Critical scaling in standard biased random walks

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The spatial coverage produced by a single discrete-time random walk, with asymmetric jump probability \( p \neq 1/2 \) and non-uniform steps, moving on an infinite one-dimensional lattice is investigated. Analytical calculations are complemented with Monte Carlo simulations. We show that, for appropriate step sizes, the model displays a critical phenomenon, at \( p = p_c \). Its scaling properties as well as the main features of the fragmented coverage occurring in the vicinity of the critical point are shown. In particular, in the limit \( p \to p_c \), the distribution of fragment lengths is scale-free, with nontrivial exponents. Moreover, the spatial distribution of cracks (unvisited sites) defines a fractal set over the spanned interval. Thus, from the perspective of the covered territory, a very rich critical phenomenology is revealed in a simple one-dimensional standard model.

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with $\hat{z} = zp^{1/n}(1 - p)^{1/n}$. From the definition of $S(z)$, the quantity $S_n$ can be obtained as $1/n!$ times the $n$th derivative of $S(z)$, evaluated at $z = 0$.

If $l^+$ and $l^-$ have common factors, a mapping exists into the corresponding case of reduced (mutually prime) lengths. Therefore, we will restrict our study to asymmetric coprime couples of step lengths. Within the latter class of RWs, one has the subclass where one of the lengths is unitary. Let us consider as representative instances of this subclass, the case $(l^+, l^-) = (2, 1)$ that admits an exact solution. In this case, the sum in Eq. (3) becomes

$$P(0, z) = \sum_{k \geq 0} \binom{3k}{k} p^k (1 - p)^{2k} z^{3k},$$

that can be reduced to

$$P(0, z) = N(y) \frac{1}{\sqrt{1 - y^2}},$$

(4) for $|y| \leq 1$, where $y^2 = (27/4)p(1 - p)^2 z^3$. Tauberian methods can be applied to evaluate $S_n$ [4]. Alternatively, the nth derivative of $S(z)$ can be calculated through Cauchy integral formula over a suitable contour circling the origin. Since $S(z)$ given by Eq. (4) has one single pole in the complex plane, at $z = 1$, then, in the limit of large $n$, one gets (after conveniently deforming the integration path) $S_n = -\ln n + O(1)$, where $c_0$ is a constant of order 1. It is noteworthy that this is the same asymptotic law found for the standard RW, with unbiased symmetric jumps to nearest neighbors (hence $l^+ = l^-$), but in 3D random lattices [4]. The fraction of different sites visited (measured over the average length of the RW) is $f_{v,n} = S_n/L_n$, where $L_n$ is the average total displacement. In the large $n$ limit, the length of the RW, for $p \neq 1/3$, is $L_n \sim |\langle s > n| = |3p - 1|^n$. Thus, asymptotically, $f_{v,n}$ becomes $f_v = [3p - 1]^{-1}$, hence, the fraction of unvisited sites is

$$f_u = 1 - f_v = 1 - [3p - 1]^{-1},$$

(5) where $P(0, 1)$ is given by Eq. (4).

Fig. 1(a) exhibits $f_u$ as a function of $p$, for $(l^+, l^-) = (2, 1)$. A transition occurs at $p_c = 1/3$, where $f_u$ vanishes as $f_u \approx A + O(\Delta^2)$, with $A \equiv p - p_c$, that can be derived exactly from Eq. (4). For $p \leq p_c$ all sites are eventually visited at least once, as expected, because, as soon as $\langle s > n = 3\Delta < 0$, the walker is biased towards the direction of unitary steps, which in turn implies full coverage of the RW length. Meanwhile, for $p > p_c$, sequences of adjacent visited sites (fragments) are interrupted by unvisited ones. Therefore, the RW undergoes a transition from a fully covered state to a fragmented one. For other instances of $(l^+, 1)$, the transition occurs at the critical probability $p_c = 1/(l^+ + 1)$, where $\langle s > n = (pl^+ + p - 1)n$ changes sign (driftless diffusion). The case $(l^+, l^-) = (3, 1)$, obtained by means of Monte Carlo (MC) simulations up to $n \approx 10^9$ time steps, is also displayed in Fig. 1(a), exhibiting similar features. In both cases, $f_u(\Delta)$ vanishes with unitary exponent (see inset of Fig. 1(a)).

For non-unitary coprime step lengths (see Fig. 1(b)) a more general scenario arises. Full coverage occurs only at the critical point $p_c = l^-/(l^+ + l^-)$, where $S_n = (pl^+ + p - 1)n$ is strictly null. Fragmented states are found both below and above $p_c$, with maximal unvisited fractions, $f_u^* = 1 - 1/l^-$ and $f_u^* = 1 - 1/l^+$, respectively. Thus, the cases $(l^+, 1)$, with $l^+ > 1$, constitute special instances where one of the states is fully covered, in accordance with the fact that the corresponding maximal unvisited fraction $f_u^*$ vanishes. Although we are not dealing with symmetric steps, notice that in the symmetric case $(1, 1)$, $f_u^* = f_u^* = 0$ and the full curve $f_u(p)$ collapses to zero, in agreement with the facts that there is no transition in such case and that full coverage occurs for any $p$.

