Higher Frobenius-Schur Indicators for Semisimple Hopf Algebras in Positive Characteristic

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Abstract. Let $H$ be a semisimple Hopf algebra over an algebraically closed field $k$ of characteristic $p > \dim_k(H)^{1/2}$. We show that the antipode $S$ of $H$ satisfies the equality $S^2(h) = uhu^{-1}$, where $h \in H$, $u = S(\Lambda_2)\Lambda_1$, and $\Lambda$ is a nonzero integral of $H$. The formula of $S^2$ enables us to define higher Frobenius-Schur indicators for the Hopf algebra $H$. This generalizes the notions of higher Frobenius-Schur indicators from the case of characteristic 0 to the case of characteristic $p > \dim_k(H)^{1/2}$. These indicators defined here share some properties with the ones defined over a field of characteristic 0. Especially, all these indicators are gauge invariants for the tensor category $\text{Rep}(H)$ of finite dimensional representations of $H$.

1. Introduction

Lichnensk-Montgomery [9] generalized the classical Frobenius-Schur (FS) indicators from group-theoretic result to the setting of a semisimple involutory Hopf algebra $H$. They also defined higher FS indicators $\nu_n(V)$ by using idempotent integral $\Lambda$ of $H$, namely,

\begin{equation}
\nu_n(V) = \chi_V(\Lambda_{(1)} \cdots \Lambda_{(n)}) \text{ for } n \geq 1,
\end{equation}

where $\chi_V$ is the character afforded by finite dimensional representation $V$ of $H$. The higher FS indicators were later extensively studied by Kashina-Sommerhäuser-Zhu for semisimple Hopf algebras over an algebraically closed field of characteristic zero [6], and by Ng-Schauenburg for semisimple quasi-Hopf algebras over the field of complex numbers [11]. The notions of higher FS indicators have been generalized to objects of a pivotal category [12, 13].

However, the notions of higher FS indicators for semisimple Hopf algebras over a field of positive characteristic seem not to be considered (except for those semisimple involutory Hopf algebras). In this paper, we consider higher FS indicators for a finite dimensional semisimple Hopf algebra $H$ over an algebraically closed field $k$ of characteristic $p > \dim_k(H)^{1/2}$. We need to point out that the Hopf algebra $H$ here is not necessarily involutory unless the characteristic $p$ is larger than a certain number (see [16, 3]).

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For the antipode $S$ of $H$, we first obtain a formula for $S^2$ as follows:

$$S^2(h) = uhu^{-1},$$

where $h \in H$, $u = S(\Lambda(2))\Lambda(1)$ and $\Lambda$ is a nonzero integral of $H$. According to the formula of $S^2$, we have an isomorphism of $H$-modules

$$j_{u,V} : V \to V^{**}, \quad j_{u,V}(v)(f) = f(u \cdot v) \text{ for } v \in V, f \in V^*,$$

which is functorial in $V$. As the element $u = S(\Lambda(2))\Lambda(1)$ is not necessarily a group-like element, the functorial isomorphism $j_u : id \to (\cdot)^{**}$ is not necessarily a tensor isomorphism. In other words, the category $\text{Rep}(H)$ of finite dimensional representations of $H$ is not necessarily pivotal with respect to the structure $j_u$. Even though, using the functorial isomorphism $j_u$ we may still define the $n$-th FS indicator $\nu_n(V)$ of $V$ to be the trace of a certain $\mathbb{k}$-linear operator as Ng-Schauenburg did in [12]. It is similar to the case of characteristic 0 that the $n$-th FS indicator $\nu_n(V)$ defined here can also be entirely described in terms of the integral $\Lambda$ of $H$ and the character $\chi_V$ of $H$-module $V$:

$$(1.2) \quad \nu_n(V) = \chi_V(u^{-1}\Lambda(1) \cdots \Lambda(n)) \text{ for } n \geq 1.$$  

Moreover, the formula (1.2) does not depend on the choice of the nonzero integral $\Lambda$ and it recovers the original formula (1.1) when the characteristic of $\mathbb{k}$ is zero and $\Lambda$ is idempotent.

Note that the formula (1.2) can be written as $\nu_n(V) = \chi_V(u^{-1}P_n(\Lambda))$ for $n \geq 1$, where $P_n$ is the $n$-th Sweedler power map of $H$. Clearly, the $n$-th Sweedler power map $P_n$ is valid for all $n \in \mathbb{Z}$, this motivates us to extend the $n$-th FS indicator from $n \geq 1$ to $n \in \mathbb{Z}$. That is, by definition, $\nu_n(V) = \chi_V(u^{-1}P_n(\Lambda))$ for all $n \in \mathbb{Z}$. We find that the higher FS indicators defined over a field of characteristic $p > \dim_{\mathbb{k}}(H)^{1/2}$ and the ones defined over a field of characteristic 0 share some common properties. For instance, it is similar to the case of characteristic 0 (see [5, 6]) that by replacing $V$ with the regular representation $H$, we reconstruct the $n$-th indicator of $H$, a notion defined by the trace of the map $S \circ P_{n-1}$. Also, it is similar to characteristic 0 case that $V$ and its dual $V^*$ have the same higher FS indicators. Especially, similar to the case of characteristic 0 that the $n$-th FS indicator $\nu_n(V)$ defined here is an invariant of the tensor category $\text{Rep}(H)$ for any $n \in \mathbb{Z}$ and any finite dimensional representation $V$ of $H$.

The paper is organized as follows: In Section 2, we present some basic results on semisimple Hopf algebras. In Section 3, we deduce the formula of $S^2$ by comparing two different forms of the character $\chi_H$ of the regular representation $H$. We investigate some properties of the element $u = S(\Lambda(2))\Lambda(1)$ and show that the integral $\Lambda$ of $H$ is cocommutative if and only if $S^2 = id$. In Section 4, we generalize the notions of higher FS indicators from characteristic 0 case to characteristic $p > \dim_{\mathbb{k}}(H)^{1/2}$ case and find that the indicators defined here share some common properties with the ones defined over a field of characteristic 0. In Section 5, we show that the $n$-th FS indicator $\nu_n(V)$ is a gauge invariant for any integer $n$ and any finite dimensional representation $V$ of $H$. 
2. Preliminaries

Throughout this paper, $H$ is a finite dimensional semisimple Hopf algebra over an algebraically closed field $\mathbb{k}$ of characteristic $p > \dim_\mathbb{k}(H)^{1/2}$. We need to stress that all results presented here are also valid for the case of characteristic 0, although we only deal with the case of characteristic $p > \dim_\mathbb{k}(H)^{1/2}$.

As a Hopf algebra, $H$ has a counit $\varepsilon$, antipode $S$, multiplication $m$ and comultiplication $\Delta$. The comultiplication $\Delta(a)$ will be written as $\Delta(a) = a(1) \otimes a(2)$ for $a \in H$, where we omit the summation sign. We denote by $\Lambda$ and $\lambda$ the left and right integrals of $H$ and $H^*$ respectively so that $\lambda(\Lambda) = 1$. Since the semisimple Hopf algebra $H$ is unimodular, the left and right integrals of $H$ are the same. We refer to [10] for basic theory of Hopf algebras.

