Solitons in a $PT$-symmetric grating-assisted co-directional coupler

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Abstract. We explore a co-directional coupling arrangement between two waveguides mediated by a $PT$-symmetric sinusoidal grating characterized by an index-modulation parameter $\kappa$ and a gain/loss parameter $g$. We show that the device supports soliton-like solutions for both the $PT$-conserving regime $g < \kappa$ and the $PT$-broken regime $g > \kappa$. In the first case the coupler exhibits a gap in wave-number $k$, and the solitons can be regarded as an extension of a previous solution found for pure index modulation. In the second case the coupler exhibits a gap in frequency $\omega$ and the solutions are entirely new.

1. Introduction
Since soliton propagation in an optical fiber was first demonstrated 40 years ago, we have seen an explosion of new ideas, materials and designs for new all-optical devices and systems based on the robustness of soliton signals [1, 2]. Soliton properties rely strongly on the interplay between structural and/or material dispersions on the one hand and refractive index nonlinearity on the other. Under the proper propagation conditions, it takes a certain distance to achieve this dispersion-nonlinearity balance, a distance that varies from a few hundred meters to several kilometers in the case of an unperturbed communication fiber. With sufficiently high signal power, gap solitons in reflective fiber Bragg gratings can achieve this balance within a few centimeters. The unique dispersion characteristics of fiber or waveguide Bragg gratings allow one to control soliton velocities from zero to the speed of light in the bare fiber or waveguide [3, 4].

While Bragg gratings with sub-micron periods provide coupling between waves propagating in opposite directions, gratings with longer pitch (from tens of microns to hundreds of microns) can be resonant also with co-propagating modes, leading to forward coupling. Such periodical structures are called long-period gratings (LPGs) [5, 6]. Although originally proposed for such purposes as band-rejection filters, due to their broad-band operation (compared with short-period Bragg gratings), LPGs are perfect devices for ultra-short optical signal processing, like, for example, temporal differentiation [7] and integration of sub-picosecond optical waveforms [8].

Inside reflective Bragg gratings strong interaction between forward and backward waves produces mode dispersion, which can be depicted in a dispersion diagram showing the relation between the frequency $\omega$ and the wave propagation constant, $k(\omega)$, when nonlinearity is neglected. As was shown by Kogelnik and Shank [9] in the case of reflective gratings with...
index modulation, their mode interaction generates a band of frequencies (stop-band), centered at the Bragg frequency $\omega_B$, in which propagation is forbidden, i.e. a forbidden frequency gap ($\omega$-gap) (Fig. 1(a)). Outside the stop-band the dispersion curves are hyperbolas. The vicinity of the stop-band is a spectral area where different types of Bragg soliton may exist [10] when nonlinearity is taken into account.

On the other hand, in the case of a reflective grating produced by gain/loss modulation the dispersion diagram has the same hyperbolic dispersion curves with the same asymptotes as before, except that the curves are rotated by 90° (Fig. 1 (b)). Instead of a stop-band in frequency, we now have a forbidden band of propagation constants, i.e. a gap in wave number ($k$-gap), centered at the resonance frequency $\omega_R$.

When it comes to band-gap diagrams for long-period gratings providing co-propagation coupling, the situation is now reversed: LPGs that couple through index modulation belong to $k$-gap devices [11, 12, 13], while those coupling through gain/loss modulation are $\omega$-gap structures.

The duality between contra-propagating ($k$-gap) and co-propagating ($\omega$-gap) mode coupling is well established for index Bragg grating and index LPGs [12, 14, 15]. Indeed the long-period co-propagation coupling was considered among the first examples of a periodic structure yielding trapping through quadratic nonlinearity [16]. The coupled-wave equations governing the long-period co-directional coupling are very similar to those of contra-directional coupling but with space and time variables interchanged. Although solitons of these models are related to the existence of a spectral gap (though in wave-number space or $k$-gap), they are more naturally named resonance solitons [12]. The duality between the description of gap solitons in reflective nonlinear Bragg index gratings and resonance solitons implies an important physical difference: solitons of the co-propagating modes cannot have zero velocity in the lab frame [13] if they are coupled through index modulation.

