Ordinal Computers

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Abstract

Can a computer which runs for time $\omega^2$ compute more than one which runs for time $\omega$? No. Not, at least, for the infinite computer we describe. Our computer gets more powerful when the set of its steps gets larger. We prove that their theory of second order arithmetic cannot be decided by computers running to countable time.

Section 1. Introduction; Undecidability of Arithmetic.

Our motivation is to build a computer that will store and manipulate surreal numbers. Hackelroad [1] and Lurie [4] examined at least two ways to compute surreals in finite time, and shown the difficulty of building a field of surreals in which $x > y$, $x = y + z$, and $x = y \times z$ are decidable. Likewise for reals. In the recursive reals algebra but not order is decidable. And so it seems that the question of whether the theory of either field under $+$, $\times$, $<$ is decidable, ought to refer to decidability by some class of computers that can compute more than finite-time Turing Machines. If we’re going to talk about whether a computer can decide facts about numbers, then let’s have a computer that can construct all the numbers we want to talk about and decide the algebra and order relations. Computers running to time $\aleph_1$ can compute all reals, and to ordinal time can compute all surreals. Now, what facts about such numbers can ordinal computers decide?
We will prove, in sections 2 and 5, that polynomials with variables in the integers and the reals cannot be decided by a computer running for countable time. This is a curious result, since polynomials over the reals can be decided by Elimination of Quantifiers, and polynomials over integers can be decided in countable time by simply checking all the possible inputs. Our theorem suggests that both methods are tight, or, more precisely, that there is no way to join them together into a single countable-time algorithm for polynomials in integers and reals. To be specific, we will show that it cannot be decided in countable time whether there exists a real number \( x \) so that for all possible choices of some 20 integers, a polynomial in the integers is zero, and \( g < n \times x < g + 1 \) – an inequality in the integers and the real.

In section 3 we define a general notion of an ordinal computer, and in section 4 we prove something about them: that what can be affirmed in time \( < i \) is equivalent to what can be defined by the sentences of order \( i \) in a language of arbitrarily-high order. This is simply a generalization of the idea that sentences involving existential quantification over the reals ought to be affirmable by a computer which ran to an arbitrary countable time, and deniable by a computer running to time \( \aleph_1 \). To stress this, we will write \( \aleph_1 \) throughout to identify both the first set nonisomorphic to \( \aleph_0 \), though this is often written as the ordinal \( \omega_1 \).

A reader interested in ordinal computers may read only the brief section 3. A reader interested in our strange theorem may skip sections 3 and 4.

We assume CH – the cardinality of the reals is the same as the first uncountable ordinal.

Our strange theorem concerns the language \((R, Z, <, +, \times)\). That is, statements with variables ranging over \( R \), with symbols \(<, +, \times\) and a predicate \( Z \) which is true exactly on the integers. As Alex Wilkie pointed out to the author, this is just the theory of second order arithmetic. Let \( A \) be an algorithm which halts in countable time on just the false statements. We intend to show \( A \) doesn’t exist.

To each algorithm \( A \) we can build a finite-time machine \( B \) which accepts real number inputs (an infinite, pre-written tape). On real inputs \( x \) and \( r \), our machine \( B \) will run for time \( \omega \) just in case \( x \) encodes a run of \( A \), starting with input \( r \) and halting after countable time. If \( x \) does not code a run of \( A \) on \( r \), then \( B \) will halt in finite time. If \( A \) halts in countable time, there is some \( x \) on which \( B \) runs for time \( \omega \). So \( A \) halts in countable time just in case \( B \) doesn’t halt in finite time.
We prove in the last section that there exists a statement in the language $(R, Z, +, \times, <)$ which is true of $x, r$, just in case $B$ runs for time $\omega$ on inputs $x, r$. The variables $x, r$ are just the free variables of this statement. The statement also contains some quantified integer variables, but no quantified real variables. Now $A$ halts on $r$ just in case for some real $x$, $B$ doesn’t halt on $x, r$. That is, there is some $x$ so that our statement (call it $\phi$) is true. So $A$ halts on statement $r$ just in case there exists a real $x$ so that $\phi$ is true of $x, r$.

