Chiral Ising Gross-Neveu criticality of a single Dirac cone: A quantum Monte Carlo study

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We perform large-scale quantum Monte Carlo simulations of SLAC fermions on a two-dimensional square lattice at half filling with a single Dirac cone with $N = 2$ spinor components and repulsive on-site interactions. Despite the presence of a sign problem, we accurately identify the critical interaction strength $U_c = 7.28 \pm 0.02$ in units of the hopping amplitude, for a continuous quantum phase transition between a paramagnetic Dirac semimetal and a ferromagnetic insulator. Using finite-size scaling, we extract the critical exponents for the corresponding $N = 2$ chiral Ising Gross-Neveu universality class: the inverse correlation length exponent $\nu^{-1} = 1.19 \pm 0.03$, the order parameter anomalous dimension $\eta_\phi = 0.31 \pm 0.01$, and the fermion anomalous dimension $\eta_\psi = 0.136 \pm 0.005$.

Introduction.—Massless Dirac fermions have been identified as the relevant low-energy quasiparticles in various condensed matter systems including graphene, topological insulators, $d$-wave superconductors, Weyl semimetals, and ultracold fermions in optical lattices [1–7]. Nonetheless, strong interactions can generate a finite mass for the Dirac fermions and spontaneously break some of the symmetries of the model. The quantum phase transitions at which this occurs are typically described by the Gross-Neveu (GN) universality classes [8]. In particular, a single Dirac cone in (2+1)D subject to on-site repulsive interactions—such as can be found on the surface of a correlated topological insulator—can develop an Ising-type ferromagnetic (FM) order, which generates a $\mathbb{Z}_2$ symmetry-breaking FM mass gap [9, 10]. For a chemical potential at the Dirac point, the quantum critical point (QCP) of the resulting transition from semimetal (SM) to insulator is believed to belong to the chiral Ising GN universality class [11–16] with $N = 2$ Dirac spinor components.

Useful insights for the $N = 2$ chiral Ising GN universality class have been obtained from several approaches including the conformal bootstrap, the functional renormalization group (fRG), and analytical field theory methods such as large-$N$ and $\epsilon$ expansions. However, these methods so far yield inconsistent results. For example, while the conformal bootstrap [17] predicts $\nu^{-1} = 0.86$, fRG [18] and the $\epsilon$ expansion [16] predict $\nu^{-1} = 1.229$ and $\nu^{-1} = 1.276$, respectively. (For other critical exponents, see Table I.) These significant discrepancies demand a resolution from numerically exact quantum Monte Carlo (QMC) simulations which have been unavailable thus far. The lack of QMC studies of this problem originates in part from fermion-doubling theorems which state that a local lattice model cannot realize a single symmetry-protected Dirac cone [19]. Indeed, all previous QMC studies of chiral Ising GN criticality have utilized local lattice models and thus could only access even numbers of Dirac cones, e.g., $N = 4$ [20–23] and $N = 8$ [24–28].

In this paper, we instead use a nonlocal lattice realiza-

|        | $\nu^{-1}$  | $\eta_\phi$ | $\eta_\psi$ |
|--------|-------------|-------------|-------------|
| this work (QMC) | $1.19 \pm 0.03$ | $0.31 \pm 0.01$ | $0.136 \pm 0.005$ |
| conf. bootstrap [17] | $0.86$ | $0.320$ | $0.134$ |
| fRG [18] | $1.229$ | $0.372$ | $0.131$ |
| $\epsilon$ expansion [16] | $1.276$ | $0.2934$ | $0.1400$ |

TABLE I. Our QMC evaluation of the critical exponents for the $N = 2$ chiral Ising GN universality class, compared with previous estimates.

