Dunford–Henstock–Kurzweil and Dunford–McShane Integrals of Vector-Valued Functions Defined on $m$-Dimensional Bounded Sets

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Abstract. In this paper, we define the Dunford–Henstock–Kurzweil and the Dunford–McShane integrals of Banach space-valued functions defined on a bounded Lebesgue measurable subset of $m$-dimensional Euclidean space $\mathbb{R}^m$. We will show that the new integrals are “natural” extensions of the McShane and the Henstock–Kurzweil integrals from $m$-dimensional closed non-degenerate intervals to $m$-dimensional bounded Lebesgue measurable sets. As applications, we will present full descriptive characterizations of the McShane and Henstock–Kurzweil integrals in terms of our integrals. Moreover, a relationship between new integrals will be proved in terms of the Dunford integral.

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1. Introduction

In the paper [19], the Hake–Henstock–Kurzweil and the Hake–McShane integrals are defined. It is proved that those integrals are “natural” extensions of the Henstock–Kurzweil and the McShane integrals from $m$-dimensional closed non-degenerate intervals to $m$-dimensional open and bounded sets, see [19, Theorems 3.1 and 3.2]. The motivation behind those new integrals is to obtain Hake-type theorems for the Henstock–Kurzweil and the McShane integrals of a Banach space-valued function defined on a closed non-degenerate interval in $m$-dimensional Euclidean space $\mathbb{R}^m$, see [19, Theorems 3.3 and 3.4].

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There are Hake-type theorems for certain Henstock–Kurzweil-type integrals of real-valued functions in [1,9,22,25,27,28].

In this paper, we define the Dunford–Henstock–Kurzweil and the Dunford–McShane integrals of Banach space-valued functions defined on a bounded subsets $G \subset \mathbb{R}^m$, such that $|G \setminus G^o| = 0$. We will show that the new integrals are also “natural” extensions of the McShane and the Henstock–Kurzweil integrals from $m$-dimensional closed non-degenerate intervals to $m$-dimensional bounded Lebesgue measurable sets, see Theorems 3.3 and 3.6.

As applications, we will present full descriptive characterizations of the McShane and the Henstock–Kurzweil integrals in terms of our integrals, see Theorems 3.5 and 3.8.

In the paper [11], D. H. Fremlin proved the following result for the case of a compact non-degenerate subinterval $I \subset \mathbb{R}$.

**Theorem 1.1.** (Fremlin’s Theorem) A function $f : I \to X$ is McShane integrable on $I$ if and only if it is Henstock–Kurzweil integrable and Pettis integrable on $I$.

Checking Fremlin’s proof, it can be seen that it still holds when $I$ is an $m$-dimensional closed non-degenerate subinterval in $\mathbb{R}^m$, c.f. [26, Theorem 6.2.6]. Using Fremlin’s Theorem, we will show a relationship between Dunford–McShane and Dunford—Henstock–Kurzweil integrals in terms of the Dunford integral, see Theorem 3.6.

### 2. Notations and Preliminaries

Throughout this paper, $X$ denotes a real Banach space with the norm $|| \cdot ||$ and $X^*$ its dual. The Euclidean space $\mathbb{R}^m$ is equipped with the maximum norm. $B_m(t,r)$ denotes the open ball in $\mathbb{R}^m$ with center $t$ and radius $r > 0$. We denote by $\mathcal{L}(\mathbb{R}^m)$ the $\sigma$-algebra of Lebesgue measurable subsets of $\mathbb{R}^m$ and by $\lambda$ the Lebesgue measure on $\mathcal{L}(\mathbb{R}^m)$. $|A|$ denotes the Lebesgue measure of $A \in \mathcal{L}(\mathbb{R}^m)$. We put:

$$\mathcal{L}(A) = \{ A \cap L : L \in \mathcal{L}(\mathbb{R}^m) \},$$

for any $A \in \mathcal{L}(\mathbb{R}^m)$.

The subset $\prod_{j=1}^{m} [a_j, b_j] \subset \mathbb{R}^m$ is said to be a closed non-degenerate interval in $\mathbb{R}^m$, if $-\infty < a_j < b_j < +\infty$, for $j = 1, \ldots, m$. Two closed non-degenerate intervals $I$ and $J$ in $\mathbb{R}^m$ are said to be non-overlapping if $I^o \cap J^o = \emptyset$, where $I^o$ denotes the interior of $I$. By $\mathcal{I}$, the family of all closed non-degenerate subintervals in $\mathbb{R}^m$ is denoted, and by $\mathcal{I}_E$, the family of all closed non-degenerate subintervals in $E \in \mathcal{L}(\mathbb{R}^m)$.

Let $E \in \mathcal{L}(\mathbb{R}^m)$. A function $F : \mathcal{I}_E \to X$ is said to be an additive interval function, if, for each two non-overlapping intervals $I, J \in \mathcal{I}_E$, such that $I \cup J \in \mathcal{I}_E$, we have:

$$F(I \cup J) = F(I) + F(J).$$

A pair $(t, I)$ of a point $t \in E$ and an interval $I \in \mathcal{I}_E$ is called an $\mathcal{M}$-tagged interval in $E$, and $t$ is the tag of $I$. Requiring $t \in I$ for the tag of
I, we get the concept of an HK-tagged interval in E. A finite collection 
\{(t_i, I_i) : i = 1, \ldots, p\} of M-tagged intervals (HK-tagged intervals) in E is
called an M-partition (HK-partition) in E, if \{I_i : i = 1, \ldots, p\} is a collection
of pairwise non-overlapping intervals in \(I_E\). Given \(Z \subset E\), a positive function
\(\delta : Z \to (0, +\infty)\) is called a gauge on Z. We say that an M-partition (HK-partition) \(\pi = \{(t_i, I_i) : i = 1, \ldots, p\}\) in E is:

- M-partition (HK-partition) of E, if \(\bigcup_{i=1}^{p} I_i = E\),
- Z-tagged if \(\{t_1, \ldots, t_p\} \subset Z\),
- \(\delta\)-fine if for each \(i = 1, \ldots, p\), we have \(I_i \subset B_m(t_i, \delta(t_i))\).

