The growth of chiral magnetic instability in a large-scale magnetic field

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The chiral magnetic effect emerges from a microscopic level, and its interesting consequences have been discussed in the dynamics of the early universe, neutron stars and quark-gluon plasma. An instability is caused by anomalous electric current along magnetic field. We investigate effects of plasma motion on the instability in terms of linearized perturbation theory. A magnetic field can inhibit magnetohydrodynamic waves to a remarkable degree and thereby affects the instability mode. We also found that the unstable mode is consisted of coupling between Alfvén and one of magneto-acoustic waves. Therefore, the propagation of a mixed Alfvén wave driven by magnetic tension is very important. The direction of unperturbed magnetic field favors the wave propagation of the instability mode, when Alfvén speed exceeds sound speed.

Subject Index MHD : dynamo : neutron stars : early universe

1. Introduction

It is widely recognized that the magnetohydrodynamics (MHD) is capable of describing a variety of astrophysical phenomena. The treatment is macroscopic one consisted of fluid motions coupled with electromagnetic forces. A set of MHD equations is scale-independent, and may be applied to the laboratory to astrophysical plasma. The electromagnetic fields are described by classical Maxwell’s equations. They have a symmetry with respect to a parity, that is, a transformation property under spatial inversion. Physical vectors can equate only vectors of the same kind. An example is Ohm’s law \( \vec{j} = \sigma \vec{E} \) with a scalar \( \sigma \), where electric field \( \vec{E} \) and electric current \( \vec{j} \) are polar vectors. Magnetic vector \( \vec{B} \) is an axial one, and is connected to the polar current vector \( \vec{j} \) as \( \nabla \times \vec{B} = 4\pi \vec{j}/c \) (e.g., [1]).

A peculiar form of an electric current arises from a microscopic level:

\[
\vec{j} = \kappa \vec{B}.
\]  

This form means that \( \kappa \) is not scalar but pseudo-scalar by the parity transformation. Possible origin of the form (1) is a quantum anomaly known as the chiral magnetic effect (e.g.,[2–6] and and references therein). There is imbalance between left-handed and right-handed particles in a quantum system, and the current flow along the magnetic field emerges in a macroscopic level. It is known that the electric current along the magnetic field causes an instability, leading to a growth of magnetic field (e.g., [7–9]). Magnetic helicity, which is an indicator of a global topology, is also changed as well as the field amplification. It has been discussed that the chiral magnetic effect leads to inverse cascade, that is, energy transfer from small to large scales. The problem is studied from various aspects[10–15]. This property is an important process of self-organization of turbulent structure.
In recent years, the chiral magnetic effect is widely discussed in relation to quark-gluon plasma in heavy-ion collision experiment [16], and astrophysical consequences in the early universe [7, 10, 17], core-collapse super-nova [18, 19], magnetar [20]. The electric current (1) is also discussed in a context of mean-field MHD dynamo (e.g., [21–23]). In the theory, the mean values of the variables can be distinguished from the fluctuating ones. Thus, the electric current (1) on a large scale is caused as an ensemble of screw-like vortices in microscopic turbulence. Direct numerical simulations of MHD dynamo have been developed with the increase of computer power. In the approach, dynamics of all scales is simultaneously followed as far as small-scale waves are numerically resolved. Thus, a model of microscopic turbulence is no longer needed there. Indeed, some chiral MHD simulations have been performed in non-relativistic framework [24, 25] and in relativistic framework [20]. Their remarkable results demonstrate the ability of the method. However computational cost may be high in the high resolution simulations. Another example of the form (1) is force-free magnetic fields (e.g., [21, 26]), in which Lorentz force vanishes \( \vec{j} \times \vec{B} = 0 \). In a stationary case, \( \kappa \) is constant along magnetic field line. The magnetosphere around a star is modeled by the force-free approximation, in which magnetic pressure is assumed to be much larger than thermal one.

The macroscopic dynamics are governed by the same equations, although the transport coefficient \( \kappa \) determined by a microscopic process is various in the magnitude. Bearing various astrophysical environments in mind, it is important to explore the instability in a wide range of parameters. In this paper, we consider normal mode analysis for linearized system of chiral MHD equations. The background state is assumed to be homogeneous with uniform magnetic field, and small perturbations propagate as MHD waves in the absence of chiral magnetic effect. We study the modification of wave propagation and the instability caused by the chiral magnetic effect in Section 2. This problem was partially studied [24], where the modification is found, but the general property is not clear. The reason will be discussed after our results. We extensively analyze it to explore relevance of the instability in various astrophysical environments. We discuss our results in Section 3.