If $V$ is a finite dimensional $H$-module, then $V$ is also called a representation of $H$ via the algebra homomorphism $\rho_V : H \to \text{End}_\mathbb{k}(V)$ given by $\rho_V(h)(v) = h \cdot v$ for $h \in H$ and $v \in V$. We will make no distinction between the two notions. The character of $V$ is the Frobenius homomorphism $\chi_V : H \to \mathbb{k}$ given by $\chi_V(h) = \text{tr}(\rho_V(h))$ for $h \in H$. The $\mathbb{k}$-linear dual space $V^*$ is also an $H$-module via $(h \cdot f)(v) := f(h^{-1} \cdot v)$ for $h \in H$, $f \in V^*$ and $v \in V$. In particular, the dual module $V^*$ has the character $\chi_{V^*} = \chi_V \circ S$.

The category $\text{Rep}(H)$ of finite dimensional representations of $H$ is a semisimple tensor category, where the monoidal structure stems from the comultiplication $\Delta$.

Recall that the dual Hopf algebra $H^*$ has an $H$-bimodule structure given by

\[(a \cdot f)(b) = f(ba), \quad (f \cdot a)(b) = f(ab)\]

for $a, b \in H$, $f \in H^*$. Moreover, $(H^*, \cdot)$ and $(\cdot, H^*)$ are free $H$-modules generated by $\lambda$, i.e., $H^* = \lambda \cdot H$ and $H^* = H \cdot \lambda$ (see [15, Corollary 2(b)]). This provides an associative and non-degenerate bilinear form $H \times H \to \mathbb{k}$ by $a \times b \mapsto \lambda(ab)$ for $a, b \in H$.

Moreover, the pair $(H, \lambda)$ is a Frobenius algebra with the Frobenius homomorphism $\lambda$ satisfying the equality (see [15, Eq.(1.1)]):

\[a = \lambda(a\Lambda_1)S(\Lambda_2) = \lambda(S(\Lambda_2)a)\Lambda_1 \quad \text{for} \quad a \in H.\]

The pair $\Lambda_1 \otimes S(\Lambda_2)$ satisfying (2.1) is called the dual basis of $H$ with respect to the Frobenius homomorphism $\lambda$.

Since the right integral $\lambda$ of $H^*$ satisfies $\lambda(ab) = \lambda(S^2(b)a)$ for all $a, b \in H$ (see [15, Theorem 3(a)]), the Hopf algebra $H$ is a symmetric algebra with a symmetric bilinear form given by

\[H \times H \to \mathbb{k}, \quad a \times b \mapsto \lambda(uab) = (\lambda \leftarrow u)(ab) = (u \rightarrow \lambda)(ab),\]

where $u$ is a unit of $H$ satisfying $S^2(h) = uhu^{-1}$ for all $h \in H$ and the Frobenius homomorphism $\lambda \leftarrow u = u \rightarrow \lambda$ holds because $\lambda(au) = \lambda(S^2(u)a) = \lambda(ua)$ for all $a \in H$. Using (2.1) we may see that the pair $\Lambda_1 \otimes u^{-1}S(\Lambda_2)$ is a dual basis of $H$ with respect to $\lambda \leftarrow u = u \rightarrow \lambda$ (see also [2, Lemma 1.4(2)]). The symmetry of the Frobenius homomorphism $\lambda \leftarrow u = u \rightarrow \lambda$ means that

\[\Lambda_1 \otimes u^{-1}S(\Lambda_2) = u^{-1}S(\Lambda_2) \otimes \Lambda_1.\]
By Wedderburn’s theorem, the semisimple Hopf algebra $H$ is isomorphic to a direct sum of full matrix algebras over $k$, namely,

$$H \cong \bigoplus_{i \in I} M_{d_i}(k).$$

Let $e_i$ be the idempotent of $H$ satisfying that $He_i \cong M_{d_i}(k)$. Then $\{e_i\}_{i \in I}$ forms a complete set of central primitive idempotents of $H$. Let $V_i$ be a simple left module (unique up to isomorphism) over the matrix algebra $M_{d_i}(k)$. Then $\dim_k(V_i) = d_i$ and $\{V_i\}_{i \in I}$ forms a complete set of simple left $H$-modules up to isomorphism. The left regular representation $H$ has the decomposition $H \cong \bigoplus_{i \in I} V_i^{\text{id}}$ as $H$-modules, so the character $\chi_H$ of the left regular representation $H$ is equal to $\sum_{i \in I} d_i \chi_i$, where each $\chi_i$ is the character of $V_i$.

For any simple $H$-module $V_i$ and any $\varphi \in \text{End}_k(V_i)$, we use the dual basis $\Lambda_{(1)} \otimes u^{-1} S(\Lambda_{(2)})$ with respect to the Frobenius homomorphism $\lambda \leftarrow u$ to define the map $I(\varphi) \in \text{End}_k(V_i)$ by

$$I(\varphi)(v) = \Lambda_{(1)} \varphi(u^{-1} S(\Lambda_{(2)}) v) \text{ for } v \in V_i.$$

Note that $I(\varphi)$ lies in $\text{End}_H(V_i) \cong k$. There exists a unique element $c_i \in k$ such that

$$I(\varphi) = c_i \text{Tr}(\varphi) \text{id}_{V_i}, \text{ for all } \varphi \in \text{End}_k(V_i).$$

Such an element $c_i$, depending only on the isomorphism class of $V_i$, is called the Schur element associated to $V_i$ (see [4, Theorem 7.2.1]). Since $H$ is semisimple, it follows from [4, Theorem 7.2.6] that the Schur element $c_i \neq 0$ in $k$ and the Frobenius homomorphism $\lambda \leftarrow u$ can be written explicitly as follows:

$$\lambda \leftarrow u = u \rightarrow \lambda = \sum_{i \in I} \frac{1}{c_i} \chi_i.$$

### 3. A formula for the square of antipodes

In this section, we will provide a formula for $S^2$ by virtue of a nonzero integral $\Lambda$ of $H$. Then we study some properties of the element $u := S(\Lambda_{(2)})\Lambda_{(1)}$. Especially, we will give a sufficient and necessary condition for $S^2 = \text{id}$ via the integral $\Lambda$.

Let $u$ be a unit of $H$ satisfying $S^2(a) = uau^{-1}$ for all $a \in H$. We fix a left integral $\Lambda$ of $H$ and a right integral $\lambda$ of $H^*$ such that $\lambda(\Lambda) = 1$. We denote $\{V_i\}_{i \in I}$ the set of all simple left $H$-modules up to isomorphism. For each $V_i$ we denote $c_i$ the Schur element of $V_i$ associated to the dual basis $\Lambda_{(1)} \otimes u^{-1} S(\Lambda_{(2)})$ of $H$ with respect to the Frobenius homomorphism $\lambda \leftarrow u$. We denote $\{e_i\}_{i \in I}$ the set of all central primitive idempotents of $H$. We first establish a relationship between the elements $u$ and $u = S(\Lambda_{(2)})\Lambda_{(1)}$.

