Deep ReLU neural network approximation of parametric and stochastic PDEs with lognormal inputs

Dinh Dũng *a, Van Kien Nguyenb, and Duong Thanh Phamc

aInformation Technology Institute, Vietnam National University, Hanoi
144 Xuan Thuy, Cau Giay, Hanoi, Vietnam
Email: dinhzung@gmail.com

bFaculty of Basic Sciences, University of Transport and Communications
No.3 Cau Giay Street, Lang Thuong Ward, Dong Da District, Hanoi, Vietnam
Email: kiennv@utc.edu.vn

cVietnamese German University,
Le Lai street, Binh Duong New City, Binh Duong Province, Vietnam
Email: duongpt@vgu.edu.vn

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Abstract

We investigate non-adaptive methods of deep ReLU neural network approximation of the solution \( u \) to parametric and stochastic elliptic PDEs with lognormal inputs on non-compact set \( \mathbb{R}^\infty \). The approximation error is measured in the norm of the Bochner space \( L_2(\mathbb{R}^\infty, V, \gamma) \), where \( \gamma \) is the tensor product standard Gaussian probability on \( \mathbb{R}^\infty \) and \( V \) is the energy space. The approximation is based on an \( m \)-term truncation of the Hermite generalized polynomial chaos expansion (gpc) of \( u \). Under a certain assumption on \( \ell_q \)-summability condition for lognormal inputs \( (0 < q < \infty) \), we proved that for every integer \( n > 1 \), one can construct a non-adaptive compactly supported deep ReLU neural network \( \phi_n \) of size not greater than \( m = O(n/\log n) \), having \( m \) outputs so that the summation constituted by replacing polynomials in the \( m \)-term truncation of Hermite gpc expansion by these \( m \) outputs approximates \( u \) with an error bound \( O \left( \frac{n}{\log n} \right)^{-1/q} \). This error bound is comparable to the error bound of the best approximation of \( u \) by \( n \)-term truncations of Hermite gpc expansion which is \( O(n^{-1/q}) \). We proved an analogous result on approximation of holomorphic functions with values in a Hilbert space \( X \) which represent solution maps to a wide range of parametric PDEs. Both these results are derived from a result on deep ReLU neural network approximation in a general Bochner space \( L_2(\mathbb{R}^\infty, X, \gamma) \). We also obtained some results on similar problems for parametric and stochastic elliptic PDEs with affine inputs, based on the Jacobi and Taylor gpc expansions.

Keywords and Phrases: High-dimensional approximation; Deep ReLU neural networks; Parametric and stochastic elliptic PDEs; Lognormal inputs.

Mathematics Subject Classifications (2010): 65C30, 65D05, 65D32, 65N15, 65N30, 65N35.

*Corresponding author
1 Introduction

The aim of the present paper is to construct deep ReLU neural networks for approximation of parametric and stochastic elliptic PDEs with lognormal or affine inputs. We investigate the convergence rate of this approximation in terms of the size of the approximating deep ReLU neural networks.

The universal approximation capacity of neural networks has been known since the 1980’s ([13, 37, 24, 6]). In recent years, deep neural networks have been rapidly developed and successfully applied to a wide range of fields. The main advantage of deep neural networks over shallow ones is that they can output compositions of functions cheaply. Since their application range is getting wider, theoretical analysis revealing reasons of these significant practical improvements attracts substantial attention [2, 19, 44, 53, 54]. In the last several years, there has been a number of interesting papers that addressed the role of depth and architecture of deep neural networks in approximating functions that possess special regularity properties such as analytic functions [21, 42], differentiable functions [49, 56], oscillatory functions [31], functions in Sobolev or Besov spaces [1, 28, 32, 57]. High-dimensional approximations by deep neural networks have been studied in [43, 52, 16, 18], and their applications to high-dimensional PDEs in [51, 22, 47, 33, 26, 27, 29]. Most of these papers used deep ReLU (Rectified Linear Unit) neural networks since the rectified linear unit is a simple and preferable activation function in many applications. The output of such a neural network is a continuous piece-wise linear function which is easily and cheaply computed. We refer the reader to the recent surveys [20, 48] for various problems and aspects of neural network approximation and bibliography.

In computational uncertainty quantification, the problem of efficient numerical approximation for parametric and stochastic partial differential equations (PDEs) has been of great interest and achieved significant progress in recent years. There is a vast number of works on this topic to mention all of them. We point out just some works [3, 5, 4, 8, 10, 11, 12, 7, 9, 15, 14, 23, 36, 58, 59] which are directly related to our paper.

Recently, a number of works have been devoted to various problems and methods of deep neural network approximation for parametric and stochastic PDEs with affine inputs on the compact set \(I := [-1,1]^\infty\) and with the condition on uniform ellipticity, such as dimensionality reduction [55], deep neural network expression rates for the Taylor generalized polynomial chaos expansion (gpc) of solutions to parametric elliptic PDEs [50], reduced basis methods [40] the problem of learning the discretized parameter-to-solution map in practice [25], Bayesian PDE inversion [46, 34, 33], etc. In particular, in [50] the authors proved dimension-independent deep neural network expression rate bounds of the uniform approximation of solution to parametric elliptic PDE based on \(n\)-term truncations of the non-orthogonal Taylor gpc expansion. The construction of approximating deep neural networks relies on weighted summability of the Taylor gpc expansion coefficients of the solution which is derived from its analyticity.

Let \(D \subset \mathbb{R}^d\) be a bounded Lipschitz domain. Consider the diffusion elliptic equation
\[
-\text{div}(a \nabla u) = f \quad \text{in} \quad D, \quad u|_{\partial D} = 0, 
\]
for a given fixed right-hand side \(f\) and a spatially variable scalar diffusion coefficient \(a\). Denote by \(V := H_0^1(D)\) the energy space and \(H^{-1}(D)\) the dual space of \(V\). Assume that \(f \in H^{-1}(D)\) (in what follows this preliminary assumption always holds without mention). If \(a \in L_\infty(D)\) satisfies the ellipticity assumption
\[
0 < a_{\min} \leq a \leq a_{\max} < \infty,
\]
by the well-known Lax-Milgram lemma, there exists a unique solution \( u \in V \) to the equation (1.1) in the weak form
\[
\int_D a \nabla u \cdot \nabla v \, dx = \langle f, v \rangle, \quad \forall v \in V.
\]

We consider diffusion coefficients having a parametric form \( a = a(y) \), where \( y = (y_j)_{j \in \mathbb{N}} \) is a sequence of real-valued parameters ranging in the set \( U^\infty \) which is either \( \mathbb{R}^\infty \) or \( \mathbb{I}^\infty \). Denote by \( u(y) \) the solution to the parametric diffusion elliptic equation
\[
-\text{div}(a(y) \nabla u(y)) = f \quad \text{in} \quad D, \quad u(y)|_{\partial D} = 0.
\tag{1.2}
\]

The resulting solution operator maps \( y \in U^\infty \mapsto u(y) \in V \). The objective is to achieve a numerical approximation of this complex map by a small number of parameters with a guaranteed error in a given norm. Depending on the nature of the modeled object, the parameter \( y \) may be either deterministic or random. In the present paper, we consider the so-called lognormal case when \( U^\infty = \mathbb{R}^\infty \) and the diffusion coefficient \( a \) is of the form
\[
a(y) = \exp(b(y)), \quad \text{with} \quad b(y) = \sum_{j=1}^\infty y_j \psi_j,
\tag{1.3}
\]

where the \( y_j \) are i.i.d. standard Gaussian random variables and \( \psi_j \in L^\infty(D) \).

Let us briefly describe the main contribution of the present paper. We investigate non-adaptive methods of deep ReLU neural network approximation of the solution \( u(y) \) to parametric and stochastic elliptic PDEs (1.2) with lognormal inputs (1.3) on non-compact set \( \mathbb{R}^\infty \) for which the uniform ellipticity assumption is not required, differing from the above mentioned works. The approximation is based on truncations of the orthonormal Hermite gpc expansion of \( u(y) \):
\[
u(y) = \sum_{s \in F} u_s H_s(y), \quad u_s \in V
\]
(see Section 3.1 for a detailed description of this expansion). The approximation error is measured in the norm of the Bochner space \( L_2(\mathbb{R}^\infty; V, \gamma) \), where \( \gamma \) is the tensor product standard Gaussian probability measure on \( \mathbb{R}^\infty \). By using the results on some weighted \( \ell_2 \)-summability of the energy norm of \( V \) of the Hermite gpc expansion coefficients of \( u(y) \) obtained in [4], we prove the following. Assume that there exists a sequence of positive numbers \( (\rho_j)_{j \in \mathbb{N}} \) such that for some \( 0 < q < \infty \),
\[
\left\| \sum_{j \in \mathbb{N}} \rho_j |\psi_j| \right\|_{L_\infty(D)} < \infty \quad \text{and} \quad \left( \rho_j^{-1} \right)_{j \in \mathbb{N}} \in \ell_q(\mathbb{N}).
\]

Let \( \delta \) be an arbitrary positive number. Then for every integer \( n > 1 \), we can construct an index sequence \( \Lambda_n := \left( s^j \right)_{j=1}^m \subset F \) with \( m = O(n/\log n) \), and a compactly supported deep ReLU neural network \( \phi_n := \left( \phi_j \right)_{j=1}^m \) of size at most \( n \) on \( \mathbb{R}^m \) so that

(i) The index sequence \( \Lambda_n \) and deep ReLU neural network \( \phi_n \) are independent of \( u \);

(ii) The input and output dimensions of \( \phi_n \) are at most \( m \);

(iii) The depth of \( \phi_n \) is \( O \left( n^{\delta} \right) \).
(iv) The support of $\phi_n$ is contained in the cube $[-T, T]^m$ with $T = O(n/\log n)$;

(v) If $\Phi_j$ is the extension of $\phi_j$ to the whole $\mathbb{R}^\infty$ by $\Phi_j(y) = \phi_j \left((y_j)^m\right)$ for $y = (y_j)_{j \in \mathbb{N}}$, then
\[ \left\| u - \sum_{j=1}^m u_j \Phi_j \right\|_{L_2(\mathbb{R}^\infty, V, \gamma)} = O\left((n/\log n)^{-1/q}\right). \quad (1.4) \]

We proved an analogous result on approximation of holomorphic functions with values in a Hilbert space $X$ which represent solution maps to a wide range of parametric PDEs. Both these results are derived from a result on deep ReLU neural network approximation in a general Bochner space $L_2(\mathbb{R}^\infty, X, \gamma)$. Notice that the error bound of this deep ReLU neural network approximation is comparable to the error bound of the best approximation of $u$ by (linear non-adaptive)$n$-term truncations of the Hermite gpc expansion as well as of the approximation by the particular truncation $S_n u := \sum_{j=1}^n u_{s_j} H_{s_j}$ which is $O(n^{-1/q})$ [4, 14]. However, deep ReLU neural network represents a continuous piecewise linear function defined on a number of polyhedral subdomains which shows that deep ReLU neural networks are easily generated and computed in numerical implementation. For some PDEs, it has been shown that deep neural networks are capable of representing solutions without incurring the curse of dimensionality, see, for instance, [39, 30, 38, 38]. Therefore deep ReLU neural networks is one of the preferable tool in numerical solving (parametric) PDEs. We refer the reader to [25, 40, 50] for further discussion on the role of neural networks in approximation parametric PDEs.

Our main idea of constructing the approximating deep ReLU neural networks $\phi_n$ as well as the proof of the error bound (1.4) is that we first approximate the solution $u$ by the truncation $S_n u := \sum_{j=1}^n u_{s_j} H_{s_j}$ with error bound $O(n^{-1/q})$. Then we construct a compactly supported deep neural network $\phi_n$ that approximate $S_n u$ with error $O(n^{-1/q})$. The construction is based on the known realizaton of approximate multiplication by deep ReLU neural networks, and some known results on Gaussian-weighted polynomial approximation on $\mathbb{R}^m$. Finally using weighted $\ell_2$-summability of the gpc Hermite expansion coefficients of the parametric solution $u(y)$ [4] we estimate the weight of the deep neural networks $\phi_n$ as $n \log n$.

We also obtained some results in manner of the items (i)–(v) on similar problems for parametric and stochastic elliptic PDEs (1.2) with affine inputs
\[ a(y) = \bar{a} + \sum_{j=1}^\infty y_j \psi_j, \quad (1.5) \]
where $U^\infty = \mathbb{I}^\infty$. The proofs of these results rely on the Jacobi and Taylor gpc expansions of $u(y)$ by extension of the construction and method in [50]. Our result improves that obtained in [50].

The paper is organized as follows. In Section 2, we present a necessary knowledge about deep ReLU neural networks. Section 3 is devoted to the investigation of non-adaptive methods of deep ReLU neural network approximation of the solution $u$ to the parameterized diffusion elliptic equation (1.2) with lognormal inputs (1.3) on $\mathbb{R}^\infty$. In Section 4, we extend the theory presented in Section 3 to the parameterized diffusion elliptic equation (1.2) with the affine inputs (1.5). Some concluding remarks are presented in Section 5.

**Notation** As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{Z}$ the integers, $\mathbb{R}$ the real numbers and $\mathbb{N}_0 := \{s \in \mathbb{Z} : s \geq 0\}$. We denote $\mathbb{R}^\infty$ the set of all sequences $y = (y_j)_{j \in \mathbb{N}}$ with $y_j \in \mathbb{R}$. 

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Denote by $F$ the set of all sequences of non-negative integers $s = (s_j)_{j \in \mathbb{N}}$ such that their support $\nu_s := \text{supp}(s) := \{j \in \mathbb{N} : s_j > 0\}$ is a finite set. We use $(e^j)_{j \in \mathbb{N}}$ for the standard basis of $\ell_2(\mathbb{N})$. For a set $G$, we denote by $|G|$ the cardinality of $G$. We use letters $C$ and $K$ to denote general positive constants which may take different values, and $C_{\alpha,\beta,...}$ and $K_{\alpha,\beta,...}$ when we want to emphasize the dependence of these constants on $\alpha, \beta, \ldots$, or when this dependence is important in a particular situation.