We now recall the definitions of the McShane and the Henstock–Kurzweil
integrals of a function \(f : W \to X\), where \(W\) is a fixed interval in \(I\). The
function \(f\) is said to be McShane (Henstock–Kurzweil) integrable on \(W\) if
there is a vector \(x_f \in X\), such that for every \(\varepsilon > 0\), there exists a gauge \(\delta\) on
\(W\), such that for every \(\delta\)-fine M-partition (HK-partition) \(\pi\) of \(W\), we have:

\[
\left\| \sum_{(t, I) \in \pi} f(t)|I| - x_f \right\| < \varepsilon.
\]

In this case, the vector \(x_f\) is said to be the McShane (Henstock–Kurzweil)
integral of \(f\) on \(W\) and we set \(x_f = (M) \int_W f d\lambda (x_f = (HK) \int_W f d\lambda)\).
The function \(f\) is said to be McShane (Henstock–Kurzweil) integrable over
a subset \(A \subset W\), if the function \(f 1_A : W \to X\) is McShane (Henstock–Kurzweil)
integrable on \(W\), where \(1_A\) is the characteristic function of the set
\(A\). The McShane (Henstock–Kurzweil) integral of \(f\) over \(A\) will be denoted
by \((M) \int_A f d\lambda ((HK) \int_A f d\lambda)\). If \(f : W \to X\) is McShane integrable on \(W\),
then by [26, Theorem 4.1.6], the function \(f\) is the McShane integrable on each
\(A \in \mathcal{L}(W)\), while by [26, Theorem 3.3.4], if \(f\) is Henstock–Kurzweil integrable
on \(W\), then \(f\) is the Henstock–Kurzweil integrable on each \(I \in \mathcal{I}_W\). Therefore,
we can define an additive interval function \(F : \mathcal{I}_W \to X\) as follows:

\[
F(I) = (M) \int_I f d\lambda, \quad (F(I) = (HK) \int_I f d\lambda), \quad \text{for all } I \in \mathcal{I}_W,
\]

which is called the primitive of \(f\). For the McShane and Henstock–Kurzweil
integrals, we refer to [2–4,6–8,11–18,20,21,23,24,26]

We now define Dunford–McShane and Dunford–Henstock–Kurzweil
integrals on a bounded Lebesgue measurable subset in \(\mathbb{R}^m\). Fix a bounded
Lebesgue measurable subset \(E \in \mathcal{L}(\mathbb{R}^m)\), such that \(E^o \neq \emptyset\), where \(E^o\) is
the interior of \(E\). A sequence \((I_k)\) of pairwise non-overlapping intervals in \(I_E\) is
said to be a division in \(E\). By \(\mathcal{P}_E\), the family of all divisions in \(E\) is denoted.
A division \((I_k) \in \mathcal{P}_E\) is said to be a division of \(E\) if:

\[
E = \bigcup_{k=1}^{+\infty} I_k.
\]

We denote by \(\mathcal{D}_E\) the family of all divisions of \(E\). Clearly, \(\mathcal{D}_E \subset \mathcal{P}_E\). By [10,
Lemma 2.43], the family \(\mathcal{D}_E^o\) is not empty, and since

\[
\mathcal{D}_E^o \subset \mathcal{P}_E^o \subset \mathcal{P}_E,
\]
it follows that $\mathcal{P}_E$ is not empty.

**Definition 2.1.** A additive interval function $F : \mathcal{I}_E \to X$ is said to be a \textit{Dunford-function}, if given a division $(I_k) \in \mathcal{P}_E$, we have:

(i) the series

$$\sum_{k:|I \cap I_k| > 0} F(I \cap I_k)$$

is unconditionally convergent in $X$, for each $I \in \mathcal{I}$,

(ii) if $(I_k) \in \mathcal{P}_E^o$, then the equality

$$F(I) = \sum_{k:|I \cap I_k| > 0} F(I \cap I_k),$$

holds for all $I \in \mathcal{I}_E$.

**Definition 2.2.** We say that the additive interval function $F : \mathcal{I}_E \to X$ has $\mathcal{M}$-\textit{negligible variation} (\textit{HKK-negligible variation}) over a subset $Z \subseteq \mathbb{R}^m$, if, for each $\varepsilon > 0$, there exists a gauge $\delta_\varepsilon$ on $Z$, such that for each $Z$-tagged $\delta_\varepsilon$-fine $\mathcal{M}$-partition (\mathcal{HK}-partition) $\pi$ in $\mathbb{R}^m$, we have:

(i) the series

$$\sum_{k:|I \cap I_k| > 0} F(I \cap I_k)$$

is unconditionally convergent in $X$, for each $(t, I) \in \pi$;

(ii) the inequality

$$\left\| \sum_{(t, I) \in \pi} \left( \sum_{k:|I \cap I_k| > 0} F(I \cap I_k) \right) \right\| < \varepsilon,$$

holds,

whenever $(I_k) \in \mathcal{P}_E^o$. We say that $F$ has $\mathcal{M}$-\textit{negligible variation} (\textit{HKK-negligible variation}) \textit{outside of} $E^o$ if $F$ has $\mathcal{M}$-negligible variation (\textit{HKK-negligible variation}) over $(E^o)^c = \mathbb{R}^m \setminus E^o$.

**Definition 2.3.** We say that a function $f : E \to X$ is \textit{Dunford–McShane (Dunford–Henstock–Kurzweil) integrable} on $E$ with the primitive $F : \mathcal{I}_E \to X$, if we have:

(i) for each $\varepsilon > 0$, there exists a gauge $\delta_\varepsilon$ on $E^o$, such that for each $\delta_\varepsilon$-fine $\mathcal{M}$-partition (\mathcal{HK}-partition) $\pi$ in $E^o$, we have:

$$\left\| \sum_{(t, I) \in \pi} (f(t)|I| - F(I)) \right\| < \varepsilon;$$

(ii) $F$ is a Dunford-function;

(iii) $F$ has $\mathcal{M}$-negligible (\textit{HKK-negligible}) variation outside of $E^o$. 

In this case, we define the Dunford–McShane (the Dunford–Henstock–Kurzweil) integral of \( f \) over \( I \in \mathcal{I}_E \) as follows:

\[
(DM) \int_I f \, d\lambda = F(I), \quad (DHK) \int_I f \, d\lambda = F(I)
\]

Clearly, if \( f : E \to X \) is Dunford–McShane (Dunford–Henstock–Kurzweil) integrable on \( E \) with the primitive \( F \) and \( E = E^o \), then \( f \) is Hake–McShane (Hake–Henstock–Kurzweil) integrable on \( E \) with the primitive \( F \).