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In this paper, we instead use a nonlocal lattice realiza-

tion of a single Dirac fermion with $N = 2$ spinor components, known as the SLAC fermion [29], subject to an on-site Hubbard repulsion. By employing a state-of-the-art auxiliary-field QMC algorithm, we identify and investigate its FM QCP for the first time. The model is not entirely sign-problem free, but the sign problem is benign at the QCP (Fig. 1). (By contrast, QMC simulations of chiral XY [30] or Heisenberg [31] quantum criticality with a single Dirac cone are sign-problem free.) Since the noise of QMC simulations is inversely proportional to the average sign, denoted $\langle \text{sign} \rangle$, the central limit theorem requires an increase in the Monte Carlo samplings and measurements by a factor of $1/\langle \text{sign} \rangle^2$ to achieve the same level of accuracy as in sign-problem-free models. For example, when $\langle \text{sign} \rangle = 1/5$, we need nearly 25 times more measurements. In this work, we have employed massively parallel QMC simulations on thousands of CPU cores to mitigate this problem. For each data point, we have taken up to several billion measurements to keep the statistical error below 0.2% and obtain highly accurate results. We have also considered an imaginary time step of $\Delta \tau = 0.05$ to keep the Trotter error well below our statistical error. This approach allows us to circumvent the sign problem and accurately extract the critical exponents of the $N = 2$ chiral Ising GN universality class (Table I), our main result.

Model.—We consider an $L \times L$ square lattice with unit lattice constant having a single linearly dispersing Dirac cone in its first Brillouin zone. The free Hamiltonian in
momentum space is given by:

\[ H_0 = \sum_p \Psi_p^\dagger (p_x \sigma_x + p_y \sigma_y) \Psi_p, \tag{1} \]

with \( \Psi_p^\dagger = (c_p^\dagger, c_{-p}^\dagger) \) where \( c_p^\dagger \) is the electron annihilation (creation) operator with momentum \( p = (p_x, p_y) \) and spin \( \sigma \), and \( \sigma_\alpha, \alpha = x, y, z \) are the Pauli matrices operating on the spin degree of freedom. We extract the real-space representation of the above Hamiltonian by performing a Fourier transformation, which yields:

\[ H_0 = \sum_i \sum_R \left( t_R c_i^\dagger c_{i+R} + \text{h.c.} \right), \tag{2} \]

where \( c_{i\sigma}^\dagger \) is the electron annihilation (creation) operator on site \( i \) with spin \( \sigma \), and \( t_R \) denotes the electron hopping amplitude between site \( i \) and \( i + R \). Here \( R = (R_x, R_y) \) enumerates all neighbors of site \( i \) along the \( x \) and \( y \) directions. The explicit form of \( t_R \) is:

\[ t_R = \frac{i(-1)^{R_x}}{2 \sin(\pi R_x/L)} \delta_{R_x,0} + \frac{(-1)^{R_y}}{2 \sin(\pi R_y/L)} \delta_{R_y,0}, \tag{3} \]

where the overall hopping amplitude has been set to unity. Note that Eq. (3) introduces electron hopping beyond nearest neighbors. We add a local repulsive Hubbard interaction,

\[ H_U = U \sum_i (n_i^\uparrow - 1/2) (n_i^\downarrow - 1/2), \tag{4} \]

where \( U > 0 \) is the interaction strength and \( n_{i\sigma} = c_{i\sigma}^\dagger c_{i\sigma} \) is the electron number operator. For sufficiently large \( U \), we expect long-range Ising FM order in the \( z \) direction, which breaks time-reversal symmetry spontaneously and gaps out the Dirac cone. At half-filling, the single-particle density of states vanishes, thus we expect a line of finite-temperature transitions that terminates at a zero-temperature QCP with finite critical interaction strength \( U_c \) [22]. We simulate this quantum phase transition in the total Hamiltonian \( H = H_0 + H_U \) using the determinant QMC formalism, hereafter referred to simply as QMC.

QMC method.—We employ a projector QMC method to analyze the quantum phase transition in our model system. In this method, the ground-state expectation value of an observable \( O \) is calculated using imaginary-time propagation of a trial wave function \( |\Psi_T\rangle \) via

\[
\langle \Psi_T | O | \Psi_T \rangle = \lim_{\Theta \to \infty} \frac{\langle \Psi_T^\dagger e^{-\beta H} O e^{-\beta H} | \Psi_T \rangle}{\langle \Psi_T^\dagger e^{-\beta H} | \Psi_T \rangle}.
\]

