Expanded-clique graphs and the domination problem

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Abstract

Given a graph $G$ such that each vertex $v_i$ has a value $f(v_i)$, the expanded-clique graph $H$ is the graph where each vertex $v_i$ of $G$ becomes a clique $V_i$ of size $f(v_i)$ and for each edge $v_i,v_j \in E(G)$, there is a vertex of $V_i$ adjacent to an exclusive vertex of $V_j$. In this work, among the results, we present two characterizations of the expanded-clique graphs, one of them leads to a linear-time recognition algorithm. Regarding the domination number, we show that this problem is $\text{NP}$-complete for planar bipartite $3$-expanded-clique graphs and for cubic line graphs of bipartite graphs.

1 Introduction

We consider finite, simple and undirected graphs. The degree, the open and the closed neighborhoods of a vertex $v$ are denoted by $d(v), N(v)$ and $N[v]$, respectively. In this text, whenever we refer to a graph by $G$, we will denote its vertex set by $\{v_1,\ldots,v_n\}$. Given a function $f : V(G) \rightarrow \mathbb{N}$ such that $f(v_i) \geq d(v_i)$ for every $v_i \in V(G)$, the expanded-clique graph $H$ of $(G,f)$ is defined as follows: for every $v_i \in V(G)$, there is a set $V_i \subseteq V(H)$ with $f(v_i)$ vertices forming a clique; and for every $v_iv_j \in E(G)$, there are $v_{i,j} \in V_i$ and $v_{j,i} \in V_j$ such that $v_{i,j}v_{j,i}$ in $E(H)$. If $f(v_i) - d_G(v_i) = k_i > 0$, then $V_i$ has $k_i$ simplicial vertices, which are denoted by $v'_{i,1},\ldots,v'_{i,k_i}$. In this case, $G$ is the root of $H$ under the $f$-expanded-clique operation. The set $V_i$ will be referred to as the expanded clique (associated with $v_i$). Note that for every $v \in V_i$, $d(v) \in \{|V_i| - 1,|V_i|\}$. We say that $H$ is an expanded-clique graph if $H$ is the expanded-clique

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graph for some graph $G$ and function $f$. If $f(v_i) = k$ for some $k \in \mathbb{N}$, then we can say that $H$ is a $k$-expanded-clique graph.

In this work, we are interested in the complexity aspects of the recognition problem of the expanded-clique graphs and of the domination problem on this class. We remark that the well-studied inflated graphs $[1, 5, 6]$ form a subclass of the clique-expanded graphs, since a graph $H$ is inflated if it is the expanded-clique graph of a pair $(G, f)$ satisfying $f(v_i) = d(v_i)$ for every $v_i \in V(G)$.

Recall that $D \subseteq V(G)$ is a dominating set of a graph $G$ if every vertex of $V(G) \setminus D$ has a neighbor in $D$. The domination problem is a classic problem in graphs having many relevant applications $[4]$. It can be stated as:

**Dominating set**

**Input:** A graph $G$ and a positive integer $\ell$.

**Question:** Is there a dominating set $S \subseteq V(G)$ so that $|S| \leq \ell$?

The text is organized as follows. In Section 2, we begin by showing that the subdivided-line graphs $[3]$ and the Sierpiński graphs $[8]$ are proper subclasses of expanded-clique graphs, and that this class is a proper subclass of the line graphs of bipartite graphs $[2]$. Next, we present two characterizations of the expanded-clique graphs, one of them leading to a linear-time recognition algorithm. Section 3 deals with the domination problem. We show that this problem is NP-complete for planar bipartite 3-expanded-clique graphs and for cubic line graphs of bipartite graphs. We also show that given an expanded-clique $H$, the domination number of $H$ plus the 2-independence number of the root $G$ of $H$ is equal to $|V(G)|$. As a consequence of this result, we derive lower and upper bounds for the dominating number of expanded-clique graphs, which lead to the fact that a dominating set of $H$ can be easily found within the ratio $1 + \frac{1}{\Delta(G)}$ of the minimum.

We conclude this section by presenting useful notation. Consider $G$ a graph. The minimum and maximum degrees of $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. We say that $G$ is a cubic graph if $G$ is 3-regular. We write $K_n$ for the complete graph with $n$ vertices. The subgraph of $G$ induced by $V'$ is denoted by $G[V']$. A vertex $v \in V$ is a simplicial vertex if $N(v)$ induces a clique. For the case of $V' \subseteq V$, denote the closed neighborhood of $V'$ in $G$ as $N[V'] = \{v \in N[v'] : \text{for all } v' \in V'\}$, and open neighborhood of $V'$ in $G$ as $N(V') = N[V'] \setminus V'$. We write $K_{p,q}$ for the complete bipartite graph where independent sets have respectively the sizes $p$ and $q$. A claw is a $K_{1,3}$ graph. A diamond is an induced cycle $C_4$ plus one chord. An odd-hole is any induced cycle $C_q$ where $q$ is an odd number greater than 4. The butterfly graph, also called hourglass graph, is the graph depicted in Figure 1.

## 2 Characterization and recognition

The subdivision of a graph $F$, $S(F)$, is the replacement of every edge $uv \in E(F)$ for a new vertex $x_{uv}$ and edges $x_{uv}u$ and $x_{uv}v$. The line graph of $F$, written $L(F)$, is the graph whose
vertex set is \( E(F) \) and in which two distinct vertices \( uv \) and \( xy \) are adjacent if and only if they are adjacent in \( F \), i.e., \( \{u, v\} \cap \{x, y\} \neq \emptyset \). We say that \( L(S(F)) \) is a subdivided-line graph \([3]\). See an example in Figure 2.

