THE GAUGE FIXING THEOREM WITH APPLICATIONS TO THE YANG-MILLS FLOW OVER RIEMANNIAN MANIFOLDS

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ABSTRACT. In 1982, Uhlenbeck [29] established the well-known gauge fixing theorem, which has played a fundamental role for Yang-Mills theory. In this paper, we apply the idea of Uhlenbeck to establish a parabolic type of gauge fixing theorems for the Yang-Mills flow and prove existence of a weak solution of the Yang-Mills flow on a compact $n$-dimensional manifold with initial value $A_0$ in $W^{1,n/2}(M)$. When $n = 4$, we improve a key lemma of Uhlenbeck (Lemma 2.7 of [29]) to prove uniqueness of weak solutions of the Yang-Mills flow on a four dimensional manifold.

1. Introduction

Let $M$ be a compact $n$-dimensional Riemannian manifold without boundary and let $E$ be a vector bundle over $M$ with compact Lie group $G$. For a connection $D_A$, the Yang-Mills functional is defined by

$$YM(A; M) = \int_M |F_A|^2 dv,$$

where $F_A$ is the curvature of $D_A$. A connection $D_A$ is called to be Yang-Mills if it is a critical point of the Yang-Mills functional; i.e. $D_A$ satisfies the Yang-Mills equation

$$D_A^* F_A = 0.$$  
(1.1)

The Yang-Mills flow equation is

$$\frac{\partial A}{\partial t} = -D_A^* F_A$$  
(1.2)

with initial condition $A(0) = A_0$, where $A_0$ is a given connection on $E$.

The Yang-Mills flow has played an important role in Yang-Mills theory. Atiyah and Bott [1] introduced the Yang-Mills flow. Donaldson ([5], [6]) proved global existence of the smooth solution to the Yang-Mills heat flow in holomorphic vector bundles over compact Kähler manifolds and used it to establish that a stable irreducible holomorphic vector bundle $E$ over a compact Kähler surface $X$ admits a unique Hermitian-Einstein connection, which was later called the Donaldson-Uhlenbeck-Yau theorem, and see different approach in [30] for the case of holomorphic vector bundles over compact Kähler manifolds. Simpson [23] generalized the Donaldson-Uhlenbeck-Yau theorem in holomorphic bundles over some non-compact Kähler manifolds. We refer to see [11], [18], [32] for further generalizations to the Yang-Mills-Higgs flow on compact or complete Kähler manifolds.

When holomorphic vector bundles are not stable, there is a conjecture of Bando and Siu [2] on the relation between the limiting bundle of the Yang-Mills flow and

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the Harder-Narashimhan filtration on Kähler manifolds. The author and Tian [12] established asymptotic behaviour of the Yang-Mills flow to prove the existence of singular Hermitian-Yang-Mills connections on higher dimensional Kähler manifolds. Daskalopoulos and Wentworth [4] settled the Bando-Siu conjecture on Kähler surfaces. Recently, Jacob [16] and Sibley [24] settled the conjecture of Bando and Siu [2] on higher dimensional Kähler manifolds by using the asymptotic result in [12].

Without the holomorphic structure of the bundle $E$ over Kähler manifolds, it is very interesting to investigate existence of the Yang-Mills flow in vector bundles over $n$-dimensional Riemannian manifolds. For the case of lower dimensional manifolds (i.e. $n = 2, 3$), Rado [19] proved global existence of the smooth solution of the Yang-Mills flow. It is well known that Yang-Mills equations in dimension four have many similarities to the harmonic map equation in dimension two, so dimension four is a critical case for Yang-Mill equations as dimension two is for harmonic maps.

Chang-Ding-Ye [3] constructed a counter-example that the harmonic map flow on $S^2$ blows up at finite time, so it was suggested that Yang-Mill flow in dimension four should blow up in finite time. However, in contrast to the setting of [3], Schlatter, Struwe and Tahvildar-Zadeh [21] proved global existence of the $m$-equivariant Yang-Mills flow on $\mathbb{R}^4$. Later, the author and Tian [13] also proved global existence of the $SO(4)$-equivariant Yang-Mills flow on $\mathbb{R}^4$. Recently, Waldron [31] established global existence of the smooth solution to the Yang-Mills flow when $\|F^+\|_{L^2(M)}$ is sufficiently small. When $n > 5$, it was known that the Yang-Mills flow could blow up in finite time (e.g. [10]).

On the other hand, Uhlenbeck [29] established a gauge fixing theorem, which has played an important role to study the moduli space of Yang-Mills connections. Since the Yang-Mills functional is gauge invariant, the Yang-Mills flow equation (1.2) is not a parabolic system. In order to investigate existence of the Yang-Mills flow, we apply the idea of Uhlenbein in [29] to establish a parabolic version of gauge fixing theorems, depending on time, such that the Yang-Mills flow is equivalent to a parabolic system, which is called the the Yang-Mills equivalent flow (see below (1.3)-(1.5)). More precisely, we have

**Theorem 1.1.** For $n \geq 4$, let $D_A$ be a smooth solution of the Yang-Mills flow (1.2) in $B_{r_0}(x_0) \times [0, t_1]$ with smooth initial value $A_0$ for some constant $t_1 > 0$, where $B_{r_0}(x_0)$ is the ball in $M$ with centre at $x_0$ and radius $r_0 > 0$. Assume that there exists a sufficiently small $\varepsilon > 0$ such that

$$\sup_{0 \leq t \leq t_1} \int_{B_{r_0}(x_0)} |F_A(x, t)|^{n/2} dv \leq \varepsilon.$$ 

Then there are smooth gauge transformations $S(t) = e^{u(t)}$ and smooth connections $D_a = S^* (D_A)$ satisfying the equation

$$\frac{\partial a}{\partial t} = -D_a^* F_a + D_a s$$

in $B_{r_0}(x_0) \times [0, t_1]$, where

$$s(t) = S^{-1}(t) \circ \frac{d}{dt} S(t).$$

Moreover, for all $t \in [0, t_1]$, we have

$$d^* a(t) = 0$$

in $B_{r_0}(x_0)$, $a(t) \cdot \nu = 0$ on $\partial B_{r_0}(x_0)$,
with Neumann boundary condition. However, in Theorem 1.1, the initial value for any \( R \) by the property that existence of a weak solution of the Yang-Mills flow with initial value \( A \) was pointed out again in [31]. We would like to point out that the weak solution equivalent to a smooth solution for \( t \) at the finite or infinite singular time \( T \) at the finite or infinite singular time \( T \) of Struwe and proved constructed by Struwe in [26] is a weak limit of smooth solutions. In this sense, we would like to point out that (1.4)-(1.5) can be obtained by using Uhlenbeck’s gauge fixing theorem might be not unique, one cannot prove (1.6) easily. Instead, we have to follow all steps of Uhlenbeck’s original proof to fix Coulomb gauges for each \( t > 0 \) along the flow to prove (1.6).

As an application of Theorem 1.1, we prove

**Theorem 1.2.** For a connection \( A_0 \) with \( F_{A_0} \in L^{n/2}(M) \) with \( n \geq 4 \), there is a solution of the Yang-Mills flow (1.3) in \( M \times [0, T_1) \) with initial value \( D_{A_0} = D_{ref} + A_0 \) for a maximal existence time \( T_1 > 0 \). For each \( t \in (0, T_1) \), the solution \( A(t) \) is gauge-equivalent to a smooth solution of the Yang-Mills flow. At the maximal existence time \( T_1 \), there is at least one singular point \( x_0 \in M \), which is characterized by the property that

\[
\limsup_{t_i \to T_1} \int_{B_R(x_0)} |F(x, t_i)|^{n/2} dv \geq \varepsilon_0
\]

for any \( R \in (0, R_0] \) for some \( R_0 > 0 \).

As a consequence of Theorem 1.2 for \( n = 4 \), it provides a new proof of local existence of a weak solution of the Yang-Mills flow with initial value \( A_0 \in H^1(M) \). When \( n = 4 \), Struwe [26] proved existence of a weak solution, which is gauge-equivalent to a smooth solution for \( t \in (0, T_1) \) with the maximal existence time \( T_1 > 0 \), to the Yang-Mills flow in vector bundles over four manifolds for an initial value \( A_0 \in H^1(M) \). The author, Tian and Yin [14] introduced the Yang-Mills flow to proved the global existence of weak solutions of the Yang-Mills flow on four manifolds. Recently, using an idea on the broken Hodge gauge of Uhlenbeck [28], the author and Schabrun [15] established an energy identity for the Yang-Mills flow at the finite or infinite singular time \( T_1 \).

It was known that Struwe [26] only proved uniqueness of weak solutions of the Yang-Mills flow with initial value \( A_0 \in H^1(M) \) under an extra condition that \( A_0 \) is irreducible; i.e. for all \( s \in \Omega^0(\text{ad}E) \)

\[
\|s\|_{L^2(M)} \leq C\|D_{A_0}s\|_{L^2(M)}.
\]

It has been an open problem about the uniqueness of weak solutions of the Yang-Mills flow in four manifolds with initial data in \( H^1(M) \) (Recently, this problem was pointed out again in [31]). We would like to point out that the weak solution constructed by Struwe in [26] is a weak limit of smooth solutions. In this sense, we solve the problem of Struwe and prove

**Theorem 1.3.** When \( n = 4 \), the weak solutions of the Yang-Mills flow (1.2) with initial value \( A_0 \in H^1(M) \) are unique.

For the proof of Theorem 1.3 we need a variant of a parabolic gauge fixing theorem for the Yang-Mills flow. However, in Theorem 1.1, \( d^*a = 0 \) in \( B_{r_0}(x_0) \) with Nuemann boundary condition \( a \cdot \nu = 0 \) on \( \partial B_{r_0}(x_0) \) might be not unique, so
the parabolic gauge fixing theorem in Theorem 1.1 is not good enough to establish uniqueness of weak solutions of the Yang-Mills flow. To overcome the difficulty, we improve a key lemma of Uhlenbeck (Lemma 2.7 of [29]) from the Neumann boundary condition to the Dirichlet boundary condition. By a special covering of $M$ and ordering each open ball, we glue local connections together to a global connection on the whole manifold $M$ to prove uniqueness of weak solutions of the Yang-Mills flow. Finally, we would like to remark that for $n \geq 5$, weak solutions of the Yang-Mills flow with initial value $A_0 \in H^1(M)$ might not be unique (see [8]).

The paper is organised as follows. In Section 2, we recall some necessary background and estimates on the Yang-Mills flow. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.2. In Section 5, we show Theorem 1.3.

2. Some results on the Yang-Mills flow for smooth initial data

2.1. Local existence of the flow. Let $D_{A_0} = D_{ref} + A_0$ be a given smooth connection in $E$, where $D_{ref}$ is a given smooth connection. We write $D_{a(t)} = D_{ref} + a(t)$. Then

$$F_{D_{a(t)}} = F(D_{ref}) + D_{ref}a(t) + a(t) \# a(t).$$

Following [26], we consider an equivalent flow

$$\frac{\partial a(t)}{\partial t} = -D_{a(t)}^* F_{D_{a(t)}} - D_{a(t)}(D_{a(t)}^* a)$$

with $a(0) = A_0$. Note that (2.1) is a nonlinear parabolic system. By the well-known theory of partial differential equations, there is a unique smooth solution of (2.1) with the initial value on $M \times [0, t_1]$ for some $t_1 > 0$. By the theory of ordinary differential equations, there is a unique solution $S \in G$ to the following initial problem:

$$\frac{d}{dt}S = -S \circ (D_{a(t)}^* a)$$

in $M \times [0, t_1]$ with initial value $S(0) = I$.

Through the gauge transformation

$$D_{a(t)} = S^* D_A = S^{-1} \circ D_A \circ S,$$

we have (e.g. see [26], [11])

$$F_{D_{a(t)}} = S^{-1} F_A S, \quad D_{a(t)}(D_{a(t)}^* a) = D_{a(t)} \circ (D_{a(t)}^* a) - D_{a(t)}^* a \circ D_{a(t)}.$$

Combining (2.1), (2.2) with above facts yields

$$\frac{d}{dt}D_A = \frac{dS}{dt} \circ D_{a(t)} \circ S^{-1} + S \circ \frac{dD_{a(t)}}{dt} \circ S^{-1} + S \circ D_{a(t)} \circ \frac{dS^{-1}}{dt}$$

$$= S^{-1} \left( -D_{a(t)}^* F_{D_{a(t)}} \right) \circ S$$

$$= -D_A^* F_A.$$

This shows that $D_A = (S^{-1})^* D_{a(t)}$ satisfies the Yang-Mills flow with $A(0) = A_0$ in $M \times [0, t_1]$ for some $t_1 > 0$ and is unique (see [14]).
2.2. Some estimates on the YM flow. We recall from [29] that

**Lemma 2.1.** Let $A(t)$ be a smooth solution to the Yang-Mills flow in $M \times [0, T]$ with initial value $A(0) = A_0$ for some $T > 0$. For each $t$ with $0 < t \leq T$, we have

$$(2.3) \quad \int_M |F_{A(t)}|^2 \, dv + \int_0^t \int_M \left| \frac{\partial A}{\partial s} \right|^2 \, dv \, ds = \int_M |F_{A_0}|^2 \, dv.$$  

Moreover, we have

**Lemma 2.2.** Let $A(t)$ be a smooth solution to the Yang-Mills flow in $M \times [0, T]$ with initial value $A(0) = A_0$, and assume that there is a constant $\varepsilon > 0$ such that

$$\sup_{0 \leq t \leq T} \max_{x \in M} \int_{B_{R_0}(x)} |F_A(\cdot, t)|^{n/2} \, dv \leq \varepsilon$$

for some positive $R_0 < 1$. Then there is a constant $C$ such that

$$(2.4) \quad \int_0^T \int_M |\nabla A F_A|^2 \, dv \, dt \leq C(1 + \frac{T}{R_0^2}) \int_M |F_{A_0}|^2 \, dv.$$  

**Proof.** Applying the Bianchi identity $D_A F_A = 0$ and the well-known Weizenb"ock formula (e.g. [12]), we have

$$D_A D^*_A F_A = \nabla_A \nabla_A F_A + F_A \# F_A + Rm \# F_A,$$

where $Rm$ denote the Riemannian curvature of $M$. Let $\{B_{R_0}(x_i)\}_{i=1}^J$ be an open cover of $M$. By using the Hölder inequality and the Sobolev inequality, we have

$$\int_M |\nabla A F_A|^2 \, dv \leq \int_M |D_A^* F_A|^2 \, dv + C \int_M |F_A|^3 + |F_A|^2 \, dv$$

$$\leq C \sum_{i=1}^J \left( \int_{B_{R_0}(x_i)} |F_A|^{n/2} \, dv \right)^\frac{2}{n} \left( \int_{B_{R_0}(x_i)} |F_A|^{\frac{2n}{n-2}} \, dv \right)^{(\frac{n-2}{n})}$$

$$+ C \int_M |D_A^* F_A|^2 + |F_A|^2 \, dv$$

$$\leq C \varepsilon^{2/n} \int_M |\nabla A F_A|^2 \, dv + C \int_M (1 + \frac{1}{R_0^2}) |F_A|^2 + |D_A^* F_A|^2 \, dv.$$  

(2.4) follows from choosing $\varepsilon$ sufficiently small and integrating in $t$. \hfill \Box

**Lemma 2.3.** Let $A(t)$ be a smooth solution to the Yang-Mills flow in $M \times [0, T]$. There exist constants $\varepsilon = \varepsilon(E) > 0$ and $R_0 > 0$ such that if

$$\sup_{0 \leq t \leq T} \max_{x \in M} \int_{B_{R_0}(x)} |F_A(\cdot, t)|^{n/2} \, dv \leq \varepsilon$$

for some positive $R_0 < 1$, then

$$(2.5) \quad \int_{B_{R_0}(x)} |F_A|^{n/2}(\cdot, t) \, dv + \int_0^t \int_{B_{R_0}(x)} |F_A|^{\frac{n+4}{2}} |\nabla A F_A|^2 \, dv$$

$$\leq \int_{B_{2R_0}(x)} |F_A|^{n/2}(0) \, dv + C \frac{1}{R_0^2} \int_0^t \int_{B_{2R_0}(x)} |F_A|^{n/2}(s) \, dv$$

for all $x_0 \in M$. 

