Exact description of non-Markovian effect in open quantum system: 
the discretized environment method

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An approach, called discretized environment method, is introduced to treat exactly non-Markovian 
effects in open quantum systems. In this approach, a complex environment described by a spectral 
function is mapped into a finite set of discretized states with an appropriate coupling to the system 
of interest. The finite set of system plus environment degrees of freedom are then explicitly followed 
in time leading to a quasi-exact description. The present approach is anticipated to be particularly 
accurate in the low temperature and strongly non-Markovian regime. The discretized environment 
method is validated on a two-level system (qubit) coupled to a bosonic or fermionic heat bath. A 
perfect agreement with the quantum Langevin approach is found. Further illustrations are made 

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I. INTRODUCTION

In the last decades, efforts have been made to de-
scribe the dynamics of open quantum systems in various 
regime of coupling to an environment at zero or finite 
temperature [1, 2]. From these efforts, a large variety 
of methods either exact or approximate have emerged 
to treat the non-Markovian effects which often play im-
portant role in the dynamics of open quantum system. 
Among the deterministic approaches, one can quote pro-
jection techniques like Nakajima-Zwanzig (NZ) or Time-
Convolutionless (TCL) [3–6]. While being rather efficient 
in some cases [7, 8], these approaches rapidly becomes 
cumbersome as the complexity of the system increases.

Recently, several stochastic methods have been pro-
posed to reformulate exactly the problem of a system 
coupled to an environment. This includes the Quan-
tum state diffusion (QSD) approach [9–13], the quan-
tum Langevin approach [14–18], the Quantum Monte-
Carlo (QMC) [19–22] or the stochastic method of Ref. 
[23]. While very promising, these approaches rapidly becomes 
cumbersome as the complexity of the system increases.

Schrödinger equations can be derived without difficul-
ties like in the QMC approach [20][21], some trajectories 
might be unstable and can lead to uncontrolled numeric.

The aim of the present work is to provide a simple 
deterministic approach which is able to describe exactly 
for a large class of situations in the dynamics of an open 
quantum system. The main hypothesis of the method is 
that the environment can be explicitly incorporated in 

II. DISCRETIZED ENVIRONMENT METHOD 
FOR OPEN QUANTUM SYSTEMS

In the present work, we consider the situation where 
a system S is linearly coupled to an environment E or 
thermal bath. The Hamiltonian $H$ is decomposed as:

$$H = \sum_i \hbar \omega_i S_i S_i + \sum_\nu \hbar \omega_\nu B_\nu^\dagger B_\nu + H_C. \quad (1)$$
The creation/annihilation operators \( \{ S_i^+, S_i \} \) (resp. \( \{ B_i^+, B_i \} \)) are associated to the system (resp. environment) degrees of freedom. \( H_C \) denotes the coupling Hamiltonian between the two subsystems and is written here as:

\[
H_C = \sum_{i,\nu} g_{i\nu} (S_i \otimes B_i^\dagger + S_i^\dagger \otimes B_i). \tag{2}
\]

In the following, we simply assume that the coupling constant are independent on the system state, i.e. \( g_{i\nu} = g_{i} \).

The difficulty in treating the system+environment problem stems from the complexity of the environment. It is usually assumed that the number of degrees of freedom of the environment is too large to follow them explicitly in time. To deal with this complexity, the standard strategy is to derive the equation of motion for the system degrees of freedom only where the effect of the environment is treated in an approximative way \[1\] [2].

### A. Direct diagonalization approach

In the present article, we start from a completely different strategy and assume that, in some cases, the environment degrees of freedom can be explicitly included in the description and that the Hamiltonian can be directly diagonalized. Such diagonalization requires a priori to introduce a basis for the full Hilbert space of the system plus environment.

