Green’s function approach to the non-equilibrium superconductivity near the critical line

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Abstract. In spite of the absent friction of super-currents, normal currents affect the superconducting condensate. The BCS approach with Bogoliubov-Valutin quasiparticles is not suited for description of the normal current near the critical line. We review an alternative theory of the non-equilibrium superconductivity dealing exclusively with normal-state quasiparticles. It is based on the Thouless T-matrix criterion, which is extended to non-equilibrium. The problem of selfconsistency of the T-matrix in the superconducting state is examined and solved with the help of the multiple-scattering theory.

1. Introduction
The non-equilibrium superconductivity is traditionally described with the help of Green’s functions. As Gor’kov has shown, the theory of Bardeen, Cooper and Schrieffer (BCS) is recovered if the set of usual single-particle functions is extended by anomalous functions correlating two creation (annihilation) operators of opposite spin. These anomalous functions vanish in the normal state while in the superconducting state they are finite and carry the essential part of physical properties being responsible for the non-dissipative current.

The presence of anomalous functions results in the gap $\Delta$ in the single-particle energy spectrum

$$E^{(1,2)}_k = \sqrt{\epsilon_k^2 + |\Delta|^2},$$

where $\epsilon_k = k^2/2m - k_F^2/2m$. Superscripts remind that there are two branches of equal energy corresponding to two spin branches in the normal state. Single-particle excitations with energies (1) are called Bogoliubov-Valutin quasiparticles. As one can see, in the BCS approach the gap is a part of the unperturbed effective Hamiltonian, accordingly it is suited for description of well developed superconducting states sufficiently far from the critical temperature.

The phenomenological theory of Ginzburg and Landau (GL) approaches the superconductivity from the other side. Its derivation is restricted to a narrow vicinity of the critical temperature, where the BCS gap is very small and its effect can be treated in the power expansion. The free energy of Ginzburg and Landau goes to the power $|\Delta|^4$.

In spite of different regions on applicability, Gor’kov [1] has recovered the GL theory from anomalous Green’s functions as the limit of small $\Delta$. In equilibrium this limit is well defined and Gor’kov approach provides sound microscopic justification of the phenomenological theory.
Figure 1. Motion of the Bogoliubov-Valutin quasiparticle in the energy band. The quasiparticle \( \Psi_k^{(1) \dagger} \) driven by a time-dependent vector potential, \( \mathbf{k} = \mathbf{k}_0 + e \mathbf{A} \), adiabatically (thin black arrows) follows the quasiparticle energy \( E^{(1)} \) in the left panel. The central and right panel show the same motion in the normal-state energy band. The Bogoliubov-Valutin quasiparticle starts as an electron of momentum \( |\mathbf{k}| > k_F \) and spin \( \uparrow \). Crossing the Fermi level it changes into a hole of momentum \( -\mathbf{k} \) with \( |-\mathbf{k}| < k_F \) and spin \( \downarrow \). The reverse conversion happens as it leaves the Fermi sphere. By a non-adiabatic Zener-type tunneling across the gap (grey thick arrow in the left panel) the Bogoliubov-Valutin quasiparticle emerges as a quasi-hole with energy \( |\mathbf{k}| < k_F \) in the \( E^{(2)} \)-branch (dashed line rotated by 180°). This Zener-type tunneling corresponds to the continuous motion of the electron with spin \( \uparrow \) (grey thick arrow in the central panel).

Out of equilibrium the situation becomes less transparent. Problems appear on both sides. The GL phenomenology does not apply since there are no well defined thermodynamical functions. Moreover, even if one constructs an approximative thermodynamical function, the state of the system is not in its minimum and the dynamics of the relaxation process towards the minimum is not covered by variational methods. The non-equilibrium version of the GL theory thus has to be derived microscopically [2, 3, 4].

A convenient microscopic picture of processes in the non-equilibrium superconductor provides the distribution of Bogoliubov-Valutin quasiparticles. Corresponding time evolution is covered by the kinetic equation of Boltzmann type in which the \( E_k^{(1,2)} \) enter as the quasiparticle energy. Bogoliubov-Valutin quasiparticles in superconductors are, however, confusing entities, see Fig. 1. Unlike in the Fermi liquid, their charge depends on momentum [5]

\[
q_k = -\frac{e \epsilon_k}{E_k}.
\]

For \( \epsilon_k \gg |\Delta| \), the Bogoliubov-Valutin quasiparticle \( \Psi_k^{(1) \dagger} \) is like an electron with spin \( \uparrow \) and momentum \( \mathbf{k} \). Reducing its momentum to \( \epsilon_k \ll -|\Delta| \) it turns into a hole of spin \( \downarrow \) and momentum \( -\mathbf{k} \). Such quasiparticle picture is justified only for changes which are slow on the time scale of the inverse BCS gap. Under these conditions, the cross element between quasiparticles on branches \( E^{(1)} \) and \( E^{(2)} \) vanishes by dephasing, because it oscillates with frequency \( E^{(1)} + E^{(2)} \), which is greater than \( 2|\Delta| \) and thus fast on the scale of changes. The adiabatic evolution of Bogoliubov-Valutin quasiparticles holds only for sufficiently large gap, which is not satisfied near the critical temperature.

For a small gap or fast perturbations the cross element has to be included. The coupling between branches results in the evolution of one quasiparticle into its counterpart, say quasiparticle goes from branch \( E^{(1)} \) into branch \( E^{(2)} \). In the dialect of the historical
semiconductor model of superconductors, this jump corresponds to the Zener tunneling across the superconducting gap. The tunneling reveals peculiarities of the Bogoliubov-Valutin representation: the quasiparticle $\Psi_{k}^{(1)\dagger}$ tunnels from its branch into the quasihole $\Psi_{-k}^{(2)}$ in the second branch.

