Fusion of Superalgebras and $D = 3, \mathcal{N} = 4$ Quiver Gauge Theories

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Abstract: For further investigating the underlying structures of the $D = 3, \mathcal{N} = 4$ Chern-Simons-matter (CSM) theories, we suggest a new concept and procedure for “fusing” two superalgebras into a single new superalgebra. The starting superalgebras may be those used in the previous construction of the double-symplectic 3-algebras in the $\mathcal{N} = 4$ CSM theories: The bosonic parts of these two superalgebras share at least one simple factor or $U(1)$ factor. We are able to provide two different methods to do the “fusion”. Several explicit examples are presented to demonstrate the “fusion” procedure. We also generalize the “fusion” procedure so that more than two superalgebras can be fused into a single one, provided some conditions are satisfied. It is shown that two or more $\mathcal{N} = 4$ theories with different gauge groups may be associated with the same “fused” superalgebra.

Keywords: Fusion, Superalgebras, Symplectic 3-Algebras, Chern-Simons-matter Theories.
1. Introduction

In the recent years, the constructions of $D = 3$, $\mathcal{N} \geq 4$ superconformal Chern-Simons-matter (CSM) theories have attracted lots of attention, because these theories are conjectured to be the dual gauge theories of multiple M2-branes [1]–[15]. It has been demonstrated that general Chern-Simons gauge theories with (or without) matter are conformally invariant at the quantum level [17, 18, 19, 20, 21]. After incorporating the extended supersymmetries into the CSM theories, these theories become extended superconformal CSM theories, and we expect that they are also conformally invariant at the quantum level.
The authors have been able to construct the $\mathcal{N} = 4$ quiver gauge theory in terms of the double-symplectic 3-algebra or the $\mathcal{N} = 4$ three-algebra\(^1\). (The double-symplectic 3-algebra is reviewed in Appendix A.) The double-symplectic 3-algebra consists of two sub symplectic 3-algebras. Denoting the generators of the two sub 3-algebras as $T_a$ and $T_{a'}$ ($a = 1, \cdots, 2R$ and $a' = 1, \cdots, 2S$), respectively, then the generators of the double-symplectic 3-algebras are the disjoint union of the two sets of generators $T_a$ and $T_{a'}$. The untwisted multiplet $\Phi_A$ and the twisted multiplet $\Phi_{\dot{A}}$ of theory take values in these two sub 3-algebras, respectively, i.e. $\Phi_A = \Phi^a_A T_a$ and $\Phi_{\dot{A}} = \Phi^{a'}_{\dot{A}} T_{a'}$. Here $A = 1, 2$ and $\dot{A} = 1, 2$ are fundamental indices of the $SU(2) \times SU(2)$ R-symmetry group. The $\mathcal{N} = 4$ action can be built up by gauging part of the full symmetry generated by the double-symplectic 3-algebra.

Recently, using two superalgebras $G$ and $G'$ whose bosonic parts share at least one simple factor or one $U(1)$ factor to construct the two sub symplectic 3-algebras, the authors have been able to derive several classes $\mathcal{N} = 4$ theories with new gauge groups and recover all known $\mathcal{N} = 4$ theories derived from ordinary Lie (2-)algebra approach as well\(^1\). (The general forms of $G$ and $G'$ are presented in Appendix B.)

In this paper we will propose the concept of “fusing” two superalgebras $G$ and $G'$ into a single closed superalgebra, which is not a direct product of $G$ and $G'$, and will present two different methods to carry out the fusion procedure. Of course, one of the motivations is that the resulting superalgebra would be useful, or at least helpful, for further investigating the underlying structures of the $\mathcal{N} = 4$ CSM theories, hopefully because of the known close relationship between the superalgebras and the double-symplectic 3-algebras in the $\mathcal{N} = 4$ CSM theories\(^1\). Also, the concept of fusing two superalgebras into a closed superalgebra and the problem of classifying these “fused” superalgebras may be mathematically interesting.

More concretely, if we identify the generators of the two sub 3-algebras $T_a$ and $T_{a'}$ with the fermionic generators of the two superalgebras $Q_a$ and $Q_{a'}$, respectively,

$$T_a = Q_a, \quad T_{a'} = Q_{a'}, \quad (1.1)$$

then one may construct the 3-brackets in terms of double graded commutators on the superalgebras\(^1\), for instances,

$$[T_a, T_b; T_c] \doteq [(Q_a, Q_b), Q_c], \quad [T_{a'}, T_{b'}; T_c] \doteq [(Q_{a'}, Q_{b'}), Q_c]. \quad (1.2)$$

Here $Q_a$ and $Q_{a'}$ are the fermionic generators of the two superalgebras $G$ and $G'$, respectively; the double graded commutator $[(Q_a, Q_b), Q_c]$ is defined by an anticommutator and a commutator. In order that the theory is physically interesting, we must require that there are nontrivial interactions between the twisted and untwisted multiplets; mathematically, we must require that

$$[T_a, T_b; T_c] \neq 0, \quad [T_{a'}, T_{b'}; T_c] \neq 0. \quad (1.3)$$

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\(^1\)The $\mathcal{N} = 4$ three-algebra is obtained from the double-symplectic 3-algebra by a “contraction”: the two structure constants $f_{abc}^{\text{cd}}$ and $f_{a'b'c'd}^{\text{ab}}$ are set to vanish, while the rest four structure constants remain the same\(^1\). However, in this paper we focus on the double-symplectic 3-algebra approach.
Taking account of (1.2), one is led to

\[ \{Q_a, Q_b, Q_c\} \neq 0, \quad \{Q_{a'}, Q_{b'}, Q_c\} \neq 0. \quad (1.4) \]

It has been proved that Eqs (1.4) can be satisfied if the bosonic parts of \( G \) and \( G' \) share at least one simple factor or one \( U(1) \) factor, provided that the common part of bosonic parts of \( G \) and \( G' \) is not a center of \( G \) and \( G' \) [3, 13]. The first equation of (1.4) implies that there must be a nontrivial anticommutator between \( Q_a \) and \( Q_{c'} \) in the sense that

\[ \{Q_a, Q_{c'}\} \neq 0, \quad (1.5) \]

provided that the \( Q_aQ_bQ_c \) Jacobi identity is obeyed:

\[ \{\{Q_a, Q_b\}, Q_c\} + \{\{Q_a, Q_{c'}\}, Q_b\} + \{\{Q_{c'}, Q_b\}, Q_a\} = 0. \quad (1.6) \]

Actually, if \( \{Q_a, Q_{c'}\} = 0 \), the last two terms of the RHS of (1.6) vanish. As a result, one must have \( \{\{Q_a, Q_b\}, Q_{c'}\} = 0 \), which contradicts the first equation of Eq. (1.4). We are led to Eq. (1.3) again if we combine the second equation of (1.4) and the \( Q_{a'}Q_bQ_{c'} \) Jacobi identity. For some special cases, one can show that the anticommutator (1.3) does not vanish by calculating it directly (Eqs. (2.28) and (3.5) are two examples).

On the other hand, according to the basic idea of supersymmetry, an anticommutator of two fermionic generators gives a linear combination of bosonic generators. It is therefore natural to introduce a set of bosonic generators \( M_{\tilde{a}} \) defined by

\[ \{Q_a, Q_{c'}\} = t^\tilde{a}_{ac'} M_{\tilde{a}}, \quad (1.7) \]

where \( t^\tilde{a}_{ac'} \) are structure constants. Having defined (1.7), if we also define \([M_{\tilde{a}}, M_{\tilde{b}}]\) and every commutator of \( M_{\tilde{a}} \) and any generator of \( G \) and \( G' \) properly, so that every Jacobi identity of the “total” superalgebra consisting of \( M_{\tilde{a}} \) and all generators of \( G \) and \( G' \) is obeyed, we say that \( G \) and \( G' \) have been “fused” into a single superalgebra which is closed. Note that one needs only to introduce the set of new bosonic generators \( M_{\tilde{a}} \) into the system for the purpose of fusion; in particular, one does not have to introduce any new fermionic generator into the system.

We demonstrate that if both \( G \) and \( G' \) are orthosymplectic or unitary superalgebras, and their bosonic parts of \( G \) and \( G' \) share at least one simple factor or \( U(1) \) factor, we can fuse them into a single superalgebra by using two distinct methods. Some explicit examples are presented to demonstrate how to construct this class of superalgebras by fusing two superalgebras. For example, we can fuse \( U(N_1|N_2) \) and \( U(N_2|N_3) \) into a single superalgebra \( U(N_2|N_1 + N_3) \). Here the common bosonic part of \( U(N_1|N_2) \) and \( U(N_2|N_3) \) is \( U(N_2) \). Conversely, we can use the sub-superalgebras \( U(N_1|N_2) \) and \( U(N_2|N_3) \) of \( U(N_2|N_1 + N_3) \) to construct the 3-algebra in the \( \mathcal{N} = 4 \) theory, providing the bosonic parts of the two sub-superalgebras share one common factor \( U(N_2) \).

We are able to work out the general structure of the “fused” superalgebras by adding several new graded commutators into the two superalgebras \( G \) and \( G' \). We also generalize our fusion procedure by showing that three or more superalgebras can be fused into a single superalgebra by introducing more bosonic generators (analogy to \( M_{\tilde{a}} \) in (1.7)).
The fusion procedure can be also generalized in another direction: by adding some fermionic generators into the system. We will call this procedure a fermionic fusion if one needs to introduce at least a set of new fermionic generators (except for new bosonic generators) for fusing two or more superalgebras into a single one (see Sec. 4.2).

We demonstrate that two or more theories with different gauge groups can be associated with the same “fused” superalgebra.

This paper is organized as follows. In section 2 and section 3, we present some explicit examples of the new superalgebra fused by two superalgebras. The general structure of the new superalgebra is worked out in section 4, and some examples for fusing three or more superalgebras into a single one are given as well. We end section 5 with conclusions and discussions. Several appendices are attached to make the paper more self-contained. In Appendixes A and B, we review the \( N = 4 \) theories based on the 3-algebras and the superalgebra realization of 3-brackets and fundamental identities (FIs), respectively. We summarize our conventions in Appendix C. The commutation relations of some superalgebras used to construct symplectic 3-algebras are given in Appendix D.

2. Fusing Two Orthosymplectic Superalgebras

As we explained in Section 1, it is possible to fuse two superalgebras whose bosonic parts share at least one simple factor or \( U(1) \) factor. In this section, we demonstrate how to “fuse” a pair of orthosymplectic superalgebras into a single superalgebra by presenting two explicit examples.

2.1 \( Sp(2N_1) \times SO(N_2) \times Sp(2N_3) \) Gauge Group

Here we choose the two superalgebras as \( G = OSp(N_2|2N_1) \) and \( G' = OSp(N_2|2N_3) \). (The commutation relations of \( OSp(M|2N) \) are given by Appendix D.2.) Namely the bosonic parts of the two superalgebras share one simple factor \( SO(N_2) \). We denote the fermionic generators and the antisymmetric tensor of \( OSp(N_2|2N_1) \) as

\[
Q_a = Q_{\bar{i}i} \quad \text{and} \quad \omega_{ab} = \omega_{\bar{i}i\bar{j}j} = \delta_{ij}\omega_{ij},
\]

where \( \bar{i} = 1, \ldots, N_2 \) is an \( SO(N_2) \) fundamental index, and \( i = 1, \ldots, 2N_1 \) an \( Sp(2N_1) \) fundamental index. For convenience, we cite the commutation relations of \( OSp(N_2|2N_1) \) here

\[
\begin{align*}
[M_{\bar{i}j}, M_{\bar{k}l}] &= \delta_{j\bar{k}}M_{\bar{i}l} - \delta_{i\bar{k}}M_{\bar{j}l} + \delta_{i\bar{j}}M_{\bar{k}l} - \delta_{ij}M_{kl}, \\
[M_{\bar{i}j}, M_{\bar{k}l}] &= \omega_{j\bar{k}}M_{\bar{i}l} + \omega_{i\bar{k}}M_{\bar{j}l} + \omega_{i\bar{j}}M_{\bar{k}l} - \omega_{ij}M_{kl}, \\
[M_{\bar{i}j}, Q_{kl}] &= \delta_{j\bar{k}}Q_{il} - \delta_{i\bar{k}}Q_{jl}, \\
[M_{\bar{i}j}, Q_{kl}] &= \omega_{j\bar{k}}Q_{il} + \omega_{i\bar{k}}Q_{jl}, \\
[Q_{\bar{i}j}, Q_{\bar{k}j}] &= k(\omega_{\bar{i}\bar{j}}M_{\bar{j}j} + \delta_{\bar{i}\bar{j}}M_{\bar{j}j}),
\end{align*}
\]

Similarly, we denote the fermionic generators and the antisymmetric tensor of \( OSp(N_2|2N_3) \) as

\[
Q_{a'} = Q_{\bar{i}i'} \quad \text{and} \quad \omega_{a'b'} = \omega_{\bar{i}i'\bar{j}j'} = \delta_{ij}\omega_{ij}.
\]
where \( i' = 1, \cdots, 2N_3 \) is an \( Sp(2N_3) \) fundamental index, which is independent of \( i \) in the sense that

\[
[M_{i'j'}, Q_{i'i}] = 0 \quad \text{and} \quad [M_{ij}, Q_{i'i}] = 0. \tag{2.4}
\]

Eqs. (2.4) are explicit examples of (B.13). The super Lie algebra \( OSp(N_2|2N_3) \) has similar expressions as that of (2.2).

To construct the corresponding \( \mathcal{N} = 4 \) theory, we can calculate the double graded commutator

\[
[\{Q_a, Q_b\}, Q_c] = f_{abc}^d Q_d \tag{2.5}
\]

and read off the structure constants from the right hand side. Using (2.2), we obtain

\[
[\{Q_{i'i}, Q_{j'j}\}, Q_{k'k}] = k\omega_{ij}(\delta_{jk}Q_{i'k'} - \delta_{ik}Q_{j'k'}). \tag{2.6}
\]

It is not difficult to read off the structure constants from the right hand side. Using (2.2), we obtain

\[
f_{abc'd} = f_{\bar{i}i, \bar{j}j, k'k', l'l'} = k\omega_{ij}(\delta_{ik}Q_{jl'} - \delta_{il}Q_{jk'}). \tag{2.7}
\]

Similarly, one can calculate \( f_{abcd} \) by using \( \{\{Q_a, Q_b\}, Q_c\} = f_{abc}^d Q_d \). A short calculation gives

\[
f_{abcd} = f_{\bar{i}i, \bar{j}j, k'k, l'l} = k[(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\omega_{ij}\omega_{kl} - \delta_{ij}\delta_{kl}(\omega_{ik}\omega_{jl} + \omega_{il}\omega_{jk})]. \tag{2.8}
\]

And \( f_{a'b'c'd'} \) have a similar expression:

\[
f_{a'b'c'd'} = f_{\bar{i}'i', \bar{j}'j', k'k', l'l'} = k[(\delta_{i'k}\delta_{j'l} - \delta_{i'l}\delta_{j'k})\omega_{i'j'}\omega_{k'l'} - \delta_{i'j'}\delta_{k'l'}(\omega_{i'k}\omega_{j'l} + \omega_{i'l}\omega_{j'k})]. \tag{2.9}
\]

Alternatively, one can read off \( k_{uv} \) and \( \tau_{a'b'}^\alpha \) from (2.2) by comparing (2.2) with (B.1) as well as (B.11). For instance,

\[
(\tau_{mn})_{\bar{i}'i', \bar{j}'j'} = \omega_{ij}(\delta_{n\bar{j}}\delta_{m\bar{i}} - \delta_{m\bar{j}}\delta_{n\bar{i}}), \tag{2.10}
\]

\[
k_{\bar{m}'\bar{n}'\bar{p}'\bar{q}'} = \frac{k}{4}(\delta_{\bar{m}'\bar{p}'}\delta_{\bar{n}'\bar{q}'} - \delta_{\bar{m}'\bar{q}'}\delta_{\bar{n}'\bar{p}'}). \tag{2.11}
\]

Similarly, we have

\[
(\tau_{pq})_{k'k', l'l'} = \omega_{k'l'}(\delta_{pk}\delta_{ql} - \delta_{pl}\delta_{qk}). \tag{2.12}
\]

Combining Eqs. (2.10)–(2.12) gives (2.7):

\[
f_{abc'd'} = k_{gh}\tau_{ab}^g \tau_{c'd'}^h = k_{\bar{m}'\bar{n}'\bar{p}'\bar{q}'}(\tau_{mn})_{\bar{i}'i', \bar{j}'j'}(\tau_{pq})_{k'k', l'l'} = f_{\bar{i}i, \bar{j}j, k'k', l'l'} = k\omega_{ij}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \tag{2.13}
\]

In this way, one can also calculate \( f_{abcd} = k_{uv}\tau_{ab}^u \tau_{cd}^v \) and \( f_{a'b'c'd'} = k_{u'v'}\tau_{a'b'}^{u'} \tau_{c'd'}^{v'} \); they are the same as (2.8) and (2.3), respectively.

