1D SCHRODINGER OPERATORS WITH SHORT RANGE INTERACTIONS: TWO-SCALE REGULARIZATION OF DISTRIBUTIONAL POTENTIALS

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Abstract. For real $L_\infty(\mathbb{R})$-functions $\Phi$ and $\Psi$ of compact support, we prove the norm resolvent convergence, as $\varepsilon$ and $\nu$ tend to 0, of a family $S_{\varepsilon \nu}$ of one-dimensional Schrödinger operators on the line of the form

$$S_{\varepsilon \nu} = -\frac{d^2}{dx^2} + \frac{\alpha}{\varepsilon^2} \Phi \left( \frac{x}{\varepsilon} \right) + \frac{\beta}{\nu} \Psi \left( \frac{x}{\nu} \right),$$

provided the ratio $\nu/\varepsilon$ has a finite or infinity limit. The limit operator $S_0$ depends on the shape of $\Phi$ and $\Psi$ as well as on the limit of ratio $\nu/\varepsilon$. If the potential $\alpha \Phi$ possesses a zero-energy resonance, then $S_0$ describes a non trivial point interaction at the origin. Otherwise $S_0$ is the direct sum of the Dirichlet half-line Schrödinger operators.

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1. Introduction

The present paper is concerned with convergence of the family of one-dimensional Schrödinger operators of the form

$$S_{\varepsilon \nu} = -\frac{d^2}{dx^2} + \frac{\alpha}{\varepsilon^2} \Phi \left( \frac{x}{\varepsilon} \right) + \frac{\beta}{\nu} \Psi \left( \frac{x}{\nu} \right), \quad \text{dom} S_{\varepsilon \nu} = W_2^2(\mathbb{R}) \quad (1.1)$$
as the positive parameters $\nu$ and $\varepsilon$ tend to zero simultaneously. Here $\Phi$ and $\Psi$ are real potentials of compact supports, and $\alpha$ and $\beta$ are real coupling constants.

Our motivation of the study on this convergence comes from an application to the scattering of quantum particles by $\delta$- and $\delta'$-shaped potentials, where $\delta$ is the Dirac delta-function. The potentials in (1.1) are a two-scale regularization of the distribution $\alpha \delta'(x) + \beta \delta(x)$ provided that the conditions

$$\int_{\mathbb{R}} \Phi(t) \, dt = 0, \quad \int_{\mathbb{R}} t \Phi(t) \, dt = -1 \quad \text{and} \quad \int_{\mathbb{R}} \Psi(t) \, dt = 1 \quad (1.2)$$

hold. Our purpose is to construct the so-called solvable models describing with admissible fidelity the real quantum interactions governed by the Hamiltonian

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The quantum mechanical models that are based on the concept of point interactions reveal an undoubted effectiveness whenever solvability together with non triviality is required. It is an extensive subject with a large literature (see e.g., [4, 7] and the references given therein).

We emphasize that all results presented here concern arbitrary potentials $\Phi$ and $\Psi$ of compact support, and the $(\alpha\delta' + \beta\delta)$-like potentials satisfying conditions (1.2) are only a special case in our considerations, the title of paper notwithstanding. It is interesting to observe that if the first condition in (1.2) is not fulfilled, then these potentials do not converge even in the distributional sense. However, surprisingly enough, the resolvents of $S_{\varepsilon\nu}$ still converge in norm.

We say that the Schrödinger operator $-\frac{d^2}{dt^2} + \alpha\Phi$ in $L^2(\mathbb{R})$ possesses a half-bound state (or zero-energy resonance) if there exists a non trivial solution $u_\alpha$ to the equation $-u'' + \alpha\Phi u = 0$ that is bounded on the whole line. The potential $\alpha\Phi$ is then called resonant. In this case, we also say that $\alpha$ is a resonant coupling constant for the potential $\Phi$. Such a solution $u_\alpha$ is unique up to a scalar factor and has nonzero limits $u_\alpha(\pm\infty) = \lim_{x \to \pm\infty} u_\alpha(x)$ (see [9, 25]). Our main result reads as follows.

Let $\Phi$ and $\Psi$ be integrable and bounded real functions of compact support. Then the operator family $S_{\varepsilon\nu}$ given by (1.1) converges as $\nu,\varepsilon \to 0$ in the norm resolvent sense, i.e., the resolvents $(S_{\varepsilon\nu} - z)^{-1}$ converge in the uniform operator topology, provided the ratio $\nu/\varepsilon$ has a finite or infinity limit.

Non-resonant case. If the potential $\alpha\Phi$ does not possess a zero-energy resonance, then the operators $S_{\varepsilon\nu}$ converge to the direct sum $S_- \oplus S_+$ of the Dirichlet half-line Schrödinger operators $S_{\pm}$.

Resonant case. If the potential $\alpha\Phi$ is resonant with the half-bound state $u_\alpha$, then the limit operator $S$ is a perturbation of the free Schrödinger operator defined by $S\phi = -\phi''$ on functions $\phi$ in $W^2_2(\mathbb{R} \setminus \{0\})$, subject to the boundary conditions at the origin

$$
\begin{pmatrix}
\phi(+0) \\
\phi'(+0)
\end{pmatrix} =
\begin{pmatrix}
\theta_\alpha(\Phi) & 0 \\
\beta \omega_\alpha(\Phi, \Psi) & \theta_\alpha(\Phi)^{-1}
\end{pmatrix}
\begin{pmatrix}
\phi(-0) \\
\phi'(-0)
\end{pmatrix}.
$$

The diagonal matrix element $\theta_\alpha(\Phi)$ is specified by the half-bound state of potential $\alpha\Phi$ and is defined by

$$
\theta_\alpha(\Phi) = \frac{u^+_\alpha}{u^-_\alpha},
$$

where $u^\pm_\alpha = u_\alpha(\pm\infty)$. The value $\omega_\alpha(\Phi, \Psi)$ depends on both potentials $\Phi$ and $\Psi$ as well as on the limit of ratio $\nu/\varepsilon$ as $\nu,\varepsilon \to 0$, and describes different kinds of the resonance interaction between the potentials $\Phi$ and $\Psi$. Three cases are to be distinguished:

(i) if $\nu/\varepsilon \to \infty$ as $\nu,\varepsilon \to 0$, then

$$
\omega_\alpha(\Phi, \Psi) = \frac{u^+_\alpha}{u^-_\alpha} \int_{R_+} \Psi(t) \, dt + \frac{u^-_\alpha}{u^+_\alpha} \int_{R_-} \Psi(t) \, dt;
$$

(ii) if $\nu/\varepsilon \to 0$ as $\nu,\varepsilon \to 0$, then

$$
\omega_\alpha(\Phi, \Psi) = \int_{R_+} \Psi(t) \, dt + \int_{R_-} \Psi(t) \, dt;
$$

(iii) if $\nu/\varepsilon \to 0$ as $\nu,\varepsilon \to 0$, then

$$
\omega_\alpha(\Phi, \Psi) = \frac{u^+_\alpha}{u^-_\alpha} \int_{R_+} \Psi(t) \, dt + \frac{u^-_\alpha}{u^+_\alpha} \int_{R_-} \Psi(t) \, dt.
$$
(ii) if the ratio \( \nu/\varepsilon \) converges to a finite positive number \( \lambda \) as \( \nu, \varepsilon \to 0 \), then
\[
\omega_\alpha(\Phi, \Psi) = \frac{1}{u_\alpha} \int_{\mathbb{R}} \Psi(t) u_\alpha^2(\lambda t) \, dt; \tag{1.6}
\]
(iii) if \( \nu/\varepsilon \to 0 \) as \( \nu \) and \( \varepsilon \) go to zero, then
\[
\omega_\alpha(\Phi, \Psi) = \frac{u_\alpha^2(0)}{u_\alpha} \int_{\mathbb{R}} \Psi(t) \, dt. \tag{1.7}
\]

The point interaction generated by conditions (1.3) may be regarded as the first approximation to the real interaction governed by the Hamiltonian \( S_{\nu,\varepsilon} \) with coupling constants \( \alpha \) lying in vicinity of the resonant values. The explicit relations between the matrix entries \( \theta_\alpha(\Phi), \omega_\alpha(\Phi, \Psi) \) and the potentials \( \Phi, \Psi \) make it possible to carry out a quantitative analysis of this quantum system, e.g., to compute approximate values of the scattering data. Of course the same conclusion holds in the non-resonant case, but then the quantum dynamics is asymptotically trivial.

