From quantum to elliptic algebras

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Abstract

It is shown that the elliptic algebra \( \mathcal{A}_{q,p}(\hat{sl}(2)_c) \) at the critical level \( c = -2 \) has a multidimensional center containing some trace-like operators \( t(z) \). A family of Poisson structures indexed by a non-negative integer and containing the \( q \)-deformed Virasoro algebra is constructed on this center. We show also that \( t(z) \) close an exchange algebra when \( p^m = q^{c+2} \) for \( m \in \mathbb{Z} \), they commute when in addition \( p = q^{2k} \) for \( k \) integer non-zero, and they belong to the center of \( \mathcal{A}_{q,p}(\hat{sl}(2)_c) \) when \( k \) is odd. The Poisson structures obtained for \( t(z) \) in these classical limits contain the \( q \)-deformed Virasoro algebra, characterizing the structures at \( p \neq q^{2k} \) as new \( \mathcal{W}_{q,p}(sl(2)) \) algebras.

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1 Introduction

The concept of $q$-deformed Virasoro and $W_N$ algebras has recently arisen in connection with some aspects of integrable systems. In particular these algebras have been introduced \cite{1,2} as an extension of the Virasoro and $W_N$ algebras identified in the quantum Calogero–Moser model \cite{3}. In the same way as Jack polynomials (eigenfunctions of the quantum Calogero–Moser model) arise as singular vectors of $W_N$ algebras \cite{3,4}, MacDonald polynomials (eigenfunctions of the trigonometric Ruijsenaars–Schneider model) are singular vectors of the $q$-deformed $W_N$ algebras.

These deformed algebras were shown to arise in fact from a procedure of construction mimicking the already known scheme \cite{5} for undeformed Virasoro and $W_N$ algebras: at the critical value $c = -N$, the quantum affine Lie algebra $U_q(\widehat{sl}(N)_c)$ has a multidimensional center \cite{6} where a $q$-deformed Poisson bracket can be defined as limit of the commutator structure \cite{7}. This Poisson bracket is the semi-classical limit of the $q$-deformed Virasoro (for $N = 2$) or $W_N$ algebras. Its quantization, performed in \cite{8}, can be considered as the most general definition of $q$-deformed $W_N$ algebras, often called $W_{q,p}(sl(N))$, $p$ being related to the quantization parameter. In fact, the construction of classical and quantized algebras was not achieved in \cite{7,8} by direct computation but using the $q$-deformed bozonization \cite{9} of $U_q(\widehat{sl}(N)_c)$. Interestingly the $q$-deformed $W_N$ algebras are characterized by elliptic structure coefficients: for instance, the $q$-deformed Virasoro algebra of \cite{1} is defined by the generating operator $T(z)$ such that

$$f_{1,2}(w/z) T(z) T(w) - f_{1,2}(z/w) T(w) T(z) = \frac{(1 - q)(1 - p/q)}{1 - p} \left( \delta\left(\frac{w}{zp}\right) - \delta\left(\frac{wp}{z}\right) \right),$$

where

$$f_{1,2}(x) = \frac{1}{1 - x} \frac{\langle x|, pq^{-1}; p^2 \rangle_{\infty}}{\langle x|, pq, p^2 q^{-1}; p^2 \rangle_{\infty}}, \quad \langle x|a_1 \ldots a_k; t \rangle_{\infty} \equiv \prod_{i=1}^{k} \prod_{n=0}^{\infty} (1 - a_i x t^n).$$

The parameters $p$ and $q$ are rewritten as $q = e^h$, $p = e^{h(1-\beta)}$: $h$ is the deformation parameter and $\beta$ is the “quantization” parameter, the semi-classical limit $\beta \to 0$ giving back the $q$-deformed Poisson bracket of \cite{7} and the limit $h \to 0$ giving back the linear Virasoro algebra.

