Canonical reduction of two-dimensional gravity for Particle Dynamics

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Abstract

We develop the formalism for canonical reduction of (1 + 1)-dimensional gravity coupled with a set of point particles by eliminating constraints and imposing coordinate conditions. The formalism itself is quite analogous to the (3 + 1)-dimensional case; however in (1 + 1) dimensions an auxiliary scalar field is shown to have an important role. The reduced Hamiltonian is expressed as a form of spatial integral of the second derivative of the scalar field. Since in (1 + 1) dimensions there exists no dynamical degree of freedom of the gravitational field (i.e. the transverse-traceless part of the metric tensor is zero), the reduced Hamiltonian is completely determined in terms of the particles’ canonical variables (coordinates and momenta). The explicit form of the Hamiltonian is calculated both in post-linear and post-Newtonian approximations.

1 Introduction

Early versions of (1 + 1)-dimensional gravity [1, 2] have in recent years led to intensive study of a wide variety of such theories, in large part because problems in quantum gravity become much more mathematically tractable in this context [3]. In contrast to (3 + 1)-dimensional gravity, the action integrals for these theories must incorporate some dynamics in the form of an auxiliary (or dilaton) field since the Einstein tensor is topologically trivial in two dimensions. Although most of these theories can be written in a generic form [4] whose general properties can then be studied [5, 6, 7], there is often much to be learned by focussing on specific models within this general class [8].

One such theory has been extensively studied in many respects, including gravitational collapse, black holes, cosmological solutions and quantization [9, 10, 11, 12]. Referred to as $R = T$ theory, the specific form of the coupling of the auxiliary field $Ψ$ to gravity is chosen so that it decouples from the classical field equations in such a way as to ensure that the evolution of the gravitational field is determined only by the matter stress-energy (and reciprocally) [9, 10]. In this manner $R = T$ theory captures in two spacetime dimensions...
the essence of classical general relativity (as opposed to classical scalar-tensor theories), and has (1 + 1)–dimensional analogs of many of its properties [10, 11]. Indeed, the theory can be understood as the $D \to 2$ limit of general relativity [13].

One of the fundamental problems of (1+1)-dimensional gravity is its relationship to Newtonian gravity. This is in general a problematic issue [7]. For a system of particles the Hamiltonian in Newtonian gravity in two dimensions is

$$H = \sum_a \frac{p_a^2}{2m_a} + \pi G \sum_a \sum_b m_a m_b \left| z_a - z_b \right|$$

where $m_a$, $z_a$ and $p_a$ are the rest mass, the coordinate and the momentum of $a$-th particle, respectively, and $G$ is the gravitational constant. We note that the $R = T$ theory has been shown to have a Newtonian limit [9, 11]. However the dynamical role of the auxiliary field has not yet been fully analysed, a task which we investigate in this paper.

More generally we shall, in the context of $R = T$ theory, formulate a general framework for deriving a Hamiltonian for a system of particles, which coincides in a slow motion, weak field limit with the Hamiltonian (1). In addition to the properties stated above, an advantage of the $R = T$ model is that it needs only one auxiliary field. As with the ADM formalism in (3 + 1)–dimensional theory, we develop a canonical reduction by eliminating constraints and imposing coordinate conditions, which are quite analogous to the ADM conditions. In its final form the reduced Hamiltonian of the system is given as a spatial integral of the second derivative of the auxiliary scalar field $\Psi$. Since in two dimensions there exists no transverse-traceless part of the metric tensor (which are the real dynamical variables of gravitational field), the scalar field $\Psi$ is given as a function of the dynamical variables ($z_a$, $p_a$) of the particles by solving the constraint equations. Then the Hamiltonian is completely determined in terms of the coordinates and momenta of the particles.

The outline of our paper is as follows. Details of the reduction process are described in section 2, where much attention is devoted to the transformation of the total generator of the whole system and the choice of the coordinate conditions, from which the reduced Hamiltonian is defined. The consistency of the canonical reduction is proved in section 3. It guarantees that the canonical equations of motion given by the reduced Hamiltonian are identical with the original geodesic equations. The explicit form of the Hamiltonian is calculated in section 4 in a post-linear approximation (a series expansion in the coupling constant $G$) and in section 5 in a post-Newtonian approximation. While the former approximation is appropriate for the analysis of fast motion, the latter is adequate for treating the slow motion, weak field case. Section 6 is reserved for concluding remarks. The role of the $\Psi$ field is discussed in the simplest case of a single static source. Finally, an appendix illustrates the relation between the equations of motion in our coordinate conditions and those in other conditions.

## 2 Canonical formalism for particle dynamics

The action integral for the gravitational field coupled with point particles is [10]

$$I = \int dx^2 \left[ \frac{1}{2\kappa} \sqrt{-g} \left\{ \Psi R + \frac{1}{2} g^{\mu\nu} \nabla_\mu \Psi \nabla_\nu \Psi \right\} \right]$$
where $\Psi$ is the auxiliary scalar field. Here $g_{\mu\nu}$, $g$, $R$ and $\tau_a$ are the metric tensor of spacetime, $\text{det}(g_{\mu\nu})$, the Ricci scalar and the proper time of $a$-th particle, respectively, and $\kappa = 8\pi G/c^4$. The symbol $\nabla_\mu$ denotes the covariant derivative associated with $g_{\mu\nu}$.

The field equations derived from the variations $\delta \Psi$ and $\delta g_{\mu\nu}$ are

\[ R - g^{\mu\nu} \nabla_\mu \nabla_\nu \Psi = R - \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Psi) = 0 \]  

\[ \frac{1}{2} \nabla_\mu \Psi \nabla_\nu \Psi - \frac{1}{4} g_{\mu\nu} \nabla^\lambda \nabla_\lambda \Psi + g_{\mu\nu} \nabla^\lambda \nabla_\lambda \Psi - \nabla_\mu \nabla_\nu \Psi = \kappa T_{\mu\nu} \]  

where

\[ T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta L_M}{\delta g^{\mu\nu}} \]

\[ = \sum_a m_a \int d\tau_a \frac{1}{\sqrt{-g}} g_{\nu\sigma} g_{\nu\rho} \frac{dz_a^\sigma}{d\tau_a} \frac{dz_a^\rho}{d\tau_a} \delta^2(x - z_a(\tau_a)) , \]  

$L_M$ being the matter Lagrangian given by the second term in the brackets on the right hand side of (2). The geodesic equation derived from the variation $\delta z^\mu_a$ is

\[ \frac{d}{d\tau_a} \left\{ g_{\mu\nu}(z_a) \frac{dz_a^\nu}{d\tau_a} \right\} - \frac{1}{2} g_{\nu\lambda,\mu}(z_a) \frac{dz_a^\nu}{d\tau_a} \frac{dz_a^\rho}{d\tau_a} = 0 . \]  

The trace of Eq.(4) is

\[ \nabla^\mu \nabla_\mu \Psi = \kappa T^\mu_\mu , \]

which yields

\[ R = \kappa T^\mu_\mu . \]  

Hence particle dynamics in $R = T$ theory may be described in terms of the equations (6) and (8), both of which are independent of the scalar field. Particulate matter (in terms of the trace of its stress-energy tensor) generates spacetime curvature via (8); the effects of curvature act back upon matter via (6).

Note that all three components of the metric tensor cannot be determined from (8), since it is only one equation. The two extra degrees of freedom are related to the choice of coordinates. If the coordinate conditions are chosen to be independent of $\Psi$, equation (8) determines the metric tensor completely. More generally however, we need to know the scalar field $\Psi$, through which the metric tensor is determined – it is this field that guarantees conservation of the stress-energy tensor via (4). So far little attention has been paid to the role of the scalar field $\Psi$.

