On irreducible representations of the exotic conformal Galilei algebra

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Received 18 October 2010, in final form 12 November 2010
Published 15 December 2010
Online at stacks.iop.org/JPhysA/44/035401

Abstract

We investigate the representations of the exotic conformal Galilei algebra. This is done by explicitly constructing all singular vectors within the Verma modules, and then deducing irreducibility of the associated highest weight quotient modules. A resulting classification of infinite-dimensional irreducible modules is presented.

PACS numbers: 03.65.Fd, 02.10.Sv

1. Introduction

Since the proposal of a nonrelativistic analogue of the AdS/CFT correspondence [1, 2], conformal invariance in nonrelativistic physics has attracted renewed interest. Originally, the correspondence was discussed for systems admitting the full Schrödinger symmetry group. It was recognized soon after which the correspondence was able to extend to a wider class of systems (see for example [3–8]). In this paper we focus on the symmetry, discussed in [8], generated by the conformal Galilei algebra (CGA) with a so-called exotic central extension. This is an example of a non-semisimple Lie algebra whose representation theory has not been explored in great detail.

The exotic central extension is crucial for the holographic construction presented in [8], although the obtained bulk spacetime is a ‘wrong-signature’ version of AdS$_7$. The same algebra also plays an important role in some models from classical mechanics with higher order time derivatives [9–11]. Furthermore, it is known that there are many physical systems admitting extended (not necessarily conformal) Galilei symmetry with the exotic central extension (see for example [12, 13] and references therein). We thus believe that the CGA with the exotic central extension is of physical importance and that the study of representations of the algebra will be useful for further investigation.
The CGA [14] is an extension of the Galilei algebra which generates the basic symmetry of nonrelativistic systems. The class of CGA we focus on is labelled by a positive half-integer \( \ell \) [15]. The smallest example \( \ell = 1/2 \) has the generators of conformal and scale transformations in addition to those of the Galilei algebra. The CGA of this case corresponds to the Schrödinger algebra without central extension. The \( \ell = 1 \) algebra comprises constant accelerations and we will study a particular case of this type. The CGA admits two types of central extension [8–11]. One exists for half-integral values of \( \ell \) and any dimension of spacetime. The other exists only in \((2+1)\) dimensions with integral values of \( \ell \), which is the motivation for naming the central extension exotic.

In this paper we study the highest weight representations, especially the Verma modules, over the exotic CGA in full detail. The classification of the irreducible modules is provided by the method similar to that for semisimple Lie algebras. After presenting the structure of the CGA in section 2, we give the Verma modules in section 3 and then proceed to construct all singular vectors and present a complete list of infinite-dimensional irreducible representations in section 4. We remark that our presentation is similar to that of [16] for the case of the Schrödinger algebra. We also remark that the vector field realization of the CGA is given in [7, 8] and the representation of an infinite-dimensional extension of CGA is studied in [17].

2. Structure of the exotic conformal Galilei algebra

For \( \ell \in \frac{1}{2}\mathbb{Z} \), the ‘spin-\( \ell \)’ class [15] of conformal extensions of the Galilei algebra is realized by the following infinitesimal action on \((d+1)\)-dimensional spacetime:

\[
H = \frac{\partial}{\partial t}, \quad D = -t \frac{\partial}{\partial t} - lx_i \frac{\partial}{\partial x_i}, \quad C = t^2 \frac{\partial}{\partial t} + 2tx_i \frac{\partial}{\partial x_i},
\]

\[
J_{ij} = -x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i}, \quad P^n_i = (-t)^n \frac{\partial}{\partial x_i},
\]

where \( n = 0, \ldots, 2\ell \) and \( i = 1, \ldots, d \).

The algebra under consideration in this paper is a conformal Galilei algebra with a central extension. This particular central extension only exists in \((2+1)\)-dimensional spacetime with the integer spin \( \ell \). Consequently this algebra is sometimes referred to as the exotic conformal Galilei algebra in the literature [8, 10, 11]. We follow the notations and conventions of [8] studying the case \( \ell = 1 \) and \( d = 2 \).

