CONSTRUCTIONS AND COHOMOLOGY OF COLOR HOM-LIE ALGEBRAS

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Abstract. The main purpose of this paper is to define representations and a cohomology of color Hom-Lie algebras and to study some key constructions and properties. We describe Hartwig-Larsson-Silvestrov Theorem in the case of Γ-graded algebras, study one-parameter formal deformations, discuss $\alpha^k$-generalized derivation and provide examples.

Introduction

Color Hom-Lie algebras are the natural generalizations of Hom-Lie algebras and Hom-Lie superalgebras. In recent years, they have become an interesting subject of mathematics and physics. A cohomology of color Lie algebras were introduced and investigated, see [26, 28], and the representations of color Lie algebras were explicitly described in [4]. As is well known, derivations and extensions of Hom-Lie algebras, Hom-Lie superalgebras and color Hom-Lie algebras are very important subjects. Color Hom-Lie algebras were studied in [34]. In the particular case of Hom-Lie superalgebras, cohomology theory was provided in [3]. This paper focusses on Γ-graded Hom-algebras with $\Gamma$ is an abelian group. Mainly, we prove a Γ-graded version of a Hartwig-Larsson-Silvestrov Theorem and we study representations and cohomology of color Hom-Lie algebras.

The paper is organized as follows. In section 1, we recall definitions and some key constructions of color Hom-Lie algebras and we provide a list of twists of color Hom-Lie algebras. Section 2 is dedicated to describe and prove the Γ-graded version Hartwig-Larsson-Silvestrov Theorem, which was proved for Hom-Lie algebras in [15, Theorem 5] and for Hom-Lie superalgebras in [5, Theorem 4.2]). In section 3, we construct a family of cohomologies of color Hom-Lie algebras, discuss representation theory in connection with cohomology and compute the second cohomology group of $(\mathfrak{sl}_2, [\,,.]_{\alpha}, \varepsilon, \alpha)$. In section 4, we study formal deformations of color Hom-Lie algebras. In last section, we study the homogeneous $\alpha^k$-generalized derivations and the $\alpha^k$-centroid of color Hom-Lie algebras and we give some properties generalizing the homogeneous generalized derivations discussed in [8]. Moreover in Proposition 5.4 we prove that the $\alpha$-derivation of color Hom-Lie algebras gives rise to a color Hom-Jordan algebras.

1. Definitions, Constructions and Examples

In the following we summarize definitions of color Hom-Lie and color Hom-associative algebraic structures (see [34]) generalizing the well known color Lie and color associative algebras.

Throughout the article we let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and $\mathbb{K}^*=\mathbb{K}\setminus\{0\}$ be the group of units of $\mathbb{K}$.

- Let $\Gamma$ be an abelian group. A vector space $V$ is said to be $\Gamma$-graded, if there is a family $(V_{\gamma})_{\gamma \in \Gamma}$ of vector spaces of $V$ such that

$$V = \bigoplus_{\gamma \in \Gamma} V_{\gamma}.$$

An element $x \in V$ is said to be homogeneous of the degree $\gamma \in \Gamma$ if $x \in V_{\gamma}, \gamma \in \Gamma$, and in this case, $\gamma$ is called the color of $x$. As usual, denote by $\mathfrak{r}$ the color of an element $x \in V$. Thus each homogeneous element $x \in V$ determines a unique group of element $\mathfrak{r} \in \Gamma$ by $x \in V_{\mathfrak{r}}$. Fortunately, We can almost drop the symbol " − " , since confusion rarely occurs. In the sequel, we will denote by $H(V)$ the set of all the homogeneous elements of $V$.

Let $V = \bigoplus_{\gamma \in \Gamma} V_{\gamma}$ and $V' = \bigoplus_{\gamma \in \Gamma} V'_{\gamma}$ be two $\Gamma$-graded vector spaces. A linear mapping $f : V \rightarrow V'$ is said to be homogeneous of the degree $\nu \in \Gamma$ if

$$f(V_{\gamma}) \subseteq V'_{\gamma+\nu}, \forall \gamma \in \Gamma.$$

if in addition, $f$ is homogeneous of degree zero, i.e. $f(V_{\gamma}) \subseteq V'_{\gamma}$ holds for any $\gamma \in \Gamma$, then $f$ is said to be even.
Remark 1.1. In particular the proposition is valid when the result of endomorphism, we construct a new color Hom-Lie algebra.

Definition 1.1. Let $\mathbb{K}$ be a field and $\Gamma$ be an abelian group. A map $\varepsilon : \Gamma \times \Gamma \to \mathbb{K}^*$ is called a skew-symmetric bi-character on $\Gamma$ if the following identities hold, for all $a, b, c$ in $\Gamma$

$$
\begin{align*}
(1) \quad & \varepsilon(a, b) \varepsilon(b, a) = 1, \\
(2) \quad & \varepsilon(a, b + c) = \varepsilon(a, b) \varepsilon(a, c), \\
(3) \quad & \varepsilon(a + b, c) = \varepsilon(a, c) \varepsilon(b, c).
\end{align*}
$$

The definition above implies, in particular, the following relations

$$
\varepsilon(a, 0) = \varepsilon(0, a) = 1, \quad \varepsilon(a, a) = \pm 1, \quad \text{for all } a \in \Gamma.
$$

If $x$ and $x'$ are two homogeneous elements of degree $\gamma$ and $\gamma'$ respectively and $\varepsilon$ is a skew-symmetric bi-character, then we shorten the notation by writing $\varepsilon(x, x')$ instead of $\varepsilon(\gamma, \gamma')$.

Definition 1.2. A color Hom-Lie algebra is a quadruple $(A, [[., .], \varepsilon, \alpha])$ consisting of a $\Gamma$-graded vector space $A$, a bi-character $\varepsilon$, an even bilinear mapping $[., .] : A \times A \to A$ (i.e. $[A_i, A_j] \subseteq A_{i+j}$ for all $a, b \in \Gamma$) and an even homomorphism $\alpha : A \to A$ such that for homogeneous elements we have

$$
[\alpha x, \alpha y] = -\varepsilon(x,y)[\alpha x, \alpha y] \quad (\varepsilon \text{-Hom-Jacobi condition}).
$$

where $\bigcirc_{x,y,z} \varepsilon(z, x)[\alpha(x), [y, z]] = 0$ ($\varepsilon$-Hom-Jacobi condition).

In particular, if $\alpha$ is a morphism of color Lie algebras (i.e. $\alpha \circ [., .] = [\alpha . \circ \alpha\alpha^{-1}]$), then we call $(A, [[., .], \varepsilon, \alpha])$ a multiplicative color Hom-Lie algebra.

Observe that when $\alpha = Id$, the $\varepsilon$-Hom-Jacobi condition reduces to the usual $\varepsilon$-Jacobi condition

$$
\bigcirc_{x,y,z} \varepsilon(z, x)[x, [y, z]] = 0
$$

for all $x, y, z \in H(A)$.

Example 1.1. A color Lie algebra $(A, [[., .], \varepsilon])$ is a color Hom-Lie algebra with $\alpha = Id$, since the $\varepsilon$-Hom-Jacobi condition reduces to the $\varepsilon$-Jacobi condition when $\alpha = Id$.

Definition 1.3. Let $(A, [[., .], \varepsilon, \alpha])$ be a color Hom-Lie algebra. It is called

1. multiplicative color Hom-Lie algebra if $\alpha [x, y] = [\alpha x, \alpha y]$.
2. regular color Hom-Lie algebra if $\alpha$ is an automorphism.
3. involutive color Hom-Lie algebra if $\alpha$ is an involution, that is $\alpha^2 = Id$.

Let $(A, [[., .], \varepsilon, \alpha])$ be a regular color Hom-Lie algebra. It was observed in [9] that the composition method using $\alpha^{-1}$ leads to a color Lie algebra.

Proposition 1.1. Let $(A, [[., .], \varepsilon, \alpha])$ be a regular color Hom-Lie algebra. Then $(A, [[., .], \alpha^{-1} = \alpha^{-1} \circ [[., .], \varepsilon])$ is a color Lie algebra.

Proof. Note that $[., .]_{\alpha^{-1}}$ is $\varepsilon$-skew-symmetric because $[., .]$ is $\varepsilon$-skew-symmetric and $\alpha^{-1}$ is linear.

For $x, y, z \in H(A)$, we have:

$$
\begin{align*}
\bigcirc_{x,y,z} \varepsilon(z, x)[x, [y, z]_{\alpha^{-1}}]_{\alpha^{-1}} &= \bigcirc_{x,y,z} \varepsilon(z, x)\alpha^{-1}[x, [y, z]_{\alpha^{-1}}] \\
&= \alpha^{-2}(\bigcirc_{x,y,z} \varepsilon(z, x)[\alpha(x), [y, z]]) \\
&= 0.
\end{align*}
$$

Remark 1.1. In particular the proposition is valid when $\alpha$ is an involution. The following theorem generalize the result of [9]. In the following starting from a color Hom-Lie algebra and an even color Lie algebra endomorphism, we construct a new color Hom-Lie algebra.
We recall in the following the definition of Hom-associative color algebra which provide a different way for constructing color Hom-Lie algebra by extending the fundamental construction of color Lie algebras from associative color algebra via commutator bracket multiplication. This structure was introduced by [34].

**Definition 1.4.** [34] A color Hom-associative algebra is a triple $(A, \mu, \alpha)$ consisting of a $\Gamma$-graded linear space $A$, an even bilinear map $\mu : A \times A \to A$ (i.e $\mu(A_0, A_0) \subset A_{a+b}$) and an even homomorphism $\alpha : A \to A$ such that

\[
\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)).
\]

In the case where $\mu(x, y) = \varepsilon(x, y)\mu(y, x)$, we call the Hom-associative color algebra $(A, \mu, \alpha)$ a commutative Hom-associative color algebra.

**Remark 1.2.** We recover classical associative color algebra when $\alpha = Id_A$ and the condition (1.1) is the associative condition in this case.

**Proposition 1.2.** [34] Let $(A, \mu, \alpha)$ be a Hom-associative color algebra defined on the linear space $A$ by the multiplication $\mu$ and an even homomorphism $\alpha$. Then the quadruple $(A, [\cdot, \cdot], \varepsilon, \alpha)$, where the bracket is defined for $x, y \in \mathcal{H}(A)$ by

\[
[x, y] = \mu(x, y) - \varepsilon(x, y)\mu(y, x)
\]

is a color Hom-Lie algebra.

**Theorem 1.1.** Let $(A, [\cdot, \cdot], \varepsilon, \alpha)$ be a color Hom-Lie algebra and $\beta : A \to A$ be an even color Lie algebra endomorphism. Then $(A, [\cdot, \cdot], \varepsilon, \beta \circ \alpha)$, where $[x, y] = \beta \circ [x, y]$, is a color Hom-Lie algebra. Moreover, suppose that $(A', [\cdot, \cdot], \varepsilon)$ is a color Lie algebra and $\alpha' : A' \to A'$ is color Lie algebra endomorphism. If $f : A \to A'$ is a Li color algebra morphism that satisfies $f \circ \beta = \alpha' \circ f$ then

\[
f : (A, [\cdot, \cdot], \varepsilon, \beta \circ \alpha) \to (A', [\cdot, \cdot], \varepsilon, \alpha')
\]

is a morphism of color Hom-Lie algebras.

**Proof.** Obviously $[\cdot, \cdot]_{\beta}$ is $\varepsilon$-skew-symmetric and we show that $(A, [\cdot, \cdot]_{\beta}, \varepsilon, \beta \circ \alpha)$ satisfies the $\varepsilon$-Hom-Jacobi condition 12.

\[
\begin{align*}
\sum_{x, y, z} \varepsilon(z, x)[\beta \circ \alpha(x), [y, z]]_{\beta} &= \sum_{x, y, z} \varepsilon(z, x)[\beta \circ \alpha(x), \beta([y, z])]_{\beta} \\
&= \beta^2 \varepsilon(z, x)[\alpha(x), [y, z]] \\
&= 0.
\end{align*}
\]

The second assertion follows from

\[
f([x, y]_{\beta}) = f([\beta(x), \beta(y)]) = [f \circ \beta(x), f \circ \beta(y)]' = [\alpha' \circ f(x), \alpha' \circ f(y)]' = [f(x), f(y)]'_{\alpha'}.
\]

Example 1.2. Let $(A, [\cdot, \cdot], \varepsilon)$ be a color Lie algebra and $\alpha$ be a color Lie algebra morphism, then $(A, [\cdot, \cdot]_{\alpha}) = \alpha \circ [\cdot, \cdot], \varepsilon, \alpha)$ is a multiplicative color Hom-Lie algebra.

**Definition 1.5.** Let $(A, [\cdot, \cdot], \varepsilon, \alpha)$ be a multiplicative color Hom-Lie algebra and $n \geq 0$. Define the nth derived Hom-algebra of $A$ by

\[
A^{(n)} = (A, [\cdot, \cdot]^{(n)}) = \alpha^{2^n-1} \circ [\cdot, \cdot], \varepsilon, \alpha^{2^n}).
\]

Note that $A^{(0)} = A$, $A^{(1)} = (A, [\cdot, \cdot]^{(1)}) = \alpha \circ [\cdot, \cdot], \varepsilon, \alpha^{2})$, and $A^{(n+1)} = (A^n)^1$.

**Corollary 1.1.** Let $(A, [\cdot, \cdot], \varepsilon, \alpha)$ be a color Hom-Lie algebra. Then the nth derived Hom-algebra of $A$

\[
A^{(n)} = (A, [\cdot, \cdot]^{(n)}) = \alpha^{2^n-1} \circ [\cdot, \cdot], \varepsilon, \alpha^{2^n})
\]

is also a color Hom-Lie algebra for each $n \geq 0$. 

