\{\beta\}\text{-expansion in QCD, its conformal symmetry limit: theory + applications}

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Abstract

The basis of the \{\beta\}\text{-expansion for the perturbative series evaluated in the $\overline{\text{MS}}$ scheme for the renormalization group invariant quantities is summarized. Comparison with a similar representation, used within the BLM-motivated Principle of Maximal Conformality, is discussed. We stress that the original \{\beta\}\text{-expansion contains a completed list of terms rather than its PMC analog. The arguments in favour of the complete \{\beta\}\text{-expansion are presented. They are based on the relations which follow from the power $\beta$-function generalization of the Crewther relation for the nonsinglet $\overline{\text{MS}}$ contributions to the Adler $D^\text{NS}$-function and to the Bjorken sum rule $C^\text{Bjp}_\text{NS}$ of the polarized lepton-nucleon scattering. The terms of the complete \{\beta\}\text{-expansion at the $O(\alpha_3^3)$ level for $D^\text{NS}$ and $C^\text{Bjp}_\text{NS}$ are presented. These perturbative results are expressed in the PMC-type form. The problem of applications of these expressions for phenomenological applications is summarized.

Keywords: Representations of the perturbative QCD series, scale-fixing prescriptions.

1. Introduction

The \{\beta\}\text{-expansion approach, discussed here, was originally proposed in [1]. The aim was to construct generalizations of the BLM approach [2] at the levels higher than the NNLO one while the first method to fix the BLM-type scale for the RG-invariant quantities was developed in [3]. This \{\beta\}\text{-expansion was used to explore multiple power $\beta$-function generalization of the Crewther relation in the $\overline{\text{MS}}$-scheme for the nonsinglet (NS) corrections to the Adler $D$-function and to the Bjorken sum rule of the polarized lepton-nucleon scattering [4]. Expanding this form of the generalized Crewther relation in powers of $\alpha_s$ and keeping the single power of the QCD $\beta$-function only, one can recover the generalized Crewther relation with the single $\beta$-function factor. The existence of this $\overline{\text{MS}}$-scheme relation was discovered at the $\alpha_3^3$-level [5] and confirmed later on in [6] at the $\alpha_3^2$ order. This relation follows from the consideration of the AVV quark current triangle diagram not only in the massless quark-parton model [7], which respects conformal symmetry, but in the case, when the insertion of higher-order QCD corrections to this triangle diagram are also taken into account [8]. Theoretical validity of the generalized $\overline{\text{MS}}$-scheme Crewther relation, presented as the additional term with factored out single power of the $\beta$-function was studied in [9, 10], where its validity in all orders of perturbation theory was investigated. More recently the \{\beta\}\text{-expansion approach was explored in [11] in relation to its analog, used in [12–16] for various applications of the Principle of Maximal Conformality (PMC) proposed in [17]. Note that the main aim of PMC, which is similar to the seBLM method in [11], is to construct a new high-order representation of the BLM approach by absorbing all terms proportional to the $\beta$-function coefficients into the scales of each integer power of the coupling $\alpha_s$ in perturbative series for the RG-invariant quantities. For the NS Adler function and the Bjorken polarized sum rule the coefficients of these modified series should respect the relations, which follow from the conformal symmetry and the original Crewther relation of [7] (for the recent theoretical studies of the consequences of the conformal symmetry in QED and QCD see [18]).
2. Comparison of the complete and incomplete (\(\beta\))-expansions for the \(D^{NS}\)-function

Following the work \[11\], let us clarify first the differences between the complete and unique \(\beta\)-expansion \[1\] and the incomplete one used in the studies of \[12-16\]. Within the complete \(\beta\)-expansion the expression for the perturbative coefficients of the \(N^0\)LO approximation of the \(D^{NS}\)-function

\[
D^{NS}(a_s) = 1 + \sum_{n=1}^{\infty} d_n a_s^n
\]

is expressed through the coefficients of the \(\beta\)-function of the colour \(SU(N_c)\) gauge group model

\[
\beta(a_s) = \mu^2 \frac{\partial a_s}{\partial \mu^2} = - \sum_{\ell=0}^\infty \beta_\ell(N_F) a_s^{\ell+2}.
\]

in the following form:

