LOG-SCALE EQUIDISTRIBUTION OF NODAL SETS IN GRAUERT TUBES

ROBERT CHANG AND STEVE ZELDITCH

Abstract. Let $M_{\tau_0}$ be the Grauert tube (of some fixed radius $\tau_0$) of a compact, negatively curved, real analytic Riemannian manifold $M$ without boundary. Let $\varphi_\lambda$ be a Laplacian eigenfunction on $M$ of eigenvalues $-\lambda^2$ and let $\varphi_\lambda^C$ be its holomorphic extension to $M_{\tau_0}$. In this article, we prove that on $M_{\tau_0} \setminus M$, there exists a dimensional constant $\alpha > 0$ and a full density subsequence $\{\lambda_j\}_{k=1}^\infty$ of the spectrum for which the masses of the complexified eigenfunctions $\varphi_{\lambda_j}^C$ are asymptotically equidistributed at length scale $(\log \lambda_j)^{-\alpha}$. Moreover, the complex zeros of $\varphi_{\lambda_j}^C$ also become equidistributed on this logarithmic length scale.

1. Introduction

Let $(M^n, g)$ be a compact, negatively curved, real analytic Riemannian manifold without boundary. Let $\Delta = \Delta_g$ be the (negative) Laplacian. We denote by $\{\varphi_j\}_{j=0}^\infty$ an orthonormal basis of eigenfunctions:

$$(\Delta + \lambda_j^2)\varphi_j = 0,$$

where (as usual) eigenvalues are enumerated in increasing order $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \uparrow \infty$. To date, no distribution law is known for real nodal sets of Laplacian eigenfunctions on $M$. But, in the manner of [Ze1], we are able to obtain a small-scale limit distribution of the complex nodal sets of the analytic continuations of eigenfunctions to the Grauert tube $M_{\tau_0}$ of $M$. By ‘small-scale’ we mean length scales that shrink logarithmically with respect to the frequency parameter $\lambda_j$. This is the smallest scale to which quantum ergodicity may be presently localized, as seen in Hezari-Riviere [HeR] and Han [Ha]. Along individual geodesics, equidistribution of complex nodal sets is proved down to the scale $\lambda^{-1}$ in [Ze3] using quite different arguments.

By a well-known theorem of Bruhat-Whitney, any real analytic manifold $M$ admits a complexification $M_C$ into which it embeds as a totally real submanifold. The metric $g$ on $M$ induces a plurisubharmonic function $\rho$ whose square root $\sqrt{\rho}: M_C \to [0, \infty)$ is called the Grauert tube function. There exists a geometric constant $\tau_0 = \tau_0(M, g)$ so that, for each $\tau \leq \tau_0$, the sublevel set

$$M_\tau := \{\zeta \in M_C: \sqrt{\rho}(\zeta) < \tau\}$$

is a strictly pseudo-convex domain in $M_C$. We call $M_\tau$ the Grauert tube of $M$ of radius $\tau$. The $(1, 1)$-form $\omega := -i\partial \bar{\partial} \rho$ endows $M_\tau$ with a Kähler metric and $(M, g) \hookrightarrow (M_\tau, \omega)$ is an isometric embedding. (The unusual sign convention that makes the Kähler form negative is
adopted from [GST1].) We write
\[ d\mu = \omega^n \quad \text{and} \quad d\mu_\tau := \frac{\omega^n}{d\sqrt{\rho} |\partial M_\tau |} = \frac{\omega^n}{d\tau}. \] (1)
for the Kähler volume form on \( M_\tau \) and the Liouville surface measure on \( \partial M_\tau \), respectively.

There exists a diffeomorphism \( E \), defined in (7), between \( M_\tau \) and the co-ball bundle \( B_\tau^*M = \{ (x,\xi) \in T^*M : |\xi|_{g_\tau} < \tau \} \). The Kähler form \( \omega \) on \( M_\tau \) is the pullback under \( E \) of the standard symplectic form on \( B_\tau^*M \). Conversely, \( E \) endows \( B_\tau^*M \) with a complex structure \( J_g \) adapted to \( g \). Definitions and background are recalled in Section 2; see also [GST2, LS1].

Every eigenfunction \( \varphi_j \) on \( M \) admits an analytic extension \( \varphi_j^C \) to the maximal Grauert tube \( M_{\tau_0} \). The analytically continued eigenfunctions are smooth on the boundaries for every \( \tau \leq \tau_0 \). The complex zero set of \( \varphi_j^C \) is the complex hypersurface
\[ Z_j := \{ \zeta \in M_{\tau_0} : \varphi_j^C(\zeta) = 0 \}. \]
The zero sets define currents \([Z_j]\) of integration in the sense that for every smooth \((n-1,n-1)\) test form \( \eta \in \mathcal{D}^{n-1,n-1}(M_{\tau_0}) \), we the pairing
\[ \langle [Z_j], \eta \rangle := \int_{Z_j} \eta = \int_{M_{\tau_0}} \frac{i}{2\pi} \partial \bar{\partial} \log |\varphi_j^C|^2 \wedge \eta \]
is a well-defined closed current\(^1\) In the special case \( \eta = f\omega^{n-1} \), the zero set defines a positive measure \(|Z_j|\) by
\[ \langle |Z_j|, f \rangle := \int_{Z_j} f\omega^{n-1}, \quad f \in C(M_{\tau_0}). \]

The limit distribution of the zero currents \([Z_j]\) has been investigated in [Ze1]. It was shown that on a compact, real analytic, negatively curved manifold, one has
\[ \frac{1}{\lambda_j} [Z_{j_k}] \rightharpoonup \frac{i}{\pi} \partial \bar{\partial} \sqrt{\rho} \quad \text{weakly as currents on } M_{\tau_0} \] (2)
along a density one subsequence of eigenvalues \( \lambda_{j_k} \). The motivating problem of this article is to obtain a similar convergence theorem on balls in \( M_{\tau_0} \setminus M \) with logarithmically shrinking radii of size
\[ \varepsilon(\lambda_j) := (\log \lambda_j)^{-\alpha} \quad \text{for some fixed } \alpha > 0 \text{ to be specified.} \]
The parameter \( \alpha \) depends only on the dimension, and is independent of the frequency \( \lambda_j \). The resulting log-scale convergence theorems, Theorem 1 and Theorem 8.2, along with their proofs, are generalizations of those in [CZ] in the setting of eigensections of ample line bundles over a compact boundaryless Kähler manifold, but have several new features.

**Theorem 1.** Let \((M,g)\) be a real analytic, negatively curved, compact manifold without boundary. Let \( \omega := -i\partial \bar{\partial} \rho \) be the Kähler form on the Grauert tube \( M_{\tau_0} \). Assume that
\[ 0 \leq \alpha < \frac{1}{2(3n-1)}, \quad \varepsilon(\lambda_j) = (\log \lambda_j)^{-\alpha}. \]

\(^1\)Since \( Z_j \) may be singular, we include background on the last statement in [A].
Then there exists a full density subsequence of eigenvalues $\lambda_{jk}$ such that for any $f \in C(M_{\tau_0})$ and for any arbitrary but fixed $\zeta_0 \in M_{\tau_0} \setminus M$, we have

$$\left| \frac{1}{\lambda_{jk} \varepsilon(\lambda_{jk})^{2n-1}} \int_{Z_{jk} \cap B(\zeta_0, \varepsilon(\lambda_{jk}))} f \omega^{n-1} \right| = \frac{1}{\varepsilon(\lambda_{jk})^{2n-1}} \int_{B(\zeta_0, \varepsilon(\lambda_{jk}))} f \frac{i}{\pi} \partial \bar{\partial} |\Im(\zeta - \zeta_0)|_{g_0} \wedge \omega^{n-1} = o(1). \quad (3)$$

Here, $\omega_0 := -i \partial \bar{\partial} |\Im(\zeta - \zeta_0)|_{g_0}^2$ denotes the flat Kähler form in local Kähler coordinates centered at $\zeta_0$, with $| \cdot |_{g_0}$ the Euclidean distance. The $o(1)$ remainder is uniform for any $\zeta_0$ lying in an ‘annulus’ $0 < \tau_1 \leq \sqrt{p}(\zeta_0) \leq \tau_0$.

Theorem 1 is deduced from a rescaled version given in Theorem 8.2. The latter theorem is stated using the holomorphic dilation introduced in Section 3.1. Briefly, define dilation operator $D^{\varepsilon(\lambda)}_{(\lambda)} : \zeta \mapsto \zeta + \varepsilon(\lambda)(\zeta - \zeta_0)$ in Kähler normal coordinates around $\zeta_0$. The zero currents $[Z_j]$ on shrinking balls $B(\zeta_0, \varepsilon(\lambda_j))$ pull back to currents $D^{\varepsilon(\lambda_j)}_{(\lambda_j)}[Z_j]$ on a fixed unit ball $B(\zeta_0, 1) \subset \mathbb{C}^n$. The normalizing factors in Theorem 1 arise from homogeneity and rescaling: $\omega^{n-1}$, $\omega_0^{n-1}$ are homogeneous of degree $2n-2$ and $\frac{i}{\pi} \partial \bar{\partial} |\Im(\zeta - \zeta_0)|_{g_0}$ is homogeneous of degree 1. The scaling of the nodal current on the left side is the same as that of its limit current $\frac{i}{\pi} \partial \bar{\partial} |\Im(\zeta - \zeta_0)|_{g_0}$.

**Remark 1.1.** In the statement of Theorem 1, the center $\zeta_0$ is arbitrary but fixed in the interior of $M_{\tau_0} \setminus M$ and only the radii of the balls are shrinking. Also, note that $\zeta_0$ must lie away from the totally real submanifold $M$ of $M_{\tau_0}$, or equivalently the zero section $0_M$ of $B^* M$. Reasons are discussed in Section 1.2.

**Remark 1.2.** The zero sets $Z_j$ may be singular but it is known that the singular set of the real nodal set is of real codimension four (see [A]). For generic metrics, all of the nodal sets are regular [U].

Knowledge of the log-scale $L^2$ masses of eigenfunctions is required to deduce Theorem 1. To state the relevant result, we need some more notation:

$$\Theta_j(\zeta) := \| \varphi_j^C \|_{L^2(\partial M, \nu(\zeta))}, \quad U_j(\zeta) := \frac{\varphi_j^C(\zeta)}{\Theta_j(\zeta)}, \quad (\zeta \in M_{\tau_0} \setminus M)$$

In words, the normalizing factor $\Theta_j(\zeta)$ is the $L^2$-norm (of the restriction $\varphi_j^C \big|_{\partial M, \nu(\zeta)}$) of $\varphi_j^C$ along the boundary of the Grauert tube of radius $\sqrt{p}(\zeta)$. The function $U_j$ is the (unrestricted) complexified eigenfunction $\varphi_j^C$ normalized by this $L^2$-norm. Finally, let

$$u_j^\tau(Z) := U_j(Z) \big|_{\partial M} = \frac{\varphi_j^C(Z) \big|_{\partial M}}{\| \varphi_j^C \big|_{\partial M} \|_{L^2(\partial M)}} \quad (Z \in \partial M, \ 0 < \tau \leq \tau_0)$$

be the restriction of $U_j$ to the Grauert tube of radius $\sqrt{p}(\zeta) = \tau$. (We denote points by $Z$ instead of $\zeta$ when working on a fixed slice $\partial M$.) The global behavior of $L^2$ masses of $U_j$ and $u_j^\tau$ are known. Specifically, [Ze1] Lemma 1.4, Lemma 4.1 proved the existence of a density one subsequence $\{ \varphi_{jk} \}$ of orthonormal basis such that

$$|U_{jk}|^2 \omega^n \to \omega^n \quad \text{and} \quad |u_{jk}^\tau|^2 d\mu_\tau \to d\mu_\tau \quad (4)$$
in the sense of weak* convergence on \( C(M_{\tau_0}) \) and on \( C(\partial M_\tau) \) for each \( 0 < \tau \leq \tau_0 \), respectively. (Recall \( \text{(1)} \) for the definitions.) Integrating over \( M_{\tau_0} \) (resp. \( \partial M_\tau \)) implies the \( L^2 \) masses of \( U_{jk} \) (resp. \( u_{jk}^\tau \)) become equidistributed in all of \( M_{\tau_0} \) (resp. \( \partial M_\tau \)). It is not known whether the convergence \( \text{(1)} \) holds at logarithmic length scales (i.e., simultaneously on all Kähler balls of logarithmically shrinking radii). Luckily, all that is needed for the proof of Theorem \( \text{(1)} \) is a uniform \( L^2 \) volume comparison theorem, which we presently state.

**Theorem 2.** Let \((M,g)\) be a real analytic, negatively curved, compact manifold without boundary. Let \( \omega := -i\partial\bar{\partial}\sqrt{\rho} \) denote the Kähler form on the Grauert tube \( M_{\tau_0} \). Assume that

\[
0 \leq \alpha < \frac{1}{2(3n-1)}, \quad \varepsilon(\lambda_j) = (\log \lambda_j)^{-\alpha}.
\]

Then there exists a full density subsequence of eigenvalues \( \lambda_{jk} \) such that for arbitrary but fixed \( \zeta_0 \in M_{\tau_0} \setminus M \), there is a uniform two-sided volume bound

\[
c \Vol_{\omega}(B(\zeta_0, \varepsilon(\lambda_{jk}))) \leq \int_{B(\zeta_0, \varepsilon(\lambda_{jk}))} |U_{jk}|^2 d\mu \leq C \Vol_{\omega}(B(\zeta_0, \varepsilon(\lambda_{jk}))). \tag{5}
\]

The constants \( c, C \) are geometric constants depending only on \( \sqrt{\rho}(\zeta_0) \); they are uniform for any \( \zeta_0 \) lying in an ‘annulus’ \( 0 < \tau_1 < \sqrt{\rho}(\zeta_0) \leq \tau_0 \).

**Remark 1.3.** Only the lower bound in the statement of Theorem 2 – used crucially in a proof by contradiction argument for Proposition 8.5 around \( \text{(11)} - \text{(12)} \) – is needed to imply Theorem \( \text{(1)} \).

Log-scale results of this kind, which we briefly recall in Section 6 were first proved in the real domain by Hezari-Rivière [HeR] and X. Han [Ha]. In the setting of a general compact, negatively curved, Kähler manifold (not necessarily real analytic), an analogous result can be found in [CZ, Theorem 2].

**Remark 1.4.** The semi-classical notation \( h := \lambda^{-1} \) is also used throughout Section 4–7 in which we write \( \delta(h) = ||\log h||^{-\alpha} = (\log \lambda)^{-\alpha} = \varepsilon(\lambda) \); see \( \text{(7)} \).

