SELF-DUALITY AND SUPERSYMMETRY

Maxim Konyushikhin and Andrei V. Smilga

SUBATECH, Université de Nantes, 4 rue Alfred Kastler,
BP 20722, Nantes 44307, France and
Institute for Theoretical and Experimental Physics,
B. Cheremushkinskaya 25, Moscow 117259, Russia

We observe that the Hamiltonian $H = \mathcal{D}^2$, where $\mathcal{D}$ is the flat 4d Dirac operator in a self-dual gauge background, is supersymmetric, admitting 4 different real supercharges. A generalization of this model to the motion on a curved conformally flat 4d manifold exists. For an Abelian self-dual background, the corresponding Lagrangian can be derived from known harmonic superspace expressions.

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I. INTRODUCTION

The main purpose of this paper is to present a simple supersymmetric quantum mechanical model which, surprisingly, did not attract much attention so far.

It was known since some time that one can treat the problem of the motion of a fermion on an even-dimensional manifold with an arbitrary gauge field background as a supersymmetric one such that, e.g., the Atiyah-Singer index of a Dirac operator can be interpreted as the Witten index of a certain supersymmetric Hamiltonian [1]. Our remark is that if the gauge field is self-dual and the 4d metric is flat, the system enjoys an extended supersymmetry with two pairs of supercharges. A similar $\mathcal{N} = 2$ supersymmetric system can be written for conformally flat 4d manifolds, though supercharges in this case are not related to $\mathcal{D}$, and the Hamiltonian does not coincide with $\mathcal{D}^2$.

In Sect. II, we present the model. In Sect. III, we analyze in some more details its simplest version (flat metric and constant self-dual Abelian field density) and derive the spectrum. In Sect. IV, we derive the component Lagrangian from a certain harmonic superspace (HSS)

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1 $\mathcal{N}$ counts the number of complex supercharges.
II. FERMIONS IN 4D SELF-DUAL BACKGROUND

Consider the Dirac operator in flat 4d Euclidean space

\[ \mathcal{D} = \sum_{\mu=0,1,2,3} D_{\mu} \gamma_{\mu}, \]

where \( D_{\mu} = \partial_{\mu} - iA_{\mu} \) and \( \gamma_{\mu} \) are Euclidean anti-Hermitian gamma–matrices,

\[ \gamma_{\mu} = \begin{pmatrix} 0 & -\sigma_{\mu}^\dagger \\ \sigma_{\mu} & 0 \end{pmatrix}, \quad \{\gamma_{\mu}, \gamma_{\nu}\} = -2\delta_{\mu\nu}, \]

with \( (\sigma_{\mu})_{\alpha\beta} = \{i, \vec{\sigma}\}_{\alpha\beta} \) and \( (\sigma_{\mu}^\dagger)^{\beta\alpha} = \{-i, \vec{\sigma}\}_{\beta\alpha} \) (\( \vec{\sigma} \) are ordinary Pauli matrices). The indices are raised and lowered, as usual, with antisymmetric Levi-Civita tensors \( \varepsilon_{\alpha\beta} = \varepsilon_{\dot{\alpha}\dot{\beta}} = -\varepsilon^{\alpha\beta} = -\varepsilon^{\dot{\alpha}\dot{\beta}} \), \( \varepsilon_{12} = 1 \). (These are more or less the conventions of [4] rotated to Euclidean space.)

The Hamiltonians we are going to construct enjoy SO(4) = SU(2) \( \times \) SU(2) covariance such that the undotted spinor index refers to the first SU(2) factor, while the dotted one to the second. The matrices \( \sigma_{\mu}, \sigma_{\mu}^\dagger \) satisfy the identities

\[ \sigma_{\mu}\sigma_{\nu} + \sigma_{\nu}\sigma_{\mu}^\dagger = \sigma_{\mu}^\dagger\sigma_{\nu} + \sigma_{\nu}^\dagger\sigma_{\mu} = 2\delta_{\mu\nu}, \]
\[ \sigma_{\mu}^\dagger\sigma_{\nu} + \sigma_{\nu}^\dagger\sigma_{\mu} = 2i\eta^{a}_{\mu\nu}\sigma_{a}, \]
\[ \sigma_{\mu}\sigma_{\nu} - \sigma_{\nu}\sigma_{\mu}^\dagger = 2i\bar{\eta}^{a}_{\mu\nu}\sigma_{a}, \]

where \( \eta^{a}_{\mu\nu}, \bar{\eta}^{a}_{\mu\nu} \) are the 't Hooft symbols,

\[ \eta^{a}_{ij} = \bar{\eta}^{a}_{ij} = \varepsilon_{aij}, \quad \eta^{a}_{0i} = -\bar{\eta}^{a}_{0i} = \bar{\eta}^{a}_{i0} = -\eta^{a}_{i0} = \delta_{ai} \]

(\( \sigma_{a} \) – Pauli matrices, indices \( a, i, j \) run from 1 to 3). They are self-dual (anti-self-dual),

\[ \eta^{a}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu
u\rho\lambda}\eta^{a}_{\rho\lambda}, \quad \bar{\eta}^{a}_{\mu\nu} = -\frac{1}{2}\varepsilon_{\mu
u\rho\lambda}\bar{\eta}^{a}_{\rho\lambda}, \]

with the convention \( \varepsilon_{0123} = -1 \). Another useful identity is

\[ \sigma_{2}\sigma_{\mu}^\dagger T_{\mu} = -\sigma_{\mu}^\dagger. \]

