Large deviations for conservative, stochastic PDE and non-equilibrium fluctuations

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High Dimensional Hamilton-Jacobi PDEs
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[G., Fehrman; arxiv, 2020].
Introduction: Large deviations in zero range process

1. Introduction: Large deviations for the zero range process
   - Fluctuations in the zero range process
   - Link to stochastic PDE

2. Two ways to the LDP
   - Scaling and criticality for the skeleton equation
   - Well-posedness of the skeleton equation
The zero range process
(could also consider simple exclusion, independent particles).

Figure: Harris, Rákos, Schütz; 2005

- Discrete $d$-dim. torus $\mathbb{T}_N^d := (\mathbb{Z}/(N\mathbb{Z}))^d$
- State space $\mathcal{M}_N := \mathbb{N}_0^d$, i.e. configurations $\eta : \mathbb{T}_N^d \rightarrow \mathbb{N}_0 :$ System in state $\eta$ if container $x$ contains $\eta(x)$ particles.
- Local jump rate function $g : \mathbb{N}_0 \rightarrow \mathbb{R}_0^+$, with $g(k) = 0$ iff $k = 0$, $g$ Lipschitz continuous.
Translation invariant, asymmetric, zero mean transition probability

\[ p(x, y) = p(x - y), \quad \sum_k kp(k) = 0. \]

Zero range process with jump rate \( g \) is the Markov jump process \( \eta_t \) on \( \mathbb{M}_N := \mathbb{N}_0^{dN} \) with generator

\[ L_N f(\eta) = \sum_{x, y \in \mathbb{T}_N^d} (f(\eta^{x,y}) - f(\eta))g(\eta(x))p(x, y), \]

where \( f : \mathbb{M}_N \to \mathbb{R}, \ \eta^{x,y} = \eta - 1_{\{x\}} + 1_{\{y\}}. \)

Invariant probability measures \( \nu_\rho \) (product structure and explicit), indexed by density \( \rho \geq 0. \)
Hydrodynamic limit?

Empirical density field

\[ \mu_t^N(x) := \frac{1}{Nd} \sum_k \delta_k \left( \frac{x}{N} \right) \eta_{tN^2}(k). \]

Assume that \( \mathcal{L}(\eta_0^N) \) is associated to a smooth profile \( \rho_0 \) with \( 0 < \rho_- \leq \rho_0 \leq \rho_+ < \infty \), that is, \( \mathcal{L}(\eta_0^N) \) around \( Nx \) converges to \( \nu_{\rho_0(x)} \) in probability.

Figure: see Zimmer et. al.
Theorem (Hydrodynamic limit - Ferrari, Presutti, Vares; 1987)

For $\delta > 0$, $G$ continuous, bounded,

$$
\lim_{N \to \infty} \mathbb{P} \left[ \left| \mu^N_t, G \right| - \left\langle \bar{\rho}_t \, dx, G \right\rangle \right] > \delta = 0
$$

where $\bar{\rho}(t, x)$ is the unique weak solution to

$$
\partial_t \bar{\rho} = \frac{1}{2} \partial_{xx} \Phi(\bar{\rho})
$$

$$
\bar{\rho}(0) = \rho_0
$$

with $\Phi$ the mean local jump rate

$$
\Phi(\rho) = \mathbb{E}_{\nu_\rho}[g(\eta(0))].
$$

E.g. $\Phi(\rho) = \rho \, |\rho|^{m-1}$.

- E.g. for constant intensity ($g(k) = 1_{k>0}$) get $\Phi(\rho) = \frac{\rho}{1+\rho}$.
- Independent particles, simple exclusion process: $\Phi(\rho) = \rho$. 
Theorem (Central limit fluctuations in non-equilibrium - Ferrari, Presutti, Vares; 1988)

