RESIDUE CLASSES FREE OF VALUES OF EULER’S FUNCTION

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Dedicated to Andrzej Schinzel on his sixtieth birthday

1. Introduction

By a totient we mean a value taken by Euler’s function $\phi(n)$. Dence and Pomerance [DP] have established

Theorem A. If a residue class contains at least one multiple of 4, then it contains infinitely many totients.

Since 1 is the only odd totient, it remains to examine residue classes consisting entirely of numbers $\equiv 2 \pmod{4}$. In this paper we shall characterize which of these residue classes contain infinitely many totients and which do not. We show that the union of all residue classes that are totient-free has asymptotic density $3/4$, that is, almost all numbers that are $\equiv 2 \pmod{4}$ are in a residue class that is totient-free. In the other direction, we show the existence of a positive density of odd numbers $m$, such that for any $s \geq 0$ and any even number $a$, the residue class $a \pmod{2^sm}$ contains infinitely many totients.

We remark that if a residue class $r \pmod{s}$ contains infinitely many totients, it is possible, using the methods of [DP] and Narkiewicz [N], to get an asymptotic formula for the number of $n \leq x$ with $\phi(n) \equiv r \pmod{s}$.

Acknowledgements. We take this opportunity to thank Sybilla Beckmann-Kazez, Andrew Granville, Robert Rumely, Andrzej Schinzel, Roy Smith, Robert Varley and Felipe Voloch for helpful discussions. Research of the second author was supported by Grant 96-01-00378 from the Russian Foundation for Basic Research. The third author is supported in part by an NSF grant.

2. Preliminary results

Totients in a residue class consisting of numbers that are $\equiv 2 \pmod{4}$ necessarily are of the form $p^k - p^{k-1}$ for some prime $p \equiv 3 \pmod{4}$ and $k \geq 1$. We begin by characterizing those residue classes which contain only finitely many totients.
Lemma 1. Suppose $s \geq 1$, $k \geq 1$, $a \equiv 2 \pmod{4}$. Then there is a number $y \equiv 3 \pmod{4}$ such that $y^k - y^{k-1} \equiv a \pmod{2^s}$.

Proof. The lemma is trivial when $s = 1$ or $k = 1$, so suppose $s \geq 2$, $k \geq 2$. It suffices to show that the congruence

$$y^k - y^{k-1} \equiv x^k - x^{k-1} \pmod{2^s}$$

has no solutions with $y, x \equiv 3 \pmod{4}$ and $x \not\equiv y \pmod{2^s}$. If such a solution exists, write $x = zy$, so that $y(1 - z^k) \equiv 1 - z^{k-1} \pmod{2^s}$. Since $z \not\equiv 1 \pmod{2^s}$, we have

$$y(1 + z + \cdots + z^{k-1}) \equiv 1 + \cdots + z^{k-2} \pmod{2^s}.$$ 

However, as $y$ and $z$ are both odd, the above congruence is impossible. □

Lemma 2. Suppose $k \geq 2$, $M \geq 1$ and $p \equiv 3 \pmod{4}$ is prime. Then there is a number $x$ with $(x, M) = 1$ and $x^k - x^{k-1} \equiv p^k - p^{k-1} \pmod{M}$.

Proof. It is sufficient to prove the existence of such $x$ for $M = r^l$ where $r$ is a prime. If $r \neq p$ we set $x = p$. If $r = p$ we look for $x = p^{k-1}u + 1$ for some number $u$. Then

$$u(p^{k-1}u + 1)^{k-1} \equiv p - 1 \pmod{p^w},$$

where $w = \max(0, l - k + 1)$. Let

$$f(U) = U(p^{k-1}U + 1)^{k-1} - p + 1.$$ 

Since $f(U) \equiv U + 1 \pmod{p}$, which has the root $-1$, and $f'(-1) \equiv 1 \pmod{p}$, Hensel’s lemma implies there is some root $u$ of (1). □

Lemma 3. Suppose $m$ is odd, $s \geq 2$, $a \equiv 2 \pmod{4}$. If the congruence

$$x^k - x^{k-1} \equiv a \pmod{m}$$

has a solution with $k \geq 1$ and $(x, m) = 1$, then the progression $a \pmod{2^s m}$ contains infinitely many totients. Otherwise the progression contains either one or no totients, according as $a = p - 1$ for some $p|m$ or not.

