An $O^*(1.0821^n)$-Time Algorithm for Computing Maximum Independent Set in Graphs with Bounded Degree 3

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Abstract. We give an $O^*(1.0821^n)$-time, polynomial space algorithm for computing Maximum Independent Set in graphs with bounded degree 3. This improves all the previous running time bounds known for the problem.

1 Introduction

The Maximum Independent Set problem is one of the extensively studied NP-hard problems in literature. Given a graph $G$, an Independent Set of the graph is any subset of vertices of $G$ such that no two vertices in the set has an edge between them. The Maximum Independent Set Problem (henceforth denoted by MIS) is to find the Independent Set in $G$ with the largest cardinality. In this paper we give an algorithm for solving Maximum Independent Set Problem in graphs with bounded degree 3 (henceforth denoted by MIS3). Johnson and Szegedy [15] showed that there can be a sub-exponential algorithm for MIS3 if and only if there is a sub-exponential algorithm for MIS in general graphs. Impagliazzo, Paturi and Zane [13] showed that the existence of a sub-exponential algorithm for MIS implies the existence of a sub-exponential algorithm for many other NP-hard problems like Vertex Cover, Clique, k-Set Cover, k-SAT and k-colorablity. Also, strong inapproximablity results are known for MIS like the one given by Hastad [12]. MIS is also known to be not fixed parameter tractable unless $P = NP$ [8]. Bourgeois [2] gave a bottom-up method that showed the improvement in the running times of MIS in low-degree graphs can be used to get improved running times for MIS in general graphs. All these factors taken together increases the significance of study of exact algorithms for MIS3.

1.1 Related work

The initial approaches for solving the MIS problem used the idea of enumerating all the maximal independent sets. In 1965, Moon and Moser [16] showed that the number of maximal independent sets in a graph is bounded by $3^{\frac{n}{2}} \approx 2^{0.528n}$. In 1977, Tarjan and Trojanowski [19] initiated the backtracking with case analysis approach to solve the MIS problem. They gave an $O^*(2^\frac{n}{2})$ (≈ $O^*(1.2600^n)$) algorithm for MIS problem in general graphs. Almost all the solutions for MIS that came thereafter has used their basic technique as backtracking with case analysis. Jian [14] gave an $O^*(1.2346^n)$ algorithm by using an improved case analysis. Robson [18] also used similar techniques to give an $O^*(1.2244^n)$ polynomial-space and an $O^*(1.2109^n)$ exponential-space algorithm.

In more recent developments, Beigel [4] used the number of edges as the complexity measure and obtained an $O^*(2^{0.114e})$ algorithm for MIS in graphs with $e$ edges. It gave a running time bound of $O^*(1.1259^n)$ on degree-3 bounded graphs (MIS3). For general graphs, he also gave a simple $O^*(1.2338^n)$ algorithm and an $O^*(1.2227^n)$ algorithm which involved complicated case analysis.
Chen, Kanj, and Jia [6] examined the complementary problem of vertex cover on low degree graphs and suggested an \(O^*(1.174^n)\) algorithm for MIS3. The same authors [7] used amortized analysis to get a better running time of \(O^*(1.1254^n)\) for MIS3. Fomin, Grandoni and Kratsch [10] gave an \(O^*(1.2210^n)\) algorithm for MIS in general graphs by using the measure and conquer approach. They assigned different weights to the vertices based on their degree to get a better complexity measure for analysis. Fomin and Hoie [9] proposed an algorithm for MIS3 which was different from all the other algorithms in their basic approach. They proved a bound of close to \(n/6\) on the path-width of degree-3 bounded graphs and used this result along with a dynamic programming approach to get a running time bound of \(O^*(2^{n/6})\) \(\approx O^*(1.1225^n)\). Fürer introduced \(m-n\) \((m\) is the number of edges and \(n\) is the number of vertices) as a complexity measure in his work [11]. He gave an \(O^*(1.1120^n)\) algorithm for MIS3. He also introduced a novel method to handle small sized separators, thereby cutting of a constant size subgraph. Bourgeois et al. [4] utilized some of the ideas given by Fürer and refined the case analysis to get a better running time of \(O^*(1.0977^n)\).

Razgon [17] did an extensive case analysis and obtained a bound of \(O^*(1.0892^n)\) for the running time of MIS3. He used the more intuitive complexity measure of the number of degree-3 vertices. The main cases that were handled were small cuts, triangles, rectangles, odd edges etc. Xiao in his work [21], introduced a new complexity measure and used a concise set of branching rules. He improved the running time of MIS3 to \(O^*(1.0855^n)\). The complexity measure used was \(\eta(G) = \sum_{v \in V} \max (\delta(v) - 2, 0)\) where \(\delta(v)\) is the degree of the vertex \(v\). Bourgeois et al. [5] gave an \(O^*(1.08537^n)\) algorithm for MIS problem in graphs of average degree-3 by doing a careful analysis of the worst-case scenarios. They used the \(m-n\) complexity measure similar to [11] and [4]. The same authors also gave a bottom-up method in [3] which propagates the improvements in time complexity for sparser instances into improvements in time complexity for denser instances. Finally, Xiao, and Nagamuchi [22] gave an \(O^*(1.0836^n)\) algorithm for MIS3. The algorithm used the idea of avoiding bottleneck cases by wisely choosing the branchings depending on the presence of different structures in the graph. The number of branching rules were few, but detailed case analysis was done on those rules to show that they cover all the cases and that they do so within the proposed time bounds.

1.2 Overview of our algorithm

We use a recursive backtracking algorithm to solve the problem. The algorithm is described as a recursive function \(MIS(G)\). During a call of \(MIS(G)\), depending on the conditions satisfied by \(G\), \(MIS(G)\) reduces \(G\) by performing some operations on it and calls \(MIS(G')\) recursively on the reduced graph \(G'\). In some cases, only one recursive call is made and such steps are called non-branching steps. But in some cases, more than one recursive calls may be made by \(MIS(G)\) and in such cases we say that a branching occurs. For example, we may select a vertex \(v\) in \(G\) and return the largest set among \(MIS(G \setminus v)\) and \(MIS(G \setminus (\{v\} \cup N(v)))) \cup \{v\}\).

For recursive backtracking based algorithms, the complexity of the algorithm is determined by the decrease of problem size during the branchings. Each branching gives a recurrence equation which can be solved by standard methods to get the complexity of the corresponding branching. The overall complexity of the algorithm is given by the complexity of the worst branching in the algorithm.
For analysis purposes, we define the complexity measure of our algorithm as
\[ \eta(G) = \sum_{v \in V} \max(\delta(v) - 2, 0) \]
where \( \delta(v) \) is the degree of \( v \) in \( G \). At the start of the algorithm, \( \eta(G) \) is equal to the number of degree-3 vertices as there are no vertices having degree \( \geq 4 \). Also, for graphs with bounded degree 2, there is a straightforward polynomial time algorithm. Since we are interested only in the exponential time complexity, \( \eta(G) \) is a valid complexity measure. By defining our complexity measure in this manner, we are able to capture the degree of vertices into the problem size. Whenever a vertex is removed, or there is a decrease in degree of a vertex, we get a decrease in problem size. We illustrate this using a simple example in figure 1. Suppose we branch on the vertex \( u \). That is, we consider two possibilities, (i) No maximum independent set has \( u \), and (ii) \( u \) is in some maximum independent set. In the first case, we remove \( u \) and recurse on the remaining graph, while in the second case, we remove \( u, a, b, c \) and recurse on the remaining graph. So, in the first case, \( \eta \) reduces by at least 4 (since \( a, b, c \) become degree-2 vertices) and in the second case \( \eta \) reduces by at least 10. We say that this branch has a branching form \((4, 10)\). Let \( T(\eta) \) denote the running time of the algorithm. Then we have the following recursive relation for the running time with respect to this branching:
\[ T(\eta) = T(\eta - 4) + T(\eta - 10) + \text{poly}(n) \]
Suppose, hypothetically, all branches in the algorithm are of the above kind. Then the running time of the algorithm will be \( (\text{poly}(n) \cdot \alpha^n) \), where \( \alpha \) is the unique positive real root of the equation \( x^{-4} + x^{-10} = 1 \). We consider all branches incurred during the execution of our algorithm, and compare the corresponding roots of the such equations. The maximum value of the root will define the running time of our algorithm.

Fig. 1. Simple example for illustrating the running time behavior for branching.

Given the above basic idea regarding computing the running time, our algorithm is based on the observation that if there were no small cycles in the graph, there exists a simple branch that causes a very high decrease in complexity measure. Figure 2 shows such a case. Note that if the graph does not have simple cycles of length at most 6 and it is 3-regular, then the neighborhood structure of adjacent vertices \( u \) and \( v \) is as shown in figure 2 where all nodes are distinct. Here we can have the following 3-way branching: (i) There is an MIS containing \( u \), (ii) there is an MIS containing \( v \), and (iii) all MIS contain \( p, q, r, s \). We will argue that in all the three cases above, there is a significant reduction of the complexity parameter \( \eta \) which is good for our analysis.

So, we remove the 3-cycles, 4-cycles, 5-cycles and 6-cycles by using some properties that holds when they are present. Removing a particular structure may give the necessary decrease in complex-
ity measure only if some other structures are not present. For example, the step in our algorithm that removes a pentagon will give the sufficient decrease in complexity measure only if it is carried out after removing the rectangles. Removing rectangle may in turn depend on removal of triangles. The algorithm mostly involves ordering these different cases and sub-cases properly taking care of these dependencies.

Our algorithm can be viewed as consisting of the following three parts:

1. In the first part, we remove some simple structures without branching.
2. In the second part we remove the triangles, rectangles and vertices with degree $\geq 4$. After the second part of the algorithm, the graph that remains should be a 3-regular graph with no rectangles or triangles.
3. In the third part, we remove pentagons and hexagons and also perform the branching step when no triangles, rectangles, pentagons or hexagons are present.

Certain simple structures such as degree-1 vertices are removed without branching. All of these simple structures can be found in polynomial time. In each of these non-branching steps, we prove that $\eta$ does not increase during that step. So, these steps do not add to the exponential time complexity of the algorithm. The removal of these structures help in decreasing $\eta$ by a larger value during the subsequent branching steps. After this we remove the triangles and rectangles. While removing triangle, we carefully choose which triangle to branch on in order to get the required decrease in $\eta$ while branching. After that removing rectangles gives the required decrease in $\eta$ during branching.

In the third part of the algorithm, the goal is to remove pentagons and hexagons. For this we have to deal separately with different sub-cases like a pentagon and hexagon intersecting in 2 edges, pentagon and pentagon intersecting in 1 edge etc. For understanding the need of removing these structures, we have given a dependency graph of these structures in figure 3. In the naming of the nodes $H$ stands for hexagon, $P$ for pentagon and $S$ for septagon and the numbers stands for number of intersecting edges. For example the node $PH2$ stands for the step of removing a pentagon and hexagon intersecting in 2 edges. Node $P$ stands for the step of removing pentagons. Node $HH2nc$ stands for the step of removing a hexagon and hexagon intersecting intersecting in 2 non-consecutive edges. Node $PPP$ stands for the step of removing the structure where two non-adjacent edges of a pentagon are each part of another pentagon. The order of these steps in the algorithm follows from this dependency graph. We also note that there should not be any cycles in
the dependency graph in order to ensure the correctness of the algorithm. After removing pentagons and hexagons, we have one more step which gives the required decrease in $\eta$ only if pentagons and hexagons are not present.

![Dependency graph for the steps performed by the algorithm in a 3-regular graph with no triangles or rectangles](image)

Fig. 3. Dependency graph for the steps performed by the algorithm in a 3-regular graph with no triangles or rectangles

To illustrate some of the above ideas, we give a few steps of our argument here. Most of the ideas that we use, should be covered in these steps. These steps are executed when the graph does not have some simpler structures. These simple structures have been removed by some other steps that we do not talk about here. All the steps will be described in detail in section 2. The details basically involves extensive case analysis using the ideas presented here.

Suppose through some branching and non-branching steps, we have removed all cycles of of length at most 6 from the graph and the graph is 3 regular. Then, a non-trivial residual graph will have the structure as shown in figure 2. We consider the following 3-way branch in this case: (i) there is an MIS including $u$, (ii) there is an MIS including $v$, and (iii) all MIS have $p, q, r, s$. In case (iii), we see that there is a significant reduction in the complexity parameter. In fact, we can argue a reduction of 26. In the first two cases, the reduction is not so high but what we can argue is that in the subsequent steps after this step, there will be significant reduction in the complexity parameter. By using these properties of the branching, we can show that all the branching forms that could result in this case will give the desirable running time.

Now, let us go back one step and see what we can do if the graph is 3-regular and does not have cycles of length 3 and 4 but has a 5-cycle (we call such cycles pentagons). Let us try to design a step that removes a pentagon. Figure 4 shows a pentagon $pqrs$. The first question we ask is whether all vertices in the figure are necessarily distinct? For instance is it possible that $g = e$? No, since the graph does not have any 4-cycle. Similarly, we argue that if the graph does not have any of the following structures shown in figure 5 then all the vertices of figure 4 are distinct. This explains our dependency graph in figure 3. The rightmost node P corresponds to removing pentagons. The directed edge from the rightmost node to nodes H denote that the hexagons should be removed from the graph before the pentagons could be removed. In other words, the removal of pentagons is dependent upon the removal of hexagons. Similarly, the edge from P to PS2 and PPP denote that the first two structures in figure 5 should be removed before we can branch to remove the pentagons and so on.

Given this, we consider the following 3-way branch: (i) there is a MIS including $q$, (ii) there is a MIS including $p$, and (iii) all MIS include $a, b, r, t$. In case (iii), $\eta$ decreases by at least 26. The first
two cases result in a residual graph that can be simplified and in the subsequent branch $\eta$ decreases significantly.

Now, we will go one step back and try to remove the structures in figure 5. The way the algorithm is presented in the appendix is that we first describe the strategy to remove simple structures before describing the strategy for the structures that depend on simple structures.

After an extensive case analysis, we observe that the worst case branching form is $(16, 24, 16, 16)$. The unique positive real root of

$$3 \cdot x^{-16} + x^{-24} = 1$$

is $1.0821$. Hence we get the running time of our algorithm to be $O^*(1.0821^n)$.

Comparison with Algorithm by Xiao and Nagamuchi [22] Our algorithm is closely related to the algorithm by Xiao and Nagamuchi [22]. In order to simplify our algorithm, we use certain ideas in their work such as simple graph structures called funnels, desks etc. However, we use them in a slightly different way in our algorithm. The main distinguishing feature of our algorithm is the removal of 5-cycles and 6-cycles. Although many of the previous works implicitly remove 3-cycles and 4-cycles, none of them have tried to remove 5-cycles or 6-cycles. In [22], the main strategy of the algorithm is to avoid bottleneck cases by wisely choosing which branching rule to use. This reduces the number of branching rules but the analysis involves a lot of case analysis. In our algorithm, instead of avoiding the bottleneck cases, the general strategy is to come up with a new branching rule for a bottleneck case using the structural properties of the graph when that case occurs.
1.3 Parametrized Algorithm for Vertex Cover

Xiao [20] gave a parametrized algorithm for the k-Vertex Cover problem for graphs with bounded degree-3. This problem is defined as follows: Given a graph with bounded degree-3 and an integer k, check if it has a vertex cover of size at most k. Let us denote this problem by $kVC_3$. Xiao [20] gave an algorithm for $kVC_3$ with running time $O(1.6651^{k-\frac{2}{5}n_0})$, where $n_0$ denotes the number of vertices in the graph with degree $\geq 2$. Using this algorithm and our algorithm for MIS3, we can obtain a parametrized algorithm for $kVC_3$ in the following manner: Let $\alpha = 1.8026$. If $n > \alpha k$, then use Xiao’s algorithm that has a running time of $O(1.6651^{k(1-2\alpha/5)})$. Otherwise, we use our algorithm for solving MIS3 and then use the complement as the vertex cover. This has a running time of $O^*(1.0821^{\alpha k})$. So, the overall running time of the parametrized algorithm is $O^*(1.1529^k)$. This is better than $O^*(1.1616^k)$ running time of the algorithm by Xiao [20]. The improvement is mainly due to the fact that we are using a faster algorithm for MIS3.

2 Algorithm and analysis

Here we give a detailed description and analysis of our algorithm. We start with the preliminaries.

Our algorithm is given in the form of a recursive function $FindMIS(G)$ which outputs a Maximum Independent Set of $G$. We do not give a pseudo code for the algorithm. Each section describes one step of the algorithm. Each step checks for a particular structure in $G$ and if that structure is present makes one or more recursive call to $FindMIS$. The correctness and analysis of the step is done in the same section itself.

