Compact $C^*$-quantum groupoids

Thomas Timmermann

timmermt@math.uni-muenster.de

University of Münster

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Quantum groupoids/Hopf bimodules — Introduction

Ingredients of a quantum groupoid:  
- a Hopf bimodule

\[
B \xrightarrow{\rho} A \xrightarrow{\Delta} A_{\sigma \star \rho} A
\]

- left/right Haar weights, antipode, modular element, . . .

Flavours of quantum groupoids:  
- finite ones [Böhm, Szlachányi, Nikshych, . . . ]
- cnt. bundles of l.c. quantum groups [Blanchard, Enock]
- measurable quantum groupoids in the setting of von Neumann algebras [Enock, Lesieur, Vallin]
- locally compact quantum groupoids in the setting of \( C^* \)-algebras?
What is a Hilbert module over a KMS-weight?

Given: $C^*$-algebra $B$ with faithful KMS-weight $\mu$

$\rightsquigarrow$ GNS-rep. $B \hookrightarrow \mathcal{L}(H_\mu), \text{ commutant } B' \subseteq \mathcal{L}(H_\mu)$

Def.: Hilbert $C^*$-module over $\mu := (H, \alpha)$, where

- $H$ Hilbert space, $\alpha$ closed subspace of $\mathcal{L}(H_\mu, H)$,
- $[\alpha H_\mu] = H$, $[\alpha^* \alpha] = B$, $[\alpha B] = \alpha$

Lemma:

- $\alpha$ is a Hilbert $C^*$-module over $B$
- $\alpha \otimes_B H_\mu \cong H$ via $\xi \otimes_B \eta \equiv \xi \eta$
- $\exists$ normal nd. representation $\rho_\alpha : B' \rightarrow \mathcal{L}(H)$ s.t.
  $$\rho_\alpha(x)\xi\eta = \xi x\eta \text{ for all } \xi \in \alpha, \eta \in H_\mu$$
Examples of Hilbert $C^*$-modules over KMS-weights

1. If $B$ commutative, then

$$\{\text{Hilbert } C^*\text{-modules over } \mu\} \rightarrow \{\text{n. nd. representations of } B'\} \uparrow \downarrow \{\text{cont. Hilbert bundles on } \widehat{B}\} \rightarrow \{\mu\text{-mb. Hilbert bundles on } \widehat{B}\}$$

2. If $B \subseteq A$ and $\phi: A \rightarrow B$ is a conditional expectation s.t. $\nu := \mu \circ \phi$ is a KMS-weight on $A$, then:

- $\exists$ GNS-map $\Lambda_\phi: A \rightarrow \mathcal{L}(H_\mu, H_\nu)$ s.t. $\Lambda_\phi(a)\Lambda_\mu(b) = \Lambda_\nu(ab)$
- $(H_\nu, \overline{\Lambda_\phi(A)})$ is a Hilbert $C^*$-module over $\mu$
What is a Hilbert bimodule over KMS-weights?

**Given:** \( C^* \)-algebras \( B, C \) with faithful KMS-weights \( \mu, \nu \)

- opposite KMS-weight \( \mu^{\text{op}} \) on \( B^{\text{op}} \) via \( b^{\text{op}} \mapsto \mu(b) \)
- GNS-representations \( B^{\text{op}} \circlearrowleft H_{\mu^{\text{op}}} \cong H_\mu \circlearrowright B \)

**Def.:** Hilbert \( C^* \)-bimodule over \( (\mu^{\text{op}}, \nu) := (H, \alpha, \beta) \), where
  - \((H, \alpha), (H, \beta)\) Hilbert \( C^* \)-modules over \( \mu^{\text{op}}, \nu \)
  - \( \alpha = [\rho_\beta(C^{\text{op}})\alpha] \) and \( \beta = [\rho_\alpha(B)\beta] \)

**Ex.:** If \( B \subseteq A, \phi: A \to B \) and \( \nu = \mu \circ \phi \) are as before, then
  - \( \exists \) opposite cond. expectation \( \phi^{\text{op}}: A^{\text{op}} \to B^{\text{op}} \) with GNS-map \( \Lambda_{\phi^{\text{op}}}: A^{\text{op}} \to \mathcal{L}(H_{\mu^{\text{op}}}, H_\nu) \)
  - \( (H_\nu, \overline{\Lambda_{\phi^{\text{op}}}(A^{\text{op}})}, \overline{\Lambda_{\phi}(A)}) \) is H.-\( C^* \)-bimodule/(\( \mu^{\text{op}}, \mu \))
The relative tensor product of Hilbert $C^*$-bimodules

**Given:** $H.-C^*$-bimod. $(H, \alpha, \beta), (K, \gamma, \delta)$ over $(\tau^{op}, \mu), (\mu^{op}, \nu)$

$$\sim H_{\tau^{op}} \xrightarrow{\alpha} H \xleftarrow{\beta} H_{\mu} = H_{\mu^{op}} \xrightarrow{\gamma} K \xleftarrow{\delta} H_{\nu}$$

