Asymptotically flat black holes and gravitational waves in three-dimensional massive gravity

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Abstract Different classes of exact solutions for the BHT massive gravity theory are constructed and analyzed. We focus in the special case of the purely quadratic Lagrangian, whose field equations are irreducibly of fourth order and are known to admit a unique (flat) maximally symmetric solution, as well as asymptotically locally flat black holes endowed with gravitational hair. The first class corresponds to a Kerr-Schild deformation of Minkowski spacetime along a covariantly constant null vector. As in the case of General Relativity, the field equations linearize so that the generic solution can be easily shown to be described by four arbitrary functions of a single null coordinate. These solutions can then be regarded as a new sort of pp-waves. The second class is obtained from a deformation of the static asymptotically locally flat black hole, that goes along the spacelike (angular) Killing vector. Remarkably, although the deformation is not of Kerr-Schild type, the field equations also linearize, and hence the generic solution can be readily integrated. It is neither static nor spherically symmetric, being described by two integration constants and two arbitrary functions of the angular coordinate. In the static case it describes “black flowers” whose event horizons break the spherical symmetry. The generic time-dependent solution appears to describe a graviton that moves away from a black flower. Despite the asymptotic behaviour of these solutions at null infinity is relaxed with respect to the one for General Relativity, the asymptotic symmetries
coincide. However, the algebra of the conserved charges corresponds to BMS\(_3\), but devoid of central extensions. The “dynamical black flowers” are then shown to possess a finite energy. The surface integrals that define the global charges also turn out to be useful in the description of the thermodynamics of solutions with event horizons.

1 Introduction

After a century, General Relativity still remains as one of the banners of what theoretical physicists regard as a heuristic theory. It has also raised one the biggest challenges of the discipline, being the suitable description of the gravitational field at a microscopic level. Although the problem is unsolved yet, there have been different interesting proposals, including supergravity, string theory, loop quantum gravity and the gauge/gravity correspondence (for reviews about these subjects, see e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]). Some of them actually pursue the more ambitious goal of constructing a single unified theory that described matter and their interactions. Since none of the proposals are still fully satisfactory, some part of the community has in parallel tried to follow a more modest approach, focusing on toy models that capture some of the key features of the gravitational field. Along this line, the same theory of General Relativity, once formulated on three spacetime dimensions turns out to be much simpler than its four-dimensional counterpart, and consequently, many things have been learned. For instance, in the case of negative cosmological constant \(\Lambda\), black holes in vacuum were shown to be obtained just from suitable identifications of the maximally symmetric AdS\(_3\) background [15], [16]. This simplicity then appears to be inherited in Strominger’s proposal for the microscopic counting of the asymptotic growth of the number of states of a black hole [17], which allows to recover the semiclassical Bekenstein-Hawking black hole entropy in terms of Cardy formula [18]. The idea relies on an old result by Brown and Henneaux [19], which states that the asymptotic symmetries of General Relativity with \(\Lambda < 0\) in three spacetime dimensions are spanned by the algebra of the two-dimensional conformal group with a precise central extension. This naturally suggests that a quantum theory of gravity in 3D would be described in terms of a conformal field theory in two dimensions. Nowadays, this result clearly appears as a precursor of the AdS/CFT correspondence [20, 21, 22]. Within this set up, there are interesting proposals to quantize the theory, as the one in [23, 24], which relies on assuming that the partition function can be holomorphically factorized. Recent different approaches have also been presented in e.g., [25, 26]. Despite the circle has not yet been completed, the lessons learned from three-dimensional General Relativity appear to provide new and good hints towards the solution of the problem in four spacetime dimensions. However, for the sake of the main topic of this book, that concerns gravitational waves, this toy model is certainly oversimplified because of the absence of local degrees of freedom. In other words, General Relativity in three dimensions has no chance of possessing propagating gravitons, and so from
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this point of view, the theory turns out to be trivial. This is a direct consequence of a purely geometrical fact: in three dimensions the Weyl tensor identically vanishes, and hence the Riemann tensor can be expressed in terms of the Ricci tensor and the curvature scalar. Thus, according to the Einstein equations, the curvature becomes completely determined by the stress-energy tensor, so that in vacuum, spacetime is necessarily of constant curvature $\Lambda$.

