On the input of a measurement process

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Abstract. It is assumed sometimes that the input of a measurement, and therefore the entity with which a measuring system interacts, is a quantity value, possibly the (true) measurand value, and from this hypothesis the model of ideal measurement as an identity process is formulated. In this paper we show that this position is based on an inappropriate superposition of quantities and quantity values, and therefore should be discarded.

1. Introduction

As any process, measurement can be firstly characterized as a black box that transforms inputs to outputs. Agreeing upon the nature of such entities is then the preliminary step of a top-down analysis or development process. Even at a so general level two incompatible positions are maintained on the input of a measurement process, a situation that might hinder a smooth, mutual understanding and therefore would be beneficially solved. Our aim in this paper is to argue about this duplicity and to show how the two positions can be reconciled.

A widespread standpoint on measurement is presented thus: «The input to the measurement system is the true value of the variable; the system output is the measured value of the variable. In an ideal measurement system, the measured value would be equal to the true value.» [1]. This position incorporates the customary intuition that measurement is generally affected by errors, which produce unwanted changes that should be minimized, so that in the ideal case in which all errors have been eliminated all changes are eliminated in turn. The immediate consequence is that an ideal measurement process is mathematically modeled as an identity function $m$, $m(x) = x$, where $x$ is the (true) value of the measured quantity.

This characterization has been further and more analytically developed [2] by decomposing the mathematical model of measurement into two basic stages, called “observation” and “restitution”, where «Observation is a mapping (a function) from the measurand value $x$ to the instrument indication $y$, i.e., $y = f(x)$, where function $f$ can be obtained by calibration. Restitution is a mapping from the instrument indication to the measurement value $\hat{x}$, which may be obtained by inverting $f$, i.e., $\hat{x} = f^{-1}(y)$. Measurement is the concatenation of the two, i.e., $\hat{x} = f^{-1}[f(x)]$, which, in the ideal uncertainty-free case, yields identity, $\hat{x} = x$.» Two interesting points emerge here. First, the term for the input entity is not “true value” but simply “value”, perhaps in accordance with the Guide to the expression of uncertainty in measurement (GUM), where the term “true value” is avoided because the word “true” “is viewed as redundant” [3] (the argument of the GUM is not so clear, given that it seems to be

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unable to distinguish between, e.g., measured and specified values; on the other hand the subject of “truth” of values is immaterial for what follows). Second, the entity that in the first quotation was generically mentioned as a variable is presented here a measurand, a crucial concept that the International Vocabulary of Metrology (VIM) defines “quantity intended to be measured” [4].

The model implied in [1] and [2] is then:

![Diagram](image)

This is simple and elegant, but – we argue here – too simple. Indeed, the assumption that measuring systems interact with (true) values critically hides the basic feature of measurement of being a process mapping properties to property values or, more specifically, quantities to quantity values (the distinction between quantities and quantity values is developed in [5]). For example, according to the VIM measurement is “a process of experimentally obtaining one or more quantity values that can reasonably be attributed to a quantity” [4]. While the measured value, $\hat{x}$, is attributed to the measurand, the input quantity in the process cannot be generally expected to be the measurand itself, in the sense that what is experimentally under measurement can be different from what is intended to be measured. In the assumption of ideality, then:

![Diagram](image)

In order to highlight the difference between these two standpoints, we will develop the hypothesis that measurement, a process of information acquisition, is presented in the quotations above as structurally analogous to a process of information transfer, and measuring systems as structurally analogous to communication systems, where in fact the basic problem is to infer the entity $x$ that was in input to the system from the channel output $y$, i.e., what was sent from what has been received. Clearly, sometimes similar can be said also about measurement, aimed at inferring a measured value $x$ for the measurand from the obtained indication value $y$. Nevertheless the analogy fails in a critical point. But let us develop it for the moment.

