Groundstate asymptotics for a class of singularly perturbed \( p \)-Laplacian problems in \( \mathbb{R}^N \)

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Abstract

We study the asymptotic behavior of positive groundstate solutions to the quasilinear elliptic equation

\[ -\Delta_p u + \varepsilon u^{p-1} - u^{q-1} + u^{l-1} = 0 \quad \text{in} \quad \mathbb{R}^N, \quad (P_\varepsilon) \]

where \( 1 < p < N, \ p < q < l < +\infty \) and \( \varepsilon > 0 \) is a small parameter. For \( \varepsilon \to 0 \), we give a characterisation of asymptotic regimes as a function of the parameters \( q, l \) and \( N \). In particular, we show that the behavior of the groundstates is sensitive to whether \( q \) is less than, equal to, or greater than the critical Sobolev exponent \( p^* := \frac{Np}{N-p} \).

Keywords: Groundstates, Liouville-type theorems, quasilinear equations, singular perturbation.

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1 Introduction

The present paper is devoted to the study of the positive solutions to the quasilinear elliptic equation

$$-\Delta_p u + \varepsilon u^{p-1} - |u|^{q-2} u + |u|^{l-2} u = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{P_\varepsilon}$$

where

$$\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u),$$

is the $p$-Laplacian operator, $1 < p < N$, $p < q < l$ and $\varepsilon > 0$ is a small parameter. Our main aim is to understand the behaviour of positive groundstate solutions to (P_\varepsilon) as $\varepsilon \to 0$.

By a solution to (P_\varepsilon) we mean a weak solution $u_\varepsilon \in W^{1,p}(\mathbb{R}^N) \cap L^l(\mathbb{R}^N)$. These solutions are constructed as critical points of the energy

$$E_\varepsilon(u) := \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx - \int_{\mathbb{R}^N} F_\varepsilon(u) dx, \quad (E_\varepsilon)$$

where

$$F_\varepsilon(u) = \int_0^u \tilde{f}_\varepsilon(s) ds,$$

and the expression $\tilde{f}_\varepsilon$ is a suitable bounded truncation of

$$f_\varepsilon(s) := -\varepsilon s^{p-1} + |s|^{q-2} s - |s|^{l-2} s. \tag{1.1}$$

Throughout the paper by groundstate solution to (P_\varepsilon) we mean a positive weak solution which has the least energy $E_\varepsilon$ amongst all the other non-trivial solutions.

In the first part of the paper, for all $1 < p < N$ and $p < q < l$, we prove the existence of a radial groundstate solution $u_\varepsilon$ of (P_\varepsilon) for all sufficiently small $\varepsilon > 0$, see Theorem 2.1, extending classical results of Berestycki and Lions [3] from the Laplacian ($p = 2$) to the $p$-Laplacian setting, for any $1 < p < N$. As a byproduct of the method [3] which is adapted to the present quasilinear context, the weak solution to (P_\varepsilon) which are found, are essentially bounded, and so by well-known uniform $C^{1,\alpha}$ estimates for the $p$-Laplacian, they decay uniformly to zero as $|x| \to \infty$. We recall
that, as in the known case $p = 2$ treated in [3], the symmetry of the solutions is achieved as a limit of a suitable (minimising) sequence of radially decreasing rearrangements constructed from a possibly non-radial minimising sequence. Theorem 2.1 in Section 6.2 summarises all the above results about the existence and basic properties of the groundstates to $(P)\varepsilon$.

We point out that for large $\varepsilon > 0$ equation $(P)\varepsilon$ has no finite energy solutions, so the restriction on the size of $\varepsilon$ is essential for the existence of the groundstates. The uniqueness (up to translations) of a spherically symmetric groundstate of $(P)\varepsilon$ is rather delicate. For $p \leq 2$, Serrin and Tang proved [33] Theorem 4] equation $(P)\varepsilon$ admits at most one positive groundstate solution. For $p > 2$ the uniqueness could be also expected but to the best of our knowledge this remains an open question. We do not study the question of uniqueness in this paper and none of our result rely on the information about the uniqueness of the groundstate to $(P)\varepsilon$.

The question of understanding the asymptotic behaviour of the groundstates $u_\varepsilon$ of $(P)\varepsilon$ as $\varepsilon \to 0$, naturally arises in the study of various bifurcation problems, for which $(P)\varepsilon$ at least in the case $p = 2$ can be considered as a canonical normal form (see e.g. [8, 39]). This problem may also be regarded as a bifurcation problem for quasilinear elliptic equations

$$-\Delta_p u = f_\varepsilon(u) \quad \text{in} \quad \mathbb{R}^N,$$

whose nonlinearity $f_\varepsilon$ has the leading term in the expansion around zero which coincides with the ones in $(P)\varepsilon$. Let us also mention that problem $(P)\varepsilon$ in the case $p = 2$ appears in the study of phase transitions [6][25][42], as well as in the study of the decay of false vacuum in quantum field theories [7].

Loosely speaking, to understand the asymptotic behaviour of the groundstates $u_\varepsilon$ as $\varepsilon \to 0$, one notes that elliptic regularity implies that locally the solution $u_\varepsilon$ converges as $\varepsilon \to 0$ to a radial solution of the limit equation (see Theorem 6.3)

$$-\Delta_p u - |u|^{p^*-2}u + |u|^{p-2}u = 0 \quad \text{in} \quad \mathbb{R}^N. \quad (P_0)$$

It is known that (here and in the rest of the paper $p^* := \frac{pN}{N-p}$ is the critical Sobolev exponent): when $q \leq p^*$ equation $(P_0)$ has no non-trivial finite energy solution, by Pohožaev’s identity (3.1); whereas for $q > p^*$ equation $(P_0)$ admits a radial groundstate solution. Existence goes back to Berestycki-Lions [3] and Merle-Peletier [23] in the case $p = 2$ and, in the context of the present paper, it is proved in the general $p$-Laplacian case (see Theorem 4.3); whereas uniqueness questions have been studied by Tang [38, Theorem 4.1], see also Remark 4.4.

In Theorem 2.8 we prove using direct variational arguments that, as expected, for $q > p^*$ solutions $u_\varepsilon$ converge as $\varepsilon \to 0$ to a non-trivial radial groundstate solution to the ‘formal’ limit equation $(P_0)$. The fact that for $q \leq p^*$ equation $(P_0)$ has no non-trivial positive solutions, suggests that for $q \leq p^*$ the solutions $u_\varepsilon$ should converge almost everywhere, as $\varepsilon \to 0$, to the trivial zero solution of equation $(P_0)$ (see estimate (2.2)). This however does not reveal any information about the ‘limiting profile’ of $u_\varepsilon$. Therefore, instead of looking at the formally obtained limit equation $(P_\varepsilon)$, we are going to show that for $q \leq p^*$ solutions $u_\varepsilon$ converge to a non-trivial limit after a rescaling. The limiting profile of $u_\varepsilon$ will be obtained from the groundstate solutions of the limit equations associated with the rescaled equation $(P)\varepsilon$, where the choice of the associated rescaling and limit equation depends on the value of $p$ and on the space dimension $N$ in a highly non-trivial way. The convergence of rescaled solutions $u_\varepsilon$ to their limiting profiles will be proved using a variational analysis similar to the techniques developed in [24] in the case $p = 2$. Note that the natural energy space for equation $(P)\varepsilon$ is the usual Sobolev space

$$W^{1,p}(\mathbb{R}^N) := \left\{ u : u \in L^p(\mathbb{R}^N) \quad \text{and} \quad \nabla u \in L^p(\mathbb{R}^N) \right\},$$

with the norm

$$||u||_{1,p} = ||u||_p + ||\nabla u||_p.$$
while for $q > p^*$ the limit equation $(P_0)$ is variationally well-posed in the homogeneous Sobolev space $D^{1,p}(\mathbb{R}^N)$ defined for $1 < p < N$ as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $||\nabla u||_{L^p}$. Since $W^{1,p}(\mathbb{R}^N) \subsetneq D^{1,p}(\mathbb{R}^N)$, it follows that no natural perturbation setting (in the spirit of the implicit function theorem or Lyapunov-Schmidt type reduction methods) is available to analyse the family of equations $(P_\varepsilon)$ as $\varepsilon \to 0$. In fact, even for $p = 2$ a linearisation of $(P_0)$ around the groundstate solution is not a Fredholm operator and has zero as the bottom of the essential spectrum in $L^2(\mathbb{R}^N)$. In the case of the $p$–Laplace equations the difficulty in applying classical perturbation methods is even more striking, as for $1 < p < 2$ the energy associated with the $p$-Laplacian is not twice Fréchet differentiable. In order to understand the limiting profile of $u_\varepsilon$ in the case $q \leq p^*$, we introduce the ‘canonical’ rescaling associated with the lowest order nonlinear term in $(P_\varepsilon)$:

$$v_\varepsilon(x) = \varepsilon^{-\frac{1}{p^*-1}} u_\varepsilon \left( \frac{x}{\varepsilon} \right).$$

(1.2)

Then $(P_\varepsilon)$ reads as

$$-\Delta_p v + v^{p-1} = v^{q-1} - \varepsilon \tilde{v}^{p-1} \quad \text{in} \quad \mathbb{R}^N,$n$$

from which we formally get, as $\varepsilon \to 0$, the limit problem

$$-\Delta_p v + v^{p-1} = v^{q-1} \quad \text{in} \quad \mathbb{R}^N.$$ 

(RO)

We recall that for $q \geq p^*$ equation $(R_0)$ has no non-trivial finite energy solutions, as a consequence of Pohozaev’s identity (3.1); whereas for $p < q < p^*$ equation $(R_0)$ possesses a unique radial groundstate solution. Existence was proved in [15] and uniqueness by Pucci-Serrin [29] Theorem 2. The particular rescaling (1.2) allows to have, when $p < q < p^*$, for both $(R_0)$ and the limit problem $(R_0)$, a variational formulation on the same Sobolev space $W^{1,p}(\mathbb{R}^N)$. This indicates that problem $(R_\varepsilon)$ could be considered as a small perturbation of the limit problem $(R_0)$. In particular in the case $p = 2$ the family of the ground states $(v_\varepsilon)$ of problem $(P_\varepsilon)$ could be rigorously interpreted as a perturbation of the groundstate solution of the limit problem $(R_0)$ using the perturbation techniques and framework developed by Ambrosetti, Malchiodi et al., see [2] and references. However, for $p \neq 2$ the Lyapunov–Schmidt reduction technique, in the spirit of [2] is not directly applicable. Instead, in this work, using a direct variational argument inspired by [24] Theorem 2.1 we prove (see Theorem 2.2) that for $p < q < p^*$ groundstate solutions $(v_\varepsilon)$ of the rescaled problem $(R_\varepsilon)$ converge to the (unique) radial groundstate of the limit problem $(R_0)$.

In the critical case $q = p^*$, the limit problem $(R_0)$ has no non-trivial positive solutions. This means that in this case the ‘canonical’ rescaling (1.2) does not accurately capture the behaviour of $(v_\varepsilon)$. In the present paper, extending the results obtained in [24] for $p = 2$, we show that for $q = p^*$ the asymptotic behaviour of the groundstate solutions to $(P_\varepsilon)$ after a rescaling is given by a particular solution of the critical Emden-Fowler equation

$$-\Delta_p U = U^{p^*-1} \quad \text{in} \quad \mathbb{R}^N.$$ 

(R_*)

It is well-known that equation $(R_*)$ admits a continuum of radial groundstate solutions. We will prove that the choice of the rescaling (and a particular solution of $(R_*)$) which provides the limit asymptotic profile for groundstate solutions $v_\varepsilon$ to equation $(P_\varepsilon)$ depends on the dimension $N$ in a non-trivial way (see Theorem 2.3).

Wrapping up, we provide a characterisation of the three asymptotic regimes occurring as $\varepsilon \to 0$, i.e. the subcritical case $q < p^*$, the supercritical case $q > p^*$ and the critical case $q = p^*$, extending the results of [23] and [24], to both a singular ($p < 2$) and degenerate ($p > 2$) quasilinear setting.

**Asymptotic notation**

Throughout the paper we will extensively use the following asymptotic notation. For $\varepsilon \ll 1$ and $f(\varepsilon), g(\varepsilon) \geq 0$, we write $f(\varepsilon) \preceq g(\varepsilon)$, $f(\varepsilon) \sim g(\varepsilon)$ and $f(\varepsilon) \simeq g(\varepsilon)$, implying that there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0$:

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we study the limit behaviour of such a branch of ground states when $\varepsilon \to 0$. As anticipated earlier, none of our subsequent results rely on the uniqueness of $u_\varepsilon$ for all $\varepsilon > 0$. As usual, $C, c, c_1, \ldots$ denote generic positive constants independent of $\varepsilon$.

2 Main results

The following theorem summarizes the existence results for the equation \((P_\varepsilon)\). The proof is a standard adaptation of the Berestycki and Lions method [53]. For completeness, we sketch the arguments in Section 3.

**Theorem 2.1.** Let $N \geq 2$, $1 < p < N$ and $p < q < l$. Then there exists $\varepsilon_* = \varepsilon_*(p,q,l) > 0$ such that for all $\varepsilon \in (0, \varepsilon_*)$, equation \((P_\varepsilon)\) admits a ground state $u_\varepsilon \in W^{1,p}(\mathbb{R}^N) \cap L^l(\mathbb{R}^N) \cap C^{1,\alpha}_{loc}(\mathbb{R}^N)$. Moreover, $u_\varepsilon(x)$ is a positive monotone decreasing function of $|x|$ and

$$u_\varepsilon(|x|) \leq Ce^{-\delta|x|}, \quad x \in \mathbb{R}^N,$$

for some $C, \delta > 0$.

For $p \leq 2$, Serrin and Tang proved [33, Theorem 4] that equation \((P_\varepsilon)\) admits at most one positive ground state solution. For $p > 2$ the uniqueness to the best of our knowledge remains an open question. As anticipated earlier, none of our subsequent results rely on the uniqueness of ground states of \((P_\varepsilon)\). In what follows, $u_\varepsilon$ always denotes ‘a’ groundstate solution to \((P_\varepsilon)\) for an $\varepsilon \in (0, \varepsilon_*)$. When we say that groundstates $u_\varepsilon$ converge to a certain limit (in some topology) as $\varepsilon \to 0$, we understand that for every $\varepsilon > 0$ a groundstate of \((P_\varepsilon)\) is selected, so that $(u_\varepsilon)_{\varepsilon \in (0, \varepsilon_*)}$ is a branch of groundstates of \((P_\varepsilon)\), which is not necessarily continuous in $\varepsilon$. In the present work we study the limit behaviour of such a branch of ground states when $\varepsilon \to 0$.

2.1 Subcritical case $p < q < p^*$

As anticipated earlier, since in the subcritical case the “ formal ” limit equation \((P_0)\) has no groundstate solutions, the family of groundstates $u_\varepsilon$ must converge to zero, uniformly on compact subsets. We describe the asymptotic behaviour of $u_\varepsilon$ performing the rescaling \((1.2)\) which transforms \((P_\varepsilon)\) into equation \((P_0)\). In Section 4 using the variational approach developed in the main part of this work we prove the following result, which extends [24, Theorem 2.1] to the case $p \neq 2$.

**Theorem 2.2.** Let $N \geq 2$, $1 < p < N$, $p < q < p^*$ and $(u_\varepsilon)$ be a family of groundstates of \((P_\varepsilon)\). As $\varepsilon \to 0$, the rescaled family

$$v_\varepsilon(x) := \varepsilon^{-\frac{1}{p-1}} u_\varepsilon \left( \frac{x}{\sqrt[2]{\varepsilon}} \right) \quad (2.1)$$

converges in $W^{1,p}(\mathbb{R}^N)$, $L^l(\mathbb{R}^N)$ and $C^{1,\alpha}_{loc}(\mathbb{R}^N)$ to the unique radial groundstate solution $v_0(x)$ of the limit equation \((P_0)\). In particular,

$$u_\varepsilon(0) \simeq \varepsilon^{-\frac{p}{p^*-1}} v_0(0). \quad (2.2)$$

2.2 Critical case $q = p^*$

In this case we show that after a suitable rescaling the correct limit equation for \((P_\varepsilon)\) is given by the critical Emden-Fowler equation

$$-\Delta_p U = U^{p^*-1} \quad \text{in } \mathbb{R}^N. \quad (R_*)$$
It is well-known by Guedda-Veron [16] that the only radial solution to \( \{ R_e \} \) is given, up to the sign, by the family of rescalings

\[
U_\lambda(|x|) := U_1(|x|/\lambda) \quad (\lambda > 0),
\]

where

\[
U_1(|x|) := \left[ \frac{\kappa^{1/p'} N^{1/p} }{1 + |x|^{p'}} \right]^{\kappa/p'},
\]

and where \( p' := \frac{p}{p-1} \) and \( \kappa := \frac{N-p}{p-1} \). Recently in [12] it has been observed that \( \pm U_\lambda \) are the only nontrivial radial solutions to \( \Delta_p u + |u|^{p-2} u = 0 \) in \( D^{1,p}(\mathbb{R}^N) \). Sciunzi [32] proved that any positive solution to \( \{ R_e \} \) in \( D^{1,p}(\mathbb{R}^N) \) is necessarily radial about some point; this combined with [16] gives a complete classification of the positive finite energy solutions to \( \{ R_e \} \).