2 Deep ReLU neural networks

In this section, we present some necessary definitions and elementary facts on deep ReLU neural networks. There is a wide variety of neural network architectures and each of them is adapted to specific tasks. As in [56], we will use such deep feed-forward neural networks that allows connections between neurons in a layer and in any preceding layers. This improves the bound of the weight of parallelization network via the weights of the component networks (see Lemma 2.2 below) in comparing with the standard deep feed-forward neural networks which allows the connection between neurons only in neighboring layers (comp. [50]).

In deep neural network approximation, we will employ the ReLU activation function that is defined by $\sigma(t) := \max\{t, 0\}$, $t \in \mathbb{R}$. We will use the notation $\sigma(x) := (\sigma(x_1), \ldots, \sigma(x_d))$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

**Definition 2.1** Let $d, L \in \mathbb{N}$, $L \geq 2$, $N_0 = d$, and $N_1, \ldots, N_L \in \mathbb{N}$. Let $W^\ell = (w^\ell_{i,j}) \in \mathbb{R}^{N_\ell \times \left(\sum_{i=1}^{\ell-1} N_i\right)}$, $\ell = 1, \ldots, L$, be $N_\ell \times \left(\sum_{i=1}^{\ell-1} N_i\right)$ matrices, and $b^\ell = (b^\ell_j) \in \mathbb{R}^{N_\ell}$. A ReLU neural network $\Phi$ with input dimension $d$, output dimension $N_L$ and $L$ layers is a sequence of matrix-vector tuples

$$\Phi = ((W^1, b^1), \ldots, (W^L, b^L)),$$

in which the following computation scheme is implemented

$$z^0 := x \in \mathbb{R}^d;$$

$$z^\ell := \sigma\left(W^\ell (z^0, \ldots, z^{\ell-1})^T + b^\ell\right), \quad \ell = 1, \ldots, L - 1;$$

$$z^L := W^L(z^0, \ldots, z^{L-1})^T + b^L.$$

We call $z^0$ the input and with an ambiguity denote $\Phi(x) := z^L$ the output of $\Phi$ and in some places we identify a deep ReLU neural network with its output. We will use the following terminology.

- The number of layers $L(\Phi) = L$ is the depth of $\Phi$;
- The number of nonzero $w^\ell_{i,j}$ and $b^\ell_j$ is the size of $\Phi$ and denoted by $W(\Phi)$;
- When $L(\Phi) \geq 3$, $\Phi$ is called a deep neural network, and otherwise, a shallow neural network.

The following two results are easy to verify from the definition above. We also refer the reader to [32, Remark 2.9 and Lemma 2.11] for further remarks and comments.
Lemma 2.2 (Parallelization) Let \( N \in \mathbb{N}, \lambda_j \in \mathbb{R}, j = 1, \ldots, N \). Let \( \Phi_j, j = 1, \ldots, N \) be deep ReLU neural networks with input dimension \( d \). Then we can explicitly construct a deep ReLU neural network denoted by \( \Phi \) so that

\[
\Phi(x) = \sum_{j=1}^{N} \lambda_j \Phi_j(x), \quad x \in \mathbb{R}^d.
\]

Moreover, we have

\[
W(\Phi) \leq \sum_{j=1}^{N} W_j \quad \text{and} \quad L(\Phi) = \max_{j=1,\ldots,N} L_j.
\]

The network \( \Phi \) is called the parallelization network of \( \Phi_j, j = 1, \ldots, N \).

Lemma 2.3 (Concatenation) Let \( \Phi_1 \) and \( \Phi_2 \) be two ReLU neural networks such that output layer of \( \Phi_1 \) has the same dimension as input layer of \( \Phi_2 \). Then, we can explicitly construct a ReLU neural network \( \Phi \) such that \( \Phi(x) = \Phi_2(\Phi_1(x)) \) for \( x \in \mathbb{R}^d \). Moreover we have

\[
W(\Phi) \leq W(\Phi_1) + W(\Phi_2) \quad \text{and} \quad L(\Phi) = L(\Phi_1) + L(\Phi_2).
\]

The network \( \Phi \) is called the concatenation network of \( \Phi_1 \) and \( \Phi_2 \).

We recall the following result, see [50, Proposition 3.3].

Lemma 2.4 For every \( \delta \in (0,1) \), \( d \in \mathbb{N}, d \geq 2 \), we can explicitly construct a deep ReLU neural network \( \Phi_P \) so that

\[
\sup_{x \in [-1,1]^d} \left| \prod_{j=1}^{d} x_j - \Phi_P(x) \right| \leq \delta.
\]

Furthermore, if \( x_j = 0 \) for some \( j \in \{1, \ldots, d\} \) then \( \Phi_P(x) = 0 \) and there exists a constant \( C > 0 \) independent of \( \delta \) and \( d \) such that

\[
W(\Phi_P) \leq C(1 + d \log(d\delta^{-1})) \quad \text{and} \quad L(\Phi_P) \leq C(1 + \log d \log(d\delta^{-1})).
\]

The statement \( \Phi_P(x) = 0 \) (\( d = 2 \)) when \( x_1 \cdot x_2 = 0 \) was proved in [50, Proposition 3.1], see also [32, Proposition C.2] and [45, Proposition 4.1]. For general \( d \) we refer to [43, Proposition 2].

Let \( \varphi_1 \) be the continuous piece-wise function with break points \( \{-2, -1, 1, 2\} \) such that \( \varphi_1(x) = x \) if \( x \in [-1,1] \) and \( \text{supp}(\varphi_1) \subset [-2,2] \). By this definition, we find that \( \varphi \) can be realized exactly by a deep neural network (still denoted by \( \varphi_1 \)) with size \( W(\varphi_1) \leq C \) for some positive constant \( C \). Similarly, let \( \varphi_0 \) be the neural network that realizes the continuous piece-wise function with break points \( \{-2, -1, 1, 2\} \) and \( \varphi_0(x) = 1 \) if \( x \in [-1,1], \text{supp}(\varphi_0) \subset [-2,2] \). Clearly \( W(\varphi_0) \leq C \) for some positive constant \( C \).

From Lemma 2.4 we obtain

Lemma 2.5 Let \( \varphi \) be either \( \varphi_0 \) or \( \varphi_1 \). For every \( \delta \in (0,1) \), \( d \in \mathbb{N}, \) we can explicitly construct a deep ReLU neural network \( \Phi \) so that

\[
\sup_{x \in [-2,2]^d} \left| \prod_{j=1}^{d} \varphi(x_j) - \Phi(x) \right| \leq \delta.
\]
Furthermore, supp(Φ) ⊂ [−2, 2]^d and there exists a constant C > 0 independent of δ and d such that
\[ W(\Phi) \leq C(1 + d \log(d\delta^{-1})) \quad \text{and} \quad L(\Phi) \leq C(1 + \log d \log(d\delta^{-1})). \quad (2.1) \]

Proof. The network Φ is constructed as a concatenation of \( \{\varphi(x_j)\}_{j=1}^d \) and \( \Phi_P \). The estimate (2.1) follows directly from Lemmas 2.3 and 2.4. □

3 Parametric PDEs with lognormal inputs

In this section, we investigate non-adaptive methods of deep ReLU neural network approximation of the solution \( u(y) \) to parametric elliptic PDEs (1.2) with lognormal inputs (1.3) on \( \mathbb{R}^\infty \) and of holomorphic maps on \( \mathbb{R}^\infty \). We construct such methods and prove convergence rates of the approximation by them. The results are derived from a general theory on deep ReLU neural network approximation in Bochner space \( L^2(\mathbb{R}^\infty, X, \gamma) \) of functions \( v \) on \( \mathbb{R}^\infty \) taking values in a Hilbert space \( X \) and satisfying some weighted \( \ell_2 \)-summability conditions of the Hermite gpc expansion coefficients of \( v \).

3.1 Approximation by truncations of the Hermite gpc expansion

We first recall a concept of infinite tensor product of probability measures. Let \( \mu(y) \) be a probability measure on \( \mathbb{U} \). We introduce the probability measure \( \mu(y) \) on \( \mathbb{U}^\infty \) as the infinite tensor product of the probability measures \( \mu(y_j) \):
\[ \mu(y) := \bigotimes_{j \in \mathbb{N}} \mu(y_j), \quad y = (y_j)_{j \in \mathbb{N}} \in \mathbb{U}^\infty. \]
The sigma algebra for \( \mu(y) \) is generated by the set of cylinders \( A := \prod_{j \in \mathbb{N}} A_j \), where \( A_j \subset \mathbb{U} \) are univariate \( \mu \)-measurable sets and only a finite number of \( A_i \) are different from \( \mathbb{U} \). For such a set \( A \), we have \( \mu(A) = \prod_{j \in \mathbb{N}} \mu(A_j) \). If \( \varphi(y) \) is the density of \( \mu(y) \), i.e., \( d\mu(y) = \varphi(y)dy \), then we write
\[ d\mu(y) := \bigotimes_{j \in \mathbb{N}} \varphi(y_j)dy_j, \quad y = (y_j)_{j \in \mathbb{N}} \in \mathbb{U}^\infty. \]
(For details on infinite tensor product of probability measures, see, e.g., [35, pp. 429–435].)

Let \( X \) be a Hilbert space. The probability measure \( \mu(y) \) induces the Bochner space \( L^2(\mathbb{U}^\infty, X, \mu) \) of \( \mu \)-measurable mappings \( v \) from \( \mathbb{U}^\infty \) to \( X \) for which the norm
\[ \|v\|_{L^2(\mathbb{U}^\infty, X, \mu)} := \left( \int_{\mathbb{U}^\infty} \|v(\cdot, y)\|^2_X d\mu(y) \right)^{1/2} < \infty. \]

In this section, we consider the lognormal case with \( \mathbb{U}^\infty = \mathbb{R}^\infty \) and \( \mu(y) = \gamma(y) \), the infinite tensor product standard Gaussian probability measure. More precisely, let \( \gamma(y) \) be the probability measure on \( \mathbb{R} \) with the standard Gaussian density:
\[ d\gamma(y) := g(y)dy, \quad g(y) := \frac{1}{\sqrt{2\pi}} e^{-y^2/2}. \]
Then the infinite tensor product standard Gaussian probability measure \( \gamma(y) \) on \( \mathbb{R}^\infty \) can be defined by

\[
d\gamma(y) := \bigotimes_{j \in \mathbb{N}} g(y_j) \, d(y_j).
\]

In this section, we make use of the abbreviation \( L_2(X) := L_2(\mathbb{R}^\infty, X, \gamma) \). Denote by \( F \) the set of all sequences of non-negative integers \( s = (s_j)_{j \in \mathbb{N}} \) such that their support \( \nu_s := \{ j \in \mathbb{N} : s_j > 0 \} \) is a finite set. A powerful strategy for approximation of functions \( v \in L_2(X) \) is based on truncations of the Hermite gpc expansion

\[
v(y) = \sum_{s \in F} v_s H_s(y), \quad v_s \in X,
\]

where

\[
H_s(y) = \bigotimes_{j \in \mathbb{N}} H_{s_j}(y_j), \quad v_s := \int_{\mathbb{R}^\infty} v(y) H_s(y) \, d\gamma(y), \quad s \in F,
\]

with \((H_k)_{k \in \mathbb{N}_0}\) being the Hermite polynomials normalized according to \( \int_{\mathbb{R}} |H_k(y)|^2 g(y) \, dy = 1 \). Notice that \((H_s)_{s \in F}\) is an orthonormal basis of \( L_2(\mathbb{R}^\infty, \gamma) := L_2(\mathbb{R}^\infty, \mathbb{R}, \gamma) \). Moreover, for every \( v \in L_2(X) \) represented by the series (3.1), the Parseval’s identity holds

\[
\|v\|_{L_2(X)}^2 = \sum_{s \in F} \|v_s\|_X^2.
\]

For \( s, s' \in F \), the inequality \( s' \leq s \) means that \( s'_j \leq s_j \) for \( j \in \mathbb{N} \). A set \( \Lambda \subset F \) is called downward closed if the inclusion \( s \in F \) yields the inclusion \( s' \in F \) for every \( s' \in F \) such that \( s' \leq s \). A sequence \((\sigma_s)_{s \in F}\) is called increasing if \( \sigma_{s'} \leq \sigma_s \) when \( s' \leq s \).

We say that \( v \in L_2(X) \) represented by the series (3.1), satisfies Assumption A if

**Assumption A** There exist a constant \( M \) and an increasing sequence \( \sigma = (\sigma_s)_{s \in F} \) of positive numbers such that \((\sigma_s^{-1})_{s \in F} \in \ell_q(F)\) for some \( q \) with \( 0 < q < \infty \) and

\[
\left( \sum_{s \in F} (\sigma_s \|v_s\|_X)^q \right)^{1/2} \leq M < \infty.
\]

Assume that \( 0 < q < \infty \) and \( \sigma = (\sigma_s)_{s \in F} \) is an increasing sequence of positive numbers. For \( \xi > 0 \), we introduce the set

\[
\Lambda(\xi) := \{ s \in F : \sigma_s^q \leq \xi \}, \quad (3.2)
\]

and the following numbers (when \( \Lambda(\xi) \) is finite):

\[
m_1(\xi) := \max_{s \in \Lambda(\xi)} |s|_1, \quad (3.3)
\]

and

\[
m(\xi) := \max \{ j \in \mathbb{N} : \exists s \in \Lambda(\xi) \ \text{such that} \ s_j > 0 \}.
\]

Sometimes in this paper, without ambiguity we will use \( m \) and \( m_1 \) instead of \( m(\xi) \) and \( m_1(\xi) \). Observe that under Assumption A, the set \( \Lambda(\xi) \) is finite and downward closed.
For a function \( v \in L^2(X) \) represented by the series (3.1), we define the truncation

\[
S_{\Lambda(\xi)}v := \sum_{s \in \Lambda(\xi)} v_s H_s.
\]  

(3.5)

Notice that if Assumption A holds, then \( m \) is finite and, therefore, the truncation \( S_{\Lambda(\xi)}v \) of the series (3.1) can be seen as a function on \( \mathbb{R}^m \).

