Negative $K$-groups of abelian categories

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Abstract

We prove that negative $K$-groups of small abelian categories are trivial.

Introduction

In the celebrated paper [Sch06], Schlichting predicted that for any small abelian category $\mathcal{A}$ and any positive integer $n$, the negative $K$-group $K_{-n}\mathcal{A}$ is trivial. He proved that it is true for $n = 1$ and any $\mathcal{A}$ and for any $n$ and any noetherian abelian category $\mathcal{A}$. The goal of this paper, we will prove this conjecture for any $n$ and any $\mathcal{A}$. We use the technique established in [Moc13] and [HM13]. The main point is making use of higher derived categories $D^n\mathcal{A}$ of $\mathcal{A}$. (For definition, see 1.5.) In [HM13], we give an interpretation of negative $K$-groups as the obstruction groups of idempotent completeness of higher derived categories. (See 1.5(2).) The main theorem is the following.

Theorem 0.1. For any abelian category $\mathcal{A}$ and for any positive integer $n \geq 2$, $D^n\mathcal{A}$ is trivial. In particular, if $\mathcal{A}$ is essentially small, the $-m$-th $K$-group $K_{-m}\mathcal{A}$ of $\mathcal{A}$ is trivial for any $m > 0$.

In section 1, we review several notions about relative exact categories established in [Moc13] and [HM13]. In section 2, we give a proof of the main theorem.

1 Relative exact categories

In this section, we recall the fundamental notions and properties for relative exact categories from [HM13] and [Moc13].

1.1 (Relative exact categories). 1) A relative exact category $\mathcal{E} = (\mathcal{E}, w)$ is a pair of an exact category $\mathcal{E}$ with a specific zero object $0$ and a class of morphisms in $\mathcal{E}$ which is closed under finite compositions. Namely $w$ satisfies the following two axioms.

(Identity axiom). For any object $x$ in $\mathcal{E}$, the identity morphism $\text{id}_x$ is in $w$.

(Composition closed axiom). For any composable morphisms $\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet$ in $\mathcal{E}$, if $a$ and $b$ are in $w$, then $ba$ is also in $w$.

For any relative exact category $\mathcal{E}$, we write $\mathcal{E}_E$ and $w_E$ for the underlying exact category and the class of morphisms of $\mathcal{E}$ respectively.

2) A relative exact functor between relative exact categories $f : \mathcal{E} = (\mathcal{E}, w) \rightarrow (\mathcal{F}, v)$ is an exact functor $f : \mathcal{E} \rightarrow \mathcal{F}$ such that $f(w) \subset v$ and $f(0) = 0$. We denote the category of relative exact categories and relative exact functors by $\text{RelEx}$.

3) We write $\mathcal{E}^w$ for the full subcategory of $\mathcal{E}$ consisting of those object $x$ such that the canonical morphism $0 \rightarrow x$ is in $w$. We consider the following axioms.

(Strict axiom). $\mathcal{E}^w$ is an exact category such that the inclusion functor $\mathcal{E}^w \hookrightarrow \mathcal{E}$ is exact and reflects exactness.

(Very strict axiom). $\mathcal{E}$ satisfies the strict axiom and the inclusion functor $\mathcal{E}^w \hookrightarrow \mathcal{E}$ induces a fully faithful functor $D^b(\mathcal{E}^w) \hookrightarrow D^b(\mathcal{E})$ on the bounded derived categories.

We denote the category of strict (resp. very strict) relative exact categories by $\text{RelEx}^\text{strict}$ (resp. $\text{RelEx}^\text{vs}$).
1.2 (Derived category). We define the derived categories of a strict relative exact category $E = (E, w)$ by the following formula

$$D_\#(E) := \text{Coker}(D_{\#}(E^w) \to D_{\#}(E))$$

where $\# = b, \pm$ or nothing. Namely $D_\#(E)$ is a Verdier quotient of $D_{\#}(E)$ by the thick subcategory of $D_{\#}(E)$ spanned by the complexes in $Ch_\#(E^w)$.

1.3 (Quasi-weak equivalences). Let $P_\# : Ch_{\#}(E) \to D_{\#}(E)$ be the canonical quotient functor. We denote the pull-back of the class of all isomorphisms in $D_{\#}(E)$ by $qw_{\#}$ or simply $qw$. We call a morphism in $qw$ a quasi-weak equivalence. We write $Ch_{\#}(E)$ for a pair $(Ch_{\#}(E), qw)$. We can prove that $Ch_{\#}(E)$ is a complicial bWaldhausen category in the sense of [TT90, 1.2.11]. In particular, it is a relative exact category. The functor $P_\#$ induces an equivalence of triangulated categories $T(Ch_{\#}(E), qw) \simeq D_{\#}(E)$ (See [Sch11, 3.2.17]). If $w$ is the class of all isomorphisms in $E$, then $qw$ is just the class of all quasi-isomorphisms in $Ch_{\#}(E)$ and we denote it by $qis$.