The above described complicated Zener-type tunneling corresponds to the usual evolution of Landau quasiparticle $\Psi_{k}^{\uparrow\dagger}$ in the normal state. Indeed, $\Psi_{k}^{(1)\dagger}=\Psi_{k}^{\uparrow\dagger}$ for $\epsilon_{k}\gg|\Delta|$ and $\Psi_{-k}^{(2)}=\Psi_{k}^{\uparrow\dagger}$ for $\epsilon_{k}\ll-|\Delta|$. In the critical region, where the system approaches the normal state and the $\Delta$ becomes very small, perturbations are fast compared to $1/|\Delta|$ so that the tunneling prevails over adiabatical evolution in the vicinity of the energy gap. In such situation the gap function $\Delta$ should not be included in the unperturbed Hamiltonian. We have to leave the Bogoliubov-Valutin representation and formulate the superconductivity in terms of the ‘normal’ Landau quasiparticles treating the gap as a perturbation. Expansion in powers of $\Delta$ is closer in spirit to the original approach of Ginzburg and Landau.

In this paper we do not benefit from the anomalous functions which are ultimately linked with the Bogoliubov-Valutin representation. Instead we formulate the theory of superconductivity with the help of the T-matrix approximation of the selfenergy. The present approach is advantageous in the critical region, because the single-particle Green function in the limit of small $\Delta$ naturally matches the normal-state function and a corresponding evolution of single-particle excitations agrees with the customary Landau quasiparticle picture.

2. Thouless criterion

The theory dealing exclusively with ‘normal’ quasiparticles was proposed by Thouless [6] soon after discovery of the Cooper pairing. He described the pairing of electrons in the ladder approximation and analyzed properties of the T-matrix. When system reaches conditions for the onset of the superconductivity, the T-matrix becomes singular. The Thouless criterion

$$\frac{1}{T_{R}(0,0)}=0$$

establishes critical temperature of the superconducting state from the retarded T-matrix in the normal state for pair energy $\Omega=0$ and pair momentum $Q=0$.

Unfortunately, the selfconsistent ladder approximation (known as the Galitskii approximation in solid state physics [7] and called standard approximation in the non-relativistic nuclear physics) fails in the superconducting state, because it does not yield the BCS gap. The Thouless criterion thus merely provides conditions in which the superconducting condensate nucleates, but it cannot be used to evaluate the magnitude of the gap even in the asymptotic region $\Delta \to 0$. In the rest of this section we generalize the Thouless criterion to non-equilibrium. In the next section we show how to overcome problems of the Galitskii approximation and evaluate the magnitude of the gap.

2.1. Galitskii approximation

Recently the Thouless criterion has been employed under non-equilibrium conditions [8]. To treat non-local contributions to the nucleating condensate, the criterion has to be expressed in the space-time variables. The ladder approximation of the T-matrix satisfies

$$T_{R}(12,34)=V(1234)+V(1256)G_{(2)}^{R}(56,78)T_{R}(78,34),$$

where $V$ is the pairing potential, $G_{(2)}$ is the two-particle propagator, numbers are cumulative arguments $1 \equiv r_{1}, t_{1}$, groups of arguments restricted to equal time are not separated by comma, and bar denotes space-time integrals. We tacitly assume that spin $\uparrow$ is associated with odd arguments and $\downarrow$ with even ones.
In the Galitskii approximation the selfenergy is the mean value of the T-matrix obtained by averaging over accessible interaction partners \( \Sigma(1,3) = -iT(12,34)G(\bar{1},\bar{2}) \). This time-contour relation yields the retarded selfenergy
\[
\Sigma^R(1,3) = T^R(12,34)G^<(\bar{4},\bar{2}) - T^<(12,34)G^A(\bar{4},\bar{2}).
\] (5)

Analytic pieces of the T-matrix \( T^R \) and \( T^< \) become singular under identical conditions. Their amplitudes of the singularity in the pole, however, essentially differ. While the retarded function has usual pole which appears with any bounded state, the pole of \( T^< \) is enhanced by the Bose-Einstein condensation. Indeed, in the equilibrium \( T^<(\Omega,\mathbf{Q}) = -2\text{Im} \ T^R(\Omega,\mathbf{Q})/(e^{\Omega} - 1) \). The distance of the pole from zero behaves as \( 1/\text{volume} \) so that the occupation of the condensation mode \( \mathbf{Q} = 0 \) is proportional to the volume of the sample, while occupations of other modes are independent of volume. To describe the effect of the condensate on the single-particle spectrum it is sufficient to keep the more divergent term
\[
\Sigma^R(1,3) \approx -T^<(12,34)G^A(\bar{4},\bar{2}).
\] (6)

The correlation part of the T-matrix satisfies
\[
T^<(12,34) = T^R(12,5\bar{6})G^<_{(2)}(5\bar{6},\bar{7}\bar{8})T^A(\bar{7}\bar{8},34). \] (7)

When the T-matrix becomes singular, its value is much larger than the potential, \( |T^R| \gg V \), therefore from (4) follows
\[
T^R(12,34) \approx V(12\bar{5}\bar{6})G^R_{(2)}(5\bar{6},\bar{7}\bar{8})T^R(\bar{7}\bar{8},34). \] (8)

Using approximation (8) in Eq. (7) we obtain
\[
T^<(12,34) = V(12\bar{5}\bar{6})G^R_{(2)}(5\bar{6},\bar{7}\bar{8})T^<(\bar{7}\bar{8},34). \] (9)

The set of equations for the Galitskii approximation is completed by the Dyson equation \( G^R = G^R_0 + G^R_0\Sigma^R G^R \) and the generalized Kadanoff-Baym equation \( G^< = G^R\Sigma^< G^A \), where \( \Sigma^<(1,3) = T^<(12,34)G^>(\bar{4},\bar{2}) \). We will focus on Eq. (9).

2.2. Linearized time-dependent Ginzburg-Landau theory

The condensate is described by the element \( T^< \). One can equally well use \( T^> \), because their Bose factor differ by unity which has to be compared with the macroscopic number of Cooper pairs in the system.

