} \approx |I_J|^{-\frac{1}{7}} \eta_1^5 \) such that
\[
\|u(t_j)\|_{L^6(|x-x_j|<C(\eta_1)N_j^{-1})}\geq c\eta_1^\frac{3}{2}, \tag{44}
\]
\[
\|\nabla u(t_j)\|_{L^2(|x-x_j|<C(\eta_1)N_j^{-1})}\geq c\eta_1^\frac{3}{4}, \tag{45}
\]
\[
\|u(t_j)\|_{L^2(|x-x_j|<C(\eta_1)N_j^{-1})}\geq c\eta_1^\frac{3}{8}N_j^{-1}. \tag{46}
\]

**Proof:** By Beinstein estimate, \( \forall N \in 2^\mathbb{Z} \), we have
\[
\|P_{<N}u\|_\infty \leq N^{\frac{5}{2}}\|P_{<N}u\|_6 \leq CN^{\frac{5}{4}},
\]
which allows us to control the \( L^{10} \) norm of low frequency by interpolation,
\[
\|P_{<N}u\|_{10} \leq \|P_{<N}u\|_\infty^{\frac{1}{10}}\|P_{<N}u\|_6^{\frac{9}{10}} \leq CN^{\frac{2}{5}},
\]
hence, using Hölder inequality in time, we have
\[
\|P_{<N}u\|_{L^{10}(I_J;L^{10})} \leq C|I_J|^{\frac{1}{10}}N^{\frac{1}{4}}.
\]
Taking \( N = N_{j0} = C|I_J|^{-\frac{1}{5}} \eta_1^{\frac{5}{8}} \), one sees that
\[
\|P_{<N_{j0}}u\|_{L^{10}(I_J;L^{10})} < \frac{\eta_1}{2},
\]
and thus
\[
\|P_{\geq N_{j0}}u\|_{L^{10}(I_J;L^{10})} > \frac{\eta_1}{2}.
\]
Using Littlewood-Paley theorem, we have
\[
\left( \frac{\eta_1}{2} \right)^{10} \leq \|P_{\geq N_{j0}}u\|_{L^{10}(I_J;L^{10})}^{10} = \int_{I_J} \|P_{\geq N_{j0}}u(t)\|_{10}^{10}dt
\]
\[
= \int_{I_J} \left( \sum_{N \geq N_{j0}} |P_Nu(t)|^2 \right)^{\frac{1}{2}} \|P_{<N}u(t)\|_{10}^{10}dt
\]
\[
= C \int_{I_J} \int_{\mathbb{R}^3} \sum_{N_{j1} \geq \cdots \geq N_{j5} \geq N_{j0}} |P_{N_{j1}}u(t)|^2 \cdots |P_{N_{j5}}u(t)|^2 dx dt.
\]
Letting \( \sigma_N = N^{\frac{1}{2}} \|P_Nu\|_{L^{\infty}_{x \in I_J \times \mathbb{R}^3}} \), we see the last line is smaller than
\[
C \sup_{N \geq N_{j0}} \sigma_N^{20} \int_{I_J} \int_{\mathbb{R}^3} \sum_{N_{j1} \geq \cdots \geq N_{j5} \geq N_{j0}} |P_{N_{j1}}u(t)|^2 |P_{N_{j2}}u(t)|^4 N_{j2}^{\frac{1}{2}}N_{j3}N_{j4}N_{j5} dx dt
\]
\[
\leq C \sup_{N \geq N_{j0}} \sigma_N^{20} \int_{I_J} \int_{\mathbb{R}^3} \sum_{N_{j1} \geq N_{j2} \geq N_{j0}} N_{j2}^{\frac{10}{2}} |P_{N_{j1}}u(t)|^2 |P_{N_{j2}}u(t)|^4 dx dt, \tag{47}
\]
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by summing \(N_5, N_4\) and \(N_3\). Using Hölder inequality and Young’s inequality, (44) can be controlled by

\[
C \sup_{N \geq N_{j_0}} \sigma_N^\frac{20}{3} \sum_{N_{1} \geq N_{2} \geq N_{j_0}} N_{1}^\frac{10}{3} \|P_{N_{1}} u\|^2_{L^\infty} \|P_{N_{2}} u\|^\frac{4}{10}_{L^\infty}
\leq C \sup_{N \geq N_{j_0}} \sigma_N^\frac{20}{3} \sum_{N_{1} \geq N_{2} \geq N_{j_0}} N_{2}^2 N_{1}^{-2} \|\nabla P_{N_{1}} u\|^2_{L^\infty} \|\nabla P_{N_{2}} u\|^\frac{4}{10}_{L^\infty}
\leq C \sup_{N \geq N_{j_0}} \sigma_N^\frac{20}{3} \sum_{N \geq N_{j_0}} \|\nabla P_{N} u\|^\frac{10}{3}_{L^\infty} (I_j; L^\infty)
\leq C \sup_{N \geq N_{j_0}} \sigma_N^\frac{20}{3} \cdot (I_j; L^\infty)
\leq C \sup_{N \geq N_{j_0}} \sigma_N^\frac{20}{3}.
\]

This implies that,

\[
\sup_{N \geq N_{j_0}} \sigma_N \geq c\eta_1^\frac{3}{2},
\]

thus there exists \(t_j \in I_j, x_j \in \mathbb{R}^3\) and \(N_j \geq N_{j_0}\) such that

\[
|P_{N_j} u(x_j, t_j)| \geq c\eta_1^\frac{3}{2} N_j^\frac{3}{2}.
\]

Now we deduce (44), (45), (46) from (48). By the definition of \(P_{N_j}\), we see that

\[
\begin{align*}
\eta_1^\frac{3}{2} N_j^\frac{3}{2} & \leq |P_{N_j} u(x_j, t_j)| \\
& = \left| \int \tilde{\phi}_{N_j} (x_j - x) u(t_j, x) dx \right| \\
& \leq \left| \int_{|x - x_j| < C(\eta_1) N_j^{-1}} \tilde{\phi}_{N_j} (x_j - x) u(t_j, x) dx \right| + \left| \int_{|x - x_j| > C(\eta_1) N_j^{-1}} \tilde{\phi}_{N_j} (x_j - x) u(t_j, x) dx \right| \\
& \leq \left( \int_{\mathbb{R}^3} |\tilde{\phi}_{N_j} (x_j - x)|^\frac{6}{\hat{p}} dx \right)^\frac{\hat{p}}{6} \left( \int_{|x - x_j| < C(\eta_1) N_j^{-1}} |u(t_j, x)|^6 dx \right)^\frac{1}{6} \\
& \quad + \left( \int_{|x - x_j| > C(\eta_1) N_j^{-1}} |\tilde{\phi}_N (x_j - x)|^\frac{6}{\hat{p}} dx \right)^\frac{\hat{p}}{6} \left( \int_{\mathbb{R}^3} |u(t_j, x)|^6 dx \right)^\frac{1}{6}.
\end{align*}
\]

Noting \(\tilde{\phi}_N (\cdot) = N^3 \tilde{\phi}(\cdot)\) and \(\tilde{\phi}\) is rapidly decreasing, one obtains

\[
(49) \leq CN_j^\frac{3}{2} \left( \int_{|x - x_j| < C(\eta_1) N_j^{-1}} |u(t, x)|^6 dx \right)^\frac{1}{6} + \frac{c}{2} \eta_1^\frac{3}{2} N_j^\frac{3}{2},
\]

