THE GEOMETRIC STRUCTURE OF $\mathcal{W}_N$-GRAVITY

C. M. HULL

Physics Department, Queen Mary and Westfield College,
Mile End Road, London E1 4NS, United Kingdom.

ABSTRACT

The full non-linear structure of the action and transformation rules for $\mathcal{W}_N$-gravity coupled to matter are obtained from a non-linear truncation of those for $w_\infty$ gravity. The geometry of the construction is discussed, and it is shown that the defining equations become linear after a twistor-like transform.
1. Introduction

Classical $\mathcal{W}$-gravity theories [1-8] are higher-spin gauge theories in two dimensions that result from gauging $\mathcal{W}$-algebras [9], which are higher-spin extensions of the Virasoro algebra. One motivation for studying two-dimensional matter coupled to $\mathcal{W}$-gravity is that such systems can be interpreted as generalisations of string theory in which the two-dimensional space-time is regarded as a world-sheet, in much the same way that matter coupled to ordinary gravity in two dimensions leads to conventional string theory. In particular, the $\mathcal{W}$-algebras play a central role in such $\mathcal{W}$-string theories, just as the Virasoro algebra plays a central role in string theory. The actions for $\mathcal{W}$-gravity coupled to matter have a complicated non-polynomial dependence on the gauge fields. In the case of gravity, this non-linear structure is best understood in terms of Riemannian geometry and this suggests that some higher spin geometry might lead to a better understanding of $\mathcal{W}$-gravity. A number of approaches to the geometry of $\mathcal{W}$-gravity theories have been considered [6,11-20]. In [18,19], the complete non-linear structure of the coupling of a scalar field on a world-sheet $M$ to $w_\infty$ gravity was given in terms of a function $\tilde{F}$ on the cotangent bundle of $M$ that satisfied a certain non-linear differential equation, which is sometimes referred to as a Monge-Ampère equation [21] or as one of Plebanski’s equations [22]. Such equations also arise in the study of $4-D$ self-dual gravity [22]; other connections between $\mathcal{W}$-algebras and gravitational instantons, which may be related, were described in [23,24]. In particular, it was shown in [18,19] that the function $\tilde{F}$ could be interpreted as giving a family of Kähler potentials for Ricci-flat metrics on $\mathbb{R}^4$, with self-dual curvature. The purpose of this paper is to extend the results of [18,19] to the case of $\mathcal{W}_N$-gravity; some of the results to be derived here were announced in [20].

It will be shown here that the coupling of a scalar field to $\mathcal{W}_N$ gravity can be given as a non-linear truncation of the action for the coupling to $w_\infty$ gravity. The lagrangian is a function $\tilde{F}$ which, in addition to satisfying the Monge-Ampère equation, satisfies an $(N + 1)$'th order non-linear partial differential equation, and
it is a non-trivial fact that this constraint is consistent with the Monge-Ampère equation. This differential constraint can be interpreted geometrically as a condition on the family of self-dual metrics on $\mathbb{R}^4$. For $\mathcal{W}_3$, the 4’th order differential equation satisfied by the Kähler potential can be written as

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} G^{\alpha\beta} \left[ T_{\alpha\mu\bar{\nu}} T_{\beta\sigma\rho} + T_{\alpha\mu\bar{\sigma}} T_{\beta\rho\nu} + T_{\bar{\beta}\rho\mu} T_{\alpha\nu\sigma} + T_{\bar{\beta}\rho\nu} T_{\alpha\mu\sigma} \right]$$ (1.1)

where $G_{\mu\bar{\nu}}$ is the Kähler metric and $T_{\mu\nu\bar{\rho}}$ is a certain third rank tensor that is given in terms of the Kähler potential $K$ by $T_{\mu\nu\bar{\rho}} = \partial_{\mu} \partial_{\nu} \partial_{\bar{\rho}} K$ in certain special coordinate systems. It is interesting to note that similar, but distinct, geometrical constraints arise in the study of ‘special geometry’, i.e. the geometry of the moduli space of Calabi-Yau manifolds, and in the geometry of $N = 2$ supersymmetric gauge multiplets in 4 and 5 dimensions [27]. For $\mathcal{W}_N$ with $N > 3$, the differential constraint can be written as a restriction on the $(N - 3)$’th covariant derivative of the curvature tensor.

**Linearised $W$-Gravity**

Before proceeding to the non-linear theories, it will be useful to review linearised $W_N$ and $w_\infty$ gravity. Consider the action for a free scalar field in two dimensions

$$S_0 = \frac{1}{2} \int d^2x \partial_\mu \phi \partial^\mu \phi$$ (1.2)

This has an infinite number of conserved currents, which include [3]

$$W_n = \frac{1}{n} (\partial \phi)^n, \quad n = 2, 3, ..., N$$ (1.3)

and these satisfy the conservation law $\bar{\partial} W_n = 0$. The current $W_2 = T$ is a component of the energy-momentum tensor and generates a Virasoro algebra. The

---

* Flat two-dimensional space $M_0$ has metric $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = 2dzd\bar{z}$, where $z = \frac{1}{\sqrt{2}} (x^1 + ix^2)$, $\bar{z} = \frac{1}{\sqrt{2}} (x^1 - ix^2)$ are complex coordinates if $M_0$ is Euclidean, while, if $M_0$ is Lorentzian, $z = \frac{1}{\sqrt{2}} (x^1 + x^2)$, $\bar{z} = \frac{1}{\sqrt{2}} (x^1 - x^2)$ are null real coordinates. $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$. 

---


currents (1.3) generate a current algebra which is a certain classical limit of the $\mathcal{W}_N$ algebra of [25] for finite $N$, and in the limit $N \to \infty$, the classical current algebra becomes the $w_\infty$ algebra [23]. Similarly, the currents $\overline{W}_n = \frac{1}{n}(\bar{\partial}\phi)^n$ generate a second copy of the $\mathcal{W}_N$ or $w_\infty$ algebra.

Adding the Noether coupling of the currents $W_n, \overline{W}_n$ to corresponding gauge fields $h_n, \bar{h}_n$ gives the linearised action

$$S = \int d^2x \left[ \partial \phi \bar{\partial} \phi + \sum_{n=2}^{N} \frac{1}{n} \left[ h_n(\partial \phi)^n + \bar{h}_n(\bar{\partial} \phi)^n \right] + O(h^2) \right]$$

(1.4)

which is invariant, to lowest order in the gauge fields, under the transformations

$$\delta \phi = \sum_{n=2}^{N} \left[ \lambda_n(z, \bar{z})(\partial \phi)^{n-1} + \bar{\lambda}_n(z, \bar{z})(\bar{\partial} \phi)^{n-1} \right]$$

$$\delta h_n = -2\bar{\partial}\lambda_n + O(h), \quad \delta \bar{h}_n = -2\partial \bar{\lambda}_n + O(h)$$

(1.5)

This gives the linearised action and transformations of $\mathcal{W}_N$ or (in the $N \to \infty$ limit) $w_\infty$ gravity. The full gauge-invariant action and gauge transformations are non-polynomial in the gauge fields.

2. Non-Linear $w_\infty$-Gravity

The non-linear structure of the coupling of a scalar field to $w_\infty$ gravity [18,19] will now be reviewed. The two-dimensional manifold $M$, which will sometimes be referred to as the world-sheet, can have any topology and has local coordinates $x^\mu$. The action is a non-polynomial function of $\partial_\mu \phi$ and can be written as

$$S = \int_M d^2x \tilde{F}(x, \partial \phi)$$

(2.1)

for some $\tilde{F}$, which has the following expansion in $y_\mu = \partial_\mu \phi$:

$$\tilde{F}(x, y) = \sum_{n=2}^{\infty} \frac{1}{n} \tilde{g}^{\mu_1 \ldots \mu_n}_{(n)}(x)y_{\mu_1}y_{\mu_2}\ldots y_{\mu_n}$$

(2.2)

where $\tilde{g}^{\mu_1 \ldots \mu_n}_{(n)}(x)$ are symmetric tensor (density) gauge fields.
The gauge fields \( \tilde{g}_{(2)}, \tilde{g}_{(3)}, \tilde{g}_{(4)}, \ldots \) are required to satisfy an infinite set of constraints, the first few of which are

\[
\det \left( \tilde{g}^{\mu\nu}_{(2)} \right) = \epsilon \tag{2.3}
\]

\[
\tilde{g}_{\mu\nu} \tilde{g}^{\mu\nu}_{(3)} = 0 \tag{2.4}
\]

\[
\tilde{g}_{\mu\nu} \tilde{g}^{\mu\nu\rho\sigma}_{(4)} = \frac{2}{3} \tilde{g}_{\mu\alpha} \tilde{g}_{\nu\beta} \tilde{g}^{\mu3\rho}_{(3)} \tilde{g}^{\nu\alpha\sigma}_{(3)} \tag{2.5}
\]

where \( \tilde{g}_{\mu\nu} \) is the inverse of \( \tilde{g}^{\mu\nu}_{(2)} \) (\( \tilde{g}_{\mu\nu} \tilde{g}^{\rho\sigma}_{(2)} = \delta_{\mu}^\rho \)) and \( \epsilon = \pm 1 \) is the signature of the world-sheet metric. For the \( n = 2 \) gauge field, the constraint (2.3) can be solved in terms of an unconstrained metric tensor \( g_{\mu\nu} \) as \( \tilde{g}^{\mu\nu}_{(2)} = \sqrt{\epsilon} gg^{\mu\nu} \), where \( g = \det[g_{\mu\nu}] \), so that the term \( \tilde{g}^{\mu\nu}_{(2)} \partial_\mu \phi \partial_\nu \phi \) becomes the standard minimal coupling to gravity. If \( \epsilon = 1 \), this metric has Euclidean signature while if \( \epsilon = -1 \) the signature is Lorentzian. Alternatively, the single constraint \( \det(\tilde{g}^{\mu\nu}_{(2)}) = \epsilon \) on the three components of \( \tilde{g}^{\mu\nu}_{(2)} \) can be solved in terms of two unconstrained functions \( h_2(x), \tilde{h}_2(x) \) which correspond to the two spin-two gauge fields of the previous section. Similarly, the constraints on the spin-\( n \) gauge field \( \tilde{g}^{\mu_1\mu_2\ldots\mu_n}_{(n)}(x) \) can be solved either in terms of tensor gauge fields satisfying algebraic trace constraints, or in terms of two unconstrained functions, which can be identified with the gauge fields \( h_n(x), \tilde{h}_n(x) \) [19].

