ON A QUANTUM WEYL CURVATURE HYPOTHESIS

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Abstract

Roger Penrose’s Weyl curvature hypothesis states that the Weyl curvature is small at past singularities, but not at future singularities. We review the motivations for this conjecture and present estimates for the entropy of our Universe. We then extend this hypothesis to the quantum regime by demanding that the initial state of primordial quantum fluctuations be the adiabatic vacuum in a (quasi-) de Sitter space. We finally attempt a justification of this quantum version from a fundamental theory of quantum gravity and speculate on its consequences in the case of a classically recollapsing universe.

1Invited contribution for a special topical collection celebrating Sir Roger Penrose’s Nobel Prize, edited by I. Fuentes and H. Ulbricht.
1 How special is our Universe?

Our Universe seems to be very special. On large scales, it is approximately homogeneous and isotropic; as indicated by the cosmic microwave background (CMB), initial anisotropies are limited by a number of order $10^{-5}$. The Universe is also very young. The observed age of about 13.8 billion years may not seem small on everyday standards, but it is surprisingly small when compared to other scales; Poincaré cycles, for example, are much bigger even for very small systems. In fact, 13.8 billion years is more or less the minimal time needed for main sequence stars, habitable planets, and life to develop.

Can one estimate quantitatively how special the Universe is? An answer can be provided by calculating its entropy in an appropriate way and comparing it with the maximum possible entropy. For ordinary matter, states of maximum entropy are homogeneous, so one might wonder whether the Universe started in a state of high entropy. That this argument is misleading comes to light when one takes into account the contribution of gravitational degrees of freedom to entropy. Gravity is universally attractive; consequently, inhomogeneous (‘condensed’) states are entropically preferred. Unfortunately, no general expression for the entropy of the gravitational field is known. But what we know is an exact formula for the entropy of a black hole, which is arguably the most condensed system in nature. This formula is the expression for the Bekenstein–Hawking entropy and is given by

$$S_{BH} = k_B \frac{A c^3}{4 G \hbar} \equiv k_B \frac{A}{(2 l_P)^2}, \quad (1)$$

where $A$ denotes the area of the black hole’s event horizon, and $l_P = \sqrt{G \hbar / c^3}$ is the Planck length. If we set Boltzmann’s constant $k_B$ equal to one (as we shall do in most of the coming equations), the Bekenstein–Hawking entropy gives the area in terms of (twice) the Planck length squared. In the special case of a spherically-symmetric (Schwarzschild) black hole with mass $M$, Eq. (1) assumes the form

$$S_{BH} \approx 1.07 \times 10^{77} k_B \left( \frac{M}{M_\odot} \right)^2, \quad (2)$$

where $M_\odot$ is the solar mass.

Equation (1) is found using the laws of black-hole mechanics. Its statistical origin is, so far, unknown, despite many attempts (and preliminary results) using current approaches to quantum gravity; see, for example, Kiefer (2012a) and the references therein. Following John Wheeler’s old idea of “it from bit”, one can divide the area $A$ into cells of size Planck-mass squared and calculate the number of ways one can attach the bits 0 and 1 to these cells. This gives a simple model to understand the possible origin of (1), and it also provides the means to calculate
statistical correction terms of the form $\propto \ln(A/l_P^2)$, which arise from Stirling’s formula for factorials (Kiefer and Kolland 2008). Except for Planck-size black holes, these corrections terms are negligible and will not be taken into account below.

