Second-order differential equation with indefinite and repulsive singularities

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Abstract—This paper concerns a second-order differential equation with indefinite and repulsive singularities. It is the first time to study differential equation containing both indefinite and repulsive singularities simultaneously. A set of sufficient conditions are obtained for the existence of positive periodic solutions. The theoretical underpinnings of this paper are the positivity of Green’s function and fixed point theorem in cones. Our results improve and extend the results in previous literatures. Finally, three examples and their numerical simulations (phase diagrams and time diagrams of periodic solutions) are given to show the effectiveness of our conclusions.

Keyword—indefinite and repulsive singularities; fixed point theorem in cones; positive periodic solution.

MSC—34B16; 34B18; 34C25.

1 Introduction

There are many works on the problem of periodic solutions for the nonlinear ordinary differential equations with deviating argument, $p$-Laplacian, neutral type operator and singularity (see e.g. \cite{1,52}). In this paper, we pay particular attention to the differential equations with singularity. In fact, the differential equations with singularity has attracted many mathematicians’ attention (see e.g. \cite{3,6,10,12,17,19,24,26,33,35,38,47,49,51,52}). By the method of upper and lower solution, Lazer and Solimini \cite{31} initially studied the periodic solutions of the following singular
equations

\[ x''(t) + \frac{1}{x^\mu(t)} = m(t), \quad (1.1) \]

and

\[ x''(t) - \frac{1}{x^\mu(t)} = m(t), \quad (1.2) \]

where \( m(t) \in C(\mathbb{R}, \mathbb{R}) \) is a periodic function and \( \mu > 0 \) is a constant. It is said that equation (1.1) has an attractive singularity and equation (1.2) has a repulsive singularity. Pino and Manásevich \[43\] studied the periodic solutions for a problem arising in nonlinear elasticity. The equations is formulated as

\[ x''(t) + f(t, x(t)) = 0. \quad (1.3) \]

Further, Pino, Manásevich and Montero, \[42, 43\] considered the equation (1.3) with singularities. Hakl and Torres \[22\] discussed equation (1.3) with attractive-repulsive singularities:

\[ x''(t) = \frac{b(t)}{x^{\nu_1}(t)} - \frac{c(t)}{x^{\nu_2}(t)} + e(t), \quad (1.4) \]

where \( b(t), c(t) \in C(\mathbb{R}, (0, +\infty)) \) are periodic functions, \( \nu_1 > \nu_2 > 0 \) are constants. The positivity of \( b(t), c(t) \) states the equation has both attractive and repulsive singularities. Bravo and Torres \[6\] investigated equation (1.3) with an indefinite singularity as follows:

\[ x''(t) = \frac{b(t)}{x^3(t)}, \quad (1.5) \]

where \( b(t) \) is a piecewise-constant sign-changing function. Motivated by the two works \[6, 22\], the mathematicians paid their attention to the singular equations containing both attractive and repulsive singularities simultaneously or indefinite singularity (see \[8, 10, 21, 24, 33, 35, 37, 47, 52\]). Thereinto, Hakl and Zamora \[24\], Godoy and Zamora \[21\] generalized equation (1.5) in a wider application. In the recent years, Chu et. al. \[16, 17, 28\], Li and Zhang \[33\], Torres \[45, 46\] obtained some existence results for positive periodic solutions of the following equation by some fixed point theorems

\[ x''(t) + a(t)x(t) = f(t, x) + e(t), \quad (1.6) \]

where \( f(t, x) \) may be singular at \( x = 0 \). Subsequently, Cheng and Cui \[8\] considered equation (1.6) with \( f(t, x) \) an indefinite singularity, i.e.,

\[ x''(t) + q(t)x(t) = \frac{b(t)}{x^\rho(t)} + e(t), \quad (1.7) \]

where \( \rho > 0 \) is a constant and \( b(t) \in C(\mathbb{R}, \mathbb{R}) \) is a periodic function. Lu et. al. \[35, 37\] by means of coincidence degree theory generalized indefinite singularity to Liénard equation.

In this paper, we study the following differential equation with indefinite and repulsive singularities simultaneously:

\[ x''(t) + p(t)x'(t) + q(t)x(t) = \frac{b(t)}{x^{\rho_1}(t)} + \frac{c(t)}{x^{\rho_2}(t)} + e(t), \quad (1.8) \]
where \( p(t), q(t), b(t) \in C(\mathbb{R}, \mathbb{R}) \) and \( c(t), e(t) \in C(\mathbb{R}, (0, +\infty)) \) are \( \omega \)-periodic functions, \( \rho_1, \rho_2 \) are positive constants. The term \( \frac{b(t)}{x^{\rho_1}(t)} \) has an indefinite singularity in view of the uncertainty of the sign of weight function \( b(t) \), even the singularity will be disappeared if \( b(t) = 0 \) in some subintervals. In addition to this, the equation has a repulsive singularity because of the term \( \frac{c(t)}{x^{\rho_2}(t)} \) and \( c(t) > 0 \). In particular, if equation (1.8) has no damping term (i.e., \( p(t) \equiv 0 \)), it reduces to the following differential equation:

\[
x''(t) + q(t)x(t) = \frac{b(t)}{x^{\rho_1}(t)} + \frac{c(t)}{x^{\rho_2}(t)} + e(t).
\]

We generalize and improve the previous works in three aspects. Firstly, equation (1.8) is investigated in a very general form containing both indefinite singularity and repulsive singularity simultaneously. Obviously, it is more general and complex than equations (1.3)-(1.7). In fact, equations (1.3)-(1.7) either do not contain indefinite singularity or single singularity. This means that the previous methods can not be directly applied to equation (1.8) any more. Secondly, noticing the term \( \frac{b(t)}{x^{\rho_1}} - \frac{c(t)}{x^{\rho_2}} \), Chu et al. [12], Cheng and Ren [10], Hakl and Torres [22] required \( \nu_1 > \nu_2 \). In fact, \( \nu_1 > \nu_2 \) implies that equation (1.4) tends to be more repulsive singularity. In this paper, we consider all the three cases, \( \rho_1 > \rho_2, \rho_1 = \rho_2 \) and \( \rho_1 < \rho_2 \). Consequently, our results generalize and improve the existing literatures. Thirdly, our results can be applied to both strong singularity and weak singularity. In fact, following Lazer and Solimini’s work, there are many good papers appearing in this field by different kinds of methods such as the fixed point theorems [8, 9, 27], lower and upper solution [1, 22, 23], Leray-Schauder alternative principle [12, 28, 32] and coincidence degree theory [5, 29, 37], and so on. Different from above methods, in this paper, we use the theory of the fixed point in cones to study the periodic solution.