It is natural to ask what happens if one of the coupling constants is zero and the family \( S_{\nu,\varepsilon} \) becomes one-parametric. For if \( \beta = 0 \) and so the \( \delta \)-like component of the short range potential is absent, then the results are in agreement with the results obtained recently in [21, 22]: the operators
\[
S_{\varepsilon} = -\frac{d^2}{dx^2} + \frac{\alpha}{\varepsilon^2} \Phi \left( \frac{x}{\varepsilon} \right), \quad \text{dom } S_{\varepsilon} = W_2^2(\mathbb{R}) \tag{1.8}
\]
converge as \( \varepsilon \to 0 \) in the norm resolvent sense towards the operator \( S \) defined by conditions (1.3) with \( \beta = 0 \), if \( \alpha \Phi \) possesses a zero-energy resonance, and to the direct sum \( S_- \oplus S_+ \) otherwise. As for the case \( \alpha = 0 \), the limit Hamiltonian, as \( \nu \to 0 \), must be associated with the \( \beta \delta(x) \)-interaction. However, we see at once that zero is a resonant coupling constant for any potential \( \Phi \) and the half-bound state \( u_0 \) is a constant function. Therefore \( \theta_0(\Phi) = 1 \), and \( \omega_0(\Phi, \Psi) = \int_{\mathbb{R}} \Psi(t) \, dt \), no matter which a formula of (1.5)–(1.7) we use. Hence, the operator \( S \) is defined by the boundary conditions
\[
\phi(+0) = \phi(-0), \quad \phi'(+0) = \phi'(-0) + \beta \phi(0) \int_{\mathbb{R}} \Psi(t) \, dt,
\]
as one should expect.

It has been believed for a long time [33] that the Hamiltonians \( S_{\varepsilon} \) given by (1.8) with \( \alpha \neq 0 \) converge as \( \varepsilon \to 0 \) in the norm resolvent sense to the direct sum \( S_- \oplus S_+ \) of the Dirichlet half-line Schrödinger operators for any potential \( \Phi \) having zero mean. If so, the \( \delta' \)-shaped potential defined through the regularization \( \varepsilon^{-2} \Phi(\varepsilon^{-1} \cdot) \) must be opaque, i.e., acts as a perfect wall, in the limit \( \varepsilon \to 0 \). However, the numerical analysis of exactly solvable models of \( S_{\varepsilon} \) with piece-wise constant \( \Phi \) of compact support performed recently by Zolotaryuk a.o. [10, 35, 37] gives rise to doubts that the limit \( S_- \oplus S_+ \) is correct. The authors demonstrated that for a resonant \( \Phi \), the limiting value of the transmission coefficient of \( S_{\varepsilon} \) is different from zero. The operators \( S_{\varepsilon} \) also arose in [2, 13, 14] in connection with the approximation of smooth planar quantum waveguides by quantum graphs. Under the assumption that the mean value of \( \Phi \) is different from zero, the authors singled out the set of resonant potentials \( \Phi \) producing a “non-trivial” (i.e., different from
A similar resonance phenomenon was also obtained in [20], where the asymptotic behaviour of eigenvalues for the Schrödinger operators perturbed by $\delta'$-like short range potentials was treated (see also [30]). The situation with these controversial results was clarified in [21,22]. Note that Seba was the first [32] who discovered the “resonant convergence” for a similar family of the Dirichlet Schrödinger operators on the half-line.

There is a connection between the results presented here and the low energy behaviour of Schrödinger operators, in particular the low-energy scattering theory. Generally, the zero-energy resonances are the reason for different “exceptional” cases of the asymptotic behaviour. Albeverio and Høegh-Krohn [6] considered the family of Hamiltonians $H_\varepsilon = -\Delta + \lambda(\varepsilon)\varepsilon^{-2}V(\varepsilon^{-1}x)$ in dimension three, where $\lambda(\varepsilon)$ was a smooth function with $\lambda(0) = 1$ and $\lambda'(0) \neq 0$. It was shown that $H_\varepsilon$ converge in the strong resolvent sense, as $\varepsilon \to 0$, to the operator that is either the free Hamiltonian $-\Delta$ or its perturbation by a delta-function depending on whether or not there is a zero-energy resonance for $-\Delta + V$. In [3], the low-energy scattering was discussed; the authors used the results of [6] and the connection between the low-energy behaviour of scattering matrix for the Hamiltonian $-\Delta + V$ in $L_2(\mathbb{R}^3)$ and for the corresponding scaled Hamiltonians $-\Delta + \varepsilon^{-2}V(\varepsilon^{-1}x)$ as $\varepsilon \to 0$ to study in detail possible resonant and non-resonant cases. Similar problem for Hamiltonians including the Coulomb-type interaction was treated in [5]. The low-energy scattering for the one-dimensional Schrödinger operator $S_1$ and its connection to the behaviour of the corresponding scaled operators $S_\varepsilon$ as $\varepsilon \to 0$ was thoroughly investigated by Bollé, Gesztesy, Klaus, and Wilk [9,10], taking into account the possibility of zero-energy resonances; in dimension two, the low-energy asymptotics was discussed in [6]. Continuity of the scattering matrix at zero energy for one-dimensional Schrödinger operators in the resonant case was established by Klaus in [26]. Relevant references in this context are also [1,15]. Simon and Klaus [25,27,28] observed the connection between the zero-energy resonances and the coupling constant thresholds, i.e., the absorption of eigenvalues. These results depend on properties of the corresponding Birman-Schwinger kernel.

We note that singular point interactions for the Schrödinger operators in dimensions one and higher have widely been discussed in both the physical and mathematical literature; see [11,12,19,24,29,31,34].

2. Preliminaries

There is no loss of generality in supposing that the supports of both $\Phi$ and $\Psi$ are contained in the interval $I = [-1,1]$. Denote by $\mathcal{P}$ the class of real integrable and bounded functions of compact support contained in $I$.

**Definition 2.1.** The resonant set $\Lambda_\Phi$ of a potential $\Phi \in \mathcal{P}$ is the set of all real value $\alpha$ for which the operator $-\frac{d^2}{dt^2} + \alpha \Phi$ in $L_2(\mathbb{R})$ possesses a half-bound state, i.e., for which there exists a non trivial $L_\infty(\mathbb{R})$-solution $u_\alpha$ to the equation

$$-u'' + \alpha \Phi u = 0.$$  \hspace{1cm} (2.1)

The half-bound state $u_\alpha$ is then constant outside the support of $\Phi$. Moreover, the restriction of $u_\alpha$ to $I$ is a nontrivial solution of the Neumann boundary value
problem
\[- u'' + \alpha \Phi u = 0, \quad t \in \mathcal{I}, \quad u'(-1) = 0, \quad u'(1) = 0. \quad (2.2)\]

Consequently, for any \( \Phi \in \mathcal{P} \) the resonant set \( \Lambda_\Phi \) is not empty and coincides with the set of all eigenvalues of the latter problem with respect to the spectral parameter \( \alpha \). In the case of a nonnegative (resp. nonpositive) potential \( \Phi \) the spectrum of \((2.2)\) is discrete and simple with one accumulation point at \(-\infty \) (resp. \( +\infty \)). Otherwise, \((2.2)\) is a problem with indefinite weight function \( \Phi \) and has a discrete and semisimple spectrum with two accumulation points at \( \pm \infty \) (see also \([20][23]\)).

We introduce some characteristics of the potentials \( \Phi \) and \( \Psi \). Let \( \theta \) be the map of \( \Lambda_\Phi \) to \( \mathbb{R} \) defined by
\[
\theta(\alpha) = \frac{u^+_\alpha}{u^-_\alpha} = \frac{u^{(1)}(+1)}{u^{(1)}(-1)}. \quad (2.3)
\]
Since the half-bound state is unique up to a scalar factor, this map is well defined. Throughout the paper, we choose the half-bound state so that \( u_\alpha(x) = 1 \) for \( x \leq -1 \). Then \( \theta(\alpha) = u^+_\alpha \), and \( u_\alpha(x) = \theta(\alpha) \) for \( x \geq 1 \). Here and subsequently, \( \theta_\alpha \) stands for the value \( \theta(\alpha) \). For our purposes it is convenient to introduce the maps:
\[
\begin{align*}
\zeta : \Lambda_\Phi &\to \mathbb{R}, \\
\zeta(\alpha) &= \theta_\alpha \int_{\mathbb{R}^+} \Psi \, dt + \theta_\alpha^{-1} \int_{\mathbb{R}^-} \Psi \, dt; \\
\kappa : \Lambda_\Phi \times \mathbb{R}^+ &\to \mathbb{R}, \\
\kappa(\alpha, \lambda) &= \theta_\alpha^{-1} \int_{\mathbb{R}} \Psi(t) u^2_\alpha(\lambda t) \, dt; \\
\mu : \Lambda_\Phi &\to \mathbb{R}, \\
\mu(\alpha) &= \theta_\alpha^{-1} u^2_\alpha(0) \int_{\mathbb{R}} \Psi \, dt. 
\end{align*}
\]

(Compare with \((1.3)\)–\((1.7)\)).

