A straight waveguide with a wire inducing resonances

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Abstract
We study a straight infinite planar waveguide with a so-called leaky wire attached to the walls of the waveguide. The wire is modelled by an attractive delta interaction supported by a finite segment. If the wire is placed perpendicularly, then the system preserves mirror symmetry which leads to an embedded eigenvalue phenomenon. We show that if we break the symmetry, the corresponding resolvent poles turn into resonances. The widths of the resonances are calculated explicitly in the lowest order perturbation term.

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1. Introduction

A waveguide with a leaky wire

This paper forms part of the line of research often called that of the Schrödinger operator with delta interactions. To explain the physical motivation, let us consider a quantum particle moving in a straight planar waveguide \( \Omega := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{R}, x_2 \in (0, \pi)\} \) of width \( \pi \). Assuming that the waveguide forms impenetrable walls at its boundaries, we exclude the possibility of tunnelling apart from in the area \( \Omega \); mathematically this corresponds to the Dirichlet boundary conditions at \( \partial \Omega \). Moreover, the interface of two different materials produces an additional jump of potential localized on a finite line \( \Sigma \) placed in the waveguide. In fact, \( \Sigma \) is defined by a finite straight line attached to the walls of \( \Omega \); to be precise, \( \Sigma = \{ (\epsilon x_2, x_2) \in \mathbb{R}^2, x_2 \in (0, \pi) \} \). If \( \epsilon = 0 \), then the potential is localized perpendicularly to \( \partial \Omega \) and, consequently, the system possesses mirror symmetry with respect to the axis \( O = \{(x, \pi/2) : x \in \mathbb{R}\} \). Therefore \( \epsilon \) determines a symmetry breaking parameter.
In fact, positivity and negativity of $\epsilon$ lead to the same physical model. Therefore, without loss of generality, we assume that $\epsilon \geq 0$. Since the potential is supported by the set $\Sigma_\epsilon$ of lower dimension, it imitates a so-called leaky quantum wire which can be modelled by a delta interaction. A large number of models with delta interactions are discussed in [2]. Some specific models of leaky quantum wires were studied in [5, 9–11, 15, 17, 20, 21].

The Hamiltonian of the system

The above described system is governed by a Hamiltonian which can be symbolically written as

$$-\Delta_\Omega - \alpha \delta(\cdot - \Sigma_\epsilon), \quad \alpha \in \mathbb{R},$$

where $\Delta_\Omega$ is the Laplace operator acting in $L^2(\Omega)$ with Dirichlet boundary conditions, $\delta(\cdot - \Sigma_\epsilon)$ stands for the Dirac delta modelling the wire and $\alpha$ describes the interaction supported by $\Sigma_\epsilon$.

In the following we mainly assume that the interaction is attractive. In this case, $\alpha$ is positive.

To give a mathematical meaning to the above formal expression we use the form-sum method. More precisely, we construct the form which intuitively preserves the properties of (1.1) and, at the same time, the operator $H_{\alpha,\epsilon}$ associated with this form is self-adjoint.

The main results of the paper

The model is attractive not just due to the applications to quantum mechanics. Mathematically, it reveals a relation between the geometry of a quantum system and its spectral properties. Our main aim is to discover how the geometry of a wire represented by the parameter $\epsilon$ affects the spectral properties of the Hamiltonian $H_{\alpha,\epsilon}$.

To get insight into the spectral structure of the model at hand, let us start our analysis from the case $\epsilon = 0$. Then due to the symmetry of the system, the Hamiltonian can be decomposed into a longitudinal component defined by the one-dimensional Laplace operator $-\Delta(1)$ with one-point interaction determined by the coupling constant $\alpha > 0$ and a transverse component determined by the Dirichlet Laplacian acting in $L^2([0, \pi])$. The latter has a quantized spectrum given by $\{k^2\}_{k \in \mathbb{N}}$. Consequently, the original Hamiltonian $H_{\alpha,0}$ is unitarily equivalent to the orthogonal sum of the operators, which can be symbolically written as

$$\bigoplus_{k \in \mathbb{N}} H_k^\alpha, \quad H_k^\alpha = -\Delta(1) - \alpha \delta(x) + k^2.$$  

The spectrum of the particular component is given by

$$\sigma(H_k^\alpha) = [k^2, \infty) \cup \{E_k\}, \quad k \in \mathbb{N},$$

where

$$E_k = -\frac{\alpha^2}{4} + k^2;$$

cf [2]. The threshold of the essential spectrum of $H_{\alpha,0}$ is determined by the ground state of $H_1^\alpha$. This means that $\sigma_{ess}(H_{\alpha,0}) = [1, \infty)$. Moreover, the same half-line determines the essential spectrum of the system if $\epsilon > 0$, i.e. we have $\sigma_{ess}(H_{\alpha,\epsilon}) = [1, \infty)$; see section 3. Furthermore, it follows from (1.3) that

$$\sigma_p(H_{\alpha,0}) = \{E_k\}_{k \in \mathbb{N}}.$$  

Therefore, the wire induces at least one discrete point of the spectrum below the threshold 1 and an infinite number of embedded eigenvalues. The latter correspond to the phenomenon of autoionization. The states located above the ionization threshold can be experimentally observed, for instance as bumps in the scattering cross section when an electron leaves a
multi-electron atom. Usually helium or, more often, barium atoms are applied in such experimental set-ups. Autoionizing systems are often described by means of so-called Fano profiles \[8, 18, 24\], and they have been discussed in various contexts. For instance see \[22, 23\] (and the references quoted therein), where various models of laser–atomic system interactions were considered.

