The application of representation theory in directed strongly regular graphs

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Abstract

The concept of directed strongly regular graphs (DSRG) was introduced by Duval in 1988 [4]. In the present paper, we use representation theory of finite groups in order to investigate the directed strongly regular Cayley graphs. We first show that a Cayley graph $C(G, S)$ is not a directed strongly regular graph if $S$ is a union of some conjugate classes of $G$. This generalizes an earlier result of Leif K. Jørgensen [8] on abelian groups. Secondly, by using induced representations, we have a look at the Cayley graph $C(N times_\theta H, N_1 \times H_1)$ with $N_1 \subseteq N$ and $H_1 \subseteq H$, determining its characteristic polynomial and its minimal polynomial. Based on this result, we generalize the semidirect product method of Art M. Duval and Dmitri Iourinski in [5] and obtain a larger family of directed strongly regular graphs. Finally, we construct some directed strongly regular Cayley graphs on dihedral groups, which partially generalize the earlier results of Mikhail Klin, Akihiro Munemasa, Mikhail Muzychuk, and Paul Hermann Zieschang in [9]. By using character theory, we also give the characterization of directed strongly regular Cayley graphs $C(D_n, X \cup Xa)$ with $X \cap X^{-1} = \emptyset$.

Keywords: Directed strongly regular graph; Representation theory; Induced representation; Cayley graph

1. Introduction

A directed strongly regular graph (DSRG, since it will appear many times, it will be abbreviated as "DSRG" in the following) with parameters $(n, k, \mu, \lambda, t)$ is a $k$-regular directed graph on $n$ vertices such that every vertex is on $t$ 2-cycles, and the number of paths of length two from a vertex $x$ to a vertex $y$ is $\lambda$ if there is an edge directed from $x$ to $y$ and it is $\mu$ otherwise. A DSRG with $t = k$ is an (undirected) strongly regular graph (SRG). Duval showed that a DSRG

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Another definition of a directed strongly regular graph can be given in terms of adjacency matrices. Let $D$ be a directed graph with $n$ vertices. Let $A = A(D)$ denote the adjacency matrix of $D$; $I = I_n$ denote the $n \times n$ identity matrix; $J = J_n$ denote the all-ones matrix. Then $D$ is a DSRG with parameters $(n, k, \mu, \lambda, t)$ if and only if (i) $JA = AJ = kJ$ and (ii) $A^2 = tI + \lambda A + \mu(J - I - A)$. Duval [4] developed necessary conditions on the parameters of $(n, k, \mu, \lambda, t)$-DSRG and calculated the spectrum of a DSRG.

**Proposition 1.1.** ([4]) A DSRG with parameters $(n, k, \mu, \lambda, t)$ with $0 < t < k$ satisfy

\[
\begin{align*}
k(k + (\mu - \lambda)) &= t + (n - 1) \mu, \\
d^2 &= (\mu - \lambda)^2 + 4(t - \mu), 4d2k - (\lambda - \mu)(n - 1), \\
\frac{2k - (\lambda - \mu)(n - 1)}{d} &\equiv n - 1(\text{mod } 2), \\
\left|\frac{2k - (\lambda - \mu)(n - 1)}{d}\right| &\leq n - 1,
\end{align*}
\]

where $d$ is a positive integer, and

\[
0 \leq \lambda < t, 0 < \mu \leq t, -2(k - t - 1) \leq \mu - \lambda \leq 2(k - t).
\]

**Proposition 1.2.** ([4]) A DSRG with parameters $(n, k, \mu, \lambda, t)$ has three distinct integer eigenvalues

\[
k > \rho = \frac{1}{2}(- (\mu - \lambda) + d) > \sigma = \frac{1}{2}(- (\mu - \lambda) - d).
\]

The multiplicities are

\[
1, \quad m_\rho = -\frac{k + \sigma(n - 1)}{\rho - \sigma}, \quad m_\sigma = \frac{k + \rho(n - 1)}{\rho - \sigma}
\]

respectively.

**Proposition 1.3.** ([4]) If $G$ is a DSRG with parameters $(n, k, \mu, \lambda, t)$, then its complement $G'$ is also a DSRG with parameters $(n', k', \mu', \lambda', t')$, where $k' = (n - 2k) + (k - 1)$, $\lambda' = (n - 2k) + (\mu - 2)$, $t' = (n - 2k) + (t - 1)$ and $\mu' = (n - 2k) + \lambda$.

**Remark 1.4.** Let $D$ be a DSRG with parameters $(n, k, \mu, \lambda, t)$, and let $A$ be the adjacency matrix of $D$. Then $A$ has minimal polynomial of degree 3, that is $(A - kI_n)(A^2 + (\mu - \lambda)A + (\mu - t)I_n) = (A - kI_n)(A - \rho I_n)(A - \sigma I_n) = 0$.

We introduce some notations of multisets. Let $A$ be a multiset together with a multiplicity function $\Delta_A : A \rightarrow \mathbb{N}$, where $\Delta_A(a)$ counting “how many times of $a$ occurs in the multiset $A$”. We say $x$ belongs to $A$ (i.e. $x \in A$) if $\Delta_A(x) > 0$. In the following, $A$ and $B$ are multisets, with multiplicity functions $\Delta_A$ and $\Delta_B$. 

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• **Union**, $A \cup B$: the union of multisets $A$ and $B$, is defined by $\Delta_{A \cup B} = \Delta_A + \Delta_B$;

• **Scalar multiplication**, $n \oplus A$: the scalar multiplication of a multiset $A$ by a natural number $n$ by, is defined by $\Delta_{n \oplus A} = n\Delta_A$.

If $A$ and $B$ are usual sets, we use $A \cup B$, $A \cap B$ and $A \setminus B$ denote the usual union, intersection and difference of $A$ and $B$. For example, if $A = \{1, 2\}$ and $B = \{1, 3\}$, then $A \cup B = \{1, 2, 3\}$, $A \cup B = \{1, 2, 3\}$ and $A \setminus B = \{2\}$.

We will always use “set” to mean the usual set (i.e., not multiset) unless otherwise indicated.

Let $G$ be a group and $e_G$ or $e$ be the identity element of $G$. For any subset $S$ of a multiplicative group $G$ and an integer $t$, we use $S^{(t)}$ to denote the set $\{a^t | a \in S\}$. A subset $S$ is called antisymmetric if $S \cap S^{(-1)} = \emptyset$.

Leif K. Jørgensen in [8] showed that a DSRG cannot be a Cayley graph of an abelian group. The goal of Section 3 is to generalize this result. By using character theory of finite groups, we show that a Cayley graph $C(G, S)$ is not a DSRG if $S$ is a union of some conjugate classes of $G$. To prove this theorem, we calculate the spectra of such Cayley graphs $C(G, S)$ in terms of the irreducible characters of $G$, we have the following theorem.

**Theorem 1.5.** Let $\text{Irr}(G) = \{\chi_1, \cdots, \chi_r\}$ be the set of irreducible characters of $G$, then the eigenvalues of Cayley graph $C(G, S)$ with $\overline{S} \in \mathcal{Z}(\mathbb{Z}[G])$ are $\lambda_1, \cdots, \lambda_r$ of multiplicities $n_1, n_2, \cdots, n_r$ respectively, where

$$\lambda_i = \frac{1}{n_i} \sum_{s \in S} \chi_i(s)$$

for $1 \leq i \leq r$, and $n_1, n_2, \cdots, n_r$ are degrees of $\chi_1, \cdots, \chi_r$ respectively.

This lemma enables us to establish the following theorem.

**Theorem 1.6.** A DSRG cannot be a Cayley graph $C(G, S)$ with $\overline{S} \in \mathcal{Z}(\mathbb{Z}[G])$.

In Section 4, a more general theorem is concerned. We show that a Cayley graph $C(G, S)$ is not a DSRG if $S = C \cup T$, where $C$ is a union of some conjugate classes of $G$ and $T$ is a subset of $G \setminus C$ such that $T \cap T^{(-1)} = \emptyset$.

Art M. Duval and Dmitri Iourinski [5] constructed DSRGs by using semidirect product. This method motivates us to investigate Cayley graphs $C(N \rtimes \theta H, S)$. In Section 5, by using induced representations, we give the characteristic polynomial and minimal polynomial of $C(N \rtimes \theta H, N_1 \times H)$ with $N_1 \subseteq N$. Based on this result, we can generalize the semidirect product construction method of Art M. Duval and Dmitri Iourinski in [5] and obtain a larger family of DSRGs. We get the following construction.
Construction 1.7. Let $\mathcal{C}(N \rtimes_\theta H, N_1 \times H)$ be a Cayley graph such that

$$\overline{N_1} = x_1N + x_2e_N$$

for some integers $d > x_1 > 0$ and $x_2 < 0$, then the Cayley graph $\mathcal{C}(N \rtimes_\theta H, N_1 \times H)$ is a DSRG with parameters

$$\left(\frac{md}{x_1m + x_2}, \frac{x_1(x_1m + x_2)}{d}, \frac{x_1(x_1m + x_2)}{d}, \frac{x_1(x_1m + x_2)}{d}\right).$$

The notation $\overline{N_1}$ appearing in the above construction will be defined in (1.2).

Let $C_n = \langle x : x^n = e \rangle$ be a (multiplicative) cyclic group of order $n$, generated by $x$. The dihedral group $D_n$ is the group of symmetries of a regular polygon, and it can be viewed as a semidirect product of two cyclic groups $C_n = \langle x \rangle$ of order $n$ and $C_2 = \langle a \rangle$ of order 2. The presentation of $D_n$ is $D_n = C_n \rtimes C_2 = \langle x, a | x^n = a^2 = e, ax = x^{-1}a \rangle$. The cyclic group $C_n$ is a normal subgroup of $D_n$. All subgroups of $C_n$ are of form $\langle x^v \rangle$, with $v$ as a positive divisor of $n$. Let $\langle x^v \rangle$ be a subgroup of $C_n$, then a transversal for $\langle x^v \rangle$ in $C_n$ is $\{ e, x^1, \cdots, x^{v-1} \}$. Let $T$ be a subset of $\{ e, x^1, \cdots, x^{v-1} \}$, we define

$$T \langle x^v \rangle = \bigcup_{x^v \in T} x^v \langle x^v \rangle,$$

where $x^v \langle x^v \rangle$ are coset of $\langle x^v \rangle$ in $C_n$, for $x^v \in T$.

