Complexity Classification Of The Six-Vertex Model

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Abstract

We prove a complexity dichotomy theorem for the six-vertex model. For every setting of the parameters of the model, we prove that computing the partition function is either solvable in polynomial time or \#P-hard. The dichotomy criterion is explicit.

Keywords: Six-vertex model; Spin system; Holant problems; Interpolation

1. Introduction

A primary purpose of complexity theory is to provide classifications to computational problems according to their inherent computational difficulty. While computational problems can come from many sources, a class of problems from statistical mechanics has a remarkable affinity to what is naturally studied in complexity theory. These are the sum-of-product computations, a.k.a. partition functions in physics.

Well-known examples of partition functions from physics that have been investigated intensively in complexity theory include the Ising model and Potts model [9, 8, 7, 11]. Most of these are spin systems. Spin systems as well as the more general counting constraint satisfaction problems (\#CSP) are special cases of Holant problems [5] (see Section 2 for definitions). Roughly speaking, Holant problems are tensor networks where edges of a graph are variables while vertices are local constraint functions; by contrast, in spin systems vertices are variables and edges are (binary) constraint functions. Spin systems can be simulated easily as Holant problems, but Freedman, Lovász and Schrijver proved that simulation in the reverse direction is generally not possible [6]. In this paper we study a family of partition functions that fit the Holant problems naturally, but not as a spin system. This is the six-vertex model.

The six-vertex model in statistical mechanics concerns crystal lattices with hydrogen bonds. Remarkably it can be expressed perfectly as a family of Holant problems with 6 parameters for the associated signatures, although in physics people are more focused on

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regular structures such as lattice graphs, and asymptotic limit. In this paper we study the partition functions of six-vertex models purely from a complexity theoretic view, and prove a complete classification of these Holant problems, where the 6 parameters can be arbitrary complex numbers.

The first model in the family of six-vertex models was introduced by Linus Pauling in 1935 to account for the residual entropy of water ice [13]. Suppose we have a large number of oxygen atoms. Each oxygen atom is connected by a bond to four other neighboring oxygen atoms, and each bond is occupied by one hydrogen atom between two oxygen atoms. Physical constraint requires that the hydrogen is closer to either one or the other of the two neighboring oxygens, but never in the middle of the bond. Pauling argued [13] that, furthermore, the allowed configuration of hydrogen atoms is such that at each oxygen site, exactly two hydrogens are closer to it, and the other two are farther away. The placement of oxygen and hydrogen atoms can be naturally represented by vertices and edges of a 4-regular graph. The constraint on the placement of hydrogens can be represented by an orientation of the edges of the graph, such that at every vertex, exactly two edges are oriented toward the vertex, and exactly two edges are oriented away from it. In other words, this is an Eulerian orientation. Since there are \((\binom{4}{2}) = 6\) local valid configurations, this is called the six-vertex model. In addition to water ice, potassium dihydrogen phosphate \(\text{KH}_2\text{PO}_4\) (KDP) also satisfies this model.

The valid local configurations of the six-vertex model are illustrated in Figure 1. There

![Figure 1: Valid configurations of the six-vertex model.](image_url)

are parameters \(\epsilon_1, \epsilon_2, \ldots, \epsilon_6\) associated with each type of the local configuration. The total energy \(E\) is given by \(E = n_1\epsilon_1 + n_2\epsilon_2 + \ldots + n_6\epsilon_6\), where \(n_i\) is the number of local configurations of type \(i\). Then the partition function is \(Z = \sum e^{-E/k_B T}\), where the sum is over all valid configurations, \(k_B\) is Boltzmann’s constant, and \(T\) is the system’s temperature. Mathematically, this is a sum-of-product computation where the sum is over all Eulerian orientations of the graph, and the product is over all vertices where each vertex contributes a factor \(c_i = e^{\epsilon_i}\) if it is in configuration \(i\) \((1 \leq i \leq 6)\) for some constant \(c\).

Some choices of the parameters are well-studied. On the square lattice graph, when modeling ice one takes \(\epsilon_1 = \epsilon_2 = \ldots = \epsilon_6 = 0\). In 1967, Elliott Lieb [12] famously showed that, as the number \(N\) of vertices approaches \(\infty\), the value of the “partition function per vertex” \(W = Z^{1/N}\) approaches \((\frac{1}{3})^{3/2}\) \(\approx 1.5390007\ldots\) (Lieb’s square ice constant). This matched experimental data \(1.540 \pm 0.001\) so well that it is considered a triumph.
There are other well-known choices in the six-vertex model family. The KDP model of a ferroelectric is to set \( \epsilon_1 = \epsilon_2 = 0 \), and \( \epsilon_3 = \epsilon_4 = \epsilon_5 = \epsilon_6 > 0 \). The Rys F model of an antiferroelectric is to set \( \epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 > 0 \), and \( \epsilon_5 = \epsilon_6 = 0 \). When there is no ambient electric field, the model chooses the zero field assumption: \( \epsilon_1 = \epsilon_2, \epsilon_3 = \epsilon_4 \), and \( \epsilon_5 = \epsilon_6 \). Historically these are widely considered among the most significant applications ever made of statistical mechanics to real substances. In classical statistical mechanics the parameters are all real numbers while in quantum theory the parameters are complex numbers in general.

In this paper, we give a complete classification of the complexity of calculating the partition function \( Z \) on any 4-regular graph defined by an arbitrary choice parameter values \( c_1, c_2, \ldots, c_6 \in \mathbb{C} \). (To state our theorem in strict Turing machine model, we take \( c_1, c_2, \ldots, c_6 \) to be algebraic numbers.) Depending on the setting of these values, we show that the partition function \( Z \) is either computable in polynomial time, or it is \#P-hard, with nothing in between. The dependence of this dichotomy on the values \( c_1, c_2, \ldots, c_6 \) is explicit.

A number of complexity dichotomy theorems for counting problems have been proved previously. These are mostly on spin systems, or \#CSP (counting Constraint Satisfaction Problems), or on Holant problems with symmetric local constraint functions. \#CSP is the special case of Holant problems where EQUALITIES of all arities are auxiliary functions assumed to be present. Spin systems are a further specialization of \#CSP, where there is a single binary constraint function (see Section 2). The six-vertex model cannot be expressed as a \#CSP problem. It is a Holant problem where the constraint functions are not symmetric. Thus previous dichotomy theorems do not apply. This is the first complexity dichotomy theorem proved for a class of Holant problems on non-symmetric constraint functions and without auxiliary functions assumed to be present.

However, one important technical ingredient of our proof is to discover a direct connection between some subset of the six-vertex models with spin systems. Another technical highlight is a new interpolation technique that carves out subsums of a partition function by assembling a suitable sublattice, and partition the sum over an exponential range according to an enumeration of the intersections of cosets of the sublattice with this range.

