Stochastic Variance Reduction via Accelerated Dual Averaging for Finite-Sum Optimization (version 2)*

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June 30, 2020

Abstract

In this paper, we introduce a simplified and unified method for finite-sum convex optimization, named Stochastic Variance Reduction via Accelerated Dual Averaging (SVR-ADA). In the nonstrongly convex and smooth setting, SVR-ADA can attain an $O\left(\frac{1}{n}\right)$-accurate solution in $O\left(n \log \log n\right)$ number of stochastic gradient evaluations, where $n$ is the number of samples; meanwhile, SVR-ADA matches the lower bound of this setting up to a $\log \log n$ factor. In the strongly convex and smooth setting, SVR-ADA matches the lower bound in the regime $n \leq O(\kappa)$ while it improves the rate in the regime $n \gg \kappa$ to $O\left(n \log \log n + \frac{n \log(1/(n\epsilon))}{\log(n/\kappa)}\right)$, where $\kappa$ is the condition number. SVR-ADA improves complexity of the best known methods without use of any additional strategy such as optimal black-box reduction, and it leads to a unified convergence analysis and simplified algorithm for both the nonstrongly convex and strongly convex settings. Through experiments on real datasets, we also show the superior performance of SVR-ADA over existing methods for large-scale machine learning problems.

1 Introduction

In this paper, we study the following composite convex optimization problem:

$$\min_{x \in \mathbb{R}^d} f(x) := g(x) + l(x) := \frac{1}{n} \sum_{i=1}^{n} g_i(x) + l(x),$$

where $g(x)$ is a convex function that is a finite sum of $n$ convex, smooth sample functions $g_i(x)$, and $l(x)$ is convex, probably nonsmooth but admitting an efficient proximal operator. In this paper, we mainly assume that each $g_i(x)$ is $L$-smooth and $l(x)$ is $\sigma$-strongly convex ($\sigma \geq 0$). If $\sigma = 0$, then the problem is nonstrongly convex. If $\sigma > 0$, then the problem is strongly convex and we define the corresponding condition number $\kappa := L/\sigma$. Instances of problem (1) appear widely in statistical learning, operational research, and signal processing. For instance, in machine learning, if $\forall i \in [n], g_i(x) := h_i(\langle \alpha_i, x \rangle)$, where $h_i : \mathbb{R} \to \mathbb{R}$ and $\alpha_i \in \mathbb{R}^d$ is the data vector, then the problem (1) is also called regularized empirical risk minimization (ERM). Important instances of ERM include ridge regression, Lasso, logistic regression, and support vector machine.

*This version was revised after Chaobing Song communicated with Guanghui Lan (george.lan@isye.gatech.edu) and Lin Xiao (Lin.Xiao@microsoft.com). This work was conducted during Chaobing Song’s visit to Professor Yi Ma’s group at UC Berkeley. The work is partially supported by the TBSI program and EECS Startup fund of Professor Yi Ma.
1.1 Context and Motivation

In the large-scale setting where \( n \) is large, first-order methods become the natural choice for solving (1) due to its better scalability. However, when \( n \) is very large, even accessing the full gradient \( \nabla g(x) \) becomes prohibitively expensive. To alleviate this difficulty, a common approach is to use a stochastic gradient \( \nabla g_i(x) \) with \( \mathbb{E}[\nabla g_i(x)] = \nabla g(x) \) to replace the full gradient \( \nabla g(x) \) in each iteration, a.k.a. stochastic gradient descent (SGD). In the stochastic setting, the goal to solve (1) becomes to find an expected \( \epsilon \)-accurate solution \( x \in \mathbb{R}^d \) satisfying \( \mathbb{E}[f(x)] - f(x^*) \leq \epsilon \), where \( x^* \) is an exact minimizer of (1). Typically, the complexity or convergence rate of such an algorithm is evaluated by the number of evaluating stochastic gradients \( \nabla g_i(x) \) \((i \in \{1, 2, \ldots, n\})\) needed to achieve the \( \epsilon \)-accurate solution.

Due to the need of only accessing stochastic gradients, SGD has a low per-iteration cost. However, SGD has a very slow convergence rate due to the constant variance \( \|\nabla g_i(x) - \nabla g(x)\| \). To improve the rate of SGD while still maintaining its low per-iteration cost, a remarkable progress in the past decade is to exploit the finite-sum structure of \( g \) in (1) to reduce the variance of stochastic gradients. In such variance reduction methods, instead of directly using \( \nabla g_i(x) \), we compute a full gradient \( \nabla g(\tilde{x}) \) of an anchor point \( \tilde{x} \) beforehand. Then we use the following variance reduced gradient

\[
\tilde{\nabla} g_i(x) := \nabla g_i(x) - \nabla g_i(\tilde{x}) + \nabla g(\tilde{x})
\]

(2)
as a proxy for the full gradient \( \nabla g(x) \) during each iteration. As a result, the amortized per-iteration cost is still the same as SGD. However, the variance reduced gradient (2) is unbiased and can reduce the variance from \( \|\nabla g_i(x) - \nabla g(x)\| \) to \( \|\nabla g_i(x) - \nabla g_i(\tilde{x})\| \). The variance \( \|\nabla g_i(x) - \nabla g_i(\tilde{x})\| \) can vanish asymptotically, thus the convergence rate of SGD can be substantially improved.

To this end, SAG [RSB12] is one of the first direct\(^{1}\) variance reduction methods to solve (1) while it uses a biased estimation of the full gradient. SVRG [IZ13] directly solves (1) and explicitly uses the unbiased estimation (2) to reduce variance. Then SAGA [DBLJ14] provides an alternative of (2) to avoid precomputing the gradient of an anchor point but with the price of an increased memory cost. Based on [SSZ14], a Catalyst approach [LMH15] is proposed to combine Nesterov’s acceleration into variance reduction methods in a black box manner. [AZ17] has proposed the first direct approach, named Katyusha (a.k.a accelerated SVRG), to combine variance reduction and a kind of Nesterov’s acceleration scheme in a principled manner. [WS16] has given a tight lower complexity bound for finite-sum stochastic optimization and shown the tightness of Katyusha (with black-box reduction [AZH16]) up to a logarithmic factor. MiG [ZSC18] follows and simplifies Katyusha by only using negative momentum to produce acceleration. Varag [LLZ19] improves Katyusha further by considering a unified approach for both the nonstrongly convex and strongly convex settings.

In Table 1, for clarity, we list the state of the art results (as well as results of this paper) for attaining an accuracy \( \epsilon \geq O\left(\frac{L}{n}\right) \). In Table 2, we give the complexity results of representative direct variance reduction methods for both the nonstrongly convex and strongly convex settings (as well as results of this paper). The literature on variance reduction is too rich to list them all here.\(^{2}\) In Table 2, we mainly list the algorithms with improved convergence results for at least one setting.

To understand where we stand with these complexity results, firstly we are particularly interested in attaining a solution with a proper accuracy such as \( \epsilon = O\left(\frac{1}{n}\right) \).\(^{3}\) To attain this accuracy, as shown in Table

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\(^{1}\)For clarification, we say an algorithm is direct if it solves the problem (1) without any reformulation, such as the dual reformulation [SSZ13], primal-dual reformulation [ZX15] or warm restart reformulation [AZH16].

\(^{2}\)For instance, when the objective (1) is strongly convex, we can also use randomized coordinate descent/ascent methods on the dual or primal-dual formulation of (1) to indirectly solve (1), such as SDCA [SSZ13] and Acc-ADCA [SSZ14], APCG [LLX14] and SPDC [ZX15]. Variance reduced methods have also been widely applied into distributed computing [RHS+15, LPLJ17] and nonconvex optimization [RHS+16, RSPS16].