Denote by \( S(\gamma_1, \gamma_2) \phi = -\phi'' \) on functions \( \phi \) in \( W^2_2(\mathbb{R} \setminus \{0\}) \) obeying the interface conditions \( \phi(+0) = \gamma_1 \phi(-0) \) and \( \phi'(+0) = \gamma_1^{-1} \phi'(-0) + \gamma_2 \phi(-0) \) at the origin. For every real \( \gamma_1 \) and \( \gamma_2 \), this operator is self-adjoint provided \( \gamma_1 \neq 0 \). Let \( S_{\pm} \) denote the unperturbed half-line Schrödinger operator \( S_{\pm} = -d^2/dx^2 \) on \( \mathbb{R}_{\pm} \), subject to the Dirichlet boundary condition at the origin, i.e., \( \text{dom} \, S_{\pm} = \{ \phi \in W^2_2(\mathbb{R}_{\pm}) : \phi(0) = 0 \} \).

In the sequel, letters \( C_j \) and \( c_j \) denote various positive constants independent of \( \varepsilon \) and \( \nu \), whose values might be different in different proofs. Throughout the paper, \( W^j_2(\Omega) \) stands for the Sobolev space and \( \| f \| \) stands for the \( L^2(\mathbb{R}) \)-norm of a function \( f \).

We start with an easy auxiliary result, which will be often used below.

**Proposition 2.2.** Assume \( f \in L^2(\mathbb{R}) \), \( z \in \mathbb{C} \setminus \mathbb{R} \), and set \( y = (S(\gamma_1, \gamma_2) - z)^{-1} f \). Then the following holds for some constants \( C_k \) independent of \( f \) and \( t \):
\[
\begin{align*}
|y(\pm 0)| &\leq C_1 \| f \|, & |y'(\pm 0)| &\leq C_2 \| f \| \\
|y(\pm t) - y(\pm 0)| &\leq C_3 t \| f \|, & |y'(\pm t) - y'(\pm 0)| &\leq C_4 t^{1/2} \| f \| 
\end{align*}
\]

for \( t > 0 \). These inequalities hold for \( y = (S_{-} \oplus S_{+} - z)^{-1} f \) too.
We first observe that \((S(\gamma_1, \gamma_2) - z)^{-1}\) is a bounded operator from \(L_2(\mathbb{R})\) to the domain of \(S(\gamma_1, \gamma_2)\) equipped with the graph norm. The latter space is continuously embedded subspace into \(W^2_2(\mathbb{R} \setminus \{0\})\). Then \(\|y\|_{W^2_2(\mathbb{R} \setminus \{0\})} \leq c_1 \|f\|\).

Owing to the Sobolev embedding theorem, we have \(\|y\|_{C^1(\mathbb{R} \setminus \{0\})} \leq c_2 \|f\|\), which establishes (2.9). Combining the previous estimates for \(y\) with the inequalities

\[
|y^{(j)}(\pm t) - y^{(j)}(\pm 0)| \leq \left| \int_0^{\pm t} |y^{(j+1)}(s)| \, ds \right|, \quad j = 0, 1,
\]

we obtain (2.7). For the case of \(S_- \oplus S_+\), the proof is similar. \(\square\)

Apparently, some versions of the next proposition are known, but we are at a loss to give a precise reference.

**Proposition 2.3.** Let \(J\) be a finite interval in \(\mathbb{R}\), and \(t_0 \in J\). Then the solution to the Cauchy problem \(v'' + qv = f\) in \(J\), \(v(t_0) = a, \quad v'(t_0) = b\) obeys the estimate

\[
\|v\|_{C^1(J)} \leq C(|a| + |b| + \|f\|_{L_\infty(J)})
\]

for some \(C > 0\) being independent of the initial data and right-hand side, whenever \(q, f \in L_\infty(J)\).

**Proof.** Let \(v_1\) and \(v_2\) be the linear independent solutions to \(v'' + qv = 0\) such that \(v_1(t_0) = 1, \quad v'_1(t_0) = 0, \quad v_2(t_0) = 0\) and \(v'_2(t_0) = 1\). Under the assumptions made on \(q\) and \(f\), these solutions belong to \(W^2_2(J)\); and consequently \(v_j \in C^1(J)\) by the Sobolev embedding theorem. Application of the variation of parameters method yields

\[
v(t) = av_1(t) + bv_2(t) + \int_{t_0}^{t} k(t, s) f(s) \, ds,
\]

where \(k(t, s) = v_1(s)v_2(t) - v_1(t)v_2(s)\). From this and the representation of the first derivative

\[
v'(t) = av'_1(t) + bv'_2(t) + \int_{t_0}^{t} \frac{\partial k(t, s)}{\partial t} f(s) \, ds
\]

we have

\[
|v(t)| + |v'(t)| \leq |a| \|v_1\|_{C^1(J)} + |b| \|v_2\|_{C^1(J)} + 2|J| \|k\|_{C^1(J \times J)} \|f\|_{L_\infty(J)}
\]

for \(t \in J\), which completes the proof. \(\square\)

We end this section with a proposition which will be useful in Sections 3 and 5.

**Proposition 2.4.** Let \(\mathbb{R}_a\) be the real line with two removed points \(-a\) and \(a\), i.e., \(\mathbb{R}_a = \mathbb{R} \setminus \{-a, a\}\). Assume \(w \in W^2_2(\mathbb{R}_a)\). There exists a function \(r \in C^\infty(\mathbb{R}_a)\) such that \(w + r \) belongs to \(W^2_2(\mathbb{R})\), \(r\) is zero in \((-a, a)\), and

\[
\max_{x \in \mathbb{R}_a} |r^{(k)}(x)| \leq C \left( |w| + |w'| + |w''| + |w'''| \right)
\]

for \(k = 0, 1, 2\), where the constant \(C\) does not depend on \(w\) and \(a\).
Proof. Let us introduce functions \( \varphi \) and \( \psi \) that are smooth outside the origin, have compact supports contained in \([0, \infty)\), and \( \varphi(+0) = 1, \varphi'(+0) = 0, \psi(+0) = 0, \psi'(+0) = 1 \). Set
\[
 r(x) = [w]_{-a} \varphi(-x-a) - [w']_{-a} \psi(-x-a) - [w]_{a} \varphi(x-a) - [w']_{a} \psi(x-a). \tag{2.10}
\]
All jumps are well defined, since \( w \in C^1(\mathbb{R}_a) \). Next, the function \( r \) is zero in \((-a, a)\) by construction. An easy computation shows that \( w + r \) is continuous on \( \mathbb{R} \) along with its derivative and consequently belongs to \( W^2_2(\mathbb{R}) \). Finally, \( (2.10) \) makes it obvious that inequality \( (2.9) \) holds.

3. Convergence of the operators \( S_{\epsilon \nu} \). The case \( \nu \varepsilon^{-1} \rightarrow \infty \).

In this section, we analyze the case of a “\( \delta \)-like” sequence that is slowly contracting relative to “\( \delta' \)-like” one. The relations between two parameters \( \varepsilon \) and \( \nu \) that lead to this case are, roughly speaking, as follows: \( \varepsilon \ll 1, \nu \ll 1, \) but \( \nu/\varepsilon \gg 1 \). It will be convenient to introduce the large parameter \( \eta = \nu/\varepsilon \). The first trivial observation is the following: if \( \nu \rightarrow 0 \) and \( \eta \rightarrow \infty \), then \( \varepsilon \rightarrow 0 \). The resonant and non-resonant cases will be considered separately.

3.1. Resonant case. We start with the analysis of the more difficult resonant case. Suppose that \( \alpha \in \Lambda_\Phi \) and set \( \zeta_\alpha = \zeta(\alpha) \), where \( \zeta \) is given by \( (2.3) \).

**Theorem 3.1.** Assume \( \Phi, \Psi \in \mathcal{P} \) and \( \alpha \) belongs to the resonant set \( \Lambda_\Phi \). Then the operator family \( S_{\epsilon \nu} \) defined by \( (1.1) \) converges to the operator \( S(\theta_\alpha, \beta \zeta_\alpha) \) as \( \nu \rightarrow 0 \) and \( \eta \rightarrow \infty \) in the norm resolvent sense.

We have divided the proof into a sequence of lemmas.