Proof. Assume that (2) has such a solution. By Lemma 1, there is a number $y \equiv 3 \pmod{4}$ such that $y^k - y^{k-1} \equiv a \pmod{2^s}$. It follows from Dirichlet’s theorem that there are infinitely many primes $p \equiv x \pmod{m}$, $p \equiv y \pmod{2^s}$, and for each we have $\phi(p^s) \equiv a \pmod{2^s m}$.

If (2) has no solution with $(x, m) = 1$, the only possible solutions of $\phi(z) \equiv a \pmod{2^s m}$ are $z = 4$, $z = p^k$ or $z = 2p^k$ where $p$ is an odd prime dividing $m$. If $z = 4$, then $a = 2$, implying (2) has the solution $x = 2, k = 2$, a contradiction. In addition, by Lemma 2, if $a \equiv p^k - p^{k-1} \pmod{m}$ for some odd prime $p$ and $k \geq 2$, then (2) has a solution with $(x, m) = 1$. Hence $z$ is either a prime or twice a prime dividing $m$. □

Using Lemma 3, it is possible to find residue classes consisting of even numbers which are free of totients. For example, the progressions $302 \pmod{1092}$ and $790 \pmod{1092}$ contain no totients. In verifying this, since $1092 = 4 \times 3 \times 7 \times 13$, one only needs to check (2) for $k$ up to 12.

In the other direction, we prove
Theorem 1. Suppose \( M = 2^s m \), where \( s \geq 2 \) and \( m \) is odd. If \( a = \phi(b) > 1 \), where \( b \) is neither prime nor twice an odd prime, then any arithmetic progression \( a \pmod{M} \) contains infinitely many totients.

Proof. If \( a \) is divisible by 4, the result follows from Theorem A. Otherwise \( a = 2 \) or \( a = p^k - p^{k-1} \) where \( p \) is an odd prime, \( k > 1 \).

If \( a = 2 \), \( M = 2^s m \), \( m \) is odd, then for any prime \( q \) such that \( q \equiv -1 \pmod{2^s} \), \( q \equiv 2 \pmod{m} \) we have \( \phi(q^2) \equiv 2 \pmod{M} \).

In the case \( a = p^k - p^{k-1} \), by Lemma 2 there is an \( x \) such that \((x, M) = 1\) and \( x^k - x^{k-1} \equiv a \pmod{M} \). For any prime \( q \equiv x \pmod{M} \) we have \( \phi(q^k) \equiv a \pmod{M} \). \( \square \)

Question. Suppose \( a \equiv 2 \pmod{4} \) is either a non-totient or a totient with exactly two pre-images \( \{p, 2p\} \) for some prime \( p \). Is \( a \) contained in a residue class containing no totients other than \( a \) itself?

The numbers 10 and 14 are the two smallest such \( a \). A short search using a computer reveals that the progression \( 14 \pmod{2^2 \times 3 \times 5 \times 13 \times 37} \) contains no totients and the progression

\[
10 \pmod{4M}, \quad M = 3 \times 7 \times 11 \times 13 \times 29 \times 31 \times 41 \times 43 \times 101 \times 151 \times 211 \times 281 \times 701
\]

contains no totients other than 10. Theorem 2 (next section) implies that for almost all such \( a \), the question may be answered in the affirmative.

3. A negative result

Theorem 2. For any \( \varepsilon > 0 \) there exist such \( m \) that at least \((1-\varepsilon)m\) residue classes \( a \pmod{4m} \), \( 0 < a < 4m \), \( a \equiv 2 \pmod{4} \) are totient-free.

Corollary. The union of all totient-free residue classes has density 3/4.

Lemma 4. For any prime \( r \geq 5 \) and for any \( k = 2, \ldots, r-2 \), the number of distinct residues \( x^k - x^{k-1} \pmod{r} \) with \((x, r) = 1\) is less than \( r - \sqrt{r}/2 \).

Remark 1. The restriction \((x, r) = 1\) is not essential as \( 0^k - 0^{k-1} = 1^k - 1^{k-1} \).

Remark 2. Surely, the estimate of Lemma 4 is very weak, and it should be \( \leq cr \), \( c < 1 \). However, Lemma 4 is sufficient to prove Theorem 2.

