Research Article

Iterative methods for solving fourth- and sixth-order time-fractional Cahn-Hilliard equation

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This paper presents analytical-approximate solutions of the time-fractional Cahn-Hilliard (TFCH) equations of fourth and sixth order using the new iterative method (NIM) and q-homotopy analysis method (q-HAM). We obtained convergent series solutions using these two iterative methods. The simplicity and accuracy of these methods in solving strongly nonlinear fractional differential equations is displayed through the examples provided. In the case where exact solution exists, error estimates are also investigated.

KEYWORDS
Cahn-Hilliard equation, fractional derivative, new iterative method, q-homotopy analysis method

MSC CLASSIFICATION
26A33; 34A12; 35R11

1 INTRODUCTION AND PRELIMINARIES

The concept of fractional calculus such as fractional derivative and fractional integral is not new. L’Hospital, in 1695, wrote a letter to Leibniz, saying “How do we calculate \( \frac{d^y}{dx^n} \) when \( n = \frac{1}{2} \)?” That is, “what will happen if we consider \( n \) to be a fraction?” Leibniz replied to L’Hospital question saying “\( d^{1/2}x \) equal to \( x\sqrt{dx : x} \). In actual fact, the reply is an apparent paradox, from this apparent paradox, one day, useful result might be drawn.” 1-3 Later, researchers discovered that fractional calculus has a wide range of applications in various fields of natural sciences and engineering such as control theory of dynamical systems, signal and image processing, financial modeling, nanotechnology, viscoelasticity, random walk, anomalous transport, and anomalous diffusion, just to name a few (for more details, see previous studies4-16).

Nonlinear partial differential equations (NPDEs) have played a vital role in various fields of engineering and natural sciences. Among such NPDEs, we have Cahn-Hilliard equation named after Cahn and Hilliard in 1958.17 This equation plays an essential role in understanding several fascinating physical phenomena, for instance, in spinodal decomposition and phase ordering dynamics. It also describes vital qualitative distinctive attribute of two-phase systems connected with phase separation processes (see previous studies17-20 for a detailed discussion). As a result of its real-world applications in the various fields mention above, researchers have investigated the mathematical and numerical solutions of this equation.20-28

Solving partial differential equations with fractional derivatives is often more difficult than solving the classical type, for its operator is defined by integral. In the recent year, researchers have developed some iterative methods for solving the nonlinear fractional differential equations, such as Adomian decomposition method,21,28,29 variational iteration method,30-32 homotopy decomposition method,33 differential transform method,34,35 perturbation iteration transformation method,36 homotopy-perturbation method,28,37 homotopy analysis method,38-40 exp-function method,41-43 wavelet method,44 Khater method,45 and residual power series method.46,47

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In this paper, we consider the time-fractional Cahn-Hilliard (TFCH) equations of the fourth and sixth order given, respectively, as follows:

\[ D_t^a u = \mu u_t + (-u_{xx} - u + u^3)_{xx}, \quad (1) \]

and

\[ D_t^b u = \mu u u_x + (u_{xx} + u - u^3)_{xxxx}, \quad (2) \]

with the initial condition

\[ u(x, 0) = f(x). \quad (3) \]

Here, \(0 < \alpha \leq 1\) stands for the order of the fractional derivative and parameter \(\mu \geq 0\). Our aim is to obtain solutions in the form of recurrence relations, using the new iteration method (NIM), which is based on the decomposition the nonlinearity term\(^{48-51}\) and q-homotopy analysis method (q-HAM), a modification of the homotopy analysis method.\(^{52-56}\)

**Definition 1.** The Gamma function is defined as\(^{57}\)

\[ \Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \, dt, \quad (4) \]

where \(\Re(z) > 0\).

**Definition 2.** The Riemann-Liouville fractional integral operator of order \(\alpha (\alpha \geq 0)\) of a function \(u(x, t) \in C_v, v \geq -1\), denoted by \(J^\alpha u(x, t)\), is defined as\(^{3,58}\)

\[ J^\alpha u(x, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \mu)^{\alpha-1} u(x, \mu) \, d\mu, \quad t > 0, \quad (5) \]

where \(J^0 u(x, t) = u(x, t)\). Then the following properties holds for function \(u(x, t)\) as follows:

a. \(J^{\alpha+\beta} u(x, t) = J^\alpha J^\beta u(x, t) = J^\beta J^\alpha u(x, t)\)

b. \(J^\alpha t^\beta = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)} t^{\alpha+\beta}\).

Here, \(\alpha, \beta \geq 0\) and \(\tau > -1\).

**Definition 3.** The (left sided) Caputo fractional derivative of a function \(u(x, t)\) of order \(\alpha\), denoted by \(D^\alpha u(x, t)\), where \(m - 1 < \alpha < m\), and \(u(x, t) \in C_m^m, m \in \mathbb{N}\) is defined as\(^{3,58,59}\)

\[ D^\alpha u(x, t) = \begin{cases} u^{(m)}(x, t), & \alpha = m, \\ J^{m-\alpha} u^{(m)}(x, t), & m - 1 < \alpha < m, \end{cases} \quad (6) \]

where

\[ J^{m-\alpha} u^{(m)}(x, t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t - \xi)^{m-\alpha-1} u^{(m)}(x, \xi) \, d\xi, \quad t > 0, \quad (7) \]

satisfies the following defined properties:

a. \(D^\alpha [\sigma u(x, t) + \rho w(x, t)] = \sigma D^\alpha u(x, t) + \rho D^\alpha w(x, t), \sigma, \rho \in \mathbb{R}\),

b. \(D^\alpha J^\alpha u(x, t) = u(x, t), \quad (8)\)

c. \(J^\alpha D^\alpha u(x, t) = u(x, t) - \sum_{j=0}^{m-1} u^j(x, 0) \frac{t^j}{j!}, \quad (9)\)

2 | ANALYSIS OF APPROXIMATE METHODS

In this section, we give a brief description of the NIM and the q-HAM.
2.1 | Fundamentals of the NIM

Consider the following functional equation

\[ u(x, t) = f(x, t) + \mathcal{L}(u(x, t)) + \mathcal{N}(u(x, t)), \]  

(8)

where \( \mathcal{L} \) and \( \mathcal{N} \) are, respectively, the linear and nonlinear operator from a Banach space \( B \) to itself, \( f(x, t) \) is a known function and \( u(x, t) \) is the unknown function. Let

\[ U_n(x, t) = \sum_{k=0}^{n} u_k(x, t). \]  

(9)

We seek for a solution \( u(x, t) \) of Equation (8) in the form of a series given as

\[ u(x, t) = \lim_{n \to \infty} U_n(x, t) = \sum_{k=0}^{\infty} u_k(x, t). \]  

(10)

Thus, Equation (9) converges absolutely and uniformly to a unique solution if the operators \( \mathcal{L} \) and \( \mathcal{N} \) are contractive.48,51 The decomposed nonlinear operator \( \mathcal{N} \) is given as

\[ \mathcal{N}(u(x, t)) = \mathcal{N}\left( \sum_{k=0}^{\infty} u_k(x, t) \right) = \mathcal{N}(u_0(x, t)) + \sum_{k=1}^{\infty} \left( \mathcal{N}\left( \sum_{i=0}^{k} u_i(x, t) \right) - \mathcal{N}\left( \sum_{i=0}^{k-1} u_i(x, t) \right) \right). \]  

(11)

In the same manner, the linear operator \( \mathcal{L} \) can be decomposed as

\[ \mathcal{L}(u(x, t)) = \mathcal{L}\left( \sum_{k=0}^{\infty} u_k(x, t) \right) = \mathcal{L}(u_0(x, t)) + \sum_{k=1}^{\infty} \left( \mathcal{L}\left( \sum_{i=0}^{k} u_i(x, t) \right) - \mathcal{L}\left( \sum_{i=0}^{k-1} u_i(x, t) \right) \right). \]  

(12)

From Equation (12), we have

\[ \mathcal{L}(u_0(x, t)) + \sum_{k=1}^{\infty} \left( \mathcal{L}\left( \sum_{i=0}^{k} u_i(x, t) \right) - \mathcal{L}\left( \sum_{i=0}^{k-1} u_i(x, t) \right) \right) = \mathcal{L}(u_0(x, t)) + \mathcal{L}(u_1(x, t)) + \mathcal{L}(u_2(x, t)) + \ldots = \sum_{k=0}^{\infty} \mathcal{L}(u_k(x, t)). \]  