The full set of constraints are generated by the following constraint on \( \tilde{F} \):

\[
\det \left( \frac{\partial^2 \tilde{F}(x, y)}{\partial y_\mu \partial y_\nu} \right) = \epsilon \tag{2.6}
\]

Expanding (2.6) in \( y \) generates the full set of constraints. This is the condition that \( \tilde{F} \) satisfies the real Monge-Ampère equation [21].
The action (2.1) is invariant under the local $w_\infty$ transformations

$$\delta \phi = \Lambda(x, \partial \phi) \quad (2.7)$$

$$\delta \tilde{g}^{(p)}_{\mu_1 \mu_2 \cdots \mu_p} = \sum_{m,n=2}^{\infty} \delta_{m+n,p+2} \left[ (m-1) \lambda_{(m)}^{(\mu_1, \mu_2, \ldots, \mu_p)} \tilde{g}^{(n)}_{\nu} - (n-1) \tilde{g}^{(n)}_{(\mu_1, \mu_2, \ldots, \mu_p)} \partial_{\nu} \lambda_{(m)} \right]$$

$$+ \frac{(m-1)(n-1)}{p-1} \partial_{\nu} \left\{ \lambda_{(m)}^{(\mu_1, \mu_2, \ldots, \mu_p)} \tilde{g}^{(n)}_{(\nu)} - \tilde{g}^{(n)}_{(\mu_1, \mu_2, \ldots, \mu_p)} \lambda_{(m)} \right\} \quad (2.8)$$

where

$$\Lambda(x^\mu, y_\mu) = \sum_{n=2}^{\infty} \lambda_{(n)}^{(\mu_1, \mu_2, \ldots, \mu_{n-1})} (x) y_{\mu_1} y_{\mu_2} \cdots y_{\mu_{n-1}} \quad (2.9)$$

for some infinitesimal symmetric tensor parameters $\lambda_{(n)}^{(\mu_1, \mu_2, \ldots, \mu_{n-1})} (x)$ which are required to satisfy the set of algebraic constraints generated by expanding

$$\tilde{F}_{\mu \nu} \frac{\partial^2}{\partial y_\mu \partial y_\nu} \Lambda(x, y) = 0 \quad (2.10)$$

in $y$, where $\tilde{F}_{\mu \nu}(x, y)$ is the inverse of the matrix $\frac{\partial^2}{\partial y_\mu \partial y_\nu} \tilde{F}(x, y)$ and can be written as

$$\tilde{F}_{\mu \nu}(x, y) = -\epsilon_{\mu \rho} \epsilon_{\nu \sigma} \tilde{F}^{\rho \sigma}(x, y) \quad (2.11)$$

Using (2.6), the constraint (2.10) can be rewritten (for infinitesimal $\Lambda$) as

$$det \left( \frac{\partial^2}{\partial y_\mu \partial y_\nu} [\tilde{F} + \Lambda](x, y) \right) = \epsilon \quad (2.12)$$

This constraint is necessary for the transformations to be a symmetry of the action [18,19]. As will be seen in the next section, the constraints (2.6),(2.10) can be solved to give a theory which, in the linearised limit, reproduces the linearised theory of the previous section.
The action is also invariant under the local symmetries with parameters \( \alpha_{(p,q)}^{\mu_1\mu_2...\mu_p} (x) \) for \( q < p \) given by

\[
\delta \tilde{g}_{(p)}^{\mu_1\mu_2...\mu_p} = \alpha_{(p,q)}^{\mu_1\mu_2...\mu_p} \\
\delta \tilde{g}_{(q)}^{\mu_1\mu_2...\mu_q} = -\frac{q}{p} \alpha_{(p,q)}^{\mu_1\mu_2...\mu_q\mu_{q+1}...\mu_p} y_{\mu_{q+1}} y_{\mu_{q+2}} \cdots y_{\mu_p}
\]

(2.13)

with all other fields inert. These are the analogues of the ‘Stuckelberg’ symmetries of [3] and reflect the reducibility of the one-boson realisation of \( w_{\infty} \). Nevertheless, most of the structure of the one-boson realisation developed in this paper carries over immediately to multi-boson realisations [28] which are non-trivial and do not have Stuckelberg symmetries. For further discussion, see [18,19,20,7,28].

3. The Solution of the Constraints

The constraint (2.6) can be given the following geometrical interpretation [18,19]. Let \( \zeta_{\mu}, \bar{\zeta}_{\bar{\mu}} \) (\( \mu = 1, 2 \)) be complex coordinates on \( \mathbb{R}^4 \). Then, for each \( x^\mu \), a solution \( \tilde{F}(x, y) \) of (2.6) can be used to define a function \( K_x(\zeta, \bar{\zeta}) \) on \( \mathbb{R}^4 \) by

\[
K_x(\zeta, \bar{\zeta}) = \tilde{F}(x^\mu, \zeta_{\mu} + \bar{\zeta}_{\bar{\mu}})
\]

(3.1)

For each \( x \), \( K_x \) can be viewed as the Kähler potential for a Kähler metric \( G^{\mu\bar{\nu}} = \partial^2 K_x / \partial \zeta_{\mu} \partial \bar{\zeta}_{\bar{\nu}} \) on \( \mathbb{R}^4 \). As a result of (2.6), each \( K_x \) satisfies the Monge-Ampère equation \( det(G^{\mu\bar{\nu}}) = \epsilon \) and so the corresponding metric is Kähler and Ricci-flat, which implies that the curvature tensor is either self-dual or anti-self-dual. In the Euclidean case (\( \epsilon = 1 \)), the metric has signature (4,0) and is hyperkähler with \( SU(2) \) holonomy, while in the Lorentzian case (\( \epsilon = -1 \)) the metric has signature (2,2) and holonomy \( SU(1,1) \). As the Kähler potential is independent of the imaginary part of \( \zeta_{\mu} \), the metric has two commuting (triholomorphic) Killing vectors, given by \( i(\partial / \partial \zeta_{\mu} - \partial / \partial \bar{\zeta}_{\bar{\mu}}) \). Thus the lagrangian \( \tilde{F}(x, y) \) corresponds to a two-parameter family of Kähler potentials \( K_{x^\nu} \) for (anti-) self-dual geometries on \( \mathbb{R}^4 \) with two Killing vectors. The parameter constraint (2.10) implies that \( \tilde{F} + \Lambda \) is also a Kähler potential for a hyperkähler metric with two Killing vectors.
Two solutions of the constraint (2.6) were discussed in [18,19] and both are related to twistor transforms. The first is the Legendre transform solution of [26]. Writing \( y_1 = \zeta \), \( y_2 = \xi \), \( \tilde{F}(x^\mu, \zeta, \xi) \) can be written as the Legendre transform with respect to \( \zeta \) of some \( \mathcal{H} \), so that

\[
\tilde{F}(x, \zeta, \xi) = \pi \zeta - \mathcal{H}(x, \pi, \xi) \tag{3.2}
\]

where the equation

\[
\frac{\partial \mathcal{H}}{\partial \pi} = \zeta \tag{3.3}
\]

gives \( \pi \) implicitly as a function of \( x, \zeta, \xi \). Taking the Legendre transform has the remarkable property of replacing the complicated non-linear equation (2.6) with the Laplace equation [26]

\[
\frac{\partial^2 \mathcal{H}}{\partial \pi^2} + \epsilon \frac{\partial^2 \mathcal{H}}{\partial \xi^2} = 0 \tag{3.4}
\]

and the general solution of this is

\[
\mathcal{H} = f(x, \pi + \sqrt{-\epsilon} \xi) + \bar{f}(x, \pi - \sqrt{-\epsilon} \xi) \tag{3.5}
\]

where \( f, \bar{f} \) are arbitrary independent real functions if \( \epsilon = -1 \) and are complex conjugate functions if \( \epsilon = 1 \). Then the general solution of (2.6) is the Legendre transform (3.2),(3.3),(3.5) and the action can be given in the first order form

\[
S = \int d^2 x \, \tilde{F}(x, y) = \int d^2 x \, \left[ \pi \partial_\tau \phi - f(x^\mu, \pi + \partial_\sigma \phi) + \bar{f}(x^\mu, \pi - \partial_\sigma \phi) \right] \tag{3.6}
\]

where \( \tau = x^1 \) and \( \sigma = -\sqrt{-\epsilon} x^2 \). The field equation for the auxiliary field \( \pi \) is (3.3) and this can be used in principle to eliminate \( \pi \) from the action, but it will not be possible to solve the equation (3.3) explicitly in general. The constraints (2.10) can be solved similarly. Expanding the functions \( f, \bar{f} \) gives the Hamiltonian
form of the \( w_\infty \) action [8]

\[
S = \int d^2x \left( \pi \partial_x \phi - \sum_{n=2}^\infty \frac{1}{n} \left[ h_n(\pi + \partial_x \phi)^n + \bar{h}_n(\pi - \partial_x \phi)^n \right] \right)
\]