Proof of Lemma 4. Let us consider the congruence

\[
(3) \quad x^k - x^{k-1} \equiv y^k - y^{k-1} \pmod{r}, \quad 1 \leq x < r, \quad 1 \leq y < r, \quad x \neq y.
\]

Let \( y \equiv xz \pmod{r} \), \( 2 \leq z < r \). Any \( z \) entails the unique solution of (3) (namely, \( x \equiv (z^{k-1} - 1)/(z - 1) \)) if \( z^{k-1} \not\equiv 1 \pmod{r} \) and \( z^k \not\equiv 1 \pmod{r} \), otherwise \( z \) does not entail any solutions. So, the number of solutions of (3) is

\[
N = r - (r - 1, k) - (r - 1, k - 1).
\]
since \((r - 1, j)\) is the number of solutions to \(z^j \equiv 1 \pmod{r}\). Now \((r - 1, k)\) and \((r - 1, k - 1)\) are coprime proper divisors of \(r - 1\). Thus, their sum is at most \(2 + (r - 1)/2\), so \(N \geq (r - 3)/2\). If the number of distinct residues \(x^k - x^{k-1} \pmod{r}\) with \((x, r) = 1\) is \(r - L\), then \(L(L - 1) \geq N\), hence \(L^2 \geq N + L > r/2\). □

Theorem 2 is equivalent to the following statement.

**Theorem 2'**. For any \(\varepsilon > 0\) there exist such odd \(m\) that for at least \((1 - \varepsilon)m\) residues \(a \pmod{m}\) the congruence (2) does not have solutions with integers \(k > 0\) and \(x\) with \((x, m) = 1\).

The equivalence of Theorems 2 and 2' follows directly from Lemma 3 and from the fact that the number of values of \(a\) in (2) of the form \(p - 1\) with \(p\) a prime factor of \(m\) is \(O(\log m)\).

**Lemma 5.** For any \(D \geq 1\) there are \(\gg_D x/\log x\) primes \(p \leq x\) for which \(D|p-1\) and no prime factor of \(p-1\) exceeds \(x^{9/20}\). The result holds for \(x\) sufficiently large depending on \(D\).

**Proof.** When \(D = 1\), this follows from the Theorem 1 of [P]. Since \(D\) is fixed and \(x \to \infty\), the general result follows by the same method. □

Remark 3. The exponent \(9/20\) in Lemma 5 is not the best possible exponent. For example, using the main theorem of [F], one can replace \(9/20\) with any number larger than \(1/(2\sqrt{e})\). However, all we shall need below is an exponent smaller than \(1/2\).

**Proof of Theorem 2'**. Let \(p_1, \ldots, p_I\) and \(q_1, \ldots, q_J\) be distinct odd primes such that

\[
\prod_i (1 - 1/p_i) < \varepsilon/4, \quad \prod_j (1 - 1/q_j) < \varepsilon/4.
\]

Set \(D = \text{lcm}(p_1 - 1, \ldots, p_I - 1, q_1 - 1, \ldots, q_J - 1)\). Let \(y\) be a sufficiently large number and let \(r_1, \ldots, r_L\) denote the primes \(\leq y\), different from all \(p_i, q_j\), for which each \(r_l - 1\) is divisible by \(D\) and by no prime \(> y^{9/20}\). By Lemma 5, \(L \gg y/\log y\). Take

\[m = \prod_i p_i \prod_j q_j \prod_l r_l.\]

By (4), the number of \(a \pmod{m}\) satisfying

\[
\exists i \ a \equiv 1 \pmod{p_i}, \quad \exists j \ a \equiv -1 \pmod{q_j}
\]

is at least \((1 - \varepsilon/2)m\). If \(a\) satisfies (5) and \(x\) is a solution of (2) with \((x, m) = 1\) then \(k \not\equiv 0 \pmod{p_i - 1}\) and \(k \not\equiv 1 \pmod{q_j - 1}\), therefore \(k \not\equiv 0 \pmod{r_l - 1}\) and \(k \not\equiv 1 \pmod{r_l - 1}\) for all \(l\). For such \(k\) we can estimate the number of possible residues \(a \pmod{r_l}\) by Lemma 4. Denote

\[n = \text{lcm}(p_1 - 1, \ldots, p_I - 1, q_1 - 1, \ldots, q_J - 1, r_1 - 1, \ldots, r_L - 1) = \text{lcm}(r_1 - 1, \ldots, r_L - 1).
\]
By construction,
\[ n \leq \prod_{p \leq y^{9/20}} p^{[\log y / \log p]} \leq \exp\{y^{9/20} \log y\}. \]

By Lemma 4, for any \( k = 1, \ldots, n \) such that for each \( l, k \not\equiv 0 \pmod{r_l - 1} \) and \( k \not\equiv 1 \pmod{r_l - 1} \), the number of \( a \pmod{m} \) for which there exists \( x \) with \( (x, m) = 1 \) satisfying (2) does not exceed
\[ m \prod_l (1 - 1/\sqrt{2r_l}) < m \exp(-L/\sqrt{2y}). \]

Thus, the number of \( a \) satisfying (5) for which a solution of (2) with \( (x, m) = 1 \) exists is less than \( mn \exp(-L/\sqrt{2y}) \leq \varepsilon m/2 \) if \( y \) is large enough. □

4. A positive result

Theorem 3. The set of all odd numbers \( m \) such that for any \( s \geq 1 \) and for any even \( a \) the residue class \( a \pmod{2^s m} \) contains infinitely many totients, has a positive lower density.

