SPATIO-TEMPORAL COEXISTENCE IN THE CROSS-DIFFUSION COMPETITION SYSTEM

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Abstract. We study a two component cross-diffusion competition system which describes the population dynamics between two biological species. Since the cross-diffusion competition system possesses the so-called population pressure effects, a variety of solution behaviors can be exhibited compared with the classical diffusion competition system. In particular, we discuss on the existence of spatially non-constant time periodic solutions. Applying the center manifold theory and the standard normal form theory, the cross-diffusion competition system is reduced to a two dimensional dynamical system around a doubly degenerate point. As a result, we show the existence of stable time periodic solutions in the system. This means spatio-temporal coexistence between two biological species.

1. Introduction. In an ecosystem, several biological species compete with each other to get common resources such as foods, habitats, water etc. A famous mathematical model which describes a competition between two biological species with migration is the following diffusion competition system:

\[
\begin{align*}
    u_t &= D_u \Delta u + (r_1 - a_1 u - b_1 v)u, \\
    v_t &= D_v \Delta v + (r_2 - a_2 u - b_2 v)v,
\end{align*}
\]

where \(u(x,t)\) and \(v(x,t)\) denote population densities of biological species at position \(x\) and at time \(t\), \(D_u\) and \(D_v\) represent diffusion coefficients of \(u\) and \(v\), respectively, \(\Delta\) means the Laplacian. The Lotka-Volterra type is adopted as the nonlinear terms. The parameters \(r_i\) (\(i = 1, 2\)) are intrinsic growth rates, \(a_1\) and \(b_2\) are intra-specific competition rates, and \(a_2\) and \(b_1\) are inter-specific competition rates. All of the parameters are positive constants. For this system, there are many literatures from theoretical point of view (cf. [15, 24]). When we consider (1) on a convex domain with the zero flux boundary conditions, the solution converges to a constant steady state as time evolves ([11, 15]). That is, the large time behavior of solution can be classified into four cases depending on the parameter values. When \(\frac{b_1}{b_2} < \frac{r_1}{r_2} < \frac{a_1}{a_2}\) (weak competition), \(\lim_{t \to \infty} (u, v) = \left( \frac{r_1 / b_2 - r_2 / b_1}{a_1 / b_2 - a_2 / b_1}, \frac{r_2 / b_2 - r_1 / b_1}{a_1 / b_2 - a_2 / b_1} \right)\). When \(\frac{a_1}{a_2} < \frac{r_1}{r_2} < \frac{b_1}{b_2}\)

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These stable to this question, Kan-on non-constant steady states when the results provide that the structure of stationary solutions for \((\text{weak competition})\) does not possess any stable non-constant steady states when the domain is convex.

However, it has been observed in nature that some biological species coexist by segregating spatially. To understand such a situation, the following cross-diffusion competition system was proposed by Shigesada, Kawasaki and Teramoto ([28]):

\[
\begin{align*}
    u_t &= \Delta \{(D_u + \alpha_1 u + \beta_1 v)u\} + (r_1 - a_1 u - b_1 v) u, \\
v_t &= \Delta \{(D_v + \alpha_2 u + \beta_2 v)v\} + (r_2 - a_2 u - b_2 v) v,
\end{align*}
\]

(2)

where coefficients \(\alpha_1\) and \(\beta_2\) indicate the population pressures from one species to itself (self-diffusion), and coefficients \(\beta_1\) and \(\alpha_2\) indicate the population pressures from one species to the other (cross-diffusion). These parameter values are nonnegative. A feature of (2) is the population pressure terms, that is, when we observe the migration term in the equation for \(u\), we have that

\[
u_t = \nabla \cdot \{(D_u + \alpha_1 u + \beta_1 v) \nabla u\} + \nabla \cdot \{u \nabla (\alpha_1 u + \beta_1 v)\}.
\]

The first term of the right hand side means diffusion whose coefficient depends on \(u\) and \(v\), and the second term indicates the directed movement that the species \(u\) moves to less populated regions of \(u\) and \(v\), namely, the species \(u\) tends to avoid crowded regions. If the parameter values in the Lotka-Volterra equations satisfy the weak competition case \(b_1/b_2 < r_1/r_2 < a_1/a_2\), the positive constant steady state \((u^*, v^*) = ((r_1 b_2 - r_2 b_1)/(a_1 b_2 - a_2 b_1), (r_2 a_2 - r_1 a_2)/(a_1 b_2 - a_2 b_1))\) is stable for suitably small values of \(\alpha_i\) and \(\beta_i\) \((i = 1, 2)\). However, when the cross-diffusion coefficients \(\alpha_2\), \(\beta_2\) and the diffusion coefficients \(D_u\), \(D_v\) in (2), spatially non-uniform steady states are formed by destabilizing the positive constant steady state \((u^*, v^*)\), which is called cross-diffusion induced instability ([25]). The non-constant steady states mean spatially segregation patterns.

In order to understand the effect of cross-diffusion, the system (2) has been intensively studied by many researchers. For example, on the existence of global solution, we refer to [4, 5, 7, 21, 23, 30] and references therein. From the viewpoint of segregation pattern formation, the existence and the nonexistence of non-constant steady states has been also studied. There are mainly three methods to treat the stationary problem for (2): the bifurcation theory ([25]), a singular limit and a singular perturbation techniques ([17, 22, 26, 27, 29]) and an elliptic approach ([19, 20]). These results provide that the structure of stationary solutions for (2) is very complicated and strongly depends on the parameter values. In addition, it is well known that the cross-diffusion system (2) is approximated by a reaction-diffusion system ([6, 12, 13]).

