Modified Variational Iteration Algorithm-II: Convergence and Applications to Diffusion Models

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Variational iteration method has been extensively employed to deal with linear and nonlinear differential equations of integer and fractional order. The key property of the technique is its ability and flexibility to investigate linear and nonlinear models conveniently and accurately. The current study presents an improved algorithm to the variational iteration algorithm-II (VIA-II) for the numerical treatment of diffusion as well as convection-diffusion equations. This newly introduced modification is termed as the modified variational iteration algorithm-II (MVIA-II). The convergence of the MVIA-II is studied in the case of solving nonlinear equations. The main advantage of the MVIA-II improvement is an auxiliary parameter which makes sure a fast convergence of the standard VIA-II iteration algorithm. In order to verify the stability, accuracy, and computational speed of the method, the obtained solutions are compared numerically and graphically with the exact ones as well as with the results obtained by the previously proposed compact finite difference method and second kind Chebyshev wavelets. The comparison revealed that the modified version yields accurate results, converges rapidly, and offers better robustness in comparison with other methods used in the literature. Moreover, the basic idea depicted in this study is relied upon the possibility of the MVIA-II being utilized to handle nonlinear differential equations that arise in different fields of physical and biological sciences. A strong motivation for such applications is the fact that any discretization, transformation, or any assumptions are not required for this proposed algorithm in finding appropriate numerical solutions.

1. Introduction

Diffusion is a basic biofunction for all living organs; all nutrient materials are transferred to cells through biomembranes through a diffusion process [1]. It is quite important from the biological point of view because different processes such as exchange of gases, absorption of some substances in the gut, and absorption of water and minerals by the roots of the plants are examples of diffusion. The movement of substances between cells also involves diffusion. Cells are surrounded by a flexible and dynamic barrier known as a membrane. These biological membranes are composed of lipids, which aggregate to form a bilayer with particular biochemical properties. The amphipathic nature of the lipid bilayer, whose tails are hydrophobic and associated with each other and whose head groups are hydrophilic and interact with the aqueous environment, is critical to its structure. The composition of the lipid bilayer is also important for the diffusion both across and within the membrane. This membrane diffusion is important for a variety of functions, some of which include regulating the fluidity of the membrane, the uptake of metabolites into the
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surface convection such as fans and pumps creating an
can be imposed when the liquid is forced to flow over
transport due to the bulk amount of fluids. Convection
random motion of the particles and the bulk motion of the
position of the energy move coming about because of the
and its mass. Since the piratical keep up their random
motion, the total heat transfer is because of the super-
position of the energy move coming about because of the
random motion of the particles and the bulk motion of the
fluid. It is customary to the utilization because of the
transport due to the bulk amount of fluids. Convection
can happen consequently freely if the fluid movement is
cased by buoyancy, caused by a change in density, or
cased by a change in the temperature of the fluid, or it
can be imposed when the liquid is forced to flow over
surface convection such as fans and pumps creating an
artificially induced convection current. In numerous
practical applications like heat loss from solar central
receivers or cooling of PV modules, natural convection
and forced convection occur at the same time.

More sophisticated models include diffusive and con-
vective transport as key elements for modeling heat and
mass transfer phenomena and continuous mechanical
processes. It ought to be noticed that convective-diffusive
transport is indispensable for predictions of various fluid
and gas flows such as pollutants spreading in the atmosphere
and pollutant transportation in groundwater and water
basins [4]. Convection-diffusion problems are governed by
typical mathematical models, which are common in gas and
fluid dynamics. Mass and heat transfer takes place not only
through diffusion but also through movement of the me-
dium. Many scientists have considered different diffusion
and convection-diffusion problems by different procedures;
for example, the wavelet-Galerkin method was used by El-
Gamal [5], while the Besel collocation method was
implemented by Yüzbaşı and Sahin [6] for the solution of
convection-diffusion problems. To investigate nonlinear
diffusion problems, a nonclassical method has been pre-
represented by Saied [4], Wang [7] employed the
Crank–Nicholson method, and Dhawan et al. [8] utilized the
finite-element method.

For the convection-diffusion models, the piecewise-an-
alysical method [9], high-order ADI method [10], least-
squares homotopy perturbation method [11], multigrid
solver [12], lattice Boltzmann model [13], stabilized finite-
element method [14], second kind Chebyshev wavelets [15],
finite element [16], finite difference method [17], decom-
position method [18], compact finite difference method [19],
meshless local Petrov–Galerkin method [20], discrete
finite volume scheme [21], discontinuous Galerkin (DG)
schemes [22], and high-order finite volume scheme [23]
have been used in the literature.

Variational iteration method which was proposed
originally by He [24] in 1999 has become popular in applied
sciences and has been extensively employed by many re-
searchers due to its promising performance in dealing with
linear and nonlinear differential equations of integer and
noninteger order. The key property of the technique is its
ability and flexibility to investigate linear and nonlinear
models conveniently and accurately. This technique has a
simpler solution procedure and can be used to handle
nonlinear differential equations that arise in different fields
of science because any small discretization, Adomian
polynomials, transformation, linearization, or any as-
sumptions are not required for this method to find the
numerical solutions [24–27]. So far, VIM has been utilized
for the numerical and analytical investigations of fractional
differential equations, oscillation equations, wave equations,
some delay differential equations, etc., and can be utilized in
a simplest way for the study of inverse problems [28],
integrodifferential equations [29], differential-difference
equations [30], and fractional calculus [31]. Many math-
ematicians have tried to develop this technique further, and
so far many modifications have been introduced. Among
these modifications, the modification where the least-
squares technology has been used by Herisancu and Marinca
[32] is very interesting. Hesameddini and Latifi-zadeh [33]
coupled it with Laplace transform, while its convergence was
proved by Salkuyeh [34]. An auxiliary parameter was in-
roduced by Inc and Yilmaz [35] to accelerate the conver-
gence speed to the exact solution. In 2010, He et al. [25],
the originator of the method himself, presented three algo-
rithms, i.e., variational iteration algorithm-I (VIA-I), algo-
rithm-II (VIA-II), and algorithm-III (VIA-III). The first
author has modified the VIA-I successfully and imple-
mented the modified version for many nonlinear PDEs
[36, 37]. Further, the first author also succeeded in modi-
ifying the algorithm-II by introducing an auxiliary parameter
recently [38].

