Experimental quantum cryptography with classical users

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Abstract

The exploit of quantum systems allows for insights that promise to revolutionise information processing, although a main challenge for practical implementations is technological complexity. Due to its feasibility, quantum cryptography, which allows for perfectly secure communication, has become the most prominent application of quantum technology. Nevertheless, this task still requires the users to be capable of performing quantum operations, such as state preparation or measurements in multiple bases. A natural question is whether the users’ technological requirements can be further reduced. Here we demonstrate a novel quantum cryptography scheme, where users are fully classical. In our protocol, the quantum operations are performed by an untrusted third party acting as a server, which gives the users access to a superimposed single photon, and the key exchange is achieved via interaction-free measurements on the shared state. Our approach opens up new interesting possibilities for quantum cryptography networks.

Quantum key distribution (QKD) is a technique that allows two users, traditionally called Alice and Bob, to exchange a cryptographic key in an information-theoretic secure way. This means that the security of the key relies on information theory and cannot be broken even by an eavesdropper with unlimited resources. Since the first QKD proposal, the BB84 protocol \cite{bb84}, much progress, both theoretical and experimental, has been made in the field. The practicality of this technology is underlined by numerous experimental and even commercial endeavors, supporting its development \cite{exp1,exp2,exp3,exp4}.

Most QKD protocols require Alice or Bob to share a quantum state, or a direct quantum channel, and to perform quantum operations, i.e. operations on quantum bits (qubits) that do not have any counterpart in classical communication, such as generation or measurement in multiple

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bases. In the endeavor of further simplifying the requirements, it is relevant to investigate how quantum the users' operation and resources need to be, while maintaining information-theoretic security.

A first step towards answering this question was made via the development of semi-quantum key distribution (SQKD) \[6, 7, 8, 9\]. In these schemes, the quantum channel is used twice such that one of the users, say Alice, is fully quantum, while the other, Bob, is restricted to a limited set of operations on qubits, namely preparation and/or measurement in a single basis, reflection and reordering. Further development has shown that Alice’s operations can be as limited as Bob’s, provided that a third party distributes entangled photons to the users and performs measurements in different bases \[10, 11\]. However, in all SQKD protocols, Alice and Bob need to provide quantum resources, as they are required to prepare single-photon quantum states. In addition, information-theoretic security was proven for only a few cases \[10, 12, 13\] and always considering the ideal scenario of perfect devices and infinite resources in the asymptotic regime.

Here we demonstrate that perfectly secure QKD is possible with purely classical users, who are not required to perform any quantum operations nor to generate quantum states. In particular, we introduce a protocol where Alice and Bob only need to perform two classical operations: the detection or reflection of a single photon. Furthermore, we prove information-theoretic security under the realistic assumption of imperfect devices and finite key.

In order to achieve this task, an untrusted server sends to Alice and Bob a single photon in an equal superposition of their respective locations. We assume that the users can communicate through a classical authenticated channel and that the server can send unauthenticated classical messages to the users. Each user can independently choose to perform two actions: “detect” (\(D\)) or “reflect” (\(R\)). In the former case, the photon travels to a detector controlled by the user; in the latter, the photon is sent back to a balanced beam splitter controlled by the server, at whose output ports two detectors, \(D_0\) and \(D_1\), are placed. When both users choose to reflect, single-photon interference occurs at the beam splitter, with the relative phase of the two interfering photon amplitudes tuned such that only detector \(D_0\) clicks. In the ideal case of perfect detection efficiency, when only one of the users chooses to detect the photon and does not find any, the photon collapses into the other user’s location. This corresponds to performing an interaction-free measurement \[14, 15, 16\], which suppresses single-photon interference at the server and allows both detectors \(D_0\) and \(D_1\) to click with equal probability. A click at detector \(D_1\), therefore, enables each user to deduce the action of the other one, thus allowing for the establishment of a “raw-key” digit. In particular, a key digit of “0” (“1”) is set when Alice chooses \(D(R)\) and Bob \(R(D)\). Other combinations are not considered, as they cannot result in a detection at \(D_1\). In our protocol, unlike the majority of QKD protocols, no use of the authenticated channel is necessary for raw-key generation. A sketch of the described scheme is outlined in Figure 1.
Figure 1: The QKD protocol with classical users. Our scheme can be summarized in the following four steps. 1) A quantum server sends single photons in superposition to the users at predetermined regular intervals, which constitute the rounds of the protocol. 2) For each round, Alice and Bob randomly choose between the two actions $D$ and $R$. 3) The server measures the photon coming from Alice/Bob and announces the following results: “0”, if detector $D_0$ clicks, “1”, if detector $D_1$ clicks, “v”, if no detector clicks, and “m”, if more than one click is observed. 4) Alice and Bob only keep the key bit if the message received from the server is “1” and they did not detect a photon, obtaining the raw key, according to the table in figure. 5) Alice and Bob communicate through an authenticated channel to verify the honesty of the server and/or the presence of an eavesdropper.

Considering that the actions of the users are chosen at random, the superposition is balanced and the server’s beam splitter is 50/50, the probability that detector $D_1$ clicks, $p(1)$, is limited to $1/8$, which is reduced by experimental imperfections, eavesdropping or the action of an adversarial server. In all other cases, the users exchange the information of their actions and detection results, which is used for verification purposes. In particular, by checking the statistics of detection in the case where each user sets $D$, Alice and Bob can verify if the received state matches the resource state that the server is supposed to send. This state, in principle, can also have a vacuum and a multi-photon component. The former is due to losses in the quantum channels connecting the server and the users, the latter is due to imperfections in the photon source. In practice, after a characterization of the losses and of the source, but prior to the start of the protocol, the server can declare the probability of sending vacuum, one or more photons. If the users do not verify these values, they assume eavesdropping and/or a dishonest server, and, consequently, discard the key.
Note that it is enough that only one user, say Alice, performs the verification with the information received from the other. This allows for a reduction in communication complexity.

The experimental set-up for the implementation of the protocol is depicted in Figure 2.

Figure 2: Experimental set-up. The regions of space occupied by Alice, Bob and the server are respectively marked in red, blue and green, whereas the path of the photons is indicated by red lines. The server uses a heralded single-photon source and a beam splitter (BS1) to produce the superposition state that is sent to Alice and Bob. Each of the users controls a switch, composed of a liquid-crystal cell (LCC) at 45°, a polarization beam splitter (PBS) and a mirror. By switching the voltage of the LCC, the users can choose to steer the photon to a detector (D) or reflect it back to the server (R). The server collects the reflected photons at a second beam splitter (BS2), where single-photon interference takes place in case both users choose to reflect. The server records the detections at D0 and D1 and announces the results to the users via a classical channel.

The single photons are provided by a source based on spontaneous parametric down-conversion (SPDC), which probabilistically generates photon pairs. One photon from each pair is used to herald the presence of the other one, which is sent to the users. Therefore, all detections in the experiment are in coincidence with the heralding detector, DH. The server sets intervals of 0.5 s, constituting the rounds of the protocol, in which Alice and Bob can decide to either detect or reflect the photons. Note that this interval can be made shorter, in the order of $10^{-8}$ s, by using ultra-fast switches and optimized bright single-photon sources [17]. At the end of each round, the server announces the result of the measurement at its detectors. The probabilistic nature of our source implies that, in each round, multiple non-simultaneous single-photon emissions can occur. In some rounds, therefore, the total number of detections is higher than one. The output rate of the source is decreased, so that the total average number of photons sent to the users is about 0.35 per round, in order to reduce the probability of multi-photon emissions.

Based on these experimental conditions, we develop a theoretical model, described in detail
in Section E of the Appendix, in which the server can send 0, 1 or 2 photons per interval with probabilities $p_0$, $p_1$ and $p_2$, respectively. In our case, these probabilities result $p_0 = 0.705$, $p_1 = 0.247$, $p_2 = 0.043$. Furthermore, we consider that the users do not keep track of the photon detection times, meaning that, at the end of each round, Alice, Bob and the server only have access to the number of detections they recorded. This makes our analysis also applicable to the case of simultaneous multi-photon emission. Our model also takes into account the finite detection efficiency of Alice’s and Bob’s detectors, $D_A$ and $D_B$, respectively, which is measured to be 58% each.

We measure the probability of the raw key generation, $p_{\text{key}}$, and the probability of error in the raw key, $p_{\text{err}}$, after $10^5$ rounds of the protocol. The results are reported in Table 1. In Section E.3 of the Appendix, we describe and report the results of the verification procedure for the experiment.

|                | Direct Method (full dataset) | Direct Method (subset) | Indirect Method (full dataset) |
|----------------|-------------------------------|------------------------|-------------------------------|
| $p_{\text{key}}$ | $1.55(3) \times 10^{-2}$      | $1.5(1) \times 10^{-2}$ | $1.5(3) \times 10^{-2}$      |
| $p_{\text{err}}$ | $7.5(8) \times 10^{-4}$      | $5(2) \times 10^{-4}$  | $3(3) \times 10^{-3}$      |

Table 1: Evaluation of key generation and error rates. The probabilities of raw-key generation, $p_{\text{key}}$ and error on a key digit, $p_{\text{err}}$, respectively, are shown per round (in our case an interval of 0.5 s). $p_{\text{key}}$ and $p_{\text{err}}$ are evaluated in three different ways: direct estimation over the full data set, direct estimation over a randomly chosen subset of $10^4$ rounds and indirect estimation. In the direct estimation, the users sacrifice a part of the raw key for verification procedure. In the indirect estimation, Alice obtains $p_{\text{key}}$ and $p_{\text{err}}$ by using the information received from Bob during the verification phase, as explained in detail in Section E.2.2 of the Appendix. This allows the parties to avoid the loss of key digits, at a price of higher uncertainty on the estimated values, which are calculated from several experimentally obtained quantities, each with its error. In the table, the numbers in parentheses are the errors on the last digits, obtained with the assumption of poissonian uncertainty on the counts.

The raw key is obtained without sifting procedure, unlike in standard QKD schemes [1, 18]. To obtain the final secret key, Alice and Bob perform standard classical post-processing through error correction and privacy amplification [2]. Based on the probabilities in Table 1, we obtain the dependence of the final secret key rate, $r$, on the number of rounds, $N$ (see Equation (C.1) from Section C of the Appendix). This dependence is plotted in Figure 3 for different values of the detection loss of $D_A$ and $D_B$, assumed to be the same. The details of how the curves were obtained are discussed in Sections D and E of the Appendix. As expected, an increase in the detection loss degrades the performance of the protocol.
Figure 3: **Secret key rate vs number of rounds, for different values of detection loss.** The black dashed curve refers to the experimental implementation, corresponding to a detection loss of 42% for each Alice and Bob. The red, cyan, blue, magenta and orange curves represent the calculated results for detection losses of 0, 3, 25, 42 and 80%, respectively. If the detection loss increases, the number of rounds for which $r$ becomes positive also increases, while the asymptotic secret key rate decreases. In the implemented case, the secret key rate becomes positive after about $4.9 \times 10^6$ rounds.

In our work, we propose and experimentally implement a novel QKD protocol allowing two classical users to establish a shared secret key using the services of an untrusted quantum server, which provides a superimposed single photon as a feasible quantum resource. We underline the applicability of our scheme by providing an information-theoretic security analysis of our protocol in the finite-key setting, which takes into account imperfect detection efficiency and multi-photon emission from the source, and by calculating the secret key rate.

Experimentally, the main challenge of the protocol is that it requires phase stability in the interferometer formed between the users and the server. This issue can be addressed by using intrinsically phase-stable schemes, like Sagnac configurations [19]. In this case, however, a quantum channel between Alice and Bob is also necessary.