A convenient notation introduced in [8] is to redefine the generators as \( P_i = P^0_i \), \( K_i = P^1_i \), \( F_i = P^2_i \), for \( i = 1, 2 \), and to define a central element \( \Theta \) in addition to those generators \( D, C, H \) and \( J_{12} \) given above, so that we have the following non-zero commutation relations:

\[
[D, H] = H, \quad [C, D] = C, \quad [C, H] = 2D,
\]

\[
[H, K_i] = -P_i, \quad [D, P_i] = P_i, \quad [C, P_i] = 2K_i,
\]

\[
[H, F_i] = -2K_i, \quad [D, F_i] = -F_i, \quad [C, K_i] = F_i,
\]

\[
[J_{12}, X_1] = X_2, \quad [J_{12}, X_2] = -X_1, \quad [K_i, K_j] = \Theta \epsilon_{ij}, \quad [P_i, F_j] = -2\Theta \epsilon_{ij},
\]

where \( X_i = P_i, K_i \) or \( F_i \) and \( \epsilon_{12} = -\epsilon_{21} = 1 \) is an antisymmetric tensor.

In order to study the representation theory of this algebra, we seek a triangular decomposition and introduce certain linear combinations of those generators given by

\[
X_{\pm} = X_1 \pm iX_2, \quad J = iJ_{12}, \quad \Theta = i\Theta.
\]
The non-zero commutators in terms of these new generators are then

\[ [J, X_\pm] = \pm X_\pm, \]
\[ [H, K_\pm] = -P_\pm, \quad [D, P_\pm] = P_\pm, \quad [C, P_\pm] = 2K_\pm, \]
\[ [H, F_\pm] = -2K_\pm, \quad [D, F_\pm] = -F_\pm, \quad [C, K_\pm] = F_\pm, \]
\[ [K_+, K_-] = -2\Theta, \quad [P_\pm, F_\mp] = \pm 4\Theta. \]

The triangular decomposition of the algebra then follows:

\[ g^+ = \{H, P_\pm, K_+\} \simeq \{H, P_\pm, K_+\} \oplus \{P_-\}, \]
\[ g^0 = \{D, J, \Theta\}, \]
\[ g^- = \{C, F_\pm, K_-\} \simeq \{C, F_\pm, K_-\} \oplus \{F_+\}. \]

Note that \( g^\pm \) is non-Abelian. Each non-Abelian part is isomorphic to the Heisenberg algebra.

### 3. Highest weight representations and Verma modules

To investigate highest weight representations of this algebra, we let \(|d, r\rangle\) be a highest weight vector such that

\[ D |d, r\rangle = d |d, r\rangle, \quad J |d, r\rangle = r |d, r\rangle, \quad \Theta |d, r\rangle = \theta |d, r\rangle, \]
\[ H |d, r\rangle = P_\pm |d, r\rangle = K_\pm |d, r\rangle = 0. \]

For fixed values of \(d\) and \(r\), the Verma module associated with this highest weight vector is then determined by \(V^{d,r} = U(g^-) |d, r\rangle\), where \(U(g^-)\) is the universal enveloping algebra of \(g^-\). Hence we are able to give a basis of \(V^{d,r}\) as \(|h, k, \ell, m\rangle = C^h K^k F^\ell F^m |d, r\rangle\).

It is also straightforward to give the action of the generators on the basis. Here we only give the action of generators from \(g^0\) and \(g^+\):

\[ D |h, k, \ell, m\rangle = (d - h - \ell - m) |h, k, \ell, m\rangle, \]
\[ J |h, k, \ell, m\rangle = (r - k - \ell + m) |h, k, \ell, m\rangle, \]
\[ H |h, k, \ell, m\rangle = -2\ell |h, k, \ell - 1, m\rangle + 4km\Theta |h, k - 1, \ell, m - 1\rangle + 2(2\ell + 2m + h - 2d - 1) |h - 1, k, \ell, m\rangle, \]
\[ K_+ |h, k, \ell, m\rangle = -2k\Theta |h, k - 1, \ell, m\rangle - h |h - 1, k, \ell, m + 1\rangle, \]
\[ P_+ |h, k, \ell, m\rangle = 4\ell \Theta |h, k, \ell - 1, m\rangle + 4hk\Theta |h - 1, k - 1, \ell, m\rangle + h(h - 1) |h - 2, k, \ell, m + 1\rangle, \]
\[ P_- |h, k, \ell, m\rangle = -4m\Theta |h, k, \ell, m - 1\rangle - 2h |h - 1, k + 1, \ell, m\rangle + h(h - 1) |h - 2, k, \ell + 1, m\rangle. \]

