Black Hole Graybody Factor and Black Hole Entropy

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Abstract

We have proposed the entropy formula of the black hole which is constructed by two intersecting D-branes with no momentum, whose compactification radii are constrained by the surface gravities in ten-dimensions. We interpret the entropy of the black hole as the statistical entropy of the effective string. We further study the behavior of the absorption cross-section of the black hole using our entropy formula.

04.70.Dy, 04.20.Gz, 04.50.+h
I. INTRODUCTION

The Bekenstein-Hawking entropies of the BPS saturated black holes have been studied in various dimensions from the nonperturbative aspects of string theory [1–3]. The black holes are constructed by the intersecting D-branes [2,4]. The entropies of the extremal or near extremal black holes are interpreted as the statistical entropies in terms of the levels of the microscopic states.

For example, the entropy of the black hole constructed by the intersecting D1-brane and D5-brane in ten-dimensions is interpreted as the statistical entropy which counts the degeneracy of the BPS saturated D-brane bound states. In this calculation, the level of the states is identified with the momentum [3]. The entropy formula in the microscopic D-brane picture is obtained for the large level of the states. Therefore, this formula is correct for the large momentum.

At the same time, the entropy of the black hole constructed by the intersecting D1-brane and D5-brane is also explained as that of the effective string living on the D5-brane [8,9]. The gas of strings has a total mass, some momentum and a winding number along the compactified direction, which are expressed in terms of the charges of the D-branes. The level of states is derived from the mass spectrum with the momentum and the winding number. Then, the level of states is finite even with no momentum. The entropy formula in terms of the microscopic states is obtained for the large level of string states. Therefore, this formula is correct for the large level of states even with no momentum.

In addition, the graybody factors for the black holes have been studied in various dimensions [3–14]. These are calculated by using the wave functions of the Klein-Gordon equation. These are proportional to the entropies of the black holes. In [8,9], the graybody factor of the effective string living on the D5-brane is shown to be the same as that calculated by using the wave functions of the Klein-Gordon equation in four- and five-dimensions.

On the other hand, the Euler numbers of the black holes have been investigated in four- and ten-dimensions [15–18]. We have calculated the Euler numbers of the black holes constructed by the intersecting D-branes in ten-dimensions [18]. These black holes have the compactification radii. In this calculation, we have proposed that the compactification radii are constrained by the surface gravities in ten-dimensions. It is necessary to avoid the singular effects of the horizon in the compactified directions. We have found that the compactification radii are the inverses of the surface gravity in the compactified directions using the Gauss-Bonnet theorem. Using this determination, we have obtained the integer valued Euler numbers which are calculated by using the ten-dimensional Gauss-Bonnet action. The Euler number is the sum of the Betti numbers. The Betti numbers are integers. Therefore, the Euler numbers must be also integers because of their definitions. We emphasize that for arbitrary compactification radii, the Euler numbers are not integers.

Moreover, we have calculated semiclassically the entropies of the black holes which are constructed by two intersecting D-branes with no momentum. The black holes have the compactification radii which are constrained by the surface gravities [15]. We have obtained that the entropies of the black holes are proportional to the product of the quantized charges,

$$S = \pi Q_1 Q_2 ,$$

where $Q_1, Q_2$ are the quantized charges of two D-branes. We have shown that the entropies are invariant under the T-duality transformation. In the BPS limit, the entropies of the
black holes constructed by two intersecting D-branes are finite. Therefore, the black holes with two charges are more important than the others, and it is necessary to study the behaviors of the black holes with two charges. However, the interpretation of the entropies (1) as the statistical entropies in term of the microscopic states has been still unclear.

The purpose of this paper is to interpret the entropy (1) of the black hole which is constructed by two intersecting D-branes with no momentum, as the statistical entropy in term of the microscopic states. The black hole has the compactified radii which are constrained by the surface gravities in ten-dimensions [18]. In order to explain the entropy of the black hole with no momentum as the statistical entropy, we consider the effective string as mentioned above [8]. The gas of strings has a total mass, some momentum, and a winding number along the compactified direction, which are expressed in terms of the charges of the D-branes, and the Newton’s constant. We interpret the entropy of the black hole as that of the effective string. The level of the states is derived from the mass spectrum of the effective string obtained by the Virasoro constraints. We consider the black hole in ten-dimensions, which has six compactified directions. The level of states is written in term of the charges, the BPS parameter, and the four-dimensional Newton’s constant $G_4$. The four-dimensional Newton’s constant depends on six compactification radii. We have proposed that the compactification radii are constrained by the surface gravities [18]. The entropy of the gas of strings is written in term of the levels of the states. We rewrite the entropy of the effective string with the compactification radii which are constrained by the surface gravities.

Consequently, the entropy of the black hole which is constructed by two intersecting D3-branes with no momentum, and has the compactification radii which are constrained by the surface gravities is shown to be the same as the statistical entropy of the effective string, written in term of the levels of the effective string states. In the BPS limit, the entropy of the black hole is finite. We emphasize that the entropy of the black hole which is constructed by two intersecting D-branes with no momentum vanishes in the BPS limit for arbitrary compactification radii.