We will prove that there is a halting problem for computers running to countable times. In section 3 we show that there is no computer which halts in countable time just in case its input corresponds to a computer that does not halt in countable time. But if we can determine in countable time whether any statement of our language is true or not, then by the equivalence shown in the last paragraph, we could determine which computers halt in countable time.

This demonstrates a class of simple formulas of second-order arithmetic not decidable in countable time. This has corollaries that can be stated without reference to ordinal computers. For instance, we prove that the theory of second-order arithmetic is not model complete (assuming CH). If it is model complete, then any formula is equivalent to an existential formula. Any existential formula of second-order arithmetic can be put in the form “for some integer values, $p$” where $p$ is a formula of $(R, <, +, \times)$. We can decide this statement in countable time by checking whether $p$ is true at any particular integer values, using elimination of quantifiers for $(R, <, +, \times)$. Unfortunately, we never expected $(R, Z, <, +, \times)$ to be model complete. The formula $\forall k \exists p, q | x - p/q | < 1/kq^2$ defines the reals whose continued fraction terms are unbounded. It seems unlikely that the complement of this set is existentially definable.

Section 2. Reducing a countable – time machine to a finite – time machine.

Let a program $A$ have finitely many instructions and keep ordinal variables. Each instruction may increment a variable, switch control as two variables are equal or not, or stop the program. That is: “$x + +,$” “if $x = y$ goto 1,” or “stop.” At a limit time-ordinal, control returns to the 0th command. At a limit time-ordinal, the value of each variable becomes the limit of the values that it has achieved.

We will construct a program $B$ which accepts a real variable $x$ iff it codes
the run of A. As a bit string, $x$ is a sequences of 1’s, the number of 1’s indicating a number, with 0’s separating numbers. B separates $x$ into three or more sequences. The first, $z$, encodes a map from $\omega$ to the timesteps of A. The second, which we will call $c$ for control, is a sequence of numbers corresponding to lines of the program A: $c_i$ is the command that was active at time $i$. For each variable $x$ that A uses, $x_i$ is the value of the variable $x$ at time $i$. How can $z$, a list of finite numbers, code a map from $\omega$ to an infinite countable ordinal? It is actually a list of a statements written in a language that B can interpret so that B accepts only those $z$ which code a map from $\omega$ to a countable ordinal. The statements of $z$ are: “$n < m$”, “$m$ is a limit ordinal,” “$m = n + 1$,” which occurs for each $n$ unless $n$ is the final element, in which case $z$ contains the fact: “$n$ is the final element”. In all of these statements $n$ and $m$ are finite numbers; $z$ codes a re-ordering of the finite numbers so that they have the same order structure as the timesteps of A. The statement “$m$ is the final element” must appear first. In this way, B can check whether or not there is a final element. Because $z$ contains explicit successor and limit statements, B can affirm, in a finite amount of time, that $n$ is a limit or that $n$ succeeds $m$. We require that all statements involving numbers less than $k$ occur before time $2k^2$. There are at most 2 statements about any particular $m$ and $n$, so there is some $z$ listing all statements about numbers less than $k$ before time $2k^2$.

When B learns that $m$ is the final element, it checks that $c_m$ is the stop command. When B learns $n < m$, it checks that $c_n$ is not the stop command. When B learns $m < n$ it checks that it hasn’t already learned $n < m$. This insures $z$ is a partial ordering. B checks that $m < n$ or $n < m$ occurs before $2(n+m)^2$. This assures that $z$ is a total order. When B learns that $m = n + 1$ it checks that there is no $l$ between $n$ and $m$. This implies that $z$ is discrete. When B learns that $m = n + 1$ it checks that $n$ is less than $m$. That is, the indices of $m$ and $n$ are in the same order as the values they encode. This all implies that $z$ represents a discrete, wellordered total order. When B learns that $m = n + 1$, it checks that $c_m$ is the correct instruction to follow $c_n$ and that $x_m$ is derived from $x_n$ by applying rule $c_n$. When it learns that $m$ is a limit ordinal, it checks that $c_m$ is 0, and that $x_m$ is the limit of $x_n$ for $n < m$. But how can B check that the variables limit properly?

