EXAMPLES OF 3-QUASI-SASAKIAN MANIFOLDS

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Dedicated to Prof. Anna Maria Pastore on occasion of her 70th birthday

Abstract. We provide a general method to construct examples of quasi-Sasakian 3-structures on a \((4n+3)\)-dimensional manifold. Moreover, among this class, we give the first explicit example of a compact 3-quasi-Sasakian manifold which is not the global product of a 3-Sasakian manifold and a hyper-Kähler manifold.

1. Introduction

The class of quasi-Sasakian manifolds was introduced by Blair in [1], and then studied by several authors (e.g. [13], [12], [9]) in order to unify the most important classes of almost contact metric manifolds, namely the Sasakian and coKähler ones, which are quasi-Sasakian manifolds of maximal and minimal rank, respectively. Moreover any quasi-Sasakian manifold is canonically endowed with a transversely Kähler foliation, so that they can be thought as an odd-dimensional analogue of Kähler manifolds.

When on a smooth manifold \(M\) there are defined three distinct quasi-Sasakian structures, with the same compatible metric, which are related to each other by certain relations similar to the quaternionic identities, one says that \(M\) is a 3-quasi-Sasakian manifold (see Section 2 for the precise definition). The class of 3-quasi-Sasakian manifolds was extensively studied a few years ago in [4] and [5], where several properties on 3-quasi-Sasakian manifolds, which do not hold for a general quasi-Sasakian structure, were proved. In particular, it was proved that the aforementioned quaternionic-like structure forces any 3-quasi-Sasakian manifold of non-maximal rank \(4l+3\) to be the local Riemannian product of a 3-c-Sasakian manifold and a hyper-Kähler manifold. Therefore a natural question arises: are there examples of 3-quasi-Sasakian manifolds which are not the global product of a 3-c-Sasakian manifold and a hyper-Kähler manifolds? In this article we give an affirmative answer to this problem. We present a general procedure to produce a large class of examples, and we prove that the 11-dimensional 3-quasi-Sasakian manifold in this class is not a global product of 3-Sasakian and hyper-Kähler manifolds.

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All manifolds considered in this paper will be assumed to be smooth and connected. For wedge product, exterior derivative and interior product we use the conventions as in Goldberg’s book [5].

2. Preliminaries

We start with a few preliminaries on almost contact metric manifolds, referring the reader to the monographs [2], [3] and to the survey [7] for further details.

An almost contact metric structure on a \((2n + 1)\)-dimensional manifold \(M\) is the data of a \((1, 1)\)-tensor \(\phi\), a vector field \(\xi\), called Reeb vector field, a 1-form \(\eta\) and a Riemannian metric \(g\) satisfying

\[
\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\]

for all \(X, Y \in \Gamma(TM)\), where \(I\) denotes the identity mapping on \(TM\). From (2.1) it follows that \(g(X, \phi Y) = -g(\phi X, Y)\), so that we can define the 2-form \(\Phi\) on \(M\) by

\[
\Phi(X, Y) = g(X, \phi Y),
\]

which is called the fundamental 2-form of the almost contact metric manifold \((M, \phi, \xi, \eta, g)\).

The manifold is said to be normal if the tensor field \(N := [\phi, \phi]_{FN} + 2d\eta \otimes \xi\) vanishes identically. Normal almost contact metric manifolds such that both \(\eta\) and \(\Phi\) are closed are called coKähler manifolds and those such that \(d\eta = c\Phi\) are called c-Sasakian manifolds, where \(c\) is a non-zero real number (for \(c = 2\) one obtains the well-known Sasakian manifolds).

The notion of quasi-Sasakian structure was introduced by Blair in his Ph.D. thesis in order to unify those of Sasakian and coKähler structures. A quasi-Sasakian manifold is defined as a normal almost contact metric manifold whose fundamental 2-form is closed. A quasi-Sasakian manifold \(M\) is said to be of rank \(2p\) (for some \(p \leq n\)) if \((d\eta)^p \neq 0\) and \(\eta \wedge (d\eta)^p = 0\) on \(M\), and to be of rank \(2p + 1\) if \(\eta \wedge (d\eta)^p \neq 0\) and \((d\eta)^{p+1} = 0\) on \(M\) (cf. [1], [13]). Blair proved that there are no quasi-Sasakian manifolds of even rank. Just like Blair and Tanno implicitly did, we will only consider quasi-Sasakian manifolds of constant (odd) rank. If the rank of \(M\) is \(2p + 1\), then the module \(\Gamma(TM)\) of vector fields over \(M\) splits into two submodules as follows:

