GENUS ONE STABLE QUASIMAP INVARIANTS FOR PROJECTIVE COMPLETE INTERSECTIONS

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ABSTRACT. By using the infinitesimally marking point to break the loop in the localization calculation as Kim and Lho, and Zinger’s explicit formulas for double $J$-functions, we obtain a formula for genus one stable quasimaps invariants when the target is a complete intersection Calabi-Yau in projective space, which gives a new proof of Kim and Lho’s mirror theorem for elliptic quasimap invariants.

1. Introduction

The moduli space of stable quotients was first constructed and studied by Marian, Oprea and Pandharipande [8]. Cooper and Zinger [5] calculated the $J$-function of stable quotients for projective complete intersections, and proved that it is related to the genus zero Gromov-Witten invariants by mirror map. Using the moduli space of stable quasimaps, which is a generalization of stable quotient by Ciocan-Fontanine, Kim and Maulik [4], Ciocan-Fontanine, Kim proved that the genus zero stable quasimap invariants (including twisted cases) are related to stable map by mirror map in [1]. Later Kim and Lho [6] obtained the formula for genus one stable quasimap invariants without markings for projective complete intersections. Combining with the result in [3] it recovered the results of Popa [9] and Zinger [10] on genus one Gromov Witten invariants of projective complete intersections by mirror map. In this paper using the double $J$-functions of complete intersections in projective space proved by Zinger [11], we give another proof of that formula.

Let $\ell$ be a nonnegative integer. $E := \bigoplus_{i=1}^{\ell} L_i$ be the direct sum of line bundles over $\mathbb{P}^{n-1}$ with degree $a_i = \deg L_i > 0$.

If $|a| := \sum_{k=1}^{\ell} a_k = n$, a transversal section $s$ of $E$ gives a Calabi-Yau manifold $X$, which is a complete intersection in $\mathbb{P}^{n-1}$.

Let $\overline{Q}_{1,0}(X,d)$ be the moduli space of genus one and degree $d$ stable quasimaps to $X$. It is a proper Deligne-Mumford stack and has perfect obstruction theory. Thus it has virtual cycle $[\overline{Q}_{1,0}(X,d)]^{\vir}$ with zero virtual dimension. Let

$$G_{1,0} := \sum_{d=1}^{\infty} q^d \deg [\overline{Q}_{1,0}(X,d)]^{\vir}.$$
Let $\overline{Q}_{1,0}(\mathbb{P}^{n-1}, d)$ be the moduli space of genus one and degree $d$ stable quasimaps to $\mathbb{P}^{n-1}$. It is a smooth Deligne-Mumford stack. Let

$$\tilde{\pi} : \tilde{C} \to \overline{Q}_{1,0}(\mathbb{P}^{n-1}, d)$$

be the universal family, $\tilde{S}$ be the universal bundle over $\tilde{C}$. Let $\tilde{\iota}$ be the closed immersion $\tilde{\iota} : \overline{Q}_{1,0}(X, d) \to \overline{Q}_{1,0}(\mathbb{P}^{n-1}, d)$. By the functoriality in \cite{7} we have

$$\tilde{\iota}_{*}[\overline{Q}_{1,0}(X, d)]^{\text{vir}} = e(\tilde{\mathcal{V}}_{1}) \cap [\overline{Q}_{1,0}(\mathbb{P}^{n-1}, d)],$$

where $\tilde{\mathcal{V}}_{1} := \bigoplus_{i=1}^{\ell} \tilde{\pi}_{*}\pi^{*}\tilde{S}^\vee \otimes a_{i}$ is locally free. Thus

$$(1.1) \quad \text{deg}[\overline{Q}_{1,0}(X, d)]^{\text{vir}} = \text{deg}(e(\tilde{\mathcal{V}}_{1}) \cap [\overline{Q}_{1,0}(\mathbb{P}^{n-1}, d)]).$$

The standard torus $T = (\mathbb{C}^*)^{n}$ action on $\mathbb{P}^{n-1}$ induces an action on the moduli space $\overline{Q}_{1,0}(\mathbb{P}^{n-1}, d)$. There are two different types of the fixed loci when applying localization method to calculate the degree on the right hand side of (1.1). One is the loop type and the other is the vertex type. To calculate the degree of the loop type, we use an infinitesimally marking to break the loop as Kim and Lho \cite{6}, see Section 2.2. But in this paper we work out the loop contribution by the double $J$-functions given by Zinger \cite{11}, which are directly related to the hypergeometric series used in calculation.

In the forthcoming papers, we will extend the calculation to the genus one stable quasimap invariants with markings, and genus two stable quasimap invariants for projective complete intersections.

Let $A_{i}(q)$ be as in Proposition \ref{2.3}, $F^{(0,0)}(\alpha_{i}, q)$ and $\Phi^{(0)}(\alpha_{i}, q)$ be as in Corollary \ref{2.2}. By the localization method, we prove the following result.

**Theorem 1.1.** For projective complete intersection Calabi-Yau $X$,

$$q \frac{d}{dq} G_{1,0} = \left\{ \sum_{i \in \mathbb{N}} \left( A_{i}(q) + \frac{1}{24} q \frac{d}{dq} \left( c_{i}(\alpha) F^{(0,0)}(\alpha_{i}, q) - \log \Phi^{(0)}(\alpha_{i}, q) \right) \right) \right\} \bigg|_{q = 0},$$

where $c_{i}(\alpha) = \sum_{k \neq i} \frac{1}{\alpha_{i} - \alpha_{k}} + \sum_{k=1}^{\ell} a_{k} \alpha_{i}$.

Let $L(q) = (1 - a^{q}q)^{-\frac{1}{q}}$ and $\mu(q) = \int_{0}^{q} \frac{L(u)-1}{u} du$, where $a^{\mathbb{Z}} = \prod_{k=1}^{\ell} a_{k}^{\mathbb{Z}}$. For $p \in \mathbb{Z}^\geq 0$, let $\tilde{I}_{p}(q) \in \mathbb{Q}[[q]]$ be defined as in Section 3. We can recover the following result of Kim and Lho from Theorem 1.1.

**Theorem 1.2.** \cite[Theorem 1.1]{6} For projective complete intersection Calabi-Yau $X$,

$$G_{1,0} = \frac{1}{2} A(q) + \frac{1}{24} \left( \sum_{i=1}^{\ell} \frac{n}{a_{i}} - \binom{n}{2} \right) \mu(q) - \frac{n(\ell+1)}{2} \log L(q),$$

$$G_{1,0} = \frac{1}{2} A(q) + \frac{1}{24} \left( \sum_{i=1}^{\ell} \frac{n}{a_{i}} - \binom{n}{2} \right) \mu(q) - \frac{n(\ell+1)}{2} \log L(q).$$
where
\[
A(q) = \frac{n}{24} (n - 1 - 2 \sum_{r=1}^{\ell} \frac{1}{a_r}) \mu(q) - \frac{3(n - 1 - \ell)^2 + (n - 2)}{24} \log(1 - a^q q) \\
- \sum_{p=0}^{n-2-\ell} \left( \frac{n - p - \ell}{2} \right) \log \hat{I}_p(q).
\]

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2. Localization

\(\mathbb{P}^{n-1}\) has a torus action induced by the standard torus \(T = (\mathbb{C}^*)^n\) action on \(\mathbb{C}^n\). The \(T\) fixed points of \(\mathbb{P}^{n-1}\) are the 1-planes spanned by \(e_i\), where \(\{e_i\}_{i=1}^n\) is the standard basis of \(\mathbb{C}^n\). We label it by \(p_i\). Let \(n\) be the set \(1 \leq i \leq n\).

Denote by \(H^*(BT, \mathbb{Q}) := \mathbb{Q}[\alpha_1, \ldots, \alpha_n]\), where \(\alpha_i = \pi_i^* c_1(\gamma)\). Here \(\pi_i : (\mathbb{P}^\infty)^n \to \mathbb{P}^\infty\) and \(\gamma^\vee \to \mathbb{P}^\infty\) are the projection onto the \(i\)-th component and the tautological line bundle respectively. Denote by \(\alpha = (\alpha_1, \ldots, \alpha_n)\). Let \(\mathbb{Q}_\alpha\) be the fractional field of \(\mathbb{Q}[\alpha_1, \ldots, \alpha_n]\).

