Exactly Solvable Models of 2d Dilaton Quantum Gravity

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ABSTRACT

We study canonical quantization of a class of 2d dilaton gravity models, which contains the model proposed by Callan, Giddings, Harvey and Strominger. A set of non-canonical phase space variables is found, forming an $SL(2,\mathbb{R}) \times U(1)$ current algebra, such that the constraints become quadratic in these new variables. In the case when the spatial manifold is compact, the corresponding quantum theory can be solved exactly, since it reduces to a problem of finding the cohomology of a free-field Virasoro algebra. In the non-compact case, which is relevant for 2d black holes, this construction is likely to break down, since the most general field configuration cannot be expanded into Fourier modes. Strategy for circumventing this problem is discussed.

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1. Introduction

Recently there has been a lot of interest in two-dimensional renormalisable models of gravity coupled to scalar fields. These are relevant for non-critical string theory [1, 2], as well as toy models for describing the formation and evaporation of black holes [3]. As shown in [4], the most general form of the action for such a model is

\[ S = - \int_M d^2 x \sqrt{-g} \left( \frac{1}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} \alpha R \phi + V(\phi) \right) , \]  

(1.1)

where \( M \) is a 2d manifold, \( g_{\mu \nu} \) is a metric on \( M \), \( \phi \) is a scalar field (dilaton), \( \alpha \) is a constant (background charge) and \( R \) is the 2d curvature scalar. We will label the time coordinate \( x^0 = t \) and the space coordinate \( x^1 = x \), while the corresponding derivatives will be denoted as \( \partial \) and \( \partial_x \), respectively.

The form (1.1) can be always achieved after suitable field redefinitions [4]. For example, the CGHS action

\[ S = -\frac{1}{8} \int_M d^2 x \sqrt{-g} e^{-2\phi} \left( R + 4 g^{\mu \nu} \partial_\mu \Phi \partial_\nu \Phi + c \right) , \]  

(1.2)

takes the form (1.1), with \( V(\phi) = \frac{c}{8} e^{\phi/\alpha} \), after the following field redefinitions

\[ \phi = \frac{1}{4\alpha} e^{-2\Phi} , \quad g_{\mu \nu} \rightarrow \frac{1}{4\alpha} e^{\phi/\alpha} g_{\mu \nu} . \]  

(1.3)

Depending on the form of the potential \( V(\phi) \) and whether the spatial section of the 2d manifold \( M \) is compact or non-compact, one can get models describing a non-critical string theory or 2d black holes. One loop perturbative analysis of (1.1) has been carried out in [4], where it was pointed out that the analysis simplify in the case when \( V(\phi) = \Lambda e^{\beta \phi} \), where \( \Lambda \) and \( \beta \) are constants.

Canonical quantization methods are more successful in exploring the nonperturbative nature of quantum gravity in four space-time dimensions [5] than the standard path-integral methods. This may naturally lead one to apply the same methods in the case of 2d gravity, more specifically, to the theory defined by the action (1.1). The canonical analysis in the case \( \beta = 0 \) and compact spatial manifold has been already carried out by the author [6], where it was demonstrated that the corresponding quantum theory is exactly solvable. This was achieved by using non-canonical phase space variables, forming an \( SL(2, \mathbb{R}) \times U(1) \) Kac-Moody algebra, which transformed the constraints into quadratic polynomials of the new variables. By using the free-field realization of the \( SL(2, \mathbb{R}) \) currents, the constraints became a free-field realization of the Virasoro algebra, whose cohomology is known [9].

In this paper we show that the same can be done in the case \( \beta \neq 0 \) and the spatial manifold is compact. Adding conformally coupled matter does not change
this result, which means that the physical Hilbert space of the CGHS model on a
circle can be obtained by solving the cohomology of a Virasoro algebra realised from
N + 2 free scalar fields, with background charges, where N is the central charge of
the conformally coupled matter. Since one can very easily construct, level by level,
the physical Hilbert space for such systems (for N = 0 there is a complete solution
\[9\]), the model is exactly solvable.