Some time ago, Roger Penrose made use of (2) to estimate the maximum possible entropy of our Universe; see Penrose (1977, 1979, 1981, 1986). For this purpose, he assumed that all matter in the observable Universe were assembled into one gigantic black hole. The size of the observable Universe is defined by the present size of the particle horizon. Using (2), Penrose obtained a value of the order $10^{123}$ (Penrose 1981). (From now on, we set $k_B = 1$.) Taking into account the fact that our Universe is presently accelerating and that we thus have to use a Schwarzschild–de Sitter solution (with a value for the cosmological constant $\Lambda$ inferred from the PLANCK data) instead of a Schwarzschild solution, this value is reduced to (Kiefer 2012b)

$$S_{\text{max}} \approx 1.8 \times 10^{121}. \quad (3)$$

But for an accelerating Universe this is, in fact, not the maximum possible entropy. For the observed value of $\Lambda$, there is a contribution from the event horizon of the de Sitter space, which will be the late-time geometry of our Universe (under the assumption of a constant $\Lambda$ and not a time-dependent dark energy). The general expression for this entropy was derived by Gibbons and Hawking (1977) and reads

$$S_{\text{EH}} = \frac{3\pi}{\Lambda l_P^2}. \quad (4)$$

Taking into account the present uncertainties in the cosmological data, Egan and Lineweaver (2010) found from this the following numerical value:

$$S_{\text{EH}} \approx 2.88 \pm 0.16 \times 10^{122}. \quad (5)$$

Comparing (5) with (3), it is clear that a future de Sitter space is entropically preferred by about one order of magnitude over a state with all matter being assembled into one gigantic black hole.

In order to calculate the probability for our Universe, the maximum value (5) must be compared with the present value for the entropy within the same region. Egan and Lineweaver (2010) present a detailed estimate of all relevant contributions to the entropy, both for the observable Universe (their Table 1) and for the matter within the event horizon (their Table 2). The dominating contributions from the gravitational side are supermassive black holes (SMBHs), followed by stellar black holes. For the region inside the event horizon, the authors present the value

$$S_{\text{SMBH}} = 1.2^{+1.1}_{-0.7} \times 10^{103} \quad (6)$$
for the entropy from supermassive black holes, while the biggest contribution to non-gravitational entropy comes from the CMB photons, with the value

$$S_{\text{CMB}} = 2.03 \pm 0.15 \times 10^{88}, \quad (7)$$

and a slightly smaller value for the entropy of relic neutrinos. We see that the non-gravitational entropy is completely negligible. (Already the entropy of the black hole in the centre of the Milky Way is about hundred times the entropy of the CMB photons.)

With these numbers, following Penrose (1981), we can estimate the probability of our Universe as follows:

$$\frac{\exp(S_{\text{SMBH}})}{\exp(S_{\max})} \approx \frac{\exp(1.2 \times 10^{103})}{\exp(2.88 \times 10^{122})} \approx \exp(-2.88 \times 10^{122}). \quad (8)$$

In the ratio of these two ‘multillions’ [a term used by Eddington for double exponentials such as $10^{10^{10}}$, see Eddington (1931, p. 450)], the huge number in the numerator is completely negligible compared to the even huger number in the denominator. A similar argument applies to the case with one black hole using (3).

As we see from (8), our Universe is very special indeed. From a pure entropic point of view, one would have expected that the Universe started from a very inhomogeneous state with black holes or already from a de Sitter-type space with large event horizon. A smooth initial state without cosmic event horizon is extremely special. Since this would correspond to vanishing Weyl tensor, Penrose came up with the hypothesis that a fundamental theory should predict vanishing Weyl tensor at past singularities. In Penrose (1986, p. 138, italics in the original), he uses the following words:

\textbf{Hypothesis (Classical):} \textit{The Weyl curvature vanishes at all past singularities, as the singularity is approached from future directions.} \[\text{[This condition can be weakened by only demanding that the Weyl tensor be finite, rather than diverging, see Penrose (2011, p. 134).]}\]

He continues by writing: ‘This has the advantage that white holes, with their unpleasant anti-thermodynamic behaviour, are excluded. …This hypothesis is time-asymmetric, as indeed could have been anticipated, since it yields the time-asymmetric Second Law.’ The Weyl curvature hypothesis (WCH) thus excludes the presence of white holes. For various aspects of the WCH, see the recent essay by Hu (2021) and the references therein.

