The entropy cones of $W_N$ and $W^d_N$ states

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ABSTRACT: The quantum entropy cones (QEC) for $W_N$ states of qubits and $W^d_N$ states of qudits are computed. These cones emerge as symmetrized quantum entropy cones (SQEC) for arbitrary $N$ and $d$. Directed graph models are presented which describe the SQEC for $W_N$ states and $W^d_N$ states.

Monogamous mutual information (MMI) is violated for all $N > 3$. 
1 Introduction

There is considerable interest in describing entropy cones in different contexts by a variety of methods. Among these are contraction maps, [1] graph and hypergraph models, [2–4] and link models [5]. One result of these strategies are entropy inequalities to be satisfied by holographic theories. The holographic entropy cones (HEC) have the feature that monogamous mutual information (MMI) [6] is satisfied. In this regard it should be emphasized that graph or hypergraph states are distinct from graph or hypergraph models. Graph and hypergraph models imply graph and hypergraph states, but not conversely, which implies that hypergraph models cones are a subset of stabilizer cones. However, there exist hypergraph states for which there is no hypergraph model.

In a series of papers Qu, et. al. [7–9] analyze graph states, hypergraph states, and stabilizer states. A brief summary of their conclusion:

1. If the rank $g$ of a hypergraph is $g > 2$, the hypergraph state $|g\rangle$ is not a stabilizer state.

2. Any stabilizer states is locally equivalent (LU) to a graph states (for $N \leq 7$ parties). Thus graph states are a subclass of stabilizer states.

3. Under LU, hypergraph states of 3 qubits split into 6 classes; one of which is not equivalent to any graph state.

4. Under SLOCC, hypergraph states of 3 qubits partition into 5 classes which cannot be converted in W-states, and one which has the same entanglement properties of a W-state.
5. A W-state of $N$-qubits is not locally maximally entangled (LME).

6. No hypergraph state of $N = 3$ qubits can be converted to a W-state of 3 qubits by SLOCC. For a $N \geq 4$ and SLOCC, this is an open question.

7. No hypergraph state of $N$-qubits can be converted to a W-state under LU.

8. As a consequence of the above, one concludes that $W_N$ states are not stabilizer states.

For large $N$, knowledge of quantum entropy cones is largely incomplete and deserves further study [10, 11]. It is in this context we study the entropy cone for $W_N$ states of $N$-qubits for any $N$. We find that all entropies are symmetrized entropies, and that therefore the resulting entropy cones are all symmetrized quantum entropy cones (SQEC). A star graph model is presented which reproduces the SQEC. An interesting feature of the model is that one leg of the star graph has negative capacity$^1$.

## 2 Entropy cones for $W_N$ states

The $W_N$ states for $N$-qubits are

$$W_N = \frac{1}{\sqrt{N}} [|00\cdots 01\rangle + |00\cdots 10\rangle + \cdots + |100\cdots 0\rangle]$$

(2.1)

Explicit calculations give the entropies for the number of parties $\leq N$, with party $N$ the purifier. These are

$$S_A = -\frac{1}{N} \ln \left( \frac{1}{N} \right) - \frac{(N - 1)}{N} \ln \left( \frac{N - 1}{N} \right)$$

$$S_{AB} = -\frac{2}{N} \ln \left( \frac{2}{N} \right) - \frac{(N - 2)}{N} \ln \left( \frac{N - 2}{N} \right)$$

$$S_{ABC} = -\frac{3}{N} \ln \left( \frac{3}{N} \right) - \frac{(N - 3)}{N} \ln \left( \frac{N - 3}{N} \right)$$

$$S_{ABCD} = -\frac{4}{N} \ln \left( \frac{4}{N} \right) - \frac{(N - 4)}{N} \ln \left( \frac{N - 4}{N} \right)$$

(2.2)

etc. The entropies which emerge from the explicit computations are automatically symmetrized, i.e. independent of specific choices for $A, B, C, \cdots$, etc. No explicit averaging is required. The relevant entropy cone is the symmetrized quantum entropy cone (SQEC) [12, 13].

