Another Classification of Incidence Scrolls

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Abstract: The aim of this paper is the computation of the degree and genus of all incidence scrolls in \( \mathbb{P}^n \). For this, we fix the dimension of a linear space which must be contained in the base of an incidence scroll. Then we will find all the incidence scrolls which have a base space of this fixed dimension. In this way, we can obtain all the incidence scrolls with a line as directrix curve, those whose base contains a plane, and so on.

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Introduction: Throughout this paper, the base field for algebraic varieties is \( \mathbb{C} \). Let \( \mathbb{P}^n \) be the \( n \)-dimensional complex projective space and \( G(l, n) \) the Grassmannian of \( l \)-planes in \( \mathbb{P}^n \). Then \( R_d^g \subset \mathbb{P}^n \) denotes a scroll of degree \( d \) and genus \( g \). We will follow the notation and terminology of [3].

It is useful to represent a scroll in \( \mathbb{P}^n \) by a curve \( C \subset G(1, n) \subset \mathbb{P}^N \). The lines which intersect a given subspace \( \mathbb{P}^r \subset \mathbb{P}^n \) are represented by the points of the special Schubert variety \( \Omega(\mathbb{P}^r, \mathbb{P}^n) \). Each \( \Omega(\mathbb{P}^r, \mathbb{P}^n) \) is the intersection of \( G(1, n) \) with a certain subspace of \( \mathbb{P}^N \). Since \( G(1, n) \) has dimension \( 2n - 2 \) and we search a curve, we must impose \( 2n - 3 \) linear conditions on \( G(1, n) \). Consequently, the choice of subspaces is not arbitrary. Any set of subspaces of \( \mathbb{P}^n \) which imposes \( 2n - 3 \) linear conditions on \( G(1, n) \) is the base of a certain incidence scroll. The background about Schubert varieties can be found in [4].

The aim of this paper is to obtain another classification of the scrolls in \( \mathbb{P}^n \) which are defined by a one-dimensional family of lines meeting a certain set of subspaces of \( \mathbb{P}^n \), a first classification is given in paper [1]. These ruled surfaces are called incidence scrolls, and such an indicated set is a base of the incidence scroll. Unless otherwise stated, we can assume that the base spaces are in general position. We will fix the dimension of a linear space which must be contained in the base of an incidence scroll. In this way, we can obtain all the incidence scrolls which have a base space of this fixed dimension, i.e., all the incidence scrolls with a line as directrix curve, those whose base contains a plane, and so on.

In section 1 we explain our notation and collect background material. Our first step will be to summarize some properties of ruled surfaces and, in particular, some general properties of incidence scrolls. For more details we refer the reader to [3] and [1] respectively. Having revised the notion of incidence scroll, we have compiled some basic properties of such a scroll. This section
contains a brief summary and the detailed proofs will appear in \[1\]. The degree of the scroll given by a general base is provided by Giambelli’s formula which appears in \[3\]. Moreover, the study of deformations of a given incidence scroll is a powerful tool in this paper in order to simplify our proofs. If the incidence scroll \( \mathbb{P}(\delta) \subset \mathbb{P}^n \) breaks up into \( \mathbb{P}(\delta_1) \subset \mathbb{P}^r \) and \( \mathbb{P}(\delta_2) \subset \mathbb{P}^s \) with \( \delta \) generators in common, then \( d = d_1 + d_2 \) and \( g = g_1 + g_2 + \delta - 1 \).

Section 2 contains our main results about incidence scrolls. We identify all the incidence scrolls. In this way, we find all incidence scrolls which contain a directrix line, those which contain a base plane and finally those which contain a base space of dimension 3. A more complete theory may be obtained by a combination of these processes. Accordingly, all the other incidence scrolls may be obtained. Moreover, we will see that the fixed base space imposes conditions on the genus of the incidence scroll.

The results on this paper belong to the Ph.D. thesis of the first author whose advisor is the second one.

1 Incidence Scrolls

A ruled surface is a surface \( X \) together with a surjective morphism \( \pi : X \rightarrow C \) to a smooth curve \( C \) such that the fibre \( X_y \) is isomorphic to \( \mathbb{P}^1 \) for every point \( y \in C \), and such that \( \pi \) admits a section. There exists a locally free sheaf \( \mathcal{E} \) of rank 2 on \( C \) such that \( X \cong \mathbb{P}(\mathcal{E}) \) over \( C \). Conversely, every such \( \mathbb{P}(\mathcal{E}) \) is a ruled surface over \( C \).