Finally, we recall the definition of the Dunford integral in the second dual \( X^{**} \) of \( X \), c.f. [5]. A function \( f : E \to X \) is said to be Dunford integrable, if \( x^* f \) is Lebesgue integrable (or, equivalently McShane integrable) for all \( x^* \in X^* \). In the case that \( f \) is Dunford integrable, by Dunford’s Lemma, for each \( A \in \mathcal{L}(E) \), there exists \( x^{**}_A \in X^{**} \) satisfying:

\[
x^{**}_A(x^*) = (M) \int_A x^* f \, d\lambda, \quad \text{for all } x^* \in X^*,
\]

and we write \( x^{**}_A = (D) \int_A f \, d\lambda \).

If \( (D) \int_A f \, d\lambda \in e(X) \subset X^{**} \), for all \( A \in \mathcal{L}(E) \), then \( f \) is called Pettis integrable, where \( e \) is the canonical embedding of \( X \) into \( X^{**} \). In this case, we write \( (P) \int_A f \, d\lambda \) instead of \( (D) \int_A f \, d\lambda \) to denote Pettis integral of \( f \) over \( A \in \mathcal{L}(E) \).

3. The Main Results

From now on, \( G \) will be a bounded subset of \( \mathbb{R}^m \), such that \( G^o \neq \emptyset \) and \( |G \setminus G^o| = 0 \). Since \( G \) is a bounded subset of \( \mathbb{R}^m \), we can fix an interval \( I_0 \in \mathcal{I} \), such that \( G \subset I_0 \). Given a function \( f : G \to X \), we denote by \( f_0 : I_0 \to X \) the function defined as follows:

\[
f_0(t) = \begin{cases} f(t) & \text{if } t \in G \\ 0 & \text{if } t \in I_0 \setminus G. \end{cases}
\]

Assume that the functions \( f : G \to X \) and \( F : \mathcal{I}_G \to X \) are given. Then, given a division \( (C_k) \in \mathcal{P}_G \), we denote:

\[
f_k = f|_{C_k} \quad \text{and} \quad F_k = F|_{I_{C_k}}, \quad \text{for each } k \in \mathbb{N}.
\]

Let us start with the following auxiliary lemmas.

**Lemma 3.1.** Let \( f : G \to X \) be a function. Then, given \( \varepsilon > 0 \), there exists a gauge \( \delta \) on \( Z = G \setminus G^o \), such that for each \( \delta \)-fine \( Z \)-tagged \( \mathcal{M} \)-partition (or \( HK \)-partition) \( \pi \) in \( I_0 \), we have:

\[
\left\| \sum_{(t, I) \in \pi} f(t)|I| \right\| < \varepsilon.
\]

**Proof.** Define a function \( g_0 : I_0 \to X \) as follows:

\[
g_0(t) = \begin{cases} f(t) & \text{if } t \in Z \\ 0 & \text{otherwise.} \end{cases}
\]
Then, by [26, Theorem 3.3.1 (or Corollary 3.3.2)], \( g_0 \) is McShane (or Henstock–Kurzweil) integrable on \( I_0 \) and
\[
\int_I g_0 \, d\lambda = 0 \quad \text{for all } I \in \mathcal{I}_I.
\]
Therefore, by [26, Lemma 3.4.2 (or Lemma 3.4.1)], given \( \varepsilon > 0 \), there exists a gauge \( \delta \) on \( Z \), such that for each \( \delta \)-fine \( \mathcal{M} \)-partition (or \( \mathcal{HK} \)-partition) \( \pi \) in \( I_0 \), we have:
\[
\left\| \sum_{(t, I) \in \pi} g_0(t) |I| \right\| < \varepsilon,
\]
and since \( g_0(t) = f(t) \) for all \( t \in Z \), the last result proves the lemma. \( \square \)

The next lemma follows immediately from Lemma 3.1 and Definition 2.2.

**Lemma 3.2.** Let \( f : G \to X \) be a function, and let \( F : \mathcal{I}_G \to X \) be an additive interval function. If \( F \) has \( \mathcal{M} \)-negligible (or \( \mathcal{HK} \)-negligible) variation outside of \( G^o \), then given \( \varepsilon > 0 \), there exists a gauge \( \delta \) on \( Z \), such that for each \( \delta \)-fine \( \mathcal{M} \)-partition (or \( \mathcal{HK} \)-partition) \( \pi \) in \( I_0 \), we have:
\[
\left\| \sum_{(t, I) \in \pi} \left( f(t) |I| - \sum_{k : |I \cap I_k| > 0} F(I \cap I_k) \right) \right\| < \varepsilon,
\]
whenever \( (I_k) \in \mathcal{D}_{G^o} \).

We now present a full characterization of the Dunford–McShane integral in terms of the McShane integral.

**Theorem 3.3.** Let \( f : G \to X \) be a function and let \( F : \mathcal{I}_G \to X \) be an additive interval function. Then, the following statements are equivalent:

(i) \( f \) is Dunford–McShane integrable on \( G \) with the primitive \( F \);

(ii) \( f_0 \) is McShane integrable on \( I_0 \) with the primitive \( F_0 \), such that \( F_0(I) = F(I) \), for all \( I \in \mathcal{I}_G \);

(iii) \( F \) is a Dunford-function and has \( \mathcal{M} \)-negligible variation outside of \( G^o \), and given any division \( (C_k) \in \mathcal{D}_{G^o} \), each \( f_k \) is McShane integrable on \( C_k \) with the primitive \( F_k \).