**Lemma 3.1** For every \( v \in L^2(X) \) satisfying Assumption A and for every \( \xi > 1 \), there holds

\[
\|v - S_{\Lambda(\xi)}v\|_{L^2(X)} \leq M\xi^{-1/q}.
\]

**Proof.** Applying the Parseval’s identity, noting (3.5), (3.2) and Assumption A, we obtain

\[
\|v - S_{\Lambda(\xi)}\|^2_{L^2(X)} = \sum_{\sigma_s > \xi^{1/q}} \|v_s\|^2_X = \sum_{\sigma_s > \xi^{1/q}} (\sigma_s \|v_s\|_X)^2 \sigma_s^{-2}
\]

\[
\leq \xi^{-2/q} \sum_{s \in F} (\sigma_s \|v_s\|_X)^2 = M^2 \xi^{-2/q}.
\]

\[
\]

3.2 Approximation by deep ReLU neural networks

In this section, we construct a deep ReLU neural network which can be used to approximate \( v \in L^2(X) \). We primarily approximate \( v \) by the truncation \( S_{\Lambda(\xi)}v \) (see (3.5)) of the series (3.1). Under the assumptions of Lemma A.2 in Appendix, \( S_{\Lambda(\xi)}v \) can be seen as a function on \( \mathbb{R}^m \), where we recall that \( m := m(\xi) \). Then we approximate \( S_{\Lambda(\xi)}v \) by its truncation \( S_{\omega,\Lambda(\xi)}^\omega v \) on a sufficiently large cube

\[
B^m_\omega := [-2\sqrt{\omega}, 2\sqrt{\omega}]^m \subset \mathbb{R}^m,
\]

where the parameter \( \omega \) depending on \( \xi \) is chosen in an appropriate way.

In what follows, for convenience we consider \( \mathbb{R}^m \) as the subset of all \( y \in \mathbb{R}^\infty \) such that \( y_j = 0 \) for \( j = m + 1, \ldots \). If \( g \) is a function on \( \mathbb{R}^m \) taking values in a Hilbert space \( X \), then \( g \) has an extension to the whole \( \mathbb{R}^\infty \) which is denoted again by \( g \), by the formula \( g(y) = g\left((y_j)_{j=0}^m\right) \) for \( y = (y_j)_{j \in \mathbb{N}} \). The tensor product of standard Gaussian probability measures \( \gamma(y) \) on \( \mathbb{R}^m \) is defined by

\[
d\gamma(y) := \bigotimes_{j=1}^m \gamma_j(y_j) = \gamma(y).
\]

For a \( \gamma \)-measurable subset \( \Omega \) in \( \mathbb{R}^m \), the spaces \( L^2(\Omega, X, \gamma) \) and \( L^2(\Omega, \gamma) \) are defined in the usual way.

Our next task is to construct deep ReLU neural networks \( \phi_s \) on the cube \( B^m_\omega \) to approximate \( H_s, s \in \Lambda(\xi) \). The network \( \Phi_{\Lambda(\xi)} := (\phi_s)_{s \in \Lambda(\xi)} \) on \( B^m_\omega \) with \( |\Lambda(\xi)| \) outputs which is constructed by parallelization is used to construct an approximation of \( S_{\omega,\Lambda(\xi)}^\omega v \) and hence of \( v \). Namely, we approximate \( v \) by

\[
\Phi_{\Lambda(\xi)}v(y) := \sum_{s \in \Lambda(\xi)} v_s \phi_s(y).
\]

(3.6)
For $\theta, \lambda \geq 0$, we define the sequence
\[
p_s(\theta, \lambda) := \prod_{j \in \mathbb{N}} (1 + \lambda s_j)^{\theta}, \quad s \in F,
\]
with abbreviation $p_s(\theta) := p_s(\theta, 1)$. We put for $(\sigma_s^{-1})_{s \in F} \in \ell_q(F)$
\[
K_q := \sum_{s \in F} \sigma_s^{-q},
\]
and for $(p_s(\theta)\sigma_s^{-1})_{s \in F} \in \ell_q(F)$
\[
K_{q, \theta} := \left(\sum_{s \in F} p_s(\theta)^q \sigma_s^{-q}\right)^{\frac{1}{q}}.
\]

Our result in this section is read as follows.

**Theorem 3.2** Let $v \in L^2(X)$ satisfy Assumption A. Let $\theta$ be any number such that $\theta \geq 4/q$. Assume that the sequence $\sigma = (\sigma_s)_{s \in F}$ in Assumption A satisfies $\sigma_{i'} \leq \sigma_i$ if $i' < i$ and $(p_s(\theta)\sigma_s^{-1})_{s \in F} \in \ell_q(F)$. Then for every $\xi > 1$, we can construct a deep ReLU neural network $\phi_{\Lambda(\xi)} := (\phi_s)_{s \in \Lambda(\xi)}$ on $\mathbb{R}^m$, $m \leq \lfloor K_{q, \theta}\xi \rfloor$, having the following properties.

(i) The deep ReLU neural network $\phi_{\Lambda(\xi)}$ is independent of $v$;
(ii) The input and output dimensions of $\phi_{\Lambda(\xi)}$ are at most $m$;
(iii) $W(\phi_{\Lambda(\xi)}) \leq C\xi \log \xi$;
(iv) $L(\phi_{\Lambda(\xi)}) \leq C\xi^{1/q}$;
(v) $\text{supp}(\phi_{\Lambda(\xi)}) \subset [-T, T]^m$, where $T := 4\sqrt{\lfloor K_{q, \theta}\xi \rfloor}$;
(vi) The approximation of $v$ by $\Phi_{\Lambda(\xi)}v$ gives the error estimate
\[
\|v - \Phi_{\Lambda(\xi)}v\|_{L^2(X)} \leq C\xi^{-1/q}.
\]

Here the constants $C = C_{M, q, \sigma, \theta}$ are independent of $v$ and $\xi$.

Let us first introduce the above mentioned function $S^\omega_{\Lambda(\xi)}v$ with a special choice of $\omega$. In this section, for $\xi > 1$, we use the letter $\omega$ only for the notation
\[
\omega := \lfloor K_{q, \theta}\xi \rfloor,
\]
where $K_{q, \theta}$ is the constant defined in Lemma A.1 in Appendix. For a function $\varphi$ defined on $\mathbb{R}$, we denote by $\varphi^\omega$ the truncation of $\varphi$ on $B^1_\omega$, i.e.,
\[
\varphi^\omega(y) := \begin{cases} 
\varphi(y) & \text{if } y \in B^1_\omega \\
0 & \text{otherwise}.
\end{cases}
\]
If $\nu_s \subset \{1, \ldots, m\}$, we put

$$H^\omega_s(y) := \prod_{j=1}^{m} H^\omega_j(y_j), \quad y \in \mathbb{R}^m.$$ 

We have $H^\omega_s(y) = \prod_{j=1}^{m} H^\omega_j(y_j)$ if $y \in B^\omega_m$ and $H^\omega_s(y) = 0$ otherwise.

For a function $v \in L_2(X)$ represented by the series (3.1), noting the truncation $S_\Lambda(\xi)v$ given by (3.5) and (3.2), we define

$$S^\omega_\Lambda(\xi)v := \sum_{s \in \Lambda(\xi)} v_s H^\omega_s.$$ 

From Lemma A.2 in Appendix one can see that for every $s \in \Lambda(\xi)$, $H_s$ and $H^\omega_s$ and therefore, $S_\Lambda(\xi)v$ and $S^\omega_\Lambda(\xi)v$ can be considered as functions on $\mathbb{R}^m$. For $g \in L_2(\mathbb{R}^m, X, \gamma)$, we have $\|g\|_{L_2(\mathbb{R}^m, X, \gamma)} = \|g\|_{L_2(\mathbb{R}^\infty, X, \gamma)}$ in the sense of extension of $g$. We will make use of these facts without mention.

To prove Theorem 3.2 we will employ a so-called technique of intermediate approximation for estimation of the approximation error in Theorem 3.2 which in our case is as follows. Suppose that the function $\Phi_\Lambda(\xi)$ is already constructed. Due to the inequality

$$\|v - \Phi_\Lambda(\xi)v\|_{L_2(X)} \leq \|v - S_\Lambda(\xi)v\|_{L_2(X)} + \|S_\Lambda(\xi)v - S^\omega_\Lambda(\xi)v\|_{L_2(\mathbb{R}^m \setminus B^\omega_m, X, \gamma)} + \|S^\omega_\Lambda(\xi)v - \Phi_\Lambda(\xi)v\|_{L_2(\mathbb{R}^m \setminus B^\omega_m, X, \gamma)} + \|\Phi_\Lambda(\xi)v\|_{L_2(\mathbb{R}^m \setminus B^\omega_m, X, \gamma)},$$

the estimate (3.10) will be done via estimates of the four terms in the right-hand side. The first term is already estimated as in Lemma 3.1. The estimates for the others will be carried out in the below corresponding lemmas (Lemmas 3.4–3.6). In order to do this we need an auxiliary lemma on estimation of the $L_2(\mathbb{R}^m \setminus B^\omega_m, \gamma)$-norm of a polynomial on $\mathbb{R}^m$.

Below in this subsection, we use letters $C$ and $K$ to denote various constants which may depend on the parameters $M, q, \sigma, \theta$, as mentioned in Theorem 3.2.

**Lemma 3.3** Let $\varphi(y) = \prod_{j=1}^{m} \varphi_j(y_j)$ for $y \in \mathbb{R}^m$, where $\varphi_j$ is a polynomial in the variable $y_j$ of degree not greater than $\omega$ for $j = 1, \ldots, m$. Then there holds

$$\|\varphi\|_{L_2(\mathbb{R}^m \setminus B^\omega_m, \gamma)} \leq C m \exp(-K\omega) \|\varphi\|_{L_2(\mathbb{R}^m, \gamma)},$$

where the constants $C$ and $K$ are independent of $\omega$, $m$ and $\varphi$.

**Proof.** The proof of the lemma relies on the following inequality which is an immediate consequence of [41, Theorem 6.3]. Let $\psi$ be a polynomial of degree at most $\ell$. Applying [41, Theorem 6.3] for polynomial $\psi(\sqrt{2}t)$ with weight $\exp(-t^2)$ (in this case $a_\ell = \sqrt{\ell}$, see [41, Page 41]) and $\kappa = \sqrt{\ell} - 1$ we get

$$\|\psi\|_{L_2(\mathbb{R} \setminus [-\sqrt{\ell}, \sqrt{\ell}], \gamma)} \leq C \exp(-K\ell) \|\psi\|_{L_2([-\sqrt{\kappa}, \sqrt{\kappa}], \gamma)}$$

for some positive number $C$ and $K$ independent of $\ell$ and $\psi$. We denote

$$I_{j} := \mathbb{R} \times \ldots \times (\mathbb{R} \setminus [-2\sqrt{\omega}, 2\sqrt{\omega}]) \times \ldots \times \mathbb{R} \subset \mathbb{R}^m.$$
Since \( \mathbb{R}^m \setminus B_\omega^m = \bigcup_{j=1}^m I_j \), we have
\[
\| \varphi \|_{L^2(\mathbb{R}^m \setminus B_\omega^m, \gamma)} \leq \sum_{j=1}^m \| \varphi \|_{L^2(I_j, \gamma)} = \sum_{j=1}^m \left( \| \varphi_j \|_{L^2(\mathbb{R} \setminus B_\omega^m, \gamma)} \prod_{i \neq j} \| \varphi_i \|_{L^2(\mathbb{R}, \gamma)} \right). \tag{3.14}
\]

Applying (3.13) for the polynomials \( \varphi_j \), for \( j = 1, \ldots, m \), whose degree is not greater than \( \omega \) we obtain
\[
\| \varphi_j \|_{L^2(\mathbb{R} \setminus B_\omega^m, \gamma)} \leq C \exp (-K \omega) \| \varphi_j \|_{L^2(\mathbb{R}, \gamma)}
\]
with some constants \( C \) and \( K \) independent of \( \omega, m \) and \( \varphi \). This together with (3.14) yields
\[
\| \varphi \|_{L^2(\mathbb{R}^m \setminus B_\omega^m, \gamma)} \leq C \exp (-K \omega) m \prod_{i=1}^m \| \varphi_i \|_{L^2(\mathbb{R}, \gamma)} = C m \exp (-K \omega) \| \varphi \|_{L^2(\mathbb{R}^m, \gamma)}.
\]

Lemma 3.4 Let the assumptions of Theorem 3.2 be satisfied. Then for every \( \xi > 1 \), we have that
\[
\| S_{\Lambda(\xi)} v - S_{\Lambda(\xi)}^\omega v \|_{L^2(\mathbb{R}^m \setminus B_\omega^m, X, \gamma)} \leq C \xi^{-1/q},
\]
where the constant \( C \) independent of \( v \) and \( \xi \).