A scroll is a ruled surface embedded in \( \mathbb{P}^n \) in such a way that the fibres \( f \) have degree 1. If we take a very ample divisor on \( X \), \( D \sim aC + b\mathcal{F} \), then the embedding \( \Phi : X \rightarrow \mathbb{P}^n = \mathbb{P}(H^0(\mathcal{O}_X(D))) \) determines a scroll when \( a = 1 \). A scroll \( \mathbb{P}(\mathcal{F}) \subset \mathbb{P}^n \) is said to be an incidence scroll if it is generated by the lines which meet a certain set \( \mathcal{B} \) of linear spaces in \( \mathbb{P}^n \), or equivalently, if the correspondent curve in \( G(1,n) \) is an intersection of special Schubert varieties \( \Omega(\mathbb{P}^r, \mathbb{P}^n) \), \( 0 \leq r < n - 1 \). Such a set is called a base of the incidence scroll and such a base will be denoted by:

\[
\mathcal{B} = \{ \mathbb{P}^{n_1}, \mathbb{P}^{n_2}, \ldots, \mathbb{P}^{n_r} \}.
\]

We will write it simply \( \mathcal{B} \) when no confusion can arise, where \( n_1 \leq n_2 \leq \cdots \leq n_r \), i.e., \( C_\mathcal{B} = \bigcap_{i=1}^r \Omega(\mathbb{P}^{n_i}, \mathbb{P}^n) \subset G(1,n) \).

Therefore, unless otherwise stated, we will work with linear spaces in general position. By general position we will mean that \( \mathbb{P}^{n_1}, \ldots, \mathbb{P}^{n_r} \in \mathcal{W} = G(n_1, n) \times \cdots \times G(n_r, n) \) is contained in a nonempty open subset of \( \mathcal{W} \). For simplicity of notation, we abbreviate it to base in general position.

**Proposition 1.1** The intersection \( C = \bigcap_{i=1}^r \Omega(\mathbb{P}^{n_i}, \mathbb{P}^n) \) of special Schubert varieties associated to linear spaces \( \mathbb{P}^{n_i}, i = 1, \ldots, r, \) in general position is an irreducible curve of \( G(1,n) \) if and only if it verifies the following equality

\[
rn - (n_1 + n_2 + \cdots + n_r) - r = 2n - 3 \quad (IS)
\]
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Proof. See [1], Proposition 2.4.

Moreover, the incidence scroll generated by a base $B$ have degree $d$ if and only if we obtain the following equality of Schubert cycles:

$$
\Omega(n_1, n) \cdots \Omega(n_r, n) = d \Omega(0, 2).
$$

Furthermore, we present one of the three main theorems of the symbolic formalism, known as Schubert calculus, for solving enumerative problems.

**Theorem 1.2 (Pieri’s formula)** For all sequences of integers $0 \leq a_0 < \cdots < a_l \leq n$ and for $h = 0, \cdots, n - l$, the following formula holds in the cohomology ring $H^*(G(l, n); \mathbb{Z})$:

$$
\Omega(a_0, \cdots, a_l)\Omega(h, n) = \sum \Omega(b_0, \cdots, b_l)
$$

where the sum ranges over all sequences of integers $b_0 < \cdots < b_l$ satisfying $0 \leq b_0 \leq a_0 < b_1 \leq \cdots < b_l \leq a_l$ and $\sum_{i=0}^{l} b_i = \sum_{i=0}^{l} a_i - (n - l - h)$.

Proof. See [1], p. 1073.

Finally, let us mention an important property of degeneration of these scrolls.

**Proposition 1.3** Let $R^d_g \subset \mathbb{P}^n$ be an incidence scroll with base $B$ in general position. Suppose that $\mathbb{P}^{n_1}$ and $\mathbb{P}^{n_2}$ lie in a hyperplane $\mathbb{P}^{n-1}$ and have in common $\mathbb{P}^{m}$, $m = n_1 + n_2 - n + 1$. Then the scroll breaks up into:

- $R^d_{g_1} \subset \mathbb{P}^n$ with base $\bar{B} = \{ \mathbb{P}^{m}, \mathbb{P}^{n_1}, \cdots, \mathbb{P}^{n_i}, \cdots, \mathbb{P}^{n_j}, \cdots, \mathbb{P}^{n_r} \}$ (which is possibly degenerate);

- $R^d_{g_2} \subset \mathbb{P}^{n-1}$ with base $\bar{B} = \{ \cdots, \mathbb{P}^{n_i-1}, \mathbb{P}^{n_i}, \mathbb{P}^{n_i+1-1}, \cdots, \mathbb{P}^{n_j-1}, \mathbb{P}^{n_j}, \cdots \}$

which have $\kappa \geq 1$ generators in common. Then, $d = d_1 + d_2$ and $g = g_1 + g_2 + \kappa - 1$.

Moreover, if $m = 0$, then the incidence scroll breaks up into a plane and an incidence scroll $R^d_{g} \subset \mathbb{P}^{n-1}$ with base $\bar{B}$ in general position.

