HEIGHTS, REGULATORS AND SCHINZEL’S DETERMINANT INEQUALITY

SHABNAM AKHTARI AND JEFFREY D. VAALER

Abstract. We prove inequalities that compare the size of an $S$-regulator with a product of heights of multiplicatively independent $S$-units. Our upper bound for the $S$-regulator follows from a general upper bound for the determinant of a real matrix proved by Schinzel. The lower bound for the $S$-regulator follows from Minkowski’s theorem on successive minima and a volume formula proved by Meyer and Pajor. We establish similar upper bounds for the relative regulator of an extension $l/k$ of number fields.

1. Introduction

Let $k$ be an algebraic number field, $k^\times$ its multiplicative group of nonzero elements, and $h : k^\times \to [0, \infty)$ the absolute, logarithmic, Weil height. If $\alpha$ belongs to $k^\times$ and $\zeta$ is a root of unity in $k^\times$, then the identity $h(\zeta \alpha) = h(\alpha)$ is well known. It follows that the height $h$ is constant on cosets of the quotient group

$$G_k = k^\times / \text{Tor}(k^\times).$$

Therefore the height is well defined as a map $h : G_k \to [0, \infty)$.

Let $S$ be a finite set of places of $k$ such that $S$ contains all the archimedean places. Then

$$O_S = \{ \gamma \in k : |\gamma|_v \leq 1 \text{ for all places } v \notin S \}$$

is the ring of $S$-integers in $k$, and

$$O_S^\times = \{ \gamma \in k^\times : |\gamma|_v = 1 \text{ for all places } v \notin S \}$$

is the multiplicative group of $S$-units in the ring $O_S$. We write

$$\text{Tor}(O_S^\times) = \text{Tor}(k^\times)$$

for the torsion subgroup of $O_S^\times$, which is also the torsion subgroup of the multiplicative group $k^\times$. As is well known, (1.2) is a finite, cyclic group of even order, and

$$\Omega_S(k) = O_S^\times / \text{Tor}(O_S^\times) \subseteq G_k$$

is a free abelian group of finite rank $r$, where $|S| = r + 1$.

In this paper we establish simple inequalities between the $S$-regulator $\text{Reg}_S(k)$ and products of the form

$$\prod_{j=1}^r ([k : \mathbb{Q}] h(\alpha_j)).$$

2000 Mathematics Subject Classification. 11J25, 11R04, 46B04.

Key words and phrases. $S$-regulator, Weil height.

This research was supported by NSA grant, H98230-12-1-0254.
where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are multiplicatively independent elements in the group $U_S(k)$.

**Theorem 1.1.** Let the multiplicative group of $S$-units $O_S^\times$ have positive rank $r$, and let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be multiplicatively independent elements in the free group $U_S(k)$. If $A \subseteq U_S(k)$ is the multiplicative subgroup generated by $\alpha_1, \alpha_2, \ldots, \alpha_r$, then

$$\text{Reg}_S(k)[U_S(k) : A] \leq \prod_{j=1}^r ([k : \mathbb{Q}] h(\alpha_j)).$$

A special case of (1.4) occurs when $S$ is the collection of all archimedean places of $k$. We write $O_k^\times$ for the ring of algebraic integers in $k$, and $O_k^\times$ for the multiplicative group of units in $O_k$. If $k$ is not $\mathbb{Q}$, and $k$ is not an imaginary quadratic extension of $\mathbb{Q}$, then the quotient group

$$\mathcal{U}(k) = O_k^\times / \text{Tor}(O_k^\times) \subseteq G_k$$

is a free abelian group of positive rank $r$, where $r + 1$ is the number of archimedean places of $k$. It is known from work of Remak [22], [23], and Zimmert [28], that the regulator Reg($k$) is bounded from below by an absolute constant. Further, Friedman [12] has shown that Reg($k$) takes its minimum value at the unique number field $k_0$ having degree 6 over $\mathbb{Q}$, and having discriminant equal to $-10051$. Thus by Friedman’s result we have

$$0.2052 \cdots = \text{Reg}(k_0) \leq \text{Reg}(k)$$

for all algebraic number fields $k$. Combining the inequalities (1.4) and (1.5) leads to the following explicit lower bound.

**Corollary 1.1.** Assume that $k$ is not $\mathbb{Q}$, and $k$ is not an imaginary quadratic extension of $\mathbb{Q}$, so that $\mathcal{U}(k)$ has positive rank $r$. Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be multiplicatively independent elements in $\mathcal{U}(k)$. If $A \subseteq \mathcal{U}(k)$ is the subgroup generated by $\alpha_1, \alpha_2, \ldots, \alpha_r$, then

$$\text{(1.6)} \quad (0.2052 \cdots)[\mathcal{U}(k) : A] \leq \prod_{j=1}^r ([k : \mathbb{Q}] h(\alpha_j)).$$

Let $k$ be an algebraic number field such that the unit group $O_k^\times$ has positive rank $r$. The inequality (1.6) implies that each collection $\alpha_1, \alpha_2, \ldots, \alpha_r$ of multiplicatively independent units must contain a unit, say $\alpha_1$, that satisfies

$$\text{(1.7)} \quad (0.2052 \cdots) \leq [k : \mathbb{Q}] h(\alpha_1).$$

A result of this sort was proposed by Bertrand [5, comment (iii), p. 210], who observed that it would follow from an unproved hypothesis related to Lehmer’s problem.