2. Information transfer and information acquisition

An ideal transmission channel has null equivocation, because it does not lose information, and null noise, because it does not generate information: under the hypothesis that the channel is ideal, the knowledge of its behavior and the output is sufficient to infer the input with certainty. For example, let us suppose that $x$ is the value of a physical quantity, say an electric tension, and that the channel attenuates the input signal at the receiver end uniformly by a factor of $k > 1$, so that its behavior is modeled as $y = f(x) = x/k$ (ideality is manifest here: infinite bandwidth and zero noise). By means of a suitable characterization of the channel (the functional analogous of calibration) the factor $k$ is evaluated, so that the input can be “reconstructed” from the output as $\hat{x} = ky$, i.e., $\hat{x} = f^{-1}(y)$, exactly as proposed above for measurement. In fact, ideal transmission “yields identity”, $\hat{x} = x$.

At the core of this interpretation there is the assumption that both the input and the output of the process (of measurement, of transmission) are values. This arises the question of the usefulness of an experimental process (ideally) implementing an identity function. In the case of transmission channels the answer is clear, and can be further highlighted by a second, parallel analogy, about information storage (“memory”) systems.
The task of an information transmission channel is to transfer information at a distant location. It operates according to a two stage process. In the first stage its input and output (channel write and read) occur in different places, $p_1$ and $p_2$, so that the model of this stage is:

$$y_{p_1} = f(x_{p_1})$$

If $f$ is known, in the second stage $f^{-1}$ is applied to $y_{p_1}$ (then in $p_2$), and the channel behavior is maximally useful because the two equated values:

$$\hat{x}_{p_1} = x_{p_1}$$

refer to different places.

The task of an information storage system is to maintain information through time. It operates according to a two stage process. In the first stage its input and output (memory write and read) occur in different times, $t_1$ and $t_2$, so that the model of this stage is:

$$y_{t_1} = f(x_{t_1})$$

If $f$ is known, in the second stage $f^{-1}$ is applied to $y_{t_1}$ (then in $t_2$), and the channel behavior is maximally useful because the two equated values:

$$\hat{x}_{t_1} = x_{t_1}$$

refer to different times.

Hence identity is justified, and in fact desirable, for information transfer in space (i.e., transmission) and in time (i.e., memorization): the value that was in input is obtained identical here (transmission) or now (memorization).

(The behavior of information transfer systems is usually modeled even more abstractly, in terms of transfer of information entities such as “symbols” and sequences of them, i.e., “messages” (according to Shannon, «The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point.» [6]), so that quantity values act as carriers for such entities. This level, implying an invertible encoding of symbols to quantity values, is immaterial here and will not be further considered.)

On the other hand, this two stage model is too rudimentary, as it neglects a crucial difference between the first and the second stage, and can be refined by acknowledging that:

- the first stage (transmission, corresponding to observation above) is experimental: it is a physical process of transmitting a signal, that eventually will be interpreted;
- the passage from the first to the second stage is based on a mathematical function $f$ that is just a model of the first stage, not the stage itself;
- the second stage (reconstruction, or “restitution” as it called above) is computational: it is a numerical process of inverting $f$ and applying $f^{-1}$.

These points are effectively presented in terms of the distinction between quantities, hereafter denoted by uppercase characters, and quantity values, denoted by lowercase characters [5]. In fact, of the first stage of an information transfer system, intended as a part of the empirical world (W), two levels of modeling can be given:

(M1) quantitative: a device that causally changes its state in response to an external perturbation, where the perturbation is an input quantity $X$ and the changing state is an output quantity $Y$ (not necessarily of different kind), such that, under specified conditions, e.g., $Y = f(X) = X/K$, the units of such quantities and therefore the numerical value of $K$ being still undetermined;

(M2) numerical: a device whose $X$-$Y$ behavior has been identified, e.g., as $Y = f(X) = X/K$, for given units of such quantities, so that the numerical value of $K$ in units of $X/Y$, $k$, is known; when the value $x$ of the quantity $X$ is in input, the value $y = f(x) = x/k$ of the quantity $Y$ is obtained in output.