1.1. Outline of proof. Theorem 2 is proved by expressing the \( L^2 \) mass of \( u_{jk}^\tau \) (resp. \( U_{jk} \)) in terms of matrix elements of Szegő-Toeplitz operators on \( \partial M_\tau \) for \( 0 < \tau \leq \tau_0 \) (resp. Bergman-Toeplitz operators on \( M_{\tau_0} \)). We show that a certain Poisson-FBI transform conjugates a (smoothed) characteristic function of the ball \( B(\zeta_0, \varepsilon(\lambda_j)) \) to a semi-classical pseudodifferential operator acting on \( L^2(M) \) whose symbol has the same properties as (but does not coincide with) the small-scale symbols used in [Ha]. This conjugation allows us to derive Proposition 7.2, a variance estimate for matrix elements in the complex domain, by relating it to the known variance estimate in the real domain of [Ha].

Once the variance estimate is proved, the comparability result of Theorem 2 follows the path in [HeR, Ha, CZ]. Namely, one chooses an appropriate covering of \( M_{\tau_0} \) and extracts a subsequence of eigenvalues of density one for which one has simultaneous asymptotic log-scale QE for the balls in every cover. The balls are ‘dense enough’ that one obtains good upper and lower bounds for eigenfunction mass in any logarithmically shrinking ball.

Lastly, to derive Theorem \( \text{(1)} \) from Theorem 2, we follow the method of [SZc, CZ] that uses well-known facts about plurisubharmonic functions. We begin by rewriting the zero current \( [Z_j] \) as \( \partial\bar{\partial} \) of plurisubharmonic functions using the Poincaré-Lelong formula \( \text{(32)} \). A standard compactness theorem yields the desired result.
1.2. **Singular behavior along the real domain.** We briefly discuss the reasons for requiring centers \( \zeta_0 \) of balls to lie in \( M_0 \backslash M \).

The key tool in studying the mass and zeros in the complex domain is the complexified Poisson operator \( P^\tau: L^2(M) \to O^{\mathbb{H}^{-1}}(\partial M_\tau) \) defined in Section 2.3. By \( O^{\mathbb{H}^{-1}}(\partial M_\tau) \) we mean the Hardy-Sobolev space of boundary values of holomorphic functions in \( M_\tau \) with the designated Sobolev regularity. This Hilbert space is the quantization of the symplectic cone \( \Sigma_\tau \subset T^*(\partial M_\tau) \) defined in Section 2.2, an \( \mathbb{R}_+ \)-bundle \( \Sigma_\tau \to \partial M_\tau \). The Poisson operator is a homogeneous Fourier integral operator with positive complex phase adapted to the homogeneous symplectic isomorphism \( \iota_\tau: T^*M \backslash 0 \to \Sigma_\tau \) of (9).

The homogeneous theory becomes singular along the zero section \( 0 \in \Sigma \), or equivalently along the totally real submanifold \( M \). This reflects the fact that the eigenfunctions \( \varphi_j \) microlocally concentrate on energy surfaces \( \{ |\xi|_g = \lambda_j \} \), the characteristic variety of \( \Delta + \lambda_j^2 \). In the semi-classical setting of \( h^2 \Delta + 1 \) (with \( h = \lambda_j^{-1} \)), the eigenfunctions concentrate on \( S^*M \). The energy level 1 is arbitrary here and depends on the choice of constant \( C \) in the semi-classical scaling \( h_j = C \lambda_j^{-1} \). One may adjust it so that eigenfunctions concentrate on any energy surface \( \partial B^{*}_\tau \) with respect to semi-classical pseudodifferential operators \( \text{Op}_{h_j}(a) \). But this scaling breaks down on the zero section.

The singularity of the theory along the zero section may be seen in Theorem 4.1. When conjugated back to the real domain, the symbols become functions of \( |\xi| \) and are singular when \( \xi = 0 \). It seems that the behavior on the zero section can be studied by using an adapted class of observables that smoothly interpolates between pseudodifferential operators when \( \tau = 0 \) and Toeplitz operators when \( \tau > 0 \). We hope to clarify this issue in the future.

1.3. **Acknowledgments.** We thank the referee for a very careful reading of the manuscript and for pointing out numerous corrections. We also thank B. Shiffman for contributing to A.

## 2. Background

### 2.1. Grauert tube and the co-ball bundle.** The readers are referred to [GSt1, GSt2, LS1, LS2] for the analysis of the complex Monge-Ampère equation, the Grauert tube function, the geometry of Grauert tubes and related topics. Here we provide only a brief summary of some notation and theorems needed for this paper, following [Ze1, Ze3].

A real analytic manifold \( (M, g) \) always possesses a complexification \( M_\mathbb{C} \), that is, a complex manifold of which \( M \) is a totally real embedded submanifold. Let \( \exp_x: T_x^*M \to M \) be the Riemannian exponential map, i.e., \( \exp_x \xi = \pi \exp t \Xi_{|\xi|_g^2} \), where \( \pi: T^*M \to M \) is the natural projection and \( \Xi_{|\xi|_g^2} \) is the Hamiltonian flow of \( |\xi|_g^2 \). The analyticity of \( M \) implies that the exponential map admits an analytic extension

\[
\exp_x^\mathbb{C}: U_x \subset T_x^*M \otimes \mathbb{C} \to M_\mathbb{C}
\]

defined in a suitable domain \( U_x \subset T_x^*M \). Its restriction to the imaginary axis (that is, the analytic extension in \( t \) of \( \exp_x(t \xi) \) to imaginary time \( t = i \)) is denoted by

\[
E: B^*_\tau M \to M_\mathbb{C}, \quad (x, \xi) \mapsto E(x, \xi) := \exp_x^\mathbb{C}(i \xi).
\]
For all $\tau > 0$ sufficiently small, $\pi$ is a diffeomorphism between the co-ball bundle $B^*_\tau M = \{(x, \xi) \in T^* \tau M : |\xi|_{g_\tau} < \tau\}$ and the subset

$$M_\tau := \{\zeta \in M_C : \sqrt{\rho}(\zeta) < \tau\} \subset M_C.$$

Here, $\sqrt{\rho}$ is known as the Grauert tube function, and its sublevel set $M_\tau$ is known as the Grauert tube (of radius $\tau$). The restriction $E|_{\partial B^*_\tau M}$ of $\pi$ to the co-sphere bundle is a CR holomorphic diffeomorphism between the two strictly pseudo-convex CR manifolds $\partial B^*_\tau M$ and $\partial M_\tau$.

The square $\rho$ of the Grauert tube function is a strictly plurisubharmonic function uniquely determined by two conditions:

- It is a solution of the Monge-Ampère equation $(\partial \overline{\partial} \sqrt{\rho})^n = \delta_M$, where $\delta_M$ is the delta-function on the real manifold $M$ with respect to the volume form $dV_g$.
- The Kähler form $\omega := -i\partial \overline{\partial} \rho$ restricts to $g$ along $M$.

If we write $r(x, y)$ for the Riemannian distance function on $M$, then $r^2(x, y)$ is real analytic in a neighborhood of the diagonal in $M \times M$. It possesses an analytic continuation $r^2(\zeta, \overline{\zeta})$ for $\zeta \in M_C$ in a sufficiently small neighborhood of the totally real submanifold $M$. The plurisubharmonic function is related to the Riemannian distance function by

$$\rho(\zeta) = \frac{1}{4} r^2(\zeta, \overline{\zeta}).$$

For the trivial case $M = \mathbb{R}^n$, we have $M_C = \mathbb{C}^n$ and $\sqrt{\rho}(\zeta) = \sqrt{-\frac{1}{4}(\zeta - \overline{\zeta})^2} = |\text{Im} \zeta|$. More examples are found in [Ze1].

### 2.2. Szegő projector.

Let $\mathcal{O}(\partial M_\tau)$ denote the space of CR holomorphic functions on $\partial M_\tau$. We use the notation

$$\mathcal{O}^{s+\frac{n+1}{4}}(\partial M_\tau) := W^{s+\frac{n+1}{4}}(\partial M_\tau) \cap \mathcal{O}(\partial M_\tau)$$

for the subspace of the Sobolev space $W^{s+\frac{n+1}{4}}(\partial M_\tau)$ consisting of CR holomorphic functions. The inner product is taken with respect to the Liouville surface measure (1). The Szegő projector

$$\Pi_\tau : L^2(\partial M_\tau) \rightarrow \mathcal{O}^0(\partial M_\tau)$$

is the orthogonal projection onto boundary values of holomorphic function. It is well-known (cf. [BoS, MS, GSt2]) that $\Pi_\tau$ is a complex Fourier integral operator of positive type, whose real canonical relation is the graph of the identity map on the symplectic cone

$$\Sigma_\tau = \{(Z ; rd^c \sqrt{\rho}(Z)) \in T^*(\partial M_\tau) : Z \in \partial M_\tau, r > 0\}$$

spanned by the contact form $d^c \sqrt{\rho} = -i(\partial - \overline{\partial}) \sqrt{\rho}$ on $\partial M_\tau$. Since $\Sigma_\tau$ is an $\mathbb{R}_+$-bundle over $\partial M_\tau$, we can define the symplectic equivalence of cones:

$$\iota_\tau : T^* M \setminus 0 \rightarrow \Sigma_\tau, \quad \iota_\tau(x, \xi) := \left( E\left(x, \tau \frac{\xi}{|\xi|}\right), |\xi| d^c \sqrt{\rho}_{E(x, \tau \frac{\xi}{|\xi|})} \right).$$

(9)
2.3. Poisson-wave operator. A key object in our analysis is the Poisson-wave operator

\[ P^\tau : L^2(M) \to O^{\frac{n-1}{4}}(\partial M_\tau). \]

(Unlike for the Szegő projector \((\mathcal{S})\), \(\tau\) appears as a superscript here because we will be considering semi-classical Poisson-wave operators, which are denoted by \(P^\tau_k\).) The Poisson-wave operator is obtained from the half-wave operator by analytic extension in the time and spatial variables. Specifically, recall that the half-wave operator is given by \(U(t) := e^{i\sqrt{-\Delta}t}\). When \(t = i\tau\) lies in the positive imaginary axis, \(P^\tau \) is a complex Fourier integral operator known as the Poisson-wave operator. As discussed in \([Bo, GSt2, GLS]\), for \(0 < \tau \leq \tau_0\) and \(y \in M\) fixed, the Poisson kernel \(P^\tau(\cdot, y) = U(i\tau, \cdot, y)\) extends to a holomorphic function on \(M_\tau\).

Take for concreteness the wave kernel on \(\mathbb{R}^n\) as an example. The Euclidean wave kernel

\[ U(t, x, y) = \int_{\mathbb{R}^n} e^{it|\xi|} e^{i\langle \xi, x-y \rangle} d\xi, \]

analytically continues to \((i\tau, x + ip) \in \mathbb{C}_+ \times \mathbb{C}^n\) by the integral formula

\[ P^\tau(x + ip, y) = \int_{\mathbb{R}^n} e^{-\tau|\xi|} e^{i\langle \xi, x-y+ip \rangle} d\xi, \]

which converges absolutely for \(|p| < \tau\).

On a general Riemannian manifold there exists a similar Lax-Hörmander parametrix for the wave kernel:

\[ U(t, x, y) = \int_{T^*_y M} e^{it|\xi|} e^{i\langle \xi, x-y \rangle} A(t, x, y, \xi) d\xi, \quad \text{for } t = i\tau, \]

where \(|\cdot|_y\) is the metric norm function at \(y\), and where \(A(t, x, y, \xi)\) is a polyhomogeneous amplitude of order 0. The holomorphic extension \(x \mapsto \zeta\) to the Grauert tube \(M_{\tau_0}\) at time \(t = i\tau\) is a Fourier integral operator with complex phase of the form

\[ P^\tau(\zeta, y) = \int_{T^*_y M} e^{-\tau|\xi|} e^{i\langle \xi, \exp_C y^{-1} \zeta \rangle} A(t, \zeta, y, \xi) d\xi. \]

The complexified exponential map \(\exp_C\) appearing in the phase function of the parametrix above is the local holomorphic extension of the Riemannian exponential map as defined in \([6]\). It is easy to see that the integral converges absolutely for \(\sqrt{\rho(\zeta)} < \tau\). We refer to \([T, Le, Ze2]\) for proofs and background. The following result is stated by Boutet de Monvel \([Bo]\); proofs are given in \([Ze2, Le]\).

**Theorem 2.1.** Let \(\iota_\tau : T^* M \setminus 0 \to \Sigma_\tau\) be the symplectic equivalence defined by \((9)\). Then the Poisson-wave operator \(P^\tau : L^2(M) \to O(\partial M_\tau)\) with the parametrix given by \((11)\) is a complex Fourier integral operator of order \(-\frac{n-1}{4}\) associated to the positive complex canonical relation

\[ \Gamma := \{(y, \eta, \iota_\tau(y, \eta)) \} \subset T^* M \times \Sigma_\tau. \]

Moreover, for any \(s\),

\[ P^\tau : W^s(M) \to O^{s+\frac{n-1}{4}}(\partial M_\tau) \]

is a continuous isomorphism.
It is helpful to introduce the framework of *adapted Fourier integral operators*. This notion is defined and discussed in the [BoGu] Appendix A.2. If $X, X'$ are two smooth real manifolds, and $\Sigma \subset T^*X \setminus 0$, $\Sigma' \subset T^*X' \setminus 0$ are two symplectic cones, then a Fourier integral operator $F$ with complex phase is adapted to a homogeneous symplectic diffeomorphism $\chi: \Sigma \to \Sigma'$ if the canonical relation of $F$ is a positive complex canonical relation whose real points consist of the graph of $\chi$ and if the symbol of $F$ is elliptic. Theorem 2.1 may be reformulated in this language as follows: $P^r$ is a Fourier integral operator with complex phase of order $-\frac{n-1}{4}$ adapted to the symplectic isomorphism $\iota_r: T^* M \setminus 0 \to \Sigma_r$ given by (9). The point of the reformulation is that one may identify the graph of $\iota_r$ with the graph of $G^{i\tau}$, where $G^t(x, \xi) = |\xi|G^t(x, \frac{\xi}{|\xi|})$ is the homogeneous geodesic flow defined on $T^* M \setminus 0$. Its analytic continuation in $t$ is also homogeneous, so we have

$$G^{i\tau}(x, \xi) = |\xi|G^{i\tau}(x, \frac{\xi}{|\xi|}).$$

It is observed in [Ze3] that $\iota_r(y, \eta) = G^{i\tau}(y, \eta)$. Thus, $G^{i\tau}$ gives a homogeneous symplectic isomorphism $G^{i\tau}: T^*M \setminus 0 \to \Sigma_r$.