Then there is some positive constant \( C \varepsilon > \) and assume that there is a \( R \)-mills flow in \( B_6 \).

**Proof.** By Proposition 3 of [12], we have

\[
\frac{\partial F_A}{\partial t} = -D_A^* D_A F_A = -\nabla_A \nabla_A F_A + F_A \# F_A + \text{Rm} \# F_A,
\]

where \( \text{Rm} \) is the Riemannian curvature. Then

\[
\frac{d}{dt} \int_{B_2 R_0(x)} |F_A|^{n/2} \phi^2 \, dv = -\frac{n}{2} \int_{B_2 R_0(x)} |F_A|^{\frac{n}{2}} \left( \nabla_A \left( \phi \frac{\partial}{\partial t} F_A \right) \right) \phi^2 \, dv
\]

\[
= -\frac{n}{2} \int_{B_2 R_0(x)} \left( \nabla_A |F_A|^{\frac{n}{2}} \phi \right) \left( \nabla_A F_A \right) \phi \, dv + \frac{n}{2} \int_{B_2 R_0(x)} |F_A|^{\frac{n}{2}} \left( \nabla_A F_A \right) \phi \, dv
\]

\[
\leq -\frac{n}{2} \int_{B_2 R_0(x)} \left( |F_A|^{\frac{n}{2}} \left| \nabla_A F_A \right|^2 + \frac{n}{2} \left| F_A \right|^{\frac{n}{2}} \left| \nabla_A F_A \right|^2 \phi \right) \phi \, dv
\]

\[
+ C \int_{B_2 R_0(x)} |F_A|^{\frac{n}{2}} \left( |F_A|^3 + |F_A|^2 + \varepsilon |\nabla_A F_A|^2 \right) \phi \, dv + C \left| \nabla \phi \right|^2 |F_A|^n \, dv.
\]

Note that

\[
\int_{B_2 R_0(x)} |F_A|^{\frac{n}{2}} |F_A|^3 \phi^2 \, dv = \int_{B_2 R_0(x)} |F_A||F_A|^{2} \phi^2 \, dv
\]

\[
\leq \left( \int_{B_2 R_0(x)} |F_A|^{n/2} \right)^{\frac{2}{n}} \left( \int_{B_2 R_0(x)} |F_A|^{\frac{n}{2}} \phi^{\frac{n}{2}} \, dv \right)^{\frac{n}{2}}
\]

\[
\leq C \varepsilon^{\frac{2}{n}} \int_{B_2 R_0(x)} \left| \nabla \left( |F_A|^2 \phi \right) \right|^2 \, dv
\]

\[
\leq C \varepsilon^{\frac{2}{n}} \int_{B_2 R_0(x)} \left| F_A \right|^{\frac{n}{2}} \left| \nabla |F_A|^2 \phi \right|^2 \phi \, dv + C \int_{B_2 R_0(x)} \left| F_A \right|^{\frac{n}{2}} \left| \nabla \phi \right|^2 \, dv.
\]

Combining above inequalities and choosing \( \varepsilon \) sufficiently small, the claim is proved.

Moreover, we have

**Lemma 2.4.** (\( \varepsilon \)-regularity estimates) Let \( A(t) \) be a smooth solution to the Yang-Mills flow in \( B_{r_0}(x_0) \times [t_0 - r_0^2, t_0] \) with initial value \( A(0) = A_0 \) for some \( r_0 > 0 \) and assume that there is \( \varepsilon > 0 \) such that

\[
\sup_{t_0 - r_0^2 \leq t \leq t_0} \int_{B_{r_0}(x_0)} |F_A(x,t)|^{n/2} \, dv \leq \varepsilon.
\]

Then there is some positive constant \( C \) such that

\[
|F_A(x_0,t_0)|^2 \leq \frac{C}{r_0^2} \int_{t_0 - r_0^2}^{t_0} \int_{B_{r_0}(x_0)} |F_A|^2 \, dv \, dt.
\]

**Proof.** By Proposition 3 of [12], we have

\[
\left( \frac{\partial}{\partial t} - \Delta_M \right) |F_A|^2 + 2|\nabla_A F_A|^2
\]

\[
\leq C(|F_A|^2 + |F_A|)
\]
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Then it is standard to apply the Scheon’s idea to get the required result (e.g. [12]), so we omit the details here.

Lemma 2.5. (Higher regularity estimates) Let $A(t)$ be a solution to the Yang-Mills flow in $M \times [0, T]$ with initial value $A(0) = A_0$. Assume that there is a constant $\varepsilon > 0$ such that

$$
\sup_{0 \leq t \leq T} \int_{B_{r_0}(x_0)} |F_A(x, t)|^{n/2} dv \leq \varepsilon
$$

for a ball $B_{r_0}(x_0)$. Then there are positive constants $C(k)$ for any integer $k \geq 1$ such that

$$
\int_{\frac{T}{2}}^T \int_{B_{\frac{r}{2}}(x_0)} |F_A|^2 + \cdots + |\nabla^k A| |F_A|^2 dv \, dt \leq C(k).
$$

Proof. By Proposition 3 of [12], we have

$$
\left( \frac{\partial}{\partial t} - \Delta_M \right) |\nabla^k A| |F_A|^2 + 2 |\nabla^{k+1} A| |F_A|^2 \leq C |\nabla^k F_A| \sum_{j=0}^k (|\nabla^{k-j} A| |F_A| + 1) \leq C(|\nabla^k F_A|^2 + 1).
$$

Then we apply the Moser’s estimate to get the required result. \(\square\)

3. Proof of Theorem 1.1

In order to prove Theorems 1.1 we need some lemmas.

Lemma 3.1. For a matrix $u_1(t)$, set $s_1(t) = e^{-u_1(t)} \circ \frac{de^{u_1(t)}}{dt}$. If $|u_1(t)|$ is bounded, then

$$
|\nabla s_1(t)| \leq C \left| \nabla \frac{du_1}{dt} \right| + C |\nabla u_1| \left| \frac{du_1}{dt} \right|.
$$

Proof. Note that

$$
\nabla s_1(t) = \nabla e^{-u_1(t)} \circ \frac{de^{u_1(t)}}{dt} + e^{-u_1(t)} \circ \nabla \frac{de^{u_1(t)}}{dt}
$$

and

$$
e^{u_1(t)} = I + u_1(t) + \frac{u_1^2}{2!} + \cdots + \frac{u_1^k}{k!} + \cdots.
$$

Then

$$
\frac{de^{u_1(t)}}{dt} = \frac{du_1}{dt} + \frac{du_1}{dt} u_1(t) + u_1(t) \frac{du_1}{dt} + \cdots + \frac{du_1}{dt} u_1^{k-1} + \cdots + u_1^{k-1} \frac{du_1}{dt} + \cdots,
$$

which implies

$$
\left| \frac{de^{u_1(t)}}{dt} \right| \leq \left| \frac{du_1}{dt} \right| + \left| \frac{du_1}{dt} \right| (e^{u_1(t)} - 1).
$$

Similarly, we have

$$
\left| \nabla e^{-u_1(t)} \right| \leq |\nabla u_1| + |\nabla u_1| (e^{u_1(t)} - 1).
$$

Then
Then we claim that there is a constant $C$ for any $p > 2$.

**Proof.**

\[
\frac{\partial}{\partial t} \nabla_{e^{u_1(t)}} = \frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial t} (t) + \frac{\partial u_1}{\partial t} + \cdots + \frac{\partial u_1}{\partial t} + \cdots.
\]

Moreover

\[
|\nabla (u^k)| = |\nabla u_1 u^k - \cdots + u^k \nabla u_1| \leq (k - 1) |\nabla u_1| |u_1|^{k-2}.
\]

Then

\[
\left| \frac{\partial u_1}{\partial t} \right| \leq \left| \nabla u_1 \right| (e^{u_1} - 1) + \left| \frac{\partial u_1}{\partial t} \right| |\nabla u_1| (1 + |u_1| + \cdots + \frac{|u_1|^{(k-2)}}{(k-2)!} + \cdots)
\]

This proves our claim.

**Lemma 3.2.** For a given function $f \in W^{1,p}(S^{n-1} \times [0,1])$, let $v$ be a solution of the heat equation on $S^{n-1} \times [0,1]$ satisfying

\[
\partial_r v = \Delta_{S^{n-1}} v + f
\]

with $v(\theta, 1) = 0$ on $S^{n-1}$. Let $\varphi(r)$ be a smooth cut-off function in $[0,1]$ with $\varphi(r) = 1$ near 1 and $\varphi(r) = 0$ for $[0, \eta]$ with some small constant $\eta > 0$. Then we have

\[
||\varphi v||_{W^{1,p}(S^{n-1} \times [0,1])} \leq C ||f||_{L^p(S^{n-1} \times [0,1])}
\]

and

\[
||\varphi v||_{W^{2,p}(S^{n-1} \times [0,1])} \leq C ||f||_{W^{1,p}(S^{n-1} \times [0,1])}
\]

for any $p > 1$.

**Proof.** This lemma was mentioned by Uhlenbeck in [28]. For completeness, we give a proof here.

By the standard $L^p$-estimate of parabolic equations [17], we have

\[
\int_{S^{n-1} \times [0,1]} |\partial_r (\varphi v) + \Delta_{S^{n-1}} (\varphi v) + |v\varphi v| |f + \partial_r \varphi v| |d\theta dr
\]

\[
\leq C \int_{S^{n-1} \times [0,1]} |f|^p d\theta dr + C \int_{S^{n-1} \times [\eta,1]} |v|^p d\theta dr.
\]

Then we claim that there is a constant $C$ such that

\[
\int_{S^{n-1} \times [\eta,1]} |v|^p d\theta dr \leq C \int_{S^{n-1} \times [0,1]} |f|^p d\theta dr.
\]

If not, there is a sequence of $k$ and solutions $v_k$ of the heat equation

\[
\partial_r v_k = \Delta_{S^{n-1}} v_k + f_k
\]
with $v_k(\theta, 1) = 0$ on $S^{n-1}$ such that
\[
\int_{S^{n-1} \times [\eta, 1]} |v_k|^p d\theta dr \geq k \int_{S^{n-1} \times [0, 1]} |f_k|^p d\theta dr.
\]
Set
\[
\tilde{v}_k = \frac{v_k}{\|v_k\|_{L^p(S^{n-1} \times [\eta, 1])}}, \quad \tilde{f}_k = \frac{f_k}{\|v_k\|_{L^p(S^{n-1} \times [\eta, 1])}}.
\]
It implies that $\|\tilde{v}_k\|_{L^p(S^{n-1} \times [\eta, 1])} = 1$ and $\tilde{v}_k$ satisfies
\[
\partial_r \tilde{v}_k = \Delta_{S^{n-1}} \tilde{v}_k + \tilde{f}_k.
\]
with $\tilde{v}_k(\theta, 1) = 0$ on $S^{n-1}$. Noting that $\|\tilde{f}_k\|_{L^p(S^{n-1} \times [\eta, 1])} \leq 1/k$, there is a function $v_\infty$ such that as $k \to \infty$, $\partial_r \tilde{v}_k$ converges to $\partial_r v_\infty$ and $\nabla^2_\theta \tilde{v}_k$ converges to $\nabla^2_\theta v_\infty$ weakly in $L^p(S^{n-1} \times [\eta, 1])$. By the Sobolev compact imbedding theorem (e.g. [17]), $\tilde{v}_k$ converges to $v_\infty$ strongly in $L^p(S^{n-1} \times [\eta, 1])$, which implies that
\[
\|v_\infty\|_{L^p(S^{n-1} \times [\eta, 1])} = 1.
\]
Moreover, $v_\infty$ also satisfies
\[
\partial_r v_\infty = \Delta_{S^{n-1}} v_\infty.
\]
with $v_\infty(\theta, 1) = 0$ on $S^{n-1}$. By the backward uniqueness of the heat equation, $v_\infty$ must be zero in $S^{n-1} \times [\eta, 1]$. This is contradicted with the fact that $\|v_\infty\|_{L^p(S^{n-1} \times [\eta, 1])} = 1$. Therefore, our claim is proved, so (3.2) holds. Similarly, (3.3) can be proved by differentiating in $r$ in (3.4).

We recall a key lemma of Uhlenbeck (Lemma 2.7 in [29]) in the following:

**Lemma 3.3.** For some $p > \frac{n}{2}$, let $A \in W^{1,p}(U)$ be a connection satisfying $d^* A = 0$ in $\bar{U} = B_1(0)$ with
\[
\|A\|_{L^p(U)} \leq k(n)
\]
for a sufficiently small $k(n)$. Let $\lambda \in W^{1,p}(U)$ satisfy $\lambda \cdot \nu = 0$ on $\partial U$. There is a small constant $\varepsilon > 0$ such that if
\[
\|\lambda\|_{W^{1,p}(U)} \leq \varepsilon,
\]
then there is a gauge transformation $S = e^a \in W^{2,p}(U)$ to solve
\[
(3.5) \quad d^* a = d^* (S^{-1} dS + S^{-1} (A + \lambda) S) = 0
\]
in $U$ with $\int_U u dx = 0$ and $\partial_r u = 0$ on $\partial U$.

Now we complete a proof of Theorem 1.1

**Proof.** Without loss of generality, we assume that $U = B_1(0)$ and denote $D_A = d + A$. At $t = 0$, it follows from Uhlenbeck’s gauge fixing theorem [29] that there is a smooth gauge transformation $S_0 = S(0)$ and a connection $D_{a(0)} = S_0^* (D_A(0)) = d + a(0)$ satisfying
\[
d^* a(0) = 0 \quad \text{in } U, \quad a(0) \cdot \nu = 0 \text{ on } \partial U
\]
and
\[
\int_U |a(0)|^p + |\nabla a(0)|^p dv \leq C(p) \int_U |F_{a(0)}|^p dv
\]
for any $p > \frac{n}{2}$. 

For any $p \in (n/2, n]$ and for the above $\varepsilon > 0$, there is a constant $\delta > 0$ such that for all $t, t' \in [0, t_1]$ with $|t - t'| \leq \delta$, we have

$$
\int_U |\nabla (A(t) - A(t'))|^p + |A(t) - A(t')|^p \, dv \leq \varepsilon^p.
$$

Next, we follow the procedure of [29] to fix a Coulomb gauge in $[0, \delta]$. Through the gauge transformation $S_0$ the induced connection $D_{\tilde{A}(t)} = S^*(0)(D_{A(t)}) = d + \tilde{A}(t)$, with $\tilde{A}(t) = S_0^{-1} dS_0 + S_0^{-1} A(t) S_0$, is also a smooth solution the Yang-Mills flow in $\bar{U} \times [0, t_1]$ with $\tilde{A}(0) = a(0)$. However, $\tilde{A}(t)$ does not satisfy the boundary condition of $\tilde{A} \cdot \nu = 0$ on $\partial U$, so we cannot apply above Lemma 3.3 to fix a Coulomb gauge for $\tilde{A}(t)$ for $t \in [0, \delta]$. In order to sort out the boundary issue, it follows from Lemma 2.6 of [29] to get that there are gauge transformations $e^{u_1(t)}$ such that

$$
e^{-u_1(t)} (D_{\tilde{A}(t)}) = e^{-u_1(t)} \circ (d + \tilde{A}(t)) \circ e^{u_1(t)} = d + a_1(t),$$

where $a_1(t) := \tilde{A}(0) + \lambda(t)$ and

$$
\lambda(t) := -\tilde{A}(0) + e^{-u_1(t)} (d e^{u_1(t)} + e^{-u_1(t)} (\tilde{A}(t))) e^{u_1(t)}.
$$

In fact, we can choose $u_1(t) = \varphi \tilde{v}$, where $\varphi(r)$ is a smooth cut-off function defined in Lemma 3.2 and $\tilde{v}$ is the solution of

$$
\left( \frac{\partial}{\partial r} - \Delta_{S^{n-1}} \right) \tilde{v} = x \cdot (\tilde{A}(t) - \tilde{A}(0)) \quad \text{for } (r, \theta) \in [0, 1] \times S^{n-1}
$$

with $\tilde{v}(1, \theta) = 0$ for all $\theta \in S^{n-1}$. Then, we have $u_1(t) = 0$ and $d e^{u_1(t)} = d u_1(t)$ on $\partial U$ for all $t \in [0, \delta]$, which imply

$$
\lambda(t) \cdot \nu = (d u_1(t) + \tilde{A}(t) - \tilde{A}(0)) \cdot \nu = 0 \quad \text{on } \partial U,
$$

which implies that the new connection $a_1(t)$ satisfies the required boundary condition $a_1(t) \cdot \nu = 0$ on $\partial U$.