We further study the graybody factor for the black hole with two large charges in four-dimensions. The graybody factor is proportional to the entropy of the black hole. We have shown that the entropies of the black holes constructed by the intersecting D-branes are invariant under T-duality transformation [15]. Therefore, the graybody factors are also invariant under the T-duality transformation.

The organization of this paper is as follows. In section 2, we review the way to calculate the absorption cross-sections of the black holes with four charges in four-dimensions, using the wave functions of the Klein-Gordon equation. In section 3, we review the way to calculate the entropy and the absorption cross-section of the effective string. We find that the entropy of the effective string is the same as that of the black hole as calculated in section 2. In section 4, we review our proposal that the compactification radii are constrained by the surface gravities. We calculate the entropy of the black hole which is constructed by two intersecting D3-branes with no momentum. In section 5, we consider the entropy of the effective string with two large charges. We show that the entropy of the black hole which is calculated in section 4 can be interpreted as that of the effective string. We further discuss the graybody factor for the black hole.

3
II. BLACK HOLE GRAYBODY FACTOR

In this section, we review the way to obtain the graybody factor for the black hole using the wave functions which are the solutions of the Klein-Gordon wave equation. This calculation is discussed in [8–14].

We consider the non-extremal black hole in four-dimensions. The black hole is obtained by dimensional reduction from the ten-dimensional black hole. We consider the black hole in ten-dimensions which is constructed by the intersecting D1-brane and D5-brane with the Kaluza Klein monopole and the momentum. The black hole has two $U(1)$ charges $r_2, r_3$, and the momentum charge $r_1$, and the Kaluza-Klein charge $r_n$, and the BPS parameter $r_0$, where $r_0, r_1, r_n \ll r_2, r_3$. The metric of the four-dimensional black hole obtained by dimensional reduction from the ten-dimensional black hole is that

$$ds^2 = -f^{-1/2}h dt^2 + f^{1/2}(h^{-1} dr^2 + r^2 d\Omega^2),$$

where

$$h(r) = (1 - \frac{r_0}{r}),$$

$$f(r) = (1 + \frac{r_1}{r})(1 + \frac{r_2}{r})(1 + \frac{r_3}{r})(1 + \frac{r_n}{r}),$$

and

$$r_1 = r_0 \sinh^2 \sigma_1, \quad r_n = r_0 \sinh^2 \sigma_n.$$

The entropy of the black hole in the semiclassical calculation is

$$S = A/4G_4 = 4\pi \sqrt{r_2r_3r_0} \cosh \sigma_1 \cosh \sigma_n/4G_4.$$

We can also obtain the black holes which have four charges in ten- or eleven-dimensions. In ten-dimensions, they are constructed by four intersecting D3-branes, or by T-duality transformation from them. In eleven-dimensions, they are constructed by three intersecting M5-branes with the momentum, or four intersecting M2-branes. These are also constructed by two intersecting M5-branes with the Kaluza-Klein monopole and momentum, or M2-brane and M5-brane.

The $l$-th partial wave equation of a massless scalar is

$$\frac{h}{r^2} (hr^2 \frac{dR}{dr}) + \left[f\omega^2 - h\frac{l(l+1)}{r^2}\right]R = 0.$$

We consider the inner region (region I), $r \ll r_2, r_3$. We have the equation

$$z \frac{d}{dz} \left[ z \frac{dR_I}{dz} \right] + [D + \frac{C}{(1-z)} + \frac{E}{(1-z)^2}]R_I = 0,$$

where

$$z \equiv h(r),$$

$$D = \omega^2 r_2 r_3 \sinh^2 \sigma_1 \sinh^2 \sigma_n,$$

$$C = \omega^2 r_2 r_3 (\sinh^2 \sigma_1 + \sinh^2 \sigma_n) + l(l+1),$$

$$E = \omega^2 r_2 r_3 - l(l+1).$$
If we propose the solution of the equation as

\[ R_I = z^\alpha (1 - z)^\beta F(z) , \]  

(7)

then we obtain the relations as

\[ E + \beta (\beta - 1) = 0 , \quad \alpha^2 + D + C + E = 0 . \]  

(8)

We choose the solution as

\[ \alpha = -i\omega \sqrt{r_2 r_3} \cosh \sigma_1 \cosh \sigma_n , \]
\[ 2\beta = 1 - \sqrt{(2l + 1)^2 - 4\omega^2 r_2 r_3} . \]  

(9)

The equation for \( F \) is

\[ z(1 - z) \frac{d^2 F}{dz^2} + [(2\alpha + 1)(1 - z) - 2\beta z] \frac{dF}{dz} - [\alpha + \beta + iD, \alpha + \beta - iD; 1 + 2\alpha; z] F = 0 . \]  

(10)

The hypergeometric equation is

\[ z(1 - z) \frac{d^2 F}{dz^2} + [(1 + A + B)z - C] \frac{dF}{dz} - ABF = 0 , \]  

(11)

and we denote the solution of this equation as \( F(A, B, C; z) \). Therefore, we obtain the solution of (5) as

\[ R_I = z^\alpha (1 - z)^\beta F(\alpha + \beta + i\sqrt{D}, \alpha + \beta - i\sqrt{D}; 1 + 2\alpha; z) . \]  

(12)

Using the asymptotics of the hypergeometric functions for \( z \to 1 \), we find that

\[ R_I \to \left( \frac{r_0}{r} \right)^\beta \frac{\Gamma(1 + 2\alpha)\Gamma(1 - 2\beta)}{\Gamma(1 + \alpha - \beta - i\sqrt{D})\Gamma(1 + \alpha - \beta + i\sqrt{D})} , \]  

(13)

for large \( r \), and \( R_I \sim z^\alpha (1 - z)^\beta \) for small \( r \).