Let \(x\) be the equivariant Chern class of the dual universal bundle. Then
\[
H^*_T(\mathbb{P}^{n-1}) \cong \mathbb{Q}[\alpha_1, \ldots, \alpha_n][x] / < \prod_{i=1}^n (x - \alpha_i) >.
\]

Let
\[
\phi_i = \prod_{k \neq i} (x - \alpha_k).
\]

Then \(\phi_i\) is the equivariant Poincaré dual of the points \(p_i \in H^*_T(\mathbb{P}^{n-1})\). The Euler class
\[
e(T_{\mathbb{P}^{n-1}})|_{p_i} = \prod_{k \neq i} (\alpha_i - \alpha_k) = \phi_i|_{p_i}
\]

The Artiyah-Bott localization theorem states that
\[
\int_{\mathbb{P}^{n-1}} \eta = \sum_{p_i} \int_{p_i} \frac{\eta|_{p_i}}{e(N_{p_i/\mathbb{P}^{n-1}})}, \text{ for all } \eta \in H^*_T(\mathbb{P}^{n-1}).
\]

Therefore
\[
\eta|_{p_i} = \int_{\mathbb{P}^{n-1}} \eta \phi_i.
\]

So for \(\eta \in H^*_T(\mathbb{P}^{n-1})\)
\[
\eta|_{p_i} = 0 \text{ for all } i \in n \iff \eta = 0.
\]
Denote by
\[ \overline{\mathcal{Q}}_{g,k|m}(\mathbb{P}^{n-1}, d) \]
the moduli space of genus \( g \) degree \( d \) quasimaps to \( \mathbb{P}^{n-1} \) with ordinary \( k \) pointed markings and infinitesimally weighted \( m \) pointed markings (see [2, Section 2]). When \( g = 1 \) and \( k = 0, m = 1 \), the genus one moduli space \( \overline{\mathcal{Q}}_{1,0|1}(\mathbb{P}^{n-1}, d) \) is a smooth Deligne-Mumford stack since the obstruction vanishes. Let
\[ \pi : \mathcal{C} \to \overline{\mathcal{Q}}_{1,0|1}(\mathbb{P}^{n-1}, d) \]
be the universal family, \( \mathcal{S} \) be the universal bundle over \( \mathcal{C} \). Let \( \mathcal{S}^\vee \) be the dual of \( \mathcal{S} \).

Let \( \iota \) be the closed immersion \( \iota : \overline{\mathcal{Q}}_{1,0|1}(X, d) \to \overline{\mathcal{Q}}_{1,0|1}(\mathbb{P}^{n-1}, d) \), by the functoriality in [7] we have
\[ \iota_*(\overline{\mathcal{Q}}_{1,0|1}(X, d)) \] \[ \equiv \mathcal{E}(\mathcal{V} \cap \overline{\mathcal{Q}}_{1,0|1}(\mathbb{P}^{n-1}, d)), \]
where \( \mathcal{V}_1 := \oplus_{i=1}^\ell \pi_* \mathcal{S}^\vee \otimes \mathcal{E}_i \) is locally free.

2.1. Localization. First we recall some facts about residues, which are from [10, Section 1.2].

If \( f \) is a rational function in \( z \) and possibly other variables and \( z_0 \in \mathbb{S}^2 \), let \( \mathcal{R}_{z=z_0} f(z) \) denote the residue of the one-form \( f(z)dz \) at \( z = z_0 \); thus,
\[ \mathcal{R}_{z=\infty} f(z) \equiv -\mathcal{R}_{w=0} \{ w^{-2} f(w^{-1}) \}. \]

If \( f \) involves variables other than \( z \), \( \mathcal{R}_{z=z_0} f(z) \) is a function of the other variables. If \( f \) is a power series in \( q \) with coefficients that are rational functions in \( z \) and possibly other variables, let \( \mathcal{R}_{z=z_0} f(z) \) denote the power series in \( q \) obtained by replacing each of the coefficients by its residue at \( z = z_0 \). If \( z_1, \ldots, z_k \) is a collection of points in \( \mathbb{S}^2 \), not necessarily distinct, we define
\[ \mathcal{R}_{z=z_1,...,z_k} f(z) \equiv \sum_{y \in \{z_1, \ldots, z_k\}} \mathcal{R}_{z=y} f(z). \]

Let \( \tilde{\pi} : \tilde{\mathcal{C}} \to \overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1}, d) \) be the universal family, \( \tilde{\mathcal{S}} \) be the universal bundle over \( \tilde{\mathcal{C}} \). Let
\[ \gamma^{(d)}_{n:a} := \bigoplus_k R^0 \tilde{\pi}_*(\tilde{\mathcal{S}}^\vee \otimes \mathcal{E}_k) \to \overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1}, d), \]
(2.4)
\[ \hat{\gamma}^{(d)}_{n:a} := \bigoplus_k R^0 \tilde{\pi}_*(\tilde{\mathcal{S}}^\vee \otimes \mathcal{E}_k(-\sigma_1)) \to \overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1}, d), \]
(2.5)
and
\[ \hat{\gamma}^{(d)}_{n:a} := \bigoplus_k R^0 \tilde{\pi}_*(\tilde{\mathcal{S}}^\vee \otimes \mathcal{E}_k(-\sigma_2)) \to \overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1}, d). \]
(2.6)
where \( \sigma_i \) is the section correspondent to the marking. Thus \( \gamma^{(d)}_{n:a}, \hat{\gamma}^{(d)}_{n:a} \) and \( \hat{\gamma}^{(d)}_{n:a} \) are \( \mathbb{T} \)-equivariant vector bundles. Let
\[ \hat{\mathcal{Z}}(x, h, q) := 1 + \sum_{d=1}^\infty q^d d \text{ev}_1 \left[ \frac{\mathcal{E}(\mathcal{V}^{(d)}_{n:a})}{h - \psi_1} \right] \in H^*_\mathbb{T}(\mathbb{P}^{n-1}[h^{-1}][[q]]), \]
(2.7)
where \( \mathcal{E} \) is an \( \mathbb{E} \)-module.
(2.8) \[ \tilde{Z}(x, h, q) := 1 + \sum_{d=1}^{\infty} q^d e^{\psi_1} \left[ \frac{e(\tilde{V}(d))}{h - \psi_1} \right] \in H^*_T(\mathbb{P}^{n-1})[h^{-1}][[q]]. \]

For \( s \in \mathbb{Z}^0 \), Let

(2.9) \[ \dot{Z}^{(s)}(x, h, q) := x^s + \sum_{d=1}^{\infty} q^d e^{\psi_1} \left[ \frac{e(\tilde{V}(d)) e^{\psi_2} x^s}{h - \psi_1} \right] \in H^*_T(\mathbb{P}^{n-1})[h^{-1}][[q]], \]

(2.10) \[ \ddot{Z}^{(s)}(x, h, q) := x^s + \sum_{d=1}^{\infty} q^d e^{\psi_1} \left[ \frac{e(\tilde{V}(d)) e^{\psi_2} x^s}{h - \psi_1} \right] \in H^*_T(\mathbb{P}^{n-1})[h^{-1}][[q]]. \]

Let

(2.11) \[ \dot{Z}(h_1, h_2, q) := \frac{[\Delta]}{h_1 h_2} + \sum_{d=1}^{\infty} q^d \left\{ e^{\psi_1} e^{\psi_2} \right\} \left[ \frac{e(\tilde{V}(d))}{(h_1 - \psi_1)(h_2 - \psi_2)} \right] \in H^*_T(\mathbb{P}^{n-1})[h_1^{-1}, h_2^{-1}][[q]], \]

where \([\Delta]\) is the equivariant Poincaré dual of the diagonal class. Denote by

(2.12) \[ \tilde{Z}^*_i(h_1, h_2, q) = \frac{1}{2h_1 h_2} \left( \dot{Z}(h_1, h_2, q) - \frac{[\Delta]}{h_1 h_2} \right)_{p_i \times p_j}. \]

Let

\[ \mathcal{Y}(x, h, q) = \sum_{d \geq 0} q^d \mathcal{Y}_d(h), \]
\[ \dot{\mathcal{Y}}(x, h, q) = \sum_{d \geq 0} q^d \dot{\mathcal{Y}}_d(h), \]
\[ \ddot{\mathcal{Y}}(x, h, q) = \sum_{d \geq 0} q^d \ddot{\mathcal{Y}}_d(h), \]

where

\[ \mathcal{Y}_d(h) = \sum_{\ell} \prod_{k=1}^{\ell} \prod_{l=1}^{a_{k\ell}} \frac{a_k x + l h}{\prod_{k=1}^{n} \prod_{k=1}^{a_{k\ell}} (x - \alpha_k + l h)}, \]

\[ \dot{\mathcal{Y}}_d(h) = \sum_{d} \prod_{k=1}^{\ell} \prod_{l=1}^{a_{k\ell}} \frac{a_k x + l h}{\prod_{k=1}^{n} \prod_{k=1}^{a_{k\ell}} (x - \alpha_k + l h) - \prod_{k=1}^{n} (x - \alpha_k)}, \]

\[ \ddot{\mathcal{Y}}_d(h) = \sum_{\ell} \prod_{k=1}^{\ell} \prod_{l=1}^{a_{k\ell}} \frac{a_k x + l h}{\prod_{k=1}^{n} \prod_{k=1}^{a_{k\ell}} (x - \alpha_k + l h) - \prod_{k=1}^{n} (x - \alpha_k)}. \]
Denote $\tilde{I}_0(q) = \hat{Y}(x, h, q)|_{x=\alpha=0}$ and $\bar{I}_0(q) = \hat{Y}(x, h, q)|_{x=\alpha=0}$.

