THE $N = 1$ SUPER HEISENBERG-VIRASORO VERTEX ALGEBRA AT LEVEL ZERO

DRAŽEN ADAMOVIĆ, BERISLAV JANDRIĆ, AND GORDAN RADOBOLJA

Abstract. We study the representation theory of the $N = 1$ super Heisenberg-Virasoro vertex algebra at level zero, which extends the previous work on the Heisenberg-Virasoro vertex algebra [9], [5] and [6] to the super case. We calculated all characters of irreducible highest weight representations by investigating certain Fock space representations. Quite surprisingly, we found that the maximal submodules of certain Verma modules are generated by subsingular vectors. The formulas for singular and subsingular vectors are obtained using screening operators appearing in a study of certain logarithmic vertex algebras [2].

Contents

1. Introduction 2
Free-field realisation and screening operators 3
Characters of $L[p, r]$ 5
The structure of Verma modules 5
Future work 6
2. Preliminaries 6
2.1. Partitions and super_partitions 6
2.2. The vertex algebra $V_{SH}(c_L, c_{L, \alpha})$ 7
2.3. Contragredient modules 10
3. Free field realisation 11
3.1. Realisation of Verma modules $V[p, r], p \in \mathbb{C} \setminus \mathbb{Z}_{<0}.$ 13
3.2. Realisation of $L[p, r], p \in \mathbb{Z}_{<0}$ as submodules of $F_{p, r}$ 14
4. Screening operators and singular vectors, case $p > 0$ 16
4.1. Screening operators 16

Date: November 25, 2020.

2010 Mathematics Subject Classification. Primary 17B69; Secondary 17B20, 17B65.

Key words and phrases. Super Heisenberg-Virasoro vertex algebra.
1. Introduction

The Heisenberg-Virasoro vertex algebra is a generalisation of the Virasoro and the Heisenberg vertex algebra. It appears as a vertex subalgebra of affine vertex-algebras and \( \mathcal{W} \)-algebras. It is shown in [8], [14] that in the case of non-zero level, the Heisenberg-Virasoro vertex algebra is isomorphic to the tensor product of the Heisenberg and the Virasoro vertex algebra. But it is relatively less known that in the case of level zero, the Heisenberg-Virasoro vertex algebra has completely different structure. The study of the level-zero case was initiated by Y. Billig in [9]. Among other results, Billig calculated characters of all irreducible, highest weight modules. In our previous papers [5]-[6], we extended Billig’s work by applying recent constructions in vertex-algebra theory and conformal field theory such as fusion rules, logarithmic modules, Whittaker modules, screening operators. We have demonstrated that the Heisenberg-Virasoro vertex algebra is also an interesting example of a vertex algebra which allows us to understand more general non-rational vertex algebras.

A natural problem is to try to find its super-analogue. This leads to the \( N = 1 \) super Heisenberg-Virasoro vertex algebra. This vertex algebra was introduced in [3], using the notion of conformal Lie superalgebras. The universal \( N = 1 \) Heisenberg-Virasoro algebra is denoted by \( V^{SH}(c_L, c_\alpha, c_{L,\alpha}) \), and is generated by two even fields (the Virasoro field \( L(z) \) and the Heisenberg field \( \alpha(z) \)) and by two odd fields \( G(z) \) and \( \Psi(z) \). Similarly to the non-super case, in the case of a non-zero level (i.e., when the central element \( C_\alpha \) acts non-trivially), this vertex algebra is isomorphic to a tensor product of the Neveu-Schwarz vertex superalgebra and the Clifford-Heisenberg algebra.
In this paper, we start with a detailed study of the vertex algebra $V^{SH}(c_L,c_L,\alpha) := V^{SH}(c_L,0,c_L,\alpha)$, i.e. the level zero case. We plan to extend results from [9], [5] and [6] to the super case. In the present paper, we calculated characters of all irreducible, highest weight modules and investigated the structure of the Verma modules and Fock spaces for $V^{SH}(c_L,c_L,\alpha)$.

Let us point out one main difference in comparison with the non-super case. Certain Verma modules for $V^{SH}(c_L,c_L,\alpha)$ contain subsingular vectors which are not present in non-super case. Due to the existence of such vectors, description of the maximal submodule of Verma modules is more complicated in the super case, and Billig’s method cannot be easily generalised. Our method for calculating the character is based on free-field realisation, and formulas for singular/subsingular vectors. We show that in the "half cases", the irreducible highest weight modules are realised as submodules of certain Fock spaces. Combining this approach with explicit formulas for singular and subsingular vectors in Fock spaces, we are able to describe all characters.

**Free-field realisation and screening operators.** In [3] (see also [15]), we obtained a free field realisation of $V^{SH}(c_L,c_L,\alpha)$ as a subalgebra of the tensor product of a rank two Heisenberg algebra $M(1)$ (generated by $c(z)$ and $d(z)$), and a fermionic algebra $F^{(2)}$ generated by $\Psi^+(z)$ and $\Psi^-(z)$. The following fields

\[
\alpha = -c_L,\alpha c(-1) \\
\tau = \sqrt{2}\left(\frac{1}{2}c(-1)\Psi^+(-\frac{1}{2}) + \frac{1}{2}d(-1)\Psi^-(-\frac{1}{2}) + \frac{c_L-3}{12}\Psi^+(\frac{3}{2}) - \Psi^+(-\frac{3}{2})\right) \\
\omega = \frac{1}{2}c(-1)d(-1) + \frac{c_L-3}{24}c(-2) - \frac{1}{2}d(-2) + \omega_{fer} \\
\Psi = -\sqrt{2}c_L,\alpha \Psi^-(\frac{1}{2})
\]

generate $V^{SH}(c_L,c_L,\alpha)$.

In this paper we discuss a free-field realisation of highest weight modules for $V^{SH}(c_L,c_L,\alpha)$, which appear to be certain $V^{SH}(c_L,c_L,\alpha)$-submodules of Fock modules over the Clifford-Heisenberg vertex algebra $M(1) \otimes F^{(2)}$. 
We introduce a suitable parametrisation (11) of highest weights:

\[ v_{p,r} := e^{-\frac{p+1}{2}d + rc}, \quad \text{where} \quad d = d - \frac{cL - 3c}{12} \]

is highest weight vector of highest weight

\[ (h_{p,r}, h_\alpha) = \left( (1 - p^2) \frac{cL - 3}{24} - rp, (1 + p)cL, \alpha \right). \]

If \( M(h, h_\alpha) \) is a highest weight module of highest weight \( (h_{p,r}, (1 + p)h_\alpha) \) we denote it by \( M[p, r] \).

We define the Fock space \( F_{p,r} := (M(1) \otimes F(2)).v_{p,r} \), and consider it as a \( V^{SH}(cL, cL, \alpha) \)-module.

We prove in Proposition 2.2 that all irreducible highest weight modules such that \( p \neq 0 \), can be realised as subquotients of the Fock spaces. More precisely:

- If \( p \notin \mathbb{Z}_{<0} \), then \( V[p, r] \cong F_{p,r} \) (cf. Proposition 3.2).
- If \( p \in \mathbb{Z}_{<0} \), then \( L[p, r] \cong \langle v_{p,r} \rangle \subset F_{p,r} \) (cf. Proposition 3.5).

Let \( a = \Psi^{-\left( -\frac{1}{2} \right)}e^{\frac{c}{2}}. \) We show that \( Q = a_0 \) is a nilpotent screening operator which commutes with the action of \( V^{SH}(cL, cL, \alpha) \).

- If \( p \in \mathbb{Z}_{>0} \) is odd, \( Qv_{p,r} - \frac{1}{2} \) is a singular vector in \( V[p, r] \) (Theorem 4.2).

In order to construct other singular and subsingular vectors in Verma modules and Fock spaces, we need more complicated screening operators. For that purpose, we consider the operators (introduced in [2]):

\[ S = \sum_{i>0} \frac{1}{i} a_{-i}a_i, \quad \text{and} \quad S^{tw} = \sum_{i>0} \frac{1}{i + \frac{1}{2}} a_{-i-\frac{1}{2}}a_{i+\frac{1}{2}}, \]

which were used for studying logarithmic vertex operator algebras. In the present paper we show that

- \( G = e_0^c - S \) and \( G^{tw} = e_0^c - S^{tw} \) are screening operators (Theorem 4.1).
- \( (G^{tw})^pv_{p,r-n} \) are singular vectors in \( V[p, r] \) when \( p > 0 \) is even (Theorem 4.6).
- \( G^nQv_{p,r-n-1/2} \) are singular vectors, and \( G^n v_{p,r-n} \) are subsingular vectors in \( V[p, r] \) when \( p > 0 \) is odd (Theorem 4.4).
Characters of $L[p, r]$. We use the free field realisation to find the $q$-characters of irreducible highest weight modules $L[p, r]$ (Theorem 6.3).

**Theorem 1.1.** Assume that $p \in \mathbb{Z} \setminus \{0\}$ and $r \in \mathbb{C}$. Then we have:

\[
\text{char}_q L[p, r] = q^{h_{p,r}} (1 - q^{\frac{|p|}{2}}) \prod_{k=1}^{\infty} \frac{(1 + q^{k-1/2})^2}{(1 - q^k)^2}, \quad \text{if } p \text{ is odd;}
\]

\[
\text{char}_q L[p, r] = q^{h_{p,r}} (1 - q^{p}) \prod_{k=1}^{\infty} \frac{(1 + q^{k-1/2})^2}{(1 - q^k)^2}, \quad \text{if } p \text{ is even.}
\]

Our proof is based on the following steps.

