EXISTENCE FOR WEAKLY COERCIVE NONLINEAR DIFFUSION EQUATIONS VIA A VARIATIONAL PRINCIPLE

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Abstract. We are concerned with the study of the well-posedness of a nonlinear diffusion equation with a monotonically increasing multivalued time-dependent nonlinearity derived from a convex continuous potential having a superlinear growth to infinity. The results in this paper state that the solution of the nonlinear equation can be retrieved as the null minimizer of an appropriate minimization problem for a convex functional involving the potential and its conjugate. This approach, inspired by the Brezis-Ekeland variational principle, provides new existence results under minimal growth and coercivity conditions.

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1 Introduction

We are concerned with the study of the well-posedness of a nonlinear diffusion equation with a monotonically increasing discontinuous nonlinearity derived from a convex continuous potential, by using a dual formulation of this equation as a minimization of an appropriate convex functional. The idea of identifying the solutions of evolution equations as the minimas of certain functionals is due to Brezis and Ekeland and originates in their papers published in 1976 (see [8] and [9]). During the past decades this approach has enjoyed much attention, as seen in the various literature and in some more recently published monograph and papers (see e.g., [2], [3], [17], [16], [24], [22], [23], [5], [6], [20]). In [20] two cases were considered, the first for a continuous potential with a polynomial growth and the second for a singular potential. The latter has provided the existence of the solution to variational inequality which models a free boundary flow.

The challenging part in this duality principle is the proof of the well-posedness of the evolution equation as a consequence of the existence of a null minimizer in the associated minimization problem (that is a solution which minimizes the functional to zero). A general receipt for proving this implication does not exist, it rather depending on the good choice of the functional and on the particularities of the potential of the nonlinearity arising in the diffusion term. This way of approaching the well-posedness of nonlinear diffusion equations by a dual formulation as a minimization problem is extremely useful especially when a direct approach by using the semigroup theory (see e.g., [4], [12]) or other classical variational results (see e.g., [19]) cannot be followed due either to the low regularity of the data or to the weak coercivity of the potential.

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In this work, the nonlinearity in the diffusion term is more general and it has a time and space dependent potential assumed to have a weak coercivity and no particular regularity with respect to time and space. The paper is organized in two parts. At the beginning we investigate the case with the potential and its conjugate depending on time and space. We prove that the minimization problem has at least one solution, unique if the functional is strictly convex. This seems to be a good candidate for the solution to the nonlinear equation, reason for which it can be viewed as a generalized or variational solution. If the admissible set is restricted by imposing a $L^\infty$-constraint on the state, then the generalized solution which minimizes the functional to zero turns out to be quite the weak solution to the nonlinear equation.

The second part concerns the case in which the potential does not depend on space. The main result establishes that the null minimizer in the minimization problem is the unique solution to the nonlinear equation, provided that the potential exhibits a symmetry at large values of the argument.

We would like to mention the benefit of such a duality approach, which allows an elegant proof of the existence for a time dependent diffusion equation, under general assumptions, by making possible its replacement by the problem of minimizing a convex functional with a linear state equation. We also stress that the existence results obtained in this way are not covered and do not follow by the general existence theory of porous media equations, as well as that of time dependent nonlinear infinite dimensional Cauchy problems.

2 Problem presentation

We deal with the problem

\[
\frac{\partial y}{\partial t} - \Delta \beta(t, x, y) \ni f \quad \text{in} \ Q := (0, T) \times \Omega,
\]

\[
- \frac{\partial \beta(t, x, y)}{\partial \nu} = \alpha \beta(t, x, y) \quad \text{on} \ \Sigma := (0, T) \times \Gamma,
\]

\[
y(0, x) = y_0 \quad \text{in} \ \Omega,
\]

where $\Omega$ is an open bounded subset of $\mathbb{R}^N$, $N \leq 3$, with the boundary $\Gamma$ sufficiently smooth, $T$ is finite and $\beta$ has a potential $j$. The notation $\frac{\partial}{\partial \nu}$ represents the normal derivative and $\alpha$ is positive.

In this paper we assume that $j : Q \times \mathbb{R} \to (-\infty, \infty]$ and has the following properties:

(\text{h}_1) \ (t, x) \to j(t, x, r)$ is measurable on $Q$, for all $r \in \mathbb{R}$,

(\text{h}_2) \ $j(t, x, \cdot)$ is a proper, convex, continuous function, a.e. $(t, x) \in Q$,

\[
\partial j(t, x, r) = \beta(t, x, r) \quad \text{for all} \ r \in \mathbb{R}, \ \text{a.e.} \ (t, x) \in Q,
\]

\[
\frac{j(t, x, r)}{|r|} \to \infty, \quad \text{as} \ |r| \to \infty, \ \text{uniformly for} \ (t, x) \in Q,
\]

\[
\frac{j^*(t, x, \omega)}{|\omega|} \to \infty, \quad \text{as} \ |\omega| \to \infty, \ \text{uniformly for} \ (t, x) \in Q,
\]
We define the conjugate \( j^* : Q \times \mathbb{R} \to (-\infty, \infty] \) by

\[
j^*(t, x, \omega) = \sup_{r \in \mathbb{R}} (\omega r - j(t, x, r)) \text{, a.e. } (t, x) \in Q.
\] (2.6)

Then, the following two relations (Legendre-Fenchel) take place (see [14], p. 6, see also [15]):

\[
j(t, x, r) + j^*(t, x, \omega) \geq r\omega \quad \text{for all } r, \omega \in \mathbb{R}, \text{ a.e. } (t, x) \in Q,
\] (2.7)

\[
j(t, x, r) + j^*(t, x, \omega) = r\omega \quad \text{iff } \omega \in \partial j(t, x, r), \text{ for all } r, \omega \in \mathbb{R}, \text{ a.e. } (t, x) \in Q.
\] (2.8)

By (2.2) it follows that \( \beta \) is a maximal monotone graph (possibly multivalued) on \( \mathbb{R} \), a.e. \( (t, x) \in Q \). Relations (2.3)-(2.4) are equivalent with the properties that \( (\beta)^{-1}(t, x, \cdot) \) and \( \beta(t, x, \cdot) \), respectively, are bounded on bounded subsets, uniformly a.e. \( (t, x) \in Q \). This means that for any \( M > 0 \) there exists \( Y_M \) and \( W_M \), independent on \( t \) and \( x \), such that

\[
\sup \{|r|; r \in \beta^{-1}(t, x, \omega), \ |\omega| \leq M\} \leq W_M,
\] (2.9)

\[
\sup \{|\omega|; \omega \in \beta(t, x, r), \ |r| \leq M\} \leq Y_M.
\] (2.10)

In fact, when \( j \) does not depend on \( t \) and \( x \), relations (2.3)-(2.4) express that

\[
D(\partial j(r)) = R(\partial j(r)) = \mathbb{R}, \quad D(\partial j^*(r)) = R(\partial j^*(r)) = \mathbb{R}
\]

(see [4], p. 9). We also recall that \( \partial j^*(t, x, \cdot) = (\partial j(t, x, \cdot))^{-1} \) a.e. \( (t, x) \in Q \).

We call weakly coercive a nonlinear diffusion term with \( j \) having the properties (2.3)-(2.4), and implicitly the corresponding equation (2.1).

We also recall that a proper, convex l.s.c. function is bounded below by an affine function, hence

\[
j(t, x, r) \geq k_1(t, x)r + k_2(t, x), \quad j^*(t, x, \omega) \geq k_3(t, x)\omega + k_4(t, x)
\] (2.11)

for any \( r, \omega \in \mathbb{R} \) and we assume that

\[
k_i \in L^\infty(Q), \ i = 1, \ldots, 4.
\] (2.12)

In fact (2.11) follows if besides (2.5) we assume that there exist \( \xi, \eta \in L^\infty(Q) \) such that

\[
\xi \in \partial j(t, x, 0), \ \eta \in \partial j^*(t, x, 0) \text{ a.e. } (t, x) \in Q.
\]

In this work we show that problem (2.1) reduces to a certain minimization problem (\( P \)) for a convex lower semicontinuous functional involving the functions \( j \) and \( j^* \). In Section 3, the existence of at least a solution to (\( P \)) is proved in Theorem 3.2, this being actually the generalized solution associated to (2.1). The uniqueness is deduced directly from (\( P \)) under the assumption of the strictly convexity of \( j \). Moreover, when a state constraint \( y \in [y_m, y_M] \) is included in the admissible set we show that the null minimization solution is the unique weak solution to (2.1) in Theorem 3.3.

In the case when \( j \) does not depend on \( x \) but on \( t \) and has the same behavior at \( |r| \) large, i.e., it satisfies the relation

\[
j(t, -r) \leq \gamma_1 j(t, r) + \gamma_2, \text{ for any } r \in \mathbb{R}, \text{ a.e. } t \in (0, T),
\] (2.13)
with $\gamma_1$ and $\gamma_2$ constants, we prove in Theorem 4.3 in Section 4 that the solution to the minimization problem is the unique weak solution to (2.1) without assuming the previous additional state constraint. This is based on Lemma 4.1 which plays an essential role in the proof of this result. We mention that stochastic porous media equations of the form (2.1) were studied under a similar assumptions in [7], by a different method.

Theorem 4.3 is the main novelty of this work since it provides existence in (2.1) for a time dependent weakly coercive $j$. With respect to the treatment of the case which assumed a polynomial boundedness of $j$ (see [20]), the present one requires a sharp analysis in the $L^1$-space.

2.1 Functional setting

First, we introduce several linear operators related to problem (2.1). Actually they represent the operator $-\Delta$ defined on various spaces. The main operators which we use, $A_{0,\infty}$ and $A$ are defined as follows:

\[
A_{0,\infty} \psi = -\Delta \psi, \quad A_{0,\infty} : D(A_{0,\infty}) = X \subset L^\infty(\Omega) \rightarrow L^\infty(\Omega),
\]

\[
X = \left\{ \psi \in W^{2,\infty}(\Omega), \frac{\partial \psi}{\partial \nu} + \alpha \psi = 0 \text{ on } \Gamma \right\}
\]

and

\[
A : D(A) = L^1(\Omega) \subset X' \rightarrow X',
\]

\[
\langle A \theta, \psi \rangle_{X',X} = \int_\Omega \theta A_{0,\infty} \psi dx, \quad \forall \theta \in L^1(\Omega), \forall \psi \in X,
\]

where by $X'$ we denote the dual of $X$, with the pivot space $L^2(\Omega)$ ($X \subset L^2(\Omega) \subset X'$).