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by choosing $C(\eta_1)$ sufficiently large and
\[
\left(\int_{\mathbb{R}^3} |u(t_j, x)|^6 dx\right)^{\frac{1}{6}} \leq \|u(t_j)\|_0 \leq C \Lambda_1^{\frac{1}{3}}.
\]
Thus we obtain (44). To see (45), we begin with (48) that
\[
\eta_1^{\frac{3}{4}} N_j^{\frac{1}{2}} < |P_{N_j} u(x_j, t_j)| = |(\Delta^{-1} \nabla) P_{N_j} \nabla u(x_j, t_j)|
\]
\[
= |K_{N_j} * \nabla u(x_j, t_j)| = \int K_{N_j} (x_j - x) \nabla u(x, t_j) dx (50)
\]
where $K_{N_j}$ is the kernel of $(\Delta^{-1} \nabla) P_{N_j}$, $K_{N_j}(x) = F^{-1} \left( \frac{\cdot}{|\cdot|^2} \phi_N(\cdot) \right) (\xi)$, and
\[
\|K_{N_j}\|_{L^2} = N_j^{\frac{3}{4}} \|F(\frac{\cdot}{|\cdot|^2} \phi(\cdot))\|_2,
\]
\[
\|K_{N_j}\|_{L^2(|x| \geq C(\eta_1) N_j^{-1})} = N_j^{\frac{3}{4}} \|F^{-1} \left( \frac{\cdot}{|\cdot|^2} \phi(\cdot) \right)\|_{L^2(|x| \geq C(\eta_1))} \leq C \eta_1^{\frac{3}{2}} N_j^{\frac{1}{2}},
\]
if $C(\eta_1)$ is large enough. Thus (50) has the bound
\[
\left(\int |K_{N_j} (x_j - x)|^2 dx\right)^{\frac{1}{2}} \left(\int |\nabla u(x, t_j)|^2 dx\right)^{\frac{1}{2}}
\]
\[
+ \left(\int |K_{N_j} (x_j - x)|^2 dx\right)^{\frac{1}{2}} \left(\int |\nabla u(x, t_j)|^2 dx\right)^{\frac{1}{2}}
\]
\[
\leq C N_j^{\frac{3}{2}} \left(\int |x - x_j| < C(\eta_1) N_j^{-1} |\nabla u(x, t_j)|^2 dx\right)^{\frac{1}{2}} + \frac{C}{2} \eta_1^{\frac{3}{4}} N_j^{\frac{1}{2}},
\]
and we have
\[
\|\nabla u(x, t_j)\|_{L^2(|x - x_j| < C(\eta_1) N_j^{-1})} \geq C \eta_1^{\frac{3}{4}}.
\]

The proof of (46) is similar. Thus we end the proof of Proposition 5.2. Now we use the radial assumption to locate the bubble at origin.

**Corollary 5.3:** Let the conditions in Proposition 5.2 be fulfilled. Assume further that $u$ is radial, then there holds that
\[
\|u(t_j)\|_{L^6(|x| < C(\eta_1) N_j^{-1})} \geq C \eta_1^{\frac{3}{4}},
\]
\[
\|\nabla u(t_j)\|_{L^2(|x| < C(\eta_1) N_j^{-1})} \geq C \eta_1^{\frac{3}{4}},
\]
\[
\|u(t_j)\|_{L^2(|x| < C(\eta_1) N_j^{-1})} \geq C \eta_1^{\frac{3}{4}} N_j^{-1},
\]

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with $t_j, N_j$ the same with Proposition 5.2.

**Proof:** We prove (51)–(53) by showing that
\[ |x_j| < C(\eta_1)N_j^{-1}, \]
since once this has been done, we can choose a new constant $\tilde{C}(\eta_1)$ large enough such that
\[ B(0, \tilde{C}(\eta_1)N_j^{-1}) \supset B(x_j, C(\eta_1)N_j^{-1}), \]
(51)–(53) then follow from (44)–(46).

Letting $S(0,|x_j|)$ be a sphere with radius $|x_j|$ and center 0. By geometrical observation, one has $O(|x_j|/C(\eta_1)N_j^{-1})$ balls that have radius $C(\eta_1)N_j^{-1}$ and center at the points on the sphere. By radial assumption and Proposition 5.2, on each ball, $u(t_j)$ has nontrivial $L^6$ norm.

Using the boundedness of $L^6$ estimate, one has
\[ O(|x_j|/C(\eta_1)N_j^{-1})(C\eta_1^{3/2})^6 \leq \|u(t)\|_6^6 \leq CA_1. \]
This gives the desired control on $|x_j|$ and concludes Corollary 5.3.

6 Proof of Proposition 3.5: In case of solitonlike solution

Applying Corollary 5.3 on each interval in the middle component $I^{(2)}$, we get a sequence of time $\{t_j\}$, $t_j \in I_j, \frac{4}{3} + 1 \leq j \leq \frac{2}{3}J_1$, such that
\[ \|\nabla u(t_j)\|_{L^2(|x| \leq C(\eta_1)N_j^{-1})} > c\eta_1^{3/2}, \quad (54) \]
\[ \|u(t_j)\|_{L^2(|x| \leq C(\eta_1)N_j^{-1})} > c\eta_1^{3/2}N_j^{-1}, \quad N_j \geq C|I_j|^{-\frac{1}{3}}\eta_1^5. \quad (55) \]

Now, we discuss two different cases according to the size of the bubble. First, if there exists $\eta_2, 0 < \eta_2 \ll \eta_1$ such that
\[ c|I_j|^{-\frac{1}{3}}\eta_1^{5} \leq N_j \leq \frac{C(\eta_1)}{\eta_2}|I_j|^{-\frac{1}{3}}, \quad \frac{J_1}{3} + 1 \leq j \leq \frac{2}{3}J_1, \quad (56) \]
we call the solution solitonlike. Otherwise there must be $j_0 \in [\frac{4}{3} + 1, \cdots, \frac{2}{3}J_1]$ such that
\[ N_{j_0} \geq \frac{C(\eta_1)}{\eta_2}|I_{j_0}|^{-\frac{1}{3}} \iff C(\eta_1)N_{j_0}^{-1} < \eta_2|I_{j_0}|^{-\frac{1}{3}}. \quad (57) \]
As a consequence, we have concentration as follows,

$$\|\nabla u(t_0)\|_{L^2(|x|<\frac{1}{3} \eta_2 |t_0|^{\frac{1}{2}})} > c\eta_1^{\frac{3}{2}}. \quad (58)$$

In this case, we call the solution is blow up solution. In this section, we aim to estimate $J_1$ in case of solitonlike solution. We follow the idea of [13] and begin the proof by showing that (55) holds for every $t \in J_1$, and $\frac{d}{J_1} + 1 \leq j \leq \frac{2}{3} J_1$.