![Figure 1: Claw, Diamond, Odd-hole and Butterfly graph](image)

Figure 1: Claw, Diamond, Odd-hole and Butterfly graph

The Sierpiński graphs were introduced by Klavžar and Milutinović as a generalization of the graph of the Tower of Hanoi problem \([7]\). Given integers \( p \geq 1 \) and \( q \geq 1 \), the Sierpiński graph \( S(p, q) \) has a vertex for each \( p \)-tuple that can be formed from \( \{1, \ldots, q\} \) and, for two distinct vertices \( u = (u_1, u_2, \ldots, u_p) \) and \( w = (w_1, w_2, \ldots, w_p) \), \( uv \in E(S(p, q)) \) if and only if there exists an \( h \in \{1, \ldots, p\} \) such that

1. \( u_t = w_t \) for \( t \in \{1, \ldots, h - 1\} \);
2. \( u_h \neq w_h \);
3. \( u_t = w_h \) and \( w_t = u_h \) for \( t \in \{h + 1, p\} \).

In Figure 3, we show examples of \( S(1, 3) \), \( S(2, 3) \), and \( S(3, 3) \).

Observe that a subdivided-line graph has no simplicial vertices, whereas the number of simplicial vertices of a Sierpiński graph is \( q \), which means that they are disjoint graph classes. In the next result, we show that both are proper subclasses of the expanded-clique graphs.

![Figure 2: An example of a line graph of subdivision.](image)
Proposition 2.1. If $H$ is a subdivided-line graph or a Sierpiński graph, then $H$ is expanded-clique.

Proof. For both cases, we need to find a root $G$ and its corresponding $f$.

For a subdivided-line graph $H = L(S(G))$, take $G$ as the root and $f(v_i) = d(v_i)$ for every $v_i \in V(G)$. Note that the vertex set of $H$ is formed by a set $V_i$ with $d(v_i)$ vertices forming a clique for every $v_i \in V(G)$. Furthermore, if $v_i v_j \in E(G)$, then the vertex $v_{ij}$ of $S(G)$, originated by the edge $v_i v_j$, becomes an edge in $H$ joining one vertex of $V_i$ to an exclusive vertex of $V_j$. Since there are no more edges, we conclude that $H$ is the expanded-clique graph of $(G, f)$.

Now, let $H$ be a Sierpiński graph $S(p, q)$. Note that if $q = 1$, then $H$ is the trivial graph. Then, we can assume that $q \geq 2$. For $p = 1$, note that $H$ is the complete graph with $q$ vertices. Then, we can choose $V(G) = \{v_1\}$ and $f(v_1) = q$. For $p > 1$, let $G = S(p-1, q)$ and $f(v_i) = q$ for every $v_i \in V(G)$. The expanded-clique graph of $(G, f)$ has $q|V(G)|$ vertices and this is the number of vertices of $S(p, q)$ because for any $(p-1)$-tuple that can be formed with $q$ elements, we can form $q$ tuples by adding one coordinate. The definition of Sierpiński graphs and the definition of expanded-clique graphs imply that these $q$ vertices form a clique. Denote by $V_i$ such clique associated with $v_i \in V(S(p-1, q))$. We can write $v_i = (u_1, \ldots, u_{p-1})$. Note that $d(v_i) \in \{q-1, q\}$. We know that every vertex of $S(p, q)$ and every vertex of a $q$-expanded-clique graph has also degree $q-1$ or $q$. If $d(v_i) = q$, then since $p \geq 2$ and $q \geq 2$, the definition implies that there are two coordinates of $v_i$ that are different. Then, for every $v_j \in N(v_i)$, there is a vertex in $V_i$ and a vertex in $V_j$ that are adjacent in $S(p, q)$. If $d(v_i) = q-1$, then all coordinates of $v_i$ are equal. Then for every $v_j \in N(v_i)$, there is a vertex in $V_i$ and a vertex in $V_j$ that are adjacent in $S(p, q)$. Since there are no more edges in $S(p, q)$ and the edges cited above are precisely the edges of the expanded-clique graph of $(G, f)$, we conclude that $H = S(p, q)$ is an

Figure 3: Examples of Sierpiński graphs.
expanded-clique graph.

Before showing that the line graphs of bipartite graphs form a superclass of the expanded-clique graphs, we recall an useful result.

**Proposition 2.2.** [2] A graph is a line graph of a bipartite graph if and only if it is (claw, diamond, odd-hole)-free.

**Theorem 2.3.** If $H$ is an expanded-clique graph, then $H$ is a line graph of a bipartite graph.

**Proof.** Let $H$ be an expanded-clique graph. By Proposition 2.2, it suffices to show that $H$ is (claw, diamond, odd-hole)-free. By the definition of expanded-clique graphs, for every $v \in V(H)$, either $N(v)$ is a clique or there is $u \in N(v)$ such that $N(v) \setminus \{u\}$ is a clique and $u$ has no neighbors in $N(v)$. Since a claw and a diamond have vertices not satisfying this property, we conclude that $H$ is (claw, diamond)-free. Now, let $C_q = u_1 \ldots u_q u_1$ be an induced cycle of $H$ where $q \geq 4$, and let $G$ be a root of $H$. By the definition of expanded-clique graphs, every vertex $u_i \in V(H)$ belongs to exactly one expanded clique $V_j$, for some $v_j \in V(G)$, and has at most one neighbor outside $V_j$. Since $C_q$ has no chords, for every vertex $u_i$ of $C_q$, one of its neighbors in $C_q$ belongs to the same expanded clique as $u_i$, and the other belongs to another expanded clique, which implies that $q$ is even and that $H$ is odd-hole-free.

