Beyond Nonexpansive Operations in Quantitative Algebraic Reasoning

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ABSTRACT
The framework of quantitative equational logic has been successfully applied to reason about algebras whose carriers are metric spaces and operations are nonexpansive. We extend this framework in two orthogonal directions: algebras endowed with generalised metric space structures, and operations being nonexpansive up to a lifting. We apply our results to the algebraic axiomatisation of the Łukaszyk–Karmowski distance on probability distributions, which has recently found application in the field of representation learning on the first page. Copyrights for components of this work owned by others than the author(s) must be honored. Abstracting with credit is permitted. To copy otherwise, or republish, to post on servers or to redistribute to lists, requires prior specific permission and/or a fee. Request permissions from permissions@acm.org.

CCS CONCEPTS
• Theory of computation → Equational logic and rewriting; Algebraic semantics; Probabilistic computation.

KEYWORDS
equational logic, quantitative reasoning, metrics, monads, probability distributions, free algebras

1 INTRODUCTION
Equational reasoning and algebraic methods are widespread in all areas of computer science, and in particular in program semantics. Indeed, initial algebra semantics and monads are cornerstones of the modern theory of functional programming and allow us to reason about inductive definitions, computational effects and specifications in a formal way (see, e.g., [8, 17, 18]). In elementary terms, this is due to the fact that many objects of interest in programming are free algebras of some algebraic theory, i.e., a signature \( \Sigma \) together with a set of equational axioms \( E \) between \( \Sigma \)-terms. Examples include:

\[ \Sigma = \{ \vee : 2 \} \quad E = \begin{cases} x \vee y = y \vee x, & x \vee x = x, \\ x \vee (y \vee z) = (x \vee y) \vee z \end{cases} \]

finite sets (free algebras of the theory of semilattices)

finite lists (free monoids), finitely supported distributions (free convex algebras) etc. Since free algebras are (up to isomorphism) term algebras—i.e., sets of \( \Sigma \)-terms modulo the congruence relation \( \equiv_E \) generated from the axioms \( E \) using the deduction rules of the syntactic apparatus of equational logic—they are easy to manipulate formally in a computer.

Objects definable as free algebras, as in the framework outlined above, are sets \( X \) equipped with operations of type \( X^n \to X \). This means it is not straightforward, or even possible, to describe objects that are sets endowed with some additional structure such as, e.g., a metric \( d : X^2 \to [0,1] \). To address this limitation, in a series of recent papers (including [3, 4, 11–13]), the authors have proposed the notion of quantitative algebras: algebras whose carriers are metric spaces.

At the syntactic level, the apparatus of equational logic is replaced by a deductive system allowing the derivation of judgments of the form \( s =_t \), where \( s, t \) are \( \Sigma \)-terms and \( \varepsilon \in [0,1] \), with the intended meaning that \( d(s, t) \leq \varepsilon \). These judgments are derived using quantitative inferences, i.e., deduction rules of the form:

\[ \{s_1 =_{t_1} t_1, \ldots, s_n =_{t_n} t_n\} \vdash s =_\varepsilon t. \]

In particular, the deductive system includes rules such as:

\[ \emptyset \vdash x =_0 x \quad \{x =_{t_1} y, y =_{t_2} z\} \vdash x =_{t_1 + t_2} z \]

corresponding to properties of metrics such as reflexivity \( (d(x, x) = 0) \), symmetry \( (d(x, y) = d(y, x)) \) and triangular inequality \( (d(x, z) \leq d(x, y) + d(y, z)) \). A quantitative theory over a signature \( \Sigma \) is generated from a set of quantitative inferences, playing the role of implicative axioms, by closing under deducibility in the apparatus. Models of quantitative theories are quantitative algebras, which are metric spaces \( (A, d) \) equipped with interpretations \( [\text{op}] : A^n \to A \) of the operations such that for each \( \text{op} \in \Sigma \)

\[ d([\text{op}](a_1, \ldots, a_n), [\text{op}](a'_1, \ldots, a'_n)) \leq \max\{d(a_i, a'_i)\}_{1 \leq i \leq n}. \]

This is equivalent to requiring that \( [\text{op}] : (A^n, d_X) \to (A, d) \) is nonexpansive (also known as \( 1 \)-Lipschitz), with \( d_X \) being the (categorical) product metric on \( A^n \). This is reflected in the deductive system by a rule called NE:

\[ \{x_i =_{y_i} y_i\}_{1 \leq i \leq n} \vdash \text{op}(x_1, \ldots, x_n) =_{\text{max}(x_1, \ldots, y_n)} \text{op}(y_1, \ldots, y_n). \]

Consider, for example, the theory of quantitative semilattices of [11] having signature \( \Sigma = \{\vee : 2\} \) and implicative axioms (we just
write \( s \equiv t \) for \( \emptyset \vdash s \equiv t \):
\[
x \lor y \equiv_0 y \lor x \quad x \lor x \equiv_0 x \quad x \lor (y \lor z) \equiv_0 (x \lor y) \lor z
\]
These just state the usual axioms of semilattices. Indeed, since in any metric space it holds that \( d(x, y) = 0 \) implies \( x = y \), the judgment \( s \equiv t \) expresses equality. From these axioms, further quantitative inferences can be obtained using the deductive apparatus, like the NE rule:
\[
\{ x = e_1, x' = e_1 \} \vdash x \lor y = \max(e_1, e_2) \quad x' \lor y'
\]
which expresses that the interpretation of the binary operation \( \lor : 2 \to 2 \) must be nonexpansive.

Given a quantitative theory over a signature \( \Sigma \) generated by a set of implicational axioms, we have a category \( \text{Alg}(\Sigma, E) \) consisting of quantitative algebras modelling the theory and their homomorphisms, i.e., nonexpansive maps \( f : (A, d_A) \to (B, d_B) \) preserving all operations \([\text{op}]\). Among the main results of [3, 11, 12] the following is of key importance:

**Theorem 3.3 in [3].** The free quantitative algebra generated by a metric space \((A, d)\) exists in \(\text{Alg}(\Sigma, E)\) and is isomorphic to the quantitative term algebra \(T_{\Sigma, E}(A, d)\).

More can be said if the implicational axioms \( E \) have a constrained form, where all the terms in their premises are variables: \( x_1 = e_1, y_1, \ldots, x_n = e_n, y_n \vdash s \equiv t \). In this case, which covers several interesting examples (e.g., quantitative semilattices), we have a stronger result:

**Theorem 4.2 in [3].** The Eilenberg–Moore category \(\text{EM}(T_{\Sigma, E})\) of the term monad \(T_{\Sigma, E}\) is isomorphic to the category \(\text{Alg}(\Sigma, E)\).

Several interesting metric spaces can be identified with free quantitative algebras. For example the collection of non-empty finite subsets of \((A, d)\), endowed with the Hausdorff metric and interpreting \(\{\_\} = \emptyset \) (union), can be shown (see [11]) to be isomorphic to the free quantitative semilattice generated by the metric space \((A, d)\).

### 1.1 Beyond Metric Spaces and Nonexpansive Maps

The main purpose of this paper is to extend the framework of [11] outlined above, while maintaining its key characteristics and properties, in order to reason equationally about additional interesting mathematical objects which do not fit the constraints of the original framework.

We immediately discuss a specific example arising from recent research in the field of learning and artificial intelligence [6], which will serve as a main motivation. Other examples are discussed in Section 5. In [6], the authors have developed new techniques for representation learning on Markov processes based on the Łukaszyk–Karmowski (ŁK for short) distance [10]. This is a distance \(d_{\text{ŁK}} : DX \times DX \to [0, 1]\) on finitely supported distributions on a set \(X\) endowed with an arbitrary map \(d : X^2 \to [0, 1]\) (i.e., \((X, d)\) is not necessarily a metric space). Even if \(d\) is a metric, the ŁK distance \(d_{\text{ŁK}}\) does not satisfy all axioms of metric spaces. Specifically, the reflexivity property is in general not satisfied: \(d_{\text{ŁK}}(\varnothing, \varnothing) \neq 0\). However, \(d_{\text{ŁK}}\) always satisfies the symmetry and triangular inequality axioms (see Equations (1) and (4) in Section 2) and, therefore, \((DX, d_{\text{ŁK}})\) is a diffuse metric space (see [6] or Section 2.3 for precise definitions). If we consider the convex algebra operation \(+_p : DX \times DX \to DX\) on probability distributions defined by
\[
(\varnothing +_p \psi)(x) = p(x) + (1 - p')\psi(x),
\]
then it can be shown (see Lemma 5.3) that \(+_p\) fails to be nonexpansive with respect to the ŁK distance:
\[
d_{\text{ŁK}}(\varnothing +_p \varnothing, \psi +_p \psi') > \max\{d_{\text{ŁK}}(\varnothing, \psi), d_{\text{ŁK}}(\varnothing, \psi')\}.
\]

Thus we have an interesting mathematical object, the diffuse metric space \((DX, d_{\text{ŁK}})\), whose underlying set \(DX\) is the free convex algebra over the set \(X\) (see, e.g., [9]), not fitting the framework of [11] due to two reasons: (1) \(d_{\text{ŁK}}\) is not a metric, and (2) the algebraic (convex algebra) operation \(+_p\) is not nonexpansive.

Our contribution is to extend the framework of [11] along two orthogonal axes in order to accommodate examples (see Section 5) such as the one just discussed.

**First extension axis:** our framework can be instantiated on structures \((X, d)\) where \(d : X^2 \to [0, 1]\) is a generalised metric such as any of the following (see Section 2 for details): an ultrametric, metric, pseudometric, quasimetric, diffuse metric or just a fuzzy relation (i.e., \(d\) unconstrained).

This first contribution is natural, yet requires some technical care. Most notably, we need to carefully distinguish in the deductive apparatus between the notions of equality (=) and zero distance (\(=_0\)). This is due to the fact that, unlike the case of metric spaces, in generalised metric spaces (e.g., pseudometric or diffuse metrics) it does not hold that \(d(x, y) = 0\) implies \(x = y\). As a consequence, the identification of \(=\) and \(=_0\) is generally unsound. Our deductive apparatus, unlike that of [11], will therefore handle both ordinary equations \((s = t)\) and quantitative equations \((s =_t t)\), connected by the following congruence principle:

\[
x = y \Rightarrow ((x =_t z \Rightarrow y =_t z) \text{ and } (z =_t x \Rightarrow z =_t y)).
\]

**Second extension axis:** our framework can deal with quantitative algebras whose operations are not nonexpansive with respect to the categorical product. The motivating example being the diffuse metric space \((DX, d_{\text{ŁK}})\) with the convex combination operation \(+_p\) discussed earlier. This is in our opinion the main conceptual and technical contribution of the paper.

To achieve this flexibility, we consider **lifted signatures** \(\widehat{\Sigma} = \{\text{op}_1 : n_1 : L_{\text{op}}, \}_l \in l\). Each operation \(\text{op}\) has an arity \(n \in \mathbb{N}\), as for standard signatures, and is further equipped with a lifting which maps any generalised metric space \((X, d)\) to a generalised metric space \((X^n, L_{\text{op}}(d))\) whose underlying set is the product set \(X^n\), subject to some technical constraints.

In this new setting, quantitative algebras for a lifted signature \(\widehat{\Sigma}\) are (generalised) metric spaces \((X, d)\) in \(G\text{Met}\) where, for each \(l \in \Sigma\), the interpretation \([\text{op}] : X^n \to X\) is nonexpansive up to \(L_{\text{op}}\), namely:

\[
[\text{op}] : (X^n, L_{\text{op}}(d)) \to (X, d) \quad \text{is nonexpansive.}
\]

At the syntactic level, our deductive apparatus replaces the NE rule of [11] with a rule denoted by \(L^\ast\text{-NE}\) (see Definition 3.11) expressing that each \(\text{op} : n : L_{\text{op}} \in \widehat{\Sigma}\) is nonexpansive up to \(L_{\text{op}}\).
The framework of [11] can be seen as a particular case of ours by taking GMet = Met and restricting all $L_{op}$ to be the standard $n$–ary (categorical) product in Met: $L_{op}(X, d) = (X^n, d^\wedge)$.