Unfortunately, this construction breaks down in the non-compact case, due to the
absence of well defined Fourier modes for a most general field configuration. Namely,
if one wants to include the black hole solutions into the quantum theory, then one
has to allow field configurations which blow up either at \(x = \pm \infty\) or at some finite \(x\).
Such configurations cannot be expanded into Fourier series. Therefore one needs an
alternative way of defining the quantum theory, which is discussed in the conclusions.

2. Canonical Analysis

The canonical formulation of (1.1) requires that the 2d manifold \(M\) has a topology
of \(\Sigma \times \mathbb{R}\), where \(\Sigma\) is the spatial manifold and \(\mathbb{R}\) is the real line corresponding to the
time direction. \(\Sigma\) can be either a circle \(S^1\) or a real line. The compact spatial topology
is relevant for string theory, while the non-compact spatial topology is relevant for
2d black holes, although we will argue at the end of the paper that the black hole
solutions are possible even in the compact case. The compact case is simpler for
analysis, due to absence of the “surface” terms. In the non-compact case, one can
assume appropriate boundary conditions at \(x = \pm \infty\), such that boundary terms do
not appear. However, in a most general case they will be present.

Derivation of the canonical form of the action (1.1) is simplified by introducing
the lapse function \(N(x, t)\) and the shift vector \(n(x, t)\) \[5\]. Then the metric \(g_{\mu\nu}\) takes
the following form

\[
g_{00} = -N^2 + n^2 g , \quad g_{01} = ng , \quad g_{11} = g ,
\]

(2.1)

where \(g(x, t)\) is a metric on \(\Sigma\). After introducing the canonical momenta for \(g\) and \(\phi\) as

\[
p = \frac{\partial L}{\partial \dot{g}}, \quad \pi = \frac{\partial L}{\partial \dot{\phi}},
\]

(2.2)

where \(L\) is the Lagrangian density of (1.1), then up to surface terms, the action
becomes

\[
S = \int dt dx \left( p \dot{g} + \pi \dot{\phi} - \frac{N}{\sqrt{g}} G_0 - n G_1 \right),
\]

(2.3)
where
\[ G_0(x) = -\frac{2}{\alpha^2} g^2 p^2 - \frac{2}{\alpha} g p \pi + \frac{1}{2} (\phi')^2 + g V(\phi) - \frac{\alpha g'}{2 g} \phi' + \alpha \phi'' \]
\[ G_1(x) = \pi \phi' - 2 p' g - pg' \quad . \]
(2.4)

The constraints \( G_0 \) and \( G_1 \) form a closed Poisson bracket algebra
\[
\{ G_0(x), G_0(y) \} = -\delta'(x - y)(G_1(x) + G_1(y)) \\
\{ G_1(x), G_0(y) \} = -\delta'(x - y)(G_0(x) + G_0(y)) \\
\{ G_1(x), G_1(y) \} = -\delta'(x - y)(G_1(x) + G_1(y)) \quad , \]
(2.5)

where the fundamental Poisson brackets are defined as
\[
\{ p(x), g(y) \} = \delta(x - y) \quad , \quad \{ \pi(x), \phi(y) \} = \delta(x - y) \quad . \]
(2.6)

\( G_1 \) generates the spatial diffeomorphisms, while \( G_0 \) generates the time translations of \( \Sigma \), in full analogy with the \( 3 + 1 \) gravity case. Note that the algebra (2.5) is isomorphic to two commuting copies of the 1d diffeomorphism algebra, which can be seen by defining the constraints as
\[
T_\pm = \frac{1}{2} (G_0 \pm G_1) \quad . \]
(2.7)

Introduction of the conformally coupled scalar matter changes the action (1.1) by
\[
S_m = -\frac{1}{2} \int_M d^2 x \sqrt{-g} g^{\mu \nu} \partial_\mu \phi^i \partial_\nu \phi_i \quad , \]
(2.8)

where \( i = 1, ..., N \). The constraints change as
\[
G_0 \rightarrow G_0 + \frac{1}{2} \pi_i^2 + \frac{1}{2} (\phi'_i)^2 \\
G_1 \rightarrow G_1 + \pi_i \phi'_i \quad , \]
(2.9)

where \( \pi_i \) are the canonically conjugate momenta for \( \phi_i \).

