Supersymmetry and Gravity in Noncommutative Field Theories

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We discuss the renormalization properties of noncommutative supersymmetric theories. We also discuss how the gauge field plays a role similar to gravity in noncommutative theories.

1. Introduction

Noncommuting coordinates have been proposed long ago as a way to get rid of divergences in quantum field theory \[1\] and more recently in the context of string theory with D-branes \[2\]. Its effects can easily be seen. If \(x\) and \(y\) have a commutation relation \([x, y] = \theta\) then at the quantum level we expect an uncertainty \(\Delta x \Delta y \sim \theta\). Together with the usual uncertainty relation \(\Delta x \Delta p_x \sim 1\) we find that \(\Delta y \sim \theta \Delta p_x\). This means that the ultraviolet (UV) regime in the \(x\)-direction produces an infrared (IR) effect in the \(y\)-direction and vice-versa. At the quantum field theory level this phenomenon manifests itself as a mixture of UV and IR divergences already at the one loop level \[3\]. If the theory is renormalized in the usual way IR divergences appear jeopardizing renormalizability at higher loop levels. It was suggested that supersymmetry would improve this situation since they are less divergent than conventional theories \[4\] and this was found to be true for the gauge field two-point function at one loop level \[5\]. A general proof soon appeared showing that the noncommutative (NC) Wess-Zumino model is free of the UV/IR mixing at all orders in perturbation theory \[6\]. So far, it is the only known model of a four dimensional renormalizable noncommutative (NC) field theory. Its low energy properties were studied in detail \[7\]. Other noncommutative supersymmetric non-gauge theories were also found to be free of UV/IR mixing. For instance, the supersymmetric nonlinear sigma model in three dimensions turns out to be renormalizable in the \(1/N\) expansion \[8,9\]. Spontaneous symmetry breaking also has troubles in the presence of noncommutativity \[10\]. In three dimensions the situation is improved and it seems that supersymmetry plays no role in this case \[9\]. For supersymmetric gauge theories up to two loop orders the mixing is also absent \[11\] but it seems that for higher loops this is no longer true.

Another interesting aspect of NC field theories is the Seiberg-Witten (SW) map. Instead of working with NC fields with its exotic properties we map them to ordinary commutative fields \[2\]. In this way a local field theory is obtained at the expense of introducing non-renormalizable interactions. Many properties of the NC fields are usually lost after performing the SW map. For instance, translations in NC directions are equivalent to gauge transformations \[12\], a feature similar to that found in general relativity where local translations are associated to general coordinate transformations. This property is completely lost after the SW map. However, we found that another aspect concerning gravity emerges: NC theories can be interpreted as ordinary theories immersed in a gravitational background generated by the gauge field \[13\]. What is interesting is that the gravity coupling is sensitive to the charge with uncharged fields coupling more strongly than charged ones.

In the next section we will review the UV/IR mixing for the simple case of a scalar field showing how the IR divergences become a source of trouble. Then, in section 3 we will discuss the
NC Wess-Zumino model proving its renormalizability to all loop orders. Section 4 is devoted to the study of the UV/IR mixing in the NC Gross-Neveu and nonlinear sigma models. In section 5 we show how supersymmetry improves the situation in the NC supersymmetric nonlinear sigma model. Aspects concerning spontaneous symmetry breaking in NC theories are discussed in section 6 and the relation between gravitation and NC theories is discussed in the last section.

2. UV/IR Mixing for the Scalar Field

Noncommutative field theories are obtained from the commutative ones by replacing the ordinary field multiplication for the Moyal product, which is defined as

\[
(\phi_1 \star \phi_2)(x) = \left[ e^{i g \theta^\mu \sigma^\nu \phi_1(x) \phi_2(y)} \right]_{\gamma = x},
\]

(1)

Then the noncommutative \(\phi^4\) model in 3 + 1 dimensions reads as

\[
L = \frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi - \frac{m^2}{2} \phi \star \phi - \frac{g^2}{4!} \phi \star \phi \star \phi \star \phi.
\]

(2)

