Quark and Nuclear Matter
in the
Linear Chiral Meson Model

J. Berges†

Center for Theoretical Physics
Massachusetts Institute of Technology
Cambridge, MA 02139, USA

D.–U. Jungnickel‡, C. Wetterich§

Institut für Theoretische Physik
Universität Heidelberg
Philosophenweg 16
69120 Heidelberg, Germany

Abstract

We present an analytical description of the phase transitions from a nucleon gas to
nuclear matter and from nuclear matter to quark matter within the same model. The
equation of state for quark and nuclear matter is encoded in the effective potential of a
linear sigma model. We exploit an exact differential equation for its dependence upon the
chemical potential $\mu$ associated to conserved baryon number. An approximate solution
for vanishing temperature is used to discuss possible phase transitions as the baryon
density increases. For a nucleon gas and nuclear matter we find a substantial density
enhancement as compared to quark models which neglect the confinement to baryons.
The results point out that the latter models are not suitable to discuss the phase diagram
at low temperature.

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§Email: Berges@ctp.mit.edu, current address: Institut für Theoretische Physik, Universität Heidelberg,
Philosophenweg 16, 69120 Heidelberg, Germany
‡Email: D.Jungnickel@thphys.uni-heidelberg.de
§Email: C.Wetterich@thphys.uni-heidelberg.de
1 Introduction

The equation of state for strongly interacting matter at nonzero density is needed for the understanding of neutron stars [1], as well as for the interpretation of heavy ion collision experiments [2]. Any analytical description of the equation of state for baryons at nonzero baryon density has to cope with the problem that the effective degrees of freedom change from nucleons at low density to quarks at high density. We attempt in this note a unified description of both the nuclear gas–liquid transition and the transition to quark matter. For this purpose we work within an effective linear meson model coupled to quarks and nucleons. It should describe the low momentum degrees of freedom of QCD for the range of temperatures and densities which are relevant for these phase transitions. Our main computational tool will be a new exact functional differential equation for the dependence of the effective action on the baryon chemical potential and an approximate solution to it. We will see many similarities but also important differences as compared to a mean field theory treatment.

Our main interest are the chiral aspects of the equation of state for quark and nuclear matter at nonzero baryon density and the order of the involved phase transitions. In this note we concentrate on the simplest possibility where only the color-singlet chiral condensate plays a role in the transitions. Our investigation should constitute a useful point of comparison for other models with more complicated condensates. In fact, in addition to the nuclear and quark matter phases a number of interesting possibilities like the formation of meson condensates or strange quark matter [3, 4, 5] have been proposed. Also an extensive discussion has focused around the symmetry of the high density state, where the spontaneous breaking of color is associated with the phenomenon of color superconductivity [6, 7, 8, 9]. Other ideas concern the spontaneous breaking of the color symmetry in the vacuum [10, 11]. In this note we adopt the working assumption that possible additional condensates have only little influence on the transitions associated with the order parameter of chiral symmetry breaking. We also concentrate mainly on the case of two quark flavors and neglect isospin violation.

Within this setting and using a rather crude approximation for the transition from quark to nucleon degrees of freedom, we find that for low temperature both the nuclear gas–liquid and the hadron–quark transitions are of first order, in accordance with indications from earlier investigations (cf. [13, 12, 14] and [9] and references therein). The phase transition from nuclear to quark matter tends to be much stronger (larger surface tension) than the gas–liquid nuclear transition. The first order character of these phase transitions would have important implications. In particular, one may combine this with information about the high temperature phase transition for vanishing baryon density: One expects [15, 12, 14] an endpoint of the first order critical line between quark and nuclear matter if the zero density, high temperature transition is a crossover (as for two flavor QCD with non–vanishing quark masses). Such an endpoint corresponds to a second order transition where a large correlation length may lead

1 The results of [12] indicate that condensates of quark Cooper pairs do not influence the behavior of the chiral condensate to a good approximation. The phenomenon of high density color superconductivity has only minor influence on the equation of state for quark matter [9]. On the other hand, for a substantial color octet condensate in the vacuum [10, 11] our assumption would not hold.

2 Isospin violation and electromagnetism are important for nuclear matter in neutron stars. Our result for the equation of state is therefore not quantitatively realistic in all respects. Isospin violation can be incorporated in our formalism without conceptual difficulties. We have already included electromagnetism phenomenologically for the quantitative description of nuclei.
to distinctive signatures in relativistic heavy ion collisions [16]. If the zero density transition turns out to be of first order in three flavor QCD, such an endpoint does not necessarily occur. (Endpoints are not excluded in this case, however, since the two first order regions could be disconnected.) The first order line for the gas–liquid nuclear transition exhibits a critical endpoint for a temperature of about 10 MeV. Signatures and critical properties of this point have been studied through measurements of the yields of nuclear fragments in low energy heavy ion collisions [17, 18].

We emphasize that our treatment of quarks and baryons is still very crude and a different picture of the high density transition, e.g. due to the inclusion of other condensates, may well appear for two flavor QCD. A general outcome of our analysis concerns the crucial importance of confinement for any understanding of the phase transitions at high density and low temperature. In fact, the contribution of a free gas of baryons to the dependence of the free energy on the chemical potential is enhanced by a factor 27 (!) as compared to the contribution from a free gas of quarks. Therefore, the binding of quarks into baryons at low density and temperature plays a crucial quantitative role which cannot be neglected by any satisfactory treatment of the high density transition. In fact, a first order transition between nuclear matter and quark matter would presumably connect an approximately free nucleon gas at low density to a quark gas at high density. Such a transition involves then a description in terms of baryons in the low density phase and cannot be understood within quark descriptions, which do not reflect the large baryonic enhancement factor. This may explain the phenomenologically unacceptable low critical densities often found in such quark descriptions.

In quantum field theory the effects of a non–vanishing baryon density in thermal equilibrium or the vacuum are described by adding to the classical action a term proportional to the chemical potential $\mu$,

$$\Delta_\mu S = 3i\mu \sum_j b_j \int_0^{1/T} dx_0 \int d^3 \vec{x} \bar{\psi}_j \gamma^0 \psi_j \equiv -3\frac{\mu}{T} B .$$  \hspace{1cm} (1)$$

The index $j$ labels all fermionic degrees of freedom which carry a non–vanishing baryon number $b_j$ and a summation over spinor indices is assumed implicitly. For a description of the fermionic degrees of freedom in terms of quarks the sum is over $N_c$ colors and $N_F$ flavors, with $b_j = 1/3$. We neglect the heavy quarks and concentrate on a two-flavor approximation where also the strange quark is omitted. For the nucleon degrees of freedom we include protons and neutrons with $b_j = 1$. For our conventions, $\mu$ corresponds to the chemical potential of quark number density. The baryon number density $n$ can be obtained from the $\mu$–dependence of the Euclidean effective action $\Gamma$, evaluated at its minimum for fixed temperature $T$ and volume $V$.

$$n \equiv \langle B \rangle \equiv -\frac{1}{3} \frac{\partial}{\partial \mu} \Gamma_{\text{min}}T \bigg|_{T,V} .$$  \hspace{1cm} (2)$$

We note that the Helmholtz free energy is $F = \Gamma_{\text{min}} T + 3\mu n V$. Our aim is a computation of the difference of $\Gamma_{\text{min}}$ between non–vanishing and vanishing $\mu$. For $T = 0$ this is dominated by fermionic fluctuations with (spatial) momenta $q^2 \leq \mu^2$. For not too large $\mu$ (say $\mu \lesssim 600$ MeV) we can therefore work with an effective model for the low momentum degrees of freedom of

\footnote{More precisely, $B$ counts the number of baryons minus antibaryons. For $T \to 0$ the factor $T/V$ is simply the inverse volume of four–dimensional Euclidean space.}
QCD. This argument generalizes to moderate temperatures, say $T \lesssim 200$ MeV. In the bosonic sector we will work with a linear meson model whereas for the fermions we keep the multiplet with lowest mass as discussed above. Our description takes into account the lightest scalar and pseudoscalar mesons as well as the lowest multiplet of vector mesons.

The minimum of the effective action corresponds to the minimum of the effective meson potential $U = \Gamma T / V$ for constant scalar meson fields. In consequence, $U$ is a function of a complex $N_F \times N_F$ scalar field matrix $\Phi$, which describes the nonets of scalar and pseudoscalar mesons, and a similar matrix for the vector mesons. For a discussion of the chiral phase transition it will be sufficient to know the dependence of $U$ on space and time independent fields which can acquire a vacuum expectation value consistent with $SU(N_F)$ symmetry. These are the real diagonal elements of $\Phi$ which we denote by $\sigma$, and similar diagonal elements $\omega$ for the zero component of the vector mesons. In the limit of vanishing current quark masses the minimum of $U$ at sufficiently high temperature or high density should occur at $\sigma = 0$ in this model. For low $T$ and $\mu$ spontaneous chiral symmetry breaking is triggered by a non-vanishing expectation value $\overline{\sigma}(\mu, T)$, corresponding to the location of the minimum of $U(\sigma, \omega; \mu, T)$. (We adopt the convention through this work that bars indicate locations of potential minima.)

The explicit breaking of chiral symmetry through non-vanishing current quark masses is described by a linear source term contained in $U$ which induces nonzero $\sigma$ even in the phase without spontaneous symmetry breaking\footnote{The zero component of the vector fields $\omega_\mu$ can acquire a nonvanishing expectation value since at nonzero chemical potential Lorentz invariance is broken.} \cite{19, 21, 22}. The baryon density $n$, energy density $\epsilon$ and pressure $p$ follow from $U(\mu, T) \equiv U(\overline{\sigma}(\mu, T), \overline{\omega}(\mu, T); \mu, T) = \epsilon - Ts - 3\mu n$ as

$$n = -\frac{1}{3} \frac{\partial}{\partial \mu} U(\mu, T); \quad p = -U(\mu, T)$$

$$\epsilon \equiv \frac{E}{V} = U(\mu, T) + 3\mu n - T \frac{\partial U}{\partial T}(\mu, T). \quad (3)$$

Here we have normalized $U(0, 0) = 0$ corresponding to vanishing pressure in the vacuum.

For fluctuations in the momentum range $q_H^2 < \overline{q}^2 < (600 \text{ MeV})^2$ we work within the linear quark meson model \cite{21, 22, 23, 24, 25, 26} in an approximation which does not describe the effects of confinement. For low momenta, i.e. $\overline{q}^2 < q_H^2$, this description therefore becomes inappropriate. Three quarks are bound into color singlet nucleons. In this momentum range we describe the fermionic degrees of freedom by baryons, while keeping the description of the bosons in terms of the scalar field $\Phi$ and corresponding vector meson fields. The use of the same bosonic fields for the whole momentum range will turn out to be an important advantage since it facilitates the computation of the free energy in different ranges of $\mu$, corresponding in turn to different baryon densities and a different picture for the relevant fermionic degrees of freedom. For nuclear matter a typical value of the “transition momentum” is $q_H \gtrsim 260$ MeV. We find that the quark–hadron phase transition is substantially influenced by the change from quark to baryon fields at $q_H$. This implies that a reliable quantitative understanding of this transition requires also a quantitative treatment of the change of effective fermionic degrees of freedom.

\footnote{For general quark or nucleon masses the diagonal elements $\sigma$ of the scalar meson matrix, and similarly for $\omega$, can differ from each other. We suppress this dependence in the notation since we will only perform calculations for the case where all diagonal elements are the same.}
Furthermore, our computation reveals that the transition from a nucleon gas to nuclear matter can be described realistically in terms of nucleon and meson degrees of freedom only if one accepts a relatively complicated form of the vacuum effective potential for the color-singlet chiral order parameter $\sigma$. In particular, one needs large higher order couplings which do not seem very natural. It is conceivable that this situation changes for a more complex vacuum with additional condensates.