For $k \geq 3$, a sequence of vertices $u_1, \ldots, u_k$ of a graph $F$ is a chain if $u_i u_{i+1} \in E(F)$ for $i \in \{1, \ldots, k-1\}$, $d(u_i) = 2$ for $i \in \{2, \ldots, k-1\}$ and it is maximal with these properties. We say that a chain is bad if $k$ is odd, $d(u_1) \geq 3$ and $d(u_k) \geq 3$. If a chain is not bad, then it is good. We say that a vertex $v \in V(F)$ is 1-simplicial if there is $u \in N(v)$ such that $N(v) \setminus \{u\}$ is a clique and $u$ has no neighbors in $N(v)$. In this case we say that $u$ is the outsider of $v$.

**Theorem 2.4.** A graph $H$ that is not a cycle is expanded-clique if and only if every vertex of $H$ is simplicial or 1-simplicial and every chain of $H$ is good.

**Proof.** We can assume that $H$ is a non-trivial connected graph. We begin by considering the case where $H$ is a path $P_k$ for $k \geq 2$. Since, in this case, every vertex of $H$ is simplicial or 1-simplicial and $H$ does not have bad chains, we have to show a pair $(G, f)$ such that $G$ is the root of $H$ under the $f$-expanded-clique operation. For $k$ even, we choose $G = P_{\frac{k}{2}} = v_1 \ldots v_{\frac{k}{2}}$ and set $f(v_i) = 2$ if $i \in \{1, \ldots, \frac{k}{2}\}$; and for $k$ odd, we choose $G = P_{\left\lfloor \frac{k}{2} \right\rfloor} = v_1 \ldots v_{\left\lfloor \frac{k}{2} \right\rfloor}$ and set $f(v_i) = 2$ if $i \in \{1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor\}$ and $f(v_{\left\lfloor \frac{k}{2} \right\rfloor}) = 1$. From now on, we can assume that $H$ is neither a cycle nor a path.

For the necessity, consider that $H$ is the expanded-clique of a pair $(G, f)$. First, suppose by contradiction that $u_1, \ldots, u_k$ is a bad chain of $H$. Since $d(u_1) \geq 3$ and $d(u_2) = 2$, we conclude that $u_1$ and $u_2$ belong to different expanded cliques of $H$. By symmetry, $u_{k-1}$ and $u_k$ belong to different expanded cliques of $H$. Furthermore, the expanded clique containing $u_2$ also contains $u_3$ and no more vertices. Then, $\{u_i, u_{i+1}\}$ is an expanded clique for every $i < k$ such that $i$ is even. Therefore, $u_{k-1}$ and $u_k$ belong to the same expanded clique of $H$, which is a contradiction. Therefore, any chain of $H$ is good.
Now, suppose by contradiction that there is \( v \in V(H) \) such that \( N(v) \) is not simplicial neither 1-simplicial. Note that \( d(v) \geq 3 \). Then, there are \( u, w \in N(v) \) such that \( uw \notin E(H) \) and let \( x \in N(v) \setminus \{u, w\} \). If \( u, w \in N_H(x) \), then \( H \) has a diamond, which is not possible by Proposition 2.2 and Theorem 2.3. If \( ux, xw \notin E(H) \), then \( H \) has a claw, which is also not possible. Therefore, every vertex of \( N(v) \setminus \{u, w\} \) is adjacent to exactly one vertex of \( \{u, w\} \).

Hence, the subgraph of \( H \) induced by \( N(v) \) has exactly two connected components, say \( C_1 \) and \( C_2 \), each one being a complete graph. Since \( v \) is not 1-simplicial, we have that \( |V(C_i)| \geq 2 \) for \( i \in \{1, 2\} \). Without loss of generality, we can assume that the expanded clique containing \( v \) is \( \{v\} \cup V(C_1) \). Now, we reach a contradiction because \( v \) has at least 2 neighbors outside the expanded clique containing it, which is not possible in an expanded-clique graph.

For the sufficiency, consider that every chain of \( H \) is good and for every \( v \in V(H) \), either \( v \) is a simplicial or a 1-simplicial vertex. We will construct a graph \( G \) and a function \( f \) such that \( H \) is the expanded-clique graph of \( (G, f) \). In order to do this, we define \( S(v) \) for every \( v \in V(H) \) as follows.

First, consider the vertices \( v \) with degree at least 3 in any order. If \( v \) is a simplicial vertex, then set \( S(v) = N(v) \); otherwise, let \( u \in N(v) \) such that \( N(v) \setminus \{u\} \) is a clique and \( u \) has no neighbors in \( N(v) \), and set \( S(v) = N(v) \setminus \{u\} \). For the vertices \( v \) with degree 2 in any exists, choose anyone having a neighbor \( w \) such that \( S(w) \) has already been defined. Denote by \( u \) the neighbor of \( v \) different of \( w \). Then, define \( S(v) = \{v, u\} \) and \( S(u) = \{v, u\} \) and repeat. Finally, if there are pendant vertices \( v \) such that \( S(v) \) has not been defined yet, define \( S(v) = \{v\} \). The assumptions on \( H \) guarantee that \( S(v) \) will be defined for every vertex \( v \in V(H) \).

Now, consider any ordering \( u_1, \ldots, u_p \) of \( V(H) \). Then, for \( i \) from 1 to \( p \), add to \( G \) a vertex \( v_i \) if \( S(u_i) \neq S(u_j) \) for every \( j \in [i - 1] \) and set \( f(v_i) = |S(u_i)| \). Finally, add to \( G \) the edge \( v_iv_j \) if there is an edge joining some vertex of \( S(u_i) \) to some vertex of \( S(u_j) \). Noting that \( S(u_i) \) is a clique for every \( u_i \in V(H) \) and that every vertex of \( S(u_i) \) has at most one neighbor outside \( S(u_i) \), we conclude that \( H \) is the expanded-clique graph of \( (G, f) \).