In the following, we introduce system and environment ground states, denoted by \( |0\rangle_S \) and \( |0\rangle_E \). These states can be considered as particle vacuum respectively for the creation operators \( S_i^+ \) and \( B_i^\dagger \):

\[
| i \rangle = S_i^\dagger | 0 \rangle_S, \quad S_i | 0 \rangle_S = 0,
| \nu \rangle = B_i^\dagger | 0 \rangle_E, \quad S_i | 0 \rangle_E = 0. \tag{3}
\]

The set of product states \( \{ |0\rangle_S \otimes |0\rangle_E \}, \{ |i\rangle \otimes |0\rangle_E \}, \{ |0\rangle_S \otimes |\nu\rangle \}, \{ |i\rangle \otimes |\nu\rangle \} \) forms a basis of the complete Hilbert space.

Two difficulties generally prevents a direct diagonalization of the Hamiltonian. The first one comes from the fact that the Hilbert space, associated to the environment, is usually of infinite size. The second one is that even if one can consider a finite but very large set of environment states, in many situations, the basis will remain rather large and diagonalization will require large numerical effort. Denoting by \( N_S \) (\( N_E \)) the number of excited states in the system (in the environment). The total Hilbert space size is \( (N_S + 1) \times (N_E + 1) \).

The specific shape of the Hamiltonian considered here is anticipated to reduce the effort necessary to perform a direct diagonalization. Indeed, only the states \( \{ |i\rangle \otimes |0\rangle_E \} \) and \( \{ |0\rangle_S \otimes |\nu\rangle \} \) are coupled to each other. Therefore, only matrices of dimension \( (N_S + 1) \times (N_E + 1) \) need to be diagonalized, reducing significantly the numerical challenge.

For compactness, in the following, we will introduce the notation:

\[
| 0 \rangle = |0\rangle_S \otimes |0\rangle_E,
| i \rangle = |i\rangle \otimes |0\rangle_E, \quad | \nu \rangle = |\nu\rangle \otimes |0\rangle_S.
\]

The state \( |0\rangle \) is a vacuum for the total system. In the reduced space, formed by the states \( \{ |i\rangle_0, |\nu\rangle_0 \} \), the Hamiltonian matrix, denoted by \( H_R \), is written as:

\[
H_R = \sum_i \hbar \omega_i |i\rangle_0 0 \langle i| + \sum_\nu \hbar \omega_\nu |\nu\rangle_0 0 \langle \nu| + \sum_{i,\nu} g_{i\nu} \{ |i\rangle_0 0 \langle \nu| + |\nu\rangle_0 0 \langle i| \}. \tag{4}
\]

Formally, the diagonalization of the Hamiltonian \( H_R \) is equivalent to define a new subset of operators \( \{ A_\alpha^+, A_\alpha \} \) in such a way that the Hamiltonian takes the form:

\[
H = \sum_\alpha \hbar \Omega_\alpha A_\alpha^+ A_\alpha \tag{5}
\]

The new operators are rather specific since they combines degrees of freedom of both the system and the environment. Denoting by \( U \) the associated unitary canonical transformation, we use the following convention:

\[
\begin{align*}
A_\alpha^+ = & \sum_i U_{i\alpha} S_i^\dagger \otimes 1_E + \sum_\nu U_{\nu\alpha} 1_S \otimes B_\nu^\dagger, \tag{6} \\
A_\alpha = & \sum_i U_{i\alpha}^* S_i \otimes 1_E + \sum_\nu U_{\nu\alpha}^* 1_S \otimes B_\nu.
\end{align*}
\]

and associate to them new vacuum \( |0_\alpha\rangle \) defined through the property \( A_\alpha |0_\alpha\rangle = 0 \). For compactness of notation, we will write \( A_\alpha = \sum_\alpha U_{i\alpha} S_i^\dagger + \sum_\nu U_{\nu\alpha} B_\nu^\dagger \). The inverse transformation to \([3]\) are given by

\[
S_i^+ = \sum_\alpha U_{i\alpha}^* A_\alpha, \quad B_\nu^\dagger = \sum_\alpha U_{\nu\alpha}^* A_\alpha. \tag{7}
\]

In the following, we will systematically use the notation \( (i, j) \) for the system, \( (\nu, \mu) \) for the environment and \( (\alpha, \beta) \) for the new states that combines both subsystems.