To derive a Hamiltonian for a system of particles, we shall utilize the canonical formalism [14]. Writing $\gamma = g_{11}, N_0 = (-g^{00})^{-1/2}, N_1 = g_{10}$, the action (2) transforms to

\[ I = \int dx^2 \left\{ \sum_a p_a z_a \delta(x - z_a(x^0)) + \pi \dot{\Psi} + \Pi \ddot{\Psi} + N_0 R^0 + N_1 R^1 \right\} \]  

(9)
where \( \pi \) and \( \Pi \) are conjugate momenta to \( \gamma \) and \( \Psi \) respectively, and

\[
R^0 = -\kappa \sqrt{\gamma} \pi^2 + 2\kappa \sqrt{\gamma} \pi \Pi + \frac{1}{4\kappa \sqrt{\gamma}} (\Psi')^2 - \frac{1}{\kappa \sqrt{\gamma}} (\frac{\Psi'}{\gamma})' - \sum_a \sqrt{\frac{p_a^2}{\gamma}} + m_a^2 \delta(x - z_a(x^0))
\]

(10)

\[
R^1 = \frac{\gamma'}{\gamma} \pi - \frac{1}{\gamma} \Pi \Psi' + 2\pi' + \sum_a \frac{p_a}{\gamma} \delta(x - z_a(x^0))
\]

(11)

Here and in the following we denote \( \partial_0 \) by a symbol (\( \dot{\cdot} \)) and \( \partial_1 \) by a symbol (\( \cdot' \)).

Taking variations \( \delta \gamma, \delta \pi, \delta N_0, \delta N_1, \delta \Psi, \delta \Pi, \delta z_a \) and \( \delta p_a \) of the action (9), we have

\[
\dot{\pi} + N_0 \left\{ \frac{3\kappa}{2} \sqrt{\gamma} \pi^2 - \frac{\kappa}{\sqrt{\gamma}} \pi \Pi + \frac{1}{8\kappa \sqrt{\gamma}} (\Psi')^2 - \sum_a \frac{p_a^2}{2\gamma^2 \sqrt{\frac{p_a^2}{\gamma} + m_a^2}} \delta(x - z_a(x^0)) \right\}
\]

\[
+ N_1 \left\{ -\frac{1}{\gamma^2} \Pi \Psi' + \frac{\pi'}{\gamma} + \sum_a \frac{p_a}{\gamma^2} \delta(x - z_a(x^0)) \right\}
\]

\[
+ N_0' \frac{1}{2\kappa \sqrt{\gamma}} \Psi' + N_1' \frac{\pi}{\gamma} = 0
\]

(12)

\[
\dot{\gamma} - N_0 (2\kappa \sqrt{\gamma} \pi - 2\kappa \sqrt{\gamma} \Pi) + N_1 \frac{\gamma'}{\gamma} - 2N_1' = 0
\]

(13)

\[
R^0 = 0
\]

(14)

\[
R^1 = 0
\]

(15)

\[
\dot{\Pi} + \partial_1 \left( -\frac{1}{\gamma} N_1 \Pi + \frac{1}{2\kappa \sqrt{\gamma}} N_0 \Psi' + \frac{1}{\kappa \sqrt{\gamma}} N_0' \right) = 0
\]

(16)

\[
\dot{\Psi} + N_0 (2\kappa \sqrt{\gamma} \pi) - N_1 (\frac{1}{\gamma} \Psi') = 0
\]

(17)

\[
\dot{p}_a + \frac{\partial N_0}{\partial z_a} \sqrt{\frac{p_a^2}{\gamma} + m_a^2} - \frac{N_0}{2\sqrt{\frac{p_a^2}{\gamma} + m_a^2}} \frac{p_a^2}{\gamma} \frac{\partial \gamma}{\partial z_a}
\]

\[
- \frac{\partial N_1}{\partial z_a} \frac{p_a}{\gamma} + N_1 \frac{p_a}{\gamma^2} \frac{\partial \gamma}{\partial z_a} = 0
\]

(18)

\[
\dot{z}_a - N_0 \frac{p_a}{\sqrt{\frac{p_a^2}{\gamma} + m_a^2}} + N_1 \frac{\pi}{\gamma} = 0
\]

(19)

In the equations (18) and (19), all metric components \( (N_0, N_1, \gamma) \) are evaluated at the point \( x = z_a \) and

\[
\frac{\partial f}{\partial z_a} = \frac{\partial f(x)}{\partial x} \bigg|_{x = z_a}
\]

It can be shown that this set of equations is equivalent to the equations (3), (4) and (6). Here we show explicitly the equivalence of (18) and (19) to the geodesic equation (6). We
express the \((\mu = 1)\) component of \((6)\) in terms of \(N_0, N_1\) and \(\gamma\) and their derivatives. Using the relations
\[
d\tau_a = dt \left\{ \frac{N_0^2}{\gamma} (N_1 + \gamma \dot{z}_a)^2 \right\}^{\frac{1}{2}}
\]
we get
\[
\frac{d}{dt} \left\{ \frac{N_1 + \gamma \dot{z}_a}{\left[ N_0^2 - \frac{1}{\gamma} (N_1 + \gamma \dot{z}_a)^2 \right]^{\frac{1}{2}}} \right\} + \frac{1}{\left[ N_0^2 - \frac{1}{\gamma} (N_1 + \gamma \dot{z}_a)^2 \right]^{\frac{1}{2}}} \left\{ \frac{N_0 \partial N_0}{\partial z_a} - \left( \frac{N_1}{\gamma} + \dot{z}_a \right) \frac{\partial N_1}{\partial z_a} \right. \\
\left. + \frac{1}{2} \left[ \frac{N_1^2}{\gamma^2} - (\dot{z}_a)^2 \right] \frac{\partial \gamma}{\partial z_a} \right\} = 0 . \tag{20}
\]
Eliminating \(p_a\) from \((18)\) and \((19)\) yields an equation identical with \((20)\).

From the expression \((9)\), the total generator (obtained from variations at the end point) is given by
\[
G = \int dx \left\{ \sum_a p_a \delta (x - z_a) \delta z_a + \pi \delta h - \Psi \delta \Pi \right\} , \tag{21}
\]
where the above form has been obtained by adding a total time derivative \(-\partial_0 (\Pi \Psi)\) to the original action \((9)\), and where the constraint equations \((14)\) and \((15)\) have been taken into account. In the above \(h = 1 + \gamma\); the advantage of using \(h\) will become clear later \([15, 16]\).

We must now identify the dynamic and gauge character of the variables which appear in the generator \((21)\). By considering the constraint equations \((14)\) and \((15)\), we see that the only linear terms there are \(\Psi' \sqrt{h - 1}\) and \(\pi'\) respectively. The equations may therefore be solved for these quantities in terms of the dynamical and gauge \((i.e.\ co-ordinate)\) degrees of freedom. By writing \(\Pi = \frac{1}{\Delta} \Pi'\), where \(1/\Delta\) is the inverse of the operator \(\Delta = \partial^2 / \partial x^2\) with appropriate boundary condition, we find that the generator becomes
\[
G = \int dx \left\{ \sum_a p_a \delta (x - z_a) \delta z_a - \left[ -\frac{1}{\kappa} \left( \frac{\Psi'}{\sqrt{h - 1}} \right) \right] \delta \left[ \frac{1}{\Delta} \left( \sqrt{h - 1} - \frac{1}{\Delta} \Pi' \right) \right] \\
- \left[ 2\pi' - \left( \frac{\Psi'}{\sqrt{h - 1}} \right) \left( \frac{1}{\Delta} \Pi' - \frac{1}{h - 1} \Pi \Psi' \right) \delta \left( \frac{1}{2\Delta} h' \right) \right] \right\} . \tag{22}
\]
where in obtaining this form, we have discarded surface terms.