In Section 6, we construct some directed strongly regular Cayley graphs on dihedral groups. In the following constructions, $v$ is a positive divisor of $n$ and $l = \frac{n}{v}$.

Construction 1.8. Let $v$ be an odd positive divisor of $n$. Let $T$ be a subset of $\{ x^1, \cdots, x^{v-1} \}$, and $X$ be a subset of $C_n$ satisfy the following conditions:

(i) $X = T \langle x^v \rangle$.

(ii) $X \cup X^{(-1)} = C_n \setminus \langle x^v \rangle$.

Then the Cayley graph $\mathcal{C}(D_n, X \cup Xa)$ is a DSRG with parameters $(2n, n - l, \frac{n-l}{2}, \frac{n-l}{2} - l, \frac{n-l}{2})$.

Construction 1.9. Let $v > 2$ be an even positive divisor of $n$. Let $T$ be a subset of $\{ x^1, \cdots, x^{v-1} \}$, and $X$ be a subset of $C_n$ satisfy the following conditions:

(i) $X = T \langle x^v \rangle$.

(ii) $X \cup X^{(-1)} = (C_n \setminus \langle x^v \rangle) \cup (x^v \langle x^v \rangle)$.

(iii) $X \cup (x^v X) = C_n$.

Then the Cayley graph $\mathcal{C}(D_n, X \cup Xa)$ is a DSRG with parameters $(2n, n, \frac{n}{2} + l, \frac{n}{2} - l, \frac{n}{2} + l)$.

Construction 1.10. Let $v$ be an odd positive divisor of $n$. Let $T$ be a subset of $\{ e, x^1, \cdots, x^{v-1} \}$ with $e \in T$, and $X, Y \subseteq C_n$ satisfy the following conditions:
(i) \( Y = T(x^v) = X \cup \langle x^v \rangle \).
(ii) \( Y \cup Y(-1) = C_n \cup \langle x^v \rangle \).

Then the Cayley graph \( C(D_n, X \cup Ya) \) is a DSRG with parameters \((2n, n, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n+1}{2})\).

The above constructions partially generalize the earlier results of Mikhail Klin, Akihiro Munemasa, Mikhail Muzychuk, and Paul Hermann Zieschang in [9].

We consider the problem of characterizing directed strongly regular Cayley graphs on dihedral groups. We give a characterization of the directed strongly regular Cayley graph \( C(D_n, X \cup Xa) \) with \( X \cap X(-1) = \emptyset \). We have been proved:

**Theorem 1.11.** A Cayley graph \( C(D_n, X \cup Xa) \) with \( X \cap X(-1) = \emptyset \) is a DSRG with parameters \((2n, |X|, \mu, \lambda, t)\) if and only if there exists a subset \( T \) of \( \{x_1, \ldots, x_{n-1}\} \) satisfies the following conditions:
(i) \( X = T(x^v) \);
(ii) \( X \cup X(-1) = C_n \setminus \langle x^v \rangle \), where \( v = \frac{n}{\mu^2 - 1} \).

2. Preliminaries

We now give some basic definitions and notations of Cayley graph, Cayley multigraph, group ring, group algebra and representation theory.

Let \( \Gamma \) be a digraph and the adjacent matrix of \( \Gamma \) is denoted by \( A(\Gamma) \). For a matrix \( A \), we denote the characteristic polynomial and the minimal polynomial of \( A \) by \( F_A(x) \) and \( M_A(x) \).

**Definition 2.1.** (Cayley graph) Let \( G \) be a finite group and \( S \subseteq G \setminus \{e_G\} \). The Cayley graph of \( G \) generated by \( S \), denoted by \( \mathcal{C}(G, S) \), is the digraph \( \Gamma \) such that \( V(\Gamma) = G \) and \( x \rightarrow y \) if and only if \( yx^{-1} \in S \), for any \( x, y \in G \).

A multigraph version of Cayley graph is Cayley multigraph.

**Definition 2.2.** (Cayley multigraph) Let \( G \) be a finite group and \( S \) be a multisubset of \( G \setminus \{e_G\} \). The Cayley multigraph of \( G \) generated by \( S \), denoted by \( \mathcal{C}(G, S) \), is the digraph \( \Gamma \) such that \( V(\Gamma) = G \) and \( x \rightarrow y \) if and only if \( yx^{-1} \in S \), for any \( x, y \in G \).

**Definition 2.3.** (Group Ring) For any group \( G \) and ring \( R \), the group ring \( R[G] \) of \( G \) over \( R \), denoted by \( R[G] \), consists of all finite formal sums of elements of \( G \), with coefficients from \( R \). i.e.,
\[
R[G] = \left\{ \sum_{g \in G} r_g g : r_g \in R, r_g \neq 0 \text{ for finite } g \right\}.
\]
The operations $+$ and $\cdot$ on $R[G]$ are given by
\[
\sum_{g \in G} r_g g + \sum_{g \in G} s_g g = \sum_{g \in G} (r_g + s_g) g,
\]
\[
\left( \sum_{g \in G} r_g g \right) \cdot \left( \sum_{g \in G} r_g g \right) = \left( \sum_{g \in G} t_g g \right),
\]
\[
t_g = \sum_{g' g'' = g} r_{g'} s_{g''}.
\]

The center of $R[G]$ is the set of elements of $R[G]$ which commute with all the elements in $R[G]$. We denote the center of $R[G]$ by $\mathcal{Z}(R[G])$.

For any multisubset $X$ of $G$, let $\overline{X}$ denote the element of the group ring $R[G]$ that is the sum of all elements of $X$, i.e.,
\[
\overline{X} = \sum_{x \in X} \Delta_X(x) x.
\]

The lemma below allows us to express a sufficient and necessary condition for a Cayley graph to be directed strongly regular in terms of group ring.

**Lemma 2.4.** A Cayley graph $C(G, S)$ is a DSRG with parameters $(n, k, \mu, \lambda, t)$ if and only if $|G| = n$, $|S| = k$, and
\[
\overline{S^2} = te + \lambda \overline{S} + \mu (G - e - \overline{S}).
\]

2.1. Representation theory

This section is based on the book J. L. Alperin and Rowen B. Bell, *Groups and representations*.

Let $F$ be a field and $G$ be a finite group, then the group ring $F[G]$ is not only a ring but also an $F$-vector space having $G$ as a basis and hence having finite dimension $|G|$. In this case, $F[G]$ becomes an algebra which we call it *group algebra*.

Let $V$ be a finite-dimensional vector space over the field $F$. We define the general linear group $GL(V)$ to be the group of all invertible linear transformations of $V$. We use notation $1_V$ to denote the *identity linear transformation* of $V$. We define the general linear group $GL_n(F)$ to be the set consisting of all invertible matrices over $F$ of order $n$.

An (finite-dimensional) $F$-linear representation $(V, \rho)$ of $G$ in $V$ is a homomorphism $\rho : G \to GL(V)$. If $V$ is a finite-dimensional vector space, the degree of $(V, \rho)$ is the dimension of $V$ and we denote it by $\deg \rho$. We also say that $V$ is a representation space of $G$ with respect to the representation $\rho$.

Let $(V, \rho)$ be an $F$-linear representation. A subspace $W$ of $V$ is $G$-invariant, if for all $u \in W$ and $g \in G$, we have $\rho(g)(u) \in W$. If $W$ is a $G$-invariant subspace, we can restrict $\rho$ on $W$ and obtain a representation $\rho|W : G \to GL(W)$. We say $\rho|W$ is a subrepresentation of $\rho$. An $F$-linear representation $(V, \rho)$ is irreducible if $V$ doesn’t have any untrivial subrepresentation.
Let \((V, \rho)\) be a representation of group \(G\) with \(\dim_F V = n\), let \(B = (e_i)\) be a basis of \(V\), and let \(\rho_B(g)\) be the matrix of \(\rho(g)\) with respect to the basis \(B\) for any \(g \in G\).

Let \(\rho\) and \(\rho'\) be two representations of the same group \(G\) in vector spaces \(V\) and \(V'\) respectively, these representations are said to be equivalent or isomorphic if there exists a linear isomorphism \(\omega : V \to V'\) such that the diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\rho(g)} & V \\
\downarrow{\omega} & & \downarrow{\omega} \\
V' & \xrightarrow{\rho'(g)} & V'
\end{array}
\]

commutes for each \(g \in G\).

Let \(\text{Irr}_F(G)\) be the set of all the non-equivalent irreducible \(F\)-representations of \(G\).

The left regular representation of \(G\) is the homomorphism \(\rho_{\text{reg}} : G \to GL(\mathbb{C}[G])\) defined by

\[
\rho_{\text{reg}}(g) \left( \sum_{h \in G} r_h h \right) = \sum_{h \in G} r_h (gh)
\]

for \(g \in G\).

2.2. Character theory of finite groups

Let \(F = \mathbb{C}\) and \((V, \rho)\) be a \(\mathbb{C}\)-linear representation of \(G\), the \(\mathbb{C}\)-character \(\chi_\rho : G \to \mathbb{C}\) of \(\rho\) is defined by

\[
\chi_\rho(g) = \text{tr}_\rho(g),
\]

where \(\text{tr}\) is the trace function. The kernel of \(\chi_\rho\) is defined by \(\mathcal{K}_{\chi_\rho} = \{ g \in G | \chi_\rho(g) = \chi_\rho(1) \}\), which is a normal subgroup of \(G\).

The character of an irreducible representation is called an irreducible character. Let \(\text{Irr}(G)\) be the set of all the irreducible \(\mathbb{C}\)-characters of \(G\).

Let \(G\) be a finite group with class number \(r\), i.e., the number of distinct conjugacy classes of \(G\). Let \(C_1, C_2, \cdots, C_r\) be the \(r\) conjugacy classes of \(G\), and \(g_1, g_2, \cdots, g_r\) be the representatives of the \(r\) conjugacy classes respectively. The following lemmas give some elementary theorems in character theory, which can be referred to [1] and [10].

**Lemma 2.5.** Let \(G\) be a finite group. Then we have the following theorems:

1. The number of all the non-equivalent irreducible \(\mathbb{C}\)-representations (characters) of \(G\) is equal to \(r\), the class number of \(G\). Equivalently,

\[
|\text{Irr}_\mathbb{C}(G)| = |\text{Irr}(G)| = r.
\]
(2) For any \( C \)-linear representation \((V, \rho) \in \text{Irr}_C(G)\), \( \chi_\rho(1) = \deg \rho = \dim V \).