In our system, the gaps between subsequent embedded eigenvalues behave as

\[ E_{k+1} - E_k = 2k + 1. \]

We show that for small parameter \( \epsilon > 0 \), the embedded eigenvalues are recovered from the essential spectrum and move to the second-sheet continuation of the resolvent, i.e. they produce resonances. Furthermore, it is proved that for \( E_n > 1 \), the pole of the resolvent is localized at

\[ z_n = E_n + V_n \epsilon + W_n \epsilon^2 + O(\epsilon^3), \]

where \( V_n \) and \( W_n \) are found explicitly. We show that \( V_n \) is real, so it does not contribute to the width of the resonance. Therefore \( V_n \) stands for the first perturbation term of the ‘energy shift’. The lowest order of the resonance widths \( \gamma_n \) can be recovered from the imaginary component of \( W_n \), to be precise, given \( n \in \mathbb{N} \) there exists \( \epsilon_n \) such that for \( \epsilon \in (0, \epsilon_n) \) we have

\[ \gamma_n = 2|\Im W_n| \epsilon^2. \]

Finally, let us mention that the problem of a waveguide with delta interaction was studied in \[13\]. The authors discussed spectral properties of the system with varying longitudinal straight line interaction. From the spectral point of view, the aforementioned model is essentially different from the waveguide with the ‘almost perpendicular’ wire which we analysed in this paper. For the related problems concerning spectral properties in the straight waveguides, we recommend seeing \[3, 6, 7, 14, 16\]. On the other hand, the resonance models with delta interactions were studied in \[12, 20, 21\].

**Notation**

- We adopt the abbreviation \( L^2_{\Sigma} \equiv L^2(\Sigma, dl) \), where \( dl \) is a linear measure on \( \Gamma \); the corresponding scalar product will be denoted as \((\cdot, \cdot)_{\Sigma}\).
- In the following, the space \( L^2([0, \pi]) \equiv L^2([0, \pi], dx) \) is frequently used; the corresponding scalar product will be denoted as \((\cdot, \cdot)\) for short.
- The symbol \( \Delta \) stands for the two-dimensional Laplace operator acting in \( L^2(\Omega) \) and satisfying the Dirichlet boundary conditions.
- We use the standard notation \( W^{n,2}_0(\Omega), n \in \mathbb{N}, \) for the Sobolev space of the trace-0 functions.
- We define \( \omega_k(\cdot) := \sin k(\cdot) \).

**2. The Hamiltonian of the system and its resolvent**

**2.1. The Hamiltonian**

The unperturbed Hamiltonian. Let us start our discussion with the ‘free’ Dirichlet waveguide. The Hamiltonian of such a system is given by the self-adjoint operator

\[ H_0 = -\Delta : \mathcal{D}(H_0) \to L^2(\Omega), \]
where the domain \( D(H_0) \) coincides with \( W^{2,2}_0(\Omega) \). Operator \( H_0 \) is associated with the following quadratic form:

\[
\mathcal{E}_0[f] = \int_{\Omega} |\nabla f|^2, \quad f \in W^{1,2}_0(\Omega).
\]

The translational symmetry of the system leads to the following decomposition:

\[
H_0 = -\Delta^{(1)} \otimes 1 + 1 \otimes -\Delta^{(1)}_D, \quad \text{onto} \quad L^2(\mathbb{R}) \otimes L^2([0, \pi]),
\]

where \( \Delta^{(1)} \) is the one-dimensional Laplace operator acting in \( L^2(\mathbb{R}) \) and \( -\Delta^{(1)}_D \) is the one-dimensional Laplace operator acting in \( L^2([0, \pi]) \) and satisfying Dirichlet boundary conditions.

The Hamiltonian of the waveguide with a wire. Now we introduce the delta potential localized at \( \Sigma_e \). The modified Hamiltonian formally corresponds to (1.1). To give a mathematical meaning to this formal expression, we employ the form-sum method. Consider the sesquilinear form

\[
\mathcal{E}_{\alpha, \epsilon}[f] = \int_{\Omega} |\nabla f|^2 - \alpha \int_{\Sigma_e} |I_\epsilon f|^2, \quad f \in W^{1,2}_0(\Omega),
\]

where \( I_\epsilon \) stands for the continuous embedding operator, embedding \( W^{1,2}_0(\Omega) \) into \( L^2(\Sigma_e) \). The form \( \mathcal{E}_{\alpha, \epsilon} \) is symmetric and semi-bounded. Moreover, the Dirac delta supported by \( \Sigma_e \) defines the Kato class measure; cf [4]. This implies the closeness of \( \mathcal{E}_{\alpha, \epsilon} \). Consequently, there exists a uniquely defined operator \( H_{\alpha, \epsilon} \) associated with the form \( \mathcal{E}_{\alpha, \epsilon} \) via the second representation theorem. This operator defines the Hamiltonian of our system.