The rest structure of this paper is as follows. In section 2, the sufficient conditions for the positivity of Green’s function of the second-order differential equation are given. Our main results are stated in Section 3. In section 4, three examples and their simulations are given to confirm our conclusions.

## 2 Positivity of Green’s function

In order to make good use of fixed point theorem to get the existence of positive periodic solution for equation (3.1), first of all we need to guarantee the invariance of the sign of Green’s function of the nonhomogeneous linear equation corresponding to equation (3.1). According to the specific situation of this paper, we consider the positivity of Green’s function.

Let \( C^1_\omega := \{ \phi(t), \phi'(t) \in C(\mathbb{R}, \mathbb{R}) : \phi(t + \omega) = \phi(t), \phi'(t + \omega) = \phi'(t), t \in \mathbb{R} \} \), equipped with the norm \( \| \phi \| = \| \phi \|_\infty + \| \phi' \|_\infty \), and \( C^1_\omega \) in the norm \( \| \cdot \| \) sense is a Banach space, here \( \| \phi \|_\infty = \max_{t \in [0, \omega]} |\phi(t)| \). For a continuous function \( \phi(t) \), we define \( \phi^* := \max_{0 \leq t \leq \omega} \phi(t) \) and \( \phi_* := \min_{0 \leq t \leq \omega} \phi(t) \).
Considering the following second-order nonhomogeneous linear differential equation,

\[
\begin{aligned}
  &\left\{ \begin{array}{l}
  u''(t) + p(t)u'(t) + l(t)u(t) = h(t), \\
  u(0) = u(\omega), \quad u'(0) = u'(\omega),
  \end{array} \right. \\
  \end{aligned}
\]

(2.1)

where \( l(t) = \frac{2(t)}{\alpha} \), \( h(t) \in C(\mathbb{R}, (0, +\infty)) \) is a continuous \( \omega \)-periodic function. The unique \( \omega \)-periodic solution of equation (2.1) is expressed as

\[
u(t) = \int_0^\omega G(t, s)h(s)ds,
\]

where \( G(t, s) \) is Green’s function of equation (2.1). Now we talk about sufficient conditions to make \( G(t, s) > 0 \) such that \( \omega \)-periodic solution of equation (2.1) is positive. To describe the problem conveniently, we assume the following condition holds:

(A) The Green’s function \( G(t, s) > 0 \) for all \((t, s) \in [0, \omega] \times [0, \omega]\).

In 2005, Wang et al. [48, Lemma 2.4] proved condition (A) can be satisfied under the following two conditions:

(A1) There are continuous \( \omega \)-periodic functions \( a_1(t) \) and \( a_2(t) \) such that \( \int_0^\omega a_1(t)dt > 0 \), \( \int_0^\omega a_2(t)dt > 0 \) and

\[
a_1(t) + a_2(t) = p(t), \quad a_1'(t) + a_1(t)a_2(t) = l(t), \quad \text{for} \ t \in \mathbb{R}.
\]

(A2) \( (\int_0^\omega p(t)dt)^2 \geq 4\omega^2 \exp \left( \frac{1}{\omega} \int_0^\omega \ln l(s)ds \right) \).

On the basis of Wang’s work, Cheng and Ren gave the following conclusion:

**Lemma 2.1.** [10, Lemma 2.2] Assume that condition (A1) holds, then condition (A) holds.

In 2012, Chu et al. [12] also proved condition (A) by using antimaximum principle. First, authors defined two functions

\[
\varsigma(p)(t) = \exp \left( \int_0^t p(s)ds \right)
\]

and

\[
\varsigma_1(p)(t) = \varsigma(p)(\omega) \int_t^\omega p(s)ds + \int_t^\omega \varsigma(p)(s)ds,
\]

and then gave the following theorem:

**Lemma 2.2.** [12, Corollary 2.6] Assume that \( \int_0^\omega l(s)\varsigma(p)(s)\varsigma_1(-p)(s)ds \geq 0 \)

and

\[
\sup_{0 \leq t \leq \omega} \left\{ \int_t^{t+\omega} \varsigma(-p)(s)ds \int_t^{t+\omega} \max\{l(s), 0\}\varsigma(p)(s)ds \right\} \leq 4
\]

hold. If \( l(t) \neq 0 \), then condition (A) holds.
In particular, as the damping term of equation (2.1) is absent (i.e., \( p(t) \equiv 0 \)), a specific value about Green’s function can be obtained, and condition \((A)\) is ensured.

Lemma 2.3. [25, Lemma 2.5] In the case \( l(t) = \xi^2 \) with \( \xi > 0 \), the expression of the Green’s function as follows,

\[
G(t, s) = \begin{cases} 
\frac{\cos \xi (t - s - \frac{\omega}{2})}{2\xi \sin \frac{\xi \omega}{2}}, & 0 \leq s < t \leq \omega, \\
\frac{\cos \xi (t - s + \frac{\omega}{2})}{2\xi \sin \frac{\xi \omega}{2}}, & 0 \leq t < s \leq \omega.
\end{cases}
\] (2.2)

If

\[
\xi < \frac{\pi}{\omega},
\] (2.3)

then condition \((A)\) holds.

