GLOBAL ANALYTIC HYPOELLIPTICITY FOR A CLASS OF QUASILINEAR SUMS OF SQUARES OF VECTOR FIELDS

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Abstract. We prove a global analytic regularity result for an operator of Hörmander rank 2 constructed from $2n$ quasilinear vector fields on a compact manifold in $\mathbb{R}^{2n+1}$.

1. Introduction

Our aim, in this work, is to prove a global analytic regularity result on a compact manifold $M$ for some quasilinear equations.

A model of such results is the following: in $C^2$ take a bounded domain $\Omega$ with strictly pseudo-convex real analytic boundary, $M$. Then one has two globally defined independent real, real analytic vector fields $X_1$ and $X_2$, namely the real and imaginary parts of a (globally defined) holomorphic vector field $L = X_1 - iX_2$ tangent to $M$.

Take a function $u$ in $C^\infty(M)$ and consider an analytic matrix function $H(x, t)$ defined on a neighborhood of $\{(x, u(x)) : x \in M\}$ in $M \times \mathbb{C}$ and set

$$Y = H(X) \text{ i.e., } Y(x) = H(x, u(x))X(x)$$

so that one obtains two $C^\infty$ vector fields, $Y_1$ and $Y_2$ on $M$.

We assume that $H(x, t)$ is invertible so that one can express $X = H^{-1}(Y)$ with $H^{-1} \in C^\omega$.

Consider the operator

$$P_u = Y_1^2 + Y_2^2 + a_1Y_1 + a_2Y_2 + b$$

with $a_j, b$ analytic and assume that $P_uu \in A(M)$. Can one conclude that $u$ is analytic on $M$ if the associated Levi form is non-degenerate? Note that $P_uu = f$ is a quasi-linear equation.

The question is global. There are known local results for more special equations (cf. [13]).
In higher dimensions, one generally does not have globally defined vector fields $X_1, \ldots, X_{2n}$ related to a CR structure on $M$ induced by the complex structure on $\mathbb{C}^n$. But one can consider a (finite) family of open sets $\{V_\ell\}_{1 \leq \ell \leq p}$ covering $M$ and analytic vector fields $\{X_{k,\ell}\}_{k=1,\ldots,2n}$ on $V_\ell$. Then we may consider

$$Y_{(\ell)} = H(X_{(\ell)})$$

where

$$X_{(\ell)} = \begin{pmatrix} X_{1,\ell} \\ \vdots \\ X_{2n,\ell} \end{pmatrix}$$

and the associated operator

$$P_{\ell,u} = \sum_{j=1}^{2n} Y_{j,\ell}^2 + a_{j,\ell} Y_{j,\ell} + b_{\ell}, \text{ with } a_{j,\ell}, b_{\ell} \text{ analytic}.$$ 

Now assume that for all $\ell$, $P_{\ell,u}u \in A(V_\ell)$.

Then the question is: is $u$ analytic on $M$ under a non-degeneracy hypothesis on the associated Levi form?

**Theorem 1.1.** Under the above hypotheses, if $Pu$ is real analytic globally on $M$ then so is any (moderately smooth) solution $u$.

A more interesting problem (as related to the complex Laplacian on forms) is to consider a system. But in this paper we consider only the scalar case. Note that from results on $C^\infty$ regularity (cf. Xu (, et al.), one need only assume that $u$ is in $C^{2,\alpha}$ in our theorem.

2. Some notation and definitions

Let $M$ be a compact, real analytic manifold of dimension $2n + 1 \geq 3$, and let $(V_j)_{j=1,\ldots,p}$ be a covering of $M$ such that, in each $V_j$, there are given $2n$ real analytic, real vector fields $X_{1,j}, \ldots, X_{2n,j}$ such that

- On $V_j \cap V_k$ every $X_{\ell,j}$ (resp. $X_{\ell,k}$) is a linear combination of the $(X_{\ell,k})_{\ell=1,\ldots,p}$ (resp. of the $(X_{\ell,j})_{\ell=1,\ldots,p}$) with real analytic coefficients.
- There exists a globally defined, real analytic real vector field $T$ such that $(X_{1,j}, \ldots, X_{2n,j}, T)$ is a basis in $V_j$ and if

$$(2.1) \quad [X_{\ell,j}, X_{m,j}] \equiv a_{\ell m j}^T T \mod (X_{\ell,j})$$

then the matrix $(a_{\ell m j}^T)$ is non-degenerate.
Remark 1. It is a result that goes back to Tanaka (9) and used to advantage in the work of the present authors in many places that under the non-degeneracy assumption (2.1), one always may modify the given vector field $T$ by adding multiples of the $X_{j,\ell}$ in such a way that
\begin{equation}
[X_{j,\ell}, T] \equiv 0 \mod (X_{1,\ell}, \ldots, X_{1,\ell}).
\end{equation}

Definition 2.1. We call such a family $(X_{\ell,j}, T)$ of systems of vector fields a compatible family.

Now, we may assume that each $(V_j)$ is the domain of a coordinate chart. So in each $V_j$ and for every $s \geq 0$, we may consider an elliptic pseudodifferential operator of order $s$ which we denote by $\Lambda^s_j$.

Let us fix a family $(\varphi_j)_{j=1,...,p}$ such that
\begin{equation}
\varphi_j \in D(V_j), \quad 0 \leq \varphi_j \leq 1, \quad \sum \varphi_j \equiv 1 \text{ on } M.
\end{equation}

Let $(\rho_j)_{j=1,...,p}$ be another family such that such that
\begin{equation}
\rho_j \in D(V_j), \quad 0 \leq \rho_j \leq 1, \quad \rho_j \equiv 1 \text{ on supp } \varphi_j.
\end{equation}

Now one has, say for $t \geq s \geq 0$,
\begin{equation}
\|v\|_t \lesssim \sum_j \|\varphi_j v\|_t \lesssim \sum_j (\|\rho_j \Lambda^s_j \varphi_j v\|_{t-s} + \|\varphi_j v\|), \forall v \in C^\infty(M)
\end{equation}
where $\| \|_t$ denotes the Sobolev norm.