Proof. See [1], Proposition 3.1.

If $m = 0$, then shall refer to this particular degeneration as join $\mathbb{P}^{n_1}$ and $\mathbb{P}^{n_2}$ (i.e., $n_1 + n_2 = n - 1$) and to the inverse as separate $\mathbb{P}^{n_1}$ and $\mathbb{P}^{n_2}$ (i.e., $n_1 + n_2 = n$).

2 Classification of Incidence Scrolls

In [1] we have obtained a classification of incidence scrolls of genus 0 and 1. We can study a new point of view to give another classification of the incidence scrolls. For this we will fix the dimension of a base space. In this way, we will obtain all the incidence scrolls with a line as directrix curve, those whose base contains a plane, and finally, those whose base contains a 3-plane. All the others may be obtained by combinations of these processes.
2.1 Incidence Scrolls with a Directrix Line

The following theorem gives all linearly normal incidence scrolls with a line as directrix. For the proof we refer the reader to [1], Proposition 4.3.

**Theorem 2.1** In \( \mathbb{P}^n \), \( n \geq 3 \), the scroll given by

\[
B_n = \{ \mathbb{P}^1, (n-1) \mathbb{P}^{n-2} \}
\]

in general position is the rational normal scroll of degree \( n-1 \) with a line as minimum directrix.

We see at once that the previous theorem gives all rational normal scrolls with a line as directrix. Then these are projective models of rational ruled surfaces \( X_e, e \geq 0 \). For each \( X_e \), the unisecant complete linear system which gives the immersion is defined by the very ample divisor \( H \sim C_0 + (e+1)f \).

### TABLE 1. INCIDENCE SCROLLS WITH A DIRECTRIX LINE

| Scroll | \( n_i \), \( i = 1, \ldots, 7 \) | Normalized | Min. Dir.(*) | \( \text{deg}(\mathcal{b}) \) |
|--------|-------------------------------|------------|-------------|----------------|
| \( R_2^0 \subset \mathbb{P}^3 \) | 3 | \( - - - - - - - \) | \( \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \) | \( \mathbb{P}^3 (\infty^5) \) | 1 |
| \( R_3^0 \subset \mathbb{P}^4 \) | 1 | 3 | \( - - - - - - - \) | \( \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} (-1) \) | \( \mathbb{P}^1 (1) \) | 2 |
| \( R_4^0 \subset \mathbb{P}^5 \) | 1 | - | 4 | \( - - - - - - - \) | \( \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} (-2) \) | \( \mathbb{P}^1 (1) \) | 3 |
| \( R_5^0 \subset \mathbb{P}^6 \) | 1 | - | - | 5 | \( - - - - - - - \) | \( \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} (-3) \) | \( \mathbb{P}^1 (1) \) | 4 |
| \( R_6^0 \subset \mathbb{P}^7 \) | 1 | - | - | - | 6 | \( - - - - - - - \) | \( \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} (-4) \) | \( \mathbb{P}^1 (1) \) | 5 |
| \( R_7^0 \subset \mathbb{P}^8 \) | 1 | - | - | - | - | 7 | \( - - - - - - - \) | \( \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} (-5) \) | \( \mathbb{P}^1 (1) \) | 6 |
| \( R_8^0 \subset \mathbb{P}^9 \) | 1 | - | - | - | - | 8 | \( - - - - - - - \) | \( \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} (-6) \) | \( \mathbb{P}^1 (1) \) | 7 |

* Number of minimum directrix curves

2.2 Incidence Scrolls with \( \mathbb{P}^2 \in B \)

**Theorem 2.2** For every \( n \geq 4 \) and \( 0 \leq i \leq \frac{n}{2} \), there is an incidence scroll of degree \( d_i^n = \binom{n-1}{2} + i - 1 \) and genus \( g_i^n = \binom{n-i-2}{2} \) given by

\[
B_i^n = \{ \mathbb{P}^2, i \mathbb{P}^{n-3}, (n-2i) \mathbb{P}^{n-2} \}
\]

in general position. The directrix curve in the plane has degree \( (n-i-1) \). Moreover, these are all incidence scrolls with base \( \mathbb{P}^2 \).

**Proof.** If a nondegenerate incidence scroll \( R \subset \mathbb{P}^n \) has a plane as base space, then the other base spaces have dimension \( n-3 \) or \( n-2 \). Fix \( \mathbb{P}^2 \in B \). Then we can vary the number of \( \mathbb{P}^{n-3} \)'s, written \( i \), between 0 and \( \frac{n}{2} \) (respectively \( \frac{n-1}{2} \)) if \( n \) is even (respectively if \( n \) is odd). Fix \( \mathbb{P}^2 \) and \( i \mathbb{P}^{n-3} \). Then the number of \( \mathbb{P}^{n-2} \)'s is determined by (IS).