## 2 Chemical potential flow equation

We employ a new method for the computation of the $\mu$–dependent part of the effective action that relies on an exact functional differential equation for $\Gamma$. This equation expresses the $\mu$–derivative of $\Gamma$ in terms of the exact field dependent fermion propagator. We start from the generating functional of the connected Green functions

$$ W[j] = \ln \int D\chi \exp \left\{ -S[\chi] - \Delta_\mu S[\chi] + \int j\chi \right\} $$

where $\chi$ stands collectively for bosonic and fermionic fields with associated sources $j$ and $S$ is the action for $\mu = 0$. For our purpose it is convenient to subtract from the effective action (defined by a Legendre transform) the $\mu$–dependent fermion bilinear (1):

$$ \Gamma[\varphi] = -W[j] + \int j\varphi - \Delta_\mu S[\varphi], \quad \varphi = \frac{\delta W}{\delta j}. $$

The $\mu$–dependence of $\Gamma$ arises only through $\Delta_\mu S$ and can be expressed by a trace over the connected two–point function. Using the manipulations of generating functions outlined in [27] in a context with fermions [28, 29, 30] one obtains the exact nonperturbative functional differential equation

$$ \frac{\partial}{\partial \mu} \Gamma = -\text{Tr} \left\{ \frac{\partial R_\mu}{\partial \mu} \left( \Gamma^{(2)} + R_\mu \right)^{-1} \right\} $$

where

$$ R_{\mu,jj'}(q,q') = 3i\mu b_j \gamma^0(2\pi)^4 \delta(q-q')\delta_{jj'}. $$

We remind that $\Gamma$ is a functional of the meson and fermion fields, and the $\mu$–derivative on the left hand side of (6) is taken for fixed fields. The exact inverse propagator $\Gamma^{(2)}$ is the second functional derivative with respect to the fields. It is a matrix in the space of internal indices and momenta and involves fermions and bosons. Since $\Delta_\mu S$ only affects fermions, the trace is over fermionic indices only and contains a momentum integration. For a configuration with constant bosonic fields and vanishing fermion fields $\Gamma^{(2)}$ does not mix bosons and fermions and is diagonal in momentum space. We therefore only need the inverse fermion propagator

$$ \Gamma^{(2)}_{jj'}(q,q') = H_{jj'}(q)(2\pi)^4 \delta(q-q') $$

in order to obtain an exact equation for the $\mu$–dependence of the effective potential

$$ \frac{\partial U}{\partial \mu} = -\sum_j 3b_j \int \frac{d^4q}{(2\pi)^4} \text{tr} \gamma^0 \left[ H(q) + 3i\mu \gamma^0 \right]_{jj}^{-1}. $$

\[\text{We mention that in the presence of a local gauge symmetry this equation is manifestly gauge invariant.}\]
Here $\text{tr}$ denotes a Lorentz trace and $H_{jj',b_{jj'}} = b_j\delta_{jj'}$ are matrices in the space of fermion species. This exact relation expresses the baryon density for arbitrary quark mass term\(^7\) (cf. eq. (30)) in terms of the exact fermionic propagators in presence of nonvanishing meson fields. We will see that the momentum integral is both ultraviolet and infrared finite such that eq. (9) is well defined.

The exact fermion propagators are not known, and we have to proceed to approximations. The advantage of our underlying exact expression remains, however, that is easy to study which are the effects of qualitative and quantitative changes in the approximations for the fermionic propagators. In particular, we will learn how the transition from quark to nucleon degrees of freedom strongly affects the form of the effective potential – a discussion that would not be possible within a mean field approximation for a given effective model either of quarks or of baryons alone. In the present work we will use a rather simple approximation both for the quark and baryon propagators. For arbitrary $\sigma$ and $\omega$ we approximate

$$H_{jj'}(q) = \left[ q_{\nu}\gamma_{\nu} + m_j(\sigma; \mu, T)\gamma^5 + ib_j\Omega\gamma^0 \right] \delta_{jj'}$$

with

$$\Omega = g_\omega(\sigma, T)\omega .$$

We use our ansatz for the fermion propagator only to compute the $\mu$–dependent contributions to the effective potential, i.e. we consider here the difference $U(\sigma, \omega; \mu, T) - U(\sigma, \omega; 0, T)$. In fact, the computation of the contributions due to a non–vanishing chemical potential allows one to use quite crude approximations in many situations. This is based on the observation that strongly interacting fermions are often successfully described as freely propagating quasi–particles. In our case they acquire an effective “constituent” mass $m\gamma_5$ through a strong Yukawa coupling to mesonic vacuum expectation values. (The matrix $\gamma_5$ appears in the mass term as a consequence of our Euclidean conventions \[28\].) Similarly the constant field $\omega$ denotes the analytic continuation of the zero component of the Euclidean $\omega$–vector–meson field with coupling $g_\omega$ to the fermions. The piece $\sim \Omega$ is the remnant of the vector coupling $\sim \omega_\mu\gamma^\mu$ in a situation where the zero component of $\omega_\mu$ can acquire an expectation value due to the breaking of Lorentz invariance\(^8\) by the nonvanishing chemical potential (1). We will later determine the values of $\omega$ self-consistently. An important simplification in our ansatz (10) is the neglect of a possible wave function renormalization $Z_\psi(q, \sigma, \omega)$ which could multiply the kinetic term $q_{\nu}\gamma^\nu$. In a more realistic setting this will certainly play a role near the transition between quarks and baryons.

We emphasize that the computation of $\partial U/\partial \mu$ according to eq. (9) does not need any information about the masses and effective self-interactions of the mesons. They determine, however, the effective potential at zero baryon density $U(\sigma, \omega; 0, T)$ and therefore influence the possible phase transitions. In fact, the meson self–interactions may turn out to be quite complicated. We do not attempt here to compute the meson masses and self-interactions by a mean field approximation, since earlier renormalization group investigations have shown \[22\] that this is probably much too crude. The advantage of our method is that the lack of knowledge about the meson interactions can be separated from computation of the $\mu$-dependence of the

\(^7\)The relation (9) holds for arbitrary $\sigma$ which corresponds to arbitrary quark mass $m_q$ through $\partial U/\partial \sigma \sim m_q$.

\(^8\)For $\mu = 0$ one therefore has $\omega = 0$. 

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potential. We will simply parametrize $U(\sigma, \omega; 0, T)$ in accordance with the symmetries and (indirect) observational knowledge.

At this point it may be useful to summarize the approximations that affect the computation of the $\mu$-dependence of the effective potential if we use the ansatz (10) in the exact relation (9). Perhaps most importantly we omit the dependence of the fermion wave function renormalization on momentum, $\sigma$, $\omega$, $\mu$ and $T$. We also neglect a possible difference in normalization of the quark kinetic term and the baryon number current. Similarly, we have not considered a possible momentum dependence of the mass term as well as the momentum dependence of the contribution $\sim \gamma^0$. Finally, we assume that $m_j$ can be taken as independent of $\omega$, and $g_\omega$ as not dependent on $\omega$ or $\mu$. In this approximation the term $\sim \Omega$ can be combined with $R_\mu$ such that $\mu$ is replaced in the propagator $(\Gamma^{(2)} + R_\mu)^{-1}$ by an effective chemical potential

$$\mu_{\text{eff}} = \mu + \frac{1}{3} \Omega(\omega, \sigma; T) = \mu + \frac{1}{3} g_\omega \omega. \quad (12)$$

With the approximation (10) the evolution equation for the $\mu$-dependence of the effective meson potential takes a very simple form

$$\frac{\partial U}{\partial \mu} = - \sum_j 3b_j \int \frac{d^4q}{(2\pi)^4} \text{tr} \left\{ i\gamma^0 \left( \gamma^j + m_j \gamma^5 + 3ib_j \mu_{\text{eff}} \gamma^0 \right)^{-1} \right\}. \quad (13)$$

The remaining trace over spinor indices is easily performed

$$\frac{\partial U}{\partial \mu} = -2 \sum_j \int \frac{d^3\vec{q}}{(2\pi)^3} K_j, \quad (14)$$

$$K_j = 6ib_j \int \frac{dq^0}{2\pi} \frac{(q^0 + 3ib_j \mu_{\text{eff}})}{(q^0 + 3ib_j \mu_{\text{eff}})^2 + \vec{q}^2 + m_j^2}. \quad (15)$$

For non-vanishing temperature the $q^0$-integration is replaced by a sum over Matsubara frequencies

$$\int \frac{dq^0}{2\pi} \rightarrow T \sum_{n \in \mathbb{Z}} \quad (16)$$

with $q^0 = 2\pi(n + 1/2)T$ and, correspondingly, $\delta(q - q') \rightarrow \delta(q - q')\delta_{nn'}/(2\pi T)$. This results in

$$K_j = 3b_j \left( e^{\sqrt{q^2 + m_j^2 - 3b_j \mu_{\text{eff}}}} + 1 \right)^{-1} - \left( e^{\sqrt{q^2 + m_j^2 + 3b_j \mu_{\text{eff}}}} + 1 \right)^{-1} \quad (17)$$

where the two terms are proportional to the fermion and anti-fermion contributions to $n$. We see explicitly that the momentum integration is finite due to the exponential suppression for large $\vec{q}^2$.

We will concentrate here mainly on $T = 0$ where the $q^0$-integration yields a step function:

$$K_j = 3b_j \Theta(9b_j^2(\mu_{\text{eff}}^2)^2 - (\vec{q}^2 + m_j^2)) \quad (18)$$

The remaining $\vec{q}$-integration is therefore cut off in the ultraviolet, $\vec{q}^2 < 9b_j^2(\mu_{\text{eff}}^2)^2 - m_j^2$, and only involves momenta smaller than the Fermi energy $3b_j \mu_{\text{eff}}^{(0)}$. As it should be, it is dominated by modes with energy $\sqrt{\vec{q}^2 + m_j^2}$ near the Fermi surface.
3 Quark and nucleon degrees of freedom

We will assume that \( \partial U / \partial \mu \) can be expressed as a simple sum of the contribution from quarks with momenta \( \vec{q}^2 > q_H^2 \) and that of baryons with momenta \( \vec{q}^2 < q_H^2 \). This is the simplest approximation which catches the effective transition from quarks to baryons as effective degrees of freedom in the relevant momentum range. It will be sufficient to demonstrate the most important effects of confinement on the \( \mu \)-dependence of the effective potential, namely that the contribution of a gas of nucleons is greatly enhanced as compared to a corresponding contribution of quarks. In a more realistic scenario the transition between quark and nucleon degrees of freedom will be less abrupt. Within our approximations part of the uncertainty related to this effective transition can be studied by allowing that \( q_H \) depends on \( \sigma \) since typically the relevant values for \( \sigma \) depend on the baryon density, being higher for a nucleon gas than for a quark gas.