$^1$Jonathan Harper alerted us to the possible relevance of negative capacities.
Define

\[
I_3(A : B : C) = [S_A + S_B + S_C] - [S_{AB} + S_{BC} + S_{AC}] + S_{ABC}
= 3[S_A - S_{AB}] + S_{ABC}
\] (2.3)

using the symmetrized properties of (2.2). One requires \( I_3 \leq 0 \) for the monogamy of mutual information (MMI) to be satisfied for a restriction to the holographic entropy cone (HEC) to leading order in the gravitational coupling [6, 14]. \( I_3 = 0 \) for \( N = 3 \), but \( I_3 > 0 \) for \( N \geq 4 \) \( W_N \) states so that these are not suitable for holographic applications to leading order in the gravitational coupling.

Explicit examples of the entropy vectors for \( W_N \) with the obvious symmetries of (SQEC) understood are

\[ N = 3 : \quad S_A = S_{AB} = \left[ \ln 3 - \frac{2}{3} \ln 2 \right] \]
\[ S_{ABC} = 0 \] (2.4)

\[ N = 4 : \quad S_A = S_{ABC} = \left[ \ln 4 - \frac{3}{4} \ln 3 \right] \]
\[ S_{AB} = \ln 2 \]
\[ S_{ABCD} = 0 \] (2.5)

\[ N = 5 : \quad S_A = S_{ABCD} = \left[ \ln 5 - \frac{4}{5} \ln 4 \right] \]
\[ S_{AB} = S_{ABC} = \left[ \ln 5 - \frac{3}{5} \ln 3 - \frac{2}{5} \ln 2 \right] \]
\[ S_{ABCDE} = 0 \] (2.6)
\[ N = 6 : \quad S_A = S_{ABCDE} \]
\[ = \left[ \ln 6 - \frac{5}{6} \ln 5 \right] \]
\[ S_{AB} = S_{ABCD} \]
\[ = \left[ \ln 3 - \frac{2}{3} \ln 2 \right] \]
\[ S_{ABC} = \ln 2 \]
\[ S_{ABCDEF} = 0. \] (2.7)

It is instructive to see examples of the entropy vectors for \( W_N \) written explicitly, to emphasize that they are already symmetrized.

\[ N = 3 : \quad \vec{S} = \{(1, 1, 1); (1, 1, 1)\} \left[ \ln 3 - \frac{2}{3} \ln 2 \right] \]
\[ S_{ABC} = 0 \] (2.8)

\[ N = 4 : \quad \vec{S} = \{(1, 1, 1, 1) \left[ \ln 4 - \frac{3}{4} \ln 3 \right]; (1, 1, 1, 1, 1) \ln 2; (1, 1, 1, 1) \left[ \ln 4 - \frac{3}{4} \ln 3 \right]\} \]
\[ S_{ABCD} = 0, \] (2.9)

etc.

From (2.2) one has
\[ \tilde{S}_l = \frac{l}{N} \ln \left( \frac{N}{l} \right) + \frac{(N - l)}{N} \ln \left( \frac{N}{N - l} \right) \] (2.10)

so that
\[ \tilde{S}_l = \tilde{S}_{N-l} \] (2.11)

with \( l = 1 \) to \( N - 1 \), and
\[ \tilde{S}_{\frac{N}{2}} = \ln 2 \] (2.12)

for \( N \) even. While for \( N \) odd,
\[ \tilde{S}_{\frac{N+1}{2}} = \tilde{S}_{\frac{N+1}{2}} \]
\[ = \frac{N + 1}{2N} \ln \left( \frac{2N}{N + 1} \right) + \frac{N - 1}{2N} \ln \left( \frac{2N}{N - 1} \right) \] (2.13)
That is (2.12) and (2.13) satisfy

$$\tilde{S}_{l+1} = \tilde{S}_l$$

(2.14)

where $\left[\frac{N}{2}\right] = \frac{N}{2}$ or $\frac{N+1}{2}$, whichever is integer. All entropies are therefore averaged or symmetrized entropies as discussed in [12, 13], so that they all belong to the symmetrized quantum entropy cones (SQEC). Since $I_3 > 0$ for $N \geq 4$, the entropies of $W_N$ do not satisfy the inequalities of the symmetrized holographic entropy cones (SHEC), which implies that SQEC $\supset$ SHEC for $W_N$ states.