2.1 Preliminaries

Following are some of the definitions that we will be using in the rest of the paper.

Definition 1 (Independent Set (IS)). Given a graph $G = (V,E)$, an Independent Set of $G$ is defined to be a subset of vertices such that there is no edge between any two vertices of this subset. Henceforth, we will use the acronym IS for Independent Set.

Definition 2 (Maximum Independent Set (MIS)). Given a graph $G = (V,E)$, a Maximum Independent Set of $G$ is an independent set with maximum cardinality. Henceforth, we will use the acronym MIS for Maximum Independent Set.

Definition 3 (Switching). Given a set $A$ and vertices $u,v$, switch$(A,u,v)$ is defined to be the set $(A \setminus \{u\}) \cup \{v\}$. Similarly for sets $B$ and $C$, switch$(A,B,C)$ is defined as $(A \setminus B) \cup C$.

Definition 4 ($N(A)$). For a given subset of vertices $A$ of a graph $G$, $N(A)$ denotes the largest subset of vertices such that each vertex $v$ in this subset is adjacent to a vertex in $A$. Also, we use $N^2(A)$ to denote $N(N(A))$. For a single vertex $u$, we may omit the set notation and write $N(u)$.

Definition 5 ($\delta_G(v)$). For any vertex $v$ of Graph $G$, $\delta_G(v)$ denotes $|N(\{v\})|$. We may omit the subscript $G$ when it is clear from the context.

Definition 6 ($\eta(G)$). For any graph $G(V,E)$, we define the complexity measure $\eta(G) = \sum_{v \in V} \max(\delta(v) - 2, 0)$. 
We use \( \eta(G) \) as the complexity measure of our algorithm. It is a valid complexity measure as there exist a polynomial time algorithm for MIS-2 and there are no vertices of degree \( \geq 4 \) initially in the given graph.

**Definition 7 (\( \gamma(G) \)).** We will define \( \gamma(G) \) inductively. When we give \( G \) as input to our algorithm \( \text{FindMIS} \), if the first recursive call is made by a branching step, then \( \gamma(G) = G \). Otherwise, the first recursive call should be made by a non-branching step. Let \( G' \) be the input of this call; i.e. the algorithm calls \( \text{FindMIS}(G') \). In this case, \( \gamma(G) = \gamma(G') \).

Less formally, we can think of \( \gamma(G) \) as the graph obtained from \( G \) after a series of zero or more non-branching steps such that no more branching steps can be done on \( \gamma(G) \).

**Definition 8 (\( S(G,A) \)).** Given a graph \( G \) and a subset of vertices \( A \) of \( G \), \( S(G,A) \) is defined to be \( \gamma(G \setminus A) \).

**Definition 9 (\( T(G,A) \)).** Given a graph \( G \) and a subset of vertices \( A \) of \( G \), \( T(G,A) \) is defined to be \( \gamma(G \setminus (A \cup N(A))) \).

**Definition 10 (\( \alpha_i(G,A) \)).** Given a graph \( G \) and a subset of vertices \( A \) of \( G \), \( \alpha_i(G,A) \) is the set of all vertices \( v \) of \( G \) such that \( \delta(v) \geq i + 2 \) in \( G \) and either \( v \notin S(G,A) \) or \( \delta_G(v) - \delta_{S(G,A)}(v) \geq i \).

**Definition 11 (\( \beta_i(G,A) \)).** Given a graph \( G \) and a subset of vertices \( A \) of \( G \), \( \beta_i(G,A) \) is the set of all vertices \( v \) of \( G \) such that \( \delta(v) \geq i + 2 \) in \( G \) and either \( v \notin T(G,A) \) or \( \delta_G(v) - \delta_{T(G,A)}(v) \geq i \).

**Definition 12 (\( \alpha(G,A) \)).** \( \alpha(G,A) \) is defined as \( \sum_i |\alpha_i| \).

Note that \( \alpha(G,A) \) is the decrease in complexity measure from \( G \) to \( S(G,A) \).

**Definition 13 (\( \beta(G,A) \)).** \( \beta(G,A) \) is defined as \( \sum_i |\beta_i| \).

Note that \( \beta(G,A) \) is the decrease in complexity measure from \( G \) to \( T(G,A) \).

**Definition 14 (\( P(G,A) \)).** Given a graph \( G \) and a subset of vertices \( A \) of \( G \), \( P(G,A) \) is defined as \( \text{FindMIS}(G \setminus A) \).

**Definition 15 (\( Q(G,A) \)).** Given a graph \( G \) and a subset of vertices \( A \) of \( G \), \( Q(G,A) \) is defined as \( \text{FindMIS}(G \setminus (A \cup N(A))) \cup A \).

**Definition 16 (\( \text{MaxSet}(A_1,A_2,...,A_n) \)).** \( \text{MaxSet}(A_1,...,A_n) \) denotes a set with highest cardinality among the sets \( A_1, A_2, \ldots, A_n \).

**Definition 17 (Branching form).** When we mention that a particular branching step has a branching of the form \( (a_1,a_2,...,a_n) \), we mean that the step has \( n \) branches and the branches are such that in the \( i \)th branch \( \eta \) decreases by at least \( a_i \).

**Definition 18 (Odd Chain).** A sequence of degree-2 vertices \( v_1, v_2, \ldots, v_k \) is called an odd chain if \( k \) is odd, \( N(v_i) = \{v_{i-1}, v_{i+1}\} \forall 1 < i < k \), \( N(v_1) = \{v_2, u\} \) and \( N(v_k) = \{v_{k-1}, w\} \) where \( u \) and \( w \) are vertices of degree \( \geq 3 \).
Definition 19 (Dominating vertex). A vertex $u$ is said to dominate another vertex $v$ if $u \in N(v)$ and $N(u) \setminus \{v\} \subseteq N(v)$. Here $v$ is called a dominated vertex and $u$ is called a dominating vertex.

Definition 20 (Fine Graph). A graph $G$ is said to be a fine graph if it contains at least one of the following:

1. a dominating vertex
2. a degree-4 vertex
3. an odd chain

Definition 21 (Bottleneck Graph). A graph $G$ is called bottleneck if it is not a fine graph.

Definition 22 (Triangle, Rectangle, Pentagon, Hexagon). 3 vertices $p$, $q$ and $r$ are said to form a triangle $pqr$ in $G$ if they form a 3-cycle in $G$. Similarly we call 4-cycles, 5-cycles and 6-cycles as rectangles, pentagons and hexagons respectively.

2.2 Non-Branching Steps

In this subsection, we remove some simple structures from the given graph, in case they exist. We do this without any branching steps. If a structure is removed in a step, then for the proceeding steps, we can assume that particular structure is not present in the graph. We may not mention this explicitly. For each of these steps, we prove the following two properties.

Property 1 (non-decreasing measure property). $\eta$ does not increase during the step

Property 2 (decreasing measure on transition property). If the graph before the step is a fine graph and the graph after the step is a bottleneck graph, then $\eta$ decreases by at least one during the step.

Fact 1. There is a polynomial time algorithm for finding a MIS of a graph with bounded degree 2.

Due to Fact 1, we can assume that the given graph $G$ contain at least one degree-3 vertex. We can assume that $G$ is connected, since MIS of each connected component can be computed independently. We can also assume that the number of vertices in $G$ is $> 10$ as we can find MIS of a graph of lesser size in constant time.

Removing Dominating Vertices If $G$ has a vertex $u$ dominating $v$, then $\text{FindMIS}(G)$ returns $P(G, \{v\})$. The correctness of this step follows from Lemma 1 below. Lemma 2 proves that properties 1 and 2 are satisfied during this step. Note that this step removes all degree-1 vertices since any degree-1 vertex dominates its neighbor.

Lemma 1. If $G$ has a vertex $u$ dominating another vertex $v$, then there exist an MIS $I$ of $G$ such that $v \notin I$.

Proof. Suppose $I'$ is an MIS of $G$ such that $v \in I'$. Then we know $N(v) \cap I' = \emptyset$. But since $N(u) \setminus \{v\} \subseteq N(v)$, we get that $(N(u) \setminus \{v\}) \cap I' = \emptyset$. So $I = (I' \setminus \{v\}) \cup u$ is an IS of $G$. Since $u \notin I'$, we get $|I| = |I'|$ and hence $I$ is an MIS of $G$. \hfill \square

Lemma 2. Properties 1 and 2 are satisfied in this step.
Proof. Since we are not adding any edges or vertices, property \(1\) is satisfied. Let \(u\) and \(v\) be the dominating and dominated vertices respectively. Suppose \(u\) is a degree-1 vertex. In that case, \(v\) should have degree at least 2 since \(G\) has more than 2 vertices. If the degree of \(v > 2\), then \(\eta\) decreases after this step. Consider the case when the degree of \(v\) is 2. Let \(w\) be the other neighbor of \(v\). If \(\delta(w) \geq 3\), then \(\eta\) decreases during the step and we are done. So, assume \(\delta(w) = 2\). (If \(\delta(v) = 1\), then \(G\) has only 3 vertices). In this case \(w\) is a dominating vertex in the resultant graph after deletion and hence the resultant graph is a fine graph. So, property \(2\) is satisfied when \(\delta(v) = 1\). Now consider the case when \(\delta(v) > 1\). Since the graph has more than 4 vertices, at least one vertex in \(\{v\} \cup N(v)\) should have degree \(\geq 3\). Hence \(\eta\) decreases during the deletion of \(v\). Hence, property \(2\) holds.

Note that from the next step onwards (i.e. when all dominating vertices have been removed) if \(G\) is a fine graph, then \(G\) should contain an odd-chain or a degree-4 vertex.

Removing degree-2 vertices Let \(v\) be a degree-2 vertex in \(G\) and let \(N(v) = \{v_1, v_2\}\). If \(v_1\) and \(v_2\) are adjacent then \(v_1\) is dominated by \(v\) and would have been removed in the previous step. So we may assume \(v_1\) and \(v_2\) are not adjacent. In this case, \(\text{FindMIS}(G)\) first calls \(\text{FindMIS}(G')\) where \(G'\) is as defined in Lemma \(3\) below and then finds the MIS of \(G\) as given by lemma \(3\). Lemma \(4\) shows that properties \(1\) and \(2\) holds in this step.

Lemma 3. Let \(G'\) be the graph constructed from \(G\) by deleting \(v\) and coalescing \(v_1\) and \(v_2\) into a single vertex \(u\). Let \(I'\) be an MIS of \(G'\). From \(I'\) we can construct an MIS \(I\) of \(G\) as:

1. If \(u \in I'\), then \(I = (I' \setminus \{u\}) \cup \{v_1, v_2\}\)
2. otherwise, \(I = I' \cup \{v\}\)

Proof. In both the cases, it follows directly that \(I\) is an IS of \(G\) and also \(|I| = |I'| + 1\). Suppose there exist an MIS \(I_1\) of \(G\) such that \(|I_1| \geq |I| + 1 = |I'| + 2\). This means that \(v_1, v_2 \in I_1\). Then \((I_1 \setminus \{v_1, v_2\}) \cup \{u\}\) gives an IS of \(G'\) with size \(|I'| + 1\). This means \(I'\) is not an MIS of \(G'\) which is a contradiction. \(\square\)

Lemma 4. Properties \(1\) and \(2\) holds for this step.

Proof. Let \(G, G', v, v_1, v_2, u\) be as defined in Lemma \(3\). Note that

\[
\eta(G') - \eta(G) = \max(\delta_G(u) - 2, 0) - \max(\delta_G(v_1) - 2, 0) - \max(\delta_G(v_2) - 2, 0) \\
= (\delta_{G'}(u) - 2) - (\delta_G(v_1) - 2 + \delta_G(v_2) - 2) \\
\]

On the other hand, \(\delta_{G'}(u) \leq \delta_G(v_1) + \delta_G(v_2) - 2\). So we get that \(\eta(G') - \eta(G) \leq 0\) and hence property \(1\) holds. We also get that property \(2\) holds except when \(\delta_{G'}(u) = \delta_G(v_1) + \delta_G(v_2) - 2\). This can happen only if \(v_1\) and \(v_2\) does not have any common neighbors except \(v\) in \(G\). So assume, that is the case. If \(v_1\) and \(v_2\) are both of degree \(\geq 3\) or if at least one of \(v_1\) and \(v_2\) has degree \(\geq 4\), then \(u\) is a vertex with degree \(\geq 4\) and hence \(G'\) is not a bottleneck in that case. The only case remaining is when at least one of \(v_1\) and \(v_2\) is of degree 2 and both have degree less than 4. Note that \(G\) does not have a dominating vertex. Suppose \(G\) is a fine graph due to the presence of a degree-4 vertex. Since, no degree-4 vertices are removed in this step, \(G'\) is also a fine graph. If \(G\) is a fine graph due to the presence of an odd chain that includes a subset of vertices \(\{v, v_1, v_2\}\), then
the length of the chain decreases by 2 which makes $G'$ a fine graph again. If $G$ is a fine graph due to the presence of an odd chain that does not include any of the vertices \( \{v, v_1, v_2\} \), then this odd chain is present also in $G'$ and this means that $G'$ is a fine graph.

Hence, if $G$ was a fine graph then $G'$ will be a fine graph. So, we have that property 2 is satisfied in all the cases. \[\square\]

**Removing Roofs** We use the idea of roofs given in [22].

**Definition 23 (roof).** A 5-cycle $u_1u_2u_3u_4u_5$ is called a roof iff $\delta(u_2) = 3$, $\delta(u_5) = 3$, $\delta(u_1) \geq 3$, $\delta(u_3) \geq 3$, $\delta(u_4) \geq 3$ and there is an edge from $u_2$ to $u_5$.

If $G$ has a roof $u_1u_2u_3u_4u_5$, then $\text{FindMIS}(G)$ returns $P(G, \{u_1\})$. The correctness of this is proved in Lemma 5 below.

**Lemma 5.** If $u_1u_2u_3u_4u_5$ is a roof in $G$, then $G$ has an MIS $I$ such that $u_1 \notin I$.

**Proof.** Let $I_1$ be an MIS of $G$ containing $u_1$. Clearly, $u_2, u_5 \notin I_1$. At least one of $u_3$ and $u_4$ is not present in $I_1$. Assume w.l.o.g that $u_3$ is not present. Now, $\text{switch}(I_1, u_1, u_2)$ is an MIS of $G$ which does not contain $u_1$. \[\square\]

Note that removing roofs decreases $\eta$ by at least 1 and hence properties 1 and 2 hold for this step.

**Removing Short 3-funnels and Desks** We use the idea of 3-funnels and desks given in [22].

**Definition 24 (3-funnel).** Suppose $v_1$ is a degree-3 vertex with $N(v_1) = \{v_2, v_3, u\}$ where the degree of $u, v_2$ and $v_3$ are at least 3. Then $u - v_1 - \{v_2, v_3\}$ is said to be a 3-funnel if $v_1v_2v_3$ is a triangle.

**Definition 25 (Short 3-funnel).** A 3-funnel $u - v - \{v_1, v_2\}$ is said to be short if $N(u) \cap N(v) = \emptyset$ and the number of non-adjacent pair of vertices in $(N(u) \setminus \{v\}) \times (N(v) \setminus \{u\})$ is at most $\delta(u)$.

**Definition 26 (Desk).** A chordless 4-cycle $pqrs$ is said to be a desk if

(i) $N(A) \cap N(B) = \emptyset$, where $A = \{p, r\}$ and $B = \{q, s\}$,
(ii) all of $p, q, r, s$ are of degree $\geq 3$, and
(iii) each of $A$ and $B$ has at most 2 neighbours outside $A \cup B$.

We remove short 3-funnels and desks without branching. For this purpose we define alternative subsets which are also given in [22].

**Definition 27 (Alternative subsets).** Two nonempty disjoint independent subsets $A$ and $B$ of vertices in a graph $G$ are said to be alternative if $|A| = |B|$ and there exist an MIS $I$ of $G$ such that either $I \cap (A \cup B) = A$ or $I \cap (A \cup B) = B$.

**Definition 28 (fold($G, A, B$)).** For any two alternative subsets $A$ and $B$ of $G$, we define fold($G, A, B$) as the graph obtained from $G$ by removing $A \cup B \cup (N(A) \cap N(B))$ and adding an edge between all non-adjacent pair of vertices in $(N(A) \setminus (B \cup N(B))) \times (N(B) \setminus (A \cup N(A)))$. 
Lemma 6. If $A$ and $B$ are alternative subsets in $G$ and $I_f$ is an MIS of $fold(G, A, B)$, then either $I_f \cup A$ or $I_f \cup B$ is an MIS of $G$.