We first consider the range of momenta with \( \vec{q}^2 \geq q_H^2 \) for which we use an effective linear quark meson model [21, 22, 23, 24, 25, 26]. Here the quark mass term \( \sim \gamma^5 \) arises through a Yukawa coupling \( h \) to the expectation value of the \( \sigma \)–field, \( m_q = h_q(\sigma; \mu, T) \sigma \). Since the quark description breaks down for small momenta, we restrict the integration over \( \vec{q}^2 \) in (14) to the range \( \vec{q}^2 > q_H^2 \). We therefore infer for the quark contribution to the \( \mu \)–dependence of the effective potential (for \( \mu > 0 \) and \( T = 0 \))

\[
\frac{\partial U^{(Q)}}{\partial \mu} = -\frac{N_c N_F}{3\pi^2} \left[ \left( \mu_{\text{eff}}^2 - h_q^2 \sigma^2 \right)^{3/2} - q_H^2 \right] \Theta \left( \mu_{\text{eff}}^2 - h_q^2 \sigma^2 - q_H^2 \right). \tag{19}
\]

For the low momentum range \( \vec{q}^2 < q_H^2 \) where the fermionic degrees of freedom are the lightest baryons rather than quarks we repeat the steps leading from (6) to (19). The trace now involves a sum over proton and neutron but no color factor. This yields a contribution (again for \( T = 0 \))

\[
\frac{\partial U^{(B)}}{\partial \mu} = -\frac{2}{\pi^2} \left\{ (9\mu_{\text{eff}}^2 - m_N^2)^{3/2} \Theta(9\mu_{\text{eff}}^2 - m_N^2) \Theta(m_N^2 + q_H^2 - 9\mu_{\text{eff}}^2) \\
+ q_H^2 \Theta(9\mu_{\text{eff}}^2 - m_N^2 - q_H^2) \right\}. \tag{20}
\]

We parametrize the nucleon mass as \( m_N(\sigma) = 3h_N(\sigma) \sigma \) and note that for \( h_N \approx h_q \) one has \( m_N \approx 3m_q \), as appropriate for nucleons described as composites of three constituent quarks. The baryon density can be directly inferred from eqs. (19), (20) as

\[
n = -\frac{1}{3} \left. \left( \frac{\partial U^{(Q)}}{\partial \mu} + \frac{\partial U^{(B)}}{\partial \mu} \right) \right|_{\sigma=\sigma, \omega=\omega} \tag{21}
\]

since the partial derivatives of the effective potential with respect to \( \sigma \) and \( \omega \) vanish at the \( \mu \)–dependent potential minimum \((\sigma, \omega)\).

We repeat that this picture is only a crude approximation to the binding of quarks into nucleons. A nucleon description should work well for \( h_N^2 \sigma^2 \) near \( \mu_{\text{eff}}^2 \), since only low momentum degrees of freedom contribute in this range. On the other hand, the quark description becomes important for \( h^2 \sigma^2 \ll \mu_{\text{eff}}^2 - q_H^2 \). In a more realistic description the \( \Theta \)–functions in (19), (20)
would become smooth. The characteristic quark–baryon transition momentum \( q_H \) typically depends on \( \sigma \). Indeed, a baryon description for the low momentum degrees of freedom is necessary for \( \sigma \) not too far from its vacuum expectation value \( \sigma_0 \). We will see that in this range of \( \sigma \) the transition momentum \( q_H \) is around 260 MeV or higher. On the other hand, baryons do not seem to be meaningful degrees of freedom in a situation of chiral symmetry restoration at \( \sigma = 0 \) such that \( q_H \) may vanish for \( \sigma = 0 \).

For \( h_N \) of the same order as \( h \), the nucleon mass is about three times the quark mass for a given value of \( \sigma \). Therefore eq. (20) results in an important enhancement of \( \partial U / \partial \mu \) in the range \( \sqrt{\mu_{\text{eff}}^2 - \frac{1}{3} q_H^2} < h_N |\sigma| < \mu_{\text{eff}} \) as compared to the contribution from the quarks. This is mainly due to the fact that more energy levels fall below the Fermi energy \( 3 \mu_{\text{eff}} \) for the baryons. More precisely, a factor \( 3^3 = 27 \) arises from the ratio \( [(9 \mu^2 - m^2_N) / (\mu_{\text{eff}}^2 - h^2_N \sigma^2)]^{3/2} \) if the subtraction of \( q_H^2 \) in eq. (19) can be neglected. An additional suppression of the quark contribution by a factor \( b_q = 1/3 \) in the coupling of the chemical potential is canceled by the color factor \( N_c = 3 \). (Neglecting strangeness, the two quark flavors are matched by the two species of nucleons. For a light strange quark one would observe an additional enhancement for the baryon contribution due to the larger number of baryons in an octet as compared to the three flavors.) This “nucleon enhancement” is one of the most important observations of the present paper. We believe that this effect is quite robust in view of a possible more precise modeling since only very simple properties of the fermion degrees of freedom play a role for our argument. In our description the large “nucleon enhancement” by a factor of about 27 is the basic mechanism which may lead to separate gas–liquid and hadron–quark phase transitions. Despite this enhancement one observes that \( \partial U / \partial \mu \) is continuous in \( \sigma \) and \( \mu \). Furthermore, for \( h_N(\sigma) = h_q(\sigma) \) the simultaneous jump of the renormalized fermion mass by a factor of three, together with a similar jump of the renormalized coupling to vector mesons (due to the factor \( b_j \) in eq. (10)) could also be accounted for by a sudden drop of the fermion wave function renormalization\(^9\) form \( Z_\psi = 1 \) for \( q^2 > q_H^2 \) to \( Z_\psi = 1/3 \) for \( q^2 < q_H^2 \). This corresponds to the continuity in \( \mu_{\text{eff}} \) which does not depend on the wave function renormalization multiplying the fermion kinetic term.

With \( \frac{\partial}{\partial \mu_{\text{eff}}} = \frac{\partial}{\partial \mu} \) we can easily rewrite eqs. (19), (20) as flow equations for \( \mu_{\text{eff}} \). In the approximation of \( \mu \)–independent Yukawa couplings \( h = h(\sigma) \), \( h_N = h_N(\sigma) \) and \( q_H = q_H(\sigma) \) these differential equations can be integrated analytically. We define

\[
U(\sigma, \omega; \mu, T) \equiv U_0(\sigma; T) + U_\omega(\sigma, \omega; T) + 2 U_\mu(\sigma, \omega; \mu, T)
\]

(22)

where \( 2 U_\mu(\sigma, \omega; \mu, T) \) entails the \( \mu \)–dependent contribution from the two lightest quarks \( (2 U^{(q)}_\mu) \) as well as proton and neutron \( (2 U^{(n)}_\mu) \). The \( \mu \)–independent part of the potential is thus given by \( U_0 + U_\omega \) with \( U_\omega \) the \( \omega \)–dependent contribution. For \( T = 0 \) we obtain

\[
U_\mu = U^{(q)}_\mu + U^{(n)}_\mu,
\]

(23)

\[
U^{(q)}_\mu(\sigma, \omega; \mu, 0) = -\frac{1}{4 \pi^2} \mu_{\text{eff}} \left( \mu_{\text{eff}}^2 - \frac{5}{2} h^2_N \sigma^2 \right) \sqrt{\mu_{\text{eff}}^2 - h^2_N \sigma^2}
\]

\(^9\)Such a drop of the effective wave function renormalization would be required for a Higgs picture of the QCD vacuum where quarks and baryons are described by the same field [10].
\[ U^{(n)}_{\mu}(\sigma, \omega; \mu, 0) = -\frac{27}{4\pi^2} \left\{ \left( \frac{\mu^2}{h_N^2} - \frac{2}{9} \sigma^2 \right) \Theta(\mu^2 - h_N^2\sigma^2) \Theta(h_N^2\sigma^2 - \frac{1}{9} q_H^2 - \mu^2) \right\} \]

In this expression the dependence on \( \mu \) and \( \omega \) appears only implicitly through \( \mu_{\text{eff}} \). For given \( h_q \) and \( h_N \) the \( \sigma \)-dependence of \( U_{\mu} \) is uniquely determined once the dependence of \( q_H \) on \( \sigma \) is fixed.

The qualitative dependence of \( q_H(\sigma) \) on \( \sigma \) can be inferred from the following argument: A crucial ingredient for the confinement of quarks in hadrons is the formation of QCD strings. Strings break because of pair production of mesons if typical quark kinetic energies become too large. Therefore baryons can only exist for sufficiently small average quark kinetic energies or momenta. Very roughly, the relevant critical kinetic energy is expected to be proportional to the pion mass \( \sqrt{q_H^2(\sigma) + h^2\sigma^2} \approx 2m_\pi(\sigma) \). The \( \sigma \)-dependence of the pion mass can be inferred from the effective potential as \( m_\pi^2(\sigma) = (\partial U/\partial \sigma + 2m_\pi^2 f_\pi)/(4\sigma) \), with \( m_\pi \) the pion mass in the vacuum and \( f_\pi \) the pion decay constant. Since \( m_\pi^2(\sigma) \) always tends to zero for small enough \( \sigma \) there should be a critical value \( \sigma_c \) for which \( q_H(\sigma_c) = 0 \). We will not use baryons for \( \sigma < \sigma_c \) and take \( q_H(\sigma < \sigma_c) = 0 \). For our purpose we will be satisfied with a crude approximation\(^{10}\) where we neglect the \( \sigma \)-dependence of \( q_H \) in the range of \( \sigma \) relevant for nuclear physics, \( \sigma > \sigma_H \)

\[ q_H(\sigma) = \frac{q_H}{\sigma_H^2 - \sigma_c^2} \sqrt{(\sigma^2 - \sigma_c^2)(2\sigma_H^2 - \sigma_c^2 - \sigma^2)} \Theta(\sigma - \sigma_c) \Theta(\sigma_H - \sigma) + q_H \Theta(\sigma - \sigma_H). \] \( \sigma_H = 25 \text{ MeV} \). We expect that the constant \( q_H \) should have the size of a typical QCD scale, i.e., around 200 MeV. On the other hand, we will find that the quantitative aspects of the quark–hadron transition depend on \( \sigma_c, \sigma_H \) and \( q_H \) which parameterize in our crude approximation the effects of confinement. This underlines that a more quantitative understanding of the effective transition from quarks to nucleons is needed before reliable statements about the hadron-quark phase transition at low temperature can be made.

It is interesting to note that for two light quark flavors \( (N_c N_F = 6) \) the \( \mu \)-dependent contribution to the potential at the origin and therefore to the energy density reads for arbitrary

\(^{10}\) For our choice \( \partial q_H(\sigma)/\partial \sigma \) is continuous at \( \sigma = \sigma_H \).
finite $h(\sigma, \mu)$

\[ e^{(0)}_{\mu} = -6U_\mu(\sigma = 0, \omega; \mu, 0) = \frac{3}{2\pi^2} \mu^4 = \left(\frac{3}{2}\right)^{7/3} \pi^{2/3} \left(\bar{n}^{(0)}\right)^{4/3}. \]  

This has the simple interpretation of the total energy of six massless s quarks with all energy levels filled up to the Fermi energy $\mu_{\text{eff}}$. Furthermore, for $\sigma$ and $\mu$ in the range relevant for nuclear physics and for sufficiently large $q_H$, i.e. $q_H^2 > 9(\mu_{\text{eff}}^2 - h_N^2\sigma^2)$, the contribution $U^{(n)}_{\mu}$ is simply the mean field result for a nucleon meson model, whereas $U^{(q)}_{\mu}$ vanishes. Our approach gives a new motivation for the approximate validity of mean field theory from the truncation of an exact flow equation. Furthermore, it offers the possibility of a systematic improvement, e.g., by taking the $\mu$–dependence of $h_N$ into account. Despite this similarity, our method goes beyond mean field theory in an important aspect: For the free energy only the difference between vanishing and non–vanishing chemical potential is described by mean field theory, whereas we do not rely on mean field results for the effective action $\Gamma$ at $\mu = 0$. Since $\Gamma(\mu = 0)$ is the generating functional for the propagators and vertices in vacuum it can, in principle, be directly related to measured properties like meson masses and decays. This is very important in practice, since mean field theory does not give a very reliable description of the vacuum properties.

