RANDOM MATRICES: PROBABILITY OF NORMALITY

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Abstract. We consider a random $n \times n$ matrix, $M_n$, whose entries are i.i.d. Rademacher random variables (taking values $\{\pm 1\}$ with probability $1/2$) and prove

$$2^{-(0.5+o(1))n^2} \leq \mathbb{P}(M_n \text{ is normal}) \leq 2^{-(0.302+o(1))n^2}.$$ 

We conjecture that the lower bound is sharp.

1. Introduction

In this paper, we investigate the following question:

How often is a random matrix normal?

We consider random matrices with i.i.d. entries. Despite the central role of normal matrices in matrix theory, to our surprise, we found no previous results concerning this natural and important question. When the entries have a continuous distribution, the problem is, of course, easy. The probability in question is zero, as the set of normal matrices, viewed as points in $\mathbb{R}^{n^2}$, is not full dimensional. However, for discrete distributions, the situation is totally different.

We are going to focus on Rademacher matrix, whose entries take values $\pm 1$ with probability $1/2$. This is the most important class among random matrices with discrete distribution. We denote the $n \times n$ Rademacher matrix by $M_n$ and by $\nu_n$ the probability that $M_n$ is normal. Throughout this paper, we assume that $n$ tends to infinity and all asymptotic notations are used under this assumption.

Clearly, the probability that $M_n$ is symmetric is $2^{-(0.5+o(1))n^2}$. Since symmetric matrices are normal, $\nu_n \geq 2^{-(0.5+o(1))n^2}$.

We conjecture that this lower bound is sharp.

Conjecture 1.1. Let $\nu_n$ be defined as above. Then,

$$\nu_n = 2^{-(0.5+o(1))n^2}.$$ 

Our main result is that $\nu_n \leq 2^{-(0.302+o(1))n^2}$. We actually proved a more general statement

Theorem 1.2. For any fixed matrix $C$

$$\mathbb{P}(M_nM_n^T = M_n^TM_n + C) \leq 2^{-(0.302+o(1))n^2}.$$ 

Setting $C = 0$, one obtains $\nu_n \leq 2^{-(0.302+o(1))n^2}$. This more general setting plays a role in our proof. In the rest of the paper we can assume, without loss of generality, that $C$ has integer entries.

There have been studies of Rademacher matrices with a similar flavor, such as estimating the probability that the matrix is singular [1, 6, 7, 2] or has double eigenvalues [8, 10]. In these cases, the conjectural bounds are of the form $2^{-(c+o(1))n^2}$, for some constant $c > 0$. While this probability is small, it is still much larger than $2^{-\Omega(n^2)}$, which enables one to exclude very rare events (those occurring with probability $2^{-\omega(n^2)}$) and then condition on their complement. It is, in fact, the strategy used to obtained the best current bounds for these problems.

The difficulty with the problem at hand is that we are aiming at a bound which is extremely small (notice that any non-trivial event concerning $M_n$ holds with probability at least $2^{-n^2}$, which is the mass of a single $\pm 1$ matrix). There is simply no non-trivial event of probability $1 - 2^{-\omega(n^2)}$ to condition
Thus, one needs a new strategy. The key of our approach is a new observation that for any given matrix, we can permute its rows and columns so that the ranks of certain submatrices follow a given pattern (see Section 2.2). The fact that there are only \( n! = 2^{O(n^2)} \) permutations works in our favor and enables us to execute a different type of conditioning. To our best knowledge, an argument of this type has not been used in random matrix theory.

2. Preliminaries

In this section we will introduce notation, definitions and some lemmas that will be used in later sections.

Claim 2.1. Let \( Q_n \) be the set of vertices of the hypercube \( \{\pm 1\}^n \). Then for any \( k \)-dimensional subspace \( S \) of \( \mathbb{R}^n \), we have:

\[
|Q_n \cap S| \leq 2^k.
\]

The above claim is well-known (see [5, 6]) and follows from the simple fact that there is a set of \( k \) coordinates that determines all other coordinates in a vector in \( S \). As a consequence, we obtain the following lemma.

Lemma 2.2. Let \( M \in M_{k \times m}(\pm 1) \) be a fixed matrix of rank \( r > 0 \), let \( c \in M_{k \times 1}(\mathbb{Z}) \) be a fixed vector and let \( x_m \in \{\pm 1\}^m \) be a random vector uniformly distributed over the sample space. Then the following holds

\[
\mathbb{P}(Mx_m = c) \leq 2^{-r}.
\]

Definition 2.3. Let \( S_n \) be the set of all permutations of \( \{1, 2, ..., n\} \). For any \( \sigma \in S_n \) and any \( n \times n \) matrix \( M \), set

\[
M_{\sigma} := S_{\sigma} M S_{\sigma}^T,
\]

where \( S_{\sigma} \) is the permutation matrix associated with \( \sigma \). Technically speaking, \( M_{\sigma} \) is created by permuting the rows and columns of \( M \) according to \( \sigma \).

Definition 2.4. Let \( M \) be a fixed \( n \times n \) matrix and let \( 1 \leq i \leq n \) be a fixed integer. We define \( R_i \) and \( C_i \) to be the top right \( i \times (n - i - 1) \) and bottom-left \( (n - i - 1) \times i \) submatrices of \( M \) respectively. Thus, \( R_i \) is formed by the last \( n - i - 1 \) entries of the first \( i \) rows of \( M \) and \( C_i \) is formed by the last \( n - i - 1 \) entries of the first \( i \) columns of \( M \). We also let \( r_i \) and \( c_i \) denote the last \( n - i \) entries of the \( i^{th} \) row and column of \( M \) (see Figure 3).