Here, we follow the approach introduced in Ref. [32] and choose an interacting trial wave function to further enhance the performance and convergence of the projector QMC algorithm. We consider a Gutzwiller-projected wave function \( |\Psi_T\rangle = e^{-g_{GW} \sum_i n_i^\uparrow n_i^\downarrow} |\mathrm{FS}\rangle \) which can be easily implemented as our trial state within QMC. Here, \( |\mathrm{FS}\rangle \) is the noninteracting Fermi sea, and \( g_{GW} \) is the Gutzwiller variational parameter whose optimal value follows the approximate relation \( g_{GW} \approx 0.17U \). We perform calculations for system sizes \( L \in \{5, 7, 9, 11, 13, 15, 17, 19\} \), and use an imaginary projection time of \( 2 \Theta = 14 \) which is long enough to obtain ground-state properties. To demonstrate this, in the Supplemental Material we compare the performance of projector QMC with a nontrivial Gutzwiller-projected state against regular projector QMC (\( g_{GW} = 0 \)) as well as finite-temperature QMC. We find that the algorithm with \( g_{GW} \neq 0 \) converges to the ground state the fastest. Moreover, an effective inverse temperature \( \beta_{\text{eff}} \) can be defined for a given projection time \( \Theta \) such that the projector QMC results are approximately equivalent to finite-temperature QMC results at temperature \( T = 1/\beta_{\text{eff}} \). We find that \( \beta_{\text{eff}} (g_{GW} = 0) \approx 2 \Theta + 11 \pm 1 \) while \( \beta_{\text{eff}} (g_{GW} = 0) \approx 2 \Theta + 7 \pm 1 \). Our projector QMC method with \( g_{GW} \neq 0 \) and \( 2 \Theta = 14 \) thus allows us to effectively reach temperatures as low as \( \beta_{\text{eff}} = 25 \pm 1 \), which is sufficient to elucidate ground-state physics.

Although QMC is an unbiased method and is very effective for studying lattice models of strongly correlated electrons, its negative sign problem hinders its application to many problems of interest [33]. Nonetheless, the sign problem in QMC depends highly on the model’s formulation, such that one may improve the energy scales that QMC can reach by choosing appropriately the Hubbard-Stratonovich (HS) decoupling of the interaction term. For the present model, the average sign is significantly higher if we decouple the interaction in the \( s_x \) or \( s_y \) channels rather than the usual \( s_z \) channel [Fig. 1(a)]. With that decoupling, Fig. 1(b) shows that the average sign of our model at the QCP is not very severe, and we can reach sufficiently low temperatures to accurately predict ground-state properties.

FM transition.—We probe FM ordering in our model by computing the spin-spin correlation function,

\[ M_{ij} = \langle s_i^z s_j^z \rangle, \tag{5} \]
FIG. 2. (a) Binder ratio \( B \) and (b) correlation ratio \( R_{1,1} \) as a function of \( U \) for various \( L \) (various symbols). The crossing point corresponds to \( U_c \). We identify \( 7.25 < U_c < 7.3 \) using these two methods. (c) FM spin susceptibility \( S_{k=0} \) and (d) equal-time fermion Green’s function \( G_{k-k'\pi} \) for various system sizes close to the critical point \((U = 7.25, 7.275, 7.3)\). The observed linear behavior on a log-log scale is consistent with the expected power-law decay \( S_{k=0} \sim L^{-(1+\eta_0)} \) and \( G_{k-k'\pi} \sim L^{-\eta_0} \) at criticality. The negative of the slope \( \eta_0 \) is included for each \( U \). We find \( \eta_0 = h - 1 \approx 0.31 \) and \( \eta_0 = h \approx 0.135 \) by taking the average across all three values of \( U \).

whose Fourier transform is the spin structure factor:

\[
S(k) = \frac{1}{L^2} \sum_{ij} e^{i \mathbf{k}(\mathbf{i}-\mathbf{j})} M_{ij},
\]

where \( s_{z,i} = \frac{1}{2} (n_{i\uparrow} - n_{i\downarrow}) \) denotes the \( z \) component of the electron spin operator at site \( i \). In the broken-symmetry phase at large \( U \), we expect long-range order at wave vector \( k = 0 \) and the condensation of the \( s_{z,i} \) operator in the thermodynamic limit. A mean-field calculation of the uniform local magnetization \( m = \langle s_{z,4} \rangle \) shows that the system undergoes a continuous phase transition from the paramagnetic SM to an FM ordered phase above a critical interaction strength \( U_c^{MF} \sim 2.2 \). To go beyond the mean-field approximation, we perform QMC simulations of the spin-spin correlation function \( \langle s_i s_j \rangle \) in real space. We expect the true \( U_c \) to be larger than the mean-field result due to the disordering effect of quantum fluctuations.

