Hydrodynamic equations of anisotropic, polarized and inhomogeneous superfluid vortex tangles

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Abstract

We include the effects of anisotropy and polarization in the hydrodynamics of inhomogeneous vortex tangles, thus generalizing the well known Hall-Vinen-Bekarevich-Khalatnikov equations, which do not take them in consideration. These effects contribute to the mutual friction force $F_{ns}$ between normal and superfluid components and to the vortex tension force $\rho_s T$. These equations are complemented by an evolution equation for the vortex line density, which takes into account these contributions.

1 Introduction

The possibility to use turbulent superfluids to explore with relative facility turbulent flows with high Reynolds numbers is an increasingly exciting perspective. Quantized vortices in superfluids have been mainly studied in two typical situations: rotating superfluids and counterflow experiments, the latter meaning the presence of a heat flux, but with zero barycentric motion. In these situations the vortices are modeled, respectively, as an array of parallel rectilinear vortices or as an almost isotropic tangle. In both cases, the mutual force between the normal component and the superfluid due to the presence of vortex lines is well known, and the so-called vortex line tension is zero [1]–[4].

However, in other situations, as rotating counterflow or non stationary Couette and Poiseuille flows, one expects a partially polarized vortex tangle, as a compromise between the orienting effect of a rotation or of a macroscopic velocity gradient, and the randomizing effect of the relative velocity of normal and superfluid components. In these cases, the mutual friction force $F_{ns}$ between these components, as well as the nonvanishing tension of the vortex lines $\rho_s T$, which are fundamental ingredients of the hydrodynamics of turbulent superfluids in the well-known Hall-Vinen-Bekarevich-Khalatnikov (HVBK) model [5, 6], are not sufficiently known. Thus, the exploration of $F_{ns}$ and $T$ for partially polarized tangles with nonvanishing average curvature of vortex lines is an open topic.

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The quantitative values of the corrections are expected to be relevant in some steady states, where a sufficiently high polarization may induce macroscopic eddies [7], and many of the proposed experiments which involve nonstationary flows, for their own sake or in view to the application of classical measurement techniques, as for instance thin oscillating wires or small-particle velocimetry, to superfluids. Thus a basic understanding of these effects is essential to obtain the form of $F_{ns}$ and $T$. This is the aim of this paper.

The structure of the paper is the following one: in Section 2 an introduction to the usual HVBK model is made, with special emphasis on the expression of the mutual friction force $F_{ns}$ and the tension $T$ in the cases of rotating helium and counterflow superfluid turbulence; in Section 3 a microscopic expression for the friction force and vortex tension is given, checking their limit of validity in HVBK model; in Section 4 a more general expression of the HVBK model is given through more exhaustive expressions for the mutual friction force $F_{ns}$ and tension $T$. The model is completed by a generalization of the Vinen’s equation, as an evolution equation for the vortex line density $L$. At least, in Section 5 an application of the model to the interesting case of rotating counterflow turbulence is made, which takes advance of the recent study on the anisotropy of the vortex tangle.

2 HVBK model. Evolution equations for the normal and the superfluid components

It is known that if a superfluid ($^{4}\text{He}$ and $^{3}\text{He}$ liquid helium, Bose-Einstein condensates, neutron stars, ...) rotates at a constant angular velocity $\Omega$, exceeding a critical value $\Omega_{c}$, an ordered array of quantized vortex lines of equal circulation $\kappa$ parallel to the rotation axis is created [1]–[4]. The quantum of vorticity is $\kappa = h/m_{4}$ in superfluid $^{4}\text{He}$, with $h$ the Planck constant and $m_{4}$ the mass of $^{4}\text{He}$ atom; in $^{3}\text{He}$ and neutron stars, which are formed by fermionic particles, the superfluidity is due to the formation of Cooper pairs, and therefore it is $\kappa = h/2m$, where $m$ is the mass of an $^{3}\text{He}$ atom or of a neutron, respectively.

In most literature, the motion of a superfluid is modeled using Landau’s two-fluid model, which regards the fluid to be made of two completely mixed components: the normal fluid and the superfluid, with densities $\rho_{n}$ and $\rho_{s}$ respectively, and velocities $v_{n}$ and $v_{s}$ respectively, with total mass density $\rho$ and barycentric velocity $v$ defined by $\rho = \rho_{s} + \rho_{n}$ and $\rho v = \rho_{s}v_{s} + \rho_{n}v_{n}$. The first component is related to thermally excited states (phonons and rotons) that form a classical Navier-Stokes viscous fluid. The second component is related to the quantum coherent ground state and it is an ideal fluid, which does not experience dissipation neither carries entropy.

If the superfluid is put in rotation with angular velocity $\Omega$ higher than $\Omega_{c}$, the ordered array of parallel quantized vortex lines is described by introducing the line density $L$, defined as the average vortex line length per unit volume, equivalent to the areal density of vortex lines, which is proportional to the angular velocity [1]–[4], namely $L = 2\Omega/\kappa$.

It is well known too that a disordered tangle of quantized vortex lines is created in the so-called counterflow superfluid turbulence [1]–[4], characterized by no matter flow but only heat transport, exceeding a critical heat flux $q_{c}$. When the turbulence is fully developed, the line density $L$ is proportional to the square of the averaged counterflow velocity vector

$$V_{ns} = [v_{ns}]_{av} = \frac{1}{\Lambda} \int v_{ns} d\Lambda$$ (2.1)
related to heat flux \((v_{ns} = v_n - v_s)\) being the microscopic counterflow velocity) \([1]-[4]\). \(L \simeq \gamma_H^2 V_{ns}^2 / \kappa^2\), the dimensionless coefficient \(\gamma_H\) being dependent on the temperature. In \((2.1)\) and in the following, capital letters denote local macroscopic velocities averaged over a small mesoscopic volume \(\Lambda\), threaded by a high density of vortex lines.

Due to the smallness of the quantum of circulation, even a relatively weak rotation or a small counterflow velocity produce a large density of vortex filaments. It is therefore possible to develop a set of macroscopic hydrodynamic equations which average over the presence of many individual vortex lines and incorporate the macroscopic effects of the vortices in the evolution equations for superfluid and normal fluid velocities.

