Greedy Maximization Framework for Graph-based Influence Functions

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Abstract—The study of graph-based submodular maximization problems was initiated in a seminal work of Kempe, Kleinberg, and Tardos (2003): An influence function of subsets of nodes is defined by the graph structure and the aim is to find subsets of seed nodes with (approximately) optimal tradeoff of size and influence. Applications include viral marketing, monitoring, and active learning of node labels. This powerful formulation was studied for (generalized) coverage functions, where the influence of a seed set on a node is the maximum utility of a seed item to the node, and for pairwise utility based on reachability, distances, or reverse ranks.

We define a rich class of influence functions which unifies and extends previous work beyond coverage functions and specific utility functions. We present a meta-algorithm for approximate greedy maximization with strong approximation quality guarantees and worst-case near-linear computation for all functions in our class. Our meta-algorithm generalizes a recent design by Cohen et al (2014) that was specific for distance-based coverage functions.

I. INTRODUCTION

Submodular maximization problems are extensively studied and used in many application domains. The aim is to compute a subset $S$ of items of a certain size that maximizes some submodular and monotone influence (valuation) function $\text{Inf}(S)$. Even for the special case of coverage problems, the problem of computing, for parameter $s$, a set $S$ of $s$ items with maximum $\text{Inf}(S)$ is NP hard and also hard to approximate. A simple and practical algorithm is Greedy, which sequentially selects the item

$$i = \arg \max_h \text{Inf}(h | S)$$

with maximum marginal influence

$$\text{Inf}(i | S) = \text{Inf}(S \cup \{i\}) - \text{Inf}(S).$$

We refer to the computed permutation of items as the greedy sequence and to the process as greedy maximization.

A classic result of Nemhauser et al [21] shows that for submodular monotone functions, any $s$ prefix of the greedy sequence of items has influence that is at least $1 - (1 - 1/s)^s \geq 1 - 1/e$ of the maximum possible. Feige [13] showed that this is the best worst-case approximation ratio we can hope for by a polynomial time algorithm. Hence the greedy sequence approximates the full Pareto front of seed set size versus influence. A very useful relaxation is approximate greedy which selects in each step an approximate maximizer that has at least $(1 - \epsilon)$ of the maximum marginal influence. Approximate greedy often scales up the computation while lowering the approximation ratio guarantee by at most $O(\epsilon)$.

We are interested here in influence functions that are expressed in terms of utility values $u_{ij}$ between our items and elements. The influence of the seed set $S$ can then be defined as the sum over elements $j$ of the maximum utility of a seed item to $j$:

$$\text{Inf}(S) = \sum_j \max_{i \in S} u_{ij}. \quad (1)$$

Coverage functions are the special case where for any elements $j$, there is $c_j > 0$ such that $u_{ij} \in \{0, c_j\}$. A natural extension replaces the maximum in (1) with another submodular aggregation function $F$ applied to the multiset $\{u_{ij} | i \in S\}$.

Graphs are a common model of representing relations between entities: Edges represent stronger affinity between their end points but more generally, affinity can be derived from the ensemble of paths connecting nodes. A sparse graph can therefore represent dense and intricate affinity relations between all pairs of nodes. With graph-based influence, items and elements are nodes in a graph and utility values are affinity. Influence is derived from utility using max aggregation [1]. Seed sets with high influence and small size optimize coverage, diversity, and information and have low redundancy. Among the many applications are selecting anchors/hubs/coresets for monitoring/analysing/sparfying the network, active learning candidates in semi-supervised learning context, or viral marketing in social networks. Graph-based influence is rooted in classic graph formulation of the set (maximum) cover problem and notions of centrality (influence of a single vertex) in social networks [2], [14]. It was popularized and extended in a seminal work of Kempe, Kleinberg, and Tardos [17].

There are several natural ways to derive utility values from graph structure (See example in Figure 1). The hugely popular Independent Cascade (IC) model of [17] uses $u_{ij} = 1$ if $j$ is reachable from $i$ and 0 otherwise. Gomez-Rodriguez et al proposed distance-based (“continuous time”) utility, where edge lengths model propagation times [13], [12]: For a threshold parameter $T$, $u_{ij} = 1$ if and only if $d_{ij} \leq T$, where $d_{ij}$ is the shortest path distance. More generally, [10] considered utility that smoothly decreases with distance. Smooth distance-based utility was studied in social network analysis [2], [22], [14], economic models [5], and general network analysis [11]. Reverse-rank utility is based on the order induced by distances.
utility with $\alpha(x) = 1/x$

| distance | $d_{BA} = 2$ | $u_{BA} = 1/2$ |
|----------|--------------|----------------|
|          | $d_{CA} = 1$ | $u_{CA} = 1$   |
|          | $d_{DA} = 5$ | $u_{DA} = 1/5$|

| reverse rank | $\pi_{AB} = 3$ | $u_{BA} = 1/3$ |
|--------------|---------------|----------------|
| survthresh   | $\tau_{AB} = 2$ | $u_{BA} = 2$   |
|              | $\tau_{DB} = 1$ | $u_{DB} = 1$   |

