INTERIOR AND BOUNDARY GRADIENT ESTIMATES FOR
NEUMANN PROBLEM OF FULLY NONLINEAR HESSIAN
EQUATIONS

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Abstract. In this paper we study the a priori gradient estimates for admissible solutions to Neumann boundary value problem of fully nonlinear Hessian equations on Riemannian manifolds. We firstly derive an interior gradient estimates for admissible solutions, then we obtain boundary gradient estimates based on the interior gradient estimates we have got.

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1. Introduction

Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(n \geq 2\) with Riemannian metric \(g\) and nonempty boundary \(\partial M\). We denote the interior of \(M\) by \(\mathring{M}\), i.e. \(\mathring{M} = M/\partial M\). In this paper, we firstly study the a priori interior gradient estimates for solutions to the fully nonlinear equation

\[
(1.1) \quad f(\lambda(\nabla^2 u)) = \psi(x, u, \nabla u), \text{ on } M,
\]

where \(\nabla\) is the Levi-Civita connection of \(M\), \(\nabla u\) is the gradient of function \(u\), \(\lambda(\nabla^2 u)\) denotes the eigenvalues of the Hessian \(\nabla^2 u\) of function \(u\) with respect to \(g\), and \(\psi \in \mathcal{C}^{3}(M \times \mathbb{R} \times \mathbb{R}^n) > 0\). The smooth function \(f\) defined on an open convex symmetric cone \(\Gamma \subset \mathbb{R}^n\) with vertex at the origin is symmetric with respect to \(\lambda \in \Gamma\). The cone also satisfies the following condition

\[
\Gamma_n \equiv \{\lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0 \} \subseteq \Gamma \neq \mathbb{R}^n.
\]

We call a function \(u\) is an admissible solution of equation \((1.1)\) if \(\lambda(\nabla^2 u) \in \Gamma\) holds on \(M\). In this work, we derive gradient estimates for admissible solutions.

For the function \(f\), we suppose it satisfies the following conditions:

\[
(1.2) \quad f_i \equiv \frac{\partial f}{\partial \lambda_i} > 0,
\]

\[
(1.3) \quad f \text{ is concave},
\]

\[
(1.4) \quad f > 0 \text{ in } \Gamma, f = 0 \text{ on } \partial \Gamma,
\]

and

\[
(1.5) \quad f \text{ is homogeneous of degree one}.
\]

These conditions are well known in the study of fully nonlinear elliptic equations since the fundamental work in [1]. We mention some results in [1] here. Let
\( F(r) = f(\lambda(r)) \) for \( r \in S^{n \times n} \) with \( \lambda(r) \in \Gamma \), where \( S^{n \times n} \) is the set of \( n \times n \) symmetric matrices. Denote \( F^{ij} := \frac{\partial^2 F}{\partial x_i \partial x_j} \). The first result is that the matrix \( \{F^{ij}\} \) has eigenvalues \( f_1, \ldots, f_n \). By (1.2), \( \{F^{ij}\} \) is positive definite for \( \lambda(r) \in \Gamma \), which means equation (1.1) is elliptic for admissible solutions. Another result is that the assumption (1.3) implies that \( F \) is concave. To derive both interior and boundary gradient estimates, we have to assume the following condition

\[
(1.6) \quad f_j(\lambda) \geq \nu_0 \left(1 + \sum f_i(\lambda)\right), \text{ for any } \lambda \in \Gamma \text{ with } \lambda_j < 0,
\]

where \( \nu_0 \) is a uniform positive constant. It is wildly used in deriving gradient estimates, see [6], [21] and [22]. We also assume that the following growth condition for \( \psi(x, z, p) \) holds in \( M \times \mathbb{R} \times \mathbb{R}^n \):

\[
(1.7) \quad |\psi_x| + |\psi_z||\nabla u| + |\psi_p||\nabla u|^2 \leq C|\nabla u|^{2+\gamma}
\]

for a positive constant \( \gamma < 1 \) and a uniform positive constant \( C \).