As an immediate future line of research, our security analysis of finite keys in the presence of experimental imperfections can be applied to show the same security levels for other cryptographic schemes, such as counterfactual quantum cryptography [20] [21] [22] [23] [24], or the key distribution based upon recently proposed two-way communication with one photon [25] [26].

In practical terms, recent progresses in bright deterministic single-photon sources [27], high-efficiency detectors [28] and fast switches [17] promise to push our scheme towards real-world applications.
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APPENDIX

A  The experimental set-up

After setting its polarization to “horizontal” (H), that is parallel to the optical table, a single photon is sent to a beam splitter that creates the superposition between Alice’s and Bob’s locations. Each of the users controls a liquid-crystal cell (LCC) at 45° and a polarization beam splitter (PBS). The phase retardation between the two axes of the LCC can be switched between 0 and π by means of a voltage signal. Consequently, the photon polarization is rotated by 0° or 90°, respectively. In the first case, the photon is transmitted by the PBS and steered to a fiber-coupled avalanche photo-diode (APD) for detection, D_A or D_B; in the second case, the photon travels back to the server. The detection efficiency of D_A and D_B is evaluated by comparison with a fully-characterized transition-edge superconducting nanowire detector. The photons going back to the server impinge onto a second beam splitter, at whose outputs two fiber-coupled APDs, D_0 and D_1, are placed. The set-up, therefore, implements a folded Mach-Zehnder interferometer. The phase between the two arms of the interferometer is set such that, when Alice and Bob both decide to reflect back the photon, detector D_0 clicks. The interferometer is passively stabilized, so that the phase is constant for about 100 s. After this time, the phase is actively re-set to the initial value by using a piezo transducer.
B The single-photon source

We use an SPDC-based single-photon source in a Sagnac configuration [29], with a 20-mm-long periodically-poled potassium tytanyl phosphate (PPKTP) crystal. The crystal is pumped by a continuous-wave single-mode diode laser at 395 nm with a maximum power of about 20 mW. The Sagnac loop is realised using a dual-wavelength polarization beam splitter and two mirrors. The crystal converts a photon at 395 nm into two photons at 790 nm and orthogonal polarizations. The produced photons are coupled into single-mode fibers: one of them is sent to the Mach-Zehnder interferometer and the other is directly sent to an APD for heralding the presence of its twin in the interferometer. The use of polarizers for both photons of each pair ensure that a defined polarization state is produced, in particular $|H⟩|V⟩$, where $H$ stands for “horizontal” and $V$ for “vertical”. The possibility of simultaneous multi-photon emission from the source is ruled out by the measurement of the heralded second-order correlation function at zero delay, $g^{(2)}(0)$ [30], which should be exactly 0 for an ideal perfect single-photon source. We obtain $g^{(2)}(0) = 0.004 ± 0.010$, measured at a total detection rate of about $15 \times 10^3$ photons per round (in our case 0.5 s) and a pump power of 7 mW. Our value of $g^{(2)}(0)$ is comparable with the lowest ones obtained in quantum optics experiments [31]. In order to minimize the contribution of higher-order emissions, we decrease the pump power such that about 0.35 heralded photons per round are sent to the interferometer, resulting into a total effective number of detections per round of 0.2, due to detection losses.

C Assumptions and notation for the security analysis

We prove the security of our key distribution protocol assuming that anything outside of Alice’s and Bob’s private laboratories, including the quantum server, is completely untrusted. After Alice and Bob receive quantum states of some form (possibly consisting of multiple qubits) from the server (or an adversary, Eve, in the middle) and perform their respective actions, they will receive a classical message from the server indicating a possible measurement outcome. However, the server is under no obligation in our proof of security to report the measurement outcome honestly, or to even perform any measurement at all. On the rounds where the server announces “1”, Alice and Bob generate the raw key of length $N_{raw}$ whenever one of them chose to detect the photon without registering any click at the detector, while the other reflected. Note that due to experimental imperfections and eavesdropping (or server’s dishonesty), server can announce “1” even if both agents reflected, or both detected vacuum, in which case they do not share the same raw key and the error is introduced. Let $N$ be the total number of rounds of the protocol and $p(1)$ be the probability of the server announcing “1” when no party received a click upon detection. Then $N_{raw} = p(1)N$ rounds are potentially used for key generation. Alice and Bob may choose to use a (small) subset of the raw key of size $\mu$ to directly estimate the statistics used to compute the secret key rate. The portion of the raw key remaining after parameter estimation step is called the sifted key, of the length $N_{sift} = N_{raw} - \mu$. Let the random variables $R_A$ and $R_B$ denote Alice’s and Bob’s respective sifted keys. Nevertheless, completing this stage does not guarantee the following requirements for the shared key to be a perfectly secure secret key:

(i) Alice and Bob share exactly the same uniformly distributed key. The parameter estimation step only sets the degree of the correlation between Alice’s and Bob’s random variables $R_A$ and $R_B$ of the sifted key (it is used to establish the effects of noise, introduced either by imperfect devices, or by Eve).
(ii) The shared key is completely uncorrelated with Eve (including the server). Again, the parameter estimation step will only give upper limits to the correlation with Eve.

Both problems are treated classically, as they are applied to classical random variables. Problem (i) is solved using standard error correction techniques (often called information reconciliation), which turn $R_A$ and $R_B$ into $\tilde{X}_A$ and $\tilde{X}_B$, such that $\tilde{X}_A = \tilde{X}_B$. Problem (ii) is solved by further applying privacy amplification techniques, resulting in random variables $X_A = X_B$ uncorrelated with Eve, giving the final secret key of length $N_{\text{sec}}$.

The security level of the key shared between Alice and Bob is given by parameter $\epsilon$, which quantifies how uncorrelated the key $X_A = X_B$ is from Eve or a dishonest server (for the formal definition, see Equation (1) from [32]). The security criterion requires $\epsilon$ to tend to zero as the number of rounds $N$ tends to infinity, thus obtaining perfectly secret key in the asymptotic scenario. One can compute the sifted key rate as

$$r' = \lim_{N \to \infty} N_{\text{sec}}/N_{\text{sif}} = S(A|C) - H(A|B).$$

Conditional Shannon entropy $H(A|B)$ can be easily computed using the probabilities $p_{i,j}$ of Alice and Bob establishing the raw key bit values $i$ and $j$, respectively. Further, the secret key rate is defined as $r = N_{\text{sec}}/N = r'(N_{\text{sif}}/N)$, which is the same as the sifted key rate in the asymptotic regime: since in order to obtain good enough statistics during the verification procedure, the number $\mu$, albeit big, is still finite, we have $N_{\text{sif}} = N - \mu \approx N$, for $N \to \infty$. In the realistic case of limited resources, however, where Alice and Bob can exchange only a finite number of keys, we must take into account the imperfect parameters. Using the security criterion given by [32], let us denote $\epsilon_{PE}$ as a given error tolerance for the parameter estimation. One can further compute $\delta$, as a function of $\epsilon_{PE}$, a confidence interval so that the observed parameters are $\delta$ close to the actual values, except with probability $\epsilon_{PE}$. Let $\epsilon$ be the desired security of the final secret key, and let $\epsilon_{EC}$ be the maximal probability that Bob computes error correction incorrectly. All of these are given by the user. Then, after $\mu$ rounds being wasted for direct parameter estimation and the proportion of qubits used $(p(1)N - \mu)/N$, it was shown in [32] that

$$r \geq \frac{p(1)N - \mu}{N} \left( S(A|C) - \frac{\text{leak}_{EC} + \Delta}{p(1)N - \mu} \right),$$

where

$$\Delta = 2\log_2 \left( \frac{1}{2(\epsilon - \epsilon_{EC} - \epsilon')} \right) + 7\sqrt{\frac{p(1)N - \mu}{\log_2(2/\epsilon')} \log_2(2/\epsilon_{PE})},$$

and $\epsilon'$ is arbitrary (chosen by the user to maximize the expression but bounded by $\epsilon - \epsilon_{EC} > \epsilon' > \epsilon_{PE} \geq 0$. In the above expression, $S(A|C)$ is minimized over all observable statistics within the given confidence interval (so that the actual statistics of the real density operator are within $\delta(\epsilon_{PE})$ of the observed statistics, except with probability $\epsilon_{PE}$). The value $\text{leak}_{EC}$ represents the number of (classical) bits exchanged between Alice and Bob during the error correction. Again, using [32], we take $\text{leak}_{EC}/(p(1)N - \mu) = (1.2)h(Q)$, where $Q = p_{\text{err}}/p(1)$ and $p_{\text{err}}$ is the probability to generate opposite key bits during the entire protocol. Note that $\mu$ will also be a function of $\epsilon_{PE}$, since the smaller that is, the larger $\mu$ will be.

In order to compute the secret key rate described above, one needs to compute $S(A|C)$ for a given system. Before we proceed to discuss the ideal and experimental scenario, let us first define some useful terminology.

Let us denote the Hilbert spaces corresponding to Alice’s and Bob’s equipments as $\mathcal{H}_A = \text{span}\{|D_c\rangle_A, |D_v\rangle_A, |D_t\rangle_A, |D'_t\rangle_A, |D'_{c_t}\rangle_A, |R\rangle_A\}$ and $\mathcal{H}_B = \text{span}\{|D_c\rangle_B, |D_v\rangle_B, |D_t\rangle_B, |D'_t\rangle_B, |D'_{c_t}\rangle_B, |R\rangle_B\}$, respectively. Here, $|D_c\rangle$ and $|D_v\rangle$ denote the states of a detector, the first corresponding to the
the purified initial state of Alice’s apparatus is always include the coin states into the macroscopic description of the apparatus states, such that states, and set the initial state of the apparatus accordingly, resulting in a proper mixture of the two exchanged key. 

A dishonest server can entangle with the photons sent to Alice and Bob to extract information about $C$, since in our set-up, Alice and Bob do not keep track of the detection times. Moreover, the latter not click because there were no photons present, or they were lost. Performing sophisticated quantum measurements, one cannot distinguish whether a detector did not click because there were no photons present, or they were lost.

The server’s Hilbert space $\mathcal{H}_S = \text{span}\{\ket{0}_S, \ket{1}_S, \ket{v}_S, \ket{m}_S\}$ consists of macroscopic orthogonal states modeling classical messages “0”, “1”, “v” (vacuum) and “m” (multiple clicks), respectively. Additionally, we denote server’s ancilla system by $C$, spanned by the Hilbert space $\mathcal{H}_C$, which a dishonest server can entangle with the photons sent to Alice and Bob to extract information about the exchanged key.