Fig. 1. Utility example for toy graph with edge lengths/lifetimes

instead of the magnitudes and by doing that, it “factors out” the effect of varying density. The special case of “reverse nearest neighbors” influence, where $u_{ij} = 1$ if $i$ is the nearest neighbor of $j$, was formalized by Korn and Muthukrishnan [18]: The influence of a seed set is the number of reverse nearest neighbors it has. Buchnik and Cohen [4] generalized it to higher order reverse near neighbors and smooth decrease with reverse rank.

A powerful extension is to use a randomized model that selects edges or edge lengths from the base graph [17], [8], [1]. Utility values are then defined as the expectation. This allows reachability or distance-based utility values to capture finer properties of the connecting paths ensemble: In particular, utility increases not only for shorter paths but also with more disjoint paths. We note here that popular spectral kernels (e.g. effective resistances or hitting probabilities of random walks) [6] share these basic qualities. The IC model of [17] selects edges from the base set with independent probabilities. Distance-based models assign edge lengths drawn from (typically exponential or Weibull) distributions [15], [12], [8], [1]. Computationally, randomized models are handled by generating multiple sets of edges using Monte Carlo simulations and averaging over simulations.

Massive data sets pose significant scalability challenges. The exact greedy algorithm is polynomial but we seek algorithms that are near linear. A fruitful research thread proposed many different heuristics and algorithms for influence maximization [19], [5], [16], [23]. Most of these algorithms can guarantee both near-linear computation and approximation quality only for small seed set sizes. The only approach that computes a full approximate greedy sequence using near-linear computation is the SKIM (sketch based influence maximization) algorithm of Cohen et al [9], which efficiently maintains samples of the marginal influence sets of nodes. SKIM was initially designed for reachability-based influence and extended to distance-based [10] and reverse-rank influence [4].

Contribution and outline

Our main contribution here is presenting a general framework for efficient greedy influence maximization that generalizes and extends this previous work.

Previous work focused on influence functions with max aggregation [1], where the contribution of an element $j$ to the influence of the seed set $S$ is equal to the maximum utility $\max_{i \in S} u_{ij}$ of a seed item. With max aggregation, the value of a seed set of videos to a user is equal to that of their favourite video in the seed set. Often, however, elements derive additional value from other seed items. In our example, user $j$ may be able to watch another video and thus values its second-favourite seed video $i$ at $u_{ij}/2$. In our toy graph example of Figure 1 when considering distance-based utility, the value of the seed set $\{B, C, D\}$ to node $A$ is $1$ with max aggregation, and is equal to $u_{CA}$, but is $u_{CA} + u_{BA}/2 = 1.25$ when the second favourite $B$ contributes half of $u_{BA}$. In Section III we define submodular top-$\ell$ aggregation functions that depend on the $\ell$ highest utility values of seed items to the element. We also show that when the utility matrix $\{u_{ij}\}$ is provided explicitly, an approximate greedy sequence can be computed in time that is near-linear in the number of nonzero entries.

In a graph setting, explicit computation of $\{u_{ij}\}$ is computationally prohibitive. SKIM and its extensions are efficient because dense utility values $\{u_{ij}\}$ are represented by a sparse structure and SKIM performs the greedy maximization using computation that is near-linear in the size of the structure. SKIM ultimately obtains utility values for a number of pairs that is near-linear in the number of nodes. In Section III we present an abstraction of two access primitives to the utility matrix that suffice for performing a “SKIM-like” influence maximization computation: reverse sorted access oracle and forward search oracle. In Section V we present the SKIM meta-algorithm that performs approximate greedy influence maximization for influence functions with submodular top-$\ell$ aggregation using near-linear computation and number of oracle calls.

In Section IV we tie back our meta-algorithm to previous work by reviewing how the two access oracles are realized for reachability, distance, and reverse rank utilities. We also present oracles for Survival threshold utility [7], which generalizes reachability utility and is inspired by survivability analysis [20]: For graph with edge weights interpreted as lifetime values, the survival threshold $\tau_{ij}$ is the maximum $t$ such that $j$ is reachable from $i$ through edges with lifetime at least $t$. We use $u_{ij} = \tau_{ij}$ (See example in Figure 1).

Our near-linear meta-algorithm for approximate greedy maximization is the first to apply to influence functions with (i) previously-studied utility and aggregation function other than maximum, (ii) smooth reverse-rank utility, and (iii) survival threshold utility.

