Renormalization constants using quark states in Landau gauge

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We show that given one $O(a)$ improvement constant, $b_m$, all the remaining quantities needed to define the renormalized and $O(a)$ improved dimension-3 quark bilinears can be obtained by studying the matrix elements of these operators between external quark states in a fixed gauge.

1. INTRODUCTION

One of the leading uncertainties in lattice calculations involves the connection between the lattice and the continuum renormalized operators. Current estimates \cite{1} show that one-loop perturbation theory for $O(a)$ improved Wilson fermions underestimates quantities like $Z_0 P / Z_0 S$ by $\sim 10\%$ at lattice scales of 2–4 GeV. Furthermore, perturbative estimates of the $O(a)$ improvement coefficients are significantly different from their non-perturbative estimates.

All the scale independent renormalization constants and the improvement coefficients in the quenched theory have been determined by imposing vector and axial ward identities on the lattice. This Ward identity method is computationally intensive and alternate methods are therefore desirable, especially to determine the scale dependent renormalization constants ($Z^0_T$ and $Z^0_P$ or $Z^0_S$).

An alternate well known method that gives all the renormalization constants \cite{2} and some of the improvement constants \cite{3,4} involves calculating the matrix elements of the quark bilinears between external quark states in a fixed gauge. This method is far more tractable computationally and the generalization to 4-fermion operators is straightforward. Here, we show how this method gives all but one ($b_m$ as discussed in section 3.1) of the $O(a)$ renormalization constants.

2. METHOD

We start by defining the notation. $O(a)$ improvement of the theory is achieved by improving the action and the operators simultaneously. The improved renormalized quark fields, $\hat{\psi}$ can be related to the lattice quark field $\psi$ by

$$\hat{\psi} = Z^{-1/2}_\psi (1 - b \psi a m_I) \times [1 - a c'_\psi (\not{D} + m_I) - a c'_{NGI} \not{\partial}] \psi,$$

where $m_I$ is an $O(a)$ improved quark mass. Thus, apart from a mass dependent renormalization constant, one needs (i) an equation of motion correction, $c'_\psi$, that does not affect position space correlation functions at finite separations, and (ii) mixing with a gauge non-invariant operator, $c_{NGI}$, that appears because the calculation is performed in a fixed gauge.

We also write the renormalized propagator as

$$\langle T[\hat{\psi}\hat{\psi}] \rangle \equiv \frac{1}{i \hat{\Sigma}_1 \hat{\not{p}} + \hat{\Sigma}_2} \equiv -i \dot{\sigma}_1 \hat{p} + \dot{\sigma}_2,$$

where $T$ is the time-ordering symbol, and $p_\mu$ is the four momentum of the quark. Neglecting logarithmic and higher order corrections in $p^{-2}$, the improved renormalized propagator at high momenta is given by

$$\hat{\Sigma}_1 = 1 + \frac{\alpha_1 m_I^2 + \beta_1}{p^2}$$

$$\hat{\Sigma}_2 = Z_m m_I \left[ 1 + \frac{\alpha_2 m_I^2 + \beta_2}{p^2} \right] + \frac{\beta'_2}{p^2},$$

where $Z_m$ is the renormalization constant of $m_I$. The important point to note in these expressions is that chiral symmetry prevents terms proportional to $m_I$ and $m_I^2$ in $\Sigma_1$ and $\Sigma_2$ respectively.
Expanding the lattice quark fields in terms of the continuum field, lattice propagator is

\[ \langle \bar{\psi} \psi \rangle = -i\sigma_1 p + \sigma_2 \]

\[ = Z_\psi (1 + 2b_\psi a)(1 + 2i\alpha_{NGI}p) \times (-i\sigma_1 p + \sigma_2) + 2a c'_\psi. \]

From this the unknown constants \( Z_\psi, Z_m, b_\psi, c_{NGI}, \) and \( c'_\psi \) can be extracted as follows. We first expand \( \sigma_1 \) and \( \sigma_2 \) at large \( p^2 \) as

\[ p^2 \sigma_1 = \sigma_1^{LO} + \frac{\sigma_1^{NLO}}{p^2} + O(p^{-4}) \]

\[ \sigma_2 = \sigma_2^{LO} + \frac{\sigma_2^{NLO}}{p^2} + O(p^{-4}). \]

where the terms dropped are yet higher order in \( p^{-2} \). From Eqns. 5-8, we note that these leading and next to leading coefficients, \( \sigma_{1,2}^{(N)LO} \), of the expansion of \( \sigma_{1,2} \) in \( p^{-2} \) have the following dependence on the quark mass:

\[ \sigma_1^{LO} = \sigma_1^{LO,0} + \sigma_1^{LO,1} m \]

\[ \sigma_1^{NLO} = \sigma_1^{NLO,0} + \sigma_1^{NLO,1} m + \sigma_1^{NLO,2} m^2 + \sigma_1^{NLO,3} m^3 \]

\[ \sigma_2^{LO} = \sigma_2^{LO,0} \]

\[ \sigma_2^{NLO} = \sigma_2^{NLO,0} + \sigma_2^{NLO,1} m + \sigma_2^{NLO,2} m^2, \]