2. Preliminaries and notations

A constraint function \( f \) of arity \( k \) is a map \( \{0, 1\}^k \rightarrow \mathbb{C} \). Fix a set of constraint functions \( \mathcal{F} \). A signature grid \( \Omega = (G, \pi) \) is a tuple, where \( G = (V, E) \) is a graph, \( \pi \) labels each \( v \in V \) with a function \( f_v \in \mathcal{F} \) of arity \( \deg(v) \), and the incident edges \( E(v) \) at \( v \) with input variables of \( f_v \). We consider all 0-1 edge assignments \( \sigma \), each gives an evaluation \( \prod_{v \in V} f_v(\sigma|_{E(v)}) \), where \( \sigma|_{E(v)} \) denotes the restriction of \( \sigma \) to \( E(v) \). The counting problem on the instance \( \Omega \) is to compute \( \text{Holant}_\Omega = \sum_{\sigma : E \rightarrow \{0, 1\}} \prod_{v \in V} f_v(\sigma|_{E(v)}) \). The Holant problem parameterized by the set \( \mathcal{F} \) is denoted by \( \text{Holant}(\mathcal{F}) \). We denote by \( \text{Holant}(\mathcal{F} | G) \) the Holant problem on bipartite graphs where signatures from \( \mathcal{F} \) and \( G \) are assigned to vertices from the Left and Right.

A spin system on \( G = (V, E) \) has a variable for every \( v \in V \) and a binary function \( g \) for every edge \( e \in E \). The partition function is \( \sum_{\sigma : V \rightarrow \{0, 1\}} \prod_{(u, v) \in E} g(\sigma(u), \sigma(v)) \). Spin systems
are special cases of \#CSP(\mathcal{F}) (counting CSP) where \mathcal{F} consists of a single binary function. In turn, \#CSP(\mathcal{F}) is the special case of Holant where \mathcal{F} contains \text{EQUALITY} of all arities.

A constraint function is also called a signature. A function \( f \) of arity \( k \) can be represented by listing its values in lexicographical order as in a truth table, which is a vector in \( \mathbb{C}^{2^k} \), or as a matrix in \( \mathbb{C}^{2^{k_1} \times 2^{k_2}} \) if we partition the \( k \) variables to two parts, where \( k_1 + k_2 = k \). A function is symmetric if its value depends only on the Hamming weight of its input. A symmetric function \( f \) on \( k \) Boolean variables can be expressed as \([f_0, f_1, \ldots, f_k] \), where \( f_w \) is the value of \( f \) on inputs of Hamming weight \( w \). For example, \((=_k)\) is the \text{EQUALITY} signature \([1, 0, \ldots, 0, 1]\) (with \( k-1 \) 0’s) of arity \( k \). We use \( \neq_2 \) to denote binary \text{DISEQUALITY} function \([0, 1, 0, 0]\). The support of a function \( f \) is the set of inputs on which \( f \) is nonzero.

Given an instance \( \Omega = (G, \pi) \) of Holant(\mathcal{F}), we add a middle point on each edge as a new vertex to \( G \), then each edge becomes a path of length two through the new vertex. Extend \( \pi \) to label a function \( g \) to each new vertex. This gives a bipartite Holant problem Holant(\( g \mid \mathcal{F} \)). It is obvious that Holant(\( \neq_2 \mid \mathcal{F} \)) is equal to Holant(\mathcal{F}).

For \( T \in \text{GL}_2(\mathbb{C}) \) and a signature \( f \) of arity \( n \), written as a column vector \( f \in \mathbb{C}^n \), we denote by \( T^{-1} f = (T^{-1})^\otimes n f \) the transformed signature. For a signature set \( \mathcal{F} \), define \( T^{-1} \mathcal{F} = \{ T^{-1} f \mid f \in \mathcal{F} \} \). For signatures written as row vectors we define \( \mathcal{F} T \) similarly. The holographic transformation defined by \( T \) is the following operation: given a signature grid \( \Omega = (H, \pi) \) of Holant(\( \mathcal{F} \mid \mathcal{G} \)), for the same bipartite graph \( H \), we get a new signature grid \( \Omega' = (H, \pi') \) of Holant(\( \mathcal{F} T \mid T^{-1} \mathcal{G} \)) by replacing each signature in \( \mathcal{F} \) or \( \mathcal{G} \) with the corresponding signature in \( \mathcal{F} T \) or \( T^{-1} \mathcal{G} \).

In this paper we focus on Holant(\( \neq_2 \mid f \)) when \( f \) has support among strings of hamming weight 2. They are the six-vertex models on general graphs. This corresponds to a set of (non-bipartite) Holant problems by a holographic reduction [16]. Let \( Z = \frac{1}{\sqrt{2}} [\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}] \). The matrix form of \( \neq_2 \) is \([0, 1] = Z^T Z \). Under a holographic transformation with bases \( Z \), Holant(\( \neq_2 \mid f \)) becomes Holant(\( \neq_2 \mid Z^\otimes 4 f \)), where \( Z^\otimes 4 f \) is a column vector \( f \) multiplied by the matrix tensor power \( Z^\otimes 4 \). The bipartite Holant problems of the form Holant(\( \neq_2 \mid f \)) naturally correspond to the non-bipartite Holant problems Holant(\( Z^\otimes 4 f \)). In general \( f \) and \( Z^\otimes 4 f \) are non-symmetric functions.

A signature \( f \) of arity 4 has the signature matrix \( M = M_{x_1x_2x_3x_4}(f) = \begin{bmatrix} f_{0000} & f_{0010} & f_{0001} & f_{0011} \\ f_{1000} & f_{1010} & f_{1001} & f_{1011} \\ f_{0100} & f_{0101} & f_{1010} & f_{1011} \\ f_{1100} & f_{1110} & f_{1101} & f_{1111} \end{bmatrix} \). If \( \{i, j, k, \ell\} \) is a permutation of \( \{1, 2, 3, 4\} \), then the \( 4 \times 4 \) matrix \( M_{x_i x_j, x_k x_\ell}(f) \) lists the 16 values with row index \( x_i x_j \in \{0, 1\}^2 \) and column index \( x_k x_\ell \in \{0, 1\}^2 \) in lexicographic order.

Let \( N = \{0, 1, 2, \ldots\}, H = \frac{1}{\sqrt{2}} [\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}] \) and \( N = [\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}] \). Note that \( N \) is the double \text{DISEQUALITY}, which is the function of connecting two pairs of edges by \( (\neq_2) \).