A set of hydrodynamical equations frequently used is the Hall-Vinen-Bekarevich-Khalatnikov (HVBK) model \([1, 5, 6, 8]-[12]\). These equations, which were derived by a number of researchers over the years, are those of the two-fluid model, modified to incorporate the presence of vortices. Here, for sake of simplicity, we consider the incompressible HVBK equations, which in an inertial frame are written \([1, 12]\)

\[
\begin{align*}
\rho_n \frac{\partial V_n}{\partial t} + \rho_n (V_n \cdot \nabla) V_n &= -\rho_n \frac{\partial}{\partial \rho} \nabla p_n - \rho_s S \nabla T + F_{ns} + \eta \nabla^2 V_n, \\
\rho_s \frac{\partial V_s}{\partial t} + \rho_s (V_s \cdot \nabla) V_s &= -\rho_s \frac{\partial}{\partial \rho} \nabla p_s + \rho_s S \nabla T - F_{ns} + \rho_s T.
\end{align*}
\]

In these equations \(p_n\) and \(p_s\) are the effective pressures acting on the normal and the superfluid component, respectively, defined as \(\nabla p_n = \nabla p + (\rho_s/2)\nabla V_{ns}^2\) and \(\nabla p_s = \nabla p - (\rho_n/2)\nabla V_{ns}^2\), \(p\) the total pressure, \(S\) the entropy, \(T\) the absolute temperature, and \(\eta\) the dynamic viscosity of the normal component. The effects of the vortices are described by \(F_{ns}\), the friction force exerted by the superfluid component on the normal component — which bears an opposite sign in the evolution equation \((2.3)\) — and \(\rho_s T\), the vortex tension force, related to the average curvature of vortices, which will be discussed below.

The expression of the mutual friction force \(F_{ns}\) in the HVBK equations is \([12]\)

\[
F_{ns}^{(HVBK)} = \rho_s \alpha \omega \times (\tilde{\omega} \times (V_{ns} - \tilde{\beta} \nabla \times \tilde{\omega})) + \rho_s \alpha' \tilde{\omega} \times (V_{ns} - \tilde{\beta} \nabla \times \tilde{\omega}),
\]

where \(\tilde{\omega} = \nabla \times V_s\) is the local averaged superfluid vorticity, \(\hat{\omega} = \tilde{\omega}/|\tilde{\omega}|\) the unit vector along \(\tilde{\omega}\), and \(\tilde{\beta}\) the vortex tension parameter, defined as \([1]\)

\[
\tilde{\beta} = \frac{\epsilon V}{\kappa \rho_s} = \frac{\kappa}{4\pi} \ln \left( \frac{c}{a_0 L^{1/2}} \right),
\]

with \(c\) a constant of the order of unity, \(a_0\) the radius of the vortex core, of about 1 Å, and \(\epsilon V\) the energy per unit length of vortex line. Coefficients \(\alpha\) and \(\alpha'\) depend on temperature and describe the interaction between the normal fluid and the vortices. They are linked to the well-known Hall-Vinen coefficients \(B\) and \(B'\) by the relations \(\alpha = B(\rho_n/2\rho)\), \(\alpha' = B'(\rho_n/2\rho)\) \([1]\). Note that this friction force is mediated through the presence of the vortex lines, as \(\tilde{\omega}\) is related to them.

In \((2.3)\) \(\rho_s T\) is the vortex tension force whose microscopic meaning will be discussed in Section 3. In the HVBK equations \(T\) is substituted with

\[
T^{(HVBK)} = (\tilde{\beta} \nabla \times \tilde{\omega}) \times \tilde{\omega} = \tilde{\beta} (\tilde{\omega} \cdot \nabla) \tilde{\omega}.
\]

From this expression it follows that in this approximation \(T\) vanishes when \(\tilde{\omega}\) is homogeneous \((\nabla \tilde{\omega} = 0)\) or when the vortex lines are parallel to each other, in which case \(\tilde{\omega}\) will be orthogonal to the gradient of \(\tilde{\omega}\).
In the regular vortex array produced by the pure rotation, the vorticity $\vec{\omega}$ equals $2\Omega$ everywhere, $\Omega$ being the angular velocity, and the mutual friction force $F_{ns}$ (2.4) assumes the Hall-Vinen expression

$$F_{ns} = 2\rho_s\alpha\hat{\Omega} \times [\hat{\Omega} \times (V_n - V_s)] + 2\rho_s\alpha'\hat{\Omega} \times (V_n - V_s),$$

(2.7)

because $\nabla \times \vec{\omega} = 0$, and $\vec{\omega} = 2\Omega$. As seen in (2.6), the vortex tension $T$ vanishes, because the vortices are straight lines.

In counterflow superfluid turbulence the vortex tangle is supposed isotropic and the mutual friction force (2.4) is expressed by [1]–[4]

$$F_{ns} = -\frac{2}{3}\rho_s\kappa_\alpha L \dot{V}_{ns},$$

(2.8)

However, the hypothesis of complete isotropy of the tangle is not confirmed by the simulation of the dynamics of the vortex tangle made by Schwarz [13–15] and in the experiments [16]. Anyway, the assumed isotropy of the tangle implies that the vortex tension $T$ vanishes because the microscopic vorticity would be isotropically distributed, in such a way that the average $\vec{\omega}$ is null everywhere.

The two quantities $F_{ns}$ and $T$ are the aim of the present paper. First, we will analyze the limit of validity of expressions (2.4) and (2.6) for $F_{ns}$ and $T$ in the HVBK model, restricted to regions with a high density of vortex lines, all pointing in the same direction, and with the same curvature vector. Thus we will pay special attention to the fluctuations in the direction of the vortex lines and to the local anisotropy and polarization of the tangle as well as on its inhomogeneity, and their consequences on $F_{ns}$ and $T$.

### 3 Expressions for $F_{ns}$ and $T$

The HVBK equations (2.2)–(2.3) have been used in several problems in superfluid hydrodynamics, for instance, to study the Taylor-Couette flow in helium II [8–10, 12] and, recently, the jump in the rotational speed observed in neutron stars [17, 18]. The numerical simulations obtained with these equations show the formation of macroscopic vortices, but they do not include the microscopic chaotic dynamics expressed, for instance, by the fluctuations of the tangents to the vortex lines.