Of particular interest are the consequences of the WCH for a recollapsing universe (as is currently disfavoured by observations, but is still a theoretical possibility). In Fig. 1 we present a diagram similar to the one presented in Penrose
Figure 1: Situation for a recollapsing universe when implementing the WCH: the big crunch is fundamentally different from the big bang because the big bang is very smooth (small Weyl tensor, low entropy), whereas the big crunch is very inhomogeneous (diverging Weyl tensor, high entropy).

(1981). The ‘stalactites’ there symbolize black holes (before evaporation); a more probable universe would have ‘stalactites’ as well as ‘stalagmites’, the latter representing white holes. (These must not be confused with primordial black holes which would correspond to ‘very long’ stalactites almost touching the big bang line.)

Vanishing Weyl tensor entails, in particular, the absence of gravitational radiation. From the WCH it then follows that all gravitational waves must be retarded. This is analogous to the Sommerfeld condition stating the absence of advanced electromagnetic radiation; see Zeh (2007). Such a condition is crucial for understanding the origin of the arrow of time.

Gravitational waves can be described by certain Weyl scalars constructed from the Weyl tensor (Newman and Penrose 1962). One of them is

$$\Psi_4 := -\frac{1}{8c^2} \left( h_+ - i h_\times \right),$$

(9)

where $h_+$ and $h_\times$ denote the two polarization states of weak gravitational waves. The Newman–Penrose quantity $\Psi_4$ describes the helicity state $s = -2$, while its complex conjugate describes $s = +2$. Weyl scalars will play a role in the quantum version of the Weyl curvature hypothesis below where we will demand that $\Psi_4$ be small.

The problem connected with the WCH is thus to understand initial conditions in cosmology. This was already emphasized by Eddington (1931). He envisaged the possibility that a low-entropy state is generated by an extremely improbable fluctuation, which is an idea dating back to Boltzmann. He called such a process anti-chance, but was unwilling to accept this possibility in reactions between
atoms or other physical systems. He saw the only possibility for such a process in the boundary conditions: “Accordingly, we sweep anti-chance out of the laws of physics—out of the differential equations. Naturally, therefore, it reappears in the boundary conditions . . . ” Among his arguments to reject the idea of an unlike fluctuation he used a concept that today is known as ‘Boltzmann brain’: if we just emerged from a gigantic fluctuation, it would be more likely that ‘we’ would just emerge as brains seeing a disorganized world rather than an ordered world such as ours. This is because observing a disorganized world (and even a partly organized world full of inconsistent documents) is immensely more probable than observing an organized world. Eddington speaks of mathematical physicists instead of Boltzmann brains: “… it is practically certain that a universe containing mathematical physicists will at any assigned date be in the state of maximum disorganisation which is not inconsistent with the existence of such creatures.” After rejecting this idea, he concluded: “We are thus driven to admit anti-chance; and apparently the best thing we can do with it is to sweep it up into a heap at the beginning of time, as I have already described.”

But can we understand this occurrence of anti-chance at the beginning of time?

2 Low entropy for early quantum perturbations

It is not surprising that we know relatively little about the early phases of our Universe. A generic state would look very different from our present approximately homogeneous and isotropic world. But given the fact that symmetries play a fundamental role in physics, one might speculate that the Universe started with a highly symmetric state. Alexei Starobinsky came up with the idea that “the universe was in a maximum symmetrical state before the beginning of the classical Friedmann expansion” (Starobinsky 1979). For this state, he chose de Sitter space, which is (as Minkowski space) a state with maximal symmetry.

Classical de Sitter space is homogeneous and isotropic and thus cannot lead to structure formation. The situation is different if quantum fluctuations are taken into account. Starobinsky suggested that the quantum fluctuations for gravitational waves (the gravitons) are initially in their ground state (the adiabatic vacuum). During the expansion, vacuum modes with large enough wavelength become excited and are no longer in their ground state. In the spirit of the inflationary universe, which was developed in the years after Starobinsky’s suggestion, one can extend this idea also to quantum scalar modes (scalar components of the metric together with the inflaton).