(3) (First orthogonality relation) Let \( \chi_1, \cdots, \chi_r \) be all the \( r \) irreducible \( C \)-characters of \( G \), then
\[
\frac{1}{|G|} \sum_{h=1}^r \chi_i(g_h)\chi_j(g_h) |C_h| = \delta_{ij}
\]
for any \( 1 \leq i, j \leq r \).

(4) (Second orthogonality relation) Let \( g_1, g_2, \cdots, g_r \) be the conjugacy class representatives of \( G \), then
\[
\frac{1}{|G|} \sum_{h=1}^r \chi_h(g_i)\chi_h(g_j) |C_i| = \delta_{ij}
\]
for any \( 1 \leq i, j \leq r \).

(5) (Character table) The character table of \( G \) is a \( r \times r \) matrix \( X = (\chi_i(g_j)) \). Let \( D_1 = \text{diag}\{ |C_1|, \cdots, |C_r| \} \) and \( D_2 = \text{diag}\{ \chi_1(1), \cdots, \chi_r(1) \} \) def. \( \text{diag}\{ n_1, \cdots, n_r \} \). Then the orthogonality relations can be rewritten by the matrices form:
\[
X D_1 X^T = |G| I_r, \quad X^T A X = |G| D_2^{-1}.
\] (2.1)

The inversion formula is given now.

**Lemma 2.6.** (Inversion formula) Let \( G \) be a finite group and \( A = \sum_{g \in G} a_g g \in C[G] \), then
\[
a_g = \frac{1}{|G|} \sum_{\chi \in \text{Irr}(G)} \chi(A_g^{-1})\chi(e), \forall g \in G.
\]
In particular, if \( G \) is an abelian group, then the above formula becomes
\[
a_g = \frac{1}{|G|} \sum_{\chi \in \text{Irr}(G)} \chi(A)\chi(g), \forall g \in G.
\]

**Corollary 2.7.** Let \( G \) be an abelian group and \( A, B \in C[G] \), then \( A = B \) if and only if \( \chi(A) = \chi(B) \) for all \( \chi \in \text{Irr}(G) \).

3. **A DSRG cannot be a Cayley graph \( C(G, S) \) with \( \overline{S} \in Z(\mathbb{Z}[G]) \)**

L. K. Jørgensen [8] proved that a DSRG cannot be a Cayley graph \( C(G, S) \) of an abelian group. This section gives a generalization of this result. We show that a DSRG cannot be a Cayley graph \( C(G, S) \), where \( S \) is a union of some conjugate classes of \( G \).

At first, we give a lemma about the group algebra \( C[G] \) and the center \( Z(\mathbb{Z}[G]) \). Let \( e \) be the identity element of \( G \). It is clear that \( e \) is the unity of the group algebra \( C[G] \).
Lemma 3.1. Let $G$ be a finite group and $\text{Irr}(G) = \{\chi_1, \cdots, \chi_r\}$ be the set of all the irreducible $\mathbb{C}$-characters of $G$. Define

$$e_i = \frac{\chi_i(1)}{|G|} \sum_{g \in G} \chi_i(g) g$$

for $1 \leq i \leq r$. Then

(1) The $r$ elements $e_1, e_2, \cdots, e_r$ form a complete family of primitive central idempotents with $e_1 + e_2 + \cdots + e_r = e$. The group algebra $\mathbb{C}[G]$ can be decomposed into the direct sum of minimal two-sided ideals (also $(\mathbb{C}[G], \mathbb{C}[G])$-bimodules), that is

$$\mathbb{C}[G] = \mathbb{C}[G]e_1 \oplus \mathbb{C}[G]e_2 \cdots \oplus \mathbb{C}[G]e_r.$$

(2) The $r$ elements $\{e_1, e_2, \cdots, e_r\}$ form a $\mathbb{C}$-basis of $\mathcal{Z}(\mathbb{C}[G])$. Then, as a vector space, we have

$$\mathcal{Z}(\mathbb{C}[G]) = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \cdots \oplus \mathbb{C}e_r.$$

The spectrum of a graph $\Gamma$ is the set of eigenvalues of $A(\Gamma)$, i.e., the adjacency matrix of $\Gamma$, together with their multiplicities. The spectrum of a graph is an important algebraic invariant. In general, we still cannot obtain a simple and explicit formula of eigenvalues of a Cayley graph $C(G, S)$. We now list some known results about the spectrum of Cayley graph $C(G, S)$.

When $G$ is an abelian group, we have:

**Theorem 3.2.** ([2], Theorem 5.4.10) Let $G$ be an abelian group, $S$ be a subset of $G \setminus \{e\}$, and $\text{Irr}(G) = \{\chi_1, \chi_2, \cdots, \chi_n\}$ be the set of irreducible characters of $G$. Then the eigenvalues of the adjacency matrix $A(C(G, S))$ are

$$\lambda_i = \sum_{s \in S} \chi_i(s)$$

for any $1 \leq i \leq n$.

In [3], L. Babai derived an expression for the spectrum of the Cayley graph $C(G, S)$ in terms of irreducible characters of the group $G$. Let $\text{Irr}(G) = \{\chi_1, \chi_2, \cdots, \chi_r\}$ be the set of irreducible characters of $G$, and $n_1, n_2, \cdots, n_r$ are degrees of $\chi_1, \cdots, \chi_r$, respectively.

**Theorem 3.3.** ([3]) Let $C(G, S)$ be a Cayley graph. Then

$$\lambda_{i,1}^t + \lambda_{i,2}^t + \cdots + \lambda_{i,n_i}^t = \sum_{g_1 \cdots g_t \in S} \chi_i(g_1 \cdots g_t)$$

for any positive integer $t$ and $1 \leq i \leq r$, where $\lambda_{i,1}, \lambda_{i,2}, \cdots, \lambda_{i,n_i}$ are eigenvalues of $C(G, S)$, of the same multiplicities $n_i$. 

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3.1. The spectra of Cayley graphs $C(G, S)$ with $S \in \mathcal{Z}(\mathbb{Z}[G])$

Let $\rho_{\text{reg}}$ be the left regular representation of $G$. If $G = \{g_1, g_2, \ldots, g_{|G|}\}$ were chosen as a basis of the group algebra $\mathbb{C}[G]$, then the matrix $\rho_{\text{reg}, G}(S)$ is an adjacent matrix of Cayley graph $C(G, S)$, for any $S \subseteq G \setminus \{e_G\}$.

Let $C_1, C_2, \ldots, C_r$ be the conjugacy classes of the group $G$, then $C_1, C_2, \ldots, C_r$ are called the conjugacy class sum of $G$. It is clear that $\{C_1, C_2, \ldots, C_r\}$ is a $\mathbb{C}$-basis of $\mathcal{Z}(\mathbb{C}[G])$, i.e.,

$$\mathcal{Z}(\mathbb{C}[G]) = C_{C_1} \oplus C_{C_2} \cdots \oplus C_{C_r}.$$ 

This shows that a subset $S$ is a union of some conjugate classes of $G$ if and only if $S \in \mathcal{Z}(\mathbb{C}[G]) \cap \mathbb{Z}[G] = \mathcal{Z}(\mathbb{Z}[G])$. From Lemma 3.1, $\{e_1, e_2, \ldots, e_r\}$ is another basis of $\mathcal{Z}(\mathbb{C}[G])$, i.e.,

$$\mathcal{Z}(\mathbb{C}[G]) = C_{e_1} \oplus C_{e_2} \cdots \oplus C_{e_r}$$

where

$$e_j = \frac{\chi_j(1)}{|G|} \sum_{g \in G} \chi_j(g) g$$

for $1 \leq j \leq r$. The following lemma gives the spectrum of the Cayley graph $C(G, S)$, where $S$ is a union of some conjugate classes of $G$.

**Theorem 3.4.** Let $C(G, S)$ be a Cayley graph with $S \in \mathcal{Z}(\mathbb{Z}[G])$ and $\text{Irr}(G) = \{\chi_1, \ldots, \chi_r\}$ be the set of irreducible characters of $G$. Then the eigenvalues of such Cayley graph are $\lambda_1, \ldots, \lambda_r$ of multiplicities $n_1^2, n_2^2, \ldots, n_r^2$ respectively, where

$$\lambda_i = \frac{1}{n_i} \sum_{s \in S} \chi_i(s)$$

for $1 \leq i \leq r$, and $n_1, n_2, \ldots, n_r$ are degrees of $\chi_1, \ldots, \chi_r$ respectively.

**Proof.** From Lemma 3.1(2), assuming

$$S = \sum_{j=1}^r \lambda_j e_j$$

for some complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$. Therefore

$$\rho_{\text{reg}}(S) = \sum_{j=1}^r \lambda_j \rho_{\text{reg}}(e_j).$$

Then we have

$$S = \sum_{j=1}^r \lambda_j e_j = \frac{1}{|G|} \sum_{j=1}^r \lambda_j n_j \sum_{g \in G} \chi_j(g) g$$

$$= \sum_{g \in G} \left( \frac{1}{|G|} \sum_{j=1}^r \lambda_j n_j \chi_j(g) \right) g = \sum_{g \in S(g)g}$$
from the equation (3.1). This gives that \((\lambda_1, \cdots, \lambda_r)^T\) satisfies the linear equation

\[
\mathcal{X}^T D_2 \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_r \end{pmatrix} = |G| \begin{pmatrix} \Delta_S(g_1) \\ \Delta_S(g_2) \\ \vdots \\ \Delta_S(g_r) \end{pmatrix},
\]

where \(g_1, g_2, \cdots, g_r\) are the representatives of the \(r\) conjugacy classes. Multiplying \(\mathcal{X}D_1\) on both sides of the above equation, we can get

\[
\mathcal{X}D_1 \mathcal{X}^T D_2 \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_r \end{pmatrix} = |G| D_2 \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_r \end{pmatrix} = |G| \mathcal{X}D_1 \begin{pmatrix} \Delta_S(g_1) \\ \Delta_S(g_2) \\ \vdots \\ \Delta_S(g_r) \end{pmatrix}
\]

by Lemma [235](5). Then

\[
\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_r \end{pmatrix} = D_2^{-1} \mathcal{X}D_1 \begin{pmatrix} \Delta_S(g_1) \\ \Delta_S(g_2) \\ \vdots \\ \Delta_S(g_r) \end{pmatrix}.
\]