Call an odd number \( m \) “good” if for any \( a \) the congruence (2) has a solution with positive integers \( k \) and \( (x, m) = 1 \). Theorem 3 has an equivalent form:

Theorem 3'. The set of all good odd numbers has a positive lower density.

Lemma 6. Suppose \( f(x, y) \) is a polynomial absolutely irreducible modulo \( p \). Then the number \( N \) of solutions modulo \( p \) of \( f(x, y) \equiv 0 \pmod{p} \) satisfies
\[ |N - (p + 1)| \leq (d - 1)(d - 2)\sqrt{p} + d, \]
where \( d \) is the total degree of \( f \).

Proof. In the case that \( f \) is non-singular over \( \overline{\mathbb{F}_p} \), we use Weil’s theorem. The extra \( d \) on the right of the inequality is an upper estimate for the number of solutions “at infinity”. If \( f \) is singular, we use the principal result of Leep and Yeomans [LY]. □

Lemma 7. Suppose \( p \) is a prime and \( L, a, s, t \) are positive integers with \( (as, p) = 1 \). Then the polynomial
\[ f(x, y) = y^L(1 - x^s) - ax^t \]
is absolutely irreducible modulo \( p \).

Proof. If \( f(x, y) \) is reducible over \( \overline{\mathbb{F}_p} \), then
\[ h(y) = y^L - \frac{ax^t}{1 - x^s} \]
is reducible over the field $k = \mathbb{F}_p(x)$. By the criterion of Capelli and Rédei (see Theorem 21 in [S]), this forces the existence of some $b$ in $k$ such that $ax^t/(1-x^s) = b^q$ for some prime $q$ dividing $L$, or $ax^t/(1-x^s) = -4b^4$, in which case 4 divides $L$. However, since $s$ is coprime to $p$, $1-x$ divides $1-x^s$ to just the first power, so neither possibility can occur. □

Remark 4. It is also possible to give a direct proof of Lemma 7. Over $\bar{k}$ we have the factorization

$$h(y) = (y - r_1z) \cdots (y - r_Lz),$$

where each $r_i \in \mathbb{F}_p$ satisfies $r_i^L = 1$, $z \in \bar{k}$, and $z^L = ax^t/(1-x^s)$. Since $h$ is reducible over $k$, there exists a product

$$(y - r_{i_1}z) \cdots (y - r_{i_j}z) \in k[y],$$

where $j < L$. In particular, the constant coefficient lies in $k$, whence $z^j \in k$. If $m$ is the smallest positive integer with $z^m \in k$, then we have $m|L$, $m < L$. Writing $u(x) = z^m$, we have

$$u(x)^{L/m} = \frac{ax^t}{1-x^s}.$$  

As $1-x$ divides $1-x^s$ to just the first power, this equation is clearly impossible.

Lemma 8. There is a number $p_0$ such that for any prime $p > p_0$, any positive integers $L \leq p^{1/10}$ and $l \leq L$ and any integer $a$ the congruence (2) has a solution with $m = p$, $k \equiv l \pmod{L}$ and $(x, p) = 1$.

Proof. We may assume $a \not\equiv 0 \pmod{p}$. To prove the lemma, it is enough to show the existence of a solution $y$ of the congruence

$$(6) \quad y^L(1-g) \equiv ag^l \pmod{p}$$

with a primitive root $g$. Indeed, we can let $x \equiv g^{-1} \pmod{p}$ and $k = l - uL$, where $u$ is such that $y \equiv g^u \pmod{p}$. We show a solution $y, g$ to (6) exists by estimating the number of solutions of

$$(7) \quad y^L(1-z^s) \equiv az^{sl} \pmod{p},$$

where $s$ is a square-free divisor of $p-1$, and using inclusion-exclusion. By Lemma 7, the polynomial $y^L(1-z^s) - az^{sl}$ is absolutely irreducible. For a square-free divisor $s$ of $p-1$, let $N_s$ be the number of solutions of (7). For $s \leq p^{1/5}$ we apply Lemma 6 and for larger $s$ we use the trivial bound $N_s \leq pL$. Write $N_s = p + E_s$. By
inclusion-exclusion, the number of solutions of (6) with a primitive root \( g \) is