Eqs. (2.7)–(2.9) satisfy the symmetry conditions (A.23), the reality conditions (A.24) and the FIs (A.14). Eqs. (2.8) and (2.9) also satisfy the constraint equations (A.25). Substituting Eqs. (2.7)–(2.9) into (A.25) and (A.31) gives the \( \mathcal{N} = 4 \) CSM theory with gauge group \( Sp(2N_1) \times SO(N_2) \times Sp(2N_3) \), which was first constructed in Ref. [3] by using an ordinary Lie algebra approach.
2.2 Fusing $OSp(N_2|2N_1)$ and $OSp(N_2|2N_3)$ into $OSp(N_2|2(N_1 + N_3))$

In this section we investigate the two superalgebras $OSp(N_2|2N_1)$ and $OSp(N_2|2N_3)$ further. We demonstrate that they can be “fused” into a single closed superalgebra by using two distinct methods. Finally we prove that the “fused” superalgebra is nothing but $OSp(N_2|2(N_1 + N_3))$.

Here the essential observation is that the anti-commutator of $Q_a$ and $Q_{b'}$ cannot vanish, i.e.

$$\{Q_a, Q_{b'}\} = \{Q_{ii'}, Q_{jj'}\} \neq 0 \quad (2.14)$$

provided that the $Q_a Q_{b'} Q_{c'}$ ($Q_{ii'} Q_{jj'} Q_{kk'}$) Jacobi identity is obeyed. Actually, if $\{Q_{ii'}, Q_{jj'}\} = 0$, then the $Q_{ii'} Q_{jj'} Q_{kk'}$ Jacobi identity

$$\{\{Q_{ii'}, Q_{jj'}\}, Q_{kk'}\} + \{\{Q_{ii'}, Q_{kk'}\}, Q_{jj'}\} + \{\{Q_{kk'}, Q_{jj'}\}, Q_{ii'}\} = 0 \quad (2.15)$$

implies that

$$[M_{j'k'} Q_{ii'}] = 0, \quad (2.16)$$

which is contradictory with the third equation of (2.2). So (2.14) must hold. According to the fundamental idea of supersymmetry, the anti-commutator of two fermionic generators must be a linear combination of bosonic generators. On the other hand, since $i$ and $j'$ are independent indices, it is natural to define

$$\{Q_{ii'}, Q_{jj'}\} = k \delta_{ij} M_{ij'}, \quad [M_{ij'}, Q_{kk'}] = [M_{j'k'}, Q_{ii'}] = 0, \quad [M_{i'i'}, Q_{jj'}] = \omega_{i'j'} Q_{jj'}, \quad [M_{i'i'}, Q_{jj}] = \omega_{ij} Q_{j'j}. \quad (2.17)$$

In the first equation, we have introduced a set of new bosonic generators $M_{ij'}$. So the first equation of (2.17) is an explicit example of (1.7). Using (2.17), it is easy to verify that the $Q_{ii'} Q_{jj'} Q_{kk'}$ Jacobi identity and the $Q_{ii'} Q_{jj'} Q_{kk'}$ Jacobi identity are obeyed. One can also define all other possible commutators involving $M_{ij'}$, i.e. $[M_{ii'}, M_{jj'}], [M_{ij'}, M_{kk'}]$ and $[M_{j'j'}, M_{kk'}]$, by requiring that the corresponding Jacobi identities are obeyed. For example, consider the $M_{i'i'}, Q_{jj'} Q_{kk'}$ Jacobi identity

$$[M_{i'i'}, \{Q_{jj'}, Q_{kk'}\}] - \{Q_{jj'}, [M_{i'i'}, Q_{kk'}]\} - \{Q_{kk'}, [M_{i'i'}, Q_{jj'}]\} = 0 \quad (2.18)$$

A short calculation gives

$$[M_{i'i'}, M_{jj'}] = \omega_{i'j'} M_{ij} + \omega_{ij} M_{i'j'}. \quad (2.19)$$

It can be seen that the commutator of two new generators gives rise an $Sp(2N_1)$ generator $M_{ij}$ of $G$ and an $Sp(2N_3)$ generator $M_{i'j'}$ of $G'$. The structure constants of the commutator (2.19) furnish a fundamental representation of $M_{ij}$ and a fundamental representation of $M_{i'j'}$. Similarly, the commutator $[M_{ij'}, M_{kk'}]$, determined by the $M_{ij'} Q_{kk'} Q_{jk'}$ Jacobi identity, is given by

$$[M_{ij'}, M_{kk'}] = \omega_{jk} M_{ik'} + \omega_{i'k} M_{j'k}. \quad (2.20)$$
The generators of \( \mathfrak{osp} \) indices; for instance, \( Q \) oscillators as follows 

\[
[M_{\nu\nu'}, M_{kk'}] = \omega_{\nu\nu'} M_{k\nu} + \omega_{\nu'k'} M_{k'\nu'}, \tag{2.21}
\]

It can be seen that \( M_{kk'} \) provide a fundamental representation of \( M_{\nu\nu'} \).

It is not difficult (though a little tedious) to verify that every Jacobi identity of the “total” superalgebra consisting of \( M_{\nu\nu'} \) and all generators of \( OSp(N_2|2N_1) \) and \( OSp(N_2|2N_3) \) is satisfied. So the five graded commutators in (2.17) must be the correct ones, and the new superalgebra “fused” by \( OSp(N_2|2N_1) \) and \( OSp(N_2|2N_3) \) is closed. The commutation relations of the “fused” superalgebra include (2.17), (2.19), (2.20), (2.21), and the commutation relations of \( OSp(N_2|2N_1) \) and \( OSp(N_2|2N_3) \). These commutation relations suggest that the “fused” superalgebra is simple, i.e. it has no proper invariant sub-superalgebra.

In this way, we have “fused” the two orthosymplectic superalgebras \( OSp(N_2|2N_1) \) and \( OSp(N_2|2N_3) \) by solving the important Jacobi identities. We now want to provide an alternative way to construct the “fused” superalgebra. The main idea is the following: Using oscillators to realize \( OSp(N_2|2N_1) \) and \( OSp(N_2|2N_3) \) first, then all commutation relations for fusing \( OSp(N_2|2N_1) \) and \( OSp(N_2|2N_3) \) can be determined straightforwardly by using oscillator algebras.

To realize \( OSp(N_2|2N_1) \), we introduce a set of bosonic oscillators and a set of fermionic oscillators as follows 

\[
[b_\bar{i}, b_\bar{j}^\dagger] = \delta_\bar{i}\bar{j}, \quad [b_\bar{i}, b_j] = [b_\bar{i}^\dagger, b_j^\dagger] = 0; \quad \{a_\bar{i}, a_j\} = \delta_\bar{i}j, \quad \{a_\bar{i}, a_j^\dagger\} = \{a_\bar{i}^\dagger, a_j\} = 0. \tag{2.22}
\]

Here \( \bar{i} = 1, \ldots, N_2 \) and \( \bar{i} = 1, \ldots, 2N_1 \). We use the invariant tensors \( \delta_{ij} \) and \( \omega_{ij} \) to lower indices; for instance,

\[
b_\bar{i}^\dagger \equiv \delta_{ij} b_j^\dagger, \quad \text{and} \quad a_i^\dagger \equiv \omega_{ij} a_j^\dagger. \tag{2.24}
\]

The generators of \( OSp(N_2|2N_1) \) can be constructed as follows 

\[
Q_{\bar{i}\bar{i}} = \sqrt{-k}(a_\bar{i}^\dagger a_\bar{i} + a_\bar{i} a_\bar{i}^\dagger), \quad M_{\bar{i}j} = b_\bar{i} b_j - b_j b_\bar{i}, \quad M_{\bar{i}\bar{i}} = -(a_\bar{i}^\dagger a_\bar{j} + a_\bar{j} a_\bar{i}^\dagger). \tag{2.25}
\]

It is straightforward to verify that (2.23) satisfy the commutation relations of \( OSp(N_2|2N_1) \) (2.3). Similarly, the generators of \( OSp(N_2|2N_3) \) can be constructed as follows 

\[
Q_{\bar{i}\bar{i'}} = \sqrt{-k}(c_{\bar{i}'} b_\bar{i}^\dagger + c_\bar{i} b_{\bar{i'}}), \quad M_{\bar{i}j} = b_\bar{i} b_j - b_j b_\bar{i}, \quad M_{\bar{i}\bar{i'}} = -(c_\bar{i}^\dagger c_{\bar{i'}} + c_{\bar{i'}} c_\bar{i}), \tag{2.26}
\]

where \( b_j \) and \( b_\bar{i}^\dagger \) are the same as that of (2.22); \( c_\bar{i}^\dagger \) and \( c_{\bar{i'}} \) satisfy \( \omega_{\bar{i}\bar{i'}} c_{\bar{i}} c_{\bar{i'}} \) (\( \bar{i}' = 1, \ldots, 2N_3 \)) are a third independent set of oscillators, satisfying 

\[
\{c_{\bar{i}}^\dagger, c_{\bar{i'}}\} = \delta_{\bar{i}\bar{i'}}, \quad \{c_{\bar{i}}^\dagger, c_{\bar{i}}\} = \{c_{\bar{i}}, c_{\bar{i'}}\} = \{c_{\bar{i'}}^\dagger, c_{\bar{i}}^\dagger\} = 0. \tag{2.27}
\]

With (2.25) and (2.26), the anticommutator of \( Q_{\bar{i}\bar{i}} \) and \( Q_{\bar{j}\bar{j'}} \) is given by 

\[
\{Q_\bar{i}, Q_{\bar{j}\bar{j'}}\} = \{Q_{\bar{i}\bar{i}}, Q_{\bar{j}\bar{j'}}\} = k \delta_{\bar{i}\bar{j}}(-c_{\bar{j}} a_\bar{i} + c_{\bar{i}} a_{\bar{j}}^\dagger). \tag{2.28}
\]
Comparing it with the first equation of (2.17), we are led to define the set of new bosonic generators $M_{ij'}$, as

$$M_{ij'} = -c_{j'}^i a_i + c_j a_{i'}^i. \tag{2.29}$$

Substituting the oscillator realizations (2.25), (2.26), and (2.29) into the commutation relations (2.17), (2.19), (2.20), and (2.21), we find that they are exactly obeyed. Namely, all the generators constructed in terms of oscillators obey exactly the same commutation relations as before. It is therefore unnecessarily to verify the Jacobi identities. In this way, we have constructed the closed superalgebra “fused” by $OSp(N_2|2N_1)$ and $OSp(N_2|2N_3)$ in terms of three independent sets of oscillators. The advantage of the oscillator-realization approach is that it shows explicitly that it is unavoidable to introduce the set of new bosonic generators $M_{ij'}$ (see (2.28) and (2.27)). Also, using oscillators one can construct the commutation relations (2.17), (2.19), (2.20), and (2.21) without any guessing work.

Let us summarize the bosonic subalgebra of the “fused” superalgebra as follows:

$$
\begin{align*}
[M_{ij}, M_{kl}] &= \delta_{jk} M_{il} - \delta_{ik} M_{jl} + \delta_{ij} M_{lk} - \delta_{il} M_{jk}, \\
[M_{ij}, M_{kl}] &= \omega_{jk} M_{il} + \omega_{ik} M_{jl} + \omega_{ij} M_{lk} + \omega_{il} M_{jk}, \\
[M_{i'j'}, M_{k'l'}] &= \omega_{j'k'} M_{i'l'} + \omega_{i'k'} M_{j'l'} + \omega_{i'l'} M_{j'k'} + \omega_{i'j'} M_{l'k'}, \\
[M_{i'l'}, M_{j'k'}] &= \omega_{l'k'} M_{i'j'} + \omega_{l'j'} M_{i'k'}, \\
[M_{i'j'}, M_{kk'}] &= \omega_{j'k'} M_{i'k'} + \omega_{ik'} M_{j'k'}, \\
[M_{i'j'}, M_{kk'}] &= \omega_{j'kk'} M_{i'k'} + \omega_{i'k'} M_{j'k'}. \tag{2.30}
\end{align*}
$$

The other commutators vanish. The first 3 lines are the Lie algebras of $SO(N_2)$, $Sp(2N_1)$, and $Sp(2N_3)$, respectively; the last 3 lines are the commutators involving the set of new generators $M_{i',j'}$ (see (2.19), (2.20), and (2.21)). So the bosonic part of the ‘fused’ superalgebra consists of four sets of generators

$$M^U = (M_{ij}, M_{i'j'}, M_{i'j'}, M_{i'j'}), \tag{2.31}$$

in which we have selected only the first three sets of generators, namely

$$M^m = (M_{ij}, M_{i'j'}, M_{i'j'}), \tag{2.32}$$

to construct the $N = 4$ CSM theory with gauge group $Sp(2N_1) \times SO(N_2) \times Sp(2N_3)$. Notice that (2.32) is an example of (2.10).

