On fundamental solutions of higher-order space-fractional Dirac equations

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Starting from the pseudo-differential decomposition \( D = (-\Delta)^{\frac{1}{2}} H \) of the Dirac operator \( D = \sum_{j=1}^{n} e_j \partial_{x_j} \) in terms of the fractional operator \( (-\Delta)^{\frac{1}{2}} \) of order 1 and of the Riesz–Hilbert type operator \( H \) we will investigate the fundamental solutions of the space-fractional type Dirac equation of Lévy–Feller type

\[
\partial_t \Phi_\alpha(x, t; \theta) = -(-\Delta)^{\frac{1}{2}} \exp\left(\frac{i\pi \theta}{2} H\right) \Phi_\alpha(x, t; \theta)
\]

involving the fractional Laplacian \( -(-\Delta)^{\frac{1}{2}} \) of order \( \alpha \), with \( 2m \leq \alpha < 2m + 2 (m \in \mathbb{N}) \), and the exponentiation operator \( \exp\left(\frac{i\pi \theta}{2} H\right) \) as the hypercomplex counterpart of the fractional Riesz–Hilbert transform carrying the skewness parameter \( \theta \), with values in the range \( |\theta| \leq \min\{\alpha - 2m, 2m + 2 - \alpha\} \). Such model problem permits us to obtain hypercomplex counterparts for the fundamental solutions of higher-order heat-type equations

\[
\partial_t F_M(x, t) = \kappa_M(\partial_x)^M F_M(x, t) (M = 2, 3, \ldots)
\]

in case where the even powers resp. odd powers \( \partial_t F^{2m} = (-\Delta)^m (M = \alpha = 2m) \) resp. \( \partial_t F^{2m+1} = (-\Delta)^m \frac{1}{2} H (M = \alpha = 2m + 1) \) of \( D \) are being considered.

KEYWORDS
fundamental solutions, Mellin–Barnes integral representations, Riesz–Hilbert transform, space-fractional Dirac equation, Wright series expansions

MSC CLASSIFICATION
30G35; 26A33; 35S30; 33E12, 35S10; 35S10

1 | INTRODUCTION

1.1 | State of art

Let \( t > 0 \) and \( x, \xi \in \mathbb{R}^n \). For real-valued functions \( \varphi \) resp. \( \psi \) with membership in the Schwarz space \( S(\mathbb{R}^n) \) resp. the space of tempered distributions \( S'(\mathbb{R}^n) \), let

\[
(F \varphi)(\xi) = \int_{\mathbb{R}^n} \varphi(x) e^{-i(x, \xi)} dx, \varphi \in S(\mathbb{R}^n)
\]

be the Fourier transform over \( \mathbb{R}^n \) and

\[
(F^{-1} \psi)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \psi(\xi) e^{i(x, \xi)} d\xi, \psi \in S(\mathbb{R}^n)
\]

its inverse.
The kernel function \( K_{a,n}(x, t) \) defined through the Fourier inversion formula \( K_{a,n}(x, t) = (\mathcal{F}^{-1} e^{-t|\xi|^a})(x) \)

\[
K_{a,n}(x, \tau) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-t|\xi|^a} e^{i(x, \xi)} d\xi (\Re(\tau) \geq 0) \tag{3}
\]

arises as the fundamental solution of the fractional heat equation of order \( a \) \((a > 0)\)

\[
\partial_t \Phi_a(x, t) = (-\Delta)^\frac{a}{2} \Phi_a(x, t). \tag{4}
\]

Indeed, for \( \tau = t \), the kernel function \( K_{a,n}(x, t) \) solves equation (4) and satisfies the initial condition

\[
K_{a,n}(x, 0) := \lim_{t \to 0^+} K_{a,n}(x, t) = \delta(x).
\]

An effective way to reduce the right-hand side of (3) to a one-dimensional integral is provided in Samko et al., p. 485, lemma 25.1

by the following Fourier inversion formula:

\[
\int_{\mathbb{R}^n} \phi(\xi) e^{i(x, \xi)} d\xi = \frac{(2\pi)^n}{|x|^\frac{n}{2}} \int_0^\infty \phi(\rho) \rho^\frac{n}{2} J_{\frac{n}{2}-1}(\rho|x|) d\rho \tag{5}
\]

underlying to radial functions \( \psi(\xi) = \phi(|\xi|) \) (see also Stein et al., theorem 3.3 of chapter IV). Moreover, a simple calculation based on the change of variable \( \rho \to \frac{\rho}{|x|} \) on the right-hand side of (5) shows that equation (3) can be cast in the form

\[
K_{a,n}(x, \tau) = \frac{1}{(2\pi)^n |x|^n} \int_0^\infty e^{-t \left( \frac{\rho}{|x|} \right)^a} \rho^\frac{n}{2} J_{\frac{n}{2}-1}(\rho|x|) d\rho (\Re(\tau) \geq 0), \tag{6}
\]

where \( J_\nu \) denotes the \( \nu \)-th Bessel function (see George et al., chapter 14).

We point out here that \( K_{a,n}(x, t) \) corresponds to higher-dimensional generalization of the so-called Lévy distribution of order \( a \) (cf. Bernstein and Faustino). In the one-dimensional case, that is, \( \rho^\frac{1}{2} J_{-\frac{1}{2}}(\rho) = \sqrt{\frac{2}{\pi}} \cos(\rho) \), when \( n = 1 \) (see George et al., subsect. 14.7), the integral representation (3) has been tackled by renowned mathematicians such as Bernstein, Lévy, Polya, Burwell, and Widder.

In higher dimensions, the interest in studying the fundamental solution for fractional diffusion problems of type (4) for values of \( a \) in the range \( 0 < a \leq 2 \) has a vast literature (see, e.g., Caffarelli and Silvestre and the references therein), especially when we are dealing with symmetric stable processes of index \( a \) (cf. Blumenthal and Getoor and Kolokoltsov) in \( \mathbb{R}^n \) through the positivity condition \( \varphi \geq 0 \Rightarrow \exp \left(-t(-\Delta)_{\frac{n}{2}}\right) \varphi \geq 0 \) of the diffusive semigroup \( \left\{ \exp \left(-t(-\Delta)_{\frac{n}{2}}\right) \right\}_{t \geq 0} \) of Markovian type.

In stark contrast, the cases where \( a > 2 \)—corresponding to a fractional interpolation of higher-order heat equations (case of \( \frac{2}{a} \in \mathbb{N} \setminus \{1\} \)—are less understood (cf. Li and Wong). In fact, the oscillator behavior of the Bessel functions appearing on the right-hand side of (6) shows in turn that the positivity condition is no longer preserved for the semigroup \( \left\{ \exp \left(-t(-\Delta)_{\frac{n}{2}}\right) \right\}_{t \geq 0} \) in case of \( a \geq 4 \), and whence, the classical methods adopted for the second-order heat equation cannot be applied in this context. We refer to Gazzola et al. for a wide overview of the problem and to Ferreira and Ferreira for further advances beyond the biharmonic case (i.e., when \( \frac{a}{2} = 2 \) treated on the paper.