We now proceed in the standard way. The quadratic terms gives the propagator. Since the Moyal product has the property \(\int dx \ f(x) g(x) = \int dx f(x) g(x)\), the propagator is the same as in the commutative case. This is a general property of noncommutative theories: the propagators are not modified by the noncommutativity. The vertices, however, are in general affected by phase factors. In this case we get

\[
-\frac{g^2}{6} \cos(\frac{1}{2} k_1 \wedge k_2) \cos(\frac{1}{2} k_3 \wedge k_4) + \cos(\frac{1}{2} k_1 \wedge k_3) \cos(\frac{1}{2} k_2 \wedge k_4) + \cos(\frac{1}{2} k_1 \wedge k_4) \cos(\frac{1}{2} k_2 \wedge k_3).
\]

(3)

We can now compute the one loop correction for the two-point function. It is easily found to be

\[
\frac{g^2}{3(2\pi)^2} \int d^4k \left( 1 + \frac{1}{2} \cos(k \wedge p) \right) \frac{1}{k^2 + m^2}.
\]

(4)

The first term is the usual one loop mass correction of the commutative theory (up to a factor 1/2) which is quadratically divergent. The second term is not divergent due to the oscillatory nature of \(\cos(k \wedge p)\). This shows that the non-locality introduced by the Moyal product is not so bad and leaves us with the same divergence structure of the commutative theory. This is also a general property of noncommutative theories \([14]\). To take into account the effect of the second term we regularize the integral using the Schwinger parametrization

\[
\frac{1}{k^2 + m^2} = \int_0^\infty d\alpha e^{-\alpha(k^2 + m^2)} e^{-\frac{x}{\alpha}}, \quad (5)
\]

where a cutoff \(\Lambda\) was introduced. We find

\[
\Gamma^{(2)} = \frac{g^2}{48\pi^2} \left[ (\Lambda^2 - m^2 \ln(\frac{\Lambda^2}{m^2}) + \ldots) + \frac{1}{2}(\Lambda_{eff}^2 - m^2 \ln(\frac{\Lambda_{eff}^2}{m^2}) + \ldots) \right],
\]

(6)

where \(\Lambda_{eff}^2 = 1/(\tilde{\Lambda}^2 + 1/\Lambda^2)\), \(\tilde{\Lambda}^2 = \theta_{\mu\nu} p_{\mu} p_{\nu}\). Note that when the cutoff is removed, \(\Lambda \to \infty\), the noncommutative contribution remains finite providing a natural regularization. Also \(\Lambda_{eff}^2 = 1/\tilde{p}^2\) which diverges either when \(\theta \to 0\) or when \(\tilde{p} \to 0\).

The one loop effective action is then

\[
\int d^4p \frac{1}{2} (\tilde{p}^2 + M^2 + \frac{g^2}{96\pi^2(\tilde{p}^2 + 1/\Lambda^2)} - \frac{g^2M^2}{96\pi^2} \ln \left( \frac{1}{M^2(\tilde{p}^2 + 1/\Lambda^2)} \right)) \phi(p) \phi(-p),
\]

(7)

where \(M\) is the renormalized mass. Let us take the limits \(\Lambda \to \infty\) and \(\tilde{p} \to 0\). If we take first \(\tilde{p} \to 0\) then \(\tilde{p}^2 << 1/\Lambda^2\) and \(\Lambda_{eff} = \Lambda\) showing that we recover the effective commutative theory. However, if we take \(\Lambda \to \infty\) then \(\tilde{p}^2 >> 1/\Lambda^2\) and \(\Lambda_{eff}^2 = 1/\tilde{p}^2\) so that we get

\[
\int d^4p \frac{1}{2} (\tilde{p}^2 + M^2 + \frac{g^2}{96\pi^2\tilde{p}^2} - \frac{g^2M^2}{96\pi^2} \ln \left( \frac{1}{M^2\tilde{p}^2} \right) + \ldots) \phi(p) \phi(-p),
\]

(8)

which is singular when \(\tilde{p} \to 0\). This shows that the limit \(\Lambda \to \infty\) does not commute with the low momentum limit \(\tilde{p} \to 0\) so that there is a mixing of UV and IR limits.

The theory is renormalizable at one loop order if we do not take \(\tilde{p} \to 0\). What about higher
loop orders? Suppose we have insertions of one loop mass corrections. Eventually we will have to integrate over small values of \( \tilde{p} \) which diverges when \( \Lambda \to \infty \). Then we find an IR divergence in a massive theory. This combination of UV and IR divergences makes the theory non-renormalizable.

