A NEW GENERALIZATION OF OSTROWSKI TYPE
INEQUALITY ON TIME SCALES

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Abstract. In this paper we first extend a generalization of Ostrowski type
inequality on time scales for functions whose derivatives are bounded and then
unify corresponding continuous and discrete versions. We also point out some
particular integral type inequalities on time scales as special cases.

1. INTRODUCTION

In 1938, Ostrowski derived the following interesting integral inequality.

**Theorem 1.1.** Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and differentiable in
\((a, b)\) and its derivative \( f' : (a, b) \to \mathbb{R} \) is bounded in \((a, b)\), that is, \( \|f'\|_\infty := \sup_{t \in (a, b)} |f'(x)| < \infty \). Then for any \( x \in [a, b] \), we have the inequality:

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \left( \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right) (b-a) \|f'\|_\infty.
\]

(1)

The inequality is sharp in the sense that the constant \( \frac{1}{4} \) cannot be replaced by a
smaller one.

2000 Mathematics Subject Classification. 26D15; 39A10; 39A12; 39A13.
Key words and phrases. Ostrowski’s inequality; generalization; time scales; Simpson inequality;
trapezoid inequality; midpoint inequality.

This paper was typeset using AMS-\LaTeX.
More recently, the authors proved the Ostrowski-Grüss type inequality on time scales [12].

**Theorem 1.3.** Let \(a, b, s, t \in \mathbb{T}\), \(a < b\) and \(f : [a, b] \to \mathbb{R}\) be differentiable. If \(f^\Delta\) is rd-continuous and
\[
\gamma \leq f^\Delta(t) \leq \Gamma, \quad \forall \ t \in [a, b].
\]
Then we have
\[
\left| f(t) - \frac{1}{b-a} \int_a^b f^\sigma(s) \Delta s - \frac{f(b) - f(a)}{(b-a)^2} \left(h_2(t, a) - h_2(t, b)\right)\right| \leq \frac{1}{4} (b-a)(\Gamma - \gamma)
\]
for all \(t \in [a, b]\).

In the present paper, by introducing a parameter, we first extend a generalization of Ostrowski type inequality on time scales for functions whose derivatives are bounded and then unify corresponding continuous and discrete versions. We also point out some particular integral type inequalities on time scales as special cases.

**2. Time scales essentials**

Now we briefly introduce the time scales theory and refer the reader to Hilger [2] and the books [2 3 8] for further details.

**Definition 2.1.** A time scale \(\mathbb{T}\) is an arbitrary nonempty closed subset of real numbers.

**Definition 2.2.** For \(t \in \mathbb{T}\), we define the **forward jump operator** \(\sigma : \mathbb{T} \to \mathbb{T}\) by \(\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}\), while the **backward jump operator** \(\rho : \mathbb{T} \to \mathbb{T}\) is defined by \(\rho(t) = \sup \{s \in \mathbb{T} : s < t\}\). If \(\sigma(t) > t\), then we say that \(t\) is right-scattered, while if \(\rho(t) < t\) then we say that \(t\) is left-scattered.

Points that are right-scattered and left-scattered at the same time are called isolated. If \(\sigma(t) = t\), the \(t\) is called right-dense, and if \(\rho(t) = t\) then \(t\) is called left-dense. Points that are both right-dense and left-dense are called dense.

**Definition 2.3.** Let \(t \in \mathbb{T}\), then two mappings \(\mu, \nu : \mathbb{T} \to [0, +\infty)\) satisfying
\[
\mu(t) := \sigma(t) - t, \quad \nu(t) := t - \rho(t)
\]
are called the **graininess functions**.

We now introduce the set \(\mathbb{T}^\kappa\) which is derived from the time scales \(\mathbb{T}\) as follows. If \(\mathbb{T}\) has a left-scattered maximum \(t\), then \(\mathbb{T}^\kappa := \mathbb{T} - \{t\}\), otherwise \(\mathbb{T}^\kappa := \mathbb{T}\). Furthermore for a function \(f : \mathbb{T} \to \mathbb{R}\), we define the function \(f^\sigma : \mathbb{T} \to \mathbb{R}\) by \(f^\sigma(t) = f(\sigma(t))\) for all \(t \in \mathbb{T}\).

