A TIMECOP’S CHASE AROUND THE TABLE

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ABSTRACT

We consider the cops and robbers game variant consisting of one cop and one robber on time-varying graphs (TVG). The considered TVGs are edge periodic graphs, i.e., for each edge, a binary string \( s_e \) determines in which time step the edge is present, namely the edge \( e \) is present in time step \( t \) if and only if the string \( s_e \) contains a 1 at position \( t \mod |s_e| \). This periodicity allows for a compact representation of an infinite TVG. We prove that even for very simple underlying graphs, i.e., directed and undirected cycles the problem whether a cop-winning strategy exists is NP-hard and \( W[1] \)-hard parameterized by the number of vertices. Our second main result are matching lower bounds for the ratio between the length of the underlying cycle and the least common multiple (lcm) of the lengths of binary strings describing edge-periodicities over which the graph is robber-winning. Our third main result improves the previously known EXPTIME upper bound for PERIODIC COP & ROBBER on general edge periodic graphs to PSPACE-membership.

1 Introduction

In general, a \textit{time-varying graph} (TVG) describes a graph that varies over time. For most applications, this variation is limited to the availability or weight of edges meaning that edges are only present at certain time-steps or the time needed to cross an edge changes over time. TVGs are of great interest in the context of graph games such as competitive diffusion games and Voronoi games \cite{2}. There are plenty of representations for TVGs in the literature which are not equivalent in general. For instance, in \cite{4} a TVG is defined as a tuple \( G = (V, E, T, \rho, \zeta) \) where \( V \) is a set of vertices, \( E \subseteq V \times V \times L \) is a set of labeled edges (with labels from a set \( L \) ), \( T \subseteq \mathbb{T} \) is the \textit{lifetime} of the graph, \( \mathbb{T} \) is the temporal domain and assumed to be \( \mathbb{N} \) for discrete systems and \( \mathbb{R}^+ \) for continuous systems, \( \rho: E \times T \to \{0, 1\} \) is the \textit{presence function} indicating whether an edge \( e \) is present in time step \( t \), and \( \zeta: E \times T \to \mathbb{T} \) is the \textit{latency function} indicating the time needed to cross edge \( e \) in time step \( t \). We call the graph \( G = (V, E) \) the \textit{underlying graph} of \( G \). The literature has not yet agreed on how the function \( \rho \) (and \( \zeta \)) are given in the input. This is of significant importance, if \( \rho \) exhibits periodicity with respect to single edges in the context of computational complexity. We say that a TVG belongs to the class of TVGs featuring \textit{periodicity of edges}, defined as class \( 8 \) in \cite{4}, if \( \forall e \in E, \forall t \in T, \forall k \in \mathbb{N}, \rho(e, t) = \rho(e, t + kp_e) \) for some \( p_e \in \mathbb{T} \), depending on \( e \), and the underlying graph \( G \) is connected. For these TVGs, the function \( \rho \) can be represented for each edge \( e \in E \) as a binary string of size \( p_e \), concatenating the values of \( \rho(e, t) \) for \( 0 \leq t < p_e \). Note that the period of the whole graph \( G \) is then the least common multiple (lcm for short) of all string lengths \( p_e \) describing edge periods. Therefore, the underlying graph \( G \) of \( G \) can have exponentially many different sub-graphs \( G_t \) representing a snapshot of \( G \) at time \( t \). This exponential blow-up is a huge challenge in determining the precise complexity of problems for TVGs featuring periodicity of edges as discussed in more detail in Section 5 and 6. Often,

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for general TVGs a representation containing all sub-graphs representing snapshots over the whole lifetime of the graph is chosen when the complexity of decision problems over TVGs are considered [1,13]. An approach to unify the representation of TVGs is given in [18], also including the existence of vertices being effected over time. This approach represents \( \rho(e,t) \) by enhancing an edge \( e = (u,v) \) with the departure time \( t_d \) at \( u \) and the arrival time \( t_a \) at \( v \), where \( t_a \) might be smaller than \( t_d \) in order to model periodicity. As for TVGs with periodicity of edges where \( \rho \) is represented as a binary string for each edge the periodicity of the TVG \( G \) might be exponential in its representation using the approach in [13] would also cause an exponential blow-up in the representation of \( G \), as a decrement of the time value could only be used after a whole period of the graph, rather than after the period of one edge. An other class of TVGs based on periodicity was considered in the field of robotics to model motion planning tasks if time dependent obstacles are present [17]. There, the availability of the vertices in the graph changes periodically and each edge needs a constant number of time steps to be crossed. An edge \( e = \{u,v\} \) is only present if in the time span needed to cross \( e \) both endpoints are \( u \) and \( v \) are still present. In [17] the periodic function describing the availability of a vertex and the function describing the time needed to cross an edge is represented by an on-line program and can hence handle values exponential in their representation. This is crucial in the PSPACE-hardness proof of the reachability problem for graphs in this class presented in [17]. There, the hardness is obtained by a reduction from the halting problem for linear space-bounded deterministic Turing machines where a configuration of the Turing machine is encoded in the time step. In the reduction, the periodicity of a single vertex as well as the time needed to cross an edge is of value exponential in the tape length-bound. Note that this representation of periodicity is exponentially more compact than in our setting and thus the result of [17] does not translate to our setting.

We will stick in the following to the model describing TVG featuring periodicity of edges where the function \( \rho(e,t) \) is represented as a binary string. For this representation, Erlebach and Spooner [8] introduced recently a variant of the famous cops and robber problem which is intensively studied for static graphs [3]. In the static setting, the game is played on a given graph and includes a single robber and a set of \( k \)-cops. The cops and robber occupy vertices and the cops choose their vertices first. Then, in each round the players alternate turns and the cops move first. Thereby, each cop can move to an adjacent vertex or pass and stay on her vertex. The same holds for the robber. We say that a graph is \( k \)-cop-winning, if there exists a strategy for the \( k \) cops in which they finally catch the robber, i.e., a cop occupies the same vertex as the robber. If the context is clear, we call a 1-cop-winning graph a cop-winning graph. If a graph is not cop-winning, we call it robber-winning. A special interest on the game of cops and robbers lied in characterizing graphs which are \( k \)-cop-winning. While for one cop, the cop-winning graphs where understood in 1978 and independently 1983 [16, 15] as those featuring a special kind of ordering on the vertex set, called a cop-win or eliminating order, the case for \( k \) cops was long open and solved in 2009 by exploiting a linear structure of a certain power of the graph [5].

In 2020 Erlebach and Spooner [8] connected the two discussed graph-theoretical topics of great interest by introducing a cops and robber game for edge periodic graphs. These are TVGs featuring periodicity of edges as discussed above. They gave an algorithm which determines if the given edge periodic graph is \( k \)-cop-winning, which runs in time \( O(\text{lcm} \cdot k \cdot |G|^{k+2}) \), where \( \text{lcm} \) is the least common multiple of all length of binary strings describing \( \rho(e,t) \) and \( G \) is the underlying graph. As \( \text{lcm} \) can be exponentially in the input size, they proposed the question of whether this problem is NP-hard. This question was answered positively for a one-cop version, also in 2020, by Morawietz, Rehs, and Weller [14] for TVGs of which the underlying graph has a constant size vertex cover or where two edges have to be removed to obtain a cycle. Moreover, they showed that the problem is \( W[1] \)-hard when parameterized by the size of the underlying graph \( G \) even in these restricted cases, that is, they showed that there is presumably no algorithm solving the problem in time \( f(|G|) \cdot n^{O(1)} \) for any computable function \( f \). In other words, the exponential growth of the running time of every algorithm solving the problem has to depend on the length of binary strings describing \( \rho(e,t) \).

