Yang-Baxter basis of Hecke algebra and Casselman’s problem (extended abstract)

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Abstract
We generalize the definition of Yang-Baxter basis of type A Hecke algebra introduced by A.Lascoux, B.Leclerc and J.Y.Thibon (Letters in Math. Phys., 40 (1997), 75–90) to all the Lie types and prove their duality. As an application we give a solution to Casselman’s problem on Iwahori fixed vectors of principal series representation of p-adic groups.

1 Introduction
Yang-Baxter basis of Hecke algebra of type A was defined in the paper of Lascoux-Leclerc-Thibon [LLT]. There is also a modified version in [Las]. First we generalize the latter version to all the Lie types. Then we will solve the Casselman’s problem on the basis of Iwahori fixed vectors using Yang-Baxter basis and Demazure-Lusztig type operator. This paper is an extended abstract and the detailed proofs will appear in [NN].

2 Generic Hecke algebra
2.1 Root system, Weyl group and generic Hecke algebra
Let $\mathcal{R} = (\Lambda, \Lambda^*, R, R^*)$ be a (reduced) semisimple root data cf. [Dem]. More precisely $\Lambda \simeq \mathbb{Z}^r$ is a weight lattice with rank $\Lambda = r$. There is a pairing $\langle , \rangle: \Lambda^* \times \Lambda \to \mathbb{Z}$. $R \subset \Lambda$ is a root system with simple roots $\{\alpha_i\}_{1 \leq i \leq r}$ and positive roots $R^+$. $R^* \subset \Lambda^*$ is the set of coroots, and there is a bijection $R \to R^*$, $\alpha \mapsto \alpha^*$. We also denote the coroot $\alpha^*_i = h_{\alpha_i}$. The Weyl group $W$ of $\mathcal{R}$ is generated by simple reflections $S = \{s_i\}_{1 \leq i \leq r}$. The action of $W$ on $\Lambda$ is given by $s_i(\lambda) = \lambda - \langle \alpha_i^*, \lambda \rangle \alpha_i$ for $\lambda \in \Lambda$. We define generic Hecke algebra $H_{t_1, t_2}(W)$ over $\mathbb{Z}[t_1, t_2]$ with two parameters $t_1, t_2$ as follows. Generators are $h_i = h_{\alpha_i}$ with relations $(h_i - t_1)(h_i - t_2) = 0$ for $1 \leq i \leq r$ and the braid relations $h_i h_j \cdots = h_j h_i \cdots$, where $m_{i,j}$ is the order of $s_i s_j$ for $1 \leq i < j \leq r$.

We need to extend the coefficients to the quotient field of the group algebra $\mathbb{Z}[\Lambda]$. An element of $\mathbb{Z}[\Lambda]$ is denoted as $\sum_{\lambda \in \Lambda} c_{\lambda} e^\lambda$. The Weyl group acts on $\mathbb{Z}[\Lambda]$. 

by \( w(e^{\lambda}) = e^{w\lambda} \). We extend the coefficient ring \( \mathbb{Z}[t_1, t_2] \) of \( H_{t_1, t_2}(W) \) to
\[
Q_{t_1, t_2}(\Lambda) := \mathbb{Z}[t_1, t_2] \otimes Q(\mathbb{Z}[\Lambda])
\]
where \( Q(\mathbb{Z}[\Lambda]) \) is the quotient field of \( \mathbb{Z}[\Lambda] \).

\[
H^{Q(\Lambda)}_{t_1, t_2}(W) := Q_{t_1, t_2}(\Lambda) \otimes_{\mathbb{Z}[t_1, t_2]} H_{t_1, t_2}(W).
\]

For \( w \in W \), an expression of \( w = s_{i_1} s_{i_2} \cdots s_{i_\ell} \) with minimal number of generators \( s_{i_\ell} \in S \) is called a reduced expression in which case we write \( \ell(w) = \ell \) and call it the length of \( w \). Then \( h_w = h_{i_1} h_{i_2} \cdots h_{i_\ell} \) is well defined and \( \{ h_w \}_{w \in W} \) forms a \( Q_{t_1, t_2}(\Lambda) \)-basis of \( H^{Q(\Lambda)}_{t_1, t_2}(W) \).

### 2.2 Yang-Baxter basis and its properties

Yang-Baxter basis was introduced in the paper [LLT] to investigate the relation with Schubert calculus. There is also a variant in [Las] for type \( A \) case. We generalize that results to all Lie types.

For \( \lambda \in \Lambda \), we define \( E(\lambda) = e^{-\lambda} - 1 \). Then \( E(\lambda + \nu) = E(\lambda) + E(\nu) + E(\lambda)E(\nu) \). In particular, if \( \lambda \neq 0 \), \( \frac{1}{E(\lambda)} + \frac{1}{E(-\lambda)} = -1 \).