**Definition 2.4.** Let \(f : \mathbb{T} \to \mathbb{R}\) be a function on time scales. Then for \(t \in \mathbb{T}^\kappa\), we define \(f^\Delta(t)\) to be the number, if one exists, such that for all \(\varepsilon > 0\) there is a neighborhood \(U\) of \(t\) such that for all \(s \in U\)
\[
\left| f^\sigma(s) - f^\Delta(t) - f^\sigma(s)\right| \leq \varepsilon |\sigma(t) - s|.
\]
We say that \(f\) is \(\Delta\)-differentiable on \(\mathbb{T}^\kappa\) provided \(f^\Delta(t)\) exists for all \(t \in \mathbb{T}^\kappa\).

**Definition 2.5.** A mapping \(f : \mathbb{T} \to \mathbb{R}\) is called **rd-continuous** (denoted by \(C_{rd}\)) provided it satisfies

(1) \(f\) is continuous at each right-dense point or maximal element of \(\mathbb{T}\).
(2) The left-sided limit \( \lim_{s \to t^-} f(s) = f(t-) \) exists at each left-dense point \( t \) of \( T \).

**Remark 2.1.** It follows from Theorem 1.74 of Bohner and Peterson \[2\] that every rd-continuous function has an anti-derivative.

**Definition 2.6.** A function \( F : T \to \mathbb{R} \) is called a \( \Delta \)-antiderivative of \( f : T \to \mathbb{R} \) provided \( F^\Delta(t) = f(t) \) holds for all \( t \in T^\kappa \). Then the \( \Delta \)-integral of \( f \) is defined by

\[
\int_a^b f(t) \Delta t = F(b) - F(a).
\]

**Proposition 2.7.** Let \( f, g \) be rd-continuous, \( a, b, c \in T \) and \( \alpha, \beta \in \mathbb{R} \). Then

1. \( \int_a^b [\alpha f(t) + \beta g(t)] \Delta t = \alpha \int_a^b f(t) \Delta t + \beta \int_a^b g(t) \Delta t \),
2. \( \int_a^b f(t) \Delta t = -\int_b^a f(t) \Delta t \),
3. \( \int_a^b f(t) \Delta t = \int_c^b f(t) \Delta t + \int_a^c f(t) \Delta t \),
4. \( \int_a^b (f(t)g^\Delta(t) \Delta t) = (f g)(b) - (f g)(a) - \int_a^b f^\Delta(t) g(\sigma(t)) \Delta t \),
5. \( \int_a^a f(t) \Delta t = 0 \).

**Definition 2.8.** Let \( h_k : T^2 \to \mathbb{R} \), \( k \in \mathbb{N}_0 \) be defined by

\[
h_0(t, s) = 1 \text{ for all } s, t \in T\]

and then recursively by

\[
h_{k+1}(t, s) = \int_s^t h_k(\tau, s) \Delta \tau \text{ for all } s, t \in T.
\]

3. **The Ostrowski Type Inequality on Time Scales**

Our main result reads as follow.

**Theorem 3.1.** Let \( a, b, s, t \in T \), \( a < b \) and \( f : [a, b] \to \mathbb{R} \) be differentiable. Then

\[
\left| (1 - \lambda)f(t) + \lambda \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f^\sigma(s) \Delta s \right|
\]

\[
\leq \frac{M}{b-a} \left( h_2 \left( a, a + \lambda \frac{b-a}{2} \right) + h_2 \left( t, a + \lambda \frac{b-a}{2} \right) \right)
\]

\[
+ h_2 \left( t, b - \lambda \frac{b-a}{2} \right) + h_2 \left( b, b - \lambda \frac{b-a}{2} \right) \right)
\]

for all \( \lambda \in [0, 1] \) and \( t \in [a + \lambda \frac{b-a}{2}, b - \lambda \frac{b-a}{2}] \cap T \), where

\[
M := \sup_{a < t < b} |f^\Delta(t)| < \infty.
\]
This inequality is sharp provided
\[
\frac{\lambda}{2}a(b-a) + \frac{\lambda^2}{4}(b-a)^2 \leq \int_a^b s \Delta s. \tag{5}
\]

**Remark 3.1.** We note that the condition (5) is trivial if \( \lambda = 0 \).

To prove Theorem 3.1 we need the following Generalized Montgomery Identity.