Since we are dealing with a reparametrization invariant system, the Hamiltonian vanishes on the constraint surface (i.e. it is proportional to the constraints). Therefore the dynamics is determined by the constraints only. Since \( G_0 \) and \( G_1 \) are irreducible, there will be \( (2 + N) - 2 = N \) local physical degrees of freedom. When \( N = 0 \), there are only finitely many global physical degrees of freedom (zero modes of \( g \) and \( \phi \)), and one is dealing with a topological field theory. When \( N \neq 0 \), these global degrees of freedom will be present, together with the local ones. In the quantum theory, this classical counting can be spoiled by the anomalies. However, when the anomalies are absent, this counting should still hold, as the subsequent analysis will show.
3. \( SL(2, \mathbb{R}) \otimes U(1) \) Variables

We now specialize to the case \( V(\phi) = \Lambda e^{\beta \phi} \). As in the case \( \beta = 0 \) \[3\], the variables \((g, p, \phi, \pi)\) are not convenient for quantization, since \( G_0 \) is a non-polynomial function of these variables. First we perform a canonical transformation in order to get rid off the exponential in \( \phi \) term

\[
\begin{align*}
g &= e^{-\beta \phi} \tilde{g} , & p &= e^{\beta \phi} \tilde{p} \\
\phi &= \tilde{\phi} , & \pi &= \tilde{\pi} + \beta \tilde{p} \tilde{g} .
\end{align*}
\] (3.1)

The constraints now become

\[
\begin{align*}
G_0(x) &= -\frac{2}{\alpha^2} (1 + \alpha \beta) g^2 p^2 - \frac{2}{\alpha} g p \pi + \frac{1}{2} (1 + \alpha \beta)(\phi')^2 + \Lambda g - \frac{\alpha g'}{2 g} \phi' + \alpha \phi'' \\
G_1(x) &= \pi \phi' - 2 p' g - p g' ,
\end{align*}
\] (3.2)

where we have dropped the tildas. As in the \( \beta = 0 \) case, we are going to look for the analogs of the \( SL(2, \mathbb{R}) \) variables introduced in \[3\]. We define

\[
(1 + \alpha \beta) J^+ = -\frac{\sqrt{2}}{g} T_- + \frac{\Lambda}{\sqrt{2}}
\]

\[
(1 + \alpha \beta) J^0 = (1 + \alpha \beta) g p + \frac{\alpha}{2} \left( \pi - \frac{\alpha g'}{2 g} \right)
\]

\[
J^- = \frac{\alpha^2}{\sqrt{2}} g
\]

\[
(1 + \alpha \beta) \frac{1}{2} P_D = \frac{1}{\sqrt{2}} \left( \pi - \frac{\alpha g'}{2 g} + (1 + \alpha \beta) \phi' \right) .
\] (3.3)

The \((J^a, P_D)\) variables satisfy an \( SL(2, \mathbb{R}) \otimes U(1) \) current algebra

\[
\begin{align*}
\{J^a(x), J^b(y)\} &= f^{ab}_c J^c(x) \delta(x-y) - \bar{\alpha}^2 \frac{\eta^{ab}}{2} \delta'(x-y) \\
\{P_D(x), P_D(y)\} &= -\delta'(x-y),
\end{align*}
\] (3.4)

where

\[
\bar{\alpha}^2 = \frac{\alpha^2}{1 + \alpha \beta} ,
\] (3.5)

and \( f^{ab}_c = 2 \epsilon^{abc} \eta_{dc} \) with

\[
\eta^{ab} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix} ,
\] (3.6)
and \( \{J, P_D\} = 0 \). Instead of using the canonical variables \((\pi_i, \phi_i)\), we introduce the left/right moving currents