Let us reveal our motivation behind this definition. Notice that if we condition on the entries in the main diagonal and the first \( k - 1 \) rows and columns of \( M_n \), then in order for \( M_n \) to be normal, \( r_k \) and \( c_k \) have to satisfy the following linear equation:

\[
C_{k-1}^T c_k - R_{k-1} r_k^T = c,
\]

where \( c^T \) is a vector in \( \mathbb{Z}^{k-1} \), determined by the entries that were conditioned upon. We can rewrite \((2.1)\) in a nicer way, as

\[
\begin{bmatrix}
R_{k-1}^T \\
C_{k-1}
\end{bmatrix}
\begin{bmatrix}
-c_k^T \\
-c_k
\end{bmatrix}
= c.
\]

As we will mainly be working with equations of the form \((2.2)\), we define \( T_k := [R_k^T \ C_k]^T \) and \( x_k := [-r_k^T \ c_k]^T \). Relation \((2.2)\) can then be rewritten as:

\[
T_{k-1}^T x_k = c.
\]

Given a deterministic matrix \( M \), the matrices \( T_i \) are well defined. We define \( \text{rank}_i(M) \) by

\[
\text{rank}_i(M) := \text{rank}(T_i).
\]
3. Equivalence classes and the Permutation Lemma

We form the following equivalence classes. For two square matrices $M$ and $N$ of size $n$

$$M \leftrightarrow N \iff \exists \sigma \in S_n \text{ such that } M\sigma = N.$$

**Definition 3.1.** Let $C$ be a fixed $n \times n$ matrix. We say that $M$ is $C$-normal if and only if $\exists \sigma \in S_n$ such that $MM^T - M^TM = C\sigma$.

**Proposition 3.2.** Let $\sigma \in S_n$, then $M$ is $C$-normal if and only if $M\sigma$ is $C$-normal.

**Proof.** For any permutation $\sigma' \in S_n$ let $S_{\sigma'}$ be the permutation matrix associated with it. Then

$$M \text{ is } C\text{-normal} \iff \exists \rho \in S_n \text{ such that } MM^T - M^TM = C_\rho$$

$$\iff S_\sigma MM^T S_{\sigma}^T - S_\sigma M^TMS_{\sigma}^T = S_\sigma C_\rho S_{\sigma}^T$$

$$\iff S_\sigma MS ST_{\sigma}^T - S_\sigma M^T S_{\sigma}^T S_\sigma MS_{\sigma}^T = S_\sigma S_{\rho} CS_{\rho}^T S_{\sigma}^T$$

$$\iff M_\sigma (M_\sigma)^T - (M_\sigma)^T M_\sigma = C_{\sigma\rho}$$

$$\iff M_\sigma \text{ is } C\text{-normal}.$$  

□

**Observation 3.3.** Note that Proposition 3.2 implies that if $M \leftrightarrow N$ and $M$ is $C$-normal, then $N$ is also $C$-normal. There are $n! = 2^{(n \log(n))} = 2^{o(n^2)}$ permutations in $S_n$, hence it is enough to bound the equivalence classes containing $C$-normal matrices. Hence Theorem 1.2 can be re-written as Theorem 3.4 below.

**Theorem 3.4.** For any fixed matrix $C$

$$P(\exists \sigma \in S_n \text{ s.t. } M_{n,\sigma} \text{ is } C\text{-normal}) \leq 2^{-(0.302+o(1))n^2}.$$  

From now on we will say that the matrix $M$ and $N$ are equivalent if they are in the same equivalence class. The key idea of our argument is that given any matrix $M$, we can find a permutation $\sigma$ such that we can tightly control the $\text{rank}_i(M_{n,\sigma})$. In particularly we want $\text{rank}_i(M_{n,\sigma})$’s to be as big as possible, so that we have many restrictions on $x_{i+1}$ in equation (2.3).

We claim that we can find $k, t$ and $\sigma \in S_n$ such that for all $1 \leq i \leq n$, $\text{rank}_i(M_{n,\sigma})$ equals $R_{k,t}(i)$, where $R_{k,t}(i)$ is defined below (see also Figure 3.1).
Figure 3.1. A graphical representation of $R_{k,t}(i)$.

We are now ready to state our permutation lemma.

**Lemma 3.5** (Permutation Lemma). Let $M$ be a fixed $n \times n$ matrix. Then there exist $k, t \in \mathbb{Z}_+$ and $\sigma \in S_n$ such that $M\sigma$ satisfy the condition (3.2) below:

\[
\text{rank}(M\sigma) = R_{k,t}(i), \quad \forall 1 \leq i \leq n.
\]

**Proof of the Permutation Lemma.**

Let $x_k := [r_k T c_k]^T$. Recall that $T_i$ is a $2(n-i-1) \times i$ matrix. By the definition of $T_i$, we obtain $T_i$ from $T_{i-1}$ in two steps (see Figure 3.2):

- First, we augment $T_{i-1}$ by $x_i'$. This increases the rank by at most 1. Call this increment $a_i$.
- Next, we delete the first and the $(n-i)^{th}$ rows of the previously obtained matrix. This decreases the rank by at most 2. Call this reduction $b_i$.