Then the main question now is the existence of a theory which is renormalizable to all loop orders. Since the UV/IR mixing appears at the level of quadratic divergences a candidate theory would be a supersymmetric one because it does not have such divergences. As we shall see this indeed happens.

### 3. Noncommutative Wess-Zumino Model

The noncommutative Wess-Zumino model in 3 + 1 dimensions \([6]\) has an interaction Lagrangian given by

\[
L_g = g(F \star A \star A - F \star B \star B + G \star A \star B + G \star B \star A - \bar{\psi} \star \psi \star A - \bar{\psi} \star \psi \star B),
\]

where \( A \) and \( B \) are bosonic fields, \( F \) and \( G \) are auxiliary fields and \( \psi \) is a Majorana spinor. The quadratic part of the Lagrangian is identical to the commutative case. The action is invariant under the usual supersymmetry transformations. The supersymmetry transformations are not modified by the Moyal product since they are linear in the fields.

As usual, the propagators are not modified by noncommutativity and the vertices are modified by phase factors. The degree of superficial divergence for a generic 1PI graph \( \gamma \) is then \( d(\gamma) = 4 - I_{AF} - I_{BF} - N_A - N_B - 2N_F - 2N_G - \frac{2}{3}N_{\psi} \), where \( N_{\gamma} \) denotes the number of external lines associated to the field \( \gamma \) and \( I_{AF} \) and \( I_{BF} \) are the numbers of internal lines associated to the mixed propagators \( AF \) and \( BF \), respectively. In all cases we will regularize the divergent Feynman integrals by assuming that a supersymmetric regularization scheme does exist.

The one loop analysis can be done in a straightforward way. As in the commutative case all tadpoles contributions add up to zero. We have verified this explicitly. The self-energy of \( A \) can be computed and the divergent part is contained in the integral

\[
16g^2 \int \frac{d^4k}{(2\pi)^4} \left( 1 + \frac{1}{2} \cos(k \cdot p) \right) \frac{(p \cdot k)^2}{(k^2 - m^2)^3}. \tag{9}
\]

The first term is logarithmically divergent. It differs by a factor 2 from the commutative case. As usual, this divergence is eliminated by a wave function renormalization. The second term is UV convergent and for small \( p \) it behaves as \( p^2 \ln(p^2/m^2) \) and actually vanishes for \( p = 0 \). Then there is no IR pole. The same analysis can be carried out for the others fields. Therefore, there is no UV/IR mixing in the self-energy as expected.

To show that the model is renormalizable we must also look into the interactions vertices. The \( A^3 \) vertex has no divergent parts as in the commutative case. The same happens for the other three point functions. For the four point vertices no divergence is found as in the commutative case. Hence, the noncommutative Wess-Zumino model is renormalizable at one loop with a wave-function renormalization and no UV/IR mixing.

To go to higher loop orders we proceed as in the commutative case. We derived the supersymmetry Ward identities for the n-point vertex function. Then we showed that there is a renormalization prescription which is consistent with the Ward identities. They are the same as in the commutative case. And finally we fixed the primitively divergent vertex functions. Then we found that there is only a common wave function renormalization as in the commutative case. In general we expect \( \varphi_R = Z^{-1/2} \varphi, m_R = Zm + \delta m, g_R = Z'^{1/2} Z'g \). At one loop we found \( \delta m = 0 \) and \( Z' = 1 \). We showed that this also holds to all orders and no mass renormalization is needed.

Being the only consistent noncommutative quantum field theory in 3 + 1 dimensions known so far it is natural to study it in more detail. As a first step in this direction we considered the non-relativistic limit of the noncommutative Wess-Zumino model \([7]\). We found the low energy effective potential mediating the fermion-fermion and boson-boson elastic scattering in the non-relativistic regime. Since noncommutativity breaks Lorentz invariance we formulated the theory in the center of mass frame of reference.
where the dynamics simplifies considerably. For the fermions we found that the potential is significantly changed by the noncommutativity while no modification was found for the bosonic sector. The modifications found give rise to an anisotropic differential cross section.

Subsequently the model was formulated in superspace and again found to be renormalizable to all loop orders \[15\]. The one and two loops contributions to the effective action in superspace were also found \[16\]. The one loop Kahlerian effective potential does not get modified by noncommutativity and the two loops non-planar contributions to the Kahlerian effective potential are leading in the case of small noncommutativity \[16\].