\[
\Gamma(TM) = \mathcal{E}^{2p+1} \oplus \mathcal{E}^{2p}, \quad p + q = n,
\]

where \(\mathcal{E}^{2q} = \{X \in \Gamma(TM) \mid i_X d\eta = 0 \text{ and } i_X \eta = 0\}\) and \(\mathcal{E}^{2p+1} = \mathcal{E}^{2p+1}(\xi)\), \(\mathcal{E}^{2p}\) being the orthogonal complement of \(\mathcal{E}^{2q}(\xi)\) in \(\Gamma(TM)\).

These modules satisfy \(\phi \mathcal{E}^{2p} = \mathcal{E}^{2p}\) and \(\phi \mathcal{E}^{2q} = \mathcal{E}^{2q}\) (13).

We now come to the main topic of our paper, i.e. 3-quasi-Sasakian geometry, which is framed into the more general setting of almost 3-contact geometry. An almost contact metric 3-structure on a smooth manifold \(M\) is the data of three almost contact structures \((\phi_1, \xi_1, \eta_1), (\phi_2, \xi_2, \eta_2), (\phi_3, \xi_3, \eta_3)\) satisfying the following relations, for any even permutation \((\alpha, \beta, \gamma)\) of \((1, 2, 3)\),

\[
\begin{align*}
\phi_\gamma &= \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta \phi_\alpha + \eta_\alpha \otimes \xi_\beta, \\
\xi_\gamma &= \phi_\alpha \xi_\beta = -\phi_\beta \xi_\alpha, \quad \eta_\gamma = \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha,
\end{align*}
\]

and a Riemannian metric \(g\) compatible with each of them. This definition was introduced, independently, by Kuo ([11]) and Udriste ([14]). In particular, they proved that necessarily \(\dim(M) = 4n + 3\). It is well known that in any almost 3-contact metric manifold the Reeb vector fields \(\xi_1, \xi_2, \xi_3\) are orthonormal with
Theorem 3.1

The following theorem assures that these three ranks coincide.

Moreover, by putting \( \mathcal{H} = \bigcap_{\alpha=1}^{3} \ker (\eta_{\alpha}) \) one obtains a 4\(n\)-dimensional horizontal distribution on \( M \) and the tangent bundle splits as the orthogonal sum \( TM = \mathcal{H} \oplus \mathcal{V} \), where \( \mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle \) is the vertical distribution.

**Definition 2.1.** A quasi-Sasakian 3-structure is an almost contact metric 3-structure \( \{ (\phi_{\alpha}, \xi_{\alpha}, \eta_{\alpha}, g) \} \in \{1, 2, 3\} \) on a smooth manifold \( M \) such that each almost contact metric structure is quasi-Sasakian. The manifold \( M \) will be called a 3-quasi-Sasakian manifold. 

In particular, a quasi-Sasakian 3-structure such that each structure is Sasakian is called a Sasakian 3-structure and the manifold is said to be a 3-Sasakian manifold. A quasi-Sasakian 3-structure such that each structure is coKähler is called a cosymplectic 3-structure and the manifold is said to be a 3-cosymplectic manifold.

Let us collect some known results on 3-quasi-Sasakian manifolds. The following theorem combines the results obtained in Theorems 3.4 and 4.2 of [4], and Theorem 3.7 of [5].

**Theorem 2.2.** Let \( (M, \phi_{\alpha}, \xi_{\alpha}, \eta_{\alpha}, g) \) be a 3-quasi-Sasakian manifold. Then the 3-dimensional distribution \( \mathcal{V} \) generated by \( \xi_1, \xi_2, \xi_3 \) is integrable. Moreover, \( \mathcal{V} \) defines a Riemannian foliation with totally geodesic leaves on \( M \), and for any even permutation \((\alpha, \beta, \gamma)\) of \((1, 2, 3)\) and for some \( c \in \mathbb{R} \)

\[ [\xi_\alpha, \xi_\beta] = c \xi_\gamma. \]

Moreover, \( c = 0 \) if and only if the structure is 3-cosymplectic.