So, now, one has
\begin{equation}
\|v\|_t \lesssim \sum_j \|\varphi_j v\|_t \lesssim \sum_j (\|\rho_j \Lambda^s_j \varphi_j v\|_{t-s} + \|\varphi_j v\|), \forall v \in C^\infty(M)
\end{equation}

and assume that
\begin{equation}
P_j u \in A(V_j), \quad \forall j.
\end{equation}

Finally let us denote by $(\ , \ )_s$ the $s-$Sobolev scalar product (in each $V_j$, when one has functions with compact support in $V_j$) and remember the following:
\begin{equation}
\forall \delta > 0, \exists C_\delta : \forall w \in C_0^\infty, \|w\|_{s,2}^2 \leq \delta \|w\|_{s+1/2}^2 + C_\delta \|w\|_{0}^2.
\end{equation}
Maximal Estimates

Our aim in this section is to prove the following:

**Theorem 3.1.** We have the following maximal estimates for \( s \geq 0 \):

\[
(3.1_s) \quad \|v\|_{s+1/2}^2 + \sum_{j,\ell} \|X_{j,\ell} \varphi v\|_s^2 \lesssim \sum_{\ell} |(\varphi_{\ell} P_{\ell} v, \varphi_{\ell} v)| + \|v\|_0^2,
\]

for \( v \in C^\infty(M) \) and

\[
(3.2_s) \quad \|v\|_{s+1}^2 + \sum_{j,\ell} \|X_{j,\ell} \varphi v\|_{s+1/2}^2 \lesssim \sum_{\ell} \|\varphi_{\ell} P_{\ell} v\|_s^2 + \|v\|_0^2
\]

for \( v \in C^\infty(M) \) and in fact

\[
(3.3_s) \quad \|v\|_{s+1}^2 + \sum_{j,\ell} \|X_{j,\ell} \varphi v\|_{s+1/2}^2 + \sum_{j,k,\ell} \|X_{j,k,\ell} \varphi v\|_s^2 \lesssim \sum_{\ell} \|\varphi_{\ell} P_{\ell} v\|_s^2 + \|v\|_0^2
\]

for \( v \in C^\infty(M) \).

**Proof.** The proof is known when written for functions with compact support in coordinate charts. This is a global version. Let us first show the statements at level \( s = 0 \). For simplicity we take \( \ell \) fixed and set \( X_{j,\ell} = X_j, j = 1, \ldots, 2n \) and \( \varphi = \varphi_{\ell} \). We have

\[
\sum_j \|X_j \varphi v\|_0^2 = \sum_j (X_j \varphi v, X_j \varphi v) \lesssim \sum_j (Y_j \varphi v, Y_j \varphi v)
\]

because \( H \) is invertible. Note that \( \lesssim \) may indicate a constant which depends on \( u \) and its first few derivatives. Now

\[
(Y_j \varphi v, Y_j \varphi v) = -(Y_j^2 \varphi v, \varphi v) + (\theta_j \varphi v, \varphi v), \quad \theta_j \in C^\infty(V_j)
\]

\[
= -([Y_j^2, \varphi] v, \varphi v) + (\varphi Y_j^2 v, \varphi v) + O(\|\varphi v\|_0 \|Y_j \varphi v\|_0).
\]

Now, using \([Y_j^2, \varphi] = 2Y_j[Y_j, \varphi] - [Y_j, [Y_j, \varphi]]\) we easily obtain

\[
(3.3) \quad |([Y_j^2, \varphi] v, \varphi v)| \lesssim C_1 \|v\|_0^2 + \frac{1}{C_0} \sum_j \|X_j \varphi v\|_0^2.
\]

Thus,

\[
\sum_j \|X_j \varphi v\|_0^2 \lesssim |(\varphi P v, \varphi v)| + C_1 \|v\|_0^2 + \frac{1}{C_0} \sum_j \|X_j \varphi v\|_0^2.
\]
Now use
\[ \| \varphi v \|_{1/2} \lesssim \sum_j \| X_j \varphi v \|_0^2 + \| \varphi v \|_0^2 \quad \forall v \in C^\infty(M) \] (3.4)

(see J. J. Kohn [7]) to obtain (3.1) in case \( s = 0 \).

Now we can deduce (3.2) from (3.1) in the following way: using (2.5) and (2.6), we have
\[
\sum_\ell \| \varphi \ell v \|_1^2 + \sum_{j, \ell} \| X_{j, \ell} \varphi \ell v \|_{1/2}^2 \lesssim \sum_\ell \| \rho \ell \Lambda_{\ell}^{1/2} \varphi \ell v \|_{1/2}^2 \\
+ \sum_{j, \ell} \| \rho \ell \Lambda_{\ell}^{1/2} X_{j, \ell} \varphi \ell v \|_0^2 + \| v \|_0^2 + \sum_{j, \ell} \| X_{j, \ell} \varphi \ell v \|_0^2 \\
\lesssim \sum_\ell \| \rho \ell \Lambda_{\ell}^{1/2} \varphi \ell v \|_{1/2}^2 + \sum_{j, \ell} \| X_{j, \ell} \rho \ell \Lambda_{\ell}^{1/2} \varphi \ell v \|_0^2 + \sum_\ell \| \varphi \ell v \|_{1/2}^2 + \sum_{j, \ell} \| X_{j, \ell} \varphi \ell v \|_0^2.
\] (3.5)

The last two sums are easy to handle. The first two sums are (from the first part of the theorem at level \( s = 0 \)), less than
\[ \sum_\ell \| (\rho \ell P \Lambda_{\ell}^{1/2} \varphi \ell v, \rho \ell \Lambda_{\ell}^{1/2} \varphi (v)) \| + \sum_\ell \| \varphi \ell v \|_{1/2}^2 \] (3.6)

Now, for simplicity we forget the index \( \ell \) and consider
\[ \rho P \Lambda^{1/2} \varphi v = [\rho P, \Lambda^{1/2} \varphi] v + \Lambda^{1/2} \varphi P v \] (3.7)

Now as we obtained (3.3) we have
\[ |(\rho P, \Lambda^{1/2} \varphi) v, \rho \Lambda^{1/2} \varphi v) \| \leq C_1 \| v \|_{1/2}^2 + \frac{1}{C_0} \sum_\ell \| X_{j, \ell} \varphi \ell v \|_{1/2}^2 \] (3.8)

By taking \( C_0 \) large enough and using (2.9) we have the desired inequality, because the term \( |(\Lambda^{1/2} \varphi P v, \rho \Lambda^{1/2} \varphi v) \| \) is less than \( C_1 \| \varphi P v \|_0^2 + \frac{1}{C_0} \sum_\ell \| \varphi v \|_0^2 \) (with \( C_0 \) large, \( C_1 \) depending on \( C_0 \) as usual).

This proves (3.2) with \( s = 0 \). To bound also the third term on the left in (3.3) for \( s = 0 \), we argue as follows: first the function \( X_{j, \ell} v \) is inserted in place of \( v \) in (3.1) with \( s = 0 \) and an error of the type \( C \sum_\ell \| v \|_0^2 \) is introduced through a bracket of the form \( ([X, X] v, X^2 v) \).

While this is a new error, it is already controlled by (3.2), which completes the proof for \( s = 0 \).