Thus, we have

\[
\text{rank}(T_i) = \text{rank}(T_{i-1}) + a_i - b_i,
\]

where $a_i \in \{0, 1\}$ and $b_i \in \{0, 1, 2\}$.

The desired permutation $\sigma$ is defined as the product of $n$ transpositions $\sigma := \prod_{i=1}^n \sigma_i$, where $\sigma_i = (i, s_i)$ for some index $s_i \geq i$. Let $M^{[i]} := M_{\prod_{k=1}^{i-1} \sigma_k}$. We will have $M_\sigma = M^{[n]}$, and $M^{[i]}$ is obtained from $M^{[i-1]}$ by applying $\sigma_i$, namely, swapping the $i^{th}$ row and column with the $s_i^{th}$ row and column.

The index $s_i$ is defined to basically maximize the quantity $a_i - b_i$ (the gain in the rank). We use the following algorithm:

- Find indices $j \geq i$ which minimize $b_i$;
- Among those $j$, pick one which maximizes $a_i$ (if we still have ties, we pick the smallest index).

Our claim is an easy consequence of Lemma 3.6 below. 

□
Lemma 3.6. The sequence $b_i$ (of $M_x$) is non-decreasing. In other words, there are indices $1 \leq j_1 \leq j_2 \leq n$ such that $b_i = 0$ for $i \in [1, j_1 - 1]$, $b_i = 1$ for $i \in [j_1, j_2 - 1]$, and $b_i = 2$ for $i \in [j_2, n]$. Furthermore, the sequence $a_i$ is non-increasing for $i \in [1, j_1 - 1]$, $i \in [j_1, j_2 - 1]$ and $i \in [j_2 - 1, n]$.

Proof of Lemma 3.6. The intuition behind this proof is that as $i$ increases, the dimensions of the $T_i$'s work in our favor. The lemma follows from two keys observations:

- $b_i \leq b_{i+1}$ for all $1 \leq i < n$.
  
  **Proof.** Suppose that the algorithm from the proof of the permutation lemma generates $b_i > b_{i+1}$ for some $i \in [1, n-1]$. It follows that at the $i^{th}$ step, the permutation $(i, s_{i+1})$ generates a new value for $b_i$ not bigger than the old value of $b_{i+1}$, which is a contradiction. To see this, let $row_1$ and $row_2$ be the rows we delete from $T_i|x'_{i+1}$ to create $T_{i+1}$. Let us replace $\sigma_i$ with $(i, s_{i+1})$ and see how $b_i$ changes: the two rows we delete at the $i^{th}$ step will be exactly $row_1$ and $row_2$ with the last coordinate removed. Since having fewer coordinates works in our favor, we know that by applying permutation $(i, s_{i+1})$ at step $i$, the rank of $T_{i+1}$ and $T_i|x'_{i+1}$ can not differ by more then $b_{i+1}$.

- If $b_i = b_{i+1}$, then $a_i \geq a_{i+1}$.
  
  **Proof.** Suppose that the algorithm from the proof of the permutation lemma generates $b_i = b_{i+1} = l$ with $a_{i+1} = 1$ and $a_i = 0$. Let us replace $\sigma_i$ with $(i, s_{i+1})$ and see how $a_i$ and $b_i$ change. First, from the previous analysis, $l$ is greater then the new $b_i$. Furthermore, the new $x'_i$ is formed by adding two coordinates to the old $x'_{i+1}$ and additionally the new $T_i$ has fewer columns than the old $T_{i+1}$. Both arguments work in our favor, so the new $a_i$ is at least the old $a_{i+1}$. This is a contradiction, which completes the proof.

Observation 3.7. If $n - i - 2 > \text{rank}(T_i)$ then, when creating $T_{i+1}$ we can always find two rows that can be deleted without reducing the rank of $T_i|x'_{i+1}$, hence $b_i$ takes value zero at least $n - k - 2$ times. As $t$ is an upper bound on the number of times $b_i$ takes the value 0, we have $k + t \geq n - 2$. Also, since $|R_{k,t}(i+1) - R_{k,t}(i)| \leq 2$, we have $k \leq 2n/3$ and $t + k/2 \geq n$. 

![Figure 3.2](image-url)
4. A recursion

In this section, we use the Permutation lemma to derive a recursive bound towards the desired result.

**Definition 4.1.** We define \( \mathcal{M}_{k,t}(C) \) to be the collection of all \( C \)-normal matrices \( M \) with \( \pm 1 \) entries which satisfy condition (3.2). Note that for the rest of the paper \( C \) will be a fixed matrix and, for simplicity, we will write \( \mathcal{M}_{k,t} \) instead of \( \mathcal{M}_{k,t}(C) \).

The following lemma allows us to exploit the fact that if \( M \) is in the form given by equation (3.2), we can control \( \mathbb{P}(M \text{ is normal}) \). Let \( D \) denote the diagonal entries of \( M \).

