On the Eigenvalue Problem of the $su(1, 1)$-Algebra and the Coupling Scheme of Two $su(1, 1)$-Spins

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Abstract

After recapitulating the eigenvalue problem of the $su(1, 1)$-algebra in the conventional form, the same problem is treated in an unconventional form, in which the eigenvalue is pure imaginary. Further, the coupling scheme of two $su(1, 1)$-spins is discussed in the framework of two possibilities, in which certain new aspects appear. Finally, the coupling scheme developed in this paper is applied to a concrete example, which will serve boson realization of the $so(4)$- and the $so(3, 1)$-algebra presented in the next paper.
§1. Introduction

Boson realization of the Lie algebra has played a central role in theoretical studies of nuclear many-body systems. We can find its example in classical review article by Klein and Marshalek. Many of these studies are restricted to compact algebras such as the $su(2)$- and the $su(3)$-algebra. In the case of non-compact algebras, the $su(1,1)$-algebra is in quite interesting position, because this algebra enables us to describe the damped and the amplified harmonic oscillation in thermal circumstance as a conservative form. However, the $su(1,1)$-algebra seems for us to be fundamental not only for applying the above-mentioned problem but also for investigating the boson realization of various Lie algebras. In the next paper, we intend to report a possible boson realization of the $so(4)$- and the $so(3,1)$-algebra. In this case, also the $su(1,1)$-algebra plays a central role. In this boson realization, we encounter two points which are clarified for the $su(1,1)$-algebra. One is the eigenvalue problem in an unconventional form, which gives us a pure imaginary eigenvalue. Second is related to the coupling of two and three $su(1,1)$-spins.

With the aim of serving the boson realization of the $so(4)$- and the $so(3,1)$-algebra, the above-mentioned two points are discussed in this paper. The first point was already investigated in Ref. in rather formal form. Then, we borrow its basic parts. Concerning the second, we already developed the idea of the coupling scheme in the Holstein-Primakoff boson representation. In the present paper, we give a form which does not depend on the concrete representation, together with some aspects.

In the next section, the eigenvalue problem in a conventional form is summarized. In §3, the eigenvalue problem is discussed in an unconventional form, in which pure imaginary eigenvalue appears. Section 4 is devoted to presenting the coupling of two $su(1,1)$-spins in two schemes, one of which gives us new aspects. In §5, a concrete example is treated. This example will play a central role in the next paper. Finally, a few comments are mentioned.

§2. Preliminary argument

First of all, let us recapitulate basic part of the eigenvalue problem of a single $su(1,1)$-spin system. Hereafter, we call a system obeying the $su(1,1)$-algebra the $su(1,1)$-spin system. The $su(1,1)$-algebra is composed of three operators which we denote as $\hat{T}_0$, $\hat{T}_\pm$. They obey the relations

\[
\hat{T}_0^* = \hat{T}_0, \quad \hat{T}_\pm^* = \hat{T}_\mp, \quad [\hat{T}_+ , \hat{T}_- ] = -2\hat{T}_0, \quad [\hat{T}_0, \hat{T}_\pm ] = \pm \hat{T}_\pm. \tag{2.1}
\]

\[
[\hat{T}_+, \hat{T}_+ ] = -2\hat{T}_0, \quad [\hat{T}_0, \hat{T}_\pm ] = \pm \hat{T}_\pm. \tag{2.2}
\]
The Casimir operator, which we denote \( \hat{T}^2 \), and its property are given by

\[
\hat{T}^2 = \hat{T}_0^2 - (1/2) \cdot (\hat{T}_- \hat{T}_+ + \hat{T}_+ \hat{T}_-) = \hat{T}_0(\hat{T}_0 + 1) - \hat{T}_+ \hat{T}_- , \quad (2.3)
\]

\[
[ \hat{T}_{\pm,0} , \hat{T}^2 ] = 0 . \quad (2.4)
\]

A typical example of \( \hat{T}_{\pm,0} \) is the following :

\[
\hat{T}_+ = \hat{c}^* \hat{d}^* , \quad \hat{T}_- = \hat{d} \hat{c} , \quad \hat{T}_0 = 1/2 + (1/2) \cdot (\hat{c}^* \hat{c} + \hat{d}^* \hat{d}) . \quad (2.5)
\]

This expression was used for the description of the damped and the amplified harmonic oscillation.[3,4]

We start in the assumption of the existence of the state \(| t(q) \rangle \) which follows the relation

\[
\hat{T}_- | t(q) \rangle = 0 , \quad \hat{T}_0 | t(q) \rangle = t | t(q) \rangle , \quad (2.6)
\]

\[
\langle t(q) | t'(q') \rangle = \delta_{t,t'} \delta_{qq'} . \quad (2.6)
\]

Here, the symbol \((q)\) denotes a set of quantum numbers additional to \( t \). For the system shown in the relation (2.7) except the vacuum \(| 0 \rangle \), the states which satisfy the relation (2.6) are classified into two types, that is, \((\sqrt{(2t-1)!})^{-1}(\hat{c}^*)^{2t-1}|0 \rangle\) and \((\sqrt{(2t-1)!})^{-1}(\hat{d}^*)^{2t-1}|0 \rangle\) \((t = 1, 3/2, 2, \cdots)\). The case \( t = 1/2 \) corresponds to \(| 0 \rangle \). This implies that, in order to discriminate between both types, the extra quantum number is necessary. In the case where we regard the sum of several \( su(1,1) \)-spin systems as a single system, it is natural that the extra quantum numbers are necessary. The relation (2.6) leads us to

\[
\hat{T}^2 | t(q) \rangle = t(t-1) | t(q) \rangle . \quad (2.7)
\]

Since \( \langle t(q) | \hat{T}_- \hat{T}_+ | t(q) \rangle \) is positive-definite and also \( \langle t(q) | \hat{T}_+ \hat{T}_- | t(q) \rangle \) vanishes, \( t \) should be positive. It is clear from the relation

\[
t = \langle t(q) | \hat{T}_0 | t(q) \rangle = (1/2) \langle t(q) | [ \hat{T}_- , \hat{T}_+ ] | t(q) \rangle
\]

\[
= (1/2) \langle t(q) | \hat{T}_- \hat{T}_+ | t(q) \rangle > 0 . \quad (2.8)
\]