2.2. The resolvent of \( H_{\alpha, \epsilon} \). The resolvent of the ‘free’ Hamiltonian. The Hamiltonian of the unperturbed waveguide does not admit any bound states. The threshold of the essential spectrum is determined by the lowest transversally quantized energy, i.e. the ground state energy of \( -\Delta^{(1)}_D \); cf (2.1). Since the waveguide width equals \( \pi \), the lowest transversal energy is 1. Therefore

\[
\sigma(H_0) = [1, \infty).
\]

Suppose \( z \in \mathbb{C} \setminus [1, \infty) \) and \( R_0(z) \) stands for the resolvent of \( H_0 \), i.e. \( R_0(z) = (H_0 - z)^{-1} \). Then \( R_0(z) \) is an integral operator with the kernel

\[
G_0(z; x, y) = \frac{i}{\pi} \sum_{k=1}^{\infty} \frac{e^{i(z-x-y)}}{\sqrt{z-x-y^2}} \omega_k(x_2)\omega_k(y_2),
\]

where \( x = (x_1, y_1) \) (analogously \( y \)), the square root function, is defined on the first Riemann sheet and \( \omega_k(\cdot) := \sin k(\cdot) \).

The resolvent of \( H_{\alpha, \epsilon} \). To reconstruct the Krein-like resolvent of \( H_{\alpha, \epsilon} \), we have to introduce embeddings of the ‘free’ resolvent into the space \( L^2(\Sigma_e) \). By means of the trace map \( I_\epsilon \), we define

\[
\tilde{R}_\epsilon(z) := I_\epsilon R_0(z) : L^2(\Omega) \rightarrow L^2(\Sigma_e).
\]

Its adjoint \( \tilde{R}_\epsilon^*(z) : L^2(\Sigma_e) \rightarrow L^2(\Omega) \) acts as \( \tilde{R}_\epsilon^*(z)f = G_0(z) \ast f \delta_{\Sigma_e} \). Moreover, we introduce the bilateral embedding

\[
R_\epsilon(z) = I_\epsilon \tilde{R}_\epsilon(z) : L^2(\Sigma_e) \rightarrow L^2(\Sigma_e).
\]
Finally, we define an operator which is a key tool for further spectral analysis, based on the generalized Birman–Schwinger argument. Let

\[ \Gamma_\epsilon(z) := I - \alpha R_\epsilon(z) : L^2(\Sigma_\epsilon) \to L^2(\Sigma_\epsilon). \]

Relying on the results of [4], we can formulate the following statement.

**Theorem 2.1.** Suppose \( z \in \mathbb{C} \setminus [1, \infty) \), and the operator \( \Gamma_\epsilon(z) \) is invertible. Then the operator

\[ R_{\alpha, \epsilon}(z) = R_0(z) + \tilde{R}_\epsilon^*(z) \Gamma_\epsilon(z)^{-1} \tilde{R}_\epsilon(z) \]  

defines the resolvent of \( H_{\alpha, \epsilon} \), i.e. \( R_{\alpha, \epsilon}(z) = (H_{\alpha, \epsilon} - z)^{-1} \).

Moreover,

\[ \dim \ker(H_{\alpha, \epsilon} - z) = \dim \ker \Gamma_\epsilon(z) \]

and the map \( h \mapsto \tilde{R}_\epsilon^*(z)h \) defines a bijection from \( \ker \Gamma_\epsilon(z) \) onto \( \ker(H_{\alpha, \epsilon} - z) \).

The above theorem allows one to ‘shift’ the eigenvalue problem for the differential operator \( H_{\alpha, \epsilon} \) to the problem of finding the zeros of the integral operator \( \Gamma_\epsilon(\cdot) \).

For a more general approach, we recommend [25].

### 2.3. Reparametrization

A disadvantage of the operator \( \Gamma_\epsilon(z) \) is the fact that it acts in the trace space \( L^2(\Sigma_\epsilon) \) which is \( \epsilon \)-dependent. To make it ‘stable’ w.r.t. \( \epsilon \), we parametrize \( \Sigma_\epsilon \) by means of \( x_2 \) instead of the length of arc \( l \). Recall that \( \Sigma_\epsilon = \{(\epsilon x_2, x_2) \in \Omega : x_2 \in (0, \pi)\} \). Using the relation \( \frac{d}{d\epsilon} = (\epsilon^2 + 1)^{1/2} \) we conclude that the map

\[ f \mapsto (\epsilon^2 + 1)^{1/4} f((\epsilon^2 + 1)^{1/2}(\cdot)) \]

defines a unitary operator \( U : L^2(\Sigma_\epsilon) \to L^2([0, \pi]) \).

Without risk of confusion, we keep the same notation \( R_\epsilon(z) \) for its unitary ‘equivalent’ \( UR_\epsilon(z)U^{-1} : L^2([0, \pi]) \to L^2([0, \pi]) \) and analogously for \( \Gamma_\epsilon(z) \).