In a well known paper Lehmer [16] posed the problem, reformulated in the language and notation developed here, of deciding if there exists a positive constant $c_0$ such that the inequality

$$\text{(1.8)} \quad c_0 \leq [k : \mathbb{Q}] h(\gamma)$$

holds for all elements $\gamma$ in $k^\times$, which are not in $\text{Tor}(k^\times)$. If $\gamma \neq 0$ is not a unit, then it is easy to show that

$$\log 2 \leq [k : \mathbb{Q}] h(\gamma).$$
Hence the proposed lower bound (1.8) is of interest for non-torsion elements \( \gamma \) in the unit group \( \mathcal{O}_k^\times \), or equivalently, for a nontrivial coset representative \( \gamma \) in \( \mathcal{U}(k) \). The inequality (1.6) provides a solution to a form of Lehmer’s problem on average. Further information about Lehmer’s problem is given in [6 section 1.6.15] and in [25].

In section 3 we give an analogous upper bound for the relative regulator associated to an extension \( l/k \) of algebraic number fields.

We will show that the inequality (1.4) is sharp up to a constant that depends only on the rank \( r \) of the group \( \mathcal{U}_S(k) \), but \textit{not} on the underlying field \( k \). Related results have been proved by Brindza [7], Bugeaud and Győry [8], Hajdu [14], and Matveev [18, 19]. More general inequalities that apply to arbitrary finitely generated subgroups of \( \mathbb{Q}_k^\times \) were obtained in [26, Theorem 1 and Theorem 2]. The inequality (1.4) that we prove here is sharper but less general, as it applies only to subgroups of a group of \( S \)-units having maximum rank.

\textbf{Theorem 1.2.} Let the multiplicative group of \( S \)-units \( \mathcal{O}_S^\times \) have positive rank \( r \), and let \( \mathfrak{A} \subseteq \mathcal{U}_S(k) \) be a subgroup of rank \( r \). Then there exist multiplicatively independent elements \( \beta_1, \beta_2, \ldots, \beta_r \) in \( \mathfrak{A} \), such that

\begin{equation}
\prod_{j=1}^{r} (|k : \mathbb{Q}| h(\beta_j)) \leq \frac{2^r (r!)^3}{(2r)!} \text{Reg}_S(k)[\mathcal{U}_S(k) : \mathfrak{A}] .
\end{equation}

We note that if \( r = 2 \) then (1.3) and (1.9) imply that the multiplicatively independent elements \( \beta_1 \) and \( \beta_2 \) contained in the subgroup \( \mathfrak{A} \subseteq \mathcal{U}_S(k) \) satisfy the inequality

\[ \text{Reg}_S(k)[\mathcal{U}_S(k) : \mathfrak{A}] \leq \big( |k : \mathbb{Q}| h(\beta_1) \big) \big( |k : \mathbb{Q}| h(\beta_2) \big) \leq \frac{4}{3} \text{Reg}_S(k)[\mathcal{U}_S(k) : \mathfrak{A}] . \]

It follows that \( \beta_1 \) and \( \beta_2 \) form a basis for the group \( \mathfrak{A} \). More generally, by using a well known lemma proved by Mahler [17] and Weyl [27] (see also [9, Chapter V, Lemma 8]), we obtain the following bound on the product of the heights of a basis for the subgroup \( \mathfrak{A} \subseteq \mathcal{U}_S(k) \).

\textbf{Corollary 1.2.} Let the multiplicative group of \( S \)-units \( \mathcal{O}_S^\times \) have positive rank \( r \), and let \( \mathfrak{A} \subseteq \mathcal{U}_S(k) \) be a subgroup of rank \( r \). Then there exists a basis \( \gamma_1, \gamma_2, \ldots, \gamma_r \) for the free group \( \mathfrak{A} \), such that

\begin{equation}
\prod_{j=1}^{r} (|k : \mathbb{Q}| h(\gamma_j)) \leq \frac{2^r (r!)^4}{(2r)!} \text{Reg}_S(k)[\mathcal{U}_S(k) : \mathfrak{A}] .
\end{equation}

\textbf{2. Preliminary results}

At each place \( v \) of \( k \) we write \( k_v \) for the completion of \( k \) at \( v \), so that \( k_v \) is a local field. We select two absolute values \( || \cdot ||_v \) and \( | \cdot |_v \) from the place \( v \). The absolute value \( || \cdot ||_v \) extends the usual archimedean or non-archimedean absolute value on the subfield \( \mathbb{Q} \). Then \( | \cdot |_v \) must be a power of \( || \cdot ||_v \), and we set

\begin{equation}
| \cdot |_v = || \cdot ||_{d_v/d} ,
\end{equation}

where \( d_v = [k_v : \mathbb{Q}_v] \) is the local degree of the extension, and \( d = [k : \mathbb{Q}] \) is the global degree. With these normalizations the height of an algebraic number \( \alpha \neq 0 \) that belongs to \( k \) is given by

\begin{equation}
h(\alpha) = \sum_v \log^+ |\alpha|_v = \frac{1}{2} \sum_v |\log |\alpha|_v| ,
\end{equation}
where \( \log^+ x = \max(0, \log x) \) for \( x > 0 \). Each sum in (2.2) is over the set of all places \( v \) of \( k \), and the equality between the two sums follows from the product formula. Then \( h(\alpha) \) depends on the algebraic number \( \alpha \neq 0 \), but it does not depend on the number field \( k \) that contains \( \alpha \). We have already noted that the height is well defined as a map

\[
h : \mathcal{G}_k \to [0, \infty).
\]

Elementary properties of the height show that the map \((\alpha, \beta) \mapsto h(\alpha \beta^{-1})\) defines a metric on the group \( \mathcal{G}_k \).

