WELL-POSEDNESS OF THE LINEARIZED PRANDTL EQUATION AROUND A NON-MONOTONIC SHEAR FLOW

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Abstract. In this paper, we prove the well-posedness of the linearized Prandtl equation around a non-monotonic shear flow in Gevrey class $2 - \theta$ for any $\theta > 0$. This result is almost optimal by the ill-posedness result proved by Gérard-Varet and Dormy, who construct a class of solution with the growth like $e^{\sqrt{k}t}$ for the linearized Prandtl equation around a non-monotonic shear flow.

1. Introduction

In this paper, we study the Prandtl equation in $\mathbb{R}_+ \times \mathbb{R}_+^2$

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial y^2} + \frac{\partial p}{\partial x} &= 0, \\
\frac{\partial x}{\partial u} + \frac{\partial y}{\partial v} &= 0, \\
u|_{y=0} = v|_{y=0} = 0 \quad \text{and} \quad \lim_{y \to +\infty} u(t, x, y) = U(t, x), \\
\end{align*}
\]

(1.1)

where $(u, v)$ denotes the tangential and normal velocity of the boundary layer flow, and $(U(t, x), p(t, x))$ is the values on the boundary of the tangential velocity and pressure of the outflow, which satisfies the Bernoulli’s law

\[
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + \frac{\partial p}{\partial x} = 0.
\]

This system introduced by Prandtl [15] is the foundation of the boundary layer theory. It describes the first order approximation of the velocity field near the boundary in the zero viscosity limit of the Navier-Stokes equations with non-slip boundary condition. One may check [14] for more introductions on the boundary layer theory.

To justify the zero viscosity limit, one of key step is to deal with the well-posedness of the Prandtl equation. Due to the lack of horizontal diffusion in (1.1), the nonlinear term $v \frac{\partial u}{\partial y}$ will lead to one horizontal derivative loss in the process of energy estimate. Up to now, the question of whether the Prandtl equation with general data is well-posed in Sobolev spaces is still open except for some special cases:

- Under a monotonic assumption on the tangential velocity of the outflow, Oleinik [14] proved the local existence and uniqueness of classical solutions to (1.1). With the additional favorable condition on the pressure, Xin and Zhang [17] obtained the global existence of weak solutions to (1.1).

- For the data which is analytic in $x, y$ variables, Sammartino and Caflisch [16] established the local well-posedness of (1.1). Later, the analyticity in $y$ variable was removed by Lombardo, Cannone and Sammartino [11]. Zhang and the third author [18] also established the long time well-posedness of (1.1) for small tangential analytic data.

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Recently, Alexandre et al. [1] and Masmoudi and Wong [12] independently develop direct energy method to prove the well-posedness of the Prandtl equation for monotonic data in Sobolev spaces. Their works might shed some light on the zero viscosity limit problem in Sobolev spaces. See also [9] for the case with multiple monotonicity regions. Recently, we also present an elementary proof by using the paralinearization technique [3].

On the other hand, Gérard-Varet and Dormy [5] proved the ill-posedness in Sobolev spaces for the linearized Prandtl equation around non-monotonic shear flows. The nonlinear ill-posedness was also established in [7, 8] in the sense of non-Lipschitz continuity of the flow. However, Gérard-Varet and Masmoudi [6] can prove the well-posedness of the Prandtl equation (1.1) for a class of data in Gevrey class $\frac{7}{4}$. In [6], the authors conjectured that their result should not be optimal. The analysis and numerics performed in [5] suggest that the optimal exponent may be $s = 2$. Indeed, Gérard-Varet and Dormy constructed a class of solution with the growth like $e^{\sqrt{k}t}$ for the linearized Prandtl equation around a non-monotonic shear flow, where $k$ is the tangential frequency.

The goal of this paper is to prove the well-posedness of the linearized Prandtl equation around non-monotonic shear flows. The nonlinear ill-posedness was also established in [7, 8] in the sense of non-Lipschitz continuity of the flow. However, Gérard-Varet and Masmoudi [6] can prove the well-posedness of the Prandtl equation (1.1) for a class of data in Gevrey class $\frac{7}{4}$. In [6], the authors conjectured that their result should not be optimal. The analysis and numerics performed in [5] suggest that the optimal exponent may be $s = 2$. Indeed, Gérard-Varet and Dormy constructed a class of solution with the growth like $e^{\sqrt{k}t}$ for the linearized Prandtl equation around a non-monotonic shear flow, where $k$ is the tangential frequency.

The main result of this paper is stated as follows.

**Theorem 1.1.** Let $\theta \in (0, \frac{1}{2}]$. Assume that $e^{(D_x)^{\frac{3}{2} + 2\theta}} u_0 \in H^{\frac{4}{3} + \theta, 1}_\mu$ with $\partial_y u_0|_{y=0} = 0$ for $k = 0, 2$. Then there exists $T > 0$ so that (1.4) has a unique solution $u$ in $[0, T]$, which satisfies (7.2). In particular, we have

$$u_\Phi \in L^\infty(0, T; H^{\frac{4}{3} + \theta, 1}_\mu).$$

Here we denote

$$f_\Phi \triangleq F^{-1}(e^{\Phi(t, \xi)} \widehat{f}(\xi)), \quad \Phi(t, \xi) \triangleq (1 - \lambda t) \langle \xi \rangle^{\frac{3}{2} + 2\theta},$$
and $H_\mu^\sigma$ is the weighted Sobolev space with $\mu = e^{\frac{y}{2}}$ which will be introduced later.

**Remark 1.1.** Li and Yang [10] proved the well-posedness of nonlinear Prandtl equation in Gevrey class 2 for the data with non-degenerate critical point and polynomial decay in $y$. They used Gérard-Varet and Masmoudi’s framework with an introduction of a new unknown $h_1 = \partial_y^2 u - \frac{\partial_y u^s}{\partial_y u^s} \partial_y u$, which is used to control the regularity of $\partial_y^2 u$. In the last section, we will explain how to obtain the well-posedness of (1.4) in Gevrey class 2 by using our framework and $h_1$. Two methods should be helpful to understand the complex structure of the Prandtl equation and provide evidence about the conjecture that the well-posedness in Gevrey class 2 is optimal.

Let us present some key ingredients of our proof.

1. Gevrey regularity estimate in monotonic domain. Motivated by [1], we will introduce the good unknown $w_1 = \partial_y \left( \frac{u}{\partial_y u^s} \right)$ to control the horizontal regularity of the solution in this domain, which satisfies

$$\partial_t w_1 + u^s \partial_x w_1 - \partial_y^2 w_1 = \partial_y F_1.$$ 

The key point is that the equation of $w_1$ does not lose the derivative.

2. Gevrey regularity estimate in non-monotonic domain. Because $w_1$ does not make sense in non-monotonic domain, motivated by [6], we introduce $h = d \partial_y h, d = (\partial_y^2 u^s)^{\frac{1}{2}}$ to control the horizontal regularity of the solution in this domain, which satisfies

$$\partial_t h + u^s \partial_x h - \partial_y^2 h + d(v \partial_y^2 u^s) = (\partial_t d - \partial_y^2 d) \partial_y u - 2 \partial_y d \partial_y^2 u.$$ 

All the terms in this equation are good except $d(v \partial_y^2 u^s)$. The key point is

$$\int_{\mathbb{R}^2} d(v \partial_y^2 u^s) h dx dy = 0. \quad (1.5)$$

So, this term is also good in the energy estimate. However, the localization in $y$ variable will destroy the cancellation structure (1.5). In particular, the energy estimate in non-monotonic domain will give rise to a new trouble term

$$\int_{\mathbb{R}^2} d(v \partial_y^2 u^s) h dx dy = 0. \quad (1.6)$$

which can be reduced to control the terms like $(w_1, \partial_y u)_{L^2}, i = 1, 2$ modulus some lower order terms. Here $w_2 = \partial_y u^s \partial_y u - \partial_y^2 u^s u$ and $\phi_3(y)$ is a cut-off function supported in non-monotonic domain. To control them, we need to use the Gevrey regularity and the following

3. Anisotropic regularity estimates. The unknowns $w_1$ and $h$ have to work in the functional spaces with different horizontal regularity. Roughly speaking,

$$\phi_3(y) h(x, y) \in L^2(0, T; H^\frac{1}{2}, 0), \quad \phi_1(y)(w_1) \in L^2(0, T; H^\frac{1}{2}, 0).$$

Here $\phi_1(y)$ is a cut-off function supported in monotonic domain.

4. The derivative gain of $w_1$ can be easily obtained by using Gevrey regularity and good structure of $w_1$. The unknown $w_2 = \partial_y u^s \partial_y u - \partial_y^2 u^s u$ satisfies an equation similar to $w_1$, but with a key trouble term in the $H^\frac{1}{2}, 0$ energy estimate, which takes

$$\left( \phi_3(y) \partial_y^2 u^s \partial_y^2 u, \phi_3(y) (w_2) \right)_{H^\frac{1}{2}, 0}.$$
The main difficulty is that one cannot deduce \( \phi_3(y) \partial_y^2 u_\Phi \in L^2(0, T; H^{3,0}) \) from \( \phi_3(y) h_\Phi \in L^2(0, T; H^{3,1}) \). However, one can prove \( \phi_3(y) w_2 \in L^2(0, T; H^{3,0}) \) by using some key structures found in [6]. As we said above, this estimate is not enough to handle (1.6). On the other hand, one can prove the same regularity as \( w_1 \) in the framework of Gevrey class \( \mathcal{F}_r \). This may be the main reason why the work [6] can achieve the well-posedness in Gevrey class \( \mathcal{F}_r \).

5. Improved regularity estimate of \( w_2 \). Compared with \( w_1, w_2 \) lose \( \frac{1}{8} \)-order derivative. However, we find that \( \varphi^{1+\theta_1} \langle D_x \rangle^{\frac{3}{4}+\theta_2} (w_2)_\Phi \in L^2(0, T; L^2) \), where \( \varphi \) is a cut-off function vanishing at critical point. Compared with the work [6], this weighted estimate is completely new, and moreover is enough to handle (1.6). The price to pay is to use Gevrey \( 2-\theta \) regularity.

6. In our framework, if we use the unknown \( h_1 \), we can easily deduce \( \phi_3(y) \partial_y^2 u_\Phi \in L^2(0, T; H^{3,0}) \), thus \( \phi_3(y) (w_2)_\Phi \in L^2(0, T; H^{3,0}) \) and avoid the Gevrey regularity loss. This will be explained in the last section.

Let us conclude the introduction with the following notations. Let \( \omega(y) \) be a nonnegative function in \( \mathbb{R}^+ \). We introduce the weighted \( L^p \) norm

\[
\|f\|_{L^p_\omega} \overset{\text{def}}{=} \|\omega(y) f(x, y)\|_{L^p}, \quad \|f\|_{L^p_{\omega,\omega}} \overset{\text{def}}{=} \|\omega(y) f(y)\|_{L^p}.
\]

The weighted anisotropic Sobolev space \( H^{s,\ell}_{\omega,\omega} \) for \( s = k + \sigma \) and \( k, \ell \in \mathbb{N}, \sigma \in [0, 1) \) consists of all functions \( f \in L^2_{\omega,\omega} \) satisfying

\[
\|f\|^2_{H^{s,\ell}_{\omega,\omega}} \overset{\text{def}}{=} \sum_{\alpha \leq k, \beta \leq \ell} \|\partial_x^\alpha (D_x)^\sigma \partial_y^\beta f\|^2_{L^2_{\omega,\omega}} < +\infty.
\]

We denote by \( H^{k,\ell}_{\omega,\omega} \) the weighted Sobolev space in \( \mathbb{R}^+ \), which consists of all functions \( f \in L^2_{\omega,\omega} \) satisfying

\[
\|f\|^2_{H^{k,\ell}_{\omega,\omega}} \overset{\text{def}}{=} \sum_{\beta \leq \ell} \|\partial_y^\beta f\|_{L^2_{\omega,\omega}}^2 < +\infty.
\]

In the case when \( \omega = 1 \), we denote \( H^{k,\ell}_{\omega} \) by \( H^{k,\ell} \), and \( H^{k,\ell}_{\omega,\omega} \) by \( H^{k,\ell}_y \) for the simplicity.