Proof. From definition of alternative subsets, we get that there exist an MIS $I$ of $G$ such that either $I \cap (A \cup B) = A$ or $I \cap (A \cup B) = B$. WLOG assume that $I \cap (A \cup B) = A$. Consider $I' = I \setminus (A \cup B \cup (N(A) \cap N(B)))$. We claim that $I'$ is an independent set of $fold(G, A, B)$. Note that the only reason why $I'$ might not be independent in $fold(G, A, B)$ is that there is an edge $(p, q)$ in $fold(G, A, B)$, where $p \in N(A) \setminus (B \cup (N(A) \cap N(B))), q \in N(B) \setminus (A \cup (N(A) \cap N(B)))$ and $p, q \in I'$. However, this cannot be the case since $A \subseteq I$.

Now we just have to prove that $I_f \cup A$ or $I_f \cup B$ is an IS in $G$. Any pair of vertices in $(N(A) \setminus (B \cup N(B))) \times (N(B) \setminus (A \cup N(A)))$ are adjacent in $fold(G, A, B)$ and so either $I_f \cap (N(A) \setminus (B \cup N(B))) = \phi$ or $I_f \cap (N(B) \setminus (A \cup N(A))) = \phi$. Hence one of $I_f \cup A$ and $I_f \cup B$ is an IS in $G$ (Note that $I_f \cap (N(A) \cap N(B)) = \phi$).

Lemma 7. For a 3-funnel $u - v - \{v_1, v_2\}$ in $G$, $\{u\}$ and $\{v\}$ are alternative subsets.

Proof. Let $I$ be an MIS such that $u, v \notin I$. Since edge $(v_1, v_2)$ is present in $G$, at most one of $v_1$ and $v_2$ can be in $I$. Assume without loss of generality that $v_1 \notin G$. But then, $switch(I, v_2, v)$ is an MIS of $G$ including $v$.

If there is a short-3-funnel $u - v - \{v_1, v_2\}$ in $G$, then $FindMIS(G)$ first calls $FindMIS(fold(G, \{u\}, \{v\}))$ and MIS of $G$ can be calculated from this in polynomial time due to lemma 6. Lemma 14 proves that properties 1 and 2 are satisfied in this step.

![Fig. 6. A 3-funnel $u - v - \{v_1, v_2\}$](image)

Lemma 8. Properties 1 and 2 holds for the case of removing short-3-funnels as given above.

Proof. Consider a graph $G$ with a 3-funnel as showed in figure 6. Suppose it is a short-3-funnel. Let $G' = fold(G, \{u\}, \{v\})$. Let $p$ be the number of non-adjacent pairs in $(N_G(u) \setminus \{v\}) \times (N_G(v) \setminus \{u\})$. By definition of short-3-funnels, $p$ is at most $\delta_G(u)$. Let $l = (\delta_G(u) - 2) + (\delta_G(v) - 2) + (\delta_G(u) - 1) + (\delta_G(v) - 1) = 2\delta_G(u) + 2\delta_G(v) - 6 = 2\delta_G(u)$. Note that $l$ is equal to the decrease in $\eta$ due to the deletion of $u$ and $v$ from $G$. Note that the $fold(G, \{u\}, \{v\})$, in addition to removing $u$ and $v$, adds edges between all non-adjacent pairs in $(N_G(u) \setminus \{v\}) \times (N_G(v) \setminus \{u\})$. This increases $\eta$ by an additive factor of $2p$. So, we get that $\eta(G) - \eta(G') = l - 2p \geq 0$. Hence $\eta$ does not increase during the step and property 1 holds.

Let us consider the case when $\eta(G) - \eta(G') = 0$. This implies $p = \delta(u)$. We will prove that in this case $G'$ either contains a dominating vertex or a degree-4 vertex. This will prove that $G'$ is a
fine graph and hence property 2 will hold. In \( G' \), \( v_1 \) is adjacent to all vertices in \( N_G(u) \setminus \{v\} \). Then, in \( G' \), \( v_2 \) should be adjacent to at least one vertex which is not in \( \{v_1\} \cup (N_G(u) \setminus v) \) so that \( v_2 \) does not dominate \( v_1 \) in \( G' \). But this means that \( v_2 \) has at least 4 neighbors in \( G' \). (Because \( v_2 \) is also adjacent in \( G' \) to all vertices in \( N_G(u) \setminus \{v\} \) and \( v_1 \).) Hence \( v_2 \) is either a degree-4 vertex or a dominating vertex in \( G' \).

Lemma 9. For a desk \( u_1u_2u_3u_4 \), \( A = \{u_1, u_3\} \) and \( B = \{u_2, u_4\} \) are alternative.

Proof. We will prove that there exist an MIS \( I \) such that \( |I \cap \{u_1, u_2, u_3, u_4\}| = 2 \) from which we get that \( I \cap \{u_1, u_2, u_3, u_4\} = A \) or \( B \). Suppose that \( |I \cap \{u_1, u_2, u_3, u_4\}| \leq 1 \). We can assume without loss of generality that \( I \cap \{u_2, u_3, u_4\} = \phi \). From definition of desks it follows that \( |N(\{u_1, u_3\}) \setminus B| \leq 2 \) and if \( u_1 \in I \) we have that \( |I \cap N(\{u_1, u_3\})| \leq 1 \). Hence, switch(\( I, N(A), A \)) is an MIS of \( G \) that contains \( A \).

If there is a desk \( u_1u_2u_3u_4 \) in \( G \) then FindMIS(\( G \)) computes FindMIS(\( fold(G, \{u_1, u_3\}, \{u_2, u_4\} ) \)) and MIS of \( G \) can be calculated from this in polynomial time due to lemma 6. Lemma 10 proves that properties 1 and 2 are satisfied in this step.

Lemma 10. Properties 1 and 2 holds for the case of removing desks as given above.

Proof. Consider the desk \( u_1u_2u_3u_4 \). Let \( D = \{u_1, u_2, u_3, u_4\} \), \( A = \{u_1, u_3\} \) and \( B = \{u_2, u_4\} \). Let \( l = \sum_{u \in D} (\delta(u) - 2) + |N(D) \setminus D| \). Note that \( 2l \) denotes the decrease in \( \eta \) on removing vertices \( D \).

Let \( p \) denote the number of non-adjacent pairs in \( (N(A) \setminus D) \times (N(B) \setminus D) \). Note that \( 2p \) is the increase in \( \eta \) due to addition of edges during the fold operation. Let \( d = \eta(G) - \eta(fold(G, A, B)) \). From the above, we get that \( d = 2l - 2p \). By property of desks, \( |N(D) \setminus D| \) is at most 4 and at least 2. When \( |N(D) \setminus D| < 4 \), we have \( p \leq 2 \) and \( l \geq \frac{1}{2}(4 + 2) = 3 \). So, we have \( d > 0 \) in this case. Hence, except when \( |N(D) \setminus D| = 4 \), properties 1 and 2 are satisfied.

Now consider the case when \( |N(D) \setminus D| = 4 \). This means \( |N(A) \setminus D| = |N(B) \setminus D| = 2 \). So, \( p \leq 4 \) here. Also, \( l \geq \frac{1}{2}(4 + 4) = 4 \). Hence, \( d \geq 0 \) and hence property 1 is satisfied. Now, suppose property 2 is not satisfied in this case. Then \( d = 0 \). This means \( l = 4 \) and \( p = 4 \). But then all vertices in \( D \) should be degree-3 (since \( l = 4 \)). In that case, all the 4 vertices in \( N(D) \setminus D \) will become degree-4 vertices in \( fold(G, A, B) \). This is because all vertices in \( N(D) \setminus D \) are degree-3 vertices and no two vertices in \( N(D) \setminus D \) are connected (since \( p = 4 \)). Hence \( fold(G, A, B) \) is a fine graph in that case. But then property 2 will hold.

Now we state a lemma that will help to analyze the branching steps.

Lemma 11. If \( G_1 \) and \( G_2 \) are two graphs without a degree-1 vertex and \( \eta(G) - \eta(G') \geq k \) where \( k \) is odd, then \( \eta(G) - \eta(G') \geq k + 1 \).

Proof. Follows from the fact that number of odd-degree vertices in a graph is even.

Now, we proceed to the branching steps.
2.3 Removing Good 3-funnels

**Definition 29 (Good 3-funnel).** A 3-funnel \( u_1 - v_1 - \{v_2, v_3\} \) is said to be good if \( v_2 \) or \( v_3 \) has degree greater than 3.

**Lemma 12.** If \( G \) has a 3-funnel \( u_1 - v_1 - \{v_2, v_3\} \), edges \( (u_1, v_2) \) and \( (u_1, v_3) \) are not present in \( G \).

**Proof.** Otherwise \( v_1 \) is a dominating vertex and we have already removed all dominating vertices. \( \square \)

**Lemma 13.** For a 3-funnel \( u_1 - v_1 - \{v_2, v_3\} \), there exist distinct \( t_1 \) and \( w_1 \) such that \( \{t_1, w_1\} \subseteq N(u_1) \setminus (N(v_2) \cup N(v_3)) \).

**Proof.** Let \( A = N(u_1) \setminus (N(v_2) \cup N(v_3)) \). We have to prove \( |A| \geq 2 \). Let \( p \) be the number of non-adjacent pairs in \((N(u_1) \setminus \{v_1\}) \times (N(v_1) \setminus \{u_1\})\). Each vertex in \( N(u_1) \setminus \{v_1\} \cup A \) can contribute at most 1 to \( p \) and each vertex in \( A \) can contribute at most 2. Therefore, \( p \leq |N(u_1)| - 1 - |A| + 2|A| = \delta(u_1) - 1 + |A| \). But if \( p \leq \delta(u_1) \), then \( u_1 - v_1 - \{v_2, v_3\} \) is a short-funnel which is not possible (since we have removed short funnels). Hence, \( p > \delta(u_1) \) which implies \( |A| \geq 2 \). \( \square \)

If \( G \) has a good 3-funnel \( u_1 - v_1 - \{v_2, v_3\} \) such that \( \delta(u_1) \geq 4 \), then we select that good 3-funnel for our branching operation. Otherwise, if \( G \) has a good 3-funnel \( u_1 - v_1 - \{v_2, v_3\} \) such that \( \delta(v_3) \geq 4 \) and at least one vertex in \( N(v_2) \setminus \{v_1, v_2\} \) is in a triangle, then we select that good 3-funnel. Otherwise, we select any good 3-funnel arbitrarily. Let \( u_1 - v_1 - \{v_2, v_3\} \) be the selected good 3-funnel. Now, \( \text{FindMIS}(G) \) returns \( \text{FindMIS}(\text{MaxSet}(Q(G, u_1), Q(G, v_1))) \). The correctness of this follows from lemma 7. In lemma 14, we prove that the branching form is either \((8,12)\) or \((10,10)\).

**Lemma 14.** The branching form in this step is \((8,12)\) or \((10,10)\).

**Proof.** Firstly, we note that a branching of the form \((7,11)\) implies a branching of the form \((8,12)\) and a branching of the form \((9,9)\) or \((9,10)\) implies a branching of the form \((10,10)\) due to lemma 11. Assume without loss of generality that \( \delta(v_3) \geq 4 \). \( G \) should have a subgraph as shown in figure 7 due to the lemmas 12 and 13. Also, we can assume that \( \{w_1, t_1\} \subseteq N(u_1) \setminus (N(v_2) \cup N(v_3)) \). Let \( A = \{u_1, w_1, t_1, v_1, v_2, v_3\} \). We have \( A \cup N(\{w_1, t_1\}) \subseteq \beta_1(G, u_1) \). Let \( B = |N(\{w_1, t_1\}) \setminus \{w_1, t_1, u_1\}| \). We can say that \( A \cap B = \phi \) due to lemmas 12 and 13. So, \( \beta(G, u_1) \geq |\beta_1(G, u_1)| \geq |A| + |B| = 6 + |B| \geq 7 \). Let \( C = \{v_1, v_3, v_2, u_2, u_3, w_3, u_1, t_1\} \). We have that \( C \subseteq \beta_1(G, v_1) \) and \( v_3 \in \beta_2(G, v_1) \). Let \( D = \beta_1(G, v_1) \setminus C \) and \( E = \beta_2(G, v_1) \setminus \{v_3\} \). So, \( \beta(G, v_1) \geq |C| + |D| + |E| + 1 = 10 + |D| + |E| \). Now, suppose that \( \delta(v_2) > 3 \). Then, \( v_2 \in E \) and hence \( \beta(G, v_1) \geq 11 \). So, in that case we get a
branching form of $(7,11)$ and we are done. So, we can assume that $\delta(v_2) = 3$ for the rest of the proof.

Now, suppose that $\delta(u_1) > 3$. Then, $u_1 \in E$ and hence $\beta(G,v_1) \geq 11$. So, in that case we get a branching form of $(7,11)$ and we are done. So, we can assume that $\delta(u_1) = 3$ for the rest of the proof. Now, $\delta(u_2) = 3$ because, otherwise we would have selected good 3-funnel $u_2 - v_2 - \{v_1, v_3\}$ instead of $u_1 - v_1 - \{v_2, v_3\}$. Now, suppose edge $(w_1,t_1)$ was present in $G$. Then $u_2$ should be part of a triangle because otherwise we would have selected good 3-funnel $u_2 - v_2 - \{v_1, v_3\}$ instead of $u_1 - v_1 - \{v_2, v_3\}$. Let $u_2p_2q_2$ be this triangle. Now, $p_2,q_2 \in \beta_1(G,v_1)$ because $u_2$ will become a dominating vertex after the deletion of $\{v_1\} \cup N(v_1)$. If $\{p_2,q_2\} \setminus C \neq \emptyset$, then $|D| \geq 1$ and hence $\beta(G,v_1) \geq 10 + |D| + |E| \geq 11$. So, we get a branching form $(7,11)$ in that case. So, assume $\{p_2,q_2\} \subseteq C$. But this implies $p_2 = w_1$ and $q_2 = t_1$ without loss of generality. Note that $u_2$ cannot have edges to any other vertex in $C$ except $w_1,t_1$ and $v_2$ since we have already removed dominating vertices and short-3 funnels. But in this case, $\delta(w_1) > 3$ and $\delta(t_1) > 3$ in order to make sure that $w_1$ and $t_1$ are not dominating vertices. But, then $w_1, t_1 \in \beta_2(G,u_1)$ and so $\beta(G,u_1) \geq |\beta_1(G,u_1)| + |\beta_2(G,u_1)| \geq 7 + 2 = 9$. Hence, we get a branching form $(9,10)$. So, if edge $(w_1,t_1)$ was present, then we get a branching form of either $(8,12)$ or $(10,10)$. So, let us assume that edge $(w_1,t_1)$ is not present for the rest of the proof.

Let $s_2$ and $s_3$ be 2 neighbours of $w_1$ other than $u_1$. Let $t_2$ and $t_3$ be 2 neighbours of $t_1$ other than $u_1$. Let $F = \{s_2,s_3,t_2,t_3\}$. We get that $F \subseteq B$. So, if $|F| \geq 3$, we have $\beta(G,u_1) \geq 6 + 3 = 9$ and hence a branching form of $(9,10)$. So assume that $|F| \leq 2$. But $|F| \geq 2$ as $s_2$ and $s_3$ are distinct. This implies without loss of generality that $s_2 = t_2$ and $s_3 = t_3$. If $\delta(s_2) > 3$, then $s_2 \in \beta_2(G,u_1)$ and $\beta(G,u_1) \geq 6 + |F| + 1 = 9$. This implies a branching form $(9,10)$. Let us assume that $\delta(s_2) = 3$. Similarly, we can also assume that $\delta(s_3) = 3$. Let $s_4$ be the remaining neighbor of $s_2$ other than $w_1$ and $t_1$. Let $s_5$ be the remaining neighbor of $s_3$ other than $w_1$ and $t_1$. $s_4,s_5 \in \beta_1(G,u_1)$ because $s_2$ and $s_3$ become degree-1 after the deletion of $u_1 \cup N(u_1)$. If $\{s_4,s_5\} \setminus A \neq \emptyset$, then $\beta(G,u_1) = 6 + |F| + |\{s_4,s_5\} \setminus A| = 6 + 2 + 1 = 9$ and we get a branching of the form $(9,10)$.

So, assume that $\{s_4,s_5\} \setminus A = \emptyset$. Then $s_4 = v_3$ and $s_5 = v_2$ without loss of generality. But then $v_3 \in \beta_2(G,u_1)$ and hence we get $\beta(G,u_1) = 6 + |F| + 1 = 9$. So, we get a branching of the form $(9,10)$.

Therefore, in all the cases we get a branching form $(8,12)$ or $(10,10)$.