Let \( \eta_1, \eta_2, \ldots, \eta_r \) be multiplicatively independent elements in \( \mathcal{U}_S(k) \) that form a basis for \( \mathcal{U}_S(k) \) as a free abelian group of rank \( r \). Then let

\[
M = \left( d_v \log \|\eta_j\|_v \right)
\]
denote the \((r + 1) \times r\) real matrix, where \( v \in S \) indexes rows and \( j = 1, 2, \ldots, r \) indexes columns. At each place \( \hat{v} \) in \( S \) we write

(2.3) \[
M(\hat{v}) = \left( d_v \log \|\eta_j\|_v \right)
\]

for the \( r \times r \) submatrix of \( M \) obtained by removing the row indexed by the place \( \hat{v} \). Then the \( S \)-regulator of \( \mathcal{O}_S^X \) (or of \( \mathcal{U}_S(k) \)) is the positive number

(2.4) \[
\text{Reg}_S(k) = \left| \det M(\hat{v}) \right|,
\]

which is independent of the choice of \( \hat{v} \) in \( S \). Using an inequality proved by A. Schinzel [24] that bounds the determinant of a real matrix, we will prove that

(2.5) \[
\text{Reg}_S(k) \leq \prod_{j=1}^{r} (|k : \mathbb{Q}| h(\eta_j)).
\]

If the better known inequality of Hadamard is used to estimate the determinant that defines the \( S \)-regulator on the right of (2.4), we obtain an upper bound that is larger than (2.5) by a factor of \( 2^r \).

Assume more generally that \( \alpha_1, \alpha_2, \ldots, \alpha_r \) are multiplicatively independent elements in \( \mathcal{U}_S(k) \), but they do not necessarily form a basis for the free group \( \mathcal{U}_S(k) \). It follows that there exists an \( r \times r \), nonsingular matrix \( B = (b_{ij}) \) with entries in \( \mathbb{Z} \), such that

(2.6) \[
\log \|\alpha_j\|_v = \sum_{i=1}^{r} b_{ij} \log \|\eta_i\|_v
\]

for each place \( v \) in \( S \) and for each \( j = 1, 2, \ldots, r \). Alternatively, (2.6) can be written as the matrix identity

(2.7) \[
(d_v \log \|\alpha_j\|_v) = (d_v \log \|\eta_j\|_v) B.
\]

If

(2.8) \[
\mathfrak{A} = (\alpha_1, \alpha_2, \ldots, \alpha_r) \subseteq \mathcal{U}_S(k)
\]

is the multiplicative subgroup generated by \( \alpha_1, \alpha_2, \ldots, \alpha_r \), we find that the index of this group is given by

(2.9) \[
[\mathcal{U}_S(k) : \mathfrak{A}] = | \det B |.
\]

This will lead to the more general inequality (1.4).
3. Relative regulators

Throughout this section we suppose that $k$ and $l$ are algebraic number fields with $k \subseteq l$. We write $r(k)$ for the rank of the unit group $\mathcal{O}_k^\times$, and $r(l)$ for the rank of the unit group $\mathcal{O}_l^\times$. Then $k$ has $r(k) + 1$ archimedean places, and $l$ has $r(l) + 1$ archimedean places. In general we have $r(k) \leq r(l)$, and we recall (see [21, Proposition 3.20]) that $r(k) = r(l)$ if and only if $l$ is a CM-field, and $k$ is the maximal totally real subfield of $l$.

The norm is a homomorphism of multiplicative groups

$$\operatorname{Norm}_{l/k} : l^\times \to k^\times.$$  

If $v$ is a place of $k$, then each element $\alpha$ in $l^\times$ satisfies the identity

$$(3.1) \quad \left[l : k\right] \sum_{w|v} \log |\alpha|_w = \log |\operatorname{Norm}_{l/k}(\alpha)|_v,$$

where the absolute values $|\ |_v$ and $|\ |_w$ are normalized as in (2.1). It follows from (3.1) that the norm, restricted to the subgroup $\mathcal{O}_l^\times$ of units, is a homomorphism

$$\operatorname{Norm}_{l/k} : \mathcal{O}_l^\times \to \mathcal{O}_k^\times,$$

and the norm, restricted to the torsion subgroup in $\mathcal{O}_l^\times$, is also a homomorphism

$$\operatorname{Norm}_{l/k} : \operatorname{Tor}(\mathcal{O}_l^\times) \to \operatorname{Tor}(\mathcal{O}_k^\times).$$

Therefore we get a well defined homomorphism, which we write as

$$\operatorname{norm}_{l/k} : \mathcal{O}_l^\times / \operatorname{Tor}(\mathcal{O}_l^\times) \to \mathcal{O}_k^\times / \operatorname{Tor}(\mathcal{O}_k^\times),$$

and define by

$$\operatorname{norm}_{l/k}(\alpha \operatorname{Tor}(\mathcal{O}_l^\times)) = \operatorname{Norm}_{l/k}(\alpha) \operatorname{Tor}(\mathcal{O}_k^\times).$$

However, to simplify notation we write

$$F_k = \mathcal{O}_k^\times / \operatorname{Tor}(\mathcal{O}_k^\times), \quad \text{and} \quad F_l = \mathcal{O}_l^\times / \operatorname{Tor}(\mathcal{O}_l^\times),$$

and we write the elements of the quotient groups $F_k$ and $F_l$ as coset representatives rather than cosets. Obviously $F_k$ and $F_l$ are free abelian groups of rank $r(k)$ and $r(l)$, respectively.

