Scaling laws of turbulent dynamos

Comportements asymptotiques des dynamos turbulentes

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Abstract

We consider magnetic fields generated by homogeneous isotropic and parity invariant turbulent flows. We show that simple scaling laws for dynamo threshold, magnetic energy and Ohmic dissipation can be obtained depending on the value of the magnetic Prandtl number.

keywords: dynamo; turbulence; magnetic field

Version française abrégée Il est à présent admis que les champs magnétiques des étoiles voire même des galaxies sont engendrés par l’écoulement de fluides conducteurs de l’électricité [1,2,3]. Ceux-ci impliquent des nombres de Reynolds cinétique, $Re$, et magnétique, $Rm$, très élevés ($Re = VL/\nu$, $Rm = \mu_0 \sigma V L$, où $V$ est l’écart-type des fluctuations de vitesse, $L$, l’échelle intégrale de l’écoulement, $\nu$, la viscosité cinétique du fluide, $\sigma$, sa conductivité électrique et $\mu_0$, la perméabilité magnétique). Aucune expérience de laboratoire ou simulation numérique directe des équations de la magnétohydrodynamique, ne permet l’étude du problème dans des régimes de paramètres, $Re$ et $Rm$, d’intérêt astrophysique. Il est donc utile de considérer des hypothèses plausibles afin de pousser plus loin l’analyse dimensionnelle qui, à partir des paramètres $V$, $L$, $\nu$, $\sigma$, $\mu_0$ et de la densité du fluide $\rho$, prédit pour le seuil de l’effet dynamo et la densité moyenne d’énergie magnétique, $B^2/2\mu_0$, saturée non linéairement au-delà du seuil,

\[ R_m^c = f(Re), \]

\[ \frac{B^2}{\mu_0} = \rho V^2 g(R_m, Re). \]

Dans le cas d’un écoulement turbulent homogène isotope, donc de vitesse moyenne nulle, et invariant par symétrie plane, donc sans hétélicité, les résultats des simulations numériques les plus performantes réalisées à ce jour montrent que $R_m^c$ augmente continuellement en fonction de $Re$ [5]. Schekochihin et al. proposent que deux scénarios extrêmes, schématisées dans la figure 1, seront susceptibles d’être observés lorsque les ordinateurs auront acquis la puissance requise pour effectuer des calculs à $Re$ plus élevé : (i) une saturation $R_m^c \rightarrow$ constante, ou alors (ii) une croissance de la forme $R_m^c \propto Re$. D’autres simulations numériques directes, réalisées avec des écoulements turbulents possédant un champ de vitesse moyen de géométrie fixée $\nabla(r) \neq 0$, semblent suivre le scénario (i) [13].
Lorsque le nombre de Prandtl magnétique, \( P_m = R_m/Re = \mu_0 \sigma \nu \), est faible, \( P_m \ll 1 \), l'échelle de dissipation Joule du champ magnétique, \( l_o = LR_m^{-3/4} \), est grande par rapport à l'échelle de Kolmogorov \( l_K = LR^{-3/4} \). Le champ magnétique se développe donc à une échelle suffisamment grande pour ne pas être affecté par la viscosité cinématique. Cette hypothèse, couramment effectuée en turbulence, permet de conclure en faveur du scénario (i). En effet, si \( \nu \) n'est pas pris en compte, l'analyse dimensionnelle impose \( R_m^c \rightarrow \) constante. Il n'est donc pas surprenant que les modélisations des grandes échelles, qui ne résolvent pas les échelles dissipatives, donnent ce résultat. Le scénario (i) sera donc toujours observé à \( P_m \) suffisamment faible sous réserve bien sûr que l'on ait dynamo.

Il est cependant utile d'analyser le scénario (ii) d'autant plus que, comme nous pouvons le remarquer, il correspond à la prédiction faite par Batchelor en 1950 [4]. En se basant sur une analogie entre l'équation de l'induction et celle de la vorticité, Batchelor avait estimé que le seuil d'une dynamo engendrée par un écoulement turbulent devait correspondre à \( P_m \) d'ordre unité, soit \( R_m^c \propto Re \). Même si nous savons aujourd'hui que l'analyse de Batchelor est discutable, il est intéressant de déterminer sous quelle hypothèse minimale sa prédiction est correcte. Supposons donc que nous nous limitions aux modes instables de champ magnétique, suffisamment localisés au sein de l'écoulement afin de ne pas être affectés par les conditions aux limites. Il est alors possible de ne pas prendre en compte l'échelle spatiale \( L \), et l'analyse dimensionnelle impose pour le seuil, \( P_m = \) constante, soit le scénario (ii) \( R_m^c \propto Re \).

