CALABI–YAU COMPLETE INTERSECTIONS IN $G_2$-GRASSMANNIANS

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Abstract. We study Calabi–Yau complete intersections defined by equivariant vector bundles on homogeneous spaces of the simple Lie group of type $G_2$.

1. Introduction

A smooth projective manifold is said to be Calabi–Yau if the first Chern class is trivial. Calabi–Yau manifolds have attracted attentions from both mathematicians and string theorists, not only because of their importance in the classification of algebraic varieties, but also because of their relation with string theory and mirror symmetry. A Calabi–Yau manifold in dimensions less than three are either an elliptic curve, an abelian surface, a K3 surface, a bielliptic surface, or an Enriques surface. In dimensions greater than two, it is not known whether the number of deformation equivalence classes (or even homeomorphism classes) of Calabi–Yau manifolds is finite or not.

One way to construct Calabi–Yau manifolds is to take nef complete intersections of line bundles on toric varieties (see e.g. [CK99] and references therein). Other known constructions include

- complete intersections of line bundles [BCFKvS98, BCFKvS00] or more general equivariant vector bundles on Grassmannians [IIM], and
- Pfaffian varieties in projective spaces [Ton04],

to name a few.

In this paper, we study Calabi–Yau manifolds in $G_2$-Grassmannians, which are complete intersections of globally generated equivariant vector bundles. Here, a $G_2$-Grassmannian is the homogeneous spaces of the simple Lie group $G$ of type $G_2$ by a maximal parabolic subgroup $P$. The main result is the following:

**Theorem 1.1.** On each $G_2$-Grassmannian, there exists a unique equivariant vector bundle $\mathcal{E}$ such that

- $\mathcal{E}$ is globally generated,
- $\mathcal{E}$ is not a direct sum of line bundles, and
- the zero locus of a general global section is a Calabi–Yau 3-fold.

Parabolic subgroups and equivariant vector bundles are described in terms of crossed Dynkin diagrams and the highest weights of the corresponding representations of $P$ respectively. The complete list of pairs $(P, \mathcal{E})$ of a parabolic subgroup $P$ and a globally generated equivariant vector bundle $\mathcal{E}$ on $G/P$ such that $\det \mathcal{E} \cong \omega_{G/P}^\vee$ and $\operatorname{rank} \mathcal{E} \leq \dim G/P - 2$ is given in Tables 1.1–1.4.
| No. | $P$   | $\mathcal{E}$ |
|-----|-------|---------------|
| 1   | $\times \times$ | $(2, 2)$ |

**Table 1.1.** Complete intersection Calabi–Yau 5-folds

| No. | $P$   | $\mathcal{E}$ |
|-----|-------|---------------|
| 1   | $\times \times$ | $(3, 0)$ |
| 2   |     | $(0, 5)$ |
| 3   | $\times \times$ | $(0, 1) \oplus (2, 1)$ |
| 4   |     | $(1, 1) \oplus (1, 1)$ |
| 5   |     | $(0, 2) \oplus (2, 0)$ |

**Table 1.2.** Complete intersection Calabi–Yau 4-folds

| No. | $P$   | $\mathcal{E}$ |
|-----|-------|---------------|
| 1   | $\times \times$ | $(1, 1)$ |
| 2   |     | $(1, 0) \oplus (2, 0)$ |
| 3   | $\times \times$ | $(1, 1)$ |
| 4   |     | $(0, 1) \oplus (0, 4)$ |
| 5   |     | $(0, 2) \oplus (0, 3)$ |
| 6   | $\times \times$ | $(0, 1)^{\oplus 2} \oplus (2, 0)$ |
| 7   |     | $(0, 1) \oplus (1, 0) \oplus (1, 1)$ |
| 8   |     | $(0, 2) \oplus (1, 0)^{\oplus 2}$ |

**Table 1.3.** Complete intersection Calabi–Yau 3-folds

| No. | $P$   | $\mathcal{E}$ |
|-----|-------|---------------|
| 1   | $\times \times$ | $(0, 2)$ |
| 2   |     | $(0, 1) \oplus (2, 0)$ |
| 3   |     | $(1, 0)^{\oplus 3}$ |
| 4   | $\times \times$ | $(1, 0) \oplus (0, 2)$ |
| 5   |     | $(0, 1)^{\oplus 2} \oplus (0, 3)$ |
| 6   |     | $(0, 1) \oplus (0, 2)^{\oplus 2}$ |
| 7   | $\times \times$ | $(0, 1)^{\oplus 2} \oplus (1, 0)^{\oplus 2}$ |

**Table 1.4.** Complete intersection K3 surfaces
Remark 1.2. The $G_2$-Grassmannian corresponding to is embedded into $\mathbb{P}^6$ as a smooth quadric 5-fold. Calabi–Yau 3-folds contained in a (not necessarily smooth) quadric 5-fold are classified in [KK16, Section 5], and Theorem 1.1 in this case immediately follows from this result.