Our aim is to prove (3.1) and deduce (3.2) from (3.1) as before. First observe that (in view of (2.5))
\[ \| v \|_{s+1/2}^2 \lesssim \sum_\ell \| \rho \ell \Lambda_{\ell}^{1/2} \varphi \ell v \|_{1/2}^2 + \| v \|_0^2 \] and
\[ \sum_{\ell} \|X_{j,\ell} \varphi v\|_s^2 \lesssim \sum_{\ell} \|X_{j,\ell} \rho \Lambda_\ell^s \varphi v\|_0^2 + \|v\|_0^2 \lesssim \sum_{\ell} \|X_{j,\ell} \rho \Lambda_\ell^s \varphi v\|_0^2 + \sum_{\ell} \|\rho \Lambda_\ell^s \varphi v\|_{1/2}^2 + \|v\|_0^2. \]

Now, we use (3.10) to obtain
\[ (3.9) \quad \|v\|_{s+1/2}^2 + \sum_{\ell} \|X_{j,\ell} \varphi v\|_s^2 \lesssim \sum_{\ell} |(\rho_\ell \Lambda_\ell^s \varphi v, \rho_\ell \Lambda_\ell^s \varphi v)|. \]

In view of (2.9), the term \( \|v\|_s^2 \) may be replaced by \( \|v\|_2^2 \).

Now we consider the first term in the second member of (3.9) and write: \( \rho_\ell \Lambda_\ell^s \varphi v = \rho_\ell [P_\ell, \Lambda_\ell^s \varphi] v + \rho_\ell \Lambda_\ell^s \varphi_\ell v. \)

So one sees that one is reduced to study \( (\rho_\ell [P_\ell, \Lambda_\ell^s \varphi] v, \rho_\ell \Lambda_\ell^s \varphi v) \), because one has easily
\[ (3.10) \quad |(\rho_\ell \Lambda_\ell^s \varphi_\ell v, \rho_\ell \Lambda_\ell^s \varphi v)| \leq C |(\varphi_\ell v, \varphi_\ell v)_s| + \delta \left\{ \sum_{\ell} \|\varphi_\ell v\|_{s+1/2}^2 + \sum_{\ell} \|X_{j,\ell} \varphi v\|_s^2 \right\} + C_\delta \|v\|_0^2. \]

Now again forget the index \( \ell \) and consider
\[ [P, \Lambda^s \varphi] = \sum_j [Y_j^2, \Lambda^s \varphi] = \sum_j 2Y_j^2 [Y_j, \Lambda^s \varphi] - [Y_j, [Y_j, \Lambda^s \varphi]] \]

Then one has, as in (3.3),
\[ |(\rho_\ell [P_\ell, \Lambda_\ell^s \varphi] v, \rho_\ell \Lambda_\ell^s \varphi_\ell v)| \leq \frac{1}{C_0} \sum_j \|X_{j,\ell} \rho_\ell \Lambda_\ell^s \varphi v\|_0^2 + C_1 \|\rho_\ell \Lambda_\ell^s \varphi v\|_0^2 \]

where \( C_1 \) depends on \( C_0 \), as usual.

Then, again using (2.9) and taking \( C_0 \) big enough, the first member of (3.10) is less than \( C (\sum_\ell |(\varphi_\ell v, \varphi_\ell v)_s| + \|v\|_0^2) \) for some \( C > 0 \).

Now, we want to prove (3.2.3) using (3.1) as we did for \( s = 0 \).

One has, as in that case,
\[ \sum_{\ell} \|\varphi_\ell v\|_{s+1}^2 + \sum_{j,\ell} \|X_{j,\ell} \varphi_\ell v\|_{s+1/2}^2 \lesssim \sum_{\ell} \|\rho_\ell \Lambda_{\ell}^{1/2} \varphi_\ell v\|_{s+1/2}^2 \]
\[ + \sum_{j,\ell} \|\rho_\ell \Lambda_{\ell}^{1/2} X_{j,\ell} \varphi_\ell v\|_s^2 + \|v\|_{s+1/2}^2 + \sum_{j,\ell} \|X_{j,\ell} \varphi v\|_s^2 \]
\[ \lesssim \sum_{\ell} \rho_\ell \Lambda_{\ell}^{1/2} \varphi_\ell v\|_{s+1/2}^2 + \sum_{j,\ell} \|X_{j,\ell} \rho_\ell \Lambda_{\ell}^{1/2} \varphi_\ell v\|_s^2 + \sum_{\ell} \|\varphi_\ell v\|_s^2 \]
\[ + \sum_{j,\ell} \|X_{j,\ell} \varphi v\|_s^2. \]
The worst terms are the first two ones, from (3.1). They are less than

\[ \sum_{\ell} \left| \langle \rho_{\ell} P_{\ell} A_{\ell}^{1/2} \varphi_{\ell} v, \rho_{\ell} A_{\ell}^{1/2} \varphi_{\ell} v \rangle \right| + \| v \|_0^2 \]

Now the end of the proof of (3.2) follows the lines of the end of the proof in the case \( s = 0 \), and the proof of (3.3) is also as before. \( \square \)

**Remark 2.** This proof of the global version only requires careful computations.

**Corollary 3.1.** Let \( T \) be a global, real analytic, non-zero vector field on \( M \) complementary to the \( X \) and satisfying (2.1). Then

\[ \| T v \|_s^2 \lesssim \sum_{\ell} \| \varphi_{\ell} P_{\ell} v \|_s^2 + \| v \|_0^2 \quad \forall v \in C^\infty(M) \]

**Remark 3.** The existence of such a global \( T \) has been shown ([10], [12]) when \( M \) is a compact real compact CR manifold.

4. **High Powers of the Vector Field \( T \).**

The overall strategy is to use the maximal estimates above with \( v \) replaced by \( T^p u \). Once one has control over high \( T \) derivatives of the solution the other derivatives follow by standard techniques.

Now the vector field \( T \) being global, dealing with just a bounded number of vector fields \( X_j \) that are only locally defined is not a very delicate issue. For instance, the above Corollary may be strengthened to

(4.1) \[ \| T v \|_s^2 + \| v \|_{s+1}^2 + \sum_{j,\ell} \| X_j,\ell \varphi_{\ell} v \|_{s+1/2}^2 + \sum_{j,k,\ell} \| X_j,\ell X_k,\ell \varphi_{\ell} v \|_s^2 \]

\[ \lesssim \sum_{\ell} \| \varphi_{\ell} P_{\ell} v \|_s^2 + \| v \|_0^2 \quad \forall v \in C^\infty(M) \]

or, in the form we will use it, for any integer \( r \),

(4.2) \[ \| T^{r+1} v \|_s^2 + \sum_{j,\ell} \| X_j,\ell \varphi_{\ell} T^r v \|_{s+1/2}^2 + \sum_{j,k,\ell} \| X_j,\ell X_k,\ell \varphi_{\ell} T^r v \|_s^2 \]

\[ \lesssim \sum_{\ell} \| \varphi_{\ell} P_{\ell} T^r v \|_s^2 + \| T^r v \|_0^2 \quad \forall v \in C^\infty(M) \]
Proposition 4.1. There exist constants $C, C_u, C_h$ such that for all $r$, \[ \frac{1}{r!} \left( \|T^{r+1}v\|^2 + \|X^2T^rv\|^2 \right) \leq 4^r C^r C_u^r C_h^r \] 

Proof. In view of (4.2), our (only) task is to commute $T^r$ past $P_\ell$ with errors which can be recursively estimated to grow ‘analytically’ since then after specializing $v$ to $u$, we know that $P_\ell u \in A(M)$.