Our main result in this work is the following theorem, which extends [24] Theorem 2.5 to the case \( p \neq 2 \).

**Theorem 2.3.** Let \( N \geq 2 \), \( 1 < p < N \), \( p^* = q < l \) and \( (u_\varepsilon) \) be a family of groundstates of \( (P_\varepsilon) \). There exists a rescaling

\[
\lambda_\varepsilon : (0, \varepsilon_*) \to (0, \infty)
\]

such that as \( \varepsilon \to 0 \), the rescaled family

\[
v_\varepsilon(x) := \lambda_\varepsilon^{\frac{N-p}{p}} u_\varepsilon(\lambda_\varepsilon x)
\]

converges in \( D^{1,p}(\mathbb{R}^N) \), \( L^1(\mathbb{R}^N) \) and \( C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N) \) to the radial groundstate solution \( U_1(x) \) of the Emden–Fowler equation \( \{ R_e \} \). Moreover,

\[
\lambda_\varepsilon \gtrsim \begin{cases} 
\varepsilon^{-\frac{p^*}{p(p-1)}} & 1 < p < \sqrt{N}, \\
\left( \varepsilon (\log \frac{\varepsilon}{\lambda_\varepsilon}) \right)^{\frac{p^*}{p(p-1)}} & p = \sqrt{N}, \\
\varepsilon^{-\frac{p}{l(p-1)+2}} & \sqrt{N} < p < N,
\end{cases}
\]

and

\[
\lambda_\varepsilon \lesssim \begin{cases} 
\varepsilon^{-\frac{p^*}{p(p-1)}} & 1 < p < \sqrt{N}, \\
\left( \varepsilon (\log \frac{\varepsilon}{\lambda_\varepsilon}) \right)^{\frac{p^*}{p(p-1)}} & p = \sqrt{N}, \\
\varepsilon^{-\frac{p}{l(p-1)+2}} & \sqrt{N} < p < N.
\end{cases}
\]

**Remark 2.4.** The lower bound (2.6) on \( \lambda_\varepsilon \) can be converted into an upper bound on the maximum of \( u_\varepsilon \),

\[
u_\varepsilon(0) \lesssim \begin{cases} 
\varepsilon^{-\frac{p^*}{p(p-1)}} & 1 < p < \sqrt{N}, \\
\left( \varepsilon (\log \frac{\varepsilon}{\lambda_\varepsilon}) \right)^{\frac{p^*}{p(p-1)}} & p = \sqrt{N}, \\
\varepsilon^{-\frac{p}{l(p-1)+2}} & \sqrt{N} < p < N.
\end{cases}
\]

For \( 1 < p < \sqrt{N} \) lower bound (2.6) and upper bound (2.7) are equivalent and hence optimal. For \( \sqrt{N} \leq p < N \), the upper bounds in (2.7) do not match the lower bounds (2.6). However, under some additional restrictions we could obtain optimal two–side d estimates.

**Theorem 2.5.** Under the assumptions of Theorem 2.3 we additionally have

\[
\lambda_\varepsilon \sim \begin{cases} 
\varepsilon^{-\frac{p^*}{p(p-1)}} & 1 < p < \sqrt{N} \text{ and } N \geq 2, \\
\left( \varepsilon (\log \frac{\varepsilon}{\lambda_\varepsilon}) \right)^{-\frac{p^*}{p(p-1)}} & p = \sqrt{N} \text{ and } N \geq 4, \\
\varepsilon^{-\frac{p}{l(p-1)+2}} & \sqrt{N} < p < \frac{N+1}{2} \text{ and } N \geq 4,
\end{cases}
\]
and

$$u_\varepsilon(0) \sim \begin{cases} \\ \varepsilon^{-\frac{p-1}{p}} & 1 < p < \sqrt{N} \text{ and } N \geq 2, \\ \varepsilon^{\frac{p}{2}} \left(\varepsilon(\log \frac{1}{\varepsilon})ight)^{\frac{p-2}{p}} & p = \sqrt{N} \text{ and } N \geq 4, \\ \sqrt{N} < p < \frac{N+1}{2} \text{ and } N \geq 4. \end{cases}$$

(2.10)

**Remark 2.6.** In the case \( p = 2 \) and \( N \geq 3 \), two-sided asymptotics of the form (2.9) were derived in [25] using methods of formal asymptotic expansions. Later, two sided bounds of the form (2.9) were rigorously established for \( p = 2 \) in [24, Theorem 2.5]. The barrier approach developed in [24, Lemma 4.8] in order to refine upper bounds on \( \lambda_\varepsilon \) in the difficult case \( \sqrt{N} \leq p < N \) cannot be fully extended to \( p \neq 2 \), see Lemma 5.11. In this difficult case, the matching upper bounds of the form (2.6) are valid for \( \sqrt{N} < p < \frac{N+1}{2} \) and \( N \geq 4 \).

**Remark 2.7.** Theorem 2.5 leaves open the following cases, where matching lower and upper bounds are not available:

- \( N \geq 4 \) and \( \frac{N+1}{2} \leq p < N \)
- \( N = 3 \) and \( \sqrt{3} \leq p < 2 \)
- \( N = 2 \) and \( \sqrt{2} \leq p < 2 \)

Note that the case \( N = 3 \) and \( p = 2 \) is not included in Theorem 2.5. However, matching bound (2.9) and (2.10) remain valid in this case. This is one of the results in [24, Theorem 2.5]. We conjecture that the restriction \( p < \frac{N+1}{2} \) is merely technical and is due to the method we use.

### 2.3 Supercritical case \( q > p^* \)

Unlike the subcritical and critical cases, for \( q > p^* \) the ‘formal’ limit equation (2.8) admits a nontrivial solution. Using a direct analysis of the family of constrained minimization problems associated with (P\(\varepsilon\)), we prove the following result, which extends [24, Theorem 2.3] to the case \( p \neq 2 \).

**Theorem 2.8.** Let \( N \geq 2 \), \( 1 < p < N \), \( p^* < q < l \) and \( (u_\varepsilon) \) be a family of groundstates of (P\(\varepsilon\)). As \( \varepsilon \to 0 \), the family \( u_\varepsilon \) converges in \( D^{1,p}(\mathbb{R}) \), \( L^1(\mathbb{R}) \) and \( C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N) \) to a groundstate solution \( u_0(x) \) of the limit equation (P\(0\)), with

$$u_0(x) \sim |x|^{-\frac{N-p}{2}} \quad \text{as} \quad |x| \to \infty.$$  

Moreover, it holds that

$$u_\varepsilon(0) \approx u_0(0),$$

and that \( \varepsilon||u_\varepsilon||_p \to 0 \).

### 2.4 Organisation of the paper

This paper is organised as follows. Section 3 is devoted to the existence and qualitative properties of groundstates \( u_\varepsilon \) to (P\(\varepsilon\)), whereas in Section 4 we deal with existence and qualitative properties of groundstates to the limiting PDE’s (P\(0\)), (P\(\lambda_0\)), (P\(\lambda_\infty\)). Both sections contain various facts about the equation (P\(\varepsilon\)) and limiting equations which are involved in our analysis. In the rest of the paper we study the asymptotic behaviour of the groundstates \( u_\varepsilon \). In Section 5 we study the most delicate critical case \( q = p^* \) and prove Theorem 2.3. In Section 6 we consider the supercritical case \( q > p^* \) and prove Theorem 2.8. In Section 7 we consider the subcritical case \( q < p^* \) and prove Theorem 2.2. For the reader convenience we have collected in the sections A and B of the Appendix some auxiliary results which have been used in the main body of the paper.
3 Groundstate solutions to \((P_\varepsilon)\)

3.1 Necessary conditions and Pohožaev’s identity

According to Pohožaev’s classical identity [26] for \(p\)-Laplacian equations, a solution to \((P_\varepsilon)\) which is smooth enough, necessarily satisfies the identity

\[
\int_{\mathbb{R}^N} |\nabla u|^p dx = p^* \int_{\mathbb{R}^N} F(u) dx,
\]

for \(1 < p < N\). Identities of this type are classical, see for instance in [28] for \(C^2\) solutions and [9] for bounded domains. In the present paper the following version of Pohožaev’s identity has been extensively used.

**Proposition 3.1.** Suppose \(f : \mathbb{R} \to \mathbb{R}\) is a continuous function such that \(f(0) = 0\), and set

\[
F(t) = \int_0^t f(s) ds.
\]

Let \(u \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)\), and \(|\nabla u|, F(u) \in L^1(\mathbb{R}^N)\) with \(u\) such that

\[-\Delta_p u = f(u),\]

holds in the sense of distribution. Then \(u\) satisfies (3.1).

**Proof.** We first assume that \(p \leq 2\). By Theorem 2.5 in [31], we have

\[u \in W^{2,p}_{\text{loc}}(\mathbb{R}^N), \quad p \leq 2.\]

Having checked the existence and local summability of the second weak derivatives in this case we argue as follows. Multiply the equation by \(x_i \partial_i u(x)\) and integrate over \(B_R = B(0, R)\) and denote by \(n(\cdot)\) the outer normal unit vector. Observe that the vector field

\[v = x_i \partial_i u |\nabla u|^{p-2} \nabla u\]

is such that \(v \in C(\mathbb{R}^N, \mathbb{R}^N)\) and \(\text{div} v \in L^1_{\text{loc}}(\mathbb{R}^N)\). By the divergence theorem (see e.g. Lemma 2.1 in [22]) we have

\[
\int_{B_R} \Delta_p u x_i \partial_i u(x) dx = \int_{\partial B_R} |\nabla u(\sigma)|^{p-2} \partial_i u(\sigma) \sigma_i \nabla u \cdot n d\sigma
- \int_{B_R} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla [x_i \partial_i u(x)] dx.
\]

Write the last integral as \(A_i + B_i\), where

\[
A_i := \int_{B_R} |\nabla u(x)|^{p-2} \partial_i u(x)^2 dx,
B_i := \frac{1}{p} \int_{B_R} \partial_i (|\nabla u(x)|^p) x_i dx.
\]

An integration by parts in \(B_i\) yields

\[
B_i = \frac{1}{p} \int_{\partial B_R} |\nabla u(\sigma)|^p \sigma_i n_i d\sigma - \frac{1}{p} \int_{B_R} |\nabla u(x)|^p dx.
\]

On the other hand we have also

\[
\int_{B_R} f(u(x)) x_i \partial_i u(x) dx
\]
\[= - \int_{B_R} F(u(x))dx + \int_{\partial B_R} F(u(x))\sigma, n_x d\sigma.\]

Summing up on \(i\) we have
\[(*) \quad N \int_{B_R} F(u(x))dx + \left(1 - \frac{N}{p}\right) \int_{B_R} |\nabla u(x)|^pdx = \int_{\partial B_R} |\nabla u(\sigma)|^{p-2} \nabla u \cdot \sigma \nabla u \cdot n d\sigma
- \frac{1}{p} \int_{\partial B_R} |\nabla u(\sigma)|^p \sigma \cdot n d\sigma + \int_{\partial B_R} F(u(x))\sigma \cdot n d\sigma.\]

The right hand side is bounded by
\[M(R) = \left(1 + \frac{1}{p}\right) R \int_{\partial B_R} |\nabla u(\sigma)|^p d\sigma + R \int_{\partial B_R} |F(u(x))|d\sigma.\]

Similarly as in Lemma 2.3 from [22], since \(F(u), |\nabla u|^p \in L^1(\mathbb{R}^N)\) there exists a sequence \(R_n \to \infty\) such that \(M(R_n) \to 0\). By using the monotone convergence theorem in (*) we obtain the conclusion in the case \(p \leq 2\).

For \(p > 2\) a regularisation argument similar to [11, p. 833] (see also [12, 17, 20]) allows to work with a \(C^{1,\alpha}_{\text{loc}}\) approximation \(u_{\varepsilon} \in C^2\) which classically solves
\[-\text{div} \left( (\varepsilon + |\nabla u_{\varepsilon}|^2)^{\frac{p-2}{2}} \nabla u_{\varepsilon} \right) = f(u) \quad \text{in } B_{2R},
\] \[u_{\varepsilon} = u \quad \text{on } \partial B_{2R}.\]

The proof can be then carried out with similar modifications of the proof given in the case \(p \leq 2\), performing the \(\varepsilon\)-limit before letting \(R \to +\infty\) along a suitable sequence \((R_n)_{n \in \mathbb{N}}\), and this concludes the proof. \(\square\)

### 3.2 Existence and variational characterisation of the groundstates

To prove the existence of ground states, we first observe that the method of Berestycki-Lions [3] although focused on the case \(p = 2\) is applicable in the present quasilinear context, we sketch the proof referring to [3] for the details. In fact, observe that \(f_\varepsilon(s) = |s|^{q-2}s - |s|^{r-2}s - \varepsilon |s|^{p-2}s\) satisfies
\[f_1 \quad -\infty < \lim_{s \to 0^+} \frac{f_\varepsilon(s)}{s^{p-1}} \leq \lim_{s \to 0^+} \frac{f_\varepsilon(s)}{s^{p-1}} = -\varepsilon < 0.\]

\[f_2 \quad -\infty \leq \liminf_{s \to +\infty} \frac{f_\varepsilon(s)}{s^{p-1}} \leq 0, \quad \text{where } p^* = \frac{pN}{N-p}.
\]

\[f_3 \quad \text{There exists } \varepsilon_* > 0 \text{ such that for all } \varepsilon \in (0, \varepsilon_*) \text{ the following property holds: there exists } \zeta > 0 \text{ such that } F_\varepsilon(\zeta) = \int_0^\zeta f_\varepsilon(s)ds > 0.\]

To prove the existence of an optimiser, one carries on with the constrained minimisation argument as in [3], based on the truncation of the nonlinearity \(f_\varepsilon\), which allows to use \(W^{1,p}(\mathbb{R}^N)\) for the functional setting. For all \(\varepsilon \in (0, \varepsilon_*)\) in the present context \(p \neq 2\) a suitable truncated function \(\tilde{f}_\varepsilon : \mathbb{R} \to \mathbb{R}\) is provided by:
\[\tilde{f}_\varepsilon(u) = \begin{cases} 0, & u < 0, \\ u^{q-1} - u^{l-1} - \varepsilon u^{p-1}, & u \in [0, 1], \\ -\varepsilon, & u > 1. \end{cases}\]

Replacing in [22] the non-linearity with the above bounded truncation \(\tilde{f}_\varepsilon(u)\) makes the minimisation problem
\[S_\varepsilon = \inf \left\{ \int_{\mathbb{R}^N} |\nabla w|^pdx; \quad w \in W^{1,p}(\mathbb{R}^N), \quad p^* \int_{\mathbb{R}^N} \tilde{f}_\varepsilon(w)dx = 1 \right\}. \quad (S_\varepsilon)\]
well-posed in $W^{1,p}(\mathbb{R}^N)$ even for supercritical $l > p^*$. Standard compactness arguments using radially symmetric rearrangements of minimising sequences allows to obtain a radially decreasing optimiser $w_\varepsilon$. If $w_\varepsilon$ is an optimiser for $(S_\varepsilon)$ then a Lagrange multiplier $\theta_\varepsilon$ exists such that
\[-\Delta_p w_\varepsilon = \theta_\varepsilon f_\varepsilon(w_\varepsilon) \quad \text{in } \mathbb{R}^N.\] (3.3)
Note that by construction $f_\varepsilon(u) \in L^\infty(\mathbb{R}^N)$ and then by a classical result of DiBenedetto, see e.g. Corollary p. 830 in [11], any solution $u \in W^{1,p}(\mathbb{R}^N)$ to the truncated problem with $f_\varepsilon$ is regular, i.e. $u \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$. Then the maximum principle implies that any solution for the truncated problem is strictly positive and solves the original problem
\[-\Delta_p w_\varepsilon = \theta_\varepsilon f_\varepsilon(w_\varepsilon) \quad \text{in } \mathbb{R}^N.\] (3.4)

The exponential decay estimate (3.10) on $w_\varepsilon$ follows by Gazzola-Serrin ([14, Theorem 8]). As a consequence of the regularity and summability, $w_\varepsilon$ satisfies both Nehari’s identity
\[
\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^p dx = \theta_\varepsilon \int_{\mathbb{R}^N} f_\varepsilon(w_\varepsilon) w_\varepsilon dx,
\] (3.5)
and Pohozaev’s identity (3.1)
\[
\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^p dx = \theta_\varepsilon p^* \int_{\mathbb{R}^N} F_\varepsilon(w_\varepsilon) dx.
\] (3.6)

The latter immediately implies that
\[
\theta_\varepsilon = S_\varepsilon.
\] (3.7)

Then a direct calculation involving (3.7) shows that the rescaled function
\[
u_\varepsilon(x) := w_\varepsilon(x/\sqrt{S_\varepsilon})
\] (3.8)
is the radial groundstate of $(P_\varepsilon)$, described in Theorem 3.2 below.