We introduce the operator

\[
A_1 \psi = -\Delta \psi, \quad A_1 : D(A_1) = L^1(\Omega) \rightarrow L^1(\Omega),
\]

\[
D(A_1) = \left\{ \psi \in W^{1,1}(\Omega); \Delta \psi \in L^1(\Omega), \frac{\partial \psi}{\partial \nu} + \alpha \psi = 0 \text{ on } \Gamma \right\},
\]

which is $m$-accretive on $L^1(\Omega)$ (see [10]). For a later use we recall that

\[
A_2 \psi = -\Delta \psi, \quad A_2 : X_2 = D(A_2) \subset L^2(\Omega) \rightarrow L^2(\Omega),
\]

\[
X_2 = \left\{ \psi \in W^{2,2}(\Omega); \frac{\partial \psi}{\partial \nu} + \alpha \psi = 0 \text{ on } \Gamma \right\},
\]

is $m$-accretive on $L^2(\Omega)$ and $\widetilde{A}_2$, its extension to $L^2(\Omega)$, defined by

\[
\widetilde{A}_2 : L^2(\Omega) \subset X'_2 \rightarrow X'_2,
\]

\[
\langle \widetilde{A}_2 \theta, \psi \rangle_{X'_2,X_2} = \int_\Omega \theta A_2 \psi dx, \quad \forall \theta \in L^2(\Omega), \forall \psi \in X_2,
\]

is $m$-accretive on $X'_2$. Here, $X'_2$ is the dual of $X_2$ with $L^2(\Omega)$ as pivot space (see these last definitions in [20]).
Finally, let us consider the Hilbert space $V = H^1(\Omega)$ endowed with the norm

$$\|\phi\|_V = \left(\|\phi\|^2 + \alpha \|\phi\|^2_{L^2(\Gamma)}\right)^{1/2},$$

which is equivalent (for $\alpha > 0$) with the standard Hilbertian norm on $H^1(\Omega)$ (see [21], p. 20). The dual of $V$ is denoted $V'$ and the scalar product on $V'$ is defined as

$$(\theta, \overline{\theta})_{V'} = \langle \theta, A^{-1}_V \overline{\theta} \rangle_{V', V}$$

(2.19)

where $A_V : V \to V'$ is given by

$$\langle A_V \psi, \phi \rangle_{V', V} = \int_{\Omega} \nabla \psi \cdot \nabla \phi \, dx + \int_{\Gamma} \alpha \psi \phi \, d\sigma,$$

for any $\phi \in V$. (2.20)

(In fact, $A_V$ is the extension of $A_2$ defined by (2.17) to $V'$.)

For the sake of simplicity, we shall omit sometimes to write the function arguments in the integrands, writing $\int_Q g \, dx \, dt$ instead of $\int_Q g(t, x) \, dx \, dt$, where $g : Q \to \mathbb{R}$. In appropriate places we indicate it as $g(t)$, to specify that $g : (0, T) \to Y$, with $Y$ a Banach space.

### 2.2 Statement of the problem

In terms of the previously introduced operators we can write the abstract Cauchy problem

$$\frac{dy}{dt}(t) + A\beta(t, x, y) \ni f(t), \text{ a.e. } t \in (0, T),$$

$$y(0) = y_0.$$ (2.21)

**Definition 1.1.** Let $f \in L^\infty(Q)$ and $y_0 \in V'$. We call a weak solution to (2.1) a pair $(y, \eta)$,

$$y \in L^1(Q) \cap W^{1,1}([0, T]; X'), \ w \in L^1(Q), \ w(t, x) \in \beta(t, x, y(t, x)) \text{ a.e. } (t, x) \in Q,$$

which satisfies the equation

$$\int_0^T \left\langle \frac{dy}{dt}(t), \psi(t) \right\rangle_{X', X} \, dt + \int_Q w(t, x)(A_{0,\infty}\psi(t))(x) \, dx \, dt = \int_0^T \langle f(t), \psi(t) \rangle_{X', X} \, dt$$

(2.22)

for any $\psi \in L^\infty(0, T; X)$, and the initial condition $y(0) = y_0$.

In literature, such a solution is called sometimes very weak or distributional solution.

We consider the minimization problem

$$\text{Minimize } J(y, w),$$

$$\text{(P)}$$
where

\[
J(y, w) = \begin{cases} 
\int_Q (j(t, x, y(t, x)) + j^*(t, x, w(t, x))) \, dx \, dt + \frac{1}{2} \|y(T)\|^2_V, \\
-\frac{1}{2} \|y_0\|^2_V - \int_Q y(t, x)(A_{0, \infty}^{-1} f(t))(x) \, dx \, dt \quad \text{if } (y, w) \in U, \\
+\infty, \quad \text{otherwise},
\end{cases} 
\]

and

\[
U = \{(y, w) \in L^1(Q) \cap \mathcal{W}^{1,1}([0, T]; X'), \ y(T) \in V', \ w \in L^1(Q), \\
j(\cdot, \cdot, \cdot), j^*(\cdot, \cdot, \cdot) \in L^1(Q), \ \ (y, w) \text{ verifies (2.24) below}\}
\]

\[
\frac{dy}{dt}(t) + Aw(t) = f(t) \text{ a.e. } t \in (0, T), 
\]

(2.24)

\[
y(0) = y_0.
\]

Here, \(\frac{dy}{dt}\) is taken in the sense of \(X'\)-valued distributions on \((0, T)\).

We see that, by the existence theory of elliptic boundary value problems (see [1]), if \(f(t) \in L^\infty(\Omega)\) then \(A_{0, \infty}^{-1} f(t) \in \bigcap_{p \geq 2} W^{2,p}(\Omega) \subset L^\infty(\Omega)\), a.e. \(t \in (0, T)\), so the last term in the expression of \(J\) makes sense.

## 3 Time and space dependent potential

In this section we consider that \(j\) and \(j^*\) depend on \(t\) and \(x\) as well, and assume \((h_1) - (h_2), \ (2.2) - (2.5), \ (2.11) - (2.12)\). We begin with an intermediate result.

**Lemma 3.1.** The function \(J\) is proper, convex and lower semicontinuous on \(L^1(Q) \times L^1(Q)\).

**Proof.** It is obvious that \(J\) is proper (because \(U \neq \emptyset\)) and convex. Let \(\lambda > 0\). For the lower semicontinuity we prove that the level set

\[
E_\lambda = \{(y, w) \in L^1(Q) \times L^1(Q); \ J(y, w) \leq \lambda\}
\]

is closed in \(L^1(Q) \times L^1(Q)\). Let \((y_n, w_n) \in E_\lambda\) such that

\[
y_n \to y \text{ strongly in } L^1(Q), \ w_n \to w \text{ strongly in } L^1(Q), \text{ as } n \to \infty. \quad (3.1)
\]

It follows that \((y_n, w_n) \in U\) is the solution to

\[
\frac{dy_n}{dt}(t) + Aw_n(t) = f(t), \text{ a.e. } t \in (0, T), 
\]

(3.2)

\[
y_n(0) = y_0
\]
and
\[
J(y_n, w_n) = \int_Q \left( j(t, x, y_n(t, x)) + j^*(t, x, w_n(t, x)) \right) dx dt \\
+ \frac{1}{2} \left\{ \|y_n(T)\|_{V'}^2 - \|y_0\|_{V'}^2 \right\} - \int_Q y_n A_{0,\infty}^{-1} f dx dt \leq \lambda. 
\] (3.3)

The convergences (3.1) imply that
\[
\int_0^T (Aw_n(t), \psi(t))_{X',X} dt = \int_Q w_n A_{0,\infty} \psi dx dt \rightarrow \int_Q w A_{0,\infty} \psi dx dt, \text{ as } n \rightarrow \infty,
\]
for any \( \psi \in L^\infty(0, T; X) \) and
\[
\int_Q y_n A_{0,\infty}^{-1} f dx dt \rightarrow \int_Q y A_{0,\infty}^{-1} f dx dt, \text{ as } n \rightarrow \infty,
\]
Therefore, by (3.2), we can write
\[
\int_0^T \left\langle \frac{dy_n}{dt}(t), \psi(t) \right\rangle_{X',X} dt = - \int_0^T w_n A_{0,\infty} \psi dx dt + \int_Q f \psi dx dt, 
\] (3.4)
for any \( \psi \in L^\infty(0, T; X) \), and we deduce that
\[
\frac{dy_n}{dt} \rightarrow \frac{dy}{dt} \text{ weakly in } L^1(0, T; X') \text{ as } n \rightarrow \infty,
\]
meaning that \( y_n \) is absolutely continuous on \([0, T] \) with values in \( X' \).

Again by (3.2) we have
\[
y_n(t) = y_0 + \int_0^t f(s) ds - \int_0^t A w_n(s) ds, \text{ for } t \in [0, T].
\] (3.5)

From here we get
\[
\int_\Omega y_n(t) \phi dx = \langle y_0, \phi \rangle_{V',V} + \int_0^t \int_\Omega f(s) \phi dx ds - \int_0^t \langle A w_n(s), \phi \rangle_{X',X} ds
\] (3.6)
for any \( \phi \in X \) and \( t \in [0, T] \). Passing to the limit we obtain
\[
l(t) = \lim_{n \rightarrow \infty} \int_\Omega y_n(t) \phi dx = \langle y_0, \phi \rangle_{V',V} + \int_0^t \int_\Omega f(s) \phi dx ds - \int_0^t \langle A w(s), \phi \rangle_{X',X} ds.
\]

We multiply this relation by \( \varphi_0 \in L^\infty(0, T) \) and integrate over \((0, T)\), to obtain that
\[
\int_0^T \varphi_0(t) l(t) dt \\
= \int_0^T \left( \langle y_0, \phi \rangle_{V',V} + \int_0^t \int_\Omega f(s) \phi dx ds - \int_0^t \langle A w(s), \phi \rangle_{X',X} ds \right) \varphi_0(t) dt.
\] (3.7)
We multiply (3.5) by $\varphi_0(t)\phi(x)$ and integrate over $(0, T) \times \Omega$. We have
\[
\int_Q \varphi_0 \phi y_n \, dx \, dt = \int_0^T \left( \langle y_0, \phi \rangle_{V', V} + \int_0^t \int_0^1 f(s) \phi ds \, dx \right) \varphi_0(t) \, dt \quad (3.8)
\]
whence we use the strong convergence $y_n \to y$ in $L^1(Q)$ to get that
\[
\int_Q \varphi_0 \phi y \, dx \, dt = \int_0^T \left( \langle y_0, \phi \rangle_{V', V} + \int_0^t \int_0^1 f(s) \phi ds \, dx \right) \varphi_0(t) \, dt \quad (3.9)
\]
Comparing (3.7) and (3.9) we deduce that
\[
\int_0^T \varphi_0(t) l(t) \, dt = \int_Q \varphi_0 \phi y \, dx \, dt \quad \text{for any } \varphi_0 \in L^\infty(0, T),
\]
hence
\[
l(t) = \lim_{n \to \infty} \int_\Omega y_n(t) \phi \, dx = \int_\Omega y(t) \phi \, dx \quad \text{for any } \phi \in X, \ t \in [0, T].
\]
Thus
\[
y_n(t) \to y(t) \quad \text{weakly in } X' \quad \text{as } n \to \infty, \quad \text{for any } t \in [0, T] \quad (3.10)
\]
and therefore
\[
y_n(T) \to y(T), \ y_n(0) \to y(0) = y_0 \quad \text{weakly in } X', \quad \text{as } n \to \infty. \quad (3.11)
\]
Letting $n \to \infty$ in (3.4) we obtain
\[
\int_0^T \left\langle \frac{dy}{dt}(t), \psi(t) \right\rangle_{X', X} \, dt + \int_0^T \int_\Omega w A_{0, \infty} \psi \, dx \, dt = \int_0^T \left\langle f(t), \psi(t) \right\rangle_{X', X} \, dt,
\]
which proves that $(y, w)$ is the solution to (2.24).

