EMBEDDING THEOREMS FOR SOLVABLE GROUPS

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Abstract. In this paper, we prove a series of results on group embeddings in groups with a small number of generators. We show that each finitely generated group $G$ lying in a variety $\mathcal{M}$ can be embedded in a 4-generated group $H \in \mathcal{MA}$ ($\mathcal{A}$ means the variety of abelian groups). If $G$ is a finite group, then $H$ can also be found as a finite group. It follows, that any finitely generated (finite) solvable group $G$ of the derived length $l$ can be embedded in a 4-generated (finite) solvable group $H$ of length $l+1$. Thus, we answer the question of V. H. Mikaelian and A.Yu. Olshanskii. It is also shown that any countable group $G \in \mathcal{M}$, such that the abelianization $G_{ab}$ is a free abelian group, is embeddable in a 2-generated group $H \in \mathcal{MA}$.

Key words. Solvable group, embedding, variety

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1. Introduction

The main aim of this paper is to study embeddings of finitely generated groups in 2- or 4-generated groups. Let $G$ stand for a finitely generated group which lies in a variety $\mathcal{M}$, and $H$ for a 4-generated group in which $G$ can be embedded. We show that $H$ can be found in the variety $\mathcal{MA}$, where $\mathcal{A}$ denote the variety of abelian groups. It follows that every finitely generated solvable group of the derived length $l$ can be embedded in a 4-generated solvable group of length $l+1$.

We also study what further properties of $G$ our embedding procedure endow $H$. Finiteness is one of them, and if $G$ is a finite $p$-group ($p$ is a prime) , $H$ can be chosen as a finite $p$-group. Also, if $G$ has finite exponent, then $H$ can be made to have finite exponent. If $G$ is a countable group such that the abelianization $G_{ab}$ is a free abelian group then $H$ can be found as a 2-generated group.

Thus, we refine the classical results on embeddings of countable groups, a brief overview of which is given below. These results refer to embeddings in 2-generated groups. We do not know whether parameter 4 can be lowered in our results. We believe that this cannot be done.

In the late 1940s, G. Higman, B. H. Neumann and H. Neumann showed that every countable group embeds in a 2-generator group, in the same paper in which they introduced and successifully applied HNN-extensions. Their method using free constructions of groups did not give similar results for varieties of groups. So then B. H. Neumann and H. Neumann in applied wreath products to prove that every countable group $G$ lying in a variety $\mathcal{M}$ can be embedded in a 2-generated group $H \in \mathcal{MA}^2$. If $G$ is finite ($p$-) group, then $H$ can be chosen as finite ($p$-) group. Also, if $G$ has finite exponent, then $H$ can be made to have finite exponent.

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It follows that every countable solvable group of the derived length \( l \) can be embedded in some 2-generated solvable group of length \( l+2 \). Note that this bound is sharp. Namely, the group \( \mathbb{Q} \) of rationals does not embed into any finitely generated metabelian group \( M \). Indeed, \( M \) is residually finite by Hall’s theorem proved in [1], but \( \mathbb{Q} \) is not. Thus we cannot lower \( l+2 \) to \( l+1 \) in the Neumann-Neumann embedding theorem.

V.H. Mikaelian and A. Yu. Olshanskii gave an explicit classification of all abelian groups that can occur as subgroups of finitely generated metabelian groups as follows.

**Theorem** (V. H. Mikaelian, A. Yu. Olshanskii [5]). Let \( A \) be an abelian group.

The following properties are equivalent.

1. \( A \) is a subgroup of a finitely generated metabelian group;
2. \( A \) is a subgroup of a finitely generated abelian-by-polycyclic group;
3. \( A \) is a subgroup of a finitely presented metabelian group;
4. \( A \) is a subgroup of a 2-generated metabelian group;
5. \( A \) is a Hall group.

Note that (3) follows from a remarkable statement independently proved by G. Baumslag and V.N. Remeslennikov: each finitely generated metabelian group embeds in some finitely presented metabelian group (see [10]).

By definition, \( A \) is a Hall group if

- \( A \) is a (finite or) countable abelian group;
- \( A = T \oplus K \), where \( T \) is a bounded torsion group (i.e., the orders of all elements in \( T \) are bounded), \( K \) is torsion free;
- \( K \) has a free abelian subgroup \( F \) such that \( K/F \) is a torsion group with trivial \( p \)-subgroups for all primes except for the members of a finite set defined by \( K \).

In [7], A. Yu. Olshanskii established a number of other embedding theorems for metabelian groups.

Every finitely generated nilpotent group satisfies the maximal condition on subgroups, that is, every subgroup is finitely generated (see [2] or [10]). Hence each non-finitely generated nilpotent group cannot be embedded in a finitely generated nilpotent group. However every finitely generated nilpotent group embeds in some 2-generated nilpotent group of sufficiently large class [8]. Similarly, every polycyclic group embeds in a 2-generated polycyclic group [9].