\begin{align*}
&d_1 = d_1[0], \\
&d_2 = \beta_0(N_F) d_2[1] + d_2[0], \\
&d_3 = \beta_0^2(N_F) d_3[2] + \beta_1(N_F) d_3[0,1] \\
&\quad \quad + \beta_0(N_F) d_3[1] + d_3[0], \\
&d_4 = \beta_0^3(N_F) d_4[3] + \beta_1(N_F) \beta_0(N_F) d_4[1,1] \\
&\quad \quad + \beta_0(N_F) d_4[0,0,1] + \beta_0^2 d_4[2] \\
&\quad \quad + \beta_1(N_F) d_4[0,1] + \beta_0(N_F) d_4[1] + d_4[0].
\end{align*}

where \(N_F\) is the number of fermion flavours and the underlined terms were neglected in similar expansions used in \[12-16\]. The reason of neglecting them is related to the fact that the authors of these works define their \(\beta\)-expansions from the traditional expressions for \(d_i\) coefficients expanded in powers of \(N_F\), namely

\begin{align*}
&d_1 = N_F^0 d_1, \\
&d_2 = N_F d_2 + d_2, \\
&d_3 = N_F^2 d_3 + N_F d_3 + d_3, \\
&d_4 = N_F^3 d_4 + N_F^2 d_4 + N_F d_4 + d_4.
\end{align*}

However, it is already known that to formulate the generalized BLM approach at the NNLO using Eqs.(7-9), it is necessary to take into account some extra information \[11,13\]. Within the approach of \[1\] extra terms, which allow one to obtain the complete \(\beta\) expansion of \(d_3\) in Eq.(5), are the analytical contributions of the multiplet of light gluinos \(n_3\) to the \(O(\alpha_s^3)\) approximations of the \(D^{NS}\)-function \[19\] and to the \(\beta\)-function of the \(SU(N_c)\) group, which was evaluated in the \(\overline{MS}\)-scheme at the three-loop level in \[20\]. The application of new degrees of freedom \(n_3\) in both \(D^{NS}(a_s)\) and \(\beta(a_s)\)-functions at the \(O(\alpha_s^3)\) level allowed splitting in the \(\beta\)-expansion of the \(\beta_1(N_F)\) and \(\beta_0(N_F)\)-dependent terms, both contributing to the \(N_F\)-term of \(d_3\) in Eq.(9). We have thus obtained \[1\] the elements in the RHS of Eqs.(3-5), which define the following new matrix representation for \(D^{NS}\):

\[
D^{NS}(Q^2) = 1 + \sum_{n=1}^{\infty} \sum_{l} d_n(l) Q^2_l B_n(N_F) 
\]

In Eq.(11), the \(B_n(N_F)\) factors are the products of the \(\beta\)-function coefficients of Eqs.(3-5), \(d_n(N_F) = d_n[l] B_n(N_F)\) are the \(N_F\)-dependent coefficients in Eqs.(7-10) while the elements \(d_n[l]\) do not depend on the numbers of flavours \(N_F\). Note that in view of the absence of an analytical result for the gluino contributions to \(D^{NS}\) at the \(O(\alpha_s^3)\) level, we are unable to get most part of the terms in the \(\beta\)-expansion of \(d_3\) in Eq.(6). Indeed, only the leading \(\beta_0^3(N_F) d_3[3]\)-contribution is known from analytical calculations of \[3\]. This information was already used in the all-order generalization of the BLM approach of \[21\], based on absorbing into the BLM scale these renormalon-type terms only. Since we are interested in the resummation of all \(\beta\)-dependent terms, we will consider only certain expressions at the \(O(\alpha_s^3)\)-level.

In the \(\overline{MS}\)-scheme, at this order of perturbation theory the elements of the \(\beta\)-expansion for \(D^{NS}\) have the following analytic form \[1\]:

\begin{align*}
&d_1[0] = \frac{3}{4} C_F, \\
&d_2[1] = \left(\frac{33}{8} - 3 \zeta_3\right) C_F, \\
&d_3[0] = \frac{C_F^2}{16} + \frac{1}{16} C_F C_A, \\
&d_3[2] = \left(\frac{151}{6} - 19 \zeta_3\right) C_F, \\
&d_3[0,1] = \left(\frac{101}{16} - 6 \zeta_3\right) C_F, \\
&d_3[1] = \left(\frac{-27}{8} - \frac{39}{4} \zeta_3 + 15 \zeta_5\right) C_F^2, \\
&\quad \quad - \left(\frac{9}{64} - 5 \zeta_3 + \frac{5}{2} \zeta_5\right) C_F C_A, \\
&d_3[0] = \frac{-69}{128} C_F^3 + \frac{71}{64} C_F^2 C_A + \frac{523}{768} C_F C_A^2.
\end{align*}