Proof. By the equality
\[
\| H_s - H_s^\omega \|_{L^2(\mathbb{R}^m \setminus B_\omega^m, \gamma)} = \| H_s \|_{L^2(\mathbb{R}^m \setminus B_\omega^m, \gamma)}, \quad s \in \Lambda(\xi),
\]
and the triangle inequality we obtain
\[
\| S_{\Lambda(\xi)} v - S_{\Lambda(\xi)}^\omega v \|_{L^2(\mathbb{R}^m \setminus B_\omega^m, X, \gamma)} \leq \sum_{s \in \Lambda(\xi)} \| \Sigma_s \|_X \| H_s - H_s^\omega \|_{L^2(\mathbb{R}^\infty, \gamma)}
\]
\[
= \sum_{s \in \Lambda(\xi)} \| \Sigma_s \|_X \| H_s \|_{L^2(\mathbb{R}^m \setminus B_\omega^m, \gamma)}.
\tag{3.15}
\]
Notice that for every \( s \in \Lambda(\xi), H_s(y) = \prod_{j=1}^m H_{s_j}(y_j), y \in \mathbb{R}^m \), where \( H_{s_j} \) is a polynomial in variable \( y_j \), of degree not greater than \( m_1(\xi) \leq \omega \). Applying Lemma 3.3 gives
\[
\| H_s \|_{L^2(\mathbb{R}^m \setminus B_\omega^m, \gamma)} \leq C m \exp (-K \omega) \| H_s \|_{L^2(\mathbb{R}^m, \gamma)} \leq C m \exp (-K \omega),
\]
where the constants \( C \) and \( K \) are independent of \( \omega, m \) and \( s \). This together with (3.15), (3.11) and the Cauchy–Schwarz inequality yields that
\[
\| S_{\Lambda(\xi)} v - S_{\Lambda(\xi)}^\omega v \|_{L^2(\mathbb{R}^m \setminus B_\omega^m, X, \gamma)} \leq C m \exp (-K \omega) \sum_{s \in \Lambda(\xi)} \| \Sigma_s \|_X
\]
\[
\leq C m \exp (-K \omega) \sup_{s \in \Xi} \sigma_s^{-1} \sum_{s \in \Lambda(\xi)} \sigma_s \| \Sigma_s \|_X
\]
\[
\leq C m \exp (-K \omega) | \Lambda(\xi) |^{1/2} \left( \sum_{s \in \Lambda(\xi)} (\sigma_s \| \Sigma_s \|_X)^2 \right)^{1/2}
\]
\[
\leq C \xi^{3/2} \exp (-K \xi) \leq C \xi^{-1/q},
\tag{3.16}
\]

\[\text{12}\]
where in the last estimates we used Assumption A and Lemmata A.1 and A.2 in Appendix.

We will now construct a deep ReLU neural network \( \phi_{\Lambda(\xi)} := (\phi_s)_{s \in \Lambda(\xi)} \) on \( \mathbb{R}^m \) for approximating \( S^\omega_{\Lambda(\xi)}v \) by the function \( \Phi_{\Lambda(\xi)}v \) defined as in (3.6).

It is well-known that for each \( s \in \mathbb{N}_0 \), the univariate Hermite polynomial \( H_s \) can be written as

\[
H_s(x) = \sqrt{s!} \sum_{\ell=0}^{s} \frac{(-1)^\ell x^{s-2\ell}}{(s-2\ell)!} = \sum_{\ell=0}^{s} a_{s,\ell} x^\ell. \tag{3.17}
\]

From (3.17) for each \( s \in \Lambda(\xi) \) we have

\[
H_s(y) = \prod_{j=1}^{m} H_{s_j}(y_j) = \sum_{\ell=0}^{s} \left( \prod_{j=1}^{m} a_{s_j,\ell_j} \right) y^\ell = \sum_{\ell=0}^{s} a_{\ell} y^\ell,
\]

where we put \( a_{\ell} := \prod_{j=1}^{m} a_{s_j,\ell_j} \) and \( y^\ell := \prod_{j=1}^{m} y_j^{\ell_j} \). Hence, we get for every \( y \in B_{\omega}^m \),

\[
S^\omega_{\Lambda(\xi)}v(y) = \sum_{s \in \Lambda(\xi)} v_s \sum_{\ell=0}^{s} a_{\ell} \left( y^\ell \right)^\omega = \sum_{s \in \Lambda(\xi)} v_s \sum_{\ell=0}^{s} a_{\ell} \left( 2^{\sqrt{\omega}} \right)^{\ell_j}, \tag{3.18}
\]

Let \( \ell \in \mathbb{F} \) be such that \( 0 \leq \ell \leq s \). For \( \ell \neq \mathbf{0} \), with an appropriate change of variables, the term \( \prod_{j=1}^{m} \left( \frac{y_j}{2^{\sqrt{\omega}}} \right)^{\ell_j} \) can be represented in the form \( \prod_{j=1}^{m} \varphi_1(x_j) \), where \( \varphi_1 \) is defined before Lemma 2.5. Hence by Lemma 2.5, for every \( \ell \) satisfying \( 0 < \ell \leq s \), with

\[
\delta_s^{-1} := \xi^{1/q+1/2} p_s(1) \left( 2^{\sqrt{\omega}} \right)^{|s|} \max_{0 \leq \ell \leq s} \{|a_{\ell}\}|, \tag{3.19}
\]

there exists a deep ReLU neural network \( \phi_{s,\ell} \) on \( \mathbb{R}^m \) such that

\[
\sup_{y \in B_{\omega}^m} \left| \prod_{j=1}^{m} \left( \frac{y_j}{2^{\sqrt{\omega}}} \right)^{\ell_j} - \phi_{s,\ell} \left( \frac{y}{2^{\sqrt{\omega}}} \right) \right| \leq \delta_s, \tag{3.20}
\]

and

\[
\text{supp} \left( \phi_{s,\ell} \left( \frac{y}{2^{\sqrt{\omega}}} \right) \right) \subset B_{\omega}^{\nu_{\ell}}. \tag{3.21}
\]

In the case when \( \ell = \mathbf{0} \), we fix an index \( j \in \nu_s \) and define the deep ReLU neural network \( \phi_{s,0}(y) := a_0 \varphi_0 \left( \frac{y_j}{2^{\sqrt{\omega}}} \right) \) for \( y \in \mathbb{R}^m \), where \( \varphi_0 \) is defined before Lemma 2.5. Then \( |a_0 - \phi_{s,0}(y)| = 0 \) for \( y \in B_{\omega}^m \). Observe that the size and depth of \( \phi_{s,0} \) are bounded by a constant. For \( \ell \neq \mathbf{0} \), the size and the depth of \( \phi_{s,\ell} \) are bounded as

\[
W(\phi_{s,\ell}) \leq C \left( 1 + |\ell|_1 \left( \log |\ell|_1 + \log \delta_s^{-1} \right) \right) \leq C \left( 1 + |\ell|_1 \log \delta_s^{-1} \right) \tag{3.22}
\]

and

\[
L(\phi_{s,\ell}) \leq C \left( 1 + \log |\ell|_1 \left( \log |\ell|_1 + \log \delta_s^{-1} \right) \right) \leq C \left( 1 + \log |\ell|_1 \log \delta_s^{-1} \right) \tag{3.23}
\]

due to the inequality \( |\ell|_1 \leq \delta_s^{-1} \). In the following we will use the convention \( |\mathbf{0}|_1 = 1 \). Then the estimates (3.22) and (3.23) holds true for all \( \ell \) with \( 0 \leq \ell \leq s \).
We define the deep ReLU neural network $\phi_s$ on $\mathbb{R}^m$ by
\[
\phi_s(y) := \sum_{0 \leq \ell \leq s} a_{\ell} \left( 2\sqrt{\omega} \right)^{|\ell|} \phi_{s,\ell} \left( \frac{y}{2\sqrt{\omega}} \right), \quad y \in \mathbb{R}^m,
\] (3.24)
which is a parallelization of the component deep ReLU neural networks $\phi_{s,\ell} \left( \frac{y}{2\sqrt{\omega}} \right)$. From (3.21) it follows
\[
\text{supp}(\phi_s) \subset B_{4\omega}^{[\nu_s]}.
\] (3.25)
We define $\phi_{\Lambda(\xi)} := (\phi_s)_{s \in \Lambda(\xi)}$ as the deep ReLU neural network realized by parallelization $\phi_s$, $s \in \Lambda(\xi)$. Consider the approximation of $S_{\Lambda(\xi)}^\omega v$ by $\Phi_{\Lambda(\xi)} v$.

**Lemma 3.5** Under the assumptions of Theorem 3.2, we have
\[
\| S_{\Lambda(\xi)}^\omega v - \Phi_{\Lambda(\xi)} v \|_{L_2(B_{\omega}^m, X, \gamma)} \leq C \xi^{-1/q},
\]
where the constants $C$ is independent of $v$ and $\xi$.

**Proof.** From (3.18), (3.20) and (3.24) and similarly to (3.16), we have that
\[
\| S_{\Lambda(\xi)}^\omega v - \Phi_{\Lambda(\xi)} v \|_{L_2(B_{\omega}^m, X, \gamma)} = \left\| \sum_{s \in \Lambda(\xi)} v_s H_s^\omega - \sum_{s \in \Lambda(\xi)} v_s \phi_s(y) \right\|_{L_2(B_{\omega}^m, X, \gamma)}
\]
\[
\leq \sum_{s \in \Lambda(\xi)} \|v_s\|_X \sum_{\ell=0}^s \left| a_{\ell} \left( 2\sqrt{\omega} \right)^{|s|} \right| \delta_s
\]
\[
\leq \xi^{-1/q - 1/2} \sum_{s \in \Lambda(\xi)} \|v_s\|_X \leq C \xi^{-1/q}.
\]

**Lemma 3.6** Under the assumptions of Theorem 3.2, we have
\[
\| \Phi_{\Lambda(\xi)} v \|_{L_2(\mathbb{R}^m \setminus B_{\omega}^m, X, \gamma)} \leq C \xi^{-1/q},
\] (3.26)
where the constant $C$ is independent of $v$ and $\xi$.

**Proof.** By (3.20) we have $|\phi_{s,\ell} \left( \frac{y}{2\sqrt{\omega}} \right)| \leq 2, \forall y \in \mathbb{R}^m$. Hence, by (3.24) we have that
\[
\| \Phi_{\Lambda(\xi)} v \|_{L_2(\mathbb{R}^m \setminus B_{\omega}^m, X, \gamma)} \leq \sum_{s \in \Lambda(\xi)} \|v_s\|_X \sum_{\ell=0}^s \left| a_{\ell} \left( 2\sqrt{\omega} \right)^{|s|} \right| \left\| \phi_{s,\ell} \left( \frac{y}{2\sqrt{\omega}} \right) \right\|_{L_2(\mathbb{R}^m \setminus B_{\omega}^m, \gamma)}
\]
\[
\leq 2 \sum_{s \in \Lambda(\xi)} \|v_s\|_X \sum_{\ell=0}^s \left| a_{\ell} \left( 2\sqrt{\omega} \right)^{|s|} \right| \|1\|_{L_2(\mathbb{R}^m \setminus B_{\omega}^m, \gamma)}.
\]
Applying Lemma 3.3 to the polynomial $\varphi(y) = 1$, we get
\[
\| \Phi_{\Lambda(\xi)} v \|_{L_2(\mathbb{R}^m \setminus B_{\omega}^m, X, \gamma)} \leq C m \sum_{s \in \Lambda(\xi)} \|v_s\|_X \sum_{\ell=0}^s \left( 4\omega \right)^{|s|/2} \exp(-K\omega) \left| a_{\ell} \right|
\]
\[
\leq C m \sum_{s \in \Lambda(\xi)} \|v_s\|_X \left( 4\omega \right)^{|s|/2} \exp(-K\omega) \sum_{\ell=0}^s \left| a_{\ell} \right|.
\]
In order to estimate the sum $\sum_{\ell=0}^{s} |a_{\ell}|$, we need an inequality for the coefficients of Hermite polynomials. By the representation (3.17) of $H_s$, $s \in \mathbb{N}_0$, there holds

$$\sum_{\ell=0}^{s} |a_{\ell}| \leq \sqrt{s!}. \tag{3.27}$$

Indeed, this inequality is obvious with $s = 0, 1, 2, 3$. When $s \geq 4$ we have $\frac{1}{r(s-2\ell)!} \leq \frac{1}{2}$ for all $\ell = 0, \ldots, \lfloor s/2 \rfloor$. Therefore,

$$\sum_{\ell=0}^{s} |a_{\ell}| \leq \sqrt{s!} \sum_{\ell=0}^{\lfloor s/2 \rfloor} \frac{2^{-\ell}}{\ell!(s-2\ell)!} \leq \frac{\sqrt{s!}}{2} \sum_{\ell=0}^{\lfloor s/2 \rfloor} 2^{-\ell} \leq \sqrt{s!}.$$

It follows from (3.27) that

$$\sum_{\ell=0}^{s} |a_{\ell}| = \sum_{\ell=0}^{s} \prod_{j=1}^{m} |a_{s,j,\ell,j}| \leq \prod_{j=1}^{m} \sum_{\ell=0}^{s_j} |a_{s,j,\ell,j}| \leq \prod_{j=1}^{m} \sqrt{s_j!}, \tag{3.28}$$

and hence,

$$\sum_{\ell=0}^{s} |a_{\ell}| \leq \prod_{j=1}^{m} \sqrt{s_j!} \leq \prod_{j=1}^{m} |s_j|^{s_j/2} \leq |s_1|^{s_1/2}. \tag{3.29}$$

By using this estimate and Lemma A.1 in Appendix, we can continue the estimation of $\|\Phi_{\Lambda(\xi)}\|_{L_2(\mathbb{R}^m \setminus B_0^\omega, X, \gamma)}$ as

$$\|\Phi_{\Lambda(\xi)}\|_{L_2(\mathbb{R}^m \setminus B_0^\omega, X, \gamma)} \leq Cm \sum_{s \in \Lambda(\xi)} \|v_s\|_X (4\omega)^{m_1} \exp(-4\omega)m_1^{m_1/2}$$

$$\leq Cm |\Lambda(\xi)|^{1/2} \left( \sum_{s \in \Lambda(\xi)} \|v_s\|_X^2 \right)^{1/2} (4\omega)^{m_1} \exp(-K\omega)m_1^{m_1}$$

$$\leq Cm \epsilon^{1/2} (4\omega)^{m_1} \exp(-K\omega)m_1^{\frac{m_1}{2}}.$$

We have from the inequality $\frac{1}{m_1} \leq \frac{1}{4}$ and Lemma A.1 in Appendix that $m_1 \leq K_q \epsilon^{1/4}$, and from Lemma A.2 in Appendix that $m \leq K_q \epsilon$. Taking account of the choice of $\omega$, we derive the estimate

$$\|\Phi_{\Lambda(\xi)}\|_{L_2(\mathbb{R}^m \setminus B_0^\omega, X, \gamma)} \leq C\epsilon^{3/2} (4K_q \epsilon)^{K_q \epsilon^{1/4}} (K_q \epsilon^{1/4})^{K_q \epsilon^{1/4}} \exp(-K K_q \epsilon),$$

which implies (3.26).