It is clear that Theorem 2.4 leads to a polynomial-time algorithm for answering whether a graph \( H \) that is not a cycle is expanded-clique. We will present in the sequel a linear-time algorithm for this problem. For completeness, our algorithm considers also the case where \( H \) is a cycle. For this purpose, we need of the following result.

**Proposition 2.5.** If \( H \) is an expanded-clique graph, then \( H \) is \( C_4 \)-free.

**Proof.** Let \( H \) be the expanded-clique graph of a pair \( (G, f) \). Suppose by contradiction that \( H \) contains an induced \( u_1u_2u_3u_4u_1 \). By the definition, the expanded cliques of \( H \) form a partition of \( V(H) \). Consider the expanded clique \( V_i \) containing \( u_1 \). We also know that \( u_1 \) has at most one neighbor outside \( V_i \). Without loss of generality, we can assume that \( u_3 \in V_i \) and that \( u_2 \) belongs to an expanded clique \( V_j \) different of \( V_i \). The same reasoning implies that \( u_3 \in V_j \). With these facts, we conclude that there are two edges in \( G \) joining the vertices \( v_i, v_j \in V(G) \) associated with \( V_i \) and \( V_j \), which contradicts the assumption that \( G \) is a simple graph.  

□
Algorithm 1: Is_EXPANDED_CLIQUE

**Input:** A graph $H$ of order $n \geq 3$ that is not a cycle (each adjacency lists is ordered by the vertex number).

**Output:** The root $G$ of $H$ if one there exists

1. if $H$ is a cycle $C_n$ then
   2. if $n == 3$ then return $((\{v_1\}, \{}), 3)$;
   3. if $n == 4$ or $n \geq 5$ odd then return NO;
   4. if $n \geq 6$ even then return $(C_{\frac{n}{2}}, 2)$;

5. sort the adjacency lists
6. for $u_i \in H.V$ do
   7. $u_i.marked = \text{FALSE}$
   8. $u_i.outsider = \text{NULL}$
   9. $u_i.current = u_i.first\_neighbor$

10. for $u_i \in H.V$ do
    11. if $u_i.marked == \text{FALSE}$ then
        12. $u_i.marked = \text{TRUE}$
        13. if $u_i.deg \geq 3$ then
            14. if IsSimp1Simp($H, u_i$) == FALSE then return NO;
            else
                15. if IsGoodChain($H, u_i$) == FALSE then return NO;

17. return $(G, f)$
Procedure IsSimp1Simp(H, u_i)

18 if u_i.outsider = NULL then
19   Let w_1, w_2, w_3 be the first, the second and the third neighbors of u_i
20   if w_1w_2 ∉ E(H) then
21     if w_1w_3 ∉ E(H) then u_i.outsider = w_1;
22     else u_i.outsider = w_2;
23   else for w ∈ u_i.Adj do
24     if w ≠ w_1 and ww_1 ∉ E(H) then
25       u_i.outsider = w
26       w.outsider = u_i
27
28 for w ∈ u_i.Adj do
29   if w ≠ u_i.outsider then
30     for z ∈ u_i.Adj do
31       if z ≠ w and z ≠ u_i.outsider then
32         if z.current ≠ w then
33           if z.outsider ≠ NULL then return FALSE;
34             z.outsider = z.current
35             z.current.outsider = z
36             z.current = z.current.next
37 for w ∈ u.Adj do
38   if w ≠ u.outsider then
39     w.marked = TRUE
40   if w.current ≠ NULL then
41     if w.outsider == NULL then
42       w.outsider = w.current
43       w.current.outsider = w
44     if w.current.next ≠ NULL then return FALSE;
45   else return FALSE;
```
Procedure IsGoodChain(H, u)

46 if u.deg == 1 then
47     W = singleton formed by the only neighbor of u
48     good_chain = TRUE
49 else
50     W = set formed by the two neighbors of u
51     good_chain = FALSE
52 q = 1
53 for w ∈ W do
54     while TRUE do
55         v = neighbor of w different of u
56         if v.deg ≥ 3 then break;
57         q = q + 1
58         v.marked = TRUE
59         if v.deg == 1 then
60             good_chain = TRUE
61             break
62         u = w
63     w = v
64 if good_chain or q is even then return TRUE;
65 return FALSE
```

Theorem 2.6. Algorithm 1 is correct.

Proof. First, consider that the input graph H is a cycle C_n for n ≥ 3. It is clear that C_3 is the 3-expanded-clique graph of the trivial graph. By Proposition 2.5, we know that C_4 is not an expanded-clique graph. Proposition 2.2 and Theorem 2.3 imply that C_n is not expanded-clique graph for n ≥ 5 odd. For n ≥ 6 even, it suffices to note that C_n is the expanded-clique graph of the pair (C_2^n, 2). These cases are considered in lines 1 to 4.

Consider now that H is connected and is not a cycle. By the characterization given in Theorem 2.4, we have to show that the algorithm returns a pair (G, f) where H is the f-expanded-clique of G if and only if every vertex of H is simplicial or 1-simplicial and every chain of H is good. In the for loop beginning in line 6, the algorithm set initial values for the variables associated with every vertex u_i of H, and current. When the marked variable becomes TRUE, the algorithm already knows to which expanded clique u_i belongs to, which prevents that an expanded clique be discovered more than once. The outsider variable will register the outsider of u_i if the algorithm reaches out that u_i is a 1-simplicial vertex with degree at least 3. The auxiliary variable current keeps the current neighbor of u_i during a search in its neighborhood. Next, in the for loop beginning in line 10, the algorithm passes through every vertex of H testing whether it is simplicial or 1-simplicial (line 14) if its degree is
at least 3. Otherwise, it verifies whether the chain containing it is good (line 16). Therefore, it suffices to show that (i) \text{IsSimp1Simp}(H, u_i) returns \text{FALSE} if and only if $u_i$ is neither simplicial nor 1-simplicial in the case where $u_i, \deg \geq 3$, and that (ii) \text{IsGoodChain}(H, u_i) returns \text{FALSE} if and only if the chain containing $u_i$ is not good in the case where $u_i, \deg \leq 2$.