Note once more that in the PMC studies of \[12-16\] an analog of the \(d_3[1]\)-term was absent (or nullified). Therefore, the remaining \(\overline{MS}\)-scheme contributions to Eq.(5) will differ from the ones presented in Eqs.(16) and (18).
3. The $\overline{MS}$-scheme generalized Crewther relation and the $[\beta]$-expansion for the Bjorken polarized sum rule

The Bjorken polarized sum rule, which is still interesting for phenomenological studies \cite{22,23}, is defined as

$$S_{Bjp} = \int_0^1 g_1^{lp-in}(x, Q^2)dx = \frac{g_A}{6} C^{Bjp}(a_s). \quad (19)$$

Its coefficient function $C^{Bjp}$ contains the NS and singlet (SI) contributions

$$C^{Bjp}(a_s) = C^{Bjp}_{NS}(a_s) + C^{Bjp}_{SI}(a_s). \quad (20)$$

The existence of the SI term at the $O(a_s^3)$ level was demonstrated in \cite{22-24}, though its analytical expression is not yet fixed by direct diagram-by-diagram calculations.

The $[\beta]$-expansion pattern is now applied to the coefficients $c_n$ of the perturbative approximation for $C^{Bjp}_{NS}(a_s)$ \cite{26}:

$$C^{Bjp}_{NS}(a_s) = 1 + \sum_{n=0}^{n=4} c_n a_s^n, \quad (21)$$

$$c_1 = c_1[0] \quad (22)$$

$$c_2 = \beta_0(N_F)c_2[1] + c_2[0] \quad (23)$$

$$c_3 = \beta_0^2(N_F)c_3[2] + \beta_0(N_F)c_3[0,1] + \beta_0^2 c_3[1][0] \quad (24)$$

$$c_4 = \beta_0^3(N_F)c_4[3] + \beta_0^2(N_F)c_3[1,1] + \beta_0^3 c_4[1][0] \quad (25)$$

These coefficients are related to similar ones, which enter into the $[\beta]$-expansion of the perturbative series for $D^{NS}$ through the multiple power $\beta$-function form of the generalization of the Crewther relation \cite{25}. Note that the application of the $\overline{MS}$-scheme generalization of the Crewther relation, considered in \cite{15}, gives the $[\beta]$-expanded expressions for the coefficients of the $C^{Bjp}_{NS}(a_s)$-series without the terms underlined in Eqs. (24-25).

We will show that the absence of these terms in the studies of \cite{12,16} contradicts the existing analytical $\overline{MS}$-scheme $O(a_s')$ results for the $D^{NS}(Q^2)$ function \cite{19,25,26} and the $\overline{MS}$-scheme generalization of the Crewther relation \cite{3,4} written down in the multiple power $\beta$-function representation of \cite{4}. This part of the talk follows from the studies of \cite{11}.

The $\overline{MS}$-scheme single $\beta$-function expression for the generalized Crewther relation has the following form:

$$D^{NS}(a_s)C^{Bjp}_{NS}(a_s) = 1 + \frac{\beta(a_s)}{a_s} K(a_s). \quad (26)$$

Here $K(a_s) = K_1 a_s + K_2 a_s^2 + K_3 a_s^3 + O(a_s^4)$ is the polynomial, where the known coefficient $K_1$ depends on the $SU(N_c)$ Casimir operator $C_F$ while the coefficients $K_2$ and $K_3$ also known analytically depend on $C_F$, $C_A$, $T_F$ and $N_F$. This form, originally discovered in \cite{5} at the $O(a_s^1)$ level, was recently confirmed by direct $O(a_s^2)$ calculations of $D^{NS}$ and $C^{Bjp}_{NS}$ performed in the colour $SU(N_c)$ gauge group theory \cite{6}. In \cite{3}, it was demonstrated that Eq. (26) can be rewritten as

$$D^{NS}(a_s)C^{Bjp}_{NS}(a_s) = 1 + \frac{\beta(a_s)}{a_s} \sum_{n=1}^{n=3} \left( \frac{\beta(a_s)}{a_s} \right)^{n-1} P_n(a_s) \quad (27)$$

