LIE GROUPS AS 3-DIMENSIONAL ALMOST CONTACT B-METRIC MANIFOLDS IN THE MAIN VERTICAL CLASSES

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Abstract. Almost contact B-metric manifolds of dimension 3 are constructed by a two-parametric family of Lie groups. The class of these manifolds in a known classification of almost contact B-metric manifolds is determined as the direct sum of the main vertical classes. The type of the corresponding Lie algebras in the Bianchi classification is given. Some geometric characteristics and properties of the considered manifolds are obtained.

Introduction

The study of the differential geometry of the almost contact B-metric manifolds has initiated in [5]. The geometry of these manifolds is a natural extension of the geometry of the almost complex manifolds with Norden metric [3, 6] in the case of odd dimension. Almost contact B-metric manifolds are investigated and studied for example in [5, 12, 15, 16, 17, 19, 20, 21].

Here, an object of special interest are the Lie groups considered as 3-dimensional almost contact B-metric manifolds. For example of such investigation see [10].

The aim of the present paper is to make a study of the most important geometric characteristics and properties of a family of Lie groups with almost contact B-metric structure of the lowest dimension 3, belonging to the main vertical classes. These classes are $F_4$ and $F_5$, where the fundamental tensor $F$ is expressed explicitly by the metric $g$, the structure $(\varphi, \xi, \eta)$ and the vertical components of the Lee forms $\theta$ and $\theta^*$, i.e. in this case the Lee forms are proportional to $\eta$ at any point. These classes contain some significant examples as the time-like sphere of $g$ and the light cone of the associated metric of $g$ in the complex Riemannian space, considered in [5], as well as the Sasakian-like manifolds studied in [7].

The paper is organized as follows. In Sec. 1 we give some necessary facts about almost contact B-metric manifolds. In Sec. 2 we construct and study a family of Lie groups as 3-dimensional manifolds of the considered type.

1. Almost contact manifolds with B-metric

Let $(M, \varphi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional almost contact B-metric manifold, i.e. $(\varphi, \xi, \eta)$ is a triplet of a tensor (1,1)-field $\varphi$, a vector field $\xi$ and its dual 1-form $\eta$ called an almost contact structure and the following identities holds:

$\varphi \xi = 0, \quad \varphi^2 = -\text{Id} + \eta \otimes \xi, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1,

where $\text{Id}$ is the identity. The B-metric $g$ is pseudo-Riemannian and satisfies

$g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y)$

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for arbitrary tangent vectors $x, y \in T_pM$ at an arbitrary point $p \in M$ [3].

Further, $x, y, z, w$ will stand for arbitrary vector fields on $M$ or vectors in the
tangent space at an arbitrary point in $M$.

Let us note that the restriction of a B-metric on the contact distribution $H = \ker(\eta)$ coincides with the corresponding Norden metric with respect to the almost complex structure and the restriction of $\varphi$ on $H$ acts as an anti-isometry on the metric on $H$ which is the restriction of $g$ on $H$.

The associated metric $\tilde{g}$ of $g$ on $M$ is given by $\tilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y)$. It is a B-metric, too. Hence, $(M, \varphi, \xi, \eta, \tilde{g})$ is also an almost contact B-metric manifold. Both metrics $g$ and $\tilde{g}$ are indefinite of signature $(n + 1, n)$.

The structure group of $(M, \varphi, \xi, \eta, g)$ is $G \times \mathcal{I}$, where $\mathcal{I}$ is the identity on span($\xi$) and $G = GL(n; \mathbb{C}) \cap O(n, n)$.

