Generalization of $C^*$-algebra methods via real positivity for operator and Banach algebras

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Dedicated with affection and gratitude to Richard V. Kadison.

Abstract. With Charles Read we have introduced and studied a new notion of (real) positivity in operator algebras, with an eye to extending certain $C^*$-algebraic results and theories to more general algebras. As motivation note that the ‘completely’ real positive maps on $C^*$-algebras or operator systems are precisely the completely positive maps in the usual sense; however with real positivity one may develop a useful order theory for more general spaces and algebras. This is intimately connected to new relationships between an operator algebra and the $C^*$-algebra it generates. We have continued this work together with Read, and also with Matthew Neal. Recently with Narutaka Ozawa we have investigated the parts of the theory that generalize further to Banach algebras. In the present paper we describe some of this work which is connected with generalizing various $C^*$-algebraic techniques initiated by Richard V. Kadison. In particular Section 2 is in part a tribute to him in keeping with the occasion of this volume, and also discusses some of the origins of the theory of positivity in our sense in the setting of algebras, which the later parts of our paper develops further. The most recent work will be emphasized.

1. Introduction

This is a much expanded version of our talk given at the AMS Special Session “Tribute to Richard V. Kadison” in January 2015. We survey some of our work on a new notion of (real) positivity in operator algebras (by which we mean closed subalgebras of $B(H)$ for a Hilbert space $H$), unital operator spaces, and Banach algebras, focusing on generalizing various $C^*$-algebraic techniques initiated by Richard V. Kadison. In particular Section 2 is in part a tribute to Kadison in keeping with the occasion of this volume, and we will describe a small part of his opus relevant to our setting. This section also discusses some of the origins of the theory of positivity in our sense in the setting of algebras, which the later parts of our paper develops further. In the remainder of the paper we illustrate our

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real-positivity theory by showing how it relates to these results of Kadison, and
also give some small extensions and additional details for our recent paper with
Ozawa [22], and for [21] with Neal.

With Charles Read we have introduced and studied a new notion of (real) posi-
tivity in operator algebras, with an eye to extending certain C*-algebraic results
and theories to more general algebras. As motivation note that the ‘completely’
real positive maps on C*-algebras or operator systems are precisely the completely
positive maps in the usual sense (see Theorem 5.2 below); however with real posi-
tivity one may develop an order theory for more general spaces and algebras that is
useful at least for some purposes. We have continued this work together with Read,
and also with Matthew Neal; giving many applications. Recently with Narutaka
Ozawa we have investigated the parts of the theory that generalize further to Ba-
nach algebras. In all of this, our main goal is to generalize certain nice C*-algebraic
results, and certain function space or function algebra results, which use positivity
or positive approximate identities, but using our real positivity. As we said above,
indeed, in the present paper we survey some of this work which is connected with work
of Kadison. The most recent work will be emphasized, particularly parts of the
Banach-algebraic paper [22]. One reason for this emphasis is that less background
is needed here (for example noncommutative topology, or our work on noncommu-
tative peak sets and peak interpolation, which we have surveyed recently in [12]
although we have since made more progress in [26]). Another reason is that we
welcome this opportunity to add some additional details and complements to [22]
(and to [21]). In particular we will prove some facts that were stated there without
proof. A subsidiary goal of Sections 6 and 7 is to go through versions for general
Banach algebras of results in Sections 3, 4, and 7 of [22] stated for Banach algebras
with approximate identities. We will also pose several open questions. The draw-
back of course is that the Banach algebra case is sometimes less
impressive and clean than the operator algebra case, there usually being a price to
be paid for generality.

Of course an operator algebra or function algebra A may have no positive el-
ements in the usual sense. However we see e.g. in Theorem 5.2 below that an
operator algebra A has a contractive approximate identity iff there is a great abun-
dance of real-positive elements; for example, iff A is spanned by its real-positive
elements. Below Theorem 5.2 we will point out that this is also true for certain
classes of Banach algebras. Of course in the theory of C*-algebras, positivity and
the existence of positive approximate identities are crucial. Some form of our ‘pos-
itive cone’ already appeared in papers of Kadison and Kelley and Vaught in the
early 1950’s, and in retrospect it is a natural idea to attempt to use such a cone
to generalize various parts of C*-algebra theory involving positivity and the exis-
tence of positive approximate identities. However nobody seems to have pursued
this until now. In practice, some things are much harder than the C*-algebra
case. And many things simply do not generalize beyond the C*-theory; that is,
our approach is effective at generalizing some parts of C*-algebra theory, but not
others. The worst problem is that although we have a functional calculus, it is
not as good. Indeed often at first sight in a given C*-subtheory, nothing seems to
work. But if one looks a little closer something works, or an interesting conjec-
ture is raised. Successful applications so far include for example noncommutative
topology (eg. noncommutative Urysohn and Tietze theorems for general operator
algebras, and the theory of open, closed and compact projections in the bidual),
lifting problems, the structure of completely contractive idempotent maps on an
operator algebra (described in Section 3 below), noncommutative peak sets, peak
interpolation, and some other noncommutative function theory, comparison theory,
the structure of operator algebras, new relationships between an operator algebra
and the $C^*$-algebra it generates, approximate identities, etc.

2. Richard Kadison and the beginnings of positivity

The first published words of Richard V. Kadison appear to be the following:

"It is the purpose of the present note to investigate the order prop-
erties of self-adjoint operators individually and with respect to con-
taining operator algebras".

This was from the paper [49], which appeared in 1950. In the early 1950s the war
was over, John von Neumann was editor of the Annals of Mathematics and was
talking to anybody who was interested about ‘rings of operators’, Kadison was in
Chicago and the IAS, and all was well with the world. In 1950, von Neumann wrote
a letter to Kaplansky (IAS Archives, reproduced in [65]) which begins as follows:

"Dear Dr. Kaplansky,

Very many thanks for your letter of February 11th and your
manuscript on "Projections in Banach Algebras". I am very glad
that you are submitting it for THE ANNALS, and I will immedi-
ately recommend it for publication.

Your results are very interesting. You are, of course, very right:
I am and I have been for a long time strongly interested in a “purely
gebraical” rather than “vectorial-spatial” foundation for theories
of operator algebras or operatorlike-algebras. To be more precise:
It always seemed to me that there were three successive levels of
abstraction - first, and lowest, the vectorial-spatial, in which the
Hilbert space and its elements are actually used; second, the purely
gebraical, where only the operators or their abstract equivalents
are used; third, the highest, the approach when only linear spaces
or their abstract equivalents (i.e. operatorially speaking, the pro-
jections) are used. [...] After Murray and I had reached somewhat
rounded results on the first level, I neglected to make a real effort
on the second one, because I was tempted to try immediately the
third one. This led to the theory of continuous geometries. In
studying this, the third level, I realized that one is led there to the
theory of “finite” dimensions only. The discrepancy between what
might be considered the “natural” ranges for the first and the third
level led me to doubt whether I could guess the correct degree of
generality for the second one...”.

It is remarkable here to recall that von Neumann invented the abstract def-
ition of Hilbert spaces, the theory of unbounded operators (as well as much of
the bounded theory), ergodic theory, the mathematical formulation of quantum
mechanics, many fundamental concepts associated with groups (like amenability),
and several other fields of analysis. Even today, teaching a course in functional
analysis can sometimes feel like one is mainly expositing the work of this one man.
However von Neumann is saying above that he had unfortunately neglected what he calls the ‘second level’ of ‘operator algebra’, and at the time of this letter this was ripe and timely for exploration.

Happily, about the time the above letter was written, Richard Vincent Kadison entered the world with a bang: a spate of amazing papers at von Neumann’s ‘second level’. Indeed Kadison soon took leadership of the American school of operator algebras. Some part of his early work was concerned with positive cones and their properties. We will now briefly describe a few of these and spend much of the remainder of our article showing how they can be generalized to nonselfadjoint operator algebras and Banach algebras. The following comprises just a tiny part of Kadison’s opus; but nonetheless is still foundational and seminal. Indeed much of $C^*$-algebra theory would disappear without this work. At the start of this section we already mentioned his first paper, devoted to ‘order properties of self-adjoint operators individually and with respect to containing operator algebras’. His memoir “A representation theory for commutative topological algebra” [51] soon followed, one small aspect of which was the introduction and study of positive cones, states, and square roots in Banach algebras. In the 1951 Annals paper [50], Kadison generalized the Banach-Stone theorem, characterizing surjective isometries between $C^*$-algebras. This result has inspired very many functional analysts and innumerable papers. See for example [38] for a collection of such results, together with their history, although this reference is a bit dated since the list grows all the time. See also e.g. [11], Section 6. In a 1952 Annals paper [52] he proved the Kadison–Schwarz inequality, a fundamental inequality satisfied by positive linear maps on $C^*$-algebras. His student Størmer continued this in a very long (and still continuing) series of deep papers. Later this Kadison–Schwarz work was connected to completely positive maps, Stinespring’s theorem and Arveson’s extension theorem (see the next paragraph and e.g. [68]), conditional expectations, operator systems and operator spaces, quantum information theory, etc. A related enduring interest of Kadison’s is projections and conditional expectations on $C^*$-algebras and von Neumann algebras. A search of his collected works finds very many contributions to this topic (e.g. [53]).

In 1960, Kadison together with I. M. Singer [57] initiated the study of nonselfadjoint operator algebras on a Hilbert space (henceforth simply called operator algebras). Five years later or so, the late Bill Arveson in his thesis continued the study of nonselfadjoint operator algebras, using heavily the Kadison-Fuglede determinant of [54] and positivity properties of conditional expectations. This work was published in [4]; it develops a von Neumann algebraic theory of noncommutative Hardy spaces. We mention in passing that we continued Arveson’s work from [4] in a series of papers with Labuschagne, again using the Kadison-Fuglede determinant of [54] as a main tool (see e.g. the survey [15]), as well as positive conditional expectations and the Kadison–Schwarz inequality. This is another example of using $C^*$-algebraic methods, and in particular tools originating in seminal work of Kadison, in a more general (noncommutative function theoretic) setting. However since this lies in a different direction to the rest of the present article we will say no more about this. In the decade after [4], Arveson went on to write many other seminal papers on nonselfadjoint operator algebras, perhaps most notably [5], in which completely positive maps and the Kadison–Schwarz inequality play a decisive
role, and which may be considered a source of the later theory of operator spaces and operator systems.

Another example: in 1968 Kadison and Aarnes, his first student at Penn, introduced strictly positive elements in a C*-algebra $A$, namely $x \in A$ which satisfy $f(x) > 0$ for every state $f$ of $A$. They proved the fundamental basic result:

**Theorem 2.1 (Aarnes–Kadison).** For a C*-algebra $A$ the following are equivalent:

1. $A$ has a strictly positive element.
2. $A$ has a countable increasing contractive approximate identity.
3. $A = zAz$ for some positive $z \in A$.
4. The positive cone $A_+$ has an element $z$ of full support (that is, the support projection $s(z)$ is 1).

The approximate identity in (2) may be taken to be commuting, indeed it may be taken to be $(z\pi_n)$ for $z$ as in (3). If $A$ is a separable C*-algebra then these all hold.

Aarnes and Kadison did not prove (4). However (4) is immediate from the rest since $s(z)$ is the weak* limit of $z\pi_n$, and the converse is easy. This result is related to the theory of hereditary subalgebras, comparison theory in C*-algebras, etc. In fact much of modern C*-algebra theory would collapse without basic results like this. For example, the Aarnes–Kadison theorem implies the beautiful characterization due to Prosser [71] of closed one-sided ideals in a separable C*-algebra $A$ as the ‘topologically principal (one-sided) ideals’ (we are indebted to the referee for pointing out that Prosser was a student of Kelley). The latter is equivalent to the characterization of hereditary subalgebras of such $A$ as the subalgebras of form $zAz$. (We recall that a hereditary subalgebra, or HSA for short, is a closed selfadjoint subalgebra $D$ satisfying $DAD \subset D$.) These results are used in many modern theories such as that of the Cuntz semigroup. Or, as another example, the Aarnes–Kadison theorem is used in the important stable isomorphism theorem for Morita equivalence of C*-algebras (see e.g. [10, 28]).

Indeed in some sense the Aarnes–Kadison theorem is equivalent to the first assertion of the following:

**Theorem 2.2.** A HSA (resp. closed right ideal) in a C*-algebra $A$ is (topologically) principal, that is of the form $zAz$ (resp. $zA$) for some $z \in A$ if it has a countable (resp. countable left) contractive approximate identity. Every closed right ideal (resp. HSA) is the closure of an increasing union of such (topologically) principal right ideals (resp. HSA’s).

Indeed separable HSA’s (resp. closed right ideals) in C*-algebras have countable (resp. countable left) approximate identities.

One final work of Kadison which we will mention here is his first paper with Gert Pedersen [55], which amongst other things initiates the development of a comparison theory for elements in C*-algebras generalizing the von Neumann equivalence of projections. Again positivity and properties of the positive cone are key to that work. This paper is often cited in recent papers on the Cuntz semigroup.

The big question we wish to address in this article is how to generalize such results and theories, in which positivity is the common theme, to not necessarily selfadjoint operator algebras (or perhaps even Banach algebras). In fact one often can, as we have shown in joint work with Charles Read, Matt Neal, Narutaka...
Ozawa, and others. In the Banach algebra literature of course there are many generalizations of $C^*$-algebra results, but as far as we are aware there is no ‘positivity’ approach like ours (although there is a trace of it in [27]). In particular we mention Sinclair’s generalization from [74] of part of the Aarnes–Kadison theorem:

**Theorem 2.3 (Sinclair).** A separable Banach algebra $A$ with a bounded approximate identity has a commuting bounded approximate identity.

If $A$ has a countable bounded approximate identity then Sinclair and others show results like $A = xA = Ay$ for some $x, y \in A$. In part of our work we follow Sinclair in using variants of the proof of the Cohen factorization method to achieve such results but with ‘positivity’.

We now explain one of the main ideas. Returning to the early 1950s: it was only then becoming perfectly clear what a $C^*$-algebra was; a few fundamental facts about the positive cone were still being proved. We recall that an unpublished result of Kaplansky removed the final superfluous abstract axiom for a $C^*$-algebra, and this used a result in a 1952 paper of Fukamiya, and in a 1953 paper of John Kelley and Vaught [58] based on a 1950 ICM talk by those authors. These sources are referenced in almost every $C^*$-algebra book. The paper of Kelley and Vaught was titled “The positive cone in Banach algebras”, and in the first section of the paper they discuss precisely that. The following is not an important part of their paper, but as in Kadison’s paper a year earlier they have a small discussion on how to make sense of the notion of a positive cone in a Banach algebra, and they prove some basic results here. Both Kadison and Kelley and Vaught have some use for the set

$$\mathfrak{F}_A = \{x \in A : \|1 - x\| \leq 1\}.$$  

In their case $A$ is unital (that is has an identity of norm 1), but if not one may take 1 to be the identity of a unitization of $A$. In [23], Charles Read and the author began a study of not necessarily selfadjoint operator algebras on a Hilbert space $H$; henceforth operator algebras. In this work, $\mathfrak{F}_A$ above plays a pivotal role, and also the cone $\mathbb{R}_+^{-} \mathfrak{F}_A$. In [24] we looked at the slightly larger cone $\mathfrak{r}_A$ of so called accretive elements (this is a non-proper cone or ‘wedge’). In an operator algebra these are the elements with positive real part; in a general Banach algebra they are the elements $x$ with $\text{Re } \varphi(x) \geq 0$ for every state $\varphi$ on a unitization of $A$. We recall that a state on a unital Banach algebra is, as usual in the theory of numerical range [27], a norm one functional $\varphi$ such that $\varphi(1) = 1$. That is, accretive elements are the elements with numerical range in the closed right half-plane. We sometimes also call these the real positive elements. We will see later in Proposition 6.10 that $\mathbb{R}_+ \mathfrak{r}_A = \mathfrak{r}_A$. That is, the one cone above is the closure of the other. We write $\mathfrak{C}_A$ for either of these cones.