Let us fix a function \( f \in L_2(\mathbb{R}) \) and a number \( z \in \mathbb{C} \) with \( \text{Im} z \neq 0 \). For abbreviation, in this section we let \( S \) stand for \( S(\theta_\alpha, \beta \zeta_\alpha) \). Our aim is to approximate both vectors \( (S_{\epsilon \nu} - z)^{-1} f \) and \( (S - z)^{-1} f \) in \( L_2(\mathbb{R}) \) by the same element \( y_{\epsilon \nu} \) from the domain of \( S_{\epsilon \nu} \). Of course, such an approximation must be uniform in \( f \) in bounded subsets of \( L_2(\mathbb{R}) \). We construct the vector \( y_{\epsilon \nu} \) in the explicit form, which allows us to estimate \( L_2(\mathbb{R}) \)-norms of the differences \( (S_{\epsilon \nu} - z)^{-1} f - y_{\epsilon \nu} \) and \( (S - z)^{-1} f - y_{\epsilon \nu} \). This is the aim of the next lemmas.

First we construct a candidate for the approximation as follows. Let us set \( y = (S - z)^{-1} f \). Write \( w_{\epsilon \nu}(x) = y(x) \) for \( |x| > \nu \) and
\[
 w_{\epsilon \nu}(x) = y(-0)(u_\alpha(x/\varepsilon) + \beta \nu h_{\epsilon \nu}(x/\nu)) + \varepsilon g_{\epsilon \nu}(x/\varepsilon) \quad \text{for} \quad |x| \leq \nu.
\]
Here \( h_{\epsilon \nu} \) and \( g_{\epsilon \nu} \) are solutions to the Cauchy problems
\[
 h'' = \Psi(t)u_\alpha(\eta t), \quad t \in \mathbb{R}, \quad h(0) = 0, \quad h'(0) = 0; \tag{3.1}
\]
\[
 \begin{cases}
 g'' - \alpha \Phi(t)g = \alpha \beta \eta y(-0)\Phi(t)h_{\epsilon \nu}(\eta t), & t \in \mathbb{R}, \\
 g(-1) = 0, \quad g'(-1) = y'(-0) + \beta y(-0) \int_{\mathbb{R}} \Psi ds & \tag{3.2}
\end{cases}
\]
respectively, and \( u_\alpha \) is the half-bound state corresponding to the resonant coupling constant \( \alpha \). Hence we can surely expect that \( y \) is a very satisfactory approximation to \( (S_{\epsilon \nu} - z)^{-1} f \) for \( |x| > \nu \), but the approximation on the support of \( \Psi \) is more subtle.
Lemma 3.2. The function $h_{\varepsilon \nu}$ possesses the following properties:
(i) there exist constants $C_1$ and $C_2$ such that
\[ \|h_{\varepsilon \nu}\|_{C^1(\mathbb{R})} \leq C_1, \quad |h_{\varepsilon \nu}(t)| \leq C_2 t^2 \] (3.3)
for all $\varepsilon, \nu \in (0, 1)$ and $t \in \mathbb{R}$;
(ii) the asymptotic relations
\[ h'_{\varepsilon \nu}(-1) = -\int_{\mathbb{R}} \Psi ds + O(\eta^{-1}), \quad h'_{\varepsilon \nu}(1) = \theta_\alpha \int_{\mathbb{R}^+} \Psi ds + O(\eta^{-1}) \] (3.4)
hold as $\nu \to 0$ and $\eta \to \infty$.

Proof. The solution $h_{\varepsilon \nu}$ and its derivative can be represented as
\[ h_{\varepsilon \nu}(t) = \int_0^t (t - s)\Psi(s)u_\alpha(\eta s) ds, \quad h'_{\varepsilon \nu}(t) = \int_0^t \Psi(s)u_\alpha(\eta s) ds. \] (3.5)
The first estimate in (3.3) follows immediately from these relations, because $\Psi$ and $u_\alpha$ belong to $L_\infty(\mathbb{R})$. By the same reason,
\[ |h_{\varepsilon \nu}(t)| \leq c_1 \left| \int_0^t |t - s| ds \right| \leq C_2 t^2. \]
Now according to our choice of the half-bound state, we see that
\[ u_\alpha(\eta t) \to u_\alpha^*(t) = \begin{cases} 1 & \text{if } t < 0, \\ \theta_\alpha & \text{if } t > 0 \end{cases} \]
in $L_{1,loc}(\mathbb{R})$, as $\eta \to \infty$. In addition, the difference $u_\alpha(\eta t) - u_\alpha^*(t)$ is zero outside the interval $[-\eta^{-1}, \eta^{-1}]$ and bounded on this interval. In view of the second relation in (3.3), this establishes the asymptotic formulas (3.4). \hfill \square

Lemma 3.3. There exist constants $C_1$ and $C_2$, independent of $f$, such that
\[ |g_{\varepsilon \nu}(t)| \leq C_1(1 + |t|)\|f\|, \quad t \in \mathbb{R}, \] (3.6)
\[ |g'_{\varepsilon \nu}(t)| \leq C_2 \|f\|, \quad t \in \mathbb{R} \] (3.7)
for all $\varepsilon$ and $\nu$ whenever the ratio of $\varepsilon$ to $\nu$ remains bounded as $\varepsilon, \nu \to 0$. In addition, the value $g'_{\varepsilon \nu}(1)$ admits the asymptotics
\[ g'_{\varepsilon \nu}(1) = \theta_\alpha^{-1} \left(y'(-0) + \beta y(-0) \int_{\mathbb{R}^+} \Psi ds \right) + O(\eta^{-1})\|f\| \] (3.8)
as $\nu \to 0, \eta \to \infty$.

Proof. From Proposition 2.3 it follows that
\[ \|g_{\varepsilon \nu}\|_{C^1(\mathbb{R})} \leq c_1(|y(-0)| + |y'(-0)|) + c_2 \eta |y(-0)| \|h_{\varepsilon \nu}(\eta^{-1} \cdot)\|_{C(\mathbb{R})}. \]
Next, in light of (3.3), we have
\[ \|h_{\varepsilon \nu}(\eta^{-1} \cdot)\|_{C(\mathbb{R})} = \max_{|t| \leq |\eta|^{-1}} |h_{\varepsilon \nu}(t)| \leq c_3 \eta^{-2}. \] (3.9)
Combining this estimate with (3.7), we deduce
\[ \|g_{\varepsilon \nu}\|_{C^1(\mathbb{R})} \leq c_4(|y(-0)| + |y'(-0)|) \leq c_5 \|f\|. \] (3.10)
Since the support of $\Phi$ lies in $I$, the function $g_{\nu}$ is linear outside $I$, namely $g_{\nu}(t) = g_{\nu}(-1)(t + 1)$ for $t \leq -1$ and $g_{\nu}(t) = g_{\nu}(1) + g'_{\nu}(1)(t - 1)$ for $t \geq 1$. Therefore estimates (3.6), (3.7) follow easily from these relations and (3.11).

Next, multiplying equation (3.2) by $u_\alpha$ and integrating on $I$ by parts yield

$$\theta_\alpha g_{\nu}'(1) - g'_{\nu}(-1) = \alpha \beta \nu y(-0) \int_{-1}^{1} \Phi(s) W_{\nu}(s) y^{-1}(s) u_\alpha(s) \, ds.$$ 

The right-hand side can be estimated by $c_6 \nu^{-1} ||f||$ provided $|\eta| \geq 1$, in view of (3.9) and Proposition 2.2. Recalling the initial conditions (3.2), we obtain (3.8).

Corollary 3.4. The function $w_{\nu}$ is bounded in $[-\nu, \nu]$ uniformly in $\varepsilon$ and $\nu$ provided the ratio $\varepsilon/\nu$ remains bounded as $\varepsilon, \nu \to 0$, and there exists a constant $C$ such that max$_{|x| \leq \nu} |w_{\nu}(x)| \leq C ||f||$.

Proof. The corollary is a direct consequence of Lemmas 3.1 and 3.3. We only note that

$$\varepsilon \max_{|x| \leq \nu} |g_{\nu}(x/\varepsilon)| \leq c_1 \varepsilon (1 + \nu/\varepsilon) ||f|| = c_1 (\varepsilon + \nu) ||f|| \leq c_2 \nu ||f||,$$

(3.11)
in view of (3.6) and the assumption that $\varepsilon \leq \nu$.

