Analytic Bethe Ansatz for 1-D Hubbard model and twisted coupled XY model

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Abstract

We found the eigenvalues of the transfer matrices for the 1-D Hubbard model and for the coupled XY model with twisted boundary condition by using the analytic Bethe Ansatz method. Under a particular condition the two models have the same Bethe Ansatz equations. We have also proved that the periodic 1-D Hubbard model is exactly equal to the coupled XY model with nontrivial twisted boundary condition at the level of hamiltonians and transfer matrices.

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1 Introduction

The 1-D Hubbard model (the Hubbard model) is one of the significant exactly solvable models in condensed matter physics. Lieb and Wu [1] succeeded in diagonalizing the hamiltonian in the frame of the coordinate Bethe Ansatz. However, the integrability of the Hubbard model was recently set up by Shastry [2, 3], Olmedilla, Wadati and Akutsu [4, 5, 6] from the viewpoint of the Quantum Inverse Scattering Method (QISM). The Yang-Baxter equation for the related R matrix was proved in [7]. In [2, 3], Shastry found the Yang-Baxter relation and constructed the transfer matrix of the coupled XY model which is equal to the Hubbard model with the help of the Jordan-Wigner transformation. The commutative family with one free parameter ensures the integrability of the system. In a different approach, Olmedilla, Wadati and Akutsu [4, 5, 6], starting from the super L-operator of the Hubbard model, solved the super-Yang-Baxter (SYB) relation and found the invertible R matrix which is the same as that given by Shastry [8] up to a scalar function. In both cases [3, 5], the hamiltonian can be derived from the transfer matrix under the periodic boundary condition.

The base of the Quantum Inverse Scattering Method is the Yang-Baxter relation. The later has closely related to the Quantum group and the Yangian. This technique provides a systematic method to deal with the integrable 1-D quantum systems and 2-D solvable statistical mechanical models. It is known that most of integrable systems can be handled in the frame of the Algebraic Bethe Ansatz (or Analytic Bethe Ansatz) method. But, up to now, there is no report about the diagonalization of the transfer matrix for the Hubbard model in the QISM approach. So, it is important to find the solution of the Hubbard model by using QISM. In [8], Shastry conjectured the eigenvalue of the transfer matrix of the coupled XY model based upon the coordinate Bethe Ansatz method. Comparing the Bethe Ansatz equations given by Lieb and Wu [1] and by Shastry [8], an extra factor appears and it shows some difference between the two models.

The motivation of this paper is to find the eigenvalues of the transfer matrices related to the Hubbard model and the coupled XY model with twisted boundary condition by a version of analytic Bethe Ansatz method (ABA). We want to give some explanation about the difference between the coupled XY and the Hubbard models. The extra factor originates from the boundary condition. In fact the Jordan-Wigner transformation does not keep the invariance of the boundary condition. We also show that the Hubbard model with periodic boundary condition is exactly equal to the coupled XY model with a special boundary condition.

The organization of this paper is to recall the Yang-Baxter relation for the Hubbard model and the parametrization of the R matrix in section 2. In section 3, we investigate the model at special cases by using the algebraic Bethe Ansatz method. Because there
are two kinds of creation operators with spin-up (or spin-down), such as $T_{21}$ and $T_{43}$ ($T_{21}$ and $T_{43}$), the general multi-particle states become very complicated. However, the special solution gives an insight about the general structure of the eigenvalue. Section 4 will attributes to an analytic Bethe Ansatz method to the Hubbard model. In our approach, the analytic property of the eigenvalue of the transfer matrix, together with the asymptotic behaviour, determines almost all unknown functions. The discussion is different from the standard analytic Bethe Ansatz method since there is not the crossing symmetry, which has played an important role [9, 10, 11, 12]. In section 5, we will apply the analytic Bethe Ansatz method developed in section 4 to the coupled XY model with twisted boundary condition. Under a special choice, it recovers the results of the Hubbard model in the level of Bethe Ansatz equations and the eigenvalue. In section 6 we show that the Hubbard model with periodic boundary condition is equal to the coupled XY model with special boundary condition by using the Jordan-Wigner transformation and some discussions are given in section 7.

2 The Hubbard model and the Super-Yang-Baxter relation

In this section we recall the definition of the SYB relation for the Hubbard model and some useful functional relations. We follow the notation in [3]. The hamiltonian of the Hubbard model is

$$H_{Hu} = -\sum_{j=1, s=\uparrow, \downarrow}^{L} (a_{m+1,s}^+ a_{m,s} + a_{m,s}^+ a_{m+1,s}) + U \sum_{m=1}^{L} (n_{m\uparrow} - 1/2)(n_{m\downarrow} - 1/2)$$  

(1)

where $a_{m,s}^+ (a_{m,s})$ stands for the $m$-th site electron creation (annihilation) operator with spin $s$. The Super-Yang-Baxter relation is [3]

$$\mathcal{R}(\mu, \nu) [\mathcal{L}_m(\mu) \otimes_s \mathcal{L}_m(\nu)] = [\mathcal{L}(\nu)_m \otimes_s \mathcal{L}(\mu)] \mathcal{R}(\mu, \nu)$$  

(2)

The super tensor product is defined by

$$[A \otimes_s B]_{j,i} = A^i_j B^a_0 (-1)^{p(i)+p(j)}p(a),$$  

(3)

where $p(1) = p(4) = -p(2) = -p(3) = 1$ are the parities. For the Hubbard model the L-operator takes the form

$$\mathcal{L}(\mu) =$$
\[
\begin{pmatrix}
-e^{h(\mu)} f_{m\uparrow} f_{m\downarrow} & f_{m\uparrow} a_{m\downarrow} & i a_{m\uparrow} f_{m\downarrow} & i a_{m\uparrow} a_{m\downarrow} e^{h(\mu)} \\
-if_{m\uparrow} a_{m\downarrow} & f_{m\uparrow} g_{m\downarrow} e^{-h(\mu)} & a_{m\uparrow} a_{m\downarrow} e^{-h(\mu)} & i a_{m\uparrow} g_{m\downarrow} \\
-a_{m\uparrow} f_{m\downarrow} & a_{m\uparrow} a_{m\downarrow} e^{-h(\mu)} & g_{m\uparrow} f_{m\downarrow} e^{-h(\mu)} & g_{m\uparrow} a_{m\downarrow} \\
-ia_{m\uparrow} a_{m\downarrow} e^{h(\mu)} & a_{m\uparrow} g_{m\downarrow} & i g_{m\uparrow} a_{m\downarrow} & -g_{m\uparrow} g_{m\downarrow} e^{h(\mu)}
\end{pmatrix}
\]

where \( \mu \) is the spectrum parameter, and the functions \( f_{ms} \) and \( g_{ms} \) are

\[
\begin{align*}
    f_{ms} &= w_4(\mu) - w_3(\mu) - \{w_4(\mu) - w_3(\mu) - i[w_4(\mu) + w_3(\mu)]\} n_{ms} \\
    g_{ms} &= w_4(\mu) + w_3(\mu) - \{w_4(\mu) + w_3(\mu) - i[w_4(\mu) - w_3(\mu)]\} n_{ms} \\
    w_4(\mu) + w_3(\mu) &= \sin(\mu + \pi/4) \\
    w_4(\mu) - w_3(\mu) &= \sin(\mu - \pi/4) \\
    \sinh(2h(\mu)) &= -\frac{U}{4} \cos(2\mu)
\end{align*}
\]