The following is known, some of it attributable to Lumer and Phillips, or implicit in the theory of contraction semigroups, or can be found in e.g. [63] Lemma 2.1. The latter paper was no doubt influential on our real-positive theory in [22].

**Lemma 2.4.** Let $A$ be a unital Banach algebra. If $x \in A$ the following are equivalent:

1. $x \in \mathfrak{r}_A$, that is, $x$ has numerical range in the closed right half-plane.
2. $\|1 - tx\| \leq 1 + t^2 \|x\|^2$ for all $t > 0$.
3. $\|\exp(-tx)\| \leq 1$ for all $t > 0$.
4. $\| (t + x)^{-1} \| \leq \frac{1}{t}$ for all $t > 0$. 

(5) \(\|1 - tx\| \leq \|1 - t^2x^2\|\) for all \(t > 0\).

**Proof.** For the equivalence of (1) and (3), see [27, p. 17]. Clearly (5) implies (2). That (2) implies (1) follows by applying a state \(\varphi\) to see \(|1 - t\varphi(x)| \leq 1 + Kt^2\), which forces Re \(\varphi(x) \geq 0\) (see [63, Lemma 2.1]). Given (4) with \(t\) replaced by \(1/t\), we have

\[
\|1 - tx\| = \|(1 + tx)^{-1}(1 + tx)(1 - tx)\| \leq \|1 - t^2x^2\|.
\]

This gives (5). Finally (1) implies (4) by e.g. Stampfli and Williams result [76, Lemma 1] that the norm in (4) is dominated by the reciprocal of the distance from \(-t\) to the numerical range of \(x\). \(\square\)

(We mention another equivalent condition: given \(\epsilon > 0\) there exists a \(t > 0\) with \(\|1 - tx\| < 1 + \epsilon t\). See e.g. [27, p. 30].)

Real positive elements, and the smaller set \(\mathfrak{A}_A\) above, will play the role for us of positive elements in a \(C^*\)-algebra. While they are not the same, real positivity is very compatible with the usual definition of positivity in a \(C^*\)-algebra, as will be seen very clearly in the sequel, and in particular in the next section.

3. Real completely positive maps and projections

Recall that a linear map \(T: A \to B\) between \(C^*\)-algebras (or operator systems) is *completely positive* if \(T(A_+) \subseteq B_+\), and similarly at the matrix levels. By a *unital operator space* below we mean a subspace of \(B(H)\) or a unital \(C^*\)-algebra containing the identity. We gave abstract characterizations of these objects with Matthew Neal in [17, 20], and have studied them elsewhere.

**Definition 3.1.** A linear map \(T: A \to B\) between operator algebras or unital operator spaces is *real positive* if \(T(\mathfrak{r}A) \subseteq \mathfrak{r}B\). It is *real completely positive*, or RCP for short, if \(T^n\) is real positive on \(M_n(A)\) for all \(n \in \mathbb{N}\).

(This and the following two results are later variants from [9] of matching material from [23] for \(\mathfrak{F}_A\).)

**Theorem 3.2.** A (not necessarily unital) linear map \(T: A \to B\) between \(C^*\)-algebras or operator systems is real completely positive in the usual sense iff it is RCP.

We say that an algebra is *approximately unital* if it has a contractive approximate identity (cai).

**Theorem 3.3** (Extension and Stinespring-type Theorem). A linear map \(T: A \to B(H)\) on an approximately unital operator algebra or unital operator space is RCP iff \(T\) has a completely positive (in the usual sense) extension \(\tilde{T}: C^*(A) \to B(H)\). Here \(C^*(A)\) is a \(C^*\)-algebra generated by \(A\). This is equivalent to being able to write \(T\) as the restriction to \(A\) of \(V^*\pi(\cdot)V\) for a \(*\)-representation \(\pi: C^*(A) \to B(K)\), and an operator \(V: H \to K\).

Of course this result is closely related to Kadison’s Schwarz inequality. In particular, if one is trying to generalize results where completely positive maps and the Kadison’s Schwarz inequality are used in the \(C^*\)-theory, to operator algebras, one can see how Theorem 3.2 would play a key role. And indeed it does, for example in the remaining results in this section.

We will not say more about unital operator spaces in the present article, except to say that it is easy to see that completely contractive unital maps on a unital operator space are RCP.
We give two or three applications from \[21\] of Theorem 3.3. The first is related to Kadison’s Banach–Stone theorem for \(C^\ast\)-algebras \[50\], and uses our Banach–Stone type theorem \[16\] Theorem 4.5.13].

Theorem 3.4. (Banach–Stone type theorem) Suppose that \(T : A \to B\) is a completely isometric surjection between approximately unital operator algebras. Then \(T\) is real (completely) positive if and only if \(T\) is an algebra homomorphism.

In the following discussion, by a projection \(P\) on an operator algebra \(A\), we mean an idempotent linear map \(P : A \to A\). We say that \(P\) is a conditional expectation if \(P(P(abP(c)) = P(a)P(b)P(c)\) for \(a, b, c \in A\).

Proposition 3.5. A real completely positive completely contractive map (resp. projection) on an approximately unital operator algebra \(A\), extends to a unital completely contractive map (resp. projection) on the unitalization \(A^1\).

Much earlier, we studied completely contractive projections \(P\) and conditional expectations on unital operator algebras. Assuming that \(P\) is also unital (that is, \(P(1) = 1\)) and that \(\operatorname{Ran}(P)\) is a subalgebra, we showed (see e.g. \[16\] Corollary 4.2.9]) that \(P\) is a conditional expectation. This is the operator algebra variant of Tomiyama’s theorem for \(C^\ast\)-algebras. A well known result of Choi and Effros states that the range of a completely positive projection \(P : B \to B\) on a \(C^\ast\)-algebra \(B\), is again a \(C^\ast\)-algebra with product \(P(xy)\). The analogous result for unital completely contractive projections on unital operator algebras is true too, and is implicit in the proof of our generalization of Tomiyama’s theorem above. Unfortunately, there is no analogous result for (nonunital) completely contractive projections on possibly nonunital operator algebras without adding extra hypotheses on \(P\). However if we add the condition that \(P\) is also ‘real completely positive’, then the question does make good sense and one can easily deduce from the unital case and Proposition 3.5 one direction of the following:

Theorem 3.6. \[21\] The range of a completely contractive projection \(P : A \to A\) on an approximately unital operator algebra is again an operator algebra with product \(P(xy)\) and \(c\text{-ai}(P(e_i))\) for some \(c\text{-ai}(e_i)\) of \(A\), iff \(P\) is real completely positive.

Proof. For the ‘forward direction’ note that \(P^{**}\) is a unital complete contraction, and hence is real completely positive as we said in above Theorem 3.3. For the ‘backward direction’ the following proof, due to the author and Neal, was originally a remark in \[21\]. By passing to the bidual we may assume that \(A\) is unital, using the BRS characterization of operator algebras \[16\] Section 2.3. If \(P(P(1))x = P(xP(1)) = x\) for all \(x \in \operatorname{Ran}(P)\) then we are done by the abstract characterization of operator algebras from \[16\] Section 2.3, since then \(P(xy)\) defines a bilinear completely contractive product on \(\operatorname{Ran}(P)\) with ‘unit’ \(P(1)\). Let \(I(A)\) be the injective envelope of \(A\). We may extend \(P\) to a completely positive completely contractive map \(\hat{P} : I(A) \to I(A)\), by \[9\] Theorem 2.6] and injectivity of \(I(A)\). We will abusively sometimes write \(P\) for \(\hat{P}\), and also for its second adjoint on \(I(A)^{**}\). The latter is also completely positive and completely contractive. Then

\[
P(P(1)\hat{\pi}) \geq P(P(1)) = P(1) \geq P(P(1))\hat{\pi}.
\]

Hence these quantities are equal. In the limit, \(P(s(P(1))) = P(1)\), if \(s(P(1))\) is the support projection of \(P(1)\). Hence \(P(z) = 0\) where \(z = s(P(1)) - P(1)\). If
y ∈ I(A)_+ with ∥y∥ ≤ 1, then P(y) ≤ P(1) ≤ s(P(1)), and so s(P(1))P(y) = P(y) = P(y)s(P(1)). It follows that s(P(1))x = xs(P(1)) = x for all x ∈ Ran( ˆ{P}). If also ∥x∥ ≤ 1, then

\[ P(P(1)x) = P(s(P(1))x) = P(zx) = P(s(P(1)))x = P(x) \]

by the Kadison-Schwarz inequality, since

\[ P(zx)P(zx)^* ≤ P(zxx^*z) ≤ P(z^2) \leq P(z) = 0. \]

Thus P(P(1)x) = x if x ∈ P(A). Similarly, P(xP(1)) = x as desired. Thus P(xy) defines a bilinear completely contractive product on Ran(P) with ‘unit’ P(1).

The main thrust of [21] is the investigation of the completely contractive projections and conditional expectations, and in particular the ‘symmetric projection problem’ and the ‘bicontractive projection problem’, in the category of operator algebras, attempting to find operator algebra generalizations of certain deep results of Størmer, Friedman and Russo, Effros and Størmer, Robertson and Youngson, and others (see papers of these authors referenced in the bibliography below), concerning projections and their ranges, assuming in addition that our projections are real completely positive. We say that an idempotent linear P : X → X is completely symmetric (resp. completely bicontractive) if I − 2P is completely contractive (resp. if P and I − P are completely contractive). ‘Completely symmetric’ implies ‘completely bicontractive’. The two problems mentioned at the start of this paragraph concern 1) Characterizing such projections P; or 2) characterizing the range of such projections. On a unital C*-algebra B the work of some of the authors mentioned at the start of this paragraph establish that unital positive bicontractive projections are also symmetric, and are precisely \(1_2(I + \theta)\), for a period 2 *-automorphism \(\theta : B \to B\). The possibly nonunital positive bicontractive projections P are of a similar form, and then q = P(1) is a central projection in M(B) with respect to which P decomposes into a direct sum of 0 and a projection of the above form \(1_2(I + \theta)\), for a period 2 *-automorphism \(\theta\) of qB. Conversely, a map P of the latter form is automatically completely bicontractive, and the range of P, which is the set of fixed points of \(\theta\), is a C*-subalgebra, and P is a conditional expectation.

One may ask what from the last paragraph is true for general (approximately unital) operator algebras A? The first thing to note is that now ‘completely bicontractive’ is no longer the same as ‘completely symmetric’. The following is our solution to the symmetric projection problem, and it uses Kadison’s Banach–Stone theorem for C*-algebras [50], and our variant of the latter for approximately unital operator algebras (see e.g. [16] Theorem 4.5.13):

**Theorem 3.7.** [21] Let A be an approximately unital operator algebra, and P : A → A a completely symmetric real completely positive projection. Then the range of P is an approximately unital subalgebra of A. Moreover, \(P^{**}(1) = q\) is a projection in the multiplier algebra M(A) (so is both open and closed).

Set D = qAq, a hereditary subalgebra of A containing P(A). There exists a period 2 surjective completely isometric homomorphism \(\theta : A → A\) such that \(\theta(q) = q\), so that \(\theta\) restricts to a period 2 surjective completely isometric homomorphism D → D. Also, P is the zero map on \(q^-A + Aq^+ + q^+Aq^-\), and

\[ P = \frac{1}{2}(I + \theta) \quad \text{on } D. \]
In fact
\[ P(a) = \frac{1}{2} (a + \theta(a)(2q - 1)), \quad a \in A. \]

The range of \( P \) is the set of fixed points of \( \theta \).

Conversely, any map of the form in the last equation is a completely symmetric real completely positive projection.

**Remark.** In the case that \( A \) is unital but \( q \) is not central in the last theorem, if one solves the last equation for \( \theta \), and then examines what it means that \( \theta \) is a homomorphism, one obtains some interesting algebraic formulae involving \( q, q^\perp, A \) and \( \theta_{|qAq} \).

For the more general class of completely bicontractive projections, a first look is disappointing—most of the last paragraph no longer works in general. One does not always get an associated completely isometric automorphism \( \theta \) such that \( P = \frac{1}{2}(I + \theta) \), and \( q = P(1) \) need not be a central projection. However, as also seems to be sometimes the case when attempting to generalize a given \( C^* \)-algebra fact to more general algebras, a closer look at the result, and at examples, does uncover an interesting question. Namely, given an approximately unital operator algebra \( A \) and a real completely positive projection \( P : A \to A \) which is completely bicontractive, when is the range of \( P \) a subalgebra of \( A \) and \( P \) a conditional expectation? This seems to be the right version of the ‘bicontractive projection problem’ in the operator algebra category. We give in [21] a sequence of three reductions that reduce the question. The first reduction is that by passing to the bidual we may assume that the algebra \( A \) is unital. The second reduction is that by cutting down to \( qAq \), where \( q = P(1) \) (which one can show is a projection), we may further assume that \( P(1) = 1 \) (one can show \( P \) is zero on \( q^\perp A + Aq^\perp \)). The third reduction is by restricting attention to the closed algebra generated by \( P \), we may further assume that \( P(A) \) generates \( A \) as an operator algebra. We call this the ‘standard position’ for the bicontractive projection problem. It turns out that when in standard position, \( \text{Ker}(P) \) is forced to be an ideal with square zero.

In the second reduction above, that is if \( A \) and \( P \) are unital, then one may show that \( A \) decomposes as \( A = C \oplus B \), where \( 1_A \in B = P(A), C = (I - P)(A) \), and we have the relations \( C^2 \subset B, CB + BC \subset C \) (see [21] Lemma 4.1 and its proof). The period 2 map \( \theta : x + y \mapsto x - y \) for \( x \in B, y \in C \) is a homomorphism (indeed an automorphism) on \( A \) iff \( P(A) \) is a subalgebra of \( A \), and we have, similarly to Theorem 3.7

**Corollary 3.8.** If \( P : A \to A \) is a unital idempotent on a unital operator algebra then \( P \) is completely bicontractive iff there is a period 2 linear surjection \( \theta : A \to A \) such that \(||I \pm \theta||_{cb} \leq 2 \) and \( P = \frac{1}{2}(I + \theta) \). The range of \( P \) is a subalgebra iff \( \theta \) is also a homomorphism, and then the range of \( P \) is the set of fixed points of this automorphism \( \theta \). Also, \( P \) is completely symmetric iff \( \theta \) is completely contractive.

