Kähler structures on $T^*G$ having as underlying symplectic form the standard one

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Abstract

For a connected Lie group $G$, we show that a complex structure on the total space $TG$ of the tangent bundle of $G$ that is left invariant and has the property that each left translation $G$-orbit is a totally real submanifold is induced from a smooth immersion of $TG$ into the complexification $G^\mathbb{C}$ of $G$. For $G$ compact and connected, we then characterize left invariant and biinvariant complex structures on the total space $T^*G$ of the cotangent bundle of $G$ which combine with the tautological symplectic structure to a Kähler structure.

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1 Introduction

In [HRS09], the first named author has developed, in collaboration with two physicists, a gauge model for quantum mechanics on a stratified space. The underlying unreduced phase space is the total space $T^*G$ of the cotangent bundle of a compact connected Lie group $G$, endowed with the tautological symplectic structure, and the reduced phase space is the singular symplectic quotient of $T^*G$ with respect to conjugation. The standard identification of $T^*G$ with the complexification $G^\mathbb{C}$ of $G$ via a choice of invariant inner product on the Lie algebra $\mathfrak{g}$ of $G$ and the standard polar decomposition map from $TG \cong G \times \mathfrak{g}$ to $G^\mathbb{C}$ turns $T^*G$ into a Kähler manifold, the Kähler structure being $G$-biinvariant. We refer to this structure as the standard structure. At the reduced level, the gauge model in [HRS09] is built on the associated singular Kähler quotient. The complex structure on $T^*G$ resulting from the identification with $G^\mathbb{C}$ has no interpretation in physics, and the question arises as to what extent the physical interpretation depends on the choice of complex structure. To attack this question, as a preliminary step, in the present paper, we classify all left invariant and all biinvariant complex structures on $T^*G$ which combine with the tautological symplectic structure to a Kähler structure. To this end, we elaborate on an approach in [Bie03] aimed at describing Kähler structures on a space of the kind $T^*G$ and at exploring their Ricci curvatures. In a sense, we globalization some of the results in [Bie03]. More precisely, we show that, given a connection 1-form and a horizontal 1-form as in Proposition 3.1 of [Bie03], when these forms satisfy the integrability conditions spelled out in that Proposition and hence determine a complex structure on $P = G \times \mathfrak{g} \cong TG$, this complex structure on $P$ is actually induced from a smooth immersion $P \to G^\mathbb{C}$. A precise statement is given as Theorem 3.1 below. In Theorem 3.2 we will, furthermore, give a criterion which characterizes those complex structures $J$ on $P$ which are $G$-biinvariant. In Theorem 5.1 for $G$ compact and connected, we then single out those complex structures on $T^*G$ which combine with the
tautological symplectic structure to a Kähler structure. In Subsection 5.3, we illustrate our approach with a class of examples more general than the standard structure on $T^*G$.

This paper is based on the second-named author’s doctoral dissertation to be submitted in partial fulfillment of the requirements for the PhD degree at the university Lille 1.

2 Gauge theory with structure group acting from the left

In the standard setup, cf. e. g. [KN63], the structure group acts on the total space of a principal bundle from the right. Below we will work with principal bundles having structure group acting from the left. For ease of exposition, and to introduce notation, we briefly explain the requisite formalism.

Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra of left invariant vector fields, the Lie bracket being written as $[\cdot, \cdot]$. We denote the Lie algebra of right invariant vector fields on $G$ by $\mathfrak{g}^r$, with Lie bracket $[\cdot, \cdot]^r : \mathfrak{g}^r \times \mathfrak{g}^r \to \mathfrak{g}$. When we identify $\mathfrak{g}$ with $\mathfrak{g}^r$ as vector spaces via the canonical identifications with the tangent space $T_eG$ to $G$ at the identity element $e$ of $G$, the bracket $[\cdot, \cdot]^r$ gets identified with the negative of $[\cdot, \cdot]$.

Given a smooth manifold $M$ and a vector space $V$, let $\mathcal{A}(M,V)$ denote the graded vector space of $V$-valued differential forms on $M$. We denote the de Rham operator by $d$. Let $\xi : P \to M$ be a principal $G$-bundle having the structure group $G$ acting on $P$ from the left, let $V$ be a right $G$-module, write $\mathcal{A}_{\text{basic}}(P,V) \subseteq \mathcal{A}(P,V)$ for the graded vector space of basic $V$-valued differential forms on $P$ and, with an abuse of notation, write the induced infinitesimal $\mathfrak{g}$-action on $V$ from (beware) the left as $[\cdot, \cdot]^r : \mathfrak{g} \times V \to V$. This infinitesimal action induces the pairing

$$[\cdot, \cdot]^r : \mathcal{A}(G,\mathfrak{g}) \times \mathcal{A}(P,V) \longrightarrow \mathcal{A}(P,V). \quad (2.1)$$

The pairing (2.1) is, in particular, defined on $\mathcal{A}(G,\mathfrak{g})$, and the right invariant Maurer-Cartan form $\varpi_G : T^*G \to \mathfrak{g}$ of $G$ satisfies the Maurer-Cartan equation or structure equation

$$d\varpi_G + \frac{1}{2}[\varpi_G,\varpi_G]^r = 0. \quad (2.2)$$

A connection form for $\xi$ is a $\mathfrak{g}$-valued 1-form $\theta : TP \to \mathfrak{g}$ which, on the vertical part of $TP$, restricts to the obvious extension of the right invariant Maurer-Cartan form $\varpi_G$ and which is $G$-equivariant in the sense that

$$\theta(xY) = \text{Ad}_x(Y),$$

for any tangent vector $Y$ to $P$ and $x \in G$. The curvature of $\theta$ is then given by the (familiar) expression

$$d\theta + \frac{1}{2}[\theta, \theta]^r \in \mathcal{A}^2(P,\mathfrak{g}), \quad (2.3)$$
necessarily a basic \( \mathfrak{g} \)-valued 2-form on \( P \). On the basic forms \( A_{\text{basic}}(P,V) \), the operator \( d^\theta \) of covariant derivative is given by

\[
d^\theta = d + [\theta, \cdot]^{-} : A_{\text{basic}}(P,V) \to A_{\text{basic}}(P,V),
\]

where the notation \([\cdot, \cdot]^-\) is slightly abused. Notice that the values of the sum \( d + [\theta, \cdot]^- \) (restricted to \( A_{\text{basic}}(P,V) \)) lie in \( A_{\text{basic}}(P,V) \) but not necessarily the values of the individual operators \( d \) or \([\theta, \cdot]^-\).

\section{Invariant complex structures on the total space of the tangent bundle of a Lie group}

\subsection{Left invariant complex structures}

We write the tangent bundle of a smooth manifold \( M \) as \( \tau_M : TM \to M \). Let \( G \) be a connected Lie group. Henceforth the terms left translation and right translation mean left translation and right translation, respectively, with respect to members of \( G \). We will say that a complex structure \( J \) on the total space \( TG \) of the tangent bundle of \( G \) is admissible when \( J \) is left invariant and when each left translation \( G \)-orbit is a totally real submanifold. Our aim is to explore such admissible complex structures.

Let \( G^C \) be the complexification of \( G \) \cite{Hoc66}; we denote the image of \( G \) in \( G^C \) by \( \mathcal{G} \). When \( G \) is compact—our main case of interest—the canonical homomorphism from \( G \) to \( G^C \) is injective, and we can identify \( G \) with \( \mathcal{G} \). For general \( G \), a left translation equivariant immersion of \( TG \) into \( G^C \), necessarily onto an open subset, plainly induces a left translation invariant complex structure on \( TG \), and when the immersion is also right translation equivariant, the complex structure on \( TG \) is biinvariant. Theorem 3.1 below says that any left translation invariant complex structure on \( TG \) arises in this manner.