As a paradigmatic example, we will analyze the analytically soluble case $(l^+, l^-) = (2, 1)$, in the vicinity of the critical point, i.e., in the limit $\Delta \to 0^+$. In order to quantitatively characterize fragment sizes, the usual computed quantities are [11]:

$$\bar{n}_\ell = \sum_{\ell \geq 1} n_\ell, \quad \langle \ell \rangle = \sum_{\ell \geq 1} n_\ell \ell^2 \sum_{\ell \geq 1} n_\ell \ell,$$

(6) where $n_\ell$ is the mean number of fragments of size $\ell$, normalized per site. Since two contiguous fragments are separated, in the (2, 1) case, by one single unvisited site, then $\bar{n}_\ell \approx f_u$, that vanishes as $\sim \Delta$ (see Fig. 1(a)). Also, straightforwardly, $\sum_{\ell \geq 1} n_\ell \ell = 1 - f_u$, that approaches 1 in the critical limit. Noticing that $n_\ell \ell$ is the probability that a given site belongs to a fragment of size $\ell$, then, $\bar{\ell} \approx \sum_{\ell \geq 1} n_\ell \ell^2$ defines the mean size of the fragments. In order to compute $\langle \ell \rangle$, the distribution of sizes of covered clusters (or fragments), $n_\ell$, was numerically built from MC simulations run up to $n \approx 10^9/\Delta$ steps and averaged over at least $10^2$ different realizations. The distributions for different values of $\Delta$ are displayed in Fig. 2. For very large $\ell$, the decay is exponential: $\sim \exp(-\ell/\lambda)$. Parameter $\lambda$, together with $\langle \ell \rangle$, are plotted as a function of $\Delta$ in the upper inset of Fig. 2.
(being $\lambda \approx (\ell)/2 \sim \Delta^{-\gamma}$, with $\gamma \approx 1.15$). Meanwhile, $n_1 \sim \Delta$, representing a finite fraction of $f_u$. In the lower inset of Fig. 2 the same distributions of the main frame are scaled. Let us employ the standard ansatz for cluster size distributions [11], defined through:

$$n_\ell(\Delta) \propto \Delta^{\omega} \phi(\Delta^{1/\sigma} \ell)/(\Delta^{1/\sigma} \ell)^{\gamma},$$

(7)

where $\phi(x)$ goes to a constant value for small $x$ and decays exponentially in the opposite limit of large $x$. The power-law decay, with exponent $\tau \approx 1.15$, that emerges in the limit of vanishing $\Delta$ is characteristic of a critical behavior and signals the coexistence of fragments of all sizes in that limit.

By means of integral approximations to the sums in Eqs. (6) and employing Eq. (7), one gets the following relations amongst critical exponents. Firstly, $1 \approx \sum_{\ell \geq 1} n_\ell \ell \approx \int_1^\infty n_\ell \ell d\ell \sim \Delta^{-2/\sigma}$, implying $\omega = 2/\sigma$. Secondly, $\Delta^{-\gamma} \sim \langle \ell \rangle \approx \int_1^\infty n_\ell \ell^2 d\ell \sim \Delta^{-3/\sigma}$, hence $\omega = 3/\sigma - \gamma$, that, together with the preceding relation, implies $\gamma = 1/\sigma$ and $\omega = 2/\sigma$. The latter equality is in good accord with the behavior of the envelope of the distributions that has slope -2 (Fig. 2). Excellent data collapse is obtained for $\omega = 2/\sigma$, with $\gamma \approx 1.15$. Additionally, since $\bar{n}_\ell \sim \Delta$, then, from $\bar{n}_\ell \approx \int_1^\infty n_\ell \ell^2 d\ell \sim \Delta^{-\gamma}$, it must be $\tau = 2 - \sigma = 2 - 1/\gamma$. From the scaled histograms, we obtained $\tau \approx 1.15 \pm 0.05$, consistent with the theoretical prediction within error bars.

At this point, it is worth comparing our results with those for another 1D critical phenomenon, namely 1D percolation (1DP) with bonds connecting nearest neighbors [12], to which many important 1D models are related (e.g., Ref. [13]). On one hand, for 1DP, $\bar{n}_\ell \sim \Delta^{2-\alpha_p}$, with $\alpha_p = 1$, as in the present problem. On the other hand, $\langle \ell \rangle \sim \Delta^{-\gamma_p}$, with $\gamma_p = 1$ and $\omega_p = 2\gamma_p = 2$, values that are close but different from those found for the present problem. Moreover, the distribution of fragment sizes is a power-law, in contrast with the pure Poissonian one for 1DP. Then, we may conclude that the present model does not belong to the 1DP universality class. Indeed, by identifying visited sites with occupied ones, the occupation probability in our problem is $f_u$, that tends to one in the critical limit. However, differently from the standard percolation problem, in the present case, unvisited sites are not independently located, e.g., if $(l^+, l^-)= (2,1)$, a sequence of two or more adjacent unvisited sites has associated a strictly null probability of occurrence. Therefore, occupation correlations arise which are absent in the standard percolation problem.