The classical examples of \( f \) are given by \( (\sigma_k/\sigma_l)^{1/(k-l)} \), \( 0 \leq l < k \leq n \), defined on the cone \( \Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1, \ldots, k \} \), where \( \sigma_k(\lambda) \) is the \( k \)-th elementary symmetric function

\[
\sigma_k(\lambda) = \sum_{i_1 < \ldots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \quad k = 1, \ldots, n,
\]

and we define that \( \sigma_0 \equiv 1 \). In (1.1), if \( f = \sigma_1^{1/n} \), it is the well known Monge-Ampère equation. When \( f = (\sigma_k)^{1/k} \), it is called \( k \)-Hessian equation.

Throughout this paper, we suppose the assumptions (1.2), (1.7) hold. Our first result is the following interior gradient estimates for admissible solutions of equation (1.1) on Riemannian manifold \( (M, g) \).

**Theorem 1.1.** Suppose \( u \in C^3(M) \) is an admissible solution of equation (1.1). For any geodesic ball \( B_r(x_0) \subset M \) with center \( x_0 \in M \) and radius \( r < 1 \), there exists a positive constant \( C \) such that

\[
|\nabla u(x_0)| \leq \frac{C}{r},
\]

where \( C \) only depends on \( |u|_{C^0} \) and other known data.

The next result of this work is the \textbf{a priori} boundary gradient estimates for admissible solutions to equation (1.1) with the Neumann boundary value condition

\[
(1.8) \quad u_\nu = \varphi(x, u), \text{ on } \partial M,
\]

where \( \nu \) is the unit inner normal vector to \( \partial M \) with respect to \( g \) and the function \( \varphi \in C^3(\partial M \times \mathbb{R}) \). In the recent breakthrough work [16] and [15] for Neumann problem, the boundary gradient estimates are established by reduction to the interior gradient estimates. Here, we follow their idea. Given a small positive constant \( \mu \), we define the boundary strip of \( \partial M \) by

\[
M_\mu := \{ x \in M, 0 < d(x, \partial M) < \mu \}.
\]

We prove that the gradient of admissible solutions can be bounded uniformly on \( M_\mu \) when \( \mu \) is sufficiently small, where the bound depends on the interior gradient estimates.

Now we state the boundary gradient estimates for admissible solutions of equation (1.1) with Neumann boundary value (1.8).
Theorem 1.2. Suppose \( u \in C^3(M) \) is an admissible solution of (1.1) with Neumann boundary value (1.8). Then, there is a small positive constant \( \mu \) such that
\[
\sup_{M_\mu} |\nabla u| \leq C (1 + \sup_{M/M_\mu} |\nabla u|),
\]
where \( C \) depends on \( |u|_{C^0}, \mu \) and other known data.

Combining Theorem 1.1 and Theorem 1.2 we can get the following gradient estimates for admissible solutions of Neumann problem (1.1) and (1.8).

Theorem 1.3. Suppose \( u \in C^3(M) \) is an admissible solution of Neumann problem (1.1) and (1.8). We have
\[
\sup_{M} |\nabla u| \leq C,
\]
where \( C \) depends on \( |u|_{C^0} \) and other known data.

Theorem 1.1 extends some previous work by Chou and Wang [3] and Chen [2], in which the interior gradient estimates for admissible solutions are carried out for equation (1.1) with \( f = \sigma_k/\sigma_l \) and \( f = (\sigma_k/\sigma_l)^{1/(k-l)} \) respectively in Euclid space. Recently, the interior gradient estimates for more general Monge-Ampère type equation have been derived in Euclid space [10] and on Riemannian manifolds [9] without the growth condition on \( \psi \).

In conformal geometry, there are interior gradient estimates for fully nonlinear equations, see [7], [8] and [11]. For the prescribed mean curvature equation, such estimates have been extensively studied, see [5]. Interior gradient estimates have also been obtained for Weingarten curvature equations in [12] and see [21] for some other high order curvature equations. Li [13] generalized the results in [12] to general curvature equations by analysis on the graphs of solutions. We remark that Wang in [23] obtained the interior gradient estimates for Weingarten curvature equations using the maximum principle with suitable choice of auxiliary function.

The rest of the paper is organized as follows: in Section 2, we prove the interior gradient estimates for fully nonlinear equation (1.1). In section 3, we derive gradient estimates on \( M_\mu \) for equation (1.1) with Neumann boundary value (1.8) via the interior gradient estimates.