We next consider the outer region (region II) as \( r \gg r_0, r_1, r_n \). The wave equation is

\[ \rho^{-2} \frac{d}{d\rho} \rho^2 \frac{dR}{d\rho} + \left[ 1 + \omega(r_2 + r_3) \rho \right] + \frac{\omega^2(r_2 r_3) - l(l + 1)}{\rho^2} R = 0 , \]  

(14)

where \( \rho = \omega r \). The solution of this equation which matches the asymptotic form in region I is

\[ R_{II} = A\rho^{-\beta} e^{\pm i\rho} F(s + 1 \pm ni, 2s + 2; \mp 2i\rho) , \]
\[ n = -\frac{\omega(r_2 + r_3)}{2} , \quad -s(s + 1) = \omega^2(r_2 r_3) - l(l + 1) , \]  

(15)

where \( F \) is the confluent hypergeometric function. We obtain that \( R_{II} \sim \frac{A}{\rho} e^{\pm i\rho} \) for large \( \rho \) and \( l = 0 \), and \( R_{II} \sim A\rho^{-\beta} \) for small \( \rho \).

Matching \( R_{II} \) to \( R_I \) in the range \( r_0 \ll r \ll r_2, r_3 \), we find that
\[ A = (\omega r_0)^\beta \frac{\Gamma(1 + 2\alpha) \Gamma(1 - 2\beta)}{\Gamma(1 + \alpha - \beta - i\sqrt{D}) \Gamma(1 + \alpha - \beta + i\sqrt{D})}. \]  

The flux per unit solid angle is

\[ F = \frac{1}{2i} (R^* h r^2 \partial_r R) \]  

The absorption probability is the ratio of the incoming flux at the horizon to the incoming flux at infinity,

\[ P = \frac{\mathcal{F}_h}{\mathcal{F}_{\infty}^{\text{incoming}}} = 4\omega^2 \sqrt{r_2 r_3 r_0} \cosh \sigma_1 \cosh \sigma_n |A|^{-2}. \]  

The s-wave absorption cross-section is

\[ \sigma_{\text{abs}} = \frac{\pi}{\omega^2} P_{l=0} = 4\pi \sqrt{r_2 r_3 r_0} \cosh \sigma_2 \cosh \sigma_n |A|^{-2} \]

\[ = 4\pi \sqrt{r_2 r_3 r_0} \cosh \sigma_2 \cosh \sigma_n \frac{\omega}{2(T_L + T_R) \left( e^{\pi \sigma_n} - 1 \right) \left( e^{\pi \sigma_n} - 1 \right)}, \]  

where we define the left and right temperatures as

\[ T_R = \frac{1}{4\pi \sqrt{r_2 r_3} \cosh(\sigma_1 + \sigma_n)}, \quad T_L = \frac{1}{4\pi \sqrt{r_2 r_3} \cosh(\sigma_1 - \sigma_n)}, \]  

and the Hawking temperature is

\[ T_H = \frac{1}{4\pi \sqrt{r_2 r_3} \cosh \sigma_1 \cosh \sigma_n}, \]

\[ \frac{2}{T_H} = \frac{1}{T_L} + \frac{1}{T_L}. \]

For small \( \omega \), the absorption cross-section coincides the area of the horizon.

**III. EFFECTIVE STRING**

In the section, we review the way to calculate the graybody factor and the statistical entropy of the effective string living on the D5-brane. This calculation is discussed in [9]. We find that the entropy of the effective string is the same as that of the black hole as calculated semiclassically in the previous section.

We consider the effective string living on the D5-brane. The gas of strings has a total mass \( m \), and has some momentum number \( n_p \), and a winding number \( n_w \), along the compactified direction with the compactification radii \( R \). We explain the entropy of the black hole as that of the effective string, which has an effective tension \( T_{\text{eff}} = 1/(2\pi\alpha'_{\text{eff}}) \) and an effective central charge \( c_{\text{eff}} = N_B + \frac{1}{2} N_f \). \( N_B \) and \( N_f \) are the numbers of bosons and fermions. The
levels of states, $N_L$ and $N_R$, are calculated from the mass spectrum of the effective string obtained by the Virasoro constraints.