Now we have the crucial relationship (in $V_j$) \[ [T, X_{j,\ell}] = \sum_k c_{jk} X_{k,\ell} \] with $c_{jk}$ real analytic functions. Since $P_\ell$ is a quadratic polynomial in the $X_{j,\ell}$, with coefficients $h(x, u)$ which are real analytic functions of the spatial variables $x$ and the solution $u(x)$, we may write \[ [T, P_\ell] = \sum [T, h(x, u) X_{k,\ell} X_{j,\ell}] = (Th)X^2 + h[T, X^2] \] \[ = (Th)X^2 + h\{\tilde{a}X^2 + \tilde{a}X + \tilde{a}\} \] with analytic functions $\tilde{a}$, $\tilde{a}$ and $\tilde{a}$. Generically, then, we may write \[ [T, P_\ell] = \sum ((T)h)X^2 \] where $h((T)h)$ denotes (at most) a first derivative (in $(x, t)$) of $h(x, u(x))$ times one of a finite collection of analytic functions of $x$, namely the coefficients of the $X$ in the bracket $[T, X]$ mentioned above in (2.1) (and possibly one derivative of this coefficient). There may also be fewer than two $X$’s on the right in (4.4). For the rest of the paper we will assume for simplicity that $h = h(u)$.

Next, we will need an expression for the more complicated bracket \[ [T^r, P_\ell] = \sum_{r'=0}^{r-1} T^{r'} [T, P_\ell] T^{r-r'-1} = \sum_{r'=0}^{r-1} T^{r'} \circ h' X^2 T^{r-r'-1} \] only we move the $h'$ to the very left yet leave $X^2$ for the moment wherever they are, which we denote by enclosing the $X^2$ in parentheses and placing them on the left. That is, \[ [T^r, P_\ell] = \sum_{r'=1}^{r} \binom{r}{r'} (T^{r'}h)(u(x))(X^2)T^{r-r'}. \] or \[ \frac{[T^r, P_\ell]}{r!} = \sum_{r'=1}^{r} \frac{(T^{r'}h)(u(x))}{r!} \frac{(X^2)T^{r-r'}}{(r-r')!}. \]
Now for the term \((T^r h)(u(x))\), we will use the Faà di Bruno formula or rather crude bounds for the results: writing

\[ D_x^k g(u(x)) = (u' D_u + D_x)_k - 1 u' D_u g(u(x)); \]

writing this crudely as

\[ D_x^k g(u(x)) = ((u' \sigma + D_x)_k - 1 u' \sigma = D_g) g, \]

i.e., \( \sigma \) becomes a ‘counter’ for the number of derivatives received by \( g \). Then this is at worst

\[ \sum_{k'} \left( \frac{k}{k'} \right) g^{(k-k')}(D_x^{k'} u^{k-k'}). \]

Finally, we must analyze expressions such as \( D^a u^b \):

\[ D^a u^b = \sum_{a_1 + \ldots + a_b = a} \binom{a}{a_1, \ldots, a_b} u^{(a_1)} \ldots u^{(a_b)} \]

where \( \binom{a}{a_1, \ldots, a_b} \) denotes the multinomial expression

\[ \binom{a}{a_1, \ldots, a_b} = \frac{a!}{a_1! \ldots a_b!(a - \sum a_j)!} = \frac{a!}{a_1! \ldots a_b!} \]

since the \( \sum a_j = a \). We have

\[ \frac{D^a u^b}{a!} = \sum_{a_1 + \ldots + a_b = a} \frac{u^{(a_1)}}{a_1!} \ldots \frac{u^{(a_b)}}{a_b!} \]

Thus we have

\[ \frac{[T^r, P]}{r!} = \sum_{r'=1}^{r} \frac{(T_{x,t}^r h(u(x,t)))}{r'} \frac{(X^2)T^{r'-r'}}{(r-r')!}. \]

and so (cf. (4.6)):

\[ \frac{(T^{r'}) h(u(x))}{r'} \sim \sum_{r' - r'' \geq 1} \frac{h^{(r'-r'')}}{(r' - r'')!} \frac{(T^{r''}(u^{r'-r''}))}{r''!} \]

\[ = \sum_{r' - r'' \geq 1} \frac{h^{(r'-r'')}}{(r' - r'')!} \sum_{\sum i' = r''} \frac{T^{r''} u^{i'}}{r''!} \ldots \frac{T^{r''-r''} u^{i'}}{r''-r''!} \]
or in all,
\begin{equation}
\frac{[T^r, P] v}{r!} = \sum_{\substack{r > r' + r'' \geq 1 \\
\sum_{j=1}^n t_j = r' \\
\sum_{j=1}^n t_j'' = r''}} \frac{h^{(r' - r'')} (r' - r'')!}{r'!} T^{r''} \cdot \frac{T^{r'''} u'}{r'''}! \cdot \frac{X^2 T^{r- r'} v}{(r - r')!}.
\end{equation}

To simplify the argument we have dropped the localizing functions, since for global arguments when these functions always appear on the left of the norms they may be brought out easily and replaced by another partition of unity, with new X's if needed; also have ignored the order of the X's and T's, indicating this by putting the X^2 in parentheses, not to indicate that they may not be present (though they may not) but that they may appear with several T's to the left and more to the right. We have also dropped all subscripts. Schematically, we may then write (4.2) together with (4.8) as
\begin{equation}
\|T^{r+1} v\|_s + \|X^2 T^r v\|_s \lesssim \|P T^r v\|_s + \|T^r v\|_s \lesssim \|T^r P v\|_s + \|T^r v\|_s^2 + \|[T^r, P] v\|_s \quad \forall v \in C^\infty(M)
\end{equation}
and so
\begin{equation}
\frac{1}{r!} \left\{ \|T^{r+1} v\|_s + \|X^2 T^r v\|_s \right\} \lesssim \frac{1}{r!} \left\{ \|T^r P v\|_s + \|T^r v\|_s \right\} +
\sum_{\substack{r > r' + r'' \geq 1 \\
\sum_{j=1}^n t_j = r' \\
\sum_{j=1}^n t_j'' = r'' \\
\sum_{j=1}^n t_j''' = (r'' + 1) - r'}} \left\| \frac{h^{(r' - r'')} (r' - r'')!}{r'!} T^{r''} \cdot \frac{T^{r'''} u'}{r'''}! \cdot \frac{X^2 T^{r- r'} v}{(r - r')!} \right\|_s.
\end{equation}

Now for our value of s, H^s is an algebra, and so the norm of the product of derivatives of copies of u may be replaced by the product of the norms, each of which will have the form of one of the terms on the left hand side of (4.10).