2. Proof of Theorem 1.1

Without loss of generality, we suppose \( 0 \in \bar{M} \). Choose a positive constant \( r < 1 \) such that \( B_r(0) \subset \bar{M} \). We prove there exists a uniform constant \( C \) depending on \( |u|_{C^0} \) such that \( |\nabla u(0)| \leq \frac{C}{r} \). Consider the following auxiliary function on \( B_r(0) \),
\[
G(x) = |\nabla u|h(u)\zeta(x),
\]
where \( \zeta(x) = r^2 - \rho^2(x) \) and \( \rho(x) \) denotes the geodesic distance from 0. We can assume \( r \) is small such that \( \rho \) is smooth and \( |\nabla \rho| = 1 \) in \( B_r(0) \). Note that \( \nabla_{ij} \rho^2(0) = 2\delta_{ij} \). For sufficiently small \( r \), we may further assume in \( B_r(0) \) that
\[
\delta_{ij} \leq |\nabla_{ij} \rho^2| \leq 3\delta_{ij}.
\]
Then we assume that \( G(x) \) attains its maximum at some point \( x_0 \in B_r(0) \). Choose smooth orthonormal local frames \( e_1, \cdots, e_n \) about \( x_0 \) such that
\[
\nabla_1 u(x_0) = |\nabla u(x_0)|,
\]
and \( \{\nabla_{ij} u\}_{2 \leq i, j \leq n} \) is diagonal at \( x_0 \). We may also assume \( \nabla_i e_j = 0 \) at \( x_0 \). We differentiate \( \log G \) at \( x_0 \) twice to get that
\[
\frac{\nabla_{i1} u}{\nabla_1 u} + \frac{\nabla_i h}{h} + \frac{\nabla_i \zeta}{\zeta} = 0
\]
and
\[
0 \geq \frac{\nabla_{ij} u}{\nabla_1 u} + \frac{\nabla_i \xi}{\nabla_1 u} \left( \frac{\nabla_{jk} u}{\nabla_1 u} \right) - 2\nabla_{ij} u \nabla_{jk} u \frac{\nabla_{ij} u}{(\nabla_1 u)^2} - \frac{\nabla_{ij} h}{h^2} - \frac{\nabla_i \zeta}{\zeta} - \frac{\nabla_j \zeta}{\zeta}.
\]
Recall the formula for interchanging order of covariant derivatives
\[
\nabla_{ijk} = \nabla_{kij} + R_{kij} \nabla_i u.
\]
We have
\[
F_{ij} \nabla_{i1} u = \psi_{x_1} + \psi_z \nabla_1 u + \psi_{p_1} \nabla_{11} u + F_{ij} P_{kij} \nabla_i u.
\]
By the growth condition of \( \psi \), we have
\[
F_{ij} \nabla_{i1} u \geq -C|\nabla_1 u|^{2+\gamma} \left( 1 + \sum F_{ii} \right) + \psi_{p_1} \nabla_{11} u,
\]
where \( C \) is a uniform positive constant. By equality (2.1) we see that
\[
F_{ij} \frac{\nabla_{i1} u \nabla_{j1} u}{(\nabla_1 u)^2} = F_{ij} \left( \frac{\nabla_i h}{h} + \frac{\nabla_i \zeta}{\zeta} \right) \left( \frac{\nabla_j h}{h} + \frac{\nabla_j \zeta}{\zeta} \right) \leq 2F_{ij} \frac{\nabla_i h \nabla_j h}{h^2} + \frac{2F_{ij} \nabla_i \zeta \nabla_j \zeta}{\zeta}.
\]
It is readily to see
\[
F_{ij} \frac{\nabla_i \zeta}{\zeta} - F_{ij} \frac{\nabla_i \zeta \nabla_j \zeta}{\zeta^2} \geq -3 \sum F_{ii} \frac{h'}{h} - 4\rho^2 F_{ij} \nabla_i \rho \nabla_j \rho.
\]
Since \( \{F_{ij}\} > 0 \), contracting (2.2) with \( \{F_{ij}\} \) and by (2.3) and (2.1) we get
\[
0 \geq -C|\nabla_1 u|^{1+\gamma} \left( 1 + \sum F_{ii} \right) - \frac{h'}{h} \psi_{p_1} \nabla_{11} u + \frac{2\rho \psi_{p_1} \nabla_{11} \rho}{h} + \frac{h'}{h} F_{ij} \nabla_{ij} u
\]
\[
+ \left( \frac{h''}{h} - 3\frac{h'^2}{h^2} \right) F_{11} \nabla_{11} u - 3 \sum F_{ii} - \frac{12\rho^2}{\zeta^2} F_{ij} \nabla_{i} \rho \nabla_{j} \rho,
\]
here and in what follows \( C \) will be a positive constant only depending on \( |u|_{C^0} \) and other known data, but may change from line to line.