1.1. Examples of twists of color Hom-Lie algebras. In this section we provide examples of color Hom-Lie algebras. We use the classification of color Lie algebra provided in [23] and the twisting principle. In Examples 1.3 and 1.4 \( \Gamma \) is the group \( \mathbb{Z}_2^3 \), and \( \varepsilon : \Gamma \times \Gamma \to \mathbb{C} \) is defined by the matrix

\[
\begin{pmatrix}
1 & -1 & -1 \\
-1 & 1 & -1 \\
-1 & -1 & 1
\end{pmatrix}.
\]

Elements of this matrix specify in the natural way values of \( \varepsilon \) on the set \( \{(1,1,0),(1,0,1),(0,1,1)\} \times \{(1,1,0),(1,0,1),(0,1,1)\} \subset \mathbb{Z}_2^3 \times \mathbb{Z}_2^3 \) (The elements \( (1,1,0),(1,0,1),(0,1,1) \) are ordered and numbered by the numbers \( 1,2,3 \) respectively). The values of \( \varepsilon \) on other elements from \( \mathbb{Z}_2^3 \times \mathbb{Z}_2^3 \) do not affect the multiplication \( [.,.] \).

**Example 1.3.** The graded analogue of \( sl(2,\mathbb{C}) \) is defined as complex algebra with three generators \( e_1, e_2 \) and \( e_3 \) satisfying the commutation relations \( e_1 e_2 + e_2 e_1 = e_3 \), \( e_1 e_3 + e_3 e_1 = e_2 \), \( e_2 e_3 + e_3 e_2 = e_1 \). Let \( sl(2,\mathbb{C}) \) be \( \mathbb{Z}_2^3 \)-graded linear space \( sl(2,\mathbb{C}) = sl(2,\mathbb{C})_{(1,1,0)} \oplus sl(2,\mathbb{C})_{(1,0,1)} \oplus sl(2,\mathbb{C})_{(0,1,1)} \) with basis \( e_1 \in sl(2,\mathbb{C})_{(1,1,0)}, e_2 \in sl(2,\mathbb{C})_{(1,0,1)}, e_3 \in sl(2,\mathbb{C})_{(0,1,1)} \). The homogeneous subspaces of \( sl(2,\mathbb{C}) \) graded by the elements of \( \mathbb{Z}_2^3 \) different from \( (1,1,0),(1,0,1) \) and \( (0,1,1) \) are zero and so are omitted. The bilinear multiplication \( [.,.] : sl(2,\mathbb{C}) \times sl(2,\mathbb{C}) \to sl(2,\mathbb{C}) \) defined, with respect to the basis \( \{e_1,e_2,e_3\} \), by the formulas

\[
\begin{align*}
[e_1, e_1] &= e_1 e_1 - e_1 e_1 = 0, \\
[e_1, e_2] &= e_1 e_2 + e_2 e_1 = e_3, \\
[e_2, e_2] &= e_2 e_2 - e_2 e_2 = 0, \\
[e_1, e_3] &= e_1 e_3 + e_3 e_1 = e_2, \\
[e_3, e_3] &= e_3 e_3 - e_3 e_3 = 0, \\
[e_2, e_3] &= e_2 e_3 + e_3 e_2 = e_1,
\end{align*}
\]

makes \( sl(2,\mathbb{C}) \) into a three-dimensional color-Lie algebra.

By using [34] Theorem 3.14, we provide this Lie color algebra with a Hom structure, for that we consider a linear even map \( \alpha : sl(2,\mathbb{C}) \to sl(2,\mathbb{C}) \) checking \( \alpha[x,y] = [\alpha(x), \alpha(y)] \) \( \forall x, y \in H(sl(2,\mathbb{C})) \) in such way \( (sl(2,\mathbb{C}),[.,.],\alpha = \alpha \circ [.,.],\varepsilon) \) is a color Hom-Lie algebra. The morphism of \( sl_2^3 \) are given with respect to the basics \( \{e_1,e_2,e_3\} \) by

\[
\begin{align*}
\alpha(e_1) &= a_1 e_1 + a_2 e_2 + a_3 e_3, \\
\alpha(e_2) &= b_1 e_1 + b_2 e_2 + b_3 e_3, \\
\alpha(e_3) &= c_1 e_1 + c_2 e_2 + c_3 e_3,
\end{align*}
\]

where \( a_i, b_i, c_i \in \mathbb{C} \).

Then, we obtain the following lists of twisted color Hom-Lie algebra \( (sl_2^3,[.,.],\alpha = \alpha \circ [.,.],\varepsilon,\alpha) \) in Table 1:

| Table 1 |
\[
\begin{array}{|c|c|c|}
\hline
[e_1, e_2]_{\alpha_1} = -e_3 & \alpha_1 = \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) & [e_1, e_2]_{\alpha_2} = -e_3 \\
[e_1, e_3]_{\alpha_1} = -e_2 & \alpha_2 = \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right) & [e_1, e_3]_{\alpha_2} = e_2 \\
[e_2, e_3]_{\alpha_1} = e_1 & [e_1, e_3]_{\alpha_2} = -e_3 & [e_2, e_3]_{\alpha_2} = -e_3 \\
\hline
[e_1, e_3]_{\alpha_3} = -e_3 & \alpha_3 = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array} \right) & [e_1, e_2]_{\alpha_4} = e_3 \\
[e_1, e_3]_{\alpha_3} = e_2 & [e_1, e_3]_{\alpha_4} = -e_2 & [e_2, e_3]_{\alpha_4} = -e_1 \\
[e_2, e_3]_{\alpha_3} = -e_1 & \alpha_4 = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) & [e_1, e_2]_{\alpha_4} = -e_2 \\
\hline
[e_1, e_3]_{\alpha_5} = e_2 & \alpha_5 = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{array} \right) & [e_1, e_3]_{\alpha_6} = e_3 \\
[e_1, e_3]_{\alpha_5} = e_3 & [e_1, e_3]_{\alpha_6} = -e_3 & [e_2, e_3]_{\alpha_6} = e_1 \\
[e_2, e_3]_{\alpha_5} = e_1 & [e_1, e_2]_{\alpha_6} = -e_2 & [e_1, e_3]_{\alpha_6} = -e_3 \\
\hline
[e_1, e_2]_{\alpha_7} = -e_2 & \alpha_7 = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{array} \right) & [e_1, e_2]_{\alpha_8} = e_2 \\
[e_1, e_3]_{\alpha_7} = e_3 & [e_1, e_3]_{\alpha_8} = -e_3 & [e_2, e_3]_{\alpha_8} = -e_1 \\
[e_2, e_3]_{\alpha_7} = -e_1 & [e_1, e_2]_{\alpha_8} = e_2 & \alpha_8 = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right) \\
\hline
[e_1, e_2]_{\alpha_9} = e_3 & \alpha_9 = \left( \begin{array}{ccc} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right) & [e_1, e_2]_{\alpha_{10}} = -e_3 \\
[e_1, e_3]_{\alpha_9} = e_1 & [e_1, e_3]_{\alpha_{10}} = -e_1 & [e_2, e_3]_{\alpha_{10}} = e_2 \\
[e_2, e_3]_{\alpha_9} = e_2 & \alpha_{10} = \left( \begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{array} \right) & [e_1, e_2]_{\alpha_{10}} = -e_3 \\
\hline
[e_1, e_2]_{\alpha_{11}} = e_1 & \alpha_{11} = \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right) & [e_1, e_3]_{\alpha_{12}} = -e_3 \\
[e_1, e_3]_{\alpha_{11}} = -e_2 & [e_1, e_3]_{\alpha_{12}} = e_1 & [e_2, e_3]_{\alpha_{12}} = -e_2 \\
[e_2, e_3]_{\alpha_{11}} = -e_2 & \alpha_{12} = \left( \begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{array} \right) & \end{array}
\]
\[ \begin{array}{|c|c|c|} \hline [e_1, e_2]_{\alpha_{13}} = e_3 & \alpha_{13} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & [e_1, e_2]_{\alpha_{14}} = e_1 \\ [e_1, e_3]_{\alpha_{13}} = -e_1 & \alpha_{14} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \\ [e_2, e_3]_{\alpha_{13}} = -e_2 & \alpha_{15} = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ [e_1, e_2]_{\alpha_{15}} = -e_1 & [e_1, e_3]_{\alpha_{15}} = -e_3 \\ [e_2, e_3]_{\alpha_{15}} = e_2 & [e_2, e_3]_{\alpha_{16}} = -e_2 \\ [e_1, e_2]_{\alpha_{17}} = e_1 & \alpha_{17} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ [e_1, e_3]_{\alpha_{17}} = -e_3 & [e_1, e_3]_{\alpha_{18}} = e_1 \\ [e_2, e_3]_{\alpha_{17}} = -e_2 & [e_2, e_3]_{\alpha_{18}} = e_3 \\ [e_1, e_2]_{\alpha_{19}} = -e_2 & \alpha_{19} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} \\ [e_1, e_3]_{\alpha_{19}} = -e_1 & [e_1, e_3]_{\alpha_{20}} = -e_2 \\ [e_2, e_3]_{\alpha_{19}} = e_3 & [e_2, e_3]_{\alpha_{20}} = -e_3 \\ [e_1, e_2]_{\alpha_{21}} = e_2 & \alpha_{21} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\ [e_1, e_3]_{\alpha_{21}} = -e_1 & [e_1, e_3]_{\alpha_{22}} = e_1 \\ [e_2, e_3]_{\alpha_{21}} = -e_3 & [e_2, e_3]_{\alpha_{22}} = e_3 \\ [e_1, e_2]_{\alpha_{23}} = -e_1 & \alpha_{23} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ [e_1, e_3]_{\alpha_{23}} = -e_2 & [e_1, e_3]_{\alpha_{24}} = e_1 \\ [e_2, e_3]_{\alpha_{23}} = e_3 & [e_2, e_3]_{\alpha_{24}} = -e_3 \\ \hline \end{array} \]

**Remark 1.3.** The morphisms algebras of \(sl_2\) viewed as a color Lie algebra are all automorphisms.

**Example 1.4.** The graded analogue of the Lie algebra of the group of plane of motions is defined as a complex algebra with three generators \(e_1, e_2\) and \(e_3\) satisfying the commutation relations \(e_1 e_2 + e_2 e_1 = e_3, e_1 e_3 + e_3 e_1 = e_2, e_2 e_3 + e_3 e_2 = 0\). The linear space \(A\) spanned by \(e_1, e_2, e_3\) can be made into a \(\Gamma\)-graded \(\varepsilon\)-Lie algebra. The grading group \(\Gamma\) and the commutation factor \(\varepsilon\) are the same as in Example 1.3. But the multiplication \([.,.]\) is different and is defined by

\[
[e_1, e_1] = e_1 e_1 - e_1 e_1 = 0, \quad [e_1, e_2] = e_1 e_2 + e_2 e_1 = e_3, \\
[e_2, e_2] = e_2 e_2 - e_2 e_2 = 0, \quad [e_1, e_3] = e_1 e_3 + e_3 e_1 = e_2, \\
[e_3, e_3] = e_3 e_3 - e_3 e_3 = 0, \quad [e_2, e_3] = e_2 e_3 + e_3 e_2 = 0.
\]

**By using** [34] **Theorem 3.14**, we provide the following color Lie algebras with a Hom structure. We consider an even linear map \(\alpha : A \rightarrow A\) checking \(\alpha[x, y] = [\alpha(x), \alpha(y)]\) for all \(x, y \in H(A)\) in such way \((A, [.,.], \alpha) = \alpha \circ [.,., \alpha])\) is a color Hom-Lie algebra. Then, in Table 2, we obtain the following color Hom-Lie algebras:

**Table 2**
Let \( A \) be an even color algebra. We assume that \( A \) is color commutative, that is for homogeneous elements \( x, y \) in \( A \), the identity \( xy = \varepsilon(x, y)yx \) holds. Let \( \sigma : A \to A \) be an even color algebra endomorphism of \( A \). Then \( A \) is color bimodule over itself, the left (resp. right) action is defined by \( x \cdot y = \sigma(x)y \) (resp. \( y \cdot x = yx \)). For simplicity, we denote the module multiplication by a dot and the color multiplication by juxtaposition. In the sequel, the elements of \( A \) are supposed to be homogeneous.

**Definition 2.1.** Let \( d \in \Gamma \). A color \( \sigma \)-derivation \( \Delta \) on \( A \) is an endomorphism satisfying

\[
\begin{align*}
(CD_1) : & \quad \Delta d(A) \subseteq A_{\gamma+d}, \\
(CD_2) : & \quad \Delta d(xy) = \Delta d(x)y + \varepsilon(d, x)\sigma(x)\Delta d(y), \forall x, y \in H(G).
\end{align*}
\]

In particular for \( d = 0 \), we have \( \Delta(xy) = \Delta(x)y + \sigma(x)\Delta(y) \), then \( \Delta \) is called even color \( \sigma \)-derivation. The set of all color \( \sigma \)-derivation is denoted by \( \text{Der}_\sigma^\varepsilon(A) = \bigoplus_{\gamma \in \Gamma} \text{Der}^\varepsilon_{\gamma}(A) \).

The structure of \( A \)-color module of \( \text{Der}_\sigma(A) \) is as usual. Let \( \Delta \in \text{Der}_\sigma(A) \), the annihilator \( \text{Ann}(\Delta) \) is the set of all \( x \in H(G) \) such that \( x \cdot \Delta = 0 \). We set \( A \cdot \Delta = \{ x \cdot \Delta : x \in H(G) \} \) to be a color \( A \)-module of \( \text{Der}_\sigma(A) \).

Let \( \sigma : A \to A \) be a fixed endomorphism, \( \Delta \) an even color \( \sigma \)-derivation \( \Delta \in \text{Der}_\sigma^\varepsilon(A) \) and \( \delta \) be an element in \( A \). Then

**Theorem 2.1.** If

\[
\sigma(\text{Ann}(\Delta)) \subseteq \text{Ann}(\Delta)
\]

holds, the map \([., .]_{\sigma} : A \cdot \Delta \times A \cdot \Delta \to A \cdot \Delta \) defined by setting:

\[
[x \cdot \Delta, y \cdot \Delta]_{\sigma} = (\sigma(x) \cdot \Delta) \circ (y \cdot \Delta) - \varepsilon(x, y)(\sigma(y) \cdot \Delta) \circ (x \cdot \Delta)
\]

where \( \circ \) denotes the composition of functions, is a well-defined color algebra bracket on the \( \Gamma \)-graded space \( A \cdot \Delta \) and satisfies the following identities for \( x, y \in H(G) \):

\[
[x \cdot \Delta, y \cdot \Delta]_{\sigma} = (\sigma(x)\Delta(y) - \varepsilon(x, y)\sigma(y)\Delta(x)) \cdot \Delta, \forall x, y \in H(G).
\]

\[
[x \cdot \Delta, y \cdot \Delta]_{\sigma} = -\varepsilon(x, y)[y \cdot \Delta, x \cdot \Delta]_{\sigma}, \forall x, y \in A.
\]
In addition, if
\[ \Delta(\sigma(x)) = \delta\sigma(\Delta(x)), \quad \forall x \in \mathcal{H}(\mathcal{G}) \]
holds, then
\[ \bigcup_{x,y,z} \varepsilon(z,x) \left( [\sigma(x) \cdot \Delta, [y \cdot \Delta, z \cdot \Delta]_{\sigma}] + \delta[x \cdot \Delta, [y \cdot \Delta, z \cdot \Delta]_{\sigma}] \right) = 0, \quad \forall x, y, z \in \mathcal{H}(\mathcal{A}) \]
for all \( x, y \) and \( z \) in \( \mathcal{H}(\mathcal{G}) \).