After the reparametrization, the kernel of the operator \( R_\epsilon(z) : L^2([0, \pi]) \to L^2([0, \pi]) \) is given by

\[ G_\epsilon(z; x_2, y_2) = \frac{i}{\pi} \sum_{k=1}^{\infty} \frac{\exp(-|x_2 - y_2|/\sqrt{z - k^2}) \omega_k(x_2)\omega_k(y_2)}{\sqrt{z - k^2}}; \]

cf (2.2).

### 3. The spectrum of \( H_{\alpha, \epsilon} \); embedded eigenvalue phenomena

**The stability of the essential spectrum**

Assume that \( \alpha > 0 \). Since the delta perturbation is compactly supported, one may expect stability of the essential spectrum with respect to the ‘free’ Hamiltonian \( H_0 \). It was shown in [4] that a finite measure preserves the essential spectrum of the Laplacian acting in \( L^2(\mathbb{R}^n) \).

The argument employed in [4] can be extended to the Laplacian acting in \( L^2(\Omega) \). Namely, both operators \( \tilde{R}_\epsilon^*(z) \) and \( R_\epsilon(z) \) are compact. Indeed, note that the operator \( R_0(z) : L^2(\Omega) \to W^{2,2}_0(\Omega) \) is bounded. Furthermore, using the standard embedding theorem (cf [1]), we conclude that the inclusion \( W^{2,2}_0(\Omega) \subset C(\Sigma_\epsilon) \) defines a compact embedding operator in the sense of the trace map. On the other hand, since the embedding \( C(\Sigma_\epsilon) \subset L^2(\Sigma_\epsilon) \) is continuous, we conclude that the trace map \( I : W^{2,2}_0(\Omega) \subset W^{1,2}_0(\Omega) \to L^2(\Sigma_\epsilon) \) is compact. This yields compactness of \( \tilde{R}_\epsilon(z) \) as well as its adjoint \( \tilde{R}_\epsilon^*(z) \). Consequently, the continuity of \( \Gamma_\epsilon(z)^{-1} \)
implies compactness of $\tilde{R}_\epsilon^*(z)\Gamma_\epsilon(z)^{-1}\tilde{R}_\epsilon(z)$. In view of the Weyl theorem, we have stability of the essential spectrum

$$\sigma_{\text{ess}}(H_{\alpha,\epsilon}) = \sigma_{\text{ess}}(H_0) = [1, \infty).$$

**The point spectrum of $H_{\alpha,0}$: poles of the resolvent**

Our first aim is to characterize the point spectrum of the Hamiltonian $H_{\alpha,0}$ which governs the system with the wire placed perpendicularly to the boundaries of $\Omega$. In fact, the problem was solved in [3], employing the generalized sum method. Moreover, relying on the symmetry argument, we provide a characterization of the spectrum of $H_{\alpha,0}$ in section 1. Now we would like to find the point spectrum of $H_{\alpha,0}$ as the poles of its resolvent.

Using (2.5) we have

$$G_0(z; x_2, y_2) = \frac{i}{\pi} \sum_{k=1}^{\infty} \frac{1}{\sqrt{z - k^2}} \omega_k(x_2)\omega_k(y_2). \quad (3.1)$$

Since the following $\{((2\pi)^{1/2})\omega_k(\cdot)\}_{k \in \mathbb{N}}$ forms an orthonormal basis in $L^2([0, \pi])$, we arrive at

$$\ker \Gamma_0(z) \neq \{0\} \iff 2\sqrt{z - k^2} - i\alpha = 0, \quad k \in \mathbb{N}. \quad (3.2)$$

Note that the latter admits solutions also for $z \in [1, \infty)$ and $\alpha > 0$. To be precise, given $k \in \mathbb{N}$ the number

$$E_k = -\frac{\alpha^2}{4} + k^2 \quad (3.3)$$

determines a solution of (3.2) and, at the same time, determines an eigenvalue of $H_{\alpha,0}$; cf [3]. In the following theorem, we summarize the above discussion.

**Theorem 3.1.** Assume that $\alpha > 0$. The eigenvalues of $H_{\alpha,0}$ take the form (3.3). Consequently, the number of the discrete spectrum points is finite and given by

$$\sharp \sigma_d(H_{\alpha,0}) = \left\{ k \in \mathbb{N} : -\frac{\alpha^2}{4} + k^2 < 1 \right\}.$$

On the other hand, for any $k \in \left\{ n \in \mathbb{N} : -\frac{\alpha^2}{4} + k^2 \geq 1 \right\}$, $E_k$ given by (3.3) constitutes an embedded eigenvalue of $H_{\alpha,0}$. Therefore, the number of embedded eigenvalues is infinite.