(13)

Considering Equations (9) to (13), from Equation (8), we have

\[ \sum_{k=0}^{\infty} u_k(x, t) = f(x, t) + \sum_{k=0}^{\infty} \mathcal{L}(u_k(x, t)) + \mathcal{N}(u_0(x, t)) + \sum_{k=1}^{\infty} \left( \mathcal{N}\left( \sum_{j=0}^{k} u_j(x, t) \right) - \mathcal{N}\left( \sum_{j=0}^{k-1} u_j(x, t) \right) \right). \]  

(14)
Then, from Equation (14), we define the iterations

\[
\begin{align*}
    u_0 &= f(x), \\
    u_1 &= \mathcal{L}(u_0) + \mathcal{N}(u_0), \\
    u_2 &= \mathcal{L}(u_1) + \mathcal{N}(u_0 + u_1) - \mathcal{N}(u_0)), \\
    u_3 &= \mathcal{L}(u_2) + (\mathcal{N}(u_0 + u_1 + u_2) - \mathcal{N}(u_0 + u_1)), \\
    & \vdots \\
    u_{m+1} &= \mathcal{L}(u_m) + \left(\mathcal{N}\left(\sum_{j=0}^{m} u_j(x, t)\right) - \mathcal{N}\left(\sum_{j=0}^{m-1} u_j(x, t)\right)\right), \quad m = 1, 2, 3, \ldots . 
\end{align*}
\]

\subsection*{2.2 Idea of the q-HAM}

Consider the nonlinear differential equation

\[
\mathcal{N}\left(D_t^q u(x, t)\right) - f(x, t) = 0, \quad (16)
\]

where \( f(x, t) \) and \( u(x, t) \) are, respectively, the known and unknown functions, \( \mathcal{N} \) is the nonlinear operator, and \( D_t^q \) is the conformable fractional derivative with respect to “t.” In order to generalize the concept of homotopy method, we construct the zeroth-order deformation equation given as

\[
(1 - qn)\mathcal{L}(\Phi(x, t; q) - u_0(x, t)) = h q \mathcal{H}(x, t)(\mathcal{N}[D_t^q \Phi(x, t; q)] - f(x, t)), \quad n \geq 1, \quad (17)
\]

where \( q \left(0 \leq q \leq \frac{1}{n}\right) \) is the embedded parameter, \( \mathcal{L} \) is the auxiliary linear operator, and \( \mathcal{H}(x, t) \) is a nonzero auxiliary function. For \( q = 0, \frac{1}{n} \), respectively, we obtain from Equation (17) the following:

\[
\Phi(x, t; 0) = u_0(x, t), \quad \Phi(x, t; \frac{1}{n}) = u(x, t). \quad (18)
\]

When \( q \) rises from 0 to \( \frac{1}{n} \), the solution \( \Phi(x, t; q) \) ranges from the initial guess \( u_0(x, t) \) to the solution \( u(x, t) \). If \( u_0, h, \mathcal{L} \), and \( \mathcal{H}(x, t) \) are chosen appropriately, then the solution \( \Phi(x, t; q) \) of Equation (17) is valid as long as \( 0 \leq q \leq \frac{1}{n} \). Hence, we obtain the Taylor series expansion for \( \Phi(x, t; q) \) as

\[
\Phi(x, t; q) = u_0(x, t) + \sum_{r=1}^{\infty} u_r(x, t) q^r, \quad (19)
\]

where

\[
u_r(x, t) = \left. \frac{1}{r!} \frac{\partial^r \Phi(x, t; q)}{\partial q^r} \right|_{q=0}.
\]

If we choose \( u_0, h, \mathcal{L} \), and \( \mathcal{H}(x, t) \) properly so that Equation (19) converges at \( q = \frac{1}{n} \), then from Equation (18), we obtain

\[
\begin{align*}
    u(x, t) &= u_0(x, t) + \sum_{r=1}^{\infty} u_r(x, t) \left(\frac{1}{n}\right)^r. 
\end{align*}
\]

We define the vector \( \bar{u}_r \) as follows:

\[
\bar{u}_r = \{u_0(x, t), u_1(x, t), \ldots , u_r(x, t)\}. \quad (22)
\]

By differentiating Equation (17) \( r \)-times (with respect to \( 1 [q] \)) \( q \epsilon \), substituting \( q = 0 \), and then multiply it by \( \frac{1}{r!} \), we obtained what is known as the \( r^\text{th} \)-order deformation equation \( 52, 54, 60 \)

\[
\mathcal{L} \left( u_r(x, t) - \Psi_r u_{r-1}(x, t) \right) = h \mathcal{H}(x, t) \mathcal{R}_r \left( \bar{u}_{r-1} \right). \quad (23)
\]
subject to the initial conditions

\[ u^{(j)}(0, x) = 0, \quad j = 0, 1, 2, 3, ..., r - 1, \quad (24) \]

where

\[ R_r (\tilde{u}_{r-1}) = \frac{1}{(r - 1)!} \left. \frac{D^{r-1} (\mathcal{N}[D^r \Phi(x, t, q)] - f(x, t, q))}{\partial q^{r-1}} \right|_{q=0}, \quad (25) \]

and

\[ \Psi_r^* = \begin{cases} 0 & r \leq 1, \\ n & \text{otherwise}. \end{cases} \quad (26) \]

From Equation (23) with \( r \geq 1 \), one can get

\[ u_r(x, t) = \Psi_r^* u_{r-1}(x, t) + hJ^r R_r(\tilde{u}_{r-1}(x, t)). \quad (27) \]

Finally, the series solutions by q-HAM is presented by

\[ u(x, t; n; h) \cong U_r(x, t; n, h) = u_0(x, t) + \sum_{j=1}^{r} u_j(x, t; n, h) \left( \frac{1}{n} \right)^j, \quad (28) \]

which gives the appropriate solution in terms of convergence parameters \( n \) and \( h \).

3 | SOLUTIONS OF FOURTH-ORDER TFCH EQUATION

In this section, we present the application of the above mentioned methods to obtain approximate solution of the fourth-order TFCH Equation (1) subject to different initial conditions.

3.1 | Case I

Consider the general form of TFCH Equation (1) as

\[ D^\alpha u = \mu u_x + 6uu_x^2 + 3u^2 u_{xx} - u_{xxx} - u_{xxxx}, \quad 0 < \alpha \leq 1, \quad (29) \]

with the initial condition

\[ u(x, 0) = f(x) = \tanh \left( \frac{x}{\sqrt{2}} \right). \quad (30) \]

The exact solution of Equation (29) when \( \alpha = 1 \) and \( \mu = 1 \) is

\[ u(x, t) = \tanh \left( \frac{x + t}{\sqrt{2}} \right). \quad (31) \]

**NIM solution:**

Applying \( J^\alpha \) to both sides of Equation (29), then Equations (29) and (30) are equivalent to the integral equation

\[ u(x, t) = f(x, t) + \mathcal{L}(u(x, t)) + \mathcal{N}(u(x, t)), \]

where

\[ u_0 = f(x) = \tanh \left( \frac{x}{\sqrt{2}} \right), \]

\[ \mathcal{L}(u) = J^\alpha (\mu u_x - u_{xx} - u_{xxxx}), \]

\[ \mathcal{N}(u) = J^\alpha \left( 6uu_x^2 + 3u^2 u_{xx} \right). \]
We obtain components of the series solution using NIM recurrent relation in Equation (15) successively as follows:

\[ u_1(x, t) = \mathcal{L}(u_0) + \mathcal{N}(u_0) \]
\[ = J^a(\mu u_0 - u_{0xx} - u_{0xxxx}) \]
\[ + J^a(6u_0u_{0x}^2 + 3u_{3}^2u_{0xx}) \]
\[ = \mu \text{sech}^2\left(\frac{x}{\sqrt{2}}\right) t^\alpha, \]
\[ \sqrt{2}\Gamma(\alpha + 1) \]

