Joint distribution in residue classes of polynomial-like multiplicative functions

by

Paul Pollack (Athens, GA) and Akash Singha Roy (Chennai)

1. Introduction. For any integer-valued arithmetic function, it is reasonable to ask how the values of \( f \) are distributed in arithmetic progressions. As stated, this problem is far too general; to get any traction, it is necessary to restrict \( f \). Let us suppose that \( f \) is multiplicative and that \( f \) is polynomial-like, in the sense that there is a polynomial \( F(T) \in \mathbb{Z}[T] \) such that \( f(p) = F(p) \) for every prime number \( p \). In this case, Narkiewicz (beginning in \[Nar67\]) has made a comprehensive study of the distribution of \( f \) in coprime residue classes. For a thorough survey of this work, see \[Nar84\, Chapter V\]. See also \[Nar12\] for a more recent contribution to this subject by the same author.

In 1982, Narkiewicz \[Nar82\] observed that his methods could be applied to study the joint distribution of several functions. We state a special case of the main theorem of \[Nar82\]. Let \( f_1, \ldots, f_K \) be a finite sequence of multiplicative, integer-valued arithmetic functions. Say that \( f_1, \ldots, f_K \) is nice if the following conditions hold:

(i) each \( f_k \) is polynomial-like for a nonconstant polynomial: there is a nonconstant polynomial \( F_k(T) \in \mathbb{Z}[T] \) such that \( f_k(p) = F_k(p) \) for all primes \( p \),

(ii) \( F_1(T) \cdots F_K(T) \) has no multiple roots.

If \( f_1, \ldots, f_K \) is a nice family, then a prime \( p \) is called good for \( f_1, \ldots, f_K \) if (a) \( p > 5 \), (b) \( p > (1 + \sum_k \deg F_k(T))^2 \), (c) \( p \) does not divide the leading coefficient of any \( F_k(T) \), and (d) \( p \) does not divide the discriminant of \( F_1(T) \cdots F_K(T) \). For any fixed nice family \( f_1, \ldots, f_K \), all but finitely many primes are good. Narkiewicz proves that if every prime divisor of \( q \) is good, and
one restricts attention to $n$ for which the values $f_1(n),\ldots,f_K(n)$ are coprime to $q$, then those values are asymptotically jointly uniformly distributed among the coprime residue classes modulo $q$. More precisely: For every choice of integers $a_1,\ldots,a_K$ coprime to $q$, we have

\[(1.1) \quad \sum_{n \leq x} 1 \sim \frac{1}{\phi(q)^K} \sum_{n \leq x} 1 \quad \text{gcd}(\prod_{k=1}^{K} f_k(n),q)=1\]
as $x \to \infty$. (It is proved along the way that the right-hand side of $(1.1)$ tends to infinity under the same hypotheses.) In particular, we get joint uniform distribution in coprime residue classes modulo $p$ for all good primes $p$.

So far everything that has been said concerns the distribution to a fixed modulus $q$. It is natural to also consider the distribution when $q$ grows with $x$. We prove a joint uniform distribution result of this kind for nice families valid when the modulus $q$ is prime or “nearly prime”. Here “nearly prime” means that $\delta(q)$ is small where

$$\delta(q) := \sum_{p \mid q} \frac{1}{p}.$$  

Our main theorem is as follows:

**Theorem 1.1.** Fix a nice sequence $f_1,\ldots,f_K$ of multiplicative functions and fix $\epsilon > 0$. Then $(1.1)$ holds, uniformly as $q,x \to \infty$ with $\delta(q) = o(1)$ and $q \leq (\log x)^{1/K-\epsilon}$, for every choice of coprime residue classes $a_1,\ldots,a_K$ mod $q$. In other words: For each $\eta > 0$, there is a positive integer $N$ (depending on $f_1,\ldots,f_K$, $\epsilon$, and $\eta$) such that the following holds. Suppose that $x > N$, that $(\log x)^{1/K-\epsilon} \geq q \geq N$, and that $\delta(q) < 1/N$. Then for every $K$-tuple of integers $a_1,\ldots,a_K$ coprime to $q$, the ratio of the LHS to the RHS in $(1.1)$ lies in $(1-\eta,1+\eta)$.

For example, let $f_1(n) = n$, $f_2(n) = \phi(n)$, and $f_3(n) = \sigma(n)$. These form a nice family. By the result of Narkiewicz quoted above, the values of $n$, $\phi(n)$, $\sigma(n)$ coprime to $p$ are uniformly distributed in coprime residue classes modulo $p$ for each fixed $p \geq 17$. It then follows from Theorem 1.1 that this equidistribution holds uniformly for $17 \leq p \leq (\log x)^{1/3-\epsilon}$.

There are two directions in which one might hope to strengthen Theorem 1.1. First, it would be desirable to weaken the condition $\delta(q) = o(1)$, e.g., by replacing it with Narkiewicz’s condition that $q$ is divisible only by good primes. Such an improvement would seem to require a substantial new idea. Second, one might hope to enlarge the range of allowable $q$ past $(\log x)^{1/K-\epsilon}$. It was proved in [LLPSR] that when $K = 1$ and $f_1(n) = \phi(n)$, one can replace $(\log x)^{1-\epsilon}$ with $(\log x)^A$ for an arbitrary $A$, provided $q$ is restricted to primes. This might seem to suggest that $(\log x)^{1/K-\epsilon}$ in Theorem 1.1 can
Polynomial-like multiplicative functions

always be replaced with \((\log x)^A\), with \(A\) arbitrary. As we now explain, this is too optimistic.