Using Theorem 2.2 we may divide 3-quasi-Sasakian manifolds in two classes according to the behaviour of the leaves of the foliation \( \mathcal{V} \): those 3-quasi-Sasakian manifolds for which each leaf of \( \mathcal{V} \) is locally \( \text{SO}(3) \) (or \( \text{SU}(2) \)) (which corresponds to take in Theorem 2.2 the constant \( c \neq 0 \)), and those for which each leaf of \( \mathcal{V} \) is locally an abelian group (this corresponds to the case \( c = 0 \)).

3. Basic properties of 3-quasi-Sasakian manifolds

For a 3-quasi-Sasakian manifold one can consider the ranks, a priori distinct, of the three quasi-Sasakian structures \( (\phi_1, \xi_1, \eta_1, g), (\phi_2, \xi_2, \eta_2, g), (\phi_3, \xi_3, \eta_3, g) \). The following theorem assures that these three ranks coincide.

**Theorem 3.1** ([4], [5]). Let \( (M, \phi_{\alpha}, \xi_{\alpha}, \eta_{\alpha}, g) \) be a 3-quasi-Sasakian manifold of dimension \( 4n + 3 \). Then the 1-forms \( \eta_1, \eta_2 \) and \( \eta_3 \) have the same rank \( 4l + 3 \), for some integer \( l \leq n \), or 1 according to \( [\xi_\alpha, \xi_\beta] = c \xi_\gamma \) with \( c \neq 0 \) or \( c = 0 \), respectively.

According to Theorem 3.1 the common rank of \( \eta_1, \eta_2, \eta_3 \) is called the rank of the 3-quasi-Sasakian manifold \( (M, \phi_{\alpha}, \xi_{\alpha}, \eta_{\alpha}, g) \)

Furthermore, for any 3-quasi-Sasakian manifold of rank \( 4l + 3 \) one can consider the following distribution

\[ \mathcal{E}^{4m} := \{ X \in \Gamma(\mathcal{H}) \mid i_X d\eta_\alpha = 0, \ \alpha = 1, 2, 3 \} \ (l + m = n) \]

and its orthogonal complement \( \mathcal{E}^{4l+3} := (\mathcal{E}^{4m})^\perp \). In [5] it was proved the following remarkable property of 3-quasi-Sasakian manifolds, which in general does not hold for a general quasi-Sasakian structure.
Theorem 3.2. Let \((M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)\) be a 3-quasi-Sasakian manifold of rank \(4l + 3\). Then the distributions \(\mathcal{E}^{4l+3}\) and \(\mathcal{E}^{4m}\) are integrable and define Riemannian foliations with totally geodesic leaves.

In particular it follows that \(\nabla \mathcal{E}^{4l+3} \subset \mathcal{E}^{4l+3}\) and \(\nabla \mathcal{E}^{4m} \subset \mathcal{E}^{4m}\). The leaves of such foliations are 3-c-Sasakian manifolds (i.e., for each \(\alpha \in \{1, 2, 3\}\), \(d\eta_\alpha = c\Phi_\alpha\)) and hyper-Kähler manifolds, respectively (cf. Theorem 5.4 and Theorem 5.6 of [5]). Thus we can state the following corollary.

Corollary 3.3. Any 3-quasi-Sasakian manifold of rank \(4l + 3\), with \(1 \leq l < n\), is the local product of a 3-c-Sasakian manifold and of a hyper-Kähler manifold.

Another strong consequence of Theorem 3.2 is the following corollary.

Corollary 3.4. Any 3-quasi-Sasakian manifold of maximal rank is necessarily 3-c-Sasakian.

Thus in the two extremal cases — maximal and minimal rank — the geometry of a 3-quasi-Sasakian manifold is well known. In the rank 1 case, the structure turns out to be 3-cosymplectic and we can refer the reader to [6] for the main properties of these geometric structures and non-trivial examples. In the rank \((4n + 3)\) case, by applying a certain homothety one can obtain a 3-Sasakian structure. Thus we shall deal with the non-trivial cases \(\text{rank}(M) \neq 1\), \(\text{rank}(M) \neq \dim(M)\).

4. A GENERAL CONSTRUCTION

Let \((M', \phi'_\alpha, \xi'_\alpha, g')\) and \((M'', J''_\alpha, g'')\) be a 3-Sasakian and a hyper-Kähler manifold, respectively. Set \(\text{dim}(M') = 4l + 3\) and \(\text{dim}(M'') = 4m\). We define a canonical 3-quasi-Sasakian structure on the product manifold \(M := M' \times M''\) in the following way.