Noting that $\tilde{A}(t) = S_0^{-1} dS_0 + S_0^{-1} A(t) S_0$, we have

$$
\tilde{A}(t) - \tilde{A}(0) = S_0^{-1} (A(t) - A(0)) S_0.
$$

Using (3.6), we have

$$
\| \tilde{A}(t) - \tilde{A}(0) \|_{W^{1, p}(U)} = \| A(t) - A(0) \|_{W^{1, p}(U)} \leq \varepsilon
$$

for any $t \in [0, \delta]$.

By the $L^p$-estimate in Lemma 3.2 we have

$$
\int_U |\nabla u_1|^q(t) \, dv \leq C \| \tilde{A}(t) - \tilde{A}(0) \|_{L^q(U)} \leq C \| \tilde{A}(t) - \tilde{A}(0) \|_{W^{1, p}(U)}
$$

for $q > n$. By the Sobolev embedding theorem, $|u_1(t)|$ is uniformly bounded for any $t \in [0, \delta]$ for a sufficiently small $\delta > 0$. Moreover, differentiating equation (3.5) in $t$ yields

$$
\frac{\partial u_1(t)}{\partial t} = \varphi \left( \frac{\partial}{\partial r} - \Delta_{S^{n-1}} \right)^{-1} (x, \frac{\partial \tilde{A}}{\partial t}).
$$

By applying the $L^p$-estimate in Lemma 3.2 again, we have

$$
\int_U |\nabla \frac{\partial u_1(t)}{\partial t}|^2 \, dv \leq C \int_U |\frac{\partial \tilde{A}}{\partial t}|^2 (\cdot, t) \, dv \leq C \int_U |\nabla A|^2 (\cdot, t) \, dv
$$

for any $t \in [0, \delta]$.\]
It can be checked that
\[ D_{a_1(t)} = d + a_1(t) = d + \tilde{A}(0) + \lambda(t) = d + e^{-u_1(t)}de^{u_1(t)} + e^{-u_1(t)}(\tilde{A}(t))e^{u_1(t)} \]
satisfies
\[ \frac{\partial a_1}{\partial t} = -D_{a_1}^*F_{a_1} + D_{a_1}s_1, \tag{3.11} \]
where \( s_1 = S_1^{-1} \circ \frac{d}{dt} S_1 \) and \( S_1(t) = e^{u_1(t)} \).

By Lemma 3.1, we have
\[ |\nabla s_1(t)| \leq C|\nabla u_1| + C|\nabla u_1| \frac{\partial u_1}{\partial t} \]
for all \( t \in [0, \delta] \) for a sufficiently small \( \delta > 0 \). By the Sobolev inequality and noticing that \( u_1(t) = 0 \) on \( \partial U \), we have
\[ \int_U |\nabla s_1(t)|^2 dv \leq C \int_U |\nabla u_1|^2 dv + C \int_U |\nabla u_1|^n dx \int_U |\nabla u_1|^{2/n} \]
\[ \leq C \int_U |\nabla u_1|^2 dv \leq C \int_U |\nabla A|^2 dv \]
since \( \int_U |\nabla u_1|^n dv \) is uniformly bounded for all \( t \in [0, \delta] \).

Next, we will fix the coulomb gauge for \( A(t) \) in \( [0, \delta] \). By using the \( L^p \)-estimate in Lemma 3.2 again, we have
\[ |u_1(t)| \leq C\|u_1(t)\|_{W^{2,p}(U)} \leq C\|\tilde{A}(t) - \tilde{A}(0)\|_{W^{1,p}(U)} \leq C\varepsilon \]
for all \( t \in [0, \delta] \). We note that
\[ \lambda(t) = e^{-u_1(t)}\tilde{A}(0)e^{u_1(t)} - \tilde{A}(0) + e^{-u_1(t)}de^{u_1(t)} + e^{-u_1(t)}(\tilde{A}(t) - \tilde{A}(0))e^{u_1(t)} \]
Then
\[ \|\lambda(t)\|_{W^{1,p}(U)} \leq C\varepsilon. \]

By using Lemma 3.3 for a sufficiently small constant \( \varepsilon > 0 \), there is a \( u_2(t) \in W^{2,p}(U) \) with \( \nabla u_2 \cdot \nu = 0 \) on \( \partial U \) and \( \int_U u_2(t) dv = 0 \) for all \( t \in [0, \delta] \) such that the gauge transformation \( S_2(t) = e^{u_2(t)} \in W^{2,p}(U) \) solves
\[ d^*a = d^*(S_2^{-1}dS_2 + S_2^{-1}(\tilde{A}(0) + \lambda(t))S_2) = 0 \] in \( U \)
with \( \alpha \cdot \nu = 0 \) on \( \partial U \), where \( D_a = d + a = (S_1S_2)^*(D_{\tilde{A}}) = S_2(S_1^*(D_{\tilde{A}})) \). It was indicated by Uhlenbeck (Lemma 2.7 of [29]) that \( u_2(t) \) smoothly depends on \( \lambda(t) \), so we choose the norm \( \|u_2(t)\|_{W^{2,p}(U)} \) sufficiently small since \( \|\lambda(t)\|_{W^{1,p}(U)} \) is very small. This implies that \( |u_2(t)| \) can be sufficiently small for \( t \in [0, \delta] \). In fact, we can verify this directly. Note that the equation (3.12) is equivalent to
\[ -d^*du_2 = d^*[(e^{u_2})^{-1}de^{u_2(t)} - du_2] + \nabla e^{-u_2} \#(\tilde{A}(0) + \lambda(t)) \#e^{u_2} + e^{-u_2}d^*\lambda(t)e^{u_2} \]
Note that \( \|\tilde{A}(0)\|_{L^n(U)} \leq C\varepsilon \leq k(n) \) and \( \|\lambda(t)\|_{W^{1,p}(U)} \leq C\varepsilon \). By the \( L^p \)-estimate, we have
\[ \|u_2(t)\|_{W^{2,p}(U)} \leq C\|\lambda(t)\|_{W^{1,p}(U)}^p + C \int_U |\nabla u_2(t)|^p dv \]
\[ \leq C\|\lambda(t)\|_{W^{1,p}(U)}^p + C \left( \int_U |\nabla u_2|^n dv \right)^{\frac{p}{n}} \left( \int_U |\tilde{A}(0)|^n + |\lambda(t)|^n dv \right)^{\frac{p}{n}}. \]
For a sufficient small \( \varepsilon > 0 \) and using the Sobolev inequality, we have for \( p > n/2 \)
\[ \|u_2(t)\|_{W^{2,p}(U)} \leq C\|\lambda(t)\|_{W^{1,p}(U)} \leq C\varepsilon. \]
In fact, through a bootstrap argument, it can be proved that \( u_2 \) is smooth in \( U \) since \( A \) is smooth in \( U \) (see also in Proposition 9.3 of [27]). By Lemma 3.1 we have

\[
C^{-1} |\frac{\partial u_2}{\partial t}| \leq |s_2| \leq C |\frac{\partial u_2}{\partial t}|
\]

and

\[
|\nabla \frac{\partial u_2}{\partial t}| \leq |\nabla s_2(t)| + C |\nabla u_2| |\frac{\partial u_2}{\partial t}|
\]

for \( t \in [0, \delta] \). Since \( \int_U u_2(t) \, dx = 0 \), we have \( \int_U \frac{\partial u_2}{\partial t} \, dv = 0 \). Since \( \int_U |\nabla u_2(t)|^n \, dv \) can be chosen to be small for \( t \in [0, \delta] \), then we have

\[
\int_U |\nabla s_2(t)|^2 \, dv \leq C \int_U |\nabla u_2|^2 \, dv + C \left( \int_U |\nabla u_2|^n \right)^{2/n} \left( \int_U |\nabla u_2|^{2n} \, dv \right)^{1/n} \leq C \int_U |\nabla s_2(t)|^2 \, dv.
\]

It implies that

\[
\int_U |s_2(t)|^2 \, dv \leq C \int_U |\nabla u_2|^2 \, dv \leq C \int_U |\nabla s_2(t)|^2 \, dv \leq C \int_U |\nabla u_2|^2 \, dv.
\]

Set \( s_2(t) = S_2^{-1} \circ \frac{dS_2}{dt} \) with \( S_2 = e^{u_2} \). Then \( D_a = S_2^* (D_{a_1}) \) satisfies (L3); i.e.

\[
(3.14) \quad \frac{\partial a}{\partial t} = -D_a^* F_a + D_a s \quad \text{in } U \times [0, \delta]
\]

with \( s = (S_1(t)S_2(t))^{-1} \circ \frac{dS_1(t)S_2(t)}{dt} = S_2^{-1}(t) S_1(t) S_2(t) + S_2^{-1}(t) \circ \frac{dS_2}{dt} \).

Using the fact that \( d^{\ast} a = 0 \) in \( U \) and \( a \cdot \nu = 0 \) on \( \partial U \), it implies from Lemma 2.5 of [29] that for all \( t \in [0, \delta_1] \)

\[
\int_U |a(\cdot, t)|^{n/2} + |\nabla a(\cdot, t)|^{n/2} \, dv \leq C \int_U |F_a(\cdot, t)|^{n/2} \, dv \leq C \varepsilon.
\]

By the Sobolev inequality, we have

\[
\|a\|_{L^\infty(U)} \leq C \|a(\cdot, t)\|_{W^{1,n/2}(U)} \leq C \varepsilon.
\]

Recalling that \( s(t) = S_2^{-1}(t) s_1(t) S_2(t) + s_2(t) \), we have

\[
\int_U \left< ds, \frac{\partial a}{\partial t} \right> \, dv = \int_U \left< \frac{\partial s}{\partial x_k}, \frac{\partial a}{\partial t} \right> \, dv = \int_U \left< s, \frac{\partial d^{\ast} a}{\partial t} \right> \, dv + \int_{\partial U} \left< s, \partial a \cdot \nu \right> = 0.
\]

By using the Hölder and the Sobolev inequality, we have

\[
(3.15) \quad \int_U \left< D_a s, \frac{\partial a}{\partial t} \right> \, dv = \int_U \left< ds, \frac{\partial a}{\partial t} \right> + \left< [a, S_2^{-1}(t) s_1(t) S_2(t) + s_2(t)], \frac{\partial a}{\partial t} \right> \, dv \leq C \left( \int_U |a|^{n/2} \, dv \right)^{2/n} \left( \int_U |s_1|^{2n} \, dv \right)^{\frac{(n-2)}{n}} + \left( \int_U |s_2|^{2n} \, dv \right)^{\frac{(n-2)}{n}} \]

\[
+ \frac{1}{C} \int_U |\frac{\partial a}{\partial t}|^2 \, dv \leq C \varepsilon \int_U |\nabla s_1|^2 + |\nabla s_2|^2 \, dv + \frac{1}{C} \int_U |\frac{\partial a}{\partial t}|^2 \, dv.
\]
Using (3.14), we know
\[
\int_U |D_a s - \frac{\partial a}{\partial t}|^2 \, dv \leq C \int_U |\nabla a F_a|^2 \, dv.
\]
Since \( s(t) = S_2^{-1}(t)s_1(t)S_2(t) + s_2(t) \) with \( S_2 = e^{u_2} \), we note
\[
|\nabla s_2| \leq |D_a s| + |\nabla s_1| + C|\nabla S_2| |s_1| + |a|(|s_1| + |s_2|).
\]
It follows from (3.13) that \( \|\nabla S_2\|_{L^r(U)} \) can be sufficiently small. Then
\[
\int_U |\nabla s_2|^2 \, dv \leq C \int_U |D_a s|^2 + |\nabla s_1|^2 + |\nabla S_2|^2 |s_1|^2 + |a|^2(|s_1|^2 + |s_2|^2) \, dv
\]
\[
\leq C \int_U |D_a s|^2 + |\nabla s_1|^2 \, dv + C \left( \int_U |\nabla S_2|^n \, dv \right)^{\frac{2}{n}} \left( \int_U |s_1|^{\frac{2n}{n-2}} \, dv \right)^{\frac{n-2}{n}}
\]
\[
+ C \left( \int_U |a|^n \, dv \right)^{\frac{2}{n}} \left( \int_U (|s_1|^{\frac{2n}{n-2}} + |s_2|^{\frac{2n}{n-2}}) \, dv \right)^{\frac{n-2}{n}}
\]
\[
\leq C \int_U |D_a s|^2 + |\nabla s_1|^2 \, dv + C \varepsilon \int_U |\nabla s_2|^2 \, dv.
\]
Choosing \( \varepsilon \) sufficiently small in (3.15), we obtain
\[
(3.16) \quad \int_U |s|^2 + |D_a s|^2 + |\frac{\partial a}{\partial t}|^2 \, dv \leq C \int_U |\nabla a F_a|^2 \, dv
\]
for any \( t \in [0, \delta] \). Then we have
\[
\int_0^{\delta} \int_U |s|^2 + |D_a s(t)|^2 + |\frac{\partial a}{\partial t}|^2 \, dv \, dt \leq C \int_0^{\delta} \int_U |\nabla a F_a|^2 \, dv \, dt.
\]
Moreover, we have \( d^* a(t) = 0 \) in \( U \) and \( a(t) \cdot \nu = 0 \) on \( \partial U \) for \( t \in [0, \delta] \).

For the above choices of \( \delta \), we assume that \( \delta \leq t_1 \). If \( \delta < t_1 \), then we repeat the above the procedure starting at \( t_0 = \delta \) instead of \( t_0 = 0 \); i.e. At \( t_0 = \delta \), there is a gauge transformation \( \hat{S} = S(\delta) = e^{u_1(\delta)} \) such that \( D_{\hat{A}_t} = d + \hat{A}(t) = \hat{S}^*(DA) \) is a smooth in \( \hat{U} \) such that at \( t = \delta \)
\[
d^* \hat{A}(\delta) = 0, \quad \text{in } U, \quad \hat{A}(\delta) \cdot \nu = 0 \text{ on } \partial U.
\]
Since \( \hat{S} \) is a smooth transformation, \( \hat{S}^*(DA) \) is also a smooth solution of Yang-Mills flow. Repeating the above procedure, we can find two new smooth \( u_1(t) \) and \( u_2(t) \) on \( [\delta, 2\delta] \) starting at \( t = \delta \) with \( u_2(\delta) = 0 \). More precisely, there is a new gauge transformation \( S_1(t) = e^{u_1(t)} \) and \( S_2(t) = e^{u_2(t)} \) for any \( t \in [\delta, 2\delta] \), with initial condition \( \hat{u}_1(\delta) = 0 \) and \( \hat{u}_2(\delta) = 0 \), and the new connection
\[
D_{a(t)} = S(t)^* (D_{A(t)}) = (e^{u_2(t)})^* \circ (e^{u_1(t)})^* \circ (\hat{S}^*(DA(t)))
\]
for \( t \in [\delta, 2\delta] \) satisfying the same equation (2.19) (or (2.22)) in \( U \times [\delta, 2\delta] \) with initial values \( \hat{u}_2(\delta) = 0 \) and \( \hat{u}_2(\delta) = 0 \).