We first consider the neutral absorption probability of the effective string. We assume that the mass levels are given by

$$m^2 = \left(2\pi R n_w T_{\text{eff}} + \frac{n_p}{R}\right)^2 + 8\pi T_{\text{eff}} N_R = \left(2\pi R n_w T_{\text{eff}} - \frac{n_p}{R}\right)^2 + 8\pi T_{\text{eff}} N_L ,$$

which is obtained from Virasoro constraint. Let the incoming scalar have energy $\omega$. Since it does not carry any charge, the numbers $n_w, n_p$ are not altered by the absorption of the scalar. Thus we have

$$2m\delta m = 8\pi T_{\text{eff}} \delta N_R = 8\pi T_{\text{eff}} \delta N_L .$$

The scalar contributes one left oscillator $\alpha_i$ and one right oscillator $\tilde{\alpha}_j$, with

$$i = j = \delta N_R = \delta N_L = \frac{m}{4\pi T_{\text{eff}}} \omega .$$

We average the absorption rate over all initial string states of a given mass. We find that the absorption cross-section, which is the absorption rate minus the emission rate, is

$$\sigma_{\text{abs}} = G_4 \delta N_L \frac{e^{\beta^*_L \delta N_L} + e^{\beta^*_R \delta N_R} - 1}{(e^{\beta^*_L \delta N_L} - 1)(e^{\beta^*_R \delta N_R} - 1)} ,$$

where

$$\beta^*_{R,L} = \frac{\partial S}{\partial N_{R,L}} = \frac{\pi}{\sqrt{N_{R,L}}} ,$$

and $G_4$ is the four-dimensional Newton’s constant. $S$ is the entropy of the effective string as

$$S = 2\pi(\sqrt{N_L} + \sqrt{N_R}) ,$$

with $c = 6$. For the right movers, the factor in the exponent is

$$\beta^*_R \delta N_R = \frac{m\omega}{4T_{\text{eff}}\sqrt{N_R}} .$$

We identify this with $\frac{\omega}{2T_R}$, then

$$T_R = 2T_{\text{eff}} \frac{\sqrt{N_R}}{m} .$$

We will identify the effective string tension as the D-string tension, as

$$T_{\text{eff}} = \frac{1}{8\pi\alpha' r_3} ,$$

where we have set $\alpha' = 1$. We obtain that
\[ T_R = \frac{1}{4\pi \sqrt{r_2 r_3} \cosh(\sigma_1 + \sigma_n)} , \]
\[ T_L = \frac{1}{4\pi \sqrt{r_2 r_3} \cosh(\sigma_1 - \sigma_n)} , \]
\[ \text{by using} \]
\[ m = \frac{r_0}{8G_4} (\cosh(2\sigma_1) + \cosh(2\sigma_n)) , \]
\[ 2\pi R n_{\omega} T_{\text{eff}} = \frac{r_0}{8G_4} \sinh(2\sigma_1) , \]
\[ \frac{n_p}{R} = \frac{r_0}{8G_4} \sinh(2\sigma_n) . \]
\[ (31) \]
These are in agreement with the temperatures (20) as discussed in previous section. The cross-section (25) is in complete agreement with the result (19). The entropy is
\[ S = 2\pi \sqrt{\frac{c_{\text{eff}}}{6}} (\sqrt{N_R} + \sqrt{N_L}) = 4\pi \sqrt{r_2 r_3} r_0 \cosh \sigma_1 \cosh \sigma_n / 4G_4 , \]
with \( c_{\text{eff}} = 6 \), which is in agreement with the entropy calculated semiclassically in the previous section.

We next consider the absorption probability of the charged particles. It was found that the wave equation for a scalar of energy \( k_0 \) and charge \( k_5 \) is obtained from that for a neutral scalar of energy \( \omega \) by the following replacement of parameters:
\[ \omega \rightarrow \omega', \quad \sigma_n \rightarrow \sigma'_n, \]
\[ (34) \]
where
\[ \omega' = \sqrt{k_0^2 - k_5^2} , \quad e^{\sigma'} = e^{\sigma} \frac{k_0 - k_5}{\omega'} . \]
\[ (35) \]
The incoming scalar carries the \( U(1) \) charge corresponding to the momentum component \( k_5 \), which is along the direction in which the effective string carries the momentum and the winding number. The variation \( \delta n_p \), whose relation to the absorbed charge is \( k_5 = \frac{\delta n_p}{R} \), while \( k_0 = \delta m \). Thus, we find that
\[ 2m k_0 = 8\pi T_{\text{eff}} \delta N_R + 2k_5 \left( 2\pi R n_{\omega} T_{\text{eff}} + \frac{n_p}{R} \right) , \]
\[ 2m k_0 = 8\pi T_{\text{eff}} \delta N_L - 2k_5 \left( 2\pi R n_{\omega} T_{\text{eff}} - \frac{n_p}{R} \right) . \]
\[ (36) \]
Using this, we obtain that
\[ \beta^*_L \delta N_L = 2\pi \sqrt{r_2 r_3} \left( k_0 \cosh(\sigma_1 - \sigma_n) + k_5 \sinh(\sigma_1 - \sigma_n) \right) = \frac{\omega'}{2T'_L} , \]
\[ \beta^*_R \delta N_R = 2\pi \sqrt{r_2 r_3} \left( k_0 \cosh(\sigma_1 + \sigma_n) - k_5 \sinh(\sigma_1 + \sigma_n) \right) = \frac{\omega'}{2T'_R} , \]
\[ (37) \]
where
\[
\frac{1}{T_L} = 4\pi \sqrt{r_2 r_3 \cosh(\sigma_1 - \sigma'_n)} , \quad \frac{1}{T_R} = 4\pi \sqrt{r_2 r_3 \cosh(\sigma_1 + \sigma'_n)} , \quad \tag{38}
\]

\[
\frac{1}{T_H} = \frac{1}{2T_L} + \frac{1}{2T_R} .
\]

We then find that the effective string absorption cross-section for charged scalars can be written as

\[
\sigma_{\text{abs}} = 4\pi \sqrt{r_2 r_3 r_0} \cosh \sigma_1 \cosh \sigma'_n \frac{\omega'}{2(T_L + T_R)} \left( \frac{e^{\sigma'_n} - 1}{e^{\sigma'_L} - 1} \right) . \quad \tag{39}
\]

The result is in agreement with that calculated by using the wave functions of the Klein-Gordon wave equation.

