Solving Graph Isomorphism Problem for a Special case

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Abstract—Graph isomorphism is an important computer science problem. The problem for the general case is unknown to be in polynomial time. The best algorithm for the general case works in quasi-polynomial time \cite{1}. The solutions in polynomial time for some special type of classes are known. In this work, we have worked with a special type of graphs. We have proposed a method to represent these graphs and finding isomorphism between these graphs. The method uses a modified version of the degree list of a graph and neighbourhood degree list \cite{2}. These special type of graphs have a property that neighbourhood degree list of any two immediate neighbours is different for every vertex. The representation becomes invariant to the order in which the node was selected for giving the representation making the isomorphism problem trivial for this case. The algorithm works in \( O(n^4) \) time, where \( n \) is the number of vertices present in the graph. The proposed algorithm runs faster than quasi-polynomial time for the graphs used in the study. Index Terms—raph theory, graph isomorphism problem raph theory, graph isomorphism problem g

I. INTRODUCTION

Graph is a popular data structure and can be used in many complex real world applications, such as social networks, networking. For example, two people on a social networking site a and b can be represented by a graph consisting nodes \( v_a \) and \( v_b \). Now, if they are friends, then their relationship can be represented by an edge \((v_a,v_b)\) between them. Graph theory is very important in many other application. There are many open problems in the graph theory. Description of graph isomorphism problem is easy, but difficult to solve. Graph isomorphism: It is a bijection on vertex set of graph \( G \) and \( H \) that preserves edges. Graph isomorphism problem is a special case of subgraph isomorphism problem which is in NP-complete complexity class.

Checking whether two Graphs are isomorphic or not is an old and interesting computational problem. The simplest, but inefficient approach for checking two graphs \( G \) and \( H \), each with vertices \( v \) is making all the permutation of the nodes and see if it is edge preserving bijection or not. Obviously, this solution takes \( O(v^v!\) \) time, where \( v \) is the number of vertices of the graph. Even for smaller values of \( v \) the problem becomes intractable because of this enormous time complexity.

Babai \cite{3} has shown the general case algorithm to be solved in quasi-polynomial time. While GI problem for a general case is a hard problem, many works have been done on different special types of graphs. These special types are important in graph theory and polynomial time solution is known for them. Kelly et. al. \cite{3} and Aho et. al \cite{4} have worked on isomorphism on trees and they have found a polynomial time algorithm for this case. Hopcraft et. al. found an algorithm for solving GI on permutation graphs \cite{5}. In fact, this problem is in log space. Lueker et. al. have worked on interval graphs \cite{6}. Colbourn et. al. have worked on color-preserving isomorphism of colored graphs with bounded color multiplicity \cite{12}. Zager et. al. have worked on this problem using similarity functions \cite{13}. In this work, first we will define the graphs that are used in this study. After that, we will discuss the algorithm for solving isomorphism for this type of graphs in polynomial time. A worked out example is also given along with the algorithm.

II. ALGORITHM

This is a simple example of GI problem. We have given two graphs \( G_1 \) and \( G_2 \). The problem is that we have to find whether these two graphs are isomorphic or not and if they are isomorphic then give the isomorphism. In practice, the the input is the adjacency matrices of these two graphs. Two isomorphic graphs have equal number of nodes, number of edges, degree sequence. But these properties are not enough to prove the isomorphism. Two graph with same degree sequence can be non-isomorphic.

Consider Fig. 1 and Fig. 2 Brute-force method is to try all the possibilities and to find the edge preserving bijection from vertex set of graph \( G_1 \) to vertex set of graph \( G_2 \). After trying all the possibilities one can find out that these two graphs are isomorphic. Using this brute-force method the isomorphism function \( I : G_1 \rightarrow G_2 \) is given in the Table 1. First, we will define the graphs that can be solved by the proposed method. Consider Fig. 1 and Fig. 2 Brute-force method is to try all the possibilities and to find the edge preserving bijection from vertex set of graph \( G_1 \) to vertex set of graph \( G_2 \). After trying all the possibilities one can find out that these two graphs are isomorphic. Using this brute-force method the isomorphism function \( I : G_1 \rightarrow G_2 \) is given in the Table 1. Permissible graphs: Let \( G(V,E) \) be a simple, connected graph ,