As mentioned above, there is a lot of literature on the existence of global solution and the existence of non-constant stationary solutions. Here, a natural question arises: Is there any dynamic coexistence between two species in (2)? Does (2) possess spatially non-constant time periodic solutions? To this question, Kan-on proved the existence of time periodic solutions for a special parameter case ([14]). The purpose of this paper is to show the existence of spatially non-constant time periodic solutions in the cross-diffusion competition system for a different parameter regime from [14].
In this paper, we focus on the one-dimensional problem for (2) and discuss the spatio-temporal coexistence of the two biological species. In what follows, we study the case where $D_u = D_v = d$, $a_1 = a_2 = b_1 = 0$ and $\beta_1 = \gamma$ for simplicity, that is,

\[
\begin{aligned}
&u_t = \{(d + \gamma v)u\}_{xx} + (r_1 - a_1 u - b_1 v)u, \quad x \in (0, L), \quad t > 0, \\
v_t = d v_{xx} + (r_2 - a_2 u - b_2 v)v, \quad x \in (0, L), \quad t > 0,
\end{aligned}
\]  

(3)

under the Neumann boundary conditions

\[
u_x(0, t) = u_x(L, t) = 0, \quad v_x(0, t) = v_x(L, t) = 0, \quad t > 0.
\]  

(4)

This paper is organized as follows: We numerically investigate global bifurcation structures of (3) with (4) in Section 2. Here, the numerical results suggest that the occurrence of periodic solutions is in a doubly degenerate point related to 1- and 2-Fourier modes. In Section 3, the center manifold theory and the standard normal form theory reduce (3) to a two dimensional dynamical system around the doubly degenerate point, and we show the existence result for time periodic solutions which bifurcate from a Hopf bifurcation point, which means spatio-temporal coexistence of two competing species.

2. Numerical study of bifurcation structure. In this section, we numerically investigate global bifurcation diagrams for the cross-diffusion competition system (3) with (4). Biologically, (3) implies that the species $u$ avoids the species $v$ while the species $v$ does not possess such an effect. When the weak competition case $\frac{b_1}{b_2} < \frac{r_1}{r_2} < \frac{a_1}{a_2}$ is considered, the positive constant steady state $(u^*, v^*) = \left(\frac{r_2 b_1 - r_1 b_2}{a_1 b_2 - a_2 b_1}, \frac{r_1 a_2 - r_2 a_1}{a_1 b_2 - a_2 b_1}\right)$ exists and is stable for the small values of $\gamma$. On the other hand, as the value of $\gamma$ increases, the cross-diffusion induced instability occurs and spatial segregation patterns appear([25]). Since the parameters $d$ and $\gamma$ may play an important role for the segregation pattern formation, we consider a relation between the stability of the positive constant steady state $(u^*, v^*)$ and the parameter values $d$ and $\gamma$ in (3).

In what follows, we focus our attention on the constant stationary solution $(u, v) = (u^*, v^*)$. Let $(\tilde{u}, \tilde{v}) = (u - u^*, v - v^*)$. Then (3) is rewritten as a system of $(\tilde{u}, \tilde{v})$, that is, the evolution equation of the perturbations is given by $U_t = AU + N(U)$. Here, though $U$ denotes $(\tilde{u}, \tilde{v})$, we drop the tildes hereafter. Thus,

\[
A = \begin{pmatrix}
(d + \gamma v)\frac{\partial^2}{\partial x^2} - a_1 u^* \gamma u^* \frac{\partial^2}{\partial x^2} - b_1 v^* \\
-\frac{\partial^2}{\partial x^2} - d \frac{\partial^2}{\partial x^2} - b_2 v^*
\end{pmatrix}, \quad N(U) = \begin{pmatrix}
(\gamma (u v)_{xx} - a_1 u^2 - b_1 v^2) \\
-\frac{\partial^2}{\partial x^2} - b_2 v^2
\end{pmatrix}.
\]

In order to make calculation simple, the “hidden symmetry” ([9]) is useful to derive the dynamical system given by the Fourier coefficients. If $(u(x, t), v(x, t))$ is a solution of (3) with (4), then we can extend it on $[0, 2L]$ as follows:

\[
\tilde{u}(x, t) = \begin{cases}
(u(x, t) & x \in [0, L], \\
u(2L - x, t) & x \in [L, 2L],
\end{cases}
\quad \text{and} \quad \tilde{v}(x, t) = \begin{cases}
v(x, t) & x \in [0, L], \\
v(2L - x, t) & x \in [L, 2L],
\end{cases}
\]

which satisfy the extended system

\[
U_t = AU + N(U), \quad x \in (0, 2L), \quad t > 0.
\]  

(5)
under the periodic boundary conditions
\[
\begin{align*}
 u(0, t) &= u(2L, t), \quad u_x(0, t) = u_x(L, t) = u_x(2L, t) = 0, \quad t > 0, \\
 v(0, t) &= v(2L, t), \quad v_x(0, t) = v_x(L, t) = v_x(2L, t) = 0, \quad t > 0.
\end{align*}
\]
This allows us to express solutions of (5) with (6) as complex Fourier expansions
\[
\begin{align*}
 u(x, t) &= \sum_{m \in \mathbb{Z}} u_m(t)e^{-\sqrt{1+m^2}kx}, \quad v(x, t) = \sum_{m \in \mathbb{Z}} v_m(t)e^{-\sqrt{1+m^2}kx}, \quad k := \frac{\pi}{L} \quad (7)
\end{align*}
\]in the function space
\[
\mathcal{H}_{\text{per}} := \{(u, v) \in H^2_{\text{per}}(0, 2L) \times H^2_{\text{per}}(0, 2L)\}, \quad (8)
\]where \(H^2_{\text{per}}(0, 2L) = \{w \in H^2(0, 2L); w(0) = w(2L), w_x(0) = w_x(L) = w_x(2L) = 0\}. Substituting (7) into (5) and using the orthogonality of trigonometric functions, we obtain the infinite dimensional dynamical system
\[
\begin{align*}
 \begin{pmatrix}
 u_m \\
 v_m
 \end{pmatrix}
 &= M_m \begin{pmatrix}
 u_m \\
 v_m
 \end{pmatrix} + \begin{pmatrix}
 F_m \\
 G_m
 \end{pmatrix}, \quad m \in \mathbb{Z}, \quad (9)
\end{align*}
\]where
\[
\begin{align*}
 M_m &= \begin{pmatrix}
 -(d + \gamma u^*)m^2k^2 + a_1u^* \\
 & -a_2v^*
 \end{pmatrix}, \quad -(\gamma m^2k^2 + b_1)u^* \\
 & -\{dm^2k^2 + b_2v^*\}, \quad (10)
\end{align*}
\]and
\[
\begin{align*}
 F_m &= -a_1 \sum_{m_1 + m_2 = m} u_{m_1}u_{m_2} - \sum_{m_1 + m_2 = m} (\gamma m^2k^2 + b_1)u_{m_1}v_{m_2}, \quad (11)
\end{align*}
\]
\[
\begin{align*}
 G_m &= -a_2 \sum_{m_1 + m_2 = m} u_{m_1}v_{m_2} - b_2 \sum_{m_1 + m_2 = m} v_{m_1}v_{m_2}. \quad (12)
\end{align*}
\]We study the dynamical system (9) in the Fourier space
\[
\mathcal{F} := \{(u_m, v_m) \in \mathbb{C}^2; (u_m, v_m) = (u_{-m}, v_{-m}), \quad \|\{(u_m, v_m)\}_{m \in \mathbb{Z}}\|_2 = \sum_{m \in \mathbb{Z}} (1 + m^2)^2\,(u_m, v_m)^2 < \infty\}, \quad (13)
\]which is equivalent to \(\mathcal{H}_{\text{per}}\) by the mapping \(\mathcal{P}: \mathcal{H}_{\text{per}} \rightarrow \mathcal{F}\), where
\[
\mathcal{P}(u, v) = \left\{ \frac{1}{2L} \int_0^{2L} (u(x, t), v(x, t))e^{-\sqrt{1+m^2}kx} \, dx \right\}_{m \in \mathbb{Z}}. \quad (14)
\]