The case \( t = 0 \) is not interesting, because, in addition to \( \hat{T}_- | t(q) \rangle = 0 \), we have \( \hat{T}_+ | t(q) \rangle = 0 \) and there does not exist any state connecting with \(| t = 0(q) \rangle \) through \( \hat{T}_+ \). Further, we note the following fact : Two positive values \( t' \) and \( t \) with \( t' + t = 1 \) give us the relation \( t'(t' - 1) = t(t - 1) \). This implies that, except the case \( t' = t = 1/2 \), two different eigenvalues of \( \hat{T}_0 \) for \(| t(q) \rangle \) and \(| t' = 1 - t(q) \rangle \) give the same eigenvalue for \( \hat{T}^2 \) in the region \( 0 < t', t < 1 \). In the general framework, it may be impossible to fix the values of \( t \), for example, in the case of the \( su(2) \)-algebra where \( s \) is fixed 0, 1/2, 1, 3/2, \cdots. It depends on the concrete cases, for example, \( t = 1/2, 1, 3/2, \cdots \) for the system (2.5).
By operating $\hat{T}_+$ successively for $(t_0 - t)$ times, we can define the following state $|q; t, t_0\rangle$:

$$
|q; t, t_0\rangle = \sqrt{\frac{\Gamma(2t)}{(t_0 - t)!\Gamma(2t + t)}} (\hat{T}_+)^{t_0-t} |t(q)\rangle,
$$

$$
\langle q; t, t_0| = \sqrt{\frac{\Gamma(2t)}{(t_0 - t)!\Gamma(2t + t)}} \langle t(q)| (\hat{T}_-)^{t_0-t},
$$

$$
\langle q; t_0|q; t, t_0\rangle = 1.
$$

(2.9)

Here, $\Gamma(z)$ denotes the gamma function ($z = 2t, t_0 + t$). The ket state $|q; t, t_0\rangle$ is an eigenstate of $\hat{T}^2$ and $\hat{T}_0$ and forms the orthogonal set:

$$
\hat{T}^2 |q; t, t_0\rangle = t(t-1)|q; t, t_0\rangle,
$$

$$
\hat{T}_0 |q; t, t_0\rangle = t_0|q; t, t_0\rangle,
$$

$$
t_0 = t, t + 1, t + 2, \ldots.
$$

(2.10a)

(2.10b)

(2.11)

Of course, the bra state $\langle q; t, t_0|$ also satisfies

$$
\langle q; t, t_0|\hat{T}^2 = t(t-1)\langle q; t, t_0|,
$$

$$
\langle q; t, t_0|\hat{T}_0 = t_0\langle q; t, t_0|.
$$

(2.12a)

(2.12b)

In the present system, we introduce the following operator:

$$
\hat{Q}_+ = \hat{T}_+ (\hat{T}_0 + \hat{T} + \varepsilon)^{-1},
$$

$$
\hat{Q}_- = (\hat{T}_0 + \hat{T} + \varepsilon)^{-1} \hat{T}_-.
$$

Here, $\varepsilon$ denote infinitesimal parameter which avoids null denominator and $\hat{T}$ is an operator obeying

$$
\hat{T} |q; t, t_0\rangle = t|q; t, t_0\rangle.
$$

(2.14)

Formally, it can be expressed in the form

$$
\hat{T} = -\hat{T}^2 \left( 1 + \sum_{n=1}^{\infty} \frac{(-)^n 2^n (2n-1)!!}{(n+1)!} \left( \hat{T}^2 \right)^n \right).
$$

(2.15)

Important property of $\hat{Q}_+$ is as follows:

$$
[\hat{T}_-, \hat{Q}_+] = 1, \quad [\hat{T}_0, \hat{Q}_+] = \hat{Q}_+, \quad [\hat{T}_+, \hat{Q}_+] = (\hat{Q}_+)^2,
$$

$$
\left[ \sqrt{\hat{T}_0 + \hat{T}} \hat{Q}_-, \hat{Q}_+ \sqrt{\hat{T}_0 + \hat{T}} \right] = 1.
$$

(2.16)

In the next problem and in the discussion on the addition of two $su(1, 1)$-spins, $\hat{Q}_+$ will play a central role.
In addition to the eigenvalue equation (2.10b), we can set up an eigenvalue problem given in the following relation:

\[ \hat{T}^0 |q; t, t^0\rangle = t^0 |q; t, t^0\rangle . \]  (2.17)

Here, \( \hat{T}^0 \) is defined as

\[ \hat{T}^0 = (1/2)(\hat{T}_+ + \hat{T}_-) . \]  (2.18)

Of course, \( t^0 \) is regarded as real. The properties of \( \hat{T}^0 \), together with \( \hat{T}^\pm \) defined in the next section, will be presented in the next section for the problem of pure imaginary eigenvalue of \( \hat{T}^0 \). With the aim of obtaining the state \( |q; t, t^0\rangle \), first, we investigate the following eigenvalue equation:

\[ \hat{T}_- |t, t^0(q)\rangle = 2t^0 |t, t^0(q)\rangle . \]  (2.19)

With the use of the relation (2.16), the normalized state \( |t, t^0(q)\rangle \) is obtained in the form

\[ |t, t^0(q)\rangle = \left(\frac{\sqrt{N(t, t^0)}}{N(t, t^0)}\right)^{-1} \exp \left(2t^0 \hat{Q}_+\right) |t(q)\rangle , \]  (2.20a)

\[ N(t, t^0) = \sum_{n=0}^{\infty} \frac{(2t^0)^n}{n!} \cdot \frac{\Gamma(2t)}{\Gamma(2t+n)} . \]  (2.20b)

Clearly, \( t^0 \) is arbitrary real number and \( N(t, t^0) \) is convergent in the region \( |t^0| < \infty \). Next, let the operator \( \hat{\Omega}_+ \), which obeys the following relation, exist:

\[ \hat{T}_+ \hat{\Omega}_+ + [\hat{T}_-, \hat{\Omega}_+] = 0 . \]  (2.21)

Then, we can verify easily that the state satisfying the eigenvalue equation (2.17), \( |q; t, t^0\rangle \), is expressed in the form

\[ |q; t, t^0\rangle = \hat{\Omega}_+ |t, t^0(q)\rangle . \]  (2.22)

The above means that our problem is reduced to find the explicit form of \( \hat{\Omega}_+ \).

Let \( \hat{\Omega}_+ \) express in the form

\[ \hat{\Omega}_+ = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} (\hat{Q}_+)^{2n} f_n(\hat{T}, \hat{T}_0 - \hat{T}) , \]  (2.23)

\[ f_0(\hat{T}, \hat{T}_0 - \hat{T}) = 1 . \]  (2.24)

Substituting the expression (2.23) into the relation (2.21) and using the definition of \( \hat{Q}_+ \) shown in the relation (2.16) and the relations (2.2)\( \sim \)(2.4), we have

\[ (\hat{T}_0 - \hat{T} + 2n)f_n(\hat{T}, \hat{T}_0 - \hat{T}) - (\hat{T}_0 - \hat{T})f_n(\hat{T}, \hat{T}_0 - \hat{T} - 1) \]
\[ -n(\hat{T}_0 - \hat{T} + 2(\hat{T} + n - 1))f_{n-1}(\hat{T}, \hat{T}_0 - \hat{T}) = 0 . \]  (n = 0, 1, 2, \ldots)  (2.25)
By operating the state given in the relation \( (2.9) \), \(|q; t, t_0\rangle\), on the relation \( (2.25) \), the following relation is obtained:

\[
\begin{align*}
(m + 2n)f_n(t, m) - mf_n(t, m - 1) - n(m + 2(t + n - 1))f_{n-1}(t, m) &= 0, \\
m &= t_0 - t, & (m = 0, 1, 2, \cdots) \\
f_0(t, m) &= 1.
\end{align*}
\]

The form \( (2.26) \) is a double recursion relation for \( n \) and \( m \) and successively from \( n = m = 0 \), we obtain \( f_n(t, m) \). Then, replacing \( t \) and \( m \) with \( \hat{T} \) and \( \hat{T}_0 - \hat{T} \), respectively, \( f_n(\hat{T}, \hat{T}_0 - \hat{T}) \) is derived. In Appendix, we will show the procedure for determining \( f_n(t, m) \) and some lower cases are given. The above is a possible idea to solve the eigenvalue equation \( (2.7) \). The state \(|q; t, t_0\rangle\) shown in the relation \( (2.22) \) may be not normalizable.