**Proposition 1.** For \( \lambda \in \Lambda \), if \( \lambda \neq 0 \), let \( h_i(\lambda) := h_i + \frac{h_{i+1} t_2}{E(\lambda)} \). Then these satisfy the Yang-Baxter relations, i.e. if we write \( [p, q] := p\lambda + q\nu \) for fixed \( \lambda, \nu \in \Lambda \), the following equations hold. We assume all appearance of \([p, q]\) is nonzero.

\[
\begin{align*}
&h_i([0, 1]) h_j([0, 1]) = h_j([0, 1]) h_i([0, 1]) & \text{if } m_{i,j} = 2 \\
&h_i([0, 1]) h_j([1, 1]) h_i([0, 1]) = h_j([0, 1]) h_i([1, 1]) h_j([1, 0]) & \text{if } m_{i,j} = 3 \\
&h_i([0, 1]) h_j([1, 1]) h_i([0, 1]) = h_j([0, 1]) h_i([1, 2]) h_j([1, 1]) h_i([1, 0]) & \text{if } m_{i,j} = 4 \\
&h_i([0, 1]) h_j([1, 2]) h_i([0, 1]) = h_j([0, 1]) h_i([3, 2]) h_j([1, 2]) & \text{if } m_{i,j} = 6
\end{align*}
\]

**Proof.** We can prove these equations by direct calculations. \( \square \)

**Remark 1.** In [Che] I. Cherednik treated Yang-Baxter relation in more general setting. There is also a related work [Kat] by S. Kato and the proof of Theorem 2.4 in [Kat] suggests a uniform way to prove Yang-Baxter relations without direct calculations.

We use the Bruhat order \( x \leq y \) on elements \( x, y \in W \) (cf. [Hum]). Following [Las] we define the Yang-Baxter basis \( Y_w \) for \( w \in W \) recursively as follows.

\[
Y_e := 1, \quad Y_w := Y_{w'}(h_i + \frac{1}{w^E(\alpha_i)}) \quad \text{if } w = w's_i > w'.
\]

Using the Yang-Baxter relation above it is easy to see that \( Y_w \) does not depend on a reduced expression of \( w \). As the leading term of \( Y_w \) with respect to the Bruhat order is \( h_w \), they also form a \( Q_{t_1, t_2}(\Lambda) \)-basis \( \{ Y_w \}_{w \in W} \) of \( H^{Q(\Lambda)}_{t_1, t_2}(W) \).
We are interested in the transition coefficients \( p(w,v) \) and \( \tilde{p}(w,v) \in Q_{t_1,t_2}(\Lambda) \) between the two basis \( \{Y_w\}_{w \in W} \) and \( \{h_w\}_{w \in W} \), i.e.

\[
Y_v = \sum_{w \leq v} p(w,v)h_w, \quad \text{and} \quad h_v = \sum_{w \leq v} \tilde{p}(w,v)Y_w.
\]

Take a reduced expression of \( v \) e.g. \( v = s_{i_1} \cdots s_{i_\ell} \) where \( \ell = \ell(v) \) is the length of \( v \) (cf. [Hum]). Then \( Y_v \) is expressed as follows.

\[
Y_v = \prod_{j=1}^\ell \left( h_{\beta_j} + \frac{t_1 + t_2}{E(\beta_j)} \right)
\]

where \( \beta_j := s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) \) for \( j = 1, \ldots, \ell \). The set \( R(v) := \{\beta_1, \ldots, \beta_\ell\} \subset R^+ \) is independent of the reduced expression of \( v \). The Yang-Baxter basis defined in [LLT] is normalized as follows.

\[
Y_v^{LLT} := \left( \prod_{j=1}^\ell \frac{E(\beta_j)}{t_1 + t_2} \right) Y_v = \prod_{j=1}^\ell \left( \frac{E(\beta_j)}{t_1 + t_2} h_{\beta_j} + 1 \right).
\]

**Remark 2.** The relation to \( K \)-theory Schubert calculus is as follows. If we set \( t_1 = 0, t_2 = -1 \) and replacing \( \alpha_i \) by \( -\alpha_i \). Then the coefficient of \( h_w \) in \( Y_v^{LLT} \) is the localization \( \psi^w(v) \) at \( v \) of the equivariant \( K \)-theory Schubert class \( \psi^w \) (cf. [LSS]).

Let \( w_0 \) be the longest element in \( W \). Define \( Q_{t_1,t_2}(\Lambda) \)-algebra homomorphism \( \Omega : H^Q_{t_1,t_2}(\Lambda) \to H^Q_{t_1,t_2}(\Lambda) \) by \( \Omega(h_w) = h_{w_0w_0w_0} \). Let \( \ast \) be the ring homomorphism on \( \mathbb{Z}[\Lambda] \) induced by \( \ast(e^\lambda) = e^{-\lambda} \) and extend to \( Q_{t_1,t_2}(\Lambda) \).

**Proposition 2.** (Lascoux [Las] Lemma 1.8.1 for type A case) For \( v \in W \),

\[
\Omega(Y_{w_0v_0}) = \ast[w_0(Y_v)]
\]

where \( W \) acts only on the coefficients.

