Abstract. Fine-scale structures in a turbulent channel flow and boundary layer at comparable Reynolds numbers are studied through the velocity gradient tensor. The joint pdf of the two invariants of the tensor that characterize the topology of the structures collapse well among these flows under the Kolmogorov scaling in both, the inner and outer layers. However, the averaged time-evolution of the invariant differs among these flows even in the inner layer. This discrepancy is caused by the different effect of the pressure field on the dynamics of these invariants. By using filtered velocity fields, large-scale structures in the outer layer have also been studied. Their topology is similar to smaller-scales, but less intermittency is observed.

1. Introduction

The topology of fine-scale structures of fluid flows is characterized by the velocity gradient tensor, $A_{ij} \equiv \partial u_i / \partial x_j$, where $(x_1, x_2, x_3)$ and $(u_1, u_2, u_3)$ are the Cartesian coordinate and the velocity field, respectively. Here we consider incompressible flows for which $A_{ii} = 0$. It is convenient to decompose the tensor into a symmetric part $S_{ij}$ and an asymmetric part $W_{ij}$ by

$$
S_{ij} = \frac{1}{2}(A_{ij} + A_{ji}), \quad W_{ij} = \frac{1}{2}(A_{ij} - A_{ji}).
$$

In general, the velocity gradient tensor is characterized by five scalar invariants, which are given by

$$
Q_A = -\frac{1}{2}A_{ij}A_{ji}, \quad R_A = -\frac{1}{3}A_{ij}A_{jk}A_{ki},
$$

$$
Q_S = -\frac{1}{2}S_{ij}S_{ji}, \quad R_S = -\frac{1}{3}S_{ij}S_{jk}S_{ki}, \quad Q_W = -\frac{1}{2}W_{ij}W_{ji}.
$$

These invariants give us fruitful information of the velocity field (Meneveau, 2011). For example, the values of $Q_A$ and $R_A$, which are related by the eigenvalues of $A_{ij}$, classify in an unambiguous way the local topology of the fine-scale structures (Chong et al., 1990). The joint probability density function (jpdf) between $Q_A$ and $R_A$ has been extensively studied using numerical and experimental data, which has been shown to have a skewed distribution. Direct numerical simulations (DNS) (Siggia, 1981; Jiménez et al., 1993) have, up to now, also revealed that turbulent flows have a preferred topology - that is, worm- or sheet-like elongated vortices at...
Table 1. Parameters of the DNS. The outer scale $\delta$ is the boundary layer thickness for the boundary layer and the half-width for the channel flow. $T$ is the time-interval used to compute the statistics. The superscript $^+$ means normalization by a wall-unit determined from the friction velocity, $u_\tau$, and the kinematic viscosity, $\nu$. The domain size ($L_x, L_y, L_z$) for the boundary layer is 4.3% of the full-length of the numerical domain of the DNS, while the full domain was used for the channel.

|            | $\delta^+$ | $L_x/\delta$ | $L_y/\delta$ | $L_z/\delta$ | $Tu_\tau/\delta$ |
|------------|-------------|---------------|---------------|---------------|------------------|
| Boundary layer | 600         | 2.4           | 4.4           | 9.2           | 8               |
| Channel     | 630         | $2\pi$        | 2             | $\pi$         | 13              |

small-scales, which is common to the different types of flows studied. The same approach has been used in this study - using new numerical data generated by DNS of a zero-pressure-gradient turbulent boundary layer and channel flow, to address the similarities and differences between these flows in their topological characteristics.

2. DNS data base

Simulations of a turbulent boundary layer were carried out with the same numerical methods described in Simens et al. (2009) for Reynolds number $R_\theta = 600$ to 2000. For this study, a short streamwise domain where the boundary layer thickness $\delta$ changes 3% is used to compute the statistics. A turbulent channel flow was also simulated by solving incompressive Navier-Stokes equation for the primitive variables. Parameters for both these simulations are summarized in Table 1. Since the numerical channel used here is too small to capture outer large-scale structures, the study is restricted to the small-scales structures which are expected to be unaffected by any artefact that may be introduced due to the small box size. The basic statistics of these flows, as functions of the wall-distance, $y$, are shown in figure 1 and are found to agree reasonably well. The small excess in the wall-normal and spanwise velocity fluctuations in the boundary layer compared to the channel flow has previously been observed in Jiménez et al. (2010).