$$= 1 + \sum_{m=1}^{m=4} \frac{\beta(a_s)}{a_s} \left( \frac{\beta(a_s)}{a_s} \right)^{n-1} P_n(a_s) \quad (28)$$

where $k + m = r$ and the coefficients $P_n[r,m]$ contain rational fractions and Riemann $\zeta$-functions of odd arguments. In Eq. (27), the known coefficients of the polynomials $P_n(a_s)$ do not depend on $C_F, T_F$ for (a more obvious clarification this property see the second expression in Eq. (27)) and are expressed by means of the coefficients of the $[\beta]$-expansion as

$$P_1(a_s) = -a_s \left[ c_2[1] + d_2[1] \right] + a_s \left[ c_3[1] + d_3[1] + d_1(c_2[1] - d_2[1]) \right] \quad (29)$$

$$P_2(a_s) = a_s \left[ c_3[2] + d_3[2] + a_s \left[ c_4[2] + d_4[2] - d_1(c_3[2] - d_3[2]) \right] \right] \quad (30)$$

Using Eq. (27), the following relations between the elements of the $[\beta]$-expansions of Eqs. (31) and Eqs. (24-25) were obtained \cite{3}:

$$0 = c_n[0] + d_n[0] + \sum_{j=1}^{n-1} d_j[0] c_{n-j}[0] \quad (31)$$

$$\sum_{j=1}^{j=3} c_j[0] = -c_2[1] - d_2[1] \quad (32)$$

3
Using Eqs. (1-4) for \( d_a[l] \) and solving then either Eq. (31) (initial Crewther relation) or Eq. (34) (from (4)) we got the elements of the relation (7) and of the conformal symmetry. However, we come to the definite theoretical conclusions. First, conclude that the coefficients of the \( \beta \)-expansion for \( C_{NS}^{Bj} \) at the \( O(\alpha_s^2) \) level:

\[
\begin{align*}
\tilde{c}_1[0] &= -\frac{3}{4} C_F \quad (34) \\
\tilde{c}_2[1] &= \frac{3}{2} C_F \quad (35) \\
\tilde{c}_2[0] &= \frac{21}{32} C_F^2 - \frac{1}{16} C_F C_A \quad (36) \\
\tilde{c}_3[2] &= -\frac{151}{24} C_F \quad (37) \\
\tilde{c}_3[0, 1] &= -\frac{59}{16} + 3\zeta_2 \quad (38) \\
\tilde{c}_3[1] &= \frac{83}{24} - \zeta_2 \quad (39) \left[ C_F^2 \right. \\
&\quad + \left. \frac{215}{64} - 6\zeta_3 + \frac{5}{2} \zeta_2 \right] C_F C_A \\
\tilde{c}_3[0] &= \frac{3}{128} C_F^3 - \frac{65}{64} C_F^2 C_A \quad (40) \left[ -\frac{523}{768} - \frac{27}{8} \zeta_3 \right] C_F C_A^2 \,.
\end{align*}
\]

Apart from the presented above analytical expressions, we come to the definite theoretical conclusions. First, we note that Eq. (31) is the consequence of the Crewther relation (7) and of the conformal symmetry. However, it does not give us a possibility to say anything about the scheme-independence of the coefficients \( d_1[0] \) and \( c_1[0] \) even within the MS-like schemes. We can only conclude that the coefficients of the PMC series are scheme-dependent but obey the scheme-independent relation, which follows from the conformal symmetry.

Second, the chain of Eqs. (33) clearly demonstrates, that the MS analytical calculations of \( D^{\text{BS}}, C_{Bj} \) and the MS-scheme generalizations of the Crewther relations of Eq. (29), (27) do not allow one to neglect (or nullify) the terms \( d_2[1], d_4[0, 1] \) in the \( \beta \)-expansion of the coefficients Eq. (5) and Eq. (6) of the \( D^{\text{BS}} \) RG-invariant function and of their analogs \( c_1[1], c_4[0, 1] \) in the perturbative expansion of the Bjorken polarized sum rule. Indeed, their absence contradicts the analytical result on the LHS of Eq. (33), obtained in (4) from the MS-scheme generalization of the Crewther relation. In view of this, the theoretical and phenomenological studies of the works (12-16), where the discussed above nonzero terms were neglected, should be reconsidered. This was done in part in (11) and we will summarize below the concrete foundations of this work.

