Some Subordination Results Associated With Certain Subclass of Analytic Meromorphic Functions

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Abstract: For functions belonging to each of the subclasses $S^*_w(\beta)$ and $C^*_w(\beta)$ of normalized analytic functions in the open unit disk $D$, which are investigated in this paper when $0 \leq \beta < 1$, the authors derive several subordination results involving the Hadamard product (or convolution) of the associated functions. A number of interesting consequences of some of these subordination results are also discussed.

Key words: Univalent functions, convex functions, subordination principle, hadamard product (or convolution), subordinating factor sequence

INTRODUCTION

Let $A$ be the class of functions $f$ normalized by:

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n$$

which are analytic in the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$.

As usual, we denote by $S$ the subclass of $A$, consisting of functions which are also univalent in $D$. We recall here the definitions of the well-known classes of starlike function and convex functions:

$$S^* = \left\{ f \in A : \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0, z \in D \right\}$$

and

$$C^* = \left\{ f \in A : \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, z \in D \right\}.$$

Let $w$ be a fixed point in $D$ and $A(w) = \{f \in H(D) : f(w) = f'(w)-1=0\}$.

In [15], Kanas and Ronning introduced the following classes $S^*_w = \{f \in A(w) : f$ is univalent in $D\}$

and $C^*_w = \{f \in A(w) : \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, z \in D \}$. Later Acu and Owa [1] studied the classes extensively.

Let $S^*_w$ denote the subclass of $A(w)$ consisting of the function of the form:

$$f(z) = \frac{a}{z-w} + \sum_{n=1}^{\infty} a_n (z-w)^n$$

($a_n \geq 0, z \in D$), where $\alpha = \text{Res}(z,w), 0 < \alpha \leq 1$ with $z \neq w$.

The class $S^*_w$ is defined by geometric property that the image of any circular arc centered at $w$ is starlike with respect to $f(w)$ and the corresponding class $C^*_w$ is defined by the property that the image of any circular arc centered at $w$ is convex.

We observe that the definitions are somewhat similar to the ones introduced by Goodman in [13,14] for uniformly starlike and convex functions, except that in this case the point $w$ is fixed.

The functions $f(z)$ in $S^*_w$ is said to be starlike functions of order $\beta$ if and only if:

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \beta \quad (z \in D)$$

for some $\beta(0 \leq \beta < 1)$. We denote by $S^*_w(\beta)$ the class of all starlike functions of order $\beta$.

Similarly, a functions $f(z)$ in $S^*_w$ is said to be convex of order $\beta$ if and only if:

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \beta \quad (z \in D)$$

for some $\beta(0 \leq \beta < 1)$.

It follows from the definitions 3 and 4 that:
\( f(z) \in S_w^*(\beta) \Leftrightarrow zf'(z) \in C_w^*(\beta) \quad (5) \)

We denote by \( C_w^*(\beta) \) the class of all convex functions of order \( \beta \).

For the function \( f(z) \) in the class \( S_w \), we define:

- \( I^0f(z) = f(z) \)
- \( I^1f(z) = (z-w)f'(z) + \frac{2\alpha}{z-w} \)
- \( I^kf(z) = (z-w)(I^{k-1}f(z))' + \frac{2\alpha}{z-w} \)

and for \( k = 1, 2, 3, \ldots \) we can write:

\[
I^k f(z) = (z-w)I^{k-1}f(z)' + \frac{2\alpha}{z-w}
\]

The differential operator \( I^* \) studied extensively by \([10,11]\) and in the case \( w = 0 \) was given by \([9]\).

We note that the class \( S_0^*(\beta) \) and various other subclasses of \( S_0^*(\beta) \) have been studied rather extensively by \([1-8,10-12,16-25]\).

Next, we will recall each of the following coefficient inequalities associated with the function classes \( S_0^*(\beta) \) and \( C_0^*(\beta) \) as well as some significant definitions which will contribute to this study.

**Definitions and preliminaries:** Theorem A \([11]\) if \( f \in S_w \), given by 2, satisfies the coefficient inequality:

\[
\sum_{n=1}^{\infty} n^{k}(n+\beta)a_n \leq \alpha(1-\beta)
\]

with \( \beta(0 \leq \beta < 1) \) and \( 0 < \alpha \leq 1 \), then \( f \in S_w^*(k, \beta) \).

**Theorem B:** If \( f \in S_w \), given by 2, satisfies the coefficient inequality:

\[
\sum_{n=1}^{\infty} n^{k+1}(n+\beta)a_n \leq \alpha(1-\beta)
\]

with \( \beta(0 \leq \beta < 1) \) and \( 0 < \alpha \leq 1 \), then \( f \in C_w^*(k, \beta) \).