Our aim here is to incorporate inhomogeneities, anisotropy, polarization, and tension of the tangle of vortex lines in an extension of the HVBK hydrodynamic equations. To this purpose, we make a critical analysis of these equations, and we perform a suitable modification of them. Furthermore, we include an additional equation for the evolution of $L$, besides the equations (2.2) and (2.3), because $L$ is related to forces and tensions.

#### 3.1 Microscopic expressions for friction force and vortex tension

To derive the expressions (2.4) and (2.6) of $F_{ns}$ and of $T$, used in the HVBK equations, and to determine their limit of validity, we consider the microscopic description of a vortex tangle in the vortex filament model by Schwarz [13–15]. In this model, a quantized vortex filament is assumed as a classical vortex line in the superfluid with a hollow core and quantized circulation $\kappa$. The vortex line is described by a vectorial function $s(\xi, t)$, $\xi$ being the arc-length, $s' = \partial s/\partial \xi$ the unit vector tangent along the vortex line, and $s'' = \partial^2 s/\partial \xi^2$ the curvature vector. The
normal component reacts to a moving vortex by producing the "microscopic" mutual friction force $-f_{MF}$, which can be written as [1]

$$f_{MF} = \alpha \rho_s \kappa s' \times [s' \times (v_{n} - v_{sl})] + \alpha' \rho_s \kappa s' \times (v_{n} - v_{sl}), \tag{3.1}$$

where $v_{sl} = v_s + v_1$ is the "local superfluid velocity", sum of the superfluid velocity at large distance from any vortex line and of the self-induced velocity, described by the tangent unit vector $s'$ and by the curvature $s''$. In the "local induction approximation", the self-induced velocity $v_1$ is approximated by [1–4]

$$v_1 \simeq v_1^{loc} = \hat{\beta} [s' \times s'']. \tag{3.2}$$

The intensity of $v_1$ is $|v_1| \simeq \hat{\beta}/R$, with $R$ the curvature radius of the vortex lines. The self-induced velocity is zero if the vortices are straight lines. Observing that $v_n - v_{sl} = v_{ns} - v_1$, being $v_{ns}$ the microscopic counterflow velocity, the mutual friction force per unit length assumes the expression

$$f_{MF} = \alpha \rho_s \kappa s' \times [s' \times (v_{ns} - v_1)] + \alpha' \rho_s \kappa s' \times (v_{ns} - v_1). \tag{3.3}$$

In the evolution equations for the velocities of normal and superfluid components, the "macroscopic" mutual friction force $F_{ns}$ per unit volume which superfluid and normal components mutually exert, is obtained by averaging (3.3) over a small volume $\Lambda$; one has

$$< f_{MF} > = \frac{\int f_{MF} d\xi}{\int d\xi} = \frac{1}{\Lambda L} \int f_{MF} d\xi, \tag{3.4}$$

where the integral is made over all the vortices contained in the volume $\Lambda$; to obtain $F_{ns}$, we must multiply the average (3.4), which denotes the average mutual friction force per unit length, by $L$, which denotes the length of vortex lines per unit of volume. One obtains

$$F_{ns} = [f_{ns}]_{av} = L < f_{MF} > = \alpha \rho_s \kappa L < s' \times [s' \times (v_{ns} - v_1)] > + \alpha' \rho_s \kappa L < s' \times (v_{ns} - v_1) >. \tag{3.5}$$

The vortex tension force $T$ arises from the microscopic form of the evolution equation of $v_s$, which, neglecting the mutual friction force, is written [1]

$$\frac{\partial v_s}{\partial t} = v_L \times \omega_{micr} + \nabla \mu, \tag{3.6}$$

where $v_L$ is the velocity of the vortex line and the term $\nabla \mu$ describes the effects of pressure and temperature gradients. Here $v_s$ denotes the microscopic velocity of the superfluid around the vortex and $\omega_{micr} = \nabla \times v_s$ the vorticity of the single vortex line.

Note that $v_L = v_{sl} = v_s + v_1$ [1], with $v_1$ the "self induced-velocity". When the average on the mesoscopic volume $\Lambda$ is taken one has

$$\frac{\partial v_s}{\partial t} = [v_s \times (\nabla \times v_s)]_{av} + [v_1 \times (\nabla \times v_s)]_{av} - \frac{\nabla p_s}{\rho} + S \nabla T, \tag{3.7}$$

where $\nabla \mu = -\nabla p_s/\rho + S \nabla T$. Furthermore, it is $v_s \times (\nabla \times v_s) = -(v_s \cdot \nabla)v_s + (1/2)\nabla v_s^2$. In the HVBK equation, one makes the approximation $[v_s \times (\nabla \times v_s)]_{av} = -(V_s \cdot \nabla)V_s$, obtaining

$$\rho_s \frac{\partial v_s}{\partial t} + \rho_s (V_s \cdot \nabla) V_s = \rho_s T - \frac{\rho_s}{\rho} \nabla p_s + \rho_s S \nabla T. \tag{3.8}$$
This allows to identify the tension $T$ as

$$T = [v_i \times (\nabla \times v_s)]_{av} = \kappa L < v_i \times s' >,$$  \hspace{1cm} (3.9)

where we have denoted the average of $v_i \times (\nabla \times v_s)$, over the small vortex tangle contained in the mesoscopic volume $\Lambda$, with angular brackets, as in (3.3).

Relation (3.9) may be rewritten in several equivalent forms, by taking into account (3.2) and some vectorial identities. For instance, $< s' \times v_i >$ may be expressed, in the local induction approximation, as

$$< s' \times v_i > = \tilde{\beta} < s' \times (s' \times s'') > = -\tilde{\beta} < s'' >,$$  \hspace{1cm} (3.10)

where it has been taken into account that $s'$ is a unit vector.

Using (3.10), expression (3.9) may be rewritten as

$$T = [v_i \times \tilde{\omega}_{micr}]}_{av} = \kappa L \tilde{\beta} < (s' \times s'') \times s' > = \kappa L \tilde{\beta} < s'' >.$$

Another useful form for $T$ may be obtained by using

$$s'' = (s' \cdot \nabla)s' = -s' \times (\nabla \times s')$$  \hspace{1cm} (3.12)

so that expression (3.10) may be rewritten as

$$< s' \times v_i > = \tilde{\beta} < s' \times (s' \times s'') > = \tilde{\beta} < s' \times (\nabla \times s') >.$$

Expressions (3.9), (3.11) and (3.13) will be used in the next sections.