The SQEC is simplical [13]. For each $N$, the facets of the SQEC satisfy the inequalities for $l$ parties [13, 15],

$$-\tilde{S}_{l-1} + 2\tilde{S}_l - \tilde{S}_{l+1} \geq 0$$

(2.15)

with

$$\tilde{S}_0 = 0,$$

(2.16)

for

$$1 \leq l \leq \left[\frac{N}{2}\right].$$

(2.17)

The extreme rays of the SQEC are those described in [13]. Thus the entropy cone of $W_N$ provide an explicit realization of a SQEC.

3 A graph model

Graph models constructed for holographic entropy cones have been in the context of undirected graphs with positive weights [1, 2, 4, 5]. However, in the application to the entropy cones of $W_N$ states we propose a model with directed star graphs with $l - 1$ legs of weight one, and one leg with weight $w < 0$, as made explicit in what follows.

Consider a star graph with $l$ legs, with $l - 1$ legs weight one, and one leg with weight $w$. Every entropy $S_I$ is given by the min-cut prescription [13]

$$S_I = \min\{|I|, l - |I| + w\}$$

(3.1)

Identify $S_I = S_l$. Then the symmetrized entropy vectors are [13]

$$\tilde{S}_l = \left(\frac{N}{l}\right)^{-1} \left[\left(\frac{N - 1}{l}\right) S_l + \left(\frac{N - 1}{N - l}\right) S_{N-l}\right]$$

$$= \frac{1}{N} \left[(N - l) \min\{l, N - 1 - l + w\} + l \min\{N - l, w + l - 1\}\right].$$

(3.2)
Equation (3.2) describes two separate star graphs, with two different weights \( w \). For the second term in (3.2)

\[
(S_l)_{1} = \frac{l}{N} \min[N - l, w_1 + l - 1],
\]

(3.3)

with the choice

\[
\min \left[ (N - l), \ln \left( \frac{N}{l} \right) \right] = \ln \left( \frac{N}{l} \right).
\]

(3.4)

That is

\[
(w_1)_l = -(l - 1) + \ln \left( \frac{N}{l} \right) < 0 \text{ for } l > 1 + \ln \left( \frac{N}{l} \right).
\]

(3.5)

So that (3.4) is satisfied for

\[
l = 1 \text{ to } \frac{N}{2} \quad N \text{ even}
\]

\[
= 1 \text{ to } \left[ \frac{N}{2} \right] \quad N \text{ odd}.
\]

(3.6)

Thus, the second term in (3.2) gives

\[
(S_l)_{1} = \left( \frac{l}{N} \right) \ln \left( \frac{N}{l} \right).
\]

(3.7)

Note that this coincides with the first term in (2.10).

Similarly, for the first term in (3.2) consider

\[
\min[l, N - 1 - l - w_2],
\]

(3.8)

with the choice

\[
(w_2)_l = l - (N - 1) + \ln \left( \frac{N}{N-l} \right) < 0 \text{ for } l < (N - 1) - \ln \left( \frac{N}{N-l} \right).
\]

(3.9)

From (3.5) and (3.9),

\[
(w_1)_{N-l} = (w_2)_l,
\]

(3.10)

so that

\[
(S_l)_{2} = \left( \frac{N-l}{N} \right) \ln \left( \frac{N}{N-l} \right)
\]

(3.11)

Putting this together with (3.2), one obtains that the star-graph model, with one leg with negative weights \( w_1 \) and \( w_2 \) respectively, when (3.5) and (3.9) are satisfied.

Assembling all the pieces from (3.2), (3.7), and (3.11) one finds that \( \tilde{S}_l \) constructed from the graph model coincides with (2.10). The novel feature of the model is that one leg of the star graph has negative capacity. This can be understood in terms of directed graphs, where negative flows are permitted.
4 Entropy cones for $W^d_N$ states

Of interest is the QEC for $W^d_N$ state of qudits, which also emerge as symmetrized from explicit calculations. Significantly these results can be understood as the coarse graining of the SQEC of $W_N$ states, which is analogous to the extensive strategy discussed for the HEC [16–19].

