UNIVALENCE OF CRITERIA FOR LINEAR FRACTIONAL DIFFERENTIAL OPERATOR $D^{\alpha,\beta}_x$ WITH THE BESSEL FUNCTIONS

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Abstract. In this paper our aim is to extend and improve the sufficient conditions for integral operators involving the normalized forms of the generalized Bessel functions of the first kind to be univalent in the open unit disk as investigated recently by (Erhan, E. Orhan, H. and Srivastava, H. (2011). Some sufficient conditions for univalence of certain families of integral operators involving generalized Bessel functions. Taiwanese Journal of Mathematics, 15 (2), pp.883-917) and (Baricz, A. and Frasin, B. (2010). Univalence of integral operators involving Bessel functions. Applied Mathematics Letters, 23 (4), pp.371–376).

1. Introduction and some preliminary results

Several applications of Bessel functions arise naturally in a wide variety of problems in applied mathematics, statistics, operational research, theoretical physics and engineering sciences. Bessel functions are series solutions to a second order differential equation that ascend in many and diverse situations. Bessel’s differential equation of order $\nu$ is defined as (see, for details, [25]):

\begin{equation}
 z^2 w'' + b z w' + [d z^2 - \nu^2 + (1-b)] w = 0 \quad b, d, \nu \in \mathbb{C} \tag{1.1}
\end{equation}

A particular solution of the differential equation (1.1), which is denoted by $w_{\nu,b,d}(z)$ is called the generalized Bessel function of the first kind of order $\nu$. In fact, we have the following familiar series representation for the function $w_{\nu,b,d}(z)$:

\begin{equation}
 w_{\nu,b,d}(z) = \sum_{n=0}^{\infty} \frac{(-d)^n}{n! \Gamma(\nu + n + \frac{b+1}{2})} \left( \frac{z}{2} \right)^{2\nu+n} (z \in \mathbb{C}). \tag{1.2}
\end{equation}

where $\Gamma(z)$ stands for the Euler gamma function. The series in (1.2) permits us to study the Bessel, the modified Bessel and the spherical Bessel functions in a unified manner. Each of these particular cases of the function $w_{\nu,b,d}(z)$ is worthy of mention here.

- For $b = d = 1$ in (1.2), we obtain the familiar Bessel function $J_\nu(z)$ defined by (see [25]; see also [6]):

\begin{equation}
 J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\nu + n + 1)} \left( \frac{z}{2} \right)^{2\nu+n} (z \in \mathbb{C}). \tag{1.3}
\end{equation}

- For $b = -d = 1$ in (1.2), we obtain the familiar Bessel function $I_\nu(z)$ defined by (see [25]; see also [6]):

\begin{equation}
 I_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(\nu + n + 1)} \left( \frac{z}{2} \right)^{2\nu+n} (z \in \mathbb{C}). \tag{1.4}
\end{equation}

Now, consider the function $u_{\nu,b,d} : \mathbb{C} \rightarrow \mathbb{C}$, defined by the transformation

\[ u_{\nu,b,d}(z) = 2^\nu \Gamma \left( \nu + \frac{b+1}{2} \right) \left( \frac{z}{\sqrt{2}} \right)^{-\nu \sqrt{2} w_{\nu,b,d}(\sqrt{2})}. \]

By using the well-known Pochhammer (or Appell) symbol, defined in terms of the Euler gamma function,

\[ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\ldots(a+n-1) \]

2010 Mathematics Subject Classification. Primary 33C10, 30C45; Secondary 30C20, 30C75.

Key words and phrases. Analytic functions, Univalent functions, Integral Operator, Generalized Bessel functions, Ahlfors-Becker univalence criteria, fractional differential operator.

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and \((a)_0 = 1\), we obtain for the function \(u_{v,b,d}(z)\) the following representation

\[
u_{v,b,d}(z) = \sum_{n=0}^{\infty} \frac{(-d/4)^n}{(v+b+1/2)_n} z^n,
\]

where \(v + \frac{b+1}{2} \neq 0, -1, -2, \ldots\). This function is analytic on \(\mathbb{C}\) and satisfies the second order linear differential equation

\[4z^2u''(z) + 2(2v + b + 1)zu'(z) + dzu(z) = 0.\]

We now introduce the function \(\varphi_{v,b,d}(z)\) defined in terms of the generalized Bessel function \(w_{v,b,d}(z)\) by

\[
\varphi_{v,b,d}(z) = z w_{v,b,d}(z)
\]

\[
= 2^{\nu} \Gamma \left( v + \frac{b+1}{2} \right) z^{1-v/2} w_{v,b,d}(\sqrt{z})
\]

\[
= z + \sum_{n=1}^{\infty} (-d)^n \frac{z^{n+1}}{4^n n!(k)_n} \left( k = v + \frac{b+1}{2} \right)
\]

Let \(A\) denote the class of analytic function \(f\) defined in the unit disk \(U = \{ z : |z| < 1 \}\) and has the form \(f(z) = z + \sum_{k=2}^{\infty} a_k z^k\). For functions \(f(z) = z + \sum_{k=2}^{\infty} a_k z^k\) and \(g(z) = z + \sum_{k=2}^{\infty} b_k z^k\), the Hadamard product (or convolution) \(f \ast g\) is defined, as usual, by \((f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k\).

This paper deals with the linear fractional differential operator \(D_{\lambda}^{\alpha,\nu}\) for complex numbers \(\alpha, \lambda\), where

\[
D_{\lambda}^{\alpha,\nu} f(z) = \left( \left[D_{\lambda}^{\alpha,\nu} + D_{\lambda}^{\alpha,\nu} + \ldots + D_{\lambda}^{\alpha,\nu} \right] f(z) \right)
\]

\[
D_{\lambda}^{\alpha,\nu} = \Gamma(2 - \alpha) z^{\alpha} D_{\lambda}^{\alpha,\nu} f(z) \ast g_{\lambda}(z), \ \alpha \neq 2, 3, 4, \ldots
\]

\[
g_{\lambda}(z) = z - (1 - \lambda) z^2 = z + \sum_{k=2}^{\infty} [1 + \lambda(k - 1)] z^k,
\]

\[
D_{\lambda}^{\alpha,\nu} f(z) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dz} \int_{0}^{z} \frac{f(t)}{(z-t)^{\alpha}} dt, 0 \leq \alpha < 1
\]

The operator \(D_{\lambda}^{\alpha,\nu}\) was introduced by [3]. Using the fractional derivative of order \(\alpha, D_{\lambda}^{\alpha,\nu}\) [14], Owa and Srivastava [15] introduced the operator \(\Omega^{\alpha} : A \rightarrow A\), which is known as an extension of fractional derivative and fractional integral, as follows

\[
\Omega^{\alpha} f(z) = \Gamma(2 - \alpha) z^{\alpha} D_{\lambda}^{\alpha,\nu} f(z), \ \alpha \neq 2, 3, 4
\]

\[
= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1) \Gamma(2 - \alpha)}{\Gamma(k+1 - \alpha)} a_k z^k
\]

\[
= \varphi(2, 2 - \alpha; z) \ast f(z)
\]

In [3], the linear fractional differential operator \(D_{\lambda}^{\alpha,\nu} : A \rightarrow A\) is defined as follows,

\[
(1.5) \quad D_{\lambda}^{\alpha,\nu} f(z) = z + \sum_{k=2}^{\infty} \left( \frac{\Gamma(k+1) \Gamma(2 - \alpha)}{\Gamma(k+1 - \alpha)} (1 + \lambda(k - 1)) \right) a_k z^k
\]

When \(\alpha = 0\), we get Al-Oboudi differential operator [2], when \(\alpha = 0\) and \(\lambda = 1\), we get Salagean differential operator [20] and when \(n = 1\) and \(\lambda = 0\), we get Owa-Srivastave fractional differential operator [15].