By (3.3) and (2.11) we can write that
\[
\int_Q \left( k_1 y_n + k_2 + k_3 w_n + k_4 \right) \, dx \, dt + \frac{1}{2} \|y_n(T)\|_{V'}^2 \\
\leq \int_Q \left( j(t, x, y_n(t, x)) + j^*(t, x, w_n(t, x)) \right) \, dx \, dt + \frac{1}{2} \|y_n(T)\|_{V'}^2 \\
\leq \frac{1}{2} \|y_0\|_{V'}^2 + \|y_n\|_{L^1(Q)} \left\| A_{0, \infty}^{-1} f \right\|_{L^\infty(Q)} + \lambda \leq C,
\]
whence, using (3.11) we get
\[
\frac{1}{2} \|y_n(T)\|_{V'}^2 \leq C + \max_{i=1, \ldots, 4} |k_i|_{L^\infty(Q)} \left( \|y\|_{L^1(Q)} + \|w\|_{L^1(Q)} + 2 \right) = C_1
\]
with $C$ and $C_1$ constants and $|k_i|_{\infty} = \|k_i\|_{L^\infty(Q)}$.

It follows that $y_n(T) \to \xi$ weakly in $V'$ as $n \to \infty$. As seen earlier, $y_n(T) \to y(T)$ weakly in $X'$, and by the uniqueness of the limit we get $\xi = y(T) \in V'$.

The function

$$\varphi : L^1(Q) \to \mathbb{R}, \varphi(z) = \int_Q j(t, x, z(t, x))dxdt$$

is proper, convex and l.s.c. (see [4], p. 56) and so by Fatou’s lemma (if $j$ would be nonnegative) we get

$$\varphi(y) \leq \liminf_{n \to \infty} \varphi(y_n) = \liminf_{n \to \infty} \int_Q j(t, x, y_n(t, x))dxdt < \infty. \quad (3.12)$$

Since $j$ is not generally nonnegative we use (2.11) and apply Fatou’s lemma for

$$\tilde{j}(t, x, r) = j(t, x, r) - k_1(t, x)r - k_2(t, x) \geq 0.$$ We get, by the strongly convergence $y_n \to y$ in $L^1(Q)$ and the continuity of $j$,

$$\int_Q (j(t, x, y(t, x)) - k_1y - k_2)dxdt = \int_Q \liminf_{n \to \infty} \tilde{j}(t, x, y_n(t, x))dxdt$$

$$\leq \liminf_{n \to \infty} \int_Q \tilde{j}(t, x, y_n(t, x))dxdt = \liminf_{n \to \infty} \int_Q j(t, x, y_n(t, x))dxdt - \int_Q (k_1y + k_2)dxdt,$$

and so (3.12) holds.

Similarly we have that $\int_Q j^*(t, x, w(t, x))dxdt < \infty$, and so, in particular, we have shown that $(y, w) \in U$.

Moreover, passing to the limit in (3.3) as $n \to \infty$ we obtain by lower semicontinuity that

$$\int_Q (j(t, x, y(t, x)) + j^*(t, x, w(t, x)))dxdt + \frac{1}{2} \|y(T)\|_{V'}^2$$

$$- \frac{1}{2} \|y_0\|_{V'}^2 - \int_Q yA_{0, \infty}^{-1}f dxdt \leq \liminf_{n \to \infty} J(y_n, w_n) \leq \lambda$$

which means that $(y, w) \in E_\lambda$. This ends the proof. \qed

**Theorem 3.2.** Problem $(P)$ has at least a solution $(y^*, w^*)$. If $j$ is strictly convex the solution to $(P)$ is unique.

**Proof.** By (2.11) we note that if $(y, w) \in U$, then

$$J(y, w) \geq -|k_1|_{\infty} \|y\|_{L^1(Q)} - |k_2|_{\infty} - |k_3|_{\infty} \|w\|_{L^1(Q)} - |k_4|_{\infty}$$

$$- \frac{1}{2} \|y_0\|_{V'}^2 - \|y\|_{L^1(Q)} \|A_{0, \infty}^{-1}f\|_{L^\infty(Q)}.$$ Let us set $d = \inf_{(y, w) \in U} J(y, w)$. We assume first that $d > -\infty$ and we shall show later that this is indeed the only case.
Let us consider a minimizing sequence \((y_n, w_n) \in U\), such that

\[
d \leq J(y_n, w_n) \leq d + \frac{1}{n},
\]

where the pair \((y_n, w_n)\) satisfies (3.2).

By (2.3)–(2.4), for any \(M > 0\), there exist \(C_M\) and \(D_M\) such that \(j(t, x, r) > M|r|\) as \(|r| > C_M\) and \(j^*(t, x, \omega) > M|\omega|\) as \(|\omega| > D_M\). Then, by (3.13) we write

\[
\int_{\{(t,x);|y_n(t,x)| \leq C_M\}} j(t, x, y_n(t, x)) dxdt + M \int_{\{(t,x);|y_n(t,x)| > C_M\}} |y_n| dxdt
\]

\[
+ \int_{\{(t,x);|w_n(t,x)| \leq D_M\}} j^*(t, x, w_n(t, x)) dxdt + M \int_{\{(t,x);|w_n(t,x)| > D_M\}} |w_n| dxdt
\]

\[
+ \frac{1}{2} \|y_n(T)\|_V^2 - \frac{1}{2} \|y_0\|_V^2 \leq d + \frac{1}{n}
\]

\[
\leq \|A_{0, \infty}f\|_{L^\infty(Q)} \left( \int_{\{(t,x);|y_n(t,x)| \leq C_M\}} |y_n| dxdt + \int_{\{(t,x);|y_n(t,x)| > C_M\}} |y_n| dxdt \right).
\]

Denoting \(\|A_{0, \infty}f\|_{L^\infty(Q)} = f_\infty\), and taking \(M\) large enough such that \(M > f_\infty\) it follows that

\[
(M - f_\infty) \int_{\{(t,x);|y_n(t,x)| > C_M\}} |y_n| dxdt + M \int_{\{(t,x);|w_n(t,x)| > D_M\}} |w_n| dxdt
\]

\[
+ \frac{1}{2} \|y_n(T)\|_V^2,
\]

\[
\leq \frac{1}{2} \|y_0\|_V^2, + f_\infty C_M \text{meas}(Q) + \int_{\{(t,x);|y_n(t,x)| \leq C_M\}} |j(t, x, y_n(t, x))| dxdt
\]

\[
+ \int_{\{(t,x);|w_n(t,x)| \leq D_M\}} |j^*(t, x, w_n(t, x))| dxdt + d + 1
\]

\[
\leq \frac{1}{2} \|y_0\|_V^2, + f_\infty C_M \text{meas}(Q) + \int_{\{(t,x);|y_n(t,x)| \leq C_M\}} \left| \tilde{j}(t, x, y_n(t, x)) \right| dxdt
\]

\[
+ \int_{\{(t,x);|w_n(t,x)| \leq D_M\}} \left| \tilde{j}^*(t, x, w_n(t, x)) \right| dxdt + d + 1
\]

\[
+ \int_{\{(t,x);|y_n(t,x)| \leq C_M\}} |k_1 y_n + k_2| dxdt + \int_{\{(t,x);|w_n(t,x)| \leq D_M\}} |k_3 w_n + k_4| dxdt,
\]

where \(\tilde{j}(t, x, r) = j(t, x, r) - k_1 r - k_2, \tilde{j}^*(t, x, \omega) = j^*(t, x, \omega) - k_3 \omega - k_4\).

Recalling (2.10) and (2.9),

\[
j(t, x, y_n(t, x)) \leq |j(t, x, 0)| + |\eta_n(t, x)| |y_n(t, x)| \leq Y_M^1 \text{ on } \{(t,x); |y_n(t,x)| \leq C_M\},
\]

\[
j^*(t, x, w_n(t, x)) \leq |j^*(t, x, 0)| + |\varpi_n(t, x)| |y(t, x)| \leq W_M^1 \text{ on } \{(t,x); |w_n(t,x)| \leq D_M\},
\]

where \(\eta_n(t, x) \in \beta(t, x, y_n)\) and \(\varpi_n(t, x) \in (\beta)^{-1}(t, x, w_n)\) a.e. on \(Q\).

Then

\[
0 \leq \tilde{j}(t, x, y_n(t, x)) \leq Y_M^1 + |k_1|_\infty C_M + |k_2|_\infty \text{ on } \{(t,x); |y_n(t,x)| \leq C_M\},
\]
\[ 0 \leq \tilde{j}^*(t, x, w_n(t, x)) \leq W_M^1 + |k_3|_\infty D_M + |k_4|_\infty \text{ on } \{(t, x); |w_n(t, x)| \leq D_M\} \]

and we deduce that

\[
(M - f_\infty) \int_{\{(t,x); y_n(t,x) > C_M\}} |y_n| \, dx \, dt + M \int_{\{(t,x); |w_n(t,x)| > D_M\}} |w_n| \, dx \, dt \quad (3.14)
\]

\[ + \frac{1}{2} \|y_n(T)\|_{V'}^2 \leq C + d. \]

Consequently, this yields

\[
\|y_n\|_{L^1(Q)} \leq C, \quad \|w_n\|_{L^1(Q)} \leq C, \quad \|y_n(T)\|_{V'} \leq C. \quad (3.15)
\]

(By \(C\) and \(C_i, i = 1, \ldots, 4\), we denote several constants independent on \(n\)).