**Our contribution.** In this work, we show, that the NP-hardness holds for even simpler classes of edge periodic graphs, namely for directed and undirected cycles. Moreover, we show that the \( W[1] \)-hardness when parameterized by \( |G| \) even holds for these restricted instances (Section 5). The quite restricted class of undirected cycles was also studied in [8] where an upper bound on the length of the cycle with respect to the \( \text{lcm} \) was given for which each edge periodic cycle is robber winning. To be more precise, for an edge periodic cycle on \( n \) vertices it holds that if \( n \geq 2 \cdot \ell \cdot \text{lcm} \), then the graph is robber winning. Here, \( \ell = 1 \) if \( \text{lcm} \) is at least two times the longest size of a binary string describing \( \rho(e,t) \) and \( \ell = 2 \), otherwise. For these upper bounds only non-matching lower bounds where given, i.e., a family of cop-winning edge periodic cycles with length \( 3 \cdot \text{lcm} \) for \( \ell = 2 \) and \( 1.5 \cdot \text{lcm} \) for \( \ell = 1 \) are given. These lower bounds left a gap of size \( 0.5 \cdot \ell \cdot \text{lcm} \). In this work, we show that the given upper bounds are tight by closing this gap by giving families of cop-winning edge periodic cycles of length \( 2 \cdot \ell \cdot \text{lcm} - 1 \) (Section 4). Finally, we improve the currently best \( \text{EXPTIME} \) upper bound shown in [8] for the problem, whether a given edge periodic graph is cop-winning to \( \text{PSPACE} \) (Section 5). We conclude with a discussion on the restricted class of directed edge periodic cycles indicating that due to the compact representation of the edge periodic graphs, which does not introduce
additional amounts of freedom, the standard complexity classes, such as NP and PSPACE might not be suitable to precisely characterize the complexity of this problem (Section 6).

2 Preliminaries

For a string $w = w_0 w_1 \ldots w_n$ with $w_i \in \{0, 1\}$, for $0 \leq i \leq n$, we denote with $w[i]$ the symbol $w_i$ at position $i$ in $w$. We write the concatenation of strings $u$ and $v$ as $u \cdot v$. For non-negative integers $i \leq j$ we denote with $[i, j]$ the interval of natural numbers $n$ with $i \leq n \leq j$.

An edge periodic (temporal) graph $G = (V, E, \tau)$ (see also [8]) consists of a graph $G = (V, E)$ (called the underlying graph) and a function $\tau : E \to (0, 1)^*$ where $\tau$ maps each edge $e$ to a sequence $\tau(e) \in [0, 1]^*$ such that $e$ exists in a time step $t \geq 0$ if and only if $\tau(e)[t] = 1$, where $\tau(e)[t] := \tau(e)[t \mod |\tau(e)|]$. For an edge $e$ and non-negative integers $i \leq j$ we inductively define $\tau(e)[i, j]^o := \tau(e)[i] \cdot \tau(e)[i+1, j]^o$ and $\tau(e)[i, j]^o = \tau(e)[j]^o$. If $\tau(e) = 1$, we call $e$ a 1-edge. Every edge $e$ exists in at least one time step, that is, for each edge $e$ there is some $t_e \in [0, |\tau(e)| - 1]$ with $\tau(e)[t_e] = 1$. We might abbreviate $i$ repetitions of the same symbol $\sigma$ in $\tau(e)$ as $\sigma^i$. Let $L_G = \{ |\tau(e)| \mid e \in E \}$ be the set of all edge periods of some edge periodic graph $G = (V, E, \tau)$ and let $\text{lcm}(L_G)$ be the least common multiple of all periods in $G$. We call an edge periodic graph $G$ with an underlying graph consisting of a single cycle an edge periodic cycle. We denote with $G(t)$ the sub-graph of $G$ present in time step $t$. We do not assume that $G$ is connected in any time step. We will discuss directed and undirected edge periodic graphs. If not stated otherwise, we assume an edge periodic graph to be undirected. We illustrate the notion of edge periodic cycles in Figure 1 showing an edge periodic cycle $G$ together with $G(t)$ for the first 5 time steps. We now define the considered cops and robbers variant on edge periodic graphs with one single cop. Here, first the cop chooses her start vertex in $G(0)$, then the robber chooses his start vertex in $G(0)$. Then, in each time step $t$, the cop and robber move to an adjacent vertex over an edge which is present in $G(t)$ or pass and stay on their vertex. Thereby, the cop moves first, followed by the robber. We say that the cop catches the robber, if there is some time step $t$ for which the cop and the robber are on the same vertex after the cop moved. If the cop has a strategy to catch the robber regardless which start vertex the robber chooses, we say that $G$ is cop-winning and call the strategy implemented by the cop a cop-winning strategy. If for all cop start vertices, there exists a start vertex and strategy for the robber to elude the cop indefinitely, we call $G$ robber-winning. The described game is a zero-sum game, i.e., $G$ is either cop-winning or robber-winning.

**PERIODIC COP & ROBBER**

**Input:** An edge periodic graph $G = (V, E, \tau)$.

**Question:** Is $G$ cop-winning?

3 It’s hard to run around a table

In this section, we show that the NP-hardness of PERIODIC COP & ROBBER already holds if the input graphs are very restricted. More precisely, we show that PERIODIC COP & ROBBER is NP-hard and W[1]-hard when parameterized by $|G|$, even for directed and undirected edge periodic cycles $G$.

**Theorem 1.** PERIODIC COP & ROBBER on directed or undirected edge periodic cycles is NP-hard, and W[1]-hard parameterized by the size of the underlying graph $G$.

Both, the undirected and directed case of Theorem 1 is shown by a reduction of the PERIODIC CHARACTER ALIGNMENT problem which was shown to be both NP-hard and W[1]-hard when parameterized by $|X|$ in [14].

**PERIODIC CHARACTER ALIGNMENT**

**Input:** A finite set $X \subseteq \{0, 1\}^*$ of binary strings.
Next, we show that starting from a yes-instance of Periodic Character Alignment instance with set of strings $X = \{x_1, \ldots, x_m\}$ in the proof of Lemma 1, for $x_j \in X$ the edge labels are defined as $\xi(x_j) := \xi(x_j[0] \cdot \xi(x_j[1]) \cdot \ldots \cdot \xi(x_j[x_j - 1] \cdot 1)$, with $\xi(c) := 0^{m0}01^{m0}$ for $c \in \{0, 1\}$. The upper chain corresponds to the vertices $r_j$ and the lower chain to the vertices $\ell_j$.

**Question:** Is there a position $i$, such that $x[i]^\circ = 1$ for all $x \in X$, where $x[i]^\circ := x[i \mod |x|]$?

We begin with considering the case of undirected edge periodic cycles and then proceed by adapting the obtained construction for directed edge periodic cycles.

**Lemma 1.** Periodic Cop & Robber on undirected edge periodic cycles is NP-hard and $W[1]$-hard parameterized by the size of the underlying graph $G$.