Consider the operator

\[ H = \frac{1}{2}\mathcal{D}^2 = -\frac{1}{2}D^2 - \frac{i}{4} F_{\mu\nu}\gamma_{\mu}\gamma_{\nu}, \]
where \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu] \) is the field strength. It is well known that nonzero eigenvalues of the Euclidean Dirac operator come in pairs \((-\lambda, \lambda)\) and hence the spectrum of the Hamiltonian \( H \) is double-degenerate for all excited states. This means that, for any external field \( A_\mu \), this Hamiltonian is supersymmetric \(^1\) admitting two different anticommuting real supercharges: \( \mathcal{D} \) and \( i \mathcal{D} \). Suppose now that the background field is self-dual, 

\[
F_{\mu \nu} = \frac{1}{2} \varepsilon_{\mu \nu \rho \delta} F_{\rho \delta} \quad \rightarrow \quad F_{\mu \nu} = \eta^{\alpha}_{\mu \nu} B_a .
\]

One can be easily convinced that in this case the Hamiltonian admits four different Hermitian square roots \( S_A \) that satisfy the extended supersymmetry algebra

\[
\{ S_A, S_B \} = 4 \delta_{AB} H .
\]

One of the choices is

\[
\begin{align*}
S_1 &= \mathcal{D} = \gamma_0 D_0 + \gamma_1 D_1 + \gamma_2 D_2 + \gamma_3 D_3 , \\
S_2 &= \gamma_0 D_3 + \gamma_1 D_2 - \gamma_2 D_1 - \gamma_3 D_0 , \\
S_3 &= \gamma_0 D_2 - \gamma_1 D_3 - \gamma_2 D_0 + \gamma_3 D_1 , \\
S_4 &= \gamma_0 D_1 - \gamma_1 D_0 + \gamma_2 D_3 - \gamma_3 D_2 .
\end{align*}
\]

Introducing the complex supercharges

\[
\begin{align*}
Q_1 &= (S_1 - iS_2)/2 , & Q_2 &= (S_3 - iS_4)/2 , \\
\bar{Q}^1 &= (S_1 + iS_2)/2 , & \bar{Q}^2 &= (S_3 + iS_4)/2 ,
\end{align*}
\]

we obtain the standard \( \mathcal{N} = 2 \) supersymmetry algebra \(^2\)

\[
\{ Q_\alpha, Q_\beta \} = 0 , & \quad \{ Q_\alpha, \bar{Q}^\beta \} = 2 \delta^\beta_\alpha H .
\]

Correspondingly, the excited spectrum of \( H \) is four-fold degenerate, while the spectrum of \( \mathcal{D} \) consists of the quartets involving two degenerate positive and two degenerate negative eigenvalues.

The algebra \(^9\) with supercharges \(^10\) holds for any self-dual field, irrespectively of whether it is Abelian or non-Abelian. Thus, the additional 2-fold degeneracy of the spectrum of the Dirac operator mentioned above should be there for a generic self-dual field. One

\(^2\) Note that, in contrast to \( \mathcal{D} \), the operator \( \mathcal{D} \gamma_5 \) is not expressed into a linear combination of \( S_A \). In other words, the \( \mathcal{N} = 1 \) supersymmetry algebra with the operators \( \mathcal{D}(1 \pm \gamma_5) \) is not a subalgebra of the \( \mathcal{N} = 2 \) algebra \(^12\).
particular example of a non-Abelian self-dual field is the instanton solution, where this degeneracy was observed back in [3] (see Eqs. (4.15) there).

To make contact with the Lagrangian (and, especially, superfield) description, it is convenient to introduce holomorphic fermion variables, which satisfy the standard anticommutation relations

$$\{\psi_\alpha, \psi_\beta\} = \{\bar{\psi}^{\dot{\alpha}}, \bar{\psi}^{\dot{\beta}}\} = 0, \quad \{\bar{\psi}^{\dot{\alpha}}, \bar{\psi}_\beta\} = \delta^{\dot{\alpha}}_\beta.$$  \hspace{1cm} (13)

One of the possible choices is

$$\psi_1 = \frac{-\gamma_0 + i\gamma_3}{2}, \quad \bar{\psi}^1 = \frac{\gamma_0 + i\gamma_3}{2},$$

$$\psi_2 = \frac{\gamma_2 + i\gamma_1}{2}, \quad \bar{\psi}^2 = \frac{-\gamma_2 + i\gamma_1}{2}. \hspace{1cm} (14)$$

Then two complex supercharges (11) are expressed in a very simple way,

$$Q_\alpha = (\sigma_\mu \tilde{\psi}^\alpha)_\mu (\hat{p}_\mu - A_\mu),$$

$$\bar{Q}^\alpha = (\psi \sigma_\mu^\dagger)^\alpha (\hat{p}_\mu - A_\mu), \hspace{1cm} (15)$$

with $\hat{p}_\mu = -i\partial_\mu$. The Hamiltonian (17) is expressed in these terms as

$$H = \frac{1}{2} (\hat{p}_\mu - A_\mu)^2 + \frac{i}{4} F_{\mu\nu} \psi \sigma_\mu^\dagger \sigma_\nu \bar{\psi}. \hspace{1cm} (16)$$