(3.7)

A related solution [18,19] that involves transforming with respect to both components of \( y_\mu \) and maintains Lorentz covariance was suggested by the results of [2] and their generalisation [3,4]. It will be useful to introduce a background ‘metric’ \( \tilde{h}^{\mu\nu}(x) \) on the cotangent space, satisfying the constraint

\[
det(\tilde{h}^{\mu\nu}) = \epsilon \]

(3.8)

This constraint can be solved in terms of an unconstrained background ‘metric’ \( h^{\mu\nu} \) by

\[
\tilde{h}^{\mu\nu} = [\epsilon det(h^{\mu\nu})]^{-1/2} h^{\mu\nu}
\]

(3.9)

(Note that \( \tilde{h}^{\mu\nu} \) only determines \( h^{\mu\nu} \) up to a Weyl rescaling.) In [18,19], this ‘metric’ was chosen to be the flat metric

\[
\tilde{h}^{\mu\nu}(x) = h^{\mu\nu}(x) = \eta^{\mu\nu}
\]

(3.10)

but here it will be useful to allow a more general choice. Different choices will give equivalent results, but a judicious choice in which \( \tilde{h}^{\mu\nu} \) transforms as a tensor density will be seen later to lead to manifestly covariant results.

\( \tilde{F}(x^\mu, y_\nu) \) is written as a transform of a function \( H(x^\mu, \pi^\nu) \) as follows:

\[
\tilde{F}(x^\mu, y_\nu) = 2\pi^\mu y_\mu - \frac{1}{2} \tilde{h}^{\mu\nu} y_\mu y_\nu - 2H(x, \pi)
\]

(3.11)

where the equation

\[
y_\mu = \frac{\partial H}{\partial \pi^\mu}
\]

(3.12)

implicitly determines \( \pi^\mu = \pi^\mu(x^\nu, y_\rho) \). The transform again linearises (2.6) and \( \tilde{F} \)
satisfies (2.6) if and only if its transform \( H \) satisfies

\[
\frac{1}{2} \tilde{h}^{\mu\nu} \frac{\partial^2 H}{\partial \pi^\mu \partial \pi^\nu} = 1 \tag{3.13}
\]

and \( \tilde{h}^{\mu\nu} \) satisfies (3.8). This is true for any ‘background metric’ \( \tilde{h}^{\mu\nu} \) satisfying (3.8).

It will be useful to introduce a zweibein \( e_\mu^a \) \( (a = 1, 2) \) such that \( h^{\mu\nu} = \eta^{ab} e_\mu^a e_\nu^b = 2 e_\mu^a e_\nu^b \) with \( e_\mu^\pm = \frac{1}{\sqrt{2}} (e_\mu^1 \pm e_\mu^2) \), and define \( \pi^1, \pi^2 \) by \( \pi^a = eae_\mu^a \pi^\mu \) where \( e = \text{det}(e^a_\mu) \), together with the null coordinates \( \pi^\pm = \frac{1}{\sqrt{2}} (\pi^1 \pm \sqrt{-\epsilon} \pi^2) \) which are independent real coordinates for Lorentzian signature \((\epsilon = -1)\) and are complex conjugate coordinates for Euclidean signature \((\epsilon = 1)\). The general solution of (3.13) can now be written as

\[
H = e \left[ \pi^+ \pi^- + f(x, \pi^+) + \bar{f}(x, \pi^-) \right] \tag{3.14}
\]

(where \( \bar{f} = f^* \) if \( \epsilon = 1 \), but \( f, \bar{f} \) are independent real functions if \( \epsilon = -1 \)). This solution can be used to write the action

\[
S = \int d^2x \left( 2 \pi^\mu y_\mu - \tilde{h}_{\mu\nu} \pi^\mu \pi^\nu - \frac{1}{2} \tilde{h}^{\mu\nu} y_\mu y_\nu - 2ef(x, \pi^+) - 2e\bar{f}(x, \pi^-) \right) \tag{3.15}
\]

where \( \tilde{h}_{\mu\nu} \) is the inverse of \( \tilde{h}^{\mu\nu} \). The field equation for \( \pi^\mu \) is (3.12), and using this to substitute for \( \pi \) gives the action (2.1) subject to the constraint (2.6) (details are given in the appendix). Alternatively, expanding the functions \( f, \bar{f} \) as

\[
f = \sum_{n=2}^{\infty} \frac{1}{n} h_n(x)(\pi^+)^n, \quad \bar{f} = \sum_{n=2}^{\infty} \frac{1}{n} \bar{h}_n(x)(\pi^-)^n \tag{3.16}
\]

gives precisely the form of the action given in [3], following the approach of [2]. The parameter constraint (2.10) is solved similarly, and the solutions can be used to
write the symmetries of (3.15) in a form similar to that given in [3]. For example, the variation of \( \phi \) given by \( \delta \phi = \Lambda(x, y) \) with \( y_\mu = \partial_\mu \phi \) becomes

\[
\delta \phi = \Lambda(x^\mu, \pi^\nu) = \Lambda(x, y(\pi)) \tag{3.17}
\]

where \( y(\pi) \) is found by solving \( \partial \tilde{F} / \partial y_\mu = 2\pi^\mu - \tilde{h}^{\mu\nu} y_\nu \). The constraint (2.10) on \( \Lambda(x, y) \) then becomes the following simple linear Laplace equation constraint on \( \Lambda(x, \pi) \):

\[
\tilde{h}^{\mu\nu} \frac{\partial^2 \Lambda}{\partial \pi^\mu \partial \pi^\nu} = 0 \tag{3.18}
\]

The calculation leading to this result is given in the appendix.

The part of \( H \) quadratic in \( \pi \) is

\[
\frac{1}{2} (\tilde{h}^{\mu\nu} \pi_\mu \pi_\nu + h_2 (\pi^+)^2 + \bar{h}_2 (\pi^-)^2) \tag{3.19}
\]

and the terms involving \( h_2, \bar{h}_2 \) consist of a background part \( (\tilde{h}^{\mu\nu}) \) and a perturbation involving \( h_2, \bar{h}_2 \). Different choices of \( \tilde{h}^{\mu\nu} \) correspond to expanding the full metric about different background metrics. The action (3.15) is invariant under spin-two transformations for any choice of background \( \tilde{h}^{\mu\nu} \); different choices lead to different transformation rules. For example, with the choice \( \tilde{h}^{\mu\nu} = \eta^{\mu\nu} \), the action (3.15) becomes precisely that of [3] and the transformations are those given in [3]. If instead \( \tilde{h}^{\mu\nu} \) is chosen to be a tensor density transforming as

\[
\delta \tilde{h}^{\mu\nu} = k^\rho \partial_\rho \tilde{h}^{\mu\nu} - 2\tilde{h}^{\rho(\mu} \partial_\rho k^{\nu)} + \tilde{h}^{\mu\nu} \partial_\rho k^\rho \tag{3.20}
\]

under spin-two transformations with parameter \( k^\mu = \lambda_2^\mu \) and \( \pi^\mu \) is also chosen to be a tensor density, then \( \tilde{h}^{\mu\nu} \) transforms as a tensor, \( \pi^a \) is a coordinate scalar the first three terms in (3.15) are manifestly coordinate invariant and the remaining terms will be invariant if the gauge fields \( h_n, \bar{h}_n \) are chosen to transform as scalars under coordinate transformations and as spin \( n \) tensors under two-dimensional
local Lorentz transformations. Then the part of $H$ quadratic in $\pi$ is given by (3.19) and the terms involving $h_2, \bar{h}_2$ can be absorbed into a shift of $\tilde{h}^{\mu\nu}$. After this shift, the action is given by (3.15),(3.16), in terms of the new shifted $\tilde{h}^{\mu\nu}$, which is again a tensor density, but now with

$$h_2 = \bar{h}_2 = 0$$

(3.21)

As a result, $\tilde{g}^{\mu\nu}_{(2)}$, the spin-two gauge field in the expansion (2.2) of $\tilde{F}$, is now given by

$$\tilde{g}^{\mu\nu}_{(2)} = \tilde{h}^{\mu\nu}$$

(3.22)

giving a formulation similar to that of [2]. This has the advantage that the invariance under diffeomorphisms is manifest, although the shift of variables leads to a formulation in which the spin-two gauge fields are no longer on an equal footing with the higher-spin ones.