Accordingly to the literature, the optimal environment to deal with the biharmonic heat equation and fractional heat equation (4) of order \( a > 4 \) can be provided by an exploitation of the notion of pseudo-Markov processes or even as composition of Brownian motions or stable processes with Brownian motions. Those approaches have been popularized in the stochastic community since fundamental the papers of Hochberg and Funaki, but we will not pursue these topics in the present paper.

On the other hand, we notice that the model problem involving the space-fractional heat equation (4) only provides faithful generalizations for higher-order heat equations of even order (i.e., \( a = 2m \), with \( m \in \mathbb{N} \)). Also, on the multidi-
mensional generalization of the Lévy–Feller approach the lack of a closed form for the underlying generator semigroup is one of the major difficulties.¹⁹

To the best of our knowledge, to derive faithful analogues of Airy like functions in higher dimensions, carrying higher-order heat operators of odd type (cf. Górskia et al.²⁰), one may consider the pseudo-differential reformulation of Dirac theory in the spirit of the seminal paper²¹ of Li et al. (see also McIntosh²² for a broad overview) in a way that the Dirac operator \( D \) is recasted in terms of the identity \( D = (-\Delta)^{\frac{1}{2}} H \). Here and elsewhere, \( H \) stands for the Riesz–Hilbert transform.

The aim of this paper is to amalgamate the study of the fundamental solutions of the polyharmonic heat-type \( \partial_t + (-\Delta)^m \) \((m \in \mathbb{N})\) and the higher-order Dirac-type operators \( \partial_t \pm iD^{2m+1} \) \((m \in \mathbb{N})\) as well. To this end, we will investigate the fundamental solutions \( \Phi_n(x, t; \theta) \) of the modified version of the space-fractional heat equation (4) carrying the fractional Laplacian \(-(-\Delta)^{\frac{2}{3}}\) of order \( \alpha \), with \( 2m \leq \alpha < 2m + 2 \) \((m \in \mathbb{N})\):

\[
\begin{align*}
\partial_t \Phi_n(x, t; \theta) &= \exp \left( \frac{i\theta}{2} \mathcal{H} \right) \Phi_n(x, t; \theta), \quad \text{for} \quad (x, t) \in \mathbb{R}^n \times (0, \infty) \\
\Phi_n(x, 0; \theta) &= \delta(x), \quad \text{for} \quad x \in \mathbb{R}^n
\end{align*}
\]

The evolution equation (7) is built upon the Cauchy problem of Lévy–Feller type considered by Gorenflo et al.²³ Here, the exponentiation operator \( \exp \left( \frac{i\theta}{2} \mathcal{H} \right) \) carrying the skewness parameter \( \theta \) in the range \(|\theta| \leq \min\{|\alpha - 2m, 2m + 2 - \alpha|\}\) mimics the fractional Hilbert transform in optics introduced by Lohmann et al.²⁴ (see also Bernstein²⁵, sect. 4 and Bernstein,²⁶ sect. 4).

The main gain of our model problem formulation against the multidimensional approaches available on the literature relies on the replacement of the parametrized measure on the unit sphere of \( \mathbb{R}^n \)—hard to compute, in general (cf. Nolan et al.²⁷)—by a rotation action in the plane parametrized by \( \theta \mapsto \exp \left( \frac{i\theta}{2} \mathcal{H} \right) \)—easy to describe through the Fourier multiplier reformulation of \( \mathcal{H} \) as \( \mathcal{H} = F^{-1} \frac{-i\phi}{|\xi|^\phi} F \) (cf. Bernstein,²⁵ sect. 4).

1.2 | Layout of the paper and main results

Let us summarize the layout of the paper: in Section 2, we establish the basic facts on Clifford algebras, Dirac operators, and fractional Riesz–Hilbert-Type transforms following Bernstein’s footsteps (cf. Bernstein²⁵,²⁶). In Section 3, we will obtain a Mellin–Barnes representation for the kernel function \( K_{a,n}(x, \tau) \) in the spherical form (6), and in Section 4, we use the framework considered previously to investigate the fundamental solution associated to the Cauchy problem (7).

Our first theorem to be proved in Section 4.1 shows that the fundamental solution \( \Phi_n(x, t; \theta) \) of (7) can be represented in terms of the kernel functions \( K_{a,n}(x, te^{i\frac{\pi}{2}}) \) represented through equation (3) (case of \( \tau = te^{i\frac{\pi}{2}} \):

**Theorem 1.** The fundamental solution \( \Phi_n(x, t; \theta) \) defined by the Cauchy-type problem (7), where

\[
2m \leq \alpha < 2m + 2 \quad \text{and} \quad |\theta| \leq \min\{|\alpha - 2m, 2m + 2 - \alpha|\} \quad (m \in \mathbb{N})
\]

is equal to

\[
\Phi_n(x, t; \theta) = \frac{1}{2}(I + \mathcal{H})K_{a,n}(x, te^{i\frac{\pi}{2}}) + \frac{1}{2}(I - \mathcal{H})K_{a,n}(x, te^{-i\frac{\pi}{2}}).
\]

**Remark 1.** We emphasize that the set of conditions

\[
2m \leq \alpha < 2m + 2 \quad \text{and} \quad |\theta| \leq \min\{|\alpha - 2m, 2m + 2 - \alpha|\} \quad (m \in \mathbb{N})
\]

leads to \(-\frac{\pi}{2} < \frac{\pi\theta}{2} < \frac{\pi}{2} \), because of \(\min\{|\alpha - 2m, 2m + 2 - \alpha|\} \leq 1\). That is equivalent to say that \(\Re(te^{\pm i\frac{\pi}{2}}) = t \cos \left( \frac{\pi\theta}{2} \right) \geq 0\) is always fulfilled under the aforementioned conditions so that the kernel functions \( K_{a,n}(x, te^{\pm i\frac{\pi}{2}}) \) appearing on the statement of Theorem 1 are well defined.
Remark 2. In case of \( \theta = 0 \), one can easily see that \( \Phi_{a}(x, t; 0) \) coincides with the fundamental solution of the space-fractional heat equation (4). In particular, from the choice, \( a = 2m \) one can moreover say that \( \Phi_{2m}(x, t; 0) \) approaches the fundamental solution of the higher-order heat operator \( \partial_{t} + (\Delta)^{m} \), already treated by Li and Wong.\(^{13} \)

The second theorem to be proved in Section 4.1 is essentially a continuation of the proof of Theorem 1 and combines analytic series expansions of Wright type (see Kilbas et al.\(^{28} \)) with the singular integral representation of the Riesz–Hilbert transform \( \mathcal{H} \). Its preparation starts in Section 3 with the reformulation of the one-dimensional integral representation (6) as a Mellin convolution integral that in turn permits us to represent \( K_{a,n}(x, r) \) as a Wright series expansion

\[
\int_{\mathbb{R}}^{1} \Psi_{1} \left[ \left( \begin{array}{c} a_1, a_1 \\ b_1, \beta_1 \end{array} \right) ; \right] = \sum_{k=0}^{\infty} \frac{\Gamma(a_1 + \alpha_1 k)}{\Gamma(b_1 + \beta_1 k)} \frac{x^k}{k!},
\]

where \( \lambda \in \mathbb{C}, a_1, b_1 \in \mathbb{C} \) and \( \alpha_1, \beta_1 \in \mathbb{R}\setminus\{0\} \).