Suppose that \(f_1, \ldots, f_K\) is a fixed nice family with \(K \geq 2\). Fix a prime \(p_0\) with \(f_1(p_0), \ldots, f_K(p_0)\) all nonzero. Let \(X := 2(\log x)^{1/(K-1)}\), and choose \(p\) to be a prime in \((2X/3, X]\). As \(x \to \infty\), there are “obviously” at least
\[
(1 + o(1))\frac{x}{p \log x}
\]
\[
\geq \frac{4}{3} x/p^K
\]
values of \(n \leq x\) having \(f_k(n) \equiv f_k(p_0) \pmod{p}\) for all \(k = 1, \ldots, K\), since \(n\) can be taken as any prime congruent to \(p_0 \pmod{p}\). This shows that equidistribution in coprime residue cannot hold up to \(X\). It is conceivable that in Theorem 1.1 uniformity holds up to \((\log x)^{1/(K-1)}-\epsilon\) (interpreted as \((\log x)^A, \text{\(A\) arbitrary, when \(K = 1\))}. Again, it would seem to require a new idea to decide this.

We conclude this introduction with a brief summary of the proof of Theorem 1.1: Split off the first several largest prime factors of \(n\), say
\[
n = mP_J \cdots P_1,
\]
where \(P^+(m) \leq P_J \leq \cdots \leq P_1\). (Here \(J\) must be chosen judiciously; we also ignore \(n\) with fewer than \(J\) prime factors.) Most of the time, \(P_J, \ldots, P_1\) will appear only to the first power in \(n\), so that \(f_k(n) = f_k(m)f_k(P_J) \cdots f_k(P_1)\). Then, given \(m\), we use the prime number theorem for progressions (Siegel–Walfisz) and character sum estimates to understand the number of choices for \(P_1, \ldots, P_J\) compatible with the congruence conditions on \(f_k(n)\).

Notation and conventions. Throughout, the letters \(p, P, r\), with or without subscripts, always denote primes whether or not this is explicitly mentioned. We use \(P^+(n)\) for the largest prime factor of \(n\), with the convention that \(P^+(1) = 1\). We write \(f(\chi)\) for the conductor of the Dirichlet character \(\chi\).

2. Preparation

2.1. Sieve lemmas. We will make frequent use of the following special case of the fundamental lemma of sieve theory, as formulated in [HR74, Theorem 7.2, p. 209].

**Lemma 2.1.** Let \(X \geq Z \geq 3\). Suppose that the interval \(I = (u, v]\) has length \(v - u = X\). Let \(\mathcal{P}\) be a set of primes not exceeding \(Z\). For each \(p \in \mathcal{P}\), choose a residue class \(a_p \pmod{p}\). The number of integers \(n \in I\) not congruent to \(a_p \pmod{p}\) for any \(p \in \mathcal{P}\) is
\[
x \left( \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right) \right) \left(1 + O \left( \exp \left( -\frac{1}{2} \frac{\log x}{\log Z} \right) \right) \right).
\]

The following application of Lemma 2.1 yields a lower bound for the “numerator” on the right-hand side of (1.1). See Scourfield’s Theorem 4 in [Sco84] for a closely related result (and compare with [Sco85]).

**Lemma 2.2.** Fix a nice arithmetic function \(f\) (meaning that \(f\) is nice when viewed as a singleton sequence). Suppose that \(q, x \to \infty\) with \(q = x^{o(1)}\) and

**Notation and conventions.** Throughout, the letters \(p, P, r\), with or without subscripts, always denote primes whether or not this is explicitly mentioned. We use \(P^+(n)\) for the largest prime factor of \(n\), with the convention that \(P^+(1) = 1\). We write \(f(\chi)\) for the conductor of the Dirichlet character \(\chi\).
\( \delta(q) = o(1) \). The number of \( n \leq x \) for which \( \gcd(f(n), q) = 1 \) eventually \(^{(1)}\) exceeds

\[
\frac{1}{20} x \prod_{\substack{p \leq x \\ \gcd(f(p), q) > 1}} \left( 1 - \frac{1}{p} \right).
\]

**Remark.** (a) With a small amount of additional effort, one could show that \(^{(2.1)}\) is the correct order of magnitude for this count of \( n \). But we will not need this.

(b) It will be useful momentarily to know that the product over \( p \) in \(^{(2.1)}\) has size at least \((\log x)^{O(1)}\). To see this, choose \( F(T) \in \mathbb{Z}[T] \) with \( f(p) = F(p) \) for all \( p \). It suffices to show that

\[
\sum_{\substack{p \leq x \\ \gcd(f(p), q) > 1}} \frac{1}{p} = o(\log \log x).
\]

Let \( S \) be the set of primes \( p \leq x \) with \( \gcd(f(p), q) > 1 \). For each prime \( r \) dividing \( q \), let \( S_r = \{ p \in (r, x] : F(p) \equiv 0 \pmod{r} \} \). Since \( F \) has \( O_f(1) \) roots modulo every prime \( r \),

\[
\sum_{r \mid q} \sum_{p \in S_r} \frac{1}{p} \ll_f \log \log x \sum_{r \mid q} \frac{1}{r} = \delta(q) \log \log x = o(\log \log x).
\]

Here the sum over \( p \in S_r \) has been estimated by partial summation and the Brun–Titchmarsh inequality. For each \( r \) dividing \( q \), there are \( O_f(1) \) primes \( p \leq r \) with \( F(p) \equiv 0 \pmod{r} \). So if we put \( S' := S \setminus \bigcup_{r \mid q} S_r \), then \( \#S' \ll_f \omega(q) \), and, writing \( p_k \) for the \( k \)th prime in the usual increasing order, we get

\[
\sum_{p \in S'} \frac{1}{p} \leq \sum_{k=1}^{\#S'} \frac{1}{p_k} \ll_f \log \log(3\omega(q)) = o(\log \log x),
\]

using the simple bound \( \omega(q) = O(\log x) \) in the last step.