In this study, we aim to discuss the convergence analysis
of the modified algorithm-II (MVIA-II) and to implement it
for finding the numerical solution of nonlinear PDEs arising
in physical and biological sciences which are modelled via
the diffusion equations. We summarize the contents of the
paper in the following sections. In Section 2, the proposed
method and its implementation are described. A conver-
gence analysis is discussed in Section 3. In Section 4, a
nonlinear diffusion equation with two convection-diffusion
models is investigated. Some concluding observations are
discussed in the Section 5.
2. Modified Iteration Algorithm-II

In this section, the main idea of the modification is illustrated by considering a nonlinear differential equation:

\[ L[\mathcal{I}(\xi)] + N[\mathcal{I}(\xi)] = k(\xi), \]  

(3)

where \( L[\mathcal{I}(\xi)] \) and \( N[\mathcal{I}(\xi)] \) denote the linear and nonlinear operators, respectively, whereas \( k(\xi) \) is a nonhomogeneous term. For an appropriate given initial condition \( \mathcal{I}_0(\xi) \), series solution \( \mathcal{I}_{m+1}(\xi) \) of equation (3) can be obtained as

\[ \mathcal{I}_{m+1}(\xi) = \mathcal{I}_m(\xi) + p \int_0^\xi \lambda(\theta) [L[\mathcal{I}_m(\theta)] + N[\mathcal{I}_m(\theta)] - k(\theta)] d\theta, \]  

(4)

where \( \mathcal{I}_n(\theta) \) is a restricted term which gives \( \delta \mathcal{I}_n(\theta) = 0 \) and \( p \) and \( \lambda(\theta) \) are the auxiliary parameter and Lagrange multiplier, respectively, which can be found optimally. The first one is used to accelerate the convergence to the exact solution [39–43], while the second one is used to construct the correction function [44]. The significant value of \( \lambda(\theta) \) can be achieved by applying \( \delta \) on both the sides of the recurrent relation (4) with respect to \( \mathcal{I}_n(\xi) \), which leads to

\[ \delta \mathcal{I}_{m+1}(\xi) = \delta \mathcal{I}_m(\xi) + p \int_0^\xi \lambda(\theta) [L[\mathcal{I}_m(\theta)] + N[\mathcal{I}_m(\theta)] - k(\theta)] d\theta, \]  

(5)

where the following Lagrange multipliers can be obtained:

\[ \lambda = -1, \quad \text{for } i = 1, \]
\[ \lambda = \xi - t, \quad \text{for } i = 2. \]

(6)

Also, the following general formula for the Lagrange multiplier in the cases \( i \geq 1 \) is available:

\[ \lambda = \frac{(-1)^i(\theta - t)^{i-1}}{(i-1)!}. \]  

(7)

After finding the value of \( \mathcal{I}_n(\theta) \), an iteration formula is constructed by using this value in the correctional function 4 as follows:

\[ \mathcal{I}_{n+1}(\xi) = \mathcal{I}_n(\xi) + p \int_0^\xi \frac{(-1)^i(\theta - t)^{i-1}}{(i-1)!} d\theta \]

\[ [L[\mathcal{I}_n(\theta)] + N[\mathcal{I}_n(\theta)] - k(\theta)] d\theta. \]

(8)

A more summarizing and concise iteration formula that can be constructed is known as the modified variational iteration algorithm-II (MVIA-II):

\[ \mathcal{I}_{n+1}(\xi) = \mathcal{I}_n(\xi) + p \int_0^\xi \frac{(-1)^i(\theta - t)^{i-1}}{(i-1)!} \]

\[ \cdot [N[\mathcal{I}_n(\theta)] - k(\theta)] d\theta. \]

(9)

It is convenient to repeat iterations many times to arrive at the given accuracy for the advanced computer technique. An exact solution \( \mathcal{I}(\xi) \) is obtained as

\[ \mathcal{I}(\xi) = \lim_{n \to \infty} \mathcal{I}_n(\xi). \]

(10)

In this technique, one does not require the discretization of the domain and linearization of the given differential equations. We simply need to calculate the Lagrange multiplier of the given differential equation by restricting the nonlinear terms and get the analytical/numerical solution of the given differential equations in the series form:

\[
\begin{align*}
\mathcal{I}_0(\xi) & \text{ is an appropriate initial approximation,} \\
\mathcal{I}_1(\xi, p) & = \mathcal{I}_0(\xi) + p \int_0^\xi \lambda(\theta)[L[\mathcal{I}_0(\theta)] + N[\mathcal{I}_0(\theta)] - k(\theta)] d\theta, \\
\mathcal{I}_{n+1}(\xi, p) & = \mathcal{I}_n(\xi, p) + p \int_0^\xi \lambda(\theta)[N[\mathcal{I}_n(\theta, p)] - k(\theta, p)] d\theta, \\
& n = 1, 2, \ldots 
\end{align*}
\]

(11)