4. Noncommutative Gross-Neveu and Nonlinear Sigma Models

Another model where renormalizability is spoiled by the noncommutativity is the $O(N)$ Gross-Neveu model. The commutative model is perturbatively renormalizable in $1+1$ dimensions and $1/N$ renormalizable in $1+1$ and $2+1$ dimensions. In both cases it presents dynamical mass generation. It is described by the Lagrangian

$$\mathcal{L} = \frac{i}{2} \bar{\psi} \not\partial \psi + \frac{g}{4N} \bar{\psi} i \not\sigma \psi,$$

where $\psi_i, i = 1, \ldots, N$, are two-component Majorana spinors. Since it is renormalizable in the $1/N$ expansion in $1+1$ and $2+1$ dimensions we will consider both cases. As usual, we introduce an auxiliary field $\sigma$ and the Lagrangian turns into

$$\mathcal{L} = \frac{i}{2} \bar{\psi} \not\partial \psi - \frac{\sigma}{2} \bar{\psi} \psi - \frac{N}{4g} \sigma^2.$$

Replacing $\sigma$ by $\sigma + M$ where $M$ is the VEV of the original $\sigma$ we get the gap equation (in Euclidean space) $M/2g - (1/2\pi)^D \int d^D k M/(k^2 + M^2) = 0$. To eliminate the UV divergence we need to renormalize the coupling constant by $1/g = 1/g_R + (2/(2\pi)^D) \int d^D k \sqrt{k^2 + \mu^2}$. In $2+1$ dimensions we find $1/g_R = (\mu - |M|)/2\pi$, and therefore only for $-1/g_R + \mu/2\pi > 0$ it is possible to have $M \neq 0$, otherwise $M$ is necessarily zero. No such a restriction exists in $1+1$ dimensions. In any case, we will focus only in the massive phase. The propagator for $\sigma$ is proportional to the inverse of the following expression

$$- \frac{iN}{2g} - iN \int \frac{d^D k}{(2\pi)^D} \frac{k \cdot (k + p) + M^2}{(k^2 - M^2)((k + p)^2 - M^2)},$$

which is divergent. Taking into account the gap equation the above expression reduces to

$$\frac{(p^2 - 4M^2)N}{2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - M^2)((k + p)^2 - M^2)},$$

which is finite. Then there is a fine tuning which is responsible for the elimination of the divergence and which might be absent in the noncommutative case due to the UV/IR mixing.

The interacting part of the noncommutative model is defined by \[8\]

$$L_i = \frac{1}{2} \sigma \star \bar{\psi} \psi - \frac{N}{4g} \sigma^2 - \frac{N}{2g} M \sigma.$$

Elimination of the auxiliary field results in a four-fermion interaction of the type $\bar{\psi}_i \psi_i \bar{\psi}_j \psi_j$. However a more general four-fermion interaction may involve a term like $\bar{\psi}_i \bar{\psi}_j \psi_i \psi_j$. This last combination does not have a simple $1/N$ expansion and we will not consider it. The Moyal product does not affect the propagators and the trilinear vertex acquires a correction of $\cos(p_1 \wedge p_2)$ with regard to the commutative case. Hence the gap equation is not modified, while in the propagator for $\sigma$ we find a divergent piece.

On the other side, the nonlinear sigma model also presents troubles in its noncommutative version. The noncommutative model is described by

$$\mathcal{L} = - \frac{1}{2} \varphi \not\partial \varphi - \frac{1}{2} \lambda \not\partial \varphi \not\partial \varphi - \frac{N}{2g} \lambda,$$

where $\varphi_i, i = 1, \ldots, N$, are real scalar fields, $\lambda$ is the auxiliary field and $M$ is the generated mass. The leading correction to the $\varphi$ self-energy is

$$- i \int \frac{d^2 \omega}{(2\pi)^2} \frac{\cos^2(\omega \wedge p)}{(\omega^2 + p^2 - M^2)} \Delta_\lambda(k),$$

where $\Delta_\lambda$ is the propagator for $\lambda$. As for the case of the scalar field this can be decomposed as a sum of a quadratically divergent part and a UV finite part. Again there is the UV/IR mixing destroying the $1/N$ expansion.
5. Noncommutative Supersymmetric Nonlinear Sigma Model