One more consequence of Pohozaev’s identity (3.6) is an expression for the total energy of the solution
\[
E_\varepsilon(u_\varepsilon) = \left(\frac{1}{p} - \frac{1}{p^*}\right) S_\varepsilon^{N/p},
\]
(see [3, Corollary 2]), which shows that $u_\varepsilon$ is indeed a ground state, i.e. a nontrivial solution with the least energy. Another simple consequence of (3.6) is that $(P_\varepsilon)$ has no nontrivial finite energy solutions for $\varepsilon \geq \varepsilon_*$. The threshold value $\varepsilon_*$ is simply the smallest value of $\varepsilon > 0$ for which the energy $E_\varepsilon$ is non-negative and can be computed explicitly.

To summarize, in the spirit of [3, Theorem 2] we have the following

**Theorem 3.2.** Let $N \geq 2$, $1 < p < N$ and $p < q < l$. Then there exists $\varepsilon_* = \varepsilon_*(p,q,l) > 0$ such that for all $\varepsilon \in (0, \varepsilon_*)$, the minimization problem $(S_\varepsilon)$ has a minimizer $w_\varepsilon \in W^{1,p}(\mathbb{R}^N) \cap L^l(\mathbb{R}^N) \cap C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$. The minimizer $w_\varepsilon$ satisfies
\[-\Delta_p w_\varepsilon = S_\varepsilon f_\varepsilon(w_\varepsilon) \quad \text{in } \mathbb{R}^N.\] (3.9)
Moreover, $w_\varepsilon(x)$ is a positive monotone decreasing function of $|x|$ and
\[w_\varepsilon(|x|) \leq C e^{-\delta |x|}, \quad x \in \mathbb{R}^N,
\] (3.10)
for some $C, \delta > 0$. The rescaled function
\[u_\varepsilon(x) := w_\varepsilon(x/\sqrt{S_\varepsilon})
\]
is a groundstate solution to $(P_\varepsilon)$.
In view of (3.2) and since we are interested only in positive solutions of $(P_\varepsilon)$, in what follows we always assume that the nonlinearity $f_\varepsilon(u)$ in $(P_\varepsilon)$ is replaced by its bounded truncation $\tilde{f}_\varepsilon(u)$ from (3.2), without mentioning this explicitly.

**Remark 3.3.** Equivalently to $(S_\varepsilon)$, we can consider minimising the quotient

$$S_\varepsilon(w) := \frac{||\nabla w||_p^p}{\left(\frac{p^*}{p^*} \int_{\mathbb{R}^N} F_\varepsilon(w) dx\right)^{(N-p)/N}}, \quad w \in \mathcal{M}_\varepsilon,$$

where

$$\mathcal{M}_\varepsilon := \left\{0 \leq w \in W^{1,p}(\mathbb{R}^N), \int_{\mathbb{R}^N} F_\varepsilon(w) dx > 0\right\}.$$

Setting $w_\lambda(x) := w(\lambda x)$, it is easy to check that $S_\varepsilon(w_\lambda) = S_\varepsilon(w)$ for all $\lambda > 0$. Therefore it holds that

$$S_\varepsilon = \inf_{w \in \mathcal{M}_\varepsilon} S_\varepsilon(w). \quad (3.11)$$

Moreover the inclusion $\mathcal{M}_{\varepsilon_2} \subset \mathcal{M}_{\varepsilon_1}$ for $\varepsilon_2 > \varepsilon_1 > 0$, (3.11) implies that $S_\varepsilon$ is a nondecreasing function of $\varepsilon \in (0, \varepsilon_*)$.

### 4 Limiting PDE’s

#### 4.1 Critical Emden-Fowler Equation

In this section, we recall some old and new results for the critical Emden-Fowler equation

$$-\Delta_p u = |u|^{p^* - 2} u, \quad u \in D^{1,p}(\mathbb{R}^N), \quad u > 0, \quad (R_*)$$

where $1 < p < N$, $p^* = pN/(N - p)$ is the critical exponent for the Sobolev embedding. We observe that any nontrivial non-negative solution to $(R_*)$ is necessarily positive as a consequence of strong maximum principle (see [10]). Solutions of $(R_*)$ are critical points of the functional

$$J(u) := \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{1}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} dx. \quad (4.1)$$

By the Sobolev embedding $D^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N)$, $J$ is defined in $D^{1,p}(\mathbb{R}^N)$. Since by [13] all the minimising sequences for

$$S_* := \inf \left\{\int_{\mathbb{R}^N} |\nabla w|^p dx; \quad w \in D^{1,p}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |w|^{p^*} dx = 1\right\}, \quad (S_*)$$

are relatively compact modulo translations and dilations, critical points for $J$ are provided by direct minimisation, after suitable rescaling of solutions $W$ to the Euler–Lagrange equation for $S_* - \Delta_p W = \theta W^{p^* - 1} \text{ in } \mathbb{R}^N. \quad (4.2)$

Indeed since

$$\int_{\mathbb{R}^N} |\nabla W|^p dx = \theta \int_{\mathbb{R}^N} |W|^{p^*} dx = \theta,$$

it follows that $S_* = \theta$, hence $W$ and by invariance the family $W_{\lambda,y}$ solve

$$-\Delta_p W_{\lambda,y} = S_* W_{\lambda,y}^{p^* - 1} \text{ in } \mathbb{R}^N. \quad (4.3)$$

Positive finite energy solutions to this equation are classified after the works of Guedda-Veron [16] and of Sciunzi [32] mentioned in the introduction, which we recall in the following
Theorem 4.1. Let $1 < p < N$. Then every radial solution $U$ to (4.4) is represented as

$$U(|x|) = U_{\lambda,0}(|x|) := \left[ \frac{\lambda^{p'/p} k^{1/p'} N^{1/p}}{\lambda^{p'} + |x|^{p'}} \right]^{k/p'},$$

(4.4)

for some $\lambda > 0$, where $p' := \frac{p}{p-1}$ and $k := \frac{N-p}{p-1}$, [16].

In fact, every solution $U$ to (4.4) is radially symmetric about some points $y \in \mathbb{R}^N$ and therefore it holds that

$$U(x) = U_{\lambda,y}(x) := \left[ \frac{\lambda^{p'/p} k^{1/p'} N^{1/p}}{\lambda^{p'} + |x-y|^{p'}} \right]^{k/p'},$$

(4.5)

for some $\lambda > 0$ and $y \in \mathbb{R}^N$, [32].

In the case $p = 2$ and $N \geq 3$ this result is classical, see [5]. Hence, the radial ground state of (R) is given by rescaling the function

$$U_{1,0}(x) := \left[ \frac{k^{1/p'} N^{1/p}}{1 + |x|^{p'}} \right]^{k/p'},$$

(4.6)

and moreover it holds that

$$||\nabla U_{\lambda,0}||_p^p = ||U_{\lambda,0}||_{p^*}^p = S^N_{p^*},$$

(4.7)

see e.g. [37]. It follows that all the positive minimizers for (S) are translations of the radial family

$$W_\lambda(x) := U_{\lambda,0}(\sqrt{S}x).$$

(4.8)

4.2 Supercritical zero mass equation

This section is devoted to the supercritical equation

$$-\Delta_p u - |u|^{q-2}u + |u|^{l-2}u = 0 \quad \text{in} \quad \mathbb{R}^N,$$

(4.9)

where $1 < p < N$ and $p^* < q < l$.

Remark 4.2. Note that by Pohozaev’s identity (3.1), equation (P0) has no solution in $D^{1,p}(\mathbb{R}^N) \cap C^{1,\alpha}_{loc}(\mathbb{R}^N)$ $q \leq p^*$.

We prove the following existence result in the spirit of Merle-Peletier [23] to the case $p \neq 2$.

Theorem 4.3. Let $N \geq 2$, $1 < p < N$ and $p^* < q < l$. Equation (P0) admits a groundstate solution $u_0 \in D^{1,p}(\mathbb{R}^N) \cap L^l(\mathbb{R}^N) \cap C^{1,\alpha}_{loc}(\mathbb{R}^N)$, such that $u_0(x)$ is a positive monotone decreasing function of $|x|$ and

$$u_0(x) \simeq |x|^{-\frac{2-N}{l-2}} \quad \text{as} \quad |x| \to \infty.$$  

(4.10)

Remark 4.4. The uniqueness result of [38] is applicable to fast decay solutions to (P0). However the regularity hypothesis H1 as stated at p. 155 in [38] would require $p^* \geq 2$, namely $p \geq \frac{2N}{N+2}$.

Proof. Following Berestycki-Lions [9] in the present ‘zero-mass case’ context we solve the variational problem in $D^{1,p}(\mathbb{R}^N)$ namely

$$S_0 := \inf \left\{ \int_{\mathbb{R}^N} |\nabla w|^p dx \middle| w \in D^{1,p}(\mathbb{R}^N), \quad p^* \int_{\mathbb{R}^N} \tilde{F}_0(w) dx = 1 \right\},$$

(S0)

where

$$\tilde{F}_0(w) = \int_0^w f_0(s) ds,$$
and \( \tilde{f}_0(s) \) is a bounded truncation of the nonlinearity

\[
f_0(s) = |s|^{q-2}s - |s|^{l-2}s,
\]
e. g.

\[
\tilde{f}_0(u) = \begin{cases} 
0, & u < 0, \\
q^{-1} - l^{-1}, & u \in [0, 1], \\
0, & u > 1.
\end{cases}
\]

The above bounded truncation makes the minimisation problem well-posed in \( D^{1,p}(\mathbb{R}^N) \). Arguing as for the positive mass case the existence of a radially decreasing optimiser \( u \) is standard.

The global boundedness of the truncation allows to use the classical result of DiBenedetto, see e.g. Corollary p. 830 in [11], to show that \( u \in C^{1,\alpha}_{loc}(\mathbb{R}^N) \). Then the maximum principle implies that any solution for the truncated problem solves in fact \((P_0)\) and is strictly positive.

Note that by Ni’s inequality A.3 and the \( C^{1,\alpha}_{loc}(\mathbb{R}^N) \) regularity it follows that \( u \in L^\infty(\mathbb{R}^N) \). By interpolation with Sobolev’s inequality this implies that \( u \in L^l(\mathbb{R}^N) \) for all \( l > p^* \).

With the lemmas below on the asymptotic decay we conclude the proof.

The following lemma about asymptotic properties of solutions is taken from [13].

**Lemma 4.5** ([13, Corollary 8.3.]). Let \( 1 < p < N \). Assume that

\[
\|V(x)\| \leq \frac{g(|x|)}{1 + |x|^p},
\]

where \( g : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is bounded and continuous and satisfies the following conditions:

\[
(C1) \int_1^\infty t^{1-N} \int_{|x|=t} \frac{g(|x|)}{|x|^p+|x|^{N-1}} d|x| \frac{dt}{|x|} < \infty.
\]

\[
(C2) \int_1^\infty g(|x|) d|x| < \infty.
\]

Assume that

\[
-\Delta_p u + V(x)u^{p-1} = 0, \quad \text{in } \mathbb{R}^N \setminus B_1(0),
\]

admits a positive supersolution. Then \((4.11)\) admits a solution which satisfies

\[
U_0 \simeq C |x|^{-\frac{N-p}{p-1}} \quad \text{as } |x| \to \infty.
\]

**Corollary 4.6.** If

\[
V(x) = \frac{c}{(1 + |x|)^{p+\delta}},
\]

and \( c \) is sufficiently small then \((4.11)\) admits a positive solution that satisfies \((4.12)\).

**Proof.** We can take

\[
g(|x|) = |x|^{-\delta}.
\]

Then \((C1), (C2)\) are elementary to check.

The decay estimate \((4.9)\) is proved in the following lemma.

**Lemma 4.7.** Let \( u_0 \in D^{1,p}(\mathbb{R}^N) \cap L^l(\mathbb{R}^N) \) be a positive radial solution of \((P)\). Then

\[
u_0 \simeq |x|^{\frac{N-p}{p-1}} \quad \text{as } |x| \to \infty.
\]
Proof. Since \( u_0 \in D^{1,p}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \) is radial then by the Ni type inequality (4.3) we have

\[
u_0 \leq c |x|^{-\frac{N-p}{p}} \quad \text{in } \mathbb{R}^N \setminus B_1(0).
\]

Ans since \( l > p^* \) then we have for some \( \delta_1 > 0 \)

\[
u_0^{l-p} \leq c |x|^{-\frac{N-p}{p}(l-p)} = \frac{c}{|x|^{p+\delta_1}} \quad \text{in } \mathbb{R}^N \setminus B_1(0),
\]

(4.14)

implying

\[
u_0^{l-p} \leq \frac{C}{(1 + |x|)^{p+\delta_1}}, \quad \text{in } \mathbb{R}^N,
\]

for sufficiently large constant \( C \) independent of \( x \). Now set

\[-\Delta_p u_0 + (u_0^{l-p})u_0^{p-1} = u_0^{q-1} \geq 0, \quad \text{in } \mathbb{R}^N,
\]

and then we have

\[-\Delta_p u_0 + \frac{C}{(1 + |x|)^{p+\delta_1}}u_0^{p-1} \geq 0, \quad \text{in } \mathbb{R}^N.
\]

As a consequence, \( u_0 \) is a supersolution of (4.11) and then by comparison principle (see Theorem B.1 in the Appendix), we obtain

\[u_0 \geq c |x|^{-\frac{N-p}{p}} \quad \text{in } |x| > 1.
\]

(4.15)

Similarly, we can set

\[-\Delta_p u_0 - (u_0^{q-p})u_0^{p-1} = -u_0^{q-1} \leq 0, \quad \text{in } \mathbb{R}^N,
\]

and since \( q > p^* \) we have for some \( \delta_2 > 0 \),

\[
u_0^{q-p} \leq c'|x|^{-\frac{N-p}{p}(q-p)} \leq \frac{c'}{|x|^{p+\delta_2}}, \quad \text{in } |x| > 1,
\]

implying

\[
u_0^{q-p} \leq \frac{C'}{(1 + |x|)^{p+\delta_2}}, \quad \text{in } \mathbb{R}^N,
\]

and hence

\[-\Delta_p u_0 - \frac{C'}{(1 + |x|)^{p+\delta_2}}u_0^{p-1} \leq 0 \quad \text{in } \mathbb{R}^N.
\]

Now since \( u_0 \in D^{1,p}_{rad}(\mathbb{R}^N) \) is a subsolution of (4.11), then by Lemma [5.2] \( u_0 \) satisfies condition (S) and hence by comparison principle Theorem [5.1] we have

\[u_0 \leq c'|x|^{-\frac{N-p}{p}} \quad \text{in } |x| > 1,
\]

(4.16)

and hence from (4.15) and (4.16) the conclusion follows.

\[\square\]

5 Proof of Theorem 2.3: critical case \( q = p^* \)

In this section we analyse the behaviour of the ground states \( u_\varepsilon \) of equation (2.3) as \( \varepsilon \to 0 \) in the critical case \( q = p^* \) and prove Theorem 2.3. Although our approach follows the ideas of [24], the present \( p \)-Laplacian setting requires substantial modifications.
5.1 Variational estimates for $S_\varepsilon$

Equivalently to the Sobolev constant $\{S_\varepsilon\}$, we consider the Rayleigh type Sobolev quotient

$$S_\varepsilon(w) := \frac{\int_{\mathbb{R}^N} |\nabla w|^p dx}{\left(\int_{\mathbb{R}^N} |w|^{p^*} dx \right)^{\frac{N}{N-p}}}, \quad w \in D^{1,p}(\mathbb{R}^N), \quad w \neq 0,$$

which is invariant with respect to the dilations $w_\lambda(x) := w(x/\lambda)$, so that

$$S_\varepsilon = \inf_{0 \neq w \in D^{1,p}(\mathbb{R}^N)} S_\varepsilon(w).$$

We define the gap

$$\sigma_\varepsilon := S_\varepsilon - S_*$$

To estimate $\sigma_\varepsilon$ in terms of $\varepsilon$, we shall use the Sobolev minimizers $W_\lambda$ from $\{S_\varepsilon\}$ as test functions for $\{S_\varepsilon\}$. Since $W_\lambda \in L^p(\mathbb{R}^N)$ only if $1 < p < \sqrt{N}$, we analyse the higher and lower dimensions separately. It is easy to check that $W_\lambda \in L^s(\mathbb{R}^N)$ for all $s > \frac{N(p-1)}{N-p}$, with

$$||W_\lambda||_s^s = \lambda^{\frac{N-s}{p} + N} ||W_1||^s_s = \lambda^{\frac{N-s}{p}(s-p')} ||W_1||^s_s,$$

and that, if $1 < p < \sqrt{N}$ then $W_\lambda \in L^p(\mathbb{R}^N)$ it holds that

$$||W_\lambda||_p^p = \lambda^p ||W_1||^p_p.$$

In the case of dimensions $p = \sqrt{N}$ and $\sqrt{N} < p < N$, given $R \gg \mu$, we introduce a cut-off function $\eta_R \in C^\infty_0(\mathbb{R})$ such that $\eta_R(r) = 1$ for $|r| < R$, $0 < \eta_R < 1$ for $R < |r| < 2R$, $\eta_R(r) = 0$ for $|r| > 2R$ and $|\eta_R(r)| \leq 2/R$. We then compute as in e.g. [35, Chapter III, proof of Theorem 2.1]

$$||\nabla (\eta_R W_\mu(x))||_p^m = S_* + O\left(\left(\frac{R}{\mu}\right)^{-\frac{N-p}{p-1}}\right),$$

$$||\eta_R W_\mu||_p^{p^*} = 1 - O\left(\left(\frac{R}{\mu}\right)^{-\frac{N-p}{p-1}}\right),$$

$$||\eta_R W_\mu||_1^1 = \mu^{-\frac{N-p}{p}(1-p') ||W_1||_1^1} \left(1 - O\left(\left(\frac{R}{\mu}\right)^{-\frac{N-p}{p-1} + N}\right)\right),$$

and

$$||\eta_R W_\mu||_p^p = \begin{cases} O\left(\mu^p \log R\right), & p = \sqrt{N}, \\ O\left(\mu^\frac{N-p}{p} R^{\frac{2-N}{p}}\right), & \sqrt{N} < p < N. \end{cases}$$

As a consequence of these expansions we get an upper estimate for $\sigma_\varepsilon$ which plays a key role in what follows.