In an overall view, such technique mimics the framework used by Górska et al. in the series of papers\(^{20,29} \) to derive, in one-dimensional case (\( n = 1 \)), signed Lévy stable laws and generalized Airy functions as well.

**Theorem 2.** Let \( \Phi_{a}(x, t; \theta) \) be the kernel function determined in Theorem 1, for values of \( a \) and \( \theta \) in the range

\[
2m \leq a < 2m + 2 \text{ and } |\theta| \leq \min\{a - 2m, 2m + 2 - a\} \text{ (} m \in \mathbb{N} \).
\]

Then, we have the following:

1. \( \Phi_{a}(x, t; \theta) \) admits the following singular integral representation

\[
\Phi_{a}(x, t; \theta) = \frac{2^{1-n}}{\pi^{n} t^{n/2}} \Re \left( e^{-i \pi \frac{n}{2}} \Gamma \left( \frac{n}{2}, \frac{x^{2}}{4} t^{2} \right) - \frac{|x|^{2} t^{2} - \frac{1}{4} x^{2}}{e^{\frac{\pi}{2}}} \right) + i \frac{2^{1-n}}{\pi^{n} t^{n/2}} \Gamma \left( \frac{n+1}{2} \right) P.V. \int_{\mathbb{R}} \Im \left( e^{-i \pi \frac{n}{2}} \Gamma \left( \frac{n+1}{2}, \frac{x^{2}}{4} t^{2} - \frac{1}{4} x^{2} \right) - \frac{|x-y|^{2} t^{2} - \frac{1}{4} (x-y)^{2}}{e^{\frac{\pi}{2}}} \right) \frac{y}{|y|^{n+1}} dy,
\]

whereby \( \Re(s) \) resp. \( \Im(s) \) stands for the real resp. imaginary part of \( s \in \mathbb{C} \).

2. In case of \( \theta = 0 \), \( \Phi_{a}(x, t; 0) \) simplifies to

\[
\Phi_{a}(x, t; 0) = \frac{2^{1-n}}{\pi^{n} t^{n/2}} \Gamma \left( \frac{n}{2}, \frac{x^{2}}{4} t^{2} \right) - \frac{|x|^{2} t^{2} - \frac{1}{4} x^{2}}{e^{\frac{\pi}{2}}}.
\]

Remark 3. The fundamental solution \( \Phi_{a}(x, t; \theta) \) provided by Theorem 1 and Theorem 2 resembles to the integral representation considered by Górska et al. in his study,\(^{20} \) (sect. 2) to exploit the Airy integral transform, formerly treated on Widder’s paper.\(^{9} \)

In particular, \( \Phi_{2m}(x, t; 0) \) (see Statement 2 of Theorem 2) resembles the well-known symmetric Lévy stable signed functions \( g \) \( 2m(u) \) (see Górska et al.\(^{20} \), sect. 3). Interesting enough, the real part of \( \Phi_{2m+1}(x, t; \pm 1) \), that yields from the substitutions \( a = 2m + 1 \) and \( \theta = \pm 1 \), may be considered as hypercomplex extensions of higher-order Airy functions \( \delta_{2m+1}(u) \) obtained in Górska et al.\(^{20} \) (sect. 4).

## 2 | PRELIMINARIES

### 2.1 | Clifford algebras and Dirac operators

Let \( e_1, e_2, \ldots, e_n \) be an orthonormal basis of \( \mathbb{R}^{n} \) satisfying the graded anti-commuting relations

\[
e_{j} e_{k} + e_{k} e_{j} = -2 \delta_{jk} \text{ for any } j, k = 1, 2, \ldots, n,
\]

and $C\ell_{0,n}$ the universal Clifford algebra with signature $(0,n)$.

The Clifford algebra $C\ell_{0,n}$ is an associative algebra with identity $e_0 := 1$, containing $\mathbb{R}$ and $\mathbb{R}^n$ as subspaces. In particular, vectors $x = (x_1, x_2, \ldots, x_n)$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ of $\mathbb{R}^n$ are represented in terms of the linear combinations

$$x = \sum_{j=1}^n x_j e_j \text{ resp. } \xi = \sum_{j=1}^n \xi_j e_j,$$

whereas the Euclidean inner product $\langle x, \xi \rangle$ between $x, \xi \in \mathbb{R}^n$:

$$\langle x, \xi \rangle = \sum_{j=1}^n x_j \xi_j \in \mathbb{R}$$

recasted in terms of anti-commutator $\langle x, \xi \rangle = -\frac{1}{2}(x \xi + \xi x)$, belongs to the center of $C\ell_{0,n}$. By the preceding relation, it follows that the square $\xi^2$ is a real number satisfying $\xi^2 = -|\xi|^2$, where $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ stands for the Euclidean norm in $\mathbb{R}^n$.

Here we would like to stress that the linear space isomorphism given by the mapping $e_{j_1} e_{j_2} \ldots e_{j_r} \mapsto dx_{j_1} dx_{j_2} \ldots dx_{j_r}$, with $1 \leq j_1 < j_2 < \ldots < j_r \leq n$, enable us to establish an isomorphism between $C\ell_{0,n}$ and the exterior algebra $\bigwedge(\mathbb{R}^n) = \bigoplus_{r=0}^n \bigwedge^r(\mathbb{R}^n)$ (cf. Vaz and da Rocha, chapter 4) so that $C\ell_{0,n}$ has dimension $2^n$.

