STABILITY OF AXISYMMETRIC CHIRAL SKYRMIONS

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ABSTRACT. We examine topological solitons in a minimal variational model for a chiral magnet, so-called chiral skyrmions. In the regime of large background fields, we prove linear stability of axisymmetric chiral skyrmions under arbitrary perturbations in the energy space, a long-standing open question in physics literature. Moreover, we show strict local minimality of axisymmetric chiral skyrmions and nearby existence of moving soliton solution for the Landau-Lifshitz-Gilbert equation driven by a small spin transfer torque.

1. INTRODUCTION AND MAIN RESULTS

We are concerned with topological solitons \( m : \mathbb{R}^2 \to S^2 \) occurring in two-dimensional chiral magnets governed by interaction energies of the form

\[
E(m) = \int_{\mathbb{R}^2} \frac{1}{2} |\nabla m|^2 + m \cdot (\nabla \times m) + \frac{h}{2} |m - \hat{e}_3|^2 \, dx.
\]

The hallmark of such systems is the helicity term \( m \cdot (\nabla \times m) \) arising from antisymmetric exchange also known as Dzyaloshinkii-Moriya interaction. Helicity suspends the Derrick-Pohozaev non-existence criterion and breaks independent \( O(2) \) invariance in the target and the domain. It enables the reduction of energy by twisting, an effect that is maximized if the horizontal magnetization field \( m = (m_1, m_2) \) is antiparallel to the level sets of \( m_3 \), see [18]. Isolated chiral skyrmions are stable critical points of \( E \) in a non-trivial homotopy class, characterized by the topological charge or degree

\[
Q(m) = \frac{1}{4\pi} \int_{\mathbb{R}^2} m \cdot \partial_1 m \times \partial_2 m \, dx.
\]

The lack of \( O(2) \) invariance leads to a specific energetic selection of degree \( Q = -1 \) in accordance with the twisting behaviour relative to the selected background state \( m(\infty) = \hat{e}_3 \). Restricting the energy to the class of axisymmetric configurations of the form

\[
m_0(r e^{i\psi}) = \left( e^{i(\psi + \frac{\pi}{2})} \sin \theta(r), \cos \theta(r) \right)
\]

with the usual identification \( \mathbb{R}^2 \cong \mathbb{C} \), the corresponding minimizing problem gives rise to a boundary value problem for a second order ordinary differential equation for the polar profile \( \theta : [0, \infty) \to \mathbb{R} \)

\[
\theta'' + 1 \frac{r}{r^2} \theta' - \frac{1}{r^2} \sin \theta \cos \theta + \frac{2}{r} \sin^2 \theta - h \sin \theta = 0
\]

with boundary conditions

\[
\theta(0) = \pi \quad \text{and} \quad \lim_{r \to \infty} \theta(r) = 0.
\]
The equation has been studied extensively in physics literature mainly based on numerical simulation, exploring the occurrence and radial stability of isolated chiral skyrmions, see [3, 4, 16]. Our main result provides a rigorous confirmation and shows the linear stability of $m_0$ with respect to arbitrary perturbations in the energy space if $h$ is sufficiently large.

**Theorem 1.** There exists a unique solution to the boundary value problem (1.4) - (1.5) provided $h \gg 1$. The field $m_0$ given by (1.3) is a linearly stable critical point of (1.1). More precisely, the Hessian $H = H(\phi, \psi)$ at $m_0$ satisfies for some $\lambda > 0$

$$H(\phi, \phi) \geq \lambda \|\phi - \phi_0\|^2_{H^1},$$

for all tangent fields $\phi \in H^1$ along $m_0$, where $\phi_0$ is the $L^2$ projection of $\phi$ onto $\text{span}\{\partial_1 m_0, \partial_2 m_0\}$, the kernel of the associated Jacobi operator, which is a Fredholm operator of index zero.

The existence result is obtained by a variational method using ingredients from [18]. Uniqueness follows from some fine analysis of the boundary value problem (1.4)-(1.5) of the polar profile $\theta$, which can be found in the appendix. The main result about the stability of $m_0$ is obtained by an adaption of the Fourier argument in [20, 11, 12]. Representing $\phi$ in terms of a moving frame along $T_{m_0} \mathbb{S}^2$, we use a Fourier expansion to eliminate the dependence on the angular coordinate and thereby the Hessian becomes a series depending only on the radial coordinate. Then we show that non-negativity depends only on the first two modes, and we prove their non-negativity by means of a decomposition argument. The spectral gap and Fredholm property are then proven by means of a variational argument. The latter one can be used to prove strict local minimality modulo translations:

**Theorem 2.** For $h \gg 1$ there exist $\mu > 0$ and $\varepsilon > 0$ such that

$$E(m) - E(m_0) \geq \mu \inf_{x \in \mathbb{R}^2} \|m(\cdot - x) - m_0\|^2_{H^1},$$

for all $m \in H^1(\mathbb{R}^2; \mathbb{S}^2)$ such that $\|m - m_0\|_{H^1} < \varepsilon$.

A major open question is whether $m_0$ minimizes the energy even globally within the homotopy class $Q = -1$. Energy minimizing chiral skyrmions have been constructed [18] by means of a concentration-compactness argument, which extend to the case where helicity is replaced by a general form of anisotropic Dzyaloshinskii-Moriya interaction [10], where axisymmetry is lost. Notably while Theorem 2 provides the strict local minimality for axisymmetric chiral skyrmions, the analog stability statement is not at hand for minimizing chiral skyrmions.

Starting from linear stability of static solitons and a perturbation we shall also consider the slow solitonic motion of chiral skyrmions driven by horizontal currents $v \in \mathbb{R}^2$ of small size. The Landau-Lifshitz-Gilbert equation with adiabatic and non-adiabatic spin transfer torques is given by

$$(1.6) \quad \partial_t m + (v \cdot \nabla) m = m \times \left[ \alpha \partial_t m + \beta (v \cdot \nabla) m - h_{\text{eff}}(m) \right],$$

where the effective field $h_{\text{eff}}(m)$ is minus the $L^2$ gradient of $E$, and $\alpha, \beta > 0$ are the Gilbert damping factor and the ratio of non-adiabaticity, respectively, see e.g. [19, 13]. We shall construct moving soliton solutions $m = \hat{m}(x - ct)$ with propagation velocities $c \in \mathbb{R}^2$. In the case $\alpha = \beta$, such solutions are obtained from stationary solutions with $c = v$, i.e., skyrmions moves parallel to the current without changing shape. For $\alpha \neq \beta$, however, skyrmions deform and are deflected according
to the skyrmion hall effect with a propagation speed $c$ implicitly determined by Thiele's equation

$$(1.7) \quad 4\pi (v - c)^{\perp} = D(\beta v - \alpha c)$$

where the dissipative tensor $D \in \mathbb{R}^{2 \times 2}$ is given by

$$D_{jk} = \int_{\mathbb{R}^{2}} \partial_{j} m \cdot \partial_{k} m \, dx,$$

see e.g. [14] and references therein.

**Theorem 3.** For $h \gg 1$ there exists $\varepsilon > 0$ such that (1.6) possesses for $|v| < \varepsilon$ a unique family of moving soliton solution $m = m(x - ct)$ in an $H^{2}$ neighborhood of $m_{0}$ with $c$ determined by (1.7).

The proof follows from an adaption of the continuation argument in [5], using the implicit function theorem on Hilbert manifolds with the spin velocity $v \in \mathbb{R}^{2}$ serving as a perturbation parameter. Ingredients are in particular the non-degeneracy and Fredholm property of the linearized equilibrium equation as a consequence of Theorem [1].

Moving soliton solutions have been observed numerically in a long time asymptotics in [13]. The dynamic stability and compactness of spin-transfer torque driven chiral skyrmions governed by a slightly modified energy functional has been discussed in [6] in the context of an almost conformal regime using regularity arguments. These results, however, are only shown on a finite time horizon depending on the size of $v$. Our result confirms the existence of global solutions in form of traveling waves with small velocities.

The remainder of the paper is organized as follows. In Section 3 we prove non-negativity of the Hessian and identify its kernel. Section 4 is devoted to the spectral gap and the Fredholm property, finishing the proof of Theorem 1. Sections 5 and 6 are devoted to the proof of Theorem 2 and Theorem 3, respectively. In the Appendix, we provide some fine properties of the polar profile $\theta$ of the axisymmetric solution, which are essential to our main results.

### 2. Preliminaries

2.1. **Function spaces.** For $k \in \mathbb{N}$ we shall consider fields

$$m \in H^{k}_{\bar{e}_{3}}(\mathbb{R}^{2}; \mathbb{S}^{2}) = \{m: \mathbb{R}^{2} \to \mathbb{S}^{2} : m - \bar{e}_{3} \in H^{k}(\mathbb{R}^{2}; \mathbb{R}^{3})\},$$

and tangent maps along $m$

$$\phi \in H^{k}(\mathbb{R}^{2}; T_{m}\mathbb{S}^{2}) = \{\phi \in H^{k}(\mathbb{R}^{2}; \mathbb{R}^{3}) : \phi \perp m \text{ a.e.}\}.$$

We also use the spaces $H^{0} = L^{2}$ and $H^{\infty} = \bigcap_{k \in \mathbb{N}} H^{k}$.

For $k \geq 2$ the corresponding spaces are uniformly Hölder continuous by Sobolev embedding. Moreover the spaces $H^{k}_{\bar{e}_{3}}(\mathbb{R}^{2}; \mathbb{S}^{2})$ are Hilbert manifolds. By virtue of the immersion theorem, the projection map

$$(2.1) \quad \pi(m + \phi) := \frac{m + \phi}{|m + \phi|} \in H^{k}_{\bar{e}_{3}}(\mathbb{R}^{2}; \mathbb{S}^{2}).$$

identifies a zero neighborhood in the Hilbert space $H^{k}(\mathbb{R}^{2}; T_{m}\mathbb{S}^{2})$, the tangent space of $H^{k}_{\bar{e}_{3}}(\mathbb{R}^{2}; \mathbb{S}^{2})$ at $m$, with an open neighborhood of $m$ in $H^{k}_{\bar{e}_{3}}(\mathbb{R}^{2}; \mathbb{S}^{2})$. 