Note that from next step onwards, if the input graph $G$ has a triangle $pqr$, then either all of $p,q$ and $r$ are degree-3 vertices or all of $p,q$ and $r$ have degree more than 3.

### 2.4 Removing Good Triangles

**Definition 30 (Good Triangle).** A triangle is said to be good if degree of all its 3 vertices are greater than 3.

We will have two sub-cases. Case 1 is performed if there are 2 good triangles having a common edge. Case 2 is performed only when case 1 does not apply.

**Case 1:** If there exist two good triangles having a common edge. In this case, a subgraph as in figure should occur with $p,q,r,s$ having degree $\geq 4$. Clearly in this case, there exist an MIS $I$ of $G$ such that either $(p \notin I \text{ and } r \notin I)$ or $(q \notin I \text{ and } s \notin I)$. Hence in this case $FindMIS(G)$ returns
$MaxSet(P(G\{p,r\}), P(G, \{q,s\}))$. Let $A = \{p, q, r, s\}$. Clearly $A \subseteq \alpha_2(G, \{p, r\})$ and $A \subseteq \alpha_2(G, \{q, s\})$. Also $A \cup \{p', r'\} \subseteq \alpha_1(G, \{p, r\})$ and $A \cup \{q', s'\} \subseteq \alpha_1(G, \{q, s\})$. So the branching in this case is of the form $(6, 10)$.

**Case 2:** In this case, a subgraph as in figure 9 should occur so that case 1 cannot be applied. In this case we select an $u_i$ among $u_1, u_2, u_3$ such that $\beta(G, u_i)$ is the maximum. Note that this can be done in polynomial time. $\text{FindMIS}(G)$ returns $MaxSet(P(G, \{u_i\}), Q(G, \{u_i\}))$. The correctness is due to the fact that any MIS of $G$, either contains $u_i$ or does not contain $u_i$. We prove in lemma 15 that the branching in this case is of the form $(6, 14)$.

**Lemma 15.** Branching in case 2 is of the form $(6, 14)$.

**Proof.** Let $A = \{u_1, u_2, u_3\}$. For all $i \in \{1, 2, 3\}$, we have $A \cup N(u_i) \subseteq \alpha_1(G, \{u_i\})$ and $u_i \in \alpha_2(G, \{u_i\})$. Hence we have $\alpha(G, u_i) \geq 6$ for all $i \in \{1, 2, 3\}$. Now we have to prove that $\exists i \in \{1, 2, 3\}$ such that $\beta(G, u_i) \geq 13$ and then by using lemma 11, we get that the branching is of the form $(6, 14)$. Let $B = \{v_1, w_1, v_2, w_2, v_3, w_3\}$. We have that $A \subseteq \beta_2(G, \{u_i\})$ and $A \cup B \subseteq \beta_1(G, \{u_i\})$ for all $i \in \{1, 2, 3\}$. If $\delta(u_i) \geq 5$ for any $i \in \{1, 2, 3\}$, we get that $u_i \in \beta_3(G, \{u_i\})$ and hence $\beta(G, \{u_i\}) \geq 13$. So let us assume $\delta(u_i) = 4$ for all $i \in \{1, 2, 3\}$. Then there should be at least one vertex in $B$ which has a neighbor outside $A \cup B$ as otherwise $G$ will have less than 10 vertices. Assume without loss of generality that $v_1$ is this vertex and let $p$ be the neighbor of $v_1$ outside $A \cup B$. Clearly $p \in \beta_1(G, \{u_1\})$ and hence $\beta(G, \{u_1\}) \geq 13$. \[\square\]

### 2.5 Removing Rectangles

Suppose $G$ has a rectangle $u_1u_2u_3u_4$. Since $G$ has no dominated vertices, desks, roofs, good 3-funnels and good triangles, without loss of generality we can assume that $v_1, v_2, v_3$ and $w_1$ as shown in figure 9 are present where all of $u_1, u_2, u_3, u_4, v_1, v_2, v_3$ and $w_1$ are distinct.
If \( \delta(u_2) = \delta(u_3) = \delta(u_4) = 3 \), then \( \text{FindMIS}(G) \) returns \( \text{MaxSet}(Q(G, \{u_2, u_4\}), Q(G, \{u_3\})) \). Otherwise \( \text{FindMIS}(G) \) returns \( \text{MaxSet}(P(G, \{u_1, u_3\}), P(G, \{u_2, u_4\})) \). The correctness of this follows from Fact 2 and lemma 16. From lemmas 11, 17, 18, 19 and 20, we get that the branching exhausts all the possible cases in this step.

Fact 2. If \( G \) has a 4-cycle \( u_1u_2u_3u_4 \) then for any MIS \( I \) of \( G \), either \( u_1, u_3 \not\in I \) or \( u_2, u_4 \not\in I \).

Lemma 16. If \( G \) has a 4-cycle \( u_1u_2u_3u_4 \) and \( \delta(u_2) = \delta(u_3) = \delta(u_4) = 3 \), then there exist an MIS \( I \) of \( G \) such that either \( u_3 \in I \) or \( u_2, u_4 \in I \).

Proof. Consider an MIS \( I \) of \( G \). If \( u_2, u_4 \not\in I \), we are done. If \( u_2, u_4 \not\in I \) then we can switch \( u_3 \) with its other neighbor to get an MIS including \( u_3 \). If \( u_2 \in I \) and \( u_4 \not\in I \), then \( u_1, u_3 \not\in I \) and hence we can switch \( u_4 \) with its other neighbor to get an MIS including \( u_2 \) and \( u_4 \). Similarly, we can argue for \( u_2 \not\in I \) and \( u_4 \in I \). \( \square \)

Lemma 17. If \( u_2 \) or \( u_4 \) is of degree \( \geq 4 \), then \( \alpha(G, \{u_1, u_3\}) \geq 9 \) and \( \alpha(G, \{u_2, u_4\}) \geq 9 \).

Proof. Assume without loss of generality that \( \delta(u_2) \geq 4 \). Let \( w_2 \) be the neighbor of \( u_2 \) that is not shown in figure 10. \( w_2 \) cannot be same as any other vertex shown in figure 10 as there are no good triangles or good 3-funnels in \( G \). We have that \( \{u_1, u_2, w_1, u_3, u_4, v_3\} \subseteq \alpha_1(G, \{u_1, u_3\}), \{u_1, u_2\} \subseteq \alpha_2(G, \{u_1, u_3\}), \{u_1, u_2, w_2, u_3, u_4\} \subseteq \alpha_1(G, \{u_2, u_4\}) \) and \( \{u_1, u_2\} \subseteq \alpha_2(G, \{u_2, u_4\}) \). If \( \delta(u_3) > 3 \), we have that \( u_3 \in \alpha_2(G, \{u_2, u_4\}) \) and if otherwise we have \( u_3 \in \alpha_1(G, \{u_2, u_4\}) \) (because \( u_3 \) becomes a degree-1 vertex after deletion of \( u_2 \) and \( u_4 \)). Hence we have that \( \alpha(G, \{u_1, u_3\}) \geq 9 \) and \( \alpha(G, \{u_2, u_4\}) \geq 9 \). \( \square \)

Lemma 18. If \( u_2, u_3 \) and \( u_4 \) are of degree 3 and \( u_4 \not\in N(v_2) \) then \( \beta(G, \{u_2, u_4\}) \geq 9 \) and \( \beta(G, \{u_3\}) \geq 9 \).

Proof. Let \( v_4 \) be the neighbor of \( u_4 \) other than \( u_1 \) and \( u_3 \). \( v_4 \) is not same as \( v_1 \) or \( v_2 \) as there are no good triangles or good 3-funnel in \( G \). \( v_4 \) is not same as \( v_2 \) by our assumption. If \( v_4 = v_3 \), then we have a short funnel \( u_2 - u_3 - \{u_4, v_3\} \). So, \( v_4 \neq v_3 \). Now, we have that \( \{u_1, u_2, w_1, u_3, u_4, v_2, v_3, v_4\} \subseteq \beta_1(G, \{u_2, u_4\}) \). Hence, \( \beta(G, \{u_2, u_4\}) \geq 9 \). Also, \( \{u_1, u_2, u_3, u_4, v_2, v_3, v_4\} \subseteq \alpha_1(G, \{u_2, u_4\}) \) and \( u_1 \in \beta_2(G, \{u_3\}) \). If \( \delta(v_3) > 3 \), we have that \( v_3 \in \beta_2(G, \{u_3\}) \) and hence in that case \( \beta(G, \{v_3\}) \geq 9 \). So, we now assume that \( \delta(v_3) = 3 \). Note that except when \( N(v_3) \setminus \{u_1, u_2, u_4, v_2, v_3\} = \phi \), we have \( \beta(G, \{u_3\}) \geq 9 \). But if \( N(v_3) \setminus \{u_1, u_2, u_4, v_2, v_3\} = \phi \), then \( \{v_2, v_4\} \cap N(v_3) \neq \phi \). Assume without loss of generality that \( v_2 \in N(v_3) \). Now, if \( \delta(v_2) = 3 \), we have that \( v_2u_2u_3v_3 \) is a desk or \( v_2u_2u_3v_3 \) is part of a roof. Hence, \( \delta(v_2) \geq 4 \) and hence \( v_2 \in \beta_2(G, \{u_3\}) \). So, we have \( \beta(G, \{u_3\}) \geq 9 \). \( \square \)
Lemma 19. If \( u_2, u_3 \) and \( u_4 \) are of degree 3 and \( u_4 \in N(v_2) \) then, \( \beta(G, \{u_2, u_4\}) \geq 9 \) and \( \beta(G, \{u_3\}) \geq 9 \).

Proof. In this case, we have that \( \delta(v_2) \geq 4 \) so that, \( u_2u_3u_4v_2 \) is not a desk or part of a roof. We have that \( \{u_1, u_2, v_1, v_3, v_4, v_2, v_3\} \subseteq \beta_1(G, \{u_2, u_4\}) \) and \( \{u_1, v_2\} \subseteq \beta_2(G, \{u_2, u_4\}) \). Hence, \( \beta(G, \{u_2, u_4\}) \geq 10 \). Also, \( \{u_1, u_2, u_3, v_4, v_2, v_3\} \cup N(v_3) \subseteq \beta_1(G, \{u_3\}) \) and \( \{u_1, v_2\} \subseteq \beta_2(G, \{u_3\}) \).

So, except when \( N(v_3) \setminus \{u_1, u_2, u_4, v_2\} = \phi \), we have \( \beta(G, \{u_3\}) \geq 9 \). But if \( N(v_3) \setminus \{u_1, u_2, u_4, v_2\} = \phi \), then \( \{u_1, v_2\} \subseteq N(v_3) \). But then \( u_1 \in \beta_3(G, \{u_3\}) \) and hence \( \beta(G, \{u_3\}) \geq 9 \). \( \square \)

Lemma 20. If \( \delta(u_2) = \delta(u_4) = 3 \) and \( \delta(u_3) \geq 4 \) then \( \alpha(G, \{u_2, u_4\}) \geq 7 \) and \( \alpha(G, \{u_1, u_3\}) \geq 11 \).

Proof. Clearly, \( \{u_1, u_2, u_3, u_4\} \subseteq \alpha_1(G, \{u_2, u_4\}) \) and \( \{u_1, u_3\} \subseteq \alpha_2(G, \{u_2, u_4\}) \). Hence, we have that \( \alpha(G, \{u_2, u_4\}) \geq 7 \). Let \( v_3 \) be the fourth neighbour of \( u_3 \). We have that \( \{u_1, u_2, v_1, v_3, u_4, v_3, v_2, v_3\} \subseteq \alpha_1(G, \{u_1, u_3\}) \) and \( \{u_1, u_3\} \subseteq \alpha_2(G, \{u_1, u_3\}) \). So, except when \( v_3 = v_1 \) or \( v_1 \), we have \( \alpha(G, \{u_1, u_3\}) \geq 11 \). So, let us assume \( v_3 = v_1 \) without loss of generality. Now, if \( \delta(v_1) \geq 3 \), we have \( v_1 \in \alpha_2(G, \{u_1, u_3\}) \) and hence \( \alpha(G, \{u_1, u_3\}) \geq 11 \). So, assume \( \delta(v_1) = 3 \). Then \( v_1 \) becomes a degree-1 vertex after the deletion of \( u_1 \) and \( u_3 \) and hence \( N(v_1) \subseteq \alpha(G, \{u_1, u_3\}) \). Also, in this case \( v_1 \) cannot have edges to any of the vertices shown except \( v_2 \). So, if there is no edge \( (v_1, v_2) \) in \( G \), then we have that \( \alpha(G, \{u_1, u_3\}) \geq 11 \). So, we assume that there is an edge from \( v_1 \) to \( v_2 \). But then \( \delta(v_2) \geq 4 \) since, otherwise \( u_3v_1v_2u_2 \) will be a desk or part of a roof. So, we have that \( v_2 \in \alpha_2(G, \{u_1, u_3\}) \) and hence \( \alpha(G, \{u_1, u_3\}) \geq 11 \). \( \square \)

2.6 Removing a vertex with degree \( \geq 4 \)

If there is a vertex \( v \) in \( G \) with \( \delta(v) \geq 4 \), then \( \text{FindMIS}(G) \) returns \( \text{MaxSet}(P(G, \{v\}), Q(G, \{v\})) \). The correctness is obvious as any MIS of \( G \) either contains \( v \) or does not contain \( v \). From lemma [21] we have that the branching in this case is in the form (6, 14).

Lemma 21. In this case \( \alpha(G, \{v\}) \geq 6 \) and \( \beta(G, \{v\}) \geq 14 \).

Proof. Clearly \( v \in \alpha_2(G, \{v\}) \) and \( |N(\{v\}) \cup \{v\}| \subseteq \alpha_1(G, \{v\}) \). But \( |N(\{v\}) \cup \{v\}| = 5 \). So \( \alpha(G, \{v\}) \geq 6 \). We also have, \( v \in \beta_2(G, \{v\}) \) and \( |N(\{v\}) \cup N^2(\{v\}) \cup \{v\}| \subseteq \beta_1(G, \{v\}) \). Since there are no rectangles and no triangles having a vertex with degree \( \geq 4 \) in \( G \), we have that \( |N(\{v\}) \cup N^2(\{v\}) \cup \{v\}| = 1 + 4 + 8 = 13 \). So, \( \beta(G, \{v\}) \geq 14 \). \( \square \)

Note that from next step onwards we can assume the input graph \( G \) to be 3-regular. Also, \( G \) should be a bottleneck graph from next step.

2.7 Removing 3-funnels

If \( G \) has a 3-funnel \( u_1 - v_1 - \{v_2, v_3\} \) such that at least one vertex in \( N(\{v_2, v_3\}) \backslash \{v_1, v_2, v_3\} \) is in a triangle, then we select this 3-funnel. Otherwise, we select any 3-funnel arbitrarily. Let \( u_1 - v_1 - \{v_2, v_3\} \) be the selected 3-funnel. Now, \( \text{FindMIS}(G) \) returns \( \text{FindMIS}(\text{MaxSet}(Q(G, u_1), Q(G, v_1))) \). The correctness of this follows from lemma [7] In lemma [22] we prove that the branching form is (8, 10).

Lemma 22. The branching form in this step is (8, 10).
Proof. Firstly, we note that a branching of the form (7, 9) implies a branching of the form (8, 10) due to lemma 11 and 13. Also, we can assume that \( \{w_1, t_1\} \subseteq N(u_1) \setminus \{N(v_2) \cup N(v_3)\} \). Let \( A = \{u_1, w_1, t_1, v_1, v_2, v_3\} \). We have \( A \cup N(\{w_1, t_1\}) \subseteq \beta_1(G, u_1) \). Let \( B = |N(\{w_1, t_1\})| \setminus \{w_1, t_1, u_1\} \). We can say that \( A \cap B = \phi \) due to lemmas 12 and 13. So, \( \beta(G, u_1) \geq |\beta_1(G, u_1)| \geq |A| + |B| = 6 + |B| \geq 7 \). Let \( C = \{v_1, v_3, v_2, u_2, u_3, u_1, w_1, t_1\} \). We have that \( C \subseteq \beta_1(G, v_1) \). Let \( D = \beta_1(G, v_1) \setminus C \). So, \( \beta(G, v_1) \geq |C| + |D| = 8 + |D| \). Now, suppose edge \( (w_1, t_1) \) was present in \( G \). Then \( u_2 \) (or \( v_3 \)) should be part of a triangle because otherwise we would have selected 3-funnel \( u_2 - v_2 - \{v_1, v_3\} \) (or \( v_3 - v_3 - \{v_1, v_3\} \)) instead of \( u_1 - v_1 - \{v_2, v_3\} \). Let \( u_2, v_2 \) be this triangle. \( p_2, q_2 \in \beta_1(G, v_1) \) because \( u_2 \) will become a dominating vertex after the deletion of \( v_1 \cup N(v_1) \). If \( \{p_2, q_2\} \setminus C \neq \phi \), then \( |D| \geq 1 \) and hence \( \beta(G, v_1) \geq 8 + |D| \geq 9 \). So, we get a branching form (7, 9) in that case. So, assume \( p_2, q_2 \in C \). But this implies \( p_2 = w_1 \) and \( q_2 = t_1 \) without loss of generality. \( u_2 \) cannot have edges to any other vertex in \( C \) except \( u_1, t_1 \) and \( v_2 \). But in this case, \( w_1 \) is a dominating vertex which is not possible. So, if edge \( (w_1, t_1) \) was present, then we get a branching form (8, 10).