In the sequel, we always consider multivector functions $(x, t) \mapsto \Psi(x, t)$ with values on the complexified Clifford algebra $\mathbb{C} \otimes C\ell_{0,n}$, written as linear combinations in terms of the $r$-multivector basis $e_{j_1} e_{j_2} \ldots e_{j_r}$, labeled by the subsets $J = \{j_1, j_2, \ldots, j_r\}$ of $\{1, 2, \ldots, n\}$, that is,

$$\Psi(x, t) = \sum_{r=0}^{n} \sum_{|J|=r} \psi_J(x, t) e_J, \text{ with } e_J = e_{j_1} e_{j_2} \ldots e_{j_r} \text{ and } \psi_J(x, t) \in \mathbb{C}. \quad (10)$$

It is well known that the Dirac operator $D$, defined as

$$D\Psi(x, t) = \sum_{j=1}^n e_j \partial_{x_j} \Psi(x, t), \quad (11)$$

factorizes the Euclidean Laplacian $\Delta = \sum_{j=1}^n \partial_{x_j}^2$, that is $D^2 = -\Delta$. Further, induction over $m \in \mathbb{N}$ shows us that

$$D^{2m} = (-\Delta)^m \text{ and } D^{2m+1} = (-\Delta)^m D. \quad (12)$$

### 2.2 Background on function spaces, distributions and Fourier transform

To extend the action of the Fourier transform (1) and its inverse (2) by component-wise application to Clifford-valued functions (10), we need to consider the $\dagger$-conjugation operation $\Psi(x, t) \mapsto \Psi(x, t)^\dagger$ on the complexified Clifford algebra $\mathbb{C} \otimes C\ell_{0,n}$ defined as

$$\Phi(x, t) \Psi(x, t) = \Psi(x, t)^\dagger \Phi(x, t)^\dagger$$

$$(\psi_J(x, t) e_J) = \psi_J(x, t) e_J^\dagger \ldots e_{j_2}^\dagger e_{j_1}^\dagger \quad (1 \leq j_1 < j_2 < \ldots < j_r \leq n).$$

$$e_j^\dagger = -e_j \quad (1 \leq j \leq n) \quad (13)$$
Henceforth, the norm $\|\Psi(x, t)\|$ of the Clifford-valued-function (10) induced by the identity $\|\Psi(x, t)\|^2 = \Psi(x, t)^* \Psi(x, t)$—which is positive definite in the view of (13) and of the anti-commuting relations (9)—allows us to define

$$\|\Psi(\cdot, t)\|_{L^p} := \left( \int_{\mathbb{R}^n} |\Psi(x, t)|^p dx \right)^{\frac{1}{p}}, \text{ if } 1 \leq p < \infty$$

$$\sup_{x \in \mathbb{R}^n} |\Psi(x, t)|, \text{ if } p = \infty$$

(14)

as the underlying norm of the Bochner type spaces $L^p(\mathbb{R}^n; \mathbb{C} \otimes C'_{0,n})$ (cf. Hytönen et al, subsect. 1.2.b)

We then consider the multi-index abbreviations

$$x^{(\mu)} := (x_1)^{\mu_1}(x_2)^{\mu_2} \cdots (x_n)^{\mu_n}, \text{ for } \mu := (\mu_1, \mu_2, \ldots, \mu_n) \in (\mathbb{N}_0)^n$$

$$\partial^{(\nu)} := \partial_1^{\nu_1} \cdots \partial_n^{\nu_n}, \text{ for } \nu := (\nu_1, \nu_2, \ldots, \nu_n) \in (\mathbb{N}_0)^n$$

$$|\mu| = \mu_1 + \ldots + \mu_n, \text{ for } \mu := (\mu_1, \mu_2, \ldots, \mu_n) \in (\mathbb{N}_0)^n$$

$$|\nu| = \nu_1 + \ldots + \nu_n, \text{ for } \nu := (\nu_1, \nu_2, \ldots, \nu_n) \in (\mathbb{N}_0)^n$$





to exploit the definition of the Schwartz space $S(\mathbb{R}^n)$ and its dual $S'(\mathbb{R}^n)$ to Clifford-valued distributions in the spirit of Hytönen et al, subsect. 2.4.c & 2.4.d)

The space of rapidly decaying functions $\Psi(\cdot, t)$ with values on $\mathbb{C} \otimes C'_{0,n}$, denoted by $S(\mathbb{R}^n; \mathbb{C} \otimes C'_{0,n})$, is the Fréchet space defined by the family of semi-norm conditions underlying to the positive constant $M > 0$:

$$\sup_{x \in \mathbb{R}^n; |\mu| + |\nu| < M} \|x^{(\mu)}(\partial^{(\nu)}\Psi(x, t))\| < \infty$$

wheras the space of $\mathbb{C} \otimes C'_{0,n}$—valued tempered distributions $S'(\mathbb{R}^n; \mathbb{C} \otimes C'_{0,n})$ consists of continuous linear functionals induced by the mapping $\Psi(\cdot, t) \mapsto \langle \Psi(\cdot, t), \Phi(\cdot, t) \rangle_{L^2}$, whereby

$$\langle \Psi(\cdot, t), \Phi(\cdot, t) \rangle_{L^2} := \int_{\mathbb{R}^n} \Psi(x, t)^* \Phi(x, t) dx$$

(15)

stands for the sesquilinear form (possibly Clifford-valued).

Notice that $S(\mathbb{R}^n; \mathbb{C} \otimes C'_{0,n})$ is a dense subset in $L^p(\mathbb{R}^n; \mathbb{C} \otimes C'_{0,n})$ (cf. Hytönen et al, subsect. 2.4.23) for any $1 \leq p < \infty$ (i.e., for values of $p \neq \infty$). On the other hand, the sesquilinear form (15) permits us not only to exploit the main features of the Fourier transform such as the mapping property $F : L^1(\mathbb{R}^n; \mathbb{C} \otimes C'_{0,n}) \rightarrow L^\infty(\mathbb{R}^n; \mathbb{C} \otimes C'_{0,n})$ (cf. Hytönen et al, definition 2.4.1) and the Plancherel theorem (cf. Hytönen et al, theorem 2.4.9) but also to extend the action of the Fourier transform (1) to an isomorphism in $S'(\mathbb{R}^n; \mathbb{C} \otimes C'_{0,n})$ as follows:

$$\langle F\Psi(\cdot, t), F\Phi(\cdot, t) \rangle_{L^2} = \langle \Psi(\cdot, t), \Phi(\cdot, t) \rangle_{L^2}, \text{ for all } \Psi(\cdot, t) \in S(\mathbb{R}^n; \mathbb{C} \otimes C'_{0,n}) \& \Phi(\cdot, t) \in S'(\mathbb{R}^n; \mathbb{C} \otimes C'_{0,n}).$$

In particular, that allows us to show that for every $\Psi(\cdot, t)$ with membership in $L^p(\mathbb{R}^n; \mathbb{C} \otimes C'_{0,n})$, the family of continuous linear functionals $\Psi(\cdot, t) \mapsto \langle \Psi(\cdot, t), \Phi(\cdot, t) \rangle_{L^2}$ always defines an element of $S(\mathbb{R}^n; \mathbb{C} \otimes C'_{0,n})$ (see Hytönen et al, example 2.4.26 for further details).