**4. Resonances**

**4.1. Resonances as the poles of the resolvent**

The Birman–Schwinger principle is based on a relation between the eigenvalues of the Hamiltonian and the poles of its resolvent. Using this argument, we have stated that the Hamiltonian $H_{\alpha,0}$ admits embedded eigenvalues which determine the zeros of $\Gamma_0(\cdot)$. A question arises: where are the poles of the resolvent localized if we ‘slightly’ break the symmetry? The aim of this section is to show that the resolvent of $H_{\alpha,\epsilon}$ admits the second-sheet analytical continuation in a certain sense and, moreover, the second-sheet continuation $\Gamma_{II}(\cdot)$ of $\Gamma_0(\cdot)$ has zeros in the lower half-plane, i.e.

$$\ker \Gamma_{II}(z) \neq \{0\}, \quad \exists \zeta < 0. \quad (4.1)$$

The resolvent poles defined by (4.1) constitute resonances.
4.2. Analytical continuation of $R_n(z)$

Let $n \in \mathbb{N}$ and $E_n = -\frac{\alpha^2}{4} + n^2$ be an embedded eigenvalue of $H_{\alpha, n}$. Henceforth we assume that $\alpha \neq 2\sqrt{n^2-k^2}$ for any $k \in \mathbb{N}$ such that $k \leq n$. In this way we exclude the case where $E_n$ reaches the threshold of the essential spectrum of $H_{\alpha, n}$; cf (1.2).

The goal of this section is to construct the analytical continuation of $R_\alpha(\cdot)$ in a neighbourhood of $E_n$. In fact, $E_n$ is localized between the neighbouring thresholds of the essential spectra of (1.2). These thresholds determine the boundaries of the largest real interval containing $E_n$ and admitting an analytical continuation of $R_\alpha(z)$ to the lower half-plane. To specify this interval, let us define

$$A_n := \{k : k^2 < E_n\}, \quad A_n^c := \mathbb{N} \setminus A_n.$$

Suppose $k_1 := \max\{k \in \mathbb{N} : k \in A_n\}$ and set $\Upsilon_n := (k_1^2, (k_1 + 1)^2)$. Then $E_n \in \Upsilon_n$. Note that at most one eigenvalue can be localized between $k^2$ and $(k + 1)^2$.

In the following we denote as $z \mapsto \sqrt{z-k^2} = \sqrt{z-k^2_1}$ and $z \mapsto \sqrt{z-k^2_2}$, respectively, the first and the second Riemann sheet of the square root function with the cut $[k^2, \infty)$. Then, for any $k \in \mathbb{N}$, the function $\mathbb{C}_+ \cup \Upsilon_n \ni z \mapsto \sqrt{z-k^2_1}$ has an analytical continuation via $\tau_{k_1}$ to a bounded open set $\Upsilon_{-\pi} = \Upsilon_- \subset \mathbb{C}_-$ with boundaries containing $\Upsilon_n$. This continuation is given by

$$\Upsilon_n \cup \Upsilon_- \ni z \mapsto \tau_{k_1}(z) = \begin{cases} \sqrt{z-k^2_1} & \text{for } k \in A_n^c, \\ \sqrt{z-k^2_2} & \text{for } k \in A_n. \end{cases}$$

It is clear from the definition that $\tau_{k_1}(\cdot)$ depends on $n$. By means of $\tau_{k_1}(\cdot)$, we can construct an analytical continuation of $R_\alpha(z)$ to $\Upsilon_n \cup \Upsilon_-$ which is given by an integral operator $R_\alpha^H(z)$ acting in $L^2([0, \pi])$ with the kernel

$$G^H_\alpha(z; x_2, y_2) := \frac{1}{\pi} \sum_{k=1}^{\infty} e^{-i\sqrt{z-k^2_1} |x_2-y_2|} \frac{\omega_k(x_2)\omega_k(y_2)}{\tau_{k_1}(z)}.$$

(4.2)

**Lemma 4.1.** Suppose $z \in \Upsilon_n \cup \Upsilon_-$. Operator $R_\alpha^H(z)$ is bounded.

**Proof.** Note that (4.2) consists of the finite number $\sharp A_n$ of bounded operators for which $\Im \tau_{k_1}(z) \leq 0$. The remaining part

$$\frac{1}{\pi} \sum_{k \in A_n^c} e^{-i\sqrt{z-k^2_1} |x_2-y_2|} \sqrt{z-k^2_2} \omega_k(x_2)\omega_k(y_2)$$

is also bounded due the analogous argument which implies boundedness of (2.5). \[\square]\n
Finally, we can construct, in an analogous way to $R_\alpha^H(z)$, the second-sheet continuation of the remaining ingredients of the resolvent $R_0(\cdot)$, $R_\alpha^S(\cdot)$ and $R_\alpha(\cdot)$. Relying on (2.3), we can build up the second-sheet continuation $R_{\alpha, e}^H(z)$ which for any $f, g \in C_0^\infty(\mathbb{R}^2)$ defines the analytic function $z \mapsto (f, R_{\alpha, e}^H(z)g)$.