We denote the pointwise orthogonal projection onto $T_m \mathbb{S}^2$ by
\[ P_m = 1 - m \otimes m. \]
The following can be proven along the lines of Lemma 3 in \([2]\):

**Lemma 1.** For $k \in \mathbb{N}$ and $m \in C^\infty(\mathbb{R}^2; \mathbb{S}^2)$, the orthogonal projection
\[ P_m : H^k(\mathbb{R}^2; \mathbb{R}^3) \to H^k(\mathbb{R}^2; T_m \mathbb{S}^2) \]
is bounded. Moreover, $C_c^\infty(\mathbb{R}^2; T_m \mathbb{S}^2)$ is dense in $H^k(\mathbb{R}^2; T_m \mathbb{S}^2)$.

Here $C_c^\infty(\Omega; T_m \mathbb{S}^2) = \{ \phi \in C_c^\infty(\Omega; \mathbb{R}^3) : \phi \perp m \}$ for open sets $\Omega \subset \mathbb{R}^2$.

**2.2. Continuous extension of the helicity term.** Integrability of the helicity term $m \cdot (\nabla \times m)$ in \([11]\) requires appropriate decay of $m$. Starting from the dense subclass $\{ m : \mathbb{R}^2 \to \mathbb{S}^2 : m - \hat{e}_3 \in C_c^\infty(\mathbb{R}^2; \mathbb{R}^3) \}$, the continuous extension of helicity to $H_{\text{ad}}^1(\mathbb{R}^2; \mathbb{S}^2)$ is obtained by adding a null Lagrangian and integration by parts, which amounts to
\[ E_H(m) = \int_{\mathbb{R}^2} (m - \hat{e}_3) \cdot (\nabla \times m) \, dx = 2 \int_{\mathbb{R}^2} (m_1 \partial_2 m_3 - m_2 \partial_1 m_3) \, dx, \]
see \([18]\). This way the energy $E$ given by \([11]\) becomes a bounded quadratic form in $m - \hat{e}_3 \in H^1(\mathbb{R}^2; \mathbb{R}^3)$.

**2.3. First variation.** Suppose $m_0 \in H_{\text{ad}}^1(\mathbb{R}^2; \mathbb{S}^2)$. For $\phi \in L^\infty \cap H^1(\mathbb{R}^2; T_{m_0} \mathbb{S}^2)$ and $|t| < \| \phi \|_{L^1}^{-1}$ the variation map
\[ m_t := \pi(m_0 + t \phi) \in H_{\text{ad}}^1(\mathbb{R}^2; \mathbb{S}^2) \]
is well-defined with $\dot{m}_0 = \phi$. We say that $m_0$ is a critical point of $E$ if the first variation
\[ \delta E(m)(\phi) = \lim_{t \to 0} \frac{E(m_t) - E(m_0)}{t} \]
vanishes for arbitrary $\phi \in L^\infty \cap H^1(\mathbb{R}^2; T_{m_0} \mathbb{S}^2)$. In physical terms the first variation is given by the effective field
\[ h_{\text{eff}}(m) = \Delta m - 2 \nabla \times m + h\hat{e}_3, \]
i.e., minus the $L^2$ gradient of the energy $\delta E(m)(\phi) = -\langle h_{\text{eff}}(m), \phi \rangle_{L^2}$ for all admissible $\phi$. The effective field gives rise to a generalized tension field
\[ \tau(m) = P_m h_{\text{eff}}(m). \]
Critical points $m_0 \in H_{\text{ad}}^1(\mathbb{R}^2; \mathbb{S}^2)$ are therefore characterized by
\[ \tau(m_0) = 0 \quad \text{or equivalently} \quad m_0 \times h_{\text{eff}}(m_0) = 0 \]
in the weak sense with the interpretation $m \times \Delta m = \nabla \cdot (m \times \nabla m)$.

**Proposition 1.** Critical points of finite energy belong to the space $H_{\text{ad}}^\infty(\mathbb{R}^2; \mathbb{S}^2)$.

**Sketch of the proof.** The claim follows from well-established strategies for the regularity of finite energy harmonic maps in two dimensions \([7]\), starting from the equation $\tau(m) = 0$, which may be written as
\[ -\Delta m + 2 \nabla \times m + h(m - \hat{e}_3) = \Lambda(m) m \]
with the Lagrange multiplier
\[ \Lambda(m) = |\nabla m|^2 + 2m \cdot (\nabla \times m) + h(1 - m_3). \]
As $|\nabla m|^2 m = B : \nabla m$ with $\text{div } B \in L^2(\mathbb{R}^2; \mathbb{R}^3)$, it follows that $\Delta m$ belongs to the local Hardy space (see e.g. [22]) modulo $L^2$ perturbations. In turn, it follows that $\Delta m \in L^1 + L^2(\mathbb{R}^2; \mathbb{R}^3)$, hence $m$ is uniformly continuous. A classical argument [15] yields $m$ is smooth. Finally a bootstrapping argument based on testing with $\Delta^k m$ and using interpolation inequalities yields an $H^k$-bound for every $k \in \mathbb{N}$. □

2.4. Second variation. Suppose $m_0$ is a critical point of finite energy and $\phi, \psi \in H^2(\mathbb{R}^2; T_{m_0}\mathbb{S}^2)$. Then $m_{st} = \pi(m_0 + s\phi + t\psi)$ is a smooth map into $H^2(\mathbb{R}^2; T_{m_0}\mathbb{S}^2)$ for $s$ and $t$ sufficiently small. The Hessian or second variation of $E$ at $m_0$ is given by

$$ (2.7) \quad \left. \frac{\partial^2}{\partial s \partial t} \right|_{s,t=0} E(m_{st}) = \delta^2 E(m)(\phi, \psi) - \delta E(m)(\langle \phi \cdot \psi \rangle m), $$

where

$$ \delta^2 E(m)(\phi, \psi) = \left. \frac{\partial^2}{\partial s \partial t} \right|_{s,t=0} E(m + s\phi + t\psi). $$

For a fixed critical point $m_0 \in H^2(\mathbb{R}^2; \mathbb{S}^2)$ we shall denote the Hessian by

$$ (2.8) \quad \mathcal{H}(\phi, \psi) = \mathcal{H}_\infty(\phi, \psi) - \Delta \mathcal{H}(\phi, \psi) $$

with

$$ \mathcal{H}_\infty(\phi, \psi) = \delta^2 E(m_0)(\phi, \psi) = \int_{\mathbb{R}^2} (\nabla \phi : \nabla \psi + 2\phi \cdot (\nabla \times \psi) + h \phi \cdot \psi) \, dx $$

and with [3,6]

$$ (2.9) \quad \Delta \mathcal{H}(\phi, \psi) = \delta E(m_0)(\langle \phi \cdot \psi \rangle m_0) = \int_{\mathbb{R}^2} \Lambda(m_0)(\phi \cdot \psi) \, dx. $$

We shall use the notation $\mathcal{H}(\phi) = \mathcal{H}(\phi, \phi)$ and similar for $\mathcal{H}_\infty$ and $\Delta \mathcal{H}$.

In view of Proposition 5 the Hessian has a bounded extension to $H^1(\mathbb{R}^2; \mathbb{R}^3)$.

2.5. Jacobi operator. Using Lemma 1 the Hessian defines the Jacobi operator

$$ \mathcal{J} : H^2(\mathbb{R}^2; T_{m_0}\mathbb{S}^2) \to L^2(\mathbb{R}^2; T_{m_0}\mathbb{S}^2) $$

by

$$ (2.10) \quad \langle \mathcal{J} \phi, \psi \rangle_{L^2} = \mathcal{H}(\phi, \psi) \quad \text{for all } \phi, \psi \in H^2(\mathbb{R}^2; T_{m_0}\mathbb{S}^2). $$

More explicitly, we have $\mathcal{J} = P_{m_0} \circ \mathcal{L}$, where

$$ (2.11) \quad \mathcal{L} \phi = -\Delta \phi + 2\nabla \times \phi + (h - \Lambda(m_0)) \phi, $$

extending $\mathcal{J}$ to $H^2(\mathbb{R}^2; \mathbb{R}^3)$ as a uniformly elliptic operator. The linearization of the tension field is expressed in terms of the Jacobi operator, i.e., for a differentiable one-parameter family $s \mapsto m_s \in H^2_E(\mathbb{R}^2; \mathbb{S}^2)$ with tangent field $\dot{m}_0 = \phi$

$$ (2.12) \quad \mathcal{L} \phi = -\frac{d}{ds} \bigg|_{s=0} \tau(m_s) \quad \text{hence } \mathcal{J} \phi = -\frac{d}{ds} \bigg|_{s=0} \left( P_{m_0} \tau(m_s) \right). $$

Jacobi fields $\phi$ are solutions of $\mathcal{J} \phi = 0$. Every smooth family $m_t$ with $|t| < \varepsilon$ of critical points $\tau(m_t) = 0$ gives rise to a Jacobi field $\phi = \dot{m}_0$. In particular, the translation invariance of the energy implies for every critical point $m_0 \in H^2_E(\mathbb{R}^2; \mathbb{S}^2)$:

$$ (2.13) \quad \mathcal{J} ((c \cdot \nabla) m_0) = 0 \quad \text{for every } c \in \mathbb{R}^2. $$

Theorem 1 states that for the axisymmetric skyrmion every Jacobi field is obtained in this manner.
2.6. Axisymmetric critical points. Letting $\hat{e}_r = (\hat{r}, 0)$ one obtains for axisymmetric $m_0$ of the form of (1.3)

$$m_0 \times h_{\text{eff}}(m_0) = -\left(\theta'' + \frac{1}{r}\theta' - \frac{1}{r^2}\sin \theta \cos \theta + \frac{2}{r} \sin^2 \theta - h \sin \theta\right) \hat{e}_r.$$ 

In particular, $m_0$ is a critical point iff the corresponding polar profile $\theta$ satisfies the ordinary differential equation (1.4), the Euler-Lagrange equation of for the radial energy, see [3, 4, 16].

$$E(\theta) = 2\pi \int_0^\infty \left(\frac{(\theta')^2}{2} + \frac{\sin^2 \theta}{2r^2} + \theta' + \frac{\sin \theta \cos \theta}{r} + h(1 - \cos \theta)\right) r \, dr.$$ 

The finite energy condition requires $\theta(r) \to 0$ (modulo $2\pi$) as $r \to \infty$. The topological constraint $Q(m_0) = -1$ imposes $\theta(0) = \pi$, i.e. (1.5).