The graded \( \mathcal{R} \) matrix reads

\[
\begin{pmatrix}
\rho_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 & i\rho_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \rho_3 & 0 & 0 & \frac{\rho_6}{i} & 0 & 0 & i\rho_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\rho_{10}}{i} & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & i\rho_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & i\rho_6 & 0 & 0 & \rho_5 & 0 & 0 & \rho_7 & 0 & 0 & \frac{\rho_6}{i} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & \frac{\rho_{10}}{i} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\rho_{10}}{i} & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \rho_7 & 0 & 0 & \rho_5 & 0 & 0 & i\rho_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_4 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & \frac{\rho_6}{i} & 0 & 0 & \rho_3 & 0 & 0 \\
0 & 0 & 0 & 0 & \rho_8 & 0 & 0 & i\rho_6 & 0 & 0 & \frac{\rho_6}{i} & 0 & 0 & \rho_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & i\rho_9 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & i\rho_9 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
where all functions are defined by

\[
\rho_1(\mu, \nu) = e^l \alpha(\mu) \alpha(\nu) + e^{-l} \gamma(\mu) \gamma(\nu),
\]

\[
\rho_4(\mu, \nu) = e^l \gamma(\mu) \gamma(\nu) + e^{-l} \alpha(\mu) \alpha(\nu),
\]

\[
\rho_9(\mu, \nu) = -e^l \alpha(\mu) \gamma(\nu) + e^{-l} \gamma(\mu) \alpha(\nu),
\]

\[
\rho_{10}(\mu, \nu) = e^l \gamma(\mu) \alpha(\nu) - e^{-l} \alpha(\mu) \gamma(\nu),
\]

\[
\rho_3(\mu, \nu) = \frac{e^l \alpha(\mu) \alpha(\nu) - e^{-l} \gamma(\mu) \gamma(\nu)}{\alpha^2(\mu) - \gamma^2(\nu)},
\]

\[
\rho_5(\mu, \nu) = \frac{-e^l \gamma(\mu) \gamma(\nu) + e^{-l} \alpha(\mu) \alpha(\nu)}{\alpha^2(\mu) - \gamma^2(\nu)},
\]

\[
\rho_6(\mu, \nu) = \frac{e^{-h}[e^l \alpha(\mu) \gamma(\mu) - e^{-l} \alpha(\nu) \gamma(\nu)]}{\alpha^2(\mu) - \gamma^2(\nu)},
\]

\[
\rho_7(\mu, \nu) = \rho_1(\mu, \nu) - \rho_3(\mu, \nu),
\]

\[
\rho_3(\mu, \nu) = \rho_4(\mu, \nu) - \rho_5(\mu, \nu),
\]

\[
l = h(\mu) - h(\nu),
\]

\[
h = h(\mu) + h(\nu).
\]

Due to the Super-Yang-Baxter relation, one can define the monodromy matrix

\[
\mathcal{T}(\mu) = \mathcal{L}_L(\mu) \cdots \mathcal{L}_2(\mu) \mathcal{L}_1(\mu)
\]

which still satisfies the Super-Yang-Baxter relation

\[
\mathcal{R}([\mathcal{T}(\mu) \otimes_s \mathcal{T}(\nu)]) = [\mathcal{T}(\nu) \otimes_s \mathcal{T}(\mu)] \mathcal{R}(\mu, \nu)
\]

The Super-Yang-Baxter relation leads to the existence of the commutative family of the transfer matrices \( t_H(\mu) = str \mathcal{T}(\mu) \) with infinitely many different value of \( \mu \). So, the Hubbard model is integrable. The infinite number of conserved quantities can be derived from the transfer matrix \( t_H(\mu) \). The derivation of \( \log [t_H(\mu)] \) at \( \mu = \pi/4 \) gives the hamiltonian of the Hubbard model with the periodic boundary condition. Before ending this section, we list some useful functional relations which will be used in the
following sections,

\[
\begin{align*}
1 &= \rho_1 \rho_4 + \rho_9 \rho_{10}, \\
2 &= \rho_1 \rho_5 + \rho_3 \rho_4, \\
1 &= \rho_3 \rho_5 - (\rho_6)^2, \\
\rho_{10} &= \rho_6(e^h\alpha(\mu)\alpha(\nu) + e^{-h}\gamma(\mu)\gamma(\nu)), \\
\rho_9 &= \rho_6(e^{-h}\alpha(\mu)\alpha(\nu) + e^{h}\gamma(\mu)\gamma(\nu)).
\end{align*}
\]

(10)

3 An algebraic analysis on the eigenvalue

In this section, we will discuss the solution of the Hubbard model in some special cases by using the algebraic Bethe Ansatz method. For the general multiparticle states, it becomes very difficult even if there are two electrons with opposite spins.

Taking the special elements of the Super-Yang-Baxter relation, one can obtain the following useful relations

\[
\begin{align*}
T_{44}(\mu)T_{43}(\nu) &= \frac{R_{44}^{44}(\nu, \mu)}{R_{44}^{44}(\nu, \mu)} T_{43}(\nu) T_{44}(\mu) - \frac{R_{44}^{34}(\mu, \nu)}{R_{34}^{44}(\nu, \mu)} T_{43}(\mu) T_{44}(\nu) \\
T_{33}(\mu)T_{43}(\nu) &= -\frac{R_{33}^{33}(\mu, \nu)}{R_{33}^{34}(\mu, \nu)} T_{43}(\nu) T_{33}(\mu) - \frac{R_{43}^{43}(\nu, \mu)}{R_{43}^{34}(\nu, \mu)} T_{43}(\mu) T_{33}(\nu)
\end{align*}
\]

(11)

(12)
\[
\begin{align*}
T_{22}(\mu) T_{43}(\nu) &= - \left( \frac{R_{23}^{32}(\mu, \nu)}{R_{24}^{31}(\mu, \nu)} - \frac{R_{23}^{41}(\mu, \nu) R_{14}^{32}(\mu, \nu)}{R_{24}^{31}(\mu, \nu) R_{14}^{41}(\mu, \nu)} \right) T_{43}(\nu) T_{22}(\mu) \\
&\quad + \left( \frac{R_{23}^{14}(\nu, \mu)}{R_{24}^{31}(\nu, \mu)} - \frac{R_{23}^{14}(\nu, \mu) R_{14}^{14}(\nu, \mu)}{R_{24}^{31}(\nu, \mu) R_{14}^{41}(\nu, \mu)} \right) T_{41}(\nu) T_{24}(\mu) \\
&\quad - \left( \frac{R_{23}^{23}(\nu, \mu)}{R_{24}^{31}(\nu, \mu)} - \frac{R_{23}^{23}(\nu, \mu) R_{14}^{23}(\nu, \mu)}{R_{24}^{31}(\nu, \mu) R_{14}^{41}(\nu, \mu)} \right) T_{42}(\nu) T_{23}(\mu) \\
&\quad - \frac{R_{23}^{14}(\mu, \nu) R_{14}^{13}(\mu, \nu)}{R_{24}^{31}(\mu, \nu) R_{14}^{41}(\mu, \nu)} T_{41}(\mu) T_{24}(\nu) \\
&\quad + \frac{R_{24}^{14}(\mu, \nu)}{R_{14}^{41}(\mu, \nu)} T_{21}(\mu) T_{44}(\nu) \\
&\quad + \frac{R_{24}^{23}(\mu, \nu)}{R_{14}^{41}(\mu, \nu)} T_{42}(\mu) T_{23}(\nu) \\
T_{11}(\mu) T_{43}(\nu) &= \frac{R_{13}^{31}(\mu, \nu)}{R_{14}^{31}(\mu, \nu)} T_{43}(\nu) T_{11}(\mu) + \frac{R_{13}^{13}(\mu, \nu)}{R_{14}^{41}(\mu, \nu)} T_{41}(\nu) T_{13}(\mu) \\
&\quad + \frac{R_{14}^{13}(\mu, \nu)}{R_{14}^{41}(\mu, \nu)} T_{21}(\mu) T_{43}(\nu) + \frac{R_{14}^{43}(\mu, \nu)}{R_{41}^{41}(\mu, \nu)} T_{41}(\mu) T_{23}(\nu) \\
&\quad - \frac{R_{14}^{14}(\mu, \nu)}{R_{14}^{41}(\mu, \nu)} T_{41}(\mu) T_{13}(\nu)
\end{align*}
\]