\section*{§3. An unconventional form of the eigenvalue problem}

In §2, we recapitulated the eigenvalue problem of the \( su(1,1) \)-spin system, in which \( \hat{T}_2 \) and \( \hat{T}_0 \) are diagonalized. Further, in associating this eigenvalue problem, we sketched the case of \( (\hat{T}_2, \hat{T}_0) \). Concerning this problem, the following eigenvalue equation can be also set up:

\[
\hat{T}_0|q; t, t_0\rangle = it_0|q; t, t_0\rangle. \quad (i = \sqrt{-1})
\]

Here, \( \hat{T}_0 \) is defined in the relation \( (2.18) \). The operators \( \hat{T}^\pm \), which are used later, are defined as

\[
\hat{T}^0 = (1/2) \cdot (\hat{T}_+ + \hat{T}_-) , \quad \hat{T}^\pm = (1/2i) \cdot (\hat{T}_+ - \hat{T}_-) \mp \hat{T}_0.
\]

They obey the relations

\[
\begin{align*}
\hat{T}^{0*} &= \hat{T}^0, \\
\hat{T}^{\pm*} &= \hat{T}^{\mp}, \\
[ \hat{T}^+, \hat{T}^- ] &= 2i\hat{T}^0, \\
[ \hat{T}^0, \hat{T}^\pm ] &= \pm i \hat{T}^\mp.
\end{align*}
\]

The Casimir operator \( \hat{T}^2 \) and its property are given by

\[
\begin{align*}
\hat{T}^2 &= (\hat{T}^0)^2 - (1/2) \cdot (\hat{T}_- \hat{T}_+ + \hat{T}_+ \hat{T}_-) = \hat{T}^0(\hat{T}^0 \mp i) - \hat{T}^\pm \hat{T}^\mp, \\
[ \hat{T}^{\pm,0} , \hat{T}^2 ] &= 0.
\end{align*}
\]

The relations \( (3.3) \)\( \sim (3.6) \) should be compared with the relations \( (2.1) \)\( \sim (2.4) \). The state \(|q; t, t_0\rangle\) satisfies

\[
\hat{T}^2|q; t, t_0\rangle = t(t - 1)|q; t, t_0\rangle.
\]
In the subsequent paper, we will encounter Eq. (3.1) at some occasions. Since \( t^0 \) is proved to be real, Eq. (3.1) does not claim the eigenvalue equation in the conventional sense.

Various properties of the relation (3.1) can be found in Ref. 4), in which, instead of \( \hat{T}_0 \), the operator \( \left( \frac{1}{2i} \right) \cdot (\hat{T}_+ - \hat{T}_-) \) was adopted. For the completeness of the paper, we borrow the results of Ref. 4). The \( su(1,1) \)-algebra in the representation \( (\hat{T}_+ \pm \hat{T}_0, 0) \) plays a central role in the case of describing “damped and amplified harmonic oscillator” in thermal circumstance.

The relation (3.7) tells us that \( t \) is a good quantum number and the state \( \vert q; t, t^0 \rangle \rangle \) satisfying the relation (3.1) may be expanded in terms of the orthogonal set \{\( \vert q; t, t^0 \rangle \rangle \}\}. First, we search the solutions of Eq. (3.1) which obey the condition

\[
\hat{T}^\mp \vert q; t, t^0 \rangle \rangle = 0 .
\]  

(3.8)

Successive use of the relation (3.4) gives us the solutions of Eq. (3.1), which we denote \( \vert \pm t(q) \rangle \rangle \), in the following form:

\[
\begin{align*}
\vert \pm t(q) \rangle \rangle & = \left( \sqrt{N_t} \right)^{-1} \exp(\pm i\hat{T}_+) t(q) \rangle \rangle , \quad t^0 = \pm t , \\
\hat{T}^0 \vert \pm t(q) \rangle \rangle & = \pm it \vert \pm t(q) \rangle \rangle , \quad \hat{T}^\mp \vert \pm t(q) \rangle \rangle = 0 .
\end{align*}
\]  

(3.9a, 3.9b)

The quantity \( N_t \) denotes the normalization constant, which will be discussed later. Further, we define the state

\[
\vert q; t, \pm t^0 \rangle \rangle = \left( \sqrt{N_{t,t^0}} \right)^{-1} (\hat{T}^\pm)^{t^0 - t} \vert \pm t(q) \rangle \rangle . \quad (t^0 \geq t)
\]  

(3.10)

Here, \( N_{t,t^0} \) denotes also the normalization constant, which will be discussed later. The relation (3.4) presents us the following relation:

\[
\hat{T}^0 \vert q; t, \pm t^0 \rangle \rangle = \pm it^0 \vert q; t, \pm t^0 \rangle \rangle .
\]  

(3.11)

The above indicates that the state (3.10) is identical with the solution of Eq. (3.1).

Our next task is to clarify the orthogonality of the set \{\( \vert q; t, t^0 \rangle \rangle \}\}. Since \( t^0 \) in Eq. (3.4) is real, i.e., \( it^0 \) is pure imaginary, it is impossible to define the orthogonality in the conventional manner. In Ref. 4), on the basis of the time-reversal conjugate, the problem was discussed and in this paper, we translate the basic part of Ref. 4) in a plain form. We define the bra state conjugate to the ket state \( \vert q; t, t^0 \rangle \rangle \) in the form

\[
\langle \langle q; t, \pm t^0 \vert = \left( \sqrt{N_{t,t^0}} \right)^{-1} \langle \langle \pm t(q) \vert (\hat{T}^\mp)^{t^0 - t} ,
\]  

(3.12)

\[
\langle \langle \pm t(q) \vert = \left( \sqrt{N_t} \right)^{-1} \langle \langle t(q) \vert \exp(\pm i\hat{T}_-) .
\]  

(3.13)
The state \( \langle q; t, \pm t^0 | \) satisfies
\[
\langle q; t, \pm t^0 | \hat{T}^0 = \pm it^0 \langle q; t, \pm t | .
\]
(3.14a)