The proof is by induction on \( i \). If \( i = 0 \), we will proof that the incidence scroll \( R_0^\mathbb{P}^2 \subset \mathbb{P}^n \) with base \( B_0^\mathbb{P}^2 = \{ \mathbb{P}^2, n \mathbb{P}^{n-2} \} \) in general position has degree \( \left( \frac{n}{2} \right) - 1 \).
and genus \((n^2 - 2)\), and the plane directrix curve has degree \((n - 1)\). To do this, we suppose that \(\mathbb{P}^2 \vee \mathbb{P}^{n-2} = \mathbb{P}^{n-1} \subset \mathbb{P}^n\), for \(\mathbb{P}^2 \in \mathcal{B}_0^2\) and any \(\mathbb{P}^{n-2} \in \mathcal{B}_0^{n-2}\). Then the incidence scroll \(R_{g_0}^{d_0} \subset \mathbb{P}^n\) degenerates into: \(R_{g_0}^{d_0} \subset \mathbb{P}^{n-1}\) and the rational normal scroll of degree \(n - 1\) with \(n - 2\) generators in common. By Proposition 1.3, it is obvious that \(d_0^n = g_0^n + n - 1\) and \(g_0^n = g_0^{n-1} + n - 3\). Since the incidence scroll \(R_0^2 \subset \mathbb{P}^4\) is given by \(\mathcal{B}_4^4 = \{5 \mathbb{P}^2\}\) (apply Proposition 1.3) to \(2 \mathbb{P}^2 \in \mathcal{B}_4^2\),

\[
\begin{align*}
d_0^1 &= 5 \Rightarrow d_0^3 = 9 \Rightarrow d_0^6 = 14 \Rightarrow d_0^n = \left(\frac{n}{2}\right) - 1; \\
g_0^1 &= 1 \Rightarrow g_0^3 = 3 \Rightarrow g_0^6 = 6 \Rightarrow g_0^n = \left(\frac{n-2}{2}\right).
\end{align*}
\]

Moreover, the degree of the plane directrix curve is \((n - 1)\) because it is the number of lines in \(\mathbb{P}^n\) which meet a \(\mathbb{P}^1\) and \(\mathbb{P}^{n-2}\) (by Giambelli’s formula).

According to Pieri’s formula, we have:

\[
\Omega(1, n)\Omega(n - 2, n)^n = \\
\Omega(0, n)\Omega(n - 2, n)^{n-1} + \Omega(1, n - 1)\Omega(n - 2, n)^{n-1} = \\
1 + \Omega(0, n - 1)\Omega(n - 2, n)^{n-2} + \Omega(1, n - 2)\Omega(n - 2, n)^{n-2} = \\
\cdots = (n - 1)\Omega(0, 1) = n - 1.
\]

Assume the theorem holds for \(i - 1 \geq 0\); we will obtain the incidence scroll \(R_{g_0}^{d_0} \subset \mathbb{P}^n\) with base \(\mathcal{B}_{i-1}^n = \{\mathbb{P}^2, (i - 1) \mathbb{P}^{n-3}, (n - 2i + 2) \mathbb{P}^{n-2}\}\) and a directrix curve in \(\mathbb{P}^2\) of degree \((n - i)\).

Separate plane and a \(\mathbb{P}^{n-2}\) in \(\mathcal{B}_{i-1}^n\). Then, we obtain an incidence scroll in \(\mathbb{P}^{n+1}\) of degree \((n - i) + i - 1\) and genus \((n - 1)\) with base \(\mathcal{B}' = \{\mathbb{P}^2, i \mathbb{P}^{n-2}, (n - 2i + 1) \mathbb{P}^{n-1}\}\), and a plane directrix curve of degree \((n - i)\). Hence, writing \((n - 1)\) instead of \(n\), we conclude the proof.

**Remark 2.3** Since we will work under the hypothesis \(n \geq 4\), this theorem gives all the incidence scrolls with a plane directrix curve. For \(n = 3\), the incidence scroll \(R_0^2 \subset \mathbb{P}^3\) with a plane directrix curve also appears in Theorem 2.1. Moreover, it is important to note that:

1. In general, if \(R_{g_0}^{d_0} \subset \mathbb{P}^n\) \((n \geq 4)\) has a plane directrix curve, then the plane which contains such a directrix is a base space (see [1], Proposition 2.5). This is not true for \(n = 4\) and \(i = 2\). In this case, we obtain the incidence scroll \(R_0^2 \subset \mathbb{P}^3\) with base \(\mathcal{B}_3^3 = \{3 \mathbb{P}^1\}\) where \(\mathbb{P}^2 \notin \mathcal{B}_3^2\) because a plane directrix curve is a hyperplane section and then any plane does not impose conditions on \(G(1, 3)\).

