Invariant measures involving local inverse iterates

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Abstract

We study new invariant probability measures, describing the distribution of multivalued inverse iterates (i.e. of different local inverse iterates) for a non-invertible smooth function $f$ which is hyperbolic, but not necessarily expanding on a repellor $\Lambda$. The methods for the higher dimensional non-expanding and non-invertible case are different than the ones for diffeomorphisms, due to the lack of a nice unstable foliation (local unstable manifolds depend on prehistories and may intersect each other, both in $\Lambda$ and outside $\Lambda$), and the fact that Markov partitions may not exist on $\Lambda$. We obtain that for Lebesgue almost all points $z$ in a neighbourhood $V$ of $\Lambda$, the normalized averages of Dirac measures on the consecutive preimage sets of $z$ converge weakly to an equilibrium measure $\mu^-$ on $\Lambda$; this implies that $\mu^-$ is a physical measure for the local inverse iterates of $f$. It turns out that $\mu^-$ is an inverse SRB measure in the sense that it is the only invariant measure satisfying a Pesin type formula for the negative Lyapunov exponents. Also we show that $\mu^-$ has absolutely continuous conditional measures on local stable manifolds, by using the above convergence of measures. Several classes of examples of hyperbolic non-invertible and non-expanding repellors, with their inverse SRB measures, are given in the end.

Mathematics Subject Classification 2000: 37D35, 37A60, 37D20.

Keywords: Hyperbolic non-invertible maps (endomorphisms), repellors, SRB measures for endomorphisms, physical and equilibrium measures.

1 Introduction

SRB measures (Sinai, Ruelle, Bowen) and physical measures have been studied for many classes of dynamical systems having some form of hyperbolicity, either uniform, partial or non-uniform. Intuitively physical measures describe the distributions of forward iterates in a neighbourhood of an attractor. For uniformly hyperbolic systems in the vicinity of an attractor, the existence of physical measures and many of their properties were proved by Bowen ([2]). SRB measures are usually defined by the absolute continuity of their conditional measures on local unstable manifolds ([18]). The term physical measures was introduced by Eckmann and Ruelle ([4]) who also proved many of their properties and gave relations to examples from physics (turbulence theory, statistical mechanics, strange attractors, etc.) Measure-theoretic entropy and Lyapunov exponents prove to be very important with regard to physical and SRB measures, as in Pesin’s entropy formula ([4], [6], [7], [18], etc.) For uniformly hyperbolic dynamical systems having an attractor $\Lambda$, physical measures are in fact SRB measures as was proved by Sinai, Ruelle, Bowen ([2], [18]). For other
systems there may exist physical measures which are not SRB (as in [1]). In [4], it was studied
mainly the case of attractors for diffeomorphisms or the case of a flow indexed with both positive
and negative parameters $t$. In such a case the inverse of the map is well defined and it is also a
smooth map. For flows we simply can take $f^t, t < 0$. One cannot do the same if the dynamical
system is not invertible.

In this paper we focus on finding physical measures giving the distribution of consecutive
preimage sets for non-invertible smooth maps (such maps will be called endomorphisms), in
the vicinity of a hyperbolic repellor. There are many examples of systems which are not invertible,
for instance the non-invertible horseshoes from [1], s-hyperbolic holomorphic maps in several
dimensions and their invariant sets ([10]), skew products having a finite iterated function system in
the base and overlaps in their fibers, hyperbolic toral endomorphisms, baker’s transformations with
overlaps, etc. By similarity to the SRB measure ([2], [18]), one natural question would be to study
the distribution of various preimages near a hyperbolic repellor $\Lambda$. The problem is that
there is no unique inverse $f^{-1}$; instead, if $f$ does not have any critical points near the $\Lambda$, we will
obtain local inverse iterates, or equivalently a multivalued inverse iterate of $f$. If $f$ is locally
d-to-1 on a basic set $\Lambda$, and if the local inverse iterates of $f$ on some open set $W$ are denoted by
$f_{W,1}^{-1}, \ldots, f_{W,d}^{-1}$, then the multivalued inverse of $f$ on $W$ is $(f_{W,1}^{-1}, \ldots, f_{W,d}^{-1})$. Knowing the behaviour
of inverse trajectories of a system may be important when we want to obtain information about
the past states of the system.

It is important to keep in mind that the map $f$ is not assumed expanding on $\Lambda$; indeed for
the expanding case a lot is known about the distribution of preimages (see [8], [16]) and the situation
is characterized by the fact that local inverse iterates decrease exponentially fast the diameter of
small balls; this guarantees that we have bounded distortion lemmas. However in the general higher
dimensional non-invertible hyperbolic case we do not have control on the distortion of small balls
under local inverse iterates; indeed they may increase in the stable direction in backward time.

Non-invertibility brings many difficulties into the setting, like not being able to apply directly
Birkhoff Ergodic Theorem for $f^{-1}$ like in the case of diffeomorphisms, the non-existence of a Markov
partition of $\Lambda$ (as $f$ is just an endomorphism, not necessarily expanding on $\Lambda$), etc. One classical
tool when dealing with endomorphisms would be to use the natural extension $\hat{\Lambda}$ of $\Lambda$ (also known
as the inverse limit), but then one loses differentiability properties near $\Lambda$, as $\hat{\Lambda}$ is not a manifold.
In general for endomorphisms, local unstable manifolds depend on whole prehistories not only on
the base points ([15]); this dependence is Holder continuous with respect to prehistories ([9]). Our
repellors will be in fact unions of global stable sets, but the overlappings and foldings of the system
introduce a complicated and very irregular dynamics. Moreover the number of preimages belonging
to $\Lambda$ of a given point may vary a priori along $\Lambda$.

For attractors/repellors $\Lambda$ for diffeomorphisms $f$ we know that there exists an SRB/inverse SRB
measure on $\Lambda$ and that $(\Lambda, f|_\Lambda)$ becomes a Bernoulli 2-sided transformation ([2], [8]). This is based
mainly on the existence of Markov partitions in the invertible case. Also for expanding maps there
exist Markov partitions ([17], [16]) and the system is isomorphic to a 1-sided Markov chain. In the
non-invertible non-expanding case we however do not have Markov partitions, as mentioned above.
The main directions and results of the paper are the following:

First we will specify what we understand by a repellor, in Definition 1. We prove that on a repellor Λ, the number of preimages belonging to Λ of any \( x \in \Lambda \) is locally constant. We also show a very important property of these sets, namely the stability under perturbations, in Proposition 3. Then we prove in Theorem 1 that the pressure of the stable potential \( \Phi^s \) along a connected repellor Λ is related to the number \( d \) of preimages of an arbitrary point, which remain in Λ.

We will define next the probability measures \( \mu^z_n := \frac{1}{d^n} \sum_{y \in f^{-n}z \cap U} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i y}, n \geq 1, z \in V \subset U \), where \( V, U \) are close enough neighbourhoods of Λ. In Theorem 2 we give the main result, namely the weak convergence of the measures \( \mu^z_n \) towards the equilibrium measure \( \mu_s \) of the potential \( \Phi^s \), for Lebesgue almost all points \( z \in V \). In this Theorem, Λ will be assumed connected (not a very restrictive assumption for our notion of repellor, as will be seen). We show then in Theorem 3 that a Pesin type formula involving the negative Lyapunov exponents can be derived for this physical measure \( \mu^- = \mu_s \). This will give also the absolute continuity of conditional measures of \( \mu^- \) on stable manifolds, by using the convergence of measures of Theorem 2 and a result of Liu (7) relating entropy, folding entropy and negative Lyapunov exponents. In fact by using the convergence of the measures \( (\mu^z_n)_n \) from Theorem 2 we show that the folding entropy \( H_{\mu^-}(\epsilon/f^{-1}\epsilon) \) is equal to \( \log d \), where \( \epsilon \) is the partition of Λ into single points. Therefore by all these properties, it follows that \( \mu^- \) can be viewed as an inverse SRB measure.

The above inverse Pesin type formula will imply in Theorem 4 that the repellor Λ with its inverse SRB measure \( \mu^- \) is not isomorphic to a one-sided Bernoulli shift. This is in contrast with the case of attractors for diffeomorphisms where the attractor, together with its SRB measure, is 2-sided Bernoulli. We show however in Theorem 5 that \( \mu^- \) has Exponential Decay of Correlations on Holder potentials.

Finally we describe some classes of examples in Section 3 among which hyperbolic toral endomorphisms, other Anosov endomorphisms, as well as new classes of non-expanding repellors which are not Anosov, together with their inverse SRB measures.

2 Main results.

First we will specify what do we understand by repellor. As a general setting throughout the paper, we consider \( f : M \rightarrow M \) a smooth (say \( C^2 \)) map on a Riemannian manifold, and Λ an \( f \)-invariant compact set in \( M \) which does not intersect the critical set \( C_f \) of \( f \). We remark that the preimages of a point from Λ do not have to remain in Λ necessarily. Also let us notice that if \( C_f \) would intersect Λ, the basic ideas would remain the same as long as we assume an integrability condition on \( \log |Df_\Lambda| \) over Λ.

