A NOTE ON A CLASS OF COSMOLOGICAL STRING BACKGROUNDS

Björn Andreas

Humboldt-Universität zu Berlin, Institut für Physik, D-10115 Berlin, Germany

We study a class of four dimensional, anisotropic string backgrounds and analyse their expansion and singularity structure. In particular we will see how O(3,3) duality acts on this class.
In the course of study of string cosmological backgrounds one class of solutions appeared over the time which was originally introduced by Mueller [1] and was studied in the string frame [2,3] and in related works [4].

In this note we will study the space of solutions in the Einstein frame which is related to the string frame by a conformal transformation. We will show that if one introduces a new time coordinate within the Einstein frame one can apply techniques known from General Relativity to get a geometrical interpretation of the expansion and singularity structure of the solutions. Furthermore we will see how O(3,3) duality acts on this space and it will be shown that depending on the anisotropy of the solution different types of singularities appear.

Now let us start and consider a string propagating in the presence of a D-dimensional background spacetime metric $G(X)$, a dilaton $\Phi(X)$ and the antisymmetric tensor field $B(X)$ with field strength $H(X)$. The string tree level effective action for these fields is given in the Einstein frame by

$$S = \int d^Dx \sqrt{-G} \left[ R - 2(\nabla \phi)^2 - e^{-4\phi} \frac{1}{12} H^2 \right] + S_f.$$  

where $S_f$ denotes that there can be moduli fields coming from compactification and higher loop effects, etc. By varying $G, B, \phi$ we obtain the equations of motion

$$R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R = T_{\mu\nu}$$

$$\nabla_\mu (e^{-4\phi} H^\mu_{\nu\lambda}) = 0$$

$$\nabla^2 \phi + e^{-4\phi} \frac{1}{12} H^2 = 0.$$  

with the energy-momentum tensor for the axion-dilaton system given by

$$T_{\mu\nu} = \frac{1}{4} (H^2_{\mu\nu} - \frac{1}{6} G_{\mu\nu} H^2) e^{-4\phi} + 2(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} G_{\mu\nu} (\nabla \phi)^2)$$

These equations can also be obtained [5,6] by the conditions of vanishing $\beta$-functions in the corresponding two-dimensional $\sigma$ model.

Now let us consider as an ansatz the anisotropic spacetime metric which has a time dependent dilaton and a spacetime metric given in the Einstein frame by (setting $B(X) = 0$)

$$ds_E^2 = -e^{-2\phi} dt^2 + \sum_{i=1}^3 (R_i(t))^2 (dX^i)^2,$$

where $R_i(t)^2$ is given by $R_i(t)^2 = e^{-2\phi} R_i^2(t)^2$ and the corresponding field equations (cf. for the stringframe [1,2]) are

$$\sum_{i<j} \frac{\dot{R}_i \dot{R}_j}{R_i R_j} = (\dot{\phi}(t))^2$$

$$-e^{2\phi} R_i^2 \left[ \frac{\dot{R}_k \dot{R}_l}{R_k R_l} + \frac{\dot{R}_k}{R_k} \dot{\phi} + \frac{\dot{R}_l}{R_l} \dot{\phi} + \frac{\dot{R}_k}{R_k} + \frac{\dot{R}_l}{R_l} \right] = e^{2\phi} R_i(t)^2 (\dot{\phi}(t))^2.$$  

1
and \( i, k, l \in \{1,2,3\} \) with \( k, l \neq i \) \((k > l)\) further \( R_i = R_i(t) \) and \( \phi = \phi(t) \). The dot denotes derivative with respect to \( t \).

A family of solutions introduced originally in the string frame \([1]\) are given in the Einstein frame by

\[
R_i(t) = \alpha_i (t - t_0)^{p_i + \frac{p}{2}} \quad \phi(t) = -\frac{1}{2} \log \beta^2 (t - t_0)^p,
\]

where \( \alpha_i, \beta, t_0 \) are arbitrary real numbers; \( p_i \) and \( p \) are real but restricted through

\[
\sum_{i=1}^{3} p_i^2 = 1, \quad \sum_{i=1}^{3} p_i = 1 - p, \quad \sum_{i<j} p_ip_j = \frac{1}{2} (p^2 - 2p).
\]

Now these equations lead to the geometrical picture that the \( p_i \) can be thought of being points on the unit 2-sphere (note that \( \sum_i p_i = 1 - p \) determines a codimension one subvariety in this sphere) and every choice of a tupel \( (p_i) \) determines a solution of the differential equation system and restricts the range of \( p \). The solutions depend on 7 parameters \( \alpha_i, \beta, t_0 \) (fixing the initial values of \( R_i \) and \( \phi \)) and 2 free coordinates on the 2-sphere. The behavior of the dilaton field depends on whether \( p = 0, p < 0 \) or \( p > 0 \).