We define as Reeb vector fields \(\xi_\alpha := \xi'_\alpha\), for each \(\alpha \in \{1, 2, 3\}\). Next, let \(\phi_\alpha\) be the \((1, 1)\)-tensor field determined by

\[
\phi_\alpha X := \begin{cases}
\phi'_\alpha X, & \text{if } X \in \Gamma(TM') \\
J''_\alpha X, & \text{if } X \in \Gamma(TM'').
\end{cases}
\]

Finally, we consider the product metric \(g := g' + g''\) and we define three 1-forms \(\eta_1, \eta_2, \eta_3\) by \(\eta_\alpha := g(\cdot, \xi_\alpha)\). From the definition it follows that the horizontal distribution \(\mathcal{H} := \bigcap_{\alpha=1}^{3} \ker(\eta_\alpha)\) coincides with \(\mathcal{H'} \oplus TM''\), where \(\mathcal{H'}\) is the horizontal distribution of the 3-Sasakian manifold \(M'\). Then on \(\mathcal{H}\) the triple \((\phi_1, \phi_2, \phi_3)\) satisfies the quaternionic relations

\[
\phi_\alpha \phi_\beta = -\phi_\beta \phi_\alpha = \phi_\gamma
\]

for a cyclic permutation \((\alpha, \beta, \gamma)\) of \(\{1, 2, 3\}\). On the other hand, \(\phi_\alpha \xi_\beta = \phi'_\alpha \xi'_\beta = \xi'_\gamma = \xi_\gamma = -\phi_\beta \xi_\alpha\). Hence

\[
\phi_\gamma = \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta \phi_\alpha + \eta_\alpha \otimes \xi_\beta,
\]

\[
\xi_\gamma = \phi_\alpha \xi_\beta = -\phi_\beta \xi_\alpha, \quad \eta_\gamma = \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha
\]

and we conclude that \(\{(\phi_\alpha, \xi_\alpha, \eta_\alpha)\}_{\alpha \in \{1, 2, 3\}}\) is an almost contact 3-structure on \(M\). By the very definition of \(g\) and \(\phi_\alpha\) then we have that \(g\) is a compatible metric.
Let us show that \( \{ (\phi_\alpha, \xi_\alpha, \eta_\alpha, g) \}_{\alpha \in \{1,2,3\}} \) is a 3-quasi-Sasakian structure on \( M \). Notice that each fundamental 2-form \( \Phi_\alpha := g(\cdot, \phi_\alpha) \) is given by

\[
\Phi_\alpha(X, Y) := \begin{cases} 
\Phi'_\alpha(X, Y), & \text{if } X, Y \in \Gamma(TM') \\
0, & \text{if } X \in \Gamma(TM'), \text{ if } Y \in \Gamma(TM'') \\
\Omega''_\alpha(X, Y), & \text{if } X, Y \in \Gamma(TM'') 
\end{cases}
\]

where \( \Phi'_\alpha \) and \( \Omega''_\alpha \) denote the fundamental 2-forms of \( (M', \phi'_\alpha, \xi'_\alpha, g') \) and \( (M'', J''_\alpha, g'') \), respectively. By using the well-known formula

\[
d\Phi_\alpha(X, Y, Z) = X(\Phi_\alpha(Y, Z)) + Y(\Phi_\alpha(Z, X)) + Z(\Phi_\alpha(X, Y)) - \Phi_\alpha([X, Y], Z) - \Phi_\alpha([Y, Z], X) - \Phi_\alpha([Z, X], Y)
\]

we see that

\[
d\Phi_\alpha(X, Y, Z) = \begin{cases} 
d\Phi'_\alpha(X, Y, Z), & \text{if } X, Y, Z \in \Gamma(TM') \\
0, & \text{if } X, Y \in \Gamma(TM'), Z \in \Gamma(TM'') \\
0, & \text{if } X \in \Gamma(TM'), Y, Z \in \Gamma(TM'') \\
d\Omega''_\alpha(X, Y, Z), & \text{if } X, Y, Z \in \Gamma(TM''). 
\end{cases}
\]