Denote

$$\Lambda^*(\xi) := \{(s, \ell) \in \mathbb{F} \times \mathbb{F} : s \in \Lambda(\xi) \text{ and } 0 \leq \ell \leq s\}. \tag{3.30}$$

Now we estimate the size and depth of the deep ReLU neural network $\phi_{\Lambda(\xi)}$.

**Lemma 3.7** Under the assumptions of Theorem 3.2, the input and output dimensions of $\phi_{\Lambda(\xi)}$ are at most $[K_q \epsilon]$, 

$$W(\phi_{\Lambda(\xi)}) \leq C\epsilon \log \epsilon, \tag{3.31}$$
and

$$L(\phi_{\Lambda(\xi)}) \leq C \xi^{1/\theta} \log \xi, \quad (3.32)$$

where the constants $C$ are independent of $v$ and $\xi$.

Proof. The input dimension of $\phi_{\Lambda(\xi)}$ is not greater than $m(\xi)$ which is at most $|K_q\xi|$ by Lemma A.2 in Appendix. The output dimension of $\phi_{\Lambda(\xi)}$ is the number $|\Lambda(\xi)|$ which is at most $|K_q\xi|$ by Lemma A.1(i) in Appendix.

By Lemma 2.2 and (3.22) the size of $\phi_{\Lambda(\xi)}$ is estimated as

$$W(\phi_{\Lambda(\xi)}) = \sum_{(s, \ell) \in \Lambda^*(\xi)} L(\phi_{s, \ell}) \leq C \sum_{(s, \ell) \in \Lambda^*(\xi)} (1 + |\ell|_1 \log \delta_s^{-1}). \quad (3.33)$$

From (3.19) we have

$$\log(\delta_s^{-1}) \leq C \left( \log \xi + \log p_s(1) + |s|_1 \log(4\omega) + \log \left( \max_{0 \leq \ell \leq s} |a_{\ell}| \right) \right). \quad (3.34)$$

Noting that $|\ell|_1 \leq |s|_1$ for all $(s, \ell) \in \Lambda^*(\xi)$, we obtain

$$\sum_{(s, \ell) \in \Lambda^*(\xi)} (1 + |\ell|_1 \log \delta_s^{-1}) \leq C \left( \log \xi \sum_{(s, \ell) \in \Lambda^*(\xi)} |s|_1 + \sum_{(s, \ell) \in \Lambda^*(\xi)} |s|_1 \log p_s(1) \right. \quad \left(3.35\right)$$

$$+ \log(2\omega) \sum_{(s, \ell) \in \Lambda^*(\xi)} |s|^2 + \sum_{(s, \ell) \in \Lambda^*(\xi)} |s|_1 \log \left( \max_{0 \leq \ell \leq s} |a_{\ell}| \right) \right)$$

$$\leq C \left( \log \xi \sum_{(s, \ell) \in \Lambda^*(\xi)} |s|_1 \log p_s(1) \right. \quad \left(3.35\right)$$

$$+ \log(2\omega) \sum_{(s, \ell) \in \Lambda^*(\xi)} |s|^2 + \sum_{(s, \ell) \in \Lambda^*(\xi)} |s|_1 \log \left( \max_{0 \leq \ell \leq s} |a_{\ell}| \right) \right).$$

For the first and second terms on the right-hand side, since $(p_s(\frac{4}{q}, 1)\sigma^{-1})_{s \in \mathbb{F}} \in \ell_q(\mathbb{F})$ from Lemma A.3 in Appendix we have

$$\log \xi \sum_{(s, \ell) \in \Lambda^*(\xi)} |s|_1 \log p_s(1) \leq \log \xi \sum_{(s, \ell) \in \Lambda^*(\xi)} p_s(2) \leq C \xi \log \xi \quad (3.36)$$

and

$$\log(2\omega) \sum_{(s, \ell) \in \Lambda^*(\xi)} |s|^2 \leq \log(2\omega) \sum_{(s, \ell) \in \Lambda^*(\xi)} p_s(2) \leq C \xi \log(2\omega) \leq C \xi \log \xi, \quad (3.37)$$

where in last inequality we note that $\omega = |K_q, 0|\xi$, see (3.11). Now we turn to the third term in (3.35). The inequalities (3.28) imply

$$\log \left( \max_{0 \leq \ell \leq s} |a_{\ell}| \right) \leq \log \left( \prod_{j=1}^m s_j! \right) \leq \sum_{j=1}^m \log(s_j!) \leq \sum_{j=1}^m s_j^2 \leq p_s(2).$$
Using Lemma A.3 in Appendix again we also obtain
\[
\sum_{(s, \ell) \in \Lambda^*(\xi)} |s|_1 p_s(2) \leq \sum_{(s, \ell) \in \Lambda^*(\xi)} p_s(3) \leq C\xi,
\]
since \((p_s(\frac{4}{q+1})\sigma_s^{-1})_{s \in F} \in \ell_q(F)\). This together with (3.36) and (3.37) yields
\[
\sum_{(s, \ell) \in \Lambda^*(\xi)} (1 + |\ell|_1 \log \delta_s^{-1}) \leq C\xi \log \xi,
\]
which combined with (3.33) gives (3.31).

By Lemma 2.2 and (3.23) the depth of \(\phi_{\Lambda(\xi)}\) is bounded as
\[
L(\phi_{\Lambda(\xi)}) = \max_{(s, \ell) \in \Lambda^*(\xi)} L(\phi_{s, \ell}) \leq C \max_{(s, \ell) \in \Lambda^*(\xi)} (1 + \log |\ell|_1 \log \delta_s^{-1}).
\]
Due to (3.34), this inequality can be modified as
\[
L(\phi_{\Lambda(\xi)}) \leq C \max_{s \in \Lambda(\xi)} (\log |s|_1) \max_{(s, \ell) \in \Lambda^*(\xi)} (\log \delta_s^{-1}). \tag{3.38}
\]
From Lemma A.1 in Appendix we obtain
\[
\max_{s \in \Lambda(\xi)} (\log |s|_1) \leq C \log \xi.
\]

We have by (3.34) that
\[
\max_{(s, \ell) \in \Lambda^*(\xi)} (\delta_s^{-1}) \leq C \left( \log \xi + \max_{s \in \Lambda(\xi)} \log p_s(1) + \log(2\omega) \max_{s \in \Lambda(\xi)} |s|_1 + \max_{s \in \Lambda(\xi)} \left( \max_{0 \leq \ell \leq s} |a_\ell| \right) \right). \tag{3.39}
\]
For the second and third terms on the right-hand side, we have by the well-known inequality \(\log p_s(1) \leq |s|_1\) and Lemma A.1 in Appendix,
\[
\max_{s \in \Lambda(\xi)} \log p_s(1) \leq \max_{s \in \Lambda(\xi)} |s|_1 \leq C\xi^{1/\theta q}
\]
and
\[
\log(2\omega) \max_{s \in \Lambda(\xi)} |s|_1 \leq C\xi^{1/\theta q} \log \xi.
\]
Now we turn to the fourth term in (3.39). From (3.29) it follows that
\[
\log \left( \max_{0 \leq \ell \leq s} |a_\ell| \right) \leq \log \left( |s|_1 |s|_1 \right) = |s|_1 \log |s|_1.
\]
Hence,
\[
\max_{(s, \ell) \in \Lambda^*(\xi)} \log \left( \max_{0 \leq \ell \leq s} |a_\ell| \right) \leq \max_{s \in \Lambda(\xi)} (|s|_1 \log |s|_1) \leq C\xi^{1/\theta q} \log \xi.
\]
This together with (3.38)–(3.2) yields (3.32).

We are now in a position to prove Theorem 3.2.

**Proof.** [Proofs of Theorem 3.2]. By (3.12) and Lemmas 3.1 and 3.4–3.6 we deduce that
\[
\|v - \Phi_{\Lambda(\xi)}v\|_{L_2(X)} \leq C\xi^{-1/q}.
\]
The claim (v) is proven. The claims (i)–(iii) follow from Lemma 3.7 and the claim (iv) from Lemma A.2 in Appendix and (3.25).
3.3 Application to parameterized elliptic PDEs with lognormal inputs

In this section, we apply the results in the previous section to deep ReLU neural network approximation of the solution \( u(y) \) to the parametric elliptic PDEs (1.1) with lognormal inputs (1.3). This is based on a weighted \( \ell_2 \)-summability of the series \( (\|u_s\|_V)_{s \in F} \) in following lemma which has been proven in [4, Theorems 3.3 and 4.2].

**Lemma 3.8** Assume that there exist a number \( 0 < q < \infty \) and an increasing sequence \( \rho = (\rho_j)_{j \in \mathbb{N}} \) of numbers such that \( (\rho_j^{-1})_{j \in \mathbb{N}} \in \ell_q(\mathbb{N}) \) and

\[
\left\| \sum_{j \in \mathbb{N}} \rho_j |\psi_j| \right\|_{L_\infty(D)} < \infty.
\]

Then we have that for any \( \eta \in \mathbb{N} \),

\[
\sum_{s \in F} (\sigma_s \|u_s\|_V)^2 < \infty,
\]

where

\[
\sigma^2_s := \sum_{\|s'\|_{\ell_\infty(\mathbb{N})} \leq \eta} \binom{s}{s'} \prod_{j \in \mathbb{N}} \rho_j^{2s_j'}.
\] (3.40)

This summability result leads to significant improvements of the convergent rate in the case when the component functions \( \psi_j \) have limited overlaps such as splines, finite elements or wavelet bases (for details, see [4]).

The following lemma is proven in [14, Lemma 5.3].

**Lemma 3.9** Let \( 0 < q < \infty, (\rho_j)_{j \in \mathbb{N}} \) be a sequence of positive numbers such that the sequence \( (\rho_j^{-1})_{j \in \mathbb{N}} \) belongs to \( \ell_q(\mathbb{N}) \). Let \( \theta \) be an arbitrary nonnegative number and \( (p_s(\theta))_{s \in F} \) the sequence given in (3.7). Let for \( \eta \in \mathbb{N} \) the sequence \( (\sigma_s)_{s \in F} \) be defined as in (3.40). Then for any \( \eta > \frac{2(\theta+1)}{q} \), we have

\[
\sum_{s \in F} p_s(\theta) \sigma_s^{-q} < \infty.
\]

Our result for the solution \( u \) to the parametric elliptic PDEs (1.1) with lognormal inputs (1.3) is read as follows.

**Theorem 3.10** Under the assumptions of Lemma 3.8, let \( 0 < q < \infty \) and \( \delta \) be arbitrary positive number. Then for every integer \( n > 1 \), we can construct a deep ReLU neural network \( \phi_{\Lambda(\xi_n)} := (\phi_s)_{s \in \Lambda(\xi_n)} \) on \( \mathbb{R}^m \) with \( m := \left\lceil K \frac{n}{\log n} \right\rceil \) for some positive constant \( K \), having the following properties.

(i) The deep ReLU neural network \( \phi_{\Lambda(\xi_n)} \) is independent of \( u \);

(ii) The input and output dimensions of \( \phi_{\Lambda(\xi_n)} \) are at most \( m \);

(iii) \( W(\phi_{\Lambda(\xi_n)}) \leq n \);
(iv) \( L(\phi_{\Lambda(\xi_n)}) \leq C_\delta n^\delta \); 

(v) \( \text{supp}(\phi_{\Lambda(\xi_n)}) \subset [-T, T]^m \), where \( T := C'_\delta \sqrt{\frac{n}{\log n}} \); 

(vi) The approximation of \( u \) by \( \Phi_{\Lambda(\xi_n)}u \) defined as in (3.6), gives the error estimate 

\[
\|u - \Phi_{\Lambda(\xi_n)}u\|_{L_2(V)} \leq C \left( \frac{n}{\log n} \right)^{-1/q}.
\]

Here the constants \( C \), \( C_\delta \) and \( C'_\delta \) are independent of \( u \) and \( n \).

Proof. To prove the theorem, we apply Theorem 3.2 to the solution \( u \). Without loss of generality, we can assume that \( \delta \leq 1/4 \). We take first the number \( \theta := 1/\delta q \) satisfying the inequality \( \theta \geq 4/q \), and then choose a number \( \eta \in \mathbb{N} \) satisfying the inequality \( \eta > \frac{2(\theta+1)}{q} \). By using Lemmas 3.8 and 3.9 one can check that for \( X = V \) and the sequence \( (\sigma_s)_{s \in \mathbb{Z}} \) defined as in (3.40), \( u \in L_2(V) \) satisfies the assumptions of Theorem 3.2. For a given integer \( n > 1 \), we choose \( \xi_n > 1 \) as the maximal number satisfying the inequality \( C\xi_n \log \xi_n \leq n \), where \( C \) is the constant in the claim (ii) of Theorem 3.2. It is easy to verify that there exist positive constants \( C_1 \) and \( C_2 \) independent of \( n \) such that \( C_1 \frac{n}{\log n} \leq \xi_n \leq C_2 \frac{n}{\log n} \). From Theorem 3.2 with \( \xi = \xi_n \), we deduce the desired results. \( \square \)

3.4 Application to approximation of holomorphic functions

The proof of the weighted \( \ell_2 \)-summability result formulated in Lemma 3.9 employs bootstrap arguments and induction on the differentiation order of derivatives with respect to the parametric variables. This approach and technique are too complicated and difficult for extension to more general parametric PDE problems, in particular, of higher regularity, in the lognormal case (comp. [3]). A different approach to a unified summability analysis of Hermite gpc expansions of various scales of function spaces, based on parametric holomorphy, has been suggested in [17]. As we will see below, it covers a wide range of parametric PDE problems. Let us briefly describe it.