**Proof.** We first sketch the idea of the construction. It is helpful to consider Figure 2 in the following. We represent each string in $X$ by an edge label. The constructed cycle will consist of two chains connected by two special edges. In the first chain, the elements in $X$ are increasingly listed in some fixed order as individual edge labels each. In the second chain the same edge labels are listed decreasingly in the same order. This will allow the cop and the robber to occupy antipolar vertices with the same edge labels on incident edges. Hence, while the cop is on one chain and the robber on the other chain, the robber can mimic the movements of the cop. The two chains are connected by two special edges for which their edge labels are complementary in one position of the labels and identical in all other positions. This will allow the cop to switch between the chains in a certain time step while the robber is trapped on his chain. In this situation, the cop will be able to catch the robber if and only if there is a position $i$, such that $x[i]^\circ = 1$ for all $x \in X$, in which case all edges of the chains will be present for some period.

We now proceed with the formal proof. Let $X$ be an instance of Periodic Character Alignment. We describe how to construct an undirected edge periodic cycle $G = (V, E, \tau)$ of Periodic Cop & Robber, where $G$ is an undirected edge periodic cycle such that $X$ is a yes-instance of Periodic Character Alignment if and only if $G$ is a yes-instance of Periodic Cop & Robber.

Let $|X| = m$ and $\{x_1, \ldots, x_m\}$ be the elements of $X$. We set $V := \{\ell_j, r_j \mid 0 \leq j \leq m\}$ and $E := \{(\ell_j, r_j) \mid 1 \leq j \leq m\} \cup \{(\ell_0, r_m), (\ell_m, r_0)\}$. Next, we set $\tau((\ell_0, r_m)) := 10^m0^{10^m}$ and $\tau((\ell_m, r_0)) := 0^{10^m}0^{10^m}$. Let $\xi(c) := 0^{m0}01^{m0}$ for $c \in \{0, 1\}$. Finally, we set $\tau((\ell_j, \ell_{j+1})) := \tau((r_{j-1}, r_j)) := \xi(x_j[0]) \cdot \xi(x_j[1]) \cdot \ldots \cdot \xi(x_j[x_j - 1])$ for each $x_j \in X$. Note that the length of each edge label is divisible by $q := 2m + 2$. For $q \geq 0$, let $T_j := [q, q + r, r, r + 1]$ denote the $i$-th time block, that is, the $q$ consecutive time steps starting from $q$. Note that the $j$-th edge label limited to the $i$-th time block $\tau((\ell_j, \ell_{j+1}))[T_j]^\circ = \tau((r_{j-1}, r_j))[T_j]^\circ$ is exactly $\xi(x_j[i])$.

Next, we show that $X$ is a yes-instance of Periodic Character Alignment if and only if $G$ is a yes-instance of Periodic Cop & Robber.

$(\Rightarrow)$ Let $i$ be a position such that $x[i]^\circ = 1$ for all $x \in X$. We describe the winning strategy for the cop. She should choose the vertex $\ell_m$ as her start vertex and should never move until the beginning of the $i$-th time block $T_i$. Since $x[i]^\circ = 1$ for all $x \in X$, $\tau((\ell_j, \ell_{j+1}))[T_i] = \tau((r_{j-1}, r_j))[T_i] = \xi(1) = 0^{10^m}01^{m0}$. Consequently, in time step $t$ only the edge $(\ell_0, r_m)$ exists and in the following $m$ time steps, all edges except $(\ell_0, r_m)$ and $(\ell_m, r_0)$ exist.

If the robber is currently on some vertex $r_j$, then the cop should move to $r_m$ in time step $t$. Otherwise, the cop should stay on $\ell_0$ in this time step. By the fact that the edge $(\ell_m, r_0)$ does not exist in time step $t$, we obtain that, at the beginning of time step $t + 1$, both players are either on vertices labeled with $r$ or labeled with $\ell$. Since all edges of the two paths $(\ell_0, \ldots, \ell_m)$ and $(r_0, \ldots, r_m)$ exist in the time steps $[t + 1, t + m]$, the cop can catch the robber in at most $m$ time steps, since neither $(\ell_0, r_m)$ nor $(\ell_m, r_0)$ exists in any of the time steps $[t + 1, t + m]$. Consequently, $G$ is a yes-instance of Periodic Cop & Robber.

$(\Leftarrow)$ Suppose that $X$ is a no-instance of Periodic Character Alignment. We describe a winning strategy for the robber. In the following, we say that the vertex $\ell_j$ is the mirror vertex of $r_j$ and vice versa. Moreover, we say...
that the robber mirrors the move of the cop at some time step $t$, if the cop is on the mirror vertex of the robber at the beginning of time step $t$ and the robber moves to the mirror vertex of the vertex the cop ends on in time step $t$.

The start vertex of the robber should be the mirror vertex of the start vertex of the cop. If it is possible, then the robber should always mirror the moves of the cop.

Note that the only move the robber cannot mirror is if the cop traverses the edge $\{\ell_m, r_0\}$ at the beginning of some $i$-th time block.

We show that the robber has a strategy to end on the mirror vertex during the $i$-th time block and evade the cop until then.

Assume w.l.o.g., that the cop moves from $\ell_m$ to $r_0$ and, thus, the robber is currently on $r_m$. Since $X$ is a no-instance of PERIODIC CHARACTER ALIGNMENT, there is at least one $x_j \in X$ with $x_j[i] = 0$. Hence, $\tau(\{\ell_{j-1}, x_j\})[T_i] = \tau(\{j_{j-1}, r_j\})[T_i] = \xi(0) = 00^m1^m$. Consequently, the cop cannot catch the robber in the first $m + 1$ time steps of the $i$-th time block. Hence, the robber should stay on this vertex until the beginning of time step $q \cdot i + m + 1$.

If the cop moves from $r_0$ to $\ell_m$ in time step $q \cdot i + m + 1$, the robber is again on the mirror vertex of the cop and is able to mirror all of the cops moves in the remaining time steps of this time block. Otherwise, the cop stays on some vertex $r_n$. In this case, the robber should move to $\ell_0$. Since the edges $\{\ell_0, r_m\}$ and $\{\ell_m, r_0\}$ do not exist in the remaining time steps of this time block, the cop cannot catch the robber in this time block. Moreover, since all edges of the path $(\ell_0, \ldots, \ell_m)$ exist in the last $m$ time steps of the $i$-th time block, the robber can move along the path $(\ell_0, \ldots, \ell_m)$ and reach the mirror vertex of the cop in at most $m$ time steps. Consequently, we can show via induction, that the robber has an infinite evasive strategy and, thus, $G$ is a no-instance of PERIODIC COP & ROBBER.

Since PERIODIC CHARACTER ALIGNMENT is $W[1]$-hard when parameterized by $|X|$ and $|V| = |E| = 2 \cdot |X| + 2$, we obtain that PERIODIC COP & ROBBER is $W[1]$-hard when parameterized by the size of the underlying graph of $G$ even on undirected edge periodic cycles.

Next, we adapt the previous construction for directed edge periodic cycles. It is helpful to consider Figure 3 in the following. In the adaption, we only have one chain listing the elements of $X$. The end vertex of this chain is connected to a new vertex $s$ which is again connected to the start vertex of the chain. The edges incident with $s$ will act as the two edges connecting the two chains in the previous construction by delaying the robber, such that the cop can catch him if all edges corresponding to $X$ are present in some time period.