**Lemma 3.2 (Generalized Montgomery Identity).** Under the assumptions of Theorem 3.1, we have
\[
(1 - \lambda)f(t) + \lambda \frac{f(a) + f(b)}{2} = \frac{1}{b-a} \int_a^b f'(s) \Delta s + \frac{1}{b-a} \int_a^b K(t,s) f'(s) \Delta s,
\]
where
\[
K(t,s) = \begin{cases} 
  s - \left( a + \frac{b-a}{\lambda} \right), & s \in [a,t), \\
  s - \left( b - \frac{b-a}{\lambda} \right), & s \in [t,b]. 
\end{cases} \tag{6}
\]

**Proof.** Integrating by parts and applying Property 2.7, we have
\[
\int_a^b K(t,s) f'(s) \Delta s \\
= \int_a^t \left( s - \left( a + \frac{b-a}{\lambda} \right) \right)f'(s) \Delta s + \int_t^b \left( s - \left( b - \frac{b-a}{\lambda} \right) \right)f'(s) \Delta s \\
= \left( t - \left( a + \frac{b-a}{\lambda} \right) \right)f(t) + \frac{\lambda}{2}(b-a)f(a) - \int_a^t f'(s) \Delta s \\
- \left( t - \left( b - \frac{b-a}{\lambda} \right) \right)f(t) + \frac{\lambda}{2}(b-a)f(b) - \int_t^b f'(s) \Delta s \\
=(b-a) \left( (1 - \lambda)f(t) + \lambda \frac{f(a) + f(b)}{2} \right) - \int_a^b f'(s) \Delta s,
\]
from which we get the desired identity. \( \Box \)

**Corollary 3.3 (Continuous case).** Let \( T = \mathbb{R} \). Then
\[
(1 - \lambda)f(t) + \lambda \frac{f(a) + f(b)}{2} = \frac{1}{b-a} \int_a^b f(s) ds + \frac{1}{b-a} \int_a^b K(t,s) f'(s) ds. \tag{7}
\]

**Remark 3.2.** This is the Montgomery identity in the continuous case, which can be found in [6].
Corollary 3.4 (Discrete case). Let $T = \mathbb{Z}$, $a = 0$, $b = n$, $s = j$, $t = i$ and $f(k) = x_k$. Then
\[
(1 - \lambda)x_i + \lambda \frac{x_0 + x_n}{2} = \frac{1}{n} \sum_{j=1}^{n} x_j + \frac{1}{n} \sum_{j=0}^{n-1} K(i, j) \Delta x_j,
\]
where
\[
K(i, 0) = -\frac{n\lambda}{2},
\]
\[
K(1, j) = j - \left(n - \frac{n\lambda}{2}\right) \text{ for } 1 \leq j \leq n - 1,
\]
\[
K(n, j) = j - \frac{n\lambda}{2} \text{ for } 0 \leq j \leq n - 1,
\]
\[
K(i, j) = \begin{cases} 
  j - \frac{n\lambda}{2}, & j \in [0, i), \\
  j - \left(n - \frac{n\lambda}{2}\right), & j \in [i, n - 1],
\end{cases}
\]
as we just need $1 \leq i \leq n$ and $1 \leq j \leq n - 1$.

Corollary 3.5 (Quantum calculus case). Let $T = q^{\mathbb{N}_0}$, $q > 1$, $a = q^m$, $b = q^n$ with $m < n$. Then
\[
(1 - \lambda)f(t) + \lambda \frac{f(q^m) + f(q^n)}{2} = \sum_{k=m}^{n-1} q^k f(q^{k+1}) + \frac{1}{q^n - q^m} \sum_{k=m}^{n-1} \left[ f(q^{k+1}) - f(q^k) \right] K(t, q^k),
\]
where
\[
K(t, q^k) = \begin{cases} 
  q^k - \left(q^m + \lambda q^m - q^n\right), & q^k \in [q^m, t), \\
  q^k - \left(q^n - \lambda q^m + q^n\right), & q^k \in [t, q^n].
\end{cases}
\]