**Proof:** It is easy to check that if:

\[
f(z) \in S_w^*(\beta) \Leftrightarrow zf'(z) \in C_w^*(\beta)
\]

Then we have \( f \in C_w^*(k, \beta) \). Hence the theorem.

In view of Theorem A and Theorem B, we now introduce the subclasses \( S_0^*(\beta) \subseteq S_0^*(\beta) \) \( C_0^*(\beta) \subseteq C_0^*(\beta) \) which consist of functions \( f \in S_w \) whose Taylor-Maclaurin coefficients \( a_n \) satisfy the inequalities 3 and 4, respectively.

In our proposed investigation of functions in the classes \( S_0^*(\beta) \) and \( C_0^*(\beta) \) we shall also make use of the following definitions and results.

**Definition 1:** (Hadamard Product or Convolution). Given two functions \( f, g \in S_w \) where \( f \) is given by 5 and \( g(z) \) is defined by:

\[
g(z) = \frac{\alpha}{z-w} + \sum_{n=0}^{\infty} b_n (z-w)^n
\]

(\( b_n \geq 0, z \in D \)). The Hadamard product (or convolution) \( f * g \) is defined (as usual) by:

\[
(f * g)(z) = \frac{\alpha}{z-w} + \sum_{n=0}^{\infty} b_n (z-w)^n
\]

\[
(z-w)^n = (g * f)(z)
\]

**Definition 2:** (Subordination Principle). For two functions \( f \) and \( g \), analytic in \( D \), we say that the function \( F(z) \) is subordinate to \( g(z) \) in \( D \) and write \( f \prec g \) or \( f(z) \prec g(z) \).

If there exists a Schwarz function \( w(z) \), analytic in \( D \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) such that \( f(z) = g(w(z)) \).

In particular, if the function \( g \) is univalent in \( D \), the above subordination is equivalent to \( f(0) = g(0) \) and \( f(D) \subseteq g(D) \).

**Definition 3:** (Subordinating Factor Sequence). A sequence \( \{b_n\}_{n=1}^{\infty} \) of complex numbers is said to be a subordinating factor sequence if, whenever \( f(z) \) of the form (2) is analytic, univalent and convex in \( D \), we have the subordination given by:

\[
\sum_{n=0}^{\infty} a_n b_n (z-w)^n \prec f(z)
\]

(\( z \in D, a_1 = 1 \))

**Theorem C:** (cf. Wilf \([26]\)). The sequence \( \{b_n\}_{n=1}^{\infty} \) is a subordinating factor sequence if and only if:

\[
\Re \left( 1 + 2 \sum_{n=0}^{\infty} b_n z^n \right) > 0, \ (z \in D)
\]
Subordination results for the classes: \( S'_w(\beta) \) AND \( ST_w(\beta) \) Our first main result (Theorem 1 below) provides a sharp subordination result involving the function class \( S'_w(\beta) \).

**Theorem 1:** Let the function \( f \) defined by 2 be in the class \( S'_w(\beta) \). Also let \( \Omega \) denote the familiar class of functions \( f \in S_w \) which are also univalent and convex in \( D \), then:

\[
\left( 1 + \frac{1 + \beta}{1 + \beta + \alpha - \alpha \beta} (f \ast g)(z) \right) < g(z) \quad (z \in D, 0 \leq \beta < 1, 0 < \alpha \leq 1) \quad \text{and} \quad \Re(f(z)) > \frac{1 + \beta + \alpha - \alpha \beta}{2(1 + \beta)} \quad (13)
\]

The following constant factor in the subordination result (13):

\[
\frac{1 + \beta}{1 + \beta + \alpha - \alpha \beta}
\]

cannot be replaced by a larger one.

**Proof:** Let \( f \in S'_w(\beta) \) and suppose that:

\[
g(z) = \frac{\alpha}{z - w} + \sum_{n=1}^{\infty} c_n (z - w)^n \in \Omega.
\]

Then we readily have:

\[
\frac{1 + \beta}{1 + \beta + \alpha - \alpha \beta} (f \ast g)(z) = \frac{1 + \beta}{1 + \beta + \alpha - \alpha \beta} \left( \frac{\alpha}{z - w} + \sum_{n=1}^{\infty} c_n (z - w)^n \right)
\]

Thus, by Definition 3, the subordination result 13 will hold true if:

\[
\frac{1 + \beta}{1 + \beta + \alpha - \alpha \beta} \left( \frac{\alpha}{z - w} + \sum_{n=1}^{\infty} c_n (z - w)^n \right)^{\infty} \tag{15}
\]

is a subordinating factor sequence (with, of course, \( a_1 = 1 \)).