2. The plane directrix curve is not necessarily the directrix curve of minimum degree. For example, for \(i = 1\) and \(n = 4\), we obtain \(R_0^3 \subset \mathbb{P}^4\) which is defined by \(\mathcal{B}_3^4 = \{\mathbb{P}^1, 3 \mathbb{P}^2\}\) with a line as minimum directrix curve (see Theorem 2.1).

3. For \(n = 4\), the plane directrix curve is not necessarily unique. For example, if \(i = 0\) and \(n = 4\), then we obtain \(R_0^7 \subset \mathbb{P}^4\) with base \(\mathcal{B}_7^4 = \{5 \mathbb{P}^2\}\) where each plane directrix curve is of type \(C_3^1 \subset \mathbb{P}^2\).
4. If we take a plane between the base spaces, then we impose conditions on the genus of the incidence scroll (the same is true for $\mathbb{P}^1$ because the incidence scroll is always rational). In the notation of Theorem 2.2 the genus of the scroll (written $g$) is subject to the condition

$$2g = (n - i - 2)(n - i - 3).$$

For example, for $g = 2$, there exists no incidence scrolls with a plane directrix curve.

5. There are scrolls with a plane directrix curve which are not incidence. For example, we can take $R_6^0 \subset \mathbb{P}^7$ with a directrix conic, a three-dimensional family of directrix quartics and a five-dimensional family of directrix quintics. It is not defined by incidences (see [1], Example 4.8).

| Scroll     | $n_i$, $i = 1, \ldots, 7$ | Min. Dir.($\star\star$) |
|------------|--------------------------|-------------------------|
| Genus 0    |                          |                         |
| $R_0^1 \subset \mathbb{P}^3$ | 3 | - - - - | $\mathbb{P}^1$ ($\infty$) |
| $R_1^1 \subset \mathbb{P}^4$ | 1 | 3 - - - - | $\mathbb{P}^1$ (1) |
| $R_0^2 \subset \mathbb{P}^5$ | - | 3 1 - - - | $C_1^5 \subset \mathbb{P}^2$ ($\infty$) |
| $R_1^2 \subset \mathbb{P}^6$ | - | 1 3 - - - | $C_1^6 \subset \mathbb{P}^2$ (1) |
| Genus 1    |                          |                         |
| $R_1^1 \subset \mathbb{P}^4$ | - | 5 - - - - | $C_1^5 \subset \mathbb{P}^2$ ($\infty$) |
| $R_1^1 \subset \mathbb{P}^6$ | - | 1 2 2 - - - | $C_1^6 \subset \mathbb{P}^2$ (2) |
| $R_1^2 \subset \mathbb{P}^7$ | - | 1 - 3 1 - - | $C_1^7 \subset \mathbb{P}^2$ (1) |
| $R_1^1 \subset \mathbb{P}^8$ | - | 1 - - 4 - - | $C_1^8 \subset \mathbb{P}^2$ (1) |
| Genus 3    |                          |                         |
| $R_1^1 \subset \mathbb{P}^{10}$ | - | 1 5 - - - | $C_1^9 \subset \mathbb{P}^2$ (1) |
| $R_1^1 \subset \mathbb{P}^{10}$ | - | 1 1 4 - - - | $C_1^9 \subset \mathbb{P}^2$ (1) |
| $R_1^2 \subset \mathbb{P}^{10}$ | - | 1 - 2 3 - - | $C_1^9 \subset \mathbb{P}^2$ (1) |
| $R_1^2 \subset \mathbb{P}^{10}$ | - | 1 - - 3 2 - | $C_1^9 \subset \mathbb{P}^2$ (1) |
| $R_1^3 \subset \mathbb{P}^{10}$ | - | 1 - - - 4 1 | $C_1^9 \subset \mathbb{P}^2$ (1) |
| $R_1^3 \subset \mathbb{P}^{10}$ | - | 1 - - - 5 - | $C_1^9 \subset \mathbb{P}^2$ (1) |

* Incidence special scroll
  (**) Number of minimum directrix curves

In Table 2 we see some examples of incidence scrolls which have a plane directrix curve. Moreover, the table contains all the incidence scrolls with a plane directrix curve of genus 0, 1 and 3.
2.3 Incidence Scrolls with $\mathbb{P}^r \in \mathcal{B}, r \geq 3$

We can continue with a similar method for obtain a new theorem which provides all the incidence scrolls which have a $\mathbb{P}^3$ between the base spaces. Without loss of generality we can assume $n \geq 5$ because the other cases appear in Theorems 2.1 and 2.2. For example, $R_3^1 \subset \mathbb{P}^4$ with $\mathbb{P}^1$ as minimum directrix curve is given by Theorem 2.1.