**Proof.** We must first show that \([.,.]_{\sigma}\) is a well-defined function. That is, if \( x_1 \cdot \Delta = x_2 \cdot \Delta \), then
\[ [x_1 \cdot \Delta, y \cdot \Delta]_{\sigma} = [x_2 \cdot \Delta, y \cdot \Delta]_{\sigma} \]
and
\[ [y \cdot \Delta, x_1 \cdot \Delta]_{\sigma} = [y \cdot \Delta, x \cdot \Delta]_{\sigma} \]
for \( x_1, x_2, y \in \mathcal{H}(\mathcal{A}) \). Now \( x_1 \cdot \Delta = x_2 \cdot \Delta \) is equivalent to \( (x_1 - x_2) \in \text{Ann}(\Delta) \). Therefore, using the assumption (2.2), we also have \( \sigma(x_1 - x_2) \in \text{Ann}(\Delta) \). Then since \( |x_1| = |x_2| \) and \( \sigma(x_1 - x_2) \in \text{Ann}(\Delta) \), we obtain:
\[ [x_1 \cdot \Delta, y \cdot \Delta]_{\sigma} = [x_2 \cdot \Delta, y \cdot \Delta]_{\sigma} \]
\[ [y \cdot \Delta, x_1 \cdot \Delta]_{\sigma} = [y \cdot \Delta, x \cdot \Delta]_{\sigma} \]
which shows (2.8). The proof of (2.9) is analogous.

Next we prove (2.2), which also shows that \( \mathcal{A} \cdot \Delta \) is closed under \([.,.]_{\sigma}\).

Let \( x, y, z \in \mathcal{H}(\mathcal{G}) \) be arbitrary. Then, since \( \Delta \) is an even color \( \sigma \)-derivation on \( \mathcal{A} \) we have:
\[ [x \cdot \Delta, y \cdot \Delta]_{\sigma}(z) = (\sigma(x) \cdot \Delta)(y \cdot \Delta)(z) - \varepsilon(x, y)(\sigma(y) \cdot \Delta)(x \cdot \Delta)(z) \]
\[ = \sigma(x)(\Delta(y \Delta)(z) - \varepsilon(x, y)(\sigma(y) \cdot \Delta)(x \cdot \Delta)(z)) \]
\[ = \sigma(x)\left( \Delta(y \Delta)(z) + \sigma(y)\Delta^2(z) \right) - \varepsilon(x, y)\sigma(y)\left( \Delta(x \Delta)(z) + \sigma(x)\Delta^2(z) \right) \]
\[ = \left( \sigma(x)\Delta(y) - \varepsilon(x, y)\sigma(y)\Delta(x) \right) \Delta(z) + \left( \sigma(x)\sigma(y) - \varepsilon(x, y)\sigma(y)\sigma(x) \right) \Delta^2(z). \]

Since \( \mathcal{A} \) is color commutative, the last term is zero. Thus (2.2) is true. The \( \varepsilon \)-skew-symmetry condition (2.5) is clear from the definition of \( \Delta \) and the definition of \([.,.]_{\sigma}\) on the formula (2.2), it is also easy to see that \([.,.]_{\sigma}\) is an even bilinear map.

It remains to prove (2.7). Using (2.2) and that \( \Delta \) is an even color \( \sigma \)-derivation on \( \mathcal{A} \), we get
\[ \varepsilon(z,x)[\sigma(x) \cdot \Delta, [y \cdot \Delta, z \cdot \Delta]_{\sigma}]_{\sigma} = \varepsilon(z,x)[\sigma(x) \cdot \Delta, \sigma(y) \cdot \Delta(z) - \varepsilon(y,z)\sigma(z) \cdot \Delta(y)] \cdot \Delta \]
\[ = \varepsilon(z,x)[\sigma^2(x)\Delta(\sigma(y)\Delta(z) - \varepsilon(y,z)\sigma(z)\Delta(y))] \cdot \Delta \]
\[ + \varepsilon(y,z)\varepsilon(x,z+y)\sigma(z)\Delta(y) - \varepsilon(x,z+y)\sigma(y)\sigma(z)\Delta(y)] \cdot \Delta \]
\[ = \varepsilon(z,x)[\sigma^2(x)\Delta(\sigma(y)\Delta(z) + \sigma^2(x)\sigma(y)\Delta^2(z)] \]
\[ - \varepsilon(y,z)\Delta(\sigma(y)\Delta(y) - \varepsilon(y,z)\sigma(z)\Delta^2(y)) \]
\[ - \varepsilon(z,x)\varepsilon(x,z+y)\sigma^2(z)\sigma(\Delta(y))) \Delta(\sigma(x)) \cdot \Delta. \]

where \( \sigma^2 = \sigma \circ \sigma \) and \( \Delta^2 = \Delta \circ \Delta \). Applying cyclic summation to the second and fourth term in (2.10) and since \( \mathcal{A} \) is color commutative, we get
\[ \bigcup_{x,y,z} \varepsilon(z,x) \left( \sigma^2(x)\Delta(\sigma(y)\Delta^2(z) - \varepsilon(y,z)\sigma^2(x)\sigma(z)\Delta^2(y)) \cdot \Delta \right) \]
\[ = \bigcup_{x,y,z} \varepsilon(z,x) \left( \sigma^2(x)\Delta(\sigma(y)\Delta^2(z) - \varepsilon(x,y)\sigma^2(y)\sigma^2(x)\Delta^2(z)) \cdot \Delta \right) \]
\[ = 0. \]
Similarly, if we apply cyclic summation to the fifth and sixth term in (2.10) and use the relation (2.6) we obtain:

\[
\circ_{x,y,z} \varepsilon(z,x) \left( -\varepsilon(x,y)\sigma^2(y)\Delta(z)\Delta(x) + \varepsilon(y,z)\varepsilon(x,y+z)\sigma^2(z)\Delta^2(y)\Delta(x) \right)
\]

\[
= \circ_{x,y,z} \varepsilon(z,x) \left( -\varepsilon(x,y)\sigma^2(y)\sigma(\Delta(z))\Delta(x) + \varepsilon(y,z)\varepsilon(x,y+z)\sigma^2(z)\Delta^2(y)\delta(\Delta(x)) \right)
\]

\[
= \delta \circ_{x,y,z} \varepsilon(z,x) \left( (-\varepsilon(x,y)\sigma^2(y)\sigma(\Delta(z))\sigma(\Delta(x)) + \varepsilon(x,y)\sigma^2(y)\sigma(\Delta(z))\sigma(\Delta(z)) \right) \cdot \Delta
\]

\[
= 0,
\]

where we again use the color commutativity of \( \mathcal{A} \). Consequently, the only terms in the right hand side of (2.10) which do not vanish when take cyclic summation are

\[
\circ_{x,y,z} \varepsilon(z,x) \sigma(x) \cdot \Delta, [y \cdot \Delta, z \cdot \Delta]_\sigma
\]

(2.11)

We now consider the other term in (2.7). First that from (2.6) we have

\[
[y \cdot \Delta, z \cdot \Delta]_\sigma = (\sigma(y)\Delta(z) - \varepsilon(y,z)\sigma(z)\Delta(y)) \cdot \Delta.
\]

Using first this and then (2.4), we get

\[
\delta[x \cdot \Delta,[y \cdot \Delta, z \cdot \Delta]_\sigma] = \delta[x \cdot \Delta,(\sigma(y)\Delta(z) - \varepsilon(y,z)\sigma(z)\Delta(y)) \cdot \Delta]_\sigma
\]

\[
= \delta \left( \sigma(x)\Delta(\sigma(y)\Delta(z)) - \varepsilon(x,y+z)\sigma(\sigma(y)\Delta(z)) - \varepsilon(y,z)\sigma(\Delta(z)) \Delta(x) \right)
\]

\[
= \delta \left( (\varepsilon(x,y)\sigma(x)\Delta^2(z)\sigma(y) + \varepsilon(y,z)\sigma(x)\sigma(\Delta(z))\Delta(\sigma(y))) - \varepsilon(x,y)\varepsilon(y,z)\sigma(\Delta(z))\sigma^2(y)\Delta(x) + \Delta(\sigma(y))\sigma^2(z)\Delta(x) \right).
\]

Using (2.6), this equal to

\[
(\varepsilon(x,y)\delta(\sigma(z))\Delta^2(z)\sigma(y) + \varepsilon(y,z)\sigma(x)\Delta(\sigma(z))\sigma(\sigma(y)) - \delta(\sigma(x)\Delta^2(y)\sigma(z) - \delta(\sigma(x)\sigma(\Delta(z))\Delta(\sigma(z)) \cdot \Delta
\]

\[
= \left( \varepsilon(x,y)\delta(\sigma(x)\Delta^2(z)\sigma(y) - \delta(\sigma(x)\Delta^2(y)\sigma(z) - \varepsilon(x,y)\varepsilon(y,z)\Delta(\sigma(z))\sigma^2(y)\Delta(x) + \Delta(\sigma(y))\sigma^2(z)\Delta(x) \right) \cdot \Delta.
\]

The first two terms of this last expression vanish after a cyclic summation, so we get

\[
\circ_{x,y,z} \varepsilon(z,x) \delta[x \cdot \Delta,[y \cdot \Delta, z \cdot \Delta]_\sigma]_\sigma
\]

(2.12)

Finally, combining this with (2.11) we deduce

\[
\circ_{x,y,z} \varepsilon(z,x) [(\sigma(x) \cdot \Delta,[y \cdot \Delta, z \cdot \Delta]_\sigma) + \delta[x \cdot \Delta,[y \cdot \Delta, z \cdot \Delta]_\sigma]_\sigma
\]

\[
= \circ_{x,y,z} \varepsilon(z,x) [\sigma(x) \cdot \Delta,[y \cdot \Delta, z \cdot \Delta]_\sigma + \circ_{x,y,z} \varepsilon(z,x) \delta[x \cdot \Delta,[y \cdot \Delta, z \cdot \Delta]_\sigma]_\sigma
\]

\[
= \circ_{x,y,z} \varepsilon(z,x) \left( \sigma^2(x)\Delta(\sigma(y)\Delta(z) - \varepsilon(y,z)\sigma^2(x)\Delta(\sigma(z)\Delta(y)) \right) \Delta
\]

\[
+ \circ_{x,y,z} \varepsilon(z,x) \left( -\varepsilon(x,y)\varepsilon(y,z)\Delta(\sigma(y))\sigma^2(x)\Delta(z) + \Delta(\sigma(z))\sigma^2(x)\Delta(y) \right) \Delta
\]

\[
= 0,
\]

as was to be shown. The proof is complete. \( \square \)
3. Cohomology and Representations of Color Hom-Lie Algebras

In this section we define a family of cohomology complexes of color Hom-Lie algebras, discuss the representations in connection with cohomology and provide an example of computation.

3.1. Cohomology of Color Hom-Lie Algebras. We extend to color Lie algebras, the concept of $A$-module introduced in [28, 30], and then define a family of cohomology complexes for color Hom-Lie algebras.

Let $(A, [\cdot, \cdot], \varepsilon, \alpha)$ be a color Hom-Lie algebra, $(M, \beta)$ be a pair of $\Gamma$-graded vector space $M$ and an even homomorphism of vector spaces $\beta : M \rightarrow M$, and

$$[\cdot, \cdot]_M : A \times M \rightarrow M$$

be an even bilinear map satisfying $[A_{\gamma_1}, M_{\gamma_2}] \subseteq M_{\gamma_1 + \gamma_2}$ for all $\gamma_1, \gamma_2 \in \Gamma$.

**Definition 3.1.** The triple $(M, [\cdot, \cdot]_M, \beta)$ is said to be an $A$-module if the even bilinear map $[\cdot, \cdot]_M$ satisfies

$$(3.13) \quad \beta([\alpha(x), \beta(m)]_M) = [\alpha(x), \beta(m)]_M$$

and

$$(3.14) \quad ([x, y], \beta(m)]_M = [\alpha(x), [y, m]]_M - \varepsilon(x, y)[\alpha(y), [x, m]]_M.$$

The cohomology of color Lie algebras was introduced in [28]. In the following, we define cohomology complexes of color Hom-Lie algebras. The set $C^n(A, M)$ of $n$-cochains on space $A$ with values in $M$, is the set of $n$-linear maps $f : A^n \rightarrow M$ satisfying

$$f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = -\varepsilon(x_i, x_{i+1})f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n), \quad \forall \ 1 \leq i \leq n - 1.$$  

For $n = 0$, we have $C^0(A, M) = M$.

The map $f$ is called even (resp. of degree $\gamma$) when $f(x_1, \ldots, x_i, \ldots, x_n) \in M_0$ for all elements $(x_1, \ldots, x_n) \in A^{\otimes n}$ (resp. $f(x_1, \ldots, x_i, \ldots, x_n) \in M_\gamma$ for all elements $(x_1, \ldots, x_n) \in A^{\otimes n}$ of degree $\gamma$).

A $n$-cochain on $A$ with values in $M$ is defined to be a $n$-cochain $f \in C^n(A, M)$ such that it is compatible with $\alpha$ and $\beta$ in the sense that $f \circ \alpha = \beta \circ f$.

Denote by $C^n_{\alpha, \beta}(A, M)$ the set of $n$-cochains:

$$C^n_{\alpha, \beta}(A, M) = \{f : A \rightarrow M : f \circ \alpha = \beta \circ f\}.$$  

We extend this definition to the case of integers $n < 0$ and set

$$C^n_{\alpha, \beta}(A, M) = \{0\} \quad \text{if } n < -1 \quad \text{and} \quad C^0(A, M) = M.$$  

A homogeneous element $f \in C^n_{\alpha, \beta}(A, M)$ is called $n$-cochain.