The Lagrangian for the commutative supersymmetric sigma model is given by

\[ L = \frac{1}{2} \partial^\mu \varphi_i \partial_\mu \varphi_i + \frac{i}{2} \bar{\psi}_i \gamma_\mu \psi_i + \frac{1}{2} F_i F_i + \sigma \varphi_i F_i \]

\[ + \frac{1}{2} \lambda \varphi_i \varphi_i - \frac{1}{2} \varphi_i \psi_i - \xi \varphi_i \varphi_i - N \frac{2 g}{\lambda} \sigma, \]

(15)

where \( F_i \), \( i = 1, \ldots, N \), are auxiliary fields. Furthermore, \( \sigma, \lambda \) and \( \xi \) are the Lagrange multipliers which implement the supersymmetric constraints.

After the change of variables \( \lambda \to \lambda + 2 M \sigma \), \( F \to F - M \varphi \) where \( M =< \sigma > \), and the shifts \( \sigma \to \sigma + M \) and \( \lambda \to \lambda + \lambda_0 \), where \( \lambda_0 =< \lambda > \), we arrive at a more symmetric form for the Lagrangian. Now supersymmetry requires \( \lambda_0 = -2 M^2 \) and the gap equation is \( (1/(2 \pi)^D) \int d^D k i/(k^2 - M^2) = 1/g \), so a coupling constant renormalization is required. We now must examine whether the propagator for \( \sigma \) depends on this renormalization. We find that the two point function for \( \sigma \) is proportional to the inverse of

\[ \frac{< \pi^2 - 4 M^2 > N}{2} \int \frac{d^D k}{(2 \pi)^D} \frac{1}{(k^2 - M^2)((k + p)^2 - M^2)}, \]

which is identical to the Gross-Neveu case. Notice that the gap equation was not used. The finiteness of the above expression is a consequence of supersymmetry.

The interacting part of the noncommutative version of the supersymmetric nonlinear sigma model is given by \( [8] \)

\[ L_i = \frac{\lambda}{2} \varphi_i \varphi_i - \frac{1}{2} F_i \varphi_i \varphi_i + \frac{\sigma \varphi_i + \varphi_i \sigma}{2} \]

\[ - \frac{1}{2} \xi \varphi_i \psi_i - \frac{1}{2} (\bar{\xi} \psi_i \varphi_i + \bar{\varphi}_i \varphi_i \psi_i) \]

\[ - \frac{N}{2 g} \lambda - \frac{N M \sigma}{g}. \]

(16)

Notice that supersymmetry dictates the form of the trilinear vertices. Also, the supersymmetry transformations are not modified by noncommutativity since they are linear and no Moyal products are required.

The propagators are the same as in the commutative case. The vertices have cosine factors due to the Moyal product. We again consider the propagators for the Lagrange multiplier fields. Now the \( \sigma \) propagator is modified by the cosine factors and is well behaved both in UV and IR regions. The propagators for \( \lambda \) and \( \xi \) are also well behaved in UV and IR regions.

The degree of superficial divergence for a generic 1PI graph \( \gamma \) is \( d(\gamma)/D = (D - (D - 1)N_\sigma/2 - (D - 2)N_\varphi/2 - D N_\gamma/2 - N_\sigma - 3 N_\xi/2 - 2 N_\lambda \), where \( N_\sigma \) is the number of external lines associated to the field \( \sigma \). Potentially dangerous diagrams are those contributing to the self-energies of the \( \varphi \) and \( \psi \) fields since, in principle, they are quadratic and linearly divergent, respectively. For the self-energies of \( \varphi \) and \( \psi \) we find that they diverge logarithmically and they can be removed by a wave function renormalization of the respective field. The same happens for the auxiliary field \( F \).

The renormalization factors for them are the same so supersymmetry is preserved in the noncommutative theory. This analysis can be extended to the \( n \)-point functions. In \( 2 + 1 \) dimensions we find nothing new showing the renormalizability of the model at leading order of \( 1/N \). However, in \( 1 + 1 \) dimensions there are some peculiarities. Since the scalar field is dimensionless in \( 1 + 1 \) dimensions any graph involving an arbitrary number of external \( \varphi \) lines is quadratically divergent. In the four-point function there is a partial cancellation of divergences but a logarithmic divergence still survives. The counterterm needed to remove it can not be written in terms of \( \int d^2 x \varphi_i \varphi_j \varphi_j \varphi_j \) and \( \int d^2 x \varphi_i \varphi_j \varphi_i \varphi_j \). A possible way to remove this divergence is by generalizing the definition of 1PI diagram. However the cosine factors do not allow us to use this mechanism which casts doubt about the renormalizability of the noncommutative supersymmetric \( O(N) \) nonlinear sigma model in \( 1 + 1 \) dimensions.