Let $\eta$ be the zero range process with constant intensity with $\eta_0^N$ in local equilibrium with smooth profile $\rho_0$. Define the fluctuation density field

$$Y_t^N = \frac{1}{\sqrt{N}} \sum_k \delta_k^N(x)[\eta_{tN^2}(k) - <\eta_{tN^2}(k)>]$$

for $t \geq 0$, $< \cdot >$ the expectation. Let $\mathcal{P}^N$ the law of $Y^N$ on $D([0, \infty), \mathscr{L}^1(\mathbb{T}))$. Then,

$$\mathcal{P}^N \rightarrow^* \mathcal{P} \text{ for } N \rightarrow \infty$$

with $\mathcal{P}$ a Gaussian, supported on $C([0, \infty), \mathscr{L}^1(\mathbb{T}))$, the martingale solution to

$$dY_t = \partial_{xx}(\Phi'(\bar{\rho}_t(x))Y_t)\,dt + \partial_x(\sqrt{\Phi(\bar{\rho}_t(x)))dW_t}$$

with initial condition $Y_0$ is Gaussian, centered and covariance

$$\mathcal{X}(\rho_0(x)) = \mathbb{E}_{\nu_{\rho_0(r)}}(\eta_0(\eta_0 - \rho_0(x))).$$
Let now $\rho_0$ constant.

For $\mu \in D([0, t_0]; M_+)$, $\rho_t = d\mu_t / dx$ let

$$l_0(\mu) = \inf \left\{ \int_0^{t_0} \int_T |g|^2 dxd\sigma : g \in L^2_{t,x}, \partial_t \rho = \partial_{xx} \Phi(\rho) + \partial_x (\Phi^{1/2}(\rho)g) \right\}.$$ 

This is called the "skeleton equation".

E.g. independent particles: $\Phi(\rho) = \rho$.

**Theorem (Large deviation principle, Kipnis, Olla, Varadhan; 1989 & Benois, Kipnis, Landim; 1995)***

For every measurable $A \subseteq D([0, t_0], M_+)$,

$$\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}[\mu^N \in \bar{A}] \leq - \inf_{\mu \in \bar{A}} l_0(\mu) \leq - \inf_{\mu \in A^\circ} l_0(\mu) \leq \liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}[\mu^N \in A^\circ].$$

Informally, take $A = \{\rho \, dx\}$,

$$\mathbb{P}[\mu^N \approx \rho \, dx] \approx \exp\{-N l_0(\rho \, dx)\}.$$
Link to stochastic PDE

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**Aim:** Continuum model reflecting not only mean behavior but also fluctuations

**Ansatz:** Langevin dynamics

\[ \partial_t \rho = \partial_{xx} (\Phi(\rho)) + "\text{fluctuations}". \]

Concretely

\[ \partial_t \rho^N = \partial_{xx} \left( \Phi(\rho^N) \right) + \frac{1}{\sqrt{N}} \partial_x \left( \sqrt{\Phi(\rho^N)} dW_t \right). \tag{\ast} \]

**Informal justification:**

1. **Physics:** Fluctuation-dissipation relation, detailed balance, “fluctuating hydrodynamics”
2. **Mean behavior / law of large numbers**

\[ \rho^N \rightarrow \bar{\rho} \quad \text{as} \quad N \rightarrow \infty. \]
3. **Central limit fluctuations:** \( Y^N := \sqrt{N}(\rho^N - \bar{\rho}) \). Then, informally \( \mathcal{L}(Y^N) \overset{\ast}{\rightarrow} \mathcal{L}(Y) \) with

\[ \partial_t Y = \partial_{xx} (\Phi'(\bar{\rho}) Y) + \partial_x \left( \sqrt{\Phi(\bar{\rho})} dW_t \right). \]
4. **Large deviations:** See below, large deviations of (\ast) are the same as for \( \mu^N \).
Large deviations:

- Zero range process [Dirr, Stamatakis, Zimmer; 2016] as special case of macroscopic fluctuation theory (MFT) [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim; 2015],
  \[
  \partial_t \rho = \partial_{xx} (\Phi(\rho)) + \frac{1}{\sqrt{N}} \partial_x \left( \sqrt{\Phi(\rho)} dW_t \right).
  \]

- Model case: Dean-Kawasaki, independent particles, \( \Phi(\rho) = \rho \), i.e.
  \[
  \partial_t \rho = \partial_{xx} \rho + \frac{1}{\sqrt{N}} \partial_x (\sqrt{\rho} dW_t).
  \]

Informally same rate function as the zero range process:

- Informally applying the contraction principle to the solution map \( F : \frac{1}{\sqrt{N}} dW \mapsto \rho \),
  \[
  I(\rho) = \inf \{ I_{dW}(g) : F(g) = \rho \}.
  \]