We recall the concept of “\((b, \xi, \varepsilon, X)\)-holomorphic functions” which has been introduced in [17]. For \( N \in \mathbb{N} \) and a positive sequence \( \varrho = (\varrho_j)_{j=1}^N \), we put

\[
S(\varrho) := \{ z \in \mathbb{C}^N : |\text{Im} z_j| < \varrho_j \ \forall j \} \quad \text{and} \quad B(\varrho) := \{ z \in \mathbb{C}^N : |z_j| < \varrho_j \ \forall j \}. \tag{3.41}
\]

Let \( X \) be a complex separable Hilbert space, \( b = (b_j)_{j \in \mathbb{N}} \) a positive sequence, and \( \xi > 0 \), \( \varepsilon > 0 \). For \( N \in \mathbb{N} \) we say that a positive sequence \( \rho = (\rho_j)_{j=1}^N \) is \((b, \xi)\)-admissible if 

\[
\sum_{j=1}^N b_j \rho_j \leq \xi. \tag{3.42}
\]

A function \( v \in L_2(X) \) is called \((b, \xi, \varepsilon, X)\)-holomorphic if

(i) for every \( N \in \mathbb{N} \) there exists \( v_N : \mathbb{R}^N \to X \), which, for every \((b, \xi)\)-admissible \( \varrho \), admits a holomorphic extension (denoted again by \( v_N \)) from \( S(\varrho) \to X \); furthermore, for all \( N < M \)

\[
v_N(y_1, \ldots, y_N) = v_M(y_1, \ldots, y_N, 0, \ldots, 0) \quad \forall (y_j)_{j=1}^N \in \mathbb{R}^N, \tag{3.43}
\]
(ii) for every $N \in \mathbb{N}$ there exists $\varphi_N : \mathbb{R}^N \to \mathbb{R}_+$ such that $\|\varphi_N\|_{L^2(\mathbb{R}^N; \gamma)} \leq \varepsilon$ and
\[
\sup_{\rho \text{ is } (b, \xi)\text{-adm.}} \sup_{z \in \mathcal{B}(\rho)} \|v_N(y + z)\|_X \leq \varphi_N(y) \quad \forall y \in \mathbb{R}^N,
\]
(iii) with $\tilde{v}_N : \mathbb{R}^\infty \to X$ defined by $\tilde{v}_N(y) := v_N(y_1, \ldots, y_N)$ for $y \in \mathbb{R}^N$ it holds
\[
\lim_{N \to \infty} \|v - \tilde{v}_N\|_{L^2(X)} = 0.
\]

The following key result on weighted $\ell_2$-summability of $(b, \xi, \varepsilon, X)$-holomorphic functions has been proven in [17, Corollary 4.9].

**Theorem 3.11** Let $v$ be $(b, \xi, \varepsilon, X)$-holomorphic for some $b \in \ell_p(\mathbb{N})$ with $0 < p < 1$. Let $\eta \in \mathbb{N}$ and let the sequence $\rho = (\rho_j)_{j \in \mathbb{N}}$ be defined by
\[
\rho_j := b^{p-1}_j \frac{\xi}{4\sqrt{\eta}} \frac{\|b\|_{\ell_p}}{\xi}.
\]
Then $v$ satisfies Assumption A for $q := \frac{p}{1-p}$, $\sigma = \sigma(\rho, \eta) := (\sigma_s)_{s \in F}$ given by (3.40), and $M := \varepsilon C_{b, \xi}$ with some positive constant $C_{b, \xi}$.

In the same way as the proof of Theorem 3.10, from Theorem 3.11 and Lemma 3.9 we derive

**Theorem 3.12** Let $v$ be $(b, \xi, \varepsilon, X)$-holomorphic for some $b \in \ell_p(\mathbb{N})$ with $0 < p < 1$, and let $\delta$ be an arbitrary positive number. Then, with the notations of Theorem 3.11, for every integer $n > 1$, we can construct a deep ReLU neural network $\phi_{\Lambda(\xi_n)} := (\phi_s)_{s \in \Lambda(\xi_n)}$ on $\mathbb{R}^m$ with $m := \lceil K \frac{n}{\log n} \rceil$ for some positive constant $K$, having the following properties.

(i) The deep ReLU neural network $\phi_{\Lambda(\xi_n)}$ is independent of $v$;
(ii) The input and output dimensions of $\phi_{\Lambda(\xi_n)}$ are at most $m$;
(iii) $W(\phi_{\Lambda(\xi_n)}) \leq n$;
(iv) $L(\phi_{\Lambda(\xi_n)}) \leq C_\delta n^\delta$;
(v) $\text{supp } (\phi_{\Lambda(\xi_n)}) \subset [-T, T]^m$, where $T := C'_{\delta} \sqrt{\frac{n}{\log n}}$;
(vi) The approximation of $u$ by $\Phi_{\Lambda(\xi_n)}u$ defined as in (3.6), gives the error estimate
\[
\|v - \Phi_{\Lambda(\xi_n)}u\|_{L^2(X)} \leq C \left( \frac{n}{\log n} \right)^{(1/p - 1)}.
\]

Here the constants $C$, $C_\delta$ and $C'_{\delta}$ are independent of $v$ and $n$.

We notice some important examples of $(b, \xi, \varepsilon, X)$-holomorphic functions which are solutions to parametric PDEs equations and which were studied in [17]. Let $b(y)$ be defined as in (1.3) and $V$ a holomorphic map from an open set in $L_\infty(D)$ to $X$. Then function compositions of the type
\[
v(y) = V(\exp(b(y)))
\]
are \((b, \xi, \varepsilon, X)\)-holomorphic under certain conditions [17, Proposition 4.11]. This allows us to apply Theorem 3.12 for deep ReLU neural network approximation of solutions \(v(y) = \mathcal{V}(\exp(b(y)))\) as \((b, \xi, \varepsilon, X)\)-holomorphic functions on various function spaces \(X\), to a wide range of parametric and stochastic PDEs with lognormal inputs. Such function spaces \(X\) are high-order regularity spaces \(H^s(D)\) and corner-weighted Sobolev (Kondrat’ev) spaces \(K^s_\kappa(D)\) for the parametric elliptic PDEs (1.1) with lognormal inputs (1.3); spaces of solutions to linear parabolic PDEs with lognormal inputs (1.3); spaces of solutions to linear elastostatics equations with lognormal modulus of elasticity; spaces of solutions to Maxwell equations with lognormal permittivity.

4 Parametric elliptic PDEs with affine inputs

The theory of non-adaptive deep ReLU neural network approximation of functions in Bochner spaces with the infinite tensor product Gaussian measure, which has been discussed in Section 3 can be generalized and extended to other situations. In this section, we present some results on similar problems for the parametric elliptic equation (1.2) with the affine inputs (1.5). The Jacobi and Taylor gpc expansions of the solution play a basic role in the proofs of these results.

4.1 Approximation by deep ReLU neural networks

For given \(a, b > -1\), we consider the infinite tensor product of the Jacobi probability measures on \(I^\infty\)
\[
d\nu_{a,b}(y) := \bigotimes_{j\in\mathbb{N}} \delta_{a,b}(y_j) \, dy_j,
\]
where
\[
\delta_{a,b}(y) := c_{a,b}(1-y)^a(1+y)^b, \quad c_{a,b} := \frac{\Gamma(a+b+2)\Gamma(a+1)\Gamma(b+1)}{2^{a+b+1}\Gamma(a+1)\Gamma(b+1)}.
\]
If \(v \in \mathcal{L}_2(X) := L_2(I^\infty, X, \nu_{a,b})\) for a Hilbert space \(X\), we consider the orthonormal Jacobi gpc expansion of \(v\) of the form
\[
v = \sum_{s \in \mathcal{F}} v_s J_s(y), \quad (4.1)
\]
where
\[
J_s(y) = \bigotimes_{j\in\mathbb{N}} J_{s_j}(y_j), \quad v_s := \int_{I^\infty} v(y)J_s(y) d\nu_{a,b}(y),
\]
and \((J_k)_{k \geq 0}\) is the sequence of Jacobi polynomials on \(I := [-1,1]\) normalized with respect to the Jacobi probability measure, i.e., \(\int_{I^\infty} |J_k(y)|^2 \, d\nu_{a,b}(y) = 1\). One has the Rodrigues’ formula
\[
J_k(y) = \frac{c_{a,b}^k}{k!2^k} (1-y)^{-a}(1+y)^{-b} \frac{d^k}{dy^k} \left( (y^2-1)^k(1-y)^a(1+y)^b \right),
\]
where \(c_{a,b}^0 := 1\) and
\[
c_{a,b}^k := \sqrt{\frac{(2k+a+b+1)\Gamma(k+a+b+1)\Gamma(a+1)\Gamma(b+1)}{\Gamma(k+a+1)\Gamma(k+b+1)\Gamma(a+b+2)}}, \quad k \in \mathbb{N}. \quad (4.2)
\]
Examples corresponding to the values \(a = b = 0\) are the family of the Legendre polynomials, and to the values \(a = b = -1/2\) the family of the Chebyshev polynomials.
Assumption B  Let $0 < q < \infty$, $c_k^{a,b}$ be defined as in (4.2) and let $(\delta_j)_{j \in \mathbb{N}}$ be a sequence of numbers strictly larger than 1 such that $(\delta_j^{-1})_{j \in \mathbb{N}} \in \ell_q(\mathbb{N})$. For $v \in \mathcal{L}_2(X)$ represented by the series (4.1), there exists a sequence of positive numbers $(\rho_j)_{j \in \mathbb{N}}$ such that $c_k^{a,b} \rho_j^{-k} \leq \delta_j^{-k}$ for $k,j \in \mathbb{N}$ and

$$\left(\sum_{s \in \mathcal{F}} (\sigma_s \|v_s\|_X)^2\right)^{1/2} \leq M < \infty,$$

where

$$\sigma_s := c_s^{-1} \prod_{j \in \mathbb{N}} \rho_j^{s_j}, \quad c_s := \prod_{j \in \mathbb{N}} c_s^{a,b}. \quad \text{(4.3)}$$

Theorem 4.1 Let $v \in \mathcal{L}_2(X)$ satisfy Assumption B. Then for every integer $n > 1$, we can construct a deep ReLU neural network $\phi_{\Lambda(\xi_n)} := (\phi_s)_{s \in \Lambda(\xi_n)}$ on $\mathbb{R}^m$ with $m := \lfloor K \frac{n}{\log n} \rfloor$ for some positive constant $K$, having the following properties.

(i) The deep ReLU neural network $\phi_{\Lambda(\xi_n)}$ is independent of $u$;

(ii) The input and output dimensions of $\phi_{\Lambda(\xi_n)}$ are at most $m$;

(iii) $W(\phi_{\Lambda(\xi_n)}) \leq n$;

(iv) $L(\phi_{\Lambda(\xi_n)}) \leq C(\log n)^2$;

(v) Let $\Phi_{\Lambda(\xi_n)} v$ be defined by the formula (3.6) with replacing $\mathbb{R}^\infty$ by $\mathbb{R}^\infty$. Then the approximation of $v$ by $\Phi_{\Lambda(\xi_n)} v$ gives the error estimate

$$\|v - \Phi_{\Lambda(\xi_n)} v\|_{\mathcal{L}_2(X)} \leq C \left( \frac{n}{\log n} \right)^{-1/q}.$$

Here the constant $C$ is independent of $v$ and $n$.

The proof of Theorem 4.1 is similar to the proof of Theorem 3.2, but simpler due to Assumption B and the compact property of $\mathbb{R}^\infty$.

We now are in position to prove Theorem 4.1.

Proof. [A sketch of proof of Theorem 4.1] Similar to the proof of Theorem 3.2, this theorem is deduced from a counterpart of Theorem 3.2 for the case $\mathbb{R}^\infty$. It states that for every $\xi > 1$, we can construct a deep ReLU neural network $\phi_{\Lambda(\xi)} := (\phi_s)_{s \in \Lambda(\xi)}$ on $\mathbb{R}^m$ with $m \leq \lfloor K q \xi \rfloor$, having the following properties.

(i) The input and output dimensions of $\phi_{\Lambda(\xi)}$ are at most $m$;

(ii) $W(\phi_{\Lambda(\xi)}) \leq C \xi \log \xi$;

(iii) $L(\phi_{\Lambda(\xi)}) \leq C(\log \xi)^2$;

(iv) The approximation of $v$ by $\Phi_{\Lambda(\xi)} v = \sum_{s \in \Lambda(\xi)} v_s \phi_s$ gives the error estimate

$$\|v - \Phi_{\Lambda(\xi)} v\|_{\mathcal{L}_2(X)} \leq C \xi^{-1/q}.$$
Here the constants $C$ are independent of $v$ and $\xi$.

