Deformed Legendre Polynomial and Its Application

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Abstract
A new kind of deformed calculus was introduced recently in studying of parabosonic coordinate representation. Based on this deformed calculus, a new deformation of Legendre polynomials is proposed in this paper, some properties and applications of which are also discussed.

1 Introduction

Parastatistics was introduced by Green as an exotic possibility extending the Bose and Fermi statistics [1] and for the long period of time the interest to it was rather academic. Nowadays it finds some applications in the physics of the quantum Hall effect [2] and (probably) it is relevant to high temperature superconductivity [3]. The paraquantization, carried out at the level of the algebra of creation and annihilation operators, involves trilinear(or double) commutation relations in place of the bilinear relations that characterize Bose and Fermi statistics. Recently, the trilinear commutation relations of single paraparticle systems was rewritten as bilinear commutation relations by virtue of the so called R-deformed Heisenberg algebra [4]. For instance, the trilinear commutation relations [5]

\[
\begin{align*}
[a, \{a^\dagger, a\}] &= 2a, \\
[a, \{a^\dagger, a^\dagger\}] &= 4a^\dagger, \\
[a, \{a, a\}] &= 0,
\end{align*}
\]

(1)

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where $a^\dagger$ and $a$ are parabose creation and annihilation operators respectively, can be replaced by

\[ [a, a^\dagger] = 1 + (p - 1)R, \quad \{R, a\} = \{R, a^\dagger\} = 0, \quad R^2 = 1, \quad (2) \]

where $R$ is a reflection operator and $p$ is the paraquantization order ($p = 1, 2, 3, ...$). Obviously, the bilinear commutation relations (2) may be treated as some kind of deformation of the ordinary Bose commutator with deformation parameter $p$.

From the experience of studying q-deformed oscillators, we know that it will be very useful if one introduces corresponding deformed calculus to analyse this parabose system. This was done recently and based on the new deformed calculus, the parabosonic coordinate representation was developed. Since special functions play important roles in mathematical physics, it is reasonable to imagine that some deformation of the ordinary special functions based on the new deformed calculus will also play similar roles in studying the parabose systems. In this paper, we introduce a new kind of deformation for the ordinary Legendre polynomials and demonstrate its properties. As an example of applications of these deformed Legendre polynomials, we discuss excitations on a parabose squeezed vacuum state and calculate norm of the excitation states.

The paper is organized as follows. In Section 2, for the sake of self-contained of the present paper, we briefly mention the basic idea of the new kind of deformed calculus. The deformed Legendre polynomials and relevant differential expressions are introduced in section 3. Sections 4 and 5 are devoted to demonstration of orthonormality of the deformed Legendre polynomials and their recursion relations respectively. In section 6, we show that the deformed Legendre polynomials can be used to normalize the excitation states on a squeezed vacuum state for single parabose mode. There are also some discussions and remarks in the last section.

## 2 Deformed calculus related to parabosonic coordinate representation

It is well-known that parabose algebra is characterized by the double commutation relations (1). If one demands that the usual relations

\[ a = \frac{x + iP}{\sqrt{2}}, \quad a^\dagger = \frac{x - iP}{\sqrt{2}} \quad (3) \]
still work for the parabose case, where \( x \) and \( P \) stand for the coordinate and momentum operator respectively, it can be proved that the most general expression for the momentum operator \( P \) in the coordinate \( x \) diagonal representation is of the form

\[
P = -i \frac{d}{dx} - i \frac{p-1}{2x} (1 - R),
\]

(4)

where \( p \) is the paraquantization order and \( R \) the reflection operator which has property \( R f(x) = f(-x) \) in the coordinate representation for any \( x \)-dependent function \( f(x) \). From (4) a new derivative operator \( D \) can be defined which acts on function \( f(x) \) as

\[
D f(x) \equiv \frac{D}{Dx} f(x) = \frac{d}{dx} f(x) + \frac{p-1}{2x} (1 - R) f(x)
\]