\[
P_i = \frac{1}{\sqrt{2}}(\pi_i + \phi_i') \quad , \quad \tilde{P}_i = \frac{1}{\sqrt{2}}(\pi_i - \phi_i') ,
\]

satisfying

\[
\{P_i(x), P_j(y)\} = -\delta_{ij} \delta'(x-y) \quad , \quad \{\tilde{P}_i(x), \tilde{P}_j(y)\} = \delta_{ij} \delta'(x-y) ,
\]

and \( \{P, \tilde{P}\} = 0 \). Now one can show that the energy-momentum tensor associated to the algebra \((3.4)\) via the Sugavara construction, together with the matter contribution

\[
S = T_g + T_m
\]

\[
T_g = \frac{1}{\alpha^2} \eta_{ab} J^a J^b - (J^0)' , \quad T_m = \frac{1}{2} P_D^2 + \frac{\bar{\alpha}}{\sqrt{2}} P_D' + \frac{1}{2} P_i^2 ,
\]

satisfies \( S = T_+ \) on the constraint surface. Therefore the constraints become

\[
J^+(x) - \lambda = 0 \\
S(x) = T_g(x) + T_m(x) = 0 ,
\]

where \( \lambda = \frac{\Lambda}{\sqrt{2(1+\alpha^2)}} \).

4. Quantum Theory

The quantum theory can be now constructed by following the approach of [6]. We promote \(J\)'s and \(P\)'s into Hermitian operators, satisfying

\[
[J^a(x), J^b(y)] = i f_{abc} J^c \delta(x-y) - \frac{k}{2} \eta^{ab} \delta'(x-y) \\
[P_I(x), P_J(y)] = -i \delta_{IJ} \delta'(x-y) , \quad I = i, D .
\]

We introduce a new constant \( k \), which is different from \( 2\pi \alpha^2 \) due to ordering ambiguities. It will be determined from the requirement of anomaly cancelation. Since the constraints are independent of the \( \tilde{P}_i \) variables, then the complete physical Hilbert space will be a tensor product of a \( \tilde{P} \) Hilbert space with the physical Hilbert space of \((J, P_I)\) variables.

In the case of \( \Sigma = S^1 \), we construct the kinematical Hilbert space as a Fock space built on the vacuum state annihilated by the positive Fourier modes of \( J \) and \( P_I \). If we define the Fourier modes as

\[
J^a(x) = \frac{1}{2\pi} \sum_n e^{in\xi} J_n^a , \quad P_I(x) = \frac{1}{\sqrt{2\pi}} \sum_n e^{in\alpha_n^I} ,
\]

(4.2)}
then (4.1) becomes
\[
[J^a_n, J^b_m] = i f_{ab}^c J^c_{n+m} + \frac{k}{2} \eta^{ab} n \delta_{n+m},
\]
\[
[\alpha^I_n, \alpha^J_m] = \delta_{IJ} n \delta_{n+m} .
\] (4.3)

The Fock space vacuum is defined as \(|j, m\rangle \otimes |p_I\rangle\), where \(|j, m\rangle\) is the \(SL(2, \mathbb{R})\) vacuum
\[
J^a_n |j, m\rangle = 0 , \quad n \geq 1
\]
\[
J^a_0 |j, m\rangle = j^a |j, m\rangle ,
\] (4.4)
while \(|p_I\rangle\) is the \(U(1)\) vacuum
\[
\alpha^I_n |p_M\rangle = 0 , \quad n \geq 1
\]
\[
\alpha^I_0 |p_M\rangle = p_I |p_I\rangle .
\] (4.5)

The quantum constraints are defined as
\[
L_n = \frac{1}{k + 2} \sum_m \eta_{ab} : J^a_{n-m} J^b_m : - inJ^0_n + \frac{1}{2} \sum_m : \alpha^I_{n-m} \alpha^I_m : + inQ_I \alpha^I_n ,
\] (4.6)
where \(S(x) = \frac{1}{2\pi} \sum_n e^{inx} L_n\), and the normal ordering is with respect to the vacuum states (4.4-5). Note that the anomaly appears in the quantum algebra of the constraints (4.6), proportional to the central charge of gravity plus matter system
\[
c = \frac{3k}{k + 2} - 6k + N + 1 + 12 Q_D^2 ,
\] (4.7)
where \(Q_D = \sqrt{\pi \alpha}\). When anomaly appears in the constraint algebra, then one has to use the Gupta-Bleuler quantization procedure, or the BRST quantization, which is more suitable in this case.