For the Hubbard model the Hilbert space consists of four states: double occupied state \( |\uparrow\downarrow\rangle \), spin-up state \( |\uparrow\rangle \), spin-down state \( |\downarrow\rangle \) and unoccupied state \( |0\rangle \). We denote them by \( |1\rangle \), \( |2\rangle \), \( |3\rangle \) and \( |4\rangle \) respectively. It is convenient to introduce the reference state

\[
|\text{vac}\rangle = |4\rangle_1 \otimes \cdots \otimes_s |4\rangle_L
\]

Using the explicit expression of the L-operator, one can find that the monodromy matrix acting on the reference state takes the form

\[
\mathcal{T}(\mu)|\text{vac}\rangle = \begin{pmatrix}
A_1(\mu) & 0 & 0 & 0 \\
T_{21}(\mu) & A_2(\mu) & 0 & 0 \\
T_{31}(\mu) & 0 & A_3(\mu) & 0 \\
T_{41}(\mu) & T_{32}(\mu) & T_{43}(\mu) & A_4(\mu)
\end{pmatrix}|\text{vac}\rangle
\]

where

\[
A_4(\mu) = [-\alpha^2(\mu) e^{h(\mu)}]^L
\]
\[ A_2(\mu) = A_3(\mu) = [\alpha(\mu)\gamma(\mu)e^{-h(\mu)}]^L \]
\[ A_4(\mu) = [-\gamma^2(\mu)e^{h(\mu)}]^L \]

Unlike the case where the nested Bethe Ansatz method is applicable, the operator \( T_{21} \) is related to operator \( T_{43} \), creating an electron with spin-down. Generally, the relation is very complicated. When they act on the reference state, however, the situation becomes simple. An algebraic calculation shows

\[ T_{21} \tilde{\mu} |\text{vac}\rangle = (-1)^L \alpha(\tilde{\mu}) \gamma(\tilde{\mu}) \cdot \prod_{j=1}^N \rho_{ij}(\mu_j, \mu) |\text{vac}\rangle \]

where the quantity with tilde is defined by

\[ e^{-2h(\tilde{\mu})} \frac{\alpha(\tilde{\mu})}{\gamma(\mu)} = e^{2h(\mu)} \frac{\alpha(\mu)}{\gamma(\mu)} \]

Further, one can show

\[ T_{21} T_{43}(\mu_1) \cdots T_{43}(\mu_n) |\text{vac}\rangle \propto T_{43}(\mu) T_{43}(\mu_1) \cdots T_{43}(\mu_n) |\text{vac}\rangle \]

It is worthy to notice that this relation is valid only for the reference state; when acting on the other states, it will be invalid. There is a similar relation between \( T_{31} \) and \( T_{42} \). Due to this relation, we can construct the special states with all spin-up (spin-down) with \( T_{43} (T_{32}) \)

\[ |\Psi_N \rangle = T_{43}(\mu_1) \cdots T_{43}(\mu_N) |\Psi_N \rangle \]

Using the commutative relations, we can find

\[ t_H(\mu)|\Psi_N \rangle = \Lambda(\mu)|\Psi_N \rangle + \text{unwanted terms.} \]

where

\[ \Lambda(\mu) = A_4(\mu) \prod_{j=1}^N \frac{\rho_1(\mu_j, \mu)}{i\rho_9(\mu_j, \mu)} - A_3(\mu) \prod_{j=1}^N \frac{-\rho_4(\mu, \mu_j)}{i\rho_9(\mu, \mu_j)} \]
\[ -A_2(\mu) \prod_{j=1}^N \frac{i\rho_{10}(\mu, \mu_j)}{\rho_1(\mu, \mu_j) - \rho_3(\mu, \mu_j)} \]
\[ + A_1(\mu) \prod_{j=1}^N \frac{-i\rho_{10}(\mu, \mu_j)}{\rho_1(\mu, \mu_j) - \rho_3(\mu, \mu_j)} \]

The vanishing unwanted terms gives the Bethe Ansatz equation

\[ [-e^{2h(\mu_j)} \frac{\alpha(\mu_j)}{\gamma(\mu_j)}]^L = 1 \]
Thus, the states $|\Psi_N\rangle$ are really the eigenstates of the transfer matrix $t(\mu)$ if the spectrum parameters are appropriately chosen to satisfy the Bethe Ansatz equation (20).

The general states with $N$ spin-down and $N - M$ spin-up are given by sums of products of the combination of $\mathcal{T}_{21}$, $\mathcal{T}_{31}$, and $\mathcal{T}_{4j}$, $j = 1, 2, 3$ acting on the reference state. Let us consider a special case $N - M = M = 1$. The general form is

$$|\Psi_{1,1}\rangle = \{ f_1(\mu_1, \mu_2)\mathcal{T}_{42}(\mu_1)\mathcal{T}_{43}(\mu_2) + f_2(\mu_1, \mu_2)\mathcal{T}_{43}(\mu_1)\mathcal{T}_{42}(\mu_2) $$

$$+ f_3(\mu_1, \mu_2)\mathcal{T}_{21}(\mu_1)\mathcal{T}_{31}(\mu_2) + f_4(\mu_1, \mu_2)\mathcal{T}_{31}(\mu_1)\mathcal{T}_{21}(\mu_2) $$

$$+ f_5(\mu_1, \mu_2)\mathcal{T}_{41}(\mu_1) + f_6(\mu_1, \mu_2)\mathcal{T}_{41}(\mu_2) \} |\text{vac}\rangle$$

We can determine all coefficients $f_j$ by requiring the r.h.s. of equation (21) to be the eigenvector of the transfer matrix. The general solution is very complicated. Fortunately, we can show by an explicit calculation that $|\Psi_{1,1}\rangle$ is the eigenstate if $f_3 = f_4 = 0$ with appropriate $f_5$ and $f_6$. Using the Super-Yang-Baxter relation, we find the eigenvalue for the case when $f_1 = -f_2$

$$\Lambda(\mu) = A_4(\mu) \prod_{j=1}^{2} \frac{\rho_1(\mu_j, \mu_1)}{i \rho_3(\mu_j, \mu)} + A_1(\mu) \prod_{j=1}^{N} \frac{-i \rho_{10}(\mu, \mu_j)}{\rho_1(\mu, \mu_j) - \rho_3(\mu, \mu_j)} $$

$$+ A_2(\mu) \left\{ \frac{\rho_4(\mu, \mu_1) \rho_{10}(\mu, \mu_2)}{\rho_3(\mu, \mu_1) [\rho_1(\mu, \mu_2) - \rho_3(\mu, \mu_2)]} $$

$$+ \frac{\rho_4(\mu, \mu_2) \rho_{10}(\mu, \mu_1)}{\rho_3(\mu, \mu_2) [\rho_1(\mu, \mu_1) - \rho_3(\mu, \mu_1)]} $$

$$- \left[ \frac{\rho_{10}(\mu, \mu_1)}{\rho_1(\mu, \mu_1) - \rho_3(\mu, \mu_1)} + \frac{\rho_4(\mu, \mu_1)}{\rho_3(\mu, \mu_1)} \right] $$

$$\times \left[ \frac{\rho_{10}(\mu, \mu_2)}{\rho_1(\mu, \mu_2) - \rho_3(\mu, \mu_2)} + \frac{\rho_4(\mu, \mu_2)}{\rho_3(\mu, \mu_2)} \right] \right\}$$