Further, the state \( \langle \pm t(q) | \) obeys
\[
\langle \pm t(q) | \hat{T}^0 = \pm it \langle \pm t(q) | ,
\]
(3.14b)

The definitions (3.10) and (3.12) for the ket and the bra states, \( | q; t, \pm t^0 \rangle \) and \( \langle q; t, \pm t^0 \| \), gives us the relation
\[
\langle q; t, \pm t^0 | q; t, \pm t^0 \rangle = 0 .
\]
(3.15)

However, the following relation should be noted :
\[
\langle q; t, \pm t^0 | q; t, \mp t^0 \rangle \neq 0 .
\]
(3.16)

For the other quantum numbers, it may be trivial. Concerning the normalization, we set up the relations
\[
N_{\pm} = 2^{-2t} , \quad \text{i.e.,} \quad \langle \pm t(q) | \pm t(q) \rangle = 1 ,
\]
(3.22)

\[
N_{\pm} = \frac{(t^0 - t)! \Gamma(t^0 + t)}{\Gamma(2t)} , \quad \text{i.e.,} \quad \langle q; t, \pm t^0 | q; t, \pm t^0 \rangle = 1 .
\]
(3.23)
In the sense mentioned above, the state $|q; t, \pm t^0\rangle$ is normalized.

The state $|q; t, \pm i t^0\rangle$ may be an eigenstate of the operators $\hat{Q}$, $\hat{T}$ and $\hat{T}^0$ with the eigenvalues $q$, $t$ and $\pm i t^0$, which gives us the eigenstate of the Hamiltonian $\hat{H} = E(\hat{Q}; \hat{T}, \hat{T}^0)$ as a function of $\hat{Q}$, $\hat{T}$ and $\hat{T}^0$:

$$\hat{H}|q; t, \pm i t^0\rangle = E(q; t, \pm i t^0)|q; t, \pm i t^0\rangle .$$

(3.24)

Here, it should be noted that the eigenvalue $E(q; t, \pm i t^0)$ is not always real. In the case where we treat the time-evolution of the system obeying the Hamiltonian $\hat{H} = E(\hat{Q}; \hat{T}, \hat{T}^0)$, the solution of the following Schrödinger equation should be investigated:

$$i \partial_\tau |\psi(\tau)\rangle = \hat{H}|\psi(\tau)\rangle . \quad (\tau; \text{the variable of time})$$

(3.25)

As a possible solution of Eq.(3.25), we can adopt

$$|q; t, \pm i t^0(\tau)\rangle = |q; t, \pm i t^0\rangle e^{-i E(q; t, \pm i t^0) \tau} .$$

(3.26)

Further, we require the bra-state of $|q; t, \pm i t^0(\tau)\rangle$ in the form

$$\langle q; t, \pm i t^0(\tau) | = \langle q; t, \pm i t^0 | e^{i E(q; t, \pm i t^0) \tau} .$$

(3.27)

In the form (3.27), $E^*$ denotes complex conjugate of $E$. The bra- and the ket-state give us

$$\langle q; t, \pm i t^0(\tau) | q; t, \pm i t^0(\tau) \rangle = e^{2(\text{Im} E(q; t, \pm i t^0)) \tau} .$$

(3.28)

Here, Im $E$ denotes the imaginary part of $E$. General solution of Eq.(3.25) is expressed as

$$|\psi(\tau)\rangle = \sum_{q,t,t^0} C_{q,t,t^0} |q; t, \pm i t^0\rangle e^{-i E(q; t, \pm i t^0) \tau} .$$

(3.29)

Here, the coefficient $C_{q,t,t^0}$ is determined by the initial condition. We require the condition

$$\sum_{q,t,t^0} |C_{q,t,t^0}|^2 = 1 .$$

(3.30)

The bra-state is defined as

$$\langle \psi(\tau) | = \sum_{q,t,t^0} C_{q,t,t^0}^* \langle q; t, \pm i t^0 | e^{i E(q; t, \pm i t^0) \tau} .$$

(3.31)

Then, the norm $\langle \psi(\tau)|\psi(\tau)\rangle$ is given in the form

$$\langle \psi(\tau)|\psi(\tau)\rangle = \sum_{q,t,t^0} |C_{q,t,t^0}|^2 e^{2(\text{Im} E(q; t, \pm i t^0)) \tau} .$$

(3.32)
Here, \( \text{Im}E(q; t, \pm it^0) \) denotes imaginary part of \( E(q; t, \pm it^0) \). Of course, the condition (3.32) gives us

\[
\langle \langle \psi(0)|\psi(0) \rangle \rangle = 1.
\] (3.33)

The relation (3.32) suggests us that our present idea has a possibility of describing the damping and amplifying of motion of the system under investigation. If the norm (3.32) is decreasing for \( \tau \), the system is under the damping and if increasing, the amplifying. In the forthcoming paper, we will discuss this problem.

§4. Coupling of two \( su(1, 1) \)-spins

Under the preparation discussed in §§ 2 and 3, we investigate the addition of two \( su(1, 1) \)-spins. The present authors gave an idea for the case of the Holstein-Primakoff boson representation and the present treatment is, in some sense, independent of the representation and contains some new aspects. We denote the operators governing two \( su(1, 1) \)-algebras as \( \hat{X}_{\pm,0} \) and \( \hat{Y}_{\pm,0} \). The sum of two \( su(1, 1) \)-algebras is defined in the following two forms :

\[
\hat{T}_{\pm,0} = \hat{X}_{\pm,0} + \hat{Y}_{\pm,0},
\] (4.1a)
\[
\hat{T}_{\pm,0} = \hat{X}_{\pm,0} - \hat{Y}_{\mp,0}.
\] (4.1b)

We call the forms (4.1a) and (4.1b) the \( a \)- and the \( b \)-type, respectively. It is easily verified that they obey

\[
[\hat{T}_+, \hat{T}_-] = -2\hat{T}_0, \quad [\hat{T}_0, \hat{T}_\pm] = \pm \hat{T}_\pm,
\] (4.2a)
\[
[\hat{T}_+, \hat{T}_-] = -2\hat{T}_0, \quad [\hat{T}_0, \hat{T}_\pm] = \pm \hat{T}_\pm.
\] (4.2b)

In the form analogous to the relation (2.6), we set up

\[
\hat{X}_-\|x(u)\| = 0, \quad \hat{X}_0\|x(u)\| = \|x(u)\|, \quad (x > 0)
\]
\[
\langle x(u)\|x'(u')\rangle = \delta_{xx'}\delta_{uu'},
\] (4.3a)
\[
\hat{Y}_-\|y(v)\| = 0, \quad \hat{Y}_0\|y(v)\| = \|y(v)\|, \quad (y > 0)
\]
\[
\langle y(v)\|y'(v')\rangle = \delta_{yy'}\delta_{vv'}.
\] (4.3b)