Now to get rid of the scale factor in front of our line element let us introduce a new time coordinate \( \tau \) that leads to a line element which can be studied by known methods from General Relativity.

\[
ds^2_E = -d\tau^2 + \sum_i R_i(\tau)^2 d(X^i)^2.
\]

The corresponding coordinate transformation can be obtained from \( \dot{t}(\tau) = t(\tau)^{-\frac{p}{2}} \) where the dot denotes now and in the following the derivative with respect to \( \tau \). A solution is given by

\[
t(\tau) = (\tau + \frac{p}{2}\tau + c)^{1+\frac{p}{2}}, \quad c = \text{const}.
\]

The solutions are given in the new time coordinate by

\[
R_i(\tau) = t(\tau)^{p_i + \frac{p}{2}}, \quad \phi(\tau) = -\frac{1}{2} \ln t(\tau)^p
\]

where we have set for simplicity \( \alpha_i, \beta = 1 \) and \( t_0 = 0 \).

If we insert our new line element into the equations of motion of our background fields we find

\[
\sum_{i<j} \frac{\dot{R}_i R_j}{R_i R_j} = p^2 \frac{\dot{t}(\tau)^2}{4t(\tau)^2}
\]

\[
-R_i^2 \left[ \frac{\dot{R}_k R_i}{R_k R_i} + \frac{\dot{R}_k}{R_k} + \frac{\ddot{R}_i}{R_i} \right] = p^2 \frac{t(\tau)^2\dot{p}_i + p\dot{t}(\tau)^2}{4t(\tau)^2}
\]
with \(i, k, l \in \{1, 2, 3\}\) and \(k, l \neq i\) and \(k > l\). The energy density \(\mu\) and the pressure \(\hat{p}\) of the dilaton matter system are given by

\[
T_{00} = \mu(\tau), \quad T_{ij} = \hat{p}(\tau)G_{ij} = \frac{p^2}{4}(\frac{i(\tau)^2}{2})G_{ij}.
\]

Now to decide which behavior our solutions have (i.e. if they describe a contracting or expanding universe and if there is an initial singularity) we have to consider the energy-momentum conservation which leads to the following equation

\[
D^\mu T_{\mu\nu} = G^\mu\lambda \partial_\lambda T_{\mu\nu} - G^{\mu\nu} \Gamma^\rho_{\mu\lambda} T_{\rho\nu} - G^{\mu\nu} \Gamma^\rho_{\nu\lambda} T_{\mu\rho} = -\dot{\mu} - (\mu + \hat{p}) \sum_i \frac{\dot{R}_i(\tau)}{R_i(\tau)} = 0
\]

Since \(\frac{\dot{R}_i}{R_i}\) measures the expansion respectively contraction of the universe with respect to the \(i\)-th coordinate, we learn

\[
\frac{\dot{R}_i}{R_i} > 0 \iff \text{Expansion}; \quad \frac{\dot{R}_i}{R_i} < 0 \iff \text{Kontraktion}
\]

Finally we have to study the Raychaudhuri equation in order to decide if we have an initial singularity [4]. This equation can be derived if one considers the energy-momentum tensor of our dilaton matter system which can be thought of as a perfect fluid and is given by

\[
T_{\mu\nu} = \hat{p}G_{\mu\nu} + (\mu + \hat{p})V_\mu V_\nu.
\]

where \(V_\mu\) is the tangential vector to a congruence of timelike curves (with \(g(V, V) = -1\)). Now every congruence of timelike geodesics converges if \(R_{\mu\nu}V^\mu V^\nu \geq 0\). The equations of motion satisfies this condition if the energy-momentum tensor satisfies

\[
T_{\mu\nu}V^\mu V^\nu - \frac{1}{2} V^\mu V_\mu T \geq 0, \quad \mu + \hat{p}_i \geq 0, \quad \mu + \sum_i \hat{p}_i \geq 0.
\]

where \(T = T_{\mu\nu}T_{\mu\nu}\). This condition will be satisfied if the energy-momentum tensor is diagonal \(T_{\mu\nu} = \text{diag}(\mu, \hat{P}_i)\) with \(\hat{P}_i = \hat{p}_i G_{ij}\); the left hand side of this equation is the source term in the contracted Einstein equations