**Lemma 4.2** (Recursion Lemma). For any \( i < j \), let \( X_{i:j} \) denote the event that \( x_k = X_k \) for \( i \leq k \leq j \). Then, for any \( 1 \leq k \leq t \leq n \) and \( 1 \leq i \leq n \), we have

\[
\sup_{D, X_{1:i-1}} \mathbb{P}(M_n \in \mathcal{M}_{k,t} | D, X_{1:i-1}) \leq \left\{ \begin{array}{ll}
2^{-R_{k,t}(i-1)} & \text{if } 2n - 2i > \text{rank}_i(M_n) \\
2^{-(n-i)^2 + o(n^2)} & \text{if } 2n - 2i \leq \text{rank}_i(M_n).
\end{array} \right.
\]

**Proof.** Note that if \( M \in \mathcal{M}_{k,t} \) then \( M \) is \( C \)-normal, and so by relation (2.3) \( T_i^T x_i = c \), where \( c \) is a vector uniquely determined by \( C, D \) and \( x_1, \ldots, x_{i-1} \). Thus, conditioned on \( x_1, \ldots, x_{i-1} \) and \( D, x_i \) belongs to a subspace \( H \) of dimension \( \text{max}\{2n - 2i - \text{rank}(T_{i-1}), 0\} \). Recall that by the Permutation Lemma, \( \text{rank}(T_{i-1}) = \text{rank}_{i-1}(M_x) = R_{k,t}(i - 1) \). Using Claim 2.1, we have

\[
\mathbb{P}(M_n \in \mathcal{M}_{k,t} | D, X_{1:i-1}) \leq \sum_{X_i \in H} \mathbb{P}(M_n \in \mathcal{M}_{k,t} | D, X_{1:i}) \mathbb{P}(X_{i+1} \mid X_i \text{ satisfies } (4.1)) \leq 2^{-(2n-2i)+\max(2n-2i-R_{k,t}(i-1),0)} \sup_{X_i \in \{\pm 1\}^{2n-2i}} \mathbb{P}(M_n \in \mathcal{M}_{k,t} | D, X_{1:i}).
\]

(1) If \( 2n - 2i > \text{rank}(T_{i-1}) \), relation (4.2) implies

\[
\mathbb{P}(M_n \in \mathcal{M}_{k,t} | D, X_{1:i-1}) \leq 2^{-R_{k,t}(i-1)} \sup_{X_i \in \{\pm 1\}^{2n-2i}} \mathbb{P}(M_n \in \mathcal{M}_{k,t} | D, X_{1:i}).
\]

(2) If \( 2n - 2i \leq \text{rank}(T_{i-1}) \) then \( 2n - 2i - 2j \leq \text{rank}(T_{i+j-1}) \) for all \( 0 \leq j \leq n - i \) as \( |R_{k,t}(i) - R_{k,t}(i - 1)| \leq 2 \). Now we can repetitively use relation (4.2):

\[
\mathbb{P}(M_n \in \mathcal{M}_{k,t} | D, X_{1:i-1}) \leq 2^{-(2n-2i)} \sup_{X_1, \ldots, X_i} \mathbb{P}(M_n \in \mathcal{M}_{k,t} | D, X_{1:i}) \leq 2^{-(n-i)^2 + o(n^2)}.
\]

\( \square \)

5. Proof of Theorem 1.2

Let \( C \) be a fixed matrix. Our goal is to bound \( \mathbb{P}(M_n \in \mathcal{M}_{k,t}) \) for each \( k \) and \( t \) (recall that \( \mathcal{M}_{k,t} \) depends also on \( C \), but since \( C \) is a fixed matrix we omit to emphasize its dependency). Note that for some specific values of \( k \) and \( t \), the problem is trivial. One can easily see from Observation 3.7 that \( \mathcal{M}_{k,t} \) is empty when \( k + t < n - 2, k \leq 2n/3 \) or \( t + k/2 > n \).

The proof will go as follows: in Sections 5.1 and 5.2 we present two different approaches. The first one provides good bounds on \( \mathbb{P}(M_n \in \mathcal{M}_{k,t}) \) when \( 2t + k \) is close to \( 2n \) while the second one provides
good bounds when $2t + k$ is far from $2n$. In Section 5.3 we combine the two results to get the desired bound through an optimization process.

5.1. The First Case.

**Lemma 5.1.** We have, for $1 \leq k \leq \frac{2n}{2}$ and $\frac{k}{2} < n - t \leq k$

$$\mathbb{P}(M_n \in \mathcal{M}_{k,t}) \leq \begin{cases} 2n^2 + k^2 + t^2 + kt - 2kn - 2nt + o(n^2) & \text{if } k \geq \frac{n}{2} \\ 2t^2 - 3k^2 + 2kn + kt - 2nt + o(n^2) & \text{if } k \leq \frac{n}{2}. \end{cases} \quad (5.1)$$

In particular we have:

$$\mathbb{P}(M_n \text{ is } C\text{-normal}) \leq 2^{-0.25n^2 + o(n^2)}. \quad (5.2)$$

**Proof of Lemma 5.1**

Let $D$ be the diagonal of $M_n$. Then,

$$\mathbb{P}(M_n \in \mathcal{M}_{k,t}) = \sum_{D \in \{\pm 1\}^n} \mathbb{P}(M_n \in \mathcal{M}_{k,t}|D)\mathbb{P}(D) \leq \sup_{D \in \{\pm 1\}^n} \mathbb{P}(M_n \in \mathcal{M}_{k,t}|D).$$