The BRST charge \(\hat{Q}\) can be constructed as
\[
\hat{Q} = c_0 (L_0 - a) + \sum_{n \neq 0} c_n L_n + \sum_n c^+_n (J^+_n - \lambda \delta_{n,0}) + \cdots ,
\] (4.8)
where \(c_n\) and \(c^+_n\) are the Fourier modes of the ghosts corresponding to the constraints (3.10). The dots correspond to the terms proportional to the ghost momenta, such that \(\hat{Q}^2 = 0\), and \(a\) is the intercept \([7]\). The nilpotency condition requires vanishing of the total central charge, which includes the ghost contributions
\[
c - 26 - 2 = 0 ,
\] (4.9)
and the intercept must satisfy

\[ a = 1 + \frac{k}{4} - \frac{1}{2}Q^2_D . \]  

Evaluation of the cohomology of the BRST charge (4.8) simplifies if one employs the Wakimoto construction for the \( SL(2, \mathbb{R}) \) algebra (4.3) \[10\]. As in the \( \beta = 0 \) case \[6\], we introduce three new variables \( \beta(x), \gamma(x) \) and \( P_L(x) \) such that

\[
J^+(x) = \beta(x) \\
J^0(x) = -\beta(x)\gamma(x) : -k_1P_L(x) \\
J^-(x) = \beta(\sigma)\gamma^2(x) : +2k_1\gamma(x)P_L(x) + k_2\gamma'(x) \ ,
\]

where

\[
[\beta(x), \gamma(y)] = -i\delta(x - y) \ , \ [P_L(x), P_L(y)] = i\delta'(x - y) \ ,
\]

whith the other commutators bieng zero. Then the expressions (4.11) satisfy the \( SL(2, \mathbb{R}) \) algebra (4.1) if

\[
k_1 = \sqrt{\frac{k + 2}{2}} \ , \ k_2 = -k \ ,
\]

where the scalar constraint now becomes

\[
\hat{S} \ =: \beta'\gamma : -\frac{1}{2} : P^2_L : + \frac{Q_L}{\sqrt{2\pi}}P'_L + \frac{1}{2} : P^2_D : + \frac{Q_D}{\sqrt{2\pi}}P'_D + \frac{1}{2} : P^2_i := 0 \ ,
\]

where

\[
Q_L = k_1 - \frac{1}{2k_1} .
\]

Note that the transformation (4.11) is also defined classically, with \( k_1 = \frac{\bar{\alpha}}{\sqrt{2}} \) and \( k_2 = -\bar{\alpha}^2 \), satisfying the algebra (3.4).

If we define \( B(x) = \beta(x) - \lambda \) and \( \Gamma(x) = \gamma(x) \), then the \( J^+ \) constraint implies that \( B = 0 \), and consequently we can drop the canonical pair \( (B, \Gamma) \) from the theory. Therefore we are left with \( P_L, P_D \) and \( P_i \) variables, obeying only one constraint

\[
S = -\frac{1}{2}P^2_L + \frac{Q_L}{\sqrt{2\pi}}P'_L + \frac{1}{2}P^2_D + \frac{Q_D}{\sqrt{2\pi}}P'_D + \frac{1}{2}P^2_i := 0 \ .
\]
5. BRST Cohomology

Now one needs to study the BRST cohomology of a Virasoro algebra

\[ L_n = \frac{i}{2} \sum_m :\alpha_{n-m} \cdot \alpha_m : + i n Q \cdot \alpha_n \]  