Hence, we have

\[
\lambda_i = \frac{1}{n_i} \sum_{j=1}^r |C_j| \Delta_S(g_j) \chi_i(g_j) = \frac{1}{n_i} \sum_{s \in S} \chi_i(s)
\]

for each \(1 \leq i \leq r\). It follows from Lemma [3.1] that

\[
\mathbb{C}[G] = \mathbb{C}[G]e_1 \oplus \mathbb{C}[G]e_2 \oplus \cdots \oplus \mathbb{C}[G]e_r
\]

is a decomposition of the group algebra into minimal two-sided ideals (also \((\mathbb{C}[G], \mathbb{C}[G])\)-bimodules). For each \(1 \leq i \leq r\), the \(\rho_{\text{reg}}(S)\) restricts to the submodule \(\mathbb{C}[G]e_i\) is

\[
\rho_{\text{reg}}(S) \big|_{\mathbb{C}[G]e_i} = \sum_{j=1}^r \left( \lambda_j \rho_{\text{reg}}(e_j) \big|_{\mathbb{C}[G]e_i} \right) = \lambda_i \rho_{\text{reg}}(e_i) \big|_{\mathbb{C}[G]e_i} = \lambda_i \mathbf{1}_{\mathbb{C}[G]e_i},
\]

this is a scalar multiplication. Thus the eigenvalues of \(\rho_{\text{reg}}(S) \big|_{\mathbb{C}[G]e_i}\) are \(\lambda_i\) with multiplicity \(n_i^2 = \dim \mathbb{C}[G]e_i\). Then the result follows. \qed
3.2. The proof of the main theorem

**Lemma 3.5.** (see [9], Lemma 3.2). Let $H$ be a regular non-empty directed graph without undirected edges, then $A = A(H)$ has at least one non-real eigenvalue.

**Theorem 3.6.** A DSRG cannot be a Cayley graph $C(G, S)$ with $S \in Z(Z[G])$.

**Proof.** Note that if $C$ is a conjugacy class of $G$, then so do $C^{(-1)}$. Define

$$S_1 = \bigcup_{\{C_i : C_i^{(-1)} \subseteq S\}} C_i \text{ and } S_2 = \bigcup_{\{C_i : C_i^{(-1)} \not\subseteq S\}} C_i,$$

then $S_1$ and $S_2$ are also a union of some conjugacy classes of $G$, and hence $S_1, S_2 \in Z(Z[G])$. Thus from Theorem 3.4, the eigenvalues of Cayley graphs $C(G, S_1)$ and $C(G, S_2)$ are \( \{ \lambda_{1,1}, \cdots, \lambda_{1,r} \} \) and \( \{ \lambda_{2,1}, \cdots, \lambda_{2,r} \} \) of the same multiplicities $n_{1,1}, n_{1,2}, \cdots, n_{1,r}$ respectively, where

$$\lambda_{1,i} = \frac{1}{n_{1,i}} \sum_{s \in S_1} \chi_i(s), \quad \lambda_{2,i} = \frac{1}{n_{2,i}} \sum_{s \in S_2} \chi_i(s),$$

for $1 \leq i \leq r$. Suppose Cayley graph $C(G, S)$ is a DSRG, then all the eigenvalues of $C(G, S)$ are real numbers. Note that $S_1 \cup S_2 = S$, then from Theorem 3.4 we have

$$\lambda_i = \frac{1}{n_i} \sum_{s \in S} \chi_i(s) = \frac{1}{n_i} \sum_{s \in S_1} \chi_i(s) + \frac{1}{n_i} \sum_{s \in S_2} \chi_i(s) = \lambda_{1,i} + \lambda_{2,i}$$

and

$$\lambda_{1,i} = \frac{1}{n_{1,i}} \sum_{C_i \subseteq S_1} \chi_i(s) = \frac{1}{2n_i} \sum_{C_i \subseteq A} \left( \chi_i(C_i) + \chi_i(C_i^{(-1)}) \right)$$

$$= \frac{1}{n_i} \sum_{C_i \subseteq A} R(\chi_i(C_i)) \in \mathbb{R}$$

for any $1 \leq i \leq r$, where $R(z)$ denotes the real component of $z$. Therefore all the eigenvalues $\{\lambda_{2,i}\}_{i=1}^r$ of $C(G, S_2)$ are real. This is a contradiction since Cayley graph $C(G, S_2)$ is a regular non-empty directed graph without undirected edges, it has at least one non-real eigenvalue. \(\square\)

Then we can get the following corollary easily since any conjugacy class of an abelian group $G$ just consist of only one element.

**Corollary 3.7.** ([8]). A DSRG cannot be a Cayley graph $C(G, S)$ of an abelian group.

4. DSRG cannot be a Cayley graph $C(G, S)$ with $S = C \cup T$

For any subset $M$ of $G$, recall the set $N_G(M) = \{ g \in G | M^g = gMg^{-1} = M \}$, is the normalizer of $M$ in $G$. Let $T$ be an antisymmetric subset of $N_G(M) \setminus (M \cup \{e\})$. 

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**Theorem 4.1.** Let $M$ be a subset of $G$ such that $M$ is closed under taking inverses, i.e., $M^{-1} = M$, and $T$ be an antisymmetric subset of $N_G(M) \setminus (M \cup \{e\})$, then Cayley graph $C(G,M \cup T)$ is not a DSRG.

**Proof.** For any $x, y \in G$, we define the sets $A_{x,y} = \{(c,t)|(c,t) \in M \times T, ct = yx^{-1}\}$ and $B_{x,y} = \{(t',c')|(t',c') \in T \times M, t'c' = yx^{-1}\}$. We assert that $|A_{x,y}| = |B_{x,y}|$. We prove this assertion by giving a one-to-one correspondence between the sets $A_{x,y}$ and $B_{x,y}$.

Indeed, if $(c,t) \in A_{x,y}$ such that $ct = yx^{-1}$, then the assumption $t \in T \subseteq N_G(M)$ implies $c' = t^{-1}ct \in M$, so $yx^{-1} = tc'$ and $(t,c') \in B_{x,y}$. Conversely, if $(t',c') \in B_{x,y}$ satisfies $t'c' = yx^{-1}$, then $c = t'c't'^{-1} \in M$. Hence the array $(c',t)$ satisfies $ct' = yx^{-1}$ and then $(c,t') \in A_{x,y}$. This gives a one-to-one correspondence between the sets $A_{x,y}$ and $B_{x,y}$.

Therefore, $A(C(G,M))$ and $A(C(G,T))$ are commutative and $A(C(G,M)) + A(C(G,T)) = A(C(G,M \cup T))$. Then the eigenvalues of $A(C(G,M \cup T))$ are the sums of the corresponding eigenvalues of $A(C(G,M))$ and $A(C(G,T))$.

If Cayley graph $C(G,M \cup T)$ is a DSRG, then all the eigenvalues of $A(C(G,M \cup T))$ are real numbers. Note that all the eigenvalues of $A(C(G,M))$ are real since $A(C(G,M))$ is a symmetric matrix, this shows that all the eigenvalues of $A(C(G,T))$ are real. But Lemma 3.5 implies that at least one eigenvalues of $A(C(G,T))$ is not real. This is a contradiction. $\square$

Let $S = C \cup T$, where $C$ is a union of some conjugate classes of $G$ and $T$ is an antisymmetric subset such that $C \cap T = \emptyset$. From Theorem 4.1, we have the following corollary.

**Corollary 4.2.** DSRG cannot be a Cayley graph $C(G,S)$ with $S = C \cup T$, where $C$ is a union of some conjugate classes of $G$ and $T$ is an antisymmetric subset of $G$ such that $C \cap T = \emptyset$.

**Proof.** Observe that the normalizer of $C$ is $G$. As the proof of Theorem 3.6 we can write $C = C_1 \cup C_2$, where $C_1 = \bigcup_{i \subseteq C} C_i$ and $C_2 = \bigcup_{i \supseteq C} C_i$. Let $C' = C_1$ and $T' = T \cup C_2 \subseteq G = N_G(C')$, then $C' = C'(-1)$ is closed under taking inverses and $T' \subseteq G \setminus (C_1 \cup \{e\})$. Then this corollary follows from the Theorem 4.1 with $M = C'$ and $T = T'$. $\square$

5. A generalization of semidirect product constructions of DSRGs

Let $G$ be a finite group, and $\mathbb{Z}[G]$ be the integral group ring. Define

$$\mathbb{Z}_{>0}[G] = \left\{ \sum_{g \in G} a_g g \mid a_g \in \mathbb{Z}, a_g \geq 0, \forall g \in G \right\}.$$  

Let $\mathcal{P}(G) = \{S : S \text{ is a multisubset of } G\}$ be the set of all the multisubsets of $G$. Define

$$\mathcal{P}_G : \mathbb{Z}_{>0}[G] \rightarrow \mathcal{P}(G), \sum_{g \in G} a_g g \mapsto \biguplus_{g \in N} a_g \oplus \{g\}. \quad (5.1)$$
This is a one-to-one correspondence between $\mathbb{Z}_{\geq 0}G$ and $\mathcal{P}(G)$.

5.1. Preliminary

Definition 5.1. (Semidirect product) Let $N$ and $H$ be two groups. Let $\theta : H \to \text{Aut}(N)$ be a given homomorphism from $H$ to $\text{Aut}(N)$. Let $N \rtimes_\theta H$ be the direct product set of $N$ and $H$, with the following operation for the product of two elements

$$(n, h)(n', h') = (n[\theta(h)(n')], hh').$$

Then $N \rtimes_\theta H$ is called the semidirect product of $N$ and $H$ with respect to the homomorphism $\theta$.

We can define an action of $H$ on $N$. Let $N \rtimes_\theta H$ be the semidirect product of $N$ and $H$, then $H$ acts on $N$ by defining $h \cdot n := hnh^{-1} = \theta(h)(n)$. Then the orbit of $n$ is the set $H \cdot n = \{h \cdot n : h \in H\}$ and hence $N$ has a unique partition consisting of orbits. There is a trivial $H$-orbit only consists of the identity element $e_N$, and the other $H$-orbits are called the untrivial $H$-orbits of $N$.