Fix $\mathbb{P}^3$. The other base spaces have dimension $\geq n - 4$ because the scroll is non-degenerate. Then the other base spaces are necessarily $j \mathbb{P}^{n-4}$'s, $i \mathbb{P}^{n-3}$'s and $n + 1 - 3j - 2i \mathbb{P}^{n-2}$'s. In fact, we see at once that

1. $\mathbb{P}^3$ imposes $(n - 4)$ conditions but we need $(2n - 3)$ conditions;
2. $0 \leq j \leq \frac{1}{3}(n + 1)$;
3. fixing $j$ between the above limits, we obtain $0 \leq i \leq \frac{1}{2}(n + 1 - 3j)$.

Lemma 2.4 In $\mathbb{P}^{n}$ ($n \geq 5$), the incidence scroll given by $\mathcal{B}_{0,0}^n = \{\mathbb{P}^3, (n + 1) \mathbb{P}^{n-2}\}$ in general position has degree $d_{0,0}^n = \binom{n+1}{3} - n - 1$ and genus $g_{0,0}^n = \binom{n}{3} + \binom{n-1}{3} - 2n + 4$. The directrix curve in $\mathbb{P}^3$ has degree $\binom{n}{2} - 1$.

Proof. We proceed by induction on $n$. If $n = 5$, then we will prove that $\mathcal{B}_{0,0}^5$ defines the scroll $R_{8}^{14} \subset \mathbb{P}^5$ and the directrix curve in $\mathbb{P}^3$ has degree 9. Suppose that 2 $\mathbb{P}^3$'s have by intersection a plane instead of a line. Since the scroll breaks up into $R_3^1 \subset \mathbb{P}^5$ and $R_3^6 \subset \mathbb{P}^4$ (see Theorem 2.2) with 5 generators in common, Proposition 1.3 shows that the scroll has degree 14 and genus 8. Moreover, the directrix curve in each $\mathbb{P}^3$ has degree equal to the number of lines in $\mathbb{P}^5$ which meet a $\mathbb{P}^2$ and 6 $\mathbb{P}^3$'s. From Pieri's formula we have:

$$
\Omega(2, 5) \Omega(3, 5)^6 = \\
\quad = \Omega(1, 5) \Omega(3, 5)^4 + \Omega(2, 4) \Omega(3, 5)^5 = \\
\quad = \Omega(0, 5) \Omega(3, 5)^4 + 2 \Omega(1, 4) \Omega(3, 5)^4 + \Omega(2, 3) \Omega(3, 5)^4 = \\
\quad = \cdots = 9 \Omega(0, 2) \Omega(3, 5) = 9 \Omega(0, 1) = 9.
$$

Suppose that the incidence scroll with base $\mathcal{B}_{0,0}^n$ in general position has degree $d_{0,0}^n$ and genus $g_{0,0}^n$ and each directrix curve in $\mathbb{P}^3$ has degree $\binom{n}{2} - 1$. Then we will proof the lemma for $n + 1$.

Let $R_9^s \subset \mathbb{P}^{n+1}$ be the incidence scroll defined by $\mathcal{B}_{0,0}^{n+1}$ in general position. We will calculate its degree and genus. If $\mathbb{P}^3 \vee \mathbb{P}^{n-1} = \mathbb{P}^n$, then the scroll breaks up into:

- $R_{(n+1)}^{\binom{n+1}{2} - 1} \subset \mathbb{P}^{n+1}$ with base $\mathcal{B}_{0}^{n+1}$ (see Theorem 2.2);
- $R_{90,0}^{\binom{n}{2}} \subset \mathbb{P}^{n}$ with base $\mathcal{B}_{0}^{0}$, by hypothesis of induction, and $\binom{n}{2} = n - 1$ generators in common.
An easy computation shows that \( d = \binom{n+2}{3} - n - 2 = d_{0,0}^{n+1} \) and \( g = \binom{n}{3} + \binom{n+1}{3} - 2n + 2 = g_{0,0}^{n+1} \). Moreover, the degree of the directrix curve in \( \mathbb{P}^3 \) is the number of lines in \( \mathbb{P}^{n+1} \) which meet a \( \mathbb{P}^2 \) and \( (n + 2) \mathbb{P}^{n-1} \)'s. It is exactly:

\[
\Omega(2, n + 1)\Omega(n - 1, n + 1)^{n+2} = \\
= \Omega(1, n + 1)\Omega(n - 1, n + 1)^{n+1} + \Omega(2, n)\Omega(n - 1, n + 1)^{n+1} = \\
= n\Omega(0, 1) + (n - 1)\Omega(0, 1) + \Omega(2, n - 1)\Omega(n - 1, n + 1)^n = \\
= \cdots = n + (n - 1) + (n - 2) + \cdots + 2 = \frac{1}{2}(n - 1)(n + 2).
\]

\[\square\]

**Lemma 2.5** In \( \mathbb{P}^n (n \geq 5) \), the incidence scroll with base \( \mathcal{B}_{0,1}^n = \{ \mathbb{P}^3, \mathbb{P}^{n-3}, (n-1) \mathbb{P}^{n-2} \} \) in general position has degree \( d_{0,1}^n = \binom{n}{3} - 1 \) and genus \( g_{0,1}^n = \binom{n-2}{3} + \binom{n-3}{3} - n + 3 \). The directrix curve in \( \mathbb{P}^3 \) has degree \( \binom{n}{2} \).