**Proposition 6.1:** Assume $u$ satisfy (55), (56), then there exists $C(\eta_1, \eta_2), c(\eta_1, \eta_2)$ such that

$$\|u(t)\|_{L^2(|x| \leq C(\eta_1, \eta_2) |t|^{\frac{1}{2}})} \geq c(\eta_1, \eta_2) |t|^{\frac{1}{2}}, \quad \forall t \in J_1 \text{ and } j \in [\frac{1}{3} J_1 + 1, \frac{2}{3} J_1]. \quad (59)$$

**Proof:** Fix $j$, from (56), we have

$$CN_1^{-5}|J_1|^{\frac{1}{2}} \geq N_j^{-1} \geq c(\eta_1) \eta_2 |J_1|^{\frac{1}{2}}. \quad (60)$$

Applying this estimate to (55), one gets

$$\|u(t_j)\|_{L^2(|x| \leq C(\eta_1) |t_j|^{\frac{1}{2}})} \geq c(\eta_1) \eta_2 |t_j|^{\frac{1}{2}}. \quad (61)$$

From (60) and by choosing $C(\eta_1, \eta_2)$ sufficiently large, we have

$$\|u(t)\|_{L^2(|x| \leq C(\eta_1, \eta_2) |t_j|^{\frac{1}{2}})} \geq \|u(t_j)\|_{L^2(|x| \leq C(\eta_1, \eta_2) |t_j|^{\frac{1}{2}})} - \frac{|J_1| \|u\|_{L^\infty (I; H^1)}}{C(\eta_1, \eta_2) |t_j|^{\frac{1}{2}}} \geq c(\eta_1) \eta_2 |t_j|^{\frac{1}{2}} - c(\eta_1, \eta_2) |t_j|^{\frac{1}{2}} \geq c(\eta_1, \eta_2) |t_j|^{\frac{1}{2}}. \quad (62)$$

This is exactly (60). Once we have gotten (59), we can follow the same way in [13] to obtain the finiteness of $J_1$. For the sake of completeness, we give the proof. First, we do some elementary computation,

$$c(\eta_1, \eta_2) |J_1| \leq \int_{|x| \leq C(\eta_1, \eta_2) |t|^{\frac{1}{2}}} |u|^2(x, t) dx \leq \int_{|x| \leq C(\eta_1, \eta_2) |t|^{\frac{1}{2}}} |x|^\frac{1}{2} |u|^2(x, t) \frac{dx}{|x|^\frac{1}{2}} \leq \left( \int_{|x| \leq C(\eta_1, \eta_2) |t|^{\frac{1}{2}}} |x|^\frac{1}{2} dx \right)^{\frac{3}{2}} \left( \int_{|x| \leq C(\eta_1, \eta_2) |t|^{\frac{1}{2}}} \frac{|u(x, t)|^6}{|x|} dx \right)^{\frac{1}{3}} \leq C(\eta_1, \eta_2) |J_1|^{\frac{3}{2}} \left( \int_{|x| \leq C(\eta_1, \eta_2) |t|^{\frac{1}{2}}} \frac{|u(x, t)|^6}{|x|} dx \right)^{\frac{1}{3}},$$

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thus, we have

$$
\int_{|x|\leq C(\eta_1, \eta_2)|I_j|^{\frac{1}{2}}} \frac{|u(x, t)|^6}{|x|} dx \geq c(\eta_1, \eta_2)|I_j|^{-\frac{1}{2}}. \tag{60}
$$

Comparing (60) with Morawetz estimate (38), one obtains,

**Corollary 6.2:** For any $I \subset I^{(2)}$, we have

$$
\sum_{\frac{1}{2} J_1 + 1 \leq j \leq \frac{2}{3} J_1; I_j \subset I} |I_j|^{\frac{1}{2}} \leq C(\eta_1, \eta_2)|I|^{\frac{1}{2}}. \tag{61}
$$

**Proof:** Noting $|I_j|^{\frac{1}{2}} < |I|^{\frac{1}{2}}$ and letting $A = C(\eta_1, \eta_2)$, (60) becomes

$$
\int_{|x| \leq A|I|^{\frac{1}{2}}} \frac{|u|^6(x, t)}{|x|} dx \geq c(\eta_1, \eta_2)|I_j|^{-\frac{1}{2}}. \tag{62}
$$

Integrating (62) on $I_j$ and summing together in $j$, we get,

$$
c(\eta_1, \eta_2) \sum_{\frac{1}{2} J_1 + 1 \leq j \leq \frac{2}{3} J_1; I_j \subset I} |I_j|^{\frac{1}{2}} \leq \int_I \int_{|x| \leq A|I|^{\frac{1}{2}}} \frac{|u|^6(x, t)}{|x|} dx dt \\
\leq CA|I|^{\frac{1}{2}} \leq C(\eta_1, \eta_2)|I|^{\frac{1}{2}},
$$

this gives (61). As a direct consequence of Corollary 6.2, we have

**Corollary 6.3:** Let $I = \bigcup_{j_1 \leq j \leq j_2} I_j$ be a union of consecutive intervals, $\frac{1}{2} J_1 + 1 \leq j_1, j_2 \leq \frac{2}{3} J_1$, then there exists $j_1 \leq j \leq j_2$ such that $|I_j| > c(\eta_1, \eta_2)|I|$.

**Proof:** From (61) we know that

$$
C(\eta_1, \eta_2)|I|^{\frac{1}{2}} \geq \sum_{j_1 \leq j \leq j_2} |I_j|^{\frac{1}{2}} \geq \sum_{j_1 \leq j \leq j_2} |I_j| (\sup_{j_1 \leq j \leq j_2} |I_j|)^{-\frac{1}{2}} = |I| (\sup_{j_1 \leq j \leq j_2} |I_j|)^{-\frac{1}{2}},
$$

and hence

$$
C(\eta_1, \eta_2)|I|^{-\frac{1}{2}} \geq (\sup_{j_1 \leq j \leq j_2} |I_j|)^{-\frac{1}{2}}. \tag{63}
$$

(63) allows us to find an interval $I_j$ such that

$$
|I_j| > c(\eta_1, \eta_2)|I|.
$$

Now, we show that the intervals $I_j$ must concentrate at some time $t_*$. 

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by the local mass conservation, we have following proof, as just like the thing that has been mentioned in [13].

On the other hand, by the local mass estimate (37), we have

\[ |I_{j_1}| \geq 2|I_{j_2}| \geq \cdots \geq 2^{k-1}|I_{j_K}|, \]

and \( \text{dist}(t_*, I_{j_k}) \leq C(\eta_1, \eta_2)|I_{j_k}|. \)

For the proof of this Proposition, one refers to [13], Proposition 3.8.

Let \( t_* \) and \( I_{j_1}, \cdots, I_{j_k}, \cdots, I_{j_K} \) be as in the Proposition 6.4 and for every \( t \in I_{j_k} \), there holds

\[ \text{Mass}(u(t), B(0; C(\eta_1, \eta_2)|I_{j_k}|^{\frac{1}{2}})) \geq c(\eta_1, \eta_2)|I_{j_k}|^{\frac{1}{2}}, \quad \forall t \in I_{j_k}, \quad 1 \leq k \leq K. \]

(64)

The point 0 can be substituted by \( x_{j_k} \) without modification to the following proof, as just like the thing that has been mentioned in [13].

By the local mass conservation, we have

\[ \text{Mass}(u(t_*), B(0; C(\eta_1, \eta_2)))|I_{j_k}|^{\frac{1}{2}} \geq c(\eta_1, \eta_2)|I_{j_k}|^{\frac{1}{2}} - \frac{|t_* - t|\|u\|_{L^\infty(I_{j_k}, \dot{H}^{1})}}{C(\eta_1, \eta_2)|I_{j_k}|^{\frac{1}{2}}}, \]

\[ \geq c(\eta_1, \eta_2)|I_{j_k}|^{\frac{1}{2}}, \quad \forall 1 \leq k \leq K. \]

Denote \( B_k = B(0; C(\eta_1, \eta_2)|I_{j_k}|^{\frac{1}{2}}) \), we rewrite the above estimate as follows,

\[ \text{Mass}(u(t_*), B_k) \geq c(\eta_1, \eta_2)|I_{j_k}|^{\frac{1}{2}}, \quad 1 \leq k \leq K. \]

(65)

On the other hand, by the local mass estimate (37), we have

\[ \text{Mass}(u(t_*), B_k) \leq C(\eta_1, \eta_2)|I_{j_k}|^{\frac{1}{2}}. \]