Proof of Theorem 3.1. By applying Lemma 3.2, we get
\[
\left| (1 - \lambda)f(t) + \lambda \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_{a}^{b} f^\sigma(s) \Delta s \right|
\]
\[
\leq \frac{1}{b - a} \left| \int_{a}^{b} K(t, s) f^\Delta(s) \Delta s \right|
\]
\[
\leq \frac{M}{b - a} \left( \int_{a}^{t} |K(t, s)| \Delta s + \int_{t}^{b} |K(t, s)| \Delta s \right)
\]
\[
= \frac{M}{b - a} \left( \int_{a}^{t} |s - \left(a + \frac{b - a}{2}\right)| \Delta s + \int_{t}^{b} |s - \left(b - \frac{b - a}{2}\right)| \Delta s \right)
\]
\[
= \frac{M}{b - a} \left( \int_{a}^{a + \lambda b - a} |s - \left(a + \frac{b - a}{2}\right)| \Delta s + \int_{a + \lambda b - a}^{a + \lambda b - a} |s - \left(a + \frac{b - a}{2}\right)| \Delta s \right)
\]
\[ \begin{align*}
&= \frac{M}{b - a} \left( \int_{a + \frac{b - a}{2}}^{a} \left( s - (a + \frac{b - a}{2}) \right) \Delta s + \int_{a + \frac{b - a}{2}}^{b} \left( s - \left( a + \frac{b - a}{2} \right) \right) \Delta s \right) \\
&\hspace{1em} + \int_{b - \frac{b - a}{2}}^{b} \left( s - \left( b - \frac{b - a}{2} \right) \right) \Delta s + \int_{b - \frac{b - a}{2}}^{b} \left( s - \left( b - \frac{b - a}{2} \right) \right) \Delta s \\
&= \frac{M}{b - a} \left( h_2 \left( a, a + \frac{b - a}{2} \right) + h_2 \left( t, a + \frac{b - a}{2} \right) \\
&\hspace{1em} + h_2 \left( t, b - \frac{b - a}{2} \right) + h_2 \left( b, b - \frac{b - a}{2} \right) \right) \\
&= \frac{1}{b - a} \left( h_2 \left( a, a + \frac{b - a}{2} \right) + h_2 \left( b - \frac{b - a}{2}, a + \frac{b - a}{2} \right) + h_2 \left( b, b - \frac{b - a}{2} \right) \right). \\
\end{align*} \]

Moreover,

\[ h_2 \left( a, a + \frac{b - a}{2} \right) = \int_{a + \frac{b - a}{2}}^{a} \left( s - \left( a + \frac{b - a}{2} \right) \right) \Delta s = \int_{a + \frac{b - a}{2}}^{a} s \Delta s - \left( a + \frac{b - a}{2} \right) \left( a - \left( a + \frac{b - a}{2} \right) \right) = \int_{a + \frac{b - a}{2}}^{a} s \Delta s + \left( a + \frac{b - a}{2} \right) \frac{b - a}{2}. \]

\[ h_2 \left( b - \frac{b - a}{2}, a + \frac{b - a}{2} \right) = \int_{a + \frac{b - a}{2}}^{b - \frac{b - a}{2}} \left( s - \left( a + \frac{b - a}{2} \right) \right) \Delta s \]
\[ \begin{align*}
\int_{a+\lambda \frac{b-a}{2}}^{b-\lambda \frac{b-a}{2}} s \Delta s - \left(a + \lambda \frac{b-a}{2}\right) \left(b - \lambda \frac{b-a}{2} - \left(a + \lambda \frac{b-a}{2}\right)\right) \\
\int_{a+\lambda \frac{b-a}{2}}^{b-\lambda \frac{b-a}{2}} s \Delta s - \left(a + \lambda \frac{b-a}{2}\right) (b-a) (1-\lambda).
\end{align*} \]

\[ h_2 \left(b, b - \lambda \frac{b-a}{2}\right) = \int_{b-\lambda \frac{b-a}{2}}^{b} \left(s - \left(b - \lambda \frac{b-a}{2}\right)\right) \Delta s \]

\[ = \int_{b-\lambda \frac{b-a}{2}}^{b} s \Delta s - \left(b - \lambda \frac{b-a}{2}\right) \left(b - \left(b - \lambda \frac{b-a}{2}\right)\right) \]

\[ = \int_{b-\lambda \frac{b-a}{2}}^{b} s \Delta s - \left(b - \lambda \frac{b-a}{2}\right) \frac{b-a}{2}. \]