Next, for a given integer $r$, we define the coboundary operator $\delta^n_r$.

**Definition 3.2.** We call, for $n \geq 1$ and for a any integer $m$, a $n$-coboundary operator of the color Hom-Lie algebra $(A, [\cdot, \cdot], \varepsilon, \alpha)$ the linear map $\delta^n : C^n_{\alpha, \beta}(A, M) \rightarrow C^{n+1}_{\alpha, \beta}(A, M)$ defined by

$$\delta^n_r(f)(x_0, \ldots, x_n) = \sum_{0 \leq s < t \leq n} (-1)^s \varepsilon(x_{s+1} + \ldots + x_{t-1}, x_t)f(\alpha(x_0), \ldots, \alpha(x_{s-1}), [x_s, x_t], \alpha(x_{s+1}), \ldots, \hat{x}_t, \ldots, \alpha(x_n))$$

$$+ \sum_{s=0}^n (-1)^s \varepsilon(\gamma + x_0 + \ldots + x_{s-1}, x_s)[\alpha^{r+n-1}(x_s), f(x_0, \ldots, \hat{x}_s, \ldots, x_n)]_M,$$

where $f \in C^n_{\alpha, \beta}(A, M)$, $\gamma$ is the degree of $f$, $(x_0, \ldots, x_n) \in H(A)^{\otimes n}$ and $\hat{x}$ indicates that the element $x$ is omitted.

In the sequel we assume that the color Hom-Lie algebra $(A, [\cdot, \cdot], \varepsilon, \alpha)$ is multiplicative.

For $n = 1$, we have

$$\delta^1 : C^1(A, M) \rightarrow C^2(A, M)$$

such that for two homogeneous elements $x, y$ in $A$

$$(3.17) \quad \delta^1_r(f)(x, y) = \varepsilon(\gamma, x)\rho(x)(f(y)) - \varepsilon(\gamma + x, y)\rho(y)(f(x)) - f([x, y]).$$
and for \( n = 2 \), we have
\[
\delta^2_f : C^2(A, M) \longrightarrow C^3(A, M)
\]
such that, for three homogeneous elements \( x, y, z \) in \( A \), we have
\[
\begin{align*}
\delta^2_f(f)(x, y, z) &= \varepsilon(\gamma, x)\rho(\alpha(x))(f(y, z)) - \varepsilon(\gamma + x, y)\rho(\alpha(y))(f(x, z)) \\
&+ \varepsilon(\gamma + x + y, z)\rho(\alpha(z))(f(x, y)) - f([x, y], \alpha(z)) \\
&+ \varepsilon(y, z)f([x, z], \alpha(y)) + f(\alpha(x), [y, z]).
\end{align*}
\]
(3.18)

**Lemma 3.1.** With the above notations, for any \( f \in C^n_{\alpha, \beta}(A, M) \), we have
\[
\delta^n_\alpha(f) \circ \alpha = \beta \circ \delta^n_\beta(f), \ \forall \ n \geq 2.
\]
Thus, we obtain a well defined map \( \delta^n : C^n_{\alpha, \beta}(A, M) \longrightarrow C^{n+1}_{\alpha, \beta}(A, M) \).

**Proof.** Let \( f \in C^n_{\alpha, \beta}(A, M) \) and \((x_0, ..., x_n) \in H(A^{\otimes n+1})\), we have
\[
\begin{align*}
\delta^n_\alpha(f) &\circ \alpha(x_0, ..., x_n) \\
&= \delta^n_\alpha(f)(\alpha(x_0), ..., \alpha(x_n)) \\
&= \sum_{0 \leq s < t \leq n} (-1)^{t-s} \varepsilon(x_0 + \ldots + x_{t-1}, x_t) f(\alpha^2(x_0), ..., \alpha^2(x_{s-1}), [\alpha(x_s), \alpha(x_t)], \alpha^2(x_{s+1}), ..., \hat{x}_t, ..., \alpha^2(x_n)) \\
&+ \sum_{s=0}^n (-1)^s \varepsilon(x_0 + \ldots + x_{s-1}, x_s) [\alpha^{n+r}(x_s), \beta \circ f(x_0, ..., \hat{x}_s, ..., x_n)]M \\
&= \sum_{0 \leq s < t \leq n} (-1)^{t-s} \varepsilon(x_0 + \ldots + x_{t-1}, x_t) f \circ \alpha(x_0), ..., \alpha(x_{s-1}), [\alpha(x_s), \alpha(x_t)], \alpha(x_{s+1}), ..., \hat{x}_t, ..., \alpha(x_n)) \\
&+ \sum_{s=0}^n (-1)^s \varepsilon(x_0 + \ldots + x_{s-1}, x_s) \beta([\alpha^{n+r-1}(x_s), f(x_0, ..., \hat{x}_s, ..., x_n)]M) \\
&= \beta \circ \delta^n_\beta(f)(x_0, ..., x_n).
\end{align*}
\]
Then \( \delta^n_\alpha(f) \circ \alpha = \beta \circ \delta^n_\beta(f) \) which completes the proof. \( \square \)

**Theorem 3.1.** Let \((A, [], [\cdot, \cdot], \varepsilon, \alpha)\) be a color Hom-Lie algebra and \((M, \beta)\) be an \( A \)-module. Then the pair \((\bigoplus_{n \geq 0} C^n_{\alpha, \beta}, \delta^n)\) is a cohomology complex. That is the maps \( \delta^n \) satisfy \( \delta^n \circ \delta^{n-1} = 0, \ \forall \ n \geq 2, \forall \ r \geq 1 \).

**Proof.** For any \( f \in C^{n-1}(A, M) \), we have
\[
\begin{align*}
\delta^{n-1}_\beta(f)(x_0, ..., x_n) &= \\
&= \sum_{s < t} (-1)^t \varepsilon(x_0 + \ldots + x_{t-1}, x_t) \delta^{n+1}(f)(\alpha(x_0), ..., \alpha(x_{s-1}), [x_s, x_t], \alpha(x_{s+1}), ..., \hat{x}_t, ..., \alpha(x_n)) \\
&+ \sum_{s=0}^n (-1)^s \varepsilon(f + x_0 + \ldots + x_{s-1}, x_s) [\alpha^{n+r-1}(x_s), \delta^{n-1}(f)(x_0, ..., \hat{x}_s, ..., x_n)]M.
\end{align*}
\]
(3.19)
From (3.19) we have
\[
\begin{align*}
\delta^{n-1}_\beta(f) &= \sum_{s < t} (-1)^t \varepsilon(x_{s+1} + \ldots + x_{t-1}, x_t) \delta^{n+1}(f)(\alpha^2(x_0), ..., \alpha^2(x_{s-1}), [\alpha(x_s), \alpha(x_t)], \alpha^2(x_{s+1}), ..., \hat{x}_t, ..., \alpha^2(x_n)) \\
&+ \sum_{s < t} (-1)^s \varepsilon(x_{s+1} + \ldots + x_{s-1}, x_s) [\alpha([x_s, x_t]), \alpha^2(x_{s+1}), ..., \hat{x}_t, ..., \alpha^2(x_n)]M.
\end{align*}
\]
(3.20)
\[
(3.22) \quad f(\alpha^2(x_0), \ldots, \alpha^2(x_{s' - 1}), [\alpha(x_{s' - 1}), [x_s, x_t]], \alpha^2(x_{s' + 1}), \ldots, \widehat{x}_{s,t}, \ldots, \alpha^2(x_n)) \\
+ \sum_{s' < s < t' < t} (-1)^{i'} \varepsilon(x_{s' + 1} + \ldots + [x_s, x_t] + \ldots + x_{t' - 1}, x_{t'})
\]
\[
(3.23) \quad f(\alpha^2(x_0), \ldots, \alpha^2(x_{s' - 1}), [\alpha(x_{s'}), \alpha(x_{t'}), \alpha^2(x_{s' + 1}), \alpha([x_s, x_t]), \ldots, \widehat{x}_{t'}, \ldots, \alpha^2(x_n)) \\
+ \sum_{s' < s < t' < t} (-1)^{i'} \varepsilon(x_{s' + 1} + \ldots x_{s - 1} + [x_s, x_t] + x_{s + 1} + \ldots + \widehat{x}_t + \ldots + x_{t' - 1}, x_{t'})
\]
\[
(3.24) \quad f(\alpha^2(x_0), \ldots, \alpha^2(x_{s' - 1}), [\alpha(x_{s'}), \alpha(x_{t'}), \alpha^2(x_{s' + 1}), \alpha([x_s, x_t]), \ldots, \widehat{x}_{t'}, \ldots, \alpha^2(x_n)) \\
+ \sum_{s' < s < t' < t} (-1)^{i'} \varepsilon(x_{s' + 1} + \ldots + x_{t' - 1}, x_{t'}) f(\alpha^2(x_0), \ldots, [x_s, x_t], \alpha(x_{t'}), \alpha^2(x_{s + 1}), \ldots, \widehat{x}_{t'}, \ldots, \alpha^2(x_n))
\]
\[
(3.25) \quad [\alpha(x_{s'}), \alpha(x_{t'}), \ldots, \widehat{x}_{t'}, \ldots, \alpha^2(x_n)) \\
+ \sum_{s' < s < t' < t} (-1)^{i'} \varepsilon(x_{s' + 1} + \ldots + \widehat{x}_t + \ldots + x_{t' - 1}, x_{t'})
\]
\[
(3.26) \quad f(\alpha^2(x_0), \ldots, \alpha^2(x_{s - 1}), [[x_s, x_t], \alpha(x_{t'}), \alpha^2(x_{s + 1}), \ldots, \widehat{x}_{t'}, \ldots, \alpha^2(x_n)) \\
+ \sum_{s' < s < t' < t} (-1)^{i'} \varepsilon(x_{s' + 1} + \ldots + x_{t' - 1}, x_{t'}) f(\alpha^2(x_0), \ldots, \alpha^2(x_{s - 1}), \alpha([x_s, x_t]), \ldots, \widehat{x}_{t'}, \ldots, \alpha^2(x_n))
\]
\[
(3.27) \quad [\alpha(x_{s'}), \alpha(x_{t'}), \ldots, \widehat{x}_{t'}, \ldots, \alpha^2(x_n)) \\
+ \sum_{s' < s < t' < t} (-1)^{i'} \varepsilon(x_{s' + 1} + \ldots + \widehat{x}_t + \ldots + x_{t' - 1}, x_{t'})
\]
\[
(3.28) \quad f(\alpha^2(x_0), \ldots, \alpha^2(x_{s - 1}), \alpha([x_s, x_t]), \alpha^2(x_{s + 1}), \ldots, [\alpha(x_{s'}), \alpha(x_{t'}), \ldots, \widehat{x}_t, \ldots, \widehat{x}_{t'}, \ldots, \alpha(x_{n})) \\
+ \sum_{s' < s < t' < t} (-1)^{i'} \varepsilon(x_{s' + 1} + \ldots + \widehat{x}_{t'}, \ldots + x_{t' - 1}, x_{t'})
\]
\[
(3.29) \quad f(\alpha^2(x_0), \ldots, \alpha^2(x_{s - 1}), \alpha([x_s, x_t]), \ldots, \widehat{x}_t, \ldots, [\alpha(x_{s'}), \alpha(x_{t'}), \ldots, \widehat{x}_{t'}, \ldots, \alpha(x_{n})) \\
+ \sum_{s' < s < t' < t} (-1)^{i'} \varepsilon(\gamma + x_0 + \ldots + x_{t' - 1}, x_{t'})
\]
\[
(3.30) \quad (\alpha^{r + n - 2}(x_s), f(\alpha(x_0), \ldots, \widehat{x}_{s}, \ldots, \alpha(x_{n})))_M \\
+ \sum_{t < s' < t} (-1)^{i'} \varepsilon(\gamma + x_0 + \ldots + x_{t' - 1}, x_{t'})
\]
\[
(3.31) \quad [\alpha^{r + n - 2}(x_s), f(\alpha(x_0), \ldots, [\widehat{x}_{s}, \ldots, \alpha(x_{n})))_M \\
+ \sum_{t < s' < t} (-1)^{i'} \varepsilon(\gamma + x_0 + \ldots + x_{t' - 1}, x_{t'})
\]
\[
(3.32) \quad [\alpha^{r + n - 2}(x_s), f(\alpha(x_0), \ldots, [x_s, x_t], \ldots, \widehat{x}_{s', t'}, \ldots, \alpha(x_{n})))_M \\
+ \sum_{t < s' < t} (-1)^{i'} \varepsilon(\gamma + x_0 + \ldots + x_{t' - 1}, x_{t'})
\]
\[
(3.33) \quad [\alpha^{r + n - 2}(x_s), f(\alpha(x_0), \ldots, [x_s, x_t], \ldots, \widehat{x}_{t', s'}, \ldots, \alpha(x_{n})))_M \\
+ \sum_{t < s' < t} (-1)^{i'} \varepsilon(\gamma + x_0 + \ldots + x_{t' - 1}, x_{t'})
\]

The identity \[\text{(3.20)}\] implies that
\[
[\alpha^{r + n - 1}(x_s), \delta^{r - 1}_{s'} f(x_0, \ldots, \widehat{x}_{s}, \ldots, x_n)]_M = [\alpha^{r + n - 1}(x_s), \sum_{t < s'} (-1)^{i'} \varepsilon(x_{s' + 1} + \ldots + x_{t' - 1}, x_{t'}) \\
f(\alpha(x_0), \ldots, \alpha(x_{s' - 1}), \alpha(x_{s'}), \alpha(x_{t'}), \alpha(x_{s' + 1}), \ldots, \widehat{x}_{t, s'}, \ldots, \alpha(x_{n})))_M
\]
Thus, by a simple calculation, we get

\[ f(\alpha(x_0), ..., \alpha(x_{s'-1}), [x_{s'}, x_{t'}], \alpha(x_{s'+1}), ..., \hat{x}_{t'}, ..., \alpha(x_n))]_M \\
= [\alpha^{r+n-1}(x_s), \sum_{s' < s < t} (-1)^{t'-1} \varepsilon(x_{s'+1} + ... + \hat{x}_s + ... + x_{t'-1}, x_{t'})] \\
\]