Let \( G \) be a group and \( g, f \in G \). Further in the paper \( g^f \) denotes \( f^{-1}gf \) (conjugate of \( g \) by \( f \)) and \([g,f]\) stands for \( g^{-1}f^{-1}gf \) (commutator of \( g \) and \( f \)). Also \([g,f;1]\) means \([g,f]\) and inductively \([g,f;k+1]\) stands for \([([g,f;k],f],k=1,2,..., By G' we denote the derived subgroup of \( G \). Then \( G_{ab} = G/G' \) is the abelianization of \( G \). \( \mathbb{Z} \) means the infinite cyclic group and \( \mathbb{Z}_n \) denotes a cyclic group of order \( n \).

Recall that the Cartesian wreath product of groups is defined as follows. Let \( A \) and \( B \) be groups and \( D \) a group of all functions \( f : B \to A \) with multiplication \((f_1f_2)(x) = f_1(x)f_2(x) \) for \( x \in B \). The group \( B \) acts on \( D \) from the left by shift automorphisms: \( f^b(x) = f(bx) \) for all \( f \in D \), \( b, x \in B \), and the associated with this action semidirect product \( D \rtimes B \) is called the Cartesian wreath product of the groups \( A \) and \( B \), denoted by \( AWrB \). The subgroup \( D \) is called base subgroup of \( A WrB \). Thus, every element of \( AWrB \) has a unique presentation as \( bf \) (\( b \in B, f \in D \)) and
the multiplication rule follows from the conjugation formula

\[ f^b(x) = f(bx) \]

in \( AWrB \) for any \( b, x \in B \) and \( f \in D \). If instead of \( D \) one takes the smaller group consisting of all functions with finite support, that is, functions taking only non-identity values on a finite set of points, then one obtains a subgroup of \( AWrB \) called the wreath product (direct wreath product); it is denoted by \( A wr B \).

2. Main results

The following question was posed by V. H. Mikaelian and A. Yu. Olshanskii in [5] and was also written by A. Yu. Olshanskii in [4](Question 18.73):

Does every finitely generated solvable group of derived length \( l \geq 2 \) embed into a \( 2 \)-generated solvable group of length \( l + 1 \)? Or at least, into some \( k \)-generated \((l + 1)\)-solvable group, where \( k = k(l) \)?

We prove the following embedding theorems that imply an affirmative answer to Mikaelian-Olshanskii’ question. In the following statements, \( \mathcal{M} \) means an arbitrary variety of groups and \( \mathcal{A} \) is the variety of abelian groups. For \( s \in \mathbb{N} \), \( \mathcal{A}_s \) means the variety of abelian groups of exponent \( s \).

**Theorem 1.** Let \( G \) be a countable group such that the abelianization \( G_{ab} \) is a free abelian group. Then \( G \) embeds in some \( 2 \)-generated subgroup \( H \) of the Cartesian wreath product \( GWr\mathbb{Z} \).

**Corollary 2.**

1. Let \( G \in \mathcal{M} \) be a countable group such that the abelianization \( G_{ab} \) is a free abelian group. Then \( G \) embeds in some \( 2 \)-generated group \( H \in \mathcal{MA} \). In particular, every finitely generated group \( G \in \mathcal{M} \) such that the abelianization \( G_{ab} \) is torsion-free, embeds in some \( 2 \)-generated group \( H \in \mathcal{MA} \).

2. Let \( G \) be a countable solvable group of derived length \( l \) such that the abelianization \( G_{ab} \) is a free abelian group. Then \( G \) embeds in some \( 2 \)-generated solvable group \( H \) of length \( l + 1 \). In particular, every finitely generated solvable group \( G \) of derived length \( l \) such that the abelianization \( G_{ab} \) is torsion-free, embeds in some \( 2 \)-generated solvable group \( H \) of length \( l + 1 \).

3. Every finitely generated group \( G \in \mathcal{M} \) has a subgroup \( K \) of finite index that can be embedded in some \( 2 \)-generated group \( H \in \mathcal{MA} \). In particular, every finitely generated solvable group \( G \) of derived length \( l \) has a subgroup \( K \) of finite index that can be embedded in some \( 2 \)-generated solvable group \( H \) of length \( l + 1 \).

**Theorem 3.** Let \( G \) be a countable group such that the abelianization \( G_{ab} \) is a direct product of a free abelian group and a finite group. Then \( G \) embeds in some \( 4 \)-generated subgroup \( H \) of the Cartesian wreath product \( GWr\mathbb{Z}^3 \).

**Corollary 4.**

1. Let \( G \in \mathcal{M} \) be a countable group such that the abelianization \( G_{ab} \) is a direct product of a free abelian group and a finite group. Then \( G \) embeds in some 2-generated subgroup \( H \in \mathcal{MA} \). In particular, every finitely generated group \( G \in \mathcal{M} \) embeds in some \( 4 \)-generated group \( H \in \mathcal{MA} \).

2. Let \( G \) be a countable solvable group of derived length \( l \) such that the abelianization \( G_{ab} \) is a direct product of a free abelian group and a finite group.
Then $G$ embeds in a 4-generated solvable group $H$ of length $l + 1$. In particular, every finitely generated solvable group $G$ of derived length $l$ embeds in some 4-generated solvable group $H$ of length $l + 1$.