If \( f \) and \( g \) have signature matrices \( M(f) = M_{x_i x_j, x_k x_\ell}(f) \) and \( M(g) = M_{x_i x_j, x_k x_\ell}(g) \), by connecting \( x_k \) to \( x_s \), \( x_\ell \) to \( x_t \), both with \text{DISEQUALITY} \( (\neq_2) \), we get a signature of arity 4 with the signature matrix \( M(f)NM(g) \) by matrix product with row index \( x_s x_j \) and column index \( x_u x_v \).
The six-vertex model is Holant(≠2 | f), where $M_{x_1, x_2, x_3}(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix}$. We also write this matrix by $M(a, x, b, y, c, z)$. When $a = x, b = y$ and $c = z$, we abridge it as $M(a, b, c)$. Note that all nonzero entries of $f$ are on Hamming weight 2. Denote the 3 pairs of ordered complementary strings by $\lambda = 0011, \overline{\lambda} = 1100, \mu = 0110, \overline{\mu} = 1001, \nu = 0101, \overline{\nu} = 1010$. The support of $f$ is the union $\{\lambda, \overline{\lambda}, \mu, \overline{\mu}, \nu, \overline{\nu}\}$ of the pairs $(\lambda, \overline{\lambda}), (\mu, \overline{\mu})$ and $(\nu, \overline{\nu})$, on which $f$ has values $(a, x), (b, y)$ and $(c, z)$. If $f$ has the same value in a pair, say $a = x$ on $\lambda$ and $\overline{\lambda}$, we say it is a twin.

The permutation group $S_4$ on $\{x_1, x_2, x_3, x_4\}$ induces a group action on $\{s \in \{0, 1\}^4 \mid \text{wt}(s) = 2\}$ of size 6. This is a faithful representation of $S_4$ in $S_6$. Since the action of $S_4$ preserves complementary pairs, this group action has nontrivial blocks of imprimitivity; namely $\{A, B, C\} = \{\{\lambda, \overline{\lambda}\}, \{\mu, \overline{\mu}\}, \{\nu, \overline{\nu}\}\}$. The action on the blocks is a homomorphism of $S_4$ onto $S_3$, with kernel $K = \{1, (12)(34), (13)(24), (14)(23)\}$. In particular one can calculate that the subgroup $S_{\{2, 3, 4\}} = \{1, (23), (34), (24), (234), (243)\}$ maps to $\{1, (AC), (BC), (AB), (ABC), (ACB)\}$. By a permutation from $S_4$, we may permute the matrix $M(a, x, b, y, c, z)$ by any permutation on the values $\{a, b, c\}$ with the corresponding permutation on $\{x, y, z\}$, and moreover we can further flip an even number of pairs $(a, x), (b, y)$ and $(c, z)$. In particular, we can arbitrarily reorder the three rows in $\begin{bmatrix} a & x \\ b & y \\ c & z \end{bmatrix}$, and we can also reverse the order of arbitrary two rows together. In the proof, after one construction, we may use this property to get a similar construction and conclusion, by quoting this symmetry of three pairs or six values.

**Definition 2.1.** A 4-ary signature is redundant iff in its 4 by 4 signature matrix the middle two rows are identical and the middle two columns are identical.

**Theorem 2.2.** [2] If $f$ is a redundant signature and the determinant

$$\det \begin{bmatrix} f_{0000} & f_{0010} & f_{0011} \\ f_{0100} & f_{0110} & f_{0111} \\ f_{1100} & f_{1110} & f_{1111} \end{bmatrix} \neq 0,$$

then Holant(≠2 | f) is #P-hard.

We use $\mathcal{A}$ and $\mathcal{P}$ to denote two classes of tractable signatures. The classes $\mathcal{A}$ and $\mathcal{P}$ are identified as tractable for #CSP [3]. Problems defined by $\mathcal{A}$ are tractable essentially by Gauss Sums (See Theorem 6.30 of [10]). The signatures in $\mathcal{P}$ are tensor products of signatures whose supports are among two complementary bit vectors. Problems defined by them are tractable by a propagation algorithm. The full version [1] contains complete definitions and characterizations of these classes.

**Theorem 2.3.** [3] Let $\mathcal{F}$ be any set of complex-valued signatures in Boolean variables. Then #CSP($\mathcal{F}$) is #P-hard unless $\mathcal{F} \subseteq \mathcal{A}$ or $\mathcal{F} \subseteq \mathcal{P}$, in which case the problem is computable in polynomial time.

**Definition 2.4.** $\mathcal{M}$ is the set of functions, whose support is composed of strings of Hamming weight at most one. $\mathcal{M}' = \{g \mid \exists f \in \mathcal{M}, g(x) = f(\overline{x})\}$, where $\overline{x}$ is the complement of $x$. 

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Note that all unary functions are in $\mathcal{M} \cap \mathcal{M}'$. Theorem 2.5 is a consequence of Theorem 2.2 in [4].

**Theorem 2.5.** Holant($\neq_2 | \mathcal{M}$) and Holant($\neq_2 | \mathcal{M}'$) are polynomial time computable.

### 3. Main theorem

**Theorem 3.1.** Let $f$ be a 4-ary signature with the signature matrix $M_{x_1,x_2,x_3,x_4}(a,x,b,y,c,z)$, then Holant($\neq_2 | f$) is #P-hard except for the following cases:

- $f \in \mathcal{P}$;
- $f \in \mathcal{A}$;
- there is a zero in each pair $(a,x), (b,y), (c,z)$;

in which cases Holant($\neq_2 | f$) is computable in polynomial time.

We prove the complexity classification by categorizing the six values $a, b, c, x, y, z$ in the following way.

1. There is a zero pair. If $f \in \mathcal{A} \cup \mathcal{P}$, then it is tractable. Otherwise it is #P-hard.
2. All values in $\{a, x, b, y, c, z\}$ are nonzero. We prove these are #P-hard.
   a. Three twins. We prove this case mainly by an interpolation reduction from redundant signatures, then apply Theorem 2.2.
   b. There is one pair that is not twin. We prove this by a reduction from Case 2a.
3. There is exactly one zero in $\{a, x, b, y, c, z\}$. All are #P-hard by reducing from Case 2.
4. There are exactly two zeros which are from different pairs. All are #P-hard by reducing from Case 2.
5. There is one zero in each pair. These are tractable according to Theorem 2.5.

By definition, in Case 1 and Case 5, $f$ may have more zero values than the stated ones.

These cases above cover all possibilities: After Case 1 we may assume that there is no zero pair. Then after Case 2 we may assume there is at least one zero and there is no zero pair. Similarly after Case 3 we may assume there are at least two zeros and there is no zero pair. So Case 4 finishes the case when there are exactly two zeros. After Case 4 we may assume there are at least three zeros, but there is no zero pair. Therefore we may assume the only case remaining is where there are exactly three zeros in three distinct pairs, and Case 5 finishes the proof.

