DIMENSION OF AUTOMORPHISMS WITH FIXED DEGREE FOR POLYNOMIAL ALGEBRAS

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Abstract. Let $K[x, y]$ be the polynomial algebra in two variables over an algebraically closed field $K$. We generalize to the case of any characteristic the result of Furter that over a field of characteristic zero the set of automorphisms $(f, g)$ of $K[x, y]$ such that $\max\{\deg(f), \deg(g)\} = n \geq 2$ is constructible with dimension $n + 6$. The same result holds for the automorphisms of the free associative algebra $K \langle x, y \rangle$. We have also obtained analogues for free algebras with two generators in Nielsen–Schreier varieties of algebras.

Introduction

Our paper is inspired by the following problem. Let $K$ be an arbitrary field of any characteristic and let $K[x, y]$ be the polynomial algebra in two variables. How many $K$-automorphisms $\varphi = (f, g)$ of $K[x, y]$ satisfying $\deg(\varphi) := \max\{\deg(f), \deg(g)\} = n$? Here $\varphi = (f, g)$ means that $f = \varphi(x)$, $g = \varphi(y)$.

In the sequel all automorphisms are $K$-automorphisms. Motivated by the problem of Arnaud Bodin [B] to determine the number of automorphisms $\varphi$ with $\deg(\varphi) \leq n$ over the finite field $\mathbb{F}_q$ with $q$ elements, in our recent paper [DY] we determined the number $p_n$ of $\mathbb{F}_q$-automorphisms of degree $n$ and found the Dirichlet series generating function of the sequence $p_n$, $n \geq 1$.

When the field $K$ is infinite, the natural translation of the problem is in the language of algebraic geometry. When the field $K$ is algebraically closed and of characteristic 0 this was considered by Bass, Connell and Wright [BCW] as a possible approach to the Jacobian conjecture. Identifying the endomorphisms $\varphi = (f_1, \ldots, f_m)$ of degree $\leq n$ of $K[X] = K[x_1, \ldots, x_m]$ with points of $K^N$ where $N = m(m+n)$ is

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m times the number of monomials of degree $\leq n$ in $K[X]$, the set $J_{1,n}$ of all endomorphisms of degree $\leq n$ with Jacobian $J(f_1, \ldots, f_m) = 1$ is a closed subvariety of $K^N$. The subset of automorphisms $A_{1,n}$ in $J_{1,n}$ is also a closed subvariety of $K^N$. In [BCW] it is proved that the Jacobian conjecture would follow if $J_{1,n}$ is irreducible and $\dim(A_{1,n}) = \dim(J_{1,n})$. Furter [F] studied in detail the case of two variables. Corollary 1.6 of [BCW] gives that the set $A_n$ of all automorphisms of degree $\leq n$ is an algebraic variety. Furter showed that for $K[x,y]$ the variety $A_n$ is of dimension $n + 6$ for any $n \geq 2$ (and of dimension 6 for $n = 1$). He also proved that unfortunately, the variety $A_n$ is irreducible for $n \leq 3$ only, that means the approach for the Jacobian conjecture suggested by [BCW] does not work. Applying the result of Moh [M] that the Jacobian conjecture is true for all endomorphisms of $K[x,y]$ with invertible Jacobian and of degree $n \leq 100$, Furter obtained that $J_n$ is reducible for all $n = 4, \ldots, 100$. His proof uses the theorem of Jung – van der Kulk [J, K] that the automorphisms of the polynomial algebra $K[x,y]$ over any field $K$ are tame and the structure of the automorphism group

$$\text{Aut}(K[x,y]) = A \ast_C B, \quad C = A \cap B,$$

where $A \ast_C B$ is the free product of the subgroup $A$ of affine automorphisms and and the subgroup $B$ of triangular automorphisms with amalgamated subgroup $C = A \cap B$.

Our first theorem transfers the result of Furter [F] on the dimension of $A_n$ to the case of any algebraically closed field of arbitrary characteristic. It requires a basic knowledge of algebraic geometry only. As in [F] and [DY] we use the theorem of Jung – van der Kulk and the decomposition $\text{Aut}(K[x,y]) = A \ast_C B$ and obtain that the set $A^{(n)}$ of all automorphisms $\varphi$ of $K[x,y]$ with $\deg(\varphi) = n \geq 2$ is a constructible subset of dimension $n + 6$ in $K^N$. As an immediate consequence we obtain that the dimension of the set $A_n$ of automorphisms of degree $\leq n$ is also equal to $n + 6$ for all $n \geq 2$. We think that, although using similar ideas, our proof is simpler than that of Furter [F].