(5)

where \( df = \frac{d}{dx} f \). Definition (5) implies that \( D \) acts on an even function \( f_e(-x) = f_e(x) \) as the ordinary derivative \( Df_e(x) = df_e(x) \), and \( D \) acts on an odd function \( f_o(-x) = -f_o(x) \) leads to

\[
Df_o(x) = df_o(x) + \frac{p-1}{2x} f_o(x).
\]

(5)

For \( p = 1 \) case, \( D \) reduces to the ordinary derivative operator \( d \). Since \( P = -iD \), Eq.(5) means that the pair \( (x, D) \) in realization of parabose algebra for a single degree of freedom plays the same role as \( (x, d) \) in realization of the ordinary Bose algebra. Like q-deformed calculus in which the q-analogue of the number system was defined by

\[
[n]_q = \frac{q^n - 1}{q-1},
\]

(7)

such that when \( q \to 1, [n]_q \to n \), in the present case, one can introduce a new kind of deformed number system which is defined by

\[
[n] = n + \frac{p-1}{2} (1 - (-1)^n).
\]

(6)

Obviously, \([2k] = 2k, [2k+1] = 2k+p\) for any integer \( k \) and when \( p \to 1, [n] \to n \). So paraquantization order \( p \) may be referred to as a deformation parameter. In terms of the number system \([n]\), basis vectors of Fock space for single mode of parabose oscillators take the usual form

\[
|n\rangle = \frac{(a^\dagger)^n}{\sqrt{[n]!}} |0\rangle, \quad a^\dagger |n\rangle = \sqrt{[n+1]|n+1\rangle}, \quad a |n\rangle = \sqrt{[n]|n-1\rangle},
\]

(7)

where \([n]! = [n][n-1]...[1],[0]! \equiv 1, \) and \( |0\rangle \) is the unique vacuum vector satisfying \( a |0\rangle = 0, aa^\dagger |0\rangle = p |0\rangle \). Generalization of the ordinary differential relation \( dx^n = nx^{n-1} \) reads

\[
Dx^n = [n]x^{n-1},
\]

(8)
which reveals the effect of the deformed derivative operator $D$ on the polynomials of $x$. It is worthy of mention that in some special situation the usual Leibnitz rule also works for the deformed operator $D$

\[ D(fg) = (Df)g + f(Dg), \tag{9} \]

where either $f(x)$ or $g(x)$ is an even function of $x$.

Of course, inversion of the deformed derivative operator $D$ also leads to a new deformed integration which may be formally written as

\[ \int Dx F(x) = \sum_{n=0}^{\infty} (-)^n \left( \int dx \frac{p-1}{2x} (1-R) \right)^n \int dx F(x) \]

\[ = \int dx F(x) - \int dx \frac{p-1}{2x} (1-R) \int dx F(x) \]

\[ + \left( \int dx \frac{p-1}{2x} (1-R) \right)^2 \int dx F(x) - \cdots. \tag{10} \]

From this expression, it is easily seen that if $F(x)$ is an odd function of $x$, its deformed integration will reduce to the ordinary integration, that is, $\int Dx F(x) = \int dx F(x)$ for $F(-x) = -F(x)$. Corresponding to Eq. (8), one has

\[ \int Dx x^n = \frac{x^{n+1}}{n+1} + c, \tag{11} \]

where $c$ is an integration constant. Eq. (10) gives a formal definition for the deformed integration in the sense of indefinite integral. For definite integral, we have

\[ \int_a^b Dx F(x) = \int_a^b dx \sum_{n=0}^{\infty} (-)^n \left( \int_a^x \frac{p-1}{2x} (1-R) \int_a^x dx \right)^n F(x) \]

\[ = \int_a^b dx F(x) - \int_a^b dx \frac{p-1}{2x} (1-R) \int_a^x dx F(x) \]

\[ + \int_a^b dx \frac{p-1}{2x} (1-R) \int_a^x dx \frac{p-1}{2x} (1-R) \int_a^x dx F(x) - \cdots. \tag{12} \]

If either $F(x)$ or $G(x)$ is an even function of $x$, one has a formula of integration by parts from Eq. (9)

\[ \int_a^b Dx \frac{DF}{Dx} G = F G \bigg|_a^b - \int_a^b Dx F \frac{DG}{Dx}. \tag{13} \]
3 Defor med Legendre polynomials and their differential expressions