3. Statistics of the invariants

A significant characteristic feature of wall-bounded turbulent flows is inhomogeneity in the wall-normal direction. The topology of flow structures at small scales also exhibits this inhomogeneity. It is typically demonstrated by the profiles of the statistics of the invariants as functions of the wall-distance, as shown in figure 2. The different flows are found to have similar mean profiles of $Q_A$. In both flows, $\langle Q_A \rangle$ is negative in the viscous sublayer and positive in the buffer layer, while it vanishes in the outer layer. Figure 2(b) compares the mean profiles of $-\langle Q_S \rangle$ and $\langle Q_W \rangle$, which are essentially averaged dissipation rate and enstrophy in these flows. To emphasize the small difference in the outer layer, $\langle Q_S \rangle$ and $\langle Q_W \rangle$ in this figure are premultiplied by $y$. In both flows, $-\langle Q_S \rangle$ and $\langle Q_W \rangle$ agree through whole the range of $y$ (Soria et al., 1997). A difference between these flows is found in the outer layer, where the profiles of $-\langle Q_S \rangle$ and $\langle Q_W \rangle$ have a small bump in the boundary layer. It is a reflection of the influence of laminar flow into the boundary layer, that enhances shear and dissipation at the laminar-turbulence boundary (Corrsin & Kistler, 1955).
Figure 1. Comparison of statistics between channel (○) and boundary layer (△). (a) Mean streamwise velocity $U$ and (b) root-mean-square of fluctuations of streamwise ($u'^+$), wall-normal ($v'^+$) and spanwise ($w'^+$) velocity component, as functions of the wall-distance $y$.

Figure 2. Mean profiles of (a) $Q_A$ and (b) $Q_S$ and $Q_W$, for channel (○) and boundary layer (△). To emphasize small differences in the outer layer, $Q_S$ and $Q_W$ in (b) are multiplied by $y$. $-\langle Q_S \rangle; -\langle Q_W \rangle$. Vertical dashed lines indicate the wall-normal ranges, labeled by I ($16 < y^+ < 50$) and II ($0.2 < y/\delta < 0.5$ in boundary layer), to be used for local averaging of statistics shown below.

In the subsequent analysis two wall-normal ranges are considered: (i) the buffer layer ($16 < y^+ < 50$) where $Q_A$ is positive, and (ii) the outer layer ($0.2 < y/\delta < 0.5$) where the difference in the mean dissipation rate and enstrophy between the channel and boundary layer is significant. These two regions are indicated as I and II in figure 2 respectively.

Figures 3(a,b) compares the jpdf of $Q_A$ and $R_A$ between the channel flow and boundary layer averaged within the domains I and II. These jpdfs share a typical feature in common to other types of turbulent flows: they have long tails in the second and forth quadrants in the $(R_A, Q_A)$ plane. The channel flow and boundary layer jpdfs agree quite well when each is scaled by the local average $\langle -Q_S \rangle$ in both domains I and II. Since normalization by $\langle -Q_S \rangle$ is equivalent to the Kolmogorov scaling, this result is natural. However, it should be noted that the boundary layer has a stronger intensity than the channel when normalized by their wall-units, as shown in figure...
2(b). Figure 3(c,d) show the jpdfs of $Q_W$ and $Q_S$. As previously shown by Blackburn et al. (1996); Chong et al. (1998), they are strongly correlated in the viscous sublayer (not shown), which leads to an additional constraint to the velocity gradient tensor in the near-wall region. They become uncorrelated far from the wall. The jpdf of $Q_S$ and $Q_W$ of the two different flows are similar near the wall, as shown in figure 3(c). A difference between these flows is found in the tail of the pdf in the outer layer though its shape is similar (see figure 3(d)). The tail is not scaled well by the Kolmogorov scale or the wall-units. In the wall-units the boundary layer has a wider tail than the channel. The anomaly of the scaling implies that the scales in the dissipation range are influenced by laminar flow even below 50% of $\delta$ of boundary layers.

4. Averaged dynamics of invariants

Lagrangian derivatives of the invariants are derived from Navier-Stokes equations (Cantwell, 1992). For $Q_A$ and $R_A$,

$$\frac{DQ_A}{Dt} = -3R_A - A_{ij}H_{ji}, \quad \frac{DR_A}{Dt} = \frac{2}{3}Q_A^2 - A_{ij}A_{jk}H_{kl},$$

\[ (2) \]
where
\[ H_{ij} = -\left( \frac{\partial^2 p}{\partial x_i \partial x_j} - \frac{\partial^2 \rho}{\partial x_i \partial x_j} \right) + \nu \frac{\partial^2 A_{ij}}{\partial x_i \partial x_j}, \]
p is the pressure and \( \delta_{ij} \) is the Kronecker’s delta. Equation (2) states that dynamics of \( Q_A \) and \( R_A \) follows an autonomous system if \( H_{ij} = 0 \). This system, so-called the Restricted Euler (RE) equation, preserves the discriminant \( D \equiv \frac{1}{A^2} R_A^3 + Q_A^3 \) of the characteristic equation of \( A_{ij} \), and its dynamics is independent of the flow geometry and Reynolds number (Cantwell, 1992). In turbulent flows, \( H_{ij} \) is considered as a stochastic forcing to the system, and some models have been suggested (Meneveau, 2011). The averaged evolution of the invariants can be obtained from experimental or numerical data by conditional average,
\[
\left\langle \frac{DQ_A}{Dt} \mid Q_A, R_A \right\rangle, \quad \left\langle \frac{DRA}{Dt} \mid Q_A, R_A \right\rangle,
\]
as proposed by Ooi et al. (1999) who investigated the evolution of the invariants by using DNS data of isotropic turbulence. Such data is also useful to assess and improve models.