Les scénarios considérés ci-dessus conduisent aussi à des prédicitions différentes pour la densité d'énergie magnétique engendrée par effet dynamo. Le scénario (i) qui consiste à ne pas prendre en compte \( \nu \) revient à négliger la dépendance en \( Re \) de \( g(R_m, Re) \) dans (2). Au voisinage du seuil, \( V \) est déterminé par \( R_m^c \sim \mu_0 \sigma VL \) et \( g(R_m) \propto R_m - R_m^c \) dans le cas d'une bifurcation supercritique. Il en résulte [12]
\[
B^2 \propto \frac{\rho}{\mu_0 (\sigma L)^2} (R_m - R_m^c).
\]
(3)

Loin du seuil pour \( P_m \ll 1 \), \( Re \gg R_m \gg R_m^c \), on peut supposer que \( B \) ne dépend plus de \( \sigma \) à condition que le champ magnétique se développe à une échelle plus grande que \( l_o \). Il en résulte alors l'équipartition entre énergie magnétique et cinétique, \( B^2/\mu_0 \propto \rho V^2 \), tel que supposé initialement par Biermann et Schlüter [15].

Un résultat complètement différent est obtenu dans le scénario (ii). Il convient de considérer les paramètres du problème sous la forme équivalente, \( B, \epsilon = V^3/L, L, \nu, \sigma, \mu_0 \) et \( \rho \). En effet, le champ magnétique à petite échelle est alimenté par la puissance par unité de masse \( \epsilon \) qui cascade depuis l'échelle intégrale, et il est donc important de conserver ce paramètre même si l'on ne prend pas en compte explicitement \( L \). L'analyse dimensionnelle conduit alors à
\[
\frac{B^2}{\mu_0} = \rho \sqrt{\epsilon c} h(P_m) = \frac{\rho V^2}{\sqrt{Re}} h(P_m),
\]
(4)
qui, pour \( P_m \sim 1 \), n'est autre que le résultat obtenu par Batchelor en supposant que la saturation correspond à l'équipartition entre l'énergie magnétique et l'énergie cinétique à l'échelle de Kolmogorov.

Revenons au cas \( P_m \ll 1 \) qui correspond aux écoulements de métaux liquides et plasmas à l'origine du champ magnétique des planètes et des étoiles (\( P_m < 10^{-5} \)). Dans ce cas, le champ magnétique se développe à des échelles a priori comprises entre \( L \) et \( l_o \) avec \( l_o \gg l_K \), et il en résulte que \( R_m^c \) ne dépend pas de variations de \( P_m \) (ou de \( Re \)) et que \( B^2/\mu_0 = \rho V^2 g(R_m) \) (scénario (i)). Intéressons nous à la puissance dissipée par effet Joule par une telle dynamo. Il faut à cet effet déterminer à quelles échelles se développe le champ magnétique. Utilisons pour cela un argument à la Kolmogorov en supposant que dans la zone inertielle, c'est à dire pour les nombres d'onde \( k \) tels que \( k l_o \ll 1 \ll kL \), la puissance spectrale \( |\mathbf{B}(k)|^2 \) est indépendante de \( L, \sigma \) and \( \nu \). Il en résulte
\[
|\mathbf{B}|^2 \propto \mu_0 \rho c \frac{\epsilon}{2} k^{-5/3}.
\]
(5)
Ceci n'est pas la seule possibilité parmi les nombreuses prédicitions relatives au spectre de la turbulence magnétodynamique, mais dans le cas présent, c'est probablement la plus simple. L'intégration sur \( k \) redonne l'équipartition \( B^2/\mu_0 \propto \rho V^2 \). La contribution dominante à l'effet Joule provient de l'échelle \( l_o \). Nous obtenons
\[
\frac{\mathbf{j}^2}{\sigma} = \frac{1}{\mu_0} \int |\mathbf{j}|^2 dk \propto \frac{1}{\mu_0} \int k^2 |\mathbf{B}|^2 dk \propto \frac{\rho}{\mu_0 \sigma} c \frac{\epsilon}{2} l_o^{4/3} \propto \rho \frac{V^3}{L},
\]
(6)

où \( \mathbf{j} \) est le vecteur densité de courant. Nous constatons donc que la dissipation Joule est du même ordre que la puissance totale disponible. Remarquons qu'il en serait de même pour une dynamo de Batchelor suivant le scénario (ii) pour \( P_m \sim 1 \), car bien que la densité d'énergie soit plus faible, l'échelle caractéristique du champ magnétique l'est également.
1. Introduction

It is now believed that magnetic fields of stars and possibly galaxies are generated by the motion of electrically conducting fluids through the dynamo process [1,2,3]. These flows involve huge kinetic, $Re$, and magnetic, $R_m$, Reynolds numbers ($Re = VL/\nu$, $R_m = \mu_0 \sigma V L$, where $V$ is the rms velocity amplitude, $L$ is the integral length scale, $\nu$ is the kinematic viscosity of the fluid, $\sigma$ is its electrical conductivity and $\mu_0$ is the magnetic permeability).