Note that besides the Hamiltonian, the number of excitation \( N \) in the total system, that is a constant of motion, also takes a simple form:

\[
N = \sum_i S_i^\dagger S_i + \sum_\nu B_\nu^\dagger B_\nu = \sum_\alpha A_\alpha^+ A_\alpha. \tag{8}
\]

### B. Information on system evolution

The introduction of new operators where the Hamiltonian takes a diagonal form gives directly access to the dynamical evolution. Using the Heisenberg picture, the evolutions of the new degrees of freedom simply read:

\[
A_\alpha^+(t) = e^{i\Omega_\alpha t} A_\alpha^+(0), \quad A_\alpha(t) = e^{-i\Omega_\alpha t} A_\alpha(0)
\]

Using the unitary transformation between the original operators and \( \{ A_\alpha^+ \} \), one can directly get

\[
S_i^+(t) = \sum_{\alpha k} U_{i\alpha}^* U_{j\alpha} e^{i\Omega_\alpha t} S_j^+(0) + \sum_{\nu} U_{i\nu}^* U_{\nu\alpha} e^{i\Omega_\alpha t} B_\nu^+(0) \equiv \sum_k M_{ij}(t) S_j^+(0) + \sum_{\nu} M_{i\nu}(t) B_\nu^+(0). \tag{9}
\]
The $M$ matrix introduced here contains all the information about the system or environment evolution.

Let us, for instance, assume that the total initial density $D(0)$ is separated as:

$$D(0) = D_S(0) \otimes D_E(0)$$  \hspace{1cm} (10)

where $D_S$ ($D_E$) are the initial system (environment) density matrix with $\text{Tr}(D_S/D_E)$. Note that more general total density could be used if necessary.

We further assume that initially

$$\langle S_i^\dagger(0)S_j(0) \rangle = \text{Tr}(S_i^\dagger S_j D(0)) = \text{Tr}(S_i^\dagger S_j D_S(0)) = \delta_{ij} n_i^0$$

and similarly

$$\langle B_i^\dagger(0)B_j(0) \rangle = \delta_{ij} n_j^0.$$  \hspace{1cm} (11)

Starting from Eq. (9), we directly get the compact expression for the two-time correlation function

$$\langle S_i^\dagger(t) S_j(t') \rangle = \sum_k M_{ik}^*(t') n_k^0 M_{jk}(t) + \sum_{\nu} M_{i\nu}^*(t') n_{\nu}^0 M_{\nu j}(t)$$  \hspace{1cm} (12)

that is often computed in open quantum systems to probe the memory effects.

C. Discretization of environment

The present approach relies on the possibility to have a discrete and finite environment. In the theory of open quantum systems, a key quantity related to the non-Markovian effects is the dissipative kernel $K(t)$, that we define here as

$$K(t) = \sum_\nu \frac{g_\nu^2}{\hbar^2 \omega_{\nu}} e^{-i \omega_{\nu} t}. \hspace{1cm} (13)$$

Most often this expression is replaced by the integral such that:

$$\sum_\nu \frac{g_\nu^2}{\hbar^2 \omega_{\nu}} e^{-i \omega_{\nu} t} \rightarrow \int_0^{+\infty} \frac{\rho(\omega) g^2(\omega)}{\hbar^2 \omega} e^{-i \omega t} d\omega, \hspace{1cm} (14)$$

where $\rho(\omega)$ is the environment density of state and $g(\omega)$ is the density-dependent coupling. Note that here it is implicitly assumed that the environment has only positive frequencies, i.e. $\omega_{\nu} \geq 0$. In practice, a special continuous function is assumed for $\rho(\omega) g^2(\omega) = J(\omega)$. A typical example is the Lorentz-Drude spectral function