\[ u_2(x, t) = \mathcal{L}(u_1) + (\mathcal{N}(u_0 + u_1) - \mathcal{N}(u_0)) \]
\[ = J^a (\mu u_{1x} - u_{1xxx} + 12u_0u_{0x}u_{1x} + 6u_1u_{0x}^2 + 6u_0u_1u_{0xx} + 3u_{3}^2u_{1xx}) \]
\[ + J^a (6u_0u_{1x}^2 + 12u_1u_{0x}u_{1x} + 3u_{3}^2u_{0xx} + 6u_0u_1u_{1xx} + 6u_1u_{1x}^2 + 3u_{3}^2u_{1xx}) \]
\[ = - \frac{\mu^2 \tanh\left(\frac{x}{\sqrt{2}}\right) \text{sech}^2\left(\frac{x}{\sqrt{2}}\right) t^\alpha}{\Gamma(2\alpha + 1)} + \frac{3\mu^2 \Gamma(2\alpha + 1) (4 \cosh(\sqrt{2}x) - 11) \tan\left(\frac{x}{\sqrt{2}}\right) \text{sech}^6\left(\frac{x}{\sqrt{2}}\right) t^{3\alpha}}{2\Gamma(\alpha + 1)^2 \Gamma(3\alpha + 1)} \]
\[ + \frac{3\mu^3 \Gamma(3\alpha + 1) (3 \cosh(\sqrt{2}x) - 4) \text{sech}^8\left(\frac{x}{\sqrt{2}}\right) t^{4\alpha}}{2\sqrt{2}\Gamma(\alpha + 1)^3 \Gamma(4\alpha + 1)}. \]

Following the same procedure, expression for \( u_m(x, t) \), \( m = 3, 4, ... \) can be obtained. The expression of the series solution given by NIM can be written in the form

\[ u(x, t) \cong U_2 = \sum_{i=0}^{2} u_i(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) \]
\[ = \tanh\left(\frac{x}{\sqrt{2}}\right) + \frac{\mu \text{sech}^2\left(\frac{x}{\sqrt{2}}\right) t^\alpha}{\sqrt{2}\Gamma(\alpha + 1)} - \frac{\mu^2 \tanh\left(\frac{x}{\sqrt{2}}\right) \text{sech}^2\left(\frac{x}{\sqrt{2}}\right) t^{2\alpha}}{\Gamma(2\alpha + 1)} \]
\[ + \frac{3\mu^2 \Gamma(2\alpha + 1) (4 \cosh(\sqrt{2}x) - 11) \tan\left(\frac{x}{\sqrt{2}}\right) \text{sech}^6\left(\frac{x}{\sqrt{2}}\right) t^{3\alpha}}{2\Gamma(\alpha + 1)^2 \Gamma(3\alpha + 1)} \]
\[ + \frac{3\mu^3 \Gamma(3\alpha + 1) (3 \cosh(\sqrt{2}x) - 4) \text{sech}^8\left(\frac{x}{\sqrt{2}}\right) t^{4\alpha}}{2\sqrt{2}\Gamma(\alpha + 1)^3 \Gamma(4\alpha + 1)} \]. \tag{32} \]

Thus, Equation (32) gives an approximate solution to problem (29). **q-HAM solution:** To apply the q-HAM, we rewrite Equation (29) as

\[ D_t^\alpha u - \mu u_x - 6u u_{xx}^2 - 3u^2 u_{xx} + u_{xxx} + u_{xxxx} = 0, \quad 0 < \alpha \leq 1. \tag{33} \]

Applying q-HAM to Equation (33), we obtain from Equation (25) the expression

\[ R_m (\tilde{u}_{m-1}) = D_t^\alpha u_{m-1} - \mu u_{(m-1)x} - 6 \sum_{k=0}^{m-1} \sum_{j=0}^{k} u_j u_{(k-j)x} u_{(m-1-k)x} \]
\[ - 3 \sum_{k=0}^{m-1} \sum_{j=0}^{k} u_j u_{(k-j)x} u_{(m-1-k)x} + u_{(m-1)x} + u_{(m-1)x}. \tag{34} \]
Using q-HAM recurrent relation in Equation (27). Then from Equations 26 and 34, we obtain the following:

\[ u_1(x, t) = \Psi_1^{*} u_0(x, t) + hJ^{\alpha} \left( R_1 \left( \bar{u}_0(x, t) \right) \right) \]
\[ = hJ^{\alpha} \left( D_t^\alpha u_0 - \mu u_0 - 6u_0u_0u_0 u_0 \right) + hJ^{\alpha} (-3u_0u_0u_0 u_0 + u_0 + u_0) \]
\[ = \frac{h\mu \text{sech}^2 \left( \frac{x}{\sqrt{2}} \right) t^\alpha}{\sqrt{2}\Gamma(\alpha + 1)} \]

\[ u_2(x, t) = \Psi_2^{*} u_1(x, t) + hJ^{\alpha} \left( R_2 \left( \bar{u}_1(x, t) \right) \right) \]
\[ = hu_1 + hJ^{\alpha} \left( D_t^\alpha u_1 - \mu u_1 - 6u_0u_0u_0 u_0 - 6u_0u_0u_0 u_0 - 6u_0u_0u_0 u_0 \right) + hJ^{\alpha} (-3u_0u_0u_0 u_0 + u_0 + u_0) \]
\[ = (n + h) u_1 - \frac{h^2 \mu^2 \text{tanh} \left( \frac{x}{\sqrt{2}} \right) \text{sech}^2 \left( \frac{x}{\sqrt{2}} \right) t^{2\alpha}}{\Gamma(2\alpha + 1)} \]

\[ u_3(x, t) = \Psi_3^{*} u_2(x, t) + hJ^{\alpha} \left( R_3 \left( \bar{u}_2(x, t) \right) \right) \]
\[ = hu_2 + hJ^{\alpha} \left( D_t^\alpha u_2 - \mu u_2 - 6u_0u_0u_0 u_0 - 6u_0u_0u_0 u_0 - 6u_0u_0u_0 u_0 \right) + hJ^{\alpha} (-3u_0u_0u_0 u_0 + u_0 + u_0) \]
\[ = (n + h) u_2 - \frac{\sqrt{2}h^3 \mu^3 (\cosh(\sqrt{2}x) - 2) \text{sech}^4 \left( \frac{x}{\sqrt{2}} \right) t^{3\alpha}}{2\Gamma(3\alpha + 1)} \]
\[ - \frac{6h^3 \mu^2 (4 \cosh(\sqrt{2}x) - 11) \text{tanh} \left( \frac{x}{\sqrt{2}} \right) \text{sech}^6 \left( \frac{x}{\sqrt{2}} \right) t^{3\alpha}}{2\Gamma(3\alpha + 1)} \]
\[ - \frac{3h^3 \mu^2 \Gamma(2\alpha + 1) (4 \cosh(\sqrt{2}x) - 11) \text{tanh} \left( \frac{x}{\sqrt{2}} \right) \text{sech}^6 \left( \frac{x}{\sqrt{2}} \right) t^{3\alpha}}{2\Gamma(\alpha + 1)^2 \Gamma(3\alpha + 1)} \]

Following the same procedure, expression for \( u_m(x, t), m = 4, 5, 6, \ldots \) can be obtained. The expression of the series solution given by q-HAM can be written in the form

\[ u(x, t; n; h) \cong U_3 = u_0(x, t) + \sum_{i=1}^{3} u_i(x, t; n; h) \left( \frac{1}{n} \right)^i \]
\[ = u_0(t, x) + \frac{u_1(t, x; n; h)}{n} + \frac{u_2(t, x; n; h)}{n^2} + \frac{u_3(t, x; n; h)}{n^3} \]
\[ = \text{tanh} \left( \frac{x}{\sqrt{2}} \right) + \frac{\mu h(3n^2 + 3nh + h^2) \text{sech}^2 \left( \frac{x}{\sqrt{2}} \right) t^\alpha - \frac{h^2 \mu^2 (3n + 2h) \text{tanh} \left( \frac{x}{\sqrt{2}} \right) \text{sech}^2 \left( \frac{x}{\sqrt{2}} \right) t^{2\alpha}}{n^3 \Gamma(2\alpha + 1)} \]
\[ + \frac{h^3 \mu^2 \text{sech}^4 \left( \frac{x}{\sqrt{2}} \right) \left( \sqrt{2} \mu \left( \cosh(\sqrt{2}x) - 2 \right) - 6(4 \cosh(\sqrt{2}x) - 11) \text{tanh} \left( \frac{x}{\sqrt{2}} \right) \text{sech}^2 \left( \frac{x}{\sqrt{2}} \right) \right) t^{3\alpha}}{2n^3 \Gamma(3\alpha + 1)} \]
\[ - \frac{3h^3 \mu^2 \Gamma(2\alpha + 1) (4 \cosh(\sqrt{2}x) - 11) \text{tanh} \left( \frac{x}{\sqrt{2}} \right) \text{sech}^6 \left( \frac{x}{\sqrt{2}} \right) t^{3\alpha}}{2n^3 \Gamma(\alpha + 1)^2 \Gamma(3\alpha + 1)} \]
Thus, Equation (35) gives an approximate solution to problem (29) in terms of convergence parameters $h$ and $n$. In the case when $n = \alpha = 1$, we choose $h = -1$, and obtain from Equation (35), the expression