1.2 Outline and Main Results

After presenting some background material in Section 2, we introduce in Section 3 our new framework for quantitative reasoning based on liftings, and we prove the soundness of the associated deductive apparatus. In Section 4, we define the term monad and we recover the key results of the framework of [11] in our new “lifted” setting. In particular, we prove that the deductive apparatus is complete, and we obtain proofs of the corresponding variants of Theorem 3.3 (free algebras exist and are term algebras) and Theorem 4.2 ($\text{EM}(\Sigma_{\mathcal{L}_o}) \cong \text{Alg}(\mathcal{E}))$) from [3]. We give examples of applications of our new apparatus in Section 5, covering in particular the interesting case of theLK diffuse metric on probability distributions.

2 BACKGROUND

2.1 Monads

We present some definitions and results regarding monads. We assume the reader is familiar with basic concepts of category theory (see, e.g., [1]). Facts easily derivable from known results in the literature are systematically marked as “Proposition” throughout the paper.

Definition 2.1 (Monad). A monad on a category $C$ is a triple $(M, \eta, \mu)$ comprising a functor $M: C \to C$ together with two natural transformations: a unit $\eta: 1_C \Rightarrow M$, where $1_C$ is the identity functor on $C$, and a multiplication $\mu: M^2 \Rightarrow M$, satisfying $\mu \circ \eta M = \mu \circ M \eta = \eta M \circ \mu M = \mu \circ \mu M$.

A monad $M$ has an associated category of $M$–algebras.

Definition 2.2 ($\mathcal{M}$–algebras). Let $(M, \eta, \mu)$ be a monad on $C$. An algebra for $M$ (or $\mathcal{M}$–algebra) is a pair $(A, a)$ where $A \in C$ is an object and $a: M(A) \to A$ is a morphism such that (1) $a \circ \eta_A = id_A$ and (2) $a \circ M\eta = a \circ \mu_A$ hold. An $\mathcal{M}$–algebra morphism between two $\mathcal{M}$–algebras $(A, a)$ and $(A', a')$ is a morphism $f: A \to A'$ in $C$ such that $f \circ a = a' \circ M(f)$. The category of $\mathcal{M}$–algebras and their morphisms, denoted by EM$(M)$, is called the Eilenberg–Moore category for $M$.

2.2 Universal Algebra

We recall basic definitions and results from universal algebra, [5] is a standard reference.

Definition 2.3 (Signature). A signature is a set $\Sigma$ containing operations symbols each with an arity $n \in \mathbb{N}$. We write $\text{op} : n \in \Sigma$ for a symbol op with arity $n$ in $\Sigma$. With some abuse of notation, we also denote with $\Sigma$ the functor $\Sigma: \text{Set} \to \text{Set}$ with the following action:

$$\Sigma(A) := \bigoplus_{\text{op} : n \in \Sigma} A^n; \quad \Sigma(f) := \bigoplus_{\text{op} : n \in \Sigma} f^\wedge.$$

Definition 2.4 ($\Sigma$–algebra). A $\Sigma$–algebra is an algebra for the functor $\Sigma$. Equivalently, it is a set $A$ equipped with a set $[\Sigma]_A$ of interpretations of the operation symbols, i.e., for every op $: n \in \Sigma$ there is a function $[\text{op}]_A : A^n \to A$ in $[\Sigma]_A$. We call $A$ the carrier set. A homomorphism between two $\Sigma$–algebras with carrier sets $A$ and $B$ is a function $f : A \to B$ preserving $[\cdot\cdot\cdot]$, i.e., satisfying $\forall \text{op} : n \in \Sigma, \forall a_1, \ldots, a_n$,

$$f([\text{op}]_A(a_1, \ldots, a_n)) = [\text{op}]_B(f(a_1), \ldots, f(a_n)).$$

The category of $\Sigma$–algebras and their homomorphisms is denoted by Alg($\Sigma$).

Definition 2.5 (Term algebra). Let $\Sigma$ be a signature and $A$ a set. We denote with $T_{\Sigma}A$ the set of terms built from $A$ using the operations in $\Sigma$, i.e., the set inductively defined as follows: for any $a \in A$, and $\text{op}(t_1, \ldots, t_n) \in T_{\Sigma}A$ for any op $: n \in \Sigma$ and $t_1, \ldots, t_n \in T_{\Sigma}A$. The set $T_{\Sigma}A$ has a canonical $\Sigma$–algebra structure with the interpretation of the operations $\text{op} : n \in \Sigma$, defined by:

$$[\text{op}](t_1, \ldots, t_n) = \text{op}(t_1, \ldots, t_n).$$

It is called the term algebra over $A$ and denoted by $T_{\Sigma}A$ (like its carrier set). We often identify elements $a \in A$ with the corresponding terms $a \in T_{\Sigma}A$.

Definition 2.6 (Term monad). The assignment $A \mapsto T_{\Sigma}A$ can be turned into a functor $T_{\Sigma}: \text{Set} \to \text{Set}$ by inductively defining, for any function $f : A \to B$, the homomorphism $T_{\Sigma}f: T_{\Sigma}A \to T_{\Sigma}B$ as follows: for any $a \in A$, $(T_{\Sigma}f)(a) = f(a)$, and $\forall \text{op} : n \in \Sigma, \forall t_1, \ldots, t_n \in T_{\Sigma}A$,

$$T_{\Sigma}f(\text{op}(t_1, \ldots, t_n)) = \text{op}(T_{\Sigma}f(t_1), \ldots, T_{\Sigma}f(t_n)).$$

This becomes a monad by defining the unit $\eta_{\Sigma}^A : A \to T_{\Sigma}A$ as mapping $a \in A$ to the term $a$ in $T_{\Sigma}A$, and the multiplication $\mu_{\Sigma}^A : T_{\Sigma}(T_{\Sigma}A) \to T_{\Sigma}A$ as mapping a term built out of terms $t(t_1, \ldots, t_n)$ to the flattened term $t(t_1, \ldots, t_n)$. We call $(T_{\Sigma}, \eta_{\Sigma}, \mu_{\Sigma}^A)$ the term monad for $\Sigma$.

Proposition 2.7. For any signature $\Sigma$, Alg($\Sigma$) $\cong$ EM($T_{\Sigma}$).

For the rest of this paper, let $X$ be a fixed countable set of variables. An interpretation of $X$ (or variable assignment) in a $\Sigma$–algebra $A = (A, \Sigma)$ is a map $i : X \to A$. The interpretation extends to arbitrary $T_{\Sigma}X$ terms by inductively defining $[[\cdot\cdot\cdot]] : T_{\Sigma}X \to A$:

$$[x]^i = i(x) \quad \text{and} \quad [\text{op}(t_1, \ldots, t_n)]^i = \text{op}([t_1]^i, \ldots, [t_n]^i).$$

In cases where $i : X \to T_{\Sigma}X$ is an interpretation in a term algebra, we write $i'$ instead of $[[\cdot\cdot\cdot]]$ to emphasize that its action is straightforward. It can be seen as a completely syntactical rewriting procedure, as $i'$ takes a term in $T_{\Sigma}X$ and replaces all occurrences of $x$ with the term $i(x)$.

Definition 2.8 (Equations and their models). An equation over $\Sigma$ is a pair of $\Sigma$–terms over $X$, i.e., an element of $T_{\Sigma}X \times T_{\Sigma}X$ which we write as $s = t$. We say a $\Sigma$–algebra $A = (A, \Sigma)$ satisfies an equation $s = t$, denoted $A \vDash s = t$, if for any $i : X \to A$, $[s]^i = [t]^i$. We write $A \vDash s = t$ when the equality holds for a particular interpretation $i$. Given a set $E$ of equations over $\Sigma$, we denote by Alg($\Sigma, E$) the full subcategory of Alg($\Sigma$) of all algebras that satisfy all equations in $E$.

Definition 2.9 (Congruence). A congruence relation on a $\Sigma$–algebra $A = (A, \Sigma, A)$ is an equivalence relation $R \subseteq A^2$ such that for every op $: n \in \Sigma$, if $(a_1, b_1) \in R, \ldots, (a_n, b_n) \in R$ then it holds that $([\text{op}]_A(a_1, \ldots, a_n), [\text{op}]_A(b_1, \ldots, b_n)) \in R$. If $R$ is a congruence
then the interpretation of each \( op \in \Sigma \) is well-defined on the set \( A/R \) of \( R \)-equivalence classes, by:

\[
\llbracket op \rrbracket_{A/R}(\{a_1\}_R, \ldots, \{a_n\}_R) = \llbracket op \rrbracket_A(a_1, \ldots, a_n)
\]

Then we have the algebra \( h/R = (A/R, \llbracket \Sigma \rrbracket_{A/R}) \).

**Definition 2.10 (Term monad, with equations).** Let \( \Sigma \) be a signature, \( E \) a set of equations over \( \Sigma \), and \( A \) a set. Denote with \( \equiv_{E,A} \) the smallest congruence on the term algebra \( T_{\Sigma,E}A \) such that \( (T_{\Sigma,E}A)/\equiv_{E,A} \in \text{Alg}(\Sigma, E) \), i.e., \( (T_{\Sigma,E}A)/\equiv_{E,A} \) satisfies all equations in \( E \). We define \( T_{\Sigma,E} \), a variant of the term monad, that sends a set \( A \) to \( T_{\Sigma,E}A/\equiv_{E,A} \). Given a function \( f : A \to B \), we define the function \( T_{\Sigma,E}f : T_{\Sigma,E}A \to T_{\Sigma,E}B \) using the already defined \( T_{\Sigma,E}f \): for any \( t \in T_{\Sigma,E}A \), \( T_{\Sigma,E}f(t)\equiv_{E,B} = T_{\Sigma,E}f(t)\equiv_{E,A} \). One can check that \( T_{\Sigma,E}f \) is well-defined and makes \( T_{\Sigma,E} \) into a functor. In fact, it is a monad with unit \( \eta_{E,A} = a \mapsto \llbracket a \rrbracket_{E,A} \) and multiplication

\[
\mu_{E,A} = \llbracket \iota(\llbracket t_1\rrbracket_{E,A}, \ldots, \llbracket t_n\rrbracket_{E,A}) \rrbracket_{E,F_{\Sigma,E}} \mapsto \llbracket \iota(\llbracket t_1\rrbracket_{E,A}, \ldots, \llbracket t_n\rrbracket_{E,A}) \rrbracket_{E,F_{\Sigma,E}}.
\]

We call \( (T_{\Sigma,E}, \eta_{\Sigma,E}, \mu_{\Sigma,E}) \) the term monad for \( (\Sigma, E) \).

**Proposition 2.11.** For any signature \( \Sigma \) and any set \( E \) of equations over \( \Sigma \), \( \text{Alg}(\Sigma, E) \equiv \text{EM}(T_{\Sigma,E}) \).

A corollary of the above proposition is that the (\( \Sigma \), \( E \))-algebra over a set \( A \) is \( (T_{\Sigma,E}A/\equiv_{E,A}, \llbracket \Sigma \rrbracket_{A/R}) \), with the canonical interpretation of operations:

\[
\llbracket op \rrbracket_{T_{\Sigma,E}A/\equiv_{E,A}}(\{a_1\}_R, \ldots, \{a_n\}_R) = \llbracket op \rrbracket_{T_{\Sigma,E}A/\equiv_{E,A}}(\llbracket a_1\rrbracket_{E,A}, \ldots, \llbracket a_n\rrbracket_{E,A}).
\]

### 2.3 Generalized Metric Spaces

**Definition 2.12 (FRel).** A fuzzy relation on a set \( A \) is a map \( d : A \times A \to [0, 1] \). A morphism between two fuzzy relations \( (A, d) \) and \((B, \Delta)\) is a map \( f : A \to B \) that is nonexpansive (also referred to as \( 1 \)-Lipschitz) namely, \( \forall a, a' \in A, \Delta(f(a), f(a')) \leq d(a, a') \). We denote by \( \text{FRel} \) the category of fuzzy relations and nonexpansive maps.