In cosmic perturbation theory, one can combine the scalar fluctuations of the metric and a scalar field into the gauge-invariant ‘Mukhanov–Sasaki variable’ $v(\eta, x)$, where $\eta$ denotes the conformal time defined by $d\eta/dt = a^{-1}$, and $a$
is the scale factor of a Friedmann–Lemaître (F-L) universe; see, for example, Brizuela et al. (2016) and the references therein. We denote the Fourier transform of \( v(\eta, x) \) by \( v_k \); we also introduce the Fourier-transformed perturbation variable of the gauge-invariant tensor perturbations \( h_{ij} \) with polarization \( \lambda \in \{+, \times\} \) by

\[
\nu_k^{(\lambda)} := \frac{a h_k^{(\lambda)}}{\sqrt{16\pi G}}.
\] (10)

We note that an important feature in the definition of these variables is the rescaling with respect to \( a \). This becomes especially relevant in quantum theory. By expression (10) we can relate the variable \( v_k^{(\lambda)} \) to the Weyl scalar (9) (similar features hold for the relation between \( v_k \) and other Weyl scalars).

For a perturbed inflationary universe, one obtains an action containing a background part with scale factor \( a \) and homogeneous field \( \phi \) plus a sum over all \( k \) with \( \eta \)-dependent oscillators described by \( v_k \) and \( v_k^{(\lambda)} \); these oscillators have the ‘frequencies’ \( S, T, T^2 \) given by

\[
S \omega_k^2(\eta) = k^2 - \frac{z''}{z}
\] (11)

for the scalar (metric and scalar-field) perturbations and by

\[
T \omega_k^2(\eta) = k^2 - \frac{a''}{a}
\] (12)

for the tensor perturbations, respectively; moreover, we have \( z := \phi' / H \), where \( H \) is the Hubble parameter, and primes denote derivatives with respect to conformal time. (We restrict here to minimally coupled fields.)

Since \( v_k \) and \( v_k^{(\lambda)} \) are quantum variables, they obey Schrödinger equations with respect to \( \eta \) (or \( t \)). Assuming now the initial condition that these states be in their adiabatic vacuum state, one has for them the wave functions

\[
\psi_k(v_k) = N_k \exp \left( -\frac{1}{2} \Omega_k^{(0)} v_k^2 \right),
\] (13)

with \( \Omega_k^{(0)} = k \), and a similar expression for the tensorial modes. With this initial condition, the solution of the Schrödinger equations for the modes are Gaussians of the form (13) with an \( \eta \)-dependent factor \( \Omega_k^{(0)}(\eta) \). These states are now sums over excited states. In the case of pure de Sitter inflation the factors in the exponent read

\[
da S \Omega_k^{(0)}(\eta) := \frac{k^3 \eta^2}{1 + k^2 \eta^2} + \frac{i}{\eta(1 + k^2 \eta^2)},
\] (14)
the expression for slow-roll inflation is more complicated (Bizuela et al. 2016). The important feature is the occurrence of an imaginary term in this expression. Quantum mechanically, the ensuing states correspond to two-mode squeezed states.

From the solutions of the Schrödinger equations one can derive the power spectra for the density perturbations and for the primordial gravitational waves. The corresponding expressions contain explicitly the Planck length and could thus be interpreted as quantum-gravitational effects (Krauss and Wilczek 2014).