### 3.2 Limit of validity of the HVBK's equations

As it can be seen by comparing (2.4) and (3.3), in expression (2.4) of the mutual friction force used in the HVBK equations the quantities $< s' \times (s' \times (v_{ns} - v_i)) >$ and $< s' \times (v_{ns} - v_i) >$ are approximated with $\tilde{\omega} \times [\tilde{\omega} \times (V_{ns} - \tilde{\beta} \nabla \times \tilde{\omega})]$ and $\tilde{\omega} \times (V_{ns} - \tilde{\beta} \nabla \times \tilde{\omega})$ respectively, and $|\nabla \times v_s|$ is approximated by $\kappa L$. The same approximations, when applied to the vortex tension $T$, yield $(\beta \nabla \times \tilde{\omega}) \times \tilde{\omega}$.

We analyze in this section the limits of validity of the approximations leading to these expressions for the mutual friction force $F_{ns}$ and tension $T$, and we will show that these limits are related to the neglect of the second order moments of the vector $s'$.

We consider first the two averaged quantities $< s' \times (s' \times v_{ns}) >$ and $< s' \times v_{ns} >$. We denote with $p = < s' >$ and $V_{ns}$ the averaged values of $s'$ and $v_{ns}$, and with $\delta s'$ and $\delta v_{ns}$ their respective fluctuations. The average value of the unit vector $s'$ is called tangle polarity [18–21], and — recall that the superfluid vorticity is quantized — is linked to the local averaged superfluid vorticity $\tilde{\omega}$ by the relation

$$p = < s' > = \frac{1}{\Lambda L} \int s'd\xi = \frac{\tilde{\omega}}{\kappa L} = \frac{\nabla \times V_s}{\kappa L}.$$

(3.14)

For a totally polarized tangle with all the tangents $s'$ parallel to each other, $|p| = 1$. In two dimensions, the polarization may be interpreted as $(n^+ - n^-)/(n^+ + n^-)$ with $n^+$ the vortices rotating in one direction and $n^-$ the vortices rotating in opposite direction. Processes producing a partial separation of $+$ and $-$ vortices will thus change $p$, and may lead to the formation of relatively large eddies [7]. Thus, the dynamics of $p$ may be certainly important and it can be influenced by the boundary conditions.
With the notation just introduced, neglecting the fluctuations of the counterflow velocity $V_{ns}$, we may write $s' = p + \delta s'$. Since $\langle \delta s' \rangle = 0$, we obtain

$$\langle s' \times (s' \times V_{ns}) \rangle = p \times (p \times V_{ns}) + \langle \delta s' \times (\delta s' \times V_{ns}) \rangle,$$

$$\langle s' \times V_{ns} \rangle = p \times V_{ns}.$$  

Note that in the equation (2.4) only the first terms in the right-hand side of equations (3.15) and (3.16) appear. Indeed, recalling equation (3.14), the first terms in the right-hand side of equations (3.15) and (3.16) can be written as

$$p \times (p \times V_{ns}) = \frac{1}{\kappa L} \hat{\omega} \times [\hat{\omega} \times V_{ns}] = \langle s' \times (s' \times V_{ns}) \rangle^{(HV BK)},$$

$$p \times V_{ns} = \frac{1}{\kappa L} \hat{\omega} \times V_{ns} = \langle s' \times V_{ns} \rangle^{(HV BK)},$$

whereas the other terms, quadratic in the fluctuations, have been neglected. In general, this will not be correct, for instance, in the limiting situations of an isotropic tangle, $p = 0$, but $\langle s' \times (s' \times V_{ns}) \rangle = (2/3)V_{ns}$ $[19]$.

Note also that in this simplified hypothesis, the second term in equation (3.15), dependent on the second moments of $s'$, can be neglected only if the orientational fluctuations of this unit vector in the small volume $\Lambda$ are very small, i.e. if most of the vortex lines in the volume have the same direction.

Furthermore, making use of equation (3.12), we obtain

$$\langle v_1 \rangle = \tilde{\beta} \langle s' \times s'' \rangle = \tilde{\beta} \langle (\nabla \times s') - [s' \cdot (\nabla \times s')]s' \rangle = \tilde{\beta} \langle (U - s's') \cdot (\nabla \times s') \rangle,$$  

(3.19)

with $U$ the unit matrix and $s's'$ the diadic product.

From equations (3.17)–(3.18) it is seen that in the HVBK equations all the second-order moments of the fluctuations in $s'$ are neglected and it is implicitly assumed that $\kappa L \simeq |\nabla \times V_s|$, that is $| \langle s' \rangle | \simeq 1$. Furthermore, we note that in the HVBK equations the quantity $\langle v_1 \rangle$ is simply approximated by $\tilde{\beta} \langle \hat{\omega} \times s' \rangle$ and therefore the quantity $| \langle s' \cdot (\nabla \times s') \rangle s' |$ is neglected.

The relation

$$\kappa L \simeq |\nabla \times V_s| = |\tilde{\omega}|,$$  

(3.20)

which is used to evaluate the line density $L$, is the most critical hypothesis underlying the HVBK equations. In fact, if in a mesoscopic region $\Lambda$ there are several vortex lines oriented in a random way, the line density $L$ in this point will be very different from $L \simeq |\nabla \times V_s|/\kappa$, which corresponds to an extreme polarization with all or almost all the lines pointing out in the same direction, pointed out by $| \langle s' \rangle | = 1$. For example, if in the small region $\Lambda$ a vortex loop is present, the average circulation relative to this loop is zero, but it is not so for the line density $L$.