In the following we prove the 5 cases to prove the main theorem.

### 4. Case 1: One zero pair

In this section we prove Case 1. Note that by renaming the variables $x_1, x_2, x_3, x_4$ we may assume the signature $f$ of arity 4 with one zero pair has the form in (4.1).

**Lemma 4.1.** Let $f$ be a 4-ary signature with the signature matrix

$$M_{x_1,x_2,x_3,x_4}(f) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha & \beta & 0 \\ 0 & \gamma & \delta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (4.1)$$
where \( \{s,t,u,v\} \) is a permutation of \( \{1,2,3,4\} \). Then Holant(\( \not\equiv 2\mid f \)) is \#P-hard unless \( f \in \mathcal{A} \) or \( f \in \mathcal{P} \), in which case the problem is computable in polynomial time.

**Proof.** By the \( S_4 \) group symmetry, we only need to prove the lemma for \( (s,t,u,v) = (1,2,4,3) \). Tractability follows from Theorem 2.3.

Let \( g(x,y) \) be the binary signature \( g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \) in matrix form. This means \( g_{00} = \alpha, g_{01} = \beta, g_{10} = \gamma \) and \( g_{11} = \delta \). We prove that \#CSP(\( g \)) \( \leq_T \) Holant(\( \not\equiv 2\mid f \)) in two steps. In each step, we begin with a signature grid and end with a new signature grid such that the Holant values of both signature grids are the same.

For step one, let \( G = (U,V,E) \) be a bipartite graph representing an instance of \#CSP(\( g \)), where each \( u \in U \) is a variable, and each \( v \in V \) has degree two and is labeled \( g \). We define a cyclic order of the edges incident to each vertex \( u \in U \), and decompose \( u \) into \( k = \deg(u) \) vertices. Then we connect the \( k \) edges originally incident to \( u \) to these \( k \) new vertices so that each vertex is incident to exactly one edge. We also connect these \( k \) new vertices in a cycle according to the cyclic order (see Figure 2b). Thus, in effect we have replaced \( u \) by a cycle of length \( k = \deg(u) \). (If \( k = 1 \) there is a self-loop.) Each of \( k \) vertices has degree 3, and we assign them \( (=3) \). Clearly this does not change the value of the partition function. The resulting graph has the following properties: (1) every vertex has either degree 2 or degree 3; (2) each degree 2 vertex is connected to degree 3 vertices; (3) each degree 3 vertex is connected to exactly one degree 2 vertex.

Now step two. For every \( v \in V \), \( v \) has degree 2 and is labeled by \( g \). We contract the two edges incident to \( v \). The resulting graph \( G' = (V',E') \) is 4-regular. We put a node on every edge of \( G' \) and assign \( \not\equiv 2 \) to the node (see Figure 2c). Next we assign a copy of \( f \) to every \( v' \in V' \) after this contraction. The input variables \( x_1, x_2, x_3, x_4 \) are carefully assigned at each copy of \( f \) as illustrated in Figure 3 so that there are exactly two configurations to each original cycle, which correspond to cyclic orientations, due to the \( \not\equiv 2 \) on it and the support set of \( f \). These correspond to the 0-1 assignment values at the original variable \( u \in U \). Moreover in each case, the value of the function \( g \) is perfectly mirrored by the value of the function \( f \) under the orientations. So we have \#CSP(\( g \)) \( \leq_T \) Holant(\( \not\equiv 2\mid f \)).

We have \( f(x_1, x_2, x_3, x_4) = g(x_1, x_3) x_1 x_3 \not\equiv 2 \) \( \chi_1 \not\equiv 3 \chi_2 \not\equiv 3 \). Hence, \( g \in \mathcal{A} \) \( \cup \) \( \mathcal{P} \) implies \( f \in \mathcal{A} \) \( \cup \) \( \mathcal{P} \).

Therefore if \( f \not\in \mathcal{A} \) \( \cup \) \( \mathcal{P} \), then \( g \not\in \mathcal{A} \) \( \cup \) \( \mathcal{P} \). Then \#CSP(\( g \)) is \#P-hard by Theorem 2.3. It follows that Pl-Holant(\( \not\equiv 2\mid f \)) is \#P-hard. This finishes the proof.

**5. Case 2: All six values are nonzero**

In this section, we handle the case \( axbycz \neq 0 \), by proving all problems in this case are \#P-hard. Firstly, we give a technical lemma for interpolation reduction. Then we prove the 3-twins case. Finally, we prove the other cases by realizing a 3-twins problem.

**Lemma 5.1.** Suppose \( \alpha, \beta \in \mathbb{C} - \{0\} \), and the lattice \( L = \{(j,k) \in \mathbb{Z}^2 \mid \alpha^j \beta^k = 1\} \) has the form \( L = \{(ns,nt) \mid n \in \mathbb{Z}\} \), where \( s,t \in \mathbb{Z} \) and \( (s,t) \neq (0,0) \). Let \( \phi \) and \( \psi \) be any numbers satisfying \( \phi^s \psi^t = 1 \). If we are given the values \( N_\ell = \sum_{j,k \geq 0, j+k \leq m} (\alpha^j \beta^k)^\ell x_{j,k} \) for \( \ell = 1,2,\ldots \left(\frac{m+2}{2}\right) \), then we can compute \( \sum_{j,k \geq 0, j+k \leq m} \phi^j \psi^k x_{j,k} \) in polynomial time.
Figure 2: The reduction from \( \#CSP(g) \) to Holant\( (\neq_2 | f) \). The circle vertices are assigned \( =d \), where \( d \) is the degree of the corresponding vertex, the diamond vertices are assigned \( g \), the triangle vertices are assigned \( f \), and the square vertices are assigned \( \neq_2 \). In the first step, we replace a vertex by a cycle, where the length of the cycle is the degree of the vertex. The vertices on the cycle are assigned \( =3 \). In the second step, we merge two vertices that are connected to the diamond with \( g \) and assign \( f \) to the new vertex.

Figure 3: Assigning input variables at one copy of \( f \): Suppose the binary function \( g \) is applied to (the ordered pair) \((u, u')\). The variables \( u \) and \( u' \) have been replaced by cycles of length \( \text{deg}(u) \) and \( \text{deg}(u') \) respectively. (In the figure, they have \( \text{deg}(u) = 5 \) and \( \text{deg}(u') = 3 \).) For the cycle \( C_u \) representing a variable \( u \), we associate the value \( u = 0 \) with a clockwise orientation, and \( u = 1 \) with a counter-clockwise orientation. We assign \( x_i \) to the edge labelled by \( i \) for \( i = 1, 2, 3, 4 \). Then by the support of \( f \), \( x_1 = 0 \) forces \( x_2 = 1 \), \( 0 \) respectively, and similarly \( x_4 = 0,1 \) forces \( x_3 = 1,0 \) respectively. Thus there is a natural 1-1 correspondence between \( u = 0 \) (respectively, \( u = 1 \)) with clockwise (respectively, counter-clockwise) orientation of the cycle \( C_u \), and similarly for \( C_{u'} \). Under this 1-1 correspondence, the value of the function \( g \) is perfectly mirrored by the value of the function \( f \).