We are also able to obtain that the dimension of the set of coordinates in $K[x,y]$ with a fixed degree $n$ ($n \geq 2$) is equal to $n + 3$.

By the theorem of Czerniakiewicz and Makar-Limanov [Cz, ML] for the tameness of the automorphisms of $K\langle x,y \rangle$ over any field $K$ and the isomorphism $\text{Aut}(K[x,y]) \cong \text{Aut}(K\langle x,y \rangle)$ which preserves the degree of the automorphisms we derive immediately that the dimension of the automorphisms of degree $n$ of the free associative algebra $K\langle x,y \rangle$ is also $n + 6$. Here $(f,g)$ is identified with a point of $K^p = 2(1 + 2 + 2^2 + \cdots + 2^n) = 2^{n+2} - 2$, because $2^m$ is the dimension of the vector
space of homogeneous polynomials of degree $m$ in $K(x,y)$. In this case the commutator criterion of Dicks \cite{D} gives that $\varphi = (f,g)$ is an automorphism if and only if $[f,g] = fg - gf = a[xy]$ for some nonzero constant $a$.

As in \cite{DY} we have an analogue of our result on the automorphisms of $K[x,y]$ in the case of free Nielsen – Schreier algebras $F(x,y)$ with two generators, i.e., if the subalgebras of $F(x,y)$ are free in the same class of algebras. For the exact result we make the additional requirement that the dimension $c_n$ of homogeneous polynomials of degree $n$ in $F(x)$ satisfies $c_n \leq c_{n+1}$ for any $n \geq 1$. Examples of such algebras are the free nonassociative algebras and free commutative nonassociative algebras. The dimension of the set of automorphisms of degree $n$ of $F(x,y)$ is

$$d_n = c_1 + c_2 + \cdots + c_n + 4 + 2\varepsilon,$$

where $\varepsilon = 1$ for unitary algebras and $\varepsilon = 0$ for nonunitary algebras. In the case of the free nonassociative algebra $K\{x,y\}$ we have obtained the generating function (see, for instance, \cite{W}) of the sequence $d_n$:

$$d(t) = \sum_{n \geq 1} d_n t^n = \frac{1 + 4(2 + \varepsilon)t - \sqrt{1 - 4t}}{2(1-t)}.$$

1. Dimension of automorphisms of polynomial algebras

Using that the automorphisms of $K[x,y]$ are tame and the structure of $\text{Aut}(K[x,y]) = A \ast_C B$ of the automorphism group of $K[x,y]$ it is easy to obtain the following well known fact, see e.g. \cite{W} or \cite{F}. A simple proof is given in \cite{DY}. Below we write the automorphisms as functions. If $\varphi = (f_1(x,y), g_1(x,y)), \psi = (f_2(x,y), g_2(x,y))$, then $\varphi \circ \psi(u) = \varphi(\psi(u)), u \in K[x,y]$, and hence

$$\varphi \circ \psi = (f_2(f_1(x,y), g_1(x,y)), g_2(f_1(x,y), g_1(x,y))).$$

**Proposition 1.1.** Define the sets of automorphisms of $K[x,y]$

$$A_0 = \{\iota = (x,y), \alpha = (y, x + ay) \mid a \in K\},$$

$$B_0 = \{\beta = (x + h(y), y) \mid h(y) \in y^2 K[y]\}.$$

Every automorphism $\varphi$ of $K[x,y]$ can be presented in a unique way as a composition

$$\varphi = (f,g) = \alpha_1 \circ \beta_1 \circ \alpha_2 \circ \beta_2 \circ \cdots \circ \alpha_k \circ \beta_k \circ \lambda,$$

where $\alpha_i \in A_0, \alpha_2, \ldots, \alpha_k \neq \iota, \beta_i \in B_0, \beta_1, \ldots, \beta_k \neq \iota, \lambda \in A$. If $\beta_i = (x + h_i(y), y)$ and $\deg(h_i(y)) = n_i$, then the degree of $\varphi$

$$n = \deg(\varphi) = \max\{\deg(f), \deg(g)\} = n_1 \cdots n_k.$$
is equal to the product of the degrees of $\beta_i$.