Now we determine the function \( h \). Let \( \delta \) be a small positive constant to be chosen and \( C_0 \) be a positive constant such that \( |u|_{C^0} + 1 \leq C_0 \). Let \( h \) be defined by
\[
h(u) = e^{\delta (u + C_0)^2}.
\]
Differentiating $h$ twice we get
\[ h' = 2\delta(u + C_0)e^{\delta(u+C_0)^2} = 2\delta(u + C_0)h \]
and
\[ h'' = 4\delta^2(u + C_0)^2 h + 2\delta h. \]

It follows that
\[ h'' - 3\frac{h'^2}{h} = 2\delta - 8\delta^2(u + C_0)^2 \geq \delta \]
when $\delta$ is sufficiently small.

By the homogeneity of $f$, we see that $F^{ij}\nabla_{ij}u = \psi \geq 0$. With the assumption $f$ is concave, we have
\[ \sum f_i(\lambda) = f(\lambda) + \sum f_i(\lambda)\left(1 - \lambda_i\right) \geq f(1) \]
where $1 = (1, \cdots, 1) \in \Gamma$. Without loss of generality, we may assume that $f$ is normalized such that $f(1) = 1$. We now derive from (2.5) that
\[ (2.6) \]
\[ 0 \geq \delta F^{11}(\nabla_1u)^2 - C|\nabla_1u|^{1+\gamma} \left(1 + \sum F^{ii}\right) - \left(\frac{3}{\zeta} + \frac{Cr\nabla_1u}{\zeta} + \frac{12r^2}{\zeta^2}\right) \sum F^{ii} \]
where we used the fact $|\nabla\rho| = 1$ and $|\psi_p| \leq C|\nabla u|$ when $|\nabla u|$ is sufficiently large.

From (2.1) we have
\[ \nabla_{11}u = -\nabla_1\left(\frac{h'}{h}\nabla_1u - \frac{2\rho}{\zeta}\nabla_1\rho\right) \]
\[ = -2\delta(u + C_0)(\nabla_1u)^2 + \frac{2\rho}{\zeta}\nabla_1u\nabla_1\rho. \]

Since $u + C_0 \geq 1$, we assume at $x_0$ that $\nabla_1u$ is sufficiently large such that $\zeta\nabla_1u > \frac{\rho\nabla_1\rho}{8(u + C_0)}$. This means that $\nabla_{11}u < 0$ at $x_0$. By the assumption (1.6) we now have
\[ F^{11} \geq \nu_0 + \nu_0 \sum F^{ii}. \]

Hence, when $\nabla_1u(x_0)$ is sufficiently large such that $|\nabla_1u(x_0)|^{1+\gamma} \geq \frac{C}{\delta\nu_0}$, from (2.6) we have
\[ 0 \geq \frac{\delta}{2} F^{11}(\nabla_1u)^2 - \left(\frac{3}{\zeta} + \frac{Cr\nabla_1u}{\zeta} + \frac{12r^2}{\zeta^2}\right) \sum F^{ii} \]
\[ \geq \frac{\sum F^{ii}}{\zeta^2} \left(\frac{\delta\nu_0}{2}(\zeta\nabla_1u)^2 - Cr\nabla_1u\zeta - 15r^2\right). \]

We therefore obtain that
\[ \zeta(x_0)\nabla_1u(x_0) \leq \frac{Cr}{\delta\nu_0}. \]

We just proved that $G(x_0)$ is bounded by some uniform positive constant $C$, which depends only on $|u|_{C^0}$ and other known data. Therefore, by $G(0) \leq G(x_0)$, we have
\[ |\nabla u(0)| \leq \frac{C}{r^2} \zeta(x_0)\nabla_1u(x_0) \leq \frac{C}{r}, \]
where $C$ depends only on $|u|_{C^0}$ and other known data. The proof of Theorem 1.1 is complete.
3. Proof of Theorem 1.2

In this section, we suppose that $\varphi$ is smoothly extended to $M \times \mathbb{R}$. In order to deal with the boundary gradient estimates for Neumann problem, the following function

$$w(x) := u(x) - \varphi(x, u)d(x)$$

has been used in the construction of auxiliary function, which ensures that the maximum point of the auxiliary function must be an interior point of $M$. This function has been used in recent pioneering work on Neumann problem of $k$-Hessian equation, see [15] and [17], and of the mean curvature equation, see [16].