2. Basic estimates for the shear flow

Let \( u^s(t, y) \) be the solution of the heat equation

\[
\begin{cases}
\partial_t u^s - \partial_y^2 u^s = 0, \\
u^s|_{y=0} = 0 \quad \text{and} \quad \lim_{y \to +\infty} u^s(t, y) = 1, \\
u^s|_{t=0} = u^s_0(y).
\end{cases}
\]

\( (2.1) \)

**Proposition 2.1.** Assume that \( \partial_y u^s_0 \in H^{3}_{y,\mu} \) and \( u^s_0(0) = 0, \partial_y^2 u^s_0(0) = 0 \). Then it holds that for any \( t \in [0, +\infty) \),

\[
E^s(t) \overset{\text{def}}{=} \|\partial_y u^s(t)\|^2_{H^{3}_{y,\mu}} + \int_0^t \|\partial_y u^s(\tau)\|^2_{H^{3}_{y,\mu}} d\tau \leq \|\partial_y u^s_0\|^2_{H^{3}_{y,\mu}} e^{Ct}.
\]

Moreover, if for \( k = 0, 1, 2, 3 \),

\[
|\partial_y^k (u^s_0 - 1)(y)| \leq c^{-1} e^{-y} \quad \text{for} \ y \in [0, +\infty),
\]
then we have
\[ |\partial^k_y (u^s(t, y) - 1)| \leq C e^{-y}, \]
for \((t, y) \in [0, 1] \times [0, +\infty)\).

Proof. Taking \(L_{y,\mu}^2\) inner product between the first equation of (2.1) and \(u^s_t\), we obtain
\[
\frac{d}{dt} \|\partial_y u^s\|^2_{L_{y,\mu}^2} + \|u^s_t\|^2_{L_{y,\mu}^2} \leq C \|\partial_y u^s\|^2_{L_{y,\mu}^2}.
\]
Taking the time derivative to the first equation of (2.1), then taking \(L_{y,\mu}^2\) inner product between the resulting equation and \(u^s_t\), we get
\[
\frac{d}{dt} \|u^s_t\|^2_{L_{y,\mu}^2} + \|\partial_y u^s_t\|^2_{L_{y,\mu}^2} \leq C \|u^s_t\|^2_{L_{y,\mu}^2}.
\]
And taking \(L_{y,\mu}^2\) inner product between the resulting equation and \(\partial^2_y u^s\), we get
\[
\frac{d}{dt} \|\partial^2_y u^s\|^2_{L_{y,\mu}^2} + \|\partial^2_y u^s_t\|^2_{L_{y,\mu}^2} \leq C \|\partial^2_y u^s\|^2_{L_{y,\mu}^2} + \frac{1}{2} \|\partial^2_y u^s_t\|^2_{L_{y,\mu}^2}.
\]
Taking the \(\partial_t \partial_y\) to the first equation of (2.1), then taking \(L_{y,\mu}^2\) inner product between the resulting equation and \(\partial^2_y u^s\), we deduce that
\[
\frac{d}{dt} \|\partial^2_y u^s\|^2_{L_{y,\mu}^2} + \|\partial^2_y u^s_t\|^2_{L_{y,\mu}^2} \leq C \|\partial^2_y u^s\|^2_{L_{y,\mu}^2} + \frac{1}{2} \|\partial^2_y u^s_t\|^2_{L_{y,\mu}^2}.
\]
Using \(\partial_t u^s = \partial^2_y u^s\), we deduce from Gronwall’s inequality that
\[
\|\partial_y u^s(t)\|^2_{L_{y,\mu}^2} + \|u^s_t(t)\|^2_{L_{y,\mu}^2} + \|\partial_t \partial_y u^s\|^2_{L_{y,\mu}^2} + \|\partial^2_y \partial_t u^s\|^2_{L_{y,\mu}^2} \leq \|\partial_y u^s_0\|^2_{H_{y,\mu}} e^{Ct},
\]
from which and \(\partial_t u^s = \partial^2_y u^s\), it follows that
\[
E^s(t) \leq \|\partial_y u^s_0\|^2_{H_{y,\mu}} e^{Ct}.
\]

For the pointwise estimates, we need to use the representation formula of the solution
\[
u^s(t, y) = \frac{1}{2\sqrt{\pi t}} \int_{0}^{+\infty} \left( e^{-\frac{(y-y')^2}{4t}} - e^{-\frac{(y+y')^2}{4t}} \right) u^s_0(y') dy'.
\]
We write
\[
u^s(t, y) - 1 = -\frac{1}{2\sqrt{\pi t}} \int_{0}^{+\infty} e^{-\frac{(y+y')^2}{4t}} u^s_0(y') dy' + \frac{1}{2\sqrt{\pi t}} \int_{0}^{+\infty} e^{-\frac{(y-y')^2}{4t}} (u^s_0(y') - 1) dy' + \left( \frac{1}{2\sqrt{\pi t}} \int_{0}^{+\infty} e^{-\frac{(y-y')^2}{4t}} dy' - 1 \right)
\triangleq I_1 + I_2 + I_3.
\]
The result is obvious for \(|y| \leq 4\). So, we assume \(y \geq 4 \geq 4t\). Thanks to \(|u^s_0(y)| \leq C\), it follows that
\[
|I_1| \leq \frac{1}{2\sqrt{\pi t}} \int_{0}^{+\infty} e^{-\frac{y^2}{4t}} e^{-2\pi t (y')^2} |u^s_0(y')| dy' \leq \frac{1}{2\sqrt{\pi t}} e^{-\frac{y^2}{4t}} \int_{0}^{+\infty} e^{-\frac{(y')^2}{4t}} |u^s_0(y')| dy' \leq C e^{-y},
\]

Thanks to $|u_0^s(y) - 1| \leq e^{-y}$, we infer that

$$|I_2| \leq e^{-y} \frac{1}{2\sqrt{\pi t}} \int_{0}^{+\infty} e^{-\frac{(y-y')^2}{4t}} e^{y-y'} |u_0^s(y') - 1| e^{y'} dy'$$

$$\leq Ce^{-y} \frac{1}{2\sqrt{\pi t}} \int_{0}^{+\infty} e^{-\frac{y'^2}{4t}} e^{y'} dy'$$

$$\leq Ce^{-y} e^t \int_{0}^{+\infty} e^{-(\xi - \sqrt{t})^2} d\xi$$

$$\leq Ce^{-y}.$$ For $I_3$, we have

$$|I_3| \leq \frac{1}{\sqrt{\pi t}} \int_{\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2} d\xi - 1 = \frac{1}{\sqrt{\pi}} \int_{\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2} d\xi.$$ If $2\sqrt{t} \leq 1$, then

$$|I_3| \leq C \int_{\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi} d\xi \leq Ce^{-\frac{y}{2\sqrt{t}}} \leq Ce^{-y},$$

and if $2\sqrt{t} \geq 1$ and $y \geq 4t$, then

$$|I_3| \leq C \int_{\frac{y}{2\sqrt{t}}}^{+\infty} e^{-2\sqrt{t}\xi} d\xi \leq \frac{C}{2\sqrt{t}} e^{-y} \leq Ce^{-y}.$$ Putting the estimates of $I_1 - I_3$ together, we deduce that

$$|u^s(t, y) - 1| \leq Ce^{-y}.$$ Thanks to $u_0^s(0) = 0$ and $\partial_y^2 u_0^s(0) = 0$, we get by integration by parts that

$$\partial_y u^s(t, y) = \frac{1}{2\sqrt{\pi t}} \int_{0}^{+\infty} \left( e^{-\frac{(y-y')^2}{4t}} + e^{-\frac{(y+y')^2}{4t}} \right) \partial_y u_0^s(y') dy',$$

$$\partial_y^2 u^s(t, y) = \frac{1}{2\sqrt{\pi t}} \int_{0}^{+\infty} \left( e^{-\frac{(y-y')^2}{4t}} - e^{-\frac{(y+y')^2}{4t}} \right) \partial_y^2 u_0^s(y') dy',$$

$$\partial_y^3 u^s(t, y) = \frac{1}{2\sqrt{\pi t}} \int_{0}^{+\infty} \left( e^{-\frac{(y-y')^2}{4t}} + e^{-\frac{(y+y')^2}{4t}} \right) \partial_y^3 u_0^s(y') dy'.$$

Then in the same derivation as in $I_2$, we have for $k = 1, 2, 3$,

$$|\partial_y^k u^s(t, y)| \leq Ce^{-y} \text{ for } (t, y) \in [0, 1] \times [0, +\infty).$$

This finishes the proof of the proposition. \qed

**Lemma 2.2.** Let $u_0^s(y)$ be as in Proposition 2.1. If $u_0^s(y)$ satisfies (1.3), then there exists $T_1 > 0$ so that for any $t \in [0, T_1]$,

$$\partial_y^2 u^s(t, y) \geq \frac{c}{2} \text{ for } y \in \left[\frac{1}{2}, 2\right],$$

$$\partial_y u^s(t, y) \geq \frac{c}{2} e^{\delta t} \text{ for } y \in [0, 1 - \delta] \cup [1 + \delta, +\infty).$$

**Proof.** We have

$$\partial_y u^s(t, y) = \partial_y u_0^s(y) + \int_{0}^{t} \partial_t \partial_y u^s(\tau, y) d\tau,$$
\[ \partial_y^2 u^s(t, y) = \partial_y^2 u_0^s(y) + \int_0^t \partial_t \partial_y^2 u^s(\tau, y) d\tau. \]

Notice that
\[ \left| \int_0^t \partial_t \partial_y u^s(\tau, y) d\tau \right| \leq Ct^{\frac{1}{2}} \| \partial_y^3 u^s \|_{L^2_{y} H^1_y}, \]
\[ \left| \int_0^t \partial_t \partial_y^2 u^s(\tau, y) d\tau \right| \leq Ct^{\frac{1}{2}} \| \partial_y^4 u^s \|_{L^2_{y} H^1_y}. \]

Then the lemma follows from Proposition 2.1 and (1.3).

\[ \square \]

3. Introduction of good unknowns

An essential difficulty solving the Prandtl equations is the loss of one derivative in the horizontal direction \( x \) induced by the term \( v \partial_y u^s \). To eliminate the trouble term \( \partial_y u^s v \) in (1.4), it is natural to introduce a good unknown \( w_1 \) defined by

\[ w_1 \overset{\text{def}}{=} \partial_y \left( -\frac{u}{\partial_y u^s} \right), \]

which is motivated by the work \[1\]. Then a direct calculation gives

\[ \begin{align*}
\partial_t w_1 + u^s \partial_x w_1 - \partial_y^2 w_1 &= \partial_y F_1, \\
\partial_y w_1 |_{y=0} &= 0 \quad \text{and} \quad \lim_{y \to +\infty} w_1 = 0, \\
w_1 |_{t=0} &= w_0(x, y),
\end{align*} \]

where \( F_1 \) is given by

\[ F_1 = u \partial_t \left( \frac{1}{\partial_y u^s} \right) - \left[ \partial_y^2, \frac{1}{\partial_y u^s} \right] u. \]

Here we used the fact that \( \partial_y^2 u = 0 \) on \( y = 0 \), which can be seen from (1.4).