Definition 1. Let \( f : M \rightarrow M \) be a smooth (for example \( C^2 \)) map on a Riemannian manifold and let Λ be a compact set which is \( f \)-invariant (i.e \( f(\Lambda) = \Lambda \)) and s.t \( f|_\Lambda \) is topologically transitive;
assume also that there exists a neighbourhood $U$ of $\Lambda$ such that $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n U$. Such a set will be called a basic set for $f$ (\cite{5}). We say that $\Lambda$ is a repellor for $f$ if $\Lambda$ is a basic set for $f$, $C_f \cap \Lambda = \emptyset$ and if there exists a neighbourhood $U$ of $\Lambda$ such that $\bar{U} \subset f(U)$.

We will call any point $y \in f^{-1}(x)$ an $f$-preimage of $x \in M$; and by $n$-preimage of $x$ we mean any point $y \in f^{-n}(x)$, for an integer $n > 0$.

**Proposition 1.** In the setting of Definition 1, if $\Lambda$ is a repellor for $f$, then $f^{-1}\Lambda \cap U = \Lambda$. If moreover $\Lambda$ is assumed to be connected, the number of $f$-preimages that a point has in $\Lambda$ is constant.

*Proof.* Let a point $x \in \Lambda$, and $y$ be an $f$-preimage of $x$ from $U$. Then $f^n y \in \Lambda, n \geq 1$. From Definition 1, since $\Lambda$ is assumed to be a repellor, the point $y$ has a preimage $y_{-1}$ in $U$; then $y_{-1}$ has a preimage $y_{-2}$ from $U$, and so on. Thus $y$ has a full prehistory belonging to $U$ and also its forward orbit belongs to $U$, hence $y \in \Lambda$ since $\Lambda$ is a basic set. So $f^{-1}\Lambda \cap U = \Lambda$.

We prove now the second part of the statement. Let a point $x \in \Lambda$ and assume that it has $d$ $f$-preimages in $\Lambda$, denoted $x_1, \ldots, x_d$. Consider also another point $y \in \Lambda$ close to $x$. If $y$ is close enough to $x$ and since $C_f \cap \Lambda = \emptyset$, it follows that $y$ also has exactly $d$ $f$-preimages in $U$, denoted by $y_1, \ldots, y_d$. Since from the first part we know that $f^{-1}\Lambda \cap U = \Lambda$, we obtain that $y_1, \ldots, y_d \in \Lambda$. In conclusion the number of $f$-preimages in $\Lambda$ of a point is locally constant. If $\Lambda$ is assumed to be connected, then the number of preimages belonging to $\Lambda$ of any point from $\Lambda$, must be constant. \(\square\)

Let us denote by $d(x)$ the number of $f$-preimages that the point $x$ has in the repellor $\Lambda$. Then from the above Proposition we know that $d(\cdot)$ is locally constant on $\Lambda$. Clearly there exist only finitely many values that $d(\cdot)$ may take on $\Lambda$. We will assume in the sequel that the number of preimages $d(\cdot)$ is constant on $\Lambda$. This happens for instance when $\Lambda$ is connected (from Proposition 1). We give the results in this setting (i.e when $\Lambda$ is connected), but in fact all we need is that $d(\cdot)$ is constant.

We will work with uniformly hyperbolic endomorphisms on $\Lambda$ (\cite{15, 11, 9}, etc.) The stable tangent spaces $E^s_x, x \in \Lambda$ depend Holder continuously on $x$ (see \cite{5, 9, 11}); the unstable tangent spaces depend on whole prehistories, i.e. we have $E^u_{\hat{x}}, \hat{x} \in \hat{\Lambda}$. Here $(\hat{\Lambda}, \hat{f})$ is the natural extension (\cite{14}), or inverse limit of the dynamical system $(\Lambda, f)$; the space $\hat{\Lambda} := \{ \hat{x} = (x, x_{-1}, x_{-2}, \ldots), f(x_{-i}) = x_{-i+1}, i \geq 1, x_0 := x \}$ is the space of full prehistories of points from $\Lambda$ and the map $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}, \hat{f}(\hat{x}) = (f(x, x_{-1}, x_{-2}, \ldots), \hat{x} \in \hat{\Lambda}$ is the shift homeomorphism. We denote also by $\pi : \hat{\Lambda} \to \Lambda$ the canonical projection given by $\pi(\hat{x}) = x, \hat{x} \in \hat{\Lambda}$. The compact topological space $\hat{\Lambda}$ can be endowed with a natural metric, but it is not a manifold.

We shall denote $Df|_{E^s_x}$ by $Df^s(x)$ and call it the stable derivative at $x \in \Lambda$; and $Df_u(\hat{x}) := Df|_{E^u_{\hat{x}}}$ is the unstable derivative at $\hat{x} \in \hat{\Lambda}$. Similarly the local stable and unstable manifolds are denoted by $W^s_r(x), W^u_r(\hat{x}), \hat{x} \in \hat{\Lambda}$, for some small $r > 0$. We call stable potential the function

$$\Phi^s(x) := \log |Jac(Df^s(x))| = \log |det(Df^s(x))|, x \in \Lambda$$

One notices that there exists a bijection between the set $\mathcal{M}(f)$ of $f$-invariant probability measures on $\Lambda$ and the set $\mathcal{M}(\hat{f})$ of $\hat{f}$-invariant probability measures on the natural extension $\hat{\Lambda}$, so
that to any measure $\mu \in \mathcal{M}(f)$ we associate the unique measure $\hat{\mu} \in \mathcal{M}(\hat{f})$ satisfying the relation $\pi_*(\hat{\mu}) = \mu$ (for example Rokhlin, [14]). It is easy to show that $h_{\hat{\mu}}(\hat{f}) = h_\mu(f)$ and that $P_f(\phi \circ \pi) = P_f(\phi), \forall \phi \in \mathcal{C}(\Lambda, \mathbb{R})$. Thus $\mu$ is an equilibrium measure for a potential $\phi$ if and only if its unique $\hat{f}$-invariant lifting $\hat{\mu}$ is an equilibrium measure for $\phi \circ \pi$ on $\hat{\Lambda}$. Next let us transpose to the setting of endomorphisms, some properties of equilibrium measures from the diffeomorphism case, by using liftings to the natural extension.

**Proposition 2.** Let $\Lambda$ be a hyperbolic basic set for a smooth endomorphism $f : M \to M$, and let $\phi$ a Holder continuous function on $\Lambda$. Then there exists a unique equilibrium measure $\mu_\phi$ for $\phi$ on $\Lambda$ such that for any $\varepsilon > 0$, there exist positive constants $A_\varepsilon, B_\varepsilon$ so that for any $y \in \Lambda, n \geq 1$,

$$A_\varepsilon e^{S_n \phi(y) - nP(\phi)} \leq \mu_\phi(B_n(y, \varepsilon)) \leq B_\varepsilon e^{S_n \phi(y) - nP(\phi)}$$

**Proof.** The shift $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$ is an expansive homeomorphism. The existence of a unique equilibrium measure for the Holder potential $\phi \circ \pi$ with respect to the homeomorphism $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$ follows from the standard theory of expansive homeomorphisms (for example [5]); let us denote it by $\hat{\mu}_\phi$. According to the discussion above there exists a unique probability measure $\mu_\phi$ with $\mu_\phi := \pi_* \hat{\mu}_\phi$, and $\mu_\phi$ is the unique equilibrium measure for $\phi$ on $\Lambda$. The uniqueness follows from the bijection between $\mathcal{M}(f)$ and $\mathcal{M}(\hat{f})$ and from the fact that $\hat{\phi} = \phi \circ \pi : \hat{\Lambda} \to \mathbb{R}$ is Holder continuous (as $\pi : \hat{\Lambda} \to \Lambda$ is Lipschitz and $\phi$ is Holder). Now, there exists a $k = k(\varepsilon) \geq 1$ such that $\hat{f}^k(\pi^{-1}B_n(y, \varepsilon)) \subset B_n - k(\hat{f}^k \hat{y}, 2\varepsilon) \subset \Lambda$, for any $y \in \Lambda$. On the other hand for any $\hat{y} \in \hat{\Lambda}$, we have $\pi(B_n(\hat{y}, \varepsilon)) \subset B_n(y, \varepsilon)$. The last two set inclusions and the $\hat{f}$-invariance of $\hat{\mu}_\phi$, together with the estimates for the $\hat{\mu}_\phi$-measure of the Bowen balls in $\hat{\Lambda}$ (from [5]) imply that there exist positive constants $A_\varepsilon, B_\varepsilon$ (depending on $\varepsilon > 0$ and $\phi$) such that the estimates from the statement hold. \qed

Next let us show that the notion of connected repellor is **stable under perturbations**; this property is important when dealing with systems having a small level of random noise, as it happens in most physical situations.

**Proposition 3.** Let $\Lambda$ be a connected repellor for a smooth map $f : M \to M$ so that $f$ is hyperbolic on $\Lambda$, and let a perturbation $g$ which is $C^1$-close to $f$. Then $g$ has a connected repellor $\Lambda_g$ close to $\Lambda$ such that $g$ is hyperbolic on $\Lambda_g$. In addition the number of $g$-preimages belonging to $\Lambda_g$ of any point of $\Lambda_g$, is the same as the number of $f$-preimages in $\Lambda$ of a point from $\Lambda$.