- $L[p, r]$ is realised as a submodule of $\mathcal{F}_{p, r}$ for $p < 0$ (cf. Proposition 3.5).
- By evaluating a spanning set of $L[p, r]$ in $\mathcal{F}_{p, r}$, we get the inequalities:

\[
\text{char}_q L[p, r] \leq q^{h_{p,r}} (1 - q^{\frac{|p|}{2}}) \prod_{k=1}^{\infty} \frac{(1 + q^{k-1/2})^2}{(1 - q^k)^2}, \quad \text{if } p \text{ is odd;}
\]

\[
\text{char}_q L[p, r] \leq q^{h_{p,r}} (1 - q^{p}) \prod_{k=1}^{\infty} \frac{(1 + q^{k-1/2})^2}{(1 - q^k)^2}, \quad \text{if } p \text{ is even.}
\]

- The opposite inequalities follow by using formulas for singular/subsingular vectors in $\mathcal{F}_{p, r}$.
- The formula for $p > 0$ follows by applying the contragradient modules, which must have the same characters.

The structure of Verma modules. A companion problem is the complete description of the structure of Verma modules and Fock spaces. Although we are able to describe the characters of irreducible modules, our methods cannot solve the question of the structure of these indecomposable modules in full generality. As it is common in Lie super-algebra theory, in general, one does not have embeddings of Verma modules. However, we prove/conjecture the following:

- The structure of the Verma modules $V[p, r]$ for $p$ even, is completely determined. In particular, every embedding $V[p, r] \hookrightarrow V[p, r]$ is injective, and the maximal submodule of $V[p, r]$ is generated by a singular vector (cf. Theorem 7.1) for which we present explicit formulas.
• If $p > 0$ is odd there exists a non-injective homomorphism $V[p, r'] \to V[p, r]$. Moreover, the maximal submodule of $V[p, r]$ is generated by a subsingular vector (20) (cf. Theorem 7.2).
• If $p < 0$ is odd, we conjecture that the all homomorphisms $V[p, r'] \to V[p, r]$ are embeddings, but we cannot prove this.
• All embedding diagrams (partially conjectured for $p$ odd) are presented in the Appendix.

Future work. In a sequel [7], we shall focus on a connection with other logarithmic vertex algebras. Among other results, we will prove the following characterisation of $V^{SH}(c_L, c_{L,\alpha})$:

Theorem 1.2. [7] We have:

$$V^{SH}(c_L, c_{L,\alpha}) = \text{Ker}_{-1,0} Q \cap \text{Ker}_{-1,0} G.$$ 

We would like to thank Maria Gorelik for useful comments related to zero-divisors in Lie superalgebras and embeddings of Verma modules.

D.A. and G.R. are partially supported by the QuantiXlie Centre of Excellence, a project cofinanced by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (KK.01.1.1.01.0004).

2. Preliminaries

2.1. Partitions and super-partitions. Recall that a partition in a set $S \subset \mathbb{Q}_{\geq 0}$ is a finite sequence $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \in S^\ell$ of length $\ell = \ell_\mu \in \mathbb{Z}_{>0}$ satisfying

(1) \hspace{1cm} \mu_1 \geq \mu_2 \geq \cdots \geq \mu_\ell.

The weight of the partition $\mu$ is defined to be $\deg_\mu = \mu_1 + \mu_2 + \cdots + \mu_\ell$. Let $\mathcal{P}$ denote the set of all partitions in $\mathbb{Z}_{>0}$.

A super-partition in the set $S \subset \mathbb{Q}_{\geq 0}$ is a finite sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in S^\ell$ of length $\ell = \ell_\lambda \in \mathbb{Z}_{>0}$ satisfying

(2) \hspace{1cm} \mu_1 > \mu_2 > \cdots > \mu_\ell.$
The weight of the superpartition $\lambda$ is defined to be $\deg\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$. Let $\mathcal{SP}$ denote the set of all super-partitions in $\frac{1}{2} + \mathbb{Z}_{\geq 0}$.

Define the partial ordering in $\mathcal{P}$ (resp. $\mathcal{SP}$) by

$$\mu < \mu' \quad \text{if} \quad \mu_1 = \mu'_1, \ldots, \mu_{r-1} = \mu'_{r-1}, \mu_r > \mu'_r.$$ 

Now we consider the set $\mathcal{P} \times \mathcal{SP}$. For $(\mu, \lambda) \in \mathcal{P} \times \mathcal{SP}$, we define:

$$\deg_{\mu,\lambda} := \deg\mu + \deg\lambda, \quad \ell_{\mu,\lambda} := \ell\mu + \ell\lambda.$$ 

For $(\mu^i, \lambda^i) \in \mathcal{P} \times \mathcal{SP}$, we define the following partial ordering:

$$(\mu^1, \lambda^1) < (\mu^2, \lambda^2) \quad \text{if} \quad \deg_{\mu^1,\lambda^1} < \deg_{\mu^2,\lambda^2}$$

or $\deg_{\mu^1,\lambda^1} = \deg_{\mu^2,\lambda^2}, \quad \ell_{\mu^1,\lambda^1} < \ell_{\mu^2,\lambda^2},$

or $\deg_{\mu^1,\lambda^1} = \deg_{\mu^2,\lambda^2}, \quad \ell_{\mu^1,\lambda^1} = \ell_{\mu^2,\lambda^2}, \quad \mu_1 < \mu_2$

or $\deg_{\mu^1,\lambda^1} = \deg_{\mu^2,\lambda^2}, \quad \ell_{\mu^1,\lambda^1} = \ell_{\mu^2,\lambda^2}, \quad \mu_1 = \mu_2, \quad \lambda_1 < \lambda_2.$

For an even element $X$ (resp. odd element $Y$) and a partition $\mu \in \mathcal{P}$ (resp. a super-partition $\lambda$), we define monomials

$$X_{-\mu} := X(-\mu_1) \cdots X(-\mu_n), \quad X_\mu := X(\mu_n) \cdots X(\mu_1),$$

$$Y_{-\lambda} := Y(-\lambda_1) \cdots Y(-\lambda_n), \quad Y_\lambda := Y(\lambda_n) \cdots Y(\lambda_1).$$

2.2. The vertex algebra $V^{\mathcal{SH}}(c_L, c_{L,\alpha})$. The notion of $N = 1$ Heisenberg-Virasoro algebra was introduced in [3]. We recall main definitions and constructions.

**Definition 2.1.** The $N = 1$ Heisenberg-Virasoro algebra $\mathcal{SH}$ is an infinite dimensional Lie algebra with even generators $L(n), \alpha(n)$, odd generators $G(n + \frac{1}{2}), \Psi(n + \frac{1}{2}), n \in \mathbb{Z}$, and three central elements $C_L, C_\alpha, C_{L,\alpha}$, subject
to the following super-commutator relations:

\[
\begin{align*}
[\alpha(m), \alpha(n)] &= \delta_{m+n,0}mC_{\alpha} \\
[L(m), \alpha(n)] &= -n\alpha(m+n) - \delta_{m+n,0}(m^2 + m)C_{L,\alpha} \\
[L(m), L(n)] &= (m-n)L(m+n) + \delta_{m+n,0}\frac{m^3 - m}{12}C_{L} \\
[\Psi(m + \frac{1}{2}), \Psi(n + \frac{1}{2})]_+ &= \delta_{m+n+1,0}C_{\alpha} \\
[\alpha(m), \Psi(n + \frac{1}{2})] &= 0 \\
[G(m + \frac{1}{2}), G(n + \frac{1}{2})]_+ &= 2L(m+n+1) + \delta_{m+n+1,0}\frac{m^2 + m}{3}C_{L} \\
[L(m), G(n + \frac{1}{2})] &= \left(\frac{m+1}{2} - n - \frac{1}{2}\right)G(m+n+\frac{1}{2}) \\
[\alpha(m), G(n + \frac{1}{2})] &= m\Psi(m+n+\frac{1}{2}) \\
[\Psi(m + \frac{1}{2}), L(n)] &= \frac{2m+n+1}{2}\Psi(m+n+\frac{1}{2}); \\
[\Psi(m + \frac{1}{2}), G(n + \frac{1}{2})]_+ &= \alpha(m+n+1) + 2m\delta_{m+n+1,0}C_{L,\alpha} \\
[SH, C_{\alpha}] = [SH, C_L] = [SH, C_{L,\alpha}] &= 0
\end{align*}
\]

Lie superalgebra $SH$ has the following triangular decomposition:

\[
SH = SH^- \oplus SH^0 \oplus SH^+, \quad \text{where}
\]

\[
\begin{align*}
SH^\pm &= \text{span}_\mathbb{C}\{L(\pm n), \alpha(\pm n), G(\pm (n - 1/2)), \Psi(\pm (n - 1/2))| \ n \in \mathbb{Z}_{>0}\} \\
SH^0 &= \text{span}_\mathbb{C}\{L(0), \alpha(0), C_L, C_{\alpha}, C_{L,\alpha}\}.
\end{align*}
\]

Let $V(c_L, c_{\alpha}, c_{L,\alpha}, h, h_{\alpha})$ denote the Verma module of highest weight $(h, h_{\alpha})$ and central charge $(c_L, c_{\alpha}, c_{L,\alpha})$. We showed (3) that there is a universal vertex algebra associated to $SH$, and it is realised as

\[
V^{SH}(c_L, c_{\alpha}, c_{L,\alpha}) \cong \frac{V(c_L, c_{\alpha}, c_{L,\alpha}, 0, 0)}{(G(-\frac{1}{2})v)}.
\]

Basis of $V^{SH}(c_L, c_{\alpha}, c_{L,\alpha})$ consists of monomials

\[
(\Psi_{-\lambda - \alpha - \mu -} G_{-\lambda -} L_{-\mu +}), \quad \mu_i^+ \neq 1, \lambda^+ \neq 1/2.
\]
The vertex operator $Y(\cdot, z)$ is uniquely determined by

\[ Y(\alpha(-1)1, z) = \alpha(z) = \sum_{n \in \mathbb{Z}} \alpha(n)z^{-n-1}, \]
\[ Y(L(-2)1, z) = L(z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}, \]
\[ Y\left(\Psi\left(-\frac{1}{2}\right)1, z\right) = \Psi(z) = \sum_{n \in \mathbb{Z}} \Psi(n + \frac{1}{2})z^{-n-1}, \]
\[ Y\left(G\left(-\frac{3}{2}\right)1, z\right) = G(z) = \sum_{n \in \mathbb{Z}} G(n + \frac{1}{2})z^{-n-2}. \]