Here is a non-exhaustive\(^1\) list of constraints on fuzzy relations that have been considered in the literature.

\[
\begin{align*}
\forall a, b \in A, \quad d(a, b) &= d(b, a) \quad (1) \\
\forall a \in A, \quad d(a, a) &= 0 \quad (2) \\
\forall a, b \in A, \quad d(a, b) = 0 \implies a = b \quad (3) \\
\forall a, b, c \in A, \quad d(a, c) \leq d(a, b) + d(b, c) \quad (4) \\
\forall a, b, c \in A, \quad d(a, c) \leq \max\{d(a, b), d(b, c)\} \quad (5)
\end{align*}
\]

Each has a somewhat standard name, (1) is symmetry, (2) is indiscernibility of identicals or reflexivity, (3) is identity of indiscernibles, (4) is triangle inequality, and (5) is strong triangle inequality. Restricting \( \text{FRel} \) to relations that satisfy a subset of the axioms above, we get many categories of interest whose objects were studied at least once in the literature.

\(^1\)A wider class of implicational constraints can be handled in our framework. We chose these five as running example since they are well-known.

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For example, metrics (\( \text{Met} \)) are fuzzy relations that satisfy axioms (1)–(4), pseudometrics (\( \text{PMet} \)) satisfy (1) and (4), and diffuse metrics (\( \text{DMet} \)) satisfy (1) and (4). Other examples include: quasimetrics (\( \text{QMet} \)), pseudquasimetrics (\( \text{PQMet} \)), metametrics (\( \text{MMet} \)), semimetrics (\( \text{SMet} \)), pseudosemimetrics (\( \text{PSMet} \)), ultrametrics (\( \text{UMet} \)). Different notions of morphisms between these objects have been considered (e.g., continuous functions, contracting maps, etc.) but, for our purposes, we will work with full subcategories of \( \text{FRel} \) and hence keep nonexpansiveness as the only condition on morphisms. This choice implies that isomorphisms of fuzzy relations are bijections that preserve distances. In the sequel, we write \( \text{GMet} \) for a category of generalized metric spaces, which can stand for any full subcategory of \( \text{FRel} \) satisfying a fixed subset of axioms (1)--(5).

All products and coproducts exist in \( \text{GMet} \) and are easy to define. Let \( ((A_i, d_i) \mid i \in I) \) be a non-empty family of generalized metric spaces. The product is \( (\Pi_{i \in I} A_i, \sup_{i \in I} d_i) \), with \( \sup_{i \in I} d_i : [(\Pi_{i \in I} A_i) \times (\Pi_{i \in I} A_i)] \to [0, 1] \) defined for \( \tilde{a}, \tilde{b} \in \Pi_{i \in I} A_i \) by:

\[
\sup_{i \in I} d_i(\tilde{a}, \tilde{b}) = \sup_{i \in I} d_i(\tilde{a}_i, \tilde{b}_i).
\]

We denote the sup–metric \( \sup_{i \in I} d_i \) just by \( d_\infty \) when the index set \( I \) is clear. The coproduct is given by \( \Pi_{i \in I} d_i : (\Pi_{i \in I} A_i) \to [0, 1] \), defined for \( a \in A_i \) and \( b \in A_k \) by:

\[
\Pi_{i \in I} d_i(a, b) = \begin{cases} 
  d_i(a, b) & \text{if } j = k \\
  1 & \text{otherwise}
\end{cases}
\]

The empty product, i.e., the terminal object, is given by \( d_1 : \{\ast\} \to [0, 1] \), defined by

\[
d_1(\ast, \ast) = \begin{cases} 
  0 & \text{if constraint (2) holds in } \text{GMet} \\
  1 & \text{otherwise}
\end{cases}
\]

The empty coproduct, i.e., the initial object, is the only possible fuzzy relation on the empty set (which vacuously satisfies all the axioms that must hold in \( \text{GMet} \)).

**Definition 2.13 (Isometric embedding).** A nonexpansive map \( f : (A, d) \to (B, \Delta) \) is an isometry if for any \( a, a' \in A, \Delta(f(a), f(a')) = d(a, a') \). An isometric embedding is an isometry that is injective.\(^2\) For any generalized metric space \( (A, d) \) and subset \( A' \subseteq A \), the inclusion \( i : (A', d|_{A'}) \to (A, d) \) is an isometric embedding.

\(^2\)In the category of metric spaces, any isometry is injective, but this is not true for all \( \text{GMet} \).
3 QUANTITATIVE REASONING WITH LIFTINGS

We introduce in this section our novel framework. In Subsection 3.1 we present the notion of liftings of signatures, the associated concept of quantitative \( \Sigma \)-algebras and define the classes \( \text{Alg}(\Sigma, S) \) definable by sets of implicational axioms. In Subsection 3.2 we define the syntactical deductive apparatus used to reason about equality and distance in quantitative \( \Sigma \)-algebras, and prove the soundness theorem, stating that the syntactic apparatus guarantees correct derivations.

3.1 Generalised Quantitative Algebras

In what follows, a given category \( \text{GMet} \) is fixed.

**Definition 3.1 (Lifting).** Given functors \( F : \text{Set} \to \text{Set} \) and \( L : \text{GMet} \to \text{GMet} \) we say that \( L \) is a lifting of \( F \) (from \( \text{Set} \) to \( \text{GMet} \)) if the following diagram commutes, where \( U \) is the expected forgetful functor:

\[
\begin{array}{ccc}
\text{GMet} & \xrightarrow{L} & \text{GMet} \\
\downarrow U & & \downarrow U \\
\text{Set} & \xrightarrow{F} & \text{Set}
\end{array}
\]

Hence, for any lifting \( L \), on objects we have \( L(A, d) = (F(A), d') \) for some \( d' \) which we denote with \( d' = L(d) \). We will interchangeably write \( L(A, d) \) and \( (F(A), L(d)) \).

**Definition 3.2.** A lifting \( L \) preserves isometric embeddings if, whenever \( f : (A, d) \to (B, \Delta) \) is an isometric embedding then \( L(f) : L(A, d) \to L(B, \Delta) \) is an isometric embedding.

Informally, this property holds when \( L \) is compatible with the operation of taking subspaces. In the rest of this paper, we will be only interested in liftings that preserve isometric embeddings and often just refer to them as liftings.

**Example 3.3.** Take as \( \text{GMet} \) the category \( \text{Met} \) of metric spaces. Consider the functor \( F = \text{id} \). Then, for any lifting \( L, (A, d) \mapsto (A, L(d)), \) so \( L(d) \) is a distance on \( A \). As examples of liftings of \( F \) preserving isometric embeddings, we list:

1. the identity: \( L(d)(a, a') = d(a, a') \),
2. the scaling: \( L(d)(a, a') = r \cdot d(a, a') \) for \( r \in (0,1) \),
3. the discrete distance: \( L(d)(a, a') = 1 \) if \( a \neq a' \).

Similarly, consider \( F = (-)^2 \), i.e., \( F(A) = A \times A \) and \( F(f) = f \times f \). In this case, \( L(d) \) is a distance on \( A \times A \). Examples of liftings preserving isometric embeddings include:

1. the standard product distance:
   \( L(d)((a_1, a'_1), (a_2, a'_2)) = \max\{d(a_1, a_2), d(a'_1, a'_2)\} \),
2. the discrete distance: \( L(d)((a_1, a'_1), (a_2, a'_2)) = 1 \) if \( (a_1, a'_1) \neq (a_2, a'_2) \).

We note, as in some of the examples above, that for any \( \text{GMet} \), if \( F \) is the \( n \)-ary product endofunctor \( (-)^n \) on \( \text{Set} \), then the \( n \)-ary product \( d_k \text{ in } \text{GMet} \) is a lifting of \( F \) preserving isometric embeddings. We refer to it as the sup-product lifting and denote it by \( L_\infty \). Accordingly, for \( n = 0 \), the lifting \( L_\infty \) maps any object to the terminal object in \( \text{GMet} \), and, for \( n = 1 \), the lifting \( L_\infty \) is the identity functor.

**Definition 3.4 (Nonexpansiveness up to lifting).** Let \( F : \text{Set} \to \text{Set} \) and \( L \) a lifting of \( F \). Let \( (A, d) \) and \( (B, \Delta) \) in \( \text{GMet} \). We say that a function \( f : F(A) \to B \) is nonexpansive up to \( L \) (or \( L \)-nonexpansive) if \( f : (F(A), L(d)) \to (B, \Delta) \) is nonexpansive (i.e., it is a morphism in \( \text{GMet} \)).

**Example 3.5.** As in the previous example, fix \( \text{GMet} = \text{Met} \) and consider \( F = \text{id} \). Consider the metric space \([0,1], d\), the unit interval with its standard Euclidean metric (i.e. \( d(x,y) = |x-y| \)), and the map \( f : F([0,1]) \to [0,1] \) defined as \( f(x) = x^2 \). If we take as lifting of \( F \) the identity lifting \( L \) from Example 3.3 (i.e., the lifting \( L_\infty \)) the function \( f \) is not \( L \)-nonexpansive because, e.g., \( \frac{1}{10} = d(\frac{1}{10}, 1) < d((\frac{1}{10})^2, 1^2) = \frac{1}{10} \). By contrast, if we take as lifting \( L \) the discrete lifting then \( f \) is trivially \( L \)-nonexpansive. In fact, any function \( f : [0,1] \to [0,1] \) is nonexpansive up to the discrete lifting.

We are now ready to introduce the concept of lifted signature, which extends the usual notion of signature \( \Sigma \) from universal algebra.

**Definition 3.6 (Lifted signature).** Given a signature \( \Sigma = \{ \text{op}_j : n_j \}_{i \in I} \) a lifting of \( \Sigma \) to \( \text{GMet} \) is a choice, for each \( i \in I \), of lifting \( L_{\text{op}_j} \) of the \( n_j \)-ary product \( (-)^{n_j} : \text{Set} \to \text{Set} \). An operation symbol \( \text{op}_j \) with arity \( n \) and associated lifting \( L_{\text{op}_j} \) is now denoted by \( \text{op}_j \). We write \( \hat{\Sigma} \) for lifted signatures \( \{ \text{op}_j : n_j : L_{\text{op}_j} \}_{i \in I} \) to clearly distinguish them from ordinary signatures.

Note that, given any signature \( \Sigma \), it is possible to obtain a lifted signature \( \hat{\Sigma} \) by choosing, for each \( \text{op}_j \), the sup-product lifting \( L_\infty \) of \( (-)^{n_j} \).