**Proof.** Since $\Lambda$ has a neighbourhood $U$ so that $U \subset f(U)$, it follows that for $g$ close enough to $f$, we will obtain $U \subset g(U)$. If $g$ is $C^1$-close to $f$, then we can take the set

$$\Lambda_g := \bigcap_{n \in \mathbb{Z}} g^n(U)$$

and it is quite standard that $g$ is hyperbolic on $\Lambda_g$ (for example [15], [9], etc.) One can form then the natural extension of the system $(\Lambda_g, g)$. We know that there exists a conjugating homeomorphism $H : \hat{\Lambda} \to \hat{\Lambda}_g$ which commutes with $\hat{f}$ and $\hat{g}$. The natural extension $\hat{\Lambda}$ is connected if $\Lambda$ is connected, from the fact that the topology on $\hat{\Lambda}$ is induced by the product topology from $\Lambda^\mathbb{N}$. Hence $\hat{\Lambda}_g$ is
developed in order to obtain estimates for the stable dimension in the non-invertible case ([11]).

As we have seen in Proposition 1 if \( \Lambda \) is a connected repellor, then the number of preimages \( g^{-n}(U) \), \( i = 1, \ldots, d \); hence \( y \) has exactly \( d \) \( g \)-preimages belonging to \( \Lambda_g \).

We will need in the sequel an estimate of the volume of a tubular unstable neighbourhood \( f^n(B_n(y, \varepsilon)) \), where \( B_n(y, \varepsilon) := \{ z \in M, d(f^iz, f^iy) < \varepsilon, i = 0, \ldots, n - 1 \} \) is a Bowen ball. The set \( f^n(B_n(y, \varepsilon)) \) is a neighbourhood in \( M \) of the local unstable manifold \( W^u_z(\hat{f}^ ny) \), for \( \hat{f}^ ny = (f^ny, f^{n-1}y, \ldots, y, \ldots) \). Such sets were used in the definition of the inverse pressure, a notion developed in order to obtain estimates for the stable dimension in the non-invertible case ([11]).

By the measure \( m(\cdot) \) on \( M \) we understand the Lebesgue measure defined on the manifold \( M \). And by \( S_n \Phi(y) \) we denote the consecutive sum \( \phi(y) + \ldots + \phi(f^{n-1}y) \) for \( y \in \Lambda, \phi \in C(\Lambda, \mathbb{R}) \).

**Lemma 1.** Let \( f : M \to M \) be a smooth endomorphism and \( \Lambda \) be a basic set on which \( f \) is hyperbolic. Then for some fixed small \( \varepsilon > 0 \) there exist positive constants \( A, B > 0 \) such that for any \( n \geq 1 \) we have:

\[
A e^{S_n \Phi^*(y)} \leq m(f^n B_n(y, \varepsilon)) \leq B e^{S_n \Phi^*(y)}
\]

**Proof.** First of all let us notice that \( S_n \Phi^*(y) = \log |\det(Df^n_s(y))|, y \in \Lambda, n \geq 1 \). From [5] we know that the stable spaces depend Hölder continuously on their base point. Thus \( \Phi^* \) is a Hölder function on \( \Lambda \), as \( C_f \cap \Lambda = \emptyset \). Thus as in proposition 1.6 from [11], we obtain a Bounded Distortion Lemma, saying that there exist positive constants \( \hat{A}, \hat{B} \) such that \( \hat{A} \leq \frac{S_n \Phi^*(y)}{e^{S_n \Phi^*(y)}} \leq \hat{B}, n \geq 1, z \in B_n(y, \varepsilon) \). Then using this Bounded Distortion Lemma, the conclusion follows similarly as in [2].

**Theorem 1.** Consider \( \Lambda \) to be a connected hyperbolic repellor for the smooth endomorphism \( f : M \to M \); let us assume that the constant number of \( f \)-preimages belonging to \( \Lambda \) of any point from \( \Lambda \) is equal to \( d \). Then \( P(\Phi^* - \log d) = 0 \).

**Proof.** As we have seen in Proposition 1 if \( \Lambda \) is a connected repellor, then the number of preimages belonging to \( \Lambda \) of any point from \( \Lambda \) is constant and equal to some integer \( d > 0 \).

In fact if the neighbourhood \( V \) of \( \Lambda \) is close enough to \( \Lambda \), then we can assume that any point \( y \in V \) has exactly \( d \) \( f \)-preimages belonging to \( U \). We want to show that there exists a neighbourhood \( V \) of \( \Lambda \) such that any point from \( V \) has exactly \( d^n \) \( n \)-preimages belonging to \( U \), for any \( n \geq 1 \). First let us assume that the metric around \( \Lambda \) is adapted to the hyperbolic structure on \( \Lambda \), i.e there is \( \lambda \in (0, 1) \) so that if \( z \in W^u_r(\hat{y}) \) and \( \hat{z} = (z, z_{-1}, \ldots) \) is the prehistory of \( z \) \( r \)-shadowing the prehistory \( \hat{y} \), then

\[
d(y, z) \geq d(y_{-1}, z_{-1}) \cdot \frac{1}{\lambda} \geq d(y_{-2}, z_{-2}) \cdot \frac{1}{\lambda^2} \geq \ldots
\]

(1)
Now consider a point \( y \in V \) and some preimage \( y_{-1} \in f^{-1}(y) \cap U \); if \( y \) is close enough to \( \Lambda \), then \( y_{-1} \in U \), and let us assume that we can continue this prehistory until we reach level \( m \). In other words \( (y, y_{-1}, \ldots, y_{-m}) \) is a finite prehistory of \( y \) with \( y_{-1}, \ldots, y_{-m} \in U \), but there exists a preimage \( \tilde{y}_{-m-1} \) of \( y_{-m} \) which escapes \( U \), so that \( y_{-m} \) has less than \( d \) preimages in \( U \). From the definition of repellor we know that \( \tilde{U} \subset f(U) \), thus there exists some preimage \( y_{-m-1} \in U \cap f^{-1}(y_{-m}) \). Then this preimage \( y_{-m-1} \) will have a full prehistory in \( U \). Since \( \Lambda \) is a basic set and \( y_{-m} \) has a full prehistory in \( U \), it follows that there exists a prehistory \( \hat{\xi} \in \Lambda \) such that \( y_{-m} \in W^u_r(\hat{\xi}) \), if \( U \) is close enough to \( \Lambda \) ([5], [15]).

Consequently \( y \in W^u_r(\hat{f}^m \hat{\xi}) \); but from (i) we have \( d(y_{-m}, \Lambda) \leq d(y_{-m}, \xi) \leq \lambda^m d(y, f^m \xi) \leq \lambda^m d(y, \Lambda) \). Recall however that a preimage of \( y_{-m} \) escapes \( U \), thus \( d(y_{-m}, \Lambda) \) must be larger than some positive fixed constant \( \chi_0 \). Therefore if \( V \) is close enough to \( \Lambda \) (and hence \( m \) is large enough) we obtain a contradiction, since we know from above that \( d(y_{-m}, \Lambda) \leq \lambda^m \cdot d(y, \Lambda) \).

Hence there must exist a neighbourhood \( V \) of \( \Lambda \) such that any point from \( V \) has exactly \( d^n \) \( n \)-preimages belonging to \( U \), for any \( n \geq 1 \).

Let us take now an \((n,\varepsilon)\)-separated set of maximal cardinality in \( \Lambda \) and denote it by \( F_n(\varepsilon) \). Hence \( B_n(y,\varepsilon/2) \cap B_n(z,\varepsilon/2) = \emptyset \), \( \forall y, z \in F_n(\varepsilon) \). From the maximality condition it follows also that \( \Lambda \subseteq \bigcap_{y \in F_n(\varepsilon)} B_n(y,2\varepsilon) \). Now from the fact that \( C_f \cap \Lambda = \emptyset \), it follows that there exists a positive constant \( \varepsilon_0 \) such that if \( y, z \in f^{-1}x \cap U, y \neq z \), then \( d(y, z) > \varepsilon_0 \). This implies that if \( y, z \in f^{-n}x \cap \Lambda, y \neq z \), then we cannot have \( z \in B_n(y,4\varepsilon) \) for small enough \( \varepsilon \).

So for a point \( y \in V \), we know that any two of its different \( n \)-preimages must belong to distinct balls of type \( B_n(\xi,2\varepsilon), \xi \in F_n(\varepsilon) \); and \( y \) must have \( d^n \) \( n \)-preimages in \( U \). If \( y_{-n} \) is an \( n \)-preimage in \( U \) of \( y \), then there exists \( \hat{\xi} \in \Lambda \) so that \( y \in W^u_r(\hat{\xi}) \) and thus \( y_{-n} \in B_n(\xi_{-n},\varepsilon) \) for some \( \xi_{-n} \in \Lambda \). But since \( F_n(\varepsilon) \) is a maximal \((n,\varepsilon)\)-separated set in \( \Lambda \), it follows that \( \xi_{-n} \in B_n(z,2\varepsilon) \) for some \( z \in F_n(\varepsilon) \). Hence \( y_{-n} \in B_n(z,3\varepsilon) \) and \( y \in f^n(B_n(z,3\varepsilon)) \) for some \( z \in F_n(\varepsilon) \). Thus we have the following geometric picture of the dynamics on the basin \( V \) of the repellor: through every point \( y \in V \) there pass \( d^n \) tubular neighbourhoods of type \( f^nB_n(z_i,3\varepsilon), z_i \in F_n(\varepsilon), i = 1, \ldots, d^n \). Let us denote such an intersection by \( V_n(z_1, \ldots, z_{d^n}) \).