**Proposition 3.1.** With the notions above, we have $u = u \sum_{i \in I} \dim_k(V_i)c_ie_i$, which is a unit of $H$. 

Proof. Note that each central primitive idempotent \( e_i \) acts as the identity on \( V_i \) and annihilates \( V_j \) for \( j \neq i \). It follows that \( \chi_j(e_i) = \dim_k(V_i) \) if \( i = j \) and 0 otherwise. By (2.4) we have

\[
\chi_j(a) = \chi_j(ae_i) = \sum_{j \in I} \frac{1}{c_j} \chi_j(c_jae_i) = (u \to \lambda(c_jae_i) = (uc_i e_i \to \lambda)(a).
\]

Thus, \( \chi_i = uc_i e_i \to \lambda \) and hence

\[
\chi_H = \sum_{i \in I} \dim_k(V_i) \chi_i = u \sum_{i \in I} \dim_k(V_i) c_i e_i \to \lambda.
\]

For any map \( \varphi \in \text{End}_k(H) \), the trace of \( \varphi \) is \( \text{tr}(\varphi) = \lambda(\varphi(S(\Lambda(2))) \Lambda(1)) \) (see [15, Theorem 2]). Taking into account that \( \varphi = L_a \), where \( L_a \) is the left multiplication operator of \( H \) by \( a \), we have

\[
\chi_H(a) = \text{tr}(L_a) = \lambda(a S(\Lambda(2))) \Lambda(1)) = (S(\Lambda(2))) \Lambda(1) \to \lambda)(a).
\]

This implies that \( \chi_H = S(\Lambda(2)) \Lambda(1) \to \lambda \). Comparing it with (3.1) and using the non-degeneracy of the Frobenius homomorphism \( \lambda \), we have

\[
S(\Lambda(2)) \Lambda(1) = u \sum_{i \in I} \dim_k(V_i) c_i e_i.
\]

Since \( p > \dim_k(H)^{1/2} \), it follows that \( p^2 > \dim_k(H) = \sum_{i \in I} \dim_k(V_i)^2 \geq \dim_k(V_i)^2 \). Hence \( p > \dim_k(V_i) \) and \( \dim_k(V_i) \neq 0 \) in \( k \) for any \( i \in I \). Thus, the element \( u \) is the same as \( S(\Lambda(2)) \Lambda(1) \) up to a central unit \( \sum_{i \in I} \dim_k(V_i) c_i e_i \).

\[\square\]

Remark 3.2. Proposition 3.1 also holds if the field \( k \) has characteristic 0. In this case, \( S^2 = \text{id} \) (see [7] or [8]) implying that \( u = S(\Lambda(2)) \Lambda(1) = S(\Lambda(2)) S^2(\Lambda(1)) = S(S(\Lambda(1)) \Lambda(2)) = \varepsilon(\Lambda) \).

Proposition 3.1 gives a formula for \( S^2 \), namely,

\[
S^2(a) = uau^{-1} \quad \text{for} \quad a \in H,
\]

where \( u = S(\Lambda(2)) \Lambda(1) \). In the sequel, we will replace \( u \) with \( u \). In this case, the equality (2.2) turns out to be

\[
\Lambda(1) \otimes u^{-1} S(\Lambda(2)) = u^{-1} S(\Lambda(2)) \otimes \Lambda(1),
\]

which is the dual basis of \( H \) with respect to the Frobenius homomorphism \( \lambda \to u \). The Schur element associated to the simple \( H \)-module \( V_i \) under the new dual basis \( \Lambda(1) \otimes u^{-1} S(\Lambda(2)) \) with respect to the Frobenius homomorphism \( \lambda \to u \) is \( \frac{1}{\dim_k(V_i)} \). Therefore, the equality (2.4) turns out to be

\[
\lambda \to u = u \to \lambda = \sum_{i \in I} \dim_k(V_i) \chi_i = \chi_H.
\]

By applying [2, Theorem 1.5] and (3.2), we obtain the expression of each central primitive idempotent \( e_i \) of \( H \) as follows:

\[
e_i = \dim_k(V_i) \chi_i(\Lambda(1))u^{-1} S(\Lambda(2)) = \dim_k(V_i) \chi_i(u^{-1} S(\Lambda(2))) \Lambda(1).
\]

HIGHER FROBENIUS-SCHUR INDICATORS 5
Let $g \in G(H)$ and $\alpha \in \text{Alg}(H, k)$ be the modular elements of $H$ and $H^*$ respectively. Recall that the Radford’s formula of $S^4$ has the form (see [14, Proposition 6]):

$$S^4(\alpha) = \alpha^{-1} \rightarrow (gag^{-1}) \leftarrow \alpha.$$  

Since $H$ is unimodular, i.e., $\alpha = \varepsilon$, the Radford’s formula of $S^4$ now becomes

$$S^4(\alpha) = gag^{-1}.$$  

The distinguished group-like element $g$ and the integral $\Lambda$ of $H$ satisfy the following useful equality (see [15, Theorem 3(d)]):

$$\Lambda_{(2)} \otimes \Lambda_{(1)} = \Lambda_{(1)} \otimes S^2(\Lambda_{(2)}) \varepsilon.$$  

After these preparations, we give some properties of the element $u$ as follows:

**Proposition 3.3.** The element $u = S(\Lambda_{(2)}) \varepsilon$ satisfies the following properties:

1. $u = \chi_H(\Lambda_{(1)}) S(\Lambda_{(2)})$.
2. $\Lambda_{(1)} u^{-1} S(\Lambda_{(2)}) = 1$.
3. $\lambda(e_i) = \dim_k(V_i) \chi_i(\Lambda_{(1)})$.
4. $u S(u) u = \varepsilon(\Lambda) \sum_{i \in I} \frac{\dim_k(V_i)}{\lambda(e_i)} e_{i'}$.
5. $S(u^{-1}) u = u S(u^{-1})$, which is the distinguished group-like element $g$ of $H$.

**Proof.** (1) It follows from (3.4) that $e_i u = \dim_k(V_i) \chi_i(\Lambda_{(1)}) S(\Lambda_{(2)})$. Thus,

$$u = \sum_{i \in I} e_i u = \sum_{i \in I} \dim_k(V_i) \chi_i(\Lambda_{(1)}) S(\Lambda_{(2)}) = \chi_H(\Lambda_{(1)}) S(\Lambda_{(2)}).$$

(2) Since $\Lambda_{(1)} \otimes u^{-1} S(\Lambda_{(2)}) = u^{-1} S(\Lambda_{(2)}) \otimes \Lambda_{(1)}$ by (3.2), we obtain the desired result by multiplying the tensor factors together.

(3) Since $e_i = \dim_k(V_i) \chi_i(\Lambda_{(1)}) u^{-1} S(\Lambda_{(2)})$, it follows that

$$e_i = u e_i u^{-1} = \dim_k(V_i) \chi_i(\Lambda_{(1)}) S(\Lambda_{(2)}) u^{-1}.$$  

Hence

$$\lambda(e_i) = \dim_k(V_i) \chi_i(\Lambda_{(1)}) \lambda(S(\Lambda_{(2)}) u^{-1}) = \dim_k(V_i) \chi_i(u^{-1}),$$

where the last equality follows from (2.1).