So, let us assume that edge \( (w_1, t_1) \) is not present for the rest of the proof. Let \( w_2 \) and \( w_3 \) be two neighbors of \( w_1 \) other than \( u_1 \). Let \( t_2 \) and \( t_3 \) be two neighbors of \( t_1 \) other than \( u_1 \). Let \( F = \{w_2, w_3, t_2, t_3\} \). We get that \( F \subseteq B \). So, if \( |F| \geq 3 \), we have \( \beta(G, u_1) \geq 6 + 3 = 9 \) and hence a branching form of (9, 8). So assume that \( |F| \leq 2 \). But we know that \( |F| \geq 2 \) as \( w_2 \) and \( w_3 \) are distinct. This implies without loss of generality that \( w_2 = t_2 \) and \( w_3 = t_3 \). Let \( w_4 \) be the remaining neighbor of \( w_2 \) other than \( w_1 \) and \( t_1 \). Let \( w_5 \) be the remaining neighbor of \( w_3 \) other than \( w_1 \) and \( t_1 \). \( w_4, w_5 \in \beta_1(G, u_1) \) because \( w_2 \) and \( w_3 \) become degree-1 after the deletion of \( u_1 \cup N(u_1) \). If \( \{w_4, w_5\} \setminus A \neq \phi \), then \( \beta(G, u_1) = 6 + |F| + |\{w_4, w_5\} \setminus A| = 6 + 2 + 1 = 9 \) and we get a branching of the form (9, 8). So, assume that \( \{w_4, w_5\} \setminus A = \phi \). Then \( w_4 = v_3 \) and \( w_5 = v_2 \) without loss of generality. Then \( w_2 = u_3, w_3 = u_2 \) and \( G \) has only 8 vertices which is not possible. Therefore, in all the possible cases we get a branching form (8, 10).

Note that from next step onwards, the input graph \( G \) cannot have any triangles.

2.8 Some Properties of a 3-regular graph without rectangles and triangles

From this section onwards, we can safely assume that the input graph \( G \) is 3-regular and does not have triangles and rectangles. We may not mention this explicitly in the lemmas that follow.

Lemma 23. If a hexagon \( pqrstuv \) occurs in \( G \), then there exist an MIS \( I \) of \( G \) such that either \( p, r, t \in I \) or \( q, s, u \in I \) or \( p, s \in I \) or \( q, t \in I \) or \( r, u \in I \).

Proof. For the sake of contradiction, assume that there is an MIS \( I_1 \) of \( G \) such that none of the cases above occur. Then there exists three adjacent vertices of the hexagon which are not in \( I_1 \). So
if we include the middle vertex among them in $I_1$ and remove its other neighbor from $I_1$, it still remains an MIS. Repeating this step as long as there are 3 adjacent vertices of the hexagon which are not in $I_1$, we end up in an MIS including one of the 5 combinations given in the lemma.

\[ \text{Lemma 24. If a pentagon } pqrst \text{ occurs in } G, \text{ then there exists an MIS } I \text{ of } G \text{ such that either } p, r \in I \text{ or } p, s \in I \text{ or } q, s \in I \text{ or } q, t \in I \text{ or } r, t \in I. \]

\[ \begin{proof} \text{For the sake of contradiction, assume that there is an MIS } I_1 \text{ of } G \text{ such that none of the cases above occur. Then there exists three adjacent vertices of the pentagon which are not in } I_1. \text{ So if we include the middle vertex among them in } I_1 \text{ and remove its other neighbor from } I_1, \text{ it still remains an MIS. Repeating this step as long as there are 3 adjacent vertices of the pentagon which are not in } I_1, \text{ we end up in an MIS including one of the 5 combinations given in the lemma.} \quad \square \end{proof} \]

\[ \text{Lemma 25. If subgraph as in Figure 12 occurs in } G, \text{ then there exist an MIS } I \text{ of } G \text{ such that either } u \in I \text{ or } v \in I \text{ or } p, q, r, s \in I. \]

\[ \begin{proof} \text{Suppose no MIS of } G \text{ contains } u \text{ or } v. \text{ Then all MIS of } G \text{ should contain } p, q, r, s \text{ because, otherwise there will be an MIS containing } u \text{ or } v. \quad \square \end{proof} \]

![Fig. 12. If no MIS contain u or v then all MIS contain p, q, r, s.](image)

**Definition 31 (Complete-pentagon property).** A vertex $u$ in $G$ is said to satisfy complete-pentagon property in $G$ iff, for all $v \in N(u)$, the following property is satisfied: for all $w \in N(v)$, there exist a pentagon including all of $u, v$ and $w$.

\[ \text{Lemma 26. For any vertex } u \text{ in the given graph } G \text{ with } N(u) = \{u_1, u_2, u_3\}, \]

1. $\eta(G) - \eta(G \setminus (\{u\} \cup N(u))) \geq 10$ and,
2. either $G \setminus (\{u\} \cup N(u))$ contains an odd chain or $u$ satisfies complete-pentagon property in $G$.

\[ \begin{proof} \quad 1. \text{All vertices in } \{u\} \cup N(u) \cup N^2(u) \text{ are either absent or degree-2 vertices in } G \setminus \{u\}. \text{ Since } G \text{ is triangle and rectangle-free, all the 10 vertices in } \{u\} \cup N(u) \cup N^2(u) \text{ are distinct. So } \eta(G) - \eta(G \setminus \{u\}) \geq 10. \]

2. Let $G' = G \setminus (\{u\} \cup N(u))$. Suppose $G'$ does not contain an odd chain. We will prove that $u$ satisfies complete-pentagon property in $G$. All vertices in $N^2(u)$ are degree-2 vertices in $G'$ and these are the only degree-2 vertices in $G'$. So any vertex in $N^2(u)$ should have an edge in $G$ to some other vertex in $N^2(u)$ in order to avoid an odd chain in $G'$. Let $v$ be any vertex in $N(u)$ and $w$ be any vertex in $N(v)$. Since $w \in N^2(u)$ and there are no triangles in $G$, we have that $w$ should have an edge to some vertex in $N^2(u) \setminus N(v)$. Hence there is a pentagon including $u, v$ and $w$ and therefore $u$ satisfies complete-pentagon property. \quad \square \end{proof} \]
**Definition 32 (X-branch).** Suppose we are performing a branching step on $G$. If one of the branches produced simply includes $u$ (i.e., it returns $Q(G, \{u\})$) and $u$ does not satisfy complete-pentagon property, then that particular branch of the branching step is called an X-branch. We denote an X-branch by the letter $X$ in a branching form. For example, the branching form $(X, 16, 16)$ denotes that the branching step produces 3 branches, one of them is an X-branch and in the other two $\eta$ decreases by at least 16.

**Lemma 27.** For any two distinct vertices $u$ and $v$ in $G$ with $\{u_1, u_2\} \subset N(u)$ and $\{v_1, v_2\} \subset N(v)$, $|\{u_1, u_2\} \cup \{v_1, v_2\}| \geq 3$.

*Proof.* Follows from the fact that $G$ does not have any triangles or rectangles. \hfill $\Box$

2.9 Pentagon and hexagon sharing two edges

In a rectangle and triangle free 3-regular graph, a hexagon and a pentagon cannot share 2 non-consecutive edges. So, a subgraph as in Figure 13 should occur.

![Fig. 13. A pentagon and hexagon sharing 2 edges](image)

In this case $\text{FindMIS}(G)$ returns $\text{MaxSet}(Q(G, \{b\}), Q(G, \{f\}))$. The correctness is proved by lemma 28. From lemma 26 it follows that the branching is of the form $(8, 10)$.

**Lemma 28.** If the subgraph as in Figure 13 occurs in graph $G$, then there exist an independent set $I$ of $G$ such that either $b \in I$ or $f \in I$.

*Proof.* Suppose for the sake of contradiction, $b$ and $f$ do not belong to any MIS of $G$. Then any MIS of $G$ will include $a, c$ and $g$. This implies that $e, d$ and $h$ are not present. But then we can switch $a$ with $e$ and then switch $c$ with $b$. Thus we have an MIS including $b$ and hence a contradiction. \hfill $\Box$

2.10 Two Hexagons sharing three edges

In a rectangle and triangle free graph, if two hexagons share three edges, then they should be consecutive edges and a subgraph as in Figure 14 occurs.

In this case $\text{FindMIS}(G)$ returns $\text{MaxSet}(Q(G, \{f\}), Q(G, \{e\}))$. The correctness follows from lemma 29. From lemma 26 it follows that the branching is of the form $(8, 10)$. 

Lemma 29. If the subgraph as in Figure 14 occurs in graph $G$, then there exists an independent set $I$ of $G$ such that either $f \in I$ or $c \in I$.

Proof. Suppose, for the sake of contradiction $c$ and $f$ do not belong to any MIS of $G$. So, any MIS of $G$ includes at least two vertices from $\{a, e, g\}$ and 2 vertices from $\{b, d, h\}$. Note that this is not possible.  

\[\square\]

2.11 Two hexagons sharing two non-consecutive edges

In a rectangle and triangle free graph, if two hexagons share two non-consecutive edges, then a subgraph as in Figure 15 occurs.

In this case $\text{FindMIS}(G)$ returns $\text{MaxSet}(Q(G, \{d\}), Q(G, \{g\}))$. The correctness follows from lemma 30. From lemma 26 it follows that the branching is of the form $8, 10$.

Lemma 30. If the subgraph as in Figure 15 occurs in graph $G$, then there exist an independent set $I$ of $G$ such that either $d \in I$ or $g \in I$.

Proof. Suppose $d$ or $g$ do not belong to any MIS of $G$. So, any MIS of $G$ includes $c$ and $e$. Then there exist an MIS $I_1$ of $G$ including $a, c$ and $e$. This implies that $h, i \notin I_1$. Now, if we remove $a, c, e$ from $I_1$ and add $b, d, f$, it is still an MIS. But this contradicts the assumption that $d$ does not belong to any MIS of $G$.  

\[\square\]
2.12 Two Pentagons sharing two edges

In a rectangle and triangle free graph, if two pentagons share two edges, then a configuration as in Figure 16 occurs.

![Fig. 16. Two pentagons sharing two edges](image)

**Lemma 31.** \( p, q, r, s, t \) are distinct from each other and distinct from \( a, b, c, d, e, f \) and \( g \).

*Proof.* Since there are no rectangles or triangles in the graph, we can say \( p \neq r \), \( p \neq q \), \( p \neq t \), \( t \neq r \), \( s \neq r \), \( q \neq s \), \( t \neq r \), \( t \neq s \) and \( t \neq q \). Also, there cannot be any edge between any of \( a, b, c, d, e, f \) and \( g \) other than the ones shown in Figure 16. \( p \neq s \) because if \( p = s \) then hexagons \( fbcdeg \) and \( pfbad \) share two non-consecutive edges. Similarly, \( r \neq q \). \( \Box \)

**Lemma 32.** There are no edges between any of \( p, q, r \) and \( s \).

*Proof.* If edge \( (p, q) \) exist then pentagon \( pfbcq \) and hexagon \( fbcdeg \) share two edges. Edge \( (p, r) \) cannot exist as that will form a rectangle. If edge \( (p, s) \) exist then hexagons \( pbcds \) and \( fbcdeg \) share three edges. Now using a symmetric argument, we can say there are no edges between any of \( p, q, r \) and \( s \). \( \Box \)

**Lemma 33.** Either \( (p, t) \) and \( (q, t) \) are not present in the graph or \( (r, t) \) and \( (s, t) \) are not present.

*Proof.* If edge \( (p, t) \) and \( (s, t) \) are both present, then pentagon \( pfbat \) and hexagon \( tabcds \) share two edges. So edges \( (p, t) \) and \( (s, t) \) cannot be present at the same time. Similarly, edges \( (r, t) \) and \( (q, t) \) cannot be present together. If edges \( (p, t) \) and \( (r, t) \) are both present, then hexagon \( ptae\) and hexagon \( rtabfg \) share two non adjacent edges. So \( (p, t) \) and \( (r, t) \) cannot be present at the same time. Similarly, \( (q, t) \) and \( (s, t) \) cannot be present at the same time. Hence, either \( (p, t) \) and \( (q, t) \) are not present or \( (r, t) \) and \( (s, t) \) are not present. \( \Box \)

In case a configuration as in Figure 17 occurs in \( G \) and edge \( (w, q) \) is present, then \( \text{FindMIS}(G) \) returns \( \text{MaxSet}(Q(G, \{f\}), Q(G, \{b\})) \). The correctness of this is proved in lemma 48. From lemma 26 it follows that in this case the branching is of the form \((8, 10)\).

In case a configuration as in Figure 17 occurs in \( G \) and edge \( (w, q) \) is not present, then \( \text{FindMIS}(G) \) returns \( \text{MaxSet}(Q(G, \{u\}), Q(G, \{b, p\}), Q(G, \{c, s\})) \). The correctness of this is proved in lemma 42. From lemmas 45, 47, 46 and 11 it follows that in this case the branching is of the form \((X, 16, 16)\).
Lemma 34. If configuration as in Figure 16 occurs in $G$, then $f$ does not satisfy complete-pentagon property in $G$.

Proof. Suppose $f$ satisfied complete-pentagon property in $G$. Then there should be a pentagon including all of $f,b$ and $c$. This is possible only if either edge $(p,q)$ is present or edge $(g,q)$ is present or $p=s$ or $p=e$ or $g=s$ or $g=e$. By lemma 31, $p$ is not equal to $e$ or $s$ and there cannot be an edge from $p$ to $q$ due to lemma 32. Edge $(g,q)$ is not present because of lemma 31. $g$ is not equal to $e$ or $s$ as there are no rectangles in $G$. Hence our assumption that $f$ satisfies complete-pentagon property is contradicted.

Lemma 35. If configuration as in Figure 16 occurs in $G$, then $g$ does not satisfy complete-pentagon property in $G$.

Proof. Follows by symmetry from lemma 34.

Lemma 36. If the configuration as in Figure 16 occurs in graph $G$, then there exist an independent set $I$ of $G$ such that either $f \in I$ or $b,g,d \in I$ or $p,s \in I$.

Proof. Suppose there is no MIS of $G$ including $f$ or including $b,g$ and $d$. Then, since $fbcdeg$ is a hexagon, there exist an MIS $I_1$ of $G$ such that either $c,g \in I_1$ or $b,e \in I_1$ (Due to lemma 23).

- Case $c,g \in I_1$: If $p \notin I_1$, then we can switch $g$ with $f$. If $s \notin I_1$ then we can switch $c$ with $d$ and then include $b$. Therefore $p,s \in I_1$.
- Case $b,e \in I_1$: If $p \notin I_1$ then we can switch $b$ with $f$. If $s \notin I_1$ then we can switch $e$ with $d$ and then include $g$. Therefore $p,s \in I_1$.

Lemma 37. If the configuration as in Figure 16 occurs in graph $G$, then there exist an independent set $I$ of $G$ such that either $g \in I$ or $e,f,c \in I$ or $q,r \in I$.

Proof. This follows from lemma 36 using symmetry.

Lemma 38. If edges $(r,t)$ and $(s,t)$ are not present then, $\beta(G,\{b,d,g\}) \geq 15$.

Proof. Let $A = \{a,b,c,d,e,f,g,t,p,q,r,s\}$ and $B = (N(r) \setminus \{g\}) \cup (N(s) \setminus \{d\})$. We have that $A \cup B \subseteq \beta(G,\{b,d,g\})$. Also, $|A| = 12$ and $B \cap A = \emptyset$ by lemmas 31 and 32. Also, $|B| \geq 3$ by lemma 27. Hence, $\beta(G,\{b,d,g\}) \geq |A| + |B| \geq 12 + 3 = 15$.