As in the classical case, any lifted signature \( \hat{\Sigma} \) gives rise to an endofunctor on \( \text{GMet} \) (denoted by \( \hat{\Sigma} \) too) lifting the endofunctor \( \Sigma \) with the following action:

\[
\hat{\Sigma}(A, d) := \bigsqcup_{\text{op}_j, \text{op}_j \in \Sigma} \hat{\text{L}}(\text{op}_j)(A, d) \quad \hat{\Sigma}(f) := \bigsqcup_{\text{op}_j, \text{op}_j \in \Sigma} \hat{\text{L}}(\text{op}_j(f)),
\]

**Definition 3.7 (Quantitative \( \hat{\Sigma} \)-algebra).** A quantitative \( \hat{\Sigma} \)-algebra is an algebra for the functor \( \hat{\Sigma} \). Equivalently, it is a generalised metric space \((A, d) \in \text{GMet}\) equipped with a set \( \hat{\text{Alg}}(\Sigma) \) of interpretations of operation symbols, as follows: every \( \text{op}_j \) is interpreted as a map \( \text{op}_j \) \( A^\Sigma \to A \) which is \( L_\infty \)-nonexpansive, i.e., such that

\[
\text{op}_j \mathbf{A} : (A^\Sigma, L_{\text{op}_j}(d)) \to (A, d) \text{ is nonexpansive}.
\]

We call \((A, d)\) the carrier space. A homomorphism between two quantitative \( \hat{\Sigma} \)-algebras with carrier spaces \((A, d) \) and \((B, \Lambda) \) is a nonexpansive map \( f : \text{op}_j \mathbf{A} \) preserving all operations, i.e., \( \text{Vop} : n : \text{op}_j \mathbf{A} \in \hat{\Sigma} \mathbf{v}_1, \ldots, n : \text{op}_j \mathbf{A} \in A \),

\[
f(\text{op}_j \mathbf{A}(a_1, \ldots, a_n)) = \text{op}_j \mathbf{B}(f(a_1), \ldots, f(a_n)).
\]

The category of quantitative \( \hat{\Sigma} \)-algebras is denoted by \( \text{Alg}(\hat{\Sigma}) \).

We remark that, in the particular case of \( \text{GMet} = \text{Met} \) and \( \hat{\Sigma} \) being the sup–product lifting of some signature \( \Sigma \), the notion of quantitative \( \hat{\Sigma} \)-algebra coincides with that of quantitative algebra for the signature \( \Sigma \) of the framework of [11].

Any quantitative \( \hat{\Sigma} \)-algebra yields a \( \Sigma \)-algebra by applying the forgetful functor to \( U : \text{GMet} \to \text{Set} \) because \( U[\text{op}_j \mathbf{A}] \) has type...
A^n → A and morphisms in Alg(Σ) are already Σ-algebra homomorphisms. We obtain the following commutative square of forgetful functors.

\[
\begin{array}{c}
\text{Alg(Σ)} \longrightarrow \text{GMet} \\
\downarrow \quad \downarrow \\
\text{Alg(Σ)} \longrightarrow \text{Set}
\end{array}
\]

Definition 3.8 (Equations). Given a quantitative Σ-algebra \( A := (A,d,[\Sigma]) \) and an equation \( e \in T_\Sigma X \times T_\Sigma X \times [0,1] \), we say that \( A \) satisfies \( e \), denoted \( A \vdash e \), if its underlying \( \Sigma \)-algebra satisfies \( e \).

Definition 3.9 (Quantitative equation). A quantitative equation in the signature \( \Sigma \) is an element \( e \in T_\Sigma X \times T_\Sigma X \times [0,1] \), i.e. a triple comprising two \( \Sigma \)-terms \( s \) and \( t \) and a real number \( \varepsilon \in [0,1] \). We denote it by \( s =_\varepsilon t \). We say that \( A := (A,d,[\Sigma]) \) satisfies \( s =_\varepsilon t \), denoted \( A \vdash s =_\varepsilon t \), if for any variable assignment \( t : X \rightarrow A \), \( d([s]) = [t] \leq \varepsilon \). We write \( A \vdash s =_\varepsilon t \) when the inequality holds for a particular assignment \( t \).

Let \( \mathcal{V}_\Sigma X = T_\Sigma X \times T_\Sigma X \times T_\Sigma X \times T_\Sigma X \times [0,1] \) denote the set of equations and quantitative equations over the signature \( \Sigma \) and variables \( X \). We use the letter \( \phi \) to range over \( \mathcal{V}_\Sigma X \).

Following [11], we will consider classes of \( \Sigma \)-algebras axiomatised by (quantitative) equational implications, rather than just (quantitative) equations. While this level of generality is not required in many applications, as several useful examples (see Section 5) are purely (quantitative) equational, it allows for a direct comparison of our results and those of [3, 11].

Definition 3.10 (Horn clauses). In the sequel, \( \mathcal{H}_2(X) = \mathcal{P}(\mathcal{V}_2 X) \times \mathcal{V}_2 X \) denotes the set of (possibly infinitary) Horn clauses over the signature \( \Sigma \) and variables \( X \). A Horn clause \( H \in \mathcal{H}_2(X) \) is written \( \bigwedge_{i \in I} \phi_i \Rightarrow \psi \) as its intended semantics is that \( \psi \) holds whenever each \( \phi_i \) holds. More formally, we say that an algebra \( A := (A,d,[\Sigma]) \in \text{Alg}(\Sigma) \) satisfies a clause \( H = \bigwedge_{i \in I} \phi_i \Rightarrow \psi \), denoted \( A \vdash H \), if for any variable assignment \( t : X \rightarrow A \), \( A \vdash t \phi_i \) whenever \( A \vdash t \phi_i \) for every \( i \). We write \( A \vdash t \phi \) whenever the implication is true for a particular assignment \( t \). We call \( H = \bigwedge_{i \in I} \phi_i \Rightarrow \psi \) basic if each premise \( \phi_i \) is a (quantitative) equation between variables: \( \phi_i \) is either of the form \( x = y \) or \( x =_\varepsilon y \), for \( x,y \in X \) and \( \varepsilon \in [0,1] \).

Given a set \( S \subseteq \mathcal{H}_2(X) \), we let \( \text{Alg}(\Sigma,S) \) denote the full subcategory of \( \text{Alg}(\Sigma) \) containing all quantitative \( \Sigma \)-algebras that satisfy all clauses in \( S \).

### 3.2 Syntactic Apparatus for Quantitative Reasoning

Following [11], we now introduce a logical apparatus for reasoning about quantitative \( \Sigma \)-algebras. We use the following notation to improve readability: for a set \( \Sigma \) of Horn clauses \( \vdash \subseteq \mathcal{H}_2(X) \) we write \( \{\phi_i\}_{i \in I} \vdash \phi \) to mean that the Horn clause \( \bigwedge_{i \in I} \phi_i \Rightarrow \phi \) belongs to the set \( \Sigma \).

Definition 3.11. A quantitative theory over \( \Sigma \) is a set of Horn clauses \( \Sigma \subseteq \mathcal{H}_2(X) \) such that conditions (I)–(VI) hold:

(I) \( \vdash \) is closed under the following inference rules for any \( \Gamma, \Gamma' \subseteq \mathcal{V}_\Sigma X \), \( \phi, \psi \in \mathcal{V}_\Sigma X \) and substitution \( \sigma : X \rightarrow \mathcal{V}_\Sigma X \):

\[
\begin{array}{c}
\Gamma \vdash \phi \\
\forall \phi \in \Gamma, \Gamma \vdash \phi \\
\Gamma \vdash \psi \\
\phi \in \Gamma \\
\Gamma \vdash \phi
\end{array}
\]

Sub

Cut

Hyp

(II) \( \vdash \) contains, for any \( \phi \in \mathcal{V}_\Sigma X \) and \( x,y,z \in X \), the clauses:

(Refl) \( \theta \vdash \theta \)

(Sym) \( x = y \vdash y = x \)

(Trans) \( x = y, y = z \vdash x = z \)

(App) \( \{x_i = y_i : i \in 1,\ldots,n\} \vdash \text{op}(\overline{x}) = \text{op}(\overline{y}) \)

(III) \( \vdash \) contains, for any \( x,y \in X, \varepsilon, \epsilon' \in [0,1] \), the clauses:

(1-bdd) \( \theta \vdash x =_1 y \)

(Max) \( x =_\varepsilon y \vdash x =_{\epsilon'} y \)

(Arch) \( x =_\varepsilon y \vdash x \leq_{\epsilon'} y \)

(IV) \( \vdash \) contains, for any \( x,y,z \in X, \epsilon \in [0,1] \), the clauses:

(Comp_{x}) \( x = y, x = z \vdash y = z \)

(Comp_{y}) \( x = y, y = z \vdash x = z \)

(V) depending on the notion of \( \text{GMet} \) used, \( \vdash \) contains an appropriate subset of the following clauses for any \( x,y,z \in X, \epsilon, \epsilon' \in [0,1] \):

\[
\begin{array}{c}
x =_\varepsilon y \lor y =_\varepsilon x \\
\theta \vdash x =_0 y \\
x \vdash 0 \vdash x =_0 y \\
x =_\varepsilon y \lor y =_{\epsilon'} x \\
x =_{\epsilon'} x \lor x =_{\epsilon'} y \\
x =_{\epsilon'} x \lor x =_{\epsilon'} y
\end{array}
\]

(VI) \( \vdash \) is closed under the following inference rule, for any \( \phi \in \mathcal{V}_\Sigma X \) and for any set \( \overline{x} \subseteq \overline{y} = \{x_1,\ldots,x_n, y_1,\ldots,y_n\} \) of up to 2n variables (not necessarily distinct):

\[
\{w =_{\Delta(x,w,z)} z : w,z \in \overline{x} \cup \overline{y}\} \vdash \text{op}(\overline{x}) =_{\delta} \text{op}(\overline{y})
\]

Condition (I) is standard and reflects the semantics of \( \vdash \) as the theory of universally quantified implications. Condition (II) includes the standard axioms of equational logic, thus (I)+(II) allows to perform equational reasoning regarding equations \( (s =_\varepsilon t) \). Condition (III) poses the constraints on quantitative equations \( (s =_\varepsilon t) \) ensuring the intended semantics: \( \Delta(s,t) \leq \epsilon \), for any fuzzy relation \( d \in \text{FRel} \). Condition (IV) adds two axioms governing the logical interplay between equality (=) and the quantitative relations \( =_\varepsilon \). It expresses the fact that equality is a congruence relation (both on the left and the right argument) for the relation \( =_\varepsilon \), for all \( \epsilon \in [0,1] \). It it possible to formulate axioms defining each category of generalised metric spaces \( \text{GMet} \). Finally, Condition (VI) expresses the property that, for any \( \phi \in \mathcal{V}_\Sigma X \), the operation \( \text{op} \) is \( \text{Lop} \)-nonexpansive. The Horn clause introduced has up to \( 2n^2 \) premises: quantitative equations of the form \( w =_{\Delta(w,z)} z \), where \( \Delta(w,z) \) is a number in \([0,1]\), for each choice of \( w,z \in \overline{x} \cup \overline{y} \). We can see these numbers as defining a fuzzy relation \( \Delta : \overline{x} \cup \overline{y} \rightarrow [0,1] \).
(\(\mathcal{X} \cup \mathcal{Y}, A\)) is a GMet space. This is therefore a constraint on the \((2n)^2\) values \(\Delta(w, z)\). If the proviso is satisfied, since \(L\) is a GMet lifting, 
\(L_{op}(\mathcal{X} \cup \mathcal{Y}, A)\) is a GMet space too and the value in the quantitative equation in the conclusion (i.e., \(\delta = L_{op}(\Delta(\mathcal{X}, \mathcal{Y}))\) is defined.

**Example 3.12.** In order to improve readability when displaying instances of the \(L\)-NE rule, we will often omit some of the \((2n)^2\) premises \(w = \Delta(w, z)\) when \(\Delta(w, z)\) is implicitly understood from the context. For instance, consider the case \(GMet = Met\) and a binary operation \(\circ : 2 \times L\) with \(L\) the sup–product lifting. An instance of the \(L\)-NE rule

\[
\begin{array}{c}
x_1 =_{\varepsilon_1} y_1, x_2 =_{\varepsilon_2} y_2 \\
\end{array}
\]

implicitly assuming all other premises to be of the form \(w =_{\varepsilon} z\), where \(w \neq z\), and \(w =_{\varepsilon} z\), otherwise, for \(w, z \in \mathcal{X} \cup \mathcal{Y}\). Thus, the fuzzy relation on \(\mathcal{X} \cup \mathcal{Y}\) described by these premises is:

\[
\begin{array}{c|c|c}
0 & x_1 & y_1 \\
\hline
1 & x_2 & y_2 \\
0 & 1 & 0
\end{array}
\]

which indeed satisfies the axioms of \(Met\). Since we are considering the sup–product lifting, \(\delta = L_{op}(\Delta((x_1, x_2), (y_1, y_2))) = \max\{\varepsilon_1, \varepsilon_2\}\). Hence all provisos of the \(L\)-NE rule are satisfied, meaning that this is a valid instance of the \(L\)-NE rule.