Figures 4(a,b) show examples of trajectories on the \( (R_A, Q_A) \) plane computed by the conditional averages (3), comparing the channel and boundary layer in the lower buffer layer, \( 16 < y^+ < 50 \) (the range indicated by I in figure 2). They share a common qualitative feature of trajectories, which has been found in different types of flow. They tend to travel clock-wise around the origin. In this range, trajectories are repelled by the vicinity of the origin, and converge towards a limit cycle in both of flows. The time scales of the cycle at different layers in a boundary layer have recently been studied by Atkinson et al. (2011). Its basin is larger in channel flow than in the boundary layer. If \( Q_A \) and \( R_A \) are large enough, trajectories move away along \( D = 0 \) line. This asymptotic behaviour is similar to that of the RE system. The length-scale of this motion will ultimately become smaller than the spatial resolution and will not be resolved by DNS, but such events are quite rare. Although there is reasonable good agreement in the the statistics of the invariants between the channel flow and the boundary layer in the buffer layer, this agreement is not carried over to the trajectories. This discrepancy is attributed to the different behaviour of pressure field penetrating deeply into the near-wall range, which has been noticed by Jiménez et al. (2010). The dynamics determined by \( H_{ij} \) without the pressure-Hessian terms are shown in figures 4(c,d). These figure shows that the viscous effect attracts trajectories to the origin, or converges them to the line \( D = 0 \). The long tail of the jpdf along this line is generated by this effect. This behaviour is consistent to a linear damping model suggested by Martín et al. (1998). Comparing the results in figures 3(c) and (d), it is found that the viscous effect is independent of the type of flow near the wall. It turns out that the discrepancy of trajectories is caused by the pressure, as shown in figures 4(e,f) where the viscous term has been removed from the \( H_{ij} \) term. A major effect of the pressure terms, or incompressibility, is to force the invariants \( Q_A < 0, R_A < 0 \) towards \( Q_A > 0, R_A < 0 \), which forms a vortical structure stretched in its axial direction. Comparing figures 3(e,f) and also their actual velocities, the effect of pressure terms is found to have a larger effect in channel flow.

5. Filtered field

The velocity gradient tensor concerns fine-scale structures in the dissipation range. However, if \( A_{ij} \) is computed from a velocity field filtered to smooth out small-scale fluctuations, its invariants characterize the topological feature at larger-scales. For example, filtered velocity fields can be used to see structures in the inertial subrange in homogeneous isotropic turbulence (Chertkov et al., 1999). In wall-bounded turbulent flows, self-similar flow structures above the buffer layer are typically observed, and they are attracting some interests because of their role in the
Figure 4. Trajectories of $Q_A$ and $R_A$ averaged in $16 < y^+ < 50$ for (a,c,e) boundary layer and (b,d,f) channel. (a,b) by equation (2), (c,d) without pressure terms, and (e,f) without viscous term. Gray contours are the same as in figure 3(a).
dynamics of these flows. To study these structures above the buffer layer, a Gaussian low-
pass filter is used to remove fluctuations whose wavelengths in the wall-parallel directions are
below 200 wall-units. Figure 5 compares a filtered wall-normal velocity field with the original
field, showing that the filtered field preserves only the large-scale fluctuation. At large-scales
the topology of flow structures are similar to those at the fine-scales, as shown in figure 6(a).
Though this type of structures is common, we can find a quantitative difference between large-
and small-scales. The contour-level at 70% of the probability in the jpdf agree, but the original
one has a wider tail, which provides some evidence that suggests that the observed structures
are more intermittent at smaller-scales. Furthermore, the time-evolution of these invariants at
large-scales differs from that at the fine-scales (figure 6(b)). At large-scales, where viscous effect
is weak, the dynamics is dominated by the pressure terms in $H_{ij}$. The limit cycle observed
in figures 3(a,b) is absent, and all the trajectories move away from the vicinity of the origin
nearly along the $D = 0$ line. This may suggest that they break up into smaller ones. However,
it should be noted that the trajectories of invariants cannot solely describe such a multi-scale
phenomenon.
6. Conclusions

The statistics and dynamics of the scalar invariants of the velocity gradient tensor have been studied by using DNS data of a turbulent channel flow and boundary layer at similar Reynolds numbers. The statistics of these flows agree in the inner layer, but dissipation rate and enstrophy in the boundary layer is stronger than channel in the outer layer. The similarity of the topology of fine-scales between these flows is found through whole the wall-normal range, and the relation between $Q_S$ and $Q_W$ also agree. The difference in the scales in the outer layer may be attributed to the invasion of potential flow into the boundary layer. It is found that the pressure effect leads to the different behaviour of $Q_A$ and $R_A$ even in the lower inner layer, $16 < y^+ < 50$.

We also compared topology and dynamics of larger-scale structures deduced from invariants for filtered velocity field with those of fine-scales. At larger-scales the topology are qualitatively similar to the small-scales, but less intermittent. The dynamics is dominated by the pressure because viscous effect is quite weak.

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