Two natural problems arise in this context. The first is the extension of these Poisson brackets constructions to the elliptic quantum algebra $A_{q,p}(\widehat{sl}(2)_c)$ \cite{10,11,12}, which is a double deformation of $U(\widehat{sl}(2)_c)$, the limit $p \to 0$ giving the quantum affine algebra $U_q(\widehat{sl}(2)_c)$. In this context we will show the existence of a multidimensional center at $c = -2$, where we will construct a set of Poisson brackets containing the $q$-deformed Virasoro algebra introduced by \cite{3}.

A second question is the analysis of the connection between $A_{q,p}(\widehat{sl}(2)_c)$ and quantized $q$-deformed Virasoro algebra, both of which depend on elliptic structure functions. Our result is that if $q^{c+2} = p^m$ for any $m \in \mathbb{Z}\backslash\{0\}$, the algebra
\(A_{q,p}(\hat{sl}(2)_c)\) contains a quadratic subalgebra which builds a natural quantization of the Poisson bracket structure of the \(q\)-deformed Virasoro algebra; in such a way we will construct a family of \(W_{q,p}(\hat{sl}(2))\) algebras in the framework of the elliptic quantum algebra \(A_{q,p}(\hat{sl}(2)_c)\).

## 2 The elliptic quantum algebra \(A_{q,p}(\hat{sl}(2)_c)\)

Consider the \(R\)-matrix of the eight vertex model found by Baxter \[13\]:

\[
R_{12}(x) = \frac{1}{\mu(x)} \begin{pmatrix} a(u) & 0 & 0 & d(u) \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ d(u) & 0 & 0 & a(u) \end{pmatrix}
\]

where the functions \(a(u), b(u), c(u), d(u)\) are given by

\[
a(u) = \frac{\snh(\lambda - u)}{\snh(\lambda)}, \quad b(u) = \frac{\snh(u)}{\snh(\lambda)}, \quad c(u) = 1, \quad d(u) = k \snh(\lambda - u) \snh(u).
\]

The function \(\snh(u)\) is defined by \(\snh(u) = -i \sin(iu)\) where \(\sin(u)\) is Jacobi’s elliptic function with modulus \(k\). The functions (2.2) can be seen as depending on:

\[
p = \exp\left(-\frac{\pi K'}{K}\right), \quad q = -\exp\left(-\frac{\pi \lambda}{2K}\right), \quad x = \exp\left(\frac{\pi u}{2K}\right),
\]

where \(K, K'\) denote the elliptic integrals \[13\]. The factor \(\mu(x)\) is given by \[12\]:

\[
\frac{1}{\mu(x)} = \frac{1}{\kappa(x^2)} \begin{pmatrix} (p^2; p^2)_{\infty} & \Theta_{p^2}(px^2) & \Theta_{p^2}(q^2) \\ (p; p)_{\infty} & \Theta_{p^2}(p^2x^2) & \Theta_{p^2}(p^2x^2) \end{pmatrix},
\]

\[
\frac{1}{\kappa(x^2)} = (q^4 x^{-2}; p, q^4)_{\infty} (q^2 x^2; p, q^4)_{\infty} (p^2 x^2; p, q^4)_{\infty} (p^2 x^2; p, q^4)_{\infty},
\]

where

\[
(x; p_1, \ldots, p_m)_{\infty} = \prod_{n_i \geq 0} (1 - x p_1^{n_1} \ldots p_m^{n_m})
\]

\[
\Theta_{p^2}(x) = (x; p^2)_{\infty} (p^2 x^{-1}; p^2)_{\infty} (p^2; p^2)_{\infty}.
\]

To avoid singularities in the functions (2.2,2.4) we will suppose \(|q|, |p| < 1\).