**Proof.** When \( \ell(v) > 0 \) there exists \( s \in S \) such that \( v = v's > v' \). Using the induction assumption on \( v' \), we get the formula for \( v \). \( \square \)

Taking the coefficient of \( h_w \) in the above equation, we get

**Corollary 1.**

\[
p(w_0w_0w_0, w_0v_0v_0) = \ast[w_0p(w,v)].
\]

### 2.3 Inner product and orthogonality

Define inner product \( (\ , )^H \) on \( H^Q_{t_1,t_2}(W) \) by \( (f,g)^H := \text{the coefficient of } h_{w_0} \) in \( fg' \), where \( g' = \sum c_w h_{w^{-1}} \) if \( g = \sum c_w h_w \). It is easy to see that \( (fh_s, g)^H = (f, gh_s)^H \) for \( f, g \in H^Q_{t_1,t_2}(W) \) and \( s \in S \). There is an involution \( \hat{\ast} : H^Q_{t_1,t_2} \to \)}
$H_{t_1,t_2}^{Q(\Lambda)}$ defined by $\hat{h}_i = h_i - (t_1 + t_2), \hat{t}_1 = -t_2, \hat{t}_2 = -t_1$. It is easy to see that $\hat{h}_s h_s = -t_1 t_2$ for $s \in S$.

The following proposition is due to A.Lascoux for the type A case [Las] P.33.

**Proposition 3.** For all $v, w \in W$,

$$(\hat{h}_v, \hat{w}_0) = \delta_{v,w}.$$ 

**Proof.** We can use induction on the length $\ell(v)$ of $v$ to prove the equation. 

We have another orthogonality between $Y_v$ and $w_0(Y_{w_0}w)$.

**Proposition 4.** (Type A case was due to [LLT] Theorem 5.1, [Las] Theorem 1.8.4.)

For all $v, w \in W$,

$$(Y_v, w_0(Y_{w_0}w)) = \delta_{v,w}.$$ 

**Proof.** We use induction on $\ell(v)$ and use the fact that if $s \in S$ and $u \in W$, then $Y_u h_s = aY_{us} + bY_s$ for some $a, b \in \mathbb{Q}_{t_1,t_2}(\Lambda)$.

### 2.4 Duality between the transition coefficients

Recall that we have two transition coefficients $p(w, v), \tilde{p}(w, v) \in \mathbb{Q}_{t_1,t_2}(\Lambda)$ defined by the following expansions.

$$Y_v = \sum_{w \leq v} p(w, v) h_w$$

$$\hat{h}_v = \sum_{w \leq v} \tilde{p}(w, v) Y_w$$

Below gives a relation between them.

**Theorem 1.** (Lascoux [Las] Corollary 1.8.5 for type A case) For $w, v \in W$,

$$\tilde{p}(w, v) = (-1)^{\ell(v)-\ell(w)} p(vw_0, ww_0).$$

**Proof.** We will calculate $(\hat{h}_v, w_0(Y_{w_0}w))$ in two ways. As $\hat{h}_v = \sum_{w \leq v} \tilde{p}(w, v) Y_w$,

$$(\hat{h}_v, w_0(Y_{w_0}w)) = \tilde{p}(w, v)$$

by the orthogonality on $Y_v$ (Proposition 4). On the other hand, as $h_i + \frac{t_1 + t_2}{E(\beta)} = \hat{h}_i - \frac{t_1 + t_2}{E(\beta)}$ for $\beta \in R$, we can expand $Y_v$ in terms of $\hat{h}_w$ as follows.

$$Y_v = \sum_{w \leq v} (-1)^{\ell(v)-\ell(w)} \ast [p(w, v)] \hat{h}_w.$$
So we have
\[
 w_0(Y_{w_0w}) = \sum_{w_0v \leq w_0w} (-1)^{\ell(v) - \ell(w)} w_0[*p(w_0v, w_0w)]h_{w_0v}.
\]

Then using the orthogonality on \( h_v \) (Proposition 3) and Corollary 1,
\[
(h_v, w_0(Y_{w_0w}))^H = (-1)^{\ell(v) - \ell(w)} w_0[*p(w_0v, w_0w)] = (-1)^{\ell(v) - \ell(w)} p(vw_0, vw_0).
\]
The theorem is proved.

2.5 Recurrence relations

Here we give some recurrence relations on \( p(w, v) \) and \( \tilde{p}(w, v) \).

**Proposition 5. (left \( p \))** For \( w \in W \) and \( s \in S \), if \( sv > v \) then
\[
p(w, sv) = \begin{cases} 
\frac{1}{E(\alpha_s)} [p(w, v) - t_1 t_2 s[p(sw, v)]] & \text{if } sw > w \\
(t_1 + t_2)(\frac{1}{E(\alpha_s)} + 1) s[p(w, v)] + s[p(sw, v)] & \text{if } sw < w.
\end{cases}
\]

**Proof.** By the definition we have \( Y_{sv} = Y_s Y_v \) from which we can deduce the recurrence formula.

We note that by this recurrence we can identify \( p(w, v) \) as a coefficient of transition between two bases of the space of Iwahori fixed vectors cf. Theorem 3 below.