Letting \( N := \log(\frac{1}{\eta_3}) \), then for \( k' > k + N \), we have that

\[ \int_{B_k'} |u(t_*, x)|^2dx \leq C(\eta_1, \eta_2)|I_{j_{k'}}| \leq C(\eta_1, \eta_2)2^{-(k'-k)}|I_{j_k}| \]

\[ \leq C(\eta_1, \eta_2)\eta_3 2^{-(k'-k-N)}|I_{j_k}|, \]

and hence,

\[ \sum_{k+N \leq k' \leq K} \int_{B_{k'}} |u(t_*, x)|^2dx \leq C(\eta_1, \eta_2)\eta_3|I_{j_k}| \sum_{k+N \leq k' \leq K} 2^{-(k'-k-N)}. \]

(66)
By the finiteness of the summation, the assumption on $\eta_1, \eta_2, \eta_3,$ and \( (64) \), we continue to estimate \( (65) \) by
\[
c(\eta_1, \eta_2)|I_{jh}| \leq \frac{1}{2} \text{Mass}(u(t_*), B_k)^2 = \frac{1}{2} \int_{B_k} |u(t_*, x)|^2 dx,
\]
and hence
\[
\int_{B_k \setminus \bigcup_{k+N \leq k' \leq K} B_{k'}} |u(t_*, x)|^2 dx \geq \int_{B_k} |u(t_*, x)|^2 dx - \int_{\bigcup_{k+N \leq k' \leq K} B_{k'}} |u(t_*, x)|^2 dx
\geq \int_{B_k} |u(t_*, x)|^2 dx - \sum_{k+N \leq k' \leq K} \int_{B_{k'}} |u(t_*, x)|^2 dx
\geq \frac{1}{2} \int_{B_k} |u(t_*, x)|^2 dx \geq c(\eta_1, \eta_2)|I_{jh}|.
\] (67)
By Hölder inequality, we further give the upper bounds of the left side as follows,
\[
\int_{B_k \setminus \bigcup_{k+N \leq k' \leq K} B_{k'}} |u(t_*, x)|^2 dx
\leq \left( \int_{B_k \setminus \bigcup_{k+N \leq k' \leq K} B_{k'}} |u(t_*, x)|^6 dx \right)^\frac{2}{3} \text{mes}(B_k)^{\frac{1}{3}}
\leq C(\eta_1, \eta_2)|I_{jh}| \left( \int_{B_k \setminus \bigcup_{k+N \leq k' \leq K} B_{k'}} |u(t_*, x)|^6 dx \right)^\frac{1}{3},
\]

hence we have
\[
\int_{B_k \setminus \bigcup_{k+N \leq k' \leq K} B_{k'}} |u(t_*, x)|^6 dx \geq c(\eta_1, \eta_2).
\] (68)
Summing \( (68) \) in $k$, we obtain
\[
\sum_{k=1}^{K} \int_{B_k \setminus \bigcup_{k+N \leq k' \leq K} B_{k'}} |u(t_*, x)|^6 dx \geq c(\eta_1, \eta_2)K.
\] (69)

Denoting $P_k := B_k \setminus \bigcup_{k+N \leq k' \leq K} B_{k'}$, then \( \{P_k\}_{k=1}^K \) overlaps at most $N$ times. Thus the left hand side of \( (69) \) is smaller than
\[
N \int_{\mathbb{R}^3} |u(t_*, x)|^6 dx.
\]
By the definition of $\eta_3$, the boundedness of $\|u(t_*)\|_6$, we have an upper bound for $K$,
\[
K \leq C(\eta_1, \eta_2, \eta_3, \Lambda_1, \Lambda_2),
\]
and this in turn gives the control of $J_1$,
\[
J_1 \leq C \exp(C(\eta_1, \eta_2, \eta_3, \Lambda_1, \Lambda_2)).
\]
7 In case of blow up solution

Our purpose of this section is to prove the boundedness of $J_1$ under the condition (57) and (58). That is, for the solution, we have concentration at some $t_0 \in I_{j_0}$, $j_0 \in \left[\frac{1}{3}, \cdots, \frac{2}{3}J_2 \right]$ such that

$$\|\nabla u(t_0)\|_{L^2(|x|<\frac{1}{2}\eta_2|I_{j_0}|^{\frac{1}{2}})} > c\eta_1^3.$$  \hfill (70)

If $t_0$ lies in the left side of $I_{j_0}$, we take $I = [t_0, b]$ where $b$ is the left end point of $I_{j_0}$; Otherwise we take $I = [a, t_0]$ with $a$ the right end point of $I_{j_0}$. Then (70) becomes

$$\|\nabla u(t_0)\|_{L^2(|x|<\eta_2|I_{j_0}|^{\frac{1}{2}})} > c\eta_1^3.$$  \hfill (71)

Assume $I = [t_0, b]$, we plan to re-solve the problem (1) forward in time. Otherwise, we do in the reverse direction. First we show that, by removing the small bubble, we remove nontrivial portion of energy.

Let $\chi$ be a smooth radial function such that $\chi(x) = 1$ as $|x| \leq 1$, and $\chi(x) = 0$ as $|x| \geq 2$. Let $\phi(x) = \chi\left(\frac{x}{N\eta_2|I_{j_0}|^{\frac{1}{2}}}\right)$ for some $N \geq 1$ to be specified later, and $w(t_0, x) = (1 - \phi(x))u(t_0, x)$, then we have

**Lemma 7.1** $E_1(w(t_0)) \leq E_1(u(t_0)) - c\eta_1^3$.

**Proof:** By noting $w(t_0) = (1 - \phi)u(t_0)$, we compute that

$$\nabla w(t_0) = (1 - \phi)\nabla u(t_0) - \nabla \phi u(t_0),$$

and thus,

$$|\nabla w(t_0)|^2 = |\nabla u(t_0)|^2 + (\phi^2 - 2\phi)|\nabla u(t_0)|^2 + |\nabla \phi|^2 |u(t_0)|^2 - 2Re(1 - \phi)\nabla \phi \bar{u}(t_0) \nabla u(t_0).$$

Integrating it on $\mathbb{R}^3$, one gets

$$\|\nabla w(t_0)\|_2^2 \leq \|\nabla u(t_0)\|_2^2 + \int_{\mathbb{R}^3} (\phi^2 - 2\phi)|\nabla u(t_0, x)|^2 dx - 2\int_{\mathbb{R}^3} |\nabla \phi(x) u(t_0, x)|^2 dx - 2Re \int_{\mathbb{R}^3} (1 - \phi) \nabla \phi \bar{u}(t_0) \nabla u(t_0)(x) dx.$$

By the trivial inequality: $\phi^2 - 2\phi \leq -\phi$ and (71), one can estimate the second term of the right side by

$$-\int_{|x| \leq N\eta_2|I_{j_0}|^{\frac{1}{2}}} |\nabla u(t_0, x)|^2 dx \leq -c\eta_1^3.$$
Now, we estimate the remaining two terms. We use Hölder inequality to control them by

\[
C \|\nabla \phi \|_2^2 \|u(t_0)\|_{L^6}^{1 \over 2} + C \|\nabla \phi \|_2 \|\nabla u(t_0)\|_{L^6}^{1 \over 2} \|u(t_0)\|_{L^6}^{1 \over 2} \\
\leq C \left( \|u(t_0)\|_{L^6}^{1 \over 2} + \|u(t_0)\|_{L^6} \right) \|\nabla u(t_0)\|_2.
\]

(72)

Now, we claim that, there must exist \( N \) which depend only on \( \eta_1 \) such that

\[
\|u(t_0)\|_{L^6}^{1 \over 2} \leq \eta_1^4.
\]

(73)

Indeed, if otherwise, we will have \( N \) annuluses, on each annulus, \( u(t_0) \) has nontrivial \( L^6 \) norm. Summing these annuluses together, we obtain

\[
N(\eta_1^4) \leq \sum_{N' \leq N \in \mathbb{N}} \|u(t_0)\|_{L^6}^{1 \over 2} \leq C,
\]

by the boundedness of \( L^6 \) estimate. This will be a contradiction if \( N \geq C \eta_1^{-24} \). Hence, one can fix \( N = C(\eta_1) \) such that (73) holds and

\[
\text{(72)} \leq C \eta_1^4.
\]

We finally obtain this Lemma by noting

\[
E_1(w(t_0)) = \frac{1}{2} \|\nabla w(t_0)\|_2^2 + \frac{1}{3} \|w(t_0)\|_6^6,
\]

and combing the above estimates together.