Following Costa and Friedman [10], the subgroup of relative units in $\mathcal{O}_l^\times$ is defined by

$$\{ \alpha \in \mathcal{O}_l^\times : \operatorname{Norm}_{l/k}(\alpha) \in \operatorname{Tor}(\mathcal{O}_k^\times) \}.$$  

Alternatively, we work in the free group $F_l$, where the image of the subgroup of relative units is the kernel of the homomorphism $\operatorname{norm}_{l/k}$. That is, we define the subgroup of relative units in $F_l$ to be the subgroup

$$(3.2) \quad E_{l/k} = \{ \alpha \in F_l : \operatorname{norm}_{l/k}(\alpha) = 1 \}.$$  

We also write

$$I_{l/k} = \{ \operatorname{norm}_{l/k}(\alpha) : \alpha \in F_l \} \subseteq F_k$$

for the image of the homomorphism $\operatorname{norm}_{l/k}$. If $\beta$ in $F_l$ represents a coset in the subgroup $F_k$, then we have

$$\operatorname{norm}_{l/k}(\beta) = \tilde{\beta}^{[l:k]}.$$  

Therefore the image $I_{l/k} \subseteq F_k$ is a subgroup of rank $r(k)$, and the index satisfies

$$(3.3) \quad [F_k : I_{l/k}] < \infty.$$
It follows that $E_{l/k} \subseteq F_l$ is a subgroup of rank $r(l/k) = r(l) - r(k)$, and we restrict our attention here to extensions $l/k$ such that $r(l/k)$ is positive.

Let $\eta_l, \eta_2, \ldots, \eta_r(l/k)$ be a collection of multiplicatively independent relative units that form a basis for the subgroup $E_{l/k}$. At each archimedean place $v$ of $k$ we select a place $\tilde{w}_v$ of $l$ such that $\tilde{w}_v|v$. Then we define an $r(l/k) \times r(l/k)$ real matrix
\[
M_{l/k} = ([l_w : Q_w] \log ||\eta_j||_w),
\]
where $w$ is an archimedean place of $l$, but $w \neq \tilde{w}_v$ for each $v|\infty$, $w$ indexes rows, and $j = 1, 2, \ldots, r(l/k)$ indexes columns. We write $l_w$ for the completion of $l$ at the place $w$, $Q_w$ for the completion of $Q$ at the place $w$, and we write $[l_w : Q_w]$ for the local degree. Of course $Q_w$ is isomorphic to $\mathbb{R}$ in the situation considered here.

As in [10], we define the relative regulator of the extension $l/k$ to be the positive number
\[
\text{Reg}(E_{l/k}) = |\det M_{l/k}|.
\]

It follows, as in the proof of [10] Theorem 1 (see also [11]), that the value of the determinant on the right of (3.5) does not depend on the choice of places $\tilde{w}_v$ for each archimedean place $v$ of $k$.

**Theorem 3.1.** Let $k \subseteq l$ be algebraic number fields such that the group $E_{l/k}$ of relative units has positive rank $r(l/k) = r(l) - r(k)$. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r(l/k)$ be a collection of multiplicatively independent relative units in $E_{l/k}$. If $\mathcal{E} \subseteq E_{l/k}$ is the multiplicative subgroup generated by $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r(l/k)$, then
\[
\text{Reg}(E_{l/k})[E_{l/k} : \mathcal{E}] \leq \prod_{j=1}^{r(l/k)} ([l : Q] h(\varepsilon_j)).
\]

The relative regulator can also be expressed as a ratio of the (ordinary) regulators $\text{Reg}(k)$ and $\text{Reg}(l)$ by using the basic identity
\[
[F_k : I_{l/k}] \text{Reg}(k) \text{Reg}(E_{l/k}) = \text{Reg}(l),
\]
which was established in [10] Theorem 1. A slightly different definition for a relative regulator was considered by Bergé and Martinet in [2], [3], and [4]. We have used the definition proposed by Costa and Friedman in [10] and [11], as it leads more naturally to the inequality (3.6). Further lower bounds for the product on the right of (3.6) follow from inequalities for the relative regulator obtained by Friedman and Skoruppa [13].

4. Schinzel’s norm

For a real number $x$ we write
\[
x^+ = \max\{0, x\}, \quad \text{and} \quad x^- = \max\{0, -x\},
\]
so that $x = x^+ - x^-$ and $|x| = x^+ + x^-$. If $x = (x_n)$ is a (column) vector in $\mathbb{R}^N$ we define
\[
\delta : \mathbb{R}^N \to [0, \infty)
\]
by
\[
\delta(x) = \max \left\{ \sum_{m=1}^{N} x_m^+, \sum_{n=1}^{N} x_n^- \right\}.
\]

The following inequality was proved by A. Schinzel [23].
Theorem 4.1. If \( x_1, x_2, \ldots, x_N \), are column vectors in \( \mathbb{R}^N \), then
\[
|\det(x_1 x_2 \cdots x_N)| \leq \delta(x_1) \delta(x_2) \cdots \delta(x_N).
\]

An upper bound that is slightly sharper than (4.2) was established by C. R. Johnson and M. Newman [15]. However, the bound obtained by Johnson and Newman does not lead to a significant improvement in the results we obtain here.