**Proposition 6. (right \( p \))** For \( w \in W \) and \( s \in S \), if \( vs > v \) then
\[
p(w, ws) = \begin{cases} 
\frac{1}{E(\alpha_s)} p(w, v) - t_1 t_2 p(ws, v) & \text{if } ws > w \\
(t_1 + t_2)(\frac{1}{E(\alpha_s)} + 1) p(w, v) + p(ws, v) & \text{if } ws < w.
\end{cases}
\]

**Proof.** We can use the equation \( Y_{vs} = Y_s Y_v \) and taking the coefficient of \( h_w \), we get the formula.

**Proposition 7. (left \( \tilde{p} \))** For \( w \in W \) and \( s \in S \), if \( sv > v \) then
\[
\tilde{p}(w, sv) = \begin{cases} 
-\frac{1}{E(\alpha_s)} \tilde{p}(w, v) + (t_2 + \frac{1}{E(\alpha_s)}) (t_2 + \frac{1}{E(-\alpha_s)}) s[\tilde{p}(sw, v)] & \text{if } sw > w \\
-\frac{1}{E(\alpha_s)} \tilde{p}(w, v) + s[\tilde{p}(sw, v)] & \text{if } sw < w.
\end{cases}
\]

**Proof.** We can prove the recurrence relation using Corollary 2 below.

**Proposition 8. (right \( \tilde{p} \))** For \( w \in W \) and \( s \in S \), if \( vs > v \) then
\[
\tilde{p}(w, vs) = \begin{cases} 
-\frac{1}{E(\alpha_s)} \tilde{p}(w, v) + (t_2 + \frac{1}{E(\alpha_s)}) (t_2 + \frac{1}{E(-\alpha_s)}) \tilde{p}(ws, v) & \text{if } ws > w \\
-\frac{1}{E(\alpha_s)} \tilde{p}(w, v) + \tilde{p}(ws, v) & \text{if } ws < w.
\end{cases}
\]

**Proof.** We can prove the recurrence relation using Corollary 2 below.
3 Kostant-Kumar’s twisted group algebra

Let $Q_{t_1,t_2}^{KK}(W) := Q_{t_1,t_2}(A)\#\mathbb{Z}[W]$ be the (generic) twisted group algebra of Kostant-Kumar. Its element is of the form $\sum_{w \in W} f_w \delta_w$ for $f_w \in Q_{t_1,t_2}(A)$ and the product is defined by

$$\left( \sum_{w \in W} f_w \delta_w \right) \left( \sum_{u \in W} g_u \delta_u \right) = \sum_{w,u \in W} f_w(w) \delta_{wu}.$$ 

Define $y_i \in Q_{t_1,t_2}^{KK}(W)$ ($i = 1, \ldots, r$) by

$$y_i := A_i \delta_i + B_i \quad \text{where} \quad A_i := \frac{t_1 + t_2 e^{-\alpha_i}}{1 - e^{\alpha_i}}, \quad B_i := \frac{t_1 + t_2}{1 - e^{-\alpha_i}}.$$ 

**Proposition 9.** We have the following equations.

1. $(y_i - t_1)(y_i - t_2) = 0$ for $i = 1, \ldots, r$.
2. $y_i y_j \cdots y_{i_{m_{i,j}}} y_{j \cdots} = y_{j \cdots} y_{i \cdots}$, where $m_{i,j}$ is the order of $s_i s_j$.

**Proof.** These equations can be shown by direct calculations. $\square$

By this proposition we can define $y_w := y_{i_1} \cdots y_{i_\ell}$ for a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$. These $\{y_w\}_{w \in W}$ become a $Q_{t_1,t_2}(A)$-basis of $Q_{t_1,t_2}^{KK}(W)$.

**Remark 3.** This operator $y_i$ can be seen as a generic Demazure-Lusztig operator. When $t_1 = -1, t_2 = q$, it becomes $y_{s_i}^q$ in Kumar’s book [Kum] (12.2.E(9)). We can also set $A_i$ which satisfies

$$A_i A_{-i} = \frac{(t_1 + t_2 e^{\alpha_i})(t_1 + t_2 e^{-\alpha_i})}{(1 - e^{\alpha_i})(1 - e^{-\alpha_i})}.$$ 

For example, if we set $A_i = \frac{t_1 + t_2 e^{\alpha_i}}{1 - e^{\alpha_i}}$ and $t_1 = q, t_2 = -1$ and replace $\alpha_i$ by $-\alpha_i$, it becomes Lusztig’s $T_{s_i}$ [Lu1]. If we set $A_i = -\frac{t_1 + t_2 e^{\alpha_i}}{1 - e^{\alpha_i}}$ and $t_1 = -1, t_2 = v$ and replace $\alpha_i$ by $-\alpha_i$, it becomes $T_{s_i}$ in [BBL].

We can define a $Q_{t_1,t_2}(A)$-module isomorphism $\Phi : Q_{t_1,t_2}^{KK}(W) \to H_{t_1,t_2}(\mathbb{A}(W))$ by $\Phi(y_w) = h_w$. Let $\Delta_{s_i} := A_i \delta_i$. Define $A(w) := \prod_{\beta \in R(w)} \frac{t_1 + t_2 e^{-\beta}}{1 - e^{\beta}}$ and $\Delta_w := A(w) \delta_w$. Then it becomes that $\Delta_{s_{i_1}} \cdots \Delta_{s_{i_\ell}} = A(w) \delta_w = \Delta_w$. In particular, $\Delta_{s_{i_\ell}}$’s satisfy the braid relations. We can show below by induction on length $\ell(w)$.