For the rest of the paper, it will be convenient to trivialize the tangent bundle of \( G \) and to play down the linear structure of the fibers. When we view the Lie algebra \( \mathfrak{g} \) (of left-invariant vector fields on \( G \)) merely as an affine manifold, we write it as \( \mathcal{A}_\mathfrak{g} \). Left translation yields the familiar \( G \)-equivariant diffeomorphism

\[
G \times \mathcal{A}_\mathfrak{g} \to TG,
\]

the left \( G \)-action on the factor \( \mathcal{A}_\mathfrak{g} \) being trivial, and we will exclusively work with \( G \times \mathcal{A}_\mathfrak{g} \) (rather than with \( TG \)). Accordingly we will say that a complex structure \( J \) on \( G \times \mathcal{A}_\mathfrak{g} \) is admissible when \( J \) is left \( G \)-invariant and when each left translation \( G \)-orbit is a totally real submanifold; we will then say that \( J \) is an admissible almost complex structure when \( J \) is not required to satisfy the integrability condition. Notice that \( \mathcal{A}_\mathfrak{g} \) is also \( G \)-equivariant relative to right translation when the \( G \)-action on \( G \times \mathcal{A}_\mathfrak{g} \) from the right is given by
right translation in $G$ and the adjoint action on $A_g$. Notice also that there is an obvious bijective correspondence between admissible (almost) complex structures on $TG$ and on $G \times A_g$.

Let $\gamma: A_g \rightarrow G^C$ be a smooth map having the property that the composite

$$A_g \xrightarrow{\gamma} G^C \xrightarrow{\pi} G \backslash G^C$$  \hspace{1cm} (3.2)

is a smooth map of maximal rank; the domain and range of $\gamma$ being smooth manifolds of the same dimension, $\gamma$ is necessarily an immersion and, furthermore, a submersion onto an open subset of $G \backslash G^C$. Then the map

$$\Pi_\gamma: G \times A_g \rightarrow G^C, \ (x, a) \mapsto x\gamma(a),$$  \hspace{1cm} (3.3)

is a left translation equivariant smooth immersion onto an open subset of $G^C$. Hence the complex structure of $G^C$ induces an admissible complex structure $J_\gamma$ on $G \times A_g$. We will refer to (3.3) as the generalized polar map associated to $\gamma$. Notice when $\gamma$ is equivariant with respect to the adjoint action and $G$-conjugation in $G^C$, the complex structure $J_\gamma$ is also right translation invariant and hence biinvariant.

**Theorem 3.1.** Let $J$ be an admissible complex structure on $G \times A_g$. There is a smooth map $\gamma_J: A_g \rightarrow G^C$, unique up to right multiplication by a constant member of $G^C$, such that the associated generalized polar map (3.3) is a left translation equivariant holomorphic immersion onto an open subset of $G^C$, and the composite $\pi \circ \gamma: A_g \rightarrow G \backslash G^C$ is necessarily an immersion. In particular, when $\gamma_J$ is $G$-equivariant relative to the adjoint action and $G$-conjugation in $G^C$, the map $\Pi_{\gamma_J}$ is also right translation equivariant, and the complex structure $J$ is then right invariant as well and hence biinvariant.

Under the circumstances of Theorem 3.1 we will say that $\gamma_J$ is an admissible map inducing $J$. Notice we do not assert that a biinvariant admissible complex structure $J$ on $G \times A_g$ is induced from a $G$-equivariant map $\gamma_J: A_g \rightarrow G^C$. In the next subsection we shall explain how a general biinvariant complex structure arises.

### 3.2 Biinvariant complex structures

The description of biinvariant complex structures on $G \times A_g \cong TG$ is more subtle. To prepare for it, let $G$ be a group, $H \subseteq G$ a subgroup, and $B$ a simply connected (left) $H$-manifold. The group $H$ acts on $\text{Map}(B, G)$ by the association

$$H \times \text{Map}(B, G) \rightarrow \text{Map}(B, G), \ (x, \gamma) \mapsto x\gamma,$$

given by the explicit expression

$$x\gamma: B \rightarrow G, \ x\gamma(b) = x\gamma(x^{-1}b)x^{-1}, \ x \in H, \ \gamma: B \rightarrow G.$$
Thus a smooth map $\gamma : B \to G$ is $H$-equivariant if and only if $\gamma$ is fixed under the $H$-action on $\text{Map}(B, G)$. We will say that a smooth map $\gamma : B \to G$ is quasi $H$-equivariant when there is a smooth map $c : H \to G$ such that

$$\gamma^{-1}(b) x \gamma(b) = c(x), \quad x \in H, \ b \in B. \quad (3.4)$$

Thus a smooth map $\gamma : B \to G$ is $H$-equivariant if and only if it is quasi $H$-equivariant relative to the constant smooth map $c : H \to G$ where $c(x) = e$ as $x$ ranges over $H$. Accordingly, we will say that a smooth left equivariant map $\Pi : G \times \mathbb{A}_g \to G^C$ is quasi right $G$-equivariant when there is a smooth map $c : G \to G^C$ such that

$$\Pi((x, Y)z) = \Pi(x, Y)c(z), \ x \in G, \ Y \in \mathbb{A}_g, \ z \in G. \quad (3.5)$$

In particular, a smooth left equivariant map $\Pi : G \times \mathbb{A}_g \to G^C$ is biinvariant if and only if it is quasi right $G$-equivariant relative to the constant smooth map $c : G \to G^C$ where $c(x) = e$ as $x$ ranges over $G$.

**Theorem 3.2.** Under the circumstances of Theorem 3.1, the left invariant complex structure $J$ on $G \times \mathbb{A}_g$ is biinvariant if and only if the smooth map $\gamma_J : \mathbb{A}_g \to G^C$ is, furthermore, quasi $G$-equivariant.

### 3.3 The standard structure

An example of a generalized polar map is the ordinary polar map. With the notation

$$\gamma_{st} : \mathbb{A}_g \to G^C, \quad \gamma_{st}(a) = \exp(ia), \ a \in \mathbb{A}_g, \quad (3.6)$$

the ordinary polar map takes the form

$$\Pi = \Pi_{st} : G \times \mathbb{A}_g \to G^C, \quad (x, a) \mapsto x\gamma_{st}(a), \ x \in G, \ a \in \mathbb{A}_g. \quad (3.7)$$

For $G$ compact, this map is a diffeomorphism and thus plainly induces an admissible complex structure on $G \times \mathbb{A}_g$; we refer to this structure as the standard complex structure on $G \times \mathbb{A}_g$ and denote it by $J_{st}$.

For general $G$, in view of the classical expression for the derivative of the exponential mapping, cf. e. g. [Hel84] (II.1.7), in terms of the $g^C$-valued 1-form

$$\phi_{st} = (d\gamma_{st})\gamma_{st}^{-1} : T\mathbb{A}_g \to g \oplus i g,$$

at $a \in \mathbb{A}_g$, the derivative

$$(d\phi_{st})_a : T_a\mathbb{A}_g \cong g \to T_{\exp(ia)}G^C \to T_eG^C = g \oplus i g$$

is given by the association

$$V \mapsto \frac{\cos(\text{ad}(a)) - \text{Id}}{\text{ad}(a)}(V) + i\frac{\sin(\text{ad}(a))}{\text{ad}(a)}(V), \ V \in g. \quad (3.8)$$
When $G$ is not compact, in general, the canonical map $G \to G^C$ is not injective, and the projection $G \to \overline{G}$ to the image $\overline{G}$ of $G$ in $G^C$ is a covering projection. Moreover, even when the complexification map $G \to G^C$ is injective (so that $G \to \overline{G}$ identifies the two groups), the topology of $\overline{G} \setminus G^C$ is in general non-trivial. This happens, for example, when $G = \text{SL}(2, \mathbb{R})$.

Furthermore, in general, the ordinary polar map $\Pi_{st}$ is not even a local diffeomorphism: Indeed, in view of (3.8), at $a \in \mathbb{A}_g$, the derivative of (3.2) is given by the association

$$T_{\mathbb{A}_g} \cong \mathfrak{g} \to \mathfrak{g}, \ V \mapsto \frac{\sin(\text{ad}(a))}{\text{ad}(a)}(V), \ V \in \mathfrak{g}. \quad (3.9)$$

Hence (3.2) is a local diffeomorphism at $a \in \mathbb{A}_g$ if and only if the linear endomorphism $\frac{\sin(\text{ad}(a))}{\text{ad}(a)}$ of $\mathfrak{g}$ is invertible. This explains why, for non-compact semisimple $G$, the adapted complex structure is not defined on all of $T\mathbb{A}_g$; see e. g. [Sző04] for details.