The noncommutative supersymmetric nonlinear sigma model can also be formulated in superspace where it possible to show that model is renormalizable to all orders of \( 1/N \) and explicitly verify that it is asymptotically free \([17]\).
6. Spontaneous Symmetry Breaking in Noncommutative Field Theory

Having seen the important role supersymmetry plays in noncommutative models it is natural to go further. Spontaneous symmetry breaking and the Goldstone theorem are essential in the standard model and the effect of noncommutativity in this setting deserves to be fully understood. In four dimensions it is known that spontaneous symmetry breaking can occur for the $U(N)$ model but not for the $O(N)$ unless $N = 2$. The $O(2)$ case was analyzed in detail [10] and the results for the $U(N)$ case have been extended to two loops [13]. Going to higher loops requires an IR regulator which can no longer be removed [19]. Due to these troubles we will consider three dimensional tor which can no longer be removed [19]. Due to UV/IR mixing in the non-planar sector. It is also fortunate that it leads to an analytic behavior in the IR so that the mass corrections vanish for $p = 0$. This mechanism does not appears in the four dimensional case.

The two point function for $\sigma$ is also analytic in the IR leading to a relation among the parameters. The divergences in the higher point functions can also be eliminated. Therefore, we have shown that this $O(N)$ model is renormalizable at one loop for any $N$ [9], in contradistinction to the four dimensional case where $N$ must be equal to 2.

A supersymmetric version of this model can be formulated in superspace. Again, the gap equation is not affected by noncommutativity. The mass corrections for the pion two point function are UV finite and free of UV/IR mixing as expected. It also vanishes for $p = 0$. Supersymmetry does not appear to be important in this situation.

7. Noncommutativity and Gravity

An important property of NC theories, which distinguishes them from the conventional ones, is that translations in the NC directions are equivalent to gauge transformations [12]. This can be seen even for the case of a scalar field which has the gauge transformation $\delta \hat{\phi} = -i[\hat{\phi}, \hat{\lambda}]$, where $[A, B]_* = A \star B - B \star A$ is the Moyal commutator. Under a global translation the scalar field transforms as $\delta_T \hat{\phi} = \xi^{\mu} \partial_\mu \hat{\phi}$. Derivatives of the field can be rewritten using the Moyal commutator as $\partial_\mu \hat{\phi} = -\partial_{\mu}^{-1}[x^\nu, \hat{\phi}]$, so that $\delta \hat{\phi} = \delta_T \hat{\phi}$ with gauge parameter $\lambda = -\partial_{\mu}^{-1}[\xi^\mu x^\nu]$. The only other field theory which has this same property is general relativity where local translations are gauge transformations associated to general coordinate transformations. This remarkable property shows that, as in general relativity, there are no local gauge invariant observables in NC theories.

As remarked in the introduction the connection between translations and gauge transformations seems to be lost after the SW map. A global translation on commutative fields can no longer be rewritten as a gauge transformation. We will show that another aspect concerning gravity emerges when commutative fields are employed. Noncommutative field theories can be interpreted
as ordinary theories immersed in a gravitational background generated by the gauge field \[ 1 \].

The action for a real scalar field in the adjoint representation of \( U(1) \) coupled minimally to a gauge field and in flat space-time is

\[
S_\varphi = \frac{1}{2} \int d^4x \, \hat{D}_\mu \hat{\varphi} \ast \hat{D}_\mu \hat{\varphi},
\]

where \( \hat{D}_\mu \hat{\varphi} = \partial_\mu \hat{\varphi} - i [\hat{A}_\mu, \hat{\varphi}] \). The SW map is given by \( \hat{A}_\mu = A_\mu - \frac{i}{2} \theta^{\alpha \beta} A_\alpha (\partial_\beta A_\mu + F_{\beta \mu}) \), \( \hat{\varphi} = \varphi - \theta^{\alpha \beta} A_\alpha \partial_\beta \varphi \), so that the action can be written, to first order in \( \theta \), as

\[
S_\varphi = \frac{1}{2} \int d^4x \left[ \partial^\mu \varphi \partial_\mu \varphi + 2 \Theta^{\mu \nu} F_{\alpha \nu} \partial^\mu \varphi \partial_\nu \varphi \right].
\] (18)

