Two Species of Vortices in Massive Gauged Non-linear Sigma Models

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Abstract

Non-linear sigma models with scalar fields taking values on \( \mathbb{C}P^n \) complex manifolds are addressed. In the simplest \( n = 1 \) case, where the target manifold is the \( S^2 \) sphere, we describe the scalar fields by means of stereographic maps. In this case when the \( U(1) \) symmetry is gauged and mass terms are allowed, the model accommodates stable self-dual vortices of two kinds with different energies per unit length and where the Higgs field winds at the cores around the two opposite poles of the sphere. Allowing for dielectric functions in the magnetic field, similar and richer self-dual vortices of different species in the south and north chart can be found by slightly modifying the potential. In the gauged \( \mathbb{C}P^2 \) model, self-dual semi-local vortices arise in the first choice of chart. The transition functions to the second or third charts break the \( U(1) \times SU(2) \) semi-local symmetry, but there is still room for standard self-dual vortices of the second species.

1 Introduction

The existence of BPS equations and a BPS bound in a non-linear sigma model with the target manifold an \( S^2 \)-sphere and \( \mathbb{R}^{1,2} \) as Minkowski space-time was investigated by B.J. Schroers in the short letter [1]. In that work, the author introduced a \( U(1) \) gauge field minimally coupled to the scalar fields via covariant derivatives, whereas a
Maxwell term allowed for the presence of planar vector bosons. This approach led to BPS solitons, which were found to be akin to baby skyrmions or $\mathbb{CP}^1$ lumps existing in other $(2 + 1)$-dimensional models. Nevertheless, one may think that in a system of this kind, suitable for describing the dynamics of ideal charged bosonic plasmas, topological defects of the Nielsen-Olesen vortex type should exist. In the guise of two-dimensional instanton these vortices were discovered by Nitta and Vinci in the $(1 + 1)$-dimensional $\mathbb{C}P^1$-sigma model with extended $\mathcal{N} = (2, 2)$ supersymmetry, see Reference [2]. Here we shall show that in a slight modification of the Schroers scenario, self-dual topological vortices do indeed exist and enjoy quite standard properties, except that there are two species of magnetic flux tubes, i.e., we shall describe in detail the Nitta-Vinci vortices in a purely bosonic model in $(3 + 1)$-dimensions.

One-dimensional topological defects of the kink type have been unveiled in massive non-linear $S^2$-sigma models in $(1 + 1)$-dimensions. In that case the search was made possible because of the Hamilton-Jacobi separability in elliptic coordinates of the Neumann problem, a solvable dynamical system which is tantamount to the search for kinks in the non-linear $S^2$-sigma model with a non-degenerate mass spectrum, see [3, 4]. The procedure also worked for finding kinks in a hybrid non-linear $S^2$-sigma/Ginzburg-Landau 2D model [5]. As in this latter case, we shall see that only a very precise choice of the potential is compatible with the existence of self-dual vortices in the massive non-linear $S^2$-sigma model. To build the $U(1)$ gauge theory out of the $O(3)$ background symmetry of the $S^2$-sphere, our proposal is to gauge the stereographic coordinates rather than the original fields. By doing so, the potential can be chosen in such a way that self-duality is guaranteed simultaneously in both the south- and north-charts of the sphere. Although the vorticial solutions corresponding to both charts have different energies per unit length, there is a local version of the Bogomolny bound that ensures their separate stability.

The BPS structure of this model is compatible with the modification of the magnetic field by a dielectric function. By means of this generalization and the choice of a dielectric factor that is well behaved in the two charts of the sphere, we find families of self-dual vortices carrying scalar profiles and magnetic fields that are susceptible to being modified almost at will. These new vortices belong to the class discovered in [6] but also appear in two species attached respectively to the north and south poles. It is interesting to point out that the Nitta-Vinci vortices correspond to the choice of a constant dielectric function. We also remark that the choice $H(|\phi|) = \frac{1}{|\phi|^2}$ as the dielectric function is very particular: in the south chart the analogue to the self-dual vortices existing in the Abelian Chern-Simons-Higgs planar gauge theory appear, see [7], obtained in this case by replacing $\mathbb{R}^2$ by $S^2$ as the scalar field space. The corresponding self-dual vortices in the north chart, however, are singular. The four-parametric family $H(|\phi|) = \frac{c_0|\phi|^2 + c_1}{b_0|\phi|^2 + b_1}$ of dielectric functions provides an ample supply of self-dual vortices that we shall describe in detail. In particular, we shall show that self-dual vortices of two species exist which are akin to those unveiled in [8] in the commutative limit but choosing $S^2$ as the target manifold.

The $S^2$ sphere as a complex manifold is the $\mathbb{C}P^1$ compactification of $\mathbb{C}$, and the stereographic version of the round metric, which is the key ingredient to giving the right properties to the vortices on the sphere, becomes precisely the Fubini-Study
metric of that Kh"aler manifold. Thus, it is natural to think that some kind of well-behaved vortices, sharing many features with $S^2$ vortices, should also exist in the higher rank non-linear $\mathbb{CP}^n$-sigma Abelian Higgs model. In the last section of the paper we shall show that this is in fact the case. There is, however, an important novelty: in one of the $\mathbb{CP}^n$-charts there exist BPS semi-local topological solitons, with or without vorticity, which are the cousins of the semi-local defects discovered in the scalar sector of Electroweak Theory when the weak angle is $\frac{\pi}{2}$ and described in References [9, 10, 11]. In the $n-1$ remaining charts, only purely vorticial self-dual vortices exist, all of them of the second species.

2 Construction of the non-linear $S^2$-sigma model

We begin with a system of three scalar fields $\Phi_a$, $a = 1, 2, 3$ taking values on a sphere: $\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = \rho^2$. The respective stereographic coordinates for the south and north charts are the complex scalar fields

$$\phi = \rho \frac{\Phi_1 + i\Phi_2}{\rho - \Phi_3}, \quad \psi = \rho \frac{\Phi_1 - i\Phi_2}{\rho + \Phi_3}.$$  

The transformation $\phi = \frac{\rho^2}{\psi^2}$ from the south to the north chart reverses the orientation. The reason for choosing this option is to deal with scalar fields coupled to the gauge field with identical electric charges in both charts. The massless Lagrangian $\mathcal{L}_\Phi$ describing the dynamics of the $\Phi$ fields, in terms of the south-chart field, becomes:

$$\mathcal{L}_\Phi = \frac{1}{2} \frac{4\rho^4}{(\rho^2 + |\phi|^2)^2} \partial_\mu \phi^* \partial^\mu \phi.$$  

The global $U(1)$ symmetry is made local following the standard procedure: a gauge field $A_\mu$ enters the system and supplements the local $U(1)$ transformation $\phi \to e^{ie\chi(x)}\phi$ with the gauge transformation $A_\mu \to A_\mu + \partial_\mu \chi$. A potential energy density yielding spontaneous symmetry breaking and the Higgs mechanism can also be introduced. All this leads to the Lagrangian of a gauged massive Abelian non-linear $S^2$-sigma model of the form

$$\mathcal{L}_S = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \frac{4\rho^4}{(\rho^2 + |\phi|^2)^2} D_\mu \phi^* D^\mu \phi - U_S(|\phi|^2).$$  