The BCS model of the interaction is point like, \( V(1234) = V\delta(1-2)\delta(2-3)\delta(3-4) \), so that \( T^<(12,34) = T^<(1,3)\delta(1-2)\delta(3-4) \) and the equation (9) simplifies as
\[
T^<(1,3) = K(1,\bar{2})T^<(\bar{2},\bar{3}), \] (10)

where \( K(1,2) = VG^R_{(2)}(11,22) \) is the integration kernel.

Following Gor’kov, we assume that the integration kernel extends over a small space and time region. This allows us to expand the gap in the lowest gradients
\[
T^<(2,3) \approx T^<(1,3) + \frac{\partial T^<(1,3)}{\partial t_1}(t_2 - t_1) + \frac{\partial T^<(1,3)}{\partial r_1}(r_2 - r_1) + \frac{1}{2} \frac{\partial^2 T^<(1,3)}{\partial r_1^2}(r_2 - r_1)^2, \] (11)

which leads to the equation
\[
\alpha T^< + \Gamma \frac{\partial T^<}{\partial t_1} + S \frac{\partial T^<}{\partial r_1} - \frac{1}{2m^*} \frac{\partial^2 T^<}{\partial r_1^2} = 0 \] (12)
with the Ginzburg-Landau (GL) parameter \( \alpha(1) = K(1, \tilde{2})/C(1) \), the relaxation rate \( \Gamma(1) = -K(1, \tilde{2})(t_2 - t_1)/C(1) \), the vector coefficient \( S(1) = K(1, \tilde{2})(r_2 - r_1)/C(1) \), and the weight of the isotropic tensor \( 1/m^*(1) = -K(1, \tilde{2})(r_2 - r_1)^2/C(1) \). The norm \( C \) is introduced for convenience of the final rearrangement.

Note that the differential equation acts only on the left argument of the correlation function. A conjugated equation can be derived for the right argument. Both equations are solved by the separable function

\[
T^<(1, 2) = \Delta^*(1)\Delta(2),
\]

where the star denotes the complex conjugation. This result agrees with numerical observations that at the singularity the T-matrix becomes separable [9]. Due to the separability Eq. (12) simplifies to the linearized time-dependent Ginzburg-Landau (TDGL) equation

\[
\alpha \Delta^* + \Gamma \frac{\partial \Delta^*}{\partial t} + S \frac{\partial \Delta^*}{\partial r} - \frac{1}{2m^*} \frac{\partial^2 \Delta^*}{\partial r^2} = 0. \tag{14}
\]

Further progress requires to specify Green’s functions. The time-contour two-particle propagator \( G_{(2)}(12, 34) = iG_\tau(1, 3)G_\downarrow(2, 4) \) has the retarded component

\[
G^{R}_{(2)}(12, 34) = -i\theta(t_1 - t_3) \left(G^G_{\tau}(1, 3)G^G_{\downarrow}(2, 4) - G^G_{\tau}(1, 3)G^G_{\downarrow}(2, 4)\right). \tag{15}
\]

We assume all functions to be diagonal in the spin. Thouless evaluated the single-particle functions in the quasi-particle approximation of the spectral function, \( A_\tau(\omega, k, r, t) \approx 2\pi\delta(\omega - \varepsilon_\tau(k, r, t)) \), where \( \varepsilon_\tau(k, r, t) \) is the energy of quasiparticle of spin \( \uparrow \) and momentum \( k \) at point \( r \) and time \( t \). Having only a single pole, the corresponding correlation functions can be expressed in the form of the Kadanoff-Baym ansatz \( G^G_{\tau}(\omega, k, r, t) \approx f_\tau(k, r, t) A_\tau(\omega, k, r, t) \), which allows for a convenient extension to non-equilibrium.

The actual evaluation of coefficients \( \alpha, \Gamma, S \) and \( m^* \) is rather lengthy and we refer the reader to details [8]. Let us list four essential steps. First, the distribution \( f \) was obtained from the (normal state) Boltzmann equation in the relaxation time approximation allowing for the electric and magnetic fields, and the temperature gradient. Second, integrals giving coefficients were evaluated in the mixed Wigner representation. Except for explicit gradients one has to include internal gradients needed to shift the center of the kernel. For example \( K(r_1, r_2) \equiv K(r, \xi) \) has the center coordinate \( r = \frac{1}{2}(r_1 + r_2) \) and the difference coordinate \( \xi = r_1 - r_2 \). In the TDGL equation we need material parameters at point \( r_1 \). The coefficient thus includes the gradient correction \( K(r_1, \xi) \approx K(\xi) - (\partial K/\partial r)\xi/2 \). Third, the gap function was scaled to the GL order parameter, \( \psi = \Delta(\sqrt{n}/k_BT)\sqrt{\int_0^\infty dx/(1/2x)(\partial^2(1/e^x + 1)/\partial x^2)} \). The density \( n \) and temperature \( T \) are generally functions of time and space, giving additional gradients. In reality, these gradient cancel with internal gradient terms obtain in the second step. The equation for the order parameter thus has simpler parameters. Fourth, following Gor’kov, the space gradient were minimized by the choice of gauge so that the vector coefficient \( S \) vanishes. Going back to the original gauge one arrives at the linearized TDGL equation

\[
\frac{1}{2m^*} \left(-i \frac{\partial}{\partial r} - m^* \mathbf{v}_n - 2e\mathbf{A}\right)^2 \psi + \alpha \psi = -\Gamma \left( \frac{\partial}{\partial t} - 2ie\phi \right) \psi. \tag{16}
\]

In order to write the GL theory in its common form, equation (16) is the complex conjugate of Eq. (14). The electrostatic potential \( \phi \) makes the theory gauge invariant. Units with \( \hbar = 1 \) are used.

Compared to studies based on the Gor’kov formalism, the Thouless approach gives a new term represented by the mean velocity of normal electrons \( \mathbf{v}_n = \mathbf{j}_n/(en) \), where \( \mathbf{j}_n \) is the
electric current in the background of the nucleating condensate. Its contribution is rather small, nevertheless it can be observed via the fast far infra-red spectroscopy \[10, 11\] and according to numerical studies it modifies a motion of kinematic vortices \[12\]. In Ref. \[12\] it is shown that the new term is purely transient, therefore in stationary situations like in the conversion region near contacts, the term \(m^* v_n\) has no effect in spite of large normal currents.