### 4. The definition of the scale-fixing prescription

To define the generalized BLM approach within the complete and unique \( \beta \)-expansion approach of (1), one should absorb all \( \beta \)-dependent terms of the \( \beta \)-expanded coefficients into the scales of the coupling constants. Following the study of (11), let us absorb all \( \beta \)-dependent terms of the coefficients in Eqs. (33), into the new scales of the related perturbative expansions of the \( D^{\text{BS}} \)-function and \( C_{NS}^{Bj} \) RG-invariant quantities. Using the solution of the RG-equation in Eq. (2) we re-express the QCD running coupling constant \( a_s(\mu^2) \) in terms of the new one \( a_s = a_s(\mu^2) \) in the following form considered in (11), namely:

\[
a_s(t) = a' - \frac{1}{11} \beta(a_s) \frac{\Delta}{11} + \beta(a_s) \frac{\Delta^2}{2!} \,.
\]

The term \( \Delta \) defines the shift of the scales as

\[
\Delta = \Delta(a_s) = \Delta_0 + a_s \beta_0 \Delta_1 + (a_s \beta_0)^2 \Delta_2 + \ldots \quad (43)
\]

where the coupling constant dependence of this shift was first introduced in (13) in the process of the first formulation of the NNLO generalization of the BLM approach. Fixing now \( Q^2 = \mu^2 \) we obtain the \( \beta \)-expansions of the transformed to the new scale coefficients \( d_a \) of the perturbative expressions for the \( D^{\text{BS}} \)-function. They have the following form (11):

\[
\begin{align*}
\Delta_t[0] &= d_1[0] \quad (44) \\
\Delta_t[1] &= \beta_0 d_2[1] + d_2[0] - \beta_0 \Delta_0 \quad (45) \\
\Delta_t[2] &= \beta_0^2(d_3[2] - 2d_2[1] \Delta_0 + \Delta_0^2) + \beta_1(d_3[0, 1] - \Delta_0) + \beta_0(d_3[1] - 2d_2[0] \Delta_0) + d_3[0] + \beta_0^2 \Delta_1 \quad (46)
\end{align*}
\]

For the sake of generality, we also present the expression for the fourth term, which due to still incomplete
analytical information on its $\beta$-expansion can not be involved in the concrete numerical studies

$$
\begin{align*}
\Delta_0 &= d_3[1] = \frac{11}{2} - 4\zeta_3 = 0.69177 \\
\tilde{\Delta}_0 &= c_2[1] = -2 \\
\Delta_1 &= \frac{1}{\beta_0^2}[\beta^2_0(d_3[1] - d_2[1])^2] \\
&+ \beta_1(d_3[1] - 2d_2[1]) + \beta_0d_3[1] - 2d_2[1]d_2[1]] \\
\tilde{\Delta}_1 &= \frac{1}{\beta_0^2}[\beta^2_0(c_3[1] - c_1[1])^2] \\
&+ \beta_1(c_3[1] - c_1[1]) + \beta_0(c_3[1] - 2c_2[0]c_1[1]).
\end{align*}
$$

(47)

The general idea of [11] is to absorb all $\beta$-dependent terms in Eqs. (54), including the ones omitted in [12, 16], namely the terms proportional to $d_3[1], d_4[0, 1]$ and $d_4[1]$. Then, we accumulate these terms in “shift” coefficients $\Delta_0, \Delta_1$ and $\Delta_2$, which defines the new BLM (PMC-type) scales of $D^N_S (a_3^\prime)$. We will present here the results of application of this procedure at the $O(a_3^\prime)^2$ level only, where all coefficients of the $\beta$-dependent terms are already determined, see Eqs. (13), (17) and Eqs. (35), (59).

5. The concrete analytical and numerical $O(a_3^\prime)$ studies

Following the studies [11], and solving Eqs. (45), (46) with respect to $\Delta_0, \Delta_1$ and similar expressions for $C_{NS}^{BIP}$ in the case of ordinary QCD, we arrive at the concrete analytical and numerical results for the parameters in the defined in Eq. (43) scale $\Delta$ of the PMC-type BLM generalization of $O(a_3^\prime)$ approximations for $D^N_S$ and $C_{NS}^{BIP}$:

$$
\begin{align*}
\Delta_0 &= -2, \quad \tilde{\Delta}_0 = -2, \\
\Delta_1 &= \frac{1}{\beta_0^2}[\beta^2_0(d_3[1] - d_2[1])^2] \\
&+ \beta_1(d_3[1] - 2d_2[1]) + \beta_0d_3[1] - 2d_2[1]d_2[1]] \\
\tilde{\Delta}_1 &= \frac{1}{\beta_0^2}[\beta^2_0(c_3[1] - c_1[1])^2] \\
&+ \beta_1(c_3[1] - c_1[1]) + \beta_0(c_3[1] - 2c_2[0]c_1[1]).
\end{align*}
$$

