A GLOBAL BRIANÇON-SKODA-HUNEKE-SZNAJDMAN
THEOREM

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Abstract. We prove a global effective membership result for polynomials on a
non-reduced algebraic subvariety of $\mathbb{C}^N$. It can be seen as a global version of a
recent local result of Sznajdman, generalizing the Briançon-Skoda-Huneke theorem
for the local ring of holomorphic functions at a point on a reduced analytic space.

1. Introduction

Let $x$ be a point on a smooth analytic variety $X$ of pure dimension $n$ and let $O_x$
be the local ring of holomorphic functions. The classical Briançon-Skoda theorem,
\cite{10}, states that if $(a) = (a_1, \ldots, a_m)$ is any ideal in $O_x$ and $\phi$ is in $O_x$, then $\phi \in (a)^r$
if
\begin{equation}
|\phi| \leq C|a|^\nu r^{-1}
\end{equation}
holds with $\nu = \min(m, n)$. The proof given in \cite{10} is purely analytic. The condition
\eqref{1.1} can be rephrased as saying that $\phi$ belongs to the the integral closure
$(a)^{\nu + r - 1}$, and thus the theorem is algebraic. Therefore it was somewhat astonishing that it
took several years before algebraic proofs were found, \cite{15,16}. Later on, Huneke, \cite{15},
proved a far-reaching algebraic generalization which contains the following statement
for non-smooth $X$.

Let $x \in X$ be a point on a reduced analytic variety of pure dimension. There is a
number $\nu$ such that if $(a) = (a_1, \ldots, a_m)$ is any ideal in $O_x$ and $\phi$ is in $O_x$, then \eqref{1.1}
implies that $\phi \in (a)^r$.

An important point is that $\nu$ is uniform with respect to both $(a)$ and $r$. The
smallest possible such $\nu$ is called the Briançon-Skoda number, and it depends on
the complexity of the singularities of $X$ at $x$. An analytic proof of this statement
appeared in \cite{3}. A variant for a non-reduced $X$ of pure dimension was formulated
and proved in \cite{24}.

Let $x$ be a point on a non-reduced analytic space $X$ of pure dimension $n$, and
let $X_{\text{red}}$ be the underlying reduced space. There is a natural mapping $O_{X,x} \to
O_{X_{\text{red}},x}$. Let $i: X \to \Omega \subset \mathbb{C}^N$ be a local embedding, and let $\mathcal{J}_{X,x}$ be the associated
local ideal in $O_{\Omega,x}$, so that $O_x = O_{X,x} = O_{\Omega,x}/\mathcal{J}_{X,x}$. We say that a holomorphic
differential operator $L$ is Noetherian at $x$ if $L\phi$ vanishes on $X_{\text{red},x}$ (or equivalently, $L\phi \in \sqrt{\mathcal{J}_{X,x}} = \mathcal{J}_{X_{\text{red}},x}$) as soon as $\phi \in \mathcal{J}_{X,x}$. Such an $L$ defines a mapping
$L: O_{X,x} \to O_{X_{\text{red}},x}$, $\phi \mapsto L\phi$.

Theorem 1.1 (Sznajdman, \cite{24}). Given $x \in X$, there is a finite set $L_\alpha$ of Noetherian
operators at $x$ and a number $\nu$ such that for each ideal $(a) = (a_1, \ldots, a_m) \subset O_{X,x}$

\begin{itemize}
  \item \textbf{The author was partially supported by the Swedish Research Council.}
\end{itemize}
and \( \phi \in \mathcal{O}_{X,x} \),

\[
|L_\alpha \phi| \leq C|a|^{\nu+r} \text{ on } X_{\text{red},x}
\]

for all \( \alpha \), implies that \( \phi \in (a)^\nu \).

Here \( |a| \) means \( |a_1| + \cdots + |a_m| \) (where \( |a_j| \) is the modulus of the image of \( a_j \) in \( \mathcal{O}_{X,x} \)), which up to constants is independent of the choice of generators of the ideal \((a)\). The condition (1.2) means that \( L_\alpha \phi \) is in the integral closure of the image in \( \mathcal{O}_{X,\text{red},x} \) of \((a)^{\nu+r}\).

Applying to \((a) = (0)\) we find that \( L_\alpha \phi = 0 \) on \( X_{\text{red},x} \) for all \( \alpha \) implies that \( \phi = 0 \) in \( \mathcal{O}_{X,x} \).

**Remark 1.2.** In \([24]\) the number \( \nu \) appears from a log resolution of \( X_{\text{red}} \) and therefore it cannot be tracked down. In the recent paper \([25]\) is given a proof in which \( \nu \), in the case when \( X \) is reduced, is expressed in terms of the pullback to \( X_{\text{red}} \) of the Fitting ideals of \( \mathcal{O}_{X,x} = \mathcal{O}_{\mathcal{O},x}/J_X \) together with the singularities of meromorphic \((n,0)\)-forms that are holomorphic in the Barlet sense. When \( X \) is not reduced additional singularities come into play, but which still can be expressed without a log resolution.

We now turn our attention to global variants. Let \( V \) be a purely \( n \)-dimensional algebraic subvariety of \( \mathbb{C}^N \). Assume that \( F_j \) are polynomials in \( \mathbb{C}^N \) of degree \( \leq d \). If the polynomial \( \Phi \) belongs to the ideal \((F_1, \ldots, F_m)\), i.e., there are polynomials \( Q_j \) such that

\[
(1.3) \quad \Phi = \sum_{j=1}^m F_j Q_j \text{ on } V,
\]

then it is natural to ask for a representation with some control of the degree. It is well-known that if \( V = \mathbb{C}^N \), then in general \( \max_j \deg F_j Q_j \) must be doubly exponential in \( d \), i.e., like \( 2^{2^d} \). However, in the Nullstellensatz, i.e., \( \Phi = 1 \), then (roughly speaking) \( d^n \) is enough, this is due to Kollár, \([17]\), and Jelonek, \([16]\). In \([14]\) Hickel proved a global effective version of the Briançon-Skoda theorem for polynomial ideals in \( \mathbb{C}^n \), basically saying that if \( |\Phi|/|F|^{\min(m,n)} \) is locally bounded, then there is a representation \((1.3)\) in \( \mathbb{C}^n \) with \( \deg F_j Q_j \leq \deg \Phi + Cd^n \). For the precise statement, see \([14]\) or \([6]\). In \([6\), Theorem A\] a generalization to polynomials on reduced algebraic subvarieties of \( \mathbb{C}^N \) appeared. Our objective in this paper is to find a generalization to a not necessarily reduced algebraic subvariety \( V \) of \( \mathbb{C}^N \) of pure dimension \( n \). If \( V \) is defined by the ideal \( J_V \subset \mathbb{C}[x_1, \ldots, x_N] \), then \((1.3)\) means that \( \Phi - (F_1 Q_1 + \cdots + F_m Q_m) \) belongs to \( J_V \).