**Definition 3.13.** Given a set of clauses \(S \subseteq H_{\Sigma}(x)\), we let \(\Sigma_{\mathcal{S}}\) denote the smallest quantitative theory containing \(S\), and refer to it as the GMet quantitative theory axiomatised by \(S\).

Our first main result is the soundness theorem, stating that if a Horn clause \(H\) is derivable in the deductive apparatus from an axiom set \(S\) of Horn clauses (i.e., \(H\) is in the quantitative theory axiomatised by \(S\)), then indeed \(H\) holds true in any \(\Sigma\)-algebra satisfying the axioms \(S\).

**Theorem 3.14 (Soundness).** Let \(A = (A, d, [\Sigma]) \in Alg(\Sigma, S)\) and \(H \in \Sigma_{\mathcal{S}}\). Then \(A \vdash H\).

**Proof.** We show each rule in Definition 3.11 is valid in \(A\).

(I) The inference rules Sub, Cut and Hyp are valid by purely logical arguments because the semantics of clauses \(\{\phi_i\}_{i \in I} \vdash_L \phi\)

are universally quantified implications:

\(\forall x (\land_{i \in I} \phi_i \Rightarrow \phi)\).

(II) The clauses Refl, Sym, Trans and App are valid because equality (=) is an equivalence relation and is trivially compatible with all operations \(op \in \Sigma\) (i.e., it is a congruence).

(III) The clauses 1-bdd, Max and Arch are valid because the distance \(d\) has type \(d : A \times A \rightarrow [0,1]\) and the interpretation of quantitative equations \(x =_{\varepsilon} y\) is \(d(i(x), i(y)) \leq \varepsilon\) for any variable assignment \(i : X \rightarrow A\).

(IV) The rules \(Comp_p\) and \(Comp_q\) are valid because equality (=) is trivially a congruence for all relations \(=_{\varepsilon}\).

(V) The clauses corresponding to axioms in \(GMet\) are valid because

\(A = (A, d, [\Sigma]) \in GMet\), and the interpretation of the Horn clauses (universally quantified implications) coincides with the axioms of \(GMet\) as stated in Section 2.3.

(VI) The inference \(L\)-NE has a proviso \(((\mathcal{X} \cup \mathcal{Y}, A) \in GMet)\) stating that the finite set \(\mathcal{X} \cup \mathcal{Y}\) of variables endowed with the distances \(\Delta(w, z)\), for \(w, z \in \mathcal{X} \cup \mathcal{Y}\), is a GMet space. Assume this as hypothesis. Since \(op : n : L_{op} \in \Sigma\), we know that \(L_{op}\) is a lifting on GMet. Therefore the set \(\Gamma\) equipped with the distance \(L_{op}(\Delta)\) is an element of GMet. Hence, the numerical value

\[\delta = L_{op}(\Delta((x_1, \ldots, x_n), (y_1, \ldots, y_n)))\]

is defined. We need to prove that the Horn clause

\[
\{ w = \Delta(w, z) | w, z \in \mathcal{X} \cup \mathcal{Y} \} \vdash op(\mathcal{x}) =_\delta op(\mathcal{y})
\]

holds in \(A\). Let \(i : X \rightarrow A\) be an assignment and assume that, for all \(w, z \in \mathcal{X} \cup \mathcal{Y}\),

\[d([w]^i, [z]^i) \leq \Delta(w, z)\]

As consequence, the map

\[f : (\mathcal{X} \cup \mathcal{Y}, A) \rightarrow (A, d) = w \mapsto [w]^i\]

is nonexpansive. Thus, the lifting

\[L_{op}(f) : L_{op}(\mathcal{X} \cup \mathcal{Y}, A) \rightarrow L_{op}(A, d)\]

i.e.,

\[L_{op}(f) : (\mathcal{X} \cup \mathcal{Y}, A) \rightarrow (A^n, L_{op}(d))\]

is also nonexpansive, and we have the following derivation which implies (6), i.e., the validity of the conclusion:

\[d([op(\mathcal{x})]^i, [op(\mathcal{y})]^i)\]

\[\leq L_{op}(d)([([x_1]^i, \ldots, [x_n]^i], [y_1]^i, \ldots, [y_n]^i])\]

\[\leq L_{op}(d)(L_{op}(f)(\mathcal{x}), L_{op}(f)(\mathcal{y}))\]

\[\leq L_{op}(\Delta(\mathcal{x}, \mathcal{y}))\]

\[\leq \delta\]

where \(\Delta\) applies the fact that \([op]\) is \(L_{op}\)-nonexpansive, (B) follows as \(L_{op}(f)\) applies \(f\) pointwise to \(n\)-ary tuples, and (C) uses nonexpansiveness of \(L_{op}(f)\) from (7).

**4 TERM MONAD AND FREE QUANTITATIVE ALGEBRAS**

Given a lifted signature \(\Sigma\) and a set of Horn clauses \(S\) axiomatising a theory \(\Sigma_{\mathcal{S}}\), we describe in Subsection 4.1 the construction of the term \((\Sigma, S)\)-algebra (denoted by \(T_{\Sigma, S}(A, d)\)) on a given GMet space \((A, d)\). We then show (Theorem 4.6) how this yields a monad \(T_{\Sigma, S}\) on GMet.

Next, in Subsection 4.2 we show two main results regarding this monad. First (Theorem 4.7), for any given \((A, d) \in GMet\), the algebra \(T_{\Sigma, S}(A, d)\) is the free algebra in \(Alg(\Sigma, S)\) generated by \((A, d)\). Second (Theorem 4.8), if all Horn clauses in \(S\) are basic (see Definition 3.10), then \(Alg(\Sigma, S) \cong EM(T_{\Sigma, S})\).

The definition of the monad \(T_{\Sigma, S}\) and the proof techniques used to establish the two theorems are inspired by those of [3, 11]). In fact, the latter can be seen as special instances, in our framework, when \(GMet = Met\) and all liftings in \(\Sigma\) are sup–product liftings.
4.1 The Term Monad

Fix a quantitative term ⊩ over a lifted signature $\Sigma$. The construction of the term monad is done via several steps. First, we consider the set of ground terms, i.e., the set of terms without variables $(T_\Sigma \emptyset)$ and we define on them a congruence $\equiv$, and a fuzzy relation $d_\Sigma$ induced by the equations and quantitative equations in $\Sigma$. We then show in Lemma 4.2 how these allow us to build a quantitative $\Sigma$-algebra over quotiented $T_\Sigma \emptyset$ terms.

**Definition 4.1.** We let $E(\vdash)$ (resp. $QE(\vdash)$) be the set of equations (resp. quantitative equations) over $T_\Sigma \emptyset$ that are conclusions of Horn clauses in $\vdash$ having no premises. Formally:

$E(\vdash) = \{ s = t \mid \emptyset \vdash s = t, \text{for } s, t \in T_\Sigma \emptyset \}$

$QE(\vdash) = \{ s =_\Sigma t \mid \emptyset \vdash s =_\Sigma t, \text{for } s, t \in T_\Sigma \emptyset \}$

Based on these, we define the following relation and fuzzy relation over $T_\Sigma \emptyset$:

$\equiv \subseteq T_\Sigma \emptyset \times T_\Sigma \emptyset \times \Delta \times [0, 1], \quad d_\Sigma (s, t) = \inf \{ \epsilon \mid s =_\Sigma t, \epsilon \in QE(\vdash) \}$

**Lemma 4.2.** The following hold:

1. The relation $\equiv$ is an equivalence relation on $\Sigma$-terms without variables and is compatible with all operations.
2. $(T_\Sigma \emptyset/\equiv, [\Sigma])$ is the free $\Sigma$-algebra on the empty set, with carrier $T_\Sigma \emptyset/\equiv$, and operations $[\Sigma]$: $\big[\text{op}([t_1], \ldots, [t_n])\big] = \text{op}([t_1], \ldots, [t_n])$.
3. The fuzzy relation $d_\Sigma$ satisfies the following properties:
   a) $d_\Sigma (s, t) \leq \epsilon$ if and only if $(s =_\Sigma t) \in E(\vdash)$
   b) $d_\Sigma$ preserves the equivalence $\equiv$, i.e., $d_\Sigma$ is well-defined on $\equiv$-equivalence classes:
      
5. $(T_\Sigma \emptyset/\equiv, d_\Sigma)$ is a $\text{GMet}$ space.

6. $\Sigma$ satisfies all the clauses in $\vdash$.

**Proof.** All points are enforced by the presence of certain rules and clauses in the syntactic proof system, and the fact that $\vdash$, being a theory, is closed under them. Item 1 follows by Refl, Sym, Trans and App. Item 2 follows from the characterisation of free $\Sigma$-algebras from Subsection 2.2. Item 3a follows from Max and Arch, and 3b from Comp$_r$ and Comp$_l$. Item 4 follows from the axioms in (V) corresponding to $\text{GMet}$.

Item 5 is enforced by the $L$-$\text{NE}$ rule. We discuss this case in greater detail. We need to show that, for any op : $n : L_\text{op} \in \Sigma$, the interpretation $[\text{op}]$ is $L_\text{op}$-nonexpansive. This means checking that:

$[\text{op}] : \{ (T_\Sigma \emptyset/\equiv)^n, L_\text{op}(d_\Sigma) \} \to (T_\Sigma \emptyset/\equiv, d_\Sigma)$

is nonexpansive, i.e., that for any $\tilde{s} = (s_1, \ldots, s_n)$ and $\tilde{t} = (t_1, \ldots, t_n)$ in $(T_\Sigma \emptyset)^n$,

$d_\Sigma ([\text{op} (\tilde{s})], [\text{op} (\tilde{t})]) = L_\text{op}(d_\Sigma (\tilde{s}, \tilde{t})). \quad (8)$

Using the Sub rule, we instantiate the $L$-$\text{NE}$ rule with premises $p = \Delta(p, q)$ for $p, q \in \{ s_1, \ldots, s_n \} \cup \{ t_1, \ldots, t_n \}$ where $\Delta(p, q) = d_\Sigma (p, q)$. This set of premises satisfies the proviso of the $L$-$\text{NE}$ rule, since $d_\Sigma$ is a $\text{GMet}$ relation (Item 4). Hence, because all premises are in $\text{QE}(\vdash)$ (Item 3a), also the quantitative equation in the conclusion of the $L$-$\text{NE}$ rule is in $\text{QE}(\vdash)$ (apply Cut):

$\text{op} (\tilde{t}) \equiv L_\text{op}(\Delta (\tilde{s}, \tilde{t})) \text{ op} (\tilde{t}) \in \text{QE}(\vdash) \quad (9)$

Now, since we have the isometric embedding

$$\left((\{ s_1, \ldots, s_n \} \cup \{ t_1, \ldots, t_n \}, \Delta) \hookrightarrow (T_\Sigma \emptyset, d_\Sigma) \right),$$

and $\text{Lop}$ preserves isometric embeddings, this implies

$L_\text{op}(\Delta (\tilde{s}, \tilde{t})) = L_\text{op}(d_\Sigma (\tilde{s}, \tilde{t}))$,

and thus by Item 3a and (9), we conclude (8) holds.

Lastly, we show Item 6. Let $i : X \to T_\Sigma \emptyset/\equiv$, be an interpretation of the variables. Define a substitution $\sigma : X \to T_\Sigma \emptyset$ mapping $x$ to one representative in $i(x)$. We first show that for any $\phi \in V_\Sigma X$,

$I \vdash^1 \phi \iff \phi (\sigma)$ (10)

where we recall that $\sigma$ denotes the extension of $\sigma$ to terms (see discussion after Proposition 2.7). We prove the case $\phi = s =_\Sigma t$ (a similar argument works when $\phi$ is not quantitative).