If \( a \) and \( b \) are nonnegative real numbers then
\[
2 \max \{a, b\} = |a + b| + |a - b|.
\]

This leads to the identity
\[
\delta(x) = \max \left\{ \sum_{m=1}^{N} x_m^+, \sum_{n=1}^{N} x_n^- \right\} = \frac{1}{2} \left| \sum_{n=1}^{N} x_n \right| + \frac{1}{2} \sum_{n=1}^{N} |x_n|.
\]

It follows easily from (4.3) that \( x \mapsto \delta(x) \) is a continuous, symmetric distance function, or norm, defined on \( \mathbb{R}^N \). Let
\[
K_N = \{x \in \mathbb{R}^N : \delta(x) \leq 1\}
\]
be the unit ball associated to the norm \( \delta \). Then \( K_N \) is a compact, convex, symmetric subset of \( \mathbb{R}^N \) having a nonempty interior.

Lemma 4.1. Let \( \delta : \mathbb{R}^N \to [0, \infty) \) be the continuous distance function defined by (4.3), and let \( K_N \) be the unit ball defined by (4.4). Then we have
\[
\text{Vol}_N(K_N) = \frac{(2N)!}{(N!)^3}.
\]

Proof. We write \( J \) for the \((N+1) \times N\) matrix
\[
J = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-1 & -1 & -1 & \cdots & -1
\end{pmatrix}.
\]

Then it is obvious that \( J \) has rank \( N \). Let
\[
D_N = \{y \in \mathbb{R}^{N+1} : y_0 + y_1 + y_2 + \cdots + y_N = 0\},
\]
so that \( D_N \) is the \( N \)-dimensional subspace of \( \mathbb{R}^{N+1} \) spanned by the columns of \( J \). Further, let
\[
B_{N+1} = \{y \in \mathbb{R}^{N+1} : \|y\|_1 = |y_0| + |y_1| + |y_2| + \cdots + |y_N| \leq 1\}
\]
denote the unit ball in \( \mathbb{R}^{N+1} \) with respect to the \( \| \cdot \|_1 \)-norm. If \( x \) is a (column) vector in \( \mathbb{R}^N \), we find that
\[
\delta(x) = \|Jx\|_1,
\]
and therefore
\[
K_N = \{x \in \mathbb{R}^N : \|Jx\|_1 \leq 1\}.
\]
It follows that
\[
\text{Vol}_N(K_N) = \int_{\mathbb{R}^N} \chi_{B_{N+1}}(Jx) \, dx \\
= |\det U| \int_{\mathbb{R}^N} \chi_{B_{N+1}}(JUx) \, dx,
\]
where \( y \mapsto \chi_{B_{N+1}}(y) \) is the characteristic function of the subset \( B_{N+1} \), and \( U \) is an arbitrary \( N \times N \) nonsingular real matrix. We select \( U \) so that the columns of the matrix \( JU \) form an orthonormal basis for the subspace \( D_N \). With this choice of \( U \) we find that
\[
\int_{\mathbb{R}^N} \chi_{B_{N+1}}(JUx) \, dx = \text{Vol}_N(D_N \cap B_{N+1}) = \frac{\sqrt{N+1}(2N)!}{2^N(N!)^3},
\]
where the second equality on the right of (4.7) follows from a result of Meyer and Pajor [20, Proposition II.7]. Because the columns of \( JU \) are orthonormal, we get
\[
\text{Vol}_N(K_N) = \frac{\sqrt{N+1}(2N)!}{2^N(N!)^3}.
\]
Next we suppose that
\[
A = (a_1 \ a_2 \ \cdots \ a_N)
\]
is an \( N \times N \) nonsingular matrix with columns \( a_1, a_2, \ldots, a_N \). Obviously the columns of \( A \) form a basis for the lattice
\[
\mathcal{L} = \{ A\xi : \xi \in \mathbb{Z}^N \} \subseteq \mathbb{R}^N.
\]
Then by Schinzel’s inequality we have
\[
|\det A| \leq \prod_{n=1}^{N} \delta(a_n).
\]
Using the geometry of numbers, we will establish the existence of linearly independent points \( \ell_1, \ell_2, \ldots, \ell_N \) in the lattice \( \mathcal{L} \), for which the product
\[
\prod_{n=1}^{N} \delta(\ell_n)
\]
is not much larger than $|\det A|$. An explicit bound on such a product follows immediately from Minkowski’s theorem on successive minima and our formula (4.5) for the volume of $K_N$.