Let us give a brief proof of these claims. For the function $v \in L_2(X)$ represented by the series (4.1) and the sequence $(\sigma_s)_{s \in F}$ given as in (4.3), we define

$$S_{\Lambda(\xi)}v := \sum_{s \in \Lambda(\xi)} v_s J_s,$$

where $\Lambda(\xi)$ is defined by the formula (3.2) for the sequence $(\sigma_s)_{s \in F}$ given as in (4.3). Then in the same way as the proof of Lemma 3.1, we prove the estimate

$$\|v - S_{\Lambda(\xi)}v\|_{L_2(X)} \leq C \xi^{-1/q}.$$

(4.4)

By Lemma A.2 in Appendix for every $s \in \Lambda(\xi)$, $J_s$ and $S_{\Lambda(\xi)}v$ can be considered as functions on $\mathbb{I}^m$. As the next step, we will construct a deep ReLU neural network $\phi_s := (\phi_s)_{s \in \Lambda(\xi)}$ on $\mathbb{I}^m$ for approximating $S_{\Lambda(\xi)}v$ by $\Phi_{\Lambda(\xi)}$. From (A.4) for each $s \in F$ we have

$$J_s(y) = \sum_{\ell=0}^s a_\ell y^\ell,$$

where $a_\ell := \prod_{j=1}^m a_{s_j, \ell_j}$ and $y^\ell := \prod_{j=1}^m y_j^{\ell_j}$. Hence, we get for every $y \in \mathbb{I}^m$,

$$S_{\Lambda(\xi)}v(y) := \sum_{s \in \Lambda(\xi)} v_s J_s(s) = \sum_{s \in \Lambda(\xi)} v_s \sum_{\ell=0}^s a_\ell y^\ell.$$

By Lemma 2.5, for every $\ell$ with $0 \leq \ell \leq s$, with

$$\delta_s^{-1} := \xi^{1/q} p_s(1) \max_{0 \leq \ell \leq s} \{|a_\ell|\},$$

there exists a deep ReLU neural network $\phi_{s, \ell}$ on $\mathbb{I}^m$ such that

$$\sup_{y \in \mathbb{I}^m} \left| y^\ell - \phi_{s, \ell}(y) \right| \leq \delta_s,$$

and the size and depth of $\phi_{s, \ell}$ are bounded as

$$W(\phi_{s, \ell}) \leq C \left(1 + |\ell|_1 \log \delta_s^{-1}\right)$$

and

$$L(\phi_{s, \ell}) \leq C \left(1 + \log |\ell|_1 \log \delta_s^{-1}\right).$$

We define the deep ReLU neural network $\phi_s$ on $\mathbb{I}^m$ by

$$\phi_s := \sum_{0 \leq \ell \leq s} a_\ell \phi_{s, \ell},$$

which is a parallelization of component networks $\phi_{s, \ell}$. We define $\Phi_{\Lambda(\xi)} := (\phi_s)_{s \in \Lambda(\xi)}$ as the deep ReLU neural network realized by parallelization $\phi_s$, $s \in \Lambda(\xi)$. Consider the approximation of $S_{\Lambda(\xi)}v$ by $\Phi_{\Lambda(\xi)}v$. By the same way as the proof of Lemma 3.5, we can prove

$$\|S_{\Lambda(\xi)}v - \Phi_{\Lambda(\xi)}v\|_{L_2(X)} \leq C \xi^{-1/q},$$

(4.5)
where the constant $C$ is independent of $v$ and $\xi$.

Let us check the claims (i)–(iv) formulated at the beginning of the proof. From (4.4) and (4.5) we deduce the claim (iv). The proof of the claim (i)–(iii) repeats the proof of Lemma 3.7 in Appendix with a slight modification. We indicate some particular differences in the proofs. There are no longer the fourth term in the right-hand side of (3.34) and the third term in the right-hand side of (3.39). Lemma A.3 in Appendix which is used in the proof follows from Lemma A.4 in Appendix. Lemma A.1(ii) in Appendix and the inequality (3.27) are replaced by the stronger Lemma A.5 and inequality (A.5) in Appendix. This helps us to receive the improved bound $L(\phi_v) \leq C (\log \xi)^2$.

4.2 Application to parameterized elliptic PDEs with affine inputs

We now apply Theorem 4.1 to the solution $u(y)$ to the parameterized elliptic PDEs (1.1) with affine inputs (1.5).

Theorem 4.2 Let $0 < q < \infty$, $c_k^{a,b}$ be defined as in (4.2) and let $(\delta_j)_{j \in \mathbb{N}}$ be a sequence of numbers strictly larger than 1 such that $(\delta_j^{-1})_{j \in \mathbb{N}} \in \ell_q(\mathbb{N})$. Let $\bar{a} \in L_\infty(D)$ and $\inf \bar{a} > 0$. Assume that there exists a sequence of positive numbers $(\rho_j)_{j \in \mathbb{N}}$ such that $c_k^{a,b} \rho_j^{-k} < \delta_j^{-k}$, $k, j \in \mathbb{N}$, and

$$
\left\| \sum_{j \in \mathbb{N}} \rho_j |\psi_j| \right\|_{L_\infty(D)} < 1.
$$

(4.6)

Then for every integer $n > 1$, we can construct a deep ReLU neural network $\phi_{\Lambda(\xi_n)} := (\phi_s)_{s \in \Lambda(\xi_n)}$ on $\mathbb{R}^m$ with $m := \lfloor K \frac{n}{\log n} \rfloor$ for some positive constant $K$, having the following properties.

(i) The deep ReLU neural network $\phi_{\Lambda(\xi_n)}$ is independent of $u$;

(ii) The input and output dimensions of $\phi_{\Lambda(\xi_n)}$ are at most $m$;

(iii) $W(\phi_{\Lambda(\xi_n)}) \leq n$;

(iv) $L(\phi_{\Lambda(\xi_n)}) \leq C(\log n)^2$;

(v) The approximation of $u$ by $\Phi_{\Lambda(\xi_n)} u = \sum_{s \in \Lambda(\xi_n)} u_s \phi_s$, where $u_s, s \in \mathcal{F}$, are the Jacobi gpc expansion coefficients of $u \in L_2(V)$, gives the error estimate

$$
\|u - \Phi_{\Lambda(\xi_n)} u\|_{L_2(V)} \leq C \left( \frac{n}{\log n} \right)^{-1/q}.
$$

Here the constant $C$ is independent of $u$ and $n$.

Proof. It has been proven in [5] that under the assumptions of the theorem, for the sequence $(\sigma_s)_{s \in \mathcal{F}}$ given as in (4.3),

$$
\sum_{s \in \mathcal{F}} (\sigma_s \|u_s\|_V)^2 < \infty.
$$

This means that Assumption B holds for $v = u$ with $X = V$. Hence, applying Theorem 4.1 to $u$, we prove the theorem. \[\square\]
We next discuss the approximation by deep ReLU neural networks for parametric elliptic PDEs with affine inputs and error measured in the uniform norm of \( L_{\infty}(I^\infty, V) \) by using \( m \)-term truncations of the Taylor gpc expansion of \( u \).

If for the sequence \( (\rho_j)_{j \in \mathbb{N}} \) of numbers strictly larger than 1 we have the condition 4.6 and if \( (\rho_j^{-1})_{j \in \mathbb{N}} \) is summable with \( 0 < q < 2 \), then the solution \( u \) to the parameterized elliptic PDEs (1.1) with affine inputs (1.5) can be decomposed in the Taylor gpc expansion

\[
 u = \sum_{s \in \mathcal{F}} t_s y_s, \quad t_s = \frac{1}{s!} \partial^s u(0)
\]

with

\[
 \left( \sum_{s \in \mathcal{F}} (\sigma_s \| t_s \| V)^2 \right)^{1/2} \leq C < \infty,
\]

where

\[
 \sigma_s := \prod_{j \in \mathbb{N}} \rho_j^{s_j},
\]

see [5, Theorem 2.1]. Moreover, the sequence \( \{\| t_s \| V \}_{s \in \mathcal{F}} \) is \( \ell_p \)-summable with \( p = \frac{2q}{2+q} < 1 \). We define

\[
 S_{\Lambda(\xi)} v := \sum_{s \in \Lambda(\xi)} t_s y_s,
\]

where \( \Lambda(\xi) \) is given by the formula (3.2). The following theorem is an improvement of [50, Theorem 3.9].

**Theorem 4.3** Let \( \bar{\alpha} \in L_{\infty}(D) \) and \( \text{ess inf} \ \bar{\alpha} > 0 \). Assume that there exists an increasing sequence \( (\rho_j)_{j \in \mathbb{N}} \) of numbers strictly larger than 1 such that the sequence \( (\rho_j^{-1})_{j \in \mathbb{N}} \) is \( \ell_q \)-summable with \( 0 < q < 2 \), and there holds the condition (4.6). Then for every integer \( n > 1 \), we can construct a deep ReLU neural network \( \phi_{\Lambda(\xi_n)} := (\phi_s)_{s \in \Lambda(\xi_n)} \) on \( \mathbb{R}^m \) with \( m := \left\lfloor K \frac{n}{\log n} \right\rfloor \) for some positive constant \( K \), having the following properties.

1. The deep ReLU neural network \( \phi_{\Lambda(\xi_n)} \) is independent of \( u \);
2. The input and output dimensions of \( \phi_{\Lambda(\xi_n)} \) are at most \( m \);
3. \( W(\phi_{\Lambda(\xi_n)}) \leq n \);
4. \( L(\phi_{\Lambda(\xi_n)}) \leq C \log n \log \log n \);
5. The approximation of \( u \) by \( \Phi_{\Lambda(\xi_n)} u := \sum_{s \in \Lambda(\xi_n)} t_s \phi_s \) gives the error estimate

\[
 \| u - \Phi_{\Lambda(\xi_n)} u \|_{L_{\infty}(I^\infty, V)} \leq C \left( \frac{n}{\log n} \right)^{(1/q-1/2)}.
\]

Here the constant \( C \) is independent of \( u \) and \( n \).
Proof. This theorem can be proven in a way similar to the proof of Theorem 4.2. Let us give a brief proof. Given \( \xi \geq 3 \), we have the Cauchy-Schwarz inequality and Lemma A.4 in Appendix that
\[
\|u - S_{\Lambda(\xi)}u\|_{L_\infty(\mathbb{R}^n, V)} \leq \sum_{s \in \Lambda(\xi)} \|t_s\|_V \leq \left( \sum_{s \in \Lambda(\xi)} (\sigma_s \|t_s\|_V)^2 \right)^{1/2} \left( \sum_{s \in \Lambda(\xi)} \sigma_s^{-q} \sigma_s^{-2q} \right)^{1/2}
\]
\[
\leq C\xi^{-(1/q-1/2)} \left( \sum_{s \in \Lambda(\xi)} \sigma_s^{-q} \right)^{1/2} \leq C\xi^{-(1/q-1/2)}.
\]

Put \( \delta := \xi^{-(1/q-1/2)} \). For every \( s \in \Lambda(\xi) \), by Lemma 2.5 there exists a deep ReLU neural network \( \phi_s \) on \( \mathbb{I}^m \) such that
\[
\sup_{y \in \mathbb{I}^m} |y^s - \phi_s(y)| \leq \delta,
\]
and the size and depth of \( \phi_s \) are bounded as
\[
W(\phi_s) \leq C (1 + |s|_1 \log \delta^{-1}) \leq C (1 + |s|_1 \log \xi)
\]
and
\[
L(\phi_s) \leq C (1 + \log |s|_1 \log \delta^{-1}) \leq C (1 + \log |s|_1 \log \xi).
\]
We define \( \Phi_{\Lambda(\xi)} := (\phi_s)_{s \in \Lambda(\xi)} \) as the deep ReLU neural network realized by parallelization of \( \phi_s, s \in \Lambda(\xi) \). Consider the approximation of \( u \) by
\[
\Phi_{\Lambda(\xi)} u := \sum_{s \in \Lambda(\xi)} t_s \phi_s(y).
\]
Then by the inclusion \( (\|t_s\|_V)_{s \in \mathbb{I}^m} \in L_p(\mathbb{I}^m), p \in (0, 1) \) and (4.7), we have
\[
\|u - \Phi_{\Lambda(\xi)} u\|_{L_\infty(\mathbb{R}^n, V)} \leq \|u - S_{\Lambda(\xi)}u\|_{L_\infty(\mathbb{R}^n, V)} + \|S_{\Lambda(\xi)}u - \Phi_{\Lambda(\xi)} u\|_{L_\infty(\mathbb{R}^n, V)}
\]
\[
\leq C\xi^{-(1/q-1/2)} + \sum_{s \in \Lambda(\xi)} \|t_s\|_V \|y_s - \phi_s\|_{L_\infty(\mathbb{R}^n, V)}
\]
\[
\leq C\xi^{-(1/q-1/2)} + C\xi^{-(1/q-1/2)} \sum_{s \in \Lambda(\xi)} \|t_s\|_V \leq C\xi^{-(1/q-1/2)},
\]
where the constants \( C \) may be different and are independent of \( u \) and \( \xi \). By the construction of \( \Phi_{\Lambda(\xi)} \) we have
\[
W(\Phi_{\Lambda(\xi)}) \leq \sum_{s \in \Lambda(\xi)} \leq W(\phi_s) \leq \sum_{s \in \Lambda(\xi)} C (1 + |s|_1 \log \xi) \leq C \left( |\Lambda(\xi)| + \log \xi \sum_{s \in \Lambda(\xi)} p_s(1) \right)
\]
\[
\leq C \left( |\Lambda(\xi)| + \log \xi \sum_{s \in \Lambda(\xi)} p_s(1) \sigma_s^{-q} \sigma_s^q \right) \leq C \xi \log \xi
\]
where in the last estimate we used Lemmas A.1(i) and A.4 in Appendix. Similarly, we have
\[
L(\Phi_{\Lambda(\xi)}) \leq \max_{s \in \Lambda(\xi)} L(\phi_s) \leq C \max_{s \in \Lambda(\xi)} (1 + \log |s|_1 \log \xi) \leq C \log \xi \log \log \xi,
\]
see Lemma A.5 in Appendix. Now following argument at the end of the proof of Theorem 3.10, we obtain the existence of \( \xi_n \) for a given \( n > 1 \).
5 Concluding remarks

We have established bounds in terms of the size \( n \) of deep ReLU neural networks for error of approximation of the solution \( u \) to parametric and stochastic elliptic PDEs with lognormal inputs by them. The method of this approximation is as follows. For given \( n \in \mathbb{N}, n > 1 \) we construct a compactly supported deep ReLU neural network \( \phi_n := (\phi_j)_{j=1}^m \) of the size \( \leq n \) on \( \mathbb{R}^m, m = O(n/\log n) \), with \( m \) outputs to approximate the \( m \)-term truncation of the Hermite gpc expansion \( \sum_{j=1}^m u_s H_s \) of \( u \) by \( u_n := \sum_{j=1}^m u_s \phi_j \). We proved that the extension of \( u_n \) to \( \mathbb{R}^\infty \) approximates \( u \) with the error bound \( O((n/\log n)^{-1/q}) \), and that the depth of \( \phi_n \) is \( O(n^\delta) \) for any \( \delta > 0 \). This result is extended to approximation of holomorphic functions which include solutions to some important problems of linear elliptic and parabolic PDEs with lognormal inputs. We also obtained similar results for approximation by deep ReLU neural networks of solutions to parametric and stochastic elliptic PDEs with affine inputs. These results are based on an \( m \)-term truncation of the Jacobi and Taylor gpc expansions of the solutions.