For each $x^\mu$, the variables $h_n(x), \bar{h}_n(x)$ parameterise the space of Kähler potentials $K_x$ (given by (3.1)) which are solutions of the Monge-Ampère equation, so that the $h_n, \bar{h}_n$ can be taken to be the moduli of self-dual metrics on $\mathbb{R}^4$ with two commuting Killing vectors. For the family of geometries labelled by the worldsheet coordinates $x^\mu$, the moduli become functions $h_n(x), \bar{h}_n(x)$ of $x^\mu$ and these functions are interpreted as the gauge fields of $w_\infty$ gravity.

4. Non-Linear $\mathcal{W}_N$ Gravity

From the discussion in the introduction, the linearised action for $\mathcal{W}_N$ gravity (i.e. the action to linear order in the gauge fields) is an $N$’th order polynomial in $\partial_\mu \phi$ given by (1.4). However, the full non-linear action is non-polynomial in the gauge fields and in $\partial_\mu \phi$, but the coefficient of $(\partial \phi)^n$ for $n > N$ is a polynomial function of the finite number of fundamental gauge fields that occur in the linearised
action. The simplest way in which this might come about would be if the action were given by (2.1),(2.2) and $\tilde{F}$ satisfies a constraint of the form

$$\frac{\partial^{N+1}\tilde{F}}{\partial y_{\mu_1}\partial y_{\mu_2}...\partial y_{\mu_{N+1}}} = 0 + O(\tilde{F}^2) \quad (4.1)$$

where the right hand side is non-linear in $\tilde{F}$ and its derivatives, and depends only on derivatives of $\tilde{F}$ of order $N$ or less. It will be shown in this section that this is indeed the case; the action for $\mathcal{W}_N$ gravity is given by (2.1) where $\tilde{F}$ satisfies (2.6) and (4.1), and the right hand side of (4.1) will be given explicitly. Just as the non-linear constraint (2.6) had an interesting geometric interpretation, it might be expected that the non-linear form of (4.1) should also be of geometric interest. It is essential that (4.1) should be consistent with the Monge-Ampère constraint (2.6).

In the last section, the action for $w_\infty$ gravity was given in terms of a function $\mathcal{H}(x^\mu, \pi, \xi)$ satisfying (3.4) or a function $H(x^\mu, \pi^\mu)$ satisfying (3.13). It follows from the results of [2-4,8] that these same actions can be used for $\mathcal{W}_N$ gravity provided that the functions $\mathcal{H}$ or $H$ are restricted to be $N$’th order polynomials in $\pi$ or $\pi^\mu$. The canonical first order form of the $\mathcal{W}_N$ gravity action is then given by (3.6) where $\mathcal{H}$ (3.5) satisfies (3.4) and

$$\frac{\partial^{N+1}\mathcal{H}}{\partial \pi^{N+1}} = 0 \quad (4.2)$$

so that expanding the functions $f, \tilde{f}$ gives the action (3.7), but with the summation now running from $n = 2$ to $n = N$ [8] so that there are only a finite number of gauge fields $h_n, \tilde{h}_n$ where $n = 2, 3, \ldots, N$.

Similarly, the covariant first order form of the action is given by (3.15),(3.16) where $H$ satisfies (3.13) and

$$\frac{\partial^{N+1}H}{\partial \pi^{\mu_1}\partial \pi^{\mu_2}...\partial \pi^{\mu_{N+1}}} = 0 \quad (4.3)$$

so that $H$ (3.14) is given by (3.16), with the summation running from $n = 2$ to $n = N$ [2-4]. Again, this leaves a finite set of gauge fields, $h_n, \tilde{h}_n$ where $n = 2, 3, \ldots, N$ for $\mathcal{W}_N$ gravity, in agreement with the linearised analysis.
It is remarkable that the constraints defining $\mathcal{W}_N$ gravity – (3.4),(4.2) or (3.13),(4.3) – are simple linear equations when written in terms of the $\pi$ variables. This can be understood in terms of the relation [26] between the transform from $\tilde{F}(x, y)$ to $H(x^\mu, \xi, \pi)$ or $H(x^\mu, \pi^\mu)$ and the Penrose transform, which translates the condition that a geometry be self-dual into a linear twistor-space condition. The Laplace equations (3.4),(3.13) become the Monge-Ampère equation (2.6) when written in terms of $\tilde{F}$ and it is this equation which characterises $w_\infty$ gravity.

The $\mathcal{W}_N$ condition, which is a complicated non-linear constraint on $\tilde{F}$, becomes the simple linear constraint (4.2) or (4.3) that the transform $H$ or $H$ is an $N$'th order polynomial in $\pi$.

The Constraints on $\tilde{F}$

The equations (3.11),(3.12) give $\tilde{F}$ implicitly in terms of the function $H$ and these can now be used to relate derivatives of $\tilde{F}$ to those of $H$. It will be useful to introduce the notation

$$
H_{\mu_1 \mu_2 \ldots \mu_n} = \frac{\partial^n H}{\partial \pi^{\mu_1} \partial \pi^{\mu_2} \ldots \partial \pi^{\mu_n}}, \quad F^{\mu_1 \mu_2 \ldots \mu_n} = \frac{\partial^n \tilde{F}}{\partial y^{\mu_1} \partial y^{\mu_2} \ldots \partial y^{\mu_n}} (4.4)
$$

and to define the inverse $H^{\mu\nu}$ of the ‘metric’ $H_{\mu\nu}(x, \pi)$, so that $H^{\mu\nu} H_{\nu\rho} = \delta^\mu_\rho$.

Differentiating (3.11) twice with respect to $y$ and using (3.12) and

$$
\frac{\partial \pi^\mu}{\partial y^\nu} = H^{\mu\nu} (4.5)
$$

gives

$$
F^{\mu\nu} = -\tilde{h}^{\mu\nu} + 2 H^{\mu\nu} (4.6)
$$

which can be used to give the ‘metric’ $H_{\mu\nu}$ in terms of $\tilde{F}$ and the metric $\tilde{h}^{\mu\nu}$:

$$
H_{\mu\nu} = 2 \left( \tilde{h}^{\mu\nu} + F^{\mu\nu} \right)^{-1} (4.7)
$$
Further differentiation yields

\[ F^{\mu\nu\rho} = -2H^{\mu\alpha}H^{\nu\beta}H^{\rho\gamma}H_{\alpha\beta\gamma} \] (4.8)

\[ F^{\mu\nu\rho\sigma} = -2H^{\mu\alpha}H^{\nu\beta}H^{\rho\gamma}H^{\sigma\delta}H_{\alpha\beta\gamma\delta} + \frac{3}{2}H_{\alpha\beta}F^{\alpha(\mu\nu F^{\rho\sigma})\beta} \] (4.9)

\[ F^{\mu\nu\rho\sigma\tau} = -2H^{\mu\alpha}H^{\nu\beta}H^{\rho\gamma}H^{\sigma\delta}H^{\tau\epsilon}H_{\alpha\beta\gamma\delta\epsilon} + 5H_{\alpha\beta}F^{\alpha(\mu\nu F^{\rho\sigma\tau})\beta} - \frac{15}{4}H_{\alpha\beta}H_{\gamma\delta}F^{\alpha(\mu\nu F^{\rho\sigma\gamma} F^{\tau})\beta\delta} \] (4.10)

and it is straightforward to extend this to any number of derivatives (see the appendix for details).

Consider first \( \mathcal{W}_3 \) gravity. For \( N = 3 \), the equation (4.3) becomes

\[ H_{\alpha\beta\gamma\delta} = 0 \] (4.11)

and using this (4.9) becomes

\[ F^{\mu\nu\rho\sigma} = \frac{3}{2}H_{\alpha\beta}F^{\alpha(\mu\nu F^{\rho\sigma})\beta} \] (4.12)

or, using (4.7),

\[ F^{\mu\nu\rho\sigma} = 3(\tilde{h}^{\alpha\beta} + F^{\alpha\beta})^{-1}F^{\alpha(\mu\nu F^{\rho\sigma})\beta} \] (4.13)

This is the required extra constraint for \( \mathcal{W}_3 \) gravity. Thus the action for \( \mathcal{W}_3 \) gravity is given by (2.1),(2.2), where \( \tilde{F} \) is a function satisfying the two constraints (2.6) and (4.13).

Similarly, for \( \mathcal{W}_4 \) gravity, \( H_{\alpha\beta\gamma\delta\epsilon} = 0 \) and (4.10) becomes

\[ F^{\mu\nu\rho\sigma\tau} = 5H_{\alpha\beta}F^{\alpha(\mu\nu F^{\rho\sigma\tau})\beta} - \frac{15}{4}H_{\alpha\beta}H_{\gamma\delta}F^{\alpha(\mu\nu F^{\rho\sigma\gamma} F^{\tau})\beta\delta} \] (4.14)

so that the \( \mathcal{W}_4 \) action is (2.1) where \( \tilde{F} \) satisfies (2.6) and (4.14), and \( H^{\mu\nu} \) is given in terms of \( \tilde{F} \) by (4.7). Similar results hold for all \( N \). In each case, taking
the transform of the linear constraint (4.3) yields an equation of the form (4.1), where the right hand side is constructed from the \( n \)th order derivatives \( F^{\mu_1 \ldots \mu_n} \) for \( 2 < n \leq N \) and from \( H_{\mu \nu} \).