### 3.4 The infinitesimal version

We view the left invariant Maurer-Cartan form of $G$ as a trivializable principal left $G$-bundle $\omega_G: TG \to \mathbb{A}_g$. Via the trivialization (3.1), this bundle comes down to the trivial principal left $G$-bundle $\text{pr}_{\mathbb{A}_g}: G \times \mathbb{A}_g \to \mathbb{A}_g$. For better readability, in the present subsection, we will write the total space $G \times \mathbb{A}_g$ as $P$. We denote the resulting foliation of $P$ given by the fibers of $\omega_G$ by $\mathcal{F}_G$, and we write the tangent bundle of $\mathcal{F}_G$ as $\tau_{\mathcal{F}_G}: TP \to P$. The vector bundle $\tau_{\mathcal{F}_G}$ is the vertical subbundle of the tangent bundle $\tau_P$ of $P$ with respect to principal left $G$-structure of $P$.

The fundamental vector field map $\mathfrak{g} \times P \to TP$ identifies the trivial vector bundle $\text{pr}_P: \mathfrak{g} \times P \to P$ on $P$ with the vertical subbundle $\tau_{\mathcal{F}_G}: TP \to P$ of $\tau_P: TP \to P$. Likewise, the obvious map is a diffeomorphism

$$G \times T\mathbb{A}_g \to P \times_{\mathbb{A}_g} T\mathbb{A}_g.$$  

Consequently the fundamental exact vector bundle sequence associated to the principal left $G$-bundle $\text{pr}_{\mathbb{A}_g}: P \to \mathbb{A}_g$ takes the form

$$0 \longrightarrow \mathfrak{g} \times P \longrightarrow TP \longrightarrow G \times T\mathbb{A}_g \longrightarrow 0. \quad (3.10)$$

The projection $\text{pr}_{\mathbb{A}_g}$ induces an isomorphism

$$\mathcal{A}(\mathbb{A}_g, \mathfrak{g}) \rightarrow \mathcal{A}_{\text{basic}}(P, \mathfrak{g}) = \mathcal{A}_{\text{basic}}(G \times \mathbb{A}_g, \mathfrak{g}) \quad (3.11)$$

of graded vector spaces onto the graded vector space of basic forms. We will say that a basic $\mathfrak{g}$-valued 1-form on $P = G \times \mathbb{A}_g$ is regular when the associated $\mathfrak{g}$-valued 1-form on $\mathbb{A}_g$ has maximal rank, that is, is an isomorphism $T_a\mathbb{A}_g \to \mathfrak{g}$ of real vector spaces for every point $a$ of $\mathbb{A}_g$. The following observation is essentially Proposition 3.1 in [Bie03], and we refer to that paper for a proof. We reproduce the wording here to introduce notation.
Proposition 3.3. (i) Let $J$ be an admissible almost complex structure on $P = G \times A_\mathfrak{g}$ and let $\tau_P = \tau_{F_G} \oplus J(\tau_{F_G})$ be the associated Whitney sum decomposition of the tangent bundle $\tau_P: TP \to P$ of $P$ as a sum of the vertical subbundle $\tau_{F_G}$ and $J(\tau_{F_G}): J(\tau_{F_G}) \to P$. Then the projection

$$TP = TF_G \oplus J(TF_G) \to TF_G,$$

combined with the projection $TF_G \cong \mathfrak{g} \times P \to \mathfrak{g}$, yields a principal left $G$-connection form $\theta_J : TP \to \mathfrak{g}$ for $\text{pr}_{A_\mathfrak{g}}$, and the $\mathfrak{g}$-valued 1-form

$$L_J = -\theta_J \circ J : TP \to \mathfrak{g}$$

(3.12)

is basic and regular.

(ii) Conversely, a left $G$-connection form $\theta : TP \to \mathfrak{g}$ for $\text{pr}_{A_\mathfrak{g}}$ and a basic regular $\mathfrak{g}$-valued 1-form $L : TP \to \mathfrak{g}$ determine a unique admissible almost complex structure $J$ on $P = G \times A_\mathfrak{g}$ such that $\theta = \theta_J$ and $L = L_J$.

(iii) Under the circumstances of (i) or (ii) above, the almost complex structure $J$ is integrable if and only if

$$F_{\theta_J} = L_J \Lambda L_J = \frac{1}{2}[L_J, L_J]^- \in A^2(TP, \mathfrak{g}),$$

(3.13)

$$d^{\theta_J} L_J = 0.$$  

(3.14)

Lemma 3.4.

(i) Let $J$ be an admissible almost complex structure on $G \times A_\mathfrak{g}$, let $(\theta, L)$ be the pair of $\mathfrak{g}$-valued 1-forms on $G \times A_\mathfrak{g}$ associated to $J$ by the construction in Proposition 3.3. Let $c_\theta : TA_\mathfrak{g} \to \mathfrak{g}$ denote the $\mathfrak{g}$-valued 1-form on $A_\mathfrak{g}$ whose extension $c_\theta^G : G \times TA_\mathfrak{g} \to \mathfrak{g}$ to $G \times TA_\mathfrak{g}$ yields, via the projection from $T(G \times A_\mathfrak{g})$ to $G \times TA_\mathfrak{g}$, a uniquely determined basic form $c_\theta^G$ on $G \times A_\mathfrak{g}$ so that

$$\theta = \theta_G + c_\theta^G.$$  

(3.15)

Likewise let $s_L : TA_\mathfrak{g} \to \mathfrak{g}$ be the $\mathfrak{g}$-valued 1-form on $A_\mathfrak{g}$ of maximal rank associated to the basic 1-form $L$ through the identification (3.11).

(ii) Let $J$ be an admissible almost complex structure on $G \times A_\mathfrak{g}$, let $(\theta, L)$ be the pair of $\mathfrak{g}$-valued 1-forms on $G \times A_\mathfrak{g}$ associated to $J$ by the construction in Proposition 3.3 and let $(c_\theta, s_L)$ be the associated pair of 1-forms on $A_\mathfrak{g}$. The $\mathfrak{g}^\mathbb{C}$-valued 1-form

$$\phi_J = c_\theta + is_L : TA_\mathfrak{g} \to \mathfrak{g}^\mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$$

(3.16)
on \(A_g\) has the property that its imaginary part \(TA_g \to \mathfrak{g}\) (component of \(\phi_J\) into \(i\mathfrak{g}\)) has maximal rank.

(ii) Every \(\mathfrak{g}^C\)-valued 1-form on \(A_g\) whose imaginary part has maximal rank arises in this manner from an almost complex structure \(J\) on \(G \times A_g\) of the kind spelled out in (i).

(iii) Under the circumstances of (i) or (ii), the almost complex structure \(J\) on \(G \times A_g\) is integrable if and only if \(\phi_J\) satisfies the integrability condition

\[
d\phi_J + \phi_J \wedge \phi_J \in \mathcal{A}^2(\mathbb{H}, \mathfrak{g}).
\]

**Proof.** The pair \((\theta, L)\) satisfies the integrability conditions (3.13) and (3.14) if and only if \(\phi_J\) satisfies the integrability condition

\[
d\phi_J + \phi_J \wedge \phi_J \in \mathcal{A}^2(\mathbb{H}, \mathfrak{g}).
\]

For illustration, and to introduce notation, suppose that \(G\) is compact (and connected), and let \((\theta_{st}, L_{st})\) be the pair arising from the standard complex structure \(J_{st}\) on \(G\); in terms of the obvious trivialization \(TA_g \cong A_g \times \mathfrak{g}\), the associated \(\mathfrak{g}^C\)-valued 1-form \(\phi_{st}: TA_g \to \mathfrak{g}^C\) on \(A_g\), cf. (3.16) and (3.8) above, is given by

\[
\varphi_{st} = (d\gamma_{st})^{-1} = c_{st} + is_{st},
\]

\[
(c_{st})_a(V) = \frac{\cos(\text{ad}(a)) - \text{Id}}{\text{ad}(a)}(V) = \sum_{j=1}^{\infty} \frac{(-1)^j}{(2j)!} \text{ad}^{2j-1}(a)(V),
\]

\[
(s_{st})_a(V) = \frac{\sin(\text{ad}(a))}{\text{ad}(a)}(V) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} \text{ad}^{2j}(a)(V),
\]

where \(a \in A_g\), \(V \in \mathfrak{g}\). Consequently, given a point \(a\) of \(A_g\) and, furthermore, \(X \in T_aG = \mathfrak{g} = \mathfrak{h}\) (identifications as real vector spaces) and \(V \in T_aA_g \cong \mathfrak{g} = \mathfrak{h}\), the values \(\theta_{st}(X, V)\) and \(L_{st}(X, V)\) are given by

\[
(\theta_{st})_{(e, a)}(X, V) = X + \frac{\cos(\text{ad}(a)) - \text{Id}}{\text{ad}(a)}(V),
\]

\[
(L_{st})_{(e, a)}(X, V) = \frac{\sin(\text{ad}(a))}{\text{ad}(a)}(V).
\]

These are exactly the expressions given as (3.5) and (3.6) in [Bie03].

