THE CHAIN FOUNTAIN AGAIN

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ABSTRACT. We propose a simple model which possesses the chain fountain effect. The solution of corresponding equations is stable.

1. INTRODUCTION

This article is devoted to popular in the Internet effect. We start from description of the effect given in [1]:

"If a chain is initially at rest in a beaker at a height $h_1$ above the ground, and the end of the chain is pulled over the rim of the beaker and down towards the ground and then released, the chain will spontaneously flow out of the beaker under gravity. Furthermore, the beads do not simply drag over the edge of the beaker but form a fountain reaching a height $h_2$ above it."

Actually, article [1] does not contain a mathematical model which would imply the chain fountain effect.

A good way to explain what is going on is to obtain the fountain as a solution of the differential equations of the chain with prescribed boundary conditions. But these equations are very complicated indeed [2].

We propose very simple and rough model of the fountain.

2. THE MAIN MODEL

Our system (see Fig. 1) consists of the pile $A$, the chain which rises up from $A$ and runs to the pile $D$. The pile $D$ is also belonged to our system. The chain is undergone to the ideal constraint which makes the chain to have the semicircle shape $BC$ and two vertical intervals $AB$ and $CD$. The semicircle $BC$ can freely slide up and down along the axis $Oy.$
One can consider this constraint as a smooth weightless semicircle tube and this tube slides along a rail which coincides with \( Oy \). The chain runs inside the tube.

![Figure 1. The shape of the fountain.](image-url)

The radius \( r > 0 \) and the height difference \( h > 0 \) between \( A \) and \( D \) are the parameters of the system. By \( \rho \) denote the mass per unit length of the chain.

Thus we have a system with two degrees of freedom. The generalized coordinates are as follows: \( y \) is the ordinate of the centre \( E \) of the semicircle and \( s \) is the length of chain’s tail rests in the pile \( D \).

The velocities of chain’s links at the segments \([C, D]\) and \([A, B]\) are equal to

\[
\mathbf{v}_C = -s\mathbf{e}_y, \quad \mathbf{v}_B = (2\dot{y} + \dot{s})\mathbf{e}_y
\]

respectively.

3. Auxiliary problems

3.1. The chain is rising up. The force \( \mathbf{R} \) draws the chain up from the pile \( A \) by its tip \( K \). See Fig. 2.

\[
\mathbf{g} = -g\mathbf{e}_y, \quad \mathbf{R} = R\mathbf{e}_y, \quad R \geq 0, \quad \mathbf{v}_K = \dot{y}\mathbf{e}_y.
\]
The system is of one degree of freedom and it is characterised by the
generalised coordinate $y$ which is the altitude of the point $K$. We consider
the motion such that $\dot{y} > 0$.

**Hypothesis 1.** We assume that the chain’s velocity has a discontinuity
at the height $\varepsilon \ll 1$ from the bottom. (The solution to the corresponding
continuum media equations has the strong discontinuity.) This discontinuity
is inside the contour $Q$.

Upper the level $\varepsilon$ all the chain’s links have the same velocity $v_K$ and below
this level the velocity of the links is continuously decreases from $v = \lambda \dot{y} e_y$
just under the level $\varepsilon$ to zero at the pile $A$.

We assume that the positive parameter $\lambda$ is sufficiently small: $\lambda < 1/2$.

The chain meets the contour $Q$ with velocity $v$.

We also assume that the height $\varepsilon$ is very small so that we can neglect the
mass of the segment $[0, \varepsilon]$ and we neglect the force the segment influences to
the contour $Q$.

**Remark 1.** Another question is for which functions $v = \psi(v_K)$ does the
fountain effect exist? In this paper we use the simplest function: $\psi(\cdot) = \lambda \cdot$.

From hypothesis 1 and from 3 it follows that the density $\rho_*$ below the
discontinuity and the density $\rho$ above the discontinuity are related as follows

$$\rho_* \lambda \dot{y} = \rho \dot{y} \implies \rho_* = \rho / \lambda.$$ 

Applying the linear momentum balance equation for variable mass sys-
tems 4 to the content of the contour $Q$ we have

$$\frac{d}{dt} (\rho y \dot{y}) = R - y \rho g + \lambda ^2 \rho_* \dot{y}^2 = R - y \rho g + \lambda \rho \dot{y}^2.$$ 

This equation can be rewritten in the energy form
\[ \frac{d}{dt}(T + V) = (F_A + R, v_K), \quad T = \frac{1}{2}\rho y \ddot{y}^2, \quad V = \frac{1}{2}\rho g y^2, \quad (3.2) \]

Here
\[ F_A = -\alpha \rho y \ddot{y}^2 e_y, \quad \alpha = \frac{1}{2} - \lambda \in \left(0, \frac{1}{2}\right); \quad (3.3) \]

and \((\cdot, \cdot)\) stands for the standard inner product.