**Lemma 7.2** We have that,

\[
E_1(w(t_0)) \leq \Lambda_1 - c \eta_1^3,
\]

\[
E_2(w(t_0)) \leq \Lambda_2 + C \eta_1^4.
\]

**Proof:** Noting Lemma 7.1, it suffices to prove

\[
E_1(u(t_0)) \leq E_1(u(0)) + C \eta_1^4,
\]

\[
E_2(u(t_0)) \leq E_2(u(0)) + C \eta_1^4.
\]

So, Let’s compute the increment of \( E_i(u(t)) \) from 0 to \( t_0 \):

\[
\int_0^{t_0} \frac{\partial}{\partial t} E_1(u(t))dt, \text{ and } \int_0^{t_0} \frac{\partial}{\partial t} E_2(u(t))dt.
\]

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From the equation (1), we see that
\[
\frac{\partial}{\partial t} E_2(u(t)) = \frac{\partial}{\partial t} \|xu(t)\|_2^2
\]
\[
= 2Im \int_{\mathbb{R}^3} x\bar{u}\nabla u(t,x) dx
\]
\[
\leq C \|xu\|_{L^\infty((0,t_0);L^2)} \|
abla u\|_{L^\infty((0,t_0);L^2)} \leq C(\Lambda_1, \Lambda_2). \quad \forall t \in [0,t_0],
\]
thus, we have
\[
\left| \int_0^t \frac{\partial}{\partial t} E_2(t) dt \right| \leq C' \eta_1^4.
\]
By noting that \( \frac{\partial}{\partial t} E_1(t) = -\frac{\partial}{\partial t} E_2(t) \), we get
\[
\left| \int_0^t \frac{\partial}{\partial t} E_1(t) dt \right| \leq C' \eta_1^4,
\]
hence, Lemma 7.2 follows.

Putting these Lemmas aside, we turn to re-solve the solution from \( t_0 \) forward. We do this by splitting \( u = v + w \) and studying the following two initial data problems:

\[
\begin{cases}
  (i\partial_t + \frac{\Delta}{2} + |x|^2)v = |v|^4v, \\
  v(x,t_0) = \phi(x)u(t_0,x).
\end{cases}
\]
(74)

\[
\begin{cases}
  (i\partial_t + \frac{\Delta}{2} + |x|^2)w = |v + w|^4(v + w) - |v|^4v, \\
  w(x,t_0) = (1 - \phi(x))u(t_0,x).
\end{cases}
\]
(75)

Now, Let’s first prove that (74) is wellposed on \([t_0, \infty)\).

**Proposition 7.3** There exists a unique solution \( v(x,t) \) to (74) satisfies

\[
\|v\|_{L^{10}(I;L^{10})} \leq C\eta_1, \quad \|v\|_{L^{10}([b,\infty);L^{10})} \leq C\eta_2^{1/2}.
\]

\[
\|A(\cdot)v\|_{L^q(I;L^r)} \leq C,
\]
where \( A \in \{J, H\} \) and \((q,r)\) are admissible pairs.

**Proof:** We begin by computing the \( L^{10} \) norm of the linear flow \( U(t-t_0)(\phi u(t_0)) \). First, by Duhamel’s formula, we observe that

\[
U(t-t_0)u(t_0) = u(t) + i \int_{t_0}^t U(t-s)|u|^4u(s)ds,
\]
from this, we see that

\[
\|U(\cdot-t_0)u(t_0)\|_{L^{10}(I;L^{10})} \leq \|u\|_{L^{10}(I;L^{10})} + \| \int_{t_0}^t U(t-s)|u|^4u(s)ds \|_{L^{10}(I;L^{10})}.
\]

Applying embedding and Strichartz, the second term is smaller than
\[
\|J(\cdot)\int_t^t U(t-s)|u|^4 u(s) ds\|_{L^{10}(I;L^{10})} \\
\leq C\|u\|_{L^{10}(I;L^{10})}^4 \|J(\cdot)u\|_{L^{10}(I;L^{10})} \\
\leq C\eta_t^4 \leq \eta_t,
\]
and hence,
\[
\|U(\cdot - t_0)u(t_0)\|_{L^{10}(I;L^{10})} \leq 2\eta_t.
\]
Noting \(\phi u(t_0)\) is a radial function in space, we have that
\[
U(t-t_0)(\phi u(t_0))(x) = \exp \{\frac{it(\Delta + |x|^2)}{2}\} (\phi u(t_0))(x) \\
= \mathcal{F}^{-1} \left( \exp \{-\frac{it(|\xi|^2 + \Delta \xi)}{2}\} \phi u(t_0)(\xi) \right) (x) \\
= \int_{\mathbb{R}^3} e^{i x \xi} e^{-\frac{it}{2}(|\xi|^2 + \Delta \xi)} \int_{\mathbb{R}^3} \hat{\phi}(t_0)(\xi - \xi_1)\hat{\phi}(\xi_1) d\xi_1 d\xi.
\]
Expanding \(|\xi|^2 = |\xi - \xi_1|^2 + 2\xi_1 (\xi - \xi_1) + |\xi_1|^2\), the above term becomes
\[
\int \int e^{i(\xi - \xi_1)(x-(t-t_0)\xi_1)} e^{-\frac{it}{2}(|\xi - \xi_1|^2 + \Delta \xi - \xi_1)} \hat{\phi}(t_0)(\xi - \xi_1) d\xi_1 e^{-\frac{it}{2}(|\xi_1|^2} \hat{\phi}(\xi_1) d\xi_1,
\]
by renaming the variable, one sees that this is exactly
\[
\int_{\mathbb{R}^3} U(t-t_0)u(t_0)(x - (t-t_0)\xi_1) e^{-\frac{it}{2}(|\xi|^2} \hat{\phi}(\xi_1) d\xi_1,
\]
and hence
\[
\|U(\cdot - t_0)(\phi u(t_0))\|_{L^{10}(I;L^{10})} \leq \|U(\cdot - t_0)u(t_0)\|_{L^{10}(I;L^{10})} \|\hat{\phi}\|_1 \\
\leq C\|U(\cdot - t_0)u(t_0)\|_{L^{10}(I;L^{10})} \leq C\eta_t.
\]
The estimate of the linear flow allows us to solve the problem in the following set,
\[
X := \left\{ v(x,t) \left| \|v\|_{L^{10}(I;L^{10})} \leq C\eta_t, \quad \|A(\cdot)v\|_{L^{10}(I;L^{10})} \leq C, \quad A \in \{J,H\} \right. \right\},
\]
donated with the metric
\[
d(u_1,u_2) = \|u_1 - u_2\|_{L^{10}(I;L^{10})} + \max_{A \in \{J,H\}} \|A(\cdot)(u_1 - u_2)\|_{L^{10}(I;L^{10})},
\]
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We omit the proof of this part since it is routine. Once we have gotten
the solution on $I = [t_0, b]$, we extend this solution beyond $I$. As a
consequence, we are left to show a finite apriori spacetime es-
timate on $[b, \infty)$. Assuming $v$ be a finite energy solution on $[b, \infty)$, we redefine
the energy of $v$ by

$$
\tilde{E}_1(t) = \frac{1}{2} \|J(t - t_0)v(t)\|_2^2 + \frac{1}{3} \cosh^2(t - t_0)\|v(t)\|_6^6;
$$

$$
\tilde{E}_1(t) = \frac{1}{2} \|H(t - t_0)v(t)\|_2^2 + \frac{1}{3} \sinh^2(t - t_0)\|v(t)\|_6^6;
$$