Figure 4: Recursive construction of the interpolation in Lemma 5.2. The circles are assigned \( f \) and the squares are assigned \( \neq_2 \).
Proof. We treat $\sum_{j,k \geq 0, j+k \leq m} (\alpha_j \beta^k)^{\ell} x_{j,k} = N_{\ell}$ (where $1 \leq \ell \leq \binom{m+2}{2}$) as a system of linear equations with unknowns $x_{j,k}$. The coefficient vector of the first equation is $(\alpha_j \beta^k)$, indexed by the pair $(j, k)$, where $0 \leq j, k \leq m$ and $j + k \leq m$. The coefficient matrix of the linear system is a Vandermonde matrix, with row index $\ell$ and column index $(j, k)$. However, this Vandermonde matrix is rank deficient. If $(j, k) - (j', k') \in L$, then columns $(j, k)$ and $(j', k')$ have the same value.

We can combine the identical columns $(j, k)$ and $(j', k')$ if $(j, k) - (j', k') \in L$, since for each coset $T$ of $L$, the value $\alpha_j \beta^k$ is constant. Thus, the sum $\sum_{j,k \geq 0, j+k \leq m} (\alpha_j \beta^k)^{\ell} x_{j,k}$ can be written as $\sum_{T} \alpha_j \beta^k \left( \sum_{j,k \geq 0, j+k \leq m, (j,k) \in T} x_{j,k} \right)$, where the sum over $T$ is for all cosets $T$ of $L$ having a non-empty intersection with the cone $C = \{(j, k) \mid 0 \leq j, k \leq m, j + k \leq m \}$. Now the coefficient matrix, indexed by $1 \leq \ell \leq \binom{m+2}{2}$ for the rows and the cosets $T$ with $T \cap C \neq \emptyset$ for the columns, has full rank. And so we can solve $\left( \sum_{j,k \geq 0, j+k \leq m, (j,k) \in T} \alpha_j \beta^k x_{j,k} \right)$ for each coset $T$ with $T \cap C \neq \emptyset$. Notice that for the sum $\sum_{j+k \leq m} \phi^j \psi^k \in T \cdot x_{j,k}$, we also have the expression $\sum_{T} \phi^j \psi^k \left( \sum_{j,k \geq 0, j+k \leq m, (j,k) \in T} x_{j,k} \right)$, since $\phi^j \psi^k$ on each coset $T$ of $L$ is also constant. The lemma follows.

Now we prove the $\#P$-hardness for the 3-twins case. In this case $a = x$, $b = y$ and $c = z$. We denote by $M(a, b, c)$ the problem defined by the signature matrix $M_{x_1 x_2 x_4 x_3}(a, b, b, c, c)$.

**Lemma 5.2.** Let $f$ be a 4-ary signature with the signature matrix $M_{x_1 x_2 x_4 x_3}(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ a & 0 & 0 & 0 \end{bmatrix}$ with $abc \neq 0$. Then Holant$(\neq_2 \mid f)$ is $\#P$-hard.

**Proof.** We construct a series of gadgets by a chain of one leading copy of $f$ and a sequence of twisted copies of $f$ linked by two ($\neq_2$)'s in between. It has the signature matrix $D_s = M(NM')^{s-1}$, for $s \geq 1$, where $M = M_{x_1 x_2 x_4 x_3}(f)$, $M' = M_{x_2 x_1 x_4 x_3}(f)$ is a permuted copy of $M$, and $N$ is the double **DISEQUALITY**. See Figure 4. This is in the right side of Holant$(\neq_2 \mid f)$.

The signature matrix of this gadget is given as a product of matrices. Each matrix is a function of arity 4. Notice that the two row indices in $M_{x_2 x_1 x_4 x_3}(f)$ exchange their positions compared with the standard one $M_{x_1 x_2 x_4 x_3}(f)$. Thus the rows of $M$ under go the permutation $(00, 01, 10, 11) \rightarrow (00, 10, 01, 11)$ to get $M'$. In other words, $M'$ is obtained from $M$ by exchanging the middle two rows. Also $NM'$ reverses all 4 rows of $M'$. So we have

$$NM' = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{bmatrix}, \quad \text{and} \quad D_s = \begin{bmatrix} 0 & 0 & a^s \\ 0 & [b \ c ]^s & 0 \\ a^s & c \ b & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We diagonalize the 2 by 2 matrix in the middle using $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ (note that $H^{-1} = H$),
and get $D_s = P \Lambda_s P$, where

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \Lambda_s = \begin{bmatrix} 0 & 0 & a^s \\ 0 & (b + c)^s & 0 \\ 0 & 0 & (b - c)^s \end{bmatrix}.$$  

The matrix $\Lambda_s$ has a good form for polynomial interpolation. Suppose we have a problem $\text{Holant}(\bm{F})$ to be reduced to $\text{Holant}(\bm{M})$. Let $F$ appear $m$ times in an instance $\Omega$. We replace each appearance of $F$ by a copy of the gadget $D_s$, to get an instance $\Omega_s$ of $\text{Holant}(\bm{M})$. We can treat each of the $m$ appearances of $D_s$ as a new gadget composed of three functions in sequence, $P$, $\Lambda_s$ and $P$, and denote this new instance by $\Omega'_s$. We divide $\Omega'_s$ into two parts. One part is composed of $m$ functions $\Lambda_s$. The second part is the rest of the functions, including $2m$ occurrences of $P$, and its signature is represented by $X$ (which is a tensor expressed as a row vector). The Holant value of $\Omega'_s$ is the dot product $\langle X, \Lambda^{\otimes m}_s \rangle$, which is a summation over $4m$ bits, that is, the values of the $4m$ edges connecting the two parts. We can stratify all 0-1 assignments of these $4m$ bits having a nonzero evaluation of $\text{Holant}_{\Omega'_s}$ into the following categories:

- There are $i$ many copies of $\Lambda_s$ receiving inputs 0011 or 1100;
- There are $j$ many copies of $\Lambda_s$ receiving inputs 0110; and
- There are $k$ many copies of $\Lambda_s$ receiving inputs 1001

such that $i + j + k = m$.