Expanding \( \tilde{F} \) in \( \partial_\mu \phi \) (2.2) gives the coefficient of the \( n \)-th order \( \partial_\mu_1 \phi \ldots \partial_\mu_n \phi \) interaction, which is proportional to \( \tilde{g}^{(n)}_{(n)} \). The constraint (4.1) implies that for \( n > N \), the coefficient \( \tilde{g}(n) \) of the \( n \)-th order interaction can be written in terms of the coefficients \( \tilde{g}(m) \) of the \( m \)-th order interactions for \( 2 \leq m \leq N \). For \( \mathcal{W}_3 \), the \( n \)-point vertex can be written in terms of 3-point vertices for \( n > 3 \), so that

\[
\tilde{g}^{\mu \nu \rho \sigma}_{(4)} = 2 \left( h^{\alpha \beta} + \tilde{g}^{\alpha \beta}_{(2)} \right)^{-1} \tilde{g}^{\alpha (\mu \nu} \tilde{g}^{\rho \sigma)}_{(3)}
\]

(4.15)

\[
\tilde{g}^{\mu \nu \rho \sigma \tau}_{(5)} = 5 \left( h^{\alpha \beta} + \tilde{g}^{\alpha \beta}_{(2)} \right)^{-1} \left( h^{\gamma \delta} + \tilde{g}^{\gamma \delta}_{(2)} \right)^{-1} \tilde{g}^{\alpha (\mu \nu | \gamma} \tilde{g}^{\rho \sigma \tau)}_{(3) 3} \tilde{g}^{\beta \delta}_{(3)}
\]

(4.16)

etc, while for \( \mathcal{W}_4 \), all vertices can be written in terms of 3- and 4-point vertices, e.g.

\[
\tilde{g}^{\mu \nu \rho \sigma \tau}_{(5)} = 5 \left( h^{\alpha \beta} + \tilde{g}^{\alpha \beta}_{(2)} \right)^{-1} \tilde{g}^{\alpha (\mu \nu} \tilde{g}^{\rho \sigma \tau)}_{(3) 4} \\
- 5 \left( h^{\alpha \beta} + \tilde{g}^{\alpha \beta}_{(2)} \right)^{-1} \left( h^{\gamma \delta} + \tilde{g}^{\gamma \delta}_{(2)} \right)^{-1} \tilde{g}^{\alpha (\mu \nu | \gamma} \tilde{g}^{\rho \sigma \tau)}_{(3) 3} \tilde{g}^{\beta \delta}_{(3)}
\]

(4.17)

These ‘factorisations’ can be illustrated in Feynman-style diagrams. (4.15) is depicted in fig. 1, where the ‘propagators’ represent contraction of indices using the metric \( H_{\mu \nu} \). Similarly, (4.16) and (4.17) are depicted in figs. 2 and 3 respectively, where the ‘summation over channels’ is not shown explicitly.

**The Constraints on \( \Lambda \)**

From the linearised analysis, it is expected that the \( \mathcal{W}_N \) gravity action should be invariant under transformations under which

\[
\delta \phi = \Lambda(x, \partial \phi)
\]

(4.18)

where \( \Lambda(x, y) \) is of the form (2.9) and satisfies constraints whose linearised forms
are
\[ \frac{\partial^2 \Lambda}{\partial y_\mu \partial y_\nu} = 0 + \ldots \] (4.19)
and
\[ \frac{\partial^N \Lambda}{\partial y_\mu_1 \partial y_\mu_2 \ldots \partial y_\mu_N} = 0 + \ldots \] (4.20)

The full non-linear form of the constraint (4.19) is given by (2.10), while the non-linear form of (4.20) will give the parameters \( \lambda_{(n)}^{\mu_1 \mu_2 \ldots \mu_{n-1}} \) for \( n > N \) in terms of the parameters \( \lambda_{(m)}^{\mu_1 \mu_2 \ldots \mu_{m-1}} \) for \( m \leq N \) and the gauge fields, so that the number of independent symmetries is the same as in the linearised theory.

The full non-linear form of these constraints will now be found by transforming the corresponding constraints in the covariant first order form of the theory. The covariant first order form of the \( w_\infty \) action, given by (3.15),(3.16) where \( H \) satisfies (3.13), is invariant under transformations given explicitly in [3] and which include (3.17) where \( \Lambda(x, \pi) \) satisfies the constraint (3.18). It follows from the results of [2-4] that the truncation to \( \mathcal{W}_N \) gravity is obtained by imposing the constraint (4.3), so that \( H \) is an \( N \)'th order polynomial in \( \pi \), together with the constraint
\[ \frac{\partial^N \Lambda}{\partial \pi_\mu_1 \partial \pi_\mu_2 \ldots \partial \pi_\mu_N} = 0 \] (4.21)
on \( \Lambda(x, \pi) \), so that \( \Lambda(x, \pi) \) is an \((N-1)\)'th order polynomial in \( \pi \).

In addition, the constraints on the gauge fields given by (4.3) or (4.13),(4.14) etc are not invariant under the \( \Lambda \) transformations, but they become invariant if the \( \Lambda \) transformations are supplemented by compensating ‘Stuckelberg’ transformations, as in [3].

Now, using the chain rule and (4.5), it is straightforward to express derivatives of \( \Lambda(x, \pi) \) with respect to \( \pi \) in terms of derivatives of \( \Lambda(x, y) \) with respect to \( y \).
For example,

\[
\frac{\partial \Lambda}{\partial \pi^\mu} = \frac{\partial \Lambda}{\partial y_\alpha} H_{\alpha \mu} \\
\frac{\partial^2 \Lambda}{\partial \pi^\mu \partial \pi^\nu} = \frac{\partial^2 \Lambda}{\partial y_\alpha \partial y_\beta} H_{\alpha \mu} H_{\beta \nu} + \frac{\partial \Lambda}{\partial y_\alpha} H_{\alpha \mu \nu} \\
\frac{\partial^3 \Lambda}{\partial \pi^\mu \partial \pi^\nu \partial \pi^\rho} = \frac{\partial^3 \Lambda}{\partial y_\alpha \partial y_\beta \partial y_\gamma} H_{\alpha \mu} H_{\beta \nu} H_{\gamma \rho} \\
+ 3 \frac{\partial^2 \Lambda}{\partial y_\alpha \partial y_\beta} H_{\alpha (\mu \nu} H_{\rho)\beta} + \frac{\partial \Lambda}{\partial y_\alpha} H_{\alpha \mu \nu \rho} \\
+ 6 \frac{\partial \Lambda}{\partial y_\alpha \partial y_\beta} H_{\alpha (\mu \nu \rho} H_{\beta)\sigma} H_{\gamma \sigma} \\
+ \frac{\partial^2 \Lambda}{\partial y_\alpha \partial y_\beta} [3 H_{\alpha (\mu \nu \rho} H_{\sigma)\beta} + 4 H_{\alpha (\mu \nu \rho} H_{\sigma)\beta}]
\] (4.22)

Consider first the case of $W_3$ gravity. In this case, $H_{\mu \nu \rho \sigma} = 0$ and the constraint on $\Lambda(x, \pi)$ given by

\[
\frac{\partial^3 \Lambda}{\partial \pi^\mu \partial \pi^\nu \partial \pi^\rho} = 0 \\
(4.23)
\]

can be rewritten in terms of $\Lambda(x, y)$ and $\tilde{F}(x, y)$ using (4.22) as

\[
\frac{\partial^3 \Lambda}{\partial y_\alpha \partial y_\beta \partial y_\gamma} = \frac{3}{2} \frac{\partial^2 \Lambda}{\partial y_\mu \partial y_\rho} \delta_\gamma (\gamma \Gamma^{\alpha \beta})^\nu H_{\mu \nu} \\
(4.24)
\]

where $H_{\mu \nu} = 2(\tilde{h}^{\mu \nu} + \tilde{F}^{\mu \nu})^{-1}$ as before. This constraint gives all of the parameters $\lambda_n$ for $n = 4, 5, 6, \ldots$ in terms of the gauge fields, the diffeomorphism parameter $\lambda^{\mu}_{(2)}$ and the spin-three parameter $\lambda^{\mu}_{(3)}$ which satisfies the tracelessness constraint $(\delta^{\mu}_{(2)})^{-1} \lambda^{\mu}_{(3)} = 0$. For example, for spin-four the constraint implies that

\[
\lambda^{\mu \nu \rho}_{(4)} = 2(\tilde{h}^{\alpha \beta} + \tilde{g}^{\alpha \beta}_{(2)})^{-1} \lambda^{\alpha (\mu \nu \rho \beta)}_{(3)} \\
(4.25)
\]

The $W_3$ constraint (4.13) gives a sequence of constraints on the gauge fields, the first two of which are given by (4.15) and (4.16). These constraints are not
invariant under the gauge field transformations, which are now given by (2.8), but
with all the parameters for $\lambda_{(s)}^{\mu\nu\cdots}$ for $s > 3$ given in terms of the $s = 2$ and $s = 3$
parameters. They do become invariant, however, if the spin two and spin three
transformations are supplemented by a compensating Stuckelberg transformation
of the form given by (2.13) with $p = 4, q = 2$. The transformations of $\tilde{g}^{\mu\nu\rho}_{(3)}$ are
unmodified, but the spin-three transformation of $\tilde{g}^{\mu\nu}_{(2)}$, which previously was zero,
now becomes modified to

$$\delta \tilde{g}^{\mu\nu}_{(2)} = \frac{1}{2} A^{(\mu\nu\rho\sigma)\gamma} y_{\rho} y_{\sigma}$$