Thus, in this situation, the right-hand side of (4) equals to

\[ \frac{1}{b-a} \left( a+\lambda \frac{b-a}{2} \right) \int_{a}^{b} s \Delta s + \int_{a+\lambda \frac{b-a}{2}}^{b-\lambda \frac{b-a}{2}} s \Delta s + \int_{b-\lambda \frac{b-a}{2}}^{b} s \Delta s \]

\[ + \left(a + \lambda \frac{b-a}{2}\right) \frac{\lambda}{2} - \left(a + \lambda \frac{b-a}{2}\right) (1-\lambda) - \left(b - \lambda \frac{b-a}{2}\right) \frac{\lambda}{2} \]

\[ = \frac{1}{b-a} \left( -2 \int_{a}^{b} s \Delta s - \int_{a}^{b} s \Delta s \right) - (a + \lambda (b-a)) (1-\lambda). \]

Starting with the left-hand side of (4), we have

\[ \left| (1-\lambda) f(t) + \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_{a}^{b} f(\sigma(s)) \Delta s \right| \]

\[ = \left| (1-\lambda) \left(b - \lambda \frac{b-a}{2}\right) + \frac{a+b}{2} - \frac{1}{b-a} \int_{a}^{b} f(\sigma(s)) \Delta s \right| \]

\[ = \left| (1-\lambda) \left(b - \lambda \frac{b-a}{2}\right) + \frac{a+b}{2} + \frac{1}{b-a} \int_{a}^{b} s \Delta s - b-a \right| \]

\[ = -\lambda \left(1-\frac{\lambda}{2}\right) (b-a) - a + \frac{1}{b-a} \int_{a}^{b} s \Delta s, \]
where we have used
\[
\int_a^b \sigma(s) \Delta s = \int_a^b (\sigma(s) + s) \Delta s - \int_a^b s \Delta s = \int_a^b (s^2) \Delta s - \int_a^b s \Delta s = b^2 - a^2 - \int_a^b s \Delta s.
\]

So, if
\[
\lambda^2 a (b - a) + \frac{\lambda^2}{4} (b - a)^2 \leq \int_a^b s \Delta s
\]
holds true, then
\[
\left| - \lambda \left( 1 - \frac{\lambda}{2} \right) (b - a) - a + \frac{1}{b - a} \int_a^b s \Delta s \right|
\]
\[
\geq -\lambda \left( 1 - \frac{\lambda}{2} \right) (b - a) - a + \frac{1}{b - a} \int_a^b s \Delta s
\]
\[
\geq \frac{1}{b - a} \left( 2 - \int_a^b s \Delta s + \int_a^b s \Delta s \right) - (a + \lambda(b - a)) (1 - \lambda),
\]
which helps us to complete our proof.

If we apply the the inequality (4) to different time scales, we will get some well-known and some new results.

**Corollary 3.6 (Continuous case).** Let \( T = \mathbb{R} \). Then our delta integral is the usual Riemann integral from calculus. Hence,
\[
h_2(t, s) = \frac{(t - s)^2}{2}, \quad \text{for all } t, s \in \mathbb{R}.
\]
This leads us to state the following inequality
\[
\left| (1 - \lambda)f(t) + \frac{\lambda f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(s) \, ds \right|
\]
\[
\leq M \left( \frac{1}{4} (b - a) \left( (1 - \lambda)^2 + \lambda^2 \right) + \frac{1}{b - a} \left( x - \frac{a + b}{2} \right)^2 \right)
\]
for all \( \lambda \in [0, 1] \) and \( a + \lambda \frac{b - a}{2} \leq t \leq b - \lambda \frac{b - a}{2} \), where \( M = \sup_{x \in (a,b)} |f'(x)| < \infty \), which is exactly the generalized Ostrowski type inequality shown in Theorem 2 of [6].