Recently nonlinear optics has been applied to new artificial optical materials that exhibit very unusual dispersion properties. Plasmonic meta-materials, materials with zero and negative
refractive index, and \( PT \)-symmetric materials have all been studied from the point of view of nonlinear solitary wave propagation [17, 18, 19].

The situation becomes especially interesting and unusual in the case of \( PT \)-symmetric gratings [20] that combine refractive index modulation with periodic loss and gain modulation of the same pitch shifted by a half period, \( \Lambda \equiv 2\pi/K \):

\[
\Delta n(z) = \Delta n_0 \cos(Kz) + i(\alpha_0/k_0) \sin(Kz),
\]

where \( \Delta n_0 \) and \( \alpha_0 \) are the amplitudes of the index and gain/loss modulations and \( k_0 \equiv 2\pi/\lambda_0 \) the wave number in vacuum. \( PT \)-symmetric co-directional couplers were first proposed by Greenberg and Orenstein [21]. They demonstrated that such a combination of two gratings makes the coupler asymmetrical, with complete asymmetry occurring when the grating amplitudes are equal: \( \Delta n_0 = \alpha_0/k_0 \), so that \( \Delta n(z) = \Delta n_0 \exp(\pm iKz) \). The coupler in such a balanced mode exhibits very unusual behavior [22]. A signal launched into an input port of WG#1, will be coupled into WG#2 with amplification provided at a certain value of the coupling coefficient (\( \kappa \)) and the grating length (\( L \)) product, \( \kappa L > 1 \) (Fig. 2(a)). At the same time, if this signal launched into an input port of WG#2, it will not be coupled into WG#1 at all. It will propagate through WG#2 as if the grating were invisible. This unique property of \( PT \)-symmetric co-directional coupling can be used to trap light if we connect the output of the receptor waveguide WG#2 with its input to form a ring, as shown in Fig. 2(b). It also can be used to design lasers with very low threshold [23] and many other applications.

\( PT \)-symmetry provides a means of controlling the dispersion characteristics of the grating-assisted coupler. With only index modulation (\( \alpha_0 = 0 \)) it is a \( k \)-gap structure, as shown in Fig. 3(a). However, as the amplitude of the gain/loss modulation in Eq. (1) gradually increases, the wave-number gap shrinks and finally disappears in the case of a balanced \( PT \)-symmetric grating (\( \alpha_0 = k_0\Delta n_0 \), Fig. 3(b)). Finally in the broken \( PT \)-symmetry situation, when the gain/loss modulation amplitude exceeds the index amplitude (\( \alpha_0 > k_0\Delta n_0 \), Fig. 3(c)), the coupler becomes an \( \omega \)-gap structure, similar to a reflective index grating. The larger the gain/loss amplitude, the larger the forbidden gap.
PT-symmetric grating-assisted co-directional couplers with a large grating period (tens or hundreds of microns) are much easier to manufacture using standard optical photolithography than similar contra-directional couplers with sub-micron periodicity, which involve complex e-beam lithography. Due to grating-assisted coupling there is considerable freedom of design, whereby WG#1 and WG#2 could be quite different, with different nonlinearity and dispersion. Also, as already mentioned, co-directional couplers can be built with rather broad spectral transmission capable of processing very short optical waveforms.

2. Coupled-mode equations for the coupler

Let us consider a nonlinear grating-assisted co-directional coupler with two dissimilar parallel waveguides WG#1 and WG#2 with different propagation constants $\beta_1(\omega)$ and $\beta_2(\omega)$, as shown in Fig. 2(a). We neglect any evanescent coupling, so optical signals propagating along the two waveguides in the same direction (z) can interact with one another only through a PT-symmetric grating with combined modulation of refractive index and loss/gain shifted by a half period, $\Lambda$, as in Eq. (1).

In linear mode the grating enables coupling between the two co-propagating modes in the vicinity of the resonance frequency, $\omega_R$, where $\beta_1(\omega_R) - \beta_2(\omega_R) = K$. We therefore write the longitudinal part of the electromagnetic field in the respective waveguides as $E_1 = A_1(z,t)e^{i(\beta_1 z - \omega t)}$ and $E_2 = A_2(z,t)e^{i(\beta_2 z - \omega t)}$, where $A_1(z,t)$ and $A_2(z,t)$ are slowly-varying functions of z and t.