It is clear now why the spinor indices in Eq. (12) are undotted, while in Eq. (14) they are dotted. The supercharges are rotated by the first SU(2) and the variables $\psi_\alpha$ by the second.\footnote{Note that complex conjugation leaves the spinors in the same representation, the symmetry group here is SO(4) rather than SO(3, 1).}

A careful distinction between two different SU(2) factors allows one to understand better the reason why the supercharges (15) satisfy the simple algebra (12) in a self-dual background. The self-dual field density $\mathcal{F}$ carries in the spinor notation only dotted indices. Therefore any expression involving $\mathcal{F}, \psi, \bar{\psi}$ is a scalar with respect to undotted SU(2). The only such scalar that can appear in the r.h.s. of the anticommutators of the supercharges $\{Q_\alpha, \bar{Q}^\beta\}$ is the structure which is proportional to $\delta^{\dot{\beta}}_\alpha$, i.e. the Hamiltonian. No other operator is allowed.

In the Abelian case, the supercharges (15) and the Hamiltonian (16) are scalar operators not carrying matrix indices anymore. This allows one to derive the Lagrangian,

$$L = \frac{1}{2} \ddot{x}_\mu \dot{x}_\mu + A_\mu(x) \dot{x}_\mu + i\psi^{\dot{\alpha}} \dot{\psi}_\alpha - \frac{i}{4} \mathcal{F}_{\mu\nu} \psi \sigma_\mu^\dagger \sigma_\nu \bar{\psi}. \hspace{1cm} (17)$$
In the non-Abelian case, the expressions (15, 16) still keep their color matrix structure, and one cannot derive the Lagrangian in a straightforward way. One of the ways to handle the matrix structure is to introduce a set of color fermion variables (say, in the fundamental representation of the group) and impose the extra constraint considering only the sector with unit fermion charge [1]. An alternative (non-Abelian) construction of the Lagrangian is presented in [6], but in this paper we consider Lagrangians only for Abelian fields.

As will be demonstrated explicitly in Sect. IV the component Lagrangian (17) follows from the superfield action written earlier by Ivanov and Lechtenfeld in the framework of harmonic superspace approach [3]. We will see that one can naturally derive in this way a σ-model type generalization of the Lagrangian (17) describing the motion over the manifold with nontrivial conformally flat metric
\[ ds^2 = \{ f(x) \}^{-2} dx_\mu dx_\mu. \]

It is written as follows
\[
L = \frac{1}{2} f^{-2} \dot{x}_\mu \dot{x}_\mu + A_\mu(x) \dot{x}^\mu + i \bar{\psi}_\gamma \dot{\psi}^\gamma \sigma_\mu \sigma_\nu \bar{\psi} - \frac{i}{4} f^2 F_{\mu\nu} \psi \sigma_\mu^\dagger \sigma_\nu \bar{\psi} + \frac{1}{4} \left\{ \frac{3}{2} (\partial_\mu f)^2 - f \partial^2 f \right\} \psi^4 + \frac{i}{2} f^{-1} \partial_\mu f \dot{x}_\nu \psi \sigma_\mu^\dagger \sigma_\nu \bar{\psi}. \tag{18}
\]

The corresponding (quantum) Noether supercharges and the Hamiltonian are
\[
Q_\alpha = f (\sigma_\mu \bar{\psi})_\alpha (\hat{p}_\mu - A_\mu) - \psi_\gamma \bar{\psi}^\gamma (\sigma_\mu \bar{\psi})_\alpha i \partial_\mu f,
\]
\[
\bar{Q}^\alpha = (\psi \sigma_\mu^\dagger)_\alpha (\hat{p}_\mu - A_\mu) f + i \partial_\mu f \left( \psi \sigma_\mu^\dagger \right)^\alpha \cdot \psi_\gamma \bar{\psi}^\gamma, \tag{19}
\]
\[
H = \frac{1}{2} f (\hat{p}_\mu - A_\mu)^2 f + \frac{i}{4} f^2 F_{\mu\nu} \psi \sigma_\mu^\dagger \sigma_\nu \bar{\psi} - \frac{1}{2} f i \partial_\mu f (\hat{p}_\nu - A_\nu) \psi \sigma_\mu^\dagger \sigma_\nu \bar{\psi} + f \partial^2 f \left\{ \psi_\gamma \bar{\psi}^\gamma - \frac{1}{2} \left( \psi_\gamma \bar{\psi}^\gamma \right)^2 \right\}. \tag{20}
\]

On the other hand, one can explicitly calculate the anticommutators of the supercharges (19) for any self-dual 4 field \( A_\mu(x) \), Abelian or non-Abelian, and verify that the algebra (12) holds. While doing this, the use of the following Fierz identity
\[
(\bar{\psi} \sigma_\mu^\dagger)^\beta (\sigma_\nu \psi)_\alpha - (\sigma_\mu \bar{\psi})_\alpha (\psi \sigma_\nu^\dagger)^\beta = \delta_\alpha^\beta \bar{\psi} \sigma_\mu^\dagger \sigma_\nu \psi, \tag{21}
\]
which can be proven using (6), is convenient.