**Theorem 2.1.** ([11, Theorem 3]) If $\ell \in \mathbb{Z}^\geq 0$, $n \in \mathbb{Z}^+$, and $a \in (\mathbb{Z}^>)^\ell$ are such that $|a| := \sum_{i=1}^\ell a_i = n$, then

$$\tilde{Z}(x, h, q) = \frac{\hat{Y}(x, h, q)}{\tilde{I}_0(q)} \in H^*_T(\mathbb{P}^{n-1})[[h^{-1}, q]],$$

and

$$\bar{Z}(x, h, q) = \frac{\hat{Y}(x, h, q)}{\bar{I}_0(q)} \in H^*_T(\mathbb{P}^{n-1})[[h^{-1}, q]].$$

### 2.2. Insertion of 0+ weighted marking

Let $\hat{ev}_1: Q_{1,0}1(\mathbb{P}^{n-1}, d) \to [\mathbb{C}^n/\mathbb{C}^*]$ be the evaluations map at the infinitesimally marking, see [2, Section 2.3]. Let

$$\tilde{\gamma} \in H^2_T([\mathbb{C}^n/\mathbb{C}^*])$$

be the lift of $\gamma \in H^2_T(\mathbb{P}^{n-1})$, where $\gamma$ is the hyperplane class.

Let

$$\langle \tilde{\gamma} \rangle_{1,0,1,d} := \int_{Q_{1,0,1}(\mathbb{P}^{n-1}, d)} e(\mathcal{V}_1)\hat{ev}_1^*\tilde{\gamma},$$

then

$$\sum_{d=1}^{\infty} q^d \langle \tilde{\gamma} \rangle_{1,0,1,d} = q \frac{d}{dq} G_{1,0}.$$  

In the rest of this section we use localization method to calculate the left hand side of the above equation. As described in [8, Section 7], the fixed loci of the $T$-action on $Q_{1,0,1}(\mathbb{P}^{n-1}, d)$ are indexed by connected decorated graphs. Such a graph can be described by set $(\text{Ver}, \text{Edg})$ of vertices, A decorated graph is a tuple

$$\Gamma = (\text{Ver}, \text{Edg}; \mu, \varrho, \eta),$$

where $(\text{Ver}, \text{Edg})$ is a graph as above and

$$\mu: \text{Ver} \to \mathbb{n}, \quad \varrho: \text{Ver} \sqcup \text{Edg} \to \mathbb{Z}^\geq 0 \quad \text{and} \quad \eta: [1] \to \text{Ver}$$

are maps such that

$$\mu(v_1) \neq \mu(v_2) \text{ if } \{v_1, v_2\} \in \text{Edg}, \quad \varrho(e) \neq 0 \forall e \in \text{Edg}.$$  

Let

$$|\Gamma| \equiv \sum_{v \in \text{Ver}} \varrho(v) + \sum_{e \in \text{Edg}} \varrho(e)$$

be the degree of $\Gamma$. Denoted by

$$\text{val}(v) \equiv (|\{e \in \text{Edg}: v \in e\}|, |\{i \in [1]: \eta(1) = v\}|).$$

for the vertices $v \in \text{Ver}.$
Let \( Z_\Gamma \) be the fixed locus of \( \overline{\mathcal{Q}}_{1,0,1}(\mathbb{P}^{n-1}, d) \) corresponds to a decorated graph \( \Gamma \). Thus
\[
Z_\Gamma \cong \prod_{v \in \text{Ver}} \mathcal{M}_{g_v, \text{val}(v)}(\mathcal{B}(v))
\]
up to a finite group quotient, where \( \mathcal{M}_{g_v,(k,m)}(\mathcal{B}(v)) \) denotes the moduli space weighted pointed stable curves with \( k \) ordinary markings and \( m \) infinitesimally markings. When \( m = 0 \), we abbreviated by \( \mathcal{M}_{g_v,k}(\mathcal{B}(v)) \). Let
\[
\pi_1 : \mathcal{C}_{\mathcal{M}_{g_v,\text{val}(v)}(\mathcal{B}(v))} \rightarrow \mathcal{M}_{g_v,\text{val}(v)}(\mathcal{B}(v))
\]
be the restriction of the universal family.

With \( b_1, b_2, r \in \mathbb{Z}^\geq 0 \), let
\[
\mathcal{F}^{(b_1,b_2)}_n(\alpha_i, q) = \sum_{d=1}^{\infty} \frac{q^d}{d!} \int_{\mathcal{M}_{0,2d}} \mathcal{G}^{(b_1,b_2)}_{n,d}(\alpha_i, q),
\]
where
\[
\mathcal{G}^{(b_1,b_2)}_{n,d}(\alpha_i, q) := \frac{\prod_{k \neq i} (\alpha_i - \alpha_k) e(\gamma_{n,d}(\alpha_i)) \psi_{b_1} \psi_{b_2}}{\prod_{k \neq i} e(R^0_{\pi_1}\mathcal{S}^\mathcal{V}(\alpha_i - \alpha_k))}.
\]

By the proof of [5, Theorem 4], we have
\[
q \frac{d}{dq} \mathcal{F}^{(b_1,b_2)}_n(\alpha_i, q) = \frac{1}{b_1!} \mathcal{F}^{(0,0)}_n(\alpha_i, q) b_1 \frac{1}{b_2!} \mathcal{F}^{(0,0)}_n(\alpha_i, q) b_2 q \frac{d}{dq} \mathcal{F}^{(0,0)}_n(\alpha_i, q).
\]

By inductions on \( b_2 \), this gives
\[
\mathcal{F}^{(0,b_2)}_n(\alpha_i, q) = \frac{1}{(b_2 + 1)!} \mathcal{F}^{(0,0)}_n(\alpha_i, q)^{b_2+1}.
\]

Thus the \( r = 0 \) case of [5, Proposition 8.3] is equivariant to
\[
\mathcal{R}_{\hbar=0} \{ e^{-\frac{\mathcal{F}^{(0,0)}_n(\alpha_i, \hbar)}{\hbar}} \mathcal{Y}(\alpha_i, \hbar, q) \} = 0.
\]

Thus there is an expansion

**Corollary 2.2.** [11, (4-9)]
\[
e^{-\frac{\mathcal{F}^{(0,0)}_n(\alpha_i, \hbar)}{\hbar}} \mathcal{Y}(\alpha_i, \hbar, q) = \sum_{m=0}^{\infty} \Phi^{(m)}(\alpha_i, q) \hbar^m.
\]

In the case of moduli space \( \overline{\mathcal{Q}}_{1,0,1}(\mathbb{P}^{n-1}, d) \), they are two types of graph, either one distinguished vertex or one loop, depending on whether the stable quismaps they describe are constant or not. The graphs with one loop are called \( A_i \)-graphs. In a graph of the \( A_i \)-type, the marked point 1 is attached to some vertex \( v_0 \in \text{Ver} \) that lies inside of the loop and is labeled \( i \).

Denote \( \mathcal{A}_i(q) \) by the total contribution from type \( A_i \) graphs, then
Proposition 2.3. For every \( i \in \mathbb{n} \),
\[
\mathcal{A}_i(q) = (\phi_i|_{p_i})^{-1}\left( q \frac{d}{dq} F_{n}(0,0)(\alpha_i, q) + \alpha_i \right)
\]
\[
\mathfrak{F}_{h_1=0} \{ \mathfrak{F}_{h_2=0} \left\{ e^{-F_n^{(0,0)}(\alpha_i, q)} \left( \frac{1}{h_1} + \frac{1}{h_2} \right) \tilde{Z}_{i*}(h_1, h_2, q) \right\} \}.
\]

Proof. Let \( \Gamma \) be a decorated graph of \( A_i \) type, then we can break \( \Gamma \) into a strand \( \Gamma_{\pm} \) at \( v_0 = \eta(1) \), where \( \pm \) are points attached to \( v_0 \). Let \( \mu(v_0) = i \). Thus \( \mathcal{Z}_\Gamma = \bar{\mathcal{M}}_{0,(2,1)|\mathcal{N}(v_0)} \times \mathcal{Z}_{\Gamma_{\pm}} \), let
\[
\pi_p : \mathcal{Z}_\Gamma \to \bar{\mathcal{M}}_{0,(2,1)|\mathcal{N}(v_0)}, \quad \text{and} \quad \pi_e : \mathcal{Z}_\Gamma \to \mathcal{Z}_{\Gamma_{\pm}}
\]
be the projections, then
\[
e(\mathcal{V}_i) = \frac{\pi^*_e e(\mathcal{V}_{n,a}^{(d_0(v_0)))})}{\pi^*_p e(\mathcal{V}_{n,a}^{(d_0(v_0)))}} \frac{\prod_{k \neq i}(\alpha_i - \alpha_k)}{\prod_{k \neq i}(R^0 \pi_{1*} \mathcal{S}^v(\alpha_i - \alpha_k))}
\]
\[
e(T_{\mu(v_0)} \mathcal{N} \mathcal{Z}_\Gamma) = \frac{\pi^*_e e(\mathcal{V}_{n,a}^{(d_0(v_0)))})}{\pi^*_p e(\mathcal{V}_{n,a}^{(d_0(v_0)))}} \frac{\prod_{k \neq i}(\alpha_i - \alpha_k)}{\prod_{k \neq i}(R^0 \pi_{1*} \mathcal{S}^v(\alpha_i - \alpha_k))}
\]
where \( h_{\pm} = c_1(L_{\pm}) \in H^*(\bar{\mathcal{M}}_{0,(2,1)|\mathcal{N}(v_0)}), L_{\pm} \) be the universal tangent line bundle at the marked point corresponding to \( \pm \).