The vanishing unwanted terms gives the Bethe Ansatz equation

$$\left[ \prod_{j=1}^{2} \frac{-\alpha(\mu_j)}{\gamma(\mu_j)} e^{2\hbar(\mu_j)} \right] \Lambda^L = 1$$

For the case $f_1 = f_2$ the eigenvalue and the Bethe Ansatz equations are given by the equations (19) and (20), respectively. It is worthy to point out that these states considered here are not complete. For example, consider the case $f_1 = f_2 = 0$, one can get similar results.
4 Analytic Bethe Ansatz for the Hubbard model

In the last section, we applied the algebraic Bethe Ansatz method to some eigenstates of the Hubbard model. For general states, the straightforward calculation becomes very complicated. In this section, however, we want to discuss an analytic Bethe Ansatz method to the same problem based on the hints given by the above results. We should generalize the standard ABA [9, 10, 11, 12] in which the crossing symmetry and asymptotic behaviour play a key role. We can not apply the same argument to the eigenvalue of the Hubbard model in which there is no such crossing symmetry for the R matrix.

Let us first investigate the special solutions and show how to generalize it from the viewpoint of ABA. One may understand that the analytic property of function $\Lambda(\mu)$ leads to the Bethe Ansatz equation (20). It is clear that $\rho_0(\mu, \mu_j) = 0$ is the simple pole of $\Lambda(\mu)$ (equation (19)). In order to keep the analytic property of the eigenvalue, the residue at such pole must be zero. The Bethe Ansatz equation is nothing but the condition of vanishing residue. Similarly the vanishing residue at the pole $\rho_1(\mu, \mu_j) = \rho_3(\mu, \mu_j)$ gives the same Bethe Ansatz equation. This property can be generalized to all kinds of states with different particles. The eigenvalue function should be analytic and has only superficial simple poles. The vanishing residues at such poles will give the Bethe Ansatz equations.

Let us discuss the general eigenvalue of the Hubbard model. The special solutions (19) and (22) together with some standard knowledge of algebraic Bethe Ansatz contain enough message about the general one. It consists of four terms which are proportional to $A_j(\mu), j = 1, 2, 3, 4$ respectively. The terms involving $A_1$ and $A_4$ are dependent only on the total number of electrons. The other terms depend on both the total number $N$ of electrons and the number $M$ of spin-up electrons as shown in the last section. Thus, the general eigenvalue should be

$$\Lambda(\mu) = A_4(\mu) \prod_{j=1}^{N} \frac{\rho_1(\mu_j, \mu)}{i \rho_0(\mu_j, \mu)} - A_3(\mu) \prod_{j=1}^{N} \frac{-\rho_4(\mu, \mu_j)}{i \rho_0(\mu_j, \mu)} \prod_{m=1}^{M} g_3(\mu, \lambda_m)$$

$$- A_2(\mu) \prod_{j=1}^{N} \frac{-i \rho_{10}(\mu, \mu_j)}{\rho_1(\mu, \mu_j) - \rho_3(\mu, \mu_j)} \prod_{m=1}^{M} g_2(\mu, \lambda_m)$$

$$+ A_1(\mu) \prod_{j=1}^{N} \frac{-i \rho_{10}(\mu, \mu_j)}{\rho_1(\mu, \mu_j) - \rho_3(\mu, \mu_j)}$$

(24)

where $g_2$ and $g_3$ are undetermined functions. The $\mu_j$ and $\lambda_m$ are free parameters. $N$ is the total number of electrons, $M$ the number of spin-up electrons.

Now, we show how the analytic property of the eigenvalue restricts the unknown functions. First, $\Lambda(\mu)$ has two sets of poles related to parameters $\mu_j$. One (case A) is
controlled by the null denominator of the first two terms of equation (24). The another (case B) is from the last two terms. For case A the position of poles is determined by

\[ e^{2h(\mu)} \frac{\alpha(\mu)}{\gamma(\mu)} = e^{2h(\mu_j)} \frac{\alpha(\mu_j)}{\gamma(\mu_j)} \]  (25)

Due to the \(i\pi\)-period of \(h(\mu)\), we can get \(\mu = \mu_j\) in the region \(0 \leq \mu_j \leq \pi\). At these poles, the functions \(\rho_1(\mu, \mu_j) - \rho_3(\mu, \mu_j)\) and \(\rho_{10}(\mu, \mu_j)\) also vanish. But, the ratio is finite. So, the singularity at these poles is dominated by the terms in the first line. The null residue requires

\[ \left[ -e^{2h(\mu_j)} \frac{\alpha(\mu_j)}{\alpha(\mu_j)} \right]^L = \prod_{m=1}^{M} g_3(\mu_j, \lambda_m) \]  (26)

For case B, the position of poles satisfies

\[ 0 = \left[ e^{-h(\mu_j) - h(\mu) \gamma(\mu_j) \alpha(\mu_j)} - e^{h(\mu_j) - h(\mu) \gamma(\mu) \alpha(\mu_j)} \right] \times \left[ e^{h(\mu) - h(\mu_j) \gamma(\mu) \alpha(\mu_j)} - e^{h(\mu) - h(\mu) \gamma(\mu) \alpha(\mu)} \right] \]  (27)

The first part is equal to \(\rho_9(\mu, \mu_j) = 0\). It is not a pole due to \(\rho_{10}(\mu, \mu_j) = 0\) at this point. The real pole locates in \(\tilde{\mu}_j\)

\[ e^{-2h(\tilde{\mu}_j)} \frac{\alpha(\tilde{\mu}_j)}{\gamma(\tilde{\mu}_j)} = e^{2h(\mu_j)} \frac{\alpha(\mu_j)}{\gamma(\mu_j)} \]  (28)

The vanishing residue at \(\mu = \tilde{\mu}_j\) gives

\[ \left[ -e^{2h(\tilde{\mu}_j)} \frac{\gamma(\tilde{\mu}_j)}{\alpha(\tilde{\mu}_j)} \right]^L = \prod_{j=1}^{N} g_2(\mu_j, \lambda_m) \]  (29)

In order to keep the analytic property of the eigenvalue, Equations (26) and (28) must be satisfied simultaneously, which leads to the following functional relation

\[ \prod_{m=1}^{M} g_3(\mu_j, \lambda_m) = \prod_{m=1}^{M} g_2^{-1}(\tilde{\mu}_j, \lambda_m) \]  (30)

We find that it is convenient to introduce the new variables \(k\) and \(k_j\)

\[ e^{ik} = -e^{2h(\mu)} \frac{\alpha(\mu)}{\gamma(\mu)} \]  (31)
In terms of the new variables the relation between the quantities with and without tilde is very simple

\[ \sin(\tilde{k}_j) = \sin(k_j) + i \frac{U}{2} \]  