Under the circumstances that condition \((A)\) holds, one can say

\[
G^* := \max_{0 \leq s, t \leq \omega} G(t, s) > G_* := \min_{0 \leq s, t \leq \omega} G(t, s) > 0,
\]

and we define

\[
\sigma := \frac{G_*}{G^* + \max_{0 \leq s, t \leq \omega} \left| \frac{\partial G(t, s)}{\partial t} \right|}, \quad \delta := \frac{\max_{0 \leq s, t \leq \omega} \left| \frac{\partial G(t, s)}{\partial t} \right|}{G_*}.
\]

3 Existence of positive periodic solution for equation (1.8)

Our main results and their proofs are presented in this section. To study equation (1.8), we make a transformation of variable \( x = y^\alpha \) and \( \alpha = \frac{1}{\rho_1 + 1} \). Then equation (1.8) is rewritten as follows:

\[
y''(t) + p(t)y'(t) + \frac{q(t)}{\alpha}y(t) = \frac{c(t)}{\alpha} y^{1-\alpha - \alpha \rho_2}(t) + \frac{c(t)}{\alpha} y^{1-\alpha}(t) + (1 - \alpha) \frac{y'(t)^2}{y(t)} + \frac{b(t)}{\alpha},
\] (3.1)

Obviously, by this transformation, it no longer contains indefinite singularity. We see that the term \((1 - \alpha) \frac{y'(t)^2}{y(t)}\) has a repulsive singularity. Now let’s consider the term \( \frac{c(t)}{\alpha} y^{1-\alpha - \alpha \rho_2}(t)\):

(i) If \( \rho_1 > \rho_2 \) (i.e. \( 0 < 1 - \alpha - \alpha \rho_2 < 1 \)), the term has no singularity, in this case, equation (3.1) has just a repulsive singularity;

(ii) if \( \rho_1 < \rho_2 \) (i.e. \( 1 - \alpha - \alpha \rho_2 < 0 \)), the term has a repulsive singularity, in this case, equation (3.1) has two repulsive singularities;

(iii) if \( \rho_1 = \rho_2 \) (i.e. \( 1 - \alpha - \alpha \rho_2 = 0 \)), same situation as (i).

Consequently, the existence of positive periodic solution for equation (1.8) is equivalent to that of equation (3.1). To prove the existence of positive periodic solutions, we need a fixed point theorem in cones (see [40]).
Lemma 3.1. Let \( X \) be a Banach space and \( K \) is a cone in \( X \). Assume that \( \Lambda_1, \Lambda_2 \) are open subsets of \( X \) with \( 0 \in \Lambda_1, \Lambda_2 \subset \Lambda_2 \). Let

\[
\mathcal{T} : K \cap (\overline{\Lambda}_2 \setminus \Lambda_1) \to K
\]

be a continuous and completely continuous operator such that

(i) \( \|\mathcal{T}y\| \leq \|y\| \) for \( y \in K \cap \partial\Lambda_2 \);

(ii) there exists \( y_0 \in K \setminus \{0\} \) such that \( y \neq \mathcal{T}y + \lambda y_0 \) for \( y \in K \cap \partial\Lambda_1 \) and \( \lambda > 0 \).

Then \( \mathcal{T} \) has a fixed point in \( K \cap (\overline{\Lambda}_2 \setminus \Lambda_1) \).

All the discussion throughout this paper takes place in Banach space \( C^1_\omega \), define a cone \( K \) in space \( C^1_\omega \) as follows:

\[
K = \{ y \in C^1_\omega : \min y(t) \geq \sigma \|y\|, |y'(t)| \leq \delta y(t), 0 \leq t \leq \omega \}.
\]

In what follows, we discuss the periodic solutions with three cases: \( \rho_1 > \rho_2, \rho_1 < \rho_2 \) and \( \rho_1 = \rho_2 \).

Theorem 3.1. Assume that condition \((A)\) and \( \rho_1 > \rho_2 \) hold. Furthermore, suppose there exist \( R > r > 0 \) such that:

\[
(H_1) \quad \frac{1}{\sigma} \left( -\frac{b^*_r}{c_*} \right)^{\frac{1}{\alpha}} \leq r \leq \left( \frac{G^{-1} \omega^\alpha}{\alpha} \right)^{\frac{1}{\alpha + \alpha \rho_2}}.
\]

\[
(H_2) \quad \frac{(1-\alpha)(G^*_r + \delta G_\alpha)}{\sigma} \delta \omega < 1.
\]

Then there is at least one positive \( \omega \)-periodic solution of equation \((3.1)\), saying \( y(t) \) satisfying

\( r \leq \|y\| \leq R \). Consequently, it is a positive periodic solution of equation \((1.8)\).

Proof. To use the fixed point in cones, take \( \Lambda_1 := \{ y \in C^1_\omega : \|y\| < r \}, \quad \Lambda_2 := \{ y \in C^1_\omega : \|y\| < R \} \),

and define an operator \( \mathcal{T} \) as

\[
(Ty)(t) = \int_0^\omega G(t,s) \left( \frac{c(s)}{\alpha} y^{1-\alpha \rho_2}(s) + \frac{e(s)}{\alpha} y^{1-\alpha}(s) + (1-\alpha) \frac{|y'(s)|^2}{y(s)} + \frac{b(s)}{\alpha} \right) ds,
\]

then a fixed point of operator equation \( y = \mathcal{T}y \) is an \( \omega \)-periodic solution of equation \((3.1)\). To this end, we divide it into four steps.