The covariant derivative and the electromagnetic tensor are defined in the usual way: $D_\mu \phi = \partial_\mu \phi - ie A_\mu \phi$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Under the change of coordinates $\phi = \frac{\rho^2}{\psi^2}$, the covariant derivatives and the potential energy density in the north chart become:

$$D_\mu \phi = -\frac{\rho^2}{(\psi^*)^2} D_\mu \psi^* , \quad U_N(|\psi|^2) = U_S\left(\frac{\rho^4}{|\psi|^2}\right)$$

in such a way that the Lagrangian in this chart now reads:

$$\mathcal{L}_N = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \frac{4\rho^4}{(\rho^2 + |\psi|^2)^2} D_\mu \psi^* D^\mu \psi - U_N(|\psi|^2).$$
2.1 Bogomolny splitting and self-dual vortices

Lagrangians such as (1) and (3) with a function of the scalar field multiplying the covariant derivative term were studied by M. A. Lohe in [12]. Other references in which models with a similar structure were investigated are, for instance, [13, 14, 15]. In a field theory encompassing scalar and gauge fields where the Lagrangian density is of the general form

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} g(|\varphi|^2) D_\mu \varphi^* D^\mu \varphi - \frac{1}{2} W^2(|\varphi|^2), \]

and both the function \( g(|\varphi|^2) \) and the potential energy density are semi-definite positive, it is possible to write the energy per unit length of static and \( x_3 \)-independent configurations à la Bogomolny

\[ E = \int d^2 x \left\{ \frac{1}{2} (B \pm W(|\varphi|^2))^2 + \frac{1}{2} g(|\varphi|^2) |D_1 \varphi \mp i D_2 \varphi|^2 \mp (W(|\varphi|^2)B + R) \right\} \]

after taking the Weyl plus axial gauge: \( A_0 = A_3 = 0 \). Here, \( B = F_{12} \) is the magnetic field and \( R = -\frac{i}{2} g(|\varphi|^2) \varepsilon_{ij} D_i \varphi^* D_j \varphi, i, j = 1, 2 \) can be rewritten as

\[ R = -\frac{i}{2} \varepsilon_{ij} \partial_i [F(|\varphi|^2)(\partial_j \ln \varphi - ieA_j)] + \frac{1}{2} e F(|\varphi|^2)B, \]

where \( F \) is a primitive of \( g \) such that \( F(0) = 0 \) in order to avoid singularities in the logarithm. Choosing a potential energy density

\[ U(\varphi) = \frac{1}{2} W^2(|\varphi|^2) = \frac{e^2}{8} (F(|\varphi|^2) - a^2)^2, \]

(4)

where \( a^2 \) belongs to the range \( 0 < a^2 < F(\infty) \) such that \( U(1) \) spontaneous symmetry breaking is ensured, the last two terms in the energy integral combine as a boundary contribution that is proportional to the magnetic flux \( \Phi_M \). Thus, finite-energy configurations comply with the inequality \( E \geq \frac{1}{2} ea^2 |\Phi_M| \), and the Bogomolny bound is saturated if and only if the first-order self-duality equations

\[ B = \pm \frac{e}{2} \left( a^2 - F(|\varphi|^2) \right) \quad , \quad D_1 \varphi \pm i D_2 \varphi = 0 \]

(5)

are satisfied. For radially symmetric fields \( \varphi(r, \theta) = f(r) e^{in\theta}, rA_\theta = n\beta(r) \) in the topological sector with winding number \( n \), the PDE system (5) becomes the ODE system of coupled equations

\[ \frac{n}{r} \frac{d\beta}{dr} = \pm \frac{e}{2} \left[ a^2 - F(f^2) \right] \quad , \quad \frac{df}{dr} = \pm \frac{n}{r} (1 - e\beta)f. \]

(6)

Boundary conditions on the solutions are dictated by regularity at the origin and energy finiteness: \( f(0) = 0, \beta(0) = 0, f(\infty) = v \) and \( \beta(\infty) = \frac{1}{e} \). \( v^2 \) is a zero of the potential, \( F(v^2) = a^2 \). The solutions are vortices and anti-vortices with magnetic flux \( e\Phi_M = 2\pi n \) and energy per unit length \( E = \pi |n| a^2 \). Beyond the radially symmetric
case, an index theorem calculation gives the dimension of the moduli space of solutions of \[5\] in each topological sector, see \[12, 16\]. The space of linear deformations of a vortex that preserve the self-duality equations is the kernel of the differential operator

\[ D = \begin{pmatrix}
    \partial_1 + eA_2 & -\partial_2 + eA_1 & e\varphi_2 & e\varphi_1 \\
    \partial_2 - eA_1 & \partial_1 + eA_2 & -e\varphi_1 & e\varphi_2 \\
    e g(|\varphi|^2) \varphi_1 & e g(|\varphi|^2) \varphi_2 & -\partial_2 & \partial_1 \\
    0 & 0 & \partial_1 & \partial_2
\end{pmatrix} \]

while, by introducing the auxiliary operator

\[ P = \begin{pmatrix}
    e\varphi_1 & e\varphi_2 & -\partial_2 & \partial_1 \\
    e\varphi_2 & -e\varphi_1 & \partial_1 & \partial_2 \\
    0 & 0 & e\varphi_2 & e\varphi_1 \\
    0 & 0 & -e\varphi_1 & e\varphi_2
\end{pmatrix} \]

it is not difficult to show that \( \ker PD^\dagger = \{0\} \), and hence \( \ker D^\dagger = \{0\} \). This vanishing theorem and the supersymmetric pairing of the non-zero eigenvalues of the Laplacians associated to \( D \) and \( D^\dagger \), give the dimension of the moduli space as

\[
\dim \ker D = \text{ind} \ker D = \text{Tr} \left\{ \frac{M^2}{DD^\dagger + M^2} \right\} - \text{Tr} \left\{ \frac{M^2}{D^\dagger D + M^2} \right\} = \frac{e}{\pi} \Phi_M = 2n,
\]

where the result comes from trace evaluation in the limit \( M^2 \to \infty \). The interpretation, \[17, 18\], is that the Higgs field of the general solution with winding number \( n \) has \( |n| \) zeroes along the plane, and the \( 2|n| \) parameters of the solution correspond to the coordinates of these zeroes. The self-dual defects are thus non-interacting solitons at the verge between two superconducting regimes. Rewriting the coupling in \[4\] as \( \lambda e^2 \), one has that for \( \lambda < 1 \) the mass of the vector boson would be greater than that of the Higgs boson and long-range inter-vortex forces would be attractive (type I superconductivity), whereas in the opposite \( \lambda \geq 1 \) case, the vortices would repel each other (type II superconductivity).

### 2.2 Self-dual vortices in the south chart

In the gauged non-linear sigma model on the sphere, the potential energy density, together with the metric function, are chosen to be

\[
g_S(|\phi|^2) = \frac{4\rho^4}{(\rho^2 + |\phi|^2)^2}, \quad U_S(|\phi|^2) = \frac{e^2}{8} \left( \frac{4\rho^2 |\phi|^2}{\rho^2 + |\phi|^2} - a^2 \right) \]

in order to find the Lagrangian \[1\] at the critical self-dual point in the south chart. It is required that \( 0 < a^2 < 4\rho^2 \) whereas \( U_S(|\phi|^2) \) vanishes along the vacuum orbit

\[
|\phi|^2 = v^2 = \frac{\rho^2 a^2}{4\rho^2 - a^2},
\]
see Figure 1(left). The Higgs mechanism occurs and provides identical masses to the vector and the Higgs bosons:

\[ m_A^2 = m_H^2 = \frac{\rho^2 a^2 e^2}{\rho^2 + v^2} = \frac{4\rho^2 - a^2 e^2 a^2}{4\rho^2} = m^2. \]

The Bogomolny equations are

\[ B = \pm \frac{e}{2} \left( a^2 - \frac{4\rho^2 |\phi|^2}{\rho^2 + |\phi|^2} \right), \quad D_1 \phi \pm i D_2 \phi = 0 \quad (8) \]

and the vortex energy per unit length is \( E = \pi n a^2 \) for a winding number \( n \). The scalar field maps the center of the vortices to the south-pole of the sphere, whereas the image of the boundary circle of the plane \( r \to +\infty \) is the parallel circle \( |\phi| = v \) in the south-chart of the sphere. The radially symmetric configurations \( \phi(r, \theta) = f(r)e^{i\theta} \) and \( rA_\theta = n\beta(r) \) are BPS \( \phi \)-vortex solutions if the functions \( f(r) \) and \( \beta(r) \) solve the ODE system (6) for the choice of \( F(|\phi|^2) \) written in formula (7). The solutions have been determined numerically by a standard shooting procedure for configurations with vorticity \( n = 1, 2, 3, 4 \) and are shown in Figure 2, together with the energy density \( \epsilon(r) \) and the magnetic field \( B(r) \). Figure 3 is a graphical representation of the \( n = 1 \) and \( n = 2 \) self-dual vortices using the original valued-on-the sphere field \( \Phi \), which is plotted as a unit vector on each point of the spatial plane.