At this stage, we need to know some properties of the undetermined functions \( g_i \) which hides already in the special solution with \( N = 2, M = 1 \). In terms of \( k_j \), equation (22) changes into

\[
\Lambda(\mu) = \left( -\alpha^2(\mu) e^{h(\mu)} \right)^L \Lambda(k)
\]

\[
\Lambda(k) = \prod_{j=1}^{2i} 2i \cos((k + k_j)/2) \cos((\tilde{k} + k_j)/2)
\]

\[
- e^{-ikL} \prod_{j=1}^{N} \frac{2i \cos((k + k_j)/2) \cos((\tilde{k} + k_j)/2)}{i \sin(k) - i \sin(k_j)}
\]

\[
\times \frac{i2 \sin(k) - i \sin(K_1) - i \sin(k_2) - U/2}{i2 \sin(k) - i \sin(k_1) - i \sin(k_2) + U/2}
\]

\[
- e^{-ikL} \prod_{j=1}^{2} \frac{2i \cos((k + k_j)/2) \cos((\tilde{k} + k_j)/2)}{i \sin(k) - i \sin(k_j) + U/2}
\]

\[
\times \frac{i2 \sin(k) - i \sin(K_1) - i \sin(k_2) + 3U/2}{i2 \sin(k) - i \sin(k_1) - i \sin(k_2) + U/2}
\]

\[
- e^{-i(k+\tilde{k})L} \prod_{j=1}^{2} \frac{2i \cos((k + k_j)/2) \cos((\tilde{k} + k_j)/2)}{i \sin(k) - i \sin(k_j) + U/2}
\]

(33)

It is clear that the eigenvalue has another simple pole \( 2 \sin(k) = \sin(k_1) + \sin(k_2) - U/2 \). The Bethe Ansatz equation ensures the analytic property of the eigenvalue. This strongly suggests that the undetermined functions have the simple poles of the form \( \sin(k) = \text{const} \). On the other hand, the analytic property of the eigenvalue requires that \( g_2 \) and \( g_3 \) must have same poles. Therefore, the general form of \( g_2(\mu, \lambda) \) and \( g_3(\mu, \lambda) \) is

\[
g_2(\mu, \lambda) = \frac{P_2(k, \lambda)}{i \sin(k) - \lambda + U/4}, \quad g_3(\mu, \lambda) = \frac{P_3(k, \lambda)}{i \sin(k) - \lambda + U/4}
\]

(34)

where the function \( P_2(k, \lambda) \) and \( P_3(k, \lambda) \) are integral functions. The general form is
\( P_r(k, \lambda) = \sum_{n=0}^\infty a_r^n(k)(\lambda)^n \). Substituting it into equation (30), we have found

\[
\begin{align*}
  a_0^2(k) &= [i \sin(k) - U/4] a_1^2(k) \\
  a_0^3(k) &= [i \sin(k) + 3U/4] a_1^3(k) \\
  a_n^2(k) &= a_n^2(k) = 0, n \geq 2 \\
  a_1^3(k) &= [a_1^2(\hat{k})]^{-1}
\end{align*}
\]

where \( \hat{k} \) is defined by \( \sin(\hat{k}) = \sin(k) - iU/2 \). Moreover, the function \( a_1^2(k) \) is analytic and has no zero in the complex plane. It will be fixed by the asymptotic behaviour of the transfer matrix. Let us assume \( U \leq 0 \) and \( \mu \rightarrow -i\infty \), then the eigenvalue approaches to

\[
\Lambda(\mu) \rightarrow e^{3L\infty} \{ [a_1^2(\infty)]^M + [a_1^2(\infty)]^{-M} \}.
\]

Comparing this with the asymptotic behaviour of \( t(\mu) \rightarrow e^{3L\infty} \), we get \( a_1^2(\infty) \) is a no-zero constant. Based upon the knowledge of analysis such as Liouville’s theorem on integral functions, we arrive at \( a_1^2(k) \) being a no-zero constant. A special form of \( \Lambda(\mu) \) under \( N = M = 1 \) fixes this constant to be unit. Finally, we arrive at the final results

\[
\Lambda(\mu) = \left( -\alpha^2(\mu) e^{h(\mu)} \right)^L \Lambda(k)
\]

\[
\Lambda(k) = \prod_{j=1}^N \frac{2i \cos((k + k_j)/2) \cos((\tilde{k} + k_j)/2)}{i \sin(k) - i \sin(k_j)}
- e^{-ikL} \prod_{j=1}^N \frac{2i \cos((k + k_j)/2) \cos((\tilde{k} + k_j)/2)}{i \sin(k) - i \sin(k_j)}
\times \prod_{m=1}^M \frac{i \sin(k) - \lambda_m - U/4}{i \sin(k) - \lambda_m + U/4}
- e^{-ikL} \prod_{j=1}^N \frac{2i \cos((k + k_j)/2) \cos((\tilde{k} + k_j)/2)}{i \sin(k) - i \sin(k_j) + U/2}
\times \prod_{m=1}^M \frac{i \sin(k) - \lambda_m + 3U/4}{i \sin(k) - \lambda_m + U/4}

- e^{-i(k + \tilde{k})L} \prod_{j=1}^N \frac{2i \cos((k + k_j)/2) \cos((\tilde{k} + k_j)/2)}{i \sin(k) - i \sin(k_j) + U/2}
\]

(37)

The parameters satisfy the following Bethe Ansatz equations

\[
e^{ik_jL} = \prod_{m=1}^M \frac{i \sin(k_j) - \lambda_m - U/4}{i \sin(k_j) - \lambda_m + U/4}
\]
\[- \prod_{m=1}^{M} \frac{\lambda_r - \lambda_m - U/2}{\lambda_r - \lambda_m + U/2} = \prod_{j=1}^{N} \frac{i \sin(k_j) - \lambda_r + U/4}{i \sin(k_j) - \lambda_m - U/4} \]  

(38)

Differentiating \(\log(\Lambda(\mu))\) at \(\mu = \pi/4\) will give the energy of the Hubbard model, which coincides with the one given in [1]

\[E = \frac{UL}{4} - \frac{NU}{2} - \sum_{j=1}^{N} \cos(k_j) \]  

(39)

We have checked that equations (37) and (38) with \(N = 2, M = 1\) coincide with the result obtained in section 3.

5 ABA for the coupled XY model

In this section we investigate the eigenvalue of the transfer matrix of the coupled XY model with twisted boundary condition.