II. INFLUENCE FUNCTIONS

We consider here influence functions of a particular form. We have $n$ items, elements, pairwise utility values $u_{ij} \geq 0$ of an item to an element, and an aggregation function $F$ that is applied to a multiset of numbers.

We define the utility

$$u_{Sj} = F(U_{Sj})$$

of a seed set $S$ of items to an element $j$ as the aggregation function $F$ applied to the multiset of pairwise values

$$U_S(j) = \{u_{ij} \mid i \in S\}.$$
Finally, the influence of a seed set $S$ of items is defined as the sum over elements $j$ of the utility $u_{Sj}$ of $S$ to the element

$$\text{Inf}(S) = \sum_j u_{Sj} = \sum_j F(U_S(j)).$$

The simplest and most common aggregation function is the maximum $F(U_S(j)) = \max_{x \in S} u_{ij}$. We define a natural class of more general aggregations that are monotone submodular functions of the $\ell$ largest values in $U_S(j)$. We start with a useful definition of a domination partial order on multisets of positive numbers:

$$A \succeq B \iff \forall i, i^\text{th}(A) \geq i^\text{th}(B),$$

where $i^\text{th}(A)$ is the $i$th largest value in $A$ when $i \leq |A|$ and is $0$ otherwise. For a parameter $\ell$, a function $F$ is submodular top-$\ell$ if and only if:

$$F(\emptyset) = 0$$

$$\forall a > 0, F(\{a\}) = a$$

$$\forall A, F(A) = F(\text{top-}\ell(A))$$

$$\forall A, B A \succeq B \implies F(A) \geq F(B).$$

Some examples of submodular top-$\ell$ functions are: max $F(A) = \max_{x \in A} a$, sum of top-$\ell$ values $F(A) = \sum_{i=1}^{\ell} i^\text{th}(A)$, or a weighted sum $F(A) = \sum_{i=1}^{\ell} \frac{1}{s} i^\text{th}(A)$.

Lemma 2.1: If $F$ is submodular top-$\ell$ then

$$\forall A, B, C A \succeq B \implies F(A \cup C) \geq F(B \cup C).$$

Proof: Note that for all $C$,

$$A \succeq B \implies A \cup C \succeq B \cup C$$

We show that when the utility matrix $u_{ij}$ is provided explicitly, the lazy approximate greedy algorithm has a guaranteed approximation ratio of $1 - 1/e - \epsilon$ using computation that is near-linear in input sparsity (number of nonzero entries).

A. Utility Digest

To efficiently compute marginal utility as seed items are added, we maintain a utility digest, uDigest[$j$], for each element $j$. The digest is a summary of $U_S$ with internal implementation that depends on the aggregation function $F$. It always suffices to store the $\ell$ largest values in $U_S(j)$, but a compact representation (e.g. histograms) suffices for some $F$. We will use the following operations:

- uDigest[$j$].init initializes an empty digest with threshold $0$.
- uDigest[$j$].thresh $\leftarrow$ inf $x \in F(U_S(j) \cup \{x\}) > F(U_S(j))$ returns the threshold value $x$ that can increase utility of the seed set.
- uDigest[$j$].marg($x$) $\leftarrow F(U_S(j) \cup \{x\}) - F(U_S(j))$ returns the marginal gain of adding a seed $i \notin S$ with utility $x = u_{ij}$.
- uDigest[$j$].AddMarg($y, x$) $\leftarrow F(U_S(j) \cup \{y, x\}) - F(U_S(j) \cup \{y\})$ For two items $h, i \notin S$ with utility $y = u_{ij}$ and $x = u_{ij}$, the marginal gain of adding $i$ if $h$ is already added to $S$.

- uDigest[$j$].val $\leftarrow F(U_S(j))$
- uDigest[$j$].update($x$): Compute a digest of $U_S(j) \cup \{x\}$ given $x$ and digest of $U_S(j)$. Updating digest with a new seed item $i$ with utility $u_{ij} = x$.

B. Approximate lazy greedy

The algorithm maintains all items $i$ in a max heap with priorities equal to their marginal utility at the time of insertion. The initial priority of $i$ is $\text{Inf}(\{i\}) = \sum_j u_{ij}$. We iterate the following until the heap is empty. We pop the item $i$ at the top of the heap and compute its exact marginal influence

$$\text{Inf}(i \mid S) = \sum_j u_{\text{Digest}}[j].\text{marg}(u_{ij}).$$

If $\text{Inf}(i \mid S)$ is at least $(1 - \epsilon)$ of the current priority of item $i$, it is added to the seed set $S$ and we update $\forall j$, $\text{uDigest}[j].\text{update}(u_{ij})$. Otherwise, if $\text{Inf}(i \mid S) > \max_h \text{Inf}(\{h\})/n^2$, item $i$ is placed back in the heap with current priority equal to $\text{Inf}(i \mid S)$.