1.4 (Consistent axiom). Let $E = (E, w)$ be a strict relative exact category. There exists the canonical functor $i^\#: E \to Ch_{\#}(E)$ where $i^\#(x)k = x$ if $k = 0$ and $0$ if $k \neq 0$. We say that $w$ (or $E$) satisfies the consistent axiom if $i^\#(w) \subset qw$. We denote the full subcategory of consistent relative exact categories in $\text{RelEx}$ by $\text{RelEx}_{\text{consist}}$.

1.5 (Higher derived categories). (cf. [HM13, 3.1, 3.2]). Let $E$ be a very strict consistent relative exact category and we denote $n$-th times iteration of $Ch$ for $E$ by $\Sigma^n E$ and $D^n(E) := D_0(\Sigma^n E)$ the $n$-th higher derived category of $E$. Then for any positive integer $n$, we have

$(1) \; K_{-n}(E) \simeq K_0(D^n(E)).$

$(2) \; K_{-n}(E)$ is trivial if and only if $D^n(E)$ is idempotent complete.

$(3) \; The \; canonical \; functor \; \Sigma E \to Ch_\# \Sigma E \; induces \; an \; equivalence \; of \; triangulated \; categories \; D E \simeq D_\#(\Sigma E).$

2 Proof of the main theorem

In this section, we prove Theorem 1.1. Let $A$ be an essentially small abelian category and $n$ a positive integer $n \geq 2$.

2.1. We have the equalities

$$D_n A = D_\#(\Sigma^n A) \simeq D \Sigma^{n-1} A$$

$$\Sigma E \rightarrow \text{Ch}_\# \Sigma E$$

where the equalities I and II just come from definitions and [1.5] (3) respectively. For simplicity, we put $B = Ch^{n-1} A$, $F = Ch^{n-2} A$ and $v = w_{\Sigma^{n-1} A}$. Then the pair $(B, v)$ is a complicial exact category with weak equivalences or a bicomplcial pair in the sense of [Sch11] or [Moc13]. Therefore $(B, v)$ is very strict by [Moc13, 3.9] and hence the functor $D_\# B^o \to D_{\#} (B)$ induced from the inclusion functor $B^o \to B$ is fully faithful for $\# \in \{b, \pm, \text{nothing} \}$ by [HM13, 1.2].

By virtue of equality (1), Theorem 1.1 follows from Proposition 2.2 below.

Proposition 2.2. The functor $D_\# B^o \to D_{\#} B$ induced from the inclusion functor $B^o \to B$ is an equivalence of triangulated categories for $\# \in \{b, \pm, \text{nothing} \}$.

Proof. For $\# = +$, we use Lemma 2.3 below. We check the condition $(*)$ in [2.3] for $C = B = D = B^o$. For any complex $b$ in $Ch F$, there is a canonical epimorphism $\text{Cone id}_b[-1] \to b$ in $Ch F$ with $\text{Cone id}_b[-1] \subset B^o$. Hence we obtain the result for $\# = +$. Since we have $D - B = (D_B)^{\text{op}}$ and $D - B^o = (D_B^{\text{op}})^{\text{op}}$, we also get the result for $\# = -$. Finally notice that $DB$ is generated by $D_B$ and $D_B^{\text{op}}$, then $DB^o \to D_B^{\text{op}}$ is essentially surjective and we complete the proof.

Lemma 2.3. (cf. [TT90, 1.9.5, 1.9.7]). Let $C$ be an abelian category and $D$ an idempotent complete strict exact subcategory of $C$. If $D$ has enough objects to resolve" in the following sense $(*)$, then the inclusion functor $Ch_\# D \to Ch_\# C$ induces an equivalence of triangulated categories $D_\# \to D_\# C$.

$(*)$ For any integer $k$, any complex $x$ in $Ch_\# C$ such that $H_i x = 0$ for $i < k$ and any epimorphism in $C$, $a \to H_{k+1} x$, then there exists an object $d$ in $D$ and a morphism $d \to a$ such that the composition is an epimorphism in $C$. \hfill $\Box$
References

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