The L-operator related to the coupled XY model is [2, 3]:

\[L_{m}(\mu) = \begin{pmatrix} e^{h(\mu)}p_{m}^{+}q_{m}^{+} & p_{m}^{-}\tau_{m}^{+} & \sigma_{m}^{+}q_{m}^{-} & e^{h(\mu)}\sigma_{m}^{-}\tau_{m}^{-} \\ p_{m}^{+}\tau_{m}^{+} & e^{-h(\mu)}p_{m}^{+}q_{m}^{-} & e^{-h(\mu)}\sigma_{m}^{-}\tau_{m}^{+} & \sigma_{m}^{+}q_{m}^{-} \\ \sigma_{m}^{+}q_{m}^{+} & e^{-h(\mu)}\sigma_{m}^{-}\tau_{m}^{-} & e^{-h(\mu)}p_{m}^{-}q_{m}^{+} & p_{m}^{-}\tau_{m}^{-} \\ e^{h(\mu)}\sigma_{m}^{+}\tau_{m}^{+} & \sigma_{m}^{+}q_{m}^{-} & p_{m}^{-}\tau_{m}^{+} & e^{h(\mu)}p_{m}^{-}q_{m}^{-} \end{pmatrix} \]  

(40)

where \(\sigma_{m}^{a}\) and \(\tau_{m}^{a}\) are two independent Pauli matrices located in \(m\)-th site. The operator \(p^{\pm}\) and \(q^{\pm}\) read

\[p_{m}^{\pm} = w_{4}(\mu) \pm w_{3}(\mu)\sigma_{m}^{z}, \]

\[q_{m}^{\pm} = w_{4}(\mu) \pm w_{3}(\mu)\tau_{m}^{z}. \]  

(41)

In references [2, 3], it was shown that this L-operator satisfies the Yang-Baxter equation

\[R(\mu, \nu)L_{m}(\mu) \otimes L_{m}(\nu) = L_{m}(\nu) \otimes L_{m}(\mu)R(\mu, \nu) \]  

(42)
The R matrix is
\[
\begin{bmatrix}
\rho_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 & \rho_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & \rho_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \rho_3 & 0 & 0 & \rho_6 & 0 & 0 & \rho_6 & 0 & 0 & -\rho_8 & 0 & 0 & 0 & 0 \\
0 & \rho_{10} & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \rho_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \rho_6 & 0 & 0 & \rho_5 & 0 & 0 & -\rho_7 & 0 & 0 & \rho_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \rho_6 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_6 & 0 & 0 & \rho_5 & 0 & 0 & \rho_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \rho_{10} & 0 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \rho_6 & 0 & 0 & \rho_5 & 0 & 0 & -\rho_7 & 0 & 0 & \rho_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_6 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \rho_9 & 0 & 0 & 0 & 0 & 0 & \rho_9 & 0 & 0 & \rho_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_9 & 0 & 0 & 0 & 0 & 0 & \rho_9 & 0 & 0 & \rho_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_9 & 0 & 0 & 0 & 0 & 0 & \rho_9 & 0 & 0 & \rho_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_1 & \end{bmatrix}
\]

This Yang-Baxter relation ensures the monodromy matrix \( T(\mu) = L_L(\mu) \otimes \cdots \otimes L_1(\mu) \) satisfying the Yang-Baxter relation. In order to simplify our calculation, we choose the ferromagnetic state (all spin-down states) as the reference state. From the explicit expression of the L-operator, we have

\[
T(\mu)|\text{vac}\rangle = \begin{bmatrix}
A_1(\mu) & 0 & 0 & 0 \\
T_{21}(\mu) & A_2(\mu) & 0 & 0 \\
T_{31}(\mu) & 0 & A_3(\mu) & 0 \\
T_{41}(\mu) & T_{42}(\mu) & T_{43}(\mu) & A_4(\mu)
\end{bmatrix} |\text{vac}\rangle
\]

where

\[
A_4(\mu) = [\alpha^2(\mu)e^{h(\mu)}]^L \\
A_2(\mu) = A_3(\mu) = [\alpha(\mu)\gamma(\mu)e^{-h(\mu)}]^L \\
A_1(\mu) = [\gamma^2(\mu)e^{h(\mu)}]^L
\]

Similarly, one can define the transfer matrix \( t(\mu) = stT(\mu) \) and find the eigenvalue of it, which will related to the periodic boundary condition. The eigenvalue of the
diagonal-of-diagonal transfer matrix of this model with periodic condition was found by Bariev [13] in terms of the coordinate Bethe Ansatz method. In order to consider twisted boundary condition, we introduce the generalized transfer matrix

\[ t^g(\mu) = T_{11}(\mu) a_{i\beta_1} + T_{22}(\mu) a_{i\beta_2} + T_{33}(\mu) a_{i\beta_3} + T_{44}(\mu) a_{i\beta_4} \]  

where

\[ a_{i\beta_1} = e^{a_\sigma N_\sigma + a_\tau N_\tau + a_0} \]
\[ a_{i\beta_2} = e^{c_\sigma N_\sigma + c_\tau N_\tau + c_0} \]
\[ a_{i\beta_3} = e^{-i\beta_2} \]
\[ a_{i\beta_4} = e^{-i\beta_1} \]

where \( a_s \) and \( c_s \) are free parameters, \( N_\sigma \) \((N_\tau)\) the total number of \( \sigma \)-spin \((\tau\)-spin). Now, we want to find the eigenvalue of \( t^g(\mu) \) by means of the analytic Bethe Ansatz method. First, by using the algebraic Bethe Ansatz method, we find the eigenvalue of the states with \( N_\tau \)-spin \((\sigma\)-spin) flipping from the reference state. After a long but direct calculation, we arrive at

\[ \Lambda_N(\mu) = e^{a_\sigma N + a_0} [e^{h(\mu)} \alpha^2(\mu)]^L \prod_{j=1}^N \frac{\rho_1(\mu_j, \mu)}{\rho_9(\mu_j, \mu)} \]
\[ + e^{c_\sigma N + c_0} [e^{-h(\mu)} \alpha(\mu) \gamma(\mu)]^L \prod_{j=1}^N \frac{\rho_4(\mu, \mu_j)}{\rho_9(\mu, \mu_j)} \]
\[ + e^{-a_\sigma N + a_0} [e^{-h(\mu)} \alpha(\mu) \gamma(\mu)]^L \prod_{j=1}^N \frac{\rho_{10}(\mu, \mu_j)}{\rho_1(\mu_j, \mu) - \rho_3(\mu_j, \mu)} \]
\[ + e^{-c_\sigma N - c_0} [e^{-h(\mu)} \alpha(\mu) \gamma(\mu)]^L \prod_{j=1}^N \frac{\rho_{10}(\mu, \mu_j)}{\rho_3(\mu, \mu_j) - \rho_1(\mu, \mu_j)} \]  

where \( s = \sigma \) \((s = \tau)\) for all \( \sigma \) \((\tau)\) spin-up states, the parameters \( \mu_j \) are determined by

\[ e^{2h(\mu_j)} \alpha(\mu_j) \gamma(\mu_j) = (-1)^{N+1} e^{(c_s - a_s) N + c_0 - a_0} \]  

From this expression and the similar arguments in the last section, we can write the
general form of the eigenvalue

\[ \Lambda(\mu) = e^{a_0 + a_2 M + a_r (N - M)} \left[ e^{h(\mu) R^2(\mu)} \right]^L \prod_{j=1}^{N} \frac{\rho_1(\mu_j, \mu)}{\rho_9(\mu_j, \mu)} \]

\[ + e^{c_0 + c_2 M + c_r (N - M)} \left[ e^{-h(\mu) R(\mu)} \right]^L \]

\[ \times \prod_{j=1}^{N} \frac{\rho_4(\mu, \mu_j)}{\rho_9(\mu, \mu_j)} \prod_{m=1}^{M} g_3(\mu, \lambda_m) \]

\[ + e^{c_0 - c_2 M - c_r (N - M)} \left[ e^{-h(\mu) R(\mu)} \right]^L \]

\[ \times \prod_{j=1}^{N} \frac{\rho_{10}(\mu, \mu_j)}{\rho_1(\mu_j, \mu) - \rho_3(\mu, \mu_j)} \prod_{m=1}^{M} g_2(\mu, \lambda_m) \]