Now we assume that the affine spaces $K^N$ are equipped with the Zariski topology. Recall that the set $\mathcal{W} \subseteq K^N$ is constructible if it is a finite union of intersections $\mathcal{U}_i \cap \mathcal{V}_i$, where $\mathcal{U}_i$ is open and $\mathcal{V}_i$ is closed in the Zariski topology. We order the monomials of degree $\leq n$ in $K[x, y]$: $u_1, \ldots, u_{N/2}$, $N = (n + 1)(n + 2)$, and identify the endomorphism $\varphi = (f, g)$ of $K[x, y]$ of degree $\leq n$,

$$f = \sum_{j=1}^{N/2} a_j u^j, \quad g = \sum_{j=1}^{N/2} b_j u^j, \quad a_j, b_j \in K,$$

with the point

$$\pi(\varphi) = (a_1, \ldots, a_{N/2}, b_1, \ldots, b_{N/2}) \in K^N.$$

Clearly, the map $\pi$ is a bijection of the set of endomorphisms of degree $\leq n$ onto $K^N$.

The following theorem is a generalization of the main results of Furter [F], who proved the zero characteristic case.

**Theorem 1.2.** Let $\mathcal{A}^{(n)}$ be the set of all automorphisms of degree $n$ of $K[x, y]$. Then the set $\pi(\mathcal{A}^{(n)})$ is a constructible subset of $K^N$ of dimension $d_n = n + 6$, $n \geq 2$.

**Proof.** Let $n \geq 2$. We identify the affine endomorphism

$$\lambda = (a_1 x + b_1 y + c_1, a_2 x + b_2 y + c_2)$$

with the point $\pi(\lambda) = (a_1, b_1, c_1, a_2, b_2, c_2) \in K^6$, the automorphism

$$\alpha = (y, x + ay) \in A_0$$

with the point $\pi_0(\alpha) = a \in K$, and the triangular automorphism

$$\beta = (x + h(y), y), \quad h(y) = h_m y^m + h_{m-1} y^{m-1} + \cdots + h_2 y^2,$$

with the point $\pi_m(\beta) = (h_m, h_{m-1}, \ldots, h_2) \in K^{m-1}$.

We fix an ordered factorization $n = n_1 \cdots n_k$, $n_i \geq 2$. We denote by $B(n_1, \ldots, n_k)$ the set of all automorphisms

$$\varphi = (f, g) = \alpha_1 \circ \beta_1 \circ \alpha_2 \circ \beta_2 \circ \cdots \circ \alpha_k \circ \beta_k \circ \lambda$$

with deg$(h_i) = n_i$, $\alpha_1 \neq \iota$ and by $C(n_1, \ldots, n_k)$ the corresponding set with $\alpha_1 = \iota$. We define a bijection $\vartheta$ between the endomorphisms

$$\varphi = \alpha_1 \circ \beta_1 \circ \alpha_2 \circ \beta_2 \circ \cdots \circ \alpha_k \circ \beta_k \circ \lambda,$$

$\alpha_1 \neq \iota$, $\lambda$ being an affine endomorphism, and the points

$$\vartheta(\varphi) = (\pi_0(\alpha_1), \ldots, \pi_0(\alpha_k), \pi_{n_1}(\beta_1), \ldots, \pi_{n_k}(\beta_k), \pi(\lambda)) \in K^M,$$

$$M = k + (n_1 - 1) + \cdots + (n_k - 1) + 6 = n_1 + \cdots + n_k + 6.$$
If the endomorphism \( \varphi \) satisfies \( \alpha_1 = \iota \), then we omit the first coordinate \( \pi_0(\alpha_1) \) and define a bijection \( \vartheta' \) between \( \varphi \) and the points of \( K^{M^{-1}} \). We define two mappings \( \xi : K^M \to K^N \) and \( \eta : K^{M^{-1}} \to K^N \) by

\[
\xi(u) = \pi(\vartheta^{-1}(u)), \quad u \in K^M, \\
\eta(v) = \pi(\vartheta^{-1}(v)), \quad v \in K^{M^{-1}}.
\]

(Although \( \vartheta, \vartheta', \xi \) and \( \eta \) depend on the factorization \( n = n_1 \cdots n_k \), we shall use the same notation for all factorizations.)