Let us assume Alice tosses a fair coin to decide whether she will detect or reflect the photon, and set the initial state of the apparatus accordingly, resulting in a proper mixture of the two states, $|D_v\rangle_A|D_v\rangle$ and $|R\rangle_A|R\rangle$, and analogously for Bob. Without the loss of generality, we can always include the coin states into the macroscopic description of the apparatus states, such that the purified initial state of Alice’s apparatus is 

$$|\phi_0\rangle_A = \frac{1}{\sqrt{2}} \left( |D_v\rangle_A + |R\rangle_A \right), \quad (C.3)$$

and analogously for Bob, making their joint state as 

$$|\phi_0\rangle_{AB} = \frac{1}{2} \left( |D_v,R\rangle_{AB} + |R,D_v\rangle_{AB} + |D_v,D_v\rangle_{AB} + |R,R\rangle_{AB} \right). \quad (C.4)$$

Note that due to possible imperfect single-photon sources, and the presence of adversaries, the number of photons present is not necessarily fixed to be one. Thus, we will use a number basis to describe the photonic states. In this paper, we will decompose the overall Fock space of the photons in Alice’s and Bob’s arms as $\mathcal{F}_f = \text{span}\{|0\rangle_f, |1\rangle_f, |v\rangle_f, |0,1\rangle_f, |2,0\rangle_f, |1,1\rangle_f, |1',1\rangle_f, |0,2\rangle_f\}$ $\oplus \mathcal{F}_k^f$, where $|0,0\rangle_f \equiv |v\rangle_f$ represents the vacuum state, $|1,0\rangle_f$ represents a photon in Alice’s arm and $|0,1\rangle_f$ to be in Bob’s arm. Similarly, $|2,0\rangle_f$, and $|0,2\rangle_f$, represent two non-simultaneous photons in Alice and Bob’s arms, respectively; whereas $|1,1\rangle_f$ and $|1',1\rangle_f$ represent the case of two non-simultaneous photons when the first one went to Alice’s arm while the second to Bob and vice-versa, respectively. $\mathcal{F}_k^f$ denotes the sub-space corresponding to the multi-photon case of $k > 2$ photons. The action of photonic creation operators $\hat{a}_f^\dagger$ and $\hat{b}_f^\dagger$, in terms of the number basis $|a,b\rangle_f$, with $a,b \in \mathbb{N}_0$ being the number of photons in Alice’s and Bob’s arms, respectively, is given by $(\hat{a}^\dagger)^a (\hat{b}^\dagger)^b |0\rangle_f = \sqrt{a! b!} |a,b\rangle_f.$
D Security Analysis - Ideal case

D.1 Extraction of the secret key

For the ideal case scenario, we assume that the server is has a perfect single-photon source, all the channels are lossless, and Alice’s and Bob’s detectors are perfect, which means they have 100% detection efficiency and zero dark counts. In this section, we assume that the users have already performed the verification procedure to check that the resource state provided by the server is the expected one and that the announcements of the servers are compatible with their actions (see section D.2). Therefore, the perfect single photon state that Alice and Bob expect to be sent is

\[ |\phi_0\rangle_f = \left( \frac{\hat{a}^\dagger + \hat{b}^\dagger}{\sqrt{2}} \right) |0,0\rangle_f = \frac{|1,0\rangle_f + |0,1\rangle_f}{\sqrt{2}}, \tag{D.1} \]

with \(|1,0\rangle_f\) and \(|0,1\rangle_f\) representing the photon located in Alice’s and Bob’s arms, respectively. However, we assume that the following entangled state is sent to Alice and Bob by the server (or Eve)

\[ |\phi_0\rangle_{fC} = |1,0\rangle_f \otimes |c_{1,0}\rangle_C + |0,1\rangle_f \otimes |c_{0,1}\rangle_C \tag{D.2} \]

where \(|c_{a,b}\rangle_C \in \mathcal{H}_C\) are not necessarily orthogonal nor normalised. Moreover, as per usual in QKD security proofs, Alice and Bob can enforce symmetry, and so, we may assume \(\langle c_{a,b}\rangle_C = 1/2\). Therefore, we can write the joint initial state as

\[ |\phi_0\rangle_{ABfC} = |\phi_0\rangle_{AB} \otimes |\phi_0\rangle_{fC} \tag{D.3} \]

\[ = \frac{1}{2} \left( |D_v,R\rangle_{AB} + |R,D_v\rangle_{AB} + |D_v,D_v\rangle_{AB} + |R,R\rangle_{AB} \right) \otimes \left( |1,0\rangle_f |c_{1,0}\rangle_C + |0,1\rangle_f |c_{0,1}\rangle_C \right). \]

Alice’s and Bob’s actions on a given initial photon state are given by

\[ |D_v,R\rangle |1,0\rangle \rightarrow |D_v,R\rangle |0,0\rangle, \]
\[ |D_v,R\rangle |0,1\rangle \rightarrow |D_v,R\rangle |0,1\rangle, \]
\[ |D_v,D_v\rangle |1,0\rangle \rightarrow |D_v,D_v\rangle |0,0\rangle, \]
\[ |D_v,D_v\rangle |0,1\rangle \rightarrow |D_v,D_v\rangle |0,0\rangle, \]
\[ |R,R\rangle |1,0\rangle \rightarrow |R,R\rangle |0,0\rangle, \]
\[ |R,R\rangle |0,1\rangle \rightarrow |R,R\rangle |0,1\rangle, \tag{D.4} \]

and, therefore

\[ |\phi_1\rangle_{ABfC} = \frac{1}{2} \left[ |D_v,R\rangle |0,0\rangle \otimes |c_{1,0}\rangle + |D_v,R\rangle |0,1\rangle \otimes |c_{0,1}\rangle + |R,D_v\rangle |1,0\rangle \otimes |c_{1,0}\rangle + |R,D_v\rangle |0,0\rangle \otimes |c_{0,1}\rangle \right. \]
\[ + \left. |D_v,D_v\rangle |0,0\rangle \otimes |c_{1,0}\rangle + |D_v,D_v\rangle |0,0\rangle \otimes |c_{0,1}\rangle + |R,R\rangle \otimes (|1,0\rangle \otimes |c_{1,0}\rangle + |0,1\rangle \otimes |c_{0,1}\rangle) \right]. \tag{D.5} \]

Above, as well as in rest of the Appendix, for simplicity we omit writing the labels of the quantum states (\(A, B, C, S\) and \(f\)), whenever it is implicitly unambiguous to which space they belong by their quantum numbers (\(D_v, 0, 0, \text{etc.}\)).

Upon leaving Alice’s and Bob’s labs, the server (or Eve) will apply a quantum instrument to the returning photon state. This can be modelled as an isometry \(\mathcal{I} : \mathcal{F}_f \otimes \mathcal{H}_C \rightarrow \mathcal{H}_S \otimes \mathcal{H}_C\), given by

\[ \mathcal{I} |a',b'\rangle_f |c_{a,b}\rangle_C = |0\rangle_S |e^{a,b}_{a',b'}\rangle_C + |1\rangle_S |f^{a,b}_{a',b'}\rangle_C + |v\rangle_S |g^{a,b}_{a',b'}\rangle_C, \tag{D.6} \]
where states from $\mathcal{H}_C$ are not necessarily normalized nor orthogonal, and $a,b$ are no longer correlated with $a',b'$ due to Alice’s and Bob’s actions given by Equation (D.4). Note that since we are assuming an ideal case, the term corresponding to the message “m” is absent from the above equation.

We are interested only in the rounds when the server announces “1” and neither Alice nor Bob detect a photon, and the users generate the key. Thus, while writing the state after the server applies $I$ on $|\phi_1\rangle_{ABC}$, we will omit writing the server’s message state $|1\rangle_S$ (corresponding to announcing a result “1”). The final density operator representing the state of the system $ABC$, conditioned on the event that the server sends the message “1” and none of the users detects a photon (only the rounds used for key generation), is

$$
|\phi_2\rangle_{ABC} = \frac{1}{\sqrt{N}} \left\{ |D_v,R\rangle \otimes \frac{1}{2} |f_{0,1}^{0,1}\rangle + |R,D_v\rangle \otimes \frac{1}{2} |f_{1,0}^{1,0}\rangle + |R,R\rangle \otimes \frac{1}{2} \left[ |f_{1,0}^{1,0}\rangle + |f_{0,1}^{0,1}\rangle \right] \right\}
\approx \frac{1}{\sqrt{N}} \left\{ |D_v,R\rangle \otimes |k_{0,0}\rangle + |R,D_v\rangle \otimes |k_{1,1}\rangle + |R,R\rangle \otimes |k_{1,0}\rangle \right\},
$$

where the states $|k_{i,j}\rangle_C$ are associated to Alice establishing the value $i$ and Bob $j$ as a key bit, are given by

$$
|k_{0,0}\rangle_C = \frac{1}{2} |f_{0,1}^{0,1}\rangle, \\
|k_{1,1}\rangle_C = \frac{1}{2} |f_{1,0}^{1,0}\rangle, \\
|k_{1,0}\rangle_C = \frac{1}{2} \left[ |f_{1,0}^{1,0}\rangle + |f_{0,1}^{0,1}\rangle \right].
$$

Note that in general there should also be a state $|k_{0,1}\rangle_C$ (corresponding to $|D_v,D_v\rangle$), but in the ideal case, due to the assumption of perfect detectors, the values $i=0$ and $j=1$ are not possible. In fact, this case would imply that both Alice and Bob detect vacuum and the server announces 1, and can only be possible in case of lossy detectors, dark counts or imperfect single-photon sources. The normalization constant $N$ is the probability to obtain the result 1, $p(1)$, when there were no clicks at the users’ detectors, and is given by

$$
N = \langle k_{0,0}|k_{0,0}\rangle + \langle k_{1,1}|k_{1,1}\rangle + \langle k_{1,0}|k_{1,0}\rangle = p(1).
$$

Let us define $p_{0,0} = p(D_v,R ; 1) = \langle k_{0,0}|k_{0,0}\rangle$ as the joint probability for the event when Alice detects vacuum and Bob reflects, and the server announces the result “1”, and analogously $p_{1,1}$, $p_{0,1}$ and $p_{1,0}$. Here, we use the semi column (;) to denote logical AND between two propositions. Therefore, we can define the probability to share the key as $p_{key} = p_{0,0} + p_{1,1}$ and the probability of an error as $p_{err} = p_{0,1} + p_{1,0}$. Further, let $Q$ denote the probability that the server announces the result “1”, given that both Alice and Bob reflected. Then, $p_{1,0} = p(1|R,R)p(R,R) = Q/4$ denotes the error in the key, $p_{err}$, since $p_{0,1} = 0$. In the ideal case, it is easy to see that

$$
p_{0,0} = \langle k_{0,0}|k_{0,0}\rangle = \frac{1}{16}, \quad p_{0,1} = \langle k_{0,1}|k_{0,1}\rangle = 0, \\
p_{1,1} = \langle k_{1,1}|k_{1,1}\rangle = \frac{1}{16}, \quad p_{1,0} = \langle k_{1,0}|k_{1,0}\rangle = \frac{Q}{4}.
$$

Note that in the ideal case, it is impossible for Alice and Bob to generate a key error corresponding to $p_{0,1}$, i.e. the case when they both detect vacuum and the server announces 1. Further, from
Equations (D.8) and (D.10), we have
\[
\text{Re} \langle k_0,0 | k_{1,1} \rangle = -\frac{p_{\text{key}}}{2} + \frac{Q}{8}, \quad p(1) = N = (1 + 2Q)/8, \quad (D.11)
\]

At this point, we compute the conditional entropy between Alice and the adversary, \( S(A|C) \), for the rounds where raw key bits are generated. Using Equation (D.7), the density operator, after dropping off-diagonal terms, with \( |k_{i,j}\rangle_C \langle k_{l,m}| \), for \((i,j) \neq (l,m)\), is
\[
\rho_{ABC} = \frac{1}{N} \left( |D_v,R\rangle_{AB} \langle D_v,R| \otimes |k_{0,0}\rangle_C \langle k_{0,0}| + |R,D_v\rangle_{AB} \langle R,D_v| \otimes |k_{1,1}\rangle_C \langle k_{1,1}| + |R,R\rangle_{AB} \langle R,R| \otimes |k_{1,0}\rangle_C \langle k_{1,0}| \right). \quad (D.12)
\]

The state \( |D_v,R\rangle \langle D_v,R| \), describing Alice detecting without a click and Bob reflecting, is associated to a shared key bit 0. Similarly, \( |R,D_v\rangle \langle R,D_v| \) is associated to a key bit 1. Whereas, \( |R,R\rangle \langle R,R| \) corresponds to errors in the key, when the two users establish opposite key bit values.