**Proof.** (i) \( \Rightarrow \) (iii) Assume that \( f \) is Dunford–McShane integrable on \( G \) with the primitive \( F \) and let \( (C_k) \) be any division of \( G^o \). Then, given \( \varepsilon > 0 \), there exists a gauge \( \delta \) on \( G^o \), such that for each \( \delta \)-fine \( \mathcal{M} \)-partition \( \pi \) in \( G^o \), we have:
\[
\left\| \sum_{(t, I) \in \pi} \left( f(t) |I| - F(I) \right) \right\| < \varepsilon.
\]
By Definition 2.3, \( F \) is a Dunford-function and has \( \mathcal{M} \)-negligible variation outside of \( G^o \). Thus, it remains to prove that each \( f_k \) is McShane
integrable on \( C_k \) with the primitive \( F_k \). Let \( \pi_k \) be a \( \delta_k \)-fine \( \mathcal{M} \)-partition of \( C_k \), where \( \delta_k = \delta|_{C_k} \). Then, \( \pi_k \) is a \( \delta \)-fine \( \mathcal{M} \)-partition in \( G^0 \) and, therefore:

\[
\left\| \sum_{(t,I) \in \pi_k} \left( f_k(t) \cdot |I| - F_k(I) \right) \right\| = \left\| \sum_{(t,I) \in \pi_k} \left( f(t) \cdot |I| - F(I) \right) \right\| < \varepsilon.
\]

This means that \( f_k \) is McShane integrable on \( C_k \) with the primitive \( F_k \).

\[\text{(iii) } \Rightarrow \text{(ii)} \] Assume that \( \text{(iii)} \) holds. Let \( \varepsilon > 0 \) and let \( (C_k) \in \mathcal{D}_{G^0} \).

Then, since each function \( f_k \) is McShane integrable on \( C_k \) with the primitive \( F_k \), by [26, Lemma 3.4.2], there exists a gauge \( \delta_k \) on \( C_k \), such that for each \( \delta_k \)-fine \( \mathcal{M} \)-partition \( \pi_k \) in \( C_k \), we have:

\[
\left\| \sum_{(t,I) \in \pi_k} \left( f_k(t) \cdot |I| - F_k(I) \right) \right\| \leq \frac{1}{2^k} \frac{\varepsilon}{4}.
\] (3.1)

Note that for any \( t \in G^0 = \bigcup_k C_k \), we have the following possible cases:

- there exists \( i_0 \in \mathbb{N} \), such that \( t \in (C_{i_0})^0 \);
- there exists \( j_0 \in \mathbb{N} \), such that \( t \in (C_{j_0})^0 \). In this case, there exists a finite set \( \mathcal{N}_t = \{ j \in \mathbb{N} : t \in (C_j)^0 \} \), such that \( t \in \bigcap_{j \in \mathcal{N}_t} C_j \) and \( t \notin C_k \), for all \( k \in \mathbb{N} \setminus \mathcal{N}_t \). Hence, \( t \in \bigcup_{j \in \mathcal{N}_t} C_j^0 \).

For each \( k \in \mathbb{N} \), choose \( \delta_k \), so that for any \( t \in G^0 \), we have:

\[ t \in (C_k)^0 \Rightarrow B_m(t, \delta_k(t)) \subset C_k \]

and

\[ t \in C_k \setminus (C_k)^0 \Rightarrow B_m(t, \delta_k(t)) \subset \bigcup_{j \in \mathcal{N}_t} C_j. \]

Since \( F \) has \( \mathcal{M} \)-negligible variation outside of \( G^0 \), there exists a gauge \( \delta_v \) on \( I_0 \setminus G^0 \), such that for each \( (I_0 \setminus G^0) \)-tagged \( \delta_v \)-fine \( \mathcal{M} \)-partition \( \pi_v \) in \( I_0 \), we have:

\[
\left\| \sum_{(t,I) \in \pi_v} \left( \sum_{k : |I \cap C_k| > 0} F(I \cap C_k) \right) \right\| < \frac{\varepsilon}{4}. \] (3.2)

By Lemma 3.2, we can choose \( \delta_v \), so that for each \( (G^0 \setminus G^0) \)-tagged \( \delta_v \)-fine \( \mathcal{M} \)-partition \( \pi \) in \( I_0 \), we have:

\[
\left\| \sum_{(t,I) \in \pi} \left( f(t) \cdot |I| - \sum_{k : |I \cap C_k| > 0} F(I \cap C_k) \right) \right\| < \frac{\varepsilon}{4}. \] (3.3)

By hypothesis, we have also that \( F \) is a Dunford-function. Therefore, we can define an additive interval function \( F_0 : \mathcal{I}_{I_0} \rightarrow X \) as follows:

\[
F_0(I) = \sum_{k : |I \cap C_k| > 0} F(I \cap C_k), \quad \text{for all } I \in \mathcal{I}_{I_0}. \] (3.4)

Clearly, \( F_0(I) = F(I) \), for all \( I \in \mathcal{I}_G \). We will show that \( f_0 \) is McShane integrable on \( I_0 \) with the primitive \( F_0 \). To see this, we first define a gauge \( \delta_0 : I_0 \rightarrow (0, +\infty) \) as follows. For each \( t \in G^0 \), we choose:
\[ \delta_0(t) = \begin{cases} \delta_{i_0}(t) & \text{if } t \in (C_{i_0})^o \\ \min \{\delta_j(t) : j \in N_i\} & \text{otherwise,} \end{cases} \]

while for \( t \in I_0 \setminus G^o \), \( \delta_0(t) = \delta_v(t) \). Let \( \pi \) be an arbitrary \( \delta_0 \)-fine \( \mathcal{M} \)-partition of \( I_0 \). Then:

\[ \pi = \pi_1 \cup \pi_2 \cup \pi_3 \cup \pi_4, \]

where

\[ \pi_1 = \{(t, I) \in \pi : (\exists i_0 \in N)[t \in (C_{i_0})^o]\} \]
\[ \pi_2 = \{(t, I) \in \pi : (\exists j_0 \in N)[t \in C_{j_0} \setminus (C_{j_0})^o]\} \]

and

\[ \pi_3 = \{(t, I) \in \pi : t \in G \setminus G^o\} \]
\[ \pi_4 = \{(t, I) \in \pi : t \in I_0 \setminus G\} \]

Hence:

\[
\left\| \sum_{(t,I) \in \pi} (f_0(t)|I| - F_0(I)) \right\| \leq \left\| \sum_{(t,I) \in \pi_1} (f(t)|I| - F(I)) \right\| + \left\| \sum_{(t,I) \in \pi_2} (f(t)|I| - F(I)) \right\| \\
+ \left\| \sum_{(t,I) \in \pi_3} (f(t)|I| - F_0(I)) \right\| + \left\| \sum_{(t,I) \in \pi_4} F_0(I) \right\|. \tag{3.5}\]