It is then interesting to look for different models of gravity in three-dimensional spacetimes that could incorporate local propagating degrees of freedom for the gravitational field. In order to do that, it is useful to recall some remarks about Riemannian geometry in three dimensions. Indeed, unlike the case of higher-dimensional spaces, the fact that the Weyl tensor identically vanishes does not mean that three-dimensional metrics are conformally flat. Indeed, in 3D, the suitable object that plays this role is the Cotton tensor

$$C_{\mu \nu} = \frac{1}{\sqrt{g}} \varepsilon^{\alpha \beta} \nabla_{\alpha} \left( R_{\beta \nu} - \frac{1}{4} g_{\beta \nu} R \right),$$

(1)

being symmetric and traceless. Thus, for any three-dimensional conformally flat spacetime the Cotton tensor vanishes, while the converse is also true but just on local patches. This tensor is a fundamental piece of the “topologically massive gravity” theory of Deser, Jackiw and Templeton [27, 28], whose Lagrangian is the one of General Relativity plus an additional term proportional to the three-dimensional Lorentz-Chern-Simons form. The field equations in vacuum are given by

$$G_{\mu \nu} + \Lambda g_{\mu \nu} - \frac{1}{\mu} C_{\mu \nu} = 0,$$

(2)

which by virtue of (1) are of third order for the metric and not invariant under parity. Nonetheless, and remarkably, upon linearization around a maximally symmetric background solution, it is simple to verify that the theory exorcises the ghosts that generically appear in theories with higher order field equations, since it possesses a healthy single propagating degree of freedom. Note that this sort of massive graviton differs from the Fierz-Pauli one, which possesses two degrees of freedom and respects parity. It is worth mentioning that since constant curvature spacetimes are conformally flat, they trivially solve the field equations of topologically massive gravity (2), and therefore, in the case of $\Lambda < 0$, BTZ black holes do. The theory in vacuum also admits interesting exact solutions for which spacetime is not of constant curvature, see e.g., [29, 30, 31, 32, 33, 34, 35, 36, 37, 38].

More recently, a different theory of massive gravity in three spacetime dimensions has been proposed by Bergshoeff, Hohm and Townsend (BHT) [39, 40]. The action is described by the Einstein-Hilbert one with cosmological constant, plus the addition of a precise combination of quadratic terms in the curvature, as in (3), but with an independent coupling. The field equations are then of fourth order and invariant under parity. Also noteworthy, once the field equations are linearized on a maximally symmetric background, they reduce to the ones of Fierz and Pauli for a massive graviton.
In the full nonlinear case, the field equations in vacuum are known to admit a wide class of interesting exact solutions, see e.g., [41, 42, 40, 43, 44, 45, 46, 47, 48, 49, 50, 51].

It is worth pointing out that, as it occurs for the Einstein-Gauss-Bonnet theory in \( d > 4 \) dimensions, which is also quadratic in the curvature [52, 53], the theory actually admits two different maximally symmetric solutions in vacuum, unless the couplings are chosen in a special way, so that the theory acquires an interesting behaviour.

In the special case in which the BHT theory admits a unique maximally symmetric vacuum of constant curvature \( \lambda \), at the perturbative level there is a single local propagating degree of freedom, being described by a “partially massless graviton” [54, 55, 56, 57, 58, 40]. This special case, in the limit of vanishing \( \lambda \), is known to enjoy very remarkable properties [58], and it is then the main focus of the remaining sections.

2 Asymptotically locally flat black holes in BHT massive gravity

Hereafter, we consider the BHT massive gravity theory in the case that admits a unique maximally symmetric (flat) background solution. The action is then described just by the terms that are purely quadratic in the curvature, which reads

\[
I[g] = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left( R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2 \right),
\]

so that the field equations become irreducibly of fourth order, and are given by

\[
2\Box R_{\mu\nu} - \frac{1}{2} \nabla_{\mu} \nabla_{\nu} R - \frac{1}{2} R g_{\mu\nu} + 4R_{\mu\sigma\nu\rho}R^{\sigma\rho} - \frac{3}{2} R R_{\mu\nu} - R_{\sigma\rho} R^{\sigma\rho} g_{\mu\nu} + \frac{3}{8} R^2 g_{\mu\nu} = 0.
\]

This three-dimensional gravity theory possesses an additional peculiar feature: unlike the case of General Relativity or topologically massive gravity, it admits asymptotically locally flat black holes in vacuum [43]. In the static case, the metric is given by

\[
ds^2 = -(br - \mu) dt^2 + \frac{dr^2}{br - \mu} + r^2 d\phi^2,
\]

so that the Ricci scalar reads

\[
R = \frac{-2b}{r}.
\]