$I \vdash^1 \phi \iff d_\Sigma ([\sigma (s)], [\sigma (t)]) \leq \epsilon \quad \text{def. of } \vdash^1$

$\iff d_\Sigma (\sigma (s), \sigma (t)) \leq \epsilon \quad \text{def. of } \equiv$

$\iff d_\Sigma (\sigma (s), \sigma (t)) \leq \epsilon \quad \text{Item 3b}$

$\iff \emptyset \vdash \sigma (s) \approx \sigma (t) \quad \text{Item 3a}$

This concludes the proof of (10). We can now derive Item 6. Suppose $(\phi_i)_{i \in I} \vdash^1 \phi$. Applying Sub, we find $(\sigma (\phi_i))_{i \in I} \vdash^1 \sigma (\phi)$. If $I \vdash^1 \phi_i$ for each $i \in I$, by (10) we have $\emptyset \vdash \sigma (\phi_i)$ for every $i \in I$. Therefore, we can apply Cut to infer $\emptyset \vdash \sigma (\phi)$, and consequently $I \vdash^1 \phi$, again using (10).

We remark that the last step of the proof of Item 5 uses the technical assumption that liftings $L_\text{op}$ preserve isometric embeddings. In contrast, the proof of Theorem 3.14 (soundness) can be carried out without this hypothesis. Therefore, this technical assumption is not needed to reason syntactically about equality and distance in quantitative algebras but is required to ensure that the construction of the term algebra (à la [11]) is valid. It is also used in the proof of Theorem 4.8.

Now, given a $\text{GMet}$ space $(A, d)$, we aim at defining a $\Sigma$-algebra over terms generated from $A$ (i.e., $T_\Sigma A$ instead of $T_\Sigma \emptyset$), taking into account the distance on $A$ given by $d$. We do so via an extension of the theory $\vdash$.

**Definition 4.3 (Theory Extension).** Given a $\text{GMet}$ space $(A, d)$, a lifted signature $\tilde{\Sigma}$ and a theory $\vdash$ over $\tilde{\Sigma}$, we define:

- a new lifted signature $\tilde{\Sigma}_A = \tilde{\Sigma} \cup \{ a : 0 : L_a \mid a \in A \}$,
- where we add a fresh constant $a$ (of arity 0) for each element $a \in A$ where $L_a = L_\Sigma$ is the 0-ary sup–product lifting from Example 3.3. Note that we can identify $T_\Sigma A (\Sigma$-terms with variables in $A)$ with $T_\Sigma A (\Sigma (A)$-terms without variables).
- a new theory $\vdash_A$ over $\tilde{\Sigma}_A$ defined as the $\text{GMet}$ theory generated by the set of clauses

$$\vdash_A \cup \{ a + a = a = (a, a) \mid (a, a) \in A \times A \},$$
i.e., all the clauses in \( \vdash \) and new ones describing the distances between the new constants in \( A \).

We refer to \( \Sigma_A \) as the signature \( \Sigma \) extended by the GMet space \( (A, d) \). Similarly, \( \vdash_A \) is the theory \( \vdash \) extended by \( (A, d) \).

In what follows, we fix an axiom set of Horn clauses \( S \) and the associated \( \Sigma \)-theory \( \vdash_S \) axiomatised by \( S \). Its extension by a GMet space \( (A, d) \) is the theory \( \vdash_{SA} \) over \( \Sigma_A \) whose term algebra (as in Lemma 4.2, Item 5) is

\[
(\Sigma_A, \emptyset/\vdash_{SA}, d_{SA}, [\Sigma_A])
\]

or, identifying \( \Sigma_A, \emptyset \) with \( \Sigma_A \), the \( \Sigma_A \)-algebra

\[
(\vdash_A/\vdash_{SA}, d_{SA}, [\Sigma_A])
\]

We can turn this into a \( \Sigma \)-algebra

\[
(\vdash_A/\vdash_{SA}, d_{SA}, [\Sigma_A])
\]

by forgetting the interpretations \([a]\) of all constants \( a \in A \). Moreover, this is also a \( (\Sigma, S) \)-algebra as it satisfies all the clauses in \( \vdash_S \), by Item 6 of Lemma 4.2. Since the set of Horn clauses \( S \) is fixed, we introduce the following shortcuts to ease the notation:

\[
\equiv_A := \equiv_{\vdash_{SA}} \quad \tilde{T}_{\Sigma,S} := \tilde{T}_{\Sigma} / \equiv_A \quad \tilde{T}_{\Sigma,S}^d := d_{\vdash_{SA}}
\]

so that, for any \( (A, d) \), \( (\tilde{T}_{\Sigma,S}^d, \tilde{T}_{\Sigma,S}) \) is a \( (\Sigma, S) \)-algebra, we refer to it as the term \( (\Sigma, S) \)-algebra generated by \( (A, d) \).

The construction of the term algebra allows us to prove the following equation3 completeness result.

**Theorem 4.4 (Completeness).** If \( \phi \in \mathcal{V}_E X \) is satisfied in all algebras of \( \text{Alg}(\Sigma, S) \), then \( \vdash_S \phi \).

**Proof.** Let \( x_1, \ldots, x_n \) be the variables appearing in \( \phi \) and \( F = \{ \overline{x}_1, \ldots, \overline{x}_n \} \) be copies of these variables. We denote by \( (F, d_1) \) the discrete generalised metric space, i.e., \( d_1(\overline{x}_i, \overline{x}_j) = 1 \) for all \( \overline{x}_i, \overline{x}_j \) (except for \( \overline{x}_i = \overline{x}_j \) when \( 2 \) holds in GMet). Recall that when extending the theory \( \vdash_S \) with \( (F, d_1) \) (see Definition 4.3), no new clause is added by the discrete metric \( \emptyset \vdash S \vdash \_ \leq \_ \) is derivable in any quantitative theory by (1-bdd). Take any substitution \( \sigma : X \rightarrow \tilde{T}_{\Sigma,F} \), \( \emptyset \) sending each variable \( x_i \) in \( \{ x_1, \ldots, x_n \} \) to the constant \( \overline{x}_i \), it is possible to show

\[
\emptyset \vdash_{SF} \sigma^*(\phi) \equiv \emptyset \vdash_S \phi.
\]

(11)

In \( \sigma^*(\phi) \), each occurrence of the variable \( x_i \) is replaced by the constant \( \overline{x}_i \). Thus, in words, (11) says that anything derivable about universally quantified variables is also derivable about constants and vice-versa. It is crucial that we did not add any clause in \( \vdash_{SF} \), we only added the set of constants \( F \) to the signature.

Consider the algebra

\[
T_F = (\Sigma, \emptyset/\vdash_F, d_{SF}, [\Sigma])
\]

that is, the term algebra (as in Lemma 4.2, Item 5) for the theory \( \vdash_{SF} \) over \( \Sigma_F \). Let \( t : X \rightarrow \tilde{T}_F / \equiv_{SF} \) be the interpretation of the variables defined as \( dx(x) = [\sigma(x)] = e_F \). In particular, \( t \) sends \( x_i \) to \([\overline{x}_i] = e_F \). By applying (10) from the proof of Lemma 4.2 and (11) from above, we have

\[
T_F \vdash t^* \phi \iff \emptyset \vdash_{SF} \sigma^*(\phi) \iff \emptyset \vdash_S \phi.
\]

(12)

We are now ready to prove the statement of the theorem. Suppose that \( \phi \in \mathcal{V}_E X \) is satisfied in all algebras of \( \text{Alg}(\Sigma, S) \). Then in particular it is satisfied by the term \( (\Sigma, S) \)-algebra generated by \( (F, d_1) \)

\[
T = (T_F / \vdash_{SF}, d_{SF}, [\Sigma])
\]

with assignment \( t \). Since \( T \) is obtained from \( T_F \) by forgetting the interpretation of the constants in \( F \), and since \( \phi \) does not contain any such constant, we have

\[
T \vdash t^* \phi \iff T_F \vdash t^* \phi.
\]

(13)

Therefore we conclude by (12) that \( \emptyset \vdash_S \phi \).

\[\Box\]

The assignment

\[
(A, d) \mapsto (\tilde{T}_{\Sigma,S}^d, \tilde{T}_{\Sigma,S}, [\Sigma])
\]

can be turned into a functor \( \tilde{T}_{\Sigma,S} \) : \( \text{GMet} \rightarrow \text{Alg}(\Sigma, S) \) by defining, for each nonexpansive map \( f : (A, d) \rightarrow (B, \Lambda) \),

\[
\tilde{T}_{\Sigma,S}(f) := [t]_{\equiv_{SA}} \mapsto [t(f), \ldots, f(\overline{x}_n)]_{\equiv_{SB}}
\]

which is equivalent to

\[
\tilde{T}_{\Sigma,S}(f) := \{ t(a_1, \ldots, a_n) \}_{\equiv_{SB}} \mapsto \{ t(f(a_1), \ldots, f(\overline{x}_n)) \}_{\equiv_{SB}}
\]

To check that \( \tilde{T}_{\Sigma,S} \) is indeed a functor, one needs to verify that \( \tilde{T}_{\Sigma,S}(f) \) is well-defined on equivalence classes, nonexpansive, and commutes with operations in \( \Sigma \), and that \( \tilde{T}_{\Sigma,S} \) preserves composition. The following lemma implies the first two properties.

**Lemma 4.5.** Let \( f : (A, d) \rightarrow (B, \Lambda) \) be an arrow in \( \text{GMet} \). For all \( s, t \in \tilde{T}_{\Sigma,A} \),

\[
[s]_{\equiv_{SA}} = [t]_{\equiv_{SB}} \Rightarrow [\tilde{T}_{\Sigma,F}(s)]_{\equiv_{SB}} = [\tilde{T}_{\Sigma,F}(t)]_{\equiv_{SB}}
\]

\[
\tilde{T}_{\Sigma,S}(d([s]_{\equiv_{SA}}, [t]_{\equiv_{SA}})) \leq e \Rightarrow [\tilde{T}_{\Sigma,F}(d([s]_{\equiv_{SA}}, [t]_{\equiv_{SA}}))]_{\equiv_{SB}} \leq e
\]

The commutation with operations and preservation of composition follow from the fact that \( \tilde{T}_{\Sigma,F} \) commutes with the operations in \( \Sigma \) and \( \tilde{T}_{\Sigma,F} \) preserves composition.

Hence \( \tilde{T}_{\Sigma,S} : \text{GMet} \rightarrow \text{Alg}(\Sigma, S) \) is indeed a functor. It can be turned, by application of the forgetful functor (every algebra in \( \text{Alg}(\Sigma, S) \) is a GMet space), to a functor of type

\[
\tilde{T}_{\Sigma,S} : \text{GMet} \rightarrow \text{GMet}
\]

The latter can be given the structure of a monad on GMet by defining unit \( \tilde{\eta}_{(A, d)} : (A, d) \rightarrow \tilde{T}_{\Sigma,S}(A, d) \) and multiplication \( \tilde{\mu}_{(A, d)} : \tilde{T}_{\Sigma,S}(\tilde{T}_{\Sigma,S}(A, d)) \rightarrow \tilde{T}_{\Sigma,S}(A, d) \) as follows:

\[
\tilde{\eta}_{(A, d)} : a \mapsto [a]_{\equiv_{SA}}
\]

\[
\tilde{\mu}_{(A, d)} : [t_1]_{\equiv_{SA}}, \ldots, [t_n]_{\equiv_{SA}} \mapsto [t_1, \ldots, t_n]_{\equiv_{SA}}
\]

It can be verified that these maps are nonexpansive and well-defined, and that they satisfy the conditions in Definition 2.1. Therefore, we can state:

---

3 We remark that a stronger statement (completeness with respect to Horn clauses) is in [11, Section 5]. We have not yet checked if their stronger completeness result holds in our new apparatus.
THEOREM 4.6. \( (\hat{T}_{Σ̃}, \hat{η}, \hat{µ}) \) is a monad on \( \text{GMet} \).