Specializing to v = u and bounding the norm of the product by the product of the norms we observe that except for the term involving derivatives of h, all other terms are of the same form since one T derivative and two X derivatives carry the same weight on the left hand side of (4.10):
The constant includes a power of $C$ for each norm that follows. Note that since the derivatives on $h$ are of that order, this constant will be included with the analyticity constant for $h$, and in the future constants with exponents comparable to the number of derivatives on a function known to be analytic will be permitted without comment.

Note that the term in the product with $(X^2)$ is analogous to the extra $T$ derivative on each of the other terms. Hence these terms are similar to the left hand side, and could be handled at once inductively except for counting the number of them, but it is simpler to iterate (4.11) directly, at least until all terms have order less than $r/2$.

Since there can be at most one term of order larger than $r/2$, after the next ‘pass’ we observe that the product will look just like the right hand side of (4.10) again, except that there will be one more norm of derivatives of $h$.

That is, applying (4.11), with $r$ replaced by $r - r'$, to the term in (4.11) with $(X^2)$, we have

\begin{equation}
(4.12) \quad \frac{1}{(r - r')!} \{||T^{r-r'+1}v||_s + ||X^2T^{r-r'}v||_s \} \lesssim \frac{1}{r!} \{||T^rPv||_s + ||T^rv||_s \} + \sum_{r-r' \geq \rho' - \rho'' \geq 1} \sum_{r-r'' \geq r'' \geq 1} C_{\rho' - \rho'' + 2} \left| \frac{h(\rho' - \rho'')}{(\rho' - \rho'')!} \right|_s \left| \frac{(X^2)T^{r-r'-\rho'}v}{(r-r'-\rho')!} \right|_s \prod_{j=1}^{r-r'-\rho'} \left| \frac{T^{\rho'_{j}}u'_j}{\rho'_{j}!} \right|_s \right|_s
\end{equation}

we find $r \geq r' + \rho' - r'' - \rho''$ and $\sum_{j=1}^{r-r''} \rho'_{j} = r'' + \rho''$ so if we set $s' = r' + \rho'$ and $s'' = r'' + \rho''$, we have a sum over $s' - s''$ and $\sum_{j+k=2} s'_{j+k} = s''$ subject to the obvious subdivisions.
That is, over \( r' + \rho' = s', r'' + \rho'' = s'' \), 

\[
(4.13) \quad \frac{1}{r!} \left\{ \| T^{r+1} v \|_s + \| X^2 T^r v \|_s \right\} \lesssim \sum_{r' \geq 0} \frac{1}{(r-r')!} \left\{ \| T^{r-r'} P v \|_s + \| T^{r-r'} v \|_s \right\} + 
\]

\[
+ \sum_{s' = r' + \rho', s'' = r'' + \rho''} \left\{ \left\| C^{r'-r''+2} h(r'-r'') \right\|_s \left\| C^{\rho'-\rho''+2} h(\rho'-\rho'') \right\|_s \right\} \times 
\]

\[
\left\| \frac{T^{s''}}{s''!} u' \right\|_s \ldots \left\| \frac{T^{s''-s'}}{s''-s'!} u' \right\|_s \left\| \frac{(X^2 T^{s'-s'}) v}{(r-s)!} \right\|_s .
\]

Note that in using the fact that \( H^s \) is an algebra, i.e., \( \| f g \|_s \leq B \| f \|_s \| g \|_s \), we have absorbed the algebra constant with the constant \( C \) inside the norms of \( h \). We further estimate the norms of derivatives of \( h \) (noting that each occurrence contains at least one such derivative) by

\[
(4.14) \quad \| C^{\ell+2} h(\ell)(x, y, u) \|_s \leq C_h^{\ell} \ell !
\]

We are nearly ready to iterate this procedure until even the last term has order less than \( r/2 \); for except for the sum (the number of terms), each term has a bound which is stable in the number of iterations, namely the last right hand side above is bounded by

\[
(4.15) \quad \sum C'_t \prod \left\{ \left\| \frac{T^k u'}{k!} \right\|_s \text{ or } \left\| \frac{X^2 T^k u}{k!} \right\|_s \right\}
\]

where the sum of the \( k + 1 \) is at most \( r \) and \( t \leq r \) is the number of terms in the product.

As for the sum, whether after a single full pass or multiple ones, the number of terms corresponds at most to the number of ways to partition \( r \) derivatives among at most \( r \) functions, generally many fewer. Denoting by \( D \) a derivative (\( r \) of them) and by \( u \) a copy of \( u \) (\( t \) of them) we are faced with the number of ways to ‘identify’ or select \( t \) items (the \( u's \)) from among \( r + t \) items (the \( D's \) and \( u's \)) with the understanding
that in an expression such as

\[
\underbrace{\underbrace{\underbrace{\underbrace{\ldots}}}}_{r \text{ D}'s \text{ and } t(\leq r) \text{ u}'s}^{r_1} u \underbrace{\underbrace{\underbrace{\underbrace{\ldots}}}}_{r_2} u \underbrace{\underbrace{\underbrace{\underbrace{\ldots}}}}_{r_3} u \cdots \underbrace{\underbrace{\underbrace{\underbrace{\ldots}}}}_{r_t} u
\]

the D’s differentiate only the first u which follows. The answer is that there are certainly not more than \(\binom{r+t}{t} \leq 2^{r+t} \leq 2^{2r} = 4^r\) ways.

Finally, since we may iterate this procedure until the maximal order of differentiation on u is 1 or 2, and bound this small number of derivatives by a constant (with at most r such terms, naturally - that’s all the derivatives there were). Thus the left hand side of (4.10) is bounded by:

\[
\frac{1}{r!} \left\{ \| T^{r+1} v \|_s^2 + \| X^2 T^r v \|_s^2 \right\} \leq 4^r C^r C_u C_h^r
\]

which clearly yields analytic growth (of T derivatives) of the solution u since \(C_u\) depends only on the first few derivatives of u and s is taken just large enough to ensure that \(H^s\) is an algebra.

\[\square\]

5. Mixed Derivatives - the case of global X

To finish the proof in the case where the vector field(s) X are globally defined it remains to show that we may estimate mixed derivatives as effectively as we did the high T derivatives. The result of Helffer and Mattera show that it \textit{would} suffice to handle pure powers of the vector field X, but mixed derivatives will invariably enter through brackets of pairs of X’s. Thus this we start by using the \textit{a priori} estimate \(3.13\) with \(v\) replaced by \(\varphi X^r\) (and later by a mixture of derivatives in X and in T). What ultimately happens is that brackets of pairs X’s will produce T’s, but at most half as many, and we will be led back to (nearly pure) powers of T. The non-linearity of the problem introduces nothing new in this overall pattern.