In light of Theorem 2.1 and the calculus of FIOs, the operator

$$A^r := (P^{r*}P^r)^{-\frac{1}{2}}: L^2(M) \to L^2(M). \quad (12)$$

is an elliptic, self-adjoint pseudodifferential operator of order $\frac{n-1}{4}$ with principal symbol $|\xi|^{-\frac{n-1}{4}}$. Equivalently, $P^{r*}P^r$ is a pseudodifferential operator of order $-\frac{n-1}{2}$ with principal symbol $|\xi|^{-\frac{n-1}{2}}$. An immediate consequence of Theorem 2.1 (12) and the symbol calculus of FIOs is the following.

**Proposition 2.2.** The operator $V^r := P^r A^r: L^2(M) \to \mathcal{O}^0(\partial M_r)$ is unitary (of order 0) with an approximate left inverse given by $V^{r*} A^r P^{r*}$. Moreover, $(A^r)^2 P^{r*} : \mathcal{O}^0(\partial M_r) \to L^2(M)$ is an approximate left inverse to $P^r$.

**2.4. Analytic continuation of eigenfunctions via the Poisson-wave kernel.** Let $\{\varphi_j\}$ be an orthonormal basis of Laplacian eigenfunctions on $(M, g)$ with eigenvalue $-\lambda_j^2$. Then the half-wave kernel $U(t, x, y) := e^{it\sqrt{-\Delta}}(x, y)$ admits the eigenfunction expansion

$$U(t, x, y) = \sum_{j=0}^{\infty} e^{it\lambda_j} \varphi_j(x) \overline{\varphi_j(y)}.$$

It follows that the holomorphic extension to $M_r \times M$ of the Poisson kernel is given by

$$P^r(\zeta, y) = U(i\tau, \zeta, y) = \sum_{j=0}^{\infty} e^{-\tau\lambda_j} \varphi_j^C(\zeta) \overline{\varphi_j(y)}, \quad (\zeta, y) \in M_r \times M.$$

We therefore obtain a formula for the analytic extension $\varphi_j^C$ of an eigenfunction $\varphi_j$ to the Grauert tube. Specifically, if $Z \in \partial M_r$ (so in particular $\sqrt{r}(Z) = \tau$), then

$$\varphi_j^C(Z) = e^{\tau\lambda_j} (P^r \varphi_j)(Z) = e^{\sqrt{r}(Z)\lambda_j} (P^r \varphi_j)(Z), \quad Z \in \partial M_r. \quad (13)$$
2.5. **Szegő-Toeplitz multiplication operators.** Let $M_0$ be a Grauert tube of some fixed radius $\tau_0$. For $0 < \tau \leq \tau_0$ we consider operators of the form

$$\Pi_\tau a \Pi_\tau : \mathcal{O}^0(\partial M_\tau) \to \mathcal{O}^0(\partial M_\tau),$$

where by an abuse of notation we write $a$ for multiplication by the symbol $a \in \mathcal{C}_\infty(\partial M_\tau)$. The operator (14) is an example of a Szegő-Toeplitz operator. More generally, such an operator of order $s$ acting on $H^2(\partial M_\tau)$ is of the form $\Pi_\tau Q \Pi_\tau$, with $Q$ a pseudodifferential operator of order $s$. For this article it suffices to take $Q = a$ to be a multiplication operator.

A Szegő-Toeplitz operator might be homogeneous or semi-classical depending on the nature of $Q$.

2.6. **Poisson conjugation of Szegő-Toeplitz operators.** The conjugation of a Toeplitz multiplication operator by the Poisson-wave FIO is studied in [Ze1, Lemma 3.1] and in [Ze3, Section 4.1].

**Lemma 2.3.** Let $a \in \mathcal{C}_\infty(M_\tau)$ and let $P^\tau$ be the Poisson-wave operator defined by (11). Then the conjugation

$$P^{\tau*} \Pi_\tau a \Pi_\tau P^\tau \in \Psi^{-\frac{n+1}{2}}(M)$$

is a pseudodifferential operator with principal symbol equal to (the homogeneous extension of) $a(x, \xi)|\xi|_g^{\frac{n+1}{2}}$. Moreover, let $V^\tau$ be the unitary operator defined in Proposition 2.2 then

$$V^{\tau*} \Pi_\tau a \Pi_\tau V^\tau \in \Psi^0(M)$$

with principal symbol equal to (the homogeneous extension of) $a(x, \xi)$.

Note that

$$V^{\tau*} \Pi_\tau a \Pi_\tau V^\tau = A^\tau P^{\tau*} \Pi_\tau a \Pi_\tau P^\tau A^\tau,$$

so that the second statement follows from Proposition 2.2 or from the first by (12).

**Remark 2.4.** The factors of $\Pi_\tau$ are redundant here because, by Theorem 2.7, $P^\tau$ maps into the range of $\Pi_\tau$.

3. **Balls and dilation in Grauert tubes**

The purpose of this section is to introduce the balls and local dilation that are relevant to the calculus of pseudodifferential operators with log-scale symbols.

**Definition 3.1.** We define Kähler balls $B(\zeta_0, \varepsilon(\lambda_j))$ in the Grauert tube to be balls with respect to the Kähler metric $\omega = -i \partial \bar{\partial} \rho$. For reasons discussed in Section 1.3, we consider Kähler balls whose centers $\zeta_0 \in M_\tau \setminus M$ do not lie on the totally real submanifold $M$. The radii $\varepsilon(\lambda_j) = (\log \lambda_j)^{-\alpha}$ shrinks logarithmically relative the frequency parameter $\lambda_j$.

We also need to introduce local dilation centered at points $\zeta_0 \in M_\tau$. When working with holomorphic or plurisubharmonic functions, we always use local holomorphic dilation. But when working with dilated symbols we may use more general dilation that are more convenient. A technical point to address is that the local dilation does not preserve the family of Kähler balls. But for centers close enough to the real domain $M$, the metric is almost Euclidean on logarithmically shrinking balls.
3.1. Holomorphic dilation. Let $\zeta_0 = E(x_0, \xi_0) \in M_0$ be fixed and consider a local Kähler normal coordinate chart around $\zeta_0$. In such a chart, the Kähler potential satisfies $\rho(\zeta, \zeta_0^*) = |\text{Im}(\zeta - \zeta_0)|^2 + O(|\text{Im}(\zeta - \zeta_0)|^2)$, so that $\partial \bar{\partial} \rho = g_0 + O(|\text{Im}(\zeta - \zeta_0)|^2)$, where $g_0$ is the standard Euclidean Hermitian metric. We denote the unit ball centered at $\zeta_0$ in this local Euclidean metric by $B(\zeta_0, 1)$.

The local holomorphic dilation of $B(\zeta_0, 1)$ in Kähler normal coordinates $\zeta$ centered at $\zeta_0 \in M_0 \setminus M$ is defined by

$$D_{\varepsilon(\lambda)}^{\zeta_0} : B(\zeta_0, 1) \to B(\zeta_0, \varepsilon(\lambda)), \quad \zeta \mapsto \zeta_0 + \varepsilon(\lambda)(\zeta - \zeta_0). \quad (15)$$

This choice of local dilation is not adapted to Grauert tube geometry in that sense that the $\varepsilon$-dilate of a point in $\partial M_\tau$ is not necessarily a point in $\partial M_\varepsilon$. But since the metric and tube function are almost Euclidean in shrinking balls one has constants $c_g, C_g > 0$ so that

$$c_g \varepsilon(\lambda) \sqrt{\rho}(\zeta) \leq \sqrt{\rho}(D_{\varepsilon(\lambda)}^{\zeta_0}(\zeta)) \leq C_g \varepsilon(\lambda) \sqrt{\rho}(\zeta)$$

provided $\sqrt{\rho}(\zeta)$ is small enough. Indeed, it suffices to verify the inequalities for the Euclidean metric, where $\sqrt{\rho}(\zeta) = |\text{Im} \zeta|$ and where $C_g = c_g = 1$.

3.2. Phase space dilation. Theorem 5.1 introduces another type of dilation, which is more conveniently expressed in terms of the usual cotangent coordinates $(x, \xi)$. The dilation in local coordinates centered at $(x_0, \xi_0) \in \partial B^*_\tau M$ is of the form

$$(x, \xi) \mapsto \left( x_0 + \frac{x - x_0}{\varepsilon(\lambda)}, \xi_0 + \frac{\tau \xi - \xi_0}{\varepsilon(\lambda)} \right), \quad (x_0, \xi_0) \in \partial B^*_\tau M. \quad (16)$$

Note that the unit vector $\hat{\xi} := \xi/|\xi|$ is scaled by the parameter $\tau = |\xi_0|_{x_0}$, with $(x_0, \xi_0)$ the fixed center of dilation.

This is closely related to, but not identical to, the dilation introduced in [Ha]. In that article one fixes a point $(x_0, \xi_0) \in S^* M = \partial B^1 M$ in the unit co-sphere bundle and dilates by

$$(x, \xi) \mapsto \left( x_0 + \frac{x - x_0}{\varepsilon(\lambda)}, \xi_0 + \frac{\hat{\xi} - \xi_0}{\varepsilon(\lambda)} \right), \quad (x_0, \xi_0) \in S^* M.$$

Both types of dilation are homogeneous in $\xi$. The one essential difference is that in (16), we allow $|\xi_0|_{x_0} = \tau$ and $\tau \hat{\xi}$ to be any positive numbers bounded away from zero; they need not be the same. Thus, we are not only localizing in the direction of co-vectors but also in their norms.

4. Poisson conjugation of log-scale Toeplitz operators to semi-classical pseudodifferential operators with log-scale symbols

In this section, we generalize the conjugation result of Lemma 2.3 in two ways. On one hand, we let the symbol depend on the frequency $\lambda$, similar to the $\delta(h)$-(micro)localized symbols (23) in the Riemannian setting. On the other hand, we consider Bergman-Toeplitz operators, realized as direct integrals of Szegő-Toeplitz operators. We show that conjugation by the FBI transform takes a decomposable, log-scale Bergman-Toeplitz operator to a semi-classical pseudodifferential operator with a log-scale symbol.
Indeed, Thus

In this semi-classical notation, the Laplacian eigenfunctions satisfy

\[ \Delta \varphi_j = h^{-2} E_j \varphi_j = \lambda_j^2 \varphi_j. \]

Here, we use the semi-classical Fourier transform

\[ \hat{f}(x, \xi, t, \tau) = \int_{\mathbb{R}^n} e^{it(x,y)} e^{-\tau|x|^2/h} f(y) dy. \]

to diagonalize \( \tau = e^{-\sqrt{-\Delta}} \). It is evident that \( P_h^\tau = P^\tau \) by changing variables \( \xi \to \xi/h \). Indeed,

\[ P_h^\tau e^{i(x,k)/h} = e^{-\tau|k|/h} e^{i(x,k)/h}. \]

Thus \( P_h^\tau \) is still the homogeneous Poisson operator \( e^{-\sqrt{-\Delta}} \).

The same change of variables is valid in the manifold setting and we continue to denote the Poisson operator in semi-classical form by \( \tau \). The semi-classical version of the zeroth order unitary operator \( V^\tau \) from Proposition 2.2 is denoted

\[ V_h^\tau := P_h^\tau (P_{\tau}\tau)^{-\frac{1}{2}} : L^2(M) \to \mathcal{O}^0(\partial M). \]

4.2. Log-scale symbols and semi-classical pseudodifferential operators. Let \( 0 \leq a \leq 1 \) be a smooth cutoff function that is equal to 1 on \( B(0,1) \subset \mathbb{C}^n \) and vanishes outside \( B(0,2) \subset \mathbb{C}^n \). We use \( \tau \) to identify \( M_{\tau} \) with \( B_{\tau}^* M \). Using local coordinates induced by \( \exp_{x_0} : T_{x_0}^* M \otimes \mathbb{C} \to M \), consider symbols that, near \( (x_0, \xi_0) \in \partial B_{\tau}^* M \), are locally of the form

\[ a_{\delta(h)}^{(x_0, \xi_0)}(x, \xi) := a \left( x_0 + \frac{x - x_0}{\delta(h)}, \xi_0 + \frac{\xi - \xi_0}{\delta(h)} \right). \]

Symbols of the type \( \tau \) satisfy the estimate

\[ |D^\beta a_{\delta(h)}^{(x_0, \xi_0)}| \leq C_{\delta(h)} \delta(h)^{|\beta|}, \]

and are said to belong to the symbol classes \( S_{\delta(h)}^0 \). More generally, a function \( b \in \mathcal{C}^\infty(T^*M) \) belongs to the symbol class \( S_{\delta(h)}^{k} \) if

\[ \sup_{(x, \xi) \in T^* M} |\partial^\beta_x \partial^\gamma_\xi b| \leq C_{\beta, \gamma} \delta(h)^{|\beta| - |\gamma|} (1 + |\xi|^2 h^{k-|\beta|})^{1/2} \]

for some constant \( C_{\beta, \gamma} \) independent of \( h \).

The semi-classical pseudodifferential operator quantizing a symbol \( a \) is defined by the usual local (semi-classical) Fourier transform formula

\[ \text{Op}_h(a)(x, y) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(x,y)} a(x, \xi, h) \, d\xi. \]
The quantization of a symbol \( b \in S_{\delta(h)}^k \) is denoted by \( \text{Op}_h(b) \in \Psi_{\delta(h)}^k \). We refer to [Ha] for a discussion of the symbol classes \( S_{\delta(h)}^k \) and [Zw] for symbol classes and quantizations in general.

### 4.3. Semi-classical Poisson conjugation of log-scale Toeplitz operators.

**Theorem 4.1.** Let \((x_0, \xi_0) \in \partial B^*_r M\) be fixed. For symbols \( a_{\delta(h)}^{(x_0, \xi_0)} \in C^\infty(M_{\tau_0}) \) of the form (18), we have

\[
P_h^* \Pi_r a_{\delta(h)}^{(x_0, \xi_0)} \Pi_r P_h^* = \text{Op}_h \left( h^{\frac{\alpha - 1}{2}} |\xi|^{-\frac{\alpha - 1}{2}} a \left( x_0 + \frac{x - x_0}{\delta(h)}, \xi_0 + \frac{\tau \xi - \xi_0}{\delta(h)} \right) \right) \in \Psi_{\delta(h)}^{\frac{n - 1}{2}} (M) \tag{21}
\]

modulo \( h\delta(h)^{-2}\Psi_{\delta(h)}^{\frac{n - 1}{2}} (M) \) and

\[
V_h^* \Pi_r a_{\delta(h)}^{(x_0, \xi_0)} \Pi_r V_h^* = \text{Op}_h \left( a \left( x_0 + \frac{x - x_0}{\delta(h)}, \xi_0 + \frac{\tau \xi - \xi_0}{\delta(h)} \right) \right) \in \Psi_0^{\frac{n - 1}{2}} (M)
\]

modulo \( h\delta(h)^{-2}\Psi_0^{\frac{n - 1}{2}} (M) \). Note that the \( \tau \)-scaling affects only \( \hat{\xi} := \xi / |\xi| \).