**Theorem 5.** Let $G$ be a group generated by a finite set $u_1, \ldots, u_m$ of elements of finite orders $l_1, \ldots, l_m$, respectively. Then $G$ embeds in some 4-generated subgroup $H$ of length $l + 1$.

**Corollary 6.** (1) Let $G \in \mathcal{M}$ be a $m$-generated group of exponent $e$. Then $G$ embeds in a 4-generated group $H \in \mathcal{MA}$ where $s = e^{m+1}$, and so has exponent $e^{m+2}$.

(2) Let $G \in \mathcal{M}$ be a finite group. Then $G$ embeds in some 4-generated finite group $H \in \mathcal{MA}$. In particular, every finite solvable ($p$-) group $G$ of derived length $l$ embeds in some 4-generated solvable ($p$-) group $H$ of length $l + 1$.

3. Proof of Theorem III

Let $G$ be a countable group such that $\bar{A} = G_{ab}$ is a free abelian group with basis $\{\bar{a}_i : i \in I \subseteq \mathbb{N}\}$. Denote by $a_i$ a preimage of $\bar{a}_i$ in $G$ and set $\bar{A} = \text{gp}(a_i : i \in I)$.

Let $C$ be an infinite cyclic group generated by $c$ and $U = G \text{ Wr } C \simeq G \text{ Wr } \mathbb{Z}$. We are to show that $G$ embeds in some 2-generator subgroup $H$ of $U$, and thus proves Theorem III. Now $D_U$ denotes the base group of $U$.

Let $s_1 < s_2 < \ldots < s_i < \ldots$ be a sequence of positive integers such that $s_i + j - s_j = s_{k+l} - s_l$ if and only if $i = k$ and $j = l$. For definiteness, we take $s_i = 2^i$ for $i = 1, \ldots,$. This sequence is called a strictly uneven sparse sequence.

For brevity, denote $c_i = c^{s_i}, i = 1, 2, \ldots,$ and set $C = \{c_i : i = 1, 2, \ldots\}$.

Suppose that $G$ is generated by a set of generating elements $\{g_j : j \in J \subseteq \mathbb{N}\}$. Let $g_j = a(j)g_j'$, where $a(j) \in A, g_j' \in G'$, $j \in J$. Then $G = \text{gp}(a_i, g_j') : i \in I, j \in J$.

Let

$$g_j' = \prod_{q=1}^{r_j} [u_{j,q}, v_{j,q}], \quad u_{j,q}, v_{j,q} \in G; \quad j \in J.$$  

Let $H = \text{gp}(c, d)$ where $d \in D_U$ is defined as follows. Let $C_1 \subseteq C, C_1 = \mathcal{A} \cup \mathcal{U} \cup \mathcal{V}$ ($|\mathcal{A}| = |I|, |\mathcal{U}| = |V| = \{|u_{j,q} : q = 1, \ldots, r_j, j \in J\} = \{|v_{j,q} : q = 1, \ldots, r_j, j \in J\}$) be a disjoint union of sets. Here $I$ is in one-to-one correspondence $\iota$ with indexes of elements in $\mathcal{A}$, the set of all elements of the form $u_{j,q}$ in one-to-one correspondence $\delta$ with the set of indexes of all elements in $\mathcal{U}$, and the set of all elements of the form $v_{j,q}$ in one-to-one correspondence $\lambda$ with the set of indexes of all elements in $\mathcal{V}$.

Then we set

$$d(c_{\iota(i)}) = a_i \quad \text{for each } i \in I;$$
$$d(c_{\delta(u_{j,q})}) = u_{j,q}, \quad \text{for each } u_{j,q};$$
$$d(c_{\lambda(v_{j,q})}) = v_{j,q} \quad \text{for each } v_{j,q};$$

and $d(c^s) = 1$ in all other cases.

First, we prove that $H$ contains all the elements $\tilde{g}_j' \in D_U$ such that $\tilde{g}_j'(1) = g_j'$ and $\tilde{g}_j'(c^s) = 1$ for $s \neq 0$. For any pair $(j, q)$, we compute by direct computation that

$$[d^{c^s(u_{j,q})}, d^{c^s(v_{j,q})}](1) = [u_{j,q}, v_{j,q}].$$


We set \( \tilde{g}_j^i = [d^{s_i(r_j,q_i)}, d^{s_i(q_i,q_j)}] \). It remains to verify that every value \( \tilde{g}_j^i \) is trivial for each \( s \neq 0 \). This statement follows because the sequence \( s_1, s_2, \ldots \), is strictly even sparse, and therefore each other non-trivial value \( d^{s_i(q_i,q_j)}(c^s) \), \( s \neq 0 \), meets the trivial value of \( d^{s_i(r_j,q_i)}(c^s) \). Then we get, by (3.1), that \( gp(\tilde{g}_j^i : j \in J)(1) = G' \), therefore \( gp(\tilde{g}_j^i : j \in J) \simeq G' \). Obviously, \( \tilde{g}_j^i \in H \) for every \( j \in J \).