Now we consider the auxiliary function

$$G(x) := |\nabla w|^2 e^{h(u)} e^{\phi(d)}$$

where $h(u) = \delta(u + C_0)^2$ and $\phi(d) = Ad$, here $C_0$ is as in Section 2 and $A$ is a positive constant to be chosen. Assume $\max_{x \in \partial M} G(x)$ is attained at $x_0$. We prove Theorem 1.2 by three cases.

For the first case that $x_0 \in \partial M$, we choose local orthonormal coordinate at $x_0$ such that $e_n = \nu$ and $e_1, \cdots, e_{n-1}$ are tangential to the boundary $\partial M$. We have at $x_0$ that

$$0 \geq (\log G)_\nu = \frac{\nabla_\nu |\nabla w|^2}{|\nabla w|^2} + h' \nabla_\nu u + A \nabla_\nu d.$$  (3.1)

Note that $\nabla_\nu d = 1$ and $d = 0$ on $\partial M$. We see that

$$\nabla_\nu w = \nabla_\nu u - \varphi \nabla_\nu d = 0.$$  

Then (3.1) becomes to

$$0 \geq \frac{2 \nabla_\alpha w \nabla_{\nu \alpha} w}{|\nabla w|^2} + h' \nabla_\nu u + A,$$

where repeated $\alpha$ means summation from 1 to $n - 1$. With the boundary condition (3.8) we have

$$\nabla_{\nu \alpha} w = \nabla_{\alpha \nu} u - \nabla_{\alpha \nu} (\varphi d) = \nabla_{\alpha \nu} u - d \nabla_{\alpha \nu} \varphi - \varphi \nabla_{\alpha \nu} d - \nabla_\alpha \varphi \nabla_\nu d - \nabla_\alpha d \nabla_\nu \varphi = \nabla_\alpha \varphi - \nabla_{\alpha \nu} u - \varphi \nabla_\alpha d - \nabla_\alpha \varphi \nabla_\nu d - \nabla_\alpha d \nabla_\nu \varphi.$$  

We should note that $|\nabla w|$ and $|\nabla u|$ are equivalent when $|\nabla u|$ is large enough. It yields that on $\partial M$

$$|\nabla_{\nu \alpha} w| \leq C |\nabla u|,$$

where $C$ only depends on $|u|_{C^0}$ and other known data. Hence, we have

$$0 \geq -C - h' |\varphi|_{C^0} + A,$$

from which we can get a contradiction if $A$ is sufficiently large. We therefore have that $G(x_0)$ is bounded.

For the second case that $x_0 \in M_\mu$, we use the maximum principle. Differentiating $\log G$ at $x_0$ twice we obtain

$$\frac{2 \nabla_k w \nabla_{jk} w}{|\nabla w|^2} + h' \nabla_j u + A \nabla_j d = 0,$$  (3.2)
and

\[
0 \geq \frac{1}{|\nabla w|^2} \left\{ 2\nabla_k w \nabla_{ijk} w + 2\nabla_{ik} w \nabla_{jk} w \right. \\
- \frac{2}{|\nabla w|^2} \nabla_k w \nabla_i w \nabla_{jk} w \nabla_{il} w \left. \right\} \\
+ h''\nabla_i u \nabla_j u + h'\nabla_{ij} u + A\nabla_{ij} d.
\]

(3.3)

Contracting (3.3) with \{F^{ij}\}, by (3.2) and \{F^{ij}\} > 0, we have

\[
0 \geq \frac{2}{|\nabla w|^2} F^{ij} \nabla_k w \nabla_{ijk} w + (h'' - 2h'^2) F^{ij} \nabla_i u \nabla_j u \\
- 2A^2 F^{ij} \nabla_i d \nabla_j d + h' F^{ij} \nabla_{ij} u + A F^{ij} \nabla_{ij} d.
\]