From (3.13) we get

\[
I_n := \int_Q j(t, x, y_n(t, x)) \, dx \, dt + \int_Q j^*(t, x, w_n(t, x)) \, dx \, dt \leq C. \quad (3.16)
\]

We continue by proving that separately each term is bounded, i.e.,

\[
\int_Q j(t, x, y_n(t, x)) \, dx \, dt \leq C_1, \quad \int_Q j^*(t, x, w_n(t, x)) \, dx \, dt \leq C_2. \quad (3.17)
\]

We write

\[
I_n = \int_{\{(t,x); |y_n(t,x)| \leq M\}} j(t, x, y_n(t, x)) \, dx \, dt + \int_{\{(t,x); |y_n(t,x)| > M\}} j(t, x, y_n(t, x)) \, dx \, dt
\]

\[ + \int_{\{(t,x); |w_n(t,x)| \leq M\}} j^*(t, x, w_n(t, x)) \, dx \, dt + \int_{\{(t,x); |w_n(t,x)| > M\}} j^*(t, x, w_n(t, x)) \, dx \, dt \leq C. \]

Therefore

\[
\int_{\{(t,x); |y_n(t,x)| > M\}} j(t, x, y_n(t, x)) \, dx \, dt + \int_{\{(t,x); |w_n(t,x)| > M\}} j^*(t, x, w_n(t, x)) \, dx \, dt \leq C + Y^1_M \text{meas}(Q) + W^1_M \text{meas}(Q) = C_3.
\]

Since \(j(t, x, y_n(t, x)) \geq k_1(t, x)y_n(t, x) + k_2(t, x)\) we deduce that

\[
\int_{\{(t,x); |w_n(t,x)| > M\}} j^*(t, x, w_n(t, x)) \, dx \, dt \leq C_4,
\]

whence

\[
\int_Q j^*(t, x, w_n(t, x)) \, dx \, dt \leq C_1. \quad (3.18)
\]

Finally, (3.16) yields

\[
\int_Q j(t, x, y_n(t, x)) \, dx \, dt \leq C_2, \quad (3.19)
\]

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with $C_1$ and $C_2$ independent of $n$.

Next, we shall show that the sequences $(y_n)_n$ and $(w_n)_n$ are weakly compact in $L^1(Q)$.

To this end we have to show that the integrals $\int_S |w_n| \, dx \, dt$, with $S \subset Q$, are equi-absolutely continuous, meaning that for every $\varepsilon > 0$ there exists $\delta$ such that $\int_S |w_n| \, dx \, dt < \varepsilon$ whenever $\text{meas}(S) < \delta$. Let $M_\varepsilon > \frac{2\varepsilon}{\delta}$, where $C_2$ is the constant in (3.17), and let $R_M$ be such that $\frac{j^*(t,x,w_n)}{|w_n|} \geq M_\varepsilon$ for $|r| > R_M$, by (2.3). If $\delta < \frac{\varepsilon}{2R_M}$ then

$$\int_S |w_n| \, dx \, dt \leq \int_{\{t,x; |w_n(t,x)| > R_M\}} |w_n| \, dx \, dt + \int_{\{t,x; |w_n(t,x)| \leq R_M\}} |w_n| \, dx \, dt$$

$$\leq M_\varepsilon^{-1} \int_Q j^*(t,x,w_n(t,x)) \, dx \, dy + R_M \delta < \varepsilon.$$ 

Hence, by the Dunford-Pettis theorem it follows that $(w_n)_n$ is weakly compact in $L^1(Q)$. In a similar way we proceed for showing the weakly compactness of the sequence $(y_n)_n$. Thus,

$$y_n \to y^* \text{ weakly in } L^1(Q), \quad w_n \to w^* \text{ weakly in } L^1(Q) \text{ as } n \to \infty,$$

$$Aw_n \to Aw^* \text{ weakly in } L^1(0,T;X'), \quad \text{as } n \to \infty,$$

by (2.15) which implies by (3.2) that

$$\frac{dy_n}{dt} \to \frac{dy^*}{dt} \text{ weakly in } L^1(0,T;X') \text{ as } n \to \infty.$$ 

Passing to the limit in

$$\int_0^T \left\langle \frac{dy_n}{dt}(t), \psi(t) \right\rangle_{X',X} \, dt + \int_Q w_n A_{0,\infty} \psi \, dx \, dt = \int_0^T \left\langle f(t), \psi(t) \right\rangle_{X',X} \, dt$$

for any $\psi \in L^\infty(0,T;X)$ we get that $(y^*,w^*)$ verifies (2.22), or equivalently (2.24), i.e.,

$$\int_0^T \left\langle \frac{dy^*}{dt}(t), \psi(t) \right\rangle_{X',X} \, dt + \int_Q w^* A_{0,\infty} \psi \, dx \, dt = \int_0^T \left\langle f(t), \psi(t) \right\rangle_{X',X} \, dt.$$ 

Next we show that

$$y_n(T) \to y^*(T) \text{ and } y_n(0) \to y(0) = y_0 \text{ weakly in } V', \text{ as } n \to \infty,$$

in a similar way as in Lemma 3.1. In order to obtain (3.9) we use the weakly compactness of $(y_n)_n$ in $L^1(Q)$.

Finally, by passing to the limit in (3.13), on the basis of the weakly lower semicontinuity of the functional $J$ on $L^1(Q) \times L^1(Q)$, we obtain that

$$J(y^*,w^*) = d.$$ 

Hence, we have got that $y^* \in L^1(Q)$, $w^* \in L^1(Q)$, $y^*(T) \in V'$ and $(y^*,w^*)$ satisfies (2.24). By (3.17) we get

$$\int_Q j(t,x,y^*(t,x)) \, dx \, dt < \infty, \quad \int_Q j^*(t,x,w^*(t,x)) \, dx \, dt < \infty.$$
With these relations we have ended the proof that \((y^*, w^*)\) belongs to \(U\) and that it is a solution to \((P)\).

Let us show now that \(d > -\infty\). Indeed, otherwise, for every \(K\) real positive, there exists \(n_K\), such that for every \(n \geq n_K\) we have \(J(y_n, w_n) < -K\). Following the computations in the same way as before we arrive at the inequality \((3.14)\) which reads now

\[
(M - f_\infty) \int_{\{(t,x); |y_n(t,x)| > C_{34}\}} |y_n| \, dx \, dt + M \int_{\{(t,x); |w_n(t,x)| > D_{34}\}} |w_n| \, dx \, dt
+ \frac{1}{2} \|y_n(T)\|_{V'}^2 \leq C - K.
\]

Since \(C\) is a fixed constant, this implies \(C - K < 0\), for \(K\) large enough, and this leads to a contradiction, as claimed.

The argument for the uniqueness proof is standard and it relies on the assumption of the strict convexity of \(j\) and on the obvious inequality

\[
J\left(\frac{y_1 + y_2}{2}, \frac{w_1 + w_2}{2}\right) = \int_Q \left(j\left(t, x, \frac{y_1 + y_2}{2}(t, x)\right) + j^*\left(t, x, \frac{w_1 + w_2}{2}(t, x)\right)\right) \, dx \, dt
+ \frac{1}{2} \left\|\frac{y_1 + y_2}{2}(T)\right\|_{V'}^2 - \frac{1}{2} \|y_0\|_{V'}^2 - \int_Q \frac{y_1 + y_2}{2} A_{0,\infty}^{-1} f \, dx \, dt
\leq \frac{1}{2} (J(y_1, w_1) + J(y_2, w_2)) - \frac{1}{2} \left\|\frac{y_1 - y_2}{2}(T)\right\|_{V'}^2,
\]

where \((y_1, w_1)\) and \((y_2, w_2)\) are two solutions to \((P)\). \(\Box\)

We call the solution to the minimization problem \((P)\) a variational or generalized solution to \((2.1)\).

One might suspect that if the minimum in \((P)\) is zero, then the null minimizer is a weak solution to \((2.1)\). We shall prove this for a slightly modified version of \((P)\), by including a boundedness constraint for the state \(y\) in the admissible set \(U\). More exactly we consider the problem

\[
\text{Minimize } \bar{J}(y, w) \text{ for all } (y, w) \in \bar{U}
\]

where

\[
\bar{J}(y, w) = \begin{cases} J(y, w), & (y, w) \in \bar{U}, \\
+\infty, & \text{otherwise}, 
\end{cases}
\]

\[
\bar{U} = \{(y, w) \in U; y(t, x) \in [y_m, y_M] \text{ a.e. } (t, x) \in Q\},
\]

with \(y_m, y_M\) two constants. We assume that

\[
y_0 \in L^\infty(\Omega), \quad y_0 \in [y_m, y_M], \quad f \in L^\infty(Q)
\]

and remark that \(\bar{U}\) is not empty (it contains e.g., \(y_0\) with \(w_0 = A_{0,\infty}^{-1} f(t)\) given by \((2.24)\)).
If we set $y_m = 0$, then the previous boundedness property is in agreement with the physical significance of $y$, that of a fluid concentration in a diffusion process, which is nonnegative.

Problem $(\tilde{P})$ has at least a solution and the proof is the same as in Theorem 3.2.

**Theorem 3.3.** Let $(y, w) \in \tilde{U}$ be a null minimizer in $(\tilde{P})$, i.e.,

$$\min(\tilde{P}) = \tilde{J}(y, w) = 0.$$  

Let us assume in addition that

$$\frac{1}{2} \|y(T)\|_{V'}^2 - \frac{1}{2} \|y_0\|_{V'}^2 - \int_Q y(t, x)(A_{0,\infty}^{-1} f(t))(x) dx dt = - \int_Q w(t, x)y(t, x)dxdt. \quad (3.21)$$  

Then

$$w(t, x) \in \beta(t, x, y(t, x)), \text{ a.e. } (t, x) \in Q,$$

and the pair $(y, w)$ is the unique weak solution to (2.1).

**Proof.** Let $(y, w)$ be the null minimizer in $(\tilde{P})$. Then

$$\tilde{J}(y, w) = \int_Q (j(t, x, y(t, x)) + j^*(t, x, w(t, x))) dx dt$$  

$$+ \frac{1}{2} \|y(T)\|_{V'}^2 - \frac{1}{2} \|y_0\|_{V'}^2 - \int_Q y(t, x)(A_{0,\infty}^{-1} f(t))(x) dx dt = 0.$$  

By (3.21) we have

$$\int_Q (j(t, x, y(t, x)) + j^*(t, x, w(t, x)) - y(t, x)w(t, x)) dx dt = 0. \quad (3.22)$$

This implies that $j(t, x, y(t, x)) + j^*(t, x, w(t, x)) - y(t, x)w(t, x) = 0$ a.e. $(t, x) \in Q$ and so

$$w(t, x) \in \beta(t, x, y(t, x)) \text{ a.e. } (t, x) \in Q,$$

as claimed. \hfill \Box

### 4 Time dependent potential

In this section we consider the case when $j$ and $j^*$ depend only on $t$ and assume $(h_1) - (h_2), \ (2.2)-(2.5), \ (2.11) \text{ and } (2.13)$, where $k_1, k_2 \in L^\infty(0, T)$.