No.1 in Table 1.3 is a complete intersection in $\text{Gr}(2,7)$ defined by $S^\vee(1) \oplus Q^\vee(1)$, which is known to be deformation-equivalent to the intersection of the image of $\text{Gr}(2,7)$ and a linear subspace of codimension 7 in $\mathbb{P}(\wedge^2 \mathbb{C}^7)$ [IIM]. On the other hand, it is shown [KK16, Theorem 7.1] that No.3 in Table 1.3 is deformation-equivalent to the Pfaffian Calabi–Yau 3-fold appearing in [Rod00]. It follows that both Pfaffian and Grassmannian Calabi–Yau 3-folds appearing in [Rod00] are deformations of $G_2$-Grassmannian Calabi–Yau 3-folds.

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2. Homogeneous vector bundles over $G/P$

Let $\mathfrak{g}$ be a complex semisimple Lie algebra of rank $r$. The corresponding simply-connected Lie group is denoted by $G$. Fix a Cartan subgroup $H \subset G$ with the associated Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and set

$$\mathfrak{g}_\alpha := \{ v \in \mathfrak{g} \mid [h,v] = \alpha(h)v \text{ for any } h \in \mathfrak{h} \} \quad (2.1)$$

for $\alpha \in \mathfrak{h}^\vee := \text{Hom}_\mathbb{C}(\mathfrak{h}, \mathbb{C})$. One has the root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \quad (2.2)$$

where

$$\Delta := \{ \alpha \in \mathfrak{h}^\vee \mid \mathfrak{g}_\alpha \neq \{0\}, \alpha \neq 0 \}.$$  

We choose a system of simple roots $\mathcal{S} := \{ \alpha_1, \ldots, \alpha_r \} \subset \Delta$. This choice is equivalent to the choice of the sets $\Delta^+$ and $\Delta^-$ of positive and negative roots.

The Dynkin diagram of $\mathfrak{g}$ is a graph whose nodes correspond to the simple roots $\alpha_i \in \mathcal{S}$ and whose edges represent the Cartan integers $\langle \alpha_i, \alpha_j^\vee \rangle$, where $\langle -, - \rangle$ is the Killing form on $\mathfrak{h}^\vee$ and $\alpha^\vee = 2\alpha / \langle \alpha, \alpha \rangle$. One has $\langle \alpha_i, \alpha_j^\vee \rangle = 2$ for all $\alpha_i \in \mathcal{S}$, and the correspondence between edges and the Cartan integers is given by

$$\alpha \quad \beta \quad \iff \quad \langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = 0,$$

$$\alpha \quad \beta \quad \iff \quad \langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = -1,$$

$$\alpha \quad \beta \quad \iff \quad \langle \alpha, \beta^\vee \rangle = -2, \quad \langle \beta, \alpha^\vee \rangle = -1,$$

$$\alpha \quad \beta \quad \iff \quad \langle \alpha, \beta^\vee \rangle = -3, \quad \langle \beta, \alpha^\vee \rangle = -1.$$
A subgroup $P$ of $G$ is said to be *parabolic* if $G/P$ is a projective variety. Conjugacy classes of parabolic subgroups are in one-to-one correspondence with subsets $S_p \subset S$ of the set of simple roots in such a way that the corresponding subalgebra $p$ is given by

$$p := l \oplus n,$$

where the *Levi part* $l$ is

$$l := h \oplus \bigoplus_{\alpha \in \text{span} S_p \cap \Delta} g_{\alpha},$$

and the *nilpotent part* $n$ is

$$n := \bigoplus_{\alpha \in \Delta^+ \setminus \text{span} S_p} g_{\alpha}.$$

Here $\text{span} S_p \subset h^\vee$ is the linear subspace spanned by $S_p$. The subset $S_p \subset S$ can be described by a *crossed Dynkin diagram*, where elements not in $S_p$ are crossed out (i.e., elements of $S_p$ correspond to uncrossed nodes). The inclusion relation of $S_p$ corresponds to the inclusion relation of $P$. For example, the Borel subgroup is the minimal parabolic subgroup, so that all the nodes are crossed out in the corresponding crossed Dynkin diagram. We write the Weyl group of $l$ as $W_P$, which is the subgroup of $W = W_G$ generated by simple reflections associated with elements of $S_p$. One has

$$\dim G/P = \# (\Delta^+ \setminus \text{span} S_p) = \dim G/B - \dim G'/B',$$

where $G'/B'$ is the full flag variety corresponding to the full subgraph of the Dynkin diagram of $G$ consisting of uncrossed nodes.