(4.26)

where

$$A^{\mu\nu\rho\sigma} = \frac{2}{3} \lambda^{\rho\sigma}_{(3)} \partial_{\alpha} \tilde{g}^{\rho\sigma}_{\alpha} (3) - \frac{10}{3} \tilde{g}^{\rho\sigma}_{\alpha} \partial_{\alpha} \lambda^{\rho\sigma}_{(3)} + \frac{4}{3} \tilde{g}^{\mu\nu\rho}_{(3)} \partial_{\alpha} \lambda^{\rho\sigma}_{\alpha} + \frac{4}{3} \lambda^{\mu\nu}_{(3)} \partial_{\alpha} \tilde{g}^{\rho\sigma}_{\alpha}$$

$$- 2 G_{\alpha\beta} \tilde{g}^{\gamma\mu}_{(2)} \tilde{g}^{\rho\sigma}_{\alpha} \partial_{\gamma} \lambda^{\beta}_{(3)} - 2 G_{\alpha\beta} \tilde{g}^{\gamma\mu}_{(2)} \lambda^{\alpha\nu}_{(3)} \partial_{\gamma} \tilde{g}^{\rho\sigma}_{\alpha} - \frac{4}{3} \lambda^{\alpha\nu}_{\gamma\beta} \partial_{\gamma} \tilde{g}^{\rho\sigma}_{\alpha} \partial_{\gamma} \lambda^{\rho\sigma}_{(3)}$$

$$- \frac{2}{3} G_{\alpha\beta} \tilde{g}^{\gamma\mu\rho}_{(2)} \partial_{\alpha} \lambda^{\beta}_{(3)} + \frac{1}{3} G_{\gamma\delta} \tilde{g}^{\rho\sigma}_{(2)} \partial_{\alpha} \left( \tilde{g}^{\mu\nu\delta}_{(3)} \lambda^{\alpha}_{(3)} \right)$$

$$+ \frac{2}{3} G_{\gamma\delta} \tilde{g}^{\rho\sigma}_{(2)} \partial_{\alpha} \left( \tilde{g}^{\gamma\mu\rho}_{(3)} \lambda^{\gamma}_{(3)} \right) - \frac{2}{3} G_{\alpha\beta} \tilde{g}^{\gamma\mu}_{(3)} \partial_{\alpha} \left( \tilde{g}^{\rho\sigma}_{\gamma} g^{\rho\sigma}_{(2)} \partial_{\gamma} \lambda^{\alpha}_{(3)} \right)$$

$$+ 2 G_{\gamma\delta} \lambda^{\gamma\mu}_{(3)} \tilde{g}^{\rho\sigma}_{(2)} \partial_{\alpha} \tilde{g}^{\sigma\rho}_{(3)} + \frac{2}{3} G_{\gamma\delta} \left[ \lambda^{\gamma\mu}_{(3)} \tilde{g}^{\rho\sigma}_{(3)} + 2 \lambda^{\gamma\mu}_{(3)} \tilde{g}^{\rho\sigma}_{(2)} \right] \partial_{\alpha} \tilde{g}^{\rho\sigma}_{(2)}$$

(4.27)

Here

$$G_{\mu\nu} = H_{\mu\nu} \big|_{y=0} = 2 \left( \tilde{h}^{\mu\nu} + \tilde{g}^{\mu\nu}_{(3)} \right)^{-1}$$

(4.28)

This unpleasant form of the transformations simplifies dramatically if we choose
a general $\tilde{h}^{\mu\nu}$, and absorb $h_2, \bar{h}_2$ into field redefinitions, so that $\tilde{h}^{\mu\nu} = \tilde{g}^{\mu\nu}_{(2)}$ and
$G_{\mu\nu} = (\tilde{g}^{\mu\nu})^{-1}$. Then the frame components $A^{abcd}$ of (4.27) are given by

$$A^{++++} = 2 \lambda^{++}_{(3)} \nabla + \tilde{g}^{++}_{(3)} - 2 \tilde{g}^{+++}_{(3)} \nabla + \lambda^{++}_{(3)}$$

$$A^{----} = 2 \lambda^{--}_{(3)} \nabla - \tilde{g}^{--}_{(3)} - 2 \tilde{g}^{---}_{(3)} \nabla - \lambda^{--}_{(3)}$$

$$A^{+++} = A^{++-} = A^{+-+} = A^{--+} = 0$$

(4.29)

In the chiral limit $\tilde{g}^{----}_{(3)} = 0$, the transformation rules of [1] are recovered.
Similarly, for \( \mathcal{W}_4 \) gravity, the constraint \( \frac{\partial^4 \Lambda}{\partial \pi^\mu \partial \pi^\nu \partial \pi^\rho \partial \pi^\sigma} = 0 \) leads to the constraint
\[
\Lambda^\alpha{}^\beta{}^\gamma{}^\delta = 3H_{\mu\nu}F^\mu(\alpha\beta\Lambda^\gamma{}^\delta)\nu - \frac{3}{4}H_{\mu\rho}H_{\nu\sigma}F^\mu(\alpha\beta F^\gamma{}^\delta)\nu \Lambda^\rho\sigma
\]
\[
+ 2H_{\mu\nu}F^\mu(\alpha\beta\gamma\Lambda^\delta)\nu - 3H_{\mu\nu}H_{\rho\sigma}F^{\pi\rho}(\alpha F^{\beta\gamma}|\sigma|\Lambda^\delta)\mu
\]
where
\[
\Lambda^{\mu_1\mu_2...\mu_n} \equiv \frac{\partial^n \Lambda}{\partial y_{\mu_1} \partial y_{\mu_2}...\partial y_{\mu_n}}
\]
(4.31)
The only independent parameters are \( \lambda^{\mu}_{(2)} \), \( \lambda^{\mu\nu}_{(3)} \) and \( \lambda^{\mu\nu\rho}_{(4)} \), and these are subject to the constraints
\[
\tilde{g}_{(2)\mu\nu}\lambda^{\mu\nu}_{(3)} = 0
\]
\[
\tilde{g}_{(2)\mu\nu}\lambda^{\mu\nu\rho}_{(4)} = \frac{2}{3}\tilde{g}^{\mu\nu\rho}_{(3)}\Lambda(3)\tilde{g}_{(2)\mu\sigma}\tilde{g}_{(2)\nu\tau}
\]
(4.32)
where \( \tilde{g}_{(2)\mu\nu} \equiv (\tilde{g}^{\mu\nu}_{(2)})^{-1} \). As in the \( \mathcal{W}_3 \) case, the transformations of the gauge fields are again modified by compensating Stuckelberg-type transformations.

The Geometry of the Constraints

To attempt a geometric formulation of these results, note that while the second derivative of \( \tilde{F} \) defines a metric, the fourth derivative is related to a curvature, and the \( n \)'th derivative is related to the \( (n - 4) \)'th covariant derivative of the curvature. The \( \mathcal{W}_3 \) constraint (4.13) can then be written as a constraint on the curvature, while the \( \mathcal{W}_N \) constraint (4.1) becomes a constraint on the \( (N - 3) \)'th covariant derivative of the curvature. One approach, motivated by that of section 3, is to introduce a second Kähler metric \( \tilde{K}_x \) on \( \mathbb{R}^4 \) (for each \( x^\mu \in M \)) given in terms of the potential \( K_x \) introduced in (3.1) by
\[
\tilde{K}_x = K_x + \tilde{h}^{\alpha\beta}\zeta_\alpha \tilde{\zeta}_\beta
\]
(4.33)
The corresponding metric is given by
\[
\tilde{G}^{\mu\nu} = \tilde{h}^{\mu\nu} + G^{\mu\nu}
\]
(4.34)
Then if \( \tilde{F} \) satisfies the \( \mathcal{W}_3 \) constraint (4.13), the curvature tensor for the metric
\[
\hat{R}^{\mu\nu\rho\bar{\sigma}} = \frac{1}{2} \hat{G}_{\alpha\bar{\beta}} \left[ T^{\alpha\mu\bar{\nu}} T^{\bar{\beta}\bar{\sigma}\rho} + T^{\alpha\mu\bar{\sigma}} T^{\bar{\beta}\bar{\nu}\rho} + T^{\bar{\beta}\bar{\nu}\mu} T^{\alpha\rho\bar{\sigma}} + T^{\bar{\beta}\bar{\sigma}\mu} T^{\alpha\rho\bar{\nu}} \right]
\]  \quad (4.35)

where

\[
T^{\mu\nu\bar{\rho}} = \frac{\partial^3 \hat{K}}{\partial \zeta_\mu \partial \zeta_\nu \partial \bar{\zeta}_\bar{\rho}}, \quad T^{\bar{\mu}\bar{\nu}\rho} = \frac{\partial^3 \hat{K}}{\partial \bar{\zeta}_{\bar{\mu}} \partial \bar{\zeta}_{\bar{\nu}} \partial \zeta_\rho}
\]  \quad (4.36)

This is similar to, but distinct from, the constraint of special geometry [27]. Note that (4.35) is not a covariant equation as the definitions (4.36) are only valid in the special coordinate system that occurs naturally in $\mathcal{W}$-gravity. However, tensor fields $T^{\mu\nu\bar{\rho}}, T^{\bar{\mu}\bar{\nu}\rho}$ can be defined by requiring them to be given by (4.36) in the special coordinate system and to transform covariantly, in which case the equation (4.35) becomes covariant, as in the case of special geometry [27]. For $\mathcal{W}_N$, this generalises to give a constraint on the $(N - 3)$'th covariant derivative of the curvature, which is given in terms of tensors that can each be written in terms of some higher order derivatives of the Kähler potential in the special coordinate system.