(3.4)

For convenience, we denote that \(\Phi(x, u) = \varphi(x, u)d(x)\). Then, \(w = u - \Phi\). By direct calculating we see

\[
\nabla_i w = (1 - \Phi_u) \nabla_i u - \Phi_{xi}.
\]

(3.5)

We can choose \(\mu\) sufficiently small such that \(\frac{1}{2} \leq 1 - \Phi_u \leq 1\). Then, \(|\nabla w|\) and \(|\nabla u|\) is equivalent in \(M_u\) when \(|\nabla u|\) is sufficiently large. Differentiating \(\nabla_k w\) again, we have

\[
\nabla_{ik} w = (1 - \Phi_u) \nabla_{ik} u - (\Phi_{uxi} + \Phi_{uu} \nabla_i u) \nabla_k u - \Phi_{xk} \nabla_{ik} u - \Phi_{xk} \nabla_{xi}.
\]

Since \(\Phi_{uu} = \varphi_{uu}d\), when \(|\nabla u|\) is sufficiently large, we have

\[
\nabla_{ik} w = (1 - \Phi_u) \nabla_{ik} u + dO(|\nabla w|^2) + O(|\nabla w|)
\]

and

\[
\nabla_k w \nabla_{ik} w = (1 - \Phi_u) \nabla_k w \nabla_{ik} u + dO(|\nabla w|^3) + O(|\nabla w|^2).
\]

(3.6)

From (3.2) we see, for any \(1 \leq i \leq n\),

\[
\nabla_k w \nabla_{ik} u = O(|\nabla u|^3).
\]

(3.7)

Differentiating \(\nabla_{jk} w\) again, we see that

\[
\nabla_{ijk} w = (1 - \Phi_u) \nabla_{ijk} u - (\Phi_{uxi} + \Phi_{uu} \nabla_i u) \nabla_{jk} u - (\Phi_{uxj} + \Phi_{uu} \nabla_j u) \nabla_{ik} u \\
- (\Phi_{ux} + \Phi_{uxj} \nabla_i u + \Phi_{xuu} \nabla_{ij} u + \Phi_{xu} \nabla_{ij} u) \nabla_{jk} u \\
- \Phi_{xk} \nabla_{ij} u - (\Phi_{xk} \nabla_{ij} u) \nabla_{jk} u - \Phi_{xk} \nabla_{ij} u - \Phi_{xk} \nabla_{ij} u.
\]

Differentiating the equation (3.1) and by the growth condition of \(\psi\), we get

\[
F^{ij} \nabla_{kij} u = \psi_{xk} + \psi_z \nabla_{k} u + \psi_{y_i} \nabla_{kl} u = O(|\nabla u|^{2+\gamma}) + O(|\nabla u|^\gamma) \nabla_{kl} u,
\]

when \(|\nabla u|\) is sufficiently large. Recall the formula for interchanging order of covariant derivatives

\[
\nabla_{ijk} u = \nabla_{kij} u + F^{ij}_{kl} \nabla_{kl} u.
\]