Lemma 39. If edges $(p,t)$ and $(q,t)$ are not present then, $\beta(G,\{e,f,c\}) \geq 15$.
Proof. This follows from lemma 38 by symmetry. □

Lemma 40. $\beta(G, \{p, s\}) \geq 15$ except when configuration as in Figure 17 occurs.

Proof. We clearly lose vertices $p, s, b, c, d, e, f$ and $g$. So, we have to lose 7 more vertices. Let $u, v$ be the remaining neighbors of $p$ and $w, x$ be that of $s$. We lose all vertices in $\{u, v, w, x\} \cup N(u) \cup N(v) \cup N(w) \cup N(x)$. Also note that $\{u, v, w, x\} \cap \{a, b, c, d, e, f, g, p, q, r, s\} = \phi$ due to lemmas 31 and 32. So, $N(\{u, v, w, x\}) \cap \{b, c, d, e, f, g\} = \phi$. There are two possible cases.

– Case 1: When $u, v, w$ and $x$ all are distinct.
   Clearly we lose $u, v, w$ and $x$. We have to lose 3 more vertices. Suppose there is a vertex in $\{u, v\}$ (say $u$) which does not have an edge to any vertex in $\{x, w\} = \phi$. Then there should be a vertex in $\{x, w\}$ (say $w$) which does not have edge to any vertex in $\{u, v\}$. Then $|(N(u) \cup N(w)) \setminus \{u, v, w, x, p, s\}| \geq 3$ by lemma 27 and we are done.
   So assume without loss of generality that edges $(u, w)$ and $(v, x)$ are present. (Note that $(u, w)$ and $(u, x)$ cannot be present together). Each of $u, v, w$ and $x$ has one edge remaining. If at least 3 of these 4 edges go to distinct vertices, we are done. The only pair of vertices which can go to same vertex are $(u, x)$ and $(v, w)$. So assume that $u$ and $x$ both have an edge to some vertex $k$ and that $v$ and $w$ both have an edge to some vertex $l$. But this is not possible as hexagons $uwxsyp$ and $xvluwk$ share two non-consecutive edges $(u, w)$ and $(x, v)$.
– Case 2: When $u = x$. Since Figure 17 does not occur, there is no edge from $v$ to $w$. Let $k$ be the remaining neighbour of $u$. Clearly, we loose $u, v, w$ and $k$. So we have to loose 3 more vertices. Since $v$ and $w$ cannot have edges to $k$ and $|(N(v) \setminus p) \cup (N(w) \setminus s)| \geq 3$ (by lemma 27), we lose 3 more vertices. □

Lemma 41. $\beta(G, \{q, r\}) \geq 15$ except when configuration as in Figure 17 occurs.

Proof. This follows from lemma 40 by symmetry. □

Fig. 17. A special case of two pentagons sharing two edges

Lemma 42. If configuration as in Figure 17 occurs in $G$, then there exist an MIS $I$ of $G$ such that either $u \in I$ or $b, p \in I$ or $c, s \in I$. 


Proof. From lemma 36, there exist an MIS \( I_1 \) of \( G \) such that either \( f \in I_1 \) or \( b, d, g \in I_1 \) or \( p, s \in I_1 \). We prove that in all these cases we can find an MIS \( I \) as given in the lemma. If \( u \) is included in some MIS of \( G \), then we are done. So, assume \( u \) is not present in any MIS of \( G \).

- Case 1: When \( f \in I_1 \)
  \( p \notin I_1 \). So, \( s \in I_1 \) since otherwise we can find an MIS containing \( u \). So, \( d \notin I_1 \). Also, \( b \notin I_1 \). So, we can switch \( c \) with \( q \). Hence, there exist an MIS of \( G \) including \( c \) and \( s \).

- Case 2: When \( b, d, g \in I_1 \)
  Since \( f \notin I_1 \) and \( u \notin I_1 \), we can switch \( p \) with \( v \). Hence, there exist an MIS of \( G \) including \( b \) and \( p \).

- Case 3: When \( p, s \in I_1 \)
  \( f, d \notin I_1 \). If \( b \in I_1 \) then \( b, p \in I_1 \) and hence \( I = I_1 \). If \( b \notin I_1 \), then we can switch \( q \) with \( c \) and hence there exist an MIS \( I \) of \( G \) such that \( c, s \in I \).

\[ \square \]

**Lemma 43.** If configuration as in Figure 17 occurs in \( G \), then edges \( (u, t), (u, q) \) and \( (v, q) \) cannot be present in \( G \).

Proof. \((u, t)\) is not present because if it is present then hexagon \( upfbat \) and pentagon \( abfge \) share 2 edges. \((u, q)\) is not present since if it is present then hexagon \( uqcbfp \) and pentagon \( uqcds \) share 2 edges. \((v, q)\) is not present because, if it is present then hexagon \( vuqcdsw \) and pentagon \( puswv \) share 2 edges.

**Lemma 44.** If configuration as in Figure 17 occurs in \( G \), then \( u \neq t, v \neq t \) and \( w \neq t \).

Proof. \( u \neq t \) follows from lemma 33 \( v \neq t \) because, if \( v = t \) then hexagon \( taedsw \) and pentagon \( abcde \) share 2 edges. By symmetry, we can say \( w \neq t \).

**Lemma 45.** If configuration as in Figure 17 occurs in \( G \), then \( u \) does not satisfy complete-pentagon property in \( G \).

Proof. Suppose \( u \) satisfied complete-pentagon property in \( G \). Then there should be a pentagon including all of \( u, s, \) and \( d \). This is possible only if either there is an edge from \( u \) to \( q \) or \( u = a \) or \( u = f \) or \( u = t \) or \( u = b \) or \( u = r \). \( u \) is not equal to any of \( a, f \) and \( b \) as \( G \) has no triangles and rectangles. There cannot be an edge from \( u \) to \( q \) due to lemma 13 \( u \neq r \) due to lemma 32 \( u \neq t \) due to lemma 44. Hence our assumption that \( u \) satisfies complete-pentagon property is contradicted.

**Lemma 46.** If configuration as in Figure 17 occurs in \( G \) and edge \( (w, q) \) is not present, then \( \beta(G, \{b, p\}) \geq 15 \).

Proof. Let \( u_1 \) be the neighbor of \( u \) that is not shown in figure 17. Let \( A = \{a, b, c, d, e, f, g, p, q, s, t, u, v, w\} \). Clearly \( A \cup \{u_1\} \subseteq \beta(G, \{b, p\}) \). From lemmas 31, 32, and 44 it follows that all the 14 vertices in \( A \) are distinct. (Note that \( u, v \) and \( w \) are distinct as there are no triangles and rectangles in \( G \).) By lemmas 31, 32, 44 and 43 we have that \( u_1 \notin A \). Now, \( \beta(G, \{b, p\}) \geq |A \cup \{u_1\}| \geq 15 \).

**Lemma 47.** If configuration as in Figure 17 occurs in \( G \) and edge \( (w, q) \) is not present in \( G \), then \( \beta(G, \{c, s\}) \geq 15 \).
Proof. Let \( A = \{a, b, c, d, e, f, q, s, p, w, u, v\} \). Clearly, \( A \subseteq \beta_1(G, \{c, s\}) \). From lemmas \([31, 32] \) it follows that \( |A| = 12 \). (Note that \( u, v \) and \( w \) are distinct as there are no triangles and rectangles in \( G \).) So, we have to loose 3 more vertices. Let \( j \) and \( k \) be the remaining neighbor of \( u \) and \( w \) respectively. Let the two remaining neighbors of \( q \) be \( m \) and \( n \). We have \( B \subseteq \beta_1(G, \{c, s\}) \). From lemmas \([31, 32, 43, 44] \) it follows that \( B \cap A = \phi \). Hence, if \( |B| \geq 3 \), we are done.

\( j = m \) and \( k = n \) (or \( j = n \) and \( k = m \)) are not possible together as then hexagon \( umqnws \) and pentagon \( upvws \) share two edges. Also \( j \neq k \) and \( m \neq n \). Hence, \( |B| \geq 3 \). \( \Box \)

Lemma 48. If configuration as in Figure 17 occurs in \( G \) and edge \((w, q)\) is present in \( G \), then there exist an MIS \( I \) of \( G \) such that either \( f \in I \) or \( b \in I \).

Proof. Assume that no MIS of \( G \) contains \( b \) and \( f \). Then any MIS of \( G \) will include \( p, g, a \) and \( c \). Let \( I_1 \) be such an MIS of \( G \). \( q, v \notin I_1 \). So \( w \) can be switched with \( s \) in \( I_1 \) and then we can switch \( c \) with \( d \) since \( e \notin I_1 \) to get an MIS \( I \). This contradicts the statement that any MIS should include \( c \). \( \Box \)

Note that if a pentagon and hexagon share three edges then two pentagons sharing two edges are formed. So, after this step, no pentagon and hexagon can share three edges.

Lemma 49. After this step, if \( u \) is a vertex in \( G \) which is in a hexagon, then \( u \) does not satisfy complete-pentagon property.

Proof. Suppose \( aubcde \) is a hexagon in \( G \) and \( u \) satisfies complete-pentagon property. Then there is a pentagon in \( G \) including all of \( u, b \) and \( c \). But, then \( G \) contains either a triangle or two pentagons intersecting in 2 edges or a pentagon and hexagon intersecting in 2 edges. Hence we get a contradiction. \( \Box \)

2.13 Pentagon and hexagon sharing one edge

In this case, a configuration as in figure 18 should occur.

\[ \text{Fig. 18. A pentagon and hexagon sharing one edge} \]

Lemma 50. \( p, q, r, s, t, u, v \) are distinct from each other and distinct from \( a, b, c, d, e, f, g, h \) and \( i \).
Proof. There can be no edge from $b$ to $i$ because if such an edge exists, then pentagons $bcfhi$ and $bcdgi$ share two edges. Symmetrically, we can say there is no edge from $e$ to $h$. There can be no edge from $a$ to $i$ because if such an edge exists, then pentagon $aedgi$ and hexagon $cfhigd$ share two edges. Symmetrically, there can be no edge from $a$ to $h$. $p \neq r$ because if $p = r$ then pentagons $pfcba$ and $abcdde$ share two edges. $p \neq s$ because if $p = s$ then pentagon $abcdde$ and hexagon $pabcfh$ share two edges. By symmetry, we can say that $p \neq t$ and $p \neq u$. $q \neq s$ because if $q = s$ then pentagon $bcfhs$ and hexagon $cfhigd$ share two edges. $q \neq t$ because if $q = t$ then hexagons $qbcfhi$ and $cfhigd$ share three edges. $q \neq u$ because if $q = u$ then pentagon $qbcdg$ and hexagon $cfhigd$ share two edges. Using symmetry, we can say $v \neq s$, $v \neq r$ and $v \neq i$. $r \neq u$ because if $r = u$ then pentagons $fcdgu$ and $fhigu$ share two edges. Other possibilities can similarly be eliminated by observing that $G$ does not have triangles and rectangles.

In this case $FindMIS(G)$ returns $MaxSet(Q(G, \{f\}), Q(G, \{c, g, h\}), Q(G, \{p, r, u\}))$. The correctness of this is proved in lemma 51. From lemmas 49, 52, 53 and 11 it follows that the branching in this case is of the form $(X, 16, 16)$.

Lemma 51. If configuration as in Figure 18 occurs in $G$, then there exist an MIS $I$ of $G$ such that either $\{c, g, h\} \subseteq I$ or $f \in I$ or $\{p, r, u\} \subseteq I$.

Proof. Suppose that for all MIS $I$ of $G$ $f \notin I$ and $\{c, g, h\} \notin I$. Then applying lemma 23 on hexagon $cfhigd$, $\exists$ MIS $I$ of $G$ such that either $\{h, d\} \subseteq I$ or $\{c, i\} \subseteq I$.

– Case 1: When there exists an MIS $I$ of $G$ such that $\{h, d\} \subseteq I$.

If $b \notin I$, then we can switch $d$ with $c$ and $u$ with $g$ to get an MIS including $c, g$ and $h$. So, $b \in I$. Now, if $p \notin I$, then we could switch $b$ with $a$ and get an MIS $I_1$ with $b \notin I_1$. So, $p \in I$. If $r \notin I$, then we can switch $h$ with $f$ to get an MIS with $f$ which violates our assumption. If $u \notin I$, then we can switch $d$ with $g$ and then switch $b$ with $c$ to get an MIS including $c, g$ and $h$ which violates our assumption. So, $\{p, r, u\} \subseteq I$.

– Case 2: When there exists an MIS $I$ of $G$ such that $\{c, i\} \subseteq I$.

This is a symmetric case of case 1. Hence, we can say any MIS contains $p, r$ and $u$.

Lemma 52. If configuration as in Figure 18 occurs in $G$, then $\beta(G, \{c, h, g\}) \geq 15$.

Proof. Let $S = \{a, b, c, d, e, f, g, h, i, q, r, s, t, u\}$. Clearly $S \subseteq \beta_1(G, \{c, h, g\})$. From lemma 50 all vertices in $S$ are distinct. So, it is sufficient to show that at least 1 vertex belongs to $\beta_1(G, \{c, h, g\}) \setminus S$. Clearly, the two remaining neighbors of $s$ belongs to $\beta_1(G, \{c, h, g\})$. Edges $(s, r)$ and $(s, t)$ cannot exist as there cannot be any rectangles in $G$. Edge $(s, u)$ cannot exist as no pentagon and hexagon can share two edges. Edge $(s, q)$ is not present as no two hexagons can share 3 edges. So, the two neighbors of $s$ are different from any of the vertices in $S$ except $q$. So, at least 1 vertex is in $\beta_1(G, \{c, h, g\}) \setminus S$.

Lemma 53. If configuration as in Figure 18 occurs in $G$, then $\beta(G, \{p, r, u\}) \geq 15$.

Proof. Let $A = \{a, b, c, d, e, f, g, h, i, p, r, u\}$ and $B = (N(r) \setminus \{f\}) \cup (N(u) \setminus \{g\})$. Clearly $A \cup B \subseteq \beta_1(G, \{a, g\})$. Due to lemma 50 we have that $|A| \geq 12$. Edges $(p, r)$ and $(p, u)$ are not present as a hexagon and a pentagon cannot share two edges. Edge $(r, u)$ is not present as two hexagons cannot share three edges. So we have $A \cap B = \emptyset$ and by lemma 27 we have that $|B| \geq 3$. Hence, $\beta(G, \{a, g\}) \geq |A| + |B| \geq 12 + 3 = 15$.
2.14 Two hexagons sharing two adjacent edges

Fig. 19. Two hexagons sharing two adjacent edges

The analysis of this step is broken into the following two sub-cases (see Figure 19): (1) \( q = t \) and (2) \( q \neq t \). We discuss these two sub-cases in the next two subsections.

**Case 1: \( q = t \)** If \( q = t \), \( \text{FindMIS}(G) \) returns \( \text{MaxSet}(Q(G, \{c\}), Q(G, \{g\})) \). The correctness of this is proved in lemma 54. From lemma 26 it follows that the branching in this case is of the form (8, 10).

**Lemma 54.** If configuration as in Figure 19 occurs in \( G \) and \( q = t \), then there exist an MIS \( I \) of \( G \) such that either \( c \in I \) or \( g \in I \).

**Proof.** Suppose that for all MIS \( I \) of \( G \), \( c \notin I \) and \( g \notin I \). Then applying lemma 23 on hexagon cghied, we get that there exists an MIS \( I \) of \( G \) such that \( h, d \in I \). So, the vertex \( q = t \) is not present in \( I \). So, we can switch \( \{f, b, d\} \) in \( I \) with \( \{a, c, e\} \) to get an MIS including \( c \). \( \square \)

**Case 2: \( q \neq t \)** In case there is an edge from \( q \) to \( t \), then \( \text{FindMIS}(G) \) returns \( \text{MaxSet}(Q(G, \{d\}), Q(G, \{a, c, e, h\})) \). The correctness of this is proved in lemma 57. From lemma 26 it follows that the branching in this case is of the form (8, 10). In case there is no edge from \( q \) to \( t \), \( \text{FindMIS}(G) \) returns \( \text{MaxSet}(Q(G, \{d\}), Q(G, \{a, c, e, h\}), Q(G, \{q, v, t\})) \). The correctness of this is proved in lemma 56. From lemmas 49, 58, 59 and 11 it follows that the branching in this case is of the form (X, 16, 16).

**Lemma 55.** \( p, q, r, s, t, u, v \) are distinct from each other and distinct from \( a, b, c, d, e, f, g, h \) and \( i \).