\section*{4 Meson interactions}

In order to discuss possible phase transitions as $\mu$ is increased beyond a critical value we need information about the effective potential for $\mu = 0$. For a vacuum without spontaneous symmetry breaking relatively accurate information about $U_0(\sigma; T) = U_0(\sigma, \omega = 0; \mu = 0; T)$ for all relevant $\sigma$ could be extracted from the knowledge of meson masses and interactions. Also the approximation (10) for the fermionic propagator would presumably be reasonable for arbitrary $\sigma$. In case of spontaneous chiral symmetry breaking the situation is more complex: The true effective potential $U_0$ becomes convex because of fluctuations which interpolate between the minima of the “perturbative” or “coarse grained” potential [31, 32]. Masses and interactions give only information about the “outer region” of the potential which is not affected by this type of fluctuations. In parallel, the simple form of the fermionic propagator (10) becomes invalid in the “inner region” for small $\sigma$ because of a complex momentum dependence [31, 32] and the breakdown of the approximation of a constant Yukawa coupling. In order to cope with these difficulties, $U_0(\sigma; T)$ should rather be associated with a coarse grained effective potential. For a suitable coarse graining scale\footnote{The coarse graining scale $k$ is chosen such that $U_k$ is approximately $k$–independent for $|\sigma|$ around $|\overline{\sigma}|$ or larger, whereas the approach to convexity for $|\sigma| < |\overline{\sigma}|$ and $k \to 0$ has not yet set in.} $k$ the effect of the omitted fluctuations with momenta smaller than $k$ is expected to be small near the $\mu$–dependent minimum of $U$. Around the minimum at $\sigma_0$ we can therefore continue to associate $U_0(\sigma; T)$ with the effective potential and relate its properties to the measured masses and decay constants. On the other hand, we do not have much information about the shape of $U_0(\sigma; T)$ for $\sigma \approx 0$. This uncertainty in the appropriate choice of $U_0(\sigma; T)$ is one of the main shortcomings of our method. In practice, we interpolate the partly known polynomial form of $U_0(\sigma; T)$ form the outer region (which includes the minimum characterizing the vacuum) to the inner region for small $\sigma$. By continuity, this
should be quite reasonable for nuclear matter since the relevant values of \( \sigma \) are not much smaller than the vacuum expectation value \( \sigma_0 \). For quark matter, the uncertainties are more important.

We investigate here the two flavor case with a potential of the form

\[
U_0(\sigma; T) \equiv 2m_\pi^2(T) \left[ \sigma^2 - \sigma_0^2(T) \right] + 2\lambda(T) \left[ \sigma^2 - \sigma_0^2(T) \right]^2 + \\
+ \frac{4}{3} \gamma_3(T) \left[ \sigma^2 - \sigma_0^2(T) \right]^3 + \frac{\gamma_4(T)}{\sigma_0^4(T)} \left[ \sigma^2 - \sigma_0^2(T) \right]^4 \\
+ \frac{4}{5} \gamma_5(T) \left[ \sigma^2 - \sigma_0^2(T) \right]^5 - 2j\sigma + c(j, T) \tag{28}
\]

where

\[
j = 2m_\pi^2(0)\sigma_0(0), \quad c(j, 0) = 2j\sigma_0(0) \tag{29}
\]

In the remainder of this work we mainly consider \( T = 0 \) and use \( \lambda \equiv \lambda(0) \), \( U(\sigma; \mu) \equiv U(\sigma; \mu, 0) \) etc. The meson field is normalized such that \( \sigma_0 = \sigma_0(0) \) is related to the pion decay constant by \( \sigma_0 = f_\pi/2 = 46.5 \text{ MeV} \). This means that the pions have a standard kinetic term (as derived from \( \mathcal{L}^{(0)}_{\text{kin}} = \text{Tr} \partial_\mu \Phi^\dagger \partial^\mu \Phi \)). Because of higher order kinetic invariants \[24\] the kinetic term for the sigma meson, \( \mathcal{L}_{\text{kin, } \sigma} = 2Z_\sigma \partial_\sigma \sigma^\dagger \partial^\mu \sigma \) can involve a wave function renormalization \( Z_\sigma \) different from the one for the pions. The potential (28) arises from a fifth order polynomial in the invariant \( \rho = \text{Tr} \Phi^\dagger \Phi = 2\sigma^2 \) with an additional source term \( -\frac{1}{3} \text{Tr} f(\Phi + \Phi^\dagger) \), where \( j \) is proportional to the renormalized current quark mass (say at 1 GeV). The only violation of the chiral \( SU_L(2) \times SU_R(2) \) symmetry arises from this source and in the chiral limit of vanishing current quark masses the last two terms in eq. (28) should be dropped. The coupling \( \lambda \) is related to the \( \sigma \)-mass \( m_\sigma \) by \( \tilde{m}_\sigma^2 = Z_\sigma m_\sigma^2 = m_\pi^2 + 4\lambda \sigma_0^2 \). We will use here \( \tilde{m}_\sigma = 510 \text{ MeV, } \lambda = 28 \). It is actually \( \tilde{m}_\sigma \) rather than the physical mass \( m_\sigma \) which is relevant for the properties of nuclear matter. One of the parameters \( \gamma_3, \gamma_4 \) or \( \gamma_5 \) can be eliminated in favor of the scale \( \mu_0 \) which characterizes the height of \( U_0 \) at the origin

\[
\frac{\mu_0^4}{2\pi^2} \equiv U(0; 0) = \sigma_0^4 \left( 2\lambda - \frac{4}{3} \gamma_3 + \frac{4}{3} \gamma_4 - \frac{4}{5} \gamma_5 \right) + 2m_\pi^2 \sigma_0^2. \tag{30}
\]

Without the complications of confinement (i.e., for \( q_\text{H} = 0 \)) the quark–hadron phase transition in the chiral limit \( (j = 0) \) would occur for \( \mu_\text{eff} = \mu_0 \).

Finally, we determine the expectation value of \( \omega \) by observing the identity

\[
\frac{\partial U_\mu}{\partial \omega} = \frac{2}{3} g_\omega \frac{\partial U_\mu}{\partial \mu}. \tag{31}
\]

It follows from first differentiating eqs. (19), (20) with respect to \( \omega \) and then performing the \( \mu \)-integration. For the \( \mu = 0 \) contribution we only take into account a \( \sigma \)- and \( T \)-dependent mass term

\[
U_\omega = -\frac{1}{2} M_\omega^2(\sigma, T) \omega^2. \tag{32}
\]

The solution of the \( \omega \)-field equations for arbitrary \( \sigma, T, \mu \) obeys

\[
\bar{\omega}(\sigma, \mu, T) = \frac{2g_\omega}{3M_\omega^2} \frac{\partial U_\mu}{\partial \mu} (\sigma, \bar{\omega}; \mu, T). \tag{33}
\]
We note that at the potential minimum $\bar{\omega}$ is proportional to the baryon density with a negative coefficient. This implies that the coupling to $\omega$ reduces the effective chemical potential [13]. In the following we will always assume that $\bar{\omega}(\sigma, \mu, T)$ is inserted such that $\mu_{\text{eff}}$ becomes a function of $\sigma$, $\mu$ and $T$. The field equation which determines the location $\bar{\sigma}$ of the minimum of the effective potential can be expressed in terms of partial $\sigma$–derivatives at fixed $\mu_{\text{eff}}$

$$\frac{\partial U_0}{\partial \sigma}(\sigma) - M_\omega(\sigma) \frac{\partial M_\omega(\sigma)}{\partial \sigma} \omega^2 + 2 \frac{\partial U_\mu}{\partial \sigma}(\sigma) \mu_{\text{eff}} = 0 . \quad (34)$$

Below we will also neglect a possible $\sigma$-dependence of $M_\omega$. The location of the minimum $\bar{\sigma}$ becomes then independent of the value of $\omega$ and only depends on $\mu_{\text{eff}}$. For this setting the coupling to vector mesons is relevant only for the relation between $\mu_{\text{eff}}$ and $\mu$.

5 Meson–baryon interactions

A crucial ingredient for any quantitative analysis is the sigma–nucleon coupling $h_N$. We first investigate if chiral symmetry and the observed value of the pion nucleon coupling place any restrictions on this coupling. For this purpose we employ a derivative expansion of the most general effective Lagrangian which is bilinear in the nucleon doublet field $\Psi_N$ and involves scalar and pseudoscalar fields contained in the $2 \times 2$ matrix $\Phi$

$$\mathcal{L} = \frac{1}{2} \left\{ \bar{\Psi}_{NR} F(\Phi \Phi^\dagger, \rho) \Phi \Psi_{NL} - \bar{\Psi}_{NL} F(\Phi^\dagger \Phi, \rho) \Phi^\dagger \Psi_{NR} + \bar{\Psi}_{NL} G_1(\Phi \Phi^\dagger, \rho) i\gamma^\mu \partial_\mu \Psi_{NL} + \bar{\Psi}_{NR} G_1(\Phi^\dagger \Phi, \rho) i\gamma^\mu \partial_\mu \Psi_{NR} \right\} \left\{ \bar{\Psi}_{NL} \Phi G_2(\Phi \Phi^\dagger, \rho) i\gamma^\mu (\partial_\mu \Phi) \Psi_{NL} + \bar{\Psi}_{NR} \Phi G_2(\Phi^\dagger \Phi, \rho) i\gamma^\mu (\partial_\mu \Phi^\dagger) \Psi_{NR} + \text{h.c.} \right\} . \quad (35)$$

Here we have imposed $\mathcal{P}$ and $\mathcal{C}$ symmetry and used $\Psi_{NL} = (1 + \gamma_5)\Psi_N/2$. With the standard decomposition

$$\Phi = \sigma \xi^2 = \sigma U , \quad \xi = \exp \left( \frac{i}{4\sigma} \vec{\tau} \vec{n} \right) , \quad N_L = \xi \Psi_{NL} , \quad N_R = \xi^\dagger \Psi_{NR} , \quad (36)$$

one finds

$$\mathcal{L} = 3h_N(\sigma) \sigma \bar{N} \gamma_5 N + Z_N(\sigma) \bar{N} \left( i\gamma^\mu \partial_\mu - \gamma^\mu v_\mu + G_A(\sigma) \gamma^\mu \gamma_5 a_\mu \right) N - i \frac{Z_N(\sigma)}{2\sigma^2} \left[ G_A(\sigma) - 1 \right] \bar{N} \gamma^\mu N \quad (37)$$

where

$$h_N(\sigma) = F(\sigma^2, 2\sigma^2)/3 , \quad Z_N(\sigma) = G_1(\sigma^2, 2\sigma^2) , \quad G_A(\sigma) = 1 - \frac{2G_2(\sigma^2, 2\sigma^2)\sigma^2}{G_1(\sigma^2, 2\sigma^2)} . \quad (38)$$
and

\[ v_\mu = -\frac{i}{2} \left( \xi^{\dag} \partial_\mu \xi + \xi \partial_\mu \xi^{\dag} \right) \]

\[ a_\mu = -\frac{i}{2} \left( \xi^{\dag} \partial_\mu \xi - \xi \partial_\mu \xi^{\dag} \right) = \frac{1}{4\sigma} \vec{p} \partial_\mu \vec{p} + \ldots . \]  

Normalisation of the baryon number current requires \( Z_N(\sigma_0) = 1 \) and we neglect the \( \sigma \)–dependence of \( Z_N \) in the following. The strength of the linear pion–nucleon coupling is fixed by \( g_A = G_A(\sigma_0) \) and bares no relation to the function \( h_N(\sigma) \). We may expand \( h_N(\sigma) \) around \( \sigma_0 \)

\[ h_N(\sigma) = h_N(\sigma_0) + \frac{g_N}{\sigma_0} (\sigma^2 - \sigma_0^2) + \ldots . \]  

With \( m_N = 3h_N(\sigma_0)\sigma_0 = 939 \text{ MeV} \) we find \( h_N(\sigma_0) = 6.73 \). Linearizing \( m_N(\sigma) = 3h_N(\sigma)\sigma \) around \( \sigma_0 \) then yields \( m_N(\sigma) = 3h_s + \epsilon_G \), with \( h = h_N(\sigma_0) + 2g_N, \epsilon_G = -6g_N\sigma_0 \). The linear sigma–nucleon coupling \( \tilde{h} \) is a free parameter which is expected to be in the vicinity of \( h_N(\sigma_0) \). We will determine it below from the properties of nuclear matter. Since \( \tilde{h} \) also appears in the scattering of nucleons a comparison with experiment may serve as a test for our model.