\(^{3}\)This is because, in the context of large-scale statistical learning, due to statistical limits [BB08, SSS08], even under some strong regularity conditions [BB08], obtaining an \( O\left(\frac{1}{n}\right) \) accuracy will be sufficient.
Table 1: Complexity results for solving the problem (1) with accuracy $\epsilon \geq O(L/n)$.

| Algorithm                  | Nonstrongly/Strongly Convex |
|----------------------------|------------------------------|
| SVRG++ [AZY16]             | $O\left(n \log \frac{1}{\epsilon}\right)$ |
| Varag [LLZ19]              | $O\left(n \log \frac{1}{\epsilon}\right)$ |
| SVR-ADA (This Paper)       | $O\left(n \log \log \frac{1}{\epsilon}\right)$ |
| Lower bound [WS16]         | $\Omega(n)$                  |

Table 2: Complexity results for solving the problem (1). (“—” means the corresponding result does not exist or is unknown.)

| Algorithm                  | Nonstrongly Convex | Strongly Convex |
|----------------------------|--------------------|-----------------|
| SAG [RSB12]                | —                  | $O\left((1_{n<\mathcal{O}(\kappa)} \cdot n\kappa + n) \log \frac{1}{\epsilon}\right)$¹ |
| SVRG [JZ13, XZ14]          | —                  | $O\left((n + \kappa) \log \frac{1}{\epsilon}\right)$ |
| SAGA [DBLJ14]              | $\frac{n + \kappa}{\epsilon}$ | $O\left((n + \kappa) \log \frac{1}{\epsilon}\right)$ |
| SVRG++ [AZY16]             | $O\left(n \log \frac{1}{\epsilon} + \frac{L}{\epsilon}\right)$ | — |
| Katyusha¹ [AZ17]           | —                  | $O\left((n + \sqrt{n\kappa}) \log \frac{1}{\epsilon}\right)$ |
| Katyusha² [AZ17]           | $O\left(\frac{n + \sqrt{nL}}{\sqrt{\epsilon}}\right)$ | — |
| Varag [LLZ19]²             | $O\left(n \log n + \frac{\sqrt{nL}}{\sqrt{\epsilon}}\right)$ | $O\left(n \log n + \sqrt{n\kappa} \min\left\{\sqrt{\frac{\epsilon}{2}}, \log \frac{1}{\epsilon}\right\}\right)$ |
| SVR-ADA² (This Paper)      | $O\left(n \log \log n + \frac{\sqrt{\kappa \kappa}}{\sqrt{\epsilon}}\right)$ | $O\left(n \log \log n + \frac{n \log \frac{1}{\epsilon}}{\log(1 + \sqrt{n/\kappa})}\right)$ ³ |
| Lower bound [WS16]         | $\Omega\left(n + \frac{\sqrt{nL}}{\sqrt{\epsilon}}\right)$ | $\Omega\left(n + \sqrt{n\kappa} \log\left(\frac{\epsilon}{2} \sqrt{\frac{\kappa}{\epsilon}}\right)\right)$ |

¹ We use the indicator function $1_E$ with value 1 when the event $E$ is true and 0 when the event $E$ is false.

² For both Varag and SVR-ADA, the complexity results are given for accuracy $\epsilon < O(L/n)$. (For $\epsilon \geq O(L/n)$, see Table 1.)

³ For this setting, the bound $O\left(n + \frac{n \log \frac{1}{\epsilon}}{\log(1 + \sqrt{n/\kappa})}\right)$ is also valid.

1, for both nonstrongly/strongly convex settings, the non-accelerated SVRG++ and accelerated Varag need $O(n \log n)$ number of iterations whereas the lower bound [WS16] implies that we may only need $\Omega(n)$ iterations. Before this work, it is not known whether the logarithmic factor gap can be further improved or not.

As shown in Table 2, in the nonstrongly convex setting, the best-known rate $O\left(n \log n + \frac{\sqrt{nL}}{\sqrt{\epsilon}}\right)$ of direct solvers is given by Varag. In the strongly convex setting, as shown in Table 2, the rate $O\left((n + \sqrt{n\kappa}) \log \frac{1}{\epsilon}\right)$ of Katyusha¹ and Varag is optimal for the case $n \leq O(\kappa)$ according to the lower bound $\Omega\left(n + \sqrt{n\kappa} \log \frac{1}{\epsilon}\right)$ given by [WS16]. However, when $n \gg \kappa$ (which is common in the statistical learning context such as $\kappa = O(\sqrt{n})$ [BE02]), the rate by the accelerated Katyusha¹ and Varag is not better than the non-accelerated methods such as SVRG and SAGA. Before this work, it is not known in the regime $n \gg \kappa$ whether the rate can be further improved or not and whether accelerated variance reduced methods can indeed be better than non-accelerated ones.

⁴The linear convergence result is irrelevant to the problem being strongly convex or not.

⁵The rate is firstly obtained by combining Katyusha¹ with black-box reduction, which is an indirect solver.
1.2 Our Approach and Contributions

We believe that the above issues with Katyusha and Varag are the result of the acceleration strategy that it adopts, which follows the idea of combining gradient descent with mirror descent [AZ17]. The mirror descent step can only exploit the strong convexity in the current iteration and cannot explore the strong convexity along the whole optimization trajectory. As a result, despite such a combination can lead to optimal dependence on $\epsilon$, it leads to suboptimal dependence on the sample size $n$ particularly when $n \gg \kappa$ in the strongly convex setting. Although no improved results are obtained after Katyusha, MiG [ZSC18] simplifies Katyusha by showing that it is enough to lead to acceleration by only using negative momentum and performing a mirror descent step in each iteration.

In this paper, we follow the simple algorithmic framework of MiG [ZSC18] but consider a different acceleration strategy based on Nesterov’s idea of combining gradient descent and dual averaging [Nes15], hence the name stochastic variance reduction via accelerated dual averaging (SVR-ADA). The superiority of dual averaging in exploiting the structure of regularizer has been elaborated in the work [Xia10] (a NeurIPS test of time award paper). By using dual averaging, [Xia10] shows it can accumulate the weight of a sparse regularizer such as $\ell_1$-norm to produce more sparse solutions than the composite mirror descent in [Lan12, DSSST10] in the online optimization context. Here, we use dual averaging to accumulates the weight (a.k.a. the strong convexity constant) of $\ell_2$-norm along the optimization trajectory. As we will show, this can lead to better complexity dependence on the sample size $n$ in the finite-sum optimization context.

To fully exploit the potential of dual averaging, we introduce a generalized estimation sequence for the finite-sum setting. The generalized estimation sequence is concise enough that it helps us introduce a simple but novel initialization strategy. This initialization strategy leads to a superlinear convergence behavior in the first $O(\log \log n)$ epochs and thus practically fills the logarithmic gap in attaining an $O(\frac{L}{n})$-accurate solution. As a result, SVR-ADA improves the state of the art convergence rates in both strongly convex and nonstrongly convex settings.

Efficiency. As shown in Table 1, to attain a solution with $\epsilon \geq O\left(\frac{L}{n}\right)$, we only need at most $O\left(n \log \log \frac{1}{\epsilon}\right)$ number of iterations, while the best known result is $O\left(n \log \frac{1}{\epsilon}\right)$.

In the nonstrongly convex setting, as shown in Table 2, to attain an accuracy $\epsilon < O\left(\frac{L}{n}\right)$, our SVR-ADA method achieves the rate

$$O\left(n \log \log n + \frac{\sqrt{nL}}{\sqrt{\epsilon}}\right),$$

which matches the lower bound up to a log log factor, while the best known result of direct solvers before is $O\left(n \log n + \frac{\sqrt{nL}}{\sqrt{\epsilon}}\right)$, attained by Varag. Practically speaking, the log log factor can be treated as a small constant: for instance when $n \leq 2^{64}$, we have $n \log n \leq 6n$. Thus, for nonstrongly convex problems, SVR-ADA can attain an $O\left(\frac{L}{n}\right)$-accurate solution with essentially $O(n)$ iterations, practically matches the lower bound!

In the strongly convex setting, when $n \leq O(\kappa)$ and $\epsilon < O\left(\frac{L}{n}\right)$, the bound of SVR-ADA becomes

$$O\left(n \log \log n + \sqrt{n\kappa} \log \frac{L}{n\epsilon}\right),$$

which matches the lower bound with the initialized error $\epsilon_0$. Meanwhile, for this setting, if we do not consider the superlinear behavior in the initial stage, then $\forall \epsilon > 0$, the rate of SVR-ADA can also be characterized by $O\left(n + \sqrt{n\kappa} \log \frac{1}{\epsilon}\right)$.

\(^6\) The strong convexity implies that $l(z)$ can be written as a sum of a convex part $l(z) - \frac{\sigma}{2} \|z\|^2$ and $\frac{\sigma}{2} \|z\|^2$. 

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Meanwhile, when $n \gg \kappa$ and $\epsilon < O(\frac{L}{n})$, the rate of SVR-ADA becomes

$$O\left(n \log \log n + \frac{n \log(L/(n\epsilon))}{\log(n/\kappa)}\right). \quad (5)$$

Again, if we do not consider the superlinear behavior, then for $n \gg \kappa$ and $\epsilon > 0$, the bound can also be characterized as $O\left(n + \frac{n \log(1/\epsilon)}{\log(n/\kappa)}\right)$. In this regime, as shown in Table 2, both non-accelerated and accelerated methods have the rate $O(n \log(\log n))$ which has not been improved after SAG [RSB12]. Thus SVR-ADA improves the best known result by a $\log(n/\kappa)$ factor. The $\log(n/\kappa)$ factor will be significant when $n$ is very large and $\kappa$ is small such as $\kappa = O(1)$ or $O(\sqrt{n})$. For the first time, our results show that for the regime $n \gg \kappa$, accelerated methods can be better than non-accelerated methods for finite-sum optimization.