By the \(\varepsilon\)-Hom-Jacobi condition, we obtain

\[ f(\alpha(x_0), ..., \alpha(x_{s'-1}), [x_{s'}, x_{t'}], \alpha(x_{s'+1}), ..., \hat{x}_{t'}, ..., \alpha(x_n))]_M \\
+ [\alpha^{r+n-1}(x_s), \sum_{s' < s < t} (-1)^{t'-1} \varepsilon(x_{s'+1} + ... + x_{t'-1}, x_{t'})] \\
\]

\[ f(\alpha(x_0), ..., \hat{x}_s, ..., \alpha(x_{s'-1}), [x_{s'}, x_{t'}], \alpha(x_{s'+1}), ..., \hat{x}_{t'}, ..., \alpha(x_n))]_M \\
+ [\alpha^{r+n-1}(x_s), \sum_{s' = s+1}^{s-1} (-1)^{s'-1} \varepsilon(\gamma + x_0 + ... + \hat{x}_s + ... + x_{s'-1}, x_{s'})[\alpha^{r+n-2}(x_s'), f(x_0, ..., \hat{x}_{s'}, ..., x_n)]_M \\
= [\alpha^{r+n-1}(x_s), \sum_{s' = s+1}^{s-1} (-1)^{s'-1} \varepsilon(\gamma + x_0 + ... + \hat{x}_s + ... + x_{s'-1}, x_{s'})[\alpha^{r+n-2}(x_s'), f(x_0, ..., \hat{x}_{s'}, ..., x_n)]_M \\
\]

By the \(\varepsilon\)-Hom-Jacobi condition, we obtain

\[ \sum_{s < t} (-1)^t \varepsilon(x_{s+1} + ... + x_{t-1}, x_t)(3.22) + (3.25) + (3.26) = 0. \]

Using (3.31) and (3.34), we obtain

\[ \sum_{s < t} (-1)^t \varepsilon(x_{s+1} + ... + x_{t-1}, x_t) + \sum_{s = 0}^{n} (-1)^s \varepsilon(\gamma + x_0 + ... + x_{s-1}, x_s) \]

Thus

\[ \sum_{s < t} (-1)^t \varepsilon(x_{s+1} + ... + x_{t-1}, x_t) + \sum_{s = 0}^{n} (-1)^s \varepsilon(\gamma + x_0 + ... + x_{s-1}, x_s) \]

By a simple calculation, we get

\[ \sum_{s < t} (-1)^t \varepsilon(x_{s+1} + ... + x_{t-1}, x_t) + \sum_{s = 0}^{n} (-1)^s \varepsilon(\gamma + x_0 + ... + x_{s-1}, x_s) = 0, \]

and

\[ \sum_{s < t} (-1)^t \varepsilon(x_{s+1} + ... + x_{t-1}, x_t) + \sum_{s = 0}^{n} (-1)^s \varepsilon(\gamma + x_0 + ... + x_{s-1}, x_s) = 0. \]

\[ = \sum_{s < t} (-1)^t \varepsilon(x_{s+1} + ... + x_{t-1}, x_t) \left( \sum_{s' < s < t' < t} (-1)^{t'-1} \varepsilon(x_{s'+1} + ... + [x_s, x_t] + ... + x_{t'-1}, x_{t'}) \right) \\
f(\alpha^2(x_0), ..., \alpha^2(x_{s'-1}), [\alpha(x_{s'}), \alpha(x_{t'})], \alpha^2(x_{s'+1}), \alpha([x_s, x_t]), ..., \hat{x}_{t'}, ..., \alpha^2(x_n)) \\
+ \sum_{s < t} (-1)^t \varepsilon(x_{s+1} + ... + x_{t-1}, x_t) \left( \sum_{s' < s < t' < t} (-1)^{t'-1} \varepsilon(x_{s'+1} + ... + \hat{x}_s + ... + x_{t'-1}, x_{t'}) \right) \\
f(\alpha^2(x_0), ..., \alpha^2(x_{s'-1}), \alpha([x_s, x_t]), \alpha^2(x_{s'+1}), ..., \hat{x}_{t'}, ..., \alpha(x_{s'}), \alpha(x_{t'}), ..., \alpha^2(x_n)) \\
= 0. \]
Similarly, we have
\[ \sum_{s<t} (-1)^t \varepsilon(x_{s+1} + \ldots + x_{t-1}, x_t) (s.t) = 0 \]
and
\[ \sum_{s<t} (-1)^t \varepsilon(x_{s+1} + \ldots + x_{t-1}, x_t) (s.t) = 0. \]
Therefore \( \delta_r^n \circ \delta_r^{n-1} = 0 \). Which completes the proof. \( \square \)

Let \( Z^n_r(\mathcal{A}, M) \) (resp. \( B^n_r(\mathcal{A}, M) \)) denote the kernel of \( \delta^n_r \) (resp. the image of \( \delta^n_r \)). The spaces \( Z^n_r(\mathcal{A}, M) \) and \( B^n_r(\mathcal{A}, M) \) are graded submodules of \( C^n_{\alpha,\beta}(\mathcal{A}, M) \) and according to Proposition 3.1, we have
(3.38) \[ B^n_r(\mathcal{A}, M) \subseteq Z^n_r(\mathcal{A}, M). \]
The elements of \( Z^n_r(\mathcal{A}, M) \) are called n-cocycles, and the elements of \( B^n_r(\mathcal{A}, M) \) are called the n-coboundaries. Thus, we define a so-called cohomology groups
\[ H^n_r(\mathcal{A}, M) = \frac{Z^n_r(\mathcal{A}, M)}{B^n_r(\mathcal{A}, M)}. \]

We denote by \( H^n(\mathcal{A}, M) = \bigoplus_{d \in \mathbb{Z}} (H^n_r(\mathcal{A}, M))_d \) the space of all \( r \)-cohomology group of degree \( d \) of the color Hom-Lie algebra \( \mathcal{A} \) with values in \( M \).

Two elements of \( Z^n_r(\mathcal{A}, M) \) are said to be cohomologeous if their residue class modulo \( B^n_r(\mathcal{A}, M) \) coincide, that is if their difference lies in \( B^n_r(\mathcal{A}, M) \).

3.2. Adjoint representations of color Hom-Lie algebras. In this section, we generalize to color Hom-Lie algebras some results from \([9]\) and \([30]\). Let \( (\mathcal{A}, [\[,\]], \varepsilon, \alpha) \) be a multiplicative color Hom-Lie algebra. We consider \( \mathcal{A} \) represents on itself via bracket with respect to the morphism \( \alpha \).

Definition 3.3. Let \( (\mathcal{A}, [\[,\]], \varepsilon, \alpha) \) be a color Hom-Lie algebra. A representation of \( \mathcal{A} \) is a triplet \( (M, \rho, \beta) \), where \( M \) is a \( \Gamma \)-graded vector space, \( \beta \in \text{End}(M)_0 \) and \( \rho: \mathcal{A} \rightarrow \text{End}(M) \) is an even linear map satisfying
(3.39) \[ \rho([x,y]) \circ \beta = \rho(\alpha(x)) \circ \rho(y) - \varepsilon(x,y)\rho(\alpha(y)) \circ \rho(x), \quad \forall \ x, y \in \mathcal{H}(\mathcal{A}). \]

Now, we discuss the adjoint representations of a color Hom-Lie algebra.

Proposition 3.1. Let \( (\mathcal{A}, [\[,\]], \varepsilon, \alpha) \) be a color Hom-Lie algebra and \( ad: \mathcal{A} \rightarrow \text{End}(\mathcal{A}) \) be an operator defined for \( x \in \mathcal{H}(\mathcal{A}) \) by \( ad(x)(y) = [x,y] \). Then \( (\mathcal{A}, ad, \alpha) \) is a representation of \( \mathcal{A} \).

Proof. Since \( \mathcal{A} \) is color Hom-Lie algebra, the \( \varepsilon \)-Hom-Jacobi condition on \( x, y, z \in \mathcal{H}(\mathcal{A}) \) is
\[ \varepsilon(z, x)[\alpha(x), [y, z]] + \varepsilon(y, z)[\alpha(z), [x, y]] + \varepsilon(x, y)[\alpha(y), [z, x]] = 0 \]
and may be written
\[ ad([x,y])(\alpha(z)) = ad(\alpha(x))(ad(y)(z)) - \varepsilon(x,y)ad(\alpha(y))(ad(x)(z)). \]
Then the operator \( ad \) satisfies
\[ ad([x,y]) \circ \alpha = ad(\alpha(x)) \circ ad(y) - \varepsilon(x,y)ad(\alpha(y)) \circ ad(x). \]
Therefore, it determines a representation of the color Hom-Lie algebra \( \mathcal{A} \). \( \square \)

We call the representation defined in the previous Proposition the adjoint representation of the color Hom-Lie algebra \( \mathcal{A} \).

Definition 3.4. The \( \alpha^s \)-adjoint representation of the color Hom-Lie algebra \( (\mathcal{A}, [\[,\]], \varepsilon, \alpha) \), which we denote by \( ad_s \), is defined by
\[ ad_s(a)(x) = [\alpha^s(a), x], \quad \forall \ a, x \in \mathcal{H}(\mathcal{A}). \]

Lemma 3.2. With the above notations, we have \( (\mathcal{A}, ad_s(\cdot), \alpha) \) is a representation of the color Hom-Lie algebra \( (\mathcal{A}, [\[,\]], \varepsilon, \alpha) \).

\[ ad_s(\alpha(x)) \circ \alpha = \alpha \circ ad_s(x). \]

\[ ad_s([x,y]) \circ \alpha = ad_s(\alpha(x)) \circ ad_s(y) - \varepsilon(x,y)ad_s(\alpha(y)) \circ ad_s(x). \]
Proposition 3.2. Associated to the \( \gamma \)

\[
\text{Proof.} \quad \text{First, the result follows from}
\]

\[
ad_s(\alpha(x))(\alpha(y)) = [\alpha^{s+1}(x), \alpha(y)]
\]

\[
= \alpha([\alpha^s(x), y])
\]

\[
= \alpha \circ ad_s(x)(y)
\]

and

\[
ad_s([x, y])(\alpha(z)) = [\alpha([x, y]), \alpha(z)] = [[\alpha^s(x), \alpha^s(y)], \alpha(z)]
\]

\[
= \varepsilon(x + y, z)\varepsilon(z, x)\alpha^s(y)[\alpha^{s+1}(x), \alpha^s(y), z]
\]

\[
+ \varepsilon(x + y, z)\varepsilon(z, y)\varepsilon(x, y)[\alpha^{s+1}(y), [z, \alpha^{s+1}(x)]
\]

\[
= [\alpha^{s+1}(x), [\alpha^s(y), z] + \varepsilon(x + y, z)\alpha^{s+1}(y), [z, \alpha^{s+1}(x)]
\]

\[
= [\alpha^{s+1}(x), \alpha^s(y), z] - \varepsilon(x, y)[\alpha^{s+1}(y), [\alpha^s(x), z]
\]

\[
= [\alpha^{s+1}(x), ad_s(y)(z)] - \varepsilon(x, y)[\alpha^{s+1}(y), ad_s(x)(z)]
\]

\[
= ad_s(\alpha(x)) \circ ad_s(y)(z) - \varepsilon(x, y)ad_s(\alpha(y)) \circ ad_s(x)(z).
\]

Then \( ad_s([x, y]) \circ \alpha = ad_s(\alpha(x)) \circ ad_s(y) - \varepsilon(x, y)ad_s(\alpha(y)) \circ ad_s(x). \) \( \square \)

The set of \( n \)-cochains on \( A \) with coefficients in \( A \) which we denote by \( C^n(\mathcal{A}, A) \), is given by

\[
C^n(\mathcal{A}, A) = \{ f \in C^n(\mathcal{A}, A) : f \circ \alpha^{\otimes n} = \alpha \circ f \}.
\]

In particular, the set of 0-cochains is given by

\[
C^0(\mathcal{A}, A) = \{ x \in \mathcal{H}(\mathcal{A}) : \alpha(x) = x \}.
\]

**Proposition 3.2.** Associated to the \( \alpha^s \)-adjoint representation \( ad_s \), of the color Hom-Lie algebra \( (\mathcal{A}, [,], \varepsilon, \alpha) \), \( D \in C^1_{\alpha, ad_s}(\mathcal{A}, A) \) is 1-cocycle if and only if \( D \) is an \( \alpha^{s+1} \)-derivation of the color Hom-Lie algebra \( (\mathcal{A}, [,], \varepsilon, \alpha) \) of degree \( \gamma \). (i.e. \( D \in (\text{Der}_{\alpha^{s+1}}(\mathcal{A}))_\gamma \).)