**Lemma 2.** PERIODIC COP & ROBBER on directed edge periodic cycles is NP-hard, and $W[1]$-hard parameterized by the size of the underlying graph.

**Proof.** Again, we reduce from PERIODIC CHARACTER ALIGNMENT. Let $X$ be an instance of PERIODIC CHARACTER ALIGNMENT. We describe how to construct an instance $G = (V, E, \tau)$ of PERIODIC COP & ROBBER, where $G$ is a directed edge periodic cycle. Let $|X| = m$ and $\{x_1, \ldots, x_m\}$ be the elements of $X$. We set $V := \{v_j | 0 \leq j \leq m\}\cup \{s\}$ and $E := \{(v_{j-1}, v_j) | 1 \leq j \leq m\}\cup \{(v_m, s), (s, v_0)\}$. Next, we set $\tau((v_m, s)) := 0^m1^m0^m$ and $\tau((s, v_0)) := 0^m0^m1^m$. Let $\xi(c) := c^m1^m1$ for all $c \in \{0, 1\}$. Finally, we set $\tau((v_{j-1}, v_j)) := \xi(x_j[0]) \cdot \xi(x_j[1]) \cdot \cdots \cdot \xi(x_j[x_j[i]-1])$ for each $x_j \in X$.

Note that the length of each edge label is divisible by $q := 2m + 2$. For $t \geq 0$, let $T_t := [q \cdot t, q \cdot (t + 1) - 1]$ denote the $t$-th time block, that is, the $q$ consecutive time steps starting from $q \cdot t$. Note that the $j$-th edge label limited to the $t$-th time block $\tau((v_{j-1}, v_j))[T_t]$ is exactly $\xi(x_j[t])$.

Next, we show that $X$ is a yes-instance of PERIODIC CHARACTER ALIGNMENT if and only if $G$ is a yes-instance of PERIODIC COP & ROBBER.

$(\Rightarrow)$ Let $t$ be a position such that $x[t] = 1$ for all $x \in X$. We describe the winning strategy for the cop. The cop should choose the vertex $v_0$ as her start vertex and should never move until the beginning $t^* := q \cdot t$ of the $t$-th time block. By construction and the fact that $x_i[t] = 1$ for each $x_i \in X$, $\tau((v_{i-1}, v_i))[T_t] = \xi(1) = 1^m1^m1$. Hence, the cop can move from vertex $v_i$ to vertex $v_{i+1}$ in time step $t^* + i$ for each $i \in [0, m - 1]$ and, thus, reach the vertex $v_m$ in time step $t^* + m - 1$. Moreover, the cop can then move to the vertex $s$ in time step $t^* + m$. By construction, $\tau((s, v_0))[t^* + j] = 0$ for each $j \in [0, m]$. Hence, the cop has a winning strategy since she started at vertex $v_0$ and moved over every vertex of $V$ while the robber was not able to traverse the edge $(s, v_0)$.

$(\Leftarrow)$ Suppose that for every position $t$, there is some $x_j \in X$ with $x_j[t] = 0$. We show that the robber has a winning strategy. For some time step, let $u_C$ and $u_R$ denote the vertex of the cop, respectively robber in this time step. We call the vertex $v_0$ safe for all vertices of $V \setminus \{v_0, s\}$, we call $v_m$ safe for $v_0$ and $s$, and we call $s$ safe for $v_0$. Let $u_C$ be the start vertex of the cop, then the robber should choose a vertex which is safe for $u_C$ as his start vertex.

**Claim 1.** Let $t^* = t \cdot q$ be the beginning of the $t$-th time block for some $t \geq 0$, let $u_C$ be the vertex of the cop at time step $t^*$ and $u_R$ be the vertex of the robber at time step $t^*$. If $u_R$ is safe for $u_C$, then the robber has a strategy such that
In order to prove Theorem 2, we give families of edge periodic cycles for i.e., the case that \( \max(G) = 3 \).

Theorem 2. For every \( k \geq 3 \) and \( \ell \in \{1, 2\} \), there exists a cop-winning edge periodic cycle \( G = (V, E, \tau) \) with \( \max(L_\ell) = k \) and \( n = 2 \cdot \ell \cdot \text{lcm}(L_\ell) - 1 \) vertices, where \( \text{lcm}(L_\ell) \geq 2k \) if \( \ell = 1 \), and \( \text{lcm}(L_2) = k \), otherwise.

In order to prove Theorem 2, we give families of edge periodic cycles for \( \ell = 1 \) and \( \ell = 2 \), each, beginning with \( \ell = 2 \), i.e., the case that \( \text{lcm}(L_\ell) > 2 \cdot \max(L_\ell) \).

Lemma 3. For every \( k \geq 2 \) there exists an edge periodic cycle \( G = (V, E, \tau) \) with \( \text{lcm}(L_\ell) = k = \max(L_\ell) \), and \( n = 4k - 1 \) vertices that is cop-winning.

For the next section, we will stick to edge periodic cycles and consider families of cop-winning undirected edge periodic cycles.

4 Sharp bounds on the length required to ensure robber-winning edge periodic cycles

In \([8]\), an upper bound on the cycle length of an edge periodic cycle in dependence of \( \text{lcm}(L_\ell) \), required to ensure an robber winning strategy, was given. Namely, for \( |V| = n \), the graph \( G \) is robber winning if \( n \geq 2 \cdot \ell \cdot \text{lcm}(L_\ell) \), where \( \ell = 1 \) if \( \text{lcm}(L_\ell) \geq \max(L_\ell) \), and \( \ell = 2 \), otherwise (\([8\), Theorem 3]). So far, these bounds where not sharp, as for instance, in \([8]\), the only lower bounds are given by cop winning strategies for families of edge periodic cycles with \( n = 1.5 \cdot \text{lcm}(L_\ell) \) for \( \ell = 1 \) (\([8\), Theorem 5]), and \( n = 3 \cdot \text{lcm}(L_\ell) \) for \( \ell = 2 \) and \( \max(L_\ell) = \text{lcm}(L_\ell) \) (\([8\), Theorem 4]). We show that both upper bounds ensuring a robber winning strategy are sharp by presenting infinite families of cop-winning edge periodic cycles with \( n = 2 \cdot \ell \cdot \text{lcm}(L_\ell) - 1 \) vertices.

Theorem 2. For every \( k \geq 3 \) and \( \ell \in \{1, 2\} \), there exists a cop-winning edge periodic cycle \( G = (V, E, \tau) \) with \( \max(L_\ell) = k \) and \( n = 2 \cdot \ell \cdot \text{lcm}(L_\ell) - 1 \) vertices, where \( \text{lcm}(L_\ell) \geq 2k \) if \( \ell = 1 \), and \( \text{lcm}(L_2) = k \), otherwise.
Figure 4: Cycle with \(4k-1\) vertices and \(\text{lcm}(L_G) = k\) with a cop winning strategy from the start vertex marked in red. Edges not drawn (depicted by dots) are 1-edges; for all other edges, \(\tau(e)\) is explicitly noted (with gray background). The clockwise [counterclockwise] distance of each vertex to the start vertex of the cop is given as a positive [negative] number.

| time step | pos. cop  | pos. robber |
|-----------|-----------|-------------|
| \(s\)     | 0         | 2k - 1      |
| \(k - 1\) | \(k\)     | 2k          |
| \(2k - 3\)| 2k - 2    | 3k - 2      |
| \(2k - 2\)| 2k - 1    | 3k - 2      |
| \(2k - 1\)| 2k        | 3k - 2      |
| \(3k - 3\)| 3k - 1    | 4k - 4      |
| \(3k + 1\)| 3k        | 0           |
| \(4k - 1\)| 4k - 2    | 0           |
| \(4k\)    | 0         |             |

Table 1: Time steps with corresponding positions of cop and robber in the edge periodic cycle depicted in Figure 4.