**Simplicity.** We follow the framework of MiG, thus we only need two-point coupling in the inner iteration rather than three-point coupling in Katyusha and Varag. Furthermore, similar to MiG, we only need to keep track of only one variable vector in the inner loop, which gives it a better edge in sparse and asynchronous settings [ZSC18] than Katyusha and Varag. In the nonstrongly convex setting, SVR-ADA is also a direct solver without any extra effort to attain the improved complexity result (3). In the strongly convex setting, SVR-ADA attains the optimal result (4) and the improved result (5) by using a natural uniform average, fixed inner number of iterations and a consistent parameter settings for all the epochs, while Katyusha $^{sc}$ and MiG $^{sc}$ use a weighted average, and Varag uses different parameter settings and inner number of iterations for the first $O(\log(n))$ epochs and the other epochs respectively.

**Unification.** SVR-ADA uses the same parameter setting for both the nonstrongly convex and strongly convex settings. The only difference is that in SVR-ADA, we set the parameter $\sigma = 0$ in the nonstrongly convex setting, while we set $\sigma > 0$ in the strongly convex setting. Meanwhile, based on a “generalized estimation sequence”, we conduct a unified convergence analysis for both settings. The only difference is that the values of two predefined sequences of positive numbers are different. Correspondingly, Katyusha $^{sc}$ and Katyusha $^{nsc}$ (as well as MiG $^{sc}$ and MiG $^{nsc}$) use different parameter settings and independent convergence analysis for both the nonstrongly convex and strongly convex settings. Varag provides a unified approach for both settings. However to adapt to both settings, the parameter settings of Varag are very complicated.

In summary, SVR-ADA improves the state of the art complexity results for finite sum problems with a more simple and unified approach. On one hand, as we have claimed, the significant improvement can be attributed to our discovery of the effectiveness of dual averaging in direct variance reduction methods. On the other hand, it is also from a simplified generalized estimation sequence we use. The simplified estimation sequence can be dated back to the high order paper [GN19] that simplified the original estimation sequence [Nes98] significantly. Then it has been used to obtain a unified high order acceleration framework [SJM19] and improve convergence rates for minimax problems [SJM20]. Finally, it is used to obtain our improved results for the finite sum setting.

### 1.3 Other Related Works

**Regarding the lower bound under sampling with replacement.** When the problem (1) is $\sigma$-strongly convex and $L$-smooth with $\kappa = L/\sigma$, [LZ18a] has provided a stronger lower bound than [WS16] such that to find an $\epsilon$-solution $x$ such that $E[\|x - x^*\|^2] \leq \epsilon$, any randomized incremental gradient methods need at least

$$\Omega\left(\left(n + \sqrt{n\kappa}\right) \log \frac{1}{\epsilon}\right) \quad (6)$$

number of iterations when the dimension $d$ is sufficiently large. Our upper bound (5) is measured by $E[f(x)] - f(x^*)$. By the strong convexity, when $n \gg \kappa$ and $\epsilon \leq O(1/n)$, if we convert to the Euclidean
distance $\mathbb{E}[\|x - x^*\|^2]$, then our rate will be $O\left( n \log \log n + \frac{n \log(\kappa/(n\epsilon))}{\log(n/\kappa)} \right)$. At first sight, when $n \gg \kappa$, our upper bound is actually better than the lower bound (6) by a $\log(n/\kappa)$ factor, which seems rather surprising. Nevertheless, by examining carefully the assumption of [LZ18a] in deriving the lower bound (6), one sees that the randomized incremental gradient methods of [LZ18a] are referred to the ones by sampling with replacement completely. As it turns out, the SVR-ADA algorithm proposed in this paper does not satisfy the assumption of [LZ18a]. SVR-ADA is based on the two-loop structure of SVRG: in the outer loop, we compute the full gradient of an anchor point; in the inner loop, we compute stochastic gradients by sampling with replacement. In the outer loop of SVRG, the step of computing full gradient can be viewed as stochastic gradient steps with 0 step size by (implicitly) sampling without replacement. Thus, SVR-ADA does not fall into the kind of randomized incremental algorithms that [LZ18a] has examined, and the lower bound (6) does not apply to SVR-ADA.

Meanwhile, despite all Katyusha, MiGc (and also Varag [LLZ19], ASVRG [SJZ+ 18]) are accelerated versions of SVRG, they cannot obtain the improved upper bound (5). We believe that it is from the simple fact we have discussed: all of them lead to acceleration by following the style of mirror descent [AZO17], which cannot effectively explore the strong convexity along the optimization path. Thus our improved rate in (5) can be attributed to the combined merits of the (implicitly) sampling without replacement of SVRG and the acceleration effect by following the style of Nesterov’s dual averaging.

Remark 1 (Sampling without Replacement) Very recently, the superiority of sampling without replacement has also been verified theoretically [HS18, GOP19, RGP20, AS20]. Particularly, [AS20] has shown that for strongly convex and smooth finite-sum problems, SGD without replacement (also known as random reshuffling) needs $O(\sqrt{n}/\sqrt{\epsilon})$ number of stochastic gradient evaluations, which is tight and significantly better than the rate $O(1/\epsilon)$ of SGD with replacement [HK14]. Meanwhile, in practice, sampling without replacement is also more widely used in training deep neural network for its better efficiency [Bot09, RR13].

Other Acceleration Variants. Besides accelerated versions of SVRG, there are a randomized primal-dual method RPDG [LZ18a], a randomized gradient extrapolation method RGEM [LZ18b], two accelerated versions Point-SAGA [Def16] and SSNM [Def16] of SAGA, and a unified approach for (random) SVRG/SAGA/SDCA/MISO [KM19]. All these methods match the lower bound (6).

2 Algorithm: Variance Reduction via Accelerated Dual Averaging

Let $[n] := \{1, 2, \ldots, n\}$. For simplicity, we only consider the Euclidean norm $\| \cdot \| \equiv \| \cdot \|_2$. We first introduce a couple of standard assumptions about the smoothness and convexity of the problem (1).

Assumption 1 $\forall i \in [n], g_i(x)$ is convex, i.e., $\forall x, y, g_i(y) \geq g_i(x) + \langle \nabla g_i(x), y - x \rangle$; $g_i(x)$ is $L$-smooth ($L > 0$), i.e., $\| \nabla g_i(y) - \nabla g_i(x) \| \leq L \| y - x \|$. By Assumption 1 and $g(x) = \frac{1}{n} \sum_{i=1}^{n} g_i(x)$, we can verify that $g(x)$ is $L$-smooth, i.e., $\forall x, y, \| \nabla g(y) - \nabla g(x) \| \leq L \| y - x \|$. Furthermore, we assume $l(x)$ satisfies:

Assumption 2 $l(x)$ is $\sigma$-strongly convex ($\sigma \geq 0$), i.e., $\forall x, y$, and $l''(x) \in \partial l(x)$, $l(y) \geq l(x) + \langle l''(x), y - x \rangle + \frac{\sigma}{2} \| y - x \|^2$.

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Footnote: The outer loop of SVRG or our algorithm cannot be interpreted as stochastic gradient steps by sampling with replacement (say, with 0 step size), as we cannot pick all the samples with probability 1 by sampling with replacement for $n$ times.
To realize acceleration with dual averaging, we recursively define the following generalized estimation sequence for the finite-sum problem (1):

\[
\psi_{s,k}(z) := \psi_{s,k-1}(z) + a_s g(y_{s,k}) + \langle \nabla_{s,k}, z - y_{s,k} \rangle + l(z),
\]

with the initialization \(\psi_{1,0}(z) := \frac{1}{2}\|z - x_0\|^2\), \(\psi_{2,0} := m\psi_{1,1}\) with \(m \in \mathbb{Z}_+\), \(k \in \{1, 2, \ldots, m\}\), and \(\psi_{s+1,0} := \psi_{s,m}\) (for \(s \geq 2\)), where \(\{a_s\}\) is a sequence of positive numbers to be specified later. Here \(\{y_{s,k}\}\) is a sequence of vectors that will be generated by our algorithm, \(\{\nabla_{s,k}\}\) is a sequence of variance reduced stochastic gradients evaluated at \(\{y_{s,k}\}\). If \(m = 1\) and \(\nabla_{s,k} = \nabla g(y_{s,k})\), then we can verify that (7) is equivalent to the classical definition of estimation sequence by Nesterov [Nes15]. In the finite-sum setting, we set \(m = O(n)\) to amortize the computational cost per epoch \(s\), where \(n\) is the number of sample functions in (1). Then we say \(\psi_{s,k}\) is the estimation sequence in the (inner) \(k\)-th iteration of the \(s\)-th epoch.