By the Recursion Lemma we have

$$\mathbb{P}(M_n \in \mathcal{M}_{k,t}|D) \leq \sum_{x_1 \in \{\pm 1\}^{2(n-1)}} \mathbb{P}(M_n \in \mathcal{M}_{k,t}|D, X_1:1)\mathbb{P}(X_1:1|D) \leq \sup_{x_1 \in \{\pm 1\}^{2(n-1)}} \mathbb{P}(M_n \in \mathcal{M}_{k,t}|D, X_1:1) \leq 2^{-R_{k,t}(1)} \sup_{x_1 \in \{\pm 1\}^{2(n-1)}, x_2 \in \{\pm 1\}^{2(n-2)}} \mathbb{P}(M_n \in \mathcal{M}_{k,t}|D, X_1:2) \ldots \leq 2^{-\sum_{i=1}^{2n-k-t} R_{k,t}(i)} 2^{-(k+t-n)^2 + o(n^2)},$$

where the latter inequality follows by the fact that if $M \in \mathcal{M}_{k,t}$, then

$$R_{k,t}(2n - k - t) = k + t - (2n - k - t) > 2n - 2(2n - k - t + 1).$$

We conclude that

$$\mathbb{P}(M_n \in \mathcal{M}_{k,t}) \leq 2^{-(k+t-n)^2 - \sum_{i=1}^{2n-k-t} R_{k,t}(i) + o(n^2)} \leq 2^{-(k+t-n)^2 + k^2/2 - (t-k)(k-(3k/2+t-n))(2n-k-2t)+o(n^2)} \leq 2^{2n^2 + k^2 + t^2 + kt - 2kn - 2nt + o(n^2)}. \quad (5.3)$$

If $k \leq \frac{n}{2}$, then the bound from (5.3) is weak so we use a slightly different approach. Suppose that $M \in \mathcal{M}_{k,t}$. By Observation 5.7 we have $t \geq n - k - 2$ so Lemma 3.6 implies:

$$a_{k+1} - b_{k+1} = a_{k+2} - b_{k+2} = \ldots = a_{n-k-2} - b_{n-k-2} = 0.$$

By Observation 5.7 we also know that if $p \leq n - k - 2$, then $b_p = 0$ which implies that

$$a_{k+1} = a_{k+2} = \ldots = a_{n-k-2} = 0.$$

It follows that $x_p$ is in the column space of $T_{p-1}$ for any $k < p \leq n - k - 2$, hence for $k+1 \leq p \leq n - k - 2$, $x_p$ belongs to a fixed subspace $G$ of dimension $k$. Using Claim 2.1 for any $k < p \leq n - k - 2$ we have:

$$\mathbb{P}(M_n \in \mathcal{M}_{k,t}|D, X_{1:p-1}) \leq \sum_{X_p \in G} \mathbb{P}(M_n \in \mathcal{M}_{k,t}|D, X_{1:p})\mathbb{P}(X_{p+1}|D, X_{1:p-1}) \leq 2^{k-2(n-p-1)} \sup_{X_p \in \{\pm 1\}^{2(n-p)}} \mathbb{P}(M_n \in \mathcal{M}_{k,t}|D, X_{1:p}, X_p \in G).$$
Now we can combine this result with the recursion lemma:

\[
\mathbb{P}(M_n \in \mathcal{M}_{k,t}|D) \leq 2^{-\frac{k^2}{2}+o(n^2)} \sum_{i=1}^{n-k} \mathbb{P}(M_n \in \mathcal{M}_{k,t}|D, X_{1:k})
\]

\[
\leq 2^{-\frac{k^2}{2}+o(n^2)} \sum_{i=1}^{n-k} \mathbb{P}(M_n \in \mathcal{M}_{k,t}|D, X_{1:n-k-2})
\]

\[
\leq 2^{-n^2-5k^2/2+3nk+o(n^2)} \sum_{i=1}^{n-k-1} R_{k,t}(i) \mathbb{P}(M_n \in \mathcal{M}_{k,t}|D, X_{1:n-k-2})
\]

\[
\leq 2^{-n^2-2k^2+2t^2-4nt+3kt+o(n^2)} 2^{-k+t-n)^2+o(n^2)}
\]

(5.4)

Note that if we maximize the bounds over all possible choices of \(k\) and \(t\) we conclude:

\[
\mathbb{P}(M_n \in \mathcal{M}_{k,t}) \leq 2^{-0.25n^2+o(n^2)}
\]

and the conclusion follows. \(\square\)

5.2. The second case.

The idea is to bound \(\mathbb{P}(M_n \in \mathcal{M}_{k,t})\) differently when \(2n - 2t - k\) is big. Let \(M \in \mathcal{M}_{k,t}\) and let \(T_1\) be defined with respect to \(M\). Recall that \(T_1\) has \(t\) columns, \(2(n-t-1)\) rows, rank \(k\) and the property that for any \(1 \leq i \leq n - t - 1\), we delete its \(i\)th and \((n-t-1+i)\)th rows, then the rank decreases by at least one. This motivates the following definition.

Definition 5.2. Let \(M\) be a fixed \(2m \times q\) matrix. We say that \(M\) has property \(\mathcal{P}\) if, for any \(1 \leq i \leq m\), by deleting both the \(i\)th row and the \((i+m)\)th row, we reduce the rank of \(M\) by at least one.

Definition 5.3. Let \(A := \{\beta \mid \mathbb{P}(M_n \text{ is } C\text{-normal}) \leq 2^{-(\beta+o(1))n^2}\}\). We define

\[
\alpha = \limsup_{\beta \in A} \beta - 0.0001.
\]

Lemma 5.1 implies that

\[
\alpha \geq 0.2499.
\]

Lemma 5.4. Given \(1 \leq k, t \leq n\) we have that:

\[
\mathbb{P}(M_n \in \mathcal{M}_{k,t}) \leq 2^{1-o(n^2)} 2^{-\frac{k^2}{2}+2-n^2+nk+o(n^2)}.
\]

Proof of Lemma 5.4. The intuition is that, given a random \(2(n-t-1)\) \(t\) matrix with \(\pm 1\) entries and rank \(k\), the probability that it has property \(\mathcal{P}\) is very small for particular values of \(k\) and \(t\).