It should be noted that at each level the involved input-output entities are of the same ontological kind:

- in M1 they are quantities;
- in M2 they are quantity values.

The fact that the same mathematical equation formally characterizes both M1 and M2 ($Y = X/K$ in the example) might lead to suppose that there is no difference between them. On the other hand, the fact that M1 and M2 are distinct is evident because the law, as identified in M1, is independent of the conventional choice of the units: even though the decision about what to transfer regards a quantity
value $x$, what is transmitted is a quantity $X$ (a question such as “are you transmitting 1.234 V or 1234 mV?” – where 1.234 V and 1234 mV are indeed distinct values – is devoid of experimental content). The mathematical entity at the level M1 is in fact a quantity equation (a “mathematical relation between quantities in a given system of quantities, independent of measurement units” [4]), independent of the values of the involved quantities and with free parameters: it is a model sufficient to describe the experimental process.

This is just a different way to present the fact that physical laws connect quantities, not quantity values, and in fact they are independent of the choice of units. In other terms, whether a physical law apply (or: is true) does not depend on the calibration of the involved devices.

Hence, again in the mentioned example, were an additive composition available for $X$ and $Y$, what could be done independently of quantity values is the validation of the linear dependence of the two quantities, i.e., of the structure of the quantity equation. A quantity $X_1$, possibly of unknown value, is applied and the obtained quantity $Y_1$ is recorded. Then a different quantity $X_2$, say $X_2 = 2X_1$ (generated through a quantity $X'$ such that $X' \approx X_1$ and then combining $X' \circ X_1$) is applied and again the obtained quantity $Y_2$ is recorded. Through purely experimental means the ratio $Y_2/Y_1 = j$ can be obtained (e.g., in the simple case $Y_2$ is a multiple of $Y_1$, by generating $j$ clones $Y'$ of $Y_1$ and showing that $(Y' \circ \cdots \circ j)$ $Y') \approx Y_1$). Then the linearity of the system behavior requires the constancy of the ratios $X_2/X_1$ and $Y_2/Y_1$, i.e., in this case such that $j = 2$: this shows that what is invariant at the experimental level, i.e., M1, in this example are ratios of quantities, not quantity values.

On the other hand, the second stage, i.e., the reconstruction $f^{-1}$, operates on quantity values: it is performed at the level M2, and therefore it is possible only when the entire stack W-M1-M2 is available, i.e.: 
– the system has been actually (i.e., experimentally) operated (W);
– its causal $X$-$Y$ behavior has been identified (M1);
– a numerical value of $k$ has been identified through system characterization (M2), and the value $y$ of the output quantity $Y$ has been obtained.

This framework can be graphically presented as follows, where continuous arrows denote actual process actions (what we do), and dotted arrows modeled actions (what we know):

\[
\begin{align*}
\text{state of the world} & \quad \text{first stage} \quad \text{state of the world} \\
\text{before the process} & \quad \text{quantitative model} \quad \text{after the process} \\
& \quad \text{of first stage: } f \\
X & \quad Y \\
& \quad \text{system characterization} \\
x & \quad y \\
& \quad \text{second stage: } f^{-1} \\
& \quad \text{W} \\
& \quad \text{M1} \\
& \quad \text{M2}
\end{align*}
\]

The vertical arrows from $x$ to the state of the world before the process represent the realization of a quantity value by means of an empirical entity, and the vertical arrows from the state of the world after the process to $y$ represent the recognition of a quantity value in an empirical entity.

This is an adequate description of an information transfer process. According to the quotations at the beginning of this paper this is also appropriate in the case of measurement: is it the case?