Let \( X \) be the closure of \( V \) in \( \mathbb{P}^N \) and let \( X_{\text{red}} \) be the underlying reduced variety. Given polynomials \( F_1, \ldots, F_m \), let \( f_j \) denote the corresponding \( d \)-homogenizations, considered as sections of \( \mathcal{O}(d)|_{X_{\text{red}}} \), and let \( J_f \) be the coherent analytic sheaf on \( X_{\text{red}} \) generated by \( f_j \). Furthermore, let \( c_\infty \) be the maximal codimension of the so-called distinguished varieties of the sheaf \( J_f \), in the sense of Fulton-MacPherson, that are contained in

\[
X_{\text{red},\infty} := X_{\text{red}} \setminus V_{\text{red}},
\]

see Section \([5]\). It is well-known that the codimension of a distinguished variety cannot exceed the number \( m \), see, e.g., \([11\) Proposition 2.6\], and thus

\[
c_\infty \leq \min(m,n).
\]

We let \( Z_f \) denote the zero variety of \( J_f \) in \( X_{\text{red}} \).
Let \( \text{reg} X \) denote the so-called \((\text{Castelnuovo-Mumford})\) regularity of \( X \subset \mathbb{P}^N \), see, e.g., [12]. We can now formulate the main result of this paper.

**Theorem 1.3** (Main Theorem). Assume that \( V \) is an algebraic subvariety of \( \mathbb{C}^N \) of pure dimension \( n \) and let \( X \) be its closure in \( \mathbb{P}^N \). There is a finite set of holomorphic differential operators \( L_\alpha \) on \( \mathbb{C}^N \) with polynomial coefficients and a number \( \nu \) so that the following holds:

(i) For each point \( x \in V \) the germs of \( L_\alpha \) are Noetherian operators at \( x \) such that the conclusion in Theorem 1.1 holds.

(ii) If \( F_1, \ldots, F_m \) are polynomials of degree \( \leq d \), \( \Phi \) is a polynomial, and

\[
|L_\alpha \Phi|/|F|^{c,\nu} \text{ is locally bounded on } V_{\text{red}}
\]

for each \( \alpha \), then there are polynomials \( Q_1, \ldots, Q_m \) such that (1.3) holds on \( V \) and

\[
\deg (F_j Q_j) \leq \max \left( \deg \Phi + \nu d^{c,\nu} \deg X_{\text{red}}, (d-1) \min(m, n+1) + \text{reg} X \right).
\]

If there are no distinguished varieties of \( J_f \) contained in \( X_{\text{red}, \infty} \), then \( d^{c,\nu} \) shall be interpreted as 0.

In case \( V \) is reduced we can choose \( L_\alpha \) as just the identity; then (ii) is precisely (part (i) of) Theorem A in [6]. If \( V = \mathbb{C}^n \) we get back Hickel’s theorem, [14] mentioned above.

**Example 1.4.** If we apply Theorem 1.3 to Nullstellensatz data, i.e., \( F_j \) with no common zeros on \( V \) and \( \Phi = 1 \), then the hypothesis (1.4) is fulfilled, and we thus get \( Q_j \) such that \( F_1 Q_1 + \cdots + F_m Q_m - 1 \) belongs to \( J_V \) and

\[
\deg (F_j Q_j) \leq \max (\nu d^{c,\nu} \deg X_{\text{red}}, (d-1) \min(m, n+1) + \text{reg} X).
\]

See [6] Section I for a discussion of this estimate in the reduced case.

**Example 1.5.** If \( f_j \) have no common zeros on \( X \), then there is a solution to \( F_1 Q_1 + \cdots + F_m Q_m = 1 \) on \( V \) such that

\[
\deg F_j Q_j \leq \max (\deg \Phi, (d-1)(n+1) + \text{reg} X).
\]

If \( X = \mathbb{P}^n \), then \( \text{reg} X = 1 \) and so we get back the classical Macaulay theorem.

**Remark 1.6.** It follows that \( L_\alpha \) is a set of Noetherian operators such that a polynomial \( \Phi \in \mathbb{C}[x_1, \ldots, x_N] \) is in \( J_V \subset \mathbb{C}[x_1, \ldots, x_N] \) if and only \( L_\alpha \Phi = 0 \) on \( V_{\text{red}} \) for each \( \alpha \). The existence of such a set is well-known, and a key point in the celebrated Ehrenpreis-Palamodov fundamental theorem, [13] and [21]; see also, e.g., [8] and [20].

**Remark 1.7.** It turns out, see Theorem 4.1 below, that the Noetherian operators \( L_\alpha \) in Theorem 1.3 have the following additional property: For each \( \alpha \) there is a finite set of holomorphic differential operators \( M_{\alpha, \gamma} \) such that

\[
L_\alpha (\Phi \Psi) = \sum_{\gamma} L_{\gamma} \Phi M_{\alpha, \gamma} \Psi
\]

for any holomorphic functions \( \Phi \) and \( \Psi \). This fact shows that set of functions that satisfy (1.2) at a point \( x \) is indeed an ideal.

By homogenization, this kind of effective results can be reformulated as geometric statements: Let \( z = (z_0, \ldots, z_N) \), \( z' = (z_1, \ldots, z_N) \), let \( f_i(z) := z_0^d F_i(z'/z_0) \) be the \( d \)-homogenizations of \( F_i \), considered as sections of \( \mathcal{O}(d) \to \mathbb{P}^N \), and let \( \varphi(z) := \frac{\varphi(z)}{z_0} \)
Then there is a representation (1.3) on $V$ with $\deg(F_j Q_j) \leq \rho$ if and only if there are section $q_i$ of $\mathcal{O}(\rho - d)$ on $\mathbb{P}^N$ such that
\begin{equation}
(1.7) \quad f_1 q_1 + \cdots + f_m q_m = z_0^{-\deg \Phi} \varphi
\end{equation}
on $X$ in $\mathbb{P}^N$; that is, the difference of the right and the left hand sides belongs to the sheaf $\mathcal{J}_X$.

To prove Theorem 1.3 we first have to define a suitable set of global Noetherian operators on $\mathbb{P}^N$. This is done in Section 4 following the ideas of Björk, [9], in the local case, starting from a representation of $\mathcal{J}_X$ as the annihilator of a tuple of so-called Coleff-Herrera currents on $\mathbb{P}^N$. The rest of the proof of Theorem 1.3 given in Section 5 follows to a large extent the proof of Theorem A in [6]. By the construction in [4] we have a residue current $R^X$ associated with $\mathcal{J}_X$ such that the annihilator ideal of $R^X$ is precisely $\mathcal{J}_X$. Following the ideas in [6] we then form the “product” $R^f \wedge R^X$, where $R^f$ is the current of Bochner-Martinelli type introduced in [11], inspired by [22]. By computations as in [24], the condition (1.4) ensures that $\phi$ annihilates this current at each point $x \in V_{\text{red}}$. If $\rho$ is large enough, this is reflected by the first entry of the right hand side of (1.5), then a geometric estimate from [11] ensures that the $\rho$-homogenization $\phi$ of $\Phi$ indeed satisfies a condition like (1.4) even at infinity. Therefore $\phi$ annihilates the current $R^f \wedge R^X$ everywhere on $\mathbb{P}^N$. For this argument it is important that the Noetherian operators extend to $\mathbb{P}^N$. The proof of Theorem 1.3 is then concluded along the same lines as in [6] by solving a sequence of $\bar{\partial}$-equations. If $\rho$ is large enough, this is reflected by the second entry in the right hand side of (1.5), there are no cohomological obstructions. We then get a global representation of $\phi$ as a member of $\mathcal{O}(\rho) \otimes (\mathcal{J}_f + \mathcal{J}_X)$. After dehomogenization we get the desired representation (1.3).