The geometry then possesses a spacelike singularity at the origin, which in the case of \( b > 0 \) and \( \mu > 0 \), is cloaked by the event horizon located at \( r_+ = \mu / b \). Note that if \( \mu = 0 \), there is a NUT right on top of the singularity which becomes null. This class of black holes turns out to be asymptotically locally flat because, in spite of the fact that the metric does not approach the Minkowski one in the asymptotic region, the Riemann tensor vanishes anyway as \( r \to \infty \). It is also worth pointing out that
spacetimes of the form (5) are conformally flat. Indeed, these black hole solutions were first found in the context of conformal gravity in vacuum, for which the field equations imply that the Cotton tensor (1) vanishes [59].

Note that the Hawking temperature

\[ T = \frac{b}{4\pi}, \]

(7)
does not depend on the integration constant \( \mu \) in (5). As we show below, the integration constant \( b \) turns out to be related to the black hole mass, which is the only nonvanishing conserved charge. Hence, the remaining integration constant \( b \) can be interpreted as a gravitational hair parameter. It should be emphasized that dealing with the nonstandard fall off of a metric as in (5), together with the fact that the field equations are of fourth order and quadratic in the curvature, makes the construction of the surface integrals that describe the corresponding conserved charges to be a somehow cumbersome task. This is explicitly carried out for a generic set up in sections 5 and 6.

In order to catch a glimpse of the answer, one can readily compute the black hole entropy through Wald’s approach [61], which turns out to be given by

\[ S = \frac{\pi b}{4G}. \]

(8)
differing from “a quarter of the area” of the event horizon. Nevertheless, this is not surprising because what one expects for General Relativity clearly does not have to apply for the theory under discussion. The mass can then be found if one assumes that the first law of thermodynamics holds, which must be the case for any suitable action principle, i.e.,

\[ \delta M = T \delta S, \]

(9)
and hence the mass can be integrated as

\[ M = \frac{b^2}{32G}. \]

(10)

The following sections are devoted to the discussion of different classes of exact solutions, including an interesting new type of pp-waves, as well as “dynamical black flowers” [62]. The asymptotic behaviour of the theory that accommodates solutions of the latter class is also analyzed, including the explicit construction of the surface integrals that define the corresponding conserved charges and their algebra. Some aspects of the thermodynamics of the solutions with event horizons are also explored.

\[ ^1 \text{In this sense, it is amusing to see that the asymptotically locally flat black hole (5) can be constructed from the conformal gluing of three copies of BTZ black holes at infinity, so that the asymptotically locally flat region is mapped to the horizon of two of them. Besides, the conformal flatness of this class of black holes makes them also to fulfill the field equations of the Poincaré gauge theory [60].} \]
3 Partially massless pp-waves on flat spacetime

Let us consider flat Minkowski spacetime in null coordinates

$$ds^2 = 2dudv + dx^2,$$

(11)

as the seed background solution of the field equations (4) to be generalized. Following Kerr and Schild [63, 64], one might look for a consistent deformation of (11) along a null Killing vector, that for simplicity is also assumed to be covariantly constant. The spacetime metric can then be written as

$$ds^2 = ds_0^2 + H(u, x)du^2,$$

(12)

where the function $H$ parametrizes the deformation. Thus, despite the field equations (4) are quadratic in the curvature, as in the case of four-dimensional General Relativity, they also exactly linearize without making any sort of approximation. Indeed, they just reduce to

$$\partial^4 H = 0,$$

(13)

which readily integrates so that the function that describes the deformation acquires a generic form given by

$$H(u, x) = A_0(u) + A_1(u)x + A_2(u)x^2 + A_3(u)x^3,$$

(14)

depending on four arbitrary functions $A_I(u)$. Therefore, the theory turns out to admit exact solutions for which the spacetime metric is given by (12) with (14), that appear to describe a wide class of pp-waves.

It is interesting to check that the Kerr-Schild approach turns out to be successful in order to generate exact solutions beyond General Relativity. Results along these lines have been previously obtained for topologically massive gravity [35, 36, 37, 38], as well as for the generic BHT massive gravity theory [44, 65]. This is also the case for different theories in higher dimensions whose field equations are quadratic in the curvature, as the one described by the Einstein-Gauss-Bonnet action in vacuum [66, 67, 68, 69].