3. Information acquisition vs information transfer

Let us consider a basic difference between information transfer and information acquisition, measurement in particular. In the first case the value $x$ is chosen by someone / something, i.e., the sender: hence it can be generally supposed to be perfectly defined, even though unknown to the receiver, before the process takes place. On the contrary, even those who endorse a metaphysics assuming that “numbers are in the world” plausibly admit that such numbers / values are not chosen
(and the analysis above shows that at most “in the world” there are ratios of values, not values). In information transfer the value $x$ is the operative starting point of the transfer process, from which the individual quantity $X$ and more generally a state of the world are realized. On the contrary, the operative starting point of measurement is the experimental interaction between a state of the world, i.e., the state of the object under measurement in its environment, and the measuring instrument, e.g., between a body generating a force and a spring balance. While many measuring systems are based on transducers (the spring, in the example), their task is neither to perform a state transition ($W$, spring elongation), nor to transform the measurand $X$ to an indication $Y$ ($M_1$, from a force to a length), nor to map the input value $x$ to an output value $y$ ($M_2$), as instead it may be said of information transfer systems. In the case of measurement, transduction is just instrumental to the actual task of acquiring information on the measurand $X$ and representing it as a measured quantity value $\hat{x}$ (neglecting uncertainties, as in the quotations above) [7]. This has nothing to do with an equality of values, $x = \hat{x}$, and in fact it is written:

$$X = \hat{x}$$

i.e., a quantity intended to be measured, $X$, is recognized equal to a quantity value, $\hat{x}$, e.g., force applied by this object) $= 1.234 \, N$.

Understanding this equality is crucial (“equality gives rise to challenging questions which are not altogether easy to answer” [8]). It is not an identity, and it is not an equality of values (the left-hand term does not depend on a unit and is not a value). Rather, it is based on the, usual but subtle, observation that different expressions may have different meanings and nevertheless refer to the same entity. For example, $\sum_{i=0}^{\infty} 1/n!$ and $\exp(1)$ are clearly different concepts, but they refer to the same number, $2.718...$. Analogously, ‘the force applied by this object’ and $1.234 \, N$ are definitely different concepts, but might be recognized as different ways to refer to the same individual quantity, sometimes called their (common) magnitude.

This recognition is the outcome of a composite process such that:

1. at the $W$ level, the state of the object under measurement is transduced to the state of the measuring instrument;
2. at the $M_1$ level, the two states are modeled as quantities $X$ and $Y$ respectively, according to a mathematical model of transduction, $Y = f(X)$;
3. in the passage between $M_1$ and $M_2$, the crucial move from an indication (as a quantity) to an indication value is accomplished, as modeled by a function $y = g(Y)$;
4. at the $M_2$ level, the values of the initial and the transduced quantities are introduced and the measured value is computed, according to $\hat{x} = f^{-1}(y)$.

By composing these functions we get $\hat{x} = f^{-1}(g(f(X)))$, where the generic concept of system characterization, $y = f(x)$, is here the result of instrument calibration and constitutes the pre-condition for the inversion of the numerical version of $f$ (if the mapping $g$ is left implicit and the notation $Y = y$ is used instead, then the previous formula becomes:

$$\text{if } f(X) = y \text{ then } X = \hat{x}, \text{ where } \hat{x} = f^{-1}(y)$$

thus highlighting the remarkably symmetric structure of the measurement process).

The process is presented by the following modified version of the previous diagram:
4. Conclusions

Measurement is loaded of stereotypes, and it is so a fundamental process that philosophical presuppositions seem to be hardly avoidable in understanding its basics. On the other hand, measurement is often perceived as arising only practical problems, so that these basics usually remain implicit. This could be the explanation of the peculiar fact that sometimes mutually incompatible positions emerge that are not reconciled and thus remain unsolved, under the hypothesis that this compresence does not actually affect the experimental operations of measurement. Whether this hypothesis is correct or not, a conceptually and terminologically consistent presentation of measurement and its results surely requires a consistent background. This applies, first of all, to a black box modeling of measurement, where then the nature of its input and output is unambiguously stated and agreed. Characterizing the nature of measurement input poses this kind of incompatibility problems, given that measurement input is alternatively intended as a (true) value of a quantity or a quantity of an object. We have argued here that the former assumes a model in which the processes of information transfer and information acquisition are mistakenly superposed: rather, measurement systems experimentally interact with quantities, not quantity values.

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