Of course, \( u \) and \( v \) denote the extra quantum numbers. With the use of the two states \( \|x(u)\| \) and \( \|y(v)\| \), we define the states for the \( a \)- and \( b \)-type as follows :

\[
|x(u), y(v)\rangle = \|x(u)\| \otimes \|y(v)\|,
\] (4.4a)
\[
|x(u), y(v)\rangle = \left(\sqrt{N_{xy}}\right)^{-1} \exp\left[\hat{X}_+(\hat{X}_0 + \hat{X} + \varepsilon)^{-1}\hat{Y}_+\right]
\times \|x(u)\| \otimes \|y(v)\|.
\] (4.4b)
Here, $N_{xy}$ denotes the normalization constant which will be discussed later. The meaning of $\hat{X}$ and $\varepsilon$ may be understood from the discussion related to the relations (2.13)~(2.15). The states $|x(u), y(v)\rangle$ and $|x(u), y(v)\rangle$ satisfy

$$
\hat{T}_+ |x(u), y(v)\rangle = 0 , \quad \hat{T}_0 |x(u), y(v)\rangle = (x + y)|x(u), y(v)\rangle , \quad (4.5a)
$$
$$
\hat{T}_- |x(u), y(v)\rangle = 0 , \quad \hat{T}_0 |x(u), y(v)\rangle = (x - y)|x(u), y(v)\rangle . \quad (4.5b)
$$

The normalization constant $N_{xy}$ is calculated in the form

$$
N_{xy} = 1 + \sum_{n=1}^{\infty} \frac{2y(2y+1) \cdots (2y+n-1)}{2x(2x+1) \cdots (2x+n-1)} = F(2y, 1, 2x; 1) . \quad (4.6)
$$

The symbol $F(2y, 1, 2x; 1)$ denotes special case of Gauss’ hypergeometric function $F(a, b, c; z)$ and it is given as

$$
F(2y, 1, 2x; 1) = \frac{\Gamma(2x) \Gamma(2x - 2y - 1)}{\Gamma(2x - 1) \Gamma(2y - 1)} = \frac{2x - 1}{2x - 2y - 1} . \quad (2x - 2y - 1 > 0) \quad (4.7)
$$

Then, combining with $x > 0$ and $y > 0$, the condition $N_{xy} > 0$ leads us to the following condition, under which the state $|x(u), y(v)\rangle$ can be normalized:

$$
2x > 2y + 1 > 1 . \quad \text{(for the b-type)} \quad (4.8)
$$

Under the above results, we obtain two states $|x(u), y(v); t\rangle$ and $|x(u), y(v); t\rangle$ which satisfy

$$
\hat{T}_- |x(u), y(v); t\rangle = 0 , \quad \hat{T}_0 |x(u), y(v); t\rangle = t|x(u), y(v); t\rangle , \quad (4.9a)
$$
$$
\hat{T}_- |x(u), y(v); t\rangle = 0 , \quad \hat{T}_0 |x(u), y(v); t\rangle = t|x(u), y(v); t\rangle . \quad (4.9b)
$$

The explicit forms are as follows:

$$
|x(u), y(v); t\rangle = \left(\sqrt{N_{xy, t(a)}}\right)^{-1} (\hat{O}_+)^{t-(x+y)}|x(u), y(v)\rangle , \quad (4.10a)
$$
$$
|x(u), y(v); t\rangle = \left(\sqrt{N_{xy, t(b)}}\right)^{-1} (\hat{O}_+)^{(x-y)-t}|x(u), y(v)\rangle . \quad (4.10b)
$$

$$
\hat{O}_+ = \hat{X}_+ (\hat{X}_0 + \hat{X} + \varepsilon)^{-1} - \hat{Y}_+ (\hat{Y}_0 + \hat{Y} + \varepsilon)^{-1} , \quad (4.11a)
$$
$$
\tilde{O}_+ = \hat{Y}_+ . \quad (4.11b)
$$

The operator $\hat{O}_+$ is closely related to $\hat{Q}_+$ defined in the relation (2.13). The quantity $\varepsilon$ is an infinitesimal parameter and the definitions of $\hat{X}$ and $\hat{Y}$ may be understood from the relations (2.14) and (2.13). The operators $\hat{O}_+$ and $\tilde{O}_+$ satisfy

$$
[ \hat{T}_- , \hat{O}_+ ] = 0 , \quad [ \hat{T}_0 , \hat{O}_+ ] = +\hat{O}_+ , \quad (4.12a)
$$
$$
[ \hat{T}_- , \tilde{O}_+ ] = 0 , \quad [ \hat{T}_0 , \tilde{O}_+ ] = -\tilde{O}_+ . \quad (4.12b)
$$
With the use of the relation (4.12), we obtain the relation (4.9). The quantities $N_{xy,t(a)}$ and $N_{xy,t(b)}$ denote the normalization constants:

$$N_{xy,t(a)} = \sum_{r=0}^{t-(x+y)} \frac{(t-(x+y))!\Gamma(2x)\Gamma(2y)}{r!\Gamma(t+x-y-r)\Gamma(2y+r)} , \quad (4.13a)$$

$$N_{xy,t(b)} = \frac{((x-y)-t)!\Gamma(x+y-t)}{\Gamma(2y)} . \quad (4.13b)$$

The form (4.10) tells us

$$t = x + y , \ x + y + 1 , \ x + y + 2 , \cdots , \quad (\text{for the } a\text{-type}) \quad (4.14a)$$

$$t = x - y , \ x - y - 1 , \cdots , t_m . \quad (\text{for the } b\text{-type}) . \quad (4.14b)$$

For the $b$-type, $x$ and $y$ obey the condition (4.8) and $t_m$ denotes the minimum of $t$ ($0 < t_m \leq 1$). The relations (4.14a) and (4.14b) serve us the coupling rule for the additions of two $su(1,1)$-spins. In Ref.7), only the case (4.14a) was discussed. Concerning $\tilde{O}_+$, we give a short comment. In the form analogous to $\tilde{O}_+$, we can define formally $\tilde{O}'_+$ as

$$\tilde{O}'_+ = \tilde{X}_+ + \tilde{Y}_+ \epsilon^{-1} - (\tilde{Y}_0 + \tilde{Y} + \epsilon)^{-1}\tilde{Y}_-.$$  

The operator $\tilde{O}'_+$ obeys the same relation as that shown in the relation (4.12b). However, we have $\tilde{O}'_+|x(u),y(v)) = 0$, and then, $\tilde{O}'_+$ cannot play the same role as that of $\tilde{O}_+$.