By construction, $w_{\nu}$ belongs to $W_2^2(\mathbb{R} \setminus \{-\nu, \nu\})$. In general, due to the discontinuity at the points $x = \pm \nu$, $w_{\nu}$ is not an element of dom $S_{\nu}$. However, the jumps of $w_{\nu}$ and the jumps of its first derivative at these points are small enough, as shown below. By Proposition 2.4, there exists the corrector function $r_{\nu}$ of the form (2.10) such that $w_{\nu} + r_{\nu}$ belongs to $W_2^2(\mathbb{R}) = \text{dom } S_{\nu}$. Set $y_{\nu} = w_{\nu} + r_{\nu}$.

Lemma 3.5. The corrector $r_{\nu}$ is small as $\nu \to 0$, $\eta \to \infty$, and satisfies the inequality

$$\max_{x \in \mathbb{R} \setminus \{-\nu, \nu\}} |r^{(k)}_{\nu}(x)| \leq C \rho(\nu, \eta) ||f||$$

for $k = 0, 1, 2$, where $\rho(\nu, \eta) = \nu^{1/2} + \eta^{-1}$.

Proof. Assume $\varepsilon$ and $\nu$ are small enough, and $\eta \geq 1$. From our choice of $u_\alpha$, we have that $u_\alpha(-\eta) = 1$, $u_\alpha(\eta) = \theta_\alpha$, and $u_\alpha'(-\eta) = 0$. Also $g'_{\nu}(\pm \eta) = g'_{\nu}(\pm 1)$, and the bounds

$$\varepsilon |g_{\nu}(\pm \eta)| \leq c_1 \nu ||f||$$

(3.12)
hold, owing to (3.11). These relations will be used repeatedly in the proof.

According to Proposition 2.4, it is sufficient to estimate the jumps of $w_{\nu}$ and $w'_{\nu}$. At the point $x = -\nu$ we have

$$[w_{\nu}(-\nu)] = y(-0) + \beta \nu y(-0)h_{\nu}(-1) + \varepsilon g_{\nu}(-\eta) - y(-\nu),$$

$$[w'_{\nu}(-\nu)] = \beta y(-0)h_{\nu}(-1) + g_{\nu}'(-1) - y'(-\nu).$$

The first of these jumps can be bounded as follows:

$$|[w_{\nu}(-\nu)| \leq |y(-0) - y(-\nu)| + \nu |\beta||g_{\nu}(-0)||h_{\nu}(-1)| + \varepsilon |g_{\nu}(-\eta)| \leq c_2 \nu ||f||,$$
by (3.3), (3.12) and Proposition 2.2. Next, taking into account (3.4) and the initial conditions for $g_{ev}$, we see that

$$[w'_{ev}]_{-\nu} = \beta y(-0) \left( - \int_{\mathbb{R}} \Psi \, ds + O(\eta^{-1}) \right) + y'(-0) + \beta y(-0) \int_{\mathbb{R}} \Psi \, ds - y'(-\nu)$$

$$= y'(-0) - y'(-\nu) + O(\eta^{-1}) y(-0), \quad \eta \to \infty.$$ 

We can now repeatedly apply Proposition 2.2 to deduce $|[w'_{ev}]_{-\nu} | \leq c_3 g(\nu, \eta) \| f \|$. Let us turn to the jumps at the point $x = \nu$. We get

$$[w_{ev}]_{\nu} = y(\nu) - \theta_\alpha y(-0) - \beta y(-0) h_{ev}(-1) - \varepsilon g_{ev}(\eta),$$

$$[w'_{ev}]_{\nu} = y'(\nu) - \beta y(-0) h'_{ev}(1) - g'_{ev}(1).$$

Recall that $y(+0) = \theta_\alpha y(-0)$, since $y \in \text{dom} \, S$. This gives

$$|[w_{ev}]_{\nu} | \leq |y(\nu) - y(+0)| + c_4 \varepsilon |y(-0)| + \varepsilon |g_{ev}(\eta)| \leq c_5 \nu \| f \|$$

by (2.7) and (3.12). Also, combining the relation $y'(+0) = \theta_\alpha^{-1} y'(-0) + \beta \zeta_\alpha y(-0)$ and asymptotic formulas (3.4), (3.8), we deduce that

$$[w'_{ev}]_{\nu} = y'(\nu) - \beta y(-0) \left( \theta_\alpha \int_{\mathbb{R}} \Psi \, ds + O(\eta^{-1}) \right)$$

$$- \left( \theta_\alpha^{-1} y'(-0) + \theta_\alpha^{-1} \beta y(-0) \int_{\mathbb{R}} \Psi \, ds + O(\eta^{-1}) \| f \| \right)$$

$$= y'(\nu) - \beta y(-0) h'_{ev}(1) - g'_{ev}(1) - y'(\nu) - y'(+0) + O(\eta^{-1}) \| f \|,$$

hence that $|[w'_{ev}]_{\nu} | \leq c_6 g(\nu, \eta) \| f \|$. This inequality completes the proof. □

Note that the resolvents $(S_{ev} - z)^{-1}$ are uniformly bounded with respect to $\varepsilon$ and $\nu$, because $S_{ev}$ are self-adjoint. To prove Theorem 3.1, it suffices to establish the following relation

$$|(S_{ev} - z)^{-1} f - (S - z)^{-1} f | = o(1) \| f \|,$$

as $\nu, \varepsilon \to 0$, for $f$ belonging to a dense subset of $L_2(\mathbb{R})$. Let us denote by $\mathcal{F}$ the set of all $L_2(\mathbb{R})$-functions vanishing in a neighbourhood of the origin. Obviously, $\mathcal{F}$ is dense in $L_2(\mathbb{R})$.

**Lemma 3.6.** Fix $z \in \mathbb{C} \setminus \mathbb{R}$ and $f \in \mathcal{F}$. Then the following estimate

$$|(S_{ev} - z)^{-1} f - y_{ev} | \leq C \varrho(\nu, \eta) \| f \|$$

holds, as $\nu \to 0$ and $\eta \to \infty$, with some constant $C$ being independent of $f$.

**Proof.** We first compute $f_{ev} = (S_{ev} - z) y_{ev}$. For the convenience of the reader we write $y_{ev}$ in the detailed form

$$y_{ev}(x) = \begin{cases} y(x) + r_{ev}(x) & \text{if } |x| > \nu, \\ y(-0) (u_{ev}(x/\varepsilon) + \nu/\beta h_{ev}(x/\nu)) + \varepsilon g_{ev}(x/\varepsilon) & \text{if } |x| \leq \nu. \end{cases} \quad (3.13)$$

Recall that $r_{ev}$ is zero in $(-\nu, \nu)$, by construction. If $|x| > \nu$, then

$$f_{ev}(x) = \left( -\frac{d^2}{dx^2} - z \right) y_{ev}(x) = f(x) - y''_{ev}(x) - z r_{ev}(x).$$
Next, for $|x| < \nu$, we have
\[
    f(x) = \left( -\frac{d^2}{dx^2} + \alpha \varepsilon^{-2} \Phi \left( \frac{x}{\varepsilon} \right) + \beta \nu^{-1} \Psi \left( \frac{x}{\nu} \right) - z \right) y(x)
\]
\[
    = \varepsilon^{-2} y(-0) \left\{ -u'' + \alpha \Phi \left( \frac{x}{\varepsilon} \right) u_0 \right\}
    + \nu^{-1} \beta y(-0) \left\{ -h'' + \Psi \left( \frac{x}{\nu} \right) u_0 \right\}
    + \varepsilon^{-1} \left\{ -g'' + \alpha \Phi \left( \frac{x}{\varepsilon} \right) g_x + \eta \alpha \beta y(-0) \Phi \left( \frac{x}{\varepsilon} \right) h_x \left( \frac{x}{\varepsilon} \right) \right\}
    + \beta^2 \nu(-0) \Psi \left( \frac{x}{\nu} \right) h_x \left( \frac{x}{\nu} \right) + \eta^{-1} \beta \Psi \left( \frac{x}{\nu} \right) g_x \left( \frac{x}{\nu} \right) - \nu y(x)
\]
\[
    = \beta \Psi \left( \frac{x}{\nu} \right) \left\{ \beta y(-0) h_x \left( \frac{x}{\nu} \right) + \eta^{-1} g_x \left( \frac{x}{\nu} \right) \right\} - \nu y(x),
\]

since $u_0, h_x$, and $g_x$ are solutions to equations (2.1), (3.1), and (3.2), respectively.