The $(0,3)$-tensor $F$ on $M$ is defined by $F(x, y, z) = g((\nabla_{\varphi} \varphi) y, z)$, where $\nabla$ is the Levi-Civita connection of $g$. The tensor $F$ has the following properties:

$$F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi).$$

A classification of the almost contact B-metric manifolds is introduced in [5], where eleven basic classes $F_i$ $(i = 1, 2, \ldots, 11)$ are characterized with respect to the properties of $F$. The special class $F_0$ is defined by the condition $F(x, y, z) = 0$ and is contained in each of the other classes. Hence, $F_0$ is the class of almost contact B-metric manifolds with $\nabla$-parallel structures, i.e. $\nabla \varphi = \nabla \xi = \nabla \eta = \nabla g = \nabla \tilde{g} = 0$.

Let $g_{ij}$, $i, j \in \{1, 2, \ldots, 2n+1\}$, be the components of the matrix of $g$ with respect to a basis $\{e_i\}_{i=1}^{2n+1} = \{e_1, e_2, \ldots, e_{2n+1}\}$ of $T_pM$ at an arbitrary point $p \in M$, and $g^{ij} - \text{the components of the inverse matrix of } (g_{ij})$. The Lee forms associated with $F$ are defined as follows:

$$\theta(z) = g^{ij}F(e_i, e_j, z), \quad \theta^*(z) = g^{ij}F(e_i, \varphi e_j, z), \quad \omega(z) = F(\xi, \xi, z).$$

In [13], the square norm of $\nabla \varphi$ is introduced by:

$$(1.1) \quad \|\nabla \varphi\|^2 = g^{ij}g^{ks}((\nabla_{e_i} \varphi) e_k, (\nabla_{e_j} \varphi) e_s).$$

If $(M, \varphi, \xi, \eta, g)$ is an $F_0$-manifold then the square norm of $\nabla \varphi$ is zero, but the inverse implication is not always true. An almost contact B-metric manifold satisfying the condition $\|\nabla \varphi\|^2 = 0$ is called an isotropic-$F_0$-manifold. The square norms of $\nabla \eta$ and $\nabla \xi$ are defined in [14] by:

$$\|\nabla \eta\|^2 = g^{ij}g^{ks}((\nabla_{e_i} \eta) e_k, (\nabla_{e_j} \eta) e_s), \quad \|\nabla \xi\|^2 = g^{ij}g^{ks}((\nabla_{e_i} \xi) e_k, (\nabla_{e_j} \xi) e_s).$$

Let $R$ be the curvature tensor of type $(1,3)$ of Levi-Civita connection $\nabla$, i.e. $R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z$. The corresponding tensor of $R$ of type $(0,4)$ is defined by $R(x, y, z, w) = g(R(x, y)z, w)$.

The Ricci tensor $\rho$ and the scalar curvature $\tau$ for $R$ as well as their associated quantities are defined by the following traces $\rho(x, y) = g^{ij}R(e_i, x, y, e_j)$, $\tau = g^{ij}\rho(e_i, e_j)$, $\rho^*(x, y) = g^{ij}R(e_i, x, y, \varphi e_j)$ and $\tau^* = g^{ij}\rho^*(e_i, e_j)$, respectively.

An almost contact B-metric manifold is called Einstein if the Ricci tensor is proportional to the metric tensor, i.e. $\rho = \lambda g$, $\lambda \in \mathbb{R}$.

Let $\alpha$ be a non-degenerate 2-plane (section) in $T_pM$. It is known from [21] that the special 2-planes with respect to the almost contact B-metric structure are: a totally real section if $\alpha$ is orthogonal to its $\varphi$-image $\varphi \alpha$ and $\xi$, a $\varphi$-holomorphic section if $\alpha$ coincides with $\varphi \alpha$ and a $\xi$-section if $\xi$ lies on $\alpha$. 
The sectional curvature $k(\alpha;p)(R)$ of $\alpha$ with an arbitrary basis $\{x,y\}$ at $p$ regarding $R$ is defined by