**Hilbert $C^*$-bimodule over $(\tau^{op}, \nu)$**

**Thm:**
- this tensor product is functorial, unital, associative
- $\exists$ natural iso. $H_{\beta \otimes \gamma} K \cong H_{\rho_{\beta} \otimes \rho_{\gamma}} K$ (Connes’ fusion)
- $(H, \alpha, \beta) \mapsto (H, \rho_\alpha, \rho_\beta)$ is a monoidal functor from $H.-C^*$-bimodules/$(\mu^{op}, \mu)$ to $H.-$bimodules/$(B'', B')$
What is a Hopf $C^*$-bimodule?

Def.: $C^*$-algebra over $\mu := (H, \alpha, A)$, where

- $(H, \alpha)$ Hilbert $C^*$-module over $\mu$
- $A \subseteq \mathcal{L}(H)$ nd. $C^*$-algebra and $\rho_\alpha(B^{op}) \subseteq M(A)$

Ex.: $B \subseteq A$, $\phi: A \to B$ and $\nu = \mu \circ \phi$ as before

$\sim (H_\nu, \Lambda_{\phi^{op}}(A^{op}), A)$ is a $C^*$-algebra over $\mu^{op}$

Def.: category of $C^*$-algebras over $(\mu^{op}, \tau)$ . . .

Def.: Hopf $C^*$-bimodule over $\mu := (H, \alpha, \beta, \Delta)$, where

- $(H, \alpha, \beta, A)$ – briefly $A_H^{\alpha, \beta}$ – is a $C^*$-algebra/$(\mu, \mu^{op})$
- a coassociative $\Delta \in \text{Mor} (A_H^{\alpha, \beta}, A_H^{\alpha, \beta} \ast A_H^{\alpha, \beta})$

Lemma: $(H, \rho_\alpha, \rho_\beta, A''', \tilde{\Delta})$ is a Hopf-von Neumann-bimodule
The fiber product of $C^*$-algebras over KMS-weights

Given: $(H, \alpha, \beta, A), (K, \gamma, \delta, B)$ $C^*$-algebras/$(\mu^{op}, \tau), (\tau^{op}, \nu)$

\[ A_{\beta \ast \gamma} B := \{ T : T^{(*)} I(\beta) \subseteq [I(\beta) B], T^{(*)} r(\gamma) \subseteq [r(\gamma) A] \} \]

is a $C^*$-algebra over $(\mu^{op}, \nu)$ if it is nondegenerate

**Lemma:**
- this fiber product is functorial, not associative, unital only on certain subcategories
- $A_{\beta \ast \gamma} B \subseteq A'' \rho_\beta \ast \rho_\gamma B''$ (fiber product of v.N.-alg.)
## Compact Hopf $C^*$-bimodules I

### Axioms:
- $B$ unital $C^*$-algebra with faithful KMS-state $\mu$
- $A$ unital $C^*$-algebra with faithful KMS-states $\nu, \nu^{-1}$
- $\rho: B \to A, \quad \sigma: B^{\text{op}} \to A$ unital embeddings
- $\phi: A \to \rho(B) \cong B, \quad \psi: A \to \sigma(B^{\text{op}}) \cong B^{\text{op}}$ faithful cond. expectations s.t. $\nu = \mu \circ \phi$, $\nu^{-1} = \mu^{\text{op}} \circ \psi$
- $\exists \, \delta = \frac{d\nu}{d\nu^{-1}} \in A \cap \rho(B)' \cap \sigma(B^{\text{op}})'$

### Lemma:
- $H := H_\nu \cong H_{\nu^{-1}}$ is a Hilbert $C^*$-module over $(\mu, \mu^{\text{op}}, \mu^{\text{op}}, \mu)$ w.r.t. $\alpha := \overline{\Lambda_\phi(A)}, \beta := \overline{\Lambda_\psi(A)}$, $\beta := \overline{\Lambda_{\phi^{\text{op}}}(A^{\text{op}})}, \alpha := \overline{\Lambda_{\psi^{\text{op}}}(A^{\text{op}})}$
- $(H, \alpha, \beta, A)$ is a $C^*$-algebra over $(\mu, \mu^{\text{op}})$
Compact Hopf $C^*$-bimodules II

Axioms:

- $\Delta$ s.t. $(H, \alpha, \beta, A, \Delta)$ is a Hopf $C^*$-bimodule
- $\phi$ is left- and $\psi$ right-invariant w.r.t. $\Delta$
- $\Delta(\delta) = \delta_\alpha \otimes \beta \delta$
- $R$ anti-automorphism of $A$ that flips $\rho, \sigma$ and $\phi, \psi$
- *strong invariance* relating $\phi, \psi$ and $R$