Since the above wave functions are pure states, they have vanishing entropy (no missing information). A positive entropy comes into play when considering interactions of the modes with other degrees of freedom (such as fields from an effective quantum field theory) or with higher-order perturbations. This is connected with the process of decoherence – the emergence of classical behaviour (Joos et al. 2003). These interactions only play a role for the excited states, not the ground states. Detailed calculations of decoherence and entanglement for the primordial fluctuations can be found in Kiefer et al. (2007). There the von Neumann entropy

\[ S = -\text{Tr}(\rho \ln \rho) \]  

was calculated. Alternatively, one can consider the ‘linear entropy’ \( S_{\text{lin}} = \text{Tr}(\rho - \rho^2) \), which is bounded between zero (pure state) and one (maximally mixed state) and can be used to quantify the degree of purity in a simpler way than (15).

The reduced density matrix \( \rho \) occurring in (15) is obtained by integrating out irrelevant degrees of freedom from a totally entangled quantum state containing \( a, \phi \), the \( v_k \) (resp. the tensorial modes), and the irrelevant degrees of freedom. To obtain (15), one has to perform the trace over \( v_k \).

The resulting entropy increases with increase in the scale factor \( a \). It does not yet lead to the large entropies presented in the first section. For this, further entropy-producing processes play a role (e.g. reheating after inflation). But the important point is that starting with a low-entropy initial state, one has enough entropy-generating capacity to generate an arrow of time. Instead of the Weyl curvature hypothesis presented in the first section, one can thus present here the following quantum version:

**HYPOTHESIS (QUANTUM):** The quantum states for the Weyl scalars describing scalar and tensor modes assume the form of adiabatic vacuum states in a (quasi-) de Sitter space, as the region of small-enough scale factors is approached from future directions.

We have here replaced ‘past singularities’ with ‘region of small-enough scale factors’ because it is generally assumed that singularities are absent in a quantum theory of gravity.
As in the classical case, this is a conjecture only. Can it be justified at a more fundamental level?

3 Justification from quantum gravity?

So far, we have discussed quantum states for primordial fluctuations in the background of a Friedmann–Lemaître universe. Since these include metric perturbations, quantum effects of gravity are already included. We would, however, expect that a truly fundamental explanation for the Quantum Weyl Hypothesis comes from an underlying (not yet known) full quantum theory of gravity, where no background exists. A very conservative approach and one especially suited for discussing conceptual issues is quantum geometrodynamics, with the Wheeler–DeWitt equation as its central equation; see, for example, Kiefer (2012a) for a detailed introduction.

For the case of a quantized F-L universe plus the above discussed primordial fluctuations described by the gauge-invariant variables $v_k$, the Wheeler–DeWitt equation reads

\[
\frac{1}{2} \left\{ a_0^{-2} e^{-2\alpha} \left[ \frac{1}{m_P^2} \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \phi^2} + 2 a_0^6 e^{6\alpha} V(\phi) \right] + \sum_{k,S,T} \left[ - \frac{\partial^2}{\partial v_k^2} + S,T \omega_k^2(\eta) v_k^2 \right] \right\} \Psi(\alpha, \phi, \{v_k\}) = 0;
\]

(16)

see, for example, Brizuela et al. (2016) and the references therein. Here, $\alpha := \ln(a/a_0)$, where $a_0$ is a reference scale, and the ‘frequencies’ $S,T \omega_k^2(\eta)$ are given by (11) and (12). We choose units where $\hbar = c = 1$ and where the Planck mass reads

\[
m_P^2 := \frac{3\pi}{2G}.
\]

(17)

We emphasize that the potential terms in (16) are asymmetric with respect to $\alpha \rightarrow -\alpha$. In contrast to almost all the other fundamental equations in physics, the Wheeler–DeWitt equation thereby distinguishes a direction in (intrinsic) time $\alpha$. Inspecting the frequencies (11) and (12), one recognizes that they do not depend on $a$ and $\phi$ for large $k$, that is, for small-wavelength modes. This is also true in the limit of small $a$. Since the $(a, \phi)$-part (‘minisuperspace part’) of (16) then decouples from the perturbation part, one can naturally impose the following initial condition on the total quantum state, with $v_k$ (and their tensorial partners) being in the adiabatic vacuum state (13), compare Zeh (2007),

\[
\Psi \xrightarrow{\alpha \rightarrow -\infty} \psi_0(\alpha, \phi) \prod_k \psi_k(v_k).
\]