**Proof of Lemma 2.2** Fix a real number \( U \geq 2 \). We start by considering all \( n \leq x \) not divisible by any \( p \leq x^{1/U} \) with \( \gcd(f(p), q) > 1 \). For large \( q, x \) and small \( \frac{\log q}{\log x}, \delta(q) \), where here and below “large” and “small” may depend on \( U \), the sieve shows that the count of such \( n \) is

\[
x \left( \prod_{\substack{p \leq x^{1/U} \\ \gcd(f(p), q) > 1}} \left( 1 - \frac{1}{p} \right) \right) (1 + O(\exp(-U/2))).
\]

We now bound from above the number of these \( n \) with \( \gcd(f(n), q) > 1 \).

\(^{(1)}\) Meaning whenever \( q, x \) are sufficiently large and \( \frac{\log q}{\log x}, \delta(q) \) are sufficiently small.
For each $n$ surviving our initial sieve but having $\gcd(f(n), q) > 1$, we factor $n = A_1A_2B$, where

$$A_1 = \prod_{p \mid n, \gcd(f(p), q) > 1} p, \quad A_2 = \prod_{p^e \mid n, \gcd(f(p^e), q) > 1} p^e, \quad \text{and} \quad B = n/(A_1A_2).$$

Then either $A_1 > 1$ or $A_2 > 1$. Also, every prime dividing $A_1$ exceeds $x^{1/U}$.

Suppose $A_2 > 1$. Since $A_2$ is squarefull, the number of $n \leq x$ with $A_2 > x^{1/2}$ is $O(x^{3/4})$, which will be negligible for our purposes. So we assume that $A_2 \leq x^{1/2}$. Given $A_2$, we count the number of possibilities for the cofactor $A_1B$. Note that $A_1B \leq x/A_2$ and that $A_1B$ is free of prime factors $p \leq x^{1/U}$ with $\gcd(f(p), q) > 1$. So the sieve shows that the number of possibilities for $A_1B$ is at most

$$\frac{x}{A_2} \left( \prod_{p \leq x^{1/U}, \gcd(f(p), q) > 1} \left( 1 - \frac{1}{p} \right) \right) (1 + O(\exp(-U/4))).$$

(We assume as usual that $q, x$ are large and $\log q, \delta(q)$ are small.) Since

$$\sum_{M \text{ squarefull}} \frac{1}{M} = \prod_p \left( 1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots \right) = \frac{\zeta(2)\zeta(3)}{\zeta(6)} = 1.943\ldots,$$

the count of $n$ with $A_2 > 1$ is bounded above by

$$0.945x \left( \prod_{p \leq x^{1/U}, \gcd(f(p), q) > 1} \left( 1 - \frac{1}{p} \right) \right) (1 + O(\exp(-U/4))).$$

Suppose now that $A_2 = 1$. Then $n = A_1B$, where $A_1 > 1$ and every prime dividing $A_1$ exceeds $x^{1/U}$. Let $p$ be a prime dividing $A_1$, and write $A_1 = pS$. Then $n = pSB \leq x$ where $SB \leq x^{1-1/U}$. Given $S$ and $B$, the number of possible $p$ (and hence possible $n$) is, by Brun–Titchmarsh, at most

$$\sum_{r \mid q} \sum_{p \leq x/(SB), F(p) \equiv 0 (\text{mod } r)} 1 \ll_f \sum_{r \mid q} \frac{x}{rSB \log(x/(SBr))} \ll \delta(q)U \frac{x}{\log x} \frac{1}{SB};$$

here we have assumed that $q \leq x^{1/(2U)}$, so that $x/(SBr) \geq (x/(SB))/r \geq x^{1/(2U)}$ for every $r \mid q$. Summing over $S$ and $B$, the number of $n$ that arise is
\[ \ll f \delta(q)U \frac{x}{\log x} \left( \sum_{p | S \Rightarrow p \in (x^{1/U}, x]} \frac{1}{S} \right) \left( \sum_{p \leq x, \text{gcd}(f(p), q) = 1} \frac{1}{B} \right) \]

\[ \leq \delta(q)U \frac{x}{\log x} \left( \prod_{x^{1/U} < p \leq x} \left( 1 - \frac{1}{p} \right)^{-2} \right) \left( \prod_{p \leq x^{1/U}, \text{gcd}(f(p), q) = 1} \left( 1 - \frac{1}{p} \right)^{-1} \right), \]

which is

\[ \ll \delta(q)U^3 \frac{x}{\log x} \prod_{p \leq x^{1/U}, \text{gcd}(f(p), q) > 1} \left( 1 - \frac{1}{p} \right) \]

\[ \ll \delta(q)U^2x \prod_{p \leq x^{1/U}, \text{gcd}(f(p), q) > 1} \left( 1 - \frac{1}{p} \right). \]

But \( \delta(q) = o(1) \), so the final expression is \( o(x \prod_{p \leq x^{1/U}, \text{gcd}(f(p), q) > 1} (1 - 1/p)) \).

Collecting estimates shows that if \( U \) is fixed sufficiently large, then eventually the number of \( n \leq x \) with \( \text{gcd}(f(n), q) = 1 \) exceeds

\[ \frac{1}{20}x \prod_{p \leq x^{1/U}, \text{gcd}(f(p), q) > 1} \left( 1 - \frac{1}{p} \right). \]

Bounding the product over \( p \leq x^{1/U} \) from below by the product over \( p \leq x \) completes the proof. \( \blacksquare \)

Our second application of the sieve is an upper bound on the count of \( n \) with few large prime factors. More precise results on this problem have been obtained by [Ten00], but the comparatively simple Lemma 2.3 below will suffice for our purposes.