### 3.6 Proof of Theorem 3.1

Let \(J\) be an admissible almost complex structure on \(P = G \times A_g\). Let \(\phi_J \in \mathcal{A}^1(\mathbb{H}, \mathfrak{g}^C)\) be the associated \(\mathfrak{g}^C\)-valued 1-form, cf. Lemma 3.4 and suppose
that $\phi_J$ satisfies the integrability condition (3.17). Then $\phi_J$ integrates to a smooth map $\gamma_J: \mathbb{A}_g \to G^C$ so that
\[
\phi_J = (d\gamma_J)\gamma_J^{-1}.
\] (3.20)

The map $\gamma_J$ is unique up to a constant in $G^C$, that is, given $\gamma_J^1$ and $\gamma_J^2$ satisfying (3.20), $\gamma_J^2 = \gamma_J^1 c$ for some $c \in G^C$. By construction, the associated generalized polar map (3.3) is holomorphic. Moreover, the composite (3.2) is a smooth map of maximal rank between smooth manifolds of the same dimension and hence an immersion. Consequently $\Pi_{\gamma_J}$ is as well an immersion.

### 3.7 Proof of Theorem 3.2

For clarity, we will momentarily proceed under somewhat more general circumstances than actually needed for the proof of the theorem. Thus, as before, consider a group $G$, a subgroup $H \subseteq G$, and a simply connected (left) $H$-manifold $B$.

**Lemma 3.5.** Given a smooth $G$-valued map $\gamma: B \to G$, the associated $1$-form $(d\gamma)\gamma^{-1} \in \mathfrak{A}^1(B, \mathfrak{g})$ is $H$-equivariant if and only if $\gamma$ is quasi $H$-equivariant, that is, if and only if there is a smooth $G$-valued function $c: H \to G$ on $H$ such that
\[
(\gamma^{-1}(b)x\gamma(b)) = c(x), \quad x \in H, \quad b \in B.
\]

**Proof.** This is routine and left to the reader. \hfill $\Box$

**Corollary 3.6.** Let $\beta \in \mathfrak{A}^1(B, \mathfrak{g})$ be an $H$-equivariant $1$-form that satisfies the integrability condition $d\beta + \beta \wedge -\beta = 0$, and let $\gamma: B \to G$ integrate $\beta$ in the sense that $\beta = (d\gamma)\gamma^{-1}$. Then there is a smooth $G$-valued function $c: B \to G$ on $B$ such that
\[
\gamma(xb) = x\gamma(b)c(x^{-1})x^{-1}, \quad x \in H, \quad b \in B.
\] (3.21)

Now we prove Theorem 3.2. Let $J$ be a biinvariant admissible complex structure on $G \times \mathbb{A}_g$. View $J$ merely as a left translation invariant complex structure, let $(\theta, L)$ denote the associated pair of $\mathfrak{g}$-valued $1$-forms, let
\[
\phi_J = \psi_\theta + i\lambda_L: T\mathbb{A}_g \to \mathfrak{g}^C = \mathfrak{g} \oplus i\mathfrak{g}
\] (3.22)
be the resulting integrable $\mathfrak{g}^C$-valued $1$-form on $\mathbb{A}_g$, cf. (3.16), and let $\gamma_J: \mathbb{A}_g \to G^C$ be a smooth map that integrates $\phi_J$; thus $\phi_J$ equals $(d\gamma_J)\gamma_J^{-1}$.
By construction, the associated generalized polar map (3.3) is holomorphic, and the composite

\[
\mathbb{A}_g \xrightarrow{\gamma_J} G^\mathbb{C} \xrightarrow{\pi} G \backslash G^\mathbb{C}
\]

is a smooth map of maximal rank.

Since \( J \) is as well right translation invariant, so is \( \phi_J \). By Corollary 3.6 there is a smooth \( G^\mathbb{C} \)-valued function \( c: G \rightarrow G^\mathbb{C} \) such that

\[
\gamma_J(zYz^{-1}) = z\gamma_J(Y)c(z^{-1})z^{-1}, \quad z \in G, \quad Y \in \mathbb{A}_g.
\]

Consequently

\[
\Pi_{\gamma_J}((x,Y)z) = (\Pi_{\gamma_J}(x,Y))c(z)z.
\]

3.8 Explicit description of the almost complex structure

The constructions being left \( G \)-invariant, it suffices to spell out an explicit expression for the (almost) complex structure \( J \) on \( G \times \mathbb{A}_g \) in Proposition 3.3 in terms of the two 1-forms \( c_\theta \) and \( s_L \) at the points of \( G \times \mathbb{A}_g \) of the kind \((e,a)\) as \( a \) ranges over \( \mathbb{A}_g \).

**Proposition 3.7.** Let \( c: T\mathbb{A}_g \rightarrow \mathfrak{g} \) and \( s: T\mathbb{A}_g \rightarrow \mathfrak{g} \) be two \( \mathfrak{g} \)-valued 1-forms on \( \mathbb{A}_g \), the 1-form \( s \) being of maximal rank. For each \( a \in \mathbb{A}_g \), the expression

\[
J_{(e,a)}(u,v) = (-s_a(v) - c_a s_a^{-1}(u + c_a(v)), s_a^{-1}(u + c_a(v))),(3.26)
\]

as \( u \) ranges over \( T_eG = \mathfrak{g} \) and \( v \) over \( T_a\mathbb{A}_g \cong \mathfrak{g} \), yields a complex structure \( J(e,a) \) on the tangent space \( T_{(e,a)}(G \times \mathbb{A}_g) \) to \( G \times \mathbb{A}_g \) at the point \((e,a)\) of \( G \times \mathbb{A}_g \), and \( G \)-left translation then yields an admissible almost complex structure \( J \) on \( G \times \mathbb{A}_g \). In particular, when \((c,s)\) is the pair \((c_\theta,s_L)\) of 1-forms on \( \mathbb{A}_g \) arising from a given admissible almost complex structure \( J \) on \( G \times \mathbb{A}_g \), the expression (3.26) yields \( J \) in terms of \( c_\theta \) and \( s_L \).

**Proof.** This is left to the reader. \( \square \)

4 Invariant Kähler forms

Recall that \( \theta_G: T(G \times \mathbb{A}_g) \rightarrow \mathfrak{g} \) denotes the trivial principal \( G \)-connection form (relative to the obvious principal left \( G \)-structure on \( G \times \mathbb{A}_g \)). For better readability, we continue to denote \( G \times \mathbb{A}_g \) by \( P \) whenever appropriate. Given a (left) \( G \)-manifold \( M \), we denote the fundamental vector field on \( M \) associated to \( X \in \mathfrak{g} \) by \( X_M \).

**Lemma 4.1.** Given a \( G \)-equivariant map \( \mu: G \times \mathbb{A}_g \rightarrow \mathfrak{g}^* \), this map \( \mu \) is a momentum for \( \omega = -d\langle \mu, \theta_G \rangle \) in the sense that

\[
i_{X_M} \omega = d\langle \mu, X \rangle, \quad X \in \mathfrak{g}.
\]
Proof. Let $X \in \mathfrak{g}$ and write $\Theta = \langle \mu, \theta_G \rangle$. Then

\[
0 = \mathcal{L}_{X_p}(\Theta) = (di_{X_p} + i_{X_p}d)(\Theta)
\]

\[
d(\Theta(X_p)) = i_{X_p}\omega
\]

\[
\Theta(X_p) = \langle \mu, \theta_G(X_p) \rangle = \langle \mu, \mu_G(X_p) \rangle = \langle \mu, X \rangle,
\]

whence the assertion.

The following is entirely classical.