2.3 Self-dual vortices in the north chart

The field redefinition \( \psi^* = \frac{\psi^2}{\phi} \) applied in formula (7), together with the subsequent changes (2), leads to the self-dual Lagrangian (3) in the \( \psi \)-chart. Thus, in this north
Figure 3: Graphical representation of the isospin vortex field $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ for the $\phi$-vortices with $n = 1$ (left) and $n = 2$ (right) for $\rho = a = 1$ in the spatial plane.

The north-chart Bogomolny equations are

$$B = \pm \frac{\epsilon}{2} \left( 4\rho^2 - a^2 - \frac{4\rho^2 |\psi|^2}{\rho^2 + |\psi|^2} \right), \quad D_1 \psi \pm i D_2 \psi = 0$$

and the BPS energy of vortices and anti-vortices per unit length is $E = \pi |n|(4\rho^2 - a^2)$.

The point of the scalar configuration space corresponding to the center of the defects is now the north-pole, whereas the fields at large distances reach the parallel

$$|\psi|^2 = w^2 = \frac{\rho^2(4\rho^2 - a^2)}{a^2}$$

in the $\psi$-chart. Because $w^2 = \frac{\rho^2}{a^2}$, this is the same parallel that we found in the $\phi$-chart and represents the global vacuum orbit of the theory. Moreover, the Higgs mechanism in this chart produces the same masses for the vector and Higgs bosons as in the south chart:

$$m^2_H = m^2_A = \frac{4\rho^2 - a^2}{\rho^2 + w^2} \rho^2 e^2 = \frac{4\rho^2 - a^2}{4\rho^2} e^2 a^2 = m^2.$$

The system is thus globally well defined.

The vacuum orbit divides the sphere into two disjoint skullcaps, the scalar field of $\phi$- and $\psi$- vortices or antivortices respectively taking values in the south and north ones. Radially symmetric configurations in this second chart $\psi(r, \theta) = \tilde{f}(r)e^{in\theta}$ and $rA_\theta = n\tilde{\beta}(r)$ are BPS-vortex solutions living in the north skullcap if $\tilde{f}(r)$ and $\tilde{\beta}(r)$ solve the ODE system (6) for the $F(|\psi|^2)$ function appearing in (9). The system is again solved numerically and the $\tilde{f}(r)$ and $\tilde{\beta}(r)$ profiles, together with the energy density $\tilde{\epsilon}(r)$ and the magnetic field $\tilde{B}(r)$, are respectively displayed in Figure 4 for the $\psi$-vortices with vorticity $n = 1, 2, 3, 4$. Figure 5 is a graphical representation of the $\Phi$ field for these solution with $n = 1$ and $n = 2$.

In sum, there exist two species of BPS vortices whose energies are respectively $E = \pi |n|a^2$ and $E = \pi |n|(4\rho^2 - a^2)$ for configurations with $n$-vorticity. From Figures 2 and
we observe that for \( a = 1 \) the first type of vortices supports thick profiles, whereas vortices of the second species are thin; the energy per unit length of thick/thin vortices is less/more concentrated in the plane. The difference in vortex energy density width, is more pronounced when the vacuum parallel \(|\phi|^2 = v^2\) approaches the North Pole, and disappears when the vacuum orbit is the Equator, \( a^2 = 2 \rho^2 \), and thick vortices become thin and vice versa if the vacuum orbit lies in the south hemisphere, \( a^2 > 2 \rho^2 \).

A subtle point is the following: although \( \phi \)- and \( \psi \)-vortices with the same winding number live in the same topological sector, both of them are separately stable. The higher-energy per unit length vortices do not decay to the lower-energy per unit length ones. The scalar field at their centres is respectively the North or the South Pole, depending on the species. In fact, BPS vortices of different species saturate either the \( \phi \)- or \( \psi \)-Bogomolny bounds, which are valid on either the south or the north charts. The circumstance that each type of vortex carries a different energy per unit length seems a bit puzzling: the BPS bounds coming from the \( \phi \)- and \( \psi \)- field Bogomolny splittings are, respectively, \( E \geq \frac{\rho}{2} a^2 \Phi_M \) and \( E \geq \frac{\rho}{2} (4 \rho^2 - a^2) \Phi_M \). Thus, if \( 2 \rho^2 \neq a^2 \), the energy of \( \psi \)-vortices contradicts the \( \phi \)-bound, and vice versa. In fact, however, the contradiction is only apparent and is due, contrary to what happens in the usual Abelian Higgs model, to the fact that finite-energy configurations can have isolated points where the scalar field goes to infinity. Recall that the target space is \( \mathbb{C} P^1 = \mathbb{C} \cup \{ \infty \} \). A radially symmetric configuration \( \phi(r, \theta) = f(r)e^{i n \infty \theta} \) such that \( \lim_{r \to 0} f(r) = \infty \) is therefore admissible. In this case, the antisymmetric sum of double derivatives of \( \ln \phi \) produces a new contribution to the \( R \) term of the Bogomolny splitting of the form

\[
\Delta R = \frac{i}{2} \bar{F}(|\phi|^2) \varepsilon_{ij} \partial_i \partial_j \ln \phi = -\pi n_{\infty} F(|\phi|^2) \delta^{(2)}(\vec{x}).
\]
The $\phi$-field Bogomolny bound in the complex plane plus the infinity point becomes

$$E \geq \pi |n_0 a^2 - n_\infty (4\rho^2 - a^2)|$$

because $\lim_{|\phi| \to +\infty} \left( \frac{4\rho^2 |\phi|^2}{\rho^2 + |\phi|^2} - a^2 \right) = 4\rho^2 - a^2$. In the previous formula, $n_0$ denotes the number of zeroes of $\phi$ in $\mathbb{C}$, and therefore the total winding number is $n = n_0 + n_\infty$. This is the global BPS bound that encompasses the bounds in both charts. If $n_\infty = 0$, there are vortices only in the south chart, $n = n_0$, and we find the BPS $\phi$-bound. $n_0 = 0$ means that $n_\infty = n$, which is tantamount to the existence of $\psi$-vortices with vorticity $n$ in the south chart, in perfect agreement with the BPS $\psi$-bound. The $\phi$- and $\psi$- Bogomolny bounds are the two local forms of the global Bogomolny bound.