Note that, if we define:

\[ \pi_1^k = \{(t, I) : (t, I) \in \pi_1, t \in (C_k)^o\}, \]
\[ \pi_2^k = \{(t, I \cap C_k) : (t, I) \in \pi_2, t \in C_k \setminus (C_k)^o, |I \cap C_k| > 0\}, \]

then \( \pi_1^k \) and \( \pi_2^k \) are \( \delta_k \)-fine \( \mathcal{M} \)-partitions in \( C_k \). Therefore, by (3.1), it follows that:

\[
\left\| \sum_{(t,I) \in \pi_1} (f(t)|I| - F(I)) \right\| = \left\| \sum_k \sum_{(t,I) \in \pi_1, t \in (C_k)^o} (f(t)|I| - F(I)) \right\| \\
\leq \sum_k \left\| \sum_{(t,I) \in \pi_1^k} (f_k(t)|I| - F_k(I)) \right\| \\
\leq \sum_{k=1}^{+\infty} \frac{\varepsilon}{2^k 4} = \frac{\varepsilon}{4} \tag{3.6}\]
and

\[ \left\| \sum_{(t,I) \in \pi_2} (f(t)|I| - F(I)) \right\| = \left\| \sum_{(t,I) \in \pi_2} \left( \sum_{j \in N_{I_t}} (f(t)|I \cap C_j| - F(I \cap C_j)) \right) \right\| \]

\[ = \left\| \sum_{(t,I) \in \pi_2} \left( \sum_{j \in N_{I_t}} (f_j(t)|I \cap C_j| - F_j(I \cap C_j)) \right) \right\| \]

\[ \leq \left\| \sum_{k} \left( \sum_{(t,I) \in \pi_2^k} (f_k(t)|I \cap C_k| - F_k(I \cap C_k)) \right) \right\| \]

\[ \leq \sum_{k=1}^{+\infty} \frac{1}{2^k} \frac{\varepsilon}{4} = \frac{\varepsilon}{4}. \] (3.7)

We have also that \( \pi_3 \) is a \((G \setminus G^\alpha)\)-tagged \( \delta_\nu \)-fine \( M \)-partition in \( I_0 \) and \( \pi_4 \) is \((I_0 \setminus G)\)-tagged \( \delta_\nu \)-fine \( M \)-partition in \( I_0 \). Therefore, (3.2) and (3.3) together with (3.4) yield:

\[ \left\| \sum_{(t,I) \in \pi_3} (f(t)|I| - F_0(I)) \right\| < \frac{\varepsilon}{4} \quad \text{and} \quad \left\| \sum_{(t,I) \in \pi_4} F_0(I) \right\| < \frac{\varepsilon}{4}. \]

The last result together with (3.5), (3.6), and (3.7) yields:

\[ \left\| \sum_{(t,I) \in \pi} (f_0(t)|I| - F_0(I)) \right\| < \varepsilon, \]

and since \( \pi \) was an arbitrary \( \delta_0 \)-fine \( M \)-partition of \( I_0 \), it follows that \( f_0 \) is \( \text{McShane integrable} \) on \( I_0 \) with the primitive \( F_0 \).

(ii) \Rightarrow (i) Assume that (ii) holds. Then, by [26, Lemma 3.4.2], given \( \varepsilon > 0 \), there exists a gauge \( \delta_0 \) on \( I_0 \), such that for each \( \delta_0 \)-fine \( M \)-partition \( \pi_0 \) in \( I_0 \), we have

\[ \left\| \sum_{(t,I) \in \pi} (f_0(t)|I| - F_0(I)) \right\| < \varepsilon. \] (3.8)

The gauge \( \delta_0 \) can be chosen, so that for each \( t \in I_0 \), we have:

\[ t \in G^\alpha \Rightarrow B_m(t, \delta_0(t)) \subset G^\alpha. \]

By Lemma 3.1, we can also choose \( \delta_0 \), so that:

\[ \left\| \sum_{(t,I) \in \pi} f_0(t)|I| \right\| < \varepsilon, \] (3.9)

whenever \( \pi \) is \((G \setminus G^\alpha)\)-tagged \( \delta_0 \)-fine \( M \)-partition in \( I_0 \).
Hence, if we define $\delta = \delta_0|_{G^o}$, then for each $\delta$-fine $\mathcal{M}$-partition $\pi$ in $G^o$, we have:

$$\left\| \sum_{(t,I) \in \pi} (f(t)|I| - F(I)) \right\| < \varepsilon.$$  

Thus, it remains to show that $F$ is a Dunford-function and has $\mathcal{M}$-negligible variation outside of $G^o$. We first show that $F$ is a Dunford-function. Let $(I_k) \in \mathcal{P}_G$. Since for any $I \in \mathcal{I}$, we have:

$$\sum_{k: |I \cap I_k| > 0} F(I \cap I_k) = \sum_{k: |I \cap I_k| > 0} F_0(I \cap I_k) = \sum_{k: |I \cap I_k| > 0} (M) \int_{I \cap I_k} f_0 d\lambda$$

$$= (M) \int_{I \cap (\bigcup_k I_k)} f_0 d\lambda,$$

the series $\sum_{k: |I \cap I_k| > 0} F(I \cap I_k)$ is unconditionally convergent in $X$. If $(I_k) \in \mathcal{D}_{G^o}$ and $I \in \mathcal{I}_G$, then:

$$\sum_{k: |I \cap I_k| > 0} F(I \cap I_k) = (M) \int_{I \cap G^o} f_0 d\lambda = (M) \int_{I \cap G} f_0 d\lambda$$

$$= (M) \int_{I} f_0 d\lambda = F_0(I) = F(I).$$

Thus, $F$ is a Dunford-function.