**Theorem 4.2.** Let $\mathcal{L} \subseteq \mathbb{R}^N$ be the lattice defined by (4.10). Then there exist linearly independent points $\ell_1, \ell_2, \ldots, \ell_N$ in $\mathcal{L}$ such that

\[ \prod_{n=1}^N \delta(\ell_n) \leq \frac{2^N(N!)^3}{(2N)!} |\det A|. \]

**Proof.** Let

\[ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N < \infty \]

be the successive minima of the lattice $\mathcal{L}$ with respect to the convex symmetric set $K_N$. Then there exist linearly independent points $\ell_1, \ell_2, \ldots, \ell_N$ in $\mathcal{L}$ such that

\[ \delta(\ell_n) = \lambda_n \quad \text{for each } n = 1, 2, \ldots, N. \]

By Minkowski’s theorem on successive minima (see [9, section VIII.4.3]) we have the inequality

\[ \text{Vol}_N(K_N) \lambda_1 \lambda_2 \cdots \lambda_N \leq 2^N |\det A|. \]

From Lemma 4.1 we get the bound

\[ \prod_{n=1}^N \delta(\ell_n) \leq \frac{2^N(N!)^3}{(2N)!} |\det A|, \]

and this proves the theorem. \[ \square \]

5. **Proof of Theorem 1.1 and Theorem 1.2**

We require the following lemma, which connects the Schinzel norm (4.1) with the Weil height.

**Lemma 5.1.** Let $\hat{v}$ be a place of the algebraic number field $k$, and let $\alpha \neq 0$ be an element of $k^\times$. Then we have

\[ \max \left\{ \sum_{v \neq \hat{v}} \log^+ |\alpha|_v, \sum_{v \neq \hat{v}} \log^- |\alpha|_v \right\} = h(\alpha). \]

**Proof.** The product formula implies that

\[ h(\alpha) = \sum_v \log^+ |\alpha|_v = \sum_v \log^- |\alpha|_v. \]

If $\log |\alpha|_{\hat{v}} \leq 0$ then

\[ \max \left\{ \sum_{v \neq \hat{v}} \log^+ |\alpha|_v, \sum_{v \neq \hat{v}} \log^- |\alpha|_v \right\} = \sum_v \log^+ |\alpha|_v = h(\alpha). \]

On the other hand, if $\log |\alpha|_{\hat{v}} \geq 0$ then

\[ \max \left\{ \sum_{v \neq \hat{v}} \log^+ |\alpha|_v, \sum_{v \neq \hat{v}} \log^- |\alpha|_v \right\} = \sum_v \log^- |\alpha|_v = h(\alpha). \]

This proves the lemma. \[ \square \]
We now prove Theorem 1.1. First we combine (2.3), (2.4), (2.7), and (2.9), and obtain the identity
\begin{equation}
\text{Reg}_S(k)[\mathcal{U}_S(k) : \mathfrak{A}] = |k : \mathbb{Q}|^r |\det(\log |\alpha_j|_v)|,
\end{equation}
where \( v \) in \( S \setminus \{\widehat{v}\} \) indexes rows, and \( j = 1, 2, \ldots, r \) indexes columns, in the matrix on the right of (5.2). We estimate the determinant in (5.2) by applying Schinzel’s inequality (4.2). Using (4.1) and (5.1) we get
\begin{equation}
|\det(\log |\alpha_j|_v)| \leq \prod_{j=1}^r \max \left\{ \sum_{v \neq \widehat{v}} \log^+ |\alpha_j|_v, \sum_{v \neq \widehat{v}} \log^- |\alpha_j|_v \right\}
\end{equation}
(5.3)
\[ = \prod_{j=1}^r h(\alpha_j). \]
The inequality (5.3) in the statement of Theorem 1.1 follows from (5.2) and (5.3).

Next we prove Theorem 1.2. Let \( \eta_1, \eta_2, \ldots, \eta_r \) be multiplicatively independent elements in \( \mathcal{U}_S(k) \) that form a basis for \( \mathcal{U}_S(k) \) as a free abelian group of rank \( r \). Let \( \widehat{v} \) be a place of \( k \) contained in \( S \), and
\[ M^{(\widehat{v})} = (d_v \log \|\eta_j\|_v) \]
the \( r \times r \) real matrix as defined in (2.3). By hypothesis \( \mathfrak{A} \subseteq \mathcal{U}_S(k) \) is a subgroup of rank \( r \). Let \( \alpha_1, \alpha_2, \ldots, \alpha_r \) be multiplicatively independent elements in \( \mathfrak{A} \) that form a basis for \( \mathfrak{A} \). As in (2.6), there exists an \( r \times r \) nonsingular matrix \( B = (b_{ij}) \) with entries in \( \mathbb{Z} \), such that
\begin{equation}
\log \|\alpha_j\|_v = \sum_{i=1}^r b_{ij} \log \|\eta_i\|_v
\end{equation}
(5.4)
for each place \( v \) in \( S \) and for each \( j = 1, 2, \ldots, r \). Alternatively, if we define the \( r \times r \) real matrix
\[ A^{(\widehat{v})} = (d_v \log \|\alpha_j\|_v), \]
where \( v \) in \( S \setminus \{\widehat{v}\} \) indexes rows and \( j = 1, 2, \ldots, r \) indexes columns, then (5.4) is equivalent to the matrix identity
\begin{equation}
A^{(\widehat{v})} = M^{(\widehat{v})} B.
\end{equation}
(5.5)
We use the nonsingular \( r \times r \) real matrix \( A^{(\widehat{v})} \) to define a lattice \( \mathcal{L}^{(\widehat{v})} \subseteq \mathbb{R}^r \) by
\[ \mathcal{L}^{(\widehat{v})} = \{ A^{(\widehat{v})} \xi : \xi \in \mathbb{Z}^r \}. \]
Then (2.1), (2.9) and (5.5), imply that
\begin{equation}
\text{Reg}_S(k)[\mathcal{U}_S(k) : \mathfrak{A}] = |\det M^{(\widehat{v})}| |\det B| = |\det A^{(\widehat{v})}|,
\end{equation}
(5.6)
which is independent of the choice of \( \widehat{v} \) in \( S \), and is also the determinant of the lattice \( \mathcal{L}^{(\widehat{v})} \). By Theorem 1.2 and (5.6), there exist linearly independent vectors \( \ell_1, \ell_2, \ldots, \ell_r \) in \( \mathcal{L}^{(\widehat{v})} \) such that
\begin{equation}
\prod_{j=1}^r \delta(\ell_j) \leq \frac{2^r (r!)^3}{(2r)!} \text{Reg}_S(k)[\mathcal{U}_S(k) : \mathfrak{A}].
\end{equation}
(5.7)
As each (column) vector \( \ell_j \) belongs to the lattice \( L^{(v)} \), it has rows indexed by the places \( v \) in \( S \setminus \{ \infty \} \). Thus \( \ell_j \) can be written as