Therefore from Lemma (i) it follows that, if we add the volumes of all sets \( f^n(B_n(z,3\varepsilon)), z \in F_n(\varepsilon) \), we obtain that each piece \( V_n(z_1, \ldots, z_{d^n}) \) is repeated at least \( d^n \) times, hence
\[
d^n m(V) \leq \sum_{z \in F_n(\varepsilon)} e^{S_\psi(\varepsilon)}
\]
Thus since this happens for any maximal \((n,\varepsilon)\)-separated set \( F_n(\varepsilon) \),
\[
m(V) \leq P_n(\Phi^s - \log d, \varepsilon),
\]
where \( P_n(\psi, \varepsilon) \) denotes in general the quantity \( \inf \{ \sum_{z \in F} e^{S_\psi(\varepsilon)}, F(\psi, \varepsilon) \text{ separated in } \Lambda \} \), for \( \phi \) a continuous real function on \( \Lambda \).

Since \( V \) is a neighbourhood of \( \Lambda \) and thus \( m(V) > 0 \), we obtain that
\[
P(\Phi^s - \log d) \geq 0
\]
We prove now the opposite inequality. Indeed let us take some maximal \((n, \varepsilon)\)-separated set \(F_n(\varepsilon)\) in \(\Lambda\) (with respect to \(f\)). Let a point \(y \in V\), where the neighbourhood \(V\) of \(\Lambda\) was constructed earlier in the proof. Then similar to the above proof of the first inequality, each \(n\)-preimage \(y_{-n}^{i} \) of \(y\) must belong to some Bowen ball \(B_n(z^{i},3\varepsilon)\), \(z^{i} \in F_n(\varepsilon)\), \(i = 1, \ldots, d^n\), hence \(y\) belongs to the (open) intersection of \(d^n\) tubular unstable neighbourhoods centered at points \(f^n(z_i), z_1, \ldots, z_{d^n} \in F_n(\varepsilon)\), i.e \(y \in \bigcap_{1 \leq i \leq d^n} f^n(B_n(z_i,3\varepsilon))\). If \(y\) would belong also to some additional tubular unstable neighbourhood \(f^n(B_n(\omega,3\varepsilon))\) for some \(\omega \in F_n(\varepsilon)\), besides the \(d^n\) neighbourhoods \(f^n(B_n(z_i,3\varepsilon)), i = 1, \ldots, d^n\), then it would follow that \(y\) has an additional \(n\)-preimage \(y_{n+1}^{d} \in B_n(\omega,3\varepsilon)\). Thus since \(B_n(\omega,3\varepsilon) \subset U\) for small \(\varepsilon > 0\) and for \(\omega \in \Lambda\), we would get a contradiction since \(y\) has at most \(d^n\) \(n\)-preimages in \(U\); here we used again that \(\Lambda\) does not intersect the critical set of \(f\). So any \(y \in V\) belongs to only \(d^n\) tubular unstable neighbourhoods of type \(f^n(B_n(z^i,3\varepsilon)), i = 1, \ldots, d^n\).

Now, as we see from Lemma II the Lebesgue measure of a tubular unstable neighbourhood \(f^n(B_n(z,3\varepsilon)), z \in \Lambda\) is comparable to \(e^{S_n \Phi^s(z)}\) (where by comparable we mean that the ratio of the two quantities is bounded below and above by positive constants which are independent of \(z, n\)). Hence we showed that by taking \(\sum_{z \in F_n(\varepsilon)} e^{S_n \Phi^s(z)}\) we cover in fact a combined volume which is less than \(Cd^n \cdot m(U)\) (for some positive constant \(C\) independent of \(n\)). From this observation it follows that

\[
P(\Phi^s - \log d) \leq \inf_n \frac{1}{n} m(U) = 0
\]

Combining the two inequalities proved above, we obtain that \(P(\Phi^s - \log d) = 0\).

We are now ready to prove the main result of the paper, namely the existence of a physical measure for the local inverse iterates in the neighbourhood \(V\) of the hyperbolic repellor \(\Lambda\). We recall that the endomorphism \(f\) is not assumed to be expanding on \(\Lambda\), instead it has both stable and unstable directions on \(\Lambda\). As seen earlier, we can restrict without loss of generality to connected repellors. Recall also that we assumed that the critical set of \(f\) does not intersect \(\Lambda\).

**Theorem 2.** Let \(\Lambda\) be a connected hyperbolic repellor for a smooth endomorphism \(f : M \to M\). There exists a neighbourhood \(V\) of \(\Lambda\), \(V \subset U\) such that if we denote by

\[
\mu^*_n := \frac{1}{n} \sum_{y \in f^{-n}z \cap U} \frac{1}{d(f(y)) \cdot \ldots \cdot d(f^n(y))} \sum_{i=1}^{n} \delta_{f_i y}, z \in V
\]

where \(d(y)\) is the number of \(f\)-preimages belonging to \(U\) of a point \(y \in V\), then for any continuous function \(g \in C(U, \mathbb{R})\) we have

\[
\int_V |\mu^*_n(g) - \mu_s(g)| dm(z) \to 0, \quad \text{as } n \to \infty,
\]

where \(\mu_s\) is the equilibrium measure of the stable potential \(\Phi^s(x) := \log |\det(Df_s(x))|, x \in \Lambda\).

**Proof.** We assume that \(U\) is the neighbourhood of \(\Lambda\) from Definition II i.e such that \(\bar{U} \subset f(U)\). As we proved in Proposition II if \(\Lambda\) is a connected hyperbolic repellor, then any point from \(\Lambda\) has exactly \(d\) \(f\)-preimages belonging to \(\Lambda\) for some positive integer \(d\). Moreover as was shown in the
beginning of the proof of Theorem 1 there exists a neighbourhood $V$ of $\Lambda$ such that any point from $V$ has $d^n$ $n$-preimages in $U$, for $n \geq 1$.

If $\Lambda$ is a hyperbolic repellor we have that all local stable manifolds must be contained in $\Lambda$. Indeed, otherwise there may exist small local stable manifolds which are not entirely contained in $\Lambda$. Let $W^s_r(x), x \in \Lambda$ one such stable manifold, with a point $y \in W^s_r(x) \setminus \Lambda$; in this case since $y \in U$ (for small $r$) and since $\hat{U} \subset f(U)$, it follows that $y$ has a full prehistory $\hat{y}$ in $U$, and from the fact that $\Lambda$ is a basic set, we obtain that $y \in W^u_{\hat{r}}(\hat{\xi})$ for some $\hat{\xi} \in \hat{\Lambda}$. But then $y = W^s_r(x) \cap W^u_{\hat{r}}(\hat{\xi})$, hence $y \in \Lambda$ from the local product structure of $\Lambda$ (since $\Lambda$ is a basic set, see for example [5]); this gives a contradiction to our assumption. Hence there exists a small $r > 0$ such that all stable manifolds of size $r$ are contained in $\Lambda$.

We shall denote by $C(U)$ the space of real continuous functions on $U$. Let us fix now a Holder continuous function $g \in C(U)$. We will apply the $L^1$ Birkhoff Ergodic Theorem (6) on $\hat{\Lambda}$ for the homeomorphism $\hat{f}^{-1}$, in order to obtain an estimate for the measure of the set of prehistories which are badly behaved. Similarly as in [5] or [9] we know that the stable distribution is Holder continuous, hence the stable potential on $\hat{\Lambda}$ is Holder too. This means that there exists a unique equilibrium measure for this potential on $\hat{\Lambda}$; so from the bijection between $\mathcal{M}(f)$ and $\mathcal{M}(\hat{f})$ it follows that there exists a unique equilibrium measure for $\Phi^s$ on $\Lambda$ denoted by $\mu_s$. This measure is ergodic and we can apply the $L^1$ Birkhoff Ergodic Theorem to the function $g \circ \pi$ on $\hat{\Lambda}$:

$$\|\frac{1}{n}(g(x) + g \circ \pi(\hat{f}^{-1}(\hat{x})) + \ldots + g \circ \pi(\hat{f}^{-n+1}(\hat{x})) - \int_{\hat{\Lambda}} g \circ \pi d\hat{\mu}_s\|_{L^1(\hat{\Lambda},\hat{\mu}_s)} \to 0 \quad (3)$$

We make now the general observation that if $f : \Lambda \to \Lambda$ is a continuous map on a compact metric space $\Lambda$, $\mu$ an $f$-invariant borelian probability measure on $\Lambda$ and $\hat{\mu}$ is the unique $\hat{f}$-invariant probability measure on $\hat{\Lambda}$ with $\pi_*(\hat{\mu}) = \mu$, then for an arbitrary closed set $\hat{F} \subset \hat{\Lambda}$, we have that

$$\hat{\mu}(\hat{F}) = \lim_n \mu(\{x_{-n}, \exists x = (x, \ldots, x_{-n}, \ldots) \in \hat{F}\}) \quad (4)$$

Let us prove (4): first denote $\hat{F}_n := \hat{f}^{-n} \hat{F}, n \geq 1$; next notice that $\hat{\mu}(\hat{F}_n) = \hat{\mu}(\hat{F})$ since $\hat{\mu}$ is $\hat{f}$-invariant. Let also $\hat{G}_n := \pi^{-1}(\pi(\hat{F}_n)), n \geq 1$. We have $\hat{F} \subset \hat{f}^n(\hat{G}_n), n \geq 0$. Let now a prehistory $\hat{z} \in \bigcap_{n \geq 0} \hat{f}^n \hat{G}_n$; then if $\hat{z} = (z, z_{-1}, \ldots, z_{-n}, \ldots)$, we obtain that

$$z_{-n} \in \pi \hat{F}_n, \forall n \geq 0,$$

hence $\hat{z} \in \hat{F}$ since $\hat{F}$ is assumed closed. Thus we obtain $\hat{F} = \bigcap_{n \geq 0} \hat{f}^n (\hat{G}_n)$.