$$u(x, t) = \tanh \left( \frac{x}{\sqrt{2}} \right) + \frac{\mu \text{sech}^2 \left( \frac{x}{\sqrt{2}} \right) t}{\sqrt{2}} - \frac{\mu^2 \tanh \left( \frac{x}{\sqrt{2}} \right) \text{sech}^2 \left( \frac{x}{\sqrt{2}} \right) t^2}{2}$$

$$+ \frac{\mu^3 \left( \cosh(\sqrt{2}x) - 2 \right) \text{sech}^4 \left( \frac{x}{\sqrt{2}} \right) t^3}{6 \sqrt{2}} + \ldots,$$

which can be expressed in the closed-form of the exact solution (31) when $\mu = 1$.

Remark 1. This agrees with the solution obtained using ADM and HPM in Ugurlu and Kaya.28

3.2 | Case II

We consider the general form of TFCH Equation (1) as

$$D_\alpha^\mu u = \mu u_x + 6uu_x^2 + 3u^2u_{xx} - u_{xxx} - u_{xxxx}, \quad 0 < \alpha \leq 1,$$

with the initial condition

$$u(x, 0) = f(x) = e^{ix}.$$  (38)

NIM solution:

Applying $J^\alpha$ to both sides of Equation (37), then Equations (37) and 38 are equivalent to the integral equation

$$u(x, t) = f(x, t) + \mathcal{L}(u(x, t)) + \mathcal{N}(u(x, t)),$$

where

$$u_0 = f(x) = e^{ix},$$

$$\mathcal{L}(u) = J^\alpha \left( \mu u_x - u_{xx} - u_{xxxx} \right),$$

$$\mathcal{N}(u) = J^\alpha \left( 6uu_x^2 + 3u^2u_{xx} \right).$$

We obtain components of the series solution using NIM recurrent relation in Equation (15) successively as follows:

$$u_1(x, t) = \mathcal{L}(u_0) + \mathcal{N}(u_0)$$

$$= J^\alpha \left( \mu u_0 - u_{0xx} - u_{0xxxx} \right)$$

$$+ J^\alpha \left( 6u_0u_{0x}^2 + 3u_0^2u_{0xx} \right)$$

$$= \frac{\lambda e^{ix}(-\lambda^3 + 9\lambda e^{2ix} - \lambda + \mu)}{\Gamma(\alpha + 1)} t^\alpha.$$
\[u_2(x, t) = J\left(\mu u_{1x} - u_{1xx} - u_{xxx} + 12u_0 u_{0x} - 6u_1 u_{0xx} + 6u_0 u_{1x} - 3u_0^2 u_{xx}\right)\]
\[\Rightarrow J\left(\mu u_{1x} + 12u_0 u_{0x} + 6u_0 u_{1x} + 6u_0 u_{1x} - 3u_0^2 u_{xx}\right)\]
\[= J = 27\lambda^7 h^2 e^{2lx} \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)^2} (50\lambda(\lambda^3 + \lambda - \mu)e^{2lx} - (\lambda^3 + \lambda - \mu)^2 - 441\lambda^2 e^{4lx}) t^{2\alpha} \]
\[+ \frac{9\lambda^5 e^{3lx} \Gamma(3\alpha + 1)}{\Gamma(\alpha + 1)^3 \Gamma(4\alpha + 1)} (75(\lambda^7 - 2\lambda^4 \mu - 2\lambda^2 \mu + \lambda^2 \mu^2 + 2\lambda^5 + \lambda^3) e^{2lx} + 6561\lambda^3 e^{6lx} - \mu^3) t^{4\alpha}. \]

Following the same procedure, expressions for \(u_m(x, t)\), \(m = 3, 4, 5, \ldots\) can be obtained. The expression of the series solution given by NIM can be written in the form

\[u(x, t) \approx U_2 = \sum_{i=0}^{2} u_i(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t)\]
\[= e^{lx} + \frac{\lambda^2 e^{lx} (-\lambda^3 + 9\lambda e^{2lx} - \lambda + \mu) t^2}{\Gamma(\alpha + 1)}\]
\[+ \frac{\lambda^2 e^{lx} (-54\lambda e^{2lx}(14\lambda^3 + 2\lambda - \mu) + (\lambda^3 + \lambda - \mu)^2 + 675\lambda^2 e^{4lx}) t^2}{\Gamma(2\alpha + 1)}\]
\[+ \frac{27\lambda^4 h^2 e^{3lx} \Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)^2 \Gamma(3\alpha + 1)} (50\lambda(\lambda^3 + \lambda - \mu)e^{2lx} - (\lambda^3 + \lambda - \mu)^2 - 441\lambda^2 e^{4lx}) t^{2\alpha} \]
\[+ \frac{9\lambda^5 e^{3lx} \Gamma(3\alpha + 1)}{\Gamma(\alpha + 1)^3 \Gamma(4\alpha + 1)} (75(\lambda^7 - 2\lambda^4 \mu - 2\lambda^2 \mu + \lambda^2 \mu^2 + 2\lambda^5 + \lambda^3) e^{2lx} + 6561\lambda^3 e^{6lx} - \mu^3) t^{4\alpha}. \]

Thus, Equation (39) gives an approximate solution to problem (37).

**q-HAM solution:**

To apply the q-HAM, we rewrite Equation (37) as

\[D^\alpha u - \mu u_x - 6u u_x^2 - 3u^2 u_{xx} + u_{xxx} + u_{xxxx} = 0, \quad 0 < \alpha \leq 1. \]

Using q-HAM recurrent relation in Equation (27). Then from Equations (26) and (34), we obtain the following

\[u_1(x, t) = \Psi^* u_0(x, t) + h J = h J = h J = -\frac{\lambda^2 e^{lx} (-\lambda^3 + 9\lambda e^{2lx} - \lambda + \mu) t^2}{\Gamma(\alpha + 1)}. \]
\[ u_2(x, t) = \Psi_2^* u_1(x, t) + hJ^a (R_2 (\bar{u}_1(x, t))) \]
\[ = h u_1 + hJ^a (D_t^\alpha u_1 - \mu u_{1x} - 6u_0 u_{1x} u_{1x} - 6u_0 u_{1x} u_{0x} - 6u_1 u_{0x} u_{0x}) \]
\[ + hJ^a (-3u_0 u_{0x} u_{1x} - 3u_0 u_{1x} u_{0xx} - 3u_1 u_{0xx} + u_{1xx} + u_{1xxx}) \]
\[ = (n + h) u_1 + \frac{\lambda^2 h^2 e^{2x} (-54 \lambda e^{2x}(14 \lambda^3 + 2 \lambda - \mu) + (\lambda^3 + \lambda - \mu)^2 + 675 \lambda^2 e^{4x})}{\Gamma(2a + 1)} l^{2a}, \]