Secondly, \( d^{s_i(r_j,q_i)}(1) = a_i \) for each \( i \in I \). Denote \( \tilde{a}_i = d^{s_i(r_j,q_i)} \) for \( i \in I \). We set \( \bar{G} = gp(\tilde{a}_i, \tilde{g}_j^i : i \in I, j \in J) \). Then \( \bar{G}(1) = G \). We must show that \( \bar{G} \simeq G(1) \), and conclude that \( \bar{G} \simeq G \).

Obviously, there is a natural homomorphism \( \mu : \bar{G} \to \bar{G}(1) \) with the image \( \bar{G}(1) \). Obviously, the restriction of \( \mu \) to \( gp(\tilde{g}_j^i : j \in J) \) is an isomorphism of \( \bar{G}' \) onto \( G' \).

Suppose that \( z = z(\tilde{a}_i, ..., \tilde{a}_j, \tilde{g}_j^i, ..., \tilde{g}_j^k) \in ker(\mu) \). Then the sum of exponents \( \sigma_i \) of \( \tilde{a}_i \) in \( z \) is 0 for each \( t = 1, ..., k \). Since the sequence \( s_1, s_2, ..., \) is strictly even sparse all other non-trivial values of \( \tilde{a}_i \), corresponding to the its occurs in \( z \) are \( \sigma_i \)-exponents of the corresponding values of \( \tilde{a}_i \), therefore are trivial. This values don’t depend from other factors of \( z \). Thus, \( z \in G' \) and therefore \( z = 1 \). Hence \( \mu \) is an isomorphism, and \( G \) embeds in \( H \).

Theorem is proved.

**Proof of Corollary 2**

Let \( G_{ab} = \bar{A} \times \bar{T} \), where \( \bar{A} \) is a free abelian group with basis \( \{ a_i : i \in I \} \), as before, and \( \bar{T} \) is a finite abelian group. Let \( a_i \) be a preimage of \( \tilde{a}_i \) in \( G \) for \( i \in I \). We define \( K = gp(a_i, G' : i \in I) \). This subgroup has a finite index in \( G \). Then we define elements \( \tilde{a}_i, \tilde{g}_j^i \) for \( i \in I \) and \( j \in J \) as above. We set \( \tilde{G}_1 = gp(\tilde{a}_i, \tilde{g}_j^i : i \in I, j \in J) \). Then \( \tilde{G}_1(1) = K \), and \( \tilde{G}_1 \simeq \tilde{G}_1(1) \). Hence \( \tilde{G}_1 \simeq K \).

This can be confirmed by the same argument as in the proof of Theorem [1].

Corollary is proved.

4. **Proof of Theorem 3**

First, we prove a number of auxiliary statements.

Let \( G \) be a group and \( V = G \wr \mathbb{Z} \) be the Cartesian wreath product of \( G \) and the infinite cyclic group \( \mathbb{Z} = gp(b) \). Denote by \( D_V \) the base group of \( V \). For any \( u \in G \), let \( u^{(0)} \in D_V \) be the constant function \( u^{(0)} : B \to G, u^{(0)}(b') = u, i \in \mathbb{Z} \). Then \( [u^{(0)}, b] = 1 \).

Lemma 7. For any element \( d \in D_V \) there is an element \( x \in D_V \) for which

\[
(4.1) \quad d = [x, b].
\]

Moreover, for any \( u \in G \) there is a unique \( x \) for which \( x(1) = u \).

**Proof.** Let \( d = (\ldots d_{-2}, d_{-1}, d_0, d_1, d_2, \ldots) \) and \( x = (\ldots x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) \), where \( d_i = d(b^i) \) and \( x_i = x(b^i) \). Then (4.1) is equivalent to the system of equations

\[
(4.2) \quad x_j^{-1}x_{j+1} = d_j, \quad j \in \mathbb{Z}.
\]

After setting \( x_0 = u, u \in U \), we uniquely compute for \( i \geq 1 \), that

\[
(4.3) \quad x_i = ud_0...d_{i-1} \quad \text{and} \quad x_{-i} = ud_{-1}^{-1}...d_{-i}^{-1}.
\]

In other words, \( d \) is a discrete (right) derivative of \( x \), and \( x \) is a discrete integral of \( d \). This integral is uniquely defined by \( d \) and its value \( x(1) = u \). We will denote it as \( I_1(d, u) \) and write \( I_1(d, u) = d \). Then we define \( I_2(d, u) = I_1(I_1(d, u), u), \ldots \), and so on. For simplicity, we keep \( u \) for all integrals.
Corollary 8. For each $u \in G$, there are a series of elements $u^{(k)} \in D_V, k = 1, 2, ...$, for which

$$(4.4) \quad [u^{(k)}, b] = u^{(k-1)} \text{ and } u^{(k)}(1) = u.$$ 

In particular,

$$(4.5) \quad [u^{(k)}, b; k] = u^{(0)} \text{ and } [u^{(i)}, b; k] = 1 \text{ for } i < k.$$ 

Proof. We define $u^{(k)} = I_k(u^{(0)}, u)$ for $k = 1, 2, ...$

Now, let $G$ be a group and $W = G Wr Z^2$ be a Cartesian wreath product, where $Z^2 = \text{gp}(b_1) \times \text{gp}(b_2)$ is the free abelian group of rank 2 with basis $\{b_1, b_2\}$. Let $D_W$ denote the base group of $W$ consisting of all functions $f : Z^2 \to G$, $f(i, j) = f(b_1^i, b_2^j) = u_{ij}, i, j \in Z$.