\[
\ell_j = \left( d_v \sum_{i=1}^{r} f_{ij} \log \| \alpha_i \|_v \right) = \left( d_v \log \| \beta_j \|_v \right),
\]

where \( F = (f_{ij}) \) is an \( r \times r \) nonsingular matrix with entries in \( \mathbb{Z} \), and \( \beta_1, \beta_2, \ldots, \beta_r \) are multiplicatively independent elements in the group \( \mathfrak{A} \). By Lemma 5.1 we have

\[
\begin{align*}
\delta(\ell_j) &= \max \left\{ \sum_{v \neq \infty} d_v \log^+ \| \beta_j \|_v, \sum_{v \neq \infty} d_v \log^{-} \| \beta_j \|_v \right\} \\
&= [k : \mathbb{Q}] \max \left\{ \sum_{v \neq \infty} \log^+ |\beta_j|_v, \sum_{v \neq \infty} \log^{-} |\beta_j|_v \right\} \\
&= [k : \mathbb{Q}] h(\beta_j).
\end{align*}
\]

The inequality \((1.9)\) in the statement of Theorem 1.2 follows from \((3.7)\) and \((5.8)\).

6. PROOF OF THEOREM 3.1

Let \( \eta_1, \eta_2, \ldots, \eta_{r(l/k)} \) be a basis for the free abelian group \( E_{l/k} \). Then there exists a nonsingular, \( r(l/k) \times r(l/k) \) matrix \( C = (c_{ij}) \) with entries in \( \mathbb{Z} \), such that

\[
\log \| \varepsilon_j \|_w = \sum_{i=1}^{r(l/k)} c_{ij} \log \| \eta_i \|_w
\]

at each archimedean place \( w \) of \( l \). As in our derivation of \((2.7)\) and \((2.9)\), the equations \((6.1)\) can be written as the matrix equation

\[
([l_w : \mathbb{Q}_w] \log \| \varepsilon_j \|_w) = ([l_w : \mathbb{Q}_w] \log \| \eta_j \|_w) C,
\]

where \( w \) is an archimedean place of \( l \), and \( w \) indexes the rows of the matrices on both sides of \((6.2)\). Let \( \mathcal{E} \) be the subgroup of \( E_{l/k} \) generated by \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r(l/k)} \). It follows from \((6.2)\) that the index of \( \mathcal{E} \) in \( E_{l/k} \) is given by

\[
[E_{l/k} : \mathcal{E}] = | \det C |.
\]

At each archimedean place \( v \) of \( k \) let \( \hat{w}_v \) be a place of \( l \) such that \( \hat{w}_v | v \). As in \((3.4)\), we write

\[
M_{l/k} = ([l_w : \mathbb{Q}_w] \log \| \eta_j \|_w),
\]

for the \( r(l/k) \times r(l/k) \) matrix, where \( w \) is an archimedean place of \( l \), but \( w \neq \hat{w}_v \) for each \( v | \infty \), \( w \) indexes rows, and \( j = 1, 2, \ldots, r(l/k) \) indexes columns. Let

\[
L(\mathcal{E}) = ([l_w : \mathbb{Q}_w] \log \| \varepsilon_j \|_w)
\]

be the analogous \( r(l/k) \times r(l/k) \) matrix, where again \( w \) is an archimedean place of \( l \), but \( w \neq \hat{w}_v \) for each \( v | \infty \), \( w \) indexes rows, and \( j = 1, 2, \ldots, r(l/k) \) indexes columns. From \((6.2)\) we get the matrix identity

\[
6.4 \quad L(\mathcal{E}) = M_{l/k} C.
\]

Then we combine \((3.5)\), \((6.2)\), \((6.3)\), and \((6.4)\), and conclude that

\[
6.5 \quad \text{Reg}(E_{l/k}) [E_{l/k} : \mathcal{E}] = | \det L(\mathcal{E}) |.
\]
To complete the proof we apply Schinzel’s inequality (1.2) to the determinant on the right of (6.5). We find that

\[
|l : \mathbb{Q}|^{-r(l/k)} |\det L(\mathcal{C})| \leq \prod_{j=1}^{r(l/k)} \left\{ \frac{1}{2} \sum_{w \neq \hat{w}_v} \log |\varepsilon_j|_w + \frac{1}{2} \sum_{w \neq \hat{w}_v} \log |\varepsilon_j|_w \right\}
\]

\[
= \prod_{j=1}^{r(l/k)} \left\{ \frac{1}{2} \sum_{w \neq \hat{w}_v} \log |\varepsilon_j|_w + \frac{1}{2} \sum_{w \neq \hat{w}_v} \log |\varepsilon_j|_w \right\}
\]

\[
\leq \prod_{j=1}^{r(l/k)} \left\{ \frac{1}{2} \sum_{v|\infty} \log |\varepsilon_j|_v + \frac{1}{2} \sum_{w \neq \hat{w}_v} \log |\varepsilon_j|_w \right\}
\]

\[
= \prod_{j=1}^{r(l/k)} h(\varepsilon_j).
\]

Combining (6.5) and the inequality (6.6), leads to the bound (3.6) in the statement of Theorem 3.1.