**Proof.** Suppose that \( \mathbb{P}^3 \vee \mathbb{P}^{n-3} = \mathbb{P}^n \). Then the incidence scroll \( R_g^d \subset \mathbb{P}^n \) with base \( \mathcal{B}_{0,1}^{n-1} \) degenerates into:

- \( R_{g_{0,0}}^{d_{0,0}^{-1}} \subset \mathbb{P}^{n-1} \) with base \( \mathcal{B}_{0,0}^{n-1} \) (see Lemma 2.4);
- \( R_{0}^{n-1} \subset \mathbb{P}^n \) with base \( \mathcal{B}_{0}^{n-1} = \{ \mathbb{P}^1, (n - 1) \mathbb{P}^{n-2} \} \) (see Theorem 2.1) and \( n - 2 \) generators in common.

Whence, we conclude that \( d = \binom{n}{3} - 1 = d_{0,1}^n \) and \( g = \binom{n-2}{3} + \binom{n-3}{3} - n + 3 = g_{0,1}^n \).

The degree of the directrix curve which is contained in \( \mathbb{P}^3 \) is given by:

\[
\Omega(2, n)\Omega(n - 3, n)\Omega(n - 2, n)^n = \\
= \Omega(0, n)\Omega(n - 2, n)^n + \Omega(1, n - 1)\Omega(n - 2, n)^n + \\
+ \Omega(2, n - 2)\Omega(n - 2, n)^n = \cdots = \\
= 1 + (n - 2) + (n - 3) + \cdots + 2 = \frac{1}{2}(n - 1)(n - 2).
\]

\[\square\]

**Lemma 2.6** In \( \mathbb{P}^n (n \geq 5) \), the incidence scroll with base \( \mathcal{B}_{1,0}^n = \{ \mathbb{P}^3, \mathbb{P}^{n-4}, (n-2) \mathbb{P}^{n-2} \} \) in general position has degree \( d_{1,0}^n = \binom{n-1}{3} \) and genus \( g_{1,0}^n = \binom{n-2}{3} + \binom{n-3}{3} - n + 4 \). The directrix curve in \( \mathbb{P}^3 \) has degree \( \binom{n-2}{2} \).

**Proof.** Separate \( \mathbb{P}^3 \) and \( \mathbb{P}^{n-4} \). Then we conclude from Proposition 1.3 that our scroll \( R_g^d \subset \mathbb{P}^n \) with base \( \mathcal{B}_{0,0}^n \) degenerates into \( R_{g_{0}}^{d-1} \subset \mathbb{P}^{n-1} \) with base \( \mathcal{B}_{0,1}^{n-1} \), hence that \( d = \binom{n-1}{3} \) and \( g = \binom{n-2}{3} + \binom{n-3}{3} - n + 4 \), by Lemma 2.5.

If we work these lemmas, then we can obtain a general theorem which will give every incidence scroll with a \( \mathbb{P}^3 \) between the base spaces. Moreover, we will obtain the degree and the genus of the incidence scroll for any \( i \) and \( j \) between suitable limits. The degree of the directrix curve of such an incidence scroll which is given by the following theorem may easily be found by referring to each particular case and using Pieri’s formula.
Theorem 2.7 For every $n \geq 5$, $0 \leq j \leq \frac{1}{3}(n+1)$ and $0 \leq i \leq \frac{1}{2}(n+1-3j)$, there is an incidence scroll of degree
\[ d_{n,i}^j = d_{0,i+j}^n + j = (n-i-2j+1) - (n-i-2j) + (i+j)(n-i-2j-1) + j - 1 \]
and genus
\[ g_{n,i}^j = g_{0,i+j}^n = (n-i-2j)^3 + (n-i-2j-1) - 2(n-i-2j) + (i+j)(n-i-2j-2) + 4 \]
with base $B_{n,i}^j = \{\mathbb{P}^3, j \mathbb{P}^{n-4}, i \mathbb{P}^{n-3}, (n+1-3j-2i) \mathbb{P}^{n-2}\}$ in general position.

Proof. We proceed by induction on $j$.