Let \( e_1 \) be the line \( \{v_1, v_0\} \), where \( v_1 \) is the nearest point to \( + \). Let \( e_2 \) be the line \( \{v_2, v_0\} \), where \( v_2 \) is the nearest point to \( - \). Let \( \Gamma_e \) be the strand obtained by \( \Gamma_{\pm} \) eliminating \( e_1, e_2 \). Therefore we have
\[
\sum_{d=1}^{\infty} q^d \int_{\mathcal{Z}_\Gamma} e(\mathcal{V}_i) \hat{e}^* \gamma \frac{e(\mathcal{V}_i) \hat{e}^* \gamma}{e(\mathcal{N} \mathcal{Z}_\Gamma)}
\]
\[
= \sum_{d=1}^{\infty} \sum_{d_0(v_0) \geq 0} q^{d_0(v_0)} \left\langle \hat{e}^{d_0(v_0)}(\mathcal{V}_{n,a}) \frac{\hat{e}^{d_0(v_0)}(\mathcal{V}_{n,a})}{\mathcal{N} \mathcal{Z}_{\Gamma_{\pm}}} \left( h_+ - \pi_e \hat{\psi}_+ \right) \left( h_- - \pi_e \hat{\psi}_- \right) \rightangle
\]
\[
= \sum_{d=1}^{\infty} \sum_{d_0(v_0) \geq 0} q^{d_0(v_0)} \left\langle \hat{e}^{d_0(v_0)}(\mathcal{V}_{n,a}) \frac{\hat{e}^{d_0(v_0)}(\mathcal{V}_{n,a})}{\mathcal{N} \mathcal{Z}_{\Gamma_{\pm}}} \left( h_+ - \pi_e \hat{\psi}_+ \right) \left( h_- - \pi_e \hat{\psi}_- \right) \rightangle
\]
\[
= \sum_{d=1}^{\infty} \sum_{d_0(v_0) \geq 0} q^{d_0(v_0)} \left\langle \hat{e}^{d_0(v_0)}(\mathcal{V}_{n,a}) \frac{\hat{e}^{d_0(v_0)}(\mathcal{V}_{n,a})}{\mathcal{N} \mathcal{Z}_{\Gamma_{\pm}}} \left( h_+ - \pi_e \hat{\psi}_+ \right) \left( h_- - \pi_e \hat{\psi}_- \right) \rightangle
\]
\[
= \sum_{d=1}^{\infty} \sum_{d_0(v_0) \geq 0} q^{d_0(v_0)} \left\langle \hat{e}^{d_0(v_0)}(\mathcal{V}_{n,a}) \frac{\hat{e}^{d_0(v_0)}(\mathcal{V}_{n,a})}{\mathcal{N} \mathcal{Z}_{\Gamma_{\pm}}} \left( h_+ - \pi_e \hat{\psi}_+ \right) \left( h_- - \pi_e \hat{\psi}_- \right) \rightangle
\]
\[
\hat{e}_i^{\mu(v_1)}(\mathcal{V}_{n,a}) \left( \begin{array}{c} -\frac{\alpha_{\mu(v_1)} - \alpha_i}{\hat{d}(e_1)} \end{array} \right)^{-b_1+1} \hat{e}_i^{\mu(v_2)}(\mathcal{V}_{n,a}) \left( \begin{array}{c} -\frac{\alpha_{\mu(v_2)} - \alpha_i}{\hat{d}(e_2)} \end{array} \right)^{-b_2+1}
\]
\[
\int_{\mathcal{Z}_{\Gamma_e}} e(\mathcal{V}_{n,a}) \left( h_1 - \hat{\psi}_1 \right) \left( h_2 - \hat{\psi}_2 \right) \left| \begin{array}{c} h_1 = \frac{\alpha_{\mu(v_1)} - \alpha_i}{\hat{d}(e_1)} - h_2 = \frac{\alpha_{\mu(v_2)} - \alpha_i}{\hat{d}(e_2)} \end{array} \right|.
\]
where
\[ \hat{\xi}_i(\nu_1)(d) = \frac{\prod_{r=1}^{\ell} \prod_{a=1}^{d}(a_r \alpha_i + \frac{d}{\ell}(\alpha_{\mu(\nu_1)} - \alpha_i))}{d \prod_{(l,m) \neq (d,k)}^{m=1} \Gamma \alpha_m + \frac{d}{\ell}(\alpha_{\mu(\nu_1)} - \alpha_i)}, \]
\[ \hat{\xi}_i(\nu_2)(d) = \frac{\prod_{r=1}^{\ell} \prod_{a=0}^{d-1}(a_r \alpha_i + \frac{d}{\ell}(\alpha_{\mu(\nu_2)} - \alpha_i))}{d \prod_{(l,m) \neq (d,k)}^{m=1} \Gamma \alpha_m + \frac{d}{\ell}(\alpha_{\mu(\nu_2)} - \alpha_i)}. \]

Denote by
\[ \mathcal{F}(b_1, b_2) = \sum_{\nu(v_0) = 0}^{\infty} \frac{\hat{\xi}_i(\nu_0)}{\nu(v_0)!} \int_{\mathcal{M}_{0,((2,1)\nu(v_0))}} \phi_{\nu_0} \tilde{e} \psi \hat{\xi}_i^n \psi_n \prod_{k \neq i}^{2} \psi_{\nu_0}^2 \left( R_{\nu_0} \pi_{1*} S^\nu(\alpha_i - \alpha_k) \right). \]

Let \(D_{1,2} \subset \mathcal{M}_{0,((2,1)\nu(v_0))}\) be the divisor whose general element is a two-component rational curve, with one component carrying the marked point 1 and \(\hat{1}\) and the other carrying the marked point 2, where \(\hat{1}\) means the infinitesimally marking. Then \(\psi_2 = D_{1,2}\) on \(\mathcal{M}_{0,((2,1)\nu(v_0))}\), and
\[ \mathcal{F}(b_1, b_2) = \mathcal{F}(b_1, b_1) \mathcal{F}_n^{(0,b_2-1)}(\alpha_i, q) = \mathcal{F}(0, 0) \mathcal{F}_n^{(0,b_2-1)}(\alpha_i, q) \mathcal{F}_n^{(0,b_2-1)}(\alpha_i, q). \]

Because \(\mathcal{M}_{0,((2,1)\nu(v_0))}\) canonical isomorphic to the universal curve
\[ \mathcal{C}_{\mathcal{M}_{0,2}\nu(v_0)} \rightarrow \mathcal{M}_{0,2}\nu(v_0), \]
we have
\[ \mathcal{F}(0, 0) = \frac{d}{dq} \mathcal{F}_n(0, 0)(\alpha_i, q) + \alpha_i. \]

By the recursion formula \([\text{[11], (7-12)]}\) and the formulas in page 484 of \([\text{[11]}\), Section 7]
\[ \frac{1}{2} \sum_{\Gamma_{\pm}} q^{d-\nu(v_0)} \hat{\xi}_i(\nu_1)(\nu(e_1)) \left( -\frac{\alpha_{\mu(\nu_1)} - \alpha_i}{\nu(e_1)} \right)^{(b_1+1)} \hat{\xi}_i(\nu_2)(\nu(e_2)) \left( -\frac{\alpha_{\mu(\nu_2)} - \alpha_i}{\nu(e_2)} \right)^{(b_2+1)} \]
\[ \int_{\mathcal{Z}_{\nu_1}} \frac{e^{(\nu(e_1)) \nu_1 \nu_2 \nu_3 \nu_4 \nu_5 \nu_6 \nu_7 \nu_8 \nu_9 \nu_{10}}}{e(NZ\Gamma_{\pm})(h_1 - \psi_1)(h_2 - \psi_2)} \bigg|_{h_1 = \frac{\alpha_{\mu(\nu_1)} - \alpha_i}{\nu(e_1)}, h_2 = \frac{\alpha_{\mu(\nu_2)} - \alpha_i}{\nu(e_2)}} \]
\[ = \mathcal{R}_{h_1 = 0} \mathcal{R}_{h_2 = 0} \mathcal{R}_{h_2 = -b_2} \bigg( (h_1)^{-b_1} (h_2)^{-b_2} \mathcal{Z}_{\nu_1}(h_1, h_2, q) \bigg). \]