Since this is an approximate greedy sequence, we have a guaranteed approximation ratio of $1 - (1 - 1/s)^e - \epsilon$ for each $s$ prefix of the sequence. The computation is near linear when $\ell$ is small:

Lemma 2.2: Approximate lazy greedy uses

$$O(m\epsilon^{-1} \log n + \epsilon^{-1} n \log^2 n)$$

computation, where $m$ is the number of nonzeros in $u_{ij}$.

Proof: Each marginal influence computation for item $i$ amounts to $2m_i$ digest operations, typically $O(\ell)$ each, where $m_i$ is the number of elements $j$ with $u_{ij} > 0$. Thus it is $O(\ell m_i)$.

Each time an item is placed back on heap, its marginal influence decreased by at least a factor of $(1 - \epsilon)$ from the value it had when previously placed on the heap. Hence, this can happen $\epsilon^{-1} \log n$ times. Therefore, the total computation for item $i$ is $O(\epsilon^{-1} m_i \log n)$ and $O(\epsilon^{-1} \log^2 n)$ for heap operations. The claim follows by summing over items $i$.

In the sequel we address settings where $n$ is dense ($m$ is much larger than the number of items and elements) or expensive to compute and show how the maximization can be performed by only accessing “relevant” entries.

III. ORACLE ACCESS TO UTILITY VALUES

We define two access oracles to the utility matrix $\{u_{ij}\}$ that allow us to perform approximate greedy maximization while only retrieving a fraction of entries: Reverse sorted access from elements, and forward search from items.

A. Reverse sorted access

The reverse sorted access oracle REVSORTEDACCESS for element $j$ returns items and utility value pairs $(i, u_{ij})$ in non-increasing order of $u_{ij}$. It supports the following operations:

- REVSORTEDACCESS[$j$].init: Initialize reverse sorted access from $j$.
- \texttt{REVSortedAccess[j].top}: Return \( u_{ij} \) of next item without popping it.
- \((i, u_{ij}) \leftarrow \texttt{REVSortedAccess[j].pop} \): pop return the next item in the sorted order of \( u_{ij} \).
- \( \texttt{REVSortedAccess[j].delete} \): Delete the data structure.

We use this oracle as the seed set \( S \) grows. We show that for all influence functions in our class and all seed sets \( S \), the marginal utility order is the same as the utility order.

**Corollary 3.1:**
\[
\forall S, \ u_{ij} \leq u_{ji} \implies (u \mid S)_{ij} \leq (u \mid S)_{ji},
\]
where
\[
(u \mid S)_{ij} = u_{Su(i,j)} - u_{S,j}.
\]

**Proof:** We apply Lemma 2.1 with \( C = U_{Sj}, \ A = \{u_{hj}\} \) and \( B = \{u_{ij}\} \).

### B. Forward search

The forward search oracle \texttt{FORWARDSEARCH} for item \( i \not\in S \) returns all elements \( j \) for which \((u \mid S)_{ij} > 0\) along with the value \( u_{ij} \). The oracle supports initialization (with respect to the seed set \( S \)) and retrieving element utility pairs:

\[
\texttt{FORWARDSEARCH}[i].init(S) \quad (j, u_{ij}) \leftarrow \texttt{FORWARDSEARCH}[i].pop.
\]

The implementation of forward search is subtle. Our influence functions allow that for some \( i, j, S \) \( u_{ij} > u_{ih} \) and \((u \mid S)_{ij} < (u \mid S)_{ih}\), so the implementation can not just use a sorted order of \( j \) by \( u_{ij} \) that is oblivious to the current seed set \( S \), but has to adapt to \( S \) to work efficiently.

### IV. Graph-based utility

In graph-based settings, the input is represented by a graph or a set of graph instances \( \{G^{(h)}(V,E^{(h)})\} \) (obtained for example from Monte Carlo simulations of a randomized model).

All instances share the same set \( V \) of nodes, which correspond to items. Our elements \( j \) are node instance pairs \((v(j),h(j))\). The edges are directed and can have associated weights \( w^{(h)} \), with interpretation that varies between definitions of utility.