**Proposition 4.2.** Given a hamiltonian $G$-manifold $(M, G, \omega, \mu)$ with $G$-action on $M$ from the left, \( \omega(X_M, Y_M) = \langle \mu, [X, Y] \rangle, \) \( X, Y \in \mathfrak{g} \) \tag{4.2}

For intelligibility, we will now recall from Proposition 3.3 that, given an admissible (almost) complex structure $J$ on $G \times \mathbb{A}_g$, the notation being that established in Proposition 3.3, the $\mathfrak{g}$-valued 1-form $s_J = L_J = -\theta_J \circ J : TP \to \mathfrak{g}$

is basic and regular and that

\[
\theta_J = \theta_G + c_J^G
\]

\[
dL_J = -[c_J^G, L_J] = [c_J^G, L_J].
\]

**Theorem 4.3.** Let $J$ be an admissible complex structure on $G \times \mathbb{A}_g$, let $\gamma_J : \mathbb{A}_g \to G^C$ be an admissible map inducing it, and let $s_J$ and $c_J$ be the two associated $\mathfrak{g}$-valued 1-forms on $\mathbb{A}_g$ characterized by the identity

\[
(d\gamma_J)\gamma_J^{-1} = c_J + is_J \in A^1(\mathbb{A}_g, \mathfrak{g}^C).
\]

Moreover let $\mu : G \times \mathbb{A}_g \to \mathfrak{g}^*$ be a $G$-equivariant map, and suppose that $\omega = -d(\mu, \theta_G)$ is symplectic. Then $J$ and $\omega$ combine to a pseudo Kähler structure on $G \times \mathbb{A}_g$ (necessarily having momentum mapping $\mu$) if and only if the two real 1-forms $\langle \mu, c_J \rangle$ and $\langle \mu, s_J \rangle$ on $\mathbb{A}_g$ are closed. Furthermore, the Kähler form $\omega$ is then given by the expression

\[
\omega = -d(\mu, \theta_J).
\]

Finally, when $f : G \times \mathbb{A}_g \to \mathbb{R}$ is the $G$-invariant extension of an integral of $\langle \mu, s_J \rangle$ on $\mathbb{A}_g$, i.e., when $f$ is a smooth real $G$-invariant function on $G \times \mathbb{A}_g$ whose restriction to $\{e\} \times \mathbb{A}_g$ satisfies the identity

\[
df = \langle \mu, s_J \rangle,
\]

the function $2f$ is a Kähler potential on $G \times \mathbb{A}_g$, that is, satisfies the identity

\[
\omega = 2i\partial \overline{\partial} f.
\]
Lemma 4.4. Under the hypotheses of Theorem 4.3, save that we do not assume that $\omega = -d(\mu, \theta_G)$ is non-degenerate, the following hold.

(i) The data $\theta_J$, $L_J$ and $\mu$ determine a smooth $G$-equivariant map

$$\Psi: G \times \mathbb{A}_g \to \text{Hom}(g, g^*)$$  \hspace{1cm} (4.6)

satisfying the identity

$$d^\theta \mu = \Psi \circ L_J: T(G \times \mathbb{A}_g) \longrightarrow g \longrightarrow g^*$$  \hspace{1cm} (4.7)

in the sense that, for any $(x, a) \in G \times \mathbb{A}_g$, the composite

$$T_{(x,a)}(G \times \mathbb{A}_g) \xrightarrow{(L_J)_{(x,a)}} g \xrightarrow{\Psi_{(x,a)}} g^*$$

coincides with $(d^\theta \mu)_{(x,a)}$.

(ii) Given $X, Y \in g$, $\omega(X_P, JY_P) = \langle X, \Psi(Y) \rangle$.

(iii) The map $\Psi$ is symmetric in the sense that $\langle X, \Psi(Y) \rangle = \langle Y, \Psi(X) \rangle$ for any $X, Y \in g$ if and only if the real 1-form $\langle \mu, s_J \rangle$ on $\mathbb{A}_g$ is closed.

Complement to Theorem 4.3 Under the circumstances of the theorem, the metric $g$ is given by the expression

$$g = \langle \Psi \theta, \theta \rangle + \langle L, d^\theta \mu \rangle + \langle \mu, [L, \theta] \rangle.$$  \hspace{1cm} (4.9)

Proof of Lemma 4.4. Since the $g$-valued 1-form $s_J$ has maximal rank everywhere, the data $\theta_J$, $L_J$ and $\mu$ determine a smooth map $\Psi: \mathbb{A}_g \to \text{Hom}(g, g^*)$ satisfying the identity

$$d^\theta \mu = \Psi \circ s_J: T\mathbb{A}_g \longrightarrow g \longrightarrow g^*$$  \hspace{1cm} (4.10)

in the sense that, for every point $a$ of $\mathbb{A}_g$, the composite

$$T_a\mathbb{A}_g \cong g \xrightarrow{(s_J)_a} g \xrightarrow{\Psi_a} g^*$$

coincides with $(d^\theta \mu)_a$. The map (4.6) is then the unique $G$-equivariant extension to all of $G \times \mathbb{A}_g$. This establishes (i).

Let $X, Y \in g$. By construction,

$$\Psi(X) = \Psi(L_J(JX_P)) = \Psi(\theta_J(X_P)) = d^\theta \mu(JX_P)$$

$$\omega(X_P, JY_P) = \langle X, d\mu(JY_P) \rangle = \langle X, d^\theta \mu(JY_P) \rangle$$

since $JY_P$ horizontal

$$= \langle X, \Psi(L_J(JY_P)) \rangle$$

$$= \langle X, \Psi(Y) \rangle,$$
whence (ii) holds.

Finally to prove (iii), we note first that the integrability condition \( \text{viz. } d^{\theta} L_J = 0 \), implies that

\[
d\langle \mu, L_J \rangle = \langle d^{\theta} \mu \wedge L_J \rangle. \tag{4.11}
\]

Let \( X, Y \in \mathfrak{g} \). Then

\[
\langle d^{\theta} \mu \wedge L_J \rangle(JX_P, JY_P) = \langle d^{\theta} \mu(JX_P), L_J(JY_P) \rangle - \langle d^{\theta} \mu(JY_P), L_J(JX_P) \rangle = \langle \Psi(X), Y \rangle - \langle \Psi(Y), X \rangle.
\]

Hence the closedness of \( \langle \mu, L_J \rangle \) implies the symmetry of \( \Psi \). Since the 1-form \( L_J \) is basic, it vanishes on vertical vectors whence the symmetry of \( \Psi \) implies the closedness of \( \langle \mu, L_J \rangle \).

**Remark 4.5.** Under the hypotheses of Theorem 4.3, when \( J \) and \( \omega \) combine to a pseudo Kähler structure, in view of Lemma 4.4 (ii), for any point \( (x, a) \) of \( G \times \mathbb{A}_g \), the linear map \( \Psi_{(x, a)} : \mathfrak{g} \to \mathfrak{g}^* \) is invertible, and the constituent \( \langle L, d^{\theta} \mu \rangle \) of (4.9) can be written as

\[
\langle L, d^{\theta} \mu \rangle = \langle \Psi^{-1} d^{\theta} \mu, d^{\theta} \mu \rangle. \tag{4.12}
\]

In particular, \( \langle L, d^{\theta} \mu \rangle \) is a symmetric bilinear form, and the metric (4.9) can be written as

\[
g = \langle \Psi \theta, \theta \rangle + \langle \Psi^{-1} d^{\theta} \mu, d^{\theta} \mu \rangle + \langle \mu, [L, \theta] \rangle. \tag{4.13}
\]

This is essentially the same expression as [Bie03] (4.2).

**Remark 4.6.** At every point of \( P = G \times \mathbb{A}_g \), the tangent space is spanned by vectors arising from fundamental vector fields \( X_P \) and vector fields of the kind \( JY_P \), as \( X \) and \( Y \) range over \( \mathfrak{g} \).

**Lemma 4.7.** Under the hypotheses of Theorem 4.3, when \( J \) and \( \omega \) combine to a pseudo Kähler structure, the metric \( g = \omega(\cdot, J \cdot) \) is given by the expression (4.9) and the Kähler form \( \omega \) by the expression (4.3).