We now introduce the linear fractional differential operator \(D_{\lambda}^{\alpha,\nu} \varphi_{v,b,d} : A \rightarrow A\)

\[
(1.6) \quad D_{\lambda}^{\alpha,\nu} \varphi_{v,b,d}(z) = 2^{\nu} \Gamma \left( v + \frac{b+1}{2} \right) D_{\lambda}^{\alpha,\nu} \left[ z^{1-v/2} u_{v,b,d}(\sqrt{z}) \right]
\]
Various general families of integral operators were introduced and studied earlier in Geometric function theory, such as (see [21], [22], [7], [23], [12], [13] and [18])

\begin{equation}
H_{\alpha_1,\alpha_2,\ldots,\alpha_m,\beta}(z) = \left[\beta \int_0^z t^{\beta-1} \Pi_{i=1}^m \left( \frac{h_i(t)}{t} \right) \frac{dt}{t} \right]^{1/2}
\end{equation}

\begin{equation}
F_{m,\gamma}(z) = \left[ (m\gamma + 1) \int_0^z \Pi_{i=1}^m \left( f_i(t) \right)^\gamma dt \right]^{1/(m\gamma+1)}
\end{equation}

\begin{equation}
G_\lambda(z) = \left[ \lambda \int_0^z t^{\lambda-1} \left( e^{g(t)} \right)^\lambda dt \right]^{1/\lambda}
\end{equation}

where the functions \( h_1, \ldots, h_m, f_1, \ldots, f_m \) and \( g \) belong to the class \( \mathcal{A} \) and the parameters \( \alpha_1, \ldots, \alpha_m \in \mathbb{C} \setminus \{0\} \) and \( \beta, \gamma, \lambda \) are complex numbers such that the integrals in (1.7), (1.8) and (1.9) exist. Here and throughout this paper every many valued function is taken with the principle branch.

Two of the most important and known univalence criteria for analytic functions defined in the open unit disk \( U \) were obtained by Ahlfors [1] and Becker [9] and by Becker (see [10]). Some extensions of these two univalence criteria were given by Pescar (see [17]) involving a parameter \( \beta \) and by Pascu (see [16]) involving two parameters \( \alpha \) and \( \beta \). Bulut [11] obtained sufficient conditions for the univalence of the integral operator

\begin{equation}
I_{\beta}^{\alpha_1,\gamma}(f_1, \ldots, f_m) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^m \left( \frac{D_{\alpha_i}^{\alpha_1,\gamma} f_i(t)}{t} \right)^{\alpha_i} dt \right\}^{1/\beta}
\end{equation}

(\( z \in U \), \( n \in \mathbb{N}_0 \), \( m \in \mathbb{N} \), \( \beta \in \mathbb{C} \) with \( \Re(\beta) > 0 \) and \( \alpha_i \in \mathbb{C} \)(\( i \in \{1, \ldots, m\} \)).

Recently, Szasz and Kupan [24] investigated the univalence of the normalized Bessel function of the first kind \( \phi_k : \mathbb{D} \to \mathbb{C} \), defined by

\[
\phi_k(z) = 2^k \Gamma(k+1) z^{1-k/2} J_k \left( z^{1/2} \right)
\]

\[= z + \sum_{n=1}^\infty \frac{(-1)^n z^{n+1}}{4^n n! (k+1)_n},\]

Families of integral operators of types (1.7) and (1.9) which involve the normalized forms of the generalized Bessel functions of the first kind have been investigated in [4-6] and [8] to obtain sufficient conditions for integral operators to be univalent in the open unit disk. The main object of this paper is to extend and improve a forementioned results of [13] and [17]. For this purpose, it is organized as follows. In Section 2, we prove inequalities of compositions of the linear fractional differential operator \( D_{\lambda}^{\alpha,\gamma} \) with the Bessel function of the first kind of order \( \nu \) in terms of generalized hypergeometric function. In Section 3, we present univalence criteria for compositions of the linear fractional differential operator \( D_{\lambda}^{\alpha,\gamma} \) with the Bessel function in (1.1), (1.9) and (1.10). In Section 4, we discuss and compare our special cases with those in [13] and [7].

2. Preliminary Lemmas

The following result can be obtained by setting \( k_{\nu,c}(z) = D_{\lambda}^{\alpha,\gamma} \varphi_{\nu,b,d}(z) \) in the known result due to [19], Theorem 1).

**Theorem 1.** Let \( \nu \neq -1, -2, -3, \ldots \) and \( b, d \in \mathbb{C} \). If \( D_{\lambda}^{\alpha,\gamma} \varphi_{\nu,b,d} \) satisfies any one of the following inequalities:

\begin{equation}
\left| \frac{z^2 (D_{\lambda}^{\alpha,\gamma} \varphi_{\nu,b,d})'(z)}{(D_{\lambda}^{\alpha,\gamma} \varphi_{\nu,b,d})'(z)} - \frac{2z (D_{\lambda}^{\alpha,\gamma} \varphi_{\nu,b,d})'(z)}{(D_{\lambda}^{\alpha,\gamma} \varphi_{\nu,b,d})'(z)} \right| < 1 \quad (z \in U),
\end{equation}
and the inequalities

By using the well known triangle inequality

Proof. The following notations will be useful in the sequel:

\[ D \]

results.

\[ \lambda \]

\[ \phi \]

\[ n, \gamma \]

\[ k \]

\[ M \]

\[ \delta \]

\[ \alpha \]

\[ z \]

\[ \lambda \]

\[ \gamma \]

\[ n \]

\[ v, b \]

\[ d \]

\[ \lambda \]

\[ \gamma \]

\[ m \]

\[ a \]

\[ k \]

\[ w \]

\[ N \]

\[ \mathcal{N} \]

\[ \mathcal{M} \]

\[ \mathcal{U} \]

The following result is mainly based on [8] and is one of the crucial facts in the proofs of our main results.

Theorem 2. If the parameters \( v, b \in \mathbb{R} \) and \( d \in \mathbb{C} \) are so constrained that

\[ k = v + \frac{b + 1}{2} > \max \left\{ 0, \frac{\beta(d)}{4(1 + \frac{1}{\lambda})^n(3 - \gamma)^n} - 1 \right\}, \]

then the function

\[ D_{\lambda}^{n, \gamma} \varphi_{v, b, d}(z) \]

satisfies the following inequality:

\[ \frac{4M k \delta(k + 1) - \alpha(2k + 1)\beta(d) + \mathcal{N} k \delta(k + 1) - \beta(d)}{4M k} \leq \left| \frac{D_{\lambda}^{n, \gamma} \varphi_{v, b, d}(z)}{z} \right| \leq \frac{32[\delta(k)]^2 - |\beta(d)|^2}{8\delta(k)[4\delta(k) - \beta(d)]} \] \( (z \in \mathbb{U}). \)