Now the above intersection is decreasing, since $\hat{f}^{n+1} \hat{G}_{n+1} \subset \hat{f}^n \hat{G}_n, n \geq 0$. Since the above intersection is decreasing, we get that $\hat{\mu}(\hat{F}) = \lim_n \hat{\mu}(\hat{f}^n \hat{G}_n) = \lim_n \hat{\mu}(\hat{G}_n) = \lim_n \hat{\mu}(\pi^{-1}(\pi(\hat{F}_n))) = \lim_n \mu(\pi(\hat{F}_n)) = \lim_n \mu(\pi \circ \hat{f}^{-n} \hat{F})$, since $\hat{\mu}$ is $\hat{f}$-invariant. Therefore we obtain (4).

For a positive integer $n$, a continuous real function $g$ defined on the neighbourhood $U$ of $\Lambda$, and a point $y$ so that $y, f(y), \ldots, f^{n-1}(y)$ are all in $U$, let us denote by

$$\Sigma_n(g, y) := \frac{g(y) + \ldots + g(f^{n-1}y)}{n} - \int g d\mu_s, n \geq 1, y \in \Lambda$$

Now, from the convergence in $L^1(\hat{\Lambda}, \hat{\mu}_s)$ norm it follows the convergence in $\hat{\mu}_s$-measure. Thus from
we obtain that for any small \( \eta > 0 \) and \( \chi > 0 \), there exists \( N(\eta, \chi) \) so that

\[
\mu_n(x_n \in \Lambda, |\Sigma_n(g, x_n)| \geq \eta) < \chi, \text{ for } n > N(\eta, \chi)
\]  

(5)

Let us consider some small \( \varepsilon > 0 \). Recall that for \( n \geq 1 \) and \( y \in \Lambda \), the Bowen ball \( B_n(y, \varepsilon) := \{z \in M, d(f^i y, f^i z) < \varepsilon, i = 0, \ldots, n-1\} \).

We shall prove that if \( y \in \Lambda \) and \( z \in B_n(y, \varepsilon) \) for \( n \) large enough, then the behaviour of \( \Sigma_n(g, z) \) is similar to that of \( \Sigma_n(g, y) \). More precisely, assume that \( \eta > 0 \) and that \( y \in \Lambda \) satisfies \( |\Sigma_n(g, y)| \geq \eta \). Then we will show that there exists \( N(\eta) \geq 1 \) so that

\[
|\Sigma_n(g, z)| \geq \frac{\eta}{2} \forall z \in B_n(y, \varepsilon), n > N(\eta)
\]  

(6)

Since \( g \) was assumed Holder, let us assume that it has a Holder exponent equal to \( \alpha \), i.e

\[
|g(x) - g(y)| \leq C \cdot d(x, y)^\alpha, \forall x, y \in U,
\]

where \( d(x, y) \) is the Riemannian distance (from \( M \)) between \( x \) and \( y \) and \( C > 0 \) is a constant.

The idea now is that, if \( z \in B_n(y, \varepsilon) \), then for some time the iterates of \( z \) follow the iterates of \( y \) close to stable manifolds, and afterwards they follow the iterates of \( y \) closer and closer to unstable manifolds. We have in both cases an exponential growth of distances between iterates, and thus we can use the Holder continuity of \( g \) on \( U \).

If \( z \in B_n(y, \varepsilon), y \in \Lambda \) then either have \( z \in W^s_\varepsilon(y) \subset \Lambda \) or there exists a positive distance between \( z \) and the local stable manifold \( W^s_\varepsilon(y) \). In the first case there exists some \( \lambda_s \in (0,1) \) such that \( d(f^i z, f^i y) < \lambda_s^i \varepsilon, i = 0, \ldots, n-1 \). This implies that, in the case when \( z \in W^s_\varepsilon(y) \), for some \( N_0 \geq 1 \) we have:

\[
|g(f^{N_0} z) + \ldots + g(f^{n-1} y) - g(f^{N_0} z) - \ldots - g(f^{n-1} z)| \leq \lambda_s^{N_0} \cdot C_0,
\]  

(7)

for some constant \( C_0 > 0 \) independent of \( n \). If \( z \in B_n(y, \varepsilon) \) but \( z \) is not necessarily on \( W^s_\varepsilon(y) \), then the iterates of \( z \) will approach exponentially some local unstable manifolds at the corresponding iterates of \( y \) and their "projections" on these unstable manifolds increases exponentially, up to a maximum value less than \( \varepsilon \) (reached at level \( n \)). More precisely there exists some \( N_0, N_1 \geq 1 \) and some \( \lambda \in (\lambda_s, 1) \) such that \( d(f^i z, f^i y) \leq \lambda^i, i = N_0, \ldots, N_1 - 1 \); notice that \( N_0, N_1, \lambda \) are independent of \( y, z, n \). Now if the iterate \( f^{N_1} z \) becomes much closer to \( W^u_\varepsilon(f^{N_1} y) \) than to \( W^s_\varepsilon(f^{N_1} y) \), it follows that all the higher order iterates will approach asymptotically the local unstable manifolds and \( d(f^j z, f^j y) \) increases exponentially. We assume that \( N_1 \) has been taken such that for some \( \lambda_u \in (\max_{M, \Lambda} (d f_A), 1) \), we have \( d(f^j z, f^j y) \leq \lambda_u \cdot d(f^{j+1} z, f^{j+1} y), j = N_1, \ldots, n - 2 \). So the maximum such distance is \( d(f^{n-1} y, f^{n-1} z) \) and we know that \( d(f^{n-1} y, f^{n-1} z) < \varepsilon \) since \( z \in B_n(y, \varepsilon) \). Hence

\[
d(f^j z, f^j y) \leq \varepsilon \lambda_u^{n-j-1}, j = N_1, \ldots, n - 1
\]

Let us take now some \( N_2 \geq 1 \) such that \( n - N_2 > N_1 \); \( N_2 \) will be determined later. Thus from the Holder continuity of \( g \) on \( U \) we obtain (for some positive constant \( C \)) that:
\[ |g(f^{N_0}z) + \ldots + g(f^{N_1-1}z) + g(f^{N_1}z) + \ldots + g(f^{n-N_2}z) + \ldots + g(f^{n-1}z) - g(f^{N_0}y) - \ldots - g(f^{N_1-1}y) - g(f^{N_1}y) - \ldots - g(f^{n-N_2}y) - \ldots - g(f^{n-1}y)| \leq C(\lambda^\alpha N_0 + \lambda^\beta u + 2N_2|g|) \]

Thus from (7) and (8) we obtain that, if \( z \in B_n(y, \varepsilon) \) then:

\[ |\Sigma_n(g, y) - \Sigma_n(g, z)| \leq \frac{1}{n} \left[ 2N_0|g| + C(\lambda^\alpha N_0 + \lambda^\beta u + 2N_2|g|) \right] \]

From above, \( N_0, N_2 \) do not depend on \( n, y, z \). Therefore we can choose some large \( N(\eta) \) so that

\[ \frac{1}{n} (2N_0|g| + C(\lambda^\alpha N_0 + \lambda^\beta u + 2N_2|g|)) < \eta/2, \text{ for } n > N(\eta) \]

This means that the relation from (9) holds. Let us denote now by:

\[ I_n(g, x) := \frac{1}{d^n} \sum_{y \in f^{-n}(x) \cap U} |\Sigma_n(g, y)|, \]

for a continuous real function \( g : U \rightarrow \mathbb{R} \), and \( x \in V \). Recall that \( V \) is the neighbourhood of \( \Lambda \), \( \Lambda \subset V \subset U \), constructed in the proof of Theorem 2 so that every point \( x \in V \) has \( d^n \) \( n \)-preimages in \( U \) for \( n \geq 1 \). For a fixed Holder continuous function \( g \) and a small \( \eta > 0 \), we will work with \( n > N(\eta) \), where \( N(\eta) \) was found above. From (9) and the discussion afterwards, we know that \( |\Sigma_n(g, z) - \Sigma_n(g, y)| \leq \eta/2 \) if \( z \in B_n(y, \varepsilon) \) and \( y \in \Lambda \).

Let us consider now an \((n, \varepsilon)\)-separated set with maximal cardinality in \( \Lambda \), denoted by \( F_n(\varepsilon) \). As in the proof of Theorem 1 it follows that any point \( y \in V \) belongs to \( d^n \) tubular neighbourhoods, i.e \( f^n(B_n(y_i, 3\varepsilon)), y_i \in F_n(\varepsilon) \) for \( 1 \leq i \leq d^n \). Let us denote as before \( V_n(y_1, \ldots, y_{d^n}) := \bigcap_{1 \leq i \leq d^n} f^nB_n(y_i, 3\varepsilon) \).