**Lemma 5.1.** It holds that

$$0 < \sigma_\varepsilon \lesssim \begin{cases} \varepsilon^{\frac{p}{p-1}} & 1 < p < \sqrt{N}, \\ \varepsilon^{\frac{(N-p)(1-p')}{(p-1)(p-1)}} ||W_1||_1^1 & \sqrt{N} < p < N, \\ \left(\varepsilon (\log \frac{1}{\varepsilon})\right)^{\frac{p}{p-1}} & p = \sqrt{N}. \end{cases}$$

Hence, $\sigma_\varepsilon \to 0$ as $\varepsilon \to 0$. 

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Proof. We first observe that since

\[ S_* \leq S_*(w) < S_*(w) = S_0, \]

it follows that \( \sigma > 0 \). We now obtain the upper bounds on \( \sigma_\varepsilon \).

Case 1 < \( p < \sqrt{N} \). Note that \( W_\mu \in M_\varepsilon \) for all sufficiently small \( \varepsilon \) and sufficiently large \( \mu \), and we have

\[ S_*>(W_\mu) \leq \frac{S_*}{(1 - \varepsilon \mu^p - \mu^{-\frac{(N-p)}{p}}l^{-p^*})\beta_1} \tag{5.7} \]

where

\[ \beta_p := \frac{P^*}{p}||W_1||_p^p, \quad \beta_1 := \frac{P^*}{1}||W_1||_1^1. \]

We now optimise the right hand side of the estimate (5.7) picking \( \mu \) such that the function

\[ \psi_\varepsilon(\mu) := \beta_p \varepsilon \mu^p + \beta_1 \mu^{-\frac{N-\varepsilon}{p}}. \]

achieves its minimum. This occurs at

\[ \mu_\varepsilon \approx \varepsilon^{-\frac{p}{(N-p)(1-p^*)}} \tag{5.8} \]

and we have

\[ \min_{\mu > 0} \psi_\varepsilon \approx \psi_\varepsilon(\mu_\varepsilon) \approx \varepsilon^{-\frac{p}{(N-p)(1-p^*)}}. \]

In the present case 1 < \( p < \sqrt{N} \), we may conclude that

\[ S_*>(W_\mu) \lesssim \frac{S_*}{(1 - \varepsilon \mu(\mu_\varepsilon))^{(N-\varepsilon)/p}} = S_* \left(1 + O(\psi_\varepsilon(\mu_\varepsilon))\right) = S_* + O\left(\frac{\varepsilon^{-p^*}}{p}\right), \tag{5.9} \]

and (5.8) is the value of \( \mu_\varepsilon \) such that the bound (5.6) is achieved on the function \( W_\mu \).

Case \( p > \sqrt{N} \). We assume here that \( R \gg \mu \). Using \( \eta_R W_\mu \) as test function and using the calculation in (5.2)-(5.5), we get

\[ S_*(\eta R W_\mu) \leq \left( S_* + O\left(\frac{R^2}{\mu} - \frac{N-\varepsilon}{p}\right) \right) \times \left(1 - \left\{ O\left(\frac{R^2}{\mu} - \frac{N-\varepsilon}{p}\right) + \varepsilon O\left(\mu^{-\frac{N-\varepsilon}{p}} R^{\frac{2N}{p-1}}\right) \right. \right. \]

\[ + \mu^{-\frac{N-\varepsilon}{p}}l^{-p^*}l^{-1}\left[1 - O\left(\frac{R^2}{\mu} - \frac{N-\varepsilon}{p}\right)\right] \left\} \right\}^{\frac{N-\varepsilon}{p}}, \]

and hence as \( \frac{R}{\mu} \to \infty \), we have

\[ S_*(\eta R W_\mu) \leq S_* \left(1 + \psi_\varepsilon(\mu, R)\right), \]

where

\[ \psi_\varepsilon(\mu, R) := \left(\frac{R^2}{\mu} - \frac{N-\varepsilon}{p}\right) + \varepsilon \mu^{-\frac{N-\varepsilon}{p}} R^{\frac{2N}{p-1}} + \mu^{-\frac{N-\varepsilon}{p}}l^{-p^*}. \tag{5.10} \]

If in particular we choose

\[ \mu_\varepsilon = \varepsilon^{-\frac{p}{(N-p)(1-p^*)}}, \quad R_\varepsilon = \varepsilon^{-\frac{1}{p^*}}. \tag{5.11} \]

we then find that

\[ \psi_\varepsilon(\mu_\varepsilon, R_\varepsilon) \sim \varepsilon^{\frac{(N-p)(1-p^*)}{p(1-p^*)}}. \]
and, similarly to the above case, the bound (5.6) is achieved on the test function \( \eta R W \mu \) provided \( \mu \) and \( R \) are as in (5.11).  

**Case** \( p = \sqrt{N} \). Again we assume that \( R \gg \mu \). Testing again against \( \eta R W \mu \) and by (5.2)-(5.5) with \( p = \sqrt{N} \), we get

\[
S_\varepsilon(\eta R W \mu) \leq \left( S_* + O\left( \left( \frac{R}{\mu} \right)^{-\frac{N-p}{p-1}} \right) \right)
\times \left( 1 - \left( O\left( \left( \frac{R}{\mu} \right)^{-\frac{N-p}{p-1}} \right) + \varepsilon O\left( \mu \log R \right) \right) + \mu^{-\frac{N-p}{p-1}} \right) \left( 1 - \left( \frac{R}{\mu} \right)^{-\frac{N-p}{p-1}} \right) \right),
\]

and then as \( \frac{R}{\mu} \to \infty \), we have

\[
S_\varepsilon(\eta R W \mu) \leq S_* \left( 1 + \psi_\varepsilon(\mu, R) \right),
\]

where

\[
\psi_\varepsilon(\mu, R) := \left( \frac{R}{\mu} \right)^{-\frac{N-p}{p-1}} + \varepsilon \mu \log R + \mu^{-\frac{N-p}{p-1}}.
\]

Choose

\[
R_\varepsilon := \varepsilon^{-\frac{1}{p}}, \quad \mu_\varepsilon := \left( \varepsilon \log \frac{1}{\varepsilon} \right)^{-\frac{p-1}{2(p-1)}},
\]

and hence

\[
\psi_\varepsilon(\mu_\varepsilon, R_\varepsilon) \sim \left( \varepsilon \log \frac{1}{\varepsilon} \right)^{-\frac{p-1}{2(p-1)}}.
\]

Thus the bound (5.6) is achieved by the test function \( \eta R_\varepsilon W \mu_\varepsilon \), where \( \mu_\varepsilon \) and \( R_\varepsilon \) are defined in (5.13). \( \square \)

### 5.2 Pohožaev estimates

For \( \varepsilon \in (0, \varepsilon_*) \), let \( w_\varepsilon > 0 \) be a family of the minimizers for (5.12) (or equivalently (3.11)). This minimizers \( w_\varepsilon \) solve the Euler Lagrange equation

\[
-\Delta_p w_\varepsilon = S_\varepsilon \left( - \varepsilon w_\varepsilon^{p-1} + w_\varepsilon^{p-1} - w_\varepsilon^{l-1} \right) \quad \text{in } \mathbb{R}^N
\]

with the original (untruncated) nonlinearity.

Our next step is to use Nehari identity combined with Pohožaev identity for (5.14) in order to obtain the following useful relations between the norms of \( w_\varepsilon \).

**Lemma 5.2.** For all \( 1 < p < N \), set \( k := \frac{(p-1)p}{p(p-1)} > 0 \). Then, it holds that

\[
||w_\varepsilon||_p^p = k\varepsilon ||w_\varepsilon||_p^p,
\]

\[
||w_\varepsilon||_p^p = 1 + (k+1)\varepsilon ||w_\varepsilon||_p^p.
\]

**Proof.** Since \( w_\varepsilon \) is a minimizer of \( S_\varepsilon \), identities (3.5)-(3.6) read

\[
1 = ||w_\varepsilon||_p^p - \varepsilon ||w_\varepsilon||_p^p, \quad 1 = ||w_\varepsilon||_p^p - \frac{p-1}{p} ||w_\varepsilon||_l^p - \frac{p-1}{l} ||w_\varepsilon||_l^p.
\]

An easy calculation yields the conclusion. \( \square \)
Lemma 5.3. For all $1 < p < N$, we have

$$\varepsilon(k + 1)\|w_{\varepsilon}\|_p^p \leq \frac{N}{N-p}S^{-1}_* \varepsilon(1 + o(1)).$$

Proof. Using that $w_{\varepsilon}$ is a minimizer for $(S_{\varepsilon})$, by Lemma 5.2 if follows that $S_* \leq S_*(w_{\varepsilon}) = \frac{\|\nabla w_{\varepsilon}\|_p^p}{\|w_{\varepsilon}\|_{p^*}^p} = \frac{S_{\varepsilon}}{(1 + (k + 1)\varepsilon\|w_{\varepsilon}\|_p^p)^{(N-p)/N}},$

namely,

$$S_*^{N/(N-p)}(1 + (k + 1)\varepsilon\|w_{\varepsilon}\|_p^p) \leq S_{\varepsilon}^{N/(N-p)}.$$

Setting $\sigma_{\varepsilon} := S_{\varepsilon} - S_*$, as $\varepsilon \to 0$ we obtain

$$S_*^{N/(N-p)}(k + 1)\varepsilon\|w_{\varepsilon}\|_p^p \leq \sigma_{\varepsilon} \frac{N}{N-p} S_*^{N/(N-p)} - 1 + o(\sigma_{\varepsilon}),$$

and this concludes the proof. 

We note that the above results allow us to understand the behavior of the norms associated with the minimizer $w_{\varepsilon}$ to $(S_{\varepsilon})$. In fact we have the following

Corollary 5.4. As $\varepsilon \to 0$, we have

$$\varepsilon\|w_{\varepsilon}\|_p^p \to 0, \quad \|w_{\varepsilon}\|_l^l \to 0, \quad \|w_{\varepsilon}\|_{p^*}^p \to 1.$$

5.3 Optimal rescaling

We are now in a position to introduce an optimal rescaling which captures the convergence of the minimizers $w_{\varepsilon}$ to the limit Emden-Fowler ground state $W_1$.

Following [35, pp.38 and 44], consider the concentration function

$$Q_{\varepsilon}(\lambda) = \int_{B_\lambda} |w_{\varepsilon}|^p dx,$$

where $B_\lambda$ is the ball of radius $\lambda$ centred at the origin. Note that $Q_{\varepsilon}(\cdot)$ is strictly increasing, with

$$\lim_{\lambda \to 0} Q_{\varepsilon}(\lambda) = 0,$$

and

$$\lim_{\lambda \to \infty} Q_{\varepsilon}(\lambda) = \|w_{\varepsilon}\|_{p^*}^p \to 1, \quad \text{as } \varepsilon \to 0,$$

by Corollary 5.4. It follows that the equation $Q_{\varepsilon}(\lambda) = Q_*$ with

$$Q_* := \int_{B_1} |W_1(x)|^p dx < 1,$$

has a unique solution $\lambda = \lambda_{\varepsilon} > 0$ for $\varepsilon \ll 1$, namely

$$Q_{\varepsilon}(\lambda_{\varepsilon}) = Q_* \quad \text{(5.16)}$$

By means of the value of $\lambda_{\varepsilon}$ implicitly defined by (5.16), we set

$$v_{\varepsilon}(x) := \frac{\lambda_{\varepsilon}^{N/(N-p)}}{\lambda_{\varepsilon}^{N/(N-p)}(k + 1)\varepsilon\|w_{\varepsilon}\|_p^p} w_{\varepsilon}(\lambda_{\varepsilon} x), \quad \text{(5.17)}$$

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and easily check that
\[ \|v_\varepsilon\|_{p^*} = |w_\varepsilon|_{p^*} = 1 + o(1), \quad \|\nabla v_\varepsilon\|_p = \|\nabla w_\varepsilon\|_p = S_* + o(1). \] (5.18)

namely \((v_\varepsilon)\) is a minimizing family for \((S_*)\). Moreover
\[ \int_{B_1} |v_\varepsilon(x)|_{p^*} dx = Q_* . \]

The following convergence lemma follows by the Concentration-Compactness Principle of P.-L. Lions \[35, Theorem 4.9\].

**Lemma 5.5.** For all \(1 < p < N\), it holds that
\[ \|\nabla(v_\varepsilon - W_1)\|_p \to 0, \]
and
\[ \|v_\varepsilon - W_1\|_{p^*} \to 0, \]
as \(\varepsilon \to 0\).

**Proof.** By (5.18), for any sequence \(\varepsilon_n \to 0\) there exists a subsequence \((\varepsilon_n)\) such that \((v_{\varepsilon_n})\) converges weakly in \(D^{1,p}(\mathbb{R}^N)\) to some radial functions \(w_0 \in D^{1,p}(\mathbb{R}^N)\). By the Concentration-Compactness Principle \[35, Theorem 4.9\] applied to \(\|v_\varepsilon\|_{p^*}^{-1} v_\varepsilon\), we have in fact that \((v_{\varepsilon_n})\) converges to \(w_0\) strongly in \(D^{1,p}(\mathbb{R}^N)\) and \(L^{p^*}(\mathbb{R}^N)\). Hence, \(|w_0|_{p^*} = 1\) and therefore \(w_0\) is a radial minimizer of \((S_*)\), that is necessarily \(w_0 \in \{W_\lambda\}_{\lambda > 0}\). Note that it also holds
\[ \int_{B_1} |w_0(x)|_{p^*} dx = Q_* . \]

As a consequence \(w_0 = W_1\). Since the sequence \((\varepsilon_n)\) was arbitrary, the whole sequence \((v_n)\) converges to \(W_1\) strongly in \(D^{1,p}(\mathbb{R}^N)\) and \(L^{p^*}(\mathbb{R}^N)\), and this concludes the proof.