The main result of this section is that a solution to $(P)$ belongs to $L^\infty(0, T; V')$ and minimizes $J$ to zero, being exactly the unique weak solution to (2.1).

To this end we need some intermediate results. The first is proved in the next lemma and the second given in Theorem 4.2 recalls one of the main results in [20].

**Lemma 4.1.** Let $(y, w) \in U$ and $y \in L^\infty(0, T; V')$. Then $yw \in L^1(0, T; V')$ and we have the formula

$$-\int_Q yw dx dt = \frac{1}{2} \|y(T)\|_{V'}^2 - \frac{1}{2} \|y_0\|_{V'}^2 - \int_Q y A_{0,\infty}^{-1} f dx dt. \quad (4.1)$$
Proof. Let \((y, w) \in U\). Then \(y, w \in L^1(Q)\), \(j(\cdot, \cdot, y(\cdot, \cdot)) \in L^1(Q)\), \(j^*(\cdot, \cdot, w(\cdot, \cdot)) \in L^1(Q)\). By (2.13) we have
\[
j(t, -y(t, x)) \leq \gamma_1 j(t, y(t, x)) + \gamma_2 \text{ a.e. on } Q,
\]
which implies that \(\int_Q j(t, x, -y(t, x))dxdt < \infty\). Next, by the relations
\[
j(t, x, y(t, x)) + j^*(t, x, w(t, x)) \geq y(t, x)w(t, x),
j(t, x, -y(t, x)) + j^*(t, x, w(t, x)) \geq -y(t, x)w(t, x)
\]
it follows that
\[
yw \in L^1(Q). \tag{4.2}
\]
Because \((y, w) \in U\) it also satisfies (2.24). Then \(y \in L^1(Q) \cap L^\infty(0, T; V')\), \(w \in L^1(Q)\). We perform a regularization by applying \((I + \varepsilon A_\Delta)^{-1}\) to (2.24), where \(A_\Delta\) denotes here the realization of the operator \(-\Delta\) on the spaces indicated in Section 2.1. We obtain
\[
\frac{dy_\varepsilon}{dt}(t) + Aw_\varepsilon(t) = f_\varepsilon(t) \text{ a.e. } t \in (0, T), \tag{4.3}
y_\varepsilon(0) = (I + \varepsilon A_V)^{-1}y_0,
\]
where
\[
y_\varepsilon(t) = (I + \varepsilon A_V)^{-1}y(t), \text{ a.e. } t \in (0, T)
\]
\[
w_\varepsilon(t) = (I + \varepsilon A_1)^{-1}w(t), \text{ a.e. } t \in (0, T)
\]
\[
f_\varepsilon(t) = (I + \varepsilon A_{0,\infty})^{-1}f(t), \text{ a.e. } t \in (0, T). \tag{4.4}
\]
According again to Brezis and Strauss (see [10]), if \(w(t) \in L^1(\Omega)\) then
\[
w_\varepsilon(t) \in W^{1,q}(\Omega), \text{ a.e. } t \in (0, T), \text{ with } 1 \leq q < \frac{N}{N-1}. \tag{4.5}
\]
Since \(\frac{N}{N-1} < N \leq 3\), we get by the Sobolev inequalities that
\[
W^{1,q}(\Omega) \subset L^{q^*}(\Omega), \quad \frac{1}{q^*} = \frac{1}{q} - \frac{1}{N},
\]
with \(\frac{N}{N-1} \leq q^* < \frac{N}{N-2}\). It follows that
\[
w_\varepsilon \in L^1(0, T; L^2(\Omega)).
\]
Next,
\[
y_\varepsilon \in L^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; V),
\]
by a similar argument as for \(w_\varepsilon\), since \(y \in L^1(Q) \cap L^\infty(0, T; V')\) and
\[
y_\varepsilon(t) = (I + \varepsilon A_1)^{-1}y(t), \text{ a.e. } t \in (0, T),
\]
too. Finally,
\[
f_\varepsilon \in L^\infty(0, T; \bigcap_{p \geq 2} W^{2,p}(\Omega)),
\]
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by the elliptic regularity.
Moreover, $A_1$ is $m$-accretive on $L^1(\Omega)$, and it follows that
\[ w_\varepsilon(t) \to w(t) \text{ strongly in } L^1(\Omega) \text{ for any } t \in [0,T] \]
and
\[ \|w_\varepsilon(t)\|_{L^1(\Omega)} \leq \|w(t)\|_{L^1(\Omega)} \text{ for any } t \in [0,T], \]
(see [10]).
For a later use, we deduce by the Lebesgue dominated convergence theorem that
\[ w_\varepsilon \to w \text{ strongly in } L^1(Q), \text{ as } \varepsilon \to 0. \quad (4.6) \]
Similarly, we have that
\[ y_\varepsilon \to y \text{ strongly in } L^1(Q), \text{ as } \varepsilon \to 0. \quad (4.7) \]
Finally,
\[ f_\varepsilon \to f \text{ weak* in } L^\infty(Q), \text{ and strongly in } L^p(Q), \ p \geq 2, \text{ as } \varepsilon \to 0. \quad (4.8) \]
By the first relation in (4.4) we still have that
\[ (I + \varepsilon A_V)^{-1}y(t) \to y(t) \text{ strongly in } V' \text{ for any } t \in [0,T]. \]
We also observe that
\[ \int_0^T \langle Aw_\varepsilon(t), \psi(t) \rangle_{X',X} dt = \int_Q w_\varepsilon A_{0,\infty} \psi dx dt \to \int_Q w A_{0,\infty} \psi dx dt \text{ as } \varepsilon \to 0, \]
for any $\psi \in L^\infty(0,T;X)$ and by (4.3)
\[ \frac{dy_\varepsilon}{dt} \to \frac{dy}{dt} \text{ weakly in } L^1(0,T;X') \text{ as } \varepsilon \to 0. \]
Passing to the limit in (4.3) tested for any $\psi \in L^\infty(0,T;X)$,
\[ \int_0^T \left\langle \frac{dy_\varepsilon}{dt}(t), \psi(t) \right\rangle_{X',X} dt + \int_Q w_\varepsilon A_{0,\infty} \psi dx dt = \int_0^T \langle f_\varepsilon(t), \psi(t) \rangle_{X',X} dt \]
we check that $(y, w)$ indeed satisfies (2.24).
Next, we assert that
\[ \int_Q j(t, y_\varepsilon(t, x)) dx dt \leq \int_Q j(t, y(t, x)) dx dt. \quad (4.9) \]
Indeed, let us introduce the Yosida approximation of $\beta$,
\[ \beta_\lambda(t, r) = \frac{1}{\lambda} (1 - (1 + \lambda \beta(t, \cdot))^{-1})r, \text{ a.e. } t, \text{ for all } r \in \mathbb{R} \text{ and } \lambda > 0. \quad (4.10) \]
We have $\beta_\lambda(t, r) = \frac{\partial j_\lambda}{\partial r}(t, r)$, where $j_\lambda$ is the Moreau approximation of $j$,

$$j_\lambda(t, r) = \inf_{s \in \mathbb{R}} \left\{ \frac{|r - s|^2}{2\lambda} + j(t, s) \right\}, \text{ a.e. } t, \text{ for all } r \in \mathbb{R},$$

that can be still written as

$$j_\lambda(t, r) = \frac{1}{2\lambda} \left| (1 + \lambda \beta(t, \cdot))^{-1}r - r \right|^2 + j(t, (1 + \lambda \beta(t, \cdot))^{-1}r).$$

The function $j_\lambda$ is convex, continuous and satisfies

$$j_\lambda(t, r) \leq j(t, r), \text{ for all } r \in \mathbb{R}, \lambda > 0,$$

$$\lim_{\lambda \to 0} j_\lambda(t, r) = j(t, r), \text{ for all } r \in \mathbb{R}.$$
which is nonnegative by (2.11). Hence, by (4.17) we have
\[
\int_Q \tilde{j}(t, y_\varepsilon(t, x)) \, dx \, dt \leq \int_Q \tilde{j}(t, y(t, x)) \, dx \, dt + \int_Q k_1(t)(y(t, x) - y_\varepsilon(t, x)) \, dx \, dt,
\]
hence
\[
\int_Q \tilde{j}(t, y_\varepsilon(t, x)) \, dx \, dt \leq \int_Q \tilde{j}(t, y(t, x)) \, dx \, dt + \delta(\varepsilon), \tag{4.19}
\]
\[
\delta(\varepsilon) = \int_Q k_1(t)(y(t, x) - y_\varepsilon(t, x)) \, dx \, dt \leq \|k_1\|_{L^\infty(0,T)} \|y - y_\varepsilon\|_{L^1(Q)} \to 0, \text{ as } \varepsilon \to 0,
\]
by (4.7). Then, (4.19) implies (4.9) as claimed.

A similar relation to (4.9) takes place for \( j^*, \)
\[
\int_Q j^*(t, w_\varepsilon(t, x)) \, dx \, dt \leq \int_Q j(t, w(t, x)) \, dx \, dt. \tag{4.20}
\]

This implies that \( j(\cdot, y_\varepsilon(\cdot, \cdot)) \in L^1(Q), \) \( j^*(\cdot, w_\varepsilon(\cdot, \cdot)) \in L^1(Q), \) for all \( \varepsilon > 0, \) and so, by the same argument as for \( yw \) we deduce that
\[
y_\varepsilon w_\varepsilon \in L^1(Q).
\]