Proof. By using the well known triangle inequality

\[ |z_1 - z_2| \geq ||z_1| - |z_2|| \]

and the inequalities

\[ (k)_n m! \geq 2k^n, \forall n \geq 2 \]

\[ (s)_n \geq s^n \]
we obtain for all \( z \in \mathbb{U} \)

\[
\left| \frac{D^{n,\gamma}_{\lambda} \varphi_{v,b,d}(z)}{z} \right| = \left| 1 + \sum_{m=1}^{\infty} \frac{(-d)^m}{m! m^m(k)_m} \left[ \frac{(1 + \frac{1}{\lambda})_m}{m \Lambda \Gamma(2 - \gamma)_m} \right]^n z^m \right|
\]

\[
\geq 1 - \sum_{m=1}^{\infty} \frac{|d|^m}{m! m^m(k)_m} \left[ \frac{(1 + \frac{1}{\lambda})_m}{m \Lambda \Gamma(2 - \gamma)_m} \right]^n
\]

\[
\geq 1 - \left( \frac{2(1 + \lambda)}{(2 - \gamma)} \right)^n \frac{|d|}{4k} \sum_{m=1}^{\infty} \frac{|d|^{m-1}}{m^m(k+1)_m} \left[ \frac{(2 + \frac{1}{\lambda})_{m-1}}{(1 + \frac{1}{\lambda})_{m-1} (3 - \gamma)_{m-1}} \right]^n
\]

\[
\geq 1 - \left( \frac{2(1 + \lambda)}{(2 - \gamma)} \right)^n \frac{|d|}{4k} \left[ 1 + \frac{1}{2} \sum_{m=2}^{\infty} \left| \frac{|d|}{4(k+1) \Gamma(3 - \gamma)^n} \right|^{m-1} \right]
\]

\[
= \frac{4M k \delta(k+1) - \alpha(2k+1)\beta(d) + N [\beta(d)]^2}{M k[4\delta(k+1) - \beta(d)]}
\]

Which is positive if

\[
32M k \delta(k+1) - 8\alpha(2k+1)\beta(d) + N [\beta(d)]^2 > 0
\]

Similarly, by using the triangle inequality

\[ |z_1 + z_2| \leq |z_1| + |z_2| \]

and the inequalities:

\[
(s)_n n! \geq 2s^n, \forall n \geq 2
\]

\[
(s)_n \geq s^n
\]

\[
(s)_n \leq (s + n - 1)^n
\]

we obtain for all \( z \in \mathbb{U} \)

\[
\left| \frac{D^{n,\gamma}_{\lambda} \varphi_{v,b,d}(z)}{z} \right| = \left| 1 + \sum_{m=1}^{\infty} \frac{(-d)^m}{m! m^m(k)_m} \left[ \frac{(1 + \frac{1}{\lambda})_m}{m \Lambda \Gamma(2 - \gamma)_m} \right]^n z^m \right|
\]

\[
\leq 1 + \sum_{m=1}^{\infty} \frac{|d|^m}{m! m^m(k)_m} \left[ \frac{(1 + \frac{1}{\lambda})_m}{m \Lambda \Gamma(2 - \gamma)_m} \right]^n
\]

\[
\leq 1 + \left( \frac{2(1 + \lambda)}{(2 - \gamma)} \right)^n \frac{|d|}{4k} + \frac{1}{2} \sum_{m=2}^{\infty} \left| \frac{|d|}{4(k+1) \Gamma(3 - \gamma)^n} \right|^{m-1}
\]

\[
= \frac{32 |\delta(k)|^2 - [\beta(d)]^2}{8\delta(k)[4\delta(k) - \beta(d)]}
\]

Thus, the proof is complete. \( \square \)

**Theorem 3.** If the parameters \( v, b \in \mathbb{R} \) and \( d \in \mathbb{C} \) are so constrained that

\[
k > \max \left\{ 0, \frac{\beta(d)}{4(1 + \frac{1}{\lambda})^n(3 - \gamma)^n} - 1 \right\},
\]

then the function

\[
D^{n,\gamma}_{\lambda} \varphi_{v,b,d}(z) : \mathbb{U} \to \mathbb{C}
\]

satisfies the following inequalities:

\[
(D^{n,\gamma}_{\lambda} \varphi_{v,b,d}(z))' - \frac{D^{n,\gamma}_{\lambda} \varphi_{v,b,d}(z)}{z} \leq \frac{N \delta(k+1)\beta(d)}{M k[4\delta(k+1) - \beta(d)]} \quad (z \in \mathbb{U}),
\]
we obtain for all $z$

\begin{equation}
\left| \frac{z (D^1_{\lambda} \varphi_{v,b,d}(z))'}{D^1_{\lambda} \varphi_{v,b,d}(z)} - 1 \right| \leq \frac{8N \delta(k+1)\beta(d)}{32M k \delta(k+1) - 8\alpha(2k+1)\beta(d) + N^2 [\beta(d)]^2} \quad (z \in \mathbb{U}),
\end{equation}

\begin{equation}
\frac{4M k \delta(k+1) - \alpha(3k + 2)\beta(d)}{M k [4\delta(k+1) - \beta(d)]} \leq \left| z \left( D^1_{\lambda} \varphi_{v,b,d}(z) \right)' \right| \leq \frac{4M k \delta(k+1) + \alpha(2k+1)\beta(d)}{M k [4\delta(k+1) - \beta(d)]} \quad (z \in \mathbb{U}),
\end{equation}

\begin{equation}
\left| z^2 \left( D^1_{\lambda} \varphi_{v,b,d}(z) \right)'' \right| \leq \frac{N \beta(d) 4\delta(k+1) + \beta(d)}{2M k [4\delta(k+1) - \beta(d)]} \quad (z \in \mathbb{U})
\end{equation}

**Proof.** We first prove the assertion (2.7) of Theorem (3) when $\lambda > 0$. Indeed, by using the following:

\begin{align*}
(n - 1)! &= (1)_{n - 1} \\
(s)_n &\geq s^n \\
(s)_n &\leq (s + n - 1)^n
\end{align*}

then we have,

\begin{align*}
(n - 1)!(k + 1)_{n - 1} &= (1)_{n - 1} (k + 1)_{n - 1} \\
&\geq (k + 1)^{n - 1}
\end{align*}

we obtain for all $z \in \mathbb{U}$

\begin{align*}
\left| \frac{(D^1_{\lambda} \varphi_{v,b,d}(z))'}{z} - \frac{(D^1_{\lambda} \varphi_{v,b,d}(z))'}{z} \right| &= \left| \sum_{m=1}^{\infty} \frac{m(-d)^m}{m!} \left( \frac{1}{k} \right)_m \left( \frac{2}{k} \right)_m \right| ^n z^m \\
&\leq \sum_{m=1}^{\infty} \frac{m[d]^m}{m!} \left( \frac{1}{k} \right)_m \left( \frac{2}{k} \right)_m \\
&\leq \left( \frac{1}{k} \right)^n \left( \frac{2}{k} \right)^n \\
&\times \frac{d}{4k} \sum_{m=1}^{\infty} \frac{|d|^{m-1}}{4^{m-1} (m-1)!} \left( \frac{2}{k} \right)_{m-1} (3)_{m-1} \\
&\leq \left( \frac{2(1 + \lambda)}{(2 - \gamma)} \right)^n \frac{d}{4k} \sum_{m=1}^{\infty} \frac{|d|^{m-1}}{4^{m-1} (k+1)^{m-1}} \left( \frac{2}{k} \right)_{m-1} (3)_{m-1} \\
&\leq \left( \frac{2(1 + \lambda)}{(2 - \gamma)} \right)^n \frac{d}{4k} \sum_{m=1}^{\infty} \frac{|d|^{m-1}}{4^{m-1} (k+1)^{m-1}} \left( \frac{2}{k} \right)_{m-1} (3)_{m-1} \\
&= \frac{N \delta(k+1)\beta(d)}{M k [4\delta(k+1) - \beta(d)]}
\end{align*}

When $\lambda = 0$ using the same technique we get

\begin{align*}
\left| \frac{(D^1_{\lambda} \varphi_{v,b,d}(z))'}{z} \right| &\leq \frac{2^n \delta(k+1)\beta(d)}{M k [4\delta(k+1) - \beta(d)]}
\end{align*}