$$k(\alpha;p)(R) = \frac{R(x,y,y,x)}{g(x,x)g(y,y) - g(x,y)^2}.$$  

It is known from [13] that a linear connection $D$ is called a natural connection on an arbitrary manifold $(M, \varphi, \xi, \eta, g)$ if the almost contact structure $((\varphi, \xi, \eta, g)$ and the B-metric $g$ (consequently also $\tilde{g}$) are parallel with respect to $D$, i.e. $D\varphi = D\xi = \tilde{D}\eta = \tilde{D}g = \tilde{D}\tilde{g} = 0$. In [20], it is proved that a linear connection $D$ is natural on $(M, \varphi, \xi, \eta, g)$ if and only if $D\varphi = Dg = 0$. A natural connection exists on any almost contact B-metric manifold and coincides with the Levi-Civita connection only the manifold belongs to $\mathcal{F}_0$.

Let $T$ be the torsion tensor of $D$, i.e. $T(x, y) = D_x y - D_y x - [x, y]$. The corresponding tensor of $T$ of type (0,3) is denoted by the same letter and is defined by the condition $T(x, y, z) = g(T(x, y), z)$.

In [17], it is introduced a natural connection $\tilde{D}$ on $(M, \varphi, \xi, \eta, g)$ in all basic classes by

$$\tilde{D}x = \nabla_x y + \frac{1}{2} \{ (\nabla_x \varphi) y + (\nabla_x \eta) y \cdot \xi \} - \eta(y) \nabla_x \xi.$$

This connection is called a $\varphi$B-connection in [18]. It is studied for the main classes $\mathcal{F}_1, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_11$ in [17, 11, 12]. Let us note that the $\varphi$B-connection is the odd-dimensional analogue of the B-connection on the almost complex manifold with Norden metric, studied for the class $W_1$ in [4].

In [19], a natural connection $\tilde{D}$ is called a $\varphi$-canonical connection on $(M, \varphi, \xi, \eta, g)$ if its torsion tensor $\tilde{T}$ satisfies the following identity:

$$\tilde{T}(x, y, z) = \tilde{T}(x, z, y) - \tilde{T}(x, \varphi y, \varphi z) + \tilde{T}(x, \varphi z, \varphi y) =$$

$$= \eta(x) \left\{ \tilde{T}(\xi, y, z) - \tilde{T}(\xi, z, y) - \tilde{T}(\xi, \varphi y, \varphi z) + \tilde{T}(\xi, \varphi z, \varphi y) \right\}$$

$$+ \eta(y) \left\{ \tilde{T}(x, \xi, z) - \tilde{T}(x, z, \xi) - \eta(x)\tilde{T}(z, \xi, \xi) \right\}$$

$$- \eta(z) \left\{ \tilde{T}(x, \xi, y) - \tilde{T}(x, y, \xi) - \eta(x)\tilde{T}(y, \xi, \xi) \right\}.$$  

It is established that the $\varphi$B-connection and the $\varphi$-canonical connection coincide if and only if $(M, \varphi, \xi, \eta, g)$ is in the class $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_8 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}$.

In [13], it is introduced and studied one more natural connection on $(M, \varphi, \xi, \eta, g)$ called $\varphi$KT-connection, which is defined by the condition its torsion to be a 3-form. There is given that the $\varphi$KT-connection exists only in the class $\mathcal{F}_3 \oplus \mathcal{F}_7$.

In [3] it is determined the class of all 3-dimensional almost contact B-metric manifolds. It is $\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_8 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}$.

2. A FAMILY OF LIE GROUPS AS 3-DIMENSIONAL $(\mathcal{F}_4 \oplus \mathcal{F}_5)$-MANIFOLDS

In this section we study 3-dimensional real connected Lie groups with almost contact B-metric structure. On a 3-dimensional connected Lie group $G$ we take a global basis of left-invariant vector fields $\{e_0, e_1, e_2\}$ on $G$.