We conclude that the gauged massive non-linear $S^2$-sigma model admits stable self-dual solutions in the form of either $\phi$- or $\psi$- vortices. One might wonder if there are also solutions given by the symbiosis of defects of both species. When the change of variables $\phi = \frac{\rho^2}{\psi}$ is applied to a configuration satisfying (8) with the plus sign, we obtain a configuration that satisfies (10) with the minus sign, and viceversa. Only a superposition of a $\phi$-vortex with a $\psi$-antivortex could be a self-dual solution of the theory, but in this case the total winding number vanishes, and there is no topological obstruction to ensure stability. Notice, however, that for $n_0 = 1, n_\infty = -1$ the global Bogomolny bound gives an energy per unit length $E = 4\pi \rho^2$, which is the area of the sphere and coincides with the energy of a fundamental $\mathbb{C}P^1$ lump in the class $N = 1$ of the homotopy group $\pi_2(S^2)$. In fact, as it is understood in [2], a different topological interpretation of these $\phi$-vortex $\psi$-antivortex pairs is possible; in it, they appear as the basic constituents of $\mathbb{C}P^1$ lumps. On the other hand, the superposition of a $\phi$-vortex with a $\psi$-vortex could also be a non self-dual solution of the Euler-Lagrange equations, but this can only be decided by a numerical evaluation of the interaction energy à la Jacobs-Rebbi [19], which we have so far not undertaken. In any case, given that the Higgs mechanism gives mass $m$ to the elementary particles, the interaction energy among defects located at distances of order $R \gg \frac{1}{m}$ is small, $E_{\text{int}} \simeq e^{-2mR}$ [20], and mixed configurations of widely separated $\phi$- and $\psi$- defects can always be considered good approximate solutions to the Euler-Lagrange equations.

### 2.4 More doubly self-dual models

We have built the non-linear Abelian-Higgs sigma model using the natural round metric on the sphere, but once the gauge field has been introduced and the potential energy density has been chosen, the symmetry is reduced from the $O(3)$ group to the $U(1)$ subgroup of rotations around the $\Phi_3$-axis. There is nothing to prevent us from considering models on the sphere with other metrics as long as they respect this reduced symmetry. There is, however, an appealing feature of the round metric that we would like to preserve: this is the fact that it gives rise to doubly self-dual models. By this we mean that the self-duality equations on both charts have almost the same form, the only difference being the value of $W(0)$. Next we shall briefly describe a possible way to produce a number of other non-linear sigma models with this type of double self-duality. Working on the south chart, the idea is to take a dimensionless
function $f(|\phi|^2)$ such that $f(0) = 0$, $f'(|\phi|^2) \geq 0, \forall |\phi|$, and $f(\infty) = q^2 \leq \infty$. We then define the $F$ part of the potential as $F(|\phi|^2) = \rho^2 \left( f(|\phi|^2) - f\left(\frac{\rho^2}{|\phi|^2}\right) + q^2 \right)$ in such a way that $F(0) = 0$, as it should be, and $F(\infty) = 2\rho^2 q^2$ is finite. Therefore, the metric and potential on the south chart are

$$g_s(|\phi|^2) = \rho^2 \left( f'(|\phi|^2) + \frac{\rho^4}{|\phi|^4} f'\left(\frac{\rho^4}{|\phi|^2}\right) \right), \quad U_s(|\phi|^2) = \frac{e^2}{8} \left( F(|\phi|^2) - a^2 \right)^2,$$

whereas the change of fields (2) gives the following metric and potential for the $\psi$ field:

$$g_n(|\psi|^2) = \rho^2 \left( f'(|\psi|^2) + \frac{\rho^4}{|\psi|^4} f'\left(\frac{\rho^4}{|\psi|^2}\right) \right), \quad U_n(|\psi|^2) = \frac{e^2}{8} \left( F(|\psi|^2) - (2\rho^2 q^2 - a^2) \right)^2.$$

Double self-duality is thus apparent. The energies per unit length of $\phi$- and $\psi$-vortices or antivortices with winding number $n$ are respectively $E = \pi |n| a^2$ and $E = \pi |n| (2\rho^2 q^2 - a^2)$, and the phenomenology is analogous to what has been described for the model with the round metric. The case of the function $f(|\phi|^2) = \ln \frac{\rho^2 + 2|\phi|^2}{\rho^2 + |\phi|^2}$ gives an interesting example where double self-duality combines with a logarithmic potential.

## 3 Dielectric functions and self-dual vortices of two species in the Abelian $S^2$-sigma Higgs model

### 3.1 The Lagrangian and the Bogomolny splitting

The kinetic energy density for the gauge field in the Lagrangians (1) and (3) that we have been dealing with in the previous section is given by the standard Maxwell term. This is the canonical choice when the model is intended to describe the interactions of fundamental quanta in vacuo, but there are many physical systems in which the Abelian-Higgs model plays the rôle of an effective theory ruling the dynamics of the excitations of some background medium. For this sort of application, it may be the case that the minimal Maxwell term has to be supplemented with a dielectric function that will account for the enhancement or screening of the forces among quanta due to the polarization of the underlying condensate. The $U(1)$ gauge symmetry requires that the dielectric factor should depend only on the scalar field modulus, and hence the Lagrangian of Section 3 changes to the form

$$\mathcal{L} = -\frac{1}{4} H(|\varphi|^2) F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} g(|\varphi|^2) D_\mu \varphi^* D^\mu \varphi - \frac{1}{2} W^2(|\varphi|^2). \quad (11)$$

Models of this type where $H(|\varphi|^2)$ is a positive definite function admit first-order Bogomolny equations, see e.g. [6, 14, 15, 8]. The arrangement

$$E = \int d^2 x \left\{ \frac{1}{2} \left( \sqrt{H(|\varphi|^2)} B \pm W(|\varphi|^2) \right)^2 + \frac{1}{2} g(|\varphi|^2) |D_1 \varphi \mp i D_2 \varphi|^2 \right\}$$

$$\mp \left( \sqrt{H(|\varphi|^2)} W(|\varphi|^2) B + R \right) \quad (12)$$
leads, together with the choice of the potential as
\[
U(\varphi) = \frac{1}{2} W^2(|\varphi|^2) = \frac{e^2}{8} \frac{(F(|\varphi|^2) - a^2)^2}{H(|\varphi|^2)},
\]
(13)
to self-duality equations of the form
\[
B = \pm \frac{e}{2} \frac{(a^2 - F(|\varphi|^2))}{H(|\varphi|^2)}, \quad D_1 \varphi \pm i D_2 \varphi = 0
\]
(14)
whose solutions are vortices and antivortices of energy density \( E = \pi |n| a^2 \) for winding number \( n \).

### 3.2 Hybrid vortices in the Abelian \( S^2 \)-sigma Higgs model with dielectric function

In practice, the dielectric function can be chosen in many different ways that can generate a broad variety of vortex profiles. For the non-linear sigma model on the sphere, a natural option is to use a function \( H(|\varphi|^2) \), leading to regular vortices with a similar structure in both the south and north charts, as indeed happened in the minimal model without dielectric function. To achieve this behaviour we choose the south chart dielectric function and self-dual potential in the form
\[
H_S(|\varphi|^2) = c_0 |\varphi|^2 + c_1 \rho^2, \quad U_S(|\varphi|^2) = \frac{e^2}{8} \frac{b_0 |\varphi|^2 + b_1 \rho^2}{c_0 |\varphi|^2 + c_1 \rho^2} \left( \frac{4 \rho^2 |\varphi|^2}{\rho^2 + |\varphi|^2} - a^2 \right)^2,
\]
where \( b_0, b_1, c_0, c_1 \) are positive real numbers, see Figure 6.

![Figure 6: Scalar potentials in the south and north charts for values](image)

Figure 6: Scalar potentials in the south and north charts for values \( c_1 = 5 \) and \( b_0 = c_0 = b_1 = 1 \) (case A) and for the values \( b_1 = 10 \) and \( b_0 = c_0 = c_1 = 1 \) (case B).