By construction of the universal vertex algebra, the basis of $V^{SH}(c_L, c_\alpha, c_{L,\alpha})$ consists of monomials

\[ (\Psi_{-\lambda-\alpha-\mu}^-G_{-\lambda+}^-L_{-\mu+})1 \]

where $\lambda^-$ is a super-partition in $\frac{1}{2} + \mathbb{Z}_{>0}$, $\lambda^+$ is a super-partition in $\frac{3}{2} + \mathbb{Z}_{>0}$, $\mu^-$ is a partition in $\mathbb{Z}_{>0}$ and $\mu^+$ a partition in $\mathbb{Z}_{>1}$. The $q$-character of $V^{SH}(c_L, c_\alpha, c_{L,\alpha})$ is thus given by

\[ \text{char}_q V^{SH}(c_L, c_\alpha, c_{L,\alpha}) = \prod_{k=1}^{\infty} \prod_{l=2}^{\infty} \frac{(1 + q^{k-1/2}) (1 + q^{l-1/2})}{(1 - q^k) (1 - q^l)} \]
\[ = (1 - q^{1/2}) \prod_{k=1}^{\infty} \frac{(1 + q^{k-1/2})^2}{(1 - q^k)^2}. \]

(5)

In [3] we showed that when $c_\alpha \neq 0$, $V^{SH}(c_L, c_\alpha, c_{L,\alpha})$ is isomorphic to the tensor product of the $N = 1$ Neveu-Schwarz vertex algebra and Heisenberg-Clifford vertex algebra $SM(1)$. In this paper we study the structure of the vertex algebra $V^{SH}(c_L, c_{L,\alpha}) := V^{SH}(c_L, 0, c_{L,\alpha})$ and its representations. For the rest of the paper we assume that $c_{L,\alpha} \neq 0$ and write $V(h, h_\alpha)$ for the Verma module $V(c_L, 0, c_{L,\alpha}, h, h_\alpha)$. Let $L(h, h_\alpha)$ denote its simple quotient.

Denote by $V[p, r]$ (resp. $L[p, r]$) the Verma module $V(h, h_\alpha)$ (resp. the irreducible highest weight module $L(h, h_\alpha)$) with the highest weight

\[ (h, h_\alpha) := (h_{p,r}, (1 + p)c_{L,\alpha}), \quad h_{p,r} = (1 - p^2) \frac{c_L - 3}{24} - rp \]

Note that $h_{p,r} + p = h_{p,r-1}$.

**Proposition 2.2.**

(1) Let $(h, h_\alpha) \in \mathbb{C}^2$ such that $h_\alpha \neq c_{L,\alpha}$. Then there exist unique $p, r \in \mathbb{C}$, $p \neq 0$ such that $h = h_{p,r}$ and $h_\alpha = (1 + p)c_{L,\alpha}$.
For every \( r \in \mathbb{C} \), \( h_{0,r} = \frac{c_r - 3}{24} \).

In [3] we obtained the determinant formula

\[
\det[V(h, h_{\alpha})](\cdot, \cdot)_{n/2} = \text{Const} \prod_{k, l \in \mathbb{Z}_{\geq 0}} \varphi_{k,l}(c_L, c_{\alpha}, c_L, c_{\alpha}, h, h_{\alpha}) p_2(\frac{a_k + b_l}{2})
\]

where \( \varphi_{k,l}(c_L, c_{\alpha}, c_L, c_{\alpha}, h, h_{\alpha}) \) denotes

\[
\frac{c_{L,\alpha}^4}{4} \left( 1 + k - \frac{h_{\alpha}}{c_{L,\alpha}} \right) \left( -1 + k + \frac{h_{\alpha}}{c_{L,\alpha}} \right) \left( 1 + l - \frac{h_{\alpha}}{c_{L,\alpha}} \right) \left( -1 + l + \frac{h_{\alpha}}{c_{L,\alpha}} \right)
\]

and \( p_2(n) \) is Kostant partition function in \( \frac{1}{2}\mathbb{Z}_{\geq 0} \). As a direct application we get the following:

**Theorem 2.3.** [3]

1. The Verma module \( V[p, r] \) is irreducible if and only if \(|p| \notin \mathbb{Z}_{>0} \).
2. If \( p \in \mathbb{Z} \setminus \{0\} \) is even, \( V[p, r] \) contains a singular vector at conformal weight \( |p| \).
3. If \( p \in \mathbb{Z} \) is odd, \( V[p, r] \) contains a singular vector at conformal weight \( |p/2| \).

### 2.3. Contragredient modules

As in [5] we use the concept of contragredient modules.

Let \( V \) be a vertex operator superalgebra, \((M, Y_M)\) a graded \( V \)-module with gradation \( M = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_{\geq 0}} M(n) \) such that \( \dim M(n) < \infty \) and let \( \gamma \in \mathbb{C} \) such that \( L(0)|M(n) \equiv (\gamma + n) \text{Id} \). The contragredient module \( M^* \) is defined as follows. For every \( n \in \frac{1}{2}\mathbb{Z}_{\geq 0} \) let \( M(n)^* \) be the dual vector space and \( M^* = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_{\geq 0}} M(n)^* \). Consider the natural pairing \( \langle \cdot, \cdot \rangle : M^* \otimes M \to \mathbb{C} \).

Define the linear map \( Y_{M^*} : V \to \text{End} M^*[[z, z^{-1}]] \) such that

\[
\langle Y_{M^*}(v, z)w', w \rangle = \langle w', Y_M(e^{zL(1)}e^{\pi i L(0)}z^{-2L(0)}v, z^{-1})w \rangle
\]
for each $v \in V$, $w \in M$, $w' \in M^\ast$. Then $(M^\ast, Y_{M^\ast})$ carries the structure of a $V$–module. Direct calculations show that

\begin{align*}
(7) \quad \langle L(n)w', w \rangle &= \langle w', L(-n)w \rangle \\
(8) \quad \langle \alpha(n)w', w \rangle &= \langle w', (-\alpha(-n) + 2c_{L,\alpha} \delta_{n,0})w \rangle \\
(9) \quad \langle G(n + 1/2)w', w \rangle &= \langle w', -iG(-n - 1/2)w \rangle \\
(10) \quad \langle \Psi(n + 1/2)w', w \rangle &= \langle w', i\Psi(-n - 1/2)w \rangle
\end{align*}

If we take $M$ to be the simple, highest weight module $L(h, h_{\alpha})$, then one gets (cf. [12]) that $L(h, h_{\alpha})^\ast$ is again a simple module and the above calculations shows:

**Lemma 2.4.** We have

1. $L(h, h_{\alpha})^\ast \cong L(h, -h_{\alpha} + 2c_{L,\alpha})$, i.e. $L[p, r]^\ast \cong L[-p, -r]$.
2. $L(h, h_{\alpha})^\ast \cong L(h, h_{\alpha})$ if and only if $h_{\alpha} = c_{L,\alpha}$ i.e. $p = 0$.

From the previous Lemma we have

$\text{char}_q L(h, h_{\alpha})^\ast = \text{char}_q L(h, h_{\alpha})$ i.e. $\text{char}_q L[p, r] = \text{char}_q L[-p, -r]$.

### 3. Free field realisation

In this section we recall the realisation from [11]. Let $L = \mathbb{Z}c + \mathbb{Z}d$ be a lattice such that $\langle c, c \rangle = \langle d, d \rangle = 0$, $\langle c, d \rangle = 2$, let $V_L = \mathbb{C}[L] \otimes M(1)$ be the corresponding lattice vertex algebra, where $M(1)$ is Heisenberg vertex algebra generated by $c(z)$ and $d(z)$.

Consider the vertex subalgebra $\Pi(0) = \mathbb{C}[\mathbb{Z}c] \otimes M(1)$ of $V_L$. As in [11], let $\Pi(0)^{1/2}$ be its simple current extension:

$$\Pi(0)^{1/2} = \Pi(0) \oplus \Pi(0).e^{\hat{g}}.$$ 

Let $F^{(2)}$ be the fermionic vertex algebra generated by fields

$$\Psi^\pm(z) = \sum_{n \in \mathbb{Z}} \Psi^\pm \left( n + \frac{1}{2} \right) z^{-n-1}$$

such that for $i = 1, 2$, $r, s \in \frac{1}{2} + \mathbb{Z}$ we have the following anti-commutator relation

$$\{ \Psi^\pm(r), \Psi^\pm(s) \} = 0, \quad \{ \Psi^+(r), \Psi^-(s) \} = \delta_{r+s,0}.$$
The vertex algebra \( F^{(2)} \) has the following Virasoro vector of central charge \( c_{\text{fer}} = 1 \):

\[
\omega_{\text{fer}} = \frac{1}{2} \left( \Psi^+ \left( -\frac{3}{2} \right) \Psi^- \left( -\frac{1}{2} \right) + \Psi^- \left( -\frac{3}{2} \right) \Psi^+ \left( -\frac{1}{2} \right) \right) \mathbf{1}.
\]

Define the following four vectors in the vertex algebra \( M(1) \otimes F^{(2)} \):

- \( \alpha = -c_{L,\alpha} c(-1) \)
- \( \tau = \sqrt{2} \left( \frac{1}{2} c(-1) \Psi^+ \left( -\frac{1}{2} \right) + \frac{1}{2} d(-1) \Psi^- \left( -\frac{1}{2} \right) + \frac{c_{L} - 3}{12} \Psi^- \left( -\frac{3}{2} \right) - \Psi^+ \left( -\frac{3}{2} \right) \right) \)
- \( \omega = \frac{1}{2} c(-1) d(-1) + \frac{c_{L} - 3}{24} c(-2) - \frac{1}{2} d(-2) + \omega_{\text{fer}} \)
- \( \Psi = -\sqrt{2} c_{L,\alpha} \Psi^- \left( -\frac{1}{2} \right) \).