Thus in \( \int_V I_n(g, x) dm(x) \), we can decompose \( V \) into the smaller pieces \( V_n(y_1, \ldots, y_{d^n}) \), for different choices of \( y_1, \ldots, y_{d^n} \in F_n(\varepsilon) \).

We can use now relation (9) in order to replace in \( \int_V I_n(g, x) dm(x) \), the term \( |\Sigma_n(g, y)| \) with \( |\Sigma_n(g, \zeta)| \), where \( x \in V \) is arbitrary, \( y \in f^{-n}x \cap U \) and \( y \in B_n(\zeta, 3\varepsilon) \) for some \( \zeta \in F_n(\varepsilon) \). Indeed let us fix some arbitrary small \( \eta > 0 \). Then we prove similarly as in (9) that if \( n > N(\eta) \), then \( |\Sigma_n(g, y)| \leq |\Sigma_n(g, \zeta)| + \eta/2 \), if \( y \in B_n(\zeta, 3\varepsilon) \) and \( \zeta \in F_n(\varepsilon) \) \( (N(\eta) \) can be assumed to be the same as in (9) without loss of generality).

So up to a small error of \( \eta/2 \) we can replace each of the terms \( |\Sigma_n(g, y)| \) with the corresponding \( |\Sigma_n(g, \zeta)| \). This implies that in the integral \( \int_V I_n(g, x) dm(x) \), on each piece of type \( V_n(y_1, \ldots, y_{d^n}) \) in \( f^n(B_n(y_j, 3\varepsilon)) \) for \( y_j \in F_n(\varepsilon) \), we integrate in fact \( |\Sigma_n(g, y_j)| \), modulo an error of \( \eta/2 \). Then we
Thus by adding the measures of these overlaps, we recover $m$ overlap to $G$ and separated. Also if $y, \mu_1$. Also will obtain that constant $B_{n,y}$ of $n > N$.

But since the balls $B_n(y, \varepsilon/2), y \in F_n(\varepsilon)$ are mutually disjoint, we have $\sum_{y \in G_2(n, \varepsilon)} \mu_s(B_n(y, \varepsilon/2)) \leq \eta \sum_{y \in G_1(n, \varepsilon)} \mu_s(B_n(y, \varepsilon/2)) + 2||g||\mu_s(z \in \Lambda, |\Sigma_n(g, z)| \geq \frac{\eta}{2} \cdot C_{\varepsilon}

But since the balls $B_n(y, \varepsilon/2), y \in F_n(\varepsilon)$ are mutually disjoint, we have $\sum_{y \in G_1(n, \varepsilon)} \mu_s(B_n(y, \varepsilon/2)) \leq 1$. Also $\mu_s(z \in \Lambda, |\Sigma_n(g, z)| \geq \eta/2) < \chi$ for $n > N(\eta/2, \chi)$, as follows from (5). Thus by using (12)
we obtain for \( n > \sup\{N(\eta), N(\eta, \chi)\} \)

\[
\int_{V} I_n(g, x) \, dm(x) \leq C_1(\eta + \eta + C_\varepsilon \cdot 2 ||g|| \chi) = 2C_1(\eta + \chi \cdot C_\varepsilon ||g||)
\]

Since \( \eta, \chi > 0 \) were taken arbitrarily, and by recalling the formula for \( I_n(g, x) \) from (10) and the definition of \( \mu_{n_k}^z \), we obtain that:

\[
\int_{V} |\mu_{n_k}^z(g) - \mu_s(g)| \, dm(z) \to 0 \quad \text{as} \quad n \to \infty
\]

Since H\ölder continuous functions \( g \) are dense in the uniform norm on \( C(U) \), we obtain the conclusion of the Theorem for all \( g \in C(U) \).

\[\Box\]

**Corollary 1.** In the same setting as in Theorem 2, it follows that there exists a borelian set \( A \subset V \) with \( m(V \setminus A) = 0 \) and a subsequence \( (n_k)_k \) such that \( \mu_{n_k}^z \to \mu_s \) (as measures on \( U \)), for any point \( z \in A \).

**Proof.** Let us fix \( g \in C(U) \). From the convergence in Lebesgue measure of the sequence of functions \( z \to \mu_{n_k}^z(g), n \geq 1, z \in V \) obtained from Theorem 2, it follows that there exists a borelian set \( A(g) \) with \( m(V \setminus A(g)) = 0 \) and a subsequence \( (n_p)_p \) so that \( \mu_{n_p}^z(g) \to \mu_s(g), z \in A(g) \). Let us consider now a sequence of functions \( (g_m)_{m \geq 1} \) dense in \( C(U) \). By applying a diagonal sequence procedure we obtain a subsequence \( (n_k)_k \) so that \( \mu_{n_k}^z(g_m) \to \mu_s(g_m), \forall z \in \cap A(g_m), \forall m \geq 1 \). We have also \( m(V \setminus \cap A(g_m)) = 0 \), since \( m(V \setminus A(g_m)) = 0, m \geq 1 \). However any real continuous function \( g \in C(U) \) can be approximated in the uniform norm by functions \( g_m \), hence it follows that \( \mu_{n_k}^z(g) \to \mu_s(g), \forall z \in A := \cap A(g_m) \). But we showed above that \( m(V \setminus A) = 0 \).

So we obtain that \( \mu_{n_k}^z \to \mu_s \) for all points \( z \in A \), where \( A \) has full Lebesgue measure in \( V \).

\[\Box\]

### 3 Applications. Examples.

In this section we will pursue further ergodic properties of the inverse physical measure constructed in Theorem 2 and give also examples. Let us first remind the notion of the Jacobian of an endomorphism, relative to an invariant probability measure, from Parry’s book (13). Let \( f : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu) \) a measure preserving endomorphism on a Lebesgue probability space. Assume that the fibers of \( f \) are countable, i.e \( f^{-1}x \) is countable for \( \mu \)-almost all \( x \in X \). It can be proved (13) that in this case \( f \) is positively non-singular, i.e \( \mu(A) = 0 \) implies \( \mu(f(A)) = 0 \) for an arbitrary measurable set \( A \subset X \). Also there exists a measurable partition \( \alpha = (A_0, A_1, \ldots) \) of \( X \) such that \( f|_{A_i} \) is injective. Then using the absolute continuity of \( \mu \circ f \) with respect to \( \mu \), we define the **Jacobian** \( J_{f, \mu} \) on each set \( A_i \), to be equal to the Radon-Nikodym derivative \( \frac{d\mu \circ f}{d\mu} \). So:

\[
J_{f, \mu}(x) = \frac{d\mu \circ f}{d\mu}(x), x \in A_i, i \geq 0
\]
This is a well defined measurable function, which is larger or equal than 1 everywhere (due to the \( f \)-invariance of \( \mu \)). Also it is easy to see that \( J_{f,\mu}(\cdot) \) is independent of the partition \( \alpha \) and that it satisfies a Chain Rule, namely \( J_{f\circ g,\mu} = J_{f,\mu} \cdot J_{g,\mu} \) if \( f, g : X \to X \) and both preserve \( \mu \). From Lemma 10.5 of [13] we also know that

\[
\log J_{f,\mu}(x) = I(\epsilon/f^{-1}\epsilon)(x),
\]

for \( \mu \)-almost every \( x \in X \), where \( \epsilon \) is the partition of \( X \) into single points, and \( I(\epsilon/f^{-1}\epsilon)(\cdot) \) is the conditional information function of \( \epsilon \) given the partition \( f^{-1}\epsilon \). Also from the definition of the Jacobian we see (7) that:

\[
\mu(fA) = \int_A J_{f,\mu}(x)d\mu(x), \tag{13}
\]

for all special sets \( A \), i.e measurable sets such that \( f|_A : A \to f(A) \) is injective. Recall that by Definition [1] \( f \) does not have any critical points in \( \Lambda \). Before proving the main result of this Section, we remind the notion of measurable partitions subordinated to local stable manifolds; for background on measurable partitions, Lebesgue spaces and conditional measures, one can use [14].

Let \( f : M \to M \) be a smooth endomorphism defined on a Riemannian manifold \( M \) which is endowed with its Borelian \( \sigma \)-algebra \( \mathcal{B} \). Let also a probability borelian measure \( \mu \) on \( M \) which is \( f \)-invariant. If \( \xi \) is a measurable partition of \( M \), then we denote by \( \xi(x) \) the unique subset of \( \xi \) containing \( x \in X \); also by \( (M/\xi, \mu_\xi) \) we denote the factor space relative to \( \xi \). To any measurable partition \( \xi \) on \( (M,\mathcal{B},\mu) \) one can attach an essentially unique collection of conditional measures \( \{\mu_C\}_{C \in \xi} \) satisfying two conditions (see [14]):

i) \((C, \mu_C)\) is a Lebesgue space

ii) for any measurable set \( B \subset M \), the set \( B \cap C \) is measurable in \( C \) for almost all points \( C \in M/\xi \) of the factor space, and the function \( C \to \mu_C(B \cap C) \) is measurable on \( M/\xi \) and \( \mu(B) = \int_{M/\xi} \mu_C(B \cap C)d\mu_\xi \).