(4) For any $a \in H$, we have $S^3(a) = S(S^2(a)) = S(u a u^{-1}) = S(u) S(a) S(u)$, we also have $S^3(a) = S^2(S(a)) = u S(a) u^{-1}$. It follows that $S(u) u$ is a central unit of $H$. The equality $u S(u) = S(u) u$ holds because $S(u) S(u) = S^2(u) = S^2(u) u^{-1}$. For the central unit $u S(u)$, we suppose that $u S(u) = \sum_{i \in I} k_i e_i$, where each scalar $k_i \neq 0$ in $k$. Then $e_i u^{-1} = \frac{1}{k_i} e_i S(u)$. We have

$$\lambda(e_i) = (u^{-1} \rightarrow \chi_H(e_i)) = \chi_H(e_i u^{-1}) = \frac{1}{k_i} \chi_H(e_i S(u)) = \frac{\dim_k(V_i)}{k_i} \chi_i(\Lambda_{(1)} S(u)) = \frac{\dim_k(V_i)}{k_i} \chi_i(S(u)).$$
\[
\begin{align*}
&= \frac{\dim_k(V_i)}{k_i} (\chi_i \circ S)(u) = \frac{\dim_k(V_i)}{k_i} (\chi_i \circ S)(\lambda_{(2)} \lambda_{(1)}) \\
&= \frac{\dim_k(V_i)}{k_i} (\chi_i \circ S)(\lambda_{(1)} \lambda_{(2)}) = \frac{\dim_k(V_i)^2 \epsilon(\lambda)}{k_i} \neq 0.
\end{align*}
\]

It follows that \( k_i = \frac{\dim_k(V_i)^2 \epsilon(\lambda)}{\lambda(e_i)} \) and \( u S(u) = \sum_{i \in I} k_i e_i = \epsilon(\lambda) \sum_{i \in I} \frac{\dim_k(V_i)^2 \epsilon(\lambda)}{\lambda(e_i)} e_i. \)

(5) Note that \( \Lambda_{(2)} \otimes \Lambda_{(1)} = \Lambda_{(1)} \otimes S^2(\Lambda_{(2)}) \) by (3.5). Applying \( S \otimes id \) to both sides of this equality and multiplying the tensor factors together, we have \( u = S(u) g \) or \( g = S(u^{-1}) u. \)

As a consequence, we obtain the following result:

**Corollary 3.4.** For any central primitive idempotent \( e_i \) of \( H \), we have \( \lambda(e_i) = \lambda(S(e_i)). \)

**Proof.** We denote \( S(e_i) = e_i^r \) for some \( i^r \in I \), then \( V_i^r \cong V_{i^r} \), or equivalently, \( \chi_i \circ S = \chi_{i^r} \) (see [2, Lemma 1.8]). By Proposition 3.3 (3) we have

\[
\lambda(S(e_i)) = \lambda(e_i^r) = \dim_k(V_{i^r}) \chi_{i^r}(u^{-1}) = \dim_k(V_i) \chi_i(S(u^{-1})).
\]

Since \( u S(u) = \epsilon(\lambda) \sum_{i \in I} \frac{\dim_k(V_i)^2 \epsilon(\lambda)}{\lambda(e_i)} e_i \), it follows that \( S(u^{-1}) = u \frac{1}{\epsilon(\lambda)} \sum_{i \in I} \frac{\lambda(e_i)}{\dim_k(V_i)} e_i. \)

Thus,

\[
\lambda(S(e_i)) = \dim_k(V_i) \chi_i(S(u^{-1})) = \frac{\lambda(e_i)}{\epsilon(\lambda) \dim_k(V_i)} \chi_i(u)
\]

\[
= \frac{\lambda(e_i)}{\epsilon(\lambda) \dim_k(V_i)} \chi_i(\lambda_{(1)} \lambda_{(2)}) = \lambda(e_i).
\]

We complete the proof. \( \square \)

If the field \( \mathbb{k} \) has characteristic 0, then the antipode \( S \) of \( H \) satisfies \( S^2 = id \) (see [7] or [8]). This further implies that the integral \( \Lambda \) of \( H \) is cocommutative (see [7, Proposition 2(b)]). The following result shows that \( \Lambda \) being cocommutative is equivalent to \( S^2 = id \) when the characteristic of the field \( \mathbb{k} \) is larger than \( \dim_k(H)^{1/2}. \)

**Proposition 3.5.** Let \( H \) be a finite dimensional semisimple Hopf algebra over the field \( \mathbb{k} \) of characteristic \( p > \dim_k(H)^{1/2}. \) The following statements are equivalent:

1. The nonzero integral \( \Lambda \) of \( H \) is cocommutative.
2. The nonzero integral \( \lambda \) of \( H^* \) is cocommutative.
3. \( S^2 = id. \)

**Proof.** It can be seen from [15, Corollary 5] that Part (2) and Part (3) are equivalent. We next show that Part (1) and Part (3) are equivalent. If \( \Lambda \) is cocommutative, then \( u = S(\Lambda_{(2)} \Lambda_{(1)}) = S(\Lambda_{(1)} \Lambda_{(2)}) = \epsilon(\Lambda). \) It follows from \( S^2(\alpha) = u \alpha u^{-1} \) that \( S^2 = id. \) Conversely, if \( S^2 = id \), then \( u = S(\Lambda_{(2)} \Lambda_{(1)}) = S(\Lambda_{(2)}) S^2(\Lambda_{(1)}) = S(S(\Lambda_{(1)} \Lambda_{(2)})) = \epsilon(\Lambda). \) By Proposition 3.3, we have \( g = S(u^{-1}) u = 1. \) Since
Since we denote the category $\text{Rep}(H)$ of finite dimensional semisimple Hopf algebras has been studied in [6]. In this section, we will generalize these indicators from characteristic 0 to the case of characteristic $p > \dim(H)^{1/2}$ and describe them via a nonzero integral $\Lambda$ of $H$. We begin with the following preparations. Let $H$ be a finite dimensional semisimple Hopf algebra over the field $\mathbb{k}$ of characteristic $p > \dim(H)^{1/2}$ with a nonzero integral $\Lambda$ and $u = S(\Lambda(2))\Lambda(1)$. Applying $\Lambda_{n-1} \otimes \text{id}$ to both sides of the equality: $\Lambda_{(2)} \otimes \Lambda(1) = \Lambda(1) \otimes S^2(\Lambda(2))g$ (see (3.5)), we have

$$\Lambda_{(2)} \otimes \cdots \otimes \Lambda_{(n)} \otimes \Lambda(1) = \Lambda(1) \otimes \cdots \otimes \Lambda(n) \otimes S^2(\Lambda(n))g.$$

