Universal quantum criticality at finite temperature for two-dimensional disordered and clean dimerized spin-1/2 antiferromagnets

D.-R. Tan and F.-J. Jiang

Department of Physics, National Taiwan Normal University, 88, Sec.4, Ting-Chou Rd., Taipei 116, Taiwan

The quantum critical regime (QCR) of a two-dimensional (2D) disordered and a 2D clean dimerized spin-1/2 Heisenberg models are studied using the first principles nonperturbative quantum Monte Carlo simulations (QMC). In particular, the three well-known universal coefficients associated with QCR are investigated in detail. While in our investigation we find the obtained results are consistent with the related analytic predictions, non-negligible finite temperature (T) effects are observed. Moreover, a striking finding in our study is that the numerical value for one of the universal coefficients we determine is likely to be different significantly from the corresponding (theoretical) result(s) established in the literature. To better understand the sources for the discrepancy observed here, apart from carrying out the associated analytic calculations not considered previously, it will be desirable as well to conduct a comprehensive examination of the exotic features of QCR for other disordered and clean spin systems than those investigated in this study.

PACS numbers:

I. INTRODUCTION

The two-dimensional (2D) quantum antiferromagnets, both with and without charge carriers, are among the most important systems in condensed matter physics. From the experimental perspective, these materials are related to the high temperature (high $T_c$) cuprate superconductors. As a result, numerous associated experiments were conducted and the obtained data have triggered many theoretical studies of these systems, including accurate determination of their low-temperature properties such as the staggered magnetization density and the spin stiffness.

Theoretically, at zero temperature and in the ordered phase, the 2D spin-1/2 Heisenberg antiferromagnet can be treated classically and this region is known as the renormalized classical regime in the literature. When the long-range order of the system is destroyed by the quantum fluctuations, a completely different portrait of the system is destroyed by the quantum fluctuations, a completely different portrait of the system is destroyed by the quantum fluctuations, and several exotic features will emerge in QCR. These special features of QCR is signaled out by the presence of several universal behavior among some physical quantities of the underlying 2D spin system.

Based on the analytic calculations using the method of large-N expansion for the effective nonlinear sigma model of the 2D Heisenberg antiferromagnet, three universal relations are established (assuming the dynamic exponent $z$ is 1): $\chi_u = \frac{\Omega}{c^2} T$, $S(\pi, \pi)/\chi_s = \Xi T$, and $c/\xi = XT$. Here $\chi_u$, $c$, $S(\pi, \pi)$, $\chi_s$, and $\xi$ are the uniform susceptibility, the spinwave velocity, the staggered structure factor, the staggered susceptibility, and the correlation length, respectively. Moreover, the coefficients $\Omega$, $\Xi$, and $X$ are universal, namely their numerical values are independent of any microscopic details. For 2D dimerized Heisenberg models with spatial anisotropy, QCR as well as the related universal coefficients should be detectable at any values of the corresponding tuning parameter. It is probable as well that systems with (certain kinds of) quenched disorder may exhibit features of QCR.

Interestingly, while universal behavior characterizing QCR is indeed observed for several numerical studies of 2D dimerized spin models, crystalline clear evidence only found at the finite temperature regions above the related quantum critical points (QCPs). In other words, when the associated calculations are conducted away from QCPs, the emergence of the exotic QCR scaling has not been established rigorously and numerically yet.

For example, as introduced in the previous paragraph regarding QCR, a plateau is supported to appear in a certain region of the inverse temperature $\beta$ when $S(\pi, \pi)/\chi_u T$ is treated as a function of $\beta$. However, such a scenario does not occur in the relevant studies when the used data were determined away from the corresponding QCPs.

At the moment, numerical studies related to QCR have been focusing on clean dimerized spin systems. The exploration of whether features of QCR, in particular the three universal quantities mentioned previously, exist for models with quenched disorder have been examined only implicitly, not systematically. Motivated by this, here we simulate a 2D spin-1/2 Heisenberg model on the square lattice with a kind of (quenched) disorder using the quantum Monte Carlo (QMC) calculations. The employed disorder distribution is based on the so-called configurational disorder introduced in Ref. [22]. Apart from this, the 2D clean dimerized plaquette quantum spin system is investigated as well for comparison and clarification.