2. Waves in a linearized system

2.1. Equations

A set of chiral MHD equations are discussed in literature (e.g., [24, 25]). The linear perturbation equations in non-relativistic dynamics are summarized here. We assume that the unperturbed state of the medium is static and homogeneous, i.e., the density \( \rho_0 \) and magnetic field \( \vec{B} = B_0 \hat{z} \), where \( \rho_0 \) and \( B_0 \) are constant. We write small perturbation of a quantity \( Q \) as \( \delta Q \), and then the perturbation equations are given by

\[
\frac{\partial}{\partial t} \delta \vec{B} = -\vec{\nabla} \times (c_0 \delta \vec{E}),
\]

\[
c_0 \delta \vec{E} = -\delta \vec{v} \times \vec{B}_0 + \eta \vec{\nabla} \times \delta \vec{B} - \kappa \delta \vec{B},
\]

\[
\frac{\partial}{\partial t} \delta \rho + \vec{\nabla} \cdot (\rho_0 \delta \vec{v}) = 0,
\]

\[
\rho_0 \frac{\partial}{\partial t} \delta \vec{v} = -\vec{\nabla} (c_0^2 \delta \rho) + \frac{1}{4\pi} (\vec{\nabla} \times \delta \vec{B}) \times \vec{B}_0,
\]

where an adiabatic relation \( \delta p = c_0^2 \delta \rho \) and the Ampère’s law \( 4\pi \delta \vec{j} = c \vec{\nabla} \times \delta \vec{B} \) are used. An electric current parallel to magnetic field is added by the chiral magnetic effect in eq.(3).
We denote a sound speed as $c_s$ and also Alfvén speed as $c_a = (B_0^2/(4\pi \rho_0))^{1/2}$. There are two kinds of restoring forces, pressure and magnetic tension on the plasma motion. Relative importance is inferred from the ratio, which corresponds to so-called plasma $\beta$; $\beta \approx (c_s/c_a)^2$.

We assume that electric resistivity $\eta$ and a coefficient $\kappa$ of chiral magnetic effect are constant. The latter is determined by chiral chemical potential, i.e., imbalance between left and right-chiral particles. As the chiral magnetic instability grows, an electric current flows along magnetic field, and the magnitude $\kappa$ eventually reduces. Our concern is the linear growth at the initial stage, so that $\kappa$ is regraded as a constant.

We assume that all perturbed quantities are proportional to the Fourier form $\exp(-i(\omega t - \vec{k} \cdot \vec{x}))$, and that the wave propagates in $x$-$z$ plane, i.e., $\vec{k} = (k_x, 0, k_z) = (k \sin \theta, 0, k \cos \theta)$. We also assume $k > 0$, but the frequency $\omega$ in general is a complex number, $\omega \equiv \omega_R + i\omega_I$. A mode with $\omega_I > 0$ grows with time. We may limit to the case of $\kappa \geq 0$, since the chiral instability depends on $\kappa^2$ as shown below, although both signs of $\kappa$ are physically allowed.

After some manipulation, perturbed equations (2)-(5) are reduced to

$$M \delta \mathbf{u} = 0,$$

where $M$ is a $5 \times 5$ matrix whose components are explicitly given by

$$M = \begin{bmatrix} k_z & 0 & 0 & \omega + i\eta k^2 & -k_z \kappa \\ 0 & k_z & 0 & \kappa k^2 k_z^{-1} & \omega + i\eta k^2 \\ \omega^2 - c_s^2 k_x^2 & 0 & -c_s^2 k_z k_x & \omega c_s^2 k_z k_x^{-1} & 0 \\ 0 & \omega & 0 & 0 & c_s^2 k_x \\ -c_s^2 k_x k_z & 0 & \omega^2 - c_s^2 k_z^2 & 0 & 0 \end{bmatrix},$$

and a vector $\delta \mathbf{u}$ is given by $\delta \mathbf{u}^T = (\delta v_x, \delta v_y, \delta v_z, B_0^{-1} \delta B_x, B_0^{-1} \delta B_y)$. The component $\delta B_z$ is determined by the Gauss law $\nabla \cdot \delta \vec{B} = 0$, i.e., $\delta B_z = -\delta B_x k_z k_x^{-1}$.