\(^4\) Anti-self-duality conditions are obtained when one interchanges \( \sigma_\mu \) and \( \sigma_\mu^\dagger \) in all the formulas. This is equivalent to the interchange of two spinor representations of SO(4).
Note that, with a nontrivial factor $f(x)$, the supercharges (19) have nothing to do with the Dirac operator $\mathcal{D}$ in a conformally flat background: the latter cannot be expressed as a linear combination of $Q_\alpha$ and $\bar{Q}^\alpha$. In addition, the Hamiltonian (20) does not coincide with $\mathcal{D}^2/2$.

The model (18-20) is a close relative to the model constructed in Ref. [7] (see Eqs. (30,31) there), which describes the motion on a three-dimensional conformally flat manifold in external magnetic field and a scalar potential. In fact, the latter model can be obtained from the former, if assuming that the metric and the vector potential $A_\mu \equiv (\Phi, \vec{A})$ depend only on three spatial coordinates $x_i$. If assuming further that the metric is flat, one is led to the Hamiltonian [8]

$$H = \frac{1}{2} \left( \hat{p} - \vec{A} \right)^2 + \frac{1}{2} \Phi^2 + \vec{\nabla} \Phi \psi \bar{\psi},$$

(22)

which is supersymmetric under the condition $F_{ij} = \epsilon_{ijk} \partial_k \Phi$ (the 3d reduction of the 4d self-duality condition). It was noticed in Ref. [7] that the effective Hamiltonian of a chiral supersymmetric electrodynamics in finite spatial volume belongs to this class with $\Phi \propto 1/|\vec{A}|$. The vector potential $\vec{A}(\vec{A})$ describes in this case a Dirac magnetic monopole such that the Berry phase appears. The three dynamical variables $\vec{A}$ (do not confuse with curly $\vec{A}$ !) have in this case the meaning of the zero Fourier harmonic of the vector potential in the original field theory. In the leading order, the metric is flat. When higher loop corrections are included, a (conformally flat !) metric on the moduli space $\{\vec{A}\}$ appears.

Performing the Hamiltonian reduction of Eq. (20) with non-Abelian $A_\mu$, a non-Abelian generalization of Eq. (22) can easily be derived. It keeps the gauge structure of Eq. (22) with matrix-valued $\vec{A}$ and $\Phi$ satisfying the condition $F_{ij} = \epsilon_{ijk} D_k \Phi$. Note that such Hamiltonian does not coincide with the non-Abelian 3d Hamiltonian derived in Ref. [9].

### III. CONSTANT FIELD

As an illustration, consider the system described by the Hamiltonian (16) in a constant self-dual Abelian background. The constant self-dual field strength $F_{\mu\nu} = \eta_{\mu\nu}^a B_a$ is parametrized by three independent components. Let us direct $B^a$ along the third axis, $B_a = (0, 0, B)$, and choose the gauge

$$A_0 = B x_3, \quad A_2 = B x_1, \quad A_1 = A_3 = 0.$$  

(23)
The Hamiltonian (16) acquires the form

\[
H = \left\{ \frac{1}{2} (\hat{p}_0 - Bx_3)^2 + \frac{1}{2} \hat{p}_3^2 + B \left( \chi_1 \hat{x}^1 - \frac{1}{2} \right) \right\} \\
+ \left\{ \frac{1}{2} (\hat{p}_2 - Bx_1)^2 + \frac{1}{2} \hat{p}_1^2 + B \left( \chi_2 \hat{x}^2 - \frac{1}{2} \right) \right\}. \tag{24}
\]

For convenience, we have introduced notations \( \chi_1 = \bar{\psi} \dot{\psi}^1, \chi_2 = \bar{\psi} \dot{\psi}^2 \). The Hamiltonian is thus reduced to the sum \( H_1 + H_2 \) of two independent (acting in different Hilbert spaces) supersymmetric Hamiltonians, each describing the 2-dimensional motion of an electron in homogeneous orthogonal to the plane magnetic field \( \vec{B} \). The bosonic sector of each such Hamiltonian corresponds to the spin projection \( \vec{s} \cdot \vec{B}/|\vec{B}| = -1/2 \), and the fermionic sector to the spin projection \( \vec{s} \cdot \vec{B}/|\vec{B}| = 1/2 \). This is the first and the simplest supersymmetric quantum problem ever considered [10]. The energy levels for each Hamiltonian are \( \varepsilon_i = B \left( n_i + \frac{1}{2} + s_i \right), n_i \geq 0 - \) integers, \( s_i = \pm \frac{1}{2} \). Each level of \( H_i \) is doubly degenerate. Besides, there is an infinite degeneracy associated with the positions of the center of the orbit along the axes 1 and 3 that are proportional to the integrals of motion \( p_2 \) and \( p_0 \). The full spectrum

\[
E = B \left( n_1 + n_2 + 1 + s_1 + s_2 \right)
\]

is thus 4-fold degenerate at each level (except for the state with \( E = 0 \)).