**Remark 4.** Since $e^{-r|\tau|^\alpha}$ belongs to $S'(\mathbb{R}^n; \mathbb{C} \otimes C'_{0,n})$ for values of $\Re(\tau) \geq 0$ (see Remark 1), one can thus infer from density arguments, depicted as above, that the kernel function $K_{\alpha,n}(x, r)$ defined, namely, equation (3) belongs to $L^p(\mathbb{R}^n; \mathbb{C} \otimes C'_{0,n})$, whenever $1 \leq p < \infty$

### 2.3 Background on Riesz–Hilbert-Type transforms

Next we review the construction of the Hilbert transform—the so-called fractional Riesz–Hilbert transform. Following the framework considered in the series of papers, $^{25,26}$ we derive the pseudo-differential representation of the so-called Riesz–Hilbert transform through the pseudo-differential identity
\[ D = (-\Delta)^{\frac{1}{2}} H, \]  

(16)

where \((-\Delta)^{\frac{1}{2}} = F^{-1}|\xi|^{a}F\) denotes the fractional Riesz operator of order \(a\) (cf. Samko et al,1 p. 483).

First, we note that in the view of the spectral identity

\[ F(D\Psi)(\xi, t) = -i\xi F\Psi(\xi, t), \text{ with } \xi = \sum_{j=1}^{n}\xi_{j}e_{j}. \]

and of the spherical decomposition

\[ -i\xi = |\xi| |\xi|^{-1} \text{, with } |\xi| = (-\xi^{2})^{\frac{1}{2}} \neq 0, \]

it is straightforward to see that \(H\) admits the pseudo-differential representation (cf. Bernstein,25 sect. 4)

\[ H = F^{-1} \frac{-i\xi}{|\xi|} F \]

(17)

Next, we are going to derive a close formula for the fractional Hilbert transform. To do so, we shall make use of the formal series expansion

\[ \exp\left(\frac{i\pi \theta}{2} H\right) = \sum_{k=0}^{\infty} \frac{1}{2^{k}k!} H^{k} \]

(18)

The next lemma corresponds to an abridged version of Bernstein,26 definition 4.1 (see eq. 4.2) and Bernstein,25 theorem 4.2 (see also Bernstein,26, theorem 4.2):

**Lemma 1.** The fractional Riesz–Hilbert transform \(\exp\left(\frac{i\pi \theta}{2} H\right)\) admits the polar representation

\[ \exp\left(\frac{i\pi \theta}{2} H\right) = \cos\left(\frac{\pi \theta}{2}\right) I + i \sin\left(\frac{\pi \theta}{2}\right) H. \]

Moreover,

\[ \exp\left(\frac{i\pi \theta}{2} H\right) : L^{p}(\mathbb{R}^{n}; \mathbb{C} \otimes C\mathcal{L}_{0,n}) \rightarrow L^{p}(\mathbb{R}^{n}; \mathbb{C} \otimes C\mathcal{L}_{0,n}) \]

is continuous and bounded for values of \(1 < p < \infty\).

**Proof.** From the formal series expansion (18) of \(\exp\left(\frac{i\pi \theta}{2} H\right)\) one can infer that

\[ F\left[\exp\left(\frac{i\pi \theta}{2} H\right) \Psi(\cdot, t)\right](\xi) = \exp\left(\frac{i\pi \theta}{2} h(\xi)\right) F\Psi(\xi, t). \]

(20)

where \(h(\xi) := \frac{-i\xi}{|\xi|}\) denotes the Fourier multiplier of \(H\).

Here, we recall that \(h(\xi)\) arising on the exponentiation representation \(\exp\left(\frac{i\pi \theta}{2} h(\xi)\right)\) is a unitary Clifford vector, that is, \((h(\xi))^{2} = 1\). By expanding now \(\exp\left(\frac{i\pi \theta}{2} h(\xi)\right)\) as a formal series expansion, the set of recursive relations

\[ (h(\xi))^{2j} = 1 \text{ and } (h(\xi))^{2j+1} = h(\xi), \]

yielding from induction over \(j\), leads to the splitting formula

\[
\exp\left(\frac{i\pi \theta}{2} h(\xi)\right) = \sum_{j=0}^{\infty} \frac{(i\pi)^{j} \theta^{2j}}{2^{j}(2j)!} (h(\xi))^{2j} + \sum_{j=0}^{\infty} \frac{(i\pi)^{2j+1} \theta^{2j+1}}{2^{j+1}(2j+1)!} (h(\xi))^{2j+1}
\]

\[
= \sum_{j=0}^{\infty} \frac{(-1)^{j} \pi^{2j} \theta^{2j}}{2^{2j}j!} + i h(\xi) \sum_{j=0}^{\infty} \frac{(-1)^{j} \pi^{2j+1} \theta^{2j+1}}{2^{2j+1}(2j+1)!}
\]
which is equivalent to
\[ \exp \left( \frac{\pi \theta}{2} \frac{\xi}{|\xi|} \right) = \cos \left( \frac{\pi \theta}{2} \frac{\xi}{|\xi|} \right) + \frac{\xi}{|\xi|} \sin \left( \frac{\pi \theta}{2} \right). \]

Finally, by taking the inverse of the Fourier transform (2) on both sides of (20), one gets from (17) the equation (19), as expected.

Moreover, the proof of continuity and boundedness of \( \exp \left( \frac{\pi \theta}{2} |\theta| \right) : L^p(\mathbb{R}^n; \mathbb{C} \otimes C_{\ell_0}^n) \to L^p(\mathbb{R}^n; \mathbb{C} \otimes C_{\ell_0}^n) \) for values of \( 1 < p < \infty \) is, up to a linearity argument, similar to the proof of Bernstein,\textsuperscript{25, theorem 4.2}

3 | MELLIN–BARNES AND WRIGHT SERIES REPRESENTATIONS OF THE FUNDAMENTAL SOLUTION

3.1 | Mellin–Barnes representation

Following the framework considered in Görska et al.,\textsuperscript{20,29} we start to reformulate \( K_{a,n}(x, \tau) \) represented through equation (6) as a Mellin convolution at the point \( r = |x| \tau^{-\frac{1}{2}} \) (cf. Butzer and Jansche,\textsuperscript{32, sect. 4}), namely,

\[ K_{a,n}(x, \tau) = \frac{1}{(2\pi)^\frac{n}{2}|x|^n} \int_0^\infty f \left( \frac{|x| \tau^{-\frac{1}{2}}}{\rho} \right) g(\rho) \frac{d\rho}{\rho}, \]

with

\[ f(\rho) = e^{-\rho^{-\alpha}} \quad \text{and} \quad g(\rho) = \rho^\frac{n}{2} + 1 J_{\frac{n}{2} - 1}(\rho). \]