The form \((22)\) of the generator is analogous what is obtained in \((3 + 1)\) dimensions under an orthogonal decomposition of the hypersurface metric and its conjugate momentum \([13, 14]\). We therefore propose adopting the coordinate conditions
\[
x = \frac{1}{2\Delta} h' \tag{23}
\]
\[
t = -\frac{\kappa}{\Delta} \left( \sqrt{h - 1} - \frac{1}{\Delta} \Pi' \right)' \tag{24}
\]
which will then allow us to identify the Hamiltonian and momentum densities as the respective coefficients of \(\delta t\) and \(\delta x\) in the canonical form of the generator. Taking a spatial
derivative of these coordinate conditions yields their differential form, which does not explicitly depend upon the coordinates \((t, x)\). This leads to the choices

\[
h = 2 \quad \rightarrow \quad \gamma = 1 \quad (25)
\]

\[
\frac{1}{\Delta} \Pi' = 0 \quad \rightarrow \quad \Pi = 0 \quad (26)
\]

which may then be inserted in equations (12) and (19) to solve for the relevant physical degrees of freedom.

Specifically, we may insert these choices into (14) and (15) to solve for the Hamiltonian and momentum densities. We then find that the generator (22) may be expressed in the canonical form

\[
G = \int dx \left\{ \sum_a p_a \delta(x - z_a) \delta z_a - T_{0\mu} \delta x^\mu \right\} \quad (27)
\]

where

\[
T_{00} = \mathcal{H} = -\frac{1}{\kappa} \Delta \Psi \quad (28)
\]

\[
T_{01} = 2\pi' \quad .
\]

Thus, \(\mathcal{H}\) is the Hamiltonian density of the system and \(T_{01}\) is the momentum density. Note that since the constraints (14) and (15) have already been imposed, \(T_{0\mu}\) is expressed in terms of the canonical variables \(z_a\) and \(p_a\) as \(T_{0\mu}(x, z_a, p_a)\), by solving these constraints. With the coordinate choices (25) and (26), they lead to

\[
\Delta \Psi - \frac{1}{4}(\Psi')^2 + \kappa^2 \pi^2 + \kappa \sum_a \sqrt{p_a^2 + m_a^2} \delta(x - z_a) = 0 \quad ,
\]

\[
2\pi' + \sum_a p_a \delta(x - z_a) = 0 \quad .
\]

Following a similar procedure as that used to transform the generator \(G\), we rewrite the action integral (9) as

\[
I = \int d^2x \left\{ \sum_a p_a \dot{z}_a \delta(x - z_a) - \left[ -\frac{1}{\kappa} \left( \frac{\Psi'}{\sqrt{h - 1}} \right)' \right] \partial_0 \left[ -\frac{\kappa}{\Delta} \left( \sqrt{h - 1} \frac{1}{\Delta} \Pi' \right)' \right] \right.
\]

\[
- \left[ 2\pi' - \left( \frac{\Psi'}{h - 1} \right)' \frac{1}{\Delta} \Pi' - \frac{1}{h - 1} \Pi \Psi' \right] \partial_0 \left( \frac{1}{2\Delta} h' \right) + N_\mu R^\mu \right\} \quad (32)
\]

where we have discarded surface terms. Eliminating the constraints (14) and (15) and imposing the coordinate conditions (23) and (24), the action integral reduces to

\[
I_R = \int dx^2 \left\{ \sum_a p_a \dot{z}_a \delta(x - z_a) - \mathcal{H} \right\} \quad .
\]

Thus the reduced Hamiltonian for the system of particles is

\[
H = \int dx \mathcal{H} = -\frac{1}{\kappa} \int dx \Delta \Psi \quad (34)
\]
where $\Psi$ is a function of $z_a$ and $p_a$ and is determined by solving the constraint equations (30) and (31). This expression is analogous to the reduced Hamiltonian in $(3 + 1)$ dimensional general relativity \[14, 15, 16\].

We pause to comment on the relationship between the integral forms (23) and (24) of the coordinate conditions and the differential forms (25) and (26). These two forms are equivalent only when one retains the appropriate boundary conditions for the integral operator $1/\triangle$. A proper treatment \[17, 18\] entails the insertion of a regulator $\exp(-\alpha|x|)$ in the left-hand sides of (23) and (24), yielding

$$g := 2e^{-\alpha|x|}(1 - \alpha|x|) - 1$$

and

$$\Pi := \frac{d}{dx} \left( -\alpha t e^{-\alpha|x|} \text{sgn}(x) \right) \left( 2e^{-\alpha|x|} - 2|x|\alpha e^{-\alpha|x|} - 1 \right)^{1/2} \kappa$$

in place of (25) and (26) respectively. We must then insert these quantities in (12) - (19), taking the limit $\alpha \to 0$ at the end of the calculation (implicitly assuming $\alpha|x| < 1$). This turns out to be equivalent to inserting (25) and (26) in these equations. The action (33) is recovered by a similar limiting procedure.

This situation is analogous with that in $(3+1)$ dimensions dimensions as discussed by ADM in \[18\] (cf. Eqs. (80), (81), (84) and (85) in section 6). We shall discuss these points further in section 6. Note that the coordinate conditions in integral form are needed only at the stage of transforming the generator and defining the Hamiltonian density. After fixing the formalism, we need only the differential forms to solve the constraints and to evaluate the Hamiltonian, a problem which is treated in the following sections.

## 3 Consistency of the canonical reduction

Following a procedure similar to the original ADM argument \[15, 17\], we shall demonstrate here that the canonical equations of motion derived from the reduced Hamiltonian (34) are identical with equations (18) and (19), which are in turn equivalent to the original geodesic equation (6).

We start from the action integral (9). The variations with respect to $p_a$ and $z_a$ lead to the equations of motion

$$\dot{z}_a = -\int d^2y N_\mu(y) \frac{\delta R^\mu(y)}{\delta p_a(t)}$$

(35)

$$\dot{p}_a = \int d^2y N_\mu(y) \frac{\delta R^\mu(y)}{\delta z_a(t)}$$

(36)

These equations are identical with equations (18) and (19). Our purpose is to prove that these equations lead, when they are combined with the constraints (14) and (15), to the equations in the reduced formalism.

Defining

$$P_0(x) \equiv -\frac{1}{\kappa} \triangle \Psi(x)$$

(37)

$$P_1(x) \equiv 2\pi'(x)$$

(38)
and imposing the coordinate conditions, the action \( I \) may be expressed as

\[
I = \int d^2x \left\{ \sum_a p_a \dot{z}_a \delta(x - z_a) - P_0 + N_\mu R^\mu \right\}.
\]  

(39)

where we have not yet imposed the constraints \( R^\mu = 0 \). By solving the constraint equations, we get

\[
P_0(x) = \mathcal{T}_{00}(x, z_a, p_a) \equiv \mathcal{H}
\]

(40)

\[
P_1(x) = \mathcal{T}_{01}(x, z_a, p_a)
\]

(41)

where \( \mathcal{T}_{0\mu}(x, z_a, p_a) \) was described in the previous section.

Expanding \( R^\mu \) in a functional Taylor series about the point \( P_\mu = T_\mu \) gives

\[
R^\mu(x) = \int d^2y \left[ P_\nu(y) - \mathcal{T}_{\nu\omega}(y, z_a, p_a) \right] \left[ \frac{\delta R^\mu(x)}{\delta P_\nu(x)} \right]_{P=\mathcal{T}} + \cdots.
\]  

(42)

where the coordinate conditions have already been imposed. Substituting this expansion into the right hand side of (39) and taking the variation of the action \( I \) with respect to \( P_\nu \), yields

\[
- \delta \nu_0 + \int d^2 y N_\mu(y) \left\{ \frac{\delta R^\mu(y)}{\delta \mathcal{T}_{\nu\omega}(x)} \right\}_{P=\mathcal{T}} = 0
\]

(43)

where the terms represented by \( \cdots \) contain \( [P_\nu - \mathcal{T}_{\nu\omega}] \) as a factor.

