Center conditions for a simple class of quintic systems

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Abstract

We obtained center conditions for a $O$-symmetric system of degree 5 for which the origin is a uniformly isochronous singular point.

Classification: Primary 34C05; Secondary 34C25
Keywords: conditions for a center; isochronicity; commutativity

1. Let us consider a planar differential system

\begin{align*}
\dot{x} &= y + xR_{n-1}(x, y), \\
\dot{y} &= -x + yR_{n-1}(x, y),
\end{align*}

where $R_{n-1}(x, y)$ is a polynomial in $x, y$ of degree $n - 1$.

System (1) has a unique singular point $O(0, 0)$ whose linear part of center type.

Orbits of system (1) move around the origin with a constant angular velocity and the origin is a uniformly isochronous singular point.

In [1] the following problem was proposed:

Problem 19.1. Identify systems (1) of odd degree which are $O$-symmetric (not necessarily quasi-homogeneous) having $O$ as a (uniformly isochronous) center.

We solve this problem for $n = 5$ and derive necessary and sufficient center conditions for the system

\begin{align*}
\dot{x} &= y + x(ax^2 + bxy + cy^2 + dx^4 + ex^3y + fxy^3 + gxy^3 + hy^4), \\
\dot{y} &= -x + y(ax^2 + bxy + cy^2 + dx^4 + ex^3y + fxy^3 + gxy^3 + hy^4),
\end{align*}

where $a, b, c, d, e, f, g, h \in \mathbb{R}$.

*The work is supported by Russian Foundation for Basic Research*
Theorem. The origin is a center of system (3) if and only if one of the following sets of conditions is satisfied:

(i) \(a = b = c = 0, f = -3(d + h);\)
(ii) \(a = c = d = f = h = 0;\)
(iii) \(a \neq 0, c = -a, f = 3b(ae - bd)/(2a^2),\)
\[g = (2a^2bd + (2a^2 - b^2)(bd - ae))/(2a^3),\]
\[h = (-2a^2d + b(bd - ae))/(2a^2).\]

Proof. Necessity:
To describe the behaviour of trajectories of (2) near the origin we construct the comparison function \[F(x, y) = (x^2 + y^2)/2 + f_3(x, y) + f_4(x, y) + \ldots,\]
where \(f_k\) is a homogeneous polynomial of degree \(k\) whose derivative is
\[\frac{dF}{dt} = D_1(x^4 + y^4) + D_2(x^6 + y^6) + D_3(x^8 + y^8) + \ldots\]

The number of the first coefficient \(D_i\) other than zero defines the multiplicity of a complex focus and the sign of this coefficient defines stability of a focus; if \(D_i = 0\) for all \(i\) the origin is a center of (2). We refer to coefficients \(D_i\) as the Poincaré–Lyapunov constants.

To find the Poincaré–Lyapunov constants of a system \(\dot{x} = p(x, y), \dot{y} = q(x, y)\) with a linear center we used computer algebra and wrote a Mathematica code that rests on the Poincaré algorithm in \(3\); see \(3\) for more details.

```mathematica
PLconst[n_] :=
Module[{dF, ff, fF, x, y, pP, qQ, dD},
  fF[2] := (x^2+y^2)/2;
  ff[i_] := Sum[ff[i-j, j]*x^(i-j)*y^j, {j, 0, i}];
  pP[1] := y;
  pP[i_] := Sum[p[i-j, j]*x^(i-j)*y^j, {j, 0, i}];
  qQ[1] := -x;
  qQ[i_] := Sum[q[i-j, j]*x^(i-j)*y^j, {j, 0, i}];
  dF[k_] := (Sum[D[fF[i], x]*pP[k+1-i], {i, 2, k}] +
    Sum[D[fF[i], y]*qQ[k+1-i], {i, 2, k}])//Expand;
  Do[
    Solve[Table[Coefficient[dF[k], x^(k-j) y^j],
      {j, 0, k}] == Table[0, {k+1}],
    Table[ff[k-j, j], {j, 0, k}]
    ]/.Rule->Set;
  Solve[Table[Coefficient[dF[k+1], x^(k+1-j) y^j],
    {j, 0, k}]]
]
```
The procedure \texttt{PLconst[n]} returns a list \{\(D_1, \ldots, D_n\}\} of the Poincaré-Lyapunov constants if we define the coefficients \(p_{ij}, q_{ij}\) \((2 \leq i + j \leq 2n + 1)\) in the Taylor series expansion of functions \(p(x, y), q(x, y)\) beforehand.