This procedure is known as the MVIA-II, where one does not require the discretization of the domain and linearization of the given differential equations. We employ this proposed procedure for finding the analytical/numerical solution of diffusion and convection-diffusion equations. When \( p = 1 \), the variational iteration algorithm given in equation (11) becomes the standard VIA-II, and the approximate solution converges to the exact one when \( n \) approaches to infinity. Accordingly, a more accurate solution can be gained after a higher iteration process. Equation (11) has two obvious advantages: one is the limited step which is needed for better accuracy, while the other is an auxiliary parameter \( (p) \) which can be optimally determined, and its value depends upon the iteration step \( (n) \). When \( n \) tends to infinity, the value of \( p \) is equal to \( p = 1 \). We give a criterion of how to suitably choose \( p \) and optimally identify \( p \) through examples. It can be seen clearly that the variation iteration algorithms are easy for the implementation. For nonlinear problems, the nonlinear terms have to be considered as restricted variations for obtaining the value of the Lagrange multiplier, and a correction functional can be easily constructed after determining the value identified value of corresponding nonlinear terms.

3. Convergence Analysis

This section is devoted to illustrate the convergence of the proposed method VIA-II having an auxiliary parameter for ensuring the convergence to an exact solution. This method can be implemented most readily in handling nonlinear differential equations, where one does not require any discretization of the domain or linearization of the given differential equations. The linear operator \( L \) is defined as \( L = \partial^2/\partial t^2 \) and \( L = \partial^2/\partial t^2 \), when it is employed to solve diffusion and convection-diffusion problems. During solving diffusion and convection-diffusion problems, the operator \( R \) can be defined as
and \(w_n\) and \(v_n\), \(n \geq 0\), are defined by

\[
\begin{align*}
  w_0(\xi, t) &= \mathcal{I}_0(\xi, t), \\
  v_0(\xi, t) &= w_0(\xi, t), \\
  w_1(\xi, t, p) &= Rv_0(\xi, t), \\
  v_1(\xi, t, p) &= v_0(\xi, t) + Rv_0(\xi, t), \\
  \mathcal{I}_1(\xi, t, p) &= Rv_0(\xi, t), \\
  v_{n+1}(\xi, t, p) &= v_0(\xi, t) + w_1(\xi, t, p), \\
  v_{n+1}(\xi, t, p) &= v_n(\xi, t, p) + Rv_n(\xi, t, p).
\end{align*}
\]

In general, for \(n \geq 1\), it can be written as

\[
\begin{align*}
  \mathcal{I}_{n+1}(\xi, t, p) &= Rv_n(\xi, t, p), \\
  v_{n+1}(\xi, t, p) &= v_n(\xi, t, p) + w_{n+1}(\xi, t, p).
\end{align*}
\]

Accordingly,

\[
\mathcal{I}(\xi, t, p) = \lim_{n \to \infty} v_n(\xi, t, p) = w_0(\xi, t) + \sum_{n=1}^{\infty} w_n(\xi, t, p).
\]

\(\mathcal{I}_0(\xi, t)\) can be chosen uninhhibitedly, but it needs to fulfill the corresponding initial-boundary conditions. The determination of appropriate initial approximation will give productive and accurate results. The \(n\)-th order truncated series \(\mathcal{I}_n(\xi, t, p) = w_0(\xi, t) + \sum_{n=1}^{\infty} w_n(\xi, t, p)\) can be used to approximate the solution. The unknown parameter \(p\) in \(\mathcal{I}_n(\xi, t, p)\) ensures that the hypothesis is fulfilled by utilizing 2-norm error of the residual function. The error analysis and convergence criteria of VIA-II with an auxiliary parameter are revealed using the following theorems [45, 46].

**Theorem 1.** The operator \(R\) defined in (12) maps a Hilbert space \(H\) to \(H\). The solution given in (15) in the form of series is as follows:

\[
\mathcal{I}(\xi, t) = \lim_{n \to \infty} v_n(\xi, t, p) = w_0(\xi, t) + \sum_{n=1}^{\infty} w_n(\xi, t, p).
\]

It converges if \(\exists p \neq 0, 0 < \beta < 1\), such that

\[
\begin{align*}
  \|Rv_0(\xi, t)\| &\leq \beta v_0(\xi, t), \\
  \|Rv_1(\xi, t, p)\| &\leq \beta Rv_0(\xi, t), \\
  \|Rv_n(\xi, t, p)\| &\leq \beta \|Rv_{n-1}(\xi, t, p)\|, \quad n = 2, 3, 4, \ldots
\end{align*}
\]

**Proof.** To prove the required result, it is enough to verify that the sequence \(\{v_n\}_{n=0}^{\infty}\) is a Cauchy sequence in the Hilbert space \(H\). For that reason, we proceed as

\[
v_0(\xi, t) = w_0(\xi, t), v_{n+1}(\xi, t, p) = v_0(\xi, t) + Rw_0(\xi, t).
\]

Further,

\[
v_{n+1}(\xi, t, p) = v_n(\xi, t, p) + Rw_n(\xi, t, p),
\]

which implies

\[
\begin{align*}
  \|v_{n+1}(\xi, t, p) - v_n(\xi, t, p)\| &\leq \beta \|Rw_n(\xi, t, p)\| \\
  &\leq \beta^2 \|Rw_{n-1}(\xi, t, p)\|.
\end{align*}
\]

Continuing in the same way, one obtains

\[
\|v_{n+1}(\xi, t, p) - v_n(\xi, t, p)\| \leq \beta^n \|Rw_0(\xi, t, p)\|.
\]

Clearly, \(\beta^n \to 0\) as \(n \to \infty\). Thus,

\[
\|v_{n+1}(\xi, t, p) - v_n(\xi, t, p)\| \to 0.
\]

