Statistics investigation of mechanical parameter of rock based on Bayesian reliability theory

Xiangnan Wu 1,2,3, Lizhi feng 1,2,3, JiWei Zhu 1,2,3

1 State Key Laboratory of Eco-hydraulics in Northwest Arid Region, Xi’an University of Technology, Xi’an 710048, China
2 Research Center of Eco-hydraulics and Sustainable Development, The New Style Think Tank of Shaanxi Universities, Xi’an 710048, China
3 Faculty of Civil Engineering and Architecture, Xi’an University of Technology, Xi’an, Shaanxi, 710048, China

*Corresponding author’s e-mail: 149966834@qq.com

Abstract. In order to improve forecast accuracy of bearing capacity of rock, the forecast method of bearing capacity of rock was constructed on Jeffreys prior using MCMC method of Bayesian theory. The proposed approach was used to estimate the parameters of Normal distribution. Numerical simulation study was used to produce the pseudo samples. The maximum likelihood parameter estimation method and Bayesian statistical theory were used to estimate the forecast value of the Normal distribution, which has been done by comparing with the theoretical value of the pseudo sample of Normal distribution. The result indicates that the forecast model of Normal distribution is optimal than the maximum likelihood estimation, and a significant improvement were noticed with the number increasing of pseudo sample. At last, the proposed method was applied to estimate the Normal distribution of bearing capacity of rock, which shows that the proposed method in this paper has higher precision and good applicability.

1. Introduction

With the development of engineering theory and engineering technology, the current engineering analysis has gradually shifted from a deterministic framework to a random framework, and the role of reliability theory in various engineering disciplines has become increasingly prominent. As the basis of underground geotechnical engineering analysis, it is extremely important to fully understand the randomness of rock mechanics parameters and to reasonably evaluate its probability distribution characteristics. The ISRM recommended standard requires at least 3-5 samples for rock uniaxial and triaxial compression tests and is characterized by means of averaging[1-2]. Although this traditional method is widely accepted, it belongs to the fixed value design method, which means that the mechanical parameters of the rock are regarded as equivalent. In fact, the rock characteristic parameters have a certain degree of randomness, which is its natural attribute. In the actual engineering stability analysis and safety design, the fixed value design method will increase the risk by neglecting the parameter variability. Using the viewpoints of uncertainty and stochastic analysis to study the parameters of rock mechanics can better reflect the nature of things. The application of reliability theory to comprehensively revise the engineering structure norms is a common development trend in the world and a fundamental change in the engineering structure norms.
The accuracy of the statistical parameters of rock mechanics will have a great impact on the reliability analysis results. Regarding the statistical properties of rock mechanics parameters, some scholars[3-6] have studied it to a certain extent, and believe that the rock mechanics parameters obey the normal distribution.

The accuracy of the statistical parameters of rock mechanics will have a great impact on the reliability analysis results. Regarding the statistical properties of rock mechanics parameters, some scholars[3-6] have studied it to a certain extent, and believe that the rock mechanics parameters obey the normal distribution. Summarize the existing research results and find that the shortcomings are as follows:

- When there is no measured data, the statistical properties of rock mechanics parameters are assumed;
- Treating the statistical parameters in the normal distribution as fixed values rather than random variables;
- It is considered that the statistical parameters in the normal distribution are independent rather than related;
- The need for more number of trials or tests. Due to the above shortcomings, the application of reliability theory in concrete practical engineering of rock mechanics is limited.

In view of the existing research deficiencies, the Bayesian reliability estimation method[7] is used to estimate the normal distribution parameters and applied to the statistical properties of rock mechanics parameters. The method proposed in this paper can well solve the shortcomings in the existing research, including the following aspects:

- Based on measured data;
- Treating the statistical parameters in the normal distribution as random variables;
- The correlation of statistical parameters in the normal distribution can be considered.