Next, by combining theorem (2.4) and the first assertion, we immediately see that the second assertion of theorem (3) holds true for all $z \in \mathbb{U}$ if

$$32M k \delta(k+1) - 8\alpha(2k+1)\beta(d) + N^2 [\beta(d)]^2 > 0$$
In order to prove the assertion (2.9) of theorem (3), we make use of the following inequalities

\[(n + 1) \leq 2^n\]

\[n! = (2)^{n-1}\]

\[\frac{1}{k} (k)_n = (k + 1)_{n-1}, \quad n \in \mathbb{N}\]

\[n!(k + 1)_{n-1} = (2)^{n-1} (k + 1)_{n-1}\]

\[\geq [2(k + 1)]^{n-1}\]

\[(s)_n \geq s^n\]

\[(s)_n \leq (s + n - 1)^n\]

we thus find that

\[|z (D_{p,\gamma}^n D_{v,b,d} z)^{\prime}| = \left| z + \sum_{m=1}^{\infty} \frac{(m + 1)(-d)^m}{m!4^m (k)_m} \left[ \frac{(1 + \frac{1}{\lambda})_m}{(\lambda)_m (2 - \gamma)_m} \right]^{2m+1} \right| \]

\[\leq 1 + \sum_{m=1}^{\infty} \frac{(m + 1)d^m}{m!4^m (k)_m} \left[ \frac{(1 + \frac{1}{\lambda})_m}{(\lambda)_m (2 - \gamma)_m} \right]^{2m+1}\]

\[= 1 + \left( \frac{2(1 + \lambda)}{\pi} \right)^{m+1} \sum_{m=1}^{\infty} \frac{(m + 1)d^m}{m!4^m (k)_m} \left[ \frac{(1 + \frac{1}{\lambda})_m}{(\lambda)_m (2 - \gamma)_m} \right]^{2m+1}\]

\[\leq 1 + \frac{N\beta(d)}{2\lambda k} \sum_{m=1}^{\infty} \frac{|d| (\frac{1}{\lambda} + m)^n (m + 1)^n}{\beta(k + 1) (1 + \frac{1}{\lambda})^n (3 - \gamma)^n}\]

\[= 4\lambda k \delta(k + 1) + \alpha(k + 2)\beta(d)\]

which is positive if

\[4\lambda k \delta(k + 1) + \alpha(k + 2)\beta(d) > 0\]

Similarly, by using the inequalities

\[|z1 - z2| \geq ||z1| - |z2||\]

\[(s)_n \geq s^n\]

\[(s)_n \leq (s + n - 1)^n\]

\[2n! \geq (n + 1)\]

we have

\[|z (D_{p,\gamma}^n D_{v,b,d} z)^{\prime}| = \left| z + \sum_{m=1}^{\infty} \frac{(m + 1)(-d)^m}{m!4^m (k)_m} \left[ \frac{(1 + \frac{1}{\lambda})_m}{(\lambda)_m (2 - \gamma)_m} \right]^{2m+1} \right| \]

\[\geq 1 - \sum_{m=1}^{\infty} \frac{(m + 1)d^m}{m!4^m (k)_m} \left[ \frac{(1 + \frac{1}{\lambda})_m}{(\lambda)_m (2 - \gamma)_m} \right]^{2m+1}\]

\[= 1 - \left( \frac{2(1 + \lambda)}{\pi} \right)^{m+1} \sum_{m=1}^{\infty} \frac{(m + 1)d^m}{m!4^m (k)_m} \left[ \frac{(1 + \frac{1}{\lambda})_m}{(\lambda)_m (2 - \gamma)_m} \right]^{2m+1}\]

\[\leq 1 - \frac{N\beta(d)}{2\lambda k} \sum_{m=1}^{\infty} \frac{|d| (\frac{1}{\lambda} + m)^n (m + 1)^n}{\beta(k + 1) (1 + \frac{1}{\lambda})^n (3 - \gamma)^n}\]

\[= 4\lambda k \delta(k + 1) - \alpha(3k + 2)\beta(d)\]

which is positive if

\[4\lambda k \delta(k + 1) - \alpha(3k + 2)\beta(d) > 0\]
We now prove the assertion (2.11) of Theorem 3 by using the following

\[(m - 1)! \geq 2^{m-2}\]
\[(k + 1)_{m-1} \geq (k + 1)^{m-1}\]
\[m + 1 \leq 2^m\]
\[4(n - 1)! \geq (n + 1)\]

Thus we have

\[|z^2 (D_n^{\gamma} \varphi_{v,b,d}(z))^n| = \sum_{m=1}^{\infty} \frac{(m + 1)_{m} (-d)^m}{m!4^m(k)_{m}} \left[ \frac{(1 + \frac{k}{n}) (2)^m}{(1 + \frac{k}{n}) (2 - \gamma)_{m}} \right]^{n} z^{m+1} \]

\[\leq \sum_{m=1}^{\infty} \frac{(m + 1)!d}{(m - 1)!4^m(k)_{m}} \left[ \frac{(1 + \frac{k}{n}) (2)^m}{(1 + \frac{k}{n}) (2 - \gamma)_{m}} \right]^{n} \]

\[= \frac{(2(1 + \lambda)}{2 - \gamma} \right)^n \frac{|d|^n}{k} \left[ \frac{1}{2} + \sum_{m=2}^{\infty} \frac{|d|^{m-1} (2 + \frac{k}{n})_{m-1} (3(n - 1))_{m-1}}{4(k + 1)_{m} (1 + \frac{k}{n})_{m} (3 - \gamma)_{m}} \right] \]

\[= \frac{N \beta(d)}{2M} \frac{k}{4\delta(k + 1) - \beta(d)} \]

Finally, by combining the inequalities (2.9) and (2.11) we deduce that (γ) holds true for all \(z \in U\). Thus the proof is completed.

**Remark 1.** Taking \(n = 0\) in the Theorem 3, we obtain a similar result to that in [8].

**Remark 2.** Taking \(n = 0\) in the Theorem 3, we obtain a similar result to that in [13].

### 3. Univalence Criteria

In our present investigation, we need these two univalence criteria which we recall here as Lemmas (1) and (2) see [17, 18].

**Lemma 1.** (see [17]). Let \(\eta\) and \(c\) be complex numbers such that

\[\Re(\eta) > 0\] and \(|c| \leq 1\) \((c \neq -1)\).

If the function \(f \in A\) satisfies the following inequality:

\[|c|z^{2\eta} + (1 + |z|^{2\eta}) \frac{zf''(z)}{\eta f'(z)} \leq 1\] \((z \in U)\),

then the function \(F_\eta\) defined by

\[(3.1)\]

\[F_\eta(z) = \left( \eta \int_0^z t^{\eta-1} f'(t) dt \right)^{1/\eta}\]

is in the class \(S\) of normalized univalent functions in \(U\).

**Lemma 2.** (see [16]). If \(f \in A\) satisfies the following inequality:

\[
\left( \frac{1 - |z|^{2\Re(\mu)}}{\Re(\mu)} \right) \frac{|zf''(z)|}{|f'(z)|} \leq 1
\]

\((z \in U; \Re(\mu) > 0)\),
then, for all $\eta \in \mathbb{C}$ such that $\Re(\eta) \geq \Re(\mu)$, the function $F_\eta$ defined by (3.1) is in the class $S$ of normalized univalent functions in $U$.

Lemma (3) below is a consequence of the above-mentioned Becker’s univalence criterion (see [18]) and the well-known Schwarz lemma.

**Lemma 3.** (see [18].) Let the parameter $\zeta \in \mathbb{C}$ and $\theta \in \mathbb{R}$ be so constrained that

$$\Re(\zeta) \geq 1, \quad \theta > 1 \quad \text{and} \quad 2\theta|\zeta| \leq 3\sqrt{3}.$$

If the function $q \in \mathcal{A}$ satisfies the following inequality:

$$|zq'(z)| \leq \theta \quad (z \in U),$$

then the function $G_\zeta : U \to \mathbb{C}$, defined by

$$G_\zeta(z) = \left[ | C \int_0^z t^{\zeta-1} \left( e^{\theta t} \right)^{\zeta} dt \right]^{1/\zeta},$$

is in the class $S$ of normalized univalent functions in $U$.