Clearly, the coordinates of the points \( \xi(u) \) and \( \eta(v) \) in \( K^N \) are polynomial functions of the coordinates of \( u \in K^M \) and \( v \in K^{M^{-1}} \), respectively. Hence \( \xi \) and \( \eta \) are morphisms. Since the sets \( \vartheta(\mathcal{B}(n_1, \ldots, n_k)) \) and \( \vartheta'(\mathcal{C}(n_1, \ldots, n_k)) \) are open subsets of \( K^M \) and \( K^{M^{-1}} \), respectively, and \( \vartheta(\mathcal{B}(n_1, \ldots, n_k)) \) and \( \vartheta'(\mathcal{C}(n_1, \ldots, n_k)) \) are constructible subsets of \( K^N \). Since the dimension of a subset of \( K^M \) and \( K^{M^{-1}} \) under a morphism cannot be bigger that the dimension of the set itself, we obtain that

\[
\dim(\pi(\mathcal{B}(n_1, \ldots, n_k))) \leq \dim(\vartheta(\mathcal{B}(n_1, \ldots, n_k))) = M, \\
\dim(\pi(\mathcal{C}(n_1, \ldots, n_k))) \leq \dim(\vartheta'(\mathcal{C}(n_1, \ldots, n_k))) = M - 1.
\]

The image \( \pi(\mathcal{A}(n)) \) of the set of automorphisms of degree \( n \) is the union of \( \xi(\mathcal{B}(n_1, \ldots, n_k)) \) and \( \eta(\mathcal{C}(n_1, \ldots, n_k)) \) on all ordered factorizations \( n = n_1 \cdots n_k, n_i \geq 2 \). Hence

\[
\dim(\pi(\mathcal{A}(n))) = \max\{\dim(\xi(\mathcal{B}(n_1, \ldots, n_k))), \dim(\eta(\mathcal{C}(n_1, \ldots, n_k)))\}
\]

\[
\leq \max\{M = n_1 + \cdots + n_k + 6 \mid n_1 \cdots n_k = n\}.
\]

Using the inequality \( n_1 + n_2 \leq n_1 n_2 \) for \( n_1, n_2 \geq 2 \), we obtain that \( n_1 + \cdots + n_k \leq n_1 \cdots n_k = n \) and \( \dim(\pi(\mathcal{A}_n)) \leq n + 6 \). We shall complete the proof if we show that \( \dim(\pi(\mathcal{B}(n))) = n + 6 \).

The elements of \( \mathcal{B}(n) \) have the decomposition \( \varphi = (f, g) = \alpha \circ \beta \circ \lambda \), where

\[
\alpha = (y, x + ay), \quad a \in K, \\
\beta = (x + h_n y^n + \cdots + h_2 y^2, y), \quad h_i \in K, \quad h_n \neq 0, \\
\lambda = (a_1 x + b_1 y + c_1, a_2 x + b_2 y + c_2), \quad a_i, b_i, c_i \in K, \quad a_1 b_2 \neq a_2 b_1.
\]

Hence

\[
f = a_1(h_n x + ay)^n + \cdots + h_2(x + ay)^2 + a_1 y + b_1(x + ay) + c_1, \\
g = a_2(h_n x + ay)^n + \cdots + h_2(x + ay)^2 + a_2 y + b_2(x + ay) + c_2.
\]