A very important issue in Bayesian estimation is to predict the posterior distribution based on prior distributions in existing experience and historical statistics. In this paper, the principle of normal distribution quantile is used to study the statistical parameters based on Bayesian theory. The idea of Bayesian quantile estimation is based on Bayesian reliability theory, but the Bayesian reliability theory for normal distribution quantile posterior estimation is a complex high-dimensional and no analytical solution. In response to the solution of this problem, the MCMC (Markov Chain Monte Carlo) method has been developed rapidly in recent years and is widely used[8]. If the posterior distribution is a high-dimensional, complex and very common distribution, it is difficult to extract samples from the posterior distribution, but the MCMC method breaks through this extremely difficult computational problem and opens the way for the extensive application of the Bayesian method. In this paper, the MCMC method based on Bayesian reliability is used to calculate the normal distribution score, and then the distribution parameters in the normal distribution are estimated, and then applied to the statistical characteristics of rock mechanics parameters, in order to provide accurate rock mechanics reliability analysis. Learn from and reference.

2. Prediction method based on Bayesian reliability theory

2.1. Normal distribution and its quantile

The probability density function and cumulative distribution function of a normal distribution are as follows:

\[
f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[ -\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2 \right]
\]

\[
F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[ -\frac{1}{2} \left( \frac{t - \mu}{\sigma} \right)^2 \right] dt
\]
Where $\mu$ is the positional parameter and $\sigma$ is the scale parameter.

From the formula (2), the normal distribution quantile $\alpha$ is

$$x_\alpha = \mu + \Phi^{-1}(\alpha)\sigma$$  (3)

### 2.2. Jeffreys criterion to determine joint prior distribution

Based on the Bayesian theory, the positional parameter $\mu$ and the scaled parameter $\sigma$ in the normal distribution are regarded as random variables. The existing information is used for parameter estimation, and the joint probability prior distribution needs to be determined in advance. For the rock mechanics parameters, the prior distribution information of the position parameters and the scale parameters in the normal distribution cannot be obtained based on the existing experience. Therefore, it is suitable to use the non-information prior distribution.

The joint probability prior distribution is determined below based on the Jeffreys criterion. Jeffreys uses the square root of the Fisher information matrix determinant as the $(\mu, \sigma)$ prior density kernel. The steps to find the non-information prior distribution using the Jeffreys criterion [9] are as follows:

- Write the logarithm of the sample likelihood function

$$L(\mu, \sigma) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{1}{2} \left(\frac{x_i - \mu}{\sigma}\right)^2\right]$$  (4)

$$\ln L(\mu, \sigma) = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2$$  (5)

- Find the Fisher Information Matrix

$$I(\mu, \sigma) = E(-\frac{\partial^2 \ln L}{\partial \mu \partial \sigma})$$  (6)

- The $(\mu, \sigma)$ non-information prior density function is

$$\pi(\mu, \sigma) = \left[\det I(\mu, \sigma)\right]^{\frac{1}{2}}$$  (7)

For a normal distribution, the Fisher information matrix is

$$I(\mu, \sigma) = -E\begin{vmatrix}
\frac{\partial^2 \ln L}{\partial \mu^2} & \frac{\partial^2 \ln L}{\partial \mu \partial \sigma} \\
\frac{\partial^2 \ln L}{\partial \mu \partial \sigma} & \frac{\partial^2 \ln L}{\partial \sigma^2}
\end{vmatrix}$$  (8)

Available by Jeffreys guidelines

$$\pi(\mu, \sigma) = \frac{1}{\sigma}$$  (9)

### 2.3. MCMC method to determine posterior distribution

On the basis of the above-mentioned normal distribution position parameter and scale parameter joint
prior distribution, the posterior distribution of equation (3) is calculated, and then the average value of
the posterior distribution is used to characterize the estimated value. Since the posterior distribution
calculation of equation (3) is a very complicated high-dimensional problem, it is difficult to obtain a
clear posterior distribution by analytical methods. The above problem can be well solved by using the
MCMC algorithm which has been developed rapidly and widely used in recent years.

The basic idea of the MCMC algorithm is to transform a complex sampling problem into a series of
simple sampling problems, rather than extracting samples directly from complex posterior distributions
(actually unable to extract samples). Construct a Markov chain whose distribution is exactly the
posterior distribution \( \pi (\theta | x) \). When the Markov chain converges, take the sequence of sample points
\( \theta_{m+1}, \ldots, \theta_n \) on the chain, And the estimate of the posterior expectation of either function from the
ergodic theorem is:

\[
\hat{E}(g(\theta)|x) = \frac{1}{n-m} \sum_{i=m+1}^{n} g(\theta_i)
\]  

(10)