**Remark 1.** Since we used hidden symmetry and \(u(x, t), v(x, t) \in \mathbb{R}\), the equality \((u_m(t), v_m(t)) = (u_{-m}(t), v_{-m}(t)) \in \mathbb{R}^2\) holds for \(m \in \mathbb{Z}\). Therefore it is sufficient to study the dynamical system (9) for \(m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\).

Computing eigenvalues of each matrix \(M_m\), we know the stability of the constant steady state \((u^*, v^*)\) according to the parameter values. In particular, when we focus on two parameters \(d\) and \(\gamma\), neutral stability curves on which at least one eigenvalue is zero are given.

**Definition 2.1.** A neutral stability curve is defined by a set of bifurcation parameters \((d, \gamma)\) for each \(m \in \mathbb{N}\) as follows;
\[
\gamma(d; m) = \frac{(dm^2k^2 + b_2v^*)m^2k^2d + u^*(a_1dm^2k^2 + (a_1b_2 - a_2b_1)v^*)}{(-dm^2k^2 + b_2v^*)v^*m^2k^2}. \quad (15)
\]
Figure 1. Neutral stability curves in \((d, \gamma)\)-plane. The horizontal axis and the vertical axis mean the value \(d\) and the value \(\gamma\), respectively. The parameter values are \(r_1 = 5\), \(r_2 = 2\), \(a_1 = 3\), \(a_2 = 1\), \(b_1 = 1\), \(b_2 = 3\) and \(L = 1\).

Figure 1 illustrates the neutral stability curves \(\gamma(d; m)\) and the stability of the constant steady state \((u^*, v^*)\) in \((d, \gamma)\)-plane. We can see from the figure that the constant steady state \((u^*, v^*)\) is destabilized when the value of \(d\) is suitably small and the value of \(\gamma\) is suitably large. On the basis of Figure 1, we numerically compute global bifurcation diagrams by using AUTO package ([8]). In numerical computation, we set \(d\) as a bifurcation parameter and fix \(\gamma\) value. Figure 2(a) displays the global bifurcation diagram for \(\gamma = 2.5\). One can see that the constant steady state \((u^*, v^*)\) is stable for large values of \(d\). When the value \(d\) decreases, the constant steady state is destabilized at \(d = 0.0262925\) due to the cross-diffusion induced instability, and stable branches of nonconstant stationary solutions which

\[(a) \; \gamma = 2.5 \]

\[(b) \; \gamma = 1.7 \]

Figure 2. Global bifurcation diagrams for (3) with (4) when the value of \(\gamma\) varies. The horizontal axis and the vertical axis mean the value \(d\) and the boundary value \(u(0)\), respectively. Solid curves and dashed curves mean stable branches and unstable ones, respectively. The marks \(\big□\) and \(\big■\) indicate a pitchfork bifurcation point and a Hopf bifurcation point, respectively. The parameter values are the same as the ones in Figure 1.
correspond to 1-Fourier mode bifurcate primarily via a supercritical pitchfork bifurcation (see also Figure 1). As the value of $d$ decreases further, these branches connect with another nonconstant stationary solution branch at $d = 0.0186885$, which emerges as a result of the secondary bifurcation on the constant steady state $(u^*, v^*)$ through a pitchfork bifurcation. We also display a typical profile of stationary solution on each branch, where solid and dashed curves respectively represent $u$ and quintuple $v$.

When the value of $\gamma$ varies from $\gamma = 2.5$ to $\gamma = 1.7$, the bifurcation structures qualitatively change as shown in Figure 2. Actually, Figure 1 shows that 1-mode branches primarily bifurcate from the constant steady state $(u^*, v^*)$ for $\gamma = 2.5$ as the value of $d$ decreases, while 2-mode branches primarily bifurcate for $\gamma = 1.7$. Between these $\gamma$ values, there is the intersection of $\gamma(d; 1)$ and $\gamma(d; 2)$. In general, it is well known that bifurcation diagrams drastically change in the vicinity of such an intersection point ([9]). We notice that there are Hopf bifurcation points (marked with ■) in the global bifurcation diagram for $\gamma = 1.7$ of Figure 2(b). Figure 3(a) presents an enlarged view of the bifurcation diagram for $\gamma = 1.7$ in the vicinity of the Hopf bifurcation points. In addition, we depict periodic solution branches (marked with ●) which bifurcate from the Hopf bifurcation points. We numerically find out that these periodic solution branches are stable and exist for $0.0152320 \leq d \leq 0.0153749$. As the value $d$ decreases, the constant steady state $(u^*, v^*)$ is destabilized and a pitchfork bifurcation occurs (PB1 in Figure 3(a)). At the point PB2, a supercritical pitchfork bifurcation again occurs. And, these branches give rise to the Hopf bifurcations (HB in Figure 3(a)). We can identify a pair of stationary solution branches appearing from PB2 and the two periodic solution branches bifurcating from HB. Figure 3(b) shows a periodic solution on the branch in Figure 3(a).

When the value of $\gamma$ crosses the intersection point of $\gamma(d; 1)$ and $\gamma(d; 2)$, bifurcation structures qualitatively vary. We call the intersection of $\gamma(d; 1)$ and $\gamma(d; 2)$ the doubly degenerate point and denote it as $(d, \gamma) = (d^*, \gamma^*)$. Moreover, these results suggest that the Hopf bifurcation points appear from the doubly degenerate point.
Therefore, we intend to capture the Hopf bifurcations and periodic solutions by the center manifold reduction in the vicinity of the doubly degenerate point in the next section.