**Remark 4.2.** Note that the factors of \( \Pi_r \) are redundant because \( P^* \) maps into the range of \( \Pi_r \). We prove only (21) as the second conjugation statement may be proved using the first statement and the composition rule for pseudodifferential operators.

**Proof of Theorem 4.1.** The proof is essentially the same as in Lemma 2.3 since the dilation has no effect on the properties of the conjugation. Indeed, conjugation by the Fourier integral operator \( P_h^* \) preserves the symbol class \( S_{\delta(h)}^k \). Since \( a_{\delta(h)}^{(x_0, \xi_0)} \) is a function on \( \partial M_r \), it defines a homogeneous symbol of order zero on \( \Sigma_r \) in the fiber direction. Under conjugation by \( P_h^* \) it goes over to a pseudodifferential operator of order zero on \( M \) whose symbol is the transport \( a_{\delta(h)}^{(x_0, \xi_0)}(\iota_\tau(x, \xi)) \) to \( T^*M \setminus 0_M \), with \( \iota_\tau \) given by (9). If \( \pi_r : \Sigma_r \to \partial M_r \) is the natural projection then

\[
\iota_\tau^* a_{\delta(h)}^{(x_0, \xi_0)}(x, \xi) = a_{\delta(h)}^{(x_0, \xi_0)}(E(x, \tau \hat{\xi})), \quad \hat{\xi} = \frac{\xi}{|\xi|}
\]

For \( \tau, \delta(h) \) small enough we may use the Euclidean approximation to the distance function. If we center the local coordinates at \((x_0, \xi_0)\) then the cutoff as a function on \( T^*M \) has the form

\[
a_{\delta(h)}^{(x_0, \xi_0)}(\iota_\tau(x, \xi)) = a \left( x_0 + \frac{x - x_0}{\delta(h)}, \xi_0 + \frac{\tau \xi - \xi_0}{\delta(h)} \right) \quad \hat{\xi} = \frac{\xi}{|\xi|} \tag{22}
\]

Thus, \( P_h^* \Pi_r a_{\delta(h)}^{(x_0, \xi_0)} \Pi_r P_h^* \) is a homogeneous pseudodifferential operator with dilated symbol.

We now provide more details. Since the calculation is local we first provide a proof in the Euclidean case.
4.3.1. Euclidean case. Write $Z = x_1 + i\tau p$ with $|p| = 1$ and centering the dilation at $Z_0 = x_0 + i\xi_0$. We do not assume $\tau = |\xi_0|$. The composition has the form

$$P^\tau_{\star} \Pi_r a_{\delta(h)}^{(x_0,\xi_0)} \Pi_r P^\tau_{\star} (x, y)$$

$$= h^{-2n} \tau^{n-1} \int_{\mathbb{R}^n \times S^{n-1} \times \mathbb{R}^n} e^{\Psi_0 / h} a \left( x_0 + \frac{x - x_0}{\delta(h)}, \xi_0 + \frac{\tau p - \xi_0}{\delta(h)} \right) d\sigma(x_1) d\xi_2 d\sigma(p) dx_1,$$

where $d\sigma(p)$ is the standard surface area measure on $S^{n-1}$. The phase is

$$\Psi_0(\xi_1, \xi_2, x_1, p; x, y, \tau) = -\tau(|\xi_1| + |\xi_2|) + i\langle \xi_1, x_1 + i\tau p - y \rangle - i\langle \xi_2, x - (x_1 - i\tau p) \rangle$$

We note that

$$\text{Re} \Psi_0 = -\tau(|\xi_1| + |\xi_2|) - \tau\langle \xi_1 - \xi_2, p \rangle \leq 0$$

with equality if and only if $\hat{\xi}_1 = -\hat{\xi}_2 = \pm p$, that is, the Schwartz kernel integral is of smooth and of order $O(h^{-\infty})$. We absorb the factor apply the complex stationary phase method to the $dx_1 d\xi_2 d\sigma(p)$ integral. The critical point equations for $\text{Im} \Psi$ in $(x_1, \xi_2)$ are

$$\begin{cases}
dx_1 \text{Im} \Psi_0 = 0 \iff \xi_1 = -\xi_2, \\
d\xi_2 \text{Im} \Psi_0 = 0 \iff x_1 = x
\end{cases}$$

The extra $dp$ integral localizes at the above point. Since the $dx_1 d\xi_2$ integral has a non-degenerate Hessian, we may eliminate the $dx_1 d\xi_2$ integrals by stationary phase, obtaining a simpler oscillatory integral

$$h^{-2n + n} \tau^{n-1} \int_{\mathbb{R}^n \times S^{n-1}} e^{\Psi_1 / h} a \left( x_0 + \frac{x - x_0}{\delta(h)}, \xi_0 + \frac{\tau p - \xi_0}{\delta(h)} \right) d\xi_1 d\sigma(p),$$

with

$$\Psi_1(\xi_1, p; x, y, \tau) = -2\tau|\xi_1| - 2\tau \langle \xi_1, p \rangle + i\langle \xi_1, x - y \rangle.$$ 

Applying the method of stationary phase (steepest descent) to the integral over $S^{n-1}$ gives the critical point equation $p = -\hat{\xi}_1$, i.e., the point where the phase is maximal. It follows that

$$P^\tau_{\star} \Pi_r a_{\delta(h)}^{(x_0,\xi_0)} \Pi_r P^\tau_{\star} (x, y)$$

$$= h^{-2n + n} \tau^{n-1} - \frac{n-1}{2} \int_{\mathbb{R}^n} e^{i\langle \xi_1, x - y \rangle / h} a \left( x_0 + \frac{x - x_0}{\delta(h)}, \xi_0 + \frac{\tau \hat{\xi}_1 - \xi_0}{\delta(h)} \right) d\xi_1$$

modulo terms of order $h\delta(h)^{-2}$ (since each derivative of the symbol pulls out a factor of $\delta(h)^{-1}$).

4.3.2. General Riemannian manifold. The proof is similar on any real analytic Riemannian manifold. In place of the integral over $\mathbb{R}^n \times S^{n-1}$ we now have an integral over $Z \in \partial M_r$ or $(x_1, sp) \in \partial B^\circ r M$ with $|p| = 1$ under the map $Z = E(x_1, sp)$. Using the parametrix (III), we have

$$P^\tau_{\star} \Pi_r a_{\delta(h)}^{(x_0,\xi_0)} \Pi_r P^\tau_{\star} (x, y)$$

$$= h^{-2n} \tau^{n-1} \int_{T_r^* M \times T_r^* M \times \partial M_r} e^{\Psi / h} a \left( x_0 + \frac{x_1 - x_0}{\delta(h)}, \xi_0 + \frac{sp - \xi_0}{\delta(h)} \right) A T d\xi_1 d\xi_2 d\mu_r(Z).$$
with
\[ \Psi = -\tau (|\xi_1|_x + |\xi_2|_y) + i\langle \xi_1, (\exp^C_y)^{-1}(Z) \rangle - i\langle \xi_2, (\exp^C_x)^{-1}(\bar{Z}) \rangle. \]

The phase is only well-defined when \( Z \) is sufficiently close to \( x \) and to \( y \), but the phase is non-stationary and the integral is exponentially decaying otherwise. The only points for which the integral is not exponentially decaying are those \( Z \) satisfying \( \text{Im}(\xi_1, (\exp^C_y)^{-1}(Z)) = \tau|\xi_1| \) (and a similar condition holds with \( y \) replaced by \( x \) and \( \xi_1 \) replaced by \( \xi_2 \)). Note that \((\exp^C_y)^{-1}(Z) \in U_x \subset T^*_x M \otimes \mathbb{C} \).

The critical set \( C_\Psi \) of the phase is defined by
\[ C_\Psi = \{(x, y, \tau; \xi_1, \xi_2, Z) : d_{\xi_1, \xi_2, Z} \Psi = 0 \}. \]

The associated canonical relation is defined by the embedding
\[ \iota_\Psi : C_\Psi \to T^* M \times T^* M, \quad (x, y, \tau; \xi_1, \xi_2, Z) \to (x, d_\xi_1 \Psi, y, -d_\xi_2 \Psi). \] (23)

The composite operator is manifestly a Fourier integral operator with complex phase, and is a pseudodifferential operator if and only if \( C_\Psi = \Delta_{T^*_x M \times T^*_y M} \) (the diagonal).

Let \( Z = E(x_1, \tau p) \). Then the critical point equations are
\[
\begin{align*}
(i) \quad d_{\xi_1} \Psi &= 0 \iff (\exp^C_y)^{-1}(Z) = -i\tau \hat{\xi}_1 \iff x_1 = y, \quad p = -i\tau \hat{\xi}_2, \\
(ii) \quad d_x \Psi &= d_Z \left( \langle (\exp^C_y)^{-1}(Z), \xi_2 \rangle - (\exp^C_x)^{-1}(\bar{Z}), \xi_1 \rangle \right) = 0, \\
(iii) \quad d_{\xi_2} \Psi &= 0 \iff (\exp^C_x)^{-1}(\bar{Z}) = -i\tau \hat{\xi}_2.
\end{align*}
\]

Equations (i) and (iii) show that
\[ Z = \exp^C_x(i\tau \hat{\xi}_2) = \exp^C_y(-i\tau \hat{\xi}_1). \]

This implies that \( Z \in \pi^{-1}_x(x) \cap \pi^{-1}_y(y) \), where \( \pi_\tau : \partial M_\tau \to M. \) Of course, these fibers are disjoint unless \( x = y \), so only in that case does there exist a solution of the critical point equation. It then follows that \( \hat{\xi}_1 = -\hat{\xi}_2 \).

To see that \( \xi_1 = -\xi_2 \) on the critical point set, we use further use (ii). There only exists a solution of the critical point equations when \( x = y \), and then we may write \( Z = u + iv \in T^*_x M \otimes \mathbb{C} \) and study the restricted critical point equation
\[ d_{\xi_2} \Psi = 0 \iff d_{u,v} \left( \langle u + iv, \xi_2 \rangle - (u + iv, \xi_1) \right) = 0. \]

Just using \( u \in T^*_x M \) already shows that \( \xi_1 = \xi_2 \) on the critical set.

To calculate (23) we may use the Euclidean approximation to the phase based at \( (x, \xi_1) \) because on \( C_\Psi \) only the first order terms in the Taylor expansion of \( \Psi \) contribute. But then it is evident that \( d_x \Psi = \xi_2 = -d_\xi_2 \Psi \big|_{y = x} = \xi_1 \), proving that the canonical relation is the diagonal.

The principal symbol of \( P^*_h \Pi, P^*_h(x, y) \) is calculated in \([Zel1]\) and the principal symbol of \( P^*_h \Pi_{\delta(h)}^{(x_0, \xi_0)} P^*_h(x, y) \) is the same multiplied by the value of \( a_{\delta(h)}^{(x_0, \xi_0)} \) at the critical point. Note that because of the symbol class we are working with, the sub-leading term is of order \( h\delta(h)^{-2} \) as each derivative of the symbol pulls out a factor of \( \delta(h)^{-1}. \) If we use \( V^*_h \) in place of \( P^*_h \) as in Proposition 2.2 then the principal symbol is the one stated in Theorem 1.1.
4.4. Comparison of symbols. We note that symbols of the form (22) are not quite the same as the log-scaled symbols \( a_{x_0}^b(x, \xi; h) \) of (25) considered in [Ha]. However, as long as \((x_0, \xi_0)\) are fixed at a positive distance from the real domain \(M\), the same symbol estimates (19) are valid. Also note that it is not necessary to multiply by a cutoff \( \varphi(|\xi|) \) to \( S^*M \) since the cutoff \( a_{x_0}^b(x, \xi; h) \) is supported in a shrinking Kähler ball around \( E(x_0, \xi_0) \). In fact, we define the sequence \( h_j \) so that eigenfunctions concentrate on the energy surface \( \partial M_{\tau_0} \) with \(|\xi_0|_{x_0} = \tau_0\). There is no difficulty as long as \( \tau_0 > 0 \). We continue to use the notation \( op_h(a) \) for semi-classical pseudodifferential operators with symbols of the form (22).

5. Decomposable Poisson-FBI transform and Bergman-Toeplitz operators

In this section we introduce a Poisson FBI transform taking \( L^2(M) \) to a weighted Hilbert space of holomorphic functions on \( M_\tau \) rather than to CR-holomorphic functions on \( \partial M_\tau \). As explained in Section 5.1, it is defined in a novel way by a direct integral of Poisson transforms \( P_\tau \), and therefore all of its main properties flow from those established above for the Poisson kernel. The main result is the conjugation Theorem 5.1.

5.1. Weighted Bergman space and Poisson-FBI transform. The Poisson kernel endows \( O^0(M_\tau) \) with a plurisubharmonic weight \( e^{-\sqrt{\rho}/h} \). We define
\[
A^2(M_\tau, h^{-\frac{n-1}{2}} e^{-2\sqrt{\rho}/h} d\mu)
\]
to be the Hilbert space of holomorphic functions on \( M_\tau \) that lie in \( L^2(M_\tau, e^{-2\sqrt{\rho}/h} d\mu) \). It is isometric to the Hilbert space
\[
H_{\sqrt{\rho}} := \{ fh^{-\frac{n-1}{2}} e^{-\sqrt{\rho}/h} : f \in A^2(M_\tau) \subset L^2(M_\tau, d\mu) \}
\]
endowed with the inner product of \( L^2(M_\tau, d\mu) \).