Furthermore, by exploiting the Mellin convolution theorem (see eq. in Butzer and Jansche,\textsuperscript{32, theorem 3}), we also have that

\[ \mathcal{M}\{K_{a,n}(x, \tau)\}(s) = \frac{1}{(2\pi)^\frac{n}{2}|x|^n} \mathcal{M}\left\{ f \left( |x| \tau^{-\frac{1}{2}} \right) \right\}(s) \mathcal{M}\left\{ g \left( |x| \tau^{-\frac{1}{2}} \right) \right\}(s), \quad (21) \]

whereby

\[ \mathcal{M}\{\phi(\rho)\}(s) = \int_0^\infty \phi(\rho) \rho^{-s} d\rho, \quad \text{with} \ s \in \mathbb{C}, \quad (22) \]

stands for the Mellin transform (cf. Butzer and Jansche,\textsuperscript{32, eq. (1.2) of p. 326}).

The next step thus consists in computing the Mellin transforms \( \mathcal{M}\{f(\rho)\}(s) \) and \( \mathcal{M}\{g(\rho)\}(s) \) and then inverting the product \( \mathcal{M}\{f(\rho)\}(s) \mathcal{M}\{g(\rho)\}(s) \) at \( \rho = |x| \tau^{-\frac{1}{2}} \) by means of the Mellin inversion formula (cf. Butzer and Jansche,\textsuperscript{32, eq. (1.3) of p. 326})

\[ \phi(\rho) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}\{\phi(\rho)\}(s) \rho^{-s} ds, \quad \text{with} \ \rho > 0 \ & \ c = \Re(s). \quad (23) \]

Thereby, from a straightforwardly application of property

\[ \mathcal{M}\{\rho^\beta \phi(\kappa \rho^\gamma)\}(s) = \frac{1}{\gamma} \kappa^{-\frac{\beta}{\gamma}} \mathcal{M}\{\phi(\rho)\} \left( \frac{s + \beta}{\gamma} \right) \]

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carrying the parameters \( \beta \in \mathbb{C}, \gamma \in \mathbb{C} \setminus \{0\} \) and \( \kappa > 0 \) (cf. Butzer and Jansche, \textsuperscript{32}, proposition 1), there holds

\[
\mathcal{M}\{f(\rho)\}(s) = \frac{1}{|\alpha|} \mathcal{M}\{e^{-\rho}\} \left( -\frac{s}{\alpha} \right) = \frac{1}{\alpha} \Gamma \left( -\frac{s}{\alpha} \right)
\]

\[
\mathcal{M}\{g(\rho)\}(s) = \mathcal{M}\left\{ J_{n-1}(\rho) \right\} \left( s + \frac{n}{2} + 1 \right) = 2^{\frac{n}{2}+s} \frac{\Gamma \left( \frac{n}{2} + \frac{s}{2} \right)}{\Gamma \left( -\frac{s}{2} \right)}
\]

so that (21) equals

\[
\mathcal{M}\{K_{\alpha,n}(\mathbf{x}, \tau)\}(s) = \frac{2^s \Gamma \left( \frac{n}{2} + \frac{s}{2} \right) \Gamma \left( -\frac{s}{\alpha} \right)}{\alpha \pi^\frac{n}{2} |\mathbf{x}|^n \Gamma \left( -\frac{s}{2} \right)}. \tag{24}
\]

On computations above, we have used the Eulerian representation of the Gamma function (see George et al, \textsuperscript{3}, chapter 13)

\[
\int_0^\infty e^{-\rho \tau^{1-}\rho} \, d\rho = \Gamma(s) \tag{25}
\]

and the Weber integral representation (cf. Samko et al, \textsuperscript{1}, p. 490, eq. 25.27) carrying the parameters \( \beta = \frac{n}{2} + s \) and \( \nu = \frac{n}{2} - 1 \) (see also Mathai et al, \textsuperscript{33}, p. 57, eq. 2.46):

\[
\int_0^\infty \rho^\beta J_\nu(\rho) \, d\rho = 2^\beta \frac{\Gamma \left( \frac{\nu+\beta+1}{2} \right)}{\Gamma \left( \frac{-\beta+1}{2} \right)}, \quad \text{for } -\Re(\nu) - 1 < \Re(\beta) < \frac{1}{2}. \tag{26}
\]

As a consequence of (21) and of the Mellin inversion formula (23), we conclude that for \( \rho = |\mathbf{x}| \tau^{1-}\frac{1}{\alpha} \),

\[
K_{\alpha,n}(\mathbf{x}, \tau) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}\{K_{\alpha,n}(\mathbf{x}, \tau)\}(s) \left( |\mathbf{x}| \tau^{1-}\frac{1}{\alpha} \right)^{-s} ds
\]

\[
= \frac{1}{\alpha \pi^\frac{n}{2} |\mathbf{x}|^n} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma \left( \frac{n}{2} + \frac{s}{2} \right) \Gamma \left( -\frac{s}{\alpha} \right)}{\Gamma \left( -\frac{s}{2} \right)} \left( |\mathbf{x}| \tau^{1-}\frac{1}{\alpha} \right)^{-s} ds,
\]

or equivalently,

\[
K_{\alpha,n}(\mathbf{x}, \tau) = \frac{1}{\alpha \pi^\frac{n}{2} |\mathbf{x}|^n} \left[ \frac{H^{1,1}_{\frac{n}{2},\frac{1}{2}} \left( \frac{|\mathbf{x}| \tau^{1-}\frac{1}{\alpha} (1, \frac{1}{\alpha})}{2} \right)^2 \left( \frac{n}{2}, \frac{1}{2} \right) (1, \frac{1}{2}) \right]
\]

using the H-function notation (Mathai et al, \textsuperscript{33}, p. 2, and Kilbas et al, \textsuperscript{28}, eq. 5.1).

By applying general existence conditions for the H-function we conclude that, in case of \( \alpha > 1 \), the integral representation (27) yields a uniformly convergent series expansion—conditions \( \mu > 0 \) and \( q \geq 1 \) carrying the parameters \( \mu = \frac{1}{2} + \frac{1}{\alpha} - \frac{1}{\alpha} \) and \( q = 2 \) (cf. Mathai et al, \textsuperscript{33}, case 1: of theorem 1.1).
3.2 Series expansion of Wright type

To represent explicitly $K_{\alpha,n}(x,\tau)$ as a convergent series expansion of Wright type, one transforms the closed path joining the endpoints $c-i\infty$ and $c+i\infty$ associated to the straight line $\Re(s) = c \ (n < c < \frac{1-n}{\alpha})$ as a loop beginning and ending at $-\infty$ and enriching all the simple poles $s = -n - 2k \ (k \in \mathbb{N}_0)$.