Using this procedure, we found the first four Poincaré-Lyapunov constants of \(2\).

\[
\begin{align*}
D_1 &= 2(a + c), \\
D_2 &= -4ab - 4bc + 3d + f + 3h, \\
D_3 &= 2(-85a^3 + 15ab^2 - 67a^2c + 15b^2c + 61ac^2 + 43c^3 - 24bd - 34ae - \\
&\quad -22ce - 12bf - 50ag - 38cg - 48bh), \\
D_4 &= 44600a^5b + 2736ab^3 + 84696a^2bc + 2736b^2c + 47688abc + 7592bc^3 - \\
&\quad -37120a^2d - 1782b^2d - 32552acd - 2704c^2d + 2364abe + 1284bce - \\
&\quad -2673de - 6120a^2f - 234bf - 3384acf + 792c^2f - 891ef + \\
&\quad +6876bg + 5076cg - 3807dg - 1269fg + 4720a^2h + 1098b^2h + \\
&\quad +31448ach + 19456c^2h - 2673eh - 3807gh.
\end{align*}
\]

It is easy to verify that the equalities \(D_i = 0, i = 1, 2, 3, 4\) are equivalent to the following relations

\[
\begin{align*}
a + c &= 0, \\
3d + f + 3h &= 0, \\
3ce - bf + 3cg - 6bh &= 0, \\
2c^2f - 3bce + 3b^2h &= 0.
\end{align*}
\]

If \(a = 0\) then our simultaneous polynomial equations have two sets of solutions indicated in (i) and (ii). If \(a \neq 0\) then, in view of the condition \(c = -a\), we see that the other three equations constitute a non degenerate linear system for determining the variables \(f, g, h\). The solution is given by (iii).

The necessity part of the theorem is proved.

Sufficiency:

Case (i). System \(\underline{2}\) now takes the form

\[
\begin{align*}
\dot{x} &= y + x(dx^4 + ex^3y + fx^2y^2 + gxy^3 + hy^4) \equiv y + xp_4(x, y), \\
\dot{y} &= -x + y(dx^4 + ex^3y + fx^2y^2 + gxy^3 + hy^4) \equiv x + yp_4(x, y).
\end{align*}
\]

This is a quasi-homogeneous system of degree 5 whose coefficients satisfy the equality \(f = -3(d + h)\) which is the necessary and sufficient center condition in the case we study \(\underline{3}\).
Case (ii). System (2) now takes the form

\[
\begin{align*}
\dot{x} &= y + x^2y(b + ex^2 + gy^2), \\
\dot{y} &= -x + xy^2(b + ex^2 + gy^2).
\end{align*}
\]

The planar differential system

\[
\dot{x} = p(x, y), \quad \dot{y} = q(x, y)
\]

is said to be reversible (in the sense of Žoladek) if its orbits are symmetric with respect to a line passing through the origin.

System (1) is reversible if there is a linear transformation \( S : \mathbb{R}^2 \to \mathbb{R}^2 \), sending a point \((x, y)\) to the point \((x', y')\) symmetric to \((x, y)\) with respect to the line \(\alpha x + \beta y = 0\) and satisfying the condition \(S(p(x, y), q(x, y)) = -(p(S(x, y)), q(S(x, y)))\).