**Theorem 3.1.** The universal vertex algebra \( V^{SH}(c_{L}, c_{L,\alpha}) \) is simple, and it is isomorphic to the vertex subalgebra of \( M(1) \otimes F^{(2)} \) generated by \( \alpha, \Psi, \tau, \omega \).

**Proof.** We have shown in [3] that the vertex algebra \( W \) generated by \( \alpha, \Psi, \tau, \omega \) is isomorphic to some quotient of \( V^{SH}(c_{L}, c_{L,\alpha}) \). Note that \( V^{SH}(c_{L}, c_{L,\alpha}) \) is isomorphic to a certain quotient of the Verma module \( V[-1,0] \). But we will prove in Theorem [6,3] that the character of \( V^{SH}(c_{L}, c_{L,\alpha}) \), given by (5), coincides with the character of \( L[-1,0] \). This proves that \( V^{SH}(c_{L}, c_{L,\alpha}) \) is simple. In particular, \( W = V^{SH}(c_{L}, c_{L,\alpha}). \) \( \square \)

Now for each \( h \in \mathbb{C} \otimes \mathbb{Z} L \), let \( e^{h} \) denote a \( M(1) \) highest weight vector in the Fock module \( M(1, h) \), which is also a \( V^{SH}(c_{L}, c_{L,\alpha}) \)-module. We introduce a parametrisation

\[
(11) \quad v_{p,r} := e^{-\frac{\mathbf{1}}{2} \bar{d} + rc}, \quad \text{where} \quad \bar{d} = d - \frac{c_{L} - 3}{12} c.
\]

Then \( v_{p,r} \) is the highest weight vector of highest weight given by (6). Also, define

\[
\mathcal{F}_{p,r} = (M(1) \otimes F^{(2)}) v_{p,r}.
\]

Now we shall identify the contragredient module \( \mathcal{F}^{*}_{p,r} \). Using [7,10] and

\[
\langle c(n) w', w \rangle = \langle w', c(-n) + 2 \delta_{n,0} w \rangle,
\]

\[
\langle d(n) w', w \rangle = \langle w', d(-n) - \delta_{n,0} \frac{c_{L} - 3}{6} w \rangle,
\]

\[
\langle \Psi^\pm(n + 1/2) w', w \rangle = \langle w', i \Psi^\pm(-n - 1/2) w \rangle.
\]
we get
\[ F_{p,r}^* \cong F_{-p,-r}. \]

3.1. **Realisation of Verma modules** \( V[p, r], p \in \mathbb{C} \setminus \mathbb{Z}_{<0}. \) In this section, we shall prove that Verma modules \( V[p, r], p \notin \mathbb{Z}_{<0} \) can be obtained by using free-field realisation.

Let \( \mathcal{W} = V^{SH}(c_L, c_{L,0}).v_{p,r} \). Take the following basis of \( F_{p,r} \) consisting of monomials:
\[ w_{\lambda^+, \lambda^-, \mu^+, \mu^-} = (\Psi_{-\lambda^+}^+ \Psi_{-\lambda^-}^- d_{-\mu^+} c_{-\mu^-})v_{p,r} \]
where \( \lambda^\pm \in SP, \) and \( \mu^\pm \in P. \) Now, since \([c(n), c(m)] = [d(n), d(m)] = 0, \) \( \{\Psi^+(r), \Psi^-(s)\} = 0, \) and \( \{\Psi^+(r), \Psi^-(s)\} = \delta_{r+s,0} \) and from the definition of \( v_{p,r}, \) one can define the following partial ordering on basis vectors:
\[ w_{\lambda^+, \lambda^-, \mu^+, \mu^-} < w_{\tilde{\lambda}^+, \tilde{\lambda}^-, \tilde{\mu}^+, \tilde{\mu}^-} \text{ if } (\lambda^+, \mu^+) < (\tilde{\lambda}^+, \tilde{\mu}^+). \]

Let us check whether the arbitrary basis vector (12) belongs to \( \mathcal{W}. \)

Let \( \deg_{\lambda^+, \mu^+} = 0 \) (then automatically \( \ell_{\lambda^+, \mu^+} = 0). \) Since
\[ \alpha_{-\mu} = (-c_{L,0})^{\ell_{\mu}} c_{-\mu}, \quad \Psi_{-\lambda} = (-\sqrt{2}c_{L,0})^{\ell_{\lambda}} \Psi_{-\lambda}, \]
we have
\[ (\Psi_{-\lambda^-} c_{-\mu^-})v_{p,r} \in \mathcal{W}. \]

Let \( S \) be the set of all basis vectors which don’t belong to \( \mathcal{W}. \) Assume that \( S \neq \emptyset. \) By the Zorn’s lemma there must exist a minimal element \( w := w_{\lambda^+, \lambda^-, \mu^+, \mu^-} \) of \( S \) with respect to the ordering ”<”. This means that for every \( w' < w, w' \in \mathcal{W}. \)

Assume first that \( \ell_{\mu^+} = l > 0. \) Let \( \mu^+ = (\mu_1, \mu_2, \ldots, \mu_l) \) and \( \tilde{\mu}^+ := (\mu_2, \ldots, \mu_l). \) Define
\[ w' = (\Psi_{-\lambda^+}^+ \Psi_{-\lambda^-}^- d_{-\tilde{\mu}^+} c_{-\mu^-})v_{p,r} \]
Then by the assumption \( w' \in \mathcal{W}. \) We have
\[ L(-\mu_1)w' = -\frac{p + \mu_1}{2}w + \ldots \]
where \( \ldots \) denotes a sum of monomials \( w_i \) such that \( w_i < w. \) Using again the assumption, we have that all \( w_i \) and \( L(-\mu_1)w' \) belong to \( \mathcal{W}, \) so if \( p + \mu_1 \neq 0 \) we have \( w \in \mathcal{W}. \)
Assume next that $\ell_{\mu^+} = 0$. Let $\lambda^+ = (\lambda_1, \lambda_2, \ldots, \lambda_g)$ and $\bar{\lambda}^+ = (\lambda_2, \ldots, \lambda_g)$.

Define

$$w' = (\Psi_+^{\bar{\lambda}^+} \Psi_-^{\lambda^+} c_{-\mu^-}) v_{p,r}$$

Then we get

$$G(-\lambda_1)w' = -\frac{\sqrt{2}}{2}(2\lambda_1 + p)w + \cdots$$

where $\cdots$ denotes a sum of monomials $w_i$ such that $w_i < w$. As before, we conclude that if $2\lambda_1 + p \neq 0$ then $w \in \mathcal{W}$.

**Proposition 3.2.** Assume that $p \in \mathbb{C} \setminus \mathbb{Z}_{<0}$. Then we have:

$$\mathcal{F}_{p,r} \cong V[p,r].$$

**Proof.** We have shown above that all basis vectors (12) belong to $\mathcal{W}$ if $p \notin \mathbb{Z}_{<0}$. Since the $q$-character of Verma module $V[p,r]$ coincides with $q$-character of the Fock space, we conclude $\mathcal{W} = V[p,r] = \mathcal{F}_{p,r}$. □

### 3.2. Realisation of $L[p,r]$, $p \in \mathbb{Z}_{<0}$ as submodules of $\mathcal{F}_{p,r}$

Let $\mathcal{C}_{p,r}$ be the subspace of $\mathcal{F}_{p,r}$ spanned by monomials $\Psi_\lambda^{\alpha_{-\mu}} v_{p,r}$, $\lambda \in SP$, $\mu \in \mathcal{P}$. We first show that there can be no singular vectors in $\mathcal{C}_{p,r}$, and then generalise the claim to $\mathcal{F}_{p,r}$.

Since we are dealing with (anti)commuting factors, we will consider $(\mu, \lambda) \in \mathcal{P} \times SP$ as an element of a partially ordered set of partitions in $\frac{1}{2}\mathbb{Z}_{>0}$. Let $\mathcal{P}(\frac{1}{2})$ denote the set of all superpartitions $(\lambda_1, \ldots, \lambda_n)$ in $\frac{1}{2}\mathbb{Z}_{>0}$ such that $\lambda_i \neq \lambda_{i+1}$ if $\lambda_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$.

We introduce a partial ordering $<$ on $\mathcal{P}(\frac{1}{2})$ by

$$\lambda < \mu \quad \text{if} \quad \text{deg}_{\lambda} < \text{deg}_{\mu},$$

$$\lambda \prec \mu \quad \text{or} \quad \text{deg}_{\lambda} = \text{deg}_{\mu}, \lambda \prec \mu.$$  

For each $\lambda \in \mathcal{P}(\frac{1}{2})$, we define monomials

$$X^\alpha,\Psi_{-\lambda} = X^\alpha,\Psi(-\lambda_1) \cdots X^\alpha,\Psi(-\lambda_{\ell(\lambda)}), \quad Y^L,G_{\lambda} = Y^L,G(\lambda_{\ell(\lambda)}) \cdots Y^L,G(\lambda_1)$$

where

$$X^\alpha,\Psi(-\lambda_i) = \begin{cases} \alpha(-\lambda_i) & \lambda_i \in \mathbb{Z}_{>0}, \\ \Psi(-\lambda_i) & \lambda_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0} \end{cases} \quad Y^L,G(\lambda_i) = \begin{cases} L(\lambda_i) & \lambda_i \in \mathbb{Z}_{>0} \\ G(\lambda_i) & \lambda_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}. \end{cases}$$
A basis of $\mathcal{C}_{p,r}$ is then given by the set of monomials

$$w_\lambda = X_{-\lambda}^\alpha \Psi v_{p,r}, \quad \lambda \in \mathcal{P}(\frac{1}{2}).$$

**Lemma 3.3.** Let $p \in \mathbb{Z}_{<0}$. Let $\lambda, \mu \in \mathcal{P}(\frac{1}{2}\mathbb{Z}_{>0})$ such that $\mu < \lambda$. Then $Y^L_G X^\alpha_{-\mu} v_{p,r} = 0$ and $Y^L_G X^\alpha_{-\lambda} v_{p,r} = \nu v_{p,r}$ for some $\nu \neq 0$.