**Proof.** The statement follows directly from the definition of the coboundary \( \delta \). \( D \) is closed if and only if

\[
\delta(D)(x, y) = -D([x, y]) + \varepsilon(\gamma, x)[\alpha^{s+1}(x), D(y)] - \varepsilon(\gamma, y)[\alpha^{s+1}(y), D(x)] = 0.
\]

So

\[
D([x, y]) = [D(x), \alpha^{s+1}(y)] + \varepsilon(\gamma, x)[\alpha^{s+1}(x), D(y)]
\]

which implies that \( D \) is an \( \alpha^{s+1} \)-derivation of \( (\mathcal{A}, [,], \varepsilon, \alpha) \) of degree \( \gamma \). \( \square \)

**3.2.1.** The \( \alpha^{-1} \)-adjoint representation \( ad_{-1} \).

**Proposition 3.3.** Associated to the \( \alpha^{-1} \)-adjoint representation \( ad_{-1} \), we have

\[
H^0(\mathcal{A}, A) = C^0(\mathcal{A}, A) = \{ x \in \mathcal{H}(\mathcal{A}) : \alpha(x) = x \}.
\]

\[
H^1(\mathcal{A}, A) = \text{Der}_{\alpha^0}(\mathcal{A}).
\]

**Proof.** For any 0-cochain \( x \in C^0(\mathcal{A}, A) \), we have

\[
\delta(x)(y) = \varepsilon(x, y)[\alpha^{-1}(y), x] = 0, \quad \forall y \in \mathcal{H}(\mathcal{A}).
\]

Therefore any 0-cochain is closed. Thus, we have

\[
H^0(\mathcal{A}, A) = C^0(\mathcal{A}, A) = \{ x \in \mathcal{H}(\mathcal{A}) : \alpha(x) = x \}.
\]

Since there is not exact 1-cochain, by Proposition 3.2 we have

\[
H^1(\mathcal{A}, A) = \text{Der}_{\alpha^0}(\mathcal{A}).
\]

Let \( w \in C^2(\mathcal{A}, A) \) be an even \( \varepsilon \)-skew-symmetric bilinear operator commuting with \( \alpha \). Considering a \( t \)-parametrized family of bilinear operations

\[
[x, y]_t = [x, y] + tw(x, y).
\]

Since \( w \) commute with \( \alpha \), \( \alpha \) is a morphism with respect to the bracket \([.,.],\) for every \( t \). If all bracket \([.,.],\) endow that \( w \) generates a deformation of the color Hom-Lie algebra \( (\mathcal{A}, [,], \varepsilon, \alpha) \). By computing the \( \varepsilon \)-Hom-Jacobi condition of \([.,.],\), this is equivalent to

\[
(3.40) \quad \otimes_{x,y,z} \varepsilon(z, x) \left( w(\alpha(x), [y, z]) + [\alpha(x), w(y, z)] \right) = 0,
\]
(3.41) \[ \cap_{x,y,z} \varepsilon(z,x)w(\alpha(x),w(y,z)) = 0. \]

Obviously, (3.40) means that \( \varepsilon \) is an even 2-cocycle with respect to the \( \alpha^{-1} \)-adjoint representation \( ad_{-1} \). Furthermore, (3.41) means that \( \varepsilon \) must itself defines a color Hom-Lie algebra structure on \( \mathcal{A} \).

3.2.2. The \( \alpha^0 \)-adjoint representation \( ad_0 \).

**Proposition 3.4.** Associated to the \( \alpha^0 \)-adjoint representation \( ad_0 \), we have

\[
\begin{align*}
H^0(\mathcal{A}, \mathcal{A}) &= \{ x \in \mathcal{H}(\mathcal{A}) : \alpha(x) = x, [x, y] = 0 \}. \\
H^1(\mathcal{A}, \mathcal{A}) &= \frac{\text{Der}_\alpha(\mathcal{A})}{\text{Hom}_\mathcal{A}(\mathcal{A})}.
\end{align*}
\]

**Proof.** For any 0-cochain, we have \( d_0(x)(y) = [\alpha^0(x), y] = [x, y] \). Therefore, the set of 0-cocycle \( Z^0(\mathcal{A}, \mathcal{A}) \) is given by

\[
Z^0(\mathcal{A}, \mathcal{A}) = \{ x \in C^0(\mathcal{A}, \mathcal{A}) : [x, y] = 0, \forall y \in \mathcal{H}(\mathcal{A}) \}.
\]

As, \( B^0(\mathcal{A}, \mathcal{A}) = \{ 0 \} \), we deduce that \( H^0(\mathcal{A}, \mathcal{A}) = \{ x \in \mathcal{H}(\mathcal{A}) : \alpha(x) = x, [x, y] = 0 \} \). For any \( f \in C^1(\mathcal{A}, \mathcal{A}) \), we have

\[
\begin{align*}
\delta(f)(x, y) &= -f([x, y]) + \varepsilon(f, x)[\alpha(x), f(y)] - \varepsilon(f + x, y)[\alpha(y), f(x)] \\
&= -f([x, y]) + \varepsilon(f, x)[\alpha(x), f(y)] + [f(x), \alpha(y)].
\end{align*}
\]

Therefore, the set of 1-cocycles is exactly the set of \( \alpha \)-derivations \( \text{Der}_\alpha(\mathcal{A}) \). Furthermore, it is obvious that any exact 1-coboundary is of the form of \( [x, .] \) for some \( x \in C^0(\mathcal{A}, \mathcal{A}) \). Therefore, we have \( B^1(\mathcal{A}, \mathcal{A}) = \text{Hom}_\mathcal{A}(\mathcal{A}) \). Which implies that \( H^1(\mathcal{A}, \mathcal{A}) = \frac{\text{Der}_\alpha(\mathcal{A})}{\text{Hom}_\mathcal{A}(\mathcal{A})} \). \( \square \)

3.2.3. The coadjoint representation \( \tilde{\alpha} \).

In this subsection, we explore the dual representations and coadjoint representations of color Hom-Lie algebras. Let \( (\mathcal{A}, [, .], \varepsilon, \alpha) \) be a color Hom-Lie algebra and \( (M, \rho, \beta) \) be a representation of \( \mathcal{A} \). Let \( M^* \) be the dual vector space of \( M \). We define a linear map \( \tilde{\rho} : \mathcal{A} \rightarrow \text{End}(M^*) \) by \( \tilde{\rho}(x) = -\varepsilon \rho(x) \). Let \( f \in M^*, x, y \in \mathcal{H}(\mathcal{A}) \) and \( m \in M \). We compute the right hand side of the identity (3.39).

\[
(\tilde{\rho}(\alpha(x)) \circ \tilde{\rho}(y) - \varepsilon(x, y)\tilde{\rho}(\alpha(y)) \circ \tilde{\rho}(x))(f)(m)
= (\tilde{\rho}(\alpha(x)) \circ \tilde{\rho}(y)(f) - \varepsilon(x, y)\tilde{\rho}(\alpha(y)) \circ \tilde{\rho}(x)(f))(m)
= -\tilde{\rho}(y)(f)(\rho(\alpha(x))(m)) + \varepsilon(x, y)\tilde{\rho}(x)(f)(\rho(\alpha(y))(m))
= f(\rho(y) \circ \rho(\alpha(x))(m)) - \varepsilon(x, y)f(\rho(x) \circ \rho(\alpha(y))(m))
= f(\rho(y) \circ \rho(\alpha(x))(m)) - \varepsilon(x, y)f(\rho(x) \circ \rho(\alpha(y))(m)).
\]

On the other hand, we set that the twisted map for \( \tilde{\rho} \) is \( \tilde{\beta} = ^\varepsilon \beta \), the left hand side of (3.39) writes

\[
\tilde{\rho}([x, y]) \circ \tilde{\beta}(f)(m) = \tilde{\rho}([x, y])(f \circ \beta)(m)
= -f \circ \beta(\rho([x, y])(m)).
\]

Therefore, we have the following Proposition:

**Proposition 3.5.** Let \( (\mathcal{A}, [, .], \varepsilon, \alpha) \) be a color Hom-Lie algebra and \( (M, \rho, \beta) \) be a representation of \( \mathcal{A} \). Let \( M^* \) be the dual vector space of \( M \). The triple \( (M^*, \tilde{\rho}, \tilde{\beta}) \), where \( \tilde{\rho} : \mathcal{A} \rightarrow \text{End}(M^*) \) is given by \( \tilde{\rho}(x) = -\varepsilon \rho(x) \), defines a representation of color Hom-Lie algebra \( (\mathcal{A}, [, .], \varepsilon, \alpha) \) if and only if

\[
(3.42) \quad \beta \circ \rho([x, y]) = \rho(x) \circ \rho(\alpha(y)) - \varepsilon(x, y)\rho(y) \circ \rho(\alpha(x)).
\]

We obtain the following characterization in the case of adjoint representation.

**Corollary 3.1.** Let \( (\mathcal{A}, [, .], \varepsilon, \alpha) \) be a color Hom-Lie algebra and \( (\mathcal{A}, ad, \alpha) \) be the adjoint representation of \( \mathcal{A} \), where \( ad : \mathcal{A} \rightarrow \text{End}(\mathcal{A}) \). We set \( \tilde{ad} : \mathcal{A} \rightarrow \text{End}(\mathcal{A}^*) \) and \( \tilde{ad}(x)(f) = -f \circ ad(x) \). Then \( (\mathcal{A}^*, \tilde{ad}, \tilde{\alpha}) \) is a representation of \( \mathcal{A} \) if and only if

\[
\alpha \circ ad([x, y]) = ad(x) \circ ad(\alpha(y)) - \varepsilon(x, y)ad(y) \circ ad(\alpha(x)), \forall x, y \in \mathcal{H}(\mathcal{A}).
\]
3.3. Example. Let $(sl^c_2, \mathbb{C})$ be a color Lie algebra such that $sl^c_2 = \bigoplus_{\gamma \in \Gamma} X_\gamma$ and $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$, with
\[ X_{(0,0)} = 0, \ X_{(1,0)} = e_1, \ X_{(0,1)} = e_2, \ X_{(1,1)} = e_3. \]
The algebra $(sl^c_2, \mathbb{C})$ has a homogeneous basis $\{e_1, e_2, e_3\}$ with degree given by
\[ |e_1| = \gamma_1 = (1, 0), \ |e_2| = \gamma_2 = (0, 1), \ |e_3| = \gamma_3 = (1, 1). \]
The bracket $[\cdot, \cdot]$ in $sl^c_2$ is given by
\[ [e_1, e_2] = e_3, \ [e_1, e_3] = e_2, \ [e_2, e_3] = e_1. \]
Then $(sl^c_2, [\cdot, \cdot])$ is a color Lie algebra. According to Theorem 11.1 the triple $(sl^c_2, [\cdot, \cdot], \alpha)$ is a color Hom-Lie algebra such that the bracket $[\cdot, \cdot], \alpha$ and the even linear map $\alpha$ are defined by
\[ [e_1, e_2]_\alpha = e_3, \ \alpha(e_1) = -e_1 \]
\[ [e_1, e_3]_\alpha = -e_2, \ \alpha(e_2) = -e_2 \]
\[ [e_2, e_3]_\alpha = -e_1, \ \alpha(e_3) = e_3. \]
Let $\psi \in C^1_{ad}(sl^c_2, sl^c_2)$. The 2-coboundary is defined by equation (3.18). Now, suppose that $\psi$ is a 2-cocycle of $sl^c_2$. Then $\psi$ satisfies
\[ \psi(\alpha(x), [y, z]) = \psi([x, y], \alpha(z)) - \varepsilon(\gamma, x) [\alpha(x), \psi(y, z)]_\alpha + \varepsilon(\gamma + x, y) [\alpha(y), \psi(x, z)]_\alpha \]
(3.43)
By plugging the following triples
\[ (e_1, e_2, e_3), \ (e_1, e_2, e_3), \ (e_2, e_1, e_3), \ (e_2, e_3, e_1), \]
\[ (e_3, e_1, e_2), \ (e_3, e_2, e_1), \cdots \ (e_3, e_3, e_2), \ (e_3, e_3, e_3). \]
respectively in (3.43).
Case 1: If $\gamma = \gamma_1 = (1, 0)$, we obtain:
- $Z^2_{\gamma_1}(sl^c_2, sl^c_2) = \{ \psi : \psi(e_i, e_i) = 0, \ \psi(e_i, e_j) = \psi(e_j, e_i), \ \forall \ i \neq j, \ i = 1, 2, 3 \}$
  \[ = \{ \psi : \psi(e_1, e_2) = a_1 e_2 + a_2 e_3, \ \psi(e_1, e_3) = a_3 e_1 + a_4 e_2 + a_1 e_3, \ \psi(e_2, e_3) = a_5 e_2 \}, \]
- $B^2_{\gamma_1}(sl^c_2, sl^c_2) = \{ \delta f : \delta f(e_i, e_i) = 0, \ \delta f(e_1, e_2) = a_2 e_3, \ \delta f(e_1, e_3) = a_5 e_2, \ \delta f(e_2, e_3) = 0, \ \forall \ i = 1, 2, 3 \}$
Then $H^2_{\gamma_1}(sl^c_2, sl^c_2) = \{ \psi : \psi(e_1, e_2) = a_1 e_2, \ \psi(e_1, e_3) = a_3 e_1 + a_1 e_3 \}.$

Case 2: If $\gamma = \gamma_2 = (0, 1)$, we obtain:
- $Z^2_{\gamma_2}(sl^c_2, sl^c_2) = \{ \psi : \psi(e_i, e_i) = 0, \ \psi(e_i, e_j) = \psi(e_j, e_i), \ \forall \ i \neq j, \ i = 1, 2, 3 \}$
  \[ = \{ \psi : \psi(e_1, e_2) = 0, \ \psi(e_1, e_3) = a_2 e_3, \ \psi(e_2, e_3) = a_5 e_1 \}, \]
- $B^2_{\gamma_2}(sl^c_2, sl^c_2) = \{ \delta f : \delta f(e_i, e_i) = 0, \ \delta f(e_1, e_2) = a_2 e_3, \ \delta f(e_1, e_3) = 0, \ \delta f(e_2, e_3) = a_5 e_1, \ \forall \ i = 1, 2, 3 \}$
Then $H^2_{\gamma_2}(sl^c_2, sl^c_2) = \{ 0 \}.$

Case 3: If $\gamma = \gamma_2 = (1, 1)$, we obtain:
- $Z^2_{\gamma_3}(sl^c_2, sl^c_2) = \{ \psi : \psi(e_i, e_i) = 0, \ \psi(e_i, e_j) = \psi(e_j, e_i), \ \forall \ i \neq j, \ i = 1, 2, 3 \}$
  \[ = \{ \psi : \psi(e_1, e_2) = 0, \ \psi(e_1, e_3) = a_1 e_1 + a_2 e_2, \ \psi(e_2, e_3) = a_3 e_1 - a_1 e_2 \}, \]
- $B^2_{\gamma_3}(sl^c_2, sl^c_2) = \{ \delta f : \delta f(e_i, e_i) = 0, \ \delta f(e_1, e_2) = 0, \ \delta f(e_1, e_3) = a_1 e_1 + a_2 e_2, \ \delta f(e_2, e_3) = a_3 e_1 - a_1 e_2, \ \forall \ i = 1, 2, 3 \}$
Then $H^2_{\gamma_3}(sl^c_2, sl^c_2) = \{ 0 \}.$

So $H^2_{\gamma_1}(sl^c_2, sl^c_2) + H^2_{\gamma_2}(sl^c_2, sl^c_2) + H^2_{\gamma_3}(sl^c_2, sl^c_2) = \{ \psi : \psi(e_1, e_2) = a_1 e_2, \ \psi(e_1, e_3) = a_3 e_1 + a_1 e_3 \}.$
4. Formal deformations of color Hom-Lie algebras

4.1. Formal deformations of color Hom-Lie algebras.

**Definition 4.1.** Let \((A, [\cdot, \cdot], \varepsilon, \alpha)\) be a color Hom-Lie algebra. A one parameter formal deformation of \(A\) is given by \(K[[t]]\)-bilinear map \([\cdot, \cdot]: (\varepsilon, \alpha)\) of the form \([\cdot, \cdot]_t = \sum_{i \geq 0} t^i [\cdot, \cdot]_i\) where \([\cdot, \cdot]_i\) is an even \(K\)-bilinear map \([\cdot, \cdot]: (\varepsilon, \alpha)\) of \(K[[t]]\)-bilinear and satisfying for all \(x, y, z \in K\)

\[
\begin{align*}
[x, y]_t &= -\varepsilon(x, y)[y, x]_t, \\
\circ_{x, y, z} &\in \varepsilon(z, x)[\alpha(x), [y, z]_t] = 0.
\end{align*}
\]

The deformation is said to be of order \(k\) if \([\cdot, \cdot]_t = \sum_{i \geq 0} t^i [\cdot, \cdot]_i\).