4.3. The spectral condition for the resonances

**Decomposition of $R_\alpha^H(z)$**. The goal of this sequence is to formulate a spectral condition for the resolvent pole located near energy $E_n$. With this aim, we will extract from (4.2) the $n$th component of $G_\alpha^H(z, \cdot, \cdot)$ ‘responsible’ for the existence of the embedded eigenvalue $E_n$ and given by

$$S_n(z) := \frac{1}{\pi} \frac{1}{\tau_{k_1}(z)} (\omega_n, \cdot)\omega_n.$$

(4.3)
Define
\[ T_ε(z) := R^H(ε, z) - S_n(z). \] (4.4)
In fact, \( T_ε(z) \) depends also on \( n \); without risk of confusion, we omit index \( n \) here. Note that in view of lemma 4.1, operator \( T_ε(z) \) is bounded. The next step is to derive the expansion of \( T_ε(z) \) with respect to \( ε \) up to the second-order perturbation term. Employing (4.2), we get
\[ T_ε(z) = T_0(z) + T_1(z)ε + T_2(z)ε^2 + O(ε^3), \] (4.5)
where
\[ T_0(z) = \frac{i}{\pi} \sum_{k \neq n} \frac{1}{τ_k(z)} (ω_n, \cdot)ω_n, \] (4.6)
and the kernels of \( T_m(z), m = 1, 2 \) are given by
\[ T_m(z)(x_1, y_1) = -\frac{i^{m-1}}{mτ} \sum_{k \in \mathbb{N}} τ_k(z)^{m-1}|x_2 - y_2|^{m}ω_k(x_1)ω_k(y_2). \] (4.7)
Since both \( T_ε(z) \) and \( T_0(z) \) are bounded, the ‘perturbant’ \( T_ε(z) - T_0(z) \) is bounded as well. Moreover, note that \( T_1(\cdot) \) does not depend on \( z \). Therefore, in the following we will write \( T_1 \).

**Lemma 4.2.** Suppose that \( U_0 \subset \mathbb{C}_+ \cup \mathbb{I}_+ \cup \mathbb{I}_- \) defines a small neighbourhood of \( E_n \) and \( z \in U_0 \). For \( ε \) sufficiently small, the operator \( I - αT_ε(z) \) is invertible.

**Proof.** Note that for \( z \in U_0 \) the function \( z \mapsto τ_k(z) \) is analytic. Using the explicit form of \( T_0(z) \) (see (4.6)), we conclude that its eigenvalues takes the form
\[ t_k(z) = \begin{cases} \frac{i}{2τ_k(z)} & \text{for } k \neq n \\ 0 & \text{for } k = n. \end{cases} \]
The corresponding eigenfunctions are determined by \( ω_k \) for \( k \in \mathbb{N} \). Combining the above statement together with (4.5) and results from perturbation theory [19], we get that the eigenvalues of \( αT_ε(z), z \in U_0 \), take the forms
\[ αt_k(z) + O(ε), \quad k \in \mathbb{N}. \]
Since the equation
\[ αt_k(z) + O(ε) = 1 \]
has no solution for \( z \in U_0 \), we can conclude that \( \ker(I - αT_ε(z)) = \{0\} \).

Relying on the above lemma, we define
\[ η_0(z, ε) := τ_ε(z) - i\frac{α}{τ} (ω_n, I - αT_ε(z))^{-1}ω_n. \] (4.8)

**Theorem 4.3.** Suppose that \( z \in U_0 \). Then
\[ \ker Γ^H_ε(z) \neq \{0\} \iff η_0(ε, z) = 0. \]

**Proof.** Using decomposition (4.4) and lemma 4.2, we conclude that for \( z \in U_0 \) we have
\[ Γ^H_ε(z) = (I - αT_ε(z))(I - α(I - αT_ε(z))^{-1}S_n(z)). \] (4.9)
Using again the fact that \( \ker(I - αT_ε(z)) = \{0\} \), we arrive at
\[ \ker Γ^H_ε(z) \neq \{0\} \iff \ker(I - α(I - αT_ε(z))^{-1}S_n(z)) \neq \{0\}. \]
Applying (4.3), we come to the conclusion that \( f \in \ker(I - α(I - αT_ε(z))^{-1}S_n(z)) \) iff
\[ f = i\frac{α}{τ} (ω_n, f)(I - αT_ε(z))^{-1}ω_n. \]
The latter is equivalent to
\[ η_0(z, ε) = 0; \]
this completes the proof.
4.4. The solution of the spectral equation

Once we have the spectral equation our aim is to recover its solutions which reproduce the resolvent poles. Since the functions $U_n \ni z \mapsto \tau_k(z), k \in \mathbb{N}$, are analytic, the spectral condition obtained in theorem 4.3 is equivalent to

$$\tau_n(z)^2 = -\frac{\alpha^2}{\pi^2} (\omega_n, (I - \alpha T_\epsilon(z))^{-1} \omega_n)^2; \quad (4.10)$$

cf (4.8).