Since $g = uS(u^{-1})$ and $S^2(\Lambda(n)) = u\Lambda(n)u^{-1}$, the above equality induces the following equality:

$$(4.1) \quad \Lambda_{(2)} \otimes \cdots \otimes \Lambda_{(n)} \otimes u^{-1}\Lambda(1) = \Lambda(1) \otimes \cdots \otimes \Lambda(n) \otimes \Lambda(n)S(u^{-1}).$$

Note that the category $\text{Rep}(H)$ of finite dimensional representations of $H$ is a semisimple tensor category. Let $j_u : \text{id} \rightarrow (-)^\ast$ be a natural isomorphism between the identity functor and the functor of taking the second dual. It is completely determined by a collection of $H$-module isomorphisms

$$j_{u,v} : V \rightarrow V^{\ast \ast}, \quad j_{u,v}(\alpha)(f) = f(u\alpha) \quad \text{for} \quad v \in V, f \in V^\ast.$$

The inverse of $j_{u,v}$ is

$$j_{u,v}^{-1} : V^{\ast \ast} \rightarrow V, \quad \alpha \mapsto j_{u,v}^{-1}(\alpha),$$

where $j_{u,v}^{-1}(\alpha) \in V$ satisfies the equality $f(j_{u,v}^{-1}(\alpha)) = \alpha(S^{-1}(u^{-1})f)$ for $f \in V^\ast$. Since $S^2(h) = uhu^{-1}$ and $u$ is not known to be a group-like element, the natural isomorphism $j_u$ is not necessarily a tensor isomorphism. Although the representation category $\text{Rep}(H)$ with respect to the structure $j_u$ is not necessarily pivotal, we may still define higher FS indicators for any finite dimensional representation of $H$ using the structure $j_u$ of $\text{Rep}(H)$.

We denote $V^{\otimes n}$ the $n$-th tensor power of $V$ where $V^{\otimes 0}$ is the trivial $H$-module $\mathbb{k}$. For any natural number $n \geq 1$, we define the following $\mathbb{k}$-linear map

$$E^n_V : \text{Hom}_H(\mathbb{k}, V^{\otimes n}) \rightarrow \text{Hom}_H(\mathbb{k}, V^{\otimes n}), \quad f \mapsto E^n_V(f),$$

where $E^n_V(f)$ is an $H$-module morphism from $\mathbb{k}$ to $V^{\otimes n}$ given by

$$\begin{align*}
E^n_V(f) : \mathbb{k} &\xrightarrow{\text{ev} \otimes \text{id}} V^\ast \otimes V^{\ast \ast} = V^\ast \otimes \mathbb{k} \otimes V^{\ast \ast} \xrightarrow{\text{id} \otimes \text{id} \otimes \text{id}} V^\ast \otimes V^{\otimes n} \otimes V^{\ast \ast} \\
&\xrightarrow{\text{ev} \otimes \text{id}} V^{\otimes (n-1)} \otimes V^{\ast \ast} \xrightarrow{j_{u^{-1}}^{-1}} V^{\otimes n}.
\end{align*}$$

4. Higher FS indicators

If the field $\mathbb{k}$ has characteristic 0, the $n$-th FS indicators of finite dimensional representations of semisimple Hopf algebras have been studied in [6]. In this section, we will generalize these indicators from characteristic 0 to the case of characteristic $p > \dim(H)^{1/2}$ and describe them via a nonzero integral $\Lambda$ of $H$. We complete the proof. $\square$
Here the maps coev\(_V\) and ev\(_V\) are the usual coevaluation morphism of \(V^*\) and evaluation morphism of \(V\) respectively. If we set \(f(1) = \sum v_1 \otimes \cdots \otimes v_n \in V^\otimes n\), the above definition of \(E^n_V(f)\) shows that

\[
E^n_V(f)(1) = \sum v_2 \otimes \cdots \otimes v_n \otimes u^{-1}v_1.
\]

(4.2)

Similar to [12], we give the definition of the \(n\)-th FS indicator of \(V\) to be the trace of the linear operator \(E^n_V\) as follows:

**Definition 4.1.** Let \(H\) be a finite dimensional semisimple Hopf algebra over the field \(\mathbb{k}\) of characteristic \(p > \dim_{\mathbb{k}}(H)^{1/2}\). For any finite dimensional representation \(V\) of \(H\), the \(n\)-th FS indicator of \(V\) is defined by

\[
\nu_n(V) = \text{tr}(E^n_V) \text{ for } n \geq 1.
\]

Similar to the characteristic 0 case, the \(n\)-th FS indicator of \(V\) defined above can also be described by a nonzero integral \(\Lambda\) of \(H\):

**Theorem 4.2.** Let \(\Lambda\) be a nonzero integral of \(H\) and \(u = S(\Lambda_{(2)})\Lambda_{(1)}\). Suppose \(\chi_V\) is the character of a finite dimensional representation \(V\) of \(H\). We have

\[
\nu_n(V) = \chi_V(u^{-1}\Lambda_{(1)}\cdots\Lambda_{(n)}) \text{ for } n \geq 1.
\]

**Proof.** We first show that the equality \(\nu_n(V) = \chi_V(u^{-1}\Lambda_{(1)}\cdots\Lambda_{(n)})\) holds for an idempotent integral \(\Lambda\). Suppose that \(\alpha\) is the following \(\mathbb{k}\)-linear map

\[
\alpha : V^\otimes n \rightarrow V^\otimes n, \quad v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto v_2 \otimes \cdots \otimes v_n \otimes v_1
\]

and \(\delta = \alpha \circ (u^{-1}\Lambda_{(1)} \otimes \Lambda_{(2)} \otimes \cdots \otimes \Lambda_{(n)})\). We have

\[
\delta(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \alpha(u^{-1}\Lambda_{(1)}v_1 \otimes \Lambda_{(2)}v_2 \otimes \cdots \otimes \Lambda_{(n)}v_n)
\]

\[
= \Lambda_{(2)}v_2 \otimes \cdots \otimes \Lambda_{(n)}v_n \otimes u^{-1}\Lambda_{(1)}v_1
\]

\[
= \Lambda_{(1)}v_2 \otimes \cdots \otimes \Lambda_{(n-1)}v_n \otimes \Lambda_{(n)}S(u^{-1})v_1 \text{ by (4.1)}
\]

\[
= \Lambda \cdot (v_2 \otimes \cdots \otimes v_n \otimes S(u^{-1})v_1).
\]

This shows that \(\delta(V^\otimes n) \subseteq \Lambda \cdot V^\otimes n = (V^\otimes n)^H\). Note that the map

\[
\Phi : \text{Hom}_H(\mathbb{k}, V^\otimes n) \rightarrow (V^\otimes n)^H, \quad f \mapsto f(1)
\]

is an \(H\)-module isomorphism. We claim that the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Hom}_H(\mathbb{k}, V^\otimes n) & \xrightarrow{E^n_V} & \text{Hom}_H(\mathbb{k}, V^\otimes n) \\
\Phi \downarrow & & \Phi \downarrow \\
(V^\otimes n)^H & \xrightarrow{\delta} & (V^\otimes n)^H.
\end{array}
\]

Indeed, for any \(f \in \text{Hom}_H(\mathbb{k}, V^\otimes n)\), we suppose that \(f(1) = \sum v_1 \otimes \cdots \otimes v_n \in V^\otimes n\).