Concerning unvisited sites, their spatial distribution was investigated through a box-counting procedure [14]. From the history of a single RW, a segment of length $L = 2^{20} \approx 10^6$ was divided into boxes of length $2^k$, with $k \geq 0$. For each $\varepsilon = 2^k/L$, the number of boxes containing unvisited sites, $N(\varepsilon)$, was computed. Outcomes, accumulated over $10^4$ realizations, are displayed in Fig. 3. The next behavior $N(\varepsilon) \sim \varepsilon^{-d_f}$, for small $\varepsilon$, means that the spatial distribution of unvisited sites constitutes a fractal set, with dimension $d_f$. Moreover, the fractal exponent is in good accord with the exact scaling relation $d_f = \tau - 1$.

![FIG. 2: Distribution of the sizes of covered fragments ($n_1$ was omitted), for $(l^+, l^-)= (2,1)$ and different values of $\Delta = 10^{-1}$, $3 \times 10^{-2}$, $\ldots$, $10^{-4}$, from a to g, respectively. Upper inset: mean size of fragments $\langle \ell \rangle$ (squares), inverse exponential rate $\lambda$ (circles), and $[10n_1]^{-1}$ (triangles) as a function of $\Delta$. Lower inset: Scaling plot of all the distributions represented in the main frame, with $\omega = 2/\sigma$ and $1/\sigma = 1.15 \pm 0.05$. Dashed lines are drawn for comparison and their slopes indicated on the figure.](image)

![FIG. 3: Scaling plot of the number of boxes $N$ containing unvisited sites as a function of $\varepsilon$ (box size in units of $L$, where $L = 2^{20}$) for $(l^+, l^-)= (2,1)$ and different values of $\Delta$ indicated on the figure. The scaling exponent is $\delta = 1/(1-d_f)$, where $d_f \approx 0.15$. Inset: original plots of the data scaled in the main frame. All solid lines are drawn for comparison and their slopes indicated on the figure.](image)
particular case in which the steps are $(l^+, l^-) = (2,1)$, which undergoes a transition from fully to partially covered states as the jump probability $p$ overcomes a critical value. The power-law distribution of sizes of covered segments, occurring in the limit $p \to p_c^+$, indicates the coexistence of fragments of all lengths, with no characteristic length scale. Moreover, the spatial distribution of scission points (unvisited sites) determines a fractal set, in contrast with other models where the deposition of cracks has common statistics (e.g., 1D percolation [12], scission model [15]). It is pertinent remarking that akin features have been observed in one-dimensional reaction-diffusion [16, 17], q-state Potts spin flipping [17] and fragmentation dynamics [18], although criticality is attained as time evolves and critical exponents are different. A possible connection remains to be investigated.

Other asymmetric instances with steps $(l^+, l^-) = (5,3)$, whose critical curves are illustrated in Fig. 4(a), display a qualitatively similar picture to the case (2,1). Meanwhile, if both steps take non-unitary coprime values (Fig. 4(b)), the same critical phenomenology is observed in both limits $\Delta \to 0^\pm$. As a further example, scaling plots are also displayed, in Fig. 4 for the case $(l^+, l^-) = (5,3)$ in the limit $\Delta \to 0^+$. In general, critical exponents related to the fractal dimension are not universal but depend on the step lengths, since distinct site occupation correlations take place.

One one hand, the asymmetric RW, seen from the present perspective, may bear interest per se because of the nontrivial criticality contained in a simple model. On the other hand, it may constitute a useful statistical paradigm for the formation of domains or fragments by a non-equilibrium process driven by biased signal propagation. Additionally, the current coverage problem may be potentially useful in technical applications, e.g., in search strategies such as for cache hit/miss ratio optimization [3].

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FIG. 4: Critical behavior for $(l^+, l^-) = (5,3)$. Results correspond to the limit $\Delta \to 0^+$, but the same exponents are found in the limit $\Delta \to 0^-$. (a) Scaling plot of the distribution of the sizes of covered fragments, values of $\Delta$ as in Fig. 2, with $\omega = 2/\sigma$ and $1/\sigma = 1.5 \pm 0.2$, $\tau \simeq 1.4$. (b) Scaling plot of the number of boxes $N$ containing unvisited sites as in Fig. 3. In this case $d_f \simeq 0.4$. The dotted line in (a) and solid lines in (b) are drawn for comparison and their slopes indicated on the figure.

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