Remarkably, for both the considered disordered and
clean models, features of QCR do emerge at their corresponding QCPs. Furthermore, non-negligible $T$ dependence for the universal quantities of QCR, which was overlooked before, is found in our investigation. The most striking result obtained here is that, the numerical values of the universal coefficient $\Omega$ determined in our study for both the considered models are likely to deviate significantly from those calculated previously in Refs. [12, 13, 14, 21]. The evidence provided here for the described variation regarding the numerical value of $\Omega$ is convincing. This finding of ours is consistent with that obtained in Ref. [23] for which the conclusion is based on a detailed study of a 2D clean bilayer quantum spin system. To better understand the sources of the discrepancy found here, apart from carrying out the analytic calculations associated with corrections not taken into account previously, a more thorough exploration of other disordered and clean spin models than those studied here is desirable.

The rest of the paper is organized as follows. After the introduction, the considered models as well as the required physical quantities for investigating the features of QCR are described. A detailed analysis, focusing on the three well-known universal coefficients of QCR, is presented then. In particular, the numerical evidence for the discrepancies associated with corrections not taken into account previously is given. Finally, a section is devoted to conclude our study shown here.

II. MICROSCOPIC MODEL AND OBSERVABLES

The 2D spin-$\frac{1}{2}$ system with a quenched disorder and the 2D clean quantum dimerized plaquette Heisenberg model studied here are given by the same form of Hamilton operator

$$H = \sum_{(ij)} J_{ij} \vec{S}_i \cdot \vec{S}_j + \sum_{(i'j')} J'_{i'j'} \vec{S}_i \cdot \vec{S}_j,$$

(1)

$$\sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j,$$

(2)

where $J_{ij}$ and $J'_{i'j'}$ are the antiferromagnetic couplings (bonds) connecting nearest neighbor spins $(ij)$ and $(i'j')$, respectively, and $\vec{S}_i$ is the spin-$\frac{1}{2}$ operator at site $i$. The quenched disorder considered in this investigation is based on the idea of configurational disorder [22]. Specifically, in our simulations for the disordered model, the probabilities of putting a pair of $J'$-bonds vertically and horizontally in a plaquette consisting of two by two spins are both 0.5. (We will still use the term configurational disorder for this employed disorder distribution in the rest of the paper). Figure 1 demonstrates the studied disordered and clean models. Here we use the convention that the couplings $J'$ and $J$ satisfy $J' > J$. As a result, each of the considered system will undergo a quantum phase transition once the ratio $J'/J$ exceeds a particular value called the critical point. These special points in the associated parameter spaces are commonly denoted by $(J'/J)_{c}$ in the literature.

To examine the well-known universal relations of QCR, particularly to understand whether these relations appear for the considered disordered system, the staggered structure factor $S(\pi, \pi, L)$, the uniform and staggered susceptibilities ($\chi_u$ and $\chi_s$), the spinwave velocity $c$, as well as the correlation length $\xi$ are measured. The staggered structure factor $S(\pi, \pi, L)$ on a finite lattice with a linear box size $L$ is defined by

$$S(\pi, \pi, L) = 3L^2 \langle (m^z_s)^2 \rangle,$$

(3)

where $m^z_s = \frac{1}{L^2} \sum_i (-1)^{i_x+i_y} S^z_i$ and the summation is over all sites. The uniform susceptibility $\chi_u$ and staggered susceptibility $\chi_s$ take the forms

$$\chi_u = \frac{\beta}{L^2} \left\langle \left( \sum_i S_i \right)^2 \right\rangle,$$