Now we need the relation which is valid after imposing the constraints. Then, by requiring \( R^\mu = 0 \), we have

\[
- \delta \nu_0 + \int d^2 y N_\mu(y) \left[ \frac{\delta R^\mu(y)}{\delta \mathcal{T}_{\nu\omega}(x)} \right]_{P=\mathcal{T}} = 0.
\]  

(44)

Insertion of (42) into (35) leads to

\[
\dot{z}_a = - \int d^2 y N_\mu(y) \int d^2 x \left\{ \frac{\partial \mathcal{T}_{\nu\omega}}{\partial p_a(t)} \left[ \frac{\delta R^\mu(y)}{\delta \mathcal{T}_{\nu\omega}(x)} \right]_{P=\mathcal{T}} + \cdots \right\}
\]

\( + (P_\nu(x) - \mathcal{T}_{\nu\omega}(x, z_a, p_a)) \frac{\partial}{\partial p_a(x)} \left[ \frac{\delta R^\mu(y)}{\delta \mathcal{T}_{\nu\omega}(x)} \right]_{P=\mathcal{T}} + \cdots \}
\]

(45)

Imposing the constraint equations \( R^\mu = 0 \) yields for (45)

\[
\dot{z}_a = \int d^2 x \frac{\partial \mathcal{T}_{\nu\omega}}{\partial p_a(t)} \int d^2 y N_\mu(y) \left[ \frac{\delta R^\mu(y)}{\delta \mathcal{T}_{\nu\omega}(x)} \right]_{P=\mathcal{T}}
\]

(46)

As a consequence of the relation (44), (46) becomes

\[
\dot{z}_a = \int d^2 x \frac{\partial \mathcal{T}_{00}}{\partial p_a}
\]

\( = \frac{\partial H}{\partial p_a} \)

(47)

In the same way we can show that the equations of motion (36) is identical with

\[
\dot{p}_a = - \frac{\partial H}{\partial z_a}
\]  

(48)

Thus the consistency of the reduced canonical formalism is proved.
4 Post-linear approximation

For a direct calculation of the reduced Hamiltonian (34), we have to solve the constraint equations (30) and (31). However, it seems quite a difficult task to get an exact solution except in the case of a single static source. So we need some approximation method.

In this section we apply an iterative method successively to the integrand of the expression (34) with the use of the equations (30) and (31). This provides us with an expansion of (34) in powers of \( \kappa \), which we refer to as the post-linear approximation.

First, substituting \( \Delta \Psi \) given by the equation (30) into the integrand and performing a partial integration, we have

\[
H = \int dx \left\{ -\frac{1}{4\kappa} (\Psi')^2 + \kappa (\chi')^2 + \sum_a \sqrt{p_a^2 + m_a^2} \delta(x - z_a) \right\}
= \sum_a \sqrt{p_a^2 + m_a^2} + \int dx \left\{ \frac{1}{4\kappa} \Psi \Delta \Psi - \kappa \chi \Delta \chi \right\} + S_1
\]

where \( \chi \) is defined by \( \chi' \equiv \pi \) and

\[
S_1 = \left[ -\frac{1}{4\kappa} \Psi \Psi' + \kappa \chi \chi' \right]_{-\infty}^{\infty}.
\]

Noting that \( \Psi \) is of order \( \kappa \), we substitute (30) for \( \Delta \Psi \) and (31) for \( \Delta \chi \) into the integrand of the second term on the right hand side of (49). Iterating twice in terms of \( \kappa \) yields the expression

\[
H = \sum_a \sqrt{p_a^2 + m_a^2} \left\{ 1 - \frac{1}{4} \Psi(z_a) + \frac{1}{32} \Psi(z_a) \psi(z_a) \right\}
+ \frac{\kappa}{2} \sum_a p_a \chi(z_a) \left\{ 1 - \frac{1}{4} \psi(z_a) \right\} + \frac{\kappa^2}{8} \sum_a \sqrt{p_a^2 + m_a^2} \chi(z_a) \chi(z_a) + S_2
+ \int dx \left\{ \frac{\kappa^3}{8} \chi^2 (\chi')^2 - \frac{\kappa}{32} \chi^2 (\psi')^2 + \frac{\kappa}{32} \psi^2 (\chi')^2 - \frac{1}{128\kappa} \psi^2 (\psi')^2 \right\}
\]

where

\[
S_2 = \left[ -\frac{1}{4\kappa} \Psi \Psi' + \kappa \chi \chi' - \frac{\kappa}{8} \left\{ \Psi (\chi')^2 - \psi \chi^2 \right\} + \frac{1}{32\kappa} \psi^2 \psi' \right]_{-\infty}^{\infty}.
\]

The last term in (51) contributes at order \( \kappa^3 \). This iteration can be continued successively and corresponds to a series expansion in \( \kappa \).

In contrast to the (3 + 1)-dimensional situation, we encounter a subtle problem when we try to extract an explicit form of the Hamiltonian under this iteration scheme. In (1 + 1) dimensions the surface terms such as \( S_1 \) and \( S_2 \) arising in the process of the calculation do not necessarily vanish and make the Hamiltonian indefinite. This is because the dimensionless potential \( G m r / c^2 \) becomes infinite at spatial infinity, in contrast to (3 + 1) dimensions where the corresponding quantity \( G m r / c^2 \) vanishes at spatial infinity thereby assuring that the associated surface terms vanish. It is therefore an important task to find a solution of \( \Psi \) and \( \chi \) which makes the surface term vanish at a given order in \( \kappa \). This is related to the choice of boundary condition.
From the expressions of $S_1$ and $S_2$, we can infer a simple choice of the boundary condition as follows.

For $f(x) \equiv \Psi^2 - 4\kappa^2 \chi^2$,

\[
f(x) = 0 \quad \text{and} \quad f'(x) = 0 \quad \text{in a region} \quad |x| >> |z_a| \quad \text{for all} \ a. \quad (53)
\]

It is easily checked that under this boundary condition, the surface terms $S_1$ and $S_2$ exactly vanish, since they are proportional to $f'$ and $(4 - \Psi)f' + f\Psi'$ respectively.

Now let us try to obtain the Hamiltonian up to $\kappa^2$ (the 2nd-post-linear approximation). First we expand $\Psi$ and $\chi$ in a power series in $\kappa$

\[
\Psi = \kappa \Psi^{(1)} + \kappa^2 \Psi^{(2)} + \ldots \quad (54)
\]

\[
\chi = \chi^{(0)} + \kappa \chi^{(1)} + \ldots \quad (55)
\]

Substituting this expansion into the equations (30) and (31), we get

\[
\begin{align*}
\triangle \Psi^{(1)} &= -\sum_a \sqrt{p_a^2 + m_a^2} \delta(x - z_a) \quad (56) \\
\triangle \Psi^{(2)} &= -(\chi^{(0)})^2 + \frac{1}{4} (\Psi^{(1)})^2 \quad (57) \\
\triangle \chi^{(0)} &= -\frac{1}{2} \sum_a p_a \delta(x - z_a) \quad (58) \\
\triangle \chi^{(1)} &= 0 \quad (59)
\end{align*}
\]

The solutions which satisfy the boundary condition (53) are

\[
\begin{align*}
\Psi^{(1)} &= -\frac{1}{2} \sum_a \left\{ \sqrt{p_a^2 + m_a^2} r_a + \epsilon p_a (x - z_a) \right\} \quad (60) \\
\chi^{(0)} &= -\frac{1}{4} \sum_a \left\{ p_a r_a + \epsilon \sqrt{p_a^2 + m_a^2} (x - z_a) \right\} \quad (61)
\end{align*}
\]

\[
\begin{align*}
\Psi^{(2)} &= -\frac{1}{2} \left( \frac{1}{4} \right)^2 \left\{ \sum_a p_a r_a + \epsilon \sum_a \sqrt{p_a^2 + m_a^2} (x - z_a) \right\} \quad (62) \\
\chi^{(1)} &= -\frac{\epsilon}{2} \left( \frac{1}{4} \right)^2 \left\{ \sum_a \sqrt{p_a^2 + m_a^2} (x - z_a) \right\} \quad (63)
\end{align*}
\]

where $r_a \equiv |x - z_a|$. 