Since \( \Phi'_\alpha \) and \( \Omega''_\alpha \) are closed, we conclude that also each \( \Phi_\alpha \) is closed. Moreover, in order to prove the normality of the 3-structure \( \{ (\phi_\alpha, \xi_\alpha, \eta_\alpha) \}_{\alpha \in \{1,2,3\}} \), it is enough to check the vanishing of \( N_{\phi_\alpha} \) on the couples of vector fields of this type:

\[
N_{\phi_\alpha}(X', Y'), \quad N_{\phi_\alpha}(X', Y''), \quad N_{\phi_\alpha}(Y', Y'''),
\]

where \( X', Y' \) are vector fields on \( M' \) and \( X'', Y''' \) are vector fields on \( M'' \). Since \( d\eta_\alpha = 0 \) on \( TM'' \), using the definitions of \( \phi_\alpha \) and \( N_{\phi_\alpha} \),

\[
N_{\phi_\alpha}(X', Y') = N_{\phi'_\alpha}(X', Y') = 0,
\]

\[
N_{\phi_\alpha}(X'', Y''') = N_{J''_\alpha}(X'', Y''') = 0,
\]

because \( M' \) is 3-Sasakian and \( M'' \) hyper-Kähler, and

\[
N_{\phi_\alpha}(X', Y''') = \phi''_\alpha(X', Y''') + [\phi_\alpha X', \phi_\alpha Y'''] - \phi_\alpha [\phi_\alpha X', \phi_\alpha Y'''] - \phi_\alpha [X', \phi_\alpha Y'''] = 0
\]

since each summand in the last equation is zero.

Therefore \( (M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g) \) is a 3-quasi-Sasakian manifold with rank \( 4l + 3 = \dim(M') \).

We say that \( f : M \to M \) is a 3-quasi-Sasakian isometry if it is an isometry of the Riemannian manifold \( (M, g) \) preserving each quasi-Sasakian structure, namely

\[
f_* \circ \phi_\alpha = \phi_\alpha \circ f_* , \quad f_* \xi_\alpha = \xi_\alpha
\]

for each \( \alpha \in \{1,2,3\} \). Notice that from (4.1) it follows that

\[
f^* \eta_\alpha = \eta_\alpha.
\]

Indeed for any \( X \in \Gamma(TM) \)

\[
f^* \eta_\alpha(X) = \eta_\alpha(f_* X) = g(f_* X, \xi_\alpha) = g(f_* X, f_* \xi_\alpha) = g(X, \xi_\alpha) = \eta_\alpha(X).
\]

Given a free and properly discontinuous action of a discrete group (in particular, a free action of a finite group) \( G \) on a 3-quasi-Sasakian manifold \( M \) by 3-quasi-Sasakian isometries, the quotient \( M/G \) is a smooth manifold of the same dimension as \( M \) and inherits a 3-quasi-Sasakian structure from \( M \).

Recall that \( f : M'' \to M'' \) is a hyper-Kähler isometry if \( f \) is an isometry of the Riemannian manifold \( (M'', g'') \) and

\[
f_* \circ J''_\alpha = J''_\alpha \circ f_*
\]
for each $\alpha \in \{1, 2, 3\}$. From (4.3) it follows that

$$f^*\Omega'' = \Omega''_\alpha.$$ 

Suppose $G$ is a finite group that acts on $M'$ by 3-Sasakian isometries and on $M''$ by hyper-Kähler isometries. Then $G$ also acts on the product manifold $M' \times M''$ by $g \cdot (p', p'') = (g \cdot p', g \cdot p'')$, $g \in G$. It is easy to check that $G$ preserves the 3-quasi-Sasakian structure on $M' \times M''$ defined above. If the action of $G$ on $M' \times M''$ is free then the quotient $(M' \times M'')/G$ is a 3-quasi-Sasakian manifold.

As an application, we consider the 3-Sasakian manifold $S^{4l+3}$. We recall how the standard 3-Sasakian structure $(\phi'_\alpha, \xi'_\alpha, \eta'_\alpha, g')$ of the sphere is defined. Let us consider the sphere $S^{4l+3}$ as an hypersurface in $\mathbb{H}^{l+1}$. Let $(J_1, J_2, J_3)$ be the standard quaternionic structure of $\mathbb{H}^{l+1}$ that is upon identification of $T_x \mathbb{H}^{l+1}$ with $\mathbb{H}^{l+1}$ the operators $J_1, J_2, J_3$ act by multiplication with $i, j, k$ on the left.