It follows from \(f(1) = f(\Lambda \cdot 1) = \Lambda \cdot f(1)\) that

\[
\sum v_1 \otimes \cdots \otimes v_n = \sum \Lambda_{(1)}v_1 \otimes \cdots \otimes \Lambda_{(n)}v_n.
\]

(4.4)
On the one hand, we have
\[(\delta \circ \Phi)(f) = \delta(f(1)) = \delta(\sum v_1 \otimes \cdots \otimes v_n)
= \Lambda \cdot (\sum v_2 \otimes \cdots \otimes v_n \otimes S(u^{-1})v_1) \text{ by (4.3)}\]

On the other hand, we have
\[(\Phi \circ E_V^n)(f) = E_V^n(f)(1) = \sum v_2 \otimes \cdots \otimes v_n \otimes u^{-1}v_1 \text{ by (4.2)}
= \sum \Lambda(2)v_2 \otimes \cdots \otimes \Lambda(n)v_n \otimes u^{-1}\Lambda(1)v_1 \text{ by (4.4)}
= \sum \Lambda(1)v_2 \otimes \cdots \otimes \Lambda(n-1)v_n \otimes \Lambda(n)S(u^{-1})v_1 \text{ by (4.1)}
= \Lambda \cdot (\sum v_2 \otimes \cdots \otimes v_n \otimes S(u^{-1})v_1).\]

We obtain that \(\delta \circ \Phi = \Phi \circ E_V^n\), or equivalently, \(E_V^n = \Phi^{-1} \circ \delta \circ \Phi\). It follows that
\[v_n(V) = \text{tr}(E_V^n) = \text{tr}_{V^{n \circ}}(\delta)
= \text{tr}_{V^{n \circ}}(\alpha \circ (u^{-1}\Lambda(1) \otimes \Lambda(2) \otimes \cdots \otimes \Lambda(n)))
= \text{tr}_V(u^{-1}\Lambda(1) \cdots \Lambda(n))
= \chi_V(u^{-1}\Lambda(1) \cdots \Lambda(n)),\]
where the equality \(\text{tr}_{V^{n \circ}}(\alpha \circ (u^{-1}\Lambda(1) \otimes \Lambda(2) \otimes \cdots \otimes \Lambda(n))) = \text{tr}_V(u^{-1}\Lambda(1) \cdots \Lambda(n))\) follows from [6, Lemma 2.3]. We have shown that \(v_n(V) = \chi_V(u^{-1}\Lambda(1) \cdots \Lambda(n))\) where \(\Lambda\) is idempotent. Since \(u^{-1}\Lambda(1) \cdots \Lambda(n)\) does not depend on the choice of the nonzero integral \(\Lambda\), the equality \(v_n(V) = \chi_V(u^{-1}\Lambda(1) \cdots \Lambda(n))\) holds for any nonzero integral \(\Lambda\) of \(H\).

**Remark 4.3.** If the field \(k\) has characteristic 0 and \(\Lambda\) is idempotent, then \(u = \varepsilon(\Lambda) = 1\). In this case, the \(n\)-th FS indicator of \(V\) is \(\chi_V(\Lambda(1) \cdots \Lambda(n))\), which is the one defined in [6, Definition 2.3].

In the rest of this section, we will extend the \(n\)-th FS indicator \(v_n(V)\) of \(V\) from \(n \geq 1\) to the case \(n \in \mathbb{Z}\). Recall that the \(n\)-th Sweedler power map \(P_n : H \to H\) is defined by
\[P_n(a) = \begin{cases} 
  a(1) \cdots a(n), & n \geq 1; \\
  \varepsilon(a), & n = 0; \\
  S(a(1)) \cdots S(a(-n)), & n \leq -1.
\end{cases}\]

From the \(n\)-th Sweedler power map \(P_n\) of \(H\), we may see that
\[v_n(V) = \chi_V(u^{-1}P_n(\Lambda)) \text{ for } n \geq 1.\]

However, this expression is well-defined for any integer \(n\). Thus, we may extend this formula from \(n \geq 1\) to any integer \(n\) stated as follows:

**Definition 4.4.** Let \(H\) be a finite dimensional semisimple Hopf algebra over the field \(k\) of characteristic \(p > \dim_k(H)^{1/2}\). For any finite dimensional representation \(V\) of \(H\) and any \(n \in \mathbb{Z}\), the \(n\)-th FS indicator of \(V\) is defined by
\[v_n(V) = \chi_V(u^{-1}P_n(\Lambda)),\]
where $u = S(\Lambda_{(2)})\Lambda_{(1)}$.

**Remark 4.5.**

(1) Note that $S(\Lambda) = \Lambda$. The $n$-th FS indicator of $V$ can be written as

\[
v_n(V) = \begin{cases} 
\chi_V(u^{-1}\Lambda_{(1)}\cdots\Lambda_{(n)}), & n \geq 1; \\
\chi_V(u^{-1}\epsilon(\Lambda)), & n = 0; \\
\chi_V(u^{-1}\Lambda_{(-n)}\cdots\Lambda_{(1)}), & n \leq -1.
\end{cases}
\]

(2) By Proposition 3.3 (4), we have

\[
u^{-1}S(u^{-1}) = \sum_{i \in I} \frac{\lambda(e_i)}{\epsilon(\Lambda) \dim_k(V_i)^2} e_i \in Z(H).
\]

It follows that

\[
v_0(V) = \epsilon(\Lambda)\chi_V(u^{-1}) = \epsilon(\Lambda)\chi_V(u^{-1}S(u^{-1})S(u))
\]

\[
= \sum_{i \in I} \frac{\lambda(e_i)}{\dim_k(V_i)^2} \chi_V(e_i S(u)) = \sum_{i \in I} \frac{\lambda(e_i)}{\dim_k(V_i)^2} \chi_V(e_i S(\Lambda_{(1)}\cdots\Lambda_{(2)}) \cdots S(\Lambda_{(2)}))
\]

\[
= \sum_{i \in I} \frac{\lambda(e_i)}{\dim_k(V_i)^2} \chi_V(e_i S(\Lambda_{(2)}) S(\Lambda_{(1)})) = \epsilon(\Lambda) \sum_{i \in I} \frac{\lambda(e_i)}{\dim_k(V_i)^2} \chi_V(e_i).
\]

(3) $\nu_{-1}(V) = \nu_1(V) = \chi_V(u^{-1}\Lambda) = \chi_V(\frac{\Lambda}{\Lambda_{(2)})}$.

(4) By [17, Proposition 3.1], $\Lambda_{(1)}\Lambda_{(2)}$ and $\Lambda_{(2)}\Lambda_{(1)}$ are both central elements of $H$, they are determined by the values that the characters $\chi_i$ for all $i \in I$ take on them. It follows from $\chi_i(\Lambda_{(1)}\Lambda_{(2)}) = \chi_i(\Lambda_{(2)}\Lambda_{(1)})$ that $\Lambda_{(1)}\Lambda_{(2)} = \Lambda(2)\Lambda(1)$. Therefore, $\nu_{-2}(V) = \nu_2(V)$.