Representations of the parabolic subalgebra $p$ correspond bijectively to representations of the Levi subalgebra $l$. Since $l$ is reductive, finite-dimensional representations are completely reducible, and irreducible representations are highest weight representations. Since $g$ and $l$ share the same Cartan subalgebra, weights of $l$ can be regarded naturally as weights of $g$. Since $G$ and $L$ share the same Cartan subgroup, the notion of integrality of weights is the same for both $g$ and $l$. The fundamental weight associated with the simple root $\alpha_i \in S$ is denoted by $\omega_i$;

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}, \quad i, j = 1, \ldots, r.$$

A weight $\lambda = \sum_{i=1}^r \lambda_i \omega_i$ is

- *integral* if $\lambda_i \in \mathbb{Z}$ for any $i = 1, \ldots, r$,
- *p-dominant* if $\lambda_i \in \mathbb{N}$ for any $i$ such that $\alpha_i \in S_p$, and
- *g-dominant* if $\lambda_i \in \mathbb{N}$ for any $i = 1, \ldots, r$.

A highest weight representation of $l$ integrates to a representation of $P$ if and only if the highest weight is integral and $p$-dominant (see e.g. [BE89, Remark 3.1.6]). $G$-equivariant bundles on $G/P$ correspond bijectively to representations of $P$. For a representation $V$ of $P$, the corresponding $G$-equivariant bundle on $G/P$ is denoted by

$$\mathcal{E}_V := G \times_P V.$$

The irreducible representations of $P$ and $G$ with highest weight $\lambda$ are denoted by $V_\lambda^P$ and $V_\lambda^G$ respectively. The $G$-equivariant vector bundle on $G/P$ associated with an integral $p$-dominant weight $\lambda$ is denoted by

$$\mathcal{E}_\lambda := \mathcal{E}_{V_\lambda^P}.$$
which is globally generated if and only if $\lambda$ is $g$-dominant. Borel–Weil–Bott theorem states that one has an isomorphism

$$H^\ell(w)(G/P, E_\lambda) \cong (V^G_{w,\lambda})^\vee$$

(2.11)
of $G$-vector spaces. Here $\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ is the Weyl vector,

$$w.\lambda := w(\lambda + \rho) - \rho$$

(2.12)
is the affine Weyl action, $w \in W_P$ is the unique element such that $w.\lambda$ is $g$-dominant, and $\ell(w)$ is the length of (the minimal representative in $W$ of) $w \in W_P$.

The Picard group $\text{Pic}(G/P)$ is isomorphic to the group $\text{Hom}(P, \mathbb{C}^\times)$ of characters of $P$. The set of weights of a representation $V$ of $P$ is denoted by $\Delta(V)$. One has

$$\text{rank } E_V = \dim V = |\Delta(V)|, \quad \det E_V \cong E_{\det V}, \quad \text{and } \Delta(\det V) = \left\{ \sum_{\lambda \in \Delta(V)} \lambda \right\}.$$  

(2.13)
The tangent bundle $T_{G/P}$ corresponds to the $P$-vector space $g/p$ with respect to the adjoint action;

$$T_{G/P} \cong E_{g/p}.$$  

(2.14)

Since the weights of the adjoint action are roots, one has

$$\Delta(g/p) = \Delta(g) \setminus \Delta(p) = \Delta^+ \setminus \text{span } S_p.$$  

(2.15)

Let $E := E_V$ be the equivariant vector bundle on $F := G/P$ associated with a representation $V$ of $P$. Assume that $E$ is globally generated and does not contain the trivial bundle as a direct summand. For a general section $s$ of $E$, the zero locus $X := s^{-1}(0)$ is a smooth complete intersection by a generalization of the theorem of Bertini [Muk92, Theorem 1.10]. One has an exact sequence

$$0 \to T_X \to T_F|_X \to N_{X/F} \to 0.$$  

(2.16)

By taking the determinant of (2.16), one obtains an isomorphism

$$\det T_X \cong \det T_F|_X \otimes \det^{-1} N_{X/F}.$$  

(2.17)

On the other hand, since $X$ is a complete intersection, the differential $ds$ of the section $s$ induces an isomorphism

$$N_{X/F} \cong E|_X.$$  

(2.18)

The Picard group $\text{Pic} F$ is the free abelian group generated by $L_i := E_{\omega_i}$ for $i$ such that $\alpha_i \in S \setminus S_p$. A sufficient condition for $\det T_X \cong \mathcal{O}_X$ is $\det V \cong \det(g/p)$, which is necessary if the restriction map $\text{Pic } F \to \text{Pic } X$ is injective.