We define an almost contact structure on $G$ by

$$\varphi e_0 = a, \quad \varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \xi = e_0;$$

$$\eta(e_0) = 1, \quad \eta(e_1) = \eta(e_2) = 0,$$  

(2.1)
where \( o \) is the zero vector field and define a B-metric on \( G \) by

\[
\begin{align*}
g(e_0, e_0) &= g(e_1, e_1) = -g(e_2, e_2) = 1, \\
g(e_0, e_1) &= g(e_0, e_2) = g(e_1, e_2) = 0.
\end{align*}
\]

We consider the Lie algebra \( \mathfrak{g} \) on \( G \), determined by the following non-zero commutators:

\[
[e_0, e_1] = -be_1 - ae_2, \quad [e_0, e_2] = ae_1 - be_2, \quad [e_1, e_2] = 0,
\]

where \( a, b \in \mathbb{R} \). We verify immediately that the Jacobi identity for \( \mathfrak{g} \) is satisfied. Hence, \( G \) is a 2-parametric family of Lie groups with corresponding Lie algebra \( \mathfrak{g} \).

**Theorem 2.1.** Let \((G, \varphi, \xi, \eta, g)\) be a 3-dimensional connected Lie group with almost contact B-metric structure determined by (2.1), (2.2), and (2.3). Then it belongs to the class \( \mathcal{F}_4 \oplus \mathcal{F}_5 \).

**Proof.** The well-known Koszul equality for the Levi-Civita connection \( \nabla \) of \( g \)

\[
2g(\nabla_{e_i} e_j, e_k) = g(\langle [e_i, e_j], e_k \rangle) + g(\langle [e_k, e_i], e_j \rangle) + g(\langle [e_j, e_k], e_i \rangle)
\]

implies the following form of the components \( F_{ijk} = F(e_i, e_j, e_k) \) of \( F \):

\[
2F_{ijk} = g(\langle [e_i, \varphi e_j] - \varphi [e_i, e_j], e_k \rangle) + g(\varphi [e_k, e_i] - [\varphi e_k, e_i], e_j) + g([e_k, \varphi e_j] - [\varphi e_k, e_j], e_i).
\]

Using (2.5) and (2.3) for the non-zero components \( F_{ijk} \), we get:

\[
\begin{align*}
F_{101} &= F_{110} = -F_{202} = -F_{220} = a, \\
F_{102} &= F_{120} = F_{201} = F_{210} = b.
\end{align*}
\]

Immediately we establish that the components in (2.6) satisfy the condition \( F = F^4 + F^5 \) which means that the manifold belongs to \( \mathcal{F}_4 \oplus \mathcal{F}_5 \). Here, the components \( F^s \) of \( F \) in the basic classes \( \mathcal{F}_s \) \((s = 4, 5)\) have the following form (see [8])

\[
\begin{align*}
F_4(x, y, z) &= \frac{1}{2} \theta_0 \left\{ x^1 (y^0 z^1 + y^1 z^0) - x^2 (y^0 z^2 + y^2 z^0) \right\}, \\
\frac{1}{2} \theta_0' &= F_{101} = F_{110} = -F_{202} = -F_{220}; \\
F_5(x, y, z) &= \frac{1}{2} \theta_0'' \left\{ x^1 (y^0 z^2 + y^2 z^0) + x^2 (y^0 z^1 + y^1 z^0) \right\}, \\
\frac{1}{2} \theta_0'' &= F_{102} = F_{120} = F_{201} = F_{210}.
\end{align*}
\]

where \( \theta_0 = \theta(e_0) \) and \( \theta_0'' = \theta^*(e_0) \) are determined by \( \theta_0 = 2a, \theta_0'' = 2b \). Therefore, the induce 3-dimensional manifold \((G, \varphi, \xi, \eta, g)\) belongs to the class \( \mathcal{F}_4 \oplus \mathcal{F}_5 \) from the mentioned classification. It is an \( \mathcal{F}_0 \)-manifold if and only if \((a, b) = (0, 0)\) holds.