(5.1)

where

\[ X^a = (X^L, X^I) \quad X \cdot Y = \eta_{ab} X^a Y^b \quad \eta_{ab} = \text{diag}(-1, 1, \ldots, 1) \]

\[ Q_a = (Q_L, Q_D, 0, \ldots, 0) \]  

(5.2)

The BRST charge is then given by the usual expression

\[ \hat{Q} = \sum_n c_n L_{-n} + \frac{i}{2} \sum_{m,n} (m - n) : c_m c_n b_{-m-n} : - c_0 a \]  

(5.3)

The normal ordering is with respect to the vacuum \(|\text{vac}\rangle = |p\rangle \otimes |0\rangle\)

\[ \alpha_n |\text{vac}\rangle = c_n |\text{vac}\rangle = b_n |\text{vac}\rangle = 0 \quad n \geq 1 \]  

(5.4)

where |\(p\rangle\) is the \(\alpha\)-modes vacuum (\(\alpha_0 |p\rangle = p |p\rangle\)), while \(|0\rangle\) is the ghost vacuum, satisfying \(b_0 |0\rangle = 0\) (the other possibility \(c_0 |0\rangle = 0\) gives symmetric results). Nilpotency of \(\hat{Q}\) implies

\[ Q^2 = -Q^2_L + Q^2_D = 2 - N/12 \quad a = N/24 \]  

(5.5)

which are the conditions for the absence of anomalies. As expected, the first condition in (5.5) is equivalent to (4.9) if \(Q_L\) takes the value (4.15).

The zero ghost number cohomology is determined by the usual Gupta-Bleuler conditions

\[ (L_n - a \delta_{n, 0}) |\psi\rangle = 0 \quad n \geq 0 \]  

(5.6)

where \(|\psi\rangle\) belongs to the \(\alpha\)-modes Fock space \(F(\alpha)\). Since the anomalies are absent, due to (5.5), then one can expect that the classical counting of the physical degrees of freedom should hold. Hence only the zero modes of gravity plus dilaton sector should survive, together with \(N\) transverse local degrees of freedom, corresponding to the \(P_i\) modes. This can be verified for the first few levels.

Clearly, the ground state \(|p\rangle\) is a solution of (5.6) if

\[ p^2 = -p^2_L + p^2_I = N/12 \]  

(5.7)

which looks like the tachyonic ground state of the usual string theory. At the first excited level, the states are of the form

\[ |\psi\rangle = \xi \cdot \alpha_{-1} |p\rangle \]  

(5.8)
which are physical if

\[ p^2 = \frac{N}{12} - 2 \quad , \quad \bar{p} \cdot \xi = 0 \quad , \tag{5.9} \]

where \( \bar{p} = p + iQ \). The norm of the state (5.8) is

\[ |\xi|^2 = \xi_a^* \xi^a \quad , \tag{5.10} \]

which for the physical states becomes

\[
|\xi|^2 = S^{IJ} \xi_I^* \xi_J \\
= \left( 1 - \frac{|\bar{p}_1|^2}{|\bar{p}_0|^2} \right) |\xi_1|^2 + \left( \delta_{ij} - \frac{p_i p_j}{|\bar{p}_0|^2} \right) \xi_i^* \xi_j - \left( \frac{p_i \bar{p}_1}{|\bar{p}_0|^2} \xi_i^* \xi_1 + c.c. \right) , \tag{5.11} \]

where \( X_L = X_0 \) and \( X_D = X_1 \). By going into a special frame \( p_0 = m, p_1 = p_i = 0 \) for \( N < 24 \) or \( p_0 = p_1 = p, p_i = 0 \) for \( N = 24 \) or \( p_0 = p_1 = 0, p_i^2 = m^2 \) for \( N > 24 \), where \( m^2 = |2 - N/12| \), it is easy to see that the Hermitian matrix \( S^{IJ} \) always has \( N \) positive eigenvalues and one zero eigenvalue. Therefore this confirms our conjecture that only the transverse modes are physical. This is different from the usual string theory \( (Q^a = 0) \), where \( S^{IJ} \) is positive definite for \( N = D-2 \leq 24 \), while for \( N > 24 \), \( S^{IJ} \) has negative eigenvalues, corresponding to the negative norm states.