6 Nuclear matter and the nuclear phase transition

Let us turn to the zero temperature properties of nuclear matter in our picture. For \( \mu = 0 \) the effective potential or free energy \( U \) has its minimum at \( \sigma_0 = f_\pi/2 = 46.5 \text{ MeV} \). The potential in the region near \( \sigma_0 \) is not altered as long as \( \mu_{\text{eff}} \) remains small enough (cf. eq. (20)). This changes as \( \mu_{\text{eff}} \) is increased beyond a critical threshold. For suitable parameters in \( U_0 \) (eq. (28)) we observe that for \( 3\mu \) somewhat below the nucleon mass a new minimum of \( U \) occurs at \( \langle \sigma_{\text{nuc}} \rangle(\mu) < \sigma_0 \), with a potential barrier between both minima. For a certain range of \( \mu \) the local minimum at \( \langle \sigma_{\text{nuc}} \rangle(\mu) \) and the global minimum at \( \sigma_0 \) coexist. As \( \mu \) increases, the value of \( U(\sigma_{\text{nuc}}(\mu)) \) is lowered whereas \( U(\sigma_0) = 0 \) remains fixed as long as the effective chemical potential is smaller than a third of the nucleon mass, \( \mu_{\text{eff}} < h_N(\sigma_0)\sigma_0 \). There is a critical value \( \mu_{\text{nuc}} \) for which the two minima at

\[ \sigma_{\text{nuc}} \equiv \sigma_{\text{nuc}}(\mu_{\text{nuc}}) \]  

and \( \sigma_0 \) are degenerate, \( U(\sigma_{\text{nuc}}, \mu_{\text{nuc}}) = U(\sigma_0, \mu_{\text{nuc}}) = 0 \). The corresponding critical potential is plotted in figure 1. Both phases have equal, vanishing pressure \( p = -U \) and can coexist. We observe that the phase transition between the vacuum (\( \sigma = \sigma_0 \)) and nuclear matter (\( \sigma = \sigma_{\text{nuc}} \)) is clearly of first order. For small temperature this corresponds to the transition between a gas of nucleons and nuclear matter which may be associated with a nuclear liquid.

For a quantitative description we concentrate mainly on large values of \( q_H \) where this transition happens in a region with \( \mu_{\text{eff}}^2 < h_N(\sigma_{\text{nuc}})\sigma_{\text{nuc}}^2 + q_H^2/9 \). In this case the transition from nucleon to quark degrees of freedom does not affect nuclear properties and the phase transition from a nucleon gas to nuclear matter. We partly recover the \( \sigma-\omega \)–model of nuclear physics [33, 34, 20], in a context where chiral symmetry breaking and constraints from meson
masses and decays are properly incorporated. For large enough \( q_H \) the critical baryon density of the nuclear liquid is given by eq. (20)

\[
n_nuc = \frac{18}{\pi^2} \left[ \mu_{\text{eff}}(\mu_{nuc}, \sigma_{nuc}) - h_N(\sigma_{nuc}) \sigma_{nuc}^2 \right]^{3/2}.
\]  

(42)

We will see below that one can identify \( n_nuc \) with the baryon density in nuclei \( n(\rho) = 1.175 \times 10^{6} \) up to small corrections. Furthermore, the baryon number independent contribution to the binding energy per nucleon in a large sample of nuclear matter is known from the mass formula for nuclei: \( \beta(\rho) = -16.3 \text{MeV} \). In our context one finds

\[
\beta(\rho) = 3\mu_{nuc} - m_N \text{ and for realistic models the gas–liquid transition should therefore occur for } \mu_{nuc} = 307.57 \text{MeV. Eq. (42) then yields a quantitative relation between the effective chemical potential in nuclear matter } \mu_{\text{eff}, nuc} = \mu_{\text{eff}}(\mu_{nuc}, \sigma_{nuc}) \text{ and the effective nucleon mass } m_N(\sigma_{nuc}) = 3h_N(\sigma_{nuc}) \sigma_{nuc} = \epsilon_G + 3h\sigma_{nuc}. \text{ For } m_N(\sigma_{nuc})/m_N = (0.6, 0.65, 0.7, 0.75, 0.8) \text{ one finds } \mu_{\text{eff}, nuc} = (206.6, 220.9, 235.4, 250.1, 264.8) \text{ MeV. Equivalently, this can be seen as a relation between } h_N(\sigma_{nuc}) \sigma_{nuc} \text{ and the coupling } g_{(\omega)} \text{ if we use with (34)}
\]

\[
\mu_{\text{eff}, nuc} = \mu_{nuc} - \frac{1}{3} \frac{g_{\omega}^2}{M_{\omega}^2} n_{nuc}.
\]  

(43)

For a \( \sigma \)-independent \( \omega \)-mass \( M_{\omega} = 783 \text{MeV} \) typical values for the above ratios for \( m_N(\sigma_{nuc})/m_N \) are \( g_{(\omega)} = (12.61, 11.68, 10.66, 9.52, 8.21) \). From the value of nuclear density we can compute the Fermi momentum \( q_{\text{nuc}} = 259 \text{MeV} \). This yields for this scenario a lower bound \( q_H > q_{\text{nuc}} = 259 \text{MeV} \). (For quantitative computations we take \( q_H = 1.2 q_{\text{nuc}} \).)

An important quantity for the equation of state is the compression modulus

\[
K = 9n^2 \frac{d^2}{dn^2} \left( \frac{\epsilon}{n} \right) = 9 \left( \frac{dp}{dn} - 2p \right) \left( \frac{n}{\rho} \right).
\]  

(44)
The value at the phase transition

\[ K_0 = 9 \frac{dp}{dn} \bigg|_{\nu_{\text{muc}}} = 27n_{\text{muc}} \frac{d\mu}{dn} \bigg|_{\nu_{\text{muc}}} = 9n_{\text{muc}} \frac{g(\omega)^2}{M_\omega^2} + \frac{1}{\mu_{\text{eff},\nu_{\text{muc}}}} \left( \frac{3\pi^2n_{\text{muc}}}{2} \right)^{2/3} + 9n_{\text{muc}} \frac{d\sigma}{dn_{\text{muc}}} \left[ \frac{\tilde{h}m_N(\sigma_{\text{muc}})}{\mu_{\text{eff},\nu_{\text{muc}}}} + n_{\text{muc}} \frac{d}{d\sigma} \left( \frac{g(\omega)^2}{M_\omega^2} \right) \right] \]

(45)

has been inferred from experiment as \( K_0 = (210-220) \text{ MeV} \) [35, 20]. This can be used to obtain additional information about the \( \mu \)-independent part \( U_0 \) of the effective potential. Neglecting the \( \sigma \)-dependence of \( M_\omega \) one finds by differentiating eq. (34)

\[ \frac{d\sigma}{dn} = -\frac{\tilde{h}m_N(\sigma_{\text{muc}})}{\mu_{\text{eff}}(\sigma_{\text{muc}})} \left[ \frac{\partial^2 U_0(\sigma)}{\partial\sigma^2} + 2\frac{\partial U_\mu}{\partial\sigma} \right|_{\mu_{\text{eff}}(\sigma)} + \frac{6}{\pi^2} \frac{\tilde{h}^2 m_N^2(\sigma)}{\mu_{\text{eff}}(\sigma)} \right]^{-1}. \]  

(46)

Combining eqs. (46) and (45) the compression modulus yields information about \( \frac{\partial^2 U_0}{\partial\sigma^2}(\sigma_{\text{muc}}) \) in addition to \( U_0(\sigma_{\text{muc}}) \) and \( \frac{\partial U_\mu}{\partial\sigma}(\sigma_{\text{muc}}) \) which are determined (for given \( m_N(\sigma_{\text{muc}}) \) and \( \tilde{h} \)) by the condition \( U(\sigma) = 0 \) and the field equation (34).

For any given value of the coupling \( \gamma_5 \) the system of equations provides a mapping between the parameters \((\tilde{h}, g(\omega), \gamma_3, \gamma_4)\) and the quantities \((n_{\text{muc}}, \beta, K_0, m_N(\sigma_{\text{muc}}))\). For a demonstration of the range of values for various quantities of interest we report our results for two parameter sets with different \( \gamma_5 \) (A and B) in tables 1–3. (For both sets \( \beta = -16.3 \text{ MeV} \) and \( n_{\text{muc}} = \pi^{(0)} \).) Agreement with nuclear properties can indeed be achieved. It is not our aim here to make a precise determination of parameters and we only mention that somewhat smaller values of \( m_N(\sigma_{\text{muc}}) \) or other (large) values of the compression modulus lead to qualitatively similar results. One finding remains common, however: for nuclear matter properties in a reasonable range we always need large values of some of the couplings \(|\gamma_3|, |\gamma_4| \) or \(|\gamma_5|\). No viable solution was found for an approximately quartic meson potential with small \(|\gamma_{3,4,5}|\). This may be a cause of worry for this class of models since earlier renormalization group studies of the effective meson potential have typically resulted in substantially smaller higher order couplings \(|\gamma_{3,4,5}|\) than the ones needed here [22].

For given parameters we can also compute the energy density and the pressure and relate it to the baryon density. This determination of the equation of state of dense nuclear matter is

|   | \( \tilde{h} \) | \( g(\omega) \) | \( \gamma_3 \) | \( \gamma_4 \) | \( \gamma_5 \) | \( \mu_0 \) MeV |
|---|---|---|---|---|---|---|
| A | 5.4 | 9.02 | -30 | 47 | -60 | 372 |
| B | 5.0 | 9.52 | 19 | 112 | 0 | 348 |
| C | 4.6 | 8.74 | 0 | 55 | 0 | 330 |

Table 1: Coupling constants for three different parameter sets. The linear sigma–nucleon coupling \( \tilde{h} \), the coupling \( g(\omega) \) of the \( \omega \)-meson to the \( u, d \)-quarks and the nucleons, the meson self–interactions \( \gamma_3, \gamma_4, \gamma_5 \) and the scale \( \mu_0 \) are defined in sections 4 and 5.
rather insensitive to aspects of QCD not treated here like the role of gluons or vector mesons (beyond the effect of $\overline{\omega} \neq 0$). The reason is that these degrees of freedom do not contribute to the difference of the effective action between zero and nonzero chemical potential. The nuclear equation of state can therefore be considered as a prediction of the model (for fixed parameters).

|      | $\overline{\sigma}_{\text{nuc}}$ MeV | $m_N(\overline{\sigma}_{\text{nuc}})$ | $K_0$/MeV | $\Sigma$/MeV | $\mu_{\text{eff,nuc}}$/MeV |
|------|------------------------------------|-------------------------------------|-----------|--------------|--------------------------|
| $A$  | 33.2                               | 0.77                                | 214       | 1            | 256                      |
| $B$  | 30.85                              | 0.75                                | 217       | 1.3          | 250                      |
| $C$  | 29.8                               | 0.755                               | $\infty$ | 2            | 259                      |

Table 2: Properties of nuclear matter at vanishing pressure for the parameters of table 1. The table shows values for the chiral order parameter $\overline{\sigma}_{\text{nuc}}$, the effective nucleon mass $m_N(\overline{\sigma}_{\text{nuc}})$, the compression modulus $K_0$ and the effective chemical potential $\mu_{\text{eff,nuc}}$ for nuclear matter. The surface tension $\Sigma$ for the droplet model of nuclei normalized to the value $\Sigma^{(n)}$ extracted from the nuclear mass formula is discussed in section 7.