(4)

and

$$\chi_s = 3L^2 \int_0^\beta \langle (m^z_s(\tau)m^z_s(0)) \rangle d\tau,$$

(5)

respectively. The quantity $\beta$ appearing above is the inverse temperature. In addition, the correlation length $\xi$
is expressed as
\[
\xi = \frac{L_1}{4\pi} \sqrt{\frac{S(\pi, \pi)}{S(\pi + 2\pi/L_1, \pi)}} - 1 + \frac{L_2}{4\pi} \sqrt{\frac{S(\pi, \pi)}{S(\pi, \pi + 2\pi/L_2)}} - 1,
\]
where the quantities \(S(\pi + 2\pi/L_1, \pi)\) and \(S(\pi, \pi + 2\pi/L_2)\) are the Fourier modes with the second largest magnitude. Finally the spinwave velocities \(c\) for both the investigated models are calculated through the temporal and spatial winding numbers squared \(\langle W_i^2 \rangle\) and \(\langle W_i^2 \rangle\) with \(i \in \{1, 2\}\).

We would also like to point out that while the same notations are used here for both the observables of the two studied spin systems, whenever a physical quantity associated with several additional calculations using \((\langle m_i^2 \rangle)\) is some constant. Moreover, \(Q_2\) is the second Binder ratio which is defined by \(Q_2 = \langle (\langle m_i^2 \rangle)^4 \rangle / \langle (\langle m_i^2 \rangle)^2 \rangle^2\). Interestingly, the zero temperature data we calculate with the \(\beta\)-doubling scheme [23] are fully consistent with the outcomes reached in Ref. [22]. Indeed, using \(a = -0.5\), \(\omega = 1.0\), \(j_c = 1.990\) (these three results are taken directly from Ref. [22]), \(\nu = 0.7115\) (the established value for this exponent), as well as the data obtained here, we have reproduced the associated universal curve of \(Q_2\) just like the one shown in Ref. [22]. With a fixed \(\nu = 0.7115\), we additionally fit the observables \(Q_2\) and \(\rho_s L [24]\) to their expected scaling formulas near QCP. The \((J'/J)_{c}\) determined from these fits associated with \(Q_2\) and \(\rho_s L\) are both given by \((J'/J)_{c} = 1.990(1)\). This strongly suggests that taking 1.990(1) as the QCP for the investigated disordered model should be beyond any doubt.

The fact that \((J'/J)_{c}\) resulting from \(Q_2\) and \(\rho_s L\) agree quantitatively with each other indicates that the dynamic exponent \(z\) related to the considered disordered system is 1. We will demonstrate shortly that this is truly the case.

Finally, the QCP of the clean plaquette model has been calculated in Ref. [27] and is given by \((J'/J)_{c} = 1.8230(3)\).

### A. The determination of spinwave velocity \(c\)

The spinwave velocities \(c\) at the critical points for the studied models are calculated using the method of winding numbers squared proposed in Refs. [23] [25]. Specifically, for a fixed box size \(L\) and a fixed \((J'/J)\), the value of \(\beta\) is adjusted in the calculations so that the temporal winding number squared \(\langle W_i^2 \rangle\) agrees quantitatively with that of the averaged spatial winding numbers squared \(\langle W_j^2 \rangle = \frac{1}{2} \sum_{i=1,2} \langle W_i^2 \rangle\). Under such a condition, the corresponding spinwave velocity \(c(L, J'/J)\) is determined by the equation \(c(L, J'/J) = L/\beta^*\), where \(\beta^*\) is the inverse temperature for which the condition described above regarding the winding numbers squared is fulfilled. Since this method is valid only when the long-range antiferromagnetic order is present in the system, the relevant simulations are done at \(J'/J = 1.988\) for the disordered simulations are not overlooked. In addition, most (a few) of the results presented here are obtained on \(L = 256\) \((L > 256)\) lattices. For comparison, some outcomes determined with smaller box sizes are shown as well.
system (which has \((J'/J)_\infty = 1.990(1)\) \[22\]). For the clean plaquette model, calculations at several selected \(J'/J < (J'/J)_\infty = 1.8230(3)\), as well as fits and interpolations are conducted in order to obtain the bulk \(c\) at the associated critical point.