A more general condition of reversibility is as follows

\[
2\alpha \beta (p(x, y)p(x', y') - q(x, y)q(x', y')) + (\beta^2 - \alpha^2)(p(x, y)q(x', y') + p(x', y')q(x, y)) = 0.
\]

It is well known that if system (1) is reversible and has a linear center at the origin then the origin is a center of this system (see [2], for example).

Obviously, system (3) is reversible because its trajectories are symmetric with respect to both coordinate axes. So, the origin is a center for system (3).

Case (iii). System (2) now takes the form

\[
\begin{align*}
(2a^3)\dot{x} &= (2a^3)y + x(ax^2 + bxy - ay^2) \\
&+ (2a^3 + 2a^2dx^2 - 2abdx + 2a^2e x y + 2a^2d y^2 - b^2d y^2 + abe y^2), \\
(2a^3)\dot{y} &= -(2a^3)x + y(ax^2 + bxy - ay^2) \\
&+ (2a^3 + 2a^2dx^2 - 2abdx + 2a^2e x y + 2a^2d y^2 - b^2d y^2 + abe y^2).
\end{align*}
\]

It turns out that system (7) is reversible. Its trajectories are symmetric with respect to each of the two perpendicular lines defined by the equation \(ax^2 + bxy - ay^2 = 0\). The appropriate linear transformation \( S \) is given by each of the two matrices

\[
S_{1,2} = \pm (4a^2 + b^2)^{-1/2} \begin{pmatrix} -b & 2a \\ 2a & b \end{pmatrix}.
\]

This fact is confirmed by the straight calculations. We used Mathematica here.

With the coordinate change \( x \mapsto x \cos \varphi + y \sin \varphi, y \mapsto -x \sin \varphi + y \cos \varphi \) where the angle \( \varphi \) is defined from the condition \(a \tan^2 \varphi + b \tan \varphi - a = 0\), system (3) becomes as follows

\[
\begin{align*}
\dot{x} &= y + x^2y(b_1 + e_1x^2 + g_1 y^2), \\
\dot{y} &= -x + xy^2(b_1 + e_1x^2 + g_1 y^2).
\end{align*}
\]
Hence the origin is a center for system (2) in this case once again.

The theorem is proved.

2. It is known that isochronism of a center of a planar polynomial system is equivalent to existence of an analytic transversal system commuting with a given system in a neighbourhood of a center (5); observe that an arbitrary polynomial system with isochronous center not necessarily commutes with a polynomial system (6, 7).

It is proved in (8) that if the systems
\[
\dot{x} = p(x, y), \quad \dot{y} = q(x, y)
\]
\[
\dot{x} = r(x, y), \quad \dot{y} = s(x, y)
\]
commute then \(\mu(x, y) = 1/(p(x, y)s(x, y) - q(x, y)r(x, y))\) is an integrating factor of both systems.

Thereby if both commuting systems are polynomial then we can find the integrating Darboux factor for the given system and integrate the latter (about the method of Darboux and the relevant definitions see (9), for example).

We now state the following fact which will be useful later.

Considering (8), assume that
\[
p(x, y) = y + xR(x, y), \quad q(x, y) = -x + yR(x, y),
\]
\[
r(x, y) = xQ(x, y), \quad s(x, y) = yQ(x, y),
\]
where \(R(x, y), Q(x, y)\) are polynomials in \(x, y\). Then the algebraic curves \(x^2 + y^2 = 0, Q(x, y) = 0\) are invariants for each of these systems.

Indeed, it is immediately obvious that \(x^2 + y^2 = 0\) is an invariant of both systems with the cofactor \(2R(x, y)\) and \(2Q(x, y)\) respectively. The curve \(Q(x, y) = 0\) is an invariant of the second system with the cofactor \(xQ_x(x, y) + yQ_y(x, y)\).