**Proof.** Both claims follow from the definition of ordering (16). If $\deg \mu < \deg \lambda$ we obviously have $Y^L_G X^\alpha_{-\mu} v_{p,r} = 0$. By the brackets in Definition 2.1 we have for $\lambda_k \geq \mu$

$$Y^L_G (\lambda_k) X^\alpha_{-\mu} v_{p,r} = \nu_k \partial_{\lambda_k} X^\alpha_{-\mu} v_{p,r}$$

where

$$\nu_k = \begin{cases} 
\mu_1 (p - \mu_1), & \mu_1 \in \mathbb{Z}_{>0}, \\
(p - \mu_1 - 1), & \mu_1 \in \frac{1}{2} + \mathbb{Z}_{>0}.
\end{cases}$$

From this follows that $Y^L_G X^\alpha_{-\mu} v_{p,r} = 0$, and $Y^L_G X^\alpha_{-\lambda} v_{p,r} = \nu v_{p,r}$ where

$$\nu = \prod (p - \mu_i - 1) \prod \lambda_i (p - \lambda_i).$$

$\nu \neq 0$ since $p \in \mathbb{Z}_{<0}$. \qed

**Lemma 3.4.** If $p \in \mathbb{Z}_{<0}$ the subspace $\mathcal{C}_{p,r}$ of $\mathcal{F}_{p,r}$ (resp. of $V[p,r]$) contains no singular vectors.

**Proof.** Assume that

$$u = \sum_{\lambda \in \mathcal{P}(\frac{1}{2})} k_\lambda X^\alpha_{-\lambda} v_{p,r},$$

is a singular vector of conformal weight $h$ (so $\deg \lambda = h$ for each $\lambda$). Let

$$\bar{\lambda} = \min \{ \lambda : k_\lambda \neq 0 \}$$

with respect to linear ordering (16) on the set of partitions in $\frac{1}{2}\mathbb{Z}$ of total degree $h$. Consider $Y^L_G u$. By Lemma 3.3 we have

$$Y^L_G X^\alpha_{-\lambda} v_{p,r} = \nu v_{p,r},$$

$$Y^L_G X^\alpha_{-\lambda} v_{p,r} = 0, \quad \text{for } \lambda > \bar{\lambda},$$

for some $\nu \neq 0$. Therefore we have $Y^L_G u = \nu v_{p,r} \neq 0$. This is a contradiction with the assumption that $u$ is a singular vector. \qed

**Proposition 3.5.** For each non-zero vector $x \in \mathcal{F}_{p,r}$, the cyclic $V^{SH}(c_L, c_{L,\alpha})$-submodule $\langle x \rangle$ contains $v_{p,r}$. In particular, $L[p,r] = \langle v_{p,r} \rangle$ if $p \in \mathbb{Z}_{<0}$. 
Proof. Let $U = \langle x \rangle$. Taking the actions of 

$$
\alpha(n) = -2nc_{L,\alpha}d(-n), \quad \Psi(n - 1/2) = -\sqrt{2}c_{L,\alpha}d\Psi + (-n-1/2)
$$

for $n \in \mathbb{Z}_{>0}$, we see that $U$ must contain some element $w \in \mathcal{C}_{p,r}$. Using Lemma 3.4 we get that $v_{p,r} \in \langle w \rangle$. Applying this result we have that each vector in $\langle v_{p,r} \rangle$ is cyclic. Hence $L[p,r] = \langle v_{p,r} \rangle$.

4. Screening operators and singular vectors, case $p > 0$

4.1. Screening operators. Let $a = \Psi^{-}(-1/2)e^{1/2c}$. We have

$$
\tau_0a = \sqrt{2}De^{1/2c}, \\
\tau_1a = \sqrt{2}e^{1/2c} \\
\tau_n a = 0 \quad (n \geq 2) \\
L(0)a = a \\
L(m)a = 0 \quad (m \geq 1)
$$

By using the commutator formula, we get that $Q = a_0 = \text{Res}_z Y(a,z)$ either commutes or anti-commutes with the generators of $V^{SH}(c_L,c_{L,\alpha})$. Hence $Q$ is a screening operator.

We also have

$$
a_1 \tau = [\tau_{-1}, a_1] 1 = (\tau_0 a)_0 1 - (\tau_1 a)_{-1} 1 = -\sqrt{2}e^{1/2c}.
$$

$$
a_n \tau = 0 \quad \forall n \geq 2.
$$

Next we notice that

$$
\{a_n, a_m\} = 0, \quad \forall n, m \in \mathbb{Z},
$$

hence $a$ is an odd vector with anti-commuting components. In [2], we studied a construction of a derivation based on the element $a$.

Define

$$
S = \sum_{i=1}^{\infty} \frac{1}{i}a_{-i}a_i = \text{Res}_{z_1}\text{Res}_{z_2}\log\left(1 - \frac{z_2}{z_1}\right)Y(a,z_1)Y(a,z_2).
$$

$$
S^{tw} = \sum_{i=0}^{\infty} \frac{1}{i + 1/2}a_{-i-1/2}a_{i+1/2}.
$$

We have:

- $S$ is an even derivation on $\nabla = \text{Ker}_{\Pi(0)^{1/2} \otimes F(2)} Q$. 

On every $\Pi(0)^{1/2} \otimes F^{(2)}$–module $(M, Y_M)$ and $v \in \nabla$, we have

$$[S, Y_M(v, z)] = Y_M(Sv, z).$$

On every $\sigma$–twisted $\Pi(0)^{1/2} \otimes F^{(2)}$–module $(M^{tw}, Y_{M^{tw}})$ and $v \in \nabla$, we have

$$[S^{tw}, Y_{M^{tw}}(v, z)] = Y_{M^{tw}}(Sv, z).$$

In particular, we have

$$[S, L(n)] = [S, \alpha(n)] = [S, \Psi(n + 1/2)] = 0 \quad \forall n \in \mathbb{Z}.$$

$$[S, G(n - 1/2)] = [S, \tau_n] = (S\tau)_n = -\sqrt{2}(a_{-1}e^{1/2})_n = -\sqrt{2}(-1/2)e^c)_n.$$

Note also that $e_0^c$ is an (even) derivation. We have:

$$[e_0^c, L(n)] = [e_0^c, \alpha(n)] = [e_0^c, \Psi(n + 1/2)] = 0.$$

$$[e_0^c, G(n - 1/2)] = (e_0^c\tau)_n = -\sqrt{2}(-1/2)e^c)_n.$$

From these considerations we conclude:

**Theorem 4.1.** Let

$$G = e_0^c - S, \ G^{tw} = e_0^c - S^{tw}.$$

• Assume that $(M, Y_M)$ is a $\Pi(0)^{1/2} \otimes F^{(2)}$–module. Then on the submodule $\overline{M} = \text{Ker}_M Q$ we have:

$$[G, Y_M(v, z)] = 0, \quad \forall v \in V^{SH}(c_L, c_L, \alpha).$$

• Assume that $(M^{tw}, Y_{M^{tw}})$ is a $\sigma$–twisted $\Pi(0)^{1/2} \otimes F^{(2)}$–module. Then

$$[G^{tw}, Y_{M^{tw}}(v, z)] = 0, \quad \forall v \in V^{SH}(c_L, c_L, \alpha).$$

Therefore $G$ and $G^{tw}$ are screening operators.

Define the Schur polynomial $S_r(\alpha) := S_r(\alpha(-1), \alpha(-2), \cdots)$ using the generating function

$$\exp \left( \sum_{n=1}^{\infty} \frac{\alpha(-n)z^n}{n} \right) = \sum_{r=0}^{\infty} S_r(\alpha)z^r.$$

In particular we have

$$S_0(\alpha) = 1, \quad S_1(\alpha) = \alpha(-1), \quad S_2(\alpha) = \frac{1}{2} (\alpha(-1)^2 + \alpha(-2))$$
4.2. (Sub)singular vectors for $p \in \mathbb{Z}_{>0}$, $p$ odd.

**Theorem 4.2.** Assume that $p \in \mathbb{Z}_{>0}$ is odd. Then

\[(19) \quad u_{p,r} = \sum_{i=0}^{p-1} \Psi(-i - \frac{1}{2}) S_{\frac{p-1}{2} - i} \left( - \frac{\alpha}{\sqrt{2}c_{L,\alpha}} \right) v_{p,r} \]

is a singular vector (of weight $h_{p,r} + p/2 = h_{p,r-1/2}$) in $V[p,r] = F_{p,r}$.

**Proof.** We know from Proposition \[3.2\] that $V^{SH}(c_{L},c_{L,\alpha}).v_{p,r} \cong V[p,r]$. By applying the screening operator $Q$ on $v_{p,r-\frac{1}{2}}$ we get

\[Q v_{p,r-\frac{1}{2}} = a_0 v_{p,r-\frac{1}{2}}.\]

Since $v_{r-\frac{1}{2}}$ is a highest weight vector, and $Q$ (anti)commutes with the action of $V^{SH}(c_{L},c_{L,\alpha})$, it is clear that $a_0 v_{p,r-\frac{1}{2}}$ is a singular vector in $V[p,r]$. The formula follows from \[18\]. \[\square\]

**Remark 1.** One can show using results from \[4\] or \[11\] that $\ker_{F-1,0} Q$ is isomorphic to the simple affine vertex superalgebra $L_1(\mathfrak{gl}(1|1))$ associated to the Lie superalgebra $\mathfrak{gl}(1|1)$. We won’t use this identification in the current paper.