In order to check that all variables limit to their appropriate values, B accepts two reals, $x$ and $x'$, for each variable $x$ used in A. $x$ is, like all of our variables, a sequence of numbers, represented by a string of 1’s, separated
by zeroes. The \( i \)th number of \( x \), \( x_i \), represents the value of the variable \( x \) at the countable-ordinal time \( z_i \). But this value may be infinite! So \( x_i \) really is the \( z \) encoding of the value of \( x \) at the \( z \) encoding of time \( i \). B wants to check that if time \( i \) limits to time \( j \), then \( x_i \) limits to \( x_j \). This seems very difficult, because in finite time B has no way of knowing that any particular sequence of ordinal numbers limits to another ordinal. Indeed, B cannot even determine what any of the infinite numbers encoded by \( z \) are, in finite time. So to check that \( x \) is continuous, B checks that \( x \) is monotone, and that \( x \) and \( x' \) are inverses. Monotonicity means that if \( m < n \) then \( x_m \leq x_n \). That \( x \) has \( x' \) as inverse means: if \( x_m = n \), then \( x'_n \leq m \); if \( x'_m = n \), then \( x_n = x_m \). The sequence \( x \) is not strictly increasing, and it will happen that \( x_m \) is the same value for many consecutive timesteps; this introduces the asymmetry between \( x \) and \( x' \).

B checks that \( c_i \), the string encoding which command is active, timesteps appropriately, by checking that if \( c_i \) is active, then the next string to be active is \( c_{i+1} \), or, if \( c_i \) is a switch on a variable value, B finds this variable value and checks whether \( c_{i+1} \) or the alternate command was active next. At any limit ordinal, B checks that the zero-th command was active. B checks that \( x_i \) behave correctly, as well, by checking that \( x_{i+1} \) is \( x_i \) unless the command active at time \( i \) is the command ”increment \( x \)”, in which case, \( x_{i+1} = x_i + 1 \).

Section 3. Ordinal computers defined.

An ordinal computer runs for ordinal time, accepts ordinal inputs, and keeps ordinal variables. It has finitely many instructions of the form “increment \( x \)” or “if \( x = y \) goto instruction \( l \)” or “stop” Minsky [3] proves that these are sufficient to compute all Turing Machines running to finite time. Actually, he proves that “increment \( x \)”, “if \( x = 0 \) goto \( l \)”, and “decrement \( x \)” are sufficient. But we can model “decrement \( x \)” with our more general goto switch in a subroutine that starts with variables \( a \) and \( b \) equal to 0. Variable \( a \) is incremented. Then \( a \) and \( b \) are incremented until \( a = y \). Variable \( b \) is returned; \( b \) is the decrement of \( a \). Our decrement subroutine, on an input without a predecessor, is the identity. But I don’t think we can model the generalized goto switch using decrement, increment, and “if \( x = 0 \) goto \( l \)”. We can model the command \( x := y \) by incrementing \( x \) until it equals \( y \).

At a limit ordinal, what happens to the internal state of the machine? Command returns to instruction 0. Variables are set equal to their limit, if they have one; otherwise they are set to zero.
The halting problem is as difficult for computers halting at infinite ordinal as it is for computers halting at finite ordinals. Consider the set of computers which halt when given themselves as input. Let A be a computer halting on exactly those computers which don’t halt on themselves. Then run A on input A. It halts iff it doesn’t. This is true if we take halting to mean halting in finite time, ordinal time, halting before 10 timesteps have gone by, or before an uncountable number of timesteps have passed. In the final section, we will show that a computer A halts in countable time just in case some statement of $(R, Z, +, \times, <)$ is true. This implies that that language is undecidable in countable time because to decide it would be to solve the countable-time halting problem in countable time.

Section 4: The computational power of a run depends on the set of its timesteps.

We will prove that all ordinals which are equivalent under re-ordering have the same computational power. Let $\aleph_n$ be the first ordinal larger as a set than $\aleph_i$ for $i < n$. We want to know if algorithm A halts before time $\aleph_i$ on input $a < \aleph_i$. There is a computer B which halts before time $\aleph_j$, for some $j < i$, on input $a$ and all inputs $b < \aleph_i$, just in case A runs to time $\aleph_i$. This will all be simpler if we set $i=1$. Then: A stops in countable time just in case B, on all real inputs $b$, does NOT halt in finite time.