It is useful to regard \( H_{\sqrt{\rho}} \) as a direct integral
\[
H_{\sqrt{\rho}} = \int_{[0, \tau_0]} H^2(\partial M_\tau) d\tau
\]
of Hilbert spaces \( H^2(\partial M_\tau) \). Here, \( \int_{[0, \tau_0]} H^2(\partial M_\tau) d\tau \) denotes the space of \( L^2 \) sections \( f(\tau) \in H^2(\partial M_\tau) \) of the Hilbert bundle, and the direct integral formula follows from Fubini’s theorem,
\[
\| f \|^2 = \int_{0}^{\tau_0} \left( \int_{\partial M_\tau} |f(Z)|^2 d\mu_\tau(Z) \right) d\tau.
\]

We then define the ‘moving Poisson operator’ or FBI transform by
\[
T_h f(\zeta) = P_{\sqrt{\rho}}(\zeta) f(\zeta) = \int_{M} P_{\sqrt{\rho}}(\zeta, y) f(y) dV(y), \quad \zeta \in M_{\tau_0}.
\]
We claim that \( T_h : L^2(M) \to H_{\sqrt{\rho}} \) is a unitary operator. To see this, we use that \( P_\tau \) is unitary from \( L^2(M) \) to each integrand, and observe that
\[
T_h = \int_{[0, \tau_0]} P_{\tau}^* d\tau
\]
is the direct integral of a family of unitary operators index by \( \tau \).
5.2. FBI conjugation theorem. Next we define Bergman-Toeplitz operators. For \( a \in C^\infty(M_{\tau_0}) \) define

\[
\widetilde{\text{Op}}_h(a) = \int_{[0,\tau_0]} \Pi_{\tau}(a|\partial M_{\tau}) \Pi_{\tau} \, d\tau.
\]

Implicitly \( H^2(\partial M_{\tau}) \perp H^2(\partial M_{\sigma}) \) if \( \tau \neq \sigma \). This is a decomposable operator.

**Theorem 5.1.** For symbols \( a_{\delta(h)}^{(x_0,\xi_0)} \in C^\infty(M_{\tau_0}) \) of the form (20), we have

\[
T^*_h \widetilde{\text{Op}}_h(a_{\delta(h)}^{(x_0,\xi_0)}) T_h
= \text{Op}_h \left( \int_0^{\tau_0} h^{n-1} |\xi|^{-\frac{n-1}{2}} a \left( x_0 + \frac{x-x_0}{\delta(h)}, \xi_0 + \frac{\tau \xi - \xi_0}{\delta(h)} \right) d\tau \right) \in \Psi_{\delta(h)}^{-\frac{n-1}{2}}(M). \tag{24}
\]

Note that (24) follows from (21) thanks to the identity

\[
T^*_h \widetilde{\text{Op}}_h(a) T_h = \int_0^{\tau_0} P^*_h \widetilde{\text{Op}}_h(a) P^*_h \, d\tau.
\]

Indeed, a multiplication operator is automatically decomposable and the Schwartz kernel is

\[
\int_{M_{\tau_0}} P^*_h(x,\zeta) a(\zeta) P_h(\zeta, y) \, d\mu(\zeta) = \int_0^{\tau_0} \left( \int_{\partial M_{\tau}} P^*_h(x,Z) a(Z) P^*_h(Z, y) \, d\mu_{\tau}(Z) \right) d\tau.
\]

By Theorem 4.1, each integrand of the \( d\mu_{\tau}(Z) \) integral in the expression above is a semi-classical pseudodifferential operator by (21). The entire \( d\tau \) integral is therefore an integral of an analytic family (in \( \tau \)) of semi-classical pseudodifferential operators on \( M \) with the prescribed principal symbol.

6. Log-scale quantum ergodicity in the real domain

A key part of our analysis is to relate log-scale quantum variance estimates in the complex domain to those in the real domain, and reduce variance estimates to the small-scale quantum ergodicity results on negatively curved Riemannian manifolds due to Hezari-Rivière [HeR] and Han [Ha]. We briefly review their results in preparation for the next section.

As before, let \( \delta(h) = |\log h|^{-\alpha} \), with the semi-classical parameter given by (17). Consider compactly supported smooth functions that, near \( z_0 = (x_0,\xi_0) \in S^*M \), can be locally expressed as

\[
a_{z_0}^b(x,\xi;h) := b \left( x_0 + \frac{x-x_0}{\delta(h)}, \xi_0 + \frac{\xi - \xi_0}{\delta(h)} \right) \phi(|\xi|_x) \in S^0_{\delta(h)}, \tag{25}
\]

where \( b \in C^\infty_c(\mathbb{R}^n \times \mathbb{R}^{n-1}) \) is some compactly supported smooth function and where \( \phi \in C^\infty((1-1/2,1+1/2)) \) is a smooth cutoff function that is identically 1 on \((1-1/4,1+1/4)\).

It is easy to see that such a function belongs to the symbol class \( S^0_{\delta(h)} \) by verifying the symbol estimate (20). The following results pertains to \( \delta(h) \)-microlocalized symbols (25).

---

\(^2\)There is a misprint in [Ha] where the support is said to be \((-\frac{1}{2}, \frac{1}{2})\) around the zero section \( 0_M \). In fact, it needs to be around \( S^*M \).
Theorem 6.1 ([Ha Theorem 1.6]). Let \((M^n, g)\) be negatively curved (not necessarily real analytic). Let 

\[ 0 < \alpha < \frac{1}{2(2n-1)}, \quad 0 \leq \beta < 1 - 2\alpha(2n - 1) \quad \text{or} \quad \alpha = 0, \beta = 1. \]

Set \(\delta(h) = |\log h|^{-\alpha}\). Then for any orthonormal basis \(\{\varphi_j\}\) of \(h^2 \Delta\), we have

\[ h^{n-1} \sum_{E_j \in [1,1+h]} \left| \langle \text{Op}_h(a^b_{2\zeta_0}) \varphi_j, \varphi_j \rangle - \int_{S^*M} a^b_{2\zeta_0} d\mu_L \right|^2 = O(\delta(h)^{2(2n-1)}|\log h|^{-\beta}). \]

Here, \(\text{Op}_h\) is a suitable semi-classical quantization, and \(d\mu_L\) is the Liouville measure.

A covering argument using balls of inverse logarithmic radii implies the next volume comparison result.

Theorem 6.2 ([Ha Corollary 1.9]; see also [HeR Lemma 3.1]). Let \((M^n, g)\) be negatively curved (not necessarily real analytic). Let 

\[ 0 < \alpha < \frac{1}{3n} \quad \text{and} \quad r(\lambda) = (\log \lambda)^{-\alpha}. \]

Then, there exists a full density subsequence such that

\[ c \text{Vol}(B(x, r_j \zeta)) \leq \int_{B(x, r_j \zeta)} |\varphi_{j \zeta}|^2 dV \leq C \text{Vol}(B(x, r_j \zeta)) \]

uniformly for all \(x \in M\), where \(c, C > 0\) depends only on \((M, g)\).

Remark 6.3. An important technical point for this article is that the proofs of the theorems hold for symbols in \(S^0_{\delta(h)}\); the precise form of \(a^b_{\zeta_0}\) is not relevant.

7. Log-scale quantum ergodicity in Grauert tubes: Proof of Theorem 2

We introduce some notation. Let

\[ \Theta_j(\zeta) := \|\varphi_j^c\|_{L^2(\partial M_{\sqrt{\rho}(\zeta)})} \]

denote the \(L^2\)-norm of \(\varphi_j^c\) restricted to the boundary of the Grauert tube of radius \(\sqrt{\rho}(\zeta)\).

Let

\[ U_j(\zeta) := \frac{\varphi_j^c(\zeta)}{\Theta_j(\zeta)} \]

denote the normalized complexified eigenfunction. We will also consider its restriction to \(\partial M_{\tau}\) for each \(0 < \tau \leq \tau_0\) fixed:

\[ u_j^c(Z) := U_j(Z) \big|_{\partial M_{\tau}} = \frac{\varphi_j^c(Z) \big|_{\partial M_{\tau}}}{\|\varphi_j^c\|_{L^2(\partial M_{\tau})}}, \quad (Z \in \partial M_{\tau}). \quad (26) \]

Note that the denominator in (26) is a constant (depending on \(\tau\)), and the numerator is a CR-holomorphic function on \(\partial M_{\tau}\).
7.1. Variance estimates in Grauert tubes. We begin with a log-scale variance estimate for symbols on $\partial M_\tau$, which parallels [CZ, Theorem 4]. Using the $E$ map (7) to identify $B^*_{\tau_0}M$ with $M_{\tau_0}$, we henceforth write

$$a^0_{\delta(h)} : = a^{(x_0, \xi_0)}_{\delta(h)} \in C^\infty(M_{\tau_0}), \quad \zeta_0 = E(x_0, \xi_0)$$

for small-scale symbols of the form (18). We write $Z$ in place of $\zeta$ when restricting to the boundary $\partial M_\tau$, so for instance

$$a^0_{\delta(h)}(\zeta) |_{\partial M_\tau} = a^0_{\delta(h)}(Z), \quad Z \in \partial M_\tau.$$

**Proposition 7.1.** Let $(M^n, g)$ be negatively curved and real analytic. Let

$$0 < \alpha < \frac{1}{2(2n - 1)}, \quad 0 \leq \beta < 1 - 2\alpha(2n - 1) \quad \text{or} \quad \alpha = 0, \beta = 1.$$

Set $\delta(h) = |\log \delta|^{-\alpha}$ as in (17). Let $\{\varphi_j\}$ be an orthonormal basis of eigenfunctions for $\Delta$. Then for every $0 < \tau \leq \tau_0$ and every $\zeta_0 \in M_\tau \setminus M$, we have

$$h^{n-1} \sum_{E_j \in [1, 1+h]} \left| \int_{\partial M_\tau} a^0_{\delta(h)}(Z)|u^*_j(Z)|^2 d\mu_\tau(Z) - \frac{1}{\mu_\tau(\partial M_\tau)} \int_{\partial M_\tau} a^0_{\delta(h)}(Z) d\mu_\tau \right|^2 = O(h^{(2n - 1)|\log h|^{-\beta}}).$$

The remainder is uniform for any $\zeta_0$ in an ‘annulus’ $0 < \tau_1 \leq \sqrt{\rho}(\zeta_0) \leq \tau_0$.

**Proof.** We use Proposition 4.1 to transport matrix elements on $\partial M_\tau$ to matrix elements of pseudodifferential operators on $L^2(M)$. Since the restriction $\varphi^C_h(Z)$ to $\partial M_\tau$ is a CR-holomorphic function, it satisfies $\Pi_\tau \varphi^C_h(Z) = \varphi^C_j(Z)$. Moreover, $e^{-2\sqrt{\rho}(Z)/h} = e^{-2\tau/h}$ on $\partial M_\tau$. Therefore,

$$\int_{\partial M_\tau} a^0_{\delta(h)}(Z)|u^*_j(Z)|^2 d\mu_\tau(Z) = \|\varphi^C_j\|_{L^2(\partial M_\tau)}^{-2} \left\langle a^0_{\delta(h)} \Pi_\tau \varphi^C_j, \Pi_\tau \varphi^C_j \right\rangle_{L^2(\partial M_\tau)}$$

$$= e^{2\tau/h} \|\varphi^C_j\|_{L^2(\partial M_\tau)}^{-2} \left\langle a^0_{\delta(h)} \Pi_\tau P^*_h \varphi_j, \Pi_\tau P^*_h \varphi_j \right\rangle_{L^2(M)}$$

$$= \frac{\langle P^*_\tau \Pi_\tau a^0_{\delta(h)} \Pi_\tau P^*_h \varphi_j, \varphi_j \rangle_{L^2(M)}}{\langle P^*_\tau \Pi_\tau P^*_h \varphi_j, \varphi_j \rangle_{L^2(M)}}. \quad (27)$$

The last equality follows from setting $a^0_{\delta(h)} \equiv 1$, which implies

$$1 = e^{2\tau/h} \|\varphi^C_j\|_{L^2(\partial M_\tau)}^{-2} \left\langle P^*_\tau \Pi_\tau P^*_h \varphi_j, \varphi_j \right\rangle_{L^2(M)}.$$

By Proposition 4.1, $P^*_\tau \Pi_\tau a^0_{\delta(h)} \Pi_\tau P^*_h$ is an $h$-pseudodifferential operator with principal symbol

$$h^{\frac{n-1}{2}} |\xi|-\frac{n-1}{2} a \left( x_0 + \frac{x-x_0}{\delta(h)}, \xi_0 + \frac{\tau \xi - \xi_0}{\delta(h)} \right).$$

By taking $a^0_{\delta(h)} \equiv 1$ in Theorem 4.1, the denominator $P^*_\tau \Pi_\tau P^*_h = P^*_\tau P^*_h$ is found to be an $h$-pseudodifferential operator with principal symbol $h^{\frac{n-1}{2}} |\xi|-\frac{n-1}{2}$. The quotient (27) may be
rewritten using Proposition 4.1

\[
\int_{\partial M_\tau} a_{\delta(h)}^0(Z) |u_j^\tau(Z)|^2 d\mu_\tau(Z)
= \left\langle \text{Op}_h \left( h^{ \frac{n+1}{2} } |\xi|^{- \frac{n+1}{2}} a \left( x_0 + \frac{x-x_0}{\delta(h)}, \xi_0 + \frac{\tau \xi - \xi_0}{\delta(h)} \right) \right), \varphi_j, \varphi_j \right\rangle_{L^2(M)} + O(h\delta(h)^{-2})
\]

\[
= \left\langle \text{Op}_h \left( h^{ \frac{n+1}{2} } |\xi|^{- \frac{n+1}{2}} \varphi_j, \varphi_j \right)_{L^2(M)} + O(h\delta(h)^{-2})
= \left\langle V_h^{\tau*} \Pi_\tau a_{\delta(h)}^0 \Pi_\tau V_h^{\tau*} \varphi_j, \varphi_j \right\rangle_{L^2(M)} + O(h\delta(h)^{-2}). (28)
\]

As noted in Remark 5.3, Theorem 6.1 applies to symbols in the symbol class \( S^0_{\delta(h)} \). But \( V_h^{\tau*} \Pi_\tau a_{\delta(h)}^0 \Pi_\tau V_h^{\tau} \in \Psi^0_{\delta(h)}(M) \), so the proof is complete. \( \square \)

**Proposition 7.2.** With the same notation and assumptions as in Proposition 7.1. For every \( \zeta_0 \in M_\tau \setminus M \) and \( a_{\delta(h)}^0 \), we have

\[
h^{-1} \sum_{E_j \in [1,1+h]} \left| \int_{M_{r_0}} a_{\delta(h)}^0(\zeta)|U_j(\zeta)|^2 d\mu(\zeta) - \int_{0}^{r_0} \int_{\partial M_\tau} a_{\delta(h)}^0(Z) |u_j^\tau(Z)|^2 d\mu_\tau(Z) d\tau \right|^2
= O(\delta(h)^{4n}|\log h|^{-2}).
\]

The remainder is uniform for any \( \zeta_0 \) in an ‘annulus’ \( 0 < \tau_1 \leq \sqrt{\rho(\zeta_0)} \leq \tau_0 \).