3. **Existence of time periodic solutions.** In the previous section, when we assume the weak competition case, the constant steady state \((u^*, v^*)\) are destabilized due to the cross-diffusion induced instability, and Hopf bifurcation points and periodic solution branches appearing from the points are numerically found. In this section, we show the existence of spatially non-constant time periodic solutions in a rigorous way.

3.1. **Center manifold reduction.** In order to apply the center manifold theory, we should check some assumptions ([3]). Let \(X := L^2(0, L) \times L^2(0, L)\). The domain of the operator \(A\) is \(D(A) = \mathcal{F}_{\text{per}}\). Then, we find out that \(A : D(A) \to X\) is a sectorial operator and \(N(U)\) is \(C^2\). Therefore, the center manifold theorem can be applied to our problem.

By applying the center manifold theorem, we can study the dynamics around the doubly degenerate point. Set

\[
\begin{pmatrix}
x_j \\
y_j
\end{pmatrix}
= T_j^{-1} \begin{pmatrix}
u_j \\
v_j
\end{pmatrix}, \quad j = 1, 2, \tag{16}
\]

where

\[
T_j := \begin{pmatrix}
T_{11}^j & T_{12}^j \\
T_{21}^j & T_{22}^j
\end{pmatrix} = \begin{pmatrix}
(\gamma^* k_j^2 + b_1)u^* - \{(d^* + \gamma^* v^*)j_2^2k_j^2 + a_1 u^*\}, & -a_2 v^*
\end{pmatrix}.
\]

We then obtain the following infinite dimensional dynamical system:

\[
\begin{align*}
\dot{x}_j &= \frac{1}{\det T_j} (T_{22}^j \tilde{F}_j - T_{12}^j \tilde{G}_j), \quad \text{for} \quad j = 1, 2 \\
\dot{y}_j &= (\text{tr } M_j)y_j + \frac{1}{\det T_j} (-T_{21}^j \tilde{F}_j + T_{11}^j \tilde{G}_j), \quad \text{for} \quad j = 1, 2 \\
\begin{pmatrix}
\dot{u}_m \\
\dot{v}_m
\end{pmatrix} &= M_m \begin{pmatrix}
u_m \\
v_m
\end{pmatrix} + \begin{pmatrix}
\tilde{F}_m \\
\tilde{G}_m
\end{pmatrix}, \quad \text{for} \quad m \in \mathbb{N}_0 \setminus \{1, 2\},
\end{align*}
\]

Here, \(\tilde{F}_m\) and \(\tilde{G}_m\) respectively denote the nonlinear terms that \((u_j, v_j) = (T_{11}^j x_j + T_{12}^j y_j, T_{21}^j x_j + T_{22}^j y_j)(j = 1, 2)\) are substituted into \(F_m\) and \(G_m\).

We now compute the reduced system near the doubly degenerate point. Set

\[
\varepsilon_0 := (d, \gamma) - (d^*, \gamma^*) \quad \text{and} \quad \mu_j := \frac{\text{tr } M_j + \sqrt{(\text{tr } M_j)^2 - 4\det M_j}}{2}.
\]

and define a neighborhood \(\mathcal{N}_\varepsilon\) of \(\mathcal{F} \times \mathbb{R}^2\) as follows:

\[
\mathcal{N}_\varepsilon := \{(u_m, v_m) \in \mathcal{F} \times \mathbb{R}^2 : \|\{(u_m, v_m)\}_{m \in \mathbb{Z}}\|_{\mathcal{F}} + |\varepsilon_0| < \varepsilon\}.
\]

Then we obtain the following theorem.

**Theorem 3.1.** For given positive constants \(r_j\), \(a_j\), \(b_j\) satisfying \(b_1/b_2 < r_1/r_2 < a_1/a_2\) (weak competition), there exists a positive constant \(\varepsilon\) such that the local center
manifold $\mathcal{M}_{loc}^\ast$ of (18) is contained in $\mathcal{N}_C$. Furthermore, the dynamics of (18) on $\mathcal{M}_{loc}^\ast$ is topologically equivalent to the following system:

$$
\begin{align*}
\dot{x}_1 &= \mu_1 x_1 + A_1 x_1 x_2 + (A_2 x_1^2 + A_3 x_2^2) x_1 + O((|x_1, x_2|^4),
\dot{x}_2 &= \mu_2 x_2 + B_1 x_1^2 + (B_2 x_1^2 + B_3 x_2^2) x_2 + O((|x_1, x_2|^4).
\end{align*}
$$

(19)

Here, the coefficients $A_j, B_j \in \mathbb{R}$ are determined explicitly by the coefficients in (18) (see Appendix).

Proof. By using the center manifold theory with suspension trick ([3, 10]) to the following system

$$
\begin{align*}
\dot{\mu}_j &= 0, \quad \text{for } j = 1, 2
\dot{x}_j &= \mu_j x_j + \frac{1}{\det T_j} (T_{2j}^{2j} \tilde{F}_j - T_{1j}^{1j} \tilde{G}_j), \quad \text{for } j = 1, 2
\dot{y}_j &= (\text{tr } M_j) y_j + \frac{1}{\det T_j} (-T_{2j}^{2j} \tilde{F}_j + T_{1j}^{1j} \tilde{G}_j), \quad \text{for } j = 1, 2
\left(\begin{array}{c}
\dot{u}_m \\
\dot{v}_m
\end{array}\right) &= M_m \left(\begin{array}{c}
u_m \\
\tilde{F}_m \\
\tilde{G}_m
\end{array}\right), \quad \text{for } m \in \mathbb{N}_0 \setminus \{1, 2\},
\end{align*}
$$