Set \( P_1^+(n) = P^+(n) \) and define, inductively,

\[ P_{j+1}^+(n) = P^+ \left( \frac{n}{(P_1^+(n) \cdots P_j^+(n))} \right). \]

Thus, \( P_j^+(n) \) is the \( j \)th largest prime factor of \( n \) (with multiple primes counted multiply), with \( P_j^+(n) = 1 \) if \( n \) has fewer than \( j \) prime factors.

**Lemma 2.3.** Let \( x \geq y \geq 10 \). Let \( J \) be an integer, \( J \geq 2 \). The number of \( n \leq x \) with \( P_j^+(n) \leq y \) is

\[ \ll J \frac{x \log y}{\log x} (\log \log x)^{J-1}. \]

**Proof.** Suppose that \( P_j^+(n) \leq y \) and write \( n = AB \), where \( A \) is the largest divisor of \( n \) composed of primes not exceeding \( y \). Then \( \omega(B) \leq \Omega(B) < J \).
Clearly, $A \leq x^{1/2}$ or $B \leq x^{1/2}$. Suppose first that $A \leq x^{1/2}$. Then $B \leq x/A$ and $\omega(B) \leq J - 1$, so that by a classical theorem of Landau (see [HW08

Theorem 437, p. 491]), given $A$ there are $\ll J \frac{x}{A \log (x/A)} (\log \log (x/A))^{J-2} \ll \frac{x}{A \log x} (\log \log x)^{J-2}$ possible $B$. Summing $1/A$ over $A$ with $P^+(A) \leq y$ introduces a factor $\prod_{p \leq y} (1 - 1/p)^{-1} \ll \log y$, which yields for this case a slightly stronger upper bound than that claimed in the lemma.

Suppose now that $B \leq x^{1/2}$. Since $A$ has no prime factors larger than $y$, the sieve shows that given $B$, the number of possible $A \leq x/B$ is $\ll \frac{x}{B} \prod_{y < p \leq x^{1/2}} (1 - 1/p) \ll \frac{x \log y}{B \log x}$. Since

$$\sum_{B \leq x} \frac{1}{B} \leq \sum_{j=0}^{J-1} \frac{1}{j!} \left( \sum_{p \leq x} \frac{1}{p^e} \right)^j \ll J (\log \log x)^{J-1},$$

the result follows. ■

2.2. Character sums of polynomials. We require estimates for (complete, multiplicative) character sums of polynomials modulo prime powers. For prime moduli, we use the following version of the Weil bound.

Lemma 2.4. Let $\mathbb{F}_q$ be a finite field, and let $\chi_1, \ldots, \chi_K$ be characters of $\mathbb{F}_q^\times$, extended to all of $\mathbb{F}_q$ by setting $\chi_k(0) = 0$. Let $F_1(T), \ldots, F_K(T) \in \mathbb{F}_q[T]$ be nonzero and pairwise relatively prime. Assume that for some $1 \leq k \leq K$, the polynomial $F_k(T)$ is not an $\text{ord}(\chi_k)$th power in $\mathbb{F}_q[T]$ or a constant multiple of such. Then

$$\left| \sum_{x \in \mathbb{F}_q} \chi_1(F_1(x)) \cdots \chi_K(F_K(x)) \right| \leq \left( \sum_{k=1}^K d_k - 1 \right) \sqrt{q},$$

where $d_k$ denotes the degree of the largest squarefree divisor of $F_k(T)$.

Lemma 2.4 is essentially [Wan97, Corollary 2.3]. It is assumed in [Wan97] that all the $\chi_k$ are nontrivial, but this assumption is not used in the proof.

Estimating the sums to proper prime power moduli requires some stage setting. Let $p^m$ be an odd prime power, where $m \geq 2$. Let $g$ be a primitive root modulo $p^m$. Let $\chi$ be the Dirichlet character modulo $p^m$ defined on integers $x$ coprime to $p$ by

$$\chi(x) = \exp \left( 2\pi i \frac{\text{ind}_g(x)}{p^{m-1}(p-1)} \right),$$

where $g^{\text{ind}_g(x)} \equiv x \pmod{p^m}$.

Let $F(T) \in \mathbb{Z}[T]$ be a nonconstant polynomial, and let $t$ be the largest nonnegative integer for which $p^t$ divides every coefficient of $F'(T)$. Let $\tilde{F}(T) \in \mathbb{F}_p[T]$ denote the mod $p$ reduction of $p^{-t}F'(T)$. (Note that $\tilde{F}(T)$ is nonzero by the choice of $t$.) Let $A \subset \mathbb{F}_p$ denote the set of roots of $\tilde{F}(T)$ in $\mathbb{F}_p$ that
are not roots of the reduction of $F(T) \mod p$. For each $\alpha \in A$, let $\nu_{\alpha}$ denote the multiplicity of $\alpha$ as a zero of $\tilde{F}(T)$, and let $M = \max_{\alpha \in A} \nu_{\alpha}$.

The following is an immediate consequence of Cochrane’s Theorem 1.2 in [Coc02]; that very general result concerns mixed additive and multiplicative character sums, but see [CLZ03, Theorem 2.1] for the specialization to multiplicative character sums.

**Lemma 2.5.** Under the above conditions, and the additional assumption that $m \geq t + 2$, we have

$$\left| \sum_{x \mod p^m} \chi(F(x)) \right| \leq \left( \sum_{\alpha \in A} \nu_{\alpha} \right) p^\frac{t}{t+1} p^{m(1-\frac{1}{t+1})}.$$ 

The proof of Theorem 1.1 depends on the following consequence of Lemmas 2.4 and 2.5, which seems of some independent interest.