**Proposition 1** The matrix \(R_{12}\) has the following properties:

- **unitarity:** \(R_{21}(x^{-1}) R_{12}(x) = 1\),
- **crossing symmetry:** \(R_{21}(x^{-1}) t_i = (\sigma^1 \otimes \mathbb{I}) R_{12}(-q^{-1} x) (\sigma^1 \otimes \mathbb{I})\),
- **antisymmetry:** \(R_{12}(-x) = - (\sigma^3 \otimes \mathbb{I}) R_{12}(x) (\sigma^3 \otimes \mathbb{I})\),

where \(\sigma^1, \sigma^2, \sigma^3\) are the Pauli matrices and \(t_i\) is the transposition in the space \(i\).
The proof is straightforward by direct calculation.

For the definition of the elliptic quantum algebra \(A_{q,p}(\hat{sl}(2)_c)\), we need to use a modified \(R\)-matrix \(R_{12}^+(x)\) defined by

\[
R_{12}^+(x) = \tau(q^{1/2}x^{-1})R_{12}(x), \quad \tau(x) = x^{-1} \frac{\Theta_{q^2}(qx^2)}{\Theta_{q^2}(qx^{-2})}.
\]  

(2.8)

The matrix \(R_{12}^+(x)\) obeys a quasi-periodicity property:

\[
R_{12}^+(-p^{2}x) = (\sigma^1 \otimes I) (R_{21}^+(x^{-1}))^{-1} (\sigma^1 \otimes I).
\]  

(2.9)

The elliptic quantum algebra \(A_{q,p}(\hat{sl}(2)_c)\) has been introduced by \([10]\). It is an algebra of operators \(L_{\pm}^{\pm}(z)\) such that \(L_{\pm}^{\pm}(z) = 0\) if \(\varepsilon \varepsilon' \neq (-1)^n\) and, defining \(L_{\pm}^{\pm}(z) = \sum_{n \in \mathbb{Z}} L_{\varepsilon \varepsilon', n} z^n\) and encapsulating them into \(2 \times 2\) matrices \(L^{\pm}(z)\), one has, with the definition \(R_{12}^{\pm}(x, q, p) = R_{12}^+(x, q, pq^{-2c})\):

\[
\begin{align*}
R_{12}^+(z/w) L_{-}^{+}(z) L_{-}^{+}(w) &= L_{-}^{+}(w) L_{-}^{+}(z) R_{12}^{++}(z/w), \\
R_{12}^+(q^{c/2}z/w) L_{+}^{+}(z) L_{-}^{+}(w) &= L_{-}^{+}(w) L_{+}^{+}(z) R_{12}^{++}(q^{-c/2}z/w), \\
q - \det L^{+}(z) &= L_{++}^{+}(q^{-1}z)L_{--}^{+}(z) - L_{-+}^{+}(q^{-1}z)L_{+-}^{+}(z) = q^{\frac{c}{2}}, \\
L_{\varepsilon \varepsilon'}^{\pm}(z) &= \varepsilon \varepsilon' L_{-\varepsilon, -\varepsilon'}^{\pm}(p^{2}q^{-\frac{c}{2}}z).
\end{align*}
\]  

(2.10)

We now state the first important result of our study.

3 The center of \(A_{q,p}(\hat{sl}(2)_c)\)

**Theorem 1** At \(c = -2\), the operators generated by

\[
t(z) = Tr(L(z)) = Tr\left(L^{+}(q^{c/2}z)L^{-}(z)^{-1}\right)
\]  

(3.1)

commute with the algebra \(A_{q,p}(\hat{sl}(2)_c)\) and then belong to its center.

The formula for \(t(z)\) in the elliptic case is exactly identical to the one in the trigonometric case \([3]\). The proof follows on similar lines using explicitly the crossing symmetry and the unitarity of the \(R\)-matrix.

We now study the specific behaviour of the exchange algebra of \(t(z)\) with \(t(w)\) in the neighborhood of \(c = -2\).