Given that by Observation 3.7 we have that the probability in question is zero unless \(n - k - 2 \leq t \leq n - k/2\). We start by making two observations.

Observation 5.5.

(a) Let \(M\) be a \(2m \times q\) matrix, then for any \(1 \leq i \leq m\), we can swap the \(i\)th row of \(M\) with the \((m+i)\)th row of \(M\) without changing its property \(\mathcal{P}\) status.

(b) Let \(M\) be a \(2m \times q\) matrix, then for any \(1 \leq i < j \leq m\) we can swap the \(i\)th row of \(M\) with the \(j\)th row of \(M\) and the \((m+i)\)th row of \(M\) with the \((m+j)\)th row of \(M\), without changing its property \(\mathcal{P}\) status.

Given a matrix \(M\) of rank \(k\), it would be more convenient to bound the probability of having property \(\mathcal{P}\) if the first \(k\) rows were linearly independent. It turns out that we only lose a factor of \(2^{o(n^2)}\) if we consider only such matrices. A precise statement is given in Claim 5.7.

Definition 5.6. We say that a matrix \(M\) of rank \(k\) has property \(\mathcal{F}_k\) if it has property \(\mathcal{P}\) and its first \(k\) rows are linearly independent.

Claim 5.7. Let \(M_{m,q}\) be a \(2m \times q\) random matrix with Rademacher entries which take the values \(\pm 1\) with probability \(1/2\). We have

\[
\mathbb{P}(M_{m,q} \text{ has property } \mathcal{P} \text{ and rank } k) \leq \mathbb{P}(M_{m,q} \text{ has property } \mathcal{F}_k) 2^{o(m^2)}.
\]
Proof of Claim 5.4. Let $M$ be a fixed matrix of dimension $2m \times q$ and rank $k$ which has property $\mathcal{P}$. We prove that we can apply the series of operations described in Observation 5.5 to reduce it to a matrix which has property $\mathcal{F}_k$. Since we have at most $(2m)! = 2^o(m^2)$ ways to permute the rows of $M$, the conclusion follows.

Suppose that there exists a fixed matrix $M$, which cannot be reduced to one with property $\mathcal{F}_k$ using only operations from Observation 5.5. Let $i \leq k$ be the biggest index such that there exists a matrix $M'$, formed by applying such operations to $M$, and its $i^{th}$ row is the first row that is not linearly independent to the previous $i-1$ rows.

If $i \leq m$, then by property $\mathcal{P}$, we know that if we delete both the $i^{th}$ row and the $(m+i)^{th}$ row from $M'$, then we decrease the rank of its row space by at least one. Since the $i^{th}$ row is in the span of the first $i-1$ rows, then we deduce that the $(m+i)^{th}$ row is linearly independent to the first $i-1$ rows. By Observation 5.5[a] we can swap the $i^{th}$ row with the $(m+i)^{th}$ row and still preserve property $\mathcal{P}$. Hence, the conclusion follows.

We conclude that

$$\Pr(M_{m,q} \text{ has property } \mathcal{P} \text{ and rank } k) \leq \Pr(M_{m,q} \text{ has property } \mathcal{F}_k)2^o(m^2).$$

$\square$

**Lemma 5.8.** Let $M_{m,q}$ be a $2m \times q$ random matrix with Rademacher entries which take the values $\pm 1$ with probability 1/2. We have

$$\Pr(M_{m,q} \text{ has property } \mathcal{F}_k) \leq 2^{(2m-k)(k-m-q)+o(m^2)}.$$

**Proof of Lemma 5.8.** We start by conditioning on the first $k$ rows of $M_{m,q}$, which we will denote by $K$. We will denote by $A^{(i_1, ..., i_j)}$ the submatrix of a matrix $A$ created by removing its $i_1^{th}$, ..., $i_j^{th}$ rows. We also write $\text{rowsp} (A)$ to denote the row space of a matrix $A$ and $\text{row}_i (A)$ to denote the $i^{th}$ row of a matrix $A$.

Note that for $p$ in range $(\max(1, k-m), m)$ we have that $p+m > k$ and property $\mathcal{F}_k$ implies that $\text{rank} \left( \text{rowsp} \left( M_{m,q}^{(p,m+p)} \right) \right) \leq k-1$. However, since $\text{rowsp} (K^{(p)}) \subseteq \text{rowsp} \left( M_{m,q}^{(p,m+p)} \right)$ and $\text{rank} (K^{(p)}) = k-1$, we have

$$\text{rowsp} \left( M_{m,q}^{(p,m+p)} \right) = \text{rowsp} (K^{(p)}),$$

which implies that for any $k < i \leq 2m$,

$$\text{row}_i (M_{m,q}) \in \text{rowsp} (K^{(p)})$$

for any $\max(1, k-m) < p \leq m$,

and therefore

$$\text{row}_i (M_{m,q}) \in \text{rowsp} \left( K^{(\max(1,k-m)+1, \ldots, m)}) \right).$$