For convenience, we define \(A_s := A_{s-1} + a_s\) with \(A_0 := 0\). Then Algorithm 1 summarizes the proposed Stochastic Variance Reduced via Accelerated Dual Averaging (SVR-ADA) method. As one may see, besides the steps about updating the estimation sequence such as Steps 2-4, 6 and 12, Algorithm 1 mainly follows the framework of the simplified MiG [ZSC18] of Katyusha. The main differences are that we have novel but effective initialization steps in Steps 3-4 and replace the mirror descent step in MiG with the Step 12, a dual averaging step:

\[
z_{s,k} := \arg\min_z \psi_{s,k}(z).
\]

To be self-contained, note that we compute the full gradient on the anchor point in Step 7 and compute the variance reduced stochastic gradient \(\tilde{\nabla}_{s,k}\) in Steps 10 and 11. The convex combination Step 9 and the dual averaging Step 12 are used to achieve acceleration. The settings for \(\tilde{x}_s\), \(z_{s,0}\) and \(\psi_{s+1,0}\) in Step 14 are derived from our analysis. Notice that we update \(\bar{x}_s\) as a natural uniform average with respect to \(\{z_{s,k}\}\) for both the nonstrongly convex and strongly convex settings, while Katyusha (as well as MiG) uses a weighted average for the strongly convex setting and a uniform average for the nonstrongly convex setting.

In our algorithm, a key new setting is \(\psi_{2,0}\) being initialized as the \(m\) times of \(\psi_{1,1}\), which, as we will see in our proof, helps cancel the term \(\frac{A_s}{\sqrt{\epsilon}}\) in the complexity result of Katyusha\textasciicircum{asc} completely. For \(s \geq 2\), from the definition of \(\psi_{s,k}\) in (7) and \(\psi_{s+1,0} := \psi_{s,m}\), one can show that the sequence \(\psi_{s,k}(z)\) is at least \(m(1 + \sigma \sum_{i=0}^{s-1} a_i)\)-strongly convex. As we will see in the proof, this accumulation of strong convexity is crucial for our algorithm to achieve better convergence than the non-accelerated ones such as SVRG and SAGA even in the regime \(n \gg \kappa\).

To be more precise, under these settings, we can prove (in Section 3 and the appendix) the following convergence guarantee for the proposed SVR-ADA algorithm:

**Theorem 1** Let \(\{\bar{x}_s\}\) be generated by Algorithm 1. Under Assumptions 1 and 2, and taking expectation on the randomness of all the history, we have \(\forall s \geq 2\),

\[
\mathbb{E}[f(\bar{x}_s)] - f(x^*) \leq \frac{\|\bar{x}_0 - x^*\|^2}{2A_s},
\]

where \(\forall s \geq 2\),

\[
A_s \geq \max \left\{ \frac{m}{2L} \left( \frac{2}{\epsilon} \right)^{2^{-(s-1)}} \frac{1}{L} \left(1 + \sqrt{\frac{3m}{2L}} \right)^{s-1} \right\}
\]

and with \(s_0 = 1 + \lceil \log_2 \log_2 (m/2) \rceil\), besides the lower bounds in (10), we also have \(\forall s \geq s_0\),

\[
A_s \geq \max \left\{ \frac{m}{32L} \left( s - s_0 + 2\sqrt{2} \right)^2, \frac{m}{4L} \left(1 + \sqrt{\frac{3m}{2L}} \right)^{s-s_0} \right\}.
\]

As we will see, \(m\) denotes the number of inner iterations in our algorithm.

see proof for details.
Algorithm 1 Stochastic Variance Reduction via Accelerated Dual Averaging (SVR-ADA)

1: **Problem:** $\min_{x \in \mathbb{R}^d} f(x) = g(x) + l(x) = \frac{1}{n} \sum_{i=1}^{n} g_i(x) + l(x)$.
2: **Initialization:** $A_0 = 0, A_1 = a_1 = \frac{L}{2}, y_{1,1} = z_{1,0} = x_0 \in \mathbb{R}^d, \psi_{1,0}(z) = \frac{1}{2} \|z - \bar{x}_0\|^2$
3: $z_{1,1} = \arg\min_z \{ \psi_{1,1}(z) := \psi_{1,0}(z) + a_1 (g(y_{1,1}) + \langle \nabla g(y_{1,1}), z - y_{1,1} \rangle + l(z)) \}$
4: $\tilde{x}_1 = z_{1,1}, z_{2,0} = z_{1,1}, \psi_{2,0} = m\psi_{1,1}$
5: **for** $s = 2, \ldots, S$ **do**
6: $A_s = A_{s-1} + \sqrt{\frac{m A_{s-1} (1+\sigma A_{s-1})}{2L}}$ and $a_s = A_s - A_{s-1}$
7: $\mu_{s-1} = \nabla g(\tilde{x}_{s-1})$
8: **for** $k = 1, 2, \ldots, m$ **do**
9: $y_{s,k} = A_s^{-1} \tilde{x}_{s-1} + \frac{a_s}{A_s} z_{s,k-1}$
10: Sample $i$ from $\{1, 2, \ldots, n\}$ uniformly at random.
11: $\tilde{z}_{s,k} = \nabla g_i(y_{s,k}) - \nabla g_i(\tilde{x}_{s-1}) + \mu_{s-1}$
12: $a_{s,k} = \arg\min_{z} \{ \psi_{s,k}(z) := \psi_{s,k-1}(z) + a_{s}(g(y_{s,k}) + \langle \nabla g(y_{s,k}), z - y_{s,k} \rangle + l(z)) \}$
13: **end for**
14: $\tilde{x}_s = \frac{A_{s-1}}{A_s} \tilde{x}_{s-1} + \frac{a_s}{m A_s} \sum_{k=1}^{m} z_{s,k}, \quad z_{s+1,0} = z_{s,m}, \quad \psi_{s+1,0} = \psi_{s,m}$
15: **end for**
16: **return:** $\tilde{x}_S$

Theorem 1 gives a unified convergence result for both the nonstrongly convex $\sigma = 0$ and the strongly convex $\sigma > 0$ settings. As we see in (9), the estimated error in the function value is simply bounded by the term about $\|\tilde{x}_0 - x^*\|^2$.

To see implications of Theorem 1 in the complexity of algorithm, by the first term in (10), whether strongly convex or not, SVR-ADA can attain an $O\left(\frac{1}{\kappa}\right)$-accurate solution in $1 + \lceil \log_2 \log_2 (m/\kappa) \rceil$ number of epochs and thus $A_{s_0} = \frac{m}{2\kappa}$. The superlinear phenomenon is by our novel initialization $\psi_{2,0} := m\psi_{1,1}$ in Step 4 of Algorithm 1. The best known convergence rate in the initial stage is by Varag [LLZ19], which has shown a linear convergence rate in the initial stage for convex-finite sums whether strongly convex or not. In contrast, the corresponding rate of SVR-ADA is superlinear. To the best of our knowledge, we have not observed any theoretical justification of superlinear phenomenon for variance reduced first order methods.

By the second term in (10), in the strongly convex setting $\sigma > 0$, we can have an accelerated linear convergence rate from the start. Note that whatever $m \leq O(\kappa)$ or $m \gg \kappa$, the contracting ratio of SVR-ADA will always be $\left(1 + \sqrt{\frac{a_s \sigma}{2\kappa}}\right)^{-1}$, which will tend to 0 as $m \to +\infty$. However, for all the existing variance reduction methods such as Katyusha$^{ac}$ and Varag, when $m \gg \kappa$, the contracting ratio will be at least a constant such as $\frac{3}{2}$ in Katyusha$^{ac}$.

Then based on the prompt decrease in the superlinear initial stage, we also provide two new lower bounds for $A_s$ in (11). By the first term in (11), whether strongly convex or not, SVR-ADA can have at least an accelerated sublinear rate. By the second term in (11), in the strongly convex setting $\sigma > 0$, SVR-ADA will maintain an accelerated linear rate.

Thus by Theorem 1, by setting $m = O(n)$, we obtain our improved convergence rates for both the nonstrongly convex and strongly convex settings in Table 2.

The generalized estimation sequence $\{\psi_{s,k}\}$ and the associated analysis is the key in proving our main result Theorem 1, while it is commonly known to be difficult to understand. So [AZO17] has proposed a linear coupling of gradient descent and mirror descent instead. Meanwhile, there are plenty of study on understanding acceleration from discretization of continuous time dynamics [SBC14]. The estimation sequence itself also has principled explanation from a continuous time perspective [DO19, SJM19]. In Section 3, we show that the estimation sequence analysis leads to a very concise, unified, and principled convergence analysis for both the strongly convex and nonstrongly convex settings.
3 Convergence Analysis

When using estimation sequence to prove convergence rate, the main task is to given the lower bound and upper bound of \( \psi_{s,k}(z_{s,k}) \), where \( z_{s,k} = \arg \min_z \psi_{s,k}(z) \). The lower bound is given in terms of the objective value at the current iterate and the estimation sequence in the previous iteration, while the upper bound is in terms of the objective value at the optimal solution. (For simplicity, we only need to give the upper bound of \( \psi_{s,m}(z_{s,m}) \).) Then by telescoping and concatenating the lower bound and upper bound of \( \psi_{s,k}(z_{s,k}) \), we prove the rate in terms of the objective difference \( f(\bar{x}_s) - f(x^*) \) (in expectation).