**Proof.** Edges \((a, g), (a, b), (a, i), (f, i), (b, h)\) and \((f, h)\) are not present since there are no triangles and rectangles in \( G \). Edges \((f, g)\) and \((b, i)\) are not present since two pentagons cannot share two edges. Edge \((a, h)\) is not present since pentagon and hexagon cannot share two edges. \( s, t, u \) are distinct and \( p, q, r \) are distinct since there are no triangles and rectangles in \( G \). \( p \neq s \) and \( r \neq u \) since no two hexagons can share three edges. \( p \neq u \) and \( r \neq s \) since there are no rectangles in \( G \). \( s \neq q, u \neq q, t \neq r \) and \( t \neq p \) since no pentagon and hexagon can share two edges. \( v \) is not equal to any vertex in \( \{p, r, s, u\} \) as there are no triangles and rectangles in \( G \). \( v \neq q \) and \( v \neq t \) since no two pentagons can share two edges. \( \square \)

**Lemma 56.** If configuration as in Figure 19 occurs in \( G \), then there exist an MIS \( I \) of \( G \) such that either \( d \in I \) or \( \{a, c, e, h\} \subseteq I \) or \( \{q, v, t\} \subseteq I \).
Proof. Suppose that for all MIS $I$ of $G$, $d \notin I$. Then applying lemma 23 on hexagon $abcdef$, we get that there exists an MIS $I$ of $G$ such that either $\{a, c, e\} \subseteq I$ or $\{b, e\} \subseteq I$ or $\{f, c\} \subseteq I$. Consider the following three cases:

- **Case 1**: When there exist an MIS $I$ including $a$, $c$ and $e$. We can switch $t$ in $I$ with $h$ to get an MIS including $a$, $c$, $e$ and $h$.
- **Case 2**: Case 1 does not apply, but there exist an MIS $I$ including $b$ and $e$. If $q \notin I$, then we can switch $b$ in $I$ with $a$ and $g$ in $I$ with $c$ to get an MIS including $a$, $c$ and $e$. Hence, $q \in I$. If $v \notin I$, then we can switch $e$ in $I$ with $d$ to get an MIS including $d$. Hence, $v \in I$. If $t \notin I$, then we can switch $g$ in $I$ with $h$ and then switch $b$ in $I$ with $c$ and $q$ in $I$ with $a$ to get an MIS including $a$, $c$ and $e$. So, $t \in I$. So $\{q, v, t\} \subseteq I$.
- **Case 3**: Case 1 does not apply, but there exist an MIS $I$ including $c$ and $f$. This is a symmetric case of Case 2.

$\square$

**Lemma 57.** If configuration as in Figure 19 occurs in $G$ and there is an edge from $q$ to $t$, then there exist an MIS $I$ of $G$ such that either $d \in I$ or $\{a, c, e, h\} \subseteq I$.

**Proof.** This follows from lemma 56 $\square$

**Lemma 58.** If configuration as in Figure 19 occurs in $G$ and $q \neq t$, then $\beta(G, \{a, e, c, h\}) \geq 16$.

**Proof.** Clearly, $\{a, b, c, d, e, f, g, h, i, p, q, r, s, t, u, v\} \subseteq \beta_1(G, \{a, c, e, h\})$ and from lemma 55 they are all distinct. $\square$

**Lemma 59.** If configuration as in Figure 19 occurs in $G$ and if $q$ and $t$ are distinct and non-adjacent, then $\beta(G, \{q, v, t\}) \geq 15$.

**Proof.** Let $A = \{a, b, c, d, e, f, g, h, i, q, v, t\}$ and $B = (N(q) \setminus \{a\}) \cup (N(v) \setminus \{d\})$. Clearly, $A \cap B \subseteq \beta(G, \{q, v, t\})$. Edges $(q, v)$ and $(v, t)$ cannot be present in $G$ as no two hexagons can share three edges. So we have that $B \cap A = \emptyset$. Also, all the vertices in $A$ are distinct due to lemma 55. Also, $|B| \geq 3$ by lemma 27. Hence, we get that $\beta(G, \{q, v, t\}) \geq |A| + |B| \geq 12 + 3 = 15$. $\square$

### 2.15 Two hexagons sharing one edge

#### Fig. 20. Two hexagons sharing one edge
In this case, (see figure 20), \( \text{FindMIS}(G) \) returns \( \text{MaxSet}(Q(G, \{b\}), Q(G, \{a, g\}), Q(G, \{c, h\})) \). The correctness of this is proved in lemma 60. From lemmas 49, 61, 62, and 11, it follows that the branching in this case is of the form \((X, 16, 16)\).

**Lemma 60.** If configuration as in Figure 20 occurs in \( G \), then there exist an MIS \( I \) of \( G \) such that either \( b \in I \) or \( \{a, g\} \subseteq I \) or \( \{c, h\} \subseteq I \).

**Proof.** Suppose that for all MIS \( I \) of \( G, b \notin I \). Then applying lemma 23 on hexagon \( abcdef \), there exists an MIS \( I \) of \( G \) such that either \( \{a, c, e\} \subseteq I \) or \( \{a, d\} \subseteq I \) or \( \{f, c\} \subseteq I \). Now, we consider each of these 3 cases:

- **Case 1:** There exists an MIS \( I \) including \( a, c \) and \( e \).
  - If \( h \notin I \), then we can switch \( w \) in \( I \) with \( g \) to get an MIS including \( a \) and \( g \). If \( h \in I \), then \( c, h \in I \).
- **Case 2:** There exists an MIS \( I \) including \( a \) and \( d \).
  - If \( g \in I \), then \( I \) is an MIS including \( a, g \). If \( g \notin I \), then we can switch \( a \) with \( f \) and then \( q \) with \( b \) to get an MIS including \( b \).
- **Case 3:** There exists an MIS \( I \) including \( c \) and \( f \).
  - If \( h \in I \) then we are done since \( I \) includes \( \{c, h\} \). Consider the case when \( h \notin I \). If \( j \in I \), then \( v \) can be switched with \( h \), to get an MIS including \( c \) and \( h \). If \( j \notin I \), then we can switch \( f \) with \( e \), and then \( p \) with \( a \) and \( w \) with \( g \) to get an MIS including \( a \) and \( g \).

\( \square \)

**Lemma 61.** If configuration as in Figure 20 occurs in \( G \), then \( \beta(G, \{a, g\}) \geq 15 \).

**Proof.** Let \( A = \{a, b, c, f, e, g, h, i, q, p, w, v\} \) and \( B = (N(p) \setminus \{a\}) \cup (N(w) \setminus \{g\}) \). Clearly \( A \cup B \subseteq \beta_1(G, \{a, g\}) \). Due to previous steps, we have that \( |A| \geq 12 \) and \( A \cap B = \emptyset \). By lemma 27, we have that \( |B| \geq 3 \). Hence, \( \beta(G, \{a, g\}) \geq |A| + |B| \geq 12 + 3 = 15 \).

\( \square \)

**Lemma 62.** If configuration as in Figure 20 occurs in \( G \), then \( \beta(G, \{c, h\}) \geq 15 \).

**Proof.** Let \( A = \{a, b, c, d, e, f, g, h, i, j\} \) and \( B = \{q, r, s, u, v, w\} \). Clearly, \( A \cup B \subseteq \beta_1(G, \{c, h\}) \). All vertices in \( A \) are clearly distinct from each other. Also, since there are no triangles or rectangles in \( G \), pentagon and hexagon cannot share 2 edges and two hexagons cannot share three edges, we have that there can be no other edges between any vertices in \( A \) other than those shown in Figure 20. So \( A \cap B = \emptyset \). We know that \( u, v \) and \( w \) are distinct from each other. We also have that \( s \neq w, s \neq u, s \neq v, q \neq w \) and \( q \neq v \) due to previous steps. Hence, if \( q \neq u \), then \( |B| \geq 5 \) and hence \( |A \cup B| \geq 15 \) and we are done. So assume, \( q = u \) for rest of the proof. Now, \( q(= u) \) becomes a degree-1 vertex after deletion of \( c \) and \( h \). So, the remaining neighbor of \( q \), say \( q_1 \) also belongs to \( \beta_1(G, \{c, h\}) \). \( q_1 \) cannot be equal to \( r, s, w \) or \( v \) as there can be no rectangles in \( G \) and no pentagon and hexagon intersecting in two edges. Also, \( q_1 \) cannot be equal to any vertex in \( A \). Hence \( \beta(G, \{c, h\}) \geq 15 \).

\( \square \)

### 2.16 Hexagon and Septagon sharing two adjacent edges

In this case (see figure 21), \( \text{FindMIS}(G) \) returns \( \text{MaxSet}(Q(G, \{d\}), Q(G, \{e, i\}), Q(G, \{c, h\})) \). The correctness of this is proved in lemma 63. From lemmas 49, 64, and 65, it follows that the branching in this case is of the form \((X, 16, 16)\).
**Lemma 63.** If configuration as in Figure 21 occurs in $G$, then there exist an MIS $I$ of $G$ such that either $d \in I$ or $\{e, i\} \subseteq I$ or $\{c, h\} \subseteq I$.

**Proof.** Suppose that for all MIS $I$ of $G$, $d \notin I$. Then applying lemma 23 on hexagon $abcdef$, there exists MIS $I$ of $G$ such that either $\{a, c, e\} \subseteq I$ or $\{b, e\} \subseteq I$ or $\{f, c\} \subseteq I$.

- **Case 1:** There exists an MIS $I$ including $a, c$ and $e$.
  Since $g, j \notin I$, we can either switch $u$ with $i$ or we can switch $t$ with $h$. So, we get an MIS $I'$ such that either $\{c, h\} \subseteq I'$ or $\{e, i\} \subseteq I'$.

- **Case 2:** There exists an MIS $I$ including $b$ and $e$.
  If $h \notin I$ then we could switch $u$ with $i$ to get an MIS including $e$ and $i$. If $h \in I$, then we could switch $b$ with $c$ to get an MIS including $c$ and $h$.

- **Case 3:** There is an MIS $I$ including $c$ and $f$.
  Symmetric as Case 2.

$\square$

**Lemma 64.** If configuration as in Figure 21 occurs in $G$, then $\beta(G, \{c, h\}) \geq 16$.

**Proof.** Let $A = \{a, b, c, d, e, g, h, i, j, r, s, t, u, w\}$. Clearly $A \subseteq \beta_1(G, \{c, h\})$ and they are all distinct due to previous steps. Also, $N(t) \subseteq \beta_1(G, \{c, h\})$ and there cannot be any edges between any vertex in $A$ and $t$. So, two more vertices other than the 14 vertices in $A$ belongs to $\beta_1(G, \{c, h\})$.

$\square$

**Lemma 65.** If configuration as in Figure 21 occurs in $G$, then $\beta(G, \{e, i\}) \geq 16$.

**Proof.** Follows by symmetry from lemma 64.

$\square$

### 2.17 Hexagon and Septagon sharing three adjacent edges

In this case (see figure 22) $FindMIS(G)$ returns $MaxSet(Q(G, \{h\}), Q(G, \{e, i\}), Q(G, \{g, f\}))$. The correctness of this is proved in lemma 66. From lemmas 69, 67, 68 and 11, it follows that the branching in this case is of the form $(X, 16, 16)$.

**Lemma 66.** If configuration as in Figure 22 occurs in $G$, then there exists an MIS $I$ of $G$ such that either $h \in I$ or $\{e, i\} \subseteq I$ or $\{g, f\} \subseteq I$. 

$\square$
Proof. Suppose that for all MIS $I$ of $G$, $h \notin I$. Then, for an MIS $I$ of $G$, either $g \in I$ or $i \in I$ (otherwise we could switch $r$ and $h$ to get an MIS including $h$). Assume without loss of generality that $g \in I$. So $a \notin I$. If $e \notin I$ then we can switch $p$ and $f$ to get an MIS including $f$ and $g$ and we are done. So assume $e \in I$. So $d \notin I$. Now we can switch $s$ and $i$ to get an MIS including $e$ and $i$.

Lemma 67. If configuration as in Figure 22 occurs in $G$, then $\beta(G, \{g, f\}) \geq 15$.

Proof. Let $A = \{b, a, f, e, d, p, t, g, q, h, r, i\}$. Let $B = (N(p) \setminus \{f\}) \cup (N(q) \setminus \{g\})$. Clearly $A \cup B \subseteq \beta_1(G, \{g, f\})$. Also $A \cap B = \phi$ due to previous steps. $|B| \geq 3$ due to lemma 27. $|A| \geq 12$ due to previous steps. Hence, $\beta(G, \{g, f\}) \geq |A| + |B| \geq 12 + 3 = 15$.

Lemma 68. If configuration as in Figure 22 occurs in $G$, then $\beta(G, \{e, i\}) \geq 15$.

Proof. Symmetric to lemma 67.

Lemma 69. If configuration as in Figure 22 occurs in $G$, then $h$ does not satisfy complete-pentagon property.

Proof. There cannot be a pentagon including $a$ as a hexagon and pentagon cannot intersect. Hence $h$ cannot satisfy complete-pentagon property.

2.18 Removing Hexagons

Fig. 23. Hexagon not intersecting with any other hexagons or pentagons
In this case (see figure 23) \( \text{FindMIS}(G) \) returns \( \text{MaxSet}(Q(G, \{b\}), Q(G, \{c\}), Q(G, \{a, d, r, s\})) \). The correctness of this follows from lemma 25. From lemmas 49, 70 and 11 it follows that the branching in this case is of the form \((X, X, 26)\).

**Lemma 70.** If configuration as in Figure 23 occurs in \( G \), then \( \beta(G, \{a, d, r, s\}) \geq 25 \).

**Proof.** Let \( A \) be the set of all the 24 vertices shown in Figure 23. Clearly, \( A \subseteq \beta_1(G, \{a, d, r, s\}) \). Since a hexagon cannot intersect with any other hexagon or pentagon and it cannot share two or three adjacent edges with a septagon, all vertices in \( A \) are distinct. Also, \( N(y) \subset \beta_1(G, \{a, d, r, s\}) \).

\( y \) cannot have edges to any of the 24 vertices in \( A \) except \( w_1, w_2, x_1 \) and \( x_2 \). From \( y \) there can be an edge to at most one vertex among \( w_1, w_2, x_1 \) and \( x_2 \) because otherwise, either a rectangle is formed or a hexagon and septagon sharing three edges are formed. So, one edge from \( y \) must go to a vertex not in \( A \), say \( y_1 \). So, one more vertex \( (y_1) \) is in \( \beta_1(G, \{a, d, r, s\}) \) in addition to the 24 vertices in \( A \). \( \square \)

### 2.19 Pentagon and Septagon Sharing Two Adjacent Edges

![Fig. 24. Pentagon and Septagon sharing two adjacent edges](image)

**Definition 33 (Good PS Property).** Consider a pentagon \( abcd e \) and a septagon \( baegihf \) in \( G \) intersecting in 2 adjacent edges. Let \( q \) and \( t \) be the neighbours of \( f \) and \( g \) respectively that are not present in the septagon. Let \( v \) be the neighbour of \( a \) that is not present in the pentagon. Then we say that \( abcd e \) and \( baegihf \) together satisfies Good PS Property in \( G \) iff, at least one among edges \( (q, v) \) and \( (t, v) \) is not present in \( G \).

**Lemma 71.** If there is a pentagon and septagon intersecting in 2 adjacent edges in \( G \), then there exist a pentagon and septagon in \( G \) satisfying Good PS Property.

**Proof.** Let \( abcd e \) and \( baegihf \) be a pentagon and septagon intersecting in 2 adjacent edges in \( G \). Then a configuration as in figure 24 occurs. Suppose they do not satisfy Good PS Property. This means edges \( (q, v) \) and \( (v, t) \) are present in \( G \). Now, pentagon \( qvabf \) and septagon \( baegihf \) intersect in 2 adjacent edges. If they satisfy good PS property, then we are done. If they do not, then \( p = r \).

Similarly we have \( u = s \), if pentagon \( utgea \) and septagon \( baegihf \) do not satisfy good PS property. But \( p = r \) and \( u = s \) cannot hold together as then \( G \) has a hexagon \( pedu \). Hence there exist at least one pentagon-septagon pair which satisfies Good PS Property. \( \square \)
We select a pentagon-septagon pair satisfying Good PS Property. Let these pentagon and septagon be \( abcd e \) and \( baegihf \) respectively. A configuration as in figure 24 should occur. In this case \( \text{FindMIS}(G) \) returns \( \text{MaxSet}(Q(G, \{a\}), Q(G, \{e, i\}), Q(G, \{b, h\})) \). The correctness of this is proved in lemma 72. From lemmas 73, 74, 75 and 11 it follows that the branching in this case is of the form \((X, 16, 16)\).

**Lemma 72.** If configuration as in Figure 24 occurs in \( G \), then there exist an MIS \( I \) of \( G \) such that either \( a \in I \) or \( \{e, i\} \subseteq I \) or \( \{b, h\} \subseteq I \).