We test (4.3) by \( A^{-1}_2 y_\varepsilon(t) \) and integrate over \((0, T). \) Since \( y_\varepsilon \in L^1(Q) \cap L^\infty(0, T; V), \)
\( w_\varepsilon \in L^1(0, T; L^2(\Omega)) \) we get \( A^{-1}_2 y_\varepsilon(t) \in X_2, \) a.e. \( t \in (0, T) \) and by (2.18)
\[
\int_0^T \left< \tilde{A}_2 w_\varepsilon(t), A^{-1}_2 y_\varepsilon(t) \right>_{X_2', X_2} \, dt = \int_Q y_\varepsilon w_\varepsilon \, dx \, dt.
\]
Then, by a few computations we deduce by (4.3) that
\[
- \int_Q y_\varepsilon w_\varepsilon \, dx \, dt = \frac{1}{2} \|y_\varepsilon(T)\|^2_{V'}, - \frac{1}{2} \|(I + \varepsilon A_V)^{-1} y_0\|^2_{V'}, - \int_Q y_\varepsilon A^{-1}_{0, \infty} f_\varepsilon \, dx \, dt. \tag{4.21}
\]
Recalling that by (4.4) we have that
\[
(I + \varepsilon A_V)^{-1} y(t) \to y(t) \text{ strongly in } V' \text{ for any } t \in [0, T]
\]
and passing to the limit in (4.21) as \( \varepsilon \to 0 \) we obtain
\[
\lim_{\varepsilon \to 0} \left( - \int_Q y_\varepsilon w_\varepsilon \, dx \, dt \right) = \frac{1}{2} \|y(T)\|^2_{V'}, - \frac{1}{2} \|y_0\|^2_{V'}, - \int_Q y A^{-1}_{0, \infty} f \, dx \, dt. \tag{4.22}
\]
Moreover, by the strongly convergence of \((y_\varepsilon)_\varepsilon\) and \((w_\varepsilon)_\varepsilon, \) (4.7) and (4.6) we get
\[
y_\varepsilon \to y \text{ a.e. in } Q, \ w_\varepsilon \to w \text{ a.e. in } Q, \text{ as } \varepsilon \to 0,
\]
which implies that
\[
y_\varepsilon w_\varepsilon \to yw \text{ a.e. in } Q, \text{ as } \varepsilon \to 0.
\]
The functions \( j \) and \( j^\ast \) are continuous and so

\[
j(t, y_\varepsilon(t, x)) \to j(t, y(t, x)), \quad j^\ast(t, w_\varepsilon(t, x)) \to j^\ast(t, w(t, x)), \quad \text{a.e. on } Q, \quad \text{as } \varepsilon \to 0.
\]

Now, by (4.9) and (4.20) we have

\[
\int_Q (j(t, y_\varepsilon(t, x)) + j^\ast(t, w_\varepsilon(t, x)) - y_\varepsilon w_\varepsilon) \, dx \, dt
\]

and we apply the Fatou lemma because \( j(t, y_\varepsilon) + j^\ast(t, w_\varepsilon) - y_\varepsilon w_\varepsilon \geq 0 \). We get, using (4.9) and (4.20) that

\[
\int_Q (j(t, y(t, x)) + j^\ast(t, w(t, x))) \, dx \, dt \leq \lim \inf_{\varepsilon \to 0} \int_Q (j(t, y_\varepsilon(t, x)) + j^\ast(t, w_\varepsilon(t, x)) - y_\varepsilon w_\varepsilon) \, dx \, dt
\]

whence, by using (4.22), we see that

\[
-\int_Q yw dx \, dt \leq \frac{1}{2} \| y(T) \|_{V'}^2 - \frac{1}{2} \| y_0 \|_{V'}^2 - \int_Q yA_{1,\infty}^{-1} f \, dx \, dt. \quad (4.23)
\]

We continue the proof by relying on the same arguments, starting this time with Fatou's lemma applied for the positive function \( j(t, x, -y_\varepsilon) + j^\ast(t, x, w_\varepsilon) + y_\varepsilon w_\varepsilon \). By similar computations we get

\[
-\int_Q yw dx \, dt \geq \frac{1}{2} \| y(T) \|_{V'}^2 - \frac{1}{2} \| y_0 \|_{V'}^2 - \int_Q yA_{1,\infty}^{-1} f \, dx \, dt,
\]

which together with (4.23) imply (1.1).

Next we recall one of the main results given in [20] in a more general case, but particularized here to the space \( L^2(Q) \).

Let us consider the problem

\[
\begin{align*}
\frac{\partial y}{\partial t} - \Delta \tilde{\beta}(t, x, y) &\ni f & \text{in } Q, \\
-\frac{\partial \tilde{\beta}(t, x, y)}{\partial \nu} &\ni \alpha \tilde{\beta}(t, x, y) & \text{on } \Sigma, \\
y(0, x) &\ni y_0 & \text{in } \Omega,
\end{align*}
\]

(4.24)
where \( \tilde{\beta}(t,x,r) = \partial \varphi(t,x,r) \) a.e. on \( Q \), for all \( r \in \mathbb{R} \), and \( \varphi : \mathbb{R} \to \mathbb{R} \) is a proper, convex, l.s.c. function satisfying \((h_1),(h_2)\), and the growth condition

\[
C_1 |r|^2 + C_1^0 \leq \varphi(t,x,r) \leq C_2 |r|^2 + C_2^0, \quad \text{for all } r \in \mathbb{R}, \ \text{a.e. } t \in (0,T)
\]

in addition. \((C_i, C_i^0 \text{ are constants}, i = 1, 2, \text{and } C_1 > 0)\).

We consider the minimization problem

\[
\text{Minimize } J_0(y,w) = \int_Q (\varphi(t,x,y(t,x)) + \varphi^*(t,x,w(t,x)) - w(t,x)y(t,x)) \, dx \, dt \quad (P_0)
\]

for all \((y,w) \in U_0\), where

\[
U_0 = \{(y,w); \ y \in L^2(Q) \cap W^{1,2}([0,T];X'_2), \ y(T) \in V', \ w \in L^2(Q), \ \varphi(\cdot,\cdot, y) \in L^1(Q), \ \varphi^*(\cdot,\cdot, w) \in L^1(Q), \ (y,w) \text{ verifies } (4.26) \text{ below}\},
\]

\[
\begin{align*}
\frac{dy}{dt}(t) + \tilde{A}_2 w(t) &= f(t) \text{ a.e. } t \in (0,T), \quad (4.26) \\
y(0) &= y_0.
\end{align*}
\]

We recall the notations \(X_2, X'_2, \tilde{A}_2\) given in Section 2.1.

In [20] it has been proved that \((P_0)\) has at least a solution and it has been established the equivalence between \((4.24)\) and \((P_0)\), resumed below (see Theorem 3.2 in [20]).

**Theorem 4.2.** Let \(y_0 \in V'\), \(f \in L^\infty(Q)\), and let the pair \((y,w) \in U_0\) be a solution to \((P_0)\). Then, \(w(t,x) \in \tilde{\beta}(t,y(t,x))\) a.e. \((t,x) \in Q\) and \((y,w)\) is the unique weak solution to \((4.24)\). Moreover,

\[
-\int_0^t \int_\Omega yw dx d\tau \quad (4.27)
\]

\[
= \frac{1}{2} \left\{ \|y(t)\|^2_{V'} - \|y_0\|^2_{V'} \right\} - \int_0^t \int_\Omega yA^{-1}_{0,\infty} f dx d\tau, \quad \text{for all } t \in [0,T].
\]

Of course, the result remains true when \(\varphi\) does not depend on \(x\).

Now, we can pass to the main result of this section which shows that a null minimizer in \((P)\) provides a unique weak solution to \((2.1)\).

**Theorem 4.3.** Under the assumptions \((h_1)-(h_2), (2.2)-(2.5), (2.11)-(2.12), (2.13)\) problem \((P)\) has a solution \((y^*, w^*)\) such that \(y^* \in L^\infty(0,T; V')\). Then, this solution is a null minimizer in \((P)\)

\[
J(y^*, w^*) = \inf_{(y,w) \in U} J(y,w) = 0 \quad (4.28)
\]

and it turns out that it is the unique weak solution to \((2.1)\).
Proof. Let us introduce the approximating problem
\[
\begin{align*}
\frac{\partial y}{\partial t} - \Delta \beta_\lambda(t, y) &= f \quad \text{in } Q, \\
- \frac{\partial \beta_\lambda(t, y)}{\partial \nu} &= \alpha \beta_\lambda(t, y) \quad \text{on } \Sigma, \\
y(0, x) &= y_0 \quad \text{in } \Omega,
\end{align*}
\]
(4.29)
where \( \beta_\lambda \) is the Yosida approximation of \( \beta \).

Let \( \sigma \) be positive and consider the approximating problem indexed upon \( \sigma \),
\[
\begin{align*}
\frac{\partial y}{\partial t} - \Delta (\beta_\lambda(t, y) + \sigma y) &= f \quad \text{in } Q, \\
- \frac{\partial (\beta_\lambda(t, y) + \sigma y)}{\partial \nu} &= \alpha (\beta_\lambda(t, y) + y) \quad \text{on } \Sigma, \\
y(0, x) &= y_0 \quad \text{in } \Omega.
\end{align*}
\]
(4.30)

The potential of \( \beta_\lambda(t, r) + \sigma r \) is
\[
\begin{align*}
j_{\lambda, \sigma}(t, r) &= j_\lambda(t, r) + \frac{\sigma}{2} r^2, 
\end{align*}
\]
(4.31)
where \( j_\lambda \) is the Moreau regularization of \( j \). By a simple computation using (4.11), (2.11), (2.5) we get that
\[
\begin{align*}
\sigma^2 |r|^2 + k_1 r + k_2 - 2 \lambda k_1^2 \leq j_{\lambda, \sigma}(t, r) \leq |r|^2 \left( \frac{1}{2\lambda} + \sigma^2 \right). 
\end{align*}
\]
(4.32)

Hence \( j_{\lambda, \sigma} \) satisfies (4.25) and we rely on Theorem 4.2 with \( \varphi(t, r) = j_{\lambda, \sigma}(t, r) \) and \( \tilde{\beta}(t, r) = \beta_\lambda(t, r) + \sigma r \) to get that (4.30) has a unique weak solution \( (y_{\lambda, \sigma}, w_{\lambda, \sigma}) \in U_0 \),
\[
\begin{align*}
y_{\lambda, \sigma} \in L^2(Q) \cap W^{1,2}([0, T]; X_2'), \quad y_{\lambda, \sigma}(T) \in V', \\
w_{\lambda, \sigma} &= \beta_\lambda(t, y_{\lambda, \sigma}) + \sigma y_{\lambda, \sigma} \in L^2(Q).
\end{align*}
\]

This solution is the null minimizer in \( (P_0) \), i.e.,
\[
\begin{align*}
J_0(y_{\lambda, \sigma}, w_{\lambda, \sigma}) &= \int_Q \left( j_{\lambda, \sigma}(t, y_{\lambda, \sigma}(t, x)) + j_{\lambda, \sigma}(t, w_{\lambda, \sigma}(t, x)) - y_{\lambda, \sigma} w_{\lambda, \sigma} \right) dx dt = 0, 
\end{align*}
\]
(4.33)
and satisfies (4.26), namely
\[
\begin{align*}
\frac{dy_{\lambda, \sigma}}{dt}(t) + \tilde{A}_2 w_{\lambda, \sigma}(t) &= f(t) \text{ a.e. } t \in (0, T), \\
y(0) &= y_0.
\end{align*}
\]
(4.34)