For a \( \delta > 0 \), there are finitely numbers \( l \) so that
\[
[0, t_1] \subset [0, l\delta].
\]
In conclusion, for any \( t \in [0, t_1) \), there are gauge transformations \( S(t) \) and connection \( D_{a(t)} = S^*(t)(DA(t)) \) satisfies equation (1.13)-(1.16) in \( U \times [0, t_1] \).

□
4. Compactness theorem and existence of weak solutions

**Theorem 4.1.** (Parabolic compact theorem) Let \( D_{A^k} \) be a sequence of smooth solutions of the Yang-Mills flow (1.2) in \( \bar{U} \times [0, t_1] \) for a uniform constant \( t_1 > 0 \), where \( U = B_{r_0}(x_0) \) for some \( r_0 > 0 \), Assume that there is a uniform constant \( \varepsilon \) such that

\[
\sup_{0 \leq t \leq t_1} \int_U |F_{A^k(x,t)}|^{n/2} dv \leq \varepsilon,
\]

and

\[
\int_0^{t_1} \int_U |\nabla A^k(x,t) F_{A^k(x,t)}|^2 dv \, dt \leq C
\]

for a uniform constant \( C > 0 \). Then, the solution \( D_{A^k} \) converges, up to a gauge transformation, to a connection \( D_A \) smoothly in \( U \times (0, t_1] \), and \( D_A \) is a solution of the Yang-Mills flow in \( U \times (0, t_1] \).

**Proof.** Let \( D_{A^k(t)} \) be a sequence of smooth solutions of the Yang-Mills flow in \( U \times [0, t_1] \) satisfying (4.1) and (4.2). By using Theorem 1.1, there are gauge transformations \( S_k(t) = e^{u_k(t)} \) and new connections \( D_{a^k} = (S_k)^* (D_{A^k}) = \mathcal{D} + a^k \) such that

\[
d^* a^k = 0 \quad \text{in} \ U, \quad a \cdot \nu = 0 \quad \text{on} \ \partial U,
\]

satisfying

\[
\int_U r_0^{-n/2} |a^k(t)|^{n/2} + |\nabla a^k(t)|^{n/2} dv \leq C \int_U |F_{a^k(t)}|^{n/2} dv \leq C\varepsilon
\]

for any \( t \in [0, t_1] \), and \( D_{a^k} \) is a solution of the equation

\[
\frac{\partial a^k}{\partial t} = -D_{a^k}^* F_{a^k} + D_{a^k} s^k
\]

in \( U \times [0, t_1] \), where

\[
s^k(t) = (S^k)^{-1}(t) \circ \frac{d}{dt} S^k(t),
\]

and

\[
\int_0^{t_1} \int_U \frac{1}{r_0^2} |s^k|^2 + |D_{a^k} s^k|^2 + |\frac{\partial a^k}{\partial t}|^2 dv \, dt \leq C \int_0^{t_1} \int_U |\nabla a^k F_{a^k}|^2 dv \, dt \leq C.
\]

Letting \( k \to \infty \), \( (a^k, s^k) \) converges to \( (a, s) \), which is a solution to

\[
\frac{\partial a}{\partial t} = -D_{a}^* F_{a} + D_{a} s
\]

in \( U \times [0, t_1] \). Moreover, we have

\[
d^* a(t) = 0 \quad \text{in} \ U, \quad a(t) \cdot \nu = 0 \quad \text{on} \ \partial U
\]

and

\[
\int_U |F_{a(t)}|^{n/2} dv \leq \liminf_{k \to 0} \int_U |F_{a^k(t)}|^{n/2} dv \leq \varepsilon.
\]

Using the condition (4.6), we have

\[
\int_U r_0^{-n/2} |a(t)|^{n/2} + |\nabla a(t)|^{n/2} dx \leq C \int_U |F_{a(t)}|^{n/2} dv \leq C\varepsilon
\]
for all \( t \in [0, t_1] \) and
\[
(4.7) \quad \int_0^{t_1} \int_{B_{r_0}} \frac{1}{r_0^2} |s|^2 + |D_\nu s|^2 + \frac{\partial a}{\partial t}^2 + |\nabla_a F_a|^2 \, dv \, dt \leq C.
\]

Using (4.5) and the identity \( D^*_a D^*_a F_a = 0 \) (see [26]), we have
\[
(4.8) \quad d^* a = a^# D^*_a F_a + a^# \nabla s.
\]

Let \( \phi \) be a cut-off function in \( C_0^\infty(B_{\frac{4}{3}r_0}) \) with \( \phi = 1 \) in \( B_{r_0/2} \). Multiplying (4.8) with \( \phi \), we have
\[
d^* d(\phi s) = a^# D^*_a F_a \phi + a^# \nabla (s \phi) + a^# \nabla \phi s + s d^* d\phi + d\phi^# ds.
\]

By the \( L^p \)-estimates, we know
\[
\int_{B_{\frac{4}{3}r_0}} |\nabla^2 (s \phi)|^2 \, dv
\leq C \left( \int_{B_{\frac{4}{3}r_0}} |a|^n \partial_a \left( \int_{B_{\frac{4}{3}r_0}} |\nabla_a F_a|^\frac{n-2}{n} \, dv \right) \right)^{\frac{n}{n-2}} + \left( \int_{B_{\frac{4}{3}r_0}} |\nabla (s \phi)|^\frac{n-2}{n} \, dv \right)^{\frac{n}{n-2}}
+ C \left( \int_{B_{\frac{4}{3}r_0}} |a|^n \, dv \right) \left( \int_{B_{\frac{4}{3}r_0}} |s|^\frac{2n}{n-2} \, dv \right)^{\frac{n-2}{n}} + C \int_{B_{\frac{4}{3}r_0}} |\nabla s|^2 + |s|^2 \, dv
\leq C \varepsilon \int_{B_{\frac{4}{3}r_0}} |\nabla^2 (s \phi)|^2 + C \int_{B_{r_0}} |\nabla_a F_a|^2 + |F_a|^2 + |\nabla s|^2 + |s|^2 \, dv.
\]

For a sufficiently small \( \varepsilon > 0 \), we obtain
\[
\int_{B_{\frac{4}{3}r_0}} |\nabla^2 s|^2 \, dv \leq C \int_{B_{\frac{4}{3}r_0}} |\nabla_a^2 F_a|^2 + |\nabla_a F_a|^2 + |F_a|^2 \, dv + C.
\]

By (4.5), we have
\[
(4.9) \quad \frac{\partial a}{\partial t} = \Delta a + a^# F_a + \nabla a^# a + D_\nu s.
\]

Then \( \partial_a, \nabla^2 a \in L^2(B_{(1-\theta)r_0} \times [\theta, t_1]) \) for any \( \theta > 0 \). Using \( d^* a = 0 \) and the fact that \( F_a \) is bounded inside \( B_{(1-\theta)r_0} \times [\theta, t_1] \), we have
\[
\sum_{j=0}^{l+1} \int_U |\nabla^j a|^2 \, dv \leq C_l \sum_{j=1}^{l} \int_U |\nabla^j F_a|^2 \, dv.
\]

It was pointed out in [26] that using (4.5) and (4.8), \( (a, s) \) is smooth in \( U \times (0, t_1) \) by a bootstrap method (In fact, it can be also proved by using Lemma 2.4 Lemma 2.5).

Using the Uhlenbeck gauge fixing theorem, at each \( t_0 > 0 \), there are gauge transformations \( S^k(t_0) \) such that \( A^k(t) := S^k(t_0)(A^k(t)) \) satisfies the Yang-Mills flow, \( d^* A^k(t_0) = 0 \) in \( U \) and \( A^k(t_0) \cdot \nu = 0 \) on \( \partial U \). By Lemma 2.4 there is a uniform constant \( C(t_0) \) depending on \( t_0 \) such that
\[
|F_{A^k}(x, t_0)| \leq C(t_0).
\]
For each integer \( l \geq 1 \), we have
\[
\int_{\mathcal{U}} |F_{\tilde{A}^k}(x,t_0)|^2 + \cdots + |\nabla^l F_{\tilde{A}^k}(x,t_0)|^2 \, dv \leq C_l(t_0).
\]

Using \( d^* \tilde{A}^k(t_0) = 0 \), we have
\[
\sum_{j=0}^{l-1} \int_{\mathcal{U}} |\nabla^j \tilde{A}^k(x,t_0)|^2 \, dv \leq C \sum_{j=1}^{l} \int_{\mathcal{U}} |\nabla^j F_{\tilde{A}^k}|^2 \, dv.
\]

Then using the Yang-Mills flow equation again, we have
\[
\sum_{j=0}^{l} \int_{\mathcal{U}} |\nabla^j \tilde{A}(x,t)|^2 \, dv \leq \sum_{j=0}^{l} \int_{\mathcal{U}} |\nabla^j \tilde{A}(x,t_0)|^2 \, dv + C \sum_{j=1}^{l-1} \int_{t_0}^{t_1} \int_{\mathcal{U}} |\nabla^j F_{\tilde{A}}|^2 \, dv
\]
for \( t \geq t_0 \). \( \tilde{A}^k \) converges to a smooth solution \( \tilde{A} \) of the Yang-Mills flow in \( U \times (0,t_1] \).
This implies that \( a \) is smooth gauge to the smooth solution \( \tilde{A} \) for \( t \geq t_0 \). \( \square \)

As a consequence of Theorem 4.1 we have

**Theorem 4.2.** For a connection \( A_0 \) with \( F_{A_0} \in L^{n/2}(M) \), there is a local weak solution of the Yang-Mills flow \((L2)\) in \( M \times [0,t_1] \) with initial value \( D_{A_0} = D_{ref} + A_0 \) for some \( t_1 > 0 \).

**Proof.** Since \( F_{A_0} \in L^{n/2}(M) \), there is a sequence of smooth connection \( \{ A^k(0) \}_{k=1}^{\infty} \), which converges strongly to \( A_0 \) in \( W^{1,\frac{n}{2}}(M) \).

Since \( A^k(0) \) is smooth, \( D_{A^k(t)} = D_{ref} + A^k(t) \) is the unique smooth solution of the Yang-Mills flow with initial value \( A^k(0) \) for a maximal existence \( T_k > 0 \). We claim that there is a uniform constant \( t_1 > 0 \) such that \( T_k \geq t_1 \) for all \( k \geq 1 \).

For any small constant \( \varepsilon > 0 \), there is a uniform constant \( r_0 > 0 \) such that for any point \( x_0 \in M \),
\[
\int_{B_{2r_0}(x_0)} |F_{A^k_0}|^{n/2} \, dx \leq \frac{\varepsilon}{2}.
\]

Since \( M \) is compact, there is a finite cover of open balls \( \{ U_i \}_{i=1}^{l} \) of \( M \) with \( U_i = B_{r_0}(x_i) \), such that at each \( x \in M \), at most a finite number \( K \) of the balls intersect. Using Lemma 2.3 and a covering argument on \( M \), we have
\[
\int_{B_{r_0}(x_0)} |F_{A^k(t)}|^{n/2} \, dv \leq \int_{B_{2r_0}(x_0)} |F_{A^k_0}|^{n/2} \, dv + \frac{CKt}{r^2_0} \max_{0 \leq s \leq t} \max_{x \in M} \int_{B_{r_0}(x)} |F_{A^k(s)}|^{n/2} \, dv
\]
which implies
\[
\max_{x \in M} \int_{B_{r_0}(x)} |F_{A^k(t)}|^{n/2} \, dv \leq \varepsilon
\]
for all \( t \in [0, \frac{r^2_0}{2KC}] \). This implies that each smooth solution \( A^k(t) \) of the Yang-Mills flow can be extended to the uniform time \( t_1 = \frac{r^2_0}{2KC} > 0 \) such that
\[
\sup_{0 \leq t \leq t_1} \int_{B_{r_0}(x_0)} |F_{A^k(t)}|^{n/2} \, dv \leq \varepsilon
\]
and
\[
\int_{0}^{t_1} \int_{M} |\nabla A^k F A^k|^2 \, dv \, dt \leq C.
\]
Then we apply Theorem 1.1 on each $U_i$ to obtain that there are gauge transformations $S_i^k(t) = e^{a_i^k(t)}$ and new connections $D_{a_i^k} = (S_i^k)^* (D_A^k) = d + a_i^k$ such that
\[ d^* a_i^k = 0 \quad \text{in } B_{r_0}(x_0), \quad a_i \cdot v = 0 \quad \text{on } \partial U_i, \]
satisfying
\[ \int_{U_i} r^{-n/2} |a_i^k(t)|^{n/2} + |\nabla a_i^k(t)|^{n/2} \, dv \leq C \int_{U_i} |F_{a_i^k(t)}|^{n/2} \, dv \leq C \varepsilon \]
for all $t \in [0, t_1]$, and $D_{a_i^k}$ is a solution of the equation
\[ \frac{\partial a_i^k}{\partial t} = -D_{a_i^k}^* F_{a_i^k} + D_{a_i^k} s_i^k \]
in $B_{r_0}(x_0) \times [0, t_1]$, where
\[ s_i^k(t) = (S_i^k)^{-1}(t) \circ \frac{d}{dt} S_i^k(t) \]
and
\[ \int_{t_0}^{t_1} \int_{U_i} \left( \frac{\partial a_i^k}{\partial t} \right)^2 + |D_{a_i^k} s_i^k|^{2} \, dv \, dt \leq \int_{t_0}^{t_1} \int_{M} |\nabla_A F_A|^2 \, dv \, dt \leq C \int_{M} |F_A|^2 \, dv. \]

In the local trivialization of $E_{U_i}$, $d + a_i^k$ can be regarded as a local representative of the connection $D_{a_i^k}$ of $E$ over $U_i$. In the overlap $U_i \cap U_j$ of two balls, $a_i^k$ and $a_j^k$ can be identified as the same by a gauge transformation through $S_{ij,k} \in G$ between $E_{U_i}$ and $E_{U_j}$ (see Lemma 3.5 of [29]) such that
\[ a_j^k = S_{ij,k}^{-1} \circ dS_{ij,k} + S_{ij,k}^{-1} \circ a_i^k \circ S_{ij,k}, \]
\[ da_j^k = dS_{ij,k} \wedge dS_{ij,k} \circ a_i^k \circ S_{ij,k} + dS_{ij,k} \circ a_i^k \circ S_{ij,k} \circ da_i^k \circ S_{ij,k} + S_{ij,k}^{-1} \circ a_i^k \circ dS_{ij,k}. \]

between $E_{U_i}$ and $E_{U_j}$. More clearly, we have
\[ D_{a_j^k} = S_{ij,k}^* (D_{a_i^k}) = S_{ij,k}^{-1} \circ D_{a_i^k} \circ S_{ij,k}, \quad s_j^k = S_{ij,k} \circ s_i^k \circ S_{ij,k}^{-1} + S_{ij,k}^{-1} \circ \frac{d}{dt} S_{ij,k}, \]
and
\[ F_{a_j^k} = S_{ij,k}^{-1} \circ F_{a_i^k} \circ S_{ij,k}, \quad D_{a_j^k} s_j^k = S_{ij,k}^{-1} \circ D_{a_i^k} s_i^k \circ S_{ij,k}. \]

These yield that
\[ \frac{d}{dt} D_{a_j^k} = S_{ij,k}^{-1} \circ D_{a_i^k} \circ S_{ij,k} = -S_{ij,k}^{-1} \circ D_{a_i^k}^* F_{a_i^k} \circ S_{ij,k} + S_{ij,k}^{-1} \circ D_{a_i^k} s_i^k \circ S_{ij,k} = -D_{a_j^k}^* F_{a_j^k} + D_{a_j^k} s_j^k. \]

This shows that the equation (4.10) is globally defined on $M$.