Recall that the coordinates of \( K^N \) are the coefficients of the endomorphisms \( \varphi = (f, g) \) of degree \( \leq n \) and the coordinates of \( K^M = K^{n+6} \).
are determined by the automorphisms \( \alpha \) and \( \beta \) and the affine endomorphism \( \lambda \). Let
\[
\varphi(\varphi) = (a, h_n, \ldots, h_2, a_1, b_1, c_1, a_2, b_2, c_2) \in K^{n+6}.
\]
Then \( \varphi \in \mathcal{B}(n) \) if and only if \( h_n \neq 0 \), \( a_1b_2 \neq a_2b_1 \). Considering the coefficients \( x^iy^j \) of the first polynomial \( f \), let \( p_i \) be the coefficient of \( x^i \), \( i = 0, 1, 2, \ldots, n \), \( q_1 \) be the coefficient of \( x^{n-1}y \) and \( q_2 \) be the coefficient of \( y \). For the second polynomial \( g \), we denote by \( r_1, r_2, r_3 \) the coefficients of \( x, y \) and the constant term. Hence, for the pair \( \varphi = (f, g) \) the corresponding coordinates are
\[
p_i = a_1h_i, \quad i = 2, \ldots, n, \quad p_1 = b_1, \quad p_0 = c_1,
\]
\[
q_1 = na_1h_n a, \quad q_2 = a_1 + b_1 a, \quad r_1 = b_2, \quad r_2 = a_2 + b_2 a, \quad r_3 = c_2.
\]
Consider the open subset \( U \) of \( K^{n+6} \) defined by the inequalities
\[
h_n \neq 0, \quad a_1b_2 \neq a_2b_1, \quad a_1 \neq 0, \quad b_2 \neq 0.
\]
Clearly, \( \varphi^{-1}(U) \subset \mathcal{B}(n) \). Let \( \mathcal{W} = \pi(\mathcal{B}(n)) \) be the image of \( \mathcal{B}(n) \) in \( K^{N} \) and let \( \mathcal{O}(\mathcal{W}) \) be the algebra of regular functions of \( \mathcal{W} \). The functions \( p_i, q_i, r_i \) are defined on the image \( \xi(U) \) of \( U \) and satisfy the following conditions there:
\[
a = \frac{q_i}{np_n}, \quad b_1 = p_1, \quad c_1 = p_0, \quad a_1 = q_2 - b_1 a = q_2 - \frac{p_1q_1}{np_n},
\]
\[
h_i = \frac{p_i}{a_1} = \frac{np_i p_n}{np_n q_2 - p_1 q_1}, \quad i = 2, \ldots, n,
\]
\[
b_2 = r_1, \quad c_2 = r_3, \quad a_2 = r_2 - b_2 a = r_2 - \frac{q_1 r_1}{np_n}.
\]
Since the coordinate functions \( a, h_n, \ldots, h_2, a_1, b_1, c_1, a_2, b_2, c_2 \) of \( \varphi(\varphi) \) are algebraically independent on \( U \) and \( \xi(U) \) contains an open subset, we obtain that the \( n + 6 \) functions \( p_0, p_1, \ldots, p_n, q_1, q_2, r_1, r_2, r_3 \) are also algebraically independent in \( \mathcal{O}(\mathcal{W}) \). Hence \( \dim(\pi(\mathcal{B}(n))) \geq n + 6 \) which completes the proof. \( \square \)

By the above result, we are also able to determine the dimension of the set of coordinates with fixed degree.

**Theorem 1.3.** Let \( C_n \) be the set of all coordinates of degree \( n \) of \( K[x, y] \). Then \( \dim(C_n) = n + 3 \) for \( n \geq 2 \) and \( \dim(C_1) = 3 \).

**Proof.** According to the well-known theorem of Jung-van der Kulk [J1, K], for a fixed coordinate \( f \in K[x, y] \) with \( \deg(f) > 1 \), two automorphisms \( (f, g) \) and \( (f, g_1) \) with \( \deg(g) < \deg(f) \) and \( \deg(g_1) < \deg(f) \) if and only if \( g_1 = cg + d \) where \( c \in K - \{0\}, d \in K \), hence \( \dim(C_n) = \dim(B_n) - 2 \), where \( B_n \) is the set of automorphisms
(f, g) of degree n with \( \deg(f) = n > \deg(g) \). We can now determine \( \dim(B_n) \) as follows: tracing back to the proof of Theorem 1.2 in the decomposition of \((f, g)\), \( \alpha_1 \) is always the identity automorphism as \( \deg(f) > \deg(g) \). Hence \( \dim(B_n) = \dim(\pi(A^{(n)})) - 1 = n + 5 \). Therefore \( \dim(C_n) = \dim(B_n) - 2 = n + 3 \).

When \( n = 1 \), obviously the set of coordinates \( C_1 := \{ax + by + c \mid a, b, c \in K, (a, b) \neq (0, 0)\} \) has dimension 3.

**Corollary 1.4.** The dimension of the set \( A_n \) of automorphisms of \( K[x, y] \) of degree \( \leq n \) is equal to \( n + 6 \) for all \( n \geq 2 \).