One may naturally wonder whether new solutions could be obtained by regarding the black hole (5) instead of Minkowski spacetime (11) as the seed background solution. This is the main subject of the next section.

4 Dynamical black flowers

Let us now regard the asymptotically locally flat black hole (5) as the background seed solution. It turns out to be useful to write the seed black hole metric in terms of a set of coordinates that is suitably adapted to null infinity. Defining the tortoise coordinate $r^*$ according to $dr^* = \frac{dr}{\mu}$, one then sets $u = t - r^*$, so that the black
hole solution in (5) acquires the form
\[ ds^2 = -(br - \mu)du^2 - 2dudr + r^2d\phi^2. \]  
(15)

We thus look for a different class of deformations as compared with the one in the previous section. Indeed, the deformation is assumed to go along the spacelike Killing vector \( \partial_\phi \), so that the metric reads
\[ ds^2 = -(br - \mu)du^2 - 2dudr + (r - \mathcal{H}(u, \phi))^2d\phi^2, \]  
(16)

where \( \mathcal{H}(u, \phi) \) stands for an arbitrary function that is periodic in \( \phi \). Quite remarkably, although the deformation is not of Kerr-Schild type, the field equations (4) also exactly linearize without the need of any approximation. Indeed, once (16) is plugged within (4), the field equations just reduce to a single one that is linear in \( \mathcal{H} \), which reads
\[ \partial_u \left( \partial_u \mathcal{H} + \frac{b}{2} \mathcal{H} \right) = 0. \]  
(17)

The general solution can then be readily integrated, so that the function that parametrizes the deformation reads
\[ \mathcal{H}(u, \phi) = \mathcal{A}(\phi) + \mathcal{B}(\phi)e^{-\frac{b}{2}u}, \]  
(18)

where \( \mathcal{A}(\phi) \) and \( \mathcal{B}(\phi) \) stand for arbitrary periodic functions of the angular coordinate. It is worth pointing out that once the deformation is switched on, the solution of the form (16) with (18) remains being conformally flat. Besides, its Ricci scalar reads
\[ R = \frac{2b}{r-\mathcal{H}}, \]  
(19)

which manifestly signals the existence of a curvature singularity located at \( r = \mathcal{H}(u, \phi) \), and hence one can set the range of the radial coordinate according to \( \mathcal{H} < r < \infty \).

This class of solutions is then naturally divided in two cases: (i) \( \mathcal{B}(\phi) = 0 \), so that spacetime is static, and (ii) \( \mathcal{B}(\phi) \neq 0 \), leading to metrics that describe dynamical spacetimes.

In case (i) the static metric given by (16) and (18), with \( \mathcal{B}(\phi) = 0 \), possesses an event horizon at \( r = r_+ = \mu/b \). Cosmic censorship then requires the horizon to surround the curvature singularity at the origin of the radial coordinate, and hence the function \( \mathcal{A}(\phi) \) turns out to be bounded from above according to:
\[ \mathcal{A}(\phi) < r_+. \]

The shape of the deformed horizon is then generically no longer spherically symmetric. Indeed, the induced metric on \( r = r_+ \) and a constant value of the coordinate \( u \) is given by \( g_{\phi\phi} = (r_+ - \mathcal{A}(\phi))^2 \), so that for an arbitrary periodic arbitrary function \( \mathcal{A}(\phi) \), spacetime appears to describe a sort of “black flower”. Note that despite the
horizon has been deformed, rigidity still holds, and hence the Hawking temperature of the black flowers agrees with the one of the undeformed spherically symmetric case, given by (7).

In the dynamical case (ii), since the additional arbitrary periodic function does not vanish, i.e., for $B(\phi) \neq 0$, the deformation $H(u, \phi)$ propagates along outgoing null rays. Hence, the generic solution is neither static nor spherically symmetric, and it is clear that at late retarded time $u \to \infty$, the configuration tends to a static black flower. Therefore, this class of solutions can be regarded as a black flower endowed with an outgoing graviton. The behaviour of this class of “dynamical black flowers” somehow resembles the one of Robinson-Trautman spacetimes for General Relativity in four dimensions [70, 71, 72, 73], which describe outgoing radiation on a Schwarzschild black hole. Nonetheless, a difference that should be stressed is that the evolution in time for these four-dimensional analogues is described by an involved fourth-order nonlinear differential equation. Further details about the behaviour of this class of dynamical black flowers can be found in [62].