4.2 Freeness and Isomorphism Theorems

We are now ready to prove that \( \hat{T}_{Σ̃}(A, d) \) is free.

THEOREM 4.7. Let \( (A, d) \in \text{GMet} \) and \( (B, Δ, [\hat{Σ}]) \in \text{Alg}(\hat{Σ}, S) \). For any nonexpansive map \( f : (A, d) \rightarrow (B, Δ) \), there exists a unique \( \hat{Σ} \)-algebra homomorphism \( f^* : \hat{T}_{Σ̃}S \rightarrow B \) such that \( f^* \circ \hat{η}(A, d) = f \).

We summarize the statement in (14).

\[
\begin{array}{c}
\text{in \text{GMet}} & \text{in \text{Alg}(\hat{Σ}, S)} \\
(A, d) & \hat{T}_{Σ̃}(A, d) & T_{Σ̃}(A, d) \\
\hat{η}(A,d) & \hat{T}_{Σ̃}(A, d) & T_{Σ̃}(A, d) \\
f & f^* & f^* \downarrow \quad U \quad \downarrow \quad \hat{η} \\
(B, Δ) & (B, Δ, [\hat{Σ}]) & (B, Δ, [\hat{Σ}]) \\
\end{array}
\]

PROOF. Let \( E = \text{E}(\ast_{Σ̃}) \) (see Definition 4.1). We organise the proof in four steps.

Step 1. By Lemma 4.2, the carrier of \( \hat{T}_{Σ̃}(A, d) \) is \( T_{Σ̃}A / Θ \) (equivalently, \( T_{Σ̃}A / \equiv_{A} \)), i.e., the free \( (Σ̃, E) \)-algebra on \( Θ \).

Step 2. The algebra \( (B, Δ, [\hat{Σ}]) \in \text{Alg}(\hat{Σ}, S) \) can be expanded to become an algebra over the extended signature with the aid of the nonexpansive map \( f : A \rightarrow B \). Namely, we interpret the added constants in \( A \) as follows:

\[
[a]_B := f(a).
\]

Since \( f \) is nonexpansive, the expanded \( B, Δ, [\hat{Σ}_A]_B \) satisfies the additional clauses on constants:

\[
\mathcal{Λ}([a]_B, [a']_B) = \mathcal{Λ}(f(a), f(a')) \leq d_A(a, a'),
\]

and therefore it is a model of the extended theory \( \ast_{Σ̃} \). Hence \( (B, Δ, [\hat{Σ}_A]_B) \in \text{Alg}(\hat{Σ}_A, S_A) \). This means that all equations in \( E = \text{E}(\ast_{Σ̃}) \) are validated in \( B \). This in turn means that \( (B, [\hat{Σ}_A]_B) \) (forgetting the metric) is a \( (Σ, E) \)-algebra.

Step 3. Combining the first two steps, we obtain a unique \( (Σ, E) \)-algebra homomorphism

\[
g^*_0 : T_{Σ̃}A / Θ \rightarrow B
\]

where \( g^*_0 \) is the homomorphic extension of the empty function \( g_0 : Θ \rightarrow B \). By identifying \( T_{Σ̃}A / Θ \) with \( T_{Σ̃}A / \equiv_{A} \), we turn \( g^*_0 \) into a function of type \( T_{Σ̃}A / \equiv_{A} \rightarrow B \), which we denote by \( f^* \). By the definition of \( g^*_0 \) we have \( f^*([a]_{A}) = [a]_B = f(a) \), which implies that \( f^* \circ \hat{η}(A, d) = f \).

Step 4. We now conclude by proving that \( f^* \) is a \( \hat{Σ} \)-algebra homomorphism in \( \text{Alg}(\hat{Σ}, S) \), namely, it is a \( Σ \)-algebra homomorphism and it is nonexpansive. The former follows from Step 3 which defined \( f^* \) as \( g^*_0 \), which is a \( (Σ, E) \)-algebra homomorphism and thus preserves all operations in \( Σ \). For the latter, take arbitrary elements \([s]_{A}, [t]_{A} \in \hat{T}_{Σ̃}(A, d)\) and assume \( T_{Σ̃}S, d([s]_{A}, [t]_{A}) = ε \), which means that \( 0 \ast_{Σ̃} s =_ε t \). We need to show that in \((B, Δ, [\hat{Σ}]_B)\),

\[
\mathcal{Λ}([s]_B, [t]_B) = \mathcal{Λ}([s]_B, [t]_B) = ε.
\]

Since we already know that \((B, Δ, [\hat{Σ}_A]_B) \in \text{Alg}(\hat{Σ}_A, S_A)\) (Step 2), we have that \((B, Δ, [\hat{Σ}_A]_B) \ast s =_ε t\), which means

\[
\mathcal{Λ}([s]_B, [t]_B) = \mathcal{Λ}([s]_B, [t]_B) = ε.
\]

It is now sufficient to observe, using the definition of \( g^*_0 \), that

\[
f^*([t]_{A}) = [t]_B.
\]

where \([t]_B\) is the extended interpretation to \( Σ_A \). Hence, from (15) and (16) we derive the desired inequality (14).

We now focus our attention on the case of theories \( \ast_{Σ} \) over \( \hat{Σ} \) generated by a set of basic Horn clauses (Definition 3.10), that is, of the form \( \bigwedge_{i=1}^n \phi_i \Rightarrow ϕ \), where each \( \phi_i \) is a (quantitative) equation \((x = y) \) or \( (x =_ε y) \) between variables.

THEOREM 4.8. Let \( \hat{Σ} \) be a lifted signature and \( S \) a set of basic Horn clauses. Then \( \text{EM}(\hat{T}_{Σ̃}, S) \cong \text{Alg}(\hat{Σ}, S) \).

PROOF. Let \( (A, d, α) \in \text{EM}(\hat{T}_{Σ̃}, S) \), we define the interpretations \( [\hat{Σ}]_α \) as follows: for any \( op : n \in Σ, \bar{a} \in A^n \),

\[
[op]_α(\bar{a}) = α([op(\bar{a})])
\]

where \([t]_B\) stands for \([t]_B(\bar{a}) \). We claim that \((A, d, [\hat{Σ}]_α) \in \text{Alg}(\hat{Σ}, S) \). First, we show \([op]_α \) is \( L_{op} \)-nonexpansive. Given \( \bar{a}, \bar{b} \in L_{op}(A, d) \), let \( Δ \) be the restriction of \( d \) on \( \bar{a} \cup \bar{b} \), we have

\[
d(\alpha([op(\bar{a})]), \alpha([op(\bar{b})])) \leq \hat{T}_{Σ̃}S, d([op(\bar{a})], [op(\bar{b})])
\]

\[
\leq L_{op}(Δ)(\bar{a}, \bar{b})
\]

\[
= L_{op}(d)(\bar{a}, \bar{b}).
\]

The first inequality holds because \( α \) is nonexpansive, the second inequality uses the rule \( L_{NE} \), and the equality is the fact that \( L_{op} \) preserves isometric embeddings.

An adaptation of the argument in the proof of Theorem 4.2 in [3] shows \((A, d, [\hat{Σ}]_α) \) satisfies the clauses in \( S \). This defines a \( \text{Functor} \) \( \hat{P} : \text{EM}(\hat{T}_{Σ̃}, S) \rightarrow \text{Alg}(\hat{Σ}, S) \) acting trivially on morphisms and sending \((A, d, α)\) to \((A, d, [\hat{Σ}]_α)\).

In the converse direction, let \( A = (A, d, [\hat{Σ}]_A) \in \text{Alg}(\hat{Σ}, S) \), we define \( \hat{α} : \hat{T}_{Σ̃}S(A, d) \rightarrow (A, d) \) inductively as follows: for any \( a \in A, \hat{α}_A([d]) = a \) and \( \text{Vop} : n \in Σ, \forall t_1, \ldots, t_n \in T_{ΣA} \)

\[
\hat{α}_A([\text{Vop}(t_1, \ldots, t_n)]) = \{ [\hat{α}_A([t_1])], \ldots, [\hat{α}_A([t_n])] \}.
\]

This defines a \( \text{Functor} \) \( \hat{P}^{-1} : \text{Alg}(\hat{Σ}, S) \rightarrow \text{EM}(\hat{T}_{Σ̃}, S) \). It acts trivially on morphisms and sends \( \hat{α} = (A, d, [\hat{Σ}]_A) \) to \((A, d, \hat{α}_A)\).

The functor \( \hat{P} \) and \( \hat{P}^{-1} \) are inverses and we conclude the desired isomorphism.

5 EXAMPLES

In Sections 3 and 4 we have introduced the new notions of lifted signatures \( \hat{Σ} \) and quantitative \( \hat{Σ} \)-algebras, the deductive apparatus to reason about them, and we stated our main results: Theorem 3.14 (soundness), Theorem 4.4 (equational completeness), Theorem 4.7 (free algebras) and Theorem 4.8 (\( \text{EM}(\hat{T}_{Σ̃}, S) \cong \text{Alg}(\hat{Σ}, S) \) for basic theories). We now show the applicability of our framework.
5.1 Applications already studied in the literature

As already pointed out, the framework of [11] can be seen as a special case of our framework when: (1) the category of generalised metric spaces considered is Met and (2) all liftings in the lifted signature \( \bar{Σ} \) are the sup–product lifting \( L_\Sigma \) (see Example 3.3). For several interesting examples of applications, more can be said.

We first recall some definitions. Given a set \( A \), we let \( D(A) \) denote the set of finitely supported probability distributions on \( A \), i.e., functions \( φ : A \to [0,1] \) such that \( \{ (a | φ(a) > 0) \} \) is finite. For a given \( a \in A \), the Dirac distribution \( δ_\phi \in D(A) \) assigns 1 to \( a \), and 0 to all other elements. Convex algebras are algebras for the signature \( \Sigma = \{ +p : 2 \}_p \in (0,1) \) and axioms

\[
E = \left\{ x +_p x = x, \quad x + y = y +_1 x, \quad (x + p) + q = x + (p + q) \mid p, q \in (0,1) \right\}.
\]

It is well-known (see, e.g., [9]) that \( D(A) \) is the carrier of the free convex algebra on the set \( A \) with operations

\[
\llbracket +p \rrbracket(φ,ψ) := a \mapsto (p · φ(a) + (1-p) · ψ(a)).
\]

The \( \text{Met} \) quantitative theory of convex algebras from [11] can be formalised in our framework by taking \( \text{GMet} = \text{Met} \), lifted signature \( \bar{Σ} = \{ +p : 2 : A \} \), and as generating set of Horn clauses the axioms \( E \) of convex algebras together with the clause:

\[
\{ x_1 =_r y_1, x_2 =_r y_2 \} \supseteq x_1 +_p x_2 = p_1 +_1 x_1 +_p y_2,
\]

known as "Kantorovich rule". Note that, since the inequality

\[
p_r + (1-p)_s \leq \max\{r,s\}
\]

holds for all \( p, r, s \in [0,1] \), the Kantorovich rule strictly subsumes (using the Max rule) the \( L\text{-NE} \) rule for \( +p \), which only states (omitting some premises, cf. Example 3.12):

\[
\{ x_1 =_r y_1, x_2 =_r y_2 \} \supseteq x_1 +_p x_2 = \max\{x_1, x_2\} +_p y_2.
\]

Hence, in quantitative \( \text{Met} \) convex algebras, the operation \( \llbracket +p \rrbracket \) is not merely \( L\text{-NE} \)-nonexpansive, as it needs to satisfy the stronger constraint of the Kantorovich rule.