Because our systems commute the Lie bracket of the vector fields \((p, q)\) and \((r, s)\) vanishes and we have
\[
p_x(x, y)r(x, y) + p_y(x, y)s(x, y) - r_x(x, y)p(x, y) - r_y(x, y)q(x, y) = 0,
\]
or
\[
xQ(x, y)(R(x, y) + xR_x(x, y)) + yQ(x, y)(1 + xR_y(x, y)) - p(x, y)(Q(x, y) + xQ_x(x, y)) - q(x, y)Q_y(x, y) = 0,
\]
or
\[
x(Q_x(x, y)p(x, y) + Q_y(x, y)q(x, y)) =
\]
\[
= (R(x, y) + xR_x(x, y))Q(x, y) +
\]
\[
+ (1 + xR_y(x, y))yQ_x(x, y) - Q(x, y)p(x, y) =
\]
\[
= (R(x, y) + xR_x(x, y))Q(x, y) +
\]
\[
+ (1 + xR_y(x, y))yQ_x(x, y) - Q(x, y)(y + xR(x, y)) =
\]
\[
= x(xR_x(x, y) + yR_y(x, y))Q(x, y).
\]

We see that the curve \(Q(x, y) = 0\) is an invariant with the cofactor \(xR_x(x, y) + yR_y(x, y)\).
In this case \( \mu(x, y) = 1/(Q(x, y)(x^2 + y^2)) \) is an integrating Darboux factor.

3. In each of the three cases we have found a non trivial polynomial system commuting with the respective system.

In case (i) such a system is

\[
\dot{x} = (1 + ex^4 - 4dx^3y + 4hxy^3 - gy^4) \equiv x(1 + q_4(x, y)), \\
\dot{y} = (1 + ex^4 - 4dx^3y + 4hxy^3 - gy^4) \equiv y(1 + q_4(x, y)).
\]

The function

\[
\mu(x, y) = \frac{1}{(x^2 + y^2)(1 + q_4(x, y))},
\]

is the integrating Darboux factor of (9) and the function

\[
H(x, y) = \frac{(x^2 + y^2)^2}{1 + q_4(x, y)}
\]

is the first rational integral of (9).

According to [4] system (1) has a center of type \( B_k \), \( 1 \leq k \leq n - 1 \) whose boundary is a finite union of \( k \) unbounded open trajectories. Using (9), in case (i) we can describe this boundary explicitly:

\[
\varrho = \frac{1}{(0 - q_4(\cos \varphi, \sin \varphi))^{1/4}},
\]

where \( c_0 = \max_{[0, 2\pi]} q_4(\cos \varphi, \sin \varphi), x = \rho \cos \varphi, y = \rho \sin \varphi \).

A straight analysis of this expression allows us to conclude that in our case a center may be of type \( B^2 \) or \( B^4 \) only.

In case (ii) system (3) commutes with the system

\[
\dot{x} = (e - g)x(x + (ex^2 + gy^2)(b + ex^2 + gy^2)), \\
\dot{y} = (e - g)y(x + (ex^2 + gy^2)(b + ex^2 + gy^2)).
\]

This permits us to find an integrating Darboux factor

\[
\mu(x, y) = \frac{1}{(x^2 + y^2)(e - g + (ex^2 + gy^2)(b + ex^2 + gy^2))}.
\]

The algebraic curves \( x^2 + y^2 = 0, e - g + (ex^2 + gy^2)(b + ex^2 + gy^2) = 0 \) are invariant curves for system (3).

If \( b = 0 \) system (3) is a system of the form (4) for which the condition \( f = -3(d + h) \) is obviously fulfilled. Then its first integral is

\[
H(x, y) = \frac{(x^2 + y^2)^2}{1 + ex^4 - gy^4}.
\]
If $b \neq 0$ then we may suppose that $b = 1$. The general case reduces to this by the change of variables $x \to x/\sqrt{b}, y \to y/\sqrt{b}$ for $b > 0$ or $x \to y/\sqrt{-b}, y \to x/\sqrt{-b}, t \to -t$ for $b < 0$.