\[ u_3(x, t) = \Psi_3^* u_2(x, t) + hJ^a (R_3 (\bar{u}_2(x, t))) \]
\[ = h u_2 + hJ^a (D_t^\alpha u_2 - \mu u_{2x} - 6u_0 u_{0x} u_{2x} - 6u_0 u_{0x} u_{1x} - 6u_0 u_{2x} u_{0x} - 6u_1 u_{1x} u_{0x} - 6u_2 u_{0x} u_{0x}) \]
\[ + hJ^a (-3u_0 u_{0x} u_{2xx} - 3u_0 u_{1x} u_{1xx} - 3u_0 u_{2x} u_{0xx} - 3u_1 u_{1xx} u_{0xx} - 3u_1 u_{0xx} u_{0xxx} - 3u_2 u_{0xx} u_{2xx} + u_{2xx} + u_{2xxxx}) \]
\[ = (n + h) u_2 + \frac{\lambda^2 h^7 (n + h) e^{2x} (-54 \lambda e^{2x}(14 \lambda^3 + 2 \lambda - \mu) + (\lambda^3 + \lambda - \mu)^2 + 675 \lambda^2 e^{4x})}{\Gamma(2a + 1)} l^{3a} \]
\[ + \frac{\lambda^3 h^6 e^{2x} ((\lambda^3 + \lambda - \mu)^3 - 99225 \lambda^3 e^{6x} + 675 \lambda^3 (79 \lambda^3 + 37) e^{4x} - 27 \lambda^3 (2269 \lambda^4 + 578 \lambda^3 + 37) e^{2x})}{\Gamma(3a + 1)} l^{3a} \]
\[ - \frac{27 \lambda^4 h^6 e^{3x} (27(\lambda^3 + \lambda - \mu) e^{2xx} - (\lambda^3 + \lambda - \mu)^2 + 441 \lambda^2 e^{4x})}{\Gamma(3a + 1)} l^{3a} \]
\[ + \frac{27 \lambda^4 h^6 e^{3x} \Gamma(2a + 1)(50 \lambda(\lambda^3 + \lambda - \mu) e^{2xx} - (\lambda^3 + \lambda - \mu)^2 + 441 \lambda^2 e^{4x})}{\Gamma(3a + 1)} l^{3a}. \]

**FIGURE 1** Case I: Comparison among new iterative method (NIM), q-homotopy analysis method (q-HAM), and exact solution for \( n = 1 \) and \( h = -1 \) [Colour figure can be viewed at wileyonlinelibrary.com]

**FIGURE 2** Case I: Comparison among new iterative method (NIM), q-homotopy analysis method (q-HAM), and exact solution in 2D for \( n = 1, \ a = 1, \ \mu = 1, \) and \( h = -1 \) [Colour figure can be viewed at wileyonlinelibrary.com]
Following the same procedure, expression for $u_m(x, t), m = 4, 5, 6, \ldots$ can be obtained. The expression of the series solution given by q-HAM can be written in the form

$$u(x, t; n; h) \cong U_3 = u_0(x, t) + \sum_{i=1}^{3} u_i(x, t; n; h) \left( \frac{1}{n} \right)^i$$

$$= e^{ix} - \frac{\lambda e^{ix}h(3n^2 + 3nh + h^2)(-\lambda^3 + 9\lambda e^{2ix} - \lambda + \mu)}{n^3\Gamma(\alpha + 1)} t^\alpha$$

$$+ \frac{\lambda^3 h^2(3n + 2h)e^{ix} (-54\lambda^2 e^{2ix}(14\lambda^3 + 2\lambda - \mu) + (\lambda^3 + \lambda - \mu)^2 + 675\lambda^2 e^{4ix})}{n^3\Gamma(2\alpha + 1)} t^{2\alpha}$$

$$+ \frac{\lambda^3 h^3 e^{ix} (\lambda^3 + \lambda - \mu^3 - 99225\lambda^3 e^{6ix} + 675\lambda^3(709\lambda^2 + 37)e^{4ix} - 27\lambda^5(2269\lambda^4 + 578\lambda^2 + 37)e^{2ix})}{n^3\Gamma(3\alpha + 1)} t^{3\alpha}$$

$$- \frac{27\mu \lambda^4 h^4 e^{3ix} (-248\lambda^3 + \lambda(275e^{2ix} - 42) + 7\mu)}{n^3\Gamma(3\alpha + 1)} t^{3\alpha}$$

$$+ \frac{27\lambda^4 h^4 e^{3ix} (50\lambda(\lambda^3 + \lambda - \mu)e^{2ix} - (\lambda^3 + \lambda - \mu)^2 - 441\lambda^2 e^{4ix})}{n^3\Gamma(\alpha + 1)^2\Gamma(3\alpha + 1)} t^{3\alpha}.$$  (41)

Thus, Equation (41) gives an approximate solution to problem (37) in terms of convergence parameters $h$ and $n$. 

**FIGURE 3** Case I: Comparison between new iterative method (NIM) and q-homotopy analysis method (q-HAM) solution for $n = 1$ and $h = -1$ [Colour figure can be viewed at wileyonlinelibrary.com]

**FIGURE 4** Case I: Comparison between new iterative method (NIM) and q-homotopy analysis method (q-HAM) solution for $n = 1$ and $h = -1$ [Colour figure can be viewed at wileyonlinelibrary.com]
FIGURE 5  Case I: Comparison between new iterative method (NIM) and q-homotopy analysis method (q-HAM) solution for \( n = 1 \) and \( h = -1 \) [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 6  Case II: Comparison between new iterative method (NIM) and q-homotopy analysis method (q-HAM) solution for \( n = 1 \) and \( h = -1 \) [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 7  Case II: Comparison between new iterative method (NIM) and q-homotopy analysis method (q-HAM) solution in 2D for \( n = \alpha = \mu = 1, h = -1, \) and \( \lambda = -0.05 \) [Colour figure can be viewed at wileyonlinelibrary.com]

3.3  Numerical results for TFCH equation of fourth order

Here, we check how accurate these two methods are for solving time-fractional Cahn-Hillard Equation (1) with different initial conditions as shown in cases I and II of Section 3. In Figures 1 to 11, one can acknowledge how closely the approx-
approximation series solution obtained by these two methods and the exact solution. In Table 1, error estimate is done for the case when the exact solution ($\mu = 1$ and $\alpha = 1$) is known.
4.1 Case I

Consider the general form of TFCH Equation (2) as

\[ D_x^\alpha u = \mu u_x - 18u u_x^2 - 36u_x^2 u_{xx} - 24u u_x u_{xxx} - 3u_x^2 u_{xxxx} + u_{xxxx} + u_{xxxxx}, \quad 0 < \alpha \leq 1, \] (42)

with the initial condition

\[ u(x, 0) = f(x) = \tanh \left( \frac{x}{\sqrt{2}} \right), \] (43)

**NIM solution:**

Applying \( J^\alpha \) to both sides of Equation (42), then Equations (42) and (43) are equivalent to the integral equation

\[ u(x, t) = f(x, t) + \mathcal{L}(u(x, t)) + \mathcal{N}(u(x, t)), \]
where
\[ u_0 = f(x) = \tanh \left( \frac{x}{\sqrt{2}} \right), \]
\[ \mathcal{L}(u) = J^u (u_{xxxxx} + u_{xxx}), \]
\[ \mathcal{N}(u) = J^u (\mu u_x - 18u_{xx} - 36u_{x}^2 - 24uu_{x} - 3u_x^3), \]