In figure 2 we have plotted the binding energy per nucleon, $\beta = \epsilon/n - m_N$, as a function of density corresponding to the parameter set $A$. For values of $n$ larger than approximately 1.7 the details of the transition from nuclear to quark degrees of freedom become important and we don’t expect our results to remain quantitatively reliable. Similarly, our results for the baryon density as a function of pressure are displayed in figure 3. Figures 2 and 3 can be combined to yield the equation of state $\epsilon(p)$ for $n < 1.5\overline{n}^{(n)}$.

Figure 2: Binding energy per nucleon $\beta$ as a function of density. Parameter values correspond to A in table 1.
Figure 3: Baryon density as a function of pressure in the vicinity of the nuclear gas–liquid transition at very low $T$. Parameters correspond to set A in table 1.

Within the " $\sigma - \omega$ model" (models A and B) the dominant repulsion between nucleons at short distance is ascribed to the exchange of $\omega$-mesons. It is not established if this repulsion is indeed sufficient. Another possible repulsion mechanism is the effective transition from nucleons to quarks at short distance. In view of the potential difficulties of the $\sigma - \omega$ model we also explore this second alternative — our model C. For this purpose it is instructive to consider an extreme scenario where the characteristic quark–baryon transition momentum $q_H$ takes on its lower bound

$$q_H(\sigma_{\text{nuc}}) = q_H = q_{\text{nuc}} = 259 \text{ MeV} \quad (47)$$

The results correspond to the set $C$ of tables 1–3. Because of the $\Theta$–function in eq. (20) the nucleon contribution to the density does not increase any more as $\mu_{\text{eff}}$ exceeds the critical value given by eq. (43). On the other hand, there is a range of $\mu_{\text{eff}}$ for which the quark fluctuations (19) do not yet contribute to the baryon density. For this range the density $n$ will not depend on any other parameter of the model and $n_{\text{nuc}} = n^{(a)}$ is guaranteed by eq. (47). Details of the potential in the vicinity of $\sigma_{\text{nuc}}$ are now affected by the transition from nucleon to quark degrees of freedom. For the simple choice (26), however, they do not depend on $\sigma_c$ or $\sigma_H$ provided both are smaller than $\sigma_{\text{nuc}}$. Because of the gap in $\mu_{\text{eff}}$ between the nucleon Fermi surface and the onset of quark fluctuations many properties become very simple. The minimum occurs within the range $\sigma_q < \sigma_{\text{nuc}} < \sigma_{nf}$. Here $\sigma_q$ corresponds to the onset of quark fluctuations

$$\sigma_q(\mu) \equiv \frac{1}{\hbar} \sqrt{\mu^2_{\text{eff}} - q_H^2(\sigma_q)} \quad (48)$$

whereas $\sigma_{nf}$ denotes the maximal value of $\sigma$ for which all nucleon levels with $q^2 \leq q_H^2$ are filled

$$\sigma_{nf}(\mu) \equiv \frac{1}{\hbar N(\sigma_{nf})} \sqrt{\mu^2_{\text{eff}} - \frac{1}{9} q_H^2(\sigma_{nf})} \quad (49)$$
For values of $\sigma\text{^{(nuc)}}(\mu)$ between $\max(\sigma_q, \sigma_H)$ and $\sigma_{nf}$ the baryon density is independent of $\mu$

$$n(\mu) = \frac{2}{3\pi^2} q_H^3 = \pi^{(n)}.$$  \hspace{1cm} (50)

This would give a natural explanation for a large compression modulus $K$ according to eq. (45). Also, for $\max(\sigma_q, \sigma_H) < \sigma < \sigma_{nf}$ one finds that $\partial U_\mu/\partial \mu$ is independent of $\sigma$ and the constant shift (43) between $\mu_{\text{eff}}(\sigma, \mu)$ and $\mu$ holds for all $\sigma$. The relation between $n_{\text{nucl}}$ and $q_H$ is such that up to a Fermi momentum $q_H(\sigma_{\text{nucl}}) = q_H$ all levels are filled with nucleons (or bound quarks). In this crude picture the higher momentum levels (corresponding to a larger baryon number in a fixed volume) would have to be filled by free (constituent) quarks. This leads to a particularly simple explanation why nuclear density is almost independent of all other parameters characterizing the state of nuclear matter at $T = 0$, like pressure, baryon number or the $Z/B$ ratio of a nucleus. Typical parameter values and corresponding characteristics of nuclear matter for this “saturation scenario” can be found as set C in tables 1 and 2.

We next discuss the equation of state for the saturation scenario. For a given value of $\sigma$ one finds in this scenario a range $\mu_{nf} < \mu_{\text{eff}} < \mu_q$ with constant $\partial U_\mu/\partial \mu$ where

$$\mu_{nf}(\sigma) = \sqrt{h_N^2(\sigma)\sigma^2 + \frac{1}{9} q_H^2(\sigma)}$$

$$\mu_q(\sigma) = \sqrt{h_Q^2\sigma^2 + q_H^2(\sigma)}.$$  \hspace{1cm} (51)

In this range $\pi$ is independent of $\mu$ and $U_\mu$ has the simple form

$$U_\mu(\sigma; \mu) = U_\mu(\sigma; \pi) - \frac{1}{\pi^2} (\mu - \pi) q_H^3(\sigma).$$  \hspace{1cm} (52)

Here $\pi$ is a fixed reference value within the interval $[\mu_{nf}, \mu_q]$, and we remind the reader that $q_H(\sigma > \sigma_H) = q_H$. We note that the location of the potential minimum at $\sigma\text{^{(nuc)}}(\mu)$ is independent of $\mu$. Using $\pi = \mu_{\text{nucl}}$, the pressure and energy density of nuclear matter are

$$p = \frac{2 q_H^3}{\pi^2} (\mu - \mu_{\text{nucl}})$$  \hspace{1cm} (53)

$$\epsilon = \frac{2}{\pi^2} q_H^3(\mu - p) = (m_N + \beta) n.$$  \hspace{1cm} (54)

The $(T = 0)$ equation of state for nuclear matter for this extreme saturation scenario

$$\frac{\partial \epsilon}{\partial p} = 0, \quad \frac{\partial n}{\partial p} = 0;$$  \hspace{1cm} (55)

implies a diverging compression modulus $K_0$ and is therefore not fully realistic. Nevertheless, it is well conceivable that the true behavior of nuclear matter is somewhere between the simple version of the $\sigma$–$\omega$ model and the extreme saturation scenario. In the language of the $\sigma$–$\omega$ model this would be expressed through the momentum–dependence of couplings and wave function renormalizations (form factors).

Despite the substantial difference in the compression modulus the three scenarios (A)–(C) all show a similar value of $\sigma_{\text{nucl}} \simeq 30$ MeV and therefore a nucleon mass in nuclear matter
around 700 MeV, as supported by some experimental evidence [36, 20]. Also the critical value $\mu_{\text{eff, nuc}} \simeq 250$ MeV is very similar for these three models. All three scenarios support the existence of a first-order transition from the vacuum to nuclear matter. Extending this result to small $T \neq 0$ this results in a first-order transition from a nucleon gas at low density to a nuclear liquid at high density, in complete analogy to the vapor-water transition.

7 Droplet model for nuclei

For a first-order gas-liquid transition we can describe sufficiently large nuclei as droplets of nuclear matter in a surrounding vacuum. For quantitative estimates of their properties we have to take into account that because of the surface tension the pressure inside the droplet is different from zero. The nucleus is at equilibrium if the pressure equals the derivative of the sum of surface and Coulomb energy

$$p = \frac{\partial E_\Sigma}{\partial V} + \frac{\partial E_c}{\partial V}$$

$$E_\Sigma = \left(36\pi\right)^{1/3} \Sigma V^{2/3}, \quad E_c = \frac{3\alpha}{5} \left(\frac{4\pi}{3}\right)^{1/3} \kappa Z^2 V^{-1/3}.$$  \hspace{1cm} (56)

The surface tension $\Sigma$ can be expressed (thin wall approximation) in terms of the potential as

$$\Sigma = 2 \left(\frac{Z_\sigma}{Z_\pi}\right)^{1/2} \int_{\sigma_0}^{\sigma_{\text{nuc}}} d\sigma \sqrt{2U(\sigma; \mu)}.$$  \hspace{1cm} (57)

We use the phenomenological relation $Z/B \simeq (2 + 0.0153 B^{2/3})^{-1}$, $\alpha = 1/137$, and $\kappa$ should be very close to one. This leads to a volume- and therefore baryon number dependent pressure

$$p = \frac{4}{3} \left(\frac{3}{\pi}\right)^{1/3} q_{\text{nuc}} \Sigma B^{-1/3} - \frac{4}{45\pi^2} \left(\frac{3}{\pi}\right)^{1/3} \alpha \kappa q_{\text{nuc}}^4 Z^2 B^{-4/3}$$

and a total binding energy per nucleon

$$\frac{E_B}{B} = 3\mu(p) - m_N - \frac{p}{n} + \frac{E_\Sigma}{B} + \frac{E_c}{B} = \beta + \frac{E_\Sigma}{B} + \frac{E_c}{B} - \frac{9}{2K_0} \frac{p^2}{n_{\text{nuc}}^2} + \ldots.$$  \hspace{1cm} (58)

Neglecting the pressure term the mass formula for nuclei yields the “experimental” values

$$\Sigma^{(n)} = 4.22 \cdot 10^4 \text{ MeV}^3$$

$$\kappa^{(n)} = 0.96.$$  \hspace{1cm} (60)

For $B = 208(12)$ one finds $p = 0.9(5.9) \cdot 10^6 \text{ MeV}^4$. The pressure therefore contributes to $E/B$ only very little, $\Delta(E/B) = -0.012(-0.53)$ MeV and can indeed be neglected for large $B$. For small $B$ eqs. (58), (59) result in an interesting correction to the mass formula, which is usually not taken into account in the droplet model for nuclei. Comparison with fig. 3 shows that our model yields a baryon density which is indeed almost independent of $B$ for large nuclei.

\footnote{We neglect here the asymmetry effect from the proton–nucleon mass difference.}
(Note that $\pi^{(a)}$ formally corresponds to $B \to \infty$.) On the other hand, for small nuclei the baryon density is enhanced. The value of the surface tension as computed from eq. (57) for different parameter sets can be found in table 2. This result is quite reasonable in view of the uncertainties, first from the proper choice of a coarse grained potential in (57) (cf. [37, 38, 39]), and second from the choice of parameters in $U_0$. In fact, the successful explanation of the small ratio $(\Sigma^{(a)})^{1/3}/n_{\text{nuc}}$ is encouraging. In summary, our simple approach gives a quite reasonable picture for nuclei. We emphasize that once $U_0(\sigma)$ is fixed, our approximations allow for a “first principle calculation” of properties of nuclei!