All positions are \(\text{after}\) moving in this time step. The time step \(s\) denotes the start configuration. Recall that the cop moves first.

**Proof.** Consider the edge periodic cycle \(G_k = (V, E, \tau)\) depicted in Figure 4 with \(|V| = 4k - 1\). This graph admits a cop winning strategy if the cop picks the highlighted vertex with index 0 as her start vertex. The vertices are indexed by positive numbers indicating their clockwise distance to the start vertex of the cop, and with negative numbers indicating their counterclockwise distance. Let the cop pick vertex 0. We consider the antipolar vertices \((2k - 1)\) and \(-(2k - 1)\) as potential start vertices of the robber. We show that if the robber picks vertex \((2k - 1)\), then the cop has a winning strategy by continuously running clockwise, starting in time step zero, and if the robber picks vertex \(-(2k - 1)\), the same applies running counterclockwise. Note that these two positions represent extrema and being able to catch the robber at these vertices implies being able to catch him at all vertices in the graph. Table 1 shows the positions of the cop and robber for these strategies for \(k \geq 4\). For each time step, the position after both players moved are depicted; \(s\) is the start configuration. We abbreviate consecutive 1-edges and only depict the time steps and positions when one of the players reaches a non-trivial edge. For the cases of \(k = 2\) and \(k = 3\) the cop catches the robber earlier than depicted in Table 1, namely in step \(t = 6\) clockwise and \(t = 8\) counterclockwise for \(k = 2\) and in step \(t = 6\) clockwise and \(t = 9\) counterclockwise for \(k = 3\) if the robber chooses the corresponding antipolar start vertices. Details on case \(k = 2\) and \(k = 3\), and a concrete example for \(k = 4\), can be found in the appendix.

For the case that \(\ell = 1\), i.e., when \(\text{lcm}(L_G) \geq 2 \cdot \max(L_G)\), we slightly adapt the family of graphs depicted in Figure 4. Note that for \(\max(L_G) = 2\) there is no edge periodic cycle \(G = (V, E, \tau)\) with \(\text{lcm}(L_G) > \max(L_G) = 2\).

**Lemma 4.** For every \(k \geq 3\) with \(k \neq 2^m\) for all \(m \in \mathbb{N}\), there exists an edge periodic cycle \(G = (V, E, \tau)\) with \(\text{lcm}(L_G) = 2 \cdot \max(L_G) = 2 \cdot k\), and \(n = 2 \cdot 2k - 1\) vertices that is cop-winning.

**Proof.** For the case \(\ell = 1\) we introduce an artificial edge label in the edge periodic cycle in Figure 4 such that the \(\text{lcm}(L_G)\) is exactly \(2k\). This edge will not affect the run of the cop. Its purpose is to introduce a factor 2 in the number of vertices compensating the missing factor 2 from the variable \(\ell\). Therefore, note that the edge \(e_{1,2}\) connecting vertex +1 and +2 is taken by the cop only once, in the clockwise run in time step 1 and in the counterclockwise run in time step \(4k - 3\). Hence, the cop only crosses the edge in an odd time step. We can write \(k = 2^t \cdot j\) where \(j\) is an
show that if the robber picks vertex 3 with a cop winning strategy if the cop picks the highlighted vertex with index 0 as her start vertex. The vertices are indexed starting in time step zero. Since for each edge

\[ \text{lcm}(L_G) = 3k \]

and \[ \text{max}(L_G) \] are divisible by \[ k \cdot \] for some \( m \in \mathbb{N} \), it holds that for the smallest possible value of \( \text{lcm}(L_G) \) with \( \text{lcm}(L_G) > \text{max}(L_G) \), we have \( \text{lcm}(L_G) \geq 3 \cdot \text{max}(L_G) \). Hence, in these cases we need a separate family of graphs.

**Lemma 5.** For every \( k = 2^m \) with \( m \geq 2 \), there exists an edge periodic cycle \( G = (V, E, \tau) \) with \( \text{lcm}(L_G) = 3 \cdot \text{max}(L_G) = 3 \cdot k \), and \( n = 6 \cdot k - 1 \) vertices that is cop-winning.

**Proof.** Consider the edge periodic cycle \( G_k = (V, E, \tau) \) depicted in Figure 5 with \( |V| = 6k - 1 \). This graph admits a cop winning strategy if the cop picks the highlighted vertex with index 0 as her start vertex. The vertices are indexed by positive numbers indicating their clockwise distance to the start vertex of the cop. Let the cop pick vertex 0. We show that if the robber picks vertex \( 3k - 1 \), then the cop has a winning strategy by continuously running clockwise, starting in time step zero. Since for each \( j \), starting from vertex 0, the label of the \( j \)-th edge clockwise is equal to the label of the \( j \)-th edge counterclockwise, the same applies running counterclockwise if the robber picks vertex \( 3k \). Note that these two positions represent extrema and being able to catch the robber at these vertices implies being able to catch him at all vertices in the graph. Suppose that the robber picks vertex \( 3k - 1 \). Since \( k = 2^m \) for some \( m \geq 2, \frac{k}{2} \cdot k \) and \( \frac{k}{2} \cdot k \) are divisible by 3. Hence for each \( j \in [1, 6k - 3] \), the cop can traverse the edge \( \{j, j + 1\} \) in time step \( j \) and, thus, reach the vertex \( 5k - 1 \) in time step \( 5k - 2 \). We show that, starting from vertex \( 3k - 1 \) and running clockwise, the robber cannot reach vertex \( 5k \) prior to time step \( 5k - 1 \). This then implies, that the cop catches the robber after at most \( 5k - 2 \) time steps. Note that the first time the robber can traverse the edge \( \{3k - 1, 3k\} \) is at time step \( k - 1 \). Hence, the robber reaches the vertex \( 3k + 2 \) not prior to time step \( k + 1 \). Since \( k \) is not divisible by 3, the robber cannot traverse the edge \( \{3k + 2, 3k + 3\} \) in time step \( k + 2 \). Thus, the robber cannot reach the vertex \( 4k - 1 \) prior to time step \( k + 4 \) and consequently, he cannot traverse the edge \( \{4k - 1, 4k\} \) prior to time step \( 3k - 1 \). Hence, the robber reaches the vertex \( \frac{7}{4}k - 1 \) not prior to time step \( \frac{7}{4}k - 2 \). Since \( k \) is not divisible by 3, the robber cannot traverse the edge \( \{\frac{7}{4}k - 1, \frac{7}{4}k\} \) in time step \( \frac{7}{4}k - 1 \). Thus, the robber cannot reach the vertex \( 5k - 1 \) prior to time step \( 4k \) and consequently, he cannot traverse the edge \( \{5k - 1, 5k\} \) prior to time step \( 5k - 1 \). Hence, the statement holds. A concrete example for \( k = 4 \) can be found in the appendix.