In these solutions we introduced a constant of integration $\epsilon$, satisfying $\epsilon^2 = 1$. We have two types of solutions corresponding to $\epsilon = 1$ and $\epsilon = -1$, which are related to each other under time reversal. This $\epsilon$ factor guarantees the invariance of the whole theory under the time reversal.

The boundary condition is checked as follows. Up to order $\kappa^3$ we have

$$
\Psi^2 - 4\kappa^2 \chi^2 = \frac{\kappa^2}{4} \sum_a \sum_b \left\{ \left[ \sqrt{p_a^2 + m_a^2} \sqrt{p_b^2 + m_b^2} - p_a p_b \right] \left( 1 - \frac{\kappa}{8} \sum_c \left[ \sqrt{p_c^2 + m_c^2} r_c + \epsilon p_c (x - z_c) \right] \right) \\
+ \frac{\kappa}{4} \sum_c \left[ \sqrt{p_c^2 + m_c^2} A_{bc} - p_b B_{bc} \right] \right\} [r_a r_b - (x - z_a)(x - z_b)] \\
+ \epsilon \frac{\kappa^2}{2} \sum_a \sum_b \left\{ \sqrt{p_a^2 + m_a^2} p_b \left( 1 - \frac{\kappa}{8} \sum_c \left[ \sqrt{p_c^2 + m_c^2} r_c + \epsilon p_c (x - z_c) \right] \right) \\
- \frac{\kappa}{8} p_a \sum_c \left[ \sqrt{p_b^2 + m_b^2} A_{bc} - p_b B_{bc} \right] \right\} [r_a (x - z_b) - (x - z_a) r_b]
$$

where

$$
A_{bc} = \sqrt{p_c^2 + m_c^2} r_{bc} + \epsilon p_c (z_b - z_c) \\
B_{bc} = p_c r_{bc} + \epsilon \sqrt{p_c^2 + m_c^2} (z_b - z_c).
$$

Since both $[r_a r_b - (x - z_a)(x - z_b)]$ and $[r_a (x - z_b) - (x - z_a) r_b]$ vanish in a region $|x| > |z_a|, |z_b|$, the boundary condition is satisfied.

The 2nd-post-linear Hamiltonian is therefore unambiguously determined to be

$$
H = \sum_a \sqrt{p_a^2 + m_a^2} \left\{ 1 - \frac{1}{4} \Psi(z_a) + \frac{1}{32} \Psi(z_a) \Psi(z_a) \right\} \\
+ \frac{\kappa}{2} \sum_a p_a \chi(z_a) \left\{ 1 - \frac{1}{4} \Psi(z_a) \right\} + \frac{\kappa^2}{8} \sum_a \sqrt{p_a^2 + m_a^2} \chi(z_a)\chi(z_a) \\
= \sum_a \sqrt{p_a^2 + m_a^2} + \frac{\kappa}{8} \sum_a \sum_b \left( \sqrt{p_a^2 + m_a^2} \sqrt{p_b^2 + m_b^2} - p_a p_b \right) r_{ab} \\
+ \frac{\kappa}{8} \sum_a \sum_b \left( \sqrt{p_a^2 + m_a^2} p_b - p_a \sqrt{p_b^2 + m_b^2} \right) (z_a - z_b) \\
+ \frac{1}{4} \left( \frac{\kappa}{4} \right)^2 \left\{ \sum_a \sqrt{p_a^2 + m_a^2} \left[ \sum_b p_b r_{ab} + \epsilon \sum_b \sqrt{p_b^2 + m_b^2} (z_a - z_b) \right]^2 \right\} \\
- \sum_a \sum_b \sqrt{p_a^2 + m_a^2} \left[ \sum_b \sqrt{p_b^2 + m_b^2} (z_a - z_b) \right] \left[ \sum_c \sqrt{p_c^2 + m_c^2} r_{ac} + \epsilon \sum_c p_c (z_a - z_c) \right] \\
+ \sum_a \sum_b \left[ \sqrt{p_a^2 + m_a^2} \sqrt{p_b^2 + m_b^2} r_{ab} - \epsilon p_a \sqrt{p_b^2 + m_b^2} (z_a - z_b) \right] \\
\times \left[ \sum_c \sqrt{p_c^2 + m_c^2} r_{bc} + \epsilon \sum_c p_c (z_b - z_c) \right] \\
- \sum_a \sum_b \left[ \sqrt{p_a^2 + m_a^2} p_b r_{ab} - \epsilon p_a p_b (z_a - z_b) \right] \left[ \sum_c p_c r_{bc} + \epsilon \sum_c \sqrt{p_c^2 + m_c^2} (z_b - z_c) \right] \right\}.
$$
As shown in the above development, the κ-expansion is the successive approximation in the background of Minkowskian space-time and the terms in each order of κ preserve relativistic forms. This approximation is therefore appropriate for describing relativistic fast-motion of the particles and can be carried out to any desired order of κ.

5 Post-Newtonian Hamiltonian and the redefinition of canonical variables

To compare \( R = T \) theory with Newtonian gravity, we need an approximation method applicable to a slow motion in a weak field. It is provided by the so-called \( c^{-1} \) expansion. All terms appropriate to the post-Newtonian approximation (up to the order of \( c^{-4} \)) are included in the post-linear Hamiltonian (64). Noting that both \( \frac{p_a^2}{m_a^2} \) and \( \sqrt{\kappa} \) are of the order of \( c^{-2} \), we find from (64)

\[
H = \sum_a m_a + \sum_a \frac{p_a^2}{2m_a} + \frac{\kappa}{8} \sum_a \sum_b m_a m_b r_{ab} + \frac{\epsilon \kappa}{8} \sum_a \sum_b (m_a p_b - m_b p_a) (z_a - z_b) - \sum_a \frac{p_a^2}{8m_a^3} + \frac{\kappa}{8} \sum_a \sum_b m_a \frac{p_b^2}{m_b} r_{ab} - \frac{\kappa}{8} \sum_a \sum_b p_a p_b r_{ab} + \frac{1}{4} \left( \frac{\kappa}{4} \right)^2 \sum_a \sum_b \sum_c m_a m_b m_c r_{ab} r_{ac} + \frac{1}{4} \left( \frac{\kappa}{4} \right)^2 \sum_a \sum_b \sum_c m_a m_b m_c (z_a - z_b) (z_a - z_c).
\]

(65)

The corresponding solutions of \( \Psi \) and \( \chi \) are

\[
\Psi = -\frac{\kappa}{2} \sum_a m_a r_a - \frac{\epsilon \kappa}{2} \sum_a p_a (x - z_a) - \frac{\kappa}{4} \sum_a \frac{p_a^2}{m_a} r_a + \frac{1}{2} \left( \frac{\kappa}{4} \right)^2 \left( \sum_a m_a r_a \right)^2 - \frac{1}{2} \left( \frac{\kappa}{4} \right)^2 \left( \sum_a m_a (x - z_a) \right)^2 - \left( \frac{\kappa}{4} \right)^2 \sum_a \sum_b m_a m_b r_a r_{ab}.
\]

\[
\chi = -\frac{\epsilon}{4} \sum_a m_a (x - z_a) - \frac{1}{4} \sum_a p_a r_a.
\]

(66)