Notice that \( w_1 \) is only well-defined in the monotonic domain. While, Lemma 2.2 tells us

\[ |\partial_y u^s(t, y)| \geq \frac{c\delta}{2} e^{-y} \quad \text{for} \quad (t, y) \in [0, T_1] \times ([0, 1 - \delta] \cup [1 + \delta, +\infty)). \]

Then it is natural to introduce a cut-off good known

\[ \omega_1 = e^{-\frac{y}{\delta}} \phi_1(y) \partial_y \left( \frac{u}{\partial_y u^s} \right) = \psi_1(y) \partial_y \left( \frac{u}{\partial_y u^s} \right), \]

where \( \phi_1(y) \in C^\infty(\mathbb{R}_+) \) with the support included in \([0, 1 - \delta] \cup [1 + \delta, +\infty)\) and \( \phi_1(y) = 1 \) in \([0, 1 - 2\delta] \cup [1 + 2\delta, +\infty] \). A direct calculation shows

\[ \begin{align*}
\partial_t \omega_1 + u^s \partial_x \omega_1 - \partial_y^2 \omega_1 &= [\psi_1(y), \partial_y^2] w_1 + \psi_1(y) \partial_y F_1.
\end{align*} \]

To control the regularity of the solution in the non-monotonic domain, we need to use the non-degenerate condition

\[ |\partial_y^2 u^s(t, y)| \geq \frac{c}{2} \quad \text{for} \quad (t, y) \in [0, T_1] \times \left[ \frac{1}{2}, 2 \right]. \]

Motivated by \[6\], we introduce a good unknown \( h \) defined by

\[ h \overset{\text{def}}{=} d\partial_y u, \]
where \( d(t, y) = \phi_3(y)(\partial_y u^s)^{-1/2} \) and \( \phi_3(y) \) is a cut-off function supported in \( [\frac{1}{2}, 2] \) and \( \phi_3(y) = 1 \) as \( y \in [\frac{1}{2}, 2] \). Then \( h \) satisfies
\[
(3.5) \quad \partial_t h + u^s \partial_y h - \partial_y^2 h + d(v \partial_y^2 u^s) = (\partial_t d - \partial_y^2 d) \partial_y u - 2\partial_y d \partial_y^2 u.
\]

To propagate the regularity of the solution from monotonic domain to non-monotonic domain, we need to introduce another good unknown \( \varpi_2 \)
\[
(3.6) \quad \varpi_2 \overset{\text{def}}{=} \psi_2(y)(\partial_y u^s \partial_y u - u \partial_y^2 u^s) \triangleq \psi_2(y)w_2,
\]
where \( \psi_2(y) \in C_0^\infty(\mathbb{R}_+) \) with the support included in \( [1 - 3\delta, 1 + 3\delta] \) and \( \psi_2(y) = 1 \) in \( [1 - 2\delta, 1 + 2\delta] \). It is easy to check that
\[
(3.7) \quad \partial_t \varpi_2 + u^s \partial_x \varpi_2 - \partial_y^2 \varpi_2 = [\psi_2(y), \partial_y^2]w_2 + \psi_2(y)w_2,
\]
where
\[
F_2 = \partial_t \partial_y u^s \partial_y u + [\partial_y u^s, \partial_y^2] \partial_y u - u \partial_t \partial_y^2 u^s - [\partial_y^2 u^s, \partial_y^2]u.
\]
In fact, \( w_1 \) and \( w_2 \) are basically equivalent in the monotonic domain by the relation
\[
w_2 = (\partial_y u^s)^2 w_1.
\]
This in particular implies that

**Lemma 3.1.** It holds that
\[
\|1_{I_1}(y)(w_1)\|_{H^{\frac{1}{2}, 0}} \leq C\|\varpi_2\|_{H^{\frac{1}{2}, 0}},
\]
\[
\|1_{I_2}(y)(w_2)\|_{H^{\frac{1}{2}, 0}} \leq C\|\varpi_2\|_{H^{\frac{1}{2}, 0}},
\]
where \( I_1 = \text{supp} \phi_1' \) and \( I_2 = \text{supp} \varphi_2' \).

Let \( a(t) \) be a critical point of \( u^s(t, y) \), i.e.,
\[
\partial_y u^s(t, a(t)) = 0.
\]
Therefore, \( a(t) \) satisfies
\[
\partial_t a(t) = -\frac{\partial_t \partial_y u^s(t, a)}{\partial_y^2 u^s(t, a)}, \quad a(0) = 1.
\]
By Proposition 2.1, there exists \( T_2 > 0 \) so that
\[
(3.8) \quad |a(t) - 1| \leq 2\delta \quad \text{for} \quad t \in [0, T_2].
\]
Then \( u \) can be represented in terms of \( w_1, w_2 \). More precisely,

**Lemma 3.2.** We can decompose \( u \) as \( u = u_1 + u_2 \), where

\[
u_1 = \begin{cases}
\partial_y u^s \int_0^y \phi_1 w_1 dy' & \text{for} \quad y < 1 - 2\delta, \\
\partial_y u^s \left( \int_0^{1-2\delta} \phi_1 w_1 dy' + \int_{1-2\delta}^y \frac{\varpi_2}{(\partial_y^2 u^s)^2} dy' \right) & \text{for} \quad 1 - 2\delta \leq y < a(t), \\
\partial_y u^s \left( \int_0^y \frac{\varpi_2}{(\partial_y^2 u^s)^2} dy' + \int_{a(t)}^{1+2\delta} \phi_1 w_1 dy' \right) & \text{for} \quad a(t) < y < 1 + 2\delta, \\
\partial_y u^s \int_{a(t)}^y \phi_1 w_1 dy' & \text{for} \quad y \geq 1 + 2\delta,
\end{cases}
\]
4. Gevrey regularity estimate of \( \overline{w}_1 \)

In what follows, let us always assume that \( T \leq \min(T_1, T_2) \).

**Proposition 4.1.** Let \( \overline{w}_1 \) be a smooth solution of (3.3) in \([0, T]\). Then it holds that for any \( t \in [0, T] \),

\[
\frac{d}{dt} \| (\overline{w}_1) \phi \|_{H^{\frac{1}{2}, 0}}^2 + (\lambda - C) \| (\overline{w}_1) \phi \|_{H^{\frac{1}{2}, 0}}^2 + \| \partial_y (\overline{w}_1) \phi \|_{H^{\frac{1}{2}, 0}}^2 \\
\leq C \left( \| u \phi \|_{H^{\frac{1}{2}, 1}}^2 + \| (\overline{w}_1) \phi \|_{H^{\frac{1}{2}, 1}} + \| (\overline{w}_2) \phi \|_{H^{\frac{1}{2}, 0}} \right).
\]

Let us begin with the estimates of source term \( F_1 \).

**Lemma 4.2.** It holds that

\[
\| \psi_1(y) \partial_y (F_1) \phi \|_{H^{\frac{1}{2}, 0}} \leq C \| u \phi \|_{H^{\frac{1}{2}, 1}} + \| (\overline{w}_1) \phi \|_{H^{\frac{1}{2}, 1}} + \| (\overline{w}_2) \phi \|_{H^{\frac{1}{2}, 0}}.
\]

**Proof.** An easy calculation gives

\[
F_1 = -2 \left( \partial_y^2 u^s \right)^2 u + 2 \partial_y^2 u^s \partial_y u.
\]

By (3.2), we get

\[
\| \psi_1(y) \partial_y (F_1) \phi \|_{H^{\frac{1}{2}, 0}} \leq C \| u \phi \|_{H^{\frac{1}{2}, 1}} + C \| \psi_1(y) \partial_y^2 u^s \phi \|_{H^{\frac{1}{2}, 0}}.
\]

Notice that

\[
\psi_1(y) \partial_y^2 u = \partial_y u \left( \partial_y w_1 - \psi_1 w_1 + 2 \psi_1 \partial_y u \partial_y u + \psi_2 \partial_y \left( \partial_y^2 u \right) \right),
\]

which along with Lemma 2.2 implies that

\[
\| \psi_1(y) \partial_y^2 u \phi \|_{H^{\frac{1}{2}, 0}} \leq C \left( \| u \phi \|_{H^{\frac{1}{2}, 1}} + \| (\overline{w}_1) \phi \|_{H^{\frac{1}{2}, 1}} + \| (\overline{w}_2) \phi \|_{H^{\frac{1}{2}, 0}} \right).
\]

Putting the above estimates together, we conclude the lemma. \(\square\)

Now we are in position to prove proposition 4.1.

**Proof.** Applying \( e^{\Phi(t, D_x)} \) to (3.3), we obtain

\[
\partial_t (\overline{w}_1) \phi + \lambda (D_x)^{\frac{1}{2} + 2\theta} (\overline{w}_1) \phi + u^s \partial_x (\overline{w}_1) \phi = \partial_y^2 (\overline{w}_1) \phi
\]

(4.1)

Making \( H^{\frac{1}{2}, 0} \) energy estimate to (4.1), we obtain

\[
\frac{1}{2} \frac{d}{dt} \| (\overline{w}_1) \phi \|_{H^{\frac{1}{2}, 0}}^2 + \lambda \| (\overline{w}_1) \phi \|_{H^{\frac{1}{2}, 0}}^2 - \| \partial_y^2 (\overline{w}_1) \phi, (\overline{w}_1) \phi \|_{H^{\frac{1}{2}, 0}} + (u^s \partial_x (\overline{w}_1) \phi, (\overline{w}_1) \phi)_{H^{\frac{1}{2}, 0}}
\]

\[
= \left[ \psi_1(y), \partial_y^2 (w_1) \phi + \psi_1(y) \partial_y (F_1) \phi \right]_{H^{\frac{1}{2}, 0}}.
\]

Thanks to \( \partial_y (\overline{w}_1) \phi \big|_{y=0} = 0 \), we get by integration by parts that

\[
- \| \partial_y (w_1) \phi \|_{H^{\frac{1}{2}, 0}}^2 = \| (\partial_y (w_1) \phi \big|_{H^{\frac{1}{2}, 0}}, (w_1) \phi \big|_{H^{\frac{1}{2}, 0}} = 0.
\]
Similarly, we have
\[
\left(\psi_1(y), \partial_y^2 (w_1), (\overline{w}_1) \phi \right)_{H^t, \omega} \\
\leq 2 \left| \psi'_1 (w_1) \phi, \partial_y (\overline{w}_1) \phi \right|_{H^{t, \omega}} + 2 \left| \psi''_1 (w_1) \phi, (\overline{w}_1) \phi \right|_{H^{t, \omega}} \\
\leq C \left( \| \psi_1 \phi \|_{H^{t, \omega}} \right)^2 + \| \partial_y (\overline{w}_1) \phi \|_{H^{t, \omega}} + \| \overline{w}_1 \phi \|_{H^{t, \omega}} \\
\leq C \left( \left( \| \psi_1 \phi \|_{H^{t, \omega}} \right)^2 \right) + \left( \| \partial_y (\overline{w}_1) \phi \|_{H^{t, \omega}} + \| \overline{w}_1 \phi \|_{H^{t, \omega}} \right). \\
\]

It follows from Lemma 4.2 that
\[
\left(\psi_1(y) \partial_y (F_1), (\overline{w}_1) \phi \right)_{H^{t, \omega}} \leq C \left( \| u \phi \|_{H^{t, 1}}^2 + \| \overline{w}_1 \phi \|_{H^{t, \omega}}^2 + \| \overline{w}_2 \phi \|_{H^{t, \omega}}^2 \right) \\\n+ \frac{1}{8} \| \partial_y (\overline{w}_1) \phi \|_{H^{t, \omega}}^2 + C \left( \| \overline{w}_1 \phi \|_{H^{t, \omega}}^2 \right). \\
\]

Summing up all the estimates, we conclude the proposition. \[\square\]

5. Gevrey Regularity Estimate of $\overline{w}_2$

First of all, we prove Gevrey regularity without weight.