We have

\[
F^{ij} \nabla_{ijk} u = O(|\nabla u|^{2+\gamma}) + O(|\nabla u|^\gamma) \nabla_{kl} u + O(|\nabla u|) \sum F^{ij}.
\]
Note that $\Phi_{uuu} = \varphi_{uuu} d$. We have
\[
F^{ij} \nabla_{ijk} w = (1 - F_u) F^{ij} \nabla_{ijk} u - 2 F^{ij} (\Phi_{uxi} + \Phi_{uux} \nabla_i u) \nabla_{jk} u
- F^{ij} \nabla_{ij} u (\Phi_{uu} \nabla_k u + \Phi_{xk} u) + (dO(|\nabla u|^3) + O(|\nabla u|^2)) \sum F^{ii}
= O(|\nabla u|^\gamma) \nabla_k u + O(|\nabla u|^{2+\gamma}) - 2 F^{ij} (\Phi_{uxi} + \Phi_{uux} \nabla_i u) \nabla_{jk} u
- F^{ij} \nabla_{ij} u (\Phi_{uu} \nabla_k u + \Phi_{xk} u) + (dO(|\nabla u|^3) + O(|\nabla u|^2)) \sum F^{ii}.
\]
Using (3.8) we derive
\[
F^{ij} \nabla_{k} w \nabla_{ijk} w = O(|\nabla u|^{3+\gamma}) - F^{ij} \nabla_{ij} u O(|\nabla u|) (d|\nabla u| + 1) + (dO(|\nabla u|^4) + O(|\nabla u|^3)) \sum F^{ii}.
\]
By (3.8), we obtain, when $|\nabla u|$ is sufficiently large, that
\[
0 \geq (h'' - 2(h')^2) F^{ij} \nabla_{ij} u \nabla_{k} u + O(|\nabla u|^{1+\gamma})
\]
(3.8)
\[
+ \left( h' - dO(1) - \frac{O(1)}{|\nabla u|} \right) F^{ij} \nabla_{ij} u
+ \left( dO(|\nabla u|^2) + O(|\nabla u|) \right) \sum F^{ii}.
\]
By our definition $h(u) = \delta(u + C_0)^2$ and the choice of $C_0$, one can see
\[
h'(u) = 2\delta(u + C_0) \geq 2\delta, h''(u) = 2\delta.
\]
We choose $\delta$ small such that
\[
h'' - 2(h')^2 = 2\delta - 8\delta^2(u + C_0)^2 > \delta.
\]
Without loss of generality we assume that $\{\nabla_{ij} u(x_0)\}$ is diagonal. Now suppose that $|\nabla u| \leq n|\nabla_1 u|$. We obtain from (3.5) that
\[
|\nabla_1 u| = O(|\nabla_1 u|) = O(|\nabla u|) = O(|\nabla w|).
\]
From (3.2) and (3.6), we have
\[
(1 - \Phi_u) \nabla_{k} w \nabla_{1k} u = -h' \nabla_1 u |\nabla w|^2 - dO(|\nabla w|^3) + O(|\nabla w|^2).
\]
Since $\{\nabla_{ij} u(x_0)\}$ is diagonal, we obtain
\[
\nabla_{1u} u = -h' |\nabla w|^2 \frac{\nabla_1 u}{(1 - \Phi_u) \nabla_1 w} - \frac{dO(|\nabla w|^3)}{(1 - \Phi_u) \nabla_1 w} + O(|\nabla w|)
\]
which implies that $\nabla_{1u} u < 0$ when $|\nabla u|$ is sufficiently large and $d < \mu$ is sufficiently small. From (1.6), we get
\[
F^{ij} \nabla_{ij} u \nabla_{ij} u = F^{ij} (\nabla_{ij} u)^2 \geq \nu_0 |\nabla_1 u|^2 \left( 1 + \sum f_i \right).
\]
Noting that $F^{ij} \nabla_{ij} u \geq 0$, we can see
\[
\left( h' - dO(1) - \frac{O(1)}{|\nabla u|} \right) F^{ij} \nabla_{ij} u \geq 0,
\]
when $|\nabla u|$ is sufficiently large and $\mu$ is sufficiently small. We now derive from (3.8) that
\[
0 \geq \left( \nu_0 \delta + dO(1) + \frac{O(1)}{|\nabla u|} \right) \sum F^{ii} + \nu_0 \delta + \frac{O(|\nabla u|^\gamma)}{|\nabla u|},
\]
when $|\nabla u|$ is sufficiently large and $\mu$ is sufficiently small. We get a contradiction from the above inequality if $|\nabla u|$ is sufficiently large and $d<\mu$ is sufficiently small. Therefore, $G(x_0)$ is bounded in this case.

For the third case $x_0 \in \partial M_\mu \setminus \partial M$, a bound for $|\nabla u(x_0)|$ can be obtained by Theorem 1.1. Therefore, $G(x_0)$ is bounded directly.

Now, we have proved that $\max_{M_\mu} G \leq C(1 + \sup_{M/M_\mu} |\nabla u|)$, where $C$ depends on $|u|_{C^0}$ and other known data. Consequently, we have proved that for admissible solutions $u$ of equation (1.1) with Neumann boundary value condition (1.8), there exists a uniform positive constant $C$ such that

$$\sup_{M_\mu} |\nabla u| \leq C(1 + \sup_{M/M_\mu} |\nabla u|)$$

holds, where $C$ depends on $|u|_{C^0}$ and other known data. Theorem 1.2 is proved.

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