In Sections 2 and 3 we present some material on residue theory that we will use.

2. Some preliminaries on residue theory

All statements in this section can be found in [7] with proofs and/or further references. Let $Y$ be a (smooth) complex manifold of dimension $N$. Given a holomorphic function $f$ on $Y$, we have the principal value current $1/f$, defined for instance as the limit
\[
\lim_{\epsilon \to 0} \chi(|f|^2 v/\epsilon) \frac{1}{f},
\]
where $\chi(t)$ is the characteristic function of the interval $[1, \infty)$ or a smooth approximand of it and $v$ is any smooth strictly positive function. The existence of this limit for a general $f$ relies on Hironaka’s theorem that ensures that there is a modification $\pi: \tilde{Y} \to Y$ such that $\pi^* f$ is locally a monomial. It is readily checked that
\begin{equation}
(2.1) \quad f \frac{1}{f} = 1, \quad f \bar{\partial} \frac{1}{f} = 0.
\end{equation}
The current $1/f$ is well-defined even if $f$ is a holomorphic section of a Hermitian line bundle over $Y$, since $a(1/af) = 1/f$ if $a$ is holomorphic and nonvanishing.

Example 2.1. If $s$ is one complex variable it is elementary to see that the principal value current $1/s^{m+1}$ exists and that
\[
\bar{\partial} \frac{1}{s^{m+1}} \wedge ds \xi = \frac{2 \pi i}{m!} \frac{\partial^m}{\partial s^m} \xi(0),
\]
for test functions $\xi$. \hfill \Box
The sheaf \( \mathcal{P}M \) of pseudomeromorphic currents consists of currents that are (finite sums of) direct images under (compositions of) modifications, simple projections and open inclusions of currents of the form
\[
(2.2) \quad \frac{\xi}{s_{\alpha_1} \cdots s_{\alpha_n}} \wedge \bar{\partial} \frac{1}{s_{\alpha_n}},
\]
where \( s \) is a local coordinate system and \( \xi \) is a smooth form with compact support.

The sheaf \( \mathcal{P}M \) is closed under \( \bar{\partial} \) (and \( \partial \)) and multiplication by smooth forms. If \( \tau \) is in \( \mathcal{P}M \) and has support on a (reduced) subvariety \( V \) and \( \eta \) is a holomorphic form that vanishes on \( V \), then \( \bar{\eta} \wedge \tau = 0 \). We also have the Dimension principle: If \( \tau \) is a pseudomeromorphic current on \( Y \) of bidegree \((\ast,p)\) that has support on a variety \( V \) of codimension \( \geq p \), then \( \tau = 0 \).

If \( \tau \) is in \( \mathcal{P}M \) and \( V \) is a (reduced) subvariety of \( Y \), then the natural restriction of \( \tau \) to the open set \( Y \setminus V \) has a canonical extension as a principal value to a pseudomeromorphic current \( 1_{Y \setminus V} \tau \) on \( Y \): If \( h \) is a holomorphic tuple with common zero set \( V \), and \( \chi \) is a smooth approximant \( \chi \) of the characteristic function of the interval \( [1, \infty) \), then
\[
(2.3) \quad 1_{Y \setminus V} \tau = \lim_{\epsilon \to 0} \chi(|h|^2/\epsilon)\tau.
\]
It follows that \( 1_Y \tau := \tau - 1_{Y \setminus V} \tau \) is pseudomeromorphic and has support on \( V \).

Notice that if \( \alpha \) is a smooth form, then \( 1_Y \alpha \wedge \tau = \alpha \wedge 1_Y \tau \). Moreover, if \( \pi : Y \to Y \) is a modification, \( \bar{\tau} \) in \( \mathcal{P}M(Y) \), and \( \tau = \pi_* \bar{\tau} \), then
\[
(2.4) \quad 1_Y \tau = \pi_* (1_{\pi^{-1} Y} \bar{\tau})
\]
for any subvariety \( V \subset Y \). For any subvarieties \( V \) and \( W \),
\[
(2.5) \quad 1_W 1_{W^c} = 1_{W \cap W^c}.
\]

Let \( Z \subset Y \) be a reduced subvariety of pure codimension \( p \) and let \( \tau \) be a pseudomeromorphic current of bidegree \((N,\ast)\) with support on \( Z \). We say that \( \tau \) has the standard extension property, SEP, with respect to \( Z \) if \( 1_Y \tau = 0 \) for each \( V \subset Z \) of positive codimension. The sheaf of such currents is denoted by \( \mathcal{W}Z \). If \( Z = Y \) we write \( \mathcal{W} \) rather than \( \mathcal{W}Y \). The subsheaf of \( \mathcal{W}Z \) of \( \bar{\partial} \)-closed currents of bidegree \((N,p)\) is called the sheaf of Coleff-Herrera currents, \( \mathcal{CH}Z \), on \( Z \).

Remark 2.2. The sheaf \( \mathcal{CH}Z \) was introduced by Björk, in a slightly different way. For the equivalence, see \[2\] Section 5.

Example 2.3. Let \([Z]\) be the Lelong current associated with \( Z \) and let \( \beta \) be a smooth form of bidegree \((p,\ast)\). Then \( \beta \wedge [Z] \) is in \( \mathcal{W}Z \). If \( \beta \) is holomorphic, then it is in \( \mathcal{CH}Z \).

See, e.g., \[2\] Example 4.2.

Proposition 2.4. If \( L \) is a holomorphic differential operator and \( \tau \) is in \( \mathcal{W}Z \), then \( \xi \mapsto \tau.L\xi \) defines a current in \( \mathcal{W}Z \).

Proof. It is a local statement so by induction it is enough to let \( L \) be a partial derivative \( \partial/\partial \xi_1 \) with respect to some local coordinate system. Let \( L \) denote the Lie derivative with respect to this vector field. Since \( \xi \) has bidegree \((0,\ast),(\partial/\partial \xi_1)\xi = L\xi \). Thus
\[
\tau.(\partial/\partial \xi_1)\xi = \tau.L\xi = \pm L\tau.\xi,
\]
and \( L\tau \) is in \( \mathcal{W}Z \) according to \[7\] Theorem 6.6.

\[1\] We adopt here the convention from \[9\]; in, e.g., \[24\] these currents are of bidegree \((0,p)\).
2.1. Almost semi-meromorphic currents. We say that a current $b$ on a smooth manifold $Y$ is almost semi-meromorphic, $b \in ASM(Y)$, if there is a modification $\pi : Y' \to Y$, a holomorphic generically non-vanishing section $\sigma$ of a line bundle $L \to Y'$ and an $L$-valued smooth form $\omega$ such that

$$b = \pi_* \frac{\omega}{\sigma},$$

where $\omega/\sigma$ denotes the principal value current. Let $ZSS(b)$, the Zariski singular support of $b$, be the smallest analytic set such that $b$ is smooth in its complement. Let $S \to Y$ be a line bundle.