For any assignment in the category with parameter $(i, j, k)$, the evaluation of $\Lambda^{\otimes m}_s$ is clearly $a^s(b + c)^sj(b - c)^sk$. We can rewrite the dot product summation and get

$$\text{Holant}_{\Omega_s} = \text{Holant}_{\Omega'_s} = \langle X, \Lambda^{\otimes m}_s \rangle = \sum_{i+j+k=m} a^s(b + c)^sj(b - c)^sk x_{i,j,k}, \quad (5.2)$$

where $x_{i,j,k}$ is the summation of values of the second part $X$ over all assignments in the category $(i, j, k)$. Because $i + j + k = m$, we also use $x_{i,j}$ to denote the value $x_{i,j,k}$. Similarly we use $x_{j,k}$ or $x_{i,k}$ to denote the same value $x_{i,j,k}$ when there is no confusion.

Generally, in an interpolation reduction, we pick polynomially many values of $s$, and get a system of linear equations in $x_{i,j,k}$. When all $a^i(b + c)^j(b - c)^k$ are distinct, for $i + j + k = m$, we get a full rank Vandermonde coefficient matrix, and then we can solve for each $x_{i,j,k}$. Once we have $x_{i,j,k}$ we can compute any function in $x_{i,j,k}$.

When $a^i(b + c)^j(b - c)^k$ are not distinct, say $a^i(b + c)^j(b - c)^k = a'^i(b + c)^j'(b - c)^k'$, we may define a new variable $y = x_{i,j,k} + x'_{j',k'}$. We can combine all $x_{i,j,k}$ with the same $a^i(b + c)^j(b - c)^k$. Then we have a full rank Vandermonde system of linear equations in these new unknowns. We can solve all new unknowns and then sum them up to get $\sum_{i+j+k=m} x_{i,j,k}$. This is one special function in $x_{i,j,k}$.

The above are two typical application methods in this kind of interpolation. Unfortunately in our case, we may have a rank deficient Vandermonde system, and the sum $\sum_{i+j+k=m} x_{i,j,k}$ does not give us anything useful. This is because if we replace $a^s(b + c)^sj(b - c)^sk$ by the constant value 1 in equation $(5.2)$, we get $\sum_{i+j+k=m} x_{i,j,k}$. Thus, $\sum_{i+j+k=m} x_{i,j,k}$
corresponds to $\Omega_0'$ with all nonzero values in $\Lambda_s$ replaced by the constant 1, i.e., we get a reduction from the problem $M(1, 1, 0)$. But $M(1, 1, 0)$ is a tractable problem, and so we do not get any hardness result by such a reduction.

To prove this lemma, there are three cases when there are 3 twins.

1. Two elements in \{a, b, c\} are equal. By the symmetry of the group action of $S_4$, without loss of generality, we may assume $b = c$. We have

$$\Lambda_s = \begin{bmatrix}
0 & 0 & 0 & s \\
0 & (2b)^s & 0 & 0 \\
0 & 0 & 0 & 0 \\
s & 0 & 0 & 0
\end{bmatrix},$$

and equation (5.2) becomes Holant$_{\Omega_s} = \sum_{i+j=m} x_{i,j} a^s (2b)^{sj}$. Note that all terms $x_{i,j,k}$ with $k \neq 0$ have disappeared. We can interpolate to get $\sum_{i+j=m} x_{i,j}$. This sum corresponds to a #P-hard problem. In fact we define $A = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}$, and then $PAP = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 \\
0 & 1 & 2 & -1 \\
1 & 0 & 0 & 0
\end{bmatrix}$. Then Holant($\neq 2 | M(1, \frac{1}{2}, \frac{1}{2})$) is #P-hard by the determinant criterion for redundant matrices, Theorem 2.2.

2. Two elements in \{a, b, c\} have the opposite value. By the symmetry of the group action of $S_4$, without loss of generality, we may assume $b = -c$. We have

$$\Lambda_s = \begin{bmatrix}
0 & 0 & 0 & s \\
0 & 0 & 0 & 0 \\
0 & 0 & (2b)^s & 0 \\
s & 0 & 0 & 0
\end{bmatrix},$$

and equation (5.2) becomes Holant$_{\Omega_s} = \sum_{i+k=m} x_{i,k} a^s (2b)^{sk}$. Similarly note that all terms $x_{i,j,k}$ with $j \neq 0$ have disappeared. We can interpolate to get $\sum_{i+k=m} x_{i,k}$. This sum corresponds to a #P-hard problem. In fact we define $B = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}$. Then $PBP = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 \\
0 & 1 & 2 & -1 \\
1 & 0 & 0 & 0
\end{bmatrix}$. This matrix defines the problem Holant($\neq 2 | M(2, 1, -1)$), up to a nonzero constant factor.

By the group action we also have $M(-1, 2, 1)$. If we link two copies of $M(-1, 2, 1)$ by $N$, we get $M(1, 5, 4)$, because $[\frac{3}{2} \frac{1}{2}]^2 = [\frac{3}{2} \frac{1}{2}]$.

Then $M(1, 5, 4) = PAP$, where $\Lambda = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}$.

There are only two nonzero values 9 and 1 in $\Lambda$. For $M(1, 5, 4)$, we have Holant$_{\Omega_s} = \sum_{0 \leq i \leq m} x_i 9^{si}$, from which we can solve all $x_i$ ($i = 0, 1, \ldots, m$), we can compute $\sum_{0 \leq i \leq m} x_i 3^{si}$. This realizes the following problem $\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}$, which gives us Holant($\neq 2 | M(1, 2, 1)$). By the symmetry of group action we also have Holant($\neq 2 | M(2, 1, 1)$), which is #P-hard by Theorem 2.2.
3. If we consider $a$, $b$ and $c$ as three nonzero complex numbers on the plane, there are two elements in \{a, b, c\} which are not orthogonal as vectors. By the symmetry of group action of $S_4$, we may assume $b$ and $c$ are not orthogonal. If $b + c = 0$ or $b - c = 0$, then it is already proved in the first two cases. So we may assume $b \neq \pm c$.

By the interpolation method, we have a system of linear equations in $x_{i,j,k}$, whose coefficient matrix $((a^i(b+c)^j(b-c)^k)^s)$ has row index $s$ and column index from \{(i, j, k) \mid i, j, k \in \mathbb{N}, i + j + k = m\}.

Let $\alpha = \frac{b+c}{a}$ and $\beta = \frac{b-c}{a}$. Then they have different norms $|\alpha| \neq |\beta|$. Indeed, if $|\alpha| = |\beta|$ then $|1 + c/b| = |1 - c/b|$ which means that $c/b \in i\mathbb{R}$ is purely imaginary, i.e., $b$ and $c$ are orthogonal.