**Proof.** The corollary follows immediately from Theorem 1.2 because

\[
A_n = \bigcup_{i=1}^{n} A^{(i)}, \quad \dim(A_n) = \max\{\dim(A^{(i)}) \mid i = 1, \ldots, n\} = n + 6.
\]

\( \square \)

## 2. The case of free Nielsen – Schreier algebras

For a background on varieties of algebras with the Nielsen – Schreier property see e.g. [MSY]. We shall use that if such a variety is defined by a homogeneous (with respect to each variable) system of polynomial identities, then the automorphisms of the finitely generated free algebras are tame. We have the following analogue of Theorem 1.2:

**Theorem 2.1.** Let \( F(x, y) \) be the free \( K \)-algebra with two generators in a Nielsen – Schreier variety defined by a homogeneous system of polynomial identities. Let \( c_n \) be the dimension of all homogeneous polynomials \( u(x) \) in one variable of degree \( n \) in \( F(x, y) \) and assume that \( 1 = c_1 \leq c_2 \leq c_3 \leq \cdots \).

(i) The dimension of the set of automorphisms of degree \( n \) of \( F(x, y) \) is

\[
d_n = c_1 + c_2 + \cdots + c_n + 4 + 2\varepsilon,
\]

where \( \varepsilon = 1 \) for unitary algebras and \( \varepsilon = 0 \) for nonunitary algebras.

(ii) For the free nonassociative algebra \( K\{x, y\} \) the generating function of the sequence \( d_n \) is

\[
d(t) = \sum_{n \geq 1} d_n t^n = \frac{1 + 4(2 + \varepsilon)t - \sqrt{1 - 4t}}{2(1 - t)}.
\]
Proof. (i) We use the following well known property of free algebras in Nielsen – Schreier varieties defined by homogeneous polynomial identities. If several homogeneous elements in the free algebra are algebraically dependent, then one of them is a polynomial of the others. This implies that $\text{Aut}(F(x, y)) = A \ast_C B$, where $A$ is the affine group if we consider unitary algebras and the general linear group when we allow nonunitary algebras, $B$ is the group of triangular automorphisms and $C = A \cap B$. Hence we have an analogue of Proposition 1.1. Following the main steps of the proof of Theorem 1.2 we obtain that the polynomials $h_n(x)$ of degree $n$ without constant and linear term depend on $c_2 + \cdots + c_n$ coordinates. Hence for a fixed factorization $n = n_1 \cdots n_k$ the dimension of the affine space $K^M$ of the endomorphisms

$$\varphi = (f, g) = \alpha_1 \circ \beta_1 \circ \alpha_2 \circ \beta_2 \circ \cdots \circ \alpha_k \circ \beta_k \circ \lambda$$

is equal to

$$M = k + \sum_{i=1}^{k} (c_2 + \cdots + c_{n_i}) + 4 + 2\varepsilon = \sum_{i=1}^{k} (c_1 + c_2 + \cdots + c_{n_i}) + 4 + 2\varepsilon$$

because $c_1 = 1$. Hence the dimension $d_n$ of the automorphisms of degree $n$ satisfies

$$d_n \leq \max \{ M = \sum_{i=1}^{k} (c_1 + c_2 + \cdots + c_{n_i}) + 4 + 2\varepsilon \}.$$

Using the inequalities $c_1 \leq c_2 \leq c_3 \leq \cdots$ we obtain that

$$(c_1 + c_2 + \cdots + c_{n_1}) + (c_1 + c_2 + \cdots + c_{n_2}) \leq c_1 + c_2 + \cdots + c_{n_1+n_2}$$

and, by induction,

$$\sum_{i=1}^{k} (c_1 + c_2 + \cdots + c_{n_i}) \leq c_1 + c_2 + \cdots + c_n.$$

As in Theorem 1.2 we reach the equality for the set of automorphisms $\varphi = \alpha \circ \beta \circ \lambda$, where $\beta$ is a triangular automorphism of degree $n$.

(ii) For the free nonassociative algebra, see e.g. [H], the number $c_n$ is equal to the Catalan number

$$c_n = \frac{1}{n} \binom{2n - 2}{n - 1}, \quad n \geq 1,$$
the generating function of the Catalan numbers satisfies the quadratic equation $c^2(t) - c(t) - t = 0$ and has the presentation
\[ c(t) = \frac{1 - \sqrt{1 - 4t}}{2}. \]

To complete the proof we use that if the generating function of the sequence $a_1, a_2, \ldots$ is $a(t)$, then the generating function of the sequence
\[ b_n = a_1 + a_2 + \cdots + a_n, \quad n = 1, 2, \ldots, \]
has the form
\[ b(t) = \frac{a(t)}{1 - t}. \]

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