As a consequence, the HVBK equations describe correctly the interaction between the normal component and the vortex tangle only in mesoscopic regions with a high density array of vortex lines, all pointing in the same direction (totally polarized) and with the same curvature vector. This fact explains the results of the numerical simulations obtained using the HVBK equations, which show the formation of macroscopic vortices and in which the microscopic chaotic dynamic behavior of the vortices does not appear.
4 Generalization of HVBK equations

In a previous paper [22] a hydrodynamical model of turbulent superfluids was formulated, which uses as fundamental fields the mass density $\rho$, the velocity $\mathbf{v}$ of the helium as a whole, the temperature $T$, the heat flux $\mathbf{q}$ and the line density $L$. In that work, only situations in which the tangle can be supposed approximately isotropic were considered, as is often made in the study of counterflow superfluid turbulence. In Ref. [19] the anisotropy of the vortex tangle was studied, restricting the study to homogeneous situations, and neglecting the influence of the vortex tension. In another paper [23] we have considered a more general situation, taking into account of inhomogeneities, anisotropy and polarization of the vortex tangle, studying the plane Couette and Poiseuille flow. For sake of simplicity, in that work, we have neglected the vortex tension $T$.

4.1 New determination of $F_{ns}$ and $T$

We want now to obtain expressions for $F_{ns}$ and $T$ overcoming some of the restrictions mentioned in the Section 3.2. Using the local-induction approximation, and neglecting the fluctuations of the relative velocity $\mathbf{V}_{ns}$, from (3.5) we deduce that $F_{ns}$ can be written [19]

$$F_{ns} = -\rho \kappa L \left[ \alpha < U - s's' > + \alpha' < \mathbf{W} \cdot s' > \right] \cdot \mathbf{V}_{ns} + \rho \kappa L \beta \left[ \alpha < s' \times s'' > + \alpha' < s'' > \right],$$

(4.1)

with $\mathbf{W}$ the Ricci tensor (a completely antisymmetric third-order tensor such that $\mathbf{W} \cdot s' = -s' \times \mathbf{V}_{ns}$).

Introducing the tensor $\Pi = \Pi^s + \Pi^a$, with [19]

$$\Pi^s \equiv \frac{3}{2} < U - s's' >, \quad \Pi^a \equiv \frac{3}{2} \frac{\alpha'}{\alpha} < \mathbf{W} \cdot s' >,$$

(4.2)

the vectors $\mathbf{I}$ and $\mathbf{J}$ [19]

$$\mathbf{I} \equiv \frac{\int s' \times s'' d\xi}{\int |s''| d\xi}, \quad \mathbf{J} \equiv \frac{\int s'' d\xi}{\int |s''| d\xi},$$

(4.3)

and $c_1 L^{1/2} = \frac{1}{2L} \int |s''| d\xi$, a characteristic measure of the vortex tangle introduced by Schwarz [15], the mutual friction force (4.1) can be written in a compact way as

$$F_{ns} = \alpha \rho \kappa L \left[ -\frac{2}{3} \Pi \cdot \mathbf{V}_{ns} + \beta c_1 L^{1/2} \left( \mathbf{I} + \frac{\alpha'}{\alpha} \mathbf{J} \right) \right].$$

(4.4)

The tensor $\Pi$ is especially useful to describe the geometrical properties of the tangle related with the orientational distribution of the vortex lines, whose local direction is indicated by the unit tangent $s'$. When the tangle is not completely polarized nor fully isotropic, it should be described by a tensor, rather than by a vector. The tensor $\Pi$ does not contain information on the curvature of the lines because it does not contain $s''$; this is furnished by the vectors $\mathbf{I}$ and $\mathbf{J}$.

The vortex tension $T$ is also linked to the curvature vector $\mathbf{J}$; indeed it results

$$T = [\mathbf{v}_1 \times (\nabla \times \mathbf{v}_s)]_{av} = \kappa L < (\tilde{\beta} s' \times s''') \times s' > = \kappa L \tilde{\beta} < s'' > = \kappa L^{3/2} c_1 \mathbf{J}.$$

(4.5)

Observe that the tension of the vortex line, as the curvature vector, is zero in pure rotation with parallel straight vortex lines and in the isotropic tangle produced in well developed
counterflow turbulence, but it is not so in the presence of simultaneous counterflow and rotation \[24, 25\] or in the first stages of the turbulence \[26, 27\] or in the transient states after sudden acceleration in plane Couette and Poiseuille flow. For instance, \(T\) could be different than zero in Kelvin helical vortex waves, which appear when a sufficiently intense counterflow is superposed to an axial rotation. In the transition from the array of straight lines to the array of helical vortex lines, a spiral tension would appear. The tension \(T\) would be a rotating quantity, as well as the curvature \(s''\) of the helical vortices. If the wavelength becomes shorter than the characteristic observational length, the average \(T\) will become equal to zero, but if the wavelength is high enough, it could lead to specific secondary flows of \(v_s\), through equation \eqref{eq:2.3}.

As a first modification to the HVBK equations we propose to take into account the fluctuations of the vector \(s'\), which appear in the tensor \(<U - s's'>\), but, for sake of simplicity, we propose to neglect the fluctuations of \(s' \times (∇ \times s') = -s''\).

To what concerns the vectors \(I\) and \(J\), and the vortex tension \(T\), with these approximations in mind, we have the following constitutive relations

\[c_1 L^{1/2} I = <s' \times s'' > ≃ <U - s's'> \cdot ∇ \times p,\]

\[c_1 L^{1/2} J = <s'' > ≃ -p \times (∇ \times p) = (p \cdot (∇p))p - \frac{1}{2} ∇^2 p.\]

The required equations for \(F_{ns}\) and \(T\) are

\[F_{ns} = -ρ_s κ L α <U - s's'> [(V_{ns} - \tilde{β} (∇ \times p)] + ρ_s κ L α [(p \times (∇ \times p)] =
\]

\[= -\frac{2}{3} ρ_s κ L α [V_{ns} - \tilde{β} (∇ \times p)],\]

\[ρ_s T = -ρ_s κ L \tilde{β} p \times (∇ \times p).\]

In Section 5, some illustration of \eqref{eq:4.8} and \eqref{eq:4.9} will be presented, with different form of the field \(Π\). But, to complete our hydrodynamical model, we must first add to equations \eqref{eq:4.8}–\eqref{eq:4.9} an evolution equation for the line density \(L\).

### 4.2 Generalized Vinen equation including polarization, anisotropy, and inhomogeneities

In expressions \eqref{eq:4.1} and \eqref{eq:4.5} — or \eqref{eq:4.8} and \eqref{eq:4.9} — for \(F_{ns}\) and \(T\), it appears \(L\); therefore, to have a full description for the evolution of the system, an evolution equation for \(L\) is needed.