As a closing remark of this section, we would like to mention that the strategy of finding exact solutions for deformations that are non null, and then not of Kerr-Schild type (see e.g. [74]), has also been successful in the context of a generic BHT massive gravity theory [65] as well as in the case of Lovelock theories in higher dimensions [75].

5 Relaxed asymptotically flat behaviour and asymptotic symmetries

Here we explore the asymptotically locally flat behaviour of the theory described by the action (3) that is able to accommodate the class of “dynamical black flowers” discussed in the previous section. As it has been the main focus of attention for a big deal of the recent literature, see, e.g., [76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88], our description is made so that the region where the asymptotic symmetries and their corresponding global charges are defined, is located towards null infinity. Following a generic criterion (see, e.g., [89, 90]), we assume that the fall off of the gravitational field should be mapped into itself at least under the Poincaré group. Furthermore, the asymptotic behaviour must be relaxed enough such that the solutions of interest, in this case given by the dynamical black flowers, fit within the set; but at the same time, the fall-off has to be sufficiently fast so as to yield a finite set of global conserved charges.

The deviations with respect to the flat Minkowski reference background, $\Delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$, with

$$d\bar{s}^2 = -du^2 - 2du dr + r^2 d\phi^2,$$ (20)

are then proposed to be asymptotically of the form
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\[ \Delta g_{uu} = h_{uu} r + f_{uu} + \cdots \]
\[ \Delta g_{u\phi} = h_{u\phi} r + f_{u\phi} + \cdots \]
\[ \Delta g_{\phi\phi} = h_{\phi\phi} r + f_{\phi\phi} + \cdots \]
\[ \Delta g_{rr} = \frac{k_{rr}}{r^3} + \cdots \]
\[ \Delta g_{ur} = \frac{k_{ur}}{r} + \cdots \]
\[ \Delta g_{\phi r} = \frac{k_{\phi r}}{r} + \cdots \]

(21)

where \( h_{\mu\nu}, f_{\mu\nu} \) and \( k_{\mu\nu} \) stand for arbitrary functions of \( u \) and \( \phi \). As it can be seen from (21), the presence of terms along the functions \( h_{\mu\nu} \), makes the fall off of the metric to be clearly relaxed as compared with the one for General Relativity in three-dimensional spacetimes at null infinity [91, 92]. Nonetheless, the asymptotic symmetries, being spanned by diffeomorphisms that maintain the asymptotic form of the metric, i.e., that fulfill

\[ \mathcal{L}_\xi g_{\mu\nu} = O(\Delta g_{\mu\nu}) , \]

remain the same. Indeed, the asymptotic behaviour of the metric in (21) is mapped into itself under diffeomorphisms spanned by asymptotic Killing vectors whose leading terms are given by

\[ \xi^u = T(\phi) + u \partial_\phi Y(\phi) + \cdots \]
\[ \xi^r = -r \partial_\phi Y(\phi) + \cdots \]
\[ \xi^\phi = Y(\phi) - \frac{1}{r} \partial_\phi \left( T(\phi) + u \partial_\phi Y(\phi) \right) + \cdots . \]

Diffeomorphisms of this sort, for which \( T(\phi) = Y(\phi) = 0 \) then form an ideal of “pure gauge” transformations, so that the quotient algebra defines the so-called BMS\(_3\) one, given by the semidirect sum of a Virasoro algebra with an abelian ideal that corresponds to supertranslations.

It is worth to highlight that relaxing the asymptotic behaviour as in (21) does not spoil the asymptotic symmetries that are known to appear in the simpler case of pure General Relativity.