Expansion of $(\omega_n, (I - \alpha T_\epsilon(z))^{-1} \omega_n)$. The next step is to expand the expression $(\omega_n, (I - \alpha T_\epsilon(z))^{-1} \omega_n)$ involved in the above equation with respect to $\epsilon$. Suppose $A$ is invertible with the bounded inverse and $C$ is bounded with the norm $\|C\| = O(\epsilon)$. Then

$$(A - C)^{-1} = A^{-1}(I + A^{-1}C + (A^{-1}C)^2 + \cdots). \quad (4.11)$$

Specify $A := I - \alpha T_\epsilon(z)$ and $C := \alpha (T_\epsilon(z) - T_0(z))$. Combining (4.11) and (4.7) together with the fact that $A\omega_n = \omega_n$, we get

$$(\omega_n, (I - \alpha T_\epsilon(z))^{-1} \omega_n) = (\omega_n, (I + C + C^2 + \cdots) \omega_n) = \frac{\pi}{2} + \alpha \epsilon (\omega_n, T_1 \omega_n) + \alpha \epsilon^2 (\omega_n, T_2(z) \omega_n) + \alpha \epsilon^2 (\omega_n, T_2^2 \omega_n) + O(\epsilon^3). \quad (4.12)$$

With the above statements, we are ready to prove the main theorem.

**Theorem 4.4.** Let $\alpha > 0$. Suppose $n^2 > 1 + \frac{\alpha^2}{4}$, i.e. the number $E_n = -\frac{\alpha^2}{4} + n^2$ determines the embedded eigenvalue of $H_{\alpha,0}$. Then the Hamiltonian $H_{\alpha,\epsilon}$ has a resolvent pole at $z_\alpha(\epsilon) = E_n + V_n \epsilon + W_n \epsilon^2 + O(\epsilon^3),$ \quad (4.13)

where

$$V_n := -\frac{\alpha^3}{\pi} (\omega_n, T_1 \omega_n), \quad W_n := -\frac{\alpha^4}{\pi^2} (\omega_n, T_1 \omega_n)^2 - \frac{\alpha^3}{\pi} (\omega_n, T_2(E_n) \omega_n) - \frac{\alpha^4}{\pi} (\omega_n, T_2^2 \omega_n).$$

**Proof.** Suppose $z \in \mathcal{U}_n$. The spectral condition (4.10) reads as

$$\tilde{\eta}_n(z, \epsilon) = 0, \quad (4.14)$$

where

$$\tilde{\eta}_n(z, \epsilon) := z - n^2 - \frac{\alpha^2}{\pi^2} (\omega_n, (I - \alpha T_\epsilon(z))^{-1} \omega_n)^2.$$

Using the expansion (4.12), we get

$$\tilde{\eta}_n(z, \epsilon) = z - n^2 - \frac{\alpha^2}{4} - \frac{\alpha^3}{\pi} (\omega_n, T_1 \omega_n) \epsilon$$

$$+ \left( -\frac{\alpha^4}{\pi^2} (\omega_n, T_1 \omega_n)^2 - \frac{\alpha^3}{\pi} (\omega_n, T_2(z) \omega_n) - \frac{\alpha^4}{\pi} (\omega_n, T_2^2 \omega_n) \right) \epsilon^2 + O(\epsilon^3).$$

Note that

$$\tilde{\eta}_n(E_n, 0) = 0.$$

Since $\mathcal{U}_n \ni z \mapsto \tilde{\eta}_n(z, \epsilon)$ is analytic (this follows from the analyticity of $R^{\mathcal{U}_n}(\cdot)$) and $\tilde{\eta}_n(z, \epsilon)$ is $C^\infty$ as a function of $\epsilon$, we can apply the implicit function theorem in view of which equation (4.14) has a unique solution in the neighbourhood $\mathcal{U}_n$ of $E_n$. The expansion (4.13) comes directly from the above expansion of $\tilde{\eta}_n(z, \epsilon)$.
5. Geometrically induced resonances: discussion

5.1. The imaginary part of the resolvent pole

Theorem 4.4 shows that the pole of the resolvent dissolving originally in the essential spectrum is slightly shifted after breaking the symmetry. Since the operator $H_{\alpha, \epsilon}$ is self-adjoint, the pole of its resolvent can either stay in the essential spectrum or turn to the lower complex half-plane. The aim of this section is to show that the latter holds for $\epsilon$ small enough. The question is appealing because the imaginary part of the pole determines the width of the resonance.

Note that the error term in (4.13) depends on the index $n$; i.e. for $n \in \mathbb{N}$ there exists $\epsilon_n > 0$ such that for any $\epsilon \in (0, \epsilon_n)$ the term $W_n \epsilon^2$ dominates w.r.t. the error term (in the sense of the real and imaginary components).

Fix $n \in \mathbb{N}$ and suppose that $\epsilon \in (0, \epsilon_n)$. Note that $T_1$ is a self-adjoint operator and, consequently, $V_n$ is real; also the first and third components of $W_n$ (see (4.13)) are real. A nontrivial imaginary component of $z_n(\epsilon)$ of the lowest order is induced by

$$(\omega_n, T_2(E_n) \omega_n) = -\frac{i}{2\pi} \sum_{k \in \mathbb{N}} \tau_k(E_n) N_{k,n}, \quad (5.1)$$

where

$N_{k,n} := \int_0^\pi \int_0^\pi |x_2 - y_2|^2 \sin(kx_2) \sin(nx_2) \sin(ky_2) \sin(ny_2) \, dx_2 \, dy$

$$= \begin{cases} -8 \frac{k^2 n^2}{(n^2 - k^2)^4} & \text{if } |k - n| \text{ odd} \\ 0 & \text{if } |k - n| \text{ even} \\ \frac{\pi^4}{12} - \frac{\pi^2}{4k^2} & \text{if } k = n. \end{cases}$$