Notice that the tensor inside the parenthesis is traceless. If we now consider this same field coupled to a gravitational background and expand the metric around the flat Minkowski metric \( \eta_{\mu \nu} \) as \( g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu} + \eta_{\mu \nu} h \), where \( h_{\mu \nu} \) is traceless, we get

\[
L = \frac{1}{2} \left( \partial^\mu \varphi \partial_\mu \varphi - h^{\mu \nu} \partial_\mu \varphi \partial_\nu \varphi + h \partial^\mu \varphi \partial_\nu \varphi \right),
\] (19)

where indices are raised and lowered with the flat metric. Since both actions have the same structure we can identify a linearized background gravitational field \( h^{\mu \nu} = \Theta^{\mu \alpha} F_{\alpha \nu} + \Theta^{\alpha \nu} F_{\mu \alpha} + \frac{1}{2} \Theta^{\mu \nu} \partial_\alpha \varphi F_{\alpha \beta} \) and \( h = 0 \). This linearized metric describes a gravitational plane wave \[ 1 \]. Then, the effect of noncommutativity on the commutative scalar field is similar to a field dependent gravitational field.

The same procedure can be repeated for the complex scalar field and we get a linearized contribution that is half of that felt by the real scalar field. Then charged fields feel a gravitational background which is half of that felt by the uncharged ones. Therefore, the gravity coupling is now dependent on the charge of the field, being stronger for uncharged fields. Notice that the gauge field has now a dual role, it couples minimally to the charged field and also as a gravitational background.

We can now turn our attention to the behavior of a charged massless particle in this background. Its geodesics is described by

\[
(1 + \frac{1}{4} \Theta^{\alpha \beta} F_{\alpha \beta}) dx^\mu dx_\mu + \Theta_{\mu \alpha} F^{\alpha \nu} dx^\mu dx^\nu = 0.
\]

If we consider the case where there is no noncommutativity between space and time, that is \( \Theta^{0i} = 0 \), and calling \( \Theta^{ij} = \epsilon^{ijk} \theta^k \), \( F^{00} = E \), and \( F^{ij} = \epsilon^{ijk} B^k \), we find to first order in \( \theta \) that

\[
(1 - \Theta^{ij}) (1 - 2 \Theta^{ij} B^j - \Theta^{ij} (\phi \times \phi)) + \Theta^{ij} B^j (\phi \times \phi) = 0,
\]

where \( \phi \) is the particle velocity. Then to zeroth order, the velocity \( \phi_0 \) satisfies \( \phi_0^2 = 1 \) as it should.

We can now decompose all vectors into their transversal and longitudinal components with respect to \( \phi_0 \), \( E = \phi_0 E_L + \phi_0 E_T \), \( B = \phi_0 B_L + \phi_0 B_T \) and \( \theta = \phi_0 \phi_L \). We then find that the velocity is

\[
\phi^2 = 1 + \Theta^{ij} (\phi_0 \phi_L - \phi_0 \phi_T).
\] (20)

Hence, a charged massless particle has its velocity changed with respect to the velocity of light by an amount which depends on \( \theta \).

We can now check the consistency of these results by going back to the original actions and computing the group velocity for planes waves. Upon quantization they give the velocity of the particle associated to the respective field. For the charged scalar field we get the equation of motion \( (1 - \Theta^{ij} B^j - \Theta^{ij} (\phi \times \phi)) + \Theta^{ij} B^j (\phi \times \phi) = 0 \). If the field strength is constant we can find a plane wave solution with the following dispersion relation \( \Theta^{ij} B^j = 1 - \Theta^{ij} \phi_L \phi_T \). We then find that the phase and group velocities coincide and are given by (20) as expected. Therefore, in both pictures, noncommutative and gravitational, we get the same results. The dispersion relation here is similar to that found for photons in a background magnetic field (20).

This work was partially supported by CAPES, CNPq and PRONEX under contract CNPq 66.2002/1998-99.

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