The Metropolis-Hastings algorithm is the most intuitive and widely used MCMC method. The basic
principles are as follows: Try to find a simple distribution that approximates the posterior distribution,
called the candidate generation density, denoted as \( q(\theta, \theta^{(i-1)}) \); Select an initial value \( \theta^{(0)} \); Let the
value of the parameter \( \theta \) at the beginning of the i-th iteration be \( \theta^{(i-1)} \); Then the lower i iterations are
as follows;

Step1: Extracting a candidate sample \( \theta^* \) from the candidate generation density \( q(\theta, \theta^{(i-1)}) \);

Step2: Calculate the acceptance probability

\[
p(\theta, \theta^{(i-1)}, \theta^*) = \min \left\{ \frac{p(\theta = \theta^* | x) q(\theta = \theta^{(i-1)}; \theta^*)}{p(\theta = \theta^{(i-1)} | x) q(\theta = \theta^{(i-1)}; \theta^{(i-1)})}, 1 \right\}.
\]

Step3: Accept \( \theta^{(i)} = \theta^* \) with probability \( p(\theta, \theta^{(i-1)}) \) and accept \( 1 - p(\theta, \theta^{(i-1)}) \) with probability \( \theta^{(i)} = \theta^{(i-1)} \);

Step4: Repeat Step1-Step3 n times to get the posterior sample \( \theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)} \);

Step5: Find the average to get

\[
\hat{E}(g(\theta)|x) = \frac{1}{n-m} \sum_{i=m+1}^{n} g(\theta_i)
\]

2.4. Normal distribution parameter estimation

Based on the Bayesian theory, the Metropolis-Hastings algorithm in the MCMC method is used to
calculate the \( x_{a1} \) and \( x_{a2} \) values respectively, and the simultaneous equations can be obtained.

\[
x_{a1} = \mu + \Phi^{-1}(\alpha_1) \sigma
\]

(11)

\[
x_{a2} = \mu + \Phi^{-1}(\alpha_2) \sigma
\]

(12)

The estimated values of the normal distribution statistical parameters for solving the above equations are:

\[
\mu = \frac{x_{a1} \cdot \Phi^{-1}(\alpha_2) - x_{a2} \cdot \Phi^{-1}(\alpha_1)}{\Phi^{-1}(\alpha_2) - \Phi^{-1}(\alpha_1)}
\]

(13)

\[
\sigma = \frac{x_{a2} - x_{a1}}{\Phi^{-1}(\alpha_2) - \Phi^{-1}(\alpha_1)}
\]

(14)
According to the statistical characteristics of rock mechanics parameters, the optional normal distribution quantiles are 0.95 and 0.99 respectively. Then based on the above principle, the mean and variance of the normal distribution rock mechanics parameters can be obtained, and the coefficient of variation can be further obtained. Application in reliability analysis.

3. Simulation study
This section uses a simulation example to verify the correctness of the proposed method and compare it with the results of the maximum likelihood estimation method to illustrate its superiority. In order to consider the generality, the pseudo samples are generated from a standard normal distribution (position parameter is 0, scale parameter is 1), and the sample sizes are taken as \( n = 10, 20, 30, 50, 100, 200 \), respectively. The accuracy of the estimation results is measured by the root mean square error RMSE[10], the K-S test[11], and the correlation coefficient \( R^2 \)[12]. The specific simulation calculation results are shown in Table 1.