No laboratory experiments, neither direct numerical simulations are possible in the range of $Re$ and $R_m$ involved in astrophysical flows. It is thus interesting to try to guess scaling laws for the magnetic field using some simple model hypothesis. We consider here the minimum set of parameters, $V$, $L$, $\nu$, $\mu_0$, $\sigma$ and $\rho$, the fluid density. We note that discarding global rotation makes our results certainly invalid for many astrophysical objects but not all of them. Rotation is indeed not assumed important for the galaxies which do not display a large scale coherent magnetic field [1,2,3].

Calling $B$ its rms value, dimensional analysis gives

$$R_m^c = f(Re),$$

for the dynamo threshold, and

$$\frac{B^2}{\mu_0} = \rho V^2 g(R_m, Re),$$

for the mean magnetic energy density in the non-linearly saturated regime. Our aim is to determine $f$ and $g$ in various regions of the parameter space ($R_m, Re$), assuming that turbulence is homogeneous, isotropic and parity invariant (thus with no mean flow and no mean magnetic field generation through an alpha effect). As already mentioned, this may look like an academic exercise compared to most natural dynamos. It is however not more academic that the concept of homogeneous and isotropic turbulence with respect to real turbulent flows. We thus expect that our simple arguments may shed some light on open problems concerning the effect of turbulence on the dynamo threshold and on the dynamic equilibrium between magnetic and kinetic energy.

The dependence of the dynamo threshold $R_m^c = f(Re)$ in the limit of large $Re$ is still an open problem, even in the case of a homogeneous isotropic and parity invariant turbulent flow. Note that parity invariance prevents the generation of a large scale magnetic field via an alpha effect type mechanism and isotropy implies zero mean flow. Recent direct numerical simulations show that $R_m^c$ keeps increasing with $Re$ at the highest possible resolution without any indication of a possible saturation [5]. Schekochihin et al. thus propose that two limit scenarios, sketched in figure 1, could be observed when computers will be able to reach higher $Re$: (i) saturation, $R_m^c \rightarrow$ constant, or (ii) increasing threshold in the form $R_m^c \propto Re$.

A lot of work has been performed on the determination of $R_m^c$ as a function of $Re$ for turbulent dynamos in the limit of large $Re$ (or small $P_m$). We recall that (ii) has been proposed by Batchelor in one of the first papers on turbulent dynamos [4]. A lot of analytical studies have been also performed, mostly following Kazantsev’s model [6] in order to show that purely turbulent flows can generate a magnetic field. Kazantsev considered a random homogeneous and isotropic velocity field, $\delta$-correlated in time and with a wave number spectrum of the form $k^{-p}$. He showed that for $p$ large enough, generation of a homogeneous isotropic magnetic field with zero mean value, takes place. This is a nice model but its validity is questionable for realistic turbulent flows. However, Kazantsev’s model has been extrapolated to large $Re$. Various predictions, $R_m^c \propto Re$ [7], $R_m^c \rightarrow$ constant $\approx 400$ for velocity spectra with $3/2 < p < 3$ and no dynamo otherwise [8], or dynamo for all possible slopes of the velocity spectrum $1 < p < 3$ [9] have been found. These discrepancies show that extrapolation of Kazantsev’s model to realistic turbulence cannot be rigorous. The calculation is possible only in the case of a $\delta$-correlated velocity field in time, and $\delta(t - t')$, which has the dimension of the inverse of time, should then be replaced by a finite eddy turn-over time in order to describe large $Re$ effects. As already noticed, its choice is crucial to determine the behavior of $R_m^c$ versus $Re$.