For any $u \in G$ we consider the subgroup $D_u = \text{gp}(u) Wr Z^2$ of $W$ generated by $b_1, b_2$ and all functions $f : Z^2 \to \text{gp}(u)$ which make up the base group $D_u \leq D_W$. Let $f(i, j) = f(b_1^i, b_2^j) = u^{a_{ij}}, a_{ij} \in Z, i, j \in Z$. Then for any $i_0 \in Z$, the set $C(i_0)(K)$ of elements of the form $K = \{u^{a_{ij}}, j \in Z\}$, will be called $i_0$-column of $D_u$, and for any $j_0 \in Z$ the set $R(j_0)(L)$ of elements of the form $L = \{u^{a_{ij}}, i \in Z\}$ will be called $j_0$-row of $D_u$. The set $K$ can be considered as element of the base group $D_{2u}$ of $V_{2u} = \text{gp}(u)Wr \text{gp}(b_2)$, and similarly the set $L$ can be treated as element of the base group $D_{1u}$ of $V_{1u} = \text{gp}(u)Wr \text{gp}(b_1)$.

Let $C(i_0)(w^{(0)})$, be a constant column corresponding to $w \in \text{gp}(u)$. By Corollary we get a series of columns $C(i_0)(w^{(t)})$ such that $C(i_0)(w^{(t)}) = C(i_0)(w^{(t-1)})$ and $C(i_0)(w^{(t)})(i_0, 1) = w, t = 1, 2, ...$. Then $[C(i_0)(w^{(t)}), b_2; t] = C(i_0)(w^{(0)})$ and $[C(i_0)(w^{(t)}), b_2; t + r] = 1$ for every $r \geq 1$.

Similarly, we get the elements $R(j_0)(w^{(t)}), t = 0, 1, 2, ...$, that satisfy the following properties: $[R(j_0)(w^{(t)}), b_1; t] = R(j_0)(w^{(0)})$ and $[R(j_0)(w^{(t)}), b_1; t + r] = 1$ for every $r \geq 1$.

Now we are ready to prove a key lemma that allows us to distinguish individual elements of a given finite set, while at the same time making other elements of this set trivial. We are dealing with the group $W = G Wr(\text{gp}(b_1) \times \text{gp}(b_2))$ defined above.

Lemma 9. Let $u_1, ..., u_t$ be a finite set of nontrivial elements of $G$. Let $u_i^{(0)} \in D_W$ be a constant function with the value $u_i$. Then there exist functions $f_i \in D_W$, all of whose values belong to $\text{gp}(u_i)$, which satisfy the following properties.

$$(4.6) \quad [f_i, b_2; i; b_1, t - i] = u_i^{(0)}, [f_j, b_2; i; b_1, t - i] = 1 \text{ for } i \neq j, i, j \in \{1, ..., t\}.$$ 

Proof. To construct $f_i$ we define its 0-row

$$R(0)(u_i^{t-i}) = (...)u_i^{a_1,0}, u_i^{a_0,0}, u_i^{a_1,0}, ...).$$ 

Then we build each column as

$$C(j)(u_i^{a_j,0})^{(i)}$$ 

and we have as a result $f_i$. By construction every $j$th column of $[f_i, b_2; i]$ is a constant function with value $u_i^{a_{j,0}}, j \in Z$. Then

$$[f_i, b_2; i; b_1, t - i] = u_i^{(0)} \text{ and } [f_q, b_2; i; b_1, t - i] = 1 \text{ for } q < i.$$ 

It happens because the columns are constructed as discrete integrals.
Let \( q > i \). By construction \( R_q(u_q^{(t-q)}, b_1; i - i) = 1 \). In other words, this is true for a 0-row that does not change during the process of differentiating columns.

Consider a more general case. Assume that \( u \in G \) and \( q \geq 0 \). For \( r \geq 0 \), we fix 0-row \( R(0)(u^r) = (u^{a_{0}^{r}}, u^{a_{0}^{r}}, u^{a_{0}^{r}}, \ldots) \). Then we expand this row to element \( f \in D_{v'} \) by adding the columns \( C(j)(u^{a_{0}^{r}})^{(s)} \) for some \( s \geq 0 \). Obviously, for \( s = 0 \) we have \([f, b_1; r + 1] = 1 \). We will prove by induction on \( s \) that this equality is true in general case.