Then by the residue theorem on \(S^2\), and the vanishing of the residue at \(\alpha\), which can be directly obtained from the expression of \(\mathcal{Z}_{\nu_1}(h_1, h_2, q)\). We have
\[ \mathcal{R}_{h_1 = 0} \mathcal{R}_{h_2 = 0} \bigg( (h_1)^{-b_1} (h_2)^{-b_2} \mathcal{Z}_{\nu_1}(h_1, h_2, q) \bigg). \]

By the residue theorem on \(S^2\), and the vanishing of the residue at \(\alpha\), which can be directly obtained from the expression of \(\mathcal{Z}_{\nu_1}(h_1, h_2, q)\). We have
\[ \mathcal{R}_{h_1 = 0} \mathcal{R}_{h_2 = 0} \bigg( (h_1)^{-b_1} (h_2)^{-b_2} \mathcal{Z}_{\nu_1}(h_1, h_2, q) \bigg). \]
Combining formula (2.19) and (2.21),
\[
\frac{1}{2} \sum_{d=1}^{\infty} q^d \int_{Z_G} \frac{e(V_1) e^\gamma}{e(N Z_G)} = (\phi_{|p_i})^{-1} \sum_{b_1 \geq 0, b_2 \geq 0} \left( (q \frac{d}{dq} F_n^{(0,0)}(\alpha_i, q) + \alpha_i) \frac{F_n^{(0,0)}(\alpha_i, q)}{b_1!} \frac{F_n^{(0,0)}(\alpha_i, q)}{b_2!} \right)
\]
\[
\mathfrak{R}_{h_1=0} \left\{ \mathfrak{R}_{h_2=0} \left\{ (-h_1)^{-b_1} (-h_2)^{-b_2} \tilde{Z}^*_n(h_1, h_2, q) \right\} \right\}.
\]
Thus
\[
\mathcal{A}_i = \frac{1}{2} \sum_{d=1}^{\infty} q^d \int_{Z_G} \frac{e(V_1) e^\gamma}{e(N Z_G)} = (\phi_{|p_i})^{-1} \left( q \frac{d}{dq} F_n^{(0,0)}(\alpha_i, q) + \alpha_i \right)
\]
\[
\mathfrak{R}_{h_1=0} \left\{ \mathfrak{R}_{h_2=0} \left\{ e^{-F_n^{(0,0)}(\alpha_i, q)}(\frac{1}{\alpha_i} + \frac{1}{\alpha_2}) \tilde{Z}^*_n(h_1, h_2, q) \right\} \right\}.
\]

When the domain curve is mapped to the fixed point $p_i$, then the correspondent decorated graph $\Gamma$ is a vertex. They will be called $B_i$-types. In a graph of the $B_i$-type, the infinitesimally marked point 1 is attached to a vertex labeled $i$.

Let $B_i(q)$ be the total contribution from type $B_i$ graphs. Then

**Proposition 2.4.** For every $i \in \mathfrak{n}$,
\[
B_i(q) = \frac{1}{24} q^d \int_{\mathfrak{M}_{1,(0,1)|d}} \left( c_i(\alpha) F_n^{(0,0)}(\alpha_i, q) - \log \tilde{\Phi}^{(0)}(\alpha_i, q) \right),
\]
where $c_i(\alpha) = \sum_{k \neq i} \frac{1}{\alpha_k - \alpha_i} + \sum_{k=1}^{\ell} \frac{1}{\alpha_k \alpha_i}$.

**Proof.** The proof is exact the same as the calculation of the vertex contribution in [9]. We sketch it as following. Let $\Gamma$ be a type $B_i$ decorated graph, it is just vertex $v$ over $p_i$. Thus $Z_G = \mathfrak{M}_{1,(0,1)|d}$. The contribution

\[
B_i(q) = \sum_{d=1}^{\infty} q^d \int_{\mathfrak{M}_{1,(0,1)|d}} \frac{e(V_1) e^\gamma}{e(T_{p_i} \mathbb{P}^{n-1})} e(V_1(\alpha_i)) Q_v
\]
\[
= q \frac{d}{dq} \left( \sum_{d=1}^{\infty} q^d \int_{\mathfrak{M}_{1,0|d}} \frac{e(V_1) e^\gamma}{e(T_{p_i} \mathbb{P}^{n-1})} e(V_1(\alpha_i)) Q_v \right)
\]
where $\mathbb{E}$ is the Hodge bundle,

\[
Q_v = \frac{1}{\prod_{k \neq i} e(H^0(C_v, C_v(D_1)|D_1)(\alpha_i - \alpha_k))},
\]
\[
\sigma_{C_v}(D_1) \cong S^v|_{C_v}.
\]
\[ e(\mathcal{V}_1(\alpha_i)) = \frac{e(E|_{p_i})}{e(E^\vee \otimes E|_{p_i})} \tilde{Q}, \]

where

\[ (2.24) \quad \tilde{Q} = \frac{1}{\prod_{i=1}^\ell e(H^0(C_v, \mathcal{O}_{C_v}(\tilde{D}_i)|_{\tilde{D}_i})(\alpha_i))}. \]

\[ \mathcal{O}_{C_v}(\tilde{D}_i) \cong \mathcal{S}^{\vee \otimes \alpha_i}|_{C_v}. \]

Let \( c_i(\alpha) \) determined by

\[ 1 + c_i(\alpha)e(\mathcal{E}) = \frac{e(\mathcal{E}^\vee \otimes T_{p_i}\mathbb{P}^{n-1})}{e(T_{p_i}\mathbb{P}^{n-1})} \frac{e(E|_{p_i})}{e(\mathcal{E}^\vee \otimes E|_{p_i})}. \]

Thus

\[ c_i(\alpha) = \sum_{k \neq i} \frac{1}{\alpha_k - \alpha_i} + \sum_{k=1}^\ell \frac{1}{a_k \alpha_i}. \]

Denote by \( F_d^{(1,0)} = Q_v \tilde{Q} \), then

\[ \int_{\overline{M}_{1,0}|d} \frac{e(\mathcal{E}^\vee \otimes T_{p_i}\mathbb{P}^{n-1})}{e(T_{p_i}\mathbb{P}^{n-1})} e(\mathcal{V}_1(\alpha_i))Q_v \]

\[ = \int_{\overline{M}_{1,0}|d} (1 + c_i(\alpha)e(\mathcal{E})) F_d^{(1,0)}. \]

For nonnegative integers \( g \) and \( m \), the above expression for \( F_d^{(1,0)} \) also defined as an element in \( H^*(\overline{M}_{g,m}|d, \mathbb{Q}_\alpha) \), which can be written as a polynomial of diagonal classes and the psi classes. By the proof in [6, Theorem 2.6], we have

\[ \int_{\overline{M}_{1,0}|d} e(\mathcal{E}) F_d^{(1,0)} = \frac{1}{24} \int_{\overline{M}_{0,2}|d} F_d^{(0,2)}, \]

\[ \sum_{d=1}^\infty \frac{q^d}{d!} \int_{\overline{M}_{1,0}|d} F_d^{(1,0)} = \frac{1}{24} \log \left( \sum_{d=0}^\infty \frac{q^d}{d!} \int_{\overline{M}_{0,3}|d} F_d^{(0,3)} \right). \]

Therefore

\[ (2.25) \quad \sum_{d=1}^\infty \frac{q^d}{d!} \int_{\overline{M}_{1,0}|d} e(\mathcal{E}) F_d^{(1,0)} \]

\[ = \frac{1}{24} \sum_{d=1}^\infty \frac{q^d}{d!} \int_{\overline{M}_{0,(2,0)|d}} F_d^{(0,2)} \]

\[ = \frac{1}{24} \sum_{d=1}^\infty \frac{q^d}{d!} \int_{\overline{M}_{0,(2,0)|d}} \prod_{k \neq i} (\alpha_i - \alpha_k) e(\mathcal{V}_n^{(d)}(\alpha_i)) \]

\[ = \frac{1}{24} F^{(0,0)}(\alpha_i, q). \]
By [11, Proposition 4.1], we have
\[ (2.26) \quad \sum_{d=0}^{\infty} \frac{q^d}{d!} \int_{\mathcal{M}_{0,3,d}} \frac{1}{\Phi^0(\alpha_i, q)} = 1. \]

Thus combining (2.25) and (2.26), we have
\[ \mathcal{B}_i(q) = 1 - 24q \frac{d}{dq} \left( c_i(\alpha) \mathcal{F}^{(0,0)}(\alpha_i, q) - \log \Phi^0(\alpha_i, q) \right). \]

\[ \square \]

Combining Proposition 2.3 and Proposition 2.4, we have

**Theorem 2.5.** For Calabi-Yau manifold \( X \subset \mathbb{P}^{n-1} \) which is complete intersection,
\[ (2.27) \quad q \frac{d}{dq} G_{1,0} = \left\{ \sum_{i \in \mathbb{N}} \left( A_i(q) + \frac{1}{24q} \frac{d}{dq} \left( c_i(\alpha) \mathcal{F}^{(0,0)}(\alpha_i, q) - \log \Phi^0(\alpha_i, q) \right) \right) \right\} \bigg|_{\alpha=0} \]

where \( c_i(\alpha) = \sum_{k \neq i} \frac{1}{\alpha_k - \alpha_i} + \sum_{k=1}^{\ell} \frac{1}{a_k \alpha_i} \).