Let us consider solutions of deformed Legendre equation based on the deformed derivative operator $D$ defined in the previous section

$$(1 - x^2)D^2 f(x) - 2xDf(x) + \mu f(x) = 0, \quad (14)$$

or according to Eq.(9), the deformed Legendre equation is equivalent to

$$D \left((1 - x^2)Df(x)\right) + \mu f(x) = 0. \quad (15)$$

In terms of the ordinary derivative notation, Eq.(14) can be rewritt en as

$$(1 - x^2)\frac{d^2}{dx^2}f(x) - \left(2x - (p - 1)(\frac{1}{x} - x)\right)\frac{d}{dx}f(x) - \frac{p-1}{2} \left(1 + \frac{1}{x^2}\right)f(x)$$

$$+ \frac{p-1}{2} \left(1 + \frac{1}{x^2}\right)f(-x) = -\mu f(x), \quad (16)$$

which will reduce to the usual Legendre equation when $p \to 1$. We find out that when the parameter $\mu$ takes eigenvalues $\mu = [n][n+1]$, $n = 0, 1, 2, 3, \ldots$, for each given paraquantization order $p$, the deformed Legendre equation has bounded solutions (eigenfunctions) within a whole closed interval $-1 \leq x \leq 1$ which form a set of orthonormal functions in the interval. In fact, it is not difficult to see that the following polynomials

$$P_n(x) = \sum_{k=0}^{[n/2]'} \frac{(-1)^k[2n-2k]!}{2^nk!(n-k)!}[n-2k]!x^{n-2k}, \quad (17)$$

where $[k]'$ in the above of summation notation $\sum$ stands for the largest integer smaller than or equal to $k$, are the desired solutions of the deformed Legendre equation for $\mu = [n][n+1]$ which will reduce to the usual Legendre polynomials when $p \to 1$. One can substitute Eq.(17) into the deformed Legendre equation and check coefficients of all powers of $x$ being zero. So the polynomials (17) may be considered as a deformation of the usual Legendre polynomials. The first few polynomials of $P_n(x)$ have the following explicit forms

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}([3]x^2 - [1]),$$
\[ P_3(x) = \frac{1}{2}([5]x^3 - [3]x), \quad P_4(x) = \frac{1}{8}([5][7]x^4 - 2[3][5]x^2 + [1][3]), \]
\[ P_5(x) = \frac{1}{8}([7][9]x^5 - 2[5][7]x^3 + [3][5]x), \]
\[ P_6(x) = \frac{1}{48}([7][9][11]x^6 - 3[5][7][9]x^4 + 3[3][5][7]x^2 - [1][3][5]). \] (18)

Also from Eq.(17) we know that \( P_n(-x) = (-)^nP_n(x) \), which means that the deformed Legendre polynomial \( P_n(x) \) has its parity \((-)^n\).

We would like to point out a differential expression for the deformed Legendre polynomials \( P_n(x) \) to conclude this short section
\[ P_n(x) = \frac{1}{2^n n!} D^n(x^2 - 1)^n. \] (19)

which is similar to Rodrigues formula for the ordinary Legendre polynomials and can be proved straightforwardly.

### 4 Orthonormality of \( P_n(x) \)

Firstly we show the orthogonality of the deformed Legendre polynomials
\[ \int_{-1}^{1} Dx P_n(x) P_m(x) = 0, \quad (n \neq m). \] (20)