Moreover, we have by (4.27) that
\[
\begin{align*}
- \int_0^t \int_\Omega y_{\lambda, \sigma} w_{\lambda, \sigma} dx dt \tau &= \frac{1}{2} \{ \| y_{\lambda, \sigma}(t) \|_{V'}^2 - \| y_0 \|_{V'}^2 \} - \int_0^t \int_\Omega y_{\lambda, \sigma} A_{0, \infty}^{-1} f dx dt, 
\end{align*}
\]
(4.35)
for all \( t \in [0, T] \). Taking into account (4.35) and (4.33) we still can write
\[
\int_Q (j_{\lambda,\sigma}(t, y_{\lambda,\sigma}(t, x)) + j^*_{\lambda,\sigma}(t, w_{\lambda,\sigma}(t, x))) dx dt
\] (4.36)
\[
+ \frac{1}{2} \left\{ \|y_{\lambda,\sigma}(T)\|_V^2 - \|y_0\|_V^2 \right\} = \int_0^T \int_\Omega y_{\lambda,\sigma} A_{0,\infty}^{-1} f dx dt.
\]

We note that
\[
\frac{j^*_{\lambda,\sigma}(t, \omega)}{|\omega|} \to \infty \text{ as } |\omega| \to \infty,
\] (4.37)
uniformly in \( \lambda \) and \( \sigma \). This happens due to (2.10) because by setting
\[
\eta_{\lambda,\sigma} = \partial j_{\lambda,\sigma}(t, r), \quad \eta_{\lambda,\sigma} = \beta(t) + \sigma r = \beta(t, (1 + \lambda \beta(t, \cdot))^{-1} r) + \sigma r,
\]
then \( \eta_{\lambda,\sigma} \) is bounded on bounded subsets \( |r| \leq M \), uniformly in \( \lambda \) and \( \sigma \), for \( \lambda \) and \( \sigma \) small (smaller than 1, e.g.).

We also note that
\[
\int_Q j_{\lambda,\sigma}(t, y_{\lambda,\sigma}(t, x)) dx dt \geq \int_Q j_{\lambda}(t, y_{\lambda,\sigma}(t, x)) dx dt
\]
\[
= \int_Q j(t, (1 + \lambda \beta(t, \cdot))^{-1} y_{\lambda,\sigma}) dx dt + \int_Q \frac{1}{2\lambda} |y_{\lambda,\sigma} - (1 + \lambda \beta(t, \cdot))^{-1} y_{\lambda,\sigma}|^2 dx dt
\]
and
\[
\int_0^T \int_\Omega y_{\lambda,\sigma} A_{0,\infty}^{-1} f dx dt = \int_Q \left( y_{\lambda,\sigma} - (1 + \lambda \beta(t, \cdot)) y_{\lambda,\sigma} \right) A_{0,\infty}^{-1} f dx dt
\]
\[
+ \int_Q (1 + \lambda \beta(t, \cdot))^{-1} y_{\lambda,\sigma} A_{0,\infty}^{-1} f dx dt
\]
\[
\leq \int_Q \frac{1}{2\lambda} |y_{\lambda,\sigma} - (1 + \lambda \beta(t, \cdot))^{-1} y_{\lambda,\sigma}|^2 dx dt + 2\lambda \int_Q (A_{0,\infty}^{-1} f)^2 dx dt
\]
\[
+ \int_Q (1 + \lambda \beta(t, \cdot))^{-1} y_{\lambda,\sigma} A_{0,\infty}^{-1} f dx dt.
\]

Plugging these in (4.36) we get after some algebra that
\[
\int_Q j(t, (1 + \lambda \beta(t, \cdot))^{-1} y_{\lambda,\sigma}) dx dt + \int_Q j^*_{\lambda,\sigma}(t, w_{\lambda,\sigma}(t, x))) dx dt
\] (4.38)
\[
+ \frac{1}{2} \left\{ \|y_{\lambda,\sigma}(T)\|_V^2 - \|y_0\|_V^2 \right\}
\]
\[
\leq \int_Q (1 + \lambda \beta(t, \cdot))^{-1} y_{\lambda,\sigma} A_{0,\infty}^{-1} f dx dt + 2\lambda \int_Q (A_{0,\infty}^{-1} f)^2 dx dt.
\]

Further we set
\[
(1 + \lambda \beta(t, \cdot))^{-1} y_{\lambda,\sigma} = z_{\lambda,\sigma}
\]
and argue as in Theorem 3.2 to deduce by the Dunford-Pettis theorem that \((z_{\lambda,\sigma})_\sigma\) and 
\((w_{\lambda,\sigma})_\sigma\) are weakly compact in \(L^1(Q)\). Recalling (3.15) we also get

\[
\|y_{\lambda,\sigma}(T)\|_{V'} \leq C \tag{4.39}
\]

independently on \(\sigma\) and \(\lambda\).

Taking into account that \(w_{\lambda,\sigma} = \beta_\lambda(y_{\lambda,\sigma}) + \sigma y_{\lambda,\sigma}\), equation (4.35) yields

\[
\frac{1}{2} \|y_{\lambda,\sigma}(t)\|_{V'}^2 + \int_0^t \left( \beta_\lambda(\tau, y_{\lambda,\sigma})y_{\lambda,\sigma} + \sigma y_{\lambda,\sigma}^2 \right) dx d\tau = \frac{1}{2} \|y_0\|_{V'}^2 + \int_0^t \left( y_{\lambda,\sigma}(\tau), A_{\sigma,\infty}^{-1} f(\tau) \right)_{V',V} d\tau
\]

for all \(t \in [0, T]\).

Taking into account that \(\beta_\lambda(t, r) \geq 0\), for all \(r \in \mathbb{R}\), and in virtue of the Gronwall lemma, we deduce that

\[
\|y_{\lambda,\sigma}\|_{L^\infty(0,T;V')} \leq C, \tag{4.40}
\]

and

\[
\sqrt{\sigma} \|y_{\lambda,\sigma}\|_{L^2(Q)} \leq C;
\]

and

\[
\int_Q j(t, z_{\lambda,\sigma}(t, x)) dx dt \leq C, \int_Q j_{\lambda,\sigma}^*(t, w_{\lambda,\sigma}(t, x)) dx dt \leq C \tag{4.41}
\]

independently on \(\sigma\) and \(\lambda\). (For getting (4.41) we recall the arguments leading to (3.18), (3.19)).

Then, (4.36) and relation (2.11) for \(j_{\lambda,\sigma}^*\) imply that

\[
\int_Q j_{\lambda,\sigma}(t, y_{\lambda,\sigma}(t, x)) dx dt \leq C \tag{4.42}
\]

independently on \(\sigma\) and \(\lambda\). Following the proof of Theorem 3.2 we deduce that

\[
\begin{align*}
z_{\lambda,\sigma} & \to z_\lambda \text{ weakly in } L^1(Q), \text{ as } \sigma \to 0, \\
w_{\lambda,\sigma} & \to w_\lambda \text{ weakly in } L^1(Q), \text{ as } \sigma \to 0, \\
\sqrt{\sigma} y_{\lambda,\sigma} & \to \zeta_\lambda \text{ weakly in } L^2(Q), \text{ as } \sigma \to 0, \\
y_{\lambda,\sigma} & \to y_\lambda \text{ weak-star in } L^\infty(0,T;V'), \text{ as } \sigma \to 0, \\
y_{\lambda,\sigma}(T) & \to \xi \text{ weakly in } V', \text{ as } \sigma \to 0, \\
A w_\lambda & \to A w_\lambda \text{ weakly in } L^1(0,T;X'), \text{ as } \sigma \to 0, \\
\frac{dy_{\lambda,\sigma}}{dt} & \to \frac{dy_\lambda}{dt} \text{ weakly in } L^1(0,T;X'), \text{ as } \sigma \to 0.
\end{align*}
\]

By (4.42) and (4.12) we have

\[
\int_Q \frac{1}{2\lambda} \left| y_{\lambda,\sigma} - (1 + \lambda \beta(t, \cdot))^{-1} y_{\lambda,\sigma} \right|^2 dx dt \leq \int_Q j_{\lambda,\sigma}(t, y_{\lambda,\sigma}(t, x)) dx dt \leq C
\]

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whence, denoting $\chi_{\lambda, \sigma} = (y_{\lambda, \sigma} - z_{\lambda, \sigma}) / \sqrt{2\lambda}$ we see that $(\chi_{\lambda, \sigma})_\sigma$ is bounded in $L^2(Q)$ and $\chi_{\lambda, \sigma} \to \chi_\lambda$ weakly in $L^2(Q)$, as $\sigma \to 0$, on a subsequence. Then

$$y_{\lambda, \sigma} - z_{\lambda, \sigma} \to \sqrt{2\lambda} \chi_\lambda \text{ weakly in } L^1(Q), \text{ as } \sigma \to 0,$$

where $\|\chi_\lambda\|_{L^1(Q)} \leq C$. Since $z_{\lambda, \sigma} \to z_\lambda$ weakly in $L^1(Q)$, it follows that $(y_{\lambda, \sigma})_\sigma$ is bounded in $L^1(Q)$, so it converges weakly and by the limit uniqueness we have

$$y_{\lambda, \sigma} \to y_\lambda \text{ weakly in } L^1(Q), \text{ as } \sigma \to 0.$$

We also have

$$y_\lambda = z_\lambda + \sqrt{2\lambda} \zeta_\lambda \text{ a.e. on } Q.$$  \hfill (4.43)

By Arzelà-Ascoli theorem (since $V'$ is compact in $X'$ because $X$ is compact in $V$) it follows that

$$y_{\lambda, \sigma}(t) \to y_\lambda(t) \text{ in } X', \text{ uniformly in } t \in [0, T], \text{ as } \sigma \to 0,$$

so $\xi = y_\lambda(T)$ and $y_\lambda(0) = y_0$.