The higher FS indicators of any simple module $V_i$ can be described as follows:

**Proposition 4.6.** For any $n \in \mathbb{Z}$ and any simple module $V_i$ with the character $\chi_i$, we have

\[
v_n(V_i) = \frac{\chi_i(P_n(\Lambda))\lambda(e_i)}{\dim_k(V_i)^2},
\]

**Proof.** Since $P_n(\Lambda) \in Z(H)$ for any $n \in \mathbb{Z}$ (see [17, Proposition 3.1]), it follows that $P_n(\Lambda) = \sum_{i \in I} \frac{\chi_i(P_n(\Lambda))}{\dim_k(V_i)} e_i$. The $n$-th FS indicator of $V_i$ is

\[
v_n(V_i) = \chi_i(u^{-1}P_n(\Lambda)) = \frac{\chi_i(P_n(\Lambda))}{\dim_k(V_i)^2} \chi_i(u^{-1}) = \frac{\chi_i(P_n(\Lambda))\lambda(e_i)}{\dim_k(V_i)^2},
\]

where the last equality follows from Proposition 3.3 (3). \qed

For any semisimple Hopf algebra over a field $k$ of characteristic 0, the finite dimensional representation $V$ and its dual $V^*$ have the same $n$-th FS indicators for all $n \geq 1$ (see [6, Section 2.3]). The following result shows that this result also holds for the $n$-th FS indicators defined for the Hopf algebra $H$ over the field $k$ of characteristic $p > \dim_k(H)^{1/2}$. 


Proposition 4.7. Let $H$ be a finite dimensional semisimple Hopf algebra over the field $\mathbb{k}$ of characteristic $p > \dim_{\mathbb{k}}(H)^{1/2}$. Let $V$ be a finite dimensional representation of $H$ with the dual $V^*$. We have $\nu_n(V) = \nu_n(V^*)$ for all $n \in \mathbb{Z}$.

Proof. Since $S(\Lambda) = \Lambda$, we have $S(P_n(\Lambda)) = P_n(\Lambda)$ for any $n \in \mathbb{Z}$. For the case $n \geq 1$, the $n$-th FS indicator of $V^*$ is

$$
\nu_n(V^*) = (\chi_{V^*})\left(u^{-1}P_n(\Lambda)\right) = (\chi_V \circ S)(u^{-1}P_n(\Lambda))
$$

$$
= \chi_V(\Lambda_{(1)} \cdots \Lambda_{(n)} S(u^{-1})) = \chi_V(\Lambda_{(2)} \cdots \Lambda_{(1)} u^{-1} \Lambda_{(1)}) \text{ by (4.1)}
$$

$$
= \chi_V(u^{-1} \Lambda_{(1)} \Lambda_{(2)} \cdots \Lambda_{(n)}) = \nu_n(V).
$$

For the case $n \leq -1$, the $n$-th FS indicator of $V^*$ is

$$
\nu_n(V^*) = (\chi_{V^*})\left(u^{-1}P_n(\Lambda)\right) = (\chi_V \circ S)(u^{-1}P_n(\Lambda))
$$

$$
= \chi_V(\Lambda_{(-n)} \cdots \Lambda_{(1)} S(u^{-1})) = \chi_V(S(u^{-1}) \Lambda_{(-n)} \cdots \Lambda_{(1)})
$$

$$
= \chi_V(S(u^{-1}) \Lambda_{(1)} u^{-1} \Lambda_{(-n)} \cdots \Lambda_{(2)}) = \chi_V(u^{-1} \Lambda_{(-n)} \cdots \Lambda_{(1)})
$$

$$
= \nu_n(V).
$$

For the case $n = 0$, we denote $S(e_i) = e_{i'}$ for any $i \in I$, then * is a permutation of $I$, $V_{i'} \cong V_i^*$ and $\lambda(e_{i'}) = \lambda(e_i)$ by Corollary 3.4. We have

$$
\nu_0(V^*) = \varepsilon(\Lambda) \sum_{i \in I} \frac{\lambda(e_i)}{\dim_{\mathbb{k}}(V_i)} \chi_V(S(e_i)) \text{ by Remark 4.5(2)}
$$

$$
= \varepsilon(\Lambda) \sum_{i \in I} \frac{\lambda(e_{i'})}{\dim_{\mathbb{k}}(V_{i'})} \chi_V(\varepsilon_{i'}) = \varepsilon(\Lambda) \sum_{i \in I} \frac{\lambda(e_i)}{\dim_{\mathbb{k}}(V_i)} \chi_V(e_i)
$$

$$
= \nu_0(V).
$$

We complete the proof. $\square$

Kashina-Sommerhäuser-Zhu has shown in [6, Proposition 2.5] that the $n$-th FS indicator of the regular representation of a semisimple Hopf algebra over a field of characteristic 0 can be described as $\text{tr}(S \circ P_{n-1})$ for $n \geq 1$. The following result shows that this formula also holds for the $n$-th FS indicators defined for the Hopf algebra $H$ over the field $\mathbb{k}$ of characteristic $p > \dim_{\mathbb{k}}(H)^{1/2}$.

Proposition 4.8. Let $H$ be a finite dimensional semisimple Hopf algebra over the field $\mathbb{k}$ of characteristic $p > \dim_{\mathbb{k}}(H)^{1/2}$. For any $n \in \mathbb{Z}$, the $n$-th FS indicator of the regular representation of $H$ can be written as $\nu_n(H) = \text{tr}(S \circ P_{n-1})$, where $P_{n-1}$ is the $(n - 1)$-th Sweedler power map of $H$.

Proof. We choose a left integral $\Lambda$ of $H$ and a right integral $\lambda$ of $H^*$ such that $\lambda(\Lambda) = 1$. For any $n \in \mathbb{Z}$, by Radford’s trace formula [15, Theorem 2], we have

$$
\text{tr}(S \circ P_{n-1}) = \text{tr}(P_{n-1} \circ S) = \lambda(S(\Lambda_{(2)}) P_{n-1} \circ S)(\Lambda_{(1)})
$$

$$
= \lambda(S(\Lambda_{(2)}) P_{n-1}(S(\Lambda_{(1)}))) = \lambda(\Lambda_{(1)} P_{n-1}(\Lambda_{(2)}))
$$
\[ = \lambda(P_n(\Lambda)) = \chi_H(u^{-1}P_n(\Lambda)) \quad \text{by (3.3)}\]
\[ = \nu_n(H).\]

We complete the proof. \(\square\)

5. Gauge invariants

In this section, we will show that the \(n\)-th FS indicator \(\nu_n(V)\) defined in Section 4 is a gauge invariant of the tensor category \(\text{Rep}(H)\) for any \(n \in \mathbb{Z}\) and any finite dimensional representation \(V\) of the semisimple Hopf algebra \(H\).