We remark that for the subcategory of uniform algebras (that is, closed unital (or approximately unital) subalgebras of \( C(K) \), for compact \( K \)), there is a complete solution to the bicontractive projection problem.

**Theorem 3.9.** Let \( P : A \to A \) be a real positive bicontractive projection on a (unital or approximately unital) uniform algebra. Then \( P \) is symmetric, and so
of course by Theorem 3.7 we have that \( P(A) \) is a subalgebra of \( A \), and \( P \) is a conditional expectation.

**Proof.** We sketch the idea, found in a conversation with Joel Feinstein. By the first two reductions described above we can assume that \( A \) and \( P \) are unital. We also know that \( B = P(A) \) is a subalgebra, since if it were not then the third reduction described above would yield nonzero nilpotents, which cannot exist in a function algebra. Thus by the discussion above the theorem, the map \( \theta(x + y) = x - y \) there is an algebra automorphism of \( A \), hence an isometric isomorphism (since norm equals spectral radius). So \( P = \frac{1}{2}(I + \theta) \) is symmetric. \( \square \)

The same three step reduction shows that we can also solve the problem in the affirmative for real completely positive completely bicontractive projections \( P \) on a unital operator algebra \( A \) such that the closed algebra generated by \( A \) is semiprime (that is, it has no nontrivial square-zero ideals). We have found counterexamples to the general question, but we have also found conditions that make all known (at this point) counterexamples go away. See [21] for details.

4. More notation, and existence of ‘positive’ approximate identities

We have already defined the cone \( r_A \) of accretive or ‘real positive’ elements, and its dense subcone \( R^+_A \). Another subcone which is occasionally of interest is the cone consisting of elements of \( A \) which are ‘sectorial’ of angle \( \theta < \frac{\pi}{2} \). For the purposes of this paper being sectorial of angle \( \theta \) will mean that the numerical range in \( A \) (or in a unitization of \( A \) if \( A \) is nonunital) is contained in the sector \( S_\theta \) consisting of complex numbers \( re^{i\rho} \) with \( r \geq 0 \) and \( |\rho| \leq \theta \). This third cone is a dense subset of the second cone \( R^+_A \) if \( A \) is an operator algebra [26, Lemma 2.15]. We remark that there exists a well established functional calculus for sectorial operators (see e.g. [43]). Indeed the advantages of this cone and the last one seems to be mainly that these have better functional calculi. For the cone \( R^+_A \), if \( A \) is an operator algebra, one could use the functional calculus coming from von Neumann’s inequality. Indeed if \( \|I - x\| \leq 1 \) then \( f \mapsto f(I - x) \) is a contractive homomorphism on the disk algebra. If \( x \) is real positive in an operator algebra, one could also use Crouzeix’s remarkable functional calculus on the numerical range of \( x \) (see e.g. [31]). If \( x \) is sectorial in a Banach algebra, one may use the functional calculus for sectorial operators [43].

A final notion of positivity which we introduced in the work with Read, which is slightly more esoteric, but which is a close approximation to the usual \( C^* \)-algebraic notion of positivity: In the theorems below we will sometimes say that an element \( x \) is *nearly positive;* this means that in the statement of that result, given \( \epsilon > 0 \) one can also choose the element in that statement to be real positive and within \( \epsilon \) of its real part (which is positive in the usual sense). In fact whenever we say ‘\( x \) is nearly positive’ below, we are in fact able, for any given \( \epsilon > 0 \), to choose \( x \) to also be a contraction with numerical range within a thin ‘cigar’ centered on the line segment \([0, 1]\) of height \( < \epsilon \). That is, \( x \) has sectorial angle \( < \arcsin \epsilon \). In an operator algebra any contraction \( x \) with such a sectorial angle is accretive and satisfies \( \|x - \text{Re} x\| \leq \epsilon \), so \( x \) is within \( \epsilon \) of an operator which is positive in the usual sense. Indeed if \( a \) is an accretive element in an operator algebra then (principal) \( n \)-th roots of \( a \) have spectrum and numerical radius within a sector \( S_{\frac{\pi}{n}} \), and hence are as close as we like (for \( n \) sufficiently large) to an operator which is positive.
in the usual sense (see Section 6). Thus one obtains ‘nearly positive elements’ by taking \( n \)-th roots of accretive elements. A nearly positive approximate identity \( e_t \) means that it is real positive and the sectorial angle of \( e_t \) converges to 0 with \( t \). (We remark that at the time of writing we do not know for general Banach algebras if roots (or \( r \)th powers for \( 0 < r < 1 \)) of accretive elements are in \( \mathbb{R}^+ \) or in the third cone in the last paragraph, or if that third cone is contained in the second cone.)

In the last paragraphs we have described several variants of ‘positivity’, which at least in an operator algebra are each successively stronger than the last. It is convenient to mentally picture each of these notions by sketching the region containing the numerical range of \( x \). Thus for the first notion, the accretive elements, one simply pictures the right half plane in \( \mathbb{C} \). One pictures the second, the cone \( \mathbb{R}^+ \mathcal{A}_a \), as a dense cone in the right half plane composed of closed disks center \( a \) and radius \( a \), for all \( a > 0 \). The third cone is pictured as increasing sectors \( S_\theta \) in \( \mathbb{C} \), for increasing \( \theta < \frac{\pi}{2} \). And the ‘nearly positive’ elements are pictured by the thin ‘cigar’ mentioned a paragraph or so back, centered on the line segment \([0, 1]\) of height \( < \epsilon \), and contained in the closed disk center \( \frac{1}{2} \) of radius \( \frac{1}{2} \).

We now list some more of our notation and general facts: We write \( \text{Ball}(X) \) for the set \( \{ x \in X : \| x \| \leq 1 \} \). For us Banach algebras satisfy \( \| xy \| \leq \| x \| \| y \| \). If \( x \in A \) for a Banach algebra \( A \), then \( ba(x) \) denotes the closed subalgebra generated by \( x \). If \( A \) is a Banach algebra which is not Arens regular, then the multiplication we usually use on \( A^{**} \) is the ‘second Arens product’ (\( \circ \) in the notation of [32]). This is weak* continuous in the second variable. If \( A \) is a nonunital, not necessarily Arens regular, Banach algebra with a bounded approximate identity (bai), then \( A^{**} \) has a so-called ‘mixed identity’ [32, 67, 34], which we will again write as \( e \). This is a right identity for the first Arens product, and a left identity for the second Arens product. A mixed identity need not be unique, indeed mixed identities are just the weak* limit points of bai’s for \( A \).

See the book of Doran and Wichmann [34] for a compendium of results about approximate identities and related topics. If \( A \) is an approximately unital Banach algebra, then the left regular representation embeds \( A \) isometrically in \( B(A) \). We will always write \( A^1 \) for the multiplier unitization of \( A \), that is, we identify \( A^1 \) isometrically with \( A + \mathbb{C} I \) in \( B(A) \). Below I will almost always denote the identity of \( A^1 \), if \( A \) is not already unital. If \( A \) is a nonunital, approximately unital Banach algebra then the multiplier unitization \( A^1 \) may also be identified isometrically with the subalgebra \( A + \mathbb{C} e \) of \( A^{**} \) for a fixed mixed identity \( e \) of norm 1 for \( A^{**} \).

We recall that a subspace \( E \) of a Banach space \( X \) is an \( M \)-ideal in \( X \) if \( E^\perp \) is complemented in \( X^{**} \) via a contractive projection \( P \) so that \( X^{**} = E^\perp \oplus \mathbb{C} \ker(P) \). In this case there is a unique contractive projection onto \( E^\perp \). This concept was invented by Alfsen and Effros, and [44] is the basic text for their beautiful and powerful theory. By an \( M \)-approximately unital Banach algebra we mean a Banach algebra which is an \( M \)-ideal in its multiplier unitization \( A^1 \). This is equivalent (see [22] Lemma 2.4) to saying that \( \| 1 - x \|_{A^{**}} = \| e - x \|_{A^{**}} \) for all \( x \in A^{**} \), unless the last quantity is \( < 1 \) in which case \( \| 1 - x \|_{A^{**}} = 1 \). Here \( e \) is the identity for \( A^{**} \) if it has one, otherwise it is a mixed identity of norm 1. A result of Effros and Ruan implies that approximately unital operator algebras are \( M \)-approximately unital (see e.g. [16] Theorem 4.8.5 (1)). Also, all unital Banach algebras are \( M \)-approximately unital.
We use states a lot in our work. However for an approximately unital Banach algebra $A$ with cai $(e_t)$, the definition of ‘state’ is problematic. Although we have not noticed this discussed in the literature, there are several natural notions, and which is best seems to depend on the situation. For example: (i) a contractive functional $\varphi$ on $A$ with $\varphi(e_t) \to 1$ for some fixed cai $(e_t)$ for $A$, (ii) a contractive functional $\varphi$ on $A$ with $\varphi(e_t) \to 1$ for all cai $(e_t)$ for $A$, and (iii) a norm 1 functional on $A$ that extends to a state on $A^1$, where $A^1$ is the ‘multiplier unitization’ above. If $A$ satisfies a smoothness hypothesis then all these notions coincide [22], but this is not true in general. The $M$-approximately unital Banach algebras in the last paragraph are smooth in this sense. Also, if $e$ is a mixed identity for $A^{**}$ then the statement $\varphi(e) = 1$ may depend on which mixed identity one considers. In this paper though for simplicity, and because of its connections with the usual theory of numerical range and accretive operators, we will take (iii) above as the definition of a state of $A$. In [22] we also consider some of the other variants above, and these will appear below from time to time. We define the state space $S(A)$ to be the set of states in the sense of (iii) above. The quasistate space $Q(A)$ is $\{t \varphi : t \in [0, 1], \varphi \in S(A)\}$. The numerical range of $x \in A$ is $W_A(x) = \{\varphi(x) : \varphi \in S(A)\}$. As in [22] we define $\tau_{A^{**}} = A^{**} \cap \tau_{(A^1)^{**}}$. There is an unfortunate ambiguity with the latter notation here and in [22] in the (generally rare) case that $A^{**}$ is unital. It should be stressed that in these papers $\tau_{A^{**}}$ should not, if $A^{**}$ is unital, be confused with the real positive (i.e. accretive) elements in $A^{**}$. It is shown in [22], Section 2] that these are the same if $A$ is an $M$-approximately unital Banach algebra, and in particular if $A$ is an approximately unital operator algebra. It is easy to see that $A^{**} \cap \tau_{(A^1)^{**}}$ is contained in the accretive elements in $A^{**}$ if $A^{**}$ is unital, but the other direction seems unclear in general.

Of course in the theory of $C^*$-algebras, positivity and the existence of positive approximate identities are crucial. How does one get a ‘positive cai’ in an algebra with cai? We have several ways to do this. First, for approximately unital operator algebras and for a large class of approximately unital Banach algebras (eg the scaled Banach algebras defined in the next section; and we do not possess an example of a Banach algebra that is not scaled yet) we have a ‘Kaplansky density’ result: $\text{Ball}(A) \cap \tau_{A^{**}} = \text{Ball}(A^{**}) \cap \tau_{A^{**}}$. See Theorem 5.8 below. (We remark that although it seems not to be well known, the most common variants of the usual Kaplansky density theorem for a $C^*$-algebra $A$ do follow quickly from the weak* density of $\text{Ball}(A)$ in $\text{Ball}(A^{**})$, if one constructs $A^{**}$ carefully.) If $A^{**}$ has a real positive mixed identity $e$ of norm 1, then one can then get a real positive cai by approximating $e$ by elements of $\text{Ball}(A) \cap \tau_{A^1}$, if one uses the fact that in an $M$-approximately unital Banach algebra $\|1 - 2e\| \leq 1$ for a mixed identity of norm 1 for $A^{**}$):

**Theorem 4.1.** [22], [23], [72]. Let $A$ be an $M$-approximately unital Banach algebra, for example any operator algebra. Then $\mathfrak{F}_A$ is weak* dense in $\mathfrak{F}_{A^{**}}$. Hence $A$ has a cai in $\frac{1}{2}\mathfrak{F}_A$.

Applied to approximately unital operator algebras (which as we said are all $M$-approximately unital) the last assertion of Theorem 4.1 becomes Read’s theorem from [72]. See also [12], [26] for other proofs of the latter result.
Remark 4.2. For the conclusion that $\mathcal{F}_A$ is weak* dense in $\mathcal{F}_A^{**}$ one may relax the $M$-approximately unital hypothesis to the following much milder condition: $A$ is approximately unital and given $\epsilon > 0$ there exists a $\delta > 0$ such that if $y \in A$ with $\|1 - y\| < 1 + \delta$ then there is a $z \in A$ with $\|1 - z\| = 1$ and $\|y - z\| < \epsilon$. Here 1 denotes the identity of unitization of $A$. This follows from the proof of [22, Theorem 5.2]. For example, $L^1(\mathbb{R})$ satisfies this condition with $\delta = \epsilon$.

Another approach to finding a ‘real positive cai’ under a countability condition from [22, Section 2] uses a slight variant of the ‘real positive’ definition. Namely for a fixed cai $\epsilon = (\epsilon_t)$ for $A$ define $S_\epsilon(A) = \{ \varphi \in \text{Ball}(A^*) : \lim_t \varphi(\epsilon_t) = 1 \}$ (a subset of $S(A)$). Define $\tau^*_A = \{ x \in A : \text{Re} \varphi(x) \geq 0 \text{ for all } \varphi \in S_\epsilon(A) \}$. If we multiply these states by numbers in $[0,1]$, we get the associated quasistate space $Q_\epsilon(A)$. Note that $\tau^*_A$ contains $\tau_A^*$. On the other hand, [22, Theorem 6.5] (or a minor variant of the proof of it) shows that if $A^{**}$ is unital then $\tau^*_A$ is never contained in $\tau_{A^{**}}$ (or in the accretive elements in $A^{**}$) unless $\tau^*_A = \tau_A^*$.

**Theorem 4.3.** [22] A Banach algebra $A$ with a sequential cai $\epsilon$ and with $Q_\epsilon(A)$ weak* closed, has a sequential cai in $\tau^*_A$.

**Proof.** We give the main idea of the proof in [22], and a few more details for the first step. Suppose that $K$ is a compact space and $(f_n)$ is a bounded sequence in $C(K, \mathbb{R})$, such that $\lim_n f_n(x)$ exists for every $x \in K$ and is non-negative. Claim: for every $\epsilon > 0$, there is a function $f \in \text{conv}\{f_n\}$ such that $f \geq -\epsilon$ on $K$. Indeed if this were not true, then there exists an $\epsilon > 0$ such that for all $f \in \text{conv}\{f_n\}$ there is a point $x$ in $K$ with $f(x) < -\epsilon$. Moreover, for all $g \in \text{conv}\{f_n\}$, if $f \in \text{conv}\{f_n\}$ with $\|f - g\| < \frac{\epsilon}{2}$, there is a point $x$ in $K$ with $g(x) < -\frac{\epsilon}{2}$. So $A = \text{conv}\{f_n\}$ and $C = C(K)_+$ are clearly disjoint. Moreover, it is well known that convex sets $E,C$ in an LCTVS can be strictly separated iff $0 \notin \overline{E-C}$, and this is clearly the case for us here. So there is a continuous functional $\psi$ on $C(K, \mathbb{R})$ and scalars $M,N$ with $\psi(g) \leq M < N \leq \psi(h)$ for all $g \in A$ and $h \in C$. Since $C$ is a cone we may take $N = 0$. By the Riesz–Markov theorem there is a Borel probability measure $m$ such that $\sup_n \int_K f_n \, dm < 0$. This is a contradiction and proves the Claim, since $\lim_n \int_K f_n \, dm \geq 0$ by Lebesgue’s dominated convergence theorem.