(20)

there exist functions $u_m = h^u_m(x_1, x_2, \mu_1, \mu_2), v_m = h^v_m(x_1, x_2, \mu_1, \mu_2)$ (m $\in \mathbb{N}_0 \setminus \{1, 2\}$), $y_1 = h^1_m(x_1, x_2, \mu_1, \mu_2)$ and $y_2 = h^2_m(x_1, x_2, \mu_1, \mu_2)$ satisfying

$$
\begin{align*}
\frac{\partial h^u_m}{\partial x_j}(0, 0, 0, 0) &= \frac{\partial h^v_m}{\partial x_j}(0, 0, 0, 0) = \frac{\partial h^u_m}{\partial \mu_j}(0, 0, 0, 0) = \frac{\partial h^v_m}{\partial \mu_j}(0, 0, 0, 0) = 0, \quad (j = 1, 2)
\frac{\partial h^v_m}{\partial x_j}(0, 0, 0, 0) &= \frac{\partial h^1_m}{\partial x_j}(0, 0, 0, 0) = \frac{\partial h^2_m}{\partial x_j}(0, 0, 0, 0) = \frac{\partial h^v_m}{\partial \mu_j}(0, 0, 0, 0) = 0, \quad (j = 1, 2)
\end{align*}
$$

such that the local invariant manifold $\mathcal{M}_{loc}^\ast$ is represented by

$$
\mathcal{M}_{loc}^\ast = \{ (\mu_1, \mu_2), ((x_1, y_1), (x_2, y_2)), ((u_m, v_m))_{m \in \mathbb{N}_0 \setminus \{1, 2\}} \} \in \mathbb{R}^2 \times \mathcal{F};
$$

$$(u_m, v_m) = (h^u_m(x_1, x_2, \mu_1, \mu_2), h^v_m(x_1, x_2, \mu_1, \mu_2)) \text{ for } m \in \mathbb{N}_0 \setminus \{1, 2\},
$$

$$(y_1, y_2) = h^l_m(x_1, x_2, \mu_1, \mu_2) \text{ for } l = 1, 2.$$}

Differentiating $h^u_m$ and $h^v_m$ with respect to $t$, we have

$$
\left(\begin{array}{c}
\dot{u}_m \\
\dot{v}_m
\end{array}\right) = \left(\begin{array}{c}
h^u_m \\
h^v_m
\end{array}\right) = \left(\begin{array}{c}
\frac{\partial h^u_m}{\partial x_1} \\
\frac{\partial h^u_m}{\partial x_2}
\end{array}\right) \left(\begin{array}{c}
\dot{x}_1 \\
\dot{x}_2
\end{array}\right) + M_m \left(\begin{array}{c}
h^v_m \\
h^v_m
\end{array}\right) = M_m \left(\begin{array}{c}
h^v_m \\
h^v_m
\end{array}\right) + \left(\begin{array}{c}
\tilde{F}_m \\
\tilde{G}_m
\end{array}\right).$$

From the above discussion, we know that $h^u_m, h^v_m, h^1_m$ and $h^2_m$ are $O((|x_1, x_2, \mu_1, \mu_2|^2))$. We therefore obtain

$$
\left(\begin{array}{c}
h^u_m \\
h^v_m
\end{array}\right) = -M_m^{-1} \left(\begin{array}{c}
\tilde{F}_m \\
\tilde{G}_m
\end{array}\right) + O((|x_1, x_2, \mu_1, \mu_2|^3)).
$$

By the similar calculations for $y_1$ and $y_2$, we have

$$
\dot{y}_j = \frac{\partial h^v_j}{\partial x_1} \dot{x}_1 + \frac{\partial h^v_j}{\partial x_2} \dot{x}_2 = (\text{tr } M_j) h^v_j + \frac{1}{\det T_j} (-T_{2j}^{2j} \tilde{F}_j + T_{1j}^{1j} \tilde{G}_j), \quad j = 1, 2.
$$

We hence obtain

$$
\dot{h}^v_j = \frac{1}{(\text{tr } M_j)(\det T_j)} (T_{2j}^{2j} \tilde{F}_j - T_{1j}^{1j} \tilde{G}_j) + O((|x_1, x_2, \mu_1, \mu_2|^3)), \quad j = 1, 2.$$
It should be remarked that to obtain the center manifold up to the cubic order rigorously, we have to compute the functions $h_1^0, h_2^0, h_3^0, h_4^0, h_5^0$ up to the square order. Finally, substituting the approximations for the functions $h_0^i, h_1^i, h_2^i, h_3^i, h_4^i, h_5^i, h_6^i$ into the third and fourth equation of (20), we obtain the reduced system (19).

For example, the quadratic term of $\dot{x}_1$ can be calculated as follows. We have

$$\bar{F}_1 = -(2\mu_1 T_{11}^2 T_{11}^2 + (\gamma k^2 + b_1)(T_{11}^2 T_{21}^2 + T_{11}^2 T_{21}^2)) x_1 x_2 + O(|(x_1, x_2)|^3),$$

$$\bar{G}_1 = -(a_2(T_{11}^2 T_{11}^2 + T_{11}^2 T_{21}^2) + 2b_2 T_{11}^2 T_{21}^2) x_1 x_2 + O(|(x_1, x_2)|^3).$$

Therefore the quadratic term of $\dot{x}_1$ is only $x_1 x_2$, and the coefficient of it is $A_1$ (see Appendix). The quadratic term $\dot{x}_2$ of $\dot{x}_2$ is also computed in the same manner.

Remark 2. Since the eigenvalues of the linearized operator are strictly negative except for zero eigenvalues, the center manifold obtained in the above is locally attractive at the doubly degenerate point.

Remark 3. We can determine the form of (19) without specific calculation. Since the system (5) is invariant under the O(2) action, i.e., $x \mapsto x + \forall \eta$ and $x \mapsto -x$, the system (19) is also invariant under $x_m \mapsto e^{\sqrt{\gamma} \text{m} \eta} x_m$ and $x_m = -x_m \mapsto x_m$. Especially, the former implies that there are no quadratic terms except $x_1 x_2$ of $\dot{x}_1$ and $x_2^2$ of $\dot{x}_2$.

3.2. Existence and stability of equilibria. In what follows, we study the following cubic truncated system of dynamical system (19):

$$\begin{align*}
\dot{x}_1 &= \mu_1 x_1 + A_1 x_1 x_2 + (A_2 x_1^2 + A_3 x_2^2) x_1, \\
\dot{x}_2 &= \mu_2 x_2 + B_1 x_1^2 + (B_2 x_1^2 + B_3 x_2^2) x_2,
\end{align*}$$

under $A_j \neq 0, B_j \neq 0$ and $\mu_j \in \mathbb{R}$, which hold generically. Hereafter, the equilibria $(x_1, 0), (0, x_2)$ and $(x_1, x_2)$ of the dynamical system (21) are referred to as 1-mode equilibrium, 2-mode equilibrium and mixed mode equilibrium, respectively.