A determinant of the matrix $M$ provides a dispersion relation:

$$0 = (\omega^2 - c_s^2 k_x^2) \omega^2 X^2 - [\omega^2 (k_x^2 + 2k_z^2) - 2c_s^2 k_z k_x^2] c_s^2 \omega X$$
$$+ [\kappa^2 \omega^4 + (c_s^4 k_x^2 - \kappa^2 c_s^2 k_z^2) \omega^{-2} - c_s^4 c_s^2 k_z^4] k_z^2,$$

(8)

where $X = \omega + i\eta k^2$. Equation (8) is a sixth degree polynomial in $\omega$. In order to understand the general feature of the solutions, we start with some limiting cases in next subsections.

2.2. Chiral magnetic instability

We first consider the case of $c_s = c_a = 0$. That is, there are no forces on plasma motions. The dispersion relation (8) becomes

$$0 = [(\omega + i\eta k^2)^2 + \kappa^2 k_z^2] \omega^4.$$

(9)

Non-zero solution is given by $\omega = -i\eta k^2 \pm i\kappa k$. The mode with $\omega = -i(\kappa k + \eta k^2)$ always decays. However, the mode with $\omega = i(\kappa k - \eta k^2)$ grows for $k < \eta^{-1} k$. That is, the long wavelength mode with $\lambda > \lambda_c \equiv |\kappa^{-1} \eta$ is unstable. Resistivity is inefficient for such a long wavelength mode. Eigenvectors of perturbation functions satisfy

$$\delta \vec{v} = (0, 0, 0), \delta \vec{B} \propto (k_z, \pm ik_x, k_x).$$

(10)

These functions mean that the disturbance is purely magnetic one, and is transverse to the wave vector, i.e., $\vec{k} \cdot \delta \vec{B} = 0$ and also $\vec{k} \cdot \delta \vec{j} = 0$. 

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Our concern is the chiral instability mode due to $\kappa \neq 0$, so that we from now on neglect the resistivity $\eta = 0$. The approximation is valid in the long wavelength mode $\lambda \gg \lambda_c$. This approximation simplifies the dispersion relation (8), which is reduced to a cubic equation of $\omega^2$.

2.3. Effect of thermal pressure

By setting $\eta = 0$ and $c_a = 0$ in eq. (8), we have a relation:

$$0 = (\omega^2 + \kappa^2 k^2)(\omega^2 - c_a^2 k^2)\omega^2. \quad (11)$$

There are two non-trivial solutions. One is chiral magnetic mode ($\omega^2 = -\kappa^2 k^2$), and the other sound wave mode ($\omega^2 = c_a^2 k^2$). The sound wave is produced by compressional motion of matter and hence longitudinal mode, i.e., $k \cdot \delta \vec{v} \neq 0$. We explicitly check this fact by the eigen-functions:

$$\delta \vec{v} \propto (k_x, 0, k_z), \quad \delta \vec{B} = 0. \quad (12)$$

As discussed in previous subsection, the chiral mode is transverse mode $k \cdot \delta \vec{B} = 0$. These two modes are completely decoupled. The growth rate of chiral magnetic mode is not affected by the pressure.

2.4. Effect of a uniform magnetic field

In the case of $c_s = 0$ and $\eta = 0$, the dispersion relation (8) is reduced to

$$0 = [(\omega^2 - c_a^2 k^2)(\omega^2 - c_a^2 k^2) + \kappa^2 k^2 \omega^2] \omega^2. \quad (13)$$

It is clear that two waves, the Alfvén mode and fast MHD mode (or fast magneto-acoustic mode) are coupled by a $\kappa$-term. Frequency of the slow one is zero ($\omega^2 = 0$) in the limit of $c_a = 0$. Two non-zero solutions are given by

$$\frac{\omega^2}{k^2} = \frac{1}{2} \left[ (1 + \cos^2 \theta)c_a^2 - \kappa^2 \pm Q^{1/2} \right], \quad (14)$$

where

$$Q = \left[ (1 - \cos \theta)^2 c_a^2 - \kappa^2 \right] \left[ (1 + \cos \theta)^2 c_a^2 - \kappa^2 \right]. \quad (15)$$