Permissible graphs: Let \( G(V,E) \) be a simple, connected
$$V : \text{Set of vertices}, \ |V| = n$$

$$E : \text{Set of edges (Ordered pair of vertices)},$$

$$E \subseteq (V \times V), \text{ and for a given } v_i \in V \text{ and}$$

$$S_i = \{x \mid (x, v_i) \in E\}, \ \forall v_j, v_k \in S, v_j \neq v_k \implies dsv(v_j) \neq dsv(v_k).$$

Where $dsv(p)$ is the neighbourhood degree list of a given vertex $p$ with the vertices. For example, $dsv(4) = \{(5,1),(3,2),(6,2),(2,3)\}$. It is a list of tuples, where the first element is the vertex and second is its degree. The dsv is sorted by the degree (the second element).

We will solve this problem using the proposed method. The input is two graphs $G_1$ and $G_2$ in their adjacency lists form, $list_1, list_2$ respectively. The algorithm will first check that whether these graphs follow the Definition of Permissible Graphs or not. Table III shows the adjacency list, $list_1$ of the graph $G_1$. This algorithm consists of 3 sub-modules:

**A. Preprocessing**

We will process on both the graphs and store the results for later use. Here $deg\_seq_i$ is list of degree sequences of the neighbours of the vertex $i$. This step will be performed on every vertex $i$, $i \in V$ for both the graphs. For understanding, $deg\_seq_4$ is shown in the Table IV.

**Algorithm 1 Preprocessing**

1. $deg\_seq_i \leftarrow 0$
2. $S = \{x \mid (x, i) \in E\}$
3. for each $v \in S$ do
4. $deg\_seq_i.append(dsv(v))$
5. end for

**B. Checking the input**

In this module the algorithm will check whether the given graphs fits in the above definition or not. After getting the $deg\_seq_i$, for all the $i \in V$, the algorithm sorts the elements

| Vertex | degree(i) | dsv(i) |
|--------|-----------|--------|
| 1      | 1         | \{(2,3)\} |
| 2      | 3         | \{(1,1),(3,2),(4,4)\} |
| 3      | 2         | \{(2,3),(4,4)\} |
| 4      | 4         | \{(5,1),(3,2),(6,2),(2,3)\} |
| 5      | 1         | \{(4,4)\} |
| 6      | 2         | \{(7,1),(4,4)\} |
| 7      | 1         | \{(6,2)\} |

| Vertex | list elements | list elements | list elements | list elements |
|--------|---------------|---------------|---------------|---------------|
| 1      | 2             | -             | -             | -             |
| 2      | 1             | 3             | 4             | -             |
| 3      | 2             | 4             | -             | -             |
| 4      | 2             | 3             | 5             | 6             |
| 5      | 4             | -             | -             | -             |
| 6      | 4             | 7             | -             | -             |
| 7      | 6             | -             | -             | -             |
TABLE IV: degree sequence list, deg_seq4 of the graph G1

| vertex v | dsv(v) of the neighbours v of the vertex 4 |
|----------|------------------------------------------|
| 2        | {(1,1),(3,2),(4,4)}                      |
| 3        | {(2,3),(4,4)}                            |
| 6        | {(7,1),(4,4)}                            |
| 5        | {(6,4)}                                  |

Algorithm 2 Whether this method is applicable or not

1: Sort the lists in deg_seqi by the length of the lists and then lexicographically.
2: Sort dsv(i) according to the order of lists in the deg_seqi.
3: result ← true
4: for each i ∈ V do
  5:    for each x, y ∈ deg_seqi do
  6:      if x = y then
  7:        result ← false
  8:      end if
  9:    end for
10: end for

of this list. In the step 1. of the Algorithm 2, lists in the deg_seqi are sorted by lexicographically (according to the second element in the tuple) and then by the length of the list. Now, we will perform this step on Vertices of the graph G1.