(i) From lines 19 to 27, the algorithm finds out the outsider of vertex $u_i$ if one there exists. Recall that the outsider of $u_i$ is the only neighbor of $u_i$ that is not adjacent to any other neighbor of $u_i$. Then, if the first two neighbors of $u_i$ are not adjacent, then one of them is the outsider of $u_i$. To discover which one is the outsider, it suffices to test whether for any of them if it is adjacent to the third neighbor of $u_i$. These tests are done in lines 19 to 22. On the other hand, if the first two neighbors of $u_i$ are not adjacent, then none of them is the outsider of $u_i$. Then, it suffices to pass through the adjacency list of $u_i$ testing whether there is some neighbor of $u_i$ non-adjacent to the first neighbor of $u_i$. This verification is done in lines 23 to 27.

From lines 28 to 36, the algorithm checks whether any neighbor of $u_i$ different of its outsider is also neighbor of every other neighbor of $u_i$, i.e., if these vertices form a clique. Since the adjacency lists are ordered, we can pass through the adjacency lists of the neighbors of $u_i$ simultaneously. However, a neighbor $z$ of $u_i$ can be a 1-simplicial vertex, in which case $z$ has a neighbor $z'$ not in the neighborhood of $u_i$. This possibility is considered in line 32. If this occurs, then such $z'$ is the outsider of $z$ and vice-versa and this information is saved in lines 34 and 35. But if this occurs more than once, then we know that $z$ is neither simplicial nor 1-simplicial, which is tested in line 33.

If $u_i$ is a simplicial or a 1-simplicial vertex, then any neighbor of $u_i$ has $d(u_i) - 1, d(u_i)$, or $d(u_i) + 1$ neighbors. When the algorithm reaches line 37, we have checked that the neighbors of $u_i$ different of its outsider form a clique $C$. Each such vertex can have at most one neighbor outside $C$. We have finished to search the neighborhood of $u_i$, but we have to complete the search at the adjacency list of each neighbor of $u_i$ to guarantee that it does not have more vertices. If it has one neighbor and its outsider is undefined yet, then such vertex is it outsider. If it has 2 or more neighbors, then $H$ is not clique-expanded. These test are done in lines 37 to 45 completing the proof of (i).

(ii) Denote by $u_1, \ldots, u_k$ the chain $C$ containing $u_i$. If $i = 1$ and $d(u_1) = 1$, then $C$ is good by the definition. If this is the case, this information is saved in line 48. We use the set $W$ to save the set of neighbors of $u_i$ in line 47 or 50. The variable $q$ is used to count the number of internal vertices of the chain. From lines 53 to 63, the algorithm finds out all vertices belonging to such chain. It accomplish this by choosing one neighbor $w$ of $u_i$ and following the path beginning in $u_i$ and containing $w$ until that some vertex with degree different from 2 is found. Then, it repeat the same for the other neighbor of $u_i$ if it exists. If the chain is good, the algorithm answers this in line 64. Otherwise, it returns bad in line 65. It is clear that this procedure returns \text{FALSE} if and only if the chain containing $u_i$ is not good in the case where $u_i, \deg \leq 2$.

\textbf{Theorem 2.7.} For a graph $H$ of order $n$ and size $m$, Algorithm 1 finishes in $O(n + m)$ steps.
Proof. The case where $H$ is a cycle is considered in lines 1 to 4 and it finishes in constant time. From now on we assume that $H$ is a connected graph different of a cycle. Consider that $H$ is represented by adjacency lists and each vertex is associated with an exclusive number from 1 to $n$.

We begin by showing that the sort of the adjacency lists can be done in linear time (line 5). Consider an array $N$ of size $n$. Each position of $N$ is a linked list initially empty. For $i$ from 1 to $n$, we go through the adjacency list of every vertex $u_i \in V(H)$. For each $w_j \in N(u_i)$, add $u_i$ to the linked list $N[j]$. This is done in $O(n + m)$ steps for all vertices of $H$. Observe that for every $i \in \{1, \ldots, n\}$, $N[i]$ is exactly the adjacency list of vertex $u_i$ and they appear in ascending order.

Next, it is clear that the for loop of lines 6 to 9 costs $O(n)$ steps. Since the for loop beginning in line 10 has $O(n)$ iterations, we have to show that the number of steps of the functions $\text{IsSimp1Simp}$ and $\text{IsGoodChain}$ have time complexity $O(n + m)$ over all calls of these procedures.

First, note that the variable $\text{marked}$ is set to $\text{false}$ only once (line 7). Note also that a call to $\text{IsSimp1Simp}(H, u_i)$ makes that the variable $\text{marked}$ of $u_i$ and of every vertex belonging to the same expanded clique as $u_i$ are set to $\text{true}$ in lines 12 and 39, respectively. Therefore, line 11 guarantees that at most once call to $\text{IsSimp1Simp}$ occurs for each expanded clique of $H$. Because of line 58, we can also conclude that at most once call to $\text{IsGoodChain}$ occurs for each chain of $H$.