Existence of axisymmetric critical points follows from a variational argument based on energy bounds obtained in [18]. Recall that to ensure absolute integrability, the helicity term $E_H = E_H(m)$ has been modified by a null-Lagrangian, i.e., has been replaced by

$$(m - \hat{e}_3) : (\nabla \times m) \text{ or } (1 - \cos \theta) \left(\theta' - \frac{\sin \theta}{r}\right),$$

respectively. For a minimizing sequence $\theta_k$ for $E$ satisfying (1.5), one may consider the corresponding sequence $m_k$ of axisymmetric fields. For $h > 1$ the compactness argument in [18] implies $H^1$ subconvergence to an axisymmetric field $m_0$ of the form (1.3). In fact, the requisite topological lower bounds, the coercivity estimate, and the upper bound in [18] based on an axisymmetric construction hold true. Moreover, it follows that $\theta_k \to \theta$ in $H^1_{\text{loc}}(0, \infty)$ and $E(\theta) = \lim_{k \to \infty} E(\theta_k)$, i.e., $\theta$ is a minimizer and satisfies (1.4). Finally,

$$\frac{d \cos \theta_k}{dr} \leq \frac{(\theta_k')^2}{2} + \frac{\sin^2 \theta_k}{2r^2} = \frac{|\nabla m_k|^2}{2}$$

is uniformly integrable near $r = 0$, hence $\cos \theta_k$ is equicontinuous near $r = 0$, thus $\theta(0) = \pi$. A similar argument implies $\theta(r) \to 0$ as $r \to \infty$, hence (1.5).

Our analysis relies on some properties of the polar profile $\theta$: its monotonicity and certain decay properties. The main results needed are summarized in the following proposition, whose proof is deferred to the appendix.

**Proposition 2.** For $h \gg 1$, solutions of (1.4), (1.5) are unique, strictly decreasing, and satisfy the following estimates for all $r \in (0, \infty)$

$$|\cos \theta - r \sin \theta| < \frac{3}{2},$$

$$r^2(\theta'(r))^2 \geq \sin^2 \theta(r),$$

and

$$h - \frac{3}{2r} \sin \theta \geq 0.$$
2.7. **Axisymmetric non-existence.** Finally we note that, relative to the given background state \( m(\infty) = \hat{e}_3 \), axisymmetric chiral skyrmions are only possible for \( Q = -1 \).

**Proposition 3.** For \( N \in \mathbb{Z} \setminus \{0, -1\} \) there are no axisymmetric critical points

\[
m(r e^{i\psi}) = \left( e^{i(\gamma - N\psi)} \sin \theta(r), \cos \theta(r) \right)
\]

for any phase \( 0 \leq \gamma < 2\pi \).

**Proof.** The usual Derrick-Pohozaev argument implies \( E_H(m) = -2E_Z(m) \) for any critical point \( m \), where \( E_H(m) \) is given by (2.2) and

\[
E_Z(m) = \int_{\mathbb{R}^2} \frac{h}{2} |m - \hat{e}_3|^2 \, dx.
\]

is the Zeeman energy. On the other hand, a straightforward calculation yields that for an axisymmetric field with \( N \neq 0 \)

\[
E_H(m) = -2 \int_0^\infty \sin^2 \theta(r) \theta'(r) \, dr \int_0^{2\pi} \sin((N + 1)\psi - \gamma) \, d\psi
\]

vanishes unless \( N = -1 \). \( \square \)

Note that for \( N = -1 \) the helicity is minimized for \( \gamma = \frac{\pi}{2} \).

3. **Linear stability**

In this section we prove non-negativity of the Hessian \( \mathcal{H} \) for large \( h \).

**Proposition 4.** For \( h \gg 1 \) the Hessian \( \mathcal{H} \) satisfies

(i) \( \mathcal{H}(\phi) \geq 0 \) for all \( \phi \in H^1(\mathbb{R}^2; T_{m_0}S^2) \)

(ii) \( \mathcal{H}(\phi) = 0 \) if and only if \( \phi \in \text{span}\{\partial_1 m_0, \partial_2 m_0\} \).

3.1. **Approximation.** Proving (i) we can assume \( \phi \in C^\infty_0(\mathbb{R}^2 \setminus \{0\}; T_{m_0}S^2) \). In fact, by Lemma 1 and the \( H^1 \) continuity of the Hessian \( \mathcal{H} \), it is sufficient to prove non-negativity for \( \phi \in H^\infty(\mathbb{R}^2; T_{m_0}S^2) \). Moreover, it follows from (A.3) that

\[
(3.1) \quad \text{span}\{\partial_1 m_0, \partial_2 m_0\}|_{x=0} = T_{m_0(0)}S^2.
\]

Since by (2.10) and (2.13)

\[
(3.2) \quad \mathcal{H}(\phi + (c \cdot \nabla)m_0, \psi) = \mathcal{H}(\phi, \psi) \quad \text{for all } c \in \mathbb{R}^2,
\]

in particular \( \mathcal{H}(\phi) = \mathcal{H}(\phi + (c \cdot \nabla)m_0) \), we can arrange \( \Phi(0) = 0 \), and the following truncation lemma applies.

**Lemma 2.** Suppose \( \phi \in H^\infty(\mathbb{R}^2; T_{m_0}S^2) \) satisfies \( \phi(0) = 0 \). Given \( 0 < \varepsilon < 1 \) there exists \( \phi_\varepsilon \in C^\infty(\mathbb{R}^2; T_{m_0}S^2) \) such that \( \phi_\varepsilon = \phi(x) \) for \( \varepsilon \leq |x| \leq 1/\varepsilon \) and \( \phi_\varepsilon \to \phi \) in \( H^1 \) as \( \varepsilon \to 0 \).
Proof. Let $\rho \in C^\infty(\mathbb{R}^2;[0, 1])$ with $\rho = 0$ near the origin and $\rho = 1$ outside $B_1$, and $\phi_x = \rho \phi$ where $\rho_x(x) = \rho(x/\varepsilon)$. Clearly $\phi_x \to \phi$ in $L^2$ and $\nabla \phi_x = \rho \nabla \phi + \nabla \rho_x \otimes \phi \to \nabla \phi$ pointwise outside the origins $\varepsilon \to 0$. Moreover

$$\|\nabla \phi_x\|^2 \leq 2\|\nabla \phi\|^2 + 2 \left( \frac{L\omega_x}{\varepsilon} \right)^2 \chi_{B_1},$$

where $L = \sup_{x \in B_1} |\nabla \rho(x)|$ and $\omega_x = \sup_{x \in B_1} |\phi(x)| \to 0$ as $\varepsilon \to 0$. Hence $|\nabla \phi_x|^2$ is tight and equiintegrable and therefore $\nabla \phi_x \to \nabla \phi$ in $L^2$ as $\varepsilon \to 0$ by Vitali’s convergence theorem. Truncation near infinity is standard. □

In the following sections we shall show that $H(\phi) > 0$ for all $\phi \in C^\infty_c(\mathbb{R}^2 \setminus \{0\}; T_{m_0} S^2) \setminus \{0\}$, proving in particular claim (i) of Proposition 4. Once we have shown that $H$ is positive semi-definite and satisfies Cauchy-Schwarz, it follows that

$$H(\phi) = 0 \quad \text{if and only if} \quad H(\phi, \psi) = 0 \quad \text{for all} \quad \psi \in H^1(\mathbb{R}^2; T_{m_0} S^2).$$

Thus if $H(\phi) = 0$ for some $\phi \in H^1(\mathbb{R}^2; T_{m_0} S^2)$, Lemma 8 below and a bootstrapping argument imply $\phi \in H^\infty(\mathbb{R}^2; T_{m_0} S^2)$. Taking into account (3.1) and (3.2), claim (ii) of Proposition 3 follows if $H(\phi) = 0$ and $\phi(0) = 0$ implies $\phi \equiv 0$ in $\mathbb{R}^2$. By virtue of the truncation lemma, this will also follow from the estimates below.

3.2. Moving frames. We project the tangent field $\phi$ onto an appropriate orthogonal frame in $T_{m_0} S^2$. For the smooth axisymmetric critical point of the form (1.3) we choose smooth tangent vector fields

$$X = (\cos \psi, \sin \psi, 0) \quad \text{and} \quad Y = (- \sin \psi \cos \theta(r), \cos \psi \cos \theta(r), - \sin \theta(r))$$
on $\mathbb{R}^2 \setminus \{0\}$, satisfying

$$m_0 = X \times Y, \quad |X| = |Y| = 1, \quad X \cdot Y = 0.$$

Hence $(X, Y)$ forms a smooth orthogonal frame for $T_{m_0} S^2$ on $\mathbb{R}^2 \setminus \{0\}$. Tangent fields $\phi \in C^\infty_c(\mathbb{R}^2 \setminus \{0\}; T_{m_0} S^2)$ may be written as

$$\phi = u_1 X + u_2 Y$$

with uniquely determined coefficient functions $u = (u_1, u_2) \in C^\infty_c(\mathbb{R}^2 \setminus \{0\}; \mathbb{R}^2)$. Using the relations

$$X \cdot \partial_r Y = \partial_r X \cdot Y = 0 \quad \text{and} \quad \partial_\psi X \cdot Y = -X \cdot \partial_\psi Y = \cos \theta,$$

by a straightforward calculation we obtain

$$|\nabla \phi|^2 = |\nabla u|^2 + 2 \left( \frac{\cos \theta}{r^2} \right) (u \times \partial_\psi u) + \frac{1}{r^2} u_1^2 + \left( \left( \theta' \right)^2 + \frac{\cos^2 \theta}{r^2} \right) u_2^2,$$

$$\phi \cdot (\nabla \times \phi) = -\frac{1}{r} \sin \theta (u \times \partial_\psi u) + \left( \theta' - \frac{1}{r} \sin \theta \cos \theta \right) u_2^2$$

and

$$\Lambda(m_0) = (\theta')^2 + \frac{1}{r^2} \sin^2 \theta + 2 \theta' + \frac{2}{r} \sin \theta \cos \theta + \frac{2}{r} \sin \theta \cos \theta + h(1 - \cos \theta).$$