Reverse-sorted access is implemented by an appropriate graph search algorithm that is executed incrementally from a node in an instance corresponding to element \( j \) and returns nodes that correspond to items in decreasing \( u_{ij} \) order. Forward search is guided by an appropriate basic search algorithm by increasing \( u_{ij} \) with the additional property that the search tree partial order on elements \( j \) preserves marginal utility order: If \( h \) is a descendant of \( j \) then for all \( S \), \((u \mid S)_{ih} \leq (u \mid S)_{ij}\). Our forward search follows the basic search tree and accesses digest structures of visited elements to compute marginal utility value \( u_{\text{Digest}[j].marg}(u_{ij}) \). The search is pruned when the marginal utility is 0. Pruning is critical to efficiency, since ultimately we perform forward searches from all items. The tree order property is critical for the pruning correctness – so that it does not prevent us from reaching elements for which \( i \) has positive marginal utility \((u \mid S)_{ij} > 0\). The total number of nodes visited in a forward search are those adjacent to nodes with positive marginal utility.

We use the digest structures to guide the forward search pruning. Our SKIM meta-algorithm will use the forward search to update the digest structures. Two useful subroutines are \texttt{MARGGAIN}, which computes the marginal influence of an item given the current seed set, and \texttt{ADDSeed} which also updates digests of elements to reflect the addition of the new item to the seed set.

#### Function \texttt{MARGGAIN(\( i \))}: Marginal influence of \( i \not\in S \)

- **Input:** item \( i \)
- **Output:** \( \text{Inf}(u_i) \)

\[
M \leftarrow 0; \quad \text{// sum of marginal contributions}
\]

\[
\text{FORWARDSEARCH}.init(S); \quad \text{// init forward search from } i \text{ with respect to } S
\]

\[
\text{while } (j, u_{ij}) \leftarrow \text{FORWARDSEARCH}.next \neq \bot \text{ do}
\]

\[
M \leftarrow u_{\text{Digest}[j].marg}(u_{ij})
\]

\[
\text{return } M
\]

#### Function \texttt{ADDSeed(\( i \))}: Update digests of all elements

- **Input:** item \( i \)

\[
M \leftarrow 0; \quad \text{// sum of marginal contributions}
\]

\[
\text{FORWARDSEARCH}.init(S); \quad \text{// init forward search from } i \text{ with respect to } S
\]

\[
\text{while } (j, u_{ij}) \leftarrow \text{FORWARDSEARCH}.next \neq \bot \text{ do}
\]

\[
M \leftarrow u_{\text{Digest}[j].marg}(u_{ij});
\]

\[
\text{uDigest}[j].update(u_{ij})
\]

\[
\text{return } M
\]

### A. Distance-based utility

Distance-based utility is defined using a non-increasing function \( \alpha \). For item \( i \) and element \( j, u_{ij} = \alpha(d^{(h)}_{ij})), \) where \( d^{(h)}_{ij} \) is the shortest path distance from \( i \) to \( j \) in instance \( h \) with edge lengths \( w^{(h)} \).

Reverse sorted access for element \( j \) is implemented by Dijkstra’s algorithm from \( v(j) \) in the transposed (reversed edges) instance \( h(j) \). Initialization places \( v(j) \) in the heap and nodes are returned by increasing \( d_{ij} \) (decreasing \( u_{ij} = \alpha(d_{ij}) \)). When incoming edges are sorted by length, the computation is dominated by the number of traversed edges, which are edges adjacent to returned nodes.

Forward search from item \( i \) is implemented by running a copy of pruned Dijkstra from node \( i \) in each instance \( h \). The computation is guided by the digest structure and pruned at nodes \( j \) with \( u_{ih(j)} < \ell^{(h)}(U_{S(h,j)}) \). Pruning correctness is established in Lemma 4.1.

### B. Reverse-rank utility

Reverse-rank utility is similarity defined as \( u_{ij} = \alpha(\pi^{(h)}_{v(j),i}), \) where the Dijkstra rank \( \pi^{(h)}_{j,i} \) is defined as the number of nodes that are at least as close to \( j \) as \( i \) is. Following Buchnik and Cohen [4], we work with approximate ranks \( \hat{\pi} \).

The graph is preprocessed to obtain all-distances sketches,
which are used to compute \( \hat{\pi}_{ij} \) from \( d_{ij} \). Approximation is necessary because search by exact reverse ranks is provably as hard as all-pairs shortest paths computation \([4]\).

The forward search implementation uses a pruned approximate reverse-rank search \([4]\), which also traverses a shortest-path tree. The correctness of pruning is established in Lemma 4.1. The reverse sorted access oracles uses an adaptation of Dijkstra on the transposed graphs that is guided by approximate ranks.

Interestingly, our framework does not handle “forward” rank utility \( u_{ij} = \alpha(\pi_{ij}) \), as we are not aware of an efficient reverse sorted access implementation that is stable when seed nodes are added.