(18)
This is a product state, which means that tracing out some of the degrees of freedom will remain ineffective, that is, it will not lead to a mixed state; thus, the entropy for the \((a, \phi)\)-variables remains zero after coarse-graining. While the state in the Wheeler–DeWitt equation (16) is timeless, a semiclassical or ‘WKB’ time comes into play after a Born–Oppenheimer type of approximation is employed; see, for example, the detailed discussion in Kiefer (2012a). In this limit, the Schrödinger equations for the modes of the last section arise as approximate equations with respect to the WKB times \(\eta\) or \(t\). For bigger values of \(a\), entanglement will emerge, and the state (18) is replaced by

\[
\Psi(\alpha, \phi, \{v_k\}) = \psi_0(\alpha, \phi) \prod_k \psi_k(v_k, \eta),
\]

where the conformal time \(\eta\) is to be understood as a function of \(\alpha\) and \(\phi\). Here, \(\psi_k(v_k, \eta)\) are the squeezed states of the last section, which are states of Gaussian form with the parameter in the exponent given by (14) or its slow-roll generalization.

There is thus an increase in entanglement entropy from small to large scale factor and thus from small to large semiclassical time \(\eta\) (or \(t\)). Within each semiclassical and decohered branch of the full quantum state, one can express entanglement in terms of thermodynamic entropy; see, for example, Peres (1995, Chap. 9). The increase in entanglement entropy could thus be seen as providing the arrow of time in our Universe.

An interesting consequence of this arises for the case of a classically recollapsing universe (Kiefer and Zeh 1995). Instead of the classical picture shown in Fig. 1 one arrives at the quantum picture sketched in Fig. 2. Since the quan-
tum theory does not distinguish between the regions with a classical big bang and a classical big crunch (they both correspond to the same region in configuration space with small \( a \)), imposing low entropy for the ‘big bang’ directly leads to low entropy for the ‘big crunch’. Imposing the quantum version of the Weyl curvature hypothesis for the region that would classically be a big-bang singularity would then automatically entail the same version for the big-crunch region. Consequently, the arrow of time would formally reverse near the classical turning point. But since semiclassical components of the universal wave function would destructively interfere there, classical systems are not expected to survive it. Every observer in this quantum universe would thus only be able to see an expanding universe (Kiefer and Zeh 1995).

This has also consequences for black holes. A time-reversed black hole is a white hole. Thus, from the point of view of the symmetric picture shown in Fig. 2, a black hole turns into a white hole after the turning point. But for real observers, who are subject to the arrow of time and experience an expanding universe, there are only black holes.

That the arrow of time may point in the direction of an expanding universe, was envisaged long ago. John Wheeler, for example, wrote (Wheeler 1962, p. 72):

> The universe is not a system with respect to which ordinary statistical considerations apply. There is no better evidence on this point than the correlation between (a) the direction of time in which entropy increases and (b) the direction of time in which the expansion of the universe is proceeding.

These considerations are, of course, speculative. But they are concrete in the sense that they arise naturally from a straightforward combination of general relativity with quantum theory, together with a particular boundary condition. One could investigate similar conceptual issues in other theories of gravity, for example when a term proportional to Weyl-tensor squared is added to the Einstein–Hilbert action; see, for example, Kiefer and Nikolić (2017) and the references therein. We leave this for future work.

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References

[1] Brizuela, D., Kiefer, C., and Krämer, M., 2016. Quantum-gravitational effects on gauge-invariant scalar and tensor perturbations during inflation: The slow-roll approximation. *Phys. Rev. D* **94**, 123527.

[2] Eddington, Sir A. S., 1931. The End of the World: from the Standpoint of Mathematical Physics. *Supplement to Nature*, **3203**, 447–453.