The evolution equation for \(L\) in counterflow superfluid turbulence was formulated by Vinen \[27\]. Assuming homogeneous turbulence, such an equation is

\[\frac{dL}{dt} = α_v V_{ns} L^{3/2} - β_v κ L^2,\]

with \(α_v\) and \(β_v\) dimensionless parameters.

A microscopic derivation of this equation was given by Schwarz \[13, 15\] on the basis of the dynamics of the vortices, neglecting however a term in the average curvature vector \(s''\). In Ref. \[28\], because we were interested to study wall effects on the evolution of \(L\), the following extension of equation \eqref{eq:4.10} was written

\[\frac{dL}{dt} ≃ α_{c_1} V_{ns} \cdot IL^{3/2} + α'_{c_1} V_{ns} \cdot JL^{3/2} - α_{c_2} c_2 L^2,\]
with \( c_1, I \) and \( J \) defined in equations (4.3), \( c_1 \) in the line below (4.3), and \( c_2 = \frac{1}{\Lambda L^2} \int |s''|^2 d\xi \).

Substituting in this equation the relations (4.6-4.7), the following evolution equation for \( L \) which takes into account the polarization and the anisotropy of the tangle is obtained

\[
\frac{dL}{dt} \simeq \alpha L V_{ns} \cdot < U - s's' > \cdot (\nabla \times \mathbf{p}) - \alpha'L V_{ns} \cdot \mathbf{p} \times (\nabla \times \mathbf{p}) - \alpha \tilde{\beta} c_2(p)L^2.
\] (4.12)

In Refs. [23], [25]–[26], [29]–[31] Vinen’s equation was modified to describe more complex situations, as for instance the coupled situation of counterflow and rotation and in Couette and Poiseuille flows. Taking in mind the equation (4.12) and the proposed equation of Ref. [23], the previous equation would become

\[
\frac{dL}{dt} \simeq \alpha L V_{ns} \cdot < U - s's' > \cdot (\nabla \times \mathbf{p}) - \alpha'L V_{ns} \cdot \mathbf{p} \times (\nabla \times \mathbf{p}) - \alpha \tilde{\beta} \kappa L^2 \left[ 1 - \sqrt{|\mathbf{p}|} \right] \left[ 1 - B \sqrt{|\mathbf{p}|} \right],
\] (4.13)

where \( B \) is a dimensionless coefficient lower than 1. A rigorous derivation of its form would require a microscopic extension of Schwarz’s model including rotation effects. To check the consistency of the obtained evolution equation (4.13) for the vortex line density equation, in the next subsection an analysis based on the formalism of linear irreversible thermodynamics will be made.

When inhomogeneities in the line density \( L \) are taken into account, the evolution equation for line density \( L \) must include a vortex density flux \( J_L \), as

\[
\frac{\partial L}{\partial t} + \nabla \cdot J_L = \sigma_L,
\] (4.14)

where \( \sigma_L \) stands for the right-hand side of equation (4.13). The flux \( J_L \) can also be expressed in terms of a convective part \( L \mathbf{v}_L \) with \( \mathbf{v}_L \) the tangle velocity and a dissipative part \( J_L^d \). The particular form of \( J_L \) is also open to the debate. One can suppose that \( J_L \) is an independent variable [32], or, more simply, supposing that \( J_L \) is a dependent field: we have found [22] for it the form \( J_L = \nu_0 \mathbf{q} \), \( \nu_0 \) being a coefficient describing the interaction between the vortex tangle and the heat flux \( \mathbf{q} \), which is linked to the counterflow velocity by the relation \( \mathbf{q} = \rho_s T S V_{ns} \).

The heat flux \( \mathbf{q} \) may also be expressed in terms of \( \nabla T \) and \( \nabla L \) as

\[
\mathbf{q} = -\frac{\eta}{\kappa L_0} \nabla T - \frac{\chi_0}{\kappa L_0} \nabla L.
\] (4.15)

Thus, the dissipative part of the vortex flux \( J_L^d \) may be expressed as

\[
J_L^d = -\frac{\nu_0 \eta_0}{\kappa L_0} \nabla T - \frac{\nu_0 \chi_0}{\kappa L_0} \nabla L.
\] (4.16)

In isothermal situations, \( J_L^d \) may be written as \( J_L^d = -D \nabla L \), with \( D \) a vortex diffusion coefficient defined by \( D = \nu_0 \chi_0 / \kappa L_0 \) [28].

### 4.3 Consequences of Onsager-Casimir reciprocity relation

In this Subsection, we show that another modification of the vortex line density evolution equation is necessary to insure the thermodynamic consistency of the evolution equations for \( L \) and for \( V_s \), according to the formalism of linear irreversible thermodynamics [31, 33, 34]. This analysis requires the presence of another term linked to the tension of vortices.
We follow the general lines of [31, 33, 34] with the aim to study the consequences of the Onsager-Casimir reciprocity relations on the evolution equations of $V_s$ and $L$ proposed in the previous Sections. According to the formalism of nonequilibrium thermodynamics one may obtain evolution equations for $V_s$ and $L$ by writing $dV_s/dt$ and $dL/dt$ in terms of their conjugate thermodynamic forces $-\rho_s V_{ns}$ and $\epsilon_V$. The evolution equation for $V_s$, neglecting inhomogeneous contributions of pressure, temperature and velocity, in an inertial frame, is written
\[
\frac{dV_s}{dt} = -F_{ns} - \rho_s T = \alpha \rho_s \kappa L \frac{2}{3} \Pi \cdot V_{ns} - \alpha \rho_s \kappa L \frac{2}{3} \Pi \cdot (\nabla \times p) + \rho_s \kappa L \beta \hat{p} \times (\nabla \times p) \quad (4.17)
\]
and the evolution equation for $L$ is written
\[
\frac{dL}{dt} = \alpha L V_{ns} \cdot <U - s's'> \cdot (\nabla \times p) - \alpha' L V_{ns} \cdot p \times (\nabla \times p) - \alpha \beta c_2(p)L^2. \quad (4.18)
\]

However, in the right-hand side of (4.18) additional contributions must be included to make (4.18) thermodynamically consistent with (4.17).