3. T-matrix approaches to the superconductivity

It is desirable to extend the above treatment from the critical point into the superconducting region so that finite values of the GL order parameter or the BCS gap are covered. To this end we have to include the effect of the condensate on the single-particle spectrum and allow for the formation of the BCS gap. Let us first examine how the Galitskii approximation fails below the critical temperature.

3.1. Missing gap in the Galitskii approximation

The approximative retarded selfenergy obtains from Eqs. (6) and (13)
\[
\Sigma^R(1, 3) = -\Delta^*(1)\Delta(3)G^A(3, 1).
\] (17)

Problems of the Galitskii approximation appear in equilibrium as well as out of equilibrium. In this section we restrict our discussion to the simple homogeneous equilibrium system. The gap function is then a constant which can be taken real and the selfenergy (17) is conveniently expressed in the energy-momentum representation
\[
\Sigma^R(\omega, \mathbf{k}) = -\Delta^2 G^A(\omega, \mathbf{k}).
\] (18)

Similarly \(\Sigma^A(-\omega, -\mathbf{k}) = -\Delta^2 G^R(\omega, \mathbf{k})\). From the Dyson equation for the retarded propagator
\[
G^R(\omega, \mathbf{k}) = \frac{1}{\omega - \epsilon_k - \Sigma^R(\omega, \mathbf{k})} = \frac{1}{\omega - \epsilon_k + \Delta^2 G^A(-\omega, -\mathbf{k})}
\] (19)
and its advanced counterpart \(G^A(-\omega, -\mathbf{k}) = 1/(\omega - \epsilon_{k^+} + \Delta^2 G^R(\omega, \mathbf{k}))\) one finds a propagator
\[
G^R(\omega, \mathbf{k}) = \frac{\omega + \epsilon_{k^+}}{2\Delta^2} - \sqrt{\left(\frac{\omega + \epsilon_k}{2\Delta^2}\right)^2 - \frac{1}{\Delta^2}} \frac{\omega + \epsilon_{k^-}}{\omega - \epsilon_k}. \tag{20}
\]

This peculiar propagator is far from the simple BCS structure. The BCS spectral function has two \(\delta\)-functions at \(\omega = \pm E_k\) given by energies (1). Taking for example \(\epsilon_k > 0\) and \(\omega > 0\), the spectral function \(-2\Im G^R(\omega, \mathbf{k})\) obtained from (20) has non-zero values in the wide interval of energies \(\omega < \sqrt{\epsilon_k^2 + 2\Delta^2}\). Instead of the energy gap we have obtained a continuous spectrum of states in the vicinity of the Fermi energy.

The missing BCS gap restricts applicability of Galitskii approximation to the normal state. In spite of the failure in the superconducting state, the conditions of the nucleation of the superconducting condensate result from the Galitskii approximation correctly.

3.2. Prang paradox

The gap problem was first noticed by Prang. Unfortunately, we were not able to trace down his original paper. His work is mentioned by Wild \[13\] who recovered Prang’s result. Prang also noticed that the gap problem is cured making the loop of the selfenergy non-selfconsistent, i.e., replacing the ‘even-argument’ Green function by its bare value in the two-particle propagator
\[
\hat{G}^R_{(2)}(12, 34) = -i\theta(t_1 - t_3) \left(G^>(1, 3)G^d>(2, 4) - G^<(1, 3)G^d<(2, 4)\right) \tag{21}
\]
and similarly the Green’s function closing the loop

\[ \hat{\Sigma}^R(1, 3) = \hat{T}^R(12, 34)G^{0<}(4, 2) - \hat{T}^<(12, 34)G^{0A}(4, 2). \] (22)

It is striking that the Galitskii approximation which is \( \Phi \)-derivable and satisfies all demands on reliable theories fails while the weird theory with non-symmetrical treatment of interacting particles provides the well established result. Tolmachev calls this unexpected result Prang’s paradox [14].

3.3. Kadanoff-Martin theory

The approximation tested by Prange is now known as the Kadanoff-Martin theory. This name might be misleading for newcomers, because Kadanoff and Martin [15] have derived another approximation with the fully selfconsistent two-particle T-matrix and the bare Green’s function in the closing relation (22). Recently Kremp et al have derived a renormalized version of the Kadanoff-Martin approximation. Their two-particle T-matrix is also fully selfconsistent, and the Green function in the closing relation includes binary correlations on the level of the ladder approximation with the gap excluded.

It is easy to see that non-selfconsistent closing relation (22) helps. The modified propagator (19) is

\[ \hat{G}^R(\omega, \mathbf{k}) = \frac{1}{\omega - \epsilon_k + \Delta^2 G^{0A}(\omega, -\mathbf{k})} = \frac{1}{\omega - \epsilon_k - \frac{\Delta^2}{\omega + \epsilon_k}} = \frac{\omega + \epsilon_k}{\omega^2 - \epsilon_k - \Delta^2}, \] (23)

where we have used \( \epsilon_{-\mathbf{k}} = \epsilon_{\mathbf{k}} \). For a given gap \( \Delta \), this propagator agrees with the BCS propagator.

The gap is given by the kernel \( K = VG(2) \). The fully selfconsistent two-particle T-matrix of the original Kadanoff-Martin theory provides incorrect gap values. It can be seen near the critical temperature, where \( G(2) \) can be expanded in powers of \( \Delta \). Gor’kov version of the BCS theory has \( G(2) = G_0G \approx G_0G_0 + G_0G_0\Delta G_0\Delta G_0 \). The second term yields the nonlinear ‘potential’ of the Ginzburg-Landau expansion \( \hat{\beta}|\Delta|^2 \), where the hat stands for normalization to \( \Delta \) rather than the Ginzburg-Landau function \( \psi \) used below. In the original Kadanoff-Martin, \( G(2) = GG \approx G_0G_0 + G_0G_0\Delta G_0\Delta G_0 + G_0\Delta G_0\Delta G_0G_0 \), which yields twice larger ‘potential’ \( \hat{\beta}_{KM}|\Delta|^2 = 2\hat{\beta}|\Delta|^2 \). All linear terms are identical, therefore the equilibrium gap obtained from the Kadanoff-Martin theory near critical temperature \( |\Delta|^2 = -\hat{\alpha}/\hat{\beta}_{KM} = -\hat{\alpha}/2\hat{\beta} \) is by factor 1/\( \sqrt{2} \) smaller than the BCS value.