**Proposition 2.6.** Let $F_1(T), \ldots, F_K(T) \in \mathbb{Z}[T]$ be nonconstant and assume that the product $F_1(T) \cdots F_K(T)$ has no multiple roots. Let $p$ be an odd prime not dividing the leading coefficient of any of the $F_k(T)$ and not dividing the discriminant of $F_1(T) \cdots F_K(T)$. Let $m$ be a positive integer, and let $\chi_1, \ldots, \chi_K$ be Dirichlet characters modulo $p^m$, at least one of which is primitive. Then

$$(2.3) \quad \left| \sum_{x \mod p^m} \chi_1(F_1(x)) \cdots \chi_K(F_K(x)) \right| \leq (D - 1)p^{m(1-1/D)},$$

where $D = \sum_{k=1}^{K} \deg F_k(T)$.

**Proof.** Take first the case when $m = 1$. When $D = 1$, the left-hand side of (2.3) vanishes and (2.3) holds. When $D \geq 2$, we apply Lemma 2.4 with $q = p$. The mod $p$ reductions of the $F_k(T)$ are nonzero (in fact, of the same degree as their counterparts in $\mathbb{Z}[T]$), and $F_1(T) \cdots F_K(T)$ is squarefree over $\mathbb{F}_p$, so that each $F_k(T)$ is squarefree and the $F_k(T)$ are pairwise relatively prime in $\mathbb{F}_p[T]$. Since some $\chi_k$ is primitive, it has order larger than 1, and so $F_k(T)$ is not an ord($\chi_k$)th power in $\mathbb{F}_q[T]$ or a constant multiple of such. Lemma 2.4 now yields (2.3).

Henceforth, we suppose that $m \geq 2$. Let $g$ be a primitive root modulo $p^m$, and let $\chi$ be the character modulo $p^m$ defined in (2.2). We can write each $\chi_k$ in the form $\chi^{A_k}$, where $0 < A_k \leq m - 1(p - 1)$. Then

$$(2.4) \quad \sum_{x \mod p^m} \chi_1(F_1(x)) \cdots \chi_K(F_K(x)) = \sum_{x \mod p^m} \chi(F(x)),$$

where

$$F(T) := F_1(T)^{A_1} \cdots F_K(T)^{A_K}.$$
Also,

\[ F'(T) = \left( \prod_{k=1}^{K} F_k(T)^{A_k-1} \right) G(T), \]

where

\[ G(T) := \sum_{k=1}^{K} \left( A_k F'_k(T) \prod_{1 \leq j \leq K \atop j \neq k} F_j(T) \right). \]

Let \( t \) be the largest integer for which \( p^t \) divides all the coefficients of \( F'(T) \). Since none of the \( F_k(T) \) are multiples of \( p \), the power \( p^t \) is also the largest power of \( p \) dividing all the coefficients of \( G(T) \) (by Gauss’s content lemma).

We claim that \( t = 0 \). Choose, for each \( k = 1, \ldots, K \), a root \( \alpha_k \) of \( F_k(T) \) from the algebraic closure \( \mathbb{F}_p \) of \( \mathbb{F}_p \). Then in \( \mathbb{F}_p \),

\[ G(\alpha_k) = (F'_k(\alpha_k) \prod_{1 \leq j \leq K \atop j \neq k} F_j(\alpha_k)) A_k, \]

and the factor in front of \( A_k \) is nonzero. But if \( t > 0 \), then \( G(T) \) induces the zero function on \( \mathbb{F}_p \), forcing each \( A_k \) to be a multiple of \( p \). Then none of the \( \chi_k \) are primitive characters modulo \( p^m \), contrary to hypothesis.

Now let \( \mathcal{A}, \nu_\alpha, \) and \( M \) be as in the discussion preceding Lemma 2.5. Then each \( \alpha \in \mathcal{A} \) is a root in \( \mathbb{F}_p \) of the mod \( p \) reduction of \( G(T) \) of multiplicity \( \nu_\alpha \). Moreover, \( M \leq \sum_{\alpha \in \mathcal{A}} \nu_\alpha \leq \deg G(T) \leq D - 1 \). The desired upper bound (2.3) follows from (2.4) and Lemma 2.5.

3. Proof of Theorem 1.1 Throughout this proof, we suppress the dependence of implied constants or implied lower/upper bounds on the constant \( \epsilon > 0 \) as well as the family \( f_1, \ldots, f_K \). We let \( F_1(T), \ldots, F_K(T) \in \mathbb{Z}[T] \) be such that \( f_k(p) = F_k(p) \) for all primes \( p \). We put

\[ J := (K + 1)D \]

where, anticipating an application of Proposition 2.6

\[ D := 1 + \sum_{k=1}^{K} \deg F_k(T). \]

It will be convenient to introduce the notation

\[ \sum_f (x; q) := \sum_{n \leq x \atop \gcd(f(n), q) = 1} 1. \]

Throughout this proof, when we say a term is ignorable, we mean that it is of smaller order than the right-hand side of (1.1), that is, \( o(\phi(q)^{-K} \sum_f (x; q)) \).
By Lemma 2.2 (with \( f = f_1 \cdots f_K \)) and the Remark following it, we find that
\[
\phi(q)^{-K} \sum_f(x; q) \geq q^{-K} x (\log x)^{o(1)} \geq x (\log x)^{K \epsilon + o(1)} / \log x \\
\geq x (\log x)^{\epsilon + o(1)} / \log x.
\]
(Here we use our assumption that \( q \leq (\log x)^{1/K - \epsilon} \).) So Lemma 2.3 allows us to discard from the left-hand side of (1.1) those \( n \) for which \( P_j^+ (n) \leq L \), where
\[
L := \exp((\log x)^{\epsilon/2}),
\]
at the cost an ignorable error. Write each remaining \( n \) in the form \( n = m P_j \cdots P_1 \), where \( P_j = P_j^+ (n) \) for each \( j \). We keep only those \( n \) where \( P_j^+ (m) < P_j < \cdots < P_1 \). Any \( n \) discarded at this step has a repeated prime factor exceeding \( L \), and there are \( O((x/L)) \) of these, which is again ignorable.