### 5.4 Rescaled equation estimates

Our next step is to obtain upper and lower estimates on the rescaling function \(\lambda_\varepsilon\), which is implicitly determined by (5.16). The rescaled function \(v_\varepsilon\) introduced in (5.17) is such that
\[ -\Delta_p v_\varepsilon = S_\varepsilon \left( - \varepsilon \lambda_\varepsilon^p |v_\varepsilon|^{p-1} + |v_\varepsilon|^{p^*-1} - \lambda_\varepsilon^{-(N-p)(\frac{1-p}{p})p} |v_\varepsilon|^{(-1)} \right), \] (\(R^*_\varepsilon\))
as \((S_*)\) is achieved by \(w_\varepsilon\). By construction, for \(v_\varepsilon\) we obtain
\[ |v_\varepsilon|_l^p = \lambda_\varepsilon^{\frac{(l-p^*)}{l-p}} |w_\varepsilon|_l^p, \quad ||v_\varepsilon||_p^p = \lambda_\varepsilon^{-p}||w_\varepsilon||_p^p. \]

Putting Lemmas 5.2 and 5.3 together we then achieve that relation
\[ \lambda_\varepsilon^{-p(l-p^*)} |v_\varepsilon|_l^p = \lambda_\varepsilon^p k\varepsilon |v_\varepsilon|_p^p \lesssim \sigma_\varepsilon, \] (5.19)

which yields the following

**Lemma 5.6.** Let \(1 < p < N\). Then
\[ \sigma_\varepsilon^{-\frac{(l-p^*)}{l-p}} \lesssim \lambda_\varepsilon \lesssim \varepsilon^{-\frac{l}{p}} \sigma_\varepsilon^{\frac{1}{p}} . \]
Proof. The statement will follow by (5.19) combined with the observation that

$$\liminf_{\varepsilon \to 0} ||v_{\varepsilon}|| > 0, \quad \liminf_{\varepsilon \to 0} ||v_{\varepsilon}||_p > 0.$$ 

The former is a consequence of Lemma 5.5 and Hölder inequalities, which yields $L^l(B_1) \subset L^{p^*}(B_1)$ since $l > p^*$, hence

$$c||v_{\varepsilon}X_{B_1}||_l \geq ||v_{\varepsilon}X_{B_1}||_{p^*} \geq ||W_1X_{B_1}||_{p^*} - ||(W_1 - v_{\varepsilon})X_{B_1}||_{p^*} = ||W_1X_{B_1}||_{p^*} - o(1).$$

Here $X_{B_1}$ is the characteristic function of $B_1$. To show the latter, by the embedding $L^{p^*}(B_1) \subset L^p(B_1)$ since $p^* > p$, we obtain

$$c||v_{\varepsilon}X_{B_1}||_{p^*} \geq ||v_{\varepsilon}X_{B_1}||_p \geq ||W_1X_{B_1}||_p - ||(W_1 - v_{\varepsilon})X_{B_1}||_p = ||W_1X_{B_1}||_p - o(1),$$

and this concludes the proof.

By (5.6) and Lemma 5.6 we obtain both an estimate from below

$$\lambda_\varepsilon \geq s\varepsilon^{-\frac{(p^*-p)}{p^*(1-p)}} \geq \begin{cases} \varepsilon^{-\frac{(p^*-p)}{p^*}} & 1 < p < \sqrt{N}, \\ \varepsilon^{-\frac{(p^*-p)}{p^*} \log \frac{1}{\varepsilon}} & \sqrt{N} < p < N, \\ \varepsilon^{-\frac{(p^*-p)}{p^*} \log \frac{1}{\varepsilon}} & p = \sqrt{N}, \end{cases}$$

and from above

$$\lambda_\varepsilon \leq \begin{cases} \varepsilon^{-\frac{p^*}{p^*}} & 1 < p < \sqrt{N}, \\ \varepsilon^{-\frac{p^*}{p^*} \log \frac{1}{\varepsilon}} & \sqrt{N} < p < N, \\ \varepsilon^{-\frac{p^*}{p^*} \log \frac{1}{\varepsilon}} & p = \sqrt{N}. \end{cases}$$

We note that in the case $1 < p < \sqrt{N}$ the above lower and upper estimates are equivalent, therefore we have the following

**Corollary 5.7.** Let $1 < p < \sqrt{N}$. Then $||v_{\varepsilon}||_l$ and $||v_{\varepsilon}||_p$ are bounded.

**Proof.** This is an immediate consequence of (5.19) - (5.21).

In the case of lower dimensions we take into account the growth of $||v_{\varepsilon}||_p$ to obtain matching bounds. In this case instead of (5.21) we use the more explicit upper bound

$$\lambda_\varepsilon \leq \varepsilon^{-1/p} \sigma_\varepsilon^{1/p} \leq ||v_{\varepsilon}||_p^{-1} \begin{cases} \varepsilon^{-\frac{(p^*-p)}{p^*} \log \frac{1}{\varepsilon}} & \sqrt{N} < p < N, \\ \varepsilon^{-\frac{(p^*-p)}{p^*} \log \frac{1}{\varepsilon}} & p = \sqrt{N}, \end{cases}$$

which follows from (5.19) and (5.6).

**5.5 A lower barrier for $p \geq 2$**

To refine the upper bound (5.21) we shall construct a lower barrier for $w_{\varepsilon}$ in the critical régimes $\sqrt{N} \leq p < N$. For $p \geq 2$ this will be done using the following uniform estimate.
Lemma 5.8. Given $\mu > 0$ and $\gamma > 0$, set

$$h(r) := r^{-\gamma}e^{-\mu r}.$$

Assume that $p \geq 2$ and that $N - 1 - 2\gamma(p - 1) \leq 0$ and $\gamma(N - p - \gamma(p - 1)) \leq 0$. Then for all $\mu > 0$ and $r > 0$,

$$- \Delta_p h + p^{p-1}(p-1)h^{p-1} \leq \mu \frac{\gamma^{p-2}(N - 1 - 2\gamma(p - 1))}{r^{p-1}}h^{p-1} + \frac{\gamma^{p-1}(N - p - \gamma(p - 1))}{r^p}h^{p-1}.$$  \hspace{1cm} (5.23)

Remark 5.9. If $p = 2$ then (5.23) becomes an equality.

Proof. By direct calculations, we have

$$- \Delta_p h + p^{p}(p-1)h^{p-1} = (p-1)\mu^2 \left\{ \mu^{p-2} - \left( \mu + \frac{\gamma}{r} \right)^{p-2} \right\} h^{p-1} + \left( \mu + \frac{\gamma}{r} \right)^{p-2} \left\{ \mu \frac{N - 1 - 2\gamma(p - 1)}{r} + \frac{\gamma(N - p - \gamma(p - 1))}{r^2} \right\} h^{p-1}.$$

For all $\mu > 0$ and $r > 0$, by monotonicity we have

$$\left\{ \mu^{p-2} - \left( \mu + \frac{\gamma}{r} \right)^{p-2} \right\} \leq 0,$$

$$\left( \mu + \frac{\gamma}{r} \right)^{p-2} \geq \left( \frac{\gamma}{r} \right)^{p-2}.$$  \hspace{1cm} (5.24)

Therefore, assuming that $N - 1 - 2\gamma(p - 1) \leq 0$ and $\gamma(N - p - \gamma(p - 1)) \leq 0$ we can estimate,

$$- \Delta_p h + p^{p}(p-1)h^{p-1} \leq \left\{ \frac{\gamma}{r} \right\}^{p-2} \left\{ \mu \frac{N - 1 - 2\gamma(p - 1)}{r} + \frac{\gamma(N - p - \gamma(p - 1))}{r^2} \right\} h^{p-1} \leq \mu \frac{\gamma^{p-2}(N - 1 - 2\gamma(p - 1))}{r^{p-1}}h^{p-1} + \frac{\gamma^{p-1}(N - p - \gamma(p - 1))}{r^p}h^{p-1},$$

uniformly for all $\mu > 0$ and $r > 0$. \hspace{1cm} (5.25)

Remark 5.10. In the case $1 < p < 2$ by monotonicity, convexity and Taylor for all $\mu > 0$ and $r > 0$ we have

$$0 \leq \left\{ \mu^{p-2} - \left( \mu + \frac{\gamma}{r} \right)^{p-2} \right\} \leq (2 - p)\mu^{p-3}\frac{\gamma}{r}.$$  \hspace{1cm} (5.26)

Similarly, we can estimate

$$\left( \mu + \frac{\gamma}{r} \right)^{p-2} \geq \mu^{p-2} - (2 - p)\mu^{p-3}\frac{\gamma}{r},$$  \hspace{1cm} (5.27)

or, alternatively,

$$\left( \mu + \frac{\gamma}{r} \right)^{p-2} \geq \left( \frac{\gamma}{r} \right)^{p-2} - (2 - p)\left( \frac{\gamma}{r} \right)^{p-3}\frac{\mu}{r}.$$  \hspace{1cm} (5.28)

Therefore, assuming that $N - 1 - 2\gamma(p - 1) \leq 0$ and $\gamma(N - p - \gamma(p - 1)) \leq 0$ we can estimate,

$$- \Delta_p h + p^{p}(p-1)h^{p-1} \leq \mu^{p-1}\frac{(2 - p)(p - 1)\gamma}{r}h^{p-1} + \left\{ \frac{\gamma}{r} \right\}^{p-2} \left\{ \mu \frac{N - 1 - 2\gamma(p - 1)}{r} + \frac{\gamma(N - p - \gamma(p - 1))}{r^2} \right\} h^{p-1}.$$

Both (5.24) and (5.25) introduce a large positive term in (5.26) which we cannot control.
To estimate the norm $||v_\varepsilon||_p$, we note that
\[-\Delta_p v_\varepsilon + S_\varepsilon \varepsilon P p v_\varepsilon^{p+1} = S_\varepsilon |v_\varepsilon|^{p-1} - S_\varepsilon \varepsilon (N-p) \frac{|v_\varepsilon|^{p+1}}{p+1} \geq -V_\varepsilon(x) v_\varepsilon^{p-1},\]
where we have set
\[V_\varepsilon(x) := S_\varepsilon \varepsilon (N-p) \frac{|v_\varepsilon|^{p+1}}{p+1} v_\varepsilon^{-p}(x).\]
By the radial decay estimate (A.3) we have
\[v_\varepsilon(x) \leq C_{N,p} |x|^{-N/p} ||v_\varepsilon||_p.\]
By (5.18) and since $\varepsilon \frac{p(l-p)}{p+1} \lesssim \sigma_\varepsilon \to 0$ Lemmas 5.1 and 5.6 yield, for sufficiently small $\varepsilon > 0$, the following decay estimate
\[V_\varepsilon(x) := S_\varepsilon \varepsilon (N-p) \frac{|v_\varepsilon|^{p+1}}{p+1} v_\varepsilon^{-p}(x) \leq S_\varepsilon \varepsilon (N-p) \frac{|v_\varepsilon|^{p+1}}{p+1} v_\varepsilon^{-p} \leq C \lambda_\varepsilon^{(N-p)} \frac{|v_\varepsilon|^{p+1}}{p+1} v_\varepsilon^{-p},\]
where $\delta := \frac{N-p}{p+1}(l-p) - p > 0$ and the constant $C > 0$ does not depend on $\varepsilon$ or $x$. Hence, for small $\varepsilon > 0$ the rescaled functions $v_\varepsilon$ to satisfy the linear inequality
\[-\Delta_p v_\varepsilon + \varepsilon \lambda_\varepsilon^p v_\varepsilon^{p-1} + V_\varepsilon(x) v_\varepsilon^{p-1} \geq 0, \quad x \in \mathbb{R}^N.\] (5.27)
The following result provides a suitable lower barrier to (5.24) below.

**Lemma 5.11.** Assume $N \geq 4$ and $2 \leq p < \frac{N+2}{2}$. Then there exists $R > 0$, independent on $\varepsilon > 0$, such that for all small $\varepsilon > 0$,
\[h_\varepsilon(x) := |x|^{-\frac{N-p}{p-1}} e^{-\frac{\varepsilon}{p} \lambda_\varepsilon |x|}\]
satisfies
\[-\Delta_p h_\varepsilon + (p-1)\varepsilon \lambda_\varepsilon^p h_\varepsilon^{p-1} + V_\varepsilon(x) h_\varepsilon^{p-1} \leq 0, \quad |x| > R.\] (5.28)

**Proof.** By Lemma 5.8 with $\gamma = \frac{N-p}{p-1}$ we conclude that there exists $R > 1$, independent of $\varepsilon > 0$, such that
\[-\Delta_p h_\varepsilon + (p-1)\varepsilon \lambda_\varepsilon^p h_\varepsilon^{p-1} + V_\varepsilon(x) h_\varepsilon^{p-1} \leq \varepsilon \left(\frac{\lambda_\varepsilon^p}{p+1} (N-1 - 2\gamma (p-1)) \right) h_\varepsilon^{p-1} + \lambda_\varepsilon \frac{\varepsilon}{p+1} \lambda_\varepsilon^{(l-p)/p} C_{p+1} h_\varepsilon^{p-1} \leq \begin{cases} \gamma^{-2} (N+1 - 2p) \varepsilon \lambda_\varepsilon^{(l-p)/p} + C \lambda_\varepsilon \frac{\varepsilon}{p+1} h_\varepsilon^{p-1} & \text{for } |x| > R. \end{cases}\]
It is convenient to denote $s := l-p > 0$. Taking into account that $\frac{N-p}{p} (l-p) + s = (1-N/p) < 0$, we can use the lower bound (5.20) on $\lambda_\varepsilon$ to estimate
\[-\gamma^{-2} (N+1 - 2p) \varepsilon \lambda_\varepsilon^{(l-p)/p} + C \lambda_\varepsilon \frac{\varepsilon}{p+1} \lambda_\varepsilon^{(l-p)/p} \leq \gamma^{-2} (N+1 - 2p) \varepsilon \lambda_\varepsilon^{(l-p)/p} + C \varepsilon \lambda_\varepsilon^{(l-p)/p} \leq 0,\]
for all sufficiently small $\varepsilon > 0$, provided that $p < (N+1)/2$, which completes the proof.\[\Box\]

**Lemma 5.12.** Assume $N \geq 4$ and $2 \leq p < \frac{N+2}{2}$. There exists $R > 0$ and $c > 0$, independent on $\varepsilon > 0$, such that for all small $\varepsilon > 0$,
\[v_\varepsilon(x) \geq c |x|^{-\frac{N-p}{p-1}} e^{-\frac{\varepsilon}{p} \lambda_\varepsilon |x|} \quad (|x| > R).\]
Proof. Define the barrier
\[ h_\varepsilon(x) := |x|^{-\frac{p-1}{p}} e^{-\frac{\varepsilon}{\lambda_\varepsilon} |x|}, \]
which satisfies
\[ -\Delta_p h_\varepsilon + \varepsilon \lambda_\varepsilon^p h_\varepsilon^{p-1} + V_\varepsilon(x) h_\varepsilon^{p-1} \leq 0, \quad |x| > R. \tag{5.29} \]
by Lemma 5.11. Note that Lemma 5.5 and Lemma A.4 in the Appendix imply
\[ ||(v_\varepsilon - W_1)_{B_R \setminus B_{R/2}}||_\infty \to 0, \]
and hence
\[ v_\varepsilon(|x|) \to W_1(|x|), \quad \text{for } |x| = R. \]
Hence for all sufficiently small \( \varepsilon > 0 \), we have
\[ v_\varepsilon(R) \geq \frac{1}{2} W_1(R), \quad \text{for } |x| = R. \]
Since \( h_\varepsilon(R) \) is a monotone decreasing function in \( \varepsilon \), then by a suitable choice of a uniform small constant \( c > 0 \) we obtain
\[ ch_\varepsilon(R) \leq \frac{1}{2} W_1(R), \]
and hence
\[ v_\varepsilon(R) \geq ch_\varepsilon(R), \quad \text{for all small } \varepsilon > 0. \]
Then the homogeneity of (5.29) implies
\[ -\Delta_p (ch_\varepsilon) + \varepsilon \lambda_\varepsilon^p (ch_\varepsilon)^{p-1} + V_0(x)(ch_\varepsilon)^{p-1} \leq 0, \quad \text{in } |x| > R, \]
for all small \( \varepsilon > 0 \). Define a function \( ch_{\varepsilon,k} \) by
\[ ch_{\varepsilon,k} = ch_\varepsilon - k^{-1} < ch_\varepsilon, \quad \text{for all } k > 0, \]
then
\[ -\Delta_p (ch_{\varepsilon,k}) + V_0(x)(ch_{\varepsilon,k})^{p-1} + \varepsilon \lambda_\varepsilon^p (ch_{\varepsilon,k})^{p-1} \leq 0, \quad \text{in } |x| > R \tag{5.30} \]
and
\[ v_\varepsilon \geq ch_\varepsilon > ch_{\varepsilon,k}, \quad \text{for } |x| = R. \]
Now, since
\[ ch_\varepsilon \to 0, \quad \text{as } |x| \to +\infty, \]
then for \( k \) large enough there exists \( R_k > R \) such that
\[ ch_{\varepsilon,k} = 0, \quad \text{for } |x| = R_k, \]
and since \( v_\varepsilon > 0 \), then
\[ v_\varepsilon > ch_{\varepsilon,k}, \quad \text{for } |x| = R_k. \]
As a consequence, from (5.29) and (5.30), using the comparison principle (see Theorem B.1 in the Appendix) we obtain
\[ v_\varepsilon \geq ch_{\varepsilon,k}, \quad \text{for } R < |x| < R_k, \]
which can be achieved for every \( k \). Since \( R_k \to \infty \) as \( k \to \infty \), the assertion follows. \( \Box \)
5.6 Critical dimensions $N \geq 4$ and $\sqrt{N} \leq p < \frac{N+1}{2}$ completed

We now apply Lemma 5.12 to obtain matching estimates for the blow-up of $||v_\varepsilon||_p$ in dimensions $N \geq 4$ and $\sqrt{N} \leq p < \frac{N+1}{2}$.