C. Reach and survival threshold utility

Reachability utilities are \( u_{ij} = 1 \) if and only if there is a path from \( i \) to \( v(j) \) in instance \( h(j) \). The more general survival threshold utilities are defined as \( u_{ij} = \tau_i(h(j)) \) where \( \tau_i(h) \) is the maximum \( t \) such that there is a path from \( i \) to \( j \) in instance \( h \) using edges with \( w_e(h) \geq t \). The reverse sorted access and the forward search oracles are similar to distance utility, with Dijkstra-like survival threshold search algorithm \([7]\) replacing Dijkstra’s algorithm. A survival-threshold search tree from source node \( i \) has the property that for any node \( j \), the lifetime of all edges \( e \) on the path from \( i \) to \( j \) has \( t_e \geq \tau_{ij} \). By definition of \( \tau_{ij} \), there must be at least one edge on the path with \( t_e = \tau_{ij} \).

D. Pruning correctness

We establish correctness of the pruning performed by the forward search for distance-based, reverse-rank, and survival-threshold utility.

Lemma 4.1: With distance-based utility, let node \( j \) be on the shortest path from \( i \) to \( r \) in instance \( h \). With (exact or approximate) reverse-rank utility, let \( j \) be on a shortest path from \( r \) to \( i \). With survival-threshold utility, let \( j \) be on a maximum survival threshold search tree path from \( i \) to \( r \) in instance \( h \). Then,

\[
\frac{u_{ij}(h,j)}{r_j} < \frac{\ell}{\ell}(U_{S(h,j)}) \implies (u \mid S)_{i(h,r)} = 0 .
\]

Proof: We start with distance utility. For a node \( a \), let \( y_a = \) be the \( \ell \)th smallest distance of \( a \) from a node in \( S \). A forward search from \( i \) is pruned at a node \( a \) if \( d_{ia} \geq y_a \).

By definition, there are at least \( \ell \) nodes \( X \subset S \) such that \( d_{ix} \leq y_j \). From triangle inequality we have

\[
\forall x \in X, d_{xr} \leq d_{sx} + d_{sj} \leq y_j + d_{jr} .
\]

Since there are at least \( \ell \) nodes in \( S \) with distance at most \( y_j + d_{jr} \) from \( r \) we have

\[
y_r \leq y_j + d_{jr} .
\]

Finally, if the search is pruned at \( j \) then \( d_{ij} \geq y_j \). Combining with \([6]\) we obtain \( y_r \leq d_{ij} + d_{jr} = d_{ir} \). Therefore, \( (u \mid S)_{i(h,r)} = 0 \).

For reverse-rank utility we apply a similar argument on a transposed graph. We work with a shortest path from \( r \) to \( i \) that contains \( j \). For a node \( a \), let \( y_a \) be the \( \ell \)th smallest rank \( a \) has to a node in \( S \). If the search is pruned at \( j \) then \( \pi_{ij} \geq y_j \) and in particular, there is a set \( X \subset S \) of \( \ell \) nodes such that \( \forall x \in X, d_{jx} \leq d_{ji} \). From the triangle inequality, for all \( x \in X, d_{rx} \leq d_{rij} + d_{jx} \). From the shortest path property, \( d_{ri} = d_{rj} + d_{ji} \). Combining, we obtain

\[
\forall x \in X, d_{rx} \leq d_{rj} + d_{ji} \leq d_{ri} = d_{rj} + d_{ji} = d_{ri} .
\]

This implies that \( \pi_{ri} \geq y_r \) and therefore \( (u \mid S)_{i(h,r)} = 0 \).

We now consider survival-threshold utility. For node \( a \), let \( y_a = \) be the \( \ell \)th largest value in \( \{ \tau_{ij} \mid a \in S \} \). Since our path from \( i \) to \( r \) is a survival-threshold search path, the minimum weight path edge between any two path nodes \( h_1, h_2 \) has weight \( \tau_{h_1,h_2} \). Let \( e \) be the critical (minimum weight) edge on the subpath from \( j \) to \( r \). Then \( \tau_{ir} = \min(e, \tau_{ij}) \). From definition, \( y_r \geq \min(y_j, e) \). If the search is pruned at \( j \) then \( \tau_{ij} \leq y_j \). Combining we obtain

\[
\tau_{ir} = \min(e, \tau_{ij}) \leq \min(y_j, e) \leq y_r
\]

and therefore \( (u \mid S)_{i(h,r)} = 0 \).

V. GREEDY MAXIMIZATION

We now present our SKIM meta-algorithm (Algorithm \([1]\)) for approximate greedy maximization. The algorithm uses the oracle calls to access utility values. It maintains approximate marginal influence values for items in a lazy priority queue \( Q \) items, which are computed from weighted samples.