- Schilder’s theorem for Brownian sheet suggests
  \[
  I_{dW}(g) = \int_0^T \int_T |g|^2 \, dx \, dt.
  \]
Get

\[ I(\rho) = \inf \left\{ \int_0^T \int_T |g|^2 \, dx \, dt : \partial_t \rho = \partial_{xx} (\Phi(\rho)) + \partial_x \left( \sqrt{\Phi(\rho)} g \right) \right\} . \]

Obstacle

\[ \partial_t \rho = \partial_{xx} (\Phi(\rho)) + \frac{1}{\sqrt{N}} \partial_x \left( \sqrt{\Phi(\rho)} dW_t \right) \]

1. not well-posed, supercritical \(\rightarrow\) no regularity structures
2. Renormalization? Does renormalization appear in rate function? E.g. compare \(\Phi^4_{2/3}\) [Hairer, Weber; 2014].

Ansatz: joint limit "small noise, ultraviolet cutoff"

\[
\partial_t \rho^{N,K} = \partial_{xx} \left( \Phi(\rho^{N,K}) \right) + \frac{1}{\sqrt{N}} \partial_x \left( \sqrt{\Phi(\rho^{N,K})} \circ dW^K_t \right)
\]

where \(W^K = \sum_{k=1}^K e_k \beta^k\) is a spectral (smooth) approximation of \(W = \sum_{k=1}^{\infty} e_k \beta^k\).

• Gives the correct rate function for \(\frac{1}{N} \ll \frac{1}{K}\).

Note: This is a particular case in which the link between Macroscopic fluctuation theory [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim; 2015] and fluctuating hydrodynamics [Landau-Lifshitz 1973, Spohn 1991] can be made rigorous.
Two ways to the LDP

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In the following concentrate on the case 
\[ \Phi(\rho) = \rho^m, \quad m \geq 1. \]

We consider stochastic PDE of the type
\[
\partial_t \rho^{N,K} = \Delta \left( (\rho^{N,K})^m \right) + \frac{1}{\sqrt{N}} \text{div} \left( (\rho^{N,K})^{\frac{m}{2}} \circ dW_t^K \right),
\]
(*) on \( \mathbb{T}^d \times (0, \infty) \), where \( W^K = \sum_{k=1}^K e_k \beta_k^K \).

Pathwise well-posedness of (*): [Lions, Souganidis; 1998ff], [Lions, Perthame, Souganidis; 2013], [Lions, Perthame, Souganidis; 2014], [G., Souganidis; 2014], [G., Souganidis; 2015], [G., Fehrman; 2017], [Dareiotis, G.; 2019].

**Two ways to the LDP:**

1. \( \Gamma \)-convergence of the rate functional: \( N \uparrow \infty \) yields LDP for (*) with rate function

\[
I^K(\rho) = \inf \left\{ \int_0^T \int_{\mathbb{T}} |g|^2 \, dx \, dt : \partial_t \rho = \partial_{xx} (\Phi(\rho)) + \partial_x \left( \sqrt{\Phi(\rho)} P^K g \right) \right\}.
\]

Then consider \( K \uparrow \infty \).

2. Joint scaling: Weak convergence approach to LDP \( (\frac{1}{N} << \frac{1}{K}) \).
Both approaches crucially depend on understanding the skeleton PDE.

The skeleton equation

\[ \partial_t \rho = \Delta \rho^m + \text{div} \left( \rho^{m/2} g(t,x) \right) \quad (\ast) \]

\[ \rho(0,x) = \rho_0(x), \]

with \( g \in L^2_{t,x} \)?

This leads to the key problem

**Problem**

1. **Existence and uniqueness of solutions to (\ast).**
2. **Stability of solutions:** Let \( g^n \rightharpoonup g \) in \( L^2_{t,x} \) with corresponding solutions \( \rho^n, \rho \). Then

   \[ \rho^n \to \rho \]

   in \( L^\infty_t L^1_x \).

- **Difficulty:** Stable a-priori bound? \( L^p \) framework does not work.
- **Do we expect non-concentration of mass / well-posedness?**
Well-posedness of the skeleton equation

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Scaling and criticality of the skeleton equation

We consider

$$\partial_t \rho = \Delta \rho^m + \text{div}(\rho^{\frac{m}{2}} g) \quad \text{on} \quad \mathbb{R}_+ \times \mathbb{R}^d$$

with \( g \in L^q(\mathbb{R}_+, t; L^p(\mathbb{R}_x^d; \mathbb{R}_x^d)) \) and \( \rho_0 \in L^r(\mathbb{R}_x^d) \).