**Proposition 5.1.** Let $\overline{w}_2$ be a solution of (3.7) in $[0,T]$. There exists $\delta > 0$ small enough so that for any $t \in [0,T]$, 
\[
\frac{d}{dt} \left( \| \overline{w}_2 \phi \|_{H^t, \omega}^2 \right) + (\lambda - C) \left( \| \overline{w}_2 \phi \|_{H^t, \omega}^2 \right) + \| \partial_y (\overline{w}_2) \phi \|_{H^t, \omega}^2 \leq C \left( \| u \phi \|_{H^{t, 1}}^2 + \| \overline{w}_1 \phi \|_{H^{t, \omega}}^2 + \| \overline{w}_2 \phi \|_{H^{t, \omega}}^2 \right). \\
\]

The proposition can be proved by following the proof of Proposition 4.1 and using the following lemma.

**Lemma 5.2.** It holds that
\[
\left(\psi_2(y) (F_2), (\overline{w}_2) \phi \right)_{H^{t, \omega}} \leq C \left( \| u \phi \|_{H^{t, 1}}^2 + \| \overline{w}_1 \phi \|_{H^{t, \omega}}^2 + \| \overline{w}_2 \phi \|_{H^{t, \omega}}^2 \right) \\\n+ \frac{1}{2} \| \partial_y (\overline{w}_2) \phi \|_{H^{t, \omega}}^2. \\
\]

**Proof.** Notice that
\[
(5.1) \quad -u \partial_t \partial^2_y u^s - [\partial^2_y u^s, \partial^2_y] u = 2 \partial^3_y u^s \partial_y u, \\
\]

therefore,
\[
\left(\psi_2(u \partial_t \partial^2_y u^s + [\partial^2_y u^s, \partial^2_y] u) \phi, (\overline{w}_2) \phi \right)_{H^{t, \omega}} \leq C \| u \phi \|_{H^{t, 1}} \| (\overline{w}_2) \phi \|_{H^{t, \omega}}. \\
\]

Similarly, we have
\[
\left(\psi_2(\partial_t \partial_y u^s \partial_y u + [\partial_y u^s, \partial_y] \phi) \phi, (\overline{w}_2) \phi \right)_{H^{t, \omega}} = -2 \left(\psi_2 \partial^2_y u^s \partial^2_y u^s \phi, (\overline{w}_2) \phi \right)_{H^{t, \omega}}. \\
\]

The estimate of this term is very tricky. The following argument was motivated by [6]. By Lemma 3.2, $\partial_y u$ can be written as $\partial_y u = \partial_y u_1 + \partial_y u_2$. Note that both $\partial_y u_1$ and $\partial_y u_2$ are discontinuous across $y = a(t)$. In particular, we have
\[
\lim_{y \to a(t)-} \partial_y u_1 - \lim_{y \to a(t)+} \partial_y u_1 = \partial^2_y u^s \frac{u(t,x,2)}{\partial_y u^s(t,2)} \equiv J. \\
\]
Then by integration by parts, we get
\[
-2(\psi_2(y)(\partial_y^2 u^s \partial_y^2 u)_{\Phi}, (\overline{w_2})_{\Phi})_{H^4,0} = -2 \int_{y>a(t)} \psi_2(y) \partial_y^2 u^s (D_x)^{\frac{1}{2}}(\partial_y^2 u_1)_{\Phi}(D_x)^{\frac{1}{2}}(\overline{w_2})_{\Phi} dxdy
\]
\[
-2 \int_{y<a(t)} \psi_2(y) \partial_y^2 u^s (D_x)^{\frac{1}{2}}(\partial_y^2 u_1)_{\Phi}(D_x)^{\frac{1}{2}}(\overline{w_2})_{\Phi} dxdy
\]
\[
-2 \int_{y>a(t)} \psi_2(y) \partial_y^2 u^s (D_x)^{\frac{1}{2}}(\partial_y^2 u_2)_{\Phi}(D_x)^{\frac{1}{2}}(\overline{w_2})_{\Phi} dxdy
\]
\[
= 2 \int_{\mathbb{R}^2_+} \psi_2(y) \partial_y^2 u^s (D_x)^{\frac{1}{2}}(\partial_y u_1)_{\Phi}(D_x)^{\frac{1}{2}}(\overline{w_2})_{\Phi} dxdy
\]
\[
+ 2 \int_{\mathbb{R}^2_+} \psi_2(y) \partial_y^3 u^s (D_x)^{\frac{3}{2}}(\partial_y u_1)_{\Phi}(D_x)^{\frac{3}{2}}(\overline{w_2})_{\Phi} dxdy
\]
\[
+ 2 \int_{\mathbb{R}^2_+} \psi_2(y) \partial_y^2 u^s (D_x)^{\frac{3}{2}}(\partial_y u_2)_{\Phi}(D_x)^{\frac{3}{2}}(\overline{w_2})_{\Phi} dxdy
\]
\[
- 2 \int_{y=a(t)} \partial_y^2 u^s (D_x)^{\frac{3}{2}}J_{\Phi}(D_x)^{\frac{3}{2}}(\overline{w_2})_{\Phi} dx
\]
\[
\Delta A_1 + \cdots + A_5.
\]

Note that \(\psi_2(y)\) vanishes in a neighborhood of \(y = a(t)\) so that \(\partial_y u_1\) behaves like \(w_1\) on the support of \(\psi_2(y)\). Thus,
\[
A_1 \leq C \|(w_1)_{\Phi}\|_{H^{\frac{1}{2},1}} \|(w_2)_{\Phi}\|_{H^{\frac{1}{2},0}}.
\]

Thanks to \(\partial_y^2 u_2 = \partial_y^3 u^s u(t,x)^{\frac{2}{2}}\), we obtain
\[
A_4 \leq C \|u_{\Phi}\|_{H^{\frac{1}{2},1}} \|(w_2)_{\Phi}\|_{H^{\frac{1}{2},0}}.
\]

Similarly, we have
\[
A_2 \leq C \|u_{\Phi}\|_{H^{\frac{1}{2},1}} \|(w_2)_{\Phi}\|_{H^{\frac{1}{2},0}}.
\]

For \(A_5\), we get by Sobolev inequality that
\[
A_5 \leq C \|u_{\Phi}\|_{H^{\frac{1}{2},1}} \|(D_x)^{\frac{1}{2}}(\overline{w_2})_{\Phi}\|_{L^2_{\nu}} \leq C \|u_{\Phi}\|_{H^{\frac{1}{2},1}} \|\overline{\partial_y(w_2)_{\Phi}}\|_{L^2_{\nu}} \|(w_2)_{\Phi}\|_{H^{\frac{1}{2},0}}
\]
\[
\leq C \|u_{\Phi}\|_{H^{\frac{1}{2},1}}^2 + \|(w_2)_{\Phi}\|_{H^{\frac{1}{2},0}}^2 + \frac{1}{16} \|\overline{\partial_y(w_2)_{\Phi}}\|_{H^{\frac{1}{2},0}}^2.
\]

It remains to estimate \(A_3\). One has
\[
w_2 = \partial_y u^s \partial_y u - u \partial_y^2 u^s = \partial_y u^s \partial_y u_1 - u_1 \partial_y^2 u^s,
\]
which gives
\[
\partial_y w_2 = \partial_y u^s \partial_y^2 u_1 - u_1 \partial_y^3 u^s.
\]
Then we may write
\[
A_3 = 2 \int_{\mathbb{R}^2_+} \psi_2(y)^2 \partial_y^2 u^s \partial_y u^s (D_x)^{\frac{1}{2}}(\partial_y u_1)_{\Phi}(D_x)^{\frac{1}{2}}(\overline{w_2})_{\Phi} dxdy
\]
Proposition 5.3. Let $w_2$ be a solution of (3.7) in $[0,T]$ and $\theta_1 > 0$ be a small constant determined later. There exists $\delta > 0$ small enough so that for any $t \in [0,T]$ and $\delta_2 > 0$,

\[
\frac{d}{dt} \| (w_2) \phi^{1+\theta_1} \|_{H^{\frac{1}{2},0}}^2 + \| \partial_y (w_2) \phi^{1+\theta_1} \|_{H^{\frac{1}{2},0}}^2 \leq C \left( \| u \|_{H^{1+\theta_1}}^2 + \| (w_1) \|_{H^{\frac{1}{2},0}}^2 + \| (w_2) \|_{H^{\frac{1}{2},0}}^2 + \| (w_2) \phi^{1+\theta_1} \|_{H^{\frac{1}{2},0}}^2 \right) + \delta_2 \| \partial_y (w_2) \phi \|_{H^{\frac{1}{2},0}}^2.
\]

Let us begin with the following estimate of source term.
Lemma 5.4. It holds that for any $\delta_2 > 0$,
\[
(F_2)\phi, (w_2)\phi \varphi^{1+\theta_1} \leq C \left( \| u \phi \|^2_{H^{1/2}_0} + \| (w_2)\phi \varphi^{1+\theta_1} \|^2_{H^{1/2}_0} \right) + \| (w_2)\phi \|^2_{H^{1/2}_0} + \| (w_2)\phi \|^2_{H^{1/2}_0} + \delta_2 \| \partial_y (w_2)\phi \|^2_{H^{1/2}_0}.
\]

Proof. By (5.1), it is easy to show that
\[
((u\partial_t \partial_y^2 u^s + [\partial_y^2 u^s, \partial_y^2 u])\phi, (w_2)\phi \varphi^{1+\theta_1})_{H^{1/2}_0} \leq C \| u \phi \|_{H^{1/2}_0} \| (w_2)\phi \varphi^{1+\theta_1} \|^2_{H^{1/2}_0}.
\]

Similar to the proof of Lemma 5.2, we have
\[
((\partial_t \partial_y u^s \partial_y u + [\partial_y u^s, \partial_y u])\phi, (w_2)\phi \varphi^{1+\theta_1})_{H^{1/2}_0}
= -2 \int_{\mathbb{R}_+^2} \varphi(t,y)^{1+\theta_1} \partial_y^2 u^s (D_x)^{1/2} (\partial_y^2 u_1) \phi (D_x)^{1/2} (w_2) \phi dx dy
= -2 \int_{y>a(t)} \varphi(t,y)^{1+\theta_1} \partial_y^2 u^s (D_x)^{1/2} (\partial_y^2 u_1) \phi (D_x)^{1/2} (w_2) \phi dx dy
- 2 \int_{y<a(t)} \varphi(t,y)^{1+\theta_1} \partial_y^2 u^s (D_x)^{1/2} (\partial_y^2 u_1) \phi (D_x)^{1/2} (w_2) \phi dx dy
= 2(1 + \theta_1) \int_{\mathbb{R}_+^2} \varphi(t,y)^{1+\theta_1} \partial_y^2 u^s (D_x)^{1/2} (\partial_y u_1) \phi (D_x)^{1/2} (w_2) \phi dx dy
+ 2 \int_{\mathbb{R}_+^2} \varphi(t,y)^{1+\theta_1} \partial_y^2 u^s (D_x)^{1/2} (\partial_y u_1) \phi (D_x)^{1/2} (w_2) \phi dx dy
- 2 \int_{y>a(t)} \varphi(t,y)^{1+\theta_1} \partial_y^2 u^s (D_x)^{1/2} (\partial_y u_1) \phi (D_x)^{1/2} (w_2) \phi dx dy
\]

Here integration by parts does not give rise to the boundary term due to $\varphi(t, a(t)) = 0$.