| \( n \) | Parameter | Maximum Likelihood Method | Bayesian Reliability Method |
|------|-----------|---------------------------|-----------------------------|
|      | \( \mu \) | \( \sigma \)       | \( \mu \) | \( \sigma \) |
| 10   | mean      | 0.1222                    | 1.1567 | 0.0941 | 1.0811 |
|      | RMSE      | 0.2653                    | 0.3577 | 0.1981 | 0.2713 |
|      | KS        | 0.3533                    | 0.2457 | 0.8329 | 0.8897 |
| 20   | mean      | 0.0757                    | 1.1023 | 0.0511 | 1.0588 |
|      | RMSE      | 0.1835                    | 0.2453 | 0.1448 | 0.1857 |
|      | KS        | 0.2635                    | 0.1977 | 0.9013 | 0.9233 |
| 30   | mean      | 0.0533                    | 1.0757 | 0.0478 | 1.0421 |
|      | RMSE      | 0.1532                    | 0.2134 | 0.1135 | 0.1577 |
|      | KS        | 0.1526                    | 0.0997 | 0.9552 | 0.9753 |
| 50   | mean      | 0.0437                    | 1.0532 | 0.0392 | 1.0387 |
|      | RMSE      | 0.1152                    | 0.1562 | 0.0826 | 0.1281 |
|      | KS        | 0.1299                    | 0.0873 | 0.9632 | 0.9873 |
| 100  | mean      | 0.0365                    | 1.0484 | 0.0278 | 1.0326 |
|      | RMSE      | 0.0761                    | 0.1392 | 0.0518 | 0.0985 |
|      | KS        | 0.0831                    | 0.0653 | 0.9767 | 0.9915 |
| 200  | mean      | 0.0265                    | 1.0361 | 0.0212 | 1.0148 |
|      | RMSE      | 0.0571                    | 0.0765 | 0.0471 | 0.0512 |
|      | KS        | 0.0674                    | 0.0521 | 0.9811 | 0.9945 |

The analysis of the estimation results of the normal distribution parameters in Table 1 can be seen as follows: The position parameters and scale parameters of the normal distribution calculated by Bayesian reliability method are closer to the true value than the maximum likelihood estimation method; The root mean square error of the normal distribution position parameter and the scale parameter calculated by the Bayesian reliability method is smaller than the maximum likelihood estimation method; The K-S test value of the normal distribution pseudo sample calculated by the Bayesian reliability method is
smaller than the maximum likelihood estimation method; The correlation coefficient of the normal distribution pseudo sample calculated by the Bayesian reliability method is larger than the maximum likelihood estimation method. Based on the above analysis results, the estimation results based on Bayesian reliability are more accurate than the maximum likelihood estimation method. The Bayesian estimation method has better performance and is more suitable for normal distribution parameter estimation. The specific reason is that Bayesian reliability based methods can consider factors that cannot be considered by maximum likelihood estimation, namely the objective randomness of positional parameters and scale parameters, and the correlation between positional parameters and scaled parameters. In addition, it is found that as the sample size increases, the Bayesian estimate and the maximum likelihood estimate are gradually decreasing, and the difference between the two is gradually reduced.

4. Engineering Applications
By investigating the measured data of rock mechanics parameters in the existing literature, the Bayesian reliability analysis method proposed in this paper is used to study the statistical properties of rock mechanics parameters. The sampling elastic modulus values of the mixed granite in a tunnel[13] are shown in Table 2.

| Sample Number | Testing results |
|---------------|----------------|
| 1             | 4.42           |
| 2             | 4.83           |
| 3             | 4.59           |
| 4             | 4.72           |
| 5             | 4.43           |
| 6             | 4.92           |
| 7             | 4.64           |
| 8             | 4.61           |
| Sample Number | Testing results |
| 9             | 4.84           |
| 10            | 4.93           |
| 11            | 4.33           |
| 12            | 4.74           |
| 13            | 4.63           |
| 14            | 4.38           |
| 15            | 4.45           |

The Bayesian reliability method proposed in this paper is used to analyze the elastic modulus test results of hybrid granite. The comparison between the distribution function and the empirical distribution function based on MCMC method and maximum likelihood estimation method is shown in Fig. 1(In the figure, the abscissa x represents a random variable, and the ordinate F(x) represents a cumulative distribution function.). The results based on the MCMC method compared with the maximum likelihood estimation method are shown in Table 3.

Figure 1 Comparison of distribution function of elasticity modulus of granitization granite
Table 3 Results of statistics of elasticity modulus of granitization granite

| Estimation Method               | Estimation parameter | RMSE  | KS   | R²   |
|--------------------------------|----------------------|-------|------|------|
| Maximum Likelihood Method      | 4.6307               | 0.1272| 0.1524| 0.9526|
| Bayesian Reliability Method    | 4.6027               | 0.1177| 0.1016| 0.9653|

The analysis of the mechanical property parameters estimation results of the normal distribution mixed granite in Fig. 1 and Table 3 can be seen as follows: The root mean square error of the mechanical parameters of the normal distribution rock calculated by the Bayesian reliability method is smaller than that of the maximum likelihood estimation method; The K-S test value of the measured samples of the normal distribution rock mechanical parameters calculated by the Bayesian reliability method is smaller than the maximum likelihood estimation method; The correlation coefficient of the normal distribution rock mechanics parameter samples calculated by the Bayesian reliability method is larger than the maximum likelihood estimation method. Based on the above analysis results, the estimation results based on Bayesian reliability are more accurate than the maximum likelihood estimation method. The Bayesian estimation method has better performance and is more suitable for statistical analysis of normal distributed rock mechanics parameters. Therefore, the Bayesian reliability-based method proposed in this paper is proposed to study the statistical properties of normal distributed rock mechanics parameters.