Let $N_1$ be a subset of $N$, we define

$$N_1^\flat = \mathcal{P}_N \left( \sum_{h \in H} hN_1h^{-1} \right),$$

where $\mathcal{P}_N$ is defined as (5.1). Therefore $N_1^\flat$ is a multisubset of $N$, and

$$\overline{N_1} = \sum_{h \in H} hN_1h^{-1}.$$ 

The following example gives an interpretation of the above notations.

Example 5.2. Considering the dihedral group $D_8 = C_4 \times C_2$, where $C_4 = \langle x \mid x^4 = 1 \rangle$. Then $C_2 \circ x^2 = \{x^2\}$ and $\{x^2\}^\flat = \mathcal{P}_{C_4}(2x^2) = \{x^2, x^2\}$. If $N_1 = \{x^1, x^2, x^3\}$, then

$$N_1^\flat = \mathcal{P}_{C_4}(2x^1 + 2x^3 + 2x^2) = \{x^1, x^1, x^2, x^2, x^3, x^3\}.$$ 

5.2. Semidirect product constructions of DSRGs

Let $N$ be a finite group of order $m$. Art M. Duval and Dmitri Iourinski in $[$5$]$ define a group automorphism $\beta \in \text{Aut}(N)$ has the $q$-orbit condition if each of its untrivial orbits contains $q$ elements. In other words, $\beta$ satisfies $\beta^q(a) = a$, for all $a \in N$, and $\beta^u(a) = a$ implies $q|u$, for all $a \neq e_N$.

We can also define the $q$-orbit condition in terms of group action. Let $\beta \in \text{Aut}(N)$ be a group automorphism, then the subgroup $\langle \beta \rangle$, i.e., the subgroup of $\text{Aut}(N)$ generated by $\beta$, acts on $N$
naturally. Then the $\beta \in \text{Aut}(N)$ has the $q$-orbit condition provided each untrivial $\langle \beta \rangle$-orbit have $q$ elements.

The following constructions of DSRGs were given by Art M. Duval and Dmitri Iourinski.

**Theorem 5.3.** Let $N$ be a finite group of order $m$. Let $\beta \in \text{Aut}(N)$ have the $q$-orbit condition. Let $H$ be the cyclic group of order $q$ with generator $b$, and define $\theta : H \to \text{Aut}(N)$ by $\theta(b^u) = \beta^u$. Let $N_1$ be a set of representatives of the nontrivial orbits of $\beta$. Then the Cayley graph $\mathcal{C}(N \times_{\theta} H, N_1 \times H)$ is a DSRG with the parameters

$$\left(\frac{mq, m - 1}{q}, \frac{m - 1}{q}, -1, \frac{m - 1}{q}\right).$$

5.3. Induced representations (modules)

We now introduce some basic concepts of induced representations (modules). Let $H \leq G$ and $(V, \rho)$ be an $F$-linear representation of $H$. We can extend it as follows

$$\left(\sum_h a_h h\right) v := \sum_h a_h (hv).$$

Then $V$ has an $F[H]$-module structure.

Since the group algebra $F[G]$ is an $(F[G], F[H])$-bimodule, we can construct the $F[G]$-module $F[G] \otimes_{F[H]} V$. We call this $F[G]$-module the induced $F[G]$-module of $V$ (or the induction of $V$ to $G$), and we denote it by $\text{Ind}^G_H V$. We denote the $F$-linear representation arising from the $\text{Ind}^G_H V$ by $\rho^G$, and we call $\rho^G$ an induced representation.

Let $B = \{v_1, \ldots, v_m\}$ be an $F$-basis of $V$ and $\{g_1, \ldots, g_d\}$ be a (left) transversal for $H$ in $G$. Let $\rho_B(g) = (a_{ij}(g))_{m \times m}$ for any $g \in H$ and

$$\hat{a}_{ij}(g) = \begin{cases} a_{ij}(g), & g \in H, \\ 0, & g \notin H. \end{cases}$$

The following lemma gives some basic propositions about induced representations.

**Lemma 5.4.** (1) $\dim_F \text{Ind}^G_H V = |G : H| \dim_F V$.

(2) The set $C = \{g_i \otimes v_j | 1 \leq i \leq d, 1 \leq j \leq m\} = \{g_1 \otimes v_1, \ldots, g_1 \otimes v_m, \ldots, g_d \otimes v_1, \ldots, g_d \otimes v_m\}$ is an $F$-basis of $\text{Ind}^G_H V$. Moreover, as an $F$-vector space we have

$$\text{Ind}^G_H V = \bigoplus_{i=1}^d g_i (1 \otimes_{F[H]} V).$$

(3) Let $\hat{\rho}_B(g) = (\hat{a}_{ij}(g))_{m \times m}$. Then

$$\rho^G_C(g) = \begin{pmatrix} \hat{\rho}_B(g_1^{-1} gg_1) & \cdots & \hat{\rho}_B(g_1^{-1} gg_d) \\ \vdots & \cdots & \vdots \\ \hat{\rho}_B(g_d^{-1} gg_1) & \cdots & \hat{\rho}_B(g_d^{-1} gg_d) \end{pmatrix}, \quad (5.3)$$
where \( \rho_B(g) \) is the matrix of representation \( \rho(g) \) with respect to the basis \( B \), and \( \rho_C^G(g) \) is the matrix of representation \( \rho^G(g) \) with respect to the basis \( C \), for any \( g \in G \).

(4) Let \( F = \mathbb{C} \) and \( (V, \rho) \) be a \( \mathbb{C} \)-linear representation of \( H \), the \( \mathbb{C} \)-character of \( \rho \) is \( \chi \). We denote the character of \( \rho^G \) by \( \chi^G \), and we call \( \chi^G \) an induced character. Then for any \( g \in G \) we have

\[
\chi^G(g) = \sum_{i=1}^{d} \chi(g_i^{-1}gg_i).
\]

(5) Let \( U \) be an \( F[G] \)-module, then \( F[G] \) can be regarded as an \( F[H] \)-module, this \( F[H] \)-module is called the restriction of \( U \) to \( H \) and we denote it by \( \text{Res}_H^G U \). Furthermore, if \( U \) is an \( \mathbb{C}[G] \)-module having character \( \chi \), then we denote the character of the \( \mathbb{C}[H] \)-module \( \text{Res}_H^G U \) by \( \chi|_H \).

Lemma 5.5. Let \( \rho_{reg} \) be the left regular representation of \( H \). Then \( \rho_{reg}^G \) is the left regular representation of \( G \).

Proof. Note that the representation space of \( \rho_{reg} \) is \( \mathbb{C}[H] \). Therefore

\[
\text{Ind}_H^G \mathbb{C}[H] = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathbb{C}[H] \cong \mathbb{C}[G]
\]

is a left \( \mathbb{C}[G] \)-module isomorphism. This gives that \( \rho_{reg}^G \) is the left regular representation of \( G \). \( \square \)

5.4. Induced representation and directed strongly regular Cayley graph \( \mathcal{C}(N \times_\theta H, N_1 \times H) \)

Let \( G = N \times_\theta H, |N| = m \) and \( |H| = d \), then the (left) transversal for \( N \) in \( G \) is \( H = \{h_1, \ldots, h_d\} \).

In this subsection, we have a look at the Cayley graph \( \mathcal{C}(N \times_\theta H, N_1 \times H_1) \), where \( N_1 \) and \( H_1 \) are subsets of \( N \) and \( H \) respectively.

Let \( \varrho = \rho_{reg} \) be the left regular representation of \( N \), then the representation space with respect to \( \rho \) is \( \mathbb{C}[N] \). Thus \( N = \{n_1, n_2, \cdots, n_m\} \) is a \( \mathbb{C} \)-basis of \( \mathbb{C}[N] \), and \( \varrho_N(g) \) is the matrix of representation \( \rho(g) \) with respect to the basis \( N \), for any \( g \in G \).

The following lemma gives the adjacent matrix of Cayley graph \( \mathcal{C}(N \times_\theta H, N_1 \times H_1) \).

Lemma 5.6. Let \( \mathcal{C}(N \times_\theta H, N_1 \times H_1) \) be the Cayley graph defined above. Then

\[
A(\mathcal{C}(N \times_\theta H, N_1 \times H_1)) = \begin{pmatrix}
\varrho_N(h_1^{-1}N_1^{-1}h_1) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \varrho_N(h_m^{-1}N_1^{-1}h_m)
\end{pmatrix} A(\mathcal{C}(H, H_1^{-1})) \otimes I_m.
\]

(5.4)

Proof. The notations are coincided with above. It follows from Lemma 5.5 that \( \varrho^G = \rho_{reg}^G \) is the left regular representation of \( G \), and \( D = \{h_1 \otimes n_1, \cdots, h_1 \otimes n_m, \cdots, h_d \otimes n_1, \cdots, h_d \otimes n_m\} \) is
a $\mathbb{C}$-basis of $\text{Ind}_G^H \mathbb{C}[N] \cong \mathbb{C}[G]$. Note that the basis $D$ can be identified with the elements of $G$, hence $g^G_D(N_1 \times H_1)$ is an adjacent matrix of Cayley graph $C(N \rtimes_{\theta} H, N_1 \times H_1)$. Therefore

$$A(C(N \rtimes_{\theta} H, N_1 \times H_1)) = g^G_D(N_1 \times H_1) = g^G_D(N_1)g^G_D(H_1). \quad (5.5)$$

For any $h \in H_1$, Lemma 5.4 (3) gives that

$$g^G_D(h) = \begin{pmatrix} \hat{g}_N(h^{-1}_1hh_1) & \cdots & \hat{g}_N(h^{-1}_1hh_d) \\ \vdots & \ddots & \vdots \\ \hat{g}_N(h^{-1}_dhh_1) & \cdots & \hat{g}_N(h^{-1}_dhh_d) \end{pmatrix} = \begin{pmatrix} \delta_{11}I_m & \cdots & \delta_{1d}I_m \\ \vdots & \ddots & \vdots \\ \delta_{d1}I_m & \cdots & \delta_{dd}I_m \end{pmatrix} \quad (5.6)$$

where

$$\delta_{ij} = \begin{cases} 1, & \text{if } h_jh_i^{-1} = h^{-1}; \\ 0, & \text{if } h_jh_i^{-1} \neq h^{-1}. \end{cases}$$

The second equality follows from the fact that $\hat{g}_N(h^{-1}_ihh_j) \neq 0$ if and only if $h^{-1}_ihh_j \in N \cap H = e$, i.e., $h_jh_i^{-1} = h^{-1}$. In this case, $\hat{g}_N(h^{-1}_ihh_j) = g_N(e) = I_m$.