Finally, we show that $F$ has $\mathcal{M}$-negligible variation outside of $G^o$. To see this, we define a gauge $\delta_v$ on $(G^o)^c$ by $\delta_v(t) = \delta_0(t)$ if $t \in I_0 \setminus G^o$, while for $t \notin I_0$, we choose $\delta_v(t)$, so that $B_{\delta_v(t)}(t) \cap I_0 = \emptyset$. Assume that $\pi_v$ is a $(G^o)^c$-tagged $\delta_v$-fine $\mathcal{M}$-partition in $\mathbb{R}^m$. Hence:

$$\pi_0 = \{(t, I \cap I_0) : (t, I) \in \pi_v, t \in I_0 \setminus G^o, |I \cap I_0| > 0\}$$

is a $\delta_0$-fine $\mathcal{M}$-partition in $I_0$. Note that $\pi_0 = \pi_a \cup \pi_b$, where:

$$\pi_a = \{(t, J) \in \pi_0 : t \in I_0 \setminus G\} \text{ and } \pi_b = \{(t, J) \in \pi_0 : t \in (G \setminus G^o)\}.$$  

Since $\pi_a$ and $\pi_b$ are $\delta_0$-fine $\mathcal{M}$-partitions in $I_0$, by (3.8) and (3.9), it follows that:

$$\left\| \sum_{(t,J) \in \pi_0} F_0(J) \right\| \leq \left\| \sum_{(t,J) \in \pi_a} F_0(J) \right\| + \left\| \sum_{(t,J) \in \pi_b} F_0(J) \right\|$$

$$\leq \left\| \sum_{(t,J) \in \pi_a} (f_0(t)|J| - F_0(J)) \right\| + \left\| \sum_{(t,J) \in \pi_b} (f_0(t)|J| - F_0(J)) \right\|$$

$$+ \left\| \sum_{(t,J) \in \pi_b} f_0(t)|J| \right\| < 3\varepsilon.$$  

(3.10)
On the other hand, we have also:

\[
\left\| \sum_{(t,J) \in \pi_0} F_0(J) \right\| = \left\| \sum_{(t,J) \in \pi_0} (M) \int_J f_0 \, d\lambda \right\|
\]

\[
= \left\| \sum_{(t,J) \in \pi_0} (M) \int_{J \cap \mathcal{G}^o} f_0 \, d\lambda + (M) \int_{J \cap \mathcal{G}^o \setminus \mathcal{G}_0} f_0 \, d\lambda + (M) \int_{J \cap \mathcal{G}^o} f_0 \, d\lambda \right\|
\]

\[
= \sum_{(t,J) \in \pi_0} (M) \int_{J \cap \mathcal{G}^o} f_0 \, d\lambda = \left\| \sum_{(t,J) \in \pi_0} \left( \sum_{k=1}^{\infty} (M) \int_{I_k \cap J} f_0 \, d\lambda \right) \right\|
\]

\[
= \left\| \sum_{(t,J) \in \pi_v} \left( \sum_{k: |J \cap I_k| > 0} F_0(J \cap I_k) \right) \right\|
\]

whenever \((I_k) \in \mathcal{G}_0\). The last result together with (3.10) yields:

\[
\left\| \sum_{(t,I) \in \pi_v} \left( \sum_{k: |I \cap I_k| > 0} F(I \cap I_k) \right) \right\| < 3\varepsilon.
\]

This means that \(F\) has \(\mathcal{M}\)-negligible variation outside of \(G^o\), and this ends the proof. \(\square\)

Theorem 3.3 together with [17, Theorem 3(c)] yields immediately the following statement.

**Corollary 3.4.** Let \(f : G \to X\) be a Dunford–McShane integrable function on \(G\) with the primitive \(F\) and let \(h : G \to X\) be a Dunford–McShane integrable function on \(G\) with the primitive \(H\). Then:

(i) \(f + h\) is Dunford–McShane integrable function on \(G\) with the primitive \(F + H\);

(ii) \(r.f\) is Dunford–McShane integrable function on \(G\) with the primitive \(r.F\), where \(r \in \mathbb{R}\).

The following theorem follows immediately from Theorem 3.3 with \(G = I_0\).

**Theorem 3.5.** Let \(f : I_0 \to X\) be a function and let \(F : \mathcal{I}_{I_0} \to X\) be an additive interval function. Then, \(f\) is McShane integrable on \(I_0\) with the primitive \(F\) if and only if we have:

(i) for each \(\varepsilon > 0\), there exists a gauge \(\delta_\varepsilon\) on \((I_0)^o\), such that for each \(\delta_\varepsilon\)-fine \(\mathcal{M}\)-partition \(\pi\) in \((I_0)^o\), we have:

\[
\left\| \sum_{(t,I) \in \pi} (f(t)|I| - F(I)) \right\| < \varepsilon;
\]

(ii) \(F\) is a Dunford-function;

(iii) \(F\) has \(\mathcal{M}\)-negligible variation on \(Z = I_0 \setminus (I_0)^o\).

Let us now present a full characterization of the Dunford–Henstock–Kurzweil integral in terms of the Henstock–Kurzweil integral.
Theorem 3.6. Let $f : G \to X$ be a function and let $F : \mathcal{I}_G \to X$ be an additive interval function. Then, the following statements are equivalent:

(i) $f$ is the Dunford–Henstock–Kurzweil integrable on $G$ with the primitive $F$;
(ii) $f_0$ is the Henstock–Kurzweil integrable on $I_0$ with the primitive $F_0$, such that:

$$F_0(I) = \sum_{k : |I \cap C_k| > 0} F(I \cap C_k), \quad \text{for all } I \in \mathcal{I}_0$$

whenever $(C_k) \in \mathcal{D}_{G^o}$, and $F$ is a Dunford-function;
(iii) $F$ is a Dunford-function and has HK-negligible variation outside of $G^o$, and given any division $(C_k) \in \mathcal{D}_{G^o}$, each $f_k$ is Henstock–Kurzweil integrable on $C_k$ with the primitive $F_k$.

Proof. By the same manner as in Theorem 3.3, it can be proved $(i) \Rightarrow (iii)$ and $(iii) \Rightarrow (ii)$.