Using Kadanoff-Martin out of equilibrium one can expect various artifacts, because the loop line of includes matching of energies in unequal approximation which harms the energy conservation. It was corrected ten years later by Patton [17]. In his unpublished thesis he proposed to use the bare functions also in the loop line of the T-matrix. This additional non-selfconsistency converts the Kadanoff-Martin theory into the approximation discussed by Prang which yields the BCS gap.

To obtain reasonable results from the Kadanoff-Martin theory even with Patton’s correction, one has to neglect all other parts of the selfenergy except for the singular term \( \hat{\Sigma}^R(1, 3) = -\Delta^3(1)\Delta(3)G^{0A}(3, 1) \). One can imagine non-physical properties which would result taking the theory literally. For example, the dressed propagator includes the full exchange-correlation selfenergy while the bare propagator depends on the mean-field electrostatic potential only. Energy bands in the mean-field approximation are far from the real ones being mostly shifted upwards. Due to the upward shift the bare Green’s function yields a much lower electron density (for the fixed chemical potential). Since the bare function corresponds to the state of spin opposite to the dressed one, the approximation has a fake spin non-symmetry similar to the
non-symmetry observed in ferromagnetic states. Like in magnetic systems, the different values of the Fermi momentum would destroy the Cooper pairing. These problems are corrected in the renormalized version of Kremp et al [16]. Our theory explained below can be simplified to the theory of Kremp et al with the Patton’s restricted selfconsistency in the T-matrix.

3.4. Selfconsistency out of equilibrium
Out of equilibrium an implementation of the T-matrix approximation with the non-selfconsistent loop is hindered by its dependence on the non-equilibrium correlation function $G^{0<}$. It is easy to write down equation for $G^{0<}$ and eventually solve it, but such function has nothing to do with the real system. In the equilibrium one needs only propagators which represent a short-time behavior and all errors are forgotten on the quasiparticle life time. The non-equilibrium requires to solve the transport equation, where errors cumulate. To obtain a meaningful distribution of particles one has to work with fully selfconsistent correlation functions.

As long as one does not introduce additional non-systematic step like $G_{0<}$ constructed from the actual selfconsistent distribution of electrons and bare spectral function, the Kadanoff-Martin theory cannot be used out of equilibrium. It is possible to introduce the requested selfconsistency following the approach of Kremp et al based on expected macroscopic properties of the condensate. Here we start from microscopic considerations.

3.5. Pauli principle
Langer [18] argues that the problem of the Galitskii approximation in the superconducting state is hidden in unbalanced treatment of direct and exchange processes. Why the problem emerges only in the superconducting state?

Let us go back to the power expansion in the second quantization and assume a group of potentials from which we aim to evaluate the selfenergy. If this group includes a combination $\Psi^\dagger_k \Psi^\dagger_k \Psi_k \Psi^\dagger_k$, it is zero in agreement with the Pauli principle. In the diagrammatic expansion, such contribution vanishes by cancelation of a direct and exchange process, as one can see from the decoupling of four operators

$$\langle \Psi^\dagger_{i'k'} \Psi^\dagger_{i'k} \Psi_{i'k} \Psi_{i'k'} \rangle = \langle \Psi^\dagger_{i'k'} \Psi_{i'k} \rangle \langle \Psi^\dagger_{i'k} \Psi_{i'k'} \rangle - \langle \Psi^\dagger_{i'k'} \Psi_{i'k} \Psi^\dagger_{i'k} \Psi_{i'k'} \rangle,$$

which is non-zero for $k \neq k'$ but vanishes when $k = k'$. Neglect of the exchange process correspond to the neglect of the second term in decoupling (24).

In the normal state the neglect of exchange processes is well justified. The second term of decoupling (24) is zero except for $k = k'$, we can thus split dual sums over momenta as

$$\frac{1}{L^3} \sum_k \frac{1}{L^3} \sum_{k'} \cdots = \frac{1}{L^3} \sum_k \frac{1}{L^3} \sum_{k \neq k'} \cdots + \frac{1}{L^6} \sum_{k = k'} \cdots$$

For the $\Psi^\dagger_k$ normalized to the plane wave $e^{ikr}$, sums over momenta are weighted with the inverse quantization volume $1/L^3$. The number of discrete k-states in the Brillouin zone is proportional to $L^3$ so that the first term remains finite while the second one vanishes in the thermodynamical limit $L \to \infty$.

In the presence of the superconducting condensate the scaling with volume breaks. The Bose-Einstein occupation factor of the state with the condensate increases linearly with the volume so that the contribution of this single state remains finite under the thermodynamical limit. The neglect of exchange diagrams than might cause serious problems, in particular the missing superconducting gap.

Keeping non-zero $\Psi^\dagger_k \Psi^\dagger_k \Psi_k \Psi_k$ inside selfenergy results in various direct or indirect self-interactions. The self-interaction is in general a weak point of all diagrammatic approaches. It
appears already in the mean-field, which includes the Coulomb action of a particle on itself. It eventually cancels with the self-exchange included in the diagrammatic Fock term.

It is often easy to avoid the self-interaction if we leave ties of diagrammatic rules. For example, the Coulomb self-interaction is absent in the Hartree potential. It should be stated that diagrammatic expansions are mostly the most convenient approach. Using the Hartree approximation as an example, we can see that it is formally a problem of a particle in the electrostatic potential, but unlike in the mean-field approximation each particle feels a different potential. For \( N \) particles one has to solve the Schrödinger equation with \( N \) different Hartree potentials. Moreover, the Hartree single-particle states do not form the orthogonal basis.