The canonical equations of motion yield

\[
\dot{z}_a = \frac{\partial H}{\partial p_a} = \frac{p_a}{m_a} - \frac{\epsilon \kappa}{4} \sum_b m_b (z_a - z_b) - \frac{p_a^3}{2m_a^3} + \frac{\kappa}{4} \sum_b m_b \frac{p_a}{m_a} r_{ab} - \frac{\kappa}{4} \sum_b p_b r_{ab}.
\]

(68)

\[
\dot{p}_a = -\frac{\partial H}{\partial z_a} = -\frac{\kappa}{4} \sum_b m_a m_b \frac{\partial r_{ab}}{\partial z_a} - \frac{\epsilon \kappa}{4} \sum_b (m_a p_b - m_b p_a) - \frac{\kappa}{8} \sum_b \left( m_a \frac{p_b^2}{m_b} + m_b \frac{p_a^2}{m_a} \right) \frac{\partial r_{ab}}{\partial z_a}.
\]

12
\[ + \frac{\kappa}{4} \sum_b p_a p_b \frac{\partial r_{ab}}{\partial z_a} - \frac{1}{2} \left( \frac{\kappa}{4} \right)^2 \sum_b \sum_c m_a m_b m_c \left( \frac{\partial r_{ab}}{\partial z_a} r_{bc} + \frac{\partial r_{ab}}{\partial z_a} r_{ac} \right) \]

\[- \frac{1}{2} \left( \frac{\kappa}{4} \right)^2 \sum_b \sum_c m_a m_b m_c (z_a - z_b) \right]. \tag{69} \]

By eliminating \( p_a \) in the equations (68) and (69), we get the equations of motion to second order

\[ m_a \ddot{z}_a = - \frac{\kappa}{4} \sum_b m_a m_b \frac{\partial r_{ab}}{\partial z_a} + \frac{\kappa}{8} \sum_b m_a m_b \left\{ 4(\dot{z}_a)^2 - 2\dot{z}_a \dot{z}_b + (\dot{z}_b)^2 \right\} \frac{\partial r_{ab}}{\partial z_a} \]

\[- \left( \frac{\kappa}{4} \right)^2 \sum_b \sum_c m_a m_b m_c \left( \frac{\partial r_{ab}}{\partial z_a} - \frac{\partial r_{bc}}{\partial z_b} \right) r_{ab} - \frac{1}{2} \left( \frac{\kappa}{4} \right)^2 \sum_b \sum_c m_a m_b m_c \left( \frac{\partial r_{ab}}{\partial z_a} r_{bc} + \frac{\partial r_{ab}}{\partial z_a} r_{ac} \right) \]

\[ + \frac{1}{2} \left( \frac{\kappa}{4} \right)^2 \sum_b \sum_c m_a m_b m_c (z_a - z_b) \]

\[ + \frac{1}{2} \left( \frac{\kappa}{4} \right)^2 \sum_b \sum_c m_a m_b m_c \left( \frac{\partial r_{ab}}{\partial z_a} - \frac{\partial r_{bc}}{\partial z_b} \right) \]

\[ - \frac{1}{2} \left( \frac{\kappa}{4} \right)^2 \sum_b \sum_c m_a m_b m_c (z_a - z_b) \] \tag{70} \]

We pause to show that these canonical equations of motion are equivalent to the geodesic equation (6). To evaluate the geodesic equation, we have to know all components of the metric tensor, of which \( g_{11} = \gamma \) is fixed and \( N_0 \) and \( N_1 \) components are determined by combining the time derivatives of the coordinate conditions (25) and (26) with the equations (13) and (16). These equations are

\[ \kappa \chi' N_0 + N_1' = 0 \]

\[ \partial_1 \left( \frac{1}{2} N_0 \Psi' + N_0' \right) = 0 \] \tag{71} \tag{72} \]

The solutions in the post-Newtonian approximation are

\[ N_0 = 1 + \frac{\kappa}{4} \sum_a m_a r_a + \frac{\epsilon \kappa}{4} \sum_a p_a (x - z_a) + \frac{\kappa}{8} \sum_a \frac{p_a^2}{m_a} r_a + \frac{1}{4} \left( \frac{\kappa}{4} \right)^2 \left( \sum_a m_a r_a \right)^2 \]

\[ + \frac{1}{4} \left( \frac{\kappa}{4} \right)^2 \left( \sum_a m_a (x - z_a) \right)^2 + \frac{1}{2} \left( \frac{\kappa}{4} \right)^2 \sum_a \sum_b m_a m_b r_a r_{ab} + \frac{1}{2} \sum_b \sum_c m_a m_b m_c (z_a - z_b) \] \tag{73} \tag{74} \]

The boundary conditions in the previous section are incorporated in these solutions through (12) and (17), which serve as consistency equations.

Since we have proved in section 2 the equivalence of the geodesic equation (6) to a set of equations (18) and (19), we have only to evaluate the latter equations. Under the coordinate conditions (25) and (26), the equations (18) and (19) become

\[ \dot{z}_a = \frac{p_a}{\sqrt{p_a^2 + m_a^2}} N_0(z_a) - N_1(z_a) \] \tag{75} \]

\[ \dot{p}_a = -\sqrt{p_a^2 + m_a^2} \frac{\partial N_0}{\partial z_a} + p_a \frac{\partial N_1}{\partial z_a} \] \tag{76} \]
which lead to the equations (68) and (69), after the insertion of the solutions (73) and (74)
and expanding in powers of $c^{-1}$.

The Hamiltonian (65) contains a term proportional to $c^{-1}$, namely,

$$\frac{\epsilon \kappa}{8} \sum_a \sum_b (m_a p_b - m_b p_a) (z_a - z_b).$$

The appearance of such a term may seem unnatural, because in $(3+1)$ dimensions terms
in odd powers of $c^{-1}$, for example $c^{-5}$ terms, are related with gravitational radiation; yet
there exists no graviton degree of freedom in $(1+1)$-dimensions. However, by a suitable
redefinition of canonical variables

$$z_a \rightarrow \tilde{z}_a = z_a$$

$$p_a \rightarrow \tilde{p}_a = p_a - \frac{\epsilon \kappa}{4} \sum_b m_a m_b (z_a - z_b)$$

this term can be eliminated. Under this redefinition the "Poisson brackets" among the
canonical variables are kept unchanged. The Hamiltonian expressed in terms of the redefined
canonical variables is then

$$H = \sum_a m_a + \sum_a \frac{\tilde{p}_a^2}{2m_a} + \frac{\kappa}{8} \sum_a \sum_b m_a m_b \tilde{r}_{ab} - \sum_a \frac{\tilde{p}_a^4}{8m_a^3} + \frac{\kappa}{8} \sum_a \sum_b m_a \frac{\tilde{p}_a^2}{m_b} \tilde{r}_{ab}$$

$$- \frac{\kappa}{8} \sum_a \sum_b \tilde{p}_a \tilde{p}_b \tilde{r}_{ab} + \frac{1}{4} \left(\frac{\kappa}{4}\right)^2 \sum_a \sum_b \sum_c m_a m_b m_c \{ \tilde{r}_{ab} \tilde{r}_{ac} - (\tilde{z}_a - \tilde{z}_b)(\tilde{z}_a - \tilde{z}_c)\}. \tag{79}$$

It is straightforward to show that the equations of motion derived from the Hamiltonian (79)
are identical with those derived from the original variables.

6 Discussion

We have performed the canonical reduction of a set of point particles coupled to a gravita-
tional field in $(1+1)$ dimensions in the context of the $R = T$ theory, and have shown how to
carry out series expansions of such equations in powers of $\kappa$ (the post-linear approximation)
and powers of $c^{-1}$ (the post-Newtonian approximation). We have obtained explicit expres-
sions for the Hamiltonian to order $\kappa^2$ and $c^{-4}$ respectively, along with an expansion of the
geodesic equation to order $c^{-4}$.