The Hodge numbers of $X$ can be computed using the exact sequence

$$0 \to N^\vee_{X/F} \to \Omega^1_F|_X \to \Omega^1_X \to 0.$$  

(2.19)
dual to (2.16). The cohomology of $\Omega^1_F|_X$ and $N^\vee_{X/F} \cong E^\vee|_X$ can be computed by tensoring $\Omega^1_F$ and $E^\vee$ with the Koszul resolution

$$0 \to \wedge^{\text{rank } E^\vee} \to \cdots \to E^\vee \to \mathcal{O}_F \to \mathcal{O}_X \to 0.$$  

(2.20)
For Calabi–Yau 3-folds $X$ listed as No.1 and No.3 in Table 1.3, one obtains $h^{0,1} = h^{0,2} = 0$, $h^{1,1} = 1$ and $h^{1,2} = 50$. It follows that the Picard groups are of rank 1. The Koszul resolution (2.20) allows us to compute $\mathcal{O}_X(i)$, which together with Riemann–Roch theorem

$$\chi(\mathcal{O}_X(i)) = \frac{1}{6} \deg X \cdot i^3 + \frac{1}{12} c_2(X) \cdot i$$

shows

$$\deg X = 42, \quad c_2(X) = 84$$

for No.1 and

$$\deg X = 14, \quad c_2(X) = 50$$

for No.3.

### 3. Rational homogeneous varieties of $G_2$

Let $G$ be the simple Lie group of type $G_2$, which has three homogeneous spaces $G/P_1$, $G/P_2$ and $G/B$ associated with the crossed Dynkin diagrams , , and respectively. The sets of weights of the irreducible representations $V^P_{(a,b)}$ with the highest weight $(a, b) := a\omega_1 + b\omega_2$ are given by

$$\Delta \left( V^P_{(a,b)} \right) = \{ (a + j, b - 2j) \mid j = 0, 1, \ldots, b \},$$

$$\Delta \left( V^P_{(a,b)} \right) = \{ (a - 2j, b + 3j) \mid j = 0, 1, \ldots, a \},$$

$$\Delta \left( V^B_{(a,b)} \right) = \{ (a, b) \}.$$

In particular, the dimensions and the determinants of these representations are given as follows:

| representation | dimension | determinant |
|----------------|-----------|-------------|
| $V^P_{(-1,3)}$ | $b + 1$   | $(a(b + 1) + b(b + 1)/2, 0)$ |
| $V^P_{(1,0)}$ | $a + 1$   | $(0, (a + 1)b + 3a(a + 1)/2)$ |
| $V^B_{(1,0)}$ | $1$       | $(a, b)$    |

The representations $\mathfrak{g}/\mathfrak{p}$ are given by

$$\mathfrak{g}/\mathfrak{p}_1 \cong V^P_{(-1,3)} \oplus V^P_{(1,0)},$$

$$\mathfrak{g}/\mathfrak{p}_2 \cong V^P_{(1,-1)} \oplus V^P_{(1,0)} \oplus V^P_{(0,1)},$$

$$\mathfrak{g}/\mathfrak{b} \cong V^B_{(2,-3)} \oplus V^B_{(1,0)} \oplus V^B_{(1,-1)} \oplus V^B_{(0,1)} \oplus V^B_{(-1,3)} \oplus V^B_{(-1,2)};$$

whose determinants are given by

$$\text{det} \left( \mathfrak{g}/\mathfrak{p}_1 \right) \cong V^P_{(3,0)},$$

$$\text{det} \left( \mathfrak{g}/\mathfrak{p}_2 \right) \cong V^P_{(0,5)},$$

$$\text{det} \left( \mathfrak{g}/\mathfrak{b} \right) \cong V^P_{(2,2)}.$$

Theorem 1.1 is an easy consequence these facts. For example, to obtain a Calabi–Yau 3-fold from $(G/P_1, \mathcal{E}_V)$, the representation $V$ must satisfy $\dim V = 2$ and $\text{det} V = V^P_{(3,0)}$ since $\text{Pic } G/P_1 \cong \mathbb{Z}$ must inject to $\text{Pic } X$. If $V$ is decomposable, then $V \cong V^P_{(1,0)} \oplus V^P_{(2,0)}$ is the only choice, and if $V$ is indecomposable, then $V \cong V^P_{(1,1)}$ is the only choice.
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