In the past two decades, many authors have determined various sufficient conditions for the univalence of various general families of integral operators such as (see [21], [22], [7], [23], [12], [13] and [18]).

In this paper we will focus on some integral operators of the following types [17], [18] and [14] involving the normalized forms of the generalized Bessel functions of the first kind as follows

$$\mathcal{H}_{v_1,\ldots,v_m,b,d,\mu_1,\ldots,\mu_m,\eta}(z) = \left[ \eta \int_0^z t^{\eta-1} \prod_{j=1}^m \left( D_{\lambda_n}^{\mu_j} \varphi_{v_j,b,d}(t) / t \right)^{1/\mu_j} dt \right]^{1/\eta},$$

$$\mathcal{F}_{v_1,\ldots,v_m,b,d,\mu_1,\ldots,\mu_m,\eta}(z) = \left( (m+1) \int_0^z \prod_{j=1}^m \left( D_{\lambda_n}^{\mu_j} \varphi_{v_j,b,d}(t) \right)^{\mu} dt \right)^{1/(m+1)},$$

$$\mathcal{G}_{v_1,\ldots,v_m,b,d,\zeta}(z) = \left[ | \zeta \int_0^z t^{\zeta-1} \left( e^{D_{\lambda_n}^{\mu_j} \varphi_{v_j,b,d}(t)} \right)^{\zeta} dt \right]^{1/\zeta}.$$

**Theorem 4.** Let the parameters $v_1,\ldots,v_m, b \in \mathbb{R}$ and $d \in \mathbb{C}$ be so constrained that

$$k_j = v_j + b + 1 > \frac{\beta(d)}{4(1 + \frac{\beta(d)}{2})^n(3 - \gamma)^n} - 1 \quad (j = 1,\ldots,m).$$

Consider the functions $D_{\lambda_n}^{\mu_j} \varphi_{v_j,b,d} : U \to \mathbb{C}$ defined by

$$D_{\lambda_n}^{\mu_j} \varphi_{v_j,b,d}(z) = 2^\mu \Gamma \left( v_j + \frac{b + 1}{2} \right) D_{\lambda_n}^{\mu_j} \left[ z^{1-v_j/2} w_{v_j,b,d}(\sqrt{z}) \right].$$

Also let

$$k = \min\{k_1,\ldots,k_m\}, \Re(\eta) > 0, \quad c \in \mathbb{C} \setminus \{-1\} \quad \text{and} \quad \mu_j \in \mathbb{C} \setminus \{0\} \quad (j = 1,\ldots,m).$$

Moreover, suppose that these numbers satisfy the following inequality:

$$|c| + \frac{8N \delta (k+1)/6 \delta (d)}{32M k \delta (k+1) - 8 (2k+1)/6 \delta (d) + N \beta(d)/2} \sum_{j=1}^m \frac{1}{|\eta\mu_j|} \leq 1.$$

Then the function $\mathcal{H}_{v_1,\ldots,v_m,b,d,\mu_1,\ldots,\mu_m,\eta}(z) : U \to \mathbb{C}$, defined by

$$\mathcal{H}_{v_1,\ldots,v_m,b,d,\mu_1,\ldots,\mu_m,\eta}(z) = \left[ \eta \int_0^z t^{\eta-1} \prod_{j=1}^m \left( D_{\lambda_n}^{\mu_j} \varphi_{v_j,b,d}(t) / t \right)^{1/\mu_j} dt \right]^{1/\eta}$$

is in the class $S$ of normalized univalent functions in $U$. 

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Proof. We begin by setting $\eta = 1$ in the function $\mathcal{H}_{v_1, \ldots, v_m, b, d, \mu_1, \ldots, \mu_m, \eta} : \mathbb{U} \to \mathbb{C}$, as follows:

$$
\mathcal{H}_{v_1, \ldots, v_m, b, d, \mu_1, \ldots, \mu_m, \eta}(z) = \int_0^z \prod_{j=1}^m \left( \frac{D^{n \gamma} \varphi_{v_j, b, d}(t)}{t} \right)^{1/\mu_j} dt.
$$

First of all, we observe that, since $D^{n \gamma} \varphi_{v_j, b, d} \in \mathcal{A}$, that is

$$
D^{n \gamma} \varphi_{v_j, b, d}(0) = (D^{n \gamma} \varphi_{v_j, b, d})'(0) - 1 = 0,
$$

we have

$$
\mathcal{H}_{v_1, \ldots, v_m, b, d, \mu_1, \ldots, \mu_m, \eta} = \mathcal{H}'_{v_1, \ldots, v_m, b, d, \mu_1, \ldots, \mu_m, \eta}(0) - 1 = 0.
$$

On the other hand, it is easy to see that

$$
\mathcal{H}'_{v_1, \ldots, v_m, b, d, \mu_1, \ldots, \mu_m, \eta}(0) = \prod_{j=1}^m \left( \frac{D^{n \gamma} \varphi_{v_j, b, d}(z)}{z} \right)^{1/\mu_j}
$$

we thus find

$$
z \frac{\mathcal{H}''_{v_1, \ldots, v_m, b, d, \mu_1, \ldots, \mu_m, \eta}}{\mathcal{H}'_{v_1, \ldots, v_m, b, d, \mu_1, \ldots, \mu_m, \eta}}(z) = \sum_{j=1}^m \frac{1}{\mu_j} \left( \frac{z (D^{n \gamma} \varphi_{v_j, b, d}(z))'}{D^{n \gamma} \varphi_{v_j, b, d}(z)} - 1 \right)
$$

Thus, by using the inequality (2.8) of Theorem (3) 7 for each $v_j = (j = 1, \ldots, m)$, we obtain

$$
\left| z \frac{\mathcal{H}''_{v_1, \ldots, v_m, b, d, \mu_1, \ldots, \mu_m, \eta}}{\mathcal{H}'_{v_1, \ldots, v_m, b, d, \mu_1, \ldots, \mu_m, \eta}}(z) \right| \leq \sum_{j=1}^m \frac{1}{\mu_j} \left| \frac{z (D^{n \gamma} \varphi_{v_j, b, d}(z))'}{D^{n \gamma} \varphi_{v_j, b, d}(z)} - 1 \right|
$$

$$
\leq \sum_{j=1}^m \frac{1}{\mu_j} \left( \frac{8N \partial (k_j + 1)^2 \beta(d)}{32M k_j \partial (k_j + 1) - 8 \alpha (2k_j + 1)^2 \beta(d) + N^2}\right)
$$

$$
\leq \sum_{j=1}^m \frac{1}{\mu_j} \left( \frac{8N \partial (k + 1)^2 \beta(d)}{32M k\partial (k + 1) - 8 \alpha (2k + 1)^2 \beta(d) + N^2}\right)
$$

$$
\left( z \in \mathbb{U}; k = \min\{k_1, \ldots, k_m\}; k_j = v_j + \frac{b+1}{2} > \frac{\beta(d)}{4(1+\frac{\beta(d)}{k(3-\gamma)^2})} - 1 \quad (j = 1, \ldots, m) \right).
$$