First, in the initial Step 3 of Algorithm 1, by the smoothness property of \( g(z) \) and the setting \( A_1 = a_1 = \frac{1}{L} \), we have Lemma 1.

**Lemma 1 (The initial step)** It follows that
\[
\psi_{1,1}(z_{1,1}) \geq A_1 f(\bar{x}_1).
\] (12)

**Proof.** See Section A.

Lemma 1 will be used to cancel the error introduced by \( f(\bar{x}_1) \) in the following iterations. After entering into the main loop, by using the smoothness and convexity property, and the careful setting of \( \{a_s\} \) and \( \{A_s\} \), we obtain the lower bound of \( \psi_{s,k}(z_{s,k}) \) in Lemma 2.

**Lemma 2 (Lower bound)** \( \forall s \geq 2, k \geq 1 \), we have
\[
\psi_{s,k}(z_{s,k})
\geq \psi_{s,k-1}(z_{s,k-1}) + a_s g(y_{s,k}) + A_s (f(y_{s,k+1}) - g(y_{s,k}) - \frac{A_{s-1}}{2A_s L} \| \nabla s,k - \nabla g(y_{s,k}) \|^2)
- A_{s-1} (\nabla s,k, \bar{x}_{s-1} - y_{s,k}) - A_{s-1} l(\bar{x}_{s-1}).
\] (13)

**Proof.** See Section B.

In Lemma 2, the term \( \| \nabla s,k - \nabla g(y_{s,k}) \|^2 \) is the variance we need to bound, of which the bound is given in Lemma 3 based on the standard derivation in [AZ17].

**Lemma 3 (Variance reduction)** \( \forall s \geq 2, k \geq 1 \), taking expectation on the randomness over the choice of \( i \) in the \( k \)-th iteration of \( s \)-th epoch, we have
\[
\mathbb{E}[\| \nabla s,k - \nabla g(y_{s,k}) \|^2] \leq 2L(g(\bar{x}_{s-1}) - g(y_{s,k}) - (\nabla g(y_{s,k}), \bar{x}_{s-1} - y_{s,k})).
\] (14)

**Proof.** See Section C.

Then by combining Lemma 2 and Lemma 3, we will find that the inner product in (13) and (14) can be canceled with each other in expectation. Therefore after combining Lemma 2 and Lemma 3, by telescoping the resulted inequality from \( k = 1 \) to \( m \) and using the definition \( \psi_{s+1,0} := \psi_{s,m} \), we have Lemma 4.

**Lemma 4 (Recursion)** \( \forall s \geq 2 \), taking expectation on the randomness over the epoch \( s \), it follows that
\[
\mathbb{E}[\psi_{s+1,0}(z_{s+1,0}) - \psi_{s,0}(z_{s,0})] \geq \mathbb{E}\left[mA_s f(\bar{x}_s) - mA_{s-1} f(\bar{x}_{s-1})\right],
\] (15)

**Proof.** See Section D.

Besides the lower bound (15), by the convexity of \( f(x) \) and optimality of \( z_{s,m} \), we can also provide the upper bound in Lemma 5.

**Lemma 5 (Upper bound)** \( \forall s \geq 2 \), taking expectation on all the history, we have
\[
\mathbb{E}[\psi_{s,m}(z_{s,m})] \leq mA_s f(x^*) + \frac{m}{2} \| \bar{x}_0 - x^* \|^2.
\] (16)

**Proof.** See Section E.

Finally, by combining Lemmas 1, 4 and 5, we prove Theorem 1 as follows.
Proof of Theorem 1. Proof. Taking expectation on the randomness of all the history and telescoping (15) from 2 to \( s(s \geq 2) \), we have
\[
\mathbb{E}[\psi_{s+1,0}(z_{s+1,0}) - \psi_{2,0}(z_{2,0})] \geq \mathbb{E}\left[mA_s f(\bar{x}_s) - mA_1 f(\bar{x}_1)\right].
\] (17)

Meanwhile, by Lemma 1 and the setting \( z_{2,0} = z_{1,1}, \psi_{2,0} = m\psi_{1,1} \), we have
\[
\psi_{2,0}(z_{2,0}) = m\psi_{1,1}(z_{2,0}) = m\psi_{1,1}(z_{1,1}) \geq mA_1 f(\bar{x}_1).
\] (18)

So combining (17) and (18), and by the setting \( \psi_s(1,0) = \psi_{s,m}(z_{s,m})(s \geq 2) \), we have
\[
\mathbb{E}[\psi_{s,m}(z_{s,m})] = \mathbb{E}[\psi_{s+1,0}(z_{s+1,0})] \geq mA_s f(\bar{x}_s).
\] (19)

Then combining Lemma 5 and (19), we have
\[
\mathbb{E}[mA_s f(\bar{x}_s)] \leq \mathbb{E}[\psi_{s,m}(z_{s,m})] \leq mA_s f(x^*) + \frac{m}{2} \| \bar{x}_0 - x^* \|^2.
\] (20)

So after simple rearrangement of (20), we obtain (9).

Then we give the lower bound of \( A_s \) by the condition in Step 6 of Algorithm 1 and \( A_1 = a_1 = \frac{1}{L} \). To show the lower bound by the first term in (10), we know that
\[
A_s = A_{s-1} + \sqrt{\frac{mA_{s-1}(1 + \sigma A_{s-1})}{2L}} \geq \sqrt{\frac{mA_{s-1}(1 + \sigma A_{s-1})}{2L}} \geq \sqrt{\frac{mA_{s-1}}{2L}},
\] (21)

so we have
\[
\frac{2LA_s}{m} \geq \left( \frac{2LA_{s-1}}{m} \right)^{1/2} \geq \left( \frac{2LA_1}{m} \right)^{2^{-(s-1)}}.
\] (22)

Then by the setting \( A_1 = \frac{1}{L} \), we have
\[
A_s \geq \frac{m}{2L} \left( \frac{2}{m} \right)^{2^{-(s-1)}}.
\] (23)

Meanwhile, for \( s \geq 2 \), we also have
\[
A_s \geq A_{s-1} + \sqrt{\frac{mA_{s-1}(1 + \sigma A_{s-1})}{2L}} \geq A_{s-1} + \sqrt{\frac{mA_{s-1}}{2L}} A_{s-1} = \left( 1 + \sqrt{\frac{m\sigma}{2L}} \right) A_{s-1}
\]
\[
\geq \left( 1 + \sqrt{\frac{m\sigma}{2L}} \right)^{s-1} A_1
\]
\[
\geq \frac{1}{L} \left( 1 + \sqrt{\frac{m\sigma}{2L}} \right)^{s-1}.
\] (24)

Thus the lower bounds in (10) are proved. Then with \( s_0 = 1 + \lceil \log_2 \log_2(m/2) \rceil \), we have
\[
A_{s_0} \geq m \left( \frac{2}{m} \right)^{2^{-(s_0-1)}} \geq m \left( \frac{2}{m} \right)^{2^{-(\log_2 \log_2(m/2))}} \geq m \left( \frac{2}{m} \right)^{2^{-\log_2 \log_2(m/2)}}
\]
\[
= \frac{m}{4L}.
\]

Meanwhile for \( s \geq s_0 + 1 \), we have
\[
A_s \geq A_{s-1} + \sqrt{\frac{mA_{s-1}(1 + \sigma A_{s-1})}{2L}} \geq A_{s-1} + \sqrt{\frac{mA_{s-1}}{2L}}.
\] (25)
Thus we can use the mathematical induction method to prove the first lower bound in (11): \( \forall s \geq s_0, A_s \geq \frac{m}{32L} \left( s - s_0 + 2\sqrt{2} \right)^2 \).

Firstly, for \( s = s_0 \), we have \( A_s \geq \frac{m}{4L} = \frac{m}{32L} \left( 2\sqrt{2} \right)^2 \).

Then assume that for a \( s \geq s_0 + 1 \), \( A_{s-1} \geq \frac{m}{32L} \left( s - s_0 + 2\sqrt{2} \right)^2 \), then

\[
A_s \geq A_{s-1} + \sqrt{\frac{m(A_{s-1})}{2L}} \geq \frac{m}{32L} \left( s - s_0 + 2\sqrt{2} \right)^2 + \frac{m}{16L} (s - s_0) + \frac{m}{32L} (2b + 1)
\]

\[
\geq \frac{m}{32L} \left( s - s_0 + 2\sqrt{2} \right)^2.
\]

Thus the first lower bound in (11) is proved.