B checks that $b$ codes a complete run of A. That is, $b$ is a bit string which encodes: 1. a map from some $\aleph_j$, for $j < i$, to the steps of A. Program B sees the steps of A streaming by, in an order rearranged to be as short as possible. 2. Which instruction of A was in command at each time. 3. The value of all the variables of A at all times. 4. An inverse for each variable, which encodes when the variable was $<$, $=$, or $>$ than each possible value.

The hard part to check is that the value of the variable at a limit time $\lim_i$ is the limit of the values at times $i$. We have already described how such a B can operate, in the last paragraph of the previous section: by checking that the variable $b$ encodes strings $x$ and $x'$ for each variable, which represent inverses, and so that $x$ is monotone and invertible, hence continuous. We described the computer in the previous section in great detail, and here it is all the same, but with “less than $\aleph_i$” replacing “countable” and “less than some $\aleph_j$ for $j < i$” replacing “finite.”

Our description of B in terms of A is entirely primitive recursive and not dependent on A, so A may be considered a variable. Indeed, there is a
primitive recursive algorithm to produce $A_n$ from $A_{n+1}$ so that $A_{n+1}$ halts on input $a$ before time $\aleph_{n+1}$ iff for some $b$, $A_n$ doesn’t halt before time $\aleph_n$. This allows us to describe the set of inputs on which a program halts in terms of a arbitrarily-high-order language. We start at the level of predicates on finite numbers.

$$P(x_0) \text{ is true.}$$

- $A_0$ stops before $\aleph_0$ iff $\exists x_0 < \aleph_0 P(x_0)$
- $A_1$ stops before $\aleph_1$ iff $\exists x_1 < \aleph_1 \forall x_0 < \aleph_0 P(x_0)$
- $A_2$ stops before $\aleph_2$ iff $\exists x_2 < \aleph_2 \forall x_1 < \aleph_1 \exists x_0 < \aleph_0, P(x_0)$
- $A_\omega$ halts before $\aleph_\omega$ iff some sentence of the form $\forall x_7 \exists x_6 \forall x_5 \exists x_4 \ldots A_0$ is true

We have reduced the set of inputs on which some computer halts to the set of $x_0$ for which some high-level statement is true, so that runs halting before some cardinal time decide sets which are of the same level in the hierarchy.

We remark that in the hierarchy above, sentences need not be so long. We can exchange the existential and universal quantifiers if we are willing to quantify over longer ordinals. For instance: $\forall x < X \exists y < Y A_0(x, y)$ is equivalent to $\exists f < Y \forall x < X A_0(x, f(x))$. So, $A_\omega$ halts before $\aleph_\omega$ iff some sentence of the form $\exists x < \aleph_\omega \forall y < \aleph_\omega, A_0(x, y)$ is true.

Section 5. Turning a finite-time machine which accepts real inputs into a polynomial.

Let us remember that in section 1 we wanted to build a computer $B$ out of a computer $A$ so that: computer $A$ stops at countable time on countably-long bit-string $r$ just in case there exists a countably-long bit-string $x$ s.t. computer $B$ doesn’t stop in finite time on input $x, r$. The string $x$ codes a map from omega to the timesteps $A$ took; for each timestep of $A$, the value of each variable, and which instruction was operating. $B$ keeps finite variables, and is allowed to increment and compare them. $B$ is also allowed to switch on the $i$th bit of its real input. $B$ was built in section 2. Now we want to code $B$ with a polynomial relation of the form $p(Z) = 0 \& q(R, Z) > 0$, following Jones and Matijasevich [2], so that $B$ halts iff its statement in the language of inequalities of polynomials is true for no integers $m$. That is: $A$ halts in countable time on $r$ iff $\exists x B$ doesn’t stop in finite time on $r, x$ iff $\exists x \forall m \phi$. 

That is, a computer running for countable time and keeping countable variables will halt on those reals r so that $\exists x \forall \phi$. That will be proven once we turn B into a polynomial. We turn to this now.