$$J(\omega) = \frac{g_0}{\pi} \frac{\omega^2}{\gamma^2 + \omega^2}, \hspace{1cm} (15)$$

that leads to an exponentially decaying dissipative kernel

$$K(t) = \frac{g_0}{2} e^{-\gamma t} + i \frac{g_0}{\pi} \gamma \int_0^{+\infty} d\omega \frac{\sin(\omega t)}{\gamma^2 + \omega^2}. \hspace{1cm} (16)$$

Memory effect will then be important if $\gamma$ is small, while for $\gamma \rightarrow +\infty$ the Markovian limit is reached.

Most importantly, we see that the continuous limit (Eq. (14)) is just the opposite situation as the one we want to consider, i.e. an environment with a discretized spectra. Therefore, to use the technique based on direct diagonalization and to make connection with more standard approaches we need to invert the scheme depicted in Eq. (14).

Let us consider a situation where the given spectral function $J(\omega)$ is introduced. Because this only constrains the product $\rho(\omega) g^2(\omega)$, there exists some flexibility in fixing $\rho(\omega)$ and $g(\omega)$. Here, we assume that the environment frequencies are uniformly distributed between 0 and a maximal frequency $\omega_{\text{max}}$ with:

$$\omega_{\nu} = \Delta \omega (n + 1/2), \hspace{0.5cm} n = 0, \cdots, N_{\text{max}}$$  \hspace{1cm} (17)

With this convention, the spacing parameter $\Delta \omega = \omega_{\text{max}}/(N_{\text{max}} + 1/2)$.

Once the density of states is supposed to be constant the coupling parameter is $g_{\nu}$ are automatically fixed by imposing the proper continuous limit. This leads to the discrete mapping of the coupling:

$$g_{\nu} = \hbar \sqrt{\Delta \omega J(\omega_{\nu})}. \hspace{1cm} (18)$$

The present methodology, that consists in (i) discretizing the environment space and (ii) performing a direct diagonalization of the discretized Hamiltonian, is the essence of the DEM. In some simple situations, the discretized environment has been employed to obtain analytical expressions for the evolution [25-27], but as far as we know it has not been directly applied as a direct numerical approach to OQS.

In the limit of small spacing with infinitely small $\Delta \omega$ and infinitely large $\omega_{\text{max}}$ the approach is exact. Of course, in practice, only finite size matrices can be diagonalized requiring both non-zero spacing and a finite boundary for the highest frequency. The two parameters $(\Delta \omega, \omega_{\text{max}})$ determine both the numerical accuracy and the numerical effort of the approach. They should be carefully chosen to describe properly a physical process and in such a way that the number of states in the environment $N_{\text{max}}$ is minimized. More precisely, $\omega_{\text{max}}$ determines the time resolution $\Delta \tau = 2\pi/\omega_{\text{max}}$ while $\Delta \omega$ defines the maximal time $\tau_{\text{max}} = 2\pi/\Delta \omega$ over which the calculation can be considered as accurate.

As an important remark, the present approach should be greatly simplified if some physical cutoff exists on the maximal energy for the states to be considered. Having in mind the Lorentz-Drude spectral function, in physical situations a system will interact with states in the vicinity of its typical energy range determined by $\gamma$. The lower $\gamma$ is, the smaller $\omega_{\text{max}}$ could be taken. The Markovian limit will be reached if $\gamma$ is very large. Therefore, we anticipate that the more the dynamics is non-Markovian, the easier it will be to use the DEM approach. It is at variance with other techniques that are usually simplified in the Markovian limit.
Independently on the γ value, a second physical cutoff can be used that is figured out from Eq. (12). We see indeed that a state in the environment has an effect on the system only if its initial population $n^0_\nu$ is non negligible. If the environment is a thermal bath, the number of states needs to be considered depends on the temperature $T$. In particular, if $\gamma \to +\infty$, only this effect will allow to truncate the environment space in a reasonable way. So, the second conclusion is that the DEM will be easier to apply at lower temperature.