The vacuum orbit occurs in these models along the circle \( |\varphi| = \sqrt{a^2 - 4 \rho^2} \), and there is always a critical point of the potential at \( \varphi = 0 \) that can be tuned to be a minimum.
or maximum with a suitable selection of the parameters. There are two special or
limiting cases: a) If \( c_0 = b_0 \) and \( c_1 = b_1 \), the function \( H \) becomes unity and we recover
the system where the Nitta-Vinci vortices arise. b) The other case appears when \( c_0 = b_1 = 0 \) and \( c_1 = b_0 \), the function is \( H(|\phi|) = \frac{1}{|\psi|} \) and we deal with the non-
linear \( S^2 \)-sigma model version of the Chern-Simons topological topological solitons
discovered in \([4]\), see also \([21]\). This dielectric function prompts a BPS potential such
that the critical point at \( |\phi| = 0 \) becomes an absolute minimum. The vacuum orbit is
the disjoint union of one point and a circle and the space of BPS topological solitons
is richer than in the other cases. As in its planar counterpart, the system encompasses
a Coulomb and a Higgs phase. In this subsection we shall describe hybrid selfdual
vortices interpolating between Nitta-Vinci and nonlinear Chern-Simons vortices.

Changing coordinates to the north chart, the metric mutates to the known round
metric on \( S^2 \), whereas the dielectric function and potential energy density
become

\[
H_N(|\psi|^2) = \frac{c_1 |\psi|^2 + c_0 \rho^2}{b_1 |\psi|^2 + b_0 \rho^2}, \quad U_N(|\psi|^2) = \frac{e^2 b_1 |\psi|^2 + b_0 \rho^2}{8 c_1 |\psi|^2 + c_0 \rho^2} \left( \frac{4 \rho^2 |\psi|^2}{\rho^2 + |\psi|^2} - (4 \rho^2 - a^2) \right)^2.
\]

It is clear that at the Chern-Simons limit the potential in the north chart becomes
infinity at \( |\psi| = 0 \); henceforth the corresponding self-dual solitons are singular.
The duality between the theories with \((b_0, b_1, c_0, c_1; a^2)\) and \((b_1, b_0, c_1, c_0; 4 \rho^2 - a^2)\) is thus
patently clear. The regular vortices of the south (north) chart in one theory are akin
to the regular vortices appearing in the north (south) chart in the other. In particular,
the energy per unit length of the vortices in the north chart is:

\[ E = \pi (4 \rho^2 - a^2) |n|. \]

The same argument developed in the microscopic scenario of the previous section
regarding the existence of a global Bogomolny bound works here.

### 3.3 Profiles of self-dual radial vortices

The ODEs giving the radially symmetric defects are of the generic form

\[
\frac{n \, d\beta}{r \, dr} = \pm \frac{e}{2} \frac{[a^2 - F(f^2)]}{H(f^2)}, \quad \frac{df}{dr} = \pm \frac{n}{r} (1 - e \beta) f
\]

where, of course, south- or north- variables and \( F \) and \( H \) functions have to be ap-
propriately substituted and finite-energy boundary conditions respected. The energy
density per unit length and the magnetic field for these solutions are:

\[
\mathcal{E}(r) = \frac{n^2}{r^2} g[f^2(r)] f^2(r)[1 - e \beta(r)]^2 + \frac{e^2}{4} \frac{(a^2 - F[f^2(r)])^2}{H[f^2(r)]^2} H[f^2(r)]
\]

\[
B(r) = \frac{e^2 a^2 - F[f^2(r)]}{2} \frac{H[f^2(r)]^2}{H[f^2(r)]}. \]

As an illustration, here we shall present the solutions for two cases, let us call them
A and B, where the parameters are chosen to be \( \rho = a = b_0 = b_1 = c_0 = 1, c_1 = 5 \)
in Case A, and \( \rho = a = b_0 = c_0 = c_1 = 1, b_1 = 10 \) in Case B. The potential energy
densities for these parameters are plotted in Figure 6 (blue lines), where a comparison
with the potentials (red lines) giving self-duality in the microscopic, non dielectric, case is also offered. Observation of these graphics reveals to us that the potentials in case A are closer to the non-dielectric self-dual potentials than the potentials in case B in both charts. In the second model, one can observe that there is a local maximum of $U$ at $\phi = 0$ in the south chart, but the potential in the south chart shows a local minimum at $\psi = 0$.

Figure 7: Profiles of $\phi$-vortices for several values of the vorticity $n$ in case A: from left to right the function $f(r)$, the function $\beta(r)$, the energy density $\epsilon(r)$ and the magnetic field $B(r)$.

Figure 8: Profiles of $\psi$-vortices for several values of the vorticity $n$ in case A: from left to right the function $\tilde{f}(r)$, the function $\tilde{\beta}(r)$, the energy density $\tilde{\epsilon}(r)$ and the magnetic field $\tilde{B}(r)$.

Figures 7 to 10 show the specific scalar and vector boson field profiles, as well as the energy densities and the magnetic fields, of the radially symmetric solutions for cases A and B with winding numbers increasing from 1 to 4. Again, these magnitudes in case A are close to the field profiles and density energies of the Nitta-Vinci vortices. Interesting new features arise in case B profiles: namely, the magnetic field presents a local minimum at $\psi = 0$ in the north chart even for solutions with vorticity $n = 1$. Thus, the maximum values of the magnetic field are attained at a ring in the plane enclosing the origin, a configuration that resembles the self-dual Chern-Simons-Higgs vortices.

4 BPS solitons in the massive gauged non-linear $\mathbb{C}P^2$-sigma model

The non-linear sigma model on the sphere is the simplest $n = 1$ representative among a hierarchy of $\mathbb{U}(1)$-gauged non-linear sigma models with $\mathbb{C}P^n$ manifolds as target spaces. In order to extend the results found in $S^2 \simeq \mathbb{C}P^1$ to other members of the hierarchy, in this section we shall analyse the next case, since it turns out that $\mathbb{C}P^2$
already exhibits the most relevant features arising for general \( n \). \( \mathbb{CP}^2 \) is a Kähler manifold of complex dimension two. A coordinate system is built from a minimal atlas with three charts. We shall call them \( V_\phi \), \( V_\psi \) and \( V_\xi \) and shall denote the complex coordinates in each chart, respectively, as \((\phi_1, \phi_2)\), \((\psi_1, \psi_2)\) and \((\xi_1, \xi_2)\). The transition functions giving the change of coordinates in the intersections between charts are:

\[
\psi_1 = \rho_2 \phi_1, \psi_2 = \rho_2 \phi_2 \quad \text{on} \quad V_\phi \cap V_\psi, \\
\xi_1 = \rho_2 \phi_1, \xi_2 = \rho_2 \phi_2 \quad \text{on} \quad V_\phi \cap V_\xi \quad \text{and} \quad \xi_1 = \rho_2 \psi_1, \xi_2 = \rho_2 \psi_2 \quad \text{on} \quad V_\psi \cap V_\xi.
\]

The Kähler potential, expressed in the coordinates of the chart \( V_\phi \), takes the form

\[
K = 4 \rho^2 \ln(1 + |\phi_1|^2 + |\phi_2|^2) \quad \text{and it is easy to check that it adopts an equivalent form, in terms of the respective coordinates, on the other two charts.}
\]