It can be seen that the Lie algebra of $SO(N_2)$ (the first line of (2.30)), the common bosonic part of $OSp(N_2|2N_1)$ and $OSp(N_2|2N_3)$, is an invariant subalgebra of (2.30). We now prove that the last five lines of (2.30) are actually the commutation relations of $Sp(2(N_1 + N_3))$. Let us begin by considering the simplest case, i.e. $N_1 = N_3 = 1$. It is convenient to define

$$N^a = i\sigma^{a'i'j'} M_{i'i'}, \quad \text{and} \quad M_{ab} = \frac{1}{2} (\sigma^{abi'j'} M_{i'j'} + \sigma^{abi'j'} M_{i'j'}). \tag{2.33}$$
Here $\sigma^a$, $\sigma^{ab}$ and $\bar{\sigma}^{ab}$ are defined as

$$\sigma^a = (\sigma^1, \sigma^2, \sigma^3, i\mathbb{I}), \quad \sigma^{a\dagger} = (\sigma^1, \sigma^2, \sigma^3, -i\mathbb{I}),$$

$$\sigma^{ab} = \frac{1}{4}(\sigma^a \sigma^b - \sigma^b \sigma^a), \quad \bar{\sigma}^{ab} = \frac{1}{4}(\sigma^a \sigma^b - \sigma^b \sigma^a),$$

(2.34)

(2.35)

where $\sigma^i$ ($i = 1, \ldots, 3$) are Pauli matrices and $\mathbb{I}$ is the 2 $\times$ 2 unit matrix. After some algebraic steps, we convert the last five lines of (2.30) into the form

$$[N^a, N^b] = 4M^{ab},$$

$$[M^{ab}, M^{cd}] = \delta^{bc} M^{ad} - \delta^{ac} M^{bd} - \delta^{bd} M^{ac} + \delta^{ad} M^{bc},$$

$$[M^{ab}, N^c] = \delta^{bc} N^a - \delta^{ac} N^b.$$  

(2.36)

The second line is the familiar Lie algebra of $SO(4)$. However, after combining the first line and the last line, the algebra turns out to be the Lie algebra of $SO(5)$. To see this, define

$$M^{a5} = -M^{5a} = \frac{i}{2} N^a \quad \text{and} \quad M^{55} = 0.$$  

(2.37)

Now (2.36) can be recast into

$$[M^{ij}, M^{kl}] = \delta^{jk} M^{il} - \delta^{ik} M^{jl} - \delta^{jl} M^{ik} + \delta^{il} M^{jk},$$

(2.38)

where $i = 1, \ldots, 5$. Eq. (2.38) is nothing but the Lie algebra of $SO(5)$.

Recall that the Lie algebra of $SO(5)$ is isomorphic to that of $Sp(4)$. So in the special case of $N_1 = N_3 = 1$, the last five lines of (2.31) are indeed the commutation relations of $Sp(2(N_1 + N_3) = Sp(4)$, hence (2.31) is nothing but the Lie algebra of $SO(N_2) \times Sp(4)$. On the other hand, in Section 2.1, we have observed that the “fused” superalgebra is simple. A superalgebra whose bosonic part is the Lie algebra of $SO(N_2) \times Sp(4)$ must be $OSP(N_2|4)$. This special case inspires us to guess that for general $N_1$ and $N_3$, the closed superalgebra “fused” by $OSP(N_2|2N_1)$ and $OSP(N_2|2N_3)$ is nothing but $OSP(N_2|2(N_1 + N_3))$. To prove it, we combine the fermionic generators $Q_{ii}$ and $Q_{i'\nu}$ as follows

$$Q_{I\bar{I}} = \begin{pmatrix} Q_{ii} \\ Q_{i'\nu} \end{pmatrix} = Q_{ii}\delta_{1\alpha} + Q_{i'\nu}\delta_{2\alpha},$$

(2.39)

where $I = 1, \ldots, 2(N_1 + N_3)$ is a collective index; $\delta_{1\alpha} = (1, 0)^T$ and $\delta_{2\alpha} = (0, 1)^T$ are “spin up” spinor and “spin down” spinor, respectively (they are not spacetime spinors).

In the oscillator realization, Eq. (2.39) takes the form

$$Q_{ii} = \sqrt{-k}(A^b_i b_i + A^b_i b_i^\dagger),$$

(2.40)

where we have combined the two independent sets of fermionic oscillators as one set:

$$A_I = \begin{pmatrix} a_i \\ c_i' \end{pmatrix}, \quad A_I^\dagger = \begin{pmatrix} a_i^\dagger \\ c_i' \end{pmatrix}, \quad \{A_I, A_J^\dagger\} = \delta_I^J, \quad \{A_I, A_J\} = \{A_I^\dagger, A_J^\dagger\} = 0.$$

$$\omega_{IJ} = \begin{pmatrix} \omega_{ij} & 0 \\ 0 & \omega_{i'j'} \end{pmatrix}, \quad A_J^\dagger \equiv \omega_{IJ} A_J^\dagger.$$  

(2.41)
With the above collective notation, all anti-commutators of the “fused” superalgebra can be summed up in the single one,

$$\{Q_{ij}, Q_{j\ell}\} = k(\delta_{ij} M_{I\ell} + \omega_{IJ} M_{i\ell}), \quad (2.42)$$

where we have defined

$$M_{I\ell} = M_{ij} \delta_{1\alpha} \delta_{1\beta} + M_{ij'} \delta_{1\alpha} \delta_{2\beta} + M_{j'\ell} \delta_{2\alpha} \delta_{1\beta} + M_{j'\ell'} \delta_{2\alpha} \delta_{2\beta}, \quad (2.43)$$

$$\omega_{IJ} = \omega_{ij} \delta_{1\alpha} \delta_{1\beta} + \omega_{j'\ell} \delta_{2\alpha} \delta_{2\beta}. \quad (2.44)$$

Eq. (2.44) is just the component formalism of $\omega_{IJ}$ defined in (2.41). On the other hand, all commutators between the bosonic generators and the fermionic generators are compacted into two commutators:

$$[M_{I\ell}, Q_{kk}] = \omega_{JK} Q_{k\ell} + \omega_{IK} Q_{k\ell},$$

$$[M_{ij}, Q_{kk}] = \delta_{jk} Q_{ik} - \delta_{ik} Q_{j\ell}. \quad (2.45)$$

With (2.43) and (2.44), after a lengthy algebra, the last five lines of (2.30) can be recast into

$$[M_{I\ell}, M_{KL}] = \omega_{IK} M_{IL} + \omega_{IK} M_{I\ell} + \omega_{IL} M_{IK} + \omega_{IL} M_{I\ell}, \quad (2.46)$$

which is nothing but the Lie algebra of $Sp(2(N_1 + N_3))$. Together with the algebra of $SO(N_2)$ (see the first commutator of (2.4), (2.42), (2.43), and (2.46) are precisely the commutation relations of $OSp(N_2|2(N_1 + N_3))$. This completes the proof. Notice that the bosonic subalgebra of $OSp(N_2|2(N_1 + N_3))$ is the Lie algebra of $SO(N_2) \times Sp(2(N_1 + N_3))$, while the Lie algebra of the gauge group is the Lie algebra of $Sp(2N_1) \times SO(N_2) \times Sp(2N_3)$. Namely, we used only the bosonic part of $OSp(N_2|2N_1)$ and the bosonic part of $OSp(N_2|2N_3)$, not the bosonic part of $OSp(N_2|2(N_1 + N_3))$, to construct the physical theory. The difference is precisely the difference between (2.31) and (2.32).

Notice that two or more theories with different gauge groups may be associated the same “fused” superalgebra. For instance, if we select the bosonic parts of $OSp(N_2|2(N_1 - \lambda))$ and $OSp(N_2|2(N_3 + \lambda))$ ($\lambda = 0, \ldots, N_1 - 1$), sharing the common factor $SO(N_2)$, as the Lie algebra of the gauge group of the $\mathcal{N} = 4$ theory, we will obtain different theories (with different gauge groups) as $\lambda$ runs from 0 to $(N_1 - 1)$. However, the corresponding “fused” superalgebras are independent of $\lambda$, since

$$\left(\begin{array}{ll} OSp(N_2|2(N_1 - \lambda)) & OSp(N_2|2(N_3 + \lambda)) \\ OSp(N_2|2(N_1 - \lambda + N_3 + \lambda)) & OSp(N_2|2(N_1 + N_3)) \end{array}\right).$$

$$\quad (2.47)$$

### 2.3 Fusing $OSp(N_2|2N_1)$ and $OSp(N_4|2N_1)$ into $OSp(N_2 + N_4|2N_1)$

In this section, we choose the superalgebras $G$ and $G'$ as $OSp(N_2|2N_1)$ and $OSp(N_4|2N_1)$, respectively. The common simple factor of their bosonic parts is the Lie algebra of $Sp(2N_1)$.
The commutation relations of $OSp(N_2|2N_1)$ are given by \( [2.2] \). We denote the fermionic generators of $OSp(N_2|2N_1)$ and $OSp(N_4|2N_1)$ as

\[
Q_a = Q_{\bar{i}i} \quad \text{and} \quad Q_{a'} = Q_{\bar{i}'i'},
\]

respectively. Here $\bar{i} = 1, \cdots, N_2$ is an $SO(N_2)$ fundamental index; $\bar{i}' = 1, \cdots, 2N_4$ is an $Sp(2N_1)$ fundamental index; $i' = 1, \cdots, N_4$ is an $SO(N_4)$ fundamental index. The structure constants of the 3-algebra $f_{abc'd'}$ are identified with the structure constants of the double graded commutator on $G$ and $G'$: $[Q_a, Q_b], [Q_c, Q_{a'}] = f_{abc'd'} Q_{d'}$. A short calculation gives

\[
f_{abc'd'} = f_{i\bar{i},j\bar{j},k'\bar{k}',l'\bar{l}'} = -k \delta_{ij} \delta_{k'l'} (\omega_{ij}^{\bar{k}l} + \omega_{ij}^{\bar{l}k}).
\]

Also, $f_{abcd}$ are given by \( (2.8) \) and $f_{a'b'c'd'}$ have a similar expression as that of \( (2.8) \). Substituting these structure constants into the action \( (4.29) \) and the supersymmetry transformations \( (A.31) \) gives the $\mathcal{N} = 4$ CSM theory with gauge group $SO(N_2) \times Sp(2N_1) \times SO(N_4)$. This theory was first constructed in Ref. [3], using an ordinary Lie algebra approach.

To fuse the two superalgebras $G$ and $G'$, we first use oscillators to realize them. The oscillator realization of $OSp(N_2|2N_1)$ is given by \( (2.22)-(2.25) \); and $OSp(N_4|2N_1)$ has a similar realization:

\[
Q_{i'i} = \sqrt{k}(a_i d_{i'} + a_i^\dagger d_{i'}), \quad M_{j'j} = d_{i'}^\dagger d_{j'} - d_{j'}^\dagger d_{i'}, \quad M_{ij} = -(a_i a_j + a_j a_i).
\]

where $a_i$ and $a_i^\dagger$ are the same as that of \( (2.23) \); $d_{i'}$ and $d_{i'}^\dagger \equiv \delta_{i'j'}d_{j'}^\dagger$ ($i' = 1, \cdots, N_4$) are a third independent set of oscillators, satisfying

\[
[d_{i'}, d_{j'}^\dagger] = \delta_{i'j'}, \quad [d_{i'}, d_{j'}] = [d_{j'}^\dagger, d_{i'}^\dagger] = 0.
\]

With a little oscillator algebra, we obtain

\[
\{Q_a, Q_{b'} \} = \{Q_{\bar{i}i}, Q_{\bar{j}'j'} \} = k \omega_{ij} M_{ij'}, \quad M_{ij} = b_i^\dagger d_{j'} - b_{j'} d_{i'}^\dagger.
\]

This provides another example for the assertion in Section 1 that if the bosonic parts of two superalgebras $G$ and $G'$ share one simple factor, then the anticommutator between their fermionic generators cannot vanish, i.e. $\{Q_a, Q_{b'} \} \neq 0$. In order to use the same technique as the previous section, we define

\[
Q_{Ii} = \begin{pmatrix} Q_{\bar{i}i} \\ Q_{i'i} \end{pmatrix} = Q_{\bar{i}i} \delta_{1\alpha} + Q_{i'i} \delta_{2\alpha},
\]

where $I = 1, \cdots, (N_2 + N_4)$ is a collective index; $\delta_{1\alpha} = (1, 0)^T$ and $\delta_{2\alpha} = (0, 1)^T$ are “spin up” spinor and “spin down” spinor respectively (they are not spacetime spinors). In the oscillator construction, Eq. \( (2.53) \) is given by

\[
Q_{fi} = \sqrt{k}(B_i^1 a_i^1 + B_i a_i^\dagger)
\]
where we have combined the two independent sets bosonic oscillators as one set:

\[ B_I = \begin{pmatrix} b_i \cr d_i \end{pmatrix}, \quad B_I^\dagger = \begin{pmatrix} b_i^\dagger \\ d_i^\dagger \end{pmatrix}, \quad [B_I, B_J^\dagger] = \delta_{IJ}, \quad [B_I, B_J] = [B_I^\dagger, B_J^\dagger] = 0. \]

\[ \delta_{IJ} = \begin{pmatrix} \delta_{ij} & 0 \\ 0 & \delta_{ij'} \end{pmatrix}, \quad B_I^\dagger \equiv \delta_{IJ} B_I^\dagger. \] (2.55)

With the above compact notation, we obtain

\[ \{Q_{II}, Q_{JJ}\} = k(\delta_{IJ} M_{ij} + \omega_{ij} M_{IJ}), \quad [M_{ij}, Q_{Kk}] = \omega^i_{jk} Q_{Kk} + \omega^j_{ik} Q_{Kj}, \] (2.56)

where we have defined

\[ M_{IJ} = M_{ij} \delta_{1\alpha} \delta_{1\beta} + M_{ij'} \delta_{1\alpha} \delta_{2\beta} - M_{ji'} \delta_{2\alpha} \delta_{1\beta} + M_{ji} \delta_{2\alpha} \delta_{2\beta}, \] (2.57)

\[ \delta_{IJ} = \delta_{ij} \delta_{1\beta} + \delta_{ij'} \delta_{2\beta}. \] (2.58)

Eq. (2.58) is the component formalism of \( \delta_{IJ} \) defined in (2.55). On the other hand, by either requiring that the \( Q_{II} Q_{JJ} Q_{Kk} \) Jacobi identity is obeyed or using straightforward oscillator algebra, we can obtain

\[ [M_{IJ}, Q_{Kk}] = \delta_{IK} Q_{Ik} - \delta_{IK} Q_{Jk}. \] (2.59)

Similarly, by either requiring that the \( Q_{II} Q_{JJ} M_{KL} \) Jacobi identity is obeyed or using a rather lengthy oscillator algebra, we can derive the commutator

\[ [M_{IJ}, M_{KL}] = \delta_{IK} M_{IJ} - \delta_{IK} M_{JL} - \delta_{IL} M_{IK} + \delta_{IL} M_{JK}, \] (2.60)

which is just the Lie algebra of \( SO(N_2 + N_3) \). Together with the algebra of \( Sp(N_1) \) (the second line of (2.52), (2.56), (2.59), and (2.60) are precisely the commutation relations of \( OSp(N_2 + N_4|2N_1) \). Although we have completed the “fusion” of \( G \) and \( G' \) in terms of their oscillator realizations, the commutation relations (2.52), (2.56), (2.59), and (2.60) must hold in general.

3. Fusing \( U(N_1|N_2) \) and \( U(N_2|N_3) \) into \( U(N_1 + N_3|N_2) \)

In this section, we select the superalgebras \( G \) and \( G' \) as \( U(N_1|N_2) \) and \( U(N_2|N_3) \), respectively. Namely, the common part of their bosonic subalgebras is the Lie algebra of \( U(N_2) \), which is not simple due to the fact that \( U(N_2) = SU(N_2) \times U(1) \). The commutation relations of \( U(M|N) \) are given by Appendix 4.1. We denote the fermionic generators of \( U(N_2|N_3) \) as \( Q_{\dot{a}'} \) and \( Q_{\dot{a}''} \), where the subscript index \( \dot{a} = 1, \ldots, N_2 \) is a fundamental index of \( U(N_2) \) and the superscript index \( \dot{a}' = 1, \ldots, N_3 \) is an anti-fundamental indices of \( U(N_3) \). Similarly, we denote the fermionic generators of \( U(N_1|N_2) \) as \( Q_{\dot{a}'} \) and \( Q_{\dot{a}''} \), with \( \dot{a}' = 1, \ldots, N_1 \).