It follows from the equation (5.5) that

$$g^G_D(H_1) = \sum_{h \in H_1} g^G_D(h) = \sum_{h \in H_1} A(C(H, \{h^{-1}\})) \otimes I_m \quad (5.7)$$

where

$$A(C(H, H_1^{-1})) \otimes I_m.$$ 

Meanwhile, we also have

$$g^G_D(N_1) = \begin{pmatrix} \hat{g}_N(h^{-1}_1N_1h_1) & \cdots & \hat{g}_N(h^{-1}_1N_1h_d) \\ \vdots & \ddots & \vdots \\ \hat{g}_N(h^{-1}_dN_1h_1) & \cdots & \hat{g}_N(h^{-1}_dN_1h_d) \end{pmatrix} = \begin{pmatrix} g_N(h^{-1}_1N_1h_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_N(h^{-1}_mN_1h_m) \end{pmatrix} \quad (5.8)$$

The second equality follows from the fact that $\hat{g}_N(h^{-1}_iN_1h_j) \neq 0$ implies that $h^{-1}_iN_1h_j = h^{-1}_iN_1h_i(h^{-1}_iN_1h_j) \subseteq N$, then $h^{-1}_iN_1h_j \in N \cap H = e$, i.e., $h_i = h_j$. 

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Therefore, (5.5), (5.7) and (5.8) show that

\[
\varphi_D(N_1 \times H_1) = \begin{pmatrix} 
\varphi_N(h_1^{-1}N_1h_1) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \varphi_N(h_m^{-1}N_1h_m)
\end{pmatrix} \mathbf{A}(\mathcal{C}(H, H_1^{(-1)})) \otimes I_m. \quad (5.9)
\]

This completes the proof.

For \( H_1 = H \), we will determine the characteristic polynomial and minimal polynomial of Cayley graph \( \mathcal{C}(N \rtimes_\theta H, N_1 \times H) \), by using the Cayley multigraph \( \mathcal{C}(N, N_1^\sharp) \), where \( N_1^\sharp \) is defined in (5.2). Throughout this section, let

\[
L = \mathbf{A}(\mathcal{C}(N \rtimes_\theta H, N_1 \times H)) \quad \text{and} \quad K = \mathbf{A}(\mathcal{C}(N, N_1^\sharp))
\]

be the adjacent matrices of the Cayley graph \( \mathcal{C}(N \rtimes_\theta H, N_1 \times H) \) and the Cayley multigraph \( \mathcal{C}(N, N_1^\sharp) \).

**Lemma 5.7.** The characteristic polynomial of the Cayley graph \( \mathcal{C}(N \rtimes_\theta H, N_1 \times H) \) is

\[
\mathcal{F}_L(x) = x^{md-m}\mathcal{F}_K(x).
\]

The minimal polynomial of the Cayley graph \( \mathcal{C}(N \rtimes_\theta H, N_1 \times H) \) satisfies

\[
\mathcal{M}_K(x)|\mathcal{M}_L(x), \mathcal{M}_L(x)|x\mathcal{M}_K(x).
\]

**Proof.** From Lemma 5.6 and the fact that \( \mathbf{A}(\mathcal{C}(H, H^{(-1)})) = J_d \), the equation (5.4) becomes

\[
L = \begin{pmatrix} 
\varphi_N(h_1^{-1}N_1h_1) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \varphi_N(h_m^{-1}N_1h_m)
\end{pmatrix} J_d \otimes I_m
\]

\[
= \begin{pmatrix} 
\varphi_N(h_1^{-1}N_1h_1) & \cdots & \varphi_N(h_1^{-1}N_1h_1) \\
\vdots & \ddots & \vdots \\
\varphi_N(h_m^{-1}N_1h_m) & \cdots & \varphi_N(h_m^{-1}N_1h_m)
\end{pmatrix}
\]

\[
= \begin{pmatrix} 
\varphi_N(h_1^{-1}N_1h_1) \\
\vdots \\
\varphi_N(h_m^{-1}N_1h_m)
\end{pmatrix} \begin{pmatrix} 
I_m & \cdots & I_m
\end{pmatrix}.
\]
Then

\[ F_L(x) = |xI_{md} - L| \]

\[ = x^{md-m} |xI_m - \left( \begin{array}{cccc} I_m & \cdots & I_m \end{array} \right) \begin{pmatrix} \varrho_N(h_1^{-1}N_1h_1) \\ \vdots \\ \varrho_N(h_d^{-1}N_1h_d) \end{pmatrix} | \]

\[ = x^{md-m} |xI_m - \sum_{j=1}^d \varrho_N(h_j^{-1}N_1h_j) | . \]

Note that

\[ \sum_{j=1}^d \varrho_N(h_j^{-1}N_1h_j) = \varrho_N(N_1^2) = A(C(N, N_1^2)) = K \]

is the adjacency matrix of Cayley multigraph \( C(N, N_1^2) \), so

\[ F_L(x) = x^{md-m} |xI_m - \varrho_E(N_1^2)| = x^{md-m} F_K(x). \]

Observe that

\[ L^j = \left( \begin{array}{cccc} \varrho_N(h_1^{-1}N_1h_1) \\ \vdots \\ \varrho_N(h_d^{-1}N_1h_d) \end{array} \right) K^{j-1} \left( \begin{array}{cccc} I_m & \cdots & I_m \end{array} \right) \]

for any \( j \geq 1 \). Note that 0 is a root of minimal polynomial \( M_L(x) \), since 0 is a root of characteristic polynomial \( F_L(x) \). Therefore we can let \( M_L(x) = xm(x) \) for some polynomial \( m(x) \), then

\[ 0 = M_L(L) = \begin{pmatrix} \varrho_N(h_1^{-1}N_1h_1) \\ \vdots \\ \varrho_N(h_d^{-1}N_1h_d) \end{pmatrix} \begin{pmatrix} m(K) \\ I_m \cdots I_m \end{pmatrix} . \]

Multiplying \( \left( \begin{array}{cccc} I_m & \cdots & I_m \end{array} \right) \) on both sides of the above equation, we can get

\[ 0 = \left( \begin{array}{cccc} I_m & \cdots & I_m \end{array} \right) M_L(K) = \left( \begin{array}{cccc} Km(K) & \cdots & Km(K) \end{array} \right) . \]

This gives that \( Km(K) = M_L(K) = 0 \) and hence

\[ M_K(x)|M_L(x). \]

On the other hand, we note that

\[ 0 = \begin{pmatrix} \varrho_N(h_1^{-1}N_1h_1) \\ \vdots \\ \varrho_N(h_d^{-1}N_1h_d) \end{pmatrix} \begin{pmatrix} M_K(K) \\ I_m \cdots I_m \end{pmatrix} = LM_K(L), \]

this gives that \( M_L(x)|M_K(x) \).
**Theorem 5.8.** If Cayley graph $C(N \rtimes_{\phi} H, N_1 \times H)$ is a DSRG with parameters $(md, k, \mu, \lambda, t)$, then Cayley multigraph $C(N, N^\flat)$ satisfies $N^\flat = \sigma N + \frac{k(k-\sigma)}{m} N$ or $N^\flat = \frac{k-\sigma}{m} N + \sigma e_N$.

**Proof.** Suppose Cayley graph $C(N \rtimes_{\phi} H, N_1 \times H)$ is a DSRG with parameters $(md, k, \mu, \lambda, t)$, then this Cayley graph $C(N \rtimes_{\phi} H, N_1 \times H)$ has 3 distinct eigenvalues $k$, $\rho$ and $\sigma$. We can get $\rho = 0$ and $\sigma < 0$ by Lemma 5.7. Then from Remark 1.4.

It follows from Lemma 5.7 that $M_K(x) = (x - k)(x - \sigma)$ or $x(x - k)(x - \sigma)$.

**Case 1.** If $M_K(x) = (x - k)(x - \sigma)$, then

$$(K - kI_m)(K - \sigma I_m) = 0.$$  

Let $X = K - \sigma I_m$, then $X$ is a nonnegative matrix and $KX = kX$. Therefore each column of $X$ is an eigenvector corresponding to simple eigenvalue $k$ of $K$ (from the Perron-Frobenius theory, can see, e.g., Horn and Johnson [6]), but the eigenspace associated with the eigenvalue $k$ has dimension one and hence each column of $X$ is a suitable multiple of $1_m$, where $I_m$ is all-ones vector. Therefore, there are some integers $b_1, b_2, \cdots, b_m$ such that $X = (b_11_m, b_21_m, \cdots, b_m1_m)$, then $1_m^T X = 1_m^T (K - \sigma I_m) = (k - \sigma)1_m^T$. This shows that $mb_1 = mb_2 = \cdots = mb_m = k - \sigma$ and hence

$$X = K - \sigma I_m = \frac{k - \sigma}{m} J_m,$$

i.e., $N^\flat = \frac{k - \sigma}{m} N + \sigma e_N$.

**Case 2.** If $M_K(x) = x(x - k)(x - \sigma)$, then

$$K(K - kI_m)(K - \sigma I_m) = 0.$$  

Let $Y = K(K - \sigma I_m)$, then $Y$ is a nonnegative matrix and $KY = kY$. Similarly, we can also obtain that $Y = K(K - \sigma I_m) = \frac{k(k-\sigma)}{m} J_m$, so

$$K^2 = \sigma K + \frac{k(k-\sigma)}{m} J_m.$$  

This gives $N^\flat = x_1 N + x_2 e_N$.  

The above lemma motivates us to get the following constructions of DSRGs, which generalize the earlier constructions of Art M. Duval and Dmitri Iourinski in [3].

**Construction 5.9.** Let $C(N \rtimes_{\phi} H, N_1 \times H)$ be a Cayley graph such that

$$N^\flat = x_1 N + x_2 e_N.$$  

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for some integers $x_1, x_2$ with $d > x_1 > 0$ and $x_2 < 0$, then the Cayley graph $C(N \rtimes \theta H, N_1 \times H)$ is a DSRG with parameters
\[
\left( m_d x_1 m + x_2, \frac{x_1 (x_1 m + x_2)}{d}, x_2 + \frac{x_1 (x_1 m + x_2)}{d}, \frac{x_1 (x_1 m + x_2)}{d} \right).
\]

Remark 5.10. The necessary conditions on the parameters of DSRG assert that $0 < x_1 < d$ and $x_2 < 0$.