We recall from Lemma 2.2 that there is a uniform constant $C$ such that
\[ \int_{t_0}^{t_1} \int_{M} |\nabla a_k F_{a_k}|^2 \, dv \, dt = \int_{t_0}^{t_1} \int_{M} |\nabla_A F_A|^2 \, dv \, dt \leq C. \]

For any $\delta > 0$, using Lemmas 2.4, 2.5 there are constants $C(\delta, l) > 0$ such that for each $l \geq 1$
\[ \int_{\delta}^{t_1} \int_{M} |\nabla a_k F_{a_k}|^2 + \cdots + |\nabla^{l} a_k F_{a_k}|^2 \, dv \, dt \leq C(\delta, l). \]
As $k \to \infty$, $a^k_t$ converges to $a_i$ for any $t \in [\delta, t_1]$ satisfying

$$\int_{U_i} \frac{1}{r_0^n} |a_i(t)|^{n/2} + |\nabla a_i(t)|^{n/2} \, dv \leq C \int_{U_i} |F_{a_i(t)}|^{n/2} \, dv$$

and $d^* a_i(t) = 0$ in $U_i$, $a_i(t) \cdot \nu = 0$ on $\partial U_i$ for $t \in [\delta, t_1]$ satisfying

$$(4.12) \quad \frac{d}{dt} a_i = -D^*_{a_i} F_{a_i} + D_{a_i} s_i$$

Moreover, we have

$$\int_{\delta}^{t_1} \int_{U_i} |D_{a_i} s_i(t)|^2 + |\frac{\partial a_i}{\partial t}|^2 \, dv \, dt \leq C \int_{\delta}^{t_1} \int_{U_i} |\nabla a_i F_{a_i}|^2 \, dv \, dt.$$

Then as $\delta \to 0$, the required result is proved.

In the local trivialization of $E_{U_i}$, $d + a_i$ can be regarded as a local representative of the connection $D_a$ of $E$ over $U_i$. In the overlap $U_i \cap U_j$ of two balls, $a_i$ and $a_j$ can be identified as the same by a gauge transformation of $S_{ij} \in G$ between $E_{U_i}$ and $E_{U_j}$ (see Lemma 3.5 of [29]) such that

$$a_j = S_{ij}^{-1} dS_{ij} + S_{ij}^{-1} a_i S_{ij},$$

$$da_j = dS_{ij}^{-1} \wedge dS_{ij} + S_{ij}^{-1} da_i S_{ij} + dS_{ij}^{-1} a_i S_{ij} + S_{ij}^{-1} a_i dS_{ij}.$$

between $E_{U_i}$ and $E_{U_j}$. Equation (4.12) is well defined in $U_i \cap U_j$ in the following:

$$\frac{d}{dt} D_{a_j} - D_{a_j} s_j = -D^*_{a_j} F_{a_j}$$

$$= -S_{ij}^{-1} \circ D^*_{a_j} F_{a_j} \circ S_{ij} = S_{ij}^{-1} \circ (\frac{dD_{a_i}}{dt} - D_{a_i} s_i) \circ S_{ij}$$

with $s_j = S_{ij}^{-1} s_i S_{ij} + S_{ij}^{-1} \circ \frac{d}{dt} S_{ij}$.

Then we have

$$|\nabla S_{ij}| + |S_{ij}^{-1}| \leq C(|a_i| + |a_j|),$$

$$|\nabla^2 S_{ij}| + |S_{ij}^{-1}| \leq C(\sqrt{(|a_i|^2 + |a_j|^2)} + |\nabla a_i| + |\nabla a_j|) \quad \text{in} \quad U_i \cap U_j.$$

Since $d^* a_i = 0$ in $U_i$, $\Delta a_i = -d^* da_i$. However, $\Delta a_j$ and $\Delta a_i$ are not gauge invariant in $U_i \cap U_j$ satisfying

$$\Delta a_j - S_{ij}^{-1} \Delta a_i S_{ij}$$

$$= \nabla^2 S_{ij}^{-1} \# \nabla S_{ij} + \nabla S_{ij}^{-1} \# \nabla^2 S_{ij} + \nabla S_{ij}^{-1} \# \nabla a_i S_{ij} + S_{ij}^{-1} \nabla a_i \# \nabla S_{ij}$$

$$+ \nabla S_{ij}^{-1} \# a_i S_{ij} + S_{ij}^{-1} a_i \# \nabla^2 S_{ij} + \nabla S_{ij}^{-1} a_i \# \nabla S_{ij}.$$
Since $D^*_a F_{a_i} = d^* da_i + da_i \# a_i + a_i \# a_i \# a_i$ and $d^* a = 0$ in $U_i$, we have

\begin{equation}
(4.13) \quad \int_0^{t_1} \int_M |\nabla^2 a|^2 dv \, dt \leq C \int_0^{t_1} \int_M (dd^* + d^* d) |a|^2 \, dv \, dt
\end{equation}

\begin{align*}
&\leq C \int_0^{t_1} \sum_{i=1}^L \int_{U_i} |d^* da_i|^2 dv + \sum_{i,j=1}^L \int_{U_i \cap U_j} |\nabla^2 S_{ij}^{-1}|^2 (|\nabla S_{ij}|^2 + |a_i|^2) dv \, dt \\
&\quad + C \int_0^{t_1} \sum_{i,j=1}^L \int_{U_i \cap U_j} |\nabla S_{ij}|^2 (|\nabla^2 S_{ij}|^2 + |\nabla S_{ij}^{-1}|^2 |\nabla a_i|^2) \, dv \, dt \\
&\quad + C \int_0^{t_1} \sum_{i,j=1}^L \int_{U_i \cap U_j} (|a_i|^2 + |a_j|^2) (|\nabla a_i|^2 + |\nabla a_j|^2 + |a_i|^4 + |a_j|^4) \, dv \\
&\leq C \int_0^{t_1} \int_M |\nabla a_i|^2 |a_i|^2 \, dv \, dt \\
&\quad + C \int_0^{t_1} \sum_{i,j=1}^L \int_{U_i \cap U_j} (|a_i|^2 + |a_j|^2) (|\nabla a_i|^2 + |\nabla a_j|^2 + |a_i|^4 + |a_j|^4) \, dv \\
\end{align*}

Using above estimates, we have

\begin{equation}
(4.14) \quad \int_0^{t_1} \int_{U_i} |\nabla a_i|^2 |a_i|^2 \, dv \, dt
\end{equation}

\begin{align*}
&\leq \sup_{0 \leq t \leq t_1} \left( \int_{U_i} |a_i|^n \, dv \right)^{\frac{2}{n}} \int_0^{t_1} \left( \int_{U_i} |\nabla a_i|^{\frac{2n}{n-2}} \, dv \right)^{\frac{n-2}{2}} \, dt \\
&\leq C \varepsilon \int_0^{t_1} \int_{U_i} |\nabla a_i|^2 + \frac{1}{r_0^2} |\nabla a_i|^2 \, dv \, dt.
\end{align*}

Similarly, we have

\begin{equation}
(4.15) \quad \int_0^{t_1} \int_{U_i} |a_i|^6 \, dv \, dt \leq \sup_{0 \leq t \leq t_1} \left( \int_{U_i} |a_i|^n \, dv \right)^{\frac{2}{n}} \int_0^{t_1} \left( \int_{U_i} |a_i|^{\frac{4n}{n-2}} \, dv \right)^{\frac{n-2}{2}} \, dt \\
\end{equation}

\begin{align*}
&\leq C \varepsilon \int_0^{t_1} \int_{U_i} |\nabla a_i|^2 + \frac{1}{r_0^4} |a_i|^4 \, dv \, dt \\
&\leq C \varepsilon \int_0^{t_1} \int_{U_i} |\nabla^2 a_i|^2 + \frac{1}{r_0^4} |\nabla a_i|^2 \, dv \, dt + \frac{1}{2} \int_0^{t_1} \int_{U_i} |a_i|^6 + \frac{1}{r_0^4} |a_i|^2 \, dv \, dt.
\end{align*}

Using (4.13)-(4.15) and choosing $\varepsilon$ sufficiently small, we have

\begin{equation}
\int_0^{t_1} \int_M |\nabla^2 a|^2 \, dv \, dt \leq C \int_0^{t_1} \int_M \frac{1}{r_0^2} |\nabla F_a|^2 + \frac{1}{r_0^2} |F_a|^2 \, dv \, dt
\end{equation}

for some constant $C > 0$.

Noting that $F_{a_i} = da_i + [a_i, a_j]$ in $U_i$, we have

\begin{equation}
(4.16) \quad \frac{\partial F_{a_i}}{\partial t} = d(\frac{\partial a_i}{\partial t}) + \frac{\partial a_i}{\partial t} \# a_i.
\end{equation}
Then

\begin{equation}
  \int_{U_i} \left| \frac{d}{dt} \frac{\partial a_i}{\partial t} \right|^2 dv \leq C \int_{U_i} \left| \frac{\partial F_{a_i}}{\partial t} \right|^2 + \left| \frac{\partial a_i}{\partial t} \right|^2 dv \leq C \int_{U_i} \left| \frac{\partial F_{a_i}}{\partial t} \right|^2 dv + C \int_{\partial U_i} \left| \frac{\partial a_i}{\partial t} \right|^2 dv + \frac{n-2}{n} \int_{U_i} \left| \nabla \left( \frac{\partial a_i}{\partial t} \right) \right|^2 dv.
\end{equation}

Since $d^*(\frac{\partial a_i}{\partial t}) = 0$ in $U_i$ and $\frac{\partial a_i}{\partial t} \cdot \nu = 0$ on $\partial U_i$, it follows from using Lemma of \[29\] that

\begin{equation}
  \int_{U_i} \left| \nabla \left( \frac{\partial a_i}{\partial t} \right) \right|^2 dv \leq C \int_{U_i} \left| d \left( \frac{\partial a_i}{\partial t} \right) \right|^2 dv.
\end{equation}

Choosing $\varepsilon$ sufficiently small, we have

\begin{equation}
  \int_{U_i} \left| \nabla \left( \frac{\partial a_i}{\partial t} \right) \right|^2 dv \leq C \int_{U_i} \left| \frac{\partial F_{a_i}}{\partial t} \right|^2 dv + C \int_{U_i} \left| \frac{\partial a_i}{\partial t} \right|^2 dv.
\end{equation}

Using equation (4.10), we have

\begin{equation}
  \frac{\partial F_{a_i}}{\partial t} = D_{a_i} \left( \frac{\partial a_i}{\partial t} \right) = -D_{a_i}D^*_{a_i} F_{a_i} + F_{a_i}(s_i).
\end{equation}

Using $D_{a_i}F_{a_i} = 0$, we have

\begin{equation}
  \int_\delta^{t_1} \int_{U_i} \left| \frac{\partial F_{a_i}}{\partial t} \right|^2 dv dt \leq C \int_\delta^{t_1} \int_{U_i} \left| \Delta_{a_i} F_{a_i} \right|^2 + \left| F_{a_i} \right|^2 \left| s_i \right|^2 dv dt \leq C \int_\delta^{t_1} \int_{U_i} \left| \nabla_{a_i} F_{a_i} \right|^2 dv dt + C \sup_{\delta \leq t \leq t_1} \max_{M} \left| F_{a_i} \right| \int_\delta^{t_1} \left( \int_{U_i} \left| s_i \right|^2 dv \right) dt \leq C(\delta, t_1) \int_{U_i} \left| \nabla_{a_i} F_{a_i} \right|^2 + \left| \nabla_{a_i} F_{a_i} \right|^2 dv dt,
\end{equation}

which implies

\begin{equation}
  \int_\delta^{t_1} \int_{U_i} \left| \nabla \left( \frac{\partial a_i}{\partial t} \right) \right|^2 dv \leq C(\delta, t_1) \int_\delta^{t_1} \int_{U_i} \left| \nabla_{a_i} F_{a_i} \right|^2 + \left| \nabla_{a_i} F_{a_i} \right|^2 dv dt.
\end{equation}
Using equation (4.10) again, we have

\[(4.20) \int_{\delta}^{t_1} \int_{U_i} |\nabla^2 s_i|^2 \, dv \, dt \leq C \int_{\delta}^{t_1} \int_{U_i} |\nabla (D_a s_i)|^2 + |\nabla ([a_i, s_i])|^2 \, dv \, dt \]

\[\leq C \int_{\delta}^{t_1} \int_{U_i} |\nabla_a F_a|^2 + |\nabla_a F_a|^2 \, dv \, dt \]

\[\leq C(\delta, t_1) \int_{\delta}^{t_1} \int_{U_i} |\nabla_a F_a|^2 + |\nabla_a F_a|^2 \, dv \, dt \]

\[+ C \int_{\delta}^{t_1} \left( \int_{U_i} |a_i|^n \, dv \right)^{2/n} \left( \int_{U_i} |\nabla_a F_a|^{2n/2} \, dv \right)^{a-2/n} \, dt \]

\[+ C \int_{\delta}^{t_1} \left( \int_{U_i} |a_i|^n \, dv \right)^{2/n} \left( \int_{U_i} |s_i|^{2n/2} \, dv \right)^{a-2/n} \, dt \]

\[+ C \int_{\delta}^{t_1} \left( \int_{U_i} |\nabla a_i|^n \, dv \right)^{2/n} \left( \int_{U_i} |s_i|^{2n/2} \, dv \right)^{a-2/n} \, dt \]

Choosing \(\varepsilon\) sufficiently small, we have

\[\int_{\delta}^{t_1} \int_{U_i} |\nabla^2 s_i|^2 \, dv \, dt \leq C(\delta, t_1) \int_{\delta}^{t_1} \int_{U_i} |\nabla_a F_a|^2 + |\nabla_a F_a|^2 \, dv \, dt,\]

An iterating argument yields that

\[(4.21) \sup_{\delta \leq t \leq t_1} \sum_{i=1}^{l} \int_{U_i} |\nabla^k a_i(t)|^2_{L^2(U_i)} + \int_{\delta}^{t_1} \int_{U_i} |\nabla^{k+1} a_i|^2 + |\nabla^k s_i|^2 \, dv \, dt \]

\[\leq C \int_{\delta}^{t_1} \int_{M} |\nabla^k F_a|^2 + \cdots + |F_a|^2 \, dv \, dt \]

for any integer \(k \geq 1\). By the Sobolev inequality, \(a(t)\) is smooth in \(M\).

\[\int_{M} |\nabla^k s_i|^2 \, dv \leq C \int_{M} |\nabla^k F_a|^2 + \cdots + |F_a|^2 \, dv \]

for \(t \in [\delta, t_1]\) and any integer \(k \geq 2\).

Using equation, we have

\[\int_{M} |\nabla^k \frac{d a}{d t}|^2 \, dv \, dt \leq C \int_{M} |\nabla^k F_a|^2 + \cdots + |F_a|^2 \, dv \]

for \(t \in [\delta, t_1]\) and any integer \(k \geq 2\).

This shows that \(a(t)\) is a smooth solution of (4.10) in \(M \times (0, t_1]\), which is smoothly gauge to a smooth solution of the Yang-Mills flow for each \(t > 0\).

\[\square\]

Proof. By the local existence result, there is a weak solution \(A(t) \in M \times [0, t_1]\) of the Yang-Mills flow with initial value \(A_0\) satisfying \(F_{A_0} \in L^{n/2}(M)\) for some \(t_1 > 0\). If \(t_1 < T_1\), \(A(t_1)\) is gauge to a smooth connection. Then we can start again at the time \(t_1\) as new initial time and extend the solution to the maximal time \(T_1 > 0\).
such that as $t_i \to T_1$, there is a constant $\varepsilon_0 > 0$ such that there is at least singular point $x_0 \in M$, which is characterized by the condition

$$\limsup_{t_i \to T_1} \int_{B_R(x_0)} |F(x, t_i)|^{n/2} \, dv \geq \varepsilon_0$$

for any $R \in (0, R_0]$ for some $R_0 > 0$.