A systematic treatment of exchange diagrams on the level of the selfconsistent T-matrix is technically impossible. Langer [18] performed an alternative way of summation of fully anti-symmetrized power expansion and obtained the gap. His approach has no followers likely because of its complexity. Author [19] has eliminated indirect self-interactions due to missing exchange terms using the idea of the Watson-Lacelace-Fadeev multiple scattering theory. Morawetz [20] followed the Pauli principle and recovered the resulting theory from the decoupling of the three-particle correlation function.

4. Restricted selfconsistency
Among the above listed T-matrix theories of superconductivity, the multiple scattering approach has the structure best suited to describe the non-equilibrium superconductors. We will follow this line.

The multiple scattering theory aims to view binary encounters of particles as completed scattering events. Such viewpoint is clearly defined in two limiting cases. First, one can recollect orders of the interaction potential into sequence of individual binary events. These events are not affected by other particles, in fact, in strongly coupled system the binary collision is terminated by intervention of a third particle which takes over one of partners. Two selected particles collide again and again as they meet in the system. Corresponding T-matrices of binary events are constructed from bare propagators. This is the structure of the Fadeev equations, which are useful for very small systems (vast majority of application are just for three particles). It was not solved for the infinite system.

Second, one can use principles of the multiple scattering theory to eliminate double-counts of standard selfconsistent approximations. Such attitude was promoted by Soven [21] who showed that the Averaged T-matrix Approximation (ATA) of the scattering on impurities is overly selfconsistent, what limits its applicability to low concentrations. He proposed a simple correction scheme based on the multiple scattering theory. Like in the ATA, the mean scattering by randomly distributed impurities is described by the selfenergy. When one evaluates the T-matrix for the collision with some impurity, the selfenergy is reduced not to include the selected impurity (or the selected lattice site). In this way he obtained the Coherent Potential Approximation (CPA) which is as simple as the ATA having number of superior properties [22]. With respect to the superconductivity it is noteworthy, that the CPA describes a formation of a gap in the split-band limit, while the ATA fails.

In spite of the reduced selfconsistency, the CPA is based on the fully selfconsistent propagator. Accordingly, it can be extended to non-equilibrium systems [23], where the full selfconsistency is required. Following [19] we adopt Soven’s approach to eliminate double-counts from the Galitskii approximation.

The Galitskii selfenergy (5) in the energy-momentum representation

\[
\Sigma^R_{\uparrow}(\omega, \mathbf{k}) = \sum_{\mathbf{Q}} \sigma^R_{\uparrow \uparrow}(\omega, \mathbf{k})
\]  

(26)
is a sum over pair momentum $\mathbf{Q}$ of components

$$\sigma^R_{\uparrow \mathbf{Q}}(\omega, \mathbf{k}) = \int d\Omega \ T^R(\Omega, \mathbf{Q})G^<_{\uparrow}(\Omega - \omega, \mathbf{Q} - \mathbf{k}) - \int d\Omega \ T^<(\Omega, \mathbf{Q})G^A_{\downarrow}(\Omega - \omega, \mathbf{Q} - \mathbf{k}).$$  \hspace{1cm} (27)$$

The Galitskii T-matrix is constructed from the single-particle Green’s functions including $G^R_{\uparrow}$ which depends on $\sigma^R_{\uparrow \mathbf{Q}}$. The Galitskii approximation is thus overly selfconsistent. When one evaluates $\sigma^R_{\uparrow \mathbf{Q}}$, this $\mathbf{Q}$-component of the selfenergy ought to be eliminated from the effective medium not to act on itself.

### 4.1. Restricted T-matrix on the time contour

Let us allow for more general conditions in which perturbations make the energy-momentum representation of limited use. Nevertheless, we assume that we are able to identify channels of the T-matrix which we label by $\mathbf{Q}$. The selfenergy on the time contour is a sum over these channels

$$\Sigma_{\uparrow}(1, 3) = \sum_{\mathbf{Q}} \sigma_{\uparrow \mathbf{Q}}(1, 3).$$  \hspace{1cm} (28)$$

The time integral run along the time contour. The subsidiary propagator

$$G_{\uparrow \mathbf{Q}}(1, 3) = G_{\uparrow}(1, 3) - G_{\uparrow}(1, 5)\sigma_{\uparrow \mathbf{Q}}(5, 7)G_{\uparrow \mathbf{Q}}(7, 3)$$  \hspace{1cm} (29)$$

is free of $\sigma_{\uparrow \mathbf{Q}}$. Indeed, from the Dyson equation

$$G_{\uparrow}(1, 3) = G^0(1, 3) + G^0(1, 5)\Sigma_{\uparrow}(5, 7)G_{\uparrow}(7, 3)$$  \hspace{1cm} (30)$$

one finds $G_{\mathbf{Q}} = G^0 + G^0(\sum_{\mathbf{Q}' \neq \mathbf{Q}} \sigma_{\mathbf{Q}'})G_{\mathbf{Q}}$. Using Eq. (29) we can construct the subsidiary propagator as a function of the dressed Green function $G$.

The T-matrix for each channel is in the ladder approximation

$$T_{\uparrow \mathbf{Q}}(1, 2) = V + VG^{(2)}_{\uparrow \mathbf{Q}}(1, 3)T_{\uparrow \mathbf{Q}}(3, 2),$$  \hspace{1cm} (31)$$

with internal $\uparrow$ lines free of the self-action

$$G^{(2)}_{\uparrow \mathbf{Q}}(1, 2) = iG_{\uparrow \mathbf{Q}}(1, 2)G_{\downarrow}(1, 2).$$  \hspace{1cm} (32)$$

Solving the ladder equation we obtain its $\mathbf{Q}$ channel. In the simple case of the homogeneous system, the $\mathbf{Q}$ component is identical to the Fourier component of the T-matrix. In the sum this component has a weight proportional to the inverse volume. Elimination of a single channel thus vanishes in the thermodynamical limit, volume $\to \infty$, unless the channel has macroscopic occupation due the Bose-Einstein condensation of Cooper pairs.