We obtain components of the series solution using NIM recurrent relation in Equation (15) successively as follows:
\[ u_1(x, t) = \mathcal{L}(u_0) + \mathcal{N}(u_0) = J^u (u_{xxxxx} + u_{xxx} + \mu u_x - 18u_{xx}) \]
\[ -J^u (36u_x^2u_{xxx} + 24u_0u_{xxx} + 3u_x^3u_{xxx}) = \frac{\mu \tanh \left( \frac{x}{\sqrt{2}} \right) \sech^2 \left( \frac{x}{\sqrt{2}} \right)}{\sqrt{2}\Gamma(\alpha + 1)} t^\alpha, \]
\[ u_2(x, t) = \mathcal{L}(u_1) + (\mathcal{N}(u_0 + u_1) - \mathcal{N}(u_0)) \]
\[ = J^u (u_{xxxxxx} + u_{xxxx} - 36u_x^2u_{xxx} - 72u_0u_{xx} - 24u_0u_{xxx} - 24u_xu_{xxx} - 24u_0u_{xxx}) \]
\[ -J^u (24u_xu_{xxx} + 36u_xu_{xxx} + 18u_0u_{xxx} + 6u_0u_{xxx} + 3u_x^3u_{xxx} - \mu u_0u_x - \mu u_0u_x) \]
\[ -J^u (72u_0u_{xx}u_{xxx} + 36u_x^2u_{xxx} - 24u_0u_{xxx} + 18u_0u_{xxx} + 18u_0u_{xxx} - \mu u_0u_x) \]
\[ -J^u (36u_xu_{xxx} + 36u_xu_{xxx} + 6u_0u_{xxx} + 24u_0u_{xxx} + 18u_0u_{xxx} + 3u_x^3u_{xxx}) \]
\[ = \mu \tanh \left( \frac{x}{\sqrt{2}} \right) \sech^8 \left( \frac{x}{\sqrt{2}} \right) \left( \mu \cosh^6 \left( \frac{x}{\sqrt{2}} \right) + (96\sqrt{2} - 2\mu)\cosh^4 \left( \frac{x}{\sqrt{2}} \right) - 585\sqrt{2}\cosh^2 \left( \frac{x}{\sqrt{2}} \right) + 630\sqrt{2} \right) \frac{\Gamma(2\alpha + 1)}{t^{2\alpha}} \]
\[ + \frac{\mu^2\Gamma(2\alpha + 1)\tanh \left( \frac{x}{\sqrt{2}} \right) \sech^4 \left( \frac{x}{\sqrt{2}} \right) \left( 3(\mu \sqrt{2} + 1428)\sech^2 \left( \frac{x}{\sqrt{2}} \right) - 2(\sqrt{2}\mu + 192) \right)}{64\Gamma(\alpha + 1)^2\Gamma(3\alpha + 1)} t^{3\alpha} \]
\[ + \frac{420\mu^2\Gamma(2\alpha + 1)\tanh \left( \frac{x}{\sqrt{2}} \right) \sech^{10} \left( \frac{x}{\sqrt{2}} \right) (5 - 13\cosh(\sqrt{2}x))}{64\Gamma(\alpha + 1)^2\Gamma(3\alpha + 1)} t^{3\alpha} \]
\[ + \frac{3\mu^3\Gamma(3\alpha + 1)\tanh \left( \frac{x}{\sqrt{2}} \right) \sech^{12} \left( \frac{x}{\sqrt{2}} \right) \left( 3773\cosh(\sqrt{2}x) - 646\cosh(2\sqrt{2}x) + 27\cosh(3\sqrt{2}x) - 3474 \right)}{16\sqrt{2}\Gamma(\alpha + 1)^3\Gamma(4\alpha + 1)} t^{4\alpha}. \]

Following the same procedure, expressions for \( u_m(x, t), m = 3, 4, 5, \ldots \) can be obtained. The expression of the series solution given by NIM can be written in the form
\[ u(x, t) \cong U_2 = \sum_{i=0}^{2} u_i(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) \]
\[ = \tanh \left( \frac{x}{\sqrt{2}} \right) + \frac{\mu \tanh \left( \frac{x}{\sqrt{2}} \right) \sech^2 \left( \frac{x}{\sqrt{2}} \right)}{\sqrt{2}\Gamma(\alpha + 1)} t^\alpha \]
\[ - \frac{\mu \tanh \left( \frac{x}{\sqrt{2}} \right) \sech^8 \left( \frac{x}{\sqrt{2}} \right) \left( \mu \cosh^6 \left( \frac{x}{\sqrt{2}} \right) + (96\sqrt{2} - 2\mu)\cosh^4 \left( \frac{x}{\sqrt{2}} \right) - 585\sqrt{2}\cosh^2 \left( \frac{x}{\sqrt{2}} \right) + 630\sqrt{2} \right)}{\Gamma(2\alpha + 1)} t^{2\alpha} \]
\[ + \frac{\mu^2\Gamma(2\alpha + 1)\tanh \left( \frac{x}{\sqrt{2}} \right) \sech^4 \left( \frac{x}{\sqrt{2}} \right) \left( 3(\mu \sqrt{2} + 1428)\sech^2 \left( \frac{x}{\sqrt{2}} \right) - 2(\sqrt{2}\mu + 192) \right)}{64\Gamma(\alpha + 1)^2\Gamma(3\alpha + 1)} t^{3\alpha} \]
\[ + \frac{420\mu^2\Gamma(2\alpha + 1)\tanh \left( \frac{x}{\sqrt{2}} \right) \sech^{10} \left( \frac{x}{\sqrt{2}} \right) (5 - 13\cosh(\sqrt{2}x))}{64\Gamma(\alpha + 1)^2\Gamma(3\alpha + 1)} t^{3\alpha} \]
\[ + \frac{3\mu^3\Gamma(3\alpha + 1)\tanh \left( \frac{x}{\sqrt{2}} \right) \sech^{12} \left( \frac{x}{\sqrt{2}} \right) \left( 3773\cosh(\sqrt{2}x) - 646\cosh(2\sqrt{2}x) + 27\cosh(3\sqrt{2}x) - 3474 \right)}{16\sqrt{2}\Gamma(\alpha + 1)^3\Gamma(4\alpha + 1)} t^{4\alpha}. \]
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Thus, Equation (45) gives an approximate solution to problem (42).

**q-HAM solution:**

To apply the q-HAM, we rewrite Equation (42) as

\[
D_t^\alpha u - \mu uu_x + 18uu_{xx} + 36u_x^2u_{xx} + 24uu_x u_{xxx} + 3u^3u_{xxxx} - u_{xoooo} = 0, \quad 0 < \alpha \leq 1. \tag{45}
\]

Applying q-HAM to Equation (45), we obtain from Equation (25) the expression

\[
R_m \left( \bar{u}_{m-1} \right) = D_t^\alpha u_{(m-1)} - \mu \sum_{k=0}^{m-1} u_k u_{(m-1-k)xx} + 18 \sum_{k=0}^{m-1} \sum_{j=0}^{k} u_j u_{(k-j)xx} u_{(m-1-k)xx}
+ 36 \sum_{k=0}^{m-1} \sum_{j=0}^{k} u_j u_{(k-j)xx} u_{(m-1-k)xx} + 24 \sum_{k=0}^{m-1} \sum_{j=0}^{k} u_j u_{(k-j)xx} u_{(m-1-k)xxx}
+ 3 \sum_{k=0}^{m-1} u_j u_{(m-1-k)xxxx} - u_{(m-1)xxxx} - u_{(m-1)xxxxx}.
\tag{46}
\]

Using q-HAM recurrent relation in Equation (27). Then from Equations (26) and (46), we obtain the following:

\[
u_1(x, t) = \Psi^*_1 u_0(x, t) + hR_1 \left( \bar{u}_0(x, t) \right)
= hJ^n \left( D_t^\alpha u_0 - \mu u_0 u_{xx} + 18u_0 u_{xx} u_{0xx} + 36u_0 u_x u_{0xx} \right)
+ hJ^n \left( 24u_0 u_x u_{0xxx} + 3u_0 u_0 u_{xxxx} - u_{0xxxx} - u_{0xxxxx} \right)
= -\frac{h \mu \tanh \left( \frac{x}{\sqrt{2}} \right) \tanh \left( \frac{\sqrt{2}}{\sqrt{2}} \right)}{\sqrt{2} \Gamma(\alpha + 1)} t^{\alpha}.
\]

\[
u_2(x, t) = \Psi^*_2 u_1(x, t) + hJ^n \left( R_2 \left( \bar{u}_1(x, t) \right) \right)
= h\nu_1 + hJ^n \left( D_t^\alpha u_1 - \mu u_0 u_{1} + \mu u_{1} u_{xx} + 18u_0 u_{xx} u_{1xx} + 18u_0 u_{1xx} u_{0xx} + 18\mu_1 u_{0xx} u_{0xx} \right)
+ hJ^n \left( 24u_1 u_x u_{0xxx} + 3u_0 u_1 u_{0xxx} + 3u_0 u_1 u_{0xxx} + 3u_0 u_1 u_{0xxx} - u_{1xxx} - u_{1xxxx} \right)
= (n + h) u_1 - \frac{h^2 \mu \tanh \left( \frac{x}{\sqrt{2}} \right) \tanh \left( \frac{\sqrt{2}}{\sqrt{2}} \right) \left( \mu \cosh \left( \frac{\sqrt{2}}{\sqrt{2}} \right) + (96\sqrt{2} - 2\mu) \cosh \left( \frac{\sqrt{2}}{\sqrt{2}} \right) \right)}{\Gamma(2\alpha + 1)} t^{2\alpha}
- \frac{h^2 \mu \tanh \left( \frac{x}{\sqrt{2}} \right) \tanh \left( \frac{\sqrt{2}}{\sqrt{2}} \right) \left( -585\sqrt{2} \cosh \left( \frac{\sqrt{2}}{\sqrt{2}} \right) + 630 \sqrt{2} \right)}{\Gamma(2\alpha + 1)} t^{2\alpha}.
\]