Thus in these coordinates the Hessian reads

$$H(\phi) = \int_0^{2\pi} \int_0^\infty \left\{ |\nabla u|^2 + 2 \left( \frac{\cos \theta}{r^2} - \sin \theta \right) (u \times \partial_\psi u) + (u \cdot \partial_r u) + g(r, \theta) u_1^2 + h(\cos \theta) \right\} r \, dr \, d\psi$$

where

$$f(r, \theta) = \frac{1}{r^2} \cos^2 \theta - (\theta')^2 - \frac{2}{r} \sin \theta \cos \theta + h \cos \theta,$$
and
\[
g(r, \theta) = \frac{1}{r^2} (\cos^2 \theta - \sin^2 \theta) - \frac{4}{r} \sin \theta \cos \theta + h \cos \theta.
\]

3.3. **Fourier expansion.** We expand \( u_i \) in a Fourier series

\[
u_i(r, \psi) = \alpha_i^{(0)}(r) + \sum_{k=1}^{\infty} (\alpha_i^{(k)}(r) \cos(k\psi) + \beta_i^{(k)}(r) \sin(k\psi)),
\]

where \( \alpha_i^{(k)}, \beta_i^{(k)} \in C^\infty_c(0, \infty) \). Due to the \( L^2 \)-orthogonality

\[
\begin{align*}
\int_0^{2\pi} u_i^2 \, d\psi &= \pi \left( 2(\alpha_i^{(0)})^2 + \sum_{k=1}^{\infty} \left( (\alpha_i^{(k)})^2 + (\beta_i^{(k)})^2 \right) \right), \\
\int_0^{2\pi} |\partial_r u_i|^2 \, d\psi &= \pi \left( 2(\alpha_i^{(k)})^2 + \sum_{k=1}^{\infty} \left( (\alpha_i^{(k)})^2 + (\beta_i^{(k)})^2 \right) \right), \\
\int_0^{2\pi} |\partial_\psi u_i|^2 \, d\psi &= \pi \sum_{k=1}^{\infty} k^2 \left( (\alpha_i^{(k)})^2 + (\beta_i^{(k)})^2 \right), \\
\text{and } \int_0^{2\pi} (u \times \partial_\psi u) \, d\psi &= \pi \sum_{k=1}^{\infty} 2k \left( \alpha_i^{(k)} \beta_i^{(k)} - \alpha_2^{(k)} \beta_1^{(k)} \right).
\end{align*}
\]

Moreover we obtain

\[
(3.3) \quad \mathcal{H}(\phi) = 2\pi \mathcal{H_0}(\alpha_1^{(0)}, \alpha_2^{(0)}) + \pi \sum_{k=1}^{\infty} \left( \mathcal{H}_k(\alpha_1^{(k)}, \beta_2^{(k)}) + \mathcal{H}_k(\beta_1^{(k)}, -\alpha_2^{(k)}) \right)
\]

where

\[
(3.4) \quad \mathcal{H}_k(\alpha, \beta) = \int_0^{2\pi} \left\{ |\alpha'|^2 + |\beta'|^2 + \left( \frac{k^2}{r^2} + f(r, \theta) \right) \alpha^2 + \left( \frac{k^2}{r^2} + g(r, \theta) \right) \beta^2 \\
+ 4k \left( \cos \theta - \frac{\sin \theta}{r} \right) \alpha \beta \right\} r \, dr.
\]

The following Lemma provides the reduction to the first two modes.

**Lemma 3.** For \( k \geq 1 \) we have \( \mathcal{H}_{k+1}(\alpha, \beta) \geq \mathcal{H}_k(\alpha, \beta) \) for all \( \alpha, \beta \in C^\infty_c(0, \infty) \). More precisely, the inequality holds pointwisely for the corresponding integrands.

**Proof.** For any \( \alpha, \beta \in C^\infty_c(0, \infty) \) we have

\[
\mathcal{H}_{k+1}(\alpha, \beta) - \mathcal{H}_k(\alpha, \beta) = \int_0^{2\pi} \left\{ \frac{2k+1}{r^2} (\alpha^2 + \beta^2) + \frac{4}{r^2} (\cos \theta - r \sin \theta) \alpha \beta \right\} r \, dr
\]

For \( k \geq 1 \), the pointwise estimate (2.10) yields the following pointwise estimate

\[
\frac{1}{r^2} ((2k+1)(\alpha^2 + \beta^2) + 4(\cos \theta - r \sin \theta) \alpha \beta) \\
\geq \frac{1}{r^2} ((2k+1)(\alpha^2 + \beta^2) - 6|\alpha \beta|) \geq \frac{3}{r^2} (|\alpha| - |\beta|)^2 \geq 0
\]

proving the claim. \( \square \)
3.4. Non-negativity of the reduced functional. It remains to consider the first two modes for \( k = 0, 1 \). To simplify this task we need the following decomposition, which is proven and called as “Hardy-type decomposition” in [11]. We modify the conditions to suit our needs.

**Lemma 4.** Let \( A : (0, \infty) \to \mathbb{R} \) be a nonnegative \( C^1 \) function and \( V \in L^1_{\text{loc}}((0, \infty), \mathbb{R}) \). Define the operator

\[
L := -\frac{d}{dr}(A(r)\frac{d}{dr}) + V(r)
\]

and consider a smooth function \( \psi : (0, \infty) \to \mathbb{R} \) satisfying \( \psi > 0 \) in \( (0, \infty) \). For every \( f \in C_c^\infty(0, \infty) \), writing

\[
g := \frac{f}{\psi} \in C_c^\infty(0, \infty),
\]

the following decomposition holds true:

\[
\int_0^\infty Lf \cdot f \, dr = \int_0^\infty \psi^2 A(r)(g')^2 \, dr + \int_0^\infty g^2 L\psi \cdot \psi \, dr.
\]

It is customary to represent admissible \( \alpha, \beta \) as

\[
\alpha = \frac{1}{r} \sin \theta \xi \quad \text{and} \quad \beta = \theta' \eta,
\]

for some uniquely determined \( \xi, \eta \in C_c^\infty(0, \infty) \).

With \( \alpha = \frac{1}{r} \sin \theta \xi \) and \( \beta = \theta' \eta \), where \( \xi, \eta \in C_c^\infty(0, \infty) \), it follows from Lemma 4 with \( A = r, V = \frac{k^2}{r} + f(r, \theta)r, \psi = \frac{1}{r} \sin \theta \) that

\[
\int_0^\infty \left\{ r|\alpha'|^2 + \left( \frac{k^2}{r} + rf(r, \theta) \right) \alpha^2 \right\} \, dr
\]

\[
= \int_0^\infty \left\{ \frac{\sin \theta^2}{r}(\xi')^2 + \frac{k^2 - 1}{r} \left( \frac{\sin \theta}{r} \right)^2 \xi^2 + 2 \frac{\sin \theta}{r} \theta' \left( \frac{\cos \theta}{r} - \sin \theta \right) \xi^2 \right\} \, dr
\]

and respectively with \( A = r, V = \frac{k^2}{r} + g(r, \theta)r \) and \( \psi = \theta' \) that

\[
\int_0^\infty \left\{ r|\beta'|^2 + \left( \frac{k^2}{r} + rg(r, \theta) \right) \beta^2 \right\} \, dr
\]

\[
= \int_0^\infty \left\{ r(\theta')^2(\eta')^2 + \frac{k^2 - 1}{r}(\theta')^2 \eta^2 + 2 \frac{\sin \theta}{r} \theta' \left( \frac{\cos \theta}{r} - \sin \theta \right) \eta^2 \right\} \, dr.
\]

Altogether we have

\[
\mathcal{H}_k(\alpha, \beta) = \tilde{\mathcal{H}}_k(\xi, \eta)
\]

\[
= \int_0^\infty \left\{ \frac{\sin^2 \theta}{r}(\xi')^2 + r(\theta')^2(\eta')^2 + \frac{k^2 - 1}{r} \left( \frac{\sin \theta}{r} \right)^2 \xi^2 + \frac{k^2 - 1}{r}(\theta')^2 \eta^2
\]

\[
+ 2 \frac{\sin \theta}{r} \theta' \left( \frac{\cos \theta}{r} - \sin \theta \right) \left( \xi^2 - 2k\xi \eta + \eta^2 \right) \right\} \, dr.
\]

**Lemma 5.** \( \tilde{\mathcal{H}}_0(\xi, \eta) > 0 \) for all \( \xi, \eta \in C_c^\infty(0, \infty) \setminus \{0\} \).
Proof. For 

\[ \mathcal{H}_0(\xi, \eta) = \int_0^\infty \left\{ \frac{\sin^2 \theta}{r}(\xi')^2 + r(\theta')^2(\eta')^2 - \frac{1}{r} \left( \frac{\sin \theta}{r} \right)^2 - \frac{1}{r}(\theta')^2 \eta^2 + 2 \frac{\sin \theta}{r} \theta' \left( \frac{\cos \theta}{r} - \sin \theta \right) (\xi^2 + \eta^2) \right\} \, dr \]

we apply the decomposition (3.5) for \( A = \frac{\sin^2 \theta}{r} \), \( V = 0 \) and \( \psi = r \) and get

\[ \int_0^\infty \frac{\sin^2 \theta}{r}(\xi')^2 \, dr = \int_0^\infty - \left( \frac{\sin^2 \theta}{r} \xi' \right) \xi \, dr \]

\[ = \int_0^\infty r^2 \frac{\sin^2 \theta}{r} \left( \frac{\xi}{r} \right)'^2 + \left( \frac{\xi}{r} \right)^2 \left( \frac{\sin^2 \theta}{r} - 2 \sin \theta \cos \theta \theta' \right) \, dr , \]

hence

\[ (3.7) \quad \int_0^\infty \left\{ \frac{\sin^2 \theta}{r} \left( (\xi')^2 - \frac{1}{r^2}( \xi^2) \right) + \frac{2}{r^2} \sin \theta \cos \theta \theta' \xi^2 \right\} \, dr \geq 0 . \]