**Step 1.** We prove \( \mathcal{T}(K \cap (\overline{\Lambda}_2 \setminus \Lambda_1)) \subset K \). In fact, for \( y \in K \cap (\overline{\Lambda}_2 \setminus \Lambda_1) \), by definitions of \( K, \Lambda_1 \) and \( \Lambda_2 \), one can say

\[
\sigma r \leq \sigma \|y\| \leq \|y\| \leq \|y\| \leq R \text{ and } |y'(t)| \leq \delta y(t) \leq \delta R.
\]
It follows from the first half of condition \((H_1)\) that

\[
\begin{align*}
& \frac{c(t)}{\alpha} y^{1-\alpha-\alpha \rho_2}(t) + \frac{e(t)}{\alpha} y^{1-\alpha}(t) + \left(1 - \alpha \right) \frac{y'(s)^2}{y(s)} + \frac{b(t)}{\alpha} \\
\geq & \frac{c(t)}{\alpha} y^{1-\alpha-\alpha \rho_2}(t) + \frac{e(t)}{\alpha} y^{1-\alpha}(t) + \frac{b(t)}{\alpha} \\
\geq & c_\ast \sigma^{1-\alpha} r^{1-\alpha} + \frac{e_\ast \sigma^{1-\alpha}}{\alpha} r^{1-\alpha} + \frac{b_\ast}{\alpha} \\
\geq & \frac{e_\ast \sigma^{1-\alpha}}{\alpha} r^{1-\alpha} + \frac{b_\ast}{\alpha} \\
\geq & 0.
\end{align*}
\]

On the basis of condition \((A)\) and \((3.2)\), we have the following two inequalities,

\[
\min_{0 \leq t \leq \omega} (Ty)(t)
= \min_{0 \leq t \leq \omega} \int_0^\omega G(t, s) \left( \frac{c(s)}{\alpha} y^{1-\alpha-\alpha \rho_2}(s) + \frac{e(s)}{\alpha} y^{1-\alpha}(s) + \left(1 - \alpha \right) \frac{y'(s)^2}{y(s)} + \frac{b(s)}{\alpha} \right) ds
\geq G_\ast \int_0^\omega \left( \frac{c(s)}{\alpha} y^{1-\alpha-\alpha \rho_2}(s) + \frac{e(s)}{\alpha} y^{1-\alpha}(s) + \left(1 - \alpha \right) \frac{y'(s)^2}{y(s)} + \frac{b(s)}{\alpha} \right) ds
\geq \sigma G_\ast \int_0^\omega \left( \frac{c(s)}{\alpha} y^{1-\alpha-\alpha \rho_2}(s) + \frac{e(s)}{\alpha} y^{1-\alpha}(s) + \left(1 - \alpha \right) \frac{y'(s)^2}{y(s)} + \frac{b(s)}{\alpha} \right) ds
+ \sigma \max_{0 \leq s, t \leq \omega} \left| \frac{\partial G(t, s)}{\partial t} \right| \int_0^\omega \left( \frac{c(s)}{\alpha} y^{1-\alpha-\alpha \rho_2}(s) + \frac{e(s)}{\alpha} y^{1-\alpha}(s) + \left(1 - \alpha \right) \frac{y'(s)^2}{y(s)} + \frac{b(s)}{\alpha} \right) ds
\geq \sigma \max_{0 \leq s, t \leq \omega} \int_0^\omega G(t, s) \left( \frac{c(s)}{\alpha} y^{1-\alpha-\alpha \rho_2}(s) + \frac{e(s)}{\alpha} y^{1-\alpha}(s) + \left(1 - \alpha \right) \frac{y'(s)^2}{y(s)} + \frac{b(s)}{\alpha} \right) ds
+ \sigma \max_{0 \leq s, t \leq \omega} \int_0^\omega \left| \frac{\partial G(t, s)}{\partial t} \right| \left( \frac{c(s)}{\alpha} y^{1-\alpha-\alpha \rho_2}(s) + \frac{e(s)}{\alpha} y^{1-\alpha}(s) + \left(1 - \alpha \right) \frac{y'(s)^2}{y(s)} + \frac{b(s)}{\alpha} \right) ds
= \sigma \|Ty\|_\infty + \sigma \|\langle Ty \rangle \|_\infty = \sigma \|Ty\|_\infty,
\]

and

\[
\|\langle Ty \rangle(t)\| = \left\| \int_0^\omega \frac{\partial G(t, s)}{\partial t} \left( \frac{c(s)}{\alpha} y^{1-\alpha-\alpha \rho_2}(s) + \frac{e(s)}{\alpha} y^{1-\alpha}(s) + \left(1 - \alpha \right) \frac{y'(s)^2}{y(s)} + \frac{b(s)}{\alpha} \right) ds \right\|
\leq \int_0^\omega \left| \frac{\partial G(t, s)}{\partial t} \right| \left( \frac{c(s)}{\alpha} y^{1-\alpha-\alpha \rho_2}(s) + \frac{e(s)}{\alpha} y^{1-\alpha}(s) + \left(1 - \alpha \right) \frac{y'(s)^2}{y(s)} + \frac{b(s)}{\alpha} \right) ds
\leq \max_{0 \leq t \leq \omega} \left| \frac{\partial G(t, s)}{\partial t} \right| \int_0^\omega \left( \frac{c(s)}{\alpha} y^{1-\alpha-\alpha \rho_2}(s) + \frac{e(s)}{\alpha} y^{1-\alpha}(s) + \left(1 - \alpha \right) \frac{y'(s)^2}{y(s)} + \frac{b(s)}{\alpha} \right) ds
= \delta G_\ast \int_0^\omega \left( \frac{c(s)}{\alpha} y^{1-\alpha-\alpha \rho_2}(s) + \frac{e(s)}{\alpha} y^{1-\alpha}(s) + \left(1 - \alpha \right) \frac{y'(s)^2}{y(s)} + \frac{b(s)}{\alpha} \right) ds
\leq \delta \int_0^\omega G(t, s) \left( \frac{c(s)}{\alpha} y^{1-\alpha-\alpha \rho_2}(s) + \frac{e(s)}{\alpha} y^{1-\alpha}(s) + \left(1 - \alpha \right) \frac{y'(s)^2}{y(s)} + \frac{b(s)}{\alpha} \right) ds
= \delta (Ty)(t).
\]