It is natural to say that $b \in ASM(Y) \otimes S$ if there is a representation (2.6), where $\omega$ is a smooth section of $L \otimes \pi^* S$. Since $\alpha \pi^* \tau = \pi^* (\pi^* \alpha \wedge \tau)$ for any smooth function (or form) $\alpha$, it follows that $b \in ASM(Y) \otimes S$ is an ordinary almost semi-meromorphic form in each local trivialization of $S$.

We will need the following results from [7].

Proposition 2.5 ([7], Theorem 5.1). If $b$ is almost semi-meromorphic on $Y$ and $L$ is a holomorphic differential operator, then $Lb$ is almost semi-meromorphic as well.

Clearly, $ZSS(Lb) \subset ZSS(b)$.

Proposition 2.6 ([7], Theorem 4.6). If $\tau$ any pseudomeromorphic current in $Y$, then there is a unique current $b\wedge \tau$ in $Y$ that coincides with $b\wedge \tau$ outside $ZSS(b)$ and such that $1_{ZSS(b)}b\wedge \tau = 0$.

It follows from (2.3) that

$$b\wedge \tau = \lim \chi_{\delta} b\wedge \tau,$$

where $\chi_{\delta} = \chi(|g|^2/\delta)$ and $g$ is a holomorphic tuple whose zero set is precisely $ZSS(b)$.

It is not hard to check, see [7, Eq. (4.6)], that if $V$ is any (reduced) subvariety, then

$$1_{V} (b\wedge \tau) = b\wedge 1_{V} \tau.$$

It follows from (2.8) that $b \in ASM(Y)$ induces a mapping

$$W_Z \to W_Z, \quad \tau \mapsto b\wedge \tau.$$

Given a $a \in ASM(Y)$ and any $\tau \in \mathcal{P} \mathcal{M}^Y$ we can define

$$\bar{\partial} a \wedge \tau := \partial (a \wedge \tau) - (-1)^{\deg a} a \wedge \partial \tau$$

The definition is made so that the formal Leibniz rule holds. Notice that if $a$ is holomorphic outside $ZSS(a)$, then the support of $\bar{\partial} a \wedge \tau$ is contained in $\text{supp} \tau \cap ZSS(a)$. In particular, if $\gamma_1, \ldots, \gamma_p$ are holomorphic sections of some line bundle $L \to Y$, then by induction we can form the $L^{-p}$-valued current

$$\bar{\partial} a \wedge \tau := \bar{\partial} (a \wedge \tau) - (-1)^{\deg a} a \wedge \bar{\partial} \tau$$

Clearly this current has support on $Z_{\gamma} = \{ \gamma_1 = \cdots = \gamma_p = 0 \}$. If in addition $Z_{\gamma}$ has codimension $p$, then (2.9) is anti-commuting in its factors. In this case we call it the Coleff-Herrera product $\mu^1$ formed by the $\gamma_j$; one can check that $\mu^1 \wedge \omega$ is in $\mathcal{CH}_{Z_{\gamma}}$ if $\omega$ is any holomorphic $(N,0)$-form. Any Coleff-Herrera current $\mu$ can be written locally as $\mu = a \mu^1 \wedge \omega$, and therefore the annihilator $\text{ann} \mu$ is the kernel of the sheaf mapping $\mathcal{O} \to \mathcal{O}/(\gamma)$, $\phi \mapsto a \phi$, and hence $\text{ann} \mu$ is coherent.
3. Global Coleff-Herrera currents on $\mathbb{P}^N$

Let $\delta_x$ be interior multiplication by the vector field

$$\sum_{j=1}^N x_j \frac{\partial}{\partial x_j}$$

on $\mathbb{C}^{N+1}$ and recall that a differential form $\xi$ on $\mathbb{C}^{N+1} \setminus \{0\}$ is projective, i.e., the pullback of a form on $\mathbb{P}^N$, if and only if $\delta_x \xi = \bar{\delta}_x \xi = 0$, where $\delta_x$ is the conjugate of $\delta_x$. We will identify forms on $\mathbb{P}^N$ and projective forms. Notice that

$$\Omega = \delta_x(dx_0 \wedge \ldots \wedge dx_N)$$

is a non-vanishing section of the trivial bundle over $\mathbb{P}^N$, realized as a $(N,0)$-form on $\mathbb{P}^N$ with values in $\mathcal{O}(N+1)$.

If $\gamma_1, \ldots, \gamma_p$ are holomorphic sections of $\mathcal{O}(r)$ such that their common zero set $Z_\gamma$ has codimension $p$, then, cf., Section 2.1 above,

$$(3.1) \quad \mu^\gamma \wedge \Omega := \bar{\partial} \frac{1}{\gamma_p} \wedge \cdot \cdot \cdot \wedge \bar{\partial} \frac{1}{\gamma_1} \wedge \Omega$$

is a global section of $\mathcal{CH}_{Z_\gamma} \otimes \mathcal{O}(-pr + N + 1)$.

Let $Z \subset \mathbb{P}^N_x$ be a reduced projective variety of pure codimension $p$.

Lemma 3.1. Let $\mu$ be a global section $\mathcal{CH}_Z \otimes (\ell + N + 1)$ and assume that $\gamma_1, \ldots, \gamma_p$ are sections of $\mathcal{O}(r)$ with common zero set $Z_\gamma$ of codimension $p$, and that

$$(3.2) \quad \gamma_1 \mu = \cdots = \gamma_p \mu = 0.$$  

If $p \leq N - 1$, then there is a holomorphic section $a$ of $\mathcal{O}(\ell + pr)$ such that

$$(3.3) \quad \mu = a \bar{\partial} \frac{1}{\gamma_p} \wedge \cdot \cdot \cdot \wedge \bar{\partial} \frac{1}{\gamma_1} \wedge \Omega.$$  

If $\text{codim} Z = N$ and $\ell + N \geq 0$ then the same conclusion holds.

In particular we see that if $p \leq N - 1$, and $\ell + pr < 0$, then $\mu = 0$.

Remark 3.2. The Coleff-Herrera product in (3.3) is defined locally as in (2.9) by expressing $\gamma_j$ in a local frame for $\mathcal{O}(r)$ as in (2.9). One can check that one can just as well form the corresponding homogeneous Coleff-Herrera product on $\mathbb{C}^{N+1} \setminus \{0\}$ and identify it with a $\mathcal{O}(-pr)$-valued current on $\mathbb{P}^N$.