Similarly to that presented in [31], we write $dV_s/dt$ and $dL/dt$ in matrix form using the equations (4.17) and (4.18), and by means of Onsager-Casimir reciprocity we obtain an additional contribution to the evolution equation for $L$. The result is
\[
\left( \frac{dV_s}{dt} \right) = L \left( -\frac{2}{3} \rho_s \kappa \Pi - \frac{2}{3} \rho_s \kappa \nabla \times p \cdot \Pi + \frac{1}{\rho_s} p \times (\nabla \times p) - \frac{2}{3} \rho_s \kappa \nabla \times p \cdot \Pi + \frac{1}{\rho_s} p \times (\nabla \times p) - \frac{\alpha \beta}{\epsilon_V} c_2(p) \right) \left( -\rho_s V \right)
\]
where for $c_2(p)$ we can choose the expression $c_2(p) = \left( 1 - \sqrt{|p|} \right) \left( 1 - B \sqrt{|p|} \right)$ found in Ref. [23].

Therefore the equation for $dL/dt$ becomes
\[
\frac{dL}{dt} = \alpha L V_{ns} \cdot <U - s's'> \cdot (\nabla \times p) - (1 + \alpha') L V_{ns} \cdot p \times (\nabla \times p) - \alpha \beta c_2(p)L^2, \quad (4.20)
\]
or, similarly,
\[
\frac{dL}{dt} = \frac{2}{3} \alpha L V_{ns} \cdot \Pi \cdot (\nabla \times p) - \frac{1}{\rho_s} V_{ns} \cdot T - \alpha \beta \left( 1 - \sqrt{|p|} \right) \left( 1 - B \sqrt{|p|} \right) L^2. \quad (4.21)
\]

The new term not contained in the evolution equation (4.18) for $L$ is the coupling term between $dL/dt$ and $-\rho_s V_{ns}$ in the matrix in (4.19), which is linked to the tension of vortices and it is null when the tension $T$ is null.

Note that, introducing the tensor
\[
\Pi' = <U - s's'> + \frac{1 + \alpha'}{\alpha} <W \cdot s'> \quad (4.22)
\]
the system (4.19) can be written
\[
\left( \frac{dV_s}{dt} \right) = L \left( -\frac{2}{3} \rho_s \kappa \Pi - \frac{2}{3} \rho_s \kappa \Pi' \cdot (\nabla \times p) - \frac{\alpha \beta}{\epsilon_V} c_2(p) \right) \left( -\rho_s V \right) \quad (4.23)
\]
5 Application to rotating counterflow turbulence

Combination of counterflow and rotation is especially interesting in the context of the present paper, because it provides intermediate situations between an isotropic tangle and a totally anisotropic array of parallel vortices. The polarization and anisotropy of the tangle depend on the ordering influence of the rotation, which tend to align the vortices parallel to the rotation axis, and the randomizing aspects of the counterflow. In this section we will apply the previous results.

5.1 Pure counterflow

In most studies of counterflow superfluid turbulence, the vortex tangle is supposed homogeneous and isotropic in the distribution of tangent vectors $s'$ with respect to the counterflow velocity. Experimental observations [16] and numerical simulations [15] show anisotropy in the vortex line distribution with vortices concentrated in planes orthogonal to $V_{ns}$, and this anisotropy grows with the counterflow. Therefore, assuming $V_{ns}$ in the direction of the $x$ axis and isotropy in planes orthogonal to it, one can choose for the tensor $\Pi$ (equation (4.2)) the following expression [19]

$$\Pi_H = \Pi^s_H = \frac{3}{2} \begin{pmatrix} 2a & 0 & 0 \\ 0 & 1-a & 0 \\ 0 & 0 & 1-a \end{pmatrix}; \quad (5.1)$$

here $a$ is the anisotropic parameter linked to the coefficients $I_\parallel$ and $I_\perp$ introduced by Schwarz [15] by the relations $I_\parallel = 2a$ and $I_\perp = 1 - a$. Being known the values of coefficient $a$, by accurate measurements of second sound, the second order moments of the unit vector $s'$ remain determined. Indeed, it results

$$<s'_x^2> = 1 - 2a, \quad <s'_y^2> = <s'_z^2> = a. \quad (5.2)$$

For the vector $I$, we propose, according to (4.6), the following constitutive relation

$$c_1 L^{1/2} I = <s' \times s''> \simeq <U - s's'> \cdot \nabla \times p, \quad (5.3)$$

The asymmetric part of the tensor $\Pi$, the curvature vector $J$ and the vortex tension $T$ in this case are zero, owing to the supposed isotropy of the tangle, which implies $<s''> = 0$. Introducing (5.1) into (4.8) and (4.13) we could obtain the influence of the anisotropy $a$ on the mutual friction force $F_{ns}$ and the evolution of $L$.

5.2 Simultaneous counterflow and rotation

Under the simultaneous influence of counterflow velocity $V_{ns}$ and rotation with angular speed $\Omega$, rotation tends to align vortex lines parallel to rotation axis, whereas counterflow velocity tends to produce a disordered (isotropic) tangle. In these situations one has partially polarized tangles, requiring the full detailed analysis presented here. We assume that the total ensemble of vortex lines is a superposition of both contributions [19]

$$\Pi' = (1-b)\Pi^s_H + b\Pi^R_R, \quad \Pi^a = c\Pi^R_R. \quad (5.4)$$

In (5.4), $b$ and $c$ are parameters between 0 and 1, describing the relative weight of the array of vortex lines parallel to $\Omega$ and the disordered tangle of counterflow: when $b = c = 0$
we recover an approximately isotropic tangle, and when \( b = c = 1 \) the ordered array (total anisotropy). In general situations these coefficients depend on \( \Omega \) and \( V_{ns} \), or on the velocity gradient and \( V_{ns} \).