6 Conserved charges from surface integrals at null infinity and black hole thermodynamics

One of the main aims of this section is the explicit construction of conserved charges that correspond to the asymptotic symmetries spanned by (22). This task is fairly simplified once the action (3) is expressed in terms of an auxiliary field \( \ell_{\mu\nu} \), which allows to reduce the field equations from fourth to second order [40, 58]. The action
(3) can then be written as
\[ I[g, \ell] = \frac{1}{16\pi G} \int d^3 x \sqrt{-g} \left( \ell^{\mu \nu} G_{\mu \nu} - \frac{1}{4} \left( \ell^{\mu \nu} \ell_{\mu \nu} - \ell^2 \right) \right), \tag{23} \]
so that field equations associated to \( \ell_{\mu \nu} \) turn out to be algebraic and read
\[ G_{\mu \nu} - \frac{1}{2} (\ell_{\mu \nu} - g_{\mu \nu} \ell) = 0. \tag{24} \]
Consequently, the auxiliary field \( \ell_{\mu \nu} \) is given by
\[ \ell_{\mu \nu} = 2 \left( R_{\mu \nu} - \frac{1}{4} g_{\mu \nu} R \right), \tag{25} \]
being proportional to the Schouten tensor\(^2\). The variation of the action (23) with respect to the metric then yields second order field equations
\[
\begin{align*}
E_{\mu \nu} &= \nabla^\alpha \nabla_\alpha \ell^{\mu \nu} - 2 \nabla_\rho \nabla^{(\mu} \ell^{\nu)\rho} + \nabla^\mu \nabla^\nu \ell + 8^{\mu \nu} \left( \nabla_\rho \nabla_\lambda \ell^{\rho \lambda} - \nabla_\alpha \nabla_\mu \ell^{\alpha \mu} \right) + 4 \ell^{(\mu} G^{\nu)}
+ \ell^{\mu \nu} R - \ell R^{\mu \nu} - g^{\mu \nu} \ell^{\rho \sigma} G_{\rho \sigma} + \ell^{\lambda \mu} \ell^{\nu \lambda} - \ell \ell^{\mu \nu} - \frac{1}{4} g^{\mu \nu} \left( \ell^{\alpha \beta} \ell^{\alpha \beta} - \ell^2 \right) = 0. \tag{26}
\end{align*}
\]
One can then follow the covariant approach in \([98, 99]\), so that the conserved charges are the form
\[ Q_\xi = \int_0^1 ds \left( \frac{1}{2} \int_{\partial \Sigma} \varepsilon_{\nu \mu \alpha} \tilde{k}_{\xi}^{[\nu \mu]} dx^\alpha \right), \tag{27} \]
where \( \tilde{k}_{\xi}^{[\nu \mu]} \) stands for the “superpotential”, whose explicit form is somewhat involved and can be found in \([62]\). For the sake of this lecture, it is enough to mention that once the asymptotic form of the gravitational field in (21) is taken into account, it is possible to use an arbitrary set of interpolating metrics that fulfill the asymptotic conditions, so that the conserved charges (27) associated to the asymptotic symmetries spanned by (22) reduce to
\[ Q_\xi = Q[T, Y] = \frac{1}{64\pi G} \int d\phi \left( (T + u \partial_\phi Y) h_{\mu \nu}^2 + 2 Y h_{\rho \phi} h_{\mu \nu} + 4 \partial_\phi Y h_{\mu \nu} + 4 Y \partial_\phi h_{\mu \nu} \right). \tag{28} \]

The algebra of the conserved charges (28) can then be directly read from the variation of the same surface integrals, because \( \{ Q_\xi, Q_\zeta \} := \delta_\xi Q_\zeta \). Thus, considering that the leading nontrivial terms of the asymptotic field equations (26) are given by

\(^2\) Note that the Schouten tensor can be regarded as a sort of “potential” for the Cotton tensor in (1).
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\[ E_{uu} = -\frac{1}{4} (\partial_u h_{uu}^2) r^{-1} + \mathcal{O} (r^{-2}) = 0 , \]

\[ E_{u\phi} = \frac{1}{4} (\partial_\phi h_{uu}^2 - 2 \partial_u (h_{uu} h_{u\phi})) r^{-1} + \mathcal{O} (r^{-2}) = 0 , \]

(29)
(30)

together with the transformation law of the relevant components of the asymptotic form of the metric under the asymptotic symmetries (22), that read

\[ \delta_\xi h_{uu} = (T(\phi) + u \partial_\phi Y(\phi)) \partial_u h_{uu} + \partial_\phi (h_{uu} Y) , \]

\[ \delta_\xi h_{u\phi} = h_{uu} \partial_\phi (T(\phi) + u \partial_\phi Y(\phi)) + (T(\phi) + u \partial_\phi Y(\phi)) \partial_u h_{u\phi} + \partial_\phi (h_{u\phi} Y) , \]