Repeating the computations in Lemma 2.4, we find

$$
\frac{d\tilde{E}_1(t)}{dt} = -\frac{2}{3} \sinh 2(t - t_0)\|v(t)\|_6^6 = \frac{d\tilde{E}_2(t)}{dt}.
$$

Integrating the second half of the equation, we have

$$
\frac{1}{2} \|H(t - t_0)v(t)\|_2^2 + \frac{1}{3} \sinh^2(t - t_0)\|v(t)\|_6^6
$$

$$
= \frac{1}{2} \|xv(t_0)\|_2^2 - \frac{2}{3} \int_{t_0}^{t} \sinh(2(\tau - t_0))\|v(\tau)\|_6^6 d\tau
$$

This implies

$$
\sinh^2(t - t_0)\|v(t)\|_6^6 \leq C\|x\phi u(t_0)\|_2^2.
$$

By Hölder and direct computation, we continue to estimate the right
side as

$$
\|x\phi u(t_0)\|_2 \leq \|\chi(\frac{\cdot}{C(\eta_1)\eta_2 I^{\frac{5}{2}}})u(t_0)\|_2
$$

$$
\leq (C(\eta_1)\eta_2)^2 I^{\frac{5}{2}} \|\chi(\cdot)\|_3\|u(t_0)\|_6
$$

$$
\leq C(\eta_1)\eta_2^2 |I|.
$$

$$
\sinh^2(t - t_0) \geq |t - t_0|^2 \leq |I|^2, \quad t \geq b,
$$

hence, from (76), we have

$$
\|v(t)\|_6^6 \leq C(\eta_1)\eta_2^4 \leq \eta_2^4, \quad \forall t \geq b.
$$

(77)

On the other hand, noting on $[b, \infty), v$ satisfies,

$$
v(t) = U(t - t_0)v(t_0) - i \int_{t_0}^{t} U(t - s)v|^{4}v(s)ds,
$$

(78)
we have
\[ \|J(\cdot)v\|_{L^6([b,\infty);L^{18})} \leq C\|J(t_0)v(t_0)\|_2 + C\|J(\cdot)v|^{4}\|_{L^{3}([b,\infty);L^{18})} \]
\[ \leq C\|J(t_0)v(t_0)\|_2 + C\|\|v\|\|_{L^{\infty}([b,\infty);L^{6})}\|v\|_{L^{6}([b,\infty);L^{18})}3}\|J(\cdot)v\|_{L^{6}([b,\infty);L^{18})} \]
\[ \leq C\|J(t_0)v(t_0)\|_2 + C\eta_2 \|J(\cdot)v|^{4}\|_{L^{6}([b,\infty);L^{18})}. \]
This implies
\[ \|J(\cdot)v\|_{L^6([b,\infty);L^{18})} \leq C\|J(t_0)v(t_0)\|_2 \leq C, \]
where, \( C \) depends only on \( \Lambda_1, \Lambda_2 \). To see this, we use Lemma2.3 to expand \( J(t_0)v(t_0) \) as
\[ \phi(x)J(t_0)u(t_0) + i\cosh t_0u(t_0)\nabla\phi(x), \]
which can be easily controlled.

Combing the bounds (77) and (79) together and using interpolation, one obtains
\[ \|v\|_{L^{10}([b,\infty);L^{10})} \leq C\|\|v\|\|_{L^{\infty}([b,\infty);L^{6})}\|J(\cdot)v\|_{L^{6}([b,\infty);L^{18})} \]
\[ \leq C\eta_2^{\frac{1}{2}}. \]
This combining with some routine arguments gives Proposition7.3.

Now, we are at the position to solve the Cauchy problem (75). Before doing this, we list the estimates that follows from Proposition7.3 and the conditions on \( u \).
\[ \|w\|_{L^{10}(I;L^{10})} \leq C\eta_1, \quad \|A(\cdot)w\|_{L^{18}(I;L^{18})} \leq C; \quad \|Bw\|_{L^{18}(I;L^{18})} \leq C, \]
\[ \|v\|_{L^{10}(I;L^{10})} \leq C\eta_1, \quad \|A(\cdot)v\|_{L^{10}(I;L^{10})} \leq C; \quad \|Bv\|_{L^{10}(I;L^{10})} \leq C, \]
\( A \in \{J, H\}, \quad B \in \{i\nabla_x, x\}, \]
here, we have used the condition that \( I \in [0, \eta_1^4] \) to get the estimate on \( Bv, Bw \). The constants above depend only on \( \Lambda_1, \Lambda_2 \).

For the sake of doing perturbation analysis and applying the induction, it’s necessary to introduce the following Lemma.

**Lemma7.4.** We have that
\[ E_1(w(b)) \leq \Lambda_1 - c\eta_1^3, \quad E_2(w(b)) \leq \Lambda_2 + C\eta_1^4. \]
Proof: Noting Lemma 7.2, we need only to prove that
\[ \left| \int_{t_0}^{b} \frac{\partial}{\partial t} E_1(w(t)) dt \right| \leq C \eta_1^4, \]
\[ \left| \int_{t_0}^{b} \frac{\partial}{\partial t} E_2(w(t)) dt \right| \leq C \eta_1^4, \]

For simplicity, denote
\[ |v + w|^4 (v + w) - |v|^4 v = |w|^4 w + F(v, w), \]

hence, \( w \) satisfies the equation
\[ (i \partial_t + \frac{\Delta}{2} + \frac{|x|^2}{2}) w = |w|^4 w + F(v, w). \]

By some basic computation, one sees that
\[ \frac{\partial}{\partial t} E_2(w(t)) = 2Im \int_{\mathbb{R}^3} x \bar{w} \nabla w dx + 2Im \int_{\mathbb{R}^3} |x|^2 \bar{w} F(v, w) dx, \]

thus, we get
\[ \left| \int_{t_0}^{b} \frac{\partial}{\partial t} E_2(w(t)) dt \right| \leq 2|t_0 - b| \| xw \|_{L^\infty(I; L^2)} \| \nabla w \|_{L^\infty(I; L^2)} \\
+ C \| xw \|^2_{L^{\frac{10}{14}}(I; L^\frac{10}{14})} (||w||^4_{L^{10}(I; L^{10})} + ||v||^4_{L^{10}(I; L^{10})}) \leq C \eta_1^4. \]

To prove the increment of the \( E_1(w(t)) \) from \( t_0 \) to \( b \), we first compute directly that
\[
\frac{\partial}{\partial t} E_1(w(t)) = \frac{\partial}{\partial t} E_2(w(t)) + Re \int_{\mathbb{R}^3} F(v, w) \bar{w}_t(x) dx \\
= \frac{\partial}{\partial t} E_2(w(t)) + Im \int_{\mathbb{R}^3} F(v, w) (\frac{1}{2} \Delta \bar{w} + \frac{1}{2} |x|^2 \bar{w} - |w|^4 \bar{w} - F(v, w)) (x) dx \]

Integrating over \([t_0, b]\) and using integration by parts, one gets
\[ \left| \int_{t_0}^{b} \frac{\partial}{\partial t} E_1(w(t)) \right| \leq C \eta_1^4 + C(||\nabla w||^2_{L^{\frac{10}{14}}(I; L^{\frac{10}{14}})) + ||xw||^2_{L^{\frac{10}{14}}(I; L^{\frac{10}{14}}))} \\
\times (||v||^4_{L^{10}(I; L^{10})} + ||w||^4_{L^{10}(I; L^{10})}) + C(||v||^{10}_{L^{10}(I; L^{10})} + ||w||^{10}_{L^{10}(I; L^{10})}) \leq C \eta_1^4. \]

This ends Lemma 7.4.
Now, for the sake of convenience, we make a small adjustment such that the increment of \( E_1 \) and the decrement \( E_2 \) take the same value. More precisely, noting Lemma 7.4, we can get

\[
E_1(w(b)) \leq \Lambda_1 - C\eta_1^4, \quad E_2(w(b)) \leq \Lambda_2 + C\eta_1^4,
\]

here, the above two constants are same.