For each $x^\mu$, the solutions to the constraints for $\mathcal{W}_N$ gravity are parameterised by the $2(N - 1)$ variables $h_n, \bar{h}_n$ for $2 \leq n \leq N$ which are then the coordinates for the $2(N - 1)$ dimensional moduli space for the self-dual geometry satisfying the $\mathcal{W}_N$ constraint. For the $x$-dependent family of solutions, the moduli become the fields $h_n(x), \bar{h}_n(x)$ on the world-sheet.

Further properties and generalisations of these actions will be given elsewhere.
APPENDIX

The background ‘metric’ \( \tilde{h}_{\mu\nu} \), which satisfies \( \det(\tilde{h}_{\mu\nu}) = \epsilon \), can be written in terms of an unconstrained ‘metric’ \( h_{\mu\nu} \) as

\[
\tilde{h}^{\mu\nu} = \sqrt{h} h^{\mu\nu} \tag{A.1}
\]

where \( h = \epsilon \det(h_{\mu\nu}) \). Then \( \tilde{F} \) can be written as

\[
\tilde{F} = \sqrt{h} \left( 2\tilde{\pi}^\mu y_\mu - \frac{1}{2} h^{\mu\nu} y_\mu y_\nu - 2\mathcal{H} \right) \tag{A.2}
\]

where

\[
\tilde{\pi}^\mu = h^{-1/2} \pi^\mu, \quad \mathcal{H} = h^{-1/2} H \tag{A.3}
\]

It is useful to introduce a zweibein \( e_\mu^a (a = 1, 2) \) (with inverse \( e^a_\mu \), and \( \epsilon = \det(e_\mu^a) \)) such that

\[
h^{\mu\nu} = \eta^{ab} e_\mu^a e^\nu_b = 2e^{(\mu}_+ e^{\nu)}_- \tag{A.4}
\]

where \( \eta^{ab} \) is the flat metric given by \( \text{diag}(\epsilon, 1) \), and

\[
e^{\pm}_\mu = \frac{1}{\sqrt{2}} \left( e^{\mu}_1 \pm \sqrt{-\epsilon} e^{\mu}_2 \right) \tag{A.5}
\]

We define flat null coordinates \( \pi^a = e_\mu^a \tilde{\pi}_\mu = \epsilon e_\mu^a \pi^\mu \), so that \( \pi^\pm \) are independent real coordinates if the signature is Lorentzian (\( \epsilon = -1 \)), and are complex conjugate coordinates (\( \pi^+ = (\pi^-)^* \)) if the signature is Euclidean (\( \epsilon = 1 \)). In either case, the flat metric in the \( \pi^\pm \) coordinate system is

\[
\eta_{ab} = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \tag{A.6}
\]

The Lagrangian \( \tilde{F}(x, y) \) is given by a transform of a function \( H(x, \pi) \). For
general $H(x, \pi)$, the second derivative

\[ H_{\mu\nu} \equiv \frac{\partial^2 H}{\partial \pi^\mu \partial \pi^\nu} = e^{-1}H_{\mu\nu}, \quad H_{\mu\nu} \equiv \frac{\partial^2 H}{\partial \tilde{\pi}^\mu \partial \tilde{\pi}^\nu} \quad \text{(A.7)} \]

can be written in terms of $H_{ab}$

\[ H_{\mu\nu} = e^{-1}H_{ab}e_\mu^a e_\nu^b, \quad H_{ab} \equiv \frac{\partial^2 H}{\partial \pi^a \partial \pi^b} = \begin{pmatrix} h & a \\ a & \bar{h} \end{pmatrix} \quad \text{(A.8)} \]

for some $a(x, \pi), h(x, \pi), \bar{h}(x, \pi)$.

The constraint (3.13) becomes

\[ \frac{1}{2} \eta^{ab}H_{ab} = \mathcal{H}_{+-} = 1 \quad \text{(A.9)} \]

and the general solution of this is

\[ H = eH = e \left[ \pi^+ \pi^- + f(\pi^+) + \bar{f}(\pi^-) \right] \quad \text{(A.10)} \]

where dependence on $x^\mu$ has been suppressed. Differentiating twice with respect to $\pi$ gives $H_{ab}$ which, in the $\pi^\pm$ frame, takes the form

\[ H_{ab} = \begin{pmatrix} h & 1 \\ 1 & \bar{h} \end{pmatrix} \quad \text{(A.11)} \]

where

\[ h = \frac{\partial^2 f}{\partial (\pi^+)^2}, \quad \bar{h} = \frac{\partial^2 \bar{f}}{\partial (\pi^-)^2} \quad \text{(A.12)} \]

and $a = 1$. It will be convenient to define

\[ \Delta = 1 - h\bar{h} \quad \text{(A.13)} \]
In the $\pi^\pm$ frame, the determinant of (A.11) is

$$det_{\pm}(H_{ab}) = -\Delta = -1 + \hbar \bar{h} \quad (A.14)$$

while in the $\pi^1, \pi^2$ frame the determinant is

$$det(H_{ab}) = -\epsilon det_{\pm}(H_{ab}) = \epsilon \Delta \quad (A.15)$$

(The sign changes result from the fact that the Jacobian for the change of coordinates from $\pi^1, \pi^2$ to $\pi^\pm$ is $-\sqrt{-\epsilon}$.) The inverse of (A.11) is

$$H_{ab} = e^{-1}e^{a}_{\mu}e^{b}_{\nu}H^{\mu\nu} = \frac{1}{1-h\bar{h}} \begin{pmatrix} -\hbar & 1 \\ 1 & -\bar{h} \end{pmatrix}, \quad H^{\mu\nu} \equiv (H_{\mu\nu})^{-1} \quad (A.16)$$

From (4.6),(A.6),(A.11), the second derivative of $\tilde{F}$ is

$$F^{\mu\nu} = ee^{\mu}_{a}e^{\nu}_{b}F^{ab} \quad (A.17)$$

where

$$F^{ab} = -\eta^{ab} + 2H^{ab} = \frac{1}{1-h\bar{h}} \begin{pmatrix} -2\hbar & 1+h\bar{h} \\ 1+h\bar{h} & -2\hbar \end{pmatrix} \quad (A.18)$$

and the inverse of this matrix is

$$\left(F^{ab}\right)^{-1} = \frac{1}{1-h\bar{h}} \begin{pmatrix} 2\hbar & 1+h\bar{h} \\ 1+h\bar{h} & 2\bar{h} \end{pmatrix} \quad (A.19)$$

The determinant of $F^{ab}$ is $-1$ in the $\pi^\pm$ coordinate system and so is $\epsilon$ in the $\pi^1, \pi^2$ coordinate system. Using (A.17), this implies that (2.6) is indeed satisfied. Now,
using (A.16),(A.6),

\[ \mathcal{H}^{ac} \eta_{cd} \mathcal{H}^{db} = \frac{1}{(1 - h \bar{h})^2} \begin{pmatrix} -2 \bar{h} & 1 + h \bar{h} \\ 1 + h \bar{h} & -2h \end{pmatrix} \]  \hspace{1cm} (A.20)

so that (A.18) gives

\[ F_{ab} = \Delta \mathcal{H}^{ac} \eta_{cd} \mathcal{H}^{db} \]  \hspace{1cm} (A.21)

and

\[ (F_{ab})^{-1} = \Delta^{-1} \mathcal{H}_{ac} \eta^{cd} \mathcal{H}_{db} \]  \hspace{1cm} (A.22)

The variation of \( \phi \) is given by a function \( \Lambda(x, \pi^\mu) \) by (3.17), where \( \Lambda \) satisfies the constraint (3.18). This can be rewritten as

\[ \eta^{ab} \frac{\partial^2 \Lambda}{\partial \pi^a \partial \pi^b} = 0 \]  \hspace{1cm} (A.23)

Now the second relation in (4.22) can be written as

\[ \frac{\partial^2 \Lambda}{\partial \pi^a \partial \pi^b} = \frac{\partial^2 \Lambda}{\partial y^c \partial y^d} \mathcal{H}_{ac} \mathcal{H}_{bd} + \frac{\partial \Lambda}{\partial y^c} \frac{\partial^3 \mathcal{H}}{\partial \pi^a \partial \pi^b \partial \pi^c} \]  \hspace{1cm} (A.24)