This shows that \(T(K \cap (\bar{X}_2 \setminus \Lambda_1)) \subset K\).
**Step 2.** We prove that $T$ is a completely continuous operator. For any $y \in K \cap (\overline{A}_2 \setminus A_1)$, we have

$$
|Ty| = \left| \int_0^\omega G(t, s) \left( \frac{c(s)}{\alpha} y^{1-\alpha} - \alpha p_2(s) + \frac{e(s)}{\alpha} y^{1-\alpha} + (1 - \alpha) \frac{|y'(s)|^2}{y(s)} + \frac{b(s)}{\alpha} \right) \, ds \right|
$$

$$
= \int_0^\omega G(t, s) \left( \frac{c(s)}{\alpha} y^{1-\alpha} - \alpha p_2(s) + \frac{e(s)}{\alpha} y^{1-\alpha} + (1 - \alpha) \frac{|y'(s)|^2}{y(s)} + \frac{b(s)}{\alpha} \right) \, ds
$$

$$
\leq G^* \left( \frac{c^*}{\alpha} R^{1-\alpha} + \frac{e^*}{\alpha} R^{1-\alpha} + (1 - \alpha) \frac{(\delta R)^2}{\sigma} + \frac{b^*}{\alpha} \right) \omega := N_1,
$$

and

$$
|(Ty)'(t)| = \left| \int_0^\omega \frac{\partial G(t, s)}{\partial t} \left( \frac{c(s)}{\alpha} y^{1-\alpha} - \alpha p_2(s) + \frac{e(s)}{\alpha} y^{1-\alpha} + (1 - \alpha) \frac{|y'(s)|^2}{y(s)} + \frac{b(s)}{\alpha} \right) \, ds \right|
$$

$$
\leq \delta G^* \left| \int_0^\omega \left( \frac{c(s)}{\alpha} y^{1-\alpha} - \alpha p_2(s) + \frac{e(s)}{\alpha} y^{1-\alpha} + (1 - \alpha) \frac{|y'(s)|^2}{y(s)} + \frac{b(s)}{\alpha} \right) \, ds \right|
$$

$$
\leq \delta G^* \left( \frac{c^*}{\alpha} R^{1-\alpha} + \frac{e^*}{\alpha} R^{1-\alpha} + (1 - \alpha) \frac{(\delta R)^2}{\sigma} + \frac{b^*}{\alpha} \right) \omega := N_2.
$$

Furthermore, for any $y_1, y_2 \in K \cap (\overline{A}_2 \setminus A_1)$ with $y_1 \neq y_2$,

$$
|Ty_1 - Ty_2| = \left| \frac{dT(\theta y_1 + (1 - \theta) y_2)}{d\theta} (y_1 - y_2) \right| \leq N_2|y_1 - y_2|,
$$

where $0 < \theta < 1$. That is to say, $T$ is a uniformly bounded and equicontinuity operator, it follows the Arzela-Ascoli theorem that $T : K \cap (\overline{A}_2 \setminus A_1) \to K$ is a continuous and completely continuous operator.

**Step 3.** We prove (i) of Lemma 3.1 i.e.,

$$
\|Ty\| \leq \|y\|, \text{ for } y \in K \cap \partial A_2.
$$

(3.3)

In view of $y \in K \cap \partial A_2$, there is

$$
\sigma R \leq y(t) \leq \|y\| \leq \|y\| = R \text{ and } |y'| \leq \delta y(t) \leq \delta R.
$$

Therefore, we arrive at

$$
\|Ty\| = \|Ty\|_{\infty} + \|(Ty)'\|_{\infty}
$$

$$
= \max_{0 \leq t \leq \omega} \int_0^\omega G(t, s) \left( \frac{c(s)}{\alpha} y^{1-\alpha} - \alpha p_2(s) + \frac{e(s)}{\alpha} y^{1-\alpha} + (1 - \alpha) \frac{|y'(s)|^2}{y(s)} + \frac{b(s)}{\alpha} \right) \, ds
$$

$$
+ \max_{0 \leq t \leq \omega} \int_0^\omega \left| \frac{\partial G(t, s)}{\partial t} \right| \left( \frac{c(s)}{\alpha} y^{1-\alpha} - \alpha p_2(s) + \frac{e(s)}{\alpha} y^{1-\alpha} + (1 - \alpha) \frac{|y'(s)|^2}{y(s)} + \frac{b(s)}{\alpha} \right) \, ds
$$

$$
\leq (G^* + \delta G^*) \left( \frac{c^*}{\alpha} R^{1-\alpha} - \alpha p_2 + \frac{e^*}{\alpha} R^{1-\alpha} + (1 - \alpha) \frac{(\delta R)^2}{\sigma} R + \frac{b^*}{\alpha} \right) \omega.
$$

(3.4)

In view of $1 - \alpha - \alpha p_2 < 1$ and $1 - \alpha < 1$, condition $(H_2)$ implies that there is a sufficiently large $R$ such that

$$
(G^* + \delta G^*) \left( \frac{c^*}{\alpha} R^{1-\alpha} - \alpha p_2 + \frac{e^*}{\alpha} R^{1-\alpha} + (1 - \alpha) \frac{(\delta R)^2}{\sigma} R + \frac{b^*}{\alpha} \right) \omega \leq R = \|y\|.
$$

8
Therefore, inequality (3.3) holds.

**Step 4.** We prove (ii) of Lemma 3.1. To do so, let $y_0 = 1$, then $y_0 \in K \setminus \{0\}$. Now we prove that

$$y \neq Ty + \lambda y_0, \quad \text{for } y \in K \cap \partial \Lambda_1, \quad \lambda > 0$$