By averaging over collision partners we obtain the component of the averaged T-matrix

$$t_{\uparrow \mathbf{Q}}(1, 3) = -iT_{\uparrow \mathbf{Q}}(1, 3)G_{\downarrow}(3, 1).$$  \hspace{1cm} (33)$$

The T-matrix explores all orders of the interaction, therefore the same process cannot influence initial and final states of the scattering. The dressed Green’s function thus reads

$$G_{\uparrow}(1, 3) = G_{\uparrow \mathbf{Q}}(1, 3) + G_{\uparrow \mathbf{Q}}(1, 5)t_{\uparrow \mathbf{Q}}(5, 7)G_{\uparrow \mathbf{Q}}(7, 3).$$  \hspace{1cm} (34)$$

The $\mathbf{Q}$ component of the selfenergy obtains from equations (29) and (34)

$$\sigma_{\uparrow \mathbf{Q}}(1, 3) = t_{\uparrow \mathbf{Q}}(1, 3) - \sigma_{\uparrow \mathbf{Q}}(1, 5)G_{\uparrow \mathbf{Q}}(5, 7)t_{\uparrow \mathbf{Q}}(7, 3).$$  \hspace{1cm} (35)$$
The set of equations on the time contour is closed. Let us double-check its structure. Starting from some iteration of the selfenergy one evaluates the dressed Green’s function via the Dyson equation (30) and the subsidiary Green’s function via (29). These functions are used to prepare the two-particle propagator (32) from which one evaluates the $Q$ channel of the two-particle T-matrix (31). Closing the upper loop of the two-particle T-matrix yields the averaged T-matrix (33) from which one obtains a higher iteration of the $Q$ component of the selfenergy via (35). Evaluating all $Q$ components the higher iteration of the selfenergy (28) is completed.

4.2. Non-equilibrium restricted T-matrix on the real time axis

Now we are ready to fulfill our basic aim to formulate a theory of non-equilibrium superconductivity which can be expanded in orders of the gap function. We convert the above set of equations into set of equations on the real axis using the generalized Kadanoff-Baym formalism in the notation of Ref. [24]. We will not write down advanced functions which are hermitian conjugates of their retarded counterparts, and the ‘greater’ correlation functions which are linked to corresponding ‘smaller’ function by the particle-hole symmetry, $\leftrightarrow$. The spin $\downarrow$ functions obtain by the interchange $\downarrow\leftrightarrow\uparrow$.

The selfenergy (28) is a sum
\[ \Sigma_\uparrow^{\left<,R\right>} = \sum_Q \sigma_\uparrow^{\left<,R\right> Q} \] (36)
with retarded components
\[ \sigma_\uparrow^{\left<,R\right> Q} = t_\uparrow^{\left<,R\right> Q} - \sigma_\uparrow^{\left<,R\right> Q} G_\uparrow^{\left<,R\right> Q} t_\uparrow^{\left<,R\right> Q} \] (37)
and correlation parts
\[ \sigma_\uparrow^{\left<,R\right> Q} = \left(1 - t_\uparrow^{\left<,R\right> Q} G_\uparrow^{\left<,R\right> Q} t_\uparrow^{\left<,R\right> Q}\right) t_\uparrow^{\left<,R\right> Q} \left(1 - G_\uparrow^{\left<,R\right> Q} G_\uparrow^{\left<,R\right> Q}\right) - t_\uparrow^{\left<,R\right> Q} G_\uparrow^{\left<,R\right> Q} G_\uparrow^{\left<,R\right> Q} t_\uparrow^{\left<,R\right> Q}. \] (38)

The T-matrix (33) has correlation parts
\[ t_\uparrow^{\left<,R\right> Q}(1,3) = T_\uparrow^{\left<,R\right> Q}(1,3) G_\uparrow^{\left<,R\right> Q}(3,1) \] (39)
and retarded
\[ t_\uparrow^{\left<,R\right> Q}(1,3) = T_\uparrow^{\left<,R\right> Q}(1,3) G_\uparrow^{\left<,R\right> Q}(3,1) - T_\uparrow^{\left<,R\right> Q}(1,3) G_\uparrow^{\left<,R\right> Q}(3,1). \] (40)

The single-particle propagator obeys the Dyson equation
\[ G_\uparrow^{\left<,R\right> Q} = G_\uparrow^{\left<,R\right> Q} + G_\uparrow^{\left<,R\right> Q} \Sigma_\uparrow^{\left<,R\right> Q} \] (41)
and the correlation part the generalized Kadanoff-Baym equation
\[ G_\uparrow^{\left<,R\right> Q} = G_\uparrow^{\left<,R\right> Q} \Sigma_\uparrow^{\left<,R\right> Q} G_\uparrow^{\left<,R\right> Q}. \] (42)

The retarded subsidiary function (29) has the retarded propagator
\[ G_\uparrow^{\left<,R\right> Q} = G_\uparrow^{\left<,R\right> Q} - G_\uparrow^{\left<,R\right> Q} \sigma_\uparrow^{\left<,R\right> Q} G_\uparrow^{\left<,R\right> Q} \] (43)
and the correlation part
\[ G_\uparrow^{\left<,R\right> Q} = \left(1 - G_\uparrow^{\left<,R\right> Q} \sigma_\uparrow^{\left<,R\right> Q} G_\uparrow^{\left<,R\right> Q}\right) G_\uparrow^{\left<,R\right> Q} \left(1 - \sigma_\uparrow^{\left<,R\right> Q} G_\uparrow^{\left<,R\right> Q}\right) - G_\uparrow^{\left<,R\right> Q} \sigma_\uparrow^{\left<,R\right> Q} G_\uparrow^{\left<,R\right> Q}. \] (44)

The correlation part of the two-particle propagator (32) is
\[ G_\uparrow^{\left<,R\right> Q}(1,2) = G_\uparrow^{\left<,R\right> Q}(1,2) G_\uparrow^{\left<,R\right> Q}(1,2). \] (45)
4.3. Non-equilibrium superconductivity

In the homogeneous system one can easily eliminate all self-actions. In the inhomogeneous case, the decomposition of the T-matrix becomes complicated. Treatment of all channels is important for small systems like quantum dots [25], but for bulk superconductors it is an unnecessary complication. The only channel, in which the self-action has to be eliminated is the condensation channel. The T-matrix approach in general allows for parallel condensation in many channels. The multi-gap structure might appear very far from equilibrium [26], but in weakly perturbed systems the situation is different. As shown in Ref. [27] the condensation in one channel blocks a condensation in all other channels. This is important for the stability of the condensate. Indeed, in the presence of persistent currents there are channels of smaller or none current, where the condensate would lead to a lower total energy of the system. Blocking of the parallel condensation makes the persistent current stable.