For every \(n \geq i\), it holds that

\[
\|v_n - v_i\| = \|(v_n - v_{n-1}) + (v_{n-1} - v_{n-2}) + \cdots + (v_{i+1} - v_i)\|.
\]

which implies

\[
\|v_n - v_i\| \leq \|Rw_{n-1}\| + \|Rw_{n-2}\| + \cdots + \|Rw_i\|,
\]

and later

\[
\begin{align*}
  \|v_n - v_i\| &\leq \beta^{n-1} \|Rw_0\| + \beta^{n-2} \|Rw_{n-2}\| + \cdots + \beta \|Rw_0\| \\
  &\leq \left(\beta^{n-1} + \beta^{n-2} + \cdots + \beta^n\right) \|Rw_0\| \\
  &= \frac{1 - \beta^{n-i}}{1 - \beta} \|Rw_0\|.
\end{align*}
\]

As \(0 < \beta < 1\), we get

\[
\lim_{n \to \infty} \|v_n - v_i\| = 0,
\]

which shows that the sequence \(\{v_n\}_{n=0}^{\infty}\) is a Cauchy sequence in a Hilbert space \(H\), which means that

\[
\mathcal{I}(\xi, t) = \lim_{n \to \infty} v_n(\xi, t, p) = w_0(\xi, t) + \sum_{n=1}^{\infty} w_n(\xi, t, p),
\]

converges.

**Lemma 1.** Let \(Q\) be a function from a Hilbert space \(H\) to \(H\), the operator \(L\) required in (3) be defined as \(L = \partial^2 / \partial t^2\), \(i = 1, 2\), and the Lagrange multiplier be defined optimally by the variation theory; then,

\[
\begin{align*}
  \left\{ L \int_0^r \lambda(\theta)Q(\xi, \theta, p)d\theta \right\} &= -Q(\xi, \theta, p).
\end{align*}
\]
Proof. Consider the linear operator $L$ used in (3) defined as $L = \partial^2/\partial t^2$ and let $\lambda(\vartheta) = 1$. Then,

$$
\left\{ L \int_0^t \lambda(\vartheta) Q(\xi, \vartheta, p) d\vartheta \right\} = \frac{\partial}{\partial t} \int_0^t \lambda(\vartheta) Q(\xi, \vartheta, p) d\vartheta
$$

$$
= -Q(\xi, \vartheta, p).
$$

(29)

Similarly, let the operator $L$ required in (3) be defined as $L = \partial^2/\partial t^2$, and let $\lambda(\vartheta) = \vartheta - t$. Then,

$$
\left\{ L \int_0^t \lambda(\vartheta) Q(\xi, \vartheta, p) d\vartheta \right\} = \frac{\partial^2}{\partial t^2} \int_0^t (\vartheta - t) Q(\xi, \vartheta, p) d\vartheta
$$

$$
= -Q(\xi, \vartheta, p)
$$

$$
= \frac{\partial}{\partial t} \left( \int_0^t \frac{\partial}{\partial \vartheta} (\vartheta - t) Q(\xi, \vartheta, p) d\vartheta + (\vartheta - t) Q(\xi, \vartheta, p) \big|_{\vartheta=t} \right)
$$

$$
= \frac{\partial}{\partial t} \int_0^t -Q(\xi, \vartheta, p) d\vartheta = -Q(\xi, \vartheta, p),
$$

(30)

which verifies the statements.

**Theorem 2.** Let the operator $L$ needed in (3) be defined as $L = \partial^2/\partial t^2$, $i = 1, 2$. If we have the series solution (15) defined as

$$
\mathcal{X}(\xi, t) = w_0(\xi, t) + \sum_{n=1}^{\infty} w_n(\xi, t, p),
$$

(31)

then $\mathcal{X}(\xi, t)$ is an exact solution to the nonlinear partial differential equation (3).

Proof. Assume that the series solution (31) converges. Then, it implies the existence of $\lim_{n \to \infty} w_n(\xi, t, p)$ and

$$
[w_0(\xi, t) - w_1(\xi, t, p)] + \sum_{j=1}^{n} [w_j(\xi, t, p) - w_{j+1}(\xi, t, p)]
$$

$$
= w_0(\xi, t) - w_{j+1}(\xi, t, p).
$$

(32)

Therefore,

$$
[w_0(\xi, t) - w_1(\xi, t, p)] + \sum_{j=1}^{\infty} [w_j(\xi, t, p) - w_{j+1}(\xi, t, p)]
$$

$$
= w_0(\xi, t) - \lim_{n \to \infty} w_{n+1}(\xi, t, p)
$$

$$
= w_0(\xi, t).
$$

(33)

One can conclude from equation (15) that

$$
\mathcal{X}(\xi, t) = w_0(\xi, t) + \sum_{n=1}^{\infty} w_n(\xi, t, p).
$$

(34)

Applying the operator $L$ on both the sides, we get

$$
L[w_0(\xi, t) - w_1(\xi, t, p)] + \sum_{j=1}^{n} L[w_j(\xi, t, p) - w_{j+1}(\xi, t, p)]
$$

$$
= L[w_0(\xi, t)] = 0.
$$

(35)

From Lemma 1 and definitions, it follows

$$
L[w_0(\xi, t) - w_1(\xi, t, p)] = L[w_0(\xi, t)] - L[Rv_0(\xi, t)].
$$

(36)

For simplicity, suppose

$$
M = L \mathcal{X}(\xi) + N \mathcal{X}(\xi) - k(\xi).
$$

(37)