\[
R_{\mu\nu} = -8\pi S_{\mu\nu},
\]

where \(S_{\mu\nu} = T_{\mu\nu} + \frac{1}{2} G_{\mu\nu} T = \frac{1}{2}(\mu - \hat{p}) + (\hat{p} + \mu) V_\mu V_\nu\). Here we used the energy-momentum tensor. With our above ansatz \(S_{\mu\nu}\) reduces to

\[
S_{00} = \frac{1}{2}(\mu + 3\hat{p}), \quad S_{0i} = 0, \quad S_{ij} = \frac{1}{2}(\mu - \hat{p}) R_i(\tau) \Omega_{ij}(X)
\]

where \(\Omega_{ij}(X)\) is the metric of the threedimensional area of constant curvature. The time-time component of \(R_{\mu\nu}\) is then

\[
R_{00} = 4\pi(\mu + 3\hat{p}).
\]
and if we use \( R_{00} = -\sum_i \frac{\ddot{R}_i(\tau)}{R_i(\tau)} \) we find the Raychandhuri equation \(^3\)

\[
\sum_i \frac{\ddot{R}_i(\tau)}{R_i(\tau)} = -4\pi(\mu + 3\dot{p}).
\]

From the energy-momentum conservation we learn that the density of the universe vanishes for \( R(\tau) \to \infty \) and grows up to infinity for \( R(\tau) \to 0 \). Since the density is a scalar we can conclude that also the scalar curvature grows up and at \( R = 0 \) we have a singularity which is no coordinate singularity and leads usually to the break down of the Einstein theory. Thus from the Raychandhuri equation we learn that if \( \mu + 3\dot{p} > 0 \) we have an initial singularity \[^{[7]}\].

Now we are able to study the structure of our model where we have

\[
R_{00} = \frac{p^2 i(\tau)^2}{2t(\tau)^2}, \quad R_{ii} = 0, \quad R = -\frac{p^2 i(\tau)^2}{2t(\tau)^2}, \quad \mu = \dot{p} = \frac{p^2 i(\tau)^2}{4t(\tau)^2}.
\]

**Duality transformation**

Since our model is independent of three coordinates we can perform a \( O(3,3) \) duality transformation (for a review on \( O(d,d) \) duality see \[^{[8]}\] and references therein). We do this in the string frame which is related to the Einstein frame by the conformal transformation \( G^S_{\mu\nu} = e^{2\phi} G^E_{\mu\nu} \) and then transform back to the Einstein frame and find the dual line element

\[
ds^2_D = -d\tau^2 + \sum_i (R^D_i)^2 (dX^i)^2,
\]

where \( d\tau^2 = e^{-2\phi_D} dt^2 \) and \((R^D_i)^2 = e^{-2\phi_D} \ddot{R}_i^{-2}\). The equations of motion are solved by

\[
R_i(\tau)^D = \ddot{\tau}(\tau) - p_i - \frac{\dot{\tau}}{2}, \quad \phi(\tau)^D = -\frac{1}{2} \log \ddot{\tau}(\tau)^2 - p.
\]

where \( \ddot{\tau}(\tau) \) is given by rescaling of the dual metric. The duality transformation \( p_i \to -p_i, \ p \to -p + 2 \) or equivalently \( R_i \to \frac{1}{R_i}, \ p \to -p + 2, \ (i = 1, 2, 3) \) relates the two string backgrounds to each other. Remember the range of \( p \) was given by \( p_{\text{min}} = 1 - \sqrt{3} \leq p \leq 1 + \sqrt{3} = p_{\text{max}} \), and therefore \( p \to -p + 2 \) maps \( p_{\text{min}} \to p_{\text{max}} \) resp. \( p_{\text{max}} \to p_{\text{min}} \), i.e. the duality transformation operates on the space of solutions. For the dual model we find

\[
R^D_{00} = \frac{(-p + 2)^2 \ddot{\tau}(\tau)^2}{2t(\tau)^2}, \quad R^D_{ii} = 0, \quad \ddot{\tau}(\tau) = (\tau(2 - \frac{p}{2}) + c) \frac{1}{\sqrt{\tau}}
\]

\[
R^D = -\frac{(-p + 2)^2 \ddot{\tau}(\tau)^2}{2t(\tau)^2}, \quad \mu_D = \dot{\mu}_D = \frac{(-p + 2)^2 \ddot{\tau}(\tau)^2}{4t(\tau)^2}
\]