By operating $\hat{T}_+$ and $\tilde{T}_+$ on the states $|x(u),y(v);t\rangle$ and $|x(u),y(v);t\rangle$ successively for $(t_0 - t)$ times, respectively, we obtain

$$|x(u),y(v);t,t_0\rangle = \sqrt{\frac{T(2t)}{(t_0-t)!\Gamma(t_0+t)}(\hat{T}_+)^{t_0-t}}|x(u),y(v);t\rangle , \quad (4.15a)$$

$$|x(u),y(v);t,t_0\rangle = \sqrt{\frac{T(2t)}{(t_0-t)!\Gamma(t_0+t)}(\tilde{T}_+)^{t_0-t}}|x(u),y(v);t\rangle . \quad (4.15b)$$

Of course, both states are normalized. We can prove that both states are eigenstates of $\hat{X}^2$ and $\hat{Y}^2$, and further, $(\hat{T}^2,\hat{T}_0)$ and $(\tilde{T}^2,\tilde{T}_0)$, respectively, with the eigenvalues $x(x-1)$, $y(y-1)$, $t(t-1)$ and $t_0$. It should be noted that $t_0$ obeys the rule (2.12) and $t$ is governed by the rule (4.14). For the $a$-type, $x$ and $y$ are positive and for the $b$-type, $x$ and $y$ are also positive with the condition (4.8). The above is the coupling scheme of two $su(1,1)$-algebras and the states $|x(u),y(v);t,t_0\rangle$ and $|x(u),y(v);t,t_0\rangle$ are monomial.

§5. An example

In the next paper, we will describe possible boson realizations of the $so(4)$- and the $so(3,1)$-algebra, in which we contact the following $su(1,1)$-algebras:

$$\hat{T}_+ = \hat{c}_+^*\hat{c}_- - (1/2)\hat{c}_0^*\hat{c}_0 + (1/2)\hat{d}_0^2 ,$$

$$\tilde{T}_+ = \hat{c}_+\hat{c}_- - (1/2)\hat{c}_0\hat{c}_0 + (1/2)\hat{d}_0 .$$
\[ \hat{T}_- = \hat{c}_- \hat{c}_+ - (1/2) \hat{c}_0^2 + (1/2) \hat{d}_0^2, \]
\[ \hat{T}_0 = 1 + (1/2)(\hat{c}_+^* \hat{c}_+ + \hat{c}_0^* \hat{c}_0 + \hat{c}_-^* \hat{c}_-) + (1/2) \hat{d}_0^2 \hat{d}_0, \]  
(5.1a)
\[ \hat{T}_+ = \hat{c}_+^* \hat{c}_-^* - (1/2) \hat{c}_0^2 - (1/2) \hat{d}_0^2, \]
\[ \hat{T}_- = \hat{c}_- \hat{c}_+ - (1/2) \hat{c}_0^2, \]
\[ \hat{T}_0 = 1/2 + (1/2)(\hat{c}_+^* \hat{c}_+ + \hat{c}_0^* \hat{c}_0 + \hat{c}_-^* \hat{c}_-) - (1/2) \hat{d}_0^2 \hat{d}_0 . \]  
(5.1b)

Here, \((\hat{c}_{\pm,0}, \hat{\epsilon}_{\pm,0})\) and \((\hat{d}_{\pm,0}, \hat{\epsilon}_{\pm,0})\) denote boson operators. As an example of the coupling scheme developed in §4, we treat the cases (5.1a) and (5.1b). The forms (5.1a) and (5.1b) can be decomposed into the forms (4.1a) and (4.1b), respectively:

\[ \hat{X}_+ = \hat{c}_+^* \hat{c}_- - (1/2) \hat{c}_0^2, \quad \hat{X}_- = \hat{c}_- \hat{c}_+ - (1/2) \hat{c}_0^2, \]
\[ \hat{X}_0 = 3/4 + (1/2)(\hat{c}_+^* \hat{c}_+ + \hat{c}_0^* \hat{c}_0 + \hat{c}_-^* \hat{c}_-) , \]
\[ \hat{Y}_+ = (1/2) \hat{d}_0^2 , \quad \hat{Y}_- = (1/2) \hat{d}_0^2 , \quad \hat{Y}_0 = 1/4 + (1/2) \hat{d}_0^2 \hat{d}_0 . \]  
(5.2)

Further, \(\hat{X}_{\pm,0}\) can be decomposed into the following form:

\[ \hat{X}_{\pm,0} = \hat{W}_{\pm,0} + \hat{Z}_{\pm,0} , \]  
(5.4)
\[ \hat{W}_+ = \hat{c}_+^* \hat{c}_- , \quad \hat{W}_- = \hat{c}_- \hat{c}_+ , \quad \hat{W}_0 = 1/2 + (1/2)(\hat{c}_+^* \hat{c}_+ + \hat{c}_0^* \hat{c}_-) , \]  
(5.5)
\[ \hat{Z}_+ = -(1/2) \hat{c}_0^2 , \quad \hat{Z}_- = -(1/2) \hat{c}_0^2 , \quad \hat{Z}_0 = 1/4 + (1/2) \hat{c}_0^2 \hat{c}_0 . \]  
(5.6)

The sets \((\hat{W}_{\pm,0})\) and \((\hat{Z}_{\pm,0})\) also obey the \(su(1,1)\)-algebras, respectively. It may be clear that the present system is the sum of three \(su(1,1)\)-algebras, \((\hat{W}_{\pm,0}), (\hat{Z}_{\pm,0})\) and \((\hat{Y}_{\pm,0})\). Then, successively, the addition is performed.

Our first task is to search the state \(|x(u)\rangle\) obeying the relation (4.3a) under the sum (5.4). It is completed by replacing \(x, y, t, (\hat{X}_{\pm,0}, \hat{X})\) and \((\hat{Y}_{\pm,0}, \hat{Y})\) in the form (4.10a) with \(w, z, x, (\hat{W}_{\pm,0}, \hat{W})\) and \((\hat{Z}_{\pm,0}, \hat{Z})\), respectively. As for \(u\) and \(v\) in the form (4.10a), we adopt \(u = s = +, -\) and \(v\) is not necessary. Later, we will show them. By denoting \(|w(s);z\rangle\) as \(|x(w(s)z)\rangle\), we obtain

\[ \|x(w(s)z)\rangle = \left[ \hat{W}_+(\hat{W}_0 + \hat{W} + \varepsilon)^{-1} - \hat{Z}_+(\hat{Z}_0 + \hat{Z} + \varepsilon)^{-1} \right]^{x-(w+z)} |w(s), z\rangle . \]  
(5.7)