In view of Theorem C, this is equivalent to the following inequality:

\[
\Re\left( \sum_{n=1}^{\infty} \frac{1 + \beta}{1 + \beta + \alpha - \alpha \beta} a_n (z - w)^n \right) > 0 \quad (z \in D) \tag{17}
\]

Now, since \( (n + \beta) \) is an increasing function of \( n \), we have:

\[
\Re\left( \sum_{n=1}^{\infty} \frac{1 + \beta}{1 + \beta + \alpha - \alpha \beta} a_n (z - w)^n \right) > 1 - \frac{2 \alpha (1 - \beta)}{1 + \beta + \alpha - \alpha \beta} r > 0 \quad (|z - w| = r < 1)
\]

where we have also made use of the assertion 7 of Theorem A. This evidently proves the inequality 17 and hence also the subordination result 13 asserted by Theorem 1.

The inequality 14 follows from 7 upon setting:

\[
g(z) = \frac{1}{z - w} \left( \frac{\alpha}{1 - (z - w)} \right)
\]

Next we consider the function:

\[
q(z) = \frac{\alpha}{z - w} - \frac{2 \alpha (1 - \beta)}{1 + \beta + \alpha - \alpha \beta} (z - w) \quad (0 \leq \beta < 1)
\]

which is a member of the class \( S'_w(\beta) \). Then, by using 13, we have:

\[
\frac{1 + \beta}{1 + \beta + \alpha - \alpha \beta} q(z) < \frac{1}{z - w} \left( \frac{\alpha}{1 - (z - w)} \right) \quad (z \in D) \tag{21}
\]

It is also easily verified for the function \( q(z) \) defined by 20 that:

\[
\min \left\{ \Re\left( \frac{1 + \beta}{1 + \beta + \alpha - \alpha \beta} q(z) \right) \right\} = -\frac{\alpha}{2} \tag{22}
\]
which completes the proof of Theorem 1.

**Corollary:** Let the function \( f \) defined by 2 be in the class \( ST_\beta \). Then the assertions 13 and 14 of Theorem 1 hold true. Furthermore, the following constant factor:

\[
\frac{1+\beta}{1+\beta + \alpha - \alpha \beta}
\]

cannot be replaced by a larger one.

By taking \( \alpha = 1 \) in the above corollary, we obtain.

**Corollary:** Let the function \( f \) defined by 2 be in the class \( ST_\beta \). Then

\[
\left( \frac{1}{2}(1+\beta) \right) (f * g)(z) \prec g(z)
\]

and

\[
\Re(f(z)) > -\frac{1}{1+\beta}
\]

The constant factor \( \left( \frac{1}{2}(1+\beta) \right) \) in the subordination result 25 cannot be replaced by a larger one.

**Subordination results for the classes:** \( C^*_\beta \) and \( CV_\beta \) Our proof of Theorem 2 below is much akin to that of Theorem 1. Here we make use of Theorem B in place of Theorem A.

**Theorem 2:** Let the function \( f \) defined by 2 be in the class \( C^*_\beta \). Then:

\[
\frac{1+\beta}{1+\beta + \alpha - \alpha \beta} (f * g)(z) \prec g(z)
\]

(\( z \in D, 0 \leq \beta < 1, 0 < \alpha \leq 1 \)) and

\[
\Re(f(z)) > \frac{1+\beta + \alpha - \alpha \beta}{2(1+\beta)}
\]

The following constant factor in the subordination result 25:

\[
\frac{1+\beta}{1+\beta + \alpha - \alpha \beta}
\]

cannot be replaced by a larger one.

**Corollary:** Let the function \( f \) defined by 2 be in the class \( CV_\beta \). Then the assertions 25 and 26 of Theorem 2 hold true. Furthermore, the following constant factor:

\[
\frac{1+\beta}{1+\beta + \alpha - \alpha \beta}
\]

cannot be replaced by a larger one.

By taking \( \alpha = 1 \) in the above corollary, we obtain.

**Corollary:** Let the function \( f \) defined by 2 be in the class \( CV_\beta \). Then

\[
\left( \frac{1}{2}(1+\beta) \right) (f * g)(z) \prec g(z)
\]

and

\[
\Re(f(z)) > -\frac{1}{1+\beta}
\]

The constant factor \( \left( \frac{1}{2}(1+\beta) \right) \) in the subordination result 27 cannot be replaced by a larger one.

**ACKNOWLEDGEMENT**

The study presented here was supported by Science Fund: 04-01-02-F0425.

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