3. Calculation

In this section we work out the explicit expression of (2.27) by using the hypergeometric series and their properties established in [9], [10] and [11].

We define power series \( L_n, \xi_n \in \mathbb{Q}_\alpha[\mathbb{x}][[q]] \) by
\[ L_n \in \mathbb{x} + q \mathbb{Q}_\alpha[\mathbb{x}][[q]], \quad \tilde{s}_n(L_n(\mathbb{x}, q)) - qa^\mathbb{a}L_n(\mathbb{x}, q)^{|a|} = \tilde{s}_n(\mathbb{x}), \]
\[ \xi_n \in q \mathbb{Q}_\alpha[\mathbb{x}][[q]], \quad \mathbb{x} + q \frac{d}{dq} \xi_n(\mathbb{x}, q) = L_n(\mathbb{x}, q), \]
where \( \tilde{s}_r(y) \) is the r-th elementary symmetric polynomial in \( \{y - \alpha_k\} \). By [11, (4-9)], \( \xi_n(\alpha_i, q) = \mathcal{F}^{(0,0)}_n(\alpha_i, q) \). Let
\[ L(q) = (1 - a^\mathbb{a}q)^{-\frac{1}{n}}, \quad \mu(q) = \int_0^q \frac{L(u) - 1}{u} du, \]
then
\[ (3.2) \quad L_n(\mathbb{x}, q) = L(q)\mathbb{x} + \sum_{d=0}^{\infty} f_d(\mathbb{x}, \alpha) q^d, \]
where \( f_d(\mathbb{x}, \alpha) \in \mathbb{Q}_\alpha[\mathbb{x}] \) with \( \mathbb{x}|f_d(\mathbb{x}, \alpha) \), and has no \( \mathbb{x} \) term with constant coefficient, \( f_d(\mathbb{x}, 0) = 0. \)

\[ (3.3) \quad \xi_n(\mathbb{x}, q) = \mu(q)\mathbb{x} + \sum_{d=1}^{\infty} g_d(\mathbb{x}, \alpha) q^d, \]
where \( g_d(\mathbb{x}, \alpha) \in \mathbb{Q}_\alpha[\mathbb{x}] \) with \( \mathbb{x}|g_d(\mathbb{x}, \alpha) \), and has no \( \mathbb{x} \) term with constant coefficient, \( g_d(\mathbb{x}, 0) = 0. \).
By the residue theorem on \(S^2\),

\[
\sum_{i=1}^{n} \sum_{k \neq i} \xi_n(\alpha_i, q)\frac{1}{\alpha_k - \alpha_i} = - \sum_i \mathcal{R}_{z=\alpha_i} \sum_{k \neq j} \frac{\mu(q)z}{(z - \alpha_j)(z - \alpha_k)}
- \sum_{d=0}^{\infty} q^d \sum_i \mathcal{R}_{z=\alpha_i} \sum_{k \neq j} \frac{g_d(z, \alpha)}{(z - \alpha_j)(z - \alpha_k)}
= \mathcal{R}_{z=\infty} \sum_{k \neq j} \frac{\mu(q)z}{(z - \alpha_j)(z - \alpha_k)}
+ \sum_{d=0}^{\infty} q^d \mathcal{R}_{z=\infty} \sum_{k \neq j} \frac{g_d(z, \alpha)}{(z - \alpha_j)(z - \alpha_k)}
= - \left( \frac{n}{2} \right) \mu(q) + \sum_{d=0}^{\infty} q^d \mathcal{R}_{z=\infty} \sum_{k \neq j} \frac{g_d(z, \alpha)}{(z - \alpha_j)(z - \alpha_k)}.
\]

By (3.3),

\[
\left( \mathcal{R}_{z=\infty} \sum_{k \neq j} \frac{g_d(z, \alpha)}{(z - \alpha_j)(z - \alpha_k)} \right)_{\alpha=0} = 0.
\]

Thus

\[
\left( \sum_{i=1}^{n} \sum_{k \neq i} \xi_n(\alpha_i, q)\frac{1}{\alpha_k - \alpha_i} \right)_{\alpha=0} = - \left( \frac{n}{2} \right) \mu(q).
\]

\[
\sum_{i=1}^{n} \xi_n(\alpha_i, q)\frac{1}{a_k \alpha_i} = \frac{1}{a_k} \sum_{i=1}^{n} \mathcal{R}_{z=\alpha_i} \sum_{j=1}^{n} \frac{\xi_n(z, q)}{z(z - \alpha_j)}
= \frac{1}{a_k} \sum_{i=1}^{n} \mathcal{R}_{z=\alpha_i} \sum_{j=1}^{n} \frac{\mu(q)}{z(z - \alpha_j)}
+ \frac{1}{a_k} \sum_{d=0}^{\infty} q^d \sum_{i=1}^{n} \mathcal{R}_{z=\alpha_i} \sum_{j=1}^{n} \frac{g_d(z, \alpha)}{z(z - \alpha_j)}
= - \frac{1}{a_k} \mathcal{R}_{z=\infty} \sum_{j=1}^{n} \frac{\mu(q)}{z(z - \alpha_j)}
- \frac{1}{a_k} \sum_{d=0}^{\infty} q^d \sum_{i=1}^{n} \mathcal{R}_{z=\infty} \sum_{j=1}^{n} \frac{g_d(z, \alpha)}{z(z - \alpha_j)}
- \frac{1}{a_k} \sum_{d=0}^{\infty} q^d \sum_{i=1}^{n} \mathcal{R}_{z=0} \sum_{j=1}^{n} \frac{g_d(z, \alpha)}{z(z - \alpha_j)},
\]

by (3.3)

\[
\frac{1}{a_k} \sum_{d=0}^{\infty} q^d \left( \sum_{i=1}^{n} \mathcal{R}_{z=0, \infty} \sum_{j=1}^{n} \frac{g_d(z, \alpha)}{z(z - \alpha_j)} \right)_{\alpha=0} = 0.
\]
Thus
\[
\left( \sum_{k=1}^{\ell} \sum_{i=1}^{n} \frac{\xi_n(\alpha_i, q)}{a_k \alpha_i} \right)_{\alpha=0} = \sum_{k=1}^{\ell} \frac{n}{a_k}.
\]
By [11, (4-10)]
\[
\Phi^{(0)}(\alpha_i, q)_{\alpha=0} = L(q)^{\frac{m-1}{2}}.
\]
Therefore
\[
(3.4)\quad\left( \sum_{i=1}^{n} \mathcal{B}_i(q) \right)_{\alpha=0} = \frac{1}{24} \left( \sum_{i=1}^{n} \left( c_i(\alpha) \mathcal{J}^{(0,0)}(\alpha_i, q) - \log \Phi^{(0)}(\alpha_i, q) \right) \right)_{\alpha=0} = \frac{1}{24} \left( \sum_{i=1}^{n} \left( \frac{n}{a_i} - \left( n \mu(q) - \frac{n(\ell + 1)}{2} \log L(q) \right) \right) \right).
\]

For each \( p \in \mathbb{N} \), let \( \sigma_p \) be the \( p \)-th elementary symmetric polynomial in \( \alpha_1, \ldots, \alpha_n \). Denote by
\[
\mathbb{Q}[\alpha]^{S_n} \equiv \mathbb{Q}[\alpha_1, \ldots, \alpha_n]^{S_n} \subset \mathbb{Q}[\alpha_1, \ldots, \alpha_n]
\]
the subspace of symmetric polynomials, by \( \mathcal{J} \subset \mathbb{Q}[\alpha]^{S_n} \) the ideal generated by \( \sigma_1, \ldots, \sigma_{n-1} \), and by
\[
\tilde{\mathbb{Q}}[\alpha]^{S_n} \equiv \mathbb{Q}[\alpha_1, \ldots, \alpha_n]_{<(\alpha_j - \alpha_k)_{j \neq k}>} \subset \mathbb{Q}[\alpha]
\]
the subalgebra of symmetric rational functions in \( \alpha_1, \ldots, \alpha_n \) whose denominators are products of \( \alpha_j - \alpha_k \), with \( j \neq k \). For each \( i = 1, \ldots, n \), let
\[
\tilde{\mathbb{Q}}[\alpha]^{S_{n-1}} \equiv \mathbb{Q}[\alpha_1, \ldots, \alpha_n]_{<(\alpha_i - \alpha_k)_{k \neq i}>} \subset \mathbb{Q}[\alpha]
\]
be the subalgebra consisting of rational functions symmetric in \( \{\alpha_k : k \neq i\} \) and with denominators that are products of \( \alpha_i - \alpha_k \) with \( k \neq i \).