Recall from [1] that a (normalized) twist for semisimple Hopf algebra \(H\) is an invertible element \(J \in H \otimes H\) that satisfies \((\varepsilon \otimes \text{id})(J) = (\text{id} \otimes \varepsilon)(J) = 1\) and
\[ (\Delta \otimes \text{id})(J)(1 \otimes 1) = (\text{id} \otimes \Delta)(J)(1 \otimes J). \]

We write \(J = J^{(1)} \otimes J^{(2)}\) and \(J^{-1} = J^{-(1)} \otimes J^{-(2)}\), where the summation is understood.

Given a twist \(J\) for \(H\) one can define a new Hopf algebra \(H^J\) with the same algebra structure and counit as \(H\), for which the comultiplication \(\Delta^J\) and antipode \(S^J\) are given respectively by
\[ \Delta^J(a) = J^{-1}\Delta(a)J, \]
\[ S^J(a) = Q^{-1}_J S(a) Q_J, \quad \text{for } a \in H, \]
where \(Q_J = S(J^{(1)})J^{(2)}\), which is an invertible element of \(H\) with the inverse \(Q^{-1}_J = J^{-(1)}S(J^{-(2)})\). With the notions above, we have the following result:

**Proposition 5.1.** Let \(H\) be a finite dimensional semisimple Hopf algebra over the field \(\mathbb{k}\) of characteristic \(p > \dim_\mathbb{k}(H)^{1/2}\) and \(V\) a finite dimensional representation of \(H\). The \(n\)-th FS indicator \(\nu_n(V)\) of \(V\) is invariant under twisting for any \(n \in \mathbb{Z}\).

**Proof.** Let \(\Lambda\) be a nonzero integral of \(H\) and \(J\) a normalized twist for \(H\). It follows from [17, Theorem 3.4] that \(P^J_n(\Lambda) = P_n(\Lambda)\), where \(P^J_n\) and \(P_n\) are the \(n\)-th Sweedler power maps of \(H^J\) and \(H\) respectively. Moreover, \(P_n(\Lambda)\) is a central element of \(H\) (see [17, Proposition 3.1]). Since \(\Delta^J(\Lambda) = Q^{-1}_J \Lambda(1) \otimes \Lambda(2)Q_J\), it follows that
\[ u^J := S^J(\Lambda(2))Q_JQ^{-1}_J \Lambda(1) = Q^{-1}_J S(Q_J) u, \]
where \(u = S(\Lambda(2))\Lambda(1)\). For \(H\)-module \(V\) with the character \(\chi_V\), we denote \(V^J\) the same as \(V\) as \(\mathbb{k}\)-linear space but thought of as an \(H^J\)-module. Then the character of \(V^J\) is also \(\chi_V\). For any \(n \in \mathbb{Z}\), we have
\[ \nu_n(V^J) = \chi_V((u^J)^{-1}P^J_n(\Lambda)) \]
\[ = \chi_V(u^{-1}S(Q^{-1}_J)Q_JP^J_n(\Lambda)) \quad \text{by (5.1)}\]
\[ = \chi_V(u^{-1}S(Q^{-1}_J)Q_JP_n(\Lambda)) \]
\[ = \chi_V(u^{-1}S(J^{-(2)})S(J^{(1)})S(J^{(2)})P^J_n(\Lambda)) \]
\[ = \chi_V(J^{-(2)}u^{-1}S(J^{-(1)})S(J^{(1)})J^{(2)}P_n(\Lambda)) \]
\[ = \chi_V(u^{-1}S(J^{-(1)})J^{(2)}P_n(\Lambda)J^{-(2)}) \]
where \( u \) is a nonzero integral of \( H \). Let \( \sigma : H \to H' \) be a Hopf algebra isomorphism. The isomorphism follows from Proposition 5.1. Therefore, \( \nu_n(V) = \nu_n(\mathcal{F}(V)) \) for any \( n \in \mathbb{Z} \) and any finite dimensional representation \( V \) of \( H \).

**Theorem 5.2.** Let \( H \) and \( H' \) be two finite dimensional semisimple Hopf algebras over the field \( \mathbb{K} \) of characteristic \( p > \dim \mathbb{K}(H)^{1/2} \). If \( \mathcal{F} : \text{Rep}(H) \to \text{Rep}(H') \) is an equivalence of tensor categories, then \( \nu_n(V) = \nu_n(\mathcal{F}(V)) \) for any \( n \in \mathbb{Z} \) and any finite dimensional representation \( V \) of \( H \).

**Proof:** Since the \( \mathbb{K} \)-linear equivalence \( \mathcal{F} : \text{Rep}(H) \to \text{Rep}(H') \) is a tensor equivalence, it follows from [11, Theorem 2.2] that \( H \) and \( H' \) are gauge equivalent in the sense that there exist a twist \( J \) of \( H \) such that \( H' \) is isomorphic to \( H^J \) as bialgebras. Let \( \sigma : H' \to H^J \) be such an isomorphism. Then \( \sigma \) is automatically a Hopf algebra isomorphism. The isomorphism \( \sigma \) induces a \( \mathbb{K} \)-linear equivalence \( (-)^\sigma : \text{Rep}(H) \to \text{Rep}(H') \) as follows: for any finite dimensional \( H \)-module \( V \), \( V^\sigma = V \) as \( \mathbb{K} \)-linear space with the \( H' \)-module action given by \( h'v = \sigma(h')v \) for \( h' \in H' \), \( v \in V \), and \( f^\sigma = f \) for any morphism \( f \) in \( \text{Rep}(H) \). Moreover, the equivalence \( \mathcal{F} \) is naturally isomorphic to the \( \mathbb{K} \)-linear equivalence \((-)^\sigma \) (see [5, Theorem 1.1]). Therefore,

\[
\nu_n(\mathcal{F}(V)) = \nu_n(V^\sigma).
\]

Let \( \Lambda' \) be a nonzero integral of \( H' \) and \( S' \) the antipode of \( H' \). Note that the map \( \sigma : H' \to H^J \) is a Hopf algebra isomorphism. It follows that \( \sigma(\Lambda') = \Lambda \), which is a nonzero integral of \( H^J \) and \( \sigma(P_n(\Lambda')) = P_n^J(\Lambda) \), where \( P_n \) and \( P_n^J \) are the \( n \)-th Sweedler power maps of \( H' \) and \( H^J \) respectively. In particular,

\[
\sigma((u')^{-1}P_n'(\Lambda')) = (u^J)^{-1}P_n^J(\Lambda),
\]

where \( u' = S'(\Lambda'_{(2)})\Lambda'_{(1)} \) and \( u^J = S^J(\Lambda_{(2)})\Lambda_{(1)} \). We have

\[
\nu_n(V^\sigma) = \chi_{V^\sigma}(u')^{-1}P_n'(\Lambda') = \chi_{V^\sigma}(\sigma((u')^{-1}P_n'(\Lambda'))) = \chi_{V^\sigma}((u^J)^{-1}P_n^J(\Lambda)) = \nu_n(V^J) = \nu_n(V),
\]

where the last equality follows from Proposition 5.1. We conclude that \( \nu_n(\mathcal{F}(V)) = \nu_n(V) \) for any \( n \in \mathbb{Z} \) and any finite dimensional representation \( V \) of \( H \). \qed
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