**IV. SURFACE GRAVITY AND COMPACTIFICATION**

In this section, we review the way to define the compactification radii by the surface gravity, as discussed in [18]. We further calculate semiclassically the entropy of the black hole constructed by two intersecting D-branes with no momentum.

We have proposed the determination of radii in the compactified directions by the surface gravity in ten-dimensions. It is necessary to avoid the singular effects of the horizon in the compactified directions. We recall the way to define the period in the time direction. If no singular effects of the horizon exist in \(t - r\) directions, then the topology of these directions is \(\mathbb{R}^2\), and the Euler number of these directions is 1. Using the Gauss-Bonnet theorem, we have found that the Euclidean time coordinate have the period which is the inverse of the surface gravity. Therefore we need to take this period in the time direction to avoid the singular effects of the horizon. Similarly, we need to fix the compactification radii to avoid the singular effects of the horizon in the compactified directions. We have found that the compactification radii are the inverses of the surface gravity in the compactified directions using the Gauss-Bonnet theorem. Using the determination of radii, we have obtained the integer valued Euler numbers which are calculated by using the ten-dimensional Gauss-Bonnet action. The Euler number is the sum of the Betti numbers. The Betti numbers are integers. Therefore, the Euler numbers must be also integers because of their definition. We emphasize that for arbitrary compactification radii, the Euler numbers are not integers.

We have proposed that the compactification radius \(\beta_i\) in the \(i\)-th direction in ten-dimensions is constrained by the following "surface gravity \(\kappa_i\)"

\[
\beta_i(r_H) = \frac{2\pi}{\kappa_i} , \quad \kappa_i(r_H) \equiv \left. \frac{1}{2} \frac{\partial_r g_{ii}}{\sqrt{g_{rr} g_{ii}}} \right|_{r=r_H} , \quad \tag{40}
\]

\[
(i = 1 \cdots 6) .
\]

We define the proper length in the \(i\)-th direction as:

\[
L_i(r) \equiv \left| \int_0^{\beta_i(r)} \sqrt{g_{ii}} dx_i \right| = 4\pi \left| \frac{\sqrt{g_{rr}}}{\partial_r (\ln g_{ii})} \right| , \quad \tag{41}
\]

\[
(i = 1, \cdots , 6) .
\]
We consider the black hole constructed by the two intersecting D3-branes in ten-dimensions with the following metric,

\[
ds_{10}^2 = (H_1 H_2)^{1/2} [H_1^{-1} H_2^{-1} (-\hbar dt^2 + dx_1^2) + dx_2^2 + H_1^{-1} (dx_3^2 + dx_4^2) + H_2^{-1} (dx_5^2 + dx_6^2) + h^{-1} dr^2 + r^2 d\Omega_4^2],
\]

where \(r_0 \ll r_i\). The four-dimensional black hole with two large charges is obtained by dimensional reduction from the ten-dimensional black hole. These proper lengths are \(L_3 = L_4 = L_5 = L_6 = 8\pi (r + r_1)^{5/4}(r + r_2)^{5/4}/(r_1 - r_2)(r - r_0)^{1/2}\), \(L_1 = L_2 = 8\pi (r + r_1)^{5/4}(r + r_2)^{5/4}r/(r - r_0)^{1/2}[r_1(r + r_2) + r_2(r + r_1)]\).

The area in the \(\theta - \phi\) directions is

\[
A_{\theta\phi} = \int_0^\pi d\theta \int_0^{2\pi} d\phi \sqrt{g_{\theta\theta} g_{\phi\phi}} = 4\pi (r + r_1)^{1/2}(r + r_2)^{1/2}r.
\]

Then we find that the entropy of two intersecting D-branes is given by

\[
S = A_8/4G_{10} \bigg|_{r=r_0} = L_1 L_2 L_3 L_4 L_5 L_6 A_{\theta\phi}/4G_{10} \bigg|_{r=r_0} = (8\pi)^6 \pi (r + r_1)^8(r + r_2)^8r^3/G_{10}(r - r_0)^3\left[ (r_1 - r_2)^4[2r_1 r_2]^2 \right]_{r=r_0}.
\]

where \(G_{10}\) is the ten-dimensional Newton’s constant. We consider the entropy for the extremal black holes, namely, \(h = 1\) and \(r_0 = 0\). We first rewrite \((r - r_0)^3 \rightarrow r^3\) in the equation (43), and we next insert \(r = r_0 = 0\). We obtain the finite entropy for the extremal black holes as

\[
S = 4(4\pi)^7 r_1^6 r_2^6/(r_1 - r_2)^4,
\]

with \(G_{10} = 1\). We define the quantized charges of \(Dp\)-branes as

\[
Q_i = r_i/c_p,
\]

where \(c_p\) is

\[
c_p = G_4 M_p |_{r\rightarrow 0}, \quad M_p = \beta_{i_1} \cdots \beta_{i_p},
\]

\[
G_4 = G_{10}/(L_1 L_2 L_3 L_4 L_5 L_6).
\]