(3.5)

by way of contradiction. If inequality (3.5) does not hold, we can assume that there exist $y_1 \in K \cap \partial \Lambda_1$ and $\lambda_0 > 0$ such that

$$y_1 = Ty_1 + \lambda_0 y_0.$$ 

To be sure, for $y_1 \in K \cap \partial \Lambda_1$, it holds that

$$\sigma r = \sigma \|y_1\| \leq y_1(t) \leq \|y_1\| \leq \|y_1\| = r$$

and $|y_1'(t)| \leq \delta y_1(t) \leq \delta r.$

Therefore, from condition $(H_1)$, we obtain

$$y_1 = Ty_1 + \lambda_0 y_0$$

$$= \int_0^\omega G(t, s) \left( \frac{c(s)}{\alpha} y_1^{1-\alpha} y_1(s) + \frac{e(s)}{\alpha} y_1^{1-\alpha} + \frac{b(s)}{\alpha} \right) ds + \lambda_0$$

$$> \int_0^\omega G(t, s) \left( \frac{c(s)}{\alpha} y_1^{1-\alpha} y_1(s) + \frac{e(s)}{\alpha} y_1^{1-\alpha} + \frac{b(s)}{\alpha} \right) ds$$

$$\geq G_* \left( \frac{c_*}{\alpha} (\sigma r)^{1-\alpha} + \frac{e_*}{\alpha} (\sigma r)^{1-\alpha} + \frac{b_*}{\alpha} \right) \omega$$

$$\geq G_* c_* \sigma^{1-\alpha} r$$

which contradicts with $y_1 \in K \cap \partial \Lambda_1$. Hence, inequality (3.5) holds.

It follows from above four steps that all the conditions of Lemma 3.1 are satisfied. Therefore, it follows that $T$ has a fixed point $y \in K \cap (\Lambda_2 \setminus \Lambda_1)$. Obviously, the fixed point is an $\omega$-periodic solution of equation (3.1). That is, equation (3.1) has at least one positive $\omega$-periodic solution $y(t)$ satisfying $r \leq \|y\| \leq R.$

**Corollary 3.1.** Assume that inequality (2.3), conditions $(H_1)$, $(H_2)$ and $\rho_1 > \rho_2$ hold. Then equation (1.9) has at least one positive $\omega$-periodic solution.

**Theorem 3.2.** Assume that conditions $(A)$, $(H_2)$ and $\rho_1 < \rho_2$ hold. Furthermore, suppose there exist $R > r > 0$ such that:

$$(H_3) \quad \frac{1}{\alpha} \left( - \frac{b_*}{c_*} \right)^{\frac{1}{\alpha}} \leq r \leq \left( \frac{G_* c_* \omega}{\alpha} \right)^{\frac{1}{\alpha}}.$$ 

Then there is at least one positive $\omega$-periodic solution of equation (3.1), saying $y(t)$ satisfying $r \leq \|y\| \leq R$. Consequently, it is a positive periodic solution of equation (1.8).
**Proof.** We follow the same strategy and notations as in the proof of Theorem 3.1. Now we prove (3.3), similar to (3.4), by using condition \((H_2)\) and \(\rho_1 < \rho_2\), one can prove that
\[
\|T y\| = \|Ty\|_\infty + \|(Ty)\|_{\infty}
\]
\[
= \max_{0 \leq t \leq \omega} \int_0^\omega G(t, s) \left( \frac{c(s)}{\alpha} y_{1-\alpha} - \alpha y_{1-\alpha}(s) + (1 - \alpha) \frac{|y'(s)|^2}{y(s)} + \frac{b(s)}{\alpha} \right) ds
\]
\[
+ \max_{0 \leq t \leq \omega} \int_0^\omega \left| \frac{\partial G(t, s)}{\partial t} \right| \left( \frac{c(s)}{\alpha} y_{1-\alpha} - \alpha y_{1-\alpha}(s) + (1 - \alpha) \frac{|y'(s)|^2}{y(s)} + \frac{b(s)}{\alpha} \right) ds
\]
\[
\leq \left( G_0 + \delta \right) \left( \frac{c}{\alpha} y_{1-\alpha} - \alpha y_{1-\alpha} + (1 - \alpha) \frac{|y'(s)|^2}{y(s)} + \frac{b(s)}{\alpha} \right) \omega
\]
\[
\leq R = \|y\|
\]
holds if \(R\) is large enough. This means that inequality (3.3) holds.

In what follows, we prove that inequality (3.5), from condition \((H_3)\) and \(\rho_1 < \rho_2\), (3.6) gives
\[
y_t = Ty + \lambda_0 y_0
\]
\[
= \int_0^\omega G(t, s) \left( \frac{c(s)}{\alpha} y_{1-\alpha} - \alpha y_{1-\alpha}(s) + (1 - \alpha) \frac{|y'(s)|^2}{y(s)} + \frac{b(s)}{\alpha} \right) ds + \lambda_0
\]
\[
> G_0 \left( \frac{c}{\alpha} y_{1-\alpha} - \alpha y_{1-\alpha} + (1 - \alpha) \frac{|y'(s)|^2}{y(s)} + \frac{b(s)}{\alpha} \right) \omega
\]
\[
\geq G_0 \frac{c \sigma}{\alpha} y_{1-\alpha} - \alpha y_{1-\alpha} + (1 - \alpha) \frac{|y'(s)|^2}{y(s)} + \frac{b(s)}{\alpha}
\]
\[
\geq \delta\|y\|_0,
\]
which contradicts with \(y_t \in K \cap \partial \Omega_1\). Then inequality (3.5) follows. Therefore, the conclusion follows from Lemma 3.1 immediately. \(\square\)

**Corollary 3.2.** Assume that inequality (2.3), conditions \((H_2)\), \((H_3)\) and \(\rho_1 < \rho_2\) hold. Then there is at least one positive \(\omega\)-periodic solution of equation (1.9).