\(^2\) \( \frac{\ddot{R}_i}{R_i} \) measures the acceleration of the expansion resp. contraction and implies

\[
\sum_i \frac{\ddot{R}_i}{R_i} < 0 \iff \mu + 3\dot{p} > 0; \quad \sum_i \frac{\ddot{R}_i}{R_i} > 0 \iff \mu + 3\dot{p} < 0.
\]
Singularities

Let us study the singularity structure. Setting $c = 0$ we obtain (using $\dot{(\tau)}^2 = (\tau(1 + p^2))^2$ and $\dot{(\tau)}^2 = (\tau(2 - p^2))^2$ for the scalar curvatures $R$, $R^D$ and the energy densities $\mu$, $\mu^D$)

$$R = -\frac{p^2}{2}(\tau(1 + p^2))^2, \quad \mu = \frac{p^2}{4}(\tau(1 + p^2))^2,$$

$$R^D = -\frac{(2 - p)^2}{2}(\tau(2 - p^2))^2, \quad \mu^D = \frac{(2 - p)^2}{4}(\tau(2 - p^2))^2.$$

$p \neq 0$:

For solutions with $p \neq 0$ both space-times are singular at $\tau = 0$ which can be checked by inserting $p_{\text{max}}$ and $p_{\text{min}}$ into the equations above. Furthermore for all $p \neq 0$ we have

$$\mu + 3\hat{p} > 0, \quad \mu^D + 3\hat{p}^D > 0$$

and the energy density has the $\frac{1}{\tau}$ behavior typical for cosmological solutions and will grow up to infinity if $\tau \to 0$.

$p = 0$:

If $p = 0$ then the dilaton is constant and the energy density vanishes. The scalar curvature and energy density are given by

$$R = 0, \quad \mu = 0, \quad R^D = -\frac{1}{2\tau^2}, \quad \mu^D = \frac{1}{\tau^2}.$$

Further $\mu + \hat{p}$ vanishes for Ricci-flat spacetimes. To get some information about singularities one has to check if the Riemannian tensor gets singular for $\tau \to 0$. In our case we find $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} \sim \frac{1}{\tau^2}$ which will be singular at $\tau = 0$. Thus the duality transformation maps a Ricci-flat spacetime onto spacetime with singular scalar curvature.

Expansion behavior

To understand the expansion behavior we have to consider $\frac{\dot{R}}{R_i} = \frac{p_i + \frac{1}{\tau^2}}{1 + \frac{1}{\tau^2}}$ and $\frac{\dot{R}^D}{R_i^D} = -\frac{p_i + \frac{1}{\tau^2}}{2 + \frac{1}{\tau^2}}$.

$p = 0$:

The Expansion behavior for $p = 0$ is simply given by

$$\frac{\dot{R}_i}{R_i} = \frac{p_i}{\tau}, \quad \frac{\dot{R}_i^D}{R_i^D} = -\frac{p_i + 1}{2\tau}.$$

Now remember $p_i$ are coordinates of points on the unit sphere with values in $[-1,1]$. Since our solutions are 4-dimensional we exclude cases with $p_1 = 1, p_i = 0 \{i = 2, 3\}$ and $p_i = 0 \{i = 1, 2, 3\}$. Taking into account that the dual expansion coefficient being positive for all $p_i$ we learn that our dual solution describes an expanding spacetime. The original solution describes also an expanding spacetime if $p_i$ is positive with respect to the i-th
coordinate and describes an contracting spacetime if \( p_i \) is negative with respect to the i-th coordinate

\[
p = 0 : \quad p_i \begin{cases} > 0 & \text{EXPANSION} \rightarrow \text{EXPANSION} \\ < 0 & \text{CONTRACTION} \rightarrow \text{EXPANSION} \end{cases}
\]

If we solve the equation system \( \sum p_i = 1, \quad \sum p_i^2 = 1 \) we will find coordinates for \( p_i \). For example if we fix \( p_3 \) then we will obtain \( p_1 \) and \( p_2 \):

\[
p_1 = \frac{1}{2} \left( 1 - p_3 + \sqrt{1 + 2p_3 - 3p_3^2} \right), \quad p_2 = \frac{1}{2} \left( 1 - p_3 \pm \sqrt{1 + 2p_3 - 3p_3^2} \right).
\]

Example: \( p = 0, \quad p_3 = \frac{1}{2} > 0, \quad p_1 = \frac{1}{4} - \sqrt{3}/4 < 0, \quad p_2 = \frac{1}{4} + \sqrt{3}/4 > 0 \)

In the example duality maps a Ricci-flat spacetime which expands in \( i = 3, 2 \) and contracts in \( i = 1 \) to a spacetime which expands in \( i = 1, 2, 3 \). Both spacetimes are singular for \( \tau \rightarrow 0 \) but with different singularity structure. The former has an infinite THREAD singularity if \( \tau \rightarrow 0 \) and the latter a POINT singularity.