Let $N$ be the outer vector field normal to the sphere. Then one can prove that the vector fields

$$(4.4) \quad \xi'_\alpha := -J_\alpha N$$

are tangent to the sphere. Moreover, for any $X \in \Gamma(TS^{4l+3})$, we decompose $J_\alpha X$ in their components tangent and normal to the sphere,

$$(4.5) \quad J_\alpha X = \phi'_\alpha X + \eta'_\alpha (X) N,$$

so obtaining, for each $\alpha \in \{1, 2, 3\}$, a tensor field $\phi'_\alpha$ and a 1-form $\eta'_\alpha$ on $S^{4l+3}$. Then one can check that the geometric structure \{$(\phi'_\alpha, \xi'_\alpha, \eta'_\alpha, g')$\}_{$\alpha \in \{1, 2, 3\}$} is a 3-Sasakian structure on $S^{4l+3}$, being $g'$ the Riemannian metric induced by the Riemannian metric $g$ of $\mathbb{H}^{l+1} \cong \mathbb{R}^{4l+4}$.

Now we consider the isometry $f$ of $\mathbb{H}^{l+1}$ given by the multiplication with $i$ on the right. Notice that $f(S^{4l+3}) = S^{4l+3}$, because for any $x \in S^{4l+3}$ one has $||f(x)|| = ||x|| = ||x|| = 1$. Hence $f$ induces an isometry on $(S^{4l+3}, g')$, again denoted by $f$. Notice that the associativity of the product in $\mathbb{H}$ implies

$$f_* \circ J_\alpha = J_\alpha \circ f_*.$$ 

Thus $f$ is a hyper-Kähler isometry. Moreover, for any $X \in \Gamma(TS^{4l+3})$, $g(f_* N, f_* X) = g(N, X) = 0$, so that $f_* N \in \Gamma(TS^{4l+3})^1 = \langle N \rangle$. Since $||N|| = 1$ and $f$ is an isometry, it follows that

$$f_* N = N.$$ 

Then by (4.4) and (4.5) we get

$$f_* \xi'_\alpha = -f_* J_\alpha N = -J_\alpha f_* N = -J_\alpha N = \xi'_\alpha,$$

and, for all $X \in \Gamma(TS^{4l+3})$,

$$f_* (\phi'_\alpha X) + \eta'_\alpha (X) N = f_* (\phi'_\alpha X) + \eta'_\alpha (X) f_* N$$

$$= f_* J_\alpha X$$

$$= J_\alpha f_* X$$

$$= \phi'_\alpha (f_* X) + \eta'_\alpha (f_* X) N,$$

from which, taking the tangential and the normal components to the sphere, it follows that $f_* \circ \phi'_\alpha = \phi'_\alpha \circ f_*$ and $f^* \eta'_\alpha = \eta'_\alpha$. Thus $f$ is a 3-Sasakian isometry of $S^{4l+3}$. Moreover, $f^4$ is the identity operator. Thus we get an action of $\mathbb{Z}_4$ on $S^{4l+3}$ by 3-Sasakian isometries.
Let $m$ be a positive integer. We denote the hyper-Kähler isometry of $\mathbb{H}^m$, $(q_1, \ldots, q_m) \mapsto (q_1 i, \ldots, q_m i)$, by $h$. The map $h$ induces a hyper-Kähler isometry on the torus $\mathbb{T}^{4m} = \mathbb{H}^m / \mathbb{Z}^{4m}$. Thus $h$ generates an action of $\mathbb{Z}_4$ on $\mathbb{T}^{4m}$ by hyper-Kähler isometries. Note, that $\mathbb{Z}_4$ acts freely on $S^{4l+3}$, but has a fixed point $[0]$ in $\mathbb{T}^{4m}$. Nevertheless, the resulting action of $\mathbb{Z}_4$ on $S^{4l+3} \times \mathbb{T}^{4m}$ is free. We will denote the 3-quasi-Sasakian manifold $(S^{4l+3} \times \mathbb{T}^{4m}) / \mathbb{Z}_4$ by $M_{l,m}$.

Concerning this example, in view of Corollary 5.1, an interesting question is the following: is $M_{l,m}$ the global product of a 3-Sasakian manifold of dimension $4l+3$ and a hyper-Kähler manifold of dimension $4m$?