**Theorem 3.3.** Assume that conditions \((A)\), \((H_2)\) and \(\rho_1 = \rho_2\) hold. Furthermore, suppose there exist \(R > r > 0\) such that one of the following two cases holds:

Case I: \(b_\ast + c_\ast \geq 0\);

Case II: \(b_\ast + c_\ast < 0\) and condition \((H_4)\) hold:
\[
(H_4) \frac{1}{\sigma} \left( -\frac{b_\ast + c_\ast}{c_\ast} \right) \omega < r \leq \frac{G_0 \omega}{\alpha},
\]
where \(b_\ast(t) := \max\{0, b(t)\}\), \(\bar{b}_\ast := \frac{1}{\omega} \int_0^\omega b_\ast(t) dt\).

Then there is at least one positive \(\omega\)-periodic solution of equation (3.1), saying \(y(t)\) satisfying \(r \leq \|y\| \leq R\). Consequently, it is a positive periodic solution of equation (1.8).

**Proof.** If \(\rho_1 = \rho_2\), equation (3.1) can be rewritten as
\[
y''(t) + p(t)y'(t) + \frac{q(t)}{\alpha} y(t) = \frac{\alpha}{\sigma} y_{1-\alpha}(t) + (1 - \alpha) \frac{|y'(t)|^2}{y(t)} + \frac{b(t) + c(t)}{\alpha},
\]
and operator \(T\) as
\[
(Ty)(t) = \int_0^\omega G(t, s) \left( \frac{c(s)}{\alpha} y_{1-\alpha} + (1 - \alpha) \frac{|y'(s)|^2}{y(s)} + \frac{b(s) + c(s)}{\alpha} \right) ds.
\]
Similar to the proof of Theorem 3.1, for \( \forall y \in K \cap (\Theta_2 \backslash \Theta_1) \), we have
\[
e(t) y^{1-\alpha}(t) + (1 - \alpha) \frac{|y'(t)|^2}{y(t)} + \frac{b(t) + c(t)}{\alpha} \\
\geq e(t) y^{1-\alpha}(t) + \frac{b(t) + c(t)}{\alpha} \\
\geq e_* \sigma^{1-\alpha} r^{1-\alpha} + \frac{b_* + c_*}{\alpha} \\
> 0.
\]

Case I: if \( b_* + c_* \geq 0 \), it is obvious that the above inequality holds;

Case II: if \( b_* + c_* < 0 \), from the first half part of condition \((H_4)\), we can get the above inequality holds as well.

Furthermore, one can prove that \( T(K \cap (\Theta_2 \backslash \Theta_1)) \subset K \) and \( T : K \cap (\Theta_2 \backslash \Theta_1) \to K \) is a continuous and completely continuous operator.

Now we prove inequality (3.3), similar to (3.4), by using condition \((H_2)\) and \( \rho_1 = \rho_2 \), one can prove that
\[
\| Ty \| = \| Ty \|_\infty + \| (Ty)' \|_\infty \\
= \max_{0 \leq t \leq \omega} \int_0^\omega G(t,s) \left( \frac{c(s)}{\alpha} y^{1-\alpha}(s) + \frac{e(s)}{\alpha} y^{1-\alpha}(s) + (1 - \alpha) \frac{|y'(s)|^2}{y(s)} + \frac{b(s)}{\alpha} \right) ds \\
+ \max_{0 \leq t \leq \omega} \int_0^\omega \frac{\partial G(t,s)}{\partial t} \left( \frac{c(s)}{\alpha} y^{1-\alpha}(s) + \frac{e(s)}{\alpha} y^{1-\alpha}(s) + (1 - \alpha) \frac{|y'(s)|^2}{y(s)} + \frac{b(s)}{\alpha} \right) ds \\
\leq (G^* + \delta G_*) \left( \frac{c^*}{\alpha} R^{1-\alpha} + \frac{(1 - \alpha) \delta^2}{\sigma} R + \frac{b^* + c^*}{\alpha} \right) \omega \\
\leq R = \| y \|
\]
holds if \( R \) is large enough. This means that inequality (3.3) holds.

In the following, we prove that inequality (3.5).

Case I: if \( b_* + c_* \geq 0 \), the inequality
\[
y_1 = Ty_1 + \lambda_0 y_0 \\
= \int_0^\omega G(t,s) \left( \frac{e(s)}{\alpha} y_1^{1-\alpha}(s) + (1 - \alpha) \frac{|y_1'(s)|^2}{y_1(s)} + \frac{b(s) + c(s)}{\alpha} \right) ds + \lambda_0 \\
> G_* \left( \frac{e_*}{\alpha} (\sigma r)^{1-\alpha} + \frac{b_* + c_*}{\alpha} \right) \omega \\
\geq G_* e_* \sigma^{1-\alpha} \omega r^{1-\alpha} \\
\geq r = \| y_1 \|
\]
can be satisfied if \( r > 0 \) is small enough.
Case II: if \( b_* + c_* < 0 \), from condition \((H_4)\), we have
\[
y_1 = \int_0^\omega G(t,s) \left( \frac{e(s)}{\alpha} y_1^{1-\alpha}(s) + (1 - \alpha) \frac{|y'_1(s)|^2}{y_1(s)} + \frac{b(s) + c(s)}{\alpha} \right) ds + \lambda_0
\]
\[
> G_* \int_0^\omega \left( \frac{e_x (\alpha r)^{1-\alpha}}{\alpha} + \frac{b^+(s)}{\alpha} + \frac{b^-(s)}{\alpha} + \frac{c_*}{\alpha} \right) ds
\]
\[
\geq G_* \int_0^\omega \left( \frac{e_x \sigma^{1-\alpha}}{\alpha} r^{1-\alpha} + \frac{b^+(s)}{\alpha} + \frac{b_-}{\alpha} + \frac{c_*}{\alpha} \right) ds
\]
\[
> G_* \int_0^\omega \frac{b^+(s)}{\alpha} ds
\]
\[
= \frac{G_* b^+ \omega}{\alpha}
\]
\[
\geq r = \|y_1\|
\]
where \( b^-(t) := \min\{0,b(t)\} \). Both cases I and II contradict with \( y_1 \in K \cap \partial \Lambda_1 \), so inequality \((3.5)\) follows.