### 5 Complexity upper bounds

The main result of this section is that the **PERIODIC COP & ROBBER** problem for **general** edge periodic graphs can be solved in polynomial space. Note that two-player games may take exponentially many turns and hence containment in PSPACE is not obvious. In our case, already the period of graphs on which the game is played is exponential in general. This prohibits a standard incremental PSPACE algorithm approach. We show that despite the potentially exponential period of the sequence of graphs \( G(t) \) we show that we can determine whether the cop has a winning strategy by sweeping through the configuration space in a way that we only consider polynomially many vertices in each step. The fact that we only consider one cop and one robber is here crucial for the polynomial bound.

**Theorem 3.** **PERIODIC COP & ROBBER** for **edge periodic graphs** is contained in PSPACE.

For general edge periodic graphs, the **PERIODIC COP & ROBBER** problem was reduced in [8] to a variant of the **AND-OR GRAPH REACHABILITY** problem [10] via an exponential time reduction. The **AND-OR GRAPH REACHABILITY** problem is a two player game where players move a token in an AND-OR graph from a source to a target. An AND-OR graph is a graph \( G = (V, E) \) where the set of vertices is partitioned into a set of **AND** vertices \( V_A \) and a set of **OR** vertices \( V_O \). If the token is on an **OR** vertex, then player 0 moves the token, otherwise player 1 moves...
Clearly, the cop made no progress towards capturing the robber in the sequence π indistinguishable, removing the sequence a cop-winning configurations violating the assumed minimality of steps.

**Proof.**

In the following, we show how to use the structural properties of the configuration graph \( CG \) to solve the AND-OR GRAPH REACHABILITY problem in polynomial space. We begin with giving an upper bound on the length of a shortest chase in \( G \).

**Lemma 6.** Let \( G = (V, E, \tau) \) be an edge periodic graph. If \( G \) is cop-winning, then the robber can be caught within \( n^2 \cdot \text{lcm}(L_G) \) rounds.

**Proof.**

Consider the configuration graph \( CG = (V_{CG}, E_{CG}) \) of \( G \). Note that two configurations of a PERIODIC COP & ROBBER game on the same temporal snapshot graph with identical positions of the cop and robber in different time steps \( t \) and \( t' \) which differ by a multiple of \( \text{lcm}(L_G) \) are indistinguishable as configurations of the game. Hence, the size of \( CG \) is bounded by \( 2n^2 \cdot \text{lcm}(L_G) \).

Now let \( \pi \) be a shortest sequence of configurations for the start vertices \( v_c \) and \( v_r \) such that \( \pi \) ends with a cop-winning configuration. If \( |\pi| > 2n^2 \cdot \text{lcm}(L_G) \), then \( \pi \) contains two indistinguishable configurations \( \pi_i \) and \( \pi_{i+1} \). Clearly, the cop made no progress towards capturing the robber in the sequence \( \pi_i, \ldots, \pi_{i+1-1} \). As \( \pi_i \) and \( \pi_{i+1} \) are indistinguishable, removing the sequence \( \pi_i, \ldots, \pi_{i+1-1} \) from \( \pi \) yields a shorter sequence of configurations ending in a cop-winning configurations violating the assumed minimality of \( \pi \).

**Proof of Theorem 4.** We will follow the ideas from [3] of reducing the PERIODIC COP & ROBBER problem to a variant of the AND-OR GRAPH REACHABILITY problem. Therefore, we will need the notion of attractors in an AND-OR graph. Let \( G = (V, E) \) be an AND-OR graph with \( V = V_c \cup V_r \). Instead of considering a single target, we consider a set \( T \) of targets and say that player 0 wins if the token finally reaches any state in \( T \). Let \( s \in V \) be the start vertex. Intuitively, the set of attractors of \( G \) is the set of vertices from which player 0 can win the game. More formally, we inductively define the set of attractors \( \text{Attr} \) of \( T \) as:

\[
\text{Attr}^0 = T,
\]

\[
\text{Attr}^{i+1} = \text{Attr}^i \cup \{v \in V_c \mid \forall \{u, v\} \in E : u \in \text{Attr}^i \} \cup \{v \in V_r \mid \exists \{u, v\} \in E : u \in \text{Attr}^i \},
\]

\[
\text{Attr} = \bigcup_{i \geq 0} \text{Attr}^i.
\]

Let \( G = (V, E, \tau) \) be the input edge periodic graph and let \( CG = (V_{CG}, E_{CG}) \) be the configuration graph of \( G \). We will also identify \( CG \) as an AND-OR graph by declaring nodes \((u_c, u_r, s, t)\) of \( CG \) with \( s = c \) as \( OR \) vertices and nodes with \( s = r \) as \( AND \) vertices. We define the set of nodes \( T = \{(u_c, u_r, s, t) \in V_{CG} \mid u_c = u_r\} \) as the target set of the AND-OR graph \( CG \). We will now prove that the ideas from [3] of solving the PERIODIC COP & ROBBER game by checking if (i) there is some vertex \( u_c \) such that for all vertices \( u_r \in V_r \), the node \((u_c, u_r, c, t) \in V_{CG} \) is an attractor in the AND-OR graph \( CG \), can be implemented in polynomial space. Note that the set of attractors in \( CG \) corresponds to the set of configurations from which the cop has a winning strategy. We use Lemma 6 to unroll the configuration graph in order to obtain a leveled directed acyclic graph (DAG) which has width \( n^2 \) in each level and through which we can sweep level by level in order to verify property (i).

By Lemma 6, we know that in order to verify property (i) it is sufficient to consider paths of length at most \( 2n^2 \cdot \text{lcm}(L_G) \) in \( CG \) (the factor 2 is due to the alternation of players). As the configuration graph is cyclic (due to modulo counting
by \(\text{lcm}(L_G)\) we unroll the graph \(n^2\) times to allow for different time steps up to \(n^2 \cdot \text{lcm}(L_G)\). The obtain DAG is big enough to contain any shortest chase starting in any time step \(t \leq \text{lcm}(L_G)\). The so obtained graph \(C'_G\) consists of the node set \(V'_{C'_G} = V \times V \times \{c, r\} \times \{n^2 \cdot \text{lcm}(L_G)\}\) and the edge set \(E'_{C'_G}\) extending \(E_{C'_G}\) as \((u_c, u_r, s, t), (u'_c, u'_r, s', t')\) \(\in E'_{C'_G}\) if and only if \(t' \in \{t, t+1\}\), \((u_c, u_r, s, t \mod \text{lcm}(L_G)), (u'_c, u'_r, s', t' \mod \text{lcm}(L_G))\) \(\in E'_{C'_G}\) and \(t, t' \leq n^2 \cdot \text{lcm}(L_G) - 1\). Verifying property (i) then corresponds to a reachability game in the AND-OR graph associated with \(C'_G\) with target set \(T = \{(u_c, u_r, s, t) \in \hat{V}'_{C'_G} \mid u_c = u_r\}\) which can be solved using the notion of attractors. Note that in \(C'_G\) only nodes with identical time steps \(t\) and \(t + 1\) are connected. Hence, in order to compute which nodes with time step \(t\) belong to the set of attractors, we need to only know which nodes with time step \(t + 1\) are attractors. Since \(C'_G\) is a DAG we can start in the level with \(t = n^2 \cdot \text{lcm}(L_G) - 1\) of \(C'_G\).