Now, note that the cost of a call to $\text{IsSimp1Simp}(H, u_i)$ is $O(d(u_i)^2)$. Since the sum of the degrees of the vertices belonging to the expanded clique containing $u_i$ is $O(d(u_i)^2)$, the total complexity of all calls to $\text{IsSimp1Simp}$ is $O(m)$. Analogously, note that the cost of a call to $\text{IsGoodChain}(H, u_i)$ is $O(q)$ where $q$ is the size of the chain containing $q$. Since every internal vertex of a chain has degree 2, the total complexity of all calls to $\text{IsGoodChain}$ is $O(m)$. Therefore, the total time complexity of Algorithm 1 is $O(n + m)$.

We conclude this section by presenting another characterization of expanded-clique graphs.

**Corollary 2.8.** A graph is expanded-clique if and only if it is (bad chain, butterfly, claw, $C_4$, diamond, odd-hole)-free.

**Proof.** Let $H$ be an expanded-clique graph. Due to Theorem 2.3, we know that $H$ is a line graph of a bipartite graph. Then, Proposition 2.2 implies that $H$ is (claw, diamond, odd-hole)-free. By Theorem 2.4, $H$ is bad chain free. Since a butterfly has a vertex that is neither simplicial nor 1-simplicial, Theorem 2.4 also implies that $H$ is butterfly free. Since Proposition 2.5 guarantees that $H$ is $C_4$ free, $H$ is (bad chain, butterfly, claw, $C_4$, diamond, odd-hole)-free.

Conversely, let $H$ be a graph that is (bad chain, butterfly, claw, $C_4$, diamond, odd-hole)-free. If $H$ is a cycle $C_k$, then $k$ is even greater then 4. It is clear that $C_{2k'}$ for $k' \geq 3$ is an expanded-clique graph. Then, consider that $H$ is not a cycle. Assume by contradiction that $H$ is not an expanded-clique graph. As $H$ does not contain no bad chains, by Theorem 2.4, $H$ contains some vertex $v$ that is neither simplicial nor 1-simplicial.

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If there are $u_1, u_2, u_3$ in $N(v)$ such that $\{u_1, u_2, u_3\}$ is an independent set, then we have a contradiction because $H$ has a claw. Since $v$ is not simplicial, there are $u_1, u_2 \in N(v)$ such that $u_1u_2 \notin E(H)$. If some neighbor of $v$ is adjacent to both $u_1, u_2$, we would have a diamond. Hence, the subgraph of $H$ induced by $N(v)$ has exactly two connected components, say $C_1$ and $C_2$, each one being a complete graph. Since $v$ is not 1-simplicial, we have that $|V(C_i)| \geq 2$ for $i \in \{1, 2\}$. Then, choose $u_1, u_2 \in V(C_1)$ and $u_3, u_4 \in V(C_2)$. These 4 vertices plus $v$ form a butterfly, which is a contradiction.

\section{The domination problem}

In this section, we deal with the \textsc{Dominating set} problem for $k$-expanded-clique graphs $H$. For $k = 2$, the root $G$ is a path or cycle and can be easily verified that if $|V(G)| \geq 4$, then $\gamma(H) = \lceil \frac{3}{2} \rceil$. For $k = 3$, the problem becomes hard as we will see in the sequel.

The \textsc{Edge dominating set} problem asks, for a graph $G$ and an integer $\ell$, whether there is a set $E' \subseteq E(G)$ so that $|E'| \leq \ell$ and every edge of $E(G) \setminus E'$ is adjacent to some edge of $E'$. It is known that the \textsc{Edge dominating set} problem is $\text{NP}$-complete for bipartite graphs with maximum degree 3 \cite{10}, which means that \textsc{Dominating set} is $\text{NP}$-complete for line graphs of bipartite graphs with maximum degree 4 ($C_1$). It is also known \cite{11, 9} that the \textsc{Dominating set} problem is $\text{NP}$-complete for planar bipartite graphs with maximum degree 3 and girth at least $k$ for a fixed $k$ ($C_2$) and for cubic graphs ($C_3$). Up to our best knowledge, for no proper subclass of these three classes, the \textsc{Dominating set} problem is known to be $\text{NP}$-complete.

We show in Theorem 3.1 that \textsc{Dominating set} is $\text{NP}$-complete for planar bipartite $3$-expanded-clique graphs, which by Theorem 2.3 is a proper subclass of classes $C_1$ and $C_2$; and in Theorem 3.2 for cubic line graphs of bipartite graphs, a proper subclass of class $C_3$. The proofs of these two results are very similar and since both reduction are done from two variations of the \textsc{Dominating set} problem, we begin by presenting a general reduction so that next we complete each proof with the necessary details.

Given a graph $G$, we denote by $\gamma(G)$ the size of a minimum dominating set of $G$.

\textbf{Reduction A.} Consider an instance $\langle G, \ell \rangle$ of \textsc{Dominating set}. Let $G'$ be the 3-expanded-clique of $G$, and let $H$ be the 3-expanded-clique of $G'$. Set $\ell' = 2|V(G)| + \ell$. Then, $G$ has a dominating set with at most $\ell$ vertices if and only if $H$ has a dominating set with at most $2|V(G)| + \ell$ vertices.

\textbf{Proof.} The pair $\langle H, \ell' \rangle$ is an instance of \textsc{Dominating set} where $H$ is the 3-expanded-clique graph of $G'$. See Figure 4 for an example.

For $u \in V(G)$, denote the expanded clique of $G'$ associated with $u$ by $\{u_1, u_2, u_3\}$, and for $i \in [3]$ and $u_i \in V(G')$, denote the expanded clique of $H$ associated with $u_i$ by $\{u_{i,1}, u_{i,2}, u_{i,3}\}$. For $u \in V(G)$, denote by $H_u$ the subgraph of $H$ induced by $\{u_{i,j} : i \in [3] \text{ and } j \in [3]\}$. See Figure 5.