When the X’s are \textit{globally defined}, for example in \(\mathbb{C}^2\),the powers of X are treated like powers of T (e.g., with respect to the use of the Fàa di Bruno formula, especially) with the one change that an additional type of term will appear: starting with \(X^{r+2} v\) there will appear as an error \(\xi\) copies of \(X^r Tv\) when two X’s bracket to give a T. And this effect, the only new feature, will be repeated until all or nearly all the
X’s are exhausted. That is, we have the new scheme
\[ X^r \to C^{r/2}r!!(X^2)T^{r/2}, \]
where we recall the definition \( r!! = r(r-2)(r-4) \ldots \sim C^{r/2}(r/2)! \) But this is not a problem, since we have just treated essentially pure powers of \( T \) above in (4.17).

Rather than write this case out in more detail, we proceed to the next section where the problem is global but the vector fields \( X \) are only locally defined. This case incorporates many of the features of a fully local proof, though fortunately not all! Note that the \( T \) vector field is still required to be globally given.

6. Mixed Derivatives - when \( X_j \) are only locally defined

When the vector fields \( X \) are only locally defined, we cannot afford to change freely from one coordinate patch to another and to another basis of \( X \)’s each time a localizing function arising from a partition of unity is differentiated, since the constants counting the number of terms and the coefficients would grow far too fast, namely roughly \( C^r \) at each step. we will need a suitable localization of high powers of the \( X \). While one might suspect otherwise, we will

We will thus work in a single coordinate patch, drop all subscripts \( \ell \) (and \( j \) and \( k \), for simplicity), and in place of \( v \) in the \emph{a priori} estimates substitute \( \Psi X^r u \), where the localizing function \( \Psi \) will be specified further below. It will not need to be differentiated very often, but the band in which it goes from being identically equal to one to being identically zero will be of a precise width, as will subsequent localizing functions which will be introduced below. The general result on families of localizing functions is given by a result of Ehrenpreis (5):

**Proposition 6.1.** For any two open sets \( \Omega_0 \subseteq \Omega_1 \), with separation \( d = \text{dist.}(\Omega_0, \Omega_1^c) \) and any natural number \( N \), there exists a universal constant \( C \) depending only on the dimension and a function \( \Psi = \Psi_{\Omega_0,\Omega_1,N} \in C_0^\infty(\Omega_1) \), \( \Psi \equiv 1 \) on \( \Omega_0 \) with

\[
(6.1) \quad |D^\beta \Psi| \leq \left( \frac{C}{d} \right)^{|\beta|+1} N^{|eta|}, \quad |\beta| \leq 2N,
\]
though in this paper we will take $N = 4$ at most; thus the analyticity to be shown in $U_0$ will be reduced to combining the bounds on $\|T^{r+1}u\|_s$

obtained in a previous section with the bounds

$$\sum \|X^2\Psi X^r u\|_s + \|T \Psi X^r u\|_s \leq C^{r+1}r!$$

with $\Psi \equiv 1$ on the set where we want to prove analyticity.

To do this, we start with the a priori estimates as before: for $v$ of compact support where the $X_j$ are defined, and any fixed $s$, we have (3.3) in the form:

$$(6.2) \quad \|X^2v\|_s^2 + \|Xv\|_{s-1/2}^2 + \|v\|_{s-1}^2 \lesssim \|Pv\|_s^2 + \|v\|_0^2.$$

Note that we have dropped all subscripts but are working with several $X$’s. Naturally we could add a term $\|Tv\|_s$ to the left hand side using the non-vanishing of the Levi form but it will not help us here as it did above in handling high powers of $T$ applied to the solution $u$.

This estimate will be applied to $v = \Psi X^r u$ and then on the right we will write $P\Psi X^r u$ in terms of $\Psi X^r Pu$ modulo an error, namely the commutator of $aX^2$ with $\Psi X^r$, suitably expanded.

Now the crucial brackets, analogous to (4.3), will be written

$$(6.3) \quad [P, \Psi X^r]v = a_u [X^2, \Psi]X^r v + a_u \Psi[X^2, X^r]v + \Psi[a_u, X^r]X^2 v,$$

where coefficients depending on the solution $u$ (those arising in $P$ and here denoted $a_u$) are subscripted with $u$ while those which depend only on the spatial variables are not subscripted. Now

$$[X^2, \Psi] = 2\Psi'X + \Psi'',$$

(and notice that at most two derivatives appear on $\Psi$ and that these will fall to the left of all other terms in the bracket and will be changed with each iteration) and

$$(6.4) \quad [X, X] = aT$$

and so

$$[X^2, X^r] = C_r \text{ terms } [X, X]X^r + \cdots$$

independent of $u$. Here underlining a coefficient indicates the number of terms of the given type which occur and the $\cdots$ denote terms arising from bringing at least the coefficient in $aT$ to the left of $X^r$, incurring additional derivatives of course on the coefficient $a$. However all of this is linear. The non-linear phenomena occur in the last term, where $a_u = a(x, u)$ is differentiated. But this proceeds as before (cf. (4.8)):
letting, for instance, \( b^{(s)}(x, u) \) denote derivatives of the function \( b \) in its arguments, all derivatives of the solution \( u \) being split off to the right,

\[
(6.5) \quad \left[ a_u, X^r \right] w = \sum_{r' \geq r'' \geq 1} \left[ a_u \right]^{(r'-r'')} \frac{X^{r''} u}{r''!} \frac{X^{r'-r''} u}{r'!} \frac{X^{r-r''}}{(r-r')!}.
\]

So, all together, (6.3) becomes:

\[
(6.6) \quad [P, \Psi X^r] v \sim 2 a_u \Psi' X^r v + a_u \Psi'' X^r v + \sum r a_u \Psi a T \Psi X^r v + \cdots
\]

Now once we specialize \( v \) to \( u \), we will take the \( H^s \) norm of everything and use the property that this space is an algebra for our choice of \( s \). The function \( \Psi \) in the product on the right has served its purpose, and we will eventually introduce a new localizing function for each term in the product (except the coefficient, which will just be estimated), though at most one of these terms can have ‘order’ even half of \( r \) and the rest will be handled inductively.