The function $Q$ becomes negative in a range of

$$\frac{1}{(1 + |\cos \theta|)^2} < \frac{c_a^2}{\kappa^2} < \frac{1}{(1 - |\cos \theta|)^2}, \quad (16)$$

and hence $\omega$ becomes a complex number. Outside the range of eq.(16), $\omega$ is either real or pure imaginary. Nature of a solution $\omega$ is thus classified to three regions in the $c_a \kappa^{-1} - \theta$ plane, as shown in Fig.1. In the region I (left part of the figure), where $c_a \kappa^{-1} < (1 + |\cos \theta|)^{-1}$, the solution is a pure imaginary i.e., $\omega_R = 0$. On the other hand, $\omega$ is real ($\omega_I = 0$), in the region III (the up-right part of the figure). That is stable wave region, in which $c_a \kappa^{-1} > (1 - |\cos \theta|)^{-1}$. In the intermediate region II, the frequency is a complex number, $\omega = \omega_R + i\omega_I$. The mode becomes oscillatory instability.

In Fig.1, we demonstrate some contours of real and imaginary parts as a function of $c_a \kappa^{-1}$ and propagation angle $\theta$. We may limit to the case of $\omega_R > 0$ and $\omega_I > 0$ only, since a pair of $\pm (\omega_R + i\omega_I)$ is always a solution. The real part $\omega_R / (k \kappa)$, a normalized phase velocity is zero in the region I of the left panel, but it increases with $c_a \kappa^{-1}$ in the region II for a
Fig. 1 Contours of $\omega_R/(k\kappa)$ (left panel) and $\omega_I/(k\kappa)$ (right panel) in the $c_a\kappa^{-1} - \theta$ plane. In the region I of the left panel, $\omega_R/(k\kappa)$ is 0, but $\omega_R/(k\kappa)$ increases with $c_a\kappa^{-1}$ in the region II. Constant lines with $\omega_R/(k\kappa) = 1, 2$ are plotted. In region III, there are two stable waves, fast MHD and Alfvén waves for a fixed velocity $\omega_R/(k\kappa)$. In the region I of the right panel, there are two kinds of growing modes. In region III of the right panel, $\omega_I/(k\kappa)$ is zero. Some constant lines are plotted for values $\omega_I/(k\kappa)$ labeled in the figure.

Fig. 2 Three dimensional display of mode coupling as a function of $\log_{10}(c_a\kappa^{-1})$, which is chosen as y axis ($-2 \leq \log_{10}(c_a\kappa^{-1}) \leq 2$). Real and imaginary parts of a mode are shown in x ($-8 \leq \text{Re}(\omega/k\kappa) \leq 8$) and z ($-1 \leq \text{Im}(\omega/k\kappa) \leq 1$) axes. Negative value of the imaginary part is plotted by a dashed line. For large $c_a\kappa^{-1}$, two modes are stable waves, fast MHD and Alfvén waves.

fixed angle $\theta$. There are two solutions in region III. They are identified as fast MHD and Alfvén waves. The former is approximated as $(\omega/k)^2 \approx c_a^2 - (\kappa/\sin\theta)^2$ for $c_a\kappa^{-1} \gg 1$, and $\omega_R/(k\kappa)$ does not so strongly depend on $\theta$. Therefore, a curve with constant velocity $\omega_R/(k\kappa)$ becomes almost vertical in the left panel of Fig.1. The other curve plotted by a horizontal dotted line represents the Alfvén wave, which is expressed as $(\omega/k)^2 \approx c_a^2 \cos^2 \theta - \kappa^2 \cot^2 \theta$ for $c_a\kappa^{-1} \gg 1$. 

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Next we discuss the imaginary part $\omega_I/(k\kappa)$. For small $c_\alpha\kappa^{-1}$, two solutions in eq.(14) are approximated as

$$\frac{\omega^2}{k^2} = \begin{cases} -\kappa^{-2}c_\alpha^4 \cos^4 \theta + \cdots & \text{for } \theta \rightarrow 0 \\ -\kappa^{-2} + (1 + \cos^2 \theta)c_\alpha^2 + \cdots & \text{for } \theta \rightarrow \pi/2 \end{cases}$$