Therefore, using [33] to compute a bound on the conditional entropy \( S(A|C) \), we have
\[
S(A|C) \geq \frac{\langle k_{0,0}|k_{0,0} \rangle + \langle k_{1,1}|k_{1,1} \rangle}{N} \left[ h \left( \frac{\langle k_{0,0}|k_{0,0} \rangle}{\langle k_{0,0}|k_{0,0} \rangle + \langle k_{1,1}|k_{1,1} \rangle} \right) - h(\lambda_0) \right], \quad (D.13)
\]
where \( h(\cdot) \) is the binary Shannon entropy, and
\[
\lambda_0 = \frac{1}{2} \left( 1 + \sqrt{\frac{(\langle k_{0,0}|k_{0,0} \rangle - \langle k_{1,1}|k_{1,1} \rangle)^2 + 4 \text{Re}^2 \langle k_{0,0}|k_{1,1} \rangle}{\langle k_{0,0}|k_{0,0} \rangle + \langle k_{1,1}|k_{1,1} \rangle}} \right). \quad (D.14)
\]

Note that in Equation (D.13), in general there exists additional term corresponding to \( \langle k_{0,1}|k_{0,1} \rangle \) and \( \langle k_{1,0}|k_{1,0} \rangle \) (when the agents establish opposite keys), but in our case it is zero, as \( \langle k_{0,1}|k_{0,1} \rangle = 0 \).

In Figure [4] we present the dependence of the secret key rate \( r \) on the total number of rounds \( N \) for different values of \( Q \) (including the one obtained from the experimental set-up). Other parameters are taken from [32] as \( \epsilon = 10^{-5}, \epsilon_{EC} = 10^{-10} \) and \( \epsilon' = 10^{-7} \). We also assume \( \epsilon_{PE} = 10^{-11} \).
Figure 4: The secret key rate $r$ is plotted against $N$, for the ideal case of perfect single-photon sources and detectors. The blue, green and magenta curves correspond to the values of $Q$ to be $0.005, 0.025$ and $0.05$, respectively. Whereas, the red curve represents the experimentally observed value of $Q$, $0.015$.

D.2 Verification

In this section, we describe the verification procedure that Alice and Bob perform in the ideal case to check for eavesdropping or server’s dishonesty. First, the users need to check whether the state they receive is the expected resource state. In order to do this, they can consider only the cases where at least one of them decided to detect the photon and registered a click at their respective detectors. For these cases, the following conditions must be satisfied:

- The total number of clicks registered by Alice and Bob at their respective detectors must be exactly one.

- The probability that Alice detects a photon must be the same as the probability that Bob detects a photon, and both must be $\frac{1}{2}$.

Note that as these cases always involve a detection at Alice’s or Bob’s detectors, they do not imply any sacrifice of key digits. After that, Alice and Bob can decide to check the honesty of the server by verifying the compatibility of its announcements with their actions, as described in table 2. This case implies the discard of some key digits. During this verification stage Alice and Bob can directly estimate $Q$ (by estimating $p_{1,0} = Q/4$) and use it to obtain the secure key rate, as explained in the previous section.
Upon possible further action of the adversary, the above state evolves to the normalized state with the ancilla, therefore not included in the analysis, i.e., $p^{T}$ implementation, the probability to emit higher numbers of photons is considered negligible and (the creation operators $\hat{a}$ states are

$$|\phi_0\rangle_f = \sqrt{p_0} |v\rangle_f + \sqrt{p_1} \int_0^T \hat{a}^\dagger(t) |v\rangle_f \, dt + \sqrt{p_2} \int_0^T \int_0^T \left( \frac{\hat{a}^\dagger(t)\hat{a}^\dagger(t')}{\sqrt{2}} |v\rangle_f \right) \, dt \, dt', \quad (E.1)$$

where $\hat{a}^\dagger(t)$ and $\hat{a}^\dagger(t')$ represent photon creation at times $t$ and $t' > t$, respectively. In our particular implementation, the probability to emit higher numbers of photons is considered negligible and therefore not included in the analysis, i.e., $p_0 + p_1 + p_2 \approx 1$. Furthermore, we consider that, instead of the above initial photon state, the untrusted server sends the following state of photons entangled with the ancilla,

$$|\phi_0\rangle_{fC} = \sqrt{p_0} |v\rangle_f |d_v\rangle_C + \sqrt{p_1} |1\rangle_f |d_1\rangle_C + \sqrt{p_2} |2\rangle_f |d_2\rangle_C, \quad (E.2)$$

where $|v\rangle_f$ is the photon vacuum state, $|1\rangle_f = \hat{a}^\dagger(t) |v\rangle_f$, $|2\rangle_f = \hat{a}^\dagger(t)\hat{a}^\dagger(t') |v\rangle_f$, and the ancilla states are $|d_i\rangle_C \in \mathcal{H}_C$. After passing through the first 50/50 beam splitter of our interferometer, described by $\hat{a}^\dagger(t) \rightarrow (\hat{a}^\dagger(t) + \hat{b}^\dagger(t))/\sqrt{2}$ and $\hat{a}^\dagger(t') \rightarrow (\hat{a}^\dagger(t') + \hat{b}^\dagger(t'))/\sqrt{2}$, the above state becomes (the creation operators $\hat{b}$ and the states $|1,0\rangle_f$, etc. are defined in (C)

$$|\phi_0\rangle_{fC} = \sqrt{p_0} |v\rangle_f |d_v\rangle_C + \sqrt{p_1} \left( |1,0\rangle_f + |0,1\rangle_f \right) |d_1\rangle_C + \sqrt{p_2} \left( |2,0\rangle_f + |1',1\rangle_f + |1',0\rangle_f + |0,2\rangle_f \right) |d_2\rangle_C. \quad (E.3)$$

Upon possible further action of the adversary, the above state evolves to the normalized state

$$|\phi_0\rangle_{fC} \rightarrow \sum_{a+b \geq 0} \sum_{a+b \leq 2} |a,b\rangle_f |c_{a,b}\rangle_C = |0,0\rangle_f \otimes |c_{0,0}\rangle_C + |1,0\rangle_f \otimes |c_{1,0}\rangle_C + |0,1\rangle_f \otimes |c_{0,1}\rangle_C + |2,0\rangle_f \otimes |c_{2,0}\rangle_C
+ |1,1\rangle_f \otimes |c_{1,1}\rangle_C + |1',1\rangle_f \otimes |c_{1',1}\rangle_C + |0,2\rangle_f \otimes |c_{0,2}\rangle_C. \quad (E.4)$$

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
Alice’s actions & $D_e$ & $D_v$ & $D_c$ & $D_v$ & $R$ & $R$ & $R$ \\
\hline
Bob’s actions & $D_v$ & $D_e$ & $R$ & $R$ & $D_c$ & $D_v$ & $R$ \\
\hline
Server’s actions & $v$ & $v$ & $v$ & $0$ & $1$ & $0$ & $v$ \\
\hline
\end{tabular}
\caption{Announcements of an honest server. The table reports the expected messages from the server for each possible configuration of Alice’s and Bob’s set-ups. By sacrificing some key digits, Alice and Bob can check if the announcements of the server follow this table and thus verify its honesty. In all cases where the server can announce either “0” or “1”, the two events must occur both with probability $\frac{1}{2}$. In the $RR$ case, the server is supposed to always announce “0”. The cases where this does not happen lead to errors in the key. The error rate can then be estimated by Alice and Bob and used to obtain the secure key rate.}
\end{table}

E Security Analysis - Experimental Implementation

E.1 Extraction of the secret key

In this section we analyze the experimental implementation of our protocol with imperfect single-photon sources and detectors, as well as the noisy and lossy channels. We assume an untrusted server that can attack before Alice and Bob perform their respective operations, as well as after (which is equivalent to allowing Eve to intercept the photons exchanged between an honest server and the agents). We consider a probabilistic single photon source, emitting vacuum state with probability $p_0$, single photons with probability $p_1$, and two non-simultaneous photons with probability $p_2$, within a time slot of interval $T$, as

\begin{align*}
|\phi_0\rangle_f &= \sqrt{p_0} |v\rangle_f + \sqrt{p_1} \int_0^T \hat{a}^{\dagger}(t) |v\rangle_f \, dt + \sqrt{p_2} \int_0^T \int_0^T \left( \frac{\hat{a}^{\dagger}(t)\hat{a}^{\dagger}(t')}{\sqrt{2}} |v\rangle_f \right) \, dt \, dt', \quad (E.1) \\
\end{align*}

where $\hat{a}^{\dagger}(t)$ and $\hat{a}^{\dagger}(t')$ represent photon creation at times $t$ and $t' > t$, respectively. In our particular implementation, the probability to emit higher numbers of photons is considered negligible and therefore not included in the analysis, i.e., $p_0 + p_1 + p_2 \approx 1$. Furthermore, we consider that, instead of the above initial photon state, the untrusted server sends the following state of photons entangled with the ancilla,

\begin{align*}
|\phi_0\rangle_{fC} &= \sqrt{p_0} |v\rangle_f |d_v\rangle_C + \sqrt{p_1} |1\rangle_f |d_1\rangle_C + \sqrt{p_2} |2\rangle_f |d_2\rangle_C, \quad (E.2) \\
\end{align*}

where $|v\rangle_f$ is the photon vacuum state, $|1\rangle_f = \hat{a}^{\dagger}(t) |v\rangle_f$, $|2\rangle_f = \hat{a}^{\dagger}(t)\hat{a}^{\dagger}(t') |v\rangle_f$, and the ancilla states are $|d_i\rangle_C \in \mathcal{H}_C$. After passing through the first 50/50 beam splitter of our interferometer, described by $\hat{a}^{\dagger}(t) \rightarrow (\hat{a}^{\dagger}(t) + \hat{b}^{\dagger}(t))/\sqrt{2}$ and $\hat{a}^{\dagger}(t') \rightarrow (\hat{a}^{\dagger}(t') + \hat{b}^{\dagger}(t'))/\sqrt{2}$, the above state becomes (the creation operators $\hat{b}$ and the states $|1,0\rangle_f$, etc. are defined in C)

\begin{align*}
|\phi_0\rangle_{fC} &= \sqrt{p_0} |v\rangle_f |d_v\rangle_C + \sqrt{p_1} \left( |1,0\rangle_f + |0,1\rangle_f \right) |d_1\rangle_C + \sqrt{p_2} \left( |2,0\rangle_f + |1',1\rangle_f + |1',0\rangle_f + |0,2\rangle_f \right) |d_2\rangle_C. \quad (E.3) \\
\end{align*}