Now we would like to make a similar argument for $p \in \{1, \ldots, \max(1, k-m)\}$, however, the difficulty with $p$ in this range is that the $\text{row}_p (M_{m,q})$ and $\text{row}_{(p+m)} (M_{m,q})$ are rows in $K$, hence we only have that

$$\text{rowsp} \left( K^{(p,m+p)} \right) \subseteq \text{rowsp} \left( M_{m,q}^{(p,m+p)} \right).$$

Note that since $\text{rank} (\text{rowsp} (K^{(p,m+p)})) = k-2$, we cannot conclude that, for example, $\text{row}_{k+1} (M_{m,q})$ is in $\text{rowsp} (K^{(p,m+p)})$. However, if $\text{row}_{k+1} (M_{m,q}) \notin \text{rowsp} (K^{(p,m+p)})$, then

$$\dim \left( \text{span} \left( \text{rowsp} (K^{(p,m+p)}) \cup \{\text{row}_{k+1} (M_{m,q})\} \right) \right) = k-1,$$

hence by property $\mathcal{F}_k$. 

$\square$
span \( \{ \text{rowsp}(K^{(p,p+m)}) \cup \{ \text{rowk+1}(M_{m,q}) \} \} = \text{rowsp}(M^{(p,m+p)}). \)

This translates as

\[
\text{row}_{k+i} \in \text{span} \left( \text{rowsp}(K^{(p,p+m)}) \cup \{ \text{rowk+1}(M_{m,q}) \} \right) \text{ for any } 2 \leq i \leq 2m - k.
\]

To make this argument rigorous, let \( F_k(c_1, c_2, \ldots, c_{k-m}) \) be the set of all \( 2m \times q \) matrices, \( M \), having the following properties. First, the entries of \( M \) are \( \pm 1 \). Second, the matrix \( M \) satisfies property \( F_k \). Finally, it has the property that \( c_p \) is the smallest integer greater than \( k \) such that \( \text{rowk+c_p}(M^{(p,m+p)}) \notin \text{rowsp}(K^{(p,m+p)}) \) for any \( 1 \leq p \leq k - m \), where by \( K_m \) we denote the top \( k \times q \) submatrix of \( M \). If for some \( 1 \leq p \leq k - m \), there is no such \( c_p \), we will define \( c_p \) to be \( 2m \).

Note that if \( M \in F_k(c_1, \ldots, c_{k-m}) \), then, for any \( k < s \leq 2m \) we have

\[
\text{row}(M) \in \bigcap_{k-m < i \leq m} \bigcap_{i \neq s} \text{rowsp}(K^{(i)}) \bigcap_{1 \leq j \leq k-m} \text{rowsp}(K^{(j,j+m)}) \bigcap_{1 \leq j \leq k-m} \text{span} \left( \text{rowsp}(K^{(j,j+m)}) \cup \{ \text{rowk+c_j}(M) \} \right),
\]

and therefore, together with Claim 2.11 implies

\[
\mathbb{P} (M_{m,q} \in F_k(c_1, \ldots, c_{k-m})|K) \leq 2^{(2m-k) \cdot (k-m-q)} \cdot \sum_{1 \leq i \leq k-m} (c_i - k-1) \\
\leq 2^{(2m-k) \cdot (k-m-q) + (k-m)}.
\]

It follows that

\[
\mathbb{P} (M_{m,q} \text{ has property } F_k) \leq \sup_K \mathbb{P} (M_{m,q} \text{ has property } F_k|K) \\
\leq \sum_{k < c_p \leq 2m} \sup_K \mathbb{P} (M_{m,q} \in F_k(c_1, \ldots, c_{k-m})|K) \\
\leq (2m-k)^{k-m} \cdot 2^{(2m-k) \cdot (k-m-q) + (k-m)} \\
\leq 2^{(2m-k) \cdot (k-m-q) + o(m^2)}.
\]

\[\square\]

Now we are ready to complete the proof of Lemma 5.4. Let \( \tilde{M} \) be the top left \( t \times t \) submatrix of \( M_n \). Then, by Claim 5.7 and Lemma 5.8

\[
\mathbb{P} (M_n \in \mathcal{M}_{k,t}) \leq \sup_{T_t} \mathbb{P} (M_n \in \mathcal{M}_{k,t}|T_t) \cdot \mathbb{P} (T_t \text{ has property } \mathcal{P} \text{ and rank } k) \\
\leq \sup_{T_t} \mathbb{P} (M_n \in \mathcal{M}_{k,t}|T_t) 2^{(2n-2t-k)(k-n)+o(n^2)}.
\]

Also, by the definition of \( \alpha \),

\[
\mathbb{P} (M_n \in \mathcal{M}_{k,t}|T_t) \leq \sup_{T_t, \tilde{M}_n} \mathbb{P} \left( M_n \in \mathcal{M}_{k,t}|T_t, \tilde{M}_n \right) \cdot \mathbb{P}(\tilde{M}_n \text{ is } C_t^T C_t - R_t R_t^T + C \text{-normal}) \\
\leq \sup_{T_t, \tilde{M}_n} \mathbb{P} \left( M_n \in \mathcal{M}_{k,t}|T_t, \tilde{M}_n \right) 2^{-\alpha t^2 + o(n^2)}.
\]