B is allowed the commands “n++” and “if n=m goto” for its finite variables n, m. It is also allowed to switch on the $i$-th bit of the real variable x: “if $x_i = 0$ goto”, where i is a finite variable stored by B. We want, however, to weaken our program so it may only switch on the $i$-th bit of x at time $ni$ for some integer n. This can be done by encoding B in an interface program. The interface is only allowed to switch on the time-indexed bit, but it successfully stores all the bits of x, and then B can switch on a stored bit. In more detail: We store x, as a binary integer, until the program to store a bit takes longer than n steps. Then we execute B on the resulting integer, replacing “if $x_i = 0$ goto” with the command to compute $2^i$ and bitwise multiply this by x, and put the result in variable y. Then “if y = 0 goto”. We watch that B stops normally. If B halts on x, then for some large enough n B will halt normally on the truncation of x. If B doesn’t halt on x, then for no n will B halt normally. So: 

There exists x so that our more powerful computer stops on x, r just in case there exists $x, n$ so that our weaker computer stops on $x, r, n$. Now change B to read off the alternate bits of a single variable as $x$ and $n$. So we have simplified B as desired.

To this more restricted program B we associate a polynomial relation; a statement in the language $(R, Z, <, \times, +)$. B halts in finite time on real number inputs $x, r$ just in case there are some finite numbers $a, b, c, d \ldots$ so that $p(B, r, x, a, b, c, d \ldots) = 0 \& q(x, g) > 0$. This is the form whose instances we will prove form an undecidable class of statements. The first thing to do is to multiply $x$ and $r$ by $2^a$ so as to get numbers with positive integer parts (let’s assume $x$ and $r$ have no positive part). We will find $g = [2^a r]$ and $h = [2^a x]$ and henceforth only deal with $g$ and $h$: \( \exists g \text{ s.t. } g < 2^a r < g + 1 \& \exists h \text{ s.t. } h < 2^a x < h + 1 \) where our integer exponentiation is, by Matijasevich’s famous proof, expressible as a polynomial relation.

The rest of the polynomial can be interpreted as checking that $a, b, c, d \ldots$ record a run of B which stops on “inputs” $g$ and $h$. If B stopped on inputs $x$ and $r$, then B will also stop on some truncation of $x$ and $r$. The integers $b, c, d \ldots$ are bit strings. Substrings of length $n$ represent the state of B. So the first thing to do is stretch $g$ and $h$ out so that their bits are separated by $n - 1$ 0’s. \( \exists i \text{ s.t. } i \) is the stretching of $g$ by a factor of $n$. \( \exists j \text{ s.t. } j \) is the stretching of $h$ by a factor of $n.
Let us immediately prove that these can be coded by polynomials: We only need prove that stretching of finite bit strings can be computed by a Turing Machine; then it can be defined by polynomials. We take from Minsky's paper the result that there is a Turing Machine which turns a bit string $b_5b_4b_3b_2b_1$ into $b_0b_0b_0b_0b_0$. How? Erase the leftmost one, and write it at the same location on a second tape. In this case, that means to turn $b_5b_4b_3b_2b_1$ into $b_0b_0b_0$, and write 10000. The number on the second tape is $2^n$. Minsky's machine $W(2,3)$ turns this into $3^n$, which $W(3,4)$ turns into $2^2n$. Then we write this on a third tape, and start over. When we are done, $b_5b_4b_3b_2b_1$ has become $b_0b_0b_0b_0b_0$.

From here on we will follow Matijasevich and Jones. We need only add a statement to take care of the commands “1: if $x_i = 0$, goto l'. That is, command 1 is active only if command 1 was previously active and $x$ is zero. But now since $x$ is finite and properly spaced, our command takes the form of Matijasevich and Jones: “Command 1 is bitwise dominated by Command 1 minus $x$''

So B halts iff $\exists a \ldots \exists j, a$ through $j$ all integers, s.t. $p(B,a,b,c,\ldots j,n) = 0 \& g < 2^n x < g + 1 \& h < 2^n r < h + 1$. Let $\phi$ be “$p(B,a,b,c,\ldots j,n) = 0 \& g < 2^n x < g + 1 \& h < 2^n r < h + 1$.” So A halts in countable time iff $\exists$ real $x$ s.t. $\forall$ integers $a \ldots j \neg \phi$.

References

[1] Hackleroad Leon, Notre Dame J Formal Logic 31:3 Summer 1990.

[2] Matijasevich and Jones, JSL 49:3 September 1984.

[3] Minsky, Marvin, Annals of Math, Second Series, 74:3 November 1961.

[4] Lurie, Jacob, unpublished manuscript.