**Proof.** Let \( X, Y \in \mathfrak{g} \). A straightforward calculation yields

\[
\langle \Psi \theta, \theta \rangle(X_P, Y_P) = g(X_P, Y_P)
\]

\[
\langle \mu, [L_J, \theta_J] \rangle(JX_P, Y_P) = g(JX_P, Y_P)
\]

\[
\langle \mu, [L_J, \theta_J] \rangle(JX_P, JY_P) = g(JX_P, JY_P)
\]

\[
\langle L_J, d^{\theta} \mu \rangle(JX_P, JY_P) = g(JX_P, JY_P),
\]

and evaluation of any of the three constituents on the right-hand side of (4.9) at argument pairs not already spelled out is zero. In view of Remark 4.6 this shows that the metric \( g = \omega(\cdot, J \cdot) \) is given by the expression (4.9).
Likewise, in view of (3.13), we find
\[ \omega = g(J \cdot, \cdot) = \langle \Psi \theta J, \theta \rangle + \langle L, \theta \rangle (J \cdot, \cdot) \]
\[ \langle \Psi \theta J, \theta \rangle = -\langle d^\theta \mu, \theta \rangle \]
\[ \langle L, d^\theta \mu \rangle = \langle \theta, d^\theta \mu \rangle \]
\[ \langle \mu, [L, \theta] (J \cdot, \cdot) \rangle = \langle \mu, \theta \wedge \theta - L \wedge L \rangle = -\langle \mu, d\theta \rangle \]
\[ g(J \cdot, \cdot) = -\langle \langle d^\theta \mu, \theta \rangle - \langle \theta, d^\theta \mu \rangle + \langle \mu, d\theta \rangle \rangle \]
\[ = -\langle \langle d\mu, \theta \rangle - \langle \theta, d\mu \rangle + \langle \mu, d\theta \rangle \rangle \]
\[ = -d\langle \mu, \theta \rangle. \]

Consequently the Kähler form \( \omega \) is given by the expression (4.3).

**Proof of Theorem 4.3.** The “Furthermore statement” has been established already in Lemma 4.7.

Suppose first that \( \langle \mu, c^G J \rangle \) and \( \langle \mu, L J \rangle \) are closed. We must show that \( J \) is compatible with \( \omega \). Since \( \langle \mu, c^G J \rangle \) is closed,
\[ d\langle \mu, \theta \rangle = d\langle \mu, \theta \rangle + d\langle \mu, c^G J \rangle \]
\[ = d\langle \mu, \theta \rangle - \omega, \]
that is, \( \langle \mu, \theta \rangle \) is a symplectic potential for \( \omega \).

Let \( X, Y \in g \). Since \( JX_P \) and \( JY_P \) are horizontal, since \( c^G J \) vanishes on vertical vector fields, and since \( [X_P, Y_P] = -[X, Y]_P \), we find
\[ \omega(JX_P, JY_P) = d\langle \mu, \theta \rangle (JY_P, JX_P) \]
\[ = \langle d\langle \mu, \theta \rangle (JY_P), \theta_J (JX_P) \rangle - \langle d\langle \mu, \theta \rangle (JX_P), \theta_J (JY_P) \rangle \]
\[ + \langle \mu, \theta_J \rangle [JX_P, JY_P] \]
\[ = \langle \mu, \theta_J \rangle [JX_P, JY_P] \]
\[ = -\langle \mu, \theta_J \rangle [X_P, Y_P] \]
\[ = \langle \mu, \theta_J [X, Y]_P \rangle \]
\[ = \langle \mu, [X, Y] \rangle. \]

In view of Proposition 4.2 we deduce
\[ \omega(JX_P, JY_P) = \omega(X_P, Y_P). \] (4.14)

Next, by Lemma 4.3 (ii),
\[ \omega(X_P, JY_P) = \langle X, \Psi(Y) \rangle \]
\[ \omega(Y_P, JX_P) = \langle Y, \Psi(X) \rangle, \]
and, since $\omega(JX_P, JY_P) = \omega(Y_P, JX_P)$, by Lemma 4.4 (iii),
\[
\omega(X_P, JY_P) = \omega(JX_P, JY_P),
\]
since the real 1-form $\langle \mu, L_J \rangle$ is closed. In view of Remark 4.6 these calculations show that the closedness of $\langle \mu, c^G_J \rangle$ and $\langle \mu, L_J \rangle$ implies that $J$ is compatible with $\omega$.

Conversely, suppose that $J$ is compatible with $\omega$. Then
\[
\omega(X_P, JY_P) = \omega(JX_P, JY_P)
\]
and, since $\omega(JX_P, JY_P) = \omega(Y_P, JX_P)$, we conclude
\[
\langle X, \Psi(Y) \rangle = \omega(X_P, JY_P) = \omega(Y_P, JX_P) = \langle Y, \Psi(X) \rangle,
\]
whence, by Lemma 4.4 (iii), the real 1-form $\langle \mu, L_J \rangle$ is closed. Finally, by Lemma 4.7,
\[
\omega = -d\langle \mu, \theta_G \rangle = -d(\langle \mu, L \circ J \rangle) = -d(\langle \mu, L \rangle \circ J)
\]

Since $\omega = -d\langle \mu, \theta_G \rangle$, we conclude that the real 1-form $\langle \mu, c^G_J \rangle$ is closed.

To establish the “Finally” assertion we recall that, by construction, cf. (3.12), $L = -\theta \circ J$. Using the fact that, relative to the decomposition $TM \otimes \mathbb{C} = T^{\text{hol}}M \oplus \overline{T^{\text{hol}}M}$ of the total space $TM \otimes \mathbb{C}$ of the complexified tangent bundle of $M = G \times \mathbb{A}_g$ into its holomorphic and antiholomorphic constituents, keeping in mind that, on $T^{\text{hol}}M$, the complex structure is given by multiplication by $i$ and on $\overline{T^{\text{hol}}M}$ by multiplication by $-i$, we find
\[
\omega = -d(\langle \mu, L \rangle) = -d(\langle \mu, L \circ J \rangle) = -d(\langle \mu, L \rangle \circ J)
\]
\[
= -d(\langle df, J \rangle) = -d(\langle \partial f \rangle \circ J + \langle \overline{\partial f}, J \rangle)
\]
\[
= -\langle \partial + \overline{\partial}, (i(\partial f) - i(\overline{\partial f})) \rangle = -i\partial \partial f + i\overline{\partial} \overline{\partial} f
\]
\[
= 2i\partial \partial f.
\]

5 The case when $G$ is compact

Suppose that $G$ is compact and connected. Pick an invariant inner product $\cdot : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ on $\mathfrak{g}$ and use it to identify $\mathfrak{g}$ with $\mathfrak{g}^*$. The induced biinvariant Riemannian metric on $G$ identifies $TG$ with $T^*G$ in a $G$-biequivariant manner. Let $\mu : G \times \mathbb{A}_g \rightarrow \mathfrak{g}^*$ be the $G$-equivariant map $G \times \mathbb{A}_g \rightarrow \mathfrak{g}^*$ which, restricted to $\{e\} \times \mathbb{A}_g$, is the adjointness isomorphism $\sharp : \mathfrak{g} \rightarrow \mathfrak{g}^*$ of the inner product on $\mathfrak{g}$ (the identity of $\mathfrak{g}$ when we identity $\mathfrak{g}^*$ with $\mathfrak{g}$ via $\sharp$). Under the resulting $G$-biequivariant identification
\[
T^*G \rightarrow TG \rightarrow G \times \mathbb{A}_g,
\]
the $G$-biinvariant 1-form $\langle \mu, \theta_G \rangle$ on $G \times \mathbb{A}_g$ corresponds to the tautological 1-form on $T^*G$, and $\mu$ is the momentum mapping for $\omega = -d\langle \mu, \theta_G \rangle$, uniquely determined by $\omega$ up to a central value. By construction, under the identification of $T^*G$ with $TG$ induced by the chosen inner product on $\mathfrak{g}$, the standard cotangent bundle symplectic structure corresponds to $\omega$.