Meanwhile, for \( s \geq s_0 + 1 \), we also have

\[
A_s \geq A_{s-1} + \sqrt{\frac{mA_{s-1}(1 + \sigma A_{s-1})}{2L}} \geq A_{s-1} + \sqrt{\frac{m\sigma}{2L} A_{s-1}} = \left( 1 + \sqrt{\frac{m\sigma}{2L}} \right) A_{s-1}
\]

\[
\geq \left( 1 + \sqrt{\frac{m\sigma}{2L}} \right)^{s-s_0} A_{s_0}
\]

\[
\geq \frac{m}{4L} \left( 1 + \sqrt{\frac{m\sigma}{2L}} \right)^{s-s_0}.
\]

Thus the second lower bound in (11) is proved.

\[\blacksquare\]

### 4 Experiments

In this section, to verify the theoretical results and show the empirical performance of the proposed SVR-ADA method, we conduct numerical experiments on large-scale datasets in machine learning. The datasets we use are a9a, covtype, mnist and cifar10, downloaded from the LIBSVM website\cite{ASZ17}. The a9a and covtype datasets are for binary classification, while mnist and cifar10 are for multi-class classification. To make comparison easier, we normalize the Euclidean norm of each data vector in the datasets to be 1. The problem we solve is the \( \ell_2 \)-norm regularized (multinomial) logistic regression problem:

\[
\min_{w \in \mathbb{R}^{d \times (c-1)}} f(w) := \frac{1}{n} \sum_{j=1}^{n} \left( - \sum_{i=1}^{c-1} y^{(i)}_j \, w^{(i)}^T \, x_j + \log \left( 1 + \sum_{i=1}^{c-1} \exp (w^{(i)}^T \, x_j) \right) \right) + \frac{\lambda}{2} \sum_{i=1}^{c-1} \| w^{(i)} \|_2^2,
\]

where \( n \) is the number of samples, \( c \in \{2, 3, \ldots\} \) denotes the number of class (for a9a and covtype, \( c = 2 \); for mnist and cifar10, \( c = 10 \)). \( \lambda \) denotes the regularization parameter, \( y_j = (y^{(1)}_j, y^{(2)}_j, \ldots, y^{(c-1)}_j)^T \) is a one-hot vector or zero vector\cite{ZSC18}, and \( w := (w^{(1)}, w^{(2)}, \ldots, w^{(c-1)}) \in \mathbb{R}^{d \times (c-1)} \) denotes the variable to optimize. For the two-class datasets “a9a” and “covtype”, we present our results by choosing the regularization parameter \( \lambda \in \{0, 10^{-8}, 10^{-4}\} \). For the ten-class datasets “mnist” and “cifar10”, we choose \( \lambda \in \{0, 10^{-6}, 10^{-3}\} \). For \( \lambda = 0 \), the corresponding problem is unregularized and thus nonstrongly convex. For this setting, we compare SVR-ADA with the state-of-the-art variance reduction methods SVRG [JZ13], Katyusha\textsuperscript{sc} [AZ17], and MiG\textsuperscript{sc} [ZSC18]. The settings \( \lambda \in \{10^{-8}, 10^{-4}\} \) for a9a and covtype (and \( \lambda \in \{10^{-6}, 10^{-3}\} \) for mnist and cifar10) correspond to the strongly convex setting with a large condition number \( \kappa \) and that with a small condition number \( \kappa \), respectively. For both settings, we compare SVR-ADA with SVRG, Katyusha\textsuperscript{sc} and MiG\textsuperscript{sc}. 

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The horizontal axis is the number of passes through the entire dataset, and the vertical axis is the optimality gap $f(x) - f(x^*)$.

All four algorithms we compare have a similar outer-inner structure, where we set all the number of iterations as $m = 2n$. For these algorithms, the common parameter to tune is the parameter $w.r.t.$ Lipschitz constant\(^{12}\), which is tuned in $\{0.0125, 0.025, 0.05, 0.1, 0.25, 0.5\}$.\(^{13}\) Following the tradition of ERM experiments, we use the number of passes of the entire dataset as the x-axis. All four algorithms are implemented in C++ under the same framework, while the figures are produced using Python.

For the datasets a9a and covtype, as shown in Figure 1, when $\lambda = 0$, SVR-ADA decreases the error promptly in the initial stage, which validates our theoretical result in attaining an $O(1/n)$-accurate solution with $\log \log n$ passes of the entire dataset. An interesting phenomenon is that the other variance reduction methods share the same behavior with SVR-ADA in empirical evaluations (in fact, MiG$^{nsc}$ is slightly faster for both a9a and covtype datasets). This poses an open problem whether or not this superlinear phenomenon in the initial stage can be theoretically justified for SVRG, Katyusha, and MiG.

As shown in Figure 1, when $\lambda = 10^{-8}$, i.e., the large condition number setting, SVR-ADA has significantly better performance than Katyusha$^{sc}$ and MiG$^{sc}$. This is partly due to the fact that the accumulation of strong convexity by dual averaging helps us better cancel the error from the randomness and allows SVR-ADA to choose a more aggressive parameter $w.r.t.$ Lipschitz constant. When $\lambda = 10^{-4}$, i.e., the small condition number setting, as shown in Figure 1, SVR-ADA is significantly better than Katyusha$^{sc}$ and MiG$^{sc}$, which validates our superior theoretical results in (5). Meanwhile, although with less theoretical guarantee, when $\lambda = 10^{-4}$, SVRG can be competitive with SVR-ADA, which inspires us to conduct in the future a tighter convergence analysis for non-accelerated methods in the small condition number setting.

As we see, despite there are some minor differences among different tasks/datasets shown in Figure 1 and Figure 2, the general behaviors are still very consistent. From both figures, our method SVR-ADA achieves state of the art and is much faster than the non-accelerated SVRG algorithm in the nonstrongly convex setting.

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\(^{10}\)The dataset url is https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/.

\(^{11}\)Zero vector denotes the class of the $j$-th sample is $c$.

\(^{12}\)For logistic regression with normalized data, the Lipschitz constant is globally upper bounded [ZX15] by $1/4$, but in practice we can use a smaller one than $1/4$.

\(^{13}\)In our experiments, due to the normalization of datasets, all the four algorithms will diverge when the parameter is less than 0.0125. Otherwise, they always converge if the parameter is less than 0.5.
and the strongly convex setting with a large conditional number. Meanwhile, in the strongly convex setting with a small condition number, SVR-ADA is still competitive with the non-accelerated SVRG algorithm and much faster than the other two accelerated algorithms of Katyusha and MiG.

5 Conclusion and Discussion

In this work, for the finite-sum convex optimization problem, we propose to combine accelerated dual averaging with stochastic variance reduction for designing more efficient algorithms. The complexity of our algorithm practically matches the theoretical lower bounds for both settings, and leads to significant improvements in performance guarantees over existing methods. It is somewhat surprising that with such a simple and unified approach, the convergence rates can be further improved for the well-studied convex finite-sum optimization problem. As a result, we believe that the general approach of this paper may have potential extensions to nonconvex optimization, distributed settings and minimax problems.

Acknowledgements

Chaobing would like to thank Yaodong Yu for pointing out the closely related recent work of [LLZ19].

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A  Proof of Lemma 1

Proof. It follows that

\[
\psi_{1,1}(z_{1,1}) \overset{(a)}{=} \psi_{1,0}(z_{1,1}) + a_1(g(y_{1,1}) + \langle \nabla g(y_{1,1}), z_{1,1} - y_{1,1} \rangle + l(z_{1,1}))
\]
\[
\overset{(b)}{=} \frac{1}{2} \|z_{1,1} - z_{1,0}\|^2 + a_1(g(y_{1,1}) + \langle \nabla g(y_{1,1}), z_{1,1} - y_{1,1} \rangle + l(z_{1,1}))
\]
\[
\overset{(c)}{=} a_1(g(y_{1,1}) + \langle \nabla g(y_{1,1}), z_{1,1} - y_{1,1} \rangle + \frac{1}{2a_1}\|z_{1,1} - y_{1,1}\|^2 + l(z_{1,1}))
\]
\[
\overset{(d)}{=} a_1(g(y_{1,1}) + \langle \nabla g(y_{1,1}), z_{1,1} - y_{1,1} \rangle + \frac{L}{2}\|z_{1,1} - y_{1,1}\|^2 + l(z_{1,1}))
\]
\[
\overset{(e)}{\geq} a_1(g(z_{1,1}) + l(z_{1,1}))
\]
\[
\overset{(f)}{=} A_1 f(\tilde{x}_1),
\]

where (a) is by definition of \(\psi_{1,1}\), (b) is by the definition of \(\psi_{1,0}\) and \(z_{1,0} = \tilde{x}_0\), (c) is by the setting \(y_{1,1} = z_{1,0}\) and simple rearrangement, (d) is by the setting \(a_1 = \frac{1}{L}\), (e) is by Lemma 6, and (f) is by the setting \(A_1 = a_1\) and \(\tilde{x}_1 = z_{1,1}\).