By the observations of the last paragraph, it suffices to prove that
\[
\sum_f(x; q, a) \sim \frac{1}{\phi(q)^K} \sum_f(x; q),
\]
where
\[
(3.1) \quad \sum_f(x; q, a) := \sum_{m \leq x} \frac{1}{\phi(q)^K} \sum_{\substack{P_1, \ldots, P_j \leq x/m \\ L_m < P_j < \cdots < P_1 \\ (\forall k) f_k (m) \prod_{j=1}^J f_k (P_j) \equiv a_k \mod q}} 1
\]
\[
= \sum_{m \leq x} \frac{1}{\phi(q)^K} \sum_{\substack{P_1, \ldots, P_j \text{ distinct} \\ P_1 \cdots P_j \leq x/m \\ (\forall j) P_j > L_m \\ (\forall k) f_k (m) \prod_{j=1}^J f_k (P_j) \equiv a_k \mod q}} 1.
\]
We now remove the distinctness restriction in the final inner sum. Estimating crudely, this incurs an error of size \( O(x/(mL)) \) in the inner sum and an error of size \( O((x \log x)/L) \) in the double sum.

For each \( k = 1, \ldots, K \), let \( u_k \) denote a value of \( f_k (m)^{-1} a_k \mod q \) and define
\[
V_m := \left\{ (v_1 \mod q, \ldots, v_J \mod q) : \ gcd(v_1 \cdots v_J, q) = 1, (\forall k) \prod_{j=1}^J F_k (v_j) \equiv u_k \mod q \right\}.
\]
Then writing \( \mathbf{v} = (v_1 \mod q, \ldots, v_J \mod q) \) we get
\[
\sum_{\mathbf{P}_1, \ldots, \mathbf{P}_J} 1 = \sum_{\mathbf{v} \in V_m} \sum_{\substack{\mathbf{P}_1, \ldots, \mathbf{P}_J \leq x/m \\ (\forall j) \mathbf{P}_j > L_m}} 1.
\]

For each \( \mathbf{v} \in V_m \), we show how to remove the right-hand congruence conditions on the \( P_j \). First we handle \( P_1 \). Noting that \( q \leq \log x = (\log L)^{2/\epsilon} \), the Siegel–Walfisz theorem (see, for example, [MV07, Corollary 11.21]) implies that for a certain positive constant \( C = C_\epsilon \),
\[
\sum_{\mathbf{P}_1, \ldots, \mathbf{P}_J} 1 = \sum_{\mathbf{P}_2, \ldots, \mathbf{P}_J} \sum_{\substack{L_m < \mathbf{P}_1 \leq x/(mP_2 \ldots P_j) \\ (\forall j) \mathbf{P}_j > L_m}} 1 + O\left(\frac{x}{mP_2 \ldots P_j} \exp\left(-C \sqrt{\log L}\right)\right).
\]

It follows that
\[
\sum_{\mathbf{P}_1, \ldots, \mathbf{P}_J} 1 = \sum_{\mathbf{P}_2, \ldots, \mathbf{P}_J} \sum_{\substack{L_m < \mathbf{P}_1 \leq x/(mP_2 \ldots P_j) \\ (\forall j) \mathbf{P}_j > L_m}} 1 + O\left(\frac{x}{mP_2 \ldots P_j} \exp\left(-\frac{1}{2} C \sqrt{\log L}\right)\right).
\]

In the same way, the congruence conditions on \( P_2, \ldots, P_J \) can be removed successively to yield
\[
\sum_{\mathbf{P}_1, \ldots, \mathbf{P}_J} 1 = \sum_{\mathbf{P}_1, \ldots, \mathbf{P}_J} \sum_{\substack{L_m < \mathbf{P}_1 \leq x/(mP_2 \ldots P_j) \\ (\forall j) \mathbf{P}_j > L_m}} 1 + O\left(\frac{x}{mP_2 \ldots P_j} \exp\left(-\frac{1}{2} C \sqrt{\log L}\right)\right).
\]

The main term on the right-hand side is independent of \( \mathbf{v} \). Keeping in mind that \( \#V_m \leq q^J \leq (\log x)^J \) for all \( m \), we deduce from (3.1) that
(3.2) \[
\sum_f(x; q, a) = \sum_{\substack{m \leq x \\gcd(\prod_{k=1}^K f_k(m), q) = 1}} \frac{\#V_m}{\phi(q)^J} \cdot \frac{1}{J!} \sum_{\substack{P_1, \ldots, P_J \leq x/m \\gcd(f(P_j), q) = 1}} 1 + O \left( x \exp \left( -\frac{1}{4} C \sqrt{\log L} \right) \right).
\]

To handle the main term, notice that
\[
\sum_{\substack{P_1, \ldots, P_J \leq x/m \\gcd(f(P_j), q) = 1}} 1 \leq J \sum_{\substack{p \mid q \\gcd(f(P_j), q) > 1}} \sum_{\substack{P_1, \ldots, P_J \leq x/m \\gcd(f(P_j), q) = 1}} 1.
\]