(17)

where two functional forms correspond to the upper and lower signs in eq.(14). There are two growing modes, and their characteristic growth rates are $\omega_I \approx c_\alpha^2 \kappa^{-1} k$ (slowly growing mode), and $\omega_I \approx k \kappa$ (rapidly growing mode). The frequency of the former vanishes in the limit of $c_\alpha = 0$. As $c_\alpha \kappa^{-1}$ increases, both growth rates approach each other, and match on the critical line $c_\alpha \kappa^{-1} = (1 + |\cos \theta|)^{-1}$. In the right panel of Fig.1, some lines with constant $\omega_I/(k\kappa)$ are plotted. In the region I, there are two branches, which are approximated by eq.(17). In intermediate region II, the growth rate strongly depends on the propagation angle $\theta$. For example, the wave perpendicular to unperturbed magnetic field, i.e., $\theta = \pi/2$ is stabilized for $c_\alpha \kappa^{-1} \geq 1$. On the other hand, the wave parallel to the magnetic field is never stabilized for any value of $c_\alpha \kappa^{-1}$.

We show the coupling of the Alfvén and fast MHD modes, which causes an unstable mode for $c_\alpha \kappa^{-1} < 1$. Figure 2 displays how the phase velocity $\omega/k$ normalized by $\kappa$ changes with the Alfvén velocity $c_\alpha$, for fixed propagation angle $\theta = \pi/4$. In the large limit of $c_\alpha \kappa^{-1}$, there are two different modes, which are described by positive velocities. There are also negative velocity modes, but they are physically the same as positive ones. These different modes represent stable Alfvén and fast MHD waves. As $c_\alpha \kappa^{-1}$ decreases, two velocities agree at a certain point($c_\alpha \kappa^{-1} \approx 3$), where two modes convert to one oscillatory growing and one oscillatory decaying modes in a region of $c_\alpha \kappa^{-1} < 3$. The velocity $\omega_R/(k\kappa)$ further goes to 0, and $\omega_R = 0$ at $c_\alpha \kappa^{-1} \approx 0.6$. Two propagating waves merge and change as standing waves for $c_\alpha \kappa^{-1} < 0.6$. Toward $c_\alpha \kappa^{-1} \rightarrow 0$ after that point, the frequencies change as $\omega_I \rightarrow \pm 0$ or $\omega_I \rightarrow \pm k \kappa$ with $\omega_R = 0$.

The perturbation amplitudes satisfy

$$\delta v_x = -\frac{c_\alpha^2 k^2}{\omega \cos \theta} \frac{\delta B_x}{B_0}, \quad \delta v_y = -\frac{c_\alpha^2 k \cos \theta}{\omega} \frac{\delta B_y}{B_0}, \quad \delta v_z = 0,$$

$$(\omega^2 - c_\alpha^2 k^2)\delta B_x = \kappa \omega k \cos \theta \delta B_y, \quad \delta B_z = -\tan \theta \delta B_x.$$  

(18)

The perturbation of the magnetic field is always perpendicular to the wave, since $\vec{k} \cdot \delta \vec{B} = 0$. In order to study the direction of the plasma motion, we consider two limiting cases. For the wave parallel to the magnetic field ($\theta = 0$), plasma motion is also transverse to wave propagation, since $\delta v_z = 0$. There is no compression of matter, $\vec{\nabla} \cdot \delta \vec{v} = 0$ in this case. As $\theta \rightarrow \pi/2$, we have $\delta v_y \rightarrow 0$ as well as $\delta v_z = 0$. This means that plasma motion is longitudinal, $\delta \vec{v} \propto \vec{k}$, and is compressional $\vec{\nabla} \cdot \delta \vec{v} \neq 0$ in this case. At intermediate angles of wave propagation, the instability mode is a mixture of properties of two limiting cases.