\(i_1, \cdots i_p\) are the directions which the D-brane wrapping on. \(M_p\) is the bare mass of \(Dp\)-brane. Then the length in the definition of the mass (15) is not the proper length \(L_i\), but the bare length \(\beta_i\). On the other hand, \(G_4\) is given by \(1/G_4 = 1/\int_{\vec{L}} \int_0^{\beta_1} \cdots \int_0^{66} dx_1 \cdots dx_6 \sqrt{g_{11} \cdots g_{66}}\), which is obtained by the dimensional reduction of the action. Then the quantized charges \(Q_1\) and \(Q_2\) for D3-branes are
\[ Q_1 = L_1 L_2 L_3 L_4 L_5 L_6 / \beta_1 \beta_3 \beta_4 r_1 |_{r=0} = (8\pi)^2 4\pi (r_1 r_2)^3 / (r_1 - r_2)^2, \]
\[ Q_2 = L_1 L_2 L_3 L_4 L_5 L_6 / \beta_1 \beta_5 \beta_6 r_2 |_{r=0} = (8\pi)^2 4\pi (r_1 r_2)^3 / (r_1 - r_2)^2, \]

with \( G_{10} = 1 \). Using these charges, we rewritten the entropy (50)
\[ S = \pi Q_1 Q_2. \]

We also obtain the finite quantized charges for two D4-branes and for the two D2-branes using the definition (47). For two D4-branes, the metric and the quantized charges are
\[ ds_{10}^2 = (H_1 H_2)^{1/2} [H_1^{-1} H_2^{-1} (-dt^2 + dx_1^2 + dx_2^2) \\
+ H_1^{-1}(dx_3^2 + dx_4^2) + H_2^{-1}(dx_5^2 + dx_6^2) + dr^2 + r^2 d\Omega_2^2], \]
\[ H_i = 1 + \frac{r_i}{r}, \quad (i = 1, 2) \]
\[ Q_1 = (8\pi)^2 (r_1 r_2)^{5/2} / (r_1 - r_2)^2, \]
\[ Q_2 = (8\pi)^2 (r_1 r_2)^{5/2} / (r_1 - r_2)^2, \]
and the entropy is rewritten as
\[ S = \pi Q_1 Q_2 / T^2, \]
where \( T \) is the temperature as \( T = \frac{1}{4\pi \sqrt{r_1 r_2}} \). For two D2-branes, the metric and the quantized charges are
\[ ds_{10}^2 = (H_1 H_2)^{1/2} [H_1^{-1} H_2^{-1} (-dt^2 + dx_1^2 + dx_2^2) \\
+ H_1^{-1}(dx_3^2 + dx_4^2) + H_2^{-1}(dx_5^2 + dx_6^2) + dr^2 + r^2 d\Omega_2^2], \]
\[ H_i = 1 + \frac{r_i}{r}, \quad (i = 1, 2) \]
\[ Q_1 = (8\pi)^2 (4\pi)^2 (r_1 r_2)^{7/2} / (r_1 - r_2)^2, \]
\[ Q_2 = (8\pi)^2 (4\pi)^2 (r_1 r_2)^{7/2} / (r_1 - r_2)^2, \]
and the entropy is rewritten as
\[ S = \pi Q_1 Q_2 / T^2. \]

In three cases, the charges \( Q_1, Q_2 \) are constrained to be same, respectively. The proper lengths generally satisfy the conditions as \( L_1 = L_2 \), and \( L_3 = L_4 = L_5 = L_6 \), if the entropy is invariant under the T-duality transformation. Therefore the ratios of charges for these branes are
\[ \frac{Q_1}{Q_2} = \frac{r_1}{r_2} \sqrt{g_{33} g_{44}} |_{r=0} = 1, \]
using the definition (15). Then this constraint is common for the quantized charges of these branes. Therefore we obtain the new constraint for the quantized charges.
V. BLACK HOLE ENTROPY WITH TWO LARGE CHARGES

In this section, we show that the entropy of the black hole which is constructed by two intersecting D-branes with no momentum, is the same as the statistical entropy of the effective string as discussed in section 3. The black hole has the compactification radii which are constrained by the surface gravities.

We consider the black hole constructed by two D3-branes. These have the finite values of quantized charges in this section. In section 3, we consider the black hole constructed by the D1-brane and the D5-brane. However, the quantized charges for the branes are not finite using the compactification radii as discussed in section 4. Further, we consider the effective strings for the black hole constructed by two D3-branes. These are living on the intersecting direction of two D-branes. The result of calculations for the black hole is same as the discussed in section 3.

The entropy of the effective strings is written in term of the levels of states. The levels are derived from the mass spectrum, which are written by the two charges \( r_2, r_3 \), the BPS parameter \( r_0 \), and the four-dimensional Newton’s constant \( G_4 \), as discussed in section 3. The four-dimensional Newton’s constant depends on the compactification radii. We have proposed that the compactification radii are constrained by the surface gravities \[18\]. We rewrite the entropy of the effective string with the compactification radii which are constrained by the surface gravities. The entropy is shown to be the same with that of the black hole as calculated semiclassically in previous section.