Analogously it follows from (3.5) with \( A = r(\theta')^2 \), \( V = 0 \) and \( \psi = r \) and (1.4) for \( \theta \) that

\[ \int_0^\infty r(\theta')^2(\eta')^2 \, dr = \int_0^\infty - \left( r(\theta')^2 \eta' \right) \eta \, dr \]

\[ = \int_0^\infty r^2 r(\theta')^2 \left( \frac{\eta'}{r} \right)^2 \, dr + \int_0^\infty \left( \frac{\eta}{r} \right)^2 (-r(\theta')^2 - 2r^2 \theta \theta'') \, dr \]

\[ = \int_0^\infty r^3 (\theta')^2 \left( \frac{\eta'}{r} \right)^2 + \left( \frac{\eta}{r} \right)^2 (r(\theta')^2 - 2 \sin \theta \cos \theta \theta' - 2h r^2 \sin \theta \theta'') \, dr \]

and thus by the estimate (2.18) for \( \theta \)

\[ (3.8) \quad \int_0^\infty \left\{ r(\theta')^2(\eta')^2 - \frac{1}{r} (\theta')^2 \eta^2 + \frac{2}{r^2} \sin \theta \cos \theta \theta' \eta^2 + \frac{\sin^2 \theta}{r} (\theta')^2 \eta^2 \right\} \, dr \]

\[ = \int_0^\infty r^3 (\theta')^2 \left( \frac{\eta'}{r} \right)^2 + 2(-\theta') \sin \theta \left( h - \frac{3}{2r} \sin \theta \right) \eta^2 \, dr \geq 0 . \]

Adding (3.7) and (3.8) we get

\[ \tilde{\mathcal{H}}_0(\xi, \eta) \geq \int_0^\infty 2 \frac{\sin^2 \theta}{r} (-\theta')(\xi^2 + \frac{1}{2} \eta^2) \, dr \geq 0 , \]

and the claim follows. \( \Box \)

Lemma 6. \( \tilde{\mathcal{H}}_1(\xi, \eta) > 0 \) for all \( \xi, \eta \in C_c^\infty(0, \infty) \setminus \{0\} \).
Proof. By partial integration we have
\[
\mathcal{H}_1(\xi, \eta) = \int_0^\infty \frac{\sin^2 \theta}{r} (\xi')^2 + r(\theta')^2(\eta')^2 + 2 \frac{\sin \theta}{r} \theta' \left( \cos \frac{\theta}{r} - \frac{\sin \theta}{r} \right) (\xi - \eta)^2 \, dr
\]
\[
= \int_0^\infty \frac{\sin^2 \theta}{r} (\xi')^2 + r(\theta')^2(\eta')^2 + \frac{2 \sin^2 \theta}{r^3} (\xi - \eta)^2 + \frac{2 \sin^2 \theta}{r} (-\theta')(\xi - \eta)^2
\]
\[
- \frac{2 \sin^2 \theta}{r^2} (\xi - \eta)\xi' + \frac{2 \sin^2 \theta}{r^2} (\xi - \eta)\eta' \, dr
\]
\[
= \int_0^\infty \left( \frac{\sin \theta}{r^{1/2}} \xi' - \frac{\sin \theta}{r^{1/2}} (\xi - \eta) \right)^2 + \left( r(\theta')^2 - \frac{\sin^2 \theta}{r(1 + 2r^2(-\theta'))} \right) (\eta')^2
\]
\[
+ \left( \frac{\sin \theta}{r^{1/2}} (1 + 2r^2(-\theta''))^{-1/2} \eta' + \frac{\sin \theta}{r^{1/2}} (1 + 2r^2(-\theta'))^{1/2} (\xi - \eta) \right)^2 \, dr.
\]

For the second term it follows from (2.17) and the monotonicity of \( \theta \)
\[
r(\theta')^2 - \frac{\sin^2 \theta}{r(1 + 2r^2(-\theta'))} (r^2(\theta')^2 - \sin^2 \theta - 2r^3(\theta')^3) > 0
\]
for every \( r \in (0, \infty) \). Hence \( \mathcal{H}_1(\xi, \eta) \geq 0 \) with equality only if \( \xi = \eta = 0 \). \( \square \)

To conclude claim (ii) of Proposition 3 we consider for \( \phi \in H^\infty(\mathbb{R}^2; T_m \mathbb{S}^2) \) with \( \phi(0) = 0 \) and \( \mathcal{H}(\phi) = 0 \) a family of approximating truncations \( \phi_k \) such that \( \mathcal{H}(\phi_k) \to 0 \) as \( \varepsilon \to 0 \), according to Lemma 2. Regarding the zeroth mode, the estimate in the proof of Lemma 3 implies that the corresponding functions \( \xi_k \) and \( \eta_k \) converge to zero in measure. Regarding the first and higher modes, the estimate in the proof of Lemma 6 implies that the corresponding functions \( \xi_k \) and \( \eta_k \) converge in measure to a constant. But according to (3.6) and
\[
\lim_{r \to 0} \frac{\sin \theta(r)}{r} = - \lim_{r \to 0} \theta'(r) = \frac{h}{2},
\]
according to Lemma 14 this constant must be zero.

4. SPECTRAL GAP AND FREDHOLM PROPERTY

In this section we complete the proof of Theorem 1.

Lemma 7.

(i) For \( h > 1 \) the form \( \mathcal{H}_\infty \) is subcritical in the sense that for some positive \( \gamma \) and \( \nu \)
\[
\mathcal{H}_\infty(\phi) - \gamma \| \phi \|^2_{L^2} \geq \nu \| \phi \|^2_{H^1}.
\]

(ii) The form \( \Delta \mathcal{H} \) is compact in the sense that for \( \phi_k \to \phi \) weakly in \( H^1(\mathbb{R}^2; \mathbb{R}^3) \)
\[
\lim_{k \to \infty} \Delta \mathcal{H}(\phi_k) = \Delta \mathcal{H}(\phi).
\]

(iii) The functional \( \mathcal{H}(\phi) - \gamma \| \phi \|^2_{L^2} \) is weakly lower semicontinuous in \( H^1(\mathbb{R}^2; \mathbb{R}^3) \).

Proof. Applying Young’s inequality to \( \phi \cdot (\nabla \times \phi) \) and using the orthogonality relation \( \| \nabla \times \phi \|_{L^2} = \| \nabla \phi \|_{L^2} \) we obtain
\[
(4.1) \quad \mathcal{H}_\infty(\phi) \geq \frac{h - 1}{h} \max\{ \| \nabla \phi \|^2_{L^2}, h \| \phi \|^2_{L^2} \} \geq \frac{h - 1}{h + 1} \| \phi \|^2_{H^1}.
\]

Claim (i) follows with \( \gamma = (h + 1)/2 \) and \( \nu = (h' - 1)/(h' + 1) \) where \( h' = (h - 1)/2 \).
By virtue of the Rellich–Kondrachov theorem $\int_{B_R} \Lambda(m_0) |\phi|^2 \, dx$ is compact for every finite $R$. Since $\Lambda(m_0) \in L^2(\mathbb{R}^2)$ by Proposition \text[13] and $\|\psi\|_{L^4} \lesssim \|\phi\|_{H^1}$, we have
\[
\left| \mathcal{H}_\infty(\psi, \psi) - \int_{B_R} \Lambda(m_0) |\phi|^2 \, dx \right| \lesssim \|\psi\|^2_{H^1} \|\Lambda(m_0)\|_{L^2(B_N)} \to 0
\]
as $R \to \infty$. Since by assumption $\sup_{t \in \mathbb{N}} \|\phi_k\|_{H^1} < \infty$, claim (ii) and (iii) follow. \hfill \Box

We also need the following regularity result for the weak equation $\mathcal{J} \phi = f$.

**Lemma 8.** Suppose $f \in L^2(\mathbb{R}^2; T_{m^0} S^2)$. If $\phi \in H^1(\mathbb{R}^2; T_{m^0} S^2)$ satisfies
\[
\mathcal{H}(\phi, \psi) = \langle f, \psi \rangle_{L^2} \quad \text{for all} \quad \psi \in H^1(\mathbb{R}^2; T_{m^0} S^2),
\]
then $\phi \in H^2(\mathbb{R}^2; \mathbb{R}^3)$.

**Proof.** Clearly $\mathcal{H}(\phi, P_{m_0} \eta) = \langle f, \eta \rangle_{L^2}$ for all $\eta \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^3)$ and
\[
\mathcal{H}(\phi, P_{m_0} \eta) = \langle \nabla \phi, \nabla (P_{m_0} \eta) \rangle_{L^2} - \langle g_0, \eta \rangle_{L^2}
\]
for some $g_0 \in L^2(\mathbb{R}^2; \mathbb{R}^3)$ bounded in terms of $m$ and $\phi$. Moreover, for smooth $\phi$ and $m$ we have $\langle \nabla \phi, \nabla (P_{m_0} \eta) \rangle_{L^2} = -\langle P_m \Delta \phi, \eta \rangle_{L^2}$ and
\[
P_m \Delta \phi = \Delta \phi - \Delta(m \cdot m) m + \nabla \cdot (\nabla m \cdot \phi) m + \langle \nabla m \cdot \nabla \phi \rangle m
\]
hence by approximation and using $\phi \cdot m_0 = 0$
\[
\langle \nabla \phi, \nabla (P_{m_0} \eta) \rangle_{L^2} = \langle \nabla \phi, \nabla \eta \rangle_{L^2} + \langle g_1, \eta \rangle_{L^2},
\]
where $g_1 \in L^2(\mathbb{R}^2; \mathbb{R}^3)$ by Proposition \text[13] Hence $-\Delta \phi = g_0 + g_1$, and the claim follows from standard $L^2$ theory for the Poisson equation. \hfill \Box

Recall that $\ker J = \{ c \cdot \nabla m_0 : c \in \mathbb{R}^2 \}$. In this context we shall use the notation
\[
\langle \phi, \nabla m_0 \rangle_{L^2} = \sum_{j=1,2} \langle \phi, \partial_j m_0 \rangle_{L^2} \hat{e}_j \in \mathbb{R}^2.
\]