It implies

$$
L[w_0(\xi, t) - w_1(\xi, t, p)] = -L \left[ p \int_0^t \lambda(\vartheta) Mv_0(\xi, t) d\vartheta \right]
$$

$$
= pMv_0(\xi, t),
$$

$$
L[w_1(\xi, t) - w_2(\xi, t, p)] = L[w_1(\xi, t)] - L[Rv_2(\xi, t)]
$$

$$
= L[Rv_0(\xi, t)] - L[Rv_1(\xi, t, p)]
$$

$$
= \left[ p \int_0^t \lambda(\vartheta) Mv_0(\xi, t) d\vartheta \right]
$$

$$
- \left[ p \int_0^t \lambda(\vartheta) Mv_1(\xi, t) d\vartheta \right]
$$

$$
= p[Rv_1(\xi, t, p) - Rv_0(\xi, t)].
$$

(38)

In a similar way, for $j \geq 2$,

$$
L[w_1(\xi, t, p) - w_{j+1}(\xi, t, p)] = p[Rv_j(\xi, t, p) - Rv_{j-1}(\xi, t, p)].
$$

(39)

Subsequently, it follows

$$
L[w_0(\xi, t) - w_1(\xi, t, p)] + \sum_{j=1}^{n} L[w_j(\xi, t, p) - w_{j+1}(\xi, t, p)]
$$

$$
+ \sum_{j=1}^{n} L[w_j(\xi, t, p) - w_{j+1}(\xi, t, p)]
$$

$$
= pMv_0(\xi, t) + p[Rv_1(\xi, t, p) - Rv_0(\xi, t)]
$$

$$
+ p[Rv_n(\xi, t, p) - Rv_1(\xi, t, p)]
$$

$$
= pMv_n(\xi, t, p)
$$

$$
= pM \left[ w_0(\xi, t) + \sum_{j=1}^{n} w_j(\xi, t, p) \right].
$$

Hence,

$$
L[w_0(\xi, t) - w_1(\xi, t, p)] + \sum_{j=1}^{n} L[w_j(\xi, t, p) - w_{j+1}(\xi, t, p)]
$$

$$
= pM \left[ w_0(\xi, t) + \sum_{j=1}^{n} w_j(\xi, t, p) \right].
$$

(41)
Meanwhile, the unknown auxiliary parameter \( p \) is a nonzero optimal number, and it is revealed that \( \mathcal{Z}(\xi, t) = w_0(\xi, t) + \sum_{n=1}^{\infty} w_n(\xi, t, p) \) is an exact solution of the nonlinear partial differential equation (3), which was the required proof.

**Theorem 3.** Let us suppose that the solution \( \mathcal{Z}(\xi, t) = w_0(\xi, t) + \sum_{n=1}^{\infty} w_n(\xi, t, p) \), given in (15), converges to the exact solution of the model equation (1). Also, assume that if the approximate solution is the truncated series \( \mathcal{Z}_N(\xi, t) = w_0(\xi, t) + \sum_{n=1}^{N} w_n(\xi, t, p) \), then the maximum error norm can be assessed as

\[
\| \mathcal{Z}(\xi, t) - \mathcal{Z}_N(\xi, t) \| \leq \frac{1 - \beta^{N+1}}{1 - \beta}\| w_0 \|. 
\]  

**Proof.** Following Theorem 1, one obtains

\[
\| v_n - v_N \| \leq \frac{1 - \beta^{N+1}}{1 - \beta}\| w_0 \|.
\]

Since \( n \to 0 \), then \( v_n = \mathcal{Z}(\tilde{\xi}, t) \) and \( 0 < \beta < 1 \), and it follows

\[
\| \mathcal{Z}(\xi, t) - \mathcal{Z}_N(\xi, t) \| \leq \frac{1 - \beta^{N+1}}{1 - \beta}\| w_0 \|,
\]

which completes the proof.

Shortly, defining

\[
\beta_j = \begin{cases} 
\| w_{j+1} \|, & \| w_j \| \neq 0, \\
0, & \| w_j \| = 0,
\end{cases}
\]

\( j = 0, 1, 2, \ldots \)

At this point, if \( 0 < \beta_j < 1 \) for \( j = 0, 1, 2, \ldots \), then \( w_0(\xi, t) + \sum_{n=1}^{\infty} w_n(\xi, t, p) \) of (3) converges to the exact solution \( \mathcal{Z}(\xi, t) \), whereas the maximum absolute error is equal to

\[
\| \mathcal{Z}(\xi, t) - \mathcal{Z}_N(\xi, t) \| \leq \frac{1 - \beta^{N+1}}{1 - \beta}\| w_0 \|.
\]

**4. Numerical Examples**

This section is dedicated to the numerical application of the proposed algorithm (MVIA-II) for different types of diffusion equations. Here, the approximate solutions to the diffusion as well as convection-diffusion equations are obtained effortlessly and smartly without any use of transformation or linearization. We assess the validity, efficiency, and accuracy of MVIA-II by solving different types of diffusion equations. The obtained approximate results are very significant and encouraging.