To explore the SM-to-FM QCP in QMC, we use two dimensionless quantities: the Binder ratio, defined here as

\[
B = \frac{\sum_{ijkl} \langle s_{z,i}s_{z,j}s_{z,k}s_{z,l} \rangle}{\left( \sum_{ij} \langle s_{z,i}s_{z,j} \rangle \right)^2},
\]

and the correlation ratio, defined as:

\[
R_{1,1} = 1 - \frac{S(k = k^*)}{S(k = 0)},
\]

where we define \( k^* \equiv \frac{2\pi}{L} (\hat{x} + \hat{y}) \). Long-range FM ordering makes \( S(k = 0) \) diverge and hence implies \( R_{1,1} \to 1 \) in the thermodynamic limit \( L \to \infty \). In the disordered SM phase, the correlation ratio vanishes in the thermodynamic limit since \( S(k \to 0) \to S(k = 0) \). At the QCP, both \( B \) and \( R_{1,1} \) are independent of \( L \) up to finite-size corrections. Therefore, we pinpoint the QCP by plotting these ratios as a function of \( U \) for various lattice sizes, and look for a crossing point of the curves. Using the Binder ratio, we identify the QCP to be \( 7.275 \leq U_c \leq 7.3 \) [Fig. 2(a)]. The correlation ratio suggests the compatible result \( 7.25 \leq U_c \leq 7.275 \) [Fig. 2(b)].

To further corroborate these results, we also measure the fermion excitation gap \( \Delta_{sp} \) as a function of \( L \) and \( U \) using the unequal-time fermion Green’s function (Fig. 3). For imaginary-time displacement \( \tau \), the single-particle excitation gap leads to a behavior \( G(k,\tau) = \langle c_k|\tau\rangle c_{k^*}(0) \sim e^{-\Delta_{sp}(L,U)\tau} \) for the fermion Green’s function. It also implies the energy spectrum near the QCP is \( \sqrt{k^2 + \Delta_{sp}^2} \), which suggests \( L\Delta_{sp}(L,U) \) is a dimensionless quantity. In the thermodynamic limit, we expect \( \Delta_{sp} \sim 0 \) in the gapless SM phase. Thus, we can estimate the position of the QCP by plotting \( L\Delta_{sp}(L,U) \) against \( 1/L \) and extrapolating to \( L = \infty \). This suggests \( 7.2 < U_c < 7.3 \), consistent with the previous two approaches. These three methods combined indicate that \( U_c \approx 7.275 \). In the Supplemental Material,
we have computed $B$ and $R_{1,1}$ using finite-temperature QMC with $\beta = L$ [25, 34, 35] for $L$ up to 15 and achieve $7.25 < U_c < 7.3$, consistent with our projector QMC results.

Critical exponents.—Having obtained a good estimate of $U_c$, we now turn to calculating universal critical exponents directly at the QCP. Those exponents describe the power-law decay of various correlation functions at the QCP. In Fig. 2(c), we plot the FM spin susceptibility, $S(k) = 0$, for interaction strengths $U = 7.25, 7.275$, and 7.3. The spin susceptibility is expected to decay as $L^{-(1+\eta_\phi)}$ at the critical point for an $L \times L$ system. Figure 2(c) shows that the finite-size effects in the two-particle spin (bosonic) sector are insignificant as all data points follow a single straight line on a log-log scale. Our results in Fig. 2(c) thus suggest the anomalous dimension of the bosonic order parameter, $\eta_\phi$, satisfies $0.282 < \eta_\phi < 0.352$. Likewise, the equal-time fermion single-particle Green’s function in momentum space, $G_{kk} = \delta_{kk}$, must decay as $L^{-\eta_\psi}$, where $\eta_\psi$ is the anomalous dimension of the fermion operator at criticality. Accordingly, Fig. 2(d) shows that $0.129 < \eta_\psi < 0.140$. We obtained these numbers by taking the last five data points ($L = 11, 13, 15, 17, 19$) for fermions. We see that $L = 9$ follows the same line while $L = 5, 7$ exhibit visible deviations. This implies that finite-size effects are more pronounced in the fermionic sector. Among the three interaction strengths used in Fig. 2(c-d), our Binder/correlation ratio analysis suggests $U_c$ is closer to 7.275. Thus we conclude $\eta_\phi \approx 0.30 \pm 0.02$ and $\eta_\psi \approx 0.135 \pm 0.005$.