Following the same procedure, expression for \( u_m(x, t), m = 3, 4, 5, ... \) can be obtained. The expression of the series solution given by q-HAM can be written in the form
\[ u(x, t; n; h) \approx U_2 = u_0(x, t) + \sum_{i=1}^{2} u_i(x, t; n; h) \left( \frac{1}{n} \right)^i \]

\[ = u_0(t, x) + \frac{u_1(t, x; n; h)}{n} + \frac{u_2(t, x; n; h)}{n^2} \]

\[ = \tanh \left( \frac{x}{\sqrt{2}} \right) - \frac{\mu h(2n + h) \tanh \left( \frac{x}{\sqrt{2}} \right) \sech^2 \left( \frac{x}{\sqrt{2}} \right) t^\alpha}{n^2 \Gamma(\alpha + 1)} \]

\[ - \frac{\mu h^2(3n + 2h) \tanh \left( \frac{x}{\sqrt{2}} \right) \sech^8 \left( \frac{x}{\sqrt{2}} \right) \left( -585 \sqrt{2} \cosh^2 \left( \frac{x}{\sqrt{2}} \right) + 630 \sqrt{2} \right) t^{2\alpha}}{n^2 \Gamma(2\alpha + 1)} \]

Thus, Equation (47) gives an approximate solution to problem (42) in terms of convergence parameter \( h \) and \( n \).

4.2 Case II

Consider the general form of TFCH Equation (2) as

\[ D^\alpha_t u = \mu uu_x - 18uu^2_{xx} - 36u^2u_{xxx} - 24uu_xu_{xxx} - 3u^2u_{xxxx} + u_{xxxxx} - 0 < \alpha \leq 1, \]

with the initial condition

\[ u(x, 0) = f(x) = e^{i\kappa}. \]

NIM solution:

Applying \( J^\alpha \) to both sides of Equation (48), then Equations (48) and (49) are equivalent to the integral equation

\[ u(x, t) = f(x, t) + \mathcal{L}(u(x, t)) + \mathcal{N}(u(x, t)), \]

where

\[ u_0 = f(x) = e^{i\kappa}, \]

\[ \mathcal{L}(u) = J^\alpha (u_{xxxxx} + u_{xxxx}) , \]

\[ \mathcal{N}(u) = J^\alpha \left( \mu uu_x - 18uu^2_{xx} - 36u^2u_{xxx} - 24uu_xu_{xxx} - 3u^2u_{xxxx} \right). \]

We obtain components of the series solution using NIM recurrent relation in Equation (15) successively as follows:

\[ u_1(x, t) = \mathcal{L}(u_0) + \mathcal{N}(u_0) \]

\[ = J^\alpha \left( u_{xxxxx} + u_{xxxx} + \mu u_0 u_{xx} - 18u_0 u^2_{xx} \right) \]

\[ - J^\alpha \left( 36u^2_{xx} u_{xx} + 24u_0 u_{xx} u_{xxxx} + 3u^2_{xx} u_{xxxx} \right) \]

\[ = \frac{\lambda e^{i\kappa} (\lambda^2 + \lambda^3(1 - 81e^{2i\kappa}) + \mu e^{i\kappa}) t^\alpha}{\Gamma(\alpha + 1)}. \]
\[ u_2(x, t) = J^a \left( \mathcal{L}(u_1) + (N(u_0 + u_1) - N(u_0)) \right) \]
\[ = J^a \left( u_{xxxx} + u_{1xxx} - 36u_{0}u_{1xx} - 72u_{0x}u_{1x} - 24u_{00}u_{1xx} - 24u_{00x}u_{1x} \right) \]
\[ - J^a \left( 24u_{1xx}u_{00x} + 36u_{00}u_{1xx} + 18u_{1xx}u_{0x} + 6u_{1xx}u_{0} + 3u_{1xx}u_{00} - u_{00}u_{1xx} - u_{00x}u_{1x} \right) \]
\[ - J^a \left( 72u_{0x}u_{1xx} + 36u_{0}u_{1xxx} + 24u_{0}u_{1xx} + 24u_{0x}u_{1xxx} + 24u_{1xx}u_{0xxx} + 18u_{00}u_{1xx} - u_{00}u_{1xx} \right) \]
\[ - J^a \left( 3u_{1xx}u_{0xxx} + 2u_{0xx}u_{1xx} + 6u_{1xx}u_{0xx} + 2u_{0xx}u_{1xx} \right) \]
\[ = \lambda^2 e^{ix} (151875\lambda^6 e^{i2x} - 1092\mu \lambda^3 e^{i2x} - 3(324\lambda^6(61\lambda^2 + 7) - \mu^2) e^{i2x} + (\lambda^2 + 2\lambda^6 + 6\mu(11\lambda^2 + 3)\lambda^3 e^{i2x}) \right) t^{2\alpha} \]
\[ \frac{\lambda^3 e^{i2\alpha} \Gamma(2\alpha + 1)(-303750\lambda^{11} e^{i2x} + 47258883\lambda^9 e^{i3x} - 649539\mu \lambda^6 e^{i2x} + 1860\mu \lambda^6 - 2\mu^3)}{t^{3\alpha}} \]
\[ + \frac{\lambda^6 e^{i2\alpha} \Gamma(3\alpha + 1)(-243\lambda^{10} e^{i2x} - 486\lambda^8 e^{i2x} + \mu \lambda^7 - 248\lambda^6 e^{i2x} + 303750\lambda^6 e^{i3x} - 1860\mu \lambda^6 e^{i2x})}{t^{\alpha}} \]
\[ + \frac{\lambda^9 e^{i2\alpha} \Gamma(3\alpha + 1)(2\mu \lambda^5 + \mu \lambda^3 + 3\lambda^2 \mu^2 e^{i2x} + 3\mu^2 e^{i2x} - 2280\mu^3 e^{i2x})}{t^{3\alpha}} \]
\[ + \frac{3\lambda^{10} e^{i2\alpha} \Gamma(3\alpha + 1)(101250\lambda^8 - 15752961\lambda^6 e^{i2x} + 1162261467\lambda^6 e^{i4x} + 50625\lambda^6)}{t^{\alpha}} \]
\[ + \frac{3\lambda^{11} e^{i2\alpha} \Gamma(3\alpha + 1)(2 \lambda^{15} - 50625\lambda^4 e^{i2x} + 18413 + 15752961\lambda^6 e^{i2x} + 81\lambda^{11} + 256\mu \lambda^{10} e^{i2x})}{t^{\alpha}} \]
\[ + \frac{3\lambda^{12} e^{i2\alpha} \Gamma(3\alpha + 1)(2 \lambda^9 + 512\mu \lambda^6 e^{i2x} + 256\mu \lambda^6 e^{i2x} - 209952\mu \lambda^6 e^{i3x} + 26873856\mu \lambda^6 e^{i2x} + 432\mu^3 e^{i2x})}{t^{4\alpha}} \]

Following the same procedure, expression for \( u_m(x, t), m = 3, 4, 5, \ldots \) can be obtained. The expression of the series solution given by NIM can be written in the form

\[ u(x, t) \cong U_2 = \sum_{n=0}^{N} u_i(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) \]
\[ = e^{ix} + \frac{\lambda^3 e^{i2\alpha}(\lambda^5 + \lambda^3(1 - 81 e^{2ix}) + \mu e^{i2x})}{\Gamma(\alpha + 1)} \]
\[ + \frac{\lambda^2 e^{i2\alpha} (151875\lambda^6 e^{i2x} - 1092\mu \lambda^3 e^{i2x} - 3(324\lambda^6(61\lambda^2 + 7) - \mu^2) e^{i2x} + (\lambda^2 + 2\lambda^6 + 6\mu(11\lambda^2 + 3)\lambda^3 e^{i2x}))}{\Gamma(2\alpha + 1)} t^{2\alpha} \]
\[ + \frac{\lambda^6 e^{i2\alpha} (2\mu \lambda^5 + \mu \lambda^3 + 3\lambda^2 \mu^2 e^{i2x} + 3\mu^2 e^{i2x} - 2280\mu^3 e^{i2x})}{\Gamma(3\alpha + 1)} t^{3\alpha} \]
\[ + \frac{3\lambda^{10} e^{i2\alpha} (2 \lambda^{15} - 50625\lambda^4 e^{i2x} + 18413 + 15752961\lambda^6 e^{i2x} + 81\lambda^{11} + 256\mu \lambda^{10} e^{i2x})}{\Gamma(4\alpha + 1)} t^{4\alpha} \]
\[ + \frac{3\lambda^{12} e^{i2\alpha} (2 \lambda^9 + 512\mu \lambda^6 e^{i2x} + 256\mu \lambda^6 e^{i2x} - 209952\mu \lambda^6 e^{i3x} + 26873856\mu \lambda^6 e^{i2x} + 432\mu^3 e^{i2x})}{\Gamma(4\alpha + 1)} t^{4\alpha} \]
Thus, Equation (50) gives an approximate solution to problem (48).