Now let $K = Q_\epsilon(A)$ and $f_n(\psi) = \text{Re} \psi(\epsilon_n)$ where $\epsilon = (\epsilon_n)$. We have $\lim_n f_n \geq 0$ pointwise on $K$, so by the last paragraph for any $\epsilon > 0$ a convex combination of the $f_n$ is always $\geq -\epsilon$ on $K$. By a standard geometric series type argument we can replace $\epsilon$ with 0 here, so that we have a real positive element, and with more care this convex combination may be taken to be a generic element in a cai.

Finally, we state a ‘new’ result, which will be proved in Corollary 5.10 below (this result was referred to incorrectly in the published version of [22] as ‘Corollary 3.4’ of the present paper).

**Corollary 4.4.** If $A$ is an approximately unital Banach algebra with a cai $\epsilon$ such that $S(A) = S_\epsilon(A)$, and such that the quasistate space $Q(A)$ is weak* closed, then $A$ has a cai in $\tau_A^*$.

We remark that we have no example of an approximately unital Banach algebra where $Q(A)$ is not weak* closed. In particular, we have found that commonly encountered algebras have this property.
5. Order theory in the unit ball

In the spirit of the quotation starting Section 2 we now discuss generalizations of well known order-theoretic properties of the unit ball of a $C^*$-algebra and its dual. Some of these results also may be viewed as new relations between an operator algebra and a $C^*$-algebra that it generates. There are interesting connections to the classical theory of ordered linear spaces (due to Krein, Ando, Alfsen, etc) as found e.g. in the first chapters of [6]. In addition to striking parallels, some of this classical theory can be applied directly. Indeed several results from [22] (some of which are mentioned below, see e.g. the proof of Theorem 5.4) are proved by appealing to results in that theory. See also [26] for more connections if the algebras are in addition operator algebras.

The ordering induced by $τ_A$ is obviously $b ≼ a$ iff $a − b$ is accretive (i.e. numerical range in right half plane). If $A$ is an operator algebra this happens when $\text{Re}(a − b) ≥ 0$.

**Theorem 5.1.** [26] If an approximately unital operator algebra $A$ generates a $C^*$-algebra $B$, then $A$ is order cofinal in $B$. That is, given $b ∈ B_+$ there exists $a ∈ A$ with $b ≼ a$. Indeed one can do this with $b ≼ a ≼ ∥b∥ + ε$. Indeed one can do this with $b ≼ Ce_ε ≼ ∥b∥ + ε$, for a nearly positive cai $(e_ε)$ for $A$ and a constant $C > 0$.

This and the next result are trivial if $A$ unital.

**Theorem 5.2.** [26] Let $A$ be an operator algebra which generates a $C^*$-algebra $B$, and let $\mathcal{U}_A = \{a ∈ A : ∥a∥ < 1\}$. The following are equivalent:

1. $A$ is approximately unital.
2. For any positive $b ∈ \mathcal{U}_B$ there exists $a ∈ \mathcal{C}_A$ with $b ≼ a$.
3. Same as (2), but also $a ∈ \frac{1}{2}\mathfrak{N}_A$ and nearly positive.
4. For any pair $x, y ∈ \mathcal{U}_A$ there exist nearly positive $a ∈ \frac{1}{2}\mathfrak{N}_A$ with $x ≼ a$ and $y ≼ a$.
5. For any $b ∈ \mathcal{U}_A$ there exist nearly positive $a ∈ \frac{1}{2}\mathfrak{N}_A$ with $−a ≼ b ≼ a$.
6. $\mathcal{C}_A$ is a generating cone (that is, $A = \mathcal{C}_A - \mathcal{C}_A$).

In any operator algebra $A$ it is true that $\mathcal{C}_A − \mathcal{C}_A$ is a closed subalgebra of $A$. It is the biggest approximately unital subalgebra of $A$, and it happens to also be a HSA in $A$ [24]. We do not know if this is true for Banach algebras.

For ‘nice’ Banach algebras $A$ the cone $\mathcal{C}_A$ has some of the pleasant order properties in items (3)–(6) in Theorem 5.2. See also [22] Section 6] for various variants on this theme. The following is a particularly clean case:

**Theorem 5.3.** [22] Section 6] Let $A$ be an $M$-approximately unital Banach algebra. Then

1. For any pair $x, y ∈ \mathcal{U}_A$ there exist $a ∈ \frac{1}{2}\mathfrak{N}_A$ with $x ≼ a$ and $y ≼ a$.
2. For any $b ∈ \mathcal{U}_A$ there exist $a ∈ \frac{1}{2}\mathfrak{N}_A$ with $−a ≼ b ≼ a$.
3. For any $b ∈ \mathcal{U}_A$ there exist $x, y ∈ \mathcal{U}_A ∩ \frac{1}{2}\mathfrak{N}_A$ with $b = x − y$.
4. $\mathcal{C}_A$ is a generating cone (that is, $A = \mathcal{C}_A - \mathcal{C}_A$).

During the writing of the present paper we saw the following improvement of part of Corollaries 6.7 and 6.8, and on some of 6.10 in the submitted version of the
paper [22]. At the galleys stage of that paper we incorporated those advances, but unfortunately slipped up in one proof. The correct version is as below.

**Theorem 5.4.** [22] Section 6] If a Banach algebra $A$ has a cai $\epsilon$ and satisfies that $Q_\epsilon(A)$ is weak* closed, then (1)–(4) in the last theorem hold, with $\frac{1}{2}\mathcal{S}_A$ replaced by $\text{Ball}(A) \cap \mathfrak{r}_A^\epsilon$, and $\preceq$ replaced by the linear ordering defined by the cone $\mathfrak{r}_A^\epsilon$, and $\mathcal{C}_A$ replaced by $\mathfrak{r}_A^\epsilon$. One may drop the three superscript $\epsilon$’s in the last line if in addition $S(A) = S_\epsilon(A)$.

**Proof.** Lemma 2.7 (1) in [22] implies that if $Q_\epsilon(A)$ is weak* closed, then the ‘dual cone’ in $A^*$ of $\mathfrak{r}_A^\epsilon$ is $\mathbb{R}^+ S_\epsilon(A)$. By the remark before [22] Proposition 6.2] a similar fact holds for the real dual cone. Since $\|\varphi\| = 1$ for states and for their real parts, the norm on the real dual cone is additive. This is known to imply, by the theory of ordered linear spaces [6, Corollary 3.6, Chapter 2], that the open ball of $A$ is a directed set. So for any pair $x, y \in \mathcal{U}_A$ there exist $z \in \mathcal{U}_A$ with $x \preceq_\epsilon z$ and $y \preceq_\epsilon z$. Applying this again to $z, -z$ there exists $w \in \mathcal{U}_A$ with $\pm z \preceq_\epsilon w$. This implies that $\frac{w+z}{2} \in \mathfrak{r}_A^\epsilon$, and $z \preceq_\epsilon a$ where $a = \frac{w+z}{2}$. This proves (1). Applying (1) to $b, -b$ we get (2). Setting $x = \frac{a+b}{2}, y = \frac{a-b}{2}$ for $a, b$ as in (2), we get (3) and hence (4). The final assertion is then obvious since if $S(A) = S_\epsilon(A)$ then $\mathfrak{r}_A = \mathfrak{r}_A^\epsilon$ and $\preceq_\epsilon$ is just $\preceq$.

Recall that the positive part of the open unit ball $\mathcal{U}_B$ of a $C^*$-algebra $B$ is a directed set, and indeed is a net which is a positive cai for $B$. The first part of this statement is generalized by Theorems 5.2 (3), 5.3 (1), and 5.4 (1). The following generalizes the second part of the statement to operator algebras:

**Corollary 5.5.** [26] If $A$ is an approximately unital operator algebra, then $\mathcal{U}_A \cap \frac{1}{2}\mathcal{S}_A$ is a directed set in the $\preceq$ ordering, and with this ordering $\mathcal{U}_A \cap \frac{1}{2}\mathcal{S}_A$ is an increasing cai for $A$.

We do not know if the second part of the last result is true for any other classes of Banach algebras.

We say a Banach algebra $A$ is scaled if every real positive linear map into the scalars is a nonnegative multiple of a state. Of course it is well known that $C^*$-algebras are scaled. Somewhat surprisingly, we do not know of an approximately unital Banach algebra that is not scaled, and certainly all commonly encountered Banach algebras seem to be scaled. Unital Banach algebras are scaled by e.g. an argument in the proof of [63, Theorem 2.2],

**Theorem 5.6.** [26, 22] If $A$ is an approximately unital operator algebra, or more generally an $M$-approximately unital Banach algebra, then $A$ is scaled.

For operator algebras, the last result implies Read’s theorem mentioned earlier.

**Proposition 5.7.** [22] If $A$ is a nonunital approximately unital Banach algebra, then the following are equivalent:

(i) $A$ is scaled.

(ii) $S(A^1)$ is the convex hull of the trivial character $\chi_0$ and the set of states on $A^1$ extending states of $A$.

(iii) The quasistate space $Q(A) = \{ \varphi|_A : \varphi \in S(A^1) \}$.

(iv) $Q(A)$ is convex and weak* compact.
If these hold then \( Q(A) = \overline{S(A)}^{w^*} \), and the numerical range satisfies
\[
\overline{W_A(a)} = \text{conv}\{0, W_A(a)\} = W_{A^*}(a), \quad a \in A.
\]

**Theorem 5.8** (Kaplansky density type result). If \( A \) is a scaled approximately unital Banach algebra then \( \text{Ball}(A) \cap \tau_A \) is weak* dense in the unit ball of \( \tau_A^{**} \).

The last result is from [22], although some operator algebra variant was done earlier with Read.

**Corollary 5.9.** If \( A \) is a scaled approximately unital Banach algebra then \( A \) has a cai in \( \tau_A \) iff \( A^{**} \) has a mixed identity \( e \) of norm 1 in \( \tau_{A^{**}} \), or equivalently with \( \|1_A - e\| \leq 1 \).

**Proof.** This is proved in [22] Proposition 6.4, relying on earlier results there, except for parts of the last assertion. For the remaining part, if \( A \) has a cai in \( \tau_A \) then a cluster point of this cai is a mixed identity of norm 1, and it is in \( \tau_{A^{**}} \) since the latter is weak* closed and contains \( \tau_A \). However by a result from [22] (see Lemma 6.18 below), an idempotent is in \( \tau_{(A^1)^{**}} \) iff it is in \( \tau_{(A^1)^{**}} \).

**Corollary 5.10.** If \( A \) is a scaled approximately unital Banach algebra with a cai \( e \) such that \( S(A) = S_e(A) \) then \( A \) has a cai in \( \tau_A \).

**Proof.** Let \( e \) be any weak* limit point of \( e \). Clearly \( \varphi(e) = 1 \) for all \( \varphi \in S_e(A) = S(A) \). If \( \varphi \in S((A^1)^{**}) \) then its restriction to \( A^1 \) is in \( S(A^1) \), hence \( \varphi(e) \geq 0 \) by the last line and Proposition 5.7. So \( e \in A^{**} \cap \tau_{(A^1)^{**}} = \tau_{A^{**}} \), and so the result follows from our Kaplansky density type theorem in the form of its Corollary 5.9.

The class of algebras \( A \) in the last Corollary is the same as the class in the last line of the statement of Theorem 5.3. Thus for such algebras, (1)-(4) in Theorem 5.3 hold, with \( \frac{1}{2} \mathfrak{S}_A \) replaced by \( \text{Ball}(A) \cap \tau_A \), and \( \mathfrak{C}_A \) replaced by \( \tau_A \). In particular, \( \tau_A \) spans \( A \).

### 6. Positivity and roots in Banach algebras

As we said in the Introduction, this section and the next have several purposes:
We will describe results from our other papers (particularly [22], which generalizes some parts of the earlier work) connected to the work of Kadison summarized in Section 2, but we will also restate the results from several sections of [22] in the more general setting of Banach algebras with no kind of approximate identity. Also we will give a detailed discussion of roots (fractional powers) in relation to our positivity (see also [23, 24, 9, 26] for results not covered here).

Thus let \( A \) be a Banach algebra without a cai, or without any kind of bai. If \( B \) is any unital Banach algebra isometrically containing \( A \) as a subalgebra, for example any unitization of \( A \), we define
\[
\mathfrak{S}_A^B = \{ a \in A : \|1_B - a\| \leq 1 \},
\]
and write \( \tau_A^B \) for the set of \( a \in A \) whose numerical range in \( B \) is contained in the right half plane. These sets are closed and convex. Also we define
\[
\mathfrak{S}_A = \bigcup_B \mathfrak{S}_A^B, \quad \tau_A = \bigcup_B \tau_A^B,
\]
the unions taken over all unital Banach algebras \( B \) containing \( A \). Unfortunately it is not clear to us that \( \mathfrak{S}_A \) and \( \tau_A \) are always convex, which is needed in most of...
we recall that the principal spectrum is contained in a sector \( S_B \) of a unital Banach algebra \( B \).

For \( \| \cdot \| \) see that the series given for \( x \) is a root of elements of \( F \) as same. As far as we know, Kelley and Vaught [58] define these that we are aware of. We will review these and show that they are the unique unitization by a theorem of Ralf Meyer (see [16] Section 2.1]). The following is another case when this happens.

Lemma 6.1. Let \( A \) be a nonunital Banach algebra.

1. Suppose that there exists a ‘smallest’ unitization norm on \( A \oplus \mathbb{C} \). That is, there exists a smallest norm on \( A \oplus \mathbb{C} \) making it a normed algebra with product \((a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda \mu)\), and satisfying \( \|(a,0)\| = \|a\|_A \) for \( a \in A \). Let \( B_0 \) be \( A \oplus \mathbb{C} \) with this smallest norm. Then \( \mathfrak{F}_{B_0} = \mathfrak{F}_A \) and \( \mathfrak{r}_{B_0} = \mathfrak{r}_A \).

2. Suppose that the left regular representation embeds \( A \) isometrically in \( B(A) \). (This is the case for example if \( A \) is approximately unital.) Define \( B_0 \) to be the span in \( B(A) \) of \( I_A \) and the isometrically embedded copy of \( A \). This has the smallest norm of any unitization of \( A \). Hence \( \mathfrak{F}_{B_0} = \mathfrak{F}_A \) and \( \mathfrak{r}_{B_0} = \mathfrak{r}_A \).