We note that \(D\) and \(J\) are diagonal on this basis. We set

\[ p = h + \ell + m \geq 0, \quad q = k + \ell - m \in \mathbb{Z}. \]

Then we can express the action of \(D\) and \(J\) on the basis as

\[ D |h, k, \ell, m\rangle = (d - p) |h, k, \ell, m\rangle, \quad J |h, k, \ell, m\rangle = (r - q) |h, k, \ell, m\rangle. \]

It is then clear that the Verma module has a weight decomposition for fixed values of \(p\) and \(q\):

\[ V^{d,r} = \bigoplus_{d-p, r-q} V^{d,r}_{p,q}. \]
4. Singular vectors and irreducible modules

A singular vector is a homogeneous element with respect to the decomposition (1). Its general form is

\[ \langle v \rangle = \sum_{\ell, m} a_{\ell, m} |p - \ell - m, q - \ell + m, \ell, m\rangle. \]

In other words, a singular vector is a linear combination of a subset of the basis of the Verma module corresponding to fixed values of \( p \) and \( q \). We have, by definition, \( h = p - \ell - m \geq 0 \) and \( k = q - \ell + m \geq 0 \) from which it follows that

\[ 0 \leq \ell + m \leq p, \quad (2) \]

and

\[ \ell \leq m + q, \quad (0 \leq q), \quad \ell + |q| \leq m. \quad (3) \]

The most general form of the singular vector for given values of \( p \) and \( q \) can then be expressed as

\[ \langle v \rangle = \sum_{\ell, m}^{p + q - 1} a_{\ell, m} |p - \ell - m, q - \ell + m, \ell, m\rangle, \quad (4) \]

where we understand that \( a_{\ell, m} = 0 \) for the pair \((\ell, m)\) not satisfying (2) or (3). The condition for \( \langle v \rangle \) being a singular vector is given by \( H \langle v \rangle = P_{\pm} \langle v \rangle = K_{\pm} \langle v \rangle = 0 \).

In order to fully understand if we can determine the coefficients for the singular vector at a specific level, we need to consider three different cases, namely \( q > 0 \), \( q = 0 \) and \( q < 0 \).

Case \( q > 0 \). In this case one can rewrite (4) as follows:

\[ \langle v \rangle = \sum_{m=0}^{\min\{p-m,q+m\}} \sum_{\ell=0}^{p-q} a_{\ell, m} |p - \ell - m, q - \ell + m, \ell, m\rangle. \quad (5) \]

It is not difficult to see that the action of \( K_{\pm} \) on (5) is calculated as follows:

\[ K_{\pm} \langle v \rangle = -\sum_{\ell=0}^{\min\{p,q-1\}} 2\theta(q - \ell)a_{\ell,0} |p - \ell, q - 1 - \ell, \ell, 0\rangle \]

\[ -\sum_{m=1}^{\min\{p-m,q-1+m\}} \sum_{\ell=0}^{p} (2\theta(q - \ell + m)a_{\ell, m} + (p - \ell - m + 1)a_{\ell, m-1}) \]

\[ \times |p - \ell - m, q - \ell + m - 1, \ell, m\rangle. \]

The condition \( K_{+} \langle v \rangle = 0 \) yields one recurrence relation

\[ 2\theta(q - \ell + m)a_{\ell, m} + (p - \ell - m + 1)a_{\ell, m-1} = 0 \quad (6) \]

for \( 1 \leq m \leq p, \quad 0 \leq \ell \leq \min\{p - m, q - 1 + m\} \) with the initial condition

\[ a_{\ell, 0} = 0, \quad 0 \leq \ell \leq \min\{p, q - 1\}. \quad (7) \]

The recurrence relation (6) is solved to give

\[ a_{\ell, m} = \frac{1}{2\theta^m} \frac{(p - \ell)! (q - \ell)!}{(p - \ell - m)!(q - \ell + m)!} a_{\ell, 0}. \quad (8) \]

From (7) and (8) one can see the following facts:

(i) if \( p < q \) then \( a_{\ell, m} = 0 \) for all possible pairs of \((\ell, m)\); thus \( \langle v \rangle = 0 \);