For \( n + m \) odd case, Eq.(20) works obviously. In fact, from the parity of \( P_n(x) \) we have \( P_n(-x)P_m(-x) = (-)^{n+m}P_n(x)P_m(x) = -P_n(x)P_m(x) \), which means that the deformed integration in Eq.(20) will reduce to an ordinary integration with an odd integrand \( P_n(x)P_m(x) \) over an interval of integration \([-1, 1]\), therefore the integration should be zero. For \( n + m \) even case, \( P_n(x) \) and \( P_m(x) \) satisfy the following equations
\[ D \left( (1 - x^2)DP_n(x) \right) + \mu_n P_n(x) = 0, \] (21)
\[ D \left( (1 - x^2)DP_m(x) \right) + \mu_m P_m(x) = 0, \] (22)
respectively, where \( \mu_n = [n][n + 1], \mu_m = [m][m + 1] \). Multiplying Eq.(21) and Eq.(22) by \( P_m(x) \) and \( P_n(x) \) respectively, and subtracting the resulting equations, then integrating it over \([-1, 1]\), we have
\[ \int_{-1}^{1} Dx \left( P_m \frac{D}{Dx} \left( (1 - x^2)\frac{D}{Dx}P_n \right) - P_n \frac{D}{Dx} \left( (1 - x^2)\frac{D}{Dx}P_m \right) \right) \]
\[ + (\mu_n - \mu_m) \int_{-1}^{1} DxP_n P_m = 0. \]  

(23)

Since the first integration in Eq.(23) satisfies condition of the deformed integration by parts either for \( n \) and \( m \) are all even or all odd, we can write it as

\[
\int_{-1}^{1} Dx \frac{D}{Dx} \left( (1 - x^2)(P_m \frac{D}{Dx} P_n - P_n \frac{D}{Dx} P_m) \right) = (1 - x^2)(P_m \frac{D}{Dx} P_n - P_n \frac{D}{Dx} P_m)|_{-1}^{1} = 0,
\]

(24)

here we have used a fact that \( P_n(x), P_m(x) \) and their deformed derivatives are all polynomials of \( x \) and are bounded at \( x = \pm 1 \) for a fixed paraquantization order \( p \). Noticing that \( \mu_n \neq \mu_m \), so Eq.(20) is proved.

Then we calculate integration

\[
N_n^2 = \int_{-1}^{1} Dx P_n(x) P_m(x).
\]

(25)

Substituting Eq.(19) into this integration, we have

\[
N_n^2 = \frac{1}{2^n n!} \int_{-1}^{1} Dx P_n(x) \frac{D^n}{Dx^n} (x^2 - 1)^n.
\]

(26)

Noticing that the integration (26) also satisfies the condition of the deformed integration by parts no matter what non-negative integer \( n \) is, we get

\[
N_n^2 = \frac{1}{2^n n!} \left( P_n(x) \frac{D^{n-1}}{Dx^{n-1}} (x^2 - 1)^n|_{-1}^{1} - \int_{-1}^{1} Dx \frac{D}{Dx} P_n(x) \frac{D^{n-1}}{Dx^{n-1}} (x^2 - 1)^n \right).
\]

It is not difficult to see that for \( m \leq n \), \( \frac{D^m}{Dx^m} (x^2 - 1)^n = 0 \) at \( x = \pm 1 \), so

\[
N_n^2 = -\frac{1}{2^n n!} \int_{-1}^{1} Dx \frac{D}{Dx} P_n(x) \frac{D^{n-1}}{Dx^{n-1}} (x^2 - 1)^n.
\]

Continuing the procedure of integration by parts, at last we arrive at

\[
N_n^2 = \frac{(-)^n}{2^n n!} \int_{-1}^{1} Dx \frac{D^n}{Dx^n} P_n(x) (x^2 - 1)^n = \frac{(-)^n[2n]!}{2^n n! 2^n n!} \int_{-1}^{1} Dx (x^2 - 1)^n. \]

(27)

Using Eq.(11) we can rewrite the integral in the right-hand side of Eq.(27) as

\[
\int_{-1}^{1} Dx (x^2 - 1)^n = 2 \sum_{k=0}^{n} (-)^{n-k} C_n^k \frac{1}{2k + 1} = 2 (-)^{n} \sum_{k=0}^{n} (-)^{k} C_n^k \frac{1}{2k + p}
\]

\[7\]
\[= 2(-)^n \sum_{k=0}^{n} C_n^k \int_0^1 dx x^{2k+p-1} = 2(-)^n \int_0^1 dx x^{p-1}(1-x^2)^n = \frac{(-)^n 2^{2n+1}(n!)^2}{[2n+1]}\]

where \(C_n^k = \frac{n!}{k!(n-k)!}\). Substituting this result into Eq.(27) we obtain

\[N_n^2 = \frac{2}{[2n+1]}, \quad (28)\]

which is called the norm of \(P_n(x)\). Combining Eqs.(20)and (28) we have

\[\int_{-1}^1 DxP_n(x)P_m(x) = \frac{2}{[2n+1]} \delta_{n,m}. \quad (29)\]

Thus we demonstrated the orthonormality of the deformed Legendre polynomials \(P_n(x)\).