8 Two flavor quark hadron phase transition

At the critical chemical potential $\mu_{\text{nuc}}$ the free energy $U(\sigma; \mu)$ shows two degenerate minima: one at $\sigma_0$, with vanishing density and the other at $\sigma_{\text{nuc}}$, where nuclear matter density $n_{\text{nuc}}$ is reached. In this section we consider densities higher than $n_{\text{nuc}}$. For sufficiently high density one may expect, and we observe, a further transition from nuclear matter to quark matter. A new first-order phase transition would be related to a third distinct minimum of the effective potential $U$.

The results of a quantitative analysis for the polynomial potential (28) with the parameter sets $A$–$C$ are reported in table 3. Using for the transition momentum the ansatz of eq. (26)

|         | $\mu_{\text{qm}}$ MeV | $\mu_{\text{eff,qm}}$ MeV | $n_{\text{qm}}/n_{\text{nuc}}$ | $\sigma_{\text{qm}}$ MeV | $M_{\text{q,qm}}$ MeV | $\Sigma_{\text{qm}}/10^6$ MeV |
|---------|------------------------|----------------------------|---------------------------------|---------------------------|------------------------|--------------------------------|
| $A$     | 975.5                  | 530                        | 8.6                            | 2.2                       | 15.8                   | 5.9                           |
| $B$     | 859.7                  | 484                        | 6.5                            | 2.2                       | 15.4                   | 4.8                           |
| $C$     | 511                    | 370                        | 2.9                            | 5                         | 34                     | 1.7                           |

Table 3: Critical quantities for the quark–hadron phase transition. The values for the chemical potentials $\mu_{\text{qm}}$ and $\mu_{\text{eff,qm}}$, the baryon density in the quark matter phase $n_{\text{qm}}$, the order parameter $\sigma_{\text{qm}}$, the effective quark mass $M_{\text{q,qm}}$ and the surface tension $\Sigma_{\text{qm}}$ for the sets $A$ and $B$ should be interpreted as an illustration of the uncertainties of a polynomial extrapolation of the potential $U_0$ to the origin $\sigma = 0$.

with $\sigma_c$ and $\sigma_H$ in a reasonable range, we find a first order transition between nuclear and quark matter. The sets $A$ and $B$ with high values of $\mu_0$ and $\pi$ lead, however, to relatively large values of the critical chemical potential $\mu_{\text{qm}}$ at which the transition from nuclear to quark matter occurs. One also finds large values of $\mu_{\text{eff,qm}}$ and the associated critical baryon density $n_{\text{qm}}$ in the quark matter phase. This sheds doubts on the reliability of this computation. One may argue that for such high values of $\mu_{\text{eff}}$ there is no good reason why a separate minimum for nuclear matter should persist. The prediction of a first order transition for the saturation scenario (C) seems more robust in this respect.

We next present a short description of the dominant effects that lead to our picture of a first order quark-hadron phase transition. This should also give an impression of the substantial uncertainties still inherent in this picture. The dominant mechanism for a possible first-order quark-hadron phase transition is the rapid decrease of $U_\mu$ at $\sigma = 0$ due to the fluctuations of massless quarks, whereas at the potential minimum which corresponds to nuclear matter
the effect of the quark fluctuations is reduced by their effective mass and by $q_H > 0$. For
$\mu$ increasing beyond $\mu_{\text{nuc}}$ the density of nuclear matter increases beyond $n_{\text{nuc}}$ and $\sigma(\text{nuc})(\mu)$ decreases (scenarios $A$, $B$). In our crude picture this continues until $\mu_{\text{eff}}$ reaches the value $\mu_{\text{inf}}(\sigma(\text{nuc}))$ (cf. eq. (51)). At the corresponding density the equation of state becomes very stiff, similar to the saturation scenario $C$ discussed in section 6 (eq. (55)). The density can further increase because of quark contributions only once $\mu_{\text{eff}}$ becomes larger than $\mu_q(\sigma(\text{nuc}))$. (For the parameter set $C$ corresponding to the saturation scenario $\mu_{\text{eff}}$ must first exceed $\mu_q(\sigma_{\text{nuc}}) = 334 \text{ MeV}$ ($\mu > 383 \text{ MeV}$) before the density can increase beyond nuclear density.) As a result of the “frozen density” the rate of decrease of $U(\mu)$ is also frozen according to eq. (3). On the other hand, the quarks always fully contribute to $\partial U/\partial \mu$ at $\sigma = 0$ (in the absence of current quark masses). For $\mu_{\text{eff}} > q_H$ and $\mu_{\text{eff}} > \mu_{\text{inf}}(\sigma_{\text{nuc}})$ the potential at $\sigma = 0$ decreases therefore faster with $\mu$ than for the nuclear matter phase at $\sigma(\text{nuc})$ (cf. eqs.(17), (18) with $q_H(\sigma = 0) = 0$ and $q_H(\sigma_{\text{nuc}}) = q_H$).

In the vicinity of $\sigma = 0$ the effect of the current quark masses should be included for a quantitative calculation. They push the minimum to positive $\sigma$ such that large enough quark masses typically destroy a possible first-order transition. In fact, at sufficiently high $\mu$ the effective potential (22) always has a new minimum near the origin at

$$\sigma(\text{qm})(\mu) \simeq \frac{m^2_\pi}{m^2_0(\mu)} \sigma_0.$$  \hspace{1cm} (61)

Here the mass parameter

$$m^2_0(\mu) = \frac{1}{4} \left. \frac{\partial^2 U}{\partial \sigma^2} \right|_{\sigma=0} = \frac{3}{4\pi^2} h^2_q \mu_{\text{eff}}^2 - \overline{m}^2,$$  \hspace{1cm} (62)

$$\overline{m}^2 \equiv 2\sigma_0^2 \left( \frac{\lambda - \gamma_3 + 4 - \gamma_5}{} \right) - m^2_\pi = \frac{\mu_0^2}{2\pi^2\sigma_0^2} + \left( \frac{4}{3} \frac{2}{\gamma_3} - \frac{6}{5} \gamma_5 \right) \sigma_0^2 - 3m^2_\pi.$$  \hspace{1cm} (63)

corresponds to the curvature of the potential at the origin and $m^2_\pi \sigma_0$ reflects the linear source term. At this minimum the effective quark mass $m(\text{qm}) = h(\sigma(\text{qm})) = h\sigma_0 \overline{m}^2/m^2_0(\mu)$, vanishes in the chiral limit $m_\pi \rightarrow 0$ and for $\mu \rightarrow \infty$. For small enough $m(\text{qm})$ we identify the corresponding phase with quark matter. For a vanishing current quark mass ($j = 0$, $m_\pi = 0$) chiral symmetry is restored in this phase. In case of a first order phase transition and, in particular, for small current quark masses, one typically finds a situation where two different local minima at $\sigma(\text{qm})$ and $\sigma(\text{nuc})$ coexist. As $\mu$ increases, the height of the potential for the quark matter phase $U(\text{qm})(\mu) = U(\sigma(\text{qm})(\mu); \mu)$ decreases faster than the one for the nuclear matter phase $U(\text{nuc})(\mu) = U(\sigma(\text{nuc})(\mu); \mu)$, where we remind that decreasing $U$ corresponds to increasing the pressure $p = -U$. This can be seen directly from eqs. (19), (20), since $\sigma(\text{qm}) < \sigma(\text{nuc})$ and $\frac{\partial U}{\partial \sigma}(\sigma, \mu) = 0$. One concludes that for large enough $\mu$ the absolute minimum of $U$ is always given by eqs. (61)–(63).

Away from the chiral limit the quark–hadron phase transition is not characterized by a change of symmetry in our model\textsuperscript{13}. It could therefore be of first order or a crossover. (A second order transition would require an additional tuning of parameters.) Our numerical

\textsuperscript{13}As mentioned in the introduction, we do not take into account in the present approach the possible spontaneous breaking of color at high density [9] or in the vacuum [10], [11].
evaluation of $U$ for the three scenarios A, B, C shows that the existence of a first-order transition depends on assumptions about $q_H(\sigma)$. For many reasonable functional forms we find indeed a first-order transition. In view of the remaining uncertainties it seems useful to establish general criteria for the occurrence of a first-order transition within our computation. Indeed, a first-order transition is guaranteed if a range of $\mu$ exists for which $m_0^2(\mu)$ is positive and substantially larger than $m_0^2$ whereas $\sigma^{(\text{nuc})}(\mu)$ remains of the same order of magnitude as $\sigma_0$. In this case one has $\sigma^{(\text{qm})}(\mu) \ll \sigma^{(\text{nuc})}(\mu)$ and the mass term at $\sigma^{(\text{qm})}(\mu)$ is well approximated by $m_0^2(\mu)$ and therefore positive. By definition the mass term at $\sigma^{(\text{nuc})}(\mu)$ is also positive. Two local minima of $U$ coexist for this range of $\mu$. As $\mu$ is increased further the mass term at $\sigma^{(\text{qm})}(\mu)$ monotonically grows (cf. eq. (62)) thus excluding a crossover. Typical values for $\overline{m}$ from an extrapolation of the polynomial potential (28) for the parameter sets (A, B, C) are $(833, 709, 7, 584.9)$ MeV. For these values a first order transition would be guaranteed for $m_0(\mu_{\text{eff}}) \gtrsim 400$ MeV or $h_q \frac{\mu_{\text{eff}}}{260 \text{MeV}} > (13.10, 11.82, 9.92)$ if $\sigma^{(\text{nuc})}(\mu_{\text{eff}})$ remains of order $\sigma_0$. We use here a vacuum constituent quark mass of 330 MeV or a Yukawa coupling $h_q = 7.1$.

In summary, we infer a first-order transition if $\sigma^{(\text{nuc})}(\mu_{\text{eff}})$ remains of order $\sigma_0$ for $\mu_{\text{eff}} = (480, 433, 363)$ MeV for the models A, B, C. For low enough values of $\overline{m}$ (as, for instance, in scenario C) the value of $\mu_{\text{eff}}$ is low enough such that $\sigma^{(\text{nuc})}(\mu_{\text{eff}})$ is not expected to be much smaller than the value in nuclear matter at low pressure, as given in table 2. A first-order transition occurs then independently of other details of the potential. On the other hand, for large values of $\mu_{\text{eff}}$ the dependence of $\sigma^{(\text{nuc})}(\mu_{\text{eff}})$ is much more difficult to assess. It depends crucially on the way how the quarks are “switched on”, as expressed in the present formalism by the functional form of $q_H(\sigma)$. A crossover or even a rather smooth transition become possible as well.

Actually, a very natural scenario seems to be a first order transition at a critical value $\mu_{\text{eff,\,qm}}$ which is lower than $\mu_q(\sigma^{(\text{nuc})})$. In this case the quarks do not contribute in the nuclear matter phase and nucleons are absent in the quark matter phase. In the following we concentrate on this scenario which can be realized for the ansatz (26) with reasonable values of $\sigma_c$ and $\sigma_H$. Typical values of $\mu_{\text{eff,\,qm}}$ for this situation are somewhat above $q_H$, say, $\mu_{\text{eff,\,qm}} \simeq (300 - 400)$ MeV. The baryon density in the quark phase at this transition would be around three times nuclear density. These values occur naturally for values of $\mu_0$ somewhat below $\mu_{\text{eff,\,qm}}$. An investigation of the coarse grained effective potential in the framework of the average action for a nonvanishing baryon chemical potential [40] finds values of $\mu_0$ only slightly above the constituent quark mass. This can be interpreted as an information about the potential $U_0$ near the origin and supports the above scenario.