If value of the result is true then we can apply our algorithm. And it can go thorough out next stage.

C. Generate UIDs

Here we will introduce an encoding technique, in which we will assign different ids to all the vertices. And after that we will compare this ids in both the graphs among all the vertices. These unique ids (UID) are independent of the order in which we calculated these UIDs. The bool array is of size |V| and it stores whether all the vertex are included or not. The array is initialized to 0 for all the elements. The count variable is initialized with total number of vertices, |V|. It will decrement its value by one as the algorithm explores the dsv of all the vertices. Function generate_UID is called for each vertex of graphs G1 and G2. This function takes count and the vertex i as its input. Bool array element will change its value to one, if that vertex is visited. The function will be called for every vertex i with initial condition set to generate_UID (n, i). Now, we will perform this operation on the vertex 4 of Graph G1.

Now, we know from Table I that vertex D from graph G2 is isomorphic to the vertex 4 from graph G1.

D. Map Isomorphism

After generating UIDs from both the graph G1 and G2 the algorithm will try to map the UIDs of graph G1 to UIDs of Graph G2 and will report the isomorphism I. Here, UID^p is UID of graph p and of jth vertex. UID^p is the list of all the UIDs of all the vertex of the graph p. The map Isomorphism function will be called for UID of every vertex of the graph G1. The iso is and array which maps vertex of one graph to the vertex of the other graph. If for all the UIDs the value of flag is 1 and all the iso arrays give the same result then the two graphs are isomorphic and the isomorphism function is given by iso. The algorithm uses the constraint given in the Definition of Permissible Graphs., which is why the algorithm revolves around the degree sequence of the vertices.

Algorithm 4 map Isomorphism(UID^1)

1: for each UID^2_j ∈ UID^2 do
2:    iso ← −1
3:    ind ← 0
4:    flag ← 0
5:    if len(UID^1) = len(UID^2) then
6:      while ind < len(UID^1) do
7:        ind ← ind + 1
8:        if iso[UID^1][ind][0] ≠ UID^2[ind][0] then
9:          flag ← 0
10:         break
11:       end if
12:     end while
13:    if UID^1[ind][1] ≠ UID^2[ind][1] then
14:      flag ← 0
15:     break
16:    else
17:      iso[UID^1][ind][0] ← UID^2[ind][0]
18:      flag ← 1
19:    end if
20: end if
21: if flag = 1 then
22:     break
23: end if
24: end for

Any vertex of a permissible graphs can not have more than one UID.

Proof. The UID consist of UID_d (degrees) and UID_n (nodes).

- UID_d[1] = v and UID_d[1] = deg(v) .
TABLE V: sorted degree sequence list, \( \text{deg}_i \) for \( i \in V \) of the graph \( G_1 \)

| vertex | \( \text{deg}_i \) | changed order of \( \text{dsv}(i) \) | remarks |
|--------|-----------------|-------------------------------|---------|
| 1      | \{ (1,1),(3,2),(4,4) \} | \{ (2,3) \} | - |
| 2      | \{ (12,3) \}, \{ (2,3),(4,4) \}, \{ (5,1),(3,2),(6,2),(2,3) \} | \{ (1,1),(3,2),(4,4) \} | - |
| 3      | \{ (1,1),(3,2),(4,4) \}, \{ (5,1),(3,2),(6,2),(2,3) \} | \{ (2,3),(4,4) \} | - |
| 4      | \{ (4,4) \}, \{ (2,3),(4,4) \}, \{ (7,1),(4,4) \}, \{ (1,1),(3,2),(4,4) \} | \{ (5),(6),(2),(2,3) \} | changed |
| 5      | \{ (5,1),(3,2),(6,2),(2,3) \} | \{ (4,4) \} | - |
| 6      | \{ (6,2) \}, \{ (5,1),(3,2),(6,2),(2,3) \} | \{ (7,1),(4,4) \} | - |
| 7      | \{ (1,1),(4,4) \} | \{ (6,2) \} | - |