\((y^a = \epsilon^a_{\mu} y_\mu)\) and taking the trace gives

\[ \eta^{ab} \frac{\partial^2 \Lambda}{\partial \pi^a \partial \pi^b} = \eta^{ab} \frac{\partial^2 \Lambda}{\partial y^c \partial y^d} \mathcal{H}_{ac} \mathcal{H}_{bd} \]  \hspace{1cm} (A.25)

using (A.9). Then the constraint (A.23) becomes

\[ (F_{ab})^{-1} \frac{\partial^2 \Lambda}{\partial y^a \partial y^b} = 0 \]  \hspace{1cm} (A.26)

using (A.22), and this is equivalent to the parameter constraint (2.10).
We now turn to the identities satisfied by the derivatives of $\tilde{F}$. From

$$\tilde{F}(x^\mu, y^\nu) = 2\pi^\mu y_\mu - \frac{1}{2}\tilde{h}^{\mu\nu} y_\mu y_\nu - 2H(x, \pi) \quad (A.27)$$

and

$$y_\mu = \frac{\partial H}{\partial \pi^\mu} \quad (A.28)$$

it follows that

$$F^\mu \equiv \frac{\partial \tilde{F}}{\partial y_\mu} = 2\pi^\mu - \tilde{h}^{\mu\nu} y_\nu \quad (A.29)$$

$$F^{\mu\nu} \equiv \frac{\partial^2 \tilde{F}}{\partial y_\mu \partial y_\nu} = 2\frac{\partial \pi^\mu}{\partial y_\nu} - \tilde{h}^{\mu\nu} \quad (A.30)$$

$$F^{\mu\nu\rho} \equiv \frac{\partial^3 \tilde{F}}{\partial y_\mu \partial y_\nu \partial y_\rho} = 2\frac{\partial^2 \pi^\mu}{\partial y_\nu \partial y_\rho} \quad (A.31)$$

$$F^{\mu_1...\mu_n} \equiv \frac{\partial^n \tilde{F}}{\partial y_{\mu_1}\ldots\partial y_{\mu_n}} = 2\frac{\partial^{n-1}\pi^{\mu_1}}{\partial y_{\mu_2}\ldots\partial y_{\mu_{n-1}}} \quad (A.32)$$

Then differentiating (A.28) gives

$$\frac{\partial y_\mu}{\partial \pi^\nu} = \frac{\partial^2 H}{\partial \pi^\mu \partial \pi^\nu} \equiv H_{\mu\nu} \quad (A.33)$$

and hence

$$\frac{\partial \pi^\mu}{\partial y_\nu} = (H_{\mu\nu})^{-1} \equiv H^{\mu\nu} \quad (A.34)$$

Differentiating this gives

$$\frac{\partial \pi^\mu}{\partial y_\nu \partial y_\rho} = \frac{\partial}{\partial \pi^\sigma} (H_{\mu\nu})^{-1} \frac{\partial \pi^\sigma}{\partial y_\rho}$$

$$= -H^{\mu\alpha} H^{\nu\beta} H^{\rho\gamma} \frac{\partial^3 H}{\partial \pi^\alpha \partial \pi^\beta \partial \pi^\gamma} \quad (A.35)$$
Substituting this in (A.31) gives (4.8). Differentiating (A.35) gives

\[
\frac{\partial \pi^\mu}{\partial y_\nu \partial y_\rho \partial y_\sigma} = - \frac{\partial}{\partial \pi^\tau} \left( H^{\mu\alpha} H^{\nu\beta} H^{\rho\gamma} \frac{\partial^3 H}{\partial \pi^\alpha \partial \pi^\beta \partial \pi^\gamma} \right) \frac{\partial \pi^\tau}{\partial y_\sigma}
\]

\[
= -H^{\mu\alpha} H^{\nu\beta} H^{\rho\gamma} H^{\sigma\delta} H_{\alpha\beta\gamma\delta} + 3H_{\kappa(\alpha\beta} H^{\gamma\delta)\lambda} H^{\kappa\lambda} H^{\mu\alpha} H^{\nu\beta} H^{\rho\gamma} H^{\sigma\delta}
\]

(A.36)

and this leads to (4.9). It is straightforward to generalise these relations to higher derivatives, and also to represent the results graphically in figures similar to figs. 1,2,3 in which \( H_{\mu_1 \mu_2 \ldots \mu_n} \) is represented as an \( n \)-point vertex and \( H^{\mu\nu} \) is represented as a propagator.

REFERENCES

1. C.M. Hull, Phys. Lett. 240B (1990) 110.

2. K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Phys. Lett. 243B (1990) 245.

3. E. Bergshoeff, C.N. Pope, L.J. Romans, E. Sezgin, X. Shen and K.S. Stelle, Phys. Lett. 243B (1990) 350.

4. C.M. Hull, Nucl. Phys. 353 B (1991) 707.

5. C.M. Hull, Phys. Lett. 259B (1991) 68 and Nucl. Phys. B364 (1991) 621; A. Miković, Phys. Lett. 260B (1991) 75.

6. K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Nucl. Phys. B349 (1991) 791 and Phys. Lett. 251B (1990) 355; E. Bergshoeff, C.N. Pope and K.S. Stelle, Phys. Lett. 249B (1990) 208.

7. C.M. Hull, in Strings and Symmetries 1991, ed. by N. Berkovits et al, World Scientific, Singapore, 1992; Lectures on W-Gravity, W-Geometry and W-Strings, Trieste Summer School Lectures 1992, to be published by World Scientific, Singapore.

8. A. Miković, Phys. Lett. 278B (1991) 51.
9. P. Bouwknegt and K. Schoutens, CERN preprint CERN-TH.6583/92, to appear in Physics Reports.

10. C.N. Pope, L.J. Romans and K.S. Stelle, Phys. Lett. **268B** (1991) 167 and **269B** (1991) 287.

11. E. Witten, in ‘Proceedings of the Texas A & M Superstring Workshop, 1990’, ed. by R. Arnowitt et al, World Scientific Publishing, Singapore, 1991.

12. A. Bilal, Phys. Lett. **249B** (1990) 56; A. Bilal, V.V. Fock and I.I. Kogan, Nucl. Phys. **B359** (1991) 635.

13. G. Sotkov and M. Stanishkov, Nucl. Phys. **B356** (1991) 439; G. Sotkov, M. Stanishkov and C.J. Zhu, Nucl. Phys. **B356** (1991) 245.

14. M. Bershadsky and H. Ooguri, Commun. Math. Phys. **126** (1989) 49.

15. P. Di Francesco, C. Itzykson and J.B. Zuber, Commun. Math. Phys. **140** (1991) 543.

16. J.M. Figueroa-O’Farrill, S. Stanciu and E. Ramos, Leuven preprint KUL-TF-92-34.

17. J.-L. Gervais and Y. Matsuo, Phys. Lett. **B274** 309 (1992) and Ecole Normale preprint LPTENS-91-351 (1991); Y. Matsuo, Phys. Lett. **B277** 95 (1992)

18. C.M. Hull, Phys. Lett. **269B** (1991) 257.

19. C.M. Hull, $\mathcal{W}$-Geometry, QMW preprint, QMW-92-6 (1992), hep-th/9211113, to be published in Commun. Math. Phys..

20. C.M. Hull, Geometry and $\mathcal{W}$-Gravity, QMW preprint, QMW-92-21 (1992), hep-th/9301074, to be published in Proceedings of the 16th John Hopkins Workshop on Current Problems in Particle Theory, Gothenborg, 1992.

21. T. Aubin, *Non-Linear Analysis on Manifolds. Monge-Ampère Equations*, Springer Verlag, New York, 1982.

22. J.F. Plebanski, J. Math. Phys. **16** (1975) 2395.
23. I. Bakas, Phys. Lett. 228B (1989) 57 and Maryland preprint MdDP-PP-90-033.

24. Q-Han Park, Phys. Lett. 236B (1990) 429, 238B (1990) 287 and 257B (1991) 105.

25. A.B. Zamolodchikov, Teor. Mat. Fiz. 65 (1985) 1205; V.A. Fateev and S. Lykanov, Intern. J. Mod. Phys. A3 (1988) 507; A. Bilal and J.-L. Gervais, Phys. Lett. 206B (1988) 412; Nucl. Phys. B314 (1989) 646; Nucl. Phys. B318 (1989) 579.

26. U. Lindstrom and M. Roček, Nucl. Phys. B222 (1983) 285; N. Hitchin, A. Karlhede, U. Lindstrom and M. Roček, Comm. Math. Phys. 108 (1987) 535.

27. S.J. Gates, Nucl. Phys. B238 (1984) 349; M. Gunaydin, G. Sierra and P.K. Townsend, Nucl. Phys. 242 (1984) 244; B. de Wit, P.G. Lauwers and A. van Proeyen, Nucl. Phys. B255 (1985) 569; A. Strominger, Commun. Math. Phys. 133 (1990) 163; L. Castellani, R. D’Auria and S. Ferrara, Phys. Lett. 241 (1990) 57 and Class. and Quantum Grav. 7 (1990) 1767.

28. C.M. Hull, in preparation.
FIGURE CAPTIONS

1) The four-point interaction in $\mathcal{W}_3$ gravity can be written in terms of three-point interactions. The diagram represents this factorisation, with the symmetrization over the indices corresponding to the ‘sum over channels’.

2) This diagram represents the factorisation of the five-point interaction into three-point interactions in $\mathcal{W}_3$ gravity, with the summation over channels suppressed.

3) This diagram represents the factorisation of the five-point interaction into three-point and four-point interactions in $\mathcal{W}_4$ gravity, with the summation over channels suppressed.
Fig. 1

Fig. 2

Fig. 3