(31)
(32)

it is found that the algebra of the conserved charges precisely agrees with the one of the asymptotic symmetries. Indeed, the algebra does not admit central extensions, since

\[ \{ Q_{\xi_1}, Q_{\xi_2} \} - Q_{[\xi_1, \xi_2]} \approx \mathcal{X}_{\xi_1, \xi_2} \approx 0 . \]

(33)

Once expanded in Fourier modes

\[ P_n = Q \left[ e^{im\phi}, 0 \right] ; \quad J_n = Q \left[ 0, e^{im\phi} \right] , \]

(34)

the algebra acquires the standard form, given by

\[ i \{ J_m, J_n \} = (m-n) J_{m+n} , \]

\[ i \{ J_m, P_n \} = (m-n) P_{m+n} , \]

\[ i \{ P_m, P_n \} = 0 . \]

(35)

The absence of central extensions agrees with the claim in [93], which relies on the vanishing cosmological constant limit of the centrally extended two-dimensional conformal algebra that was found in [94] (see also [43]) for BHT massive gravity in the special case (with a unique maximally symmetric vacuum) from a holographic approach.

6.1 Global charges of (rotating) black holes and dynamical black flowers

Since the global charges associated to spacetimes whose asymptotic behaviour is described by (21) have already been found to be given by (28), one can readily evaluate them for the solutions of interest.

**Static asymptotically locally flat black holes.** The mass of the solution described by the line element in (5) is directly obtained from (28), by taking into account that the only relevant nonvanishing deviation from the flat background is given by \( h_{uu} = -b \). The suitable surface integral that reproduces the expected result for the
mass is then given by,

\[ M = Q (\partial_u) = \frac{b^2}{32G}, \]  

(36)
in agreement with the result obtained in eq. (10) of the introduction, that came from integrating the first law of thermodynamics. It is worth pointing out that (36) is actually the only nonvanishing global charge (28) associated to the full set of asymptotic symmetries in (22). Therefore, the remaining integration constant \( \mu \) can indeed be interpreted as a gravitational hair parameter.

**Asymptotically flat rotating black holes.**- The theory described by the action (3) has also been shown to admit asymptotically locally flat rotating black hole solutions [43, 95]. It is useful to express the solution in the set of coordinates of ref. [94], so that once written in the outgoing null coordinates defined in section 4, the metric reads

\[ ds^2 = -NFdu^2 - 2N_r^2du dr + \left( r^2 + r_0^2 \right) (d \phi + N_\phi du)^2, \]  

(37)

with

\[ F = br - \mu; \ N = \frac{(8r + a^2b^2)^2}{64 (r^2 + r_0^2)}; \ N_\phi = -\frac{a}{2} \left( \frac{br - \mu}{r^2 + r_0^2} \right); \ r_0^2 = \frac{a^2}{4} \left( \mu + \frac{a^2b^2}{16} \right). \]  

(38)

Note that when the rotation parameter \( a \) vanishes, the spacetime metric reduces to the static black hole solution in (5). In this case, the relevant deviations with respect to the flat background are given by \( h_{uu} = -b \) and \( h_{u\phi} = -\frac{4b}{r} \), and hence the mass and the angular momentum can be directly obtained by virtue of (28). They are explicitly given by

\[ M = Q (\partial_u) = \frac{b^2}{32G}, \]  

(39)

\[ J = Q (\partial_\phi) = \left( \frac{b^2}{32G} \right) a = Ma, \]  

(40)

which is in full agreement with what one recovers from the vanishing cosmological constant limit of BHT massive gravity in the special case that admits a unique maximally symmetric vacuum [94, 93]. It is worth pointing out that the integration constant \( \mu \) remains as a gravitational hair parameter, because it is simple to verify that the mass and the angular momentum in (39) and (40) are indeed the only nonvanishing global charges associated to the full set of asymptotic symmetries.

**Dynamical black flowers.**- In this case, despite spacetime is neither spherically symmetric nor static, the solution described by (16), (18) also fits within the set of asymptotic conditions (21). It is amusing to see that actually their global charges have already been obtained. This is because the deformation with respect to the static black hole does not alter the values of \( h_{uu} \) and \( h_{u\phi} \). Therefore, the global charges coincide with the ones of the asymptotically locally flat black hole, being given by \( M = \frac{b^2}{32G} \) and \( J = 0 \). It should be pointed out that, even though there is
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outgoing radiation, the total energy at null infinity remains constant, i.e., no “news” are present [96, 97]. It is also worth noting that, once restricted to the static black flower ($\mathcal{B}(\phi) = 0$), both the constant $\mu$ and the periodic function $\mathcal{A}(\phi)$ become pure hair. In this sense, from the mode expansion, one might interpret the static black flower as a black hole solution endowed with an infinite number of gravitational hair parameters.