The condition that \( p \mid f(P_1) \) puts \( P_1 \) in a certain (possibly empty) set of \( O(1) \) residue classes modulo \( p \). Removing these congruence conditions by the Siegel–Walfisz theorem (exactly as above) we find that (with \( C \) as above)
\[
\sum_{\substack{P_1, \ldots, P_J \leq x/m \\gcd(f(P_j), q) = 1}} 1 \ll \delta(q) \sum_{\substack{P_1, \ldots, P_J \leq x/m \\gcd(f(P_j), q) = 1}} 1 + \frac{x}{m} \exp \left( -\frac{1}{4} C \sqrt{\log L} \right).
\]

and so
\[
\sum_{\substack{P_1, \ldots, P_J \leq x/m \\gcd(f(P_j), q) = 1}} 1 \ll \delta(q) \sum_{\substack{P_1, \ldots, P_J \leq x/m \\gcd(f(P_j), q) = 1}} 1 + \frac{x}{m} \exp \left( -\frac{1}{4} C \sqrt{\log L} \right).
\]

Since \( \delta(q) = o(1) \), we have
\[
\sum_{\substack{P_1, \ldots, P_J \leq x/m \\gcd(f(P_j), q) = 1}} 1 = (1 + O(\delta(q))) \sum_{\substack{P_1, \ldots, P_J \leq x/m \\gcd(f(P_j), q) = 1}} 1 + O \left( x \exp \left( -\frac{1}{4} C \sqrt{\log L} \right) \right),
\]

which (considering possible orderings of \( P_1, \ldots, P_J \)) in turn is equal to
\[
(1 + O(\delta(q))) \frac{J!}{J!} \sum_{\substack{P_{J} \ldots < P_1 \\gcd(f(P_j), q) = 1}} 1 \frac{1}{x/m} \exp \left( -\frac{1}{4} C \sqrt{\log L} \right).
\]
The following Claim will be established at the end of this section as an application of Proposition 2.6.

Claim. \( \#V_m \sim q^J / \phi(q)^K \), uniformly in \( m \).

We insert the estimate of the Claim, together with the last display, into (3.2). Since \( \delta(q) = o(1) \), we have \( \frac{q^J}{\phi(q)^K} \left(1 + O(\delta(q))\right) = 1 + o(1) \). We find that up to an ignorable error, \( \sum_f(x; q, a) \) is equal to

\[
(1 + o(1)) \frac{1}{\phi(q)^K} \sum_{m \leq x} \sum_{\text{gcd}(\prod_{k=1}^K f_k(m), q) = 1} \mathbf{1}.
\]

We can view the double sum as counting those numbers \( n \leq x \) with \( \text{gcd}(f(n), q) = 1 \) and certain extra constraints: Namely, the \( J \)th largest prime factor of \( n \) exceeds \( L \) and none of the largest \( J \) prime factors are repeated. But (by reasoning as at the start of this proof) dropping the extra constraints incurs an ignorable error. So up to an ignorable error, \( \sum_f(x; q, a) \) is equal to \( \frac{1 + o(1)}{\phi(q)^K} \sum_f(x; q) \). By definition of ignorable,

\[
\sum_f(x; q, a) \sim \frac{1}{\phi(q)^K} \sum_f(x; q),
\]

and we have seen already that this suffices to complete the proof of Theorem 1.1.

Proof of the Claim. Using \( \chi_0 \) for the trivial character modulo \( q \), orthogonality yields

\[
(3.3) \quad \phi(q)^K \#V_m = \sum_{\chi_1, \ldots, \chi_K \text{ mod } q} \left(\prod_{k=1}^K \bar{\chi}_k(u_k)\right) \left(\sum_{x_1, \ldots, x_J \text{ mod } q} \chi_0 \left(\prod_{j=1}^J x_j\right) \cdot \prod_{k=1}^K \chi_k \left(\prod_{j=1}^J F_k(x_j)\right)\right) = \sum_{\chi_1, \ldots, \chi_K \text{ mod } q} \left(\prod_{k=1}^K \bar{\chi}_k(u_k)\right) S_{\chi_1, \ldots, \chi_K}^J,
\]

where

\[
S_{\chi_1, \ldots, \chi_K} := \sum_{x \text{ mod } q} \chi_0(x) \chi_1(F_1(x)) \cdots \chi_K(F_K(x)).
\]

The number of \( x \text{ mod } q \) where one of \( x, F_1(x), \ldots, F_K(x) \) has a common factor with \( q \) is \( \ll q \delta(q) = o(q) \), and so the tuple \( \chi_1, \ldots, \chi_K \) of trivial characters makes a contribution \( \sim q^J \) to (3.3). So to complete the proof, it suffices to
show that
\[
(3.4) \quad \sum_{\chi_1, \ldots, \chi_K \text{ mod } q \text{ not all trivial}} |S_{\chi_1, \ldots, \chi_K}|^J
\]
\(\text{has size } o(q^J).\)