We first use \( U(N_1|N_2) \) and \( U(N_2|N_3) \) to construct the 3-algebra in the \( \mathcal{N} = 4 \) theory. It is useful to define

\[ Q_a = \begin{pmatrix} Q_{\dot{a}'} \\ -Q_{\dot{a}''} \end{pmatrix} = Q_{\dot{a}'} \delta_{1\lambda} - Q_{\dot{a}''} \delta_{2\lambda}, \quad Q_a' = \begin{pmatrix} Q_{\dot{a}'} \\ -Q_{\dot{a}''} \end{pmatrix} = Q_{\dot{a}'} \delta_{1\alpha} - Q_{\dot{a}''} \delta_{2\alpha}, \] (3.1)
where $\delta_{1\lambda} = (1,0)^T$ and $\delta_{2\lambda} = (0,1)^T$ are “spin up” spinor and “spin down” spinor, respectively. As usual, the structure constants of the 3-algebra $f_{ab'c'd'}$ can be read off from the double graded commutator: $\{Q_a, Q_b\}, Q_{c'} = f_{ab'c'd'} Q_{d'}$. A short calculation gives

$$f_{ab'c'd'} = -k(\hat{\delta}_i^j \hat{\delta}_b^d \delta_{a'd'} + \hat{\delta}_i^j \delta_{a'b'} \hat{\delta}_d^c + \hat{\delta}_i^j \delta_{a'b'} \delta_{a'd'} + \hat{\delta}_i^j \delta_{a'b'} \delta_{a'd'}).$$

Similarly, we obtain the structure constants $f_{abcd}$,

$$f_{abcd} = f_i^j, k \bar{f}^i_{jk}, m \bar{f}^j_{km}, n \bar{f}^k_{in}, \bar{f}^l_{ij} \delta_{k\ell} \delta_{l\rho} \hat{\delta}_{ij} \bar{\delta}_{k\ell} \bar{\delta}_{l\rho} = 0. (3.2)$$

where

$$f_i^j, k \bar{f}^i_{jk}, m \bar{f}^j_{km}, n \bar{f}^k_{in}, \bar{f}^l_{ij} \delta_{k\ell} \delta_{l\rho} \hat{\delta}_{ij} \bar{\delta}_{k\ell} \bar{\delta}_{l\rho} = 0. (3.3)$$

And $f_{a'b'c'd'}$ have a similar expression as that of (3.3). Substituting these structure constants into (A.29) and (A.31) gives the $\mathcal{N} = 4$, $U(N_1) \times U(N_2) \times U(N_3)$ CSM theory. This theory was first derived in Ref. [4], using an ordinary Lie algebra approach.

To fuse $U(N_1|N_2)$ and $U(N_2|N_3)$, let us first construct them by the following three independent sets of oscillators:

$$\{a_i, a_i^\dagger\} = \hat{\delta}_i^j, \quad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0;$$

$$[b_\mathbf{a}, b_\mathbf{b}] = \delta_\mathbf{a}^\mathbf{b}, \quad [b_\mathbf{a}, b_\mathbf{b}] = [b_\mathbf{a}^\dagger, b_\mathbf{b}^\dagger] = 0;$$

$$[c_{\mathbf{i}}, c_{\mathbf{j}}^\dagger] = \delta_{\mathbf{i} \mathbf{j}}, \quad [c_{\mathbf{i}}, c_{\mathbf{j}}] = [c_{\mathbf{i}}^\dagger, c_{\mathbf{j}}^\dagger] = 0. (3.5)$$

Here $i = 1, \ldots, N_1$, $\mathbf{a} = 1, \ldots, N_2,$ and $i' = 1, \ldots, N_3$. In terms of the oscillators, the generators of $U(N_1|N_2)$ are given by

$$Q_{i}^\mathbf{a} = \sqrt{-k} a_i b_{i}^\dagger, \quad Q_{i}^\mathbf{a} = \sqrt{-k} a_i b_{i}^\dagger, \quad M_{i}^\mathbf{a} = a_i a_i^\dagger, \quad M_{i}^\mathbf{a} = b_i b_i^\dagger. (3.6)$$

A little algebra shows that they indeed obey the commutation relations of $U(N_1|N_2)$. Similarly, the generators of $U(N_2|N_3)$ are given by

$$Q_{i}^{\mathbf{a}'} = \sqrt{-k} b_{i} c_{i}'^\dagger, \quad Q_{i}^{\mathbf{a}'} = \sqrt{-k} c_{i}' b_{i}^\dagger, \quad M_{i}^{\mathbf{a}'} = a_i a_i^\dagger, \quad M_{i}^{\mathbf{a}'} = b_i b_i^\dagger. (3.7)$$

Now it is easy to calculate the non-trivial anticommutators between the fermionic generators of $U(N_1|N_2)$ and $U(N_2|N_3)$; they are given by

$$\{Q_{i}^\mathbf{a}, Q_{i}^{\mathbf{a}'}\} = k\delta_{\mathbf{a} \mathbf{a}'} (c_{\mathbf{i}'} a_{\mathbf{i}}^\dagger) \equiv k \delta_{\mathbf{a} \mathbf{a}'} M_{i}^{\mathbf{a}'};$$

$$\{Q_{i}^{\mathbf{a}}, Q_{i}^{\mathbf{a}'}\} = k\delta_{\mathbf{a} \mathbf{a}'} (a_{\mathbf{i}} c_{\mathbf{i}'}^\dagger) \equiv k \delta_{\mathbf{a} \mathbf{a}'} M_{i}^{\mathbf{a}'} (3.8)$$

Combining them with (3.1), we have

$$\{Q_{i}, Q_{i}^{\mathbf{a}}\} = -k (\delta_{\mathbf{a} \mathbf{a}'} M_{i}^{\mathbf{a}'} \delta_{1\lambda} \delta_{2\beta} + \delta_{\mathbf{a} \mathbf{a}'} M_{i}^{\mathbf{a}'} \delta_{2\lambda} \delta_{1\beta}). (3.9)$$
This is the third example that if the bosonic subalgebras of two superalgebras $G$ and $G'$ have a common part, then their fermionic generators have nontrivial anticommutators.

We now combine the two independent sets fermionic oscillators as one set:

$$
A_I = \left( \begin{array}{c} a_i \\ c_i' \end{array} \right), \quad A_I^\dagger = \left( \begin{array}{c} a_i^\dagger \\ c_i'^\dagger \end{array} \right), \quad \{A_I, A_J^\dagger\} = \delta_{IJ}^J, \quad \{A_I, A_J\} = \{A_I^\dagger, A_J^\dagger\} = 0,
$$

$$
\delta_{IJ}^J = \left( \begin{array}{cc} \delta_{iJ}^j & 0 \\ 0 & \delta_{i'J'}^{j'} \end{array} \right),
$$

(3.10)

where $I = 1, \ldots, (N_1 + N_3)$ is a collective index. We now are able to define

$$
Q_\bar{u}^I = \left( Q_\bar{u}_i^I \right) = \sqrt{-k} b_\bar{u} A_I^\dagger, \quad \bar{Q}_I^\bar{u} = \left( \bar{Q}_i^I \right) = \sqrt{-k} A_I b_{\bar{u}}^\dagger.
$$

(3.11)

As a result, we can put the known anticommutator and commutator in the compact forms

$$
\{Q_\bar{u}^I, \bar{Q}_J^\bar{v}\} = k \left( \delta_{\bar{u}J}^J M_{\bar{v}I}^J + \delta_{\bar{u}J}^J M_{\bar{v}I}^K \right), \quad [M_\bar{u}^\bar{v}, Q_\bar{u}^I] = \delta_{\bar{u}J}^J Q_\bar{v}^I, \quad [M_\bar{u}^\bar{v}, \bar{Q}_I^\bar{u}] = -\delta_{\bar{u}J}^J \bar{Q}_I^\bar{v},
$$

(3.12)

where we have defined

$$
M_{\bar{I}J} = \left( \begin{array}{cc} M_{\bar{i}J} & M_{\bar{i}J'} \\ M_{\bar{j}'I} & M_{\bar{j}'I} \end{array} \right).
$$

(3.13)

By either requiring that the $Q_I^J Q_J^J Q_{K\bar{K}}^K$ Jacobi identity is obeyed or using oscillator algebra, we can obtain

$$
[M_{\bar{I}J}, Q_\bar{u}^K] = -\delta_{\bar{I}K}^J Q_\bar{u}^J, \quad [M_{\bar{I}J}, \bar{Q}_K^\bar{u}] = \delta_{\bar{K}J}^J \bar{Q}_I^\bar{u}.
$$

(3.14)

Similarly, by either requiring that the $\bar{Q}_I^\bar{u} Q_J^J M_{K\bar{L}}^L$ Jacobi identity is obeyed or using oscillator algebra, one can derive the following commutator

$$
[M_{\bar{I}J}, M_{K\bar{L}}^L] = \delta_{\bar{J}K}^J M_{I\bar{L}}^L - \delta_{\bar{I}L}^J M_{K\bar{J}}^L.
$$

(3.15)

We recognize that it is the Lie algebra of $U(N_1 + N_3)$. Together with the algebra of $U(N_2)$, we find that (3.12), (3.14), and (3.15) furnish the commutation relations of $U(N|N_1 + N_3)$. Notice that the “fused” superalgebra $U(N|N_1 + N_3)$ can be used to construct the $\mathcal{N} = 6$ theory [7].

4. Generalizations of Fusing Superalgebras

In this section, we shall work out the general structure of the superalgebra “fused” by two superalgebras $G$ and $G'$ whose bosonic parts share at least one simple factor or $U(1)$ factor, and work out some generalizations such as “fusing” three or more superalgebras. The general commutation relations of $G$ and $G'$ are given by (B.1) and (B.2), respectively.
4.1 General Structure of Superalgebras Fused by 2 Superalgebras

Given two superalgebras $G$ and $G'$, what interests us most is that the bosonic parts of the superalgebras $G$ and $G'$ share at least one simple factor or $U(1)$ factor, while we do *not* identify $G$ and $G'$. Schematically, we have $M^g = (M^\alpha, M^g)$ and $M'^{g'} = (M'^\alpha, M^g)$, with $M^g$ the set of generators of the common bosonic part of $G$ and $G'$, i.e. $M^a \cap M'^{a'} = M^g \neq \emptyset$, but we exclude the possibility that $Q_a = Q_a'$ and $M^\alpha = M^\alpha = \emptyset$. Equivalently, we must require that [13]

$$\kappa(Q_a, Q_{b'}) = \omega_{ab'} = 0. \quad (4.1)$$

(The forms $\kappa$ are defined in Appendix [3]; for instance, $\omega_{ab} = \kappa(Q_a, Q_b).$)

Recall that in Section 1 we have defined

$$\{Q_a, Q_{b'}\} = \tilde{t}^\alpha_{ab'}M_\alpha. \quad (4.2)$$

(See Eq. (1.7).) Comparing the first equation of (2.17) with the above equation, we have

$$M_\alpha = M_{\alpha'}, \quad (\tilde{t}^\alpha)_{\alpha', \beta'} = (\tilde{t}^\beta)_{\alpha', \beta'} = k\delta_{\beta'}^\alpha \delta_{\beta'}^{\alpha'}. \quad (4.3)$$

Eqs. (2.52) and (3.9) are the other two explicit examples of (4.2). Using (4.2) and the third equations of (B.6) and (B.12), the $Q_a Q_b Q_{c'}$ Jacobi identity (4.6) can be converted into

$$\tau^g_{ab} k_{gh} \tau^{gd'} c' Q_{d'} + t^a_{bc}[M_\alpha Q_b] + t^b_{ac'}[M_\alpha Q_a] = 0. \quad (4.4)$$

Notice that the first term is a linear combination of the set of generators $Q_{d'}$. We are therefore led to define

$$[M_\alpha Q_a] = t^{\alpha}_{\alpha d'} Q_{d'}. \quad (4.5)$$

Roughly speaking, the new bosonic generators $M_\alpha$ must “rotate” the set of fermionic generators $Q_a$ into $Q_{d'}$. This means that the last term of the right hand side of $[M_\alpha Q_a] = t^{\alpha}_{\alpha d'} Q_{d'} + t^{\alpha}_{\alpha d} Q_d$ vanishes, i.e. $t^{\alpha}_{\alpha d} = 0$. A proof on that $t^{\alpha}_{\alpha d} = 0$ can be found in Ref. [13]. The last equation of (2.17) is an example of (4.5). With Eq. (1.5), the $Q_a Q_b Q_{c'}$ Jacobi identity (4.6) becomes

$$[k_{gh} \tau^g_{ab} \tau^{hb'} c' + t^a_{bc} t^{b'}_{ab'} + t^b_{bc} t^{d'}_{ab'}] Q_{d'} = 0. \quad (4.6)$$

This is a non-linear constraint on these structure constants. Now the key point is that we have to define $t^{\alpha}_{\alpha d'}$ carefully so that the identity (4.6) is obeyed. Similarly, requiring that the $Q_{c'} Q_{d'} Q_a$ Jacobi identity is obeyed leads us to define

$$[M_\alpha Q_{c'}] = t^{\alpha}_{\alpha b'} Q_b. \quad (4.7)$$

Namely, the new bosonic generators $M_\alpha$ “rotate” the set of fermionic generators $Q_{c'}$ into $Q_b$. The first equation of the second line of (2.17) is an example of (1.7). With (4.7), the $Q_{a'} Q_{b'} Q_{c'}$ Jacobi identity can be converted into

$$[k_{gh} \tau^{g}_{c'} d' R^h_{ab} + t^a_{bc} t^{b'}_{ab'} + t^b_{bc} t^{d'}_{ab'}] Q_b = 0. \quad (4.8)$$
The equation in the bracket is essentially equivalent to that of (4.6). This is also consistent with the requirement that using either \([\{Q_a, Q_b\}, Q_c]\) or \([\{Q_c, Q_d\}, Q_a]\) to calculate the structure constants \(f_{abc'd'}\) gives the same result

\[
f_{abc'd'} = k_{gh} \tau^g_{ab} \tau^h_{c'd'}.
\] (4.9)

All other unknown commutators, \([M^\tilde{a}, M^\tilde{b}]\), \([M^u, M^\tilde{a}]\), and \([M^u', M^\tilde{a}]\) can be determined by requiring that the \(M^\tilde{a}Q_aQ_{b'}, M^uQ_aQ_{b'},\) and \(M^u'Q_aQ_{b'}\) Jacobi identities are obeyed, respectively. For instance, let us consider the \(M^\tilde{a}Q_aQ_{b'}\) Jacobi identity

\[
[M^\tilde{a}, \{Q_a, Q_{b'}\}] - \{[M^\tilde{a}, Q_a], Q_{b'}\} - \{[M^\tilde{a}, Q_{b'}], Q_a\} = 0.
\] (4.10)

After some algebraic steps, we obtain the equation

\[
(t_{\tilde{a}})_{ab}[M^\tilde{a}, M^\tilde{b}] = t^{\tilde{a}c} \tau^w_{ab} k_{wuv} M^v + (t_{\tilde{a}})_{a} \tau^{u'v'}_{cb} k_{u'v'} M^{v'},
\] (4.11)

which determines the commutator \([M^\tilde{a}, M^\tilde{b}]\), since \((t_{\tilde{a}})_{ab}\) is generally consisted of by invertible invariant tensors (the last equation of (4.3) is such an example). Eq. (2.19) is an explicit example which can be derived from Eq. (4.11). Though it seems that both \(M^u\) and \(M^u'\) appear in the right hand side of (1.11), their common generators \(M^g\) are generally absent in the right hand side of the equation

\[
[M^\tilde{a}, M^\tilde{b}] = f^{\tilde{a}ig} M^i M^j.
\] (4.12)

For instance, in Eq. (2.19), the common generators \(M_{ij}\) do not appear in the right hand side. Similarly, by using Jacobi identities, we can obtain

\[
(t_{\tilde{a}})_{ab}[M^u, M^\tilde{b}] = (\tau^{uc'}_{b'} t^{\tilde{a}c'}_{ab'} k_{uab} + \tau^{uc}_{b'c} t^{\tilde{a}}_{ab} k_{uab}) M^\tilde{b},
\] (4.13)

\[
(t_{\tilde{a}})_{ab}[M^u', M^\tilde{b}] = (\tau^{uc}_{a'b'} t^{\tilde{a}c}_{ab'} k_{uab} + \tau^{uc'}_{ab'} t^{\tilde{a}}_{a'b'} k_{uab}) M^\tilde{b}.
\] (4.14)

Eqs. (2.20) and (2.21) are two explicit examples which can be derived from (4.13) and (4.14), respectively.