5. Conclusion
Based on the Bayesian reliability theory, this paper proposes a statistical method for predicting the statistical properties of rock mechanics parameters. Monte Carlo method is used to generate pseudo-samples, Bayesian estimation and maximum likelihood estimation are performed respectively, and the estimation results of the two are compared. The following conclusions are drawn:

- The numerical simulation results show that compared with the maximum likelihood estimation, Bayesian estimation is more accurate for normal distribution parameters. As the number of pseudo-samples of normal distribution increases, the error of the Bayesian estimate becomes smaller. Under large samples, the difference between the Bayesian estimate and the maximum likelihood estimate is small.

- The application results of rock mechanics parameters show that the Bayesian estimation method has higher precision and better performance, and is more suitable for estimating the statistical parameters of normal distributed rock mechanics parameters.

- In order to improve the accuracy of rock mechanics reliability analysis and better serve the engineering practice, the Bayesian reliability method proposed in this paper is proposed to study the statistical properties of normal distributed rock mechanics parameters.

Acknowledgments
Shaanxi Provincial Department of Education Natural Science Special Project(16JK1550), School Doctoral Research Fund (107-400211414)

Reference:
[1] Fairhurst C E, Hudson J A. Draft ISRM suggested method for the complete stress-strain curve for intact rock in uniaxial compression [J]. International Journal of Rock Mechanics & Mining Science & Geomechanics Abstracts, 1999, 36(3):281-289.
[2] Eberhardt E. The Complete ISRM Suggested Methods for Rock Characterization, [J]. International Journal of Rock Mechanics & Mining Sciences, 2009, 46(8):1396-1397.
[3] Ruffolo R M, Shakoor A. Variability of unconfined compressive strength in relation to number of test samples[J]. Engineering Geology, 2009, 108(1):16-23.
[4] Yan C F, Chen H K, Zhang J H. Bayes analysis of the probability distribution of rock mechanics parameters[J]. Journal of Civil, Architectural and Environmental Engineering, 1997(2):65-71.
[5] Jiang Q, Cui J, Feng X T, et al. Stochastic statistics and probability distribution estimation of mechanical parameters of basalt [J]. Rock and Soil Mechanics, 2017, 38(3):784-792.
[6] Rafiee R, Ataei M, Khalokakaie R, et al. Determination and Assessment of Parameters Influencing Rock Mass Cavability in Block Caving Mines Using the Probabilistic Rock Engineering System[J]. Rock Mechanics & Rock Engineering, 2015, 48(3):1207-1220.
[7] Hamada MS, Wilson AG, Reese CS, et al. Bayesian Reliability [M]. Springer, 2008.
[8] Han Ming. Bayesian statistics and its applications [M]. Tongji University Press, 2015.
[9] Miladinavic B, Tsokos CP, Ordinary, Bayes, empirical Bayes, and non-parametric reliability analysis for the modified Gumbel failure model [J]. Nonlinear Analysis, 2009, 71:1426–1436.
[10] Ouarda TBMJ, Charron C, Shin JY, et al. Probability distributions of wind speed in the UAE [J]. Energy Conversion & Management, 2015, 93: 414-434.
[11] Sulaiman MY, Akaak AM, Wahab MA, et al. Wind characteristics of Oman [J]. Energy, 2002, 27(1):35-46.
[12] Zhou JZ, Erdem E, Li G, et al. Comprehensive evaluation of wind speed distribution models: A case study for North Dakota sites [J]. Energy Conversion and Management, 2010, 51(7):1449-1458.
[13] Huang X Y, Wei L P. Random—Fuzzy Statistics Analysis of Rock Mechanics Parameters for Tunnels [J]. Journal of Northern Jiaotong University, 1999, 23(1):38-41.