Now we show that there exist subsingular vectors in $V[p,r]$ for $p > 0$ odd.

\[\mathcal{M}_{p,r} := (\Pi(0)^{1/2} \otimes F^{(2)}).v_{p,r}\]

is an (untwisted) $\Pi(0)^{1/2} \otimes F^{(2)}$–module. Let

\[\overline{\mathcal{M}_{p,r}} = \ker_{\mathcal{M}_{p,r}} Q.\]

Then $\mathcal{G}$ is a screening operator on $\overline{\mathcal{M}_{p,r}}$. 
Proposition 4.3. Let $p \in \mathbb{Z}_{>0}$ be odd. Then
\begin{equation}
(20) \quad w_{p,r} = \left( S_p \left( -\frac{\alpha}{c_{L,\alpha}} \right) + \sum_{k=1}^{p-1} \frac{1}{k} \left( \sum_{i \geq 0} \Psi(i + k - \frac{p}{2}) S_i \left( -\frac{\alpha}{c_{L,\alpha}} \right) \right) \left( \sum_{j \geq 0} \Psi(j - k - \frac{p}{2}) S_j \left( -\frac{\alpha}{c_{L,\alpha}} \right) \right) \right) v_{p,r}
\end{equation}
is a non-trivial subsingular vector (of weight $h_{p,r} + p = h_{p,r-1}$) in $V[p,r] = \mathcal{F}_{p,r}$.

Proof. Note the following facts:
\begin{itemize}
    \item $[Q, G] = 0$, $Q^2 = 0$.
    \item $v_{p,r-1}$ is a singular vector for every $r \in \mathbb{C}$.
    \item $Qv_{p,r-1}$ is a singular vector in $\overline{M}_{p,r}$ for every $r \in \mathbb{C}$.
    \item $GQv_{p,r-1}$ is a singular vector for every $r \in \mathbb{C}$. It is non-trivial since it has the form
    \[ (\Psi^-(1/2)e^{c/2})_0e_0v_{p,r-1} + \cdots \]
    where $\cdots$ denotes the sum of monomials containing the product of three fermionic generators $\Psi^-(j_1)\Psi^-(j_2)\Psi^-(j_3)$, and
    \[ (\Psi^-(1/2)e^{c/2})_0e_0v_{p,r-1} \neq 0. \]

Let $w_{p,r} = Gv_{p,r-1}$. The arguments above show that $Qw_{p,r} \neq 0$, but for $n \geq 1$ it holds that
\[ L(n)w_{p,r}, \alpha(n)w_{p,r}, \Psi \left( n - \frac{1}{2} \right) w_{p,r}, G \left( n - \frac{1}{2} \right) w_{p,r} \in \overline{M}_{p,r}. \]
Hence $w_{p,r}$ is a singular vector in $\overline{M}_{p,r}$, and therefore subsingular in $\overline{M}_{p,r}$. The proof follows. \qed

Theorem 4.4. Assume that $p > 0$ odd. Then we have the following family of (sub)singular vectors in $V[p,r] = \mathcal{F}_{p,r}$:
\begin{itemize}
    \item Singular vector $\upsilon_{p,r}^{(n)} = G^nQv_{p,r-n-1/2}$, $n \in \mathbb{Z}_{\geq 0}$.
    \item Subsingular vectors $\omega_{p,r}^{(n)} = G^n v_{p,r-n}$, $n \in \mathbb{Z}_{>0}$.
\end{itemize}
Note that
\[ G\left(\frac{\pi}{2}\right)w_{p,r}^{(1)} = [G\left(\frac{\pi}{2}\right), G]v_{p,r-1} = \left( [G\left(\frac{\pi}{2}\right), e_0^s] - [G\left(\frac{\pi}{2}\right), S] \right) v_{p,r-1} = \sum_{i=0}^{\pi-1} \frac{\Psi(-i - \frac{1}{2})}{c_{L,\alpha}} S_{p-\frac{1}{2}+i} v_{p,r-1} = u_{p,r}^{(0)}. \] (21)

From (20) and (21) it follows that all (sub)singular vectors in Theorem 4.4 belong to a submodule \(\langle w_{p,r}^{(1)} \rangle\).

4.3. **Singular vector for \( p \in \mathbb{Z}_{>0}, p \text{ even} \).** Let \( p \in 2\mathbb{Z}_{>0} \). As in [1], we can construct twisted \( \Pi(0)^{1/2} \)–modules.

Note that \( \sigma = e^{\pi id(0)} \) is an automorphism of the vertex operator algebra \( \Pi(0)^{1/2} \) of order two, and \( \Pi(0)^{1/2}e^{-\frac{p}{2}+1} + r, r \in \mathbb{C} \), is a \( \sigma \)–twisted \( \Pi(0)^{1/2} \)–module.

**Theorem 4.5.** Assume that \( p \in \mathbb{Z}_{>0} \) is even. Then

\[ u_{p,r} = \left( S_p \left( -\frac{\alpha}{c_{L,\alpha}} \right) + \sum_{k=0}^{\pi-1} \frac{1}{k + \frac{1}{2}} \left( \sum_{i \geq 0} \Psi(i + k - \frac{p-1}{2}) S_i \left( -\frac{\alpha}{2c_{L,\alpha}} \right) \right) \left( \sum_{j \geq 0} \Psi(j - k - \frac{p+1}{2}) S_j \left( -\frac{\alpha}{2c_{L,\alpha}} \right) \right) \right) v_{p,r} \]

is a singular vector (of weight \( h_{p,r} + p = h_{p,r-1} \)) in \( V[p, r] = F_{p,r} \).

**Proof.** Since \( (\Pi(0)^{1/2} \otimes F(2)), v_{p,r-1} = \sigma = e^{\pi id(0)} \)–twisted \( \Pi(0)^{1/2} \otimes F(2) \)–module, operator \( G^{tw} \) gives a screening operator which commutes with the action of the vertex algebra \( V^{SH}(c_L, c_{L,\alpha}) \). Therefore \( G^{tw} v_{p,r-1} \) is a singular vector. Direct calculation shows that

\[ G^{tw} v_{p,r-1} = e_0^s v_{p,r-1} - S^{tw} v_{p,r-1} = S_p(c) v_{p,r} - \sum_{j=0}^{\infty} \frac{1}{j + \frac{1}{2}} a_{-j} \frac{1}{2} a_{j+\frac{1}{2}} v_{p,r-1}. \]

The proof follows. \( \square \)

**Theorem 4.6.** Assume that \( p \in \mathbb{Z}_{>0} \) is even. Then we have the following family of singular vectors in \( V[p, r] = F_{p,r} \):

\[ u_{p,r}^{(n)} := (G^{tw})^n v_{p,r-n}, \quad n \in \mathbb{Z}_{>0}. \]

Note next that the screening operator \( G^{tw} \) is injective, and therefore it gives an inclusion of the Verma module \( V[p, r-1] = F_{p,r-1} \) into \( V[p, r] \). We have the following conclusion:
Corollary 4.7. Assume that $p \in \mathbb{Z}_{>0}$ is even. The Verma module $V[p,r]$ contains a non-trivial submodule isomorphic to the Verma module $V[p,r-1]$.

Free field realisation in case $p \in \mathbb{Z}_{>0}$ even is shown in Figure 3.

5. Relations in $V[p,r]$, $p < 0$

We first present an explicit derivation of a singular vector in $V[p,r]$, $p < 0$. Our analysis is analogous to that of [6, Section 4].

By direct calculations in $\Pi(0)\otimes F(2)$ we find

$$\frac{1}{\sqrt{2}}\Psi^{-}\left(-\frac{1}{2}\right)G(-\frac{3}{2})e^{-c} = \left(\frac{1}{2}c(-1)\Psi^{-}\left(-\frac{1}{2}\right)\Psi^{+}\left(-\frac{1}{2}\right) + \cos \right)$$

$$- \frac{cL - 15}{12} \Psi^{-}\left(-\frac{3}{2}\right)\Psi^{-}\left(-\frac{1}{2}\right) + \Psi^{+}\left(-\frac{3}{2}\right)\Psi^{-}\left(-\frac{1}{2}\right) \right) e^{-c}$$

$$L(-2)e^{-c} = \left(\frac{1}{2}c(-1)d(-1) - \frac{1}{2}d(-2) + \frac{cL - 27}{24}c(-2) + \frac{1}{2}\Psi^{+}\left(-\frac{3}{2}\right)\Psi^{-}\left(-\frac{1}{2}\right) + \frac{1}{2}\Psi^{+}\left(-\frac{1}{2}\right) \right) e^{-c}$$

so we have

$$\left( L(-2) - \frac{cL - 27}{24}c(-2) - \frac{1}{\sqrt{2}}\Psi^{-}\left(-\frac{1}{2}\right)G(-\frac{3}{2}) - \frac{cL - 15}{12} \Psi^{-}\left(-\frac{3}{2}\right)\Psi^{-}\left(-\frac{1}{2}\right) \right) e^{-c} =$$

$$= -\frac{1}{2}L(-1)(d(-1) - \Psi^{-}\left(-\frac{1}{2}\right)\Psi^{+}\left(-\frac{1}{2}\right)) e^{-c}$$

Define

$$R := \left( L(-2) - \frac{cL - 27}{24}c(-2) - \frac{1}{\sqrt{2}}\Psi^{-}\left(-\frac{1}{2}\right)G(-\frac{3}{2}) + \frac{cL - 15}{12} \Psi^{-}\left(-\frac{3}{2}\right)\Psi^{-}\left(-\frac{1}{2}\right) \right) e^{-c}.$$ 

Since $(L(-1)a)_0 = 0$ in every VOA, we conclude that $R_0v_{p,r} = 0$. 
Proposition 5.1. Let $p \in \mathbb{Z}_{<0}$, $r \in \mathbb{C}$. Then $u_{p,r} = \Phi(p,r)v_{p,r}$, where

$$
\Phi(p,r) = \sum_{i=1}^{-p} \left( L(-i) + \frac{c_L - 27}{24c_L,\alpha}(-i) \right) S_{-p-i} \left( \frac{\alpha}{c_L,\alpha} \right) +
+ S_{-p} \left( \frac{\alpha}{c_L,\alpha} \right) \left( L(0) + \frac{c_L - 3}{24c_L,\alpha}(0) \right) +
+ \frac{1}{2c_L,\alpha} \sum_{i=0}^{-p-1} \sum_{k=0}^{-p-i-1} \Psi(-i - \frac{1}{2}) G \left( -k - \frac{1}{2} \right) S_{-p-i-k-1} \left( \frac{\alpha}{c_L,\alpha} \right) +
- \frac{c_L - 15}{24c_L^2,\alpha} \sum_{i=0}^{-p-1} \sum_{k=0}^{-p-i-1} i\Psi(-i - \frac{1}{2}) \Psi \left( -k - \frac{1}{2} \right) S_{-p-i-k-1} \left( \frac{\alpha}{c_L,\alpha} \right).
$$

is a non-trivial singular vector (of weight $h_{p,r} - p = h_{p,r+1}$) in $V[p,r]$.