5.1 Kähler structures on $T^*G$

Let $J$ be the standard complex structure on $G \times \mathbb{A}_g$, cf. Subsection 3.3. In view of (3.19), the $G$-invariant 1-form $\langle \mu, s^G_{st} \rangle$ is zero, and $\langle \mu, s^G_{st} \rangle_a(V) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} a \cdot (\text{ad}^{2j}(a)(V)) = a \cdot V, \quad (5.1)$

where $a \in \mathbb{A}_g, V \in \mathfrak{g}$. Thus $\langle \mu, s^G_{st} \rangle$ is plainly closed, indeed

$\langle \mu, s^G_{st} \rangle = df, \quad f(a) = \frac{1}{2} a \cdot a, \quad a \in \mathbb{A}_g. \quad (5.2)$

Theorem 4.3 entails that $J$ and $\omega$ combine to a pseudo Kähler structure, and this structure is actually positive, that is, an ordinary Kähler structure. This structure coincides with the Kähler structure induced from the standard Kähler structure on $G^C$ (relative to the chosen inner product on $\mathfrak{g}$) via the ordinary polar map $G \times \mathbb{A}_g \rightarrow G^C$, and we will refer to this structure on $G \times \mathbb{A}_g$ as the standard Kähler structure relative to the chosen inner product · on $\mathfrak{g}$; cf. Example 4.3 in [Bie03], [Hal02]. We denote the $G$-biinvariant extension of the function $f$ introduced in (5.2) above to all of $G \times \mathbb{A}_g$ still by $f$; in view of Theorem 4.3 the function $2f$ is actually a Kähler potential. We note that, given the point $a$ of $\mathbb{A}_g$,

$$(d^2 \mu)_{(e,a)} = \Psi_{(e,a)}^{-1} : \mathfrak{g} \rightarrow \mathfrak{g}^*, \quad (5.3)$$

$$(\Psi_{(e,a)}^{-1} : \mathfrak{g} \rightarrow \mathfrak{g}^*)$$

and is plainly symmetric; here the number of 1’s is equal to the rank of $\mathfrak{g}$.

More generally, the following hold.

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Theorem 5.1. On $T^*G$, identified with $G \times \mathbb{A}_g$ by means of the chosen invariant inner product on $\mathfrak{g}$ and by means of left translation, let $J$ be an admissible complex structure, let $\gamma_J: \mathbb{A}_g \rightarrow G^C$ be an admissible map inducing $J$, and let $s_J$ and $c_J$ be the two associated $\mathfrak{g}$-valued $1$-forms on $\mathbb{A}_g$ characterized by the identity

$$(d\gamma_J)\gamma_J^{-1} = c_J + is_J \in A^1(\mathbb{A}_g, \mathfrak{g}^C).$$

Furthermore, as before, let $\mu$ denote the $G$-equivariant map $G \times \mathbb{A}_g \rightarrow \mathfrak{g}^*$ which, restricted to $\{e\} \times \mathbb{A}_g$, is the adjointness isomorphism $\mathfrak{g} \rightarrow \mathfrak{g}^*$ of the chosen inner product on $\mathfrak{g}$, and let $\omega = -d\langle \mu, c_J \rangle$, so that $\mu$ is a momentum mapping for $\omega$. Finally, take the complexification $G^C$ to be endowed with the standard Kähler structure relative to the chosen inner product on $\mathfrak{g}$.

The following are equivalent.

(i) The pieces of structure $J$ and $\omega$ combine to a Kähler structure on $G \times \mathbb{A}_g$.
(ii) The two real $1$-forms $\langle \mu, c_J \rangle$ and $\langle \mu, s_J \rangle$ on $\mathbb{A}_g$ are closed.
(iii) The associated generalized polar map $\Pi_{\gamma_J}: G \times \mathbb{A}_g \rightarrow G^C$ made explicit above as (3.3) preserves the symplectic (and hence Kähler) structures. The metric $g$ on $G \times \mathbb{A}_g$ is then given by the expression (4.9), and an integral of $\langle \mu, s_J \rangle$ yields a Kähler potential.

Indeed, Theorem 4.3 implies at once that (i) and (ii) are equivalent, with “pseudo Kähler structure” substituted for “Kähler structure” in (i). By construction, the complex structure $J$ on $G \times \mathbb{A}_g$ is induced from the complex structure on $G^C$ via the generalized polar map $\Pi_{\gamma_J}$. The closedness of the 1-form $\langle \mu, c_J \rangle$ on $\mathbb{A}_g$ is equivalent to $\Pi_{\gamma_J}$ being compatible with the symplectic structures. We justify this claim in the next subsection.

Remark 5.2. While Theorem 4.3 entails that, in the statement of Theorem 5.1, (i) and (ii) are equivalent, with “pseudo Kähler structure” substituted for “Kähler structure” in (i), under the circumstances of Theorem 5.1, the positivity of the resulting pseudo Kähler structure on $G \times \mathbb{A}_g$ is automatic since this structure is induced from the standard Kähler structure on $G^C$ via the associated generalized polar map.

5.2 Factorization of the generalized polar map

Let $\chi = (\chi_G, \chi_g): \mathbb{A}_g \rightarrow G \times \mathbb{A}_g$ be a smooth map whose component $\chi_g: \mathbb{A}_g \rightarrow \mathbb{A}_g$ is a local diffeomorphism, necessarily onto an open subset of $\mathbb{A}_g$, and let $\gamma_\chi: \mathbb{A}_g \rightarrow G^C$ be the composite of $\chi$ with the ordinary polar map $\Pi_{st}$, that is, $\gamma_\chi$ is given by the expression

$$\gamma_\chi(a) = \chi_G(a)\gamma_{st}(\chi_g(a)) = \chi_G(a)\exp(i\chi_g(a)), \quad a \in \mathbb{A}_g.$$  \hfill (5.6)

In terms of the map

$$\Pi_\chi: G \times \mathbb{A}_g \rightarrow G \times \mathbb{A}_g, \quad \Pi_\chi(x, a) = (x\chi_G(a), \chi_g(a)), \quad (5.7)$$

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the generalized polar map $\Pi_\chi$ associated to $\gamma_\chi$ factors as

$$G \times \mathbb{A}_g \xrightarrow{\Pi} G \times \mathbb{A}_g \xrightarrow{\Pi_{\text{st}}} G^C.$$  \hspace{1cm} (5.8)

Via the maps

$$\Pi_{\chi g} : G \times \mathbb{A}_g \longrightarrow G \times \mathbb{A}_g, \quad \Pi_{\chi g}(x, a) = (x, \chi_g(a)),$$

$$\Pi_{\chi G} : G \times \mathbb{A}_g \longrightarrow G \times \mathbb{A}_g, \quad \Pi_{\chi G}(x, a) = (x\chi_G(a), a),$$

(5.7) plainly decomposes as

$$G \times \mathbb{A}_g \xrightarrow{\Pi_{\chi G}} G \times \mathbb{A}_g \xrightarrow{\Pi_{\chi g}} G \times \mathbb{A}_g,$$  \hspace{1cm} (5.9)

and $\Pi_{\chi G}$ is simply a gauge transformation. Since $\mathfrak{g}$ is supposed to be compact, every smooth $\gamma_J : \mathbb{A}_g \rightarrow G \times \mathbb{A}_g$ of the kind in Theorem 3.1 has the form $\gamma_\chi$ for some $\chi$ since the ordinary polar map $\Pi_{\text{st}}$, cf. (3.7), is a diffeomorphism.

**Proposition 5.3.** Let $J = J_\gamma$ be the induced admissible complex structure on $G \times \mathbb{A}_g$, and let $(\theta, L)$ denote the pair of $\mathfrak{g}$-valued 1-forms on $G \times \mathbb{A}_g$ associated to $J$ by the construction in Proposition 3.3 Then

$$\theta = \Pi^*\chi \theta_{\text{st}} : T(G \times \mathbb{A}_g) \longrightarrow \mathfrak{g}$$  \hspace{1cm} (5.10)

$$L = \Pi^*\chi L_{\text{st}} : T(G \times \mathbb{A}_g) \longrightarrow \mathfrak{g}.$$  \hspace{1cm} (5.11)

Moreover, the constituents $c_\chi$ and $s_\chi$ of the resulting $\mathfrak{g}^C$-valued 1-form

$$\phi(\theta, L) = (d\gamma_\chi)\gamma_\chi^{-1} = c_\chi + is_\chi$$
on $\mathbb{A}_g$ given as (3.16) above take the form

$$c_\chi = (d\chi_G)\chi_G^{-1} + \text{Ad}_{\chi_G}(c_{\text{st}} \circ d\chi_g)$$  \hspace{1cm} (5.12)

$$s_\chi = \text{Ad}_{\chi_G}(s_{\text{st}} \circ d\chi_g).$$  \hspace{1cm} (5.13)

Indeed, (5.7) is a morphism of trivial principal left $G$-bundles, spelled out on the total spaces. The naturality of the constructions implies the identities (5.10) and (5.11). In view of (5.6), with the standard notation $d\chi_g : T\mathbb{A}_g \rightarrow T\mathbb{A}_g$ for the derivative of the map $\chi_g : \mathbb{A}_g \rightarrow \mathbb{A}_g$, the identities (5.10) and (5.11) imply the identities (5.12) and (5.13).