Proof. If \( B \) is any unital Banach algebra containing \( A \), and \( a \in \mathfrak{F}_{B_0} \) then \( a \in \mathfrak{F}_A \). So \( \mathfrak{F}_{B_0} = \mathfrak{F}_A \). A similar argument shows that \( \mathfrak{r}_{B_0} = \mathfrak{r}_A \), using Lemma 2.4.

We now discuss roots (that is, \( r \)'th powers for \( r \in [0,1] \)) in a subalgebra \( A \) of a unital Banach algebra \( B \). Actually, we only discuss the principal root (or power); we recall that the principal \( r \)th power, for \( 0 < r < 1 \), is the one whose spectrum is contained in a sector \( S_0 \) of angle \( \theta < 2r\pi \). There are several ways to define these that we are aware of. We will review these and show that they are the same. As far as we know, Kelley and Vaught [58] were the first to define the square root of elements of \( \mathfrak{F}_A \), but their argument works for \( r \)'th powers for \( r \in [0,1] \). If \( \|1 - x\| \leq 1 \), define

\[
x^r = \sum_{k=0}^{\infty} \binom{r}{k} (-1)^k (1 - x)^k, \quad r > 0.
\]

For \( k \geq 1 \) the sign of \( \binom{r}{k} (-1)^k \) is always negative, and \( \sum_{k=1}^{\infty} \binom{r}{k} (-1)^k = -1 \). Thus the series above converges absolutely, hence converges in \( A \). Indeed it is now easy to see that the series given for \( x^r \) is a norm limit of polynomials in \( x \) with no constant
term. Using the Cauchy product formula in Banach algebras in a standard way, one deduces that $(x^n)^r = x^n$ for any positive integer $n$.

**Proposition 6.3 (Esterle).** If $A$ is a Banach algebra then $\mathfrak{F}_A$ is closed under $r$th powers for any $r \in [0,1]$.

**Proof.** Let $x \in A \cap \mathfrak{F}_B$ where $B$ is a unital Banach algebra containing $A$. We have $1_B - x^r = \sum_{k=1}^{\infty} \left( \frac{1}{k!} \right) (-1)^k (1_B - x)^k$, which is a convex combination in $\text{Ball}(B)$. So $x^r \in A \cap \mathfrak{F}_B \subseteq \mathfrak{F}_A$. \hfill □

From [37] Proposition 2.4 if $x \in \mathfrak{F}_A$ then we also have $(x^t)^r = x^{tr}$ for $t \in [0,1]$ and any real $r$. One cannot use the usual Riesz functional calculus to define $x^r$ if 0 is in the spectrum of $x$, since such $r$th powers are badly behaved at 0. However if 0 is in the spectrum of $x$, and $x \in \mathfrak{r}^+_A$, one may define $x^r = \lim_{t \to 0+} (x + \epsilon 1_B)^r$ where the latter is the $r$th power according to the Riesz functional calculus. We will soon see that this limit exists and lies in $A$, and then it follows that it is independent of the particular unital algebra $B$ containing $A$ as a subalgebra (since all unitization norms for $A$ are equivalent). A second way to define $r$th powers for $r \in [0,1]$ in Banach algebras is found in [61], following the ideas in Hilbert space operator case from the Russian literature from the 50’s [62]. Namely, suppose that $B$ is a unital Banach algebra containing $A$ as a subalgebra, and $x \in A$ with numerical range in $B$ excluding all negative numbers. Since the numerical range is convex, it follows that this numerical range is in fact contained in a sector (i.e. a cone in the complex plane with vertex at 0) of angle $\leq \pi$. Since this is the case we are interested in, we will assume that the numerical range of $x$ is in the closed right half plane. (This is usually not really any loss of generality, since $x$ and hence the just mentioned cone can be ‘rotated’ to ensure this.) Thus the numerical range of $x$ is contained inside a semicircle, namely the one containing the right half of the circle center 0 radius $R > 0$. We enlarge this semicircle to a slightly larger ‘slice’ of this circle of radius $R_1$; thus let $\Gamma$ be the positively oriented contour which is symmetric about the $x$-axis, and is composed of an arc of the circle slightly bigger that the right half of the circle, and two line segments which connect zero with the arc. Let $\Gamma_x$ be $\Gamma$ but with points removed that are distance less than $\epsilon$ to the origin. One defines $x^t$ to be the limit as $\epsilon \to 0$ of $\frac{1}{2\pi i} \int_{\Gamma_x} \lambda^t (1_B - \lambda)^{-1} d\lambda$. The latter integral lies in $A + \mathbb{C} 1_B$, by the usual facts about such integrals. If $A$ is nonunital and $\chi_0$ is the character on $A + \mathbb{C} 1_B$ annihilating $A$ then $\chi_0(x^t)$ is the limit of $\frac{1}{2\pi i} \int_{\Gamma_x} \lambda^t (1 - \chi_0(\lambda))^{-1} d\lambda$, which is $\frac{1}{2\pi i} \int_{\Gamma} \lambda^t d\lambda = 0$. So $x^t \in A$. Note that $x^t$ is independent of the particular unitization $B$ used, using the fact that all unitization norms are equivalent. If in addition $x$ is invertible then $0 \notin \text{Sp}_B(x)$, so that we can replace $\Gamma$ by a curve that stays to one side of 0, so that $x^t$ is the $r$th power of $x$ as given by the Riesz functional calculus. In fact it is shown in [61] Proposition 3.1.9] that $x^r = \lim_{t \to 0+} (x + \epsilon 1_B)^r$ for $t > 0$, giving the equivalence with the definition at the start of this discussion. In addition we now see, as we discussed earlier, that the latter limit exists, lies in $A$, and is independent of $B$. By [61] Corollary 1.3], the $r$th power function is continuous on $\mathfrak{r}^+_A$, for any $r \in (0,1)$. Principal $n$th roots of accretive elements are unique, for any positive integer $n$ (see [61]).

A final way to define $r$th powers $x^r$ for $r \in [0,1]$ and $x \in \mathfrak{r}_A$, is via the functional calculus for sectorial operators [43] (see also e.g. [79] IX, Section 11 for some of the origins of this approach). Namely, if $B$ is a unitization of $A$ (or a unital
Banach algebra containing \( A \) as a subalgebra) and \( x \in \mathfrak{r}^{B}_{A} \), view \( x \) as an operator on \( B \) by left multiplication. This is sectorial of angle \( \leq \frac{\pi}{2} \), and so we can use the theory of roots (fractional powers) from e.g. [43, Section 3.1] (see also [78]). Basic properties of such powers include: \( x^{*}x^{t} = x^{s+t} \) and \( (cx)^{t} = c^{t}x^{t} \), for positive scalars \( c, s, t, \) and \( t \to x^{t} \) is continuous. There are very many more in e.g. [43] Proposition 3.1.9] shows that \( x^{t} = \lim_{\epsilon \to 0^+} (x+\epsilon I)^{t} \) for \( r > 0 \), the latter power with respect to the usual Riesz functional calculus. It is easy to see from the last fact that the definitions of \( x^{r} \) in the last paragraph coincide if \( x \in \mathfrak{r}_{A} \) and \( r > 0 \); so that again \( x^{r} \) is in (the copy inside \( B(B) \) of) \( A \). Another formula we have occasionally found useful is \( x^{r} = \frac{\sin(\epsilon r)}{\epsilon} \int_{0}^{\infty} s^{r-1} (s+x)^{-1} x ds \), the Balakrishnan formula (see e.g. [43, 79]).

We now show that if \( x \in \mathfrak{F}_{A} \) then the definitions of \( x^{r} \) given in the last paragraphs and in Proposition 6.3 coincide, if \( r < 0 \). We may assume that \( 0 < r \leq 1 \) and work in a unital algebra \( B \) containing \( A \). Let \( y = \frac{1}{1+r} (x+\epsilon I) \). Then \( \|1_B - y\| < 1 \), and so \( y^{r} \) as defined in the last paragraphs equals \( \sum_{k=0}^{\infty} \binom{r}{k} (1) (1_B - y)^{k} \) since both are easily seen to equal the \( r \)th power of \( y \) as given by the Riesz functional calculus. However \( \sum_{k=0}^{\infty} \binom{r}{k} (1) (1_B - y)^{k} \) converges uniformly to \( \sum_{k=0}^{\infty} \binom{r}{k} (1 - x)^{k} \), as \( \epsilon \to 0^{+} \), since the norm of the difference of these two series is dominated by

\[
\sum_{k=1}^{\infty} \binom{r}{k} (-1)^{k} \frac{1}{1+\epsilon} \| (1 - x)^{k} \| \leq \frac{\epsilon}{1+\epsilon} \to 0,
\]

using the fact that for \( k \geq 1 \) the sign of \( \binom{r}{k} (-1)^{k} \) is always negative. Also, with the definition of powers in the last paragraphs we have \( y^{r} = (x+\epsilon I)^{r} \to x^{r} \) as \( \epsilon \to 0^{+} \). Thus the definitions of \( x^{r} \) given in the last paragraphs and in Proposition 6.3 coincide in this case.

If \( A \) is a subalgebra of a unital Banach algebra \( B \) then we define the \( \mathfrak{F} \)-transform on \( A \) to be \( \mathfrak{F}(x) = x(1_B + x)^{-1} = 1_B - (1_B + x)^{-1} \) for \( x \in \mathfrak{r}_{A} \). This is a relative of the well known Cayley transform in operator theory. Note that \( \mathfrak{F}(x) \in \mathfrak{r}(x) \) by the basic theory of Banach algebras, and it does not depend on \( B \), again because all unitization norms for \( A \) are equivalent. The inverse transform takes \( y \) to \( y(1_B - y)^{-1} \).

For operator algebras we have \( \| \mathfrak{F}(x) \| \leq \| x \| \) and \( \| \mathfrak{F}(x) \| \leq \| x \| \) for \( x \in \mathfrak{r}_{A} \). For Banach algebras this is not true; for example on the group algebra of \( \mathbb{Z}_{2} \).

Unless explicitly said to the contrary, the remaining results in this section are generalizations to general Banach algebras of results from [22]. The main results here in the operator algebra case were proved earlier by the author and Read (some are much sharper in that setting).

**Lemma 6.4.** If \( A \) is a subalgebra of a unital Banach algebra \( B \) then \( \mathfrak{F}(\mathfrak{r}^{B}_{A}) \subset \mathfrak{F}^{B}_{A} \) and \( \mathfrak{F}(\mathfrak{r}_{A}) \subset \mathfrak{F}_{A} \).

**Proof.** This is because by a result of Stampfel and Williams [76, Lemma 1],

\[
\|1_B - x(1_B + x)^{-1}\| = \|(1_B + x)^{-1}\| \leq d^{-1} \leq 1
\]

where \( d \) is the distance from \(-1\) to the numerical range in \( B \) of \( x \). \( \square \)

The following was stated in [22] without proof details.

**Proposition 6.5.** If \( A \) is a unital Banach algebra and \( x \in \mathfrak{r}_{A} \) and \( \epsilon > 0 \) then \( x + \epsilon I \in C^{\mathfrak{F}}_{A} \) where \( C = \epsilon + \frac{\|x\|}{\epsilon} \).
PROOF. We have
\[ \|1 - C^{-1}(x + \epsilon 1)\| = C^{-1} \|(C - \epsilon)1 - x\| = C^{-1} \left\| \frac{\|x\|^2}{\epsilon} \right\| 1 - \frac{\epsilon}{\|x\|^2} x. \]
By Lemma 2.4 (2), this is dominated by \( C^{-1} \frac{\|x\|^2}{\epsilon}(1 + \frac{\epsilon^2}{\|x\|^2}) = 1. \)

It follows easily from Proposition 6.5 that \( \mathbb{R}^+ \mathfrak{F}_A = \tau_A \) if \( A \) is unital. For nonunital algebras we use a different argument:

**Proposition 6.6.** If \( A \) is a subalgebra of a unital Banach algebra \( B \) then \( \mathbb{R}^+ \mathfrak{F}_A = \tau_A^B \) and \( \mathbb{R}^+ \mathfrak{F}_A = \tau_A. \)

**Proof.** If \( x \in \tau_A^B \) and \( t \geq 0 \), then \( tx(1_B + tx)^{-1} \in \mathfrak{F}_A^B \) by Lemma 6.4. By elementary Banach algebra theory, \((1_B + tx)^{-1} \to 1_B \) as \( t \to 0 \). So \( x = \lim_{t \to 0^+} \frac{1}{t} tx(1_B + tx)^{-1} \), from which the results are clear.

**Remark.** There is a numerical range lifting result that works in quotients of Banach spaces with ‘identity’ or of approximately unital Banach algebras, if one takes the quotient by an \( M \)-ideal (see [30] and the end of Section 8 in [22]). This may be viewed as a noncommutative Tietze theorem, as explained in the last paragraph of Section 8 in [22]. As a consequence one can lift a real positive element in such a quotient \( A/J \) to a real positive in \( A \). This again is a generalization of a well known \( C^* \)-algebraic positivity results since as pointed out by Alfsen and Effros (and Effros and Ruan), \( M \)-ideals in a \( C^* \)-algebra (or, for that matter, in an approximately unital operator algebra) are just the two-sided closed ideals (with a cai). See e.g. [16] Theorem 4.8.5.

**Lemma 6.7.** Let \( A \) be a Banach algebra. If \( x \in \tau_A \), then \( \|x^t\| \leq \frac{2\sin(t\pi)}{\pi(1-t)} \|x\|^t \) if \( 0 < t < 1 \). If \( A \) is an operator algebra one may remove the 2 in this estimate.

To prove this and the next corollary: by the above we may as well work in any unital Banach algebra containing \( A \), and this case was done in [22]. In the operator algebra case a recent paper of Drury [35] is a little more careful with the estimates for the integral in the Balakrishnan formula mentioned above for \( x^t \), and obtains
\[ \|x^t\| \leq \frac{\Gamma(t/2)}{2\sqrt{\pi}} \Gamma(t/(1-t)/2) \]
if \( 0 < t < 1 \) and \( \|x\| \leq 1 \). Drury states this for matrices \( x \), but the same proof works for operators on Hilbert space.

**Lemma 6.8.** There is a nonnegative sequence \( (c_n) \) in \( c_0 \) such that for any Banach algebra \( A \), and \( x \in \mathfrak{F}_A \) or \( x \in \text{Ball}(A) \cap \tau_A \), we have \( \|x^\frac{1}{n} x - x\| \leq c_n \) for all \( n \in \mathbb{N} \).

**Remark 6.9.** If \( A \) is a Banach algebra and \( x \in \mathfrak{F}_A \) or \( x \in \text{Ball}(A) \cap \tau_A \) is nonzero then \( \limsup_n \|x^\frac{1}{n}\| \leq 1 \) is the same as saying \( \lim_n \|x^\frac{1}{n}\| = 1 \). For \( \|x\| \leq \|x^\frac{1}{n} x - x\| + \|x^\frac{1}{n} x\| \leq c_n + \|x^\frac{1}{n}\| \|x\|, \quad n \in \mathbb{N} \), where \( (c_n) \) in \( c_0 \) as in Lemma 6.8. This property holds if \( A \) is an operator algebra by the last assertion of Lemma 6.7.
A Banach algebra $A$ with a left bai (resp. right bai, bai) in $\mathfrak{r}_A$ has a left bai (resp. right bai, bai) in $\tilde{\mathfrak{r}}_A$. And a similar statement holds with $\mathfrak{r}_A$ and $\tilde{\mathfrak{r}}_A$ replaced by $\mathfrak{r}_B^R$ and $\tilde{\mathfrak{r}}_B^R$ for any unital Banach algebra $B$ containing $A$ as a subalgebra.