In addition, we calculate the graybody factor for the black hole. The graybody factor is proportional to the entropy of the black hole. We have shown that the entropies of the black holes constructed by the intersecting D-branes are invariant under the T-duality transformation \[18\]. Therefore, the graybody factors for the black holes are also invariant under the T-duality transformation.

We consider the black hole with two large charges, no Kaluza-Klein charge and no momentum, which is reviewed in section 2, with \( r_1, r_n = 0 \) (\( \sigma_1, \sigma_n = 0 \)). The temperatures are

\[
T_L = T_R = \frac{1}{4\pi \sqrt{r_2 r_3}} .
\] (58)

The levels of states are

\[
\sqrt{cN_R/6} = \sqrt{cN_L/6} = \left( \frac{1}{8\pi T_{\text{eff}}} \right)^{1/2} m = \sqrt{r_2 r_3} r_0 / 8G_4 ,
\] (59)

from \[32\] with \( \sigma_1, \sigma_n \rightarrow 0 \). Therefore, the entropy of the effective string is shown to be the same as the entropy of the black hole as calculated semiclassically in four-dimensions, namely

\[
S = 2\pi (\sqrt{cN_R/6} + \sqrt{cN_L/6}) = 4\pi \sqrt{r_2 r_3} r_0 / 4G_4 ,
\] (60)

with \( c = 6 \). Using the proper lengths \( L_i \), where \( i = 1, \cdots 6 \), the four-dimensional Newton’s constant is written as

\[
G_4 = G_{10}/L_1 L_2 L_3 L_4 L_5 L_6 .
\] (61)
We consider the entropy of the extremal black hole. In the calculation of the proper length, we consider the black hole constructed by the two intersecting D3-branes in ten-dimensions as the following metric,

\[
\begin{align*}
    ds_{10}^2 &= (H_1 H_2)^{1/2} [H_1^{-1} H_2^{-1} (-dt^2 + dx_1^2) + dx_2^2]
    + H_1^{-1} (dx_3^2 + dx_4^2) + H_2^{-1} (dx_5^2 + dx_6^2) + dr^2 + r^2 d\Omega_4^2,
    \\
    H_i &= 1 + \frac{r_i}{r} \quad (i = 1, 2)
\end{align*}
\] (62)

as discussed in the previous section with \( r_0 = 0 \). The four-dimensional black hole with two large charges is obtained by dimensional reduction from the ten-dimensional black hole. We introduce the stretched horizon \( r = a \) in order to obtain the finite state levels. We suppose that the event horizon and the mass are varied because of the quantum correction of the gravity. Then the mass for the effective string \( m \) is written as

\[
    m = \frac{r_2 + r_3 + \epsilon}{8G_4} - m_{\text{bare}},
\] (63)

where \( m_{\text{bare}} \) is the bare mass, and the the term \( \epsilon \) is the quantum effect for the bare mass. Then, the levels of the states are

\[
    \sqrt{N_L} = \sqrt{N_R} = \left( \frac{1}{8\pi T_{\text{eff}}} \right)^{1/2} m
    = 2\sqrt{r_1 r_2} \epsilon |_{8G_4/r \to a}
    = 2\sqrt{r_1 r_2} e L_1 L_2 L_3 L_4 L_5 L_6 / 8G_10 \bigg|_{r \to a}
    = 2(8\pi)^6 \epsilon \sqrt{r_1 r_2} (r + r_1)^{15/2} (r + r_2)^{15/2}
    \bigg/ 8G_{10} r \left[ (r_1 - r_2)^4 [r(r_1 + r_2) + 2r_1 r_2]^2 \right] r \to a,
\] (64)

where the proper length \( L_i \) are the same as that in (43) with \( r_0 \to 0 \). In this calculation, we have used the fact that the compactification radii are constrained by the surface gravities. The entropy is rewritten as

\[
    S = 2\pi \sqrt{c N_R / 6 + c N_L / 6} = 4\pi \sqrt{c r_2 r_3 / 6 \epsilon / 4G_4} \bigg|_{r \to a}
    = (8\pi)^6 \pi \sqrt{c r_1 r_2 / 6 \epsilon (r + r_1)^{15/2} (r + r_2)^{15/2}}
    / G_{10} r \left[ (r_1 - r_2)^4 [r(r_1 + r_2) + 2r_1 r_2]^2 \right] r \to a.
\] (65)

Then we obtain the finite entropy which is the same as the entropy (46) with \( a \sim \epsilon \) and \( \epsilon \to 0 \). Therefore, the entropy of the black hole with two charges, no momentum, and with the compactification radii which are constrained by the surface gravities, is shown to be the same as the statistical entropy of the effective string, which is written in term of the levels of the effective string states. In the BPS limit, the entropy of the black hole which we have just
considered is finite. We emphasize that for arbitrary compactification radii, the entropies of the black holes with two charges and no momentum vanish in the BPS limit.