TABLE VI: UID of vertex 4 of Graph \( G_1 \)

| vertex | 4 | 2 | 5 | 6 | 3 | 2 | -2 | -4 | 7 | 4 | 2 | 142 |
|--------|---|---|---|---|---|---|-----|-----|---|---|---|-----|
| degree | 4 | 2 | 1 | 2 | 3 | 2 | -2 | -4 | 1 | 4 | 3 | 4 |

TABLE VII: UID of vertex D of Graph \( G_2 \)

| vertex | D | -2 | C | B | E | F | -2 | 2 | D | A | F | E | D | 2 | D | 6 | D | F | E | D | -2 |
|--------|---|----|---|---|---|---|-----|----|---|---|---|---|---|----|----|----|---|---|---|----|----|
| degree | D | -2 | 1 | 2 | 2 | 3 | -2 | 4 | 1 | 4 | 3 | 4 | 1 | 2 | 4 | -2 | -1 | 2 | -1 | -1 | 3 | -1 | -1 | -2 |

- \( UID_n[2] = -2 \) and \( UID_2[2] = -2 \)
- The next \( deg(v) \) elements will be of \( dsn(v) \).
- Between any two \( i \) and \( j \) such that, \( UID_n[i] = -2 \), \( UID_2[i] = -2 \) and \( UID_2[j] = -2 \) and there is no \( k \), where \( i < k < j \), such that \( UID_n[k] = -2 \), \( UID_2[k] = -2 \):

\[
\text{Theorem:} \text{ There is no graph with two adjacent vertices having degree sequence} \\
\text{\( \{ (2,3) \} \) in which both vertices have degree 2.} \\
\text{Proof:} \text{ We will prove it by example. Consider Fig. 5 number}
\]

![Fig. 3: Graph \( G_3 \)](image_url)

of vertices is 8 and number of edges is 19. The graph has a triangle in it. According to Euler’s formula \([14]\) this graph is non-planar graph \( (3v - 6 \leq e) \). Once can easily verify that this graph can also be solved using the proposed method.

III. TIME COMPLEXITY AND PRACTICAL RESULTS

1) The Preprocessing step: Degree of each vertex can be found in \( O(n^2) \) and after that their degree sequence can be calculated in \( O(n^3) \). Overall, this step takes \( O(n^3) \) time.

2) Checking the input: Time complexity \( O(n^4) \).

3) Generate UIDs: The maximum length of any UID is of the order of the number of \( O(n^2) \). This function will be called \( n \) times, making this step run in \( O(n^3) \) time.

4) Try to fit Isomorphism: For finding the isomorphism, the Algorithm 4 will compare UID of one graph with all the UIDs of the other graph. This is total \( n \times n^2 \) comparisons. This is \( O(n^3) \). The algorithm will be called \( n \) times. So, the overall complexity becomes \( n \times n^3 \) making the complexity \( O(n^4) \). Overall, the algorithm runs in \( O(n^4) \) time.

The code was written in python and c++. In the Table VIII the number of permissible graphs for number of vertices ranging from 1 to 9 is given. We generated these graphs using SAGE \([15]\) and nauty \([16]\). For \( n = 9, 10.66\% \) of the graphs lie under the Definition of Permissible Graphs. There are only 0.0171% of graphs are tree for \( n = 9 \). The Theorem \([11,13]\) states that there are non-planar graphs, which can also be solved by the proposed method.

IV. CONCLUSION

The proposed algorithm solves GI problem for a special case. Before running any slower algorithm it is better to check the graphs using this algorithm. The time complexity is \( O(n^4) \), but these bounds are loose bounds. Because generating UID and matching the UIDs can be done faster, if we have
used fast methods to sort them using the property that every value is bounded by $v$, making the bounds tighter. More importantly, this is polynomial time solution for this type of graphs. This algorithm is embarrassingly parallel making it useful in practical purposes.

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