**Proof of Theorem 5.1.** By construction

$$\theta_{\text{st}} = \theta_G + c_{\text{st}}^G$$

$$\theta = \Pi^*\chi \theta_{\text{st}} = \theta_G + c_\chi^G$$

$$c_\chi = c_J = (d\chi_G)\chi_G^{-1} + \text{Ad}_{\chi_G}(c_{\text{st}} \circ d\chi_g)$$

$$\Pi^*\chi \langle \mu, \theta_{\text{st}} \rangle = \langle \mu, \Pi^*\chi \theta_{\text{st}} \rangle = \langle \mu, \theta_G + c_\chi^G \rangle$$

$$\Pi^*\chi \omega = -\Pi^*\chi d\langle \mu, \theta_{\text{st}} \rangle = -d\Pi^*\chi \langle \mu, \theta_{\text{st}} \rangle = -d\langle \mu, \theta_G + c_\chi^G \rangle$$

$$= \omega - d\langle \mu, c_\chi^G \rangle = \omega - d\langle \mu, c_J^G \rangle.$$
Hence \( \Pi^* \omega = \omega \) if and only if \( \langle \mu, c^G \rangle \) is closed. Consequently the closedness of the 1-form \( \langle \mu, c^J \rangle \) on \( G \times A_g \) is equivalent to \( \Pi^* \gamma J \) being compatible with the symplectic structures. \( \square \)

**Corollary 5.4.** Given \( \chi = (\chi_G, \chi_g) : A_g \to G \times A_g \) such that the component \( \chi'_G : A_g \to A_g \) is a local diffeomorphism, the induced admissible complex structure \( J = J_\gamma \) on \( G \times A_g \) combines with the symplectic structure \( \omega = -d\langle \mu, \theta_G \rangle \) (the standard structure relative to the chosen invariant inner product on \( g \)) to a Kähler structure on \( G \times A_g \) if and only if the 1-forms \( \langle \mu, c^J \rangle \) and \( \langle \mu, s^J \rangle \) on \( A_g \) are closed.

### 5.3 Non-standard examples of biinvariant Kähler structures

For the special case where \( \chi_G \) has constant value \( e \), the conditions in Corollary 5.4 take the form

\[
\begin{align*}
\langle c_{st} \circ d\chi_g \rangle &= 0 \\
\langle s_{st} \circ d\chi_g \rangle &= 0
\end{align*}
\]

The \( g \)-valued 1-forms \( c_{st} \circ d\chi_g \) and \( s_{st} \circ d\chi_g \) have the form

\[
\begin{align*}
(c_{st})_a(V) &= \cos(\text{ad}(a)) - \text{Id}(V) \\
(s_{st})_a(V) &= \frac{\sin(\text{ad}(a))}{\text{ad}(a)}(V)
\end{align*}
\]

Hence

\[
\begin{align*}
\langle \mu, c^J \rangle_a(V) &= a \cdot ((c^J)_a(V)) \\
&= \frac{-1}{2} a \cdot ([\chi'_g(a), c^J_a(V)]) \\
&\quad + \frac{1}{4} a \cdot ([\chi'_g(a), [\chi'_g(a), c^J_a(V)]]]) \\
&\pm \ldots \\
\langle \mu, s^J \rangle_a(V) &= a \cdot ((s^J)_a(V)) \\
&= a \cdot ((d\chi_g)_a(V)) - \frac{1}{2} a \cdot ([\chi'_g(a), [\chi'_g(a), (d\chi_g)_a(V)]]]) \pm \ldots
\end{align*}
\]

For example, when \([\chi'_g(a), a] = 0 \) for every \( a \in g \), the 1-form \( \langle \mu, c^J \rangle \) is even zero rather than just closed, and

\[
\langle \mu, s^J \rangle_a(V) = a \cdot ((d\chi_g)_a(V)).
\]

To construct non-standard examples, the idea is now to rescale the identity of \( A_g \) by a \( G \)-invariant scalar valued function on \( A_g \): Let \( \varphi : \mathbb{R} \to \mathbb{R}^+ \) be a smooth function of the (single) variable \( x \) such that the smooth function

\[
\chi : \mathbb{R} \to \mathbb{R}, \; \chi(x) = x \varphi(x^2),
\]

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is a local diffeomorphism. Notice that $\chi$ is then a diffeomorphism onto its image. We do not require that $\chi$ be onto. Possible examples, beyond $\chi(x) = x$, are $\chi(x) = \sinh(x)$ or $\chi(x) = \arctan(x)$. Pick $\Phi$ such that $\Phi' = \frac{1}{2}\varphi$ and let $\Xi(x) = \Phi(x^2)$ and

$$f(x) = x\chi(x) - \Xi(x) = x^2\varphi(x^2) - \Xi(x) = x^2\varphi(x^2) - \Phi(x^2);$$

then

$$\Xi'(x) = 2x\Phi'(x^2) = x\varphi(x^2) = \chi(x).$$
$$\chi'(x) = 2x\varphi'(x^2) + \varphi(x^2).$$
$$f'(x) = x\chi'(x).$$

Define $\chi_\theta: \mathbb{A}_\theta \rightarrow \mathbb{A}_\theta$ by

$$\chi_\theta(a) = \varphi(||a||^2)a.$$

(5.14)

Then $[\chi_\theta(a), a]$ is zero for every $a \in g$, whence the 1-form $\langle \mu, c_\chi \rangle$ is zero. Moreover

$$(d\chi_\theta)_\mu(V) = 2\varphi'(||a||^2)(a \cdot V) a + \varphi(||a||^2)V$$
$$\langle \mu, s_\chi \rangle_a(V) = a \cdot (\langle d\chi_\theta \rangle_a(V))$$
$$= (2||a||^2\varphi'(||a||^2) + \varphi(||a||^2)) a \cdot V$$
$$= \chi'(||a||) a \cdot V.$$}

Now, define $F: \mathbb{A}_\theta \rightarrow \mathbb{R}$ by

$$F(a) = ||a||^2\varphi(||a||^2) - \Phi(||a||^2) = ||a||\chi(||a||) - \Xi(||a||), \ a \in \mathbb{A}_\theta.$$}

Then $dF = \langle \mu, s_\chi \rangle$ whence, in particular, $\langle \mu, s_\chi \rangle$ is closed. The resulting complex structure $J$ on $G \times \mathbb{A}_\theta$ is biinvariant and combines with the symplectic structure $\omega (= -d\langle \mu, \theta_G \rangle)$, the tautological structure with respect to the chosen inner product on $g$, to a biinvariant Kähler structure.

As a consistency check we note that, in the special case where the function $\varphi$ has constant value $1$,

$$\chi(x) = x, \ \Xi(x) = \frac{1}{2}x^2, \ f(x) = x^2 - \frac{1}{2}x^2 = \frac{1}{2}x^2, \ F(a) = \frac{1}{2}a \cdot a.$$}

These are the corresponding identities in the standard case, cf. [5,2].

For example, with $\varphi(x^2) = \frac{\arctan(x)}{x}$, the resulting map $\chi_\theta: \mathbb{A}_\theta \rightarrow \mathbb{A}_\theta$ given by $\chi_\theta(a) = \varphi(||a||^2)a$ (a $\in \mathbb{A}_\theta$) is a diffeomorphism onto its image but is not onto. The resulting generalized polar map $\Pi: G \times \mathbb{A}_\theta \rightarrow GC$ is then a biinvariant Kähler diffeomorphism onto a proper open subset of $GC$ (endowed with the standard structure); in particular, $\Pi$ is not onto.

More generally, we can rescale the identity of $\mathbb{A}_\theta$ with a smooth function $\varphi(i_1(\cdot), \ldots, i_\ell(\cdot))$ of the invariants $i_1, \ldots, i_\ell$ of $g$, that is, consider a local diffeomorphism $\chi_\theta: \mathbb{A}_\theta \rightarrow \mathbb{A}_\theta$ of the kind

$$\chi_\theta(a) = \varphi(i_1(a), \ldots, i_\ell(a))a, \ a \in \mathbb{A}_\theta.$$
This class of examples raises the following question: Suppose that $G$ is simple. Are there biinvariant Kähler structures on $G \times A_g$ having underlying symplectic structure the tautological one (relative to an invariant inner product on $g$) that are distinct from those arising from rescaling the identity?

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