Additionally, by the Recursion Lemma,

\[
\mathbb{P} \left( M_n \in \mathcal{M}_{k,t}|T_t, \tilde{M}_n \right) \leq \sup_{D, X_1, \ldots, X_t} \mathbb{P} \left( M_n \in \mathcal{M}_{k,t}|D, X_1, t \right) \\
\leq 2^{- \left( \text{rank}(T_t) + \ldots + \text{rank}(T_{2n-k-t}) \right)(k+t-n)^2} \\
\leq 2^{k^2/2 + n^2 + t^2 - 2nk + 2kt - 2nt + o(n^2)}.
\]
Finally, we conclude
\[ P(M_n \in \mathcal{M}_{k,t}) \leq 2^{(2n-2t-k)(k-n)-\alpha t^2+k^2/2+n^2+t^2-2nk+2kt-2nt+o(n^2)} \leq 2^{(1-\alpha)t^2-k^2/2-n^2+nk+o(n^2)}. \]

\[\Box\]

5.3. Proving the Theorem 1.2.

In this section, we put everything together to conclude the proof of Theorem 1.2 using a case analysis to maximize \( \alpha \). We define \( f, g_1 \) and \( g_2 \) by the following formulas:

\[ f(\alpha, n, k, t) := (1-\alpha)t^2 - k^2/2 - n^2 + nk \]

\[ g_1(n, k, t) := t^2 - 3k^2 + 2kn + kt - 2nt \]

\[ g_2(n, k, t) := n^2 + k^2 + t^2 + kt - 2kn - 2nt. \]

By Lemma 3.1 and Lemma 3.3 we have

\[ P(M_n \in \mathcal{M}_{k,t}) \leq \begin{cases} \min (2g_1(n, k, t) + o(n^2), 2f(\alpha, n, k, t) + o(n^2)) & \text{if } k \leq \frac{n}{2} \\ \min (2g_2(n, k, t) + o(n^2), 2f(\alpha, n, k, t) + o(n^2)) & \text{if } k \geq \frac{n}{2}. \end{cases} \]

For fixed \( k \), both \( g_1 \) and \( g_2 \) are decreasing functions of \( t \), while \( f \) is increasing in \( t \). It follows that the worst lower bound for \( \alpha \) is achieved in one of the six situations:

\[ \begin{cases} f(\alpha, n, k, t) = g_1(n, k, t), \ t = n - k/2 \text{ or } t = n - k & \text{when } k \leq n/2 \\ f(\alpha, n, k, t) = g_2(n, k, t), \ t = n - k/2 \text{ or } t = k & \text{when } k \geq n/2. \end{cases} \]

Since all the equations are homogeneous, we will assume that \( n = 1 \) and we will analyze each of these extreme cases.

**Case 1.** \( t = 1 - k \) and \( k \leq 0.5 \). Relations (5.5) and (5.6) become

\[ f(\alpha, 1, k, 1 - k) = -\alpha - (1 - 2\alpha)k + (1 - \alpha)k^2 \]

\[ g_1(1, k, 1 - k) = -1 + 3k - 3k^2, \]

which implies

\[ \alpha \geq \min_{k \in [0, 0.5]} \left( \max \left( \frac{1-k}{2-k}, 1+3k^2-3k \right) \right) \geq 0.425. \]

**Case 2.** \( t = 1 - k/2 \) and \( k \leq 0.5 \). Relations (5.6) implies

\[ -\alpha \leq g_1(1, k, 1 - k/2) = -1 - 13k^2/4 + 3k \leq -0.307 \]

**Case 3.** \( t = 1 - k/2 \) and \( k \geq 0.5 \). Relations (5.7) implies

\[ -\alpha \leq g_2(1, k, 1 - k/2) = \frac{3k^2}{4} - k \leq -0.3125 \text{ since } k \leq 2/3. \]

**Case 4.** \( t = k \) and \( k \geq 0.5 \). Relations (5.5) and (5.7) become

\[ f(\alpha, 1, k, k) = (1/2 - \alpha)k^2 + k - 1 \]

\[ g_2(1, k, k) = 1 + 3k^2 - 4k, \]

which implies

\[ \alpha \geq \min_{k \in [0.5, 1]} \left( \max \left( \frac{-1+k^2/2+k}{1-k^2}, 1+3k^2-4k \right) \right) \geq 0.323. \]
Case 5. $f(\alpha, 1, k, t) = g_1(1, k, t)$ and $k \leq 0.5$. Since $f(\alpha, 1, k, t) = g_1(1, k, t)$ we get

$$k = \frac{t + 1 + \sqrt{(t+1)^2 - 5(4t - 2 - 2\alpha t^2)}}{5}$$

Hence,

$$\alpha \geq \min_t f\left(\alpha, 1, \frac{t + 1 + \sqrt{(t+1)^2 - 5(4t - 2 - 2\alpha t^2)}}{5}, t\right),$$

which leads to

$$\alpha \geq 0.302.$$

Case 6. $f(\alpha, 1, k, t) = g_2(1, k, t)$ and $k \geq 0.5$. Since $f(\alpha, 1, k, t) = g_2(1, k, t)$ we get

$$k = \frac{3 - t - \sqrt{(t-3)^2 - 3(4t - 2 - 2\alpha t^2)}}{3}$$

Hence,

$$\alpha \geq \min_t f\left(\alpha, 1, \frac{3 - t - \sqrt{(t-3)^2 - 3(4t - 2 - 2\alpha t^2)}}{3}, t\right),$$

which leads to

$$\alpha \geq 0.307.$$

It follows immediately from the above cases that

$$\nu_n \leq 2^{-(0.302+o(1))n^2}.$$

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