Unfortunately, this method cannot be used for a construction of singular vectors when $p < 0$ is odd.

6. The $q$-character of $L[p,r]$ 

Lemma 6.1. We have:

$$
\text{char}_q L[p,r] \geq q^{h_{p,r}} \left( 1 - q |p| \right) \prod_{k=1}^{\infty} \frac{(1 + q^{k-1/2})^2}{(1 - q^k)^2}, \quad \text{if } p \text{ is odd};
$$

$$
\text{char}_q L[p,r] \geq q^{h_{p,r}} \left( 1 - q |p| \right) \prod_{k=1}^{\infty} \frac{(1 + q^{k-1/2})^2}{(1 - q^k)^2}, \quad \text{if } p \text{ is even}.
$$

Proof. It suffices to prove the statements for $p < 0$. The proof for $p > 0$ then follows by using contragredient modules.

We have

$$(G_{-\lambda^+, \Psi_{-\lambda^-} - \mu^-}, \alpha)_{-\mu^-} v_{p,r} = \nu(\Psi_{-\lambda^+, \Psi_{-\lambda^-} - \mu^+, c^-}) v_{p,r} + \cdots$$

where

$$
\nu = (-c_L,\alpha)^{-\mu^-} \left( -\sqrt{2c_L,\alpha} \right)^{\ell_{\lambda^-}} \left( \prod_{i=1}^{\ell_{\mu^+}} \frac{\mu_i^+ + p}{-2} \right) \left( \prod_{k=1}^{\ell_{\lambda^+}} \frac{2\lambda_i^++p}{-\sqrt{2}} \right)
$$

and $\cdots$ denotes a sum of basis vectors $w_i$ of the type (12) such that $w_i < (\Psi_{-\lambda^+, \Psi_{-\lambda^-} - \mu^+, c^-})$ where we consider the ordering (3) with respect to pair $(\mu^+, \lambda^+)$. Therefore, the set of vectors

$$
(23) \quad (G_{-\lambda^+, \Psi_{-\lambda^-} - \mu^-}, \alpha)_{-\mu^-} v_{p,r}, \quad \mu_i^+ \neq -p, \quad \lambda_i^+ \neq -p/2
$$
is linearly independent in $L[p, r] \subset \mathcal{F}_{p, r}$ if $p$ is odd. Likewise vectors
\begin{equation}
(G_{-\lambda} \Psi_{-\lambda} - L_{-\mu} - \alpha_{-\mu}) v_{p, r}, \quad \mu_i^+ \neq -p
\end{equation}
are linearly independent if $p$ is even.

The $q$–character of the subspace of $L[p, r]$ spanned by vectors (23) is
\begin{equation}
q^{h_{p, r}} (1 - q^{\frac{|p|}{2}}) \prod_{k=1}^{\infty} \frac{(1 + q^{k-1/2})^2}{(1 - q^k)^2}
\end{equation}
while the $q$–character of the space spanned by (24) is
\begin{equation}
q^{h_{p, r}} (1 - q^{|p|}) \prod_{k=1}^{\infty} \frac{(1 + q^{k-1/2})^2}{(1 - q^k)^2}
\end{equation}
which proves both inequalities.

\begin{lemma}
We have:
\begin{align*}
\text{char}_q L[p, r] &\leq q^{h_{p, r}} (1 - q^{\frac{|p|}{2}}) \prod_{k=1}^{\infty} \frac{(1 + q^{k-1/2})^2}{(1 - q^k)^2}, \quad \text{if } p \text{ is odd;}
\text{char}_q L[p, r] &\leq q^{h_{p, r}} (1 - q^{|p|}) \prod_{k=1}^{\infty} \frac{(1 + q^{k-1/2})^2}{(1 - q^k)^2}, \quad \text{if } p \text{ is even.}
\end{align*}
\end{lemma}

\begin{proof}
We prove the claim for $p > 0$. The case $p < 0$ follows again by using contragredient modules.

Assume that $p > 0$ is odd. We have constructed families of singular and subsingular vectors in Theorem 4.4 of types
\begin{align*}
u^{(n)}_{p, r} &= \Psi(-\frac{p}{2})(\alpha(-p))^n v_{p, r} + \cdots, \\
u^{(n)}_{p, r} &= (\alpha(-p))^n v_{p, r} + \cdots.
\end{align*}
Now we can eliminate all basis vectors (12) containing either $\Psi(-\frac{p}{2})$ or $\alpha(-p)$ as a factor from the spanning set of $L[p, r]$. Therefore, the set of vectors
\begin{align*}
(\Psi^+_{-\lambda} + \Psi^-_{-\lambda} - d_{-\mu} + c_{-\mu}) v_{p, r}, \quad \lambda_i^+ \neq \frac{p}{2}, \quad \mu_i^+ \neq p
\end{align*}
spans $L[p, r]$ which gives the claimed inequalities.

If $p > 0$ is even, we have a family of singular vectors from Theorem 4.4 of type
\begin{equation}
u^{(n)}_{p, r} = (\alpha(-p))^n v_{p, r} + \cdots
\end{equation}
so we can eliminate vectors (12) containing $\alpha(-p)$ as a factor. Again, we obtain the wanted character inequality.
\end{proof}
Using these two lemmas, we get:

**Theorem 6.3.** Assume that \( p \in \mathbb{Z} \setminus \{0\} \) and \( r \in \mathbb{C} \). Then we have:

\[
\text{char}_q L[p, r] = q^{h_{p,r}} (1 - q^{p/2}) \prod_{k=1}^{\infty} \frac{(1 + q^{k-1/2})^2}{(1 - q^k)^2}, \text{ if } p \text{ is odd};
\]

\[
\text{char}_q L[p, r] = q^{h_{p,r}} (1 - q^{|p|}) \prod_{k=1}^{\infty} \frac{(1 + q^{k-1/2})^2}{(1 - q^k)^2}, \text{ if } p \text{ is even}.
\]

7. **The structure of \( V[p, r] \)**

From Theorem 6.3 it follows that the \( q \)-character of the maximal submodule in \( V[p, r] \) equals the \( q \)-character of the Verma module \( V[p, r \pm 1] \) if \( p \) is even, and \( V[p, r \pm \frac{1}{2}] \) if \( p \) is odd (where the sign is opposite the sign of \( p \)). In three out of these four cases, this submodule is actually isomorphic to the Verma module.

**Theorem 7.1.** Assume that \( p \in \mathbb{Z} \setminus \{0\} \) is even. The maximal submodule of \( V(h_{p,r}, (1 + p)c_{L,\alpha}) \) is isomorphic to \( V(h_{p,r} + |p|, (1 + p)c_{L,\alpha}) \).

**Proof.** From Theorem 6.3 the maximality of the submodule \( \langle u_{p,r} \rangle \) in Corollary 4.7 follows, which proves the claim for \( p > 0 \).

Assume now that \( p < 0 \). Let us fix the PBW basis

\[
(\Psi_{-\lambda} - G_{-\lambda} + \alpha_{\mu} - L_{-\mu^+}) v_{p,r}
\]

of Verma modules \( V[p, r] \) and ordering (3) with respect to \((\mu^+, \lambda^+)\). In Proposition 5.1 we constructed a singular vector of the type

\[
u_{p,r} = \Phi(p,r)v_{p,r} = L(p)v_{p,r} + \cdots
\]

where \( \cdots \) denotes a sum \( \sum w_i \) of basis elements (25) of weight \(-p\) such that \( w_i < L(p)v_{p,r} \). In particular, for each such monomial \( w_i \) we have \( \text{deg}_{\lambda^-, \mu^-} w_i > 0 \). We show that \( u_{p,r} \) is not annihilated by any element of \( U(SH^-) \), i.e., \( U(SH)u_{p,r} \) is isomorphic to a Verma module. Note first that

\[
(\Psi_{-\lambda} - G_{-\lambda} + \alpha_{-\mu} - L_{-\mu^+}) L(p)v_{p,r} = (\Psi_{-\lambda} - G_{-\lambda} + \alpha_{-\mu} - L_{-\hat{\mu}^+}) v_{p,r} + \cdots
\]

where \( \hat{\mu}^+ = (\mu_1^+, \ldots, -p, \ldots \mu_l^+) \), and \( \cdots \) denotes a sum of basis elements of equal \( \text{deg}_{\lambda^+, \mu^+} \) and lower \( \ell_{\lambda^+, \mu^+} \). Let

\[
0 \neq X = \sum K_{\lambda^\pm, \mu^\pm} (\Psi_{-\lambda} - G_{-\lambda} + \alpha_{-\mu} - L_{-\mu^+}) \in U(SH^-)
\]
be an arbitrary element of an universal enveloping algebra of $\mathcal{SH}^-$. Recall brackets from Definition 2.1. We use the fact that if $w_i$ is a basis element as in (25) such that $\deg_{\lambda^-\mu^-} w_i > 0$, then either $Xw_i = 0$ or $Xw_i$ is a linear combination of basis (25) monomials $w'_i$ such that $\deg_{\lambda^-\mu^-} w'_i > 0$. Then we have

$$Xu_{p,r} = XL(p)v_{p,r} + \cdots = \sum K_{\lambda^\pm\mu^\pm} (\Psi_{-\lambda^-} G_{-\lambda^+} + \alpha_{-\mu^-} L_{-\mu^+}) v_{p,r} + \cdots$$

where $\cdots$ again denotes a sum $\sum w_i$ of monomials (25) such that $w_i < (\Psi_{-\lambda^-} G_{-\lambda^+} + \alpha_{-\mu^-} L_{-\mu^+}) v_{p,r}$. Therefore, $Xu_{p,r} \neq 0$ and we conclude that $\mathcal{SH}^-$ acts freely on $u_{p,r}$, hence $\langle u_{p,r} \rangle \cong V[p, r+1]$. □

**Remark 2.** In the proof of the previous Theorem, we actually show that $\Phi(p, r)$ is not a zero divisor. In order to prove this, we use an explicit expression for $\Phi(p, r)$. But we hope that this is a part of a more general theory (cf. [13]).