Upon possible further action of the adversary, the above state evolves to the normalized state

\begin{align*}
|\phi_0\rangle_{fC} \rightarrow \sum_{a+b \geq 0} \sum_{a+b \leq 2} |a,b\rangle_f |c_{a,b}\rangle_C &= |0,0\rangle_f \otimes |c_{0,0}\rangle_C + |1,0\rangle_f \otimes |c_{1,0}\rangle_C + |0,1\rangle_f \otimes |c_{0,1}\rangle_C + |2,0\rangle_f \otimes |c_{2,0}\rangle_C \\
&\quad + |1',1\rangle_f \otimes |c_{1',1}\rangle_C + |1',0\rangle_f \otimes |c_{1',0}\rangle_C + |0,2\rangle_f \otimes |c_{0,2}\rangle_C. \quad (E.4) \\
\end{align*}
where $|c_{a,b}\rangle_C \in \mathcal{H}_C$ (not necessarily orthogonal, nor normalized states) are associated to the cases when there are $a$ and $b$ photons entering Alice’s and Bob’s arms, respectively. Nevertheless, the states $|c_{a,b}\rangle_C$ are arbitrary and contain any number of photons. Therefore, the overall state before the photon(s) enter Alice’s and Bob’s labs is

$$|\phi_0\rangle_{ABFC} = |\phi_0\rangle_{AB} \otimes |\phi_0\rangle_{FC}$$

$$= \frac{1}{2} \left( |D_v,D_v\rangle + |D_v,R\rangle + |R,D_v\rangle + |R,R\rangle \right) \otimes \left( |0,0\rangle \otimes |c_{0,0}\rangle + |1,0\rangle \otimes |c_{1,0}\rangle + |0,1\rangle \otimes |c_{0,1}\rangle + |2,0\rangle \otimes |c_{2,0}\rangle + |1,1\rangle \otimes |c_{1,1}\rangle + |1',1\rangle \otimes |c_{1',1}\rangle + |0,2\rangle \otimes |c_{0,2}\rangle \right).$$ (E.5)

Let us denote Alice’s and Bob’s respective detectors’ efficiencies as $p^A_t$ and $p^B_t$, with $p^A_t = 1 - p^A_t$ and $p^B_t = 1 - p^B_t$. Therefore, in contrast to Alice’s and Bob’s actions in the ideal case (given by Equation (D.4)), the individual actions (say, for Alice) in the practical scenario are

$$|D_v\rangle |0\rangle \to |D_v\rangle |0\rangle,$$

$$|D_v\rangle |1\rangle \to \left( \sqrt{p^A_t} |D_v\rangle + \sqrt{p^B_t} |D_c\rangle \right) |0\rangle,$$

$$|D_v\rangle |2\rangle \to \left( p^A_t |D_t D_t'\rangle + p^B_t |D_R D_R'\rangle \right) |0\rangle,$$ (E.6)

where primed and unprimed states of the apparatuses correspond to at times $t'$ and $t$, respectively. Therefore, upon applying $U_1$, given in terms of Alice’s and Bob’s local actions described by (E.6), we obtain the state $|\phi_1\rangle_{ABFC} = U_1 |\phi_0\rangle_{ABFC}$.

Following this, the adversary will apply a quantum instrument to the returning photon state which, as before, can be modelled as an isometry, whose action is described as

$$I(a', b')_f |c_{a,b}\rangle_C = |0\rangle_S |e_{a',b'}^{a,b}\rangle_C + |1\rangle_S |f_{a',b'}^{a,b}\rangle_C + |v\rangle_S |g_{a',b'}^{a,b}\rangle_C + |m\rangle_S |h_{a',b'}^{a,b}\rangle_C,$$ (E.7)

where states $|e_{a',b'}^{a,b}\rangle_C, |f_{a',b'}^{a,b}\rangle_C, |g_{a',b'}^{a,b}\rangle_C, |h_{a',b'}^{a,b}\rangle_C \in \mathcal{H}_C$ are again not necessarily normalized, nor orthogonal. Note that, due to the action of $U_1$, the photon numbers $a, b$ are no longer correlated to $a', b' \in \{0, 1, 2\}$; nevertheless, we still have $a' + b' \leq 2$.

Again, by straightforward algebra we get $|\phi_2\rangle_{ABSC} = I |\phi_1\rangle_{ABFC}$. However, we are only interested in the key-generation rounds, i.e., we condition to the event when the server announces “1” and neither Alice nor Bob receives a click. Hence, omitting writing the message state $|1\rangle_S$, the final density operator (without the off-diagonal terms) of the system $ABC$ is

$$\rho_{ABC} = \frac{1}{N} \left[ |D_v,R\rangle_{AB} \langle D_v,R| \otimes |k_{0,0}\rangle_C \langle k_{0,0}| + |R,D_v\rangle_{AB} \langle R,D_v| \otimes |k_{1,1}\rangle_C \langle k_{1,1}| + |D_t,R\rangle_{AB} \langle D_t,R| \otimes |k_{1,1}\rangle_C \langle k_{1,1}| + |R,D_t\rangle_{AB} \langle R,D_t| \otimes |k_{1,1}\rangle_C \langle k_{1,1}| + |D_t D'_t R\rangle_{AB} \langle D_t D'_t R| \otimes |k_{0,0}\rangle_C \langle k_{0,0}| + |D_t D'_t R\rangle_{AB} \langle D_t D'_t R| \otimes |k_{2,2}\rangle_C \langle k_{2,2}| + |D_t D'_t R\rangle_{AB} \langle D_t D'_t R| \otimes |k_{1,1}\rangle_C \langle k_{1,1}| + |D_t D'_t R\rangle_{AB} \langle D_t D'_t R| \otimes |k_{k_{0,0}}\rangle_C \langle k_{k_{0,0}}|\right].$$ (E.8)
Note that, as before, we use commas in the states from $\mathcal{H}_A \otimes \mathcal{H}_B$ to separate the quantum numbers defining Alice’s and Bob’s apparatus states: $|D_tD'_t,R\rangle_{AB}$ means that Alice opted to detect, unsuccessfully (due to finite detection efficiency) the two photons present in her lab, while Bob set his apparatus to reflect, etc. The states $|k_{i,j}\rangle$, etc. are associated to the cases when Alice establishes the value $i$ and Bob $j$ as a key bit, and are given by

\[
|k_{0,0}\rangle = \frac{1}{\sqrt{2}} \left[ |f_{0,0}\rangle + |f_{0,1}\rangle + |f_{0,2}\rangle \right], \quad |k_{1,1}\rangle = \frac{1}{\sqrt{2}} \left[ |f_{0,0}\rangle + |f_{1,0}\rangle + |f_{2,0}\rangle \right],
\]

\[
|k_{0,1}\rangle = \frac{1}{\sqrt{2}} \sqrt{P_{\ell}} |f_{0,0}\rangle, \quad |k_{1,0}\rangle = \frac{1}{\sqrt{2}} \left[ |f_{0,0}\rangle + |f_{1,0}\rangle + |f_{2,0}\rangle \right] + |f_{1,1}\rangle + |f_{1,1}'\rangle + |f_{0,2}\rangle, \quad (E.9)
\]

The normalization constant $\mathcal{N}$ is, again, the probability to obtain the result “1”, when there were no clicks at the agents’ detectors, given by

\[
\mathcal{N} = \langle k_{0,0}|k_{0,0}\rangle + \langle k_{0,1}|k_{0,1}\rangle + \langle k_{0,2}|k_{0,2}\rangle + \langle k_{1,1}|k_{1,1}\rangle + \langle k_{1,1}|k_{1,1}\rangle + \langle k_{1,2}|k_{1,2}\rangle + \langle k_{1,3}|k_{1,3}\rangle + \langle k_{0,1}|k_{0,1}\rangle + \langle k_{0,1}|k_{0,1}\rangle + \langle k_{0,2}|k_{0,2}\rangle + \langle k_{0,3}|k_{0,3}\rangle + \langle k_{1,0}|k_{1,0}\rangle + \langle k_{1,0}|k_{1,0}\rangle + \langle k_{1,1}|k_{1,1}\rangle + \langle k_{1,2}|k_{1,2}\rangle + \langle k_{1,3}|k_{1,3}\rangle.
\]

In $\rho_{ABC}$, given by Equation (E.8), the state $|D_t,R\rangle \langle D_t,R|$, as in the ideal case, describes Alice detecting without a click and Bob reflecting, and is associated to a shared key bit of 0. However, $|D_t,R\rangle \langle D_t,R|$, $|D'_t,R\rangle \langle D'_t,R|$ and $|D_tD'_t,R\rangle \langle D_tD'_t,R|$ also correspond to a shared key bit of 0, and are a consequence of Alice’s imperfect detector and multi-photon events. Similarly, $|R,D_t\rangle \langle R,D_t|$, $|R,D'_t\rangle \langle R,D'_t|$ and $|R,D_tD'_t\rangle \langle R,D_tD'_t|$ are associated to a key bit 1. The remaining states correspond to errors, i.e., when the two users establish opposite key bit values. From the definitions of $k_{ij}$ and $\mathcal{N}$, we know that

\[
\frac{\langle k_{0,0}|k_{0,0}\rangle + \langle k_{0,1}|k_{1,1}\rangle + \langle k_{0,2}|k_{2,2}\rangle + \langle k_{3,0}|k_{3,0}\rangle}{\mathcal{N}} = p(D_t,R \lor D'_t,R \lor D_tD'_t,R \lor D_tD'_t,R|1). \quad (E.11)
\]

Here, by $p(P|C)$ we denote the conditional probability that the proposition $P$ holds (in the above case, Alice detects and observes no clicks, while Bob reflects), given that the condition $C$ is satisfied (in the above case, the server announces “1”). Therefore, using the following terminology for different probabilities (to be used in parameter estimation described in the next section), the
Therefore, in addition to different probabilities obtained from the experiment, we also need to consider the expression (D.14) for the probability to share the key is given by

\[
p_{\text{key}} = \left[ (k_{0,0}|k_{0,0}) + (k_{1,0}|k_{1,0}) + (k_{0,1}|k_{0,1}) + (k_{1,1}|k_{1,1}) + (k_{0,0}|k_{0,0}) \right] + \left[ (k_{1,1}|k_{1,1}) + (k_{1,1}|k_{1,1}) + (k_{0,1}|k_{0,1}) + (k_{1,1}|k_{1,1}) \right] + \left[ (k_{0,0}|k_{0,0}) + (k_{1,0}|k_{1,0}) \right] + \left[ (k_{0,1}|k_{0,1}) + (k_{1,1}|k_{1,1}) \right] + \left[ (k_{0,0}|k_{0,0}) + (k_{1,0}|k_{1,0}) \right] + \left[ (k_{0,1}|k_{0,1}) + (k_{1,1}|k_{1,1}) \right]
\]

\[
= p_{0,0} + p_{1,0} + p_{0,1} + p_{1,0} + p_{0,1} + p_{1,1} + p_{0,1} + p_{1,1}
\]

\[
= p(D_v, R \lor D_r, R \lor D_t') \lor D_t' \lor R, D_v \lor R, D_t \lor R, D_t' \lor R, D_t' \lor R ; 1)
\]

where \( p(D_v, R \lor D_r, R \lor D_t' \lor D_t' ; 1) \) represents the joint probability of the following event: Alice detects vacuum, Bob reflects, and the server announces the result "1"; and analogously for the other term. Recall that, for simplicity, we use the semi colon to denote logical AND between two propositions, instead of introducing the additional parenthesis for the first one and the standard symbol \( \land \). The probability of the error in the raw key is given by

\[
p_{\text{err}} = \left[ (k_{0,1}|k_{0,1}) + (k_{1,1}|k_{1,1}) + (k_{0,1}|k_{0,1}) + (k_{1,1}|k_{1,1}) + (k_{0,1}|k_{0,1}) + (k_{1,1}|k_{1,1}) + (k_{0,0}|k_{0,0}) \right]
\]

\[
= p_{0,1} + p_{1,0} + p_{0,1} + p_{1,0} + p_{0,1} + p_{1,0} + p_{0,1} + p_{1,0}
\]

\[
= p(D_v, D_v \lor D_t, D_v \lor D_v, D_v \lor D_t, D_v \lor D_t, D_v \lor D_v, D_v \lor D_t, D_v \lor D_v ; 1) + p(R, R ; 1)
\]

where \( p(D_v, D_v \lor D_t, D_v \lor D_v, D_v \lor D_t, D_v \lor D_t, D_v \lor D_v, D_v \lor D_v ; 1) \) represents the joint probability of the following event: Alice and Bob both detect vacuum, and that the server announces the result "1"; and analogously for the other term. Note that, the probabilities \( \tilde{p}_{i,j} \) can be observed from the experiment directly.