In the next Section we shall show that the answer is negative, at least in the case $l = 1$ and $m = 1$. Namely we will prove that the 3-quasi-Sasakian manifold

$$M_{1,1} := (S^7 \times \mathbb{T}^4) / \mathbb{Z}_4$$

is not topologically equivalent to the product of a 7-dimensional compact 3-Sasakian manifold and a 4-dimensional compact hyper-Kähler manifold.

5. The manifold $M_{1,1} = (S^7 \times \mathbb{T}^4) / \mathbb{Z}_4$

Let $M$ be a compact Riemannian manifold and $G$ a finite group freely acting on $M$. Denote by $\rho_M$ the corresponding group homomorphism from $G$ to $\text{Aut}(M)$. Then from the Hodge theory we obtain the isomorphism

$$H^* (M / G) \cong H^* (M)^G := \{ x \in H^* (M) \mid \rho(a)^* x = x, \text{ for all } a \in G \}. $$

Indeed, every harmonic form on $M / G$ lifts a $G$-periodic harmonic form on $M$ and every $G$-periodic form on $M$ defines a periodic form on $M / G$. Here it is important that the projection $M \to M / G$ is a local diffeomorphism and the Laplacian $\Delta$ is defined locally.

Now, let $M$ and $N$ be two compact manifolds with $G$-action given by $\rho_M : G \to \text{Aut}(M)$ and $\rho_N : G \to \text{Aut}(N)$. We will write $\rho : G \to \text{Aut}(M \times N)$ for the corresponding action on the product $M \times N$. If $\omega$ is a $q$-form on $M$ and $\sigma$ is a $p$-form on $N$, then $pr_M^* \omega \wedge pr_N^* \sigma$ is a $(p+q)$-form on $M \times N$. Moreover,

$$\rho(a)^* (pr_M^* \omega \wedge pr_N^* \sigma) = pr_M^* \rho_M(a)^* \omega \wedge pr_M^* \rho_N(a)^* \sigma$$

for $a \in G$. By Künneth theorem we have

$$H^k (M \times N) = \bigoplus_{p+q=k} H^q (M) \otimes H^p (N).$$

From the above we see that $H^q (M) \otimes H^p (N)$ is a $G$-invariant subspace of $H^k (M \times N)$. Therefore

$$H^k (M \times N)^G = \bigoplus_{q+p=k} (H^q (M) \otimes H^p (N))^G.$$

Let us now specialize to the case of $M = S^7$ and $N = \mathbb{T}^4$ with the action of $\mathbb{Z}_4$ on $S^7$ and $\mathbb{T}^4$ defined in the previous section. Note that since the isometry $f : S^7 \to S^7$ was orientation preserving, the induced action of $\mathbb{Z}_4$ on $H^7 (S^7) \cong \mathbb{R}$ is trivial. It is also clear that $\mathbb{Z}_4$ acts trivially on $H^0 (S^7) \cong \mathbb{R}$. Thus for any $0 \leq k \leq 4$

$$H^0 (S^7) \otimes H^k (\mathbb{T}^4) \cong H^k (\mathbb{T}^4), \quad H^7 (S^7) \otimes H^k (\mathbb{T}^4) \cong H^k (\mathbb{T}^4).$$
Let us denote the Betti numbers of $M_{1,1}$ by $b_k$ and we write $\tilde{b}_k$ for $\dim H^k(T^4)$. Then, from (5.1), (5.2), and (5.3) it follows that

$$
\begin{align*}
\tilde{b}_0 &= \tilde{b}_4 = 0, \\
\tilde{b}_1 &= \tilde{b}_2 = 1, \\
\tilde{b}_2 &= \tilde{b}_3 = \tilde{b}_4 = 1, \\
\end{align*}
$$