In Ref. [19], supposing negligible the anisotropy due to the counterflow, as compared to the one produced by the angular velocity along the first axis, and assuming isotropy in the plane orthogonal to this axis, we obtained

\[
\Pi^s = \begin{pmatrix}
1 - b & 0 & 0 \\
0 & 1 + \frac{b}{2} & 0 \\
0 & 0 & 1 + \frac{b}{2}
\end{pmatrix}, \quad \Pi^a = \frac{3}{2} \alpha' c \begin{pmatrix}
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix},
\]

with the following interpretation of \( b \) and \( c \) in terms of moments of the tangent vector \( s' \)

\[
<s'_x> = c, \quad <s'_x^2> = \frac{1 + 2b}{3}, \quad <s'_y^2> = <s'_z^2> = \frac{1 - b}{3}.
\]

A microscopic evaluation of coefficients \( b \) and \( c \) was made in Ref. [19], based on a paramagnetic analogy, which reflects the competition between the orienting effects of \( \Omega \) and the randomizing effects of \( V_{ns} \), respectively analogous to the orienting effects of a magnetic field \( H \) on magnetic dipoles \( \mu \) and the randomizing effects of thermal excitations. In rotating counterflow, the rotation \( \Omega \) orients the vortices along its direction, in an analogous way to \( H \), whereas the counterflow \( V_{ns} \) plays a disordering role. We found, using the Langevin model of paramagnetism,

\[
<s'_x> = \coth x - \frac{1}{x},
\]

and

\[
<s'_x^2> = 1 + \frac{2}{x} \left[ \frac{1}{x} - \coth x \right],
\]

with \( x \) proportional to \( \Omega \kappa / V_{ns}^2 \). Similar situations would be found in Couette and Poiseuille flow where the velocity gradient, instead than a rotation, contributes to orient the vortices [35, 36] with \( \Omega \) replaced by the local shear rate.

A particular illustration of the combination of (5.5) and (4.8) could be provided by the decay of small perturbations of \( V_n - V_s \). According to (2.2) and (2.3) and the assumption \( T = 0 \) and \( \nabla T \approx 0 \), one would have, in a linear approach

\[
\frac{\partial (V_n - V_s)}{\partial t} = -\frac{2}{3} \alpha \frac{\rho}{\rho_n} \kappa L (1 - b) (V_n - V_s) + \eta \nabla^2 V_n,
\]

being \( V_n - V_s \) along \( x \) axis. The first term on the right hand side, describes \( \frac{\rho}{\rho_s \rho_n} F_{ns} \), taking into account (4.8) and (5.5).

For long-wave perturbations, \( \nabla^2 V_n \) will be small as compared to the first term and the decay of \( V_n - V_s \) to its steady state value will be exponential, with a relaxation time depending on the anisotropy parameter \( b \). This time could provide a measurement of \( b \) independent of the measurement provided by the attenuation of the second sound along different axis of the system. An analogous analysis could be carried out to explore the anisotropy parameter \( a \) introduced in (5.1) for tangles in pure counterflow.

A second particular illustration may underline the role of the term in \( I \) in expression (4.4) for \( F_{ns} \). This would be reflected in the temperature gradient needed to maintain a steady state value of \( V_{ns} \), at constant pressure. Under these conditions, one has

\[
-2 \rho_n S \nabla T + 2 F_{ns} + \eta \nabla^2 V_n = 0.
\]
Thus, neglecting, for simplicity, the term in $\eta \nabla^2 V_n$ and using (4.4) for $F_{ns}$, we have

$$\alpha \kappa L \left[ \frac{2}{3} \mathbf{\Pi} \cdot \mathbf{V}_{ns} + \beta c_1 L^{1/2} \mathbf{I} \right] = S \nabla T. \quad (5.11)$$

It is clear that the presence of $\mathbf{I}$ modifies the relation between $\mathbf{V}_{ns}$ and $\nabla T$; this could allow to measure the influence of the corresponding term.

These two simple illustrations show that the generalized equation for $F_{ns}$ considered in (4.4) is indeed expected to have specific applications.

6 Conclusions

Summarizing, in this work we propose to substitute in the HVBK equations the expression of the mutual friction force $F_{ns}$ (equation (2.4)) with the equation (4.8), to take into account of the second-order moment of $s'$, allowing in this way that not all the vortex lines in the small volume element under consideration have the same direction. Analogously, the expression (2.6) for the tension $\mathbf{T}$ has been replaced by equation (4.9). The coefficients $b$ and $c$ appearing in (5.5) can be related to the counterflow velocity and to the angular velocity by the relations (5.6), with $< s'_p >$ and $< s'_x >$ expressed by (5.7) and (5.8). For the vectors $\mathbf{I}$ and $\mathbf{J}$ we have chosen the constitutive relations (4.6) and (4.7).

Another important modification consists in adding to the evolution equations for $\mathbf{V}_n$ and $\mathbf{V}_s$ an evolution equation for $L$ including the effects of polarization anisotropy and inhomogeneities. In fact, in a general situation, it is not correct in (3.20) to substitute $\kappa L$ with the modulus of the curl of $\mathbf{V}_s$. Relations (4.13) and (4.14) avoid this simplification.

In Section 5, we have concentrate our attention on rotating counterflow; another illustration of the possible physical consequences of the tension $\mathbf{T}$, may be found in steady Poiseuille flows in an isothermal situation. There, addition of (2.2) and (2.3) leads for the equation describing the velocity profile of the normal component

$$- \nabla p + \eta \nabla^2 \mathbf{V}_n + \rho_s \mathbf{T} = 0. \quad (6.1)$$

Thus, if $\mathbf{T} = 0$, the $\mathbf{V}_n$ profile will have the typical parabolic form of Newtonian fluid. Modifications of fluids velocity profile have been studied by Godfrey and Barenghi [36]. From (6.1), these modifications would be related to the form of $\mathbf{T}$. A completely polarized array of parallel rectilinear vortices or a completely isotropic vortex tangle would have $\mathbf{T} = 0$ and would not modify the form of the $\mathbf{V}_n$ profile. However, net tension different from zero could arise at relatively low fluxes due to the influence of vortices pinned to the walls, which in the presence of the flow would have a curvature opposite to the velocity of the flow. In particular, they would have an influence on a small-amplitude oscillating flow along a cylinder with pinned vortices.

Finally, to mention yet another situation where the effects of the polarization are important, and for which experimental data have been recently available, is the eddy formation in two-dimensional counterflow in the presence of a transverse cylinder [7]. The presence of the cylinder seems to separate clockwise and counterclockwise vortices, thus modifying the polarization. The non vanishing polarization turns out to be able to produce macroscopic eddies near the cylinder (downflow with respect to the velocity of the superfluid component), as a collective effect of the vortices. In this example, small changes in the polarization produce dramatic changes in the hydrodynamic flow.
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