III. APPLICATIONS

A. Two-level system coupled to heat-bath

As a proof of the method accuracy, we consider the special case where the system only contains one state associated to $(S^1_0, S_0)$ and with excitation energy $\hbar \Omega$. The system is considered to be coupled with a heat-bath at various temperatures. This situation is actually similar to the case of a two-level system that has been used in Ref. [28] to derive the exact master equation including non-Markovian effects. Note that, the system frequency is renormalized by a counter-term to avoid the unwanted shift of the energy induced by the coupling [2]. The system frequency is set to:

$$\hbar \nu' = \hbar \Omega + \sum_\nu \frac{g^2_\nu}{\hbar^2 \omega_\nu}. \quad (19)$$

Both bosonic and fermionic system+environment are considered. The fermionic or bosonic nature only enters through the initial occupancies of the bath through:

$$n^0_\nu(T) = \frac{1}{\exp(\frac{\hbar \omega_\nu}{k_B T}) + \varepsilon}, \quad (20)$$

where $\varepsilon = -1$ (+1) for bosons (for fermions). We assume that the spectral function is the Lorentz-Drude function given by Eq. (15). In the following, we will use $\Omega, \Omega^{-1},$ and $\hbar \Omega$ as the units, respectively, for the frequencies, time, and energies.

Before presenting the results obtained with the DEM, it is interesting to illustrate the two contributions that will lead to the natural cutoff in the bath frequencies.

Assuming a typical physical situation where the state is coupled to a large set of bath states $\gamma/\Omega = 12$ and moderate temperature $k_B T/(\hbar \Omega) = 1$, the occupation probabilities for bosons or fermions as well as the function $J(\omega)/(g_0 \omega)$ are shown as a function of $\omega$ in Fig. 1. We see that for not too high temperature, even if the spectral function permits the coupling to high frequency states, these states are not necessarily needed to be considered due to the strong cut-off induced by their initial occupancy. In particular, even if $\gamma$ is rather large, a rather small cutoff in the bath frequencies can be used. In practice, for $k_B T/(\hbar \Omega) \leq 1$, a cutoff frequency $\omega_{\text{max}}/\Omega = 10$ and 250 equidistant states insure a converged results.

![FIG. 1: (color online) Illustration of the cutoff induced by the temperature. The quantity displayed here are $J(\omega)/(g_0 \omega)$ (solid line), $n^0_\nu(T)$ for bosons (dotted line), and fermions (dashed line) at $\gamma/\Omega = 12$ and $k_B T/(\hbar \Omega) = 1.$](image)

In the following we compare the DEM results with the those obtained using a more standard approach based on the quantum Langevin equation where the environment dynamics is semi-analytically deduced using the Laplace transform technique [30]. The latter approach is exact under the condition that the imaginary part of the memory kernel $K(t)$ can be neglected. This assumption is valid a priori for weak and intermediate coupling $g_0$. In Fig. 2 the two approaches are compared either in the case of bosonic systems+bath (left) or fermionic systems+bath (right). We see that in both cases, the two approaches agrees very well with each other.