Given $f \in \mathcal{F}$, we choose the number $\nu$ so small that $f(x) = 0$ for $|x| \leq \nu$. Thus $f(x) = (S_x - z)y(x)$, and consequently $y(x) = (S_x - z)^{-1}(f(x) - q(x))$, where $q(x) = r'' + z r(x) + z y(x) - \beta \Psi \left( \nu^{-1} \cdot \right) \left( \beta y(-0) h_x \left( \nu^{-1} \cdot \right) + \eta^{-1} g_x \left( \nu^{-1} \cdot \right) \right)$, and $\chi_{\nu}$ is the characteristic function of $[\nu, \nu]$. Owing to Lemmas 3.2 and 3.3, we have
\[
    |y(-0)| \left| \Psi \left( \frac{x}{\varepsilon} \right) h_x \left( \frac{x}{\varepsilon} \right) \right| \leq c_1 \| h \|_{C(\mathbb{R})} \| f \| \| \chi_{\nu}(x) \| \leq c_2 \| f \| \| \chi_{\nu}(x) \|,
    \eta^{-1} \left| \Psi \left( \frac{x}{\nu} \right) g_x \left( \frac{x}{\nu} \right) \right| \leq c_3 \eta \| \chi_{\nu}(x) \| \max_{x \in [-\nu, \nu]} \left| \chi_{\nu}(x) \right| \leq c_5 \| f \| \| \chi_{\nu}(x) \|,
\]
and hence $\| q \| \leq c_{\nu}(\nu, \eta) \| f \|$, in view of Corollary 3.3 and Lemma 3.5. Note also that $\| \chi_{\nu} \| = (2\nu)^{1/2}$. Therefore
\[
    \| (S_x - z)^{-1} f - y(x) \| = \| (S_x - z)^{-1} q(x) \| \leq \| (S_x - z)^{-1} \| \| q \| \| f \| \leq C_{\nu}(\nu, \eta) \| f \|.
\]

**Proof of Theorem 3.1.** We start with the observation that
\[
    y_{\nu} - y = r_{\nu} + (w_{\nu} - y) \chi_{\nu}.
\]
Thus $\| y_{\nu} - y \| \leq c_{\nu}(\nu, \eta) \| f \|$, by Corollary 3.4 and Lemma 3.5. Form this and Lemma 3.6, we deduce for $z \in \mathbb{C} \setminus \mathbb{R}$ and $f \in \mathcal{F}$ that
\[
    \| (S_x - z)^{-1} f - (S - z)^{-1} f \| \leq \| (S_x - z)^{-1} f - y_{\nu} \| + \| y_{\nu} - (S - z)^{-1} f \| \leq C_{\nu}(\nu, \eta) \| f \|,
\]
as $\nu \to 0, \eta \to \infty$. The proof is completed by noting that $\mathcal{F}$ is dense in $L_2(\mathbb{R})$.

**3.2. Non-resonant case.** Here we prove the following theorem:

**Theorem 3.7.** Suppose the potential $\alpha \Phi$ is not resonant; then the operators $S_{\nu}$ converge to the direct sum $S_- \oplus S_+$ of the Dirichlet half-line Schrödinger operators as $\nu \to 0$ and $\eta \to \infty$ in the norm resolvent sense.

As a matter of fact, this result is implicitly contained in the previous proof. In the non-resonant case, equation (2.1) admits only one $L_\infty(\mathbb{R})$-solution which is trivial. Additionally, for each $f \in L_2(\mathbb{R})$, the function $y = (S_- \oplus S_+ - z)^{-1} f$ satisfies the condition $y(0) = 0$. Roughly speaking, the proof of Theorem 3.7 can
be derived from the previous one with $u_\alpha$ and $h_{z\nu}$ replacing the zero functions and $y(\pm 0)$ replacing 0 in the corresponding formulas.

**Proof.** In this case the approximation $y_{\nu\varepsilon}$ is rather simpler than (3.18). Whereas $y(0) = 0$, we set

$$y_{\nu\varepsilon}(x) = \begin{cases} y(x) + r_{\nu\varepsilon}(x) & \text{if } |x| > \nu, \\ \varepsilon g(x/\varepsilon) & \text{if } |x| \leq \nu. \end{cases}$$

Here $y = (S_- \oplus S_+ - z)^{-1} f$, $r_{\nu\varepsilon}$ is a $W^2_2$-corrector of the form (2.10) as above, and $g$ is a solution to the boundary value problem

$$g'' - \alpha \Phi(t)g = 0, \quad t \in \mathbb{R}, \quad g'(-1) = y'(-0), \quad g'(1) = y'(0).$$

Such a solution exists, since $\alpha$ is not an eigenvalue of (2.2). In addition, $g$ is linear outside $I$, so it satisfies the inequalities of the form (3.6), (3.7) and (3.14).

Reasoning as in the proof of Lemma 3.6, we deduce that

$$|y(\pm 1) - \varepsilon g(\pm 1)| \leq |y(\pm 1) - \varepsilon g(\pm \eta)| \leq c_1 \nu \|f\|,$$

$$|y'(\pm 1) - \varepsilon g'(\pm 1)| \leq |y'(\pm 1) - \varepsilon g'(\pm \eta)| \leq c_2 \nu^{1/2} \|f\|,$$

provided $\eta \geq 1$, and hence that

$$\max_{x \in \mathbb{R} \setminus \{-\nu, 0, \nu\}} |y^{(k)}_{\nu\varepsilon}(x)| \leq C \nu^{1/2} \|f\|, \quad k = 0, 1, 2, \quad (3.15)$$

by Proposition 2.4. Furthermore $(S_{\nu\varepsilon} - z)y_{\nu\varepsilon} = f\chi_{\nu\varepsilon} - q_{\nu\varepsilon}$ with

$$q_{\nu\varepsilon}(x) = r_{\nu\varepsilon}''(x) + zr_{\nu\varepsilon}(x) + (\varepsilon z \chi_{\nu\varepsilon}(x) - \beta \eta^{-1} \Psi(\frac{x}{\varepsilon})) g(\frac{x}{\varepsilon}),$$

by calculations as in the proof of Lemma 3.6. Also $\|q_{\nu\varepsilon}\| \leq c_3 \nu^{1/2} \|f\|$, in view of (3.14) and (3.15). Whenever $f$ belongs to $\mathcal{F}$, we have $(S_{\nu\varepsilon} - z)y_{\nu\varepsilon} = f - q_{\nu\varepsilon}$ as soon as $\nu$ is small enough. This implies $\|(S_{\nu\varepsilon} - z)^{-1} f - y_{\nu\varepsilon}\| \leq c_4 \nu^{1/2} \|f\|$. The norm resolvent convergence of $S_{\nu\varepsilon}$ towards $S_- \oplus S_+$ now follows precisely as in the proof of Theorem 3.1. □

4. CONVERGENCE OF THE OPERATORS $S_{\nu\varepsilon}$. THE CASE $\nu \sim \varepsilon$.

In this short section we apply the results of our recent work [23] to the case $\nu \varepsilon^{-1} \rightarrow \lambda$ and $\lambda > 0$. The parameters $\varepsilon$ and $\nu$ are in this case connected by the asymptotic relation $\nu \varepsilon = \lambda \varepsilon + o(\varepsilon)$ as $\varepsilon \rightarrow 0$. Let us consider the operator family

$$H_{\lambda} = \begin{cases} S(\theta_\alpha, \beta \xi(\alpha, \lambda)) & \text{if } \alpha \in \Lambda_\Phi, \\ S_- \oplus S_+ & \text{otherwise} \end{cases} \quad (4.1)$$

for $\lambda > 0$, where $\xi$ is given by (2.4). For convenience, we shall write $S_{\nu\varepsilon}(\Phi, \Psi)$ for $S_{\nu\varepsilon}$, and $\xi(\alpha, \lambda; \Phi, \Psi)$ for $\xi(\alpha, \lambda)$ indicating the dependence of $S_{\nu\varepsilon}$ and $\xi$ on potentials $\Phi$ and $\Psi$.