**Theorem 2.** For $w \in W$, we have

$$\Phi(\Delta_w) = Y_w.$$
Proof. If \( w = s_i, \Delta_{s_i} = A_i\delta_i = y_i - B_i \). Therefore \( \Phi(\Delta_{s_i}) = h_i - B_i = h_i + \frac{t_i + s_i}{2} = Y_{s_i} \). If \( s_i w > w \), by induction hypothesis we can assume \( \Phi(\Delta_{u}) = Y_{u} = \sum_{u \leq w} p(u, w)h_u \). As \( \Phi \) is a \( Q_{t_1, t_2}(\Lambda) \)-isomorphism, it follows that \( \Delta_{u} = \sum_{u \leq w} p(u, w)y_u \). Then \( \Delta_{s_i w} = \Delta_{s_i} \Delta_{w} = A_i\delta_i \sum_{u \leq w} p(u, w)y_u = \sum_{u \leq w} s_i[p(u, w)]A_i\delta_i y_u = \sum_{u \leq w} s_i[p(u, w)][y_i - B_i]y_u = \sum_{u \leq s_i w} p(u, s_i w)y_u \). We used the recurrence relation (Proposition 5) for the last equality. Therefore \( \Phi(\Delta_{s_i w}) = \sum_{u \leq s_i w} p(u, s_i w)h_u = Y_{s_i w} \). The theorem is proved.

Corollary 2. (Explicit formula for \( \tilde{p}(w, v) \))

Let \( v = s_{i_1} \cdots s_{i_k} \) be a reduced expression. Then we have

\[
\tilde{p}(w, v) = \frac{1}{A(w)} \sum_{\epsilon = (\epsilon_1, \cdots, \epsilon_k) \in \{0, 1\}^k, s_i^{\epsilon_i} \cdots s_i^{r_i} = w} \prod_{j=1}^\ell C_j(\epsilon)
\]

where for \( \epsilon = (\epsilon_1, \cdots, \epsilon_k) \in \{0, 1\}^k \),
\[
C_j(\epsilon) := s_i^{\epsilon_i} s_i^{r_i} \cdots s_{i_{j-1}}^{r_{i_{j-1}}} (\delta_{i_{j-1}} A_i + \delta_{i_{j-1}} B_i).
\]

Proof. Taking the inverse image of the map \( \Phi \), the equality \( h_v = \sum_{w \leq v} \tilde{p}(w, v)Y_w \) becomes
\[
y_v = \sum_{w \leq v} \tilde{p}(w, v)\Delta_w = \sum_{w \leq v} \tilde{p}(w, v)A(w)\delta_w.
\]

As \( v = s_{i_1} \cdots s_{i_k} \) is a reduced expression, \( y_v = y_{s_{i_1}} \cdots y_{s_{i_k}} = (A_i\delta_i + B_i\delta_i) \cdots (A_i\delta_i + B_i\delta_i) \). By expanding this we get the formula.

\[\square\]

Remark 4. Using Theorem 1, we also have a closed form for \( p(w, v) \). We have another conjectural formula for \( p(w, v) \) using \( \lambda \)-chain cf. [Nar].

Example 1. Type \( A_2 \). We use notation \( A_{-1} = \ast(A_1), B_{-1} = \ast(B_1), B_{12} = \frac{t_1 + t_2}{1 - (t_1 + t_2)} \).

When \( v = s_1s_2s_1, w = s_1, \) then \( \epsilon = (1, 0, 0), (0, 0, 1) \) and

\[
\tilde{p}(s_1, s_1s_2s_1) = (A_1B_12B_{-1} + B_1B_2A_1)/A_1 = B_12B_{-1} + B_1B_2 = B_2B_{12}.
\]

When \( v = s_1s_2s_1, w = s_2, \) then \( \epsilon = (0, 1, 0) \) and

\[
\tilde{p}(s_2, s_1s_2s_1) = (B_1A_2B_{12})/A_2 = B_1B_{12}.
\]

When \( v = s_1s_2s_1, w = e, \) then \( \epsilon = (0, 0, 0), (1, 0, 1) \) and

\[
\tilde{p}(\epsilon, s_1s_2s_1) = B_1B_2B_1 + A_1B_{12}A_{-1}.
\]
4 Casselman’s problem

In his paper [Cas] B. Casselman gave a problem concerning transition coefficients between two bases in the space of Iwahori fixed vectors of a principal series representation of a $p$-adic group. We relate the problem with the Yang-Baxter basis and give an answer to the problem.