Let us simplify the theory neglecting the restricted selfconsistency in all but condensate channel. As shown above, the Bose-Einstein condensation leads to enhancement of a single element of the correlation part of the T-matrix. This element is separable, therefore

\[ T^< = \Delta^* \Delta + \tilde{T}^<, \]  \hspace{1cm} (49)

where the second term is a regular part. The retarded T-matrix has no singularity and satisfies the ladder equation of the Galitskii theory, \( T^R = V + VG^R T^R \). The regular correlation is also identical to the Galitskii theory \( \tilde{T}^< = T^R G^<(2) T^A \).

Again, like in the Galitskii theory the two-particle function is constructed from dressed functions \( G^<(2) = G^>(1,2) G^< \) and \( G^R(2) = -i\theta(t_1 - t_2) \left( G^>(1,2) - G^< \right) \).

Since the full selfconsistency blocks condensation, the \( \tilde{T}^< \) remains a regular function.

According to separation (49), the selfenergy splits into the ‘superconducting’ and regular part

\[ \Sigma^R(1,3) = -\Delta^*(1)\tilde{G}^A(3,1)\Delta(3) \] (50)

and

\[ \Sigma^< (1,3) = \Delta^*(1)\tilde{G}^>(3,1)\Delta(3) + \tilde{\Sigma}^< (1,3) \] (51)

with the regular retarded Green’s function \( \tilde{G}^R = G^0 R + G^0 \tilde{\Sigma}^R \tilde{G}^R \) and the correlation part \( \tilde{G}^< = \tilde{G}^R \Sigma^< \tilde{G}^A \). The dressed Green’s function includes the superconducting selfenergy

\[ G^R(1,3) = \tilde{G}^R(1,3) - \tilde{G}^R(1,5)\Delta^*(5)\tilde{G}^A(7,5)\Delta(7)G^R(7,3) \] (52)

and

\[ G^<(1,3) = G^R(1,5) \left( \tilde{\Sigma}^< (5,7) + \Delta^*(5)\tilde{G}^>(7,5)\Delta(7) \right) G^A(7,3). \] (53)
Equation for the singular part of $T^<$ implies the equation for the gap function

$$\Delta^*(1) = V \tilde{G}^R_{(2)}(1, 2) \Delta^*(2), \quad (54)$$

where the two-particle propagator $\tilde{G}^R_{(2)}(1, 2) = -i \theta(t_1 - t_2) \left( \tilde{G}^>(1, 2) - \tilde{G}^<_{(2)}(1, 2) \right)$ has restricted selfconsistency $\tilde{G}^<_{(2)}(1, 2) = \tilde{G}^<_{(1)}(1, 2)$.

The gap equation (54) corresponds to the Thouless criterion with the restricted selfconsistency on the gap in the two-particle propagator. This form extends applicability of the Thouless approach to non-equilibrium superconducting states. If we neglect the regular selfenergy or approximate it by energy-independent correction to the single-particle energy, $\epsilon_k + \tilde{\Sigma}^R(\omega, k) \approx \epsilon_k$ and $\tilde{\Sigma}^< = 0$, equation (54) becomes identical to the BCS gap equation.

In the above treatment we have not used the anomalous Green functions. We have avoided this useful tool to emphasis that the present approach is not based on them. For the single-valued condensate the singular separable part of the T-matrix leads to a separable of the two-particle propagator in which one can easily recognize the product of anomalous functions, see [28].

### 4.4. Time-dependent Ginzburg-Landau equation

In the limit of small gap $\Delta \to 0$, the present theory simplifies to the time-dependent Ginzburg-Landau theory. The gradient expansion of the integral kernel in the gap equation (54) yields linear terms which are identical to those obtained above from the Thouless criterion. Unlike in the Thouless theory, the two-particle propagator $\tilde{G}^R_{(2)}(1, 2)$ in (54) depends on the gap $\Delta$. Keeping terms to the second order of $\Delta \propto \psi$ one arrives at the TDGL equation

$$\frac{1}{2m^*} \left( -i \frac{\partial}{\partial t} - m^* \mathbf{v}_n - 2e \mathbf{A} \right)^2 \psi + \alpha \psi + \beta |\psi|^2 \psi = -\Gamma \left( \frac{\partial}{\partial t} - 2i \phi \right) \psi. \quad (55)$$

Individual terms of this equation are conveniently derived from the Thouless criterion (linear terms) and the BCS theory (the cubic term). The T-matrix approach derive all these terms from a single microscopic ground.

### 5. Conclusions

We have discussed the effect of the normal current on the formation of the superconducting condensate. The BCS approach to this effect is troubled by the fact that the Bogoliubov-Valutin quasiparticle energies do not turn into single-electron energy bands in the zero-gap limit. In this representation the normal current is difficult to describe as it requires non-adiabatic processes.

The representation problem was circumvented by the Thouless criterion which deals with fully selfconsistent Green’s function of the normal state. While the effect of the normal current on nucleation of the condensate is covered in the non-equilibrium Galitskii approximation, this approach fails to establish the amplitude of the condensate. We have shown that using the multiple-scattering corrections to the Galitskii approximation one obtains a theory applicable to the superconducting state and the gap amplitude results in agreement with the BCS theory.

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