In summary, we have defined the commutation relations (4.2), (4.5), (4.7), (4.11), (4.13), and (4.14) for fusing \(G\) and \(G'\), by requiring that the corresponding Jacobi identities are obeyed. If all Jacobi identities of the “total” superalgebra consisting of \(M_\tilde{a}\) and all generators of \(G\) and \(G'\) are satisfied, we say that the superalgebra “fused” by \(G\) and \(G'\) is closed. Notice that in fusing \(G\) and \(G'\), we have not introduced any fermionic generators; the set of bosonic generators \(M_\tilde{a}\) are the only ones introduced for the fusion. An alternate approach is that one can use oscillators to construct the generators of \(G\) and \(G'\) first, then use straightforward oscillator algebra to derive the commutation relations (4.2), (4.5), (4.7), (4.11), (4.13), and (4.14), as we did in Sections 2.2, 2.3, and 3.

We can combine the two sets of fermionic generators \(Q_a\) and \(Q_{b'}\) into one set

\[
Q_I = \begin{pmatrix} Q_a \\ Q_{a'} \end{pmatrix}.
\] (4.15)
Eqs. (2.39), (2.53), and (3.11) are explicit examples of the above combination. As a result, all commutation relations of the “fused” superalgebra can be put into the compact form

\[ \{Q_I, Q_J\} = \tau_{IJ}^U k_{UV} M^V, \quad [M^U, Q_I] = -\tau_{IJ}^U \omega^{JK} Q_K, \quad [M^U, M^V] = f^{UVW} M^W. \quad (4.16) \]

For instance, Eqs. (2.43), (2.45), and (2.46) are concrete examples of the first, second, and third equations of (4.16), respectively.

The \( M^9 M^h \alpha \) and \( M^9 M^h \alpha \) Jacobi identities are nontrivial and always obeyed due to the fact that \( M^u \cap M^{\alpha} = M^9 \neq \emptyset \) (see Appendix [3]), so that the second and third FIs of (A.14) are always satisfied, even one cannot fuse \( G \) and \( G' \) into a single closed superalgebra.

### 4.2 Fusing More Than Two Superalgebras

It is straightforward to generalize the “fusion” procedure to fuse three or more superalgebras. We may have to add new fermionic generators to fuse three or more superalgebras. Consider for example the following superalgebras

\[ (G_1, G_2, G_3) = (U(N_1 | N_2), U(N_2 | N_3), U(N_3 | N_4)). \quad (4.17) \]

Here \( G_1 \) and \( G_2 \), whose bosonic parts share a common part \( U(N_2) \), satisfy the conditions as \( G \) and \( G' \) do (see Sec. 3); \( G_2 \) and \( G_3 \) also satisfy the conditions as \( G \) and \( G' \) of Sec. 3 do, but their bosonic parts share a common part \( U(N_3) \). However, the superalgebra \( G_1 \) is independent of \( G_3 \), in the sense that every generator of \( G_1 \) commutes or anticommutes with every generator of \( G_3 \). Using (4.17) to construct the 3-algebra in the \( N = 4 \) quiver gauge theory gives the quiver diagram for the gauge group [3]

\[ U(N_1) - U(N_2) - U(N_3) - U(N_4). \quad (4.18) \]

The three copies of multiplets are in the bifundamental representations of \( U(N_1) \times U(N_2) \), \( U(N_2) \times U(N_3) \), and \( U(N_3) \times U(N_4) \), respectively.

To fuse \( G_1 \sim G_3 \), let us try to utilize their oscillator realizations. Recall that \( G_1 \) and \( G_2 \) are constructed in terms three independent oscillators in Eqs. (3.6), and their generators are given by (3.6) and (3.7), respectively. To construct \( G_3 \) in terms of oscillators, we need to introduce a fourth independent set of oscillators

\[ [d_\dot{a}, d^\dot{a}] = \delta_\dot{a}^\dot{b}, \quad [d_\dot{a}, d_\dot{b}] = [d^\dot{a}, d^\dot{b}] = 0. \quad (4.19) \]

Now the generators of \( G_3 \) can be constructed in terms of (4.19) and the third set of oscillators of (3.5):

\[ Q_{\dot{a}} \dot{\imath} = \sqrt{-k} d_\dot{a} c^{\dot{\imath} \dagger}, \quad \bar{Q}_{\dot{a}} \dot{\imath} = \sqrt{-k} c_{\dot{a}} d^{\dot{\imath} \dagger}, \quad M_\dot{\imath} \dot{\jmath} = a_\imath a_j \dagger, \quad M_\dagger \dot{\imath} = -d_\dot{a} d^{\dot{\imath}}. \quad (4.20) \]

Let us now pick up three fermionic generators from \( G_1 \), \( G_2 \), and \( G_3 \) respectively, and consider the Jacobi identity of these 3 generators:

\[ \{\{\bar{Q}_{\dot{a}} \dot{\imath}, Q_{\dot{b}} \dot{\jmath}\}, Q_{\dot{c}} \dot{k}\} + \{\{Q_{\dot{c}} \dot{k}, Q_{\dot{a}} \dot{\imath}\}, \bar{Q}_{\dot{b}} \dot{\jmath}\} + \{\{Q_{\dot{b}} \dot{\jmath}, Q_{\dot{c}} \dot{k}\}, Q_{\dot{a}} \dot{\imath}\} = 0. \quad (4.21) \]
Without any calculating, we notice immediately that the last term must vanish since $G_1$ is independent of $G_3$. A short calculation shows that the first two terms add up to be zero. The explicit expression of the first term is given by

$$[(\bar{Q}_i^{\hat{a}}, Q_i^{\hat{c}}), \bar{Q}_k^{\hat{c}'}] = \delta^{\hat{a}}_v \delta^{\hat{c}'}_v (\sqrt{-k} b_i d^{\hat{a} \dagger}) \equiv \delta^{\hat{a}}_v \delta^{\hat{c}'}_v \bar{Q}_i^{\hat{a}},$$

(4.22)

where we have defined a set of fermionic generators $\bar{Q}_i^{\hat{a}}$. Similarly, the $Q_u^{\hat{i}} Q_v^{\hat{c}} Q_u^{\hat{c}'}$ Jacobi identity is also obeyed, and we must introduce another set of fermionic generators $Q_u^{\hat{i}}$ defined by the equation

$$[(\bar{Q}_u^{\hat{i}}, \bar{Q}_v^{\hat{c}}), Q_u^{\hat{c}'}] = -\delta^{\hat{i}}_u \delta^{\hat{c}'}_v (\sqrt{-k} d_u b^i) \equiv -\delta^{\hat{i}}_u \delta^{\hat{c}'}_v \bar{Q}_u^{\hat{i}}.$$

(4.23)

Namely, we must introduce the fourth set of fermionic generators

$$Q_u = \begin{pmatrix} \bar{Q}_i^{\hat{a}} \\ -Q_u^{\hat{i}} \end{pmatrix}$$

(4.24)

into the system. According to Sec. 1, this is a typical fermionic fusion\(^2\), since we have introduced a set of new fermionic generators $Q_u^{\hat{i}}$ to fuse the three superalgebras. It is easy to verify that $\bar{Q}_i^{\hat{a}}$ and $Q_u^{\hat{i}}$ obey the commutation relations of $\mathcal{U}(N_1|N_4)$; for instance,

$$\{Q_u^{\hat{i}}, \bar{Q}_j^{\hat{c}}\} = k(\delta^{\hat{c}}_u \delta^{\hat{i}}_j M_{tj}^{\hat{c}} + \delta^{\hat{i}}_j M_{u}^{\hat{c}}).$$

(4.25)

Therefore, to obtain a fermionic fusion of $G_1 \sim G_3$, it is necessarily to introduce the fermionic generators of $\mathcal{U}(N_1|N_4)$ hence the the superalgebra $\mathcal{U}(N_1|N_4)$ into the system. Let us denote $\mathcal{U}(N_1|N_4)$ as $G_4$. Then the fermionic fusion of the superalgebras $G_1 \sim G_3$ (see (4.17)), is the same as the fusion of the four superalgebras

$$(G_1, G_2, G_3, G_4) = (\mathcal{U}(N_1|N_2), \mathcal{U}(N_2|N_3), \mathcal{U}(N_3|N_4), \mathcal{U}(N_1|N_4)).$$

(4.26)

These superalgebras form a closed “loop”

$$G_1 \quad G_4 \quad G_2 \quad G_3$$

(4.27)

in which the bosonic parts of every adjacent pair share one common part. The superalgebra fused by $G_1 \sim G_4$ is $\mathcal{U}(N_1 + N_3|N_2 + N_4)$. If we use (4.28) to construct the $\mathcal{N} = 4$ theory \(^{13}\), in accordance with (4.27), the resulting quiver diagram for the gauge group is

$$\begin{array}{c}
U(N_1) - U(N_2) \\
| \\
U(N_4) - U(N_3)
\end{array}$$

(4.28)

\(^2\)The definition of fermionic fusion is essentially different from that of the fusion in Sec. 1 (see also Sec. 4), since in defining the fusion, we have not introduced any new fermionic generators. The fermionic fusion may be not interesting as the fusion itself, since in principal, by adding sufficient fermionic generators, one may fuse any two or more superalgebras into a single close superalgebra. However, it is still interesting to ask that what are the minimum numbers of fermionic generators needed for fusing two or more superalgebras into a single closed one.
The four copies of multiplets are in the bifundamental representations of $U(N_1) \times U(N_2)$, $U(N_2) \times U(N_3)$, $U(N_3) \times U(N_4)$, and $U(N_4) \times U(N_1)$, respectively.

We see that the theory constructed by using (4.20) is completely different from the one constructed by using (4.17). However, they have the same underlying structure, in the sense that both (4.17) and (4.20) can be “fused” into the same superalgebra $U(N_1 + N_3 | N_2 + N_4)$. We have to emphasize that the Lie algebra of the gauge group represented by either (4.18) or (4.28) is just a proper subalgebra of the bosonic part of $U(N_1 + N_3 | N_2 + N_4)$, not its full bosonic part.

Also, if we select the bosonic parts of $U(N_1 + N_3 | N_2 - \lambda)$ and $U(N_1 + N_3 | N_4 + \lambda)$ ($\lambda = 0, \ldots, N_2 - 1$), sharing the common factor $U(N_1 + N_3)$, as the Lie algebra of the gauge symmetry, we will obtain a set of different $\mathcal{N} = 4$ theories as $\lambda$ runs from 0 to $N_2 - 1$; the corresponding quiver diagrams are given by

$$U(N_2 - \lambda) - U(N_1 + N_3) - U(N_4 + \lambda), \quad \lambda = 0, \ldots, N_2 - 1.$$  

(4.29)

However, all the corresponding “fused” superalgebras are the same, since

$$\left( U(N_1 + N_3 | N_2 - \lambda) \text{ fusing } U(N_1 + N_3 | N_4 + \lambda) \right) = U(N_1 + N_3 | N_2 - \lambda + N_4 + \lambda) = U(N_1 + N_3 | N_2 + N_4).$$

(4.30)

So the theories depending on $\lambda$ are not only different form each other, but also different from the two theories constructed by using (4.17) and (4.20) respectively. But all these theories are associated with the same “fused” superalgebra $U(N_1 + N_3 | N_2 + N_4)$.

Let us now try to “fuse” the superalgebras in (4.17) in an alternative approach. On one hand, fusing $G_1$ and $G_2$ first gives $U(N_1 + N_3 | N_2)$. We now must fuse $U(N_1 + N_3 | N_2)$ and $G_3 = U(N_3 | N_4)$. On the other hand, fusing $G_2$ and $G_3$ first gives $U(N_3 | N_2 + N_4)$. Hence we must fuse $U(N_3 | N_2 + N_4)$ and $G_1 = U(N_1 | N_2)$. The two fusions must give the same final result, i.e.

$$\left( U(N_1 + N_3 | N_2) \text{ fusing } U(N_3 | N_4) \right) = \left( U(N_3 | N_2 + N_4) \text{ fusing } U(N_1 | N_2) \right).$$

(4.31)

Let us look at the left hand side. We see that the common part, the Lie algebra of $U(N_3)$, is a proper subalgebra of the Lie algebra of $U(N_1 + N_3)$ of the bosonic part of $U(N_1 + N_3 | N_2)$, provided that $N_1 \neq 0$. As a result, we encounter an interesting complication. However, by using a little oscillator algebra, we find that it is necessarily to add $U(N_1 | N_4)$ into the left hand side, and fusing these three superalgebras gives exactly $U(N_1 + N_3 | N_2 + N_4)$, which also can be derived by fusing $U(N_1 | N_4)$ and the two superalgebras in the right hand side.

Consider the general line-like (not a closed loop) case of unitary superalgebras

$$(G_1, \ldots, G_n) = (U(N_1 | N_2), U(N_2 | N_3), \ldots, U(N_{n-1} | N_n), U(N_n | N_{n+1})).$$

(4.32)

where the bosonic parts of any adjacent pair $G_i$ and $G_{i+1}$ ($i = 1, \ldots, n-1; n \geq 3$) share a common part $U(N_{i+1})$; any pair of superalgebras are independent if they not adjacent ($G_1$
and $G_n$ are also independent). They can be fused into

$$
U \left( \sum_{k=1}^{1+\frac{n}{2}} N_{2k-1} \right| \sum_{k=1}^{\frac{n}{2}+1} N_{2k} \right),
$$

(4.33)

where $\lfloor \frac{n}{2} \rfloor$ is the integer part of $\frac{n}{2}$. In (4.32), if $n$ is even and the bosonic parts of $G_1$ and $G_n$ share the Lie algebra of $U(N_1) = U(N_{n+1})$, then (4.32) becomes a closed loop and the resulting “fused” superalgebra is also (4.33). However, if $n$ is odd, it seems that one cannot fuse these superalgebras forming a closed loop into a single closed superalgebra. Also, their bosonic parts cannot be selected as the Lie algebra of gauge group of the $\mathcal{N} = 4$ theories [5]. For example, consider the simplest case

$$(G_1, G_2, G_3) = (U(N_1|N_2), U(N_2|N_3), U(N_3|N_1)),
$$

(4.34)

which is a closed loop with odd number of superalgebras, in the sense that the bosonic parts of every pair superalgebras share one common part. If we pick up three fermionic generators from $G_1$, $G_2$, and $G_3$, respectively, then their Jacobi identity cannot be obeyed. Also, if we select the bosonic parts of (4.34) as the Lie algebra of gauge group of the $\mathcal{N} = 4$ theory, then at least two multiplets (out of the three multiplets) are in the dotted or undotted representation of the $SU(2) \times SU(2)$ R-symmetry group simultaneously. As a result, at least one $QQQ$ Jacobi identity cannot be obeyed, so the bosonic parts of $G_1$, $G_2$, and $G_3$ cannot be selected as the Lie algebra of gauge group of the $\mathcal{N} = 4$ theory [5].