**Proof.** Rewrite the integral over \( M_{r_0} \) as an iterated integral:

\[
\int_{M_{r_0}} a_{\delta(h)}^0(\zeta)|U_j(\zeta)|^2 d\mu(\zeta) = \int_{0}^{r_0} \int_{\partial M_\tau} a_{\delta(h)}^0(Z) |u_j^\tau(Z)|^2 d\mu_\tau(Z) d\tau.
\]

We make two observations. First, for the outer integral it suffices to integrate over \( \tau \in \left[ \sqrt{\rho(\zeta_0)} - 2\delta(h), \sqrt{\rho(\zeta_0)} + 2\delta(h) \right] \) thanks to the choice \{18\} of symbols. Second, the inner integral may be replaced by matrix elements of \( V_h^{\tau*} \Pi_\tau a_{\delta(h)}^0 \Pi_\tau V_h^{\tau} \) at the cost of \( O(h\delta(h)^{-2}) \) in light of (28):

\[
\int_{M_{r_0}} a_{\delta(h)}^0(\zeta)|U_j(\zeta)|^2 d\mu(\zeta) = \int_{\sqrt{\rho(\zeta_0)} - 2\delta(h)}^{\sqrt{\rho(\zeta_0)} + 2\delta(h)} \left( \left\langle V_h^{\tau*} \Pi_\tau a_{\delta(h)}^0 \Pi_\tau V_h^{\tau*} \varphi_j, \varphi_j \right\rangle d\tau + O(h\delta(h)^{-2}) \right)
= \int_{\sqrt{\rho(\zeta_0)} - 2\delta(h)}^{\sqrt{\rho(\zeta_0)} + 2\delta(h)} \left\langle V_h^{\tau*} \Pi_\tau a_{\delta(h)}^0 \Pi_\tau V_h^{\tau*} \varphi_j, \varphi_j \right\rangle d\tau + O(h\delta(h)^{-1}).
\]

We now subtract \( \int_{0}^{r_0} \int_{\partial M_\tau} a_{\delta(h)}^0(Z) d\mu_\tau(Z) d\tau \) from both sides of the equality and then square both sides. The error is then of order \( h^2\delta(h)^{-2} \), which we move to the left-hand side of the
equality to conserve space:
\[
\left| \int_{M} a_{\delta(h)}^\circ(\zeta) |U_j(\zeta)|^2 \, d\mu(\zeta) - \int_0^{T_0} \int_{\partial M} a_{\delta(h)}^\circ(Z) \frac{d\mu_r(Z)}{\mu_r(\partial M_r)} \, d\tau \right|^2 + O(h^2 \delta(h)^{-2})
\]
\[
= (4\delta(h))^2 \left| \int_{\sqrt{\rho}(\zeta) - 2\delta(h)}^{\sqrt{\rho}(\zeta) + 2\delta(h)} \left( V^r_{\Pi_r a_{\delta(h)}^\circ} \Pi_r V^r_{\phi_j, \phi_j} - \int_{\partial M_r} a_{\delta(h)}^\circ(Z) \frac{d\mu_r(Z)}{\mu_r(\partial M_r)} \right) \frac{d\tau}{4\delta(h)} \right|^2
\]
\[
\leq (4\delta(h))^2 \int_{\sqrt{\rho}(\zeta) - 2\delta(h)}^{\sqrt{\rho}(\zeta) + 2\delta(h)} \left| V^r_{\Pi_r a_{\delta(h)}^\circ} \Pi_r V^r_{\phi_j, \phi_j} - \int_{\partial M_r} a_{\delta(h)}^\circ(Z) \frac{d\mu_r(Z)}{\mu_r(\partial M_r)} \right|^2 \frac{d\tau}{4\delta(h)}
\]
\[
= 4\delta(h) \int_{\sqrt{\rho}(\zeta) - 2\delta(h)}^{\sqrt{\rho}(\zeta) + 2\delta(h)} \left| V^r_{\Pi_r a_{\delta(h)}^\circ} \Pi_r V^r_{\phi_j, \phi_j} - \int_{\partial M_r} a_{\delta(h)}^\circ(Z) \frac{d\mu_r(Z)}{\mu_r(\partial M_r)} \right|^2 \, d\tau.
\]

For the inequality we used that \( \frac{d\tau}{4\delta(h)} \) is a probability measure on the interval \([\sqrt{\rho}(\zeta) - 2\delta(h), \sqrt{\rho}(\zeta) + 2\delta(h)]\), so Jensen's inequality applies. Performing the Cesàro sum and using Proposition 7.2 we find
\[
\left| \sum_{E_j \in [1, 1+h]} \left( \int_{M} a_{\delta(h)}^\circ(\zeta) |U_j(\zeta)|^2 \, d\mu(\zeta) - \int_0^{T_0} \int_{\partial M} a_{\delta(h)}^\circ(Z) \frac{d\mu_r(Z)}{\mu_r(\partial M_r)} \, d\tau \right)^2 \right|
\]
\[
\leq 4\delta(h) \int_{\sqrt{\rho}(\zeta) - 2\delta(h)}^{\sqrt{\rho}(\zeta) + 2\delta(h)} C \delta(h)^{2(2n-1)} |\log h|^{-\beta} \, d\tau
\]
\[
= O(\delta(h)^{4n} |\log h|^{-\beta}) + O(h^2 \delta(h)^{-2}).
\]

This completes the proof. \( \square \)

7.2. Proof of Theorem 2 using Proposition 7.2. We now have enough tools to tackle the key volume comparison estimate Theorem 2 which is a Grauert tube analogue of Theorem 6.2. The proof uses the covering argument of [HeR §3.2, Ha §5.2, CZ §4.2]. In what follows we revert to using \( \lambda \)-notation. Recall from (17) that the semi-classical \( h \)-notation in Proposition 7.1 7.2 in particular we have \( \delta(h) = |\log h|^{-\alpha} = (\log \lambda)^{-\alpha} = \varepsilon(\lambda) \).

Proof of Theorem 2 Let \( \tau_0, \tau_1 \) be fixed with \( 0 < \tau_1 < \tau_0 \). In what follows we work with centers \( \zeta_k \) that lie in the fixed ‘annulus’ \( M_{\tau_0} \setminus M_{\tau_1} \), on which the errors remain uniform estimates. As in [Ha Lemma 5.1], for every \( \varepsilon(\lambda) \), there exists a log-good cover
\[
\mathcal{U}_\lambda := \{ B(\zeta_k, \varepsilon(\lambda)) \}_{k=1}^{R(\varepsilon(\lambda))}
\]
of \( M_{\tau_0} \setminus M_{\tau_1} \) by balls of radii \( c\varepsilon(\lambda) \) such that
(i) The number \( R(\varepsilon(\lambda)) \) of elements in the covering satisfies \( c_1 \varepsilon(\lambda)^{-2n} \leq R(\varepsilon(\lambda)) \leq c_2 \varepsilon(\lambda)^{-2n} \), where \( c_1, c_2 \) are independent of \( \varepsilon(\lambda) \).
(ii) Any \( B(\zeta, \varepsilon(\lambda)) \subset M_{\tau_0} \setminus M_{\tau_1} \) is covered by at most \( c_3 \) (independent of \( \varepsilon(\lambda) \)) number of elements of \( \mathcal{U}_\lambda \).
(iii) Any \( B(\zeta', \varepsilon(\lambda)) \subset M_{\tau_0} \setminus M_{\tau_1} \) contains at least one element of \( \{ B(\zeta_k, \frac{1}{3}\varepsilon(\lambda)) \}_{k=1}^{R(\varepsilon(\lambda))} \).
Indeed, this follows from Markov’s inequality
\[ P(\lambda_j \in [\lambda, \lambda + 1], \quad 1 \leq k \leq R(\varepsilon(\lambda))) \]
Set
\[ X_{j,k} := \left| \int_{M_{\tau_0}} a_{\varepsilon(\lambda_j)}(\zeta)|U_j|^2 \, d\mu - \int_0^{\tau_0} \int_{\partial M_{\tau}} \frac{a_{\varepsilon(\lambda_j)}(\zeta)}{\mu_\tau(\partial M_{\tau})} \, d\mu_\tau \, d\tau \right|^2. \]
(The two subscripts \( j, k \) correspond to the subscript \( j \) for the eigenvalue \( \lambda_j \) and the subscript \( k \) for the points \( \zeta_k \).) Also, let \( \beta' > 0 \) be a parameter to be chosen later and define ‘exceptional sets’ by
\[ \Lambda_k := \left\{ j : \lambda_j \in [\lambda, \lambda + 1], \quad X_{j,k} \geq \varepsilon(\lambda)^{4n}(\log \lambda)^{-\beta'} \right\}. \]
We claim
\[ \frac{\#\Lambda_k}{\lambda^{n-1}} \leq C(\log \lambda)^{-\beta+\beta'}. \]
Indeed, this follows from Markov’s inequality \( P(X_{j,k} \geq x) \leq x^{-1}EX_{j,k} \). We view \( X_{j,k} \) as real-valued random variables index by \( j \). The probability measure is the normalized counting measure on the set of indices \( j \) satisfying (29). Thanks to Proposition 7.2 for all such \( j \) the expected value of this random variable is
\[ \mathbb{E}X_{j,k} = \mathcal{O}(\varepsilon(\lambda)^{4n}(\log \lambda)^{-\beta}), \]
with the error is uniform in \( \zeta_k \in M_{\tau_0} \setminus M_{\tau_1} \) for \( k = 1, 2, \ldots, R(\varepsilon(\lambda)) \). Finally, setting \( x = \varepsilon(\lambda)^{4n}(\log \lambda)^{-\beta'} \) in the inequality yields (30).
Moreover, the union
\[ \Lambda := \bigcup_{k=1}^{R(\varepsilon(\lambda))} \Lambda_k \]
of the exceptional sets satisfies
\[ \frac{\#\Lambda}{\lambda^{n-1}} \leq CR(\varepsilon(\lambda))(\log \lambda)^{-\beta+\beta'} = C\varepsilon(\lambda)^{-2n}(\log \lambda)^{-\beta+\beta'} = C(\log \lambda)^{2\alpha-\beta+\beta'}. \]
Recall from Proposition 7.2 that \( 0 < \beta < 1 - 2\alpha(2n - 1) \), so \( \beta' > 0 \) can always be chosen small enough such that the quantity (31) tends to zero whenever \( 2\alpha - (1 - 2\alpha(2n - 1)) < 0 \). This corresponds to the range of \( \alpha \) in the statement of Theorem 2.
Consider now the ‘generic set’
\[ \Sigma := \{ j : \lambda_j \in [\lambda, \lambda + 1] \} \setminus \Lambda, \]
which is by construction a subsequence of full density:
\[ \frac{\#\Sigma}{\lambda^{n-1}} \geq 1 - C\varepsilon(\lambda)^{-2n}(\log \lambda)^{-\beta+\beta'} \to 1. \]
If \( j \in \Sigma \), then we must have
\[ \left| \int_{M_{\tau_0}} a_{\varepsilon(\lambda_j)}(\zeta)|U_j|^2 \, d\mu - \int_0^{\tau_0} \int_{\partial M_{\tau}} \frac{a_{\varepsilon(\lambda_j)}(\zeta)}{\mu_\tau(\partial M_{\tau})} \, d\mu_\tau \, d\tau \right|^2 \leq \varepsilon(\lambda)^{4n}(\log \lambda)^{-\beta'}. \]
simultaneously for all \( k = 1, 2, \ldots, R(\varepsilon(\lambda)) \), that is,
\[
\int_{M_{\varepsilon(\lambda)}} a^{C_{\varepsilon(\lambda)}}(\zeta) |U_j|^2 \, d\mu \leq C \text{Vol}_w(B(\zeta, \varepsilon(\lambda))) + o(\varepsilon(\lambda)^{2n}(\log \lambda)^{-\beta'/2}).
\]
If \( \zeta' \in M_{\varepsilon(\lambda)} \) is an arbitrary point, then the ball \( B(\zeta', \varepsilon(\lambda)) \) is contained in at most \( c_2 \) number (independent of \( \lambda \)) of elements of the log-good cover \( \mathcal{U}_\lambda \), whence we obtain the upper bound
\[
\int_{B(\zeta', \varepsilon(\lambda))} |U_j|^2 \, d\mu \leq C \sum_{\ell=1}^{c_2} \text{Vol}_w(B(\zeta_{k\ell}, \varepsilon(\lambda))) + o(\varepsilon(\lambda)^{2n}(\log \lambda)^{-\beta'/2}) \leq C \text{Vol}(B(\zeta', \varepsilon(\lambda))).
\]
The constant \( C = C(M, g) \) is independent of \( \zeta' \) throughout.

It remains to extract another full density subsequence \( \Sigma' \) using symbols of the form \( b^{C_0}(\zeta) := b(\zeta/\varepsilon) \) in local coordinates centered at \( \zeta_0 \). Here, \( 0 \leq b \leq 1 \) is taken to be a smooth cut-off function that equals 1 on \( B(0, 1/6) \subset \mathbb{C}^n \) and vanishes outside \( B(0, 1/3) \subset \mathbb{C}^n \).

Repeating the same arguments, we see that for \( j \in \Sigma' \), we have
\[
\int_{B(\zeta', \varepsilon(\lambda)/3)} |U_j|^2 \, d\mu \geq c \text{Vol}(B(\zeta, \varepsilon(\lambda)/6)) - o(\log \lambda)^{-\beta'/2})
\]
simultaneously for all \( k = 1, 2, \ldots, R(\varepsilon(\lambda)) \). Let \( \zeta' \in M_{\varepsilon(\lambda)} \) be arbitrary. Every ball \( B(\zeta', \varepsilon(\lambda)) \) contains at least one element \( B(\zeta', \varepsilon(\lambda)/3) \in \mathcal{U}_\lambda \) of the log-good cover, whence
\[
\int_{B(\zeta', \varepsilon(\lambda))} |U_j|^2 \, dV \geq c \text{Vol}(B(\zeta, \varepsilon(\lambda)/3)) \geq c \text{Vol}(B(\zeta', \varepsilon(\lambda)))
\]
Again, it is easy to verify that \( c = c(M, g) \) is independent of \( \zeta' \). This is the statement of the volume lower bound.