Lemma 5.13. If $N \geq 4$ and $\sqrt{N} < p < \frac{N+1}{2}$, then $||v_\varepsilon||_p \gtrsim \left( \frac{1}{\varepsilon \lambda_e} \right)^{\frac{2-N}{p}}$.

Proof. Since $\sqrt{N} < p < \frac{N+1}{2}$, we directly calculate from Lemma 5.12:

$$||v_\varepsilon||_p^p \geq \int_{\mathbb{R}^N \setminus B_R} |v_\varepsilon|^p dx \geq \int_{\mathbb{R}} r^{N-1} |v_\varepsilon|^{\frac{N-p}{p+1}} e^{-\frac{N-p}{p+1} \varepsilon \lambda_e} r^p dr,$$

and as $\varepsilon \to 0$ (i.e. $\frac{1}{\varepsilon \lambda_e} \to \infty$), we have

$$||v_\varepsilon||_p^p \geq c^p \int_{\mathbb{R}} r^{\frac{2-N}{p+1}-1} dr \geq \left( \frac{C}{\varepsilon \lambda_e} \right)^{\frac{2-N}{p}},$$

and this completes the proof.

As an immediate consequence of the above result, by (5.22), we obtain an upper estimate of $\lambda_\varepsilon$ which matches the lower bound of (5.20) in dimensions $N \geq 4$ and $\sqrt{N} < p < \frac{N+1}{2}$.

Corollary 5.14. If $N \geq 4$ and $\sqrt{N} < p < \frac{N+1}{2}$, then $\lambda_\varepsilon \lesssim \varepsilon^{-\left((\log \frac{1}{\varepsilon}) \right) - \frac{(p^* - p)}{p(p^* - 1)}}$.

We now move to consider the case $p = \sqrt{N}$.

Lemma 5.15. If $N \geq 4$ and $p = \sqrt{N}$ then it holds that $||v_\varepsilon||_p^p \gtrsim \log(\frac{1}{\varepsilon \lambda_e})$.

Proof. Since $p = \sqrt{N}$, by Lemma 5.12 we immediately get

$$||v_\varepsilon||_p^p \geq \int_{\mathbb{R}^N \setminus B_R} |v_\varepsilon|^p dx \geq \int_{\mathbb{R}} r^{\frac{2-N}{p+1}-1} dr = c^p \int_{\mathbb{R}} r^{-1} dr \geq \log(\frac{C}{\varepsilon \lambda_e}),$$

and this concludes the proof.

Corollary 5.16. If $N \geq 4$ and $p = \sqrt{N}$ then it holds that $\lambda_\varepsilon \lesssim \left( \varepsilon (\log \frac{1}{\varepsilon}) \right)^{-\frac{(p^* - p)}{p(p^* - 1)}}$.

Proof. By (5.19) and (5.8) we get

$$C \varepsilon^p \lambda_e^p \log \frac{1}{\varepsilon \lambda_e} \leq \left( \varepsilon (\log \frac{1}{\varepsilon}) \right)^{-\frac{(p^* - p)}{p(p^* - 1)}}.$$
for some $\delta_{1,2} \geq 0$ and $\varepsilon$ small enough, by (5.20) and (5.21). It follows that

$$\log \frac{1}{\sqrt{\varepsilon} \lambda_{\varepsilon}} \sim \log \frac{1}{\varepsilon}.$$  

Hence,

$$\lambda_{\varepsilon}^{p} \lesssim \left(\varepsilon \left(\log \frac{1}{\varepsilon}\right)^{\frac{d/(p^{*}) - 1}{d/(p - 1)}}\right) \left(\varepsilon \left(\log \frac{1}{\varepsilon}\right)^{\frac{d - p}{d - p^{*}}}ight),$$

and

$$\lambda_{\varepsilon} \lesssim \left(\varepsilon \left(\log \frac{1}{\varepsilon}\right)^{\frac{d - p}{d - p^{*}}}ight),$$

and this concludes the proof.

5.7 Improved estimates

In this section we aim to improve some of the estimates just obtained. The sharp upper estimates of $\lambda_{\varepsilon}$ yields the following

Corollary 5.17. Let $1 < p < N$. Then

$$||v_{\varepsilon}||_{l} = O(1).$$

The boundedness of the $L^{l}$ norm also allows one to reverse estimates of $||v_{\varepsilon}||_{p}$ via (5.19).

Corollary 5.18.

$$||v_{\varepsilon}||_{p} = \begin{cases} O(1), & 1 < p < \sqrt{N}, \\ O\left(\varepsilon^{\frac{p - p^{*}}{p - 1 + s}}\right), & \sqrt{N} < p < \frac{N + 1}{2}, N \geq 4, \\ O(\log \frac{1}{\varepsilon}), & p = \sqrt{N}, N \geq 4. \end{cases}$$

We now prove that the $L^{l}$ bound also implies an $L^{\infty}$ bound.

Lemma 5.19. It holds that

$$||v_{\varepsilon}||_{\infty} = O(1).$$

Proof. We start observing that by (P_{\varepsilon}) $v_{\varepsilon}$ is a positive solution to the inequality

$$-\Delta_{p} v_{\varepsilon} - V_{\varepsilon}(x) v_{\varepsilon}^{p - 1} \leq 0, \quad x \in \mathbb{R}^{N},$$

with

$$V_{\varepsilon}(x) := S_{\varepsilon} v_{\varepsilon}^{p^{*} - p}(x).$$

By Lemma A in the Appendix, we obtain

$$|v_{\varepsilon}(x)| \leq C_{l}||v_{\varepsilon}||_{l} |x|^{-N/l} \quad x \neq 0,$$

which combined with Corollary 5.17 yields

$$V_{\varepsilon}(x) \leq S_{\varepsilon} C_{l}^{p^{*} - p} ||v_{\varepsilon}||_{l}^{p^{*} - p} |x|^{-N(p^{*} - p)/l} \leq C_{*} |x|^{-pp^{*}/l},$$

for some uniform constant $C_{*} > 0$ independent on $\varepsilon$ or $x$. Hence, $v_{\varepsilon}$ is a positive solution to the inequality

$$-\Delta_{p} v_{\varepsilon} - V_{\varepsilon}(x) v_{\varepsilon}^{p - 1} \leq 0, \quad x \in \mathbb{R}^{N},$$

with $V_{\varepsilon}(x) = C_{*} |x|^{-pp^{*}/l} \in L^{s}_{\text{loc}}(\mathbb{R}^{N})$ for some $s > N/p$, since $l > p^{*}$. With these preliminaries in place, one can invoke here the result on local boundedness Theorem 7.1.1 in [30] p.154 for
On the other hand, we have

\[ \int_{B_{\varepsilon}(0)} -\Delta_p v_\varepsilon(x) dx \leq \int_{B_{\varepsilon}(0)} V_\varepsilon(x)v_\varepsilon^{p-1}(x) dx, \]

and by the divergence theorem, taking into account the monotonicity of \( v_\varepsilon \) in \( |x| \) we have

\[
\int_{B_{\varepsilon}(0)} -\Delta_p v_\varepsilon(x) dx = \int_{\partial B_{\varepsilon}(0)} -|\nabla v_\varepsilon(x)|^{p-2}\nabla v_\varepsilon(x) \cdot \nu d\sigma = \int_{\partial B_{\varepsilon}(0)} |\nabla v_\varepsilon(x)|^{p-1} d\sigma = C_1|\nabla v_\varepsilon(x)|^{p-1}|x|^{N-1}.
\]

On the other hand

\[
\int_{B_{\varepsilon}(0)} V_\varepsilon(x)v_\varepsilon^{p-1}(x) dx = C_2 \int_0^{\varepsilon} r^{-\frac{N}{p}+N-1} v_\varepsilon^{p-1}(r) dr \leq C_2|v_\varepsilon(0)|^{p-1} \int_0^{\varepsilon} r^{-\frac{N}{p}+N-1} dr = C_3|v_\varepsilon(0)|^{p-1}|\varepsilon|^{-\frac{N}{p}+N},
\]

since \( -\frac{N}{p} + N > 0 \). Hence

\[
|\nabla v_\varepsilon(x)|^{p-1} \leq \frac{C_4}{|x|^{N-1}} \int_{B_{\varepsilon}(0)} V_\varepsilon(r)v_\varepsilon^{p-1}(r) dr \leq C_5|v_\varepsilon(0)|^{p-1}|x|^{1-pp'/l},
\]

(5.34)

for some \( C_4, C_5 > 0 \) independent of \( \varepsilon \) and \( x \). Integrating again from 0 to \( x_0 \) after writing (5.34) in this form

\[
-\frac{d}{dr} v_\varepsilon(|x|) \leq C_6|v_\varepsilon(0)||x|^{1-pp'/l}/(p-1),
\]

we have

\[
v_\varepsilon(0) \leq v_\varepsilon(x_0) + C_7v_\varepsilon(0)|x_0|^{\frac{p-1}{pp'/l}}, \]

(5.35)

for some \( C_7 \) independent of \( \varepsilon \) and \( x \). Then we choose \( A \) small enough such that if \( |x_0| \leq A \) then we have

\[
v_\varepsilon(0)(1 - C_7A^{\frac{p-1}{pp'/l}}) \leq v_\varepsilon(0)(1 - C_7|x_0|^{\frac{p-1}{pp'/l}}) \leq v_\varepsilon(x_0).
\]

Then we have

\[
C_8v_\varepsilon(0) \leq v_\varepsilon(x_0), \quad \text{for all } x_0, |x_0| < A,
\]

where \( C_8 = 1 - C_7A^{\frac{p-1}{pp'/l}} \). Then by taking the power \( l \) and integrating we obtain

\[
\int_{|x|<A} C_9|v_\varepsilon(0)|^l dx \leq \int_{|x|<A} |v_\varepsilon(x)|^l dx.
\]

which by Corollary 5.17 immediately concludes the proof.

The classical elliptic regularity theory for the \( p \)-Laplacian implies, as a consequence of the \( L^\infty \) bound, the following
Corollary 5.20. It holds that \( v_{\varepsilon} \to U_1 \) in \( C^{1,\alpha}_{{\text{loc}}} (\mathbb{R}^N) \) and \( L^s(\mathbb{R}^N) \) for any \( s \geq p^* \). In particular, \( v_{\varepsilon}(0) \simeq W_1(0) \).

Proof. As a consequence of the \( L^\infty \) bound of Lemma 5.19 and the convergence of \( v_{\varepsilon} \) to the Sobolev groundstate \( U_1 \) in \( D^{1,p}(\mathbb{R}^N) \) via the compactness result in Lemma A.3 we obtain the convergence in \( L^s(\mathbb{R}^N) \) for any \( s \geq p^* \). Since we can write \( R^\ast_{\varepsilon} \) in the form

\[
-\Delta_p v_{\varepsilon} = f(v_{\varepsilon}),
\]

and by Lemma 5.19 we have

\[
\|f(v_{\varepsilon})\|_{L^\infty_{\text{loc}}(\mathbb{R}^N)} < C,
\]

uniformly with respect to \( \varepsilon \), then by [11, Theorem 2] we have

\[
\|v_{\varepsilon}\|_{C^{1,\alpha}_{{\text{loc}}}(\mathbb{R}^N)} < C,
\]

uniformly with respect to \( \varepsilon \). It follows that by the classical Arzelá-Ascoli theorem that for a suitable sequence \( \varepsilon \to 0 \) we have

\[
v_{\varepsilon} \to U_1 \quad \text{in} \quad C^{1,\alpha'}_{{\text{loc}}}(\mathbb{R}^N),
\]

where \( \alpha < \alpha' \).

Proof of Theorem 2.3. The proof follows immediately from Corollary 5.20 and Lemma 5.6 which yield the upper and lower estimates on \( \lambda_{\varepsilon} \).

Proof of Theorem 2.5. The proof follows from the sharp upper bound on \( \lambda_{\varepsilon} \) in Corollaries 5.14-5.16. In particular since from Corollary 5.20 we have

\[
u_{\varepsilon}(0) \sim \lambda_{\varepsilon} \frac{N_p}{p} v_{\varepsilon}(0),
\]

then by the sharp estimate of \( \lambda_{\varepsilon} \) we have the exact rate of the groundstate \( u_{\varepsilon}(0) \) in the present critical case

\[
u_{\varepsilon}(0) \sim \begin{cases} 
\varepsilon \frac{N_p}{p} \frac{N_p}{p^*} & 1 < p < \sqrt{N}, \ N \geq 2, \\
\varepsilon \frac{N_p}{p - p^*} \frac{N_p}{p^*} & \sqrt{N} < p < \frac{N+p+1}{2}, \ N \geq 4, \\
(\varepsilon \log \frac{1}{\varepsilon})^{\frac{N_p}{p}} & p = \sqrt{N}, \ N \geq 4.
\end{cases}
\]

(5.36)

6 Proof of Theorem 2.8: supercritical case \( q > p^* \)

In this section, we consider the supercritical case \( q > p^* \) and prove Theorem 2.8 formulated in the Introduction, which essentially says that for \( q > p^* \) ground state solutions \( u_{\varepsilon} \) converge as \( \varepsilon \to 0 \) to a non-trivial radial ground state solution of the “formal” limit equation \( (P_0) \). This result extends [24, Theorem 2.3] to \( p \neq 2 \).

6.1 The limiting PDE

From the results of Section 4 we know that for \( q > p^* \) the limit equation

\[
-\Delta_p u - |u|^{q-2} u + |u|^{l-2} u = 0 \quad \text{in} \quad \mathbb{R}^N,
\]

(\( P_0 \))
admits a positive radial groundstate solution $u_0 \in D^{1,p}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, which is fast decaying, namely such that

$$u_0 \simeq |x|^{-\frac{N-p}{2}} \text{ as } |x| \to \infty. \quad (6.1)$$

Further, it is known that $u_0 \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$.

By construction the groundstate $u_0$ admits a variational characterization in the Sobolev space $D^{1,p}(\mathbb{R}^N)$ via the rescaling

$$u_0(x) := w_0 \left( \frac{x}{\sqrt{S_0}} \right),$$

where $w_0$ is a positive radial minimizer of the constrained minimization problem

$$S_0 := \inf \left\{ \int_{\mathbb{R}^N} |\nabla w|^p \, dx \mid w \in D^{1,p}(\mathbb{R}^N), \quad p^* \int_{\mathbb{R}^N} \tilde{F}_0(w) \, dx = 1 \right\}, \quad (S_0)$$

where

$$\tilde{F}_0(w) = \int_0^w \tilde{f}_0(s) \, ds,$$

and $\tilde{f}_0(s)$ is a truncation of the nonlinearity

$$f_0(s) = |s|^{q-2} s - |s|^{l-2} s,$$

as described in Section 4. Then the minimization problem $(S_0)$ is well defined on $D^{1,p}(\mathbb{R}^N)$. The minimizer $w_0$ satisfies the Euler-Lagrange equation

$$-\Delta_p w_0 = S_0 (w_0^{q-1} - w_0^{l-1}).$$

Moreover, $w_0$ satisfies Nehari’s identity

$$\int_{\mathbb{R}^N} |\nabla w_0|^p \, dx = S_0 \left( \int_{\mathbb{R}^N} |w_0|^q \, dx - \int_{\mathbb{R}^N} |w_0|^l \, dx \right),$$

which yields

$$1 = ||w_0||_q^q - ||w_0||_l^l. \quad (6.2)$$

From the Pohožaev identity

$$\int_{\mathbb{R}^N} |\nabla w_0|^p \, dx = S_0 p^* \left( \frac{1}{q} \int_{\mathbb{R}^N} |w_0|^q \, dx - \frac{1}{l} \int_{\mathbb{R}^N} |w_0|^l \, dx \right),$$

we have

$$1 = \frac{p^*}{q} ||w_0||_q^q - \frac{p^*}{l} ||w_0||_l^l. \quad (6.3)$$

Hence from $(6.2)$ and $(6.3)$ we obtain the relation

$$||w_0||_q^q - ||w_0||_l^l = \frac{p^*}{q} ||w_0||_q^q - \frac{p^*}{l} ||w_0||_l^l = 1,$$

from which we obtain the expressions

$$||w_0||_q^q = \frac{q(l - p^*)}{p^*(l-q)}, \quad ||w_0||_l^l = \frac{l(q - p^*)}{p^*(l-q)}.$$
6.2 Energy estimates and groundstate asymptotics

The relations between $S_{\varepsilon}$ and $S_{0}$ is provided by introducing the convenient scaling-invariant quotient

$$S_{0}(w) := \frac{\int_{\mathbb{R}^{N}} |\nabla w|^{p} \, dx}{\left(p^{*} \int_{\mathbb{R}^{N}} \tilde{F}_{0}(w) \, dx\right)^{(N-p)/N}}, \quad w \in \mathcal{M}_{0}, \quad (6.4)$$

where

$$\mathcal{M}_{0} := \left\{ w \in D^{1,p}(\mathbb{R}^{N}), \int_{\mathbb{R}^{N}} \tilde{F}_{0}(w) \, dx > 0 \right\}.$$

Note that, by a rescaling argument, this is equivalent to (59):

$$S_{0} = \inf_{w \in \mathcal{M}_{0}} S_{0}(w).$$

**Lemma 6.1.** For all $1 < p < N$, it holds that

$$0 < S_{\varepsilon} - S_{0} \to 0, \quad \text{as} \quad \varepsilon \to 0.$$

**Proof.** To show that $S_{0} < S_{\varepsilon}$, simply note that

$$S_{0} \leq S_{0}(w_{\varepsilon}) < S_{\varepsilon}(w_{\varepsilon}) = S_{\varepsilon}. \quad (6.5)$$

To estimate $S_{\varepsilon}$ from above we test $\phi_{\varepsilon}$ with the minimizer $w_{0}$. By (6.1), we have $w_{0} \in L^{p}(\mathbb{R}^{N})$ if and only if $1 < p < \sqrt{N}$. We break the proof by analysing the higher and lower dimensions separately.