Similarly, one can also consider the line-like case of orthosymplectic superalgebras,

$$(OSp(M_1|2N_1), OSp(M_1|2N_2), OSp(M_2|2N_2), \ldots, OSp(M_n|2N_n), OSp(M_n|2N_{n+1}), OSp(M_{n+1}|2N_{n+1}))
$$

(4.35)

where the bosonic parts of any adjacent pair of superalgebras share one and only one common simple factor; any pair of superalgebras are independent if they not adjacent ($(OSp(M_1|2N_1)$ and $OSp(M_{n+1}|2N_{n+1})$ are independent as well). One can fuse them into

$$
OSp \left( \sum_{k=1}^{n+1} M_k \right| 2 \sum_{k=1}^{n+1} N_k \right).
$$

(4.36)

If the number of all superalgebras in (4.35) is even and the bosonic parts of the first and last superalgebras ($(OSp(M_1|2N_1)$ and $OSp(M_{n+1}|2N_{n+1})$) share one simple common factor $SO(M_1) = SO(M_{n+1})$, then (4.35) becomes a closed loop and the resulting “fused” superalgebra is also (4.36).

If the bosonic parts of three or more superalgebras share one common part, they can be also fused into a single closed superalgebra. For example, let us consider

$$(G_1, \ldots, G_n) = (OSp(M|2N_1), \ldots, OSp(M|2N_i), \ldots, OSp(M|2N_n)),
$$

(4.37)

i.e. the bosonic parts of the $n \geq 3$ orthosymplectic superalgebras share the Lie algebra of $SO(M)$. They can be fused into

$$
OSp \left( M \left| 2 \sum_{i=1}^{n} N_i \right. \right).
$$

(4.38)
Similarly, one can fuse
\[(G_1, \ldots, G_n) = (OSp(M_1|2N), \ldots, OSp(M_i|2N), \ldots, OSp(M_n|2N)) \quad (4.39)\]
and \[(G_1, \ldots, G_n) = (U(N|N_1), \ldots, U(N|N_i), \ldots, U(N|N_n)) \quad (4.40)\]
into
\[OSp\left(\sum_{i=1}^{n} M_i \mid 2N\right) \quad \text{and} \quad U\left(N \mid \sum_{i=1}^{n} N_i\right), \quad (4.41)\]
respectively. However, by the same reason that the bosonic parts of (4.34) cannot be selected as the Lie algebra of gauge group of the \(\mathcal{N} = 4\) quiver gauge theory, the bosonic parts of (4.37), (4.39) or (4.41) cannot be used to construct the \(\mathcal{N} = 4\) quiver gauge theory, though their “fused” superalgebras (4.38) and (4.41) can be used to construct the \(\mathcal{N} = 4\) GW theories and \(\mathcal{N} = 5\) theories.

It is also possible to “fuse” the superalgebras whose bosonic parts forming a more complicated mesh-like diagram. For example, consider the following seven superalgebras
\[(G_1, G_2, G_3, G_4, G_5, G_6, G_7) = (OSp(M|2N_1), OSp(M|2N_2), OSp(M|2N_3), OSp(M|2N), \]
\[OSp(M_1|2N), OSp(M_2|2N), OSp(M_3|2N)). \quad (4.42)\]
The bosonic parts of \(G_1 \sim G_4\) share the Lie algebra of \(SO(M)\), while the bosonic parts of \(G_5 \sim G_7\) share the Lie algebra of \(Sp(2N)\). Therefore the bosonic parts of the seven superalgebras form the mesh-like diagram:
\[
\begin{array}{ccc}
Sp(2N_1) & \rightarrow & SO(M_3) \\
| & | & |
\end{array}
\begin{array}{ccc}
Sp(2N_2) & \rightarrow & SO(M) - Sp(2N) - SO(M_2) \\
| & | & |
\end{array}
\begin{array}{ccc}
Sp(2N_3) & \rightarrow & SO(M_1)
\end{array}
\]
These seven superalgebras (4.42) can be fused into the superalgebra
\[OSp\left(M + \sum_{i=1}^{3} M_i \mid 2(N + \sum_{i=1}^{3} N_i)\right), \quad (4.44)\]
which can be used to construct the \(\mathcal{N} = 4\) GW theories and \(\mathcal{N} = 5\) theories.

5. Conclusions and Discussion

We have developed a fusion procedure to “fuse” two superalgebras \(G\) and \(G'\), whose bosonic parts share at least one simple factor or \(U(1)\) factor, into a single closed superalgebra. The fermionic generators of the “fused” superalgebra are a disjoint union of the fermionic generators of \(G\) and \(G'\); in fusing \(G\) and \(G'\), one needs only to introduce a set of new bosonic generators for closing the “fused” superalgebra. The generic structure of the superalgebra
“fused” by two superalgebras has been worked out, and the fusion procedure has been generalized so that one can fuse more than two superalgebras. Two different methods were introduced to do the fusion. We have constructed several classes of the “fused” superalgebras in Sec. 2, 3 and 4. For instance, in Sec. 3, we have fused $U(N_1|N_2)$ and $U(N_2|N_3)$ into $U(N_1 + N_3|N_2)$. Here the common part of the bosonic parts of $U(N_1|N_2)$ and $U(N_2|N_3)$ is $U(N_2)$. It seems all classical superalgebras admit “fusions”, i.e., two orthosymplectic (unitary) superalgebras can be fused into a single closed orthosymplectic (unitary) superalgebra, provided that certain conditions are satisfied.

We have also generalized the fusion procedure to a fermionic fusion procedure, by allowing one to add minimum numbers of fermionic generators as well as bosonic generators into the system, such that two or more superalgebras may be fused into a closed one (see Sec. 4.2).

It is particularly interesting to note that even if two or more $\mathcal{N} = 4$ theories have completely different gauge groups and different numbers of multiplets, they may have the same underlying “fused” superalgebra structure, in the sense that the corresponding two or more sets of the superalgebras, used to construct the 3-algebras that generate the gauge groups, can be “fused” into the same single closed superalgebra, respectively. For instance, the $\mathcal{N} = 4$ quiver gauge theories whose quiver diagrams are given by (4.18), (4.28), and (4.29), respectively, are associated with the same “fused” superalgebra $U(N_1 + N_3|N_2 + N_4)$.

It would be nice to explore the physical significance of this relationship.

We have also discovered that some superalgebras cannot be fused into a single closed superalgebra even if the bosonic parts of any pair of them share one common factor: (4.34) is such an example. Interestingly, the bosonic parts of (4.34) cannot be selected as the Lie algebra of Lie group of the $\mathcal{N} = 4$ theory.

We are not sure that whether all the superalgebras in our recent work [13] used to construct the symplectic superalgebras in the $\mathcal{N} = 4$ theories can be fused into single superalgebras or not. For instance, can we fuse $(\text{OSp}(N_1|2), G_3)$ into a single closed superalgebra? Here the common part of the bosonic parts of the two superalgebras is $Sp(2) \cong SU(2)$. It would be nice to achieve a complete classification of the “fused” superalgebras. It can be seen that the Lie algebras of the gauge groups of the $\mathcal{N} = 4$ quiver gauge theories have extremely rich structures, hence may inspire some further non-trivial physical and mathematical problems to study.

6. Acknowledgement

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A. A Review of the $\mathcal{N} = 4$ Theory Based on 3-Algebras

In this Appendix, we review the general $\mathcal{N} = 4$ quiver theory constructed in terms of the double-symplectic 3-algebra or the $\mathcal{N} = 4$ three-algebra [12, 13]. The generators
of the double-symplectic 3-algebra are the disjoint union of that two sub symplectic 3-algebras, whose generators are denoted as $T_a$ and $T_{a'}$, respectively, where $a = 1, \cdots , 2R$ and $a' = 1, \cdots , 2S$. The symplectic 3-algebra is a complex vector space equipped with the 3-bracket

$$[T_I, T_J; T_K] = f_{IJK}^d T_d + f_{IJK}^{d'} T_{d'}$$

$\equiv g_{IJK}^L T_L$, (A.1)

where $T_I$ can be a primed or an unprimed generator, i.e.

$$T_I = (T_a \text{ or } T_{a'}) .$$

(A.2)

The 3-bracket is required to satisfy the fundamental identity (FI):

$$[T_I, T_J; [T_M, T_N; T_K]] = [[T_I, T_J; T_M], T_N; T_K] + [T_M, [T_I, T_J; T_N]; T_K] + [T_M, T_N; [T_I, T_J; T_K]] .$$

(A.3)

Substituting (A.1) into (A.3), we see that the structure constants must obey the identity

$$g_{MNK}^O g_{IJO}^L = g_{IJM}^O g_{ONK}^L + g_{IJN}^O g_{MOK}^L + g_{IJK}^O g_{MNO}^L .$$

(A.4)

To define two symplectic 3-algebras, we introduce two invariant anti-symmetric tensors

$$\omega_{ab} = \omega(T_a, T_b) \text{ and } \omega_{a'b'} = \omega(T_{a'}, T_{b'})$$

(A.5)

into the two sub 3-algebras, respectively, and denote their inverses as $\omega^{bc}$ and $\omega^{b'c'}$, satisfying $\omega_{ab} \omega^{bc} = \delta^c_a$ and $\omega_{a'b'} \omega^{b'c'} = \delta^{c'}_{a'}$. We will use the antisymmetric tensors $\omega$ to lower or raise the indices. The unprimed and primed vectors are required to be symplectic orthogonal, that is,

$$\omega(T_a, T_{b'}) = \omega(T_{b'}, T_a) = 0 .$$

(A.6)

Finally, we assume that the 3-brackets satisfy the two conditions

$$[T_I, T_J; T_K] = [T_J, T_I; T_K] ,$$

(A.7)

$$\omega([T_I, T_J; T_K], T_L) = \omega([T_K, T_L; T_I], T_J) .$$

(A.8)

The 3-algebra defined by Eqs. (A.1)–(A.8) is called a double-symplectic 3-algebra [13].

Taking account of (A.7) and $T_I = (T_a \text{ or } T_{a'})$, we notice that (A.1) gives six independent 3-brackets. Since $T_a$ and $T_{a'}$ span two symplectic sub 3-algebras respectively, we must have

$$[T_a, T_b; T_c] = f_{abc}^d T_d \text{ and } [T_{a'}, T_{b'}; T_{c'}] = f_{a'b'c'}^{d'} T_{d'} .$$

(A.9)

Comparing (A.3) with (A.1), we note that

$$f_{abc}^{d'} = f_{a'b'c'}^d = 0 .$$

(A.10)

Using (A.6), (A.7), (A.8), and (A.10), it is not difficult to prove that

$$f_{abc}^{d'} = f_{a'b'c'}^{d'} = f_{ab'c}^d = f_{ba'c}^{d'} = 0 .$$

(A.11)
Combining (A.1) and (A.11), we learn that the rest four 3-brackets are given by

\[ [T_a, T_b; T_c] = f_{abc}^d T_d', \quad [T_d', T_b; T_c] = f_{d'b'c}^d T_d, \]  
(A.12)

\[ [T_a, T_b'; T_c] = f_{ab'c}^d T_d', \quad [T_d', T_b; T_c'] = f_{ba'c}^d T_d'. \]  
(A.13)

On account of the symmetry conditions (A.7) and (A.8), Eq. (A.4) may be decomposed into eight independent FIs. The four FIs not involving \( f_{ab'c} \) are the follows

\[ f_{abc}^g f_{gfh} + f_{ab'c}^g f_{egc} - f_{efc}^g f_{abg} = 0, \]

\[ f_{abc}^g f_{gfh} + f_{ab'c}^g f_{egc} - f_{efc}^g f_{abg} = 0, \]  
(A.14)

\[ f_{ab'c}^g f_{gfh} + f_{ab'c}^g f_{egc} - f_{efc}^g f_{abg} = 0, \]

\[ f_{ab'c}^g f_{gfh} + f_{ab'c}^g f_{egc} - f_{efc}^g f_{abg} = 0. \]

The other four FIs involving \( f_{ab'c} \) are the follows

\[ f_{ac'b}^d f_{bf'd} = f_{def} f_{ab'd} + f_{def} f_{ab'd} + f_{def} f_{ab'd}, \]

\[ f_{ac'b}^d f_{bf'd} = f_{def} f_{ab'd} + f_{def} f_{ab'd} + f_{def} f_{ab'd}, \]  
(A.15)

\[ f_{ac'b}^d f_{bf'd} = f_{def} f_{ab'd} + f_{def} f_{ab'd} + f_{def} f_{ab'd}, \]

\[ f_{ac'b}^d f_{bf'd} = f_{def} f_{ab'd} + f_{def} f_{ab'd} + f_{def} f_{ab'd}. \]

We assume that the \( \mathcal{N} = 4 \) invariance under the transformation [12]

\[ \delta_\Lambda \Phi = \Lambda^{ab} [T_a, T_b; \Phi] + \Lambda^{a'b'} [T_a', T_b'; \Phi], \]  
(A.16)

where the 3-algebra valued superfield \( \Phi \) can be an untwisted superfield \( \Phi = \Phi_A T_a \) or a twisted superfield \( \Phi = \Phi_A' T_a'. \) The infinitesimal parameters \( \Lambda^{ab} \) and \( \Lambda^{a'b'} \) are independent of superspace coordinates. The symmetry (A.16) will be gauged later. One may try to add the term

\[ \Lambda^{a'b'} [T_a, T_b'; \Phi] \equiv \delta_{\Lambda_3} \Phi, \]  
(A.17)

to the right hand side of (A.16). However, using the first equation of (A.13), we obtain

\[ \delta_{\Lambda_3} \Phi_A = \Lambda^{a'b'} [T_a, T_b'; \Phi_A T_c] = (\Lambda^{a'b'} f_{ab'c}^d \Phi_A^c) T_d'. \]  
(A.18)

The most right hand side indicates that \( \delta_{\Lambda_3} \Phi_A \neq (\delta_{\Lambda_3} \Phi)_{A}^c T_c, \) conflicting with the assumption \( \Phi_A = \Phi_A T_a. \) We therefore must require that \( \delta_{\Lambda_3} \Phi_A = 0. \) This can be fulfilled by setting either \( \Lambda^{a'b'} = 0 \) or \( f_{ab'c}^d = 0. \)

- If we set \( \Lambda^{a'b'} = 0, \) then Eq. (A.17) does not play any role in constructing the theory; only the symmetry defined by (A.16) will be gauged.