Proof. Let us introduce a trivial vector bundle $E$ of rank $p$ with global holomorphic frame elements $e_1, \ldots, e_p$ and let $e_1^*, \ldots, e_p^*$ be the dual frame for $E^*$. We then have mapping interior multiplication $\delta_\gamma : \Lambda^{p+1}E \to \Lambda^pE$ by the section $\gamma := \gamma_1 e_1^* + \cdots + \gamma_p$ of $E^*$. We also consider the exterior algebra of $E \oplus T^*\mathbb{P}^N$ so that $d\bar{x}_j \wedge e_j = -e_j \wedge d\bar{x}_j$ etc. Then both $\delta_\gamma$ and $\bar{\partial}$ extend to mappings on currents with values in $\Lambda^pE$, and

$$(3.4) \quad \delta_\gamma \bar{\partial} = -\bar{\partial} \delta_\gamma.$$  

Let $e = e_1 \wedge \ldots \wedge e_p$.

Recall that $H^{N,k}(\mathbb{P}^N, \mathcal{O}(\nu)) = 0$ if either $1 \leq k \leq N - 1$ or $k = N$ and $\nu \geq 1$. If $p \leq N - 1$, or $\ell + N + 1 \geq 1$, we can therefore find a global solution to $\bar{\partial}w_{p-1} = \mu \wedge e$. In view of (3.4) and (3.2) we have that

$$\bar{\partial} \delta_\gamma w_{p-1} = -\delta_\gamma \bar{\partial} w_{p-1} = -\delta_\gamma (\mu \wedge e) = 0.$$  




Thus we can successively solve
\begin{equation}
(3.5) \quad \bar{\partial} w_{p-1} = \mu \wedge e, \quad \bar{\partial} w_{p-2} = \delta_{\gamma} w_{p-1}, \ldots, \bar{\partial} w_0 = \delta_{\gamma} w_1.
\end{equation}
Then $a \wedge \Omega := \delta_{\gamma} w_0$ is a $\bar{\partial}$-closed, and thus a holomorphic, $(N, 0)$-form with values in $\mathcal{O}(\ell + pr + N + 1)$. Altogether,
\begin{equation*}
(\delta_{\gamma} - \bar{\partial}) w = a \wedge \Omega - \mu \wedge e
\end{equation*}
if $w = w_0 + \cdots + w_{p-1}$. As in [2] Examples 3.1 or 3.2] we can find a global current $U$ such that
\begin{equation*}
(\delta_{\gamma} - \bar{\partial}) U = 1 - \mu \wedge e.
\end{equation*}
Thus
\begin{equation*}
(\delta_{\gamma} - \bar{\partial})(aU \wedge \Omega - w) = \mu - a \mu \wedge \Omega.
\end{equation*}
Since the right hand side is in $\mathcal{CH}_Z$ it now follows from [2 Lemma 3.6] that it must vanish.

Example 3.3. Given a global section $\mu$ of $\mathcal{CH}_Z \otimes \mathcal{O}(\ell)$ one can always find $\gamma_j$ such that $[3.2]$ holds. In fact, for a large enough $r_0$ there are homogeneous forms $g'_1, \ldots, g'_m$ of degree $r_0$ that generate the homogeneous ideal $J_Z \subset \mathbb{C}[x_0, \ldots, x_N]$. If $g_1, \ldots, g_p$ are generic linear combinations of the $g'_j$, then $Z_g = \{g_1 = \cdots = g_p = 0\}$ has codimension $p$, $Z_g \supset Z$, and (expressed in a local frame) $dg_1 \wedge \ldots \wedge dg_p \neq 0$ on $Z_{reg}$. If $\gamma_j = j^{m_j+1}$ and $m_j$ are large enough, then $[3.2]$ holds. \hfill $\square$

4. Björk-type representation of global Coleff-Herrera currents

As usual we identify smooth sections $\psi$ of the line bundle $\mathcal{O}(\ell)$ by $\ell$-homogeneous smooth functions on $\mathbb{C}^{N+1} \setminus \{0\}$. Notice that then each $\partial/\partial x_j$, $j = 0, \ldots, N$, induces a differential operator $\mathcal{O}(\ell) \to \mathcal{O}(\ell + 1)$. We say that a finite sum
\begin{equation}
L = \sum_{\alpha} v_{\alpha} \frac{\partial^\alpha}{\partial x^\alpha}
\end{equation}
is a holomorphic differential operator of degree $r$ if the coefficients $v_{\alpha}$ are holomorphic sections of $\mathcal{O}(r + |\alpha|)$. Such an $L$ maps $\mathcal{O}(\ell) \to \mathcal{O}(\ell + r)$ for each $\ell$. The order of $L$ is the maximal occurring $|\alpha|$ as usual.

Consider the affinization $\mathbb{C}^N \simeq \{x_0 \neq 0\}$. Notice that there is a one-to-one correspondence between smooth sections of $\mathcal{O}(\ell)$ over $\mathbb{C}^N$ and smooth functions in $\mathbb{C}^N$, via the frame $[x_0, \ldots, x_N] \to x_0^\ell$ for $\mathcal{O}(\ell)$ over $\mathbb{C}^N$. More concretely, given the section $\phi$ one gets the associated function by just letting $x_0 = 1$. Conversely, given $\Phi$, then $\phi(x) = x_0^\ell \Phi(x'/x_0)$. In this way a differential operator of degree $r$ gives rise to a differential operator
\begin{equation*}
L = \sum_{|\alpha'| \leq M} V_{\alpha'}(x') \frac{\partial^{\alpha'}}{\partial x'^{\alpha'}}
\end{equation*}
where $V_{\alpha'}(x')$ are polynomials of degree at most $r + |\alpha'|$. Notice however, that the resulting affine $L$ will depend on $\ell$ unless $L(x_0 \phi) = x_0 L \phi$ for all $\phi$. For instance, the differential operator $L = \partial/\partial x_0$, that has order $1$ and degree $-1$, induces
\begin{equation*}
L = \ell - \sum_{j=1}^N x_j \frac{\partial^j}{\partial x_0^j}.
\end{equation*}
Notice that $L$, as well as an associated affine differential operator $L$, act on smooth $(0, \ast)$-forms as well.
The following statement is a global version of a construction due to Björk, [9]. A similar result is obtained in [25, Theorem 4.2].