7. Local Regularity in High Powers of \( X \); new localizing functions

More precisely, we restate (6.3) after specialization and introduction of the \( H^s \) norm:

\[
\frac{\|[P, \Psi X^r] u\|_s}{r!} \leq C_u \left\{ \frac{\|\Psi' X^{r+1} u\|_s}{r!} + \frac{\|\Psi'' X^r u\|_s}{r!} + \frac{\|\Psi T X^r u\|_s}{r!} + \cdots \right\}
\]

\[
+ \sum_{r' \geq r'' \geq 1} \frac{\|[P, \Psi X^r] u\|_s}{(r'-r'')!} \frac{\|X^{r''} u\|_s}{r''!} \frac{\|X^{r'-r''} u\|_s}{r'!} \frac{\|X^{r-r''} X^2 u\|_s}{(r-r')!} \|H^s\|
\]

We treat the functions \( X^2 u \) and \( u' \) similarly - they are equivalently handled by the \textit{a priori} estimate - and \textit{for convenience only} we suppose that the term with \( X^2 u \) is of highest order - i.e., \( r - r' \geq r''_j \forall j \).
Noting that $\text{supp} \Psi \subset U_{1/r}$ and bounding the norm of the coefficients by $C^{r'-r''}$,

\begin{equation}
(7.1) \quad \left\| \frac{[P, \Psi X^r] u}{r!} \right\|_s \leq C_u \left\{ \left\| \frac{\Psi' X^{r+1} u}{r!} \right\|_s + \left\| \frac{\Psi'' X^r u}{r!} \right\|_s + \frac{r}{r!} \left\| \frac{\Psi X^r u}{r!} \right\|_s + \cdots \right\} \\
+ \sum_{r' \geq r'' \geq 1} \sum_{\sum_{j=1}^{r'-r''} r_j = r''} C_{r''} \left\| \prod_{j=1}^{r'-r''} X_j^{r_j} u' \right\|_{H^s(U_{1/r})} \left\| \frac{\Psi X^{r-r'} X^2 u}{(r-r')!} \right\|_{H^s}.
\end{equation}

Again, we note that the number of terms in the product is $r' - r''$ and hence the constant arising from the algebraicity of $H^s$ will be absorbed with the analyticity constant for the coefficients $a_u$. Thus we restate (7.1) with this observation, writing $\Psi X^2$ in place of $X^2 \Psi$ on the left, modulo terms on the right, and associating powers of $r$ with derivatives of $\Psi$ or with powers of $T$, and taking $Pu = 0$:

\begin{equation}
(7.2) \quad \left\| \frac{\Psi X^2 X^r u}{r!} \right\|_s \leq C_u \left\{ \sum_{j=1}^{2} \frac{1}{r!} \left\| \frac{\Psi^{(j)} X^{r+2-j} u}{(r-j)!} \right\|_s + \frac{1}{r} \left\| \frac{\Psi X^r u}{r!} \right\|_s + \cdots \right\} \\
+ \sum_{r' \geq r'' \geq 1} \sum_{\sum_{j=1}^{r'-r''} r_j = r''} C_h \left\| \prod_{j=1}^{r'-r''} X_j^{r_j} u' \right\|_{H^s(K)} \left\| \frac{\Psi X^2 X^{r-r'} u}{(r-r')!} \right\|_{H^s}.
\end{equation}

As we iterate the terms on the right without $T$, the order will drop and we will control the coefficients and the sum below. The term with $T$ is slightly different, but we may always write,

$$\frac{1}{r!} \left\| \frac{\Psi T X^r u}{(r-2)!} \right\|_s = \frac{1}{r} \left\| \frac{\Psi X^2 X^{r-2} T u}{(r-2)!} \right\|_s,$$

and reapply (7.2) with $Tu$ in place of $u$ but with $r$ decreased by two. Thus we gradually increase the number of $T$ vector fields, with $T^\sigma$ being balanced by $\frac{1}{\sigma!!}$ before the norm, where

$$\sigma!! = \sigma(\sigma-2)(\sigma-4) \ldots,$$

preserving the balance between remaining powers of $X$ and the large factorial in the denominator, and using up two $X$’s for each $T$ until
there are essentially only powers of $T$, a situation we have treated above. (Of course there will be a ‘zig-zag’ effect - sometimes pairs of $X$’s will generate a $T$ and other times the $X$’s will differentiate the coefficients and produce the terms at the end of (7.2) above, so both effects will be combined.)

And as with the estimates of pure $T$ derivatives above, iterating the ‘principal’ term (here the last one - the one with $X^2X^r-r'u$) will lead to a sum with the same bounds for the number of its terms (cf. (4.16)), and with one new norm of derivatives of a coefficient $a_u$. Even when $Ψ$ has not been differentiated, it will be prudent to change to a new localizing function, one better geared to the number of derivatives appearing under the norm. For there are fewer derivatives now, and it would create significant difficulties to have $Ψ'$ contribute a factor of $r$ when the denominator contains only $(r-r')!$ for rather general $r'$.

8. The Localizing Functions

The first localizing function, $Ψ = Ψ_r$, satisfies:

\[(8.1) \quad Ψ_r \equiv 1 \text{ on } U_0, \quad Ψ_r \in C_0^\infty(U_{1/r}), \quad |Ψ_r^{(k)}| \leq ckr^k, \quad k \leq p(s),\]

(cf. (8.3)), where we have set, for $a \geq 0$:

\[(8.2) \quad U_a = \{(x, t) \in U_1 : \text{dist}((x, t), U_0) < a(\text{dist}(U_0, U_1^c))\}.\]

When the first localizing function needs to be replaced but, say, $\tilde{r}$ derivatives of $u$ remain to be estimated, we shall localize it with a function identically equal to one on $U_{1/r}$, the support of $Ψ$, but dropping to zero in a band of width $\frac{1}{r} \times (1 - \frac{1}{r})(\text{dist}(U_0, U_1^c)) = \frac{1}{r}$ times the remaining distance to the complement of $U_1$, i.e., supported in

\[(8.3) \quad U_{\frac{1}{r}+\frac{1}{r}}(1-\frac{1}{r}) = U_{\frac{1}{r}+(\frac{1}{r}-1)} = U_{1-(1-\frac{1}{r})(1-\frac{1}{r})}.\]

We shall denote such a function by $\rho \Psi_{\sigma}$. That is, $\rho \Psi_{\sigma}$ satisfies:

\[(8.4) \quad \rho \Psi_{\sigma} \equiv 1 \text{ on } U_{\rho}, \quad \rho \Psi_{\sigma} \in C_0^\infty(U_{\rho+\frac{1}{\rho}(1-\rho)} \subseteq U_1).\]

Derivatives of $\rho \Psi_{\sigma}$ satisfy, with universal constant $C$:

\[(8.5) \quad |D^k (\rho \Psi_{\sigma})| \leq C^k \left(\frac{\sigma}{1-\rho}\right)^k, \quad k \leq p(s).\]
uniformly in $\rho, \sigma$, where $p(s)$ will be a small number depending on the $s$ necessary to make $H^s$ an algebra in the given dimension. Of course any other (fixed) bound for $k$ would do.

9. Taking a localizing function out of the norm

While it is true that we could just write $\|\Psi w\|_s \leq c\|\Psi\|_s\|w\|_s$, for $s > 1$, to do so would incur at least two derivatives on $\Psi$ with no gain on $w$. To avoid this difficulty, we use the following finer estimates of the $H^s$ norm of product of functions.