Alternatively, we can use the scaling hypothesis and data collapse near (but away from) the QCP to simultaneously obtain estimates of the critical exponents as well as $U_c$. Scaling forms for bosonic and fermionic correlation functions can be used to extract $\eta_\phi$ and $\eta_\psi$. We begin with the spin structure factor. At $\beta = \infty$ or $\beta = L$ and near the QCP, scaling analysis reveals that [36]:

$$L^{1+\eta_\phi} S_{kk=0}(L, U) = (1 + \alpha_1 L^{-\nu_1}) f_1(uL^{1/\nu}),$$

(9)

where $u = U - U_c$, $\nu$ is the correlation length exponent, and $f_1$ is a smooth scaling function of $uL^{1/\nu}$. The term proportional to $L^{-\nu_1}$ is an effective correction-to-scaling term which can be ignored for large systems. For $S_{kk=0}$ we find that those corrections are negligible and we achieve satisfactory results by keeping the leading scaling term. Such a simplified scaling hypothesis, namely $L^{1+\eta_\phi} S_{kk=0}(L, U) = f_1((U - U_c)L^{1/\nu})$, allows the following data-collapse method to extract the critical exponents. By plotting all available data points in the $L^{1+\eta_\phi} S_{kk=0}(L, U)$ combination against $(U - U_c)L^{1/\nu}$ and using $U_c$, $\nu$, and $\eta_\phi$, we can achieve a single smooth curve rather than scattered data points, we can identify both the critical exponents $\nu$ and $\eta_\phi$ and the critical point $U_c$ [Fig. 4(a)]. This method yields $U_c \approx 7.280$, $\nu^{-1} \approx 1.19$, and $\eta_\phi \approx 0.310$. Again, finite-size effects are minimal here: we see in Fig. 4(a) that data points for systems as small as $L = 7$ also collapse to the fitting curve.

Additionally, near the QCP, the correlation ratio $R_{1,1}$ behaves as a universal function of $(U - U_c)L^{1/\nu}$ and $L^2/\beta$ where $z$ is the dynamical critical exponent and $\beta$ the inverse temperature. Here emergent Lorentz symmetry at the QCP implies $z = 1$. In Fig. 4(c), data collapse of $R_{1,1}$ yields the estimates $U_c \approx 7.265$ and $\nu^{-1} \approx 1.17$. We can also plot $L^{1+\eta_\psi} S(k = 0) = 0$ against $R_{1,1}$ to extract $\eta_\psi \approx 0.320$ [Fig. 4(b)]. The main advantage of this method compared to that used in Fig. 4(a) is that neither $U_c$ nor $\nu$ need to be determined.

Similarly, to compute the fermion anomalous dimension $\eta_\psi$, we can utilize the following scaling hypothesis in the proximity of the QCP:

$$L^{\eta_\psi} G_{kk} = \delta_{kk}(L, U) = (1 + \alpha_2 L^{-\nu_2}) f_2(uL^{1/\nu}),$$

(10)

where:

$$G_{kk} = \delta_{kk}(L, U) \equiv \frac{1}{L^2} \sum_{ij} e^{ikr(i-j)} \langle c_i \bar{c}_j \rangle,$$

(11)

and $f_2$ is another smooth scaling function. Applying data collapse to $G_{kk}(L, U)$ yields satisfactory results, especially for $L \geq 9$ [Fig. 4(d)]. We find $\eta_\psi \approx 0.141$, $U_c \approx 7.280$, and $\nu^{-1} \approx 1.19$.