\( p \neq 0 \):

Now let us come to the case \( p \neq 0 \) where the range for \( p \) and \( p_i \) is \( 1 - \sqrt{3} \leq p \leq 1 + \sqrt{3} \) and \( -1 < p_i < 1 \) respectively. Since the denominator of \( \frac{R_i}{R} \) and \( \frac{R_D}{R_i} \) is always positive we are left to check the nominator. We find that we can distinguish the following cases:

A: STATIC \( \rightarrow \) EXPANSION:

\[
\left( p_i + \frac{p}{2} \right) = 0 \rightarrow -\left( p_i + \frac{p}{2} \right) + 1 > 0
\]

B: CONTRACTION \( \rightarrow \) EXPANSION:

\[
\left( p_i + \frac{p}{2} \right) < 0 \rightarrow -\left( p_i + \frac{p}{2} \right) + 1 > 0
\]

C: EXPANSION \( \rightarrow \) EXPANSION:

\[
0 < \left( p_i + \frac{p}{2} \right) < 1 \rightarrow -\left( p_i + \frac{p}{2} \right) + 1 > 0
\]

D: EXPANSION \( \rightarrow \) CONTRACTION:

\[
\left( p_i + \frac{p}{2} \right) > 1 \rightarrow -\left( p_i + \frac{p}{2} \right) + 1 < 0.
\]

Conclusion

In this note we showed that one can get a geometrical interpretation of the space of solutions which is parametrised by \( p \) and \( p_i \) using methods known from classical General Relativity. The duality transformation \( p_i \rightarrow -p_i, p \rightarrow -p + 2 \) acting on this space maps for \( p = 0 \) and \( p_i > 0 \) expanding to expanding backgrounds and for \( p = 0 \) and \( p_i < 0 \) contracting to expanding backgrounds and if \( p \neq 0 \) the cases A-D appear. Furthermore
it has been shown that depending on the possible combinations of A-D there are backgrounds with THREAD singularities as well as POINT or PANCAKE singularities as a consequence of having an anisotropic spacetime.

Acknowledgement: I would like to thank C.Curio and D.Lüst for discussions. The work has been supported by NAFOEG.

References

1. M. Mueller Nucl.Phys.B337(1990) 37;
2. D. Lüst, preprint CERN-TH.6850/93;
3. A.A. Tseytlin and C. Vafa Nucl.Phys.B372(1992) 443;
4. E. Kiritsis, C. Kounnas and D. Lüst Int.J.Mod.Phys.A9(1994) 1361; I. Bakas Nucl.Phys.B428(1994) 374; I. Antoniades, S. Ferrara and C. Kounnas Nucl.Phys.B421(1994) 343; M. Gasperini and G. Veneziano Phys.Rev.D50(1994) 2519; A.A. Tseytlin Phys.Lett.B334(1994) 315; M. Gasperini and R. Ricci Class.Quant.Grav.12(1995) 677; N.A. Batakis and A.A. Kehagias Nucl.Phys.B449(1995) 248; A. Kehagias and A. Lukas Nucl.Phys.B477(1996) 549; A. Lukas, B. A. Ovrut and D. Waldram Phys.Lett.B393(1997) 65; A. Lukas, B. A. Ovrut and D. Waldram Nucl.Phys.B495(1997) 365; E. Kiritsis and C. Kounnas gr-qc/9701003 publ.in Nucl.Phys.Proc.Suppl.41(1995) 365; J.D. Barrow and K.E. Kunze Phys.Rev.D56(1997) 741; M.P. Dabrowski, A.L. Larsen hep-th/9706020;
5. M.B.Green, J.Schwarz and E.Witten: Superstring theory Cambridge University Press, 1987;
6. C.G.Callan, D.Friedan, E.J.Martinec und M.J.Perry, Nucl.Phys.B262,593;
7. S.W.Hawking und G.F.R.Ellis: The large scale structure of space-time, Cambridge Monographs on Mathematical Physics;
8. A. Giveon, M. Porrati and E. Rabinovici Phys.Rept.244(1994) 77;