We have considered in this reduction the dynamical role of the auxiliary scalar field $\Psi$.
Despite the decoupling of $\Psi$ from the classical gravity/matter field equations, the Hamil-
tonian of the system is defined in terms of $\Psi$. One of the essential points in this analysis
is the choice of the coordinate conditions, which we chose in the forms of (23) and (24)
through the transformation of the generator. We note that our coordinate conditions are
quite analogous to the ADM coordinate conditions in $(3+1)$-dimensional gravity. These
coordinate conditions (frequently used for particle dynamics [13, 14]) are

\[
x^i = g_i - \frac{1}{4\Delta} g^T_{i,i}
\]

\[
t = -\frac{1}{2\Delta} \pi^{ii}
\]

where

\[
g_i = \frac{1}{\Delta} \left( g_{ij,j} - \frac{1}{2\Delta} g_{jk,jki} \right)
\]

\[
g^T = g_{ii} - \frac{1}{\Delta} g_{ij,ij}
\]

and \( \pi^{ij} \) is a conjugate field to \( g_{ij} \), \( \Delta \) being the Laplacian operator in three dimensions.

The differential forms of the conditions (80) and (81) are

\[
\Delta g_{ij,j} - \frac{1}{4} g_{jk,ijk} - \frac{1}{4} \Delta g_{jj,i} = 0
\]

\[
\pi^{ii} = 0.
\]

The first condition (84) relates space-derivatives of spatial components \( g_{ij} \) of the metric tensor. In \((1 + 1)\) dimensions there is only one spatial component \( \gamma = g_{11} \); from (23) the condition analogous to (84) is \( \partial_t \gamma = 0 \). Therefore we chose the condition (25). Since \( \pi^{ij} \) in the second condition (85) is the conjugate momenta to \( g_{ij} \), the corresponding condition in \((1+1)\)-dimension might at first appear to be \( \pi = 0 \). However, the form of the generator (21) indicates that when the variation \( \delta h \) is related to one of the coordinate conditions, the variation of the conjugate \( \pi \) can no longer be exploited for the other condition. The candidates would be \( \delta \Pi \) or \( \delta \Psi \). Judging from the fact that \( \pi^{ij} \) is essentially a time-derivative of \( g_{ij} \) and \( \Pi \) is also a time-derivative of \( \gamma \), we have no choice to take other than \( \delta \Pi \).

We turn now to a discussion of the consistency of the full set of equations (12) - (19). As discussed in previous sections, to determine the Hamiltonian for a system of particles, we have only to solve the constraint equations (14) and (15) for \( \Psi \) and \( \pi \). All informations on the dynamics of particles are included in the constraint equations. Equations (13) and (16) are used to determine \( N_0 \) and \( N_1 \), and equations (18) and (19) lead to the equations of motion. The remaining two equations (12) and (17) are not necessary in describing the dynamics of the particles. However they are necessary in providing a full description of the gravitational field.

To check the consistency of the whole formalism, consider substituting the post-linear solutions of \( \Psi, \pi, N_0 \) and \( N_1 \) into (12) and (17). After some calculation, it is straightforward to see that (12) is consistently satisfied but (17) is not. The reason is as follows. In the general solution to the constraint equation (14) (or (30)) \( \Psi \) may contain in general an \( x \)-independent function \( f(t) \). Since all other equations except (17) contain only spatial derivatives of \( \Psi \), \( f(t) \) does not contribute to either the Hamiltonian or to the equations of motion. The function \( f(t) \) is necessary only for the consistency of (17). The boundary condition and the consideration on the surface term in the section 3 can be applied with \( \Psi \) replaced by \((\Psi - f(t))\), the latter function having no explicit dependence on \( t \). After lengthy
and complicated calculation of the equation (17), we can determine the explicit form of \( f(t) \) to the order \( \kappa^2 \).

To illustrate this situation, let us consider a single static source at the origin. In this case the fundamental equations are

\[
-\frac{1}{\kappa} \triangle \Psi = \kappa \pi^2 - \frac{1}{4\kappa} (\Psi')^2 + M \delta(x) \tag{86}
\]

\[
\pi' = 0 \tag{87}
\]

\[
\kappa \pi N_0 + N'_1 = 0 \tag{88}
\]

\[
\partial_1 \left( \frac{1}{2} N_0 \Psi' + N'_0 \right) = 0 \tag{89}
\]

\[
\dot{\pi} + N_0 \left[ \frac{3\kappa}{2} \pi^2 + \frac{1}{8\kappa} (\Psi')^2 \right] + N_1 \pi' + \frac{1}{2\kappa} N'_0 \Psi' + N'_1 \pi = 0 \tag{90}
\]

\[
\dot{\Psi} + 2\kappa N_0 \pi - N'_1 \Psi = 0 . \tag{91}
\]

Solutions to the constraint equations (86) and (87), which satisfy the same boundary condition in the section 3, are

\[
\Psi = -\kappa M^2 |x| + f(t) \tag{92}
\]

\[
\pi = \chi' = -\frac{\epsilon}{4} M \quad (\chi = -\frac{\epsilon M}{4} x) . \tag{93}
\]

Here we included \( f(t) \) in \( \Psi \).

The solutions of \( N_0 \) and \( N_1 \) are

\[
N_0 = e^{\frac{\epsilon M}{4} |x|} \tag{94}
\]

\[
N_1 = \epsilon \left( \frac{x}{|x|} \right) \left( e^{\frac{\epsilon M}{4} |x|} - 1 \right) . \tag{95}
\]

Equation (90) is satisfied, whereas (91) leads to

\[
\dot{f}(t) = \frac{\epsilon \kappa M}{2} . \tag{96}
\]

Then the solution of \( \Psi \) becomes

\[
\Psi = -\kappa M^2 |x| + \epsilon \kappa M^2 t . \tag{97}
\]

This shows that even for a static source the auxiliary field is not static.

The Hamiltonian is

\[
H = -\frac{1}{\kappa} \int dx \triangle \Psi = M . \tag{98}
\]
Since
\[ g_{00} = -N_0^2 + N_1^2 = 1 - 2e^{\kappa M(x)} < -1 \]
this solution has no event horizon.

The canonical formalism we have derived can be utilized in a number of ways. The most obvious of these is to obtain an explicit solution for the \( N \)-body problem in \((1 + 1)\) dimensions (the preceding discussion illustrates the solution when \( N = 1 \)). In the large-\( N \) limit with fixed proper distance between the most widely separated particles one might expect to recover the fluid collapse problem studied in ref. More generally, one could study the \( N \)-body problem where gravity is coupled to additional matter fields. These all remain interesting subjects for future investigation.

**Appendix**

**Relation between the equations of motion in the coordinate conditions \( \gamma = 1, \Pi = 0 \) and those in the conditions \( g_{00} = -g_{11}^{-1}, g_{01} = 0 \)**

One of the most popular coordinate conditions in \((1 + 1)\) dimensions is the one for which the metric tensor has the form
\[
 g_{\mu\nu} = \begin{pmatrix} -\alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}. 
\] (99)

The difference between this condition and the condition we adopted in this paper is that the former is independent of the auxiliary scalar field \( \Psi \) but the latter is not. This \( \Psi \)-dependence of the coordinate condition played an important role in the canonical reduction and the general expression of the Hamiltonian.