- If we set \( f_{ab'c}^d = 0, \) then (A.8) implies that \( f_{ab'd} c = 0. \) As a result, we have \( \delta_{\Lambda_3} \Phi_A = 0 \) as well. Note that after setting \( f_{ab'c}^d = f_{ab'd} c = 0, \) the four FIs (A.13) are satisfied automatically. We call the new 3-algebra obtained from the double-symplectic 3-algebra by setting \( f_{ab'd} c = f_{ab'd} c = 0 \) an \( \mathcal{N} = 4 \) three-algebra.
The antisymmetric tensor $\omega_{cd}$ is invariant under the transformations:

$$
\delta_{\Lambda_1} \omega_{cd} = \Lambda^{ab}(f_{abc}^\ e\omega_{ed} + f_{abd}^\ e\omega_{ce}) = 0, \quad (A.19)
$$

$$
\delta_{\Lambda_2} \omega_{cd} = \Lambda^{a'b'}(f_{a'b'c}^\ e\omega_{ed} + f_{ab'd}^\ e\omega_{ce}) = 0. \quad (A.20)
$$

Eqs (A.19) and (A.20) are nothing but $f_{abcd} = f_{abdc}$ and $f_{a'b'cd} = f_{ab'dc}$, respectively. Similarly, by considering the invariance of $\omega_{c'd'}$, we obtain $f_{a'b'c'd'} = f_{a'b'c'd'}$ and $f_{abc'd'} = f_{abcd'}$. Note that these equations are consistent with Eq. (A.16). In the case of double symplectic 3-algebra, using (A.6), it is not difficult to prove that $\omega$ are also invariant under the transformation (A.17), i.e.

$$
\delta_{\Lambda_3} \omega_{cd} = \delta_{\Lambda_3} \omega_{c'd'} = 0. \quad (A.21)
$$

Note that in proving (A.21), we have not set $\Lambda^{ab'} = 0$. In the case of $\mathcal{N} = 4$ three-algebra, Eqs. (A.21) are satisfied automatically due to the fact that $f_{abc'd'} = f_{ba'c'd'} = 0$.

Plugging $\Phi = \Phi^a T_a$ and $\Phi = \Phi^d T_{d'}$ into (A.16), respectively, we see that only four structure constants \footnote{Since $f_{abc'd'} = f_{c'd'ab}$ (see (A.23)), there are only three independent structure constants.} $f_{abc}^\ d$, $f_{a'b'c'}^\ d$, $f_{abc'd'}$ and $f_{a'b'c'}^\ d$. are needed in defining the symmetry transformation. Indeed, later we will see that only the above four structure constants appear in the action and the law of supersymmetry transformations (see (A.29) and (A.31)), while $f_{abc'd'}$ and $f_{ba'c'd'}$ (the two structure constants of the rest two 3-brackets (A.13)) do not appear in the action at all.

In summary, the four structure constants (A.22) enjoy the following symmetry properties \footnote{Since $f_{abc'd'} = f_{c'd'ab}$ (see (A.23)), there are only three independent structure constants.}

$$
f_{abcd} = f_{bacd} = f_{badc} = f_{cdab},
$$

$$
f_{abc'd'} = f_{bac'd'} = f_{bad'c'} = f_{c'd'ab},
$$

$$
f_{a'b'c'd'} = f_{b'a'c'd'} = f_{b'a'd'c'} = f_{c'd'a'b'}. \quad (A.23)
$$

To guarantee the positivity of theory, they are required to obey the reality conditions \footnote{Since $f_{abc'd'} = f_{c'd'ab}$ (see (A.23)), there are only three independent structure constants.}

$$
f^{a_a b} c d = f^{b_b d} a c, \quad f^{a'a'_b b' c'} d' = f^{b'b' d} a' c', \quad f^{a'a'_b b' c'} d' = f^{b'b' d} a' c'. \quad (A.24)
$$

To achieve the closure of the $\mathcal{N} = 4$ algebra, one must impose the linear constraints \footnote{Since $f_{abc'd'} = f_{c'd'ab}$ (see (A.23)), there are only three independent structure constants.}

$$
f_{(abc)d} = 0 \quad \text{and} \quad f_{(a'b'c')d'} = 0. \quad (A.25)
$$

It is natural to require the three independent structure constants to be invariant under the symmetry transformation (A.16), i.e.

$$
\delta_{\Lambda} f_{abcd} = \delta_{\Lambda} f_{abc'd'} = \delta_{\Lambda} f_{a'b'c'd'} = 0. \quad (A.26)
$$

A short calculation shows that Eqs. (A.26) are equivalent to the four FIs (A.14). Therefore Eqs. (A.26) do not involve the rest four FIs (A.13) at all.
Using the double-symplectic 3-algebra or the $\mathcal{N} = 4$ three-algebra, we have been able to construct the $\mathcal{N} = 4$ quiver gauge theory in a superspace approach [12]. In the theory, the un-twisted multiplets $(Z^A_a, \psi^a_A)$ and the twisted multiplets $(Z^A_{\hat{a}}, \psi^A_{\hat{a}})$ obey the following reality conditions

\begin{align}
Z^A_a &= \omega_{ab}\epsilon^{AB}Z^B_b, \quad \tilde{\psi}^A_a = \omega_{ab}\epsilon^{AB}\tilde{Z}^B_b, \\
Z^A_{\hat{a}} &= \omega_{a'b'}\epsilon^{AB}\bar{Z}^{B'}_{b'}, \quad \tilde{\psi}^A_{\hat{a}} = \omega_{a'b'}\epsilon^{AB}\bar{Z}^{B'}_{b'},
\end{align}

where $A, \hat{A} = 1, 2$ are the undotted and dotted indices of the $SU(2) \times SU(2)$ R-symmetry group, respectively. The $\mathcal{N} = 4$ Lagrangian is given by

\begin{align}
\mathcal{L} &= \frac{1}{2}(-D_\mu \tilde{Z}^A_a D^\mu Z^a_A - D_\mu \tilde{Z}^A_{\hat{a}} D^\mu Z^A_{\hat{a}} + i \tilde{\psi}^A_a \gamma^\mu D_\mu \psi^a_A + i \tilde{\psi}^A_{\hat{a}} \gamma^\mu D_\mu \psi^A_{\hat{a}}) \\
&\quad + i f_{abcd}Z^a_A Z^{Ab'} B^{d'}_B \psi^c_B \psi^d_B + i f_{a'b'd'} \bar{Z}^A_{\hat{a}} \bar{\psi}^A_{\hat{a}} \bar{B}^{d'}_{\hat{B}} + 4Z^a_A \bar{Z}^A_{\hat{a}} \bar{\psi}^a_0 \bar{\psi}^A_{\hat{a}} \\
&\quad + \frac{1}{2} \epsilon^{\mu\nu\lambda}(f_{a'b'd'} A^b_{\alpha'} A^d_{\mu'} A^c_{\alpha} + 2 \frac{1}{3} f_{a'b'd'} A^b_{\alpha'} A^c_{\alpha'} A^d_{\mu'} ) \\
&\quad + \frac{1}{2} \epsilon^{\mu\nu\lambda}(f_{a'b'c'd'} A^b_{\alpha'} A^d_{\mu'} A^c_{\mu'} A^\lambda_{\alpha'}) \\
&\quad + \epsilon^{\mu\nu\lambda}(f_{a'b'd'} A^b_{\alpha'} A^d_{\mu'} A^c_{\mu'} A^\lambda_{\alpha'}) \\
&\quad + \frac{1}{12} \epsilon^{\mu\nu\lambda}(f_{a'b'c'd'} A^b_{\alpha'} A^d_{\mu'} A^c_{\mu'} A^\lambda_{\alpha'}) \\
&\quad - \frac{1}{4} \epsilon^{\mu\nu\lambda}(f_{a'b'c'd'} A^b_{\alpha'} A^d_{\mu'} A^c_{\mu'} A^\lambda_{\alpha'})
\end{align}

where the gauge fields and the covariant derivatives are defined as

\begin{align}
D_\mu Z^A_a &= \partial_\mu Z^A_a - \hat{A}_b^a Z^B_b, \quad \hat{A}_b^a = A^b_{ac} f_{ac} + A^b_{a'c'} f_{a'c'}, \\
D_\mu \tilde{Z}^A_{\hat{a}} &= \partial_\mu \tilde{Z}^A_{\hat{a}} - \hat{A}_b^a \tilde{Z}^B_b, \quad \hat{A}_b^a = A^b_{ac} f_{ac} + A^b_{a'c'} f_{a'c'}.
\end{align}

Here $A^b_{ac}$ and $A^b_{a'c'}$ are independent Hermitian tensors, provided that the two sub 3-algebras are not identical. The $\mathcal{N} = 4$ supersymmetry transformations read

\begin{align}
\delta Z^A_a &= i \epsilon_A^A \hat{A}^a_A, \\
\delta Z^A_{\hat{a}} &= i \epsilon_A^A \hat{A}^a_A, \\
\delta \psi^a_A &= -\gamma^\mu D_\mu Z^c_{\hat{a}} \epsilon_A^A Z^c_{\hat{a}} - \frac{1}{3} f_{a'b'c''d'} Z^b_{\hat{B}} Z^c_{\hat{B}} Z^d_{\hat{C}} \epsilon_A^A C + f_{a'c'd'} B^c_{\hat{B}} \epsilon_A^A B, \\
\delta \psi^A_{\hat{a}} &= -\gamma^\mu D_\mu \bar{Z}^c_{\hat{a}} \epsilon^A_{\hat{A}} \bar{Z}^c_{\hat{a}} - \frac{1}{3} f_{a'c'd'} Z^b_{\hat{B}} Z^c_{\hat{B}} Z^d_{\hat{C}} \epsilon^A_{\hat{A}} C + f_{a'd'c'} \bar{B}^c_{\hat{B}} \epsilon^A_{\hat{A}} B, \\
\delta \hat{A}_b^a &= i \epsilon^{AB} \gamma_a^b \psi^B_{\hat{A}} f_{ab} + i \epsilon^{AB} \gamma_{a'}^b \bar{Z}^a_{\hat{A}} f_{a'b'}, \\
\delta \hat{A}_b^a &= i \epsilon^{AB} \gamma_a^b \psi^B_{\hat{A}} f_{ab} + i \epsilon^{AB} \gamma_{a'}^b \bar{Z}^a_{\hat{A}} f_{a'b'},
\end{align}

where the supersymmetry parameter $\epsilon_A^B$ obeys the following reality condition

\begin{align}
\epsilon^A_B &= -\epsilon^{BC} \epsilon_A^C B.
\end{align}
The closure of the above \( N = 4 \) algebra has been verified in Ref. [12].

One can generalize the construction of this Appendix by introducing a symplectic 3-algebra containing three or more \((n \geq 3)\) symplectic sub 3-algebras, and by letting that \(n\) multiplets take values in these \(n\) sub 3-algebras respectively. One then can realize these \(n\) sub 3-algebras in terms of \(n\) superalgebras respectively.

**B. A Review of the Superalgebra Realization**

In this Appendix, we review the superalgebra realization of the four sets of 3-brackets (A.9) and (A.12) and the four sets of FIs (A.14); we also comment on the rest two 3-brackets (A.13) and four sets of FIs (A.13) [13].

As we mentioned in Section 1, we used two superalgebras \(G\) and \(G'\) to realize the two sub algebras of the double-symplectic 3-algebras [13]. Here \(G\) and \(G'\) are given by

\[
[M^u, M^v] = f_{uv}^w M^w, \quad [M^u, Q_a] = -\tau_a^w \omega^{bc} Q_c, \quad \{Q_a, Q_b\} = \tau_{ab}^w M^w, \quad (B.1)
\]

and

\[
[M'^u, M'^v] = f_{uv'}^w M'^w, \quad [M'^u, Q_{a'}] = -\tau_{a'}^{u'} \omega'^{b'c'} Q_{c'}, \quad \{Q_{a'}, Q_{b'}\} = \tau_{ab'}^{u'} k_{u'} M'^u', \quad (B.2)
\]

respectively, where \(a = 1, \cdots, 2R\) and \(a' = 1, \cdots, 2S\). The invariant antisymmetric tensors are defined as

\[
\omega_{ab} = \kappa(Q_a, Q_b), \quad \omega_{a'b'} = \kappa(Q_{a'}, Q_{b'}), \quad (B.3)
\]

and their inverses are denoted as \(\omega^{ab}\) and \(\omega'^{a'b'}\) satisfying \(\omega^{ab} \omega_{bc} = \delta^a_c\) and \(\omega'^{a'b'} \omega_{b'c'} = \delta^{a'}_{c'}\).

We will use \(\omega\) to raise or lower indices. The invariant symmetric forms are defined as \(k^{uv} = -\kappa(M^u, M^v)\) and \(k^{u'v'} = -\kappa(M'^u, M'^v)\); their inverse are denoted as \(k_{uv}\) and \(k_{u'v'}\), satisfying \(k_{uv} k^{vw} = \delta^w_u\) and \(k_{u'v'} k^{w'u'} = \delta^{w'}_{u'}\). The forms \(\kappa\) are invariant [3, 22] in the sense that

\[
\kappa([A, B], C) = \kappa(A, [B, C]), \quad \kappa([A', B'], C') = \kappa(A', [B', C']), \quad (B.4)
\]

where \(A = Q_a\) or \(M^u\), and \(A' = Q_{a'}\) or \(M'^{u'}\).

If we set

\[
T_a \doteq Q_a, \quad T_{a'} \doteq Q_{a'}, \quad (B.5)
\]

the four 3-brackets can be constructed in terms of the double graded commutators

\[
[T_a, T_b; T_c] \doteq \{\{Q_a, Q_b\}, Q_c\}, \quad [T_{a'}, T_{b'}; T_{c'}] \doteq \{\{Q_{a'}, Q_{b'}\}, Q_{c'}\}, \quad (B.6)
\]

The right hand sides of the last two equations of (B.6) are required to satisfied two crucial conditions. First, in order that there are nontrivial interactions between the twisted and untwisted multiplets, one must require that

\[
f_{abc} a' \neq 0, \quad f_{a'b'c'} d \neq 0. \quad (B.7)
\]
Secondly, in accordance with Eqs. (A.11), we must require that

\[ f_{abc}^d = f_{a'bc'}^{d'} = 0. \]  

(B.8)

In Ref. [13], we have proved that if the bosonic parts of \(G\) and \(G'\) share at least one simple or \(U(1)\) factor, the requirements (B.7) and (B.8) can be fulfilled, provided that the common bosonic part of \(G\) and \(G'\) is not a center of \(G\) and \(G'\). Denoting the generators of the common bosonic part as \(M^9\), i.e. schematically, \(M^9 = M^u \cap M'^u\), we have

\[ M^u = (M^\alpha, M^9), \quad M'^u = (M'^\alpha, M^9). \]  

(B.9)

(Here \(\alpha\) is not an index of spacetime spinor. We hope this will not cause any confusion.) And we assume that we do not identify the two superalgebras \(G\) and \(G'\): schematically, we exclude the possibility that \(Q_a = Q_{a'}\) and \(M^\alpha = M'^\alpha = \emptyset\). Hence it is natural to require that

\[ [M^\alpha, Q_a] = [M'^\alpha, Q_a] = 0. \]  

(B.10)

With the decompositions (B.9), the anticommutators in (B.1) and (B.2) can be written as

\[ \{Q_a, Q_b\} = \tau^g_{ab}k_{gh}M^h, \quad \{Q_a', Q_b'\} = \tau^g_{a'b'}k_{gh}M^h, \]  

(B.11)

where we have decomposed the invariant quadratic form \(k_{uv}\) as \(k_{uv} = (k_{\alpha\beta}, k_{gh})\). Using (B.1), (B.2), and (B.11), the structure constants of 3-brackets in (B.4) can be easily read off; they are given by the tensor products

\[ f_{abcd} = k_{uv}\tau^u_{ab}\tau^v_{cd}, \quad f_{a'b'c'd'} = k_{uv}\tau^u_{a'b'}\tau^v_{c'd'}, \quad f_{abc'd'} = k_{gh}\tau^g_{ab}\tau^h_{cd'}, \]  

(B.12)

where we have used (B.10). By the \(M^\alpha M^9Q_{a'}\), \(M'^\alpha M^9Q_{a}\), and \(M^\alpha M'^\alpha Q_{a}\) Jacobi identities, we learn that

\[ [M^\alpha, M^9] = [M'^\alpha, M^9] = [M^\alpha, M'^\alpha] = 0. \]  

(B.13)

The structure constants (B.12) possess the desired symmetry properties (A.23) and obey the real conditions (A.24). The \(Q_aQ_bQ_c\) Jacobi identity of (B.1) implies that the first equation of (A.23) is obeyed, i.e. \(f_{(abc)d} = 0\). Similarly, \(f_{(a'b'c'd')d'} = 0\) is equivalent to the \(Q_{a'}Q_{b'}Q_{c'}\) Jacobi identity of (B.2).