1. The spinwave velocity of the disordered system

The \(\langle W^2 \rangle\) and \(\langle W^2 \rangle\) as functions of \(\beta\) for the studied disordered system are shown in Fig. 3. The calculations are done at \(J'/J = 1.988\) and the outcomes presented in the top and bottom panels of the figure are obtained with \(L = 24\) and \(L = 48\), respectively. The corresponding values of \(c\) estimated conservatively from these two simulated results match each other nicely. Indeed, while the calculated result of \(c\) for \(L = 24\) is given by \(c = 1.934(5)J\), the \(c\) determined from the data of \(L = 48\) is found to be \(c = 1.931(9)J\). We have additionally performed simulations at \(J'/J = 1.986\) with \(L = 48\). The outcome of \(c\) from the simulations associated with \(J'/J = 1.986\) agrees remarkably well with that of \(J'/J = 1.988\). Therefore it should be accurate to use \(c = 1.931(9)J\) as the bulk value of \(c\) right at the critical point.

2. The spinwave velocity of the clean plaquette model

To determine the bulk \(c\) at the critical point of the 2D clean plaquette model, a more thorough calculation is performed. In particular we carry out simulations with various box sizes \(L\) at several selected \(J'/J \leq 1.8227\) close to the critical point \((J'/J)_\infty = 1.8227(3)\). The obtained results are shown in Fig. 4. The bulk \(c(J'/J)\) of each considered \(J'/J\) is determined by applying the following two ansätze

\[
a_0 + a_1/L^2, \quad b_0 + b_1/L^2 + b_2/L^3
\]

(7) (8)

to fit the corresponding data. This strategy of calculating the bulk values of \(c\) is inspired by the results demonstrated in Ref. \[23\]. The uncertainty for the bulk \(c\) of every used \(J'/J\) is the standard deviation deriving from considering Gaussian noises in the associated weighted \(\chi^2\)-squared fits. With the outcomes from the fits employing ansatz (8), the \(c\) corresponding to \((J'/J)_\infty\) is estimated by interpolation based on a linear fit of the form \(a(J'/J) + b\). With such a procedure, the spinwave velocity \(c\) at the critical point is found to be \(c = 2.163(4)J\). Here the quoted uncertainty is not determined directly from the interpolation, but is calculated with conservation assuming that for \((J'/J)_\infty\) a similar statistic as those of the \(J'/J\) shown in Fig. 4 is reached.

B. The universal coefficient Ω

Theoretically the universal coefficient \(\Omega\) is given by \(\chi_u = (\Omega/c^2)^{2d/z-1}\) at the critical point, where \(d\) is dimensionality of the system (which is 2 here) and \(z\) is the associated dynamic exponent. The \(z\) associated with
With these two formulas, the numerical values of Ω are discussed, if no confusion arises. Whenever the results from the fits employing the second ansatz are excluded in the associated fits using the first ansatz, the observed discrepancy should be treated as a result of not taking some corrections into account in the analysis. Indeed, to the best of our knowledge, we are not aware of other formulas besides those employed here for the fits. As we will demonstrate later, such a scenario for z and Ω occurs for the clean plaquette model as well.

It is intriguing to notice that when the first ansatz is considered, as the magnitude of the determined z increases (This occurs when more and more data calculated at high temperatures are excluded in the fits), the value of b obtained comes toward 0.27185. In other words, z and Ω are correlated. Since the difference between the z found here and that estimated in Ref. [22] is only at few percent level, it is unlikely that such deviations are due to the fact that the z calculated here for the studied disordered model is a new one other than that found in [22]. Instead, the observed discrepancy should be treated as a result of not taking some corrections into account in the analysis. Indeed, to the best of our knowledge, we are not aware of other formulas besides those employed here for the fits. As we will demonstrate later, such a scenario for z and Ω occurs for the clean plaquette model as well.