### 4.1 The gauged Abelian \( \mathbb{CP}^2 \)-sigma model in the reference chart \( V_\phi \)

To formulate an Abelian gauge theory on \( \mathbb{CP}^2 \) let us begin by working on the \( V_\phi \) chart. Gauging of the scalar non-linear \( \mathbb{CP}^2 \)-sigma model gives rise to the Lagrangian

\[
\mathcal{L}_\phi = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} g_{pq} D_\mu \phi_p D^\mu \phi_q^* - U_\phi(|\phi_1|^2, |\phi_2|^2),
\]

where \( g_{pq} = \frac{\partial^2 K}{\partial \phi_p \partial \phi_q^*} \) is the standard Fubini-Study metric on \( \mathbb{CP}^2 \):

\[
g_{11} = \frac{4 \rho^2 (\rho^2 + |\phi_2|^2)}{(\rho^2 + |\phi_1|^2 + |\phi_2|^2)^2}, \quad g_{12} = -\frac{4 \rho^2 \phi_1 \phi_2}{(\rho^2 + |\phi_1|^2 + |\phi_2|^2)^2}, \\
g_{22} = \frac{4 \rho^2 (\rho^2 + |\phi_1|^2)}{(\rho^2 + |\phi_1|^2 + |\phi_2|^2)^2}, \quad g_{12} = -\frac{4 \rho^2 \phi_1 \phi_2}{(\rho^2 + |\phi_1|^2 + |\phi_2|^2)^2}.
\]

The covariant derivatives are \( D_\mu \phi_p = \partial_\mu \phi_p - ie A_\mu \phi_p \) and, to keep things simple, here we have discarded the possibility of introducing a dielectric function. We choose the
potential energy density in (16) in the form that generalizes the potential function entering the $S^2$-sigma model.

$$ U_\phi = \frac{e^2}{8} \left( \frac{4 \rho^2 (|\phi_1|^2 + |\phi_2|^2)}{\rho^2 + |\phi_1|^2 + |\phi_2|^2 - a^2} \right)^2. \quad (17) $$

The Lagrangian corresponds to a semi-local theory in which the $U(1)$ local invariance is accompanied by a global $SU(2)$ symmetry. The vacuum orbit, the set of zeroes of $U_\phi$, $\phi_1 = v_1$, $\phi_2 = v_2$, is the $S^3$ sphere:

$$ |v_1|^2 + |v_2|^2 = \frac{\rho^2 a^2}{4 \rho^2 - a^2}. $$

It is well known that $S^3$ is a Hopf bundle, i.e., it is a manifold fibered on $S^2$ with fibre $S^1$ and Hopf index 1, see e.g. References [22, 23]. Moreover, the winding number of the map from the $S^1_\infty$ circle enclosing the spatial plane to the $S^1$ fiber provided by the gauge field at infinity classifies the configuration space in $n \in \mathbb{Z}$ disconnected subspaces. In the $n = 0$ subspace, one sets a particular point of $S^3$ as the vacuum, for instance $(v_1 = \frac{\rho \alpha}{\sqrt{4 \rho^2 - a^2}}, v_2 = 0)$. The Higgs mechanism is worked out and gives mass to the physical fields. One thus finds that the mass spectrum includes, along with a degenerate couple encompassing the Higgs scalar meson and a massive vector boson, another complex Goldstone boson. The masses are:

$$ m_A^2 = m_H^2 = e^2 a^2 \left( 1 - \frac{a^2}{4 \rho^2} \right)^2 = m^2, \quad m_G^2 = 0. $$

Both the Higgs and the Goldstone fields are coupled to the gauge field through the covariant derivative terms. All this refers to elementary quanta; the other perspective is about solitons living in sectors of the configuration space with $n \neq 0$.

In this respect, we look at the static part of the energy per unit length. Working in the simultaneous temporal and axial gauge, and focusing on static and $x_3$-independent configurations, one writes:

$$ E = \int dx^2 \left\{ \frac{1}{2} B^2 + \frac{1}{2} g_{pq} D_i \phi_p D_i \phi_q^* + \frac{e^2}{8} \left( \frac{4 \rho^2 (|\phi_1|^2 + |\phi_2|^2)}{\rho^2 + |\phi_1|^2 + |\phi_2|^2 - a^2} \right)^2 \right\}. \quad (18) $$

On one hand, we have that:

$$ \frac{1}{2} (B^2 + W^2) = \frac{1}{2} \left( B \pm W \right)^2 = BW $$

$$ W = \frac{e}{2} \left( \frac{4 \rho^2 (|\phi_1|^2 + |\phi_2|^2)}{\rho^2 + |\phi_1|^2 + |\phi_2|^2 - a^2} \right). $$

On the other hand, we also split the covariant derivative terms in a similar manner

$$ \frac{1}{2} g_{pq} D_k \phi_p D_k \phi_q^* = \frac{1}{2} g_{pq} (D_1 \phi_p \pm i D_2 \phi_p) (D_1 \phi_q^* \pm i D_2 \phi_q^*) = R $$

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where the last term can be conveniently recast in the form:

\[ R = \frac{i}{2} \varepsilon_{ij} M_{pq} (\partial_i \ln |\phi_p| - i e V_{ip})(\partial_j \ln |\phi_q| + i e V_{jq}) \quad , \quad V_{ip} = A_i - \frac{1}{e} \partial_i \alpha_p . \]

Here \( \alpha_p \) is the phase of \( \phi_p \) and \( M_{pq} = g_{pq} \phi_p \phi^*_q \) (no sum in \( p \) and \( q \)) is a real symmetric matrix. Upon discarding total derivative terms, we write \( R = e \varepsilon_{ij} V_{ip} S_{jp} \), where

\[ S_{jp} = M_{pq} \partial_j \ln |\phi_p| = 2 \rho^2 \partial_j \frac{|\phi_p|^2}{\rho^2 + |\phi_1|^2 + |\phi_2|^2} , \]

and we find as the final outcome of all these manipulations that:

\[ R = 2 e \rho^2 \frac{|\phi_1|^2 + |\phi_2|^2}{\rho^2 + |\phi_1|^2 + |\phi_2|^2} B . \]

Therefore, we finish with a self-dual theory on the chart \( V_\alpha \) where the Bogomolny bound

\[ E \geq \frac{1}{2} e a^2 |\Phi_M| , \quad e\Phi_M = e \int d^2 x B = 2\pi n \]

is saturated if the Bogomolny equations

\[ B = \pm \frac{e}{2} \left( a^2 - \frac{4 \rho^2 (|\phi_1|^2 + |\phi_2|^2)}{\rho^2 + |\phi_1|^2 + |\phi_2|^2} \right) \quad , \quad D_1 \phi_p \pm i D_2 \phi_p = 0, p = 1, 2 \]

are satisfied. As required by the mixing of \( SU(2) \) and \( U(1) \) symmetries, these first-order equations are of the semi-local type. Although the equation for the magnetic field is more complicated here than in the standard examples, the results found in references such as \[9, 10, 11\] give a solid guarantee that semi-local vortices enjoying stability against decay to the vacuum and filling, for winding number \( n \), a moduli space of complex dimension \( 2n \), will also exist in the present situation.