A different problem about turbulent dynamos has been considered more recently. It concerns the effect of turbulent fluctuations on a dynamo generated by a mean flow. The problem is to estimate to which extent the dynamo threshold computed as if the mean flow were acting alone, is shifted by turbulent fluctuations. This question has been addressed only recently [10] and should not be confused with dynamo generated by random flows with zero mean. It has been shown that weak turbulent fluctuations do not shift the dynamo threshold of the mean flow at first order. In addition, in the case of small scale fluctuations, there is no shift at second order either, if the fluctuations have no helicity. This explains why the observed dynamo threshold in Karlsruhe and Riga experiments [11] has been found in good agreement with the one computed as if the mean flow were acting alone, i.e. neglecting turbulent fluctuations. Recent direct numerical simulations have shown that in the presence
The magnetic Prandtl number, $P_m = \frac{R_m}{Re} = \frac{\mu_0 \sigma \nu}{\sigma}$, is small, $P_m \ll 1$, the Ohmic dissipative scale, $l_\sigma = LR_m^{-3/4}$ is much larger than the Kolmogorov length $l_K = LR_e^{-3/4}$. Thus, if there is dynamo action, the magnetic field grows at scales much larger than $l_K$ and does not depend on kinematic viscosity. This hypothesis is currently made for large scale quantities in turbulence and if correct, scenario (i) should be followed. If $\nu$ is discarded, $R_m^c = \text{constant}$ indeed follows from dimensional analysis. It is thus not surprising that numerical models that do not resolve viscous scales, all give this result, although the value of the constant seems to be strongly dependent on the flow geometry and on the model. We conclude that if dynamo action is observed for $P_m \ll 1$, the dynamo threshold is

$$R_m^c \rightarrow \text{constant when } Re \rightarrow \infty.$$  

However, we emphasize that no clear-cut demonstration of dynamo action by homogeneous isotropic and parity invariant turbulence exists for $P_m \ll 1$. Experimental demonstrations as well as direct numerical simulations all involve a mean flow and analytical methods extrapolated to $P_m \ll 1$ are questionable.

It may be instructive at this stage to recall the study on turbulent dynamos made more than half a century ago by Batchelor [4]. Using a questionable analogy between the induction and the vorticity equations, he claimed that the dynamo threshold corresponds to $P_m = 1$, i.e. $R_m^c \propto Re$, using our choice of dimensionless parameters (scenario (ii)).

It is now often claimed that Batchelor’s criterion $P_m > 1$ for the growth of magnetic energy in turbulent flows is incorrect. However, the weaker criterion $P_m > \text{constant}$ (scenario (ii)) has not yet been invalidated.
by direct numerical simulations or by an experimental demonstration without mean flow. It is thus of interest to
determine the minimal hypothesis for which Batchelor’s predictions for dynamo onset is obtained using dimensional
arguments. To wit, assume that the dynamo eigenmodes develop at small scales such that the threshold does not
depend on the integral scale \( L \). Then, discarding \( L \) in our set of parameters, dimensional analysis gives at once
\( P_m = P_m^c = \text{constant for the dynamo threshold, i.e.} \)
\[
R_m^c \propto \text{Re}. 
\]  
(10)

It has been sometimes claimed that a non zero mean flow is necessary to get a dynamo following scenario (i).
However, we note that even for a slow dynamo, i.e., growing on a diffusive time scale, the largest scales look
stationary for a dynamo mode at wave length \( l_s \). For Kolmogorov turbulence, we indeed have, \( \mu_0 \sigma l_s^2/(L/V) \propto R_m^{-1/2} \ll 1 \). This remains true for a \( k^{-p} \) spectrum for \( p < 3 \).

3. Mean magnetic energy density

Dimensional arguments can be also used to determine scaling laws for the mean magnetic energy density. For
\( P_m \ll 1 \) (scenario (i)), discarding \( \nu \) gives
\[
\frac{B^2}{\mu_0} = \rho V^2 g_0(R_m), 
\]  
(11)
where \( g_0 \) is an arbitrary function. Close to threshold, the \( \text{rms} \) velocity \( V \) is given by \( \mu_0 \sigma V L \sim R_m^c \). In the case of
a supercritical bifurcation, \( g_0(R_m) \propto R_m - R_m^c \), and we obtain [12]
\[
B^2 \propto \frac{\rho}{\mu_0(\sigma L)^2} (R_m - R_m^c). 
\]  
(12)
Far from threshold, \( \text{Re} \gg R_m \gg R_m^c \), one could assume that \( B \) no longer depends on \( \sigma \) provided that the magnetic
field mostly grows at scales larger than \( l_s \). We then obtain equipartition between magnetic and kinetic energy
densities,
\[
B^2/\mu_0 \propto \rho V^2, 
\]  
(13)
as assumed by Biermann and Schlüter [15].