Here we have used the fact that the function $\phi : \left( \frac{\beta(d)}{4(1+\frac{\beta(d)}{k(3-\gamma)^2})} - 1, \infty \right) \to \mathbb{R}$, defined by

$$
\phi(x) = \frac{8N \partial x^2 \beta(d)}{32M x \partial (x + 1) - 8 \alpha (2x + 1)^2 \beta(d) + N^2},
$$

is decreasing and, consequently, we have

$$
\frac{8N \partial (k_j + 1)^2 \beta(d)}{32M k_j \partial (k_j + 1) - 8 \alpha (2k_j + 1)^2 \beta(d) + N^2}\leq \frac{8N \partial (k + 1)^2 \beta(d)}{32M k\partial (k + 1) - 8 \alpha (2k + 1)^2 \beta(d) + N^2}, \quad (j = 1, \ldots, m).
$$

Finally, by using the triangle inequality and the assertion of Theorem (4), we obtain

$$
\left| |z|^{2\eta} (1 - |z|^{2\eta}) \frac{z \mathcal{H}''_{v_1, \ldots, v_m, b, d, \mu_1, \ldots, \mu_m, \eta}}{\eta \mathcal{H}'_{v_1, \ldots, v_m, b, d, \mu_1, \ldots, \mu_m, \eta}}(z) \right| \leq |c| + \frac{8N \partial (k + 1)^2 \beta(d)}{32M k\partial (k + 1) - 8 \alpha (2k + 1)^2 \beta(d) + N^2} \sum_{j=1}^m \frac{1}{\eta \mu_j} \leq 1,
$$

which, in view of Lemma (4), implies that $\mathcal{H}_{v_1, \ldots, v_m, b, d, \mu_1, \ldots, \mu_m, \eta} \in \mathcal{S}$.

This evidently completes the proof of Theorem (4).
Moreover, suppose that the functions \( D(3.5) \) in Theorem (4), we immediately arrive at the following application of Theorem (4).

Let us consider the function \( F(3.6) \). The key tools in the proof are Lemma (2) and the inequality (2.8). Upon setting \( \tilde{v} \). Also let \( v \in A \). Then the function \( \tilde{F}(3.3) \) of normalized univalent functions in \( A \).

Our second result in this section provides sufficient conditions for an integral operator of the type (1.3). The second key tool is the proof of Lemma (2) and the inequality (2.8) of Theorem (3).

**Theorem 5.** Let the parameters \( v_1, \ldots, v_m, b, c, d, \eta \) and \( k_j (j = 1, \ldots, m) \) be prescribed as in Theorem (3). Also let

\[
|\mu| = \frac{1}{m} \left( \frac{32 M k \delta(k+1) - 8 \alpha(2k+1) \beta(d) + N |\beta(d)|^2}{8 N \delta(k+1) \beta(d)} \right) \eta \mu(\mu)
\]

Then the function \( \tilde{F}(3.6) \) of normalized univalent functions in \( U \).

**Proof.** Let us consider the function \( \tilde{F}(3.6) \) of normalized univalent functions in \( U \).

Observe that \( \tilde{F}(3.6) \) is prescribed as in Theorem (3), the assertion of Theorem (3) and the fact that

\[
\frac{8 N \delta(k+1) \beta(d)}{32 M k_j \delta(k_j+1) - 8 \alpha(2k_j+1) \beta(d) + N |\beta(d)|^2} \leq \frac{8 N \delta(k+1) \beta(d)}{32 M k \delta(k+1) - 8 \alpha(2k+1) \beta(d) + N |\beta(d)|^2} (j = 1, \ldots, m),
\]

Upon setting

\[
\mu_1 = \ldots = \mu_m = \mu
\]
in Theorem (4), we immediately arrive at the following application of Theorem (4).

**Corollary 1.** Let the parameters \( v_1, \ldots, v_m, b, c, d, \eta \) and \( k_j (j = 1, \ldots, m) \) be prescribed as in Theorem (3). Also let

\[
k = \min\{k_1, \ldots, k_m\} \quad \text{and} \quad \mu \in \mathbb{C}\setminus\{0\}
\]

Moreover, suppose that the functions \( D(3.5) \) in Theorem (4), we immediately arrive at the following application of Theorem (4).

Let \( \tilde{v} \) be prescribed as in Theorem (3), the assertion of Theorem (3) and the fact that

\[
\frac{8 N \delta(k_j+1) \beta(d)}{32 M k_j \delta(k_j+1) - 8 \alpha(2k_j+1) \beta(d) + N |\beta(d)|^2} \leq \frac{8 N \delta(k+1) \beta(d)}{32 M k \delta(k+1) - 8 \alpha(2k+1) \beta(d) + N |\beta(d)|^2} (j = 1, \ldots, m),
\]
we have
\[
\frac{1 - |z|^{2\Re(\mu)}}{\Re(\mu)} \left| \zeta \tilde{F}_{\nu_{1}, \ldots, v_{m} b, d, m, \mu}(z) \right| \leq \frac{|\mu|}{\Re(\mu)} \sum_{j=1}^{m} \left| z \left( \frac{\partial^n \varphi_{v_j b, d}(z)}{\partial^n \varphi_{v_j b, d}(z)} \right)^j - 1 \right|
\]
\[
\leq \frac{m|\mu|}{\Re(\mu)} \left( \frac{8N \delta(k+1)\beta(d)}{32M k \delta(k+1) - 8\alpha(2k+1)\beta(d) + N [\beta(d)]^2} \right) \leq 1 \quad (z \in \mathbb{U}).
\]
Now since
\[
\Re(m\mu + 1) > \Re(\mu) \quad (m \in \mathbb{N})
\]
and the function \( F_{\nu_{1}, \ldots, v_{m} b, d, m, \mu}(z) \) can be rewritten in the form:
\[
F_{\nu_{1}, \ldots, v_{m} b, d, m, \mu}(z) = \left( (m\mu + 1) \int_0^z t^{m\mu} \prod_{j=1}^m \left( \frac{\partial^n \varphi_{v_j b, d}(t)}{t} \right)^\mu \, dt \right)^{1/(m\mu + 1)}
\]
which in view of Lemma (2), implies that \( F_{\nu_{1}, \ldots, v_{m} b, d, m, \mu}(z) \in \mathcal{S} \). This evidently completes the proof of Theorem (5). 

Choosing \( m = 1 \) in Theorem (5), we have the following result.

**Corollary 2.** Let the parameter \( v, b \in \mathbb{R} \) and \( d \in \mathbb{C} \) be so constrained that
\[
k = v + \frac{b+1}{2} > \frac{\beta(d)}{4(1 + \frac{3}{\lambda})^n(3 - \gamma)^n} - 1.
\]
Consider the function \( \frac{D^n \varphi_{v, b, d}}{D^n \varphi_{v, b, d}} : \mathbb{U} \to \mathbb{C} \) defined by (1.6). Moreover, suppose that \( \Re(\mu) > 0 \) and
\[
|\mu| \leq \left( \frac{32M k \delta(k+1) - 8\alpha(2k+1)\beta(d) + N |\beta(d)|^2}{8N \delta(k+1)\beta(d)} \right) \Re(\mu).
\]
Then the function \( \varphi_{v, b, d, \mu}(z) : \mathbb{U} \to \mathbb{C} \), defined by
\[
\varphi_{v, b, d, \mu}(z) = \left( (\mu + 1) \int_0^z (D^n \varphi_{v, b, d}(t))^\mu \, dt \right)^{1/(\mu + 1)},
\]
is in the class \( \mathcal{S} \) of normalized univalent functions in \( \mathbb{U} \).

By applying Lemma (3) and the inequality (3.9) of Theorem (3), we easily get the following result.