\[
N = \sum_{s | p-1} \frac{\mu(s)N_s}{s} \geq p \prod_{q | p-1} \left(1 - \frac{1}{q}\right) - \sum_{s | p-1} \frac{E_s}{s} \geq \phi(p-1) - \sum_{s \leq p^{1/5}} (L + sl)^2 \sqrt{p}/s - \sum_{s > p^{1/5}} p^{9/10} \geq \frac{1}{2} \phi(p-1)
\]

provided \( p \) is sufficiently large. \( \square \)

**Corollary.** Suppose \( p_1 < p_2 < \cdots < p_r \) are odd primes larger than \( p_0 \), \( m = p_1 \cdots p_r \) and for any \( j \geq 2 \)

\[
(p_j - 1, \text{lcm}(p_i - 1 : 1 \leq i < j)) \leq p_j^{1/10}.
\]

Then \( m \) is good.

**Proof.** Let \( a \) be arbitrary. Set \( n_j = \text{lcm}(p_i - 1 : 1 \leq i < j) \) and \( P_j = p_1 \cdots p_j \) for each \( j \). We construct numbers \( x_j, k_j \) inductively as follows. Choose \( x_1, k_1 \) so that \( (x_1, p_1) = 1 \) and \( x_1^{k_1} - x_1^{k_1-1} \equiv a \pmod{p_1} \). For \( j = 2, \cdots, r \), Lemma 8 implies the existence of numbers \( x_j, k_j \) for which \( (x_j, P_j) = 1, x_j \equiv x_{j-1} \pmod{P_{j-1}} \), \( k_j \equiv k_{j-1} \pmod{n_j} \) and \( x_j^{k_j} - x_j^{k_j-1} \equiv a \pmod{P_j} \). The pair \( (x_r, k_r) \) satisfies (2) with \( (x_r, m) = 1 \). \( \square \)

Call an odd number \( m \) “forbidden” if \( m = p_1 \cdots p_j \) where \( p_1 \leq \cdots \leq p_j \) are primes and

\[
(p_j - 1, \text{lcm}(p_i - 1 : 1 \leq i < j)) > p_j^{1/10}.
\]

**Lemma 9.** The number of forbidden numbers in \( (x, 2x] \) is \( O(x/\log^5 x) \).

Theorem 3' follows easily from Lemma 9. Take some \( P \geq p_0 \). Then for \( x \geq 2P \) there are \( \gg x/\log P \) positive integers without prime factors \( \leq P \). If \( m \) in \( (x, 2x] \) is not good, the Corollary to Lemma implies \( m \) is divisible by a forbidden number \( > P^2 \). By Lemma 9, there are \( \ll x/\log^4 P \) such numbers. Therefore, for sufficiently large \( P \) and \( x \geq 2P \) we get \( \gg x/\log P \) good numbers not exceeding \( x \).

**Proof of Lemma 9.** There is a constant \( c > 0 \) so that whenever \( n \geq 10 \), the number of divisors of \( n \) is \( \leq n^{c/\log \log n} \). By standard estimates from the distribution of “smooth” numbers (see [HT]), the number of integers in \( (x, 2x] \) with all prime factors \( \leq x^{20c/\log \log x} \) is \( O(x/\log^5 x) \). Thus, we have to estimate the number \( N \)
of forbidden integers $m \in (x, 2x]$ such that $p_j > x^{20c/\log \log x}$. Denoting $l = m/p_j$, 
$n = \text{lcm}(p_i - 1 : 1 \leq i < j) = \text{lcm}(p - 1 : p|l)$, we have 
$$(p_j - 1, n) > x^{2c/\log \log x}.$$ 
For fixed $l$ there are at most $x^{c/\log \log x}$ divisors of $n$, and for any $d|n$ there are at most $2x/(dl)$ numbers $p_j > 1$ for which $lp_j \leq 2x$ and $p_j \equiv 1 \pmod{d}$. Summing over all divisors $d > x^{2c/\log \log x}$, we find that $l$ generates at most 
$$\sum_d 2x/(dl) < \sum_d 2x/(lx^{2c/\log \log x}) \leq 2x/(lx^{c/\log \log x})$$ 
forbidden numbers. Further, taking the sum over $l$, we obtain the required inequality $N \ll x/\log^5 x$. \hfill $\square$

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