For $\omega$ near to $\omega_R$ we use the Taylor expansion for $\beta_1$ and $\beta_2$:

$$\beta_r(\omega) = \beta_r(\omega_R) + \beta'_r(\omega_R)(\omega - \omega_R) + \frac{1}{2}\beta''_r(\omega_R)(\omega - \omega_R)^2 + \ldots , \quad (2)$$

($r = 1, 2$), in which the coefficients are related to the group velocity and dispersion by $V_{Gr} = 1/\beta'_r(\omega_R)$ and $\beta_{2r} = \beta''_r(\omega_R)$.

The coupled-wave equations for $A_1$ and $A_2$, including nonlinearity, then take the form

$$i \left( \frac{\partial}{\partial z} + \frac{1}{V_{Gr}} \frac{\partial}{\partial t} \right) A_1 + \frac{1}{2} \beta_{21} \frac{\partial^2 A_1}{\partial t^2} + (\kappa - g)A_2 e^{-2i\delta z} + \gamma|A_1|^2 A_1 = 0 \quad (3a)$$
\[ i \left( \frac{\partial}{\partial z} + \frac{1}{V_{G2}} \frac{\partial}{\partial t} \right) A_2 + \frac{1}{2} \beta_{22} \frac{\partial^2 A_2}{\partial t^2} + (\kappa + g) A_1 e^{+2i\delta z} + \gamma_2 |A_2|^2 A_2 = 0 \]  

(3b)

where \( \kappa = \frac{1}{2} k_0 \Delta n_0 \), \( g = \frac{1}{2} \alpha_0 \), and \( \delta \) is the detuning parameter \( \delta(\omega) \equiv \frac{1}{2}(\beta_1(\omega) - \beta_2(\omega) - K) \), with \( \delta(\omega_R) = 0 \).

In the following we will neglect dispersion in the waveguides themselves, setting \( \beta_{12} = \beta_{22} = 0 \): dispersion is provided by the grating itself.

The phase factors \( e^{+2i\delta z} \) can be removed by defining \( F_1(z,t) = A_1(z,t)e^{i\delta z} \) and \( F_2(z,t) = A_2(z,t)e^{-i\delta z} \), giving the following equations for \( F_1 \) and \( F_2 \):

\[ i \left( \frac{\partial}{\partial z} + \frac{1}{\Delta V_G} \frac{\partial}{\partial t} \right) F_1 + \delta F_1 + (\kappa - g) F_2 + \gamma_1 |F_1|^2 F_1 = 0 \]  

(4a)

\[ i \left( \frac{\partial}{\partial z} - \frac{1}{\Delta V_G} \frac{\partial}{\partial t} \right) F_2 - \delta F_2 + (\kappa + g) F_1 + \gamma_2 |F_2|^2 F_2 = 0, \]  

(4b)

where \( \Delta V_G \) is defined by

\[ \frac{1}{\Delta V_G} = \frac{1}{2} \left( \frac{1}{V_{G1}} - \frac{1}{V_{G2}} \right) \]

and \( F_1 \) and \( F_2 \) are now regarded as functions of \( z \) and \( T \), where

\[ T = t - \frac{1}{2} \left( \frac{1}{V_{G1}} + \frac{1}{V_{G2}} \right) z. \]

As a final simplification we can scale \( T \) according to \( T = \tau/\Delta V_G \), so that the equations become

\[ i \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial \tau} \right) F_1 + \delta F_1 + (\kappa - g) F_2 + \gamma_1 |F_1|^2 F_1 = 0 \]  

(5a)

\[ i \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial \tau} \right) F_2 - \delta F_2 + (\kappa + g) F_1 + \gamma_2 |F_2|^2 F_2 = 0, \]  

(5b)

It is these equations which when linearized give the band-structure diagrams of Fig. 3. In order to solve them in the nonlinear case it is convenient to write the complex \( F_r \) in polar form: \( F_r = a_r e^{i\varphi_r} \), with \( a_r \) real and positive. We then have four equations for the unknown functions \( a_r \) and \( \varphi_r \):

\[ \frac{\partial}{\partial z} a_1 = (\kappa - g) a_2 \sin \Phi \]  

(6a)

\[ \frac{\partial}{\partial z} a_2 = -(\kappa + g) a_1 \sin \Phi \]  

(6b)

\[ \frac{\partial}{\partial z} \varphi_1 = \delta + (a_2/a_1)(\kappa - g) \cos \Phi + \gamma_1 a_1^2 \]  

(6c)

\[ \frac{\partial}{\partial z} \varphi_2 = -\delta + (a_1/a_2)(\kappa + g) \cos \Phi + \gamma_2 a_2^2 \]  

(6d)

where \( \Phi = \varphi_1 - \varphi_2 \).