Remark 4. The similar dynamical system has already been studied by Armbruster et al. in [1, 2]. They studied the following dynamical system defined on $\mathbb{R}^4 = \mathbb{C} \times \mathbb{C}$:

$$\begin{align*}
\dot{z}_1 &= \bar{z}_2 + (\nu_1 + e_{11}|z_1|^2 + e_{12}|z_2|^2), \\
\dot{z}_2 &= \pm z_1^2 + z_2 (\nu_2 + e_{21}|z_1|^2 + e_{22}|z_2|^2),
\end{align*}$$

where $\nu_j$ are bifurcation parameters and $e_{jk}$ are real coefficients. The above system can be derived from the Kuramoto-Sivashinsky equation ([16]) under the periodic boundary condition. It is known that the system (22) exhibits complicated dynamics: travelling waves, modulated traveling waves and heteroclinic cycles. If we consider the system (3) under the periodic boundary condition, then we also obtain the above system (22).

3.2.1. Trivial equilibrium. The dynamical system (21) has the trivial solution $(x_1, x_2) = (0, 0)$ for any $\mu_1, \mu_2$. Linearized eigenvalues are $\mu_1, \mu_2$, thus the stability is determined by their sign. We note that the trivial solution corresponds to the constant stationary solution $(u(x, t), v(x, t)) \equiv (u^*, v^*)$.

3.2.2. 1-mode equilibria. Although we cannot directly find the 1-mode equilibrium from the reduced system (21), if we consider the case where fixed $\mu_2 \neq 0$, then there exists a function $h(x_1, \mu_1) = x_2$ for sufficiently small $0 < |\mu_1| < 1$ such that $h(0, 0) = \partial h/\partial x_1(0, 0) = \partial h/\partial \mu_1(0, 0) = 0$ by center manifold theory. It can be easily seen that $h(x_1) = -B_1/\mu_2 x_1^2 + O(|x_1|^3)$ and $\dot{x}_1 = \mu_1 x_1 + (A_2 - A_1 B_1/\mu_2) x_1^2 + O(|x_1|^3)$.
O(\|x_1\|^5), hence the 1-mode equilibria bifurcate from the trivial solution through the pitchfork bifurcation at \( \mu_1 = 0 \), and the stability is determined by the sign of \( A_2 - A_1 B_1 / \mu_2 \).

3.2.3. 2-mode equilibria. The dynamical system (21) has the 2-mode equilibria \((x_1, x_2) = (0, \pm 2x)\) for \( \mu_2 B_3 < 0 \), where \( x_2 = \sqrt{-\mu_2 / B_3} \). Indeed, by setting \( x_1 = 0 \) we have \( \hat{x}_2 = \mu_2 x_2 + B_3 x_2^3 \), and therefore the equilibria bifurcate from the trivial equilibrium through the pitchfork bifurcation at \( \mu_2 = 0 \). Note that the 2-mode equilibria correspond to the non-uniform stationary solution expressed by

\[
\begin{align*}
  u(x, t) &= u^* + 2T_{11}^2 \hat{x}_2 \cos \left( \frac{2\pi x}{L} \right) + o(\varepsilon^3), \\
  v(x, t) &= v^* + 2T_{21}^2 \hat{x}_2 \cos \left( \frac{2\pi x}{L} \right) + o(\varepsilon^3),
\end{align*}
\]

which exhibits the segregation pattern since \( T_{11}^2 T_{21}^2 < 0 \) holds. The linearized matrix around the 2-mode equilibria are

\[
\begin{pmatrix}
  \mu_1 + A_1 \hat{x}_2 + A_3 \hat{x}_2^2 & 0 \\
  0 & -2\mu_2
\end{pmatrix},
\]

and the stability is determined by the sign of eigenvalues.

3.2.4. Mixed mode equilibria. We can find that the mixed mode equilibria bifurcate from the 2-mode equilibrium \((x_1, x_2) = (0, \hat{x}_2)\) through the pitchfork bifurcation. Letting \( y = x_2 - \hat{x}_2 \) and \( \mu = \mu_1 - \hat{\mu}_1 \) \((\hat{\mu}_1 := -(A_1 + A_3 \hat{x}_2) \hat{x}_2)\), we have the following system with the suspension trick:

\[
\begin{align*}
  \dot{x}_1 &= \mu x_1 + (A_1 + 2A_3 \hat{x}_2)x_1 y + A_2 x_1^3 + A_3 x_1 y^2, \\
  \dot{\mu} &= 0, \\
  \dot{y} &= (\mu_2 + 3B_3 \hat{x}_2^3) y + (B_1 + B_2 \hat{x}_2) x_1^2 + 3B_3 \hat{x}_2 y^2 + B_2 x_1^2 y + B_3 y^3.
\end{align*}
\] (23)

By center manifold theory, we can approximate the center manifold \( y = h(x_1, \mu) = \gamma x_1^2 + O(\|(x_1, \mu)\|^3) \), where \( \gamma = -(B_1 + B_2 \hat{x}_2) / (\mu_2 + 3B_3 \hat{x}_2^2) \). Substituting it into the first component of the system (23), we obtain the reduced equation

\[
\dot{x}_1 = \mu x_1 + \{(A_1 + 2A_3 \hat{x}_2) \gamma + A_2\} x_1^3 + O(\|x_1\|^4).\] (24)

It yields that the mixed mode equilibria

\[
(x_1, x_2) = (\pm \hat{x}_1, \hat{x}_2) := \left( \pm \frac{\mu}{(A_1 + 2A_3 \hat{x}_2) \gamma + A_2}, \hat{x}_2 \right)
\]
bifurcate from the 2-mode equilibrium \((x_1, x_2) = (0, \hat{x}_2)\) through the pitchfork bifurcation at \( \mu_1 = \hat{\mu}_1 \). In the case of \((x_1, x_2) = (0, -\hat{x}_2)\), we obtain a similar result. However, it is complicated to detect directly the Hopf bifurcation point from the mixed mode equilibria obtained in this subsection.