**Case 1:** $p < \sqrt{N}$. Using $w_{0}$ as a test function for (59), we obtain

$$S_{\varepsilon} \leq S_{\varepsilon}(w_{0}) \leq \frac{S_{0}}{\left(1 - \frac{p^{*}}{p} \|w_{0}\|_{L^{p}(\mathbb{R}^{N})}^{p}\right)^{\frac{N-p}{N}}} \leq S_{0} + O(\varepsilon), \quad (6.6)$$

which proves the statement for $1 < p < \sqrt{N}$.

In the cases $p = \sqrt{N}$ and $\sqrt{N} < p < N$, given $R > 1$ we pick a cut-off function $\eta_{R} \in C_{0}^{\infty}(\mathbb{R})$ such that $\eta_{R}(r) = 1$ for $r < R$, $0 < \eta_{R} < 1$ for $R < \eta | x | < 2R$, $\eta_{R} = 0$ for $| r | > 2R$ and $\| \eta_{R} \|_{L^{p}(\mathbb{R})} \leq 2/R$. By (6.1), for $s > \frac{N}{N-p}$ we obtain

$$\int_{\mathbb{R}^{N}} |\nabla (\eta_{R}w_{0})(x)|^{p} \, dx = S_{0} + O(R^{-\frac{N}{p-s}}),$$

$$\int_{\mathbb{R}^{N}} |\eta_{R}w_{0}(x)|^{p} \, dx = \|w_{0}\|_{L^{p}(\mathbb{R}^{N})} \left(1 - O(R^{-\frac{N-p}{p}})\right),$$

$$\int_{\mathbb{R}^{N}} |\eta_{R}w_{0}(x)|^{p} \, dx = \left\{ O(\log(R)), \quad p = \sqrt{N}, \right.$$

Case $p = \sqrt{N}$. Let $R = \varepsilon^{-1}$. Testing (59) with $\eta_{R}w_{0}$ and since $q > p^{*}$, we get

$$S_{\varepsilon} \leq S_{\varepsilon}(\eta_{R}w_{0}) \leq \left(S_{0} + O(R^{-\frac{N}{p-s}})\right)^{\frac{p^{*}}{q}} \left(\left(\frac{p^{*}}{q} \|w_{0}\|_{q}^{q} \left(1 - O(R^{-\frac{N-p}{p}})\right) - \frac{p^{*}}{p} O(\log R) - \frac{p^{*}}{q} \|w_{0}\|_{q}^{q} \left(1 - O(R^{-\frac{N-p}{p}})\right)\right)^{\frac{N-p}{N}}\right)^{\frac{p}{q}}$$

$$= \frac{S_{0} + O(\varepsilon^{-\frac{N}{p-s}})}{\left(1 - o(\varepsilon^{-\frac{N}{p-s}}) - O(\varepsilon \log \frac{1}{\varepsilon})\right)^{\frac{N-p}{N}}} \leq S_{0} + O\left(\varepsilon \log \frac{1}{\varepsilon}\right),$$

deprecated
Hence for every \( s > p \). We test (5.2) with \( \eta R w_0 \) and as \( q > p^* \), we obtain

\[
S_\varepsilon \leq S_{\varepsilon}(\eta R w_0) \leq \left( S_0 + O((R)^{-\frac{N-p}{p-q}}) \right). \\
\left( \frac{p^*}{q} ||w_0||_q^q (1 - O(R^{-\frac{N-p}{p-q}})) - \frac{\varepsilon p^*}{p} O((R)^{-\frac{N-p}{p-q}}) - \frac{p^*}{l} ||w_0||_l (1 - O((R)^{-\frac{N-p}{p-q}})) \right)^{\frac{q}{p}} \\
\leq \frac{S_0 + O((\varepsilon R)^{-\frac{N-p}{p-q}})}{\left( 1 - O((\varepsilon R)^{-\frac{N-p}{p-q}}) \right)^{\frac{q}{p}}} \leq S_0 + O((\varepsilon R)^{-\frac{N-p}{p-q}}),
\]

which completes the proof. \( \square \)

**Lemma 6.2.** It holds that \( ||w_\varepsilon||_{\infty} \leq 1 \) and \( ||w_\varepsilon||_{s} \lesssim 1 \) for all \( s > p^* \).

**Proof.** Note that by (3.8) we have \( ||w_\varepsilon||_{\infty} = ||u_\varepsilon||_{\infty} \leq 1. \)

By Sobolev’s inequality and Lemma 6.1 we have

\[
||w_\varepsilon||_{p^*} \leq S_{\varepsilon}^{-1} ||\nabla w_\varepsilon||_{p^*} = S_{\varepsilon}^{-1} S_{\varepsilon} = S_{\varepsilon}^{-1} S_0 (1 + o(1)).
\]

Hence for every \( s > p^* \),

\[
||w_\varepsilon||_{s} \leq ||w_\varepsilon||_{p^*},
\]

which concludes the proof. \( \square \)

**Lemma 6.3.** For all \( 1 < p < N \), we have

\[
\varepsilon ||w_\varepsilon||_p^p \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

**Proof.** Observing that \( w_\varepsilon \) is an optimiser to (3.3), it follows that

\[
1 = p^* \int_{\mathbb{R}^N} F_\varepsilon(w_\varepsilon) dx = p^* \int_{\mathbb{R}^N} F_0(w_\varepsilon) - p^* \frac{\varepsilon}{p} ||w_\varepsilon||_p^p. \tag{6.7}
\]

Hence

\[
S_0(w_\varepsilon) = \frac{\int_{\mathbb{R}^N} ||\nabla w_\varepsilon||_p^p dx}{\left( p^* \int_{\mathbb{R}^N} F_0(w_\varepsilon) dx \right)^{(N-p)/N}} = \frac{S_{\varepsilon}}{\left( 1 + \frac{\varepsilon}{p} ||w_\varepsilon||_p^p \right)^{(N-p)/N}}.
\]

If by contradiction we had limsup\( \limsup_{\varepsilon \to 0} \varepsilon ||w_\varepsilon||_p^p = m > 0 \), then by Lemma 6.1 for any sequence \( \varepsilon_n \to 0 \), we would obtain

\[
S_0 \leq S_0(w_{\varepsilon_n}) \leq S_0 \left( \frac{1 + o(1)}{1 + \frac{\varepsilon_n}{p} m} \right) < S_0,
\]

and this, as it is clearly a contradiction, concludes the proof. \( \square \)

**Theorem 6.4.** Let \( 1 < p < N \) and \( q > p^* \). As \( \varepsilon \to 0 \), the family of groundstates \( w_\varepsilon \) converges to a groundstate \( w_0 \) in \( D^{1,p}(\mathbb{R}^N) \), \( L^1(\mathbb{R}^N) \) and \( C^{1,\alpha}_{loc}(\mathbb{R}^N) \) to (7.4). In particular

\[
w_\varepsilon(0) \to w_0(0).
\]

Furthermore \( w_0 \) is fast decaying, namely

\[
w_0(x) \sim |x|^{-\frac{N-p}{p}} \quad \text{as} \quad |x| \to \infty.
\]
Proof. Since the family $w_\varepsilon$ is bounded in $D^{1,p}(\mathbb{R}^N)$ then there exists a subsequence $w_{\varepsilon_n}$ such that

$$w_{\varepsilon_n} \rightharpoonup \tilde{w} \text{ in } D^{1,p}(\mathbb{R}^N)$$

and $w_{\varepsilon_n} \to \tilde{w}$ a.e. in $\mathbb{R}^N$, as $n \to \infty$

where $\tilde{w} \in D^{1,p}(\mathbb{R}^N)$ is a radial function. By Sobolev’s inequality, the sequence $(w_{\varepsilon_n})$ is bounded in $L^p(\mathbb{R}^N)$. Using Lemma A.3 and Sobolev’s inequality, we also obtain a uniform bound

$$|w_{\varepsilon_n}(x)| \leq C|x|^{-\frac{(N-p)}{p}}||\nabla w_{\varepsilon_n}||_p \leq C|x|^{-\frac{(N-p)}{p}}S_0,$$

for $\varepsilon$ sufficiently small. Using Lemma A.5 and Lemma 6.2 we conclude that

$$w_{\varepsilon_n} \to \tilde{w} \text{ in } L^p(\mathbb{R}^N \setminus B_r(0))$$

for $r > 0$ and $s \in (p^*, \infty)$.

Taking into account Lemma 6.3 and 6.7 we also obtain

$$\int_{\mathbb{R}^N} F_0(\tilde{w})dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} F_0(w_{\varepsilon_n})dx = \lim_{n \to \infty} (1 + p^\varepsilon_n \varepsilon |w_{\varepsilon_n}|^p) = 1.$$

By the weak lower semicontinuity property of the norm we also have that

$$||\nabla \tilde{w}||_p^p \leq \liminf_{n \to \infty} ||\nabla w_{\varepsilon_n}||_p^p = S_0,$$

i.e. $\tilde{w}$ is a minimizer for (6.9). We now claim that

$$\nabla w_{\varepsilon_n} \to \nabla \tilde{w} \text{ a.e. on } \mathbb{R}^N,$$

and then by Brezis-Lieb Lemma 4, $(w_{\varepsilon_n})$ converges strongly to $\tilde{w}$ in $D^{1,p}(\mathbb{R}^N)$. In fact, arguing as in [22] Theorem 3.3 (see also [21] Proposition 2.3), define a bounded function

$$T := \begin{cases} s & \text{if } |s| \leq 1, \\ \frac{s}{|s|} & \text{if } |s| > 1, \end{cases}$$

and consider a sequence $(B_k)$ of open subsets of $\mathbb{R}^N$ such that $\bigcup_{k=1}^{\infty} B_k = \mathbb{R}^N$. Then if

$$\lim_{n \to \infty} \int_{B_k} (||\nabla w_{\varepsilon_n}||^{p-2}\nabla w_{\varepsilon_n} - ||\nabla \tilde{w}||^{p-2}\nabla \tilde{w}) \cdot \nabla T(w_{\varepsilon_n} - \tilde{w})dx \to 0,$$

for every $k$, then

$$\nabla w_{\varepsilon_n} \to \nabla \tilde{w} \text{ a.e. on } B_k,$$

and hence by a Cantor diagonal argument, (6.8) is satisfied.

To show (6.3), we introduce a cut-off function

$$\rho(x) := \begin{cases} 1 & \text{if } |x| \leq k, \\ 0 & \text{if } |x| \geq k + 1, \end{cases}$$

and since

$$||\nabla w_{\varepsilon_n}||^{p-2}\nabla w_{\varepsilon_n} - ||\nabla \tilde{w}||^{p-2}\nabla \tilde{w}) \cdot \nabla T(w_{\varepsilon_n} - \tilde{w}) \geq 0,$$

then

$$0 \leq \int_{B_k} (||\nabla w_{\varepsilon_n}||^{p-2}\nabla w_{\varepsilon_n} - ||\nabla \tilde{w}||^{p-2}\nabla \tilde{w}) \cdot \nabla T(w_{\varepsilon_n} - \tilde{w})dx$$

$$\leq \int_{B_{k+1}} (||\nabla w_{\varepsilon_n}||^{p-2}\nabla w_{\varepsilon_n} - ||\nabla \tilde{w}||^{p-2}\nabla \tilde{w}) \cdot \nabla T(w_{\varepsilon_n} - \tilde{w}) \rho(x)dx$$

$$\leq \int_{B_{k+1}} (||\nabla w_{\varepsilon_n}||^{p-2}\nabla w_{\varepsilon_n} - ||\nabla \tilde{w}||^{p-2}\nabla \tilde{w}) \cdot \left(\rho T(w_{\varepsilon_n} - \tilde{w}) + \rho dx \right)dx$$

$$+ \int_{B_{k+1}} (||\nabla w_{\varepsilon_n}||^{p-2}\nabla w_{\varepsilon_n} - ||\nabla \tilde{w}||^{p-2}\nabla \tilde{w}) T(w_{\varepsilon_n} - \tilde{w}) \rho dx \to 0,$$
as $n \to \infty$. In fact

$$
\left| \int_{B_{k+1}} \left( |\nabla w_{x_n}|^{p-2} \nabla w_{x_n} - |\nabla w_0|^{p-2} \nabla \right) \nabla \left( \rho T(w_{x_n} - \bar{w}) \right) dx \right|
$$

$$
\leq \left| \int_{\mathbb{R}^N} \left| \nabla w_{x_n} \right|^{p-2} \nabla w_{x_n} \nabla \left( \rho T(w_{x_n} - \bar{w}) \right) dx \right| + \left| \int_{\mathbb{R}^N} \left| \nabla \bar{w} \right|^{p-2} \nabla \bar{w} \nabla \left( \rho T(w_{x_n} - \bar{w}) \right) dx \right|
$$

$$
= \left| \int_{\mathbb{R}^N} f_\varepsilon(w_{x_n}) \rho T(w_{x_n} - \bar{w}) dx \right| + \left| \int_{\mathbb{R}^N} f(\bar{w}) \rho T(w_{x_n} - \bar{w}) dx \right| \to 0,
$$

by local compactness. Moreover, by Hölder’s inequality and since $T$ is bounded and $w_{x_n} - \bar{w} \to 0$ a.e. on $\mathbb{R}^N$, then by dominated convergence theorem, we have

$$
\left| \int_{B_{k+1}} \left( |\nabla w_{x_n}|^{p-2} \nabla w_{x_n} - |\nabla \bar{w}|^{p-2} \nabla \bar{w} \right) T(w_{x_n} - \bar{w}) \nabla \rho dx \right|
$$

$$
\leq C \left( \int_{B_{k+1}} \left| T(w_{x_n} - \bar{w}) \right|^p |\nabla \rho|^p dx \right)^{\frac{1}{p}} \to 0,
$$

and hence (6.9) follows. As a consequence $(w_{x_n})$ converges to $\bar{w}$ in $D^{1,p}(\mathbb{R}^N)$ and in $L^s(\mathbb{R}^N)$ for any $s \geq p^*$, where $\bar{w}$ is a minimizer of $(S_0)$ satisfying the constraint. Similarly to the proof of Corollary 5.20, by elliptic regularity we conclude that $(w_{x_n})$ converges to $w_0$ in $C^{1,\alpha}_{loc}(\mathbb{R}^N)$. The decay follows from Lemma 4.7. This concludes the proof. 

**Proof of Theorem 2.8.** The statement follows directly from Theorem 6.3 and Lemma 6.3.

**Remark 6.5.** Observe that in the range of the parameters where the uniqueness of minimizer of $(P_0)$ holds, we have $w = w_0$ is the fast decaying groundstate constructed in Section 4.

### 7 Proof of Theorem 2.2: subcritical case $p < q < p^*$

In this section, we consider the subcritical case $p < q < p^*$ and prove Theorem 2.2 showing that after the canonical rescaling $w_{\varepsilon} \to \varepsilon$ groundstate solutions $w_{\varepsilon}$ converge as $\varepsilon \to 0$ to the unique non-trivial radial ground state solution of the limit equation $(R_0)$. This result extends Theorem 2.1 to $p \neq 2$.

Since by Pohožaev’s identity the equation $(P_1)$ has no positive finite energy solutions, to understand the asymptotic behaviour of the ground states $w_{\varepsilon}$ we consider the rescaling in $(1.2)$, which transforms $(P_1)$ into $(R_\varepsilon)$, whose limit problem as $\varepsilon \to 0$ is $(R_0)$.