3. Soliton solutions

Solitary waves correspond to solutions of Eqs. (6) that depend on a single variable. The choice of variable will determine whether the solutions belong to the \( PT \)-symmetric regime \( g < \kappa \) or the broken-symmetry regime \( g > \kappa \).
3.1. **PT-symmetric regime** $g < \kappa$, **gap in wave-number**

A natural choice (cf. Ref. [12]) is

$$\zeta = \frac{\tau - qz}{\sqrt{1 - q^2}}, \quad (7)$$

where the free parameter $q$ is the “velocity” of the soliton in the $(\tau,z)$ plane.

Defining the scaled amplitudes $u_1$ and $u_2$ by

$$a_1(\zeta) = \frac{u_1(\zeta)}{\sqrt{1 - q}} \quad \text{and} \quad a_2(\zeta) = \frac{u_2(\zeta)}{\sqrt{1 + q}}, \quad (8)$$

Eqs. (6) then become

$$\frac{du_1}{d\zeta} = (\kappa - g)u_2 \sin \Phi \quad (9a)$$
$$\frac{du_2}{d\zeta} = (\kappa + g)u_1 \sin \Phi \quad (9b)$$

$$\frac{d\varphi_1}{d\zeta} = \sqrt{\frac{1 + q}{1 - q}} \left( \frac{u_2}{u_1} (\kappa - g) \cos \Phi + \gamma_1 \frac{u_1^2}{1 - q} \sqrt{\frac{1 + q}{1 - q}} \right) \quad (9c)$$
$$\frac{d\varphi_2}{d\zeta} = \sqrt{\frac{1 - q}{1 + q}} \left( \frac{u_1}{u_2} (\kappa + g) \cos \Phi - \gamma_2 \frac{u_2^2}{1 + q} \sqrt{\frac{1 - q}{1 + q}} \right) \quad (9d)$$

From Eqs. (9a) and (9b) we see that

$$\frac{u_1^2}{\kappa - g} = \frac{u_2^2}{\kappa + g} + C_1 \quad (10)$$

We will take the constant $C_1$ to be zero, which will certainly be the case for solutions that go to zero at infinity. Then, since $u_1$ and $u_2$ are both real and positive, this implies that we must have $g < \kappa$, as advertised, and

$$u_2 = u_1 \sqrt{\frac{\kappa + g}{\kappa - g}} \quad (11)$$

It is convenient to define $\eta = u_1^2$, in which case Eq. (9a) becomes

$$\frac{d\eta}{d\zeta} = 2\alpha \eta \sin \Phi, \quad (12)$$

where $\alpha = \sqrt{\kappa^2 - g^2}$.

Combining Eqs. (9c) and (9d) we obtain the following equation for $\Phi$:

$$\frac{d\Phi}{d\zeta} = 2q\Delta + 2\alpha \cos \Phi - \theta \eta \quad (13)$$

where $\Delta = \delta/\sqrt{1 - q^2}$ and

$$\theta = \frac{\kappa + g}{\kappa - g} \frac{\gamma_2}{1 + q} \sqrt{\frac{1 - q}{1 + q}} - \frac{\gamma_1}{1 - q} \sqrt{\frac{1 + q}{1 - q}}.$$

Equations (12) and (13) are very similar to those considered by Conti and Trillo [10] when studying the existence of gap solitons in a completely different physical set-up. We give here a brief account of their equations and their solutions, which can readily be adapted to the present
situation. Similar equations for were also derived by Kazantseva et al. [17] for yet another physical set-up, but for the restricted case $\Delta = 0$.