Using the fact that for values of $\alpha > 1$ the intersection of the simple poles of $\Gamma\left(\frac{n}{2} - s\right)$ and $\Gamma\left(-\frac{s}{a}\right)$ yields an empty set ($ak \neq -n - 2k$ for all $k \in \mathbb{N}_0$), there holds by a straightforward application of the standard residue theorem, we can evaluate the contour integral appearing on the right-hand side of (27) as an infinite sum of the residues at $s = -n - 2k \ (k \in \mathbb{N}_0)$.

Namely, we have

$$K_{\alpha,n}(x,\tau) = \frac{1}{\alpha \pi^\frac{n}{2} |x|^n} \sum_{k=0}^{\infty} \lim_{\epsilon \to -n-2k} \left( s + n + 2k \right) \frac{\Gamma\left(\frac{n}{2} + \frac{s}{2} + k\right) \Gamma\left(-\frac{s}{a}\right)}{\Gamma\left(-\frac{s}{2}\right)} \left( \frac{|x|^{\frac{s}{2} - \frac{n}{2}}}{2} \right)^{-s}$$

$$= \frac{1}{\alpha \pi^\frac{n}{2} |x|^n} \sum_{k=0}^{\infty} \lim_{\epsilon \to -n-2k} \left( s + n + 2k \right) \frac{2\Gamma\left(-\frac{s}{a}\right)}{\Gamma\left(-\frac{s}{2}\right)} \left( \frac{|x|^{\frac{s}{2} - \frac{n}{2}}}{4} \right)^{n+k}$$

After a straightforward simplification, we recognize that, for values of $\alpha > 1$, the kernel function $K_{\alpha,n}(x,\tau)$ reduces to a Wright series expansion of the type (8). Namely, one gets the closed representation

$$K_{\alpha,n}(x,\tau) = \frac{2^{1-n}}{\alpha \pi^\frac{n}{2} \tau^\frac{n}{2}} \Psi_1 \left[ \left( \frac{n}{2}, \frac{2}{a} \right) \left( \frac{n}{2}, 1 \right) - \frac{|x|^{\frac{s}{2} - \frac{n}{2}}}{4} \right]. \quad (28)$$

Remark 5. From the substitution $\tau = te^{\frac{\pi i}{\alpha}}$ on (28). It thus follows then

$$K_{\alpha,n}(x,te^{\frac{\pi i}{\alpha}}) = \frac{2^{1-n}}{\alpha \pi^\frac{n}{2} t^\frac{n}{2}} e^{-\frac{i\pi a}{\alpha}} \Psi_1 \left[ \left( \frac{n}{2}, \frac{2}{a} \right) \left( \frac{n}{2}, 1 \right) - \frac{|x|^{\frac{s}{2} - \frac{n}{2}}}{4} e^{-\frac{i\pi a}{\alpha}} \right]$$

$$= \frac{2^{1-n}}{\alpha \pi^\frac{n}{2} t^\frac{n}{2}} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{n}{2} + \frac{2k}{a}\right)}{\Gamma\left(\frac{n}{2} + k\right)} \left( \frac{|x|^{\frac{s}{2} - \frac{n}{2}}}{4} \right)^{k} e^{-\frac{i\pi a}{\alpha} (n+k)}$$

Remark 6. For $\alpha = 2$ and $\tau = t$, one has that $K_{2,n}(x,t)$, determined as above, simplifies to

$$K_{2,n}(x,t) = \frac{1}{(4\pi t)^\frac{n}{2}} e^{-\frac{|x|^2}{4t}}.$$

the so-called heat kernel.
4  PROOF OF THE MAIN RESULTS

4.1  Proof of Theorem 1

To proof Theorem 1 one has to consider, as in Li et al.\textsuperscript{21} and McIntosh,\textsuperscript{21,22} the projection operators $\chi_{\pm}(\xi) (\xi \neq 0)$ defined, namely,

$$\chi_{\pm}(\xi) = \frac{1}{2} \left( 1 \pm \frac{i\xi}{|\xi|} \right)$$

(29)

(corresponding to the Fourier multipliers of $1$)

yield straightforwardly from the fact that the Fourier multiplier $h(\xi)$, which, in terms of the projection operators (Theorem 1) First note that in the view of Lemma 1, the Cauchy problem (7) on the Fourier space takes the form

$$\partial_t F(\Phi_\alpha(\cdot, t; \theta))(\xi) = -|\xi|^\alpha \left( \cos \left( \frac{\pi \theta}{2} \right) + \frac{\xi}{|\xi|} \sin \left( \frac{\pi \theta}{2} \right) \right) F(\Phi_\alpha(\cdot, t; \theta))(\xi), \quad \text{for} \quad (\xi, t) \in \mathbb{R}^n \times (0, \infty)$$

(32)

$$F(\Phi_\alpha(\cdot, t; \theta))(\xi) = 1, \quad \text{for} \quad \xi \in \mathbb{R}^n$$

which, in terms of the projection operators $\chi_{\pm}(\xi)$ defined, namely, equation (29), may be reformulated as

$$\partial_t F(\Phi_\alpha(\cdot, t; \theta))(\xi) = -|\xi|^\alpha \left( e^{i \frac{\pi \theta}{2} \chi_{-}(\xi)} + e^{-i \frac{\pi \theta}{2} \chi_{+}(\xi)} \right), \quad \text{for} \quad (\xi, t) \in \mathbb{R}^n \times (0, \infty)$$

(32)

Thus, a solution of (32) is provided, for values of $t \geq 0$, by the formal exponentiation formula

$$F(\Phi_\alpha(\cdot, t; \theta)) = \exp \left( -t |\xi|^\alpha \left( e^{i \frac{\pi \theta}{2} \chi_{-}(\xi)} + e^{-i \frac{\pi \theta}{2} \chi_{+}(\xi)} \right) \right)$$

(33)

$$= \sum_{k=0}^{\infty} \frac{(-1)^k k! |\xi|^\alpha}{k!} \left( e^{i \frac{\pi \theta}{2} \chi_{-}(\xi)} + e^{-i \frac{\pi \theta}{2} \chi_{+}(\xi)} \right)^k$$

Application of the projection properties (30) carrying the idempotents $\chi_{\pm}(\xi)$ results, after a straightforwardly computation based on induction arguments, into the identity

$$\left( e^{i \frac{\pi \theta}{2} \chi_{-}(\xi)} + e^{-i \frac{\pi \theta}{2} \chi_{+}(\xi)} \right)^k = \left( e^{i \frac{\pi \theta}{2} \chi_{-}(\xi)} + e^{-i \frac{\pi \theta}{2} \chi_{+}(\xi)} \right)^k$$

for all $k \in \mathbb{N}_0$.