**Proof.** If $(e_t)$ is a left bai (resp. right bai, bai) in $\mathfrak{r}_A$, let $b_t = \tilde{\mathfrak{r}}(e_t) \in \tilde{\mathfrak{r}}_A$. By the proof in [22] Corollary 3.9, $(b_t^{\frac{1}{n}})$ is a left bai (resp. right bai, bai) in $\tilde{\mathfrak{r}}_A$. □

**Remark 6.11.** If the bai in the last result is sequential, then so is the one constructed in $\tilde{\mathfrak{r}}_A$.

We imagine that if a Banach algebra has a cai in $\mathfrak{r}_A$ then under mild conditions it has a cai in $\tilde{\mathfrak{r}}_A$. We give a couple of results along these lines, that are not in [22].

**Corollary 6.12.** Suppose that $A$ is a Banach algebra with the property that there is a sequence $(d_n)$ of scalars with limit 1 such that $\|x^n\| \leq d_n$ for all $n \in \mathbb{N}$ and $x \in \tilde{\mathfrak{r}}_A$ (this is the case for operator algebras by Lemma 6.7). If $A$ has a left bai (resp. right bai, bai) in $\mathfrak{r}_A$ then $A$ has a left cai (resp. right cai, cai) in $\tilde{\mathfrak{r}}_A$. And a similar statement holds with $\mathfrak{r}_A$ and $\tilde{\mathfrak{r}}_A$ replaced by $\mathfrak{r}_B^R$ and $\tilde{\mathfrak{r}}_B^R$ for any unital Banach algebra $B$ containing $A$ as a subalgebra.

**Proof.** For the first case, let $(f_s)_{s \in A} = (b_t^{\frac{1}{n}})$ be the left bai in $\tilde{\mathfrak{r}}_A$ from Corollary 6.10. Note that $\|f_s\| \leq d_n$ and so it is easy to see that $\|f_s\| \to 1$ by the Remark after Lemma 6.8. If there is a contractive subnet of $(f_s)$ we are done, so assume that there is no contractive subnet. So for every $s \in A$ there is an $s' \geq s$ with $\|f_{s'}\| > 1$. Let $\Lambda_0 = \{s \in A : \|f_s\| > 1\}$. A straightforward argument shows that $\Lambda_0$ is directed, and that $(f_s)_{s \in \Lambda_0}$ is a subset of $(f_s)_{s \in A}$ which is a left bai in $\tilde{\mathfrak{r}}_A$. Then $(\frac{1}{\|f_s\|}f_s)_{s \in \Lambda_0}$ is in $\tilde{\mathfrak{r}}_A$ since $\|f_s\| > 1$. So $(\frac{1}{\|f_s\|}f_s)_{s \in \Lambda_0}$ is a left cai in $\tilde{\mathfrak{r}}_A$. The other cases are similar. □

The hypothesis in the next result that $A^{**}$ is unital is, by [7] Theorem 1.6, equivalent to there being a unique mixed identity (we thank Matthias Neufang for this reference).

**Proposition 6.13.** Let $A$ be a Banach algebra such that $A^{**}$ is unital and $A$ has a real positive cai, or more generally suppose that there exists a real positive cai for $A$ and a bai for $A$ in $\tilde{\mathfrak{r}}_A$ with the same weak* limit. Then $A$ has a cai in $\tilde{\mathfrak{r}}_A$. This latter cai may be chosen to be sequential if in addition $A$ has a sequential bai.

**Proof.** That the second hypothesis is more general follows by Corollary 6.10 since a subnet of the ensuing bai for $A$ in $\tilde{\mathfrak{r}}_A$ has a weak* limit. Note that if $(f_s)_{s \in A}$ is a bai in $\tilde{\mathfrak{r}}_A$ with $\|f_s\| \to 1$ then either there is a subnet of $(f_s)$ consisting of contractions, in which case this subnet is a cai in $\tilde{\mathfrak{r}}_A$, or $\Lambda_0 = \{s \in A : \|f_s\| \geq 1\}$ is a directed set and $(\frac{1}{\|f_s\|}f_s)_{s \in \Lambda_0}$ is a cai in $\tilde{\mathfrak{r}}_A$.

Next, suppose that $(e_t)$ is a cai in $\mathfrak{r}_A$, and $(f_s)$ is a bai in $\tilde{\mathfrak{r}}_A$ and they have the same weak* limit $f$. By a re-indexing argument, we can assume that they are indexed by the same directed set. Then $e_t - f_t \to 0$ weakly in $A$. If $E = \{x_1, \cdots , x_n\}$ is a finite subset of $A$ define $F_{s,E}$ to be the subset

\[ \{(e_t - f_t, e_t x_1 - x_1, x_1 e_t - x_1, f_t x_1 - x_1, x_1 f_t - x_1, e_t x_2 - x_2, \cdots , x_1 f_t - x_1) : t \geq s\}, \]

of $A^{4m+1}$. Since $(A^{4m+1})^{**}$ is the 1 direct sum of $4m+1$ copies of $A^*$, it is easy to see that 0 is in the weak closure of $F_{s,E}$ (since $e_t - f_t \to 0$ weakly and $e_t x_k \to x_k$, for ...
etc). Thus by Mazur’s theorem 0 is in the norm closure of the convex hull of \( F_{s,E} \). For each \( n \in \mathbb{N} \) there are a finite subset \( t_1, \ldots, t_K \) (where \( K \) may depend on \( n, s, E \)), and positive scalars \( \alpha_k^{n,s,E} \) with sum 1, such that if \( r_{n,s,E} = \sum_{k=1}^{K} \alpha_k^{n,s,E} c_{t_k} \) and \( w_{n,s,E} = \sum_{k=1}^{K} \alpha_k^{n,s,E} f_{t_k} \), then \( \| r_{n,s,E} x_k - x_k \|, \| w_{n,s,E} x_k - x_k \|, \| r_{n,s,E} - w_{n,s,E} \| \), are each less than \( 2^{-n} \) for all \( k = 1, \cdots, m \).

Note that \( (r_{n,s,E}) \) is then a cai in \( \tau_A \), and \( (w_{n,s,E}) \) is a bai in \( \overline{S}_A \). Since \( r_{n,s,E} - w_{n,s,E} \to 0 \) with \( n \), it follows that \( \| w_{n,s,E} \| \to 1 \) with \( (n,E) \). So as in the last paragraph one may obtain from \( (w_{n,s,E}) \) a cai in \( \overline{S}_A \).

If we have a sequential cai in \( \tau_A \) then it follows from e.g. Sinclair’s Aarnes-Kadison type theorem (see the lines after Theorem 2.3; alternatively one may use our Aarnes-Kadison type theorem \([1,3]\) below) that \( A = \overline{x \mathbb{A} x} \) for some \( x \in A \). Given a cai \( (f_t) \) in \( \overline{S}_A \), choose \( t_1 < t_2 < \cdots \) with \( \| f_{t_k} x - x \| + \| x f_{t_k} - x \| < 2^{-k} \). Then it is clear that \( (f_{t_k}) \) is a sequential cai in \( \overline{S}_A \).

**Remark 6.14.** It follows that under the conditions of the last result, one may improve \([22]\) Corollary 6.10 in the way described after that result (using the fact in the remark after \([22]\) Corollary 2.10).

**Corollary 6.15.** If \( A \) is a Banach algebra then \( \tau_A \) is closed under \( r \)-th powers for any \( r \in [0,1] \). So is \( \tau_A^B \) for any unital Banach algebra \( B \) isometrically containing \( A \) as a subalgebra.

**Proof.** We saw in the proof of Proposition 6.6 that if \( x \in \tau_A^B \) then \( x = \lim_{t \to 0^+} \frac{1}{t}(1 + tx)^{-1} \), and \( tx(1 + tx)^{-1} \in \overline{S}_A \). Thus by \([61]\) Corollary 1.3 we have that \( x^r = \lim_{t \to 0^+} \frac{1}{t}(tx(1 + tx)^{-1})^r \) for \( 0 < r < 1 \). By Proposition 6.3 and its proof, the latter powers are in \( \mathbb{R}^+ \overline{S}_A \), so that \( x^r \in \mathbb{R}^+ \overline{S}_A = \tau_A^B \subset \tau_A \).

In an operator algebra, if \( x \) is sectorial of angle \( \theta \leq \frac{x}{2} \) then \( x^t \) has sectorial angle \( \leq t \theta \). Indeed this is what allows us to produce ‘nearly positive elements’, as discussed in Section 3. The following, which we have not seen in the literature, may be the best one has in a Banach algebra, and this disappointment means that some of the theory from \([23, 24, 26]\) will not generalize to Banach algebras.

**Corollary 6.16.** If \( x \) is sectorial of angle \( \theta \leq \frac{x}{2} \) in a unital Banach algebra then \( x^t \) has sectorial angle \( \leq t \theta + (1-t)\frac{x}{2} \).

**Proof.** This is Corollary 6.15 if \( \theta = \frac{x}{2} \). Suppose that \( W_B(x) \subset S_\theta \). Then \( e^{\pm i(x^t-\theta)} x \) is accretive. Hence \( (wx)^t \) is accretive where \( w = e^{\pm i(x^t-\theta)} \). By \([43]\) Lemma 3.1.4 with \( f(z) = wz \) we have \( (wx)^t = wz^t \). So \( wz^t \) is accretive. Reversing the argument above we see that

\[
W_B(x) \subset e^{i(x^t-\theta)} S_x \cap e^{-i(x^t-\theta)} S_x = S_{i\theta + (1-t)\frac{x}{2}}
\]

as desired.

We learned the Hilbert space operator version of the last proof from Charles Batty.

**Proposition 6.17.** If \( A \) is a Banach algebra and \( x \in \tau_A \) then \( ba(x) = ba(\overline{S}_x) \), and so \( \overline{x \mathbb{A}} = \overline{\overline{S}_x} \mathbb{A} \).

**Proof.** We said earlier that \( \overline{S}_x \) is in \( ba(x) \) and is independent of the particular unital Banach algebra containing \( A \). Thus this result follows from the unital case considered in \([22]\) Proposition 3.11. 

Lemma 6.18. If $p$ is an idempotent in a Banach algebra $A$ then $p \in \mathfrak{F}_A$ iff $p \in \mathfrak{r}_A$.

Proof. This is clear from the unital case considered in [22, Lemma 3.12]. □

Proposition 6.19. If $A$ is a Banach algebra and $x \in \mathfrak{r}_A$, then ba$(x)$ has a bai in $\mathfrak{F}_A$. Hence any weak$^*$ limit point of this bai is a mixed identity residing in $\mathfrak{F}_A^{**}$. Indeed $(x^\#)^*$ is a bai for ba$(x)$ in $\mathfrak{r}_A$, and $(\mathfrak{F}(x)^\#)^*$ is a bai for ba$(x)$ in $\mathfrak{F}_A$.

Proof. If $x \in \mathfrak{r}_A$ then the proof of [22, Proposition 3.17] shows that ba$(x)$ has a bai in $\mathfrak{F}_A^B$, and hence any weak$^*$ limit point of this bai is a mixed identity residing in $\mathfrak{F}_A^{**} \subset \mathfrak{F}_A^{**}$. Indeed $(x^\#)^*$ is a bai for ba$(x)$ in $\mathfrak{r}_A^B$.

The following new observation is a simple consequence of the above which we will need later.

Corollary 6.20. If $A$ is a nonunital Banach algebra and if $E$ and $F$ are subsets of $\mathfrak{r}_A$ then $EA = EB$, $AF = BF$, and $EAF = EBF$, where $B$ is any unitization of $A$.

Proof. The first follows from the following fact: if $x \in \mathfrak{r}_A$ then

$$x \in \overline{xA} = \overline{ba(x)A} = \overline{xB},$$

since by Cohen factorization $x \in ba(x) = ba(x)^2 \subset \overline{xA}$. The other two are similar. □

We now turn to the support projection of an element, encountered in the Aarnes–Kadison theorem (2.1). In an operator algebra or Arens regular Banach algebra things are cleaner (see [23, 24, 9]). For a Banach algebra $A$ and $x \in \mathfrak{r}_A$, we write $s(x)$ for the weak* Banach limit of $(x^\#)$ in $A^{**}$. That is $s(x)(f) = \lim_n f(x^\#)$ for $f \in A^*$, where $\lim$ is a Banach limit. It is easy to see that $xs(x) = s(x)x = x$, by applying these to $f \in A^*$. Hence $s(x)$ is a mixed identity of ba$(x)^{**}$, and is idempotent. By the Hahn–Banach theorem it is easy to see that $s(x) \in \text{conv}\{\{x^\#: n \in \mathbb{N}\}\}$. In $x \in \mathfrak{r}_A^B$ then by an argument after [22, Proposition 3.17] we have $s(x) \in \mathfrak{F}_B^{**} \cap A^{**} = \mathfrak{F}_A^{**} \subset \mathfrak{F}_A^{**}$. If ba$(x)$ is Arens regular then $s(x)$ will be the identity of ba$(x)^{**}$.

We call $s(x)$ above a support idempotent of $x$, or a (left) support idempotent of $xA$ (or a (right) support idempotent of $A^*x$). The reason for this name is the following result.

Corollary 6.21. If $A$ is a Banach algebra, and $x \in \mathfrak{r}_A$ then $x\overline{A}$ has a left bai in $\mathfrak{F}_A$ and $x \in x\overline{A} = s(x)A^{**} \cap A$ and $(xA)^{**} = s(x)A^{**}$. (These products are with respect to the second Arens product.)

Proof. The proof of [22, Corollary 3.18] works, and gives that $x\overline{A}$ has a left bai in $\mathfrak{F}_A^B$ if $x \in \mathfrak{r}_A^B$.

As in [23, Lemma 2.10] and [22, Corollary 3.19] we have:

Corollary 6.22. If $A$ is a Banach algebra, and $x,y \in \mathfrak{r}_A$, then $x\overline{A} \subset y\overline{A}$ iff $s(y)s(x) = s(x)$. In this case $x\overline{A} = A$ iff $s(x)$ is a left identity for $A^{**}$. (These products are with respect to the second Arens product.)

As in [23, Corollary 2.7] we have:
Corollary 6.23. Suppose that \( A \) is a subalgebra of a Banach algebra \( B \). If \( x \in A \cap r_B \), then the support projection of \( x \) computed in \( A^{**} \) is the same, via the canonical embedding \( A^{**} \cong A^{\perp \perp} \subset B^{**} \), as the support projection of \( x \) computed in \( B^{**} \).