The two equations of Ref. [10], in our notation, are

$$\frac{d\eta}{d\zeta} = 2\eta \sin \Phi,$$  \hspace{1cm} (14)

$$\frac{d\Phi}{d\zeta} = 2\Delta + 2\cos \Phi - \eta.$$  \hspace{1cm} (15)

These equations can be derived from the conserved Hamiltonian

$$H = (2\cos \Phi + 2\Delta - \eta/2)\eta$$  \hspace{1cm} (16)

in the sense that $d\eta/d\zeta = -\partial H/\partial \Phi$ and $d\Phi/d\zeta = \partial H/\partial \eta$. As a result, $H$ is a constant of the motion.

There are two basic types of soliton supported by these equations, those where $\eta$ goes to zero at infinity, and those where it tends to a finite constant value. These values are given by the stationary points $(\eta_s, \Phi_s)$ of the equations, where $d\eta/d\zeta = d\Phi/d\zeta = 0$.

It is worth noting that $H_s \equiv H(\eta_s, \Phi_s) = \eta_s^2/2$. Then the equation $H = H_s$ can be solved for $\cos \Phi$, giving

$$\cos \Phi = -\Delta + \frac{\eta^2 + \eta_s^2}{4\eta},$$  \hspace{1cm} (17)

from which $\sin \Phi$ in Eq. (14) can be derived, giving an equation in terms of $\eta$ only.

As can readily be seen from Eqs. (14) and (15), one stationary point is $\eta_s = 0$, $\cos \Phi_s = -\Delta$, which requires $|\Delta| \leq 1$. These solutions, which were termed bright (B) solitons in Ref. [10], are given by a suitable choice of the arbitrary additive constant in $\zeta$ as

$$\eta_B = \frac{4(1 - \Delta^2)}{\cosh(2\sqrt{1 - \Delta^2} \, \zeta) - \Delta},$$  \hspace{1cm} (18)

$$\Phi_B = \arccos \left\{ \frac{1 - \Delta \cosh(2\sqrt{1 - \Delta^2} \, \zeta)}{\cosh(2\sqrt{1 - \Delta^2} \, \zeta) - \Delta} \right\}.$$  \hspace{1cm} (19)

The other stationary points are given by $\eta_s = 2(\Delta - 1)$ and $\Phi_s = \pi$, which, since $\eta$ must be positive, are only valid for $\Delta > 1$, i.e. for $\delta > \sqrt{1 - \beta^2}$, outside the so-called dynamical gap [10]. They give rise to two solutions, termed dark (DK) and antidark (AK), which have the same relative phase $\Phi$, but different amplitudes. In fact, as can be seen from Eq. (17), $\eta = \eta \eta_s$ and $\eta = \eta_s/y$ give the same value of $\cos \Phi$. The AK solutions are

$$\eta_{AK} = 2(\Delta - 1) \sqrt{\Delta} \cosh(2\sqrt{\Delta - 1} \, \zeta) + \frac{1}{\sqrt{\Delta} \cosh(2\sqrt{\Delta - 1} \, \zeta) - 1},$$  \hspace{1cm} (20)

$$\Phi_{AK} = \arccos \left\{ -\Delta + (\Delta - 1) \frac{\Delta \cosh^2(2\sqrt{\Delta - 1} \, \zeta) + 1}{\Delta \cosh^2(2\sqrt{\Delta - 1} \, \zeta) - 1} \right\}.$$  \hspace{1cm} (21)

They represent a localized peak above the constant background $\eta_s = 2(\Delta - 1)$.

As has been mentioned, the dark soliton solutions, which were given incorrectly in Ref. [10], have $\Phi_{DK} = \Phi_{AK}$ and

$$\eta_{DK} = 2(\Delta - 1) \sqrt{\Delta} \cosh(2\sqrt{\Delta - 1} \, \zeta) - \frac{1}{\sqrt{\Delta} \cosh(2\sqrt{\Delta - 1} \, \zeta) + 1}.$$  \hspace{1cm} (22)
They represent a localized dip below the same constant background.