5 Recursion relations of \(P_n(x)\)

There are some definite relations between neighbouring deformed Legendre polynomials and their derivatives which are called recursion relations of \(P_n(x)\). The main recursion relations are the following three:

\[[n+1]P_{n+1}(x) - [2n+1]xP_n(x) + [n]P_{n-1}(x) = 0, \quad (30)\]

\[DP_{n+1}(x) - xDP_n(x) - [n+1]P_n(x) = 0, \quad (31)\]

\[xDP_n(x) - DP_{n-1}(x) - [n]P_n(x) = 0. \quad (32)\]

It is straightforward to prove these relations by virtue of the definition (17) and the deformed differential relation (8). For instance, one can substitute (17) into (30) and obtain a polynomial of \(x\), then calculate coefficient for each power of \(x\) and find that all of these coefficients will be zero, thus relation (30) is true. Similarly, one can convince oneself that relations (31) and (32) also work. Based on the above three relations, one has more recursion relations for the deformed Legendre polynomials

\[DP_{n+1}(x) - DP_{n-1}(x) - [2n+1]P_n(x) = 0, \quad (33)\]

\[(x^2 - 1)DP_n(x) - [n]xP_n(x) + [n]P_{n-1}(x) = 0. \quad (34)\]

Obviously, these recursion relations will reduce to ones of the usual Legendre polynomials when \(p \to 1\).
The above recursion relations are very useful for calculating deformed integrations with the deformed Legendre polynomials such as
\[ \int_{-1}^{1} D x x P_m(x) P_n(x). \]

2 From Eq. (30), one has
\[ x P_m(x) = \frac{[m + 1]}{[2m + 1]} P_{m+1}(x) + \frac{[m]}{[2m + 1]} P_{m-1}(x), \]
so that
\[ \int_{-1}^{1} D x x P_m(x) P_n(x) = \frac{[m + 1]}{[2m + 1]} \int_{-1}^{1} D x P_{m+1}(x) P_n(x) \]
\[ + \frac{[m]}{[2m + 1]} \int_{-1}^{1} D x P_{m-1}(x) P_n(x). \]
Furthermore, by virtue of the orthonormality relation (29), one gets
\[ \int_{-1}^{1} D x x P_m(x) P_n(x) = \begin{cases} 
\frac{2[n]}{[2n+1][2n+3]}, & m = n - 1, \\
\frac{[n+1]}{[2n+1][2n+3]}, & m = n + 1, \\
0, & m - n \neq \pm 1.
\end{cases} \]

6 Some application and discussion
As an application of the deformed Legendre polynomials \( P_n(x) \), we want to point out that there exist some states in parabose radiation field whose normalizing factors are related to \( P_n(x) \). To see this, let us consider excitations on a parabose squeezed vacuum state for a single mode case. Denoting \( |r, 0\rangle = S(r)|0\rangle \) as the parabose squeezed vacuum state (\( r \) is a real number), where \( S(r) = (sech r)^p/2 exp \left( \frac{1}{2} (a^\dagger)^2 tanh r \right) \), we introduce
\[ |r, m\rangle = (a^\dagger)^m |r, 0\rangle \quad (m = 1, 2, 3, ...) \] (35)
as such kind of excitation states. It is easily to see that the parabose squeezed vacuum state \( |r, 0\rangle \) is normalized (\( \langle r, 0|r, 0\rangle = 1 \)) and the parabose squeezed excitation states have not been normalized. In order to normalize these states, we would like to prove by induction
\[ \langle r, m|r, m\rangle = [m]!(cosh r)^m P_m(cosh r), \] (36)
where $P_m(cosh r)$ is the deformed Legendre polynomial with argument $x = cosh r$. Firstly, we have

$$
a| r, 0 \rangle = (sech r)^{p/2} e^{\alpha/2 \tanh r} |0\rangle
$$

$$= (sech r)^{p/2} e^{\alpha/2 \tanh r} (a + a^\dagger \tanh r) |0\rangle = a^\dagger \tanh r | r, 0 \rangle.$$