(ii) if \( 0 < q \leq p \) then \( a_{\ell, m} = 0 \) for \( 0 \leq \ell \leq q - 1 \).
We study the case (ii) further. From (4) and (8) we have

\[ |v\rangle = \sum_{\ell=0}^{p-\ell} \sum_{m=-\ell}^{\ell} \left( \frac{-1}{2\theta} \right)^m \frac{(p-\ell)!}{(p-\ell-m)!(q-\ell+m)!} a_{\ell,0} |p-\ell-m, q-\ell+m, \ell, m\rangle. \]

We define

\[ |v^\ell\rangle = \sum_{m=-\ell}^{\ell} \left( \frac{-1}{2\theta} \right)^m \frac{1}{(p-\ell-m)!(q-\ell+m)!} |p-\ell-m, q-\ell+m, \ell, m\rangle. \]

Then we have \(|v\rangle = \sum_{\ell=0}^{p-\ell} |v^\ell\rangle|\alpha_\ell\rangle\) and \(|v^\ell\rangle|\alpha_\ell\rangle| q \leq \ell \leq \left[ \frac{p+q}{2} \right] \) is a set of linearly independent vectors in the kernel of \( K_* \), i.e. \( K_* |v^\ell\rangle = 0 \). We now calculate the action of \( P_* \) on \(|v^\ell\rangle\):

\[ P_* |v^\ell\rangle = \sum_{m=-\ell}^{\ell} \left( \frac{-1}{2\theta} \right)^m \frac{1}{(p-\ell-m)!} \theta^m (p-\ell-m)!(q-\ell+m)! \times \{ 4\ell\theta |p-\ell-m, q-\ell+m, \ell-1, m\rangle + 4(p-\ell-m)(q-\ell+m)\theta |p-\ell-m-1, q-\ell+m-1, \ell, m\rangle + (p-\ell-m)(p-\ell-m-1)|p-\ell-m-2, q-\ell+m, \ell, m+1\rangle \}. \]

Looking at the highest values of \( m \) the vector \(|0, p+q-2\ell, \ell-1, p-\ell\rangle\) on the RHS has nonvanishing coefficients for all possible values of \( \ell \). It follows that the condition \( P_* |v\rangle = 0 \) implies \( \alpha_\ell = 0 \). We thus have \(|v\rangle = 0 \).

We therefore have shown that there are no singular vectors for \( q > 0 \).

Case \( q = 0 \). In this case we have the recurrence relation (6) from the previous case but no initial conditions. We solve (6) to have

\[ a_{\ell,m} = \left( \frac{1}{2\theta} \right)^{m-\ell} \frac{(p-2\ell)!}{(p-\ell-m)!(m-\ell)!} a_{\ell,\ell}. \]

The condition \( P_* |v\rangle = 0 \) yields the relation

\[-4(m+1)\theta a_{\ell,m+1} - 2(p-\ell-m)a_{\ell,m} + (p-\ell-m+1)(p-\ell-m)a_{\ell-1,m} = 0.\]

Substitution of (9) into this yields the recurrence relation

\[ 4\ell\theta a_{\ell,\ell} - (p-2\ell+2)(p-2\ell+1)a_{\ell-1,\ell-1} = 0. \]

This is solved to give the expression

\[ a_{\ell,\ell} = \left( \frac{1}{4\theta} \right)^{\ell} \frac{p!}{\ell!(p-2\ell)!} a_{0,0}. \]

Substitution of (10) into (9) determines the coefficients as follows:

\[ a_{\ell,\ell} = \left( \frac{1}{2\theta} \right)^{m+\ell} \frac{1}{\ell!(m-\ell)!}(p-\ell-m)! a_{0,0}. \]

The condition \( P_* |v\rangle = 0 \) yields another recurrence relation

\[ 4(\ell+1)\theta a_{\ell+1,m} + 4(p-\ell-m)(m-\ell)\theta a_{\ell,m} + (p-\ell-m+1)(p-\ell-m)a_{\ell-1,m} = 0. \]

It is easy to verify that (11) satisfies this relation. Next we look at the condition \( H |v\rangle = 0 \). It gives the recurrence relation containing \( d \):

\[-2(\ell+1)a_{\ell+1,m} + (p-\ell-m)(p+\ell+m-2d-1)a_{\ell,m} + 4(m-\ell+1)(m+1)\theta a_{\ell,m+1} = 0. \]
Substitution of (11) into the left-hand side gives the expression
\[
\left(-\frac{1}{2}\right)^{m+\ell} \frac{1}{\ell![(m-\ell)!p!(p-\ell-m-1)!]}(p-2d-3).
\]
Setting this equal to zero we obtain the condition for \(d\):
\[
p = 2d + 3 \in \mathbb{Z}_+.
\]
(12)