References

[1] D. Allcock and J. D. Vaaler, A Banach space determined by the Weil height, Acta Arith., 136 (2009), 279–298.

[2] A.-M. Bergé and J. Martinet, Sur les minorations géométriques des régulateurs, Séminaire de Théorie des Nombres de Paris 1987–1988, (C. Goldstein, ed.), Birkhäuser Verlag, Boston, 1990, 23–50.

[3] A.-M. Bergé and J. Martinet, Minorations de hauteurs et petits régulateurs relatifs, Séminaire de Théorie des Nombres de Bordeaux 1987–1988, Univ. de Bordeaux, 1989.

[4] A.-M. Bergé and J. Martinet, Notions relatives de régulateur et de hauteur, Acta Arith., 54 (1989), 155–170.

[5] D. Bertand, Duality on tori and multiplicative dependence relations, J. Austral. Math. Soc., (Series A) 62 (1997), 198–216.

[6] E. Bombieri and W. Gubler, Heights in Diophantine Geometry, Cambridge U. Press, New York, 2006.

[7] B. Brindza, On the generators of S-unit groups in algebraic number fields, Bull. Austral. Math. Soc., 43 (1991), 325–329.

[8] Y. Bugeaud and K. Györö, Bounds for the solutions of unit equations, Acta Arith., 64 (1996), 67–80.

[9] J. W. S. Cassels, An Introduction to the Geometry of Numbers, Springer, New York, 1971.

[10] A. Costa and E. Friedman, Ratios of regulators in totally real extensions of number fields, J. Number Theory, 37 (1991), 288–297.

[11] A. Costa and E. Friedman, Ratios of regulators in extensions of number fields, Proc. Amer. Math. Soc., 119 (1993), 381–390.

[12] E. Friedman, Analytic formulas for the regulator of a number field, Invent. Math. 98 (1989), 599–622.

[13] E. Friedman and N.-P. Skoruppa, Relative regulators of number fields, Invent. Math. 135, (1999), 115–144.

[14] L. Hajdu, A quantitative version of Dirichlet’s S-unit theorem in algebraic number fields, Publ. Math. Debrecen, 42 (1993), 239–246.

[15] C. R. Johnson and M. Newman, A surprising Determinantal Inequality for Real Matrices, Math. Ann., 247 (1980), pp. 179–186.

[16] D. H. Lehmer, Factorization of certain cyclotomic functions, Annals of Math. 34 (1933), 461–479.

[17] K. Mahler, On Minkowski’s theory of reduction of positive definite quadratic forms, Quart. J. Math. Oxford, 9 (1936), 259–263.
[18] E. M. Matveev, On linear and multiplicative relations, *Russian Acad. Sci. Sb. Math.*, 78 (1994), 411–425.

[19] E. M. Matveev, On the index of multiplicative groups of algebraic numbers, *Mat. Sb.* 196 (2005), 59–70; translation in *Sb. Math.* 196 (2005), 1307–1318.

[20] M. Meyer and A. Pajor, Sections of the unit ball of $L_p^n$, *J. Func. Anal.* 80(1988), 109–123.

[21] W. Narkiewicz, *Elementary and Analytic Theory of Algebraic Numbers*, 3rd ed. Springer-Verlag, Berlin, 2010.

[22] R. Remak, Über die Abschätzung des absoluten Betrages des Regulator eines algebraischen Zahlkörpers nach unten, *J. Reine Angew. Math.*, 167 (1932), 360–378.

[23] R. Remak, Über Grössenbeziehungen zwischen Diskriminanten und Regulator eines algebraischen Zahlkörpers, *Compositio Math.*, 10 (1952), 245–285.

[24] A. Schinzel An inequality for determinants with real entries, *Colloq. Math.*, XXXVIII (1978), 319–321.

[25] C. J. Smyth, The Mahler measure of algebraic numbers: a survey, in *Number Theory and Polynomials*, ed. J. McKee and C. J. Smyth, London Math. Soc. Lecture Notes 352, Cambridge U. Press, New York, 2008.

[26] J. D. Vaaler, Heights on groups and small multiplicative dependencies, *Trans. Amer. Math. Soc.*, 366 (2014), no. 6, 3295–3323.

[27] H. Weyl, On geometry of numbers, *Proc. Lond. Math. Soc.*, (2) 47 (1942), 268–289.

[28] R. Zimmert, Ideale kleiner Norm in Idealklassen und eine Regulatorabschätzung, *Invent. Math.*, 62 (1981), 367–380.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OREGON 97403 USA
E-mail address: akhtari@uoregon.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, AUSTIN, TEXAS 78712 USA
E-mail address: vaaler@math.utexas.edu