For $j = 0$, we will prove that the incidence scroll with base $B_{0,i}^0$, in general position, has degree $d_{n,i}^0 = (n-i+1) - (n-i) + i(n-i-1)$ and genus $g_{n,i}^0 = (n-i)^3 + (n-i)^2 - 2(n-i) + 4i(n-i-2)$. To do this, we proceed by induction on $i$. The theorem is true for $i = 0$, by Lemma 2.4. Supposing the theorem true for $i-1 \geq 0$, we prove it for $i$. Suppose that $\mathbb{P}^3 \vee \mathbb{P}^{n-3} = \mathbb{P}^1$, for any $\mathbb{P}^{n-3} \in B_{0,i}^0$. Then the incidence scroll $R_{n,i}^j \subset \mathbb{P}^n$ with base $B_{0,i}^j$ in general position degenerates into: $R_{n,i}^j \subset \mathbb{P}^{n-1}$ with base $B_{0,i}^j$ (by hypothesis of induction) and $R_{0,i}^j \subset \mathbb{P}^{n-1}$ with base $B_{n,i} = \{\mathbb{P}^1, (n-i) \mathbb{P}^{n-1}\}$ (see Theorem 2.3) and $(n-i-1)$ generators in common.

Whence, we find that $d = d_{0,i-1}^0 + (n-i) = d_{n,i}^0$ and $g = g_{n,i-1}^0 + (n-i-1) = g_{n,i}^0$.

Supposing the theorem true for $j-1 \geq 0$, we prove it for $j$. Separating $\mathbb{P}^3$ and $\mathbb{P}^{n-3}$ in $B_{n,i}^j$, we obtain a base $B = \{\mathbb{P}^3, j \mathbb{P}^{n-3}, (i-1) \mathbb{P}^{n-2}, (n+1-3(j-1)-2i) \mathbb{P}^{n-1}\}$ in general position which defines $R_{g_{j-1,i}}^j \subset \mathbb{P}^{n+1}$. Write $n$ instead $n+1$ and $i$ instead $i-1$. Then the incidence scroll $R_{n,i}^j \subset \mathbb{P}^n$ with base $B_{n,i}^j$ in general position has degree $d = d_{g_{j-1,i}}^j + 1$ and genus $g = g_{n,j-1,i}^n$. Therefore, $d = d_{0,i+j}^0 + j$ and $g = g_{n,i+j}^0$, by hypothesis of induction.

Remark 2.8 The determination of the degree of the directrix curve in $\mathbb{P}^3$, written $\text{deg}(C)$, is more subtle than in the other cases because it is $\Omega(2, n)\Omega(n-4, n)\Omega(n-3, n)\Omega(n-2, n)\Omega(n+1-3j-2i)$. From what has already been proved it may be concluded that for every $j, i$ between suitable limits,
\[ \text{deg}(C) = (n-i-2j) + i + j - 1. \]

If we now apply this argument again, with $\mathbb{P}^3$ replaced by $\mathbb{P}^r$, $r \geq 4$, then we can obtain a general theorem which will give all incidence scrolls with $\mathbb{P}^r$ between the base spaces. Its formulation is very complicated because we must work with, at least, three index: $i, j, k$. But the proof is similar to that of Theorem 2.7 for all cases. Therefore we can obtain degree and genus of the incidence scroll with base
\[ B_{i_1, \cdots, i_{r-1}}^n = \{\mathbb{P}^r, i_1 \mathbb{P}^{n-r-1}, \cdots, i_{r-2} \mathbb{P}^{n-4}, i_{r-1} \mathbb{P}^{n-3}, (n+r-2 ri_1 - \cdots - 3i_{r-2} - 2i_{r-1}) \mathbb{P}^{n-2}\} \]
for any $i_1, \cdots, i_{r-1}$ between suitable limits.
TABLE 3. INCIDENCE SCROLLS WITH $\mathbb{P}^3 \in B$

| Scroll | $n_i$, $i = 1, \ldots, 4$ | Directrix in $\mathbb{P}^3$ |
|--------|------------------------|--------------------------|
| $R_1^1 \subset \mathbb{P}^9$ | - | $C^3_3 \subset \mathbb{P}^3$ |
| $R_2^2 \subset \mathbb{P}^6$ | 1 | $C^2_2 \subset \mathbb{P}^3$ |
| $R_3^3 \subset \mathbb{P}^8$ | 1 | $C^3_3 \subset \mathbb{P}^3$ |
| $R_4^4 \subset \mathbb{P}^7$ | 2 | $C^3_3 \subset \mathbb{P}^3$ |
| $R_5^5 \subset \mathbb{P}^7$ | - | $C^3_3 \subset \mathbb{P}^3$ |
| $R_6^6 \subset \mathbb{P}^7$ | - | $C^3_3 \subset \mathbb{P}^3$ |

* Incidence special scroll

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