Similar to the case of partitions subordinated to unstable manifolds ([18]) we can say (as in [7]), that a measurable partition \( \xi \) of \( (M,\mathcal{B},\mu) \) is subordinate to local stable manifolds if for \( \mu \)-almost all \( x \in M \) one has \( \xi(x) \subset W^s_r(x) \) and if \( \xi(x) \) contains an open neighbourhood of \( x \) inside \( W^s_r(x) \) (where \( r > 0 \) is sufficiently small). We can define now the absolute continuity of conditional measures on stable manifolds as in [7]:

**Definition 2.** In the above setting, we say that \( \mu \) has absolutely continuous conditional measures on local stable manifolds if for every measurable partition \( \xi \) subordinated to local stable manifolds, we have for \( \mu \) almost all \( x \in M \) that \( \mu_\xi \ll m^s_\xi \), where \( \mu_\xi \) is the conditional measure of \( \mu \) on \( \xi(x) \) and \( m^s_\xi \) denotes the induced Lebesgue measure on \( W^s_r(x) \).

By the result of Liu ([7]), we know that there exists at least one measurable partition subordinated to local stable manifolds.

Now, by Oseledec Theorem ([8]) we have that for any \( f \)-invariant Borel probability measure \( \mu \) on \( M \), and for \( \mu \)-almost every point \( x \in M \) there exists a finite collection of numbers, called
Lyapunov exponents of $f$ at $x$ with respect to $\mu$, $-\infty \leq \lambda_1(x) < \lambda_2(x) < \ldots < \lambda_q(x) < \infty$, and a unique collection of tangent subspaces of $T_x M$, $V_1(x) \subset \ldots \subset V_q(x)(x) = T_x M$ so that

$$\lim_{n \to \infty} \frac{1}{n} \log |Df^n_x(v)| = \lambda_i(x), \forall v \in V_i(x) \setminus V_{i-1}(x), 1 \leq i \leq q(x), |v| = 1$$

We also denote by $m_i(x) := \dim V_i(x) - \dim V_{i-1}(x)$ the multiplicity of $\lambda_i(x)$. As we saw before, if $\Lambda$ is a connected repellor for $f$ then $f|_\Lambda$ is constant-to-1. We are now ready to prove the following:

**Theorem 3.** Let $\Lambda$ be a connected hyperbolic repellor for a smooth endomorphism $f : M \to M$ on a Riemannian manifold $M$; assume that $f$ is $d$-to-1 on $\Lambda$. Then there exists a unique $f$-invariant probability measure $\mu^-$ on $\Lambda$ satisfying an inverse Pesin entropy formula:

$$h_{\mu^-}(f) = \log d - \int_{\Lambda} \sum_{i, \lambda_i(x) < 0} \lambda_i(x)m_i(x)d\mu^-(x)$$

In addition the measure $\mu^-$ has absolutely continuous conditional measures on local stable manifolds.

**Proof.** Notice that from the above properties of Lyapunov exponents, the derivative $Df^n_{s,x}$ for large $n$, takes into consideration all the vectors $v \in V_i(x)$ for those $i$ for which $\lambda_i(x) < 0$, i.e., for which we have contraction in the long run. Thus if $\mu$ is an $f$-invariant probability measure supported on $\Lambda$, we have by the Chain Rule and Birkhoff Theorem that

$$\int_{\Lambda} \Phi^s d\mu = \int_{\Lambda} \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \Phi^s(f^i x) d\mu(x) = \int_{\Lambda} \lim_{n \to \infty} \frac{1}{n} \log |\det(Df^n_{s,x})| d\mu(x) = \int_{\Lambda} \sum_{i, \lambda_i(x) < 0} \lambda_i(x)m_i(x)d\mu(x)$$

(14)

It follows that the inverse Pesin entropy formula from the statement of the Theorem is satisfied for $\mu = \mu_s$ since $\mu_s$ is the equilibrium measure of $\Phi^s$ and we showed in Theorem 1 that $P(\Phi^s - \log d) = 0$. If the inverse Pesin entropy formula would be satisfied for another invariant measure $\mu$, then we would have $h_{\mu}(f) = \log d - \int_{\Lambda} \sum_{i, \lambda_i(x) < 0} \lambda_i(x)m_i(x)d\mu(x)$, hence:

$$P(\Phi^s - \log d) \geq h_{\mu_s} - \log d + \int_{\Lambda} \Phi^s d\mu = 0$$

However again from Theorem 1 we know that $P(\Phi^s - \log d) = 0$, thus $\mu$ is an equilibrium measure for $\Phi^s$. But $\Phi^s$ is Holder continuous and thus it has a unique equilibrium measure. Therefore if $\mu^- := \mu_s$, we have

$$\mu = \mu_s = \mu^-$$

We want now to show the absolute continuity of conditional measures of $\mu^-$ on local stable manifolds. For this we will use Corollary 1 and results from [7]. Indeed we know that $\Lambda$ is a connected hyperbolic repellor and thus $f$ is $d$-to-1 for some integer $d \geq 1$ in a neighbourhood $V$ of $\Lambda$. We constructed the measures $\mu^z_n$, $z \in V, n \geq 1$, $\mu^z_n := \frac{1}{n} \sum_{y \in f^{-n}z} \frac{1}{n} \sum_{i=1}^{n} \delta f^iy$; and we showed in
Corollary 1 that there exists a subset $A \subset V$, having full Lebesgue measure and a subsequence $(\mu_{n_k}^z)_k$ converging weakly towards $\mu^- := \mu_z$ for every $z \in A$. Now in (13) we can take only special sets whose boundaries have $\mu^-$-measure equal to zero. For such a set $B$ we have that $\mu_{n_k}(B) \to \mu^-(B)$. But then from the definition of $\mu_{n_k}^z$ it follows that $\mu^-(f(B)) = d\mu^-(B)$ for any such special set with boundary of measure zero. As these sets form a sufficient collection (5), we obtain that the Jacobian $J_{f, \mu^-}$ is constant $\mu^-$-almost everywhere and equal to $d$. Hence from Lemma 10.5 of [13], if $\epsilon$ denotes the partition of $M$ into single points, we deduce that the conditional information function

$$I(\epsilon/f^{-1}\epsilon)(x) = \log J_{f, \mu^-}(x) = \log d$$

for $\mu^-$-almost all $x \in \Lambda$; thus

$$H_{\mu^-}(\epsilon/f^{-1}\epsilon) = \int I(\epsilon/f^{-1}\epsilon)(x)d\mu^-(x) = \log d$$

Then since $h_{\mu^-} = \log d - \int_{\Lambda} \sum_{\lambda_i(x) < 0} \lambda_i(x)d\mu^-(x)$, it follows that

$$h_{\mu^-} = H_{\mu^-}(\epsilon/f^{-1}\epsilon) - \int_{\Lambda} \sum_{\lambda_i(x) < 0} \lambda_i(x)m_i(x)d\mu^-(x)$$

Hence from [7] we obtain that $\mu^-$ has absolutely continuous conditional measures on local stable manifolds.

The question whether a measure-preserving dynamical system is 2-sided or 1-sided Bernoulli is an important one and has been solved in a number of cases (see for example [8], [3], [16]). In our case we show that the inverse SRB measure $\mu^-$ on the repellor $\Lambda$, does not form a 1-sided Bernoulli system, by contrast with the usual SRB measures on attractors of diffeomorphisms (which is 2-sided Bernoulli).

**Theorem 4.** Let $f$ as above and $\Lambda$ a connected repellor as in Theorem 3 so that $f$ is not invertible on $\Lambda$. Then $(\Lambda, f|_{\Lambda}, \mathcal{B}(\Lambda), \mu^-)$ cannot be one-sided Bernoulli.

**Proof.** Let $(\Sigma^+_m, \sigma_m, \mathcal{B}_m, \mu_p)$ a one-sided Bernoulli shift on $m$ symbols ([8]), where $\mathcal{B}_m$ denotes the $\sigma$-algebra of sets generated by cylinders in $\Sigma^+_m$, $\sigma_m$ is the shift map, and $\mu_p$ is the $\sigma_m$-invariant measure associated to a probability vector $p = (p_1, \ldots, p_m)$.

We know from Proposition 1 that if $\Lambda$ is connected, then the number of $f$-preimages belonging to $\Lambda$ is constant, and denote it by $d$; we assumed that $d > 1$. If $(\Lambda, f|_{\Lambda}, \mathcal{B}(\Lambda), \mu^-)$ would be isomorphic to a one-sided Bernoulli system $(\Sigma^+_m, \sigma_m, \mathcal{B}_m, \mu_p)$, then $d = m$ since the number of preimages is constant everywhere, for both systems. But then from the Variational Principle for entropy, we would obtain:

$$h_{\mu^-} = h_{\mu_p} \leq h_{\text{top}}(\sigma_m) = \log m = \log d$$

(15)

On the other hand since $\mu^-$ satisfies the Pesin formula on $\Lambda$, we get that $h_{\mu^-} = \log d - \int \Phi^s d\mu^-$. But $\Phi^s < 0$ and $C_f \cap \Lambda = \emptyset$, hence $h_{\mu^-} > \log d$. This gives a contradiction to (15).

\[ \square \]
We prove now that, in spite of not being 1-sided Bernoulli, the inverse SRB measure $\mu^-$ has strong mixing properties on the repellor $\Lambda$.