Here and hereafter, we omit the normalization constant for any state. The state \(|w(s), z\rangle\) is obtained by the form (4.4a):

\[ |w(s), z\rangle = \|w(s)\rangle \otimes \|z\rangle , \]  
(5.8)
\[ \hat{W}_-\|w(s)\rangle = 0 , \quad \hat{W}_0\|w(s)\rangle = w\|w(s)\rangle , \]  
(5.9)
\[ \hat{Z}_-\|z\rangle = 0 , \quad \hat{Z}_0\|z\rangle = z\|z\rangle . \]  
(5.10)
The states \( \| w(s) \rangle \) and \( \| z \rangle \) are concretely expressed as

\[
\| w(s) \rangle = \begin{cases} (\hat{c}_+)^{2w-1}\| 0 \rangle, & (s = +, w = 1, 3/2, 2, 5/2, \cdots) \\ (\hat{c}_-)^{2w-1}\| 0 \rangle, & (s = -, w = 1/2, 1, 3/2, 2, \cdots) \end{cases} \quad (5.11)
\]

\[
\| z \rangle = \begin{cases} \| 0 \rangle, & (z = 1/4) \\ \hat{c}_0\| 0 \rangle, & (z = 3/4) \end{cases} \quad (5.12)
\]

Here, \( \| 0 \rangle \) and \( \| 0 \rangle \) denote the vacuums for \( \hat{c}_+ \) and \( \hat{c}_0 \), respectively. The state \( \| x(w(s)z) \rangle \) satisfies the relation (4.3a):

\[
\hat{X}_-\| x(w(s)z) \rangle = 0, \quad \hat{X}_0\| x(w(s)z) \rangle = x\| x(w(s)z) \rangle . \quad (5.13)
\]

In the present case, the rule (4.14a) gives us

\[
x = w + z, \ w + z + 1, \ w + z + 2, \cdots . \quad (5.14)
\]

In the same form as the above, we can derive the state \( \| y \rangle \) satisfying the relation (4.3b):

\[
\| y \rangle = \begin{cases} \| 0 \rangle, & (y = 1/4) \\ \hat{d}_0\| 0 \rangle, & (y = 3/4) \end{cases} \quad (5.15)
\]

Here, \( \| 0 \rangle \) denotes the vacuum for \( \hat{d}_0 \). In the present case, \( v \) in the relation (4.3b) is not necessary. The state \( \| y \rangle \) satisfies the relation (4.3b):

\[
\hat{Y}_-\| y \rangle = 0, \quad \hat{Y}_0\| y \rangle = y\| y \rangle . \quad (5.16)
\]

Thus, we obtain \( | x(w(s)z)y; t, t_0 \rangle \) and \( | x(w(s)z)y; t, t_0 \rangle \) in the following form:

\[
| x(w(s)z)y; t, t_0 \rangle = (\hat{T}_+)^{t-t_0} \cdot \left[ \hat{X}_+\hat{X}_0 + x + \varepsilon \right]^{-1} - \hat{Y}_+\hat{Y}_0 + y + \varepsilon \right]^{-1} \right]^{1-(x+y)} \\
\times \left[ \hat{W}_+\hat{W}_0 + w + \varepsilon \right]^{-1} - \hat{Z}_+\hat{Z}_0 + z + \varepsilon \right]^{-1} \right]^{x-(w+z)} \\
\times | w(s), z, y \rangle , \quad (5.17a)
\]

\[
| x(w(s)z)y; t, t_0 \rangle = (\hat{T}_+)^{t-t_0} \cdot (\hat{Y}_+)^{(y-x)-t} \cdot \exp \left[ \hat{X}_+\hat{X}_0 + x + \varepsilon \right]^{-1}\hat{Y}_+ \right] \\
\times \left[ \hat{W}_+\hat{W}_0 + w + \varepsilon \right]^{-1} - \hat{Z}_+\hat{Z}_0 + z + \varepsilon \right]^{-1} \right]^{x-(w+z)} \\
\times | w(s), z, y \rangle , \quad (5.17b)
\]

\[
| w(s), z, y \rangle = | w(s) \rangle \otimes | z \rangle \otimes | y \rangle . \quad (5.18)
\]

It may be natural that, following the given values of \( s = +, - \), \( z = 1/4, 3/4 \) and \( y = 1/4, 3/4 \), the values of \( w, x \) and \( t \) can be changed. In the next paper, we will describe the
so(4)- and the so(3, 1)-algebra, in which we encounter the case \( (s = -, z = 1/4, x = w+1/4) \):

\[
|w + 1/4(w(-)1/4)y; t, t_0) = (\tilde{T}_+)^{t-t_0} \left[ \hat{X}_+ (\hat{X}_0 + w + 1/4 + \varepsilon)^{-1} - \hat{Y}_+ (\hat{Y}_0 + y + \varepsilon)^{-1} \right]^{t-w-1/4-y} |w(-), 1/4, y),
\]

(5.19a)

\[
|w + 1/4(w(-)1/4)y; t, t_0) = (\tilde{T}_+)^{t-t_0} \cdot (\hat{Y}_+)^{w+1/4-y-t} \cdot \exp \left[ \hat{X}_+ (\hat{X}_0 + w + 1/4 + \varepsilon)^{-1} - \hat{Y}_+ \right] |w(-), 1/4, y),
\]

(5.19b)

Of course, the notations used in this paper are different from those used in the next paper.

\section{6. Discussion}

Finally, we investigate the structure of the states \(|x(w(s)z)y; t, t_0)\) and \(|x(w(s)z)y; t, t_0)\) (hereafter, we call these two states simply the “states (5.17)”). Formally, the “states (5.17)” are specified by seven quantum numbers. However, the present system consists of four kinds of boson operators and the orthogonal set can be specified by four quantum numbers. Mainly, we discuss the meaning of this formal discrepancy. First, we investigate \((\hat{Y}_\pm, 0)\) composed of \((\hat{d}_0^*, \hat{d}_0)\). In this case, only one kind of boson operator is used, and then, the orthogonal set is given by

\[
\|N\) = (\hat{d}_0^*)^N \|0) . \quad (N = 0, 1, 2, \cdots)
\]

(6.1)

On the other hand, the framework of the \(su(1, 1)\)-algebra enable us to introduce the state

\[
\|y, y_0) = (\hat{Y}_+)^{y_0-y} \|y) . \quad (y_0 = y + n, \ n = 0, 1, 2, \cdots)
\]

(6.2)

Here, \(\|y)\) is given in the relation (5.13). Then, we have

\[
\|y = 1/4, y_0) = (\hat{Y}_+)^y \|0) \sim (\hat{d}_0^*)^{2n} \|0) . \quad (N = 2n)
\]

\[
\|y = 3/4, y_0) = (\hat{Y}_+)^y \hat{d}_0^* \|0) \sim (\hat{d}_0^*)^{2n+1} \|0) . \quad (N = 2n + 1)
\]

(6.3)