Now, we make an induction assumption in order to solve the problem (82). We assume that:

Let \( t' \in \mathbb{R} \) and \( W(t', x) \) satisfy

\[
E_1(W(t')) \leq \Lambda_1 - C\eta_1^4, \quad E_2(W(t')) \leq \Lambda_2 + C\eta_1^4.
\]

Then the Cauchy problem of (11) with prescribed data \( W(t') \) at \( t' \) is solvable on \([t' - \eta_1^4, t' + \eta_1^4]\), and the solution \( W \) satisfies

\[
\|W\|_{L^{10}([t' - \eta_1^4, t' + \eta_1^4], L^{10})} \leq C(\Lambda_1 - C\eta_1^4, \Lambda_2 + C\eta_1^4).
\]

By this assumption and Lemma 7.4, we see that the solution of

\[
\begin{cases}
iW_t + \Delta^2 W = -\frac{|x|^2}{2} W + |W|^4 W, \\
W(b) = w(b)
\end{cases}
\]

satisfies the estimate

\[
\|W\|_{L^{10}([b - \eta_1^4, b + \eta_1^4], L^{10})} \leq C(\Lambda_1 - C\eta_1^4, \Lambda_2 + C\eta_1^4) \leq C(\Lambda_1, \Lambda_2).
\]

Subtracting \( W \) from \( w \), we are left to solve the perturbation problem with respect to \( \Gamma = w - W \) on \([b, \eta_1^4]\),

\[
\begin{cases}
(i\partial_t + \Delta^2 + \frac{|x|^2}{2})\Gamma = |v + W + \Gamma|^4(v + W + \Gamma) - |v|^4v - |W|^4W, \\
\Gamma(b) = 0.
\end{cases}
\]

(82)

8 Solving the perturbation problem

Our task of this section is to solve (82) with the help of (81). To insure the smallness of the nonlinear flow, we split \([b, \eta_1^4]\) into finite subintervals such that on each subinterval, \( W \) is small, so that we can solve (82) on every subinterval. Before doing this, we re-estimate \( v \) on \([b, \infty)\).
Lemma 8.1: Excepting for (81), \(v\) satisfies
\[
\|A(\cdot)v\|_{L^\infty([b,\infty);L^\infty)} \leq c(\eta_2), \quad A \in \{J, I\}.
\]

Proof: Taking \(J\) as an example, one sees
\[
J(t)v(t) = U(t-t_0)J(t_0)v(t_0) - i \int_{t_0}^t U(t-s)J(s)|v|^4v(s)ds.
\]
For the linear term, we estimate directly. From decay estimate (13),
\[
\|U(t-t_0)J(t_0)v(t_0)\|_{L^\infty} \leq C|t-t_0|^{-\frac{3}{2}}\|J(t_0)v(t_0)\|_1,
\]
\[
\leq C|t-t_0|^{-\frac{3}{2}}(\|J(t_0)u(t_0)\|_1 + \|\cosh t_0u(t_0)\nabla_x\phi\|_1)
\]
\[
\leq C|t-t_0|^{-\frac{3}{2}}(\|\phi\|_2(\|J(t_0)u(t_0)\|_2 + \|\cosh t_0u(t_0)\|_6\|\nabla_x\phi\|_5)
\]
By noting \(\phi(x) = \chi(\frac{x}{\eta_2 C(\eta_1)|J|^\frac{1}{2}})\), one has
\[
\|U(t-t_0)J(t_0)v(t_0)\|_{L^\infty} \leq C(\eta_1)(\frac{|J|^\frac{1}{2}\eta_2}{|t-t_0|})^{\frac{3}{2}}. \tag{83}
\]
On the other hand,
\[
\|U(t-t_0)J(t_0)v(t_0)\|_{L^2} \leq \|J(t_0)v(t_0)\|_{L^2}
\]
\[
\leq \|J(t_0)u(t_0)\|_2\|\phi\|_\infty + \|\nabla\phi\|_3\cosh t_0u(t_0)\|_6
\]
\[
\leq C. \tag{84}
\]
By interpolation and (83) and (84), we have that
\[
\|U(t-t_0)J(t_0)v(t_0)\|_{L^\frac{10}{3}} \leq \|U(t-t_0)J(t_0)v(t_0)\|_{L^\infty}^{\frac{2}{3}}\|U(t-t_0)J(t_0)v(t_0)\|_2^{\frac{5}{3}}
\]
\[
\leq C(\eta_1)(\frac{|J|^\frac{1}{2}\eta_2}{|t-t_0|})^{\frac{5}{3}},
\]
and thus
\[
\|U(t-t_0)J(t_0)v(t_0)\|_{L^\frac{10}{3}([b,\infty);L^\frac{10}{3})} \leq C(\eta_1)|J|\eta_2^2 \int_b^\infty \frac{dt}{|t-t_0|^2}
\]
\[
\leq C(\eta_1)\eta_2^2. \tag{85}
\]
To estimate the nonlinear term, we denote \(t_1 = t_0 + \eta_2|I|\), and split it into two parts,
\[
\int_{t_0}^{t_1} U(t-s)J(s)|u|^4u(s)ds\|_{L^\frac{10}{3}([b,\infty);L^\frac{10}{3})} + \int_{t_1}^{\infty} U(t-s)J(s)|u|^4u(s)ds\|_{L^\frac{10}{3}([b,\infty);L^\frac{10}{3})},
\]
For the first part, we use $L^p - L^{p'}$ estimate to control it by
\[
\| \int_{t_0}^{t_1} |t - s|^{-\frac{3}{5}} \| J(s) |u|^4 u(s) \|_{L^\frac{40}{19}} \, ds \|_{L^\frac{10}{7}([b, \infty))}.
\]
Since for $s \in [t_0, t_1]$, $t > b$, $|t - s| \sim |t - t_0|$, we see the first part is smaller than
\[
C \| |t - t_0|^{-\frac{3}{5}} \int_{t_0}^{t_1} \| J(s) |v|^4 v(s) \|_{L^\frac{40}{19}} \, ds
\]
\[
\leq C |t - t_0|^{-\frac{3}{5}} \| |t - t_0|^{-\frac{3}{5}} \int_{t_0}^{t_1} \| J(s) |v|^4 v(s) \|_{L^\frac{40}{19}} \, ds
\]
\[
\leq C \eta_2^{\frac{10}{7}}.
\]
For the second part, we use Strichartz estimate to control it by
\[
C \| J(\cdot) |u|^4 u \|_{L^\frac{40}{19}([t_1, \infty))} \leq C \| \|v\|_{L^\frac{10}{7}([t_1, t_1]; L^\frac{10}{7})} \| \| J(\cdot) v \|_{L^\frac{10}{7}([t_1, t_1]; L^\frac{10}{7})}.
\]
At this moment, we follow the same way in proving (80) to get
\[
\|v\|_{L^\frac{10}{7}([t_1, \infty); L^\frac{10}{7})} \leq c(\eta_2),
\]
thus finally, (86) $\leq c(\eta_2)$. Hence we get Lemma 8.1.