Note that not all components of (5.1) contribute to the imaginary part of $z_n(\epsilon)$. To be precise, $\tau_k(E_n)$ is purely real for $k \in A_n$. Otherwise, for $k \in A_n^c$, it is purely imaginary and, consequently, the corresponding components are not employed in the imaginary part of the resonance pole. This, in view of (5.1), implies

$$\Im z_n(\epsilon) = -\frac{4\alpha^2}{\pi^3} S_n \epsilon^2 + O(\epsilon^3), \quad (5.2)$$

where

$$S_n := \sum_{k \in \mathbb{N}} \tau_k(E_n) \frac{k^2 n^2}{(n^2 - k^2)^4}$$

and $A_n^c := \{ k \in A_n : n - k \text{ is odd} \}$. Since $\tau_k(E_n) > 0$ for all $k \in A_n^c$, we obtain $\Im z_n(\epsilon) < 0$.

Formula (5.2) explicitly reveals the lowest order of the resonance width:

$$\gamma_n = \frac{2\alpha^2}{\pi^3} S_n \epsilon^2.$$

5.2. The resonance energy

The real component of the resonance pole determines the resonance energy. Fix $n \in \mathbb{N}$ and suppose that $\epsilon$ is small enough. The lowest order of the real component associated with $z_n(\epsilon)$ is given by

$$E_n = \frac{\alpha^3}{\pi} (\omega_n, T_1 \omega_n) \epsilon.$$
To derive a more explicit expression we calculate
\[
(\omega_n, T_1(\omega_n)) = \sum_{k \in \mathbb{N}} M_{k,n},
\]
where
\[
M_{k,n} := \int_0^\pi \int_0^\pi |x_2 - y_2| \sin(kx_2) \sin(ny_2) \sin(ny_2) \sin(kx_2) \, dx_2 \, dy_2.
\]

\[=
\begin{cases}
-\frac{\pi}{4} n^2 + k^2 & \text{if } k \neq n \\
\frac{21}{48} \pi + \frac{11}{6} \pi^3 & \text{if } k = n.
\end{cases}
\]

5.3. Comments on the repulsive potential

Suppose that the coupling constant \(\alpha < 0\); i.e. the delta potential localized on \(\Sigma_\epsilon\) has a repulsive character. Moreover, assume at the beginning that \(\epsilon = 0\). Note, that in this case, equation (3.2) does not admit any solution (recall that \(\Im \sqrt{z - k^2} > 0\)) and, consequently, embedded eigenvalues do not appear. However, for \(\alpha < 0\) the analogous equation, but for the second-sheet continuation of the square root function, i.e.

\[2\sqrt{z - k^2} - i\alpha = 0,
\]

has a solution at \(z_k = -\frac{\alpha^2}{4} + k^2\). Let us emphasize that the above solution does not determine an eigenvalue of \(H_{\alpha,0}\). It states a pole of the resolvent living on the second sheet. An analogous phenomenon occurs for the repulsive one-point interaction in a one-dimensional system; cf [2], chapter 1.3. If we break the symmetry, we may expect the pole to move slightly, but since \(H_{\alpha,\epsilon}\) is a self-adjoint operator, the pole has to stay on the second sheet. In this case this means that its imaginary part is nonnegative. Indeed, repeating the procedure employed for \(\alpha > 0\), after obvious changes, we get

\[\Im \tilde{z}_n(\epsilon) = -\frac{4\alpha^2}{\pi^3} \tilde{S}_n \epsilon^2 + O(\epsilon^3),
\]

where

\[\tilde{S}_n := \sum_{k \in A_n} \sqrt{E_n + k^2} \frac{k^3 n^2}{(n^2 - k^2)^2}.
\]

Since \(\sqrt{E_n - k^2} \ll 0\) we have \(\Im \tilde{z}_n(\epsilon) > 0\) for \(\epsilon\) small enough.

5.4. Open questions

The first natural question arising here concerns the higher dimensional models. For example, suppose that the geometry of the waveguide is defined by \(\Omega := \{(x_1, x_2, x_3) : x_1, x_2 \in \mathbb{R}, x_3 \in (0, \pi)\}\) and the delta interaction is supported by \(\Sigma_\epsilon := \{(0, \epsilon x_3, x_3)\} \subset \Omega\). Since the interaction is supported by a set of codimension 2, the model is essentially different from the one studied in this paper. The resolvent needs a certain kind of renormalization and this makes the resonance analysis more involved.

The other interesting question concerns the spectral properties of the system if \(\epsilon\) is large. In particular, if \(\epsilon\) goes formally to infinity, then \(\Sigma_\epsilon\) is becoming parallel to the walls of \(\Omega\). This suggests that the eigenvalues of \(H_{\alpha,\epsilon}\) are densely localized and in the limiting case we get an additional component of the essential spectrum.
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