4 Poisson algebra of \(t(z)\)

By virtue of Theorem 1 the elements \(t(z)\) and \(t(w)\) are mutually commuting at the critical level \(c = -2\). This implies a natural Poisson structure on the algebra generated by them: if \([t(z), t(w)] = (c+2)\ell(z, w) + a(c+2)\), then a Poisson bracket
is yielded by \( \{t(z)_{cr}, t(w)_{cr}\} = \ell(z, w)_{cr} \) ("cr" means that all expressions are taken at \( c = -2 \)). We have the following result:

**Theorem 2** Under its natural Poisson bracket at \( c = -2 \) the algebra generated by \( t(z) \) is closed. One has indeed (we suppress the subscript "cr" and define \( x = z/w \)):

\[
\{t(z), t(w)\} = -(\ln q) \left( x^{-1} \frac{d}{dx} \ln \tau(q^{1/2}x^{-1}) - x \frac{d}{dx} \ln \tau(q^{1/2}x) \right) t(z)t(w).
\]

(4.1)

**Proof:** From the definition of the element \( t(z) \), one has

\[
t(z)t(w) = L(z)_{i_1}^{i_1} L(w)_{i_2}^{i_2} = L^+(q^w z)_{i_1}^{j_1}(L^- (z)^{-1})_{j_1}^{i_1} L^+(q^w z)_{i_2}^{j_2}(L^- (w)^{-1})_{j_2}^{i_2}.
\]

(4.2)

The exchange relations of (2.10) and the properties of the matrix \( R_{12}(x) \) given in proposition 3 allow us to move the matrices \( L^+(q^w z), L^- (w)^{-1} \) to the left of the matrices \( L^+(q^w z), L^- (z)^{-1} \). One obtains

\[
t(z)t(w) = \mathcal{Y}(z/w)_{j_1 j_2}^{i_1 i_2} L(w)_{i_2}^{j_2} L(z)_{i_1}^{j_1},
\]

(4.3)

where the matrix \( \mathcal{Y}(z/w) \) is factorized in the following way:

\[
\mathcal{Y}(z/w) = T(z/w) R(z/w).
\]

(4.4)

The matrix factor \( R(z/w) \) depends only on the matrix (2.1):

\[
R(z/w) = \left( \left( R_{12}(w/z) R_{12}(q^{-c^{-1}} z/w) R_{12}(w/z) \right)^{i_2} R_{12}(q^w z/w)^{i_2} \right)^{i_2},
\]

(4.5)

while the numerical prefactor \( T(z/w) \) contains only a \( \tau \) dependence:

\[
T(z/w) = \frac{\tau(q^{1/2}z/w)\tau(q^{-c+1/2}w/z)}{\tau(q^{-c-3/2}z/w)\tau(q^{1/2}w/z)}.
\]

(4.6)

One easily checks the nice behaviour of \( T(z/w) \) and \( R(z/w) \) at \( c = -2 \):

\[
T(z/w)_{cr} = 1, \quad R(z/w)_{cr} = \mathbb{I}_2 \otimes \mathbb{I}_2 \implies \mathcal{Y}(z/w)_{cr} = \mathbb{I}_2 \otimes \mathbb{I}_2.
\]

(4.7)

One then computes the Poisson structure from the exchange algebra (4.3) in the neighborhood of \( c = -2 \). From equations (4.3) and (4.7) one writes

\[
t(z)t(w) = t(w)t(z) + (c + 2) \left( \frac{d\mathcal{Y}}{dc}(z/w) \right)_{j_1 j_2}^{i_1 i_2} L(w)_{i_2}^{j_2} L(z)_{i_1}^{j_1} + o(c + 2)
\]

(4.8)

and therefore

\[
\{t(z), t(w)\} = \left( \frac{d\mathcal{Y}}{dc}(z/w) \right)_{j_1 j_2}^{i_1 i_2} L(w)_{i_2}^{j_2} L(z)_{i_1}^{j_1} \bigg|_{cr}.
\]

(4.9)
The equations (4.4) and (4.7) imply
\[ \frac{dY}{dc}(x) \bigg|_{cr} = \frac{dT}{dc}(x) \bigg|_{cr} \otimes I_2 + \frac{dR}{dc}(x) \bigg|_{cr}. \] (4.10)