**4.1. Test Problem 1.** Consider the following nonlinear diffusion equation for the slow diffusion process [19]:

\[
\frac{\partial \mathcal{Z}}{\partial t} = \frac{\partial}{\partial \xi} \left( \mathcal{Z}^2 \frac{\partial \mathcal{Z}}{\partial \xi} \right), \quad 0 < \xi < 1, t > 0,
\]

with the initial condition:

\[
\mathcal{Z}(\xi, 0) = \frac{\xi + a}{2c},
\]

where \( a \) and \( c \) are the arbitrary coefficients. The exact solution of equation (48) is

\[
\mathcal{Z}(\xi, t) = \frac{\xi + a}{2\sqrt{c^2 - t}}, \quad t < c^2.
\]

For our start, we solve the above system of the diffusion equation for the slow diffusion process by MVIA-II. The correction functional of MVIA-II for equation (48) is

\[
\mathcal{Z}_{n+1}(\xi, t, p) = \mathcal{Z}_n(\xi, t, p) + p \int_0^t \frac{\partial}{\partial \xi} \left( \mathcal{Z}_n(\xi, \theta, p) \frac{\partial \mathcal{L}(\xi, \theta, p)}{\partial \xi} \right) d\xi.
\]

The value of \( \lambda(\xi) \) can be found with the help of the variational principle [47]. We find the value \( \lambda(\xi) = -1 \) of \( \lambda(\xi) \). Usage of the obtained value of \( \lambda(\xi) \) in (51) gives the below iterative formula:

\[
\mathcal{Z}_{n+1}(\xi, t, p) = \mathcal{Z}_n(\xi, t, p) - \int_0^t \frac{\partial \mathcal{L}(\xi, \theta, p)}{\partial t} \left( \mathcal{Z}_n(\xi, \theta, p) \frac{\partial \mathcal{L}(\xi, \theta, p)}{\partial \xi} \right) d\xi.
\]

A more summarizing and concise iteration formula can be constructed using the iterations (11):

\[
\mathcal{Z}_{n+1}(\xi, t, p) = \mathcal{Z}_0(\xi, t, p) - \int_0^t \frac{\partial}{\partial \xi} \left( \mathcal{Z}_n(\xi, \theta, p)^2 \right) \frac{\partial \mathcal{L}(\xi, \theta, p)}{\partial \xi} d\xi.
\]

Starting with a proper initial approximation,

\[
\mathcal{Z}_0(\xi, t) = \frac{\xi + a}{2c}.
\]

Other iterations can be obtained by utilizing the iteration formula (53), by supposing \( a = c = 1 \), as
\[ \mathcal{I}_1(\xi, t, p) = \frac{(pt + 2)(\xi + 1)}{4}, \]
\[ \mathcal{I}_2(\xi, t, p) = \frac{\xi}{2} + \frac{pt(pt + 4)(\xi + 1)(p^2t^2 + 4pt + 8)}{128} + \frac{1}{2}. \]

The threshold value of \( n \) for this case is \( n = 4 \). To find a proper value of \( p \) for \( \mathcal{I}_4(\xi, t, p) \), the following residual function is defined:

\[ r_4(\xi, t, p) = \frac{\partial \mathcal{I}_4(\xi, t, p)}{\partial t} - \frac{\partial}{\partial \xi} \left( \frac{1}{\mathcal{I}_4(\xi, t, p)} \frac{\partial \mathcal{I}_4(\xi, t, p)}{\partial \xi} \right). \]

The 2-norm of the residual function (56) in the 4th iteration with respect to \( p \) for \( (\xi, t) \in [0, 1] \times [0, 1] \) is

\[ e_4(p) = \left[ \frac{1}{(11)^2} \sum_{i=0}^{10} \sum_{j=0}^{10} \left| r_4\left( i, \frac{j}{10}, -100 \right) \right|^2 \right]^{1/2}. \]  

The residual function (56) can be used to approximate \( e_4(p) \), while the optimal value of \( p \) can be determined by minimizing \( e_4(p) \). The value of \( p \) is equal to 1.000000601028068 when the minimum value 3.21205524320124664\( \times 10^{-6} \) of \( e_4(p) \) is reached. In this example, we choose a good initial approximation, and the value of \( p \) is near 1, showing the initial approximation is of great importance for a fast convergence. Results are obtained with the usage of this optimal value of \( p \) in the space-time domain \( (\xi, t, p) \) in the Table 1 for various values of \( \xi \) and \( t \).

### Table 1: The comparison of results for Test Problem 4.1 for different values of \( t \) and \( \xi \)

| \( t \) | \( \xi \) | Exact solution | Approximate solution | CPU time in seconds |
|---|---|---|---|---|
| 0.01 | 0.5 | 0.552771 | 0.552771 | 0.050035 | 0.99 |
| 0.01 | 0.9 | 0.954786 | 0.954786 | 0.050035 | 0.99 |
| 0.1 | 0.5 | 0.579751 | 0.579751 | 0.088120 | 0.38 |
| 0.1 | 0.9 | 1.001388 | 1.001388 | 0.088120 | 0.38 |

The approximate solution for different times \( t = 0.01, t = 0.05, t = 0.10, t = 0.20, \) and \( t = 0.30 \) is visualized in Figure 1 for Test Problem 4.1.

Comparative analysis of the proposed MVIA-II with respect to CFD6 in terms of spatial convergence rate is given in Table 2. It can be seen from this table that the MVIA-II is more accurate than the CFD6 [19].

### 4.2. Test Problem 2.

Consider the convection-diffusion equation from [15] of the following form:

\[ \frac{\partial \mathcal{I}}{\partial t} + a(\xi) \frac{\partial \mathcal{I}}{\partial \xi} = b(\xi) \frac{\partial^2 \mathcal{I}}{\partial \xi^2} + f(\xi, t), \]

with initial/boundary conditions:

\[ \mathcal{I}(0, t) = 0, \mathcal{I}(0, 0) = 0, \mathcal{I}(1, t) = e^t. \]

The exact solution of equation (59) is

\[ \mathcal{I}(\xi, t) = e^t. \]

For our start, we solve the system of the convection-diffusion equation by the MVIA-II, and for comparison purposes, we take \( a(\xi) = -\xi/6, b(\xi) = \xi^2/12, \) and \( f(\xi, t) = 0. \) Constructing the correction functional of MVIA-II for equation (59) as

\[ \delta \mathcal{I}_{n+1}(\xi, t, p) = \delta \mathcal{I}_n(\xi, t, p) + \int_0^t \lambda(\xi) \left[ \frac{\partial \mathcal{I}_n(\xi, \theta, p)}{\partial \theta} + \frac{x}{6} \frac{\partial^2 \mathcal{I}_n(\xi, \theta, p)}{\partial \xi^2} \right] d\theta. \]