We are now in a position to adapt these results to our own situation. The equations (14) and (15) can be brought to the form of (12) and (13) by the substitutions

$$\zeta \rightarrow \alpha \zeta, \quad \Delta \rightarrow q \Delta / \alpha, \quad \eta \rightarrow \eta \theta / \alpha.$$  

Thus the bright soliton solutions to Eqs. (12) and (13) are

$$\eta_B = \frac{4}{\theta} \frac{(\alpha^2 - \tilde{\Delta}^2)}{\alpha \cosh(2\sqrt{\alpha^2 - \tilde{\Delta}^2} \zeta) - \tilde{\Delta}},$$

$$\Phi_B = \arccos \left( \frac{\alpha - \tilde{\Delta} \cosh(2\sqrt{\alpha^2 - \tilde{\Delta}^2} \zeta)}{\alpha \cosh(2\sqrt{\alpha^2 - \tilde{\Delta}^2} \zeta) - \tilde{\Delta}} \right),$$

where $$\tilde{\Delta} = q \Delta = q \delta / \sqrt{1 - q^2}$$, which are valid as long as $$|\tilde{\Delta}| < \alpha$$. Moreover the overall factor $$\theta$$ in Eq. (24) must be positive, since $$\eta = u_1^2$$ is positive.

The anti-dark solutions, with $$\tilde{\Delta} > \alpha$$, are now

$$\eta_{AK} = \frac{2}{\theta} (\tilde{\Delta} - \alpha) \frac{\sqrt{\Delta} \cosh(2\sqrt{\alpha^2 (\tilde{\Delta} - \alpha)} \zeta) + \sqrt{\alpha}}{\sqrt{\Delta} \cosh(2\sqrt{\alpha^2 (\tilde{\Delta} - \alpha)} \zeta) - \sqrt{\alpha}},$$

$$\Phi_{AK} = \arccos \left\{ \frac{\tilde{\Delta} + 1}{\alpha} \left( \frac{\Delta \cosh^2(2\sqrt{\alpha^2 (\tilde{\Delta} - \alpha)} \zeta) + \alpha}{\Delta \cosh^2(2\sqrt{\alpha^2 (\tilde{\Delta} - \alpha)} \zeta) - \alpha} \right) \right\},$$

representing a localized peak above the background value $$\eta_s = 2(\tilde{\Delta} - \alpha)/\theta$$. The dark solutions are trivially derived from these.

In the limit as $$\tilde{\Delta} \rightarrow \alpha$$ the $$\eta_B$$ and $$\eta_{AK}$$ solutions merge, to give a Lorentzian shape, namely

$$\eta_B \rightarrow \frac{\alpha}{\theta} \left( \frac{8}{1 + 4\alpha^2 \zeta^2} \right),$$

$$\Phi_B \rightarrow \arccos \left( \frac{1 - 4\alpha^2 \zeta^2}{1 + 4\alpha^2 \zeta^2} \right),$$

while $$\eta_{DK} \rightarrow 0$$.

3.2. Broken PT-symmetry regime $$g > \kappa$$, gap in frequency

As we have seen, the choice of variables (7) meant that soliton solutions could only be found for the PT-symmetric regime $$g < \kappa$$. However, such solutions can be found in the broken-symmetry regime if we essentially interchange the roles of $$z$$ and $$\tau$$, defining $$\zeta$$ instead as

$$\zeta = \frac{z - v \tau}{\sqrt{1 - v^2}}.$$

where the free parameter $$v$$, with $$|v| < 1$$, now has the meaning of a velocity.

Defining the scaled amplitudes $$u_1$$ and $$u_2$$ by

$$a_1(\zeta) = \frac{u_1(\zeta)}{\sqrt{1 - \nu}} \quad \text{and} \quad a_2(\zeta) = \frac{u_1(\zeta)}{\sqrt{1 + \nu}},$$

we obtain the following equations:

\[
\begin{align*}
\frac{du_1}{d\zeta} &= (\kappa - g)u_2 \sin \Phi \\
\frac{du_2}{d\zeta} &= -(\kappa + g)u_2 \sin \Phi \\
\frac{d\varphi_1}{d\zeta} &= \sqrt{\frac{1+v}{1-v}} \delta + \frac{u_2}{u_1}(\kappa - g) \cos \Phi + \gamma_1 \frac{u_1^2}{1-v} \sqrt{\frac{1+v}{1-v}} \\
\frac{d\varphi_2}{d\zeta} &= -\sqrt{\frac{1-v}{1+v}} \delta + \frac{u_1}{u_2}(\kappa + g) \cos \Phi + \gamma_2 \frac{u_2^2}{1+v} \sqrt{\frac{1-v}{1+v}}
\end{align*}
\] (32a, 32b, 32c, 32d)