\[
\text{Attr}_r^{n^2 \cdot \text{lcm}(L_G) - 1} := \{(u_c, u_r, r, n^2 \cdot \text{lcm}(L_G) - 1) \mid u_c = u_r\},
\]

\[
\text{Attr}_r^t := \{(u_c, u_r, r, t) \mid \forall ((u_c, u_r, r, t), (u'_c, u'_r, c, t + 1)) \in E'_{C'_G} : (u_c, u'_r, c, t + 1) \in \text{Attr}_r^{t+1}\} \cup \{(u_c, u_r, r, t) \mid u_c = u_r\}, \text{for } n^2 \cdot \text{lcm}(L_G) - 2 \geq t \geq 0,
\]

\[
\text{Attr}_c^t := \{(u_c, u_r, c, t) \mid \exists ((u_c, u_r, c, t), (u'_c, u'_r, r, t)) \in E'_{C'_G} : (u'_c, u'_r, r, t) \in \text{Attr}_r^t\} \cup \{(u_c, u_r, c, t) \mid u_c = u_r\}, \text{for } n^2 \cdot \text{lcm}(L_G) - 1 \geq t \geq 0.
\]

For each level \(n^2 \cdot \text{lcm}(L_G) - 1 \geq t \geq 0\) we only need to keep the last level \((t + 1)\) if existent\(^3\) of \(C'_G\) in memory in order to compute the sets \(\text{Attr}_r^t\) and \(\text{Attr}_r^{t+1}\) of nodes \((u_c, u_r, s, t)\) in \(C'_G\) from which the cop has a winning strategy where \(s\) equals the sub-script. Note that \(\bigcup_{0 \leq t \leq n^2 \cdot \text{lcm}(L_G) - 1} \text{Attr}_r^t \cup \text{Attr}_r^{t+1} = \text{Attr}\). In order to verify property (i) we only need to keep the current and latest sets \(\text{Attr}_r^t, \text{Attr}_r^t, \text{Attr}_r^{t+1}, \text{Attr}_r^{t+1}\) in memory yielding a polynomial space algorithm.

6 Discussion

While we improved the currently known upper bound for the PERIODIC COP & ROBBER problem on edge periodic graphs from EXPTIME to PSPACE and improved the lower bounds, of being NP-hard, to include also the very restrictive classes of directed and undirected edge periodic cycles, a gap in the complexity of PERIODIC COP & ROBBER remains. It is worth noticing that on one side the chosen representation of edge periodic graphs is quite compact, as a natural proof for a cop-winning strategy might be of exponential length in the input size, since the periodicity of the whole graph is the least common multiple of the periodicity of each edge, which prevents the use of a simple guess & check approach for NP-membership. On the other side, the chosen representation is still exponentially larger than the representation by on-line programs used in [17] where PSPACE-completeness for the reachability problem on a related but different class of periodic TVGs was obtained.

If we consider directed edge periodic cycles, then determining whether the given cycle is cop-winning boils down to deterministically simulating the chase starting from a (guessed) cop vertex and time step, as the optimal strategies for the cop and robber are both to keep running whenever possible (without bumping into the cop). For the robber the optimal start vertex is directly behind the cop. Since \(\text{lcm}(L_G)\) can be exponentially large in the size of \(G\) the only known upper bound on the number of steps in the simulation of the chase starting in time step \(t\) is exponential in the size of \(G\), while the chase does not reveal any complexity. The simulation could even be performed by a log-space Turing-Machine being equipped with a clock which allows for modulo queries of logarithmic size. To better understand the precise complexity of PERIODIC COP & ROBBER on directed edge periodic cycles, the theoretical analysis of potential families of cycles with shortest cop-winning strategies of exponential size would be of great interest and might indicate the necessity for a new complexity class consisting of simple simulation problems with exponential duration time.

References

[1] Sandeep Bhadra and Afonso Ferreira. Complexity of connected components in evolving graphs and the computation of multicast trees in dynamic networks. In Proceedings of the 2nd International Conference on Ad-Hoc, Mobile, and Wireless Networks, volume 2865 of Lecture Notes in Computer Science, pages 259–270. Springer, 2003.

\(^3\)Note that we can easily compute the snapshot \(G(n^2 \cdot \text{lcm}(L_G)) = G(0)\) by drawing all edges with \(\tau(e)[0]^n = 1\); and from \(G(t)\) for some time step \(t\), the snapshot \(G(t - 1)\) by shifting the pointer in each \(\tau(e)\) one step to the left. Therefore, we can compute from each level \(t + 1\) of \(C'_G\) the level \(t\) in polynomial time and space.
[2] Niclas Boehmer, Vincent Froese, Julia Henkel, Yvonne Lasars, Rolf Niedermeier, and Malte Renken. Two influence maximization games on graphs made temporal. CoRR, abs/2105.05987, 2021.

[3] Anthony Bonato. The game of cops and robbers on graphs. American Mathematical Soc., 2011.

[4] Arnaud Casteigts, Paola Flocchini, Walter Quattrociocchi, and Nicola Santoro. Time-varying graphs and dynamic networks. Int. J. Parallel Emergent Distributed Syst., 27(5):387–408, 2012.

[5] Nancy E. Clarke and Gary MacGillivray. Characterizations of k-copwin graphs. Discret. Math., 312(8):1421–1425, 2012.

[6] Luca de Alfaro, Thomas A. Henzinger, and Orna Kupferman. Concurrent reachability games. Theor. Comput. Sci., 386(3):188–217, 2007.

[7] Bolin Ding, Jeffrey Xu Yu, and Lu Qin. Finding time-dependent shortest paths over large graphs. In Proceedings of the 11th International Conference on Extending Database Technology, volume 261 of ACM International Conference Proceeding Series, pages 205–216. ACM, 2008.

[8] Thomas Erlebach and Jakob T. Spooner. A game of cops and robbers on graphs with periodic edge-connectivity. In Proceedings of 46th International Conference on Current Trends in Theory and Practice of Informatics, volume 12011 of Lecture Notes in Computer Science, pages 64–75. Springer, 2020.

[9] Niloy Ganguly, Andreas Deutsch, and Animesh Mukherjee. Dynamics on and of complex networks. Applications to Biology, Computer Science, and the Social Sciences, 2009.

[10] Petter Holme. Modern temporal network theory: a colloquium. The European Physical Journal B, 88(9):1–30, 2015.

[11] Petter Holme and Jari Saramäki. Temporal networks. Physics reports, 519(3):97–125, 2012.

[12] Neil Immerman. Number of quantifiers is better than number of tape cells. J. Comput. Syst. Sci., 22(3):384–406, 1981.

[13] Othon Michail and Paul G. Spirakis. Traveling salesman problems in temporal graphs. Theor. Comput. Sci., 634:1–23, 2016.

[14] Nils Morawietz, Carolin Rehs, and Mathias Weller. A timecop’s work is harder than you think. In Proceedings of the 45th International Symposium on Mathematical Foundations of Computer Science, volume 170 of LIPIcs, pages 71:1–71:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020.

[15] Richard J. Nowakowski and Peter Winkler. Vertex-to-vertex pursuit in a graph. Discret. Math., 43(2-3):235–239, 1983.

[16] Alain Quilliot. Jeux et pointes fixes sur les graphes. PhD thesis, Ph. D. Dissertation, Université de Paris VI, 1978.

[17] Klaus Sutner and Wolfgang Maass. Motion planning among time dependent obstacles. Acta Informatica, 26(1-2):93–122, 1988.