In Section 2 we mentioned the paper of Kadison and Pedersen [55] initiating the development of a comparison theory for elements in \( C^* \)-algebras generalizing the von Neumann equivalence of projections. Again positivity and properties of the positive cone are key to that work. Admittedly their algebras were monotone complete, but many later authors have taken up this theme, with various versions of equivalence or subequivalence of elements in general \( C^* \)-algebras (see for example [10] or [66, 3] and references therein). Indeed recently the study of Cuntz equivalence and subequivalence within the context of the Elliott program has become one of the most important areas of \( C^* \)-algebra theory. In [19] Neal and the author began a program of generalizing basic parts of the theory of comparison, equivalence, and subequivalence, to the setting of general operator algebras. In that paper we focused on comparison of elements in \( R^+ \mathfrak{H}_A \), but we proved some lemmas in [26] that show that everything should work for elements in \( r_A \). In particular, we follow the lead of Lin, Ortega, Rørdam, and Thiel [66] in studying these equivalences, etc., in terms of the roots and support projections \( s(x) \) discussed in this section above, or in terms of module isomorphisms of (topologically) principal modules of the form \( xA \) studied below. There is a lot more work needed to be done here, our paper was simply the first steps. Also, we have not tried to see if any of this generalizes to larger classes of Banach algebras. Much of our theory in [19] depends on facts for \( n \)th roots of real positive elements. Thus we would expect that a certain portion of this theory generalizes to Banach algebras using the facts about roots summarized in Section 6.

7. Structure of ideals and HSA’s

We recall that an element \( x \) in an algebra \( A \) is \textit{pseudo-invertible} in \( A \) if there exists \( y \in A \) with \( yxy = x \). The following result (which is the non-approximately unital case of [22, Theorem 3.21]) should be compared with the \( C^* \)-algebraic version of the result due to Harte and Mbekhta [45, 46], and to the earlier version of the result in the operator algebra case (see particularly [23, Section 3], and [26, Subsection 2.4] and [25]).

Theorem 7.1. Let \( A \) be a Banach algebra, and \( x \in r_A \). The following are equivalent:

(i) \( s(x) \in A \),
(ii) \( xA \) is closed,
(iii) \( Ax \) is closed,
(iv) \( x \) is pseudo-invertible in \( A \),
(v) \( x \) is invertible in \( ba(x) \).

Moreover, these conditions imply

(vi) \( 0 \) is isolated in, or absent from, \( Sp_A(x) \).

Finally, if \( ba(x) \) is semisimple then (i)–(vi) are equivalent.

Proof. The first five equivalences are just as in [22, Theorem 3.21]; as is the assertions regarding (vi), since there we may assume \( A \) is unital by definition of spectrum and because of the form of (v). \( \square \)
The next results (which are the non-approximately unital cases of results in [22, Section 3]) follow from Theorem 7.1 just as the approximately unital cases did in [22], which in turn often rely on earlier arguments from e.g. [23].

**Corollary 7.2.** If $A$ is a closed subalgebra of a unital Banach algebra $B$, and if $x \in r_A^2$, then $x$ is invertible in $B$ iff $1_B \in A$ and $x$ is invertible in $A$, and iff $ba(x)$ contains $1_B$; and in this case $s(x) = 1_B$.

**Corollary 7.3.** Let $A$ be a Banach algebra. A closed right ideal $J$ of $A$ is of the form $xA$ for some $x \in r_A$ iff $J = qA$ for an idempotent $q \in \mathcal{F}_A$.

**Corollary 7.4.** If a nonunital Banach algebra $A$ contains a nonzero $x \in r_A$ with $xA$ closed, then $A$ contains a nontrivial idempotent in $\mathcal{F}_A$. If a Banach algebra $A$ has no left identity, then $xA \neq A$ for all $x \in r_A$.

In [14] we generalized the concept of hereditary subalgebra (HSA), an important tool in $C^*$-algebra theory, to operator algebras, and established that the basics of the $C^*$-theory of HSA’s is still true. Now of course HSA’s need not be selfadjoint, but are still norm closed approximately unital inner ideals in $A$, where by the latter term we mean a subalgebra $D$ with $DAD \subset D$. Generalizing Theorem 2.2 above, we showed in [23, 24] that HSA’s and right ideals with left cais in operator algebras are manifestations of our cone $r_A$, or if preferred, $\mathcal{F}_A$ or the ‘nearly positive’ elements. We now discuss some aspects of this in the case of Banach algebras from [22], and mention some of what is still true in that setting. In particular we study the relationship between HSA’s and one-sided ideals with one-sided approximate identities. Some aspects of this relationship is problematic for general Banach algebras (see [22, Section 4]), but it works much better in separable algebras. As we said around Theorem 2.3, our work is closely related to the results of Sinclair and others on the Cohen factorization method (see e.g. [74, 37]), which does include some similar sounding but different results.

We define a right $\mathcal{F}$-ideal (resp. left $\mathcal{F}$-ideal) in a Banach algebra $A$ to be a closed right (resp. left) ideal with a left (resp. right) bai in $\mathcal{F}_A$ (or equivalently, by Corollary 7.10 in $r_A$). Henceforth in this section, by a hereditary subalgebra (HSA) of $A$ we will mean an inner ideal $D$ with a two-sided bai in $\mathcal{F}_A$ (or equivalently, by Corollary 7.10 in $r_A$). Perhaps these should be called $\mathcal{F}$-HSA’s to avoid confusion with the notation of [14, 23] where one uses cais’s instead of bai’s, but for brevity we shall use the shorter term. And indeed for operator algebras (the setting of [14, 23]) the two notions coincide, and also right and left $\mathcal{F}$-ideals are just the $r$-ideals and $\ell$-ideals of those papers (see Corollary 7.10). Note that a HSA $D$ induces a pair of right and left $\mathcal{F}$-ideals $J = DA$ and $K = AD$. Using the proof in [22, Lemma 4.2] we have:

**Lemma 7.5.** If $A$ is a Banach algebra, and $z \in \mathcal{F}_A$, set $J = zA$, $D = zA^2$, and $K = A^2z$. Then $D$ is a HSA in $A$ and $J$ and $K$ are the induced right and left $\mathcal{F}$-ideals mentioned above.

At this point we have to jettison $\mathcal{F}_A$ and $r_A$ as defined at the start of Section 0 if $A$ is not approximately unital, because the remaining results are endangered if $\mathcal{F}_A$ and $r_A$ are not closed and convex. Indeed most results in Sections 4 and 7 of [22] would seem to need $\mathcal{F}_A$ and $r_A$ to be replaced by $\mathcal{F}_A^1$ and $r_A^1$ for a fixed unital Banach algebra $B$ containing $A$ as a subalgebra. That is, we need to fix a particular unitization of $A$, not consider all unitizations simultaneously. Of course if $A$ is an
operator algebra then there is a unique unitization, hence all this is redundant. (As we said early in Section 6 we could also fix this problem by redefining \( \mathcal{F}_A = \cap_B \mathcal{F}_B^A \) and \( r_A = \cap_B r_B^A \), the intersections taken over all \( B \) as above. Everything below would then look cleaner but may be less useful.) Thus we define a right \( \mathcal{F}_B \)-ideal (resp. left \( \mathcal{F}_B \)-ideal) in \( A \) to be a closed right (resp. left) ideal with a left (resp. right) bai in \( \mathcal{F}_B^A \) (or equivalently, by Corollary 6.10 in \( r_B^A \)). Note that one-sided \( \mathcal{F}_B \)-ideals in \( A \) are exactly subalgebras of \( A \) which are one-sided \( \mathcal{F}_B \)-ideals in \( A + \mathbb{C} 1_B \) in the sense of [22] Section 4).

We define a \( B \)-hereditary subalgebra (or \( B \)-HSA for short) of \( A \) to an inner ideal \( D \) in \( A \) with a two-sided bai in \( \mathcal{F}_B^D \) (or equivalently, by Corollary 6.10 in \( r_B^D \)). Note that \( B \)-HSA’s in \( A \) are exactly subalgebras of \( A \) which are HSA’s in \( A + \mathbb{C} 1_B \) in the sense of [22] Section 4).

Again a \( B \)-HSA \( D \) induces a pair of right and left \( \mathcal{F}_B \)-ideals \( J = DA \) and \( K = AD \). Lemma 7.5 becomes: If \( z \in \mathcal{F}_A^A \), set \( J = zA, D = Az \), and \( K = A^2z \). Then \( D \) is a \( B \)-HSA in \( A \) and \( J \) and \( K \) are the induced right and left \( \mathcal{F}_B \)-ideals mentioned above.

Because of the facts at the end of the second last paragraph, and because of Corollary 6.20 in the following four results we can assume that \( A \) is unital, in which case the proofs are in [22]. These results are all stated for a Banach algebra with unitization \( B \), but they could equally well be stated for a closed subalgebra of a unital Banach algebra \( B \).

**Lemma 7.6.** Suppose that \( J \) is a right \( \mathcal{F}_B \)-ideal in aBanach algebra with unitization \( B \). For every compact subset \( K \subset J \), there exists \( z \in J \cap \mathcal{F}_A^B \) with \( K \subset zJ \subset zA \).

Applying this lemma gives the first assertion in the following result, taking \( K = \{ \frac{1}{n} e_n \} \cup \{ 0 \} \), where \( (e_n) \) is the left bai. Taking \( K = \{ \frac{d_n}{m_n ||d_n||} \} \cup \{ 0 \} \) where \( \{ d_n \} \) is a countable dense set gives the second.

**Corollary 7.7.** Let \( A \) be a Banach algebra with unitization \( B \). The closed right ideals of \( A \) with a countable left bai in \( r_B^A \) are precisely the (topologically) ‘principal right ideals’ \( zA \) for some \( z \in \mathcal{F}_A^B \) which is also in the ideal. Every separable right \( \mathcal{F}_B \)-ideal is of this form.

**Corollary 7.8.** Let \( A \) be a Banach algebra with unitization \( B \). The right \( \mathcal{F}_B \)-ideals in \( A \) are precisely the closures of increasing unions of right ideals in \( A \) of the form \( zA \) for some \( z \in \mathcal{F}_A^B \).

We say that a right module \( Z \) over \( A \) is algebraically countably generated (resp. algebraically finitely generated) over \( A \) if there exists a countable (resp. finite set) \( \{ x_k \} \) in \( Z \) such that every \( z \in Z \) may be written as a finite sum \( \sum_{k=1}^n x_k a_k \) for some \( a_k \in A \). Of course any algebraically finitely generated is algebraically countably generated.

**Corollary 7.9.** Let \( A \) be a Banach algebra with unitization \( B \). A right \( \mathcal{F}_B \)-ideal \( J \) in \( A \) is algebraically countably generated as a right module over \( A \) iff \( J = qA \) for an idempotent \( q \in \mathcal{F}_A^B \). This is also equivalent to \( J \) being algebraically countably generated as a right module over \( A + \mathbb{C} 1_B \).

The following was not stated in [22].

**Corollary 7.10.** If \( A \) is an operator algebra, a closed subalgebra of a unital operator algebra \( B \), then right and left \( \mathcal{F}_B \)-ideals in \( A \) are just the \( r \)-ideals and
\(\ell\)-ideals in \(A\) of \([14, 23]\), and \(B\)-HSA’s in \(A\) are just the HSA’s in \(A\) of those references.

\textbf{Proof.} By Corollary 7.8 a right \(\mathfrak{F}^B\)-ideal is the closure of an increasing union of right ideals in \(A\) of the form \(zA\) for \(z \in \mathfrak{F}_A\). However this is the characterization of \(r\)-ideals from \([23]\). Similarly for the left ideal case. A similar argument works for the HSA case using Corollary 7.14; alternatively, if \(D\) is a \(B\)-HSA then \(D\) has a bai from \(\mathfrak{F}_A\). By Corollary 6.12 \(D\) has a cai. \(\square\)

If \(A\) is a Banach algebra with unitization \(B\) it would be nice to say that the right \(\mathfrak{F}^B\)-ideals in \(A\) are precisely the sets of form \(EA\) for a subset \(E \subset \mathfrak{F}^B_A\) (or equivalently, \(E \subset r\mathfrak{F}^B_A\)). One direction of this is obvious: just take \(E\) to be the bai in \(\mathfrak{F}^B_A\) (resp. \(r\mathfrak{F}^B_A\)). However the other direction is false in general Banach algebras, although it does hold in operator algebras \([23]\) and commutative Banach algebras.

(Another characterization of closed ideals with bai’s in commutative Banach algebras may be found in \([60]\).)

That \(EA\) is a right \(\mathfrak{F}^B\)-ideal in \(A\) if \(A\) is a commutative Banach algebra and \(E \subset \mathfrak{F}^B_A\) follows from Theorem 7.1 in \([22]\) after noting that by Corollary 6.20 we may replace \(A\) by \(A + C1_B\). The key part of the proof of Theorem 7.1 in \([22]\) is to show that for any finite subset \(G\) of \(E\) there exists an element \(z_G \in \mathfrak{F}^B_A \cap EA\) with \(\overline{GA} = z_GA\). Indeed one can take \(z_G\) to be the average of the elements in \(G\). Then the net \((z_G^n)\), indexed by the finite subsets \(G\) of \(E\) and \(n \in \mathbb{N}\), is easily seen to be a left bai in \(EA\) from \(\mathfrak{F}^B_A\). An application of this: for such subsets \(E\) of an operator algebra or commutative Banach algebra \(A\), the Banach algebra generated by \(E\) has a bai in \(\mathfrak{F}^B_A\). This follows from the argument above since the \(z_G\) above are in the convex hull of \(E\), hence the bai \((z_G^n)\) is in the Banach algebra generated by \(E\). In particular, if \(A\) is generated as a Banach algebra by \(r\mathfrak{F}^A_A\), then \(A\) has a bai, and this bai may be taken from \(r\mathfrak{F}^A_A\). (The present paragraph is a summary of the results in \([22]\) Section 7, and a generalization of these results to the case that \(A\) is not approximately unital.)

Unless explicitly said to the contrary, all the remaining results in this section are again generalizations to general Banach algebras of results from \([22]\). Some of these were proved earlier in the operator algebra case by the author and Read. Again for their proofs we can assume that \(A\) is unital, and appeal to the matching results in \([22]\).

\textbf{Lemma 7.11.} Let \(A\) be a Banach algebra with unitization \(B\). Let \(D\) be a closed subalgebra of \(A\). If \(D\) has a bai from \(\mathfrak{F}^B_D\), then for every compact subset \(K \subset D\), there is an \(x \in \mathfrak{F}^B_D\) such that \(K \subset xDx \subset xAx\).

As in the proof sketched for Corollary 7.11 this leads to:

\textbf{Theorem 7.12.} Let \(A\) be a Banach algebra with unitization \(B\), and let \(D\) be an inner ideal in \(A\). Then \(D\) has a countable bai from \(\mathfrak{F}^B_D\) (or equivalently, from \(r\mathfrak{F}^D_D\)) iff there exists an element \(z \in \mathfrak{F}^B_D\) with \(D = zA z^*\). Thus \(D\) is of the form in Lemma 7.5 and such \(D\) has a countable commuting bai from \(\mathfrak{F}^B_D\), namely \((z \frac{1}{2})\). Any separable inner ideal in \(A\) with a bai from \(r\mathfrak{F}^B_D\) is of this form.