Via rescaling (“zooming in”):

- \( p = q = 2 \) is critical.
- \( r = 1 \) is critical, \( r > 1 \) is supercritical.
Apriori-bounds and energy space

Consider

\[ \partial_t \rho = \Delta \rho^m + \text{div}(\rho^\frac{m}{2} g) \quad \text{on} \quad \mathbb{R}_+ \times \mathbb{T}^d \quad (*) \]

with \( g \in L^2(\mathbb{R}_+; L^2(\mathbb{R}_x^d; \mathbb{R}_x^d)) \).

\( L^1 \) estimate only gives

\[ \int_{\mathbb{T}^d} \rho(t,x) \, dx = \int_{\mathbb{T}^d} \rho_0(x) \, dx. \]

Use entropy-entropy dissipation: Evolution of entropy given by \( \int_{\mathbb{T}^d} \log(\rho) \rho \). Informally gives

\[ \int_{\mathbb{T}^d} \log(\rho) \rho \, dx \bigg|_0^t + \int_0^t \int_{\mathbb{T}^d} (\nabla \rho^\frac{m}{2})^2 \lesssim \int_0^t \int_{\mathbb{T}^d} g^2. \]

Caution: Can only be true for non-negative solutions.

Non-standard weak solutions, rewriting (\( * \)) as

\[ \partial_t \rho = 2\text{div}(\rho^\frac{m}{2} \nabla \rho^\frac{m}{2}) + \text{div}(\rho^\frac{m}{2} g) \quad \text{on} \quad \mathbb{R}_+ \times \mathbb{T}^d \]

Conclusion: Have to prove uniqueness within this class of solutions.
**Ansatz for uniqueness:** Show that every weak solution is a renormalized entropy solution (extending the concepts of DiPerna-Lions, Ambrosio to nonlinear PDE).

Let $\rho$ be a weak solution to

$$
\partial_t \rho = 2 \text{div}(\rho \frac{m}{2} \nabla \rho \frac{m}{2}) + \text{div}(\rho \frac{m}{2} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d.
$$

Let

$$\chi(t, x, \xi) = 1_{0 < \xi < \rho(x, t)} - 1_{\rho(x, t) < \xi < 0}.$$

Then, informally,

$$
\partial_t \chi = m \xi^{m-1} \Delta_x \chi - g(x, t)(\partial_\xi \xi \frac{m}{2}) \nabla_x \chi + (\nabla_x g(x, t)) \xi^\frac{m}{2} \partial_\xi \chi + \partial_\xi \rho
$$

with $\rho$ parabolic defect measure

$$
\rho = \delta(\xi - \rho) 4 \frac{\xi^m}{\xi^{m-1}} |\nabla \rho \frac{m}{2}|^2.
$$

Or on "fluid level", informally, for $F$ convex,

$$
\partial_t F(\rho) - \Delta_x F^1(\rho) + g(x, t) \nabla_x F^2(\rho) + (\nabla_x g)(x, t) F^3(\rho) = -\int_\xi F''(\xi) \rho \leq 0
$$

where $(F^m)' = mf(\xi) \xi^{m-1}$, $(F^2)' = f(\xi)(\partial_\xi \xi \frac{m}{2})$, $F^3 = f(\xi) \xi^\frac{m}{2}$. 
How to make that rigorous? Take convolution
\[ \rho^\varepsilon = \varphi^\varepsilon \ast_x \rho. \]

Commutator errors,
\[
\partial_t \rho^\varepsilon = \varphi^\varepsilon \ast \partial_t \rho = \varphi^\varepsilon \ast (\Delta \rho^m + \text{div}(\rho^m g)) \\
= \Delta(\varphi^\varepsilon \ast \rho^m) + \text{div}(\varphi^\varepsilon \ast (\rho^m g)) \\
= \Delta(\rho^\varepsilon)^m + \text{div}(\rho^\varepsilon)^m g \\
+ \Delta(\varphi^\varepsilon \ast \rho^m) - \Delta(\rho^\varepsilon)^m \\
+ \text{div}(\varphi^\varepsilon \ast (\rho^m g)) - \text{div}(\varphi^\varepsilon \ast (\rho^m g)).
\]

Note: Additional commutator errors by commuting convolution and nonlinearities!