$(ii) \Rightarrow (i)$ Assume that $(ii)$ holds. Then, by Lemma 3.4.1 in [26] and by Lemma 3.1, given $\varepsilon > 0$, there exists a gauge $\delta_0$ on $I_0$, such that for each $\delta_0$-fine HK-partition $\pi$ in $I_0$, we have:

$$\left\| \sum_{(t, I) \in \pi} (f_0(t)|I| - F_0(I)) \right\| < \varepsilon, \quad (3.12)$$

and for each $\delta_0$-fine $(G \setminus G^o)$-tagged HK-partition $\pi$ in $I_0$, we have:

$$\left\| \sum_{(t, I) \in \pi} f_0(t)|I| \right\| < \varepsilon. \quad (3.13)$$

We can also choose $\delta_0$, so that $B_m(t, \delta_0(t)) \subset G^o$, for all $t \in G^o$. Hence, if we define $\delta = \delta_0|G^o$, then for each $\delta$-fine HK-partition $\pi$ in $G^o$, we have:

$$\left\| \sum_{(t, I) \in \pi} (f(t)|I| - F(I)) \right\| < \varepsilon.$$

Thus, it remains to show that $F$ has HK-negligible variation outside of $G^o$. To see this, define a gauge $\delta_v$ on $(G^o)^c$ by $\delta_v(t) = \delta_0(t)$ if $t \in I_0 \setminus G^o$, while for $t \notin I_0$, we choose $\delta_v(t)$, so that $B_m(t, \delta_v(t)) \cap I_0 = \emptyset$. Assume that $\pi_v$ is a $(G^o)^c$-tagged $\delta_v$-fine HK-partition in $\mathbb{R}^m$. Hence:

$$\pi_0 = \{(t, I \cap I_0) : (t, I) \in \pi_v, t \in I_0 \setminus G^o, |I \cap I_0| > 0\}$$
is a $\delta_0$-fine HK-partition in $I_0$. Note that $\pi_0 = \pi_a \cup \pi_b$, where

$$\pi_a = \{(t, J) \in \pi_0 : t \in I_0 \setminus G\} \text{ and } \pi_b = \{(t, J) \in \pi_0 : t \in (G \setminus G^o)\}.$$
Since $\pi_a$ and $\pi_b$ are $\delta_0$-fine $HK$-partitions in $I_0$, by (3.12) and (3.13), it follows that:

$$
\left\| \sum_{(t,J) \in \pi_0} F_0(J) \right\| \leq \left\| \sum_{(t,J) \in \pi_a} F_0(J) \right\| + \left\| \sum_{(t,J) \in \pi_b} F_0(J) \right\|
$$

$$
\leq \left\| \sum_{(t,J) \in \pi_a} (f_0(t)|J| - F_0(J)) \right\| + \left\| \sum_{(t,J) \in \pi_b} (f_0(t)|J| - F_0(J)) \right\|
$$

$$
+ \left\| \sum_{(t,J) \in \pi_b} f_0(t)|J| \right\| < 3\varepsilon.
$$

On the other hand, by (3.11), we have also:

$$
\left\| \sum_{(t,J) \in \pi_0} F_0(J) \right\| = \left\| \sum_{(t,J) \in \pi_0} \left( \sum_{k:|J \cap C_k| > 0} F(J \cap C_k) \right) \right\|
$$

$$
= \left\| \sum_{(t,I) \in \pi_v} \left( \sum_{k:|I \cap C_k| > 0} F(I \cap C_k) \right) \right\|,
$$

whenever $(C_k) \in \mathcal{D}_{G^o}$. It follows that:

$$
\left\| \sum_{(t,I) \in \pi_v} \left( \sum_{k:|I \cap C_k| > 0} F(I \cap C_k) \right) \right\| < 3\varepsilon.
$$

This means that $F$ has $HK$-negligible variation outside of $G^o$, and this ends the proof.

It easy to see that Theorem 3.6 together with [26, Theorem 3.3.6] yields the following statement.

**Corollary 3.7.** Let $f : G \to X$ be a Dunford–Henstock–Kurzweil integrable function on $G$ with the primitive $F$ and let $h : G \to X$ be a Dunford–Henstock–Kurzweil integrable function on $G$ with the primitive $H$. Then:

(i) $f + h$ is Dunford–Henstock–Kurzweil integrable function on $G$ with the primitive $F + H$;

(ii) $r.f$ is Dunford–Henstock–Kurzweil integrable function on $G$ with the primitive $r.F$, where $r \in \mathbb{R}$.

The next theorem follows immediately from Theorem 3.6 with $G = I_0$.

**Theorem 3.8.** Let $f : I_0 \to X$ be a function and let $F : I/I_0 \to X$ be a Dunford-function. Then, $f$ is Henstock–Kurzweil integrable on $I_0$ with the primitive $F$, if and only if we have:

(i) for each $\varepsilon > 0$, there exists a gauge $\delta_\varepsilon$ on $(I_0)^o$, such that for each $\delta_\varepsilon$-fine $HK$-partition $\pi$ in $(I_0)^o$, we have:

$$
\left\| \sum_{(t,I) \in \pi} (f(t)|I| - F(I)) \right\| < \varepsilon;
$$
(ii) $F$ has HK-negligible variation on $Z = I_0 \setminus (I_0)^o$.

Finally, we are going to prove a relationship between the Dunford–Henstock–Kurzweil and Dunford–McShane integrals in terms of the Dunford integral.

**Theorem 3.9.** Let $f : I_0 \to X$ be a function and let $F : \mathcal{I}_{I_0} \to X$ be an additive interval function. Then, the following statements are equivalent:

(i) $f$ is Dunford integrable and Dunford–Henstock–Kurzweil integrable on $I_0$ with the primitive $F$;

(ii) $f$ is Dunford–McShane integrable on $I_0$ with the primitive $F$.

**Proof.** (i) $\Rightarrow$ (ii) Assume that (i) holds. We first claim that:

$$\int_I f d\lambda = x_I^* \in e(X), \quad \text{for all } I \in \mathcal{I}_{I_0}. \quad (3.14)$$

To see this, let $I$ be an arbitrary closed non-degenerate interval in $\mathcal{I}_{I_0}$. Then:

$$x_I^*(x^*) = (M) \int_I x^* f d\lambda, \quad \text{for all } x^* \in X^*,$$

and since McShane integrable functions are Henstock–Kurweil integrable, it follows that:

$$x_I^*(x^*) = (HK) \int_I x^* f d\lambda, \quad \text{for all } x^* \in X^*. \quad (3.15)$$

On the other hand, by Theorem 3.6, $f$ is Henstock–Kurzweil integrable on $I_0$ with the primitive $F$ and, therefore, by [26, Theorem 6.1.1], it follows that:

$$(HK) \int_I x^* f d\lambda = x^* \left( (HK) \int_I f d\lambda \right), \quad \text{for all } x^* \in X^*.$$ 

The last result together with (3.15) yields that (3.14) holds.