As $\partial_y^2 u_2 = \partial_y^3 u_s \frac{u(t,x)}{\partial_y^2 u(t,x)}$, we have
\[
B_2 + B_4 \leq C \| u \phi \|_{H^{1/2}_0} \| (w_2)\phi \varphi^{1+\theta_1} \|^2_{H^{1/2}_0}.
\]

Thanks to $|\partial_y \varphi| \leq C$ and $\text{supp} \partial_y \varphi \subset [1 - 2\delta, 1 + 2\delta]$, we get
\[
B_1 \leq C \left| \int_{1-2\delta}^{a(t)} \varphi(t)^{1+\theta_1} \partial_y^2 u^s (D_x)^{1/2} (\partial_y u_1) \phi (D_x)^{1/2} (w_2) \phi dx dy \right|
+ C \left| \int_{1+2\delta}^{a(t)} \varphi(t)^{1+\theta_1} \partial_y^2 u^s (D_x)^{1/2} (\partial_y u_1) \phi (D_x)^{1/2} (w_2) \phi dx dy \right|
\]

\begin{align*}
\equiv B_{11} + B_{12}.
\end{align*}
By Lemma 3.2, \( \partial_y u_1 \) can be expressed as

\[
(5.2) \quad \partial_y u_1 = \begin{cases} 
\partial_y^2 u^s \left( \int_0^{1-2\delta} \phi_1 w_1 dy' + \int_y^{1-2\delta} \frac{\bar{w}_2}{(\partial_y u^s)^2} dy' \right) + \frac{\bar{w}_2}{\partial_y u^s} & \text{for } 1 - 2\delta \leq y < a(t), \\
\partial_y^2 u^s \left( \int_{1-2\delta}^{1+2\delta} \frac{\bar{w}_2}{(\partial_y u^s)^2} dy' + \int_2^{1+2\delta} \phi_1 w_1 dy' \right) + \frac{\bar{w}_2}{\partial_y u^s} & \text{for } a(t) < y < 1 + 2\delta.
\end{cases}
\]

Then we have

\[
B_{11} \leq C \left| \int_{\mathbb{R}} \int_{1-2\delta}^{a(t)} \varphi \theta_1 (\partial_y^2 u^s)^2 (D_x)^{\frac{3}{2}} \left( \int_0^{1-2\delta} \phi_1 w_1 dy' \right) \langle D_x \rangle^{\frac{3}{2}} \langle \bar{w}_2 \rangle \Phi \, dx \, dy \right| \\
+ C \left| \int_{\mathbb{R}} \int_{1-2\delta}^{a(t)} \varphi \theta_1 (\partial_y^2 u^s)^2 (D_x)^{\frac{3}{2}} \left( \int_y^{1-2\delta} \frac{\bar{w}_2}{(\partial_y u^s)^2} dy' \right) \langle D_x \rangle^{\frac{3}{2}} \langle \bar{w}_2 \rangle \Phi \, dx \, dy \right| \\
+ C \left| \int_{\mathbb{R}} \int_{1-2\delta}^{a(t)} \varphi \theta_1 (\partial_y^2 u^s) \langle D_x \rangle^{\frac{3}{2}} \langle \bar{w}_2 \rangle \Phi \langle D_x \rangle^{\frac{3}{2}} \langle \bar{w}_2 \rangle \Phi \, dx \, dy \right| \\
\triangleq D_1 + D_2 + D_3.
\]

It is easy to get

\[
D_1 \leq C \| (\bar{w}_1) \Phi \|_{H^{\frac{1}{2},0}} \| (\bar{w}_2) \Phi \|_{H^{\frac{3}{2},0}}.
\]

For \( y \in [1 - 2\delta, a(t)] \), \( \varphi \) and \( \partial_y u^s \) behaves like \( |y - a(t)| \). So,

\[
D_3 \leq C \int_{1-2\delta}^{a(t)} \frac{\varphi \theta_1}{(\partial_y u^s)^2} dy \| (D_x)^{\frac{3}{2}} \langle \bar{w}_2 \rangle \Phi \|_{L^\infty L^2}^2 \\
\leq C \int_{1-2\delta}^{a(t)} \frac{1}{|y - a|} dy \| (D_x)^{\frac{3}{2}} \partial_y (\bar{w}_2) \Phi \|_{L^2} \| (D_x)^{\frac{3}{2}} \langle \bar{w}_2 \rangle \Phi \|_{L^2} \\
\leq C \| \partial_y (\bar{w}_2) \Phi \|_{H^{\frac{3}{2},0}} \| (\bar{w}_2) \Phi \|_{H^{\frac{3}{2},0}}.
\]

Similarly, we have

\[
D_2 \leq C \int_{1-2\delta}^{a(t)} \varphi \theta_1 \int_{1-2\delta}^{y} \frac{1}{(\partial_y u^s)^2} dy' dy \| (D_x)^{\frac{3}{2}} \langle \bar{w}_2 \rangle \Phi \|_{L^\infty L^2}^2 \\
\leq C \| \partial_y (\bar{w}_2) \Phi \|_{H^{\frac{3}{2},0}} \| (\bar{w}_2) \Phi \|_{H^{\frac{3}{2},0}}.
\]

Here we used

\[
\int_{1-2\delta}^{a(t)} \varphi \theta_1 \int_{1-2\delta}^{y} \frac{1}{(\partial_y u^s)^2} dy' dy \leq C \int_{1-2\delta}^{a(t)} \varphi \theta_1 \int_{1-2\delta}^{y} \frac{1}{|y - a|} dy' dy \\
\leq C \int_{1-2\delta}^{a(t)} \frac{1}{|y - a|} dy \leq C.
\]

This shows that for any \( \delta_2 > 0 \),

\[
B_{11} \leq C \left( \| (\bar{w}_1) \Phi \|_{H^{\frac{1}{2},0}}^2 + \| (\bar{w}_2) \Phi \|_{H^{\frac{3}{2},0}}^2 \right) + \delta_2 \| \partial_y (\bar{w}_2) \Phi \|_{H^{\frac{3}{2},0}}^2.
\]

The same argument shows that

\[
B_{12} \leq C \left( \| (\bar{w}_1) \Phi \|_{H^{\frac{1}{2},0}}^2 + \| (\bar{w}_2) \Phi \|_{H^{\frac{3}{2},0}}^2 \right) + \delta_2 \| \partial_y (\bar{w}_2) \Phi \|_{H^{\frac{3}{2},0}}^2.
\]

Thus, we obtain

\[
B_1 \leq C \left( \| (\bar{w}_1) \Phi \|_{H^{\frac{1}{2},0}}^2 + \| (\bar{w}_2) \Phi \|_{H^{\frac{3}{2},0}}^2 \right) + 2\delta_2 \| \partial_y (\bar{w}_2) \Phi \|_{H^{\frac{3}{2},0}}^2.
\]
Now we deal with $B_3$. Similar to $A_3$ in Lemma 5.2, we obtain

$$B_3 = \int_{\mathbb{R}_+^2} \varphi^{1+\theta_1}(D_x) \frac{1}{2} \varphi(D_y u_1) (D_x)^{\frac{1}{2}} (D_y u_1) dxdy$$

$$- \int_{\mathbb{R}_+^2} \varphi^{1+\theta_1} \frac{1}{2} \varphi D_y u_1 (D_x)^{\frac{1}{2}} (D_y u_1) dxdy$$

$$- \int_{\mathbb{R}_+^2} \varphi^{1+\theta_1} \frac{1}{2} \varphi D_y u_1 (D_x)^{\frac{1}{2}} (D_y u_1) dxdy$$

$$- (1 + \theta_1) \int_{\mathbb{R}_+^2} \varphi^{1+\theta_1} \frac{1}{2} \varphi D_y u_1 (D_x)^{\frac{1}{2}} (D_y u_1) dxdy$$

$$- 2 \int_{\mathbb{R}_+^2} \varphi^{1+\theta_1} \frac{1}{2} \varphi D_y u_1 (D_x)^{\frac{1}{2}} (D_y u_1) dxdy$$

$$+ 2 \int_{\mathbb{R}_+^2} \varphi^{1+\theta_1} \frac{1}{2} \varphi D_y u_1 (D_x)^{\frac{1}{2}} (D_y u_1) dxdy$$

$$\triangleq B_3 + \cdots + B_{35}.$$

Similar to $A_{31}, A_{32}, A_{34}, A_{35}$ in Lemma 5.2, we have

$$B_{31} + B_{32} + B_{34} + B_{35} \leq - \left( \frac{c^2}{4} - C_1 \delta \right) ||(D_y u_1) \varphi^{1+\theta_1}||^2_{H^{\frac{1}{2}}_0} + C ||(w_2) \varphi^{1+\theta_1}||^2_{H^{\frac{1}{2}}_0}.$$

Similar to $B_1$, we have

$$B_{33} \leq C \left[ \int_{\mathbb{R}} \int_{1-126} \varphi^{1+\theta_1} \frac{1}{2} \varphi D_y u_1 (D_x)^{\frac{1}{2}} (D_y u_1) dxdy \right]$$

$$+ C \left[ \int_{\mathbb{R}} \int_{0}^{1+24} \varphi^{1+\theta_1} \frac{1}{2} \varphi D_y u_1 (D_x)^{\frac{1}{2}} (D_y u_1) dxdy \right] \triangleq E_1 + E_2.$$

We get by (5.2) that

$$E_1 \leq C \left[ \int_{\mathbb{R}} \int_{0}^{1+24} \varphi^{1+\theta_1} \frac{1}{2} \varphi D_y u_1 (D_x)^{\frac{1}{2}} (D_y u_1) dxdy \right]$$

$$+ C \left[ \int_{\mathbb{R}} \int_{0}^{1+24} \varphi^{1+\theta_1} \frac{1}{2} \varphi D_y u_1 (D_x)^{\frac{1}{2}} (D_y u_1) dxdy \right]$$

$$= E_{11} + E_{12} + E_{13}.$$

It is easy to see that

$$E_{11} \leq C \left[ \int_{\mathbb{R}} \int_{0}^{1+24} \varphi^{1+\theta_1} \frac{1}{2} \varphi D_y u_1 (D_x)^{\frac{1}{2}} (D_y u_1) dxdy \right]$$

and

$$E_{13} \leq C \int_{0}^{1+24} \varphi^{1+\theta_1} \frac{1}{2} \varphi D_y u_1 (D_x)^{\frac{1}{2}} (D_y u_1) dxdy.$$
Similarly, we have
\[ E_2 \leq C ( \| \langle w_1 \rangle \phi \|_{H^{\frac{1}{2},0}} + \| \langle w_2 \rangle \phi \|_{H^{\frac{3}{2},0}} ) + \delta_2 \| \partial_y (\langle w_2 \rangle) \phi \|_{H^{\frac{3}{2},0}}. \]
Thus, we get
\[ B_{33} \leq C ( \| \langle w_1 \rangle \phi \|_{H^{\frac{1}{2},0}} + \| \langle w_2 \rangle \phi \|_{H^{\frac{3}{2},0}} ) + 2\delta_2 \| \partial_y (\langle w_2 \rangle) \phi \|_{H^{\frac{3}{2},0}}. \]
Summing up, we obtain
\[ B_3 \leq - \left( \frac{C^2}{4} - C_1 \delta \right) \| (\partial_y u_1) \phi \varphi^{\frac{1}{2}} \|_{H^{\frac{1}{2},0}} + C ( \| (w_2) \phi \varphi^{\frac{1}{2}} \|_{H^{\frac{3}{2},0}} + \| \langle w_1 \rangle \phi \|_{H^{\frac{3}{2},0}} + \| \langle w_2 \rangle \phi \|_{H^{\frac{3}{2},0}} \right) \]
\[ + 2\delta_2 \| \partial_y (\langle w_2 \rangle) \phi \|_{H^{\frac{3}{2},0}}. \]
Summing up the estimates of \( B_1, B_2, B_3 \) and taking \( \delta \) small enough, we conclude the lemma. \( \square \)

Now we are in position to prove Proposition 5.3.