5. Uniqueness of weak solutions in dimension four

In this section, we will prove uniqueness of the weak solutions to the Yang-Mills flow on four manifolds. Firstly, we improve the Lemma 2.7 of Uhlenbeck [29] (see also [27]) in the following:

**Lemma 5.1.** For a constant $p > \frac{n}{2}$, let $A \in W^{1,p}(U)$ be a connection satisfying $d^* A = 0$ in $\bar{U} = B_1(0)$ with

$$\|A\|_{L^n(U)} \leq \varepsilon_1$$

for a sufficiently small $\varepsilon_1 > 0$. Then there is a small constant $\varepsilon_2 > 0$ such that if

$$\|\lambda\|_{W^{1,p}(U)} \leq \varepsilon_2,$$

then there is a gauge transformation $S = e^u$ with $u \in W^{2,p}(U) \cap W^{1,p}_0(\bar{U})$ to solve

$$d^* a = d^* (S^{-1} dS + S^{-1} (A + \lambda) S) = 0$$

in $U$ with $u = 0$ satisfying

$$\|u\|_{W^{2,p}(U)} \leq C\|\lambda\|_{W^{1,p}(U)} \leq C \varepsilon_2.$$

Moreover, if $A$ and $\lambda$ are smooth in $\bar{U}$, then $S$ is smooth in $\bar{U}$.

**Proof.** Note that the equation (5.1) is equivalent to

$$-d^* du = d^*[ (e^u)^{-1} de^u - du ] + \nabla e^u \#(A + \lambda) \# e^u + e^{-u} d^* \lambda e^u.$$  

By a similar proof to Proposition 9.2 of [27], the existence of a solution of (5.3) can be also proved by the following iterations. Let $u^{k-1}$ be a smooth function with $u^{k-1} = 0$ on $\partial U$ satisfying

$$\|u^{k-1}\|_{W^{2,p}(U)} \leq \eta$$

for a small constant $\eta$. By the Sobolev inequality, the norm $\|\nabla u^{k-1}\|_{W^{1,q}(U)}$ for $q > n$ is very small, so this implies that $|u^{k-1}|$ can be small when $\eta$ is sufficiently small. For the above given $u^{k-1}$, there is a smooth solution $u^k$ of

$$-d^* du^k = d^*[ (e^{u^{k-1}})^{-1} de^{u^{k-1}} - du^{k-1}] + \nabla e^{u^{k-1}} \#(A + \lambda) \# e^{u^{k-1}} + e^{-u^{k-1}} d^* \lambda e^{u^{k-1}}$$

with boundary condition $u^k = 0$ on $\partial U$.

Note that $\|A\|_{L^n(U)} \leq \varepsilon_1$ and $\|\lambda\|_{W^{1,p}(U)} \leq \varepsilon_2$. By the $L^p$-estimate of elliptic equations (e.g. [27]) and Hölder’s inequality, we have

$$\|u^k\|_{W^{2,p}(U)} \leq C \int_U (e^{\|u^{k-1}\|} - 1) |\nabla^2 u^{k-1}|^p \, dx + C \|\lambda\|_{W^{1,p}(U)}^p + C \left( \int_U |\nabla u^{k-1}|^{n/p} \, dx \right)^{\frac{n/p}{n/p - 1}} \left( \int_U |A|^{n} + |\lambda|^n \, dx \right)^{\frac{1}{n}} + C \left( \int_U |\nabla u^{k-1}|^{n/p} \, dx \right)^{\frac{n/p}{n/p - 1}} \left( \int_U |A|^{n} + |\lambda|^n \, dx \right)^{\frac{1}{n}}.$$
Letting \( \varepsilon_1, \varepsilon_2 \) and \( \eta \) be sufficiently small, and using the Sobolev inequality, we have
\[
\|u^k\|_{W^{2,p}(U)} \leq \frac{1}{2}\|u^{k-1}\|_{W^{2,p}(U)} + C\|\lambda\|_{W^{1,p}(U)} \leq \eta,
\]
where we choose \( C\varepsilon_2 \leq \frac{1}{2}\eta \). Letting \( k \to \infty \), \( u^k \) converges to \( u \) weakly in \( W^{2,p}(U) \cap W_0^{1,p}(U) \) and \( u \) is a solution of \( (5.1) \). Using the \( L^p \)-estimate again in \( (5.4) \), we obtain \( (5.2) \). Moreover, through a bootstrap argument, it can be proved that \( u \) is smooth in \( \bar{U} \) if \( A \) and \( \lambda \) are smooth in \( \bar{U} \) (see also in Proposition 9.2 of \([27]\)). \( \square \)

Let \( \{x_i \in M | i = 1, \ldots, L \} \) be a finite number of points in \( M \) such that \( \{B_{r_0}(x_i)\}_{i=1}^L \) covers \( M \) and for each \( i \) there are at most finite number \( l \) of different \( j \)’s ball \( B_{r_0}(x_j) \) with \( B_{r_0}(x_i) \cap B_{r_0}(x_j) \neq \emptyset \). For simplicity, set \( U_i = B_{r_0}(x_i) \). For the proof of uniqueness, we need to give an order of all open balls \( U_i \) in the following:

We choose \( U_1 \) as the first ball and define the second group of open balls \( \{U_{2,j}\}_{j=1}^{L_2} \) satisfying
\[
\partial U_1 \subset \bigcup_{j=1}^{L_2} U_{2,j}, \quad U_{2,j} \cap U_{2,j+1} \neq \emptyset, \quad U_{2,1} \cap U_{2,L_2} \neq \emptyset.
\]

Then we pick the second ball in the second group and then order them according to the above fact. By induction, we define the \( k \)-th group of open balls \( \{U_{k,j}\}_{j=1}^{L_k} \) by
\[
\partial(U_1 \cup \bigcup_{j=1}^{L_2} U_{2,j} \cdots \bigcup_{j=1}^{L_{k-1}} U_{k-1,j}) \subset \bigcup_{j=1}^{L_k} U_{k,j}, \quad U_{k,j} \cap U_{k,j+1} \neq \emptyset, \quad U_{k,1} \cap U_{k,L_k} \neq \emptyset.
\]
We order all balls until the final ball \( U_L \) such that for each \( i \), \( U_i \cap U_{i+1} \neq \emptyset \).

From now on, we always assume that \( n = 4 \); i.e. \( M \) is a four dimensional manifold.

**Theorem 5.2.** Let \( D_A = d + A \) be a smooth solutions of the Yang-Mills flow \([22]\) in \( M \times [0, T] \) with smooth initial value \( A_0 \) for some \( T > 0 \). Let \( \{U_i\}_{i=1}^L \) be the above open cover with an order. There is a uniform constant \( t_1 = \frac{\sqrt{2}}{2C_1} \) for some \( C_1 > 0 \) such that for each \( i \), there is a gauge transformation \( S_i(t) = e^{u_i(t)} \) and a new connection \( D_{a_i} = S_i^*(D_A) = d + a_i \) satisfying
\[(5.5) \quad d^*a_i = 0 \quad \text{in} \quad U_i
\]
and \( D_a \) is a smooth solution of the equation
\[(5.6) \quad \frac{\partial a_i}{\partial t} = -D_{a_i}^*F_{a_i} + D_{a_i}s_i
\]
in \( U_i \times [0, t_1] \), where
\[s_i(t) = S_i^{-1}(t) \circ \frac{d}{dt}S_i(t).
\]
Moreover, for all \( i = 1, \ldots, L \), we have
\[(5.7) \quad \sup_{0 \leq t \leq t_1} \int_{U_i} r_0^{-2}|a_i(t)|^2 + |\nabla a_i(t)|^2 dv \leq \varepsilon_1
\]
and
\[(5.8) \quad \sum_{i=1}^{J} \int_0^{t_1} \int_{U_i} |\nabla^2 a_i|^2 + |\nabla s_i|^2 dv dt \leq C \sum_{i=1}^{J} \int_0^{t_1} \int_{U_i} |D_{a_i}^*F_{a_i}|^2 dv dt.
\]
Moreover, \( a_i \) and \( a_{i+1} \) can be glued to a global connection by the gauge transformation \( S_{j(i+1)}(0) \), which does not depending \( t \), on the boundary \( \partial(U_{j=1}U_j) \cap U_{i+1} \).
Proof. By Uhlenbeck’s gauge fixing theorem in [29], there is a constant ε such that if
\[ \int_{B_{2r_0}(x_i)} |F_{A(0)}|^2 \, dv \leq \varepsilon \]
for each \( i = 1, \ldots, J \), then there are smooth gauge transformations \( S_i(0) \) such that connections \( D_{a_i(0)} = S_i(0)^* (D_{A(0)}) = d + a_i(0) \) satisfy
\[ d^* a_i(0) = 0 \quad \text{in} \quad B_{2r_0}(x_i), \quad a_i(0) \cdot x = 0 \quad \text{on} \quad \partial B_{2r_0}(x_i) \]
and
\[ \int_{B_{2r_0}(x_i)} \frac{|a_i(0)|^2}{(2r_0)^2} + |\nabla a_i(0)|^2 \, dv \leq C \int_{B_{2r_0}(x_i)} |F_{A(0)}|^2 \, dv \leq C \varepsilon \leq \frac{\varepsilon_1}{2}, \]
where \( \varepsilon_1 \) is the constant given in Lemma 5.1.

Since \( A \) is a smooth connection in \( M \times [0, T] \) with \( T \geq \frac{C}{\varepsilon_1} \), for a sufficiently small \( \varepsilon \), there is a constant \( \delta > 0 \) such that for all \( t, \tilde{t} \in [0, T] \) with \( |t - \tilde{t}| < \delta \), we have
\[ \|A(t) - A(\tilde{t})\|_{W^{1,p}(M)} \leq \varepsilon \leq \varepsilon_2 \]
for some \( p > 2 \).

Let \( U_1 \) be the first open set of the above cover of \( M \). Thus, for \( t \in [0, \delta] \), we have
\[ \|S_1(0)^* (A(t)) - S_1(0)^* (A(0))\|_{W^{1,p}(U_1)} = \|A(t) - A(0)\|_{W^{1,p}(U_1)} \leq \varepsilon. \]

By Lemma 5.1, for each \( t \in [0, \delta] \), there are a gauge transformation \( S_1(t) = e^{u_1(t)} \) and a new connection \( a_1(t) = S_1(t)^* (S_1(0)^* (A(t))) \) in \( U_1 \) satisfying equations (5.5)–(5.10) with \( S_1(t) \big|_{\partial U_1} = I \) and \( s_1(t) = S_1^{-1}(t) \circ \frac{\partial}{\partial t} S_1(t) = 0 \) on \( \partial U_1 \). Moreover, there is a constant \( C > 0 \) such that \( \|a_1(t)\|_{L^2(U)} \leq C \varepsilon \) for \( t \in [0, \delta_1] \).

Using \( D_{a_1(t)} D_{a_1(t)}^* F_{a_1(t)} = 0 \) and \( d^* a_1(t) = 0 \) in \( U_1 \) for \( t \in [0, \delta_1] \), we have
\[ D_{a_1(t)}^* D_{a_1(t)} s_1 = [s_{a_1}, \frac{\partial a_1}{\partial t}] = a_1 \# D_{a_1}^* F_{a_1} + a_1 \# D_{a_1} s_1. \]

By using (5.9), the Hölder inequality and the Sobolev inequality with the fact that \( s_1(t) = 0 \) on \( \partial U_1 \), we have
\[
\begin{align*}
\int_0^\delta \|D_{a_1} s_1\|_{L^2(U_1)}^2 \, dt & = \int_0^\delta \|D_{a_1} s_1\|_{L^2(U_1)}^2 \, dt \leq C \int_0^\delta \int_{U_1} \langle D_{a_1}^* D_{a_1} s_1, s_1 \rangle \, dv \, dt \\
& \leq \frac{1}{4} \int_0^\delta \|D_{a_1} s_1\|_{L^2(U_1)}^2 + \frac{1}{2} \int_0^\delta \|D_{a_1}^* F_{a_1}\|_{L^2(U_1)}^2 \, dt \\
& + C \int_0^\delta \left( \int_{U_1} |a_1|^4 \, dx \right)^{1/2} \left( \int_{U_1} |s_1|^4 \, dx \right)^{1/2} \, dt \\
& \leq \frac{1}{2} \int_0^\delta \|D_{a_1} s_1\|_{L^2(U_1)}^2 + \frac{1}{2} \int_0^\delta \|D_{a_1}^* F_{a_1}\|_{L^2(U_1)}^2 \, dt
\end{align*}
\]
for a sufficiently small \( \varepsilon \). Then it implies that
\[ \int_0^\delta \|s_1\|_{H^1(U_1)}^2 \, dt \leq C \int_0^\delta \|D_{a_1}^* F_{a_1}\|_{L^2(U_1)}^2 \, dt. \]

Let \( U_2 \) be the second open set of \( \{U_i\}_{i=1}^J \) with \( U_1 \cap U_2 \neq \emptyset \). For \( 0 \leq t \leq \delta \), set
\[ \tilde{a}_1(t) := \begin{cases} S_{12}(0)(a_1(t)), & \text{for } x \in U_1 \cap U_2 \\ S_2(0)(A(t)), & \text{for } x \in U_2 \setminus U_1, \end{cases} \]
where \( S_{12}(0) = S_{2}(0)(S_{1}^{-1}(0)) \) is a gauge transformation in \( U_{1} \cap U_{2} \) from \( U_{1} \) to \( U_{2} \).