We now claim that for any $(I_k) \in \mathcal{P}_{I_0}$ the series

$$\sum_{k=1}^{+\infty} (D) \int_{I_k} f d\lambda \quad (3.16)$$

is norm convergent in $e(X)$. Indeed, since $F$ is a Dunford-function and the equality

$$\sum_{k=1}^{+\infty} (D) \int_{I_k} f d\lambda = \sum_{k=1}^{+\infty} e(F(I_k))$$

holds, it follows that (3.16) is norm convergent in $e(X)$.

By virtue of [18, Lemma 6], $f$ is Pettis integrable. Thus, we have $f$ is Pettis integrable and Henstock–Kurweil integrable on $I_0$ with the primitive $F$. Therefore, by Fremlin’s Theorem [26, Theorem 6.2.6], we obtain that $f$ is McShane integrable on $I_0$ with the primitive $F$. Furthermore, by Theorem 3.3, it follows that $f$ is Dunford–McShane integrable on $I_0$ with the primitive $F$.

(ii) $\Rightarrow$ (i) Assume that $f$ is Dunford–McShane integrable on $I_0$ with the primitive $F$. Then, by Definition 2.3, it follows that $f$ is Dunford–Henstock–Kurzweil integrable on $I_0$ with the primitive $F$. 

By Theorem 3.3, it follows that $f$ is McShane integrable on $I_0$ with the primitive $F$. Hence, $f$ is Pettis integrable and, therefore, $f$ is Dunford integrable, and this ends the proof. \hfill $\square$

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**References**

[1] Boccuto, A., Skvortsov, V.: A Hake-type theorem for integrals with respect to abstract derivation bases in the Riesz space setting. Math. Slovaca **65**(6) (2015)

[2] Boccuto, A., Candeloro, D., Sambucini, A.R.: Henstock multivalued integrability in Banach lattices with respect to pointwise non atomic measures, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **26**(4), 363–383 (2015)

[3] Bongiorno, B.: The Henstock-Kurzweil integral, Handbook of measure theory, vol. I, II, pp. 587–615. North-Holland, Amsterdam (2002)

[4] Cao, S.S.: The Henstock integral for Banach-valued functions. SEA Bull. Math. **16**, 35–40 (1992)

[5] Diestel, J., Uhl, J.J.: Vector measures. Math. Surveys, vol. 15. American Mathematical Society, Providence, RI (1977)

[6] Di Piazza, L., Musial, K.: A characterization of variationally McShane integrable Banach-space valued functions. Ill. J. Math. **45**, 279–289 (2001)

[7] Di Piazza, L., Marraffa, V.: The McShane, PU and Henstock integrals of Banach valued functions. Czechoslov. Math. J. **52**(3), 609–633 (2002)

[8] Di Piazza, L., Preiss, D.: When do McShane and Pettis integrals coincide? Ill. J. Math. **47**(4), 1177–1187 (2003)

[9] Faure, C.A., Mawhin, J.: The Hakes property for some integrals over multidimensional intervals. Real Anal. Exch. **20**(2), 622–630 (1994/1995)

[10] Folland, G.B.: Real Analysis, Modern techniques and their applications. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication, 2nd edn. Wiley, New York (1999)

[11] Fremlin, D.H.: The Henstock and McShane integrals of vector-valued functions. Ill. J. Math. **38**, 471–479 (1994)

[12] Fremlin, D.H., Mendoza, J.: The integration of vector-valued functions. Ill. J. Math. **38**, 127–147 (1994)

[13] Fremlin, D.H.: The generalized McShane integral. Ill. J. Math. **39**, 39–67 (1995)

[14] Gordon, R.A.: The McShane integral of Banach-valued functions. Ill. J. Math. **34**, 557–567 (1990)

[15] Gordon, R.A.: The Denjoy extension of the Bohner, Pettis, and Dunford integrals. Studia Math. **T.XCII**, 73–91 (1989)

[16] Gordon, R.A.: The Integrals of Lebesgue, Denjoy, Perron, and Henstock. American Mathematical Society, Providence, RI (1994)

[17] Guoju, Y., Schwabik, Š.: The McShane integral and the Pettis integral of Banach space-valued functions defined on $\mathbb{R}^m$. Ill. J. Math. **46**, 1125–1144 (2002)

[18] Guoju, Y.: On Henstock–Kurzweil and McShane integrals of Banach space-valued functions. J. Math. Anal. Appl. **330**, 753–765 (2007)
[19] Kaliaj, S.B.: The new extensions of the Henstock-Kurzweil and the McShane integrals of vector-valued functions. Mediterr. J. Math. 15(22) (2018). https://doi.org/10.1007/s00009-018-1067-2

[20] Kaliaj, S.B.: Some remarks on descriptive characterizations of the strong McShane integral. Math. Bohem. 144(4), 339–355 (2019)

[21] Kurzweil, J., Schwabik, Š.: On the McShane integrability of Banach space-valued functions. Real Anal. Exch. 2, 763–780 (2003/2004)

[22] Kurzweil, J., Jarnik, J.: Differentiability and integrability in $n$ dimensions with respect to $\alpha$-regular intervals. Results Math. 21, 138–151 (1992)

[23] Lee, P.Y.: Lanzhou Lectures on Henstock Integration, Series in Real Analysis, vol. 2. World Scientific Publishing Co., Inc., Singapore (1989)

[24] Lee, P.Y., Výborný, R.: The Integral: An Easy Approach after Kurzweil and Henstock, Australian Mathematical Society Lecture Series 14. Cambridge University Press, Cambridge (2000)

[25] Mawhin, J., Pfeffer, W.F.: The Hakes property of multidimensional generalized Riemann integral. Czech. Math. J. 40(115), 690–694 (1990)

[26] Schwabik, Š., Guoju, Y.: Topics in Banach Space Integration, Series in Real Analysis, vol. 10. World Scientific, Hackensack (2005)

[27] Skvortsov, V., Tulone, F.: A version of Hake’s theorem for Kurzweil–Henstock integral in terms of variational measure (2019). https://doi.org/10.1515/gmj-2019-2074

[28] Singh, S.P., Rana, I.K.: The Hakes theorem and variational measure. Real Anal. Exch. 37(2), 477–488 (2011/2012)

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