*Proof.* Suppose that for all MIS \( I \) of \( G \), \( a \notin I \). Then applying lemma 24 on pentagon \( abcd e \), there exists an MIS \( I \) of \( G \) such that either \( \{b, e\} \subseteq I \) or \( \{b, d\} \subseteq I \) or \( \{c, e\} \subseteq I \). Consider the following cases:

- **Case 1:** There exists an MIS \( I \) including \( b \) and \( e \).
  
  
  Since \( f \notin I \), we can find an MIS including one vertex among \( h \) and \( i \). So, there exist an MIS either including \( \{b, h\} \) or including \( \{e, i\} \).

- **Case 2:** There exists an MIS \( I \) including \( b \) and \( d \).
  
  If \( i \notin I \), we can switch \( p \) with \( h \) to get an MIS including \( b \) and \( h \). If \( i \in I \), we can switch \( d \) with \( e \) to get an MIS including \( e \) and \( i \).

- **Case 3:** There exists an MIS \( I \) including \( c \) and \( e \).
  
  Symmetric to Case 2.

\[ \square \]

**Lemma 73.** In this case \( a \) does not satisfy complete-pentagon property in \( G \).

*Proof.* Since \( abcd e \) and \( baegihf \) satisfies Good PS Property, w.l.o.g we can assume that edge \( (q, v) \) is not present. \( g \notin N(f) \) because, otherwise two pentagons sharing 2 edges will be formed. \( d \notin N(f) \) as there are no rectangles in \( G \). Also, edge \( (h, v) \) is not present as that would result in a hexagon \( higeav \). So there is no pentagon including all of \( a, b \) and \( f \) in \( G \). Hence, \( a \) does not satisfy complete-pentagon property in \( G \).

\[ \square \]

**Lemma 74.** If configuration as in Figure 24 occurs in \( G \), then \( \beta(G, \{e, i\}) \geq 15 \).

*Proof.* Clearly, \( \{a, b, c, d, e, f, g, h, i, p, u, t, v\} \in \beta_1(G, \{e, i\}) \) and all of them are distinct due to previous steps. Also, \( N(u) \subseteq \beta_1(G, \{e, i\}) \). \( u \) cannot have edges to any of \( p, t \) and \( v \). So, two more vertices are in \( \beta_1(G, \{e, i\}) \) in addition to the 13 vertices listed above.

\[ \square \]

**Lemma 75.** If configuration as in Figure 24 occurs in \( G \), then \( \beta(G, \{b, h\}) \geq 15 \).

*Proof.* This follows from lemma 73 due to symmetry.

\[ \square \]

### 2.20 A pentagon having two non-adjacent edges each of which is part of another pentagon

In this case a configuration as in figure 25 occurs. In case \( p = v \), \( \text{FindMIS}(G) \) returns \( \text{MaxSet}(Q(G, \{h\}), Q(G, \{e, k\}), Q(G, \{g, q\})) \). The correctness of this is proved in lemma 78. From lemmas 26, 80, 81 and 11 it follows that the branching in this case is of the form \((10, 16, 18)\).

In the remaining case, \( \text{MIS}(G) \) returns \( \text{MaxSet}(Q(G, \{h\}), Q(G, \{e, k\}), Q(G, \{g, q\})) \). The correctness of this is proved in lemma 76. From lemmas 77, 79, 81 and 11 it follows that the branching in this case is of the form \((X, 16, 16)\).
Lemma 76. If configuration as in Figure 25 occurs in G, then there exist an MIS I of G such that either \( h \in I \) or \( \{e, k\} \subseteq I \) or \( \{g, q\} \subseteq I \).

Proof. Suppose that for all MIS \( I \) of G, \( h \notin I \). Then applying lemma 24 on pentagon \( edfg \), there exists an MIS \( I \) of G such that either \( \{e, g\} \subseteq I \) or \( \{f, e\} \subseteq I \) or \( \{g, d\} \subseteq I \). Consider the following cases:

- **Case 1**: There exists an MIS \( I \) including \( e \) and \( g \).
  - If \( k \in I \), we have an MIS including \( e \) and \( k \). If \( q \in I \), we have an MIS including \( g \) and \( q \). If \( k \) and \( q \) are not present, then we can switch \( c \) with \( b \), then \( e \) with \( d \) and then \( g \) with \( h \) to get an MIS including \( h \).

- **Case 2**: There exists an MIS \( I \) including \( f \) and \( e \).
  - If \( k \in I \), we have an MIS including \( e \) and \( k \). If \( k \notin I \), we can switch \( e \) with \( h \) to get an MIS including \( h \).

- **Case 3**: When there is an MIS \( I \) including \( g \) and \( d \).
  - If \( q \in I \), then \( \{g, q\} \subseteq I \). If \( q \notin I \), then we can switch \( a \) in \( I \) with \( b \) and then switch \( d \) with \( e \) to obtain an MIS including \( g \) and \( e \). We can then use case 1.

\( \square \)

Lemma 77. If configuration as in Figure 25 occurs in G and \( p \neq v \) then \( h \) does not satisfy complete-pentagon property.

Proof. There cannot be an edge from any of \( i \) or \( j \) to \( a \) as two pentagons cannot intersect in 2 edges in G. Edge \( (a, v) \) is not present as \( p \neq v \). Edge \( (a, f) \) is not present as there are no rectangles in G. Hence there cannot be any pentagon including all of \( h, e \) and \( a \). So \( h \) does not satisfy complete-pentagon property.

\( \square \)

Lemma 78. If configuration as in Figure 25 occurs in G and \( p = v \), then there exist an MIS I of G such that either \( h \in I \) or \( \{e, k\} \subseteq I \) or \( \{g, q\} \subseteq I \).

Proof. Assume that for all MIS \( I \) of G, \( \{g, q\} \not\subseteq I \) and \( h \notin I \) and \( \{e, k\} \not\subseteq I \). From lemma 76, there exist an MIS \( I \) such that either \( h \in I \) or \( \{e, k\} \subseteq I \) or \( \{g, q\} \subseteq I \). So, there exist an MIS \( I \) including \( e \) and \( k \) and not including \( b \). If \( q \notin I \), then we can switch \( c \) with \( b \) to get an MIS including \( e, k \) and \( b \) which is a contradiction. So \( q \in I \). If \( g \notin I \), then we can switch \( r \) with \( c \), \( e \) with \( a \) and then \( k \) with \( h \) to get an MIS including \( h \) which is a contradiction. So, \( g \in I \). This implies that \( \{g, q\} \subseteq I \) which is a contradiction.

\( \square \)

Lemma 79. If \( p \neq v \), then \( \beta(G, \{e, k\}) \geq 16 \).
Proof. Clearly \( \{a, b, c, d, e, f, g, h, i, j, k, p, u, v\} \subseteq \beta_1(G, \{e, k\}) \) and all of them are distinct due to previous steps and the fact that \( p \neq v \). Also, \( N(v) \subseteq \beta_1(G, \{e, k\}) \). \( v \) cannot have edges to any of the above 14 vertices due to previous steps. So, two more vertices are in \( \beta_1(G, \{e, k\}) \) in addition to the 14 vertices listed above.

Lemma 80. If \( p = v, \) then \( \beta(G, \{e, k, b\}) \geq 17. \)

Proof. Clearly \( \{a, b, c, d, e, f, g, h, i, j, k, p( = v), u, q, r\} \subseteq \beta_1(G, \{e, k, b\}) \) and all of them are distinct due to previous steps. Also, \( N(v) \subseteq \beta_1(G, \{e, k, b\}) \). \( v \) cannot have edges to any of the above 15 vertices due to previous steps. So, two more vertices are in \( \beta_1(G, \{e, k, b\}) \) in addition to the 15 vertices listed above.

Lemma 81. \( \beta(G, \{g, q\}) \geq 15. \)

Proof. Let \( A = \{a, b, c, d, e, f, g, h, i, j, k, s, t\} \). Clearly \( A \subseteq \beta_1(G, \{g, q\}) \). Also, \( N(q) \subseteq \beta_1(G, \{g, q\}) \). All the 14 vertices in \( A \) are distinct and \( q \) cannot have edges to any of the vertices in \( A \) except \( t \) due to previous steps. Hence \( \beta_1(G, \{g, q\}) \geq 15. \)

2.21 Removing Pentagons

After the step in subsection 2.20, a pentagon \( P \) can intersect with at most two pentagons and if it intersects with two pentagons, then each of them share one edge with \( P \) and those edges are adjacent in \( P \). It is also possible that \( P \) does not share edges with any pentagon or that it shares edge with only one pentagon. But \( P \) cannot intersect with any hexagon. So if a pentagon occurs in \( G \) after the step in subsection 2.20, then a configuration as in Figure 26 occurs where all the vertices shown in it are distinct.

In this case \( \text{FindMIS}(G) \) returns \( \text{MaxSet}(Q(G, \{p\}), Q(G, \{q\}), Q(G, \{r, t, a, b\})) \). The correctness of this follows from lemma 25. From lemmas 82, 11 and 83, it follows that the branching in this case is of the form \((X, X, 26)\).

Lemma 82. In this case, \( p \) and \( q \) do not satisfy complete-pentagon property.

Proof. Clearly there is no pentagon including \( p, q \) and \( b \) or including \( q, p \) and \( a \). Hence \( p \) and \( q \) does not satisfy complete-pentagon property.

\( \Box \)
Lemma 83. In this case \( \beta(G, \{r, t, a, b\}) \geq 25 \).

Proof. Let \( A \) be the set of all the 22 distinct vertices shown in Figure 26. Clearly, \( A \subseteq \beta_1(G, \{r, t, a, b\}) \). Also, \( N(h) \cup N(g) \subseteq \beta_1(G, \{r, t, a, b\}) \), \( h \) or \( g \) cannot have edges to any of the 22 vertices in \( A \) except \( u, v, w \) and \( x \). Let \( B = \{u, v, w, x\} \) and \( C = N\{h, g\} \setminus \{a\} \). We have that \( |C| = 4 \), \( C \subseteq \beta_1(G, \{r, t, a, b\}) \) and \( C \cap A \subseteq B \). So, if \( |C \cap B| \leq 1 \), then we are done. So assume that \( |C \cap B| > 1 \). |C \cap B| \leq 2 because, there are no rectangles or hexagons in \( G \). So, we get that \( |C \cap B| = 2 \) and hence \( |C \setminus B| = 2 \). So we have to lose just 1 more vertex to prove the lemma. Also, without loss of generality, we can assume that edge \((h, x)\) is present because \( C \cap B \neq \phi \). Now, \( x \) becomes a degree \(-1\) vertex after deletion of \( \{r, t, a, b\} \). So we loose all vertices in \( N(x) \). In this case, \( x \) cannot have edges to any of the vertices in \( A \) other than \( m \) and \( n \). Since \( x \) cannot have edges to both \( m \) and \( n \), we loose 1 more vertex. \( \square \)

2.22 No Hexagons and No Pentagons

After step 2.21, there are no hexagons or pentagons in \( G \). So, a configuration as in Figure 27 occurs for a graph where all the vertices shown are distinct. Also, note that no vertex in \( G \) satisfies complete-pentagon property.

![Graph with no hexagon or pentagon](image)

In this case, \( \text{FindMIS} (G) \) returns \( \text{MaxSet}(Q(G, \{u\}), Q(G, \{v\}), Q(G, \{p, q, r, s\})) \). The correctness of this follows from lemma 25. From lemma 84 it follows that the branching in this case is of the form \((X, X, 26)\).

Lemma 84. In this case \( \beta(G, \{p, q, r, s\}) \geq 26 \).

Proof. Let \( A \) be the set of all the 22 distinct vertices shown in Figure 27. Clearly, \( A \subseteq \beta_1(G, \{p, q, r, s\}) \). Also, \( N(h) \cup N(g) \cup N(f) \cup N(e) \subseteq \beta_1(G, \{p, q, r, s\}) \). \( h \) or \( g \) cannot have edges to any of the 22 vertices in \( A \) except \( i, j, k, l, m, n, o, t \). Let \( B = \{i, j, k, l\} \) and \( C = \{m, n, o, t\} \). From set \( \{e, f\} \) there can be an edge to at most one vertex in \( B \) because, otherwise hexagon is formed. Similarly, there can be at most one edge from \( \{e, f\} \) to \( C \), from \( \{g, h\} \) to \( B \), and from \( \{g, h\} \) to \( C \). So, out of the 8 edges remaining from \( \{e, f, g, h\} \), 4 should go to vertices outside \( A \) and those 4 should be distinct as there are no hexagons or rectangles in \( G \). So, \( \beta(G, \{p, q, r, s\}) \geq 22 + 4 = 26 \). \( \square \)
Lemma 85. In an X-branch one of the following 2 happens:

1. \( \eta \) decreases by 12 or
2. \( \eta \) decreases by 10 and the next branching on this branch is of the form (6, 14) or (8, 12) or (10, 10).

Proof. In an X-branch, we branch by including a vertex \( u \) in the MIS, where \( u \) does not satisfy the complete-pentagon property. From lemma 26, it follows that \( \eta(G) - \eta(G \setminus \{u\}) \geq 10 \). It also follows that \( G \setminus \{u\} \) has an odd chain. This means \( G \setminus \{u\} \) is a fine graph.

From our discussions in the above subsections we have two observations:

1. In any non-branching step, if the initial graph before the step was a fine graph, then either the graph remains a fine graph after the step or \( \eta \) decreases by at least 1 during the step.
2. In any fine graph, the branchings that can take place are limited to branchings of the form (6, 14), (8, 12) and (10, 10).

From the first observation it follows that either \( \eta(G \setminus \{u\}) - \eta(G \setminus \{u\}) \geq 1 \) or \( \gamma(G \setminus \{u\}) \geq 1 \), then by lemma 11 it follows that \( \eta(G) - \eta(G \setminus \{u\}) \geq 12 \). If \( \gamma(G \setminus \{u\}) \) is a fine graph, then from the second observation it follows that, the next branching will be of the form (6, 14) or (8, 12) or (10, 10). \( \square \)

Due to lemma 85 in all the steps in which we said a branching of the form (X, 16, 16) will occur, we can say that a branching of the form (12, 16, 16) or (16, 24, 16, 16) or (18, 22, 16, 16) or (20, 20, 16, 16) occurs. Here, we just combined the next branch together with the X-branch. Similarly wherever we said that a branching of the form (X, X, 26) occurs, we can say a branching of the form one among (12, 12, 26), (12, 16, 24, 26), (12, 18, 22, 26), (12, 20, 20, 26), (16, 24, 18, 22, 26), (16, 24, 20, 24, 26), (18, 22, 22, 20, 26), (16, 24, 16, 24, 26), (18, 22, 18, 22, 26) and (20, 20, 20, 20, 26) occurs.

From the discussions in above subsections we have that the branchings in our algorithm are limited to branches of the form (8, 10), (10, 10), (8, 12), (6, 14), (X, 16, 16), (10, 16, 18) and (X, X, 26). So, by converting the X-branches as given above, we get that at any step in the algorithm, the branching that occurs has to be of the form of one among (8, 10), (10, 10), (8, 12), (6, 14), (12, 16, 16), (16, 24, 16, 16), (18, 22, 16, 16), (20, 20, 16, 16), (10, 16, 18), (12, 12, 26), (12, 16, 24, 26), (12, 18, 22, 26), (12, 20, 20, 26), (16, 24, 18, 22, 26), (16, 24, 20, 20, 26), (18, 22, 20, 20, 26), (16, 24, 16, 24, 26), (18, 22, 18, 22, 26) and (20, 20, 20, 20, 26). So, the overall complexity of the algorithm can be calculated by solving the recurrence equations corresponding to these branching forms. The worst complexity among them is given by the recurrence equation corresponding to the branching form (16, 24, 16, 16) which is less than 1.0821\(^n\). Hence the overall time complexity of the algorithm is \( O^*(1.0821^n) \).

3 Conclusion and Future Work

We improve the current state of art in exact algorithms for the maximum independent set problem for graphs with bounded degree 3. Our algorithm becomes a part of theoretical quest to get better
exact algorithms for an NP-hard problem. Each attempt to get a better exact algorithm, introduces some new idea. For instance, our algorithm points out that cycles are bad for recursive backtracking algorithms. If the graph does not have cycles, then a simple algorithm should run in time much better than $O^*(2^n)$. Much of the ingenuity of our algorithm lies in devising ways to remove small cycles. Some open questions are: Can we get better exact algorithms for graphs with larger degree bounds? Are there ideas other than removing simple cycle that gives us better exact algorithms? Can we get a sub-exponential time algorithm?

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