The intersection \( \Gamma = \Sigma \cap \Sigma' \) is again a full density subsequence. By construction, every \( j \in \Gamma \) satisfies the two-sided bound:
\[
c \text{Vol}_w(B(\zeta', \varepsilon(\lambda))) \leq \int_{B(\zeta', \varepsilon(\lambda))} |U_j|^2 \, d\mu \leq C \text{Vol}(B(\zeta', \varepsilon(\lambda))) \quad \text{for all } \zeta' \in M_{\varepsilon(\lambda)}.
\]
This completes the proof of Theorem 2. \( \square \)

8. Log-scale equidistribution of complex zeros: Proof of Theorem 4
Recall from the previous section the two key objects of study:
\[
\Theta_j(\zeta) := \| \varphi_{j}^{C} \|_{L^2(M, \pi_{\zeta}(\zeta))} \quad \text{and} \quad U_j(\zeta) := \frac{\varphi_{j}^{C}(\zeta)}{\Theta_j(\zeta)}.
\]
By the Poincaré-Lelong formula [GH, p.388, Lemma], the current of integration \([Z_j]\) over the zero set \( Z_j = \{ \zeta \in M_{\varepsilon(\lambda)} : \varphi_{j}^{C}(\zeta) = 0 \} \) is given by the identity
\[
\frac{i}{2\pi} \bar{\partial} \log |U_j|^2 = \frac{i}{2\pi} \bar{\partial} \log |\varphi_{j}^{C}|^2 - \frac{i}{2\pi} \partial \bar{\partial} \log \Theta_j^2 = |Z_j| - \frac{i}{2\pi} \partial \bar{\partial} \log \Theta_j^2. \tag{32}
\]
To study the currents \([Z_j]\) at logarithmic length scales, let \( D_{\varepsilon(\lambda)}^{\partial j} \) denote the corresponding pullback operator corresponding to the local holomorphic dilation map \((15)\). This allows us
to work not on shrinking balls $B(\zeta_0, \varepsilon(\lambda_j))$ but on a fix-sized ball $B(\zeta_0, 1)$, which is more convenient. The (normalized) small-scale version of (32) becomes

$$\frac{i}{2\pi \lambda_j \varepsilon(\lambda_j)} \partial \overline{\partial} D_{\varepsilon(\lambda_j)}^{\alpha^*} \log |U_j|^2 = \frac{1}{\lambda_j \varepsilon(\lambda_j)} D_{\varepsilon(\lambda_j)}^{\alpha^*} [Z] - \frac{i}{2\pi \lambda_j \varepsilon(\lambda_j)} \partial \overline{\partial} D_{\varepsilon(\lambda_j)}^{\alpha^*} \log \Theta_j^2 \quad \text{as currents on } B(\zeta_0, 1). \quad (33)$$

We used the fact that the local dilation map $D_{\varepsilon(\lambda_j)}^{\alpha}$, being holomorphic, commutes with $\partial \overline{\partial}$.

**Remark 8.1.** The $\lambda_j^{-1}$ normalization is already present in (2), due to [Ze1]. Here there is an additional factor of $\varepsilon(\lambda_j)^{-1}$, which comes from the proof of Proposition 8.4, specifically (40).

**8.1. Proof of Theorem 1 using Theorem 2.** We rescale the convergence statement (3) as in (33), so that the various objects are defined on a fixed-sized ball $B(\zeta_0, 1)$ that does not change with respect to the frequency $\lambda$.

We point out a subtlety involving the parameter $\alpha > 0$ in the proof of Theorem 1 using Theorem 2. Namely, if a full density subsequence satisfies volume comparison (5) at length scale $\varepsilon(\lambda_j) = (\log \lambda_j)^{-\alpha}$, then it satisfies the zeros distribution result (3) at a coarser length scale $\varepsilon'(\lambda_j) := (\log \lambda_j)^{-\alpha'}$ for any $\alpha' < \alpha$. This inequality is strict – see the argument around (41) – (42). To emphasize the role of the two scales, we restate Theorem 1–2 as follows.

**Theorem 8.2.** Let $(M, g)$ be a real analytic, negatively curved, compact manifold without boundary. Let $\omega := -i \partial \overline{\partial} \rho$ denote the Kähler form on the Grauert tube $M_\rho$. Assume that

$$0 \leq \alpha' < \frac{1}{2(3n - 1)}, \quad \varepsilon'(\lambda_j) = (\log \lambda_j)^{-\alpha'}.$$

Then there exists a full density subsequence of eigenvalues $\lambda_{jk}$ such that for arbitrary but fixed $\zeta_0 \in M_\rho \setminus M$, there is a uniform two-sided volume bound

$$c \text{Vol}_\omega(B(\zeta_0, \varepsilon'(\lambda_{jk}))) \leq \int_{B(\zeta_0, \varepsilon'(\lambda_{jk}))} |U_j|^2 d\mu \leq C \text{Vol}_\omega(B(\zeta_0, \varepsilon'(\lambda_{jk}))). \quad (34)$$

The constants $c, C$ are geometric constants depending only on $\sqrt{\rho(\zeta_0)}$; they are uniform for $\zeta_0$ lying in an ‘annulus’ $0 < \tau_1 \leq \sqrt{\rho}(\zeta_0) \leq \tau_0$.

Moreover, for any $\alpha$ satisfying

$$0 \leq \alpha < \alpha' < \frac{1}{2(3n - 1)}, \quad \varepsilon(\lambda_j) = (\log \lambda_j)^{-\alpha},$$

the full density subsequence satisfying (34) also satisfies

$$\frac{1}{\lambda_{jk} \varepsilon(\lambda_{jk})} D_{\varepsilon(\lambda_j)}^{\alpha^*} [Z] - \frac{i}{\pi} \partial \overline{\partial} |\text{Im}(\zeta - \zeta_0)|_{g_0} \quad \text{as currents on } B(\zeta_0, 1). \quad (35)$$

Here, $D_{\varepsilon(\lambda_j)}^{\alpha^*}$ denote pullback by the local holomorphic dilation (15) and $g_0$ denotes the flat metric. Equivalently, for every test form $\eta \in \mathcal{D}^{(n - 1, n - 1)}(B(\zeta_0, 1))$, $\eta \wedge \frac{1}{\lambda_{jk} \varepsilon(\lambda_{jk})} D_{\varepsilon(\lambda_j)}^{\alpha^*} [Z] = \int_{B(\zeta_0, 1)} \eta \wedge \frac{i}{\pi} \partial \overline{\partial} |\text{Im}(\zeta - \zeta_0)|_{g_0} + o(1).$
Remark 8.3. By a partition of unity argument, Theorem [8.2] for general test forms supported on Kähler balls implies Theorem [1] for test forms on \( M_{n_0} \) of the form \( f \omega^{n-1} \) with \( f \in C(M_{n_0}) \).

The volume comparison (34) has already been proved in the previous section. Comparing what is left to prove – namely (35) – with the identity (33), we see that it suffices to establish the following Propositions 8.4–8.5.

Proposition 8.4. For the entire sequence of eigenvalues \( \lambda_j \), for every \( \zeta_0 \in M_{n_0} \setminus M \), we have

\[
\frac{i}{2\pi \lambda_j \varepsilon(\lambda_j)} \partial \bar{\partial} D_{\varepsilon(\lambda_j)}^{\mathcal{G}_0} \log \Theta^2 \to \frac{i}{\pi} \partial \bar{\partial} |\text{Im}(\zeta - \zeta_0)|_{g_0} \quad \text{as currents on } B(\zeta_0, 1).
\]

Here, \( | \cdot |_{g_0} \) denotes the Euclidean distance.

Proposition 8.5. There exists a full density subsequence of eigenvalues \( \lambda_{j_k} \) such that, for every \( \zeta_0 \in M_{n_0} \setminus M \), we have

(i) \( (\lambda_j \varepsilon(\lambda_j))^{-1} \log D_{\varepsilon(\lambda_j)}^{\mathcal{G}_0} |U_j|^2 \to 0 \) strongly in \( L^1(B(\zeta_0, 1)) \);
(ii) \( (\lambda_j \varepsilon(\lambda_j))^{-1} \partial \bar{\partial} \log D_{\varepsilon(\lambda_j)}^{\mathcal{G}_0} |U_j|^2 \to 0 \) weakly in \( \mathcal{D}(n-1, n-1)'(B(\zeta_0, 1)) \).

8.2. Proof of Proposition 8.4 using pseudodifferential operators. Using (13), we see

\[
\varphi_j^C(\zeta) = e^{\lambda_j \sqrt{\mathcal{G}_0}} (P \sqrt{\mathcal{G}_0} \varphi_j)(\zeta), \quad \zeta \in M_{n_0}.
\]

Therefore,

\[
D_{\varepsilon(\lambda_j)}^{\mathcal{G}_0} \Theta_j(\zeta) = D_{\varepsilon(\lambda_j)}^{\mathcal{G}_0} \left| \varphi_j^C \right|_{\partial M \sqrt{\mathcal{G}_0}}^2 \left| \partial M \sqrt{\mathcal{G}_0} \right|_{L^2(\partial M \sqrt{\mathcal{G}_0})}^2
= \left| \varphi_j^C \right|_{\partial M \sqrt{\mathcal{G}_0}}^2 \left| \partial M \sqrt{\mathcal{G}_0} \right|_{L^2(\partial M \sqrt{\mathcal{G}_0})}^2
= \left\langle \Pi D_{\varepsilon(\lambda_j)}^{\mathcal{G}_0} \sqrt{\mathcal{G}_0} \varphi_j^C, \Pi D_{\varepsilon(\lambda_j)}^{\mathcal{G}_0} \sqrt{\mathcal{G}_0} \varphi_j^C \right\rangle_{L^2(\partial M \sqrt{\mathcal{G}_0})}
= e^{2\lambda_j D_{\varepsilon(\lambda_j)}^{\mathcal{G}_0} \sqrt{\mathcal{G}_0}} \left\langle \Pi D_{\varepsilon(\lambda_j)}^{\mathcal{G}_0} \sqrt{\mathcal{G}_0} \varphi_j^C, \Pi D_{\varepsilon(\lambda_j)}^{\mathcal{G}_0} \sqrt{\mathcal{G}_0} \varphi_j^C \right\rangle_{L^2(M)}.
\]

The last equality follows from (36).

The operators

\[
A(\varepsilon(\lambda_j), \sqrt{\mathcal{G}_0}) := P D_{\varepsilon(\lambda_j)}^{\mathcal{G}_0} \sqrt{\mathcal{G}_0} \Pi D_{\varepsilon(\lambda_j)}^{\mathcal{G}_0} \sqrt{\mathcal{G}_0} P D_{\varepsilon(\lambda_j)}^{\mathcal{G}_0} \sqrt{\mathcal{G}_0} \in \Psi^{-\frac{n-1}{2}}(M)
\]
forms an analytic family in the parameter \( \sqrt{\mathcal{G}_0} \in (0, \tau_0] \) with \( A(\varepsilon(\lambda_j), \sqrt{\mathcal{G}_0}) \to \text{Id} \) as \( \sqrt{\mathcal{G}_0} \to 0 \). It is easy to see using the Schur-Young test that \( (1 + \Delta)^{-\frac{n+1}{2}} A(\varepsilon(\lambda_j), \sqrt{\mathcal{G}_0}) \in \Psi^{-n}(M) \) is a uniformly upper bounded family of operators on \( L^2(M) \) (see [Ze1] (34)). Therefore, writing \( A(\varepsilon(\lambda_j), \sqrt{\mathcal{G}_0}) = (1 + \lambda_j)^{\frac{n+1}{2}} (1 + \Delta)^{-\frac{n+1}{2}} A(\varepsilon(\lambda_j), \sqrt{\mathcal{G}_0}) \), we find

\[
\left| \frac{1}{\lambda_j} \log \left\langle P D_{\varepsilon(\lambda_j)}^{\mathcal{G}_0} \sqrt{\mathcal{G}_0} \Pi D_{\varepsilon(\lambda_j)}^{\mathcal{G}_0} \sqrt{\mathcal{G}_0} P D_{\varepsilon(\lambda_j)}^{\mathcal{G}_0} \sqrt{\mathcal{G}_0} \varphi_j, \varphi_j \right\rangle_{L^2(M)} \right| \leq C \frac{\log \lambda_j}{\lambda_j}
\]

(38)
for some $C$ independent of $\varepsilon$. Combining (37) and (38) gives

$$\frac{1}{2\pi \lambda_j \varepsilon (\lambda_j)} \log D^{\varphi \ast}_{\varepsilon (\lambda_j)} \Theta_j (\zeta)^2 = \frac{1}{\pi \varepsilon (\lambda_j)} D^{\varphi \ast}_{\varepsilon (\lambda_j)} \sqrt{\rho (\zeta)} + O (\lambda_j^{-1} \log \lambda_j).$$  \hspace{1cm} (39)

Recall from Section 2 that the Grauert tube function $\rho$ is related to the complexified Riemannian distance function $r$ on $M \times \overline{M}$ by

$$\rho (\zeta) = - \frac{1}{4} r^2 (\zeta, \overline{\zeta}), \quad \zeta = \exp^C (i \xi) \in M_n.$$

Taylor expanding the metric yields $\sqrt{\rho (\zeta)} = |\text{Im} (\zeta - \zeta_0)|_{g_0} + O (|\text{Im} (\zeta - \zeta_0)|^2_{g_0})$, in which $|\cdot|_{g_0}$ denotes the flat metric. This gives rise to the $\lambda_j \to \infty$ asymptotics

$$D^{\varphi \ast}_{\varepsilon (\lambda_j)} \sqrt{\rho (\zeta)} = \varepsilon (\lambda_j) |\text{Im} (\zeta - \zeta_0)|_{g_0} + O (\varepsilon (\lambda_j)^2), \quad \zeta = \exp^C (i \xi) \in M_n.$$  \hspace{1cm} (40)

The statement of Proposition 8.4 is now an immediate consequence of (39) and (40).

8.3. **Proof of Proposition 8.5 using subharmonic function theory.** Proposition 8.5 is modeled after arguments that have appeared in [SZe, Ze1, CZ]. Given $\zeta_0 \in M_n \setminus M$, consider the family of plurisubharmonic functions

$$v_j := \frac{1}{\lambda_j \varepsilon (\lambda_j)} \log D^{\varphi \ast}_{\varepsilon (\lambda_j)} |\varphi^C_j|^2 \in \text{PSH} (B (\zeta_0, 1)).$$

(The functions $v_j$ are indeed subharmonic because $\varphi^C_j$ are holomorphic by construction.) We claim

(i) \{ $v_j$ \} is uniformly bounded above on $B (\zeta_0, 1)$;

(ii) $\limsup_{j \to \infty} v_j (\zeta) \leq 2 \sqrt{\rho (\zeta)}$ on $B (\zeta_0, 1)$.