The value of \( \lambda(\xi) \) may be determined easily with the help of the variational theory:

\[ \delta \mathcal{I}_{n+1}(\xi, t, p) = \delta \mathcal{I}_n(\xi, t, p) + \delta p \int_0^t \lambda(\xi) \left[ \frac{\partial \mathcal{I}_n(\xi, \theta, p)}{\partial \theta} + \frac{x}{6} \frac{\partial^2 \mathcal{I}_n(\xi, \theta, p)}{\partial \xi^2} \right] d\theta. \]

Ignoring the restricted terms,
starting with a proper initial approximation, 
$$\mathcal{L}_0(\xi, t) = \xi^3.$$  
(67)

Other iterations can be obtained by utilizing the iteration formula (66):
$$\mathcal{L}_1(\xi, t, p) = \xi^3 (pt + 1),$$  
$$\mathcal{L}_2(\xi, t, p) = \xi^3 + \frac{pt^3 (pt + 2)}{6},$$  
$$\mathcal{L}_3(\xi, t, p) = \xi^3 + \frac{pt^3 (p^2 + 3pt + 6)}{6},$$  
$$\cdots.$$  

The threshold value of $n$ is $n = 10$ and is used to find a proper value of $p$ for $\mathcal{L}_n(\xi, t, p)$. The following residual function is defined:
$$r_{10}(\xi, t, p) = \frac{\partial \mathcal{L}_{10}(\xi, t, p)}{\partial t} - \frac{1}{\mathcal{L}_{10}(\xi, t, p)^2} \frac{\partial^2 \mathcal{L}_{10}(\xi, t, p)}{\partial \xi^2}. $$  
(69)

The 2-norm of residual function (69) for the 4th iteration with respect to $p$ for $(\xi, t) \in [0, 1] \times [0, 1]$ is
$$e_{10}(p) = \left[ \frac{1}{11} \sum_{i=0}^{10} \sum_{j=0}^{10} r_{10}(\xi, t, p) \right]^{1/2}. $$  
(70)

The residual function (69) is utilized to approximate $e_0(p)$ for finding the optimal value of $p$, which can be found by minimizing $e_0(p)$. The value of $p$ is found to be $1.00000002955466$ when the minimum value of $e_0(p)$ is $2.9866043846213265 \times 10^{-8}$. With the usage of this optimal
value of \( p \) in \( Z_{\xi, t, p} \) in the space-time domain \((\xi, t) \in [0, 1] \times [0, 1]\), the behavior of exact and present solutions can be perceived in Figures 2 and 3.

Further, to show how the numerical solution converges to an exact solution, we compute \( \beta_n \) values for this nonlinear diffusion problem:
\[ \beta_0 = \frac{[\mathcal{I}_1(\xi, t, p)]}{[\mathcal{I}_0(\xi, t, p)]} = 0.30001803084204 < 1, \]
\[ \beta_1 = \frac{[\mathcal{I}_2(\xi, t, p)]}{[\mathcal{I}_0(\xi, t, p)]} = 0.034615752701361 < 1, \]
\[ \beta_2 = \frac{[\mathcal{I}_3(\xi, t, p)]}{[\mathcal{I}_0(\xi, t, p)]} = 0.003345779402775 < 1, \]
\[ \beta_3 = \frac{[\mathcal{I}_4(\xi, t, p)]}{[\mathcal{I}_3(\xi, t, p)]} = 0.000250098190015 < 1, \]

which show that for \( n \geq 0 \), the values of \( \beta_n \) are less than one, which provide the proof as well that the MVIA-II is convergent.

In Table 3, we compare the results generated by the MVIA-II with the exact one and with the method reported in [48] for Test Problem 4.2. From this table, we can conclude that the proposed algorithm produces better results than the results of a technique based on Hermite interpolant multiscale functions given in [48].

The behavior of the exact and approximate solutions obtained by the MVIA-II is given in Figure 2. It can be revealed from this figure that the MVIA-II provides accurate and precise results. The absolute error for times \( t = 0.3, \ t = 0.6, \) and \( t = 0.9 \) is shown in Figure 3, and superb accuracy has been archived in this case as well.

4.3. Test Problem 3. Consider the convection-diffusion equation from [48] of the following form:

\[ \frac{\partial \mathcal{I}}{\partial t} + a(\xi) \frac{\partial \mathcal{I}}{\partial \xi} = b(\xi) \frac{\partial^2 \mathcal{I}}{\partial \xi^2} + f(\xi, t), \]  

with initial/boundary conditions:

\[ \mathcal{I}(\xi, 0) = e^{-\xi}, \mathcal{I}(0, t) = e^{-0.09t}, \mathcal{I}(1, t) = e^{-1-0.09t}. \]  

The exact solution of equation (72) is

\[ \mathcal{I}(\xi, t) = e^{-t-0.09t}. \]

For our start, we solve the convection-diffusion equation by the MVIA-I, and for comparison purposes, we use \( a(\xi) = -0.1, b(\xi) = 0.01 \), and \( f(\xi, t) = 0 \). Constructing the correction functional of MVIA-II for equation (72) as

\[ \mathcal{I}_{n+1}(\xi, t, p) = \mathcal{I}_n(\xi, t, p) + p \int_0^t \lambda(\xi) \]  

\[ \left\{ \frac{\partial \mathcal{I}_n(\xi, t, p)}{\partial t} - 0.1 \frac{\partial \mathcal{I}_n(\xi, t, p)}{\partial \xi} - 0.01 \frac{\partial^2 \mathcal{I}_n(\xi, t, p)}{\partial \xi^2} \right\} d\xi. \]