As for the four sets of FIs in (A.14), one can prove that they are equivalent to the \(M^uM^aQ_{a'}\), \(M^uM^aQ_{a'}\), \(M^9M^hQ_{a}\), and \(M^9M^hQ_{a'}\) Jacobi identities, respectively. For instance, using Eqs. (B.6), one of equations in (A.3) can be converted into

\[ \{\{Q_a, Q_b\}, \{Q_c, Q_d\}, Q_{a'}\} = \{\{\{Q_a, Q_b\}, Q_c\}, Q_d\}, Q_{a'}\} + \{\{Q_c, Q_d\}, \{Q_a, Q_b\}, Q_{a'}\} \]  

(B.14)

\[ \]  

\[ \text{(More generally, one can decompose } M'^u \text{ into } M'^u = (M'^\alpha, \tilde{M}^9), \text{ where } \tilde{M}^9 = T^g_{gh}M^h, \text{ with } T^g_{gh} \text{ a complex non-singular linear transformation matrix [13]. Here we set } T^g_{gh} = \delta^g_{gh} \text{ for simplicity.}} \]
A short calculation shows that it is equivalent to the second FI of (A.14). On the other hand, using (B.1), (B.11), (B.10), and (B.13), we can convert (B.14) into the \[
M^g M^h Q_{a'} = 0. \tag{B.15}
\]

In this realization, the Lie algebra of gauge group is the bosonic subalgebras of the superalgebras (B.1) and (B.2); specifically, it is spanned by the set of generators \[M_\alpha, M_g, M_{a'}\] . \tag{B.16}

The representations of the bosonic subalgebras of (B.1) and (B.2) are determined by the fermionic generators \[Q_a\] and \[Q_{a'}\] , respectively. The classification of the gauge groups of the \[N=4\] quiver gauge theories can be found in Ref. [13, 5, 10]. In particular, in Ref. [13], the authors have able to construct a number of classes of \[N=4\] theories with new gauge groups, using the approach described in this appendix.

Let us now comment on the two 3-brackets (A.13) and the four FIs (A.15). In the case of double-symplectic 3-algebra, if \[G\] and \[G'\] can be ‘fused’ into a closed superalgebra, one can construct the rest two 3-brackets Eqs. (A.13) in analogy to Eqs. (B.6), i.e.

\[
[T_a, T_{b'}; T_c] = \{Q_a, Q_{b'}\}, \quad [T_{a'}, T_b; T_c] = \{Q_{a'}, Q_b\}. \tag{B.17}
\]

In summary, we have

\[
T_I \doteq Q_I, \quad [T_I, T_J; T_K] \doteq \{Q_I, Q_J\}, \tag{B.18}
\]

where

\[Q_I = (Q_a \text{ or } Q_{a'}). \tag{B.19}\]

Note that (A.8) is obeyed by the construction

\[
\omega([T_I, T_J; T_K], T_L) = \kappa([\{Q_I, Q_J\}, Q_K], Q_L), \tag{B.20}
\]

and one can also prove that Eqs. (A.11) and (A.10) are obeyed [13].

Recall that in Sec. (1), in “fusing” \[G\] and \[G'\], we have defined the anticommutator of \[Q_a\] and \[Q_{b'}\] as (see Eq. (1.7))

\[
\{Q_a, Q_{b'}\} = \tilde{\epsilon}_{ab'} M_{\tilde{a}}, \tag{B.21}
\]

where \[M_{\tilde{a}}\] are a set of bosonic generators, and \[\tilde{\epsilon}_{ab'}\] are structure constants of the anticommutator. The structure constants of the commutators involving \[M_{\tilde{a}}\] can be found in Sec. 4. Now one can prove that the four FIs (A.13) involving \[f_{ab'cd}\] are equivalent to the four Jacobi identities relating the bosonic generators \[M_{\tilde{a}}\] defined in Eq. (B.21). In summary, using (B.18), one can construct the FI (A.3) as follows

\[
\{Q_I, Q_J\}, \{Q_M, Q_N\}, Q_K\} = \{\{Q_I, Q_J\}, Q_M, Q_N\}, Q_K\} + \{Q_M, \{Q_I, Q_J\}, Q_N\}, Q_K\} + \{Q_M, Q_N\}, \{Q_I, Q_J\}, Q_K\}. \tag{B.22}
\]
which can be converted into the \( \text{MMQ} \) Jacobi Identities of the fused superalgebra. By (B.18) and (B.22), we learn that the double-symplectic 3-algebra indeed can be constructed in terms of the fused superalgebra. However, if the superalgebras \( G \) and \( G' \) cannot be fused into a closed superalgebra, we are not sure whether one can construct (A.13) and (A.15) in terms of \( G \) and \( G' \). It would be nice to answer this question.

In the case of \( N = 4 \) three-algebra, the two 3-brackets vanish identically:

\[
[T_a, T_{b'}, T_c] = [T_a', T_{b'}, T_c'] = 0.
\]

As a result, they cannot be constructed in terms of the double graded commutators (B.17) of the “fused” superalgebra, since the structure constants of \([Q_a, Q_{b'}, Q_c] = f_{ab'c'd} Q_d\) and \([Q_a, Q_{b'}, Q_{c'}] = f_{ab'c'd} Q_d\) do not vanish on account of the \( Q_a Q_{b'} Q_c \) Jacobi identity and the \( Q_a Q_{b'} Q_{c'} \) Jacobi identity, respectively. If both \( G \) and \( G' \) are unitary superalgebras or orthosymplectic superalgebras, by direct calculation (without consulting the Jacobi identities), one can show that both \([Q_a, Q_{b'}, Q_c] \) and \([Q_a, Q_{b'}, Q_{c'}] \) are not zero (see Sec. 2 and Sec. 3).

C. Conventions and Useful Identities

The conventions and useful identities are adopted from our previous paper [12].

C.1 Spinor Algebra

In 1 + 2 dimensions, the gamma matrices are defined as

\[
(\gamma_\mu)_\alpha^\gamma (\gamma_\nu)^\gamma_\beta + (\gamma_\nu)_\alpha^\gamma (\gamma_\mu)^\gamma_\beta = 2\eta_{\mu\nu}\delta_\alpha^\beta. \quad (C.1)
\]

For the metric we use the \((-,-,+,+)\) convention. The gamma matrices in the Majorana representation can be defined in terms of Pauli matrices: \((\gamma_\mu)_\alpha^\beta = (i\sigma_2, \sigma_1, \sigma_3)\), satisfying the important identity

\[
(\gamma_\mu)_\alpha^\gamma (\gamma_\nu)^\gamma_\beta = \eta_{\mu\nu}\delta_\alpha^\beta + \epsilon_{\mu\nu\lambda}(\gamma_\lambda)_\alpha^\beta. \quad (C.2)
\]

We also define \(\epsilon^{\mu\nu\lambda} = -\epsilon_{\mu\nu\lambda}\). So \(\epsilon_{\mu\nu\lambda}\epsilon^{\rho\sigma\lambda} = -2\delta_\mu^\rho\). We raise and lower spinor indices with an antisymmetric matrix \(\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta}\), with \(\epsilon_{12} = -1\). For example, \(\gamma^\mu_\alpha = \epsilon^{\alpha\beta}_\mu \psi_\beta\) and \(\gamma^\mu_\beta = \epsilon_{\beta\gamma}(\gamma_\mu)_\alpha^\gamma\), where \(\psi_\beta\) is a Majorana spinor. Notice that \(\gamma^\mu_\alpha = (1, -\sigma^3, \sigma^1)\) are symmetric in \(\alpha\beta\). A vector can be represented by a symmetric bispinor and vice versa:

\[
A_\alpha^\beta = A_\mu \gamma^\mu_\alpha \gamma^\mu_\beta, \quad A_\mu = -\frac{1}{2} \gamma^\mu_\alpha A_\alpha^\beta. \quad (C.3)
\]

We use the following spinor summation convention:

\[
\psi_\chi = \psi_\alpha^\chi_\alpha, \quad \psi_\gamma_\mu_\chi = \psi_\alpha^\gamma(\gamma_\mu)_\alpha^\beta \chi_\beta, \quad (C.4)
\]

where \(\psi\) and \(\chi\) are anti-commuting Majorana spinors. In 1 + 2 dimensions the Fierz transformation reads

\[
(\lambda_\chi)\psi = -\frac{1}{2}(\lambda\psi)\chi - \frac{1}{2}(\lambda_\gamma\psi)\gamma_\mu\chi. \quad (C.5)
\]
C.2 $SU(2) \times SU(2)$ Identities

We define the 4 sigma matrices as

$$\sigma^a_A = (\sigma^1, \sigma^2, \sigma^3, i1),$$

(C.6)

by which one can establish a connection between the $SU(2) \times SU(2)$ and $SO(4)$ group. These sigma matrices satisfy the following Clifford algebra:

$$\sigma^a_A \sigma^{b\dagger}_C + \sigma^{b}_A \sigma^{a\dagger}_C = 2\delta^{ab}\delta^A_C,$$

(C.7)

$$\sigma^{a\dagger}_A \sigma^{b}_C + \sigma^{b\dagger}_A \sigma^{a}_C = 2\delta^{ab}\delta^A_C.$$  

(C.8)

We use anti-symmetric matrices

$$\epsilon_{AB} = -\epsilon^{AB} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \epsilon_{A\dot B} = -\epsilon^{A\dot B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(C.9)

to raise or lower un-dotted and dotted indices, respectively. For example, $\sigma^{a\dagger}A\dot B = \epsilon_{AB}\sigma^{a\dagger}_B$ and $\sigma^{a\dagger}B\dot A = \epsilon^{BC}\sigma^a_C\dot A$. The sigma matrix $\sigma^a$ satisfies a reality condition

$$\sigma^{a\dagger}A\dot B = -\epsilon^{BC}\epsilon_{AB}\sigma^a_B \text{ or } \sigma^{a\dagger}A\dot B = -\sigma^aB\dot A.$$  

(C.10)

The antisymmetric matrix $\epsilon_{AB}$ satisfies an important identity

$$\epsilon_{AB}\epsilon^{CD} = -(\delta^C_A \delta^D_B - \delta^D_A \delta^C_B),$$

(C.11)

and $\epsilon_{A\dot B}$ satisfies a similar identity.

The parameter for the $\mathcal{N} = 4$ supersymmetry transformations is defined as $\epsilon^{A\dot B} = \epsilon_0\sigma^{aAB}$.

D. The Commutation Relations of Superalgebras

These commutation relations of superalgebras are adopted from our previous paper [13].

D.1 $U(M|N)$

The commutation relations of $U(M|N)$ are given by

$$[M_{\bar{a}^\dagger}, M_{\bar{b}^\dagger}] = \delta_{\bar{a}}^{\bar{b}} M_{\bar{b}^\dagger} - \delta_{\bar{b}}^{\bar{a}} M_{\bar{a}^\dagger}, \quad [M_{\nu^\dagger}, M_{\nu'^\dagger}] = \delta_{\nu}^{\nu'} M_{\nu'^\dagger} - \delta_{\nu'^\dagger} M_{\nu^\dagger},$$

$$[M_{\bar{a}^\dagger}, Q_{\bar{a}^\dagger}] = \delta_{\bar{a}}^{\bar{a}} Q_{\bar{a}^\dagger}, \quad [M_{\nu^\dagger}, Q_{\nu^\dagger}] = -\delta_{\nu^\dagger} Q_{\nu^\dagger},$$

$$[M_{\nu^\dagger}, Q_{\nu'^\dagger}] = -\delta_{\nu}^{\nu'} Q_{\nu'^\dagger}, \quad [M_{\nu^\dagger}, Q_{\nu'^\dagger}] = \delta_{\nu}^{\nu'} Q_{\nu'^\dagger},$$

$$\{Q_{\bar{a}^\dagger}, Q_{\nu^\dagger}\} = k(\delta^{\nu^\dagger}_{\nu^\dagger} M_{\bar{a}^\dagger} + \delta_{\nu^\dagger} M_{\nu^\dagger}).$$

(D.1)

where $Q_{\bar{a}^\dagger}$ carries a $U(M)$ fundamental index $\bar{a} = 1, \cdots, M$ and a $U(N)$ anti-fundamental index $\nu^\dagger = 1, \cdots, N$. Here we have

$$Q_{\bar{a}^\dagger} = \begin{pmatrix} \bar{Q}_{\bar{a}^\dagger} \\ -\bar{Q}_{\bar{a}^\dagger} \end{pmatrix} = \bar{Q}_{\bar{a}^\dagger} \delta_{1\alpha} - \bar{Q}_{\bar{a}^\dagger} \delta_{2\alpha},$$

(D.2)
In the second equation of (D.2), we have introduced a “spin up” spinor $\chi_{1\alpha}$ and a “spin down” spinor $\chi_{2\alpha}$, i.e.,

$$\chi_{1\alpha} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \delta_{1\alpha} \text{ and } \chi_{2\alpha} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \delta_{2\alpha}. \quad (D.3)$$

And the anti-symmetric tensor $\omega_{ab}$ and its inverse read

$$\omega_{a'b'} = \begin{pmatrix} 0 & \delta_{\bar{u}i} \delta_{ij} \\ -\delta^\bar{v}_k \delta_{ij} & 0 \end{pmatrix}, \quad \omega_{b'c'} = \begin{pmatrix} 0 & -\delta^\bar{j}_k \delta_{\bar{u}v} \\ \delta^\bar{i}_k \delta_{\bar{u}v} & 0 \end{pmatrix}. \quad (D.4)$$

With (D.2) and (D.4), the superalgebra (D.1) takes the form of (B.1) or (B.2).

**D.2 OSp(M|2N)**

The super Lie algebra $OSp(M|2N)$ reads

$$[M_{ij}, M_{kl}] = \delta_{jk}M_{il} - \delta_{ik}M_{jl} + \delta_{il}M_{jk} - \delta_{jl}M_{ik},$$
$$[M_{ij}, M_{kl}] = \omega_{jk}M_{il} + \omega_{ik}M_{jl} + \omega_{il}M_{jk} + \omega_{jl}M_{ik},$$
$$[M_{ij}, Q_{kl}] = \delta_{jk}Q_{il} - \delta_{ik}Q_{jl},$$
$$[M_{ij}, Q_{kl}] = \omega_{jk}Q_{il} + \omega_{ik}Q_{jl},$$
$$\{Q_{\bar{i}k}, Q_{\bar{j}l}\} = k(\omega_{\bar{i}k}M_{\bar{j}l} + \omega_{\bar{j}l}M_{\bar{i}k}). \quad (D.5)$$

where $i = 1, \cdots , M$ is an $SO(M)$ fundamental index, and $\bar{i} = 1, \cdots , 2N$ an $Sp(2N)$ fundamental index. Here we have

$$Q_a = Q_{\bar{i}i} \text{ and } \omega_{ab} = \omega_{\bar{i}i, \bar{j}j} = \delta_{\bar{i}j}\omega_{ij}. \quad (D.6)$$

Now the superalgebra (D.2) also takes the form of (B.1) or (B.2).

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5Here the index $\alpha$ is not a spacetime spinor index. We hope this will not cause any confusion.
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