4.1 Principal series representations of $p$-adic group and Iwahori fixed vector

We follow the notations of M. Reeder [Re1, Re2]. Let $G$ be a connected reductive $p$-adic group over a non-archimedian local field $F$. For simplicity we restrict to the case of split semisimple $G$. Associated to $F$, there is the ring of integer $O$, the prime ideal $p$ with a generator $w$, and the residue field with $q = |O/p|$ elements. Let $P$ be a minimal parabolic subgroup (Borel) of $G$, and $A$ be the maximal split torus of $P$ so that $A \simeq (F^*)^r$ where $r$ is the rank of $G$. For an unramified quasi-character $\tau$ of $A$, i.e. a group homomorphism $\tau : A \to \mathbb{C}^*$ which is trivial on $A_0 = A \cap K$, where $K = G(O)$ is a maximal compact subgroup of $G$. Let $T = \mathbb{C}^* \otimes X^*(A)$ be the complex torus dual to $A$, where $X^*(A)$ is the group of rational characters on $A$, i.e. $X^*(A) = \{\lambda : A \to F^*, \text{ algebraic group homomorphism}\}$. We have a pairing $\langle, \rangle : A/A_0 \times T \to \mathbb{C}^*$ given by $\langle a, z \otimes \lambda \rangle = z^{\val(\lambda(a))}$. This gives an identification $T \simeq X_{nr}(A)$ of $T$ with the set of unramified quasi-characters on $A$ (cf. [Bum] Exercise 18,19).

Let $\Delta \subset X^*(A)$ be the set of roots of $A$ in $G$, $\Delta^+$ be the set of positive roots corresponding to $P$ and $\Sigma \subset \Delta^+$ be the set of simple roots. For a root $\alpha \in \Delta$, we define $e_\alpha \in X^*(T)$ by

$$e_\alpha(\tau) = \langle h_\alpha(w), \tau \rangle$$

for $\tau \in T$ where $h_\alpha : F^* \to A$ is the one parameter subgroup (coroot) corresponding to $\alpha$.

**Remark 5.** As the definition shows, $e_\alpha$ is defined using the coroot $\alpha^* = h_\alpha$. So it should be parametrized by $\alpha^*$, but for convenience we follow the notation of [Re1]. Later we will identify $e_\alpha (\alpha \in \Delta = R^*)$ with $e^\alpha (\alpha \in R = \Delta^*)$ by the map $*: \Delta \to R$ of root data.

$W$ acts on right of $X_{nr}(A)$ so that $\tau^w(a) = \tau(aww^{-1})$ for $a \in A$, $\tau \in T$ and $w \in W$. The action of $W$ on $X^*(T)$ is given by $(we_\alpha)(\tau) = e_{w\alpha}(\tau) = e_\alpha(\tau^w)$ for $\alpha \in \Delta$, $\tau \in T$ and $w \in W$.

The principal series representation $I(\tau)$ of $G$ associated to an unramified quasicharacter $\tau$ of $A$ is defined as follows. As a vector space over $\mathbb{C}$ it consists of locally constant functions on $G$ with values in $\mathbb{C}$ which satisfy the left relative invariance properties with respect to $P$ where $\tau$ is extended to $P$ with trivial value on the unipotent radical $N$ of $P = AN$.

$$I(\tau) := \text{Ind}_P^G(\tau) = \{f : G \to \mathbb{C} \text{ loc. const. function } |f(pg) = \tau^{\val(p)} f(g) \text{ for } \forall p \in P, \forall g \in G\}.$$
Here $\delta$ is the modulus of $P$. The action of $G$ on $I(\tau)$ is defined by right translation, i.e. for $g \in G$ and $f \in I(\tau)$, $(\pi(g)f)(x) = f(xg)$.

Let $B$ be the Iwahori subgroup which is the inverse image $\pi^{-1}(P(\mathbb{F}_q))$ of the Borel subgroup $P(\mathbb{F}_q)$ of $G(\mathbb{F}_q)$ by the projection $\pi : G(\mathbb{O}) \to G(\mathbb{F}_q)$. Then we define $I(\tau)^B$ to be the space of Iwahori fixed vectors in $I(\tau)$, i.e.

$$I(\tau)^B := \{ f \in I(\tau) \mid f(bg) = f(g) \text{ for } \forall b \in B, \forall g \in G \}.$$  

This space has a natural basis $\{ \varphi^\tau_w \}_{w \in W}$. $\varphi^\tau_w \in I(\tau)^B$ is supported on $PwB$ and satisfies

$$\varphi^\tau_w(pwb) = \tau \delta^{1/2}(p) \text{ for } \forall p \in P, \forall b \in B.$$ 

### 4.2 Intertwiner and Casselman’s basis

From now on we always assume that $\tau$ is regular i.e. the stabilizer $W_\tau = \{ w \in W \mid \tau^w = \tau \}$ is trivial. The intertwining operator $A^\tau_w : I(\tau) \to I(\tau^w)$ is defined by

$$A^\tau_w(f)(g) := \int_{N_w} f(wng)dn$$

where $N_w := N \cap w^{-1}Nw$, with $N_-$ being the unipotent radical of opposite parabolic $P_-$ which corresponds to the negative roots $\Delta^-$. The integral is convergent when $|e_\alpha(\tau)| < 1$ for all $\alpha \in \Delta^+$ such that $w\alpha \in \Delta^-$ (cf. [Bum] Proposition 63), and may be meromorphically continued to all $\tau$. It has the property that for $x, y \in W$ with $\ell(xy) = \ell(x) + \ell(y)$, then

$$A^\tau_y A^\tau_x = A^\tau_{xy}.$$ 

The Casselman’s basis $\{ f^\tau_w \}_{w \in W}$ of $I(\tau)^B$ is defined as follows. $f^\tau_w \in I(\tau)^B$ and

$$A^\tau_y f^\tau_w(1) = \begin{cases} 1 & \text{if } y = w \\ 0 & \text{if } y \neq w. \end{cases}$$