A completely different result is obtained in scenario (ii). Let us first recall that according to Batchelor’s analogy
between magnetic field and vorticity [4], the magnetic field should be generated mostly at the Kolmogorov scale,
\( l_K = L \text{Re}^{-5/4} \), where the velocity gradients are the strongest. He then assumed that saturation of the magnetic
field takes place for \( (B^2)/\mu_0 \propto \rho v_K^2 = \rho V^2/\sqrt{\text{Re}} \), where \( v_K \) is the velocity increment at the Kolmogorov scale,
\( v_K^2 = \sqrt{\nu} \epsilon = V^3/L \) is the power per unit mass, cascading from \( L \) to \( l_K \) in the Kolmogorov description of
turbulence.

This can be easily understood, \( \epsilon = V^3/L \) being the power per unit mass available to feed the dynamo, it may
be a wise choice to keep it, instead of \( V \) in our set of parameters, thus becoming \( B, \rho, \epsilon, L, \nu, \mu_0 \) and \( \sigma \). Then, if
we consider dynamo modes that do not depend on \( L \), we obtain at once
\[
\frac{B^2}{\mu_0} = \rho \sqrt{\nu \epsilon} h(P_m) = \frac{\rho V^2}{\sqrt{\text{Re}}} h(P_m) 
\]  
(14)
for saturation, where \( h(P_m) \) is an arbitrary function of \( P_m \). Close to dynamo threshold, \( P_m \approx P_m^c \), we have
\( h(P_m) \propto P_m - P_m^c \), if the bifurcation is supercritical. Only the prefactor \( \rho V^2/\sqrt{\text{Re}} \) of (14) is the kinetic energy
at Kolmogorov scale, that was assumed to be in equipartition with magnetic energy in Batchelor’s prediction.
This class of dynamos being small scale ones, it is not surprising that the inertial range of turbulence screens the
magnetic field from the influence of integral size, thus \( L \) can be forgotten. We emphasize that a necessary condition
for Batchelor’s scenario is that the magnetic field can grow below the Kolmogorov scale, i.e. its dissipative length
\( l_s \) should be smaller than \( l_K \), thus \( P_m > 1 \).

There is obviously a strong discrepancy between (13) and (14). The prefactors in these two laws are the upper
and lower limits of a continuous family of scalings that are obtained by balancing the magnetic energy with the
kinetic energy at one particular length scale within the Kolmogorov spectrum. It is not known if one of them is
selected by turbulent dynamos.
4. Ohmic losses

Ohmic losses due to currents generated by dynamo action give a lower bound to the power required to feed a dynamo. In order to evaluate them, it is crucial to know at which scales the magnetic field grows. Assuming that a dynamo is generated in the case $P_m \ll 1$ (scenario (i)), we want to give a possible guess for the power spectrum $|\hat{B}|^2$ of the magnetic field as a function of the wave number $k$ and the parameters $\rho, \epsilon, L, \nu, \mu_0$ and $\sigma$. Far from threshold, $Re \gg R_m \gg \hat{R}_m$, the dissipative lengths are such that $l_k \ll l_\sigma \ll L$. For $k$ in the inertial range, i.e. $kl_\sigma \ll 1 \ll kL$, we may use a Kolmogorov type argument and discard $L, \sigma$ and $\nu$. Then, only one dimensionless parameter is left, and not too surprisingly, we get

$$|\hat{B}|^2 \propto \mu_0 \rho \epsilon^{7/3} k^{-5/3}.$$  (15)

This is only one possibility among many others proposed for MHD turbulent spectra within the inertial range, but it is the simplest. Integrating over $k$ obviously gives the equipartition law (13) for the magnetic energy. It is now interesting to evaluate Ohmic dissipation. Its dominant part comes from the current density at scale $l_\sigma$. We have

$$\frac{j^2}{\sigma} = \frac{1}{\sigma} \int |\hat{j}|^2 dk \propto \frac{1}{\mu_0 \sigma} \int k^2 |\hat{B}|^2 dk \propto \frac{\rho}{\mu_0 \sigma} \epsilon^{7/3} l_\sigma^{4/3} \propto \rho \frac{V^3}{L}.$$  (16)

We thus find that Ohmic dissipation is proportional to the total available power which corresponds to some kind of optimum scaling law for Ohmic dissipation. Although, this does not give any indication that this regime is achieved, we note that the above scaling corresponds to the one found empirically from a set of numerical models [16]. Their approximate fit, $(B^2/\mu_0)/(j^2/\sigma) \propto L/V$, indeed results from equations (15, 16).

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