Pick $G_{\varepsilon} : \mathbb{R} \to \mathbb{R}$ a be a bounded truncated function such that

$$
G_{\varepsilon}(w) = \frac{1}{q} |w|^q - \frac{1}{p} |w|^p + \frac{\varepsilon^{-\frac{4}{p'}} - 1}{l} |w|^l,
$$

for $0 < w \leq \varepsilon^{- \frac{1}{p'}}$, $G_{\varepsilon}(w) \leq 0$ for $w > \varepsilon^{- \frac{1}{p'}}$ and $G_{\varepsilon}(w) = 0$ for $w \leq 0$. For $\varepsilon \in [0, \varepsilon^*)$, we set

$$
S_{\varepsilon} := \inf \left\{ \int_{\mathbb{R}^N} |\nabla w|^p dx \bigg| w \in W^{1,p}(\mathbb{R}^N), \quad p^* \int_{\mathbb{R}^N} G_{\varepsilon}(w) dx = 1 \right\}.
$$

a well-defined family of constrained minimisation problems, which share, together with the limit problem $(S_0)$, the same functional setting $W^{1,p}(\mathbb{R}^N)$. By Theorem 3.2 $(S_1)$ possesses a radial positive minimizer $w_{\varepsilon}$ for every $\varepsilon \in [0, \varepsilon^*)$. The rescaled function

$$
v_{\varepsilon}(x) := \frac{w_{\varepsilon}(\frac{x}{\sqrt{S_{\varepsilon}}})}{\sqrt{S_{\varepsilon}}},
$$

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is a radial groundstate of $\{\mathcal{R}_0\}$.

We estimate $\{S'_0\}$ by means of the dilation invariant representation

$$S'_0(w) := \frac{\int_{\mathbb{R}^N} |\nabla w|^p \, dx}{\left( p^* \int_{\mathbb{R}^N} G_\varepsilon(w) \, dx \right)^{(N-p)/N}}, \quad w \in \mathcal{M}_\varepsilon,$$

where $\mathcal{M}_\varepsilon := \{0 \leq w \in W^{1,p}(\mathbb{R}^N), \int_{\mathbb{R}^N} G_\varepsilon(w) \, dx > 0\}$. We have

$$S'_\varepsilon = \inf_{w \in \mathcal{M}_\varepsilon} S'_\varepsilon(w),$$

and for $\varepsilon$ small enough we have

$$S'_0 \leq S'_0(w_\varepsilon) < S'_\varepsilon(w) = S'_\varepsilon.$$  \hspace{1cm} (7.1)

This follows by observing that by definition $p^* \int_{\mathbb{R}^N} G_\varepsilon(w_\varepsilon) \, dx = 1$ and $G_\varepsilon(s)$ is a decreasing as a function of $\varepsilon$ for each $s > 0$, we have $w_\varepsilon \in \mathcal{M}_0$, and the second inequality follows again by monotonicity. Observe that by continuity $w_0 \in \mathcal{M}_\varepsilon$ for sufficiently small $\varepsilon$. As a consequence, by testing $\{S'_0\}$ with $w_0$, that for $\varepsilon$ small enough, we have that

$$S'_\varepsilon \leq S'_0(w_0) = \frac{S'_0}{\left( 1 - \frac{\beta}{\varepsilon} \right)} \left( \frac{1}{(N-p)/N} \right) \leq S'_0 + O(\varepsilon^{-1}).$$

Hence $S'_\varepsilon \to S'_0$. Reasoning as in Lemma 5.2, we obtain that

$$||w_\varepsilon||_q = \frac{(l-p^*)q}{(l-q)p^*} \frac{q(l-p)}{p(l-q)} ||w_\varepsilon||_p.$$  \hspace{1cm} (7.2)

Plugging in this identity into the definition of $S'_0(w_\varepsilon)$ and using the convergence of $S'_\varepsilon$ to $S'_0$, one can easily check that

$$\lim_{\varepsilon \to 0} ||w_\varepsilon||_p^p = \frac{p(p^*-q)}{p^*(q-p)}, \quad \lim_{\varepsilon \to 0} ||w_\varepsilon||_q^q = \frac{q(p^*-p)}{p^*(q-p)}.$$  \hspace{1cm} (7.2)

Therefore $p^* \int_{\mathbb{R}^N} G_0(w_\varepsilon) \, dx \to 1$ as $\varepsilon \to 0$. We have then achieved that a rescaling $\lambda_\varepsilon \to 1$ exists such that $p^* \int_{\mathbb{R}^N} G_0(\bar{w}_\varepsilon) \, dx = 1$ and $S'_\varepsilon(\bar{w}_\varepsilon) \to S'_0$ for $\bar{w}_\varepsilon(x) := w_\varepsilon(\lambda_\varepsilon x)$. It follows that $(\bar{w}_\varepsilon)$ is a minimizing one parameter family for $(S'_0)$ that satisfies the constraint used in the method which yields Theorem 3.2. By Theorem 3.2 we conclude that for a suitable sequence $\varepsilon_n \to 0$, it holds $w_{\varepsilon_n} \to \bar{w}$ strongly in $W^{1,p}(\mathbb{R}^N)$, and since $(\lambda_\varepsilon)$, it holds that $w_{\varepsilon_n} \to \bar{w}$, where $\bar{w}$ is the minimizer of $(S'_0)$ satisfying the constraint. By the uniqueness of minimizer of $(\mathcal{R}_0)$, we have $\bar{w} = w_0$. An obvious modification of the proof of Lemma 6.19 using $||w_\varepsilon||_p$ yields that $||w_\varepsilon||_\infty \lesssim 1$ as $\varepsilon \to 0$. By standard elliptic regularity, similarly to the proof of Corollary 5.20, we conclude that $w_\varepsilon$ converges to $w_0$ in $L^p(\mathbb{R}^N)$ for any $s \geq p$ and in $C^{1,\alpha}_{loc}(\mathbb{R}^N)$, and therefore the proof of Theorem 2.2 is complete.

A Radial functions

We recall that for $u \in L^1(\mathbb{R}^N)$, the radially decreasing rearrangement of a function $u$ is denoted by $u^*$ and it is such that for any $\alpha > 0$ it holds that

$$|x \in \mathbb{R}^N : u(x)^* \geq \alpha| = |x \in \mathbb{R}^N : |u(x)| \geq \alpha|,$$
and as a consequence, since $u_n \to u$ is bounded, for all $x, y \in \mathbb{R}^N \setminus B_r(0)$ and a uniform constant $C > 0$ we have that

$$|u_n(x) - u_n(y)| \leq C r^{(1-N)/p} |x - y|^p.$$ \hfill (A.4)

Namely, $(u_n)_{n \in \mathbb{N}}$ is bounded in $C^{0,p'}(\mathbb{R}^N \setminus B_r(0))$ and by the locally compact embedding it is strongly convergent to $u$ in $L^\infty(\mathbb{R}^N \setminus B_r(0))$. This and (A.3) yield the convergence in $L^\infty(\mathbb{R}^N \setminus B_r(0))$. \hfill $\square$
B Comparison principle for the $p$-Laplacian

Let $G \subseteq \mathbb{R}^N$ be a domain. We say that $0 \leq v \in W^{1,p}_loc(G)$ satisfies condition (S) if:

(S) there exists $(\theta_n)_{n \in \mathbb{N}} \subset W^{1,\infty}_c(\mathbb{R}^N)$ such that $0 \leq \theta_n \to 1$ a.e. in $\mathbb{R}^N$ and

$$\int_G R(\theta_n v, v) \to 0, \quad \text{as} \quad n \to +\infty,$$

where $R$ is defined by

$$R(w, v) := |\nabla w|^p - \nabla \left( \frac{w^p}{vp-1} \right) |\nabla v|^{p-2} \nabla v. \quad (B.1)$$

Notice that if $G$ is bounded and $v \in W^{1,p}_loc(G)$ then condition (S) is trivially satisfied with $\theta \equiv 1$ in $G$. In case of an unbounded domain $G$, condition (S) ensures that the subsolution $v$ is sufficiently small at infinity, in order to respect the comparison principle (see [19]).

Using condition (S), we formulate a version of comparison principle for a $p$-Laplacian with a general negative potential (see e.g. [19][27][34]).

**Theorem B.1** (Comparison principle for $p$-Laplacian). Let $0 < u \in W^{1,p}_loc(G) \cap C(\bar{G})$ be a supersolution and $v \in W^{1,p}_loc(G) \cap C(\bar{G})$ a sub-solution to the equation

$$-\Delta_p u + \nabla u |\nabla u|^{p-2} u = 0 \quad \text{in} \quad G, \quad (B.2)$$

where $V \in L^{\infty}_loc(G)$. If $G$ is an unbounded domain, assume in addition that $\partial G \neq \emptyset$ and $v^+$ satisfies condition (S). Then $u \geq v$ on $\partial G$ implies $u \geq v$ in $G$.

Below we prove a simple sufficient condition for assumption (S) to hold.

**Lemma B.2.** If $0 \leq v \in D^{1,p}_rad(\mathbb{R}^N)$ then $v$ satisfies (S).

**Proof.** Following [19][34], define

$$\eta_R(r) = \begin{cases} 1, & 0 \leq r \leq R \\ \log \frac{R^2}{\log R}, & R \leq r \leq R^2 \\ 0, & r \geq R^2, \end{cases}$$

and note that $|\eta_R| \leq 1$ a.e. in $\mathbb{R}^N$ and $|\eta'_R| \leq \frac{c}{\log R} r^{-1}$. We are going to show that

$$\int_{\mathbb{R}^N} R(\eta_R v, v) \to 0 \quad \text{as} \quad R \to \infty.$$

Using the Picone’s identity [1][10] and inequalities [34] Lemma 7.4], it is straightforward to deduce the inequalities

$$R(\eta_R v, v) \leq c_1 |v(\eta_R)'|^p, \quad (1 < p \leq 2), \quad (B.3)$$

$$R(\eta_R v, v) \leq c_2 |\eta_R v|^p |v(\eta_R)'|^2 + c_3 |v(\eta_R)'|^p, \quad (p > 2). \quad (B.4)$$

**Case** $1 < p \leq 2$. Using (B.3) and Ni’s decay estimate Lemma A.3 on $v \in D^{1,p}_rad(\mathbb{R}^N)$,

$$v \leq c |x|^{-\frac{N-p}{p}},$$

by a direct calculation we obtain

$$\int_{\mathbb{R}^N} R(\eta_R v, v) dx \leq c_1 \int_{R}^{R^2} |v(\eta_R)'|^p r^{N-1} dr \leq c \int_{R}^{R^2} r^{-\frac{N-p}{p}} \frac{1}{\log R} r^{-1} \left| r^{N-1} dr \right.$$

$$\leq \frac{C}{(\log R)^{p-1}} \to 0 \quad \text{as} \quad R \to \infty. \quad (B.5)$$
Case $p > 2$. By Hölder and (B.3) we conclude

$$\int_0^{+\infty} |\eta_R v'|^{p-2} |v(\eta_R)'| 2^{N-1} \, dr \leq \left( \int_0^{+\infty} |\eta_R v'|^{p-2} |v(\eta_R)'| 2^{N-1} \, dr \right)^{\frac{p}{2}} \left( \int_0^{+\infty} |v(\eta_R)'|^{p-2} |v(\eta_R)'| 2^{N-1} \, dr \right)^{\frac{2}{p}}$$

$$\leq c \|v\|_{D^{1,p}(\mathbb{R}^N)}^{p-2} \left( \int_{-R}^{R^2} |v(\eta_R)'|^{p-2} |v(\eta_R)'| 2^{N-1} \, dr \right)^{\frac{2}{p}} \to 0 \quad \text{as } R \to \infty.$$

Taking into account (B.4) and once again (B.5), the conclusion follows.

Remark B.3. While the statement of Lemma [B.2] is sufficient for our purposes, it is far from optimal. See [19, Appendix B] for constructions of radial functions $v \notin D^{1,p}(\mathbb{R}^N)$ which satisfy assumption (S).
References

[1] W. Allegretto and Y. X. Huang, *A Picone's identity for the p-Laplacian and applications*, Nonlinear Anal. 32 (1998), no. 7, 819–839. ↑

[2] A. Ambrosetti and A. Malchiodi, *Perturbation methods and semilinear elliptic problems on \( \mathbb{R}^n \)*, Progress in Mathematics, vol. 240, Birkhäuser Verlag, Basel, 2006. ↑

[3] H. Berestycki and P.-L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal. 82 (1983), no. 4, 313–345. ↑, 3, 5, 9, 10, 12, 34

[4] H. Brézis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. 88 (1983), no. 3, 486–490. ↑

[5] L. A. Caffarelli, B. Gidas, and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev exponents*, Comm. Pure Appl. Math. 42 (1989), no. 3, 271–297. ↑

[6] J. W. Cahn and J. E. Hilliard, *Free energy of a nonuniform system. III. nucleation in a 2-Component incompressible fluid*, J. Chem. phys. 31 (1959), no. 3, 688–699. ↑

[7] S. Coleman, *Fate of the false vacuum: Semiclassical theory*, Physical Review D 15 (1977), no. 10, 2929. ↑

[8] M. C. Cross and P. C. Hohenberg, *Pattern formation outside of equilibrium*, Reviews of modern physics 65 (1993), no. 3, 851–1112. ↑

[9] M. Degiovanni, A. Musesti, and M. Squassina, *On the regularity of solutions in the Pucci-Serrin identity*, Calc. Var. Partial Differential Equations 18 (2003), no. 3, 317–334. ↑

[10] J. I. Díaz and J. E. Saá, *Existence et unicité de solutions positives pour certaines équations elliptiques quasilineaires*, C. R. Acad. Sci. Paris Sér. I Math. 305 (1987), no. 12, 521–524. ↑

[11] E. DiBenedetto, *Local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. 7 (1983), no. 8, 827–850. ↑, 10, 13, 27

[12] A. Farina, C. Mercuri, and M. Willem, *A Liouville theorem for the p-Laplacian and related questions*, arXiv:1711.1152v2 (2017). ↑, 9

[13] M. Fraas and Y. Pinchover, *Isolated singularities of positive solutions of p-Laplacian type equations in \( \mathbb{R}^4 \)*, J. Differential Equations 254 (2013), no. 3, 1097–1119. ↑

[14] F. Gazzola and J. Serrin, *Asymptotic behavior of ground states of quasilinear elliptic problems with two vanishing parameters*, Ann. Inst. H. Poincaré Anal. Non Linéaire 19 (2002), no. 4, 477–504. ↑

[15] F. Gazzola, J. Serrin, and M. Tang, *Existence of ground states and free boundary problems for quasilinear elliptic operators*, Adv. Math. 19 (2000), no. 1-3, 1–30. ↑

[16] M. Guedda and L. Véron, *Local and global properties of solutions of quasilinear elliptic equations*, J. Differential Equations 64 (1987), no. 2, 355–360. ↑

[17] ———, *Quasilinear elliptic equations involving critical Sobolev exponents*, Nonlinear Anal. 13 (1989), no. 1, 879–902. ↑

[18] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. I*, Rev. Mat. Iberoamericana 1 (1985), no. 1, 145–201. ↑

[19] V. Liskevich, S. Lyakhova, and V. Moroz, *Positive solutions to nonlinear p-Laplace equations with Hardy potential in exterior domains*, J. Differential Equations 232 (2007), no. 1, 212–252. ↑, 35, 36

[20] C. Mercuri, G. Riey, and B. Sciunzi, *A regularity result for the p-Laplacian near uniform ellipticity*, SIAM J. Math. Anal. 48 (2016), no. 3, 2059–2075. ↑

[21] C. Mercuri and M. Squassina, *Global compactness for a class of quasi-linear elliptic problems*, Manus. Math. 140 (2013), no. 1, 119–144. ↑

[22] C. Mercuri and M. Willem, *A global compactness result for the p-Laplacian involving critical nonlinearities*, Discrete Contin. Dyn. Syst. 28 (2010), no. 2, 469–493. ↑, 9, 31

[23] F. Merle and L. A. Peletier, *Asymptotic behaviour of positive solutions of elliptic equations with critical and supercritical growth. II. The nonradial case*, J. Funct. Anal. 105 (1992), no. 1, 1–41. ↑

[24] V. Moroz and C. B. Muratov, *Asymptotic properties of ground states of scalar field equations with a vanishing parameter*, J. Eur. Math. Soc. (JEMS) 16 (2014), no. 5, 1081–1109. ↑, 3, 4, 5, 6, 7, 14, 27, 32

[25] C. Muratov and E. Vanden-Eijnden, *Breakup of universality in the generalized spinodal nucleation theory*, Journal of Statistical Physics 114 (2004), no. 3-4, 605–623. ↑, 3, 4, 7

[26] S. Pohožaev, *On the eigenfunctions of the equation \( \Delta u + \lambda f(u) = 0 \)*, Dokl. Akad. Nauk SSSR 165 (1965), 36–39. ↑

[27] A. Poliakovsky and I. Shafrir, *A comparison principle for the p-Laplacian*, Elliptic and parabolic problems (Rolduc/Gaeta, 2001), (2002), pp. 243–252. ↑

[28] P. Pucci and J. Serrin, *A general variational identity*, Indiana Univ. Math. J. 35 (1986), no. 3, 681–703. ↑