**Corollary 3.7 (Discrete case).** Let \( T = \mathbb{Z} \), \( a = 0 \), \( b = n \), \( s = j \), \( t = i \) and \( f(k) = x_k \). Thus, we have
\[
\left| (1 - \lambda) x_i + \frac{x_0 + x_n}{2} - \frac{1}{n} \sum_{j=1}^n x_j \right| \leq M \left( \frac{1}{4} \left( \frac{a + b}{2} \right)^2 + \frac{2\lambda^2 - 2\lambda + 1}{4} n^2 - 1 \right)
\]
for all \( i \in \left[ \frac{n}{2}, n - \frac{n}{2} \right] \cap \mathbb{T} \), where \( M = \max_{1 \leq i \leq n-1} |\Delta x_i| < \infty \).
Proof. In this situation, it is known that
\[ h_k(t, s) = \binom{t-s}{k}, \quad \text{for all} \quad t, s \in \mathbb{Z}. \]
Therefore,
\[
\begin{align*}
    h_2\left( a, a + \lambda \frac{b-a}{2} \right) &= \left( -\frac{n\lambda}{2} \right) = \frac{n\lambda}{2} \left( \frac{n\lambda}{2} + 1 \right), \\
    h_2\left( t, a + \lambda \frac{b-a}{2} \right) &= \left( i - \frac{n\lambda}{2} \right) = \frac{(i - \frac{n\lambda}{2} - 1)}{2}, \\
    h_2\left( t, b - \lambda \frac{b-a}{2} \right) &= \left( i - n + \frac{n\lambda}{2} \right) = \frac{(i - n + \frac{n\lambda}{2} - 1)}{2}, \\
    \text{and} \\
    h_2\left( t, b - \lambda \frac{b-a}{2} \right) &= \left( \frac{n\lambda}{2} \right) = \frac{n\lambda}{2} \left( \frac{n\lambda}{2} - 1 \right),
\end{align*}
\]
Thus, we get the desired result. □

**Corollary 3.8.** (Quantum calculus case). Let \( T = q^{\mathbb{N}_0}, q > 1, a = q^m, b = q^n \) with \( m < n \). Then
\[
\begin{align*}
    \left| (1 - \lambda)f(t) + \lambda \frac{f(q^n) + f(q^m)}{2} - \frac{1}{q^n - q^m} \int_{q^m}^{q^n} f^\sigma(s) \Delta s \right| \\
    \leq \frac{M}{(1 + q)(q^n - q^m)} \left( 2t^2 - (1 + q)(q^m + q^n)t \\
    + \left( \left( 2\lambda^2 - \frac{3}{2}\lambda + 1 \right) (q^{2m+1} + q^{2n+1}) - \lambda(3 - 2\lambda)q^{m+n+1} + \frac{\lambda}{2}(q^m - q^n)^2 \right) \right)
\end{align*}
\]
for all \( t \in \left[ q^m + \lambda \frac{q^n - q^m}{2}, q^n - \lambda \frac{q^n - q^m}{2} \right] \cap T \)
where
\[
M = \sup_{t \in (q^m, q^n)} \left| \frac{f(qt) - f(t)}{(q-1)t} \right|.
\]
Proof. In this situation, one has
\[
\begin{align*}
    h_k(t, s) = \prod_{\nu=0}^{k-1} \frac{t - q^\nu s}{q^\nu}, \quad \text{for all} \quad t, s \in T \\
\end{align*}
\]
and
\[
\begin{align*}
    f^\Delta(t) &= \frac{f(qt) - f(t)}{(q-1)t}.
\end{align*}
\]
Therefore,
\[
\begin{align*}
    h_2\left( q^m, q^m + \lambda \frac{q^n - q^m}{2} \right) &= \frac{\lambda}{2} \left( q^n - q^m \right) \frac{(q^m - (1 - \frac{\lambda}{2})) q^{m+1} - \frac{\lambda}{2} q^{n+1}}{1 + q},
\end{align*}
\]
\[ h_2 \left( t, q^n + \frac{\lambda q^n - q^m}{2} \right) = \frac{\left[ t - (1 - \frac{\lambda}{2}) q^n - \frac{\lambda}{2} q^m \right] \left[ t - (1 - \frac{\lambda}{2}) q^{m+1} - \frac{\lambda}{2} q^{n+1} \right]}{1 + q}, \]

\[ h_2 \left( t, q^n - \frac{\lambda q^n - q^m}{2} \right) = \frac{\left[ t - (1 - \frac{\lambda}{2}) q^n - \frac{\lambda}{2} q^m \right] \left[ t - (1 - \frac{\lambda}{2}) q^{m+1} - \frac{\lambda}{2} q^{n+1} \right]}{1 + q}, \]

and

\[ h_2 \left( q^n - \frac{\lambda q^n - q^m}{2} \right) = \frac{\lambda (q^n - q^m) \left[ q^n - (1 - \frac{\lambda}{2}) q^{n+1} - \frac{\lambda}{2} q^{m+1} \right]}{1 + q}. \]

Thus, we get the result. \( \square \)

4. Some particular Ostrowski type inequalities on time scales

In this section we point out some particular Ostrowski type inequalities on time scales as special cases, such as: rectangle inequality on time scales, trapezoid inequality on time scales, midpoint inequality on time scales, Simpson inequality on time scales, averaged midpoint-trapezoid inequality on time scales and others.