B  Proof of Lemma 2

Proof. As \(l(z)\) is \(\sigma\)-strongly convex, by the definition of the sequence \(\{\psi_{s,k}(z)\}\), \(\psi_{s-1,m}(z)\) is \(m + \sigma m \sum_{s=1}^{s-1} a_i = m(1 + \sigma A_{s-1})\)-strongly convex. Furthermore, we also know that \(\psi_{s,k}(z)(k \geq 0)\) is also at least \((m + \sigma m A_{s-1})\)-strongly convex. So it follows that

\[
\psi_{s,k}(z_{s,k}) \overset{(a)}{=} \psi_{s,k-1}(z_{s,k}) + a_s(g(y_{s,k}) + \langle \nabla g(y_{s,k}), z_{s,k} - y_{s,k} \rangle + l(z_{s,k}))
\]
\[
\overset{(b)}{\geq} \psi_{s,k-1}(z_{s,k-1}) + \frac{m(1 + \sigma A_{s-1})}{2} \|z_{s,k} - z_{s,k-1}\|^2
\]
\[
+ a_s(g(y_{s,k}) + \langle \nabla g(y_{s,k}), z_{s,k} - y_{s,k} \rangle + l(z_{s,k})),
\]

where (a) is by the definition of \(\psi_{s,k}\) and (b) is by the optimality condition of \(z_{s,k-1}\) and the \((m + \sigma m A_{s-1})\)-strong convexity of \(\psi_{s,k-1}\). Then we have

\[
\overset{(a)}{=} a_s(g(y_{s,k}) + A_s\left(\langle \nabla g(y_{s,k}), z_{s,k} - y_{s,k} + \frac{A_{s-1}}{A_s} \tilde{x}_{s-1}\rangle \right)
\]
\[
- A_{s-1}\langle \nabla g(y_{s,k}), \tilde{x}_{s-1} - y_{s,k} \rangle + a_s l(z_{s,k})
\]
\[
\overset{(b)}{\geq} a_s(g(y_{s,k}) + A_s\left(\langle \nabla g(y_{s,k}), y_{s,k+1} - y_{s,k} \rangle - A_{s-1}\langle \nabla g(y_{s,k}), \tilde{x}_{s-1} - y_{s,k} \rangle \right)
\]
\[
+ A_s l(y_{s,k+1}) - A_{s-1} l(\tilde{x}_{s-1})
\]

where (a) is by the fact that \(A_s = A_{s-1} + a_s\) and simple rearrangement and (b) is by \(y_{s,k+1} = \frac{A_{s-1}}{A_s} \tilde{x}_{s-1} + \frac{a_s}{A_s} z_{s,k}\) (which is by our definition of the sequence \(\{y_{s,k}\}\)) and the convexity of \(l(z)\).

Meanwhile, by our setting in Step 5 of Algorithm 1, \(A_s = A_{s-1} + \sqrt{m A_{s-1}(1 + \sigma A_{s-1})} + \frac{2A_s}{A_{s-1}} L \geq \left(1 + \frac{A_s}{A_{s-1}}\right)L\).
Then by combining (29) and (30), it follows that
\[
\psi_{s,k}(z_{s,k}) - \psi_{s,k-1}(z_{s,k-1}) \\
\geq a_s g(y_{s,k}) + A_s \left( \nabla s_{s,k} ; y_{s,k+1} - y_{s,k} \right) - A_{s-1} \left( \nabla s_{s,k} ; \bar{x}_{s-1} - y_{s,k} \right) \\
+ A_s l(y_{s,k+1}) - A_{s-1} l(\bar{x}_{s-1}) + \frac{m(1 + \sigma A_{s-1})}{2} \| z_{s,k} - z_{s,k-1} \|^2
\]
\[
\geq a_s g(y_{s,k}) + A_s \left( \nabla s_{s,k} ; y_{s,k+1} - y_{s,k} \right) - A_{s-1} \left( \nabla s_{s,k} ; \bar{x}_{s-1} - y_{s,k} \right) \\
+ A_s l(y_{s,k+1}) - A_{s-1} l(\bar{x}_{s-1}) + \frac{m A_s^2 (1 + \sigma A_{s-1})}{2} \| y_{s,k+1} - y_{s,k} \|^2
\]
\[
(a) \quad \text{is by (31). Then we have}
\]
\[
\left\langle \nabla s_{s,k} , y_{s,k+1} - y_{s,k} \right\rangle + \left( 1 + \frac{A_s}{A_{s-1}} \right) \frac{L}{2} \| y_{s,k+1} - y_{s,k} \|^2 + l(y_{s,k+1}) \\
\geq g(y_{s,k+1}) - g(y_{s,k}) + l(y_{s,k+1}) - A_{s-1} \frac{A_s}{2A_s L} \| \nabla s_{s,k} - \nabla g(y_{s,k}) \|^2
\]
\[
= f(y_{s,k+1}) - g(y_{s,k}) - A_{s-1} \frac{A_s}{2A_s L} \| \nabla s_{s,k} - \nabla g(y_{s,k}) \|^2
\]
(32)
where (a) is by Lemma 6 and the Young’s inequality \( \langle a, b \rangle \geq -\frac{1}{2} \| a \|^2 - \frac{1}{2} \| b \|^2 \). So we have
\[
\psi_{s,k}(z_{s,k}) - \psi_{s,k-1}(z_{s,k-1}) \\
\geq a_s g(y_{s,k}) + A_s \left( f(y_{s,k+1}) - g(y_{s,k}) - \frac{A_{s-1}}{2A_s L} \| \nabla s_{s,k} - \nabla g(y_{s,k}) \|^2 \right) \\
- A_{s-1} \left( \nabla s_{s,k} , \bar{x}_{s-1} - y_{s,k} \right) - A_{s-1} l(\bar{x}_{s-1}).
\]
C Proof of Lemma 3

Proof. Taking expectation on the randomness over the choice of $i$, we have

$$
\mathbb{E}[\|\nabla_{s,k} - \nabla g(y_{s,k})\|^2] = \mathbb{E}[\|\nabla g_i(y_{s,k}) - \nabla g_i(\bar{x}_{s-1}) + \mu_{s-1} - \nabla g(y_{s,k})\|^2]
$$

Taking expectation on the randomness over the choice of $i$. Taking expectation on the randomness over the choice of $i$.

$$
\leq \mathbb{E}[\|\nabla g_i(y_{s,k}) - \nabla g_i(\bar{x}_{s-1})\|^2] - \mathbb{E}[\|g(\bar{x}_{s-1}) - \nabla g(y_{s,k})\|^2]
$$

is by Lemma 6.

D Proof of Lemma 4

Proof. By Lemma 2 and taking expectation on the randomness over the choice of $i$, we have

$$
\mathbb{E}[\psi_{s,k}(z_{s,k}) - \psi_{s,k-1}(z_{s,k-1})]
$$

$$
\geq a_s g(y_{s,k}) + A_s \left( f(y_{s,k+1}) - g(y_{s,k}) - \frac{A_{s-1}}{2A_s} \|\nabla_{s,k} - \nabla g(y_{s,k})\|^2 \right)
$$

$$
- A_{s-1} \langle \nabla_{s,k}, \bar{x}_{s-1} - y_{s,k} \rangle - A_{s-1}l(\bar{x}_{s-1})
$$

$$
\geq \mathbb{E} \left[ a_s g(y_{s,k}) \right]
$$

$$
+ A_s \left( f(y_{s,k+1}) - g(y_{s,k}) - A_{s-1}(g(\bar{x}_{s-1}) - g(y_{s,k}) - \langle \nabla g(y_{s,k}), \bar{x}_{s-1} - y_{s,k} \rangle) \right)
$$

$$
- A_{s-1} \langle \nabla_{s,k}, \bar{x}_{s-1} - y_{s,k} \rangle - A_{s-1}l(\bar{x}_{s-1})
$$

$$
\geq \mathbb{E} \left[ A_s f(y_{s,k+1}) \right] - A_{s-1}f(\bar{x}_{s-1}),
$$

where (a) is by Lemma 3, and (b) is by $\mathbb{E}[\nabla_{s,k}] = \nabla g(y_{s,k})$, $A_s = A_{s-1} + a_s$ and $f(x) = g(x) + l(x)$. Summing (34) from $k = 1$ to $m$, by the setting for $s \geq 2$, $\psi_{s+1,0} := \psi_{s,m}$ and $z_{s+1,0} := z_{s,m}$, we have

$$
\mathbb{E}[\psi_{s+1,0}(z_{s+1,0}) - \psi_{s,0}(z_{s,0})] = \mathbb{E}[\psi_{s,m}(z_{s,m}) - \psi_{s,0}(z_{s,0})]
$$

$$
\geq \mathbb{E} \left[ A_s \sum_{k=1}^{m} f(y_{s,k+1}) - mA_{s-1}f(\bar{x}_{s-1}) \right]
$$

$$
\geq \mathbb{E} \left[ mA_s f(\bar{x}_s) - mA_{s-1}f(\bar{x}_{s-1}) \right],
$$

where (a) is by the convexity of $f(z)$ and the fact of $\bar{x}_s = \frac{A_{s-1}}{A_s} \bar{x}_{s-1} + \frac{a_s}{mA_s} \sum_{k=1}^{m} z_{s,k}$ (which is in turn by the definition of $\bar{x}_s = \frac{A_{s-1}}{A_s} \bar{x}_{s-1} + \frac{a_s}{mA_s} \sum_{k=1}^{m} z_{s,k}$ and the definition of $y_{s,k}$.)
E. Proof of Lemma 5