It might be instructive to explicitly associate this degeneracy with the action of supercharges (15). Let us assume for definiteness \( B > 0 \). One can represent \( Q_\alpha \) as

\[
Q_1 = \sqrt{2B} \left( b \chi_1 + a^\dagger \hat{x}^2 \right), \quad Q_2 = \sqrt{2B} \left( a \chi_1 - b^\dagger \hat{x}^2 \right) \tag{26}
\]

where \( a^\dagger, b^\dagger \) and \( a, b \) are the creation and annihilation operators,

\[
a = \frac{1}{\sqrt{2B}} \left( \hat{p}_1 - iBx_1 + ip_2 \right), \quad b = \frac{1}{\sqrt{2B}} \left( \hat{p}_3 - iBx_3 + ip_0 \right) \tag{27}
\]

\[
[a, a^\dagger] = 1, \quad [b, b^\dagger] = 1. \tag{28}
\]

In these notations, the Hamiltonian (24) takes a very simple form

\[
H = B \left\{ a^\dagger a + b^\dagger b + \chi_1 \hat{x}^1 + \chi_2 \hat{x}^2 \right\}. \tag{29}
\]

Obviously, the energy levels of the Hamiltonian (24) are defined by two integrals of motion \( p_{2,0} \), two oscillator excitation numbers \( n_{1,2} \) and two spins \( s_{1,2} \), as in Eq. (25). For each \( p_2, p_0, \)
there is a unique ground zero energy state \(|0\rangle\) annihilated by all supercharges. A quartet of excited states can be represented as

\[
|n_1, n_2\rangle, \quad Q_1^\dagger|n_1, n_2\rangle, \quad Q_2^\dagger|n_1, n_2\rangle, \quad Q_1^\dagger Q_2^\dagger|n_1, n_2\rangle \tag{30}
\]

where the state

\[
|n_1, n_2\rangle \equiv \chi_1 \cdot (a^\dagger)^{n_1} (b^\dagger)^{n_2} |0\rangle
\]
of energy \(E = B(n_1 + n_2 + 1)\) is annihilated by both \(Q_1\) and \(Q_2\).

For each \(p_2, p_0\), there are \(N\) such quartets at the energy level \(E = BN\).

**IV. FROM HARMONIC SUPERSPACE TO COMPONENTS**

In this section, we derive the Hamiltonian \((20)\) in the HSS approach. To make the paper self-consistent, we present in the Appendix its salient features and definitions in application to quantum mechanical problems. The relevant superfield action was written in \([3]\), and we show here that the corresponding component Lagrangian coincides with \([18]\).

The corresponding supercharges \([19]\) and the Hamiltonian \((20)\) involve an Abelian self-dual gauge field \(A_\mu(x)\). The non-Abelian case is treated in a separate publication \([6]\).

Let us introduce a doublet of superfields \(q^{+\dot{\alpha}}\) with charge +1 \((D^0 q^+ = q^+)\) satisfying the constraints \((A11)\). The index \(\dot{\alpha}\) is the fundamental representation index of an additional external group SU(2). The solution for these constraints in the analytical basis is [see Eq.\((A12)\)]

\[
q^{+\dot{\alpha}} = x^{\alpha\dot{\alpha}} (t_\Lambda) u^\alpha_+ - 2\theta^+ \chi^{\dot{\alpha}} (t_\Lambda) - 2\bar{\theta}^+ \bar{\chi}^{\dot{\alpha}} (t_\Lambda) - 2i\theta^+ \bar{\theta}^+ \partial_\Lambda x^{\dot{\alpha}} (t_\Lambda) u^-_\alpha. \tag{31}
\]

We impose now the additional pseudoreality condition

\[
q^{+\dot{\alpha}} = \varepsilon^{\dot{\alpha}\beta} \tilde{q}^{+\beta}, \tag{32}
\]
the field \(\tilde{q}_+^+\) being defined in Eq.\((A15)\). It implies

\[
x^{\alpha\dot{\alpha}} = -(x_{\alpha\dot{\alpha}})^*, \quad \chi^{\dot{\alpha}} = (\chi_{\dot{\alpha}})^* \equiv \bar{\chi}^{\dot{\alpha}}. \tag{33}
\]

Let us go back now to the central basis \(\{t, \theta_\alpha, \bar{\theta}^\beta, u^\pm_\gamma\}\). The solution can be presented as \(q^{+\dot{\alpha}} = u^+_\alpha q^{\alpha\dot{\alpha}}\) where \(q^{\alpha\dot{\alpha}}\) does not depend on \(u^\pm_\alpha\) (the latter follows from the constraint
\(D^+ q^{+\hat{a}} = 0\) and the definition \(D^{++} = u_\alpha^+ \frac{\partial}{\partial \alpha} \). It is convenient to go over to the \(4d\) vector notation, introducing

\[ q_\mu = -\frac{1}{2} (\sigma_\mu)_{\alpha\bar{\alpha}} q^{\alpha\bar{\alpha}}, \quad q^{+\hat{a}} = -q_\mu (\sigma_\mu^+)_{\hat{a}\alpha} u_\alpha^+ . \tag{34} \]

Now, \(q_\mu\) is a vector with respect to the group \(SO(4) = SU(2) \times SU(2)\), with the first factor representing the \(N = 2\) R-symmetry group and the second one being the extra global \(SU(2)\) group rotating the dotted “flavor” indices.