To obtain the secret key rate, we again use the bound given in [33], as

\[
S(A|C) \geq \frac{(k_{0,0}|k_{0,0}) + (k_{1,1}|k_{1,1})}{N} \left( h\left[ \frac{(k_{0,0}|k_{0,0})}{(k_{0,0}|k_{0,0}) + (k_{1,1}|k_{1,1})} \right] - h(\lambda_0) \right)
\]

\[
+ \frac{(k_{0,0}|k_{0,0}) + (k_{1,1}|k_{1,1})}{N} \left( h\left[ \frac{(k_{1,1}|k_{1,1})}{(k_{0,0}|k_{0,0}) + (k_{1,1}|k_{1,1})} \right] - h(\lambda_1) \right)
\]

\[
+ \frac{(k_{0,0}|k_{0,0}) + (k_{1,1}|k_{1,1})}{N} \left( h\left[ \frac{(k_{0,0}|k_{0,0})}{(k_{0,0}|k_{0,0}) + (k_{1,1}|k_{1,1})} \right] - h(\lambda_2) \right)
\]

\[
+ \frac{(k_{0,0}|k_{0,0}) + (k_{1,1}|k_{1,1})}{N} \left( h\left[ \frac{(k_{0,0}|k_{0,0})}{(k_{0,0}|k_{0,0}) + (k_{1,1}|k_{1,1})} \right] - h(\lambda_3) \right)
\]

\[
+ \frac{(k_{0,1}|k_{0,1}) + (k_{1,0}|k_{1,0})}{N} \left( h\left[ \frac{(k_{0,1}|k_{0,1})}{(k_{0,1}|k_{0,1}) + (k_{1,0}|k_{1,0})} \right] - h(\lambda_4) \right)
\]

with \( \lambda_i \)'s defined in the analogous way as in Equation (D.14). The first four terms in \( S(A|C) \) correspond to the keys shared between Alice and Bob, while the last term corresponds to errors in the key. However, we estimate the lower bound on \( S(A|C) \) by considering only the first term since its contribution to the entropy is far larger than that of any of the other terms. From the expression (D.14) for \( \lambda_0 \), we see that minimizing \( S(A|C) \) essentially means minimizing \( \text{Re} (k_{0,0}|k_{1,1}) \). Therefore, in addition to different probabilities obtained from the experiment, we also need to estimate \( \text{Re} (k_{0,0}|k_{1,1}) \). We proceed by computing the lower bound for \( \text{Re}^2 (k_{0,0}|k_{1,1}) \), i.e., for
Notice that the lower it is, the closer to $1/2 \lambda_0$ is, i.e., the closer to 1 the $h(\lambda_0)$ is, and the worst case scenario for $S(A|C)$, has the lowest value.

Let us use the following notation for simplification,

$$|x\rangle = |f_{1,0}^{1.0}\rangle + |f_{2,0}^{2.0}\rangle, \quad |y\rangle = |f_{0,1}^{0.1}\rangle + |f_{0,2}^{0.2}\rangle, \quad |z\rangle = |f_{1,1}^{1.1}\rangle + |f_{1,2}^{1.2}\rangle. \tag{E.15}$$

We can rewrite $|k_{0,0}\rangle$ and $|k_{1,1}\rangle$ from Equation (E.9), to obtain $\text{Re} \langle k_{0,0}|k_{1,1}\rangle$ as

$$\text{Re} \langle k_{0,0}|k_{1,1}\rangle = \frac{1}{4} \left[ \langle f_{0,0}^{0.0}|f_{0,0}^{0.0}\rangle + \text{Re} \langle x|f_{0,0}^{0.0}\rangle + \text{Re} \langle f_{0,0}^{0.0}|y\rangle + \text{Re} \langle x|y\rangle \right]. \tag{E.16}$$

Looking at the error term, $\langle k_{1,1}|k_{1,0}\rangle = Q/4$, and with straightforward substitution from the above into Equation (E.16), with $\langle f_{0,0}^{0.0}|f_{0,0}^{0.0}\rangle = 4\langle k_{0,1}|k_{0,1}\rangle = 4p_{0,1}$, we get

$$\langle k_{0,0}|k_{1,1}\rangle = \frac{Q}{8} + \frac{p_{0,1}}{2} - \frac{1}{8} \left[ \langle x|x\rangle + \langle y|y\rangle + \langle z|z\rangle \right] - \frac{1}{4} \left[ \langle x|z\rangle + \langle y|z\rangle + \langle f_{0,0}^{0.0}|z\rangle \right]. \tag{E.17}$$

In the ideal case, with no vacuum or multi-photon pulses, when $\langle x|x\rangle = 4\langle k_{0,0}|k_{0,0}\rangle = 4p_{0,0}$ and $\langle y|y\rangle = 4\langle k_{1,1}|k_{1,1}\rangle = 4p_{1,1}$, we recover Equation (D.11). By writing $\langle x|z\rangle = \langle x|x\rangle e^{r_{x,z}}$, we have

$$\text{Re} \langle x|z\rangle = |\langle x|z\rangle| \cos \varphi_{x,y} = |\langle x|y\rangle| |\langle z|z\rangle| \cos \chi_{x,z} \cos \varphi_{x,z} = \sqrt{\langle x|x\rangle} \sqrt{\langle z|z\rangle} \cos \theta_{x,z}, \tag{E.18}$$

where $\chi_{x,z}$ denotes the angle between $|x\rangle$ and $|z\rangle$ and $\cos \theta_{x,z} \equiv |\cos \chi_{x,z}| \cos \varphi_{x,z}$, and analogously for $\text{Re} \langle y|z\rangle$ and so on. Therefore, the final expression for $\text{Re} \langle k_{0,0}|k_{1,1}\rangle$ is

$$\text{Re} \langle k_{0,0}|k_{1,1}\rangle = \frac{Q}{8} + \frac{p_{0,1}}{2} - \frac{1}{8} \left[ \langle x|x\rangle + \langle y|y\rangle + \langle z|z\rangle \right] - \frac{1}{4} \left[ \sqrt{\langle x|x\rangle} \sqrt{\langle z|z\rangle} \cos \theta_{x,z} \right. \
 \left. - \frac{1}{4} \left[ \sqrt{\langle y|y\rangle} \sqrt{\langle z|z\rangle} \cos \theta_{y,z} \right]. \tag{E.19} \right.$$

To obtain $\langle x|x\rangle$ and $\langle y|y\rangle$, consider again $|k_{0,0}\rangle$ and $|k_{1,1}\rangle$ from Equation (E.9)

$$\langle k_{1,1}|k_{1,1}\rangle = \frac{1}{4} \left[ \langle f_{0,0}^{0.0}|f_{0,0}^{0.0}\rangle + \langle x|x\rangle + 2\text{Re} \langle f_{0,0}^{0.0}|x\rangle \right], \quad \langle k_{0,0}|k_{0,0}\rangle = \frac{1}{4} \left[ \langle f_{0,0}^{0.0}|f_{0,0}^{0.0}\rangle + \langle y|y\rangle + 2\text{Re} \langle f_{0,0}^{0.0}|y\rangle \right]. \tag{E.20}$$

Note that $\langle f_{0,0}^{0.0}|f_{0,0}^{0.0}\rangle = 4\langle k_{0,1}|k_{0,1}\rangle = 4p_{0,1}, \langle k_{0,0}|k_{0,0}\rangle = p_{0,0}$ and $\langle k_{1,1}|k_{1,1}\rangle = p_{1,1}$. Therefore, solving the quadratic equations obtained from (E.20), we get the following positive roots of $\sqrt{\langle x|x\rangle}$ and $\sqrt{\langle y|y\rangle}$.

$$\sqrt{\langle x|x\rangle} = 2 \left[ -\sqrt{p_{0,1}} \cos \theta_{x,f} + \sqrt{p_{1,1}} - (1 - \cos^2 \theta_{x,f}) p_{0,1} \right],$$

$$\sqrt{\langle y|y\rangle} = 2 \left[ -\sqrt{p_{0,1}} \cos \theta_{y,f} + \sqrt{p_{0,0}} - (1 - \cos^2 \theta_{y,f}) p_{0,1} \right]. \tag{E.21}$$

Analogously, for $\langle z|z\rangle$ we have

$$\langle z|z\rangle \beta + 2 \left[ \sqrt{\langle x|x\rangle} \cos \theta_{x,z} + \sqrt{\langle y|y\rangle} \cos \theta_{y,z} + 2\sqrt{p_{0,1}} \cos \theta_{f,z} \right] \sqrt{\langle z|z\rangle} \gamma$$

$$+ 4 \left[ p_{0,1} - p_{1,0} \right] + \left[ \langle x|x\rangle + \langle y|y\rangle + 2\sqrt{\langle x|x\rangle} \sqrt{\langle y|y\rangle} \cos \theta_{x,y} \right]$$

$$+ 4\sqrt{p_{0,1}} \left[ \sqrt{\langle x|x\rangle} \cos \theta_{x,f} + \sqrt{\langle y|y\rangle} \cos \theta_{y,f} \right] = 0, \tag{E.22}$$

where $\cos \theta_{x,z} \equiv |\cos \chi_{x,z}| \cos \varphi_{x,z}$ and analogously for $\cos \theta_{y,z}, \cos \theta_{f,z}$, etc. Again, solving the above quadratic equation, we can obtain the positive root of $\sqrt{\langle z|z\rangle}$. 

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E.2 Parameter estimation

Here, we briefly explain how to compute the relevant probabilities, \(p_{0,0}, p_{1,1}\) and \(p_{0,1}\), to compute \(S(A|C)\) in Equation (E.14), to eventually obtain the secret key rate in Equation (C.1).

Due to the nature of this protocol, in the ideal case, one can only expect \(p(1) = 1/8\), which is further reduced in the experimental case of imperfect detectors, etc. Therefore, it is useful if these probabilities could be computed without sacrificing any key-generation rounds. Below, we discuss the case with direct estimation where Alice and Bob use part of the key to obtain these probabilities, as well as the case of indirect estimation where no key-generation rounds are wasted.

E.2.1 Direct estimation

Here, we sacrifice \(\mu\) instances of the total \(N_{\text{raw}}\) key-generation rounds, to directly compute the relevant probabilities. However, since Alice’s and Bob’s detectors are imperfect, they cannot compute \(p_{0,0} = p(D_v,R; 1)\) and \(p_{1,1} = p(R,D_v; 1)\) directly, as they cannot differentiate the event \(D_v,R\) from the events \(D_v,R\), \(D'_v,R\) and \(D_vD'_v,R\), and analogously for \(R,D_v\). However, they can obtain \(\tilde{p}_{0,0} = p_{0,0} + p_{0,0}^2 + \tilde{p}_{0,0} = p(D_v,R \lor D'_v,R \lor D_vD'_v,R; 1)\) directly, and also \(\tilde{p}_{1,1}\).