The matrix $((a^i(b+c)^j(b-c)^k)^s)$, after dividing the $s$th row by $a^{sm}$, has the form $((\alpha^j\beta^k)^s)$, which is a Vandermonde matrix with row index $s$ and column index from \{(j, k) \mid j, k \in \mathbb{N}, j + k \leq m\}. Define $L = \{(j, k) \in \mathbb{Z}^2 \mid \alpha^j\beta^k = 1\}$. This is a sublattice of $\mathbb{Z}^2$. Every lattice has a basis. There are three cases depending on the rank of $L$.

(a) $L = \{(0, 0)\}$. All $\alpha^j\beta^k$ are distinct. It is an interpolation reduction in full power. We can realize $[\begin{array}{llll} 0 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}]$. This corresponds to Holant($\neq_2$| $M(2, 1, 1)$), which is #P-hard by Theorem 2.2.

(b) $L$ contains two vectors $(j_1, k_1)$ and $(j_2, k_2)$ independent over $\mathbb{Q}$. Then the nonzero vectors $j_2(j_1, k_1) - j_1(j_2, k_2) = (0, j_2k_1 - j_1k_2)$ and $k_2(j_1, k_1) - k_1(j_2, k_2) = (k_2j_1 - k_1j_2, 0)$ are in $L$. Hence, both $\alpha$ and $\beta$ are roots of unity, but this contradicts $|\alpha| \neq |\beta|$.

(c) $L = \{(ns, nt) \mid n \in \mathbb{Z}\}$, where $s, t \in \mathbb{Z}$ and $(s, t) \neq (0, 0)$. We know that $s + t \neq 0$, otherwise we get $|\alpha| \neq |\beta|$. By Lemma 5.1, for any numbers $\phi$ and $\psi$ satisfying $\phi^s\psi^t = 1$, we can compute $\sum_{j+k \leq m} \phi^j\psi^k x_{j,k}$ efficiently.

Define $A = [\begin{array}{llll} 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \psi & 0 \\ 1 & 0 & 0 & 0 \end{array}]$, and we have $2PAP = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \phi+\psi & 0 & 2 \\ 0 & \phi-\psi & \phi+\psi & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$. We get Holant($\neq_2$| $M(2, \phi + \psi, \phi - \psi)$).

i. $t = 0$. Without loss of generality $s > 0$. Let $\phi = 1$ and $\psi = 1/2$. We get $M(4, 3, 1)$, from which we can get $M(1, 4, 3)$ by the $S_4$ group symmetry. This is #P-hard by the same proof method as we prove $M(1, 5, 4)$ is #P-hard in Case 2.

ii. $t > 0$ and $s \geq 0$. Let $\phi = \psi + 2$. We need $f(\psi) = (\psi + 2)^s\psi^t = 1$. Because $f(0) = 0 < 1$ and $f(1) \geq 1$, there is a root $\psi_0 \in (0, 1]$. We get $M(2, 2\psi_0+2, 2)$, which is #P-hard by Case 1.

iii. $t > 0$, $s < 0$ and $|t| > |s|$. Let $\phi = \psi + 2$. $\psi^{|t|} = (\psi + 2)^{|s|}$ has a solution $\psi_0$ in $(1, \infty)$. We get $M(2, 2\psi_0+2, 2)$, which is #P-hard by Case 1.

iv. $t > 0$, $s < 0$ and $|t| < |s|$. Let $\psi = \phi + 2$. $\phi^{|t|} = (\phi + 2)^{|s|}$ has a solution $\phi_0$ in $(1, \infty)$. We get $M(2, 2\phi_0+2, -2)$, which is #P-hard by Case 2.

We finish this section by proving the other no zero cases can realize 3-twins.
Lemma 5.3. Let $f$ be a 4-ary signature with the signature matrix $M_{x_1x_2,x_4x_3}(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix}$ with $abcxyz \neq 0$. Then Holant($\neq 2$) $f$ is $\#P$-hard.

Proof. Note that $M_{x_4x_3,x_1x_2}(f) = \begin{bmatrix} 0 & 0 & x \\ 0 & b & z \\ 0 & c & y \\ a & 0 & 0 \end{bmatrix}$. Connecting two copies of $f$ back by double Disequality $N$, we get the gadget whose signature has the signature matrix

$$M_{x_1x_2,x_4x_3}(f)NM_{x_4x_3,x_1x_2}(f) = \begin{bmatrix} 0 & 0 & 0 & ax \\ 0 & 2bc & by + cz & 0 \\ 0 & by + cz & 2yz & 0 \\ ax & 0 & 0 & 0 \end{bmatrix}.$$

If $by + cz \neq 0$, we have realized a function $M(ax,ax,2bc,2yz,by + cz,by + cz)$ of two twins, with all nonzero values. We can use $M(2bc,2yz,ax,ax,by + cz,by + cz)$ to construct the following function by the same gadget

$$M(4bcyz,4bcyz,2ax(by + cz),2ax(by + cz),a^2x^2 + (by + cz)^2,a^2x^2 + (by + cz)^2).$$

If furthermore $a^2x^2 + (by + cz)^2 \neq 0$, we get a nonzero 3-twins function and we can finish the proof by Lemma 5.2. If this process fails, we get a condition that either $by + cz = 0$ or $i(ax + by + cz) = 0$ or $-i(ax + by + cz) = 0$. Recall the symmetry among the 3 pairs $(a,x),(b,y),(c,z)$. If we apply this process with a permuted form of $M$, we will get either $ax + cz = 0$ or $ax + iby + cz = 0$ or $ax - iby + cz = 0$. There is one more permutation of $M$ which gives us either $ax + by = 0$ or $ax + by + icz = 0$ or $ax + by - icz = 0$.

We claim that, when $axbycz \neq 0$, the 3 Boolean disjunction conditions can not hold simultaneously. Hence, one of three constructions will succeed and give us $\#P$-hardness.

To prove the claim, we assume that all 3 disjunction conditions hold. Then we get 3 conjunctions, each a disjunction of 3 linear equations. Each equation is a homogeneous linear equation on $(ax,by,cz)$. The 3 equations in the first conjunction all have the form $\alpha \cdot ax + 1 \cdot by + 1 \cdot cz = 0$ where $\alpha \in \{0, i, -i\}$. Similarly the 3 equations in the second and third conjunction all have the form $1 \cdot ax + \beta \cdot by + 1 \cdot cz = 0$ and $1 \cdot ax + 1 \cdot by + \gamma \cdot cz = 0$ respectively. If at least one equation holds in each of the 3 sets of linear equations with nonzero solution $(ax,by,cz)$, the following determinant

$$\det \begin{bmatrix} \alpha & 1 & 1 \\ 1 & \beta & 1 \\ 1 & 1 & \gamma \end{bmatrix} = 0,$$  \hspace{1cm} (5.3)

for some $\alpha, \beta, \gamma \in \{0, i, -i\}$. However, there are no choices of $\alpha, \beta, \gamma \in \{0, i, -i\}$ such that Equation (5.3) holds: The determinant is $\alpha\beta\gamma - 2 - \alpha - \beta - \gamma$. For $\alpha, \beta, \gamma \in \{0, i, -i\}$, the norm $|2 + \alpha + \beta + \gamma| \geq 2$, but $|\alpha\beta\gamma| = 0$ or 1.