From Eqs. (32a) and (32b) we see that

\[
\frac{u_1^2}{g - \kappa} = \frac{u_2^2}{g + \kappa},
\] (33)

taking the additive constant to be zero, as before. Then, since \(u_1\) and \(u_2\) are both real and positive, this implies that we must now have \(g > \kappa\), and

\[
u_2 = u_1 \sqrt{\frac{g + \kappa}{g - \kappa}},
\] (34)

Again defining \(\eta\) as \(\eta = u_1^2\), Eq. (32a) becomes

\[
\frac{d\eta}{d\zeta} = -2\alpha \eta \sin \Phi,
\] (35)

where \(\alpha = \sqrt{g^2 - \kappa^2}\).

Combining Eqs. (32c) and (32d) we obtain the following equation for \(\Phi\):

\[
\frac{d\Phi}{d\zeta} = 2\Delta - 2\alpha \cos \Phi - \theta \eta,
\] (36)

where in this case \(\Delta = \delta/\sqrt{1-v^2}\) and \(\theta\) is the same as previously defined, with \(q \to v\).

The substitutions required to bring Eqs. (14) and (15) to the form of (35) and (36) are

\[
\zeta \to \alpha \zeta, \quad \Delta \to \Delta/\alpha, \quad \eta \to \eta\theta/\alpha, \quad \Phi \to \Phi + \pi.
\] (37)

Thus the soliton solutions in this case take essentially the same form as those of the previous section, Eqs. (24) - (29), except that \(\Phi\) is shifted by \(\pi\) and \(\Delta\) is replaced by \(\Delta\). However, it is very important to recall that \(\alpha\) is now defined as \(\alpha = \sqrt{g^2 - \kappa^2}\), and the variable \(\zeta\) is defined as in Eq. (30) rather than (7). The parameter \(\theta\) must still be positive.

4. Discussion

The \(PT\)-symmetric case, \(g < \kappa\), seems a straightforward extension of the pure index coupling situation investigated by Wabnitz [12], except that in that paper the nonlinear terms were different, allowing for a true soliton solution. It is worth noting that the lab. frame velocity of the soliton in this case, dictated by the choice of variable \(\zeta \propto \tau - qz\) is

\[
V_s = \frac{2V_{G1}V_{G2}}{V_{G1}(1-q) + V_{G2}(1+q)},\]
(38)
which lies between $V_{G1}$ and $V_{G2}$, as might be expected.

The broken $PT$ case, $g > \kappa$, has some unusual features. First, there is no sign of the exponential growth one might expect when the $PT$ symmetry is broken, and second, the choice of variable $\zeta \propto z - v \tau$, with $\tau = \Delta V_G T$ and $T = t - \frac{1}{2}(1/V_{G1} + 1/V_{G2})z$, gives the lab. frame velocity of the soliton as

$$V_s = \frac{2vV_{G1}V_{G2}}{V_{G2}(1 + v) - V_{G1}(1 - v)}.$$  \hfill (39)

This has the nice property that the soliton can be stationary, when $v = 0$, the mechanism for which may be related to the stopping of photons at the exceptional point in a $PT$-symmetric medium[24]. At first sight it appears that $V_s$ has the less attractive property that it is unlimited if $v$ is allowed to vary freely, even becoming infinite when $V_{G2} = ((1 - v)/(1 + v))V_{G1}$. However, it is worth noting that the way that $V_s$ enters the expression for $\zeta$ according to

$$\zeta = -\frac{v \Delta V_G}{\sqrt{1 - v^2}} \left( t - \frac{z}{V_s} \right),$$  \hfill (40)

so that as $V_s$ increases, the soliton peak becomes wider and less well defined as a function of $z$, becoming a function of $t$ only when $V_s \to \infty$.

Recall that the choice of variable for $g > \kappa$ was dictated by the fact that we took the constant of integration $C_1$ to be zero in Eq. (33), as we did in Eq. (10), in order that solutions could be found that went to zero at infinity. However, if $C_1 \neq 0$ other forms of solution can be found for both $g < \kappa$ and $g > \kappa$, along the lines of Refs. [25] and [26].

We should also mention that a soliton solution can be found [27] at the exceptional point $g = \kappa$ for specially constrained values of the parameters of the waveguides and their couplings. The soliton velocity in this case has the same general features as in the case $g > \kappa$.

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