In the present paper, we have been concerned about the parametric approximability for parametric and stochastic elliptic PDEs. Therefore, the results themselves do not yield a practically realizable approximation since they do not cover the approximation of the gpc expansion coefficients which are functions of the spatial variable. Moreover, the approximant \( u_n \), as we can see, is not a real deep ReLU networks, but just a combination of the gpc Hermite coefficients and the components of a deep ReLU network. Naturally, it would be desirable to study the problem of full neural network approximation of the solution \( u \) to parametric and stochastic elliptic PDEs by combining the spatial and parametric domains based on fully discrete approximation in [3, 14]. We will discuss this problem in a forthcoming paper.

A Appendix: Auxiliary results

Lemma A.1 Let \( \theta > 0, \xi > 1 \) and \((s_s)_{s \in F} \) be a sequence of numbers strictly larger than 1. Then we have the following.

(i) Assume that \((s_s^{-1})_{s \in F} \in \ell_q(F)\). The set \( \Lambda(\xi) \) is finite and it holds

\[ |\Lambda(\xi)| \leq K_{q,\xi}, \]

where \( K_q := \sum_{s \in F} s^{-q} \) is as in (3.8).

(ii) Assume that \((p_s(\theta)s_s^{-1})_{s \in F} \in \ell_q(F)\) for some \( \theta > 0 \). There holds

\[ m_1(\xi) \leq K_{q,\theta} \xi^{\frac{1}{q}}, \]

where \( K_{q,\theta} := \left( \sum_{s \in F} p_s(\theta)s^{-q} \right)^{\frac{1}{q}} \) is as in (3.9).

Proof. Notice that \( 1 \leq s_s^{-q} \xi \) for every \( s \in \Lambda(\xi) \). This implies (i):

\[ |\Lambda(\xi)| = \sum_{s \in \Lambda(\xi)} 1 \leq \sum_{s \in \Lambda(\xi)} \xi s^{-q} \leq K_q \xi \]
Moreover, we have that $1 \leq s_j$ for every $j \in \nu_s$. Hence, we derive the inequality

$$\max_{s \in \Lambda(\xi)} |s|^{\theta_q} \leq \sum_{s \in \Lambda(\xi)} \left( \prod_{j \in \nu_s} (1 + s_j) \right)^{\theta_q} \leq \sum_{s \in \Lambda(\xi)} p_s(\theta)^q \xi \sigma_s^{-q} \leq K_{q,\theta} \xi \tag{A.1}$$

which proves (ii).

By this definition we have

$$\bigcup_{s \in \Lambda(\xi)} \nu_s \subset \{1, 2, \ldots, m(\xi)\} \tag{A.2}$$

**Lemma A.2** Let $\theta > 0$, $0 < q < \infty$ and $(\sigma_s)_{s \in \mathbb{F}}$ be an increasing sequence of numbers strictly larger than 1. Assume that $(\sigma_s^{-1})_{s \in \mathbb{F}} \in \ell_q(\mathbb{F})$ and $\sigma_{e_{i'}} \leq \sigma_{e_i}$ if $i' < i$. Then there holds

$$m(\xi) \leq K_q \xi, \tag{A.2}$$

where $K_q$ is the constant given in Lemma A.1(i).

**Proof.** Noting (3.4), there is a $s \in \Lambda(\xi)$ such that $s_{m(\xi)} > 0$. Then we have $e^{m(\xi)} \leq s$. Since $\Lambda(\xi)$ is downward closed, we have $e^{m(\xi)} \in \Lambda(\xi)$. From the definition (3.2) of $\Lambda(\xi)$ and the assumption in the lemma, we obtain

$$\sigma_{e_1}^q \leq \sigma_{e_2}^q \leq \ldots \leq \sigma_{e^{m(\xi)}}^q \leq \xi.$$ 

Thus, $e_1, \ldots, e^{m(\xi)}$ belong to $\Lambda(\xi)$. This yields the inequality $|\Lambda(\xi)| \geq m(\xi)$ which together with the inequality $|\Lambda(\xi)| \leq K_q \xi$ in Lemma A.1(i) proves (A.2). The inclusion (A.1) can then be obtained directly from (3.4).

**Lemma A.3** Let $\theta \geq 0$, $0 < q < \infty$, and $\Lambda^*(\xi)$ be defined in (3.30). Assume that $(p_s(\frac{\theta+1}{q}, 1) \sigma_s^{-1})_{s \in \mathbb{F}} \in \ell_q(\mathbb{F})$. There holds

$$\sum_{(s, \ell) \in \Lambda^*(\xi)} p_s(\theta) \leq C \xi.$$ 

**Proof.** We have

$$\sum_{(s, \ell) \in \Lambda^*(\xi)} p_s(\theta) = \sum_{s \in \Lambda(\xi)} \sum_{\ell=0}^{s} p_s(\theta) \leq \xi \sum_{\sigma_{s,\xi}^{-q} \geq 1} \sum_{\ell=0}^{s} p_s(\theta) \sigma_s^{-q}$$

$$= \xi \sum_{\sigma_{s,\xi}^{-q} \geq 1} \left( \prod_{j=1}^{m}(1 + s_j) \right) p_s(\theta) \sigma_s^{-q} \leq \xi \sum_{s \in \mathbb{F}} p_s(\theta + 1) \sigma_s^{-q} \leq C \xi.$$ 

The following lemma is a direct consequence of [14, Lemma 6.2].

**Lemma A.4** Let $0 < q < \infty$ and $\theta$ and $\lambda$ be arbitrary nonnegative real numbers. Assume that $\rho = (\rho_j)_{j \in \mathbb{N}}$ be a sequence of numbers strictly larger than 1 such that $(\rho_j^{-1})_{j \in \mathbb{N}} \in \ell_q(\mathbb{N})$. Then for the sequences $(\sigma_s)_{s \in \mathbb{F}}$ and $(p_s(\theta, \lambda))_{s \in \mathbb{F}}$ given as in (4.3) and (3.7), respectively, we have

$$\sum_{s \in \mathbb{F}} p_s(\theta, \lambda) \sigma_s^{-q} < \infty.$$ 

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Proof. Notice that $c_{a,b}^{s} \leq (1 + \lambda' s)^{\theta'}$ for $s \in \mathbb{N}_0$ with some $\lambda' > 0$ and $\theta' > 0$ depending on $a, b$. Hence, for any $\theta, \lambda \geq 0$, we get

$$p_s(\theta, \lambda)\sigma_s^{-q} = p_s(\theta, \lambda)c_{s}^{q}(\rho^{-s})^q \leq p_s(\theta, \lambda)p_s(q\theta', \lambda')(\rho^{-s})^q \leq p_s(\theta^*, \lambda^*)(\rho^{-s})^q,$$

where $\theta^* := \theta + q\theta'$ and $\lambda^* := \max(\lambda, \lambda')$. We derive that

$$\sum_{s \in \mathbb{F}} p_s(\theta, \lambda)\sigma_s^{-q} \leq \sum_{s \in \mathbb{F}} p_s(\theta^*, \lambda^*)(\rho^{-s})^q.$$

Now applying [14, Lemma 6.2] to the right-hand side we obtain the desired result.

Lemma A.5 Let $0 < q < \infty$, $c_{a,b}^{s}$ be defined as in (4.2) and let $(\delta_j)_{j \in \mathbb{N}}$ be a sequence of numbers strictly larger than 1 such that $(\delta_j^{-1})_{j \in \mathbb{N}} \in \ell_q(\mathbb{N})$. Assume that there exists a sequence of positive number $(\rho_j)_{j \in \mathbb{N}}$ such that $c_{k,j}^{a,b} \rho_j^{-k} < \delta_j^{-k}$, $k, j \in \mathbb{N}$. For the sequence $(\sigma_s)_{s \in \mathbb{F}}$ given as in (4.3), and $\xi > 1$, let $m_1(\xi)$ be the number defined by (3.3). Then we have for every $\xi > 1$,

$$m_1(\xi) \leq C \log \xi,$$

(A.3)

with the constant $C$ independent of $\xi$.

Proof. The proof relies on Lemma A.1 and a technique from the proof of [50, Lemma 2.8(ii)]. Fix a number $p$ satisfying $0 < p < q$ and let the sequence $(\beta_s)_{s \in \mathbb{F}}$ be given by

$$\beta_s^{-1} := \begin{cases} \max(\sigma_s^{-1}, j^{-1/p}) & \text{if } s = e^j, \\ \sigma_s^{-1} & \text{otherwise.} \end{cases}$$

Notice that the sequence $(\alpha_s^{-1})_{s \in \mathbb{F}}$ defined by

$$\alpha_s^{-1} := \begin{cases} j^{-1/p} & \text{if } s = e^j, \\ 0 & \text{otherwise}, \end{cases}$$

belongs to $\ell_q(\mathbb{F})$. On the other hand, from Lemma A.4 one can see that the sequence $(\sigma_s^{-1})_{s \in \mathbb{F}}$ belongs to $\ell_q(\mathbb{F})$. This implies that the sequence $(\beta_s^{-1})_{s \in \mathbb{F}}$ belongs to $\ell_q(\mathbb{F})$. Hence, by Lemma A.1 the set $\Lambda_{\beta}(\xi) := \{ s \in \mathbb{F} : \beta_s^\xi \leq \xi \}$ is finite. Notice also that $(\beta_s)_{s \in \mathbb{F}}$ is increasing and $\Lambda_{\beta}(\xi)$ is downward closed. Put $n := |\Lambda_{\beta}(\xi)|$. Then the set $\Lambda_{\beta}(\xi)$ contains $n$ largest elements of $(\beta_s)_{s \in \mathbb{F}}$. Therefore by the construction of $(\beta_s)_{s \in \mathbb{F}}$ we have

$$\min_{s \in \Lambda_{\beta}(\xi)} \beta_s^{-1} = \beta_{s_n}^{-1} \geq n^{-1/p}.$$ 

Since $c_{k,j}^{a,b} \rho_j^{-k} \leq \delta_j^{-k}$, $k, j \in \mathbb{N}$ and $(\delta_j)_{j \in \mathbb{N}}$ be a sequence of numbers strictly larger than 1 and $(\delta_j^{-1})_{j \in \mathbb{N}} \in \ell_q(\mathbb{N})$, there exists $\delta < 1$ such that $c_{k,j}^{a,b} \rho_j^{-k} \leq \delta$ for $k, j \in \mathbb{N}$. Therefore have for $r > 1$,

$$\sup_{|s|_{1}=r} \beta_s^{-1} = \sup_{|s|_{1}=r} \sigma_s^{-1} \leq \delta^r.$$ 

Let $\tilde{r} > 1$ be an integer such that $n^{-1/p} > \delta^r$. Then one can see that

$$\max_{s \in \Lambda_{\beta}(\xi)} |s|_{1} < \tilde{r}.$$
For the function $g(t) := \delta^t$, its inverse is defined as $g^{-1}(x) = \frac{\log x}{\log \delta}$. Hence we get $\tilde{r} < g^{-1}(n^{-1/p})$, and consequently,
\[
\max_{s \in \Lambda_{\beta}(\xi)} |s|_1 < g^{-1}(n^{-1/p}) \leq C \log n = C \log |\Lambda_{\beta}(\xi)|.
\]
By Lemma A.1 we obtain the inequality $|\Lambda_{\beta}(\xi)| \leq C \xi$ which together with the inclusion $\Lambda(\xi) \subset \Lambda_{\beta}(\xi)$ proves (A.3).

**Lemma A.6** Let the Jacobi polynomial $J_s$ be written in the form
\[
J_s(y) = \sum_{\ell=0}^{s} a_{s,\ell} y^{\ell},
\]
then
\[
\sum_{\ell=0}^{s} |a_{s,\ell}| \leq K_{a+b} 6^{s}.
\]

**Proof.** It is well-known that for each $s \in \mathbb{N}$, the univariate Jacobi polynomial $J_s$ can be written as
\[
J_s(y) = \frac{\Gamma(a+s+1)}{s!\Gamma(a+b+s+1)} \sum_{m=0}^{s} \binom{s}{m} \frac{\Gamma(a+b+s+m+1)}{\Gamma(a+m+1)} \left( \frac{y-1}{2} \right)^m,
\]
where $\Gamma$ is the gamma function. From $\left(\frac{y-1}{2}\right)^2 = \left(\frac{y^2-1}{2}\right)$ we see that $|a_{s,\ell}|$ is equal to the coefficient of $(y-2)^\ell$. Therefore, choosing $y = 3$ we get
\[
\sum_{\ell=0}^{s} |a_{s,\ell}| = \frac{\Gamma(a+s+1)}{s!\Gamma(a+b+s+1)} \sum_{m=0}^{s} \binom{s}{m} \frac{\Gamma(a+b+s+m+1)}{\Gamma(a+m+1)}.
\]
Let
\[
B(x, y) := \int_{0}^{1} t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}
\]
be the beta function. It is decreasing in $x$ and in $y$. Hence for $m \leq s$,
\[
\frac{\Gamma(a+b+s+m+1)}{\Gamma(a+m+1)} = \frac{\Gamma(b+s)}{B(a+m+1, b+s)} \leq \frac{\Gamma(b+s)}{B(a+s+1, b+s)} = \frac{\Gamma(a+b+2s+1)}{\Gamma(a+s+1)}.
\]
This gives
\[
\sum_{\ell=0}^{s} |a_{s,\ell}| \leq \frac{\Gamma(a+b+2s+1)}{s!\Gamma(a+s+1)} \sum_{m=0}^{s} \binom{s}{m} = 2^{s} \frac{\Gamma(a+b+2s+1)}{s!\Gamma(a+b+s+1)}.
\]
By using Stirling’s formula for the gamma function $\Gamma(x+\alpha) \sim \Gamma(x)x^\alpha$, we get
\[
\sum_{\ell=0}^{s} |a_{s,\ell}| \leq C(2e)^s \frac{\Gamma(a+b+s+1)}{s! \Gamma(a+b+s+1)} \leq C(2e)^s \left( \frac{a+b+1}{s} + 1 \right) \leq K_{a+b} 6^{s}.
\]

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