**q-HAM solution:** To apply the q-HAM, we rewrite Equation (48) as

\[ D_t^\alpha u - \mu uu_x + 18uu_{xx}^2 + 36u_x^2u_{xxx} + 24uu_xu_{xx} + 3u^2u_{xxx} - u_{xxxx} - u_{xxxxx} = 0, \quad 0 < \alpha \leq 1. \]  

(51)
**FIGURE 15** Case I: Comparison between new iterative method (NIM) and q-homotopy analysis method (q-HAM) solution for $n = 1$ and $h = -1$ [Colour figure can be viewed at wileyonlinelibrary.com]

**FIGURE 16** Case I: Comparison between new iterative method (NIM) and q-homotopy analysis method (q-HAM) solution for $n = 1$ and $h = -1$ [Colour figure can be viewed at wileyonlinelibrary.com]

**FIGURE 17** Case II: Comparison between new iterative method (NIM) and q-homotopy analysis method (q-HAM) solution for $n = 1$ and $h = -1$ [Colour figure can be viewed at wileyonlinelibrary.com]
FIGURE 18  Case II: Comparison between new iterative method (NIM) and q-homotopy analysis method (q-HAM) solution in 2D for $n = \alpha = \mu = 1, h = -1, \text{and } \lambda = 0.1$ [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 19  Case II: Comparison between new iterative method (NIM) and q-homotopy analysis method (q-HAM) solution for $n = 1$ and $h = -1$ [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 20  Case II: Comparison between new iterative method (NIM) and q-homotopy analysis method (q-HAM) solution for $n = 1$ and $h = -1$ [Colour figure can be viewed at wileyonlinelibrary.com]
FIGURE 21  Case II: Comparison between new iterative method (NIM) and q-homotopy analysis method (q-HAM) solution in 2D for \( n = \alpha = 1, \mu = 0.5, h = -1, \) and \( \lambda = 0.01 \)  [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 22  Case II: Comparison between new iterative method (NIM) and q-homotopy analysis method (q-HAM) solution for \( n = 1 \) and \( h = -1 \)  [Colour figure can be viewed at wileyonlinelibrary.com]

Using q-HAM recurrent relation in Equation (27). Then from Equations (26) and (46), we obtain the following:

\[
\begin{align*}
\psi_1(x, t) &= \Psi_1^\alpha u_0(x, t) + hR_1 \left( \bar{u}_0(x, t) \right) \\
&= h^{J_1^\alpha} \left( D_1^\alpha u_0 - \mu u_0 u_0x + 18u_0u_0xxu_0xxx + 36u_0u_0xxu_0xxx \right) \\
&\quad + h^{J_1^\alpha} \left( 24u_0u_0u_0xxx + 3u_0u_0u_0xxxx - u_0xxxx - u_0xxxxx \right) \\
&= \frac{\lambda^3 h e^{3\lambda x} \left( \lambda^3 + 3 \lambda^2 (1 - 32\lambda x) + \mu e^{2\lambda x} \right) t^\alpha}{\Gamma(\alpha + 1)}
\end{align*}
\]

\[
\begin{align*}
\psi_2(x, t) &= \Psi_2^\alpha u_1(x, t) + hR_2 \left( \bar{u}_1(x, t) \right) \\
&= h u_1 + h^{J_1^\alpha} \left( D_1^\alpha u_1 - \mu u_1 u_1x + 18u_0u_0xxu_1xxx + 18u_1u_1xxu_0xxx + 18u_1u_1xxxu_0xx \right) \\
&\quad + h^{J_1^\alpha} \left( 36u_0u_0u_1xxu_1xxx + 18u_0u_0u_1xxxu_0xx + 24u_0u_0u_1xxxu_0xx + 24u_0u_0u_1xxxu_0xx \right) \\
&\quad + h^{J_1^\alpha} \left( 24u_0u_0u_1xxx + 3u_0u_0u_1xxxx + 3u_0u_0u_1xxxxx + 3u_1u_1u_0xxxx - u_1xxxx - u_1xxxxx \right) \\
&= (n + h) u_1 + \frac{\lambda^2 h^2 e^{2\lambda x} \left( 151875 \lambda^5 e^{4\lambda x} - 1092 \mu \lambda^3 e^{3\lambda x} - 3(324 \lambda^6 (61 \lambda^2 + 7) - \mu^2) e^{2\lambda x} \right) t^{2\alpha}}{\Gamma(2\alpha + 1)} \\
&\quad + \frac{\lambda^3 h^2 e^{3\lambda x} \left( (\lambda^2 + 1)^2 \lambda^6 + 6\mu (11 \lambda^2 + 3) \lambda^3 e^{2\lambda x} \right) t^{2\alpha}}{\Gamma(2\alpha + 1)}.
\end{align*}
\]
Following the same procedure, expression for \( u_m(x, t) \), \( m = 4, 5, 6, \ldots \) can be obtained. The expression of the series solution given by q-HAM can be written in the form

\[
u(x, t; n; h) \approx U_2 = u_0(x, t) + \sum_{i=1}^{2} u_i(x, t; n; h) \left( \frac{1}{n} \right)^i
\]

\[
= u_0(t, x) + \frac{u_1(t; x; n; h)}{n} + \frac{u_2(t; x; n; h)}{n^2}
\]

\[
= e^{ix} - \frac{\lambda h (2n + h) e^{i\lambda x}}{n^2(\alpha + 1)} e^{i\mu x} \left( \frac{\lambda^3 (1 - 81e^{2i\lambda x}) + \mu e^{i\mu x}}{n^2(\alpha + 1)} - \frac{2(324\lambda^6(61\lambda^2 + 7) - \mu^2)e^{i\lambda x}}{n^2(\alpha + 1)} \right) + \frac{\lambda^2 h^2 (3n + 2h) e^{i\lambda x}}{n^2(2\alpha + 1)} \left( 151875\lambda^6 e^{4i\lambda x} - 1092\mu \lambda^3 e^{3i\lambda x} - 3(324\lambda^6(61\lambda^2 + 7) - \mu^2)e^{2i\lambda x} \right) t^{2\alpha}
\]

Thus, Equation (52) gives an approximate solution to problem (48) in terms of convergence parameters \( h \) and \( n \).

4.3 | Numerical results for TFCH equation of sixth order

Here, we present the numerical simulation of cases I and II of Section 4 to demonstrate the effectiveness of the two iterative methods used for solving TFCH Equation (2) subject to different initial conditions. Figures 12 to 22 show that the \( U_2 \)-solutions obtained by these two iterative methods are graphically and numerically indistinguishable.

5 | CONCLUDING REMARKS

In conclusion, we have studied iterative methods for constructing approximate solutions to the time-fractional nonlinear Cahn-Hilliard equations of fourth and sixth order using different initial conditions. We used NIM and q-HAM to obtain approximate series solution and present the graphical representation of the obtained results for different fractional order. We observed that the fraction factor \( \frac{1}{n} \) and the parameter \( h \) highly increase the convergence of the chances of the q-HAM. As shown in our examples, the two iterative methods do not require any transformations or perturbations. Therefore, these methods are considerably efficient, powerful, and easy to implement when compared with other numerical methods for constructing approximated solutions to the linear and nonlinear fractional differential equations. Our aim in this paper is not to conclude that one method is better than the other, but rather conclude that both methods provide a good approximate solution even in some cases, we can obtain the exact solutions.

This work does not have any conflicts of interest.

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