**Proposition 5.** For $h \gg 1$ there exists a $\lambda > 0$ increasing in $h$ such that
\[
\mathcal{H}(\phi) \geq \lambda \|\phi\|^2_{H^1} \quad \text{for all} \quad \phi \in H^1(\mathbb{R}^2; T_{m_0} S^2) \text{ with } \langle \phi, \nabla m_0 \rangle_{L^2} = 0.
\]

$J : H^2(\mathbb{R}^2; T_{m_0} S^2) \to L^2(\mathbb{R}^2; T_{m_0} S^2)$ is a Fredholm operator of index 0 with
\[
\text{ran } J = \{ f \in L^2(\mathbb{R}^2; T_{m_0} S^2) : \langle f, \nabla m_0 \rangle_{L^2} = 0 \}.
\]

**Proof.** For $k = 0, 1, 2$ we define the Hilbert spaces
\[
\mathbb{H}^k = \{ \phi \in H^k(\mathbb{R}^2; T_{m_0} S^2) : \langle \phi, \nabla m_0 \rangle_{L^2} = 0 \}.
\]
We let $I_0 := \inf \{ \mathcal{H}(\phi) : \phi \in \mathbb{H}^1, \|\phi\|_{L^2} = 1 \} \geq 0$ and suppose $I_0 = 0$. Consider
\[
I_\gamma := \inf \{ \mathcal{H}_\gamma(\phi) : \phi \in \mathbb{H}^1, \|\phi\|_{L^2} = 1 \}
\]
where
\[
\mathcal{H}_\gamma(\phi) = \mathcal{H}(\phi) - \gamma \|\phi\|^2_{L^2},
\]
which is an $H^1$ norm and therefore weakly lower semicontinuous for some positive $\gamma$ by virtue of Lemma \text[13]. If $I_\gamma = 0$, then $\mathcal{H}(\phi) \geq \gamma \|\phi\|^2_{L^2}$ which implies $I_0 > 0$. Hence $I_\gamma < 0$, and we claim the infimum is attained. From Lemma \text[13] (i) we also obtain
\[
\mathcal{H}(\phi) \geq \mathcal{H}_\gamma(\phi) \geq \nu \|\phi\|^2_{H^1} - \mu \|\phi\|^2_{L^2}
\]
for $\nu, \mu > 0$, and hence
\[
\mathcal{H}_\gamma(\phi) \geq \nu \|\phi\|^2_{H^1} - \mu \|\phi\|^2_{L^2}.
\]
with \( \nu > 0 \) and \( \mu = \| \Lambda(\mathbf{m}_0) \|_{L_\infty}^2 \). Since \( \mathcal{H}_\gamma(\phi) \geq \nu \| \phi \|_{H_1}^2 - \mu \) for \( \| \phi \|_{L^2} = 1 \), there exists for \( I_\gamma \) a minimizing sequence

\[
\phi_k \rightharpoonup \phi_* \quad \text{weakly in } H^1(\mathbb{R}^2; T\mathbf{m}_0; S^2).
\]

Clearly \( \phi_* \in \mathbb{H}^1 \) and \( \| \phi_* \|_{L^2}^2 \leq \liminf_{k \to \infty} \| \phi_k \|_{L^2}^2 = 1 \). Since \( I_\gamma < 0 \) and \( \mathcal{H}_\gamma(0) = 0 \), it follows from weak lower semicontinuity that \( \phi_* \neq 0 \). Therefore

\[
\| \phi_* \|_{L^2}^2 I_\gamma \leq \mathcal{H}_\gamma(\phi_*) \leq \liminf_{k \to \infty} \mathcal{H}_\gamma(\phi_k) = I_\gamma
\]

which implies that \( \| \phi_* \|_{L^2}^2 = 1 \) and \( I_\gamma = H_\gamma(\phi_*) \). Hence \( \mathcal{H}(\phi) \geq \mathcal{H}(\phi_*) \) for all \( \| \phi \|_{L^2} = 1 \), and therefore

\[
\mathcal{H}(\phi) \geq \mathcal{H}(\phi_*) \| \phi \|_{L^2}^2 \quad \text{for all } \phi \in \mathbb{H}^1 \setminus \{0\},
\]

a contradiction, since \( \mathcal{H}(\phi_*) > 0 \). Therefore \( I_0 > 0 \), and together with the lower bound (1.2), the claim follows with \( \lambda = \nu / (1 + \mu / I_0) \).

It now follows from the Riesz representation theorem that for \( f \in \mathbb{H}^0 \) there exists a unique \( \phi \in \mathbb{H}^1 \) with \( \mathcal{H}(\phi, \psi) = \langle f, \psi \rangle_{L^2} \) for all \( \psi \in \mathbb{H}^1 \), while Lemma 8 implies that \( \phi \in \mathbb{H}^2 \) and hence \( \mathcal{J} \phi = f \). Since \( \mathcal{J} : \mathbb{H}^2 \to \mathbb{H}^0 \) is bounded, it is an isomorphism, and the claim follows.

\[ \square \]

### 5. Local Minimality

In this section we prove Theorem 2. The key observation is that the energy difference can be expressed in terms of the extended Hessian. This is in the spirit of [17, 8], but with stronger conclusions.

**Lemma 9.** For \( \mathbf{m} \in H^1_\xi(\mathbb{R}^2; S^2) \) we have \( E(\mathbf{m}) - E(\mathbf{m}_0) = \frac{1}{2} \mathcal{H}(\mathbf{m} - \mathbf{m}_0) \).

**Proof.** Define \( \xi = \mathbf{m} - \mathbf{m}_0 \in H^1(\mathbb{R}^2; \mathbb{R}^3) \). Then

\[
E(\mathbf{m}) - E(\mathbf{m}_0) = \frac{1}{2} \mathcal{H}_\infty(\xi) + \mathcal{H}_\infty(\xi, \mathbf{m}_0 - \hat{e}_3)
\]

and using that \( \tau(\mathbf{m}_0) = 0 \)

\[
\mathcal{H}_\infty(\xi, \mathbf{m}_0 - \hat{e}_3) = \int_{\mathbb{R}^2} \Lambda(\mathbf{m}_0) \mathbf{m}_0 \cdot \xi \, dx.
\]

It follows that

\[
E(\mathbf{m}) - E(\mathbf{m}_0) = \frac{1}{2} \mathcal{H}_\infty(\xi) - \Delta \mathcal{H}(\xi) + \int_{\mathbb{R}^2} \Lambda(\mathbf{m}_0) \mathbf{m}_0 \cdot \xi \, dx
\]

\[
= \frac{1}{2} \mathcal{H}(\xi) + \frac{1}{2} \int_{\mathbb{R}^2} \Lambda(\mathbf{m}_0)(\mathbf{m} + \mathbf{m}_0) \cdot (\mathbf{m} - \mathbf{m}_0) \, dx
\]

where the integrand of the last term vanishes identically. \[ \square \]

The requisite orthogonality can be established by an appropriate translation.

**Lemma 10.** There exist \( \varepsilon > 0 \) and \( c > 0 \) such that for \( \mathbf{m} \in H^1(\mathbb{R}^2; S^2) \) with \( \| \mathbf{m} - \mathbf{m}_0 \|_{H^1} < \varepsilon \) there exists a unique \( x \in \mathbb{R}^2 \) such that

\[
\langle \mathbf{m}(\cdot - x) - \mathbf{m}_0, \nabla \mathbf{m}_0 \rangle_{L^2} = 0 \quad \text{and} \quad \| \mathbf{m}(\cdot - x) - \mathbf{m}_0 \|_{H^1} \leq c \varepsilon.
\]
Proof. Existence and uniqueness follow from the implicit function theorem applied to the $C^1$ mapping $F(x, \xi) = (m \cdot -x - m_0, \nabla m_0)_{L^2}$, where $\xi = m - m_0$, in the vicinity of $(x, \xi) = (0,0)$ in $H^1(\mathbb{R}^2; \mathbb{R}^3) \times \mathbb{R}^2$. The Jacobian $[\partial_x F(0,0)]_{jk} = -\langle \partial_j m_0, \partial_k m_0 \rangle_{L^2}$ is non-singular in $\mathbb{R}^{2 \times 2}$ since the topological degree $Q(m_0) \neq 0$. Hence there exists a $C^1$ mapping $\xi \mapsto x(\xi)$ such that $F(x(\xi), \xi) = 0$ for $\|\xi\|_{H^1} < \varepsilon$. Implicit differentiation implies $|x(\xi)| \lesssim \|\xi\|_{H^1}$. Hence the estimate follows from translation invariance of the $H^1$ norm and the fact that $\|m_0(x) - m_0\|_{H^1} \lesssim |x|$ according to Proposition [1].

Proof of Theorem [2]. By virtue of Lemma [10], we can assume $(\xi, \nabla m_0)_{L^2} = 0$ where $\xi = m - m_0$. We decompose $\xi = \xi^T + \xi^\perp$ where $\xi^T = P_m \xi$, hence

$$\xi^\perp = (\xi \cdot m_0) m_0 = -\frac{1}{2} |\xi|^2 m_0,$$

and in particular

$$\|\xi^\perp\|_{L^2} = \frac{1}{4} \|\xi\|^2_{L^4} \leq c \|\xi\|^2_{H^1}.$$}

It follows from Lemma [9] and Proposition [5]

$$E(m) - E(m_0) \geq \frac{1}{2} \left( \lambda \|\xi^T\|^2_{H^1} + \mathcal{H}(\xi^\perp) + 2 \mathcal{H}(\xi^T, \xi^\perp) \right).$$