Throughout this section, we always assume \( T \) is a time scale; \( a, b \in T \) with \( a < b \); \( f : [a, b] \to \mathbb{R} \) is differentiable. We denote

\[ M = \sup_{a < x < b} |f^\Delta(x)|. \]

**Corollary 4.1.** Under the assumptions of Theorem 3.1 with \( \lambda = 1 \) and \( t = \frac{a + b}{2} \in T \). Then we have the trapezoid inequality on time scales

\[ \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f^\sigma(s)\Delta s \right| \leq M \frac{h_2 \left( a, \frac{a + b}{2} \right) + h_2 \left( b, \frac{a + b}{2} \right)}{b - a}. \]  

(8)

**Remark 4.1.** If we take \( \lambda = 0 \) in Theorem 3.1 then Theorem 1.2 is recaptured. Therefore, Theorem 3.1 may be regarded as a generalization of Theorem 1.2.

**Corollary 4.2.** Under the assumptions of Theorem 3.1 with \( \lambda = \frac{1}{3} \). Then we have the following integral inequality on time scales

\[ \left| \frac{1}{6} (f(a) + f(b) + 4f(t)) - \frac{1}{b - a} \int_a^b f^\sigma(s)\Delta s \right| \]

\[ \leq \frac{M}{b - a} \left( h_2 \left( a, \frac{5a + b}{6} \right) + h_2 \left( t, \frac{5a + b}{6} \right) \right) + h_2 \left( t, \frac{a + 5b}{6} \right) + h_2 \left( b, \frac{a + 5b}{6} \right) \]  

(9)

for all \( t \in \left[ \frac{5a + b}{6}, \frac{a + 5b}{6} \right] \cap T. \)
Remark 4.2. If we choose \( t = \frac{a+b}{2} \) in (9), we get the Simpson inequality on time scales
\[
\left| \frac{1}{6} \left( f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right) - \frac{1}{b-a} \int_a^b f^\sigma(s) \Delta s \right| \\
\leq \frac{M}{b-a} \left( h_2 \left( a, \frac{5a+b}{6} \right) + h_2 \left( \frac{a+b}{2}, \frac{5a+b}{6} \right) \\
+ h_2 \left( \frac{a+b}{2}, \frac{a+5b}{6} \right) + h_2 \left( b, \frac{a+5b}{6} \right) \right).
\]

Corollary 4.3. Under the assumptions of Theorem 3.1 with \( \lambda = \frac{1}{2} \). Then we have the following integral inequality on time scales
\[
\left| \frac{1}{2} \left( \frac{f(a) + f(b)}{2} + f(t) \right) - \frac{1}{b-a} \int_a^b f^\sigma(s) \Delta s \right| \\
\leq \frac{M}{b-a} \left( h_2 \left( a, \frac{3a+b}{4} \right) + h_2 \left( t, \frac{3a+b}{4} \right) \\
+ h_2 \left( t, \frac{a+3b}{4} \right) + h_2 \left( b, \frac{a+3b}{4} \right) \right)
\]
for all \( t \in \left[ \frac{3a+b}{4}, \frac{a+3b}{4} \right] \cap \mathbb{T} \).