M. Reeder characterizes this using the action of affine Hecke algebra (cf. [Re2] Section 2). The affine Hecke algebra $\mathcal{H} = \mathcal{H}(G, B)$ is the convolution algebra of $B$ bi-invariant locally constant functions on $G$ with values in $\mathbb{C}$. By the theorem of Iwahori-Matsumoto it can be described by generators and relations. The basis $\{ T_w \}_{w \in \mathbb{W}_uf}$ consists of characteristic functions $T_w := ch_{BwB}$ of double coset $BwB$. Let $\mathcal{H}_W$ be the Hecke algebra of the finite Weyl group $W$ generated by the simple reflections $s_\alpha$ for simple roots $\alpha \in \Sigma$. As a vector space $\mathcal{H}$ is the tensor product of two subalgebras $\mathcal{H} = \Theta \otimes \mathcal{H}_W$. The subalgebra $\Theta$ is commutative and isomorphic to the coordinate ring of the complex torus $T$ with a basis $\{ \theta_a \mid a \in A / A_0 \}$, where $\theta_a$ is defined as follows (cf. [Lan2]). Define $A^- := \{ a \in A \mid |\alpha(a)|_F \leq 1 \forall \alpha \in \Sigma \}$. For $a \in A$, choose $a_1, a_2 \in A^-$ such that $a = a_1a_2^{-1}$. Then $\theta_a = q^{(\ell(a_1) - \ell(a_2))/2}T_{a_1}T_{a_2}^{-1}$ where for $x \in G$, $\ell(x)$ is the length function defined by $q^{\ell(x)} = [BxB : B]$ and $T_x \in \mathcal{H}$ is the characteristic function of $BxB$. 

9
By Lemma (4.1) of \cite{Re1}, there exists a unique $f^\tau_w \in I(\tau)_w \cap I(\tau)^B$ for each $w \in W$ such that

1. $f^\tau_w(w) = 1$ and
2. $\sigma(\theta_a)f^\tau_w = \tau^w(a)f^\tau_w$ for all $a \in A$.

Here $I(\tau)_w \coloneqq \{ f \in I(\tau) \mid \text{support of } f \text{ is contained in } \bigcup_{x \geq w} PxP \}$.

### 4.3 Transition coefficients

Let

$$f^\tau_w = \sum_{w \leq v} a_{w,v}(\tau) \varphi^\tau_v$$

and

$$\varphi^\tau_v = \sum_{w \leq v} b_{w,v}(\tau) f^\tau_w.$$

The Casselman’s problem is to find an explicit formula for $a_{w,v}(\tau)$ and $b_{w,v}(\tau)$.

To relate the results in Sections 2 and 3 with the Casselman’s problem, in this subsection we specialize the parameters $t_1 = -q^{-1}$, $t_2 = 1$ and take tensor product with the complex field $\mathbb{C}$. For example, the Yang-Baxter basis $Y_w$ will become a $Q_{t_1,t_2}(\Lambda) \otimes \mathbb{C}$ basis in $H_{t_1,t_2}(W) _{\mathbb{C}} = H_{t_1,t_2}(W) \otimes \mathbb{C}$. The generic Demazure-Lusztig operator defined in Section 3 will become

$$y_i = A_i \delta_i + B_i \text{ where } A_i := \frac{-q^{-1} + e^{-\alpha_i}}{1 - e^{\alpha_i}}, B_i := \frac{-q^{-1} + 1}{1 - e^{\alpha_i}}.$$

Then $(y_i + q^{-1})(y_i - 1) = 0$.

**Theorem 3.** We identify $e^\alpha$ with $e_\alpha$ (cf. Remark 4). Then,

$$a_{w,v}(\tau) = \tilde{p}(w,v)(\tau)|_{t_1 = -q^{-1}, t_2 = 1}$$

and

$$b_{w,v}(\tau) = p(w,v)(\tau)|_{t_1 = -q^{-1}, t_2 = 1}.$$

**Proof.** $b_{w,v}$’s satisfy the same recurrence relation (Proposition 5 with $t_1 = -q^{-1}, t_2 = 1$) as $p(w,v)$’s (cf. \cite{Re2} Proposition (2.2)). The initial condition $b_{w,w} = p(w,w) = 1$ leads to the second equation. The first equation then also holds. Note that the $b_{y,y}$ in \cite{Re2} is our $b_{w,y}$.