**Lemma 3.1.** Let \( f(z, \alpha) \in \mathcal{J} \mathbb{Q}[\alpha][z] \). Then for \( \frac{f(\alpha_j, \alpha)}{\prod_{k \neq j} (\alpha_j - \alpha_k)^m} \) with \( m \geq 0 \), we have
\[
(3.5)\quad\left( \sum_{j=1}^{n} \frac{f(\alpha_j, \alpha)}{\prod_{k \neq j} (\alpha_j - \alpha_k)^{m+1}} \right)_{\alpha=0} = 0.
\]

**Proof.** By the residue theorem on \( S^2 \), when \( m = 0 \),
\[
\left( \sum_{j=1}^{n} \frac{f(\alpha_j, \alpha)}{\prod_{k \neq j} (\alpha_j - \alpha_k)} \right)_{\alpha=0} = \left( \sum_{j=1}^{n} \mathcal{R}_{z=\alpha_j} \left\{ \frac{f(z, \alpha)}{\prod_{k=1}^{n} (z - \alpha_k)} \right\} \right)_{\alpha=0} = - \left( \mathcal{R}_{\infty} \left\{ \frac{f(z, \alpha)}{\prod_{k=1}^{n} (z - \alpha_k)} \right\} \right)_{\alpha=0} = 0.
\]
When \( m \geq 1 \),

\[
\left. \left( \sum_{j=1}^{n} \frac{f(\alpha_j, \alpha)}{\prod_{k \neq j} (\alpha_j - \alpha_k)^{m+1}} \right) \right|_{\alpha=0} = 0.
\]

\[
= \left. \left( \sum_{j=1}^{n} \mathcal{H}_{z=\alpha_j} \left\{ \sum_{i=1}^{n} (z - \alpha_i)^m f(z, \alpha) \prod_{k=1}^{n} (z - \alpha_k)^{m+1} \right\} \right) \right|_{\alpha=0}
\]

\[
= - \left. \mathcal{H}_{\infty} \left\{ \sum_{i=1}^{n} (z - \alpha_i)^m f(z, \alpha) \prod_{k=1}^{n} (z - \alpha_k)^{m+1} \right\} \right|_{\alpha=0}
\]

\[
= 0.
\]

\[\square\]

**Lemma 3.2.** Let \( d, m \in \mathbb{Z}_{\geq 0} \) satisfy, \( m \neq (n - 1)(d + 1) \). Then

\[
(3.6) \quad \left. \left( \sum_{j=1}^{n} \frac{\alpha_j^m}{\prod_{k \neq j} (\alpha_j - \alpha_k)^{d+1}} \right) \right|_{\alpha=0} = 0.
\]

**Proof.** By the residue theorem on \( S^2 \), when \( d = 0 \)

\[
\left. \left( \sum_{j=1}^{n} \frac{\alpha_j^m}{\prod_{k \neq j} (\alpha_j - \alpha_k)} \right) \right|_{\alpha=0} = 0.
\]

\[
= \left. \left( \sum_{j=1}^{n} \mathcal{H}_{z=\alpha_j} \left\{ \prod_{k=1}^{n} (z - \alpha_k)^m \right\} \right) \right|_{\alpha=0}
\]

\[
= - \left. \mathcal{H}_{\infty} \left\{ \prod_{k=1}^{n} (z - \alpha_k)^m \right\} \right|_{\alpha=0}
\]

\[
= 0.
\]

When \( d \geq 1 \),

\[
\left. \left( \sum_{j=1}^{n} \frac{\alpha_j^m}{\prod_{k \neq j} (\alpha_j - \alpha_k)^{d+1}} \right) \right|_{\alpha=0} = 0.
\]

\[
= \left. \left( \sum_{j=1}^{n} \mathcal{H}_{z=\alpha_j} \left\{ \sum_{i=1}^{n} (z - \alpha_i)^d z^m \prod_{k=1}^{n} (z - \alpha_k)^{d+1} \right\} \right) \right|_{\alpha=0}
\]

\[
= - \left. \mathcal{H}_{\infty} \left\{ \sum_{i=1}^{n} (z - \alpha_i)^d z^m \prod_{k=1}^{n} (z - \alpha_k)^{d+1} \right\} \right|_{\alpha=0}
\]

\[
= 0.
\]

\[\square\]
Let
\[
F(w, q) \equiv \sum_{d=0}^{\infty} q^d \frac{\prod_{k=1}^{\ell} \prod_{r=1}^{d} (a_k w + r)}{\prod_{r=1}^{d} (w + r)^n} \in \mathbb{Q}(w)[[q]],
\]
and
\[
\tilde{F}(w, q) \equiv \sum_{d=0}^{\infty} q^d \frac{\prod_{k=1}^{\ell} \prod_{r=0}^{d} (a_k w + r)}{\prod_{r=1}^{d} ((w + r)^n - w^n)} \in \mathbb{Q}(w)[[q]].
\]

These are power series in \(q\) with constant term 1 whose coefficients are rational functions in \(w\) which are regular at \(w = 0\). We denote the subgroup of all such power series by \(\mathcal{P}\) and define
\[
\mathbf{D} : \mathbb{Q}(w)[[q]] \rightarrow \mathbb{Q}(w)[[q]], \quad \mathbf{M} : \mathcal{P} \rightarrow \mathcal{P}
\]
by
\[
\mathbf{D} H(w, q) \equiv \left\{ 1 + \frac{q}{w} \frac{d}{dq} \right\} H(w, q), \quad \mathbf{M} H(w, q) \equiv \mathbf{D} \left( \frac{H(w, q)}{H(0, q)} \right).
\]

For \(s \in \mathbb{Z}^0\), let
\[
\tilde{I}_s(q) \equiv \mathbf{M}^s \tilde{F}(0, q), \quad \bar{I}_s(q) \equiv \mathbf{M}^s \bar{F}(0, q).
\]

By \([\text{4.14}]\), for \(p \geq 0\),
\[
\mathbf{M}^p \tilde{F}(w, q) = \mathbf{M}^{p+\ell} \bar{F}(w, q).
\]

Let
\[
\mathbf{D}^0 \tilde{\mathbf{Y}}(x, h, q) = \frac{\tilde{\mathbf{Y}}(x, h, q)}{I_0(q)}, \quad \mathbf{D}^s \tilde{\mathbf{Y}}(x, h, q) = \frac{1}{I_s(q)} \left\{ x + h \frac{q}{w} \frac{d}{dq} \right\} \mathbf{D}^{s-1} \tilde{\mathbf{Y}}(x, h, q)
\]
for all \(s \in \mathbb{Z}^+\) and \((\tilde{\mathbf{Y}}, \tilde{I}) = (\mathbf{Y}, \mathbf{I}), (\tilde{\mathbf{Y}}, \bar{I}), (\bar{\mathbf{Y}}, \bar{I})\). For \(r, s, s' \geq 0\), there exists \(\mathcal{C}_{s, s'}^{(r)} \in \mathbb{Q}[\alpha_1, \ldots, \alpha_n][[q]]\), such that
\[
\sum_{s'=0}^{s'} \sum_{r=0}^{s'} \mathcal{C}_{s, s'}^{(r)}(q) x^{s-r} h^{-s'} = \mathbf{D}^s \mathbf{Y}(x, h, q).
\]

\[
\mathbf{D}^s \tilde{\mathbf{Y}}(x, h, q)|_{\alpha_0 = 0} = x^s \mathbf{D}^s \bar{F}(x/h, q), \quad \text{where}
\]
\[
\mathbf{D}^0 \bar{F}(w, q) = \frac{\bar{F}(w, q)}{I_0(q)}, \quad \mathbf{D}^s \bar{F}(w, q) = \frac{1}{I_s(q)} \left\{ 1 + \frac{q}{w} \frac{d}{dq} \right\} \mathbf{D}^{s-1} \bar{F}(w, q) \quad \forall s \in \mathbb{Z}^+,
\]
with \((\tilde{\mathbf{Y}}, \tilde{F}, \tilde{I}) = (\mathbf{Y}, \mathbf{F}, \mathbf{I}), (\bar{\mathbf{Y}}, \bar{F}, \bar{I}), (\bar{\mathbf{Y}}, \bar{F}, \bar{I})\).
Theorem 3.3. [11] Theorem 4] If \( \ell \in \mathbb{Z}^{\geq 0} \), \( n \in \mathbb{Z}^{+} \), and \( a \in (\mathbb{Z}^{\geq 0})^{\ell} \) are such that \( |a| = \sum_{i=1}^{\ell} = n \), then

\[
(3.15) \quad \hat{Z}(h_1, h_2, q) = \frac{1}{h_1 + h_2} \sum_{s_1, s_2, r \geq 0 \atop s_1 + s_2 + r = n - 1} (-1)^r \sigma_r \hat{Z}^{(s_1)}(x, h_1, q) \hat{Z}^{(s_2)}(x, h_2, q),
\]

where \( \sigma_r \in \mathbb{Q}_\alpha \) is the \( r \)-th elementary symmetric polynomial in \( \alpha_1, \ldots, \alpha_n \). There exists \( \tilde{C}_r^{(s)} \in \mathbb{Q}[\alpha][[q]] \) such that

\[
\hat{Z}^{(s)}(x, h, q) = \hat{\gamma}^{(s)}(x, h, q) := \sum_{r=0}^{s-r} \sum_{s'=0}^{s} \tilde{C}_r^{(s)}(a, s'-r; a)(q) h^{s-r-s'} \mathcal{D}^{s'} \hat{\gamma}(x, h, q),
\]

where \( (\hat{Z}, \ell^*) = (\hat{Z}, 0), (\hat{Z}, \ell^*), \hat{\gamma} = \hat{Z}, \hat{Z} \).