In this appendix we shall compare the equations of motion in these two coordinate conditions. Expanding \( \alpha \) and the energy-momentum tensor (5) in a power series of \( c^{-1} \) we obtain
\[
 \alpha = 1 + 2\phi - g_{00}^{(4)} + \cdots \] (100)

\[
 T_{00} = \sum_a \frac{m_a}{\sqrt{1 - \alpha^{-2}(\dot{z}_a)^2}} \alpha^{3/2} \delta(x - z_a)
 = \sum_a m_a \left\{ 1 + 3\phi + \left( \frac{1}{2}(\dot{z}_a)^2 + \cdots \right) \right\} \delta(x - z_a) 
\] (101)

\[
 T_{01} = -\sum_a \frac{m_a \alpha^{-1/2} \dot{z}_a}{\sqrt{1 - \alpha^{-2}(\dot{z}_a)^2}} \delta(x - z_a)
 = \sum_a m_a \{ -\dot{z}_a + \cdots \} \delta(x - z_a) 
\] (102)
\[ T_{11} = \sum_a m_a \alpha^{-5/2}(z_a)^2 \delta(x - z_a) \]
\[ = \sum_a m_a (z_a)^2 \{1 + \cdots\} \delta(x - z_a) \]  
(103)

where the number in the upper parenthesis denotes the order of \( c^{-1} \) and \( \phi \equiv -\frac{1}{2}g_{00}^{(2)} \).

The post-Newtonian equation of motion of the reference reads

\[ m_1 \ddot{z}_1 = -m_1 \phi' - m_1 \psi' - 4m_1 \phi \phi' + 3m_1 \dot{\phi} \dot{z}_1 + 3m_1 \phi' (\dot{z}_1)^2 . \]  
(104)

Here \( \phi \) and \( \psi \) satisfy the equations

\[ \phi'' = \frac{\kappa}{2} T^{(0)}_{00} \]
\[ = \frac{\kappa}{2} \sum_a m_a \delta(x - z_a) \]  
(105)
\[ \psi'' = -\dot{\phi} - 2(\phi')^2 + \frac{\kappa}{2} \left(T^{00(2)} - T^{11(2)}\right) \]
\[ = -\dot{\phi} - 2(\phi')^2 - \frac{\kappa}{2} \sum_a m_a \left\{ \frac{\kappa}{4} \sum_b m_b r_{ab} + \frac{1}{2} (\dot{z}_a)^2 \right\} \delta(x - z_a) . \]  
(106)

The solutions are

\[ \phi = \frac{\kappa}{4} \sum_a m_a r_a \]  
(107)
\[ \psi = \frac{\kappa}{8} \sum_a m_a z_a (x - z_a) r_a - \frac{\kappa}{4} \sum_a m_a (\dot{z}_a)^2 r_a \]
\[ - \left(\frac{\kappa}{4}\right)^2 \sum_a \sum_b m_a m_b r_a r_b + 2 \left(\frac{\kappa}{4}\right)^2 \sum_a \sum_b m_a m_b r_{ab} r_a \]
\[ - \frac{\kappa^2}{16} \sum_a \sum_b m_a m_b r_{ab} r_a - \frac{\kappa}{8} \sum_a m_a (\dot{z}_a)^2 r_a . \]  
(108)

The equation of motion (104) for a system of two particles becomes

\[ m_1 \ddot{z}_1 = -\frac{\kappa}{4} m_1 m_2 \frac{\partial r_{12}}{\partial z_1} - \frac{\kappa^2}{8} m_1 m_2 (m_1 + m_2) (z_1 - z_2) \]
\[ + \frac{\kappa}{8} m_1 m_2 \left\{ (\dot{z}_1)^2 - 6\dot{z}_1 \dot{z}_2 + 3(\dot{z}_2)^2 \right\} \frac{\partial r_{12}}{\partial z_1} . \]  
(109)

To investigate the relation between this equation of motion and the equation obtained in our canonical formalism, we consider coordinate transformations \( x^\mu \to \bar{x}^\mu \), which connect two forms of the line element :

\[ ds^2 = -\alpha dt^2 + \frac{1}{\alpha} dx^2 \]  
(110)
\[ = -(N_0^2 - N_1^2) d\bar{x}^2 + 2 N_1 dt d\bar{x} + d\bar{x}^2 , \]  
(111)
The explicit form of the transformations is given by

\[
t = t - \frac{\epsilon \kappa}{8} \sum_a m_a (x - z_a)^2 + \cdots
\]  

(112)

\[
x = x + \frac{\kappa}{8} \sum_a m_a (x - z_a) \left| x - z_a \right| + \cdots.
\]  

(113)

For a system of two particles the above transformation (113) lead s to

\[
z_1 = \bar{z}_1 + \frac{\kappa}{8} m_2 (\bar{z}_1 - \bar{z}_2) \left| \bar{z}_1 - \bar{z}_2 \right| + \cdots
\]

\[
z_2 = \bar{z}_2 - \frac{\kappa}{8} m_1 (\bar{z}_1 - \bar{z}_2) \left| \bar{z}_1 - \bar{z}_2 \right| + \cdots
\]

\[
z_1 - z_2 = (z_1 - z_2) \left\{ 1 + \frac{\kappa}{8} (m_1 + m_2) \left| z_1 - z_2 \right| + \cdots \right\}
\]

\[
r_{12} = \ell_{12} \left\{ 1 + \frac{\kappa}{8} (m_1 + m_2) \ell_{12} + \cdots \right\}.
\]

Since the second term of the right hand side of the transformation (112) contributes to \( t \) as a correction of the order \( c^{-3} \), in the post-Newtonian approximation we can set \( t = \ell \). Then we have

\[
\dot{z}_1 = \dot{\bar{z}}_1 + \frac{\kappa}{8} m_2 \left\{ (\dot{\bar{z}}_1 - \dot{\bar{z}}_2) \left| \bar{z}_1 - \bar{z}_2 \right| + (\dot{\bar{z}}_1 - \dot{\bar{z}}_2)(\dot{\bar{z}}_1 - \dot{\bar{z}}_2) \frac{\partial \ell_{12}}{\partial \bar{z}_1} \right\} + \cdots
\]

\[
\ddot{z}_1 = \ddot{\bar{z}}_1 + \frac{\kappa}{4} m_2 \left\{ (\ddot{\bar{z}}_1 - \ddot{\bar{z}}_2) \left| \bar{z}_1 - \bar{z}_2 \right| + (\ddot{\bar{z}}_1 - \ddot{\bar{z}}_2) \frac{\partial \ell_{12}}{\partial \bar{z}_1} \right\} + \cdots
\]

\[
\frac{\partial \ell_{12}}{\partial \bar{z}_1} = \frac{\partial}{\partial \bar{z}_1} \left\{ \ell_{12} \left\{ 1 + \frac{\kappa}{8} (m_1 + m_2) \ell_{12} + \cdots \right\} \right\}
\]

\[
= \left\{ \frac{1}{4} \left( -\frac{\kappa}{4} (m_1 + m_2) \ell_{12} + \cdots \right) \right\} \left\{ \frac{\partial \ell_{12}}{\partial \bar{z}_1} + \frac{\kappa}{4} (m_1 + m_2) \frac{\partial \ell_{12}}{\partial \bar{z}_1} \ell_{12} + \cdots \right\}
\]

\[
= \frac{\partial \ell_{12}}{\partial \bar{z}_1} + \mathcal{O}(\kappa^2).
\]

By the use of these relations the equation of motion (109) is transformed to

\[
m_1 \ddot{z}_1 = -\frac{\kappa}{4} m_1 m_2 \frac{\partial \ell_{12}}{\partial \bar{z}_1} + \frac{\kappa}{8} \left\{ 4(\dot{\bar{z}}_1)^2 - 2\dot{\bar{z}}_1 \dot{\bar{z}}_2 + (\dot{\bar{z}}_2)^2 \right\} \frac{\partial \ell_{12}}{\partial \bar{z}_1} - \frac{\kappa^2}{16} m_1 m_2 (m_1 + m_2) (\dot{\bar{z}}_1 - \dot{\bar{z}}_2) \]  

(114)

This equation is identical with the equation of motion (70) we obtained in the canonical formalism.

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