### 4.1.1 Radially symmetric semi-local topological solitons

The radial ansatz for the fields

\[ \phi_1(r) = f(r)e^{i n \theta} , n \in \mathbb{Z}^+ \quad , \quad r A_\theta(r) = n \beta(r) \]
\[ \phi_2(r) = |h(r)|e^{i(\omega + i \theta)} , \quad l = 0, 1, \cdots , n \quad , \quad \omega \in \mathbb{R}^+ \quad , \]

converts the PDE Bogomolny system \[21\] in the first-order ODE system

\[ \frac{n}{r} \frac{d \beta}{dr} = \pm \frac{e}{2} \left[ a^2 - \frac{4 \rho^2 (f^2(r) + |h(r)|^2)}{\rho^2 + f^2(r) + |h(r)|^2} \right] \]
\[ \frac{df}{dr} = \frac{n}{r} f(r) [1 - \beta(r)] \quad , \quad \frac{dh}{dr} = \frac{n}{r} |h(r)| \left[ \frac{l}{n} - \beta(r) \right] . \]

The solutions of this system \[22\]-(\[23\]) complying with the asymptotic conditions

\[ \lim_{r \to \infty} f(r) = v_1 \quad , \quad \lim_{r \to \infty} h(r) = 0 \quad , \quad \lim_{r \to +\infty} \beta(r) = 1 \]
\[ f(0) = 0 \quad , \quad |h(0)| = |h_0| \delta_{l,0} \quad , \quad \beta(0) = 0 \]
are the BPS solitons of the system in the $V_\phi$ (reference) chart. The choice $h_0 = 0$ gives rise to the Nitta-Vinci vortices in the south chart of $S^2$ embedded in the reference chart of $\mathbb{CP}^2$. Setting $h_0 > 0$, cousins of the planar semi-local topological solitons arise in the gauged $\mathbb{CP}^n$ model. In fact, there exists a limit value of $h_0$ for which $\mathbb{CP}^1$-lumps appear where the magnetic flux is homogeneously spread throughout the plane. The topology behind their existence lies in the $S^2$ base space of the Hopf fibration. To illustrate these points, in Figures 11 and 12 we show two self-dual solutions for $h_0 = 0.1$ and $h_0 = 0.4$ obtained by means of the same shooting procedure as applied before. In the first case, the soliton profiles, the energy density per unit length and the magnetic field are quite close to their counterparts in the Nitta-Vinci vortices. For higher $h_0$, we see that the profiles of the solutions, together with the energy density per unit length and magnetic field, are less concentrated and tend slowly to their vacuum values.

Figure 11: Profiles of the functions $f(r)$, $\beta(r)$ and $h(r)$ (left), the energy density (middle) and the magnetic field $B(r)$ for vorticity $n = 1$ and the choice $h_0 = 0.1$.

Figure 12: Profiles of the functions $f(r)$, $\beta(r)$ and $h(r)$ (left), the energy density (middle) and the magnetic field $B(r)$ for vorticity $n = 1$ and the choice $h_0 = 0.4$.

4.2 The gauged Abelian $\mathbb{CP}^2$-sigma model in the second chart

The next task is to write the Lagrangian (16) as an Abelian theory in the chart $V_\psi$. In order to do so, we first observe that the Khähler potential becomes

$$K = 4\rho^2 \left[ \ln \left( 1 + \frac{|\psi_1|^2}{\rho^2} + \frac{|\psi_2|^2}{\rho^2} \right) - \ln \frac{\psi_1^* \psi_1}{\rho^2} \right]$$

in the new chart. Thus, the Fubini-Study metric remains formally identical. The changes in the covariant derivatives

$$D_\mu \phi_1 = -\frac{\rho^2}{\psi_1^2} (\partial_\mu \psi_1 + i e A_\mu \psi_1) , \quad D_\mu \phi_2 = -\frac{\rho^2}{\psi_1^2} (\partial_\mu \psi_1 + i e A_\mu \psi_1) + \frac{\rho}{\psi_1} \partial_\mu \psi_2$$

17
seem to be more drastic but are explained as follows: the gauge transformations \( \phi_p \rightarrow e^{i\alpha} \phi_p, p = 1, 2 \) of the \( \phi \)-fields correspond, after the application of the transition functions on \( V_\phi \cap V_\psi \), to the local phase redefinitions \( \psi_1 \rightarrow e^{-i\alpha} \psi_1 \) and \( \psi_2 \rightarrow \psi_2 \) for the \( \psi \)-fields. The \( \psi_1 \) field couples to the gauge field with opposite charge to the two \( \phi_p \) fields, but \( \psi_2 \) remains as a neutral field. Consequently, the covariant derivatives on the chart \( V_\psi \) are defined as:

\[
D_\mu \psi_p = \partial_\mu \psi_p - iN_p eA_\mu \psi_p , \quad \text{where} \quad N_1 = -1 \quad \text{and} \quad N_2 = 0
\]

and have the right holomorphic transformation properties under the change of chart:

\[
D_\mu \phi_p = \frac{\partial \phi_p}{\partial \psi_q} D_\mu \psi_q ; \quad \frac{\partial \phi_1}{\partial \psi_1} = -\frac{\rho^2}{\psi_1^2} , \quad \frac{\partial \phi_1}{\partial \psi_2} = 0 , \quad \frac{\partial \phi_2}{\partial \psi_1} = -\frac{\rho^2}{\psi_1^2} , \quad \frac{\partial \phi_2}{\partial \psi_2} = \frac{\rho}{\psi_1} .
\]

This is all that is needed to ensure that the Khäler structure of the kinetic term is preserved, leading us to the Lagrangian in the chart \( V_\psi \):

\[
\mathcal{L}_\psi = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} g_{pq} D_\mu \psi_p D^\mu \psi_q^* - U_\psi (|\psi_1|^2, |\psi_2|^2) . \tag{24}
\]

\( g_{pq} \) is the Fubini-Study metric defined in terms of the \( \psi \) fields, and the new potential energy density is

\[
U_\psi = \frac{e^2}{8} \left( \frac{4\rho^2 |\psi_1|^2}{\rho^2 + |\psi_1|^2 + |\psi_2|^2} - (4\rho^2 - \alpha^2) \right)^2 . \tag{25}
\]

Alternatively, one can derive the Lagrangian \( \mathcal{L}_\psi \) from the Lagrangian \( \mathcal{L}_\phi \) by applying the transition functions in a direct but lengthy calculation.

The potential energy density \( U_\psi \) is only invariant under the Abelian subgroup

\[
\begin{pmatrix}
\psi_1' \\
\psi_2'
\end{pmatrix} = \begin{pmatrix}
e^{i\alpha} & 0 \\
0 & e^{-i\alpha}
\end{pmatrix} \begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
\]

of the global \( SU(2) \) non-Abelian symmetry. Because \( \psi_2 \) is neutral the local \( U(1) \) transformation does not act on this field and the system is still gauge invariant in the \( V_\psi \) chart. The transition functions break the semi-local \( U(1) \times SU(2) \) invariance, leaving us with only a semi-local \( U(1)_{\text{local}} \times SU(2)_{\text{global}} \) symmetry. The vacuum orbit in this \( V_\psi \) chart, the set of zeroes of \( U_\psi, \psi_1 = w_1, \psi_2 = w_2 \), is the \( \mathbb{H}^3 \) one-sheet hyperboloid:

\[
|w_1|^2 - \frac{4\rho^2 - \alpha^2}{\alpha^2} \cdot |w_2|^2 = \rho^2 \cdot \frac{4\rho^2 - \alpha^2}{\alpha^2} . \tag{26}
\]

Note that even though the hyperboloid \( \mathbb{H}^3 \) is an open space in \( \mathbb{C}^2 \) it is accommodated within \( \mathbb{CP}^2 \) through the "infinite line": \( \mathbb{CP}^2 = \mathbb{C}^2 \cup \mathbb{CP}^1 \). We stress, however, that only points complying with \( \mathbb{H}^3 \) and living in \( V_\psi \), i.e., \( |w_1| < +\infty, |w_2| < +\infty \), are bona fide vacua of the system. Like \( S^3, \mathbb{H}^3 \) is also a 3D fibered space with the \( S^1 \)-circle as fibre. The base, however, is one sheet in the 2D hyperboloid living in the \( \mathbb{R}^3 \) subspace of \( \mathbb{C}^2 \) where \( \text{Im} w_1 = 0 \). Despite the differences in charges of the fields and vacuum orbit geometry, the Higgs mechanism gives rise to a spectrum formed
by a couple of massive particles, the Higgs and vector bosons, and a complex (but
neutral) Goldstone boson. By choosing the vacuum, e.g., on the Equatorial circle of
\(H^3\), \((w_1 = \rho \cdot \sqrt{4\rho^2 - a^2}, w_2 = 0)\), we find the same mass spectrum as in the \(V_\phi\) chart:

\[
m^2_A = m^2_H = \frac{e^2}{4} a^2 \left(1 - \frac{a^2}{4\rho^2}\right)^2 = m^2 , \quad m^2_G = 0 .
\]

We remark that, despite being “neutral”, the Goldstone field \(\psi_2\) is coupled to the \(A_\mu\)
vector boson through the mixed terms induced by the Fubini-Study metric, as can be
checked from perturbations around the vacuum chosen.