6.2 Thermodynamics

The generic expression for the conserved charges in (27) also turns out to be useful in a different sense when event horizons are present. Indeed, for black holes that possess a global Killing vector that generates the horizon, like it is the case of the static black flowers in (16), with (18) and $\mathcal{B}(\phi) = 0$, one can evaluate the surface charge for $\xi = \partial_u$ in order to deduce some of their thermodynamical properties. Indeed, since the superpotential $\tilde{k}_{\xi}^{[\mu\nu]}$ associated to a Killing vector $\xi$ is conserved [98, 99], one obtains that

$$\delta \left( Q_\xi \bigg|_{r=\infty} + Q_\xi \bigg|_{r=r_+} \right) = 0 \, . \quad (41)$$

For static black flowers it is remarkable that the superpotential can be precisely evaluated at an arbitrary value of the radial coordinate. Indeed, its only nonvanishing contribution is given by

$$\tilde{k}_{\xi}^{[ur]} = \frac{b\delta b}{32\pi G}, \quad (42)$$

which does depend neither on the radial coordinate nor on the arbitrary function $\mathcal{A}(\phi)$. Thus, as explained above, evaluating the global charge at infinity, one obtains the mass, i.e.,

$$Q_\xi \bigg|_{r=\infty} = \frac{b^2}{32G} = M, \quad (43)$$

while once the variation of the charge is evaluated at the horizon, $r = r_+$, one finds that

$$\delta Q_\xi \bigg|_{r=r_+} = \int_{\Sigma_h} \tilde{k}_{\xi}^{[ur]} \, d\phi = -\frac{b\delta b}{16G} = -T \delta S \, . \quad (44)$$

Therefore, the conservation law (41) just reduces to the first law of thermodynamics in the canonical ensemble, which reads $\delta \mathcal{F} = 0$, where $\mathcal{F} = M - TS$ stands for the Helmholtz free energy. One then concludes that the black hole, or generically, the black flower entropy that comes from Wald’s formula [61], given by (8), can also be written as

$$S = -2\pi \int_0^1 ds \left( \int_{\Sigma_h} \hat{\xi}_{\mu\nu} \hat{\epsilon}_{[\mu\nu]} \, d\phi \right), \quad (45)$$

where $\hat{\epsilon}_{\mu\nu}$ denotes the binormal to the bifurcation surface $\Sigma_h$, and is normalized according to $\hat{\epsilon}_{\mu\nu} \hat{\epsilon}^{\mu\nu} = -2$. 

It is also simple to verify that the results of this section naturally extend to the case of asymptotically locally flat rotating hairy black holes. Indeed, as a curiosity, it is worth pointing out that since the horizon of the solution in (37) possesses a vanishing angular velocity, i.e., \( \Omega_+ = 0 \), according to (41), the first law is recovered, but in the canonical ensemble:

\[
M = TdS - \Omega_+ dJ = TdS.
\]

(46)

7 Acknowledgments

The results presented here rely on our preprint [100], as well as the more recent work in [62]. We thank Oscar Fuentealba, Hernán González, Javier Matulich and Alfredo Pérez for enlightening discussions and especially Glenn Barnich for his valuable collaboration. C.T. is a Laurent Houart postdoctoral fellow. R.T. wishes to thank the Physique théorique et mathématique group of the Université Libre de Bruxelles, and the International Solvay Institutes for the warm hospitality. This work is partially funded by the Fondecyt grants N° 1130658, 1121031, 11130260, 3140125. The work of G.B. is partially supported by research grants of the F.R.S.-FNRS and IISN-Belgium as well as the “Communauté française de Belgique - Actions de Recherche Concertées”. The work of D.T. is partially supported by the ERC Advanced Grant “SyDuGraM”, by a Marina Solvay fellowship, by FNRS-Belgium (convention FRFC PDR T.1025.14 and convention IISN 4.4514.08) and by the “Communauté Française de Belgique” through the ARC program. The Centro de Estudios Científicos (CECs) is funded by the Chilean Government through the Centers of Excellence Base Financing Program of Conicyt.

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