Since z = 1 is beyond doubt for the considered disordered system, to accurately estimate the numerical value of Ω, particularly to understand its dependence on (finite) temperature, it is helpful to investigate the quantity \( \chi_u c^2 / T \) as a function of the inverse temperature \( \beta \). Such a study is inspired by the fact that a flat plateau should appear if the data of \( \chi_u c^2 / T \) are plotted against \( \beta \) since \( z = 1 \). Remarkably, a very flat plateau indeed emerges when \( \chi_u c^2 / T \) is treated as a function of \( \beta \), see Fig. 8. While it is clear that the quantity \( \chi_u c^2 / T \) receives mild corrections from terms taking some forms in \( T \), the quality of flatness shown in Fig. 8 strongly indicates that the value of the universal coefficient \( \Omega \) is larger than 0.27185 (which is the horizontal line in Fig. 8). The \( L = 120 \) data of \( \chi_u c^2 / T \) obtained at \( J' / J \) = 1.989 and 1.991 are demonstrated in Fig. 8 as well. The quantitative agreement between the \( \chi_u c^2 / T \) data of \( L = 120 \) and \( L = 256 \) rules out the possibility that the deviations of \( \Omega \) and \( z \) from their expected values are due to finite-size effects.

Aside from the results associated with \( (J'/J)_c \), The \( \chi_u c^2 / T \) as functions of \( \beta \) for both \( J'/J = 1.989 \) and 1.991 are shown in Fig. 9. As can be seen from the figure, flat plateaus well above 0.27185 show up as well. The results presented in Fig. 9 exclude the scenario that the observed discrepancy is due to the uncertainty of the critical point.

2. The results of clean system

While it is well-established that \( z = 1 \) for the considered 2D plaquette model, it will be useful to conduct a calculation like that done in the previous subsection to determine the dynamic exponent \( z \) associated with the studied clean system. Interestingly, a scenario like...
the one of the investigated disordered model is observed. Specifically, the values of $z$ obtained here are slightly above the theoretical prediction $z = 1$, see Fig. 11). Just like what has been argued previously, since the deviations found are only at few percent level, these deviations should be treated as consequences resulting from (minor) corrections not taken into account in the analysis.

Similar to the analysis done in the previous subsection, we have also investigated the size-convergence quantity $\chi_u c^2/T$ as a function of $\beta$ for the 2D clean dimerized plaquette model [24]. The considered data are determined at the expected critical point $(J'/J)_c = 1.8230$, as well as at $J'/J = 1.8228, 1.8232$ in order to take into account the effects from the uncertainties of $(J'/J)_c$. The resulting outcomes are depicted in Figs. 11 and 12.

Interestingly, while moderate $T$-dependence for $\chi_u c^2/T$ definitely appears, as shown in the figures, one sees clearly that flat plateaus emerge as well. Furthermore, by comparing the results presented in Figs. 8, 9, 11 and 12 the values of $\chi_u c^2/T^2$ for which all the plateaus take place match each other very well and are statistically above 0.27185.

In summary, the outcomes obtained here that $\Omega$ is quantitatively different from its theoretical prediction 0.27185 is convincing. In particular, based on the results of fits with a fixed $z = 1$ (in the formula (9)), the numerical value of $\Omega$ we estimate conservatively is about 0.306(10). A (slightly) larger number is reached if one uses the outcomes calculated by considering only the lower temperature data for the fits. To arrive at a more accurate determination of $\Omega$ requires better understanding of its analytic expression. This is beyond the scope of our study presented here.

For the analysis done in the following (sub)sections, the assumption $z = 1$ will be employed.

![Graph](image)

**FIG. 5:** $\chi_u c^2$ as a function of $T$ for the considered disordered model. The data are obtained at the critical point $(J'/J)_c = 1.990$ with $L = 256$.

![Graph](image)

**FIG. 6:** The results of $z$ (top panel) and $b$ (bottom panel) for the considered disordered system. These outcomes are obtained from the fits using the ansatz $a + b T^{2/z - 1}$. The horizontal $(x)$ axes stand for the minimum values of $\beta$ used in the fits. The solid lines in both panels are the corresponding theoretical predictions.