2.5. Magnetohydrodynamical effects

In previous subsections, we have separately considered the effects of plasma motion driven by thermal pressure or magnetic tension on the chiral instability. We here consider a combined effect by $c_\alpha \neq 0$ and $c_a \neq 0$. The dispersion relation (8) is a cubic equation of $\omega^2$, so that a pair ($\pm \omega$) is always a solution of it. It is also easy to understand the fact that there is at least one solution of $\omega^2 > 0$, i.e, a stable wave.
Figure 3 shows the maximum growth rate $\omega_1/(k\kappa)$ among four solutions in $c_s\kappa^{-1} - c_a\kappa^{-1}$ plane, for the propagation angle $\theta = \pi/4$. Stable wave-propagation region is expressed by upper part of ‘ν’-shape. It is natural that there is a different nature that depends on the dominant force. In high $\beta$ region (lower right part of Fig.3), there is an unstable mode. The growth rate is $\omega_1/(k\kappa) \approx 1$ in the limit of $c_a\kappa^{-1} = 0$, irrespective of $c_s\kappa^{-1}$. Plasma motion driven by dominant pressure force does not affect the instability, as discussed in subsection 2.3. As $c_a\kappa^{-1}$ increases with a fixed value of $c_s\kappa^{-1}(>1.5)$, $\omega_1/(k\kappa)$ decreases and becomes 0 at $c_a\kappa^{-1} \approx c_s\kappa^{-1}$. However, unstable bound of $\omega_1 > 0$ shifts to a larger value of $c_a\kappa^{-1}$ for a small $c_s\kappa^{-1}$, that is, a fat part of Fig.3, where $c_a\kappa^{-1} > c_s\kappa^{-1}$.

![Color contour of maximum growth rate Im($\omega/(k\kappa)$) in $c_s\kappa^{-1} - c_a\kappa^{-1}$ plane. The propagation angle of the perturbation is $\theta = \pi/4$. All modes become stable propagation waves in upper ‘ν’-shaped region (a blue region in the figure).]

In order to study the unstable mode, we show in Fig.4 how the frequency of a mode changes with $c_a\kappa^{-1}$ for a fixed $c_s\kappa^{-1}$. In the case of $c_s\kappa^{-1} = 0.75$ (left panel of Fig.4), all modes become stable waves for $c_a\kappa^{-1} \geq 3$. Their phase velocities $\omega/k$ characterize the waves, so that we identify the fast MHD, Alfvén and slow MHD waves according to the absolute value of velocity. It is also found that the unstable mode is caused by a coupling of the fast and Alfvén waves as $c_a\kappa^{-1}$ decreases. The slow one is always decoupled, and is stable wave. This situation is the same as that considered in previous subsection ($c_s = 0$), where the slow mode is decoupled as $\omega = 0$.

In the right panel of Fig.4, we show the case of $c_s\kappa^{-1} = 2$. Like the previous case, three stable waves are identified for $c_a\kappa^{-1} \geq 3$. As $c_a\kappa^{-1}$ decreases, the coupling occurs between the slow and Alfvén modes. The fast mode is always decoupled. This point differs from that for $c_s\kappa^{-1} = 0.75$. Unstable region in Fig.3 changes by a MHD mode which the Alfvén mode couples with.
The Alfvén mode couples with slow one in a high $\beta$ region ($c_s > c_a$), whereas it couples with fast one in a low $\beta$ region ($c_a > c_s$). It is interesting to observe the behavior in a small $c_a\kappa^{-1}$ region in the left panel of Fig. 4 (the case of $c_s\kappa^{-1} = 0.75$). The phase velocity $\omega R_0/(k\kappa)$ of the instability mode sharply decreases around $c_a\kappa^{-1} = 0.75$. At that point, velocity of slow mode sharply increases. That is, wave nature is exchanged. The unstable mode originates from a coupling of fast and Alfvén waves in a low $\beta$ region, but the nature changes like slow one in a high $\beta$ region. At the same time, stable mode behaves like the slow one in large $c_a$ region, but behaves like the fast one in small $c_a$ region.

![Fig. 4](image)

**Fig. 4** Three dimensional display of mode coupling as a function of $c_a\kappa^{-1}$, which is chosen as $y$ axis ($0 \leq c_a\kappa^{-1} \leq 4$). Real and imaginary parts of a mode are shown in $x$ ($-4 \leq \text{Re}(\omega/k\kappa) \leq 4$) and $z$ ($0 \leq \text{Im}(\omega/k\kappa) \leq 1$) axes. Left panel is for $c_a\kappa^{-1} = 0.75$, while right one is for $c_a\kappa^{-1} = 2$. For large $c_a\kappa^{-1}$, all modes are stable waves characterized by a real frequency. By their propagation velocity, they are identified as F(fast MHD), S(slow MHD) and A(Alfvén) waves, as labeled in the figure.