**Proposition 9.1.** If $\Psi, \tilde{\Psi}$ are two smooth, compactly supported functions with $\tilde{\Psi} \equiv 1$ on $\text{supp} \Psi$ then for every $s \leq p \in \mathbb{Z}^+$,

\[
\|\Psi D^p u\|_s \leq C^2_{s, \text{supp} \Psi} \sup_{q \leq s} \|D^q \Psi\|_{L^\infty} \|\tilde{\Psi} D^{p-q} u\|_s
\]

and

\[
\|\Psi D^p u\|_s \leq C^2_{s, \text{supp} \Psi} \sup_{q \leq s} \|D^q \Psi\|_{L^\infty} \|D^{p-q} u\|_{H^s(\text{supp} \Psi)}.
\]

Thus removing a localizing function from an $H^s$ norms, while incurring up to 2 derivatives on it, does not increase the total number of derivatives being measured, and thus should have minimal impact on the estimates.

Next, we need to confront the effect of these few derivatives on a localizing function $\Psi$, which may have been chosen with a high number of derivatives ($r$ of them) on $u$ in mind, and hence which adds a factor of $r$ each time a derivative lands on it, when the factorial in the denominator of the corresponding term may be far smaller, e.g., $(r - r')!$ for relatively large $r'$.

There are in fact several ways to handle this; one is to emphasize that at the level of (4.7) one could endeavor to keep two derivatives to the left of the big bracket whenever possible so that using Proposition 9.1 those derivatives would serve to bring the Sobolev norms on the right up to $H^s$ again, or one can proceed as follows, the method used in the second author’s earlier work [13]: since the number of terms in the product in (7.1) is $r' - r''$, with

\[
r = (r - r') + \sum_{j=1}^{r' - r''} (r_j'' + 1), \text{ with } r - r' \geq \max \{r_j'' + 1\}
\]
and so

\[ (r - r')(r' - r'' + 1) \geq r \]  \hspace{1cm} \text{(9.4)}

or

\[ \left( \frac{r}{r - r'} \right)^k \leq (r' - r'' + 1)^k, \forall k \]  \hspace{1cm} \text{(9.5)}

a relationship we will use only for small values of \( k \) but note that this factor, \((r' - r'' + 1)^k\), can be absorbed in the bound of derivatives of the coefficients \( a_u \) in (7.1).

The first time we remove a localizing function from an \( H^s \) norm, in (7.2) for instance, the couple of derivatives that will fall on \( \Psi \) will produce powers of \( r \) in view of (8.5), since initially \( \rho = 0 \). These will be balanced against \((r-1)!\) thanks to (9.5) with small powers of \((r' - r'' + 1)\), increasing \( C_h \) slightly in (7.2). We will see at the very end that the slightly different denominators in (8.5) will make little difference in the bounds.

Furthermore, upon the next iteration of (7.2), the new right hand side will have the same form. That is, there will again be a product of lower order terms (the same ones plus new ones), a second copy of \( a_u \) with derivatives which will give possibly another constant, \( C \) in front of the supremum and another copy of \( C_h \) to its appropriate power, though these constants pass into the norms of the corresponding terms, just as in the treatment of powers of \( T \) above. But notice that the number of terms in the product increases at each pass (to at most \( r \)) and the the order of the top order term decreases. Thus this sequence of constants will not contribute in the end more than \( C^r \), which is also to be expected.

Handling the sum is as before as well, and we will not comment on it further except to recall (4.16).

When the last term on the right no longer has maximal order, we turn our attention to any of the other terms of highest order and proceed as before. The factorials have been adjusted so that the behavior that will in the end guarantee analyticity is that

\[
\frac{||\Psi X^2 X^r u||_{H^s}}{(r-1)!} \leq C^{r+1} + C^{r/2} \frac{||X^2 T^{r/2} u||_{H^s(U_1)}}{(r/2)!}
\]
which will be the evident outcome of the repeated iterations of (7.2) taking the precise localizing functions into account and which, together with the previous results on (nearly) pure $T$ derivatives will complete the proof.

We should remark at the end that what was true for the first localizing function, namely (9.5), will be a little different on the next pass, since the next localizing function may bring not a factor of $r - r'$ with each derivative it receives but rather the factor (cf. (8.5))

$$\frac{r - r'}{1 - \frac{1}{r}} = (r - r') \left(\frac{r}{r - 1}\right)$$

so that, passing from $r - r'$ to $r - r' - t'$ we encounter instead of just

$$\frac{r}{r - r'} \leq r' - r'' + 1$$

an extra factor of $r/r - 1$, possibly to the $p(s)$-th power; and this may keep occurring as the order of the leading term keeps decreasing. For instance, after a few iterations, the analogous ‘extra’ factors from (8.5) will be

$$\left(\frac{r}{r - 1}\right) \left(\frac{r - r_1}{r - r_1 - 1}\right) \left(\frac{r - r_1 - r_2}{r - r_1 - r_2 - 1}\right) \ldots$$

or even the $p(s)$-th power of such a product. But there cannot be more than $r$ terms in the product and each factor is far less than 2, leading to an easily acceptable constant $C^r$ in the end.

The same procedure works at any stage. We have already seen that expanding the term of highest order leads to a new product, but of the same form with one new norm of derivatives of a coefficient $a_u$, and the total number of terms, as with the $T$ derivatives, never exceeds $4^r$, which is certainly acceptable; and the factor $(r' - r'' + 1)^k$ just above is immediately attached to the $a_u^{(r' - r'')}$ which occurs with that product (cf. (6.6)).

This means that we may remove localizing functions from the $H^s$ norms easily and replace them with new localizing functions, identically one on the support of the old one and supported in a larger open set such that a derivative of the new function is proportional to the number of derivatives still to be estimated in that term in the sense of (8.3). And while there will appear a number of copies of the (analytic) coefficients $a$, the sum of the number of derivatives they receive, and the powers
of the corresponding constants arising from the algebraicity of $H^s$, is equal to the total decrease in derivatives on the terms of highest order taken step by step, which is also reflected in the number of norms in the product. Thus the total number of derivatives appearing on coefficients will be, in the end, equal to the total number of terms in the product of norms - and since each contains a copy of $u$ with one or two derivatives, this number is comparable to $r$. Thus this product will be bounded by $C_{u,h}^r$ for suitable $C_{u,h}$ depending only on the first couple of derivatives of $u$ and on the coefficients $a_u$ (and the dimension and the initial open sets).

We are not quite home. For high $X$ derivatives, in addition to being ‘used up’ as above, will also flow to half as many $T$ derivatives, though in $\mathcal{U}_1$, due to the bracketting $[X, X] = T$ (cf. (6.4)), and the number of terms and the sums proceed exactly as in the estimation of $T$ derivatives above, in ways that have nothing to do with the local versus global behavior. There appear new norms of derivatives of the coefficient functions, exactly as before, and one slightly new feature which is the mixture between $X$ and $T$ derivatives which is inevitable but has been seen before in many of the authors’ earlier works.

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