### C. The universal coefficient $\Xi$

Theoretically, a calculation with $z = 1$ for the $O(N)$ nonlinear sigma model using the large-$N$ expansion predicts that up to the order of $1/N$, the quantity $\Xi$, which is defined as $S(\pi, \pi)/(\chi_s T)$, is a universal number given by 1.09 for $N = 3$ (which is the case here). The observables $S(\pi, \pi)/(\chi_s T)$ as functions of $\beta$ for the considered models are shown in Fig. 13. In both panels of Fig. 13 the solid lines represent the theoretical value 1.09. In addition, an uncertainty of few percent (of 1.09, dashed lines) is included in both panels as well. The results shown in Fig. 13 imply that although non-negligible $T$-dependence for $S(\pi, \pi)/(\chi_s T)$ does appear for these models, the Monte Carlo data agree very well with the associated theoretical predictions.

Most of the data shown in the bottom panel of Fig. 13 which are associated with the clean plaquette model, are determined from the results obtained on $L = 256$ lattices. For this model, we have performed simulations with $L = 256$ and $L = 512$ for the largest value of $\beta$ considered ($\beta = 20$). The agreement between the results
FIG. 7: The results of $z$ (top panel) and $b$ (Ω, bottom panel) for the considered disordered system. These outcomes are obtained from the fits using the ansatz $bt^2/\xi^3$. The horizontal (x) axes stand for the minimum values of $\beta$ used in the fits. The solid lines in both panels are the corresponding theoretical predictions.

FIG. 8: $\chi_\ast c^2/T$ as functions of $\beta$ for the studied disordered model. The data are calculated at $J'/J = 1.989$ and $L = 256$. The horizontal solid line is the theoretical prediction $\sim 0.27185$.

FIG. 9: $\chi_\ast c^2/T$ as functions of $\beta$ for the studied disordered model. The data are calculated at $J'/J = 1.990$ and $1.989$ with $L = 256$. The horizontal solid line is the theoretical prediction $\sim 0.27185$.

D. The universal coefficient $X$

The final universal coefficient studied here is associated with $c/(T\xi)$ and is predicted to be 1.04 in theory. For the investigated disordered system, the associated $L = 120$ and $L = 256$ data of $c/(T\xi)$ as functions of $\beta$ are presented in Fig. 15. In the figure besides the data of $c/(T\xi)$, the related theoretical value and few percent error for it are also shown as the solid line and dashed lines, respectively. Similar to the scenario found in our analysis of $S(\pi, \pi)/(\chi_s T)$, a noticeable dependence on $T$ for the
IV. DISCUSSIONS AND CONCLUSIONS

Using the first principles nonperturbative QMC simulations, we have investigated the exotic characteristics of QCR related to both a 2D spin system with configurational disorder and a 2D clean dimerized spin-$\frac{1}{2}$ Heisenberg model. These unique properties of the considered models result from the interplay of the thermal and the quantum fluctuations. We firstly reconfirm that the dynamic exponent $z$ for the disordered model studied here is 1. With this result, as well as the fact that $z = 1$ for the clean dimerized plaquette model, the three universal coefficients associated with QCR, namely $\Omega$, $\Xi$, and $X$ are calculated. We find our Monte Carlo data of both the disordered and the clean systems are consistent with the analytic results based on the large-$N$ calculations of the $O(N)$ nonlinear sigma model. It is interesting to notice that while quantum systems with certain kinds of quenched disorder, such as the configurational disorder employed in this study, violate the Harris criterion [22, 30–33], 2D disordered spin-$\frac{1}{2}$ models with bond dilution fulfill this principle [25, 34–37]. The results presented here seem to imply the scenario that disordered systems which violate the Harris criterion conform the theoretical predictions of QCR. It will be compelling to investigate whether for models satisfy the Harris crite-
FIG. 13: $S(\pi, \pi)/(\chi_sT)$ as functions of $\beta$ for the disordered model (top panel) and the clean dimerized system (bottom panel) investigated in this study. In both panels, data sets are determined at the corresponding critical points and the horizontal solid lines are the theoretical predictions 1.09. Most of the outcomes shown in the bottom panel are from the results of simulations with $L = 256$.