Note that \( S_{1}(t) \in W^{2,p}(U_{1}) \) for \( p > 2 \) and \( S_{1}(t) = I \) on \( \partial U_{1} \). Then \( \tilde{a}_{2} \in W^{1,p}(U_{2}) \) satisfies

\[
\|\tilde{a}_{2}(t) - a_{2}(0)\|_{W^{1,p}(U_{2})} \leq C \varepsilon \leq \varepsilon_{2}
\]

for \( t \in [0, \delta] \), where we used the fact that \( \|S_{12}(0)\|_{W^{1,4}(U_{1} \cap U_{2})} \leq C \varepsilon \). Using Lemma 5.1 with \( \lambda(t) = \tilde{a}_{2}(t) - a_{2}(0) \) in \( U_{2} \), there is a gauge transformation \( S_{2}(t) \in W^{2,p}(U_{2}) \) such that \( a_{2}(t) = S_{2}^{\ast}(\tilde{a}_{2}(t)) \) and \( d^\ast a_{2}(t) = 0 \) in \( U_{2} \) with \( S_{2}(t) = I \) on \( \partial U_{2} \) satisfying

\[
\partial_{t}a_{2} = -D_{a_{2}}^\ast F_{a_{2}} + D_{a_{2}}s_{2}
\]

in \( U_{2} \times [0, \delta] \), where \( s_{2} = (S_{2}(t) \circ S_{2}(0))^{-1}\frac{d}{dt}(S_{2}(t) \circ S_{2}(0)) \) in \( U_{2} \setminus (U_{1} \cap U_{2}) \) and

\[
s_{2} = (S_{2}(t) \circ S_{2}(0) \circ S_{1}(t) \circ S_{1}(0))^{-1}\frac{d}{dt}(S_{2}(t) \circ S_{12}(0) \circ S_{1}(t) \circ S_{1}(0)) \text{ in } U_{1} \cap U_{2}.
\]

Moreover, there is a gauge transformation \( S_{12}(t) = S_{2}(t) \circ S_{12}(0) \) in the intersection of \( U_{1} \cap U_{2} \) such that

\[
S_{2} = S_{12}^{-1}\frac{d}{dt}S_{12} + S_{12}^{-1}s_{1}S_{12}, \quad D_{a_{2}} = S_{12}^{-1}D_{a_{1}}S_{12} \quad \text{in } U_{1} \cap U_{2}.
\]

Using \( D_{a_{2}}^\ast D_{a_{2}}^\ast F_{a_{2}} = 0 \) and \( d^\ast a_{2} = 0 \) in \( U_{2} \), we have

\[
D_{a_{2}}^\ast D_{a_{2}}s_{2} = [s_{a_{2}}, \frac{\partial a_{2}}{\partial t}] = a_{2}\#D_{a_{2}}^\ast F_{a_{2}} + a_{2}\#D_{a_{2}}s_{2}
\]

in \( U_{2} \times [0, \delta] \). Using the fact that \( S_{2}(t) = I \) on \( \partial U_{2} \), we obtain \( \frac{d}{dt}S_{12} = 0 \) on \( \partial U_{2} \cap U_{1} \) and \( s_{2}(t) = 0 \) on \( \partial U_{2} \setminus U_{1} \), and since \( S_{1}(t) = I \) and \( s_{1}(t) = 0 \) on \( \partial U_{1} \), we note that \( S_{12}^{-1}\frac{d}{dt}S_{12} = s_{2} \) on \( \partial U_{1} \cap U_{2} \). For simplicity, we set

\[
w(x, t) := \begin{cases} 
S_{12}^{-1}\frac{d}{dt}S_{12} & \text{for } x \in U_{1} \cap U_{2}, \\
s_{2}(t) & \text{for } x \in U_{2} \setminus U_{1}.
\end{cases}
\]

It follow from 5.11 that

\[
\int_{U_{2}} |\nabla w|^{2} \leq C \int_{U_{2}} |\nabla s_{2}|^{2} + C \int_{U_{1} \cap U_{2}} |\nabla s_{1}|^{2} + |\nabla S_{12}|^{2}|s_{1}|^{2} dv
\]

\[
\leq C \int_{U_{2}} |\nabla s_{2}|^{2} dv + C \int_{U_{1}} |\nabla s_{1}|^{2} dv + C \left( \int_{U_{1} \cap U_{2}} |\nabla S_{12}|^{4} dv \right)^{1/2} \left( \int_{U_{1} \cap U_{2}} |s_{1}|^{4} dv \right)^{1/2}
\]

\[
\leq C \int_{U_{2}} |\nabla s_{2}|^{2} dv + C \int_{U_{1}} |\nabla s_{1}|^{2} + \frac{1}{r_{0}^{4}} |s_{1}|^{2} dv.
\]
Using (5.12) with \( w = 0 \) on \( \partial U_2 \), we have
\[
\int_0^\delta \int_{U_2} |D_{a_2} s_2|^2 \, dv \, dt = \int_0^\delta \int_{U_2 \setminus (U_1 \cap U_2)} |D_{a_2} s_2|^2 \, dv \, dt + \int_0^\delta \int_{U_1 \cap U_2} |D_{a_2} s_2|^2 \, dv \, dt.
\]
\[
= \int_0^\delta \int_{U_2} \langle D_{a_2}^* D_{a_2} s_2, w \rangle \, dv \, dt + \int_0^\delta \int_{U_1 \cap U_2} \langle D_{a_2} s_2, S_{12}^{-1} D_{a_1} s_1 S_{12} \rangle \, dv \, dt,
\]
\[
\leq \int_0^\delta \int_{U_2} \langle a_2 # D_{a_2}^* F_{a_2} + a_2 # D_{a_2} s_2, w \rangle \, dv \, dt
\]
\[
+ \frac{1}{8} \int_0^\delta \int_{U_1 \cap U_2} |D_{a_2} s_2|^2 \, dv \, dt + C \int_0^\delta \int_{U_1 \cap U_2} |D_{a_1} s_1|^2 \, dv \, dt
\]
\[
\leq \frac{1}{4} \int_0^\delta \| D_{a_2} s_2 \|^2_{L^2(U_2)} \, dt + \frac{1}{2} \int_0^\delta \| D_{a_2}^* F_{a_2} \|^2_{L^2(U_2)} \, dt.
\]
Also, we note from (5.10) that
\[
\int_0^\delta \| s_2 \|^2_{L^2(U_2)} \, dt \leq C \int_0^\delta \| w \|^2_{L^2(U_2)} + \| s_1 \|^2_{L^2(U_1)} \, dt.
\]
\[
\leq C \int_0^\delta \| s_1 \|^2_{H^1(U_1)} + \| D_{a_2} s_2 \|^2_{L^2(U_1)} \, dt.
\]
Then it implies that
\[
\int_0^\delta \| s_2 \|^2_{H^1(U_2)} \, dt \leq C \int_0^\delta \| D_{a_2} F_{a_2} \|^2_{L^2(U_2)} + \| D_{a_1}^* F_{a_1} \|^2_{L^2(U_1)} \, dt.
\]
Let \( U_3 \) be the third open set of \( \{ U_i \}_{i=1} \) with \( U_1 \cap U_2 \cap U_3 \neq \emptyset \). For \( 0 \leq t \leq \delta \), set
\[
\tilde{a}_3(t) := \begin{cases}
S_{13}^*(0)(a_1(t)), & \text{for } x \in (U_1 \cap U_3) \setminus U_2, \\
S_{23}^*(0)(a_2(t)), & \text{for } x \in U_2 \cap U_3, \\
S_3^*(0)(A(t)), & \text{for } x \in U_3 \setminus (U_1 \cup U_2),
\end{cases}
\]
where \( S_{13}(0) = S_3(0)(S_1^{-1}(0)) \) is a gauge transformation in \( U_1 \cap U_3 \) from \( U_1 \) to \( U_3 \) and \( S_{23}(0) = S_3(0)(S_2^{-1}(0)) \) is a gauge transformation in \( U_2 \cap U_3 \) from \( U_2 \) to \( U_3 \). Then \( \tilde{a}_3 \in W^{1,p}(U_2) \) satisfies
\[
\| \tilde{a}_3(t) - a_3(0) \|_{W^{1,p}(U_3)} \leq C \varepsilon \leq \varepsilon_2
\]
for \( t \in [0, \delta] \). Using Lemma 5.2 with \( \lambda(t) = \tilde{a}_3(t) - a_3(0) \) in \( U_2 \), there is a gauge transformation \( S_3(t) \in W^{2,p}(U_2) \) with \( S_3(t) = I \) on \( \partial U_3 \) such that \( a_3(t) = S_2^*(\tilde{a}_3(t)) \) and \( \partial^t a_3(t) = 0 \) in \( U_3 \) satisfying
\[
\frac{\partial a_3}{\partial t} = -D_{a_3}^* F_{a_3} + D_{a_3} s_3
\]
in $U_2 \times [0, \delta]$. There is a gauge transformation $S_{23}(t) = S_3(t) \circ S_{23}(0)$ in the intersection of $U_2 \cap U_3$ such that
\begin{equation}
(5.14) \quad s_3 = S_{23}^{-1} \frac{d}{dt} S_{23} + S_{23}^{-1} s_2 S_{23}, \quad D_{a_3} = S_{23}^{-1} D_{a_2} S_{23} \quad \text{in } U_2 \cap U_3,
\end{equation}
and there is a gauge transformation $S_{13}(t) = S_3(t) \circ S_{13}(0)$ in the intersection of $U_1 \cap U_3$ such that
\begin{equation}
(5.15) s_3 = S_{13}^{-1} \frac{d}{dt} S_{13} + S_{13}^{-1} s_1 S_{13}, \quad D_{a_3} = S_{13}^{-1} D_{a_1} S_{13} \quad \text{in } U_1 \cap U_3 \setminus U_2.
\end{equation}
Using $D^*_{a_3} D^*_{a_3} F_{a_3} = 0$ and $d^* a_3 = 0$ in $U_3$, we have
\begin{equation}
(5.16) \quad D^*_{a_3} D_{a_3} s_3 = [s a_3, \frac{\partial a_3}{\partial t}] = a_3 \# D^*_{a_3} F_{a_3} + a_3 \# D_{a_3} s_3
\end{equation}
in $U_2 \times [0, \delta]$. Using the fact that $S_3(t) = I$ on $\partial U_3$, we obtain $\frac{d}{dt} S_{13} = 0$ on $\partial U_3 \cap U_1 \setminus U_2$, $\frac{d}{dt} S_{23} = 0$ on $\partial U_3 \cap U_2$ and $s_3(t) = 0$ on $\partial U_3 \setminus (U_1 \cup U_2)$, and since $S_1(t) = I$ and $s_1(t) = 0$ on $\partial U_1$, we note that $S_{13}^{-1} \frac{d}{dt} S_{13} = s_3$ on $\partial U_1 \cap U_3$. We set
\begin{equation*}
w(x, t) := \begin{cases} 
S_{13}^{-1} \frac{d}{dt} S_{13}, & \text{for } x \in U_3 \cap U_1 \setminus U_2 \\
S_{23}^{-1} \frac{d}{dt} S_{23}, & \text{for } x \in U_3 \cap U_2 \\
s_3(t), & \text{for } x \in U_3 \setminus (U_1 \cup U_2),
\end{cases}
\end{equation*}
Using (5.16) with $w = 0$ on $\partial U_3$, we have
\begin{align*}
\int_0^\delta \int_{U_3} |D_{a_3} s_3|^2 dv dt &= \int_0^\delta \left( \int_{U_3 \setminus (U_1 \cup U_2)} + \int_{U_3 \cap U_1 \setminus U_2} + \int_{U_3 \cap U_2} \right) |D_{a_3} s_3|^2 dv dt \\
&= \int_0^\delta \int_{U_3} \langle D^*_{a_3} s_3, D_{a_3} s_3 \rangle dv dt + \int_0^\delta \int_{U_3 \cap U_2} \langle D_{a_3} s_3, S_{23}^{-1} D_{a_2} S_{23} \rangle dv dt \\
&\quad + \int_0^\delta \int_{U_3 \cap U_1 \setminus U_2} \langle D_{a_3} s_3, S_{13}^{-1} D_{a_1} S_{13} \rangle dv dt \\
&\leq \int_0^\delta \int_{U_3} \langle a_3 \# D^*_{a_3} F_{a_3} + a_3 \# D_{a_3} s_3, w \rangle dv dt + \frac{1}{8} \int_0^\delta \int_{U_3} |D_{a_3} s_3|^2 dv dt \\
&\quad + C \int_0^\delta \left( \int_{U_1} |D_{a_1} s_1|^2 dv + \int_{U_2} |D_{a_2} s_2|^2 dv \right) dt \\
&\leq \frac{1}{4} \int_0^\delta \|D_{a_2} s_2\|_{L^2(U_2)}^2 dt + \frac{1}{2} \int_0^\delta \|D_{a_2} F_{a_2}\|^2_{L^2(U_2)} dt \\
&\quad + C \int_0^\delta \left( \int_{U_3} |a_3|^4 dv \right)^{1/2} \left( \int_{U_3} |w|^4 dv \right)^{1/2} dt \\
&\quad + C \int_0^\delta \left( \int_{U_1} |D_{a_1} s_1|^2 dv + \int_{U_2} |D_{a_2} s_2|^2 dv \right) dt \\
&\leq \frac{1}{2} \int_0^\delta \|D_{a_3} s_3\|^2_{L^2(U_3)} + \frac{1}{2} \int_0^\delta \|D^*_{a_3} F_{a_3}\|^2_{L^2(U_3)} dt \\
&\quad + C \int_0^\delta \left( \int_{U_1} |D_{a_1} s_1|^2 dv + \frac{1}{r_0^2} |s_1|^2 dv + \int_{U_2} |D_{a_2} s_2|^2 dv + \frac{1}{r_0^2} |s_2|^2 dv \right) dt,
\end{align*}
where we have used the fact that
\begin{align*}
\int_{U_3} |\nabla w|^2 dv &\leq C \int_{U_1} |\nabla s_1|^2 + \frac{1}{r_0^2} |s_1|^2 dv + C \int_{U_2} |\nabla s_2|^2 + \frac{1}{r_0^2} |s_2|^2 dv + C \int_{U_3} |\nabla s_3|^2 dv.
\end{align*}
Repeating the above procedure, we have

\[(5.17)\int_0^t \sum_{i=1}^J \int_{U_i} \left( \frac{1}{r_0^2} |s_i|^2 + |D_\alpha s_i|^2 \right) dv dt \leq C \int_0^t \sum_{i=1}^J \|D_\alpha F_i\|_{L^2(U_i)}^2 dt \]

for all \(i = 1, \ldots, J\).

For the above construction, \(a_i\) and \(a_{i+1}\) are same by the gauge transformation \(S_{i(i+1)}(0)\), which does not depending \(t\), on the boundary \(\partial U_i \cap U_{i+1}\) and similarly, \(a^j\) with \(j \leq i\) and \(a^{i+1}\) are same by the gauge transformation \(S_{j(i+1)}(0)\), which does not depending \(t\), on the boundary \(\partial(U_{j+1}^j U_j) \cap U_{i+1}\). Therefore \(a\) is globally well defined in \(M \times [0, \delta]\).

We rewrite the flow equation (5.6) as

\[(5.18)\frac{\partial a}{\partial t} = \triangle a + \nabla a \# a + a \# a \# a + D_\alpha s\]

with initial value \(a(0) = A_0\). Let \(\phi\) be a cut-off function in \(U\). Multiplying (5.18) by \(\phi^2 a\) and using Young’s inequality, we have

\[(5.19)\frac{1}{2} \frac{d}{dt} \int_U \frac{1}{r_0^2} |a|^2 \phi^2 dv + \int_U \frac{1}{r_0^2} |\nabla a|^2 \phi^2 dv \]

\[\leq \frac{1}{4} \frac{1}{r_0^2} \int_U |\nabla a|^2 \phi^2 dv + C \frac{1}{r_0^2} \int_U |a|^2 |\nabla \phi|^2 + |a|^4 \phi^2 dv \]

\[+ \frac{1}{r_0^2} \int_U (|s||a|^2 \phi^2 + |s||a||\nabla \phi|\phi) dv \]

\[\leq \frac{1}{4} \frac{1}{r_0^2} \int_U |\nabla a|^2 \phi^2 dv + C \frac{1}{r_0^2} \int_U |a|^2 |\nabla \phi|^2 + (|a|^4 + |s|^2) \phi^2 dv \]

Multiplying (5.19) by \(\phi^2 \triangle a\), we obtain

\[(5.20)\frac{1}{2} \frac{d}{dt} \int_U |\nabla a|^2 \phi^2 dv + \int_U |\triangle a|^2 \phi^2 dv \]

\[\leq \frac{1}{4} \int_U |\triangle a|^2 \phi^2 dv + C \int_U (|\nabla a|^2 |a|^2 \phi^2 + |a|^6 \phi^2) dv \]

\[+ C \int_U |\nabla a|^2 |\nabla \phi|^2 dv + C \int_U |D_\alpha s|^2 dv.\]

By integration by parts twice and using Young’s inequality, we have

\[(5.21)\int_U |\nabla^2 a|^2 \phi^2 dv \leq \int_U |\triangle a|^2 \phi^2 dv + \frac{1}{2} \int_U |\nabla^2 a|^2 \phi^2 dv \]

\[+ C \int_U |\nabla a|^2 |\nabla \phi|^2 dv.\]