Let it is true for \( s - 1 \). The value \( u^{a_{ij}} \) of the function \([f, b_1; r+1] \) in any point \((i, j)\) can be computed as follows. There is a \( \mathbb{Z} \)-linear function \( L(\alpha_{i,j}, \alpha_{i+1,j}, \ldots, \alpha_{i+r+1,j}) \). This function does not depend from \( s \). By our assumptions, the value of this function for any \( s \) is 0 for \( j = 0 \) and any \( i \). By the assumption of induction, for \( s - 1 \) the value of this function is zero for every \( j \).

Then this is true for \( j = 1 \) and \( j = -1 \). Indeed, if the 1-row for \( s \) is
\[
(..., u^{\beta_{1,-1}}, u^{\beta_{0,1}}, u^{\beta_{1,1}}, ...)\]
and 0-row is
\[
(..., u^{\gamma_{-1,1}}, u^{\gamma_{0,1}}, u^{\gamma_{1,1}}, ...)\]
then 0-row for \( s - 1 \) is
\[
(..., u^{\beta_{-1,1} - \gamma_{-1,1}}, u^{\beta_{0,1} - \gamma_{0,1}}, u^{\beta_{1,1} - \gamma_{1,1}}, ...)\]

Then by the assumption of induction
\[
L(\beta_{i,j} - \gamma_{i,j}) = L(\beta_{i,j}) - L(\gamma_{i,j}) = -L(\gamma_{i,j}) = 0.
\]

Similarly, this can be proved for \( j = -1 \). Continuing, we will get that this is true for \( s \) and each \( j \).

We proceed directly to the proof of the Theorem 3.

Let \( G \) be a countable group such that the abelianization \( G_{ab} \) is a direct product of a free abelian group \( A \) with a basis \( \{a_{i} : i \in I \subset \mathbb{N} \} \) and a finite abelian group \( \bar{U} = \text{gp}(\bar{a}_{1}, ..., \bar{a}_{k}) \). Let \( a_{i} \) denote a preimage of \( \bar{a}_{i} \) and \( u_{j} \) denote a preimage of \( \bar{u}_{j} \) in \( G \). Let \( A = \text{gp}(a_{i} : i \in I) \) and \( U = \text{gp}(u_{1}, ..., u_{t}) \).

Consider the Cartesian wreath product \( W = G \wr R \), where \( C = \text{gp}(c) \) is an infinite cyclic group and \( B \) is a free abelian group with base \( \{b_{1}, b_{2} \} \). Then \( W \simeq G \wr \mathbb{Z} \). By \( D_{W} \), we denote the base group of the group \( W \).

First, we will do the same as in the proof of Theorem 1. Let \( s_{1} < ... < s_{t} < ... \) be a strictly uneven sparse sequence of positive integers, i.e., \( s_{i+1} - s_{i} = s_{k+l} - s_{l} \) if and only if \( i = k \) and \( j = l \). For definiteness, we take \( s_{i} = 2^{i} \) for \( i = 1, 2, ... \). For brevity, denote \( c_{i} = c^{s_{i}}, i = 1, 2, ... \), and set \( C = \{c_{i} : i = 1, 2, ... \} \).

Suppose that \( G \) is generated by a set of elements \( \{g_{m} : m \in M \subset \mathbb{N} \} \) such that \( G' = \text{gp}(g_{m}, m_{r}') = \{g_{m}, g_{m}' : m, m' \in M \} \). Then \( G = \text{gp}(a_{i}, u_{j}, g_{m}): i \in I, j \in \{1, ..., t \}, m, m' \in M \).

Let \( H = \text{gp}(c, b_{1}, b_{2}, d) \) where \( d \in D_{W} \) is defined as follows.

Let \( c_{1} \subset C \) be a disjoint union \( \{c_{1}, ..., c_{t} \} \cup I \cup M \) where \( |I| = |I|, |M| = |M| \). Here \( I \) is in one-to-one correspondence \( i \) with the set of indexes of elements in \( I \), \( M \) is in one-to-one correspondence \( m \) with the set of indexes of elements in \( M \).

Let \( D_{W} \) be the base group in \( W = G \wr R \wr \text{gp}(b_{1}) \wr \text{gp}(b_{2}) \) and \( u^{(0)} \in D_{W} \). Denote a constant function with the value \( u \in G \). Let \( f(j) \in D_{W}, j = 1, ..., t \), where...
are the elements constructed in Lemma 9. When constructing \( d\) we use \( f(j), j = 1, \ldots, t; a_i^{(0)}, i \in I, \) and \( g_m^{(0)}, m \in M.\) All values of \( d\) belong to \( D_H.\)

Then we set

\[
d(c_j) = f(j), \quad j = 1, \ldots, t;
\]

\[
d(c_{i(i)}) = a_i^{(0)}, \quad i \in I;
\]

\[
d(c_{\mu(m)}) = g_m^{(0)}, \quad m \in M.
\]

and we set \( d(c^s) = 1\) in all other cases when \( s \not\in C_1.\)

Then \( G^{(0)} = \text{gp}(u_j^{(0)}, a_i^{(0)}, g_m^{(0)} : j = 1, \ldots, t; i \in I, m \in M) \simeq G.\) For any \( u^{(0)},\) one has \( u^{(0)}, b_2 = u^{(0)}, b_2 = 1.\) We note also that \( G^{(0)} = \text{gp}(u_j^{(0)}, a_i^{(0)}, g_m^{(0)} : j = 1, \ldots, t; i \in I, m, m' \in M)\)

For any \( h \in D, h(1)\) means \( h(1, 1, 1).\) We write \( \hat{h}(1)\) when all other values \( h(c^s, b_i^s, b_k^s)\) are trivial.