They can then compute \(\tilde{p}_{0,0}^1 = \langle k_0^1|k_0^1\rangle\), \(\tilde{p}_{0,0}^2 = \langle k_0^2|k_0^2\rangle\) and \(\tilde{p}_{0,0}^3 = \langle k_0^3|k_0^3\rangle\), to eventually obtain \(p_{0,0}^1\). From Equation (E.9) one has

\[
\begin{align*}
p_{0,0}^1 &= p(D_v,R; 1) = \frac{p_d^A}{4} \left( \langle \ell_0^1, \ell_0^0 \rangle \right), \\
p_{0,0}^2 &= p(D_v,D'_v; 1) = \frac{p_d^A}{4} \langle \ell_0^1 | \ell_0^1 \rangle,
\end{align*}
\]

(E.23)

Even though Alice and Bob cannot compute the above probabilities, they can estimate them by looking at the events corresponding to the clicks, using the expressions

\[
\begin{align*}
p(D_v,D'_v; 1) &= \frac{p_d^A}{4} \langle \ell_0^1 | \ell_0^1 \rangle,
\end{align*}
\]

(E.24)

Therefore, we can write \((p_{0,0}^1 + p_{0,0}^2)\) and \(p_{0,0}^3\) as

\[
\begin{align*}
p_{0,0}^1 + p_{0,0}^2 &= \left( \frac{p_d^A}{p_d^A} \right) p(D_v,R \lor D'_v,R; 1), \\
p_{0,0}^3 &= \left( \frac{p_d^A}{p_d^A} \right)^2 p(D_v,D'_v,R; 1),
\end{align*}
\]

(E.25)

where only \(p(D_v,D'_v,R; 1)\) can be obtained using the rounds when Alice gets double clicks in her detector. However, \(p(D_v,R \lor D'_v,R \lor D_vD'_v,R \lor D'_vD_v,R; 1)\), corresponding to a single click in Alice’s detector, can also be obtained directly. Hence,

\[
\begin{align*}
p(D_v,R \lor D'_v,R; 1) &= p(D_v,R \lor D'_v,R \lor D_vD'_v,R \lor D'_vD_v,R; 1) - p(D_vD'_v,R; 1) - p(D_v,D'_v,R; 1) - p(D_v,D'_v,R; 1).
\end{align*}
\]

(E.26)

Also, we have

\[
\begin{align*}
p(D_vD'_v,R; 1) &= \frac{p_d^A p_d^A}{4} \langle \ell_0^1 | \ell_0^1 \rangle = p(D_vD'_v,R; 1).
\end{align*}
\]

(E.27)
Therefore, the required probabilities $p_{0,0}$ and $p_{1,1}$ are

\[
\begin{align*}
    p_{0,0} &= \tilde{p}_{0,0} - \left( \frac{p^A_v}{p^A_d} \right) p(D_c, R \vee D'_c, R \vee D'_c, R \vee D_d D'_d, R; 1) + \left( \frac{p^A_v}{p^A_d} \right)^2 p(D_c D'_c, R; 1), \\
    p_{1,1} &= \tilde{p}_{1,1} - \left( \frac{p^B_v}{p^B_d} \right) p(R, D_c \vee R, D'_c \vee R, D_d D'_d \vee R, D_c D'_c; 1) + \left( \frac{p^B_v}{p^B_d} \right)^2 p(D_c D'_c, R; 1).
\end{align*}
\]  

(E.28)

Additionally, to compute $p_{0,1}$, required to estimate $\text{Re} \langle k_0 | k_1 \rangle$ from Equation (E.19), we use $p_{0,1} = \tilde{p}_{0,1} - p_{0,1}^4 - p_{0,1}^2 - p_{0,1}^3 - p_{0,1}^5 - p_{0,1}^6$. Again, using straightforward algebra, we have

\[
\begin{align*}
    p_{0,1} &= \tilde{p}_{0,1} - \left( \frac{p^A_v}{p^A_d} \right) p(D_c, D_v \vee D_c, D'_c \vee D'_c, D_v \vee D_c D'_c, D_v \vee D'_c, D_v; 1) \\
    &\quad - \left( \frac{p^B_v}{p^B_d} \right) p(D_c, D_v \vee D_c, D'_c \vee D'_c, D_v \vee D_c D'_c, D_v \vee D'_c, D_v; 1) \\
    &\quad - 3 \left( \frac{p^A_v}{p^A_d} \right)^2 p(D_c D'_c, D_v; 1) - 3 \left( \frac{p^B_v}{p^B_d} \right)^2 p(D_v, D_c D'_c; 1) - 3 \left( \frac{p^A_v p^B_v}{p^A_d p^B_d} \right) p(D_c D'_c \vee D'_c, D_c; 1).
\end{align*}
\]  

(E.29)

Using the direct estimation method to compute all the relevant probabilities, we obtain the secret key rate $r$ (from Equation (C.1)) in Figure 5. We consider the implemented number of rounds, $10^5$, as a subset of a larger implementation and, therefore, use them to estimate the secret key rate. The probability of server announcing “1” during these rounds is, $p(1) = 0.0162$. Therefore, the amount of keys wasted during the parameter estimations is 1620 bits.

![Figure 5: Secret key rate, $r$, vs number of rounds, $N$, for the case of imperfect single-photon sources and detectors. The probability necessary for the plot are obtained from the experimental data.](image)

The probabilities of Equations (E.28) and (E.29) are the following: $p_{0,0} = (7.3 \pm 0.3) \times 10^{-3}$, $p_{1,1} = (5.5 \pm 0.3) \times 10^{-3}$, $p_{0,1} = (1.1 \pm 0.9) \times 10^{-4}$ and $p_{1,0} = (5.1 \pm 0.7) \times 10^{-4}$.

We assume $\epsilon = 10^{-5}$, $\epsilon_{EC} = 10^{-10}$ and $\epsilon_{PE} = 10^{-11}$. The value $\epsilon$ is a factor in the min-entropy expression used for the key rate computation and may actually be set by the user arbitrarily to maximize the key rate (see Lemma 1 from [32]). However, for our evaluations we simply set...
\( \epsilon' = 10^{-7} \) (optimizing this could only improve our results). For parameter estimation, we take \( \epsilon_{PE} = 10^{-11} \) and assume a confidence interval \( \delta = 10^{-4} \), given our experimental errors. The calculated secret key rate corresponds to the minimum lower bound of the entropy \( S(A|C) \) (see Equation [E.14]) over the confidence interval of the experimental probabilities. This minimum occurs for the highest value of the error probability \( p_{err} \) and the lowest of \( p_{key} \), and therefore represents the worst possible key rate within our experimental uncertainty.

We also stress that these results are lower-bounds. The actual key rate could be significantly higher. Indeed, to compute these lower bounds we took advantage of the strong sub-additivity of von Neumann entropy by actually discarding several components of the entropy function (components which would only have increased Eve’s uncertainty - thus by discarding them, we are giving an unrealistic advantage to the adversary causing the key rate to drop). Such a method gives a worst-case computation and the actual key rate will be better.

### E.2.2 Indirect estimation

To avoid wasting the rounds used for key-generation (when “1” was announced without any clicks at Alice’s and Bob’s detectors), we can use the remaining rounds (when “0”, “v” or “m” was announced or “1” was announced with click(s) at Alice’s and Bob’s detectors) for parameter estimation. For these cases, Alice and Bob can communicate over an authenticated channel to convey their respective action choices and resulting states to each other. Therefore, they can communicate for the non-useful rounds where server announces “0”, “v” or “m”, as well as the rounds where any of them detects a photon in case the server announces “1”. This method can be applied also in the ideal case described in section [D] but we present it only once for brevity.

We know that \( p_{0,0} = p(D_v,R;1) = p(D_v,R) - p(D_v,R;0) - p(D_v,R;m) \), where \( p(D_v,R) = p(D,R) - p(D_c,R) - p(D_v,R) - p(D_c,R) - p(D_v,R) - p(D_c,R) - p(D_v,R) - p(D_c,R) \). Note that, \( p(D,R) \) is the probability of Alice choosing to detect and Bob to reflect. Since Alice and Bob choose their actions at random, ideally \( p(D,D) = p(D,R) = p(R,D) = p(R,R) = 1/4 \). However, considering the finite sample size and the inefficiency of switching between the actions, Alice and Bob do not take these probabilities to be 1/4 but compute them considering only the non-useful rounds. Therefore, we have

\[
\begin{align*}
 p_{0,0} &= p(D,R) - p(D_c,R) - p(D_c,R) - p(D_v,R) - p(D_v,R) - p(D_v,R) - p(D_v,R) - p(D_v,R) \\
 &\quad - p(D_v,R) - p(D_v,R) - p(D_v,R) - p(D_v,R;0) - p(D_v,R;v) - p(D_v,R;m). \quad (E.30)
\end{align*}
\]

Note that Alice and Bob cannot directly compute all the quantities from the above expression, say, \( p(D_v,R;0) \), \( p(D_v,R;1) \), etc. They can compute \( p(D_v,R), p(D_v,R) \) and \( p(D_v,R) \) analogously as in the previous subsection, Equations (E.25). However, \( p(D_v,R \cup D_v,R \cup D_v,R \cup D_v,R) \) can be computed directly. We use \( p(D_v,R \cup D_v,R \cup D_v,R \cup D_v,R;0) \), directly observable, to estimate \( p(D_v,R;0) \). Therefore,

\[
\begin{align*}
 p(D_v,R;0) &= p(D_v,R \cup D_v,R \cup D_v,R \cup D_v,R;0) - p(D_v,R;0) - p(D_v,R;0) - p(D_v,R;0), \quad (E.31)
 p(D_v,R;0) \), \( p(D_v,R;0) \) and \( p(D_v,R;0) \), etc., can again be computed in the same way as before.
Therefore, the final expressions for $p_{0,0}$ and $p_{1,1}$, in terms of probabilities computed indirectly, are

\[
p_{0,0} = p(D,R) - p(D,c,R \lor D',R \lor D_lD'_c,R \lor D_lD'_e,R) - p(D,cD'_e,R) - p(D,v,R \lor D_e,R \lor D'_e,R \lor D_lD'_e,R; 0) \\
- p(D_v,R \lor D_e,R \lor D'_e,R \lor D_lD'_e,R; v) - p(D_v,R \lor D_e,R \lor D'_e,R \lor D_lD'_e,R; m) \\
+ \left(\frac{p_A}{p_d}\right)^2 [p(D,c,R \lor D'_e,R \lor D_lD'_c,R \lor D_lD'_e,R; 0) + p(D,c,R \lor D'_e,R \lor D_lD'_c,R \lor D_lD'_e,R; v) \\
+ p(D_c,R \lor D'_e,R \lor D_lD'_c,R \lor D_lD'_e,R; m) - p(D,c,R \lor D'_e,R \lor D_lD'_c,R \lor D_lD'_e,R)] \\
- \left(\frac{p_A}{p_d}\right)^2 [p(D,cD'_e,R; 0) + p(D,cD'_e,R; v) + p(D,cD'_e,R; m) - p(D,cD'_e,R)], \\
\text{(E.32)}
\]

\[
p_{1,1} = p(R,D) - p(R,D_c \lor R,D'_c \lor R,D_lD'_c \lor R,D_lD'_e) - p(R,D_cD'_e) - p(R,D_v \lor R,D_e \lor R,D'_e \lor R,D_lD'_e; 0) \\
- p(D_v \lor R,D_e \lor R,D'_e \lor R,D_lD'_e; v) - p(D_v \lor R,D_e \lor R,D'_e \lor R,D_lD'_e; m) \\
+ \left(\frac{p_B}{p_d}\right)^2 [p(R,D_c \lor R,D'_c \lor R,D_lD'_c \lor R,D_lD'_e; 0) + p(R,D_c \lor R,D'_c \lor R,D_lD'_c \lor R,D_cD'_e; v) \\
+ p(R,D_c \lor R,D'_c \lor R,D_lD'_c \lor R,D_cD'_e; m) - p(R,D_c \lor R,D'_c \lor R,D_lD'_c \lor R,D_cD'_e)] \\
- \left(\frac{p_B}{p_d}\right)^2 [p(R,D_cD'_e; 0) + p(R,D_cD'_e; v) + p(R,D_cD'_e; m) - p(R,D_cD'_e)], \\
\text{(E.33)}
\]