**Proof.** The assumption on $N_1^2$ implies that $K = x_1 J_m + x_2 I_m$, then
\[
\mathcal{M}_K(x) = (x - (x_1 m + x_2))(x - x_2),
\]
where $x_1 m + x_2 = k = |N_1||H| \neq 0$. Recall that $L = A(C(N \rtimes \theta H, N_1 \times H))$, then $\mathcal{M}_L(x) = x(x - k)(x - x_2)$ by Lemma 5.7. Similar to the proof of Case 2 of Theorem 5.8 we can obtain that $L(L - x_2 I) = \frac{k(k - x_2)}{m} J$, then
\[
L^2 = \frac{k(k - x_2)}{md} J + x_2 L = \frac{x_1 (x_1 m + x_2)}{d} J + x_2 L
= \frac{x_1 (x_1 m + x_2)}{d} I + \left( x_2 + \frac{x_1 (x_1 m + x_2)}{d} \right) L + \frac{x_1 (x_1 m + x_2)}{d} (J - I - L).
\]
Thus Cayley graph $C(N \rtimes \theta H, N_1 \times H)$ is a DSRG with parameters
\[
\left( m_d x_1 m + x_2, \frac{x_1 (x_1 m + x_2)}{d}, x_2 + \frac{x_1 (x_1 m + x_2)}{d}, \frac{x_1 (x_1 m + x_2)}{d} \right).
\]

We now give some applications of Construction 5.9

**Corollary 5.11.** Let $N$ be a finite group of order $m$. Let $\beta \in \text{Aut}(N)$ have the $q$-orbit condition. Let $H$ be the cyclic group of order $q$ with generator $b$, and define $\theta : H \to \text{Aut}(N)$ by $\theta(b^n) = \beta^n$. Let $O_1, O_2, \cdots, O_s$ be all the distinct nontrivial $H$-orbits of $N$.

Let $r$ be an integer such that $r < q$, and let $\widetilde{O}_i$ be a subset of $O_i$ with $|\widetilde{O}_i| = r$ for $1 \leq i \leq s$. Let $N_1 = \bigcup_{i=1}^s \widetilde{O}_i$, then the Cayley graph $C(N \rtimes \theta H, N_1 \times H)$ is a DSRG with parameters
\[
\left( m q, (m - 1)r, \frac{(m - 1)r^2}{q}, \frac{(m - 1)r^2}{q}, r, \frac{(m - 1)r^2}{q} \right).
\]

**Proof.** We note that
\[
\overline{N}_1^2 = \bigcup_{i=1}^s \overline{\widetilde{O}_i} = \bigcup_{i=1}^s r \oplus \overline{\widetilde{O}_i} = r \oplus (N \setminus \{e_N\}),
\]
so $\overline{N}_1 = rN - re_N$. From Construction 5.9 with $x_1 = r, x_2 = -r, C(N \rtimes \theta H, N_1 \times H)$ is a DSRG with parameters
\[
\left( m q, (m - 1)r, \frac{(m - 1)r^2}{q}, \frac{(m - 1)r^2}{q}, r, \frac{(m - 1)r^2}{q} \right).
\]
Remark 5.12. For \( r = 1 \), the parameters become

\[
\left( m q, m - 1, \frac{m - 1}{q}, \frac{m - 1}{q} - 1, \frac{m - 1}{q} \right),
\]

this is the semidirect product constructions of Art M. Duval and Dmitri Iourinski in [5].

The following corollary shows that the \( q \)-obit condition is not necessary.

**Corollary 5.13.** Let \( G = N \rtimes_H H \) be the semidirect product of \( N \) and \( H \) with respect to the homomorphism \( \theta \). Let \( \mathcal{O}_1 = \{e_N\}, \mathcal{O}_2, \ldots, \mathcal{O}_s \) be all the distinct \( H \)-orbits of \( N \), and \( d_i = |\mathcal{O}_i| \) for \( 1 \leq i \leq s \). Suppose \( d_1 \leq d_2 \leq \cdots \leq d_s \) and \( 1 < d_i | d_i \) for any \( 2 \leq i \leq s \).

Let \( r \) be an integer such that \( r < d_2 \), and let \( \mathcal{O}_i \) be a subset of \( \mathcal{O}_i \) with \( |\mathcal{O}_i| = \frac{d_i}{d_2} \) for \( 2 \leq i \leq s \). Let \( N_1 = \bigcup_{i=2}^{s} \mathcal{O}_i \). Then the Cayley graph \( C(N \rtimes_H H, N_1 \times H) \) is a DSRG with parameters

\[
\left( |G|, \frac{r |H| (|N| - 1)}{d_2}, \frac{r^2 |H| (|N| - 1)}{d_2^2}, \frac{r^2 |H| (|N| - 1)}{d_2^2} - \frac{r |H|}{d_2}, \frac{r^2 |H| (|N| - 1)}{d_2^2} \right).
\]

**Proof.** Note that

\[
N_i^2 = \bigcup_{i=2}^{s} \mathcal{O}_i^2 = \bigcup_{i=2}^{s} \frac{|H| |d_i|}{d_2} \mathcal{O}_i = \frac{r |H|}{d_2} \bigcup_{i=2}^{s} \mathcal{O}_i = \frac{r |H|}{d_2} \bigcup_{i=2}^{s} \mathcal{O}_i = \left( \bigcup_{i=2}^{s} \mathcal{O}_i \right) \mathcal{O}_1 = \frac{r |H|}{d_2} (N \setminus \{e_N\}).
\]

Hence \( N_i^2 = \frac{r |H|}{d_2} (N - e_N) \). Then this result follows from Construction 5.9 with \( x_1 = \frac{r |H|}{d_2}, x_2 = -\frac{r |H|}{d_2} \). \( \square \)

There are some groups \( N, H \) and homomorphism \( \theta \) satisfy the conditions of Corollary 5.13.

**Example 5.14.** If \( \beta_i \in \text{Aut}(N_i) \) has the \( q_i \)-orbit condition for \( i = 1, \cdots, l \), and \( q_i | q_{i+1} \) for each \( i = 1, \cdots, l - 1 \), then we define

\[
\beta = \prod_{j=1}^{l} \beta_i \in \text{Aut} \left( \prod_{j=1}^{l} N_j \right).
\]

Let \( H \) be the cyclic group of order \( q_i \) with generator \( b \), and define \( \theta : H \to \text{Aut} \left( \prod_{j=1}^{l} N_j \right) \) by \( \theta (b^n) = \beta^n \). Assume \( (n_1, n_2, \cdots, n_l) \neq e \) is a representative of an untrivial \( H \)-orbit \( \mathcal{O} \). Let \( |\mathcal{O}| = r \). There is a largest \( j \) such that \( n_j \neq e_{N_j} \). Therefore \( \beta^{q_i} (n_1, n_2, \cdots, n_l) = e \) and \( r | q_j \). On the other hand, note that \( \beta_j^r (n_j) = e_{N_j} \), it follows that \( q_j | r \). Therefore \( r = q_j \). This gives that the length of each untrivial \( H \)-orbit belongs to the set \( \{ q_1, q_2, \cdots, q_l \} \). Observe that there exist \( H \)-orbits of size \( q_1 \) and \( q_i \), then from Corollary 5.13 we can construct DSRGs with parameters

\[
\left( mq_1, \frac{r q_i (m - 1)}{q_1}, \frac{r^2 q_i (m - 1)}{q_1^2}, \frac{r^2 q_i (m - 1)}{q_1^2} - \frac{r q_i}{q_1}, \frac{r^2 q_i (m - 1)}{q_1^2} \right)
\]

for any \( r < q_i \), where \( m = \prod_{j=1}^{l} |N_j| \). We give a simple example.
6. Directed strongly regular Cayley graphs on dihedral groups

6.1. Preliminary

We recall $C_n = \langle x \rangle$ is a cyclic multiplicative group of order $n$. Dihedral groups $D_n$ is a semidirect product of two cyclic groups $C_n = \langle x \rangle$ of order $n$ and $C_2 = \langle a \rangle$ of order $2$. Any subset
$S$ of $D_n$ can be written by the form $S = X \cup Y a$ with $X, Y \subseteq C_n$.

We now give a characterization of the Cayley graph $C(D_n, X \cup Y a)$ to be directed strongly regular.

**Lemma 6.1.** The Cayley graph $C(D_n, X \cup Y a)$ is a DSRG with parameters $(2n, |X| + |Y|, \mu, \lambda, t)$ if and only if $X$ and $Y$ satisfy the following conditions:

(i) $(\overline{X + X^{-1}})\overline{Y} = (\lambda - \mu)\overline{Y} + \mu \overline{C_n}$; 

(ii) $X^2 + \overline{Y} Y^{-1} = (t - \mu)e + (\lambda - \mu)\overline{X} + \mu \overline{C_n}$.

**Proof.** Note that 

$$(X + Ya)^2 = X^2 + Xa + Ya X + Ya a = X^2 + \overline{Y} Y^{-1} + (X Y + Y^{-1} a)$$

Thus, from Lemma 2.4, the Cayley graph $C(D_n, X \cup Y a)$ is a DSRG with parameters $(2n, |X| + |Y|, \mu, \lambda, t)$ if and only if

$$X^2 + \overline{Y} Y^{-1} + (X + X^{-1} a) = t e + \lambda (X + Y a) + \mu (D_n - (X + Y a) - e)$$

$$= (t - \mu) e + (\lambda - \mu) \overline{X} + \mu \overline{C_n}$$

This is equivalent to conditions (6.1) and (6.2).

When $Y = X$, we have the following:

**Lemma 6.2.** The Cayley graph $C(D_n, X \cup X a)$ is a DSRG with parameters $(2n, 2|X|, \mu, \lambda, t)$ if and only if $t = \mu$ and

$$(X + X^{-1} X) = (\lambda - \mu)\overline{X} + \mu \overline{C_n}.$$  

(6.3)

**Remark 6.3.** (6.3) also implies that

$$(X + X^{-1} X)^{(-1)} = (\lambda - \mu) \overline{X^{-1}} + \mu \overline{C_n}.$$  

Therefore, the sum of the above equation and (6.3) gives that

$$(X + X^{-1})^2 = (\lambda - \mu) (X + X^{-1}) + 2 \mu \overline{C_n}.$$  

(6.4)

6.2. Some constructions of directed strongly regular Cayley graphs on dihedral groups

In the following constructions, $v$ is a positive divisor of $n$ and $l = n/v$.