Then our system takes the form
\[
\begin{align*}
\dot{x} &= y + x^2y(1 + ex^2 + gy^2) \equiv X_1(x, y), \\
\dot{y} &= -x + xy^2(1 + ex^2 + gy^2) \equiv Y_1(x, y).
\end{align*}
\]

The functions
\[
C_1 = x^2 + y^2, \quad C_2 = e - g + (ex^2 + gy^2) + (ex^2 + gy^2)^2
\]
are invariants for (11) with the cofactors
\[
L_1 = 2xy(1 + ex^2 + gy^2), \quad L_2 = 2xy(1 + 2(ex^2 + gy^2)).
\]

Moreover, if $e \neq g$ the function
\[
C_3 = \exp\left(\int_0^{ex^2+gy^2} \frac{dt}{e - g + t + t^2}\right)
\]
is invariant with the cofactor $L_3 = 2xy$.

We have $2L_1 - L_2 - L_3 = 0$. Then the function
\[
H(x, y) = \frac{C_2^2}{C_2C_3} = \frac{(x^2 + y^2)^2}{(e - g + (ex^2 + gy^2) + (ex^2 + gy^2)^2) \exp\left(\int_0^{ex^2+gy^2} \frac{dt}{e - g + t + t^2}\right)}
\]
is the first Darboux integral of (11).

Let us remark that
\[
\int \frac{dt}{e - g + t + t^2} = \frac{2}{\sqrt{4(e - g) - 1}} \arctan \frac{1 + 2t}{\sqrt{4(e - g) - 1}}
\]
for $4(e - g) - 1 > 0$ and
\[
\int \frac{dt}{e - g + t + t^2} = -\frac{2}{1 + 2t}
\]
for $4(e - g) - 1 = 0$ and
\[
\int \frac{dt}{e - g + t + t^2} = \frac{1}{\sqrt{1 - 4(e - g)}} \ln \frac{1 + 2t - \sqrt{1 - 4(e - g)}}{1 + 2t + \sqrt{1 - 4(e - g)}}
\]
for $4(e - g) - 1 < 0$.

If $e = g$ the function
\[
C_3 = \exp\frac{x^2+y^2}{2(x^2+y^2)}
\]
is invariant with the cofactor for \( L_3 = -2exy \).

We have \( 2L_1 - L_2 + \frac{1}{e}L_3 = 0 \). Then the function

\[
H(x, y) = \frac{C_1^2}{C_2C_3^{1/e}} = \frac{(x^2 + y^2)}{(1 + ex^2 + ey^2)} \exp\left(\frac{1}{e} \frac{1 + x^2}{x^2 + y^2}\right)
\]

is the first Darboux integral of (11) for \( e = g \).

Since (11) has a unique finite singular point at the origin, the phase portraits are obtained by studying the points at infinity. A standard inspection of the location and types of such points on the equator of the Poincaré sphere allows us to conclude that (11) has phase portraits of two types only: a center is of type \( B^2 \) when \( eg \geq 0 \) or of type \( B^4 \) when \( eg < 0 \).

In case (iii) a commuting system and first integral may be found on considering that system (11) is equivalent to (11).

Observe that for \( d = e = 0 \) system (11) is a quasi-homogeneous \( O \)-symmetric cubic system of the form

\[
\dot{x} = y + x(ax^2 + bxy - ay^2), \\
\dot{y} = -x + y(ax^2 + bxy - ay^2).
\]

It commutes with the system

\[
\dot{x} = x + x(bx^2 - 2axy), \\
\dot{y} = y + y(bx^2 - 2axy),
\]

and has the first integral

\[
H(x, y) = \frac{x^2 + y^2}{1 + bx^2 - 2axy}.
\]

Summarizing we conclude that the system under consideration has phase portraits of two types only: a center is of type \( B^2 \) or of type \( B^4 \).

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