Combining our results directly obtained at the QCP and those extracted from data collapse in the vicinity of the QCP, we obtain a consistent set of critical exponent estimates with error bars that reflect the totality of our
results (Table I). In the Supplemental Material, we have investigated the impact of corrections to scaling on the critical exponents we extract. Although the quality of data collapse increases significantly upon introducing the associated free parameters $\alpha$ and $\omega$ in Eqs. (9-10), we find that the exponent values remain unchanged within the statistical error bar.

**Summary and outlook.**—In summary, we applied a projector QMC method with Gutzwiller-projected trial state to study the quantum phase transition from paramagnetic Dirac semimetal to ferromagnetic insulator in a model of a single two-component Dirac fermion in (2+1)D subject to an on-site repulsive Hubbard interaction $U$. We also performed finite-temperature QMC calculations for the same model. Both methods yield consistent results, from which we conclude that the phase transition is continuous and happens at $U_c = 7.28 \pm 0.02$ in units of the fermion hopping amplitude. Besides determining the position of the QCP, our main result is a numerically exact determination of the critical exponents of the associated $N = 2$ chiral Ising GN universality class; the inverse correlation length exponent $\nu = 1.19 \pm 0.03$, the order parameter anomalous dimension $\eta_0 = 0.31 \pm 0.01$, and the fermion anomalous dimension $\eta = 0.136 \pm 0.005$.

Recently, the adiabatic QMC method \cite{37} has been introduced to exponentially enhance the average sign without compromising the accuracy of the method. As a future direction, it would be interesting to apply this algorithm to our model Hamiltonian and study the robustness of our results at considerably lower temperatures. Additionally, we restricted our study in this work to the $N = 2$ Ising GN universality class. Replacing complex fermions with real (Majorana) fermions in our model and introducing nearest-neighbor quartic interactions will give rise to an Ising GN universality class with half as many fermionic degrees of freedom, the $N = 1$ chiral Ising GN universality class. This can be taken as an effective model of interacting Majorana surface states in the 3D topological superfluid $^3$He-B \cite{38,39}. Previous works on this universality class using the conformal bootstrap \cite{17,40}, fRG \cite{18,41}, and perturbative RG \cite{14-16,42-44} have proposed that $\mathcal{N} = 1$ spacetime supersymmetry emerges at the (2+1)D QCP. A numerical verification of this prediction would be of high value.

**Acknowledgments.**—A.V. acknowledges useful discussions with Christian Mendli. S.M.T. and A.V. were supported by Iran Science Elites Federation (ISEF). J.M. was supported by NSERC Discovery Grants #RGPIN-2020-06999 and #RGPAS-2020-00064; the Canada Research Chair (CRC) Program; CIFAR; the Government of Alberta’s Major Innovation Fund (MIF); the University of Alberta; the Tri-Agency New Frontiers in Research Fund (NFRF, Exploration Stream) and the Pacific Institute for the Mathematical Sciences (PIMS) Collaborative Research Group program. This research was enabled in part by support provided by Calcul Québec (www.calculquebec.ca), Compute Ontario (www.computeontario.ca), WestGrid (www.westgrid.ca), and Compute Canada (wwwcompute.canada).
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SUPPLEMENTAL MATERIAL

Determinant QMC algorithm

Here we discuss the implementation of the determinant quantum Monte Carlo (QMC) algorithm in more detail. By introducing a small parameter \( \Delta \tau \) through \( M \Delta \tau = \beta \), where \( M \gg 1 \) is the number of imaginary time steps, due to the Trotter-Suzuki formula the following expression can be considered as the density matrix:

\[
\rho = \left( e^{-\Delta \tau (H_0 + H_U)} \right)^M = \left( e^{-\Delta \tau H_0} e^{-\Delta \tau H_U} e^{-\Delta \tau H_0} \right)^M + \mathcal{O}(\Delta \tau^3).
\]  

(S1)