3.2.5. Mixed mode equilibrium with the Hopf instability. In this subsection, we compute a equilibrium which has the Hopf instability. Assume \( x_1 x_2 \neq 0 \) and set \( x_1 = \rho x_2 \) \( (\rho \in \mathbb{R} \setminus \{0\}) \). From the stationary problem, we have

\[
0 = \mu_1 + A_1 x_2 + \rho^2 A_2 x_2^2 + A_3 x_2^2, \quad 0 = \mu_2 + \rho^2 B_1 x_2 + \rho^2 B_2 x_2^2 + B_3 x_2^2.
\] (25)

The linearized matrix \( M \) is given by

\[
M = \begin{pmatrix}
  2A_2 x_2^2 & x_1(A_1 + 2A_3 x_2) \\
  2x_1(B_1 + B_2 x_2) & 2B_3 x_2^3 - B_1 x_1^2 / x_2
\end{pmatrix}
\] (26)
thus the necessary conditions for the Hopf bifurcation are
\[ \text{tr} \, M = 2(\rho^2 A_2 + B_3)x_2 - \rho^2 B_1 = 0, \quad \det \, M = -2A_1B_1x_1^2 + O(|x_2|^3) > 0. \] (27)
We therefore obtain the Hopf bifurcation point as follows:
\[ x_1 = x_1^* := \rho x_2^*, \quad x_2 = x_2^* := \frac{\rho^2 B_1}{2(2\rho^2 A_2 + B_3)}. \]
\[ \mu_1 = \mu_1^* := -x_2^*(A_1 + A_3x_2^* + \rho^2 A_2x_2^*), \quad \mu_2 = \mu_2^* := -x_2^* \{ B_3x_2^* + \rho^2 (B_1 + B_2x_2^*) \}. \]
The solution of the system (3) corresponding to the mixed mode equilibrium \((x_1, x_2) = (x_1^*, x_2^*)\) is expressed by
\[
\begin{align*}
  u(x, t) &= u^* + 2T_1^1x_1^* \cos \left( \frac{\pi x}{L} \right) + 2T_1^2x_2^* \cos \left( \frac{2\pi x}{L} \right) + o(\varepsilon^3), \\
  v(x, t) &= v^* + 2T_2^1x_1^* \cos \left( \frac{\pi x}{L} \right) + 2T_2^2x_2^* \cos \left( \frac{2\pi x}{L} \right) + o(\varepsilon^3).
\end{align*}
\]
We remark that we should take \( \rho \) such that the equilibrium \((x_1^*, x_2^*)\) are sufficiently small with \( \rho \neq \pm \sqrt{-B_3/A_2} \). Then the bifurcation parameters \( \mu_1 \) and \( \mu_2 \) tend to 0 as \( \rho \to 0 \).

3.3. Derivation of the normal form for the Hopf bifurcation. We now compute a coefficient which determines the stability of the periodic solution bifurcating from the mixed mode equilibria through the Hopf bifurcation. By using standard normal form theory, we will transform the reduced system (21) into the normal form for the Hopf bifurcation ([18]).

Set \( \xi_1 = x_1 - x_1^* \) and \( \xi_2 = x_2 - x_2^* \) in the system (21), we obtain
\[
\frac{d}{dt} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = M_h \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \zeta_1(\xi_1, \xi_2) \\ \zeta_2(\xi_1, \xi_2) \end{pmatrix}, \tag{28}
\]
where
\[
M_h = \begin{pmatrix} 2A_2(x_1^*)^2 & x_1^*(A_1 + 2A_3x_2^*) \\ 2x_1^*(B_1 + B_2x_2^*) & -2A_2(x_1^*)^2 \end{pmatrix} \tag{29}
\]
and
\[
\zeta_1(\xi_1, \xi_2) = 3A_2x_1^*\xi_1^2 + (A_1 + 2A_3x_2^*)\xi_1\xi_2 + A_3x_1^*\xi_2^2 + A_2\xi_1^3 + A_3\xi_1\xi_2^2, \tag{30}
\]
\[
\zeta_2(\xi_1, \xi_2) = (B_1 + B_2x_2^*)\xi_1^2 + 2B_2x_1^*\xi_1\xi_2 + 3B_3x_2^*\xi_2^2 + B_2\xi_1^3 + B_3\xi_2^3. \tag{31}
\]
The eigenvalues of \( M_h \) are \( \pm \sqrt{-1} \omega \in \sqrt{-1}\mathbb{R} \), where
\[
\omega = \sqrt{\det M_h} = \sqrt{-4A_2^2(x_1^*)^4 - 2(x_1^*)^2(A_1 + 2A_3x_2^*)(B_1 + B_2x_2^*)}. \tag{32}
\]
We then obtain the following Lemma:

**Lemma 3.2.** The dynamical system (33) is transformed into the normal form for the Hopf bifurcation:
\[
\dot{z} = \lambda z + C_h|z|^2z + O(|z|^4), \quad z(t) \in \mathbb{C}, \quad \lambda \in \mathbb{C}, \quad C_h \in \mathbb{C} \tag{33}
\]
by an invertible parameter dependent change of complex coordinate.

**Proof.** We define the eigenvectors \( \mathbf{v}_+ \) and \( \mathbf{v}_- \) by
\[
\mathbf{v}_+ = \frac{1}{2\lambda M_h^{12}} \begin{pmatrix} M_h^{12} \\ -M_h^{11} + \lambda \end{pmatrix}, \quad \mathbf{v}_- = \begin{pmatrix} M_h^{11} - \lambda \\ M_h^{12} \end{pmatrix}.
\]
satisfying \( M_h v_+ = \lambda v_+ \), \( M_h^2 v_- = \bar{\lambda} v_- \) and \( \langle v_+, v_- \rangle = 1 \). Here, we denote \( \lambda = \sqrt{-1} \omega, M_h^3 = 2 A_3(x_1)^2, M_h^4 = x_1^2(A_1 + 2 A_3 x_2^2) \) and \( \langle \cdot, \cdot \rangle \) is inner product in \( \mathbb{C}^2 \):

\[
\langle a, b \rangle = \pi_1 b_1 + \pi_2 b_2.
\]

Let \( z(t) \in \mathbb{C} \) be a new variable defined by \( z = \langle v_-, (\xi_1, \xi_2)^t \rangle \). Then the function \( z(t) \) satisfies

\[
\dot{z}(t) = \sqrt{-1} \omega z + \langle v_-, (\xi_1, \xi_2)(\xi_1, \xi_2)^t \rangle
\]

with

\[
\begin{pmatrix}
\xi_1 \\
\xi_2
\end{pmatrix} = z v_+ + \bar{z} \bar{v}_+ + \frac{1}{2 \omega M_h^{12}} \left( (\omega + \sqrt{-1} M_h^{12}) z + (\omega - \sqrt{-1} M_h^{11}) \bar{z} \right).
\]