Proof. \( \forall s \geq 2 \), taking expectation on the choice of \( i \) in the \( k \)-th iteration of the \( s \)-th epoch, we have \( \forall z \),

\[
\mathbb{E}[\psi_{s,k}(z)] = \mathbb{E}[\psi_{s,k-1}(z) + a_s(g(y_{s,k}) + \langle \nabla g(y_{s,k}), z - y_{s,k} \rangle + l(z))]
\]

\[
= \psi_{s,k-1}(z) + a_s(g(y_{s,k}) + \langle \nabla g(y_{s,k}), z - y_{s,k} \rangle + l(z))
\]

\[
\leq \psi_{s,k-1}(z) + a_s(g(z) + l(z))
\]

\[
= \psi_{s,k-1}(z) + a_s f(z), \tag{36}
\]

where (a) is by the fact \( \mathbb{E}[\nabla s,k] = \nabla g(y_{s,k}) \), and (b) is by the convexity of \( g(z) \). Then taking expectation from the randomness of the epoch \( s \) and telescoping (36) from \( k = 1 \) to \( m \), we have

\[
\mathbb{E}[\psi_{s,m}(z)] \leq \psi_{s,0}(z) + ma_s f(z)
\]

\[
= \begin{cases} 
\psi_{s-1,m}(z) + ma_s f(z), & s \geq 3 \\
m\psi_{1,1}(z) + ma_2 f(z), & s = 2.
\end{cases} \tag{37}
\]

Then taking expectation from the randomness of all the history from \( s = 3 \) and telescoping (37) from \( s = 3 \) to \( S \), we have

\[
\mathbb{E}[\psi_{s,m}(z)] \leq \psi_{2,m}(z) + m \sum_{i=3}^{s} a_i f(z). \tag{38}
\]

Meanwhile taking expectation from the randomness of epoch \( s = 2 \), we have

\[
\mathbb{E}[\psi_{2,m}(z)] \leq m\psi_{1,1}(z) + ma_2 f(z)
\]

\[
= m\psi_{1,0}(z) + a_1(g(y_{1,1}) + \langle \nabla g(y_{1,1}), z - y_{1,1} \rangle + l(z))) + ma_2 f(z)
\]

\[
\leq m\left(\frac{1}{2}\|z - \bar{x}_0\|^2 + a_1(g(z) + l(z))\right) + ma_2 f(z)
\]

\[
= m(a_1 + a_2) f(z) + m \frac{2}{2} \|z - \bar{x}_0\|^2, \tag{39}
\]

where (a) is by the convexity of \( g(z) \) and \( \psi_{1,0}(z) = \frac{1}{2}\|z - \bar{x}_0\|^2 \).

So combining (38) and (39), we have

\[
\mathbb{E}[\psi_{s,m}(z)] \leq m \sum_{i=1}^{s} a_s f(z) + m \frac{2}{2} \|z - \bar{x}_0\|^2
\]

\[
= m A_s f(z) + m \frac{2}{2} \|z - \bar{x}_0\|^2, \tag{40}
\]

where (a) is by our setting \( a_s = A_s - A_{s-1} \) and \( A_0 = 0 \).

Then by (40) and the optimality of \( z_{s,m} \), we have \( \psi_{s,m}(z_{s,m}) \leq \psi_{s,m}(\bar{x}^*) \) and thus

\[
\mathbb{E}[\psi_{s,m}(z_{s,m})] \leq \psi_{s,m}(\bar{x}^*) \leq m A_s f(\bar{x}^*) + m \frac{2}{2} \|\bar{x}^* - \bar{x}_0\|^2. \tag{41}
\]
An Auxiliary Lemma

By Assumption 1 and [Nes98], we have Lemma 6.

**Lemma 6** Under Assumption 1, \( \forall x, y, \)
\[
g(y) \leq g(x) + \langle \nabla g(x), y - x \rangle + \frac{L}{2} \|y - x\|^2
\]  
(42)

and \( \forall i \in [n], \forall x, y, \)
\[
\|\nabla g_i(y) - \nabla g_i(x)\|^2 \leq 2L(g_i(y) - g_i(x) - \langle \nabla g_i(x), y - x \rangle).
\]  
(43)

Under Assumption 1, Lemma 6 are classical results in convex optimization. For completeness, we provide the proof of Lemma 6 here.

**Proof.** [Proof of Lemma 6] By Assumption 1, \( \forall i \in [n], g_i(x) \) satisfies \( \forall x, y, \|\nabla g_i(x) - \nabla g_i(y)\| \leq L\|x - y\|. \) As a result, we have
\[
\|\nabla g(x) - \nabla g(y)\| = \left\| \frac{1}{n} \sum_{i=1}^{n} \nabla g_i(x) - \frac{1}{n} \sum_{i=1}^{n} \nabla g_i(y) \right\|
\]
\[
\leq \frac{1}{n} \sum_{i=1}^{n} \|g_i(x) - g_i(y)\|
\]
\[
\leq L\|x - y\|.
\]  
(44)

The we have
\[
g(y) = g(x) + \int_0^1 \langle \nabla g(x + \tau(y - x)), y - x \rangle d\tau
\]
\[
= g(x) + \langle \nabla g(x), y - x \rangle + \int_0^1 \langle \nabla g(x + \tau(y - x)) - \nabla g(x), y - x \rangle d\tau.
\]  
(45)

Then it follow that
\[
g(y) - g(x) - \langle \nabla g(x), y - x \rangle \leq \left| \int_0^1 \langle \nabla g(x + \tau(y - x)) - \nabla g(x), y - x \rangle d\tau \right|
\]
\[
\leq \int_0^1 |\langle \nabla g(x + \tau(y - x)) - \nabla g(x), y - x \rangle| d\tau
\]
\[
\leq \int_0^1 \|\nabla g(x + \tau(y - x)) - \nabla g(x)\| \|y - x\| d\tau
\]
\[
\leq \int_0^1 L\|y - x\|^2 d\tau
\]
\[
= \frac{L}{2} \|y - x\|^2.
\]  
(46)

Thus we obtain (42).
Then denote \( \forall i \in [n], \phi_i(y) = g_i(y) - g_i(x) - \langle \nabla g_i(x), y - x \rangle \). Obviously \( \phi_i(y) \) is also \( L \)-smooth. One can check that \( \nabla g_i(x) = 0 \) and so that \( \min_y \phi_i(y) = \phi_i(x) = 0 \), which implies that

\[
\phi_i(x) \leq \phi_i(y) - \frac{1}{L} \nabla \phi_i(y)
\]

\[
= \phi_i(y) + \int_0^1 \langle \nabla \phi_i(y - \frac{\tau}{L} \nabla \phi_i(y)), -\frac{1}{L} \nabla \phi_i(y) \rangle d\tau
\]

\[
= \phi_i(y) + \langle \nabla \phi_i(y), -\frac{1}{L} \nabla \phi_i(y) \rangle + \int_0^1 \langle \nabla \phi_i(y - \frac{\tau}{L} \nabla \phi_i(y)) - \nabla \phi_i(y), -\frac{1}{L} \nabla \phi_i(y) \rangle d\tau
\]

\[
\leq \phi_i(y) - \frac{1}{L} \| \nabla \phi_i(y) \|^2 + \int_0^1 L \| \frac{\tau}{L} \nabla \phi_i(y) \| \| \frac{1}{L} \nabla \phi(y) \| d\tau
\]

\[
\leq \phi_i(y) - \frac{1}{2L} \| \nabla \phi_i(y) \|^2.
\]

(47)

Then we have \( \| \nabla \phi_i(y) \|^2 \leq 2L(\phi_i(y) - \phi_i(x)) \). Then by the definition of \( \phi_i(y) \), we obtain (43).