**Remark 4.1.** The \(\varepsilon\)-skew symmetry of \([\cdot, \cdot]_t\) is equivalent to the \(\varepsilon\)-skew symmetry of \([\cdot, \cdot]_1\) for \(i \in \mathbb{Z}_{\geq 0}\).

Condition (4.45) is called deformation equation of the color Hom-Lie algebra and it is equivalent to

\[
\bigcirc_{x, y, z} \sum_{i, j, k \geq 0} \varepsilon(z, x) t^{i+j}[\alpha(x), [y, z]_i] = 0
\]
i.e

\[
\bigcirc_{x, y, z} \sum_{i, s \geq 0} \varepsilon(z, x) t^i [\alpha(x), [y, z]_i] s_i = 0
\]
or

\[
\sum_{s \geq 0} t^s \bigcirc_{x, y, z} \sum_{i, s \geq 0} \varepsilon(z, x) [\alpha(x), [y, z]_i] s_i = 0
\]

which is equivalent to the following infinite system

\[
\bigcirc_{x, y, z} \sum_{i, k \geq 0} \varepsilon(z, x) [\alpha(x), [y, z]_i] s_i = 0, \quad \forall \ s = 0, 1, 2 \cdots
\]

In particular, for \(s = 0\), we have \(\bigcirc_{x, y, z} \varepsilon(z, x) [\alpha(x), [y, z]_0] = 0\), which is the \(\varepsilon\)-Hom-Jacobi condition of \(A\).

The equation for \(s = 1\), leads to \(\delta^2([\cdot, \cdot]_1)(x, y, z) = 0\). Then \([\cdot, \cdot]_1\) is a 2-cocycle.

For \(s \geq 2\), the identity (4.46) is equivalent to

\[
\delta^2([\cdot, \cdot]_s)(x, y, z) = -\sum_{p+q=s} \bigcirc_{x, y, z} \varepsilon(z, x) [\alpha(x), [y, z]_p] q = 0.
\]

4.2. Equivalent and trivial deformations.

**Definition 4.2.** Let \((A, [\cdot, \cdot], \varepsilon, \alpha)\) be a multiplicative color Hom-Lie algebra. Given two deformations \(A_t = (A, [\cdot, \cdot]_t, \varepsilon, \alpha)\) and \(A'_t = (A, [\cdot, \cdot]'_t, \varepsilon, \alpha')\) of \(A\) where \([\cdot, \cdot]_t = \sum_{i \geq 0} t^i [\cdot, \cdot]_i\) and \([\cdot, \cdot]'_t = \sum_{i \geq 0} t^i [\cdot, \cdot]'_i\) with \([\cdot, \cdot]_0 = [\cdot, \cdot]'_0 = [\cdot, \cdot]\). We say that \(A_t\) and \(A'_t\) are equivalent if there exists a formal automorphism \(\phi_t : A[[t]] \rightarrow A[[t]]\) that may be written in the form \(\phi_t = \sum_i \phi_i t^i\), where \(\phi_i \in \text{End}(A)_0\) and \(\phi_0 = Id\) such that

\[
\begin{align*}
\phi_t([x, y]_t) &= [\phi_t(x), \phi_t(y)]' \seteq \\
\phi_t(\alpha(x)) &= \alpha'(\phi_t(x))
\end{align*}
\]

A deformation \(A_t\) of \(A\) is said to be trivial if and only if \(A_t\) is equivalent to \(A\). Viewed as an algebra on \(A[[t]]\).

**Definition 4.3.** Let \((A, [\cdot, \cdot], \varepsilon, \alpha)\) be a color Hom-Lie algebra and \([\cdot, \cdot]_1 \in Z^2(A, A)\).

The 2-cocycle \([\cdot, \cdot]_1\) is said to be integrable if there exists a family \((\varepsilon, \alpha)_{i \geq 0}\) such that \([\cdot, \cdot]_t = \sum_{i \geq 0} t^i [\cdot, \cdot]_i\) defines a formal deformation \(A_t = (A, [\cdot, \cdot]_t, \varepsilon, \alpha)\) of \(A\).

**Theorem 4.1.** Let \((A, [\cdot, \cdot], \varepsilon, \alpha)\) be a color Hom-Lie algebra and \(A_t = (A, [\cdot, \cdot]_t, \varepsilon, \alpha)\) be a one parameter formal deformation of \(A\), where \([\cdot, \cdot]_t = \sum_{i \geq 0} t^i [\cdot, \cdot]_i\). Then

(1) The first term \([\cdot, \cdot]_1\) is a 2-cocycle with respect to the cohomology of \((A, [\cdot, \cdot], \varepsilon, \alpha)\).
(2) there exists an equivalent deformation \( \mathcal{A}'_t = (\mathcal{A}, [,], \varepsilon, \alpha') \), where \([,]'_t = \sum_{i \geq 0} t^i [,]'_t \) such that \([,]'_1 \in Z^2(\mathcal{A}, \mathcal{A}) \) and \([,]'_1 \not\in B^2(\mathcal{A}, \mathcal{A}) \).

Moreover, if \( H^2(\mathcal{A}, \mathcal{A}) = 0 \), then every formal deformation is trivial.

### 4.3. Deformation by composition

In the sequel, we give a procedure of deforming color Lie algebras into color Hom-Lie algebras using the following Proposition:

**Proposition 4.1.** Let \((\mathcal{A}, [.,], \varepsilon)\) be a color Lie algebra and \(\alpha_t\) be an even algebra endomorphism of the form \(\alpha_t = \alpha_0 + \sum_{i \geq 1} t^i \alpha_i\), where \(\alpha_i\) are linear maps on \(\mathcal{A}\), \(t\) is a parameter in \(\mathbb{K}\) and \(p\) is an integer. Let \([.,]' = \alpha_t \circ [.,] \), then \((\mathcal{A}, [.,]', \varepsilon, \alpha_t)\) is a color Hom-Lie algebra which is a deformation of the color Lie algebra viewed as a color Hom-Lie algebra \((\mathcal{A}, [.,], \varepsilon, \text{Id})\).

Moreover, the \(n\)th derived Hom-algebra

\[
\mathcal{A}^n_t = (\mathcal{A}, [.,]'_t^n) = \alpha_t^{2^n-1} \circ [.,]'_t, \varepsilon, \alpha_t^{2^n}
\]

is a deformation of \((\mathcal{A}, [.,], \varepsilon, \text{Id})\).

**Proof.** The first assertion follows from Theorem 1.1. In particular for an infinitesimal deformation of the identity \(\alpha_t = \text{Id} + t \alpha_1\), we have \([.,]' = [.,] + t \alpha_1 \circ [.,] \).

The proof of the \(\varepsilon\)-Hom-Jacobi condition of the \(n\)th derived Hom-algebra follows from Theorem 1.1. In case \(n = 1\) and \(\alpha_t = \text{Id} + t \alpha_1\) the bracket is

\[
[.,]'_1 = \text{Id} + t \alpha_1 \circ \text{Id} + t \alpha_1 \circ [.,] = [.,] + 2t \alpha_1 \circ [.,] + t^2 \alpha_1 \circ [.,]
\]

and the twist map is \(\alpha_1^2 = (\text{Id} + t \alpha_1)^2 = \text{Id} + 2t \alpha_1 + t^2 \alpha_1\). Therefore we get another deformation of the color Lie algebra viewed as a color Hom-Lie algebra \((\mathcal{A}, [.,]'_t, \varepsilon, \text{Id})\). The proof in the general case is similar. \(\square\)

**Remark 4.2.** More generally if \((\mathcal{A}, [.,], \varepsilon, \alpha)\) is a multiplicative color Hom-Lie algebra where \(\alpha\) may be written of the form \(\alpha = \text{Id} + t \alpha_1\), then the \(n\)th derived Hom-algebra

\[
\mathcal{A}^n_t = (\mathcal{A}, [.,]'_t^n) = \alpha^n \circ [.,]'_t, \varepsilon, \alpha^{n+1}
\]

gives a one parameter formal deformation of \((\mathcal{A}, [.,]'_t, \varepsilon, \alpha)\). But for any \(\alpha\) one obtains just new color Hom-Lie algebra.

### 5. Generalized \(\alpha^k\)-derivations of color Hom-Lie algebras

The purpose of this section is to study the homogeneous \(\alpha^k\)-generalized derivations and homogeneous \(\alpha^k\)-centroid of color Hom-Lie algebras generalizing the homogeneous generalized derivations discussed in [8]. In Proposition 5.2 we prove that the \(\alpha\)-derivation of color Hom-Lie algebras gives rise to a Hom-Jordan color algebras.

We need the following definitions:

**Definition 5.1.** Let \(\text{Pl}_\gamma(A) = \{D \in \text{Hom}(A, A) : D(\mathcal{A}_\gamma) \subset \mathcal{A}_{\gamma+\mu} \text{ for all } \gamma, \mu \in \Gamma\}\).

Then \(\left(\text{Pl}(A) = \bigoplus_{\gamma \in \Gamma} \text{Pl}_\gamma(A), [.,], \varepsilon, \alpha\right)\) is a color Hom-Lie algebra with the color Lie bracket

\[
[D_\gamma, D_\mu] = D_\gamma \circ D_\mu - \varepsilon(\gamma, \mu)D_\mu \circ D_\gamma
\]

for all \(D_\gamma, D_\mu \in \mathcal{H}(\text{Pl}(A))\) and with \(\alpha : A \to A\) is an even homomorphism.

A homogeneous \(\alpha^k\)-derivation of degree \(\gamma\) of \(A\) is an endomorphism \(D \in \text{Pl}_\gamma(A)\) such that

\[
[D, \alpha] = 0,
\]

\[
D([x, y]) = [D(x), \alpha^k(y)] + \varepsilon(\gamma, x)[\alpha^k(x), D(y)]
\]

for all \(x, y \in A\).

We denote the set of all homogeneous \(\alpha^k\)-derivations of degree \(\gamma\) of \(A\) by \(\text{Der}_{\alpha^k}^\gamma(A)\). The space

\[
\text{Der}(A) = \bigoplus_{k \geq 0} \text{Der}_{\alpha^k}(A)
\]
provided with the color-commutator is a color Lie algebra. Indeed, the fact that Der$_{\alpha^k}(A)$ is $\Gamma$-graded implies that Der$(A)$ is $\Gamma$-graded

$$(\text{Der}(A))\gamma = \bigoplus_{k \geq 0} (\text{Der}_{\alpha^k}(A))\gamma, \quad \forall \gamma \in \Gamma.$$  

**Definition 5.2.** (1) An endomorphism $D \in \text{Pl}_\gamma(A)$ is said to be a homogeneous generalized $\alpha^k$-derivation of degree $\gamma$ of $A$, if there exist two endomorphisms $D', D'' \in \text{Pl}_\gamma(A)$ such that

$$[D, \alpha] = 0, \quad [D', \alpha] = 0, \quad [D'', \alpha] = 0$$

(5.48)

$$D''([x, y]) = [D(x), \alpha^k(y)] + \varepsilon(\gamma, x)[\alpha^k(x), D'(y)]$$

for all $x, y$ in $A$.

We denote the set of all homogeneous generalized $\alpha^k$-derivations of degree $\gamma$ of $A$ by $\text{GDer}_{\alpha^k}(A)$. The space

$$\text{GDer}(A) = \bigoplus_{k \geq 0} \text{GDer}_{\alpha^k}(A).$$

(2) We call $D \in \text{Pl}_\gamma(A)$ a homogeneous $\alpha^k$-quasi-derivation of degree $\gamma$ of $A$, if there exists an endomorphism $D' \in \text{Pl}_\gamma(A)$ such that

$$[D, \alpha] = 0, \quad [D', \alpha] = 0$$

(5.49)

$$D'([x, y]) = [D(x), \alpha^k(y)] + \varepsilon(\gamma, x)[\alpha^k(x), D(y)]$$

for all $x, y$ in $A$.

We denote the set of all homogeneous $\alpha^k$-quasi-derivations of degree $\gamma$ of $A$ by $\text{QDer}_{\alpha^k}(A)$. The space

$$\text{QDer}(A) = \bigoplus_{k \geq 0} \text{QDer}_{\alpha^k}(A).$$

(3) If $C(A) = \bigoplus_{k \geq 0} C_{\alpha^k}(A), \forall \gamma \in \Gamma$, with $C_{\alpha^k}(A)$ consisting of $D \in \text{Pl}_\gamma(A)$ satisfying

$$D([x, y]) = [D(x), \alpha^k(y)] + \varepsilon(\gamma, x)[\alpha^k(x), D(y)]$$

(5.50)

for all $x, y$ in $A$, then $C(A)$ is called the $\alpha^k$-centroid of $A$.

We denote the set of all homogeneous $\alpha^k$-centroids of degree $\gamma$ of $A$ by $C_{\alpha^k}(A)$.

**Proposition 5.1.** Let $(A, [\cdot, \cdot], \varepsilon, \alpha)$ be a multiplicative color Hom-Lie algebra. If $D_\gamma \in \text{GDer}_{\alpha^k}(A)$ and $D_{\gamma'} \in C_{\alpha^{k'}}(A)$, then $\Delta_{\gamma'} D_\gamma \in \text{GDer}_{\alpha^{k+k'}}(A)$ is of degree $(\gamma + \gamma')$.