We found that a conservative value of \( \Delta \tau = \frac{1}{20} \) can give rise to reliable results. The interaction Hamiltonian \( H_U \) contains quartic terms which can be decoupled using a classical discrete field, a procedure known as the Hubbard–Stratonovich (HS) transformation. Generally, the average sign depends on the precise transformation, such that by choosing a suitable HS transformation we can mitigate the sign problem. The class of HS transformations we consider is:

\[
e^{-\Delta \tau U(n_{\uparrow} - 1/2)(n_{\downarrow} - 1/2)} = \frac{1}{2} e^{-\Delta \tau \sum_{n_{\uparrow} \neq \downarrow} \lambda s_i} \sum_{n_{\uparrow} = 1} e^{-2\lambda s_i \cdot \vec{l}},
\]  

(S2)
on a given lattice site \( i \) and for a given imaginary time step, where \( \lambda \) is defined via \( \cosh \lambda = \exp(\frac{U \Delta \tau}{2}) \), \( \vec{s}_i \) is the spin operator on site \( i \), and \( \vec{l} \) is a unit vector in an arbitrary direction. The freedom to choose \( \vec{l} \) means that we can decompose the interactions in different channels. While the HS transformation originally introduced by Hirsch [45, 46] corresponds to the \( s_z \) channel, we found that the average sign is significantly higher in the \( s_x \) or \( s_y \) channels. Indeed, with an \( s_z \) decoupling, Fig. 1(a) in the main text shows that the average sign of our model at the QCP is not severe and we can approach the ground state by accessing reasonably low temperatures.

To speed up calculations, instead of summing explicitly over Wick contractions, we use the following closed forms for 2-point and 4-point connected correlation functions of a fermion bilinear \( A = c^\dagger A c \):

\[
\langle \hat{A}^2 \rangle_s^C = \text{tr} [G A^2 - (G A)^2],
\]

(S3)

\[
\langle \hat{A}^4 \rangle_s^C = \text{tr} [G A^4 - 4 G G A^3 - 3 (G A)^2 + 12 (G A)^2 G A^2 - 6 (G A)^4],
\]

(S4)

where we define the fermion Green’s function matrix as

\[
G = I - \langle c c^\dagger \rangle_s.
\]

(S5)

Here \( I \) is the identity matrix, \( c \) is a vector of annihilation operators, and \( A \) is a matrix which depends on the observable. In particular, this is useful for calculating the Binder ratio, which involves both 2-point and 4-point correlators:

\[
B = \frac{\langle \hat{S}_z^4 \rangle}{\langle \hat{S}_z^2 \rangle^2} = \frac{\sum_{ijkl} \langle s_i \delta_{jz} s_j \delta_{kz} s_k \delta_{lz} \rangle}{\left( \sum_{ij} \langle s_i \delta_{jz} \rangle \right)^2}.
\]

(S6)

Here \( \hat{S}_z \) is the total spin:

\[
\hat{S}_z = (c^\dagger \cdots c^\dagger_{Lz} c \cdots c^\dagger_{Lz}) (\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})
\]

(S7)

and 0 and 1 are the empty and identity \( L^2 \times L^2 \) matrices, respectively. Without relations (S3-S4), we would have had to sum over four different indices and all possible Wick contractions of \( \langle s_z i s_z j s_z k s_z l \rangle \) to find the value of \( \langle \hat{S}_z^4 \rangle_s \). Due to efficient matrix manipulation routines, our approach is significantly faster than performing this summation.

Finite-size scaling analysis

According to the scaling hypothesis, the ferromagnetic spin structure factor \( S \) and the equal-time fermion Green’s function \( G \) should exhibit the following behavior in the vicinity of the QCP:

\[
S_{k=0}(L, U, \beta) = L^{-(1+\eta_\phi)} (1 + \alpha_1 L^{-\omega_1}) \times f_1 \left( \frac{U - U_c}{L^{1/\nu}} \right),
\]

(S8)

\[
G_{k=0}(L, U, \beta) = L^{-\eta_\psi} (1 + \alpha_2 L^{-\omega_2}) \times f_2 \left( \frac{U - U_c}{L^{1/\nu}} \right).
\]

(S9)