Assume that \(\chi_1, \ldots, \chi_K\) are Dirichlet characters modulo \(q\), not all of which are trivial. Factor \(q = \prod_{p \mid q} p^{e_p}\). Each character \(\chi_k\), for \(k = 0, 1, \ldots, K\), admits a unique decomposition of the form \(\chi_k = \prod_{p \mid q} \chi_{k,p}\), where \(\chi_{k,p}\) is a Dirichlet character modulo \(p^{e_p}\). By the type of the tuple \(\chi_1, \ldots, \chi_K\), we mean the \(\omega(q)\)-element sequence \(\{f_p\}_{p \mid q}\) of positive integers, where for each \(p\),
\[
f_p := \text{lcm}[f(\chi_{1,p}), \ldots, f(\chi_{K,p})].
\]
Write \(q = q_0q_1\), where \(q_1\) is the unitary divisor of \(q\) supported on the primes \(p \mid q\) for which \(f_p > 1\). Note that \(q_1 > 1\), since not all of \(\chi_1, \ldots, \chi_K\) are trivial. By the Chinese remainder theorem,
\[
S_{\chi_1, \ldots, \chi_K} = \prod_{p \mid q} \left( \sum_{x \mod p^{e_p}} \chi_{0,p}(x)\chi_{1,p}(F_1(x)) \cdots \chi_{K,p}(F_K(x)) \right),
\]
from which we see that
\[
|S_{\chi_1, \ldots, \chi_K}| \leq q_0 \prod_{p \mid q_1} \left| \sum_{x \mod p^{e_p}} \chi_{0,p}(x)\chi_{1,p}(F_1(x)) \cdots \chi_{K,p}(F_K(x)) \right|
\]
\[
= q_0 \prod_{p \mid q_1} \frac{p^{e_p}}{f_p} \left| \sum_{x \mod f_p} \chi_{0,p}(x)\chi_{1,p}(F_1(x)) \cdots \chi_{K,p}(F_K(x)) \right|.
\]
At least one of \(\chi_{1,p}, \ldots, \chi_{K,p}\) has conductor \(f_p\), and so the remaining sum over \(x\) may be estimated by Proposition \[2.6\] yielding
\[
|S_{\chi_1, \ldots, \chi_K}| \leq q(D - 1)^{\omega(q_1)} \prod_{p \mid q_1} f_p^{-(K+1)}.
\]
(If none of the \(F_k(T)\) are multiples of \(T\), we apply Proposition \[2.6\] with the polynomials \(T, F_1(T), \ldots, F_k(T)\); otherwise, the sum over \(x\) is unchanged if we remove the term \(\chi_{0,p}(x)\) and we apply the proposition with \(F_1(T), \ldots, F_k(T)\). Keep in mind that since \(\delta(q) = o(1)\), all the prime factors of \(q\) are large, so the nondivisibility conditions on \(p\) in Proposition \[2.6\] are certainly satisfied.) Hence (since \(J = (K + 1)D\)) we have
\[
|S_{\chi_1, \ldots, \chi_K}|^J \leq q^J(D - 1)^{\omega(q_1)J} \prod_{p \mid q_1} f_p^{-(K+1)}. \quad \text{(3.4)}
\]
There are no more than \((\prod_{p \mid q_1} f_p)^K\) tuples \(\chi_1, \ldots, \chi_K\) sharing this type, so that the contribution from all such tuples to (3.4) is at most
\[
q^J(D - 1)^{\omega(q_1)J} \prod_{p \mid q_1} f_p^{-1}. \quad \text{(3.4)}
\]
Summing \(f_p\) over all powers of \(p\), for \(p \mid q_1\), reveals that the contribution from all types corresponding to a given \(q_1\) is at
most
\[ q^J(D - 1)^{\omega(q_1)J} \frac{q_1}{\phi(q_1)} \prod_{p | q_1} p^{-1} \leq q^J(D - 1)^{\omega(q_1)J} 2^{\omega(q_1)} \prod_{p | q_1} p^{-1}. \]

Finally, summing over all unitary divisors \( q_1 \) of \( q \) with \( q_1 > 1 \) bounds (3.4) by
\[ q^J \left( \prod_{p | q} \left( 1 + \frac{2(D - 1)^J}{p} \right) - 1 \right) \leq q^J(\exp(2(D - 1)^J \delta(q)) - 1) = o(q^J). \]

Collecting the estimates completes the proof of the Claim. □

**Acknowledgements.** We thank the referee for helpful comments.

The first-named author (P.P.) is supported by the US National Science Foundation (NSF), award DMS-2001581.

**References**

[Coc02] T. Cochrane, *Exponential sums modulo prime powers*, Acta Arith. 101 (2002), 131–149.

[CLZ03] T. Cochrane, C. L. Liu, and Z. Y. Zheng, *Upper bounds on character sums with rational function entries*, Acta Math. Sinica (Engl. Ser.) 19 (2003), 327–338.

[HR74] H. Halberstam and H.-E. Richert, *Sieve Methods*, Academic Press, London, 1974.

[HW08] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 6th ed., Oxford Univ. Press, Oxford, 2008.

[LLPSR] N. Lebowitz-Lockard, P. Pollack, and A. Singha Roy, *Distribution mod \( p \) of Euler’s totient and the sum of proper divisors*, Michigan Math. J., to appear.

[MV07] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory. I. Classical Theory*, Cambridge Stud. Adv. Math. 97, Cambridge Univ. Press, Cambridge, 2007.

[Nar67] W. Narkiewicz, *On distribution of values of multiplicative functions in residue classes*, Acta Arith. 12 (1967), 269–279.

[Nar82] W. Narkiewicz, *On a kind of uniform distribution for systems of multiplicative functions*, Litovsk. Mat. Sb. 22 (1982), no. 1, 127–137.

[Nar84] W. Narkiewicz, *Uniform Distribution of Sequences of Integers in Residue Classes*, Lecture Notes in Math. 1087, Springer, Berlin, 1984.

[Nar12] W. Narkiewicz, *Weak proper distribution of values of multiplicative functions in residue classes*, J. Austral. Math. Soc. 93 (2012), 173–188.

[Sco84] E. J. Scourfield, *Uniform estimates for certain multiplicative properties*, Monatsh. Math. 97 (1984), 233–247.

[Sco85] E. J. Scourfield, *A uniform coprimality result for some arithmetic functions*, J. Number Theory 20 (1985), 315–353.

[Ten00] G. Tenenbaum, *A rate estimate in Billingsley’s theorem for the size distribution of large prime factors*, Quart. J. Math. 51 (2000), 385–403.

[Wan97] D. Wan, *Generators and irreducible polynomials over finite fields*, Math. Comp. 66 (1997), 1195–1212.