B. Non-Markovian dynamics of three-level system

As a second proof of the DEM feasibility, we consider a system formed by three levels coupled to a heat bath. The three levels are denoted by $|L\rangle$, $|0\rangle$, and $|U\rangle$. The labels ”L”, ”0” and ”U” stands here for lower, intermediate, and upper levels. The three levels have, respectively,

The present agreement completely validates the DEM. In addition, the small number of states ($N_{\text{max}} = 250$) to be considered for the environment shows that the present method can be numerically handled without any difficulty.
energies equal to \(-\hbar \Omega_L, 0, \text{ and } \hbar \Omega_U\). We further introduce the positive quantity \(\hbar \bar{\Omega} = \frac{\hbar (\Omega_U + \Omega_L)}{2}\). In the following, as in Ref. [24] we assume \(\hbar \Omega_U = \hbar \Omega_L = \hbar \Omega\). Frequencies, energies and time will be given in terms of \(\Omega, \hbar \Omega\) and \(\Omega^{-1}\) units, respectively. This case corresponds to the system considered in Ref. [24], where the non-Markovian effects have been investigated at zero temperature using the QSD. Here, we show that the DEM approach can be applied without difficulty to the finite temperature case. We introduce the two creation operators:

\[
S^\dagger_L = |L\rangle\langle 0|, \quad S^\dagger_U = |U\rangle\langle 0|.
\]

The system+environment Hamiltonian is taken as:

\[
H = \hbar \Omega(S^\dagger_U S_U - S^\dagger_L S_L) + \sum_\nu \hbar \omega_\nu B^\dagger_\nu B_\nu + \sum_\nu g_\nu (S_L + S_U) \otimes B^\dagger_\nu + \text{H.c.} \quad (21)
\]

1. Influence of lower state

Compared to the previous two-level system case, the presence of an additional lower state can significantly enrich the dynamics. For instance, considering the zero temperature case and a uniformly spaced environment with constant coupling \(g_\nu = g\) and a spacing \(\Delta \omega\), when a single occupied state is coupled to the environment, we expect a typical decay width \(\Gamma \simeq 2\pi g^2 / \Delta \omega\), leading therefore to a reduction of the lifetime with increasing the interaction. Now, when two occupied levels are coupled to the same doorway states, the Dicke super-radiance effect can take place and leads to a reduction of the decay width (increase of the lifetime) as the coupling strength increases [31]. Here, we consider a slightly different situation where the coupling is not uniform but, as previously, imposed by the spectral function given in Eq. (15). Since the bath only has positive frequencies \(\hbar \omega > 0\), the symmetry between the lower and upper levels with respect to the ground state energy is broken due to the coupling to the environment.

As an illustration of (i) the evolution of such 3-level system, (ii) the symmetry breaking, and (iii) the effect of the additional lower state compared to the 2-level system case, the evolution obtained with the DEM and different coupling strengths are shown in Fig. 3 at \(k_B T / \hbar \Omega = 1.0\). In this figure, both lower and upper states are assumed to be initially unoccupied. In all cases, the evolution of the upper state is compared to the 2-level system case of previous section. In the weak coupling regime \(g_0 = 0.001\) (not shown), the dynamic of the upper level obtained in the 2- and 3-levels systems coincide. However, as can be seen from Fig. 3 when \(g_0\) increases, the presence of an additional level at lower energy significantly affects the evolution. A marked reduction of the asymptotic occupation of the upper level is observed. It is worth mentioning that in the 2-level case, the asymptotic occupation essentially reflects the statistical bosonic character of the bath that imposes its temperature to the system (see Ref. [30]). Here, significant deviation from this pic-
FIG. 3: (color online) Evolution of the upper (dashed line) and lower level (solid line) occupation probabilities in the 3-level system coupled to a bosonic bath for the coupling strengths $g_0 = 0.1$ (a) and 0.5 (b). Both levels are assumed to be initially unoccupied, $n_U(t_0) = n_L(t_0) = 0$. In all cases, $k_B T/\hbar \Omega = 1.0$ and $\gamma/\Omega = 12$. The 2-level system case presented in previous section where the excited state is initially unoccupied is also shown as a reference (dotted line).

ture is seen. The asymptotic limit modification is also accompanied by the appearance of coherent oscillations between the lower and upper level that could be understood as an indirect coupling of the $|U\rangle$ and $|L\rangle$ levels through the heat-bath.