By [5.1], we observe that the helicity terms in $\mathcal{H}_\infty(\xi^\perp)$ and $\Delta \mathcal{H}(\xi^\perp)$ cancel, so that taking into account [5.2]

$$\mathcal{H}(\xi^\perp) = \int_{\mathbb{R}^2} |\nabla \xi^\perp|^2_{L^2} + \frac{h}{4} \frac{\langle \xi^\perp \cdot m_0 \rangle^2 - |\nabla m_0|^2}{|\xi|^4} \leq \|\xi^\perp\|^2_{H^1} - C \|\xi\|^2_{H^1},$$

where $C$ denotes a positive constant that only depends on $m_0$. Considering $\mathcal{H}(\xi^T, \xi^\perp)$, we estimate the leading order term

$$\left| \int_{\mathbb{R}^2} \nabla \xi^T : \nabla \xi^\perp \ dx \right| \leq \frac{1}{4} \int_{\mathbb{R}^2} \left| \nabla (P_{m_0} \xi) : \nabla (|\xi|^2 m_0) \right| \ dx \leq \frac{1}{4} \int_{\mathbb{R}^2} \left| \xi^T \right| \left| \nabla \xi^T \right| \left| \nabla m_0 \right| + \left| \nabla (|\xi|^2 m_0) \right| \left| \nabla (P_{m_0} \xi) \right| \ dx \leq C (\|\nabla \xi\|_{L^2} \|\xi\|^2_{L^4} + \|\xi\|^3_{L^6}) \leq C \|\xi\|^3_{H^1},$$

taking into account the orthogonality of $P_{m_0}(\nabla \xi)$ and $m_0$. Improved estimates are valid for the remaining terms, hence $\|\mathcal{H}(\xi^T, \xi^\perp)\| \leq C \|\xi\|^2_{H^1}$. It follows that for $\|\xi\|_{H^1} < \varepsilon < 1$

$$E(m) - E(m_0) \geq \frac{1}{2} (\min\{\lambda, 1\} - C \varepsilon) \|\xi\|^2_{H^1},$$

which implies the claim for $\varepsilon$ sufficiently small. □

6. Solitonic motion

In this section we prove Theorem [3]. Passing to a moving frame $m = m(x - ct)$ for some unknown $c \in \mathbb{R}^2$, the Landau-Lifshitz-Gilbert equation [1.6] with spin velocity $v \in \mathbb{R}^2$ becomes a stationary operator equation

$$F(m, c, v) = 0$$

where

$$F(m, c, v) = -\tau(m) + m \times [(v - c) \cdot \nabla m] + (\beta v - \alpha c) \cdot \nabla m.$$
Lemma 11. For \( F = F(m, c, v) \) and \( \| m - m_0 \|_{H^2} \) sufficiently small we have
\[
F = 0 \iff P_{m_0}F = 0.
\]

Proof. Necessity of \( P_{m_0}F = 0 \) is obvious. Now suppose \( P_{m_0}F = 0 \). Then it follows from the projection property of \( P_{m_0} \) that
\[
|F|^2 = \langle F, (P_m - P_{m_0})F \rangle \leq \|P_m - P_{m_0}\|_{L^\infty} |F|^2,
\]
hence \( F = 0 \) provided \( \|P_m - P_{m_0}\|_{L^\infty} < 1 \), which can also be estimated in terms of \( \|m - m_0\|_{sup} \lesssim \|m - m_0\|_{H^2} \) by Sobolev embedding. \( \square \)

Assuming \( h \gg 1 \), we aim to construct \((m, c)\) for small \( v \) in the vicinity of the stationary solution \( m_0 \) and \( c = 0 \). To this end, we introduce the operator
\[
\mathcal{F} : H^2_{\ell_1}(\mathbb{R}^2; \mathbb{S}^2) \times \mathbb{R}^2 \times \mathbb{R}^2 \to L^2(\mathbb{R}^2; T_{m_0}(\mathbb{S}^2)) \times \mathbb{S}^2
\]

between Hilbert manifolds defined by
\[
\mathcal{F}(m, c, v) = \left( \frac{P_{m_0}F(m, c, v)}{m(0)} \right).
\]

By means of the implicit function theorem on Hilbert manifolds \( \mathbb{H} \), we solve \( \mathcal{F}(m, c, v) = (0, -\hat{e}_3) \) for small \( v \). Since \( \mathcal{F}(m_0, 0, 0) = (0, -\hat{e}_3) \), we only have to show invertibility of the differential
\[
\partial_{(m, c)} \mathcal{F}(m_0, 0, 0) : H^2(\mathbb{R}^2; T_{m_0}(\mathbb{S}^2)) \times \mathbb{R}^2 \to L^2(\mathbb{R}^2; T_{m_0}(\mathbb{S}^2)) \times T_{m_0(0)}(\mathbb{S}^2)
\]
given by
\[
\partial_{(m, c)} \mathcal{F}(m_0, 0, 0) = \begin{pmatrix} J & S \\ \delta_0 & 0 \end{pmatrix},
\]

where \( S \) is the vector field given by
\[
Sc = -[m_0 \times (c \cdot \nabla)m_0 + \alpha (c \cdot \nabla)m_0].
\]

Lemma 12. Suppose \( f \in L^2(\mathbb{R}^2; T_{m_0}(\mathbb{S}^2)) \) and \( V \in T_{m_0(0)}(\mathbb{S}^2) \). Then
\[
J u + Sc = f \quad \text{and} \quad u(0) = V
\]
has a unique solution
\[
u \in H^2(\mathbb{R}^2; T_{m_0}(\mathbb{S}^2)) \quad \text{and} \quad c \in \mathbb{R}^2.
\]

Proof. Proposition \( \mathbb{S} \) implies that \( c \in \mathbb{R}^2 \) is determined by the Fredholm condition
\[
f - Sc \perp \ker J = \operatorname{span}\{\partial_1 m_0, \partial_2 m_0\},
\]
which may be written as a linear \( 2 \times 2 \) system \( Ac + b = 0 \) with
\[
b_j := \langle f, \partial_j m_0 \rangle_{L^2} \quad \text{and} \quad A_{jk} := 4\pi \epsilon_{jk} + \alpha \langle \partial_j m_0, \partial_k m_0 \rangle_{L^2}.
\]

Here we have used that \( Q(m_0) = -1 \) while
\[
\langle \partial_j m_0, m_0 \times \partial_k m_0 \rangle_{L^2} = -4\pi Q(m_0) \epsilon_{jk}.
\]

It follows from Cauchy-Schwarz that \( \det A > (4\pi)^2 \) for all \( \alpha > 0 \), which implies unique solvability. Then the equation for \( u \)
\[
Ju = f - Sc
\]
has a solution \( u_0 \) which is unique up to an element in \( \ker J \). Thanks to [3.1], the second equation

\[
\mathbf{u}(0) = \mathbf{u}_0(0) + \sum_{j=1,2} \lambda_j \partial_j \mathbf{m}_0(0) = \mathbf{V}
\]

selects a unique solution \( \mathbf{u} = \mathbf{u}_0 + \sum_{j=1,2} \lambda_j \partial_j \mathbf{m}_0 \).

We conclude that for \( h \gg 1 \) there exists \( \varepsilon > 0 \) and a smooth map

\[
v \mapsto (\mathbf{m}(v), c(v)) \in H^2_{e_3}(\mathbb{R}^2; S^2) \times \mathbb{R}^2
\]

with \( \mathbf{m}(0) = \mathbf{m}_0 \) and \( c(0) = 0 \) such that \( F(\mathbf{m}(v), c(v), v) = 0 \) for \( |v| < \varepsilon \).

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**Appendix A. Properties of the polar profile**

Here we provide the proof of Proposition 2.

**Lemma 13.** If \( \theta \) is a solution of [5.4], then

\[
0 < \theta(r) < \pi \quad \text{for all } r \in (0, \infty)
\]

and \( \theta \) is monotonically decreasing on \( (0, \infty) \) for \( h \gg 1 \).

**Proof.** We make a fine analysis of the ordinary equation in the spirit of the argument for the radial solution of a Ginzburg-Landau-type equation in [7]. The rescaled function \( \phi(t) = \pi - \theta(e^{-t}) \) satisfies the second order differential equation

\[
\phi''(t) = -\sin \phi(t)f(t), \quad \text{where } f(t) := he^{-2t} - 2e^{-t}\sin \phi - \cos \phi
\]

with the boundary conditions

\[
\lim_{t \to -\infty} \phi(t) = \pi, \quad \lim_{t \to +\infty} \phi(t) = 0.
\]

We choose \( t_0 = \frac{1}{2} \log \frac{1}{h} \) so that \( f(t) > 0 \) for any \( t < t_0 \). Moreover, for \( h \) large enough we have

\[
|\cos \phi(t) - 1| < \varepsilon, \quad |\sin \phi(t)| < \varepsilon \quad \text{and} \quad \phi(t) \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \quad \text{for all } t < t_0
\]

with some small \( \varepsilon > 0 \), e.g. \( \varepsilon = \frac{1}{4} \). We have \( \phi(t) < \pi \) for all \( t \in (-\infty, +\infty) \).

Otherwise there exists a \( t^* < t_0 \) with \( \phi(t^*) \in (\pi, 2\pi) \) and \( \phi'(t^*) < 0 \). Then \( \phi''(t^*) > 0 \) and \( \phi \) will be oscillating around \( 2\pi \) as \( t \to -\infty \).

We choose \( t_1 = \frac{1}{4} \log 2h > t_0 \) so that \( f(t) < 0 \) for all \( t > t_1 \). Then either \( \phi(t) > 0 \) for all \( t \geq t_1 \) or \( \phi(t) < 0 \) for all \( t \geq t_1 \). If this is not the case, then there exists a \( t^* > t_1 \) such that \( \phi(t^*) = 0 \), and hence \( \phi''(t^*) = 0 \). Then \( \phi'(t^*) \) must be nonzero according to the uniqueness theorem for ordinary differential equations. If \( \phi'(t^*) > 0 \), then \( \phi(t) > 0 \) and \( \phi'(t) > 0 \) for \( t > t^* \) and very close to \( t^* \). Hence \( f(t) < 0 \) one can deduce that \( \phi \) will be oscillating around \( \pi \) as \( t \to \infty \), which is a contradiction. Similarly, \( \phi'(t^*) < 0 \) also leads to a contradiction.