Remark 4.3. If we choose \( t = \frac{a+b}{2} \) in (10), we get the averaged mid-point-trapezoid inequality on time scales
\[
\left| \frac{1}{2} \left( \frac{f(a) + f(b)}{2} + f \left( \frac{a+b}{2} \right) \right) - \frac{1}{b-a} \int_a^b f^\sigma(s) \Delta s \right| \\
\leq \frac{M}{b-a} \left( h_2 \left( a, \frac{3a+b}{4} \right) + h_2 \left( \frac{a+b}{2}, \frac{3a+b}{4} \right) \\
+ h_2 \left( \frac{a+b}{2}, \frac{a+3b}{4} \right) + h_2 \left( b, \frac{a+3b}{4} \right) \right).
\]

Corollary 4.4. Under the assumptions of Theorem 3.1 with \( t = \frac{a+b}{2} \in \mathbb{T} \). Then we have the following integral inequality on time scales
\[
\left| (1 - \lambda) f \left( \frac{a+b}{2} \right) + \lambda f(a) + f(b) - \frac{1}{b-a} \int_a^b f^\sigma(s) \Delta s \right| \\
\leq \frac{M}{b-a} \left( h_2 \left( a, a + \lambda \frac{b-a}{2} \right) + h_2 \left( \frac{a+b}{2}, a + \lambda \frac{b-a}{2} \right) \\
+ h_2 \left( \frac{a+b}{2}, b - \lambda \frac{b-a}{2} \right) + h_2 \left( b, b - \lambda \frac{b-a}{2} \right) \right)
\]
for all \( \lambda \in [0, 1] \).
Remark 4.4. If we choose $\lambda = 0$ in (11), we get the mid-point inequality on time scales

$$\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_{a}^{b} f'(t) \Delta t \right| \leq \frac{M}{b-a} \left( h_{2} \left( \frac{a+b}{2}, a \right) + h_{2} \left( \frac{a+b}{2}, b \right) \right).$$

Acknowledgements

This work was supported by the Science Research Foundation of Nanjing University of Information Science and Technology and the Natural Science Foundation of Jiangsu Province Education Department under Grant No.07KJD510133.

References

[1] R. Agarwal, M. Bohner and A. Peterson, Inequalities on time scales: A survey, Math. Inequal. Appl., 4(4) (2001), 535-557.
[2] M. Bohner and A. Peterson, Dynamic Equations on Time Series, Birkhäuser, Boston, 2001.
[3] M. Bohner and A. Peterson, Advances in Dynamic Equations on Time Series, Birkhäuser, Boston, 2003.
[4] M. Bohner and T. Mathewa, The Grüss inequality on time scales, Communications in Mathematical Analysis, 3 (1) (2007), 1-8.
[5] M. Bohner and T. Mathewa, Ostrowski inequalities on time scales, J. Inequal. Pure Appl. Math., 9 (1) (2008), Art. 6, 8 pp.
[6] S. S. Dragomir, P. Cerone and J. Roumelitis, A new generalization of Ostrowski integral inequality for mappings whose derivatives are bounded and applications in numerical integration and for special means, Appl. Math. Lett., 13 (1) (2000), 19-25.
[7] S. Hilger, Ein Maßkettenkalkül mit Anwendung auf Zentrumsmanigfaltigkeiten, PhD thesis, Univesi. Würzburg, 1988.
[8] V. Lakshmikantham, S. Sivasundaram, and B. Kaymakcalan, Dynamic Systems on Measure Chains, Kluwer Academic Publishers, 1996.
[9] W. J. Liu, Q. L. Xue and S. F. Wang, Several new perturbed Ostrowski-like type inequalities, J. Inequal. Pure Appl. Math., 8(4) (2007), Art.110, 6 pp.
[10] W. J. Liu, C. C. Li and Y. M. Hao, Further generalization of some double integral inequalities and applications, Acta. Math. Univ. Comenianae, 77 (1)(2008), 147-154.
[11] W. J. Liu, Several error inequalities for a quadrature formula with a parameter and applications, Comput. Math. Appl., accepted.
[12] W. J. Liu and Q. A. Ngô, An Ostrowski-Grüss type inequality on time scales, arXiv: 0804.3231v1.
[13] D. S. Mitrovič, J. Pecarić and A. M. Fink, Inequalities for Functions and Their Integrals and Derivatives, Kluwer Academic, Dordrecht, (1994).
[14] D. S. Mitrovič, J. Pecarić and A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic, Dordrecht, (1993).
[15] H. Roman, A time scales version of a Wirtinger-type inequality and applications, Dynamic equations on time scales, J. Comput. Appl. Math., 141 (1/2) (2002), 219-226.
[16] F.-H. Wong, S.-L. Yu, C.-C. Yeh, Andersons inequality on time scales, Applied Mathematics Letters, 19 (2007), 931-935.
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