\textbf{Claim 1.} For every $u \in V(G)$, it holds that $\gamma(H_u) = 3$. 

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Proof of Claim 1. We know that $\gamma(H_u) \geq 3$ because $\Delta(H_u) = 3$ and $|V(H_u)| = 9$. On the other hand, note that any set formed by one vertex of each $K_3$ is a dominating set, and, therefore, $\gamma(H_u) = 3$. See Figure 5-(ii). ■

Claim 2. If $D$ is a dominating set of $H$, then $|V(H_u) \cap D| \geq 2$ for every $u \in V(G)$. Furthermore, if $|V(H_u) \cap D| = 2$ for some $u \in V(G)$, then $V(H_u) \setminus N[V(H_u) \cap D]$ has only one vertex and such vertex is dominated by some vertex in $H_v$ whose $|V(H_v) \cap D| \geq 3$. 

Figure 4: Graph resulting from polynomial transformation.
Proof of Claim 2. For any \( u \in V(G) \), denote by \( U_3 \) the subset of \( V(H_u) \) having neighbors only in \( H_u \) and write \( U_2 = V(H_u) \setminus U_3 \). Since \( \Delta(H_u) = 3 \) and \( |U_3| = 6 \), we conclude that \( |V(H_u) \cap D| \geq 2 \). Now, consider that \( V(H_w) \cap D = \{x, y\} \) for some \( w \in V(G) \). Observe that if \( x \in W_2 \), then \( D \) would not be a dominating set of \( H \). Therefore, by symmetry, we can assume that \( x, y \) are the vertices depicted in Figure 5-(iii) and \( V(H_w) \setminus N[V(H_w) \cap D] \) has only one vertex and such vertex is dominated by some vertex in \( H_v \) for \( v \neq w \). Since a vertex of \( V_2 \) belongs to \( D \), we have that \( |V(H_w) \cap D| \geq 3 \). \( \blacksquare \)

Now, we shall prove that \( G \) has a dominating set with at most \( \ell \) vertices if and only if \( H \) has a dominating set with at most \( 2|V(G)| + \ell \) vertices.

\( \Rightarrow \) Consider that \( D \) is a dominating set of \( G \) with \( |D| \leq \ell \). Starting with \( D' \) empty, for each \( v \in D \), add to \( D' \) the vertices of \( H_v \) with degree 2. Now, for each \( u \in V(G) \setminus D \), let \( v \in D \) such that \( uv \in E(G) \). Denote by \( x \) the vertex of \( H_u \) having a neighbor in \( H_v \). Then, at to \( D' \) the two vertices of degree 3 in \( H_u \) having a common neighbor with \( x \). Observe that \( |D'| = 2(|V(G)| - |D|) + 3|D| = 2|V(G)| + |D| \leq 2|V(G)| + \ell \) and, furthermore, \( D' \) is a dominating set of \( H \).

\( \Leftarrow \) Now, let \( D' \) be a dominating set of \( H \) such that \( |D'| \leq 2|V(G)| + \ell \). By Claim 2, we know that \( |V(H_u) \cap D'| \geq 2 \) for every \( u \in V(G) \). Therefore, the number of vertices \( v \in V(G) \) such that \( |V(H_v) \cap D'| \geq 3 \) is at most \( \ell \). Claim 2 also says that if \( |V(H_u) \cap D'| = 2 \) for \( u \in V(G) \), then \( H_u \) has a vertex dominated by some vertex in \( H_v \) where \( |V(H_v) \cap D_H| \geq 3 \), which means that \( uv \) is an edge of \( G \). Therefore, choosing set \( D \subseteq V(G) \) composed by vertices \( v \) such that \( |V(H_v) \cap D_H| \geq 3 \) we have a dominating set \( D \) of \( G \) with size at most \( \ell \). \( \square \)

**Theorem 3.1.** Dominating set is NP-complete for planar bipartite 3-expanded-clique graphs with girth at least \( k \) for a fixed \( k \).

**proof.** Since Dominating set belongs to NP for general graphs, that condition holds for our particular case. For the hardness part, we consider the Dominating set problem restricted to planar bipartite graphs with maximum degree 3 and girth at least \( k \) for a fixed \( k \) since this
version is \( \text{NP-complete} \) [11]. Let \( \langle G, \ell \rangle \) be an instance of this problem. Observe that graph \( H \) constructed by applying Reduction A to \( \langle G, \ell \rangle \), is a planar bipartite 3-expanded-clique graph, which means that DOMINATING SET is \( \text{NP-complete} \) for planar 3-expanded-clique graphs. □

**Theorem 3.2.** DOMINATING SET is \( \text{NP-complete} \) for cubic line graphs of bipartite graphs.

**proof.** As note in the previous result, this problem belongs to \( \text{NP} \). We consider the DOMINATING SET problem restricted to cubic graphs since this version is also \( \text{NP-complete} \) [9]. Let \( \langle G, \ell \rangle \) be an instance of this problem. The graph \( H \) constructed by applying Reduction A to \( \langle G, \ell \rangle \) is a 3-expanded-clique graph. By Theorem 2.3, \( H \) is a line graph of a bipartite graph. It is easy to see that \( H \) is a cubic graph, which means that DOMINATING SET is \( \text{NP-complete} \) for planar 3-expanded-clique graphs. □

**Proposition 3.3.** Let \( H \) be the expanded-clique graph of a graph \( G \) and function \( f \). If \( S \) is a dominating set of \( H \), then there is a dominating set \( S' \) of \( H \) such that \( |S'| \leq |S| \) and \( |S' \cap V_i| \leq 1 \) for every \( v_i \in V(G) \).