Now we are at the position to solve the perturbation problem (82). By induction assumption, we see that there exists constant $C = C(\Lambda_1, \Lambda_2)$ such that
\[
\|W\|_{L^\frac{10}{7}([b, \eta_1^4]; L^\frac{10}{7})} \leq C,
\]
\[
\|A(\cdot) W\|_{L^\frac{10}{7}([b, \eta_1^4]; L^\frac{10}{7})} \leq C, A \subset \{J, H\}.
\]
This allows to split $[b, \eta_1^4]$ into finite subintervals
\[
[b, \eta_1^4] = \bigcup_{j=1}^{K} I_j = \bigcup_{j=1}^{K} [b_{j-1}, b_j], b_0 = b, b_K = \eta_1^4,
\]
and such that
\[
\|W\|_{L^\frac{10}{7}(I_j; L^\frac{10}{7})} \sim \nu, \quad \|A(\cdot) W\|_{L^\frac{10}{7}(I_j; L^\frac{10}{7})} \sim \varepsilon.
\]
Then
\[
K \leq \max( (\frac{C}{\nu})^\frac{10}{7}, (\frac{C}{\varepsilon})^\frac{10}{7} ).
\]
If (82) has been solved on $[b_0, b_{j-1}]$, and
\[
\|A(b_{j-1}) \Gamma(b_{j-1}) \|_2 \leq C_{j-1}^{-1} c(\eta_2)^{1 - \frac{44}{78}}.
\]
then we can solve (82) on \([b_{j-1}, b_j]\) by proving the solution map
\[
\Phi(\Gamma(t)) = U(t-b_{j-1})\Gamma(b_{j-1})-i \int_{b_{j-1}}^t U(t-s)(|v+W+\Gamma|^4(v+W+\Gamma)-|v|^4v-|W|^4W)(s) \, ds,
\]
is contractive on the closed set
\[
X := \left\{ \Gamma \in L^{10}(I_j; L^{10}), A(\cdot)\Gamma \in L^{\frac{10}{3}}(I_j; L^{\frac{10}{3}}), \text{ and} \right\}
\]
\[
\|\Gamma\|_X = \|\Gamma\|_{L^{10}(I_j; L^{10})} + \max_{A \in \{J, H\}} \|A(\cdot)\Gamma\|_{L^{\frac{10}{3}}(I_j; L^{\frac{10}{3}})} \leq C^j c(\eta_2)^{1 - \frac{2j}{10}}
\]
donated with the metric
\[
d(u_1, u_2) = \|u_1 - u_2\|_X,
\]
and complete one step of iteration by estimating \(\|A(b_j)u(b_j)\|_2\) from Strichartz estimate. This is feasible since we can choose the absolute constants \(\varepsilon, \nu,\) and the constant \(c(\eta_2)\) small enough. The proof is routine and is omitted. Now we have a finite energy solution \(\Gamma\) on \([b, \eta_4^4]\) such that
\[
\|\Gamma\|_{L^{10}([b, \eta_4^4]; L^{10})} = \sum_{j=1}^K \|\Gamma\|_{L^{10}(I_j; L^{10})} \leq \sum_{j=1}^K (C^j c(\eta_2)^{1 - \frac{2j}{10}})^{10} \leq C.
\]

To conclude the proof of proposition 3.5 in case of blow up solution, we collect all the estimates to get
\[
\|u\|_{L^{10}(I(3); L^{10})} \leq \|u\|_{L^{10}([b, \eta_4^4]; L^{10})} \leq \|v\|_{L^{10}([b, \eta_4^4]; L^{10})} + \|W\|_{L^{10}([b, \eta_4^4]; L^{10})} + \|\Gamma\|_{L^{10}([b, \eta_4^4]; L^{10})} \leq C(\Lambda_1 \Lambda_2, \eta_1, \eta_2).
\]

Thus, \(J_1\) can be controlled by
\[
O\left(\frac{C(\Lambda_1 \Lambda_2, \eta_1, \eta_2, \eta_3)}{\eta_1}\right)^{10}.
\]
In the same way, \(J_2\) also be controlled, and thus
\[
\|u\|_{L^{10}([-\eta_4^4, \eta_4^4]; L^{10})} \leq C(\Lambda_1, \Lambda_2, \eta_1, \eta_2, \eta_3).
\]
which closes the induction and finally gives proposition 3.5.

Finally, we give some comments about this paper. In this paper, we consider the energy critical Schrödinger equation with repulsive harmonic potential which is quite different from the one without potential: We have no positive conserved quantity, but decay quantities which are not time-translation invariant. We solve these difficulties by first using the decay estimates to reduce the global problem to a problem on finite time interval, then completing the analysis by doing induction on a very small interval.

Now, let’s introduce some open problem left by this paper. One remaining problem is to generalize the result to the higher dimensional case, which is hopeful in view of the recent work in [13] and will be discussed elsewhere. Another interesting problem is how to remove the radial assumption. Because in this case, the equation is not scaling invariance, there is no hope to follow the same method in [7]. There are some other challenging problems concerning the energy critical equation with focusing nonlinearity and repulsive potential, or defocusing nonlinearity and attractive potential, which remains completely open.

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References

[1] J. Bourgain, Global well-posedness of defocusing 3D critical NLS in the radial case, JAMS12(1999), 145-171.

[2] R. Carles, Nonlinear Schrödinger equations with repulsive harmonic potential and applications, SIAM J.Math.Anal.35(2003), no.4, 823-843.

[3] R. Carles, Global existence results for nonlinear Schrödinger equations with quadratic potentials, From Homepage of R. Carles.

[4] T. Cazenave, An introduction to nonlinear Schrödinger equations, Textos de Métodos Matemáticos 26, Instituto de Matemática UFRJ, 1996.

[5] T. Cazenave, F.B. Weissler, Critical nonlinear Schrödinger Equation, Non. Anal. TMA, 14(1990), 807-836.
[6] M.Christ, J.Colliander, T.Tao, Ill-posedness for nonlinear Schrödinger and wave equation \url{http://arxiv:math.AP/0311048v1}.

[7] J. Collander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Global well-posedness and Scattering for the energy-critical nonlinear Schrödinger equation, \url{http://arxiv:math.AP/0402129v1}, 2004.

[8] R.P. Feynman and A.R. Hibbs, Quantum mechanics and path integrals (International series in Pure and Applied Physics), Maidenhead, Berksh.: McGraw-Hill Publishing Company, Ltd., 365, 1965.

[9] R.T. Glassey, On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations, J. Math. Phys. 18(1977), 1794-1797.

[10] J.Ginibre, G.Velo, Scattering theory in the energy space for a class of nonlinear Schrödinger equations, J. Math. Pure. Appl. 64(1985), 363-401.

[11] M. Grillakis, On nonlinear Schrödinger equations Comm. Partial Differential Equations 25(2000), no 9-10, 1827-1844.

[12] M. Grillakis, Regularity and asymptotic behavior of the wave equation with a critical nonlinearity, Ann. of Math. 132(1990), 485-509.

[13] T.Tao, Global well-posedness and scattering for the higher-dimensional energy-critical nonlinear Schrödinger equation for radial data, \url{http://arxiv:math.AP/0402130v1}.