Pseudoreality condition \([32]\) implies that the superfield \(q_\mu\) is real. The latter is expressed in components as follows,

\[ q_\mu = x_\mu + \theta \sigma_\mu \chi + \bar{\theta} \sigma_\mu \bar{\chi} - \frac{i}{2} \bar{x}_\mu \bar{\theta} \sigma_\mu \sigma_\nu \theta + \frac{i}{2} \bar{\theta} \sigma_\mu \chi \theta^2 - \frac{i}{2} \theta \sigma_\mu \bar{\chi} \bar{\theta}^2 - \frac{1}{4} x_\mu \theta^4 , \tag{35} \]

where \(\theta^2 \equiv \theta^a \theta_a, \bar{\theta}^2 \equiv \bar{\theta}^\alpha \bar{\theta}_\alpha, \theta^4 \equiv \theta^2 \bar{\theta}^2\).

The classical \(N = 2\) SUSY invariant action for the superfield \(q_\mu\) can now be written. It consists of two parts, \(S = S_{\text{kin}} + S_{\text{int}}\). The kinetic part,

\[ S_{\text{kin}} = \int dt \, d^4 \theta d\mu R'_{\text{kin}}(q^{+\hat{a}}, \bar{q}^{+\check{\beta}}, u_\gamma^+) = \int dt \, d^4 \theta R_{\text{kin}}(q_\mu) , \tag{36} \]

depends on an arbitrary function \(R_{\text{kin}}(q_\mu)\). Plugging \([35]\) into \([36]\) and adding/subtracting proper total derivatives, we obtain

\[ S_{\text{kin}} = \int dt \left\{ \frac{1}{2} g(x) \dot{x}_\mu \dot{x}_\mu + \frac{1}{2} \bar{g}(x) (\dot{\bar{\chi}}^{\hat{a}} \dot{\chi} - \dot{\bar{\chi}}^{\hat{a}} \dot{\chi}) + \frac{1}{8} \partial^2 g(x) \chi^4 - \frac{i}{4} \partial_\mu g(x) \dot{x}_\nu \chi \sigma_\mu^\nu \chi \right\} , \tag{37} \]

where \(g(x) = \frac{1}{2} \partial^2 \bar{R}_{\text{kin}}(x)\) and \(\chi^4 = \chi^{\hat{a}} \chi^{\hat{b}} \chi^{\check{\beta}} \chi^{\check{\delta}}\).

To couple \(x_\mu\) to an external gauge field, one should add the interaction term \(S_{\text{int}}\) that represents an integral over analytic superspace,

\[ S_{\text{int}} = \int dt \, du \, d\bar{\theta}^+ d\theta^+ R^{+\check{\beta}}_{\text{int}}(q^{+\hat{a}}(t_A, \theta^+, \bar{\theta}^+), u_\gamma^+) . \tag{38} \]

We choose \(R^{+\check{\beta}}_{\text{int}}\) (it carries the charge 2) satisfying the condition \(\tilde{R}^{+\check{\beta}}_{\text{int}} = -R^{+\check{\beta}}_{\text{int}}\) [the involution operation \(\tilde{X}\) being defined in Eqs. \([A13]\), \([A14]\)] such that the action \([38]\) is real.

To do the integral over \(\theta^+\) and \(\bar{\theta}^+\), introduce \(x^{+\hat{a}} = -x_\mu (\sigma_\mu^+)_{\hat{a}\alpha} u_\alpha^+ \equiv x^{\hat{a}} u_\alpha^+ \) [see Eq. \([34]\)]. Then

\[ S_{\text{int}} = \int dt \, du \left\{ 2i (\sigma_\mu^+)_{\hat{a}\alpha} \partial_+ \hat{a} R^{+\check{\beta}}_{\text{int}} u_\alpha^- \cdot \dot{x}_\mu - 4 \chi^{\hat{a}} \dot{\chi} \partial_+ \hat{a} \partial_+ \check{\beta} R^{+\check{\beta}}_{\text{int}} \right\} . \tag{39} \]
with
\[ \partial_{\dot{+}} R^{++}_{\text{int}}(x,u) \equiv \frac{\partial R^{++}_{\text{int}}(x^{\dot{+}}, u^\pm)}{\partial x^{\dot{+}}}. \] (40)

Now, define the gauge field,
\[ A_\mu(x) \equiv \int du \left\{ 2i (\sigma_\mu^\dagger \dot{A}_\alpha) \partial_{\dot{+}} R^{++}_{\text{int}}(x^\dot{+}, u^-_\alpha) \right\}. \] (41)

As the action (39) is real, the field \( A_\mu(x) \) is also real. It has zero divergence, \( \partial \mu A_\mu = 0 \).