$\square$
6. Case 3: Exactly one zero

**Lemma 6.1.** Let \( f \) be a 4-ary signature with the signature matrix

\[
M_{x_1x_2x_4x_3}(f) = \begin{bmatrix}
0 & 0 & 0 & a \\
0 & b & c & 0 \\
0 & z & y & 0 \\
x & 0 & 0 & 0
\end{bmatrix},
\]

where there is exactly one of \( \{a, b, c, x, y, z\} \) that is zero, then \( \text{Holant}(\not= 2 \mid f) \) is \#P-hard.

**Proof.** Without loss of generality, we can assume that \( b = 0 \). Note that \( M_{x_4x_1x_2}(f) = \begin{bmatrix}
0 & 0 & x \\
0 & c & y \\
0 & 0 & z \\
0 & 0 & 0
\end{bmatrix} \). Connecting a copy of \( f \) with this via \( N \), we get a signature \( g \) with signature matrix

\[
M_{x_1x_2x_4x_3}(f)NM_{x_3x_4x_1x_2}(f) = \begin{bmatrix}
0 & 0 & 0 & ax \\
0 & c^2 & cy & 0 \\
0 & cy & y^2 + z^2 & 0 \\
ax & 0 & 0 & 0
\end{bmatrix}.
\]

If \( y^2 + z^2 \neq 0 \), by Lemma 5.3, \( \text{Holant}(\not= 2 \mid g) \) is \#P-hard. Thus \( \text{Holant}(\not= 2 \mid f) \) is \#P-hard. Otherwise, we have

\[
y^2 + z^2 = 0.
\]

Similarly, \( M_{x_3x_4x_1x_2}(f)NM_{x_4x_3x_2x_1}(f) \) gives us

\[
y^2 + cz = 0.
\]

\( M_{x_4x_3x_1x_2}(f)NM_{x_2x_1x_4x_3}(f) \) gives us

\[
y^2 + c^2 = 0.
\]

From these equations, we get \( c^2 = z^2 = cz = -y^2 \). This gives us \( z = c \) and \( y = \pm ic \), and

\[
M = M_{x_1x_2x_4x_3}(f) = \begin{bmatrix}
0 & 0 & 0 & a \\
0 & 0 & c & 0 \\
0 & 0 & \pm ic & 0 \\
x & 0 & 0 & 0
\end{bmatrix}.
\]

For this matrix \( M \), we may construct \( MNMT = \begin{bmatrix}
0 & 0 & 0 & ax \\
0 & 0 & c^2 & 0 \\
0 & c^2 & \pm 2ic^2 & 0 \\
ax & 0 & 0 & 0
\end{bmatrix} \). Now we may repeat the construction from the beginning using \( MNMT \) instead of \( M \). Because \( (c^2)^2 + (\pm 2ic)^2 \neq 0 \), we get a function of 6 nonzero values. By Lemma 5.3, \( \text{Holant}(\not= 2 \mid f) \) is \#P-hard.

\[\square\]
7. Case 4: Exactly two zeros from distinct pairs

Lemma 7.1. Let \( f \) be a 4-ary signature with the signature matrix

\[
M_{x_1x_2,x_4x_3}(f) = \begin{bmatrix}
0 & 0 & a \\
0 & b & c & 0 \\
0 & z & y & 0 \\
x & 0 & 0 & 0
\end{bmatrix},
\]

where there are exactly two zero entries in \( \{a, b, c, x, y, z\} \) and they are from distinct pairs, then \( \text{Holant}(\neq_2 | f) \) is \#P-hard.

Proof. Recall from Section 2 that we can arbitrarily reorder the three rows in \( \begin{bmatrix} a & x \ b & y \ c & z \end{bmatrix} \), and we can also reverse arbitrary two rows. Thus, we can assume that \( ax \neq 0, bz \neq 0 \) and \( c = y = 0 \). Note that \( M_{x_2x_4,x_1x_3}(f) = \begin{bmatrix} 0 & 0 & 0 & a \\
0 & 0 & b & c \\
0 & z & 0 & 0 \\
x & 0 & 0 & 0 \end{bmatrix} \) and \( M_{x_3x_4,x_1x_2}(f) = \begin{bmatrix} 0 & 0 & 0 & x \\
0 & 0 & 0 & 0 \\
0 & b & z & 0 \\
0 & 0 & a & 0 \end{bmatrix} \). Take two copies of \( f \). If we connect the variables \( x_4, x_3 \) of the first function with the variables \( x_3, x_4 \) of the second function using \( (\neq_2) \), we get a signature \( g \) with the signature matrix

\[
M_{x_1x_2,x_4x_3}(f)NM_{x_3x_4,x_1x_2}(f) = \begin{bmatrix}
0 & 0 & 0 & ax \\
0 & b^2 & bz & 0 \\
0 & bz & z^2 & 0 \\
ax & 0 & 0 & 0
\end{bmatrix}.
\]

By Lemma 5.3, \( \text{Holant}(\neq_2 | g) \) is \#P-hard. Thus \( \text{Holant}(\neq_2 | f) \) is \#P-hard.

8. Case 5: One zero in each pair

Lemma 8.1. If there is one zero in each pair of \((a, x), (b, y), (c, z)\), then \( \text{Holant}(\neq_2 | f) \) is computable in polynomial time.

Proof. We will list the three strings of weight 2 where \( f \) may be nonzero, by the symmetry of the group action of \( S_4 \). We may assume the first string is \( \xi = 0011 \). The second string \( \eta \), being not complementary to \( \xi \) and of weight two, we may assume it is 0101.

The third string \( \zeta \), being not complementary of either \( \xi \) or \( \eta \), and of weight two, must be either 0110 or 1001. Hence, \( \xi = 0 0 1 1 \) \( \eta = 0 1 0 1 \) \( \zeta = 1 0 0 1 \).

Then \( f(x_1, x_2, x_3, x_4) = \text{Is-Zero}(x_1) \cdot g(x_2, x_3, x_4) \) or \( \text{Is-One}(x_4) \cdot h(x_1, x_2, x_3) \), where \( h \in \mathcal{M} \) and \( g \in \mathcal{M}' \). Note that the Is-Zero and Is-One are both unary functions and both belong to \( \mathcal{M} \cap \mathcal{M}' \). By Theorem 2.5, \( \text{Holant}(\neq_2 | f) \) is computable in polynomial time.

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