From the previous theorem we conclude that Theorem 4.6 lists all singular (and subsingular) vectors in $V[p, r]$, $p \in \mathbb{Z}_{>0}$ even, and we get a chain of submodules

$$V[p, r - i - 1] \subseteq V[p, r - i], \quad i \in \mathbb{Z}_{\geq 0}.$$  

Similarly, for $p \in \mathbb{Z}_{<0}$ even we have a chain

$$V[p, r + i + 1] \subseteq V[p, r + i], \quad i \in \mathbb{Z}_{\geq 0}.$$  

**Theorem 7.2.** The maximal submodule in $V[p, r]$, $p > 0$ odd is generated by a subsingular vector $w_{p,r}$ given by (24).

**Proof.** Let $M = \langle w_{p,r} \rangle$ and consider $V[p, r]/M$. In the proof of Lemma 6.2 we have shown that

$$\text{char}_q V[p, r]/M \leq q^{h_{p,r}} (1 - q^{\frac{|p|}{2}}) \prod_{k=1}^{\infty} \frac{(1 + q^{k-1/2})^2}{(1 - q^k)^2}.$$  

But, by Theorem 6.3, the right hand side is equal to $\text{char}_q L[p, r]$. Since $\text{char}_q V[p, r]/M \geq \text{char}_q L[p, r]$, we conclude that $V[p, r]/M = L[p, r]$ i.e. $M$ is the maximal submodule. □

**Remark 3.** It can be shown that the maximal submodule in $V[p, r]$, $p < 0$ odd is isomorphic to $V[p, r + \frac{1}{2}]$ and generated by a singular vector of the type $u_{p,r} = G(\frac{p}{2}) v_{p,r} + \cdots$. 

$N = 1$ HEISENBERG-VIRASORO ALGEBRA AT LEVEL 0 25
Structure of Verma modules in case \( p \in \mathbb{Z}_{<0} \) is shown in Figure 1. Free field realisation of Verma modules for \( p \in \mathbb{Z}_{>0} \) odd is shown in Figure 2.

Appendix A. Figures

Below we present embedding diagrams for the Verma module \( V[p, r] \) and its dual \( \mathcal{F}_{-p,-r} \), \( p \in \mathbb{Z}_{>0} \). We should mention that in the present paper we don’t present all proofs for the structure of \( V[p, r] \) for \( p \in \mathbb{Z}_{>0} \) odd, since it requires a very subtle analysis of indecomposable modules. In our forthcoming papers we shall investigate in more details the indecomposable modules and tensor categories related to these structures (cf. [7]).

\[
\begin{align*}
\bullet & \quad v_{p,r} \\
\bullet & \quad u_{p,r}^{(1)} \\
\bullet & \quad u_{p,r}^{(2)} \\
& \quad \vdots \\
\bullet & \quad u_{p,r}^{(n)}(5/2) \\
\bullet & \quad u_{p,r}^{(n)}(2) \\
\bullet & \quad u_{p,r}^{(n)}(3/2) \\
\bullet & \quad u_{p,r}^{(n)(1)} \\
\bullet & \quad u_{p,r}^{(n)(1/2)} \\
\bullet & \quad u_{p,r} \\
\hline
\end{align*}
\]

(a) \( V[p, r], p \in \mathbb{Z}_{<0} \) even. 
(b) \( V[p, r], p \in \mathbb{Z}_{<0} \) odd.

Singular vector \( u_{p,r}^{(n)} \) generates \( V[p, r + n] \) if \( p < 0 \). 
Singular vector \( u_{p,r}^{(n)} \) generates \( V[p, r + n], n \in \frac{1}{2}\mathbb{Z}_{>0} \).

Figure 1. Structure of Verma modules for \( p < 0 \).
\[ N = 1 \text{ Heisenberg-Virasoro Algebra at Level 0} \]

Figure 2. Realisation of \( \mathcal{F}_{p,r} \cong V[p,r] \), \( p > 0 \) odd.

Figure 3. Realisation of \( \mathcal{F}_{p,r} \cong V[p,r] \), \( p > 0 \) even.
Figure 4. $F_{p,r} \cong V[-p, -r]^*$, $p < 0$ odd.
$Gw_{p,r-1}^{(n+1/2)} = w_{p,r}^{(n-1/2)}$ and $Qw_{p,r-1/2}^{(n+1/2)}$ are subsingular vectors.
v_{p,r-1} \quad \bullet \\
\downarrow \\
\circ \quad G^w \quad \bullet \\
\downarrow \\
w^{(1)}_{p,r-1} \quad \circ \\
\downarrow \\
w^{(2)}_{p,r-1} \quad \circ \\
\downarrow \\
w^{(3)}_{p,r-1} \quad \circ \\
\downarrow \\
\vdots

Figure 5. \mathcal{F}_{p,r} \cong V[-p,-r]^*, p < 0 even.
Subsingular vector \( w^{(n-1)} \) generates \( \text{Ker}\mathcal{F}_{p,r}(G^w)^n \), \( n \in \mathbb{Z}_{>0} \).

References

[1] D. Adamović, Realizations of simple affine vertex algebras and their modules: the cases \( \widehat{sl}(2) \) and \( \widehat{osp}(1,2) \), Communications in Mathematical Physics, March 2019, Volume 366, Issue 3, pp 1025–1067;
[2] D. Adamović, A. Milas, On W-Algebras Associated to (2, p) Minimal Models and Their Representations International Mathematics Research Notices 2010 (2010) 20: 3896-3934
[3] D. Adamović, B. Jandrić, G. Radobolja, On the \( N = 1 \) super Heisenberg-Virasoro vertex algebra, to appear in Lie Groups, Number Theory, and Vertex Algebras, Proceedings of the Conference "Representation Theory XVI", June 24-29, IUC-Dubrovnik, Contemporary Mathematics.
[4] D. Adamović, V. Pedić, On fusion rules and intertwining operators for the Weyl vertex algebra, Journal of Mathematical Physics 60 (2019), no. 8, 081701, 18 pp. arXiv:1903.10248 [math.QA]
[5] D. Adamović, G. Radobolja, Free field realization of the twisted Heisenberg-Virasoro algebra at level zero and its applications, Journal of Pure and Applied Algebra 219 (10) 2015, pp. 4322-4342.
[6] D. Adamović, G. Radobolja, Self-dual and logarithmic representations of the twisted Heisenberg-Virasoro algebra at level zero, Communications in Contemporary Mathematics Vol. 21, no. 02, 1850008 (2019); arXiv:1703.00531 [math.QA].
[7] D. Adamović, G. Radobolja, in preparation
[8] E. Arbarello, C. De Concini, V. Kac, C. Procesi, Moduli spaces of curves and representation theory, Communications in Mathematical Physics 117 (1988), 1–36.

[9] Y. Billig, Representations of the twisted Heisenberg-Virasoro algebra at level zero, Canad. Math. Bull 46, no. 4 (2003) 529–537, arXiv:math/0201314

[10] Y. Billig, Energy-momentum tensor for the toroidal Lie algebras, arXiv:math/0201313

[11] T. Creutzig, N. Genra, S. Nakatsuka, Duality of subregular W-algebras and principal W-superalgebras, arXiv:2005.10713

[12] I. B. Frenkel, Y. Huang, J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, Mem. Am. Math. Soc. 104 (1993).

[13] M. Gorelik, private communications

[14] H. Guo, Q. Wang, Twisted Heisenberg-Virasoro vertex operator algebra, Glasnik Matematički Vol. 54(74)(2019), 369–407

[15] B. Jandrić, Vertex algebras associated to representations of the $N = 1$ Super Heisenberg-Virasoro and Schrödinger-Virasoro algebra, Ph.D. dissertation (in Croatian), University of Zagreb (2019)

[16] Q. Jiang, C. Jiang, Representations of the twisted Heisenberg-Virasoro algebra and the full toroidal Lie algebras, Algebra Colloquium, 2007, 14: 117–134.

[17] V. G. Kac, Vertex algebras for beginners, second ed., University Lecture Series, vol. 10, American Mathematical Society, Providence, RI, 1998.

[18] J. Lepowsky and H. Li, ”Introduction to vertex operator algebras and their representations,” in Progress in Mathematics (Birkhäuser Boston, Inc., Boston, MA, 2004), Vol. 227.

[19] A. Meurman, A. Rocha-Caridi, Highest Weight Representations of the Neveu-Schwarz and Ramond Algebras, Communications in Mathematical Physics 107 (1986), 263–294.