where, omitting terms of \( O(a^2) \),

\[ \sigma_1^{LO,0} = Z_\psi \]

\[ \sigma_1^{LO,1} = 2a Z_\psi (b_\psi - c_{NGI}) \]

\[ \sigma_1^{NLO,0} = -2Z_\psi (\beta_1 - 2a c_{NGI} \beta_2) \]

\[ \sigma_1^{NLO,1} = -2a Z_\psi [\beta_1 b_\psi - c_{NGI}(2\beta_1 - \beta_2)] \]

\[ \sigma_1^{NLO,2} = -Z_\psi (1 + \alpha_1) \]

\[ \sigma_1^{NLO,3} = -aZ_\psi [b_\psi (1 + c_{NGI}) - 2c_{NGI}(1 + 2\alpha_1 - \alpha_2)] \]

\[ \sigma_2^{LO,0} = 2a Z_m (Z_\psi c_{NGI} + c'_\psi) \]

\[ \sigma_2^{NLO,0} = 2a Z_m Z_\psi c_{NGI} \beta_1 \]

\[ \sigma_2^{NLO,1} = Z_m Z_\psi \]

\[ \sigma_2^{NLO,2} = 2a Z_m Z_\psi [b_\psi + c_{NGI}(1 + \alpha_1)]. \]

The five renormalization and improvement constants can now be obtained once \( \alpha_1 \) is eliminated using \( \sigma_{1,2}^{LO,0}, \sigma_{1,2}^{LO,1}, \sigma_{1}^{NLO,2}, \sigma_{2}^{LO,0}, \sigma_{2}^{NLO,1}, \) and \( \sigma_{2}^{NLO,2} \). Once the improved propagator is known, the remaining constants are obtained as discussed in [4].

Figure 1. Plot of \( \sigma_2 \) versus \( p^2 \) before (diamonds) and after (squares) subtraction of \( O(a^2 p^2) \) artefact for \( \kappa = 0.1344 \).

Figure 2. Plot of \( \sigma_2 \) versus \( p^{-2} \) after subtraction of \( O(a^2 p^2) \) artefact for \( \kappa = 0.1344 \).
3. EXAMPLE IMPLEMENTATION

3.1. Lattice details
To illustrate the feasibility of this method, we present a preliminary analysis of 50 32 \times 4 quenched configurations at \( \beta = 6.2 \). We take \( c_{SW} = 1.614 \) from the ALPHA collaboration [5] to define the \( O(a) \) improved action. Simulations are done at seven values of \( \kappa = 0.131, 0.1321, 0.1333, 0.1339, 0.1344, 0.1348, \) and 0.1350. The critical value of the hopping constant, \( \kappa_c = 0.135899 \), and the fourth root of the plaquette, \( u_0 = 0.88510 \) are taken from [1]. For the \( O(a) \) improved definition of quark mass we take
\[
m_I = \ln \left[ 1 + \left( \frac{1}{2\kappa u_0} - \frac{1}{2\kappa_c u_0} \right)^2 \right]
\] (23)
since the value of \( b_m \) is close to its tadpole improved tree level value [1]. We scale all lattice fermion fields by \( \sqrt{2\kappa u_0} \). The lattice momentum components are \( p \equiv \sin(2\pi j/32) \) and we average over momentum combinations equivalent under the hypercubic lattice symmetry group. Our fits use the seventy distinct momentum combinations with \( j \leq 4 \).

3.2. \( O(a^2) \) subtraction and determination of the constants
According to Eqs. (7) and (8), \( p^2 \sigma_1 \) and \( \sigma_2 \) are supposed to go to constants at large \( p^2 \), however, the data, exemplified in Figure 1, show a linear behavior in \( a^2 p^2 \) at large \( p^2 \). These \( O(a^2) \) terms are removed by fitting to the large momentum behavior.

To extract the desired coefficients \( \sigma_{1,2}^{(N)LO} \) we now fit these subtracted \( p^2 \sigma_1 \) and \( \sigma_2 \) as a function of \( p^{-2} \) (see Figure 2 for an example). The data show that one needs to keep at least terms up to \( p^{-4} \) in Eqs. (7, 8) to obtain a reasonable fit. The resulting intercepts and slopes obtained are then fit against \( m_I \) (e.g., see Figure 3) to obtain the various expressions defined in Eqs. (13–22). From these fits, we find
\[
\begin{align*}
Z_\psi &= 0.925(6)u_0 \\
Z_m &= 1.03(4)u_0^{-1} \\
c_{NGI} &= -0.02(5) \\
b_\psi &= -0.53(3)u_0^{-1}
\end{align*}
\] (24–27)

Figure 3. Slope of \( O(a^2) \)-corrected \( \sigma_2 \) with respect to \( p^{-2} \) plotted versus \( m_I \).

\[
c_\psi' = 0.27(4)
\] (28)

4. DISCUSSION
We have shown that once \( b_m \), or equivalently, \( m_I \) is known, all other \( O(a) \) improvement constants needed to define the renormalized propagator can be determined. In previous work [3], the constant \( c_{NGI} \) was left undetermined. Using perturbation theory Sharpe [6] has shown that the effect of this term is small. We show that this constant can be determined non-perturbatively and its value is indeed small.

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