Given a transformation $f : M \to M$ we say that an $f$-invariant probability $\mu$ has **Exponential Decay of Correlations** on Holder potentials ([2]) if there exists some $\lambda \in (0,1)$ such that for every $n \geq 1$:

$$|\int \phi \cdot \psi \circ f^n d\mu - \int \phi d\mu \cdot \int \psi d\mu| \leq C(\phi, \psi) \lambda^n,$$

for any Holder maps $\phi, \psi \in C(M, \mathbb{R})$, where $C(\phi, \psi)$ depends only on the potentials $\phi, \psi$.

**Theorem 5.** Let a repellor $\Lambda$ for a smooth endomorphism as in Theorem 2 and let $\mu^-$ be the unique inverse SRB measure associated. Then $\mu^-$ has Exponential Decay of Correlations on Holder potentials.

**Proof.** Since we have a uniformly hyperbolic structure for the endomorphism $f$ on $\Lambda$, we can associate to it a Smale space structure on the natural extension $\hat{\Lambda}$ ([17]). Therefore on $\hat{\Lambda}$ there exist Markov partitions of arbitrarily small diameter ([17]). Now these Markov partitions imply the existence of a semi-conjugacy $h$ with a 2-sided mixing Markov chain $\Sigma_A$. We have therefore the Lipschitz continuous maps $h : \Sigma_A \to \hat{\Lambda}$, and $\pi : \hat{\Lambda} \to \Lambda$

such that $\pi \circ \hat{f} = f \circ \pi, h \circ \sigma_A = \hat{f} \circ h$, where $\sigma_A$ is the shift homeomorphism.

Now, since the stable potential $\Phi^s$ on $\Lambda$ is Holder continuous, it follows that $\Psi^s := \Phi^s \circ \pi \circ h : \Sigma_A \to \mathbb{R}$ is Holder continuous and to the unique equilibrium measure $\mu_s$ of $\Phi^s$ it corresponds the unique equilibrium measure $\nu$ of $\Psi^s$ on $\Sigma_A$, s.t $\mu_s = (\pi \circ h)_* \nu$. We have that $P_f(\Phi^s) = P_{\sigma_A}(\Psi^s)$ and $h_{\mu_s}(f) = h_{\nu}(\sigma_A)$. Also notice that

$$\int_{\Lambda} \phi d\mu_s = \int_{\Sigma_A} \phi \circ \pi \circ h \ d\nu, \phi \in C(\Lambda)$$

Now we do have Exponential Decay of Correlations for Holder potentials for $(\Sigma_A, \nu)$ (for example [2]); so the same holds for $f|_{\Lambda}$ and the equilibrium measure $\mu_s$. Recalling that we denoted $\mu^- := \mu_s$, we obtain the conclusion.

**Examples:**

1. **Toral endomorphisms.** Let us take an integer valued $m \times m$ matrix $A$ with $\det(A) \neq 1$. This matrix induces a toral endomorphism $f_A : \mathbb{T}^m \to \mathbb{T}^m$. This toral endomorphism transforms the unit square into a parallelogram in $\mathbb{R}^m$ of area (Lebesgue measure) equal to $|\det(A)|$, and whose corners are points having only integer coordinates. Thus when we project to $\mathbb{T}^m$, we obtain that $f_A$ is $|\det(A)|$-to-1. If all eigenvalues of $A$ have absolute values different from 1, then $f_A$ is hyperbolic on the whole torus $\mathbb{T}^m$.

   Theorem 2 can be applied in this case, since $\mathbb{T}^m$ is a connected hyperbolic repellor for $f_A$, and we obtain a physical measure for the multivalued inverse iterates of $f_A$. In this case the inverse
SRB measure $\mu^-$ is in fact the Haar measure on $T^m$ since the stable potential is constant. Also from Theorem 3 we obtain that a Pesin type formula holds for the negative Lyapunov exponents.

2. Anosov endomorphisms. Theorem 2 and 3 can be applied also in the case of Anosov endomorphisms on a Riemannian manifold $M$, since $M$ can be viewed as a hyperbolic repellor. In general the stable potential is not constant and $\mu^-$ is not necessarily absolutely continuous with respect to the Lebesgue measure on $M$. We obtain again that the asymptotic distribution of preimages for Lebesgue almost every point in $M$ is equal to the equilibrium measure $\mu^-=\mu_s$, and that the inverse SRB measure $\mu^-$ has absolutely continuous conditional measures on local stable manifolds.

3. Non-Anosov hyperbolic non-expanding repellors for products. Let us take for instance $f : \mathbb{P}C^1 \to \mathbb{P}C^1, f([z_0 : z_1]) = [z_0^2 : z_1^2]$, and $g : T^2 \to T^2, g$ being induced by the matrix $A = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$. We see easily that $A$ has one eigenvalue in $(0,1)$ and another larger than 1, so $g$ is hyperbolic. We take the product

$$F : \mathbb{P}C^1 \times T^2 \to \mathbb{P}C^1 \times T^2, F([z_0 : z_1], (x,y)) = (f([z_0 : z_1]), g(x,y)) \text{ and } \Lambda := S^1 \times T^2$$

Then $\Lambda$ is a connected hyperbolic non-Anosov repellor for the smooth endomorphism $F$ and we can apply Theorems 2 and 3.

4. Perturbations. According to Proposition 3, if $f$ is hyperbolic on a connected repellor $\Lambda$ and if an endomorphism $g$ is a $C^1$ perturbation of $f$, then $g$ has a connected hyperbolic repellor denoted $\Lambda_g$ which is close to $\Lambda$. We can form then a large class of examples by perturbing known examples, like the ones above. Then since $g$ is hyperbolic on $\Lambda_g$ we can again apply Theorems 2 and 3 this time for inverse SRB measures which might be more complicated than in the original (unperturbed) example. For instance, let us take $F : \mathbb{P}C^1 \times T^2 \to \mathbb{P}C^1 \times T^2$ given by

$$F([z_0 : z_1], (x,y)) = ([z_0^2 : z_1^2], f_A(x,y)),$$

where $f_A$ is the toral endomorphism induced by the matrix $A = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$. As can be seen, $F$ has a connected hyperbolic repellor $\Lambda := S^1 \times T^2$. Consider the following perturbation of $F$, $F_\varepsilon : \mathbb{P}C^1 \times T^2 \to \mathbb{P}C^1 \times T^2$ given by:

$$F_\varepsilon([z_0 : z_1], (x,y)) = \left( [z_0^2 + \varepsilon z_1^2 \cdot e^{2\pi i (2x+y)} : z_1^2], (2x + y + \varepsilon \sin(2\pi (x+y)), 2x + 2y + \varepsilon \cos^2(4\pi x)) \right)$$

It can be seen that $F_\varepsilon$ is well defined as a smooth endomorphism on $\mathbb{P}C^1 \times T^2$ and that it is a $C^1$ perturbation of $F$. It follows from Proposition 3 that $F_\varepsilon$ has a connected hyperbolic repellor $\Lambda_\varepsilon$ (on which $F_\varepsilon$ has both stable as well as unstable directions), and that $\Lambda_\varepsilon$ is close to $\Lambda$. However $\Lambda_\varepsilon$ is different from $\Lambda$, and it has a complicated structure with self-intersections; its projection on the second coordinate is $T^2$. For this repellor $\Lambda_\varepsilon$ we can apply Theorem 2 to get a physical measure $\mu^-_{\varepsilon}$ for the local inverse iterates of $F_\varepsilon$. This physical measure $\mu^-_{\varepsilon}$ is the equilibrium measure of the non-constant stable potential

$$\Phi^s_\varepsilon([z_0 : z_1], (x,y)) := \log |\det(DF_\varepsilon)([z_0 : z_1], (x,y))|, \text{ for } ([z_0 : z_1], (x,y)) \in \Lambda_\varepsilon.$$
We know from Theorem 3 that the conditional measures of the inverse SRB measure $\mu^-_{\varepsilon}$ on the local stable manifolds (which are contained in the repellor $\Lambda_{\varepsilon}$), are absolutely continuous with respect to the induced Lebesgue measures.

Also a Pesin type formula is true for the measure-theoretic entropy $h_{\mu^-_{\varepsilon}}$ of $\mu^-_{\varepsilon}$, and the negative Lyapunov exponents (which are non-constant if $\varepsilon \neq 0$).

Similarly one can perturb other connected hyperbolic repellors to obtain new dynamical systems for which Theorems 2 and 3, as well as Corollary 1 can be applied.

Another observation is that one can form repellors quite easily. We need only the existence of families of stable/unstable cones in some open set $U$ and the topological condition $\overline{U} \subset f(U)$. Then one can form the basic set $\Lambda := \cap_{n \in \mathbb{Z}} f^n(U)$, on which we have a hyperbolic structure. The inverse SRB measure $\mu^-$ supported on $\Lambda$ can be approximated Lebesgue almost everywhere on $U$, like in Theorem 2 and will have good ergodic properties as found in Theorem 5. However it may be difficult to describe this measure explicitly, especially in the non-Anosov case, since $(\Lambda, \mu^-)$ is not 1-sided Bernoulli.

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