(50)

Second, consider the singularity of the \( \Lambda(\mu) \) at the poles related to the parameters \( \mu_j \). As done in the Hubbard model, the null residue condition requires the following relation

\[ \prod_{m=1}^{M} g_3(\mu, \lambda_m) = \prod_{m=1}^{M} g_2^{-1}(\bar{\mu}, \lambda_m) \]  

(51)

This equation is same as that in the Hubbard model (equation (34)). So, we can use the results of the Hubbard model.

\[ g_2(\mu, \lambda) = \frac{c i \sin(k) - \lambda - U/4}{c i \sin(k) - \lambda + U/4} \]

\[ g_3(\mu, \lambda) = \frac{1 i \sin(k) - \lambda - U/4}{c i \sin(k) - \lambda + U/4} \]  

(52)

here we have used the same definition of \( k \) as one in the Hubbard model. One should note that in this case, the constant \( c \) in the above equation not being 1 as in the Hubbard model. Taking \( N = M = 1 \) in equation(50) and comparing with equation
(48), one can get \( c = -1 \). Thus, we obtain the final results

\[
\Lambda(\mu) = [e^{\lambda(\mu)}\alpha^2(\mu)]L\Lambda(k)
\]

\[
\Lambda(k) = (-1)^Ne^{a_0+a_{\sigma}M+a_{\tau}(N-M)}\prod_{j=1}^{N} \frac{2\cos((k + k_j)/2) \cos((\tilde{k} + k_j)/2)}{i\sin(k) - i\sin(k_j)}
\]

\[
+e^{c_0+c_{\sigma}M+c_{\tau}(N-M)}\prod_{j=1}^{N} \frac{2\cos((k + k_j)/2) \cos((\tilde{k} + k_j)/2)}{i\sin(k) - i\sin(k_j)}
\]

\[
\times (-1)^{L+M}e^{-ikL} \prod_{m=1}^{M} \frac{i\sin(k) - \lambda_m - U/4}{i\sin(k) - \lambda_m + U/4}
\]

\[
+e^{-c_0-c_{\sigma}M-c_{\tau}(N-M)}\prod_{j=1}^{N} \frac{2\cos((k + k_j)/2) \cos((\tilde{k} + k_j)/2)}{i\sin(k) - i\sin(k_j) + U/2}
\]

\[
\times (-1)^{L+M+N}e^{-ikL} \prod_{m=1}^{M} \frac{i\sin(k) - \lambda_m + 3U/4}{i\sin(k) - \lambda_m + U/4}
\]

\[
+e^{-a_0-a_{\sigma}M-a_{\tau}(N-M)}e^{-i(k+k)L} \prod_{j=1}^{N} \frac{2\cos((k + k_j)/2) \cos((\tilde{k} + k_j)/2)}{i\sin(k) - i\sin(k_j) + U/2}
\]

The Bethe Ansatz equations are

\[
(-1)^{M+N+1+L}e^{ikL} = e^{c_0+(c_\sigma-c_{\sigma})M+(c_\tau-a_\tau)(N-M)}
\]

\[
\times \prod_{m=1}^{M} \frac{i\sin(k_j) - \lambda_m - U/4}{i\sin(k_j) - \lambda_m + U/4}
\]

\[
\prod_{j=1}^{N} \frac{i\sin(k_j) - \lambda_{\tau} + U/4}{i\sin(k_j) - \lambda_{\tau} - U/4} = (-1)^{N+1}e^{2(c_0+c_{\sigma}M+c_{\tau}(N-M))}
\]

\[
\times \prod_{m=1}^{M} \frac{\lambda_{\tau} - \lambda_m - U/2}{\lambda_{\tau} - \lambda_m + U/2}
\]

These are the exact solution of the coupled XY model with twisted boundary condition. It is of interest that it recovers the results of the Hubbard model when \( a_\sigma = a_\tau = c_\sigma = -c_\tau = -i\pi/2 \) and \( a_0 = iL\pi, c_0 = i\pi \). This gives a precise relation between the two models and make it clear why the extra factor appears among the Bethe Ansatz equations by Lieb and Wu \[\text{[1]}\] and by Shastry \[\text{[8]}\]. When \( a_s = c_s = 0, s = \sigma, \tau, 0 \), they reduced into the periodic case. The correspondence between our notation and that in \[\text{[8]}\] is \( 2i\sin(k_j) = z_j^{-1} - z_j \).
6 Twisted boundary condition

In this section, we will discuss the boundary condition related to our generalized transfer matrix (46) and prove the equality of the periodic Hubbard model to the twisted coupled XY model in terms of the transfer matrices and the hamiltonians.

First, we derive the hamiltonian related to the twisted boundary condition by using the Jordan-Wigner transformation on operators.

After a straightforward calculation, we arrive at

\[
H = \sum_{m=1}^{L-1} \left( \sigma_{m+1}^+ a_m^- + \sigma_m^+ \sigma_{m+1}^- + \tau_{m+1}^+ \tau_m^- + \tau_m^+ \tau_{m+1}^- \right) + \frac{U}{4} \sum_{m=1}^{L} \sigma_m^z \tau_m^z \\
+ \exp\{-\epsilon[a_0 + c_0 + (a_\sigma + c_\sigma)N_\sigma + (a_\tau + c_\tau)N_\tau]\} \sigma_{1}^\pm \\
+ \exp\{\epsilon[a_0 + c_0 + (a_\sigma + c_\sigma)N_\sigma + (a_\tau + c_\tau)N_\tau]\} \sigma_{1}^- \\
+ \exp\{-\epsilon'[a_0 - c_0 + (a_\sigma - c_\sigma)N_\sigma + (a_\tau - c_\tau)N_\tau]\} \tau_{1}^+ \\
+ \exp\{\epsilon'[a_0 - c_0 + (a_\sigma - c_\sigma)N_\sigma + (a_\tau - c_\tau)N_\tau]\} \tau_{1}^-
\]

where

\[
\epsilon = \begin{cases} 
1 & \text{spin up} \\
-1 & \text{spin down}
\end{cases}, \quad \epsilon' = \begin{cases} 
1 & \text{spin up} \\
-1 & \text{spin down}
\end{cases}
\]

This means that the hamiltonian (55) gives the coupled XY model discussed in the last section with the twisted boundary condition

\[
\sigma_{L+1}^\pm = e^{-\epsilon(a_\sigma + c_\sigma)} \exp\{\pm \epsilon[a_0 + c_0 + (a_\sigma + c_\sigma)N_\sigma + (a_\tau + c_\tau)N_\tau]\} \sigma_{1}^\pm \\
\tau_{L+1}^\pm = e^{-\epsilon'(a_\sigma - c_\sigma)} \exp\{\pm \epsilon[a_0 - c_0 + (a_\sigma - c_\sigma)N_\sigma + (a_\tau - c_\tau)N_\tau]\} \tau_{1}^\pm
\]

Comparing the Bethe Ansatz equations (37), (38), (53) and (54), we find that they are same if the free parameters are fixed

\[
a_\sigma = a_\tau = c_\sigma = -c_\tau = -\frac{i\pi}{2}, \quad a_0 = iL\pi, \quad c_0 = i\pi
\]