\textbf{Proof.} Recall that \( w_2 \) satisfies
\[ (5.3) \quad \partial_t w_2 + u^s \partial_x w_2 - \partial^2_y w_2 = F_2. \]
Applying \( e^{\Phi(t,D_x)} \) to (5.3), we get
\[ (5.4) \quad \partial_t (w_2) \phi + \lambda (D_x)^{\frac{1}{2}} \theta (w_2) \phi + u^s \partial_x (w_2) \phi - \partial^2_y (w_2) \phi = (F_2) \phi. \]
Taking \( \langle D_x \rangle^{\frac{1}{2}} \) on both sides of (5.4) and taking \( L^2 \) inner product with \( \langle D_x \rangle^{\frac{1}{2}} (w_2) \phi \varphi^{1+\theta_1} \), we obtain
\[ \frac{1}{2} \frac{d}{dt} \| (w_2) \phi \varphi^{\frac{1+\theta_1}{2}} \|_{H^{\frac{1}{2},0}}^2 + \lambda \| (w_2) \phi \varphi^{\frac{1+\theta_1}{2}} \|_{H^{\frac{3}{2},0}}^2 + \int_{\mathbb{R}^2_+} u^s \langle D_x \rangle^{\frac{1}{2}} \partial_x (w_2) \phi \langle D_x \rangle^{\frac{1}{2}} (w_2) \phi \varphi^{1+\theta_1} dxdy \]
\[ - \int_{\mathbb{R}^2_+} \langle D_x \rangle^{\frac{1}{2}} \partial^2_y (w_2) \phi \langle D_x \rangle^{\frac{1}{2}} (w_2) \phi \varphi^{1+\theta_1} dxdy \]
\[ \leq \int_{\mathbb{R}^2_+} \varphi^{\theta_1} \partial_t \varphi \langle D_x \rangle^{\frac{1}{2}} (w_2) \phi \|_{L^2}^2 dxdy + \langle (F_2) \phi, (w_2) \phi \varphi^{1+\theta_1} \rangle_{H^{\frac{1}{2},0}}. \]
We get by integration by parts that
\[ \int_{\mathbb{R}^2_+} u^s \langle D_x \rangle^{\frac{1}{2}} \partial_x (w_2) \phi \langle D_x \rangle^{\frac{1}{2}} (w_2) \phi \varphi^{1+\theta_1} dxdy = 0, \]
and
\[ - \int_{\mathbb{R}^2_+} \langle D_x \rangle^{\frac{1}{2}} \partial^2_y (w_2) \phi \langle D_x \rangle^{\frac{1}{2}} (w_2) \phi \varphi^{1+\theta_1} dxdy \]
\[ = \| \partial_y (w_2) \phi \varphi^{\frac{1+\theta_1}{2}} \|_{H^{\frac{3}{2},0}}^2 + (1 + \theta_1) \int_{\mathbb{R}^2_+} \varphi^{\theta_1} \partial_y \varphi \langle D_x \rangle^{\frac{1}{2}} \partial_y (w_2) \phi \langle D_x \rangle^{\frac{1}{2}} (w_2) \phi dxdy \]
\[ \geq \| \partial_y (w_2) \phi \varphi^{\frac{1+\theta_1}{2}} \|_{H^{\frac{3}{2},0}}^2 - C \| \partial_y (\langle w_2 \rangle) \phi \|_{H^{\frac{3}{2},0}} \| \langle w_2 \rangle \phi \|_{H^{\frac{3}{2},0}} \]
\[ \geq \| \partial_y (w_2) \phi \varphi^{\frac{1+\theta_1}{2}} \|_{H^{\frac{3}{2},0}}^2 - C \| \langle w_2 \rangle \phi \|_{H^{\frac{3}{2},0}}^2 - 2\delta_2 \| \partial_y (\langle w_2 \rangle) \phi \|_{H^{\frac{3}{2},0}}^2. \]
Here we used \( |\partial_y \varphi| \leq C \) and \( \psi_2(y) = 1 \) for \( y \in \text{supp} \partial_y \varphi \). Similarly, we have
\[ \int_{\mathbb{R}^2_+} \varphi^{\theta_1} \partial_t \varphi \| \langle D_x \rangle^{\frac{1}{2}} (w_2) \phi \|_{L^2}^2 dxdy \leq C \| \langle w_2 \rangle \phi \|_{H^{\frac{3}{2},0}}^2. \]
6. Gevrey regularity estimate of $h$

**Proposition 6.1.** Let $h$ be a solution of $(3.5)$ in $[0, T]$. Then it holds that

$$\frac{d}{dt} \| h_{\phi} \|_{L^2}^2 + \lambda \| h_{\phi} \|_{H^{1+\theta,0}}^2 + \| \partial_y h_{\phi} \|_{L^2}^2 \leq C \left( \| h_{\phi} \|_{L^2}^2 + \| u_{\phi} \|_{H^{1+\theta,0}}^2 + \| u_{\phi} \|_{H^{1+\theta,0}}^2 + \| (w_2)_{\phi} \|_{H^{1+\theta,0}}^2 \right).$$

The proposition follows from the following two lemmas.

**Lemma 6.2.** Let $h$ be a smooth solution of $(3.5)$ in $[0, T]$. Then it holds that

$$\frac{d}{dt} \| h_{\phi} \|_{L^2}^2 + \lambda \| h_{\phi} \|_{H^{1+\theta,0}}^2 + \| \partial_y h_{\phi} \|_{L^2}^2 \leq C \left( \| h_{\phi} \|_{L^2}^2 + \| u_{\phi} \|_{H^{1+\theta,0}}^2 \right) + 2 \left( \phi_3(y) u_{\phi}, \phi_3(y) u_{\phi} \right)_{L^2}.$$

**Proof.** Applying $e^{\Phi(t, D_z)}$ on $(3.5)$ and making $H^{3,0}$ energy estimate, we obtain

$$\frac{d}{dt} \| h_{\phi} \|_{L^2}^2 + \lambda \| h_{\phi} \|_{H^{1+\theta,0}}^2 - ((\partial_y^2 h)_{\phi}, h_{\phi})_{L^2} \leq - (u \partial_x h_{\phi}, h_{\phi})_{L^2} + ((\partial d - \partial_y^2 d)(\partial_y u)_{\phi}, h_{\phi})_{L^2} - 2((\partial_y d\partial_y^2 u)_{\phi}, h_{\phi})_{L^2}$$

$$- (d(v\partial_y^2 u_{\phi})_{\phi}, h_{\phi})_{L^2}.$$

Thanks to $h_{\phi}|_{y=0} = 0$, we get by integration by parts that

$$(u \partial_x h_{\phi}, h_{\phi})_{L^2} = 0, \quad ((\partial_y^2 h)_{\phi}, h_{\phi})_{L^2} = \| (\partial_y h)_{\phi} \|_{L^2}^2,$$

and

$$-2((\partial_y d\partial_y^2 u)_{\phi}, h_{\phi})_{L^2} = -2((\partial_y d\partial_y^2 u)_{\phi}, h_{\phi})_{L^2} + 2((\partial_y d\partial_y^2 u)_{\phi}, \partial_y h_{\phi})_{L^2} \leq C(\| u_{\phi} \|_{H^{1+\theta,0}}^2 + \| h_{\phi} \|_{L^2}^2) + \frac{1}{16} \| (\partial_y h)_{\phi} \|_{L^2}^2.$$

After some calculations, we have

$$\partial d - \partial_y^2 d = \frac{\partial_y^3 u_{\phi} \partial_y^3}{(\partial_y^2 u_{\phi})^{3/2}} - \frac{\partial_y^4}{(\partial_y^2 u_{\phi})^{1/2}} - \frac{3}{4} \frac{(\partial_y^3 u_{\phi})^2 \partial_y^3}{(\partial_y^2 u_{\phi})^{5/2}},$$

which gives

$$((\partial d - \partial_y^2 d)(\partial_y u)_{\phi}, h_{\phi})_{L^2} \leq C \| u_{\phi} \|_{H^{1+\theta,0}}^2 \| h_{\phi} \|_{L^2}.$$

Using $\partial_x u + \partial_y v = 0$, we get by integration by parts that

$$- (d(v\partial_y^2 u_{\phi})_{\phi}, h_{\phi})_{L^2} = - (\phi_3(y)\partial_y^2 u_{\phi}^{1/2} - (\partial_y^2 u_{\phi}^{1/2})(\partial_y u)_{\phi})_{L^2}$$

$$= - (\phi_3(y)\partial_y v_{\phi}, \phi_3(y)(\partial_y u)_{\phi})_{L^2}$$

$$= 2(\phi_3(y)\partial_y v_{\phi}, \phi_3(y) u_{\phi})_{L^2} + (\phi_3(y)\partial_y v_{\phi}, \phi_3(y) u_{\phi})_{L^2}$$

$$= 2(\phi_3(y)\partial_y v_{\phi}, \phi_3(y) u_{\phi})_{L^2}.$$

This completes the proof of the lemma. 

