On large families of bundles over algebraic surfaces

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Abstract: The aim of this note is to construct sequences of vector bundles with unbounded rank and discriminant on an arbitrary algebraic surface. This problem, on principally polarized abelian varieties with cyclic Neron-Severi group generated by the polarization, was considered by Nakashima in connection with the Douglas-Reinbacher-Yau conjecture on the Strong Bogomolov Inequality. In particular we show that on any surface, the Strong Bogomolov Inequality $SBI_l$ is false for all $l > 4$.

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1 Introduction

Let $S$ a smooth complex projective algebraic surface. For a vector bundle $E$ on $S$ with Chern classes $c_1$ and $c_2$, the discriminant $\Delta$ is defined by the following formula:

$$\Delta = 2rc_2 - (r - 1)c_1^2.$$

The well known Bogomolov inequality [3] asserts that for semi-stable bundles the discriminant is non-negative. In [10] the author of that paper posed the following problem:

**Problem 1.1** Construct a sequence of $\mu$-stable vector bundles $E_m$ with effectively computable discriminants $\delta_m$ such that $\delta_m$ go to infinity with $m$.

in connection with the following conjecture from [6]:

**Conjecture 1.2** Let $S$ a simply connected surface with trivial or ample canonical bundle. Then the Chern classes of any stable vector bundle with nontrivial moduli space obey the following improved Bogomolov inequality:

$$2rc_2 - (r - 1)c_1^2 - \frac{r^2}{12}c_2(S) \geq 0.$$

The above conjecture is false as proved in [4], [10] and [11]. In [10] and [11] the counterexamples were constructed using the method of elementary transformation on arbitrary surfaces as in the conjecture. In [4] there are two families of counterexamples: on $K3$ surfaces using the specific methods for such surfaces from [7] and on surfaces in $\mathbb{P}^3$ of degree $d \geq 7$ using Horrocks’s theory of monads and methods from [5]. Also, in [12] is introduced the following

**Definition 1.3** A sequence $E_m$ for $m \geq 1$ of $\mu$-stable vector bundles with the rank $r_m$ and discriminant $\Delta_m$ tending to infinity is called a large family. If $r_m = O(m^s)$ and $\Delta_m = O(m^t)$, the large family is of order $(s, t)$.

and is proved that on abelian varieties with cyclic Neron-Severi group there exists large families of orders $(1, 2)$ and $(1, 3)$.

The aim of this note is to construct large families of vector bundles of various orders on arbitrary algebraic surfaces using the ideas from [1], [2] and [8]. In Section 2 we describe the method of Li-Qin from [8]. Our main result
concerning the construction of large families on arbitrary algebraic surfaces is
Theorem 3.2 and is proved in Section 3. In particular our method shows
that on any surface, the Strong Bogomolov Inequality $SBI_l$ is false for all
$l > 4$. This fact and the relation with the original Douglas-Reinbacher-Yau
Conjecture is described in Section 4. The main ingredient for our examples
is the Serre construction, the results are valid on any surface and the bounds
are very explicit. Through the paper we shall denote by $\sim$ the equality up
to terms of lower degree and by surface we always mean a smooth projective
one over $\mathbb{C}$.

2 Notations and preliminaries

Let $S$ a smooth projective algebraic surface and $L$ a very ample polarization
on $S$. In [1] for the rank-2 case and in [8] for arbitrary rank, the authors
uses the Serre construction to obtain $L$-stable vector bundles with prescribed
Chern classes, provided that $c_2$ is sufficiently great. In fact we have the
following existence theorem from [8] which is a generalization of the similar,
rank-2 case result, from [1]:

Theorem 2.1 Let $L$ eventually rescaled such that $r \cdot L^2 > K_X \cdot L$ and
$\alpha = (r - 1)[1 + \max(p_g, h^0(S, \mathcal{O}_S(rL - c_1 + K_S))) + 4(r - 1)^2 \cdot L^2]$
$+(r - 1)c_1 \cdot L - \frac{r(r-1)}{2} \cdot L^2$. Let $c_2 \geq \alpha$. Then there exists an $L$-stable rank $r$
vector bundle $E$ with Chern classes $c_1$ and $c_2$. Moreover we have $h^2(X, \text{ad}(E)) = 0$.

For the convenience of the reader we recall the main steps of the proof.
- First of all, for $Z$ a reduced 0-cycle and two line bundles $L$, $L'$ on $S$, it is well
known that there exists a locally free extension in $\text{Ext}^1(\mathcal{O}_S(L) \otimes \mathcal{I}_Z, \mathcal{O}_S(L'))$
iff $Z$ satisfies the Cayley-Bacharach property with respect to the linear system
$\mathcal{O}_S(L - L' + K_S)$.
- This fact is generalized by Li-Qin in the following:

Proposition 2.2 Consider $r - 1$ line bundles $L_1, ..., L_{r-1}$ and 0-cycles
$Z_1, ..., Z_{r-1}$ on $S$; let $W = \bigoplus(\mathcal{O}_S(L_i) \otimes \mathcal{I}_{Z_i})$. Then, there is a locally free
extension in $\text{Ext}^1(W, \mathcal{O}_S(L'))$ iff for any $i = 1...(r - 1)$, $Z_i$ satisfies the
Cayley-Bacharach property with respect to the linear system $\mathcal{O}_S(L_i - L' + K_S)$.
- The proposition above is used by choosing $L_i = L$ for all $i$'s, $L' = c_1 +$
$(r - 1)L$ and a convenient 0-cycle $Z$ of appropriate length which produce a
locally free extension $E$ with the desired Chern classes.
- The next step is the proof of the stability of $E$ which is a consequence of some properties of the chosen 0-cycle $Z$.
- The fact that $h^2(X, ad(E)) = 0$, which means that the component of $E$ in the moduli space is smooth, follows again from the properties of the 0-cycle $Z$ using some ideas from [9].

The above Theorem 2.1 will be used in the next section to produce large families of various orders on arbitrary algebraic surfaces.

3 The construction of large families

Let’s fix an algebraic surface $S$. In what follows, we shall denote by $L_0$ a fixed polarization on $S$ and by $L = aL_0$ a high multiple of $L_0$. We shall consider the first Chern class of the form $c_1 = bL_0$ with $r \cdot a - b$ having a constant value; $r$ will be the rank of the bundle we want to obtain.

Remark 3.1 For $r \cdot a - b$ fixed but higher than a certain bound, the Riemann-Roch formula applied for $\mathcal{O}(rL - c_1 + K)$ gives:

$$h^0(S, \mathcal{O}_S(rL - c_1 + K_S)) =$$
$$\chi(\mathcal{O}_S) + \frac{1}{2}(rL - c_1 + K_S)(rL - c_1) =$$
$$\frac{r^2L^2}{2} + \frac{rL(-2c_1 + K_S)}{2} + \text{constant terms.}$$

From the above remark, it follows that for large ranks,

$$\max(p_g, h^0(S, \mathcal{O}_S(rL - c_1 + K_S))) =$$
$$\frac{r^2L^2}{2} + \frac{rL(-2c_1 + K_S)}{2} + \text{constant terms.}$$

From the existence Theorem 2.1 the constant $\alpha$ and hence $c_2$ can be taken as

$$4(r - 1)^3a^2L_0^2 + \frac{r(r - 1)a^2L_0^2}{2}$$

up to terms of lower degree. So, $2rc_2$ and the discriminant $\Delta$ have the following asymptotic formulas:

$$2rc_2 \sim 8r(r - 1)^3a^2L_0^2 + r^2(r - 1)a^2L_0^2,$$

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\[ \Delta = 2rc_2 - (r - 1)c_1^2 \sim 8r(r - 1)^3a^2L_0^2. \]

Consider now, the parameters \( r \) and \( a \) varying asymptotically as:

\[ r \sim m^s \]

and

\[ a \sim m^x \]

for natural \( m \) and real positive \( s, x \). Therefore,

\[ \Delta \sim m^{4s+x} \]

and we obtain the following:

**Theorem 3.2** For any smooth projective algebraic surface and any pair \( (s, t) \) of positive real numbers with \( t > 4s \) there exists a large family of order \( (s, t) \), i.e. a sequence of stable vector bundles \( E_m \) such that their ranks and discriminants satisfy \( r_m = O(m^s) \) and \( \Delta_m = O(m^t) \).

### 4 Connection with the Strong Bogomolov Inequality

In [12], Nakashima introduces the definition of the Strong Bogomolov Inequality \( SBI \):

**Definition 4.1** For a positive real number \( l \), a polarized algebraic surface \( (S, H) \) satisfy the Strong Bogomolov Inequality \( SBI_l \) if there is a positive constant \( \sigma \) depending only on the surface and the polarization such that for any stable vector bundle \( E \) of rank \( r \) one has the inequality:

\[ \Delta(E) \geq r^l \sigma. \]

Of course, the above inequality is a strengthening of the usual Bogomolov’s one \( \Delta \geq 0 \) and **Conjecture 1.2** is a form of \( SBI_2 \).

The following obviously proposition from [12] explain the relation between the notion of large family and the definition of the Strong Bogomolov Inequality \( SBI_l \):

**Proposition 4.2** Let \( S \) an algebraic surface with a large family of order \( (s, t) \) such that \( t < 4s \). Then the Strong Bogomolov Inequality \( SBI_l \) is false on \( S \).
From the proposition above and our Theorem 3.2 we obtain the following:

**Corollary 4.3** For any polarized algebraic surface $(S, H)$ and any $l > 4$ the Strong Bogomolov Inequality $SBI_l$ is false for $(S, H)$.

*Proof:* By Theorem 3.2 one obtain a large family of order $(s, 4s + x)$. For any $l > 4$, say $l = 4 + \epsilon$, we want to fulfill the condition $t < ls$. But this means $x < \epsilon s$ which is obviously achieved for an appropriate $x$. QED.

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