**Theorem 4.1.** Assume that \( Z \subset \mathbb{P}^N \) has pure codimension \( p \), that \( \mu \) is a global section of \( \mathcal{H}_Z \otimes \mathcal{O}(r) \), and assume that \( p \leq N - 1 \) or \( r + 1 \geq 0 \). Let \( I = \text{ann} \mu \). There is a multiindex \( m = (m_1, \ldots, m_p) \), a number \( \rho \), and for each \( \alpha \leq m \) there are holomorphic differential operators \( L_\alpha \) and \( M_{m-\alpha} \), such that \( \deg L_\alpha + \deg M_{m-\alpha} = \rho \), and a global meromorphic \((0,n)\)-form \( \tau \) with values in \( \mathcal{O}(-\rho) \), not identically polar on any irreducible component of \( Z \), such that the following hold:

(i) For any global holomorphic section \( \phi \) of \( \mathcal{O}(\ell) \) and any test form \( \xi \) of bidegree \((0,n)\) with values in \( \mathcal{O}(-r-\ell) \) we have

\[
\phi \mu . \xi = \sum_{\alpha \leq m} \int_Z \tau \wedge L_\alpha \phi \wedge M_{m-\alpha} \xi.
\]

(ii) For each point \( x \in Z \), a germ \( \psi \in \mathcal{O}_x \) is in \( I_x \) if and only if

\[
L_\alpha \psi \in \sqrt{I_x}, \quad \alpha \leq m.
\]

(iii) For each \( \alpha \leq m \) there are holomorphic differential operators \( M_{\alpha, \gamma} \), \( \gamma \leq \alpha \), such that

\[
L_\alpha (\phi \psi) = \sum_{\gamma \leq \alpha} L_\gamma \phi M_{\alpha, \gamma} \psi
\]

for all holomorphic sections \( \phi \) and \( \psi \) of \( \mathcal{O}(\ell) \) and \( \mathcal{O}(\ell') \).

**Proof.** To begin with we choose \( g_1, \ldots, g_p, m := (m_1, \ldots, m_p) \), and \( a \) as in Example 3.3 and Lemma 3.1 so that

\[
\mu = a \mu g^{m+1} \wedge \Omega.
\]

After a projective transformation on \( \mathbb{P}^N \), i.e., a linear change of variables on \( \mathbb{C}^{N+1} \), we may assume that each irreducible component of \( Z \) intersects the affine space \( \mathbb{C}^N := \{ x_0 \neq 0 \} \). Then the affinizations \( G_j \) of \( g_j \) are polynomials in \( \mathbb{C}^N \) such that \( dG_1 \wedge \ldots \wedge dG_p \) is nonvanishing on \( Z_{\text{reg}} \cap \mathbb{C}^N \), cf., Example 3.3. Let \( x' = (x_1, \ldots, x_N) \). After possibly a linear transformation of \( \mathbb{C}^N \), we may assume that the polynomial

\[
H := \det \frac{\partial G}{\partial \eta}
\]

is generically nonvanishing on \( Z \cap \mathbb{C}^N \), where

\[
x' = (\zeta, \eta) = (\zeta_1, \ldots, \zeta_n, \eta_1, \ldots, \eta_p).
\]

Let us introduce the short hand notation

\[
\bar{\partial} \frac{1}{G^{m+1}} = \bar{\partial} \frac{1}{G_1^{m_1+1}} \wedge \ldots \wedge \bar{\partial} \frac{1}{G_p^{m_p+1}}.
\]

We first look for a representation of the Coleff-Herrera current

\[
\tilde{\mu} = \bar{\partial} \frac{1}{G^{m+1}} \wedge d\eta \wedge d\zeta
\]

at points \( x \) on \( Z' := Z \cap \mathbb{C}^N \cap \{ H \neq 0 \} \). Locally at such a point we can make the change of variables

\[
w = G(\zeta, \eta), \quad z = \zeta.
\]
If $\Xi$ is a smooth $(0,n)$-form with small support, and $\Phi$ is holomorphic, with the notation $m! = m_1! \cdots m_p!$ and $\partial_w^\alpha = \partial^{(\alpha)} / \partial w^\alpha$, etc, in view of Example 2.1 we then have

$$\Phi \tilde{\mu} \Xi = \int \partial_w G_{m+1} \wedge d\eta \wedge d\zeta \wedge \Phi \Xi = \pm \int \partial_w G_{m+1} \wedge dw \wedge \Xi \Phi =$$

$$\pm \int_{w=0} \frac{(2\pi i)^p}{m!} dw \wedge \partial_w^m \left( \Xi \Phi \right) = \pm \sum_{\alpha \leq m} \int_{w=0} \frac{(2\pi i)^p}{(m-\alpha)!} dw \wedge \partial_w^{m-\alpha} \left( \Xi \Phi \right).$$

Now, notice that

$$\partial_w = \Gamma H \partial_n,$$

so that

$$\partial_n = (\partial_n G) \partial_w$$

where $\Gamma$ is a matrix of polynomials. It is readily checked that

$$\tilde{L}_\alpha := H^{2|\alpha} \left( \frac{\Gamma}{H} \partial_n \right)^{\alpha}$$

has a holomorphic extension across $H = 0$. Let us define

$$M_\beta \Xi = \pm \frac{(2\pi i)^p}{\beta!(m-\beta)!} H^{1+|m|+2|\beta|} \left( \frac{\Gamma}{H} \partial_n \right)^{\beta} \Xi.$$  

Then also $M_\beta$ is holomorphic across $H = 0$.

With $T = dz = d\zeta$, we have that

$$\Phi \tilde{\mu} \Xi = \int_{\zeta'} \sum_{\alpha \leq m} \frac{T}{H^{3|m|+1}} M_{m-\alpha} \Xi \wedge \tilde{L}_\alpha \Phi$$

for $\Xi$ with support close to $x$. We claim that if $\Phi$ is a germ of a holomorphic function at $x$, then $\Phi \tilde{\mu}_x = 0$ if and only if $\tilde{L}_\alpha \Phi = 0$ on $Z_x$ for all $\alpha \leq m$. In fact,

$$\Phi \tilde{\mu}_x = 0 \iff \Phi \tilde{\partial} \frac{1}{G_{m+1}} |_{z} = 0 \iff \Phi \tilde{\partial} \frac{1}{w^{m+1}} |_{z} = 0 \iff \partial_w^\alpha \Phi = 0 \text{ on } Z_x, \, \alpha \leq m \iff L_\alpha \Phi = 0 \text{ on } Z_x, \, \alpha \leq m.$$

Now, for each $\alpha \leq m$, let us homogenize the coefficients in $\tilde{L}_\alpha$ to obtain $\tilde{L}_\alpha$ for some fixed degree, and then let us homogenize $M_{m-\alpha}$ to $M_{m-\alpha}$ so that the sum of their degrees is a fixed number $\rho$. Let $\tau'$ be the homogenization of $T = d\zeta$, i.e.,

$$\tau' = \tilde{\partial} \frac{x_1}{x_0} \wedge \cdots \wedge \frac{x_n}{x_0}$$

if $x = (x_0, \ldots, x_N) = (x_0, \zeta, \eta)$. Finally let us homogenize $H^{3|m|+1}$ to $h$ so that $\tau := \tau'/h$ takes values in $O(-\rho)$. We possibly get some factors $x_0$ in the denominator, but since $Z$ has no irreducible component in $\{x_0 = 0\}$ this is acceptable.