[3] Egan, C. A. and Lineweaver, C. H., 2010. A Larger Estimate of the Entropy of the Universe. *Astrophys. J.*, **710**, 1825–1834.

[4] Gibbons, G. W. and Hawking, S. W., 1977. Cosmological Event Horizons, Thermodynamics, and Particle Creation. *Phys. Rev. D* **15**, 2738–2751.

[5] Hu, B.-L., 2021. Weyl Curvature Hypothesis in light of Quantum Backreaction at Cosmological Singularities or Bounces. [arXiv:2110.01104](https://arxiv.org/abs/2110.01104) [gr-qc].

[6] Joos, E., Zeh, H. D., Kiefer, C., Giulini, D., Kupsch, J., Stamatescu, I.-O., 2003. *Decoherence and the Appearance of a Classical World in Quantum Theory*. 2nd ed. Berlin: Springer.

[7] Kiefer, C., 2012a. *Quantum Gravity*. 3rd ed. Oxford: Oxford University Press.

[8] Kiefer, C., 2012b. Can the Arrow of Time be understood from Quantum Cosmology? In: *The Arrows of Time*, ed. by L. Mersini-Houghton and R. Vaas, pp. 191–203. Heidelberg: Springer.

[9] Kiefer, C. and Kolland, G., 2008. Gibbs’ paradox and black-hole entropy. *Gen. Rel. Grav.*, **40**, 1327–1339.

[10] Kiefer, C. and Nikolić, B., 2017. Notes on semiclassical Weyl gravity. In: *Fundamental Theories of Physics*, **187**, 127–143.

[11] Kiefer, C. and Zeh, H. D., 1995. Arrow of time in a recollapsing quantum universe. *Phys. Rev. D* **51**, 4145–4153.

[12] Kiefer, C., Lohmar, I., Polarski, D., and Starobinsky, A. A., 2007. Pointer states for primordial fluctuations in inflationary cosmology. *Class. Quantum Grav.*, **24**, 1699–1718.

[13] Krauss, L. M. and Wilczek, F., 2014. Using Cosmology to Establish the Quantization of Gravity. *Phys. Rev. D*, **89**, 047501.
[14] Newman, E. and Penrose, R., 1962. An Approach to Gravitational Radiation by a Method of Spin Coefficients. J. Math. Phys., 3, 566–578; Errata: Ibid. 4, 998 (1963).

[15] Penrose, R., 1977. Space-time singularities. In: Proceedings of the first Marcel Grossmann meeting on general relativity, ed. by R. Ruffini, pp. 173–181. Amsterdam: North Holland.

[16] Penrose, R., 1979. Singularities and time-asymmetry. In: General relativity – an Einstein centenary survey, ed. by S. W. Hawking and W. Israel, pp. 581–638. Cambridge: Cambridge University Press.

[17] Penrose, R., 1981. Time-asymmetry and quantum gravity. In: Quantum gravity 2: A second Oxford symposium, ed. by C. J. Isham, R. Penrose, and D. W. Sciama, pp. 244–272. Oxford: Clarendon Press.

[18] Penrose, R., 1986. Gravity and state vector reduction. In: Quantum concepts in space and time, ed. by R. Penrose and C. J. Isham, pp. 129–146. Oxford: Oxford University Press.

[19] Penrose, R., 2011. Cycles of Time – An Extraordinary New View of the Universe. New York: Alfred A. Knopf.

[20] Peres, A. (1995). Quantum Theory: Concepts and Methods Dordrecht: Kluwer.

[21] Starobinsky, A. A., 1979. Spectrum of relict gravitational radiation and the early state of the universe. JETP Lett., 30, 682–685.

[22] Wheeler, J. A., 1962. The Universe in the light of general relativity. Monist, 47:1, 40–76.

[23] Zeh, H. D., 2007. The Physical Basis of the Direction of Time. 5th ed. Berlin: Springer.