We can deal with above nonlinear terms in (5.20). By using Hölder’s inequality and the Sobolev inequality, we get

\[(5.22)\int_U \phi^2 |a|^2 |\nabla a|^2 dv \leq C \left( \int_U |a|^4 dv \right)^{1/2} \left( \int_U |\phi \nabla a|^4 dv \right)^{1/2} \]

\[\leq C \varepsilon \int_U |\nabla (\phi \nabla a)|^2 dv \leq C \varepsilon \int_U \phi^2 |\nabla^2 a|^2 dv + C \frac{1}{r_0^2} \int_U |\nabla a|^2 dv \]
and
\begin{equation}
\int_U |\mathbf{a}|^2 \, dv \leq C \left( \int_U |\mathbf{a}|^4 \, dv \right)^{1/2} \left( \int_U \phi^2 |\mathbf{a}|^4 \, dv \right)^{1/2}
\end{equation}
\begin{equation}
\leq C \varepsilon \int_U |\nabla(\phi |\mathbf{a}|^2)|^2 \, dv \leq C \varepsilon \int_U \phi^2 |\mathbf{a}|^4 + \phi^2 |\mathbf{a}|^2 |\nabla \mathbf{a}|^2 \, dv.
\end{equation}
Using (5.19)-(5.23) with a sufficiently small \( \varepsilon \), we have
\begin{equation}
\frac{d}{dt} \int_{U} \frac{1}{r_0^6} |\mathbf{a}|^2 + |\nabla \mathbf{a}|^2 \phi^2 \, dv + \int_{U} \frac{1}{r_0^8} |\nabla^2 \mathbf{a}|^2 \phi^2 \, dv
\leq C \frac{1}{r_0^6} \int_{U} \frac{1}{r_0^2} \phi^2 |\mathbf{a}|^2 + C \int_{U} \frac{1}{r_0^2} |s|^2 + |D_a s|^2 \, dv
\end{equation}
Integrating in \( t \), using (5.17) and Lemma 2.3 we have
\begin{equation}
\int_{U} |\nabla \mathbf{a}(\cdot, t)|^2 \phi^2 + \frac{1}{r_0^2} |\mathbf{a}(\cdot, t)|^2 \phi^2 \, dv
\leq \int_{U} |\nabla \mathbf{a}(0)|^2 + \frac{1}{r_0^2} |\mathbf{a}(0)|^2 \, dv + C \int_{0}^{t} \int_{U} |\nabla \mathbf{a}(\cdot, s)|^2 \phi^2 \, dv
d + \frac{C t}{r_0^2} \sup_{0 \leq s \leq t} \int_{U} |\nabla \mathbf{a}(\cdot, s)|^2 + \frac{1}{r_0^2} |\mathbf{a}(\cdot, s)|^2 \, dv
\leq \frac{1}{2} \varepsilon + \frac{C t}{r_0^2} \sup_{0 \leq s \leq t} \sup_{1 \leq k \leq J} \int_{U} |\nabla \mathbf{a}_k(\cdot, t)|^2 + \frac{1}{r_0^2} |\mathbf{a}_k(\cdot, t)|^2 \, dv \leq \varepsilon.
\end{equation}
for all \( t \in [0, \delta] \) with \( \delta \leq \frac{2 t_1}{2C_1} \). By a covering argument, we have
\begin{equation}
\sup_{0 \leq t \leq \delta} \sup_{1 \leq k \leq J} \int_{U_i} |\nabla \mathbf{a}_k(\cdot, t)|^2 + \frac{1}{r_0^2} |\mathbf{a}_k(\cdot, t)|^2 \, dv \leq \varepsilon.
\end{equation}
At \( t = \delta \), \( a(x, \delta) \in W^{1, p}(M) \) satisfies \( \|a(x, \delta)\|_{L^p(U)} \leq \varepsilon_1 \). We continue the above procedure starting at \( t = \delta \) again and prove that (5.24)-(5.26) are true for \( t \leq 2 \delta \).
By induction, we can prove (5.24)-(5.26) for all \( t \in [0, \frac{2 t_1}{2C_1}] \).
\( \square \)

**Remark:** For an initial value \( A_0 \in H^1(M) \), the result of Theorem 5.2 also holds.

Now we give a proof of Theorem 1.3.

**Proof.** Let \( \{U_i\}_{i=1}^J \) be the above open cover with an order in Theorem 5.2. For simplicity, we assume that \( r_0 = 1 \). Let \( D_{A_k} \), \( k = 1, 2 \), be two weak solutions of the Yang-Mills flow with the same initial value \( A_0 \in H^1(M) \). According to the proof of existence, \( D_{A_k} \) is the limit of smooth solutions, and hence we can apply Theorem 5.2 to have the following property: on each ball \( U_i \) with \( 1 \leq i \leq J \), there are gauge transformations \( S_i^k(t) = e^{a_i^k(t)} \) and new connections \( D_{a_i^k} = (S_i^k)^* (D_{A_k}) = d + a_i^k \) such that
\begin{equation}
a_i^k = 0 \quad \text{in} \quad U_i, \quad u_i^k(t) = 0 \quad \text{on} \quad \partial U_i,
\end{equation}
all \( t \in [0, t_1] \), and \( D_{a_i^k} \) for \( k = 1, 2 \) in \( U_i \) are solutions of the equation
\begin{equation}
\frac{\partial a_i^k}{\partial t} = -D_{a_i^k}^* F_{a_i^k} + D_{a_i^k} s_i^k
\end{equation}
in \( U_i \times [0, t_1] \), where
\begin{equation}
s_i^k(t) = (S_i^k)^{-1}(t) \circ \frac{d}{dt} S_i^k(t).
\end{equation}
For all $i = 1, \ldots, L$, we have

\begin{equation}
(5.28) \quad \sup_{0 \leq t \leq t_1} \int_{U_i} r_0^{-2} |a^k_i(t)|^2 + |\nabla a^k_i(t)|^2 \, dv \leq \varepsilon_1
\end{equation}

and

\begin{equation}
(5.29) \quad \int_0^{t_1} \sum_{i=1}^J \int_{U_i} |\nabla^2 a^k_i|^2 + \frac{1}{r_0^2} |s^k_i|^2 + |\nabla s^k_i|^2 \, dv \, dt \leq C \int_0^{t_1} \int_M |D_{a^k_i}^* F_{a^k_i}|^2 \, dv \, dt
\end{equation}

For the above construction, $a^k_i$ and $a^k_{i+1}$ are same by the gauge transformation $S_{(i+1)}(0)$, which does not depending $t$, on the boundary $\partial U_i \cap U_{i+1}$ and similarly, $a_j$ with $j \leq i$ and $a_{i+1}$ are same by the gauge transformation $S_{(i+1)}(0)$, which does not depending $t$, on the boundary $\partial(U_j \cap U_{i+1})$. Through this property, we define a global connection $a^k$ for $k = 1, 2$ in the following:

\begin{equation}
(5.30) \quad a^k = a^k_j \text{ in } U_j, \quad a^k = a^k_{j-1} \text{ in } U_{j-1} \setminus U_j, \ldots, \quad a^k = a^k_1 \text{ in } U_1 \setminus (\cup_{j=2}^J U_j)
\end{equation}

and

\begin{equation}
(5.31) \quad s^k = s^k_j \text{ in } U_j, \quad s^k = s^k_{j-1} \text{ in } U_{j-1} \setminus U_j, \ldots, \quad s^k = s^k_1 \text{ in } U_1 \setminus (\cup_{j=2}^J U_j).
\end{equation}

On the boundary $\partial(U_j \cap U_{i+1})$, there is a same gauge transformation $S_{(i+1)}(0)$ such that

\begin{equation}
(5.32) \quad a^k_{i+1}(t) = S_{(i+1)}^{-1}(0) a^k_i(t) S_{(i+1)}(0),
\end{equation}

\begin{equation}
(5.33) \quad s^k_{i+1}(t) = S_{(i+1)}^{-1}(0) s^k_i(t) S_{(i+1)}(0).
\end{equation}

For simplicity, we set $b = a^2 - a^1$ and $\sigma = s^1 - s^2$ on $M$. Since $\int_M \sigma(t) \, dv$ is a constant section in $t$, which does not depend on $x$, we can assume $\int_M \sigma(t) \, dv = 0$ for each $t$. In fact, if not, there is a gauge transformation $S(t) \in G$ satisfying

\begin{equation}
(5.34) \quad \frac{dS}{dt} = S \circ \left( \int_M \sigma(t) \, dv \right)
\end{equation}

with $S(0) = I$. Note that (5.34) can be solvable through the limit of smooth sections from using Theorem 5.2. Since $S(t)$ depends only on $t$, $S^*(a^k_i)$ for $k = 1, 2$ also satisfy (5.27)-(5.33).

It follows from using $D^*_{a^k_i} D_{a^k_i}^* F_{a^k_i} = 0$ and $d^* a^k = 0$ in $U_i$ with $k = 1, 2$ that

\begin{equation}
(5.35) \quad d^* \sigma_i = [s a^2_i, D^*_{a^2_i} F_{a^2_i}] - [s a^1_i, D^*_{a^1_i} F_{a^1_i}] - d^*([a^2_i, s^2_i] - [a^1_i, s^1_i])
\end{equation}

\begin{align*}
&= [s(a^2_i - a^1_i) + s a^1_i, D^*_{a^2_i} F_{a^2_i}] + [s a^2_i, D^*_{a^2_i} F_{a^2_i} - D^*_{a^1_i} F_{a^1_i}] + *([s a^1_i, ds^2_i] - [s a^1_i, ds^1_i]) \\
&= b_i \# D^*_{a^2_i} F_{a^2_i} + a^1_i \# \left( \frac{\partial b_i}{\partial t} - d a_i + [b_i, s^2_i] + [a^1_i, \sigma_i] \right) + b_i \# \nabla s^2_i + a^1_i \# \nabla \sigma_i.
\end{align*}
For simplicity, we denote $U_j = U_j \setminus \cup_{j'=j+1}^J U_j$, since the set $\cup_{j'=j+1}^J U_j$ is empty. Using (5.35), Sobolev’s inequalities and the Hölder inequality, we have

\[ \| \sigma \|_{H^1(M)}^2 \leq C \| \sigma \|_{L^2(M)}^2 = C \int_M \langle d^* \sigma, \sigma \rangle \, dv \]

\[ = C \int_{U_j} \langle d^* \sigma_j, \sigma_j \rangle \, dv + \cdots + C \int_{U_1 \setminus (\cup_{j'=2}^J U_j)} \langle d^* \sigma_1, \sigma_1 \rangle \, dv \]

\[ \leq C \sum_{i=1}^J \| b \|_{L^4(U_i \setminus \cup_{j'=i+1}^J U_j)}^2 \| D_{a_i}^* F_{a_i} \|_{L^2(U_i \setminus \cup_{j'=i+1} U_j)}^2 \]

\[ + C \sum_{i=1}^J \| a_i \|_{L^2(U_i \setminus \cup_{j'=i+1}^J U_j)} \| \sigma \|_{L^4(U_i \setminus \cup_{j'=i+1} U_j)}^2 \| \sigma \|_{L^2(U_i \setminus \cup_{j'=i+1} U_j)} \]

\[ + \frac{1}{4} \| s_i \|_{H^1(U_i \setminus \cup_{j'=i+1} U_i)}^2 + \frac{1}{4} \| \sigma \|_{L^4(U_i \setminus \cup_{j'=i+1} U_i)}^2 \]

\[ \leq C (\| D_{a_i}^* F_{a_i} \|_{L^2(M)}^2 + \| s_i \|_{H^1(M)}^2) \| b \|_{H^1(M)}^2 + C \varepsilon \| \frac{\partial b}{\partial t} \|_{L^2(M)}^2 . \]

By choosing $\tilde{T}$ sufficiently small, it follows from (5.29) that

\[ \int_0^\tilde{T} \| D_{a_i}^* F_{a_i} \|_{L^2(M)}^2 + \| s_i \|_{H^1(M)}^2 d t \leq C \varepsilon . \]

It follows from above two inequalities that

\[ \| \sigma \|_{L^2(0, \tilde{T}; H^1(M))}^2 \leq C \varepsilon \left( \| \frac{\partial b}{\partial t} \|_{L^2(0, \tilde{T}; L^2(M))}^2 + \| b \|_{L^\infty(0, \tilde{T}; H^1(M))}^2 \right) . \]

Using $d^* b_i = 0$ in $U_i$, we have

\[ \frac{\partial b_i}{\partial t} = - D_{a_i}^* F_{a_i} + D_{a_i}^* F_{a_i} + d \sigma_i + [a_i, s_i^2] - [a_i, s_i^2] \]

\[ = -(d^* d + d d^*) b_i + b_i \# \nabla a_i^2 + a_i^2 \# \nabla b_i \]

\[ + a_i^2 \nabla a_i^2 \nabla a_i^2 + d \sigma_i + [b_i, s_i^2] + [a_i, s_i^2] \]

in $U_i$, where we have used that $\Delta = -(d^* d + d d^*)$. 

Using (5.31) and Hölder’s inequality, we have
\[
\begin{align*}
(5.38) \quad & \int_M \frac{\partial b}{\partial t}^2 + |\Delta b|^2 \ dv + \frac{d}{dt} \int_M |\nabla b|^2 \ dv = \int_M \frac{\partial b}{\partial t} - \Delta b \ dv \\
& = \int_{U_j} \frac{\partial b}{\partial t} - \Delta b \ dv + \cdots + C \int_{U_1 \cup (\cup_{j=2} U_j)} \frac{\partial b_1}{\partial t} - \Delta b_1 \ dv \\
& \leq C \sum_{i=1}^J \left( \int_{U \cup (\cup_{j=1} U_j)} |b_i|^4 \ dv \right)^{1/2} \left( \int_{U \cup (\cup_{j=1} U_j)} |\nabla a_i|^4 \ dv \right)^{1/2} \\
& \quad + C \sum_{i=1}^J \left( \int_{U \cup (\cup_{j=1} U_j)} |a_i|^4 \ dv \right)^{1/2} \left( \int_{U \cup (\cup_{j=1} U_j)} |\nabla b_i|^4 \ dv \right)^{1/2} \\
& \quad + C \sum_{i=1}^J \left( \int_{U \cup (\cup_{j=1} U_j)} |b_i|^4 \ dv \right)^{1/2} \left( \int_{U \cup (\cup_{j=1} U_j)} |a_i|^8 + |a_i|^8 \ dv \right)^{1/2} \\
& \quad + C \sum_{i=1}^J \left( \int_{U \cup (\cup_{j=1} U_j)} |a_i|^4 \ dv \right)^{1/2} \left( \int_{U \cup (\cup_{j=1} U_j)} |s_i|^4 \ dv \right)^{1/2} \\
& \quad + C \sum_{i=1}^J \int_{U \cup (\cup_{j=1} U_j)} |\nabla \sigma_i|^2 \ dv \\
& \leq C \|b\|_{H^1(M)}^2 \left( \|a\|_{H^2(M)}^2 + \|a^2\|_{H^2(M)}^2 + \|s\|_{H^1(M)}^2 \right) \\
& \quad + C \varepsilon \|b\|_{H^2(M)}^2 + C \int_M |\nabla \sigma|^2 \ dv.
\end{align*}
\]
Integrating in $t$ and using (5.29), (5.36), we obtain
\[
\begin{align*}
\| \frac{\partial b}{\partial t} \|_{L^2(0,\tilde{T};L^2(M))}^2 + \| b \|_{L^\infty(0,\tilde{T};H^1(M))}^2 + \| b \|_{L^2(0,\tilde{T};H^2(M))}^2 \\
& \leq C \| b \|_{L^\infty(0,\tilde{T};H^1(M))} \int_0^{\tilde{T}} \left( \| a \|_{H^2(M)}^2 + \| a^2 \|_{H^2(M)}^2 + \| s \|_{H^1(M)}^2 \right) \ dt \\
& \quad + C \varepsilon \| b \|_{L^2(0,\tilde{T};H^2(M))}^2 + C \| \sigma \|_{L^2(0,\tilde{T};H^1(M))}^2 \\
& \leq C \varepsilon \left( \| \frac{\partial b}{\partial t} \|_{L^2(0,\tilde{T};L^2(M))}^2 + \| b \|_{L^\infty(0,\tilde{T};H^1(M))}^2 + \| b \|_{L^2(0,\tilde{T};H^2(M))}^2 \right).
\end{align*}
\]
Choosing $\varepsilon$ sufficiently small, we obtain that $a^1 = a^2$ and $s^1 = s^2$ in $M$. This proves our claim.

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