Here, we assumed the dynamical critical exponent \( \nu = 1 \), which is a consequence of emergent Lorentz symmetry at the QCP. Based on this observation, we have two ways to make the scaling functions \( f_1 \) and \( f_2 \) depend on a single scaling variable. First, by considering the zero-temperature limit \( \beta = \infty \), and second, at finite temperature but by considering isotropic scaling in spacetime, \( \beta = L \). Then by setting all data points on a single plot with \( (U - U_c) L^{1/\nu} \) on the horizontal axis, we can tune \( U_c, \eta_\phi, \eta_\psi, \nu \), and the \( \alpha \) and \( \omega \) parameters such that all data points collapse on a single curve. With the finite-temperature method with \( \beta = L \) scaling, large system sizes are typically required to reach the quantum critical regime and achieve sufficient accuracy for the critical exponents. For the system sizes we are able to reach, statistical noise in the finite-temperature data remains an obstacle to obtaining accurate exponents. By contrast, the projector QMC method allows us to reach the
The fitting parameters \( \alpha \) and \( \omega \) are chosen separately for each data set such that we obtain the best data collapse. Here we used \( U_c = 7.280 \), \( 1/\nu = 1.19 \), \( \eta_\phi = 0.310 \), and \( \eta_\omega = 0.136 \) for all plots. The non-universal fitting parameters are: \( \alpha_1 = 1.04 \), \( \omega_1 = 2.22 \), \( \alpha_2 = 1.001 \), \( \omega_2 = 2.64 \), \( \alpha_3 = -88.25 \), \( \omega_3 = 3.96 \), \( \alpha_4 = 29.27 \), and \( \omega_4 = 3.96 \).

As we discussed in the main text, the fitting parameters \( \alpha \) and \( \omega \) in Eqs. (S8-S9) are not important for large system sizes (larger than \( L = 7 \)). In Fig. S1, we show the effect of including these correction terms. We can achieve a good data collapse even for system sizes as small as \( L = 5 \) by choosing the following parameters for all four plots: \( U_c = 7.28 \), \( \nu^{-1} = 1.19 \), \( \eta_\phi = 0.31 \), and \( \eta_\omega = 0.136 \). We note that \( \alpha \) and \( \omega \) are not universal parameters and are chosen separately for each data set in Fig. S1, such that we can get the best data collapse.

**Projector vs finite-temperature QMC**

Although we have employed the projector QMC algorithm described in the main text as our main method, we have also performed finite-temperature determinant QMC calculations to corroborate our results. The finite-temperature determinant QMC method is based on discretizing the evolution operator in the computation of the finite-temperature statistical average \( \langle O \rangle = \frac{\text{Tr}[e^{-\beta H}O]}{\text{Tr}[e^{-\beta H}]} \), using a symmetric Suzuki-Trotter decomposition scheme. Then, we can employ an HS transformation as previously discussed to handle the on-site Hubbard interaction. We consider \( \beta = L \) scaling for \( L = 5, 7, 9, 11, 13, 15, 17 \). We extract the transition point using the Binder and correlation ratios. As Fig. S2 suggests, the transition lies somewhere close to \( U = 7.3 \), which is consistent with the projector QMC results in the main text.

Finally, we compare the relative ability of projector QMC vs finite-temperature QMC to converge to the ground state, which is our main interest. In Fig. S3,
we compute various quantities as a function of inverse temperature $\beta$ using three methods: finite-temperature QMC, regular projector QMC, and projector QMC with a Gutzwiller-projected trial wave function. For the projector QMC plots, we define $\beta \equiv 2\Theta$. We see that among all three methods, projector QMC with a Gutzwiller-projected trial state approaches ground-state properties the fastest. We can define an effective inverse temperature $\beta_{\text{eff}}$ for a given $\Theta$ by comparing the value of physical quantities we obtain in projector QMC with those from finite-temperature QMC at $T = 1/\beta$. This reveals that $\beta_{\text{eff}}(g_{GW} \neq 0) \approx 2\Theta + 11 \pm 1$ and $\beta_{\text{eff}}(g_{GW} = 0) \approx 2\Theta + 7 \pm 1$. Thus, the projector QMC computations in the main text with projection time $2\Theta = 14$ translate effectively to finite-temperature QMC with $\beta_{\text{eff}} = 25 \pm 1$. This is sufficient to accurately approximate zero-temperature critical properties.