Long rainbow arithmetic progressions

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Abstract

Define $T_k$ as the minimal $t \in \mathbb{N}$ for which there is a rainbow arithmetic progression of length $k$ in every equinumerous $t$-coloring of $[tn]$ for all $n \in \mathbb{N}$. Jungić, Licht (Fox), Mahdian, Nešetřil and Radoičić [6] proved that $\left\lfloor \frac{k^2}{4} \right\rfloor \leq T_k \leq k(k-1)^2/2$. We almost close the gap between the upper and lower bounds by proving that $T_k \leq k^2 e^{(\ln \ln k)^2(1+o(1))}$. Conlon, Fox and Sudakov [4] have independently shown a stronger statement that $T_k = O(k^2 \log k)$.

1 Introduction

An equinumerous coloring of any set of objects is a coloring in which each color is used the exact same number of times in the coloring. Given a coloring of $[n]$, an arithmetic progression in $[n]$ is rainbow if each term of the arithmetic progression has a different color. We denote a $k$-term arithmetic progression by the shorthand $k$-AP.

Jungić, Licht (Fox), Mahdian, Nešetřil and Radoičić [6] defined $T_k$ to be the minimal $t$ so that every equinumerous $t$-coloring of $[tn]$ contains a rainbow $k$-AP for every $n \in \mathbb{N}$. They proved the bounds $\left\lfloor \frac{k^2}{4} \right\rfloor \leq T_k \leq k(k-1)^2/2$ for every $k \geq 3$, and furthermore they conjectured that $T_k = \Theta(k^2)$. Little is known about exact values of $T_k$: Axenovich and Fon-Der-Flass [2] and Jungić and Radoičić [7] independently proved that $T_3 = 3$, and this remains the only value of $k$ for which $T_k$ is known exactly. Variations of this problem to understand the anti-van der Waerden numbers have been considered by Butler et al. [3].

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Quite recently, Geneson [5] proved the upper bound $T_k = k^{2 + o(1)}$. Geneson [5] achieved this improvement by making a more careful study of the possible divisors of the differences $a_{j+1} - a_j$, where $a_j$ and $a_{j+1}$ have the same color, and by utilizing the Kővári-Sós-Turán theorem. In this note, we improve the upper bound in [5] to almost match the lower bound of [6].

**Theorem 1.** $T_k \leq k^2 e^{(1+o(1)) (\log \log k)^2}$, as $k \to \infty$.

A stronger result, that $T_k = O(k^2 \log k)$, was obtained independently by Conlon, Fox and Sudakov [4]. Compared to our proof, their method considers fewer $k$-APs, but they are able to obtain a better bound because they overcount each $k$-AP only once. Our proof is self-contained.

## 2 Proof of Theorem 1

Let $t$ be the minimum number so that there is a rainbow $k$-AP for every equinumerous $t$-coloring of $[tn]$ for every $n \geq 1$. From the bounds of [6], we may assume that $\lfloor \frac{k^2}{4} \rfloor \leq t < k^3$. Furthermore, the number of $k$-APs in $[tn]$ is greater than

$$\frac{tn(tn - 3(k - 1))}{2(k - 1)}.$$

**Proof of Theorem 1.** Let $D := \{d \in [tn/10k, tn/2k] : \nu_k(d) < (1 + o(1)) \ln \ln k\}$, where $\nu_k(d)$ is the number of prime divisors of $d$ that are at most $k$, counted with multiplicity. Here the $o(1)$ term is as $k \to \infty$.

**Lemma 2.** $|[tn/10k, tn/2k] \setminus D| = o(tn/k)$.

**Proof of Lemma 2.** A simple modification of an argument for Turán’s result [8] that almost all integers at most $n$ have about $\ln \ln n$ prime factors (see, for instance, Alon and Spencer [11, pp. 45–46]) shows that the number of integers that are at most $tn/2k$ and which have more than $(1 + o(1)) \ln \ln k$ prime divisors at most $k$ is $o(tn/k)$; we omit the details.

Let $A$ be the set of $k$-APs in $[tn]$ with difference in the set $D$. We have that

$$|A| \geq \frac{t^2 n^2}{11k}. \quad (1)$$

We count the number of non-rainbow $k$-APs in $A$. Each such non-rainbow $k$-AP contains a monochromatic pair $(a, b)$. There are $tn$ choices for $a$, and given a choice of $a$, there are at most $n$ choices for $b$.

We claim that for any pair $(a, b)$, $a, b \in [tn]$, the number of $k$-APs in $A$ containing $(a, b)$ is bounded by $k \cdot e^{(1+o(1))(\log \log k)^2}$. Indeed, either $b - a$ has a representation of the form
\( b - a = dm \), with \( d \in D \) and \( m \leq k \), or it has no such representation. In the case that \( b - a \) has such a representation, then \( b - a \) has at most \( \log_2 k + (1 + o(1)) \ln \ln k \) prime factors at most \( k \) (with \( \log_2 k \) factors coming from \( m \) and \( (1 + o(1)) \ln \ln k \) factors coming from \( d \)). Therefore, the number of ways to factorize \( b - a = dm \) is the number of ways to select at most \( (1 + o(1)) \ln \ln k \) prime factors among all the \( \log_2 k + (1 + o(1)) \ln \ln k \) prime factors that \( b - a \) has. That number is bounded above by

\[
\left( \frac{\log_2 k + (1 + o(1)) \ln \ln k}{\ln \ln k} \right) \leq e^{(1+o(1)) (\ln \ln k)^2}.
\]

Finally, given \( m \), there are at most \( k \) choices for the positions of \( a \) and \( b \) in a \( k \)-AP. This implies that the number of non-rainbow \( k \)-APs containing both \( a \) and \( b \) is at most \( ke^{(1+o(1)) (\log \log k)^2} \).

Therefore, there are at most \( (tn)(n)(ke^{(1+o(1)) (\log \log k)^2}) \) non-rainbow \( k \)-APs in \( A \). Combining this with the bound for the number of \( k \)-APs in \( A \) from (1), an upper bound for \( T_k \) is given by the smallest \( t \) satisfying

\[
tn^2 ke^{(1+o(1)) (\log \log k)^2} \leq \frac{t^2 n^2}{11 k}.
\]

It suffices to take \( t = k^2 e^{(\log \log k)^2 (1+o(1))} \), completing the proof. \( \square \)

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