The following lemma is devoted to the most trouble term $((\phi_3(y)\partial_y v_{\phi}, \phi_3(y) u_{\phi})_{L^2}$. The argument is motivated by [6].
Lemma 6.3. It holds that 

\[(\phi_3'(y)v_\phi, \phi_3(y)u_\phi)_{L^2} \leq C \left( \|u_\phi\|_{H^{3/4+\theta}}^2 + \|(w_1)_\phi\|_{H^{3/2}}^2 + \|(w_2)_\phi\|_{H^{3/4+\theta}}^2 + \|(w_2)\phi\|_{H^{3/4+\theta}}^{1+\theta} \right).\]

Proof. Let \(\text{supp} \phi_3' = E_1 \cup E_2\), where \(E_1 = [\frac{1}{2}, \frac{3}{4}]\) and \(E_2 = [\frac{7}{4}, 2]\). Then we write

\[
\left| \int_{\mathbb{R}^2_+} \phi_3' \psi \phi_3 u_\phi dx dy \right| \leq \int_{\mathbb{R}} \int_{E_1} \phi_3' \psi_0 \int_0^y \partial_x u_\phi dy' u_\phi dx dy \\
+ \int_{\mathbb{R}} \int_{E_2} \phi_3' \psi_0 \int_0^y \partial_x u_\phi dy' u_\phi dx dy \\
\triangleq J_1 + J_2.
\]

In \(E_1\), \(u\) can be expressed as \(u = \partial_y u^s \int_0^y \psi \cdot w_1 dy'\) so that

\[
J_1 \leq \int_{\mathbb{R}} \int_{E_1} \phi_3' \psi_0 \int_0^y \partial_x u^s \partial_y u^s \int_0^y (\psi_0)\phi_3 u_\phi dx dy \\
\leq C \|\psi_0\|_{H^{3/2}} \|u_\phi\|_{H^{3/4+\theta}}.
\]

In \(E_2\), \(u\) can be expressed as \(u = \partial_y u^s \int_2^y \psi \cdot w_1 dy' + \partial_y u^s \int_2^y \psi \cdot u(t, x) dy'\) so that

\[
J_2 \leq \int_{\mathbb{R}} \int_{E_2} \phi_3' \psi_0 \int_0^y \partial_x u^s \partial_y u^s \int_2^y \psi \cdot w_1 dy' \partial_y u^s \int_2^y \psi \cdot u(t, x) dy' dx dy \\
+ \int_{\mathbb{R}} \int_{E_2} \phi_3' \psi_0 \int_0^y \partial_x u^s \partial_y u^s \int_2^y \psi \cdot u(t, x) dy' dx dy \\
\triangleq J_{21} + J_{22}.
\]

Similar to \(J_1\), we have

\[
J_{21} \leq C \|\psi_0\|_{H^{3/2}} \|u_\phi\|_{H^{3/4+\theta}}.
\]

Recall that \(a(t)\) is a critical point of \(u^s\). We decompose \(\int_0^y \partial_x u_\phi dy'\) into the following three parts

\[
\int_0^y \partial_x u_\phi dy' = \int_0^{a(t)} \partial_x u_\phi dy' + \int_{a(t)}^{\frac{7}{4}} \partial_x u_\phi dy' + \int_{\frac{7}{4}}^y \partial_x u_\phi dy'.
\]

Then we have

\[
J_{22} \leq \int_{\mathbb{R}} \int_{E_2} \phi_3' \psi_0 \int_0^{a(t)} \partial_x u_\phi dy' \partial_y u^s \int_2^y \psi \cdot u(t, x) dy' dx dy \\
+ \int_{\mathbb{R}} \int_{E_2} \phi_3' \psi_0 \int_{a(t)}^{\frac{7}{4}} \partial_x u_\phi dy' \partial_y u^s \int_2^y \psi \cdot u(t, x) dy' dx dy \\
+ \int_{\mathbb{R}} \int_{E_2} \phi_3' \psi_0 \int_{\frac{7}{4}}^y \partial_x u_\phi dy' \partial_y u^s \int_2^y \psi \cdot u(t, x) dy' dx dy \\
\triangleq K_1 + K_2 + K_3.
\]

By Lemma 3.2, we get

\[
K_1 \leq \int_{\mathbb{R}} \int_{E_2} \phi_3' \psi \int_0^{1-28} \partial_x u_\phi dy' \partial_y u^s \int_2^y \psi \cdot u(t, x) dy' dx dy
\]
result, we obtain
\[ K \leq K_{11} + K_{12} + K_{13}. \]

As in \( J_1 \), we have
\[ K_{11} + K_{12} \leq C \| (\overline{w}) \|_{H^{\frac{1}{2},0}} \| u \|_{H^{\frac{1}{2},1}}. \]

For \( K_{13} \), let us first estimate
\[
\left\| \int_{1-2\delta}^{a(t)} \langle D_x \rangle^{\frac{1}{2} - \theta} (\partial_y u^s) \int_{1-2\delta}^{y'} \frac{\overline{w}_2}{(\partial_y u^s)^2} dy'' \phi dy' \right\|_{L^2_x}
\leq \int_{1-2\delta}^{a(t)} \partial_y u^s \int_{1-2\delta}^{y'} \left\| \langle D_x \rangle^{\frac{1}{2} - \theta} (\overline{w}_2) \phi \right\|_{L^2_x} dy'' dy'
\leq \int_{1-2\delta}^{a(t)} \partial_y u^s \int_{1-2\delta}^{y'} \left\| \langle D_x \rangle^{\frac{1}{2} - \theta} (\overline{w}_2) \phi \right\|_{L^2_x} \left( \partial_y u^s \right)^{2 \varphi (\alpha + \theta_1)} dy'' dy'
\leq \int_{1-2\delta}^{a(t)} \left| y' - a(t) \right| \left( \int_{1-2\delta}^{a(t)} \partial_y u^s \left( \partial_y u^s \right)^{2 \varphi (\alpha + \theta_1)} dy'' \right)^{\frac{1}{2}} dy'' \left\| \langle D_x \rangle^{\frac{1}{2} - \theta} (\overline{w}_2) \phi \right\|_{L^2_x} \left( \partial_y u^s \right)^{2 \varphi (\alpha + \theta_1)} \left\| \langle D_x \rangle^{\frac{1}{2} - \theta} (w_2) \phi \right\|_{L^2_x}^{\frac{1 + \theta_1}{2}}
\leq C \left\| \langle D_x \rangle^{\frac{5}{8}} (\overline{w}_2) \phi \right\|_{L^2_x} \left\| \langle D_x \rangle^{\frac{1}{2} - \theta} (w_2) \phi \right\|_{L^2_x}^{\frac{1 + \theta_1}{2}}.
\]

Here \( \alpha = 1 - 8 \theta \) and take \( \frac{\alpha (1 + \theta_1)}{2} < \frac{1}{2} \) to ensure that \( \int_{1-2\delta}^{a(t)} \left| y' - a(t) \right| \left( \int_{1-2\delta}^{a(t)} \partial_y u^s \left( \partial_y u^s \right)^{2 \varphi (\alpha + \theta_1)} dy'' \right)^{\frac{1}{2}} dy'' \left\| \langle D_x \rangle^{\frac{1}{2} - \theta} (\overline{w}_2) \phi \right\|_{L^2_x} \left( \partial_y u^s \right)^{2 \varphi (\alpha + \theta_1)} \left\| \langle D_x \rangle^{\frac{1}{2} - \theta} (w_2) \phi \right\|_{L^2_x}^{\frac{1 + \theta_1}{2}} \leq C. \)

As a result, we obtain
\[ K_{13} \leq C \left( \| u \|_{H^{\frac{1}{2},1}}^2 + \| (\overline{w}) \|_{H^{\frac{1}{2},0}}^2 + \| (\overline{w}) \|_{H^{\frac{1}{2},0}}^2 + \| (w) \phi \|_{H^{\frac{1}{2},0}}^2 \right). \]

Then we have
\[ K_1 \leq C \left( \| u \|_{H^{\frac{1}{2},1}}^2 + \| (\overline{w}) \|_{H^{\frac{1}{2},0}}^2 + \| (\overline{w}) \|_{H^{\frac{1}{2},0}}^2 + \| (w) \phi \|_{H^{\frac{1}{2},0}}^2 \right). \]

For \( K_2 \), by Lemma 3.2, we have
\[
K_2 \leq \left| \int_{\mathbb{R}} \int_{E_2} \phi_3 \phi_3' \int_{1+2\delta}^{a(t)} \partial_x \partial_y u^s \int_{1+2\delta}^{y'} \frac{\overline{w}_2}{(\partial_y u^s)^2} dy'' \phi dy' \partial_y u^s \frac{u(t, x, 2)}{\partial_y u^s(t, 2)} \right| \left| \int_{\mathbb{R}} \int_{E_2} \phi_3 \phi_3' \int_{1+2\delta}^{a(t)} \partial_y u^s \int_{1+2\delta}^{y'} \frac{\overline{w}_2}{(\partial_y u^s)^2} dy'' \phi dy' \partial_y u^s \frac{u(t, x, 2)}{\partial_y u^s(t, 2)} \right| \]
\[
+ \left| \int_{\mathbb{R}} \int_{E_2} \phi_3 \phi_3' \int_{1+2\delta}^{a(t)} \partial_x \partial_y u^s \int_{1+2\delta}^{y'} \frac{\overline{w}_2}{(\partial_y u^s)^2} dy'' \phi dy' \partial_y u^s \frac{u(t, x, 2)}{\partial_y u^s(t, 2)} \right| \]
\[
+ \left| \int_{\mathbb{R}} \int_{E_2} \phi_3 \phi_3' \int_{1+2\delta}^{a(t)} \partial_y u^s \int_{1+2\delta}^{y'} \frac{\overline{w}_2}{(\partial_y u^s)^2} dy'' \phi dy' \partial_y u^s \frac{u(t, x, 2)}{\partial_y u^s(t, 2)} \right| \]
\[
+ \left| \int_{\mathbb{R}} \int_{E_2} \phi_3 \phi_3' \int_{1+2\delta}^{a(t)} \partial_y u^s \int_{1+2\delta}^{y'} \frac{\overline{w}_2}{(\partial_y u^s)^2} dy'' \phi dy' \partial_y u^s \frac{u(t, x, 2)}{\partial_y u^s(t, 2)} \right| \]
\[
+ \left| \int_{\mathbb{R}} \int_{E_2} \phi_3 \phi_3' \int_{1+2\delta}^{a(t)} \partial_y u^s \int_{1+2\delta}^{y'} \frac{\overline{w}_2}{(\partial_y u^s)^2} dy'' \phi dy' \partial_y u^s \frac{u(t, x, 2)}{\partial_y u^s(t, 2)} \right| \]
\[
+ \left| \int_{\mathbb{R}} \int_{E_2} \phi_3 \phi_3' \int_{1+2\delta}^{a(t)} \partial_y u^s \int_{1+2\delta}^{y'} \frac{\overline{w}_2}{(\partial_y u^s)^2} dy'' \phi dy' \partial_y u^s \frac{u(t, x, 2)}{\partial_y u^s(t, 2)} \right|.\]
Similar to $K_{13}$, we have
\[ K_{21} \leq C(\|u\|_{H^\frac{3}{4},0}^2 + \|(w_1)\|_{H^\frac{3}{4},0}^2 + \|(w_2)\|_{H^\frac{3}{4},0}^2 + \|(w_2)\|_{H^\frac{3}{4},0}^2 + \|(w_2)\|_{H^\frac{3}{4},0}^2). \]

Similar to $K_{12}$, we have
\[ K_{22} + K_{24} \leq C(\|\overline{w}_1\|_{H^\frac{3}{4},0}^2 + \|u\|_{H^\frac{3}{4},0}^2). \]

Summing up the estimates of $K_1, K_2, K_3$, we deduce that
\[ J_{22} \leq C(\|u\|_{H^\frac{3}{4},1}^2 + \|\overline{w}_1\|_{H^\frac{3}{4},0}^2 + \|\overline{w}_2\|_{H^\frac{3}{4},0}^2 + \|u\|_{H^\frac{3}{4},0}^2 + \|w_2\|_{H^\frac{3}{4},0}^2). \]

Putting the above estimates together, we conclude the lemma.

7. Proof of Theorem 1.1

Let us first recover the regularity of $u$ from $w_1$ and $h$.