We next consider the absorption cross-section of the extremal black hole with two large charges. The absorption cross-section in this range is

$$\sigma_{\text{abs}} = \frac{\pi}{\omega^2} P_{l=0} = 4\pi \sqrt{r_3} \frac{\omega}{2(T_L + T_R)} \frac{\left(e^{\frac{\omega}{T_H}} - 1\right)}{\left(e^{\frac{\omega}{T_L}} - 1\right)} \frac{\left(e^{\frac{\omega}{T_R}} - 1\right)}{\left(e^{\frac{\omega}{T_H}} - 1\right)}$$

$$= 4\pi \sqrt{r_3} e G_4 \frac{\omega}{2(T_L + T_R)} \frac{\left(e^{\frac{\omega}{T_H}} - 1\right)}{\left(e^{\frac{\omega}{T_L}} - 1\right)} \frac{\left(e^{\frac{\omega}{T_R}} - 1\right)}{G_10 L_1 L_2 L_3 L_4 L_5 L_6}$$

$$= 4SG_4 \frac{\omega}{2(T_L + T_R)} \frac{\left(e^{\frac{\omega}{T_H}} - 1\right)}{\left(e^{\frac{\omega}{T_L}} - 1\right)} \frac{\left(e^{\frac{\omega}{T_R}} - 1\right)}{G_10}, \quad (66)$$

with $T_H = T_L = T_R = \frac{1}{4\pi \sqrt{r_3}}$. We have shown that the entropy of the black hole constructed by the intersecting D-branes is invariant under the T-duality transformation [18]. Therefore, the cross-section is also invariant under the T-duality transformation. In the extremal case, we obtain the cross-section for the black holes constructed by two D3-branes as

$$\sigma_{\text{abs}} = 4SG_4 \frac{\omega}{2(T_L + T_R)} \frac{\left(e^{\frac{\omega}{T_H}} - 1\right)}{\left(e^{\frac{\omega}{T_L}} - 1\right)} \frac{\left(e^{\frac{\omega}{T_R}} - 1\right)}{G_10}$$

$$= 4\pi Q_1 Q_2 G_4 \omega \frac{\left(e^{\frac{\omega}{T_H}} - 1\right)}{2(T_L + T_R)} \frac{\left(e^{\frac{\omega}{T_L}} - 1\right)}{\left(e^{\frac{\omega}{T_R}} - 1\right)} \frac{\left(e^{\frac{\omega}{T_H}} - 1\right)}{G_10}, \quad (67)$$

These are proportional to the products of the quantized charges.

VI. CONCLUSION

We have proposed the entropy formula of the black hole (1). The black hole is constructed by two intersecting D-branes with no momentum, and has the compactification radii which are constrained by the surface gravities in ten-dimensions. We have interpreted the entropy of the black hole as the statistical entropy of the effective string. The gas of the strings has a total mass, and has some momentum and a winding numbers along the compactified direction.

First, we have considered the black hole with four charges in ten-dimensions. The black hole is constructed by the intersecting D1-brane and D5-brane with momentum and the Kaluza-Klein monopole. We have reviewed the way to obtain the graybody factor using the wave functions of the Klein-Gordon equation. The graybody factor is shown to be proportional to the area of the black hole horizon.

Second, we have reviewed the way to calculate the graybody factor of the effective string living on the D5-brane [3]. The gas of strings has a total mass, some momentum, and a winding number along the compactified direction. We have explained the entropy using the effective string, which has an effective tension $T_{\text{eff}} = 1/(2\pi \alpha'_{\text{eff}})$ and an effective central
charge $c_{\text{eff}} = N_B + \frac{1}{2} N_f$. We have found that the graybody factor of the effective string is the same as that for the black hole by using the wave functions of the Klein-Gordon equation.

Next, we have considered the entropy of the black hole which is constructed by two intersecting D3-branes with no momentum and winding number. We have proposed that the black hole has the compactification radii which are constrained by the surface gravities. The entropy is proportional to the product of two quantized charges of D-branes. In this paper, we are able to show that the entropy of the black hole is the same as the statistical entropy of the effective string, written in term of the levels of the effective string states. In the BPS limit, the entropy of our black hole is finite. We emphasize that for arbitrary compactification radii, the entropy of the black hole which is constructed by two intersecting D-branes with no momentum vanishes in the BPS limit.

Further, we have studied the graybody factor with two large charges. The graybody factor is proportional to the entropy of the black hole. We have shown that the entropies of the black holes constructed by the intersecting D-branes are invariant under T-duality transformation $\text{[18]}$. Therefore, the graybody factors are also shown to be invariant under the T-duality transformation.

We have proposed that the compactification radii of the black holes are identified with the inverse of the surface gravities. Using this determination, we have correctly obtained the integer valued Euler numbers which are calculated by using the ten-dimensional Gauss-Bonnet action. Moreover, we have obtained the entropy formula of the black hole with no momentum. In this paper, we have further obtained the interpretation of the entropy of the black hole as the statistical entropy written in term of the level of states.

In the BPS limit, the entropies, the temperatures of the black holes are finite with two charges in our results. The temperatures of the black hole with four charges vanish in the BPS limit with charges fixed. However, the cross-sections and the entropies are infinite, then it is rarely that the black holes become these states. It is natural that they emit the charged particles and become the black holes with two charges. Therefore, we think that the black holes with two charges are very special, and we further need to study the behavior of the black holes with two charges.

ACKNOWLEDGMENTS

We thank Y. Kitazawa for discussions and for carefully reading the manuscript and suggesting various improvements.
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