**Proof.** Let $D_\gamma \in \text{GDer}_{\alpha^k}(A)$. Then for all $x, y \in H(A)$, there exist $D_{\gamma'}, D''_{\gamma} \in \text{Pl}_\gamma(A)$ such that

$$D''_{\gamma}([x, y]) = [D_\gamma(x), \alpha^k(y)] + \varepsilon(\gamma, x)[\alpha^k(x), D_{\gamma'}(y)].$$

Now, let $\Delta_{\gamma'} \in C_{\alpha^{k'}}(A)$ then we have:

$$\Delta_{\gamma'} D''_{\gamma}([x, y]) = \Delta_{\gamma'}([D_\gamma(x), \alpha^k(y)] + \varepsilon(\gamma, x)[\alpha^k(x), D_{\gamma'}(y)])$$

$$= [\Delta_{\gamma'} D_\gamma(x), \alpha^{k+k'}(y)] + \varepsilon(\gamma + \gamma', x)[\alpha^{k+k'}(x), \Delta_{\gamma'} D_{\gamma'}(y)].$$

Then $\Delta_{\gamma'} D_\gamma \in \text{GDer}_{\alpha^{k+k'}}(A)$ of degree $(\gamma + \gamma')$. \hfill $\square$

**Proposition 5.2.** Let $D_\gamma \in C_{\alpha^k}(A)$, then $D_\gamma$ is an $\alpha^k$-Quasi-derivation of $A$.

**Proof.** Let $x, y \in H(A)$, we have

$$[D_\gamma(x), \alpha^k(y)] + \varepsilon(\gamma, x)[\alpha^k(x), D_\gamma(y)] = [D_\gamma(x), \alpha^k(y)] + [D_\gamma(x), \alpha^k(y)]$$

$$= 2D_\gamma([x, y])$$

$$= D_\gamma([x, y]).$$

Then $D_\gamma$ is an $\alpha^k$-Quasi-derivation of degree $\gamma$ of $A$. \hfill $\square$
Definition 5.3. If $QC(A) = \bigoplus_{\gamma \in \Gamma} QC_{\alpha}^\gamma(A)$ and $QC_{\alpha}^\gamma(A)$ consisting of $D \in P_{\gamma}(A)$ such that for all $x, y$ in $A$ 

$$[D(x), \alpha^k(y)] = \varepsilon(\gamma, x)[\alpha^k(x), D(y)],$$

then $QC(A)$ is called the $\alpha^k$-quasi-centroid of $A$.

Proposition 5.3. Let $D_\gamma \in \mathcal{H}(QC_{\alpha}^\gamma(A))$ and $D_\mu \in \mathcal{H}(QC_{\alpha}^\gamma(A))$. Then $[D_\gamma, D_\mu]$ is an $\alpha^{k+k'}$-generalized derivation of degree $(\gamma + \mu)$.

Proof. Assume that $D_\gamma \in \mathcal{H}(QC_{\alpha}^\gamma(A)), D_\mu \in \mathcal{H}(QC_{\alpha}^\gamma(A))$. Then for all $x, y \in \mathcal{H}(A)$, we have 

$$[D_\gamma(x), \alpha^k(y)] = \varepsilon(\gamma, x)[\alpha^k(x), D_\gamma(y)]$$

and 

$$[D_\mu(x), \alpha^{k'}(y)] = \varepsilon(\mu, x)[\alpha^{k'}(x), D_\mu(y)].$$

Hence 

$$[[D_\gamma, D_\mu](x), \alpha^{k+k'}(y)] = [[D_\gamma \circ D_\mu - \varepsilon(\gamma, \mu)D_\mu \circ D_\gamma](x), \alpha^{k+k'}(y)]$$

$$= [D_\gamma \circ D_\mu(x), \alpha^{k+k'}(y)] - \varepsilon(\gamma, \mu)[D_\mu \circ D_\gamma(x), \alpha^{k+k'}(y)]$$

$$= \varepsilon(\gamma + \mu, x)[\alpha^{k+k'}(x), D_\gamma \circ D_\mu(y)] - \varepsilon(\gamma, \mu)[\alpha^{k+k'}(x), D_\mu \circ D_\gamma(y)]$$

$$= \varepsilon(\gamma + \mu, x)[\alpha^{k+k'}(x), [D_\gamma, D_\mu](y)] + [[D_\gamma, D_\mu](x), \alpha^{k+k'}(y)],$$

which implies that $[[D_\gamma, D_\mu](x), \alpha^{k+k'}(y)] + [[D_\gamma, D_\mu](x), \alpha^{k+k'}(y)] = 0.$

Then $[D_\gamma, D_\mu] \in GDer_{\alpha^{k+k'}}(A)$ and is of degree $(\gamma + \mu)$.

5.1. Hom-color algebra and derivations.

Definition 5.4. Let $(A, \mu, \alpha)$ be a Hom-color algebra.

1. The Hom-associator of $A$ is the trilinear map $as_\alpha : A \times A \times A \rightarrow A$ defined as

\begin{equation} \tag{5.51}
as_\alpha = \mu \circ (\mu \otimes \alpha - \alpha \otimes \mu).
\end{equation}

In terms of elements, the map $as_\alpha$ is given by

$$as_\alpha(x, y, z) = \mu(\mu(x, y), \alpha(z)) - \mu(\alpha(x), \mu(y, z))$$

for all $x, y, z \in \mathcal{H}(A)$.

2. Let $A$ be a Hom-algebra over a field $K$ of characteristic $\neq 2$ with an even bilinear multiplication $\circ$.

If $A$ is graded by the abelian group $\Gamma$, $\varepsilon : \Gamma \times \Gamma \rightarrow K^*$ and $\alpha : A \rightarrow A$ be an even linear map, then $(A, \circ, \varepsilon, \alpha)$ is a Hom-Jordan color algebra if the identities

\begin{enumerate}
\item[(HJCJ1)]: $x \circ y = \varepsilon(x, y)y \circ x$
\item[(HJCJ2)]: $\varepsilon(w, x + z)as_\alpha(x \circ y, \alpha(z), \alpha(w)) + \varepsilon(x, y + z)as_\alpha(y \circ w, \alpha(z), \alpha(x))$
\end{enumerate}

$$+ \varepsilon(y, w + z)as_\alpha(w \circ x, \alpha(z), \alpha(y)) = 0$$

are satisfied for all $x, y, z$ and $w$ in $\mathcal{H}(A)$.

The identity $(HJCJ2)$ is called the color Hom-Jordan identity.

Observe that when $\alpha = 1d$, the color-Hom-Jordan identity $(HJCJ2)$ reduces to the usual color Jordan identity.

Proposition 5.4. Let $(A, [\cdot, \cdot], \varepsilon, \alpha)$ be a multiplicative color Hom-Lie algebra, with the operation $D_\gamma \bullet D_\mu = D_\gamma \circ D_\mu - \varepsilon(\gamma, \mu)D_\gamma \circ D_\mu$ for all $\alpha$-derivations $D_\gamma, D_\mu \in \mathcal{H}(P(A))$, the triple $(P(A), \cdot, \varepsilon, \alpha)$ is a Hom-Jordan color algebra.

Proof. Assume that $D_\lambda, D_\delta, D_\mu, D_\gamma \in \mathcal{H}(P(A))$, we have

$$D_\lambda \bullet D_\delta = D_\lambda \circ D_\delta - \varepsilon(\lambda, \theta)D_\delta \circ D_\lambda$$

$$= \varepsilon(\lambda, \theta)(D_\theta \circ D_\lambda - \varepsilon(\lambda, \theta)D_\lambda \circ D_\theta)$$

$$= \varepsilon(\lambda, \theta)D_\theta \bullet D_\lambda.$$
Since

\[
((D_\lambda \bullet D_\theta) \bullet \alpha(D_\mu)) \bullet \alpha^2(D_\gamma)
\]

\[
= D_\lambda D_\theta \alpha(D_\mu) \alpha(D_\gamma) + \varepsilon(\lambda + \theta + \mu, \gamma) \alpha^2(D_\gamma) D_\lambda D_\theta \alpha(D_\mu)
\]

\[
+ \varepsilon(\lambda + \theta, \mu) \alpha(D_\mu) D_\lambda D_\theta \alpha^2(D_\gamma) + \varepsilon(\lambda + \theta, \mu) \varepsilon(\lambda + \theta + \mu, \gamma) \alpha^2(D_\gamma) \alpha(D_\mu) D_\lambda D_\theta
\]

\[
+ \varepsilon(\lambda, \theta) \varepsilon(\lambda + \theta, \mu) \alpha(D_\mu) D_\theta D_\lambda \alpha^2(D_\gamma) + \varepsilon(\lambda, \theta) \varepsilon(\lambda + \theta, \mu) \varepsilon(\lambda + \theta + \mu, \gamma) \alpha^2(D_\gamma) \alpha(D_\mu) D_\theta D_\lambda
\]

and

\[
\alpha(D_\lambda \bullet D_\theta) \bullet (\alpha(D_\mu) \bullet \alpha(D_\gamma)) = \alpha(D_\lambda D_\theta) \alpha(D_\mu) \alpha(D_\gamma) + \varepsilon(\lambda + \theta, \mu + \gamma) \alpha(D_\mu) \alpha(D_\gamma) \alpha(D_\lambda D_\theta)
\]

\[
+ \varepsilon(\mu, \gamma) \alpha(D_\lambda D_\theta) \alpha(D_\gamma) \alpha(D_\mu)
\]

\[
+ \varepsilon(\lambda, \theta) \varepsilon(\lambda + \theta, \mu + \gamma) \alpha(D_\mu) \alpha(D_\gamma) \alpha(D_\lambda D_\theta)
\]

\[
+ \varepsilon(\lambda, \theta) \varepsilon(\lambda + \theta, \mu) \alpha(D_\mu) \alpha(D_\gamma) \alpha(D_\lambda D_\theta)
\]

\[
+ \varepsilon(\lambda, \theta) \varepsilon(\mu, \gamma) \varepsilon(\lambda + \theta, \mu + \gamma) \alpha(D_\gamma) \alpha(D_\mu) \alpha(D_\lambda D_\theta)
\]

We have

\[
\varepsilon(\gamma, \lambda + \mu) \alpha(D_\lambda \bullet D_\theta, \alpha(D_\mu), \alpha(D_\gamma))
\]

\[
= \varepsilon(\gamma, \lambda + \mu) \varepsilon(\lambda + \theta + \mu, \gamma) \alpha^2(D_\gamma) D_\lambda D_\theta \alpha(D_\mu) + \varepsilon(\lambda + \theta, \mu) \alpha(D_\mu) D_\lambda D_\theta \alpha^2(D_\gamma)
\]

\[
+ \varepsilon(\lambda, \theta) \varepsilon(\lambda + \theta + \mu, \gamma) \alpha^2(D_\gamma) D_\lambda D_\theta \alpha(D_\mu) + \varepsilon(\lambda, \theta) \varepsilon(\lambda + \theta, \mu) \alpha(D_\mu) D_\lambda D_\theta \alpha^2(D_\gamma)
\]

\[
- \varepsilon(\lambda + \theta, \mu + \gamma) \alpha(D_\mu) \alpha(D_\gamma) \alpha(D_\lambda D_\theta)
\]

\[
- \varepsilon(\lambda, \theta) \varepsilon(\lambda + \theta, \mu + \gamma) \alpha(D_\mu) \alpha(D_\gamma) \alpha(D_\lambda D_\theta)
\]

\[
- \varepsilon(\lambda, \theta) \varepsilon(\mu, \gamma) \varepsilon(\lambda + \theta, \mu + \gamma) \alpha(D_\gamma) \alpha(D_\mu) \alpha(D_\lambda D_\theta)
\]

Therefore, we get

\[
\varepsilon(\gamma, \lambda + \mu) \alpha(D_\lambda \bullet D_\theta, \alpha(D_\mu), \alpha(D_\gamma)) + \varepsilon(\lambda, \theta) \varepsilon(\mu, \gamma) \alpha(D_\mu) \alpha(D_\gamma) \alpha(D_\lambda D_\theta)
\]

\[
+ \varepsilon(\theta, \gamma + \mu) \alpha(D_\gamma \bullet D_\lambda, \alpha(D_\mu), \alpha(D_\theta)) = 0,
\]

and so the statement holds.

Corollary 5.1. Let \((\mathcal{A}, [\cdot, \cdot], \varepsilon, \alpha)\) be a multiplicative color Hom-Lie algebra, with the operation

\[
D_\gamma \bullet D_\mu = D_\gamma \circ D_\mu + \varepsilon(\gamma, \mu) D_\mu \circ D_\gamma
\]

for all \(D_\gamma, D_\mu \in \mathcal{H}(QC(\mathcal{A}))\), the quadruple \((QC(\mathcal{A}), \bullet, \varepsilon, \alpha)\) is a Hom-Jordan color algebra.

Proof. We need only to show that \(D_\gamma \bullet D_\mu \in QC(\mathcal{A})\), for all \(D_\gamma, D_\mu \in \mathcal{H}(QC(\mathcal{A}))\).

Assume that \(x, y \in \mathcal{H}(\mathcal{A})\), we have

\[
[D_\gamma \bullet D_\mu(x, \alpha^{k+s}(y))] = [D_\gamma \circ D_\mu(x, \alpha^{k+s}(y)) + \varepsilon(\gamma, \mu)[D_\mu \circ D_\gamma(x, \alpha^{k+s}(y))]
\]

\[
= \varepsilon(\gamma + \mu, x)[D_\mu(x, D_\gamma(\alpha^{k+s}(y)))] + \varepsilon(\mu, x)[D_\mu(x, D_\gamma(\alpha^{k+s}(y)))]
\]

\[
= \varepsilon(\gamma + \mu, x)[\alpha^{k+s}(x), D_\mu \circ D_\gamma(y)] + \varepsilon(\mu, x)[\alpha^{k+s}(x), D_\gamma \circ D_\mu(y)]
\]

\[
= \varepsilon(\gamma + \mu, x)[\alpha^{k+s}(x), D_\gamma \bullet D_\mu(y)].
\]

Hence \(D_\gamma \bullet D_\mu \in QC(\mathcal{A})\).
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