Obviously, \((G, \varphi, \xi, \eta, g)\) belongs to \( \mathcal{F}_4 \), \( \mathcal{F}_5 \) and \( \mathcal{F}_0 \) if and only if the parameters \( \theta_0, \theta_0'' \) and \( \theta_0 = \theta_0'' \) vanish, respectively.

According to the above, the commutators in (2.3) take the form

\[
[e_0, e_1] = -\frac{1}{2}(\theta_0' e_1 + \theta_0 e_2), \quad [e_0, e_2] = \frac{1}{2}(\theta_0 e_1 - \theta_0'' e_2), \quad [e_1, e_2] = 0,
\]

in terms of the basic components of the Lee forms \( \theta \) and \( \theta^* \).

According to Theorem 2.1 and the consideration in [9], we can remark that the Lie algebra determined as above belongs to the type \( \text{Bia}(VII_h) \), \( h > 0 \) of the Bianchi classification (e.g. [1] [2]).
Using (2.4) and (2.3), we obtain the components of $\nabla$:

$$
\begin{align*}
\nabla_{e_1}e_0 & = be_1 + ae_2, \\
\nabla_{e_1}e_1 & = -be_0, \\
\nabla_{e_1}e_2 & = ae_0,
\end{align*}
$$

(2.9)

The rest of non-zero components of $R$, $\rho$ and $\rho^*$ are determined by (2.10) and the properties $R_{i\bar{k}l} = R_{i\bar{j}k}, R_{i\bar{j}k} = -R_{i\bar{j}kl}, \rho_{i\bar{j}k} = \rho_{kij}$.

Taking into account (1.3), (2.2), (2.3) and (2.9), we have

$$
\|\nabla\varphi\|^2 = -2 \|\nabla \eta\|^2 = -2 \|\nabla \xi\|^2 = \theta_0^2 - \theta_0^2.
$$

(2.11)

Proposition 2.2. The following characteristics are valid for $(G, \varphi, \xi, \eta, g)$:

1. The $\varphi$-B-connection $\hat{D}$ (respectively, $\varphi$-canonical connection $\hat{D}$) is zero in the basis $\{e_0, e_1, e_2\}$.
2. The manifold is an isotropic-$F_0$-manifold if and only if the condition $\theta_0 = \pm \theta_0^0$ is valid.
3. The manifold is flat if and only if it belongs to $F_0$.
4. The manifold is Ricci-flat (respectively, $\ast$-Ricci-flat) if and only if it is flat.
5. The manifold is scalar flat if and only if the condition $\theta_0 = \pm \sqrt{3} \theta_0^0$ holds.
6. The manifold is $\ast$-scalar flat if and only if it belongs to either $F_4$ or $F_5$.

Proof. Using (1.3), (2.1) and (2.9), we get immediately the assertion (1). Equation (2.11) imply the assertion (2). The assertions (3), (4) and (6) holds, according to (2.10). On the 3-dimensional almost contact B-metric manifold with the basis $\{e_0, e_1, e_2\}$, bearing in mind the definitions of the Ricci tensor $\rho$ and the $\rho^*$, we have

$$
\rho_{i\bar{j}k} = R_{i\bar{j}k0} + R_{i\bar{j}k1} - R_{2\bar{j}k2} \\
\rho_{i\bar{j}k}^* = R_{kij2} + R_{2\bar{j}k1}.
$$

By virtue of the latter equalities, we get the assertion (1).

According to (2.6), (2.10) and (2.11), we establish the truthfulness of the following

Proposition 2.3. The following properties are equivalent for the studied manifold $(G, \varphi, \xi, \eta, g)$:

1. it belongs to $F_5$;
2. it is Einstein;
3. it is a hyperbolic space form with $k = -\frac{1}{4} \theta_0^2$;
4. it is $\ast$-scalar flat;
5. the Lee form $\theta$ vanishes.
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