**Construction 6.4.** Let $v$ be an odd positive divisor of $n$. Let $T$ be a subset of $\{x^1, \cdots, x^{v-1}\}$, and $X$ be a subset of $C_n$ satisfy the following conditions:

(i) $X = T \langle x^v \rangle$.

(ii) $X \cup X^{-1} = C_n \setminus \langle x^v \rangle$.

Then the Cayley graph $C(D_n, X \cup X a)$ is a DSRG with parameters $(2n, n-l, \frac{n-l}{2}, \frac{n-l}{2} - l, \frac{n-l}{2})$. 

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Proof. Note that \(X + X^{(-1)} = -\bar{X} + \frac{n+1}{2}C_n\). The result follows from Lemma 6.2 directly.

\[\square\]

Remark 6.5. The directed strongly regular Cayley graphs \(C(D_n, X \cup aX)\) constructed above satisfy \(X \cap X^{(-1)} = \emptyset\).

Construction 6.6. Let \(v > 2\) be an even positive divisor of \(n\). Let \(T\) be a subset of \(\{x^1, \ldots, x^{v-1}\}\), and \(X\) be a subset of \(C_n\) satisfy the following conditions:

\(i\) \(X = T(x^v)\).

\(ii\) \(X \cup X^{(-1)} = (C_n \setminus \langle x^v \rangle) \cup \langle x^\bar{v}(x^v) \rangle\).

\(iii\) \(X \cup (x^\bar{v}X) = C_n\).

Then the Cayley graph \(C(D_n, X \cup Xa)\) is a DSRG with parameters \((2n, n, \frac{n}{2} + l, \frac{n}{2} - l, \frac{n}{2} + l)\).

Proof. Note that \(|X| = \frac{n}{2}\) and \(X + X^{(-1)} = \bar{C}_n - \langle x^v \rangle + x^\bar{v} \langle x^v \rangle\). Thus \(X(\bar{X} + X^{(-1)}) = -l\bar{X} + \frac{n+1}{2}C_n + l\bar{v}X = -l\bar{X} + \frac{n+1}{2}C_n + l\bar{v}C_n - l\bar{X} = (\frac{n}{2} + l)\bar{C}_n - 2l\bar{X}\). The result follows from Lemma 6.2 directly.

\[\square\]

Remark 6.7. The directed strongly regular Cayley graphs \(C(D_n, X \cup aX)\) constructed above satisfy \(X \cap X^{(-1)} \neq \emptyset\).

Construction 6.8. Let \(v\) be an odd positive divisor of \(n\). Let \(T\) be a subset of \(\{e, x^1, \ldots, x^{v-1}\}\) with \(e \in T\), and let \(X, Y \subseteq C_n\) satisfy the following conditions:

\(i\) \(Y = T(x^v) = X \cup \langle x^v \rangle\).

\(ii\) \(Y \cup Y^{(-1)} = C_n \cup \langle x^v \rangle\).

Then the Cayley graph \(C(D_n, X \cup Ya)\) is a DSRG with parameters \((2n, n, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n+1}{2})\).

Proof. We have \(|Y| = |X| + l = \frac{n+1}{2}, \bar{X} + \bar{X}^{(-1)} = \bar{C}_n - \langle x^v \rangle\) and \(\bar{Y} (\bar{X} + \bar{X}^{(-1)}) = -l\bar{Y} + \frac{n+1}{2}C_n\). Meanwhile, \(\bar{X}^2 + \bar{Y} \bar{Y}^{(-1)} = \bar{X} (\bar{X} + \bar{X}^{(-1)}) + \langle x^v \rangle (\bar{X} + \bar{X}^{(-1)}) + l\langle x^v \rangle = -l\bar{X} + \frac{n+1}{2}C_n\). The result follows from Lemma 6.1 directly.

\[\square\]

Remark 6.9. It is known that the automorphism group of dihedral group \(D_n\) is

\[\text{Aut}(D_n) = \{\gamma_{s,s'} \mid 0 \leq s, s' \leq n - 1, (s, n) = 1\},\]

where \(\gamma_{s,s'}\) is defined by \(\gamma_{s,s'}(x) = x^s, \gamma_{s,s'}(a) = x^{s'}a\).

Let \(\alpha \in \text{Aut}(D_n)\), then it is easy to see that \(\alpha\) is a graph isomorphism from \(C(D_n, X \cup Ya)\) to \(C(D_n, \alpha(X) \cup \alpha(Ya))\). Furthermore, this asserts that the Cayley graphs \(C(D_n, X \cup Ya)\) and \(C(D_n, X^{(s)} \cup Y^{(s)}x^{s'}a)\) are isomorphic for any \(0 \leq s, s' \leq n - 1\) and \((s, n) = 1\).
6.3. The characterization of directed strongly regular Cayley graphs $C(D_n, X \cup Xa)$ with $X \cap X^{(-1)} = \emptyset$

Let $\zeta_n$ be a fixed primitive $n$-th root of unity, then $\text{Irr}(C_n) = \{ \chi_j | 0 \leq j \leq n-1 \}$, where $\chi_j$ is defined by $\chi_j(x^i) = \zeta_n^{ij}$ for $0 \leq i, j \leq n-1$. The character $\chi_0$ is called the principal character of $C_n$ and the characters $\chi_1, \chi_2, \ldots, \chi_{n-1}$ are called the nonprincipal characters of $C_n$.

Let $X$ be an antisymmetric subset of $C_n$, i.e., $X \cap X^{(-1)} = \emptyset$. We now give the characterization of the directed strongly regular Cayley graphs $C(D_n, X \cup Y \alpha)$ with $X \cap X^{(-1)} = \emptyset$.

**Theorem 6.10.** A Cayley graph $C(D_n, X \cup Xa)$ with $X \cap X^{(-1)} = \emptyset$ is a DSRG with parameters $(2n, 2|X|, \mu, \lambda, t)$ if and only if there exists a subset $T$ of $\{ x^1, \ldots, x^{v-1} \}$ satisfies the following conditions:

(i) $X = T\langle x^v \rangle$;
(ii) $X \cup X^{(-1)} = C_n \setminus \langle x^v \rangle$, where $v = \frac{n}{n-1}$.

**Proof.** Let $U = X \cup X^{(-1)}$, then $U$ is a subset of $C_n$ and $\Delta_U(g) \in \{ 0, 1 \}$ for any $g \in U$. Hence $U = \overline{X} + \overline{X^{(-1)}}$. It follows from Construction 6.4 that a Cayley graph $C(D_n, X \cup Xa)$ which satisfies the conditions (i) and (ii) is a DSRG with $X \cap X^{(-1)} = \emptyset$.

Conversely, suppose that $C(D_n, X \cup Xa)$ is a DSRG with parameters $(2n, 2|X|, \mu, \lambda, t)$. It follows from Lemma 6.2 and 6.3 that

$$\overline{U}^2 = (\lambda - \mu)\overline{U} + 2\mu\overline{C_n}. \tag{6.5}$$

Therefore, $\chi(\overline{U}) \in \{ 0, \lambda - \mu \}$ for any nonprincipal characters $\chi \in \text{Irr}(C_n)$. Define the set

$$U = \{ j : 1 \leq j \leq n-1, \, \chi_j(\overline{U}) = \lambda - \mu \}. \tag{6.6}$$

Then from the inversion formula (2.6), we have

$$\Delta_U(g) = \frac{1}{n} \sum_{\chi \in \text{Irr}(C_n)} \chi(\overline{U})\overline{\chi(g)} = \frac{\lambda - \mu}{n} \sum_{j \in U} \overline{\chi_j(g)} + \frac{2|X|}{n}. \tag{6.6}$$

Note that $e \not\in U$, hence $\Delta_U(e) = 0$ and then

$$\Delta_U(e) = \frac{\lambda - \mu}{n} |U| + \frac{2|X|}{n} = 0.$$ 

This gives that $2|X| = (\mu - \lambda)|U|$. Then (6.6) becomes

$$\Delta_U(g) = \frac{\mu - \lambda}{n} \left( |U| - \sum_{j \in U} \overline{\chi_j(g)} \right). \tag{6.7}$$

We assert that $\frac{n}{n-\lambda}$ is an integer. Indeed, selecting some $g \in U$, the above equation shows that $|U| - \sum_{j \in U} \overline{\chi_j(g)} = \frac{n}{n-\lambda} \in \mathbb{Q}$. Note that $|U| - \sum_{j \in U} \overline{\chi_j(g)}$ also lies in the ring $\mathbb{Z}[\zeta_n]$, which
is the ring of integers of the $n$-th cyclotomic field $\mathbb{Q}(\zeta_n)$. Therefore

$$\frac{n}{\mu - \lambda} \in \mathbb{Q} \cap \mathbb{Z}[\zeta_n] = \mathbb{Z}.$$  

The equation (6.7) also implies that

$$\Delta_U(g) = 0 \iff |U| - \sum_{j \in U} \chi_j(g) = 0 \iff \chi_j(g) = 1 \text{ for } j \in U \iff g \in \bigcap_{j \in U} K_{\chi_j} \overset{\text{def}}{=} R,$$

where $R$ is some subgroup of $C_n$. This shows that $U = C_n - R$. Note that

$$U^2 = (C_n - R)^2 = (n - 2|R|)C_n^\perp + |R|R = (n - |R|)C_n^\perp - |R|U,$$

so (6.8) gives that $|R| = \mu - \lambda$, $n - |R| = 2\mu$ and $|X| = \frac{n - |R|}{2} = \mu$. Then $R = \langle x^{\frac{n}{2\mu}} \rangle = \langle x^{\mu} \rangle$, proving (ii). In this case, from Lemma 6.2, (6.3) becomes

$$(\lambda - \mu)X + \mu C_n^\perp = XU = X(C_n - \langle x^{\mu} \rangle) = \mu C_n - \overline{X} \langle x^{\mu} \rangle,$$

i.e., $(\mu - \lambda)X = \overline{X} \langle x^{\mu} \rangle$. This asserts that $X$ is a union of some cosets of $\langle x^{\mu} \rangle$ in $C_n$, therefore $X = T\langle x^{\mu} \rangle$ for some subset $T$ of $\{x^1, \cdots, x^{v-1}\}$, proving (i). The result follows. \qed

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