**Theorem 6.** Let the parameters \( k, b \in \mathbb{R} \) and \( d, \zeta \in \mathbb{C} \) be so constrained that
\[
k := v + \frac{b+1}{2} > \frac{\beta(d)}{4(1 + \frac{3}{\lambda})^n(3 - \gamma)^n} - 1.
\]
Consider the generalized Bessel function \( \frac{D^n \varphi_{v, b, d}}{D^n \varphi_{v, b, d}} \) defined by (1.6). If \( |\zeta| \geq 1 \) and
\[
|\zeta| \leq \frac{3\sqrt{3}M k |\delta(k+1) - \beta(d)|}{8 \lambda M k \delta(k+1) + 2 \alpha(k+2)\beta(d)},
\]
then the function \( \varphi_{v, b, d, \zeta} : \mathbb{U} \to \mathbb{C} \), defined by
\[
\varphi_{v, b, d, \zeta}(z) = \left( \zeta \int_0^z t^{\zeta-1} \left( e^{D^n \varphi_{v, b, d}(t)} \right)^\zeta \, dt \right)^{1/\zeta}
\]
is in the class \( \mathcal{S} \) of normalized univalent functions in \( \mathbb{U} \).

**Remark 3.** Taking \( n = 0 \) in the above results, we obtain the same results as (19).
4. Special Cases

Taking into account the above results, we have the following particular cases.

4.1. Bessel Functions. Choosing $b = d = 1$, in (1.1) or (1.2), we obtain the Bessel function $J_\nu(z)$ of the first kind of order $\nu$ defined by (1.3). We observe also that

$$D_{\nu}^{\gamma}\mathcal{J}_2(z) = D_{\nu}^{\gamma}\left(\frac{3\sin\sqrt{z}}{\sqrt{z}} - 3\cos\sqrt{z}\right),$$
$$D_{\nu}^{\gamma}\mathcal{J}_1(z) = D_{\nu}^{\gamma}\left(\sqrt{z}\sin\sqrt{z}\right) \text{ and } D_{\nu}^{\gamma}\mathcal{J}_{-1/2}(z) = D_{\nu}^{\gamma}(z\cos\sqrt{z}).$$

**Corollary 3.** Let the function $J_\nu: \mathbb{U} \to \mathbb{C}$ be defined by

$$J_\nu(z) = 2^\nu \Gamma(\nu + 1) z^{1-\nu/2} J_\nu(\sqrt{z}).$$

Also let the following assertions hold true:

1. Let $\nu_1, \ldots, \nu_m > -1.25$ ($m \in \mathbb{N}$). Consider the functions $D_{\nu_j}^{\gamma}J_{\nu_j}: \mathbb{U} \to \mathbb{C}$ defined by

$$D_{\nu_j}^{\gamma}J_{\nu_j}(z) = 2^\nu \Gamma(\nu_j + 1) z^{1-\nu_j/2} D_{\nu_j}^{\gamma}J_{\nu_j}(\sqrt{z}) \quad (j = 1, \ldots, m).$$

Let $\nu = \min\{\nu_1, \ldots, \nu_m\}$ and let the parameters $\eta, c, \mu_1, \ldots, \mu_m$ be as in Theorem 4. Moreover, suppose that these numbers satisfy the following inequality:

$$|c| + \frac{N}{4M} \frac{\delta(v+2)\beta(1)}{(v+1)\delta(v+2) - \alpha(v+3)\beta(1) + \left(N \frac{[\beta(1)]^2}{8} \right) / 8} \sum_{j=1}^{m} \frac{1}{|\eta \mu_j|} \leq 1.$$

Then the function $\mathcal{H}_{v_1, \ldots, v_m, \mu_1, \ldots, \mu_m, \eta}(z): \mathbb{U} \to \mathbb{C}$, defined by

$$\mathcal{H}_{v_1, \ldots, v_m, \mu_1, \ldots, \mu_m, \eta}(z) = \left[ \eta \int_0^z t^{\eta-1} \prod_{j=1}^{m} \left( \frac{D_{\nu_j}^{\gamma}J_{\nu_j}(t)}{t} \right)^{1/\mu_j} dt \right]^{1/\eta},$$

is in the class $\mathcal{S}$ of normalized univalent functions in $\mathbb{U}$. In the particular case when

$$|c| + \frac{28}{233} \frac{1}{|\eta \mu|} \leq 1,$$

the function $\mathcal{H}_{1/2, \mu, \eta}(z): \mathbb{U} \to \mathbb{C}$, defined by

$$\mathcal{H}_{1/2, \mu, \eta}(z) = \left[ \eta \int_0^z t^{\eta-1} \left( D_{\nu_j}^{\gamma}(\frac{3\sin\sqrt{t}}{\sqrt{t}} - 3\cos\sqrt{t}) \right) \left( \frac{\sqrt{t} \sin\sqrt{t}}{t} \right)^{1/\mu_j} dt \right]^{1/\eta},$$

is in the class $\mathcal{S}$ of normalized univalent functions in $\mathbb{U}$.

2. Let $\nu_1, \ldots, \nu_m > -1.25$ ($m \in \mathbb{N}$) and consider the normalized Bessel functions $J_{\nu_j}: \mathbb{U} \to \mathbb{C}$ defined by (1.1). Also let $\nu = \min\{\nu_1, \ldots, \nu_m\}$ and $R(\mu) > 0$ and suppose that these that numbers satisfy the following inequality:

$$|\mu| \leq \frac{1}{m} \left( \frac{4M (v+1) \delta(v+2) - \alpha(v+3)\beta(1) + \left(N \frac{[\beta(1)]^2}{8} \right) / 8}{N \delta(v+2)\beta(1)} \right) R(\mu).$$

Then the function $F_{v_1, \ldots, v_m, \mu}(z): \mathbb{U} \to \mathbb{C}$, defined by

$$F_{v_1, \ldots, v_m, \mu}(z) = \left[ \left( \mu + 1 \right) \prod_{j=1}^{m} \left( D_{\nu_j}^{\gamma}J_{\nu_j}(t) \right)^{\mu_j} \right]^{1/(m \mu + 1)},$$

is in the class $\mathcal{S}$ of normalized univalent functions in $\mathbb{U}$. In the particular case when

$$|\mu| \leq \frac{89}{20} R(\mu),$$

the function $F_{1/2, \mu}(z): \mathbb{U} \to \mathbb{C}$, defined by

$$F_{1/2, \mu}(z) = \left[ \left( \mu + 1 \right) \int_0^z \left( D_{\nu_j}^{\gamma} \left( \sqrt{t} \sin\sqrt{t} \right) \right)^{\mu_j} dt \right]^{1/(\mu + 1)},$$
4.2. Modified Bessel Functions. Let \( \zeta \in \mathbb{C} \) and \( \nu > -1.25 \) and consider the normalized Bessel function \( J_{\nu}(z) \) given by (4.3). If \( \Re(\zeta) \geq 1 \) and
\[
|\zeta| \leq \frac{3\sqrt{3}M (\nu + 1) [4\delta(v + 2) - \beta(1)]}{8M (\nu + 1) \delta(v + 2) + 2 \alpha(v + 3)\beta(1)},
\]
then the function \( G_{v,\zeta} : U \to \mathbb{C} \), defined by
\[
(4.4)
G_{v,\zeta}(z) = \left[ \int_0^z t^{\nu-1} \left( e^{D_{\lambda}^{n,\gamma}(\sqrt{t}\sin\sqrt{t})} \right)^{\zeta} dt \right]^{1/\nu},
\]
is in the class \( S \) of normalized univalent functions in \( U \). In the particular case when \( |\zeta| \leq 1.8959... \), the function \( G_{v,\zeta}(z) : U \to \mathbb{C} \), defined by
\[
(4.12)
G_{v,\zeta}(z) = \left[ \int_0^z t^{\nu-1} \left( e^{D_{\lambda}^{n,\gamma}(\sqrt{t}\sin\sqrt{t})} \right)^{\zeta} dt \right]^{1/\nu},
\]
is in the class \( S \) of normalized univalent functions in \( U \).