Let us prove the connection (58) by using the Jordan-Wigner transformation on transfer matrix. The Jordan-Wigner transformation on operators is defined by

\[
\begin{pmatrix}
\sigma_m^+ \\
\sigma_m^-
\end{pmatrix} = V_{m}^2 \begin{pmatrix}
a_m^+ \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
v_m^2
\end{pmatrix}
\begin{pmatrix}
0 \\
a_m^+
\end{pmatrix}
\]

\[
\begin{pmatrix}
\tau_m^+ \\
\tau_m^-
\end{pmatrix} = V_{m}^2 \begin{pmatrix}
a_m^+ \\
0
\end{pmatrix}
\begin{pmatrix}
u_m^2 \\
u_m^2
\end{pmatrix}
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\begin{pmatrix}
u_m^2 \\
u_m^2
\end{pmatrix}
\begin{pmatrix}
a_m^+ \\
a_m^+
\end{pmatrix}
\]

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with the definition
\[ v_{ms} = \exp\left\{ i\frac{\pi}{2} \sum_{k=1}^{m-1} (n_{ks} - 1) \right\} \]
\[ u_{ms} = \exp\left\{ i\frac{\pi}{2} (n_{ks} - 1) \right\} \]
\[ r_{m} = \exp\left\{ i\frac{\pi}{2} \sum_{k=m+1}^{L} (n_{k\uparrow} - 1) \right\} \]

Under the transformation, the L-operator changes into \[ \mathcal{L}_m(\mu) = V_{m+1} L_m(\mu) V_m \] (62)

where
\[ V_{m+1} = V_m (U_m \uparrow \otimes U_m \uparrow) \]
\[ = V_m U_m \uparrow \otimes V_m U_m \downarrow \] (63)
\[ U_{m,s} = \text{dia}(u_{m,s}, u_{m,s}^{-1}) \]

Substituting equation (62) into \( t_H(\mu) \), we obtain
\[ t_H(\mu) = T_{11}(\mu) - T_{22}(\mu) - T_{33}(\mu) + T_{44}(\mu) \]
\[ = e^{\beta_1} T_{11}(\mu) + e^{\beta_2} T_{22}(\mu) + e^{\beta_3} T_{33}(\mu) + e^{\beta_4} T_{44}(\mu) \] (64)

where
\[ e^{\beta_1} = \exp\left\{ -i\frac{\pi}{4} \sum_{j=1}^{L} (\sigma_j^z + \tau_j^z - 2) \right\} = e^{-\beta_4} \]
\[ e^{\beta_2} = \exp\left\{ -i\frac{\pi}{4} \sum_{j=1}^{L} (\sigma_j^z - \tau_j^z) \right\} = e^{-\beta_3} \] (65)

This is exactly equal to equation (46) with condition (58). This finished our proof.

It is worthy to point out that Wadati et al \[ \text{[5]} \] also applied the Jordan-Wigner transformation to the L-operator and the R matrix. In the derivation of the Hamiltonian of the Hubbard model, they impose the periodic boundary condition. But, they did not consider the relation between the boundaries.

In the rest of this section, we give another independent proof in terms of the Hamiltonians. The periodic Hubbard model is
\[ H = -\sum_{m=1,s}^{L-1} (a_{m+1,s}^+ a_{m,s} + a_{m,s}^+ a_{m+1,s}) + \sum_{m=1,s}^{L} \left( n_{m\uparrow} - \frac{1}{2} \right) \left( n_{m\downarrow} - \frac{1}{2} \right) - \sum_{s=\uparrow,\downarrow} (a_{1,s}^+ a_{L,s} + a_{L,s}^+ a_{1,s}) \] (66)
where, we have use the periodic condition
\[ a_{L+1,s}^+ = a_{1,s}^+, a_{L+1,s} = a_{1,s}. \] (67)

Using the Jordan-Wigner transformation, we can obtain
\[
\begin{align*}
    a_{m+1,\uparrow}^+ a_{m,\uparrow} + a_{m,\uparrow}^+ a_{m+1,\uparrow} &= - (\sigma_{m+1}^+ \sigma_m^- + \sigma_m^+ \sigma_{m+1}^-) \\
    a_{m+1,\downarrow}^+ a_{m,\downarrow} + a_{m,\downarrow}^+ a_{m+1,\downarrow} &= - (\tau_{m+1}^+ \tau_m^- + \tau_m^+ \tau_{m+1}^-) \\
    a_{1,\uparrow}^+ a_{L,\uparrow} + a_{L,\uparrow}^+ a_{1,\uparrow} &= \exp\left\{ \frac{i\pi}{2} \sum_{j=1}^L (\sigma_j^z - \frac{1}{2}) \right\} (\sigma_L^- \sigma_1^+ + \sigma_1^- \sigma_L^+) \\
    a_{1,\downarrow}^+ a_{L,\downarrow} + a_{L,\downarrow}^+ a_{1,\downarrow} &= \exp\left\{ \frac{i\pi}{2} \sum_{j=1}^L (\tau_j^z - \frac{1}{2}) \right\} (\tau_L^- \tau_1^+ + \tau_1^- \tau_L^+) 
\end{align*}
\] (68)

The Hamiltonian changes into
\[
\begin{align*}
    H &= \sum_{m=1}^{L-1} (\sigma_{m+1}^+ \sigma_m^- + \sigma_m^+ \sigma_{m+1}^-) - \exp\left\{ \frac{i\pi}{2} \sum_{j=1}^L (\sigma_j^z - \frac{1}{2}) \right\} (\sigma_L^- \sigma_1^+ + \sigma_1^- \sigma_L^+) \\
    &\quad + \sum_{m=1}^{L-1} (\tau_{m+1}^+ \tau_m^- + \tau_m^+ \tau_{m+1}^-) - \exp\left\{ \frac{i\pi}{2} \sum_{j=1}^L (\tau_j^z - \frac{1}{2}) \right\} (\tau_L^- \tau_1^+ + \tau_1^- \tau_L^+) \\
    &\quad + \frac{U}{4} \sum_{m=1}^L \sigma_m^z \tau_m^z
\end{align*}
\] (69)

Therefore, under the Jordan-Wigner transformation, the Hamiltonian of the Hubbard model becomes one of the coupled XY model with twisted boundary condition
\[
\begin{align*}
    \sigma_{L+1}^\pm &= \exp\left\{ \pm \frac{i\pi}{2} \sum_{j=1}^L (\sigma_j^z - 1) \right\} \sigma_1^\pm \\
    \tau_{L+1}^\pm &= \exp\left\{ \pm \frac{i\pi}{2} \sum_{j=1}^L (\tau_j^z - 1) \right\} \tau_1^\pm 
\end{align*}
\] (70)

which is coincident with equations (57) and (58).

## 7 Concluding remarks

In this paper, we have found the eigenvalues of the Hubbard model and the coupled XY model with twisted boundary condition by using the analytic Bethe Ansatz method. We have shown how they are equal on the levels Hamiltonian and the transfer matrix by
using the Jordan-Wigner transformation. The power expansion of \( \log(t^q(\mu)) \) in terms of \( \mu \) will give explicitly the infinite number of conserved quantities. In this level we claim the coupled XY model with twisted boundary condition is integrable.

It is worthy to point out that we consider here only a set of special boundary conditions. It is not difficult to generalize to other kinds of twisted boundary conditions. For the open boundary, one should consider the solution of the reflection equations. This will be related to the surface critical behaviour of the system.

Note: After finishing this paper, we was told by Prof. Wadati that Ramos and Martin [14] found the eigenvalue of the Hubbard model by using the algebraic Bethe Ansatz method, which recovers partly our results from the different approach. In [14], they also notice the effect of boundary condition, but they did not discuss it in detail.

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