In summary there exists one singular vector in \(V^d\) if \(d\) satisfies condition (12) and it is given by (up to overall factor)
\[
|\psi_d\rangle = \sum_{m=0}^{\min[m, p-m]} \sum_{\ell=0}^{\frac{p}{2}} \left(-\frac{1}{2}\right)^{m+\ell} \frac{1}{\ell![(m-\ell)!p!(p-\ell-m-1)!]} C^{p-\ell-m} K^{-\ell} F^\ell F_m^m |d, r\rangle.
\]

It is then possible to show by induction that the singular vector has the closed form
\[
|\psi_d\rangle = (2\theta C - K_m F_\ell)^d |d, r\rangle.
\]

Case \(q < 0\). In this case the most general form (4) of the singular vector yields
\[
|\psi\rangle = \sum_{\ell=0}^{\frac{p}{2}} \sum_{m=\ell+|q|}^{p-\ell} a_{\ell,m} |p - \ell - m, m - \ell - |q|, \ell, m\rangle.
\]
The condition \(K_m |\psi\rangle = 0\) gives us the recurrence relation
\[
P_+ |\psi\rangle = 0 \quad \text{for } 0 \leq \ell \leq \frac{p}{2}, \ell + |q| \leq m \leq p - \ell - 1.
\]
On the other hand, the condition \(P_- |\psi\rangle = 0\) yields one initial condition and recurrence relation, since
\[
P_- |\psi\rangle = -\left(\sum_{m=\ell+|q|}^{p-\ell} 4(m+1)\theta a_{0,m+1} + \sum_{m=\ell+|q|}^{p-\ell} 2(p-m)\theta a_{0,m}\right) |p - m - 1, m - |q| + 1, 0, m\rangle
\]
\[
+ \sum_{\ell=1}^{\frac{p-\ell}{2}} \left[4(\ell + |q|)\theta a_{\ell+|q|} - (p - |q| - 2\ell + 2)(p - |q| - 2\ell + 1)\theta a_{\ell-1,\ell+1+|q|}\right]
\]
\times |p - |q| - 2\ell, 0, \ell, \ell - 1 + |q|\rangle
\]
\[
- \sum_{\ell=1}^{\frac{p-\ell}{2}} \left[4(m+1)\theta a_{\ell+1} + 2(p-\ell-m)\theta a_{\ell,m}\right]
\]
\times |p - \ell - m + 1, m - \ell - |q| + 1, \ell, m\rangle.
\]
(14)

Now we only inspect the important relations. Looking at \(m = |q| - 1\) we have the initial condition \(a_{0,|q|} = 0\). The first line in (14) gives the recurrence relation
\[
P_+ |\psi\rangle = 0 \quad \text{for } |q| \leq m \leq p - 1.
\]
\[
\text{Recursive use of (15) with } a_{0,|q|} = 0 \text{ shows that } a_{0,m} = 0 \text{ for } 0 \leq m \leq p.
\]
From the last line of (14) we obtain the recurrence relation
\[
4(m+1)\theta a_{\ell,m+1} + 2(p-\ell-m)\theta a_{\ell,m} - (p-\ell-m+1)(p-\ell-m)\theta a_{\ell-1,m} = 0.
\]
(16)

Substitution of (13) into (16) gives the relation
\[
2(\ell + |q|)\theta a_{\ell+1} + (p-\ell-m)(m-\ell-|q|+2)\theta a_{\ell-1,m+1} = 0.
\]
(17)
One can show by relation (17) that $a_{\ell, m} = 0$ for all possible pairs of $(\ell, m)$. We thus conclude that there are no singular vectors for $q < 0$.

The results of the preceding discussion can be nicely summarized in the following theorem.