Consider now, for every \( p \in (0,1) \), the lifting \( L^p_\Sigma \) of the binary product defined as follows:

\[
L^p_\Sigma : (A,d) \mapsto (A \times A, L^p_\Sigma (d))\]

\[
L^p_\Sigma(d)((a_1, a_2), (b_1, b_2)) = d^K(\llbracket +p \rrbracket(δ_{a_1}, δ_{a_2}), \llbracket +p \rrbracket(δ_{b_1}, δ_{b_2})].
\]

where \( d^K \) is the well-known Kantorovich distance over distributions \( D(A) \). This lifting is easily seen to preserve isometric embeddings.

Then it can be shown that the (Met) quantitative theory of convex algebras, axiomatised above, can also be presented as the theory over the lifted signature \( \bar{Σ}_K = \{ +p : 2 : L^p_\Sigma \}_p \in (0,1) \), taking as generating set of Horn clauses only the set \( E \) of axioms of convex algebras. In other words, we have cast the Kantorovich rule as a \( L\text{-NE} \) rule, by choosing the appropriate lifting \( L^p_\Sigma \) for every operation \( +p \). Note that the remaining clauses are just the purely equational axioms of the (Set) theory of convex algebras.

The same applies in several other interesting examples. For example, also the (Met) quantitative theory of convex semilattices of [15, 16] can be presented as the (Met) quantitative theory with generating clauses just the equational axioms of convex semilattices, by choosing the appropriate liftings in the lifted signature.

5.2 No constraints on algebraic operations

Among the variants of the framework of [11] that have been considered in the literature, the work of [2] is relevant in our discussion. Indeed, the authors have observed that certain fixed-point operations on metric spaces fail to be nonexpansive (up to the sup–product lifting) and, as such, cannot be cast in the framework of [11]. The solution adopted in [2] is to drop entirely all constraints on the interpretation of the operations \([ \text{op} \] and allow arbitrary maps \([ \text{op} : A^n \to A] \).

This approach can be seen as a particular instance of our framework by taking \( \text{GMet} = \text{Met} \) and lifting signatures \( \bar{Σ} \) where for all \( n : L_\text{op} \in \bar{Σ} \) the lifting \( L_\text{op} \) is the “discrete” lifting defined as follows:

\[
L_\text{op}(d)((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \begin{cases} 0 & \text{if } \forall i, a_i = b_i \\ 1 & \text{otherwise} \end{cases}
\]

Indeed, with this choice of lifting, the \( L\text{-NE} \) rule

\[
(\bar{x} \cup \bar{y}, A) \in \text{GMet} \quad \delta = L_\text{op}(\Delta(\bar{x}, \bar{y})
\]

is rendered useless, as it can always be substituted with instances of the clause 1-bdd, if \( \bar{x} \neq \bar{y} \), or with instances of the clause 0 \( \circ \text{op}(\bar{x}) =_{\text{op}} (\bar{y}) \)

Therefore our free algebra and isomorphism theorems from Section 4 hold for the theory developed in [2] and for further variants that can be conceived. Such results could not be automatically derived from the original framework of [11] only allowing for \( L\text{-NE} \)-nonexpansive operations.

5.3 The Łukaszyk–Karmowski distance on probability distributions

In this subsection we develop our main example, already presented in the introduction: the axiomatisation of the Łukaszyk–Karmowski distance \( d_{LK} \) on probability distributions [10]. The distance \( d_{LK} \) has very recently found application in the field of representation learning and it is at the core of the definition of the MICo (“matching under independent couplings”) behavioural distance on Markov processes of [6].

Recall that a diffuse metric space \( (A, d) \in \text{DMet} \) is a set \( A \) with a fuzzy relation \( d : A \times A \to [0,1] \) satisfying symmetry and triangular inequality, i.e., for all \( a, b, c \in A \):

\[
d(a, b) = d(b, a) \quad d(a, c) \leq d(a, b) + d(b, c).
\]

The notion of diffuse metric has been introduced in [6, §4.2]. The following diagrams depict some diffuse metric spaces \((A, d)\) with finite.

- \[
\begin{array}{ccccc}
0 & ½ & 1 \\
\bullet & & a \\
\end{array}
\]

- \[
\begin{array}{ccccc}
0 & 1 \\
\bullet & & a \quad 0 \\
\end{array}
\]

Definition 5.1. Let \((A, d)\) be a diffuse metric space. The Łukaszyk–Karmowski distance is the fuzzy relation \( d_{LK} \) on the set of finitely
supported probability distributions $\mathcal{D}(A)$ defined for any $\varphi, \psi \in \mathcal{D}(A)$ by

$$d_{LK}(\varphi, \psi) = \sum_{x \in \text{supp}(\varphi)} \sum_{y \in \text{supp}(\psi)} \varphi(x) \cdot \psi(y) \cdot d(x, y).$$

**Proposition 5.2.** For any diffuse metric space $(A, d)$, the space $(\mathcal{D}(A), d_{LK})$ is a diffuse metric space.

Recall from Subsection 5.1 that convex algebras are algebras for the signature $\Sigma = \{+p : 2\}_{p \in (0, 1]}$ satisfying the axioms $E$, and that the free convex algebra generated by $A$ is $\mathcal{D}(A)$. We now observe, however, that on probability distributions equipped with the Łukaszyk–Karmowski distance the operation $[+p]$ generally fails to be nonexpansive up to the $\text{sup} \cdot$ product lifting $L_X$.

**Lemma 5.3.** There exists a diffuse metric space $(A, d)$ such that the following map is not nonexpansive:

$$[+p]: (\mathcal{D}(A), d_{LK}) \times (\mathcal{D}(A), d_{LK}) \to (\mathcal{D}(A), d_{LK})$$

**Proof.** Fix the DMet space $A = \{a, b\}$ with $d(a, a) = d(b, b) = \frac{1}{2}$ and $d(a, b) = d(b, a) = 1$. Take the Dirac distributions $\delta_a, \delta_b \in \mathcal{D}(A)$. We have $d_{LK}(\delta_a, \delta_a) = d_{LK}(\delta_b, \delta_b) = \frac{1}{2}$, and

$$d_{LK}(\Sigma \cdot \frac{1}{2}(\delta_a, \delta_b), \Sigma \cdot \frac{1}{2}(\delta_a, \delta_b)) = \frac{3}{4}.$$

Recall that $d_{LK}$ is the sup-product lifting of $d_{LK}$. Hence, $[+p]$ is not nonexpansive:

$$\frac{1}{2} = \max\{d_{LK}(\delta_a, \delta_a), d_{LK}(\delta_b, \delta_b)\} = d_{LK}(\Sigma \cdot \frac{1}{2}(\delta_a, \delta_b), \Sigma \cdot \frac{1}{2}(\delta_a, \delta_b)) < d_{LK}(\Sigma \cdot \frac{1}{2}(\delta_a, \delta_b), \Sigma \cdot \frac{1}{2}(\delta_a, \delta_b)) = \frac{3}{4}.$$

We now introduce a new lifting $L_{LK}^p$ of the binary product ensuring that $[+p]$ is $L_{LK}^p$-nonexpansive. For every $p \in (0, 1)$, we define the $\text{DMet}$ lifting of the binary product:

$$L_{LK}^p : (A \times A, d_{LK}(d)) \to (A \times A, L_{LK}^p(d))$$

$$(a_1, a_2, (b_1, b_2)) = L_{LK}^p([+p]\cdot \delta_{a_1, a_2}, [+p]\cdot \delta_{b_1, b_2})$$

**Lemma 5.4.** The lifting $L_{LK}^p$ preserves isometric embeddings.

**Lemma 5.5.** For every DMet space $(A, d)$, the operation $[+p]: (\mathcal{D}(A) \times \mathcal{D}(A) \to (\mathcal{D}(A) is $L_{LK}^p$-nonexpansive.

We can then consider the following $\text{DMet}$ lifting of the signature $\Sigma$ of convex algebras: $\Sigma_{LK} := \{+p : L_{LK}^p\}_{p \in (0, 1]}$, and the quantitative $\Sigma_{LK}$-theory $\Sigma_{LK}$ generated by the set $E$ of axioms of convex algebras. In this theory the $L\text{-NE}$ rule for $+p$ takes the following form (omitting some premises, cf. Example 3.12): $$(x_1 = \xi_{t_1}, x_1 \cdot x_2 = \xi_{t_1}, y_1 x_1 = y_2, y_2 \cdot x_2 = \xi_{y_1} y_1 + \xi_{y_2} y_2)$$ with $\delta = p^2\xi_{t_1} + (1 - p)^2\xi_{t_2} + n + (1 - p)^2\xi_{y_2} + (1 - p)^2\xi_{y_2}$.

By application of Theorem 4.7 we know that $\text{Alg}(\Sigma_{LK}, E)$ has free algebras on $(A, d)$, for every DMet space $(A, d)$, and that these are term algebras $\Sigma_{LK}$-algebras $(A, d)$ on which we can reason syntactically. The following theorem states that these term algebras are isomorphic to $(\mathcal{D}(A), d_{LK}, [\Sigma])$, the collection of finitely supported probability distributions, with $L_K$ distance and standard convex algebras operations.

**Theorem 5.6.** The free algebra in $\text{Alg}(\Sigma_{LK}, E)$ on a DMet space $(A, d)$ is $(\mathcal{D}(A), d_{LK}, [\Sigma])$.

Hence we can say that the theory $\Sigma_{LK}$ axiomatises convex algebras $(\mathcal{D}(A), [\Sigma])$ with the $L_K$ distance.

**6 CONCLUSION**

We have presented an extension of the quantitative algebra framework of [11] allowing us to reason on generalised metric spaces and on algebraic operations that are nonexpansive up to a lifting. This has allowed, as an illustrative example, the axiomatisation of the Łukaszyk–Karmowski distance on probability distributions.

Envisioned applications of our new framework include the types of applications already explored for the framework of [11], but now covering more examples (generalised metric spaces and operations that are not non-expansive). For example, in [3], [4] and [15] the authors axiomatise bisimilarity distances for process algebras describing Markov process and nondeterministic probabilistic transition systems. One can now obtain, for instance, a complete axiomatisation of the bisimilarity distance of [6] on Markov processes based on the $L_K$ diffuse distance. In addition, the treatment of fixed-point operators in [2] as functions that are potentially not nonexpansive (see Section 5.2) can be applied to other types of systems, e.g., nondeterministic probabilistic transition systems with fixed-points.

Another direction of future work is to explore if, and how, the recent theoretical results developed for the framework of [11] can be adapted and generalised to our setting. For example, Birkhoff-style variety theorems [12] (see also [14]), tensor products of theories [4] and techniques to handle Banach signatures [13].

In another direction, one can look for further generalisations. For example, it would be interesting to investigate how our treatment of $\text{GMet}$ compares with the general relational apparatus of [7] and find a way to lift their more general arities. Another interesting possibility is to consider liftings of the entire signature functor $\Sigma(A) := \prod_{\alpha, \beta \in \Sigma} A^n$ rather than just liftings of each of the operations.

From a foundational standpoint, the question of what classes of monads (e.g., finitary ones) can be constructed as term monads for quantitative theories is still open.

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**REFERENCES**

[1] Steve Awodey. 2010. Category Theory. Oxford University Press.

[2] Giorgio Bacci, Giovanni Bacci, Kim G. Larsen, and Radu Mardare. 2018. A Complete Quantitative Deduction System for the Bisimilarity Distance on Markov Chains. Logical Methods in Computer Science 14, 4 (2018). https://doi.org/10.23638/LMCS-14(4:15)2018

[3] Giorgio Bacci, Radu Mardare, Prakash Panangaden, and Gordon D. Plotkin. 2018. An Algebraic Theory of Markov Processes. In Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018, Oxford, UK, July 9-12, 2018.