Let us the define the global current

$$\tilde{\mu} := 1_Z \mu^{g^{m+1}} \wedge \Omega$$

in $\mathbb{P}^N$. In view of (4.5) it takes values in $O(r - \deg \alpha)$. At each point $x \in Z'$ it is the $r - \deg \alpha$-homogenization of our previous $\tilde{\mu}$ but the global current is not necessarily $\tilde{\partial}$-closed. However,

$$a \tilde{\mu} = a 1_Z \mu^{g^{m+1}} \wedge \Omega = 1_Z a \mu^{g^{m+1}} \wedge \Omega = a 1_Z \mu = \mu,$$

since $\mu$ has support on $Z$.
For holomorphic sections $\phi$ of $\mathcal{O}(\ell - \deg a)$ and test forms $\xi$ of bidegree $(0, n)$ with support in $\mathbb{P}^N \setminus \{h = 0, x_0 = 0\}$ and values in $\mathcal{O}(-r - \ell)$ we have

\begin{equation}
(4.10)\quad \phi \tilde{\mu} \xi = \int_Z \sum_{\alpha \leq m} \tau \wedge M_{m-\alpha} \xi \wedge \tilde{L}_\alpha \phi.
\end{equation}

By Proposition 2.6 $\tau \wedge \tilde{L}_\phi \wedge [Z]$ is a global section of $\mathcal{W}_Z \otimes \mathcal{O}(e + \ell)$ and thus the integrals in the right hand side of (4.10) exist as a principal values for any test form $\xi$. In view of Proposition 2.4 the right hand side of (4.10) defines the action on $\xi$ of a global section of $\mathcal{W}_Z \otimes \mathcal{O}(e + \ell)$. Since $\{h = 0, x_0 = 0\} \cap Z$ has positive codimension on $Z$ it follows by the SEP that the equality (4.10) holds for all $\xi$.

Define the holomorphic differential operators $L_\alpha$ by the equality

\begin{equation}
(4.11)\quad L_\alpha \phi = \tilde{L}_\alpha (a \phi).
\end{equation}

Then (4.11) follows from (4.10). Thus (i) is proved.

For $x \in Z' = Z \setminus \{h = 0, x_0 = 0\}$ we have, by (4.8) and (4.11), that

\begin{equation}
(4.12)\quad \phi \mu_x = 0 \text{ if and only if } L_\alpha \phi = 0 \text{ on } Z_x, \alpha \leq m.
\end{equation}

Again since $\{h = 0, x_0 = 0\} \cap Z$ has positive codimension on $Z$, it follows by continuity and the SEP that (4.12) holds for all $x \in Z$. Thus (ii) is proved.

To see (iii), just notice that

\begin{equation}
\tilde{L}_\alpha (\Phi \Psi) = \sum_{\gamma \leq \alpha} L_{\gamma} \Phi c_{\alpha, \gamma} L_{\alpha-\gamma} \Psi,
\end{equation}

where $c_{\alpha, \gamma}$ are binomial coefficients. After homogenization and replacing $\phi$ by $a \phi$ we get (iii) with $L_{\alpha, \gamma} = c_{\alpha, \gamma} L_{\alpha-\gamma}$. \hfill \Box

Remark 4.2. One can check, cf., [5, Section 5], that $\phi 1_Z \tilde{\mu} = 0$ if and only if $\phi$ is in the intersection of the primary ideals of $(g^{m+1})$ associated with the irreducible components of $Z$. \hfill \Box

Let $\mu$ be a global section of $\mathcal{C}H_Z \otimes \mathcal{O}(r)$ in $\mathbb{P}^N$ and let $b$ be a global almost semi-meromorphic current of bidegree $(0, *)$ with values in $\mathcal{O}(r_1)$. Then $b \mu$ is a section of $\mathcal{W}_Z \otimes \mathcal{O}(r + r_1)$. Let us also assume that $ZSS(b) \cap Z$ has positive codimension in $Z$. Consider a representation of $\mu$ as in Theorem 4.1. In view of Theorem 2.5 we can define differential operators $\tilde{M}_\gamma$ with almost semi-meromorphic coefficients so that

\begin{equation}
\tilde{M}_\gamma \xi = M_\gamma (b \xi).
\end{equation}

For test forms $\xi$ of bidegree $(0, *)$ with values in $\mathcal{O}(-r - \ell)$ and with support outside $ZSS(b)$, and any global holomorphic section $\phi$ of $\mathcal{O}(\ell)$ we have

\begin{equation}
(4.13)\quad \phi b \mu \xi = \sum_{\alpha \leq m} \int_Z \tau \wedge L_\alpha \phi \wedge \tilde{M}_{m-\alpha} \xi.
\end{equation}

In view of Propositions 2.6 and 2.4 the right hand side defines a global section of $\mathcal{W}_Z \otimes \mathcal{O}(r + r_1)$. Since $Z \cap ZSS(b)$ has positive codimension in $Z$, it follows that (4.13) holds globally.
5. Proof of Theorem 1.3

Given a vector bundle \( E \to \mathbb{P}^N \), let \( \mathcal{O}(E) \) denote the associated locally free analytic sheaf. We can find a locally free resolution

\[
0 \to \mathcal{O}(E_N) \to \cdots \to \mathcal{O}(E_1) \to \mathcal{O}(E_0) \to \mathcal{O}^{\mathbb{P}^N}/\mathcal{J}_X \to 0
\]

of \( \mathcal{O}^{\mathbb{P}^N}/\mathcal{J}_X \), where \( E_0 \) is a trivial line bundle and

\[
E_k = \bigoplus_i r_k \mathcal{O}(-d_k^i)
\]

for suitable positive numbers \( d_k^i \) such that

\[
\deg X = \max_{k,i} (d_k^i - k) + 1,
\]

see, e.g., [6]. In [4, Sections 3 and 6] was introduced an associated current

\[
R^X_\alpha = R^X_\alpha + \cdots + R^X_N
\]

with support on \( Z \), where \( R^X_\alpha \) are \((0,k)\)-currents that take values in \( E_\alpha \), and with the property that

\[
(5.1) \quad \phi R^X_\alpha = 0 \text{ if and only if } \phi \in \mathcal{J}_X.
\]

Proposition 5.1. There is a bundle \( F = \bigoplus_i r_i F_i \mathcal{O}(d_i^F) \), a global section \( \mu \) of \( CH_\alpha \otimes F \), and an almost semi-meromorphic section \( b \) of \( \text{Hom}(F, \bigoplus_{i=p}^{N+1} E_i) \) such that

\[
(5.2) \quad R^X \wedge \Omega = b\mu.
\]

in \( \mathbb{P}^N \).