The value of \( \lambda(\xi) \) may be determined easily with the help of the variational theory:

\[ \delta \mathcal{I}_{n+1}(\xi, t, p) = \delta \mathcal{I}_n(\xi, t, p) + \delta p \int_0^t \lambda(\xi) \]

\[ \left\{ \frac{\partial \mathcal{I}_n(\xi, t, p)}{\partial t} - 0.1 \frac{\partial \mathcal{I}_n(\xi, t, p)}{\partial \xi} - 0.01 \frac{\partial^2 \mathcal{I}_n(\xi, t, p)}{\partial \xi^2} \right\} d\xi. \]

Ignoring the restricted terms, the stationary conditions are \( \lambda^p(\xi) = 0 \) and \( 1 + \lambda(\xi) = 0 \), from which we obtain the value \( \lambda(\xi) = -1 \), \( \lambda(\xi) = 0 \). Using this obtained value of \( \lambda(\xi) \), equation (75) gives the below iterative formula:

\[ \mathcal{I}_{n+1}(\xi, t, p) = \mathcal{I}_n(\xi, t, p) - \int_0^t \left\{ \frac{\partial \mathcal{I}_n(\xi, t, p)}{\partial t} - 0.1 \frac{\partial \mathcal{I}_n(\xi, t, p)}{\partial \xi} - 0.01 \frac{\partial^2 \mathcal{I}_n(\xi, t, p)}{\partial \xi^2} \right\} d\xi. \]

A more summarizing iteration can be constructed using the iterative formula (11):

\[ \mathcal{I}_{n+1}(\xi, t, p) = \mathcal{I}_0(\xi, t, p) - p \int_0^t \left\{ -0.1 \frac{\partial \mathcal{I}_n(\xi, t, p)}{\partial \xi} - 0.01 \frac{\partial^2 \mathcal{I}_n(\xi, t, p)}{\partial \xi^2} \right\} d\xi. \]

Starting with a proper initial approximation,

\[ \mathcal{I}_0(\xi, t) = e^{-\xi}. \]

The other iterations can be obtained by using the iterative scheme (78), as

\[ \mathcal{I}_1(\xi, t, p) = \frac{e^{-\xi}(9pt - 100)}{100}, \]

\[ \mathcal{I}_2(\xi, t, p) = \frac{e^{-\xi}(81p^2 t^2 + 1800pt + 20000)}{20000}, \]

\[ \cdots \]

The threshold value of \( n \) in this example is \( n = 5 \) and is used to find a proper value of \( p \) for \( \mathcal{I}_5(\xi, t, p) \). The following residual function is defined:

\[ r_6(\xi, t, p) = \frac{\partial \mathcal{I}_6(\xi, t, p)}{\partial t} - 0.1 \frac{\partial \mathcal{I}_6(\xi, t, p)}{\partial \xi} - 0.01 \frac{\partial^2 \mathcal{I}_6(\xi, t, p)}{\partial \xi^2}. \]

The 2-norm of residual function (81) for the 6th iteration with respect to \( p \) for \( (\xi, t) \in [0, 1] \times [0, 1] \) is equal to

\[ e_6(p) = \left[ \frac{1}{11} \sum_{i=0}^{10} \sum_{j=0}^{10} r_6(i, j, p) \right]^{1/2}. \]
The residual function (81) is used to approximate $e_6(p)$ for finding the optimal value of $p$, which can be found by minimizing $e_6(p)$. The value of $p$ is found to be 1 when the minimum value of $e_6(p)$ is $2.3804226378429283 \times 10^{-22}$. In this example, we choose a good initial approximation, and the value of $p$ is 1, showing that the initial approximation is of great importance for fast convergence. With the usage of this optimal value of $p$ in $Z_4(\xi, t, p)$ in the space-time domain $(\xi, t) \in [0, 1] \times [0, 1]$, the approximate solution is obtained. Further,
which shows that for \( n \geq 0 \), the values of \( \beta_n \) are less than one, which provide the proof as well that the proposed algorithm (MVIA-II) is convergent.

The numerical results of the MVIA-II for Test Problem 4.3 are reported in Table 4 along with the exact solution and the results in the recent literature. It can be noticed from the table that the MVIA-II gives more accurate results in comparison with the approaches given in [15, 48].

The accuracy of the suggested MVIA-II is also verified from Figure 4, whereas in Figure 5, we have shown the absolute error for various time instants \( t = 0.3 \), \( t = 0.5 \), and \( t = 1.0 \).

Figure 5 is the evidence of better accuracy of the proposed MVIA-II.

5. Conclusion

The primal purpose of this paper is to study convergence analysis of a modified variational iteration algorithm-II (MVIA-II) and its applications in physical and biological sciences. The numerical results and theoretical study of the convergence analysis point out that the proposed algorithm can solve nonlinear problems efficiently and accurately. The numerical, as well as graphical results, generated by the MVIA-II are compared with the results of the compact finite difference method, the second kind Chebyshev wavelets procedure, and a procedure based on Hermite interpolant multiscaling functions, which revealed that the MVIA-II is of high accuracy and yields accurate results. This modified algorithm facilitates computational work for solving linear as well as nonlinear problems arising in engineering and applied sciences. High-accuracy solutions can be achieved in a few iterations of the proposed algorithm compared to earlier methods reported in the literature. We hope that the achieved results will be useful for further studies in scientific research, especially in physical and biological sciences.

Data Availability

Data will be provided on request to the first author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.
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