Now we assume \( \phi(t) > 0 \) for any \( t \geq t_1 \), then \( \phi'(t) < 0 \) for \( t > t_1 \). We claim \( \phi'(t) < 0 \) for any \( t \in \mathbb{R} \); then it follows that \( \phi(t) \in (0, \pi) \). If \( \phi' \) vanishes at some points, set \( s_1 = \inf\{t \in \mathbb{R}, \phi'(t) = 0\} \). Since \( \phi(t) < \pi \) for any \( t \in \mathbb{R} \) and \( \phi(-\infty) = \pi \) we can easily deduce that \( -\infty < s_1 < t_1 \). If \( \phi(s_1) \in (2k\pi, (2k+1)\pi) \) for some non-negative integer \( k \), then \( \phi''(s_1) > 0 \) and it follows by the choice of \( t_1 \)
that \( \sin \phi(s_1) \cos \phi(s_1) = \sin \phi(s_1) e^{-s_1(he^{-s_1} - 2 \sin \phi(s_1))} + \phi''(s_1) > 0 \) and hence \( \phi(s_1) \in (2k\pi, 2k\pi + \frac{\pi}{2}) \). Therefore there exists some \( s_2 < s_1 \) such that \( \phi(s_2) = 4k\pi + \pi - \phi(s_1) \). On \( [s_2, s_1] \), since \( \phi' < 0 \) and \( \sin \phi > 0 \) we get \( \sin \phi \cos \phi - \phi'' > 0 \) and hence \( \sin^2 \phi(s_2) - (\phi'(s_2))^2 > 0 \) which yields a contradiction.

For (A.2) we adapt the argument for the exponential decay of axisymmetric solution (1.4). According to the rescaled equation (A.1) we have

\[
\theta(r) = \frac{\alpha}{\sqrt{r}} e^{\sqrt{h} r} + o(e^{-\sqrt{h} r}) \quad \text{as } r \to +\infty
\]

where \( \alpha \) is a parameter depending on \( h \), and

\[
\theta(r) = \pi - \frac{h}{2r} + o(r) \quad \text{as } r \to 0.
\]

**Proof.** For (A.2) we adapt the argument for the exponential decay of axisymmetric solution in (1.4). The axisymmetric solution is continuous for \( r > 0 \) and it satisfies the equation \( \theta'' + \frac{\omega}{r} \theta = F(\theta, r) \) where

\[
F(\theta, r) = \frac{1}{r^2} \sin \theta \cos \theta - 2 \frac{1}{r} \sin^2 \theta + h \sin \theta, \quad r \in (0, \infty).
\]

Hence \( \frac{1}{r}(r\theta')' = \theta'' + \frac{\omega}{r} \theta \) is continuous and \( \theta \) is a \( C^2 \) function for \( r > 0 \). Now \( v(r) = r \theta'(r) \) satisfies the equation

\[
\left( \frac{1}{2} v'' \right)' = (v')^2 + v' = \left( q(r) - \frac{1}{4r^2} \right) v^2
\]

where

\[
q(r) = \frac{F(\theta(r))}{\theta'(r)} = \frac{\sin \theta}{\theta} \left[ \frac{1}{r} \left( \frac{1}{r} \cos \theta - 2 \sin \theta \right) + h \right].
\]

and \( q(r) - \frac{1}{4r^2} > 1 \) for large enough \( r \). Thus \( w = v^2 \) satisfies the inequality \( w'' \geq cw \) for some constant \( c > 0 \). It implies the exponential decay of \( w \), hence of \( \theta \), i.e.,

\[
\theta(r) = \frac{\alpha}{\sqrt{r}} e^{\sqrt{h} r} + o(e^{-\sqrt{h} r})
\]

where \( \alpha, h \) are parameters depending on \( h \) and \( h > 0 \). Plugging this ansatz into the (A.3) we get \( \beta = \sqrt{h} \).

Let us now turn to (A.3). According to the rescaled equation (A.1) we have

\[
\sin^2 \phi \cos \phi + 2e^{-t} \sin \phi - he^{-2t} \geq \frac{1}{2}
\]

for \( t > 0 \) and hence

\[
\left( \frac{1}{2} \phi'' \right)' = (\phi')^2 + \phi' \frac{\sin \phi}{\phi} \left( \cos \phi + 2e^{-t} \sin \phi - he^{-2t} \right) \geq c \phi^2
\]

for some \( c > 0 \), which implies the exponential decay of \( \phi \) and therefore

\[
\theta(r) = \pi - \alpha \sqrt{h} + o(r \sqrt{h}) \quad \text{as } r \to 0.
\]
where $\tilde{\alpha}_h, \tilde{\beta}_h$ are parameters depending on $h$ and $\tilde{\beta}_h > 0$. Inserting this into (1.4) yields $\tilde{\alpha}_h = h/2$ and $\tilde{\beta}_h = 1$.

**Lemma 15.** We have the following estimates for $h$ sufficiently large

(A.4) \[ |\cos \theta - r \sin \theta| < \frac{3}{2}, \]

(A.5) \[ r^2(\theta'(r))^2 \geq \sin^2 \theta(r) \]

and

(A.6) \[ h - \frac{3}{2r} \sin \theta \geq 0 \]

for all $r \in (0, \infty)$.

In fact, as $h \to \infty$, we have $h - \frac{3}{2} \sin \theta \geq 0$ for constants $c$ approaching 2.

**Proof.** We consider the function $f(r) = r \sin(\theta(r)) - \cos(\theta(r))$. By Lemma 13 there exists a unique $r^* > 0$ so that $\theta(r^*) = \frac{\pi}{2}$. First we need an estimate of $r^*$. It follows from Young’s inequality that

\[ |E_H(m)| = \left| \int_{\mathbb{R}^2} (m - \hat{e}_3) \cdot (\nabla \times m) \, dx \right| \leq \int_{\mathbb{R}^2} \frac{1}{2} |\nabla m|^2 + \frac{1}{2} |m - \hat{e}_3|^2 \, dx \]

From the (axisymmetric) upper bound in [13] we know that $E(m) < 4\pi$ and $1 - \cos \theta > 1$ in $(0, r^*)$. Then

\[ 4\pi > E(m) \geq \int_{\mathbb{R}^2} \frac{1}{2} |\nabla m|^2 + \frac{h}{2} |m - \hat{e}_3|^2 \, dx - |E_H(m)| \]

\[ \geq \frac{h - 1}{2} \int_{\mathbb{R}^2} |m - \hat{e}_3|^2 \, dx = (h - 1) \int_{\mathbb{R}^2} (1 - m_3) \, dx \]

\[ = 2\pi(h - 1) \int_0^\infty (1 - \cos \theta) \, r \, dr > 2\pi(h - 1) \int_0^{r^*} r \, dr \]

\[ = \pi(h - 1)(r^*)^2 \]

and hence

\[ r^* < \frac{2}{\sqrt{h - 1}} < \frac{1}{2} \]

for $h > 17$. Now we have $f(r^*) = r^* < 1$, $f(r) > -1$ for all $r > 0$ and $f(r) \downarrow -1$ as $r \to +\infty$ by (A.2). If $f$ is bigger than 1 somewhere in $(r^*, \infty)$, there would exist a $\tilde{r} \in (r^*, \infty)$ such that $f(\tilde{r}) > 1$, $f'(\tilde{r}) = 0$. But for $h$ sufficiently large we can always deduce $f''(\tilde{r}) > 0$, which leads to a contradiction. Hence $f(r) < 1$ in $(r^*, \infty)$ and

\[ |\cos \theta - r \sin \theta| < 1 + r^* < \frac{3}{2} \]

For the second inequality we consider the function $g(r) = r^2(\theta'(r))^2 - \sin^2 \theta(r)$. It is clear that $g(0) = 0$ and $\lim_{r \to \infty} g(r) = 0$ by (A.2). According to the (1.4) we have

\[ g'(r) = 2r(\theta'(r))^2 + 2r^2 \theta''(r)\theta'(r) - 2 \sin \theta(r) \cos \theta(r)\theta'(r) \]

\[ = 2r \sin \theta(r)\theta''(r)(hr - 2 \sin \theta(r)). \]

If $g(r) < 0$ at some point in $(0, \infty)$, there would exist two points $0 < r_1 < r_2 < \infty$ so that $g'(r_1) = g'(r_2) = 0$. But this is impossible since the function $r \mapsto hr - 2 \sin \theta(r)$ has only one zero-point in $(0, \infty)$ due to the monotonicity of $\theta$. 

By \((A.3)\) we know that \(r \mapsto hr - \frac{1}{2} \sin \theta(r)\) is positive in the near of 0. If the map is negative at some point, it would have two zero points in \((0, \infty)\), which is impossible due to the monotonicity of \(\theta\). Therefore \(hr - \frac{1}{2} \sin \theta \geq 0\) and the last inequality follows.

Using the properties above we deduce the uniqueness.

**Lemma 16.** The solution of \((1.4)-(1.2)\) is unique for \(h \gg 1\).

**Proof.** Suppose there exist two different solutions \(\theta_1\) and \(\theta_2\). We choose \(0 < \varepsilon \ll 1\) so that \(\theta_1(r_1) = \theta_2(r_2) = \varepsilon\) with \(r_1, r_2 \gg 1\). Without loss of generality, we may assume \(r_1 > r_2\). Define \(\tilde{\theta}_2(r) = \theta_2(r - r_1 + r_2)\) for \(r \geq r_1\). Then

\[
\tilde{\theta}_2''(r) = -\frac{1}{r - r_1 + r_2} \tilde{\theta}_2' + \frac{1}{(r - r_1 + r_2)^2} \sin \tilde{\theta}_2 \cos \tilde{\theta}_2 - \frac{2}{r - r_1 + r_2} \sin^2 \tilde{\theta}_2 + h \sin \tilde{\theta}_2.
\]

If \(\tilde{\theta}_2'(r_1) > \theta_1'(r_1)\), then by \(\tilde{\theta}_2(r_1) = \theta_1(r_1)\) and the Taylor’s formula we would have \(\tilde{\theta}_2(r) > \theta_1(r)\) for \(r > r_1\) very close to \(r_1\). Using the decay property \((A.2)\) for \(r \gg 1\) and the monotonicity of trigonometric functions on \((0, \frac{\pi}{2})\) we deduce

\[
\tilde{\theta}_2''(r) - \theta_1''(r) = h(\sin \tilde{\theta}_2 - \sin \theta_1) + o(\sin \tilde{\theta}_2) > 0.
\]

Thus we obtain \(\tilde{\theta}_2(r) > \theta_1(r)\) for \(r > r_1\) and \(\tilde{\theta}_2'(r) - \theta_1'(r)\) is strictly increasing. It contradicts the fact \(\theta_1' \to 0\) and \(\theta_1' \to 0\) as \(r \to \infty\). For \(\tilde{\theta}_2'(r_1) \leq \theta_1'(r_1)\) a similar argument applies.

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