Thus, (33) simplifies to

$$F(\Phi_\alpha(\cdot, t; \theta)) = \sum_{k=0}^{\infty} \frac{(-1)^k k! |\xi|^\alpha}{k!} \left( e^{i \frac{\pi \theta}{2} \chi_{-}(\xi)} + e^{-i \frac{\pi \theta}{2} \chi_{+}(\xi)} \right)^k$$

$$= \chi_{-}(\xi) \exp \left( -te^{i \frac{\pi \theta}{2} |\xi|^\alpha} \right) + \chi_{+}(\xi) \exp \left( -te^{-i \frac{\pi \theta}{2} |\xi|^\alpha} \right).$$
Now taking the inverse $F^{-1}$ of the Fourier transform (see equation 2) on both sides of the preceding equation, there holds from equations (3) and (31)

$$
\Phi_\alpha(x, t; \theta) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \chi_-(\xi) \exp \left( -t e^{\frac{i|\xi|^2}{2}} \right) e^{i(x, \xi)} d\xi + \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \chi_+(\xi) \exp \left( -t e^{-\frac{i|\xi|^2}{2}} \right) e^{i(x, \xi)} d\xi
$$

$$
= \frac{1}{2} (I + H)K_{\alpha,n}(x, te^{\frac{i|\xi|^2}{2}}) + \frac{1}{2} (I - H)K_{\alpha,n}(x, te^{-\frac{i|\xi|^2}{2}}),
$$
as desired. \qed

Remark 7 (see also Remark 1). By combining Remark 4 and Lemma 1, one can easily infer that the components of $\Phi_\alpha(x, t; \theta)$, say $\frac{1}{2} (I + H)K_{\alpha,n}(x, te^{\frac{i|\xi|^2}{2}})$, are well-defined distributions with membership in the Bochner type spaces $L^p(\mathbb{R}^n; \mathbb{C} \otimes C_c^{0,n})$ in case where we restrict to values of $p$ in the range $1 < p < \infty$ ($p \neq 1$ and $p \neq \infty$).

Remark 8 (see also Remark 2). In the view of the set of identities

$$
D^{2m} = (-\Delta)^{\frac{2m}{2}} \exp \left( \frac{i\pi 0}{2} H \right) \quad \text{and} \quad \pm iD^{2m+1} = (-\Delta)^{\frac{2m+1}{2}} \exp \left( \pm \frac{i\pi 1}{2} H \right)
$$

that yield straightforwardly from equation (12) and from Lemma 1 one can say that the fundamental solutions $\Phi_\alpha(x, t; \theta)$ of (7) encompasses the fundamental solutions $K_{2m,n}(x, t)$ (see equation 3) of the polyharmonic heat operator $\partial_t + (-\Delta)^m (\Phi_{2m}(x, t; 0) = K_{2m,n}(x, t))$ as well as the fundamental solutions $\Phi_{2m+1}(x, t; -1)$ and $\Phi_{2m+1}(x, t; 1)$ of the higher-order Dirac-type operators $\partial_t - iD^{2m+1}$ and $\partial_t + iD^{2m+1}$, respectively.

4.2 Proof of Theorem 2

To prove Theorem 2, we will use the fact that the Riesz–Hilbert transform $H$, as represented through equation (17), may be expressed as a linear combination involving the Riesz operators $R_j = F^{-1} \frac{\xi_j}{|\xi|} F$:

$$
H = \sum_{j=1}^{n} e_j R_j.
$$

(34)
on which each $R_j$ ($j = 1, 2, \ldots, n$) is a singular integral operator uniquely determined by the kernel functions

$$
E_j(x) = \frac{\Gamma \left( \frac{n+1}{2} \right)}{\pi^{\frac{n+1}{2}} |x|^{n+1}} x_j.
$$

(35)
In particular, the spectral property (cf. Stein et al., p. 224)

$$
(\text{FE}_j)(\xi) = \frac{-i\xi_j}{|\xi|}
$$

(36)

again with framework developed in Section 3 will be of foremost interest to derive a closed formula for $\Phi_\alpha(x, t; \theta)$.

Proof. (Theorem 2)

Proof of Statement 1.

First, recall that in the view of Stein et al. (p. 225, theorem 1.1) and of the identity (36) involving the Riesz kernels (35), one can easily infer, by linearity arguments, that is a convolution-type operator represented in $L^p(\mathbb{R}^n; C_c^{0,n})$ ($1 < p < \infty$) through the singular integral operator identity

$$
\mathcal{H}\Psi(x, t) = \frac{\Gamma \left( \frac{n+1}{2} \right)}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} \Psi(x - y, t) \frac{y}{|y|^{n+1}} dy, \quad \text{with} \quad y = \sum_{j=1}^{n} y_j e_j.
$$

(37)
Then, from an easy algebraic manipulation, one gets that \( \Phi_\alpha(x, t; \theta) \), as obtained in Theorem 1, admits the singular integral representation

\[
\Phi_\alpha(x, t; \theta) = \Re K_{\alpha, n}(x, te^{i\pi \theta}) + i \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \, P.V. \int_{\mathbb{R}^n} \Im K_{\alpha, n}(x - y, te^{i\pi \theta}) \frac{y}{|y|^{n+1}} dy,
\]

(38)

with

\[
\Re K_{\alpha, n}(x, te^{i\pi \theta}) = \frac{K_{\alpha, n}(x, te^{i\pi \theta}) + K_{\alpha, n}(x, te^{-i\pi \theta})}{2},
\]

\[
\Im K_{\alpha, n}(x - y, te^{i\pi \theta}) = \frac{K_{\alpha, n}(x - y, te^{i\pi \theta}) - K_{\alpha, n}(x - y, te^{-i\pi \theta})}{2i}.
\]

Thus, in the view of Wright series representation obtained in Remark 5, the proof of Statement 1 is then immediate.

**Proof of Statement 2.**

For the proof of Statement 2 of Theorem 2, we note that for \( \theta = 0 \) there holds

\[
\Re K_{\alpha, n}(x, t) = K_{\alpha, n}(x, t) \quad \text{and} \quad \Im K_{\alpha, n}(x - y, t) = 0
\]

so that (38) simplifies to

\[
\Phi_\alpha(x, t; 0) = \frac{2^{1-n}}{\alpha \sqrt{\pi t}} I_1 \left[ \left( \frac{\alpha}{2}, \frac{\alpha}{2} \right), 1 \right] - \left[ \frac{|x|^{2 \gamma - 2}}{4} \right] \text{[case of } \gamma = \tau = t \text{ in equation (28)].}
\]

\[
\square
\]

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**CONFLICT OF INTEREST**

The author declare no potential conflict of interests.

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