After a long calculation, using various tricks in elliptic functions theory, one shows that:
\[ \frac{dR}{dc}(x) \bigg|_{cr} = 0, \] (4.11)
\[ \frac{dT}{dc}(x) \bigg|_{cr} = -\ln(q) \left( x^{-1} \frac{d}{dx} \ln(\tau(q^{1/2}x)) - x \frac{d}{dx} \ln(\tau(q^{1/2}x)) \right). \] (4.12)

From equations (4.9-4.12) formula (4.1) of Theorem 2 immediately follows. \( \square \)

The structure of the Poisson bracket (4.1) derives wholly from the \( \tau \) factor: so any dependence in \( p \) is absent in its structure function.

From the equation (4.1) and the definition of \( \tau(x) \), one gets easily
\[ \{ t(z), t(w) \} = -2 \ln q \left[ \sum_{n \geq 0} \left( \frac{2x^2q^{4n+2}}{1-x^2q^{4n+2}} - \frac{2x^{-2}q^{4n+2}}{1-x^{-2}q^{4n+2}} \right) + \sum_{n > 0} \left( -\frac{2x^2q^{4n}}{1-x^2q^{4n}} + \frac{2x^{-2}q^{4n}}{1-x^{-2}q^{4n}} \right) - \frac{x^2}{1-x^2} + \frac{x^{-2}}{1-x^{-2}} \right] t(z) t(w), \] (4.13)

where \( x = z/w \). Interpretation of the formula (4.13) must now be given in terms of the modes \( t_n \) of \( t(z) \), defined by:
\[ t_n = \oint_{C} \frac{dz}{2\pi iz} z^{-n} t(z). \] (4.14)

The structure function \( f(z/w) \) which defines the Poisson bracket (4.13) is periodic with period \( q^2 \) and has simple poles at \( z/w = \pm q^k \) for \( k \in \mathbb{Z} \). In particular it is singular at \( z/w = \pm 1 \). As a consequence, the expected definition of the Poisson bracket \( \{ t_n, t_m \} \) as a double contour integral of (4.13) must be made more precise. Deformation of, say, the \( w \)-contour while the \( z \)-contour is kept fixed may induce the crossing of singularities of \( f(z/w) \) which in turn modifies the computed value of the Poisson bracket. In particular the singularity at \( z/w = \pm 1 \) implies that one cannot identify a double contour integral with its permuted. As a consequence the quantity
\[ \oint_{C_1} \frac{dz}{2\pi iz} \oint_{C_2} \frac{dw}{2\pi i w} z^{-n} w^{-m} f(z/w) t(z) t(w) \] is not antisymmetric under the exchange \( n \leftrightarrow m \) and cannot be taken as a Poisson bracket. This leads us to define the Poisson bracket as:

**Definition 1**
\[ \{ t_n, t_m \} = \frac{1}{2} \left( \oint_{C_1} \frac{dz}{2\pi iz} \oint_{C_2} \frac{dw}{2\pi i w} + \oint_{C_2} \frac{dz}{2\pi iz} \oint_{C_1} \frac{dw}{2\pi i w} \right) z^{-n} w^{-m} f(z/w) t(z) t(w). \] (4.15)
Such a procedure guarantees the antisymmetry of the postulated Poisson structure due to the property \( f(z/w) = -f(w/z) \).

The presence of singularities at \( z/w = \pm q^k \) where \( k \neq 0 \) introduces a dependence of the Poisson bracket (4.15) on the domains of integration. If we choose the contours \( C_1 \) and \( C_2 \) to be circles of radii \( R_1 \) and \( R_2 \) respectively, the following proposition holds:

**Proposition 2** For any \( k \in \mathbb{Z}^+ \) such that \( R_1/R_2 \in [q^{+k}, q^{+(k+1)}] \), Definition 4 defines a consistent (that is “antisymmetric and obeying the Jacobi identity”) Poisson bracket whose specific form, depending on \( k \), is:

\[
\{t_n, t_m\}_k = (-1)^{k+1} 2 \ln q \oint_{C_1} \frac{dz}{2\pi iz} \oint_{C_2} \frac{dw}{2\pi i w} \cdot \sum_{s \in \mathbb{Z}} \frac{q^{(2k+1)s} - q^{-(2k+1)s}}{q^s + q^{-s}} \left( \frac{z}{w} \right)^{2s} \left\{ z^n w^{-m} t(w) t(z) \right\}. \tag{4.16}
\]

We observe that the form of the Poisson brackets (4.16) is similar to the form of the Poisson bracket obtained by (3). In particular our Poisson bracket at \( k = 1 \) is the one in (3) where the purely central term is multiplied by \( t(z) t(z) \). However one has to remember that in (3) a particular representation of \( U_q(\hat{sl}(2)_c) \) in terms of quasi-bosons is used. It is possible that an analogous bosonization of \( A_{q,p}(\hat{sl}(2)_c) \) leads to a degeneracy of such terms as \( t(z) t(z) \) giving the result of (3), unlucky at this time a bosonized version of \( A_{q,p}(\hat{sl}(2)_c) \) is available only at \( c = 1 \) (4), using bosonized vertex operators constructed in (5).

## 5 Quadratic subalgebras in \( A_{q,p}(\hat{sl}(2)_c) \)

We now turn to the task of identifying possible connections between \( A_{q,p}(\hat{sl}(2)_c) \) and \( W_{q,p}(\hat{sl}(2)) \). We first prove:

**Theorem 3** If \( p, q, c \) are connected by the relation \( p^m = q^{c+2} \), \( m \in \mathbb{Z} \), the operators \( t(z) \) realize an exchange algebra with all generators \( L^\pm(w) \) of \( A_{q,p}(\hat{sl}(2)_c) \):

\[
t(z)L^+(w) = F(m, q^{\frac{z}{w}}) L^+(w) t(z), \quad t(z)L^-(w) = F(m, -p^{\frac{z}{w}}) L^-(w) t(z), \tag{5.1}
\]

where

\[
F(m, x) = \prod_{s=1}^{2m} q^{-1} \frac{\Theta_{q^s}(x^2 q^2 p^{-s}) \Theta_{q^s}(x^{-2} q^2 p^s)}{\Theta_{q^s}(x^{-2} p^s) \Theta_{q^s}(x^2 p^{-s})} \quad \text{for } m > 0, \tag{5.2a}
\]

\[
F(m, x) = \prod_{s=0}^{2m-1} q \frac{\Theta_{q^s}(x^2 p^s) \Theta_{q^s}(x^{-2} p^{-s})}{\Theta_{q^s}(x^{-2} q^2 p^s) \Theta_{q^s}(x^2 q^2 p^{-s})} \quad \text{for } m < 0. \tag{5.2b}
\]
The proof is easy to perform using the definition of $A_{q,p}(\hat{sl}(2)_c)$ and the properties (especially the quasi-periodicity) of $R$.

Remark 1: For $m = 0$, the relation can be realized in two ways: either $e = -2$, which is the case studied in chapter 3 and leads directly to a central $t(z) \ (F(m, x) = 1)$; or $q = \exp\left(\frac{2\pi i}{e+2}\right)$, hence $|q| = 1$, which we have decided not to consider owing to the singularities in the elliptic functions defining $A_{q,p}(\hat{sl}(2)_c)$. Hence $m = 0$ will be disregarded from now on.