Note that the so called “discrete” states \[8, 9\], arise when

\[ \xi_+ = -\xi_- \quad , \quad \xi_i = 0 \quad , \tag{5.12} \]

where \( \xi_\pm = \frac{1}{\sqrt{2}}(\xi_0 \pm \xi_1) \), so that

\[ |\xi|^2 = 2|\xi_\pm|^2 > 0 \quad . \tag{5.13} \]

The physical state condition then implies

\[ \xi_+ (\bar{p}_+ - \bar{p}_-) = 0 \rightarrow \bar{p}_+ = \bar{p}_- \rightarrow p_\pm = -iQ_\pm \text{ or } p_\pm = iQ_\mp \quad . \tag{5.14} \]

However, the meaning of these states is not clear, since they have imaginary, but fixed, momenta. See [8] for a more detailed discussion.

6. Conclusions

We have solved non-perturbatively a class of 2d dilaton gravity theories, defined by a potential \( V(\phi) = \Lambda e^{\beta \phi} \). Note that the \( \beta \neq 0 \) case is essentially the same as the \( \beta = 0 \) case analysed in [6], since a set of non-canonical variables, forming an \( SL(2, \mathbb{R}) \otimes U(1) \) algebra, was found. In terms of these variables the constraints look the same in both cases. The \( SL(2, \mathbb{R}) \) variables are gauge independent generalization
of the KPZ currents [1], which were originally found in the chiral gauge for $\beta = 0$ theory. The Wakimoto construction (4.11) and the subsequent free-field expression for $S$ (4.16) is the canonical quantization analog of the DDK construction [2].

We have given only a partial analysis of the BRST cohomology for the zero ghost number sector. This analysis supports our conjecture about the physical Hilbert space, which was based on the classical counting of the physical degrees of freedom and the absence of the anomalies. We have also concluded that the physical Hilbert space is well defined for any $N$, which conflicts with the conclusions of [3, 14]. There are several possible explanations for this discrepancy. First, we have not done the full cohomology analysis. Second, their results are semi-classical, and third, the theories may not be the same, since they are working in the path-integral quantization scheme.

The key question now is whether this exactly solvable model contains the phenomena of interest, i.e. formation and evaporation of 2d black holes. The 2d black hole classical solutions appear in the non-compact case, however we will argue now that one can have black hole solutions even in the compact case. As Mann et al. have shown [11], the original black hole solutions correspond to a one sided colaps of a 1d dust, so that the singularity is at $x = \pm \infty$. However, a symmetric collaps produces a black hole solution symmetric with respect to the origin, with the singularity at the origin $x = 0$, and the horizon at $x = \pm x_h, x_h < \infty$ [11]. By restricting this solution to an interval $[-L, L]$, such that $L > x_h$, we will obtain a black hole solution on a compact interval.

A more serious objection to our solvable model as a model of quantum black holes is the fact that we have used the Fourier modes of our fields to define the quantum theory. A function can be expanded into a Fourier series only if it is piecewise continuous on $[-L, L]$. However, the black hole metric blows up at the horizon, and therefore cannot be expanded into a Fourier series. Hence by using the Fourier modes we are restricting the phase space of our model to the space of piecewise continuous solutions, which does not contain the black hole solutions. Strictly speaking, this means that our model does not describe the phenomenon of interest. However, one can hope that the Fourier modes construction can be viewed as a some kind of a discrete approximation to the full theory. Clearly, a further study is necessary.

Therefore one should find a way of defining the quantum theory without using the Fourier modes. In the canonical quantization approach, one way would be to recast the scalar constraint $S$ into a Schrodinger type equation. This would require defining an extrinsic time variable, in analogy with the 4d quantum gravity [12].

If any of the proposed methods works, then a physical Hilbert space can be
constructed, and therefore the quantum mechanics would stabilize a 2d black hole, in
analogy with the Hydrogen atom, which is classically unstable.

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