For the case $\nu = \varepsilon$, it was proved in [23] that operators $S_{\nu\varepsilon}(\Phi, \Psi)$ converge to $H_1$ in the norm resolvent sense, as $\varepsilon \rightarrow 0$. Moreover, this result is stable under a small perturbation the potential $\Psi$. If a sequence of potentials $\Psi_k$ of compact support is uniformly bounded in $L^\infty(\mathbb{R})$ and $\Psi_k \rightarrow \Psi$ in $L^1(\mathbb{R})$ as $\varepsilon \rightarrow 0$, then

$$S_{\nu\varepsilon}(\Phi, \Psi_{\varepsilon}) \rightarrow H_1$$

in the sense of the norm resolvent convergence. Note that all estimates containing $\Psi$ in the proofs of Theorems 4.1 and 5.1 in [23] remain true
Theorem 5.1. Suppose \( \mu \) and the map \( S \) converges to \( 5.1 \). non-resonant cases will be treated separately. note that if \( \varepsilon \) relative to the \( \Phi \)-shaped one. Therefore that \( \nu \varepsilon \)\( \lambda \varepsilon \) and \( \varepsilon \)\( \lambda \varepsilon \). We discuss in this section the case of the fast contracting \( \Psi \)-shaped \( d \) potential

Convergence of the operators \( 5.1 \). Let us next guess \( y \)\( \varepsilon \)\( \lambda \varepsilon \)\( \gamma \varepsilon \)\( \lambda \varepsilon \). Observe also that \( \varepsilon \)\( \lambda \varepsilon \)\( \gamma \varepsilon \)\( \lambda \varepsilon \). Hence both operators \( S_\varepsilon,\lambda \varepsilon (\Phi, \Psi) \) and \( S_\varepsilon,\lambda \varepsilon (\Phi, \Psi) \) converge to the same limit \( \overline{H}_\lambda \). We have proved:

**Theorem 4.1.** If the ratio \( \nu / \varepsilon \) tends to a finite positive number \( \lambda \) as \( \nu, \varepsilon \to 0 \), then \( S_\varepsilon \nu \) converge to the operator \( \overline{H}_\lambda \) defined by (4.1) in the norm resolvent sense.

5. Convergence of the operators \( S_\varepsilon \nu \). The case \( \nu \varepsilon^{-1} \to 0 \).

We discuss in this section the case of the fast contracting \( \Psi \)-shaped potential relative to the \( \Phi \)-shaped one. Therefore that \( \nu \varepsilon^{-1} \to 0 \) as \( \nu, \varepsilon \to 0 \). First we note that if \( \varepsilon \to 0 \) and \( \eta \to 0 \), then \( \nu \to 0 \). As in Section 3 the resonant and non-resonant cases will be treated separately.

5.1. Resonant case. Let us consider the operator \( S(\theta_\alpha, \beta \mu_\alpha) \), where \( \mu_\alpha = \mu(\alpha) \) and the map \( \mu: \Lambda_\Phi \to \mathbb{R} \) is given by (2.5).

**Theorem 5.1.** Suppose \( \Phi, \Psi \in \mathcal{P} \) and \( \alpha \in \Lambda_\Phi \); then the operator family \( S_\varepsilon \nu \) converges to \( S(\theta_\alpha, \beta \mu_\alpha) \) in the norm resolvent sense, as \( \varepsilon, \eta \to 0 \).

Given \( f \in L_2(\mathbb{R}) \) and \( z \in \mathbb{C} \setminus \mathbb{R} \), we write \( y = (S - z)^{-1} f \). Note that \( y \) satisfies the conditions

\[
y(+0) = \theta_\alpha y(-0), \quad y'(+0) = \theta_\alpha^{-1} y'(-0) + \beta \mu_\alpha y(-0). \tag{5.1}
\]

Let us next guess \( y_\varepsilon \nu \) has the form

\[
y_\varepsilon(x) = \begin{cases} y(x) + r_\varepsilon(x) & \text{for } |x| > \varepsilon, \\ y(-0)u_\alpha(x/\varepsilon) + \varepsilon g_\varepsilon(x/\varepsilon) + \beta \varepsilon h_\varepsilon(x/\nu) & \text{for } |x| \leq \varepsilon, \end{cases} \tag{5.2}
\]

where \( g_\varepsilon \) and \( h_\varepsilon \) are solutions to the Cauchy problems

\[
\begin{align*}
g''(t)g &= \beta y(-0) \eta^{-1}(\Psi^{-1})u_\alpha(t), & t \in \mathbb{R}, \\
g(-1) &= 0, & g'(-1) = y'(-0), \tag{5.3}
\end{align*}
\]

\[
\begin{align*}
h'' &= \Psi(t)g_\varepsilon(\eta t), & t \in \mathbb{R}, & h(-1) = 0, & h'(-1) = 0 \tag{5.4}
\end{align*}
\]
respectively. As above, \( u_\alpha \) is the half-bound state for the potential \( \alpha \Phi \), and \( r_{\alpha \nu} \) adjusts this approximation so as to obtain an element of \( \text{dom} S_{\alpha \nu} \). According to Proposition 2.4, there exists a corrector function that vanishes in \( (\nu, \varepsilon, \alpha) \).

**Lemma 5.2.** If the ratio of \( \nu \) to \( \varepsilon \) remains bounded as \( \nu, \varepsilon \to 0 \), then there exists a constant \( C \) such that for all \( f \in L_2(\mathbb{R}) \)

\[
\| g_{\alpha \nu} \|_{C(\mathcal{I})} \leq C \| f \|.
\]

In addition, \( g'_{\alpha \nu}(1) = g'(0) + O(\eta) \| f \| \) as \( \varepsilon, \eta \to 0 \).

**Proof.** Our proof starts with the observation that the right-hand side of equation (5.3) contains a \( \delta \)-like sequence, namely

\[
\eta^{-1} \Psi(\eta^{-1} t) \to \left( \int_{\mathbb{R}} \Psi(\eta^{-1} t) \right) \delta(x) \quad \text{in } W_2^{-1}(\mathcal{I})
\]

as \( \eta \to 0 \). Let \( v_\alpha \) be the solution of (2.1) obeying the initial conditions \( v_\alpha(-1) = 0 \) and \( v'_\alpha(-1) = 1 \). Then \( g_{\alpha \nu} \) can be represented as \( g_{\alpha \nu} = y'(-0)v_\alpha + \beta y(-0)\hat{g}_{\alpha \nu} \), where \( \hat{g}_{\alpha \nu} \) solves the equation \( g'' - \alpha \Phi g = \eta^{-1} \Psi(\eta^{-1})u_\alpha \) and satisfies zero initial conditions at \( t = -1 \). Next, \( \hat{g}_{\alpha \nu} \) converges in \( W_2^1(\mathcal{I}) \) to the solution \( \hat{g} \) of the problem

\[
g'' - \alpha \Phi(t)g = u_\alpha(0) \left( \int_{\mathbb{R}} \Psi(\eta^{-1} t) \right) \delta(x), \quad t \in \mathcal{I}, \quad g(-1) = 0, \quad g'(-1) = 0,
\]

which is clear from the explicit representation of \( \hat{g}_{\alpha \nu} \) of the form (2.8). Thus the convergence in \( W_2^1(\mathcal{I}) \) implies the uniform convergence of \( \hat{g}_{\alpha \nu} \) to \( \hat{g} \) in \( \mathcal{I} \), and consequently \( \hat{g}_{\alpha \nu} \) is uniformly bounded in \( \varepsilon \) and \( \nu \) provided \( \eta < c \). From this we see that \( \| g_{\alpha \nu} \|_{C(\mathcal{I})} \leq |y'(-0)| \| v_\alpha \|_{C(\mathcal{I})} + |\beta| |y(-0)| \| \hat{g}_{\alpha \nu} \|_{C(\mathcal{I})} \leq C \| f \| \), by (2.6).

Multiplying equation (5.3) by \( u_\alpha \) and integrating on \( \mathcal{I} \) by parts yield

\[
\theta_{\alpha} g'_{\alpha \nu}(1) - y'(-0) = \beta y(-0) \eta^{-1} \int_{-1}^{1} \Psi(\eta^{-1} s)u_\alpha^2(s) \, ds.
\]

Since \( u_\alpha(t) = u_\alpha(0) + O(t) \) as \( t \to 0 \), we have

\[
g'_{\alpha \nu}(1) = \theta_{\alpha}^{-1} \left( y'(-0) + \beta y(-0)u_\alpha^2(0) \int_{\mathbb{R}} \Psi(\eta^{-1} s) \, ds \right) + O(\eta) \| f \|
\]

\[= \theta_{\alpha}^{-1} y'(-0) + \beta \mu_\alpha y(-0) + O(\eta) \| f \|, \quad \eta \to 0,
\]

by (5.6) and (2.5). Therefore the asymptotic relation for \( g'_{\alpha \nu}(1) \) follows from (5.1).

**Lemma 5.3.** There exist constants \( C_1 \) and \( C_2 \), independent of \( f \), such that

\[
|h_{\alpha \nu}(t)| \leq C_1 (1 + |t|) \| f \|, \quad t \in \mathbb{R},
\]

\[
|h'_{\alpha \nu}(t)| \leq C_2 \| f \|, \quad t \in \mathbb{R}
\]

for all \( \varepsilon \) and \( \nu \) whenever the ratio of \( \nu \) to \( \varepsilon \) is small enough.