Passing to the limit in (4.34) we get that $(y_\lambda, w_\lambda)$ satisfies

$$\begin{align*}
\frac{dy_\lambda}{dt}(t) + A w_\lambda(t) & = f(t) \text{ a.e. } t \in (0, T), \\
y(0) & = y_0.
\end{align*}$$  \hfill (4.44)

Passing to the limit in (4.38) as $\sigma \to 0$, using the weak lower semicontinuity property we get

$$\begin{align*}
\int_Q (j(t, z_\lambda(t, x))) + j_\lambda^*(t, w_\lambda(t, x)))dxdt & \\
+ \frac{1}{2} \left\{ \|y_\lambda(T)\|_{V'}^2 - \|y_0\|_{V'}^2 \right\} - \int_Q y_\lambda A_{0, \infty}^{-1} f dxdt - 2\lambda \int_Q (A_{0, \infty}^{-1} f)^2 dxdt & \leq 0.
\end{align*}$$  \hfill (4.45)

We repeat again the arguments developed in Theorem 3.2 and deduce by the Dunford-Pettis theorem that $(z_\lambda)_\lambda$ and $(w_\lambda)_\lambda$ are weakly compact in $L^1(Q)$. It still follows that

$$\|z_\lambda(T)\|_{V'} \leq C, \int_Q j(t, z_\lambda(t, x))dxdt \leq C, \int_Q j_\lambda^*(t, w_\lambda(t, x))dxdt \leq C$$  \hfill (4.46)

independently on $\lambda$ (recall (3.15), (3.18), (3.19)). Passing to the limit in (4.40) as $\sigma \to 0$ we get

$$\|y_\lambda\|_{L^\infty(0, T; V')} \leq C$$

where $C$ are several constants independent on $\lambda$. 24
By (4.43) we get that $z_j \to z^*$ weakly in $L^1(Q)$, as $\lambda \to 0$,

$w_\lambda \to w^*$ weakly in $L^1(Q)$, as $\sigma \to 0$,

$y_\lambda \to y^*$ weak-star in $L^\infty(0,T;V')$, as $\sigma \to 0$,

$y_\lambda(T) \to y^*(T)$ weakly in $V'$, as $\sigma \to 0$,

$Aw_\lambda \to Aw^*$ weakly in $L^1(0,T;X')$, as $\sigma \to 0$,

$\frac{dy_\lambda}{dt} \to \frac{dy^*}{dt}$ weakly in $L^1(0,T;X')$, as $\sigma \to 0$.

By (4.43) we get that $z^* = y^*$ a.e. on $Q$ and by (4.46) we obtain

$$\|y^*(T)\|_{V'} \leq C, \qquad \int_Q j(t, y^*(t,x))dxdt \leq C, \qquad \int_Q j^*(t, w^*(t,x))dxdt \leq C. \tag{4.47}$$

The first inequality is obvious. For the second (if $j(t,r) \geq 0$) Fatou’s lemma yields

$$\int_Q j(t, y^*(t,x))dxdt = \int_Q \liminf_{\lambda \to 0} j(t, z_\lambda(t,x))dxdt \leq \liminf_{\lambda \to 0} \int_Q j(t, z_\lambda(t,x))dxdt \leq C. \tag{4.48}$$

If $j$ is not positive, we use again (2.11) and denoting $\tilde{j}(t,r) = j(t,r) - k_1r - k_2 \geq 0$ we write

$$\int_Q \tilde{j}(t, y^*(t,x))dxdt \leq \liminf_{\lambda \to 0} \int_Q \tilde{j}(t, z_\lambda(t,x))dxdt,$$

whence we get

$$\int_Q j(t, y^*(t,x))dxdt - \int_Q (k_1 y^* + k_2)dxdt \leq \liminf_{\lambda \to 0} \int_Q j(t, z_\lambda(t,x))dxdt - \int_Q (k_1 y^* + k_2)dxdt,$$

i.e., (4.48). In what concerns the third inequality in (4.47), we can write by (4.46)

$$\int_Q \frac{1}{2\lambda} |w_\lambda - (1 + \lambda\beta^{-1}(t,\cdot))^{-1}w_\lambda|^2 dxdt + j^*(t, (1 + \beta^{-1}(t,\cdot))^{-1}w_\lambda) \tag{4.49}$$

$$\leq \int_Q j^*_\lambda(t, w_\lambda(t,x))dxdt \leq C.$$

Recalling that $j^*(t,x,\omega) \geq k_3(t,x)\omega + k_4(t,x)$ we get that

$$k_3 \int_Q (1 + \beta^{-1}(t,\cdot))^{-1}w_\lambda dxdt \leq C + k_4 \text{meas}(Q),$$

hence $((1 + \beta^{-1}(t,\cdot))^{-1}w_\lambda)_\lambda$ is bounded in $L^1(Q)$. Then

$$\int_Q \frac{1}{2\lambda} |w_\lambda - (1 + \lambda\beta^{-1}(t,\cdot))^{-1}w_\lambda|^2 dxdt \leq C + \int_Q (k_3(1 + \beta^{-1}(t,\cdot))^{-1}w_\lambda + k_4)dxdt \leq C_1.$$
It follows that

\[(1 + \lambda \beta^{-1}(t, \cdot))^{-1}w_\lambda \to w^* \text{ weakly in } L^1(Q), \text{ as } \lambda \to 0.\]

Then, we passing to the limit in (4.49) as \(\lambda \to 0\) (if \(j^*\) is nonnegative). Otherwise we use again (2.11) for \(j^*\).

Passing to the limit in (4.44) and (4.45) as \(\lambda \to 0\) we get

\[
\frac{dy^*}{dt}(t) + Aw^*(t) = f(t) \text{ a.e. } t \in (0, T),
\]

\[
y(0) = y_0.
\]

and, again by the weak lower semicontinuity,

\[
\int_Q (j(t, y^*(t, x)) + j^*(t, w^*(t, x))) dxdt \geq \frac{1}{2} \left\{ ||y^*(T)||^2_{V'} - ||y_0||^2_{V'} \right\} - \int_Q y^* A_0^{-1} f dx d\tau \leq 0.
\]

We have got that \((y^*, w^*) \in U, y^* \in L^\infty(0, T; V')\) and so by Lemma 4.1 it follows that \(y^* w^* \in L^1(Q)\). Replacing the sum of the last two terms on the right-hand side in (4.51) by (4.1) we get

\[
\int_Q (j(t, y^*(t, x)) + j^*(t, w^*(t, x)) - y^*(t, x)w^*(t, x)) dxdt \leq 0.
\]

Recalling (2.7) we obtain

\[
\int_Q (j(t, y^*(t, x)) + j^*(t, w^*(t, x)) - y^*(t, x)w^*(t, x)) dxdt = 0
\]

which eventually implies that

\[
j(t, y^*(t, x)) + j^*(t, w^*(t, x)) - y^*(t, x)w^*(t, x) = 0 \text{ a.e. on } Q.
\]

Therefore, we conclude that \(w^*(t, x) \in \beta(t, y^*(t, x)) \text{ a.e. on } Q\), by the Legendre-Fenchel relations.

On the other hand, due again to (4.1) in Lemma 4.1, relation (4.52) means in fact that \((y^*, w^*)\) realizes the minimum in \((P)\), as claimed in (4.28).

The uniqueness follows directly by (2.1) using the monotony of \(\beta\).

Indeed, let \((y, \eta)\) and \((\tilde{y}, \tilde{\eta})\) be two solutions to (2.11) corresponding to the same data, where \(\eta(t, x) \in \beta(t, x, y(t, x)), \tilde{\eta}(t, x) \in \beta(t, x, \tilde{y}(t, x))\) a.e. on \(Q\), \((y, \eta)\) and \((\tilde{y}, \tilde{\eta})\) belong to \(U\) and \(y, \tilde{y} \in L^\infty(0, T; V')\). We write the equations satisfied by their difference

\[
\frac{d(y - \tilde{y})}{dt}(t) + A(\eta - \tilde{\eta})(t) \equiv 0 \text{ a.e. } t \in (0, T),
\]

\[
(y - \tilde{y})(0) = 0.
\]
multiply the equation by $A_{0,\infty}^{-1}(y - \tilde{y})(t)$ and integrate it over $(0, t)$ obtaining

$$\frac{1}{2}\|y - \tilde{y})(t)\|^2_{V'} + \int_0^t \int_{\Omega} (\eta - \tilde{\eta})(y - \tilde{y}) \, dx \, dt = 0.$$ 

But $\beta(t, r)$ is maximal monotone, hence we get $\|y(t) - \tilde{y}(t)\|^2_{V'} \leq 0$, whence $y(t) = \tilde{y}(t)$ for all $t \in [0, T]$.

We remark now that if $(y^*, \eta^*)$ is the solution to (2.1), with $\eta^*(t, x) \in \beta(t, y^*(t, x))$ a.e. on $Q$ and $y^* \in L^\infty(0, T; V')$, then it is a unique solution to ($P$), because by Lemma 4.1, $J(y, w) \geq 0$ for any $(y, w) \in U$ and the minimum is realized at $(y^*, \eta^*)$ since $J(y^*, \eta^*) = 0$. So, we conclude that (2.1) is equivalent with the minimization problem ($P$).

## 5 Conclusions

This paper deals with the application of the Brezis-Ekeland principle for a nonlinear diffusion equation with a monotonically increasing time depending nonlinearity which provides a potential having a weak coercivity property. The result states that the solution of the nonlinear equation can be retrieved as the null minimizer of an appropriate minimization problem for a convex functional involving the potential of the nonlinearity.

This approach is useful because it allows the existence proof in cases in which, due to the generality of the nonlinearity, standard methods do not apply. Also it can lead to a simpler numerical computation of the solution to the equation by replacing its direct determination by the numeric calculus of the minimum of a convex functional with a linear state equation.

With respect to the literature concerning existence results for (2.1), Theorem 4.3 provides existence under very general conditions on the nonlinear function $\beta$. As regards the assumption (2.13) it can be equivalently expressed as

$$\limsup_{|r| \to \infty} \frac{j(t, -r)}{j(t, r)} < \infty,$$

(see [7]). Since in specific real problems the solution to (2.1) is nonnegative, and so $\beta$ is defined on $[0, \infty)$, this condition is achieved by extending in a convenient way the function $\beta$ on $(-\infty, 0)$. For instance, conditions (2.3)-(2.4) are satisfied for $\beta$ of the form

$$\beta(t, x, r) = \text{sgn}(r) \log(|r| + a(t, x)), \quad a \geq a_0 > 0$$

or

$$\beta(t, x, r) = \text{sgn}(r) \exp(a(t, x)r^2), \quad a \geq a_0 > 0.$$ 

Concerning possible applications, we remark that problem (2.1) can be obtained by a change of variable in an equation of the form

$$\frac{\partial(m(t, x)y)}{\partial t} - \Delta \beta_0(y) \ni f$$
which is associated to various physical models as for example: fluid diffusion in saturated-unsaturated deformable porous media with the porosity \( m \) time and space dependent (see appropriate problems in [14], [11]), or to absorption-desorption processes in saturated porous media in which \( m \) is the absorption-desorption rate of the fluid by the solid. The Robin boundary condition arising in (2.1) was chosen because of its relevance in these physical models. Also, evolution equations with nonautonomous operators can be associated to models in which the boundary conditions are of time dependent nonhomogeneous Dirichlet type, or nonlocal as in population dynamics (see [13]).

In all these problems the coefficient \( m \) or the coefficients in the boundary conditions may have a very low regularity which makes not possible the approach of (2.1) by the nonlinear semigroup method in the time-dependent case given in [12].

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