Now we compute $\tilde{b}_1$, $\tilde{b}_2$, and $\tilde{b}_3$. Note that from the above equations and Poincaré duality for $M_{1,1}$, we get $\tilde{b}_1 = b_1 = b_{10} = \tilde{b}_3$. Thus it is enough to compute $\tilde{b}_2$. The cup product on $H^\ast(T^4)$ induces the $\mathbb{Z}_4$-invariant isomorphism

$$
\Lambda^\ast H^1(T^4) \rightarrow H^\ast(T^4)
$$

where $\Lambda^\ast V$ stands for the exterior algebra of a vector space $V$. Thus

$$
\tilde{b}_k = \dim(\Lambda^k H^1(T^4))^{\mathbb{Z}_4} = \dim(\Lambda^k H^1(T^4))^{h^\ast},
$$

where $h: T^4 \rightarrow T^4$ was defined in the previous section. Let $x_1, x_2, x_3, x_4$ be the coordinate functions on $T^4$ induced from $\mathbb{H}$. Denote by $\theta_1, \theta_2, \theta_3,$ and $\theta_4$ be the dual 1-forms. Then the classes $[\theta_1], [\theta_2], [\theta_3],$ and $[\theta_4]$ give a basis of $H^1(T^4)$. The matrix of $h^\ast$ in this basis is

$$
A := \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0
\end{pmatrix}
$$

The eigenvalues of $A$ over $\mathbb{C}$ are $i$ and $-i$. Since 1 is not among the eigenvalues there is no element in $H^1(T^4)$ which is $h^\ast$-invariant. Thus $b_1 = 0$. The matrix of $h^\ast$ in the basis

$$
[\theta_1] \wedge [\theta_1], \quad [\theta_1] \wedge [\theta_2] + [\theta_2] \wedge [\theta_1], \quad [\theta_2] \wedge [\theta_3] - [\theta_3] \wedge [\theta_2], \quad [\theta_3] \wedge [\theta_4] + [\theta_4] \wedge [\theta_3]
$$

of $\Lambda^2 H^1(T^4)$ is diag $(1, -1, 1, -1, 1, 1)$. Thus $\tilde{b}_2 = 4$. Using that $b_1 = b_3 = b_4 = 0$, we get from (5.4)

$$
b_0 = b_4 = b_7 = b_{11} = 1, \quad b_1 = b_3 = b_5 = b_6 = b_8 = b_{10} = 0, \quad b_2 = b_9 = 4.
$$

Thus the Poincaré polynomial of $M_{1,1}$ is

$$
P(t) := 1 + 4t^2 + t^4 + t^7 + 4t^9 + t^{11} = (1 + t^7)(1 + 4t^2 + t^4).
$$

Suppose $M_{1,1} \cong M' \times M''$, where $M'$ is a 7-dimensional 3-Sasakian manifold and $M''$ a 4-dimensional hyper-Kähler manifold. Denote by $P'$ and $P''$ the Poincaré polynomial of $M'$, respectively of $M''$. Then by Künneth theorem

$$
P(t) = P'(t)P''(t).
$$

We will write $p_1 \leq p_2$ for two polynomials with non-negative coefficients if all the coefficients of $p_2 - p_1$ are non-negative. We also write $p_1 < p_2$ if $p_1 \leq p_2$ and $p_1 \neq p_2$. It is obvious that if $p_1 \leq p_2$ then $p_1p \leq p_2p$, and if $p_1 < p_2$ then $p_1p < p_2p$ for any non-zero polynomial $p$ with non-negative coefficients.

With this notation we have $P''(t) \geq 1 + t^7$, since $M''$ is a compact orientable 7-dimensional manifold. Let us recall the following well-known result.
Theorem 5.1. If $M^4$ is a compact four-dimensional hyper-Kähler manifold, then $M^4$ is either a K3 surface or a four-dimensional torus.

Proof. From [15, Theorem 8.1] it follows that $b_1(M^4)$ is even. Moreover, since every hyper-Kähler manifold is Calabi-Yau, $M^4$ has a trivial canonical bundle. Therefore, by the Kodaira classification (cf. [10, Section 6A]) $M^4$ is either a K3 surface or a 4-torus. □

If $M'' \cong T^4$, then $P''(t) = 1 + 4t + 6t^2 + 4t^3 + t^4$. If $M''$ is a K3 surface then $P''(t) = 1 + 22t^2 + t^4$. Thus in both cases $P''(t) > 1 + 4t^2 + t^4$. Therefore

$$P(t) = P'(t)P''(t) > (1 + t^7)(1 + 4t^2 + t^7) = P(t),$$

which gives a contradiction to our assumption $M_{1,1} \cong M' \times M''$. So we finally proved

Theorem 5.2. There exist an 11-dimensional compact 3-quasi-Sasakian manifold of rank 7 which is not a global product of a 7-dimensional 3-Sasakian manifold and a 4-dimensional hyper-Kähler manifold.

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