Proof. The kernel \( K \) of \( \mathcal{O}(E_\alpha^*) \to \mathcal{O}(E_\alpha^* + 1) \) is coherent, and for a large enough integer \( d_F \), therefore \( K \otimes \mathcal{O}(d_F) \) is generated by global sections \( g_1, \ldots, g_r \). We get induced surjective sheaf mappings \( \oplus_1^r \mathcal{O} \to K \otimes \mathcal{O}(d_F) \) and \( \oplus_1^r \mathcal{O}(-d_F) \to K \). Let

\[
g: \mathcal{O}(E_\alpha) \to \bigoplus_1^r \mathcal{O}(d_F) =: F
\]

be the dual of the composed mapping \( \oplus_1^r \mathcal{O}(-d_F) \to K \to \mathcal{O}(E_\alpha^*) \). As in the proof of [24, Proposition 3.2] one verifies that \( \mu = gR^X_\alpha \) is a (vector-valued) Coleff-Herrera current. The rest of the proof precisely follows the proof of [24, Proposition 3.2], and we omit the details. \( \square \)

First assume that \( p = \text{codim } Z \leq N-1 \). Let us consider \( \mu \) as an \( r_F \)-tuple of Coleff-Herrera currents, and let \( L_\alpha, \alpha \leq m \), be a (tuple of) Noetherian operators obtained from Theorem 1.1. Moreover, let \( \tilde{M}_\alpha \) be the associated differential operators with almost semi-meromorphic coefficients so that (4.13) holds.

Given sections \( a_1, \ldots, a_m \) of a Hermitian line bundle over some open set \( U \subset \mathbb{P}^N \), we can form the associated residue current of Bochner-Martinelli type \( R^a \) in \( U \) with support on \( Z_a \), see, e.g., [6, Example 2.1]. In fact, there is an almost semi-meromorphic current \( U^a \) explicitly formed by the \( f_a \), with \( ZSS(U^a) = Z_a \), such that

\[
R^a = \lim_{\delta \to 0} R^{a,\delta}, \quad \text{where}
\]

\[
R^{a,\delta} = 1 - \chi_\delta + \bar{\partial}\chi_\delta \wedge U^a,
\]

\( \chi_\delta = \chi (|a|^2/\delta) \) and \( \chi \) is a function as in (2.3) above. As in [6, Section 2.5] we can form the product

\[
(5.3) \quad R^a \wedge R^X \wedge \Omega := \lim_{\delta \to 0} R^{a,\delta} \wedge R^X \wedge \Omega.
\]
If $\Phi$ is holomorphic and $\Phi R^\ell \wedge R^X \wedge \Omega = 0$ at $x$, then $\Phi$ is in $(a)_x + \mathcal{J}_{X,x}$, see [6] Sections 2.5 and 2.7.

Let $L_\alpha$ be the holomorphic differential operators from Theorem 4.1 associated with $\mu$.

Using the representation (4.13) of $R^X \wedge \Omega = b\mu$, the main step of the proof of [24] Theorem 1.2 precisely gives

**Proposition 5.2.** There is a number $\nu$, such that if $x \in X_{\text{red}}$, $(a) = (a_1, \ldots, a_m) \subset \mathcal{O}_{X,x}$ is a local ideal, and $\phi \in \mathcal{O}_{X,x}$, then $|L_\alpha \phi| \leq C|a|^\nu$ on $X_{\text{red},x}$ for all $\alpha \leq m$.

**Proof.** Using the representation (4.13) of $R^X \wedge \Omega = b\mu$, the main step of the proof of [24] Theorem 1.2 precisely gives this statement in a neighborhood of a fixed point $x$ for some number $\nu_x$. By compactness we get a uniform number $\nu$. □

So far we have thus obtained the number $\nu$ and the differential operators $L_\alpha$ in Theorem 1.3. Now let $F_j$ be polynomials as in Theorem 1.3 let $f_j$ be the $d$-homogenizations considered as section of $\mathcal{O}(d)$ over $X_{\text{red}}$ and let $\mathcal{J}_f$ be the associated ideal sheaf as in the introduction.

**Lemma 5.3.** Let $\Phi$ be a polynomial such that (1.4) holds and let $\phi$ be the $\rho$-homogenization of $\Phi$. If

$$
(5.4) \quad \rho \geq \deg \Phi + \nu d^\infty \deg X_{\text{red}},
$$

then $|L_\alpha \phi| \leq C|f|^\nu$ for all $\alpha$.

**Proof.** Let $\pi: \tilde{X} \to X_{\text{red}}$ be the normalization of the blow-up of $X_{\text{red}}$ along $\mathcal{J}_f$ and let $\sum r_j W_j$ be the exceptional divisor, where $W_j$ are the irreducible components and $r_j$ the corresponding multiplicities. Notice that if $\psi$ is a holomorphic section of some $\mathcal{O}(\ell)$, then $|\psi| \leq C|f|^\nu$ if and only if $\pi^* \psi$ vanishes to order at least $\nu r_j$ on $W_j$ for each $j$.

If (1.4) holds on $V_{\text{red}}$, then $\pi^* (L_\alpha \phi)$ vanishes to order $\nu r_j$ on each $W_j$ that is not fully contained in $\pi^{-1}(X_{\text{red},\infty})$. Notice that

$$
\phi = x_0^{\rho - \deg \Phi} \varphi,
$$

where $\varphi$ is the $\deg \Phi$-homogenization of $\Phi$ and thus holomorphic. If $W_j$ is contained in $\pi^{-1}(X_{\text{red},\infty})$, then $\phi$ vanishes at least to order $\rho - \deg \Phi$ on $W_j$. Since $L_\alpha$ does not involve the derivative $\partial/\partial x_0$ also $L_\alpha \phi$ vanishes to order $\rho - \deg \Phi$ on $W_j$. By the geometric estimate in [11], cf., [6] Eq. (6.2)], we have that

$$
r_j \leq d^{\text{codim } \pi(W_j)} \deg X_{\text{red}}.
$$

If (5.4) holds, therefore $\pi^* (L_\alpha \phi)$ vanishes, at least, to order $\nu r_j$ on $W_j$ for each $j$. Thus the lemma follows. □

With the same hypotheses as in Lemma 5.3 it follows from the lemma and Proposition 5.2 that

$$
(5.5) \quad \phi R^\ell \wedge R^X \wedge \Omega = 0.
$$

If in addition

$$
\rho \geq (d - 1) \min(m, n + 1) + \text{reg } X,
$$

we can now solve a sequence of global $\bar{\partial}$-equations in $\mathbb{P}^N$ and get a global solution $q_j$ to $\phi = f_1 q_1 + \cdots + f_m q_m$, cf., [6] Lemma 4.3. The fact that $X$ is not reduced plays
no role here. After dehomogenization we obtain the desired representation of \( \Phi \), and so the proof of Theorem 1.3 is complete in case \( p \leq N - 1 \).

Now assume that \( p = \text{codim } Z = N \) so that \( X_{\text{red}} \) is a finite set in \( \mathbb{C}^N \cong \mathbb{P}^N \setminus \{ x_0 \} \). If necessary we multiply \( \mu \) by a suitable power of \( x_0 \) to be able to apply Theorem 1.1. We then get the global, in \( \mathbb{C}^N \), \( \mathcal{L}_\alpha \) that form a complete set of Noetherian operators at each point \( x \in X_{\text{red}} \). Part (ii) is trivial, since the image of any ideal \( (\alpha) \subset \mathcal{O}_{X,x} \) in \( \mathcal{O}_{X_{\text{red}},x} \) is just either \( (0) \) or \( (1) = \mathcal{O}_{X_{\text{red}},x} \).

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