The $W^d_N$ states for $N$ qudits are

$$|W^d_N\rangle = \frac{1}{\sqrt{N(d-1)}} \sum_{i=1}^{d-1} ([i00\cdots0] + [0i0\cdots0] + \cdots + [00\cdots0i])$$ (4.1)

Explicit calculations give the entropies for the number of parties < $N(d-1)$, and party $N(d-1)$ the purifier. The entropies are

$$S_A = -\frac{1}{N^*} \ln \left( \frac{1}{N^*} \right) - \frac{(N^*-1)}{N^*} \ln \left( \frac{N^*-1}{N^*} \right)$$

$$S_{AB} = -\frac{2}{N^*} \ln \left( \frac{2}{N^*} \right) - \frac{(N^*-2)}{N^*} \ln \left( \frac{N^*-2}{N^*} \right)$$

$$S_{ABC} = -\frac{3}{N^*} \ln \left( \frac{3}{N^*} \right) - \frac{(N^*-3)}{N^*} \ln \left( \frac{N^*-3}{N^*} \right)$$

$$S_{ABCD} = -\frac{4}{N^*} \ln \left( \frac{4}{N^*} \right) - \frac{(N^*-4)}{N^*} \ln \left( \frac{N^*-4}{N^*} \right)$$ (4.2)

etc., where

$$N^* = N(d-1)$$ (4.3)

These entropies are automatically symmetrized, including an average over the $d$-states of qudits at fixed $N$, so that the relevant entropy cone is the SQEC [13, 16].

The entropy cones for $W_N$ states is obtained from (4.2), (4.3) by setting $d = 2$. Explicit values for entropies of $W^d_N$ states are obtained from allowed choices of $N^*$, i.e. from $N$ and $d$ in (4.3).

Coarse grained entropies

From (4.3) one observes that for fixed $d > 2$, not all integer values of $N^*$ are available. For example, for $d = 2$ (qubits), (4.3) permits any integer value. However for $d = 3$ (qutrits), $N^* = 2N$, so that only even values of $N^*$ are present in (4.2).

In this context, the entropy cones of $W^d_N$ states are obtained from a coarse-graining of $W_N$ states. This feature carries over to graph models.
Graph models

Star graph models for the SQEC of $W_N$ states are discussed in Sec. 3. One can then use coarse graining to describe a similar graph model for the SQEC of $W^d_N$ states. Using coarse graining, the appropriate graph weights for the $W^d_N$ entropy cones are

$$(w_1)_\ell \rightarrow (w_1)_\ell^* = -(\ell - 1) + \ln \left( \frac{N^*}{\ell} \right)$$

(4.4)

and

$$(w_2)_\ell \rightarrow (w_2)_\ell^* = \ell - (N^* - 1) + \ln \left( \frac{N^*}{N^* - \ell} \right)$$

(4.5)

With these identifications, the graph model presented in Sec. 3 is now applicable to the SQEC of $W^d_N$ states. The coarse graining implies that the weights satisfy

$$(w_1)_\ell^* \subseteq (w_1)_\ell$$

(4.6)

and

$$(w_2)_\ell^* \subseteq (w_2)_\ell$$

(4.7)

but not conversely. That is, they are a subset of the set of weights $(w_1)_\ell$ and $(w_2)_\ell$.

5 Concluding remarks

The main result of this paper is that the quantum entropy cone of $W_N$ states and $W^d_N$ states can be computed explicitly, and that these emerge a priori symmetrized, providing the entropies for the symmetrized quantum entropy cone (SQEC). Graph models capture these results.

Rota [14] shows that holographic systems require MMI for the validity of semi-classical geometry\,\footnote{Matt Headrick also has made this point [6]. [private communication]}, which rules out $W_N$ for $N > 3$. Akers and Rath [20] argue that to leading order, holography needs tri-partite entanglement. Since $I_3 = 0$ for $W_3$, this presents one possibility. However Akers, et. al. [21] indicate that the HEC inequalities may no longer be satisfied once general quantum corrections to holography are considered. It might be fruitful to consider $W_N$ states. However, if bulk entropies obey MMI, that implies the boundary entropies also obey MMI [21], which may limit the application of $W_N$ states.

Entropies for stabilizer states are studied in [22].
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