**Lemma 7.1.** It holds that
\[ \|u\|_{H^\frac{3}{4},1} \leq C(\|\overline{w}_1\|_{H^\frac{3}{4},0} + \|h\|_{H^\frac{3}{4},0}). \]

**Proof.** The proof is split into two steps.

**Step 1.** Estimate of $\|\partial_y u\|_{H^\frac{1}{4},0}$

First of all, we have
\[ \|\partial_y u\|_{H^\frac{1}{4},0} \leq \|1_{[0,\frac{3}{4}]}(y)\partial_y u\|_{H^\frac{1}{4},0} + \|1_{[\frac{1}{4},\frac{7}{4}]}(y)\partial_y u\|_{H^\frac{1}{4},0} + \|1_{[\frac{7}{4},+\infty]}(y)\partial_y u\|_{H^\frac{1}{4},0} \]
\[ \triangleq I_1 + I_2 + I_3. \]

For $y \in [0,\frac{3}{4}] \cup [\frac{7}{4},+\infty)$, we have
\[ \partial_y u(y) = \partial_y (\frac{1}{\partial_y u^*})u(y), \]
which gives
\[ I_1 + I_3 \leq C\|\overline{w}_1\|_{H^\frac{3}{4},0} + C\|u\|_{H^\frac{1}{4},0}^2. \]

For $y \in [\frac{3}{4},\frac{7}{4}]$, using $\partial_y u = h(\partial_y u^*)^{-\frac{1}{2}}$, we get
\[ I_2 \leq C\|h\|_{H^\frac{3}{4},0}. \]

**Step 2.** Estimate of $\|u\|_{H^\frac{1}{4},0}$
We have
\[ \| u_\Phi \|_{H^{1/2} + \theta, 0} \leq 1_{[0, 2]}(y) u_\Phi\|_{H^{1/2} + \theta, 0} + 1_{[2, \infty]}(y) u_\Phi\|_{H^{1/2} + \theta, 0} + 1_{[2, \infty]}(y) u_\Phi\|_{H^{1/2} + \theta, 0} \]
\[ \triangleq I_4 + I_5 + I_6. \]
For \( y \in [0, \frac{3}{4}] \), we have
\[ u(y) = \partial_y^s \left( \int_0^y w_1 dy' \right), \]
which gives
\[ I_4 \leq C \| (\overline{w}_1) \|_{H^{1/2} + \theta, 0}. \]
For \( y \in [\frac{3}{4}, 1] \), we have
\[ \bar{u}(y) = \int_{\frac{3}{4}}^y \frac{h}{(\partial_y^2 u_0)^{1/4}} dy' + u(t, x, \frac{3}{4}), \]
from which and the estimate of \( I_4 \), we deduce that
\[ I_5 \leq C \| (\overline{w}_1) \|_{H^{1/2} + \theta, 0} + \| h \Phi \|_{H^{1/2} + \theta, 0}. \]
For \( y \in [\frac{1}{2}, +\infty) \), we have
\[ \bar{u}(y) = \partial_y^s \left( \int_{\frac{1}{2}}^y w_1 dy' \right) + \partial_y^s \frac{u(t, x, \frac{1}{2})}{\partial_y u_0(t, x, \frac{1}{2})}, \]
from which and the estimate of \( I_5 \), we deduce that
\[ I_6 \leq C \| (\overline{w}_1) \|_{H^{1/2} + \theta, 0} + \| h \Phi \|_{H^{1/2} + \theta, 0}. \]
Now, the inequality follows by putting the estimates of \( I_1 - I_6 \) together.

Now we are in position to prove Theorem 1.1.

**Proof.** The approximate solution can be easily constructed by adding the viscous term \(-\epsilon^2 \partial_y^2 u\) to (1.4). So, we just present the uniform estimate. For this end, we introduce
\[ E(t) \triangleq \| (\overline{w}_1) \|_{H^{1/2} + \theta, 0}^2 + \| (\overline{w}_2) \|_{H^{1/2} + \theta, 0}^2 + \| (w_2) \|_{H^{1/2} + \theta, 0}^2 + \| h \Phi \|_{L^2}, \]
\[ D(t) \triangleq \| \partial_y (\overline{w}_1) \|_{H^{1/2} + \theta, 0}^2 + \| \partial_y (\overline{w}_2) \|_{H^{1/2} + \theta, 0}^2 + \| \partial_y (w_2) \|_{H^{1/2} + \theta, 0}^2 + \| \partial_y h \Phi \|_{L^2}^2, \]
\[ G(t) \triangleq \| (\overline{w}_1) \|_{H^{1/2} + \theta, 0}^2 + \| (\overline{w}_2) \|_{H^{1/2} + \theta, 0}^2 + \| (w_2) \|_{H^{1/2} + \theta, 0}^2 + \| h \Phi \|_{L^2}^2. \]
Choosing \( \lambda \) large enough and \( \delta_2 \) suitably small, we infer from Proposition 4.1, Proposition 5.1, Proposition 5.3, Proposition 6.1 and Lemma 7.1 that
\[ \frac{d}{dt} E(t) + \lambda G(t) + D(t) \leq C E(t). \]
Then Gronwall’s inequality gives
\[ E(t) + \lambda \int_0^t G(s) ds + \int_0^t D(s) ds \leq E(0) e^{Ct} \]
for any \( t \in [0, T] \).
8. Note on well-posedness in Gevrey class 2

Let us explain how to use a new unknown $h_1 = \partial_y^2 u - \frac{\partial^3 u_s}{\partial y^3 u_s^2} \partial_y u$ introduced in [10] to obtain the well-posedness of (1.4) in Gevrey class 2 in our framework. It is easy to verify that $h_1$ satisfies the following equation

$$\partial_t h_1 + u^s \partial_x h_1 + \partial_x w_2 - \partial_y^2 h_1 = \partial_t \left( \frac{\partial^3 u_s}{\partial y^3 u_s^2} \partial_y u + \left[ \frac{\partial^3 u_s}{\partial y^3 u_s^2}, \partial_y^2 u \right] \partial_y u \right).$$

The unknown $h_1$ is well-defined in non-monotonic domain. It is easy to show that

$$(\overline{h}_1)_\phi \in L^\infty(0,T; L^2) \cap L^2(0,T; H^{\frac{1}{2},0}), \quad \overline{h}_1 = \phi_3(y) h_1,$$

if $(\overline{w}_2)_\phi \in L^2(0,T; H^{\frac{1}{2},0})$. On the other hand, if we know that $(\overline{h}_1)_\phi \in L^2(0,T; H^{\frac{1}{2},0})$ which will imply $\partial_y^2 u_\phi \in L^2(0,T; H^{\frac{1}{2},0})$ because of $\partial_y u_\phi \in L^2(0,T; H^{\frac{1}{2},0})$ by Lemma 7.1, we can show that $(\overline{w}_2)_\phi \in L^2(0,T; H^{\frac{1}{2},0})$ by following the proof of Proposition 5.1. More precisely, we can deduce that

$$\frac{d}{dt} \| (\overline{w}_1)_\phi \|_{H^{\frac{1}{2},0} \rightarrow 0} + (\lambda - C) \| (\overline{w}_1)_\phi \|_{H^{\frac{1}{2},0} \rightarrow 0} + \| \partial_y (\overline{w}_1)_\phi \|_{H^{\frac{1}{2},0} \rightarrow 0}^2 \leq C \left( \| u_\phi \|_{H^{\frac{1}{2},0} \rightarrow 0}^2 + \| (\overline{w}_2)_\phi \|_{H^{\frac{1}{2},0} \rightarrow 0}^2 + \| L_1 \|_{H^{\frac{1}{2},0}}^2 \right),$$

and

$$\frac{d}{dt} \| h_\phi \|_{L^2} + \lambda \| h_\phi \|_{H^{\frac{1}{2},0} \rightarrow 0} + \| \partial_y h_\phi \|_{L^2}^2 \leq C \left( \| h_\phi \|_{L^2}^2 + \| u_\phi \|_{H^{\frac{1}{2},1} \rightarrow 0}^2 + \| (\overline{w}_1)_\phi \|_{H^{\frac{1}{2},0} \rightarrow 0}^2 + \| (\overline{w}_2)_\phi \|_{H^{\frac{1}{2},0} \rightarrow 0}^2 \right),$$

Thus, we can close the energy estimates in Gevrey class 2.

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References

[1] R. Alexandre, Y. Wang, C.-J. Xu and T. Yang, Well-posedness of the Prandtl equation in Sobolev spaces, J. Amer. Math. Soc., 28(2015), 745-784.
[2] H. Bahouri, J. Y. Chemin and R. Danchin, Fourier analysis and nonlinear partial differential equations, Grundlehren der mathematischen Wissenschaften 343, Springer-Verlag Berlin Heidelberg, 2011.
[3] D. Chen, Y. Wang and Z. Zhang, Well-posedness of the Prandtl equation with monotonicity in Sobolev Spaces, submitted.
[4] W. E, Boundary layer theory and the zero-viscosity limit of the Navier-Stokes equation, Acta Math. Sin., 16 (2000), 207-218.
[5] D. Gérard-Varet and E. Dormy, On the ill-posedness of the Prandtl equation, J. Amer. Math. Soc., 23 (2010), 591-609.
[6] D. Gérard-Varet and N. Masmoudi, Well-posedness for the Prandtl system without analyticity or monotonicity, *Ann. Sci. Éc. Norm. Supér.*, 48 (2015), 1273-1325.

[7] D. Gérard-Varet and T. Nguyen, Remarks on the ill-posedness of the Prandtl equation, *Asymptot. Anal.*, 77 (2012), 71-88.

[8] Y. Guo and T. Nguyen, A note on Prandtl boundary layers, *Comm. Pure Appl. Math.*, 64 (2011), 1416-1438.

[9] I. Kukavica, N. Masmoudi, V. Vicol and T. K. Wong, On the local well-posedness of the Prandtl and the hydrostatic Euler equations with multiple monotonicity regions, *SIAM J. Math. Anal.*, 46 (2014), 3865-3890.

[10] W. Li and T. Yang, Well-posedness in Gevrey space for the Prandtl equations with non-degenerate critical points, arXiv:1609.08430.

[11] M. C. Lombardo, M. Cannone and M. Sammartino, Well-posedness of the boundary layer equations, *SIAM J. Math. Anal.*, 35 (2003), 987-1004.

[12] N. Masmoudi and T. K. Wong, Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods, *Comm. Pure Appl. Math.*, 68 (2015), 1683-1741.

[13] G. Métivier, *Para-differential calculus and applications to the Cauchy problem for nonlinear systems*, Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series, 5, Edizioni della Normale, Pisa, 2008.

[14] O. A. Oleinik and V. N. Samokhin, *Mathematical models in boundary layer theory*, Applied Mathematics and Mathematical Computation 15 Chapman & Hall/CRC, Boca Raton, Fl., 1999.

[15] L. Prandtl, Über Flüssigkeitsbewegung bei sehr kleiner Reibung, *Verhandlung des III Intern. Math.-Kongresses, Heidelberg*, 1904, 484-491.

[16] M. Sammartino and R. E. Caflisch, Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations, *Comm. Math. Phys.*, 192 (1998), 433-461.

[17] Z. Xin and L. Zhang, On the global existence of solutions to the Prandtl’s system, *Adv. Math.*, 181 (2004), 88-133.

[18] P. Zhang and Z. Zhang, Long time well-posedness of Prandtl system with small and analytic initial data, *J. Functional Analysis*, 270(2016), 2591-2615.

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