Lemma 3.4.

\[
\left( \sum_{i=1}^{n} (\phi_i|_{p_i})^{-1} \left( q \frac{d}{dq} F_n^{(0,0)}(\alpha_i, q) + \alpha_i \right) \right)
\]

\[
\mathcal{R}_{h_1=0} \left\{ \mathcal{R}_{h_2=0} \left\{ e^{-F_n^{(0,0)}(\alpha_i, q)(\frac{1}{h_1} + \frac{1}{h_2})} \tilde{Z}_{ii}^{\ast}(h_1, h_2, q) \right\} \right\} \bigg|_{\alpha=0}
\]

\[
= \left( \sum_{i=1}^{n} (\phi_i|_{p_i})^{-1} \left( q \frac{d}{dq} F_n^{(0,0)}(\alpha_i, q) + \alpha_i \right) \right) \alpha_i^{n-1} \mathcal{R}_{h_1=0} \left\{ \mathcal{R}_{h_2=0} \left\{ e^{-F_n^{(0,0)}(\alpha_i, q)(\frac{1}{h_1} + \frac{1}{h_2})} \mathcal{F}(\alpha_i/h_1, \alpha_i/h_2, q) \right\} \right\} \bigg|_{\alpha=0},
\]

where

\[
\mathcal{F}(w_1, w_2, q) = \sum_{p=0}^{n-1-\ell} \frac{M^p \bar{F}(w_1, q) M^{n-1-p} \bar{F}(w_2, q)}{\bar{I}_p(q)} \frac{M^{n-1-p} \bar{F}(w_2, q)}{\bar{I}_p(q)}
\]

\[
+ \sum_{p=1}^{\ell} \frac{M^{n-1+p} \bar{F}(w_1, q) M^{n-p} \bar{F}(w_2, q)}{\bar{I}_{n-1+p}(q)} \frac{M^{n-p} \bar{F}(w_2, q)}{\bar{I}_{n-p}(q)}.
\]

Proof. By Theorem 3.3,

\[
\sum_{s_1, s_2, r \geq 0 \atop s_1 + s_2 + r = n - 1} (-1)^r \sigma_r \alpha_i^{s_1+s_2} + 2(h_1 + h_2)h_1 h_2 \tilde{Z}^{\ast}_{ii}
\]

\[
= \sum_{s_1, s_2 \geq 0 \atop s_1 + s_2 = n - 1} \hat{\gamma}^{(s_1)}(\alpha_i, h_1, q) \hat{\gamma}^{(s_2)}(\alpha_i, h_2, q)
\]

\[
+ \sum_{s_1, s_2 \geq 0, r > 0 \atop s_1 + s_2 = n - 1} (-1)^r \sigma_r \hat{\gamma}^{(s_1)}(\alpha_i, h_1, q) \hat{\gamma}^{(s_2)}(\alpha_i, h_2, q).
\]
By the definition of \( \tilde{y}^{(s_1)}(x, h_1, q) \) and \( \tilde{y}^{(s_2)}(x, h_1, q) \), and (5.3), the \( q \) coefficient of

\[
\mathcal{R}_{h_1=0} \left\{ \mathcal{R}_{h_2=0} \left\{ \frac{1}{2(h_1 + h_2)h_1h_2} \left\{ e^{-\mathcal{F}(0,0)(\alpha_i, q)(\frac{1}{n_1} + \frac{1}{n_2})} \right\} \sigma_i \tilde{y}^{(s_1)}(\alpha_i, h_1, q) \tilde{y}^{(s_2)}(\alpha_i, h_2, q) \right\} \right\}
\]

satisfies the condition in Lemma 3.1 for \( r > 0 \). Thus

\[
\left( \sum_{i=1}^{n} (\phi_i|p_i)^{-1} \left( q \frac{d}{dq} \mathcal{F}(0,0)(\alpha_i, q) + \alpha_i \right) \mathcal{R}_{h_1=0} \left\{ \mathcal{R}_{h_2=0} \left\{ e^{-\mathcal{F}(0,0)(\alpha_i, q)(\frac{1}{n_1} + \frac{1}{n_2})} \right\} \right\} \right) \bigg|_{\alpha=0} = \left( \sum_{\substack{s_1,s_2 \geq 0 \\text{ s.t. } s_1 + s_2 = n-1}} \tilde{y}^{(s_1)}(\alpha_i, h_1, q) \tilde{y}^{(s_2)}(\alpha_i, h_2, q) = -\alpha_i^{n-1} \mathbb{F}(\alpha_i/h_1, \alpha_i/h_2, q) \right) \right) \right) \bigg|_{\alpha=0}.
\]

By (3.13) and (3.3), the \( q \) coefficient of

\[
\mathcal{R}_{h_1=0} \left\{ \mathcal{R}_{h_2=0} \left\{ \frac{1}{2(h_1 + h_2)h_1h_2} \left\{ e^{-\mathcal{F}(0,0)(\alpha_i, q)(\frac{1}{n_1} + \frac{1}{n_2})} \right\} \right\} \right\} \bigg|_{\alpha=0}
\]

is of the form as in Lemma 3.1 and Lemma 3.2. Thus

\[
\left( \sum_{i=1}^{n} (\phi_i|p_i)^{-1} \left( q \frac{d}{dq} \mathcal{F}(0,0)(\alpha_i, q) + \alpha_i \right) \mathcal{R}_{h_1=0} \left\{ \mathcal{R}_{h_2=0} \left\{ e^{-\mathcal{F}(0,0)(\alpha_i, q)(\frac{1}{n_1} + \frac{1}{n_2})} \right\} \right\} \right) \bigg|_{\alpha=0} \]

\[
= \left( \sum_{i=1}^{n} (\phi_i|p_i)^{-1} \left( q \frac{d}{dq} \mathcal{F}(0,0)(\alpha_i, q) + \alpha_i \right) \alpha_i^{n-1} \mathcal{R}_{h_1=0} \left\{ \mathcal{R}_{h_2=0} \right\} \right) \bigg|_{\alpha=0}.
\]

Lemma 3.5. [5] Lemma 5.4]

\[
\mathcal{R}_{h_1=0} \mathcal{R}_{h_2=0} \left\{ e^{-\mu(q)\alpha_i(h_1^{-1}+h_2^{-1})} \right\} \frac{1}{h_1h_2(h_1 + h_2)} \mathbb{F}(\alpha_i/h_1, \alpha_i/h_2, q) = \alpha_i^{-1} L(q)^{-1} q \frac{d}{dq} \mathcal{A}(q),
\]
where
\[
A(q) = \frac{n}{24} (n-1-2 \sum_{r=1}^{\ell} \frac{1}{a_r}) \mu(q) - \frac{3(n-1-\ell)^2 + (n-2)}{24} \log(1 - a^\ell q) \\
- \sum_{p=0}^{n-2-\ell} \left( \frac{n-p-\ell}{2} \right) \log \dot{I}_p(q).
\]

By Lemma 3.4, Lemma 3.5 and (3.1)

\[
(3.16)
\]

\[
\left( \sum_{i=1}^{n} \mathcal{A}_i \right) \bigg|_{\alpha=0} = \left( \sum_{i=1}^{n} (\phi_i|_p)^{-1} \left( q \frac{d}{dq} \mathcal{F}_n^{(0,0)}(\alpha_i, q) + \alpha_i \right) \right) \mathcal{R}_{h_1=0} \mathcal{R}_{h_2=0} \left\{ e^{-\mathcal{F}_n^{(0,0)}(\alpha_i, q)} \left( \frac{1}{h_1} + \frac{1}{h_2} \right) \tilde{Z}_{n}^{*}(h_1, h_2, q) \right\} \bigg|_{\alpha=0} = \frac{1}{2} q \frac{d}{dq} A(q).
\]

**Theorem 3.6.** For projective complete intersection Calabi-Yau $X$,

\[
G_{1,0} = \frac{1}{2} A(q) + \frac{1}{24} \left( \sum_{i=1}^{\ell} \frac{n}{a_i} - \left( \frac{n}{2} \right) \mu(q) - \frac{n(\ell+1)}{2} \log L(q) \right),
\]

where
\[
A(q) = \frac{n}{24} (n-1-2 \sum_{r=1}^{\ell} \frac{1}{a_r}) \mu(q) - \frac{3(n-1-\ell)^2 + (n-2)}{24} \log(1 - a^\ell q) \\
- \sum_{p=0}^{n-2-\ell} \left( \frac{n-p-\ell}{2} \right) \log \dot{I}_p(q).
\]

**Proof.** The proof is just combining Theorem 2.5, (3.16) and (3.4). \qed

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