(48)

(49)

(50)

(51)

Eqs. (50), (51) contain noticeable contributions of the terms omitted in [12, 16], that are proportional to $d_3[1]$ and $c_3[1]$. Note that for normalization, used here, we have $a_3 = 4/\pi, \beta_0 = 11/4 - N_F/6$ and $\beta_1 = 51/8 - 19N_F/24$. The approximate numerical expressions for the coefficients of the $\beta$-expansion for the $D^N_S$ and $C_{NS}^{BIP}$ RG-invariant functions read

$$
\begin{align*}
d_1[0] &= 1, \quad c_1[0] = -1 \quad (52) \\
d_2[1] &= 0.69, \quad c_2[1] = -2 \quad (53) \\
d_2[0] &= 0.838, \quad c_2[0] = 0.917 \quad (54) \\
d_3[2] &= 3.103, \quad c_3[2] = -8.39 \quad (55) \\
d_3[0, 1] &= -1.2, \quad c_3[0, 1] = -0.108 \quad (56) \\
d_3[1] &= 13.925, \quad c_3[1] = -10 \quad (57) \\
d_3[0] &= -35.87, \quad c_3[0] = 35.04 \quad (58)
\end{align*}
$$

(52)

(53)

(54)

(55)

(56)

(57)

(58)

We note poor convergence of the $O(a_3^\prime)$ approximations of the perturbative series constructed from the respective conformal symmetry coefficients $d_3[0]$ and $c_3[0]$. Indeed, the concrete result for the normalized $NS$ contribution to the $e^+e^- R$-ratio, which is related to the $D^N_S (Q^2)$-function, has the following form [11]:

$$
R^N_S(s) = 1 + a_s(s_{PMC}) + 0.0833 a_3^\prime(s_{PMC})
$$

(59)

$$
- 35.872 a_3^\prime(s_{PMC}) + O(a_3^\prime)
$$

(60)

where the scale is defined through the solution of Eq. (58) and Eq. (50). We will present it for $N_F = 3$ numbers of active flavours with $\beta_0 = 2.25$ and $\beta_1 = 4$. It has the following expression:

$$
s_{PMC} = s \cdot \exp[0.69 - 0.25\beta_1 a_3^\prime(s)]
$$

(61)

Note that at the NLO we reproduce the standard BLM coefficient, which is rather small. However, the value of the NNLO coefficient is negative and huge. A similar feature was already observed in the case of applications of the first generalization of the BLM approach based on resummation of the $N_F$-dependent corrections [3]. This result of [3] was confirmed in [11]. Applying the same procedure to $C_{NS}^{BIP}$ in [11] we got

$$
C_{NS}^{BIP}(Q^2) = 1 - a_s(Q^2_{PMC}) + 0.917 a_3^\prime(Q^2_{PMC}) + 35.04 a_3^\prime(Q^2_{PMC}) + O(a_3^\prime)
$$

(62)

where

$$
Q^2_{PMC} = Q^2 \cdot \exp[-2 - 0.08\beta_3 a_3^\prime(Q^2)]
$$

(63)

Similar results were previously obtained at the NNLO for the Bjorken polarized sum rule within the procedure of [3] in [27].

6. Conclusion

We would like to emphasize that the proposed in [11] and used later on in [4, 11] $\beta$-expansion approach allows one to fix the special terms $d_3[0]$ and $c_3[0]$ of
the \( \overline{\text{MS}} \)-scheme series for the \( e^+e^- \) characteristic \( R^{\text{NS}}(s) \) and for the Bjorken polarized sum rule. They satisfy the relations, which follow from the conformal symmetry. However, leaving only these terms in the \( O(a_s^2) \) approximations for the special generalizations of the BLM procedure one gets huge coefficients related with \( O(a_s^2) \) level. In view of this, the direct applications of theoretically interesting PMC-type (or seBLM-type) approximations in the phenomenological studies should be treated with care.

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