In order to estimate the critical value $\mu_{\text{eff,\,qm}}$ for the quark hadron phase transition in this scenario we equate the pressure in the quark matter phase (for the approximation $m_\pi = 0$)

$$\rho^{(\text{qm})} = \frac{1}{2\pi^2} \left( \mu_{\text{eff}}^4 - \mu_0^4 \right)$$  \hspace{1cm} (64)

with the one in the nuclear matter phase

$$\rho^{(\text{nuc})} = \frac{2}{\pi^2} q_H^3 \mu_{\text{eff}} - \frac{2}{3\pi^2} q_H^3 \sqrt{m_N^2 + q_H^2 + \overline{p}}.$$  \hspace{1cm} (65)

Here $m_N$ and $\overline{p}$ are the nucleon mass and pressure corresponding to $\mu_{\text{eff}} = \mu_{\text{nuc}}$, respectively. For $q_H$ not much larger than $q_{\text{nuc}}$ one may neglect $\overline{p}$. Inserting two typical sets of values,
\( \mu_0 = 320 \text{ MeV}, \ q_H = 1.05(1.2)q_{\text{nuc}}, \ \overline{m}_N = 0.7(0.6)m_N, \) one obtains \( \overline{\mu}_{\text{eff,qm}} = 390(440) \text{ MeV}. \) This corresponds to a critical baryon density in the quark matter phase

\[
n_{\text{qm}} = 3.4(4.9)n_{\text{nuc}}.
\]  

(66)

For the scenarios (A) and (B) the existence of two minima found in our computation may well be an artefact of our inaccurate treatment of the transition from quarks to nucleons. Indeed, there are reasonable forms of \( q_H(\sigma) \) for which the nuclear matter minimum has reached small values of \( \overline{\sigma}_\text{nuc} \) already for substantially smaller \( \mu_{\text{eff}} \). This would favor a smooth transition. One should remember, though, that the estimate of \( \mu_{\text{eff}} \) depends crucially on the value of \( \overline{m} \) in eq. (62), which is only poorly known. In fact, the observed meson masses and decays contain only very limited information about the behavior of \( U_0 \) near the origin. We do not expect a polynomial expansion of \( U_0 \) around \( \sigma_0 \) to lead to a very good approximation of the potential in the vicinity of the origin. It is certainly possible to extrapolate a form of \( U_0 \) which is compatible with nuclear physics constraints in the region \( 0.6\sigma_0 < \sigma < 1.5\sigma_0 \) to a wide range of parameters \( \mu_0 \) and \( \overline{m} \) characterizing the behavior of \( U_0 \) near \( \sigma = 0 \). Furthermore, a possible \( \sigma \)–dependence of \( g(\omega) \) or \( M_\omega \) would substantially affect the ratio \( \mu_{\text{eff}}/\mu \). In particular, a smaller value of \( g(\omega)/M_\omega \) for the quarks (near \( \sigma = 0 \)) would enhance the effective chemical potential for given \( \mu \) in the quark phase, thereby shifting the transition to lower values of \( \mu \). We conclude that the spread in the values in table 3 (especially those corresponding to sets A and B) should be considered as an illustration of the uncertainties still inherent in the polynomial extrapolation rather than as actual predictions (which we expect closer to eq. (66) in case of a first order transition). This uncertainty is reduced significantly once independent information about the behavior of \( U_0 \) near \( \sigma = 0 \) becomes available as, for example, from ref. [40].

We have also computed the surface tension \( \Sigma_{\text{qm}} \) for the quark–hadron transition at the critical \( \mu_{\text{qm}} \). It turns out to be much larger than the one between the nucleon gas and nuclear matter. The quantitative value is given in table 3. The surface tension depends, however, strongly on the details of the transition from quark to nucleon degrees of freedom (e.g., \( \sigma_\text{c} \) and \( \sigma_H \)). Stability of nuclear matter requires the critical chemical potential \( \mu_{\text{qm}} \) for a possible quark–hadron transition to be above \( \mu_{\text{nuc}} \) as realized for our parameters. At the quark hadron phase transition the quark mass \( M_{\text{q,qm}} \) in quark matter is much smaller than in nuclear matter. Nevertheless, it is substantially larger than the current quark mass. We quote the value of the order parameter \( \overline{\sigma}_{\text{qm}} = \overline{\sigma}_{\text{qm}}(\mu_{\text{qm}}) \) for the quark phase in table 3 together with the corresponding quark mass. For \( \mu = \mu_{\text{qm}} \) the values in the nuclear matter phase are \( \overline{\sigma}_{\text{nuc}}(\mu_{\text{qm}}) \sim 24 \text{ MeV}, \ M_{\text{q,qm}}^{(\text{nuc})} \sim (165 - 170) \text{ MeV}. \) Since \( \mu_{\text{qm}} \) may exceed the effective strange quark mass, the strange quarks could play a role for this transition in real QCD.

To summarize this section, our first computation exhibits a first-order phase transition between nuclear and quark matter at a critical \( \mu_{\text{qm}} \) which is a few times nuclear density. If such a transition really occurs, our values for the density seem much more realistic than the very low values typically obtained in simple quark model computations. This underlines the importance of the correct treatment of the baryons in the nuclear matter phase. We have also seen, however, that the uncertainties remain very substantial and a smooth behavior remains also conceivable. The unknowns we have encountered will be present in any realistic mean field-type treatment of the transition between quark and nuclear matter. Whereas high density quark matter can perhaps be dealt with rather reliably at high enough baryon density, the main problem concerns the behavior of nuclear matter at high density for which confinement effects
(the binding to baryons) cannot be neglected. Any discussion of a phase transition needs knowledge about both phases concerned. The high density nuclear matter phase therefore needs always to be understood quantitatively and a pure quark model cannot give a reliable description (unless the “binding of baryons” is somehow incorporated). This raises substantial doubts about the applicability of many mean field statements about this transition in simple quark models.

9 Conclusions

We have presented here a new method for the computation of the dependence of the free energy on the chemical potential. It is based on an approximate solution to an exact functional differential equation. This method allows us to put mean field theory into a more systematic context. Chiral symmetry is explicitly implemented and phenomenological information about pion masses and decay constants is taken into account. Expressions which are close to mean field theory describe the difference in the free energy between vanishing and non–vanishing chemical potential. They can be considered as the leading order in a series of systematic truncations of the exact differential equation (6). On the other hand, the free energy for vanishing chemical potential is not reliably described by mean field theory. Many relevant characteristics of this quantity can, however, be inferred from observation.

Perhaps the most important new feature in our approach is that quark and nucleon fluctuations can be treated simultaneously within the same computation of the free energy. Only this allows a simultaneous description of the nuclear gas-liquid transition and the transition from nuclear to quark matter. A method which can deal both with nucleons and quarks is crucial for any quantitative treatment of a possible phase transition from nuclear to quark matter. Actually, for this transition the main difficulty lies in the understanding of the “low density phase”. This phase is nuclear matter at a critical density of perhaps several times nuclear density, where standard nuclear physics is not of much help and the binding of quarks into nucleons nevertheless remains an important ingredient. Simple quark models not accounting for this binding, like NJL-type models, are insufficient for a description of this transition.

We take here a very simple approximation where we use quark and nucleon degrees of freedom in their appropriate momentum ranges. For high momenta, \( \vec{q}^2 > q_{H}^2 \), the quark meson model gives a useful approximation. For small momenta, \( \vec{q}^2 < q_{H}^2 \), the effects of confinement have to be taken into account and we describe the carriers of baryon number as nucleons. Our simple model leads to a unified description of the nucleon gas, nuclear matter and transition to quark matter. The appearance of three phases of strongly interacting matter is related to three distinct minima of the effective potential for the \( \sigma \)-field. (Typically only two coexist simultaneously.) This rich structure is a consequence of the fact that more energy levels fall below the Fermi energy for nucleons than for quarks. This results in a substantial enhancement of the density or, equivalently, the \( \mu \)-dependence of the free energy due to low momentum nucleon fluctuations. The enhancement of the effect of a Fermi-gas of massive nucleons as compared to a gas of quarks with constituent masses only about a third of the nucleon mass is actually a huge factor of 27. This large enhancement relies only on the different effective masses and, therefore, Fermi surfaces of nucleons and quarks. Consequently, this property is rather independent of other more detailed features of the model.
The use of nucleon degrees of freedom and the corresponding large density enhancement factor as compared to quark degrees of freedom is crucial for any realistic description of the liquid–gas nuclear transition. It also shifts the transition from nuclear to quark matter to a larger chemical potential and baryon density, as compared to a description of the fermionic fluctuations in terms of quarks alone. This results in a substantial pressure at the coexistence between quark and nuclear matter. Standard Nambu–Jona-Lasinio type quark models fail to take into account this “density enhancement” and, therefore, typically predict a transition to quark matter at too low densities.

The gas–liquid nuclear transition can be described within the present approach in a quantitative way, at least for low temperature. We present here two classes of models, one similar to the $\sigma - \omega$-model widely used in nuclear physics, the other a “saturation model” where the short distance repulsion between nucleons arises not only from vector-meson exchange but also reflects the transition to quark degrees of freedom at high density. For a suitable choice of parameters of the effective meson masses and interactions in vacuum our models give a successful description of the nuclear droplet model, consistent with the observed nuclear density, the binding energy per nucleon, the compression modulus and the nucleon mass in nuclear matter. They explain why nuclear density is approximately independent of the baryon number of a nucleus. We also obtain a realistic value for the nuclear surface tension and we have computed small corrections to the baryon density and the mass formula for nuclei due to nonvanishing pressure. Isospin violation and electromagnetism can be incorporated easily in our model. This should give a reliable equation of state for neutron stars in the region of moderate densities. Our model predicts coupling constants which directly enter the effective nucleon–nucleon potential. Comparison with nucleon scattering experiments will provide an interesting test in the future.

A potential shortcoming of these models are the large meson self-interactions in vacuum which are needed for a realistic description. Within a linear $\sigma$-model they correspond to large coefficients of terms $\sim \sigma^6, \sigma^8$ or $\sigma^{10}$ which have not been found in previous renormalization group studies. It remains to be seen if a more complex structure of vacuum expectation values in the nuclear matter phase and in the vacuum – including spontaneous breaking of color – can ease this problem. The generalization of our approach to additional order parameters is straightforward. One may consider our quantitative results as a “prototype calculation” for understanding the properties of nuclei from QCD. It can be adapted to more complex settings. Particularly important will be a reliable computation of the vacuum-effective potential.

Within our approximations we find a first order phase transition between nuclear and quark matter at high density and vanishing temperature. The critical density is typically three to five times nuclear matter density. We emphasize, however, that some important information is still missing for a quantitative understanding of the quark–hadron transition: The first problem concerns the appropriate formulation of a coarse grained effective potential and a determination of its shape for the vacuum. This is needed since first-order transitions require a non-convex coarse-grained potential whereas the inclusion of long wavelength fluctuations leads to a convex effective potential. Within the linear quark meson model we have addressed this issue in the context of the average action [40]. The second loose end is a more detailed understanding of the change from quark to baryon effective degrees of freedom. This concerns primarily the behavior of nuclear matter at densities much larger than nuclear density. It is therefore very relevant for a quantitative description of the quark–hadron transition. Furthermore, the neglected strange
quarks could play a relevant role at sufficiently high density. Finally, the presence of additional “color symmetry breaking” expectation values are a crucial ingredient for the understanding of states with high baryon density. Again, this could influence the nature of the transition from nuclear to quark matter [41]. In view of all these uncertainties we conclude that no definite statements about the nature of this transition can be made at present. Both a genuine first-order phase transition and a relatively smooth change remain possible. The uncertainties mentioned here are common for other analytical approaches and constitute a major difficulty for a quantitative understanding of the quark–hadron phase transition.

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