An immediate corollary is:

**Theorem 4** When $p^m = q^{e+2}$, $t(z)$ closes a quadratic subalgebra:

$$t(z)t(w) = \mathcal{Y}_{p,q,m}\left(\frac{w}{z}\right) t(w)t(z) \quad (5.3)$$

where

$$\mathcal{Y}_{p,q,m}(x) = \begin{cases} 
\prod_{s=1}^{2|m-1} x^2 \Theta_{q^4}(x^{-2}p^s) \Theta_{q^{4}}(x^2q^{2}p^s) & \text{for } m > 0, \\
\prod_{s=1}^{2|m} x^2 \Theta_{q^4}(x^{-2}p^s) \Theta_{q^{4}}(x^2q^{2}p^s) & \text{for } m < 0.
\end{cases} \quad (5.4)$$

The proof is obvious from (5.1) and the definition (2.7) of $\Theta_a(x)$.

Remark 2: When $m = 1$ the exchange function in (5.3) is exactly the square of the exchange function in the quantized $q$-deformed Virasoro algebra proposed in 

Remark 3: As an additional connection we notice that all functions $\mathcal{Y}_{p,q,m}(x)$ obey the Feigin-Frenkel identities [8] for the exchange function of 

$$\mathcal{Y}(xq^2) = \mathcal{Y}(x), \quad \mathcal{Y}(xq) = \mathcal{Y}(x^{-1}) \quad (5.5)$$

Our exchange algebras then appear as natural candidates for $\mathcal{W}_{q,p}(sl(2))$ algebras generalizing the one of 

First of all we state the following theorem:

**Theorem 5** For $p = q^{2k}$, $k \in \mathbb{Z}\{0\}$, one has

$$F(m, x) = 1 \quad \text{for } k \text{ odd}, \quad (5.6)$$

$$F(m, x) = q^{-2m} x^{-4m} \left[ \frac{\Theta_{q^4}(x^2q^2)}{\Theta_{q^4}(x^2)} \right]^{4m} \quad \text{for } k \text{ even}. \quad (5.7)$$
Hence when \( k \) is odd \( t(z) \) is in the center of the algebra \( \mathcal{A}_{q,p}(\hat{\mathfrak{sl}}(2)_c) \), while when \( k \) is even \( t(z) \) is not in a (hypothetical) center of \( \mathcal{A}_{q,p}(\hat{\mathfrak{sl}}(2)_c) \). However in both cases, one has \( t(z), t(w) = 0 \).

**Proof:** Theorem 5 is easily proved using the explicit expression for \( F(m, x) \) and the definition (2.7) of \( \Theta \)-functions. The case \( k = 0 \) is excluded since it would lead to \( p = 1 \) and singularities in the definition of \( \mathcal{A}_{q,p}(\hat{\mathfrak{sl}}(2)_c) \).

This now allows us to define Poisson structures even though \( t(z) \) is not in the center of \( \mathcal{A}_{q,p}(\hat{\mathfrak{sl}}(2)_c) \) for \( k \) even. They are obtained as limits of the exchange algebras (5.3). Since the initial non-abelian structure for \( t(z) \) is closed, the exchange algebras (5.3) are natural quantizations of the Poisson algebras which we obtain.

**Theorem 6** Setting \( q^{2k} = p^{1-\frac{4}{k}} \) for any integer \( k \neq 0 \), one defines in the limit \( \beta \to 0 \) the following Poisson structures (\( x = z/w \)):

\[
\{ t(z), t(w) \} \equiv \lim_{\beta \to 0} \frac{1}{\beta} \left[ t(z)t(w) - t(w)t(z) \right]
\]

\[
= 2km \ln q \left\{ \frac{x^2}{1 - x^2} + \frac{x^{-2}}{1 - x^{-2}} + \sum_{n=0}^{\infty} \left[ \frac{2x^2 q^{4n}}{1 - x^2 q^{4n}} - \frac{2x^2 q^{4n+2}}{1 - x^2 q^{4n+2}} \right] - \frac{2x^{-2} q^{4n}}{1 - x^{-2} q^{4n}} + \frac{2x^{-2} q^{4n+2}}{1 - x^{-2} q^{4n+2}} \right\} t(z)t(w) \text{ for } k \text{ odd}, \quad (5.8a)
\]

\[
= -2km(2m - 1) \ln q \left\{ \frac{x^2}{1 - x^2} + \frac{x^{-2}}{1 - x^{-2}} + \sum_{n=0}^{\infty} \left[ \frac{2x^2 q^{4n}}{1 - x^2 q^{4n}} - \frac{2x^2 q^{4n+2}}{1 - x^2 q^{4n+2}} \right] - \frac{2x^{-2} q^{4n}}{1 - x^{-2} q^{4n}} + \frac{2x^{-2} q^{4n+2}}{1 - x^{-2} q^{4n+2}} \right\} t(z)t(w) \text{ for } k \text{ even}. \quad (5.8b)
\]