The eigenvalues of the Casimir operator \(\hat{Y}^2\) for the states \(\|y = 1/4, y_0)\) and \(\|y = 3/4, y_0)\) are equal to the value \((-3/16)\). In the case \((\hat{Z}_\pm, 0)\), the situation is the same as the case \((\hat{Y}_\pm, 0)\). From the above argument, we learn that the quantum numbers \(y\) and \(z \) (= 1/4, 3/4) are used for discriminating between the even- and the odd-boson number
states. Next, we investigate the case \( (\hat{W}_{\pm 0}) \) composed of two kinds of bosons \( (\hat{c}_+, \hat{c}_-) \) and \( (\hat{c}_-^*, \hat{c}_-^*) \). In this case, the orthogonal set is given by
\[
\|m_+, m_-\| = (\hat{c}_+^*)^{m_+} (\hat{c}_-^*)^{m_-} \|0\| \quad (m, n = 0, 1, 2, 3, \cdots) \quad (6.4)
\]
In the framework of the \( su(1,1) \)-algebra, we can introduce the following state:
\[
\|w(s)\| = (\hat{w})^{w_0 - w} \|w(s)\| \quad (w_0 = w + l, \quad l = 0, 1, 2, \cdots) \quad (6.5)
\]
Here, \( \|w(s)\| \) is given in the relation (5.11). Then, we have
\[
\|w(\pm), w_0\| = (\hat{c}_+^*)^{l+2w-1}(\hat{c}_-^*)^l \|0\| \quad (m_+ = l + 2w - 1, \quad m_- = l) \quad (6.6)
\]
\[
\|w(-), w_0\| = (\hat{c}_+^*)^l(\hat{c}_-^*)^{l+2w-1} \|0\| \quad (m_+ = l, \quad m_- = l + 2w - 1) \quad (6.7)
\]
The state (6.6) and (6.7) tell us that the set (6.4) is classified in terms of \( (m_+ > m_-, \quad m_+ \leq m_-) \) and for this classification, \( s \) is used. The eigenvalue of \( \hat{W}^2 \) for the state (6.5) is equal to \( w(w-1) \) in two groups. From the above argument, we know the role of the quantum numbers \( s, z \) and \( y \). In the “states (5.17)”, \( w, x, t \) and \( t_0 \) play the basic role, but, \( s, z \) and \( y \) restrict to the ranges of \( w, x \) and \( t \), which can be seen in various relations.

In conclusion, we give some remarks. In this paper, we discussed three problems: (i) The eigenvalue problem in an unconventional form, (ii) the coupling scheme of the addition of two \( su(1,1) \)-spins and (iii) a concrete example. These will serve boson realizations of the \( so(4) \)- and the \( so(3,1) \)-algebra developed in the next paper.

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Appendix A

--- The procedure for determining \( f_n(t, m) \) ---

First, we note that it may be enough to treat the function \( f_n(t, m) \) in the form of a polynomial in the \( n \)-th degree for \( m \). If it is accepted, the relation (2.26) can be expressed in the form
\[
\begin{align*}
n(3\alpha_n - \alpha_{n-1})m^n \\
+ \left[(3n-1)\beta_n - n\beta_{n-1} - \left(\frac{n(n-1)}{2}\alpha_n + 2n(n-1+t)\alpha_{n-1}\right)\right]m^{n-1} \\
+ \cdots + 2n(\gamma_n - (n-1+t)\gamma_{n-1}) = 0 \quad \text{(A.1)}
\end{align*}
\]
Here, \( f_n(t, m) \) is expressed as

\[
f_n(t, m) = \alpha_n m^n + \beta_n m^{n-1} + \cdots + \gamma_n . \tag{A.2}
\]

Since we have the relation (A.1) for any value of \( m \), the coefficient of any term \( m^k \) \((k = n, n-1, \cdots, 0)\) should be vanished. Then, we obtain the recursion formulae for \((\alpha_n, \beta_n, \cdots, \gamma_n)\)

Under the above idea, we set up \( f_n(t, m) \) in the form

\[
f_n(t, m) = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} g_n^{(n-r)}(t) \left( \frac{m}{3} \right)^r . \tag{A.3}
\]

Substituting the expansion (A.3) into the relation (2.25) and following the above-mentioned procedure, we have the following formulae:

\[
g_n^{(0)}(t) - g_n^{(0)}(t-1) = 0 , \tag{A.4}
\]

\[
(3n - r)g_n^{(r)}(t) - 3(n - r)g_n^{(r)}(t-1)
\]

\[
+ (n - r) \sum_{k=1}^{r} \frac{(-1)^k}{3^k(1 + k)} \cdot \frac{r!}{k!(r - k)!} g_n^{(r-k)}(t) - 2r(t + n - 1)g_n^{(n-1)}(t) = 0 .
\]

\[
(r = 1, 2, \cdots, n) \tag{A.5}
\]

Since \( g_0^{(0)}(t) = f_0(t, m) = 1 \), the relation (A.4) gives us

\[
g_n^{(0)}(t) = 1 . \tag{A.6}
\]

For \( r = 1 \), the relation (A.5) leads us to

\[
(3n - 1)g_n^{(1)}(t) - 3(n - 1)g_n^{(1)}(t-1) - \frac{n - 1}{6} g_n^{(0)}(t) - 2(t + n - 1)g_n^{(0)}(t-1) = 0 . \tag{A.7}
\]

The relations (A.6) and (A.7) determine \( g_n^{(1)}(t) \) in the form

\[
g_n^{(1)}(t) = \frac{13}{30} n + \left( t - \frac{13}{30} \right) . \tag{A.8}
\]

For \( r = 2 \), we have

\[
(3n - 2)g_n^{(2)}(t) - 3(n - 2)g_n^{(2)}(t-1)
\]

\[
+ (n - 2) \left( - \frac{1}{3} g_n^{(1)}(t) + \frac{1}{27} g_n^{(0)}(t) \right) - 4(t + n - 1)g_n^{(1)}(t-1) = 0 . \tag{A.9}
\]

The solution of the relation (A.9) is given as

\[
g_n^{(2)}(t) = \frac{169}{900} n^2 + \left( \frac{13}{15} t + \frac{181}{540} \right) n + \left( t^2 - \frac{11}{15} t + \frac{1591}{1350} \right) . \tag{A.10}
\]
In such a manner as the above, we obtain the form of \( g_n^{(r)}(t) \) \((r = 3, 4, \cdots, n - 1)\). The case \( r = n \), the relation (A.5) gives us

\[
2ng_n^{(n)}(t) - 2n(t + n - 1)g_{n-1}^{(n-1)}(t) = 0 . \tag{A.11}
\]

The relation (A.11) determines \( g_n^{(n)}(t) \) in the form

\[
g_n^{(n)}(t) = \frac{\Gamma(t + n)}{\Gamma(t)} . \tag{A.12}
\]

Concerning \( g_n^{(r)}(t) \), the above examples suggest us the following form :

\[
g_n^{(r)}(t) = \sum_{k=0}^{r} p_{n,r}^{(k)}(t)n^k , \tag{A.13}
\]

\[
p_{n,r}^{(k)}(t) = \sum_{l=0}^{k} q_{n,r,k}^{(l)} t^l . \tag{A.14}
\]

The relations for obtaining \( p_{n,r}^{(k)}(t) \), that is, \( q_{n,r,k}^{(l)} \) are omitted.

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