The Bogomolny splitting, worked according to the standard procedure, confirms
that the Lagrangian \([24]\) is again at the self-dual point. The Bogomolny bound is
in this \(V_\psi\) chart: \(E \geq \frac{e}{2} (4\rho^2 - a^2)|\Phi_M|\) where, of course, \(4\rho^2\) is greater than \(a^2\).
Note that it is the same bound as in the north chart of the \(S^2\)-sigma model. The
Bogomolny equations guaranteeing that this bound is saturated are:

\[
B = \pm \frac{e}{2} \left(4\rho^2 - a^2 - \frac{4\rho^2|\psi_1|^2}{\rho^2 + |\psi_1|^2 + |\psi_2|^2}\right) \quad D_1\psi_p \pm iD_2\psi_p = 0, p = 1, 2 . \quad (27)
\]

In particular, the equation between the covariant derivatives of \(\psi_2\) in \([27]\) is the
Cauchy-Riemann equation: the self-dual \(\psi_2\) field solutions are holomorphic (+ sign)
or antiholomorphic (− sign) functions. The energy per unit length finiteness requires,
for configurations reaching a bona fide vacuum \(|w_2| < +\infty\) at infinity, a constant \(\psi_2\)
value at long distances. Not allowing for the presence of poles, the Liouville theorem,
ensuring that an entire and bounded function in \(\mathbb{C}\) is constant, requires that \(\psi_2\) take
a constant value all along the plane. Once this constant value is substituted in the
equation for the magnetic field, \([27]\) reduces to a system of two equations with the
same structure as those giving the self-dual vortices on the north chart of the sphere
in Section 4. It then follows that, with very slight modifications, the \(S^2\) solutions
found in that section can be embedded in the \(V_\psi\) chart, becoming self-dual vortices of
the non-linear \(\mathbb{C}P^2\) sigma model of the second species. The situation is reminiscent
of that studied on the sphere in that regular vortices on \(V_\psi\), with a zero of \(\psi_1\) at
their cores, are singular solutions on the chart \(V_\phi\), whereas regular semi-local vortices
on \(V_\phi\), where \(\phi_1\) is nonzero at infinity, are singular on the chart \(V_\psi\). In any case,
the vortices on \(V_\psi\) are stable solutions, the reason being that the \(\psi_1\) field winds at
infinity around a two-dimensional throat of \(H^3\) whose radius \(\sqrt{(4\rho^2 - a^2)(\rho^2 + |w_2|^2)}\)
is positive regardless of the value of \(|w_2|\). Specifically, the choice \(\psi_2 = 0\) leads to
exactly the same vortices as those existing in the north chart in the \(S^2\)-model. Other
constant values of \(\psi_2\) merely give rise to a redefinition of the parameters in the
solutions.

4.3 Final comments

The transition between the charts \(V_\phi\) and \(V_\xi\) works similarly, and leads to couplings
\(D_\mu \xi_p = \partial_\mu \xi_p - iN_p e A_\mu \xi_p\) with \(N_1 = 0, N_2 = -1\) and a self-dual Lagrangian \(L_\xi\), which
adopts exactly the form \([24], [25]\) with the substitutions \(\psi_1 \to \xi_2, \psi_2 \to \xi_1\). The
self-dual vorticial solutions in this third chart therefore belong to the same species as those found in the $V_0$ chart.

The results described along this paper about the existence and structure of self-dual topological defects in the massive gauged $\mathbb{C} P^2$-sigma model suggest a general picture which should be valid for other higher-rank gauged $\mathbb{C} P^n$-sigma non-linear systems. In the $\mathbb{C} P^n$ theory there are $n$ scalar fields and $n+1$ charts, let us call them $V_0, V_1, \ldots, V_n$. Starting with the same coupling for all the fields on the chart $V_0$, we can construct on this chart a self-dual model with semi-local $SU(n)_{\text{global}} \times U(1)_{\text{local}}$ symmetry group, whereas in the other $n$ charts the transition functions would break the global symmetry to a $SU(n-1)_{\text{global}}$ subgroup acting on the neutral fields, and only one field would couple to the $U(1)_{\text{gauge}}$ group through the covariant derivative. There are thus semi-local topological defects on the chart $V_0$ and one-complex-component vortices analogous to those found on the sphere on the remaining ones. The vacuum orbit is $S^{2n-1}$ for the fields on $V_0$ and $\mathbb{H}^{2n-1}$ for $V_1, V_2, \ldots, V_n$. The stability of the solutions is guaranteed, in the case of $V_0$ by the known arguments for semi-local vortices, and in the other charts because the vorticity of the charged field is due to its asymptotic winding around the throat of $\mathbb{H}^{2n-1}$. Some modifications of this scenario could be considered for cases in which the assignation of couplings to the fields on $V_0$ is different. Let us assume, for instance, that there are on $V_0$ $r$ fields with coupling $N_r e$ and $n-r$ fields with coupling $N_{n-r} e$. The symmetry group in this chart is $SU(r)_{\text{global}} \times SU(n-r)_{\text{global}} \times U(1)_{\text{gauge}}$, and there are two types of semi-local vortices corresponding to those found on the sphere on the remaining ones. All the other charts fit into two different classes. There are $r$ charts such that the transition functions break the symmetry to $SU(r-1)_{\text{global}} \times SU(n-r)_{\text{global}} \times U(1)_{\text{gauge}}$, where the fields complying with the global $SU(r-1)$ are the neutral ones, those supporting the global $SU(n-r)$ symmetry have coupling $-(N_r - N_{n-r}) e$, and the remaining field has coupling $-N_r e$. The solutions existing in these charts are of two kinds: there exist embedded $S^2$ vortices, where only the field with $-N_r e$ coupling winds, and there are also defects in which this one-complex-component field is mixed with an $SU(n-r)_{\text{global}} \times U(1)_{\text{gauge}}$ semi-local vortex, with winding numbers proportional to the charges; in all these cases, the neutral fields remain frozen at constant values.

In the remaining $n-r$ charts we observe the reciprocal situation, where the symmetry is $SU(r)_{\text{global}} \times SU(n-r-1)_{\text{global}} \times U(1)_{\text{gauge}}$ and the $SU(n-r-1)$ corresponds to neutral fields, the charged fields have couplings $-(N_{n-r} - N_r) e$ for the global $SU(r)$, and there is a final charged field which couples to $A_\mu$ with intensity $-N_{n-r} e$.

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After the first version of this work had been completed and sent to the arXives, we learned that vorticial solutions on $S^2$ like those reported here had been previously found in [2] as the partons making two-dimensional $\mathbb{C} P^1$ lumps in a $\mathcal{N} = (2,2)$ supersymmetric theory. We thank the authors of [2] for pointing out their results to us, which have prompted us to generalize and enlarge the scope of our previous work.
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