[18] Klaus Wehmuth, Artur Ziviani, and Eric Fleury. A unifying model for representing time-varying graphs. In 2015 IEEE International Conference on Data Science and Advanced Analytics, DSAA 2015, Campus des Cordeliers, Paris, France, October 19-21, 2015, pages 1–10. IEEE, 2015.

[19] Zhensheng Zhang. Routing in intermittently connected mobile ad hoc networks and delay tolerant networks: Overview and challenges. IEEE Commun. Surv. Tutorials, 8(1-4):24–37, 2006.
A Details on Lemma 3

We explicitly give the edge periodic cycles for $k = 2$, $k = 3$, and $k = 4$ in the proof of Lemma 3. For $k = 2$ and $k = 3$ the chase of the cop will be shorter than described in Table 1 and for $k \geq 4$ the chase will be exactly as described in general in Table 1. The edge periodic cycle for $k = 2$ is depicted in Figure 6 and the chase is described in Table 2. For $k = 3$ the edge periodic cycle is depicted in Figure 7 and the chase is described in Table 3. Finally, for $k = 4$, the edge periodic cycle is depicted in Figure 8 and the explicit chase is described in Table 4. Note that Table 4 is identical to Table 1 if we set $k = 4$ in Table 1.

Figure 6: Edge periodic cycle for the case $k = 2$ in Lemma 3 with $4 \cdot k - 1 = 7$ vertices and $\text{lcm}(L_G) = 2$ with a cop winning strategy from the start vertex marked in red. Edges without edge label are 1-edges; for all other edges, $\tau(e)$ is explicitly noted (with gray background).

| time step | pos. cop | pos. robber |
|-----------|----------|-------------|
| 0         | 0        | 3           |
| 1         | 1        | 4           |
| 2         | 2        | 5           |
| 3         | 3        | 6           |
| 4         | 4        | 0           |
| 5         | 5        | 0           |
| 6         | 6        | 0           |

Table 2: Time steps with corresponding positions of cop and robber in the edge periodic cycle depicted in Figure 6. All positions are after moving in this time step. The time step $s$ denotes the start configuration. Recall that the cop moves first.
Figure 7: Edge periodic cycle for the case $k = 3$ in Lemma 3 with $4 \cdot k - 1 = 11$ vertices and $\text{lcm}(L_G) = 3$ with a cop winning strategy from the start vertex marked in red. Edges without edge label are 1-edges; for all other edges, $\tau(e)$ is explicitly noted (with gray background).

| time step | pos. cop | pos. robber |
|-----------|----------|-------------|
| 0         | 1        | 5           |
| 1         | 2        | 5           |
| 2         | 3        | 6           |
| 3         | 4        | 7           |
| 4         | 5        | 7           |
| 5         | 6        | 7           |
| 6         | 7        | 8           |

Table 3: Time steps with corresponding positions of cop and robber in the edge periodic cycle depicted in Figure 7. All positions are after moving in this time step. The time step $s$ denotes the start configuration. Recall that the cop moves first.

Figure 8: Edge periodic cycle for the case $k = 4$ in Lemma 3 with $4 \cdot k - 1 = 15$ vertices and $\text{lcm}(L_G) = 4$ with a cop winning strategy from the start vertex marked in red. Edges without edge label are constant 1-edges; for all other edges, $\tau(e)$ is explicitly noted (with gray background).
| time step | pos. cop | pos. robber |
|-----------|----------|-------------|
| 8         | 0        | 7           |
| 0         | 1        | 7           |
| 1         | 2        | 7           |
| 2         | 3        | 7           |
| 3         | 4        | 8           |
| 4         | 5        | 9           |
| 5         | 6        | 10          |
| 6         | 7        | 10          |
| 7         | 8        | 10          |
| 8         | 9        | 11          |
| 9         | 10       | 12          |
| 10        | 10       | 13          |
| 11        | 10       | 14          |
| 12        | 11       | 0           |
| 13        | 12       | 0           |
| 14        | 13       | 0           |
| 15        | 14       | 0           |
| 16        | 0        | 🕵️‍♂️       |

| time step | pos. cop | pos. robber |
|-----------|----------|-------------|
| 8         | 0        | 8           |
| 0         | 14       | 8           |
| 1         | 13       | 8           |
| 2         | 12       | 8           |
| 3         | 11       | 7           |
| 4         | 10       | 6           |
| 5         | 9        | 5           |
| 6         | 8        | 4           |
| 7         | 7        | 3           |
| 8         | 6        | 2           |
| 9         | 5        | 1           |
| 10        | 4        | 1           |
| 11        | 3        | 1           |
| 12        | 2        | 0           |
| 13        | 1        | 14          |
| 14        | 1        | 13          |
| 15        | 1        | 12          |
| 16        | 11       | 🕵️‍♂️       |
| 17        | 14       | 11          |
| 18        | 13       | 11          |
| 19        | 12       | 11          |
| 20        | 11       | 🕵️‍♂️       |

Table 4: Time steps with corresponding positions of cop and robber in the edge periodic cycle depicted in Figure 8. Note that the position of the cop and robber are as described in Table II for the general case of $k \geq 4$. All positions are after moving in this time step. The time step $s$ denotes the start configuration. Recall that the cop moves first.
B Details on Lemma 5

We explicitly give the edge periodic cycle for $k = 4$ in the proof of Lemma 5. The edge periodic cycle is depicted in Figure 9 and the explicit chase is described in Table 5.

![Diagram of the edge periodic cycle](image)

Figure 9: Cycle with $23 = 6k - 1$ vertices and $\text{lcm}(L_G) = 12 = 3k$ with a cop winning strategy from the start vertex 0 where $k = 4$. Edges without an explicit label are 1-edges.

| time step | pos. cop | pos. robber |
|-----------|----------|-------------|
| 8         | 0        | 11          |
| 0         | 1        | 11          |
| 1         | 2        | 11          |
| 2         | 3        | 11          |
| 3         | 4        | 12          |
| 4         | 5        | 13          |
| 5         | 6        | 14          |
| 6         | 7        | 14          |
| 7         | 8        | 14          |
| 8         | 9        | 15          |
| 9         | 10       | 15          |
| 10        | 11       | 15          |
| 11        | 12       | 16          |
| 12        | 13       | 17          |
| 13        | 14       | 17          |
| 14        | 15       | 18          |
| 15        | 16       | 19          |
| 16        | 17       | 19          |
| 17        | 18       | 19          |
| 18        | 19       |             |

| time step | pos. cop | pos. robber |
|-----------|----------|-------------|
| 8         | 0        | 12          |
| 0         | 22       | 12          |
| 1         | 21       | 12          |
| 2         | 20       | 12          |
| 3         | 19       | 11          |
| 4         | 18       | 10          |
| 5         | 17       | 9           |
| 6         | 16       | 9           |
| 7         | 15       | 9           |
| 8         | 14       | 8           |
| 9         | 13       | 8           |
| 10        | 12       | 8           |
| 11        | 11       | 7           |
| 12        | 10       | 6           |
| 13        | 9        | 6           |
| 14        | 8        | 5           |
| 15        | 7        | 4           |
| 16        | 6        | 4           |
| 17        | 5        | 4           |
| 18        | 4        |             |

Table 5: Time steps with corresponding positions of cop and robber in the edge periodic cycle depicted in Figure 9. All positions are after moving in this time step. The time step $s$ denotes the start configuration. Recall that the cop moves first.