Therefore, Lemma 3.1 shows equation \((3.1)\) has a positive \( \omega \)-periodic solution \( y(t) \) satisfying
\[
r \leq \|y\| \leq R.
\]

**Corollary 3.3.** Assume that equation \((2.3)\), condition \((H_2)\) and \( \rho_1 = \rho_2 \) hold. Then equation \((1.9)\) has at least one positive \( \omega \)-periodic solution \( y(t) \) with \( r \leq \|y\| \leq R \) if one of the following two cases holds:

Case I: \( b_* + c_* \geq 0 \);

Case II: \( b_* + c_* < 0 \) and condition \((H_4)\) hold.

**Remark 3.1.** Corollary 3.3 generalizes the results in literature [8].

### 4 Example

In this section, we give three examples and the simulations to illustrate our results.

**Example 4.1.** Consider differential equation with indefinite and repulsive singularities:
\[
x'' + \frac{1}{40} x = \frac{1 + 2 \cos(3t)}{x^2} + \frac{e^{2 \sin(3t)}}{x} + 10 + \cos(3t).
\]

Comparing equation \((4.1)\) with equation \((1.9)\), we see that \( q(t) = \frac{1}{40}, b(t) = 1 + 2 \cos(3t), b_* = -1, c(t) = e^{2 \sin(3t)}, c_* = e^{-2}, e(t) = 10 + \cos(3t), e_* = 9, \rho_1 = \frac{3}{2} > \rho_2 = \frac{13}{10} \), so we can apply Corollary 3.1

In view of \( \alpha = \frac{1}{11 \pi}, \gamma = \frac{2}{5}, q(t) = \frac{1}{11}, \xi = \sqrt{\frac{q(t)}{\alpha}} = \frac{1}{4}, \omega = \frac{2 \pi}{3}, \rho = 1 + \frac{\pi}{3} \), we know \( \xi < \frac{\pi}{2} \), equation \((2.3)\) hold. From equation \((2.2)\), we can obtain \( G_* = \frac{2 \sqrt{2 + \sqrt{3}}}{\sqrt{2 - \sqrt{3}}}, G^* = \frac{4}{\sqrt{2 - \sqrt{3}}}, \max_{0 \leq s,t \leq \omega} \left| \frac{\partial G(t,s)}{\partial t} \right| = \frac{\sqrt{2 - \sqrt{3}}}{2} \),
\[
\sigma = \frac{G_*}{G^* + \max_{0 \leq s,t \leq \omega} \left[ 2G(t,s) \right]} = \frac{4 \sqrt{2 + \sqrt{3}}}{10 - \sqrt{3}}, \delta = \frac{\max_{0 \leq s,t \leq \omega} \left| \frac{\partial G(t,s)}{\partial t} \right|}{G_*} = \frac{2 - \sqrt{3}}{4 \sqrt{2 + \sqrt{3}}},
\]

\[
\alpha = \frac{1}{11 \pi}, \gamma = \frac{2}{5}, q(t) = \frac{1}{11}, \xi = \sqrt{\frac{q(t)}{\alpha}} = \frac{1}{4}, \omega = \frac{2 \pi}{3}, \rho = 1 + \frac{\pi}{3} \).
Conditions \((H_1)\) and \((H_2)\) can be satisfied by calculation. So by using Corollary 3.1, equation (4.1) has at least one positive \(\frac{2\pi}{3}\)-periodic solution.

![Figure 1](image1.png)  

Figure 1: Phase diagram of the \(\frac{2\pi}{3}\)-periodic solution and its time series diagram. (a) Phase diagram of the periodic solution with initial value \((399.93015, 0)\). (b) Time series diagram of the periodic solution.

**Example 4.2.** Consider differential equation with indefinite and repulsive singularities:

\[
x'' + \frac{1}{40} x = \frac{1 + 2 \cos(3t)}{x^2} + \frac{e^{2 \sin(3t)}}{x^2} + 10 + \cos(3t).
\]  

(4.2)

Since \(\rho_1 = \frac{3}{2} < \rho_2 = 2\), we can apply Corollary 3.2. Similar to Example 4.1, we can get equation (2.3), conditions \((H_2)\) and \((H_3)\) hold. So by using Corollary 3.2, equation (4.2) has at least one positive \(\frac{2\pi}{3}\)-periodic solution.

![Figure 2](image2.png)  

Figure 2: Phase diagram of the \(\frac{2\pi}{3}\)-periodic solution and its time series diagram. (a) Phase diagram of the periodic solution with initial value \((399.8941, 0)\). (b) Time series diagram of the periodic solution.
Example 4.3. Consider differential equation with indefinite and repulsive singularities:

\[ x'' + \frac{1}{40}x = \frac{1 + 2 \cos(3t)}{x^2} + \frac{e^{2 \sin(3t)}}{x^2} + 10 + \cos(3t). \]  

(4.3)

Obviously, \( \rho_1 = \frac{3}{2} = \rho_2 \) and \( b_+ + c_+ < 0 \). Moreover,

\[ b_+ = \frac{1}{\omega} \int_0^\omega b^+(t) dt = \frac{3}{2\pi} \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} (1 + 2 \cos(3t)) dt = \frac{2}{3} + \frac{\sqrt{3}}{\pi}. \]

By Example 4.1 we can get equation (2.3), conditions (H_2) and (H_4) hold. So by using Case II of Corollary 3.3, equation (4.3) has at least one positive \( \frac{2\pi}{3} \)-periodic solution.

Figure 3: Phase diagram of the \( \frac{2\pi}{3} \)-periodic solution and its time series diagram. (a) Phase diagram of the periodic solution with initial value (399.9045, 0). (b) Time series diagram of the periodic solution.

Data Availability Statement

My manuscript has no associated data. It is pure mathematics.

Conflict of interest statement

The authors declare that there is no conflict of interests regarding the publication of this article.

Contributions

We declare that all the authors have same contributions to this paper.
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