Commutator estimate using non-standard (optimal) regularity \( \rho^\varepsilon \frac{m}{2} \in L^2_t H^{\frac{1}{2}}_x \)

Additional renormalization step to compensate low time integrability \( \rho^\varepsilon \frac{m}{2} g \in L^1_t L^1_x \).
Theorem

A function \( \rho \in L_t^\infty L_x^1 \) is a weak solution to

\[
\partial_t \rho = 2\text{div}(\rho^m \nabla \rho^m) + \text{div}(\rho^m g)
\]

if and only if \( \rho \) is a renormalized entropy solution.

Uniqueness for renormalized entropy solutions (variable doubling)

- Additional errors from space-inhomogeneity (with little regularity)
- Note: Entropy dissipation measure

\[
q(x, \xi, t) = \delta(\xi - \rho(x, t))4 \frac{\xi^m}{\xi^{m-1}} |\nabla \rho^m|^2
\]

does not satisfy

\[
\lim_{|\xi| \to \infty} \int_{t,x} q(x, \xi, t) \, dx \, dt = 0.
\]

- Established arguments [Chen, Perthame; 2003] not applicable.
Theorem (The skeleton equation)

Let $g \in L^2([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$, $\rho_0 \in L^1(\mathbb{T}^d)$ non-negative and $\int \rho_0 \log(\rho_0) dx < \infty$, $m \in [1, \infty)$.

1. There is a unique weak solution

$$\partial_t \rho = \Delta \rho^m + \text{div}(\rho^{m/2} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d$$

(*)

For two weak solutions $\rho^1, \rho^2 \in L^\infty([0, T]; L^1(\mathbb{T}^d))$ we have

$$\| \rho^1 - \rho^2 \|_{L^\infty([0, T]; L^1(\mathbb{T}^d))} \leq \| \rho_0^1 - \rho_0^2 \|_{L^1(\mathbb{T}^d)}.$$

2. Let $\{g_n\}_{n \in \mathbb{N}} \subseteq L^2([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$ with

$$\lim_{n \to \infty} g_n = g \text{ weakly in } L^2([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$$

and let $\rho_n \in L^1([0, T]; L^1(\mathbb{T}^d))$ be the corresponding solutions with control $g_n$. Then,

$$\lim_{n \to \infty} \rho_n = \rho \text{ strongly in } L^1([0, T]; L^1(\mathbb{T}^d))$$

where $\rho \in L^1([0, T]; L^1(\mathbb{T}^d))$ is the solution with control $g$. 
Consider
\[ d\rho^N = \Delta (\rho^N)^m \, dt + \frac{1}{\sqrt{N}} \text{div} \left( \Phi_{n(N)}^{\frac{1}{2}} (\rho^N) \circ dW^{K(N)}(t) \right). \]

**Theorem (Large deviation principle)**

Let \( K(N), n(N) \rightarrow \infty \) with \( \frac{K(N)^3}{N} \rightarrow 0 \) for \( N \rightarrow \infty \). For \( \rho_0 \in L^{m+1}(\mathbb{T}^d) \) and \( \rho \in L^\infty([0, T]; L^1(\mathbb{T}^d)) \) let
\[
I_{\rho_0}(\rho) := \inf \left\{ \frac{1}{2} \int_0^T \| g(s) \|_{L_x^2}^2 \, ds : \ g \in L^2_{t,x}, \ \partial_t \rho = \Delta \rho^m + \text{div}(\rho^m g) \right\}.
\]

Then,
1. For all \( \rho_0 \in L^{m+1}(\mathbb{T}^d), \ \rho \mapsto I_{\rho_0}(\rho) \) is a good rate function on \( L^\infty([0, T]; L^1(\mathbb{T}^d)) \).
2. The family \( \{\rho^N\} \) satisfies the large deviation principle on \( L^\infty([0, T]; L^1(\mathbb{T}^d)) \) with rate function \( I_{\rho_0} \), uniformly on compact subsets of \( L^{m+1}(\mathbb{T}^d) \).
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