In addition, we define the function \( \eta(z, \bar{z}) \in \mathbb{C} \) as

\[
\eta(z, \bar{z}) = \langle v_-, (\xi_1(zv_+ + \bar{z} \bar{v}_+), \xi_2(zv_+ + \bar{z} \bar{v}_+))^t \rangle,
\]

and we let \( \eta_{jl} (2 \leq j + l \leq 3) \) be the coefficients of the Taylor expansion of \( \eta(z, \bar{z}) \). Then, the function \( \eta(z, \bar{z}) \) is expressed by the following:

\[
\eta(z, \bar{z}) = \sum_{2 \leq j + l \leq 3} \frac{1}{j! l!} \eta_{jl} z^j \bar{z}^l + O(|z|^4), \quad \eta_{0l} = \frac{\partial^{j+l}}{\partial z^j \partial \bar{z}^l} \eta(0, 0).
\]

Furthemore, we define \( C_h \) as follows:

\[
C_h := \eta_{02} |\eta_{11}(2\lambda + \bar{\lambda})| + \frac{|\eta_{11}|^2}{\lambda} + \frac{|\eta_{02}|^2}{2(2\lambda - \lambda)} + \frac{|\eta_{21}|}{2}.
\]

Then the dynamical system (33) is transformed into (33) by using standard normal form theory (see Section 3.5 of [18]). Note that the transversality condition for the eigenvalues of \( M \) at the Hopf instability point holds generically.

From the above lemma, we obtain the following theorem which is the main result in this paper.

**Theorem 3.3.** If the sign of real part of \( C_h \) is negative (resp. positive) then the cross-diffusion competition system (3) has a locally asymptotically stable (resp. unstable) small amplitude time-periodic solution which bifurcates from the mixed mode stationary solution corresponding to the equilibrium \((x_1, x_2) = (x_1^*, x_2^*)\) of (21) through the Hopf bifurcation.

Using the explicit form of the coefficients listed in Appendix, we can compute the stability coefficient \( C_h \).

### 3.4. Case study.
Using the explicit form of the coefficients listed in Appendix, we can compute the stability coefficient \( C_h \). Although it is difficult to estimate the coefficient \( C_h \) in general, therefore we consider a typical situation in this subsection. We set the coefficients of (3) as \( r_1 = 5.0, r_2 = 2.0, a_1 = 3.0, a_2 = 1.0, b_1 = 1.0, b_2 = 3.0 \) and \( L = 1.0 \). Then the doubly degenerate point is

\[
(d, \gamma) = (d^*, \gamma^*) = \left( \frac{\sqrt{105} - 5}{32 \pi^2}, \frac{477 + \sqrt{105}}{2(45 - \sqrt{105}) \pi^2} \right).
\]
and the coefficients of the reduced system (21) are
\[
A_1 = \frac{13(62301\sqrt{105} + 324175)}{150880}, \tag{35}
\]
\[
A_2 = \frac{13(190774973965\sqrt{105} - 197222969649)}{115875840}, \tag{36}
\]
\[
A_3 = -\frac{13(861829010331286957\sqrt{105} + 8725227724004924055)}{4508108274432000}, \tag{37}
\]
\[
B_1 = \frac{13(4323445 - 424489\sqrt{105})}{104960}, \tag{38}
\]
\[
B_2 = \frac{13(6323055441825139\sqrt{105} - 66425598202278135)}{5463545856000}, \tag{39}
\]
\[
B_3 = -\frac{13(11139493\sqrt{105} + 32359635)}{63000}. \tag{40}
\]
Hence the necessary condition for the Hopf bifurcation \(A_1B_1 < 0\) is satisfied. Furthermore, if we set \(\rho \approx 2.878767\cdots\) then we have the Hopf bifurcation point \((d, \gamma) = (0.01537, 1.7)\), and therefore we obtain \(\mu_1^* \approx -0.0184679, \mu_2^* \approx 0.0131182, \omega \approx 0.0179201\) and \(\text{Re} \{C_h\} \approx -5.18326 \times 10^6 < 0\). Thus, we conclude that the locally asymptotically stable time-periodic solution bifurcates from the non-trivial stationary solution through the supercritical Hopf bifurcation.

**Appendix.** We express the coefficients of the normal form (21) explicitly.

\[
A_1 = \frac{1}{\det T_1} (T_{12}^* f_1 - T_{11}^* g_1), \quad B_1 = \frac{1}{\det T_2} (T_{22}^* f_1 - T_{21}^* g_1).
\]

\[
f_1' = -2a_1 T_{11}^* T_{11}^* + p_1 (T_{11}^* T_{21}^* + T_{12}^* T_{21}^*),
\]

\[
g_1' = \tilde{G}_1^* \tilde{G}_2^* + \tilde{G}_2^*.
\]

\[
f_2' = -2a_1 (T_{11}^*)^2 + 2a_1 T_{11}^* T_{21}^*,
\]

\[
F_0' = 2a_1 (T_{11}^*)^2.
\]
\[G_1^4 = -2b_2 T_2^2 \tilde{h}_{0,2} - a_2 (T_1^2 \tilde{h}_{0,2} + T_2^2 \tilde{h}_{0,2}), \quad G_2^5 = -2b_2 T_2^2 \tilde{h}_4 - a_2 (T_1^2 \tilde{h}_4 + T_2^2 \tilde{h}_4),\]
\[
\tilde{h}_{0,2} = -\frac{1}{2} (M_{12}^0 - M_{12}^1)/\det M_0, \quad \tilde{h}_4 = -\frac{1}{2} (a_{21}^1 F_2^1 - M_{12}^2 G_2)/\det M_0, \]
\[
\tilde{h}_1^j = -\frac{1}{2} (M_{12}^j F_2^1 - M_{12}^1 G_2^j)/\det M_j, \quad \tilde{h}_2^j = -\frac{1}{2} (a_{21}^j F_2^1 - M_{12}^1 G_2^j)/\det M_j, \]
\[
p_j = -\frac{1}{2} (\gamma^j k^2 + b_1). \quad \tilde{h}_0^j = (T_1^2 F_2^1 - T_1^1 G_2^j)/\det M_j, \]
\[
M_{11}^j = -\{(d^j + \gamma^j v^*) j^2 k^2 + a_1 u^*\}, \quad M_{12}^j = -\{(d^j + \gamma^j v^*) j^2 k^2 + b_1 u^*\}.
\]

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