We obtain weighted samples by assigning random rank \( r_j \) values to elements. The \( \text{REVSORTEDACCESS}[j] \) oracles are used, together with a heap structure \( Q \) elements on elements, to return item element pairs by non-increasing order of

\[
\frac{u_{ij}(h) \mid S)_{ij}}{r_j} (u \mid S)_{ij} .
\]

The heap is prioritized by \( \frac{u_{ij}(h) \mid S)_{ij}}{r_j} (u \mid S)_{ij} \), where \( x(j) \) is the next item to be returned by \( \text{REVSORTEDACCESS}[j] \).

The algorithm works with a threshold value \( \tau \) which decreases during the execution. The sample \( A_i \) for each item \( i \) contains all elements \( j \) such that

\[
\frac{u_{ij}(h) \mid S)_{ij}}{r_j} (u \mid S)_{ij} \geq \tau .
\]

We also maintain an estimate of the marginal influence \( \text{Inf}(i \mid S) \). The estimate is an inverse probability estimate that is computed from all elements satisfying \([8]\). The probabilities are the inclusion probability of \( j \in A_i \) over the random selection of \( r_j \). The probability that an element \( j \) satisfies \([8]\) is \( \min\{1, (u \mid S)_{ij}/\tau\} \) and its contribution to the marginal influence of \( i \) is \( (u \mid S)_{ij} \). The estimate is therefore

\[
\text{Inf}(i \mid S) = \sum_{j \in A_i} \min\{1, (u \mid S)_{ij}/\tau\} .
\]

The threshold \( \tau \) is decreased and samples and estimates are accordingly updated until at least one item \( i \) satisfies

\[
\text{Inf}(i \mid S) = \sum_{j \in A_i} \max\{(u \mid S)_{ij}, \tau\} .
\]
Efficient maintenance of the samples and estimates requires careful data structures and updates. The algorithm maintains inverted samples: index[j] for an element j contains all items such that j was sampled for i. The items are added using the reverse sorted access oracle REVSORTEDACCESS[j] and kept in that order by decreasing uij/rj. Note that this is the same order as decreasing (u | S)ij for all S. (Note that rj is fixed here, so the order of uij stays the same.) The inverted sample of j, index[j], is logically partitioned into three segments. The first segment of index[j] contains H entries, defined as those with utility (u | S)ij ≥ τ. The second contains M entries which have utility (u | S)ij < τ but satisfy (8), and the last includes L entries. L entries are those that entered as M or H, but no longer satisfy (8). They are kept because they may become useful later as τ decreases. The algorithm maintains indices HM[j] and ML[j] to the position of each segment. It also maintains the sum Est.H of H entries and the number Est.M of M entries, which suffice to compute the estimate as Est.H + τ · Est.M.

As said, the sample data structures are modified when τ is decreased and when an item is added to the seed set. The former results in new entries getting retrieved from the oracle and entries being “upgraded” from M/L to H and from L to M/H. This is done by the function MoveUp. The latter results in decreased marginal utility values (u[S]ij and in entries being “downgraded” and is handled by MoveDown.

To facilitate working with marginal utility (u[S]ij and computing it from uij, the algorithm maintains a utility digest uDigest[j] for each element j (see Section II-A). The digest is initially empty and is kept current by the algorithm as seeds are added. The digest support the forward search oracles.

The running time (computation) bound analysis follows that of [10] with some adaptations to our general aggregation functions and use of oracles.

We first bound the number of calls in forward searches. An element j can be returned by a forward search from i when uij is larger than the fth largest value in U[S]. The total number of calls by forward searches is the sum over elements of the number of times top-f set is modified as we add seed items. For analysis, we partition the greedy sequence into phases such that the marginal influence decreases by at most 1 − ε in each phase. Our approximate greedy sequence has the property that at each step we make a near uniform selection from all items with marginal influence that is at least 1 − O(ε) of the maximum. Thus, the expected number of updates in each phase is at most ρ · ln n′, where n′ ≤ n is the number of items in the phase. The algorithm terminates when the maximum marginal influence decreases below 1/n2 of its initial value. Therefore, the total number of phases is O(ε−1 log n) and the total number of calls of the forward search oracle is O((nε−1 log2 n)).

The algorithm simplifies for coverage problems with uniform utility to follow the basic SKIM design [9]. In this case, marginal contributions are either 0 or 1 and the reverse sorted access oracles REVSORTEDACCESS[j] are invoked sequentially by increasing rj, removing the need to concurrently
maintain active oracles for many elements. The samples are uniform and do not need to be broken up to segments and estimates correspond to samples size. The additional machinery was introduced in [10] in order to handle smooth distance-based utility functions.