Theorem:

- $\exists$ regular $C^*$-pseudo-multiplicative unitary

\[
H \otimes_\alpha H \cong \hat{\beta} \otimes_\rho H \xrightarrow{V} \alpha \otimes \rho_\beta H \cong H \otimes_\beta H
\]

- $\Lambda(\psi(a) \otimes_\rho \omega) \mapsto \Delta(a)(\Lambda_{\psi_{\text{op}}}(1^{\text{op}}) \otimes_\rho \omega)$

- $\exists$ anti-unitary $I: H \to H$, $\Lambda_{\nu^{-1}}(a) \mapsto \Lambda_{\nu}(R(a)^*)$

- $(V, \lambda^{i/4}IJ_\phi)$ is a weak $C^*$-pseudo-Kac system
C*-pseudo-multiplicative unitaries I

Def.: C*-pseudo-multiplicative unitary: = \((H, \hat{\beta}, \alpha, \beta, V)\), where

- \((H, \hat{\beta}, \alpha, \beta)\) is a Hilbert C*-trimodule/(\(\mu^{\text{op}}, \mu, \mu^{\text{op}}\))
- \(V: H_{\hat{\beta}} \otimes_\alpha H \rightarrow H_{\alpha} \otimes_\beta H\) is a unitary and a morphism of Hilbert C*-modules over \((\mu, \mu^{\text{op}}, \mu, \mu^{\text{op}})\)
- \(V_{12} V_{13} V_{23} = V_{23} V_{12}\) (\(V_{ij}: \text{op. on } H? \otimes? H? \otimes? H\))

Prop.: \((H, \rho_{\hat{\beta}}, \rho_\alpha, \rho_\beta, V)\) is a pseudo-multiplicative unitary in the sense of [Vallin, Enock, Lesieur]

Ex. ▶ cont. bundles of multiplicative unitaries [Blanchard]
▶ locally compact groupoids
▶ tracial conditional expectations
▶ external tensor product, direct sum, restriction, . . .
**C*-pseudo-multiplicative unitaries II**

\[
\begin{array}{cccc}
H & r(\alpha) & \rightarrow & H_\beta \otimes H \\
\downarrow A(V) & & & \downarrow \text{V} \\
H & r(\beta)^* & \leftarrow & H_\alpha \otimes H \\
\end{array}
\]

**Def.:** $V$ regular $\iff [I(\alpha)^* V r(\alpha)] = [\alpha \alpha^*]$

**Thm.:** $V$ regular $\Rightarrow (H, \alpha, \beta, A(V), \Delta)$ and $(H, \widehat{\beta}, \alpha, \widehat{A}(V), \widehat{\Delta})$

are Hopf $C^*$-bimodules, where $\Delta, \widehat{\Delta}$ are given by

$\Delta: a \mapsto V(1_{\beta \otimes \alpha} a) V^*$, $\widehat{\Delta}: \widehat{a} \mapsto V^*(\widehat{a} \otimes \beta 1) V$

**Ex.:** If $V$ is the unitary of $(B, \mu, A, \rho, \sigma, \phi, \psi, \delta, \Delta, R)$, then

$A(V) = A$, $(H, \widehat{\beta}, \alpha, \widehat{A}(V), \widehat{\Delta})$ is the *dual* Hopf $C^*$-bimod.

**Rmk.:** representations/corepresentations of $V$ lead to *universal* $C^*$-algebras $A_u(V), \widehat{A}_u(V)$
C*-pseudo-Kac systems and duality for coactions

Def.: Coaction of a Hopf C*-bimodule \((H, \alpha, \beta, A, \Delta)\) over \(\mu := \)
\begin{itemize}
  \item C*-algebra \(C^\gamma_K\) over \(\mu\) with
  \item \(\delta \in \text{Mor}(C^\gamma_K, C^\gamma_K \ast A^\alpha_{H, \beta})\) s.t. \((\delta \ast \text{id}) \circ \delta = (\text{id} \ast \Delta) \circ \delta\)
\end{itemize}

Ex.: actions on and Fell bundles of l.c. Hausdorff groupoids

Def.: C*-pseudo-Kac system := C*-pseudo-multiplicative unitary \((H, \widehat{\beta}, \alpha, \beta, V)\) and unitary \(U: H \rightarrow H\) s.t. . . .

Def./Prop.: \(\exists\) “red. crossed product/dual coaction” functors
\[
\left\{\text{coactions of } (A(V)_{H, H}^{\alpha, \beta}, \Delta)\right\} \overset{\cong}{\rightleftarrows} \left\{\text{coactions of } (\widehat{A}(V)_{H, H}^{\beta, \alpha}, \widehat{\Delta})\right\}
\]

Thm.: Every well-behaved coaction is equivariantly Morita equivalent to its bidual