At first we prove by direct computation that \( H\) contains all elements of the form \( \tilde{g}_{m,m'}^{(0)}(1)\) (remind that \( c_i\) means \( c^{s_i}\))

\[
\tilde{g}_{m,m'}^{(0)}(1) = [d^{q_{u,m'}}, d^{q_{u,m'}}], \quad m, m' \in M.
\]

Similarly we get

\[
\hat{u}^{(0)}(1) = [d, b_2; j, b_1, t - j]^{c_j}, \quad j = 1, 2, \ldots, t.
\]

If \( |I| \geq 2,\) we cannot get \( \tilde{a}_i^{(0)}\) in the similar way. Instead we will use the following elements:

\[
\tilde{a}_i^{(0)}(1) = (d^{c(i)}, \quad i \in I).
\]

We have

\[
\tilde{G} = \text{gp}(\tilde{a}_i^{(0)}(1), \tilde{g}_{m,m'}^{(0)}(1), \tilde{a}_i^{(0)}(1) : j = 1, \ldots, t; m, m' \in M, i \in I) \simeq G.
\]

Let

\[
\tilde{G} = \text{gp}(\tilde{a}_i^{(0)}(1), \tilde{g}_{m,m'}^{(0)}(1), \tilde{a}_i^{(0)}(1) : j = 1, \ldots, t; m, m' \in M, i \in I) \leq H.
\]

There is a natural homomorphism (projection) \( \nu\) of \( \tilde{G}\) onto \( \tilde{G}.\) In fact, \( \nu\) is an isomorphism. Indeed, suppose that for some word \( z\) we have

\[
z(\tilde{a}_1^{(0)}(1), \ldots, \tilde{a}_k^{(0)}(1), \tilde{a}_1^{(0)}(1), \ldots, \tilde{a}_i^{(0)}(1), \tilde{g}_{m,m'}^{(0)}(1), \ldots, \tilde{g}_{m,m'}^{(0)}(1)) \in \ker(\mu).
\]

Since \( a_1, \ldots, a_k\) induce a part of base of the free abelian group \( G_{ab}\) every exponent sum \( \sigma_i\) of \( a_i^{(0)}(1), i = 1, \ldots, k,\) in \( z\) is 0. Then every other value corresponding to entries of \( \tilde{a}_i\) in \( z\) is trivial. Then \( g\) is independent of \( \tilde{a}_i(1)\) for each \( i \in I.\) Therefore, \( g\) has only trivial values outside of 1. It follows that \( \tilde{G} \simeq G.\)

Theorem is proved.

**Remark 10.** If the group \( G\) is finitely generated, then the proof of Theorem can be carried out without introducing elements of the form \( a_i, i \in I.\)
5. Proof of Theorem [3]

Let us see what can be said if a group $G$ is generated by a finite set of elements of finite orders.

Let $G$ be a group and $V = G \Wr \mathbb{Z}$ be a wreath product of $G$ and $\mathbb{Z} = \text{gp}(b)$. As usual $D_V$ means the base group of $V$.

**Lemma 11.** Let $u \in G$ be an element of a finite order $l$. Suppose, that every component of $d \in D_V$ belongs to $\text{gp}(u)$ and $d$ is a non-constant periodic function. This means that there is a number $r > 0$ (period) such that $d(b^{i+r}) = d(b^i)$ for all $i$. Let $d(1) = u^{s_i}, i_0 \in \mathbb{N}$.

Let $x \in D_V$ satisfy the condition $[x, b] = d$ and $x(1) = u^{i_0}$. Then $x$ is periodic with period $lr$.

**Proof.** Let $d = (... d_{-2}, d_{-1}, d_0, d_1, d_2, ...)$ and $x = (... x_{-2}, x_{-1}, x_0, x_1, x_2, ...)$, where $d_i = d(b^i)$ and $x_i = x(b^i)$. We find $x$ as in the proof of Lemma [7].

Then for any $i \geq 0$, by \[ u \]
\[ x_{i+lr} = \prod_{j=0}^{lr-1} u_{i_0} d_j = u_{i_0} \quad \text{and} \quad x_{i-lr} = \prod_{j=-1}^{-lr} u_{i_0} d_{j}^{-1} = u_{i_0}, \]

because, the product of all elements from $l$ periods is 1. Therefore $x(i + r) = x(i)$ for all $i$.

\[ \square \]

**Corollary 12.** Suppose that the conditions of Lemma [11] are satisfied. Let $V_{lr} = G \Wr \mathbb{Z}_{lr}$ be the wreath product of $G$ with the cyclic group $\mathbb{Z}_{lr} = \text{gp}(b_{lr})$ of order $lr$. By $D_{V_{lr}}$, we denote the base subgroup of $V_{lr}$. Since $d$ and $x$ are $lr$-periodic they can be considered as elements of $D_{V_{lr}}$. Then $d = [x, b_{lr}]$ in $V_{lr}$.