Function NextSeed

Output: The item $i$ which maximizes $\text{Est} \cdot H[i] + \tau \cdot \text{Est} \cdot M[i]$, if happy with estimate. Otherwise $\perp$.

while true do
  if $\max \text{prior}_i < k$ then return $\perp$ else
    Remove maximum priority $i$ from Qitems;
    $I_i \leftarrow \text{Est} \cdot H[i] + \tau \cdot \text{Est} \cdot M[i]$;
    if $I_i \geq k$ and $I_i \geq \max \text{prior}_i$ then
      $I_i \leftarrow \text{MargGain}(i)$;
      if $I_i \geq (1 - \tau) / \tau$ then return $(i, I_i)$ else
        Place $i$ with priority $I_i$ in Qitems;
      return $\perp$
  else
    Place $i$ with priority $I_i$ in Qitems;

Function MoveDown $(j, x)$

Output: Update estimate contribution of element $j$ when item $h$ with utility $x = u_{ij}$ is added to $S$ to have $S^+ = S \cup \{h\}$.

// For each item $i \notin S$, the marginal utility values $(u_{|S^+}ij, u_{|S}ij)$ can be computed from $u_{\text{Digest}}(j)$ (computed for $S$) using $x = u_{ij}$ and $u_{ij}$:
$(u_{|S}ij) \equiv u_{\text{Digest}}(j) \cdot \text{marg}(u_{ij})$,
$(u_{|S^+}ij) \equiv u_{\text{Digest}}(j) \cdot \text{addM}(u_{ij})$.

$y \leftarrow 0$; $t \leftarrow 1$; $\text{HM}[y] \leftarrow \perp$;
$z \leftarrow \text{index}[j] - 1$; if $\text{ML}[j] \neq \perp$ then $z = \text{ML}[j]$; // last non-L position in index[j]

ML[y] $\leftarrow \perp$ while $y \leq z$ do
  $t \leftarrow \text{index}[j][y]$;
  if $(u_{|S}ij) \geq \tau$ then // entry was $H$
    $\text{Est} \cdot H[i] \leftarrow (u_{|S}ij)$;
    if $(u_{|S^+}ij) \geq \tau$ then // is H
      $\text{Est} \cdot H[i] \leftarrow (u_{|S^+}ij)$
    else
      $\text{Est} \cdot M[i] \leftarrow (u_{|S^+}ij)$
      if $\text{HM}[y] = \perp$ then $\text{HM}[y] = y$
      else if $(u_{|S}ij) = 0$ then // truncate
        if $t = \perp$ then $t = y$ else $\text{ML}[y] = y$
      else
        $\text{ML}[y] = \perp$ then $\text{ML}[y] = y$
      else $\text{ML}[y] = \perp$ then $\text{ML}[y] = y$
      $\text{Est} \cdot M[i] \leftarrow (u_{|S}ij)$
    $\text{Est} \cdot M[i] \leftarrow (u_{|S}ij)$
  else
    $\text{Est} \cdot M[i] \leftarrow \perp$;
    if $(u_{|S}ij) = 0$ then // truncate
      if $t = \perp$ then $t = y$ else $\text{ML}[y] = y$
    else
      $\text{ML}[y] = \perp$ then $\text{ML}[y] = y$
    $\text{Est} \cdot M[i] \leftarrow \perp$;
    if $(u_{|S}ij) \geq \tau$ then // entry was M
      if $t \neq \perp$ then truncate index[j] from $t$ on. else // clean tail
        $t \leftarrow \text{index}[j][t]$;
        while $t \geq 0$ and $i \leftarrow \text{index}[j][t] + 1$ has $(u_{|S^+}ij) = 0$ do
          $t \leftarrow t - 1$
        truncate index[j] at position $t + 1$ on
      Remove element $j$ from Qhml
    UpdateReclassThresh(j) // Update Qhml
  $y \leftarrow y + 1$
if $t \neq \perp$ then truncate index[j] from $t$ on. else // clean tail
  $t \leftarrow \text{index}[j][t]$;
  while $t \geq 0$ and $i \leftarrow \text{index}[j][t] + 1$ has $(u_{|S^+}ij) = 0$ do
    $t \leftarrow t - 1$
  truncate index[j] at position $t + 1$ on
Remove element $j$ from Qhml
UpdateReclassThresh(j) // Update Qhml

Function UpdateReclassThresh(j)

Output: Update priority of element $j$ in Qhml

$c \leftarrow 0$;
if $\text{HM}[j] \neq \perp$ then
  $c \leftarrow \text{index}[j][\text{HM}[j]]$;
if $\text{ML}[j] \neq \perp$ then
  $c \leftarrow \text{index}[j][\text{ML}[j]] + \text{max} \{c, (u_{|S}ij) / r_j\}$
if $c > 0$ then
  update priority of $j$ in Qhml to $c$

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