**Proof:** We note that

\[
\{ t(z), t(w) \} = \frac{d\gamma_{p,q,m}}{d\beta} \left( \frac{w}{z} \right) \bigg|_{\beta=0} t(z)t(w) = \frac{d\ln \gamma_{p,q,m}}{d\beta} \left( \frac{w}{z} \right) \bigg|_{\beta=0} t(z)t(w), \quad (5.9)
\]

the two equalities coming from the fact that \( \gamma_{p,q,m} = 1 \) when \( q^{2k} = p \). The proof is then obvious from (5.3.4) and the definition of \( \Theta \)-functions as absolutely convergent products (for \( |q| < 1 \)), hence the series in (5.8) are convergent and define univocally a structure function for \( \{ t(z), t(w) \} \).

The formula (5.8) coincides exactly with the Poisson structure of \( t(z) \) at \( c = -2 \), provided one reabsors \( km \) and \( -km(2m - 1) \) into the definition of the classical limit as \( \beta \to km\beta \) for \( k \) odd and \( \beta \to km(2m - 1)\beta \) for \( k \) even. So Theorem 6 provides us with an immediate interpretation of the quadratic structures (5.3). Since we have seen that the Poisson structures derived from (4.13) contained in particular the \( q \)-deformed Virasoro algebra (up to the delicate point of the central extension which is not explicit in (4.13)), the quadratic algebras (5.3) are inequivalent (for
different values of $m$) quantizations of the classical $q$-deformed Virasoro algebra, globally defined on the $\mathbb{Z}$-labeled 2-dimensional subsets of parameters defined by $p^m = q^{c+2}$. They are thus generalized $\mathcal{W}_{q,p}(sl(2))$ algebras at $c = -2 + \frac{\ln p}{\ln q}$. In such a frame the closed algebraic relation (5.1) may acquire a crucial importance as a $q$-deformation of the Virasoro-current commutations relations. A better understanding of the undeformed limit $q \to 1$ would help us to clarify this interpretation if one could indeed identify the standard Virasoro–Kac-Moody structure in such a limit. The difficulty lies in the correct definition of this limit for the generators $L^\pm(z)$ and $t(z)$ which should be consistent with such an interpretation. As for the previously mentioned central-extension problem, a help could come from an explicit bosonization of the elliptic algebra, as was done for $U_q(\hat{sl}(N)_c)$ in [11].

6 Conclusion

We have studied some aspects of the elliptic quantum algebra $A_{q,p}(\hat{sl}(2)_c)$, in order to show its importance as a generalization of the quantum affine algebra $U_q(\hat{sl}(2)_c)$. We have seen that the introduction of $A_{q,p}(\hat{sl}(2)_c)$ permits to incorporate in the context of a deformed affine algebra the $q$-deformed Virasoro algebra introduced by [1], which is the symmetry of trigonometric Ruijsenaars–Schneider model. This is obtained on the particular surface of the space of parameters of $A_{q,p}(\hat{sl}(2)_c)$ given by the equation $p = q^{c+2}$. We expect that the other $\mathcal{W}_{q,p}(sl(2))$ algebras constructed on the surfaces $p^m = q^{c+2}$, $m \neq 1$, will provide us with the mathematical structure required to study other relativistic integrable models, characterizing elliptic quantum algebras as a general framework for the description of symmetries in (quantum) relativistic mechanics.

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