From this most of the following generalization of the Aarnes–Kadison theorem (see Theorem 7.1) is immediate. By a \textit{strictly real positive element} in (v) below, we mean an element \(x \in A\) such that \(\text{Re } \varphi(x) > 0\) for all states \(\varphi\) of \(A\) which do
not vanish on \( A \). In \([23, 26]\) we generalized some basic aspects of strictly positive elements in \( C^*\)-algebras to operator algebras. The following is mostly in \([22, 26]\), and relies on ideas from \([23]\).

**Corollary 7.13** (Aarnes–Kadison type theorem). If \( A \) is a Banach algebra then the following are equivalent:

(i) There exists an \( x \in r_A \) with \( A = xAx \).

(ii) There exists an \( x \in r_A \) with \( A = xA = Ax \).

(iii) There exists an \( x \in r_A \) with \( s(x) \) a mixed identity for \( A^{**} \).

If \( B \) is a unitization of \( A \) then items (i), (ii), or (iii) above hold with \( x \in r_B \) iff

(iv) \( A \) has a sequential bai from \( r_B \).

The approximate identity in (iv) may be taken to be commuting, indeed it may be taken to be \((x^n)\) for the last mentioned element \( x \). If \( A \) is separable and has a bai \( r_B \) then \( A \) satisfies (iv) and hence all of the above. Moreover if \( A \) is an operator algebra then (i)–(iv) are each equivalent to:

(v) \( A \) has a strictly real positive element, and any of these imply that the operator algebra \( A \) has a sequential real positive cai.

Again, \( r \) can be replaced by \( \mathfrak{r} \) throughout this result, or in any of the items (i) to (v).

The proof of Corollary 7.13 is mostly in \([22, 26]\), and relies partly on ideas from \([23]\). In the operator algebra case, if (ii) holds with \( x \in \mathfrak{r}_A \) then \( (\frac{1}{2}x^n) \) is a cai for \( A \) in \( \frac{1}{2}\mathfrak{r}_A \) by \([23]\) Section 3], and \( s(x) = 1_{A^{**}} \). So \( x \) is a strictly real positive element by \([23]\) Lemma 2.10]. Conversely, if an operator algebra \( A \) has a strictly real positive element then it is explained in the long discussion before \([26]\) Lemma 3.2 how to adapt the proof of \([23]\) Lemma 2.10 to show that (iv) holds, hence (ii), and hence \( A \) has a sequential real positive cai by e.g. \([24]\) Corollary 3.5], or by our earlier Corollary 7.10.

**Corollary 7.14**. The \( B \)-HSA’s in a Banach algebra \( A \) with unitization \( B \) are exactly the closures of increasing unions of HSA’s of the form \( zAz \) for \( z \in \mathfrak{r}_A^B \).

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**References**

[1] J. F. Aarnes and R. V. Kadison, *Pure states and approximate identities*, Proc. Amer. Math. Soc. **21** (1969), 749–752.

[2] E. M. Alfsen, *Compact convex sets and boundary integrals*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 57, Springer-Verlag, New York-Heidelberg, 1971.

[3] P. Ara, F. Perera, and A. S. Toms, *K-Theory for operator algebras. Classification of \( C^*\)-algebras*, (Notes from the Summer School on Operator Algebras and Applications, held in Santander, July 21-25 2008), Aspects of operator algebras and applications, p. 1–71, Contemp. Math., **534**, Amer. Math. Soc., Providence, RI, 2011.

[4] W. B. Arveson, *Analyticity in operator algebras*, *Amer. J. Math.* **89** (1967), 578–642.

[5] W. B. Arveson, *Subalgebras of \( C^*\)-algebras*, Acta Math. **123** (1969), 141–224.

[6] L. Asimow and A. J. Ellis, *Convexity theory and its applications in functional analysis*, London Mathematical Society Monographs, 16, Academic Press London-New York, 1980.

[7] J. Baker, A. T. Lau, and J. Pym, *Module homomorphisms and topological centres associated with weakly sequentially complete Banach algebras*, J. Funct. Anal. **158** (1998), 186–208.

[8] C. J. K. Batty and D. W. Robinson, *Positive one-parameter semigroups on ordered Banach spaces*, Acta Appl. Math. **2** (1984), 221–296.
[9] C. A. Bearden, D. P. Blecher and S. Sharma, On positivity and roots in operator algebras, J. Integral Equations Operator Th. 79 (2014), 555–566.
[10] B. Blackadar, Operator algebras. Theory of C*-algebras and von Neumann algebras, Encyclopaedia of Mathematical Sciences Vol. 122, Operator Algebras and Non-commutative Geometry, III, Springer-Verlag, Berlin, 2006.
[11] D. P. Blecher, Multipliers, C-modules, and algebraic structure in spaces of Hilbert space operators, in “Operator algebras, quantization, and noncommutative geometry”, 85–128, Contemp. Math., 365, Amer. Math. Soc., Providence, RI, 2004.
[12] D. P. Blecher, Noncommutative peak interpolation revisited, Bull. London Math. Soc. 45 (2013), 1100–1106.
[13] D. P. Blecher et al, Roots in Banach algebras, In preparation.
[14] D. P. Blecher, D. M. Hay, and M. Neal, Hereditary subalgebras of operator algebras, J. Operator Theory 59 (2008), 333–357.
[15] D. P. Blecher and L. E. Labuschagne, Von Neumann algebraic Hp theory, Function Spaces: Fifth Conference on Function Spaces, Contemp. Math. Vol. 435, Amer. Math. Soc. (2007), 89–114.
[16] D. P. Blecher and C. Le Merdy, Operator algebras and their modules—an operator space approach, Oxford Univ. Press, Oxford (2004).
[17] D. P. Blecher and M. Neal, Metric characterizations of isometries and of unital operator spaces and systems, Proc. Amer. Math. Soc. 139 (2011), 985–998.
[18] D. P. Blecher and M. Neal, Open projections in operator algebras I: Comparison Theory, Studia Math. 208 (2012), 117–150.
[19] D. P. Blecher and M. Neal, Open projections in operator algebras II: compact projections, Studia Math. 209 (2012), 203–224.
[20] D. P. Blecher and M. Neal, Metric characterizations II, Illinois J. Math. 57 (2013), 25–41.
[21] D. P. Blecher and M. Neal, Completely contractive projections on operator algebras, Preprint 2015.
[22] D. P. Blecher and N. Ozawa, Real positivity and approximate identities in Banach algebras, Pacific J. Math. 277 (2015), 1–59. DOI 10.2140/pjm.2015.277.1
[23] D. P. Blecher and C. J. Read, Operator algebras with contractive approximate identities, J. Functional Analysis 261 (2011), 188–217.
[24] D. P. Blecher and C. J. Read, Operator algebras with contractive approximate identities II, J. Functional Analysis 261 (2011), 188–217.
[25] D. P. Blecher and C. J. Read, Operator algebras with contractive approximate identities III, Preprint 2013 (ArXiv version arXiv:1308.2723v2).
[26] D. P. Blecher and C. J. Read, Order theory and interpolation in operator algebras, Studia Math. 225 (2014), 61–95.
[27] F. F. Bonsall and J. Duncan, Numerical ranges of operators on normed spaces and of elements of normed algebras, London Mathematical Society Lecture Note Series 2, Cambridge University Press, London-New York, 1971.
[28] L. G. Brown, P. Green, and M. A. Rieffel, Stable isomorphism and strong Morita equivalence of C*-algebras, Pacific J. Math. 71 (1977), no. 2, 349–363.
[29] M.-D. Choi and E. G. Effros, Injectivity and operator spaces, J. Funct. Anal. 24 (1977), 156–209.
[30] C. K. Chui, P. W. Smith, R. R. Smith, and J. D. Ward, L-ideals and numerical range preservation, Illinois J. Math. 21 (1977), 365–373.
[31] M. A. Crouzeix, A functional calculus based on the numerical range: applications, Linear Multilinear Algebra 56 (2008), no. 1-2, 81–103.
[32] H. G. Dales, Banach algebras and automatic continuity, London Mathematical Society Monographs. New Series, 24, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 2000.
[33] P. G. Dixon, Approximate identities in normed algebras II, J. London Math. Soc. 17 (1978), 141–151.
[34] R. S. Doran and J. Wichmann, Approximate identities and factorization in Banach modules, Lecture Notes in Mathematics, 768, Springer-Verlag, Berlin-New York, 1979.
[35] S. Drury, Principal powers of matrices with positive definite real part, Linear Multilinear Algebra 63 (2015), 296–301.
[36] E. G. Effros and E. Størmer, Positive projections and Jordan structure in operator algebras, Math. Scand. 45 (1979), 127–138.

[37] J. Esterle, Injection de semi-groupes divisibles dans des algèbres de convolution et construction d’homomorphismes discontinus de $C(K)$, Proc. London Math. Soc. 36 (1978), 59–85.

[38] R. J. Fleming and J. E. Jamison, Isometries on Banach spaces: function spaces, Monographs and Surveys in Pure and Appl. Math. 129, Chapman and Hall/CRC press (2003).

[39] Y. Friedman and B. Russo, Contractive projections on $C_0(K)$, Trans. AMS 273 (1982), 57–73.

[40] Y. Friedman and B. Russo, Conditional expectation without order, Pacific J. Math. 115 (1984), 351–360.

[41] Y. Friedman and B. Russo, Solution of the contractive projection problem, J. Funct. Anal. 60 (1985), 56–79.

[42] T. W. Gamelin, Uniform Algebras, Second edition, Chelsea, New York, 1984.

[43] M. Haase, The functional calculus for sectorial operators, Operator Theory: Advances and Applications, 169, Birkhäuser Verlag, Basel, 2006.

[44] P. Harmand, D. Werner, and W. Werner, $M$-ideals in Banach spaces and Banach algebras, Lecture Notes in Math., 1547, Springer-Verlag, Berlin–New York, 1993.

[45] R. Harte and M. Mbekhta, On generalized inverses in $C^*$-algebras, Studia Math. 103 (1992), 71–77.

[46] R. Harte and M. Mbekhta, On generalized inverses in $C^*$-algebras II, Studia Math. 106 (1993), 129–138.

[47] D. M. Hay, Closed projections and peak interpolation for operator algebras, Integral Equations Operator Theory 57 (2007), 491–512.

[48] G. J. O. Jameson, Ordered linear spaces, Lecture Notes in Mathematics, Vol. 141, Springer-Verlag, Berlin-New York, 1970.

[49] R. V. Kadison, Order properties of bounded self-adjoint operators, Proc. Amer. Math. Soc. 2 (1951), 505–510.

[50] R. V. Kadison, Isometries of operator algebras, Ann. of Math. 54 (1951), 325–338.

[51] R. V. Kadison, A representation theory for commutative topological algebra, Mem. Amer. Math. Soc. (1951). no. 7.

[52] R. V. Kadison, A generalized Schwarz inequality and algebraic invariants for operator algebras, Ann. of Math. 56 (1952), 494–503.

[53] R. V. Kadison, Non-commutative conditional expectations and their applications, in Operator algebras, quantization, and noncommutative geometry, 143 179, Contemp. Math., 365, Amer. Math. Soc., Providence, RI, 2004.

[54] R. V. Kadison and B. Fuglede, Determinant theory in finite factors, Ann. of Math. 55 (1952), 520–530.

[55] R. V. Kadison and G. Pedersen, Equivalence in operator algebras, Math. Scand. 27 (1970), 205–222.

[56] R. V. Kadison and J. R. Ringrose, Fundamentals of the theory of operator algebras, Vol. 1, Graduate Studies in Mathematics, Amer. Math. Soc. Providence, RI, 1997.

[57] R. V. Kadison and I. M. Singer, Triangular operator algebras. Fundamentals and hyperreducible theory, Amer. J. Math. 82 (1960), 227–259.

[58] J. L. Kelley and R. L. Vaught, The positive cone in Banach algebras, Trans. Amer. Math. Soc. 74 (1953), 44–55.

[59] A. T. Lau and R. J. Loy, Contractive projections on Banach algebras, J. Funct. Anal. 254 (2008), 2513–2533.

[60] A. T. Lau and A. Ulger, Characterization of closed ideals with bounded approximate identities in commutative Banach algebras, complemented subspaces of the group von Neumann algebras and applications, Trans. Amer. Math. Soc. 366 (2014), 4151–4171.

[61] C-K. Li, L. Rodman, and I. M. Spitkovsky, On numerical ranges and roots, J. Math. Anal. Appl. 282 (2003), 329–340.

[62] V. I Mačajev and Ju. A. Palant, On the powers of a bounded dissipative operator (Russian), Ukrain. Mat. Z. 14 (1962), 329–337.

[63] B. Magajna, Weak* continuous states on Banach algebras, J. Math. Anal. Appl. 350 (2009), 252–255.

[64] R. T. Moore, Hermitian functionals on $B$-algebras and duality characterizations of $C^*$-algebras, Trans. Amer. Math. Soc. 162 (1971), 253–265.
[65] J. von Neumann, *John von Neumann: selected letters*, Edited by Miklos Rdei, History of Mathematics 27 Amer. Math. Soc., Providence, RI; London Math. Soc., London, 2005.

[66] E. Ortega, M. Rørdam, and H. Thiel, *The Cuntz semigroup and comparison of open projections*, J. Funct. Anal. 260 (2011), 3474–3493.

[67] T. W. Palmer, *Banach algebras and the general theory of *-algebras*, Vol. I. Algebras and Banach algebras, Encyclopedia of Math. and its Appl., 49, Cambridge University Press, Cambridge, 1994.

[68] V. I. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Math., 78, Cambridge University Press, Cambridge, 2002.

[69] G. K. Pedersen, *Factorization in C*-algebras*, Exposition. Math. 16 (1998), 145–156.

[70] G. K. Pedersen, *C*-algebras and their automorphism groups, Academic Press, London (1979).

[71] R. T. Prosser, *On the ideal structure of operator algebras*, Mem. Amer. Math. Soc. No. 45 (1963).

[72] C. J. Read, *On the quest for positivity in operator algebras*, J. Math. Analysis and Applns. 381 (2011), 202–214.

[73] A. G. Robertson and M.A. Youngson, *Positive projections with contractive complements on Jordan algebras*, J. London Math. Soc. (2) 25 (1982), 365–374.

[74] A. M. Sinclair, *Bounded approximate identities, factorization, and a convolution algebra*, J. Funct. Anal. 29 (1978), 308–318.

[75] A. M. Sinclair and A. W. Tullo, *Noetherian Banach algebras are finite dimensional*, Math. Ann. 211 (1974), 151–153.

[76] J. G. Stampfli and J. P. Williams, *Growth conditions and the numerical range in a Banach algebra*, Tohoku Math. J. 381 (1968), 417–506.

[77] E. Størmer, *Positive projections with contractive complements on C*-algebras*, J. London Math. Soc. (2) 26 (1982), 132–142.

[78] B. Sz.-Nagy, C. Foias, H. Bercovici, and L. Kerchy, *Harmonic analysis of operators on Hilbert space*, Second edition, Universitext. Springer, New York, 2010.

[79] K. Yosida, *Functional analysis*, Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[80] M. A. Youngson, *Completely contractive projections on C*-algebras*, Quart. J. Math. Oxford Ser. (2) 34 (1983), 507–511.

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