**Remark 4.** Baricz and Frasin proved that the following general integral operators [7]:
\[
\mathcal{H}_{v_1,...,v_m,\mu_1,...,\mu_m}(z), \quad \mathcal{F}_{v_1,...,v_m}(z) \quad \text{and} \quad G_{v,\zeta}(z)
\]
defined by (4.2), (4.3) and (4.4), respectively, are actually univalent for all \( \nu, v_1, ..., v_m > -0.69098... \).

From Corollary [3], by taking \( n = 0 \) we see that our results (with \( \nu, v_1, ..., v_m > -1.25 \)) are stronger than the Baricz-Frasin results for the same integral operators (see, for details, [7]).

**4.2. Modified Bessel Functions.** Taking \( b = 1 \) and \( d = -1 \) in (1.1) or (1.2), we obtain the modified Bessel function \( I_{\nu}(z) \) of the first kind of order \( \nu \) defined by (1.3). We observe also that
\[
D_{\lambda}^{n,\gamma} I_{3/2}(z) = D_{\lambda}^{n,\gamma} \left( 3 \cos \sqrt{z} - \frac{3 \sinh \sqrt{z}}{\sqrt{z}} \right),
\]
\[
D_{\lambda}^{n,\gamma} I_{1/2}(z) = D_{\lambda}^{n,\gamma} (\sqrt{z} \sinh \sqrt{z}) \quad \text{and} \quad D_{\lambda}^{n,\gamma} I_{-1/2}(z) = D_{\lambda}^{n,\gamma} (\sqrt{z} \cosh \sqrt{z}).
\]

**Corollary 4.** Let the function \( D_{\lambda}^{n,\gamma} I_{\nu} : U \to \mathbb{C} \) be defined by
\[
I_{\nu}(z) = 2 \Gamma(\nu + 1) z^{1-\nu/2} I_{\nu}(\sqrt{z}).
\]
Also let the following assertions hold true:

(1) Let \( \nu_1, ..., \nu_m > -1.25 \) \((m \in \mathbb{N})\). Consider the functions \( I_{\nu_j} : U \to \mathbb{C} \) defined by
\[
(4.5)
D_{\lambda}^{n,\gamma} I_{\nu_j}(z) = 2^{\nu_j} \Gamma(\nu_j + 1) z^{1-\nu_j/2} D_{\lambda}^{n,\gamma} I_{\nu_j}(\sqrt{z}) \quad (j = 1, ..., m).
\]
Let \( \nu = \min\{\nu_1, ..., \nu_m\} \) and let the parameters \( \eta, c, \mu_1, ..., \mu_m \) be as in Theorem 1. Moreover, suppose that these numbers satisfy the following inequality:
\[
|\eta| + \frac{N \delta(v + 2)\beta(1)}{4M (\nu + 1) \delta(v + 2) - \alpha(2v + 3)\beta(1) + (N \beta(1))^2} \sum_{j=1}^{m} \frac{1}{\eta\mu_j} \leq 1.
\]
Then the function \( H_{v_1,...,v_m,\mu_1,...,\mu_m,\eta}(z) : U \to \mathbb{C} \), defined by
\[
(4.6)
H_{v_1,...,v_m,\mu_1,...,\mu_m,\eta}(z) = \left[ \eta \int_0^z t^{\nu-1} \prod_{j=1}^{m} \left( \frac{D_{\lambda}^{n,\gamma} I_{\nu_j}(t)}{t} \right)^{1/\mu_j} dt \right]^{1/\eta},
\]
is in the class \( S \) of normalized univalent functions in \( U \). In the particular case when
\[
|\eta| + \frac{28}{233} \frac{1}{\eta \mu} \leq 1,
\]
the function \( H_{3/2,\mu,\eta}(z) : U \to \mathbb{C} \), defined by
\[
H_{3/2,\mu,\eta}(z) = \left[ \eta \int_0^z t^{\nu-1} \left( D_{\lambda}^{n,\gamma} \left( \frac{3 \cosh \sqrt{t}}{t} - \frac{3 \sinh \sqrt{t}}{t}\sqrt{t} \right) \right)^{1/\mu} dt \right]^{1/\eta},
\]
is in the class \( S \) of normalized univalent functions in \( U \).
(2) Let $\nu_1, \ldots, \nu_m > -1.25$ ($m \in \mathbb{N}$) and consider the normalized modified Bessel functions $\mathcal{I}_{\nu_j} : \mathbb{U} \to \mathbb{C}$ defined by (4.3). Let $\nu = \min\{\nu_1, \ldots, \nu_m\}$ and $\Re(\mu) > 0$ and suppose that these numbers satisfy the following inequality:

$$|\mu| \leq \frac{1}{m} \left( 4\mathcal{M} \left( v + 1 \right) \delta(v + 2) - \alpha(2v + 3)\beta(1) + \left( N \left( \beta(1) \right)^2 \right) / 8 \Re(\mu). \right)$$

Then the function $\mathcal{F}_{\nu_1, \ldots, \nu_m, \mu}(z) : \mathbb{U} \to \mathbb{C}$, defined by

$$(4.7) \quad \mathcal{F}_{\nu_1, \ldots, \nu_m, \mu}(z) = \left( m \mu + 1 \int_0^z \prod_{j=1}^m \left(D_{\nu_j}^{\alpha, \gamma} \mathcal{I}_{\nu_j}(t)\right)^\mu dt \right)^{1/(m \mu + 1)},$$

is in the class $\mathcal{S}$ of normalized univalent functions in $\mathbb{U}$. In the particular case when

$$|\mu| \leq \frac{89}{20} \Re(\mu),$$

the function $\mathcal{F}_{1/2, \mu}(z) : \mathbb{U} \to \mathbb{C}$, defined by

$$\mathcal{F}_{1/2, \mu}(z) = \left( \mu + 1 \int_0^z \left(D_{\nu_j}^{\alpha, \gamma} \left( \sqrt{t} \sinh \sqrt{t} \right) \right)^\mu dt \right)^{1/(\mu + 1)},$$

is in the class $\mathcal{S}$ of normalized univalent functions in $\mathbb{U}$.

(3) Let $\zeta \in \mathbb{C}$ and $\nu > -1.25$ and consider the normalized modified Bessel functions $\mathcal{I}_{\nu}(t)$ given by (2.3). If $\Re(\zeta) \geq 1$ and

$$|\zeta| \leq \frac{3\sqrt{3} \mathcal{M} \left( v + 1 \right) |\delta(v + 2) - \beta(1)|}{8 \mathcal{M} \left( v + 1 \right) \delta(v + 2) + 2 \alpha(v + 3)\beta(1)},$$

then the function $\mathcal{G}_{\nu, \zeta}(z) : \mathbb{U} \to \mathbb{C}$, defined by

$$\mathcal{G}_{\nu, \zeta}(z) = \left( \zeta \int_0^z t^{\zeta - 1} \left(D_{\nu}^{\alpha, \gamma} \mathcal{I}_{\nu}(t)\right)^\zeta dt \right)^{1/\zeta},$$

is in the class $\mathcal{S}$ of normalized univalent functions in $\mathbb{U}$. In the particular case when $|\zeta| \leq 1.1809...$, the function $\mathcal{G}_{-1/2, \zeta}(z) : \mathbb{U} \to \mathbb{C}$, defined by

$$\mathcal{G}_{-1/2, \zeta}(z) = \left( \zeta \int_0^z t^{\zeta - 1} \left(D_{\nu}^{\alpha, \gamma} \left( t \cosh \sqrt{t} \right) \right)^\zeta dt \right)^{1/\zeta},$$

is in the class $\mathcal{S}$ of normalized univalent functions in $\mathbb{U}$.

ACKNOWLEDGMENT

The present investigation was supported under (The scientific and Research Project of University of Damman) Number 2013180.

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