FIG. 14: $S(\pi, \pi)/(\chi_sT)$ as functions of $\beta$ for the studied disordered model with $J'/J = 1.2$ (top panel) and $J'/J = 1.8$ (bottom panel). In both panels, the horizontal solid lines are the theoretical prediction 1.09. These results are calculated using 8000 (4000) MC sweeps for the thermalization (measurement).

rion, the corresponding values of the three universal coefficients of QCR remain the same as the known ones in the literature.

While the numerical data obtained from the QMC simulations are in good agreement with the corresponding analytic predictions, non-negligible dependence on $T$ is observed for these three universal coefficients. Furthermore, for both the considered models, the estimated values of $\Omega$, which is related to $\chi_u c^2$, are different statistically from the analytic and numerical ones established in the literature (except that determined in Ref. [23]). The difference between the values of $\Omega$ estimated here ($\Omega = 0.306(10)$) and the theoretical result previously known ($\Omega \sim 0.27$) is more than 10 percent, which cannot be accounted for by the potential systematic uncertainties resulting from the calculations of $c$ conducted in this study. Among the relevant studies associated with $\Omega$, only the dedicated work of Ref. [23] agrees with ours. It is also interesting to notice that the $\Omega$ estimated in Ref. [23] is somewhat (slightly) larger than what has been calculated here. We attribute this to the fact that data with temperatures (lattice sizes) lower (larger) than ours were used in that work for the related analysis. Indeed, in our investigation with a fixed $z = 1$, the magnitude of $\Omega$ is increasing when more and more data of higher $T$ are excluded in the fits. Aside from that, finite-temperature effect clearly shows up for $\chi_u c^2/T$, as can be seen in Figs. 8 and 11. Such an effect to some extent will influence the determination of $\Omega$ if the formula $\chi_u c^2 = \Omega T$ is used to extract the value of $\Omega$.

It is intriguing that while the analytic outcome of $\Omega$ including both the leading and subleading contributions deviates significantly from its numerical estimations obtained in this study and in Ref. [23], the theoretical prediction of $\Omega$ by considering only the leading term is in better agreement with our results and that of Ref. [23]. In summary, the numerical evidence reached here for the described discrepancy is quite convincing. A detailed study of $\chi_u$ for the clean bilayer spin-1/2 model demonstrates such a deviation as well [23]. To shed light on this deviation, besides conducting analytic studies associated with corrections not considered before, it will be desirable as well to simulate other disordered and clean dimerized models other than those investigated here.
FIG. 15: $c/(T\xi)$ as functions of $\beta$ for the disordered model investigated in this study. The data sets are determined at the corresponding critical point $(J'/J)_c = 1.990$ and the horizontal solid line is the theoretical prediction 1.04.

V. ACKNOWLEDGMENTS

We thank A. W. Sandvik for bringing Ref. [23], in which a detailed investigation of $\Omega$ based on a clean bilayer spin model was conducted, to our attention. This study is partially supported by Ministry of Science and Technology of Taiwan.