The dynamics towards equilibrium is rather complex and no general rules can be figured out. For instance, we see that at $g_0 = 0.1$ (panel (a) of Fig. 3) for very short time the two levels evolves together and then separates from each others towards different asymptotics. Comparing with the 2-level case, we clearly see that the decay time increases due to the effect of the indirect coupling through the heat-bath, as could be expected from the Dicke super-radiance effects. As $g_0$ increases (panel (b) of Fig. 3), the transition time to equilibrium of the upper level increases due to the extra oscillations that appeared. We see that the occupation of the lowest level first follows the 2-level system case and then suddenly deviates from it.

2. Three-level system dynamics in strong non-Markovian limit

In previous applications, the results have been obtained for rather large $\gamma/\Omega$ values where the memory function, Eq. (16), decays rapidly. As $\gamma$ decreases, the evolution is anticipated to be more and more affected by the time-nonlocal nature of the system+environment evolution. As noted above, smaller $\gamma$ values can be easily described by the DEM method since it leads to a second natural cutoff in the oscillator frequencies to be considered.

Having in mind the picture in Fig. 3, it is actually anticipated that the physics towards equilibrium will encounter a transition between two limits. At $(\hbar \gamma/k_B T) \gg 1$, that is the situations considered previously, the number of doorway states is essentially restricted by initial thermal occupations. When $(\hbar \gamma/k_B T) \ll 1$, the small width of the spectral function around zero frequency will significantly cut the decay channel phase-space and largely influence the evolution. Empirically, we have found that the decay properties are unchanged from $(\hbar \gamma/k_B T) = 12$ down to $(\hbar \gamma/k_B T) \simeq 2$ and starts to be modified for lower values of $\gamma$, as the system enters the strongly non-Markovian regime.

FIG. 4: (color online) Evolution of the upper level occupation probability in the 3-level system coupled to a bosonic bath for two different temperatures (a) $T/\hbar \Omega = 0.1$ and (b) $T/\hbar \Omega = 1.0$. Both upper and lower levels are initially unoccupied, $n_U(t_0) = n_L(t_0) = 0$. In both panels, the different curves correspond to different $\gamma/\Omega$ values: 0.5 (dashed line) and 2.0 (solid curve). In all cases, the coupling strength $g_0$ is set to 0.1.

In the DEM, the total system is treated and one can formally take any $\gamma$. In Figs. 3 and 5, where the upper level is either assumed to be initially unoccupied or occupied while the lower level is always unoccupied, the effect of reducing $\gamma$ value is illustrated for two different temperatures (a) $T/\hbar \Omega = 0.1$ and (b) $T/\hbar \Omega = 1.0$. Both upper and lower levels are initially unoccupied, $n_U(t_0) = n_L(t_0) = 0$. In both panels, the different curves corresponds to different $\gamma/\Omega$ values: 0.5 (dashed line) and 2.0 (solid curve). In all cases, the coupling strength $g_0$ is set to 0.1.
FIG. 5: (color online) The same as figure [1] except that the upper level is initially occupied $n_U(t_0) = 1$ while $n_L(t_0) = 0$.

IV. CONCLUSION

A direct approach is presented to treat a class of system+environment Hamiltonian exactly. The approach, called Discretized Environment Method, relies on the possibility to accurately discretize the environment into a limited set of states able to mimic the complexity stemming from both the coupling between the system and the environment on one side and the large density of state of the environment on the other side. It is anticipated that the approach is particularly accurate in the low temperature and strongly non-Markovian limit. The DEM is illustrated in the case of 2-level system coupled to a bosonic or fermionic heat-bath and is compared to the quantum Langevin approach based on Laplace transform method where an exact solution can be obtained analytically. A perfect agreement is found between both approach, giving evidence that DEM can be a valuable tool to treat the open quantum systems. A second illustration is given through the case of a three-level system coupled to the heat-bath both in the almost Markovian and strongly non-Markovian regimes.

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