Let $u \in G, u \neq 1$ and $u^l = 1$. Then

\[ u^{(1)} = (... u^{-2}, u^{-1}, u^0, u^1, u^2, ...) \]

is obviously $l$-periodic. By Lemma [11], $u^{(2)}$ is $l^2$-periodic, and so on.

We can consider the elements $u^{(0)}, u^{(1)}, ..., u^{(t)}$ as elements of $D_{V_{lr}}$, the base subgroup of $V_{lr} = G \Wr \mathbb{Z}_{lr}$, where $\mathbb{Z}_{lr} = \text{gp}(b_{lr})$ is the cyclic group of order $l^t$.

It follows that we have the following finite analogue of Corollary [8].

**Corollary 13.** For each $u \in G, u^l = 1$, there are a series of elements $u^{(k)} \in D_{V_{lr}}, k = 1, 2, ..., t$ for which

\[ [u^{(k)}, b] = u^{(k-1)} \quad \text{and} \quad u^{(k)}(1) = u. \]

In particular,

\[ [u^{(k)}, b; k] = u^{(0)} \quad \text{and} \quad [u^{(i)}, b; k] = 1 \quad \text{for} \quad i < k. \]

**Lemma 14.** Let $u_1, ..., u_m \in G$ be elements of finite orders $l_1, ..., l_m$, respectively. Let $s = \text{lcm}(l_1^{m_1}, ..., l_m^{m_m})$. Let $V_s = G \Wr \mathbb{Z}_s$ be the wreath product of $G$ with the cyclic group $\mathbb{Z}_s = \text{gp}(b_s)$ of order $s$. By $D_{V_s}$ we denote the base subgroup of $V_s$. Then each of the elements $u_i^{(i)}$, $i = 1, ..., m$ that can be constructed by Corollary [13] can be considered as element of $D_{V_s}$. These elements have the following properties:

\[ [u_i^{(k)}, b_s] = u_i^{(k-1)}, \quad [u_i^{(k)}, b_s; k] = u_i^{(0)} \quad \text{for} \quad k = 1, ..., m \]
and

\[(5.4) \quad [u^{(i)}, b_s; k] = 1 \text{ for } i < k.\]

The following lemma is an analogue of Lemma 9.

**Lemma 15.** Let \(u_1, \ldots, u_m \in G\) be elements of finite orders \(l_1, \ldots, l_m\), respectively. Let \(s = \text{lcm}(l_1^n, \ldots, l_m^n)\). Let \(W_s = G \wr Z_s^2\) be the wreath product of \(G\) with the direct product \(\text{gp}(b_1, s) \times \text{gp}(b_2, s)\) of two cyclic groups of order \(s\) each. Let \(u_i^{(0)} \in D_{W_s}\) be a constant function with the value \(u_i\). Then there exist functions \(f_i \in D_{W_s}\), all of whose values belong to \(\text{gp}(u_i)\), which satisfy the following properties.

\[(5.5) \quad [f_i, b_2, s; i; b_1, s; t - i] = u_i^{(0)} \quad [f_j, b_2, s; i; b_1, s; t - i] = 1 \text{ for } i \neq j, \quad i, j \in \{1, \ldots, m\}.\]

The proof completely repeats the proof of Lemma 9, taking into account Lemma 14.

We proceed directly to the proof of Theorem 5. We keep the notation introduced above.

Now \(G\) is a group generated by a finite set \(u_1, \ldots, u_m\) of elements of finite orders \(l_1, \ldots, l_m\), respectively. We can assume that \(m \geq 2\). By Lemma 15, we construct the wreath product \(W_s = G \wr Z_s^2\) and elements \(f_i \in D_{W_s}, i = 1, \ldots, m\), which satisfy the equalities (5.5).

Let \(W_s = G \wr (Z_s \times Z_s^2)\), where \(Z_s = \text{gp}(c_s)\) and \(Z_s^2 = \text{gp}(b_1, s) \times \text{gp}(b_2, s)\). Let \(s_1 < s_2 < \ldots < s_m\) be a strictly uneven sparse sequence of positive integers such that \(s_{i+j} - s_j = s_{k+l} - s_l\) if and only if \(i = k\) and \(j = l\). For definiteness, we take \(s_i = 2^i\) for \(i = 1, \ldots, m\). Since \(s \geq 2^m\) this property is valid modulo \(s^2 \geq 2^{m+1}\).

The rest of the proof completely repeats the arguments of the proof of Theorem 3.

Theorem is proved.

Proof of Corollary 6. Now we can take in the proof of Theorem 5 instead of the active group \(Z_s \times Z_s^2\) in \(W_s\), the group \(Z_{s+1} \times Z_s^2\), and get the statement 1). The statement 2) follows directly from 1).

Corollary is proved.

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