**Remark 6.** There is also a direct proof that does not use recurrence relation cf. \cite{NN}.

**Corollary 3.** We have a closed formula for $a_{w,v}(\tau)$ and $b_{w,v}(\tau)$ by Corollary 2 and Theorem 1.
Corollary 4. For \( v \in W \), we have

\[ \sum_{w \leq v} b_{w,v} = \prod_{\beta \in R(e)} \frac{1 - q^{-1} e^\beta}{1 - e^\beta}, \]

and

\[ \sum_{w \leq v} b_{w,v}(-q^{-1})^\ell(w) = \prod_{\beta \in R(v)} \frac{1 - q^{-1}}{1 - e^\beta}. \]

Proof. When \( t_1 = -q^{-1}, t_2 = 1 \), we can specialize \( h_i \) to 1 and we get the first equation from the definition of \( Y_v \), since \( 1 + \frac{(1-q^{-1}) e^\beta}{1-e^\beta} = \frac{1-q^{-1} e^\beta}{1-e^\beta} \). We can also specialize \( h_i \) to \(-q^{-1}\) and \(-q^{-1} + \frac{(1-q^{-1}) e^\beta}{1-e^\beta} = \frac{1-q^{-1}}{1-e^\beta} \) gives the second equation. \( \Box \)

Remark 7. The left hand side of the first equation in Corollary 4 is \( m(e, v^{-1}) \) in \([BN]\). So this gives another proof of Theorem 1.4 in \([BN]\).

4.4 Whittaker function

M.Reeder \([Re2]\) specified a formula for the Whittaker function \( W_\tau(f^\pi_v) \) and using \( b_{w,v} \), he got a formula for \( W_\tau(\varphi^\pi_w) \). For \( a \in A \), let \( \lambda_a \in X^*(T) \) be

\[ \lambda_a(z \otimes \mu) = z^{val(\mu(a))} \text{ for } z \in \mathbb{C}^*, \mu \in X^*(A). \]

Formally the result of M.Reeder \([Re2]\) Corollary (3.2) is written as follows. For \( w \in W \) and \( a \in A^- \),

\[ W(\varphi_w)(a) = \delta^{1/2}(a) \sum_{w \leq y} b_{w,y} y \left[ \lambda_a \prod_{\beta \in R^+ - R(y)} \frac{1 - q^{-1} e^\beta}{1 - e^\beta} \right] \in \mathbb{C}[T]. \]

Then using Corollary 3, we have an explicit formula of \( W(\varphi_w)(a) \).

4.5 Relation with Bump-Nakashuji’s work

Now we explain the relation between this paper and Bump-Nakashuji \([BN]\). First of all, the notational conventions are slightly different. Especially in the published \([BN]\) the natural base and intertwiner are differently parametrized. The natural basis \( \phi_w \) in \([BN]\) is our \( \varphi_{w^{-1}} \). The intertwiner \( M_w \) in \([BN]\) is our \( A_{w^{-1}} \) so that if \( \ell(w_1 w_2) = \ell(w_1) + \ell(w_2), M_{w_1, w_2} = M_{w_1} \circ M_{w_2} \) while \( A_{w_1, w_2} = A_{w_2} A_{w_1} \).

In the paper \([BN]\), another basis \( \{ \psi_w \}_{w \in W} \) for the space \( I(\tau)^B \) was defined and compared with the Casselman’s basis. They defined \( \psi_w := \sum_{v \geq w} \varphi_v \) and expand this as \( \psi_w = \sum_{v \geq w} m(w, v) f_v \) and conversely \( f_w = \sum_{v \geq w} \tilde{m}(w, v) \psi_v \). They observed that the transition coefficients \( m(w, v) \) and \( \tilde{m}(w, v) \) factor under certain condition. Let \( S(w, v) := \{ \alpha \in R^+ | w \leq s_{\alpha} v < v \} \) and \( S'(w, v) := \{ \alpha \in R^+ | w < s_{\alpha} w \leq v \} \). Then the statements of the conjectures are as follows.
Conjecture 1. (BN Conjecture 1.2) Assume that the root system $R$ is simply-laced. Suppose $w \leq v$ and $|S(w, v)| = \ell(v) - \ell(w)$, then

$$m(w, v) = \prod_{\alpha \in S(w, v)} \frac{1 - q^{-1}z^\alpha}{1 - z^\alpha}.$$ 

Conjecture 2. (BN Conjecture 1.3) Assume that the root system $R$ is simply-laced. Suppose $w \leq v$ and $|S'(w, v)| = \ell(v) - \ell(w)$, then

$$\tilde{m}(w, v) = (-1)^{\ell(v) - \ell(w)} \prod_{\alpha \in S'(w, v)} \frac{1 - q^{-1}z^\alpha}{1 - z^\alpha}.$$ 

Proposition 10. Conjecture 1.2 and Conjecture 1.3 in BN are equivalent.

Proof. We can show $m(w, v) = \sum_{w \leq z \leq v} p(z, v)$ and $\tilde{m}(w, v) = \sum_{w \leq z \leq v} (-1)^{\ell(v) - \ell(z)} \tilde{p}(w, z)$. Then it follows by the Theorem 1 that $\tilde{m}(w, v) = (-1)^{\ell(v) - \ell(w)} m(vw_0, wv_0)$. As $S'(w, v) = S(vw_0, wv_0)$ we get the desired conclusion. \hfill \square

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