We discuss how the stable wave region changes with the propagation angle $\theta$. Figure 5 shows the region for $\theta = \pi/8, \pi/4$ and $3\pi/8$. The instability is almost unchanged in a high $\beta$ region ($c_a\kappa^{-1} > c_a\kappa^{-1}$), since $c_a$ is unimportant. However, the growth rate significantly depends on the angle $\theta$ in a low $\beta$ region ($c_a\kappa^{-1} > c_a\kappa^{-1}$), where the Alfvén wave propagation affects the instability. As discussed in subsection 2.4, unstable region diminishes with the increase of $\theta$ for $c_a\kappa^{-1} > 1$. The Alfvén mode velocity goes to 0 in orthogonal direction to unperturbed magnetic field, $\theta \to \pi/2$. Accordingly, the growth is suppressed in a low $\beta$ region with $c_a\kappa^{-1} > 1$.

A peculiar thing should be noted in the limit of $\theta = \pi/2$ ($\cos \theta = 0$). The behavior of $\cos \theta = 0$ differs from that of $\cos \theta \approx 0$. The dispersion relation for $\cos \theta = 0$ is analytically expressed, and shows that there is always one growing mode, for any values of $c_a \neq 0$ and $c_s \neq 0$. In an exactly perpendicular direction, the Alfvén wave propagation is prohibited, and the instability grows irrespective of magnetic field strength.

### 3. Summary and Discussion

The chiral magnetic instability is inherent in an electromagnetic field with the electric current parallel to the magnetic field. We have taken into account of the plasma motion in order to study its relevance in various environments. Assuming that disturbances are small, linearized
Fig. 5  Stable wave region in a parameter space of $c_s/\kappa$ and $c_a/\kappa$. A region between two curves of ‘$\nu$’-shape denotes stable wave propagation for a fixed angle. Three curves are plotted for $\theta = \pi/8, \pi/4, 3\pi/8$. Stable wave region increases with the increase of propagation angle $\theta$.

equations of chiral MHD are examined. All modes are described by solving the resultant dispersion relation, no matter whether they are stable or unstable. The analysis of it is not new, as mentioned in Introduction. We should therefore discuss previous results[24], and especially insufficient points of previous analysis. The dispersion relation depends on three parameters: a set of independent ones is chosen as $c_a/\kappa$, $c_s/\kappa$ and $\theta$ in this paper. In Ref. [24], they fixed a ratio of $c_a/c_s$, and plotted the phase velocity of MHD waves as a function of propagation angle $\theta$ for $(c_s/\kappa)^2 = 0.1, 1, 10$. We found that the choice is not good to grasp whole structure of normal frequencies, as inferred from Fig.3. Their parameters are also limited to stable wave region, and growth rates are never discussed. The relation to the instability is unclear, although modified MHD waves are shown by weak chiral magnetic effect. Our choice is fixing the propagation angle at first, and all normal frequencies were calculated in a wider parameter space. The growth rates were illustrated in three dimensional representation. Thus, we have successfully explored the mode coupling in the unstable region.

We next summarize our findings. When chiral magnetic effect is small enough, small disturbances are described by three waves, i.e, Alfvén, fast and slow MHD waves. An unstable mode originates from a coupling of the Alfvén and one of magneto-acoustic waves. The coupling condition is determined by matching the phase velocities, and therefore the counterpart is slow one for high $\beta$ plasma ($c_a > c_s$), and fast one for low $\beta$ plasma ($c_s < c_a$). Astrophysically, the high $\beta$ plasma is relevant to the early universe, and core-collapse super-nova and neutron star, while low $\beta$ plasma is relevant to a force-free magnetosphere. The Alfvén wave plays a very important role, since the magnetic perturbations are always transverse waves, both in pure Alfvén mode and in pure chiral mode.

The unstable mode grows regardless of the propagation direction in high $\beta$ plasma, where pressure is a dominant force and does not hinder the growth. As a value of $\beta \approx (c_s/c_a)^2$ decreases, magnetic tension becomes important force on plasma motion. Accordingly, wave
propagation velocity depends on the direction. In a low $\beta$ regime with $c_0 > \kappa$, three stable waves like in ordinary MHD appear by mismatching their phase velocities. The propagation of unstable mode perpendicular to the background magnetic field is strongly constrained, i.e., disturbances propagate as stable waves in the direction.

In this way, we found that the wave propagation and growth of the unstable mode are significantly affected in the presence of the magnetic field on a large scale. Especially, in a low $\beta$-regime, the direction parallel to the field is favored for unstable wave propagation, that is, the instability anisotropically grows. The situation may be related with structure formation with a larger coherent length of magnetic field, but the issue is a non-linear process and is beyond the scope of this paper.

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