[1] S. Chakravarty, B. I. Halperin, and D. R. Nelson, Phys. Rev. Lett. 60, 1057 (1988).
[2] J. D. Reger and A. P. Young, Phys. Rev. B 37, 5493 (1988).
[3] S. Chakravarty, B. I. Halperin, and D. R. Nelson, Phys. Rev. B 39, 2344 (1989).
[4] J. Oitmaa, C. J. Hamer, and Zheng Weihong, Phys. Rev. B 50, 3877 (1994).
[5] C. J. Hamer, Zheng Weihong, and J. Oitmaa, Phys. Rev. B 50, 6877 (1994).
[6] U.-J. Wiese and H.-P. Ying, Z. Phys. B 93, 147 (1994).
[7] A. W. Sandvik and D. J. Scalapino, Phys. Rev. B 51, 9403 (1995).
[8] B. B. Beard and U.-J. Wiese, Phys. Rev. Lett. 77 (1996) 5130.
[9] A. W. Sandvik, Phys. Rev. B 56, 18 (1997).
[10] F.-J. Jiang, F. Kampfer, M. Nystedt, and U.-J. Wiese, Phys. Rev. B 78, 214406 (2008).
[11] F.-J. Jiang and U.-J. Wiese, Phys. Rev. B 83, 155120 (2011).
[12] A. V. Chubukov and S. Sachdev, Phys. Rev. Lett. 71, 169 (1993).
[13] A. V. Chubukov and S. Sachdev, Phys. Rev. Lett. 71, 2680 (1993).
[14] Alexander Sokol, Rodney L. Glenister, and Rajiv R. Singh, Phys. Rev. Lett. 72, 1549 (1994).
[15] A. V. Chubukov, S. Sachdev, and J. Ye, Phys. Rev. B 49, 11919 (1994).
[16] A. W. Sandvik, A. V. Chubukov, and S. Sachdev, Phys. Rev. B 51, 16483 (1995).
[17] M. Troyer, H. Kanzaki, and K. Ueda, Phys. Rev. Lett. 76, 3822 (1996).
[18] Matthias Troyer, Masatoshi Imada, and Kazuo Ueda, J. Phys. Soc. Jpn. 66, 2957 (1997).
[19] Jae-Kwon Kim and Matthias Troyer, Phys. Rev. Lett. 80, 2705 (1998).
[20] Y. J. Kim, R. J. Birgeneau, M. A. Kastner, Y. S. Lee, Y. Endoh, G. Shirane, and K. Yamada, Phys. Rev. B 60, 3294 (1999).
[21] Y. J. Kim and R. J. Birgeneau, Phys. Rev. B 62, 6378 (2000).
[22] Dao-Xin Yao, Jonas Gustafsson, E. W. Carlson, and Anders W. Sandvik, Physical Review B, 82, 172409 (2010).
[23] A. Sen, H. Suwa, and A. W. Sandvik, Phys. Rev. B 92, 195145 (2015).
[24] A. W. Sandvik, Phys. Rev. B 66, R14157 (1999).
[25] A. W. Sandvik, Phys. Rev. B 66, 024418 (2002).
[26] The quantity $\rho_s$ (spin stiffness) is defined as $\rho_s = \sum_{i=1,2} \langle W_i^2 \rangle$, where $W_i$ is the winding number in the spatial $i$-direction.
[27] S. Wenzel and W. Janke, Phys. Rev. B 79, 014410 (2009).
[28] F.-J. Jiang, Phys. Rev. B 83, 024419 (2011).
[29] The $\chi_u c^2/T$ data of the clean plaquette model determined at $(J'/J)_c$ are examined carefully and indeed those shown in this study are size convergent. It is anticipated that the results associated with $(J'/J)_c = 1.8228$ and 1.8232 should be size convergent as well.
[30] Nvens Ma, Anders W. Sandvik, and Dao-Xin Yao, Phys. Rev. B 90, 104425 (2014).
[31] A. B. Harris, J. Phys. C 7, 1671 (1974).
[32] J. T. Chayes, L. Chayes, D. S. Fisher, and T. Spencer, Phys. Rev. Lett. 57, 2999 (1986).
[33] O. Motrunich, S.C. Ma, D.A. Huse, and D.S. Fisher, Phys. Rev. B 61, 1160 (2000).
[34] O. P. Vajk and M. Greven, Phys. Rev. Lett. 89, 177202 (2002).
[35] Rong Yu, Tommaso Roscilde, and Stephan Haas, Phys. Rev. Lett. 94, 197204 (2005).
[36] A. W. Sandvik, Phys. Rev. Lett. 96, 207201 (2006).