Notice $\sup_{B (\zeta_0, 1)} D^{\varphi \ast}_{\varepsilon (\lambda_j)} |U_j|^2 = \sup_{B (\zeta_0, \varepsilon (\lambda_j))} |U_j|^2$. To prove the first statement, it suffices to obtain a uniform upper bound on each slice $\partial M_\tau \cap B (\zeta_0, \varepsilon (\lambda_j))$ that is independent of $\tau$. Since $u_j^\tau \in \mathcal{O}^{\mathbb{C}^{\text{inh}}} (\partial M_\tau)$, we see (cf. [Ze1 §5.1])

$$\sup_{\partial M_\tau \cap B (\zeta_0, \varepsilon (\lambda_j))} |U_j|^2 \leq \sup_{\partial M_\tau} |u_j^\tau|^2 \leq \lambda_j^n \|u_j^\tau\|_{L^2 (\partial M_\tau)} = \lambda_j^n.$$

Rewriting the left-hand side as $U_j = \varphi^C_j / \|\varphi^C_j\|_{L^2 (\partial M_\tau)}$, taking the logarithm, dividing by $\lambda_j$, and finally using the limit formula of Proposition 8.4 finishes the proof of (i) and (ii).

It follows from a standard compactness theorem on plurisubharmonic functions [Ho, Theorem 4.1.9] that either $v_j \to - \infty$ locally uniformly, or there exists a subsequence that is convergent in $L^1_{\text{loc}} (B (\zeta_0, 1))$. The first possibility is easily ruled out. Indeed, if it were true, then

$$\frac{1}{\lambda_j \varepsilon (\lambda_j)} \log D^{\varphi \ast}_{\varepsilon (\lambda_j)} |U_j|^2 \leq -1 \quad \text{on } B (\zeta_0, 1) \text{ for all } \lambda_j \gg 1$$

$$\iff |U_j|^2 \leq e^{-\lambda_j \varepsilon (\lambda_j)} \quad \text{on } B (\zeta_0, \varepsilon (\lambda_j)) \text{ for all } \lambda_j \gg 1,$$

contradicting the mass comparison assumption [34].

**Remark 8.6.** By a covering argument similar to the proof of Theorem 2, it is easy to see that if a sequence $\{U_j\}$ satisfies volume comparison [34], then it satisfies volume comparison at all coarser length scales $\varepsilon (\lambda_j) = (\log \lambda_j)^{-\alpha}$ for $\alpha' < \alpha < \frac{1}{2(3n-1)}$. 
Therefore, \( v_j \) has a subsequence, which we continue to denote by \( v_j \), that converges in \( L^1 \) to \( v \in L^1(B(\zeta_0, 1)) \). By passing to yet another subsequence if necessary, we may assume that the convergence to \( v \) is pointwise almost everywhere. The upper-semicontinuous regularization
\[
v^*(\zeta) := \limsup_{\eta \to \zeta} v(\eta) \leq 2\sqrt{p(\zeta)}
\]
of \( v \) is then a plurisubharmonic function on \( B(\zeta_0, 1) \) and \( v_j \to v^* \) pointwise almost everywhere.\(^3\) The upper bound of \( 2\sqrt{p(\zeta)} \) follows from claim (ii) above.

Set
\[
\psi := v^* - 2\sqrt{p} \leq 0 \quad \text{on} \quad B(\zeta_0, 1).
\]
Assume for purposes of a contradiction that \( \|\lambda_j^{-1} \varepsilon(\lambda_j)^{-1} \log D_{\varepsilon(\lambda_j)}^* U_j^2 \|_{L^1(B(\zeta_0, 1))} \geq \delta > 0 \). It follows that
\[
W_\delta := \{ \zeta \in B(\zeta_0, 1) : \psi(\zeta) < -\delta/2 \}
\]
is an open set with nonempty interior. The shape of \( W_\delta \) is unknown – it may have a very small inradius – but it is a fixed (independent of \( \lambda_j \)) open set. To gain control over this unknown set \( W_\delta \), we make use of the volume comparison assumption (41) that takes place at the finer scale \( \varepsilon'(\lambda_j) = (\log \lambda_j)^{-\alpha'} \) for \( \alpha' < \alpha \). From this assumption we know
\[
\int_{B(\zeta', \varepsilon'(\lambda_j))} |U_j|^2 \omega^n \geq c \Vol(B(\zeta_0, \varepsilon'(\lambda_j))) \quad \text{for all} \quad \zeta' \in M_{\tau_0} \setminus M.
\]
Rescaling yields
\[
\int_{B(\zeta', \varepsilon'(\lambda_j)^{-1}(\lambda_j))} D_{\varepsilon(\lambda_j)}^* |U_j|^2 \omega^n \geq c \Vol(B(\zeta_0, \varepsilon'(\lambda_j)^{-1}(\lambda_j))). \tag{42}
\]
Notice in the above integral the radii \( \varepsilon'(\lambda_j)^{-1}(\lambda_j) = (\log \lambda_j)^{-(\alpha'-\alpha)} \) of the domain of integration shrinks to 0. Therefore, there exists \( \zeta' \in M_{\tau_0} \setminus M \) for which \( B(\zeta', \varepsilon'(\lambda_j)^{-1}(\lambda_j)) \subset W_\delta \) for all \( \lambda_j \) sufficiently large.

On one hand, from the definition (41), we know that on all of \( W_\delta \) – and in particular on \( B(\zeta', \varepsilon'(\lambda_j)^{-1}(\lambda_j)) \) – we have the upper bound \( \lambda_j^{-1} \varepsilon(\lambda_j)^{-1} \log D_{\varepsilon(\lambda_j)}^* U_j^2 \leq -\delta/2 \), i.e.,
\[
D_{\varepsilon(\lambda_j)}^* |U_j|^2 \leq e^{-\delta \lambda_j \varepsilon(\lambda_j)} \quad \zeta \in B(\zeta', \varepsilon'(\lambda_j)^{-1}(\lambda_j)) \quad \lambda_j \gg 1. \tag{43}
\]
Clearly, the exponential decay upper bound (43) is incompatible with the logarithmic lower bound (42) as \( \lambda_j \to \infty \). This shows by way of contradiction that the original assumption
\[
\|\lambda_j^{-1} \varepsilon(\lambda_j)^{-1} \log D_{\varepsilon(\lambda_j)}^* U_j^2 \|_{L^1(B(\zeta_0, 1))} \geq \delta > 0
\]
does not hold, thereby proving Proposition 8.5 (i), from which Proposition 8.5 (ii) is an immediate consequence. Combining (43), Proposition 8.4 and Proposition 8.5 (ii), we obtain the zeros distribution statement of Theorem 8.2
\[
\frac{1}{\lambda_j \varepsilon(\lambda_j)} D_{\varepsilon(\lambda_j)}^* [Z_{\lambda_j}] \to \frac{i}{\pi} \partial \overline{\partial} \text{Im}(\zeta - \zeta_0)|_{g_0} \quad \text{as} \quad \text{currents on} \quad B(\zeta_0, 1)
\]
for a full density subsequence satisfying volume comparison at the finer scale \( \alpha' \). This concludes the proof of Theorem 8.1

\(^3\)A similar argument is used in [SZ20, Lemma 1.4], which gives further details. See also [KL] for background.
APPENDIX A. CURRENTS OF INTEGRATION OVER SINGULAR VARIETIES

In general, the zero set $X$ of a holomorphic function on a complex manifold $V$ is called a complex analytic variety (which could also be the common zeros of finitely many holomorphic functions). See for instance [W]. It has a decomposition into a regular set $R(X)$ and a lower-dimensional singular set $S(X)$, i.e., $X = R(X) \cup S(X)$ where $R(X)$ is a manifold and $\dim S(X) < \dim X$ (see [Ki, Theorem 2.1.8]). In [Ki, Theorem 3.1.1] it is proved that if $X$ a $k$-dimensional complex subvariety of a complex manifold $V$ and $u \in A^2_k(V)$ is a smooth $(2k)$-form then

$$[X](u) := \int_X u = \int_{R(X)} i^* u$$

is a closed current (due to Lelong [L]). King used Federer’s geometric measure theory [F] to study such currents. A modern exposition can be found in [D, Example 1.16].

A.1. Shiffman’s Appendix. We asked B. Shiffman for further references on currents of integration over singular analytic varieties. He wrote the following addition to the Appendix, and refers to [S, Lemma A.2] for an elementary proof.

Here is a simpler way to show that $[X] = [Z_f]$ is a well-defined current: It suffices to show that the set $R(X)$ of smooth points has finite volume in a neighborhood $U$ of a singular point $z_0$. By the Weierstrass preparation theorem applied to $f$, it follows that projections from $X \cap U$ to coordinate hyperplanes have finite fibers of bounded cardinality (for good coordinates) and therefore $\text{Vol}(R(X) \cap U) = \int_{R(X) \cap U} \omega^\dim X < \infty$.

The fact that Poincare-Lelong holds at the singular points follows from the fact that the singular set $S(X)$ has Hausdorff $(2n-3)$-dimensional measure 0, and therefore $\|\partial \bar{\partial} \log |f|\|(S) = 0$, since the total variation measure of a current of order zero and dimension $p$ vanishes on sets of Hausdorff $p$-measure zero. (In fact, $S(X)$ is a subvariety of real codimension 4).

REFERENCES

[Bo] L. Boutet de Monvel, Convergence dans le domaine complexe des séries de fonctions propres, C. R. Acad. Sci. Paris Sér. A-B 287 (1978), no. 13, A855–A856. MR0551763

[BoGu] L. Boutet de Monvel and V. Guillemin, The spectral theory of Toeplitz operators, Annals of Mathematics Studies, 99, Princeton University Press, Princeton, NJ, 1981. MR0620794

[BoS] L. Boutet de Monvel and J. Sjöstrand, Sur la singularité des noyaux de Bergman et de Szegö, in Journées: Équations aux Dérivées Partielles de Rennes (1975), 123–164. Astérisque, 34-35, Soc. Math. France, Paris. MR0590106

[CZ] R. Chang and S. Zelditch, Log-scale equidistribution of zeros of quantum ergodic eigensections, to appear in Ann. Henri Poincaré. https://arxiv.org/abs/1708.02333

[D] J.-P. Demailly, Analytic methods in algebraic geometry, Surveys of Modern Mathematics, 1, International Press, Somerville, MA, 2012. MR2978333

[F] H. Federer, Geometric measure theory, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969. MR0257325

[GH] P. Griffiths and J. Harris, Principles of algebraic geometry, Wiley-Interscience, New York, 1978. MR0507725

[GLS] E. Leichtnam, F. Golse, and M. Stenzel, Intrinsic microlocal analysis and inversion formulæ for the heat equation on compact real-analytic Riemannian manifolds. Ann. Sci. Ecole Norm. Sup. (4) 29 (1996), no. 6, 669-736.

[GSt1] V. Guillemin and M. Stenzel, Grauert tubes and the homogeneous Monge-Ampère equation, J. Differential
28 ROBERT CHANG AND STEVE ZELDITCH

[GSt2] V. Guillemin and M. Stenzel, Grauert tubes and the homogeneous Monge-Ampère equation, II. J. Differential Geom. 35 (1992), no. 3, 627–641. MR1163451

[Ha] X. Han, Small scale quantum ergodicity in negatively curved manifolds, Nonlinearity 28 (2015), no. 9, 3263–3288. MR3403398

[HeR] H. Hezari and G. Rivière, \(L^p\) norms, nodal sets, and quantum ergodicity, Adv. Math. 290 (2016), 938–966. MR3451943

[Ho] L. Hörmander, The analysis of linear partial differential operators. I, second edition, Grundlehren der Mathematischen Wissenschaften, 256, Springer-Verlag, Berlin, 1990. MR1065993

[Ki] J. R. King, The currents defined by analytic varieties, Acta Math. 127 (1971), no. 3-4, 185–220. MR0393550

[Kl] M. Klimek, Pluripotential Theory, Clarendon Press, Oxford, 1991.

[L] P. Lelong, Intégration sur un ensemble analytique complexe, Bull. Soc. Math. France 85 (1957), 239–262. MR0095967

[Le] G. Lebeau, The complex Poisson kernel on a compact analytic Riemannian manifold. http://www.math.ucla.edu/~hitrik/lebeau.pdf

[LS1] L. Lempert and R. Szöke, Global solutions of the homogeneous complex Monge-Ampère equation and complex structures on the tangent bundle of Riemannian manifolds, Math. Ann. 290 (1991), no. 4, 689–712. MR1119947

[LS2] L. Lempert and R. Szöke, The tangent bundle of an almost complex manifold, Canad. Math. Bull. 44 (2001), no. 1, 70–79. MR1816050

[MS] A. Melin and J. Sjöstrand, Fourier integral operators with complex-valued phase functions, in Fourier integral operators and partial differential equations (Colloq. Internat., Univ. Nice, Nice, 1974), 120–223. Lecture Notes in Math., 459, Springer, Berlin. MR0431289

[S] B. Shiffman, Introduction to the Carlson-Griffiths equidistribution theory, in Value distribution theory (Joensuu, 1981), 44–89, Lecture Notes in Math., 981, Springer, Berlin. MR0699133

[SZe] B. Shiffman and S. Zelditch, Distribution of zeros of random and quantum chaotic sections of positive line bundles. Comm. Math. Phys. 200 (1999), no. 3, 661-683.

[T] F. Trèves, Introduction to pseudodifferential and Fourier integral operators. Vol. 2, Plenum Press, New York, 1980. MR0597145

[U] K. Uhlenbeck, Generic properties of eigenfunctions. Amer. J. Math. 98 (1976), no. 4, 1059-1078. MR0464332

[W] H. Whitney, Complex analytic varieties, Addison-Wesley Publishing Co., Reading, MA, 1972. MR0387634

[Ze1] S. Zelditch, Complex zeros of real ergodic eigenfunctions, Invent. Math. 167 (2007), no. 2, 419–443. MR2270460

[Ze2] S. Zelditch, Pluri-potential theory on Grauert tubes of real analytic Riemannian manifolds, I, in Spectral geometry, 299–339, Proc. Sympos. Pure Math., 84, Amer. Math. Soc., Providence, RI. MR2985323

[Ze3] S. Zelditch, Ergodicity and intersections of nodal sets and geodesics on real analytic surfaces, J. Differential Geom. 96 (2014), no. 2, 305–351. MR3178442.

[Zw] M. Zworski, Semiclassical analysis, Graduate Studies in Mathematics, 138. American Mathematical Society, Providence, RI (2012).

E-mail address: hchang@math.northwestern.edu
E-mail address: zelditch@math.northwestern.edu

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON IL, 60208-2730, USA