**Proof.** As in the proof of Lemma 3.3 equation (5.4) gives

\[
h_{\alpha \nu}(t) = t \int_{-1}^{1} \Psi(s)g_{\alpha \nu}(\eta s) \, ds - \int_{-1}^{1} s \Psi(s)g_{\alpha \nu}(\eta s) \, ds \quad \text{for } t \geq 1
\]
and \( h_{ev}(t) = 0 \) for \( t \leq -1 \). If \( |\eta| \leq 1 \), then (5.7), (5.8) follow from (5.5).

Lemmas 5.2 and 5.3 have the following corollary.

**Corollary 5.4.** The function \( y_{ev} \) is bounded in \([-\varepsilon, \varepsilon]\) uniformly in \( \varepsilon \) and \( \nu \) provided \( \nu/\varepsilon \leq 1 \), and \( \max_{|x| \leq \varepsilon} |y_{ev}(x)| \leq C\|f\| \) with some constant \( C \) being independent of \( f \).

The function \( w_{ev} = y_{ev} - r_{ev} \) and its first derivative have the jumps at \( x = \pm \varepsilon \):

\[
[w_{ev}]_{-\varepsilon} = y(-0) - y(-\varepsilon), \quad [w'_{ev}]_{-\varepsilon} = y'(-0) - y'(-\varepsilon),
\]

\[
[w_{ev}]_{\varepsilon} = y(\varepsilon) - \theta_{0}y(-0) - \varepsilon g_{ev}(1) - \beta \nu \varepsilon h_{ev}(\eta^{-1}), \quad [w'_{ev}]_{\varepsilon} = y'(\varepsilon) - y'_{ev}(1) - \beta \varepsilon h'_{ev}(\eta^{-1}).
\]

In view of (2.7), (5.3), (5.7) and (5.1), we conclude that three of the jumps can be bounded by \( c_{i}\varepsilon^{1/2}\|f\| \). As for the last one, we have

\[
\left| [w'_{ev}]_{\varepsilon} \right| \leq \left| |y'_{ev}(\varepsilon) - y'_{ev}(+0)| + c_1\eta\|f\| + \varepsilon |\beta| |h'_{ev}(\eta)| \right| \leq c_1(\varepsilon^{1/2} + \eta)\|f\|,
\]

by (5.8) and Lemma 5.2. We can now repeatedly apply Proposition 2.4 to deduce

\[
\max_{x \in R, |x| \leq -\varepsilon} \| r^{(k)}_{ev}(x) \| \leq C(\varepsilon, \eta)\|f\| \tag{5.9}
\]

for \( k = 0, 1, 2 \), where \( \sigma(\varepsilon, \eta) = \varepsilon^{1/2} + \eta \).

**Proof of Theorem 5.1** Let us fix \( f \in \mathcal{F} \) and write \( f_{ev} = (S_{ev} - z)y_{ev} \). As in the proof of Lemma 3.6, we compute \( f_{ev}(x) = f(x) - r''_{ev}(x) - zr_{ev}(x) \) for \( |x| > \varepsilon \). Next, for \( |x| < \varepsilon \), we have

\[
f_{ev}(x) = \left( -\frac{\partial^2}{\partial z^2} + \alpha \varepsilon^{-2} \Phi \left( \frac{z}{\varepsilon} \right) + \beta \varepsilon^{-1} \Psi \left( \frac{z}{\varepsilon} \right) - z \right) y_{ev}(x)
\]

\[
= \varepsilon^{-2}y(-0)\left\{-u''_{\alpha} + \alpha \Phi \left( \frac{z}{\varepsilon} \right) u_{\alpha} \right\}
\]

\[
+ \varepsilon^{-1}\left\{-g''_{\alpha} + \alpha \Phi \left( \frac{z}{\varepsilon} \right) g_{ev} + \beta \eta^{-1} \varepsilon^{1} \Psi \left( \frac{z}{\varepsilon} \right) u_{\alpha} \left( \frac{z}{\varepsilon} \right) \right\}
\]

\[
+ \beta \eta^{-1}\left\{-h''_{ev} + \Psi \left( \frac{z}{\varepsilon} \right) g_{ev} \left( \frac{z}{\varepsilon} \right) \right\}
\]

\[
+ \alpha \eta \Phi \left( \frac{z}{\varepsilon} \right) h_{ev} \left( \frac{z}{\varepsilon} \right) + \beta \varepsilon \Psi \left( \frac{z}{\varepsilon} \right) h_{ev} \left( \frac{z}{\varepsilon} \right) - zy_{ev}(x)
\]

\[
= \left\{ \alpha \eta \Phi \left( \frac{z}{\varepsilon} \right) + \beta \varepsilon \Psi \left( \frac{z}{\varepsilon} \right) \right\} h_{ev} \left( \frac{z}{\varepsilon} \right) - zy_{ev}(x),
\]

since \( u_{\alpha}, g_{ev} \) and \( h_{ev} \) are solutions to equations (2.1), (5.3) and (5.4) respectively.

Since \( f \in \mathcal{F} \), we can choose the number \( \varepsilon \) so small that \( f \) is zero in \((-\varepsilon, \varepsilon)\). Then \( f_{ev} = f - q_{ev} \), where

\[
q_{ev} = r''_{ev} + zr_{ev} + zy_{ev} \chi_{\varepsilon} - \left( \alpha \eta \Phi(\varepsilon^{-1} \cdot) + \beta \varepsilon \Psi(\nu^{-1} \cdot) \right) h_{ev}(\nu^{-1} \cdot).
\]

Here \( \chi_{\varepsilon} \) is the characteristic function of \([-\varepsilon, \varepsilon]\). Consequently, we conclude from Lemma 5.3 that

\[
\eta \left| \Phi \left( \frac{z}{\varepsilon} \right) h_{ev} \left( \frac{z}{\varepsilon} \right) \right| \leq c_{3} \eta \chi_{\varepsilon}(x) \max_{|x| \leq \varepsilon} |h_{ev} \left( \frac{z}{\varepsilon} \right)|
\]

\[
\leq c_{2} \eta(1 + \eta^{-1})\|f\| \chi_{\varepsilon}(x) \leq c_{0}\|f\| \chi_{\varepsilon}(x),
\]

\[
\varepsilon \left| \Psi \left( \frac{z}{\varepsilon} \right) h_{ev} \left( \frac{z}{\varepsilon} \right) \right| \leq c_{4} \varepsilon \chi_{\varepsilon}(x) \max_{|x| \leq \varepsilon} |h_{ev} \left( \frac{z}{\varepsilon} \right)| \leq c_{5}\|f\| \chi_{\varepsilon}(x),
\]
hence that \(\|q_{\varepsilon \eta}\| \leq c\sigma(\varepsilon, \eta)\|f\|\), in view of Corollary 5.4 and estimate (5.9). Thus
\[
y_{\varepsilon \eta} = (S_{\varepsilon \eta} - z)^{-1}f + (S_{\varepsilon \eta} - z)^{-1}q_{\varepsilon \eta},
\]
and therefore
\[
\|(S_{\varepsilon \eta} - z)^{-1}f - y_{\varepsilon \eta}\| \leq \|(S_{\varepsilon \eta} - z)^{-1}\|\|q_{\varepsilon \eta}\| \leq c_6\sigma(\varepsilon, \eta)\|f\|.
\]

By arguments that are completely analogous to those presented in the proof of Theorem 3.1 we conclude that
\[
\|(S_{\varepsilon \eta}(\alpha, \beta \mu) - z)^{-1}f - y_{\varepsilon \eta}\| \leq C\sigma(\varepsilon, \eta)\|f\|,
\]
and finally that operators \(S_{\varepsilon \eta}\) converge to \(S(\alpha, \beta \mu)\) in the norm resolvent sense as \(\varepsilon\) and \(\eta\) tend to zero.

5.2. Non-resonant case. Assume \(\alpha\) does not belongs to the resonant set \(\Lambda_\Phi\), and write
\[
y_{\varepsilon \eta} = (S_{\varepsilon \eta} - \varepsilon g(x/\varepsilon) + \beta \nu \varepsilon h(x/\nu))\]
for \(|x| \leq \varepsilon\), where \(g\) and \(h\) are solutions to the problems
\[
g'' - \alpha \Phi(t)g = 0, \quad g'(-1) = y'(-0), \quad g'(1) = y'(0);
\]
\[
h'' = \Psi(t)g(\eta t), \quad h(-1) = 0, \quad h'(-1) = 0
\]
respectively. As above, the corrector function \(r_{\varepsilon \eta}\) is of the form (2.10) and provides the inclusion \(y_{\varepsilon \eta} \in W^2_2(\mathbb{R})\). The rest of the proof is similar to the proof of Theorem 5.1.

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