The field strength is expressed as
\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -2\eta_\mu^\nu \int du \partial_{\dot{+}} \partial_{\dot{+}} R^{++}_{\text{int}}(x^\dot{+}, u^-_\alpha) (\sigma_\alpha^\dagger) \dot{\gamma} \] (42)
(the identities (3) were used). It is obviously self-dual. With the definitions (41) and (42) in hand, one can represent the interaction term (39) as
\[ S_{\text{int}} = \int dt \left\{ A_\mu(x) \dot{x}_\mu - i \frac{4}{\int} F_{\mu \nu} \chi \sigma_\mu^\dagger \sigma_\nu \bar{\chi} \right\}. \] (43)

Adding this to the kinetic term in (37) [where one can get rid of the factor \( g(x) \) in the fermion kinetic term by introducing canonically conjugated \( \psi_\dot{\alpha} = f^{-1}(x) \chi_\dot{\alpha}, \bar{\psi}^{\dot{\alpha}} = f^{-1}(x) \bar{\chi}^{\dot{\alpha}} \) with \( f(x) = g^{-1/2}(x) \)], one can explicitly check that the Lagrangian \( L = L_{\text{kin}} + L_{\text{int}} \) coincides, up to a total derivative, with (18). The action is invariant under supersymmetry transformations,
\[ x_\mu \rightarrow x_\mu + f \epsilon_\mu \psi + f \bar{\epsilon} \bar{\psi}, \]
\[ f \psi_\dot{\alpha} \rightarrow f \psi_\dot{\alpha} + i \dot{x}_\mu (\bar{\epsilon} \sigma_\mu)_\dot{\alpha}, \]
\[ f \bar{\psi}^{\dot{\alpha}} \rightarrow f \bar{\psi}^{\dot{\alpha}} - i \dot{x}_\mu (\sigma_\mu^\dagger \epsilon)^{\dot{\alpha}}. \] (44)

The Noether classical supercharges expressed in terms of \( \psi_\dot{\alpha}, \bar{\psi}^{\dot{\alpha}}, x_\mu \) and their canonical momenta,
\[ p_\mu = f^{-2} \dot{x}_\mu + A_\mu - i \frac{2}{\int} f \partial_\nu f \psi \sigma_\mu^\dagger \sigma_\nu \bar{\psi}, \] (45)
are
\[ Q_\alpha = f \left( \sigma_\mu \bar{\psi} \right)_\alpha (p_\mu - A_\mu) - i \partial_\nu f \psi \sigma_\nu \bar{\psi} \] (46)
\[ \bar{Q}^\alpha = [\text{complex conjugate}]. \]

The quantum supercharges are obtained from the classical ones by Weyl ordering procedure [11]. This gives (19). The anticommutator \( \{ Q_\alpha, \bar{Q}^\alpha \} \) gives the quantum Hamiltonian (20).

As was noticed, the field \( A_\mu \) naturally obtained in the HSS framework satisfies the constraint \( \partial_\mu A_\mu = 0 \). This does not really impose a restriction, however, because gauge transformations of \( A_\mu \) that shift it by the gradient of an arbitrary function amount to adding a total derivative in the Lagrangian (43).
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Appendix: Harmonic superspace in quantum mechanics

In this appendix, we introduce some basic HSS notations and definitions (see Ref. [2] for detailed explanations) in application to quantum mechanical systems.

Consider the ordinary $\mathcal{N} = 2$ superspace

$$\mathbb{R}^{1|4} = \{t, \theta_\alpha, \bar{\theta}^\beta\} ,$$

with $\theta_\alpha$ and $\bar{\theta}^\beta = \varepsilon^{\beta\gamma} \bar{\theta}^\gamma = (\theta^\beta)^\dagger$ belonging to the fundamental representation of SU(2).

Introduce the supercharges

$$Q^\alpha = \frac{\partial}{\partial \theta_\alpha} + i\bar{\theta}^\alpha \frac{\partial}{\partial t} , \quad \bar{Q}_\alpha = \frac{\partial}{\partial \bar{\theta}^\alpha} + i\theta_\alpha \frac{\partial}{\partial t} \quad (A2)$$

and superderivatives

$$D^\alpha = \frac{\partial}{\partial \theta_\alpha} - i\bar{\theta}^\alpha \frac{\partial}{\partial t} , \quad \bar{D}_\alpha = \frac{\partial}{\partial \bar{\theta}^\alpha} - i\theta_\alpha \frac{\partial}{\partial t} . \quad (A3)$$

The supercharges form the $\mathcal{N} = 2$ SUSY algebra, while the superderivatives anticommute with $Q^\alpha$ and $\bar{Q}_\beta$,

$$\{Q^\alpha, \bar{Q}_\beta\} = 2\delta^\alpha_\beta i\partial_t , \quad \{D^\alpha, \bar{D}_\beta\} = -2\delta^\alpha_\beta i\partial_t . \quad (A4)$$

To proceed to harmonic superspace $\mathbb{H}\mathbb{R}^{1+2|4} = \mathbb{R}^{1|4} \times S^2$, we introduce a set of two complex coordinates $u^{+\alpha}$. Introduce also $u^{-\alpha} = (u^{+\alpha})^*$ and impose the condition

$$u^{+\alpha} u^{-\alpha} = 1 . \quad (A5)$$

Then $u^{+\alpha}$ parametrize the R-symmetry group SU(2). The differential operators

$$D^{++} = u^+_\alpha \frac{\partial}{\partial u^{-}_\alpha} , \quad D^{--} = u^-\alpha \frac{\partial}{\partial u^{+}_\alpha} , \quad D^0 = u^+_\alpha \frac{\partial}{\partial u^{+}_\alpha} - u^-\alpha \frac{\partial}{\partial u^-\alpha} . \quad (A6)$$

Our convention follows the convention in Ref. [12], but differs from the convention of Ref. [3] by the change of time direction $t \rightarrow -t$. With this, we reproduce the correct sign in the kinetic term for the spinor field in Eq. (37).
are called *harmonic derivatives*. The U(1) charge operator $D^0$ plays a special role. The functions of zero U(1) charge live on the coset $S^2 = \text{SU}(2)/\text{U}(1)$. The coordinates $u^+_\alpha$ have charge 1, the coordinates $u^-_\alpha$ have charge -1, etc.