We can analogously estimate $p_{0,1}$ by computing $\hat{p}_{1,0}$ as

\[
\hat{p}_{1,0} = p(R,R; 1) = p(RR) - p(R,R; 0) - p(R,R; v) - p(R,R; m). \\
\text{(E.34)}
\]

Therefore,

\[
p_{0,1} = p(1) - \hat{p}_{0,0} - \hat{p}_{1,1} - \hat{p}_{1,0} - \left(\frac{p_A}{p_d}\right)^2 p(D,c,D_v \lor D,c,D'_e \lor D'_e,D_c \lor D_lD'_c,D_v \lor D_lD'_e,D_c; 1) \\
- \left(\frac{p_B}{p_d}\right)^2 p(D_v,D_c \lor D_l,D'_e \lor D'_e,D_c \lor D_v,D_cD'_e \lor D_v,D_lD'_e; 1) \\
- 3 \left(\frac{p_A}{p_d}\right)^2 p(D,cD'_e,D_v; 1) - 3 \left(\frac{p_B}{p_d}\right)^2 p(D_v,D_cD'_e; 1) - 3 \left(\frac{p_A}{p_d}\right)^2 \left(\frac{p_B}{p_d}\right)^2 p(D,c,D'_e \lor D'_e,D_c; 1). \\
\text{(E.35)}
\]

Note that, to compute $p_{\text{key}} = \hat{p}_{0,0} + \hat{p}_{1,1}$ using the indirect method, we have

\[
\hat{p}_{0,0} = p(D,R) - p(D,c,R \lor D'_e,R \lor D_lD'_c,R \lor D_lD'_e,R) - p(D,cD'_e,R) - p(D_v,R \lor D_e,R \lor D'_e,R \lor D_lD'_e,R; 0) \\
- p(D_v,R \lor D_e,R \lor D'_e,R \lor D_lD'_e,R; v) - p(D_v,R \lor D_e,R \lor D'_e,R \lor D_lD'_e,R; m), \\
\text{(E.36)}
\]

\[
\hat{p}_{1,1} = p(R,D) - p(R,D_c \lor R,D'_c \lor R,D_lD'_c \lor R,D_lD'_e) - p(R,D_cD'_e) - p(R,D_v \lor R,D_e \lor R,D'_e \lor R,D_lD'_e; 0) \\
- p(R,D_v \lor R,D_e \lor R,D'_e \lor R,D_lD'_e; v) - p(R,D_v \lor R,D_e \lor R,D'_e \lor R,D_lD'_e; m). \\
\text{(E.37)}
\]

From our experimental data, we obtain $p_{0,0} = (8 \pm 2) \times 10^{-3}$, $p_{1,1} = (6 \pm 2) \times 10^{-3}$, $p_{0,1} = (3 \pm 2) \times 10^{-3}$ and $p_{1,0} = (0.5 \pm 2) \times 10^{-3}$. All these values are compatible with those obtained with the direct estimation within experimental uncertainties, which can be reduced by employing a larger sample and improving the single-photon sources and detectors.
E.3 Verification

As for the ideal case, Alice and Bob need to verify that the state received from the server matches their expectation. In order to do that, they must estimate the probabilities $p_0$, $p_1$, and $p_2$ of Equation (E.3) and verify that they correspond to what the server declared. This task can be again performed by considering the cases where one or both users got click(s) upon choosing to detect the incoming photon(s). Considering, for instance, the cases when they both chose to detect, one can use Equations (E.5) and (E.6) to obtain the expression for $|\phi_1\rangle_{ABFC} = U_1|\phi_0\rangle_{ABFC}$, to see that

\[
p(D_c D'_c, D_v) = \left(\frac{p_B}{2}\right)^2 \langle c_2,0|c_2,0 \rangle,\quad p(D_v, D_c D'_c) = \left(\frac{p_B}{2}\right)^2 \langle c_0,2|c_0,2 \rangle,
\]

\[
p(D_c, D'_c \lor D'_c, D_c) = \frac{p_A^2 p_B^2}{4} \langle c_{1,1}'|c_{1,1}' \rangle + \langle c_{1,1}'|c_{1,1}' \rangle + \langle c_{0,2}|c_{0,2} \rangle, \tag{E.38}
\]

where $p(D_c, D'_c, D_v)$ is the probability that Alice observed two clicks and Bob none, when they both chose to detect. Similarly, $p(D_c, D'_c \lor D'_c, D_c)$ corresponds to the case when Alice and Bob observed one click each. Note that the logical OR operation is between $D_c$ and $D'_c$ and $D'_c, D_c$, and not just between $D'_c$ and $D'_c$ – it is the probability of the union of two events, each describing both Alice’s and Bob’s states of the apparatus. Hence, they can easily obtain $p_2$ using

\[
p_2 = \langle c_2,0|c_2,0 \rangle + \langle c_{1,1}'|c_{1,1}' \rangle + \langle c_{1,1}'|c_{1,1}' \rangle + \langle c_{0,2}|c_{0,2} \rangle, \tag{E.39}
\]

as well as verify if $\langle c_{1,1}'|c_{1,1}' \rangle \approx \langle c_{1,1}'|c_{1,1}' \rangle \approx \langle c_{2,0}|c_{2,0} \rangle \approx \langle c_{0,2}|c_{0,2} \rangle$. Analogously, they can obtain $p_1 = \langle c_{1,0}|c_{1,0} \rangle + \langle c_{0,1}|c_{0,1} \rangle$ by noticing that

\[
p(D_c, D_v \lor D'_c, D'_c \lor D_c D'_c, D_v \lor D_c D'_c, D_v) = \left(\frac{p_A^2}{4}\right) \langle c_{1,0}|c_{1,0} \rangle + \left(\frac{p_A^2 p_B^2}{4}\right) \langle c_{1,1}'|c_{1,1}' \rangle + \langle c_{1,1}'|c_{1,1}' \rangle + 2 \langle c_{2,0}|c_{2,0} \rangle, \tag{E.40}
\]

\[
p(D_v, D_c \lor D'_c, D_c \lor D'_c D_c \lor D_v, D_c D'_c \lor D_v, D'_c D'_c) = \left(\frac{p_B^2}{4}\right) \langle c_{0,1}|c_{0,1} \rangle + \left(\frac{p_A^2 p_B^2}{4}\right) \langle c_{1,1}'|c_{1,1}' \rangle + \langle c_{1,1}'|c_{1,1}' \rangle + 2 \langle c_{0,2}|c_{0,2} \rangle, \tag{E.41}
\]

where $p(D_c, D_v \lor D_c D'_c \lor D'_c D_c \lor D_c D'_c, D_v \lor D_c D'_c, D_v)$ is the probability that Alice observed 1 click and Bob none, when they both chose to detect.

We apply the described procedure and obtain $p_0 = 0.72$, $p_1 = 0.16$, $p_2 = 0.12$. Our source emits on average 0.35 photons per round (0.5 s). Assuming poissonian statistics of emission with that average, we should obtain $p_0 = 0.70$, $p_1 = 0.25$, $p_2 = 0.04$. The discrepancy between the measured and the expected values can be attributed to a deviation from poissonian statistics, due to fluctuations in the source parameters (temperature, pump wavelength, etc.) while taking the measurement, which lasted about 30 hours.

Except for that verification, Alice and Bob can also check if the server’s announcements are compatible with their actions, as in Table 2. In this case, however, experimental imperfections allow the server for multiple announcements for the same actions of the users. If all the experimental parameters, such as losses, splitting ratios of beam splitters, etc., are characterized prior of the beginning of the protocol, Alice and Bob can still verify the honesty of the server by comparing the statistics of the server announcements with what they expect from the properties of the set-up.
F Dependence on detection efficiency

In this section, we discuss the dependence of the secret key rate on the detection efficiencies of Alice’s and Bob’s detectors, \( p_A^d \) and \( p_B^d \), respectively. Note that, since we only consider the first term in Equation (E.14) to estimate a bound on \( S(A|C) \), we only need to compute the probabilities \( p_{0,0}, p_{1,0}, p_{0,1}, p_{1,0} \) and \( p(1) \). However, \( p_{0,0}, p_{1,0}, p_{0,1}, p_{1,0} \) are independent of \( p_A^d \) and \( p_B^d \), and it is only \( p(1) \) that has this dependence. Therefore, using the experimental data corresponding to \( p_A^\ell = 1 - p_A^d = 0.42 \) and \( p_B^\ell = 1 - p_B^d = 0.42 \), we can rewrite \( p(1) \) with the explicit dependence on the general parameters, \( \tilde{p}_A^\ell \) and \( \tilde{p}_B^\ell \), as

\[
\mathcal{N}(\tilde{p}_A^\ell, \tilde{p}_B^\ell) = \langle k_{0,0}|k_{0,0} \rangle + \sqrt{\frac{\tilde{p}_A^\ell}{p_A^\ell}} \left( \langle k_{0,0}^1|k_{0,0}^1 \rangle + \langle k_{0,0}^2|k_{0,0}^2 \rangle \right) + \left( \frac{\tilde{p}_A^\ell}{p_A^\ell} \right) \langle k_{0,0}^3|k_{0,0}^3 \rangle
\]

\[
+ \langle k_{1,1}|k_{1,1} \rangle + \sqrt{\frac{\tilde{p}_B^\ell}{p_B^\ell}} \left( \langle k_{1,1}^1|k_{1,1}^1 \rangle + \langle k_{1,1}^2|k_{1,1}^2 \rangle \right) + \left( \frac{\tilde{p}_B^\ell}{p_B^\ell} \right) \langle k_{1,1}^3|k_{1,1}^3 \rangle
\]

\[
+ \langle k_{0,1}|k_{0,1} \rangle + \langle k_{1,0}|k_{1,0} \rangle + \sqrt{\frac{\tilde{p}_A^\ell}{p_A^\ell}} \langle k_{0,1}^1|k_{0,1}^1 \rangle + \sqrt{\frac{\tilde{p}_B^\ell}{p_B^\ell}} \langle k_{0,1}^2|k_{0,1}^2 \rangle + \left( \frac{\tilde{p}_B^\ell}{p_B^\ell} \right) \langle k_{0,1}^3|k_{0,1}^3 \rangle
\]

\[
+ \sqrt{\frac{\tilde{p}_A^\ell}{p_A^\ell}} \sqrt{\frac{\tilde{p}_B^\ell}{p_B^\ell}} \left( \langle k_{0,1}^4|k_{0,1}^4 \rangle + \langle k_{0,1}^5|k_{0,1}^5 \rangle \right) + \left( \frac{\tilde{p}_A^\ell}{p_A^\ell} \right) \langle k_{0,1}^6|k_{0,1}^6 \rangle
\]

\[
+ \left( \frac{\tilde{p}_A^\ell}{p_A^\ell} \right) \langle k_{0,1}^7|k_{0,1}^7 \rangle + \left( \frac{\tilde{p}_B^\ell}{p_B^\ell} \right) \langle k_{0,1}^8|k_{0,1}^8 \rangle
\]

\[
= p(1)(\tilde{p}_A^\ell, \tilde{p}_B^\ell).
\]

Moreover, \( p_{err} = p(1) - p_{key} \) is also modified accordingly, to be used in computing \( Q = p_{err}/p(1) \) to obtain the secret key rate (C.1).