**Theorem 1.** $V^{d,r}$ has precisely one singular vector iff $2d + 3 \in \mathbb{Z}_+$ and it is given by

$$|v_i⟩ = (2\Theta C - K_+ F_3)^{2d+3} |d, r⟩.$$ (18)

As a remark, we note that we are able to define a bilinear form $(\, , )$ on $V^{d,r}$, following Shapovalov [18], such that

$$⟨d,r|d,r⟩ ≡ (|d,r⟩ , |d,r⟩ ) = 1,$$ $$⟨A|d,r⟩ , B|d,r⟩ = (|d,r⟩ , ω(A)|d,r⟩ ),$$ (19)

where $ω$ is the involutive algebra anti-automorphism defined by $ω(D) = D, ω(J) = J, ω(\Theta_1) = \Theta_1, ω(C) = H, ω(K_+) = K_-, ω(P_+) = F_+$. It is straightforward to see that the singular vectors are orthogonal to all vectors in $V^{d,r}$ with respect to the bilinear form (19).

The Verma modules without the singular vectors have no invariant submodules of highest weight type. In the remainder of the paper, we therefore study $V^{d,r}$ with $2d + 3 \in \mathbb{Z}_+$. Let $I^d = U(g^-) |v_i⟩$ where $|v_i⟩$ is the singular vector (18). Then $I^d$ is an invariant submodule and we consider the quotient module $\tilde{V}^{d,r} = V^{d,r} / I^d$. The highest weight vector in $\tilde{V}^{d,r}$, denoted by $|d,r⟩$, is defined by

$$D|d,r⟩ = d|d,r⟩, J|d,r⟩ = r|d,r⟩, \Theta|d,r⟩ = \Theta|d,r⟩, H|d,r⟩ = P_+|d,r⟩ = K_+|d,r⟩ = (2\Theta C - K_+ F_3)^{2d+3} |d, r⟩ = 0.$$ (20)

The basis of $\tilde{V}^{d,r}$ has the form of $|h, k, \ell, m⟩ = C^{h} K^{k} F^{\ell} F_{m}^{m} |d, r⟩$. However, because of (20) the vector $C^{2d+3} |d, r⟩$ is not independent. Next we look for the singular vectors in $\tilde{V}^{d,r}$ by the same procedure as the earlier discussion in this section. The singular vector has the form of

$$|v⟩ = \sum_{\ell, m} a_{\ell, m} |p - \ell - m, m - \ell, \ell, m⟩.$$  

We note that $a_{0, 0}$ is missing from the summation. As seen from the previous discussion, nonvanishing $a_{0, 0}$ is crucial for the existence of the singular vector. We thus conclude that there is no singular vector in $\tilde{V}^{d,r}$, and hence arrive at the main result of this paper.

**Theorem 2.** All irreducible highest weight modules over the exotic conformal Galilei algebra are listed as follows:

(i) the Verma module $V^{d,r}$ for $2d + 3 \notin \mathbb{Z}_+$,
(ii) the quotient module $\tilde{V}^{d,r} \subseteq V^{d,r}$ for $2d + 3 \in \mathbb{Z}_+$.

where $d, r \in \mathbb{R}$. All irreducible modules given are infinite dimensional.

5. Concluding remarks

In this paper we have determined all irreducible highest weight modules of the conformal Galilei algebra with exotic central extension. This algebra has attracted attention recently due to its application to ‘exotic’ physical systems [8–13]. It was suggested in [8] that the results of their paper relating to the exotic conformal Galilei algebra may be beneficial in the further study of such systems, and our view is that understanding the representation theory of the
symmetry algebra of these systems will enable future developments. The current paper is an
important step in developing this stratagem.

An immediate venture would be to investigate hierarchies of partial differential equations
(PDEs) associated with the singular vectors obtained in theorem 1 of this paper. Such PDEs
arise via the vector field realization of the generators in terms of differential operators,
and their action on an appropriate function space related to the Verma modules (see [19, 20] for
the semisimple case). Indeed, an analogue of this was done explicitly in [16] for the Schrödinger
algebra in (1+1) dimensions, where a hierarchy of generalized heat/Schrödinger equations was
found.

Finally, it would be desirable to provide a generalized formalism for describing the
representations of families of non-semisimple Lie algebras which contain structures such as the
Schrödinger algebras, the conformal Galilei algebras and their central extensions. Establishing
such a comprehensive framework that recovers our results and those of [16] (amongst others)
would be a worthwhile program that could result in a broad class of (nonrelativistic) physical
systems.

Acknowledgments

The majority of this work was done while NA was visiting the School of Mathematics and
Physics at the University of Queensland as a Raybould Fellow. We appreciate the support of
this program.

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