**Remark 2.** It is just for convenience we interpret formula (3.2) by means of the force \(F_A\). One can consider \(F_A\) as an artificial force.

Equation (3.1) can also be rewritten in the Lagrangian form
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = Q, \quad L = T - V \]

the generalised force \(Q\) is calculated as usual
\[ Q = \left( F_A, \frac{\partial v_K}{\partial \dot{y}} \right). \]

### 3.2. The chain is falling down.

In this section we assume that the force \(R\) is small such that the chain falls into the pile \(A\), \(\dot{y} < 0\), see Fig. 2 again.

We also employ the variable mass systems theory but with standard hypothesis: there is no discontinuities inside the contour \(Q\), the chain leaves it with the velocity \(v_K\) and stops instantly at the pile \(A\). The contour \(Q\) contains all the vertical part of the chain.

In this case the equation is as follows
\[ \rho y \ddot{y} = R - \rho g. \]

This equation can be rewritten in the form
\[ \frac{d}{dt}(T + V) = (R, v_K) + (G, v_K), \]

with
\[ G = \frac{1}{2}\rho \dot{y}^2 e_y, \quad (3.4) \]

or in the Lagrangian form
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = \dot{Q}, \quad \dot{Q} = \left( G + R, \frac{\partial v_K}{\partial \dot{y}} \right). \]

Note that remark 2 concerns also to the force \(G\).
4. THE EQUATIONS OF THE MAIN MODEL

The equation of the system, we started with, are built in the same manner as it is done in the previous section.

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = Q_s, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = Q_y.
\] (4.1)

Here \( L = T - V \); the kinetic energy \( T \) is calculated in usual way:

\[
T = \frac{1}{2} \rho \left[ (y + h)\dot{s}^2 + r\pi (\dot{s} + \dot{y})^2 + y(2\ddot{y} + \dot{s})^2 \right];
\]

and so the gravity force potential is

\[
V = \rho g(y^2 + r\pi y - hs).
\]

In accordance with the observation above and formulas (3.3), (3.4) we write

\[
F_A = -\alpha \rho |v_B|^2 e_y, \quad F_D = \frac{1}{2} \rho |v_C|^2 e_y,
\] (4.2)

recall that \( 0 < \alpha < 1/2 \).

The following generalized force corresponds to forces (4.2)

\[
Q_s = \left( F_A, \frac{\partial v_B}{\partial \dot{s}} \right) + \left( F_D, \frac{\partial v_C}{\partial \dot{s}} \right),
\]

\[
Q_y = \left( F_A, \frac{\partial v_B}{\partial \dot{y}} \right) + \left( F_D, \frac{\partial v_C}{\partial \dot{y}} \right).
\]

So that

\[
Q_s = -\rho \left( \alpha (2\dot{y} + \dot{s})^2 + \frac{1}{2} \dot{s}^2 \right), \quad Q_y = -2\alpha \rho (2\ddot{y} + \dot{s})^2.
\]

Now the Lagrange equations (4.1) are completely defined. This system has a stationary solution

\[
\dot{s}(t) = \sqrt{\frac{2gh}{2\alpha + 1}}, \quad y(t) = \frac{1}{2} \left( \beta h - \pi r \right), \quad \beta = \frac{2(1 - 2\alpha)}{2\alpha + 1} > 0.
\] (4.3)

This solution is physically reasonable (i.e. \( y(t) > 0 \)) provided \( \beta h > r\pi \). Thus to obtain the fountain effect one should choose the initial conditions and parameters \( h, r \) properly.

Solution (4.3) corresponds to the chain fountain; it has three negative characteristic exponents and one zero characteristic exponent.

The right-hand side of system (4.1) does not contain the term \( s(t) \) thus by the change \( p = \dot{s} \) this system is reduced to the third order system. Consequently solution (4.3) is exponentially stable with respect to the variables \( y, \dot{s}, \ddot{y} \), and therefore, it is Lyapunov stable with respect to the variables \( s, y, \dot{s}, \ddot{y} \).

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