(37)

Using the R-deformed commutation relation (2) for paraboson and noticing that $R |0\rangle = |0\rangle$, we find

$$\langle r, 1 | r, 1 \rangle = \langle r, 0 | aa^\dagger | r, 0 \rangle = \langle r, 0 | 1 + a^\dagger a + (p - 1)R | r, 0 \rangle$$

$$= p + \tanh^2 r \langle r, 1 | r, 1 \rangle = [1]! \cosh r P_1(cosh r).$$

(38)

Then supposing Eq.(36) is true for $n \leq m$, that is,

$$\langle r, n - 1 | r, n - 1 \rangle = [n - 1]! (\cosh r)^{n-1} P_{n-1}(cosh r),$$

(39)

we show that Eq.(36) works. In fact, using the following relations

$$\left[ a, (a^\dagger)^n \right] = (a^\dagger)^{n-1} \left( n + \frac{p - 1}{2} (1 - (-)^n)R \right),$$

$$\left[ a^\dagger, a^n \right] = -a^{n-1} \left( n + \frac{p - 1}{2} (1 - (-)^n)R \right),$$

(40)

we have

$$\langle r, m | r, m \rangle = \langle r, 0 | a^{m-1} a a^\dagger (a^\dagger)^{m-1} | r, 0 \rangle$$

$$= \langle r, 0 | a^{m-1} \left( 1 + a^\dagger a + (p - 1)R \right) (a^\dagger)^{m-1} | r, 0 \rangle$$

$$= \langle r, m - 1 | r, m - 1 \rangle + (p - 1)(-)^{m-1} \langle r, m - 1 | r, m - 1 \rangle$$

$$+ \langle r, 0 | (a^\dagger a^{m-1} + [m - 1] a^{m-2}) \left( (a^\dagger)^{m-1} a + [m - 1] (a^\dagger)^{m-2} \right) | r, 0 \rangle$$

$$= \tanh^2 r \langle r, m | r, m \rangle - [m - 1]^2 \langle r, m - 2 | r, m - 2 \rangle$$

$$+ \left( 2[m - 1] + 1 + (p - 1)(-)^{m-1} \right) \langle r, m - 1 | r, m - 1 \rangle,$$

or

$$\langle r, m | r, m \rangle = -\cosh^2 r [m - 1]^2 \langle r, m - 2 | r, m - 2 \rangle$$

$$+ \cosh^2 r (2[m - 1] + 1 + (p - 1)(-)^{m-1}) \langle r, m - 1 | r, m - 1 \rangle.$$  (41)
Substituting (39) into (41), we get
\[
\langle r, m | r, m \rangle = -\cosh^m r [m - 1]! [m - 1] P_{m-2}(\cosh r) + \cosh^{m+1} r [m - 1]! [2[m - 1] + 1 + (p - 1)(-)^{m-1}] P_{m-1}(\cosh r).
\]
Noticing that \(2[m - 1] + 1 + (p - 1)(-)^{m-1} = [2m - 1]\) and using the recursion relation (30) for \(P_n(x)\), we finally arrive at (36). Thus we see that the deformed Legendre polynomials indeed can be used to normalize the excitation states on a squeezed vacuum state for a single para-bose mode.

In summary, we introduced a new kind of deformation for the usual Legendre polynomials and discussed their main properties in this paper. These deformed Legendre polynomials may have some applications in studying para-bose systems. For instance, they can be used to describe excitations on a para-bose squeezed vacuum state. Comparing with the case of ordinary Legendre polynomials, one will naturally ask a question: is there any generating function of the deformed Legendre polynomials and what it is? Work for answering these questions is on progress.

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