One can define now harmonic projections $D^\pm = u^\pm_\alpha D^\alpha$, $\bar{D}^\pm = u^\pm_\bar{\alpha} \bar{D}^\alpha$. It is convenient to go over in the *analytic basis* in HSS,

$$\mathbb{HR}^{1+2|4} = \{ t_A, \theta^\pm, \bar{\theta}^\pm, u^\pm_\alpha \}, \quad (A7)$$

where

$$t_A = t + i (\theta^+ \bar{\theta}^- + \theta^- \bar{\theta}^+) , \quad \theta^\pm = u^\pm_\alpha \theta^\alpha, \quad \bar{\theta}^\pm = u^\pm_\bar{\alpha} \bar{\theta}^\bar{\alpha}. \quad (A8)$$

In this basis, the covariant spinor derivatives $D^+$, $\bar{D}^+$ are just

$$D^+ = \frac{\partial}{\partial \theta^-}, \quad \bar{D}^+ = - \frac{\partial}{\partial \theta^-}, \quad (A9)$$

while the operator $D^{++}$ acquires the form

$$D^{++} = u^+_\alpha \frac{\partial}{\partial u^-_\alpha} + \theta^+ \frac{\partial}{\partial \theta^-} + \bar{\theta}^+ \frac{\partial}{\partial \bar{\theta}^-} + 2i \theta^+ \bar{\theta}^+ \frac{\partial}{\partial t_A}. \quad (A10)$$

The derivative operators $D^+$, $\bar{D}^+$, $D^{++}$ (anti)commute with each other and with supercharges. Because of this, it is possible to consider a superfield $q^+$ with U(1) charge +1 satisfying

$$D^+ q^+ = 0, \quad \bar{D}^+ q^+ = 0, \quad D^{++} q^+ = 0. \quad (A11)$$

In the analytic superspace coordinates, the first and the second equations mean that $q^+$ depend only on $\theta^+$ and $\bar{\theta}^+$, but not on $\theta^-$ and $\bar{\theta}^-$. This is the so-called *superfield analyticity condition*. When expanding the field $q^+(t_A, \theta^+, \bar{\theta}^+, u^\pm_\alpha)$ over spinor coordinates and the harmonics, one obtains an infinite set of physical fields $\Phi(t_A)$. However, imposing also the condition $D^{++} q^+ = 0$ drastically reduces the number of such fields, making it finite. In the analytic basis, the solution of the constraints (A11) reads

$$q^+ = x^\alpha(t_A) u^+_\alpha - 2\theta^+ \chi(t_A) - 2\bar{\theta}^+ \bar{\chi}(t_A) - 2i \theta^+ \bar{\theta}^+ \partial_A x^\alpha(t_A) u^-_\alpha \quad (A12)$$

with the factors $-2$ introduced for convenience.

The constraints (A11) admit an involution symmetry $q^+ \rightarrow \tilde{q}^+$ which commutes with SUSY transformations [2, 3]. This involution acts just as the ordinary complex conjugation *except* its action on the harmonics $u^\pm_\alpha$, which is

$$\tilde{u}^\pm_\alpha = u^{\pm \alpha}, \quad \tilde{u}^{\pm \alpha} = -u^{\pm \alpha}. \quad (A13)$$
This gives
\[ \tilde{t}_A = t_A, \quad \tilde{\theta}^\pm = \bar{\theta}^\pm, \quad \tilde{\bar{\theta}}^\pm = -\theta^\pm, \]  
(A14)
and hence
\[ \tilde{q}^+ = [x_\alpha(t_A)]^* u^\alpha_+ - 2\theta^+ \chi^+(t_A) + 2\bar{\theta}^+ \chi^+(t_A) - 2i\theta^+ \bar{\theta}^+ \partial_A [x_\alpha(t_A)]^* u^-_\alpha. \]  
(A15)

It is straightforward to see that the field \( \tilde{q}^+ \) satisfies the same constraints (A11) as the field \( q^+ \). The involution symmetry was used in the main text to impose the pseudoreality condition (32) on the field \( q^{+\bar{\alpha}} \).

The invariant actions involve the harmonic integral \( \int du \). To find such integral of any function \( f(u^\pm_\alpha) \), one should expand \( f \) in the harmonic Taylor series and, for each term, do the integrals using the rules
\[ \int du 1 = 1, \quad \int du u^\pm_\alpha \cdots u^\pm_{\alpha_k} u^-_{\alpha_{k+1}} \cdots u^-_{\alpha_{k+\ell}} = 0, \]  
(A16)
where the integrand is symmetrized over all indices. The values of the integrals of all other harmonic monoms (for example, \( \int du u^+_\alpha u^-_\beta = \frac{1}{2} \varepsilon_{\alpha\beta} \)) follow from (A16) and the definition (A5).

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