**Proof.** Let \( S \) be a dominating set of \( H \). Apply the following process. If \( |S \cap V_i| \leq 1 \) for every \( v_i \in V(G) \), then \( S' = S \) and stop. Otherwise, there is \( v_i \in V(G) \) such that \( |S \cap V_i| \geq 2 \). Let \( v_{i,j} \in S \cap V_i \). If \( |S \cap V_j| \geq 1 \), then \( S \setminus \{v_{i,j}\} \) is also a dominating set of \( H \). Then, redefine \( S \) as \( S \setminus \{v_{i,j}\} \) and repeat. If \( |S \cap V_j| = 0 \), then \( (S \setminus \{v_{i,j}\}) \cup \{v_{j,i}\} \) is also a dominating set of \( H \). Then, redefine \( S \) as \( (S \setminus \{v_{i,j}\}) \cup \{v_{j,i}\} \) and repeat. Observe that this process eventually finishes with a set \( S' \) such that \( |S'| \leq |S| \) and \( |S' \cap V_i| \leq 1 \) for every \( v_i \in V(G) \). □

**Proposition 3.4.** If \( H \) is the expanded-clique graph of a pair \( (G, f) \) where \( f(v) > d_G(v) \) for every \( v \in V(G) \), then \( \gamma(H) = |V(G)| \).

**Proof.** Note that for every \( v_i \in V(G) \), \( V_i \) has a simplicial vertex, which implies that any dominating set \( D \) of \( H \) satisfies \( D \cap V_i \neq \emptyset \). On the other hand, if we choose a simplicial vertex of each set \( V_i \), we form a dominating set of \( H \), which means that \( \gamma(H) = |V(G)| \). □

A 2-independent set in a graph \( G \) is a subset \( I \) of the vertices such that the distance between any two vertices of \( I \) in \( G \) is at least three. We denote by \( \alpha_2(G, f) \) the maximum cardinality of a set \( S \) such that every \( v \in S \) satisfies \( f(v) = d_G(v) \) and \( S \) is a 2-independent set of \( G \).

**Theorem 3.5.** Let \( H \) be the expanded-clique graph of a pair \( (G, f) \). Then, \( \gamma(H) + \alpha_2(G, f) = |V(G)| \).

**Proof.** Let \( S \) be a dominating set of \( H \). By Proposition 3.3, there is a dominating set \( S' \) of \( H \) such that \( |S'| \leq |S| \) and \( |S' \cap V_i| \leq 1 \) for every \( v_i \in V(G) \). It is clear that \( |S'| \leq n \). Note that for any \( v_i \in V(G) \), if \( f(v_i) > d_G(v_i) \), then \( |S' \cap V_i| = 1 \). Therefore, if \( |S' \cap V_i| = 0 \), then \( f(v_i) = d_G(v_i) \). Observe that if \( f(v_i) = d_G(v_i) \), \( S' \cap V_i = \emptyset \) and \( v_iv_j \in E(G) \), then vertex \( v_{j,i} \) belongs to \( S \). Hence, the vertices \( v_i \) with \( |S' \cap V_i| = 0 \) form a 2-independent set \( T \) in \( G \) with cardinality \( |T| = |V(G)| - |S'| \) containing only vertices \( v_i \) such that \( f(v_i) = d_G(v_i) \). Since \( |S'| \leq |S| \), we conclude that \( \alpha_2(G, f) \geq |V(G)| - \gamma(H) \).
Now, let $T$ be a 2-independent set of $G$ containing only vertices $v_i$ such that $f(v_i) = d_G(v_i)$ and set $S = \emptyset$. If $v_i \in T$, then for every $v_j \in N_G(v_i)$, add $v_{j,i}$ to $S$. If $v_j$ is not in $T$ neither has a neighbor in $T$, then choose a vertex of $V_j$ and add it to $S$. Since no vertex of $G$ has more than one neighbor in $T$, we have that $S$ is a dominating set of $H$ with $|V(G)| - |T|$ vertices. Therefore, $\gamma(H) \leq |V(G)| - \alpha_2(G, f)$. 

**Theorem 3.6.** If $H$ is the $\Delta$-expanded-clique graph of $G$, then $\frac{|V(G)|\Delta}{\Delta + 1} \leq \gamma(H) \leq |V(G)|$.

**Proof.** The upper bound follows from Proposition 3.3. For the lower bound, let $S$ be a minimum dominating set of $H$. Using Proposition 3.3 again, we can assume that $|S \cap V_j| \leq 1$ for every $v_j \in V(G)$. Denote by $X$ the set formed by the vertices $v_i \in V(G)$ such that $V_i \cap S = \emptyset$. Note that for every $v_i \in X$, we have that every $d(v_i) = \Delta(G)$, that every vertex $v_k \in N(v_i)$ is such that $|V_k \cap S| = 1$, and that $N(v_i) \cap N(v_j) = \emptyset$. On the other hand, for every $v_i \in S \setminus N[X]$, it holds that $|V_i \cap S| = 1$. From these facts, we can write

$$\frac{|S|}{|V(G)|} = \frac{\sum_{v_i \in X} \Delta + \sum_{v_i \in S \setminus N[X]} 1}{\sum_{v_i \in X} (\Delta + 1) + \sum_{v_i \in S \setminus N[X]} 1} \geq \frac{\Delta}{\Delta + 1},$$

which means that the lower bound also holds. 

A consequence of Proposition 3.4 and Corollary 3.6 is that given a $k$-expanded-clique graph $H$, a set containing one vertex of each expanded clique of $H$ is a dominating set which is at most $\frac{1+\Delta}{\Delta}$ from a minimum.

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